Soft Topological Space in Virtue of Semi* Open Sets

Paulraj Gnanachandra\(^1\)*, Lellis Thivagar\(^2\) and Muneesh Kumar Arumugam\(^1\)

\(^1\)Centre for Research and Post Graduate Studies in Mathematics, Ayya Nadar Janaki Ammal College(Autonomous), Sivakasi-626 124, Tamil Nadu, India.

\(^2\)School of Mathematics, Madurai Kamaraj University, Madurai-625021, Tamil Nadu, India.

\(*\)Corresponding author

Article Info

Keywords: Soft semi*-closure, Soft semi*-connected, Soft semi*-compact soft semi*-interior, Soft generalized closure, Soft generalized interior.

2010 AMS: 06D72, 54A05.

Received: 22 March 2020

Accepted: 15 December 2020

Available online: 23 December 2020

Abstract

The ultimate purpose of this research article is to originate and examine some new kind of open sets in soft topological spaces such as soft semi*-open and soft semi*-closed sets using generalized closure operator with illustrating counter examples.

1. Introduction

In our day-to-day life, we look out problems with unreliabilities. To handle the lack of unreliability and to solve the problems related to uncertainty, a short time ago numberless theories have been developed like Rough Sets, Fuzzy Sets and Vague Sets. However, these methodologies have their own risks. To circumvent these difficulties, Molodtsov [5] developed Soft set theory to deal with unreliability. The development of Soft Set theory is whistle stop now-a-days. Soft set theory has a wider application and its progress is very rapid in different fields [see [19], [20] and [10]]. The approach of Soft topological spaces was codified by Shabir et al. [12]. Many researchers defined some basic notions on soft topology and studied many properties see [4], [13], [17], [16], [22], [7], [8] and [9]]. In this milieu, we define penetration of soft semi*-open and soft semi*-closed sets in soft topological spaces and then these are used to study properties of semi* - interior, semi* - closure of soft sets in soft topological spaces. Further the behavior of these concepts under various soft functions has obtained. Also we introduce and study soft semi*-connectedness and soft semi*-compactness using soft semi* - open sets.

2. Preliminaries

We roll call the following definitions with illustrated examples for the outpouring of this article.

Let \( \mathcal{U} \) indicates initial universe set and let \( \mathcal{E} \) be parameters proportionate to \( \mathcal{U} \). Let \( \mathcal{P}(\mathcal{U}) \) denote the power set of \( \mathcal{U} \), and let \( A \subseteq \mathcal{E} \). A subset \( A \) of a space \( (X, \tau) \) is said to be generalized closed [15] (briefly \( g \)-closed), if \( cl(A) \subseteq \mathcal{U} \) whenever \( A \subseteq \mathcal{U} \) and \( \mathcal{U} \) is open. The intersection of all \( g \)-closed sets containing \( A \) is called the \( g \)-closure of \( A \) and denoted by \( cl^*(A) \) [21]. A subset \( A \) of a space \( (X, \tau) \) is said to be generalized open if its complement is generalized closed and union of all \( g \)-open sets contained in \( A \) called the \( g \)-interior of \( A \) and is denoted by \( int^*(A) \). A subset \( S \) of a topological space \( (S, \tau) \) is said to semi*-open if \( S \subseteq (cl^*(int(S)) \) [18]. The complement of a semi*-open set is semi*-closed. It is well known that a subset \( S \) is semi*-closed if and only if \( int^*(cl(S)) \subseteq S \) [3].

**Definition 2.1.** [5] A soft set \( \mathcal{F}_A \) on the universe \( \mathcal{U} \) is defined by the set of ordered pairs \( \mathcal{F}_A = \{(x, f_A(x)) | x \in \mathcal{E}, f_A(x) \in \mathcal{P}(\mathcal{U}) \} \) where \( \mathcal{E} \) is a set of parameters, \( A \subseteq \mathcal{E} \), \( \mathcal{P}(\mathcal{U}) \) is the power set of \( \mathcal{U} \) and \( f_A : A \rightarrow \mathcal{P}(\mathcal{U}) \) such that \( f_A(x) = \emptyset \) if \( x \notin A \). Here \( f_A \) is called an approximate function of the soft set \( \mathcal{F}_A \). The value of \( f_A(x) \) may be arbitrary, some of them may be empty and some may have non-empty intersection. Note that the set of all soft sets over \( \mathcal{U} \) is denoted by \( SS(\mathcal{U}) \).

For illustration, we consider an example which we present below.
Example 2.2. Suppose $\mathbb{U}$ = set of all real numbers on the closed interval $[a, b]$. 
$\mathcal{E}$ = set of parameters. Each parameter is a word or a sentence. 
$\mathcal{E}$ = {Compact, Closed, Connected, Open}
In this case, to define a soft set means to point out closed set, connected set and so on. Let we consider below the same example in more detail. $\mathbb{U} = \{x : a \leq x \leq b\}$ and $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ where

\begin{align*}
e_1 & \rightarrow \text{‘compact’},
e_2 & \rightarrow \text{‘closed’},
e_3 & \rightarrow \text{‘connected’},
e_4 & \rightarrow \text{‘open’}.
\end{align*}

Suppose that

\begin{align*}
f(e_1) & = \{A \subseteq [a, b] : \text{Every open cover for } A \text{ in } [a, b] \text{ has finite subcover} \},
f(e_2) & = \{[\alpha, \beta] \subseteq [a, b] : \alpha, \beta \in R\},
f(e_3) & = \{A \subseteq [a, b] : \text{Separation does not exist for } A \text{ in } [a, b]\}
f(e_4) & = \{[\alpha, \beta] \subseteq [a, b] : \alpha, \beta \in R\}
\end{align*}

$\mathcal{F}_\mathbb{U} \rightarrow$ parametrized family of subsets of the set $\mathbb{U}$. Consider the mapping $f$ in which $f(e_1) \rightarrow$ subsets of $\mathbb{U}$ which are compact whose functional value is the set $\{A \subseteq [a, b] : \text{Every open cover for } A \text{ in } [a, b] \text{ has finite subcover} \}$. Hence the soft set $\mathcal{F}_\mathbb{U}$ is the collection of approximations given below:

