SHARP BOUNDS FOR BOLTZMANN AND LANDAU COLLISION OPERATORS

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ABSTRACT. The aim of the work is to provide a stable method to get sharp bounds for Boltzmann and Landau operators in weighted Sobolev spaces and in anisotropic spaces. The results and proofs have the following main features and innovations:

• All the sharp bounds are given for the original Boltzmann and Landau operators. The sharpness means the lower and upper bounds for the operators are consistent with the behavior of the linearized operators. Moreover, we make clear the difference between the bounds for the original operators and those for the linearized ones. It will be useful for the well-posedness of the original equations.

• According to the Bobylev’s formula, we introduce two types of dyadic decompositions performed in both phase and frequency spaces to make full use of the interaction and the cancellation. It allows us to see clearly which part of the operator behaves like a Laplace type operator and which part is dominated by the anisotropic structure. It is the key point to get the sharp bounds in weighted Sobolev spaces and in anisotropic spaces.

• Based on the geometric structure of the elastic collision, we make a geometric decomposition to capture the anisotropic structure of the collision operator. More precisely, we make it explicit that the fractional Laplace-Beltrami operator really exists in the structure of the collision operator. It enables us to derive the sharp bounds in anisotropic spaces and then complete the entropy dissipation estimates.

• The structures mentioned above are so stable that we can apply them to the rescaled Boltzmann collision operator in the process of the grazing collisions limit. Then we get the sharp bounds for the Landau collision operator by passing to the limit. We remark that our analysis used here will shed light on the investigation of the asymptotics from Boltzmann equation to Landau equation.

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1. Introduction

The aim of the present work is to provide a stable method to give a complete description of the behavior of the Boltzmann and Landau collision operators. We remark that it is related closely to the derivation of the Landau equation from Boltzmann equation and also the asymptotics of the Boltzmann equation from short-range interactions to long-range interactions.

We first recall that the Boltzmann equation reads:

\[ \partial_t f + v \cdot \nabla_x f = Q(f, f), \]

where \( f(t, x, v) \geq 0 \) is a distribution function of colliding particles which, at time \( t \geq 0 \) and position \( x \in \mathbb{T}^3 \), move with velocity \( v \in \mathbb{R}^3 \). We remark that the Boltzmann equation is one of the fundamental equations of mathematical physics and a cornerstone of statistical physics.

The Boltzmann collision operator \( Q \) is a bilinear operator which acts only on the velocity variables \( v \), that is,

\[ Q(g, f)(v) \overset{\text{def}}{=} \int_{\mathbb{R}^3} \int_{SS^2} B(v - v_\ast, \sigma)(g' f' - g f) d\sigma dv_\ast. \]

Here we use the standard shorthand \( f = f(t, x, v) \), \( g_\ast = g(t, x, v_\ast) \), \( f' = f(t, x, v') \), \( g'_\ast = g(t, x, v'_\ast) \) where \( (v, v_\ast) \) and \( (v', v'_\ast) \) are the velocity of particles before and after the collision. Here \( v' \) and \( v'_\ast \) are given by

\[ v' = \frac{v + v_\ast}{2} + \frac{|v - v_\ast|}{2} \sigma, \quad v'_\ast = \frac{v + v_\ast}{2} - \frac{|v - v_\ast|}{2} \sigma, \quad \sigma \in SS^2. \]

The representation follows the parametrization of the set of solutions of the physical laws of elastic collision:

\[ v + v_\ast = v' + v'_\ast, \]
\[ |v|^2 + |v_\ast|^2 = |v'|^2 + |v'|^2. \]

The nonnegative function \( B(v - v_\ast, \sigma) \) in the collision operator is called the Boltzmann collision kernel. It is always assumed to depend only on \( |v - v_\ast| \) and \( \frac{v - v_\ast}{|v - v_\ast|} \cdot \sigma \). Usually, we introduce the angle variable \( \theta \) through \( \cos \theta = \frac{v - v_\ast}{|v - v_\ast|} \cdot \sigma \). Without loss of generality, we may assume that \( B(v - v_\ast, \sigma) \) is supported in the set \( 0 \leq \theta \leq \frac{\pi}{2} \), i.e., \( \frac{v - v_\ast}{|v - v_\ast|} \cdot \sigma \geq 0 \). Otherwise, \( B \) can be replaced by its symmetrized form:

\[ \tilde{B}(v - v_\ast, \sigma) = |B(v - v_\ast, \sigma) + B(v - v_\ast, -\sigma)| 1_{\left| \frac{v - v_\ast}{|v - v_\ast|} \cdot \sigma \right| \geq 0}. \]

Here, \( 1_A \) is the characteristic function of the set \( A \). In this paper, we consider the collision kernel satisfying the following assumptions:

(A1). The cross-section \( B(v - v_\ast, \sigma) \) takes a product form as

\[ B(v - v_\ast, \sigma) = \Phi(|v - v_\ast|) b(\cos \theta), \]

where both \( \Phi \) and \( b \) are nonnegative functions.

(A2). The angular function \( b(t) \) satisfies for \( \theta \in [0, \pi/2] \)

\[ K \theta^{-1-2s} \lesssim \sin \theta b(\cos \theta) \lesssim K^{-1} \theta^{-1-2s}, \quad \text{with } 0 < s < 1, \ K > 0. \]

(A3). The kinetic factor \( \Phi \) takes the form

\[ \Phi(|v - v_\ast|) = |v - v_\ast|^\gamma, \]

where the parameter \( \gamma \) verifies that \( \gamma + 2s > -1 \).
Remark 1.1. For inverse repulsive potential, there holds that \( \gamma = \frac{p-5}{p-1} \) and \( s = \frac{1}{p+1} \). It is easy to check that \( \gamma + 4s = 1 \) which explains the assumption \( \gamma + 2s > -1 \). Generally, the case \( \gamma > 0 \), \( \gamma = 0 \), and \( \gamma < 0 \) correspond to so-called hard, Maxwellian, and soft potentials respectively.

Remark 1.2. If we replace the assumption \( \text{(1.5)} \) by

\[
K\theta^{-1-2s}\left(1 - \psi\left(\frac{\sin(\theta/2)}{\epsilon}\right)\right) \leq \sin \theta b(\cos \theta) \leq \theta^{-1-2s}\left(1 - \psi\left(\frac{\sin(\theta/2)}{\epsilon}\right)\right),
\]

where \( \psi \), a non-negative and smooth function with compact support in a ball, is defined in \( \text{(1.23)} \), then the mathematical problem of the asymptotics of Boltzmann equation from short-range interactions to long-range interactions can be formulated by the limit that the parameter \( \epsilon \) in \( \text{(1.7)} \) goes to zero. We remark that for fixed \( \epsilon \), \( \text{(1.7)} \) corresponds to the famous assumption: Grad's cut off assumption for the cross-section \( B \).

The solutions of the Boltzmann equation \( \text{(1.1)} \) enjoy the fundamental properties of conserving the mass, the momentum and the kinetic energy, that is, for all \( t \geq 0 \),

\[
\int_{\mathbb{T}^3 \times \mathbb{R}^3} f(t,x,v)\phi(v) \, dv \, dx = \int_{\mathbb{T}^3 \times \mathbb{R}^3} f(0,x,v)\phi(v) \, dv \, dx, \quad \phi(v) = 1, v, |v|^2.
\]

Moreover, if we define

\[
H(f)(t) \overset{\text{def}}{=} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f \log f \, dv \, dx,
\]

then the celebrated the Boltzmann's \( H \)-theorem predicts that the entropy is decreasing over time, which formally is

\[
\frac{d}{dt} H(f)(t) = -D(f)(t) \overset{\text{def}}{=} -\int_{\mathbb{T}^3 \times \mathbb{R}^3} Q(f,f) \log f \, dv \, dx \leq 0.
\]

We mention that the function \( D(f) \) is called entropy dissipation.

Before introducing the Landau equation, let us first give the definition of the grazing collisions limit. In this case, the collision kernel satisfies the assumption as

(B1). The rescaled Boltzmann’s cross-section \( B^\epsilon(\nu - \nu_*, \sigma) \) verifies \( B^\epsilon(\nu - \nu_*, \sigma) = \Phi([\nu - \nu_*]) b^\epsilon(\cos \theta) \).

The kinetic factor \( \Phi \) satisfies \( \text{(1.6)} \) and the angular function \( b^\epsilon(t) \) satisfies for \( \theta \in [0, \pi/2] \),

\[
\sin \theta b^\epsilon(\cos \theta) = K' \epsilon^{2s-2} \psi\left(\frac{\sin(\theta/2)}{\epsilon}\right) \theta^{-1-2s},
\]

with \( 0 < s < 1 \). Here \( K' \) is a positive constant and the function \( \psi \) is defined in \( \text{(1.23)} \).

Remark 1.3. The assumption \( \text{(1.8)} \) means that the deviation angles between relative velocities before and after collisions are restrictly less than \( \epsilon \). This means the angular function \( b^\epsilon \) is really concentrating on the grazing collisions if \( \epsilon \) is sufficiently small.

Then mathematically the grazing collisions limit is defined by the process that the parameter \( \epsilon \) goes to zero. Thanks to the full Taylor expansion, by taking the limit \( \epsilon \to 0 \), the Boltzmann collision operator \( \mathcal{Q}^\epsilon \) with rescaled cross-section \( B^\epsilon \) will be reduced to the Landau collision operator \( \mathcal{Q}_L \) defined as

\[
\mathcal{Q}_L(g, h) \overset{\text{def}}{=} \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} a(\nu - \nu_*) \nabla_v h(\nu) - \nabla_v g(\nu_*) h(\nu) \, dv_* \right\},
\]

Here the nonnegative matrix \( a \) is given by

\[
a_{ij}(v) = \Lambda \left( \delta_{ij} - \frac{v_i v_j}{|v|^2} \right) |v|^\gamma + 2, \quad \gamma \in [-3, 1],
\]

where \( \Lambda \) is a positive constant and can be calculated by

\[
\Lambda = \frac{\pi}{8} K' \int_0^{\pi/2} \psi(\theta) \theta^{1-2s} \, d\theta.
\]
Then the Landau equation can be written by
\[ \partial_t f + v \cdot \nabla_x f = Q_L(f, f). \]
(1.10)

We remark that the equation was proposed by Landau in 1936 to model the behavior of a dilute plasma interacting through binary collisions. We also mention that the Landau equation possesses all the properties known for the Boltzmann equation, namely the mass, momentum and energy conservation and the $H$-theorem.

1.1. Motivation and short review of the problem. The justification of the derivation of the Landau equation in the sense that the solution to the Boltzmann equation with rescaled cross-section (see assumption (B1)) will converge to the solution to the corresponding Landau equation in the grazing collisions limit had been proved by several authors. We refer readers to [21] and the references therein to check details. However, the physical problem of the justification is formulated as a higher-order correction to the limit. In other words, we should establish some kind of the asymptotical formula to the solutions in the limit process. Suppose that $f^e_B$ and $f_L$ represent the solutions to Boltzmann and Landau equations. In [13], for the homogeneous case, that is, the solution does not depend on the position variable $x$, the author showed that the following asymptotical formula
\[ f^e_B = f_L + O(\epsilon) \]
holds globally or locally in Sobolev spaces for almost physical potentials except for Coulomb potential. This shows that the Landau equation is a good approximation to the Boltzmann equation when the parameter $\epsilon$ is small enough which gives the validity of the Landau equation. However it is very difficult to extend the similar result to the inhomogeneous case. The main obstruction is the lack of a complete description of the asymptotical behavior of collision operator in the limit process since the behavior of the operator is very sensitive to the parameter $\epsilon$; we recall that the Boltzmann collision operator behaves like the fractional Laplace operator while the limiting operator is an explicit Laplace type operator. The same problem happens when we study the asymptotics of the Boltzmann equation from short-range interactions to long-range interactions under the assumption (1.7). We remark that the investigation of the second asymptotics is related closely to the construction of approximated solutions to the equation with long-range interactions and also to the jump phenomenon of the spectral gap of the linearized collision operator (see (1.15) for the definition) for the soft potentials ($-2s \leq \gamma < 0$) when $\epsilon$ goes to zero.

Motivated by these two asymptotical problems, in the present work, we try to find out some stable structure which enable us to obtain new and sharp bounds for the collision operator. Before going further, let us give a short review on the estimates of the Boltzmann collision operator. For simplicity, we only address the estimates in the maxwellian molecular case, that is, $\gamma = 0$.

The lower bounds of the operator are related closely to the energy functional $\langle Q(g, f), f \rangle_v$. By change of variables, we have
\[ \langle Q(g, f), f \rangle_v = \int_{v, v^*, \sigma} B(|v - v^*|, \sigma) g_* f(f' - f) d\sigma dv_* dv. \]

It is easy to check
\[ \langle Q(g, f), f \rangle_v = -\frac{1}{2} \int_{v, v^*, \sigma} B(|v - v^*|, \sigma) g_* (f'^2 - f^2) d\sigma dv_* dv + \frac{1}{2} \int_{v, v^*, \sigma} B(|v - v^*|, \sigma) g_* (f'^2 - f^2) d\sigma dv_* dv. \]

In [1], the authors gave the first coercivity estimate of $\mathcal{E}_g(f)$ which can be stated as
\[ \mathcal{E}_g(f) \geq C_g \|f\|_{H^1}^2 - \|g\|_{L^1} \|f\|_{L^2}^2, \]
where \( C_g \) is a constant depending on the lower bound of \( \| g \|_{L^1} \) and the upper bounds of \( \| g \|_{L^1_L} \) and \( \| g \|_{L^1 \log L} \) (see the definitions in Section 1.3). It gives a positive answer to the conjecture that the Boltzmann collision operator behaves like the fractional Laplace operator, that is,

\[
-Q(g, \cdot) \sim C_g (-\Delta)^s + L.O.T.
\]

This conjecture was further confirmed by the upper bound for the collision operator. Mathematically, it reads that if \( a + b = 2s \) with \( a, b \in \mathbb{R} \),

\[
|\langle Q(g, h), f \rangle_v| \lesssim \| g \|_{L^1_L} \| h \|_{H^s} \| f \|_{H^s},
\]

which was proved in [3]. The weighted Sobolev space \( H^s \) is defined in Section 1.3. This upper bound is sharp in the sense of the free choice of taking derivatives for functions \( h \) and \( f \). For the general potentials, we refer readers to [3] [10] [14] on the lower and upper bounds in weighted Sobolev spaces.

Combining the lower and upper bounds, one may find

\[
C_g \| f \|_{H^s}^2 - \| g \|_{L^1} \| f \|_{L^2} \leq (-Q(g, f), f)_v \lesssim \| g \|_{L^1_L} \| f \|_{H^s}^2.
\]

It shows that the additional condition for \( f \) is required in the upper bound. The reason lies in that some anisotropic structure hides in the operator which can not be observed in weighted Sobolev spaces. It coincides with the behavior of the linearized collision operator \( \mathcal{L}_B \) defined by

\[
\mathcal{L}_B f \overset{\text{def}}{=} -\mu^{-1} Q(\mu, \mu \frac{f}{L}) + Q(\mu \frac{f}{L}, \mu), \quad \mu = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|\theta|^2}{2}},
\]

in the physical paper [9] and the behavior of the Landau operator described in [20]. Indeed, in [9], the authors show that \( \mathcal{L}_B \) is a self-adjoint operator and has explicit eigenvalues and eigenfunctions. In particular, the eigenfunction \( E(\nu) \) takes the form of

\[
E(\nu) = f(|\nu|^2) Y(\sigma),
\]

where \( \nu = |\nu| \sigma, f \) is a radical function and \( Y \) is a real spherical harmonic. In [20], Villani proved that

\[
Q_L(f, f) = 3 \nabla \cdot (\nabla f + vf) - (P_{ij}(f) \partial_{ij} f + \nabla \cdot (vf)) + \Delta_{SS} f,
\]

where \( P_{ij}(f) = \int_{\mathbb{R}^3} f v_i v_j dv \) and \( (-\Delta_{SS}) \) is the Laplace-Beltrami operator on the unit sphere. The special form of the eigenfunction of \( \mathcal{L}_B \) and the Laplace-Beltrami operator in the expression of \( Q_L \) indicate that there should be some anisotropic structure inside the operator.

Mathematically the first attempt to capture the anisotropic structure of the operator were due to [5] and [11] (see also [6] and [12]). In fact, they introduce two types of anisotropic norms to describe the behavior of the operator which are

\[
\| f \|_{K}^2 \overset{\text{def}}{=} \| f \|_{L^2}^2 + \int_{v, v', \sigma} b(\cos \theta) \mu_*(f' - f)^2 d\sigma d\nu d\sigma d\nu
\]

and

\[
\| f \|_{N'}^2 \overset{\text{def}}{=} \| f \|_{L^2}^2 + \int_{v', v''} \langle v \rangle^{s+1/2} \langle v'' \rangle^{s+1/2} \frac{|f''|^2}{d(v, v'')} d(v, v'') d\sigma'(f)
\]

where \( d(v, v') = \sqrt{|v - v'|^2 + \frac{1}{4} (|v|^2 - |v'|^2)^2} \). We remark the second term in \( \| f \|_{N'}^2 \) is exactly the term \( \varepsilon_\mu(f) \) defined in the energy functional \( \langle Q(g, f), f \rangle_v \). Then the estimates can be stated as

\[
\| f \|_{L^2}^2 - \| f \|_{L^1_L}^2 \lesssim \langle \mathcal{L}_B f, f \rangle_v \lesssim \| f \|_{L^2}^2.
\]

Moreover, for the original collision operator \( Q(g, h) \), we have the upper bound:

\[
|\langle Q(g, h), f \rangle_v| \lesssim \| g \|_{L^2} \| h \|_{L^1} \| f \|_{L^2}.
\]

We remark that these two anisotropic norms are useful but are given in an implicit way which helps less to understand the anisotropic property of the operator. Let us check the asymptotical behavior of the norms defined in (1.16) and (1.17) in the process of the grazing collisions limit. For \( \varepsilon_\mu(f) \), on one
hand, it is stable in the limit since it is given in an implicit way. On the other hand, we have no idea on the limit of this quantity. The similar problem occurs or the situation is even worse for $\mathcal{A}(f)$ because of the rescaling \((1.8)\) to the cross-section.

Very recently, two groups gave an explicit description of the anisotropic behavior of the linearized operator. Both of them started with the same point, that is, the well understanding of the behavior of the linearized Landau operator $\mathcal{L}_L$. In fact, we have

$$\mathcal{L}_L \sim (-\Delta + |v|^2/4) + (-\Delta SS^2) \sim (-\Delta + |v|^2/4) + |D_v \times v|^2,$$

recalling that $-\Delta SS^2 = \sum_{1 \leq i < j \leq 3} (v_i \partial_j - v_j \partial_i)^2$.

In [2], the authors show that

$$\langle \mathcal{L}_B f, f \rangle + \|f\|_{L^2}^2 \sim \|f\|_{L^2}^2 + \|f\|_{H^s}^2 + \|D_v \times v\|_{L^2}^2,$$

where $|D_v \times v|f$ is a pseudo-differential operator with the symbol $|\xi \times v|f$. Thanks to [9], by comparing the eigenvalues between Boltzmann and Landau operators, in [16], the authors show that

$$\mathcal{L}_B \sim \mathcal{L}_L^s$$

and

$$\langle \mathcal{L}_B f, f \rangle + \|f\|_{L^2}^2 \sim \|f\|_{L^2}^2 + \|f\|_{H^s}^2 + \|(-\Delta SS^2)^{s/2} f\|_{L^2}^2.$$

We remark that the strategy to obtain the descriptions \((1.19)\) and \((1.21)\) of the operator depends heavily on the linearized structure, for instance, the symmetric property of the operator and the fine properties of the Maxellian state $\mu$. Hence the strategy cannot be generalized to the original collision operator and the rescaled operator under the assumption \((1.7)\) or \((1.8)\).

The short review can be summarized as follows:

1. The previous results on the description of the operator are given in a implicit way or in a unstable way. It means that the anisotropic structure hidden in the operator is still mysterious and not captured well.

2. The upper bound of the collision operator is far away from the sharpness. For instance, recalling \((1.20)\), the typical upper bound for the operator should be in the form of

$$|\langle Q(g,h), f \rangle | \lesssim C(g) \|L_{L,a}^{s/2} + 1\|_{L^2} \|L_{L,b}^{s/2} + 1\| f \|_{L^2}^2,$$

where $a, b \geq 0$ such that $a + b = 2s$.

3. For the linearized collision operator $\mathcal{L}_L^s$ with rescaled cross-section $B_L^c$ under the assumption \((1.7)\) or \((1.8)\), we have no available results on the complete description of the behavior of the operator. And we also have no idea on the sharp bounds of the original collision operator $Q^c$ in the process of the limit.

We end this subsection by the remark that points (1) and (2) are related closely to the well-posedness problem for the original equation \((1.1)\). And the point (3) is related to the investigation of two types of asymptotics mentioned before.

1.2. **The new strategy: dyadic and geometric decompositions.** In this subsection, we will explain our new strategy to catch the structure of the operator. Roughly speaking, it relies on the two types of dyadic decomposition performed in both phase and frequency spaces and the geometric structure of the elastic collision.
1.2.1. Dyadic decompositions in Phase and Frequency spaces. We first introduce two types of dyadic decomposition. Let \( B_\frac{3}{4} \) \( \left\{ \xi \in \mathbb{R}^3 \mid |\xi| \leq \frac{3}{4} \right\} \) and \( C \) \( \left\{ \xi \in \mathbb{R}^3 \mid \frac{3}{4} \leq |\xi| \leq \frac{3}{2} \right\} \). Then one may introduce two radical functions \( \psi \in C_0^\infty(B_\frac{3}{4}) \) and \( \varphi \in C_0^\infty(C) \) which satisfy

\[
\psi, \varphi \geq 0, \quad \text{and} \quad \psi(\xi) + \sum_{j=0}^{\infty} \varphi(2^{-j} \xi) = 1, \quad \xi \in \mathbb{R}^3.
\]

The first decomposition is performed in the phase space. We introduce the dyadic operator \( \mathcal{P}_j \) defined as

\[
\mathcal{P}_{-1} f(x) = \psi(x) f(x), \quad \mathcal{P}_j f(x) = \varphi(2^{-j} x) f(x), \quad (j \geq 0).
\]

We also introduce the operators relating to \( \mathcal{P}_j \):

\[
\tilde{\mathcal{P}}_j f(x) = \sum_{|k-j| \leq N_0} \mathcal{P}_k f(x) \quad \text{and} \quad \tilde{\mathcal{U}}_j f(x) = \sum_{k \leq j} \mathcal{P}_k f(x).
\]

Here \( N_0 \) is an integer which will be chosen in the later. Then we have for any \( f \in L^2(\mathbb{R}^3) \), it holds

\[
f = \mathcal{P}_{-1} f + \sum_{j=0}^{\infty} \mathcal{P}_j f.
\]

We set

\[
\Phi_k^Y(v) \overset{\text{def}}{=} \left\{ \begin{array}{ll}
|v|^j \varphi(2^{-k} |v|), & \text{if} \quad k \geq 0; \\
|v|^j \psi(|v|), & \text{if} \quad k = -1.
\end{array} \right.
\]

Then we may derive

\[
\langle Q(g, h), f \rangle_v = \sum_{k=1}^{\infty} \langle Q_k(g, h), f \rangle_v = \sum_{k \geq -1} \sum_{j \geq 1} \langle Q_k(\mathcal{P}_j g, h), f \rangle_v,
\]

where

\[
Q_k(g, h) = \int_{\mathbb{S}^2} b(\cos \theta)(g^\prime h - g h) d \sigma d v.
\]

For \( Q_k \), relative velocity \( v - v_* \) is localized in the ring \( \left\{ \frac{3}{4} 2^k \leq |v - v_*| \leq \frac{3}{2} 2^k \right\} \). Suppose \( g \) is localized in the ring \( \left\{ \frac{3}{4} 2^j \leq |v_*| \leq \frac{3}{2} 2^j \right\} \). Then thanks to the fact \( \frac{\sqrt{7}}{2} |v - v_*| \leq |v' - v_*| \leq |v - v_*| \), we have

- If \( j \leq k - N_0 \), then \( |v|, |v'| \in \left[ \frac{3}{4} 2^{2 - N_0} 2^k, \frac{3}{2} 2^{1 - N_0} 2^k \right] \);
- If \( j \geq k + N_0 \), then \( |v| \in \left( \frac{3}{4} 2^{1 - N_0} 2^j, \frac{3}{2} 2^{1 - N_0} 2^j \right] \) and \( |v'| \in \left( \frac{3}{4} 2^{1 - N_0} 2^j, \frac{3}{2} 2^{1 - N_0} 2^j \right) \);
- If \( |j - k| < N_0 \), then \( |v| \leq 22^{k+2 N_0}, |v'| \leq 22^{k+2 N_0} \).

