SCATTERING MATRICES AND GENERALIZED FOURIER TRANSFORMS IN LONG-RANGE $N$-BODY PROBLEMS

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ABSTRACT. We give a definition of scattering matrices based on the asymptotic behaviors of generalized eigenfunctions and show that these scattering matrices are equivalent to the ones defined by wave-operator approach in long-range $N$-body problems. We also define generalized Fourier transforms by the asymptotic behaviors of outgoing solutions to nonhomogeneous equations and show that they are equivalent to the definition using wave operators. We also prove that the adjoint operators of the generalized Fourier transforms are given by Poisson operators.

1. INTRODUCTION

Scattering matrices play an important role in the study of long-time asymptotic behaviors of the solutions to Schrödinger equations. Scattering matrices are defined in two different ways. In the time dependent viewpoint, the scattering matrices are defined using wave operators and the Fourier transforms. On the other hand, in the stationary viewpoint, they are defined using the asymptotic behaviors of generalized eigenfunctions at infinity. In this paper we prove that both the definitions are equivalent in long-range $N$-body problems. We also give a definition of the generalized Fourier transforms using the asymptotic behaviors of outgoing solutions to nonhomogeneous equations. We prove that they are equivalent to the ones using wave operators, and that the adjoint operators are given by Poisson operators.

Before we consider the $N$-body problems, it is instructive to recall the results for two-body problems, that is, the cases with decaying potentials. Let the potential $V(x) \in C^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$ satisfy

$$|\partial^\gamma V(x)| = O(|x|^{-\mu-|\gamma|}),$$

for $\mu > 0$ as $|x| \to \infty$. For short-range potentials, that is, when $\mu > 1$, as time $t$ tends to $\pm \infty$ the asymptotic behaviors of the solutions $e^{-it\tilde{H}} \psi$, $\tilde{H} := -\Delta + V$, $\psi \in L^2(\mathbb{R}^n)$ to the Schrödinger equations are given by the free evolution $e^{it\Delta} \psi_\pm$, $\psi_\pm \in L^2(\mathbb{R}^n)$, where $\Delta$ is the Laplacian on $\mathbb{R}^n$. In other words,

$$\|e^{-it\tilde{H}} \psi - e^{it\Delta} \psi_\pm\| \to 0,$$

as $t \to \pm \infty$. On the contrary, for any $\psi_\pm \in L^2(\mathbb{R}^n)$ there exists $\psi \in L^2(\mathbb{R}^n)$ such that

$$\|e^{it\Delta} \psi_\pm - e^{-it\tilde{H}} \psi\| \to 0,$$

as $t \to \pm \infty$.

The wave operators $W_\pm : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are defined by $W_\pm \psi_\pm := \psi$. The wave operators $W_\pm$ are partial isometries from $L^2(\mathbb{R}^n)$ to $\mathcal{H}_{ac}(\tilde{H})$, where $\mathcal{H}_{ac}(\tilde{H})$ is the absolutely continuous subspace of $\tilde{H}$.
The scattering operator $S$ is defined as a map $S\psi_- := \psi_+$. Let $F_0$ be the Fourier transform. Then, $\hat{S} := F_0SF_0^*$ commutes with any bounded Borel functions of $|\xi|^2$, and therefore, there exist $\hat{S}(\lambda) \in \mathcal{L}(L^2(\mathbb{S}^{n-1}), L^2(\mathbb{S}^{n-1}))$, a.e. $\lambda > 0$ such that

$$(\hat{S}f)(\lambda, \omega) = (\hat{S}(\lambda)f(\lambda))(\omega), \quad \xi = \sqrt{\lambda}\omega, \quad \omega \in \mathbb{S}^{n-1},$$

a.e. $\lambda > 0$ for any $f(\xi) \in L^2(\mathbb{R}^n)$ (see e.g. Reed-Simon [16]). Here $(f(\lambda))(\omega) := f(\lambda, \omega)$. The operators $\hat{S}(\lambda)$ are called scattering matrices. Thus the scattering matrices give the correspondence between the data as $t \to \pm \infty$.

On the other hand, there is another definition of the scattering matrix which is known to be equivalent to the definition as above. The another definition is based on the asymptotic behaviors of the generalized eigenfunctions at infinity (see Melrose [15] and Yafaev [18]). The generalized eigenfunctions are solutions to the Hamilton-Jacobi equation

$$H\psi(x) = -\frac{1}{2}\Delta \psi(x) + V(x)\psi(x),$$

with $|\psi| \to \infty$ as $|x| \to \infty$. The operators $\hat{S}(\lambda)$ are called scattering matrices. Thus the scattering matrices give the correspondence between the data as $t \to \pm \infty$.

The scattering matrices $\hat{S}(\lambda)$ and $\Sigma(\lambda)$ are equivalent in the sense that the following equation holds (see e.g. Reed-Simon [16] and Melrose [15]).

$$(1.2) \quad \hat{S}(\lambda) = i^{n-1}\Sigma(\lambda).$$

The scattering operator $\Sigma(\lambda)$ is defined as a map $\Sigma(\lambda) : a_- \mapsto a_+.$

In the case of long-range two-body problems, that is, when $1 \geq \mu > 0$, we need to modify both the definitions of $\hat{S}(\lambda)$ and $\Sigma(\lambda)$. In the case of long-range potentials the free evolution $e^{it\Delta}\psi_\pm, \psi_\pm \in L^2(\mathbb{R}^n)$ is replaced by $e^{-iS_\pm(D,t)}\psi_\pm$, where $S_\pm(\xi, t)$ are solutions to the Hamilton-Jacobi equation

$$\frac{\partial S_\pm}{\partial t}(\xi, t) = |\xi|^2 + V(\nabla_\xi S_\pm(\xi, t)),$$

and

$$e^{-iS_\pm(D,t)}\psi := F_0^*(e^{-iS_\pm(\xi, t)}(\hat{F}_0\psi))(\xi)).$$

On the other hand $e^{\pm i|\nabla|x|}$ in (1.2) is replace by $e^{\pm iK(x, \lambda)}$, where $K(x, \lambda)$ is a suitably chosen solution to the eikonal equation

$$|\nabla K(x, \lambda)|^2 + V(x) = \lambda.$$

Then, we have the relation (1.2) even in the long-range case (see Gâtel-Yafaev [4]).

We now turn to the $N$-body problems. We consider the generalized $N$-body Schrödinger operators. The $N$-body Schrödinger operator is a special case of the generalized $N$-body Schrödinger operators.

Set $X := \mathbb{R}^n, \quad n \in \mathbb{N}$, and let $\{X_a \subseteq X : a \in \mathcal{A}\}$ be a finite family of linear subspaces of $X$ which is closed under intersections, and includes $X$ and $X_{\text{max}} := \{0\}$. We endow $\mathcal{A}$ with a semi-lattice structure by

$$a \leq b \text{ if } X_a \supseteq X_b.$$

We denote the orthogonal complement of $X_a$ by $X^a$. We denote by $\Pi^a$ and $\Pi_a$ the orthogonal projections of $X$ onto $X^a$ and $X_a$ respectively. We use the same...
notations $\Pi^a$ and $\Pi_a$ for the corresponding orthogonal projections of the dual space of $X$. We define for all $x \in X$, $x_a = \Pi_a x$ and $x^a = \Pi^a x$. If $a \leq b$, we define

\begin{equation}
    x^b_a := \Pi^a x^b.
\end{equation}

We also define $\nabla_a = \Pi_a \nabla$ and $\nabla^a = \Pi^a \nabla$. The operators $-\Delta_a$ and $-\Delta^a$ denote the Laplacian in $X_a$ and $X^a$ respectively.

We define $C_a := X_a \cap S^{n_a - 1}$. Thus $C_a$ is a sphere of dimension $n_a - 1$. We also define the singular part of $C_a$ by

\begin{equation}
    C_{a,\text{sing}} := \bigcup_{b \preceq a} (C_b \cap C_a).
\end{equation}

$C_{a,\text{sing}}$ corresponds to the directions in which the particles collide. The regular part $C'_a$ is the complement of $C_{a,\text{sing}}$:

\begin{equation}
    C'_a := C_a \setminus C_{a,\text{sing}}.
\end{equation}

A generalized $N$-body Schrödinger operator is an operator of the form

\begin{equation}
    H := -\Delta + \sum_{a \in A} V_a(x^a),
\end{equation}

where $\Delta$ is the Laplacian in $X$ and $V_a$ are real-valued functions on $X^a$ satisfying the following condition. There exists $\mu > 0$ such that for any $a \in A$, $V_a(x^a) = V^a_s(x^a) + V^a_l(x^a)$, where

1. $V^a_s(x^a)$ is compactly supported and $V^a_s(-\Delta^a + 1)^{-1}$ is compact.
2. $V^a_l(x^a) \in C^\infty(X^a)$ and for any $\gamma \in \mathbb{N}^{n_a}$

\begin{equation}
    |\partial^\gamma V_a(x^a)| = (|x^a|^{-\mu} - |\gamma|),
\end{equation}

where $n_a := \dim X_a$.

Then, $H$ is a self-adjoint operator on $L^2(X)$.

We also define the operators $H^a$ as

\begin{equation}
    H^a := -\Delta^a + \sum_{b \leq a} V_b(x^b).
\end{equation}

The set of thresholds of a subsystem $a \in A$ is defined as

\begin{equation}
    T^a := \bigcup_{b < a} \sigma_{pp}(H^b),
\end{equation}

where $\sigma_{pp}(A)$ is the set of eigenvalues of $A$, and $b < a$ means $b \leq a$ and $b \neq a$. We also set

\begin{equation}
    T(H) := T^{a_{\text{max}}}.
\end{equation}

We label the eigenvalues of $H^a$ counted with multiplicities, by integers $m$, and we call the pairs $\alpha = (a, m)$ channels. We denote the eigenvalue of the channel $\alpha$ and the corresponding normalize eigenfunction by $E_\alpha$ and $u_\alpha$ respectively. We say that a channel $\alpha$ is a non-threshold channel if $E_\alpha \notin T^a$. When $\alpha$ is a non-threshold channel, the eigenfunction $u_\alpha$ is a Schwartz function (see Froese-Herbst [2]).

When $\mu > 1$ and $\alpha$ is a non-threshold channel, the following strong limit exists:

\begin{equation}
    W^\pm_\alpha := s - \lim_{\epsilon \to \pm \infty} e^{itH} J_\alpha e^{-it(\Delta_a + E_\alpha)} (J_\alpha v)(x),
\end{equation}

where $(J_\alpha v)(x) := u_\alpha(x^a)v(x_a)$. 
When \( \alpha \) and \( \beta \) are non-threshold channels corresponding to \( a \in \mathcal{A} \) and \( b \in \mathcal{A} \) respectively, the scattering operator \( S_{\beta \alpha} \) is defined as
\[
S_{\beta \alpha} := (W_{\beta}^+) W_{\alpha}^-.
\]

We also define the Fourier transform \( F_{\alpha} \) as
\[
F_{\alpha} : L^2(X_\alpha) \to L^2((E_\alpha, \infty); L^2(C_\alpha)),
\]
by
\[
(1.7) \quad F_{\alpha} u(\lambda, \omega) := (2\pi)^{-n/2} 2^{-1/2} \chi_{(n_\beta - 2)/2} \int e^{-i\lambda \omega} x u(x) dx, \quad \omega \in C_\alpha
\]
where \( \chi_{\beta} \) := \lambda - E_\alpha \cdot \hat{S}_{\beta \alpha} := F_{\beta} S_{\beta \alpha} F_{\alpha} \) is decomposable, namely for a.e. \( \lambda > \max\{E_\beta, E_\beta\} \) there exist bounded operators \( \hat{S}_{\beta \alpha}(\lambda) \in \mathcal{L}(L^2(C_\alpha); L^2(C_\beta)) \) such that
\[
(\hat{S}_{\beta \alpha} f)(\lambda, \hat{x}_b) = (\hat{S}_{\beta \alpha}(\lambda) f)(\hat{x}_b), \quad \hat{x}_b := x_b/|x_b| \in C_\beta,
\]
for any \( f \in L^2((E_\alpha, \infty); L^2(C_\alpha)) \) (see e.g. [16]).

The other definition comes from the asymptotic behaviors of generalized eigenfunctions even in the \( N \)-body problems.

Let \( \beta \) be a non-threshold channel. When \( \mu > 1 \) and \( V^\alpha_a = 0 \) for any \( a \in \mathcal{A} \), Vasy [19] proved the following. For \( \lambda \in (E_\alpha, \infty) \setminus T(H) \) and \( g \in C_0^\infty(C'_\alpha) \), for some \( s > 1/2 \) there exists a unique generalized eigenfunction \( u \in H^{\infty, -s}(X) \) of \( H \), and \( u \) has the form
\[
(1.8) \quad u = u_\alpha(x^a) u_\beta(x^b) e^{-i\sqrt{\lambda r} r_a} r_b^{-(n_\beta - 1)/2} + R(\lambda + i0) f,
\]
where \( r_a := |x_a|, \quad v_\beta \in C_\infty(X_\beta), \lim_{r_a \to 0} v_\beta(r_a, \hat{x}_a) = g(\hat{x}_a), R(z) := (H - z)^{-1} \) and \( H^{\infty, -s}(X) \) is a weighted Sobolev space. Vasy [19] defined the Poisson operator in order that the following holds: \( P_{\alpha, +}(\lambda) g = u \).

Let \( \beta \) be a non-threshold channel and set
\[
(\pi_\beta u)(x_b) := \int_{X_\beta} u(x^b, x_b) \tilde{u}_\beta(x^b) dx^b.
\]
Vasy [19] also proved that for a generalized eigenfunction \( u \in H^{\infty, -s}(X), \ s > 1/2 \) of \( H \) and for \( \lambda > E_\beta, \pi_\beta u \) has the following distributional asymptotic behavior:
\[
\int_{C_b} (\pi_\beta u)(r_b \omega_b) h(\omega_b) d\omega_b
\]
\[
= r_b^{-(n_\beta - 1)/2} (e^{-i\sqrt{\lambda r} r_a} Q_{\beta, -}(\pi_\beta u, h) + e^{i\sqrt{\lambda r} r_a} Q_{\beta, +}(\pi_\beta u, h) + r_b^{-i} v) \quad \text{for } h \in C_0^\infty(C'_b), \ \delta > 0, \ v \in C_\infty((1, \infty)) \cap L^\infty((1, \infty)) \text{ and } Q_{\beta, \pm}(\pi_\beta u, \cdot)
\]
define a distribution on \( C_\infty(C'_b) \).

Vasy [19] defined the scattering matrix \( \Sigma_{\beta \alpha}(\lambda) \) as
\[
\Sigma_{\beta \alpha}(\lambda)(g) := Q_{\beta, +}(\pi_\beta(P_{\alpha, +}(\lambda) g), h).
\]
Vasy [19] also proved the following: if \( \alpha \) and \( \beta \) are non-threshold channels, then for \( \lambda > \max\{E_\alpha, E_\beta\}, \ \lambda \notin T(H) \),
\[
\hat{S}_{\beta \alpha}(\lambda) = C_{\beta \alpha}(\lambda) \Sigma_{\beta \alpha}(\lambda) R, \ C_{\beta \alpha}(\lambda) := e^{i\pi(n_\alpha + n_\beta - 2)/4} (\lambda_\alpha/\lambda_\beta)^{1/2},
\]
where \( R g(\hat{x}_a) := g(-\hat{x}_a) \) for \( g \in C_0^\infty(C'_\alpha) \).

Isozaki [10] and Hassell [5] proved similar results for 2-cluster to 3-cluster scattering matrices in three-body problems and for the free channel scattering matrices respectively using different methods.
In this paper we generalize (1.9) for long-range potentials. More precisely, we obtain a result similar to (1.9) for $1 \geq \mu > 1/2$ where, $\mu$ is as in (1.5). In the long-range case we use the solutions to Hamilton-Jacobi equations in the definition of wave operators, and use the solutions to eikonal equations in the asymptotic behaviors of generalized eigenfunctions of $H$. These solutions are related by the Legendre transform.