$\{\text{(compact, } [a, b] : \text{Every open cover for } A \text{ in } [a, b] \text{ has finite subcover } )\}, \{\text{Closed, } ([\alpha, \beta] \subseteq [a, b] : \alpha, \beta \in R)\}, \{\text{Connected, } (A \subseteq [a, b] : \text{separation does not exist for } A \text{ in } [a, b]\}$

Definition 2.3. [12] Let $\xi$ be a collection of soft sets over a universe $\mathbb{U}$ with a fixed set $\mathcal{E}$ of parameters, then $\xi \subseteq SS(\mathbb{U})_E$ is called a soft topology on $\mathbb{U}$ with a fixed set $\mathcal{E}$ if

i. $\xi_E, \mathbb{U}_E$ belong to $\xi$.

ii. The union of any number of soft sets in $\xi$ belongs to $\xi$.

iii. The intersection of any finite number of soft sets in $\xi$ belongs to $\xi$.

The pair $(\mathbb{U}_E, \xi)$ is called a soft topological space.

Definition 2.4. [1] Let $\mathbb{U}$ be a universe and $\mathcal{E}$ a set of parameters. Then the collection $SS(\mathbb{U})_E$ of all soft sets over $\mathbb{U}$ with parameters from $\mathcal{E}$ is called a soft class.

Definition 2.5. [1] Let $\mathbb{U}$ be a universe and $\mathcal{E}$ be two soft classes. Then a function $f : SS(\mathbb{U})_E \rightarrow SS(\mathbb{V})_{E'}$ and its inverse are defined as

i. Let $\mathcal{L}_\mathbb{U}$ be a soft set in $SS(\mathbb{U})_E$ where $\mathcal{A} \subseteq \mathcal{E}$. The image of $\mathcal{L}_\mathbb{U}$ under a function $f$ is a soft set in $SS(\mathbb{V})_{E'}$ such that $f(\mathcal{L}_\mathbb{U})(\beta) = (\bigcup_{\alpha \in \mathcal{L}_\mathbb{U}} \beta^{\gamma_{\alpha} \mathcal{A}}(\alpha))$ for $\beta \in B = p(A) \subseteq E'$.

ii. Let $\mathcal{G}_\mathbb{U}$ be the soft set in $SS(\mathbb{V})_{E'}$ such that $\mathcal{G}_\mathbb{U} \subseteq \mathcal{E}$ such that $f^{-1}(\mathcal{G}_\mathbb{V})(\alpha) = \alpha^{-1}((\exists \beta \in B) \supseteq E)$ for $\alpha \in p^{-1}(\beta) \subseteq E$.

3. Semi* -open and semi* -closed soft sets

In this chunk, we expound soft semi* -closure and soft semi* -interior of a soft set are defined in terms of soft semi* -closed and soft semi* -open sets.

Definition 3.1. In a soft topological space $(\mathbb{U}_E, \xi)$ a soft set

i. $\xi_E$ be termed as semi* -open soft set if there exists an open soft set $\mathcal{H}_\mathbb{U}$ such that $\mathcal{H}_\mathbb{U} \subseteq \xi_E \subseteq \text{cl}^* (\mathcal{H}_\mathbb{U})$.

ii. $\xi_E$ be termed as semi* -closed soft set if there exists a closed soft set $\mathcal{K}_\mathbb{U}$ such that $\mathcal{K}_\mathbb{U} \subseteq \xi_E \subseteq \text{int}^* (\mathcal{K}_\mathbb{U})$.

We denote the set of all semi* -closed Soft sets (respectively, Semi* -open Soft sets) over $\mathbb{U}$ by $S^*\text{C}(\mathbb{U})_E$ (respectively, $S^*\text{OSS}(\mathbb{U})_E$)

Theorem 3.2. Let $\xi_E$ be a soft set in a soft topological space $(\mathbb{U}_E, \xi)$. Then the subsequent are equivalent:

1. $\xi_E$ is a semi* -closed soft set.
2. $\text{int}^* (\text{cl}(\xi_E)) \subseteq \xi_E$.
3. $\text{cl}^* (\text{int}(\xi_E)) \subseteq \xi_E$.
4. $\xi_E$ is a semi* -open soft set.

Proof. (1) $\Rightarrow$ (2): If $\xi_E$ is a semi* -closed soft set, then there exists a closed soft set $\mathcal{H}_\mathbb{U}$ such that $\mathcal{H}_\mathbb{U} \subseteq \xi_E \subseteq \text{cl}^* (\mathcal{H}_\mathbb{U})$. Also $\text{cl}(\xi_E)$ is a smallest closed soft set that contains $\xi_E$. Therefore, $\text{cl}(\xi_E) \subseteq \text{cl}(\mathcal{H}_\mathbb{U})$ which implies $\text{int}^* (\text{cl}(\xi_E)) \subseteq \text{int}^* (\mathcal{H}_\mathbb{U}) \subseteq \xi_E$.

(2) $\Rightarrow$ (3): Assume that $\text{int}^* (\text{cl}(\xi_E)) \subseteq \xi_E$. Now $\xi_E \subseteq \text{int}^* (\text{cl}(\xi_E))$. This implies $\xi_E \subseteq \text{cl}^* (\text{cl}(\xi_E))$. This implies $\xi_E \subseteq \text{cl}^* (\text{int}(\xi_E))$. Hence $\text{cl}^* (\text{int}(\xi_E)) \subseteq \xi_E$.

(3) $\Rightarrow$ (4): Take $\mathcal{H}_\mathbb{U} = \text{int}(\xi_E)$. Then $\mathcal{H}_\mathbb{U}$ is an open soft set such that $\text{int}(\xi_E) \subseteq \text{cl}^* (\mathcal{H}_\mathbb{U})$ and hence $\mathcal{H}_\mathbb{U} \subseteq \xi_E \subseteq \text{cl}^* (\mathcal{H}_\mathbb{U})$, where $\mathcal{H}_\mathbb{U}$ is an open soft set. Therefore $\xi_E$ is a semi* -open soft set.
Theorem 3.3. In a soft topological space $(\mathcal{U}_E, \tau)$, every open soft set in a soft topological space is a semi*-open soft set.