The second decomposition is performed in the frequency space. In fact, it is the standard Littlewood-Paley theor. We denote \( \tilde{m} \overset{\text{def}}{=} \mathcal{F}^{-1} \psi \) and \( \tilde{\varphi} \overset{\text{def}}{=} \mathcal{F}^{-1} \varphi \), where they are the inverse Fourier Transform of \( \varphi \) and \( \psi \). If we set \( \phi_j(x) \overset{\text{def}}{=} 2^{-3j} \varphi(2^{j} x) \), then the dyadic operators \( \tilde{\mathcal{F}}_j \) can be defined as follows

\[
\tilde{\mathcal{F}}_{-1} f(x) = \int_{\mathbb{R}^3} \tilde{m}(x - y) f(y) dy, \quad \tilde{\mathcal{F}}_j f(x) = \int_{\mathbb{R}^3} \phi_j(x - y) f(y) dy, \quad (j \geq 0).
\]

We also introduce the operators relating to dyadic operator \( \tilde{\mathcal{F}}_j \):

\[
\tilde{\mathcal{F}}_j f(x) = \sum_{|k-j| \leq 3 N_0} \tilde{\mathcal{F}}_k f(x) \quad \text{and} \quad \mathcal{F}_j f(x) = \sum_{k \geq j} \tilde{\mathcal{F}}_k f.
\]

Then for any \( f \in \mathcal{S}'(\mathbb{R}^3) \), it holds

\[
f = \tilde{\mathcal{F}}_{-1} f + \sum_{j=0}^{\infty} \tilde{\mathcal{F}}_j f.
\]
We recall that the Bochner’s formula of the operator can be stated as

\begin{equation}
\langle F(Q_k(g, h), f) \rangle = \int_{\sigma \in \mathbb{S}^2, \eta, \xi \in \mathbb{R}} b\left(\frac{\xi}{|\xi|}\right) \cdot \sigma \left[ (\langle F_{\mathcal{P}}^\prime \rangle_{\eta, \xi} - F_{\mathcal{P}}^\prime(h, \mathcal{F} f)) \right] (\mathcal{F} g)(\eta_*) (\mathcal{F} h)(\xi - \eta_*) (\mathcal{F} f)(\xi) d\sigma d\eta_* d\xi,
\end{equation}

where \( F f \) denotes the Fourier transform of \( f \) and \( \xi^- = \frac{|\xi|}{2} \).

Suppose functions \( g \) and \( h \) are localized in rings \( \left\{ \frac{2}{3} 2^p \leq |\xi| \leq \frac{2}{3} 2^p \right\} \) and \( \left\{ \frac{2}{3} 2^l \leq |\xi| \leq \frac{2}{3} 2^l \right\} \) respectively in the frequency space. Due to the equality, we have \( \frac{2}{3} 2^p \leq |\eta_*| \leq \frac{2}{3} 2^p \) and \( \frac{2}{3} 2^l \leq |\xi - \eta_*| \leq \frac{2}{3} 2^l \). Then

- If \( l \leq p - N_0 \), then \( |\xi| \in \left[ \frac{3}{4} - \frac{8}{9} 2^{-N_0} \right] 2^p, \frac{8}{9} \left( 1 + 2^{-N_0} \right) 2^p \). This implies \( |\eta_* - \xi^-| \) verifies \( |\xi^+| \in \left[ \frac{\sqrt{2}}{2} \left( \frac{3}{4} - \frac{8}{9} 2^{-N_0} \right), \frac{8}{9} \left( 1 + 2^{-N_0} \right) 2^p \right] \). Notice that \( |\eta_* - \xi^-| = |(\eta_* - \xi^-) + |\xi^+| \), then one has \( |\eta_* - \xi^-| \in \left[ \frac{\sqrt{2}}{2} \left( \frac{3}{4} - \frac{8}{9} 2^{-N_0} \right), \frac{8}{9} \left( 1 + 2^{-N_0} \right) 2^p \right] \).
- If \( l \geq p + N_0 \), then \( |\xi| \in \left[ \frac{3}{4} - \frac{8}{9} 2^{-N_0} \right], \frac{8}{9} \left( 1 + 2^{-N_0} \right) 2^l \) and \( |\eta_* - \xi^-|, |\eta_*| \leq \left( 1 + \frac{2}{3} 2^{-N_0} \right) 2^l \).
- If \( l = p - N_0 \), then \( |\xi| \leq \frac{2}{3} 2^p + N_0 \). Now let \( |\xi| \in \left[ \frac{2}{3} 2^m, \frac{2}{3} 2^m \right] \). In the case of \( m - p \leq 2N_0 \), one has \( |\xi| \in \left[ 2^{-N_0} \frac{2}{3} 2^p, 2^{-N_0} \frac{2}{3} 2^p \right] \). In the case of \( m - p > 2N_0 \), one has \( |\eta_*|, |\eta_* - \xi^-| \in \left[ \frac{1}{2} 2^{-N_0} \frac{2}{3} 2^p, \frac{1}{2} 2^{-N_0} \frac{2}{3} 2^p \right] \).

Thanks to the observations in the above and the fact

\[ \langle Q_k(g, h), f \rangle = \int_{\sigma \in \mathbb{S}^2, \varepsilon, v \in \mathbb{R}} \Phi_{h}^\prime(\langle v - \varepsilon_* \rangle) b(\cos \theta)(\bar{g} h) (|v' - f|) d\sigma d\varepsilon_* d\nu, \]

there exists an integer \( N_0 \in \mathbb{N} \) such that

\begin{equation}
\sum_{k \geq -1} \sum_{j \geq -1} \langle Q_k(\mathcal{P}_j g, h), f \rangle = \sum_{j \geq -1} \langle Q_k(\mathcal{P}_j g, \mathcal{P}_j h), \mathcal{P}_j f \rangle + \sum_{j \geq -1} \langle Q_k(\mathcal{P}_j g, \mathcal{P}_j h), \mathcal{P}_j f \rangle
\end{equation}

and

\[ \langle Q_k(g, h), f \rangle = \sum_{l \geq -1} \langle Q_k(\tilde{g} p g, \tilde{g} h), f \rangle = \sum_{l \geq -1} \langle Q_k(\tilde{g} p g, \tilde{g} h), f \rangle \]

where

\begin{align}
\mathfrak{m}_k,p,l & \quad \text{def} = \int_{\sigma \in \mathbb{S}^2, \varepsilon, v \in \mathbb{R}} \Phi_{\mathcal{P}_j}^\prime(\langle v - \varepsilon_* \rangle) b(\cos \theta)(\bar{g} h) (|v' - f|) d\sigma d\varepsilon_* d\nu, \\
\mathfrak{m}_k,p,l & \quad \text{def} = \int_{\sigma \in \mathbb{S}^2, \varepsilon, v \in \mathbb{R}} \Phi_{\mathcal{P}_j}^\prime(\langle v - \varepsilon_* \rangle) b(\cos \theta)(\bar{g} h) (|v' - f|) d\sigma d\varepsilon_* d\nu, \\
\mathfrak{m}_k,p,l,m & \quad \text{def} = \int_{\sigma \in \mathbb{S}^2, \varepsilon, v \in \mathbb{R}} \Phi_{\mathcal{P}_j}^\prime(\langle v - \varepsilon_* \rangle) b(\cos \theta)(\bar{g} h) (|v' - f|) d\sigma d\varepsilon_* d\nu, \\
\mathfrak{m}_k,p,l,m & \quad \text{def} = \int_{\sigma \in \mathbb{S}^2, \varepsilon, v \in \mathbb{R}} \Phi_{\mathcal{P}_j}^\prime(\langle v - \varepsilon_* \rangle) b(\cos \theta)(\bar{g} h) (|v' - f|) d\sigma d\varepsilon_* d\nu.
\end{align}

We remark that here we use the fact that the Fourier transform maps a radical function into a radical function.
By simple calculation, we arrive at

$$\langle Q_f (g, f) , v \rangle = \sum_{l \leq p - N_0} m_{k,p,l}^1 + \sum_{l \geq 1} m_{k,l}^2 + \sum_{p \geq 1} m_{k,p}^3 + \sum_{m < p - N_0} m_{k,p,m}^4$$

where

$$m_{k,l}^2 \overset{\text{def}}{=} \int_{\sigma \in S^2, v_0 \in R^3} \Phi_k^l (|v - v_*|) b(\cos \theta) (\sigma f \phi (\tilde{g} h) [ (\tilde{g} f) ' - \tilde{g} f] ) d\sigma dv_* dv,$$

$$m_{k,p}^3 \overset{\text{def}}{=} \int_{\sigma \in S^2, v_0 \in R^3} \Phi_k^p (|v - v_*|) b(\cos \theta) (\sigma p \phi (\tilde{g} h) [ (\tilde{g} p f) ' - \tilde{g} p f] ) d\sigma dv_* dv,$$

$$m_{k,p,m}^4 \overset{\text{def}}{=} \int_{\sigma \in S^2, v_0 \in R^3} (\tilde{g} p \phi (\tilde{g} p) [ (v - v_*|) b(\cos \theta) (\sigma p \phi (\tilde{g} h) [ (\tilde{g} m f) ' - \tilde{g} m f] ) d\sigma dv_* dv.$$

Now the estimate of the functional $\langle Q_f (g, f) , v \rangle$ is reduced to the estimates of the terms in the right-hand sides of (1.26) and (1.27).

Let us give some remarks on these two decompositions:

1. Due to (1.25), we introduce the dyadic decomposition in the frequency space which makes use of the interaction and the cancellation between the different parts of frequency of the functions $h$ and $f$. It will enable us to obtain the sharp bounds of the operator in Sobolev spaces in the sense of the free choice of taking derivatives for functions $h$ and $f$ (see (1.13)). However it is not enough to get the sharp bounds considering the fact that additional weight is paid in the upper bound compared to that in the lower bound (see (1.14)). Obviously it is caused by the anisotropic structure of the operator.

2. To clarify where the additional weight comes from, we introduce the dyadic decomposition in the phase space. By careful analysis, we can distinguish which part of the operator is the worst term that brings the additional weight to $h$ and $f$. In fact, the worst situation happens in the case that the functions $h$ and $f$ are localized in the same region both in phase and frequency spaces and at the same time the relative velocity $|v - v_*|$ is far away from the zero. In other words, in such a situation the collision operator is dominated by the anisotropic structure. It is the key point to obtain the estimate (1.22) in anisotropic spaces.

3. These two types of dyadic decompositions are consistent with the new profile of the weighted Sobolev spaces (see Theorem 5.1 in Section 5).

4. Based on the decompositions, we obtain the sharp bounds for the operator in weighted Sobolev spaces as follows:

$$| \langle Q_f (g, f) , v \rangle | \lesssim C(k) \| h \|_{H^a_1} \| f \|_{H^b_2} ,$$

where $a, b \in [0, 2s], w_1, w_2 \in R$ verifying $a + b = 2s$ and $w_1 + w_2 = \gamma + 2s$. Compared to the previous work [3, 4, 10, 14], here we have the freedom of choosing the weight functions for $h$ and $f$.

5. The decompositions are very stable in the grazing collisions limit. In fact, we can apply them to the rescaled Boltzmann operator to get the upper bounds. By taking the grazing collisions limit, these upper bounds turn to be the sharp upper bounds of the Landau operator in weighted Sobolev spaces. The reader may check details in Section 4.

1.2.2. Key observation: geometric decomposition. Now we turn to the explanation of the key observation which enables us to obtain the new sharp bounds for the original collision operator. For the simplicity, we only focus on the maxwellian molecular case ($\gamma = 0$).

We revisit the quantity $E_k (f)$. In particular, we look for a new decomposition for the term $f' - f$ contained in $E_k (f)$. Our main observation is due to geometric structure of the collision which is depicted schematically as follows:
Set $u = v - v_*$, then $v' = v_* + u^+$ and $v = v_* + u$, where $u^+ = \frac{u + |u|\sigma}{2}$. Now assuming $u = rt$ with $r = |u|$ and $\tau \in SS^2$, we infer

$$v = v_* + rt, v' = v_* + r \frac{\tau + \sigma}{2}.$$ 

Let $\zeta = \frac{r + \sigma}{|r + \sigma|} \in SS^2$. Then we have the geometric decomposition:

$$f(v') - f(v) = (f(v_* + \frac{|\tau + \sigma|}{2} r \zeta) - f(v_* + r \zeta)) + (f(v_* + r \zeta) - f(v_* + rt))$$

(1.28)

where $T_v f \overset{\text{def}}{=} f(v + \cdot)$. Applying (1.28) to the functional $\langle Q(g, h), f \rangle$, we then have

$$\langle Q(g, h), f \rangle = \int_{\sigma \in SS^2, \nu, \nu_e \in R^3} b(\cos \theta) g_*(T_v h)(\nu) (f(v_* + u^+) - f(v_* + |u| \frac{u^+}{|u^+|}) d\sigma d\nu, du$$

$$= \langle \vartheta g, h \rangle$$

$$+ \int_{\sigma \in SS^2, \nu, \nu_e \in R^3} b(\cos \theta) g_*(T_v h)(\nu) ((T_v f)(r \zeta) - (T_v f)(rt)) d\sigma d\nu, du.$$

Notice that $|u^+ - |u| \frac{u^+}{|u^+|} | \sim \theta^2 |u|$. Then by technical argument (see the proof in Theorem 1.2), the operator $\vartheta g$ behaves like

$$\vartheta g \sim C (v)^{(-\Delta)^{3/2}}.$$

Recalling the rough behavior of $Q(g, \cdot)$ (see (1.12)), we may regard $\vartheta g$ as the lower order term. Now we concentrate on the functional $\langle \vartheta g, h, f \rangle$. By (1.3), it is easy to check

$$\langle \vartheta g, h, f \rangle = \int_{\sigma \in SS^2, \nu, \nu_e \in R^3} b(\sigma \cdot \tau) 1_{0 < \tau \leq 0} g_*(T_v h)(\nu)(T_v f)(r \zeta) - (T_v f)(rt)) r^2 d\sigma d\nu, d\tau dr.$$

For fixed $v_*, \tau$ and $r$, if $\tau$ is chosen to be the polar direction, one has

$$d\sigma = \sin \theta d\theta dS^1, d\zeta = \sin \phi d\phi dS^1,$$

where $\theta = 2\phi$. From which, we deduce

$$d\sigma = 4 \cos \phi d\zeta.$$
Then by change of variables from $\sigma$ to $\zeta$, we derive
\[
\langle g_h f, f \rangle_v = \int_{\zeta \in \text{SS}^2} b(2(\zeta \cdot \tau)^2 - 1)1_{\zeta \cdot \tau \in \sqrt{2}/2}4(\zeta \cdot \tau)g_*(T_v, h)(\tau) \\
\times ((T_v f)(r\tau) - (T_v f)(r\zeta)) r^2 d\zeta dr dv_*.
\]
(1.29)

By the symmetric property of $\tau$ and $\zeta$, we get
\[
\langle g_h f, f \rangle_v = \frac{1}{2} \int_{\zeta \in \text{SS}^2, \nu \in \mathbb{R}^3} b(2(\zeta \cdot \tau)^2 - 1)1_{\zeta \cdot \tau \in \sqrt{2}/2}4(\zeta \cdot \tau)g_*(T_v, h)(\tau) \\
\times ((T_v f)(r\tau) - (T_v f)(r\zeta)) r^2 d\zeta dr dv_*.
\]

Thus to give the estimate of $\langle g_h, f \rangle$, it suffices to consider the functional
\[
\varpi(h, f) = \int_{\zeta, \tau \in \text{SS}^2} (h(\tau) - h(\zeta))(f(\zeta) - f(\tau))H(\zeta \cdot \tau)d\zeta d\tau,
\]
where $H(\zeta \cdot \tau) = b(2(\zeta \cdot \tau)^2 - 1)4(\zeta \cdot \tau)1_{\zeta \cdot \tau \in \sqrt{2}/2}$. Recall that the assumption (1.5) is equivalent to $b(\sigma \cdot \tau) \sim |\sigma - \tau|^{-2 - 2s}$, then by the fact $|\sigma - \tau| \sim |\zeta - \tau|$, we get
\[
H(\zeta \cdot \tau) \sim |\zeta - \tau|^{-2 - 2s}1_{|\zeta - \tau|^2 \leq 2 - \sqrt{2}}.
\]

Let us first consider the lower bound of the operator. Thanks to Lemma 5.5 the $L^2$ profile of the fractional Laplace-Beltrami operator, roughly speaking, we derive
\[
- \varpi(f, f) + \|f\|^2_{L^2(\text{SS}^2)} \sim \int_{\zeta, \tau \in \text{SS}^2} \frac{|f(\sigma) - f(\tau)|^2}{|\sigma - \tau|^{2+2s}} d\sigma d\tau + \|f\|^2_{L^2(\text{SS}^2)} \\
\sim \|(-\Delta_{\text{SS}^2})^{s/2} f\|_{L^2(\text{SS}^2)}^2 + \|f\|^2_{L^2(\text{SS}^2)}.
\]
(1.30)

Then we are led to
\[
\langle -\varpi, f, f \rangle_v \gtrsim \int_{v_*} g_* \|(-\Delta_{\text{SS}^2})^{s/2} T_v f\|^2_{L^2} dv_* - L.O.T,
\]
which almost yields
\[
\langle -Q(g, f), f \rangle_v \gtrsim C_g \|(-\Delta_{\text{SS}^2})^{s/2} f\|^2_{L^2} - L.O.T.
\]

Now what remains is to check the commutator estimate between the translation operator $T_v$ and the fractional Laplace-Beltrami operator $(-\Delta_{\text{SS}^2})^{s/2}$. It is purely a technical problem and highly non-trivial. We conclude that the final stage is to prove the following inequality:
\[
\|(-\Delta_{\text{SS}^2})^{s/2} T_v f\|_{L^2} \lesssim \|v_* f\|_{L^2} \|(-\Delta_{\text{SS}^2})^{s/2} f\|_{L^2} + \|f\|_{H^1}.
\]
(1.31)

It is easy to check that it holds for $s = 0$ and $s = 1$. Unfortunately because $-\Delta_{\text{SS}^2}$ does not commutate with $V$, the standard real interpolation method cannot be applied to derive (1.31). To solve the problem, we develop a new interpolation theory (which is of independently interest) to overcome the difficulty. We refer readers to check the theory in Section 5. Then by combining the coercivity estimate in Sobolev spaces in (1.14), roughly speaking, we derive
\[
\langle -Q(g, f), f \rangle_v \gtrsim C_g \|(-\Delta_{\text{SS}^2})^{s/2} f\|^2_{L^2} + \|f\|^2_{H^1} - L.O.T.
\]

We remark that it is sharp for the lower bounds of the operator in anisotropic spaces since it is consistent with the behavior of the linearized operator (1.18).

Next we turn to the upper bounds. We will show that our new observation makes the estimate (1.22) promising. Indeed, by Cauchy-Schwartz inequality, we have
\[
|\varpi(h, f)| \lesssim \left( \int_{\zeta, \tau \in \text{SS}^2} \frac{|h(\sigma) - h(\tau)|^2}{|\sigma - \tau|^{2+2s}} d\sigma d\tau \right)^{1/2} \left( \int_{\zeta, \tau \in \text{SS}^2} \frac{|f(\sigma) - f(\tau)|^2}{|\sigma - \tau|^{2+2s}} d\sigma d\tau \right)^{1/2}.
\]
(1.32)

Thanks to the Addition Theorem (see the statement in Section 5), we deduce
\[
|\varpi(h, f)| \lesssim \|1 - \Delta_{\text{SS}^2}\|_{L^2} \|1 - \Delta_{\text{SS}^2}\|_{L^2}^{b/2} f\|_{L^2},
\]
which implies
\[ \langle -\mathcal{Q}_g h, f \rangle_v \lesssim \left( \int_{\nu_*} g_* \| (-\Delta_{SS})^{a/2} T_{\nu_*} h \|_{L^2_{\nu_*}}^2 \, d\nu_* \right)^{1/2} \left( \int_{\nu_*} g_* \| (-\Delta_{SS})^{b/2} T_{\nu_*} f \|_{L^2_{\nu_*}}^2 \, d\nu_* \right)^{1/2} + L.O.T. \]

From which together with the sharp upper bounds for the operator in weighted Sobolev spaces and (1.31), we finally arrive at (1.22), the sharp upper bound of the operator in anisotropic space.

Some remarks are in order:

1. The geometric decomposition plays essential role to catch the anisotropic structure of the operator in the lower and upper bounds. Since it does not use the symmetry and regularity properties of the function \( g \), it is more robust than the previous work.
2. The geometric decomposition is stable in the process of the grazing collisions limit. Actually we can give an explicit description of the asymptotical behavior of the anisotropic structure in the limit (see Lemma (1.1)). Roughly speaking, in the process of the limit, the behavior of collision operator depends on the parameter \( \epsilon \). If the eigenvalue of \( -\Delta_{SS} \) is less than \( \epsilon^{-2} \), then the operator behaves like \( -\Delta_{SS} \). While if the eigenvalue of \( -\Delta_{SS} \) is bigger than \( \epsilon^{-2} \), then the operator behaves like \( \epsilon^{2s-2} (-\Delta_{SS})^s \). It reflects the strong connection between Boltzmann and Landau collision operators.
3. We remark that the geometric decomposition can also be applied to the operator in frequency space (recalling (1.25)) to catch the anisotropic structure. It will give an alternative proof to the lower and upper bounds of the operator in the anisotropic spaces. But it only works for the maxwellian case (\( \gamma = 0 \)) because of the simplicity of (1.25) in this case.

1.3. Notations and main results. Let us first introduce the function spaces which we shall use throughout the paper.

1. For integer \( N \geq 0 \), we define the Sobolev space

\[ H^N \overset{\text{def}}{=} \left\{ f(v) : \sum_{|\alpha| \leq N} \| \partial_\alpha^p f \|_{L^2} < +\infty \right\}. \]

2. For real number \( m, l \), we define the weighted Sobolev space

\[ H^m_l \overset{\text{def}}{=} \left\{ f(v) : \| f \|_{H^m_l} = \| \langle D \rangle^m \langle v \rangle^l f(v) \|_{L^2} < +\infty \right\}, \]

where the multi-index \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) with \( |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \) and \( \langle v \rangle = (1 + |v|^2)^{1/2} \). Here \( \langle D \rangle \) is the pseudo-differential operator with the symbol \( a(\xi) \) defined by

\[ a(D)f(x) \overset{\text{def}}{=} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i(x-y)\xi} a(\xi) f(y) dy \, d\xi. \]

3. The general weighted Sobolev space \( W^{N,p}_l \) with \( p \in [1, \infty) \) is defined as follows

\[ W^{N,p}_l \overset{\text{def}}{=} \left\{ f(v) : \| f \|_{W^{N,p}_l} = \sum_{|\alpha| \leq N} \left( \int_{\mathbb{R}^3} |\partial^\alpha f(v)|^p \langle v \rangle^l \, dv \right)^{1/p} < +\infty \right\}. \]

In particular, if \( N = 0 \), we introduce the weighted \( L^p_l \) space as

\[ L^p_l \overset{\text{def}}{=} \left\{ f(v) : \| f \|_{L^p_l} = \left( \int_{\mathbb{R}^3} |f(v)|^p \langle v \rangle^l \, dv \right)^{1/p} < +\infty \right\}. \]

4. The \( L \log L \) space is defined by

\[ L \log L \overset{\text{def}}{=} \left\{ f(v) : \| f \|_{L \log L} = \int_{\mathbb{R}^3} |f| \log(1 + |f|) \, dv < +\infty \right\}. \]
Next we list some notations which will be used in the paper. We write \( a \lesssim b \) to indicate that there is a uniform constant \( C \), which may be different on different lines, such that \( a \leq Cb \). We use the notation \( a \sim b \) whenever \( a \lesssim b \) and \( b \lesssim a \). The notation \( a^+ \) means the maximum value of \( a \) and 0. The weight function \( W_l \) is defined by \( W_l(v) \equiv \langle v \rangle^l \). We denote \( C(\lambda_1, \lambda_2, \cdots, \lambda_n) \) by a constant depending on the parameters \( \lambda_1, \lambda_2, \cdots, \lambda_n \).

