Theory of Superdualities and the Orthosymplectic Supergroup

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Abstract

We study the dualities for sigma models with fermions and bosons. We found that the generalization of the SO($m, m$) duality for $D = 2$ sigma models and the Sp($2n$) duality for $D = 4$ sigma models is the orthosymplectic duality OSp($m, m|2n$). We study the implications of this and we derive the most general $D = 2$ sigma model, coupled to fermionic and bosonic one-forms, with such dualities. To achieve this we generalize Gaillard-Zumino analysis to orthosymplectic dualities, which requires to define embedding of the superisometry group of the target space into the duality group. We finally discuss the recently proposed fermionic dualities as a by-product of our construction.
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1 Introduction

In the last twenty years dualities have played a major role in understanding non-perturbative aspects of superstring theory. They have indeed unveiled relations between different compactifications of superstring/M-theory, showing that such realizations can be seen as different descriptions of a same microscopic dynamics. It has also been conjectured long ago that superstring dualities are encoded in the global symmetries of the low-energy effective supergravity theory [1].

Until recently dualities have been characterized as mappings between bosonic backgrounds (bosonic dualities) which do not affect the fermions. Among them, T-dualities [2, 3] are mappings between “large”– and “small”–radius compactifications of superstring theory and are realized as non-local redefinitions of the world-sheet bosonic fields (i.e. coordinates on the target space-time). A condition for such a redefinition to be feasible is that the background moduli (supergravity fields) be independent of the “dualized” coordinates.

Berkovits and Maldacena, in [4], introduced a generalization of the bosonic T-duality, called “fermionic” (or super-) T-duality, which also involves the fermionic modes and which is realized as a non-local redefinition of the world-sheet fermions (fermionic coordinates in the background supermanifold). This duality can be consistently defined on superstring backgrounds in which the fields do not explicitly depend on the “dualized” super-coordinates. These ideas have been investigated so far mainly from the world-sheet point of view on specific space-time backgrounds [5, 6, 7] Therefore a general, background independent, characterization of “fermionic” dualities is still missing.

In the present paper we make progress in this direction by defining a supersymmetric sigma model which is globally invariant with respect to a super-duality group.

We consider a globally supersymmetric sigma model in two dimensions, coupled to a set of scalar and fermion one-forms behaving as field strengths of scalar and fermion “0-form fields”. The sigma model scalar fields span a supermanifold $\mathcal{SM}^{(x|y)}$ of the form $\hat{G}/\hat{H}$. We define, generalizing the Gaillard-Zumino construction [8], the most general coupling of the super-sigma model fields to the bosonic and fermionic one-forms which allows to promote the super-isometry group $\hat{G}$ of $\mathcal{SM}^{(x|y)}$ to global super-symmetry of the whole model, namely to a super-duality. This requires an embedding of $\hat{G}$ in the supergroup $\text{OSp}(m,m|4n)$, and of $\hat{H}$ inside $\text{OSp}(m|2n) \times \text{OSp}(m|2n)$, where $m$ and $2n$ are the numbers of bosonic and fermionic one-forms respectively.

The form of the sigma model is suggested by the Pure Spinor Formulation of string theory
where the bosonic and fermionic degrees of freedom are treated on the same footing. The fields appearing in our sigma models are divided into two sets of fields: the proper scalars and fermions parameterizing a homogenous supermanifold and 0-form fields (part of which can be interpreted as Matzner-Missner fields). They are identified with the coordinates of a superspacetime. On specific backgrounds, for example on $AdS_p \times S^{10-p}$, part of the bosonic and fermionic coordinates on the associated superspace, are treated as the proper fields and the others, as the 0-form fields.

By coupling the model to supergravity background and eliminating the auxiliary fields, one can recast the sigma model in the form discussed here. In that case, the couplings acquire a physical interpretation.

The paper is organized as follows. In section 2 we consider the case of a bosonic sigma model in $D = 2p$ dimensions, with a homogeneous target space of the form $G/H$, coupled to a number $n$ of $p$-form field strengths, and review the Gaillard-Zumino construction of the general form of this coupling which allows to promote the isometry group $G$ to a global on-shell symmetry of the full model. This construction requires the isometry group $G$ to be embedded in the groups $Sp(2n, \mathbb{R})$ and $SO(n, n)$ for even and odd $p$ respectively. We will then specialize our discussion, in Section 3, to $D = 2$ and consider a generic bosonic sigma model coupled to one-form field strengths, which may be seen as originating from a dimensional reduction of the bosonic sector of a $D = 4$ supergravity, as it is explained in Section 4. In addition, we discuss the relation between our sigma model and the Green-Schwarz sigma model on a given supergravity background. In the last section, we further elaborate on this identification.

In Section 5 we extend this analysis to a $D = 2$ super-sigma model coupled to a generic number of scalar and fermion one-form field strengths ($p = 1$). Our construction in this more general setting requires the sigma model super-manifold $\hat{G}/\hat{H}$ to be embedded in $OSp(m, m|4n)/[OSp(m|2n) \times OSp(m|2n)]$. In Section 6 we explicitly define such embedding on the case in which $S\mathcal{M}^{(x|y)} = \frac{OSp(p,p|4r)}{SO(p) \times SO(p) \times U(2r)} \times \frac{OSp(q,q|2s)}{SO(q) \times SO(q) \times U(s)}$ and the number of bosonic and fermionic one-forms are $m = 2pq + 4rs$ and $n = 2ps + 4qr$ respectively. In Section 5.1 we discuss in detail the action of the superdualities on the fields of the model and work out an explicit example.
2 Electric/magnetic dualities in $D = 4$ and $D = 2$ bosonic field theories

We first review the bosonic set up of electric/magnetic duality rotations in $D=4$ and $D=2$. In both cases, the relevant Lagrangian is constructed with a set of scalars parameterizing a coset manifold $G/H$ to which we add with a set of $\frac{D-2}{2}$-forms (vector fields for $D=4$, “0-form” scalars for $D=2$) on whose electric/magnetic field strengths (two-forms for $D=4$, one-forms for $D=2$) the group $G$ acts by means of linear transformations (symplectic in the $D=4$ case, pseudo-orthogonal in $D=2$ case). Hence, given an element $g$ of the isometry group $G$, there must be an embedding $g \rightarrow \Lambda_g \in \text{Sp}(2n_V, \mathbb{R})$ for $D = 4$ and $g \rightarrow \Lambda_g \in \text{SO}(n_S, n_S)$ for $D=2$, which leads to the construction of the kinetic terms for the vectors in $D=4$ and for the 0-form scalars in $D=2$ by the using the so called Gaillard-Zumino (GZ) formula. The transformations of $\text{Sp}(2n_V, \mathbb{R})$ and of $\text{SO}(n_S, n_S)$, which are not in the image of $G$ through this embedding, correspond to the possible non-trivial duality transformations leading to new theories.

2.1 $D = 4$ bosonic supergravity and its dualities

In $D = 4$ ungauged supergravity, with $N_Q$ supercharges, the bosonic Lagrangian admits the following general form

\[
\mathcal{L}^{(4)} = \sqrt{\det g} \left[ -2R[g] - \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^b h_{ab}(\phi) + \text{Im} N_{\Lambda \Sigma} F^\Lambda_{\mu \nu} F^\Sigma_{\rho \sigma} \epsilon^{\mu \nu \rho \sigma} \right] + \frac{1}{2} \text{Re} N_{\Lambda \Sigma} F^\Lambda_{\mu \nu} F^\Sigma_{\rho \sigma} \epsilon^{\mu \nu \rho \sigma}
\]

where $\phi^a$ denotes the whole set of $n_S$ scalar fields parameterizing the scalar manifold $\mathcal{M}^{D=4}_{\text{scalar}}$ which, for $N_Q > 8$, is necessarily a coset manifold:

\[
\mathcal{M}^{D=4}_{\text{scalar}} = \frac{G}{H}
\]

(2.1)

For $N_Q \leq 8$, condition (2.2) is not implied by supersymmetry. However $\mathcal{N} = 2$ supergravity, i.e. for $N_Q = 8$, a large variety of homogeneous special Kähler manifolds fall into the set up of the present general discussion. The fields $\phi^a$ have $\sigma$–model interactions dictated by the metric $h_{ab}(\phi)$ of $\mathcal{M}^{D=4}_{\text{scalar}}$.

The theory includes also $n_V = n$ vector fields $A^\Lambda_{\mu}$ for which

\[
F^\pm |^\Lambda_{\mu \nu} \equiv \frac{1}{2} \left( F^\Lambda_{\mu \nu} \pm i \ast F_{\mu \nu} \right), \text{ where } \ast F_{\mu \nu} \equiv \frac{\sqrt{\det g}}{2} \epsilon_{\mu \nu \rho \sigma} F^{\rho \sigma},
\]

\[\]

(2.3)
denote the self-dual (respectively antiself-dual) parts of the field-strengths: $^*\mathcal{F}_{\pm|\Lambda} = \mp i \mathcal{F}_{\pm|\Lambda}$. As displayed in eq.\textsuperscript{(2.1)} they are non-minimally coupled to the scalars via the symmetric complex matrix

$$\mathcal{N}_{\Lambda\Sigma}(\phi) = i \text{Im} \mathcal{N}_{\Lambda\Sigma} + \text{Re} \mathcal{N}_{\Lambda\Sigma}$$

(2.4)

Following the notations and the conventions of \cite{10}, it can be shown that the isometry group $\mathcal{G}$, global symmetry of the sigma model action, can be promoted to global on-shell symmetry of the theory, provided the following symplectic embedding of $\mathcal{G}$ is defined:

$$\mathcal{G} \mapsto \text{Sp}(2n, \mathbb{R}) ; \quad n = n_V \equiv \# \text{ of vector fields} \quad (2.5)$$

which associates with each element of $\mathcal{G}$ a symplectic electric-magnetic duality transformation on the field strengths $F_{\mu\nu}^\Lambda$ plus their magnetic duals. The latter therefore define a symplectic representation $\mathcal{W}$ of $\mathcal{G}$.

More specifically, the embedding (2.5) implies that each element $\xi \in \mathcal{G}$ is represented by means of a suitable real symplectic matrix:

$$\xi \mapsto \Lambda_\xi \equiv \begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix}$$

(2.6)

satisfying the defining relation:

$$\Lambda_\xi^T \begin{pmatrix} 0_{n \times n} & 1_{n \times n} \\ -1_{n \times n} & 0_{n \times n} \end{pmatrix} \Lambda_\xi = \begin{pmatrix} 0_{n \times n} & 1_{n \times n} \\ -1_{n \times n} & 0_{n \times n} \end{pmatrix} \quad (2.7)$$

which implies the following relations on the $n \times n$ blocks:

$$A_\xi^T C_\xi - C_\xi^T A_\xi = 0$$
$$A_\xi^T D_\xi - C_\xi^T B_\xi = 1$$
$$B_\xi^T C_\xi - D_\xi^T A_\xi = -1$$
$$B_\xi^T D_\xi - D_\xi^T B_\xi = 0$$

(2.8)

Under an element of the duality group the field strengths transform as follows:

$$\begin{pmatrix} \mathcal{F}_{\pm|\Lambda} \\ \mathcal{G}_{\pm} \end{pmatrix}' = \begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix} \begin{pmatrix} \mathcal{F}_{\pm|\Lambda} \\ \mathcal{G}_{\pm} \end{pmatrix},$$

(2.9)

\footnote{Note that, in this conventions, the physical domain of the kinetic terms for the vector fields is defined by the condition $\text{Im} \mathcal{N} < 0.$}
where, by their own definitions:

\[ \mathcal{G}^+ = \mathcal{N} \mathcal{F}^+ \quad ; \quad \mathcal{G}^- = \overline{\mathcal{N}} \mathcal{F}^- \quad (2.10) \]

Consistency of eq. (2.10) with the transformation (2.9) is guaranteed by the symplectic property of \( \Lambda_\xi \), provided the complex symmetric matrix \( \mathcal{N} \) transforms as follows:

\[ \mathcal{N}' = (C_\xi + D_\xi \mathcal{N}) (A_\xi + B_\xi \mathcal{N})^{-1} \quad (2.11) \]

The condition (2.5), which defines the duality action of the isometry group of the scalar manifold, also holds in \( D > 4 \) even dimensions, with \( D/2 \) even as well. In this case \( n \) denotes the number of \( (D-2)/2 \)-forms (vector fields in \( D = 4 \) and rank-three antisymmetric tensor fields in \( D = 8 \)). For \( D \)-even but \( D/2 \) odd, \( n \) still refers to the number of \( (D-2)/2 \)-forms (scalar fields in \( D = 2 \), rank-two antisymmetric tensor fields in \( D = 6 \) and rank-four antisymmetric tensor fields in \( D = 10 \)), but the action on their field strengths and their duals is defined by the embedding of \( \mathcal{G} \) in the pseudo-orthogonal group \( \text{SO}(n,n) \).

