GEOMETRIC CHARACTERIZATION OF $q$-PSEUDOCONVEX DOMAINS IN $\mathbb{C}^n$

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Abstract. In this paper, we investigate the notion of $q$-pseudoconvexity to discuss and describe some geometric characterizations of $q$-pseudoconvex domains $\Omega \subset \mathbb{C}^n$. In particular, we establish that $\Omega$ is $q$-pseudoconvex, if and only if, for every boundary point, the Levi form of the boundary is semipositive on the intersection of the holomorphic tangent space to the boundary with any $(n-q+1)$-dimensional subspace $E \subset \mathbb{C}^n$. Furthermore, we prove that the Kiselman’s minimum principal holds true for all $q$-pseudoconvex domains in $\mathbb{C}^p \times \mathbb{C}^n$ such that each slice is a convex tube in $\mathbb{C}^n$.

1. Introduction

We study in this paper the notion of $q$-pseudoconvexity from a geometric point of view. We consider smoothly $q$-pseudoconvex domains $\Omega$ in $\mathbb{C}^n$ which are defined by smooth $q$-subharmonic function $\rho$ such that $d\rho \neq 0$ on $\partial \Omega$. We will prove in Section 2 that, for $q$-pseudoconvex domains $\Omega \subset \mathbb{C}^n$ such that $2 \leq q \leq n$, the function $-\log d(z, \mathbb{C}\Omega)$ is $q$-subharmonic. Note that by taking in account our convention about this notion, according to [5], this isn’t generally the case for 1-pseudoconvex domains. By considering the function

$$\delta_\Omega(z, E) = \sup\{r > 0, z + B_E(r) \subset \Omega\},$$

which is the distance from $z$ to $\partial \Omega$ in the multi-complex direction supported by a $q$-dimensional subspace $E$ of $\mathbb{C}^n$, we obtain a new understanding of the concept of $q$-pseudoconvexity. We will see here that the function $-\log d(z, \mathbb{C}\Omega)$ is one of the most important tools in studying $q$-pseudoconvexity. In addition, we will show that the concepts of the weak $q$-pseudoconvexity and the strong $q$-pseudoconvexity are equivalent and we say simply $q$-pseudoconvexity. By using the function $\delta_\Omega(z, E)$, we legitimate the $q$-pseudoconvexity of the Hartogs domains when $n \geq 2q + 2$.

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In Section 3, we shall give a full rigorous proof of the local property of $q$-pseudoconvexity, that differs of such given in [5]. Furthermore, we characterize the $q$-pseudoconvexity by the Levy form of the defining function and we prove that $\Omega$ is $q$-pseudoconvex, if and only if, for every boundary point, the Levi form of the boundary is semi-positive on the intersection of the holomorphic tangent space to the boundary with any $(n - q + 1)$-dimensional subspace $E \subset \mathbb{C}^n$.

In Section 4, we attempt to show that the Kiselman’s minimum principle holds true for all $q$-pseudoconvex domains in $\mathbb{C}^p \times \mathbb{C}^n$ such that each slice is a convex tube in $\mathbb{C}^n$.

Now, let’s give the definition of a $q$-subharmonic function.

**Definition 1.1.** A function $u : \Omega \to [-\infty, +\infty]$, $u \not\equiv -\infty$, is called $q$-subharmonic if for every $(n - q + 1)$-dimensional complex subspace $E \subset \mathbb{C}^n$, the restriction $u|_{E \cap \Omega}$ is subharmonic. This means that for all compact set $K \subset E \cap \Omega$ and for every continuous harmonic function $h$ on $K$ such that $u \leq h$ on $\partial K$, we have $u \leq h$ on $K$.

Observe here that $n$-subharmonic functions are usual plurisubharmonic functions and 1-subharmonic functions are usual subharmonic functions. Further details about the notion of $q$-subharmonic functions and their properties can be obtained from [5] or [8].

The set of $q$-subharmonic functions on $\Omega$ will be denoted $q\text{-Sh}(\Omega)$.

**Example 1.1.** Consider in $\mathbb{C}^n$ the Riez kernel [6], $K(\alpha, z)$ defined by the expression

$$(1.2) \quad K(\alpha, z) = -\frac{|z|^{2(\alpha - q)}}{H_q(\alpha)}$$

where $H_q(\alpha) = \frac{\pi^{2n/2}2^{2\alpha}\Gamma(\alpha)}{\Gamma(q - \alpha)}$ and $1 \leq \alpha < q \leq n$.

For every $q$-dimensional subspace $E \subset \mathbb{C}^n$, an easy computation far from the origin of the Laplacian $\Delta K|_E$ of the restriction on $E$ of the function $K(\alpha, .)$ defined by (1.2), yields up to a positive constant

$$(1.3) \quad \Delta K|_E(\alpha, z) = -K|_E(\alpha - 1, z).$$

Then (1.3) implies that $K$ is $(n - q + 1)$-subharmonic on $\mathbb{C}^n$. In case $q = n$ and $\alpha = 1$, $K$ is the Newton kernel.

We may introduce the notion of a $q$-pseudoconvex domain in $\mathbb{C}^n$ where $n \geq 2$, by considering an integer $1 \leq q \leq n$ and a smoothly domain $\Omega \subset \mathbb{C}^n$ with a defining function $\rho$ such that $d\rho \not\equiv 0$ on $\partial \Omega$ and we define this notion as the following:

**Definition 1.2.** We say that $\Omega$ is $q$-pseudoconvex if there is a neighborhood $U$ of $\overline{\Omega}$ and a $q$-subharmonic function $\rho : U \to \mathbb{R} \cup \{-\infty\}$ such that $d\rho \not\equiv 0$ on $\partial \Omega$ and $\Omega = \{z \in \mathbb{C}^n / \rho(z) < 0\}$.

**Example 1.2.** Consider an example of $3$-pseudoconvex domain in $\mathbb{C}^5 = \mathbb{C}^3 \times \mathbb{C}^2$, which is a variant of the Kohn-Nirenberg example [4] of a pseudoconvex
domain in $\mathbb{C}^2$: 
$$
\Omega = \{(z', z, w) \in \mathbb{C}^5 : 3|z'|^2 - |z|^4 - |z_3|^4 + \Re(w) + |z|^{2k} + t|z|^2\Re(z^{2k-2}) < 0\}
$$
where $t \in \mathbb{R}$ and $k \in \mathbb{N}$, $k \geq 2$, are fixed parameters. We can easily check that if $|t| \leq \frac{k^2 - 2}{2k}$, then the restriction on every 3-complex subspace $E \subset \mathbb{C}^5$, of the defining function of $\Omega$ given by $\rho(z_1, z_2, z_3, z, w) = 3|z|^2 - |z_2|^4 - |z_3|^4 + \Re(w) + |z|^{2k} + t|z|^2\Re(z^{2k-2})$ is subharmonic. Which means that $\Omega$ is a 3-pseudoconvex domain in $\mathbb{C}^5$.