Proof. Let $S_A$ be an open soft set. Since $S_A$ is an open set, $\text{int}(S_A) = S_A$. Now $\text{int}(S_A) \subseteq \text{cl}^*(\text{int}(S_A))$. Hence $S_A$ is a semi*-open soft set.

Theorem 3.4. Every closed soft set in a soft topological space $(\mathcal{U}_E, \tau)$ is a semi*-closed soft set.

Proof. Let $S_A$ be a closed soft set. Since $S_A$ is closed, $\text{cl}(S_A) = S_A$. Now $\text{cl}^*(\text{int}(S_A)) = \text{int}^*(\text{int}(S_A))$. Hence $S_A$ is a semi*-closed soft set.

Theorem 3.5. Every semi*-open soft set is a semi-open soft set.

Proof. Let $S_A$ be a semi*-open soft set. Then $\text{int}^*(\text{int}(S_A)) \subseteq \text{int}(S_A)$. Also we see that, $\text{cl}^*(\text{int}(S_A)) \subseteq \text{cl}(S_A)$. Hence $S_A$ is a semi-open soft set.

Corollary 3.6. Every semi*-closed soft set is a semi-closed soft set.

Theorem 3.7. The arbitrary union of semi*-open soft sets is a semi*-open soft set.

Proof. Let $\{S_A\}_{\lambda}$ be a collection of semi*-open soft sets in a soft topological space. Then there exist open soft sets $(\mathbb{U}_E)_\lambda$ such that $(\mathbb{U}_E)_\lambda \subseteq (S_A)_\lambda \subseteq \text{cl}^*(\mathbb{U}_E)_\lambda$ for each $\lambda$. Hence $(\text{cl}^*(\mathbb{U}_E)_\lambda) \subseteq (\mathbb{U}_E)_\lambda \subseteq \text{cl}^*(\mathbb{U}_E)_\lambda$. Therefore, $\bigcup S_A$ is a semi*-open soft set.

Corollary 3.8. The arbitrary intersection of semi*-closed soft sets is a semi*-closed soft set.

Theorem 3.9. Let $S_C$ be a semi*-open soft set and $S_C \subseteq S_D \subseteq \text{cl}^*(S_C)$, then $S_D$ is also a semi*-open soft set.

Proof. Let $S_C$ be a semi*-open soft set. Then there exists an open soft set $(\mathbb{U}_E)_\lambda$ such that $(\mathbb{U}_E)_\lambda \subseteq (S_C)_\lambda \subseteq \text{cl}^*(\mathbb{U}_E)_\lambda$. By our assumption $\mathbb{U}_E \subseteq S_D$ and $\text{cl}^*(S_C)_\lambda \subseteq \text{cl}^*(\mathbb{U}_E)_\lambda$. Therefore $S_D$ is also semi*-open soft set.

Theorem 3.10. If a semi*-closed soft set $L_A$ is such that $\text{int}^* L_A \subseteq \text{cl}^*(L_A)$, then $S_D$ is also semi*-closed.

Proof. Similar to the above theorem.

Definition 3.11. Let $S_C$ be a soft set in a soft topological space.

(i) The soft semi*-closure of $S_C$ is $ss^*cl(S_C) = \cap \{S_F : S_F \subseteq S_C \text{ and } S_F \in S^*CSS(\mathcal{U}_E)\}$ is a soft set.

(ii) The soft semi*-interior of $S_C$ is $ss^*\text{int}(S_C) = \cup \{S_F : S_F \supseteq S_C \text{ and } S_F \in S^*OSP(\mathcal{U}_E)\}$ is a soft set.

In short, $ss^*\text{cl}(S_C)$ is the smallest semi*-closed soft set containing $S_C$ and $ss^*\text{int}(S_C)$ is the largest semi*-open soft set contained in $S_C$.

Theorem 3.12. Let $S_C$ be a soft set in a soft topological space $(\mathcal{U}_E, \tau)$. Then the soft point $\ell_F \in ss^*\text{cl}(S_C)$ if and only if every soft semi*-open set containing $\ell_F$ intersects $S_C$.

Proof. We transform each implication to its contrapositive by $\ell_F \notin ss^*\text{cl}(S_C)$ if and only if there exists a soft semi*-open set $\mathbb{U}_E$ containing $\ell_F$ that does not intersect $S_C$.

Suppose that $\ell_F \notin ss^*\text{cl}(S_C)$. Then $\ell_F \notin (ss^*\text{int}(S_C))^c$. Then $(ss^*\text{int}(S_C))^c$ is a soft semi*-open set containing $\ell_F$ that does not intersect $S_C$. Conversely, if there exists a soft semi*-open set $\mathbb{U}_E$ containing $\ell_F$ which does not intersect $S_C$, then $(ss^*\text{int}(S_C))^c$ is a soft semi*-open set containing $S_C$. By the definition of soft semi*-closure, $ss^*\text{cl}(S_C)$ is contained in $(ss^*\text{int}(S_C))^c$. Hence $\ell_F$ cannot be in $ss^*\text{cl}(S_C)$.

Theorem 3.13. Let $S_C$ and $S_D$ be two soft sets in a soft topological space. Then

(i) $S_C \in S^*CSS(\mathcal{U}_E)$ if and only if $S_C = ss^*\text{cl}(S_C)$.

(ii) $S_C \in S^*OSP(\mathcal{U}_E)$ if and only if $S_C = ss^*\text{int}(S_C)$.

(iii) $(ss^*\text{cl}(S_C))^c = ss^*\text{int}(S_C)$.

(iv) $(ss^*\text{int}(S_C))^c = ss^*\text{cl}(S_C)$.

(v) $S_C \subseteq S_D$ implies $ss^*\text{int}(S_C) \subseteq ss^*\text{int}(S_D)$.