Our first result is on the sharp upper bounds of the collision operator in weighted Sobolev spaces.

**Theorem 1.1.** Suppose \( w_1, w_2 \in \mathbb{R} \) verifying \( w_1 + w_2 = \gamma + 2s \) and \( a, b \in [0, 2s] \) verifying \( a + b = 2s \). Then for smooth functions \( g, h \) and \( f \), we have

1. If \( \gamma + 2s > 0 \),
   
   \[
   |\langle Q(g, h), f \rangle_v| \lesssim (\|g\|_{L^1_{\gamma+2s, (-w_1)^+ + (-w_2)^+}} + \|g\|_{L^2}) \|h\|_{H^s_{w_1}} \|f\|_{H^s_{w_2}},
   \]

2. If \( \gamma + 2s = 0 \),
   
   \[
   |\langle Q(g, h), f \rangle_v| \lesssim (\|g\|_{L^1_{\lambda_1}} + \|g\|_{L^2}) \|h\|_{H^s_{w_1}} \|f\|_{H^s_{w_2}},
   \]

   where \( w_3 = \max\{\delta, (-w_1)^+ + (-w_2)^+\} \) with \( \delta > 0 \) which is sufficiently small,

3. If \( -1 < \gamma + 2s < 0 \),
   
   \[
   |\langle Q(g, h), f \rangle_v| \lesssim (\|g\|_{L^1_{\lambda_1}} + \|g\|_{L^2_{(-\gamma+2s)}}) \|h\|_{H^s_{w_1}} \|f\|_{H^s_{w_2}},
   \]

   where \( w_3 = \max\{-(\gamma + 2s), \gamma + 2s + (-w_1)^+ + (-w_2)^+\} \).

**Remark 1.4.** The estimates \((1.33), (1.35)\) are sharp in weighted Sobolev spaces considering that we have freedom of choosing derivatives and the weights for functions \( h \) and \( f \). We refer readers to the very recent work \cite{19} \cite{18} to check the upper bounds of the operator by using the Bobylev's formula (see \cite{12}) or the Random transform. For the Maxwellian case (\( \gamma = 0 \)), the estimate can be stated as

\[
|\langle Q(g, h), f \rangle_v| \leq C(a, b) \|g\|_{L^1_{2s+(-w_1)^+ + (-w_2)^+}} \|h\|_{H^s_{w_1}} \|f\|_{H^s_{w_2}},
\]

where \( a + b = 2s \) and \( w_1 + w_2 = 2s \). Here we remove the restriction \( a, b \in [0, 2s] \) and improve the estimate in \cite{3}.

**Remark 1.5.** The estimates are not sharp with respect to the function \( g \). For instance, for the hard potentials, we can remove the \( L^2 \) norm of \( g \) in the estimate. For \( \gamma > -\frac{5}{2} \), we can replace the weighted \( L^2 \) norm by weighted \( L^2 \) norm of \( g \) in the estimates. We do not want to pursue such kind of estimates since weighted \( L^1 \) and \( L^2 \) norms are generally used in the energy estimates for the equation.

**Remark 1.6.** A direct application of the estimates is to prove \cite{11} holds for solutions to the spatially homogeneous Boltzmann and Landau equations for \( s \in (0, \frac{1}{2}] \) if we follow the argument in \cite{14}. Then it completes the study of the grazing collisions limit in the homogeneous case.

Next we will state our new coercivity estimates for the operator:

**Theorem 1.2.** Suppose the function \( g \) is a non-negative smooth function verifying that

\[
\|g\|_{L^1} > \delta \quad \text{and} \quad \|g\|_{L^1} + \|g\|_{L^\infty} \log L < \lambda.
\]

Then for sufficiently small \( \eta \), there exist three constants \( C_1(\delta, \lambda, \eta), C_2(\lambda, \delta) \) and \( C_3(\delta, \lambda, \eta) \) such that

1. If \( \gamma + 2s \geq 0 \),
   
   \[
   (-Q(g, f), f)_v \gtrsim C_1(\delta, \lambda, \eta) \left( \|(-\Delta_{SS})^{s/2} f\|_{L_{\gamma}^{2s+\delta}}^2 + \|f\|_{H_{\gamma}^s}^2 \right)
   - \eta C_2(\delta, \lambda) \|f\|_{L_{\gamma}^{2s+\delta}}^2 - C_3(\delta, \lambda, \eta) \|f\|_{L_{\gamma}^{2s+\delta}}^2,
   \]

where \( \lambda, \delta, \eta \) are fixed.
Remark 1.8. Suppose that $f$ is a non-negative function verifying the condition (5.17) of the operator in Theorem 1.3.\nw = \max\{\gamma + 2s, 2\}.

Remark 1.9. Compared with the previous work on the regularity of fractional derivative, the fractional Laplace-Beltrami derivative, $\|(-\Delta_{SS})^{s/2} f\|_{L^2_{t+2s}}$, can also be bounded below under the additional assumption that $f \in L^2_{t+2s}$. We comment that the strong connection between $\|(-\Delta_{SS})^{s/2} f\|_{L^2_{t+2s}}$ and $\|f\|_{L^2_{t+2s}}$ in the theorem is a bad news to the well-posedness problem of (1.1) since it will cause the trouble to close the energy estimates to the equation, in particular, in the case of $\gamma + 2s > 0$.

As a direct consequence, now we can complete the entropy dissipation estimates as follows:

**Theorem 1.3.** Suppose that $f$ is a non-negative function verifying the condition (1.37). Then it holds

\[ D(f) + \|f\|_{L^1_0} \gtrsim C(\lambda, \delta)(\sqrt{\|f\|_{L^2_{t+2s}}^2} + \|(-\Delta_{SS})^{s/2} f\|_{L^2_{t+2s}}^2) , \]

where $w = \max\{\gamma + 2s, 2\}$.

**Remark 1.10.** Suppose that $f$ is a solution to the homogeneous Boltzmann equation with $f_0 \in L^1_{t+2s} \cap L\log L$, then we obtain that for $\gamma + 2s \leq 2$, there exists a constant $C(f_0)$ such that

\[ D(f) + \|f_0\|_{L^1_0} \gtrsim C(f_0)(\sqrt{\|f\|_{L^2_{t+2s}}^2} + \|(-\Delta_{SS})^{s/2} f\|_{L^2_{t+2s}}^2) . \]

**Remark 1.11.** For $\gamma + 2s \geq 2$, we still can catch the Sobolev regularity:

\[ D(f) + \|f\|_{L^1_0} \gtrsim C(\lambda, \delta)\sqrt{\|f\|_{L^2_{t+2s}}^2} . \]

The additional assumption $f \in L^1_{t+2s}$ is compulsory to get the anisotropic regularity.

**Remark 1.12.** Compared with the entropy production estimate in (12), our results do not need additional regularity assumption on $f$ for soft potentials.

Finally let us give the sharp bounds of the operator in the anisotropic spaces:

**Theorem 1.4.** Suppose $a, b \in [0, 2s]$ verifying $a + b = 2s$, $a_1, b_1 \in [0, s]$ verifying $a_1 + b_1 = s$ and $w_1, w_2 \in \mathbb{R}$ verifying $w_1 + w_2 = \gamma + s$. Then for smooth functions $g, h$ and $f$, there hold for any $\delta > 0$, if $-1 - 2s < \gamma < -2s$,

\[ \langle -Q(g, f), f \rangle_v \gtrsim C_4(\gamma, \delta, \eta)\left( \|(-\Delta_{SS})^{s/2} f\|_{L^2_{t+2s}}^2 + \|f\|_{H^s_{t+2s}}^2 \right) - \eta C_2(\delta, \lambda) \|f\|_{L^2_{t+2s}}^2 \]

\[ -C_3(\delta, \lambda, \eta) \|g\|_{L^p_{t+2s+3p-3}} \|f\|_{L^2_{t+2s}}^2 , \]

with $p > \frac{3}{\gamma + 2s + 3}$.\n
**Remark 1.7.** Here $(-\Delta_{SS})^{s/2}$ is the fractional Laplace-Beltrami operator. One may check the definition of the operator in (5.17) and (5.20).

**Remark 1.8.** Compared to the lower bound of the functional $\langle \mathcal{L}_B f, f \rangle$, we cannot control $\langle -Q(g, f), f \rangle_v$ from the below by $\|f\|_{L^2_{t+2s}}^2$. In fact, it is false

\[ \langle -Q(g, f), f \rangle_v \gtrsim \|f\|_{L^2_{t+2s}}^2 - L.O.T. \]

Suppose it is true, then combining with upper bound (see Remark 1.14), we may derive

\[ \|f\|_{L^2_{t+2s}}^2 \lesssim \left( \|(-\Delta_{SS})^{s/2} f\|_{L^2_{t+2s}}^2 + \|f\|_{H^s_{t+2s}}^2 \right) , \]

It is obvious that the radical function does not verify such kind of the estimate. Then we get the contradiction. It shows on one hand, the lower bounds in the theorem are sharp in anisotropic spaces. On the other hand, the behavior of the original operator is different from that of the linearized operator $\mathcal{L}_B$ (recalling (1.21)).
(1) if \( \gamma > 0 \)
\[
|\langle Q(g, h), f \rangle| \lesssim (\|g\|_{L^1_{1/2}} + \|g\|_{L^1_{1/2+\gamma/2}} + \|g\|_{L^2}) (\|(-\Delta_{SS})^{a/2} h\|_{L^2_{\gamma/2}} + \|h\|_{H^{\gamma/2}_1}) \times (\|(-\Delta_{SS})^{b/2} f\|_{L^2_{\gamma/2}} + \|f\|_{H^{b/2}_1}) + \|h\|_{H^{a/2}_1} \|f\|_{H^{b/2}_1}.
\]

(2) if \( \gamma = 0 \),
\[
|\langle Q(g, h), f \rangle| \lesssim (\|g\|_{L^1_{2a/3}} + \|g\|_{L^1_{2a/3-\gamma/2}} + \|g\|_{L^2}) (\|(-\Delta_{SS})^{a/2} h\|_{L^2_{1/2}} + \|h\|_{H^{a/2}_1}) \times (\|(-\Delta_{SS})^{b/2} f\|_{L^2_{1/2}} + \|f\|_{H^{b/2}_1}) + \|h\|_{H^{a/2}_1} \|f\|_{H^{b/2}_1}.
\]

(3) if \( \gamma < 0 \),
\[
|\langle Q(g, h), f \rangle| \lesssim (\|g\|_{L^1_{1-\gamma/2}} + \|g\|_{L^1_{1-\gamma/2-\gamma/2}} + \|g\|_{L^2}) (\|(-\Delta_{SS})^{a/2} h\|_{L^2_{1/2}} + \|h\|_{H^{a/2}_1}) \times (\|(-\Delta_{SS})^{b/2} f\|_{L^2_{1/2}} + \|f\|_{H^{b/2}_1}) + \|h\|_{H^{a/2}_1} \|f\|_{H^{b/2}_1}.
\]

Remark 1.13. Thanks to the interpolation inequality
\[
\|h\|_{H^{b/2}_1} \leq \|h\|_{H^{a/2}_1} \|h\|_{L^{2a/1-\gamma/2}},
\]
and by taking \( a_1 = a/2, b_1 = b/2 \) and \( w_1 = \gamma/2 + a/2, w_2 = \gamma/2 + b/2 \), we derive that
\[
|\langle Q(g, h), f \rangle| \lesssim C(g) (\|(-\Delta_{SS})^{a/2} h\|_{L^2_{1/2}} + \|h\|_{H^{a/2}_1} + \|h\|_{L^{2a/1-\gamma/2}}) \times (\|(-\Delta_{SS})^{b/2} f\|_{L^2_{1/2}} + \|f\|_{H^{b/2}_1} + \|f\|_{L^{2a/1-\gamma/2}}) + \|h\|_{H^{a/2}_1} \|f\|_{H^{b/2}_1}.
\]
where in the equivalence we use the facts \[1.20\] and \[1.21\]. Then \[1.22\] is proved and our results are sharp in anisotropic spaces.

Remark 1.14. By taking \( a = b = s, a_1 = s, a_2 = 0, w_1 = \gamma/2, w_2 = \gamma/2 + s \), we can deduce that for any \( \eta > 0 \),
\[
|\langle Q(g, h), f \rangle| \lesssim C(g) (\|(-\Delta_{SS})^{s/2} h\|_{L^2_{1/2}} + \eta^{-1} \|h\|_{H^{\gamma/2}_1}) \times (\|(-\Delta_{SS})^{s/2} f\|_{L^2_{1/2}} + \|f\|_{H^{a/2}_1} + \eta \|f\|_{L^{2a/1-\gamma/2}}).
\]
Thanks to the symmetry property for functions \( h \) and \( f \) in the estimates, we also have
\[
|\langle Q(g, h), f \rangle| \lesssim C(g) (\|(-\Delta_{SS})^{s/2} h\|_{L^2_{1/2}} + \|h\|_{H^{a/2}_1} + \eta \|h\|_{L^{2a/1-\gamma/2}}) \times (\|(-\Delta_{SS})^{s/2} f\|_{L^2_{1/2}} + \eta^{-1} \|f\|_{H^{\gamma/2}_1}).
\]
We remark that both estimates improve the previous upper bounds in the two senses: the first one is that we only need to assume that one of \( h \) and \( f \) in the space \( L^2_{1/2+\gamma/2} \); the second one is the free choice of the constant \( \eta \) in the theorem which enables us to prove that \[1.38\] is false.

1.4. Organization of the paper. In Section 2, based on two types of the dyadic decompositions performed both in phase and frequency spaces, we give a complete proof to the sharp bounds of the collision operator in weighted Sobolev spaces.

Based on the geometric decomposition, we give a proof for the sharp lower and upper bounds of the collision operator in the anisotropic spaces in Section 3.

In Section 4, we show that the strategy is so stable that we can generalize all the estimates to the Landau collision operator by taking the grazing collisions limit. We also show some asymptotic behavior of the anisotropic structure of collision operator in the process of the grazing collisions limit.
In Section 5, we list some important lemmas which are of independent interest to the proof of the main theorems. We first give some auxiliary lemmas on the new profiles of the weighted Sobolev Spaces, new version of the interpolation theory and the basic properties of the real spherical harmonics. Then in the next we give some equivalent profiles of the fractional Laplace-Beltrami operator. Finally we give the commutator estimate between the Laplace-Beltrami operator and the translation operator, i.e. the proof of (1.31).

At the end of the paper, we give the conclusions and perspectives.

2. Upper bound for the collision operator in weighted Sobolev Spaces

In this section, we will make full use of dyadic decompositions which are performed in both Phase and Frequency spaces to give the precise estimates of the collision operator in weighted Sobolev spaces. Of course these estimates are not optimal. But they are still interesting. In fact, they have two advantages. The first one is that we have the freedom of choosing derivatives and weights for the functions compared to the previous work. The second one is that two types of decompositions used in the proof enable us to find out where the additional weight come from in the upper bound. Then this observation will be used in the next sections to improve the upper bound of the operator in the anisotropic spaces.

Before going further, we first prove some useful propositions.

Proposition 2.1. Suppose $\varpi \in (0, 1]$. Then for $|y| \neq 0$, one has

$$
||x|^{2\varpi} - |y|^{2\varpi}| \lesssim \begin{cases} 
|x - y||y|^{2\varpi - 1}, & \text{if } 0 < \varpi \leq \frac{1}{2}; \\
|x - y||y|^{2\varpi - 1} + |x - y|^2, & \text{if } \frac{1}{2} < \varpi \leq 1.
\end{cases}
$$

Proof. We first treat the case $2\varpi \leq 1$. Suppose that $|x| \leq b|y|$ with $0 < b < 1$. Then one has

$$(1 - b)|y| \leq |x - y| \leq (1 + b)|y|,$$

which implies

$$||x|^{2\varpi} - |y|^{2\varpi}| \lesssim |y|^{2\varpi} \lesssim |x - y||y|^{2\varpi - 1}.$$ 

Next we handle the case $|x| > b|y|$. We have

$$(1 - \theta)|x| + \theta|y| \geq |(1 - \theta)b + \theta||y|.$$ 

From which together with the fact

$$||x|^{2\varpi} - |y|^{2\varpi}| \lesssim |x - y| \int_0^1 [(1 - \theta)|x| + \theta|y||y|^{2\varpi - 1} d\theta,$$

we get the desired result.

When $2\varpi > 1$, the proposition is easily followed from (2.1) and the fact

$$(1 - \theta)|x| + \theta|y| \leq |y| + |x - y|.$$ 

□

Proposition 2.2. Suppose $\varpi \in (0, 1]$, $N \in \mathbb{N}$ and $\Phi_k^\gamma$ is defined in (1.24).

(1) Set $A_k^\gamma(v) \overset{\text{def}}{=} (\mathcal{F}^{\Omega}_p \Phi_k^\gamma)(v)|v|^{2\varpi}$. Then if $k \geq 0$, we have

$$\|A_k^\gamma\|_{L^\infty} \lesssim 2^{k(\gamma + \frac{1}{2} - N)} 2^{-pN} \|\Phi_0^\gamma\|_{H^{N+2}} \|\psi\|_{W^{2,\infty}_N}.$$ 

If $\gamma = 0$ and $k = -1$,

$$\|A_{-1}^\gamma\|_{L^\infty} \lesssim 2^{-pN} \|\psi\|_{H^{N+2}} \|\phi\|_{W^{2,\infty}_N}.$$
(2) Set $B_{-1}^0(v) = \tilde{\Phi}_k^Y(\xi)(v)\|v\|^{2\varnothing - (\tilde{\Phi}_k^Y(\xi)(v))}. Then there exists a constant $\eta$ depending only on $\gamma$ and $\varnothing$ such that if $\gamma + 2\varnothing > 0$, then

$$|B_{-1}^0| \leq |B_1| + |B_2|,$$

where

$$\|B_1\|_{L^2} \leq 2^{-(\eta + \frac{1}{2})p} \quad \text{and} \quad \|B_2\|_{L^\infty} \leq 2^{-\eta p},$$

and if $-1 < \gamma + 2\varnothing \leq 0$, then

$$\|B_{-1}^0\|_{L^2} \lesssim 2^{-(\eta + \frac{1}{2})p}.$$ 

Proof. (i). For $k \geq 0$, recall that $\Phi_k^Y(\nu) = |v|^p \varnothing(2^{-k} v)$, then by direct calculation, we have

$$\|\tilde{\Phi}_k^Y(\xi) = 2^{(\gamma + 3)k} \|\tilde{\Phi}_k^Y(2^k \xi),$$

which yields

$$\|\tilde{\Phi}_k^Y(\xi)_{L^\infty} \lesssim \|(-\Delta) \|\tilde{\Phi}_k^Y(\xi)_{L^2} + \|\tilde{\Phi}_k^Y(\xi)_{L^2} \|_{H^2} \lesssim \|\Phi_k^Y(2^k \xi)_{H^{N+2}} \|_{W^{2,\infty}}.$$ 

From which, we obtain

$$\|B_{-1}^0\|_{L^\infty} \lesssim 2^{k(\gamma + \frac{3}{2} - N) - 2^{-pN} \|\Phi_k^Y(2^k \xi)_{H^{N+2}} \|_{W^{2,\infty}}.$$ 

If $\gamma = 0$ and $k = -1$, by the definition, we have

$$\|B_{-1}^0\|_{L^\infty} \lesssim \|(-\Delta) \|\Phi_k^Y(\xi)_{L^2} + \|\tilde{\Phi}_k^Y(\xi)_{L^2} \|_{H^2} \lesssim 2^{-pN} \|\Phi_k^Y(2^k \xi)_{H^{N+2}} \|_{W^{2,\infty}}.$$ 

(ii). Next we treat the case $k = -1$. Let $\tilde{\Phi}_k = \sum_{|k-p| \leq N} \Phi_k$. Then by the definition of $\tilde{\Phi}_k$, we have

$$|B_{-1}^0(v) = \int \tilde{\Phi}_p(v - y)\Phi_{-1}^Y(y)|v|^{2\varnothing - |y|2\varnothing)dy|.$$ 

Thanks to Proposition [2.1] it can be reduced to bound the terms $B_{-1}^1$ and $B_{-1}^2$ which are defined by

$$B_{-1}^1 = \int |\tilde{\Phi}_p(v - y)\Phi_{-1}^Y(y)|v - y|dy,$$

and $B_{-1}^2 = \int |\tilde{\Phi}_p(v - y)\Phi_{-1}^Y(y)|v - y|\varnothing dy.$

We remark the reader that the term $B_{-1}^2$ is only needed to be considered in the case of $1/2 < \varnothing \leq 1.$

We begin with the estimate of $B_{-1}^1$. Observe that

$$B_{-1}^1 = \int |\tilde{\Phi}_p(v - y)\Phi_{-1}^Y(y)|v - y|dy$$

$$+ \int |\tilde{\Phi}_p(v - y)\Phi_{-1}^Y(y)|v - y|\varnothing dy$$

$$\|B_{-1}^1\| \lesssim \left(\int |\tilde{\Phi}_p(v - y)|^2 dy\right)^{1/2} 2^{-(\gamma + 2\varnothing - 1) + \frac{3}{2}p} \leq 2^{-(\gamma + 2\varnothing + \frac{1}{2})p}.$$ 

We turn to give the estimate to $B_{-1}^{1,2}$. To make full of the structure, we separate the estimate into several cases.

- Case 1: $\gamma + 2\varnothing > 0$. Then it holds

$$\|B_{-1}^{1,2}\|_{L^\infty} \lesssim 2^{(1 - \eta_1)p} \int |\tilde{\Phi}_p(v - y)|\Phi_{-1}^Y(y)|v - y|dy \lesssim 2^{-\eta_1 p},$$

where $\eta_1 = min\{1, \gamma + 2\varnothing\}$. 


• Case 2: \( -\frac{1}{2} < \gamma + 2\omega \leq 0 \). Then by Young inequality, it follows

\[
\|B_{1,2}^{1.2}\|_{L^2} \lesssim 2^{-\gamma} \left( \int |\tilde{p}(v-y)|^2 |y|^{2(2\omega - 1)} \right)^{1/2} d y
\]

\[
\lesssim 2^{-\gamma} 2^{-\gamma + 2(\omega + 1)p} \lesssim 2^{-\delta_2},
\]

where \( \delta_2 = \gamma + 3 + \frac{1}{2} \).

• Case 3: \( -1 < \gamma + 2\omega \leq -\frac{1}{2} \). We have

\[
\|B_{1,2}^{1.2}\|_{L^2} \lesssim 2^{-\gamma} \left( \int |\tilde{p}(v-y)|^2 |y|^{2(2\omega - 1)} \right)^{1/2} d y
\]

\[
\lesssim 2^{-\gamma + 2(\omega + 1)p} \lesssim 2^{-\delta_3},
\]

where \( \delta_3 = \delta_4 = (\gamma + 2\omega + 1)/2 \).

The similar argument can be applied to \( B_{2,1}^2 \) with \( 2\omega \geq 1 \). Suppose

\[
B_{2,1}^2 = \int |\tilde{p}(v-y)|^2 |y|^{2\omega} d y + \int |\tilde{p}(v-y)|^2 |y|^{2\omega} d y
\]

\[
= B_{2,1}^{1.2} \quad \text{def} = B_{2,1}^{1.2} + B_{2,1}^{1.2}.
\]

It is easy to check

\[
\|B_{2,1}^{1.2}\|_{L^2} \lesssim \left( \int |\tilde{p}(v-y)|^2 |y|^{2\omega} d y \right)^{1/2} 2^{-\gamma} \lesssim 2^{-\delta_5}\]

Similarly, the estimate of \( B_{2,1}^{1.2} \) falls in several cases.

• Case 1: \( \gamma \geq 0 \). Then it holds

\[
\|B_{2,1}^{1.2}\|_{L^\infty} \lesssim \int |\tilde{p}(v-y)|^2 |y|^{2\omega} d y \lesssim 2^{-\delta_4},
\]

where \( \delta_5 = 2\omega \).