In [2], Dinh introduced the notion of $p$-pseudoconcavity of a closed subset $X$ of a complex manifold $\mathbb{C}^n$ as follows:

We say that $X$ is $p$-pseudoconcave if for every open set $U \subset V$ and every holomorphic map $f$ from a neighborhood of $\overline{U}$ into $\mathbb{C}^p$, we have $f(X \cap U) \subset \mathbb{C}^p \setminus \Omega$ where $\Omega$ is the unbounded component of $\mathbb{C}^p \setminus f(X \cap \partial \Omega)$.

As it is mentioned above, $n$-pseudoconvex domains are just the usual pseudoconvex domains which are domains of holomorphy with smooth boundary. In addition, strictly $q$-pseudoconvex domains are defined at the boundary by smooth strictly $q$-subharmonic functions.

**Definition 1.3.** A function $u \in q\text{-Sh}(\Omega)$ is said to be strictly $q$-subharmonic if $u \in L^1_{loc}(\Omega)$ and if for every point $x_0 \in \Omega$ there exist a neighborhood $\omega$ of $x_0$ and $c > 0$ such that $u - c|z|^2$ is $q$-subharmonic in $\omega$.

**Remark 1.1.** By induction on $1 \leq k \leq q$, we can show that a function $u$ is strictly $q$-subharmonic on $\Omega$ means that for every point $x_0 \in \Omega$, there exist $c > 0$ and a neighborhood $\omega$ of $x_0$ such that

$$(dd^c u)^k \wedge \beta^{n-k} \geq c\beta^n \quad \omega \quad \forall \ k = 1, \ldots, q,$$

where $\beta$ is the Kahler form on $\mathbb{C}^n$.

**Definition 1.4.** Let $\Omega \subset \mathbb{C}^n$ be an open subset and a function $\psi : \Omega \rightarrow [-\infty, +\infty]$. Then $\psi$ is said to be an exhaustion, if all sub-level sets $\Omega_c = \{z \in \Omega / \psi(z) < c\}, c \in \mathbb{R}$, are relatively compact. Furthermore, we say that

1. $\Omega$ is weakly $q$-pseudoconvex, if there exists a smooth $q$-subharmonic exhaustion function $\psi \in q\text{-Sh}(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega})$;
2. $\Omega$ is strongly $q$-pseudoconvex, if there exists a smooth strictly $q$-subharmonic exhaustion function $\psi \in q\text{-Sh}(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega})$.

The main results of this paper are the followings:

**Theorem 2.2.** Let $2 \leq q \leq n$ be a nonnegative integer, $\Omega$ be an open subset in $\mathbb{C}^n$ and $E$ be a $(n-q+1)$-dimensional complex subspace. Then the following properties are equivalent:

1. $\Omega$ is strongly $q$-pseudoconvex;
2. $\Omega$ is weakly $q$-pseudoconvex;
3. $\Omega$ has a $q$-subharmonic exhaustion function;
(4) the function \((z, \xi_1, \ldots, \xi_{n-q+1}) \mapsto -\log \delta_\Omega(z, \xi_1, \ldots, \xi_{n-q+1})\) is \(q\)-subharmonic on \(\Omega \times E^{n-q+1}\);

(5) the function \(z \mapsto -\log d(z, \partial \Omega)\) is \(q\)-subharmonic on \(\Omega\).

**Theorem 3.2.** Let \(2 \leq q \leq n\) be a nonnegative integer. An open subset \(\Omega \subset \mathbb{C}^n\) with smooth boundary is \(q\)-pseudoconvex, if and only if, for every \((n-q+1)\)-dimensional complex subspace \(E \subset \mathbb{C}^n\), the Levi form \(L_{\partial \Omega, z}|_{E \cap \mathcal{T}_{\partial \Omega, z}}\) is semi-positive at every point of \(\partial \Omega\).

In case \(q = n\), Theorem 2.2 and Theorem 3.2 were proved in [1].

**Theorem 4.1.** Let \(\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{C}^p \times \mathbb{C}^n\) be a \(q\)-pseudoconvex domain such that each slice

\[
\Omega_\zeta = \{z \in \mathbb{C}^n; (\zeta, z) \in \Omega\}, \quad \zeta \in \mathbb{C}^p,
\]

is a convex tube \(\omega_\zeta + i\mathbb{R}^n, \omega_\zeta \subset \mathbb{C}^p\). Then, for every \(q\)-subharmonic function \(v(\zeta, z)\) on \(\Omega\) that does not depend on \(z\), the function \(u(\zeta) = \inf_{z \in \Omega} v(\zeta, z)\) is \(q\)-subharmonic or locally \(-\infty\) on \(\Omega_2 = \text{pr}_{\mathbb{C}^n}(\Omega)\).

In case \(q = n\), Theorem 4.1 was proved in [3].

### 2. Geometric characterizations of \(q\)-pseudoconvex domains

In this section, we will discuss some characterizations of \(q\)-pseudoconvex domains in \(\mathbb{C}^n\).

Let \(E \subset \mathbb{C}^n\) be a \(q\)-dimensional subspace. We denote by \(B_E(r)\) the ball in \(E\) of center 0 and radius \(r\), when \(E = \mathbb{C}^q, B_{\mathbb{C}^q}(r)\) will be simply denoted \(B(r)\). For \(r_0 > 0\) and \(z_0 \in \Omega\), we denote by \(z_0 + B_E(r_0)\) the set of points of the form \(z_0 + t_1\xi_1 + \cdots + t_q\xi_q\), where \((t_1, \ldots, t_q) \in B_E(1)\) and \(\{\xi_1, \ldots, \xi_q\}\) is an orthonormal basis of \(E\). We also denote \(S_E(r)\) the sphere of center 0 and of radius \(r\) in \(E\). For any \(z \in \Omega\), we put

\[
(2.4) \quad \delta_\Omega(z, E) = \sup\{r > 0, z + B_E(r) \subset \Omega\}.
\]

The expression \((2.4)\) is the distance from \(z\) to \(\partial \Omega\) in the multi-complex direction supported by \(E\).

If \(\{\xi_1, \ldots, \xi_q\}\) is an orthonormal basis of \(E\), then we will sometimes denote the distance from \(z\) to \(\partial \Omega\) by \(\delta_\Omega(z, \xi_1, \ldots, \xi_q)\). So we have

\[
(2.5) \quad \delta_\Omega(z, \xi_1, \ldots, \xi_q) = \sup\{r > 0 / z + t_1\xi_1 + \cdots + t_q\xi_q \in \Omega, (t_1, \ldots, t_q) \in B(r)\}.
\]

We will need the following elementary proposition to characterize \(q\)-subharmonic functions.