### 2.1.1 Symplectic \( \text{Sp}(2m, \mathbb{R}) \) embeddings and the Gaillard-Zumino formula for the period matrix \( \mathcal{N} \)

Focusing on the isometry group of the canonical metric defined on \( \overline{\mathcal{G}} / \mathbb{R} \), we must consider the embedding:

\[ \iota_\xi : \overline{\mathcal{G}} \rightarrow \text{Sp}(2n, \mathbb{R}) \quad (2.12) \]

This is an homomorphism between finite dimensional Lie groups and as such it can be determined explicitly. What we just need to know is the dimension of the symplectic group, namely the number \( n \) of \( \frac{D-2}{2} \)-forms appearing in the theory. In \( D = 4 \), for example \( n \) coincides with the number of vector fields \( n_{V} \), without supersymmetry, the dimension \( n_S \) of the scalar manifold\(^2\) (namely the possible choices of \( \overline{\mathcal{G}} / \mathbb{R} \)) and the number of vectors \( n_{V} \) are unrelated so that the possibilities covered by eq. (2.12) are infinitely many. In supersymmetric theories with number of the supercharges is greater or equal to eight, instead, the two numbers \( n_S \) and \( n_{V} \) are related (\( n_S \) being the number of scalar fields in the same supermultiplets as the vector fields), so that there are finitely many cases to be studied corresponding to the possible embeddings of given groups \( \mathcal{G} \) into a symplectic group \( \text{Sp}(2n, \mathbb{R}) \) of a given \( n \).

\(^2\)We restrict here and in the following to the manifold spanned by the scalar fields which couple to the vectors through the kinetic matrix \( \mathcal{N} \).
Apart from the details of the specific case considered once a symplectic embedding is given there is a general formula one can write down for the period matrix $\mathcal{N}$ that guarantees symmetry ($\mathcal{N}^T = \mathcal{N}$) and the required transformation properties.

The real symplectic group $\text{Sp}(2n, \mathbb{R})$ is defined as the set of all real $2n \times 2n$ matrices with an algebraic condition

$$\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \Lambda^T C \Lambda = C \quad (2.13)$$

where

$$C \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.14)$$

We can change the basis of the fundamental symplectic representation to a complex one, defined by the action of the Cayley matrix:

$$C_{\text{Sp}} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i1 \\ 1 & -i1 \end{pmatrix} \quad (2.15)$$

which is adapted for $\text{Sp}$ groups. In this new basis the generic symplectic matrix $\Lambda$ in (2.13) will become a complex matrix $S$ defined as follows:

$$S \equiv C \Lambda C^{-1} = \begin{pmatrix} T & V^* \\ V & T^* \end{pmatrix} \quad (2.16)$$

where

$$T = \frac{1}{2} (A - iB) + \frac{1}{2} (D + iC) \quad ; \quad V = \frac{1}{2} (A - iB) - \frac{1}{2} (D + iC) \quad (2.17)$$

In this complex basis the $U(n)$ transformations inside $\text{Sp}(2n, \mathbb{R})$ have a block diagonal form:

$$h \in U(n) \ ; \ h = \begin{pmatrix} W & 0 \\ 0 & W^* \end{pmatrix}, \quad WW^\dagger = W^\dagger W = 1 \quad (2.18)$$

$^3$The matrix $\mathcal{N}$ is named period matrix because of its meaning when the considered four-dimensional supergravity is obtained by compactification of 10D type IIB supergravity on a Calabi-Yau three-fold. In that case the matrix $\mathcal{N}$, in full analogy with the period matrix of Riemann surfaces, is obtained by considering the matrix whose entries are the periods of the cohomology three-forms on a basis of homology three-cycles. The symplectic transformations are those which leave invariant the intersection matrix of the three-cycles.
The basic idea, to obtain the general formula for the matrix $\mathcal{N}$, is that the symplectic embedding of the isometry group $\mathcal{G}$ will be such that the isotropy subgroup $\mathcal{H} \subset \mathcal{G}$ gets embedded into the maximal compact subgroup $\text{U}(n)$, namely:

$$\mathcal{G} \xrightarrow{\iota_S} \text{Sp}(2n, \mathbb{R}); \quad \mathcal{H} \xrightarrow{\iota_S} \text{U}(n) \subset \text{Sp}(2n, \mathbb{R}) \quad (2.19)$$

If this condition is realized let $\mathbb{L}(\phi)$ be a parametrization of the coset $\mathcal{G}/\mathcal{H}$ by means of coset representatives. By this we mean the following. Let $\phi^I$ be local coordinates on the manifold $\mathcal{G}/\mathcal{H}$: To each point $\phi \in \mathcal{G}/\mathcal{H}$ we assign an element $\mathbb{L}(\phi) \in \mathcal{G}$ in such a way that if $\phi' \neq \phi$, then no $h \in \mathcal{H}$ can exist such that $\mathbb{L}(\phi') = \mathbb{L}(\phi) \cdot h$. In other words for each equivalence class of the coset (labelled by the coordinate $\phi$) we choose one representative element $\mathbb{L}(\phi)$ of the class. Relying on the symplectic embedding of eq. (2.19) we obtain the symplectic representation of the coset representative in the real basis:

$$\mathbb{L}(\phi) \rightarrow \mathcal{S}_{\text{Sp}}(\phi) = \begin{pmatrix} A(\phi) & B(\phi) \\ C(\phi) & D(\phi) \end{pmatrix}, \quad (2.20)$$

Since the coset representative is acted from the right and from the left by elements of different groups, namely $\mathcal{G}$ and $\mathcal{H}$ respectively, we may use different bases for the the two indices of its matrix representation and choose the real basis for the rows and the complex one for the columns. We then define the following mixed representation of $\mathbb{L}(\phi)$:

$$\mathcal{S}_{\text{mr}}(\phi) \equiv \begin{pmatrix} \mathbf{f}(\phi) & \mathbf{f}(\phi)^* \\ \mathbf{h}(\phi) & \mathbf{h}(\phi)^* \end{pmatrix} = \mathcal{S}_{\text{Sp}}(\phi) \mathcal{C}_{\text{Sp}}^{-1}, \quad (2.21)$$

where

$$\mathbf{f}(\phi) = \frac{1}{\sqrt{2}} \left( A(\phi) - i B(\phi) \right), \quad \mathbf{h}(\phi) = \frac{1}{\sqrt{2}} \left( C(\phi) - i D(\phi) \right). \quad (2.22)$$

From the relations between the real blocks $A, B, C, D$ of a symplectic matrix, expressed by the general equations (2.8), one can verify that the $\mathbf{f}$ and $\mathbf{h}$ blocks satisfy the following conditions:

$$-i \mathbf{1} = \mathbf{f}^\dagger \mathbf{h} - \mathbf{h}^\dagger \mathbf{f}, \quad \mathbf{f}^T \mathbf{h} - \mathbf{h}^T \mathbf{f} = \mathbf{0}, \quad (\mathbf{f} \mathbf{f}^\dagger)^T = \mathbf{f} \mathbf{f}^\dagger, \quad (\mathbf{h} \mathbf{h}^\dagger)^T = \mathbf{h} \mathbf{h}^\dagger, \quad \mathbf{f} \mathbf{h}^\dagger - \mathbf{h} \mathbf{f}^\dagger = i \mathbf{1}. \quad (2.23)$$

The action of an isometry $\xi \in \mathcal{G}$, represented by the real symplectic matrix $\Lambda_\xi$ defined in eq. (2.6), on a point $\phi^I$ of the manifold is then described as follows:

$$\Lambda_\xi \mathcal{S}_{\text{mr}}(\phi) = \mathcal{S}_{\text{mr}}(\xi(\phi)) \begin{pmatrix} W(\xi, \phi) & \mathbf{0} \\ \mathbf{0} & W(\xi, \phi)^* \end{pmatrix}, \quad (2.24)$$
where $\xi(\phi)$ denotes the image of the point $\phi$ through $\xi$ and $W(\xi, \phi)$ is a suitable $U(n)$ compensator depending on both $\xi$ and $\phi$. Combining eq.s (2.24), (2.22), with eq (2.6) we immediately obtain:

\[
\begin{align*}
\mathbf{f}(\xi(\phi)) &= [A_\xi \mathbf{f}(\phi) + B_\xi \mathbf{h}(\phi)] W(\xi, \phi)^*, \\
\mathbf{h}(\xi(\phi)) &= [C_\xi \mathbf{f}(\phi) + D_\xi \mathbf{h}(\phi)] W(\xi, \phi)^*,
\end{align*}
\]

(2.25)

If we define the $n \times n$ matrix $\mathcal{N}$ as follows:

\[
\mathcal{N}(\phi) \equiv \mathbf{h}(\phi) \mathbf{f}(\phi)^{-1} = [C(\phi) - iD(\phi)] [A(\phi) - iB(\phi)]^{-1},
\]

(2.26)

it is straightforward to verify that, under a generic isometry $\xi$, it transforms as in eq. (2.11).

It is also an immediate consequence of the last of eq.s (2.23), satisfied by $\mathbf{f}$ and $\mathbf{h}$, that the matrix in eq.(2.26) is symmetric

\[
\mathcal{N}^T = \mathcal{N}
\]

(2.27)

Eq. (2.26) is the master-formula derived by Gaillard and Zumino [8]. It explains the structure of the gauge field kinetic terms in all $\mathcal{N} \geq 3$ extended supergravity theories and also in those $\mathcal{N} = 2$ theories where the special Kähler manifold $\mathcal{SM}$ is a homogeneous of the form $\mathcal{G}/\mathcal{H}$. In the following, we will derive the same formula for the orthogonal embeddings.

Notice that, at the origin of the manifold, $\phi = 0$, $\mathcal{S}_\text{Sp}$ is the $2n \times 2n$ identity matrix, namely $A(0) = D(0) = 1$ and $B(0) = C(0) = 0$. From (2.26) and (2.22) we find that $\mathcal{N}(0) = -i \mathbf{1}$. One can verify that $\text{Im}(\mathcal{N}(\phi))$ is negative definite for any $\phi$.