Our definition of the Poisson operator $P_{\alpha,+}(\lambda)$ is similar to the one in Vasy [19]. However, we obtain the asymptotic behaviors of $u := P_{\alpha,+}(\lambda)g, g \in C_{0}^{\infty}(C_{a}^{\alpha})$ in a way different from Vasy [19]. The reason for that is as follows. For short-range potentials the asymptotic behavior of the part corresponding to a non-threshold channel $\alpha$ of a generalized eigenfunction is expected to be as

$$u_{\alpha}(x^{a})\Psi_{\alpha}(x_{a})e^{\pm i\sqrt{\lambda_{\alpha}r_{a}}(n_{a}-1)/2},$$

where $\Psi_{\alpha} \in L^{2}(C_{a})$. However, for long-range potentials the factor $e^{\pm i\sqrt{\lambda_{\alpha}r_{a}}}$ in the asymptotic behavior as above is replaced by $e^{\pm iK_{\alpha}(x_{a},\lambda_{a})}$, where $K_{\alpha}(x_{a},\lambda_{a})$ is a solution to an eikonal equation. Since $K_{\alpha}(x_{a},\lambda_{a})$ depends not only on $r_{a}$ but also on $x_{a}$ unlike $\sqrt{\lambda_{a}}r_{a}$, we can not reduce the study of the asymptotic behavior of $u$ to an ordinary differential equation for $r_{a}$ as in Vasy [19]. Instead, we define a distribution $Q_{\alpha}^{\pm}(u)$ on $C_{0}^{\infty}(C_{a}^{\alpha})$ as follows:

$$(Q_{\alpha}^{\pm}(u))(h) := \lim_{\rho \to +\infty} \rho^{-1} \int_{|x_{a}|<\rho} e^{\mp iK_{\alpha}(x_{a},\lambda_{a})}(n_{a}-1)/2 h(x_{a})(\pi_{\alpha}u)(x_{a})dx_{a},$$

for any $h \in C_{0}^{\infty}(C_{a}^{\alpha})$. The existence of the limit in (1.10) is not obvious. We prove the existence of the limit in (1.10) and show that $Q_{\alpha}^{\pm}(u) \in L^{2}(C_{a})$.

Using $Q_{\beta}^{+}$ we define the scattering matrix as

$$\Sigma_{\beta\alpha}(\lambda) := Q_{\beta}^{+}(P_{\alpha,+}(\lambda)g).$$

Then, we obtain the relation (1.9) in which the definition of $S_{\beta\alpha}(\lambda)$ is also modified.

We also give a definition of the generalized Fourier transforms, and show that they are equivalent to the one defined by the wave operator approach. We also prove that the adjoint operators of generalized Fourier transforms are given by the Poisson operators $P_{\alpha,\pm}(\lambda)$.

The content of this paper is as follows. In section 2 we give some preliminaries and the main results. In section 3 we introduce the generalized Fourier transforms for decaying potentials. In section 4 we introduce the wave operators and the scattering matrices for decaying potentials, and give the relation between the scattering matrices and the adjoint operators of the generalized Fourier transforms. In section 5 we define the Poisson operators for $N$-body Schrödinger operators. In section 6 we study the asymptotic behaviors of the generalized eigenfunctions and the solutions to nonhomogeneous equations for decaying potentials. In section 7 we introduce the outgoing and incoming properties and the uniqueness theorem for nonhomogeneous equations. In section 8 we define the scattering matrices and the generalized Fourier transforms for $N$-body Schrödinger operators using the asymptotic behaviors of the generalized eigenfunctions, and outgoing or incoming solutions to nonhomogeneous equations respectively. In section 9 we study the asymptotic behaviors of the functions in the range of the resolvent and the Poisson operators. In section 10 we prove the equivalence of the two definitions of the scattering matrices. In section 11 we
prove the equivalence of the two definitions of the generalized Fourier transforms and show that the adjoint operators of the generalized Fourier transforms are given by the Poisson operators.

2. SOME PRELIMINARIES AND THE MAIN RESULTS

In this section we use the notations in section 1. We define

$$X^\varepsilon_a := \{x \in X : \text{dist}(x, X_a) < \varepsilon\},$$

$$Z_a := X_a \setminus \bigcup_{b \not\in A} X_b,$$

$$Y_a := X \setminus \bigcup_{b \not\in A} X_b,$$

$$Z_a^\varepsilon := X_a \setminus \bigcup_{b \not\in A} X^\varepsilon_b,$$

$$Y_a^\varepsilon := X \setminus \bigcup_{b \not\in A} X^\varepsilon_b.$$

The directions in which the clusters of a collide are removed in $Z_a$.

We assume the potentials $V_a$ obey the following.

Assumption 2.1. There exists $1 \geq \mu > 1/2$ such that for any $a \in A$, $V_a(x^a) = V^s_a(x^a) + V^l_a(x^a)$, where

1. $V^s_a(x^a)$ is compactly supported and $V^s_a(-\Delta^a + 1)^{-1}$ is compact.
2. $V^l_a(x^a) \in C^\infty(X^a)$ and for any $\gamma \in \mathbb{N}^{n_a}$
   $$|\partial^\gamma V_a(x^a)| = O(|x^a|^{-\mu - |\gamma|}).$$

Let $H$ and $H^a$ be as in (1.4) and (1.6) respectively.

Set $I_a(x) := \sum_{b \not\in A} V_b(x^b)$, $I^l_a(x) := \sum_{b \not\in A} V^l_b(x^b)$, and let $\chi_0 \in C^\infty(\mathbb{R})$ be a function which is 0 for $x < 1$ and 1 for $x > 2$. We define the modified potentials as follows:

$$\tilde{I}_a(x_a) := I^l_a(x_a) \prod_{a < b} \chi_0 \left(\frac{\langle x_b \rangle^3}{\langle x_a \rangle^3} \log (\langle x_a \rangle)\right) \chi_0 \left(\frac{|x_a|}{K}\right),$$

where $K > 0$ is a constant and $x_a$ is defined by (1.3). Let $\mu_0$ be a constant such that $\mu_0 \in (0, \mu)$. Then if we choose $K$ large enough, we have

$$\partial^\gamma_{x_a} \tilde{I}_a(x_a) = O \left(|x_a|^{-\mu_0 - |\gamma|}\right),$$

as $|x_a| \to \infty$ for any $\gamma \in \mathbb{N}^{n_a}$.

In section 2.4 we show that there exists $S_{a,\pm}(\xi_a, t) \in C^\infty(X_a \setminus \{0\} \times \mathbb{R}_\pm)$ satisfying the following: For any compact set $A \in X_a \setminus \{0\}$ there exists $T > 0$ such that

$$\frac{\partial S_{a,\pm}}{\partial t}(\xi_a, t) = |\xi_a|^2 + \tilde{I}_a(\nabla \xi_a S_{a,\pm}(\xi_a, t)).$$

The time-dependent definition of the scattering matrices is based on the wave operators. If $\alpha$ is a non-threshold channel, then there exist wave operators

$$W_{\alpha}^{\pm} := s - \lim_{t \to \pm \infty} e^{itH} J_{\alpha} e^{-i(S_{a,\pm}(D_a t) + E_a t)},$$

where $e^{-i(S_{a,\pm}(D_a t) + E_a t)} f := F_{\alpha}^s(e^{-i(S_{a,\pm}(\xi_a t) + E_a t)}(F_{\alpha} f)(|\xi_a|^2, \xi_a)), \xi_a := \xi_a / |\xi_a|.$
The scattering operator is defined as

\[ S_{\beta\alpha} := (W_\beta^+)^* W_\alpha^- . \]

Then \( \hat{S}_{\beta\alpha} := F_\beta S_{\beta\alpha} F_\alpha^* \) is decomposable, namely for a.e. \( \lambda > \max\{E_\alpha, E_\beta\} \) there exist bounded operators \( \hat{S}_{\beta\alpha}(\lambda) \in L(L^2(C_a); L^2(C_b)) \) such that

\[ (\hat{S}_{\beta\alpha}(\lambda) f)(\hat{x}_b) = (\hat{S}_{\beta\alpha}(\lambda) f(\lambda))(\hat{x}_b). \]

The stationary definition of the scattering matrices comes from the asymptotic behaviors of generalized eigenfunctions. Let \( L^{2,l}(\mathbb{R}^n) \) denote the Hilbert space of all measurable functions on \( \mathbb{R}^n \) such that

\[ \|f\|^2 = \int (1 + |x|)^{2l} |f(x)|^2 dx < \infty. \]

We also need the spaces \( B(\mathbb{R}^n) \) and \( B^*(\mathbb{R}^n) \) of functions. We set

\[ \rho_j := 2^j, \quad (j \in \mathbb{N}, j \geq 0) \]

\[ \Omega_j := \{ x \in \mathbb{R}^n : |x| < 1 \}, \quad \Omega_j := \{ x \in \mathbb{R}^n : 2^{j-1} < |x| < 2^j \}, \quad (j \in \mathbb{N}, j \geq 1). \]

Let \( B(\mathbb{R}^n) \) be the set of functions \( u \) such that

\[ \|u\|_{B(\mathbb{R}^n)} := \sum_{j=0}^{\infty} \rho_j^{1/2} \|u\|_{L^2(\Omega_j)} < \infty. \]

Then the dual space \( B^*(\mathbb{R}^n) \) of \( B(\mathbb{R}^n) \) is the set of functions \( u \) such that

\[ \|u\|_{B^*(\mathbb{R}^n)} := \sup_{j \geq 0} \rho_j^{-1/2} \|u\|_{L^2(\Omega_j)} < \infty. \]

Moreover, there exists a constant \( C > 0 \) such that

\[ C^{-1}\|u\|_{B^*(\mathbb{R}^n)} \leq \left( \sup_{\rho > 1} \rho^{-1} \int_{|x| < \rho} |u(x)|^2 dx \right)^{1/2} \leq C\|u\|_{B(\mathbb{R}^n)}. \]

The relation between \( L^{2,l}(\mathbb{R}^n) \), \( B(\mathbb{R}^n) \) and \( B^*(\mathbb{R}^n) \) is as follows: when \( l > 1/2 \)

\[ L^{2,l}(\mathbb{R}^n) \subset B(\mathbb{R}^n) \subset L^{2,1/2}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \]

\[ \subset L^{2,-1/2}(\mathbb{R}^n) \subset B^*(\mathbb{R}^n) \subset L^{2,-l}(\mathbb{R}^n). \]

Since we can assume \( X_a = \mathbb{R}^n \) for some \( n \in \mathbb{N} \), we can define \( L^{2,l}(X_a) \), \( B(X_a) \) and \( B^*(X_a) \) in the same way as above.

Let \( \alpha \) be a non-threshold channel. We show in section \( \S \) that for \( \lambda \in (E_\alpha, \infty) \setminus \mathcal{T}(H) \) and \( g \in C_0^\infty(C_a^0) \) there exists a unique function \( u \in B^*(X) \) such that \( (H - \lambda)u = 0 \), and \( u - e^{-iK_a(x_a, \lambda_a)} r_a^{-(n_a - 1)/2} u_a(x_a^a) g(x_a) \) is outgoing, where \( K_a(x_a, \lambda_a) \) is the solution to the eikonal equation \( |\nabla_a K_a(x_a, \lambda_a)|^2 + \tilde{I}_a(x_a) = \lambda_a \) in section \( \S \) with \( V \) replaced by \( \tilde{I}_a \) (for the definition of outgoing and incoming properties see section \( \S \)). We define the Poisson operator \( P_{\alpha,+}(\lambda) : C_0^\infty(C_a^0) \to B^*(X) \) by \( P_{\alpha,+}(\lambda) g \) := \( u \). There is also a unique function \( u \in B^*(X) \) such that \( (H - \lambda)u = 0 \), and \( u - e^{iK_a(x_a, \lambda_a)} r_a^{-(n_a - 1)/2} u_a(x_a^a) g(x_a) \) is incoming, and we define \( P_{\alpha,-}(\lambda) \) by \( P_{\alpha,-}(\lambda) g \) := \( u \).

Let \( \beta \) be a non-threshold channel. In section \( \S \) we show that for a generalized eigenfunction \( u \in B^*(X) \) and \( h \in C_0^\infty(C_b^0) \) the limit

\[ Q_{\beta}^\pm(u)(h) := \lim_{\rho \to \infty} \rho^{-1} \int_{|x_b| < \rho} h(\hat{x}_b) e^{\pm iK_b(x_b, \lambda_b)} r_b^{-(n_b - 1)/2} (\pi_{\beta}u)(x_b) dx_b, \]
exists. We can see that $Q^\pm_\alpha(u)$ defines a distribution on $C'_b$ and actually $Q^\pm_\alpha(u) \in L^2(C_b)$. We define the scattering matrix $\Sigma_{\beta\alpha}(\lambda) : C^\infty_0(C'_a) \to L^2(C_b)$ by

$$\Sigma_{\beta\alpha}(\lambda)g := Q^\pm_{\beta}(P_{\alpha,+}(\lambda)g).$$

One of our main result is the following

**Theorem 2.2.** Let $\alpha$ and $\beta$ be non-threshold channels. Then,

$$\hat{\Sigma}_{\beta\alpha}(\lambda)g = e^{i\pi(n_+ + n_3)/4} \lambda_\alpha^{1/4} \lambda_\beta^{-1/4} \Sigma_{\beta\alpha}(\lambda)Rg,$$

for $g \in C^\infty_0(C'_a)$ and a.e. $\lambda > \max\{E_\alpha, E_\beta\}$ where $(Rg)(\hat{x}_a) := g(-\hat{x}_a)$.

We also give the stationary definition of the generalized Fourier transforms. For $\lambda \in \sigma_{ess}(H) \setminus (\sigma_{pp}(H) \cup \mathcal{T}(H))$ the resolvent $R(\lambda \pm i\epsilon) := (H - \lambda \mp i\epsilon)^{-1}$, $\epsilon > 0$ is extended to $R(\lambda \pm i0)$ as an operator in $\mathcal{C}(B(X), B^*(X))$, where $\sigma_{ess}(A)$ is the set of essential spectra of $A$ (see [13]). Let $f \in L^{2,l}(X)$, $l > 1/2$, $\alpha$ be a non-threshold channel, and set $u = R(\lambda + i0)f$ or $u = R(\lambda - i0)f$. Then, the limit $Q^\pm_\alpha(u)(h)$ as $(2.3)$ exits. We define the generalized Fourier transforms $G^\pm_\alpha(\lambda) : L^{2,l}(X) \to L^2(C_a)$ by

$$G^+_\alpha(\lambda)f = D^+_\alpha(\lambda)Q^+_\alpha(R(\lambda + i0)f),$$

and

$$G^-_\alpha(\lambda)f = D^-_\alpha(\lambda)RQ^-_\alpha(R(\lambda - i0)f),$$

where $D^\pm_\alpha(\lambda) := e^{\pm i\pi(n_+ - 3)/4} \lambda_\alpha^{-1/4}$.

The generalized Fourier transforms are related to the wave operators and Poisson operators as in the following theorem.

**Theorem 2.3.** Let $\alpha$ be a non-threshold channel. Then, for $f \in L^{2,l}(X)$, $l > 1/2$ and $h \in C^\infty_0(C'_a)$ we have

$$G_\alpha(W^+_\alpha)^*(\lambda) = G^+_\alpha(\lambda)f, \quad G_\alpha(W^-_\alpha)^*(\lambda) = G^-_\alpha(\lambda)f,$$

for a.e. $\lambda > E_\alpha$.

$$(G^+_\alpha(\lambda))^* = -\hat{D}_\alpha^{-}(-\lambda)P_{\alpha,-}(\lambda),$$

$$(G^-_\alpha(\lambda))^* = -\hat{D}_\alpha^{+}(-\lambda)P_{\alpha,+}(\lambda)R,$$

for a.e. $\lambda > E_\alpha$, where $\hat{D}_\alpha(\lambda) := 2^{-1}i\pi^{-1/2}\lambda_\alpha^{-1/4}e^{\pm i\pi(n_+ - 3)/4}$.

3. Generalized Fourier transforms for decaying potentials

In this section we suppose $n \in \mathbb{N}$, and that the potential $V(x)$ is a real valued function such that $V \in C^\infty(\mathbb{R}_n)$, and for some $\mu > 0$,

$$|\partial^\gamma V(x)| = \mathcal{O}(|x|^{-\mu-|\gamma|}) \text{ as } |x| \to +\infty,$$

for any multi-index $\gamma$.