**Proposition 2.1.** Let \(v : \Omega \to [-\infty, +\infty[\) be an upper semi continuous function and suppose that \(1 \leq q \leq n\). Then \(v\) is \(q\)-subharmonic, if and only if, for every \((n-q+1)\)-dimensional complex subspace \(E \subset \mathbb{C}^n\), for any closed ball \(B = z_0 + B_E(1) \subset \Omega\) and any polynomial \(P \in \mathbb{C}[t_1, \ldots, t_{n-q+1}]\) such that

\[
v(z_0 + t_1\eta_1 + \cdots + t_{n-q+1}\eta_{n-q+1}) \leq \Re P(t_1, \ldots, t_{n-q+1})
\]
whenever $|t_1|^2 + \cdots + |t_{n-q+1}|^2 = 1$
then $v(z_0) \leq \Re P(0)$, where $\{\eta_1, \ldots, \eta_{n-q+1}\}$ is any orthonormal basis of $E$.

Proof. It is clear that the condition is necessary. Indeed, the function

$$ (t_1, \ldots, t_{n-q+1}) \mapsto \Re P(t_1, \ldots, t_{n-q+1}) $$

is pluriharmonic and hence the function $(t_1, \ldots, t_{n-q+1}) \mapsto v(z_0 + t_1 \eta_1 + \cdots t_{n-q+1} \eta_{n-q+1}) - \Re P(t_1, \ldots, t_{n-q+1})$ is subharmonic in a neighborhood of $B_E(1)$, so it satisfies the maximum principal on $B_E(1)$. To prove the sufficiency, let $v = \lim v_\mu$ be a strictly decreasing sequence of continuous functions on $\partial B$ such that $v = \lim v_\mu$ on $\partial B$.

Without loss of generalities, we may assume that $v_\mu$ is smooth on a small neighborhood of $S_E$ and

$$ v_\mu(z_0 + t_1 \eta_1 + \cdots + t_{n-q+1} \eta_{n-q+1}) = \Re P_\mu(t_1, \ldots, t_{n-q+1}) $$

whenever $|t_1|^2 + \cdots + |t_{n-q+1}|^2 = 1$

where $P_\mu \in \mathbb{C}[t_1, \ldots, t_{n-q+1}]$. Then, we have

$$ v(z_0 + t_1 \eta_1 + \cdots + t_{n-q+1} \eta_{n-q+1}) \leq \Re P_\mu(t_1, \ldots, t_{n-q+1}) $$

whenever $|t_1|^2 + \cdots + |t_{n-q+1}|^2 = 1$.

and thanks to (2.6), we get

$$ v(z_0) \leq \Re P_\mu(0) $$

$$ \leq \frac{1}{\text{area}(S_E)} \int_{S_E} \Re P_\mu(\xi) d\sigma(\xi) $$

$$ = \frac{1}{\text{area}(S_E)} \int_{S_E} v_\mu(z_0 + t_1 \eta_1 + \cdots t_{n-q+1} \eta_{n-q+1}) d\sigma(t). $$

If we take the limit of (2.7) when $\mu \to +\infty$, then we find that $v$ satisfies the mean value inequality. $\square$

In the following theorem, we give some characterizations of $q$-pseudoconvex domains.

**Theorem 2.2.** Let $2 \leq q \leq n$ be a nonnegative integer, $\Omega$ be an open subset in $\mathbb{C}^n$ and $E$ be a $(n-q+1)$-dimensional complex subspace. Then, the following properties are equivalent:

1. $\Omega$ is strongly $q$-pseudoconvex;
2. $\Omega$ is weakly $q$-pseudoconvex;
3. $\Omega$ has a $q$-subharmonic exhaustion function;
4. the function $(z, \xi_1, \ldots, \xi_{n-q+1}) \mapsto -\log d_\Omega(z, \xi_1, \ldots, \xi_{n-q+1})$ is $q$-subharmonic on $\Omega \times E^{n-q+1}$;
5. the function $z \mapsto -\log d(z, C\Omega)$ is $q$-subharmonic on $\Omega$.

We say that $\Omega$ is a $q$-pseudoconvex domain, when one of these properties holds.
Proof. We have to prove the following sequence of implications:

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1)

$\bullet$ It is clear by definitions, that implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

$\bullet$ For the implication (3) $\Rightarrow$ (4), we use Proposition 2.1. Consider in $\Omega \times E^{n-q+1}$ a ball of the form

$$B = (z_0, \xi^1, \ldots, \xi^{n-q+1}) + B_E(1)(\eta^1, \ldots, \eta^{n-q+1}, \alpha^1, \ldots, \alpha^{n-q+1})$$

where, for all $j = 1, \ldots, n-q+1$, $\xi^j = (\xi^j_1, \ldots, \xi^j_{n-q+1})$, $\eta^j = (\eta^j_1, \ldots, \eta^j_{n-q+1})$, $\alpha^j = (\alpha^j_1, \ldots, \alpha^j_{n-q+1})$ are vectors in $E$ and $B_E(1)(\eta^1, \ldots, \eta^{n-q+1}, \alpha^1, \ldots, \alpha^{n-q+1})$ is defined by the set

$$\{(t_1\eta^1_1, \ldots, t_{n-q+1}\eta^{n-q+1}_1, t_1\alpha^1_1, \ldots, t_{n-q+1}\alpha^{n-q+1}_1), (t_1, \ldots, t_{n-q+1}) \in B(1)\}.$$

Consider also a polynomial $P \in \mathbb{C}[t_1, \ldots, t_{n-q+1}]$ such that

$$- \log \delta(z_0 + t_1\eta^1_1 + \cdots + t_{n-q+1}\eta^{n-q+1}_1), \xi^1 + t_1\alpha^1_1, \ldots, \xi^{n-q+1} + t_{n-q+1}\alpha^{n-q+1}) \leq \Re P(t_1, \ldots, t_{n-q+1})$$

for $|t_1|^2 + \cdots + |t_{n-q+1}|^2 = 1$.

We have to show that the inequality (2.8) holds for $|t_1|^2 + \cdots + |t_{n-q+1}|^2 < 1$. Consider the holomorphic function $h : E \times E \to \mathbb{C}^n$ defined by

$$h(t, w) = z_0 + \sum_{j=1}^{n-q+1} t_j \eta^j + w_j \exp(-P(t_1, \ldots, t_{n-q+1}))(\xi^j + t_j \alpha^j).$$

By (2.9), we have for all $t \in \hat{B}$, $f(t, 0) = z_0 + \sum_{j=1}^{n-q+1} t_j \eta^j \in pr_1(\hat{B})$, where $pr_1 : E \times \mathbb{C}^n \to \mathbb{C}^n$ is the first projection. Hence we may deduce

$$h(\hat{B}_E \times \{0\}) = pr_1(\hat{B}) \subset \Omega.$$ 

Equation (2.8) implies that $|\exp(-P)| \leq \delta$ on $\partial B$, which leads to deduce that the following assertion holds

$$h(\partial(B_E \times B_E)) \subset \Omega.$$

We want to conclude that $h(\hat{B}_E \times B_E) \subset \Omega$. Let $I$ be the set of radii $r \geq 0$ such that $h(\hat{B}_E \times rB_E) \subset \Omega$. Then, $I$ is an open interval $]0, \text{sup}_K \psi [$. Since any $q$-dimensional complex subspace of $E \times E$ is isomorphic to $\{0\} \times E$ or $E \times \{0\}$, we may deduce that $\psi \circ h$ is a $q$-subharmonic function on a neighborhood of $\hat{B}_E \times rB_E$. The maximum principle applied with respect to $t = (t_1, \ldots, t_{n-q+1})$ implies that $\psi \circ h(t, w) \leq c$ on $\hat{B}_E \times rB_E$. Hence $h(\hat{B}_E \times rB_E) \subset \Omega_c \subset \Omega$ and $h(\hat{B}_E \times (R + \varepsilon)B_E) \subset \Omega$ for some $\varepsilon > 0$, a contradiction.