### 2.2 $D = 2$ bosonic sigma model and its dualities

Let us now consider the general form of a Lagrangian in $D = 2$. Here we have two types of scalars, namely the proper scalars $\phi^a$ and the twisted scalars or scalar 0-forms $\pi^\alpha$. This distinction is important. The proper scalars appear in the Lagrangian under the form of a usual $\sigma$-model, as coordinates on the target manifold $\mathcal{G}/\mathcal{H}$, while the scalar 0-forms appear only covered by derivatives and in two terms, one symmetric, one antisymmetric. The coefficients of these two terms are matrices depending on the proper scalars. Explicitly the Lagrangian has the form (see [11] for a general review) in the conformal gauge $g_{\mu\nu} = \delta_{\mu\nu}$:

\[
S_{(D=2)} = \int d^2 x \left\{ -\frac{1}{2} h_{ab}(\phi) \partial_\mu \phi^a \partial_\mu \phi^b + \frac{1}{2} \kappa \left[ -\partial_\mu \pi^\alpha \gamma_{\alpha\beta}(\phi) \partial_\mu \pi^\beta + \partial_\mu \pi^\alpha \theta_{\alpha\beta}(\phi) \partial_\nu \pi^\beta \epsilon^{\mu\nu} \right] \right\}
\]

(2.28)
where $\kappa$ is a normalization parameter that can always be reabsorbed into the definition of the 0-forms $\pi^\alpha$ and where, according to the general theory for the dimensions $D = 4\nu + 2$ (see section 2.4 of [10]) if $G$ is the isometry group of the $\sigma$-model metric $h_{ab}(\phi)$, then there is a pseudo-orthogonal embedding:

$$G \hookrightarrow \text{SO}(m, m)$$

where $m$ is the total number of the scalar-forms $\pi^\alpha$. Hence for each element $\xi \in G$ we have its representation by means of a suitable pseudo-orthogonal matrix:

$$\xi \mapsto \Lambda \equiv \begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix}$$

which satisfies the defining equation:

$$\Lambda^T \begin{pmatrix} 0_{m\times m} & 1_{m\times m} \\ 1_{m\times m} & 0_{m\times m} \end{pmatrix} \Lambda = \begin{pmatrix} 0_{m\times m} & 1_{m\times m} \\ 1_{m\times m} & 0_{m\times m} \end{pmatrix}$$

(2.30)

implying the following relations on the $m \times m$ blocks:

$$\begin{align*}
A_\xi^T C_\xi + C_\xi^T A_\xi &= 0 \\
A_\xi^T D_\xi + C_\xi^T B_\xi &= 1 \\
B_\xi^T C_\xi + D_\xi^T A_\xi &= 1 \\
B_\xi^T D_\xi + D_\xi^T B_\xi &= 0
\end{align*}$$

(2.31)

Let us now introduce the $D = 2$ analogue of the $D = 4$ period matrix $\mathcal{N}$. It is the following $m \times m$ matrix:

$$\mathcal{M} \equiv \theta + \gamma$$

(2.32)

which also deserves the name of period matrix. Indeed if we were to think of the considered $D = 2$ theory as the result of a compactification of $D = 10$ supergravity on a Calabi-Yau four-fold, $\mathcal{M}$ could be interpreted as the matrix of periods of the cohomology four-forms on a basis of homology four-cycles. The pseudo-orthogonal character arises from preservation of the intersection matrix of such cycles which in this case is symmetric rather than antisymmetric. Under the group $G$ the matrix $\mathcal{M}$ transforms as follows:

$$\mathcal{M}' = (C_\xi + D_\xi \mathcal{M}) (A_\xi + B_\xi \mathcal{M})^{-1}$$

(2.33)

and

$$-\mathcal{M}' = (C_\xi - D_\xi \mathcal{M}^T) (A_\xi - B_\xi \mathcal{M}^T)^{-1}$$

(2.34)
2.2.1 Pseudo-orthogonal $\text{SO}(m, m)$ embeddings and the Gaillard-Zumino formula for the period matrix $\mathcal{M}$

Let us now repeat the Gaillard-Zumino construction of the kinetic matrix $\mathcal{M}$ for the case of pseudo-orthogonal embeddings. The maximal non-compact real section of the $D_m$ Lie algebra is $\mathfrak{so}(m, m)$ which exponentiates to the group $\text{SO}(m, m)$.

In the symplectic case, relevant to $D = 4$ theories, we used two bases, related by a Cayley transformation, where the matrices of both the group and the algebra were either symplectic real or symplectic complex and pseudo-unitary at the same time. The reason for this double presentation of the group/algebra elements was that the two bases have complementary virtues. In the former, the duality rotations are simply and linearly realized on the electric-magnetic field strengths, yet the maximal compact subalgebra of the duality-algebra is not realized by block-diagonal matrices. In the latter, the maximal compact subalgebra is block-diagonal, but the action on the physical field strengths is not the simplest.

The same situation occurs in the pseudo-orthogonal case, relevant for $D = 2$ theories. Also here it is convenient to use two bases, the first optimizing the form of the electric-magnetic duality rotations, the second in which the maximal compact subalgebra $\mathfrak{so}(m) \oplus \mathfrak{so}(m) \subset \mathfrak{so}(m, m)$ has a block-diagonal representation. The difference is that in the $\mathfrak{so}(m, m)$ case both bases are real and the Cayley transformation relating them is also a real matrix. This goes hand in hand with the fact that the period matrix $\mathcal{M}$ of $D = 2$ twisted scalars is a real matrix while the kinetic matrix of $D = 4$ one-forms is a symmetric complex matrix.

In the $D = 4$ case the separation between generalized coupling constants and generalized theta-angles is performed by singling out the real and imaginary part of $\mathcal{N}$. In $D = 2$ the same separation is performed by splitting $\mathcal{M}$ into its symmetric and antisymmetric parts.

The two bases are defined by giving the form of the pseudo-orthogonal invariant metric $\mathbb{C}$. In the first basis, which we name off-diagonal, $\mathbb{C}$ has the following appearance:

$$
\mathbb{C}_{\text{off}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0_{m \times m} & 1_{m \times m} \\ 1_{m \times m} & 0_{m \times m} \end{pmatrix}
$$

In the second basis, named diagonal, the invariant matrix is the following:

$$
\mathbb{C}_{\text{dia}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \begin{pmatrix} 1_{m \times m} & 0_{m \times m} \\ 0_{m \times m} & -1_{m \times m} \end{pmatrix}
$$

The change from the off-diagonal to the diagonal basis is performed by the following
generalized Cayley matrix:
\[ C_{so} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} ; \quad C_{so} C_{so} = 1 ; \quad C_{so} = C_{so}^T \] (2.37)

which satisfies the relation:
\[ C_{so} C_{off} C_{so} = C_{dia} \] (2.38)

Let us define elements of the SO(m,m) group and of the so(m,m) Lie algebra in the off-diagonal basis:

\[ \text{SO}(m,m) \ni \Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow \Lambda^T C_{off} \Lambda = C_{off} \] (2.39)

\[ \text{so}(m,m) \ni \mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow \mathcal{L}^T C_{off} + C_{off} \mathcal{L} = 0 \]

By explicit evaluation of the Lie algebra conditions we find:
\[ \mathcal{D} = -A^T ; \quad \mathcal{B} = -B^T ; \quad \mathcal{C} = -C^T \] (2.40)

Let us now consider the image of the Lie algebra element \( \mathcal{L} \) in the diagonal-basis:

\[ S \equiv C_{so} \mathcal{L} C_{so} = \frac{1}{2} \begin{pmatrix} (A + B + C + D) & (A - B + C - D) \\ (A + B - C - D) & (A - B - C + D) \end{pmatrix} \] (2.41)

If we impose the conditions \( A = D \) and \( B = C \) we obtain the subalgebra \( \text{so}(m) \times \text{so}(m) \subset \text{so}(m,m) \):

\[ \text{so}(m) \times \text{so}(m) \ni \begin{pmatrix} A + B & 0 \\ 0 & A - B \end{pmatrix} ; \quad A = -A^T ; \quad B = -B^T \] (2.42)

The basic strategy to obtain the general formula for the kinetic matrix \( \mathcal{M} \) of m-twisted scalars coupled to a \( \mathcal{G}/\mathcal{H} \) sigma-model is completely analogous to that employed in \( D = 4 \) theories. There must exist a pseudo-orthogonal embedding

\[ \boxed{\mathcal{G} \rightarrow \text{SO}(m,m)} \] (2.43)

such that the isotropy subgroup \( \mathcal{H} \subset \mathcal{G} \) gets embedded into the maximal compact subgroup \( \text{SO}(m) \times \text{SO}(m) \), namely:

\[ \mathcal{G} \rightarrow \text{SO}(m,m) ; \quad \mathcal{H} \rightarrow \text{SO}(m) \times \text{SO}(m) \] (2.44)
Relying on the orthogonal embedding of eq. (2.44) we obtain the following pseudo-orthogonal representation of the coset representative \( \mathbb{L} \):

\[
\mathbb{L}(\phi) \longrightarrow \begin{pmatrix} \mathbf{A}(\phi) & \mathbf{B}(\phi) \\ \mathbf{C}(\phi) & \mathbf{D}(\phi) \end{pmatrix} \equiv \mathbf{O}(\phi) \in \text{SO}(m, m)
\]

\[
\mathbf{O}^T(\phi) \mathbf{C}_{\text{off}} \mathbf{O}(\phi) = \mathbf{C}_{\text{off}} \quad (2.45)
\]

Next in full analogy with equation (2.21) let us introduce the mixed-basis representation of the coset representative \( \mathbf{O}_{\mathbf{mr}}(\phi) \equiv \begin{pmatrix} \mathbf{f}(\phi) & \tilde{\mathbf{f}}(\phi) \\ \mathbf{h}(\phi) & \tilde{\mathbf{h}}(\phi) \end{pmatrix} \equiv \mathbf{O}(\phi) \mathbf{C}_{\text{so}}, \quad (2.46)
\]

where, by explicit evaluation we have:

\[
\mathbf{f}(\phi) = \frac{1}{\sqrt{2}} (\mathbf{A} + \mathbf{B}) ; \quad \tilde{\mathbf{f}}(\phi) = \frac{1}{\sqrt{2}} (\mathbf{A} - \mathbf{B})
\]

\[
\mathbf{h}(\phi) = \frac{1}{\sqrt{2}} (\mathbf{C} + \mathbf{D}) ; \quad \tilde{\mathbf{h}}(\phi) = \frac{1}{\sqrt{2}} (\mathbf{C} - \mathbf{D}) \quad (2.47)
\]

and, from the pseudo-orthogonal relations imposed on the \( \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \) blocks we deduce the following relations:

\[
\mathbf{f}^T \mathbf{h} + \mathbf{h}^T \mathbf{f} = 1 \quad ; \quad \tilde{\mathbf{f}}^T \mathbf{h} + \tilde{\mathbf{h}}^T \mathbf{f} = 0 \quad (2.48)
\]

that are the pseudo-orthogonal counterpart of the symplectic relations (2.23). Then in full analogy with the symplectic \( D = 4 \) case the period matrix \( \mathcal{M}(\phi) \) can be defined as:

\[
\mathcal{M}(\phi) \equiv \mathbf{h}(\phi) \mathbf{f}(\phi)^{-1} = [\mathbf{C}(\phi) + \mathbf{D}(\phi)] [\mathbf{A}(\phi) + \mathbf{B}(\phi)]^{-1} \quad (2.49)
\]

By the same token as in the previous case, the period matrix \( \mathcal{M}(\phi) \) given by the generalized Gaillard-Zumino formula (2.49) transforms correctly under the action of the group \( \mathcal{G} \), namely as in eq.s (2.33).

### 3 Electric/magnetic superdualities in \( D = 2 \) Bose-Fermi field theories

We can now consider the generalization of the \( D = 2 \) action (2.28) to the case where the fields are both of Bose and Fermi type. It has the following form (we have choosen, without loosing in generality, the conformal gauge for the worldsheet metric):

\[
\mathcal{S}_{(D=2)} = \int d^2x \left\{ -\frac{1}{2} H_{AB}(\Phi) \partial_\mu \Phi^A \partial^\mu \Phi^B \\
+ \frac{1}{2} \kappa \left[ -\partial_\mu \Pi^\Gamma \Gamma_{\Sigma\Lambda}(\Phi) \partial^\mu \Pi^\Lambda + \partial_\mu \Pi^\Gamma \Theta_{\Sigma\Lambda}(\Phi) \partial^\mu \Pi^\Lambda \epsilon^{\mu\nu} \right]\right\} \quad (3.1)
\]
where Φ are super-coordinates parameterizing a supermanifold \( SM^{(x|y)} \), which can be, in particular, a supercoset \( \hat{G}/\hat{H} \), while \( \Pi^\Sigma \) are a set of \((m|2n)\) fields of which the former \( m \) are bosonic, while the latter \( 2n \) are fermionic. The notation is as follows:

\[
A = \left\{ a_1, \ldots, x, \bar{a}_1, \ldots, \bar{x} \right\} \quad \text{(3.2)}
\]

\[
\Lambda = \left\{ \alpha_1, \ldots, m, \bar{\alpha}_1, \ldots, \bar{2n} \right\} \quad \text{(3.3)}
\]

where we have used the convention that unbarred indices are bosonic, while barred ones are fermionic. Furthermore \( x \) is the bosonic dimension of the supermanifold \( SM^{(x|y)} \), while \( y \) denotes its fermionic dimension. Let us also note that, taking into account the statistics of the fields, we have the following graded symmetry and graded antisymmetry of the super-matrices entering action (3.1):

\[
H_{AB}(\Phi) = (-)^{AB} H_{BA}(\Phi) \quad \Rightarrow \quad \begin{cases} 
H_{ab} = H_{ba} \\
H_{\bar{a}\bar{b}} = -H_{\bar{b}\bar{a}} \\
H_{\bar{a}b} = -H_{\bar{b}a} \\
\Gamma_{\alpha\beta} = \Gamma_{\beta\alpha} \\
\Gamma_{\bar{\alpha}\bar{\beta}} = -\Gamma_{\bar{\beta}\bar{\alpha}} \\
\Gamma_{\bar{\alpha}\beta} = \Gamma_{\bar{\beta}\alpha} \\
\Theta_{\alpha\beta} = -\Theta_{\beta\alpha} \\
\Theta_{\bar{\alpha}\bar{\beta}} = \Theta_{\bar{\beta}\bar{\alpha}} \\
\Theta_{\bar{\alpha}\beta} = -\Theta_{\bar{\beta}\alpha}
\end{cases} \quad \text{(3.4)}
\]

Let us now consider the supergroup \( \hat{G} \) of super-isometries of the \( \sigma \)-model super-metric \( H_{AB}(\Phi) \). In the case \( SM^{(x|y)} \) is a supercoset, \( \hat{G} \) coincides with the numerator supergroup, yet what we are about to say has a wider range of validity: It is not necessary that \( SM^{(x|y)} \) be a homogeneous supermanifold, it is sufficient for it to have some (super)-group of isometries \( \hat{G} \) that can be continuous or even discrete.