• Case 2: \( \gamma + 2\omega > 0 \) and \( \gamma < 0 \). Use the fact \( |\tilde{p}(v-y)|^2 |y|^{2\omega} \leq 2^{-\gamma p} \) , then one has

\[
\|B_{2,1}^{1.2}\|_{L^\infty} \lesssim 2^{-\gamma p} \int |\tilde{p}(v-y)|^2 |y|^{2\omega} d y \lesssim 2^{-\delta_6},
\]

where \( \delta_6 = \gamma + 2\omega \).

• Case 3: \( \gamma + 2\omega \leq 0 \) and \( -\frac{3}{2} < \gamma \leq 0 \). By Young inequality, we have

\[
\|B_{2,1}^{1.2}\|_{L^2} \lesssim \left( \int |\tilde{p}(v-y)|^2 |y|^{2\omega} d y \right)^{1/2}, \quad \left( \int |\tilde{p}(v-y)|^2 |y|^{2\omega} d y \right) \lesssim 2^{-2\omega} = 2^{-\delta_7}.
\]

where \( \delta_7 = \omega - \frac{1}{2} \).

• Case 4: \( \gamma + 2\omega \leq 0 \) and \( \gamma \leq -\frac{3}{2} \). Again by Young inequality, we obtain

\[
\|B_{2,1}^{1.2}\|_{L^2} \lesssim \left( \int |\tilde{p}(v-y)|^2 |y|^{2\omega} d y \right)^{1/2} \left( \int |\tilde{p}(v-y)|^2 |y|^{2\omega} d y \right) \lesssim 2^{-2\omega} 2^{-\gamma},
\]

where \( \delta_8 = \delta_9 = (\gamma + 2\omega + 1)/2 \).

Notice that there are only two types of the estimates, \( L^2 \) and \( L^\infty \) estimates, in the proof, then the proposition is easily followed by patching together all the estimates.

\( \square \)
2.1. **Estimates of $\mathfrak{M}^1_{k,p,l}$ and $\mathfrak{M}^2_{k,p,m}$ defined in (1.27).** Now we want to give the estimates to $\mathfrak{M}^1_{k,p,l}$.

**Lemma 2.1.** Suppose $N \in \mathbb{N}$. For $k \geq 0$, it holds

$$|\mathfrak{M}^1_{k,p,l}| \lesssim 2^{k(\gamma + \frac{1}{2} - N)}(2^{-p(N-2s)}2^{2s(l-p)} + 2^{-(N-\frac{3}{2})p}2^{l(l-p)})$$

$$\times \|\Phi_0\|_{H^{N+2}}\|\varphi\|_{W^{2,\infty}_N}\|\mathfrak{S}_{p}\|_{L^1} \|\mathfrak{S}_1\|_{L^2} \|\tilde{\mathfrak{S}}_p\|_{L^2} \|\mathfrak{S}_p f\|_{L^2}.$$  

If $k = -1$, we have

1. If $\gamma = 0$,

$$|\mathfrak{M}^1_{-1,p,l}| \lesssim (2^{-p(N-2s)}2^{2s(l-p)} + 2^{-(1-N)p}2^{1})$$

$$\times \|\varphi\|_{H^{N+2}}\|\varphi\|_{W^{2,\infty}_N}\|\mathfrak{S}_{p}\|_{L^1} \|\mathfrak{S}_1\|_{L^2} \|\tilde{\mathfrak{S}}_p\|_{L^2} \|\mathfrak{S}_p f\|_{L^2}.$$  

2. If $\gamma + 2s > 0$ and $\gamma > -\frac{3}{2}$,

$$|\mathfrak{M}^1_{-1,p,l}| \lesssim 2^{(\gamma - p)2^{2s}}\|\mathfrak{S}_{p}\|_{L^1} \|\mathfrak{S}_1\|_{L^2} \|\tilde{\mathfrak{S}}_p\|_{L^2} \|\mathfrak{S}_p f\|_{L^2},$$

3. If $\gamma + 2s > 0$ and $\gamma \leq -\frac{3}{2}$,

$$|\mathfrak{M}^1_{-1,p,l}| \lesssim 2^{(\gamma - p)2^{2s}}\|\mathfrak{S}_{p}\|_{L^1} + 2^{(\gamma - 2^{s})\frac{1}{2}}\|\mathfrak{S}_{p}\|_{L^2} \|\mathfrak{S}_1\|_{L^2} \|\tilde{\mathfrak{S}}_p\|_{L^2} \|\mathfrak{S}_p f\|_{L^2},$$

4. If $\gamma + 2s \leq 0$ and $\gamma > -\frac{3}{2}$,

$$|\mathfrak{M}^1_{-1,p,l}| \lesssim 2^{(\gamma - 2^{s})\frac{1}{2}}\|\mathfrak{S}_{p}\|_{L^2} \|\mathfrak{S}_1\|_{L^2} \|\tilde{\mathfrak{S}}_p\|_{L^2} \|\mathfrak{S}_p f\|_{L^2},$$

5. If $\gamma + 2s \leq 0$,

$$|\mathfrak{M}^1_{-1,p,l}| \lesssim 2^{(\gamma - 2^{s})\frac{1}{2}}\|\mathfrak{S}_{p}\|_{L^2} \|\mathfrak{S}_1\|_{L^2} \|\tilde{\mathfrak{S}}_p\|_{L^2} \|\mathfrak{S}_p f\|_{L^2},$$

where $\eta$ is a constant which depends only on $\gamma$ and $s$ and varies for different case. Functions $\psi$ and $\varphi$ are defined in (1.23).

**Proof.** We recall that the indexes $l$ and $p$ of $\mathfrak{M}^1_{k,p,l}$ verify the condition $l < p - N_0$. To make full use of the cancellation of frequency and handle the singularity caused by the angular function, we make the following decomposition:

$$\mathfrak{M}^1_{k,p,l} = \mathfrak{D}^1_{k} + \mathfrak{D}^2_{k},$$

where

$$\mathfrak{D}^1_{k} \overset{\text{def}}{=} \int_{\sigma \in S^2, v_*, v \in \mathbb{R}^3} \langle \mathfrak{S}_{p}\Phi_k \rangle (|v - v_*|)b(\cos \theta)(\mathfrak{S}_{p}\varphi)_* \langle (\mathfrak{S}_1 h) - (\mathfrak{S}_1 h)' \rangle (\tilde{\mathfrak{S}}_p f)' d\sigma dv_* dv,$$

$$\mathfrak{D}^2_{k} \overset{\text{def}}{=} \int_{\sigma \in S^2, v_*, v \in \mathbb{R}^3} \langle \mathfrak{S}_{p}\Phi_k \rangle (|v - v_*|)b(\cos \theta)(\mathfrak{S}_{p}\varphi)_* \langle (\mathfrak{S}_1 h)'(\tilde{\mathfrak{S}}_p f)' - (\mathfrak{S}_1 h)\tilde{\mathfrak{S}}_p f\rangle d\sigma dv_* dv.$$

The proof falls in several steps.

**Step 1: Estimate of $\mathfrak{D}^1_{k}$.** Observe the facts

$$(\mathfrak{S}_1 h)(v) - (\mathfrak{S}_1 h)(v')$$

(2.3) $$= (v - v') \cdot (\nabla (\mathfrak{S}_1 h))(v') + \frac{1}{2} \int_0^1 (1 - \kappa)(v - v') \otimes (v - v') : (\nabla^2 (\mathfrak{S}_1 h))(\kappa(v)) d\kappa,$$

where $\kappa(v) = v' + \kappa(v - v')$, and

$$\int_{\sigma \in S^2, v \in \mathbb{R}^3} \Gamma(|v - v_*|)b(\frac{v - v_*}{|v - v_*|}) \omega(|v' - v|)(v - v')\rho(v') d\sigma dv$$

$$= 4 \int_{\sigma \in S^2, v \in \mathbb{R}^3} \Gamma(|T_{\sigma}(v') - v_*|)b(\frac{T_{\sigma}(v') - v_*}{|T_{\sigma}(v') - v_*|}) \omega(|v' - T_{\sigma}(v')|) \frac{T_{\sigma}(v') - v'}{|v' - v_*|} \rho(v') d\sigma dv' = 0,$$
where \( w \) and \( \rho \) are smooth functions and \( T_{\sigma}(v') \) represents the transform such that \( T_{\sigma}(v') = v \). We refer the readers to [1] or [7] to check the change of variable from \( v \) to \( v' \). Then in order to get the optimal estimate, we follow the idea of [10] to introduce the function \( \psi \) (defined in [1.23]) to decompose \( \mathcal{D}^1_k \) into angular cutoff part and angular non cutoff part, that is,

\[
\mathcal{D}^1_k = \mathcal{D}^{1,1}_k + \mathcal{D}^{1,2}_k,
\]

where

\[
\mathcal{D}^{1,1}_k \overset{\text{def}}{=} \int_0^1 \int_{\sigma \in \mathbb{S}^2} |v - v_*|^2 \psi(2^l (v' - v)) (\mathfrak{F} p g)_* \times |v - v_*|^2 \psi(2^l (v' - v)) (\mathfrak{F} p f)' d\sigma d v_* d v d k,
\]

\[
\mathcal{D}^{1,2}_k \overset{\text{def}}{=} \int_0^1 \int_{\sigma \in \mathbb{S}^2} |v - v_*|^2 \psi(2^l (v' - v)) (\mathfrak{F} p g)_* \times |v - v_*|^2 \psi(2^l (v' - v)) (\mathfrak{F} p f)' d\sigma d v_* d v d k,
\]

where \( A^s_k \) is defined in the Proposition 2.2 with \( \omega = s \).

**Step 1.1: Estimate of \( \mathcal{D}^{1,1}_k \)** We divide the estimate into three cases.

**Case 1:** \( k \geq 0 \). By Cauchy-Schwarz’s inequality, one has

\[
|\mathcal{D}^{1,1}_k| \lesssim \|A^s_k\|_{L^\infty} \left( \int_0^1 \int_{\sigma \in \mathbb{S}^2} |(\mathfrak{F} p g)_*| |b(2^l (v' - v))| |(\mathfrak{F} p f)'| |(\mathfrak{F} p f)'|^{2} d\sigma d v_* d v d k \right)^{1/2}.
\]

where we use the fact \( |v - v'| = |v - v_*| \sin(\theta/2) \).

Then we follow the change of variables: \((v_* , v) \to (v_* , u_1 = v')\) and \((v_* , v) \to (v_* , u_2 = \kappa(v))\). Thanks to the fact

\[
\frac{\partial u_2}{\partial v} = (1 - \frac{\kappa}{2}) \{ (1 - \frac{\kappa}{2}) \frac{v - v_*}{|v - v_*|} - \sigma \},
\]

we derive

\[
|\mathcal{D}^{1,1}_k| \lesssim \|A^s_k\|_{L^\infty} \left( \int_0^1 \int_{\sigma \in \mathbb{S}^2} |(\mathfrak{F} p g)_*| |b(2^l (v' - v) - \sin(\theta/2)| |(\mathfrak{F} p f)'| |(\mathfrak{F} p f)'|^{2} d\sigma d v_* d v d k \right)^{1/2}.
\]

where we use the fact \( |v - v_*| \sim |u_1 - v_*| \sim |u_2 - v_*| \). It is not difficult to check

\[
|v - v_*|^{2-2s} \int_{\sigma \in \mathbb{S}^2} b(\cos(\theta)) \sin^{2}(\theta/2) \psi(2^l |v - v_*| \sin(\theta/2)) d\sigma
\]

\[
\lesssim |u_2 - u_*|^{2-2s} \int_0^{\theta} b(\cos(\theta)) \sin(\theta) d\theta d\sigma
\]

\[
\lesssim 2^{-(2-s)l},
\]

where \( \theta \) verifies \( \cos(\theta) = \frac{u_2 - u_*}{|u_2 - u_*|} \) and \( \theta/2 \leq \theta \leq \theta \). By Bernstein inequalities [5.1, 5.3], it is easy to derive

\[
|\mathcal{D}^{1,1}_k| \lesssim 2^{2sl} \|A^s_k\|_{L^\infty} \|\mathfrak{F} p g\|_{L^2} \|\mathfrak{F} h\|_{L^2} \|\mathfrak{F} p f\|_{L^2}
\]

\[
\lesssim 2^{(s+\frac{1}{2} - N)2 - p(N-2s)} 2^{2s(l-p)} \|\Phi_0\|_{H^{N-2s}} \|\varphi\|_{W^{N-2s}} \|\mathfrak{F} p g\|_{L^1} \|\mathfrak{F} h\|_{L^2} \|\mathfrak{F} p f\|_{L^2}.
\]
Case 2: $k = -1$ and $\gamma = 0$. Thanks to Proposition 2.2, we easily get
\[
|\mathcal{D}_{-1}^{1,1}| \lesssim 2^{-p(N-2s)2s(l-p)}\|\psi\|_{L^2}\|\phi\|_{W^{2s,\infty}}\|\tilde{\delta}_p g\|_{L^2}\|\tilde{f}\|_{L^2}\|\tilde{\delta}_p f\|_{L^2}.
\]

Case 3: $k = -1$ and $\gamma \neq 0$. For the general case, following the decomposition in Proposition 2.2
\[
A_{-1}^s = B^s_{-1} + \tilde{\delta}_p \Phi_{-1}^{Y+2s},
\]
one has
\[
\mathcal{D}_{-1}^{1,1} = \mathcal{D}_{-1,1}^{1,1} + \mathcal{D}_{-1,2}^{1,1}.
\]
We first have
\[
|\mathcal{D}_{-1,2}^{1,1}| \lesssim \left( \int_0^1 \int_{\sigma \in SS^2, v_1, v_2 \in \mathbb{R}^3} \left| (\tilde{\delta}_p \Phi_{-1}^{Y+2s}) (|u_1|) \right|^2 b(\cos \theta) \sin^2(\theta/2) |\psi(2^l (v' - v))| \right.
\]
\[
	imes |v - v_s|^{2-2s} (|\tilde{\delta}_p f|)^2 d\sigma d v_1 d v_2 d\varphi \left. \right) \int_0^1 \int_{\sigma \in SS^2, v_1, v_2 \in \mathbb{R}^3} \left| (\tilde{\delta}_p g)_{*} \right|^2 b(\cos \theta)
\]
\[
	imes \sin^2(\theta/2) |\psi(2^l (v' - v))| |v - v_s|^{2-2s} (|\nabla^2 \tilde{f} h(\kappa(v))|) |d\sigma d v_1 d v_2 d\varphi \right)^{1/2}.
\]

By change of variables, we derive
\[
|\mathcal{D}_{-1,2}^{1,1}| \lesssim \left( \int_0^1 \int_{\sigma \in SS^2, u_1, u_2 \in \mathbb{R}^3} \left| (\tilde{\delta}_p \Phi_{-1}^{Y+2s}) (|u_1|) \right|^2 b(\cos \theta) \sin^2(\theta/2) |\psi(2^l |u_1| \sin(\theta/2))| \right.
\]
\[
	imes |u_1|^{2-2s} (|\tilde{\delta}_p f| (u_2))^2 d\varphi du_1 du_2 d\varphi \left. \right) \int_0^1 \int_{\sigma \in SS^2, u_1, u_2 \in \mathbb{R}^3} \left| (\tilde{\delta}_p g)_{*} \right|^2 b(\cos \theta)
\]
\[
	imes \sin^2(\theta/2) |\psi(2^l (v - v_s) \sin(\theta/2))| |u_1 - u_1|^{2-2s} (|\nabla^2 \tilde{f} h(u_3)|) |d\varphi u_1 d\varphi u_2 d\varphi \right)^{1/2}.
\]

Then we have
\[
|\mathcal{D}_{-1,2}^{1,1}| \lesssim 2^{2s} \|\tilde{\delta}_p \Phi_{-1}^{Y+2s}\|_{L^2} \|\tilde{\delta}_p g\|_{L^2} \|\tilde{f}\|_{L^2} \|\tilde{\delta}_p f\|_{L^2}.
\]

Due to the fact (see [5]) that for $\gamma > -3$, there holds for all integer $k$
\begin{equation}
|\{D^k \mathcal{F} (\Phi_{-1}^{Y}) \}(\xi)| \lesssim \langle \xi \rangle^{-3-3-\gamma-k},
\end{equation}
which yields
\[
\|\tilde{\delta}_p \Phi_{-1}^{Y+2s}\|_{L^2} \lesssim \int_{\mathbb{R}^3} |\psi_p(\xi)|^2 |\xi|^{-2(\gamma+2s)-6} d\xi
\]
\[
\lesssim 2^{-2(\gamma+2s+\frac{6}{2})}.
\]

We deduce
\[
|\mathcal{D}_{-1,2}^{1,1}| \lesssim 2^{2s} 2^{-2(\gamma+2s+\frac{6}{2})} \|\tilde{\delta}_p g\|_{L^2} \|\tilde{f}\|_{L^2} \|\tilde{\delta}_p f\|_{L^2}.
\]

Due to the Bernstein inequalities (5.1) (5.3), it follows if $\gamma + 2s > 0$, then
\[
|\mathcal{D}_{-1,2}^{1,1}| \lesssim 2^{2s} 2^{-(\gamma+2s)p} \|\tilde{\delta}_p g\|_{L^1} \|\tilde{f}\|_{L^2} \|\tilde{\delta}_p f\|_{L^2}.
\]
And if $\gamma + 2s \leq 0$, then
\[
|\mathcal{D}_{-1,2}^{1,1}| \lesssim 2^{2s} 2^{-(\gamma+2s+\frac{6}{2})} \|\tilde{\delta}_p g\|_{L^2} \|\tilde{f}\|_{L^2} \|\tilde{\delta}_p f\|_{L^2}.
\]

Next we turn to the estimate of $\mathcal{D}_{-1,1}^{1,1}$. Notice that $B_{-1}$ can be separated into two parts $B_1$ and $B_2$, which separately can be controlled in $L^2$ space and in $L^\infty$ space in Proposition 2.2. Thus we may copy
the argument for $\mathcal{D}_{-1,1}^{1,1}$ to that for $\mathcal{D}_{-1,1}^{1,1}$ when $B_1$ is bounded in $L^2$ space and apply the argument for $\mathcal{D}_{-1,2}^{1,1}$ to that for $\mathcal{D}_{-1,1}^{1,1}$ when $B_2$ is controlled in $L^\infty$ space. Finally we are led to in the case of $\gamma_+ 2s > 0$,

$$|\mathcal{D}_{-1,1}^{1,1}| \lesssim 2^{2sl} (|B_1| L^2 + |B_2| L^\infty) \|F_p g\|_{L^2} \|F_{\tilde{h}} h\|_{L^2} \|\tilde{F}_p f\|_{L^2} .$$

While in the case of $\gamma_+ 2s \leq 0$, one has

$$|\mathcal{D}_{-1,1}^{1,1}| \lesssim 2^{(2s-\frac{1}{2})l/2 (l-p)} 2^{-\eta p} \|F_p g\|_{L^2} \|F_{\tilde{h}} h\|_{L^2} \|\tilde{F}_p f\|_{L^2} .$$

\textbf{Step 1.2: Estimate of $\mathcal{D}_{k}^{1,2}$.} We separate the bound into several cases. We begin with the case $k \geq 0$. \textit{Case 1: $k \geq 0$.} By Cauchy-Schwarz inequality, one has

$$|\mathcal{D}_{k}^{1,2}| \lesssim \|A_k^s\|_{L^\infty} \left( \int_0^1 \int_{\sigma \in S^2} |\tilde{F}_p g| \|b(\cos \theta)| 1 - \psi(2^l (\nu - \nu))| d\sigma d\nu d\nu dk \right)^{\frac{1}{2}} \left( \int_0^1 \int_{\sigma \in S^2} |\tilde{F}_p g| \|b(\cos \theta)| 1 - \psi(2^l (\nu - \nu))| d\sigma d\nu d\nu dk \right)^{\frac{1}{2}}$$

$$\times |v - \nu|^{-2s} |(\tilde{F}_p f)(\nu)|^2 d\sigma d\nu d\nu dk \right)^{\frac{1}{2}} \left( \int_0^1 \int_{\sigma \in S^2} |\tilde{F}_p g| \|b(\cos \theta)| 1 - \psi(2^l (\nu - \nu))| d\sigma d\nu d\nu dk \right)^{\frac{1}{2}}$$

$$\times |v - \nu|^{-2s} |(\tilde{F}_p f)(\nu)|^2 d\sigma d\nu d\nu dk \right)^{\frac{1}{2}} \left( \int_0^1 \int_{\sigma \in S^2} |\tilde{F}_p g| \|b(\cos \theta)| 1 - \psi(2^l (\nu - \nu))| d\sigma d\nu d\nu dk \right)^{\frac{1}{2}}$$

Follow the change of variables: $(\nu, \nu) \rightarrow (\nu, \nu_2 = \nu')$ and the fact (2.4), then we get

$$|\mathcal{D}_{k}^{1,2}| \lesssim \|A_k^s\|_{L^\infty} \left( \int_0^1 \int_{\sigma \in S^2} |\tilde{F}_p g| \|b(\cos \theta)| 1 - \psi(2^l |v - \nu| |\sin(\theta/2))| d\sigma d\nu d\nu dk \right)^{\frac{1}{2}} \left( \int_0^1 \int_{\sigma \in S^2} |\tilde{F}_p g| \|b(\cos \theta)| 1 - \psi(2^l |v - \nu| |\sin(\theta/2))| d\sigma d\nu d\nu dk \right)^{\frac{1}{2}}$$

$$\times |v - \nu|^{-2s} |(\tilde{F}_p f)(\nu)|^2 d\sigma d\nu d\nu dk \right)^{\frac{1}{2}} \left( \int_0^1 \int_{\sigma \in S^2} |\tilde{F}_p g| \|b(\cos \theta)| 1 - \psi(2^l |v - \nu| |\sin(\theta/2))| d\sigma d\nu d\nu dk \right)^{\frac{1}{2}}$$

Notice

$$|v - \nu|^{-2s} \int_{\sigma \in S^2} b(\cos \theta) (1 - \psi(2^l |v - \nu| |\sin(\theta/2))| d\sigma$$

$$\lesssim \left\{ \begin{array}{ll} |v - \nu|^{-2s} \int_{|v - \nu|^{-1}}^{\frac{\pi}{2}} b(\cos \theta) \sin \theta d\theta, \\
|u_2 - \nu|^{-2s} \int_{|u_2 - \nu|^{-1}}^{\frac{\pi}{2}} b(\cos \theta) \sin \theta d\theta \\
\end{array} \right.$$
where $\bar{\theta}$ verifies $\cos \bar{\theta} = \frac{u_2 - v_2}{|u_2 - v_2|} \cdot \sigma$ and $\theta/2 \leq \bar{\theta} \leq \theta$. Finally we get the estimates to $\mathcal{D}_{k}^{1,2}$, that is, for any $N \in \mathbb{R}^+$,

$$
|\mathcal{D}_{k}^{1,2}| \lesssim 2^{(y + \frac{1}{2} - N)k} 2^{2s(l-p)} 2^{(2s-N)p} \|\phi\|_{H^{N+\varepsilon}} \|\Phi_{0}\|_{H^{N+\varepsilon}} \|\widehat{\phi} p\|_{L^1} \|\widehat{\phi} h\|_{L^2} \|\widehat{\phi} f\|_{L^2}.
$$

**Case 2:** $k = -1$. Following the similar argument used in the previous step, we conclude that in the case $\gamma = 0$,

$$
|\mathcal{D}_{-1}^{1,2}| \lesssim 2^{2s(l-p)} 2^{(2s-N)p} \|\psi\|_{H^{N+\varepsilon}} \|\phi\|_{W^{2,\infty}_N} \|\widehat{\phi} p\|_{L^1} \|\widehat{\phi} h\|_{L^2} \|\widehat{\phi} f\|_{L^2},
$$

in the case of $\gamma + 2s > 0$,

$$
|\mathcal{D}_{-1}^{1,2}| \lesssim 2^{2s(l-p)} 2^{(2s-N)p} \|\psi\|_{H^{N+\varepsilon}} \|\phi\|_{W^{2,\infty}_N} \|\widehat{\phi} p\|_{L^1} \|\widehat{\phi} h\|_{L^2} \|\widehat{\phi} f\|_{L^2},
$$

and in the case of $\gamma + 2s \leq 0$,

$$
|\mathcal{D}_{-1}^{1,2}| \lesssim 2^{(2s-\frac{1}{2})} 2^{2s(l-p)} \|\psi\|_{L^1} \|\phi\|_{W^{2,\infty}_N} \|\widehat{\phi} p\|_{L^1} \|\widehat{\phi} h\|_{L^2} \|\widehat{\phi} f\|_{L^2}.
$$

**Step 2:** Estimate of $\mathcal{D}_{k}^2$. Thanks to the Cancellation Lemma in [1], we obtain

$$
\mathcal{D}_{k}^2 = |\int_{\theta, \in [0, \pi], \nu, \nu, \nu, \nu, \nu, \nu \in \mathbb{R}^3 \int_{\nu, \nu, \nu, \nu, \nu, \nu} \Phi_{k}^\gamma \left( \frac{1}{\cos \frac{\nu}{2}} - \left( \Phi_{k}^\gamma \right) \left( \frac{1}{2} \cos \theta \right) \right) b(\cos \theta) \sin \theta
\times \left( \widehat{\phi} p \right) \left( \widehat{\phi} h \right) \left( \widehat{\phi} f \right) d\theta d\nu d\nu d\nu d\nu d\nu.
$$

Notice that

$$
C_k(\theta, \xi) = \frac{1}{\cos^3 \frac{\theta}{2}} \mathcal{F} \left( \left( \Phi_{k}^\gamma \right) \left( \frac{1}{\cos \frac{\theta}{2}} \right) \right)(\xi) - \mathcal{F} \left( \left( \Phi_{k}^\gamma \right) \left( \frac{1}{2} \cos \theta \right) \right)(\xi)
= \phi_p(\xi) 2^{(\gamma+3)k} \left( \mathcal{F} \left( \Phi_{k}^\gamma \right)(\xi) \cos \frac{\theta}{2} 2^k \xi - \mathcal{F} \left( \Phi_{k}^\gamma \right)(\xi) 2^k \xi \right),
$$

where we use (2.2). We split the estimate into several cases.