$\bullet$ The implication (4) $\Rightarrow$ (5): we have

$$- \log d(z, E) = \sup_{\xi_1, \ldots, \xi_{n-q+1} \in B_E, E \subset \mathbb{C}^n} (- \log \delta(z, \xi_1, \ldots, \xi_{n-q+1})).$$
Assertion (2.12) implies that $-\log d(z, \mathcal{C})$ is a continuous function on $\Omega$ and satisfies the mean value inequality:

- The implication (5) $\implies$ (1). It is clear that

$$u(z) = |z|^2 + \max(\log d(z, \mathcal{C})^{-1}, 0)$$

is a strictly $q$-subharmonic continuous exhaustion function. Replace $|z|^2$ by $M|z|^2$, if necessary, where $M > 0$ is sufficiently big we get

(2.13)

$$u(z) = M|z|^2 + \max(\log d(z, \mathcal{C})^{-1}, 0).$$

Applying the Richberg’s theorem for the function defined by (2.13), we may conclude the existence of $\Psi \in C^\infty(\Omega)$ strictly $q$-subharmonic such that $u \leq \Psi \leq u + 1$. Then $\Psi$ is the required exhaustion function. \hfill \Box

**Example 2.1.** Consider in $\mathbb{C}^4$

$$\Omega = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4; 3|z_1 + z_2 + z_3 + z_4|^2 - 2|z_3 + z_4|^2 - 2|z_4|^2 < 0\}.$$ 

A direct calculation shows that the complex Hessian of the defining function of $\Omega$, given by $\rho(z) = 3|z_1 + z_2 + z_3 + z_4|^2 - 2|z_3 + z_4|^2 - 2|z_4|^2$, is not positive. Hence $\rho$ is not plurisubharmonic and so $\Omega$ is not pseudoconvex. However, we can easily check that the restriction of $\rho$, on each complex subspace $\{z_j = z_k = 0\}$, $1 \leq j \neq k \leq 4$, is subharmonic. So $\Omega$ has a 3-subharmonic exhaustion function, which leads to conclude by Theorem 2.2 that $\Omega$ is a pseudoconvex.

**Proposition 2.3.** (1) Let $\Omega \subseteq \mathbb{C}^n$ be a $q$-pseudoconvex domain ($\rho \geq 0$). Then $\Omega \times \Omega'$ is a $q$-pseudoconvex domain of $\mathbb{C}^n \times \mathbb{C}^n$. Furthermore, if $F : \mathbb{C}^n \to \mathbb{C}^n$ is a map defined by $F(z) = F(z', w) = f(w)$ where $f : \mathbb{C}^n \to \mathbb{C}^n$ is a unitary transformation, then the inverse image $F^{-1}(\Omega')$ is $q$-pseudoconvex.

(2) If $(\Omega_n)_{n \in I}$ is a family of $q$-pseudoconvex open subsets of $\mathbb{C}^n$, the interior of the intersection $\Omega = (\cap_{n \in I} \Omega_n)^o$ is $q$-pseudoconvex.

(3) If $(\Omega_j)_{j \in \mathbb{N}}$ is a non decreasing sequence of $q$-pseudoconvex open subsets of $\mathbb{C}^n$, then $\Omega = \cup_{j \in \mathbb{N}} \Omega_j$ is $q$-pseudoconvex.

Proof. (1) If we have for all $c \in \mathbb{R}$ and for all $c' \in \mathbb{R}$, $\Omega_c = \{z \in \mathbb{C}^n / \psi_1(z) < c\} \subseteq \Omega$ and $\Omega_{c'} = \{w \in \mathbb{C}^n / \psi_2(w) < c'\} \subseteq \Omega'$, where $\psi_1$ and $\psi_2$ are smooth $q$-subharmonic exhaustion functions, then we can write $(\Omega \times \Omega')_{c+c'} = \{z, w \in \mathbb{C}^n \times \mathbb{C}^n / \psi_1(z) + \psi_2(w) < c + c'\} \subseteq \Omega \times \Omega'$ and $(F^{-1}(\Omega'))_{c+c'} = \{z, w \in \mathbb{C}^n \times \mathbb{C}^n / \psi_1(z) + \psi_2(f(w)) < c + c'\} \subseteq F^{-1}(\Omega')$. The second assertion holds since $\psi_2 \circ f$ is $q$-subharmonic because $f$ is a unitary transformation. So $(z, w) \mapsto \psi_1(z) + \psi_2(w)$ and $z \mapsto \psi_1(z) + \psi_2(f(z))$ are exhaustion functions of $\Omega \times \Omega'$ and $F^{-1}(\Omega')$ respectively.

(2) We have $-\log d(z, \mathcal{C}) = \sup_{s \in I} - \log d(z, \mathcal{C}_s)$, so the function $z \mapsto -\log d(z, \mathcal{C})$ is $q$-subharmonic.

(3) We have $-\log d(z, \mathcal{C}) = \lim_{j \to +\infty} - \log d(z, \mathcal{C}_j)$ and this limit is $q$-subharmonic. \hfill \Box
2.1. Further examples

**Example 2.2.** Let \((f_{i,j})_{1 \leq i \leq N, 1 \leq j \leq N'}\) be a finite family of analytic functions on \(\mathbb{C}^n\) such that for all \(i = 1, \ldots, N\), \(\dim \text{Vect}\{f_{i,j}, j = 1, \ldots, N'\} \geq n - q + 1\). Recall here that for all \(i = 1, \ldots, N\), the dimension of each subspace \(V_i = \text{Vect}\{f_{i,j}, j = 1, \ldots, N'\}\), depends on the functions \(f_{i,j}, j = 1, \ldots, N'\). For all \(1 \leq j \leq N\), let
\[
P_j = \{z \in \mathbb{C}^n; |f_{i,j}(z)|^2 + \cdots + |f_{i,j}(z)|^2 - |f_{i,j}(z)|^2 - \cdots - |f_{N,j}(z)|^2 < 1\}
\]
where \((f_{i,j})_{j=1,\ldots,n-q+1}\) is an independent subfamily of \((f_{i,j})_{1 \leq i \leq N, 1 \leq j \leq N'}\). Put \(P = \bigcup_{j=1}^{N'} P_j\), then \(P\) is a \(q\)-pseudoconvex domain. In case \(\dim \text{Vect}\{f_{i,j}, i = 1, \ldots, N\} = n - q + 1\) \(N = 1\) (which means that \(q = n\) then \(P\) is a polyhedron and it is pseudoconvex.