The oscillations in the asymptotic behaviors of the functions $\hat{R}(\lambda \pm i0)f$, $f \in \mathcal{B}(\mathbb{R}_n)$ are given by the solutions to the eikonal equations, where $\hat{R}(z) := (\hat{H} - z)^{-1}$, $\hat{H} := -\Delta + V$. The solutions to the eikonal equations are given by the following lemma.

**Lemma 3.1** ([13] Lemma 2.1, [8] Theorem 4.1). There exists a real valued function $Y(x, \lambda) \in C^\infty(\mathbb{R}_n \times \mathbb{R}_+)$, $\mathbb{R}_+ := (0, \infty)$ satisfying the following properties:
Lemma 3.2 (\cite{4} Theorem 3.5, \cite{8}). For any $f \in B(\mathbb{R}^n)$ and $\lambda > 0$, there exist $a_{\pm} \in L^2(\mathbb{S}^{n-1})$ such that, as $x \to \infty$,

\[(\tilde{R}(\lambda \pm i0)f)(x) = \pi^{1/2} \lambda^{-1/4} a_{\pm}(\pm \hat{x}) w_{\pm}(x, \lambda) + o_{av}(|x|-(n-1)/2),\]

where $\hat{x} := x/|x|$, and $f(x) = o_{av}(|x|-(n-1)/2)$ means that

\[
\lim_{\rho \to +\infty} \rho^{-1} \int_{|x| \leq \rho} |f(x)|^2 dx = 0.
\]

We define the mapping $F_{\pm}(\lambda) : B(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1})$ by $(F_{\pm}(\lambda)f)(\hat{x}) = a_{\pm}(\hat{x})$. It is known that $F_{\pm}(\lambda)$ is bounded from $B(\mathbb{R}^n)$ into $L^2(\mathbb{S}^{n-1})$ (see \cite{4}). Set $\mathcal{H} = L^2(\mathbb{R}^n; L^2(\mathbb{S}^{n-1}))$. For any $f \in B(\mathbb{R}^n)$ we define the mapping $F_{\pm} : B(\mathbb{R}^n) \to \mathcal{H}$ by $(F_{\pm}f)(\lambda) = F_{\pm}(\lambda)f$. Then, $F_{\pm}$ is uniquely extended to the partial isometry with initial set $\mathcal{H}_{ac}(\tilde{H})$ and final set $\hat{\mathcal{H}}$, where $\mathcal{H}_{ac}(\tilde{H})$ is the absolutely continuous subspace $\tilde{H}$ (see \cite{4} Theorem 5.2 and \cite{13} Theorem 2.5).

We have explicit representations for $\mathcal{F}_{\pm}(\lambda) := (F_{\pm}(\lambda))^*$ on $a \in C^\infty(\mathbb{S}^{n-1})$. Let $\eta \in C^\infty(\mathbb{R})$ be a function satisfying the following condition: there exists $\kappa > 0$ such that

\[
\eta(t) = \begin{cases} 
0 & (t < \kappa) \ 
1 & (t > 2\kappa). 
\end{cases}
\]

Then, the functions

\[
u_{\pm}(x, \lambda) = \eta(r)a(\pm \hat{x}) w_{\pm}(x, \lambda),
\]

belong to the space $B^*(\mathbb{R}^n)$. We set $g_{\pm}(\lambda) = (-\Delta + V - \lambda) u_{\pm}(\lambda)$. Then, by straightforward calculations we can see $g_{\pm}(\lambda) \in L^{2,l}(\mathbb{R}^n)$ for some $l > 1/2$.

Lemma 3.3 (\cite{4} Lemma 3.7, Corollary 3.8, \cite{13} Theorem 3.4). Let $a \in C^\infty(\mathbb{S}^{n-1})$, $w_{\pm}$ be defined by (3.3) and $u_{\pm}$, $g_{\pm}$ be as above. Then we have

\[
\pm 2i\pi^{1/2} \lambda^{1/4} \mathcal{F}_{\pm}(\lambda)a = u_{\pm}(\lambda) - \tilde{R}(\lambda \mp i0)g_{\pm}(\lambda).
\]
Moreover, for any \( a \in L^2(S^{n-1}) \) we have \( \mathcal{F}_1^a(\lambda)a \in H^2_{loc}(\mathbb{R}^n) \cap B^s(\mathbb{R}^n) \) and
\[
(-\Delta + V - \lambda)\mathcal{F}_1^a(\lambda)a = 0.
\]

We introduce the following class of symbols of pseudodifferential operators. Let \( S^{m,1}, m, l \in \mathbb{R}, \) be the symbol class of \( C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \)-functions \( p(x, \xi) \) satisfying the following condition: for any \( \alpha, \beta \in \mathbb{N}^n \) there exists \( C_{\alpha,\beta} > 0 \) such that
\[
|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leq C_{\alpha,\beta} \langle x \rangle^{l-|\alpha|} \langle \xi \rangle^m, \forall (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n.
\]
The corresponding pseudodifferential operators are defined by the Weyl quantization:
\[
(\text{Op}(p))\psi(x) = (2\pi)^{-n} \int \int e^{i(x-y)\cdot \xi} p(\xi/2, \xi/2) \psi(y) dyd\xi,
\]
for \( \psi \in \mathcal{S}(\mathbb{R}^n) \). We also define the right quantization of \( p \) by
\[
(\text{Op}^r(p))\psi(x) = (2\pi)^{-n} \int \int e^{i(x-y)\cdot \xi} p(\xi, \xi) \psi(y) dyd\xi,
\]
for \( \psi \in \mathcal{S}(\mathbb{R}^n) \).

We need the following micro-local resolvent estimate.

**Lemma 3.4.** Let \( \lambda > 0, s > 1/2, t > 1 \) and \( p_{\pm} \in S^{0,0} \) be a symbol satisfying the following condition: there exists \( 0 < \epsilon \) such that \( p_{\pm}(x, \xi) = 0 \) if \( \pm \hat{x} \cdot \xi < 1 - \epsilon \), where \( \hat{x} := x/|x| \) and \( \hat{\xi} := \xi/|\xi| \). Then there exists \( C > 0 \) such that
\[
\|\text{Op}(p_{\pm})R(\lambda \pm i0)f\|_{s-t} \leq C\|f\|_s.
\]

This lemma is essentially due to Skibsted [17] (see also Isozaki [9] Theorem 2.2). The estimate for \( R(\lambda + i0) \) is obtained by the argument similar to the one below [9] Theorem 2.2 for the decaying potential \( V \), with \( s \in [9] \) Theorem 2.2 replaced by \( s - t \). The estimate for \( R(\lambda - i0) \) follows from the similar propagation estimate as \( t \to -\infty \). Note that we can assume \( s - t > -1/2 \) since otherwise there exists \( t' > 1 \) such that \( t > t', s - t' > -1/2 \) and \( \langle x \rangle^{s-t} = \langle x \rangle^{-(t'-t)} \langle x \rangle^{s-t'} \).

We also have the micro-local estimate for \( u_{\pm}(\lambda) \).

**Lemma 3.5.** Let \( p_{\pm}(x, \xi) \) be a symbol such that \( p_{\pm}(x, \xi) = 0 \) for \( (x, \xi) \) satisfying one of the following conditions for some \( \epsilon > 0 \)

(i) \( |\xi|^2 - \lambda| < \epsilon \),
(ii) \( \pm \hat{x} \cdot \xi > 1 - \epsilon \) for some \( \epsilon > 0 \),
(iii) \( \xi \in \text{supp} \ a(\pm) \).

Then for any \( m \in \mathbb{N} \) there exists a constant \( C \) such that
\[
\|x|^m \text{Op}(p_{\pm})u_{\pm} \| \leq C,
\]
where \( u_{\pm} \) is defined in [33].

**Proof.** Since a support of a symbol does not change by a choice of quantization except an error in \( S^{-\infty,-\infty} \), we can consider the right quantization \( \text{Op}^r(p_{\pm}) \) instead of the Weyl quantization. We can write
\[
\text{Op}^r(p_{\pm})u_{\pm} = \lim_{\mu \to 0_+} \int \chi_0(\mu \xi) \chi_0(\mu y) e^{i(x-y)\cdot \xi \pm iK(y)} p_{\pm}(y, \xi) a(\pm \hat{y}) \eta(y) |y|^{-(n-1)/2} dyd\xi.
\]
Since there exists a constant $\hat{C} > 0$ such that

$$|\xi \mp \sqrt{\lambda y}| > \hat{C}^{-1}(|\xi| + 1),$$

for $(y, \xi) \in \text{supp} p_z(y, \xi) a(\pm \hat{y})$ and we have

$$i|\xi \mp \sqrt{\lambda y}|^{-2}(\xi \mp \sqrt{\lambda y}) \cdot \nabla_y e^{-iy \lambda \mp \sqrt{\lambda y}|y|} = e^{-iy \lambda \mp \sqrt{\lambda y}|y|},$$

noting that $|\partial_y^n a(\pm \hat{y})| = O(|y|^{-|\alpha|})$ as $|y| \to \infty$, we obtain the result by integration by parts. \hfill \Box

4. Wave operators and scattering matrices for decaying potentials

In this section we suppose $n \in \mathbb{N}$, and \cite{13} for the potential $V$ and we use the notations in section 3. To define the wave operator we need the solutions $S_\pm(\xi, t)$ to the Hamilton-Jacobi equations obtained by the Legendre transformation of $K(x, \lambda)$. As in \cite{13} Lemma 6.1 we have

**Lemma 4.1** (\cite{13} Lemma 6.1). There exist $x_\pm(\xi, t), \lambda_\pm(\xi, t) \in C^\infty((\mathbb{R}^n \setminus \{0\} \times \mathbb{R}_\pm)$ satisfying the following condition: For any compact set $\Lambda \subset \mathbb{R}^n \setminus \{0\}$ there exist positive constants $T, C$ such that for $\xi \in \Lambda$ and $\pm \sigma > T$ we have

$$\xi = \pm \frac{\partial K}{\partial x}(x_\pm(\xi, t), \lambda_\pm(\xi, t)), \quad t = \pm \frac{\partial K}{\partial \lambda}(x_\pm(\xi, t), \lambda_\pm(\xi, t)),$$

$$|x_\pm(\xi, t) - 2\xi| \leq C(1 + |t|)^{-\mu}, \quad |\lambda_\pm(\xi, t) - |\xi|^2| \leq C(1 + |t|)^{-\mu}.$$

**Remark 4.2.** Although only $x_+(\xi, t)$ and $\lambda_+(\xi, t)$ are considered in \cite{13}, the result for $x_-(\xi, t)$ and $\lambda_-(\xi, t)$ is obtained in the same way.

**Lemma 4.3** (\cite{13}). Let us define

$$S_\pm(\xi, t) = x_\pm(\xi, t) \xi + \lambda_\pm(\xi, t) t \mp K(x_\pm(\xi, t), \lambda_\pm(\xi, t)).$$

Then, for any compact set $\Lambda \subset \mathbb{R}^n \setminus \{0\}$ there exists $T > 0$ such that $\nabla_x S_\pm(\xi, t) = x_\pm(\xi, t), \quad (\partial S_\pm/\partial t)(\xi, t) = \lambda_\pm(\xi, t)$ and

$$\frac{\partial S_\pm(\xi, t)}{\partial t} = |\xi|^2 + V(\nabla_x S_\pm(\xi, t)),$$

for $\xi \in \Lambda$ and $\pm \sigma > T$.

Let $F_\pm : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be defined by

$$(F_\pm f)(\xi) = 2^{1/2} |\xi|^{-n/2} (F_{\pm} f)(|\xi|^2, \xi/|\xi|).$$

Then $F_\pm$ is a partial isometry on $L^2(\mathbb{R}^n)$ with the initial set $\mathcal{H}_{ac}(\hat{H})$ and the final set $L^2(\mathbb{R}^n)$.

Let $F_0$ be the ordinary Fourier transformation:

$$(F_0 f)(\xi) := (2\pi)^{-n/2} \int e^{-ix \cdot \xi} f(x) dx.$$

Then, we have the following lemma.

**Lemma 4.4** (\cite{13} Theorem 7.3). The wave operators

$$W_\pm = s \lim_{t \to \pm \infty} e^{it\hat{H}} e^{-is_\pm(\xi, t)} f,$$

exist and we have $W_+ = F_+^* F_0$. Here $e^{-i s_\pm(\xi, t)} f := F_0[e^{-i s_\pm(\xi, t)} F_0 f]$. Moreover, we have the intertwining property: for any bounded Borel function $\varphi$ on $\mathbb{R}$ we have $\varphi(\hat{H}) W_\pm = W_\pm \varphi(-\Delta)$. 

Remark 4.5. This result was proved in [13] for \( S_+ (\xi, t) \), but the one for \( S_- (\xi, t) \) is obtained in the same way. The intertwining property follows from \( \mathbf{F}_0 \varphi (\hat{H}) = \varphi (|\xi|^2) \mathbf{F}_0^\pm \) and \( \varphi (|\xi|^2) \mathbf{F}_0 = \mathbf{F}_0 \varphi (\Delta) \), where \( |\xi|^2 \) is the multiplication operator by \( |\xi|^2 \).

The scattering operator \( S \) is defined by \( S := W_+^* W_- \). By the intertwining property \( \hat{S} := \mathbf{F}_0 \hat{S} \mathbf{F}_0^* \) is decomposable (see [16]). Denoting the fibers of \( \hat{S} \) by \( \hat{S} (\lambda) \), Lemma 4.6 implies \( \hat{S} (\lambda) : \mathcal{F}_- (\lambda) f \mapsto \mathcal{F}_+ (\lambda) f \), for any \( f \in \mathcal{B} (\mathbb{R}^n) \).

We have the relation between the asymptotic behaviors of the generalized eigenfunctions and the scattering matrices.

Lemma 4.6 ([1]). Let \( w_\pm (x, \lambda) \) be defined by (3.1). Then for any \( a \in C^\infty (\mathbb{S}^{n-1}) \) we have

\[
(F_+ (\lambda) a) (x) = C (\lambda) \left( a (\tilde{x}) w_+ (x, \lambda) - (\mathcal{R} \hat{S} (\lambda) a) (\tilde{x}) w_- (x, \lambda) \right) + o_{av} (|x|^{- (n-1)/2}),
\]

and

\[
(F_- (\lambda) a) (x) = C (\lambda) \left( (\hat{S} (\lambda) a (\tilde{x})) w_+ (x, \lambda) - (\mathcal{R} a) (\tilde{x}) w_- (x, \lambda) \right) + o_{av} (|x|^{- (n-1)/2}),
\]

where \( C (\lambda) := -i 2^{-1} \pi^{-1/2} \lambda^{-1/4} \) and \( (\mathcal{R} a) (\tilde{x}) := a (\tilde{x}) \).

5. Poisson operators

In this section we use the notations in sections 1 and 2. Let \( \alpha \) be a non-threshold channel.

Set \( g \in C_0^\infty (C_0') \) and

\[
v_{\alpha}^\pm (\lambda, x_a) := \eta (r_a) g (\tilde{x}_a) r_a^{- (n_0 - 1)/2} e^{\pm i K_a (x_a, \lambda_a)},
\]

for some \( \kappa > 0 \) and

\[
K_a (x_a, \lambda_a) := \sqrt{\lambda_a r_a - Y_a (x_a, \lambda_a)},
\]

respectively, where \( \lambda_a := \lambda - E_a, \lambda > E_a \) and \( Y_a (x_a, \lambda_a) \) is the function obtained in Lemma 3.4 with \( \lambda \) and \( V (x) \) replaced by \( \lambda_a \) and \( \tilde{I}_a (x_a) \) respectively.

We can easily see that \( (H - \lambda) (J_a v_{\alpha}^\pm) \in L^{2, l} (X), \ l > 1/2 \). Thus, we can define the Poisson operator \( P_{\alpha, \pm} (\lambda) : C_0^\infty (C_0') \rightarrow \mathcal{B}^* (X) \) by

\[
P_{\alpha, \pm} (\lambda) \varphi = J_a v_{\alpha}^\mp - R (|\lambda \pm i0| (H - \lambda)) (J_a v_{\alpha}^\pm).
\]

6. Asymptotic behaviors of generalized eigenfunctions and solutions to nonhomogeneous equations for decaying potentials

In this section we suppose \( n \in \mathbb{N}, V \in C^\infty (\mathbb{R}^n) \) satisfy (5.1), and use the notations in section 3.