(vi) $S_C \subseteq S_D$ implies $ss^*\text{cl}(S_C) \subseteq ss^*\text{cl}(S_D)$.

(vii) $ss^*\text{cl}(S_C) = \emptyset$. Then $ss^*\text{cl}(S_C) = \emptyset$. Then $ss^*\text{cl}(S_C) = \emptyset$.

(viii) $ss^*\text{int}(S_C) = \emptyset$. Then $ss^*\text{int}(S_C) = \emptyset$.

(ix) $ss^*\text{int}(S_C) = ss^*\text{int}(S_C) \cap SS^*\text{int}(S_D)$.

(x) $ss^*\text{cl}(S_C) \cap SS^*\text{cl}(S_C) = ss^*\text{cl}(S_C) \cap SS^*\text{cl}(S_D)$.

(xi) $ss^*\text{int}(S_C) \cap SS^*\text{int}(S_C) = SS^*\text{int}(S_C) \cap SS^*\text{int}(S_D)$.
(xii) \( ss^*cl(ss^*cl(G_C)) = ss^*cl(G_C) \).

(xiii) \( ss^*int(ss^*int(G_C)) = ss^*int(G_C) \).

Proof.

(i) Let \( G_C \) be a semi*-closed soft set. Then it is a smallest semi*-closed soft set containing itself. Then by the definition of soft semi*-closure, we have \( G_C = ss^*cl(G_C) \).

Conversely let \( G_C = ss^*cl(G_C) \). since \( ss^*cl(G_C) \) is the intersection of all soft semi*-closed sets and by using Corollary 3.8, \( ss^*cl(G_C) \in S^{CSS}(U_E) \). Hence \( G_C \in S^{CSS}(U_E) \).

(ii) Let \( G_C \) be a semi*-open soft set. Then it is a largest semi*-open soft set contained in itself. Then by the definition of soft semi*-interior \( G_C = ss^*int(G_C) \) Conversely let \( G_C = ss^*int(G_C) \) As \( ss^*int(G_C) \) is the union of all soft semi*-open sets and by using Theorem 3.7, \( ss^*int(G_C) \in S^{OSS}(U_E) \). This implies \( G_C \in S^{OSS}(U_E) \).

(iii) \( ss^*int(G_C) = \bigcup \{ (H_D)^c : H_D \text{ is a semi*-closed soft set and } (G_C)^c \subseteq H_D \} \). That is \( ss^*int(G_C) = \bigcup \{ (H_D)^c : H_D \text{ is a semi*-closed soft set and } (G_C)^c \subseteq H_D \} \). This implies \( ss^*int(G_C) = [ss^*cl(G_C)]^c \). Hence \( ss^*int(G_C) \) is \( ss^*cl(G_C) \).

(iv) Similar to (iii).

(v) \( ss^*int(G_C) \subseteq G_C \subseteq X_D \). As \( ss^*int(G_C) \) is the largest semi*-open soft set contained in \( X_D \), \( ss^*int(G_C) \subseteq ss^*cl(X_D) \).

(vi) \( X_D \subseteq ss^*cl(X_D) \). This implies \( G_C \subseteq ss^*cl(X_D) \). Hence \( G_C \subseteq ss^*cl(X_D) \). As \( ss^*cl(G_C) \) is the smallest semi*-closed soft set containing \( G_C \), \( ss^*cl(G_C) \subseteq ss^*cl(X_D) \).

(vii) Since \( \Phi_C \) and \( U_C \) are semi*-closed soft set by (i), \( ss^*cl(\Phi_C) = \Phi_C \) and \( ss^*cl(U_C) = U_C \).

(viii) Similar to (vii).

(ix) \( G_C \in X_D \subseteq G_C \subseteq X_D \).

Hence by (v)

\[ ss^*int(G_C) \subseteq ss^*int(G_C) \subseteq ss^*int(X_D) \]. This implies

\[ ss^*int(G_C) \subseteq ss^*int(G_C) \subseteq ss^*int(X_D) \]. Let \( \ell, \varphi \not\subseteq g \) \( ss^*int(G_C) \subseteq ss^*int(X_D) \). Then \( \ell, \varphi \not\subseteq \bigcup \{ (H_B)_{\lambda} \lambda \in \Lambda \} \) where \( (H_B)_{\lambda} \subseteq S^{CSS}(U_E) \) such that \( (H_B)_{\lambda} \not\subseteq G_C \) for all \( \lambda \in \Lambda \). This implies \( \ell, \varphi \not\subseteq \bigcup \{ (H_B)_{\lambda} \lambda \in \Lambda \} \) where \( (H_B)_{\lambda} \subseteq S^{CSS}(U_E) \) such that \( (H_B)_{\lambda} \not\subseteq G_C \) for all \( \lambda \in \Lambda \). Hence \( \ell, \varphi \not\subseteq \bigcup \{ (H_B)_{\lambda} \lambda \in \Lambda \} \) where \( (H_B)_{\lambda} \not\subseteq S^{CSS}(U_E) \) such that \( (H_B)_{\lambda} \not\subseteq G_C \) for all \( \lambda \in \Lambda \). This implies

\[ \ell, \varphi \not\subseteq ss^*int(G_C) \neq ss^*int(X_D) \].

Hence \( ss^*int(G_C) \subseteq ss^*int(G_C) \subseteq ss^*int(X_D) \).

Therefore \( ss^*int(G_C) \subseteq ss^*int(G_C) \subseteq ss^*int(G_C) \).

(x) \( G_C \subseteq G_C \subseteq G_C \).

Hence by (vi), \( ss^*cl(G_C) \subseteq ss^*cl(G_C) \) and \( ss^*cl(G_C) \subseteq ss^*cl(G_C) \).

This implies

\[ ss^*cl(G_C) \subseteq ss^*cl(G_C) \subseteq ss^*cl(X_D) \].

(xi) \( G_C \subseteq G_C \subseteq G_C \).

Then by (v), \( ss^*int(G_C) \subseteq ss^*int(G_C) \) and \( ss^*int(G_C) \subseteq ss^*int(G_C) \).