In papers [12, 4], it has been shown that the GS action (or the corresponding pure spinor action) displays a similar structure on \( AdS_5 \times S^5 \) background. In that case, 4 bosonic coordinates (out of the 10 of the spacetime) and 8 fermionic coordinates can be taken as the fields \( \Pi^\Lambda \) of our sigma model. The other coordinates enter the couplings \( \Gamma \) and \( \Theta \) and they can be view as the coordinates of the manifold \( SM^{(x|y)} \).

The important point to stress is that, in full analogy to purely bosonic theories the symmetries of the \( \Phi \) sector of the Lagrangian can be extended to a duality symmetry of
the equations of motion and Bianchi identities of the complete theory (including also the II.s) if and only if the following two conditions are satisfied:

a There exists an orthosymplectic embedding:

\[ \hat{\mathcal{G}} \hookrightarrow \text{OSp}(m, m|4n) \]  

(3.5)

b The kinetic matrix:

\[ \hat{\mathcal{M}}(\Phi) \equiv \hat{\Gamma}(\Phi) + \hat{\Theta}(\Phi) \]  

(3.6)

is acted on by the OSp(m, m|4n) realization of \( \hat{\mathcal{G}} \) with suitable fractional linear transformations.

Let us prove the above statements in some detail.

3.1 Orthosymplectic duality symmetries

Let us define the required orthosymplectic embedding of the isometry supergroup. Each element \( \hat{\xi} \in \hat{\mathcal{G}} \) is mapped into a graded matrix:

\[ \hat{\xi} \mapsto \hat{\Lambda}_\xi \equiv \begin{pmatrix} \hat{A}_\xi & \hat{B}_\xi \\ \hat{C}_\xi & \hat{D}_\xi \end{pmatrix} \]  

(3.7)

that satisfies the defining equation:

\[ \hat{\Lambda}_\xi^T \hat{\mathcal{C}}_{\text{off}} \hat{\Lambda}_\xi = \hat{\mathcal{C}}_{\text{off}}, \quad \hat{\mathcal{C}}_{\text{off}} = \begin{pmatrix} 0_{(m+2n)\times(m+2n)} & \Omega_{(m+2n)\times(m+2n)} \\ \Omega_{(m+2n)\times(m+2n)} & 0_{(m+2n)\times(m+2n)} \end{pmatrix} \]  

(3.8)

where \( \Omega \) is the invariant metric for an OSp(m|2n) superalgebra. For instance one can choose:

\[ \Omega_{(m+2n)\times(m+2n)} = \begin{pmatrix} \eta_{m\times m} & 0_{m\times 2n} \\ 0_{m\times 2n} & \epsilon_{2n\times 2n} \end{pmatrix} \]

\[ \eta^T = \eta; \quad \eta \eta = 1; \quad \text{signature} = (+)^m \Rightarrow \eta = 1_m \]

(3.9)

but there are also other choices discussed in the sequel.

The above definition implies the following relations on the supermatrix blocks:

\[ \hat{A}^S_T \Omega \hat{C}_\xi + \hat{C}^S_T \Omega \hat{A}_\xi = 0 \quad \hat{A}^S_T \Omega \hat{D}_\xi + \hat{C}^S_T \Omega \hat{B}_\xi = \Omega \]

\[ \hat{B}^S_T \Omega \hat{D}_\xi + \hat{D}^S_T \Omega \hat{B}_\xi = 0 \quad \hat{B}^S_T \Omega \hat{C}_\xi + \hat{D}^S_T \Omega \hat{A}_\xi = \Omega \]  

(3.10)
where the superscript $ST$ denotes the usual conjugation for supermatrices. Notice that the matrix $\Omega$ satisfies the following important properties

$$
\Omega^{ST} = \Omega \begin{pmatrix} 1_m & 0_{m \times 2n} \\ 0_{2n \times m} & -1_{2n} \end{pmatrix} ; \quad \Omega^2 = \begin{pmatrix} 1_m & 0_{m \times 2n} \\ 0_{2n \times m} & -1_{2n} \end{pmatrix}
$$

which we assume in any case. In addition, we have $(\Omega^{ST})^{ST} = \Omega$ and $\Omega^{ST} = \Omega^3$. They will be very useful in establishing the form of the orthosymplectic duality symmetries. Indeed eq. (3.11) can be used to restate the symmetry properties (3.4) of the coupling matrices $\Gamma$ and $\Theta$ appearing in the action (3.1) in the following way:

$$
\hat{\Gamma}^{ST} = \hat{\Gamma} \Omega^2 ; \quad \hat{\Theta}^{ST} = -\hat{\Theta} \Omega^2
$$

The reader can check that the above formulae are formally analogous to those for the purely bosonic case, the main difference being that everyone wears a hat and is a graded matrix. Before proceeding some comments are in order. Customarily graded matrices are written in block-form so that the off-diagonal blocks are fermionic and the diagonal ones are bosonic. This is not necessary. Graded matrices can also be written in block-forms where each block is in turn a graded matrix. This has been used in the above discussion. There always exists a change of basis that reduces any such matrix to the standard form where the fermionic entries are grouped into off-diagonal blocks. In the usual description the orthosymplectic group $\text{OSp}(m, m|4n)$ would be defined as the set of graded matrices leaving the following metric invariant:

$$
\hat{C}_{\text{dia}}^\prime = \begin{pmatrix} \eta & 0 & 0 & 0 \\ 0 & -\eta & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & -\epsilon \end{pmatrix}.
$$

By exchanging the second with the third row in the above matrix, we get:

$$
\hat{C}_{\text{dia}} = \mathcal{P} \hat{C}_{\text{dia}}^\prime \mathcal{P} = \begin{pmatrix} \eta & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & -\eta & 0 \\ 0 & 0 & 0 & -\epsilon \end{pmatrix} , \quad \mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

It is also very simple to work out the transformation connecting the bases defined by $\hat{C}_{\text{dia}}$
and by $\hat{\mathcal{C}}_{\text{off}}$, respectively. It is given by the following generalized super-Cayley matrix:

\[
\mathcal{C}_{\text{OSp}} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1
\end{pmatrix}
\]  

(3.15)

Indeed, we have:

\[
\mathcal{C}_{\text{OSp}} \hat{\mathcal{C}}_{\text{dia}} \mathcal{C}_{\text{OSp}}^{-1} = \hat{\mathcal{C}}_{\text{off}} \equiv \begin{pmatrix}
0 & 0 & \eta & 0 \\
0 & 0 & 0 & \epsilon \\
\eta & 0 & 0 & 0 \\
0 & \epsilon & 0 & 0
\end{pmatrix}
\]  

(3.16)

So we can easily go from one basis to the other by means of these transformations and we can define the orthosymplectic group as in eq. (3.8).

This being clarified let us proceed with the discussion of the fractional linear transformations of the matrix $\hat{\mathcal{M}}$. To this effect consider the following situation. Suppose that we have two supermatrices $\hat{\mathcal{X}}$ and $\hat{\mathcal{Y}}$ which have the same linear fractional transformation under a supermatrix $\left( \begin{array}{cc} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{array} \right)$:

\[
\begin{align*}
\hat{\mathcal{X}}' &= (\hat{C} + \hat{D} \hat{\mathcal{X}}) \left( \hat{A} + \hat{B} \hat{\mathcal{X}} \right)^{-1}, \\
\hat{\mathcal{Y}}' &= (\hat{C} + \hat{D} \hat{\mathcal{Y}}) \left( \hat{A} + \hat{B} \hat{\mathcal{Y}} \right)^{-1}
\end{align*}
\]  

(3.17) (3.18)

and let us formulate the following question: Which linear transposition relation between $\hat{\mathcal{X}}'$ and $\hat{\mathcal{Y}}'$ will imply, as consistency conditions, the orthosymplectic relations (3.10) on the supermatrix blocks $\hat{A}, \hat{B}, \hat{C}, \hat{D}$? Such a question is relevant for the issue of duality rotations since it is precisely in this way that the symplectic or pseudo-orthogonal character of the duality transformations is established in bosonic theories, by comparing the action of the latter on self-dual and anti-self dual field strengths. Eqs (3.17)-(3.18) are consistent with the orthosymplectic conditions (3.10) if and only if:

\[
\hat{\mathcal{Y}} = - \Omega \hat{\mathcal{X}}^{ST} \Omega^{ST}.
\]  

(3.19)

Indeed, by calculating the super-transposed of the transformed $\hat{\mathcal{Y}}$ we get:

\[
\begin{pmatrix} \hat{\mathcal{Y}}' \end{pmatrix}^{ST} = - \Omega \hat{\mathcal{X}}' \Omega^{ST} = - \left( \hat{A}^{ST} + \hat{\mathcal{Y}}^{ST} \hat{B}^{ST} \right)^{-1} \left( \hat{C}^{ST} + \hat{\mathcal{Y}}^{ST} \hat{D}^{ST} \right)
\]  

(3.20)
Combining (3.20) with the transformation law of $\hat{\mathbf{X}}$ and using the property $\Omega^{ST} \Omega = 1$, we obtain the relation:

$$\left(\hat{C} + \hat{D}\hat{\mathbf{X}}\right) \left(\hat{A} + \hat{B}\hat{\mathbf{X}}\right)^{-1} = -\Omega^{ST} \left(\hat{A}^{ST} + \hat{Y}^{ST}\hat{B}^{ST}\right)^{-1} \left(\hat{C}^{ST} + \hat{Y}^{ST}\hat{D}^{ST}\right) \Omega \quad (3.21)$$

Multiplying the above relation to the right by $\left(\hat{A} + \hat{B}\hat{\mathbf{X}}\right) \Omega$ and to the left by the following supermatrix $\left(\hat{A}^{ST} + \hat{Y}^{ST}\hat{B}^{ST}\right) \Omega$, we obtain:

$$0 = \left(\hat{A}^{ST} \Omega \hat{C} + \hat{C}^{ST} \Omega \hat{A}\right) + \hat{Y}^{ST} \left(\hat{D}^{ST} \Omega \hat{B} + \hat{B}^{ST} \Omega \hat{D}\right) + \left(\hat{C}^{ST} \Omega \hat{B} + \hat{A}^{ST} \Omega \hat{D}\right) \hat{\mathbf{X}} + \hat{Y}^{ST} \left(\hat{D}^{ST} \Omega \hat{A} + \hat{B}^{ST} \Omega \hat{C}\right) \hat{\mathbf{X}} \quad (3.22)$$

which is satisfied given the relation (3.19) if and only if the orthosymplectic conditions (3.10) are fulfilled by the $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ blocks.

Let us now consider the following case of matrix $\hat{\mathbf{X}}$:

$$\hat{\mathbf{X}} = \Omega \hat{\mathcal{M}} = \Omega \left(\hat{\Gamma} + \hat{\Theta}\right) \quad (3.23)$$

The corresponding $\hat{\mathcal{Y}}$ has the following expression:

$$\hat{\mathcal{Y}} = -\Omega \hat{\mathcal{M}}^{ST} \Omega^2 = -\Omega \left(\hat{\Gamma} - \hat{\Theta}\right) \quad (3.24)$$

the last equality following from eq. (3.12).