Proposition 6.1. Let \( \lambda > 0 \). Suppose \( u \in \mathcal{B}^* (\mathbb{R}^n) \) satisfy \( (\hat{H} - \lambda) u \in L^{2, l} (U), \ l > 1/2 \). Here \( U \) is a conic region, namely there exists \( C > 0 \) such that for any \( c > 1 \) and \( x \in U, |x| > C \) we have \( cx \in U, \) and \( L^{2, l} (U) := \{ f : \| f \|^2 := \int_U (|x|)^{2l} |f(x)|^2 dx < \infty \} \). Then for \( h \in C_0^\infty (U) \) the following limit exists:

\[
\lim_{\rho \rightarrow \infty} \rho^{-1} \int_{|x| < \rho} e^{\pm i K (x, \lambda) r^{- (n-1)/2}} h (\tilde{x}) u (x) dx,
\]
where \( U' := U \cap S^{n-1} \). The limit is equal to
\[
\lim_{\rho \to \infty} \rho^{-1} \int_{|x| < \rho} e^{\pm iK(x, \lambda)} r^{-(n-1)/2} h(x) \eta(r) \psi_1(\hat{H}) u(x) dx = 0,
\]
\[
\lim_{\rho \to \infty} \rho^{-1} \int_{|x| < \rho} e^{\mp iK(x, \lambda)} r^{-(n-1)/2} h(x) \eta(r) ((\psi(\hat{H}) - \psi(-\Delta)) u(x) dx = 0.
\]

**Lemma 6.2.** We have
\[
\lim_{\rho \to \infty} \rho^{-1} \int_{|x| < \rho} e^{\mp iK(x, \lambda)} r^{-(n-1)/2} h(x) \eta(r) \psi_1(\hat{H}) u(x) dx = 0,
\]
\[
\lim_{\rho \to \infty} \rho^{-1} \int_{|x| < \rho} e^{\pm iK(x, \lambda)} r^{-(n-1)/2} h(x) \eta(r) ((\psi(\hat{H}) - \psi(-\Delta)) u(x) dx = 0.
\]

**Proof.** Let \( \phi(x) \in C^\infty_0(\mathbb{R}^n) \) be a function such that \( \phi(x) h(\hat{x}) = h(\hat{x}) \) for \( |x| > 1 \), and \( \phi(x) = 1 - \phi(x) \). Then, \( h(\hat{x}) \eta(r) \psi_1(\hat{H})(\phi(x) u(x)) \in L^{2,l}(\mathbb{R}^n) \) for any \( l > 0 \) and \( h(\hat{x}) \eta(r) \psi_1(\hat{H})(\phi(x) u(x)) = h(\hat{x}) \eta(r) \psi_2(\hat{H})(\hat{H} - \lambda) \phi(\hat{H}) u \in L^{2,l}(\mathbb{R}^n) \) for \( l > 1/2 \).

Thus, (6.1) holds.

We can easily confirm (6.2) using Hellfer-Sjöstrand formula:
\[
\psi(A) := \frac{1}{2\pi i} \int \partial_z \Psi(z)(z - A)^{-1} dz \wedge d\bar{z},
\]
where \( \Psi \) is the almost analytic extension of \( \psi \) (see e.g. [6]).

Set \( t_0(x, \xi) = \eta(|x|) \psi(|\xi|^2) \). By Lemma 6.2 we only need to prove for \( u_0 := Op(t_0) u \) and \( v_\pm \) the existence of the limit
\[
\lim_{\rho \to \infty} \rho^{-1} \int_{|x| < \rho} u_0(x) v_\pm(x) dx.
\]

We consider the case of \( v_+ \) and denote \( v_+ \) by \( v \). The case of \( v_- \) is similar.

Let \( \chi_\pm \in C^\infty(\mathbb{R}) \) be functions such that \( \chi_+(s) = 1 \) for \( s > \sqrt{\lambda}/2 \), \( \chi_+(s) = 0 \) for \( s < -\sqrt{\lambda}/2 \), and \( \chi_+ + \chi_- = 1 \). Set \( t_\pm := \chi_\pm (\hat{x} \cdot \xi) t_0(x, \xi) \) and \( T_\pm := Op(t_\pm) \).

Then we can decompose \( u_0 \) as \( u_0 = u_+ + u_- \) where \( u_\pm := T_\pm u \).

Set \( D^\pm := -i\partial_x - i(n - 1)/2r \mp \partial_x K \).

**Lemma 6.3.** There exists \( \hat{s}_\pm \in S^{0,0} \) such that
\[
D^\pm u_\pm = r^{-2} \Delta_0 Op(\hat{s}_\pm) u_\pm + \nabla \cdot \hat{s}_\pm u_\pm + \hat{u}_\pm,
\]
where \( \Delta_0 \) is the Laplace-Beltrami operator on \( S^{n-1} \), and \( \hat{u}_\pm \in L^{2,1/3}(U) \). Here, \( \hat{s}_\pm \) is an operator written as \( \hat{s}_\pm = P_\pm \hat{\eta}(r) Op(\hat{s}_\pm) \), where \( \hat{\eta} \) is a function satisfying (3.3) with \( \eta \) and \( \kappa \) replaced by \( \hat{\eta} \) and \( 2^{-3}\kappa \) respectively, \( \hat{s}_\pm \) is a \( n \)-dimensional vector whose elements are symbols in \( S^{0,0} \) and \( P \) is an orthonormal projection onto the tangent space on the sphere \( S^{n-1} \), that is, \( P A = A - |x|^{-2} x \cdot A \) for any vector \( A \).
Proof. We shall prove the case of $\mathcal{D}^+ u_+$. The proof for $\mathcal{D}^- u_-$ is similar.

Let $\eta_1(t) \in C^\infty(\mathbb{R})$ be a function such that $\eta_1(t)\eta(t) = \eta(t)$ and $\eta(t) = 0$ for $t < \kappa/2$, where $\kappa$ is as in (6.3). Let $\tilde{\eta}_s$ be the multiplication operator by $\eta_1(|x|)$.

Using the the well-known equality

\begin{equation}
\Delta = \partial_x^2 + (n-1)r^{-1}\partial_r + r^{-2}\Delta_0,
\end{equation}

by a straightforward calculation we obtain

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\end{equation}

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\begin{equation}
\Delta = \partial_x^2 + (n-1)r^{-1}\partial_r + r^{-2}\Delta_0 = \partial_x^2 + (n-1)r^{-1}\partial_r + r^{-2}\Delta_0 |\nabla Y|^2 - |\partial_r Y|^2.
\end{equation}

In the following we denote by $u_j$ functions belonging to $L^{2,1/3}(U)$.

By (6.4) we have

\begin{equation}
\mathcal{D}^- \mathcal{D}^+ u_+ = r^{-2}\Delta_0 \eta_1 u_+ + u_1.
\end{equation}

Let $\chi_0$ be supported in $(-2\sqrt{\chi}/3, \infty)$ and satisfy $\chi_0 = 1$ on $\text{supp} \chi_0$. We take $\psi \in C^\infty_0(\mathbb{R})$ such that $\psi \psi = \psi$.

Let $\eta_{j+1}(t) \in C^\infty(\mathbb{R})$, $j \in \mathbb{N}$ be functions such that $\eta_{j+1}(t)\eta_j(t) = \eta_j(t)$ and $\eta_{j+1}(t) = 0$ for $t < 2^{-j-1} \kappa$. Let $\tilde{\eta}_{j+1}$ be the multiplication operator by $\eta_{j+1}(|x|)$.

Then choosing $\kappa$ large enough, on $\text{supp} (\hat{\chi}_+(\tilde{x} \cdot \xi)\hat{\psi}(|\xi|^2)\eta_2(|x|))$ the principal symbol of $\eta_3 \mathcal{D}^-$ is elliptic. Thus, we can construct the parametrix of $\eta_3 \mathcal{D}^-$ there, that is, there exists a symbol $s \in S^{0,0}$ such that $s\#d^- = 1 + w$ on $\text{supp} (\hat{\chi}_+(\tilde{x} \cdot \xi)\hat{\psi}(|\xi|^2)\eta_2(|x|))$ where $d^-$ is the symbol of $\eta_3 \mathcal{D}^-$ and $w \in S^{0,-\infty}$ (for the principal symbol, ellipticity and the construction of the parametrix see e.g. [7]). We set

$s := s\hat{\chi}_+(\tilde{x} \cdot \xi)\hat{\psi}(\xi)\eta_2(|x|)$.

Then applying $\text{Op}(\hat{s})$ to the both sides of (6.5) we obtain

\begin{equation}
\mathcal{D}^+ u_+ = \text{Op}(\hat{s})(r^{-2}\Delta_0 \eta_1 u_+) + u_2
\end{equation}

where $u_2 \in L^{2,l}(U)$ $l > 1/6$.

Set $\nabla \perp := \nabla - |x|^{-2}x(x \cdot \nabla)$. Then we have

\begin{equation}
\Delta = \partial_x^2 + (n-1)r^{-1}\partial_r + \nabla \perp \nabla \perp.
\end{equation}

Comparing (6.7) and (6.3) we can see that $\nabla \perp \nabla \perp = r^{-2}\Delta_0$. Thus, a calculation of pseudodifferential operators yields

\begin{equation}
\text{Op}^w(\hat{s})(r^{-2}\Delta_0 \eta_1 u_+) = r^{-2}\Delta_0 \text{Op}^w(\hat{s})\eta_1 u_+ + \nabla \cdot \tilde{S} u_+ + u_3
\end{equation}

where $\tilde{S}$ is an operator as in the Lemma 6.3.

By (6.6) and (6.8) we obtain the result. \qed

Lemma 6.4. We have

\[\int_{|x|=r} (\mathcal{D}^+ u_+) \, \tilde{\psi} dS \to 0\]

as $r \to \infty$ where $\int_{|x|=r} f \, dS$ means the integration of $f$ on $\{x : |x| = r\}$. 

Proof. In the following we denote by $F_j(r)$ functions such that $\int_1^\infty |F_j(r)|dr < \infty$. We can easily see that the following holds (for similar calculations see e.g. [8]).

\[
\int_{|x|=r} \left( \mathcal{D}^+ u_+ \right) \bar{v} dS = \int_{|x|=r} \left( \mathcal{D}^+ \mathcal{D}^+ u_+ \right) \bar{v} dS + F_1(r)
\]

(6.9)

\[
- i \partial_r \int_{|x|=r} \left( \mathcal{D}^+ u_+ \right) \bar{v} dS = \int_{|x|=r} \left( \mathcal{D}^+ \mathcal{D}^+ u_+ \right) \bar{v} dS + F_1(r)
\]

\[
= \int_{|x|=r} \left( \mathcal{D}^- \mathcal{D}^+ u_+ \right) \bar{v} dS - 2 \sqrt{\lambda} \int_{|x|=r} \left( \mathcal{D}^+ u_+ \right) \bar{v} dS
\]

\[
+ 2 \int_{|x|=r} \left( \partial_r Y \right) \left( \mathcal{D}^+ u_+ \right) \bar{v} dS + F_1(r).
\]

By (6.4) and $|\nabla Y(x)|^2 = O(|x|^{-2\mu}), |\partial_r Y(x)|^2 = O(|x|^{-2\mu})$ we have

\[
\int_{|x|=r} \left( \mathcal{D}_- \mathcal{D}_+^* \eta_+ u_+ \right) \bar{v} dS = F_2(r).
\]

By Lemma 6.3 we can see that the third term on the right-hand side of (6.9) is integrable with respect to $r$. As a result, setting

\[
\phi(r) := \int_{|x|=r} \left( \mathcal{D}^+ u_+ \right) \bar{v} dS,
\]

we have $\partial_r \phi(r) = -2i \sqrt{\lambda} \phi(r) + F_3(r)$.

Thus, setting $\phi_1(r) = e^{2i \sqrt{\lambda} r} \phi(r)$ there exists a limit $\lim_{r \to \infty} \phi_1(r)$. However, since for some $0 < \epsilon < 1$ we have $\int_1^\infty r^{-\epsilon} |\phi(r)| dr < \infty$, we obtain $\lim_{r \to \infty} \phi_1(r) = 0$, and therefore, $\lim_{r \to \infty} \phi(r) = 0$.

\[\square\]

**Lemma 6.5.** The limit $\lim_{r \to \infty} \int_{|x|=r} u_+ \bar{v} dS$ exists and the limit is equal to

\[
2^{-1} i \lambda^{-1/2} \left( \left( \mathcal{H} - \lambda \right) u, v \right) - \left( u, (\mathcal{H} - \lambda) v \right),
\]

where $(u, v) := \int_{\mathbb{R}^n} u(x) \bar{v}(x) dx$.

**Proof.** We have by Green's formula for $\{ x : |x| \leq r \}$

\[
\int_{|x| < r} \left( \Delta u_+ \bar{v} - u_+ (\Delta \bar{v}) \right) dx = \int_{|x|=r} \left( (i \mathcal{D}^+ u_+) \bar{v} - u_+ (i \mathcal{D}^+ \bar{v}) \right) dS
\]

(6.10)

\[
+ 2i \int_{|x|=r} \left( \partial_r K \right) u_+ h \bar{v} dS.
\]

By Lemma 6.3 and that $\mathcal{D}_r^+ v = 0$ for $|x|$ large enough, the first term on the right-hand side converges to 0 as $r \to \infty$.

As for the left-hand side we have

(6.11)

\[
\int_{|x| < r} \left( \Delta u_+ \bar{v} - u_+ (\Delta \bar{v}) \right) dx = \int_{|x| < r} \left( \left( \Delta - V + \lambda \right) u_+ \bar{v} - u_+ \left( \left( \Delta - V + \lambda \right) \bar{v} \right) \right) dx.
\]

As in the proof of Lemma 6.3 by the the form of $t_+$ we have

(6.12)

\[
(- \Delta + V - \lambda) T_+ = \nabla \cdot \mathcal{S}_1 + S_0 + T_+ (- \Delta + V - \lambda),
\]

where $\mathcal{S}_1$ has the same property as $\mathcal{S}$ in Lemma 6.3 and $S_0 := Op(s_0)$, $s_0 \in S^0_{0,-3/2}$. Since we also have $\bar{v} := (- \Delta + V - \lambda) v \in L^2_l(\mathbb{R}^n)$ for some $l > 1/2$, the right-hand
side of (6.11) converges to
\[
-(\nabla \cdot \tilde{S}_1 u, v) - (S_0 u + T_+ \tilde{u}, v) + (u_+, \tilde{v})
\]
(6.13)
\[
= (u, \tilde{S}_1^\perp \cdot \nabla \perp v) - (u, S_0 v) - (\tilde{u}, T_+ v) + (u, T_+ \tilde{v}),
\]
where \(\tilde{u} = (-\Delta + V - \lambda)u\) and \((v_1, v_2)\) is the inner product of \(v_1\) and \(v_2\). Therefore, the limit \(\lim_{r \to \infty} 2i \int_{|x| = r} \partial_r \tilde{K} u_+ \tilde{v} dS\) exists.

Moreover, taking the adjoint of (6.12) we have
\[
(-\Delta + V - \lambda)T_+ = \tilde{S}_1^\perp \cdot \nabla \perp - S_0 + T_+ (-\Delta + V - \lambda).
\]
Thus, the right-hand side of (6.13) is written as
\[
(6.14)
(u, (-\Delta + V - \lambda)T_+ v) - (\tilde{u}, T_+ v).
\]
We can remove \(T_+\) in (6.14), because \((1 - T_+) v \in S(\mathbb{R}^n)\) by Lemma 6.3 and therefore,
\[
\lim_{r \to \infty} 2i \int_{|x| = r} \partial_r \tilde{K} u_+ \tilde{v} dS = (u, \tilde{v}) - (\tilde{u}, v).
\]
\[\square\]

**Lemma 6.6.**

\[
\lim_{\rho \to \infty} \rho^{-1} \int_{|x| < \rho} u_- \tilde{v} dx = 0.
\]

**Proof.** We have by a straightforward calculation
\[
(6.15)
-i \partial_r \left( \int_{|x| = r} u_- \tilde{v} dS \right)
\]
\[
= \int_{|x| = r} (D^- \eta_1 u_-) \tilde{v} dS - 2\sqrt{\lambda} \int_{|x| = r} u_- \tilde{v} dS + \int_{|x| = r} (\partial_r Y) u_- \tilde{v} dS + F(r),
\]
where \(\int_1^{\infty}|F(r)| dr < \infty\).