Therefore

\[ ss^*int(G_C) \subseteq ss^*int(G_C) \subseteq ss^*int(G_C) \].

(xii) Since \( ss^*cl(G_C) \subseteq S^{CSS}(U_E) \), by (i) \( ss^*cl(ss^*cl(G_C)) = ss^*cl(G_C) \).

(xiii) Since \( ss^*int(G_C) \subseteq S^{OSS}(U_E) \), by (ii) \( ss^*int(ss^*int(G_C)) = ss^*int(G_C) \).

4. Functions using soft semi*-open sets

On this spot, we elucidate generalizations of soft functions in soft topological spaces and investigate their properties.
Definition 4.1. A soft function \( f : SS(U) \rightarrow SS(V) \) is said to be

(i) soft semi*-continuous if for each soft open set \( \mathcal{G} \subseteq \mathcal{V} \), the inverse image \( f^{-1}(\mathcal{G}) \) is a semi*-open soft set of \( U \).

(ii) soft semi*-open function if for each soft closed set \( \mathcal{L}_A \subseteq U \), the image is a semi*-open soft set of \( V \).

(iii) soft semi*-closed function if for each closed soft set \( \mathcal{K}_D \subseteq U \), the image \( f(\mathcal{K}_D) \) is a semi*-closed soft set of \( V \).

(iv) soft semi*-irresolute if for each soft open set \( \mathcal{G}_C \subseteq \mathcal{V} \), the inverse image \( f^{-1}(\mathcal{G}_C) \) is a semi*-open soft set of \( U \).

Definition 4.2. A soft function \( f : SS(U) \rightarrow SS(V) \) is semi*-continuous if for each closed soft set \( \mathcal{K}_D \subseteq \mathcal{V} \), the image \( f^{-1}(\mathcal{K}_D) \) is a semi*-closed soft set of \( U \).

Theorem 4.3. A soft function \( f : SS(U) \rightarrow SS(V) \) is semi*-continuous if and only if \( f(\text{ss}^*\text{cl}(\mathcal{L}_A)) \subseteq \text{cl}(f(\mathcal{L}_A)) \) for every soft set \( \mathcal{L}_A \subseteq \mathcal{U} \).

Proof. Assume that \( f \) is semi*-continuous. Let \( \mathcal{G} \subseteq \mathcal{V} \) be any soft closed set. Then \( f^{-1}(\mathcal{G}) \subseteq \mathcal{U} \) is a semi*-closed soft set. Therefore, \( f^{-1}(\mathcal{G}) \subseteq \text{cl}(f^{-1}(\mathcal{G})) \). This implies \( f(\text{ss}^*\text{cl}(\mathcal{L}_A)) \subseteq \text{cl}(f(\mathcal{L}_A)) \).

Conversely, assume that \( f(\text{ss}^*\text{cl}(\mathcal{L}_A)) \subseteq \text{cl}(f(\mathcal{L}_A)) \) for every soft set \( \mathcal{L}_A \subseteq \mathcal{U} \). Then \( f^{-1}(\mathcal{G}) \subseteq \text{ss}^*\text{cl}(f^{-1}(\mathcal{G})) \) for each soft closed set \( \mathcal{G} \subseteq \mathcal{V} \). Therefore, \( f \) is semi*-continuous.

Theorem 4.4. A soft function \( f : SS(U) \rightarrow SS(V) \) is semi*-continuous if \( f^{-1}(\text{int} \mathcal{G}_C) \subseteq \text{ss}^*\text{int}(f^{-1}(\mathcal{G}_C)) \) for each soft closed set \( \mathcal{G}_C \subseteq \mathcal{V} \).

Proof. Suppose \( f : SS(U) \rightarrow SS(V) \) is semi*-continuous. Let \( \mathcal{L}_A \subseteq \mathcal{U} \) be any soft set. Then \( f^{-1}(\text{int} \mathcal{G}_C) = f^{-1}(\text{cl}(\mathcal{G}_C)) = \text{int}(f^{-1}(\mathcal{G}_C)) \subseteq \text{ss}^*\text{int}(f^{-1}(\mathcal{G}_C)) \).

Conversely, assume that \( f^{-1}(\text{int} \mathcal{G}_C) \subseteq \text{ss}^*\text{int}(f^{-1}(\mathcal{G}_C)) \) for some soft closed set \( \mathcal{G}_C \subseteq \mathcal{V} \). Then \( f^{-1}(\mathcal{G}_C) = f^{-1}(\text{int} \mathcal{G}_C) \subseteq \text{ss}^*\text{int}(f^{-1}(\mathcal{G}_C)) \). Therefore, \( f \) is semi*-continuous.
forms a soft semi-open cover of \( U_c \), which is also semi-open compact. So, there exists a finite sub collection \( \Delta \) of \( \Lambda \) which also covers \( U_c \). That is \( \bigcup_{\lambda \in \Lambda} (\{\mathcal{L}_c\}_\lambda) = U_c \). This implies \( \bigcup_{\lambda \in \Lambda} (\{\mathcal{L}_c\}_\lambda)^c = \emptyset \). This is a contradiction to the finite intersection property. Hence \( \bigcap_{\lambda \in \Lambda} (\{\mathcal{L}_c\}_\lambda) \neq \emptyset \).

Conversely, assume that each family of semi-closed soft sets in \( U_c \) have the finite intersection property. If possible let us assume \((U_c, \tau)\) is not semi-open compact. Then there exists a soft semi-open cover \( \{\{\mathcal{L}_c\}_\lambda\}_{\lambda \in \Lambda} \) of \( U_c \) such that for every finite sub collection \( \Delta \) of \( \Lambda \) we have \( \bigcup_{\lambda \in \Delta} (\{\mathcal{L}_c\}_\lambda) \neq U_c \). Hence \( \bigcap_{\lambda \in \Lambda} (\{\mathcal{L}_c\}_\lambda) \neq \emptyset \).