Consider now the action (3.1) and in analogy with the bosonic case let us define the 1-form field strengths of the twisted Bose/Fermi scalars as follows:

$$F^A = d\Pi^A ; \quad F^{A\pm} = \frac{1}{2} \left(F^A \pm \star F^A\right)$$

$$G^\Sigma = \frac{1}{2} \delta \mathcal{L} \delta F^A; \quad G^{\Sigma \pm} = \frac{1}{2} \left(G^\Sigma \pm \star G^\Sigma\right)$$

From the above definitions we immediately derive the following relations:

$$G^+ = \left(\hat{\Gamma} + \hat{\Theta}\right) F^+ = \hat{\mathcal{M}} F^+$$

$$G^- = -\left(\hat{\Gamma} - \hat{\Theta}\right) F^- = -\hat{\mathcal{M}}^{ST} \Omega^2 F^+ \quad (3.25)$$

We can assume the following realization of the $\hat{\mathcal{G}}$ superisometries on the electric and magnetic field strengths, we find:

$$\forall \xi \in \hat{\mathcal{G}} : \quad \iota_\xi \begin{pmatrix} F^\pm \\ \Omega G^\pm \end{pmatrix} = \begin{pmatrix} \hat{\mathcal{A}}_\xi & \hat{\mathcal{B}}_\xi \\ \hat{\mathcal{C}}_\xi & \hat{\mathcal{D}}_\xi \end{pmatrix} \begin{pmatrix} F^\pm \\ \Omega G^\pm \end{pmatrix} \quad (3.26)$$

which are consistent with the relation (3.25) if the matrices $\hat{\mathbf{X}}$ and $\hat{\mathcal{Y}}$, as defined in eq.s (3.23) and (3.24), transform according to eq.(3.17) and (3.18) under $\begin{pmatrix} \hat{\mathcal{A}}_\xi & \hat{\mathcal{B}}_\xi \\ \hat{\mathcal{C}}_\xi & \hat{\mathcal{D}}_\xi \end{pmatrix}$. This concludes the discussion of duality symmetries in general Bose/Fermi theories.
3.2 The Gaillard-Zumino formula in the orthosymplectic case

Having clarified the general form of the duality covariant Bose-Fermi sigma–models, we can now focus on the case where the supermanifold $\mathcal{SM}(x|y) \equiv \hat{G}/\hat{H}$ is a homogeneous super-coset. In this case we can easily construct the orthosymplectic generalization of the Gaillard-Zumino formula. What is important to notice is that the basis $\hat{C}_{\text{dia}}$ is the one in which the subalgebra

$$\text{osp}(m|2n) \times \text{osp}(m|2n) \subset \text{osp}(m,m|4n)$$  \hspace{1cm} (3.27)

is diagonally embedded:

$$\begin{pmatrix} \text{osp}(m|2n)_I & 0 \\ 0 & \text{osp}(m|2n)_{II} \end{pmatrix} \in \text{osp}(m,m|4n)$$  \hspace{1cm} (3.28)

Correspondingly, given a coset representative $\hat{L}(\Phi)$ of $\hat{G}/\hat{H}$, we can consider its orthosymplectic embedding in the off-diagonal basis:

$$\hat{L}(\Phi) \rightarrow \begin{pmatrix} \hat{A}(\Phi) & \hat{B}(\Phi) \\ \hat{C}(\Phi) & \hat{D}(\Phi) \end{pmatrix} \equiv \hat{O}(\Phi) \in \text{OSp}(m,m|4n),$$

$$\hat{O}^T(\Phi) \hat{C}_{\text{off}} \hat{O}(\Phi) = \hat{C}_{\text{off}}$$  \hspace{1cm} (3.29)

As before, we also require that the embedding of $\hat{G}$ is such that its subgroup $\hat{H}$ gets embedded into $\text{osp}(m|2n) \times \text{osp}(m|2n) \subset \text{osp}(m,m|4n)$. Then in full analogy with equation (2.46) let us introduce the mixed-basis representation of the coset representative.

$$\hat{O}_{\text{mr}}(\Phi) \equiv \begin{pmatrix} \hat{f}(\Phi) & \hat{\tilde{f}}(\Phi) \\ \hat{h}(\Phi) & \hat{\tilde{h}}(\Phi) \end{pmatrix} = \hat{O}(\Phi) \hat{C}_{\text{so}}^{-1},$$  \hspace{1cm} (3.30)

where, by explicit evaluation we have:

$$\hat{f}(\Phi) = \frac{1}{\sqrt{2}} \left( \hat{A}(\Phi) + \hat{B}(\Phi) \right) ; \quad \hat{\tilde{f}}(\Phi) = \frac{1}{\sqrt{2}} \left( \hat{A}(\Phi) - \hat{B}(\Phi) \right)$$

$$\hat{h}(\Phi) = \frac{1}{\sqrt{2}} \left( \hat{C}(\Phi) + \hat{D}(\Phi) \right) ; \quad \hat{\tilde{h}}(\Phi) = \frac{1}{\sqrt{2}} \left( \hat{C}(\Phi) - \hat{D}(\Phi) \right)$$  \hspace{1cm} (3.31)

From the orthosymplectic relations imposed on the $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ blocks we deduce the following relations:

$$\hat{f}^{ST} \Omega \hat{h} + \hat{h}^{ST} \Omega \hat{f} = \Omega \quad ; \quad \hat{f}^{ST} \Omega \hat{h} + \hat{h}^{ST} \Omega \hat{f} = 0$$  \hspace{1cm} (3.32)

that are the orthosymplectic counterpart of the symplectic and pseudorthogonal relations (2.23)-(2.48).
Then the matrix $\hat{X} \equiv \Omega \hat{M}$ is defined by the obvious generalization of eq. (2.49)

$$\Omega \hat{M}(\Phi) \equiv \hat{h}(\Phi) \hat{f}(\Phi)^{-1} = \left[ \hat{C}(\Phi) + \hat{D}(\Phi) \right] \left[ \hat{A}(\Phi) + \hat{B}(\Phi) \right]^{-1}$$

(3.33)

and by the same token as in the previous cases it transforms correctly under the action of the supergroup $\hat{G}$.

4 $\text{SO}(m, m)$ embeddings from dimensional reduction $4D \rightarrow 2D$

Let us now come back to the purely bosonic theories and recall a phenomenon which was discovered in the context of dimensional reduction and provides a challenging suggestion also for Bose/Fermi theories.

As we stressed in section 2.1, as long as there is no gauging, the general form of the bosonic part of a four-dimensional supergravity Lagrangian is given by eq. (2.1). Let us name $U_{4D}$ the duality group which is an isometry of the sigma-model part of that Lagrangian and acts by symplectic duality symmetries on the vector fields. Then, following the discussion of [13], we can perform the dimensional reduction $D = 4 \rightarrow D = 2$. There are two possible routes one can follow [14]:

**Ehlers:** The Ehlers route consists of two steps. In the first step one performs the dimensional reduction to $D = 3$ and then dualizes all-vector fields to scalars. In the second step one goes down to $D = 2$. Following this route one arrives at a standard sigma model of the following form:

$$S_{Ehlers} = \int d^2 x \, h_{I J}(\Upsilon) \, \partial^\mu \Upsilon^I \, \partial^\mu \Upsilon^J$$

(4.1)

where $h_{I J}(\Upsilon)$ is the invariant metric of a non-compact coset manifold $U_{2D}/H_{3D}$. The four dimensional duality group is a subgroup of the Ehlers group $U_{2D} \subset U_{3D}$ and we have the following general Lie algebra decomposition:

$$\text{adj}(U_{3D}) = \text{adj}(U_{4D}) \oplus \text{adj}(\text{SL}(2, \mathbb{R})) \oplus W_{(W),2}$$

(4.2)

where $W$ is the symplectic representation of $U_{4D}$ to which the electric and magnetic field strengths of the vector fields are assigned in order to construct the duality symmetries of the four-dimensional Lagrangian (2.1).
Matzner-Missner: The second dimensional reduction route, named after Matzner-Missner consists of stepping down directly from $D = 4$ to $D = 2$, where the scalars remain scalars and the vector fields yield 0-form scalars. The final action is of the form (2.28). More precisely it is the following:

\[
S_{\text{MM}}^{(D=2)} = \int d^2x \left\{ -\frac{1}{2} h_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b \right. \\
+ \left. \frac{1}{2} \left[ -\nabla_\mu \pi^A \delta_{AB} \nabla_\nu \pi^B + \nabla_\mu \pi^A \varpi_{AB \sigma} \nabla_\nu \pi^B \epsilon_{\mu\nu} \right] \right\}
\]

where $h_{ab}(\phi)$ is the invariant metric of the coset $U_{4D}/H_{4D} \times \text{SL}(2, \mathbb{R})/\text{SO}(2)$, the second factor coming from the dimensional reduction of Einstein Gravity. The symmetry $U_{4D} \times \text{SL}(2, \mathbb{R})$ is realized by isometries on the sigma-model part of the Lagrangian and as duality transformations on 0-form-scalars through an embedding in $\text{SO}(m,m)$ which we recall below.

Let $n$ be the number of vector fields appearing in the $D = 4$ supergravity action (2.1). Then the dimension of the symplectic representation $W$ of $U_{4D}$ appearing in eq. (4.2) is $2n$ and the number of twisted scalars appearing in the Matzner-Missner Lagrangian (4.4) is also $2n$. Correspondingly the duality group is $\text{SO}(2n, 2n)$ and the embedding

\[
\text{Sp}(2n, \mathbb{R}) \hookrightarrow \text{SO}(2n, 2n)
\]

was described in [13]. As follows $\forall \xi \in U_{4D}$ let \( \begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix} \in \text{Sp}(2n, \mathbb{R}) \) be its representation by means of symplectic matrices that satisfy the defining condition:

\[
\begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

Then we have:

\[
\begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix} \mapsto O_\xi \equiv \begin{pmatrix} A_\xi \otimes \mathbf{1}_2 & B_\xi \otimes \epsilon \\ C_\xi \otimes \epsilon & A_\xi \otimes \mathbf{1}_2 \end{pmatrix} \in \text{SO}(2n, 2n)
\]

where $\mathbf{1}_2$ is the identity matrix in two-dimension and $\epsilon = -\epsilon^T$ denotes an antisymmetric $2 \times 2$ matrix such that $\epsilon^2 = -\mathbf{1}_2$.

The constructed matrix $O_\xi$ satisfies the pseudo-orthogonality conditions in the form:

\[
O_\xi^T \begin{pmatrix} 0 & 1_n \otimes \mathbf{1}_2 \\ 1_n \otimes \mathbf{1}_2 & 0 \end{pmatrix} O_\xi = \begin{pmatrix} 0 & 1_n \otimes \mathbf{1}_2 \\ 1_n \otimes \mathbf{1}_2 & 0 \end{pmatrix}
\]
Correspondingly the period matrix $\mathcal{M}$ in two dimensions is related to the period matrix $\mathcal{N}$ in four dimensions through the following formula:

$$\mathcal{M} = \text{Im}\mathcal{N} \otimes \mathbf{1}_2 - \text{Re}\mathcal{N} \otimes \epsilon \quad (4.8)$$

which yield the result displayed in the Lagrangian (4.4), namely:

$$(\gamma_{\alpha\beta}) = \text{Im}\mathcal{N} \otimes \mathbf{1}_2 = (\text{Im}\mathcal{N}_{\Lambda\Sigma} \delta_{AB})$$

$$(\theta_{\alpha\beta}) = -\text{Re}\mathcal{N} \otimes \mathbf{1}_2 = -(\text{Re}\mathcal{N}_{\Lambda\Sigma} \epsilon_{AB})$$

The main relevant point is that the Matzner-Missner (4.4) and the Ehlers Lagrangian (4.1) can be mapped into one another by a suitable duality transformation in $\text{SO}(2n,2n)/U_{4D}$.