Setting \(\phi(r) := e^{-2i\sqrt{\lambda} r} \int_{|x| = r} u_- \tilde{v} dS\), by (6.15) and Lemma 6.3 we can see that
\[
\int_1^{\infty} r^{-\epsilon} |\partial_r \phi(r)| dr < \infty,
\]
for some \(0 < \epsilon < 1\). Thus there exists a constant \(C > 0\) such that
\[
|\phi(r)| < r^\epsilon \left( \int_1^{r} r_1^{-\epsilon} |\partial_r \phi(r_1)| dr_1 + |\phi(1)| \right) < Cr^\epsilon.
\]
Integrating both sides of (6.15) we can see that there exists a constant \(\tilde{C}\) such that
\[
\left| \int_{|x| < \rho} u_- \tilde{v} dx \right| \leq \tilde{C} (|\phi(\rho)| + \rho^\epsilon) \leq \tilde{C} (C + 1) \rho^\epsilon,
\]
from which the lemma follows. \(\square\)
Proof of Proposition 6.1. Proposition 6.1 follows immediately from Lemmas 6.5 and 6.6. □

7. Uniqueness theorem for nonhomogeneous equations and the outgoing (incoming) property

In the following we use the notations in section 1, 2 and 3. In this section we introduce the Isozaki’s uniqueness theorem for nonhomogeneous N-body Schrödinger operators. First, we need the definition of a class of symbols of pseudodifferential operators. For \( k > 0 \) and \( \tau \in \mathbb{R} \), we introduce the following.

**Definition 7.1** (111 Definition 1.1). Let \( n \in \mathbb{N} \). \( \mathcal{R}_k^\pm(\tau) \) is the set of \( C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \)-functions \( p(x, \xi) \) such that

\[
|\partial_\tau^\alpha \partial_\xi^\beta p(x, \xi)| \leq C_\alpha\beta(x)^{-|\gamma|}(\xi)^{-k},
\]

for \( 0 \leq |\gamma| \leq k \), \( 0 \leq |\gamma_0| \leq k \) and on \( \text{supp} p(x, \xi) \)

\[
\inf_{x, \xi} \pm \xi > \pm \tau.
\]

In Isozaki’s uniqueness theorem, outgoing and incoming properties are the conditions of the uniqueness. We define the outgoing and incoming properties as follows.

**Definition 7.2.** Let \( 1/2 < l \leq 1 \) and \( n \in \mathbb{N} \).

1. A function \( u \in L^{2-l}(\mathbb{R}^n) \) is outgoing (resp., incoming), if there exist \( k_0 > 0 \), \( 0 \leq l_0 < 1/2 \) and \( \varepsilon > 0 \) such that \( \text{Op}(p)u \in L^{2-l_0}(\mathbb{R}^n) \) for any \( p \in \mathcal{R}_k^{00}(\varepsilon) \) (resp., \( p \in \mathcal{R}_k^{00}(\varepsilon^*) \)).

2. A function \( u \in L^{2-l}(\mathbb{R}^n) \) is strictly outgoing (resp., strictly incoming), if there exists \( k_0 > 0 \) such that for any \( \varepsilon > 0 \) there exists \( 0 \leq l_0 < 1/2 \) satisfying the following condition: \( \text{Op}(p)u \in L^{2-l_0}(\mathbb{R}^n) \) for any \( p \in \mathcal{R}_k^{00}(1-\varepsilon) \) (resp., \( p \in \mathcal{R}_k^{00}(1+\varepsilon) \)).

**Remark 7.3.** Since we can assume \( X_a = \mathbb{R}^n \) for some \( n \in \mathbb{N} \), we can define the outgoing and incoming properties for \( \tilde{v} \in L^{2-l}(X_a) \), \( 1/2 < l \leq 1 \) in the same way as above. When we consider operators for \( X_a \), we write as \( \mathcal{R}_k^{\pm} \). By Lemma 6.4 we can see that \( \tilde{R}_a(\lambda + i0)v := (\tilde{H}_a - \lambda - i0)^{-1}v \) (resp., \( \tilde{R}_a(\lambda - i0)v := (\tilde{H}_a - \lambda + i0)^{-1}v \) ), \( v \in L^{2-l}(X_a) \), \( l > 1/2 \) is strictly outgoing (resp., strictly incoming), where \( \tilde{H}_a := -\Delta_a + \tilde{I}_a \).

The outgoing and incoming properties can be written using the Graf’s vector field. Let us introduce the following class of functions.

**Definition 7.4** (11). (1) Let \( \mathcal{V} \) be the set of \( C^\infty(X) \)-functions \( v \) on \( X \) such that for any \( \alpha \in \mathbb{N}^{\text{dim} X} \) and \( k \in \mathbb{N} \) there exists \( C_{\alpha,k} \) satisfying the following inequality:

\[
|\partial^\alpha(x \cdot \nabla)^k v(x)| \leq C_{\alpha,k}.
\]

(2) Let \( \mathcal{V}_+^1 \) be the set of positive \( C^\infty(X) \)-functions \( r \) on \( X \) such that

\[
r(x)^2 - |x|^2 \in \mathcal{V}.
\]

We need the following differential operator.
Definition 7.5 ([3] Lemma 2.1). Let \( \lambda \in \sigma_{ess}(H) \setminus (\sigma_{pp}(H) \cap T(H)) \) and \( \epsilon > 0 \) be given. Then there exist an open neighborhood \( N_\lambda \) of \( \lambda \) and \( r \in V^1_\lambda \) such that with \( A \) given as the self-adjoint operator on \( \mathcal{H} := L^2(X) \) by

\[
A := (\omega \cdot D + D \cdot \omega)/2, \quad \omega = r\nabla r, \quad D := -i\nabla,
\]

(1) \( i[H, A] \) defined as a form on \( \mathcal{D}(H) \cap \mathcal{D}(A) \) extends to a symmetric operator on \( \mathcal{D}(H) \), and in fact

\[
i[H, A] = \sum_{a \in A} v_a \nabla^a, \quad v_a \in \mathcal{V}.
\]

(2) \( \varphi(H)i[H, A]\varphi(H) \geq 2d(\lambda)(1 - \epsilon)\varphi(H)^2 \) for all real-valued \( \varphi \in C_0^\infty(N_\lambda) \), where

\[
d(\lambda) := \inf\{\lambda - t : t \in T(H), t < \lambda\}.
\]

Let \( Y \) be the operator of multiplication by \( Y(x) := \langle x \rangle \) on \( \mathcal{H} \).

Definition 7.6. With \( B \) given as the self-adjoint operator on \( \mathcal{H} \)

\[
B := r^{-1/2}A_{r^{-1/2}} = (\nabla \cdot D + D \cdot \nabla r)/2,
\]

we let \( \mathcal{D} \) be the domain

\[
\mathcal{D} := \bigcap \mathcal{D}(Q),
\]

where the intersection is over all polynomials \( Q \) in \( Y \) and \( B \).

Definition 7.7. We define for any \( m \in \mathbb{R} \) the class \( \mathcal{O}^m(Y) \) of operators \( P \) with the properties

1) \( \mathcal{D}(P) \) and \( \mathcal{D}(P^*) \) contain \( \mathcal{D} \), and \( P \) and \( P^* \) restricted to \( \mathcal{D} \) map into itself.

2) For any \( n \in \mathbb{N} \), \( \alpha, \beta \in \mathbb{R} \) such that \( \alpha + \beta = n - m \), \( Y^\alpha \text{ad}_n(P, B)Y^\beta \) extends to a bounded operator on \( \mathcal{H} \).

Here \( \text{ad}_0(P, B) := P \), \( \text{ad}_n(P, B) := [\text{ad}_{n-1}(P, B), B] \), \( n \geq 1 \).

Let for any \( m \in \mathbb{R} \), \( \mathcal{F}^m \) be the class of \( C^\infty \)-functions on \( \mathbb{R} \) such that

\[
|f^{(k)}(t)| \leq C_k(1 + |t|)^{m-k}, \quad \forall k \geq 0.
\]

As in [3] Lemma 2.3, we have the following lemma.

Lemma 7.8. If \( f \in \mathcal{F}^m \) for some \( m < 0 \) and \( a \in A \), we have \( f(H^a) \in \mathcal{O}^0(Y) \).

Lemma 7.9 ([12]). Let \( A \) and \( B \) be lower-semibounded self-adjoint operators, \( \mathcal{D}(A) = \mathcal{D}(B) \), and \( (A - B)(A + i)^{-1} \) is a bounded operator. Then, we have

\[
\|F(A < R_1)F(B > R_2)\| = O(R_2^{-1/2}),
\]
as \( R_2 \to \infty \), where \( F(t \geq r) \) is a function such that \( F(t \geq r) = 1 \) for \( t \geq r \) and \( F(t \geq r) = 0 \) for \( t \leq r \) for \( r \in \mathbb{R} \).

Proof. Set

\[
G(t, R_2) := e^{-tB}F(B > R_2)F(A < R_1)F(B > R_2)e^{-tB}.
\]

Then, there exists \( C > 0 \) such that

\[
-\frac{d}{dt}G(t, R_2) = 2Re^{-tB}F(B > R_2)(B - A)F(A < R_1)F(B > R_2)e^{-tB}
\]

(7.1)

\[
+ 2e^{-tB}F(B > R_2)(B - A)AF(A < R_1)F(B > R_2)e^{-tB}
\]

\[
\leq Ce^{-2tR_2}.
\]

Noticing \( G(0, R_2) = F(B > R_2)F(A < R_1)F(B > R_2) \) and integrating (7.1) with respect to \( t \), we obtain the result. \( \square \)
Set
\[ F_m^+(a) := \{ f \in F_m : \text{supp } f \subset (a, \infty) \}, \]
\[ F_m^-(a) := \{ f \in F_m : \text{supp } f \subset (-\infty, a) \} \]

As in the proof of [3, Theorem 2.12], we have the following lemma.

**Lemma 7.10.** Let \( \tau \in \mathbb{R} \). Then for any \( F_{\pm} \in \mathcal{F}_m^+ (\tau) \), \( \text{Op}(p_{\pm}), \ p_{\pm} \in \mathcal{R}_m^\pm (\tau) \) and \( s > 0 \) we have
\[ X^s P_{\pm} F_{\pm} (B) X^s \in \mathcal{L}(\mathcal{H}). \]

The outgoing and incoming properties can be stated using \( B \) in Definition 7.6.

**Lemma 7.11.** Let \( 0 \leq l_0 < 1/2 < l \leq 1, n \in \mathbb{N} \) and \( u \in L^{2,-l}(\mathbb{R}^n) \). Assume that there exists \( q \in C_0^{\infty}(\mathbb{R}) \) such that \( (1 - q(-\Delta))u \in L^{2,-l_0}(\mathbb{R}^n) \). Then \( u \) is outgoing (resp., incoming) if and only if there exists \( \epsilon > 0 \) and \( 0 \leq l_1 < 1/2 \) such that \( F_+(B) u \in L^{2,-l_1}(\mathbb{R}^n) \) (resp., \( F_+(B) u \in L^{2,-l_1}(\mathbb{R}^n) \)) for any \( F_- \in \mathcal{F}_m^0(\epsilon) \) (resp., \( F_+ \in \mathcal{F}_m^0(-\epsilon) \)).

**Proof.** We prove that there exists \( \epsilon' > 0 \) such that \( \text{Op}(p) u \in L^{2,-l_0}(\mathbb{R}^n) \) for any \( p \in \mathcal{R}_m^{\epsilon'}(\epsilon) \), assuming that \( F_+(B) u \in L^{2,-l_1}(\mathbb{R}^n) \) for any \( F_- \in \mathcal{F}_m^0(\epsilon) \) and \( k_0 > 0 \). The proofs for the inverse case and the converse statements are similar.

Let \( \epsilon' > 0 \) satisfy \( \epsilon' < \epsilon \) and let \( \sigma > 0 \) be a number such that \( \epsilon' + 3\sigma < \epsilon \). Let \( f_+ \in \mathcal{F}_m^0(\epsilon' + \sigma) \) and \( f_- \in \mathcal{F}_m^0(\epsilon' + 2\sigma) \) satisfy \( f_+(t) + f_-(t) = 1 \). Then we have
\[ \text{Op}(p)u = \text{Op}(p)f_+(B)u + \text{Op}(p)f_-(B)u. \]

By Lemma 7.10 we have \( \text{Op}(p)f_+(B)u \in L^{2,-l_1}(\mathbb{R}^n) \). By the assumption we can also see that \( \text{Op}(p)f_-(B)u \in L^{2,-l_1}(\mathbb{R}^n) \) which completes the proof. \( \square \)

The following lemma is the Isozaki’s uniqueness theorem.

**Lemma 7.12** ([11 Theorem 1.3]). Let \( 1/2 < l \leq 1 \) and \( \lambda \in \sigma_{\text{ess}}(H) \setminus (\sigma_p(H) \cup \mathcal{T}(H)) \). Suppose that \( u \in L^{2,-l}(X) \) satisfies \( (H - \lambda)u = 0 \) and \( u \) is outgoing or incoming. Then \( u = 0 \).

The following Lemma is useful to confirm the outgoing and incoming properties.

**Lemma 7.13.** Let \( 0 \leq l_0 < 1/2 < l \leq 1 \) and \( u_\alpha(x^\alpha) \) be an eigenfunction of \( H^\alpha \) corresponding to a non-threshold channel \( \alpha \). If \( v \in L^{2,-l}(X_{\alpha}) \) is outgoing (resp., incoming) and there exists \( q \in C_0^{\infty}(\mathbb{R}) \) such that \( (1 - q(-\Delta_{\alpha}))v \in L^{2,-l_0}(X_{\alpha}) \). Then \( (J_{\alpha}v)(x) \) is outgoing.

**Proof.** We prove only the outgoing case. The incoming case is proved in the same way.

Let \( k_0 > 0, \epsilon > 0, 0 \leq l_1 < 1/2 \) and \( \text{Op}(p)v \in L^{2,-l_1}(X_{\alpha}) \) for any \( p \in \mathcal{R}_m^{l_0,\alpha}(\epsilon) \). Let \( \epsilon' > 0 \) be a number such that \( \epsilon' < \epsilon \), and \( \sigma > 0 \) be a number such that \( \epsilon' + 4\sigma < \epsilon \). We shall prove that \( F_-(B)(J_{\alpha}v) \in L^{2,-l_2}(X) \) for any \( F_- \in \mathcal{F}_m(\epsilon') \), where \( l_2 := \max\{l_0, l_1\} \). Let \( \varphi \in C_0^{\infty}(\mathbb{R}) \) be a function such that \( \varphi(t) = 1 \) near \( E_\alpha \), where \( E_\alpha \) is the eigenvalue corresponding to the channel \( \alpha \). Then we have \( \varphi(H^\alpha)u_\alpha = u_\alpha \). Let \( \psi \in C_0^{\infty}(\mathbb{R}) \) be a function such that \( \psi \varphi = \varphi \). Let \( f \in C^{\infty}(\mathbb{R}) \) satisfy \( f(t) = 1 \) for \( t > 2 \) and \( f(t) = 0 \) for \( t < 1 \).

By Lemma 7.9 for \( C > 0 \) large enough \( K_C := f(-\Delta^a/C)\psi(H^a) \) on \( L^2(X_{\alpha}) \) satisfies \( \|K_C\| < 1/2 \). Setting \( f_0 := 1 - f \) we have \( \varphi(H^a) = (K_C + f_0(-\Delta^a/C))\varphi(H^a) \), and therefore, \( \varphi(H^a) = (1 - K_C)^{-1}f_0(-\Delta^a/C)\varphi(H^a) \). Thus we obtain
\[ u_\alpha = (1 - K_C)^{-1}f_0(-\Delta^a/C)u_\alpha. \]
We denote by $w_j$ functions such that $w_j \in L^{2,-1,2}(X)$. Since by Lemma 7.8 we have $K_C \in \mathcal{O}p^0(Y)$, we have $(1 - K_C)^{-1} \in \mathcal{O}p^0(Y)$. Therefore, there exist $F'_{j,-} \in \mathcal{F}-(\epsilon')$, $j = 1, 2$, such that

\[
(7.2) \quad F_{-}(B)(1 - K_R)^{-1} f_0(-\Delta^a/R)(J_\alpha v) = \sum_{j=1}^{2} K'_{j}F'_{j,-}(B)f_0(-\Delta^a/R)(J_\alpha v) + w_1.
\]

Let us prove that the first term in the right-hand side of \((7.2)\) belongs to $L^{2,-1,2}(X)$. By the assumption we can see that

\[ u_\alpha v = q(-\Delta_\alpha)(J_\alpha v) + w_2, \]

where $q \in C_0^\infty(\mathbb{R})$ is as in the assumption.