So, by hypothesis \( \bigcap_{\lambda \in \Lambda} (\{\mathcal{L}_c\}_\lambda) \neq \emptyset \). Which implies \( \bigcup_{\lambda \in \Lambda} (\{\mathcal{L}_c\}_\lambda) \neq U_c \). This is a contradiction to our assumption. Therefore \((U_c, \tau)\) is a semi-open compact soft topological space.

**Theorem 5.4.** A soft topological space \((U_c, \tau)\) is semi-open compact if and only if for every family \( \Psi \) of soft sets with finite intersection property, \( \bigcap_{\Psi \in \Psi} ss^c cl(\{\mathcal{L}_c\}) \neq \emptyset \).

**Proof.** Let \((U_c, \tau)\) be a semi-open compact soft topological space. If possible let us assume that \( \bigcap_{\Psi \in \Psi} ss^c cl(\{\mathcal{L}_c\}) = \emptyset \). So \( \bigcup_{\Psi \in \Psi} (ss^c cl(\{\mathcal{L}_c\}))^c = U_c \). Hence \( \bigcap_{\Psi \in \Psi} (ss^c cl(\{\mathcal{L}_c\}))^c = \emptyset \). Which implies \( \{\{\mathcal{L}_c\}_\lambda\}_{\lambda \in \Lambda} \) forms an open soft cover for \( U_c \).

Then \( U_c = \bigcup_{\Psi \in \Psi} (ss^c cl(\{\mathcal{L}_c\})) \). Therefore \( \bigcap_{\Psi \in \Psi} (ss^c cl(\{\mathcal{L}_c\})) = \emptyset \). This is a contradiction to the finite intersection property. Hence \( \bigcap_{\Psi \in \Psi} ss^c cl(\{\mathcal{L}_c\}) \neq \emptyset \).

Conversely, assume that \( \bigcap_{\Psi \in \Psi} ss^c cl(\{\mathcal{L}_c\}) \neq \emptyset \). For every family \( \Psi \) of soft sets with finite intersection property, suppose that \((U_c, \tau)\) is not semi-open compact. Then there exists a family \( \Gamma \) of semi-open soft sets covering \( U_c \) without a finite subcover. So for every finite sub family \( \Theta \) of \( \Gamma \) we have \( \bigcap_{\Theta \in \Theta} ss^c cl(\{\mathcal{L}_c\}) = \emptyset \). This implies \( \bigcap_{\Theta \in \Theta} (ss^c cl(\{\mathcal{L}_c\}))^c = \emptyset \). This implies \( \{\{\mathcal{L}_c\}_\lambda\}_{\lambda \in \Lambda} \) is a family of soft sets with finite intersection property. Now \( \bigcup_{\Theta \in \Theta} ss^c cl(\{\mathcal{L}_c\}) \). This implies \( \bigcap_{\Theta \in \Theta} ss^c cl(\{\mathcal{L}_c\}) = \emptyset \). Hence \( \bigcap_{\Theta \in \Theta} ss^c cl(\{\mathcal{L}_c\}) = \emptyset \). This is a contradiction. Therefore \((U_c, \tau)\) is semi-open compact soft topological space.

**Theorem 5.5.** Semi-continuous image of a soft semi-open compact space is soft compact.

**Proof.** Let \( f : SS(U) \to SS(V) \) be a semi-continuous function where \((U_c, \tau)\) is a semi-compact soft topological space and \((V, \delta)\) is another soft topological space. Let \( \{\{\mathcal{L}_c\}_\lambda\}_{\lambda \in \Lambda} \) be a semi-open cover of \( V \). Since \( f \) is semi-continuous, \( f^{-1}(\{\{\mathcal{L}_c\}_\lambda\}_{\lambda \in \Lambda}) \) forms a soft semi-open cover for \( U_c \). This implies there exists a finite subset \( \Delta \) of \( \Lambda \) such that \( f^{-1}(\{\mathcal{L}_c\}_{\lambda \in \Delta}) \) forms a semi-open cover of \( U_c \).

**Theorem 5.6.** Semi-closed subspace of a semi-open compact soft topological space is soft semi-open compact.

**Proof.** Let \( U_c \) be a semi-open compact soft topological space \((U_c, \tau)\) and \( \{\{\mathcal{L}_c\}_\lambda\}_{\lambda \in \Lambda} \) be a semi-open cover for \( V \). As \( V \) is semi-open compact soft set \( V \) is a semi-open soft set. Hence \( \bigcap_{\lambda \in \Lambda} (\{\mathcal{L}_c\}_\lambda) \bigcap \{\{\mathcal{L}_c\}_\lambda\} \) forms a soft semi-open cover for \( U_c \). As \( U_c \) is soft semi-open compact \( \Lambda \) has a finite sub family \( \Delta \) such that \( \{\{\mathcal{L}_c\}_{\lambda \in \Delta}\} \) covers \( U_c \). Then \( V = \bigcap_{\lambda \in \Lambda} (\{\mathcal{L}_c\}_\lambda) \bigcap \{\{\mathcal{L}_c\}_\lambda\} \).

**Theorem 5.7.** Semi-irresolute image of a semi-open compact space is semi-open compact.

**Proof.** Let \( f : SS(U) \to SS(V) \) be a semi-irresolute function where \((U_c, \tau)\) is a semi-open compact soft topological space and \((V, \delta)\) is another soft topological space. Let \( \{\{\mathcal{L}_c\}_\lambda\}_{\lambda \in \Lambda} \) be a semi-open cover for \( V \). As \( f \) is semi-irresolute function \( f^{-1}(\{\{\mathcal{L}_c\}_\lambda\}) \) is a semi-open set for every \( \lambda \in \Lambda \). Hence \( \{f^{-1}(\{\mathcal{L}_c\})\}_{\lambda \in \Lambda} \) forms a semi-open cover for \( U_c \). Since \((U_c, \tau)\) is a semi-open compact, there exists a finite sub family \( \Delta \) of \( \Lambda \) such that \( f^{-1}(\{\mathcal{L}_c\}_{\lambda \in \Delta}) \) covers \( U_c \). Hence \( \{f^{-1}(\{\mathcal{L}_c\})\}_{\lambda \in \Delta} \) forms a finite sub cover of \( f(U_c) \). Hence \( f(U_c) \) is semi-open compact.