**Case 1:** $k \geq 0$ or $k = -1$ with $\gamma = 0$. Thanks to the mean value theorem, we have

$$
\|C_k(\theta, \cdot)\|_{L^2} \lesssim \theta^2 2^{(y + \frac{1}{2} - N)k} 2^{(1-N)p} \|\Phi_{k}^\gamma \|_{H^{N+\varepsilon}} \|\phi\|_{W^{2,\infty}_N}
$$

and

$$
\|C_{-1}(\theta, \cdot)\|_{L^2} \lesssim \theta^2 2^{(y + \frac{1}{2} - N)k} 2^{(1-N)p} \|\psi\|_{H^{N+\varepsilon}} \|\phi\|_{W^{2,\infty}_N}.
$$

Thus by Bernstein inequalities (5.1,5.3), we get for any $N \in \mathbb{R}$,

$$
|\mathcal{D}_{k}^2| \lesssim \int_{\sigma} b(\cos \theta) \|C_k(\theta, \cdot)\|_{L^2} \|\Phi_{k}^\gamma \|_{L^\infty} \|\widehat{\phi} p\|_{H^{N+\varepsilon}} \|\widehat{\phi} h\|_{L^2} \|\widehat{\phi} f\|_{L^2} d\sigma
\lesssim 2^{(y + \frac{1}{2} - N)k} 2^{(1-N)p} \|\phi\|_{W^{2,\infty}_N} \|\Phi_{0}\|_{H^{N+\varepsilon}} \|\widehat{\phi} p\|_{L^1} \|\widehat{\phi} h\|_{L^2} \|\widehat{\phi} f\|_{L^2}
\lesssim 2^{(y + \frac{1}{2} - N)k} 2^{(1-N)p} \|\phi\|_{W^{2,\infty}_N} \|\Phi_{0}\|_{H^{N+\varepsilon}} \|\widehat{\phi} p\|_{L^1} \|\widehat{\phi} h\|_{L^2} \|\widehat{\phi} f\|_{L^2}
$$

and

$$
|\mathcal{D}_{-1}^2| \lesssim 2^{(1-N)p} 2^{\frac{3}{2}} \|\phi\|_{W^{2,\infty}_N} \|\psi\|_{H^{N+\varepsilon}} \|\widehat{\phi} p\|_{L^1} \|\widehat{\phi} h\|_{L^2} \|\widehat{\phi} f\|_{L^2}.
$$

**Case 2:** $k = -1$ with general potentials. Due to the fact (2.6), it is easy to check that

$$
|C_{-1}(\theta, \xi)| \lesssim \theta^2 |\xi| \|\phi_p(\xi)\|_{L^\infty} \lesssim 2^{-(\gamma+3)p} \theta^2.
$$
Lemma 2.2. Suppose $N \in \mathbb{N}$. For $k \geq 0$, it holds

$$|\mathcal{M}_{k,p,m}^4| \lesssim 2^{s(m-p)2\left(\frac{\gamma}{2} - 2 - N\right)k - 2p(N - 2s - \gamma)} \|\Phi_0^\gamma\|_{H^{N,s}} \|\psi\|_{W^{N,\infty}_s} \|\tilde{s}_p g\|_{L^1} \|\tilde{s}_p h\|_{L^2} \|\tilde{s}_m f\|_{L^2}.$$  

For $k = -1$, we have

1. if $\gamma = 0$,
   $$|\mathcal{M}_{-1,p,m}^4| \lesssim 2^{2s(m-p)2\left(\frac{\gamma}{2} - 2 - 2s - \gamma\right)} \|\psi\|_{H^{N,s}} \|\phi\|_{W^{N,\infty}_s} \|\tilde{s}_p g\|_{L^1} \|\tilde{s}_p h\|_{L^2} \|\tilde{s}_m f\|_{L^2};$$
2. if $\gamma + 2s > 0$,
   $$|\mathcal{M}_{-1,p,m}^4| \lesssim 2^{2s(m-p)2\left(N - 2s - \gamma\right)\gamma} \|\psi\|_{H^{N,s}} \|\phi\|_{W^{N,\infty}_s} \|\tilde{s}_p g\|_{L^1} \|\tilde{s}_p h\|_{L^2} \|\tilde{s}_m f\|_{L^2};$$
3. if $\gamma + 2s \leq 0$,
   $$|\mathcal{M}_{-1,p,m}^4| \lesssim 2^{2s(m-p)2\left(N - 2s - \gamma\right)\gamma} \|\psi\|_{H^{N,s}} \|\phi\|_{W^{N,\infty}_s} \|\tilde{s}_p g\|_{L^1} \|\tilde{s}_p h\|_{L^2} \|\tilde{s}_m f\|_{L^2}.$$ 

Here $\eta$ is a constant which depends on $\gamma$ and $s$ and varies for different cases. Functions $\psi$ and $\phi$ are defined in (1.23).

Proof. Noticing the fact $m < p - N_0$, we follow the similar decomposition used in Lemma 2.1 to get

$$\mathcal{M}_{k,p,m}^4 = \int_{\sigma \in \mathbb{S}^2, v^2, v \in \mathbb{R}^3} \left(\tilde{s}_p \Phi_k^\gamma\right)((v - v^2) b(\cos \theta) \psi(2^m (v' - v))) (\tilde{s}_p g) + (\tilde{s}_p h)$$

$$\times \left[ (\tilde{s}_m f') - \tilde{s}_m f \right] d\sigma dv + \int_{\sigma \in \mathbb{S}^2, v^2, v \in \mathbb{R}^3} \left(\tilde{s}_p \Phi_k^\gamma\right)((v - v^2) b(\cos \theta)$$

$$\times \left[ (1 - \psi(2^m (v' - v))) (\tilde{s}_p g) + (\tilde{s}_p h) \right] (\tilde{s}_m f') - \tilde{s}_m f \right] d\sigma dv + dv.$$ 

The above is defined as $\mathcal{E}_{k}^1 + \mathcal{E}_{k}^2$.

Observe the fact

$$(\tilde{s}_m f')(v') - (\tilde{s}_m f)(v)$$

$$= (v' - v) \cdot (\nabla \tilde{s}_m f)(v) + \frac{1}{2} \int_0^1 (1 - \kappa)(v' - v) \otimes (v' - v) : (\nabla^2 \tilde{s}_m f)(\kappa(v)) d\kappa,$$

where $\kappa(v) = v + \kappa(v' - v)$, then we have the further decomposition:

$$\mathcal{E}_{k}^1 = \mathcal{E}_{k}^{1,1} + \mathcal{E}_{k}^{1,2},$$

$$\mathcal{E}_{k}^2 = \mathcal{E}_{k}^{2,1} + \mathcal{E}_{k}^{2,2},$$

$$\mathcal{E}_{k}^3 = \mathcal{E}_{k}^{3,1} + \mathcal{E}_{k}^{3,2},$$

$$\mathcal{E}_{k}^4 = \mathcal{E}_{k}^{4,1} + \mathcal{E}_{k}^{4,2}.$$
where

\[
\mathcal{E}_{k}^{1,1} = \int_{\sigma \in SS^{2}, v \in \mathbb{R}^{3}} \left( \tilde{\mathcal{F}} \rho \Phi^{\gamma}_{k}((v - v_{*})b(\cos \theta)\psi(2^{m}(v' - v))(\tilde{\mathcal{F}} \rho g)_{*}(\tilde{\mathcal{F}} \rho h)\right.
\]
\[\times (v' - v) : (\nabla \tilde{\mathcal{F}} m_{f})(v)d\sigma dv_{*}dv,
\]
\[
\mathcal{E}_{k}^{1,2} = \int_{0}^{1} \int_{\sigma \in SS^{2}, v \in \mathbb{R}^{3}} A^{k}_{*}((v - v_{*})b(\cos \theta)\psi(2^{m}(v' - v))(\tilde{\mathcal{F}} \rho g)_{*}(\tilde{\mathcal{F}} \rho h)
\]
\[\times |v - v_{*}|^{-2s}[(v' - v) \otimes (v' - v) : (\nabla^{2} \tilde{\mathcal{F}} m_{f})(\kappa(v))]|d\sigma dv_{*}dvdk.
\]

It is not difficult to check the main structures of \(\mathcal{E}_{k}^{1,2}\) and \(\mathcal{E}_{k}^{2}\) are almost as the same as those of \(\mathcal{D}_{k}^{1,1}\) and \(\mathcal{D}_{k}^{2}\). We conclude that for any \(N \in \mathbb{R}^{+}\),

\[
|\mathcal{E}_{k}^{1,2}| + |\mathcal{E}_{k}^{2}| \lesssim \left\{ \begin{array}{ll}
2^{(\gamma + \frac{1}{2} - N)p}2^{2s(m-p)}\|\Phi_{0}\|_{H^{N+2}}\|\varphi\|_{W_{x,v}^{N+2}}\|\tilde{\mathcal{F}} \rho g\|_{L^{1}}\|\tilde{\mathcal{F}} \rho h\|_{L^{2}}\|\tilde{\mathcal{F}} m_{f}\|_{L^{2}}, & \text{if } k \geq 0,
2^{2s(m-p)}2^{2s(N-p)}\|\psi\|_{H^{N+2}}\|\varphi\|_{W_{x,v}^{N+2}}\|\tilde{\mathcal{F}} \rho g\|_{L^{1}}\|\tilde{\mathcal{F}} \rho h\|_{L^{2}}\|\tilde{\mathcal{F}} m_{f}\|_{L^{2}}, & \text{if } k = -1 \text{ and } \gamma = 0,
2^{2s(m-p)}2^{-\eta p}\|\tilde{\mathcal{F}} \rho g\|_{L^{1}}\|\tilde{\mathcal{F}} \rho h\|_{L^{2}}\|\tilde{\mathcal{F}} m_{f}\|_{L^{2}}, & \text{if } k = -1 \text{ and } \gamma + 2s > 0,
2^{(2s - \gamma)}2^{-\eta p}\|\tilde{\mathcal{F}} \rho g\|_{L^{1}}\|\tilde{\mathcal{F}} \rho h\|_{L^{2}}\|\tilde{\mathcal{F}} m_{f}\|_{L^{2}}, & \text{if } k = -1 \text{ and } -1 < \gamma + 2s \leq 0.
\end{array} \right.
\]

Now we only need to give the bounds to \(\mathcal{E}_{k}^{1,1}\). Thanks to the fact

\[
\int_{SS^{2}} b\left( \frac{v - v_{*}}{|v - v_{*}|} \cdot \sigma \right)(v' - v)\psi(2^{m}|v - v'|)d\sigma
\]
\[= \int_{SS^{2}} b\left( \frac{v - v_{*}}{|v - v_{*}|} \cdot \sigma \right)\frac{v - v'}{|v - v'|} \frac{v - v_{*}}{|v - v_{*}|} |v - v'| \psi(2^{m}|v - v'|) \frac{v - v_{*}}{|v - v_{*}|} d\sigma
\]
\[= \int_{SS^{2}} b\left( \frac{v - v_{*}}{|v - v_{*}|} \cdot \sigma \right) 1 - \frac{v - v_{*}}{|v - v_{*}|} \cdot \sigma \frac{1}{2} \psi(2^{m}|v - v'|)d\sigma (v - v_{*}),
\]

one has

\[
2^{(2s - \gamma)} |v - v_{*}|^{-2s} \int_{SS^{2}} b\left( \frac{v - v_{*}}{|v - v_{*}|} \cdot \sigma \right)(v' - v)\psi(2^{m}|v - v'|)d\sigma
\]
\[\lesssim 2^{(2s - \gamma)} |v - v_{*}|^{-2s} \int_{SS^{2}} b\left( \frac{v - v_{*}}{|v - v_{*}|} \cdot \sigma \right) \frac{1}{|v - v_{*}|^{2s}} \psi(2^{m-1}|v| \sqrt{1 - \frac{(v')^{2}}{|v|^{2}}})d\sigma |v - v_{*}|
\]
\[\lesssim 2^{(2s - \gamma)} |v - v_{*}|^{-2s} \int_{0}^{\theta^{1-2s}} \delta^{1-2s}d\theta |v - v_{*}|
\]
\[\lesssim |v - v_{*}|^{-1}.
\]

In other words, set

\[
U(v) \overset{\text{def}}{=} 2^{(2s - \gamma)} |v|^{-2s} \int_{SS^{2}} b\left( \frac{v}{|v|} \cdot \sigma \right) 1 - \frac{v}{|v|} \cdot \sigma \frac{1}{2} \psi(2^{m-1}|v| \sqrt{1 - \frac{(v')^{2}}{|v|^{2}}})d\sigma,
\]

then \[(2.3)\] yields \(|U(v)| \lesssim |v|^{-1} \). In fact, one may check that \(U\) verifies for \(v \in \mathbb{R}^{3} \setminus \{0\}\) and the multi-index \(\alpha\),

\[
|\partial^\alpha U(v)| \lesssim |v|^{-1-|\alpha|}.
\]

Due to this observation, we have

\[
|\mathcal{E}_{k}^{1,1}| = 2^{(2s - \gamma)} \int_{v_{*}} A^{k}_{*}((v - v_{*})/(\tilde{\mathcal{F}} \rho g)_{*}(\tilde{\mathcal{F}} \rho h)(v)U(v - v_{*}) \cdot \nabla \tilde{\mathcal{F}} m_{f}(v)dv_{*}dv.
\]

We divide the estimate into three cases.
Due to the estimate (2.6) and (2.11), we deduce that

By the virtue of (2.10) and

we split

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Case 2: $k$

Case 1: $k$

$E$

$E$

$E$

$E$

$E$

By the definition, we have

$2\lesssim_{k}^{2} m \|A_{k}^{-\frac{1}{2}}\|_{L^{\infty}} \|\tilde{S}_{p} g\|_{L^{2}} \|\tilde{S}_{p} h\|_{L^{2}} \|\tilde{S}_{m} f\|_{L^{2}}$

$\lesssim 2^{2s(m-p)}2^{(\gamma+\frac{1}{2}-N)k}2^{-p(N-2s-\frac{1}{2})}\|\Phi_{0}^{y}\|_{H^{N+2}}\|\varphi\|_{W_{N}^{s,2}}\|\tilde{S}_{p} g\|_{L^{1}} \|\tilde{S}_{p} h\|_{L^{2}} \|\tilde{S}_{m} f\|_{L^{2}}$

In the case of $s < 1/2$, one has

$|\mathcal{E}_{k}^{1,1}| \lesssim 2^{(2s-2)m}\|A_{k}^{-\frac{1}{2}}\|_{L^{2}} \|\tilde{S}_{p} g\|_{L^{2}} \|\tilde{S}_{p} h\|_{L^{2}} \|\tilde{S}_{m} f\|_{L^{2}}$

$\lesssim 2^{2s(m-p)}2^{(\gamma+\frac{1}{2})k}2^{-p(N-2s-\frac{1}{2})}\|\Phi_{0}^{y}\|_{H^{N+2}}\|\varphi\|_{W_{N}^{s,2}}\|\tilde{S}_{p} g\|_{L^{1}} \|\tilde{S}_{p} h\|_{L^{2}} \|\tilde{S}_{m} f\|_{L^{2}}$

where we use the Hardy inequality to derive

$\|A_{k}^{-\frac{1}{2}}\|_{L^{2}} \leq \|\tilde{S}_{p} \Phi_{k}^{y}\|_{H^{1-2s}} \lesssim 2^{(\gamma+\frac{1}{2}-N)k}2^{-pN}\|\Phi_{0}^{y}\|_{H^{N+2}}\|\varphi\|_{W_{N}^{s,2}}$

Case 2: $k = -1$ and $\gamma = 0$. In this case, we only need to copy the argument in Case 1 to get

$|\mathcal{E}_{-1}^{1,1}| \lesssim 2^{2s(m-p)}2^{-p(N-2s-\frac{1}{2})}\|\varphi\|_{H^{N+2}}\|\varphi\|_{W_{N}^{s,2}}\|\tilde{S}_{p} g\|_{L^{1}} \|\tilde{S}_{p} h\|_{L^{2}} \|\tilde{S}_{m} f\|_{L^{2}}$

Case 3: $k = -1$ with general potentials. Following the decomposition

$A_{-1}^{s} = \tilde{S}_{p} \Phi_{-1}^{y+2s} + B_{-1}^{1}$

we split $\mathcal{E}_{-1}^{1,1}$ into two parts $\mathcal{E}_{-1}^{1,1}$ and $\mathcal{E}_{-1}^{1,2}$ defined by

$\mathcal{E}_{-1}^{1,1} = 2^{(2s-2)m}\int_{v,v_{-}}(\tilde{S}_{p} \Phi_{-1}^{y+2s})(|v-v_{-}|)(\tilde{S}_{p} g)(\tilde{S}_{p} h)(U(v-v_{-}) \cdot \nabla \tilde{S}_{m} f(v))d v_{+}dv$

$\mathcal{E}_{-1}^{1,2} = 2^{(2s-2)m}\int_{v,v_{-}}B_{-1}^{s}(|v-v_{-}|)(\tilde{S}_{p} g)(\tilde{S}_{p} h)(U(v-v_{-}) \cdot \nabla \tilde{S}_{m} f(v))d v_{+}dv$

Notice that

$U(v) = U(v)\psi(v) + U(v)(1 - \psi(v)) \overset{\text{def}}{=} U_{1}(v) + U_{2}(v)$

By the virtue of (2.10) and $U_{1}(v) = U(v)||v||^{-1}\psi(v)$, we have for $\delta > 0$,

$(\mathcal{F} U_{1})(\xi) \lesssim \int_{\mathbb{R}^{d}}\langle \xi - \zeta \rangle^{-3}\langle \zeta \rangle^{-2}d \zeta \lesssim \delta^{-1}\langle \xi \rangle^{-2+\delta}$

where we use Mihlin-Hormander multiplier theorem and the fact (2.6). Then by Plancherel equality and the fact $|U_{2}| \lesssim 1$, we have

$|\mathcal{E}_{-1}^{1,1}| \lesssim 2^{(2s-2)m}\int_{v,v_{-}}|\mathcal{F}((\tilde{S}_{p} \Phi_{-1}^{y+2s})(U_{1}))(\xi)\mathcal{F}(\tilde{S}_{p} g)(\xi)\mathcal{F}(\tilde{S}_{p} h\nabla \tilde{S}_{m} f)(\xi)|d \xi$

$+ 2^{(2s-2)m}\|\tilde{S}_{p} \Phi_{-1}^{y+2s}\|_{L^{2}} \|\tilde{S}_{p} g\|_{L^{2}} \|\tilde{S}_{p} h\|_{L^{2}} \|\tilde{S}_{m} f\|_{L^{2}}$

$\lesssim 2^{(2s-2)m}\|\mathcal{F}(\tilde{S}_{p} \Phi_{-1}^{y+2s})(U_{1})\|_{L^{\infty}} \|\mathcal{F}(\tilde{S}_{p} g)\|_{L^{2}} \|\mathcal{F}(\tilde{S}_{p} h\nabla \tilde{S}_{m} f)\|_{L^{2}}$

Due to the estimate (2.6) and (2.11), we deduce that

$|\mathcal{F}(\tilde{S}_{p} \Phi_{-1}^{y+2s})(U_{1})| \lesssim \int_{\xi \in \mathbb{R}^{d}}|\mathcal{F}(\tilde{S}_{p} \Phi_{-1}^{y+2s})(\xi)\mathcal{F}(U_{1})(\xi - \zeta)|d \zeta$

$\lesssim \delta^{-1}\int_{\xi \in \mathbb{R}^{d}}|\varphi(\xi)|\xi^{-2}(\xi - \zeta)^{\delta-2+\delta}d \zeta$

$\lesssim \delta^{-2}2^{-p(2s+3)}\int_{|\zeta| \leq 2^{p}}\zeta^{-2+\delta}d \zeta$

$\lesssim \delta^{-2}2^{-p(2s+3)}\frac{2^{h+2s+3}}{2h}$
Then if \( \gamma + 2s > 0 \), we choose \( \delta = (\gamma + 2s)/5 \) to get
\[
|\mathcal{E}_{-1}^{1,1}| \lesssim \delta^{-2} 2^{(2s-2)m_2-\frac{p(\gamma+2s-2)}{2s-2}} \|\hat{S} p g\|_{L^2} \|\hat{S} p h\|_{L^2} \|\nabla \hat{S} m f\|_{L^\infty} + 2^{-\frac{(\gamma+2s+1)}{2}} \|\hat{S} p g\|_{L^2} \|\hat{S} p h\|_{L^2} \|\nabla \hat{S} m f\|_{L^2} \lesssim (\gamma + 2s)^{-2} 2^{2s-2m_2} - \frac{1}{2}(\gamma + 2s)p \frac{1}{2}(1-m_2) \|\hat{S} p g\|_{L^2} \|\hat{S} p h\|_{L^2} \|\nabla \hat{S} m f\|_{L^2}.
\]
As for the case of \(-1 < \gamma + 2s \leq 0\), we choose \( \delta = (\gamma + 2s + 1)/5 \) to get
\[
|\mathcal{E}_{-1}^{1,1}| \lesssim (\gamma + 2s + 1) - \frac{2^{2s-1}m_2}{2} - \frac{1}{2}(\gamma + 2s+1)p \|\hat{S} p g\|_{L^2} \|\hat{S} p h\|_{L^2} \|\nabla \hat{S} m f\|_{L^2}.
\]
Now we begin to give the estimate to \( \mathcal{E}_{-1}^{1,2} \). If \(-1 < \gamma + 2s \leq 0\), thanks to Proposition 2.2, Hölder inequality and Young inequality, we have for \( \delta = \min\{1, \frac{1}{2}, \frac{1}{2}\} \),
\[
|\mathcal{E}_{-1}^{1,2}| \lesssim \|B^*_1 U_2\|_{L^2} \|\hat{S} p g\|_{L^2} \|\nabla \hat{S} m f\|_{L^2} \|\hat{S} p h\|_{L^2} + \|(B^*_1 U_1) * \hat{S} p g\|_{L^2} \|\nabla \hat{S} m f\|_{L^2} \|\hat{S} p h\|_{L^2} \lesssim 2^{2s-2m_2} \|B^*_1\|_{L^\infty} \|\hat{S} p g\|_{L^2} \|\nabla \hat{S} m f\|_{L^2} \|\hat{S} p h\|_{L^2} + \|U_1\|_{L^\infty} \|\nabla \hat{S} m f\|_{L^\infty} \|\hat{S} p h\|_{L^2} \lesssim 2^{-\frac{(\gamma+p-1)}{2}} 2^{2s-1m_2} \|\hat{S} p g\|_{L^2} \|\nabla \hat{S} m f\|_{L^\infty} \|\hat{S} p h\|_{L^2} \|\nabla \hat{S} m f\|_{L^\infty}.
\]
While in the case of \( \gamma + 2s > 0 \), with the help of Proposition 2.2, the similar argument can be applied to deduce for \( \delta = \min\{1, \frac{1}{2}, \frac{1}{2}\} \),
\[
|\mathcal{E}_{-1}^{1,2}| \lesssim 2^{2s-2m_2} \|B^*_1\|_{L^\infty} \|\hat{S} p g\|_{L^2} \|\nabla \hat{S} m f\|_{L^2} \|\hat{S} p h\|_{L^2} (\|\nabla \hat{S} m f\|_{L^2} + \|U_1\|_{L^\infty} \|\nabla \hat{S} m f\|_{L^\infty} \|\hat{S} p h\|_{L^2}) \lesssim 2^{-\frac{(\gamma+p-1)}{2}} 2^{2s-1m_2} \|\hat{S} p g\|_{L^2} \|\nabla \hat{S} m f\|_{L\infty} \|\hat{S} p h\|_{L^2} \|\nabla \hat{S} m f\|_{L^\infty}.
\]
Now patch together all the estimates, then we are led to the desired results. \( \square \)

### 2.2. Estimates of \( \mathfrak{M}_{k,l}^2 \) and \( \mathfrak{M}_{k,p}^3 \) defined in (1.27).
Since \( \mathfrak{M}_{k,l}^2 \) enjoys almost the same structure as that of \( \mathfrak{M}_{k,p}^3 \), it suffices to give the estimate to \( \mathfrak{M}_{k,p}^3 \).