We can easily see that there exist $p_+ \in \mathcal{R}_{+}^{k_0}(\epsilon' + \sigma)$ and $p_- \in \mathcal{R}_{-}^{k_0}(\epsilon' + 2\sigma)$ such that the following holds: there exits $C' > 0$ such that $|\xi| < C'$ on $\text{supp } p_\pm$, and

\[ f_0(-\Delta^a/C)q(-\Delta_\alpha)(J_\alpha v) = Op(p_+)(J_\alpha v) + Op(p_-)(J_\alpha v) + w_3. \]

By Lemma 7.10 we have $F'_{j,-}(B)Op(p_+)(J_\alpha v) \in L^{2,-1,2}(X)$. Thus, we only need to prove $Op(p_-)(J_\alpha v) \in L^{2,-1,2}(X)$, in order to prove that the right-hand side of \((7.2)\) belongs to $L^{2,-1,2}(X)$.

Let $\chi \in C_0(\mathbb{R})$ be a homogeneous function of degree 0 for $|x| > 1$ such that $\chi(x) = 1$ for $x \in \{ x : |x^a| > 2\delta|x|, |x| > 1 \}$ and $\chi(x) = 0$ for $x \in \{ x : |x^a| < \delta|x|, |x| > 1 \}$. Then it is easy to see that

\[ Op(p_-)(J_\alpha v) = Op(p_-(x, \xi)\chi(x))(J_\alpha v) + w_4. \]

Choosing $\delta$ sufficiently small, we can assume $(x^a \cdot \xi^a)/|x| < \sigma$ on

\[ \text{supp } (p(x, \xi)\chi(x)). \]

Thus, there exists $\tilde{p}_-(x, \xi, \alpha) \in \mathcal{R}_{+}^{k_0}(\epsilon' + 3\sigma)$ such that

\[ Op(p_-(x, \xi)\chi(x))(J_\alpha v) = Op(p_-(x, \xi)\chi(x))Op(\tilde{p}_-)(J_\alpha v) + w_5. \]

By the assumption that $Op(p)v \in L^{2,-1,2}(X)$ for any $p \in \mathcal{R}_{+}^{k_0}(\epsilon)$, we have $Op(\tilde{p}_-)(J_\alpha v) \in L^{2,-1,2}(X)$, and therefore, $Op(\tilde{p}_-)(J_\alpha v) \in L^{2,-1,2}(X)$. Thus, we obtain

$Op(\tilde{p}_-)(J_\alpha v) \in L^{2,-1,2}(X)$ which completes the proof. \qed

8. Scattering matrices and generalized Fourier transforms

In the following we use the notations in section 1, 2, and 3. In this section we define the scattering matrices and the generalized Fourier transforms.

**Lemma 8.1.** Let $\alpha$ be a non-threshold channel and $u \in \mathcal{B}^*(X)$. Then, there exists a constant $C$ such that

\[ \|\pi_\alpha u\|_{\mathcal{B}^*(X)} \leq C\|u\|_{\mathcal{B}^*(X)}. \]
Proof. There exist $C_1, C_2, C_3 > 0$ such that

$$\|\pi_{\alpha}u\|_{B^s(\mathbb{R})}^2 \leq C_1 \sup_{\rho > 1} \rho^{-1} \int_{|x| < \rho} \int |u(x)|^2 dx a$$

$$\leq C_2 \sup_{\rho > 1} \rho^{-1} \sum_{j \geq 0} \rho^{-1} \int_{|x| < \rho} \int x \in \Omega_j |u(x)|^2 dx a$$

$$\leq C_2 \sup_{\rho > 1, j \geq 0} (\rho + \rho_j)^{-1} \int_{|x| < 2(\rho + \rho_j)} |u(x)|^2 dx \leq C_3 \|u\|_{B^s(\mathbb{R})}^2.$$  

Let $\lambda > 0$, $u \in B^s(\mathbb{R})$, $h \in C_0^\infty(C'_a)$ and $\alpha$ be a non-threshold channel. Then, by Lemma 8.1 there exist $C, C' > 0$ such that

$$\lim_{\rho \to \infty} \rho^{-1} \int_{|x| < \rho} e^{\pm iK_a(x, \lambda)} \rho^{-1} (n_a - 1) h(\rho a)(\pi_{\alpha}u)(x_a) dx_a$$

$$\leq C \|a\|_{B^s(\mathbb{R})} \|h\|_{L^2(C'_a)}$$

$$C \|u\|_{B^s(\mathbb{R})} \|h\|_{L^2(C'_a)}.$$  

By Proposition 6.1 and (8.1) we can define a distribution $Q_\alpha^\pm(u)$ on $C'_a$ by

$$(Q_\alpha^\pm(u))(h) = \lim_{\rho \to \infty} \rho^{-1} \int_{|x| < \rho} e^{\pm iK_a(x, \lambda)} \rho^{-1} (n_a - 1) h(\rho a)(\pi_{\alpha}u)(x_a) dx_a,$$

for any $h \in C_0^\infty(C'_a)$ and by Riesz theorem we can see that $Q_\alpha^\pm(u) \in L^2(C'_a)$. Since the Lebesgue measure of $C_{\alpha,\text{sing}}$ is 0, we can extend $Q_\alpha^\pm(u)$ to $C_a$ so that $Q_\alpha^\pm(u) \in L^2(C_a)$.

Set

$$(\vec{\nu}_\alpha^\pm(\lambda))(x_a) = \vec{\nu}_{\alpha, \pm}(\lambda, x_a) := \eta(r_a) h(\rho a)(\pi_{\alpha}u)(x_a) e^{\pm iK_a(x, \lambda)} a,$$

with $\eta$ as in (3.4) for some $\kappa > 0$. Applying Proposition 6.1 we have

$$\lim_{\rho \to \infty} \rho^{-1} \int_{|x| < \rho} e^{\pm iK_a(x, \lambda)} \rho^{-1} (n_a - 1) h(\rho a)(\pi_{\alpha}u)(x_a) dx_a$$

$$= 2^{-1} i \lambda_a^{-1/2} \left\{ (\tilde{H}_a - \lambda_a)(\pi_{\alpha}u), v_{\alpha, \pm} \right\} a$$

$$- (\pi_{\alpha}u, (\tilde{H}_a - \lambda_a)v_{\alpha, \pm} a),$$

where $\tilde{H}_a := -\Delta_a + I_a$ and $(v_1, v_2)_a := \int_{X_a} v_1(x_a) v_2(x_a) dx_a$.

Let $u \in B^s(\mathbb{X})$ be a generalized eigenfunction of $H$ with an eigenvalue $\lambda$. Then by (8.2) we have

$$\lim_{\rho \to \infty} \rho^{-1} \int_{|x| < \rho} \int_{X_a} e^{\pm iK_a(x, \lambda)} \rho^{-1} (n_a - 1) h(\rho a)(\pi_{\alpha}u)(x_a) dx_a$$

$$= 2^{-1} i \lambda_a^{-1/2} \left\{ (\tilde{H}_a - \lambda_a)(\pi_{\alpha}u), v_{\alpha, \pm} \right\} a - (\pi_{\alpha}u, (\tilde{H}_a - \lambda_a)v_{\alpha, \pm} a)$$

$$= 2^{-1} i \lambda_a^{-1/2} \left\{ \pi_{\alpha}u((\tilde{I}_a - I_a)u), v_{\alpha, \pm} a - (\pi_{\alpha}u, (\tilde{H}_a - \lambda_a)v_{\alpha, \pm} a)$$

$$= -2^{-1} i \lambda_a^{-1/2} (u, (H - \lambda)J_a v_{\alpha, \pm}).$$
Lemma 9.1. Next we shall define the generalized Fourier transforms. Let $f \in L^{2l}(X)$ for some $l > 1/2$. In the similar way as (8.3) we obtain

$$\lim_{\rho \to \infty} \rho^{-1} \int_{|x_a| < \rho} e^{\pm iK_a(x_a, \lambda)} f(x_a) dx_a \approx 2^{-1} i\lambda^{-1/2}\{|f, J_\alpha v_{\alpha, \pm}\}.$$ 

For a non-threshold channel $\alpha$ we define the generalized Fourier transform as the map

$$G_\alpha^\pm(\lambda) : L^{2l}(X) \to L^2(C_\alpha),$$

given by

$$(\gamma g)(\hat{x}_a) := g(-\hat{x}_a)$$

and

$$D_\alpha^\pm(\lambda) := \pi^{-1/2} \lambda^{1/4} e^{\pm i\pi(n_\alpha - 3)/4}.$$

9. Asymptotic behaviors of functions in the range of the resolvent and Poisson operators

In the following we use the notations in section 1 2 3 5 7 and 8. In this section we study the $Q_\beta^+(u)$, where $u = P_{\alpha, \pm} g$ for $g \in C_0^\infty(C_\alpha')$ or $u = R(\lambda + i0)f$ for $f \in L^{2l}(X)$, $l > 1/2$.

Set for $\epsilon > 0$, $Y_{\alpha, \epsilon} := Y_\alpha \cap \{x \in X : |x|^s < \epsilon |x|\}$, where $Y_\alpha$ is defined as in (2.2). Let $J_{\alpha, \epsilon} \in C^\infty(X)$ be homogeneous of degree zero outside $\{x \in X : |x| = 1\}$ and supp $J_{\alpha, \epsilon} \in Y_{\alpha, \epsilon}$. Set also $d(\lambda) := \inf \{\lambda - t : t \in \mathcal{T}(H), t < \lambda\}$. We need the following lemma.

Lemma 9.1 (8 Theorem 3.5]). Let $\lambda \in (\inf \sigma(H^s), \infty) \setminus \mathcal{T}(H)$, where $\sigma(A)$ is the spectra of $A$. Then, for any $\epsilon < \sqrt{d(\lambda)}$ (resp., $\epsilon > -\sqrt{d(\lambda)}$) there exist $\epsilon > 0$, $C > 0$ and a neighborhood $N_\lambda$ of $\lambda$ such that the following holds: for any $m \in \mathbb{N}$, $s' > s > m - 1/2$, $p_- \in \mathcal{R}_{-\epsilon'(\tau)}$ (resp., $p_+ \in \mathcal{R}_{-\epsilon'(\tau)}$) and $J_{\alpha, \epsilon}$ we have

$$\|Y^{s-m} Op(p_-) J_{\alpha, \epsilon} R(z)^m Y^{-s'}\|_{\mathcal{L}(H)} \leq C,$$

(resp., $\|Y^{s-m} Op(p_+) J_{\alpha, \epsilon} R(z)^m Y^{-s'}\|_{\mathcal{L}(H)} \leq C,$)

uniformly in Re $z \in N_\lambda$ and Im $z > 0$ (resp., Im $z < 0$).

Proposition 9.2. Let $f \in L^{2l}(X)$, $l > 1/2$ and $\lambda \in \sigma_{ess}(H) \setminus (\sigma_{pp}(H) \cap \mathcal{T}(H))$. Then, for any non-threshold channel $\alpha$ such that $\lambda > E_{\alpha}$, we have $Q_\alpha^-(R(\lambda + i0)f) = 0$ and $Q_\alpha^+(R(\lambda - i0)f) = 0$. 

where $(v_1, v_2) := \int_X v_1(x)\bar{v_2}(x)dx$, and we used that $(H - \lambda)u = 0$ in the second equality.

Let $\alpha$ and $\beta$ be non-threshold channels. Now we define the scattering matrix as the map

$$\Sigma_{\beta\alpha}(\lambda) : C_0^\infty(C_\alpha') \to L^2(C_b),$$

given by

$$\Sigma_{\beta\alpha}(\lambda) g := Q_\beta^+(P_{\alpha, +}(\lambda) g).$$
Proof. We shall prove only the case of $Q^-_\alpha(R(\lambda+i0)f)$. The case of $Q^+_{\alpha}(R(\lambda-i0)f)$ is similar.

Set $u := R(\lambda + i0)f$. For some $l'$ and any $\epsilon > 0$ we have $(\hat{H}_\alpha - \lambda_0)\pi_\alpha u \in L^{2,l'}(Z_\alpha^\epsilon)$, where $Z_\alpha^\epsilon$ is defined as (2.24). Let $\tilde{\chi}_{+,-,0}$ be functions such that $\tilde{\chi}_-(s) = 1$ for $s < \sqrt{d(\lambda)/3}$, $\tilde{\chi}_-(s) = 0$ for $s > 2\sqrt{d(\lambda)/3}$, and $\tilde{\chi}_+ + \tilde{\chi}_- = 1$. We define $T_\alpha$ by

$$
\tilde{\alpha} := \tilde{\chi}_+(\hat{\alpha},\tilde{\varepsilon})\eta(r)a_\lambda(|\eta|^2), \quad \text{where } \eta_\alpha = 1 \text{ near } \lambda_\alpha \text{ and } \eta \text{ is as in (3.31) for some } \kappa > 0.
$$

Define $u_\pm := \hat{T}_\alpha^\pm(\pi_\alpha u)$. Then, as in the proof of Proposition 6.1 we obtain

$$
\lim_{\rho \to \infty} \rho^{-1} \int_{|x_\rho| < \rho} e^{iK_a(x_\rho,\lambda_\alpha)} u_\rho u(x_\rho) dx_\rho = 0, \quad h \in C_0^\infty(C_a')
$$

and

$$
\lim_{\rho \to \infty} \rho^{-1} \int_{|x_\rho| < \rho} e^{iK_a(x_\rho,\lambda_\alpha)} u_\rho u(x_\rho) dx_\rho = 0, \quad h \in C_0^\infty(C_a').
$$

As for $u_-$, for any $\epsilon_1 > 0$ let $\chi_1, \chi_2 \in C^\infty(X)$ be functions such that $\chi_1(x) = 1$ on $\{x \in X : |x| > \epsilon_1 x_\alpha\}$, $\chi_1(x) = 0$ on $\{x \in X : |x| > \epsilon_1 x_\alpha\}$, and $\chi_1 + \chi_2 = 1$. We also denote by $\chi_j$ the operator of multiplication by $\chi_j(x)$.

Then, we have

$$
u_- = \pi_\alpha(\chi_1 \hat{T}_\alpha^- u) + \pi_\alpha(\chi_2 \hat{T}_\alpha^- u).
$$

By the exponential decay of $u_\alpha$ we can see that $\pi_\alpha(\chi_2 \hat{T}_\alpha^- u) \in L^{2,l}(X_\alpha)$ for any $l > 0$.

As for $\pi_\alpha(\chi_1 \hat{T}_\alpha^- u)$ by Lemma 9.1 for sufficiently small $\epsilon_1$ we have

$$
\pi_\alpha(\chi_1 \hat{T}_\alpha^- u) \in L^{2,-l}(X_\alpha),
$$

for some $0 \leq l < 1/2$. Therefore, we have $Q^-_\alpha(u) = 0$ \hfill \square

We can see that for $g \in C_0^\infty(C_a')$ the incoming wave of $P_{\alpha,\pm}(\lambda)g$ consists only of the wave from the channel $\alpha$.

**Proposition 9.3.** Let $\alpha$ and $\beta$ be non-threshold channels and $\alpha \neq \beta$. Then, $Q^-_\alpha(P_{\alpha,\pm}(\lambda)g) = g$ and $Q^-_\beta(P_{\alpha,\pm}(\lambda)g) = 0$.

**Proof.** By Proposition 9.2 the part $R(\lambda \pm i0)(H - \lambda)(J_\alpha v_\alpha^\pm)$ of $P_{\alpha,\pm}(\lambda)g$ does not contribute to $Q^-_\alpha(P_{\alpha,\pm}(\lambda)g)$.