6. Soft semi-connectedness

Here, we come out with semi-connectedness in soft topological spaces put into action with semi-open soft sets and scrutinize its basic properties.

**Definition 6.1.** [5] Two soft sets \( \mathcal{L}_A \) and \( \mathcal{J}_B \) are said to be disjoint if \( \mathcal{L}_A(a) \cap \mathcal{J}_B(b) = \emptyset \) for all \( a \in A, b \in B \).

**Definition 6.2.** A soft semi-separation of soft topological \((U_c, \tau)\) is a pair \( \mathcal{L}_A \), \( \mathcal{J}_B \) of disjoint non null semi-open sets whose union is \( U_c \). If there does not exist a soft semi-separation of \( U_c \), then the soft topological space is said to be soft semi-connected otherwise soft semi-disconnected.

**Example 6.3.** Consider the soft topological space \((U_c, \tau)\), where \( U = \{h_1, h_2\} \), \( E = \{e_1, e_2\} \), and \( \tau = \{\emptyset, U_c, \{e_1, h_1\}, \{e_2, h_1\}, \{e_1, h_2\}, \{e_2, h_2\}, \{e_1, h_1, h_2\}, \{e_2, h_1, h_2\}\} \). The semi-open soft sets are \( \emptyset, U_c, \{e_1, h_1\}, \{e_2, h_1\}, \{e_1, h_2\}, \{e_2, h_2\}, \{e_1, h_1, h_2\}, \{e_2, h_1, h_2\}\). Here there does not exist a soft semi-separation of \( U_c \). Therefore, \((U_c, \tau)\) is soft semi-connected.

**Theorem 6.4.** If the soft sets \( \mathcal{L}_A \) and \( \mathcal{J}_B \) form a soft semi-separation of \( U_c \) and if \( V \) is a soft semi-connected subspace of \( U_c \) then \( V \cap \mathcal{L}_A \neq \emptyset \) or \( V \cap \mathcal{J}_B \neq \emptyset \).

**Proof.** Given \( \mathcal{L}_A \) and \( \mathcal{J}_B \) form a soft semi-separation of \( U_c \). Since \( \mathcal{L}_A \) and \( \mathcal{J}_B \) are disjoint semi-open soft sets \( \mathcal{L}_A \cap \mathcal{J}_B \) are also semi-open soft sets and their soft union gives \( \mathcal{J}_B \). That is they would constitute a soft semi-separation of \( \mathcal{J}_B \). This is a contradiction. Hence one of \( \mathcal{L}_A \cap \mathcal{J}_B \) is empty. Therefore \( V \) is entirely contained in one of them.
Theorem 6.5. Let $V_E$ be a soft semi*-connected subspace of $U_E$ and $\mathcal{K}_D$ be a soft set in $U_E$ such that $V_E \subseteq \mathcal{K}_D \subseteq \text{cl}(V_E)$ then $\mathcal{K}_D$ is also soft semi*-connected.

Proof. Let the soft set $\mathcal{K}_D$ satisfies the hypothesis. If possible, let $\mathcal{F}_A$ and $\mathcal{S}_C$ form a soft semi*-separation of $\mathcal{K}_D$. Then by the theorem 5.4, $V_E \subseteq \mathcal{F}_A$ or $V_E \subseteq \mathcal{S}_C$. Let $V_E \subseteq \mathcal{F}_A$. This implies $\text{cl}(V_E) \subseteq \mathcal{S}(\mathcal{F}_A)$. Since $\text{cl}(V_E) \subseteq \mathcal{F}_A$ and $\mathcal{S}_C$ are disjoint, $V_E$ cannot intersects $\mathcal{S}_C$. This is a contradiction. Hence $\mathcal{K}_D$ is soft semi*-connected.

Theorem 6.6. A soft topological space $(U_E, \mathcal{F}_E)$ is soft semi*-disconnected if and only if there exists a non null proper soft subset of $U_E$ which is both soft semi*-open and soft semi*-closed.

Proof. Let $U_E$ be a soft semi*-disconnected. Then there exist non null soft subsets $\mathcal{K}_D$ and $\mathcal{K}_C$ such that $\text{ss}(\mathcal{K}_D) \cap \mathcal{K}_C = \emptyset$. This implies $\text{cl}(\mathcal{K}_D) \cap \mathcal{K}_C = \emptyset$, and $\text{cl}(\mathcal{K}_D) \subseteq \mathcal{K}_D$. This is a contradiction to the fact that $\mathcal{K}_D \subseteq \text{cl}(\mathcal{K}_D)$. Hence $\mathcal{K}_D$ is semi*-connected. Let $\mathcal{K}_D$ and $\mathcal{K}_C$ be soft semi*-open and soft semi*-closed sets whose union is $U_E$. Since $f$ is semi*-irresolute soft function $f^{-1}(\mathcal{K}_D)$ and $f^{-1}(\mathcal{K}_D)$ are semi*-open and semi*-closed sets. Also $\mathcal{K}_D \subseteq \mathcal{K}_D$. This implies $\mathcal{K}_D \subseteq \mathcal{K}_D$ and $\mathcal{K}_C$ are also semi*-closed soft sets.

Conversely, let $\mathcal{K}_D$ be a non null proper soft subset of $U_E$ which is both semi*-open and semi*-closed. Now let $\mathcal{K}_D \subseteq \mathcal{K}_D$ is non null proper subset of $U_E$ which is also both semi*-open and semi*-closed. This implies $U_E$ can be expressed as the soft union of two semi*-separated soft sets $\mathcal{K}_D$ and $\mathcal{K}_C$. Hence $U_E$ is semi*-disconnected.

Theorem 6.7. Semi*-irresolute image of a soft semi*-connected soft topological space is soft semi*-connected.