**Example 2.3.** Consider \(n\) and \(q\) such that \(n \geq 2q + 2\) and \(\omega \subset \mathbb{C}^{n-q}\) be a \(q\)-pseudoconvex domain. Let \(u : \omega \to [-\infty, +\infty[\) be an upper semi-continuous function. Consider the Hartogs domain
\[
\Omega = \{(z_1, \ldots, z_{n-q+1}, z') \in \mathbb{C}^{n-q+1} \times \omega; \frac{1}{2} \log(|z_1|^2 + \cdots + |z_{n-q+1}|^2) + u(z') < 0\}.
\]
Then \(\Omega\) is \(q\)-pseudoconvex, if and only if, \(u\) is \(q\)-subharmonic. Indeed, to see the necessary condition, using notation (2.5), we may observe that \(u(z') = -\log \delta_{\Omega_j}(0, z'), (\xi_1, \ldots, \xi_{n-q+1})\) where \(\{\xi_1, \ldots, \xi_{n-q+1}\}\) is the canonical basis of \(\mathbb{C}^{n-q+1}\). Conversely, assume that \(u\) is \(q\)-subharmonic and continuous. If \(\psi\) is a \(q\)-subharmonic exhaustion function of \(\omega\), then, since \(u\) is continuous and since \(x \mapsto \frac{1}{|x|}\) is convex and increasing on \([-\infty, 0[\), then
\[
\psi(z') + \left|\frac{1}{2} \log(|z_1|^2 + \cdots + |z_{n-q+1}|^2) + u(z')\right|^{-1}
\]
is a \(q\)-subharmonic exhaustion function of \(\Omega\). If \(u\) is not assumed to be continuous, we may replace \(u\) by \(u * \chi_{\varepsilon}\) and write \(\Omega = \Omega_{\varepsilon}\) where
\[
\Omega_{\varepsilon} = \{(z_1, \ldots, z_{n-q+1}, z'), d(z', \mathbb{C} \omega) > \varepsilon, \frac{1}{2} \log(|z_1|^2 + \cdots + |z_{n-q+1}|^2) + u * \chi_{\varepsilon} < 0\}.
\]
We may conclude by application of property (3) of Proposition 2.3.

3. Levi form of the boundary of \(q\)-pseudoconvex domains

In this section we shall characterize the \(q\)-pseudoconvexity by the Levi form of the boundary \(\partial \Omega\). The holomorphic tangent space is by definition the largest complex subspace which is contained in the tangent space \(T_{\partial \Omega}\) to the boundary: \(hT_{\partial \Omega} = T_{\partial \Omega} \cap JT_{\partial \Omega}\), where \(J\) is the almost complex structure that is the...
operator of multiplication by \( i = \sqrt{-1} \). The holomorphic tangent space \( hT_{\partial \Omega, z} \) is the complex hyperplane of vectors \( \xi \in \mathbb{C}^n \) such that
\[
(3.14) \quad d' \rho(z) \xi = \sum_{1 \leq j \leq n} \frac{\partial \rho}{\partial z_j} \xi_j = 0.
\]
The Levi form on \( hT_{\partial \Omega} \) is defined at every point \( z \in \partial \Omega \) by
\[
(3.15) \quad L_{\partial \Omega, z}(\xi) = \frac{1}{|\nabla \rho(z)|} \sum_{1 \leq j, k \leq n} \frac{\partial^2 \rho}{\partial z_j \partial z_k} \xi_j \xi_k, \quad \xi \in hT_{\partial \Omega, z}.
\]

Let’s begin this section by showing that \( q \)-pseudoconvexity of an arbitrary domain in \( \mathbb{C}^n \) is a local property of the boundary. An other proof of this fact was given in [5].

**Proposition 3.1.** Let \( \Omega \subset \mathbb{C}^n \) be a domain such that every point \( z_0 \in \partial \Omega \) has a neighborhood \( U \) such that \( U \cap \Omega \) is \( q \)-pseudoconvex. Then \( \Omega \) is \( q \)-pseudoconvex.

**Proof.** Let \( z_0 \in \partial \Omega \) and let \( U \cap \Omega \) be a neighborhood of \( z_0 \). Since \( U \cap \Omega \) is \( q \)-pseudoconvex then it is defined in a neighborhood of \( \partial(\Omega \cap U) \) by a \( q \)-subharmonic function \( \rho \). Let \( V \) be a neighborhood of \( \partial \Omega \), then the function defined by \( w = \sup_{r>0, U \subset V} \rho_{U \cap B(0, r)} \) is \( q \)-subharmonic on \( V \). Let \( \chi \) be an increasing convex function such that
\[
(3.16) \quad \forall r \geq 0, \ \chi(r) > \sup_{(\Omega \setminus V) \cap B(0, r) \cap U} \rho_{U \cap B(0, r)}.
\]
Since the function \( z \mapsto \sum_{j=1}^n |z_j|^2 - (n-q+1)|z_n|^2 \) is \( q \)-subharmonic, then by (3.16) the function
\[
\psi(z) = \max \left( \chi \left( \sum_{j=1}^n |z|^2 - (n-q+1)|z_n|^2 \right), w(z) \right)
\]
coincides with \( \chi \left( \sum_{j=1}^n |z|^2 - (n-q+1)|z_n|^2 \right) \) in a neighborhood of \( \Omega \setminus V \). Hence \( \psi \) is an exhaustion \( q \)-subharmonic on \( \Omega \). \( \square \)

**Theorem 3.2.** Let \( 2 \leq q \leq n \). An open subset \( \Omega \subset \mathbb{C}^n \) with smooth boundary is \( q \)-pseudoconvex if and only if, for every \( (n-q+1) \)-dimensional complex subspace \( E \subset \mathbb{C}^n \), the Levi form \( L_{\partial \Omega, z}|E \cap hT_{\partial \Omega, z} \) is semipositive at every point of \( \partial \Omega \).