**Lemma 2.3.** If \( k \geq 0 \), we have
\[
|\mathfrak{M}_{k,l}^2| \lesssim 2^{(\gamma + 2s)k} 2^{2s} \|g\|_{L^1} \|\hat{S} f h\|_{L^2} \|\hat{S} f f\|_{L^2},
\]
\[
|\mathfrak{M}_{k,p}^3| \lesssim 2^{(\gamma + 2s)k} 2^{2sp} \|g\|_{L^1} \|\hat{S} f h\|_{L^2} \|\hat{S} f f\|_{L^2}.
\]
If \( k = -1 \), we have
\[
|\mathfrak{M}_{-1,l}^2| \lesssim \begin{cases} 2^{2s} \|g\|_{L^1} \|\hat{S} f h\|_{L^2} \|\hat{S} f f\|_{L^2}, & \text{if } \gamma + 2s > 0, \\ 2^{2s} \|g\|_{L^2} \|\hat{S} f h\|_{L^2} \|\hat{S} f f\|_{L^2}, & \text{if } \gamma + 2s \leq 0. \end{cases}
\]
\[
|\mathfrak{M}_{-1,p}^3| \lesssim \begin{cases} 2^{2sp} \|g\|_{L^1} \|\hat{S} f h\|_{L^2} \|\hat{S} f f\|_{L^2}, & \text{if } \gamma + 2s > 0, \\ 2^{2sp} \|g\|_{L^2} \|\hat{S} f h\|_{L^2} \|\hat{S} f f\|_{L^2}, & \text{if } \gamma + 2s \leq 0. \end{cases}
\]

**Proof.** We introduce the function \( \psi \) defined in (1.23) to decompose the quantity into two parts: angular cutoff part and angular non cutoff part.

\[
\mathfrak{M}_{k,p}^3 = \int_{\sigma \in S^2, \nu, \nu \in \mathbb{R}^3} \Phi_k^\prime (|v| - |v_\ast|) b(\cos \theta) \psi(2^p (v' - v)) (\hat{S} p g) (\hat{S} p h) \times \left[ \langle \hat{S} p f \rangle' - \hat{S} p f \right] d\sigma d\nu_\ast dv + \int_{\sigma \in S^2, \nu, \nu \in \mathbb{R}^3} \Phi_k^\prime (|v| - |v_\ast|) |b(\cos \theta)| \times [1 - \psi(2^p (v' - v)) (\hat{S} p g) (\hat{S} p h) (\hat{S} p f)' - \hat{S} p f] d\sigma d\nu_\ast dv.
\]

By using Taylor expansion (2.8), the facts (2.5), (2.9) and Hölder inequality, we deduce for \( k \geq 0 \),
\[
|\mathfrak{M}_{k,p}^3| \lesssim 2^{(\gamma + 2s)k} 2^{2sp} \|g\|_{L^1} \|\hat{S} f h\|_{L^2} \|\hat{S} f f\|_{L^2},
\]
and
\[
|\mathfrak{M}_{k,p}^3| \lesssim \begin{cases} 
2^{2sp} \| g \|_{L^1} \| \tilde{\mathcal{P}}_p h \|_{L^2} \| \tilde{\mathcal{P}}_p f \|_{L^2}, & \text{if } \gamma + 2s > 0, \\
2^{2sp} \| g \|_{L^1} \| \tilde{\mathcal{P}}_p h \|_{L^2} \| \tilde{\mathcal{P}}_p f \|_{L^2}, & \text{if } \gamma + 2s \leq 0.
\end{cases}
\]

The similar results hold for $\mathfrak{M}_{k,l}^2$. We complete the proof of the lemma. \qed

2.3. Proof of Theorem 1.1

Now we are ready to give the proof to Theorem 1.1.

Proof. Set $a_1, b_1 \in \mathbb{R}$ verifying $a_1 + b_1 = 2s$. Then by Lemma 2.1, Lemma 2.2 and Lemma 2.3, we conclude that for $k \geq 0$ or $\gamma = 0$ with $k \geq -1$,
\[
\sum_{l \leq -1} |\mathfrak{M}_{k,l}^1| + \sum_{m < p - N_0} |\mathfrak{M}_{k,p,m}^1| \lesssim C(a_1, b_1) \| g \|_{L^1} \| h \|_{H^{\alpha_1}} \| f \|_{H^{\alpha_1}},
\]
\[
\sum_{l \geq 1} |\mathfrak{M}_{k,l}^2| + \sum_{p \geq -1} |\mathfrak{M}_{k,p}^3| \lesssim 2^{(\gamma + 2s)k} \| g \|_{L^1} \| h \|_{H^{\alpha_1}} \| f \|_{H^{\alpha_1}},
\]

Then we are led to if $k \geq 0$ or $\gamma = 0$ with $k \geq -1$,
\[
|\langle Q_k(g, h), f \rangle_v| \lesssim C(a_1, b_1) 2^{(\gamma + 2s)k} \| g \|_{L^1} \| h \|_{H^{\alpha_1}} \| f \|_{H^{\alpha_1}}.
\]

Set $a, b \in [0,2s]$ verifying $a + b = 2s$. If $k = -1$, then by Lemma 2.1, Lemma 2.2 and Lemma 2.3, we have
\[
\sum_{l \leq -1} |\mathfrak{M}_{k,l}^1| + \sum_{l \geq 1} |\mathfrak{M}_{k,l}^2| + \sum_{p \geq -1} |\mathfrak{M}_{k,p}^3| + \sum_{m < p - N_0} |\mathfrak{M}_{k,p,m}^1| \lesssim (\| g \|_{L^1} + \| g \|_{L^2}) \| h \|_{H^\alpha} \| f \|_{H^\alpha},
\]
which yields
\[
|\langle Q_{-1}(g, h), f \rangle_v| \lesssim (\| g \|_{L^1} + \| g \|_{L^2}) \| h \|_{H^\alpha} \| f \|_{H^\alpha}.
\]

Now we are in a position to give the upper bound for the collision operator in weighted Sobolev space. Set $\omega_1 + \omega_2 = \gamma + 2s$. Recalling (1.26), we infer from (2.12) and (2.13),
\[
|\langle Q(g, h), f \rangle_v| \lesssim \sum_{k \geq N_0} 2^{(\gamma + 2s)k} \| \mathcal{U}_{k-N_0} g \|_{L^1} \| \tilde{\mathcal{P}}_k h \|_{H^\alpha} \| \tilde{\mathcal{P}}_k f \|_{H^\alpha}
\]
\[
+ \left( \sum_{j \geq k + N_0} 2^{(\gamma + 2s)k} \| \mathcal{P}_j g \|_{L^1} \| \tilde{\mathcal{P}}_j h \|_{H^\alpha} \| \tilde{\mathcal{P}}_j f \|_{H^\alpha} \right)
\]
\[
+ \left( \sum_{j \geq -1} \| \mathcal{P}_j g \|_{L^1} \| \tilde{\mathcal{P}}_j h \|_{H^\alpha} \| \tilde{\mathcal{P}}_j f \|_{H^\alpha} \right)
\]
\[
+ ((\| g \|_{L^1} + \| g \|_{L^2})\| \mathcal{U}_{N_0} h \|_{H^\alpha} \| \mathcal{U}_{N_0} f \|_{H^\alpha})
\]
\[
def \leq \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3.
\]

For the term $\mathcal{U}_1$, thanks to Lemma 5.1, it follows
\[
\mathcal{U}_1 \lesssim \| g \|_{L^1} \| h \|_{H^{\alpha_1}} \| f \|_{H^{\alpha_2}}.
\]

For the term $\mathcal{U}_2$, we separate the estimate into three cases. If $\gamma + 2s > 0$, we have
\[
\mathcal{U}_2 \lesssim \sum_{j \geq -1} 2^{(\gamma + 2s)j} ((\| \mathcal{P}_j g \|_{L^1} + \| \mathcal{P}_j g \|_{L^2})\| \tilde{\mathcal{P}}_j h \|_{H^\alpha} \| \tilde{\mathcal{P}}_j f \|_{H^\alpha}
\]
\[
\lesssim (\| g \|_{L^1} + \| g \|_{L^2}) \| h \|_{H^{\alpha_1}} \| f \|_{H^{\alpha_2}}.
\]
In the case of $\gamma + 2s = 0$, it holds for any $\delta > 0$,
\[
\mathcal{U}_2 \lesssim \sum_{j=-1}^{+\infty} (\|g\|_{L^1_{\delta}} + \|g\|_{L^2}) \|\mathcal{P}_j h\|_{H^\alpha} \|\mathcal{P}_j f\|_{H^b} \\
\lesssim (\|g\|_{L^1_{\delta}} + \|g\|_{L^2}) \|h\|_{H^\alpha_{w_1}} \|f\|_{H^b_{w_2}}.
\]

While in the case of $\gamma + 2s < 0$, we have
\[
\mathcal{U}_2 \lesssim \sum_{j=-1}^{+\infty} (\|\mathcal{P}_j g\|_{L^1} + \|\mathcal{P}_j g\|_{L^2}) \|\mathcal{P}_j h\|_{H^\alpha} \|\mathcal{P}_j f\|_{H^b} \\
\lesssim (\|g\|_{L^1} - (\gamma + 2s) \|g\|_{L^2}) \|h\|_{H^\alpha_{w_1}} \|f\|_{H^b_{w_2}}.
\]

Now we turn to the term $\mathcal{U}_3$. We first claim that it holds
\[
\|\mathcal{U}_{k+N_0} h\|_{H^\alpha} \lesssim 2^{k(-w_1)^+} \|h\|_{H^\alpha_{w_1}}.
\]

By the definition of $\mathcal{U}$, we have
\[
(\mathcal{U}_{k+N_0} h)(\nu) = \left[ \sum_{j \geq k+N_0} \varphi(2^{-j} \nu) + \psi(\nu) \right] h(\nu) \overset{\text{def}}{=} \tilde{\psi}_{k+N_0}(\nu) h(\nu).
\]

Thanks to the fact if $w_1 \geq 0$,
\[
\partial^\alpha (\tilde{\psi}_{k+N_0}(\nu)^{-w_1}) \lesssim \langle \nu \rangle^{-w_1 - |\alpha|} \lesssim \langle \nu \rangle^{-|\alpha|},
\]
and if $w_1 < 0$,
\[
\partial^\alpha (2^{kw_1} \tilde{\psi}_{k+N_0}(\nu)^{-w_1}) \lesssim \langle \nu \rangle^{-|\alpha|},
\]
by Lemma 5.3 we deduce that
\[
\|\mathcal{U}_{k+N_0} h\|_{H^\alpha} \lesssim \|\tilde{\psi}_{k+N_0}(\nu)^{-w_1} \langle \nu \rangle^{w_1} h(\nu)\|_{H^\alpha} \\
\lesssim 2^{k(-w_1)^+} \|h\|_{H^\alpha_{w_1}},
\]
which complete the proof to the claim. Now we apply the claim to the estimate of $\mathcal{U}_3$. It is easy to check
\[
\mathcal{U}_3 \lesssim (\|g\|_{L^1_{\gamma+2s(-w_1)+(-w_2)^+}} + \|g\|_{L^2}) \|h\|_{H^\alpha_{w_1}} \|f\|_{H^b_{w_2}}.
\]

Now patching together the estimates for $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$, we obtain the desired results in the theorem.

For the special case $\gamma = 0$, recalling the estimate (2.12) and following the similar argument used before, we will easily obtain (1.36).

3. LOWER AND UPPER BOUNDS FOR THE BOLTZMANN COLLISION OPERATOR IN ANISOTROPIC SPACES

In this section, we will give the proof to the lower and upper bounds for the collision operator in anisotropic spaces. The main idea is to use the geometric decomposition explained in the introduction and also the $L^2$ profile of the fractional Laplacian-Beltrami operator on the unit sphere in Section 5.
3.1. **Lower bounds of the operator in anisotropic spaces.** In this subsection, we shall give the proof to the coercivity estimates of the collision operator based on the geometric decomposition.

**Proof of Theorem 1.2** We shall divide the proof into several steps. It is standard to take the following decomposition:

\[
\langle -Q(g, f), f \rangle_v = -\int_{\mathbb{R}^3} dv_* d\nu \int_{\mathbb{S}^2} B(|v - v_*|, \sigma) g_* f (f' - f) d\sigma
\]  
\[
= -\frac{1}{2} \int_{\mathbb{R}^3} dv_* d\nu \int_{\mathbb{S}^2} B(|v - v_*|, \sigma) g_* (f' - f)^2 d\sigma
\]  
\[
+ \frac{1}{2} \int_{\mathbb{R}^3} dv_* d\nu \int_{\mathbb{S}^2} B(|v - v_*|, \sigma) g_* (f' f - f f') d\sigma.
\]

By change of variables, we notice that

\[
|\mathcal{L}| = |\mathbb{S}^1| \int_{\mathbb{R}^3} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin \theta \left( \frac{1}{\cos^3 \frac{\theta}{2}} B(\frac{|v - v_*|}{\cos \theta}, \cos \theta) - B(|v - v_*|, \cos \theta) \right) g_* f^2 d\theta dv_* d\nu
\]
\[
\lesssim \int_{v_* \neq v} |v - v_*|^\gamma g_* f^2 d\nu dv \overset{\text{def}}{=} \mathcal{R},
\]

which can be bounded by the following lemma:

**Lemma 3.1.** For smooth functions \( g \) and \( f \), there exists a sufficiently small constant \( \eta \) and a universal constant \( a \in (0, 1) \) such that

1. If \( \gamma \geq 0 \),
   \[
   |\mathcal{R}| \lesssim \|g\|_{L^1} \|f\|_{L^2}^2 + \|g\|_{L^1} \|f\|_{L^2}^2 ,
   \]
2. If \(-2\delta < \gamma < 0\),
   \[
   |\mathcal{R}| \lesssim \eta \frac{|\gamma|}{2\delta} \|f\|_{L^2}^2 + \eta \|f\|_{H^\delta}^2 ,
   \]
3. If \( \gamma + 2\delta = 0 \),
   \[
   |\mathcal{R}| \lesssim \left( \|g\|_{L^1} + \exp(\eta^{-1}((1 - a) \|g\|_{L^{1+2a\delta}} + a \|g\|_{L^{1+2a\delta}})) \right) \|f\|_{L^2}^2 + \eta \|f\|_{H^\delta}^2 ,
   \]
4. If \(-1 \leq \gamma + 2\delta < 0 \) and \( p > 3/2 \),
   \[
   |\mathcal{R}| \lesssim \eta \frac{3}{|\gamma + 2\delta|} \|g\|_{L^p} \|f\|_{L^p}^2 + \eta \|f\|_{H^\delta}^2 .
   \]

**Proof.** It is easy to check that if \( \gamma > 0 \), we have

\[
|\mathcal{R}| \lesssim \|g\|_{L^1} \|f\|_{L^2}^2 + \|g\|_{L^1} \|f\|_{L^2}^2 .
\]

For \( \gamma < 0 \), we observe that

\[
|\mathcal{R}| = \int_{v_* \neq v} |v - v_*|^{\gamma} \langle v \rangle^{\gamma} \langle v_* \rangle^{\gamma} |G_*|^2 (f(v)^{\gamma/2})^2 d\nu dv,
\]
where \( G = g(v)^{\gamma} \), \( F = f(v)^{\gamma/2} \).
In the case of $-2\omega < \gamma < 0$, by using the Hardy inequality, we get
\[
|\mathcal{R}| \lesssim \|g\|_{L^\gamma} \|f\|_{H^{3/2}}^2.
\]
Thanks to the condition $\gamma + 2\omega > 0$, by interpolation, we may derive
\[
|\mathcal{R}| \lesssim \eta^{\frac{\gamma}{\gamma+2\omega}} \frac{\gamma + 2\omega}{2\omega} \|g\|_{L^\gamma} \|f\|_{H^{3/2}}^2 + \eta \|f\|_{H^{3/2}}^2.
\]
For the case of $\gamma + 2\omega = 0$, we have
\[
|\mathcal{R}| \lesssim \|g\|_{L^\gamma} \|f\|_2^2 + M \|f\|_{L^2}^2 + \int_{\mathbb{R}} |v-v_+|^{2\omega} (1|v-v_+| \leq 1) G_{1|g| \geq M} F^2 dv_+ dv
\]
where $G = g(v)^{2\omega}$, $F = f(v)^{-\omega}$ and the Hardy inequality is used in the last step. Choose $a \in (0, 1)$, then we get
\[
\|G_{1|g| \geq M}\|_{L^1} \lesssim (\log M)^{-(1-a)} \|G|\log g|^{1-a}\|_{L^1} \lesssim (\log M)^{-(1-a)} \|g\|_{L^\gamma}^{1-a} \|g\|_{L^2}^{a}.
\]
It yields
\[
|\mathcal{R}| \lesssim (M + \|g\|_{L^\gamma}) \|f\|_2^2 + (\log M)^{-(1-a)} \|g\|_{L^\gamma} \|g\|_{L^2}^a \|f\|_{H^{3/2}}^2.
\]
Thus we have
\[
|\mathcal{R}| \lesssim \|g\|_{L^\gamma} + \exp((\eta^{-1} \|g\|_{L^\gamma} \|g\|_{L^2}^a)^{1/2})\|f\|_2^2 + \eta \|f\|_{H^{3/2}}^2.
\]
In particular, if $\omega \in (0, 1)$, we have for $a \in (0, 1)$
\[
|\mathcal{R}| \lesssim \|g\|_{L^\gamma} + \exp((\eta^{-1} ((1-a) \|g\|_{L^\gamma} + a \|g\|_{L^2}^{1/2}))^{1/2})\|f\|_2^2 + \eta \|f\|_{H^{3/2}}^2.
\]
If $\omega = 1$, for any $\delta > 0$, we may obtain
\[
|\mathcal{R}| \lesssim \|g\|_{L^\gamma} + \exp((\eta^{-1} (\|g\|_{L^\gamma} + \|g\|_{L^2}^{1/2}))^{2\delta})\|f\|_2^2 + \eta \|f\|_{H^{3/2}}^2.
\]
Finally we handle the case $-1 < \gamma + 2\omega < 0$. By Hardy-Littlewood-Sobolev inequality, we have
\[
|\mathcal{R}| \lesssim M \|f\|_2^2 + \|G_{1|g| \geq M}\|_{L^p} \|F\|_{H^{3/2}}^2.
\]
Since $p > 3/2$, we have
\[
\|G_{1|g| \geq M}\|_{L^\gamma} \|g\|_{L^\gamma} \lesssim M^{-(\frac{\gamma+2\omega+3}{p})} \|G\|_{L^p}^{\frac{\gamma+2\omega+3}{p}},
\]
Thus we infer
\[
|\mathcal{R}| \lesssim \eta^{-\frac{3}{(\gamma+2\omega+3)p}} \|g\|_{L^\gamma} \|g\|_{L^2} \|f\|_2^2 + \eta \|f\|_{H^{3/2}}^2.
\]
From now on, we will focus on the estimate of the elliptic part $\mathcal{E}_g^R$. We begin with two useful lemmas to deal with the simple case $\gamma = 0$ and then extend the results to the general cases.

**Lemma 3.2.** Suppose that $g$ is a non-negative function. Then
\[
\mathcal{E}_g^R(f) \gtrsim \mathcal{E}_4(g)(\|(-\Delta_{SS})^{3/2} f\|_2^2 + \|f\|_{H^2}^2) - \eta \|g\|_{L^1} \|f\|_{L^2}^2 - (\|g\|_{L^1} \eta^{-1} \mathcal{E}_3(g)^{-1} + 1) \|f\|_{L^2}^2,
\]
where
\[
\mathcal{E}_4(g) \overset{\text{def}}{=} \min\{\mathcal{E}_3(g), \|g\|_{L^1}\} \|g\|_{L^1}^{-1} \mathcal{E}_3(g)^{-1} + 2.
\]
Here \( \mathcal{C}_3(g) \overset{\text{def}}{=} \min\{\mathcal{C}_1(g), \mathcal{C}_2(g), 1\} \) and \( \mathcal{C}_1(g) \) and \( \mathcal{C}_2(g) \) are defined as follows:

\[
\mathcal{C}_1(g) \overset{\text{def}}{=} 2\sin^2 \varepsilon \left[ |g|_{L^1} - \frac{|g|_{L^1}}{r} - \sup_{|A|<4\varepsilon(2r)^2+\frac{1}{2}} \int_A g(v) dv \right],
\]

where \( \varepsilon \) and \( r \) are chosen in such a way that this quantity is positive, and

\[
\mathcal{C}_2(g) \overset{\text{def}}{=} 2\theta^2 \sup_{|\xi|=1} \left| \frac{\sin^2(\theta(\xi))}{\theta^2 |\xi|^2} \right| \left[ |g|_{L^1} - \frac{|g|_{L^1}}{r} - \sup_{|A|<4\varepsilon(2r)^2+\frac{1}{2}} \int_A g(v) dv \right],
\]

where \( \theta \) and \( r \) are chosen in such a way that this quantity is positive.