In order to imitate the same mechanism of duality at the level of Bose/Fermi theories it is convenient to rewrite the embedding (4.4) suitable for generalization to orthosymplectic groups. This is easily done by changing the basis both of the symplectic group and of the pseudo-orthogonal one. Suppose that $n$ be even and observe that

$$\begin{pmatrix} 1 & 0 \\ 0 & \eta_{2n} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \eta_{2n} \end{pmatrix} = \begin{pmatrix} 0 & \eta_{2n} \\ \eta_{2n} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -\epsilon_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \epsilon_n \end{pmatrix} = \begin{pmatrix} 0 & \epsilon_n \\ \epsilon_n & 0 \end{pmatrix}$$

where $\epsilon_n^T = -\epsilon_n$ is an antisymmetric matrix such that $\epsilon_n^2 = -1$ and $\eta_{2n}^T = \eta_{2n}$ is a symmetric one such that $\eta_{2n}^2 = 1$. Hence by means of the transformation:

$$\begin{pmatrix} 1 & 0 \\ 0 & -\epsilon_n \end{pmatrix} \begin{pmatrix} A_{\xi} & B_{\xi} \\ C_{\xi} & D_{\xi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \epsilon_n \end{pmatrix} = \begin{pmatrix} X_{\xi} & Y_{\xi} \\ W_{\xi} & Z_{\xi} \end{pmatrix} = \mathcal{G}_{\xi}$$

we obtain a symplectic matrix which satisfies the relations:

$$\mathcal{G}_{\xi}^T \begin{pmatrix} 0 & \epsilon_n \\ \epsilon_n & 0 \end{pmatrix} \mathcal{G}_{\xi} = \begin{pmatrix} 0 & \epsilon_n \\ \epsilon_n & 0 \end{pmatrix}$$

rather than in the form (4.5). Similarly starting from any matrix $\mathcal{O}$ which satisfies the pseudo-ortogonality conditions on the form (4.7), by applying the following transformation:

$$\begin{pmatrix} 1 & 0 \\ 0 & \eta_{2n} \end{pmatrix} \mathcal{O} \begin{pmatrix} 1 & 0 \\ 0 & \eta_{2n} \end{pmatrix} \equiv \mathcal{D}$$

we obtain a new one which satisfies them in the form:

$$\mathcal{D}^T \begin{pmatrix} 0 & \eta_{2n} \\ \eta_{2n} & 0 \end{pmatrix} \mathcal{D} = \begin{pmatrix} 0 & \eta_{2n} \\ \eta_{2n} & 0 \end{pmatrix}$$

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Consider now the case where \( \eta_{2n} = \epsilon_n \otimes \epsilon_2 \) and apply the transformation (4.13) to the result of the embedding \( \text{Sp}(2n, \mathbb{R}) \hookrightarrow \text{SO}(2n, 2n) \) namely to the matrix \( O_\xi \) in eq.(4.6). By direct calculation we find:

\[
O_\xi = \begin{pmatrix}
X \otimes 1_2 & Y \otimes 1_2 \\
W \otimes 1_2 & W \otimes 1_2
\end{pmatrix}
\] (4.15)

This result shows that the embedding of the product group \( \text{Sp}(2n, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \) into \( \text{SO}(2n, 2n) \), by changing basis is just an instance of a general embedding \( \text{Sp}(2p, \mathbb{R}) \times \text{Sp}(2q, \mathbb{R}) \hookrightarrow \text{SO}(2pq, 2pq) \) defined, for even \( p \), as follows:

\[
\text{Sp}(2p, \mathbb{R}) \ni (X \xi Y \xi W \xi Z \xi) \rightarrow \begin{pmatrix}
X \otimes 1_{2q} & Y \otimes 1_{2q} \\
W \otimes 1_{2q} & W \otimes 1_{2q}
\end{pmatrix} \in \text{SO}(2pq, 2pq) \quad (4.16)
\]

\[
\text{Sp}(2q, \mathbb{R}) \ni \Lambda \rightarrow \begin{pmatrix}
1_p \otimes \Lambda & 0 \\
0 & 1_p \otimes \Lambda
\end{pmatrix} \in \text{SO}(2pq, 2pq) \quad (4.17)
\]

where the \( \text{SO}(2pq, 2pq) \) invariant metric is:

\[
\begin{pmatrix}
0 & \epsilon_p \otimes \epsilon_{2q} \\
\epsilon_p \otimes \epsilon_{2q} & 0
\end{pmatrix}
\]

and \( \Lambda^T \epsilon_{2q} \Lambda = \epsilon_{2q} \)

while \( \begin{pmatrix}
X \xi \\
W \xi
\end{pmatrix} \) satisfies eq.(4.12).

Formulated as in eq.(4.17), the embedding can be extended to the orthosymplectic case giving rise to new Bose/Fermi analogues of the Ehlers/Matzner-Misner dual descriptions of the same physical system. This is what we study in the next section.

### 5 Orthosymplectic \( \text{OSp}(m, m|4n) \) embeddings

Finally, we can merge the symplectic embeddings and the orthogonal ones. In order to present the problem for the orthosymplectic embeddings, we proceed as follows. We first consider the direct product of two supergroups and a representation carrying an orthosymplectic structure. Then, we compute the duality group needed to determine the GZ formula and the sigma model. However, for reader’s convenience, we do it separately for the orthogonal subgroups as an example.

We want to embed the product of

\[
\frac{\text{OSp}(p, p|4r)}{\text{SO}(p) \times \text{SO}(p) \times \text{U}(2r)} \times \frac{\text{OSp}(q, q|2s)}{\text{SO}(q) \times \text{SO}(q) \times \text{U}(s)}
\] (5.1)

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into the coset
\[
\frac{\text{OSp}(2pq + 4rs, 2pq + 4rs|4ps + 8qr)}{\text{SO}(2pq + 4rs) \times \text{SO}(2pq + 4rs) \times \text{U}(2ps + 4qr)} \tag{5.2}
\]
whose super-isometry group acts on the representation \((4pq + 8rs|4ps + 8qr)\) with \(4pq + 8rs\) bosons and \(4ps + 8qr\) fermions. The embedding of the isometry group of (5.1) into the superisometry group of (5.2) is such that the fundamental representation of the latter is the tensor product of the fundamental representations of the two factor groups the former.

The representation carries an orthosymplectic structure and implements the duality relations for bosons, for fermions and their mixings.

Since the general result is rather cumbersome, we first exploit the embedding for the bosonic subsectors. Now, we embed
\[
\text{SO}(p, p) \times \text{SO}(q, q) \mapsto \text{SO}(2pq, 2pq), \quad \text{Sp}(4r) \times \text{Sp}(2s) \mapsto \text{SO}(4rs, 4rs) \tag{5.3}
\]
in order to obtain a linear representation of \(\text{SO}(n, n)\) with \(n = 2pq\) or \(n = 4rs\). Then, we cast everything into a bigger representation of \(\text{SO}(2pq + 4rs, 2pq + 4rs)\).

We first decompose \(\Lambda_{2p} \in \mathfrak{so}(p, p)\) (we work at algebra level) into
\[
\Lambda_{2p} = \begin{pmatrix} A_p & B_p \\ C_p & -A_p^T \end{pmatrix}, \quad B_p^T = -B_p, \quad C_p^T = -C_p.
\]
which satisfies \(\Lambda_{2p}^T \eta_{2p} + \eta_{2p} \Lambda_{2p} = 0\) where
\[
\eta_{2p} = \begin{pmatrix} 0 & 1_p \\ 1_p & 0 \end{pmatrix}
\]
and we denote by \(M_{2q}\) a matrix of \(\mathfrak{so}(q, q)\). We get the new \(4pq \times 4pq\) matrix of \(\mathfrak{so}(2pq, 2pq)\)
\[
\begin{pmatrix}
A_p \otimes 1_{2q} + 1_p \otimes M_{2q} & B_p \otimes 1_{2q} \\
C_p \otimes 1_{2q} & -A_p^T \otimes 1_{2q} - 1_p \otimes M_{2q}^T
\end{pmatrix}
\tag{5.4}
\]
where the invariant tensor is
\[
\eta_{4pq} = \begin{pmatrix} 0 & 1_p \otimes \eta_{2q} \\ 1_p \otimes \eta_{2q} & 0 \end{pmatrix}
\]
In the same way, we decompose \(\Lambda_{4r} \in \mathfrak{sp}(4r)\) into
\[
\Lambda_{2r} = \begin{pmatrix} A_{2r} & B_{2r} \\ C_{2r} & \epsilon_{2r} A_{2r}^T \end{pmatrix}, \quad B_{2r}^T = \epsilon_{2r} B_{2r} \epsilon_{2r}, \quad C_{2r}^T = \epsilon_{2r} C_{2r} \epsilon_{2r}.
\]
which satisfies \( \Lambda_{2r}^T \epsilon_{4r} + \epsilon_{4r} \Lambda_{2r} = 0 \) where
\[
\epsilon_{4r} = \begin{pmatrix} 0 & \epsilon_{2r} \\ \epsilon_{2r} & 0 \end{pmatrix}
\] (5.5)
and we denote by \( M_{2s} \) a matrix of \( \mathfrak{sp}(2s) \). We get a matrix of \( \mathfrak{so}(4rs, 4rs) \) by setting
\[
\begin{pmatrix}
A_{2r} \otimes 1_{2s} + 1_{2r} \otimes M_{2s} & B_{2r} \otimes 1_{2s} \\
C_{2r} \otimes 1_{2s} & -A_{2r}^T \otimes 1_{2s} - 1_{2r} \otimes M_{2s}^T
\end{pmatrix}
\] (5.6)
where the invariant tensor is now
\[
\epsilon_{4rs} = \begin{pmatrix} 0 & \epsilon_{2r} \otimes \epsilon_{2s} \\ \epsilon_{2r} \otimes \epsilon_{2s} & 0 \end{pmatrix}.
\]

In this way we can construct the GZ kinetic term given the matrix (5.7) for the bosonic fields whose duality is now described by the
\[
\frac{\mathfrak{so}(2pq + 4rs, 2pq + 4rs)}{\mathfrak{so}(2pq + 4rs) \times \mathfrak{so}(2pq + 4rs)}.
\]

This is the kinetic term for the bosonic fields of the sigma model. The next step is to construct the duality for the fermion kinetic terms, namely we have to embed the bosonic subgroups as follows
\[
\begin{pmatrix}
A_p \otimes 1_{2q} + 1_p \otimes M_{2q} & 0 & B_p \otimes 1_{2q} & 0 \\
0 & A_{2r} \otimes 1_{2s} + 1_{2r} \otimes M_{2s} & 0 & B_{2r} \otimes 1_{2s} \\
C_p \otimes 1_{2q}^T & 0 & -A_p^T \otimes 1_{2q} - 1_p \otimes M_{2q}^T & 0 \\
0 & C_{2r} \otimes 1_{2s} & 0 & -A_{2r}^T \otimes 1_{2s} - 1_{2r} \otimes M_{2s}^T
\end{pmatrix}
\]

In this way we can construct the GZ kinetic term given the matrix (5.7) for the bosonic fields whose duality is now described by the
\[
\begin{pmatrix}
\eta_{2p} & 0 \\
0 & \epsilon_{4r}
\end{pmatrix}, \quad
\begin{pmatrix}
0 & \epsilon_{2r} \\
\epsilon_{2r} & 0
\end{pmatrix}
\]

where the invariant tensor is now
\[
\epsilon_{4rs} = \begin{pmatrix} 0 & \epsilon_{2r} \otimes \epsilon_{2s} \\ \epsilon_{2r} \otimes \epsilon_{2s} & 0 \end{pmatrix}.
\]

However, it is rather straightforward to obtain the complete solution for the embedding of the supergroups
\[
\text{OSp}(p, p|4r) \times \text{OSp}(q, q|2s) \rightarrow \text{OSp}(2pq + 4rs, 2pq + 4rs|4ps + 8qr) \quad (5.8)
\]

The symplectic algebra in the first factor \( \mathfrak{sp}(4r) \) must satisfy a condition on the rank, which has to be multiple of two due to the present decomposition.