**Proof.** Consider a \( (n-q+1) \)-dimensional complex subspace \( E \subset \mathbb{C}^n \). Without loss of generalities we may assume \( E = \{ \xi_1 = \cdots = \xi_q = 0 \} \). Let \( \delta(z) = d(z, \Omega) \), \( z \in \Omega \), then the function \( \rho = -\delta \) is smooth near \( \partial \Omega \). Suppose that \( \Omega \) is \( q \)-pseudoconvex, then the function \( -\log(-\rho) \) is \( q \)-subharmonic which means that for all \( z \in \Omega \) near \( \partial \Omega \) and for all \( \xi \in E \), we have
\[
(3.17) \quad \sum_{q+1 \leq j, k \leq n} \left( \frac{1}{|\rho|} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} + \frac{1}{\rho^2} \frac{\partial \rho}{\partial z_j} \frac{\partial \rho}{\partial \bar{z}_k} \right) \xi_j \bar{\xi}_k \geq 0.
\]
As we have

\[ \sum_{q+1 \leq j, k \leq n} \frac{1}{\rho^2} \frac{\partial \rho}{\partial z_j} \frac{\partial \rho}{\partial z_k} \xi_j \xi_k = \left| \sum_{q+1 \leq j \leq n} \frac{1}{\rho} \frac{\partial \rho}{\partial z_j} \xi_j \right|^2, \]

then inequality (3.17) gives that

\[ \sum_{q+1 \leq j, k \leq n} \frac{\partial^2 \rho}{\partial z_j \partial z_k} \xi_j \xi_k \geq 0 \quad \text{whenever} \quad \sum_{q+1 \leq j \leq n} \frac{\partial \rho}{\partial z_j} \xi_j = 0 \]

and this is also true at the limit on \( \partial \Omega \), which means that \( \rho \) is \( q \)-subharmonic. Conversely, suppose that \( \Omega \) is not \( q \)-pseudoconvex, then by Theorem 2.2, the function \( -\log(\delta) \) is not \( q \)-subharmonic in any neighborhood of \( \partial \Omega \). Hence there exist a \((n-q+1)\)-dimensional subspace \( E \subset \mathbb{C}^n \) and an orthonormal basis \( \{\xi_1, \ldots, \xi_{n-q+1}\} \subset E \) such that the Laplacian of the function

\[ (t_1, \ldots, t_{n-q+1}) \mapsto \log(\delta(z + t_1 \xi_1 + \cdots + t_{n-q+1} \xi_{n-q+1})) \]

is strictly positive at point \((t_1, \ldots, t_{n-q+1}) = (0, \ldots, 0)\) for some \( z \) in the neighborhood of \( \partial \Omega \). By Taylor’s formula, we have

\[ (3.18) \quad \log(\delta(z + t_1 \xi_1 + \cdots + t_{n-q+1} \xi_{n-q+1})) = \log(\delta(z)) + \sum_{1 \leq j \leq n-q+1} \Re(a_j t_j + b_j t_j^2) + c_j |t_j|^2 + o(|t|^2), \]

where \( a_j, b_j \in \mathbb{C} \) and \( c_j = \left( \frac{\partial^2 \log(\delta(z + t_1 \xi_1 + \cdots + t_{n-q+1} \xi_{n-q+1}))}{\partial t_j^2} \right)_{|t_j=0} > 0 \). Let \( z_0 \in \partial \Omega \) such that \( \delta(z) = |z - z_0| \) and put

\[ (3.19) \quad h(t_1, \ldots, t_{n-q+1}) = z + \sum_{1 \leq j \leq n-q+1} t_j \xi_j + \exp \left( \sum_{1 \leq j \leq n-q+1} a_j t_j + b_j t_j^2 \right) (z_0 - z). \]

We have \( h(0) = z_0 \), write \( \delta(z + t_1 \xi_1 + \cdots + t_{n-q+1} \xi_{n-q+1}) = \delta(z + t \xi) \) as

\[
\delta(z + t \xi) = \delta \left[ z + t \xi + \exp \left( \sum_{1 \leq j \leq n-q+1} a_j t_j + b_j t_j^2 \right) (z_0 - z) \right.
\]
\[
- \exp \left( \sum_{1 \leq j \leq n-q+1} a_j t_j + b_j t_j^2 \right) (z_0 - z) \left. \right] = \left| h(t) - z_0 - \exp \left( \sum_{1 \leq j \leq n-q+1} a_j t_j + b_j t_j^2 \right) (z_0 - z) \right|.
\]
and use the triangle inequality, by (3.18) and (3.19) we get
\[
\delta(h(t)) \geq \delta(z + t\xi) - \delta(z) \geq \delta(z) \left| \exp\left( \sum_{1 \leq j \leq n-q+1} a_j t_j + b_j t_j^2 \right) - 1 \right|
\]
\[
\geq \delta(z) \exp \left( \sum_{1 \leq j \leq n-q+1} \Re(a_j t_j + b_j t_j^2) \right) \exp\left( \sum_{1 \leq j \leq n-q+1} (c_j |t_j|^2) \right)
\]
\[
- \delta(z) \exp \left( \sum_{1 \leq j \leq n-q+1} a_j t_j + b_j t_j^2 \right)
\]
\[
\geq \delta(z) \left| \exp \left( \sum_{1 \leq j \leq n-q+1} a_j t_j + b_j t_j^2 \right) \right| \left[ \exp \left( \sum_{1 \leq j \leq n-q+1} \frac{c_j |t_j|^2}{2} \right) - 1 \right]
\]
\[
\geq \delta(z) \frac{|t|^2}{6}
\]
when $|t|$ is sufficiently small and $c = \min_{1 \leq j \leq n-q+1} c_j$. Since $h(\delta(0)) = \delta(z_0) = 0$, we get at $t = 0$ for all $1 \leq j \leq n-q+1$,
\[
\frac{\partial \delta(h(t))}{\partial t_j} = \sum_{1 \leq k \leq n-q+1} \frac{\partial \delta}{\partial z_k}(z_0) \frac{\partial h}{\partial t_j}(0) = 0
\]
and
\[
\frac{\partial^2 \delta(h(t))}{\partial t_j \partial t_l} = \sum_{1 \leq k,l \leq n-q+1} \frac{\partial^2 \delta}{\partial z_k \partial z_l}(z_0) \frac{\partial h}{\partial t_j}(0) \frac{\partial h}{\partial t_l}(0) > 0.
\]
Therefore $\nabla h(0) \in hT_{\partial\Omega,z}\cap E$ and $L_{\partial\Omega,z}(\nabla h(0)) < 0$. \qed

**Definition 3.1.** Consider $2 \leq q \leq n$. The boundary $\partial\Omega$ is said to be weakly (resp. strongly) $q$-pseudoconvex, if for every $z \in \partial\Omega$ and every $(n-q+1)$-dimensional complex subspace $E \subset \mathbb{C}^n$, $L_{\partial\Omega,z}$ is semi-positive (resp. positive definite) on $E \cap hT_{\partial\Omega,z}$.