If the function \( g \) verifies the conditions (1.37), then due to the definition of \( \mathcal{C}_4(g) \), there exists a constant \( C(\delta, \lambda, \eta) \) depending only on \( \delta, \lambda \) and \( \eta \) such that

\[
(3.2) \quad \mathcal{C}_4(g) \geq C(\delta, \lambda, \eta).
\]

**Proof.** By the geometric decomposition (1.28), one has

\[
\mathcal{E}_8^0(f) \overset{\text{def}}{=} \frac{1}{2} \int_{\nu,\nu^+} g_* b(\cos \theta) (|T_{\nu^+} f|)^2 d\sigma d\nu^+ d\nu
\]

\[
- \int_{\nu,\nu^+} g_* b(\cos \theta) \left( |f(\nu^+) - f(\nu) + |u|^+ \right) d\sigma d\nu^+ d\nu
\]

\[
(3.3) \quad \overset{\text{def}}{=} \mathcal{E}_1^0 - \mathcal{E}_2^0.
\]

**Step 1: Estimate of \( \mathcal{E}_1^0 \).** By change of variables, we have

\[
\mathcal{E}_1^0 \geq \int_{\tau,\tau^+} g_* (\sigma \cdot \tau)_1_{\sigma\tau \geq 0} (|T_{\tau^+} f|)^2 r^2 d\sigma d\tau dr d\nu^+ d\nu^-
\]

For fixed \( \nu^+, \tau \in SS^2 \) and \( r \), if \( \tau \) is chosen to be the polar direction, one has

\[
d\sigma = \sin \theta d\theta dSS^1, \quad d\zeta = \sin \phi d\phi dSS^1,
\]

where \( \theta = 2\phi \). From which, we deduce

\[
d\sigma = 4\cos \phi d\zeta.
\]

From which together with the facts \( b(\tau \cdot \sigma) \sim |\sigma - \tau|^{-2+2s} \) and \( |\sigma - \tau| \sim |\zeta - \tau| \), we get

\[
\mathcal{E}_1^0 \geq \int_{\nu,\nu^+,\zeta} g_* |\zeta - \tau|^{-2+2s} (|T_{\nu^+} f|)^2 r^2 d\zeta d\tau dr d\nu^+ d\nu^-
\]

\[
\geq \int_{\nu,\nu^+,\zeta} g_* |\zeta - \tau|^{-2+2s} (|T_{\nu^+} f|)^2 r^2 d\zeta d\tau dr d\nu^+ d\nu^-
\]

Thanks to Lemma [5.6] and Lemma [5.10] we obtain

\[
\mathcal{E}_1^0 \geq \int_{\nu} g_* \|(-\Delta SS^2)^{s/2} T_{\nu} f\|_{L^2}^2 d\nu^+ d\nu^+ - \|g\|_{L^1} \|f\|_{L^2}^2
\]

\[
\geq \|g\|_{L^1} \|(-\Delta SS^2)^{s/2} f\|_{L^2}^2 - \|g\|_{L^1} \|f\|_{H^s}^2.
\]
Step 2: Estimate of $\mathcal{E}_2^0$. We introduce the dyadic decomposition in the frequency space. Set

$$\mathcal{E}_{2,k}^0 \overset{\text{def}}{=} 2 \int_{u,v,\sigma} g_\ast \varphi_k(u) b(\cos \theta) \left( f(v_\ast + u^+) - f(v_\ast + |u| u^+) \right)^2 d\sigma dv_\ast du$$

$$= 2 \sum_{l,p} \int_{u,v,\sigma} g_\ast \varphi_k(u) b(\cos \theta) \left( (\tilde{\mathcal{F}}_l f)(v_\ast + u^+) - (\tilde{\mathcal{F}}_l f)(v_\ast + |u| u^+) \right)$$

$$\times ((\tilde{\mathcal{F}}_p f)(v_\ast + u^+) - (\tilde{\mathcal{F}}_p f)(v_\ast + |u| u^+)) d\sigma dv_\ast du$$

$$= 2 \sum_{l,p} \mathcal{E}_{l,p}^1.$$

It is easy to check that

$$(3.4) \quad \mathcal{E}_{l,p} = \mathcal{E}_{l,p}^1 + \mathcal{E}_{l,p}^2,$$

where

$$\mathcal{E}_{l,p}^1 = \int_{u,v,\sigma} g_\ast \varphi_k(u) b(\cos \theta) \left( (\tilde{\mathcal{F}}_l f)(v_\ast + u^+) - (\tilde{\mathcal{F}}_l f)(v_\ast + |u| u^+) \right)$$

$$\times (\tilde{\mathcal{F}}_p f)(v_\ast + u^+) d\sigma dv_\ast du,$$

$$\mathcal{E}_{l,p}^2 = \int_{u,v,\sigma} g_\ast \varphi_k(u) b(\cos \theta) \left( (\tilde{\mathcal{F}}_l f)(v_\ast + u^+) - (\tilde{\mathcal{F}}_l f)(v_\ast + |u| u^+) \right)$$

$$\times (\tilde{\mathcal{F}}_p f)(v_\ast + |u| u^+) d\sigma dv_\ast du.$$

By the symmetric property of $\mathcal{E}_{l,p}$, without loss of the generality, we assume $l \leq p$.

Step 2.1: Estimate of $\mathcal{E}_{l,p}^1$. We introduce the function $\psi$ to split $\mathcal{E}_{l,p}^1$ into two parts: $\mathcal{E}_{l,p}^{1,1}$ and $\mathcal{E}_{l,p}^{1,2}$ which are defined by

$$\mathcal{E}_{l,p}^{1,1} = \int_{u,v,\sigma} g_\ast \varphi_k(u) b(\cos \theta) \left( (\tilde{\mathcal{F}}_l f)(v_\ast + u^+) - (\tilde{\mathcal{F}}_l f)(v_\ast + |u| u^+) \right)$$

$$\times (\tilde{\mathcal{F}}_p f)(v_\ast + u^+) \psi(2^{k/2} 2^{l/2} \sqrt{1 - \frac{u}{|u|} \cdot \sigma}) d\sigma dv_\ast du,$$

$$\mathcal{E}_{l,p}^{1,2} = \int_{u,v,\sigma} g_\ast \varphi_k(u) b(\cos \theta) \left( (\tilde{\mathcal{F}}_l f)(v_\ast + u^+) - (\tilde{\mathcal{F}}_l f)(v_\ast + |u| u^+) \right)$$

$$\times (\tilde{\mathcal{F}}_p f)(v_\ast + u^+) \left( 1 - \psi(2^{k/2} 2^{l/2} \sqrt{1 - \frac{u}{|u|} \cdot \sigma}) \right) d\sigma dv_\ast du.$$

Observe that

$$| (\tilde{\mathcal{F}}_l f)(v_\ast + u^+) - (\tilde{\mathcal{F}}_l f)(v_\ast + |u| u^+) |$$

$$\lesssim \int_0^1 d\xi |\nabla (\tilde{\mathcal{F}}_l f)(v_\ast + u^+ (\kappa + (1 - \kappa) \cos^{-1}(\theta/2))) | u^+\|1 - \cos^{-1}(\theta/2)\|.$$  

We have

$$|\mathcal{E}_{l,p}^{1,1}| \lesssim \left( \int_{k,u,v,\sigma} |g_\ast \varphi_k(u) b(\cos \theta) \nabla (\tilde{\mathcal{F}}_l f)(v_\ast + u^+ (\kappa + (1 - \kappa) \cos^{-1}(\theta/2))) |^2 d\sigma dv_\ast du \right)^{\frac{1}{2}}$$

$$\times |u^+\|1 - \cos^{-1}(\theta) | (1, q)_{L^2}^{2 - k/2 - l/2} \int d\xi d\sigma d v_\ast d u \right)^{\frac{1}{2}}$$

$$\times \left( \int_{k,u,v,\sigma} |g_\ast \varphi_k(u) b(\cos \theta) u^+\|1 - \cos^{-1}(\theta/2)\| (\tilde{\mathcal{F}}_p f)(v_\ast + u^+) |^2$$

$$\times 1_{|q| \leq 2 - k/2 - l/2} \int d\xi d\sigma d v_\ast d u \right)^{\frac{1}{2}}.$$
Let \( \tilde{u} = u^+(\kappa + (1 - \kappa) \cos^{-1}(\theta/2)) \). Then by change of variable \( u \to \tilde{u} \), one gets

\[
(3.6) \quad \left| \frac{d \tilde{u}}{d u} \right| = \left| \frac{d u^+}{d u} \right| \sim 1.
\]

Moreover, we have \( |u| \sim |\tilde{u}| \). Thanks to this observation, we get

\[
|\mathcal{E}_{l, p}^{1, 1}| \lesssim 2^{k} \left[ \int_{k, u, v, \sigma} |g_s| |\nabla (\tilde{s}_l f)|^2 (v_\ast + \tilde{u}) b(\cos \theta) \theta^2 1_{|\theta| \leq 2^{-k/2-1/2}} d\kappa d\sigma d v_\ast d \tilde{u} \right]^{1/2}
\]

\[
\times \left( \int_{k, u, v, \sigma} |g_s| |(\tilde{s}_p f)(v_\ast + \tilde{u})|^2 b(\cos \theta) \theta^2 1_{|\theta| \leq 2^{-k/2-1/2}} d\kappa d\sigma d v_\ast d \tilde{u} \right)^{1/2}
\]

\[
\lesssim 2^{(s-1)k} 2^{-(s-1)/2} \|g\|_{L^1} \|\tilde{s}_p f\|_{L^2} \|\nabla \tilde{s}_l f\|_{L^2}
\]

\[
\lesssim 2^{(l-p)s/2} 2^{ks} \|g\|_{L^1} \|\tilde{s}_p f\|_{H^{\nu/2}} \|\nabla \tilde{s}_l f\|_{H^{\nu/2}}.
\]

Now we turn to give the estimate to \( \mathcal{E}_{l, p}^{1, 2} \). By Cauchy-Schwartz inequality and the change of variables, we also have the following estimate:

\[
\mathcal{E}_{l, p}^{1, 2} \lesssim 2^{(l-p)s/2} 2^{ks} \|g\|_{L^1} \|\tilde{s}_p f\|_{H^{\nu/2}} \|\nabla \tilde{s}_l f\|_{H^{\nu/2}},
\]

which implies

\[
\mathcal{E}_{l, p}^{1} \lesssim 2^{(l-p)s/2} 2^{ks} \|g\|_{L^1} \|\tilde{s}_p f\|_{H^{\nu/2}} \|\nabla \tilde{s}_l f\|_{H^{\nu/2}}.
\]

**Step 2.2: Estimate of \( \mathcal{E}_{l, p}^2 \).** The similar argument can be applied to \( \mathcal{E}_{l, p}^{2} \) to get

\[
\mathcal{E}_{l, p}^{2} \lesssim 2^{(l-p)s/2} 2^{ks} \|g\|_{L^1} \|\tilde{s}_p f\|_{H^{\nu/2}} \|\nabla \tilde{s}_l f\|_{H^{\nu/2}},
\]

which yields

\[
|\mathcal{E}_{l, p}| \lesssim 2^{(l-p)s/2} 2^{ks} \|g\|_{L^1} \|\tilde{s}_p f\|_{H^{\nu/2}} \|\nabla \tilde{s}_l f\|_{H^{\nu/2}}.
\]

Then we arrive at

\[
\mathcal{E}_{2, k}^0 \leq 4 \sum_{l \leq p} |\mathcal{E}_{l, p}| \lesssim 2^{k^2} \|g\|_{L^1} \|f\|_{H^{\nu/2}}^2.
\]

Suppose \( |v_\ast| \sim 2^j \) and \( |u| \sim 2^k \). Then thanks to the fact \( |u| \sim |u^+| \), we have

- **Case 1:** \( j \leq k - N_0 \). Then \( |v_\ast + u^+|, |v_\ast + u| \frac{|u^+|}{|u|} | \sim 2^k; \)
- **Case 2:** \( j \geq k + N_0 \). Then \( |v_\ast + u^+|, |v_\ast + u| \frac{|u^+|}{|u|} | \sim 2^j; \)
- **Case 3:** \( |j - k| < N_0 \). Then \( |v_\ast + u^+|, |v_\ast + u| \frac{|u^+|}{|u|} | \leq 2^{k+N_0}, |v'| \leq 2^{k+N_0}. \)

It follows

\[
\mathcal{E}_2^0(g, f) = \sum_k \mathcal{E}^0_{2, k}(g, f)
\]

\[
= \sum_{k < j - N_0} \mathcal{E}_{2, k}^0 (\mathcal{P}_j g, \mathcal{P}_j f) + \sum_{j < k - N_0} \mathcal{E}_{2, k}^0 (\mathcal{P}_j g, \mathcal{P}_j f) + \sum_k \mathcal{E}_{2, k}^0 (\mathcal{P}_k g, \mathcal{P}_{k+N_0} f)
\]

Then

\[
|\mathcal{E}_2^0| \lesssim \sum_{k < j - N_0} 2^{ks} \|\mathcal{P}_j g\|_{L^1} \|\mathcal{P}_j f\|_{H^{\nu/2}}^2 + \sum_{j < k - N_0} 2^{ks} \|\mathcal{P}_j g\|_{L^1} \|\mathcal{P}_j f\|_{H^{\nu/2}}^2
\]

\[
+ \sum_k 2^{ks} \|\mathcal{P}_k g\|_{L^1} \|\mathcal{P}_{k+N_0} f\|_{H^{\nu/2}}^2
\]

\[
\lesssim \|g\|_{L^1} \|f\|_{H^{\nu/2}}^2.
\]

where we use Lemma 5.1.
The proof is inspired by [5]. Without loss of generality, we assume that the function negative. By Bobylev’s equality, we have

\[
\begin{aligned}
\|g\|_{L^2}^2 &= \|f\|_{L^2}^2 - \|g\|_{L^2}^2 - \|f\|_{H^s}^2 + \|f\|_{H^{s+1}}^2 \\
&\geq \|g\|_{L^2}^2 \|\beta\|_{L^2}^2 - \|g\|_{L^2}^2 - \|f\|_{H^s}^2 + \|f\|_{H^{s+1}}^2.
\end{aligned}
\]

Thanks to Corollary 3 and Proposition 2 in [1], we deduce that

\[
\|g\|_{L^2}^2 \geq \varepsilon_3(g)\|f\|_{H^s}^2.
\]

From which together with the previous lower bound, we are led to the desired result. □

**Lemma 3.3.** Suppose the angular function \( h \) verifies the conditions \( \int_0^{\pi/2} b(\cos \theta) \sin \theta \theta^2 d\theta < \infty \) and

\[
(3.7) \quad 1 + \int_{\sigma \in \mathbb{S}^2} b(\tau \cdot \sigma) \min(\xi^2|\tau - \sigma|^2, \delta) \, d\sigma \sim 1 + \int_{\sigma \in \mathbb{S}^2} b(2(\tau \cdot \sigma)^2 - 1) \min(\xi^2|\tau - \sigma|^2, \delta) \, d\sigma \sim W^2(\xi),
\]

where \( \tau \in \mathbb{S}^2 \) and \( W \) is a radial function satisfying \( W(|\xi|) \lesssim W(|\xi|) W(|\xi|) \) and \( W(\xi) \leq \langle \xi \rangle \). Then for any smooth function \( g \), it hold

\[
|\varepsilon_0^0(f)| \lesssim \|g\|_{L^2} \varepsilon_0^0(f) + \|W^2 g\|_{L^2} \|W(D) f\|_{L^2}^2.
\]

If \( g \) is a non-negative function verifying the condition \( (1.37) \), then there exist constants \( C(\lambda, \delta) \) and \( C(\lambda) \) such that

\[
C(\lambda, \delta) \varepsilon_0^0(f) - C(\lambda) \|f\|_{L^2}^2 \lesssim \varepsilon_0^0(f) \lesssim C(\lambda) (\varepsilon_0^0(f) + \|f\|_{L^2}^2),
\]

in other words, \( \varepsilon_0^0(f) + \|f\|_{L^2}^2 \sim \varepsilon_0(f) + \|f\|_{L^2}^2. \)

**Remark 3.1.** We remark that \( (3.7) \) holds under the assumption \( (1.5) \) or \( (1.7) \) or \( (1.8) \). Indeed we have

\[
W(\xi) = \begin{cases} 
\varepsilon(\xi)^2, & \text{under assumption } (1.5); \\
\varepsilon(\xi)^2 + \varepsilon^{s-1}(1 - \varepsilon(\xi)), & \text{under assumption } (1.7); \\
\varepsilon(\xi)^2 + \varepsilon^{s-1}(1 - \varepsilon(\xi) \langle \xi \rangle^s), & \text{under assumption } (1.8).
\end{cases}
\]

We recall that the function \( \psi \) is defined in \( (1.23) \). It is easy to check that for all the cases, the symbol function \( W \) satisfies the properties: \( W(|\xi||\xi|) \lesssim W(|\xi|) W(|\xi|) \) and \( W(\xi) \leq \langle \xi \rangle \).

**Proof.** The proof is inspired by [5]. Without loss of generality, we assume that the function \( g \) is non-negative. By Bobylev’s equality, we have

\[
\varepsilon_0^0(f) = \frac{1}{(2\pi)^3} \int_{\xi, \sigma} b(\xi) \cdot \sigma \Re((\hat{g}(0) - \hat{g}(\xi^-))(\hat{f}(\xi)^* \hat{f}(\xi))) \, d\xi \, d\sigma.
\]

We recall that \( \xi^- = \frac{\xi - |\xi|\sigma}{2} \) and \( \xi^+ = \frac{\xi + |\xi|\sigma}{2} \).

It implies

\[
\|\mu\|_{L^1} \varepsilon_0^0(f) = \|g\|_{L^1} \varepsilon_0^0(f) - \frac{1}{(2\pi)^3} I_1 + \frac{2\|\mu\|_{L^1}^2}{(2\pi)^3} I_2,
\]

where

\[
I_1 = \int_{\xi, \sigma} b(\xi) \cdot \sigma \Re((\hat{g}(0) - \hat{g}(\xi^-))(\hat{f}(\xi)^* \hat{f}(\xi))) \, d\xi \, d\sigma,
\]

and

\[
I_2 = \int_{\xi, \sigma} b(\xi) \cdot \sigma \Re((\hat{g}(0) - \hat{g}(\xi^-))(\hat{f}(\xi)^* - \hat{f}(\xi))(\hat{f}(\xi)^* \hat{f}(\xi))) \, d\xi \, d\sigma
\]

\[
\overset{\text{def}}{=} I_{2,1} + I_{2,2}.
\]
Thanks to the fact $\hat{\mu}(0) - \hat{\mu}(\xi^-) = \int_v (1 - \cos(v \cdot \xi^-)) \mu(v) dv$, we have

$$|I_1| = \left| \int_{\sigma, v, \xi} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) (1 - \cos(v \cdot \xi^-)) \mu(v) \text{Re}(\hat{f}(\xi)^+ \hat{f}(\xi)) d\sigma d\xi dv \right|$$
$$\lesssim \left( \int_{\sigma, v, \xi} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) (1 - \cos(v \cdot \xi^-)) \mu(v) |\hat{f}(\xi)^+|^2 d\sigma d\xi dv \right)^{1/2}$$
$$\times \left( \int_{\sigma, v, \xi} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) (1 - \cos(v \cdot \xi^-)) \mu(v) |\hat{f}(\xi)|^2 d\sigma d\xi dv \right)^{1/2}.$$

Observe that

$$(1 - \cos (v \cdot \xi^-)) \lesssim |v|^2 |\xi^-|^2 \leq |v|^2 |\xi |^2 \left| \frac{\xi}{|\xi|} - \sigma \right|^2 \sim |v|^2 |\xi |^2 \left| \frac{\xi}{|\xi|} + \sigma \right|^2$$

and

$$\frac{\xi}{|\xi|} \cdot \sigma = 2 \left( \frac{\xi}{|\xi|} + \sigma \right)^2 - 1.$$

Then by change of variables $\xi \to \xi^+$, the assumption (3.7) and the property $W(|\xi||\xi|) \lesssim W(|\xi|) W(|\xi|)$, we have

$$|I_1| \lesssim \int_{v, \xi} W^2(|v||\xi|) |\hat{f}(\xi)|^2 \mu(v) dv d\xi$$
$$\lesssim \|W^2 \mu\|_{L^1} \|W(D) f\|_{L^2}^2.$$

Notice that $\text{Re}(\hat{g}(0) - \hat{g}(\xi^-)) = \int_v (1 - \cos(v \cdot \xi^-)) g(v) dv$. The similar argument can be applied to get

$$|I_{2,1}| \lesssim \|W^2 g\|_{L^1} \|W(D) f\|_{L^2}^2.$$

Next, by Cauchy-Schwartz inequality, one has

$$|I_{2,1}| \lesssim \left( \int_{\xi, \sigma} b(\cos \theta) |\hat{g}(0) - \hat{g}(\xi^-)|^2 |\hat{f}(\xi)|^2 d\sigma d\xi \right)^{1/2}$$
$$\times \left( \int_{\xi, \sigma} b(\cos \theta) |\hat{f}(\xi) - \hat{f}(\xi^+)|^2 d\sigma d\xi \right)^{1/2} \overset{\text{def}}{=} (I_{2,1})^{1/2} (I_{2,1})^{1/2}.$$

Observe that $\hat{g}(0) - \hat{g}(\xi^-) = \int_v (1 - e^{-iv \cdot \xi^-}) g(v) dv$, then it holds

$$I_{2,1} \lesssim \int_{v, w, \xi} b(\cos \theta) g(v) g(w) [(1 - e^{-iv \cdot \xi^-})^2 + |1 - e^{-iw \cdot \xi^-}|^2] |\hat{f}(\xi)|^2 d\sigma d\xi dw dv$$
$$\lesssim \|g\|_{L^2} \|W^2 g\|_{L^1} \|W(D) f\|_{L^2}^2.$$

Thanks to (3.3), we have

$$\frac{1}{(2\pi)^3} \|\mu\|_{L^1} I_{2,1} = \mathcal{E}_{\mu}^0(f) - \frac{1}{(2\pi)^3} I_1,$$

which implies

$$I_{2,1} \lesssim \mathcal{E}_{\mu}^0(f) + \|W^2 \mu\|_{L^1} \|W(D) f\|_{L^2}^2.$$

We get

$$|I_{2,1}| \lesssim \eta \|g\|_{L^2} \mathcal{E}_{\mu}^0(f) + \eta^{-1} (\|W^2 g\|_{L^1} + \|g\|_{L^1} \|W^2 \mu\|_{L^1}) \|W(D) f\|_{L^2}^2.$$

Combining the above estimates, we arrive at

$$\|\mu\|_{L^1} \mathcal{E}_{\mu}^0(f) - C(\eta) \|W^2 g\|_{L^1} \|W(D) f\|_{L^2}^2$$
$$\lesssim (1 - \eta) \|g\|_{L^1} \mathcal{E}_{\mu}^0(f) \lesssim \|\mu\|_{L^1} \mathcal{E}_{\mu}^0(f) + C(\eta) \|W^2 g\|_{L^1} \|W(D) f\|_{L^2}^2,$$

(3.9)
which is enough to derive the first inequality in the lemma. Moreover, if the function \( g \) verifies the condition (3.9), then by the computation in (11) and the assumption (3.7), we have

\[
\mathcal{E}_g^0(f) + \|f\|_{L^2}^2 \gtrsim C_3(g)\|W(D)f\|_{L^2}^2.
\]

From which together with (3.9), we get the equivalence in the lemma. \( \square \)

In the next lemma, we will show that the lower bound of \( \mathcal{E}_g^Y(f) \) can be reduced to the lower bound in the maxwellian case.

**Lemma 3.4.** Suppose that the angular function \( b \) verifies the same conditions in Lemma 3.3 and \( g \) is a non-negative function verifying the condition (3.7). Then there exists a constant \( C(\lambda, \delta) \) such that

\[
\mathcal{E}_g^0(W_{y/2}f) \leq C(\lambda, \delta)(\|f\|_{L^2}^2 + \mathcal{E}_g^Y(f)).
\]

**Proof.** Let \( \chi \) be a radial and smooth function such that \( 0 \leq \chi \leq 1, \chi = 1 \) on \( B_1 \) and \( \text{Supp} \chi \subset B_2 \). We set \( \chi_R(v) = \langle v \rangle \chi \). We recall the notation: \( W_{y}(v) = \langle v \rangle \).

**Case 1:** \( |v| \) is sufficiently large. It is easy to check

\[
\mathcal{E}_g^Y(f) \geq \int_{v,v_*,\sigma} \langle v \rangle^Y (g\chi_{\pi})_* b(\cos\theta)(f' - f)^2(1 - \chi_{R})^2 d\sigma dv_* dv.
\]

Thanks to the inequality \( (a - b)^2 \geq \frac{1}{2} a^2 - b^2 \), we obtain

\[
\mathcal{E}_g^Y(f) \geq \frac{1}{2} \int_{v,v_*,\sigma} (g\chi_{\pi})_* b(\cos\theta)((W_{y/2}(1 - \chi_{R}))(f)' - W_{y/2}(1 - \chi_{R})f)' \right)^2 d\sigma dv_* dv
- 2\int_{v,v_*,\sigma} (g\chi_{\pi})_* b(\cos\theta) f'^2 ((W_{y/2}(1 - \chi_{R}))(f)' - W_{y/2}(1 - \chi_{R})f)' \right)^2 d\sigma dv_* dv.
\]

Suppose that \( \kappa(v) = v + \kappa(v' - v) \) with \( \kappa \in [0, 1] \). It is easy to check that \( \frac{\sqrt{2}}{2} |v - v_*| \leq |v' - v_*| \leq |\kappa(v) - v_*| \leq |v - v_*| \). Since now \( |v_*| \leq R/4 \), then if \( |v| \geq R \), we have \( |v| \sim |\kappa(v)| \sim |v - v_*| \). Similarly if \( |v'| \geq R \), we have \( |v'| \sim |\kappa(v')| \sim |v - v_*| \).