Since the remaining part is $J_\alpha v_\alpha^\pm$, we can easily see that the proposition holds. \hfill \square

### 10. Equivalence of the Scattering Matrices

In the following we use the notations in section 11, 12, 13, and 14. In this section we prove Theorem 1.22.

**Proof of Theorem 1.22.** Let $f_1 \in C^\infty(X_\alpha)$ and $f_2 \in C^\infty(X_\beta)$ satisfy $f_1 \in C_0^\infty(X_\alpha)$, $f_2 \in C_0^\infty(X_\beta)$, $\tilde{f}_1(\alpha,\cdot) \in C_0^\infty(C_a')$ for any $\lambda$ such that $\text{supp } \tilde{f}_1(\lambda,\cdot) \neq \emptyset$, and $\tilde{f}_2(\lambda,\cdot) \in C_0^\infty(C_a')$ for any $\lambda > 0$ such that $\text{supp } \tilde{f}_2(\lambda,\cdot) \neq \emptyset$. Here $f_1(\lambda,\omega) = \tilde{f}_1(\lambda,\omega) := (F_\alpha f_1)(\lambda,\omega)$, $\omega \in C_a$ and $f_2(\lambda)(\omega_0) = \tilde{f}_2(\lambda,\omega_0) := (F_\beta f_2)(\lambda,\omega_0)$, $\omega_0 \in C_b$, where $F_\alpha$ is defined by (1.78).
We only need to prove

\[
\int_{\max\{E_a, E_b\}}^{\infty} (S_{\beta \alpha}(\lambda) \hat{f}_1(\lambda), \hat{f}_2(\lambda))_{K_b} d\lambda \\
= \int_{\max\{E_a, E_b\}}^{\infty} e^{i\pi(n_a + n_b - 2)/4} \lambda^{1/4} \lambda^{-1/4} (\Sigma_{\beta \alpha}(\lambda) R \hat{f}_1(\lambda), \hat{f}_2(\lambda))_{K_b} d\lambda,
\]

(10.1)

where \( K_b := L^2(C_b) \) and \((v_1, v_2)_M \) is the inner product of \( v_1 \) and \( v_2 \) in a Hilbert space \( M \).

(i) First, we consider the case such that \( \alpha = \beta \).

Let \( \varphi \in C_0^\infty(C_b^\prime) \) be a function satisfying \( \varphi(\hat{\xi}_b) \hat{f}_2(\lambda, \hat{\xi}_a) = \hat{f}_2(\lambda, \hat{\xi}_a) \) for any \( \lambda \) such that \( \text{supp} \hat{f}(\lambda, \cdot) \neq 0 \), and \( \psi \in C_0^\infty(\mathbb{R}) \) be a function satisfying \( \psi(t) = 1 \) for any \( t > 0 \) such that \( \text{supp} \hat{f}_2(\sqrt{t}, \cdot) \neq \emptyset \). Let \( \eta(t) \in C_0^\infty(\mathbb{R}) \) be a function as in (3.4) for some \( \kappa > 0 \), and \( \theta_+(t) \in C_0^\infty(\mathbb{R}) \) be a function such that \( \theta_+(t) = 0 \) for \( t < 1 - 2\epsilon \) and \( \theta_+(t) = 1 \) for \( t > 1 - \epsilon \), where \( 1 > \epsilon > 0 \) is determined later.

Set \( D_a := -i \nabla_a \). Then, we have

\[
W^+_\beta f_2 = s - \lim_{t \to +\infty} e^{itH} \varphi(D_b) \psi(|D_b|^2) J_\beta e^{-i(S_{\beta, \pm}(D_b, t) + \lambda_b t)} f_2 \\
= s - \lim_{t \to +\infty} e^{itH} T^+_b J_\beta e^{-i(S_{\beta, \pm}(D_b, t) + \lambda_b t)} f_2 \\
= s - \lim_{t \to +\infty} e^{itH} T^+_b J_\beta e^{-ith_b \hat{W}^+_\beta} f_2 \\
= T^+_\beta J_\beta \hat{W}^+_\beta f_2 + i \int_0^\infty e^{isH} G^+_\beta e^{-is h_b} \hat{W}^+_\beta f_2 ds \\
= T^+_\beta J_\beta \hat{W}^+_\beta f_2 + i \int_0^\infty e^{isH} G^+_\beta e^{-is h_b} \hat{W}^+_\beta f_2 ds,
\]

(10.2)

where \( h_\beta := \bar{H}_b + E_\beta \), \( \hat{W}^+_\beta := s - \lim_{t \to +\infty} e^{itH_b} e^{-iS_{\beta, \pm}(D_b, t)} \), \( \tilde{h}_\beta := -\Delta_b + E_\beta \), \( T^+_b := Op(t^+_b) \), \( \tau^+_\beta := \varphi(\xi_b) \psi(|\xi_b|^2) \eta(|\xi_a|) \theta_+(\hat{x}_a \cdot \hat{\xi}_b) \), and \( G^+_\beta := HT^+_b J_\beta - T^+_b J_\beta h_\beta \).

Here \( S_{\beta, \pm}(\xi_b, t) \) is the function in Lemma 4.3 obtained by replacing \( V \) by \( \tilde{V}_b \).

Since \( \varphi \in C_0^\infty(C_b^\prime) \) and \( \hat{x}_a \cdot \hat{\xi}_b < 1 - \epsilon \) on \( supp \tilde{\theta}_+(\hat{x}_a \cdot \hat{\xi}_b) \), we have \( \hat{x}_a \in C_0^\infty \) on \( supp \tilde{t}^+_b \) for sufficiently small \( \epsilon \). We also have \( \hat{x}_b \cdot \hat{\xi}_b < 1 - \epsilon \) on \( supp \tilde{\theta}^+_\beta(\hat{x}_b \cdot \hat{\theta}_b) \), where \( \tilde{\theta}^+_\beta(t) := \frac{t}{\tilde{\theta}_+(t)} \). Therefore, using Lemma 3.4 we can see that if \( v \in B^*(X_a) \) is strictly outgoing or incoming, then \( G^+_\beta v \in L^2_l(X) \) for some \( l > 1/2 \).

In the same way we have

\[
W^\alpha f_1 = T_a J_a \hat{W}^\alpha f_1 - i \int_{-\infty}^0 e^{isH} G_a \hat{W}^\alpha e^{-is h_a} f_1 ds,
\]

where \( T_a := Op(t_a) \), \( t_a := \varphi(\xi_a) \psi(|\xi_a|^2) \eta(|x_a|) \theta_+(\hat{x}_a \cdot \hat{\xi}_a) + \theta_-(\hat{x}_a \cdot \hat{\xi}_a) \), \( \theta_+(t) := \theta_+(t) \), and \( G_a := HT_a J_a - T_a J_a h_a \). Here \( \psi \in C_0^\infty(\mathbb{R}) \) be a function satisfying \( \psi(t) = 1 \) for any \( t > 0 \) such that \( \text{supp} \hat{f}_1(\sqrt{t}, \cdot) \neq \emptyset \).

We also have

\[
\Omega^\alpha \hat{W}^\alpha f_1 = T_a J_a \hat{W}^\alpha f_1 + i \int_0^\infty e^{isH} G_a \hat{W}^\alpha e^{-is h_a} f_1 ds,
\]

where \( \Omega^\alpha := s - \lim_{t \to +\infty} e^{itH} J_a e^{-ith_a} \mathbf{1}_{Z_a} \tilde{p}_a^+ \), with \( Z_a \) defined as (2.1) (see Theorem 6.10.1]). Here \( \tilde{p}_a^+ \) is the asymptotic velocity for \( \bar{H}_a \). Note that Ran \( \hat{W}^\alpha = \)
where (v, ǫ is the operator corresponding to F, where \( \hat{1}_{\mathbb{Z}}(p^+_1) \) (see [1] Theorem 6.15.2)). Thus, we obtain

\[
(\Omega^+_\alpha \hat{W}^-_\alpha - W^-_\alpha) f_1 = i \int_{-\infty}^{\infty} e^{isH} G_\alpha \hat{W}^-_\alpha e^{-is\tilde{h}_0} f_1 ds.
\]

Since \( W^+_\alpha = \Omega^+_\alpha \hat{W}^+_\alpha \), we have

\[
S_{\alpha} - S_\alpha = (W^+_\alpha)^*(W^-_\alpha - \Omega^+_\alpha \hat{W}^-_\alpha),
\]

where \( S_\alpha := (\hat{W}^+_\alpha)^* \hat{W}^-_\alpha \). Therefore, by (10.2) we obtain

\[
(S_{\alpha} f_1, f_2) = (\text{see } 1, \text{ Theorem 6.15.2}). \quad \text{Thus, we obtain}
\]

\[
(\hat{W}^+_\alpha e^{-ith_0} f_1, e^{isH} G_\alpha^{\prime} \hat{W}^+_\alpha e^{-i(s+t)\tilde{h}_0} f_2)_\mathcal{H},
\]

where \( \mathcal{H} := L^2(\mathbb{X}) \).

Here we note \( \hat{W}^\pm_\alpha \) and \( \hat{F}^\pm_\alpha \) where \( (\hat{F}^\pm_\alpha f)(\lambda) := (\hat{F}^\pm_\alpha f)(\lambda - E_\alpha) \). Here \( \hat{F}^\pm_\alpha \) is the generalized Fourier transform corresponding to \( F^\pm_\alpha \) defined in section 3 with \( V \) replaced by \( \hat{I}_a \). Thus, we have

\[
\hat{W}^\pm_\alpha e^{-ith_0} f_1 = \hat{F}^\pm_\alpha e^{-it\lambda} \hat{f}_1.
\]

The second term on the right-hand side of (10.3) is calculated as

\[
(G_\alpha \hat{W}^-_\alpha e^{-ith_0} f_1, e^{isH} G_\alpha^{\prime} \hat{W}^+_\alpha e^{-i(s+t)\tilde{h}_0} f_2)_\mathcal{H}
\]

\[
= (e^{-it\lambda} \hat{f}_1, \tilde{F}_{\alpha,-}(\lambda) G_\alpha^{\prime} G_\alpha^{\prime} \hat{W}^+_\alpha e^{-i(s+t)\tilde{h}_0} f_2)_{\tilde{H}_a}
\]

\[
= \int_{E_a} e^{-it\lambda} ((G_\alpha^{\prime})^* e^{-isH} G_\alpha \hat{F}_{\alpha,-}(\lambda) f_1(\lambda), \hat{W}^+_\alpha e^{-i(s+t)\tilde{h}_0} f_2)_{\tilde{H}_a} d\lambda
\]

\[
= \int_{E_a} \int_{E_a} e^{-it\lambda+i(s+t)\lambda'} \cdot (\hat{F}_{\alpha,+}(\lambda') (G_\alpha^{\prime})^* e^{-isH} G_\alpha \hat{F}_{\alpha,-}(\lambda) f_1(\lambda), f_2(\lambda'))_{\tilde{H}_a} d\lambda d\lambda'.
\]

Here \( \tilde{H}_a := L^2((E_a, \infty); L^2(C_a)) \), and \( \tilde{F}_{\alpha, \pm}(\lambda) := \tilde{F}_{\alpha, \pm}(\lambda - E_\alpha) \), where \( \tilde{F}_{\alpha, \pm}(\lambda - E_\alpha) \) is the operator corresponding to \( F_{\pm}(\lambda) \) in section 4 with \( V \) replaced by \( \hat{I}_a \).

Inserting the convergent factor \( e^{-\epsilon s^*} \), integrating with respect to \( s \) and taking the limit as \( \epsilon \to 0^+ \) we obtain

\[
-i \int_{E_a} \int_{E_a} e^{-it(\lambda - \lambda')} (R(\lambda' + i0) G_\alpha \tilde{F}_{\alpha,-}(\lambda) \hat{f}_1(\lambda), G_\alpha^{\prime} \tilde{F}_{\alpha,+}(\lambda') \hat{f}_2(\lambda')) d\lambda d\lambda',
\]

where \( (v_1, v_2) := \int_X v_1(x) \bar{v}_2(x) dx \).
We again insert the factor $e^{-|t|}$, integrate with respect to $t$, and take the limit as $\epsilon \to 0$. Then, the second term on the right-hand side of (10.3) is written as

$$2\pi i \int_{E_a}^{\infty} (R(\lambda + i0)G_\alpha \hat{F}_{\alpha,-}^*(\lambda) \hat{f}_1(\lambda), G_\alpha^+ \hat{F}_{\alpha,+}^*(\lambda) \hat{f}_2(\lambda))d\lambda.$$ 

In the similar way the first term on the right-hand side of (10.3) is written as

$$-2\pi i \int_{E_a}^{\infty} (G_\alpha \hat{F}_{\alpha,-}^*(\lambda) \hat{f}_1(\lambda), T_a^+ J_a \hat{F}_{\alpha,+}^*(\lambda) \hat{f}_2(\lambda))d\lambda.$$ 

Noting $\hat{H}_\alpha - \lambda_\alpha \hat{F}_{\alpha,-}^*(\lambda) = 0$ we obtain

$$G_\alpha^+ \hat{F}_{\alpha,+}^*(\lambda) = \{(H - \lambda)T_a^+ J_a - T_a^+ J_a (\hat{H}_\alpha - \lambda_\alpha)\} \hat{F}_{\alpha,-}^*(\lambda) = (H - \lambda)T_a^+ J_a \hat{F}_{\alpha,-}^*(\lambda).$$

Therefore, we have

$$(S_{\alpha f_1}, f_2) - (S_{\alpha f_1}, f_2)$$

$$= 2\pi i \int_{E_a}^{\infty} (G_\alpha \hat{F}_{\alpha,-}^*(\lambda) \hat{f}_1(\lambda), R(\lambda - i0)(H - \lambda)T_a^+ J_a \hat{F}_{\alpha,+}^*(\lambda) \hat{f}_2(\lambda))d\lambda$$

$$- 2\pi i \int_{E_a}^{\infty} (G_\alpha \hat{F}_{\alpha,-}^*(\lambda) \hat{f}_1(\lambda), T_a^+ J_a \hat{F}_{\alpha,+}^*(\lambda) \hat{f}_2(\lambda))d\lambda.$$ 

We can replace $T_a^+ \hat{F}_{\alpha,+}^*(\lambda) \hat{f}_2(\lambda)$ in (10.6) by $-C_\alpha^+(\lambda)T_a^+ w_\alpha^+(\lambda)$ where

$$(w_\alpha^+(\lambda))(x_a) = w_\alpha^+(\lambda, x_a) := \eta(r_a) \hat{f}_2(\lambda, \hat{x}_a)e^{iK_a(x_a, \lambda_\alpha)}/(n_a - 1/2),$$

and

$$C_\alpha^+(\lambda) := 2^{-1}i\eta^{1/2} \lambda_\alpha^{-1/4} e^{-ir(\eta - 3)/4},$$

with $\eta$ being as in (3.4) for some $\kappa$. To see that, set $F_1^+ := J_a T_a^+ \hat{R}_a(\lambda_\alpha - i0) \hat{w}_\alpha^+(\lambda)$, where $\hat{H}_\alpha := -\Delta_\alpha + \hat{T}_a, \hat{w}_\alpha^+(\lambda) := (\hat{H}_\alpha - \lambda_\alpha)w_\alpha^+(\lambda)$ and $\hat{R}_a(\lambda_\alpha - i0) := (\hat{H}_\alpha - \lambda + i0)^{-1}$. Then, by Lemma 7.13 $F_1$ is incoming, and therefore, by Lemma 7.2 we have

$$F_1^+ - (H - i0)F_1^+ = 0.$$ 

Since we have

$$\hat{F}_{\alpha,+}^*(\lambda) \hat{f}_2(\lambda) = -C_\alpha^+(\lambda)(w_\alpha^+(\lambda) - F_1^+),$$

$T_a^+ \hat{F}_{\alpha,+}^*(\lambda) \hat{f}_2(\lambda)$ in (10.6) can be replaced by $-C_\alpha^+(\lambda)T_a^+ w_\alpha^+(\lambda)$. Set $\bar{T}_a^+ := 1 - T_a^+$. Since by Lemma 3.5 we have $\bar{T}_a^+ w_\alpha^+(\lambda) \in S(X_a)$, by Lemma 3.4 $F_2^+ := J_a \bar{T}_a^+ w_\alpha^+(\lambda)$ is incoming, and therefore

$$F_2^+ - (H - i0)F_2^+ = 0.$$ 

Thus, we can remove $T_a^+$ in front of $w_\alpha^+(\lambda)$. Therefore, we can rewrite (10.6) as

$$(S_{\alpha f_1}, f_2) - (S_{\alpha f_1}, f_2)$$

$$= -\bar{C}_\alpha(\lambda) \int_{E_a}^{\infty} (R(\lambda + i0)G_\alpha \hat{F}_{\alpha,-}^*(\lambda) \hat{f}_1(\lambda), (H - \lambda)J_a w_\alpha^+(\lambda))d\lambda$$

$$+ \bar{C}_\alpha(\lambda) \int_{E_a}^{\infty} (G_\alpha \hat{F}_{\alpha,-}^*(\lambda) \hat{f}_1(\lambda), J_a w_\alpha^+(\lambda))d\lambda,$$

where $\bar{C}_\alpha(\lambda) := \pi^{1/2} \lambda_\alpha^{-1/4} e^{ir(\eta - 3)/4}$. 