Proof. Let $f : \mathcal{SS}(U_E) \to \mathcal{SS}(V_E)$ be a semi*-irresolute soft function where $(U_E, \mathcal{F}_E)$ is a semi*-connected soft topological space. Our aim is to prove $f(U_E)$ is soft semi*-connected. Suppose assume that $f(U_E)$ is soft semi*-disconnected. Let $\mathcal{K}_D$ and $\mathcal{K}_C$ be non null disjoint semi*-open soft sets whose union is $f(U_E)$. Since $f$ is semi*-irresolute soft function $f^{-1}(\mathcal{K}_D)$ and $f^{-1}(\mathcal{K}_D)$ are semi*-open soft sets. Also they form a soft semi*-separation for $U_E$. This is a contradiction to the fact that $U_E$ is soft semi*-connected. Hence $f(U_E)$ is soft semi*-connected.

Theorem 6.8. Semi*-continuous function of a soft semi*-connected soft topological space is soft connected.

Proof. Let $f : \mathcal{SS}(U_E) \to \mathcal{SS}(V_E)$ be a semi*-continuous function where $(U_E, \mathcal{F}_E)$ is a semi*-connected soft topological space and $(V_E, \mathcal{F}_E)$ is a soft topological space. Our aim is to prove $f(U_E)$ is soft connected. Suppose assume that $f(U_E)$ is soft disconnected. Let $f(U_E) = \mathcal{K}_D \cup \mathcal{K}_C$ be a soft separation that is $\mathcal{K}_D$ and $\mathcal{K}_C$ are disjoint soft sets whose union is $f(U_E)$. This implies $f^{-1}(\mathcal{K}_D)$ and $f^{-1}(\mathcal{K}_D)$ form a soft semi*-separation of $U_E$. This is a contradiction. Hence $f(U_E)$ is soft connected.

7. Conclusion

Topology and Soft sets are playing vital role in Pure and Applied Mathematics and gives more applications in real life using various Mathematical tools. Recently scientists have studied soft set theory, which is originated by a Mathematician Molodtsov and easily applied to the theory of uncertainties. In the present work, we have continued the study of soft sets and soft topological spaces. We investigate the behavior of Soft Semi*-open and Soft Semi*-closed sets, which is a step forward to further investigate the strong base of soft topological spaces. Further we planned to introduce and investigate soft semi*-separation Axioms using soft semi*-open and soft semi*-closed sets. We assure that the belongings in this paper will help researchers move into the new direction and promote the future work in soft topological spaces.

Acknowledgement

The authors thank Dr. T.M. Al-Shami, Professor, Department of Mathematics, Sana’a University, Yemen for his keen interest about this article and valuable suggestions.

References

[1] A. Kharal, B. Ahmad, Mappings on soft classes, New Math. Nat. Comput. 7(3) (2011), 471-481.
[2] A. Robert, S. Pious Misser, A new class of nearly open Sets, Int. J. Math. Arch., 3(7)(2012), 2575-2582.
[3] A. Robert, S. Pious Misser, On semi*-closed sets, Asian J. Eng. Math., 1(4) (2012), 173-176.
[4] C. Gündüz Aras, S. Bayramov, On the Tietze extension theorem in soft topological spaces, Proc. Inst. Math. Mech., 43(1) (2017), 105-115.
[5] D. Molodtsov, Soft set theory-first results, Comput. Math. Appl., 37 (1999), 19-31.
[6] E. Peyghan, B. Samadi, A. Tayebi, On soft connectedness, arXiv:1202.1668V1 [math.GN], 8 Feb 2012.
[7] I. Demir, An approach to the concept of soft vieteros topology, Int. J. Anal. Appl., 12 (2016), 198-206.
[8] I. Demir, O.B. Ozbakir, I. Yildiz, A contribution to the study of soft proximity spaces, Filomat, 31 (2017) 2023 - 2034.
[9] I. Demir, O. B. Ozbakir, An extension of Lowen’s uniformity to the fuzzy soft sets, Konuralp J. Math., 6 (2018) 321 - 331.
[10] I. Zorlutuna, M. Akdag, W.K. Min, A. R. Roy, Remarks on soft topological spaces, Ann. Fuzzy Math. Inform., 3 (2) (2012), 171-185.
[11] K. Kannan Soft generalized closed soft topological spaces, J. Theoret. Appl. Tech., 37 (2012), 17-21.
[12] M. Shahr, M. Naz, On soft topological spaces, Comp. Math. Appl., 61(2011), 1786-1799.
[13] M.E. El-Shafei, M. Abo-Ellhameyel, T.M. Al-Shami, Two notes on “On soft Hausdorff spaces”, Ann. of Fuzzy Math. Inform., 16 (3) (2018), 333-336.
[14] N. Levine, Semi-open sets and Semi-continuity in topological spaces. Amer. Math. Monthly, 70(1) (1963), 36-41.
[15] N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo, 19 (1970), 89-96.
[16] P.K.Maji, R.Biswa, R.Roy Soft set theory, Computer and Mathematics with Applications 45 (2003) 555-562.
[17] S. Hussain, A. Ahmad, Some Properties of Soft Topological Spaces, Comput. Math. Appl., 62 (2011), 4048-4067.
[18] S. Pious Misser, A Robert On semi*-open sets, International Journal of Mathematics and soft computing, 2(2)(2012)95-105.
[19] S.M. Khalil, M. Ulzqra, M. Abd-Elghani, A.F. Al-Musawi, Soft Algebra and Soft-Baire in Fuzzy Soft Setting, Advances in Fuzzy Systems, Volume 2018, Article ID 5731682, 10 pages. 7(3)(2011), 471-481.
[20] S.M. Khalil, Decision making using algebraic operations on soft effect matrix as new category of similarity measures and study their application in medical diagnosis problems, Journal of Intelligent and Fuzzy Systems, 37(2019), 1865-1877.
[21] W. Dunham, A New Closure Operator for Non-T1 Topologies, Kyungpook Math. J. 22 (1982), 55-60.
[22] W.K. Min, A Note on Soft Topological Spaces, Comput. Math. Appl., 62(2011), 3524-3528.