**Example 3.1.** Consider in $\mathbb{C}^3$, $\Omega = \{ \rho < -1 \}$ where $\rho(z) = 3(|z_1|^2 + |z_2|^2) - 2|z_3|^2$. Then, it is clear that $\Omega$ is 2-pseudoconvex and $0 \not\in \Omega$. Further, by (3.14) and (3.15), at every point $z \in \partial\Omega$, the holomorphic tangent space to $\partial\Omega$ is given by the equation $3\overline{z}_1\xi_1 + 3\overline{z}_2\xi_2 - 2\overline{z}_3\xi_3 = 0$ and the Levi form on $hT_{\partial\Omega,z}$ is given by
\[
L_{\partial\Omega,z}(\xi) = \frac{3(|\xi_1|^2 + |\xi_2|^2) - 2|\xi_3|^2}{\sqrt{1 + 6|z_3|^2}}.
\]
An easy computation yields that for all $j = 1, 2, 3$ we have $L_{\partial\Omega,z,E_j \cap hT_{\partial\Omega,z}} \geq 0$ where $E_j = \{ \xi_j = 0 \}$. Indeed, we may chose $z \in \partial\Omega$ such that $z_3 \neq 0$. For all $\xi \in E_j \cap hT_{\partial\Omega,z}$, we have
\[
\frac{|\xi_3|^2}{|\xi_3|^2} = \left( \frac{2}{3} \right) + \frac{6|z_1|^2}{9|z_2|^2}.
\]
Hence, \( \frac{3|\xi|^2 - 2|\xi|^2}{\sqrt{1+|\xi|^2}} \geq 0 \) on \( E_1 \cap hT_{\partial \Omega} \). Similarly, we prove that for all \( \xi \in E_2 \cap hT_{\partial \Omega} \), we have \( L_{\partial \Omega} (\xi) \geq 0 \). Finally, it is obvious that \( L_{\partial \Omega} \) is positive definite on \( E_3 \cap hT_{\partial \Omega} \) but semi-positive on \( E_j \cap hT_{\partial \Omega} \), \( j = 1, 2 \) so \( \partial \Omega \) is weakly \( 2 \)-pseudoconvex.

**Example 3.2.** For any \( C < 0 \), let consider \( \Omega_C = \{ z \in \mathbb{C}^n, K(\alpha, z) < C \} \), where \( K \) is the \( (n-q) \)-subharmonic function given by (1.2). It is clear that \( z \mapsto K(\alpha, z) \) is smooth near the boundary \( \partial \Omega_C \). For all \( 1 \leq j \leq n \), an easy computation yields,

\[
\frac{\partial K}{\partial z_j} = -(\alpha-q)\bar{z}_jK(\alpha-1, z)
\]

and

\[
\begin{align*}
-\frac{1}{H_q(\alpha)} \frac{\partial^2 K}{\partial z_j \partial z_j} &= (\alpha-q)|z|^{2(\alpha-q-2)} \left( |z|^2 + (\alpha-q-1)|z_j|^2 \right) \quad \text{if } j = k \\
-\frac{1}{H_q(\alpha)} \frac{\partial^2 K}{\partial z_j \partial \bar{z}_k} &= (\alpha-q)(\alpha-q-1)z_j\bar{z}_k|z|^{2(\alpha-q-2)} \quad \text{if } j \neq k.
\end{align*}
\]

Let \( E \subset \mathbb{C}^n \) be a \( q \)-dimensional subspace. Without loss of generalities, we may assume that \( E \) is given by the equations \( \xi_{q+1} = \cdots = \xi_n = 0 \). Hence, we find that, at every point \( z \in \partial \Omega_C \), the intersection of the holomorphic tangent space to \( \partial \Omega \) with \( E \), is given by the equation \( \sum_{1 \leq j \leq q} \bar{z}_j \xi_j = 0 \) and the Levi form on \( hT_{\partial \Omega} \cap E \) is given by

\[
L_{\partial \Omega,z|hn\partial \Omega,z\cap E}(\xi) = \frac{q-\alpha}{H_q(\alpha)|\nabla K(\alpha, z)|} \left( q|\xi|^2|z|^{2(\alpha-q-1)} + (\alpha-q-1)|z|^2|\xi|^{2(\alpha-q-2)} \left| \sum_{j=1}^q z_j \xi_j \right|^2 \right),
\]

where \( |\nabla K(\alpha, z)| = (q-\alpha)|z||K(\alpha-1, z)| \) is the modulus of the complex gradient of \( K \). By the Cauchy-Schwartz inequality we find that

\[
L_{\partial \Omega,z|hn\partial \Omega,z\cap E}(\xi) \geq \frac{q-\alpha}{H_q(\alpha)|\nabla K(\alpha, z)|} (q+(\alpha-q-1))|\xi|^2|z|^{2(\alpha-q-1)} \geq 0.
\]

The last inequality holds true on \( E \cap hT_{\partial \Omega} \) for every \( q \)-dimensional complex subspace \( E \subset \mathbb{C}^n \).

4. **Kiselman’s minimum principale for \( q \)-subharmonic functions**

Let \( v \) be a \( q \)-subharmonic function on \( \Omega \times \Omega' \subset \mathbb{C}^n \times \mathbb{C}^p \). The partial minimum function on \( \Omega \) defined by

\[
u(\zeta) = \inf_{z \in \Omega} v(\zeta, z)\]
need not be \( q \)-subharmonic. Indeed, consider the following counterexample of a 2-subharmonic function in \( \mathbb{C}^3 \times \mathbb{C} \) given by
\[
v(z_1, z_2, z_3, z_4) = |z_4 + z_3 + z_2 + z_1|^2 - |z_3 + z_2 + z_1|^2 = |z_4|^2 + 2\Re(z_4(z_3 + z_2 + z_1)).
\]
We have \( u(z_1, z_2, z_3) = -|z_1 + z_2 + z_3|^2 \) and it is clear that \( u \) is not \( q \)-subharmonic for \( q = 2, 3 \).

However, the minimum property holds true when \( v(\zeta, z) \) depends only on \( \Re(z) \).

**Theorem 4.1.** Let \( \Omega = \Omega_1 \times \Omega_2 \subset \mathbb{C}^p \times \mathbb{C}^n \) be a \( q \)-pseudoconvex domain such that each slice
\[
\Omega_\zeta = \{ z \in \mathbb{C}^n ; (\zeta, z) \in \Omega \}, \quad \zeta \in \mathbb{C}^p
\]
is a convex tube \( \omega_\zeta + i\mathbb{R}^n, \omega_\zeta \subset \mathbb{C}^p \). Then, for every \( q \)-subharmonic function \( v(\zeta, z) \) on \( \Omega \) that does not depend on \( \Im(z) \), the function \( u(\zeta) = \inf_{z \in \Omega} v(\zeta, z) \) is \( q \)-subharmonic or locally \( \equiv -\infty \) on \( \Omega_2 = \text{pr}_{\mathbb{C}^n}(\Omega) \).