By Lemma 7.13 and (8.3) we can see
\[ (\tilde{S}_\alpha(\lambda)\tilde{f}_1(\lambda), \tilde{f}_2(\lambda)) = \tilde{C}_\alpha(\lambda)\{(T_a\tilde{\mathcal{F}}^\ast_{\alpha,-}(\lambda)\tilde{f}_1(\lambda), \tilde{w}^+_\alpha(\lambda))_a \\
- ((\tilde{H}_a - \lambda_\alpha)T_a\tilde{\mathcal{F}}^\ast_{\alpha,-}(\lambda)\tilde{f}_1(\lambda), \tilde{w}^+_\alpha(\lambda))_a\}, \]
where \((v_1, v_2)_a := \int_{X_a} v_1(x_a)v_2(x_a)dx_a\) and \(\tilde{S}_\alpha(\lambda)\) is the fiber of \(F_\alpha S_a F^\ast_\alpha\) (see 16).

Therefore, we obtain
\[ (S_\alpha f_1, f_2) = \int_{E_a} \tilde{C}_\alpha(\lambda)\{(T_a\tilde{\mathcal{F}}^\ast_{\alpha,-}(\lambda)\tilde{f}_1(\lambda), \tilde{w}^+_\alpha(\lambda))_a \\
- ((\tilde{H}_a - \lambda_\alpha)T_a\tilde{\mathcal{F}}^\ast_{\alpha,-}(\lambda)\tilde{f}_1(\lambda), \tilde{w}^+_\alpha(\lambda))_a\}d\lambda \]
(10.8)

\[ (S_\alpha f_1, f_2) = \int_{E_a} \tilde{C}_\alpha(\lambda)\{(J_aT_a\tilde{\mathcal{F}}^\ast_{\alpha,-}(\lambda)\tilde{f}_1(\lambda), (H - \lambda)J_a\tilde{w}^+_\alpha(\lambda))_a \\
- ((\tilde{H}_a - \lambda_\alpha)J_aT_a\tilde{\mathcal{F}}^\ast_{\alpha,-}(\lambda)\tilde{f}_1(\lambda), (H - \lambda)J_a\tilde{w}^+_\alpha(\lambda))_a\}d\lambda. \]
(10.9)

In the same way as in (10.5) we have
\[ G_{\alpha}\tilde{\mathcal{F}}^\ast_{\alpha,-}(\lambda) = (H - \lambda)T_aJ_a\tilde{\mathcal{F}}^\ast_{\alpha,-}(\lambda). \]

Thus, by (10.7) and (10.8) we have
\[ (S_\alpha f_1, f_2) = \tilde{C}_\alpha(\lambda)\int_{E_a} \{(J_aT_a\tilde{\mathcal{F}}^\ast_{\alpha,-}(\lambda)\tilde{f}_1(\lambda), (H - \lambda)(J_a\tilde{w}^+_\alpha(\lambda))_a \\
- (R(\lambda + i0)G_{\alpha}\tilde{\mathcal{F}}^\ast_{\alpha,-}(\lambda)\tilde{f}_1(\lambda), (H - \lambda)(J_a\tilde{w}^+_\alpha(\lambda))_a\}d\lambda. \]

By Lemma 7.13 \(F^-_1 := J_aT_a\tilde{R}(\lambda_\alpha + i0)(\tilde{H}_a - \lambda_\alpha)\tilde{w}^-_\alpha(\lambda)\) is outgoing, where
\[ (\tilde{w}^-_\alpha(\lambda)(x_a) = \tilde{w}^-_\alpha(\lambda, x_a) := \eta(r_a)\tilde{f}_1(\lambda, -\xi_a)e^{-iK_a(x_a, \lambda_\alpha)}f^-_{\alpha}(\eta_a^{-1}/2). \]
Thus, we have
\[ (S_\alpha f_1, f_2) = (S_\alpha f_1, f_2) = 0. \]

By Lemma 7.13 \(F^-_1 := J_a\tilde{T}_a\tilde{w}^-_\alpha(x_a, \lambda_\alpha)\) is outgoing. Thus, we have
\[ (S_\alpha f_1, f_2) = 0. \]

By (10.10) and (10.11) we can replace \(T_a\tilde{\mathcal{F}}^\ast_{\alpha,-}(\lambda)\tilde{f}_1(\lambda)\) by \(C^-_{\alpha}(\lambda)\tilde{w}^-_\alpha(\lambda)\), where
\[ C^-_{\alpha}(\lambda) := 2^{-1}i\pi^{-1/2}\lambda^{-1/4}e^{i\pi(a_\alpha - 3)/4}. \]
Thus, by (8.3) we obtain
\[ (S_\alpha f_1, f_2) = (S_\alpha f_1, f_2) = \int_{E_a} (\Sigma_{\alpha\alpha}(\lambda)\mathcal{R}\tilde{f}_1(\lambda), \tilde{f}_2(\lambda))d\lambda \]
\[ = e^{i\pi(a_\alpha - 1)/2} \int_{E_a} (\Sigma_{\alpha\alpha}(\lambda)\mathcal{R}\tilde{f}_1(\lambda), \tilde{f}_2(\lambda))d\lambda, \]
and therefore, \( 10.11 \) for \( \alpha = \beta \).

(ii) Next we consider \( \beta \alpha, \beta \neq \alpha \). Since \( (\Omega^+)^* \Omega^+ = 0 \)(see e.g. \( 11 \) Theorem 6.15.3), we have

\[
S_{\beta \alpha} = (W^+_{\beta})^*(W^-_{\alpha} - \Omega^+_\alpha \tilde{W}^-_\alpha).
\]

Hence, in the same way as above we obtain

\[
(S_{\beta \alpha} f_1, f_2) = 2\pi i \int_{\max\{E_{\alpha}, E_{\beta}\}} \{(R(\lambda + i0)G_{\alpha, \beta} \tilde{f}_1(\lambda), G^+_{\beta, \alpha}(\lambda) \tilde{f}_2(\lambda)) - (G_{\alpha, \beta} \tilde{f}_1(\lambda), T^+_\beta J^+_\beta \tilde{f}_2(\lambda))\}d\lambda
\]

\[
= 2\pi i \int_{\max\{E_{\alpha}, E_{\beta}\}} \{(R(\lambda + i0)G_{\alpha, \beta} \tilde{f}_1(\lambda), (H - \lambda)T^+_\beta J^+_\beta \tilde{f}_2(\lambda)) - (G_{\alpha, \beta} \tilde{f}_1(\lambda), T^+_\beta J^+_\beta \tilde{f}_2(\lambda))\}d\lambda.
\]

We can replace \( T^+_\beta \tilde{f}_2(\lambda) \) in \(10.12\) by \(-C^+_{\beta}(\lambda)w^+_{\beta}(\lambda)\) where

\[
C^+_{\beta}(\lambda) := 2^{-1}i\pi^{-1/2}\lambda^{-1/4}e^{-i\pi(n_\beta - 3)/4},
\]

and \( (w^+_{\beta}(\lambda))(x_0) = w^+_{\beta}(\lambda, x_0) := \eta(\beta\lambda)\tilde{f}_2(\lambda, \hat{x}_0)e^{iK_\beta(x_0, \lambda)}\tilde{r}_\beta^{-(n_\beta - 1)/2} \).

We shall prove

\[
(G_{\alpha, \beta} \tilde{f}_1(\lambda), J^+_\beta w^+_{\beta}(\lambda)) = \{(H - \lambda)T^+_\alpha J^+_\alpha \tilde{f}_1(\lambda), J^+_\beta w^+_{\beta}(\lambda)) \}
\]

\[
= (T^+_\beta J^+_\beta \tilde{f}_2(\lambda), (H - \lambda)J^+_\beta w^+_{\beta}(\lambda)).
\]

When \( a = b \) and \( \alpha \neq \beta \), by \(10.9\), \( \pi^+ J^+_{a} = 0 \) and \( \pi(\Delta, \alpha)J_{\alpha} = 0 \), \(10.12, 10.13\) holds.

When \( a \neq b \), \( J^+_a \tilde{f}_1(\lambda) \) exponentially decays on \( Y^+_{b, e} \) for any \( \epsilon > 0 \), or \( J^+_b \tilde{f}_2(\lambda) \) exponentially decays on \( Y^+_{a, e} \) for any \( \epsilon > 0 \), where \( Y^+_{a, e} \) is defined as \( 2.2 \).

In the case (a) let \( \chi_j \in C^\infty(X) \), \( j = 1, 2 \) be functions satisfying the following:
there exists \( \epsilon' > 0 \) such that \( \chi_1 = 1 \) on \( \bigcup_{\epsilon \in E} \{x : |x| > 1, 2\epsilon'|x| > |x'|\} \), \( \chi_1 = 0 \) on \( \bigcup_{\epsilon \in E} \{x : |x| > 1, \epsilon'|x| < |x'|\} \), and \( \chi_1 + \chi_2 = 1 \). We also denote by \( \chi_j \) the multiplication operator by \( \chi_j \).

Then, for \( \epsilon' \) sufficiently small \( \chi_j J^+_b \tilde{f}_2(\lambda) \) decays exponentially, and \( J^+_a \tilde{f}_1(\lambda) \) decays exponentially on \( \text{supp} \chi_2 \). Thus we can see that

\[
(\Delta T^+_a J^+_\alpha \tilde{f}_1(\lambda), \chi_j T^+_\beta J^+_\beta \tilde{f}_2(\lambda)) = (T^+_\beta J^+_\beta \tilde{f}_2(\lambda), \Delta \chi_j T^+_\beta J^+_\beta \tilde{f}_2(\lambda)), \quad j = 1, 2
\]

and therefore, \(10.13\) holds.

In the same way we can see that in the case (b) \(10.13\) holds.

Thus, by \(10.12\) we obtain

\[
(S_{\beta \alpha} f_1, f_2) = \tilde{C}_\beta(\lambda) \int_{\max\{E_{\alpha}, E_{\beta}\}} \{(G_{\alpha, \beta} \tilde{f}_1(\lambda), (H - \lambda)J^+_\beta w^+_{\beta}(\lambda)) - (R(\lambda + i0)G_{\alpha, \beta} \tilde{f}_1(\lambda), (H - \lambda)J^+_\beta w^+_{\beta}(\lambda))\}d\lambda.
\]
As in the case of $\hat{S}_{\alpha\alpha}(\lambda)$, we can replace $T_{\alpha}^{-1}F_{\alpha,-}(\lambda)f_1(\lambda)$ by $C_{\alpha}^{-}(\lambda)w_{\alpha}^{-}(\lambda)$, and therefore, we obtain \(11.1\). \(\square\)

11. Equivalence and adjoint operators of generalized Fourier transform

In the following we use the notations in sections 1, 2, 3 and 10.

**Proof of Theorem 11.1 (1).** Let $f \in L^2(X)$, $l > 1/2$ and $f_1 \in C^\infty(X_a)$ satisfy $\hat{f}_1 \in C^\infty(X_a)$ and $\hat{f}_1(\lambda, \cdot) \in C^\infty(C_a)$ for any $\lambda$ such that $\text{supp} \, \hat{f}_1(\lambda, \cdot) \neq 0$. Here $(\hat{f}_1(\lambda))(\lambda, x_a) := (F_{\alpha}f_1)(\lambda, \hat{x}_a)$.

Then, we have

$$((f, T_{\alpha}^+F_{\alpha,+}(\lambda)f_1)) - i \int_{-1}^{1} \{ (f, T_{\alpha}^+J_{\alpha}F_{\alpha,+}^*(\lambda)f_1(\lambda)) - (f, R(\lambda - i0)G_{\alpha}^+F_{\alpha,+}^*(\lambda)f_1(\lambda)) \} \, d\lambda,$$

As in the proof of Theorem 2.2 we can rewrite this as

$$\int_{-1}^{1} \{ (f, T_{\alpha}^+J_{\alpha}F_{\alpha,+}^*(\lambda)f_1(\lambda)) - (f, R(\lambda - i0)(H - \lambda)T_{\alpha}^+J_{\alpha}F_{\alpha,+}(\lambda)f_1(\lambda)) \} \, d\lambda.$$

As in the proof of Theorem 2.2 we can replace $T_{\alpha}^+F_{\alpha,+}(\lambda)f_1(\lambda)$ by $-C_{\alpha}^+(\lambda)\hat{w}_{\alpha}^+(\lambda)$, where

$$(\hat{w}_{\alpha}^+(\lambda))(x_a) = \hat{w}_{\alpha}^+(\lambda, x_a) := \eta(r_a)f_1(\lambda, \hat{x}_a)e^{iK_{\alpha}(x_a, \lambda_a)}r_{\alpha}^{-(n_a - 1)/2}$$

Thus, we obtain

$$((f, T_{\alpha}^+F_{\alpha,+}(\lambda)f_1)) = \hat{C}_{\alpha}^+(\lambda)\int_{-1}^{1} \{ (f, J_{\alpha}w_{\alpha}^+(\lambda)) - (f, R(\lambda - i0)(H - \lambda)J_{\alpha}w_{\alpha}^+(\lambda)) \} \, d\lambda,$$

where $\hat{C}_{\alpha}^+(\lambda) := 2^{-i\pi - 1/2}\lambda_{\alpha}^{-1/4}e^{i\pi(n_a - 3)/4}$. Therefore, by \(\text{Sec. 3}\) we have $(F_{\alpha}(W_{\alpha}^+)^*f)(\lambda) = G_{\alpha}^+(\lambda)f$.

In the same way we obtain

$$(F_{\alpha}(W_{\alpha}^-)^*f, \hat{f}_1) = \hat{C}_{\alpha}^-(\lambda)\int_{-1}^{1} \{ (f, J_{\alpha}w_{\alpha}^-)(\lambda)) - (f, R(\lambda + i0)(H - \lambda)J_{\alpha}w_{\alpha}^-)(\lambda)) \} \, d\lambda,$$

where $(w_{\alpha}^-)(\lambda))(x_a) = w_{\alpha}^-\lambda, x_a) := \eta(r_a)f_1(\lambda, \hat{x}_a)e^{-iK_{\alpha}(x_a, \lambda_a)}r_{\alpha}^{-(n_a - 1)/2}$ and $\hat{C}_{\alpha}^-(\lambda) := 2^{-i\pi - 1/2}\lambda_{\alpha}^{-1/4}e^{-i\pi(n_a - 3)/4}$. Therefore, by \(\text{Sec. 3}\) we have $(F_{\alpha}(W_{\alpha}^-)^*f)(\lambda) = G_{\alpha}^-(\lambda)f$. \(\square\)
Proof of Theorem 2.3 (2). By (11.2) and Theorem 2.3 (1) we can see that

\[(G^\alpha (\lambda))^* = \bar{C}_\alpha^\times (\lambda)P_{\alpha,-}(\lambda).\]

In the same way by (11.3) and Theorem 2.3 (1) we can see that

\[(G^-\alpha (\lambda))^* = C^\times_\alpha (\lambda)P_{\alpha,+}(\lambda)R.\]

Theorem 2.3 (2) follows from (11.4) and (11.5).

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