A supermatrix \( \hat{M} \) of \( \mathfrak{osp}(p, p|4r) \) satisfies the condition \( \hat{M}^{ST} \Omega + \Omega \hat{M} = 0 \), where
\[
\Omega = \begin{pmatrix} \eta_{2p} & 0 \\ 0 & \epsilon_{4r} \end{pmatrix}, \quad
\epsilon_{4r} = \begin{pmatrix} 0 & \epsilon_{2r} \\ \epsilon_{2r} & 0 \end{pmatrix}
\]

(5.9)
Decomposed in blocks \( \hat{\mathcal{M}}_{2p,4r} \) appears as follows

\[
\hat{\mathcal{M}}_{2p,4r} = \begin{pmatrix}
A_p & B_p & -\delta^T \epsilon_{2r} & -\beta^T \epsilon_{2r} \\
C_p & -A_p^T & -\gamma^T \epsilon_{2r} & -\alpha^T \epsilon_{2r} \\
\alpha & \beta & A_{2r} & B_{2r} \\
\gamma & \delta & C_{2r} & \epsilon_{2r} A_{2r}^T \epsilon_{2r}
\end{pmatrix}
\]  
(5.10)

where \( A_p, A_{2r} \) are arbitrary even matrices and \( \alpha, \delta \) are arbitrary odd matrices. \( B_p \) and \( C_p \) are antisymmetric matrices, while \( B_{2r} \) and \( C_{2r} \) satisfy

\[
B_{2r}^T \epsilon_{2r} + \epsilon_{2r} B_{2r} = 0, \quad C_{2r}^T \epsilon_{2r} + \epsilon_{2r} C_{2r} = 0.
\]

Finally, \( \beta \) and \( \gamma \) are arbitrary odd matrices. Now, we perform a change of basis such that a generic supermatrix appear in the left-upper block of the \( \mathfrak{osp}(p,p|4r) \) matrix as follows

\[
\mathcal{O}^{-1} \hat{\mathcal{M}}_{2p,4r} \mathcal{O} = \begin{pmatrix}
A_p & -\delta^T \epsilon_{2r} & B_p & -\beta^T \epsilon_{2r} \\
C_p & -\gamma^T \epsilon_{2r} & -A_p^T & -\alpha^T \epsilon_{2r} \\
\alpha & \beta & A_{2r} & B_{2r} \\
\gamma & \delta & C_{2r} & \epsilon_{2r} A_{2r}^T \epsilon_{2r}
\end{pmatrix}
\]  
(5.11)

where the supermatrices

\[
\hat{A}_{p,2r} = \begin{pmatrix}
A_p & -\delta^T \epsilon_{2r} \\
\alpha & A_{2r}
\end{pmatrix}, \quad \hat{B}_{p,2r} = \begin{pmatrix}
B_p & -\beta^T \epsilon_{2r} \\
\beta & B_{2r}
\end{pmatrix},
\]

\[
\hat{C}_{p,2r} = \begin{pmatrix}
C_p & -\gamma^T \epsilon_{2r} \\
\gamma & C_{2r}
\end{pmatrix}, \quad \hat{D}_{p,2r} = -\Omega_{p,2r}^{-1} \hat{A}_{p,2r}^{ST} \Omega_{p,2r} = \begin{pmatrix}
-A_p^T & -\alpha^T \epsilon_{2r} \\
\delta & \epsilon_{2r} A_{2r}^T \epsilon_{2r}
\end{pmatrix}
\]

with

\[
\Omega_{p,2r} = \begin{pmatrix}
1_p & 0 & 0 & 0 \\
0 & 0 & 1_p & 0 \\
0 & 1_{2r} & 0 & 0 \\
0 & 0 & 0 & 1_{2r}
\end{pmatrix}, \quad \mathcal{O} = \begin{pmatrix}
1_p & 0 & 0 & 0 \\
0 & 0 & 1_p & 0 \\
0 & 1_{2r} & 0 & 0 \\
0 & 0 & 0 & 1_{2r}
\end{pmatrix}
\]

\( \hat{A}_{p,2r} \) a supermatrix of \( \mathfrak{gl}(p|2r) \) and two matrices \( \hat{B}_{p,2r} \) and \( \hat{C}_{p,2r} \) of \( \mathfrak{osp}(p|2r) \), respectively. (Notice that counting the parameters of the matrix \( \hat{M} \) we have: \( p(2p - 1) + 2r(1 + 4r) \) bosons and \( 8pr \) fermions. They can be decomposed into \( p^2 + 4r^2 \) bosons and \( 2pr \) fermions from \( \mathfrak{gl}(p|2r) \) and \( p(p - 1) + 2r(2r + 1) \) from the two matrices \( \mathfrak{osp}(p|2r) \)).

To construct the embedding we multiply tensorially the matrix \( \mathcal{O}^{-1} \hat{M}_{2p,4r} \mathcal{O} \) with the matrix \( \hat{N}_{2q,2s} \) belonging to \( \mathfrak{osp}(q,q|2s) \) as follows

\[
\hat{A}_{2m,4n} = \begin{pmatrix}
\hat{A}_{p,2r} \otimes 1_{2q,2s} + 1_{p,2r} \otimes \hat{N}_{2q,2s} & \hat{B}_{p,2r} \otimes 1_{2q,2s} \\
\hat{C}_{p,2r} \otimes 1_{2q,2s} & \hat{D}_{p,2r} \otimes 1_{2q,2s} + 1_{p,2r} \otimes \hat{N}_{2q,2s}
\end{pmatrix}
\]  
(5.13)
where \( m = 2pq + 4rs \) and \( n = ps + 2rq \). The matrix \( \hat{\Lambda}_{2m,4n} \) is an element of \( \text{osp}(m, m|4n) \) in the basis where the metric appears in the form

\[
\Omega = \begin{pmatrix}
0 & \Omega_{p,2r} \otimes \Omega_{q,2s} \\
\Omega_{p,2r} \otimes \Omega_{q,2s} & 0
\end{pmatrix}
\]

We have derived the complete embedding of the direct product of supergroups into a matrix \( \hat{\Lambda}_{2m,4n} \) of \( \text{OSp}(m, m|2n) \), and we are now in position to compute the GZ formula (2.49).

### 5.1 An example

Let us consider a D=2 model in the Elhers frame described by the following supercoset

\[
\text{OSp}(3,3|6) \rightarrow \text{SO}(3) \times \text{SO}(3) \times \text{SO}(2) \times \text{SU}(3) .
\]

(5.14)

The bosonic subgroup of \( \text{OSp}(3,3|4) \) is \( \text{SO}(3,3) \) and \( \text{Sp}(8) \) which are non-compact. The maximal compact subgroup is \( \text{SO}(3)^2 \times \text{U}(3) \). It also contains 36 fermions organized in a representation \((6,6)\).

The corresponding Matzner-Missner model is described by the following supercoset. We start from the coset

\[
\text{OSp}(2,2|2) \times \text{OSp}(1,1|4) \rightarrow \frac{\text{SO}(2) \times \text{SO}(2) \times \text{SO}(2)}{\text{U}(2)} \oplus \frac{1}{2} \left[ ((4|2), (0|0)) \otimes ((0|0), (2|4)) \right] ,
\]

(5.15)

where the first two factors describe the proper scalar and fermionic fields while the linear representation define the bosonic and fermionic 0-forms. This example fits our discussion in the previous section for \( p = 1, q = 2, r = s = 1 \).

The bosonic part of the sigma model supercoset is

\[
\frac{\text{SO}(2)}{\text{SO}(2) \times \text{SO}(2)} \times \text{SO}(1,1) \oplus \frac{1}{2} (4,2) ,
\]

(5.16)

It has \((6 - 1 - 1) + 1 + 4 = 9\) degrees of freedom which are the fields of \( \text{SO}(3,3)/\text{SO}(3) \times \text{SO}(3) \). The other bosonic part is described by

\[
\frac{\text{Sp}(2)}{\text{SO}(2)} \times \frac{\text{Sp}(4)}{\text{U}(2)} \oplus \frac{1}{2} (2,4) ,
\]

(5.17)

The counting of degrees of freedom gives \((3 - 1) + (10 - 4) + 4 = 12\) which coincides with the degrees of freedom of \( \text{Sp}(6)/\text{SO}(2) \times \text{SU}(3) \). Now, we are ready to fuse them into a
supergroups as in (6.13) where the factor 1/2 divides the number of bosonic degrees of freedom. Summing up the bosonic degrees of freedom we have
\[(6 - 1 - 1) + (3 - 1) + 1 + (10 - 4) + 1/2(4 \times 2 + 2 \times 4) = 21\]
which are the bosonic degrees of freedom of the original coset (6.10). It is easy to check that also the fermionic degrees of freedom work. There are 16 fermions in the coset \((4 \times 2 + 2 \times 4)\) and 20 fermions in the linear representation.

The duality group is easily found
\[
\frac{\text{OSp}(8, 8|20)}{\text{OSp}(8|10) \times \text{OSp}(8|10)},
\]
which has \((164|160)\) degrees of freedom. This contains the embedding of the isometry transformations and new dualities that we are going to explore in the next section.

6 Superdualities

Having exploited the structure of the sigma model with the embedding of the supergroup \(\text{OSp}(m, m|4n)\), we clarify now the structure of the duality transformations. In particular we are interested in the superdualities, namely those transformations of the supergroup which are not in the embedded \(G\). These transformations generate new sigma models related by dualities.

Let us first recall some basic facts about dualities in the usual framework. The subgroup \(O(m, m)\) is generated by the following three set of transformations

(a) The field redefinitions are generated by the matrix \(A\) of \(GL(m, \mathbb{R})\) and embedded into the \(O(m, m)\) as follows
\[
g_A = \begin{pmatrix}
A & 0 \\
0 & (A^T)^{-1}
\end{pmatrix}
\]

(b) The second set are the shifts of the \(B\) field and they are embedded into the duality group as follows
\[
g_B = \begin{pmatrix}
1 & \Theta \\
0 & 1
\end{pmatrix}
\]
where \(\Theta\) is an \(m \times m\) antisymmetric matrix and \(I\) is the identity matrix in \(m\) dimension. This subgroup is a parabolic subgroup.

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(c) The factorized dualities are given by
\[ g_D = \begin{pmatrix} 1 - e^i & e^i \\ e^i & 1 - e^i \end{pmatrix} \] (6.3)

where \( e^i \) is a diagonal matrix will all entries equal to zero except \( \delta_{ii} \) (therefore it satisfies \( e^i e^i = e^i \)).

It can be easily checked that these transformations generate the complete group and in particular they are enough to describe all duality transformations. Combining the inversions with the parabolic subgroup of point \( b \), we generate a second \( SO(m) \). Therefore, the total number of parameters is \( m^2 + m(m-1) = m(2m-1) \) which are the parameters of the group \( SO(m,m) \). The coset \( SO(m,m)/SO(m) \times SO(m) \) describes \( m^2 \) moduli of the kinetic terms for the bosons in the action (3.1).

Let us now move to the fermionic part of the action. We have already established that the duality group is \( Sp(4n) \) and therefore, we would like to repeat also in the present context the analysis

(a) Field redefinitions are generate by a matrix \( A \) of \( GL(2n,\mathbb{R}) \) and they are embedded into the \( Sp(4n,\mathbb{R}) \) as follows
\[ g_A = \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} \] (6.4)

(b) The second set are the shifts of the WZ terms by a symmetric matrix and they are embedded into the duality group as follows
\[ g_B = \begin{pmatrix} 1 & \Theta \\ 0 & 1 \end{pmatrix} \] (6.5)

where \( \Theta \) is a symmetric matrix and 1 is the identity matrix in \( 2n \) dimension.

(c) The factorized dualities are given by
\[ g_D = \begin{pmatrix} 1 - e^i & -e^i \\ e^i & 1 - e^i \end{pmatrix} \] (6.6)

where \( e^i \) is a diagonal matrix will all entries equal to zero except \( \delta_{ii} \) (therefore it satisfies \( e^i e^i = e^i \)). Notice the minus sign in the right-upper corner. This is needed in order to be embedded in the duality group.
It can be checked that these transformations generate the entire duality group. Indeed the
inversions combined with the shifts at point \((b)\) generate a second symplectic subgroup
\(\text{Sp}(2n)\). In total one has \((2n)^2 + 2n(2n + 1)\) parameters which generates the total number
of parameters of \(\text{Sp}(4n)\). Notice that the coset \(\text{Sp}(4n)/\text{Sp}(2n) \times \text{Sp}(2n)\) has dimension
\((2n)^2\) and it describes all possible kinetic terms for the fermions. (A way to interpret those
terms in the context of string theory is to compare them with the pure spinor formulation
of string theory. From that point of view, the kinetic terms of fermions can be identified
with the RR fields).

Notice however, that the transformations generated by \(\text{SO}(m, m)\) and \(\text{Sp}(4n)\) are bosonic
transformations. Therefore, we have to explore the transformations generated by the
fermionic dualities.

Then, finally we have to cast everything in the supergroup. Therefore, we have to analyze
the off-diagonal supermatrices to see if they are still dualities of the action. In that case
the complete moduli space is captured by the supercoset
\[
\frac{\text{OSp}(m, m|4n)}{\text{OSp}(m|2n) \times \text{OSp}(m|2n)}
\]
which has \(m^2 + (2n)^2\) bosonic components and \(4nm\) fermionic components.