**Proof.** The idea of the proof is inspired from [1]. Consider a \((n-q+1)\)-complex subspace of \( \mathbb{C}^p \times \mathbb{C}^n \) such that \( L = \{ \zeta_1 = \cdots = \zeta_{q+1} = \zeta_{q+2} = \cdots = \zeta_n = 0 \} \) and \( q = s + t \). The hypothesis implies that \( v(\zeta, z) |_{L \cap \Omega} \) is convex in \( x = \Re(z) \). We may, first, assume that \( v \) is smooth, \( q \)-subharmonic in \( (\zeta, z) \) and \( v(\zeta, z) |_{L \cap \Omega} \) is strictly convex in \( x \) and
\[
\lim_{x \to a \in \Omega \cup \{ \infty \}} v(\zeta, x) = +\infty \text{ for every } \zeta \in \omega'.
\]
Then the function \( x \mapsto v_{L \cap \Omega}(\zeta, x) \) has a unique minimum point \( x = g(\zeta) \) solution of the equations \( \frac{\partial}{\partial z_k} = 0 \). As the matrix \( \left( \frac{\partial^2 v}{\partial z_k \partial z_l} \right) \) is positive definite, the implicit function theorem shows that \( g \) is smooth. Let \( B \) a ball contained in \( \Omega \) defined by the parametrization
\[
L \simeq \mathbb{C}^{n-q+1} \ni (w_1, \ldots, w_{n-q+1}) \mapsto \zeta_0 + w_1 a_1 + \cdots + w_{n-q+1} a_{n-q+1}
\]
where \( a_1, \ldots, a_{n-q+1} \in \mathbb{C}^n \) and \( w = (w_1, \ldots, w_{n-q+1}) \in B_{n-q+1} \). There exists a holomorphic function \( f \) on the unit ball \( B_{E}(1) \) whose real part solves the Dirichlet problem
\[
(4.21) \quad \Re f(t_1, \ldots, t_{n-q+1}) = g(\zeta_0 + t_1 a_1 + \cdots + t_{n-q+1} a_{n-q+1}).
\]
Since the function
\[
(w_1, \ldots, w_{n-q+1}) \mapsto v(\zeta_0 + w_1 a_1 + \cdots + w_{n-q+1} a_{n-q+1}, f(w_1, \ldots, w_{n-q+1}))
\]
is subharmonic, we get the mean value inequality
\[
v(\zeta_0, f(0)) \leq \frac{1}{\text{area}(S_E)} \int_{S_E} v(\zeta_0 + t_1 a_1 + \cdots + t_{n-q+1} a_{n-q+1}, f(t_1, \ldots, t_{n-q+1}))d\sigma(t)
\]
\[
= \frac{1}{\text{area}(S_E)} \int_{S_E} v(\zeta_0 + t_1 a_1 + \cdots + t_{n-q+1} a_{n-q+1}, g(t_1, \ldots, t_{n-q+1}))d\sigma(t).
\]
The last equality holds since we have, by (4.21), \( \Re f = g \) on \( \partial B_{n-q+1} \) and \( v(\zeta, z) = v(\zeta, \Re(z)) \) by hypothesis. We have

\[
\text{(4.22) } u(\zeta_0) \leq v(\zeta_0, f(0)) \quad \text{and} \quad u(\zeta) = v(\zeta, g(\zeta))
\]
	herefore, we see by (4.22) that \( u \) satisfies the mean value inequality, thus \( u_{|\mathcal{L}^+\Omega'} \) is subharmonic.

Let now extend the result to an arbitrary \( q \)-subharmonic function \( v \). We may suppose \( n - q + 1 \leq p \leq n \). Let \( \psi(\zeta, z) \) a positive continuous \( q \)-subharmonic function on \( \Omega \) which depends only on \( \Re(z) \) and is an exhaustion of \( \Omega \cap (\mathbb{C}^p \times \mathbb{R}^n) \), we may choose such a function as

\[
\psi(\zeta, z) = \max \left( \sum_{j=1}^{p} \zeta_j + \sum_{j=1}^{n} |z_j|^2 - \sum_{j=n-q+2}^{n} |\zeta_j|^2 - \sum_{j=n-q+2}^{n} |z_{n-j}|^2 - \log \delta_{\Omega}(\zeta, z, L) \right).
\]

There is an increasing sequence \( C_j \to +\infty \) such that each function obtained from (4.23) and defined by \( \psi_j = (C_j - \psi \ast \rho_j^{-1}) \) is an exhaustion of a \( q \)-pseudoconvex open set \( \Omega_j \Subset \Omega \) whose slices are convex tubes and such that \( d(\Omega_j, \mathcal{C}\Omega) > \frac{2}{j} \). Let

\[
\text{(4.24) } v_j(\zeta, z) = v \ast \rho_j^{-1}(\zeta, z) + \frac{1}{j} |\Re(z)|^2 + \psi_j(\zeta, z),
\]

then (4.24) gives a decreasing sequence of \( q \)-subharmonic functions on \( \Omega_j \) satisfying the previous conditions. As \( v = \lim v_j \), we see that \( u = \lim u_j \) is \( q \)-subharmonic.

As we see, it is clear that the image \( F(\Omega) \) of a \( q \)-pseudoconvex domain \( \Omega \) by a holomorphic map \( F \) need not be \( q \)-pseudoconvex. Indeed, Consider the domain \( \Omega \) defined as the following

\[
\Omega = \{ (z', z_5) = (z_1, \ldots, z_5) \in \mathbb{C}^5; \log |z_1| + v(z_2, z_3, z_4, z_5) < 0 \},
\]

where \( v \) is the function given by example (4.20). If \( \Omega' \subset \mathbb{C}^4 \) is the image of \( \Omega \) by the projection map \( (z', z_5) \mapsto z' \), then we have

\[
\Omega' = \{ (z_1, z_2, z_3, z_4) \in \mathbb{C}^4; \log |z_1| + u(z_2, z_3, z_4) < 0 \},
\]

where the function \( u \) is given by \( u(z_2, z_3, z_4) = \inf_{z_5 \in \mathbb{C}} v(z_2, z_3, z_4, z_5) \). It is clear that \( \Omega' \) is not \( 2 \)-pseudoconvex. However, we have the following result.

\[ \text{Proposition 4.2.} \text{ Let } \Omega \subset \mathbb{C}^p \times \mathbb{C}^n \text{ be a } q \text{-pseudoconvex open set such that all slices } \Omega_\zeta, \zeta \in \mathbb{C}^p, \text{ are convex tubes in } \mathbb{C}^n. \text{ Then the projection } \Omega' \text{ of } \Omega \text{ on } \mathbb{C}^p \text{ is } q \text{-pseudoconvex.} \]

\[ \text{Proof.} \text{ Let } v \text{ be a } q \text{-subharmonic function on } \Omega \text{ equal to the function } \psi \text{ defined in the proof of Theorem 2.2. Then } u \text{ is a } q \text{-subharmonic exhaustion function of } \Omega'. \]
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