The supergroup \(\text{OSp}(m, m|4n)\) can be decomposed into \(\text{GL}(m|2n)\) (which has \(m^2 + (2n)^2\)
bosonic parameters and \(4nm\) fermions) plus two copies of \(\text{OSp}(m|2n)\) with \(m(m - 1)/2 +\n(2n + 1)\) bosonic and \(2nm\) fermions. The present discussion uses the form of the super-
matrices presented above where we have decomposed the matrix \((2m + 4n) \times (2m + 4n)\)
into \((m + 2n) \times (m + 2n)\) supermatrix blocks.

So, according to the previous discussion, we have:

(a) The field redefinitions are generated by a generic supermatrix \(\hat{A}\) of \(\text{GL}(m|n)\) and
are embedded into the supergroup are follows
\[
g_A = \begin{pmatrix}
\hat{A} & 0 \\
0 & \Omega_{(m+2n)\times(m+2n)}(\hat{A}^T)^{-1}\Omega_{(n+2m)\times(n+2m)}^T
\end{pmatrix}
\]
where the matrix \(\Omega_{(n+2m)\times(n+2m)}\) is defined in (6.7).

(b) Shift by an orthosymplectic matrix \(\hat{B}\) described by the parabolic subgroup
\[
g_B = \begin{pmatrix}
1_{(m+2n)\times(m+2n)} & \hat{B}_{(m+2n)\times(m+2n)} \\
0 & 1_{m+2n}
\end{pmatrix}
\]
The matrix \(\hat{B}_{(m+2n)\times(m+2n)}\) is an element of the orthosymplectic subgroup \(\text{OSp}(m|2n)\).
The inversions are purely fermionic transformations and are obtained by the combination of the orthogonal ones \([6.3]\) and the symplectic ones \([6.6]\).

By combining the parabolic subgroup at point \((b)\) with the inversions, one finds the other parabolic subgroup \(\text{OSp}(m|2n)\) and therefore the above transformations generate the complete duality group as in the bosonic cases.

### 6.1 Fermionic Inversions

As the last section, we present a specific example obtained from duality which has an interesting interpretation from string theory point of view.

We start from the simplest example of sigma model with the choice for couplings \([3.4]\)

\[
\Gamma_{\Sigma, \Lambda}(\Phi) = \left( \delta_{\alpha\beta}, 0, 0 \right), \quad \Theta_{\Sigma, \Lambda}(\Phi) = \left( 0, 0, \delta_{\bar{\alpha}\bar{\beta}} \right). \tag{6.10}
\]

This choice corresponds to a sigma model with a simple kinetic term for the boson fields \(\Pi^\alpha\) and a WZ term for the fermions \(\Pi^{\bar{\alpha}}\). This situation resembles the starting quadratic action for Green-Schwarz string theory (or the Pure Spinor Formulation) on a flat background. The bosonic fields are identified with the coordinates \(x^m\) and the fermions are identified with the superspace coordinates. Notice that in order that the WZ term produces a quadratic term in the fermions one needs a non-trivial background (see for example \([15]\)).

The matrix \(\hat{\mathcal{M}}\) has the form

\[
\hat{\mathcal{M}} = \begin{pmatrix} \delta_{\alpha\beta} & 0 \\ 0 & \delta_{\bar{\alpha}\bar{\beta}} \end{pmatrix} \tag{6.11}
\]

We perform a duality transformation with only fermionic parameters. Since we need the group element which satisfies the equation

\[
\begin{pmatrix} 0 & \hat{\mathcal{B}} \\ \hat{C} & 0 \end{pmatrix}^{ST} \begin{pmatrix} \Omega & 0 \\ 0 & \Omega \end{pmatrix} \begin{pmatrix} 0 & \hat{\mathcal{B}} \\ \hat{C} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix} \tag{6.12}
\]

we get that \(\hat{\mathcal{C}}^{ST} = \Omega \hat{\mathcal{B}}^{-1} \Omega^{-1}\). Therefore, we use the duality transformation given by

\[
\Lambda = \begin{pmatrix} 0 & \hat{\mathcal{B}} \\ \Omega (\hat{\mathcal{B}}^{ST})^{-1} \Omega^{-1} & 0 \end{pmatrix} \tag{6.13}
\]

The matrix \(\hat{\mathcal{B}}\) is a supermatrix which must be inverted in order to get the contribution in the left-bottom quadrant. To simplify the example, we consider a supermatrix \((1|2) \times (1|2)\)
with the entries
\[
\hat{\mathcal{B}} = \begin{pmatrix} b & \beta \\ \bar{\beta} & \bar{b} \end{pmatrix}, \quad \hat{\mathcal{B}}^{-1} = \begin{pmatrix} b^{-1} + b^{-1} \beta \bar{B} \bar{\beta} & -\beta \mathbf{B} \\ -\mathbf{B} \bar{\beta} & b \mathbf{B} \end{pmatrix}, \tag{6.14}
\]
where \( b \in \mathbb{R}, \beta, \bar{\beta} \) are vectors (with fermionic components) and \( \bar{b} \) and \( \mathbf{B} = (\bar{b} \bar{\beta} - \bar{\beta} \beta)^{-1} \) are \( 2 \times 2 \) matrices. Computing the supertranspose, we get
\[
\left(\hat{\mathcal{B}}^{ST}\right)^{-1} = \begin{pmatrix} b^{-1} + b^{-1} \beta \bar{B} \bar{\beta} & -\beta^T \mathbf{B}^T \\ \mathbf{B}^T \bar{\beta}^T & b \mathbf{B}^T \end{pmatrix}, \tag{6.15}
\]
Since we are interested into the inversion transformations due to the fermionic parameters we compute the expression by performing an OSp-transformation on \( \hat{\mathcal{M}} \)
\[
\hat{\mathcal{M}} \rightarrow \Omega^{-1}(\hat{\mathcal{C}} + \hat{\mathcal{D}} \hat{\mathcal{M}})(\hat{\mathcal{A}} + \hat{\mathcal{B}} \hat{\mathcal{M}})^{-1} = \Omega^{-1} \hat{\mathcal{C}} \hat{\mathcal{B}}^{-1} \hat{\mathcal{M}}^{-1}, \tag{6.16}
\]
and inserting the result in (6.13), we get
\[
\hat{\mathcal{M}} \rightarrow \left(\hat{\mathcal{B}}^{ST}\right)^{-1} \Omega^{-1} \hat{\mathcal{B}}^{-1} \hat{\mathcal{M}}^{-1}, \tag{6.17}
\]
In addition, if we set \( \bar{\beta} = 0, b = 1, \bar{b} = \mathbf{1}_2 \) to simplify (without losing in generality) the result, this yields
\[
\hat{\mathcal{M}} = \begin{pmatrix} 1 & -\beta \\ \beta^T & \mathbf{1}_2 + \beta^T \beta \end{pmatrix}, \quad \hat{\mathcal{M}}^{-1} = \begin{pmatrix} 1 & -\beta \\ \beta^T & \mathbf{1}_2 + \beta^T \beta \end{pmatrix}. \tag{6.18}
\]
The duality transformation has produced two additional couplings: the coupling between bosons \( \Pi^\alpha \) and fermions \( \Pi^{\bar{\alpha}} \) and the coupling fermion-fermion. The first one has a fermionic nature since it must provide a fermionic coupling. Notice that one can be tempted to interpret this coupling as induced by the target-space gravitinos represented here by the vector \( \beta \). On the other hand, the coupling fermion-fermion produces a new term in the kinetic term – notice that the matrix \( \beta^T \beta \) is antisymmetric due to the fermionic nature or \( \beta \) – which can be interpreted as RR coupling.

The duality transformation of the form (6.16) are discussed for the first time in works \[\text{[4, 12]}\]. Here, we see that they enter our scheme completely and indeed our derivation could produce other non-trivial background to be studied.

\[\text{[4] In order to identify correctly the RR fields in the fermion-fermion couplings one has to use the Pure Spinor Formulation of String Theory. Following the discussion given in [10], one way to see these coupling is indeed the contribution at the quadratic level of a RR field in the supergravity background.}\]
Conclusions and outlook

We have formulated in full generality the theory of superdualities. We have found that the correct construction leads to the orthosymplectic dual itty supergroup $OSp(m,m|4n)$. We have also insisted on the interpretation of the sigma model as a model for superstrings on background with RR fields and with fermions. This can work in $D=2$, but the interpretation in higher dimensions is puzzling and we have not found a proper interpretation from supergravity compactifications except that in the case of twisted supersymmetry where some gravitinos are interpreted as ghosts. In addition, it has not be explored the implications of the superdualities at the level of physical meaningful models. Finally, the application of or formalism to double geometry and T-folds [17] and double field theory [18] might be quite interesting.

Bibliography

[1] C. M. Hull and P. K. Townsend, “Unity of superstring dualities,” Nucl. Phys. B 438 (1995) 109 [arXiv:hep-th/9410167].

[2] T. H. Buscher, “Path Integral Derivation of Quantum Duality in Nonlinear Sigma Models,” Phys. Lett. B 201 (1988) 466; T. H. Buscher, “A Symmetry of the String Background Field Equations,” Phys. Lett. B 194 (1987) 59.

[3] A. Giveon, M. Porrati and E. Rabinovici, “Target space duality in string theory,” Phys. Rept. 244 (1994) 77 [arXiv:hep-th/9401139].

[4] N. Berkovits and J. Maldacena, “Fermionic T-Duality, Dual Superconformal Symmetry, and the Amplitude/Wilson Loop Connection,” JHEP 0809 (2008) 062 [arXiv:0807.3196 [hep-th]].

[5] O. Chandia, “A Note on T-dualities in the Pure Spinor Heterotic String,” JHEP 0904 (2009) 104 [arXiv:0902.2729 [hep-th]].

[6] I. Adam, A. Dekel and Y. Oz, “On Integrable Backgrounds Self-dual under Fermionic T-duality,” JHEP 0904 (2009) 120 [arXiv:0902.3805 [hep-th]].

[7] N. Beisert, “T-Duality, Dual Conformal Symmetry and Integrability for Strings on $AdS_5 \times S^5$,” Fortsch. Phys. 57 (2009) 329 [arXiv:0903.0609 [hep-th]].
[8] M. K. Gaillard and B. Zumino, “Duality Rotations For Interacting Fields,” Nucl. Phys. B 193 (1981) 221.

[9] N. Berkovits, “Super-Poincare covariant quantization of the superstring,” JHEP 0004, 018 (2000) [arXiv:hep-th/0001035].

[10] P. Fre, “Gaugings and other supergravity tools of p-brane physics,” [arXiv:hep-th/0102114]

[11] P. Fre, “Lectures on Special Kahler Geometry and Electric–Magnetic Duality Rotations,” Nucl. Phys. Proc. Suppl. 45BC (1996) 59 [arXiv:hep-th/9512043].

[12] N. Beisert, R. Ricci, A. A. Tseytlin and M. Wolf, “Dual Superconformal Symmetry from AdS5 x S5 Superstring Integrability,” Phys. Rev. D 78, 126004 (2008) [arXiv:0807.3228 [hep-th]].

[13] P. Fre’, F. Gargiulo, K. Rulik and M. Trigiante, ”The general pattern of Kac Moody extensions in supergravity and the issue of cosmic billiards”, Nucl. Phys. B 741 (2006) 42 [arXiv:hep-th/0507249].

[14] P. Breitenlohner and D. Maison, “On The Geroch Group,” Annales Poincare Phys. Theor. 46 (1987) 215; H. Nicolai, “Two-dimensional gravities and supergravities as integrable system”.

[15] N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov and B. Zwiebach, “Superstring theory on AdS(2) x S(2) as a coset supermanifold,” Nucl. Phys. B 567, 61 (2000) [arXiv:hep-th/9907200].

[16] N. Berkovits and P. S. Howe, “Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring,” Nucl. Phys. B 635, 75 (2002) [arXiv:hep-th/0112160].

[17] C. M. Hull, “A geometry for non-geometric string backgrounds,” JHEP 0510, 065 (2005) [arXiv:hep-th/0406102].

[18] C. Hull and B. Zwiebach, “Double Field Theory,” [arXiv:0904.4664 [hep-th]].