By the Mean Value Theorem, we get

\[
\left| \int_{v,v_*,\sigma} (g\chi_{\pi})_* b(\cos\theta) f'^2 ((W_{y/2}(1 - \chi_{R}))(f)' - W_{y/2}(1 - \chi_{R})f)' \right|^2 d\sigma dv_* dv
\leq \int_{v,v_*,\sigma} (g\chi_{\pi})_* b(\cos\theta) f'^2 ((\kappa(\langle v \rangle))(\langle v \rangle)^{Y-2}|v - v_*|^2 |\langle v \rangle - |v - v_*||\right)^2 d\sigma dv_* dv
\leq \int_{v,v_*,\sigma} (g\chi_{\pi})_* b(\cos\theta) f'^2 \|W_{y}f\|_{L^2}^2 d\sigma dv_* dv
\leq \|g\|_{L^2}^2 \|f\|_{L^2}^2.
\]

Thus we arrive at

\[
\int_{v,v_*,\sigma} (g\chi_{\pi})_* b(\cos\theta)((W_{y/2}(1 - \chi_{R}))(f)' - W_{y/2}(1 - \chi_{R})f)' \right)^2 d\sigma dv_* dv \lesssim \|g\|_{L^2}^2 \|f\|_{L^2}^2 + \mathcal{E}_g^Y(f).
\]

that is

\[
\mathcal{E}_g^0((1 - \chi_{R})W_{y/2}f) \lesssim \|g\|_{L^2}^2 \|f\|_{L^2}^2 + \mathcal{E}_g^Y(f).
\]

(3.11)
Case 2: $|v|$ is bounded. Let $A, B$ be the subsets in $B_{3R}$. We denote $\chi_A$ and $\chi_B$ by the mollified characteristic functions corresponding to the sets $A$ and $B$. Then it yields
\[
\int_{v, v', v''} b(\cos \theta)(g\chi_B, s)^2 (\chi_A' - \chi_A)^2 d\sigma d v' d v
\leq \int_{v, v', v''} b(\cos \theta)(g\chi_B, s)^2 (\chi_A' - \chi_A)^2 1_{|v - v'| \leq 4R} d\sigma d v' d v
\leq \|\nabla \chi_A\|_{L^\infty}^2 \int_{v, v', v''} b(\cos \theta)(g\chi_B, s)^2 |v - v'|^2 d\sigma d v' d v
\leq R^2 \|\nabla \chi_A\|_{L^\infty}^2 \int_{v, v', v''} b(\cos \theta)(g\chi_B, s)^2 1_{|v'| \leq 6R} d\sigma d v' d v
\leq \|\nabla \chi_A\|_{L^\infty}^2 R^2 \min(1, M) \|g\|_{L^1} \|f\|_{L^{2,1}}^2.
\]
(3.12)

With the help of Lemma 2.1 in [1] and replacing (35) in [1] by (3.12), we conclude that if $\gamma < 0$,
\[
R^\gamma \int_{v, v', v''} (g\chi_B, s)^2 b(\cos \theta)((\chi_R f)' - \chi_R f)^2 d\sigma d v' d v \leq R^{-\gamma} \|g\|_{L^1} \|f\|_{L^{2,1}}^2 + \mathcal{C}_g(f),
\]
and if $\gamma > 0$,
\[
r_0^{-\gamma} \int_{v, v', v''} (g\chi_B, s)^2 b(\cos \theta)((\chi_A f)' - \chi_A f)^2 d\sigma d v' d v \leq r_0^{-\gamma} R^2 \|g\|_{L^1} \|f\|_{L^{2,1}}^2 + \mathcal{C}_g(f),
\]
where $\chi_A = \chi(\frac{v - v_0}{r_0})$, $\chi_B = \chi_{3R} - \chi(\frac{v - v_0}{3r_0})$ with $v_j \in B_{3R}$ and $r_0$ will be chosen later. Notice that
\[
(\chi_A f)' - \chi_A f = ((\chi_A W_{7/2} f)' - (\chi_A W_{7/2} f))W_{-7/2} + (\chi_A W_{7/2} f)'(W_{-7/2} f)' - W_{-7/2} f).
\]
By slight modification, we may derive that if $\gamma < 0$,
\[
\int_{v, v', v''} (g\chi_B, s)^2 b(\cos \theta)((\chi_R W_{7/2} f)' - (\chi_R W_{7/2} f))^2 d\sigma d v' d v
\leq R^{2-2\gamma} \|g\|_{L^1} \|f\|_{L^{2,1}}^2 + \mathcal{C}_g(f),
\]
(3.13)
and if $\gamma > 0$,
\[
r_0^{-2\gamma} R^2 \|g\|_{L^1} \|f\|_{L^{2,1}}^2 + r_0^{-\gamma} \mathcal{C}_g(f).
\]
(3.14)

By finite covering theorem, there exists an integer $N$ such that
\[
B_{2R} \subset \bigcup_{j=1}^N |v - v_j| \leq r_0 \quad \text{and} \quad N \sim \left(\frac{R}{r_0}\right)^3.
\]
(3.15)

Observe that
\[
\|g\chi_{R/8}\|_{L^1} \geq \|g\|_{L^1} - R^{-1} \|g\|_{L^1}
\]
and
\[
\|g\chi_{B_j}\|_{L^1} \geq \|g\|_{L^1} - (3R)^{-1} \|g\|_{L^1} - M(6r_0)^3 - (\log M)^{-1} \|g\|_{L^1}.\log L.
\]
Then choose $R = \frac{4A}{3\delta} + 1$, $M = e^{A/\delta}$ and $r_0 = \frac{1}{6} e^{-A/\delta}$ and it follows
\[
N \sim 6^3 \left(\frac{4A}{3\delta} + 1\right)^3 e^{A/\delta} \equiv C_1(\delta, \lambda)
\]
and
\[
\|g\chi_{R/8}\|_{L^1} \geq \delta/4, \|g\chi_{B_j}\|_{L^1} \geq \delta/4.
\]
Then there exists a constant $C(\lambda, \delta)$ such that
\[
\mathcal{E}_3(\gamma x R_{\frac{1}{3}}) \geq C(\lambda, \delta), \quad \mathcal{E}_3(\gamma x R_{\frac{1}{3}}) \geq C(\lambda, \delta).
\]
Thanks to (3.3) and (3.10) in the proof of Lemma 3.3 we may rewrite (3.11) as:
\[
(3.16) \quad \mathcal{E}_\mu^0((1 - \chi) \lambda) \mathcal{W}_{\gamma/2}f) \lesssim C(\lambda, \delta)(\|f\|_{L_{\gamma/2}}^2 + \mathcal{E}_g^\gamma(f)),
\]
\[
(3.17) \quad \mathcal{E}_\mu^0(\chi_A \lambda) \mathcal{W}_{\gamma/2}f) \lesssim C(\lambda, \delta)(\|f\|_{L_{\gamma/2}}^2 + \mathcal{E}_g^\gamma(f)), \quad \text{if } \gamma > 0,
\]
\[
(3.18) \quad \mathcal{E}_\mu^0(\chi_R \lambda) \mathcal{W}_{\gamma/2}f) \lesssim C(\lambda, \delta)(\|f\|_{L_{\gamma/2}}^2 + \mathcal{E}_g^\gamma(f)), \quad \text{if } \gamma < 0.
\]
We conclude that (3.16) and (3.18) yield the desired result for the soft potentials. For $\gamma > 0$, thanks to the fact
\[
\mathcal{E}_\mu^0(\chi_A \lambda) \mathcal{W}_{\gamma/2}f) \geq \frac{1}{2} \int_{\sigma \in SS^2, \nu, \nu_*, \in R^3} \mu_b(\cos \theta) \chi_A^2 ((\mathcal{W}_{\gamma/2}f)' - (\mathcal{W}_{\gamma/2}f))^2 d\sigma d\nu_* d\nu
\]
\[
- \int_{\sigma \in SS^2, \nu, \nu_*, \in R^3} \mu_b(\cos \theta)(\chi_A - \chi_A')^2 ((\mathcal{W}_{\gamma/2}f))^2 d\sigma d\nu_* d\nu,
\]
together with (3.17), we have
\[
\int_{\sigma \in SS^2, \nu, \nu_*, \in R^3} \mu_b(\cos \theta) \chi_A^2 ((\mathcal{W}_{\gamma/2}f)' - (\mathcal{W}_{\gamma/2}f))^2 d\sigma d\nu_* d\nu
\]
\[
\lesssim C(\lambda, \delta)(\|f\|_{L_{\gamma/2}}^2 + \mathcal{E}_g^\gamma(f)).
\]
From which together with (3.15) and (3.16), we are led to the desired result for the hard potentials. We complete the proof of the lemma.

We are now in a position to complete the proof of Theorem 1.2. The desired results in the theorem are easily derived from Lemma 3.1, Lemma 3.2 and Lemma 3.4.

Finally we give the proof to Theorem 1.3.

Proof. Following the computation in [1], we first have
\[
D(f) = \int_{\sigma, \nu, \nu_*} |\nu - \nu_*|^\gamma b(\cos \theta) f_* (f \log \frac{f}{f_*} - f + f') d\nu_*, d\sigma
\]
\[
- \int_{\sigma, \nu, \nu_*} |\nu - \nu_*|^\gamma b(\cos \theta) (f - f') d\nu_* d\sigma
\]
\[
\geq \int_{\sigma, \nu, \nu_*} |\nu - \nu_*|^\gamma b(\cos \theta) f_* (\sqrt{f'} - \sqrt{f})^2 d\nu_* d\sigma
\]
\[
- \int_{\sigma, \nu, \nu_*} |\nu - \nu_*|^\gamma b(\cos \theta) f_* (f - f') d\nu_* d\sigma,
\]
where we use the inequality $x \log \frac{x}{y} - x + y \geq (\sqrt{x} - \sqrt{y})^2$. By the estimates in Theorem 1.2 we arrive at the desired result for the case $\gamma \geq 0$.

For the soft potentials, $\gamma < 0$, we observe that
\[
D(f) = \frac{1}{4} \int_{\sigma, \nu, \nu_*} |\nu - \nu_*|^\gamma b(\cos \theta) (f' f_*' - f f_*) \log \frac{f' f_*}{f_* f} d\nu_* d\sigma
\]
\[
\geq \frac{1}{4} \int_{\sigma, \nu, \nu_*} \langle \nu - \nu_* \rangle^\gamma b(\cos \theta) (f' f_*' - f f_*) \log \frac{f' f_*}{f_* f} d\nu_* d\sigma
\]
\[
= \int_{\sigma, \nu, \nu_*} \langle \nu - \nu_* \rangle^\gamma b(\cos \theta) f_* (\sqrt{f'} - \sqrt{f})^2 d\nu_* d\sigma
\]
\[
- \int_{\sigma, \nu, \nu_*} \langle \nu - \nu_* \rangle^\gamma b(\cos \theta) f_* (f - f') d\nu_* d\sigma.
\]
Then the desired result is followed by the estimates in Theorem 1.2. We end the proof of the theorem.

\[ \square \]

### 3.2. Proof of Theorem 1.4

Now we are ready to give the complete proof to Theorem 1.4.

**Proof.** To give sharp bound to the operator in anisotropic space, we only need to give an alternative estimate to $\mathcal{M}^2_{k,l}$ and $\mathcal{M}^3_{k,p}$. By using the geometric decomposition (1.28), we easily get

\[ \mathcal{M}^3_{k,p} = \mathcal{M}^3_{k,p} + \mathcal{M}^3_{k,p}, \]

where

\[ \begin{align*}
\mathcal{M}^3_{k,p} &= \int_{\sigma \in \mathbb{S}^2, \nu_r, \nu_r \in \mathbb{R}^3} \Phi_k^\gamma(|u|) b(\sigma \cdot \tau)(\tilde{s}_p g) \ast (T_{\nu_r} \tilde{s}_p h)(\tau) \\
&\times ((T_{\nu_r} \tilde{s}_p f)(\tau) - (T_{\nu_r} \tilde{s}_p f)(\tau)) r^2 d\sigma d\nu_r d\nu_r, \\
\mathcal{M}^3_{k,p} &= \int_{\sigma \in \mathbb{S}^2, \nu_r, \nu_r \in \mathbb{R}^3} \Phi_k^\gamma(|\nu_r - \nu_r|) b(\cos \theta)(\tilde{s}_p g) \ast (\tilde{s}_p h) \\
&\times ((\tilde{s}_p f)(\nu_r + u^+ - (\tilde{s}_p f)(\nu_r + |u| u^+)) d\sigma d\nu_r d\nu_r.
\end{align*} \]

For the term $\mathcal{M}^3_{k,p}$, by change of variables: $\sigma \to \zeta$, it follows

\[ \begin{align*}
\mathcal{M}^3_{k,p} &= \int_{\zeta, \zeta \in \mathbb{S}^2, \nu_r, \nu_r \in \mathbb{R}^3} \Phi_k^\gamma(r) b(2(\zeta \cdot \tau)^2 - 1)(\tilde{s}_p g) \ast (T_{\nu_r} \tilde{s}_p h)(\tau) \\
&\times ((T_{\nu_r} \tilde{s}_p f)(\tau) - (T_{\nu_r} \tilde{s}_p f)(\tau)) r^2 4(\zeta \cdot \tau) d\zeta d\nu_r d\nu_r \\
&= \int_{\nu_r, \nu_r \in \mathbb{R}^3} \int_{\tau, \tau \in \mathbb{R}} \int_{\zeta, \zeta \in \mathbb{S}^2} \Phi_k^\gamma(r) (\tilde{s}_p g) \ast r^2(\tilde{s}_p h)(\tau) d\tau d\nu_r d\nu_r \\
&\times ((T_{\nu_r} \tilde{s}_p f)(\tau) - (T_{\nu_r} \tilde{s}_p f)(\tau)) d\zeta d\tau.
\end{align*} \]

Thanks to Corrolary 5.1 and Lemma 5.10, we infer for $a, b \in [0, 2s]$ with $a + b = 2s$,

\[ |\mathcal{M}^3_{k,p}| \lesssim 2^{\gamma k} \int_{\nu_r} |(\tilde{s}_p g) \ast (1 - \Delta_{SS^2})^{a/2} (T_{\nu_r} \tilde{s}_p h)\|_{L^2} \| (1 - \Delta_{SS^2})^{b/2} (T_{\nu_r} \tilde{s}_p f)\|_{L^2} d\nu_r \\
\lesssim 2^{\gamma k} \|\tilde{s}_p g\|_{L^2_{\nu_r}} (\|(-\Delta_{SS^2})^{a/2} \tilde{s}_p h\|_{L^2} + \|\tilde{s}_p h\|_{H^s}) (\|(-\Delta_{SS^2})^{b/2} \tilde{s}_p f\|_{L^2} + \|\tilde{s}_p f\|_{H^s}).
\]

From which together with Lemma 5.8 we deduce

\[ \sum_p |\mathcal{M}^3_{k,p}| \lesssim 2^{\gamma k} \|\tilde{s}_p g\|_{L^2_{\nu_r}} (\|(-\Delta_{SS^2})^{a/2} h\|_{L^2} + \|h\|_{H^s}) (\|(-\Delta_{SS^2})^{b/2} f\|_{L^2} + \|f\|_{H^s}).
\]

For the term $\mathcal{M}^3_{k,p}$, we may follow the argument used to bound $\mathcal{E}_{1,p}$ (see (3.4)) to get

\[ |\mathcal{M}^3_{k,p}| \lesssim 2^{(\gamma + s)k} 2^{2p} \|\tilde{s}_p g\|_{L^2} \|\tilde{s}_p h\|_{L^2} \|\tilde{s}_p f\|_{L^2}, \]

which implies

\[ \sum_p |\mathcal{M}^3_{k,p}| \lesssim 2^{(\gamma + s)k} \|\tilde{s}_p g\|_{L^2} \|h\|_{H^s} \|f\|_{H^s}, \]

where $a_1 + b_1 = s$ with $a_1, b_1 \in \mathbb{R}$.

We finally arrive at for $k \geq 0$

\[ \begin{align*}
\sum_p |\mathcal{M}^3_{k,p}| + \sum_l |\mathcal{M}^2_{k,l}| &\lesssim 2^{\gamma k} \|\tilde{s}_p g\|_{L^2_{\nu_r}} (\|(-\Delta_{SS^2})^{a/2} h\|_{L^2} + \|h\|_{H^s}) (\|(-\Delta_{SS^2})^{b/2} f\|_{L^2} + \|f\|_{H^s}) \\
&\quad + 2^{(\gamma + s)k} \|\tilde{s}_p g\|_{L^2} \|h\|_{H^s} \|f\|_{H^s}.
\end{align*} \]

Thanks to Lemma 2.1 and Lemma 2.2, we also have for $k \geq 0$,

\[ \sum_{l \leq p - N_0} |\mathcal{M}^1_{k,p,l}| + \sum_{m < p - N_0} |\mathcal{M}^4_{k,p,m}| \lesssim 2^{\gamma k} \|\tilde{s}_p g\|_{L^2} \|h\|_{H^s} \|f\|_{H^s}. \]
Now we are in a position to give the bound to the collision operator in anisotropic space. We conclude that for for $k \geq 0$,

$$
\langle (Q_k(g, h), f) \rangle \quad \lesssim \quad 2^{(\gamma + s)k} \| g \|_{L^1} \| h \|_{H^{\alpha_1}} \| f \|_{H^\gamma} \\
+ 2^{\gamma k} \| g \|_{L^2} \left( \| (-\Delta_{SS})^{a/2} h \|_{L^2} + \| h \|_{H^\gamma} \right) \left( \| (-\Delta_{SS})^{b/2} f \|_{L^2} + \| f \|_{H^\gamma} \right),
$$

(3.19)

and

$$
\langle (Q_{-1}(g, h), f) \rangle \quad \lesssim \quad (\| g \|_{L^1} + \| g \|_{L^2}) \| h \|_{H^\gamma} \| f \|_{H^\gamma}.
$$

Recalling (1.26), we rewrite it by

$$
(Q(g, h), f) = \sum_{k \geq N_0 - 1} \langle Q_k(g, h), f) \rangle + \sum_{j \geq k + N_0} \langle Q_{j}(g, h), f) \rangle
$$

$$
+ \sum_{|j-k| \leq N_0} \langle Q_k(g, h), f) \rangle
$$

$$
= \Omega_4 + \Omega_5 + \Omega_6.
$$

Thanks to (3.19) and (3.20), we will give the estimates term by term.

Suppose $\gamma + \omega_2 = \gamma + s$. It is not difficult to check

$$
|\Omega_4| \lesssim \sum_{k \geq N_0 - 1} \left( 2^{(\gamma + s)k} \| g \|_{L^1} \| h \|_{H^{\alpha_1}} \| f \|_{H^\gamma} + 2^\gamma \| g \|_{H^{\gamma + \omega_2}} \right)
$$

$$
\left( \| (-\Delta) h \|_{L^2} + \| h \|_{H^\gamma} \right) \left( \| (-\Delta) f \|_{L^2} + \| f \|_{H^\gamma} \right),
$$

which yields

$$
|\Omega_4| \lesssim \| g \|_{L^1} \left( \| (-\Delta) h \|_{L^2} + \| h \|_{H^\gamma} \right)
$$

$$
\left( \| (-\Delta) f \|_{L^2} + \| f \|_{H^\gamma} \right) + \| h \|_{H^{\gamma + \omega_2}} \| f \|_{H^\gamma}.
$$

For the term $\Omega_5$, it holds

$$
|\Omega_5| \lesssim \sum_{j \geq k + N_0, k \geq 0} \left( 2^{(\gamma + s)k} \| g \|_{L^1} \| h \|_{H^{\alpha_1}} \| f \|_{H^\gamma} \right)
$$

$$
\left( \| (-\Delta) h \|_{L^2} + \| h \|_{H^\gamma} \right) \left( \| (-\Delta) f \|_{L^2} + \| f \|_{H^\gamma} \right),
$$

$$
\sum_{j \geq k + N_0} \left( \| g \|_{L^1} + \| g \|_{L^2} \right) \| h \|_{H^\gamma} \| f \|_{H^\gamma}.
$$

From which, we obtain

(1) if $\gamma > 0$

$$
|\Omega_5| \lesssim \left( \| g \|_{L^1} + \| g \|_{L^2} \right) \left( \| (-\Delta) h \|_{L^2} + \| h \|_{H^\gamma} \right)
$$

$$
\left( \| (-\Delta) f \|_{L^2} + \| f \|_{H^\gamma} \right) + \| h \|_{H^{\gamma + \omega_2}} \| f \|_{H^\gamma}.
$$

(2) if $\gamma = 0$, for any $\delta > 0$,

$$
|\Omega_5| \lesssim \left( \| g \|_{L^1} + \| g \|_{L^2} \right) \left( \| (-\Delta) h \|_{L^2} + \| h \|_{H^\gamma} \right)
$$

$$
\left( \| (-\Delta) f \|_{L^2} + \| f \|_{H^\gamma} \right) + \| h \|_{H^{\gamma + \omega_2}} \| f \|_{H^\gamma}.
$$

(3) if $\gamma < 0$,

$$
|\Omega_5| \lesssim \left( \| g \|_{L^1} + \| g \|_{L^2} \right) \left( \| (-\Delta) h \|_{L^2} + \| h \|_{H^\gamma} \right)
$$

$$
\left( \| (-\Delta) f \|_{L^2} + \| f \|_{H^\gamma} \right) + \| h \|_{H^{\gamma + \omega_2}} \| f \|_{H^\gamma}.
$$
Finally we turn to give the estimate to $\mathcal{U}_6$, it holds

$$\|\mathcal{U}_6\| \lesssim \sum_{|j-k| \leq N_0, k \geq 0} \left( 2^{(\gamma + s)k} \|\mathcal{D}_j g\|_{L^1} \|\mathcal{A}_{k+N_0} f\|_{H^\gamma} + \|\mathcal{A}_{k+N_0} f\|_{H^\gamma} \right) \times \left( \|(-\Delta_{SS}^{a/2} \mathcal{A}_{k+N_0} f)\|_{L^2} + \|\mathcal{A}_{k+N_0} f\|_{H^\gamma} \right)$$

Then by Lemma 5.8 and (2.14), we are led to

(1) if $\gamma > 0$

$$\|\mathcal{U}_6\| \lesssim \left( \|g\|_{L^2_{l+2s}} + \|g\|_{L^1_{l+2s}, (-w_i^a + \epsilon(-w_2^a)} + \|g\|_{L^2} \right) \left( \|(-\Delta_{SS}^{a/2} h)\|_{L^2_{l+2s}} + \|h\|_{H^\gamma} \right) \times \left( \|(-\Delta_{SS}^{b/2} f)\|_{L^2_{l+2s}} + \|f\|_{H^\gamma} \right) + \|h\|_{H^\gamma} \|f\|_{H^\gamma}$$

(2) if $\gamma = 0$,

$$\|\mathcal{U}_6\| \lesssim \left( \|g\|_{L^2_{l+2s}} + \|g\|_{L^1_{l+2s}, (-w_i^a + \epsilon(-w_2^a)} + \|g\|_{L^2} \right) \left( \|(-\Delta_{SS}^{a/2} h)\|_{L^2_{l+2s}} + \|h\|_{H^\gamma} \right) \times \left( \|(-\Delta_{SS}^{b/2} f)\|_{L^2_{l+2s}} + \|f\|_{H^\gamma} \right) + \|h\|_{H^\gamma} \|f\|_{H^\gamma}$$

(3) if $\gamma < 0$,

$$\|\mathcal{U}_6\| \lesssim \left( \|g\|_{L^2_{l+2s}} + \|g\|_{L^1_{l+2s}, (-w_i^a + \epsilon(-w_2^a)} + \|g\|_{L^2} \right) \left( \|(-\Delta_{SS}^{a/2} h)\|_{L^2_{l+2s}} + \|h\|_{H^\gamma} \right) \times \left( \|(-\Delta_{SS}^{b/2} f)\|_{L^2_{l+2s}} + \|f\|_{H^\gamma} \right) + \|h\|_{H^\gamma} \|f\|_{H^\gamma}$$

Then the theorem follows by patching together all the above estimates to $\mathcal{U}_1, \mathcal{U}_5$ and $\mathcal{U}_6$. We complete the proof to the theorem. \hfill $\square$

4. **GRAZING COLLISIONS LIMIT AND SHARP BOUNDS FOR THE LANDAU COLLISION OPERATOR**

In this section, we will show that the strategy used to give the bounds for the Boltzmann collision operator is robust and then it can be applied to capture the intrinsic structure of the collision operator in the process of the grazing collisions limit. Before giving the estimates, we first introduce the special function $W^\epsilon$ defined by

$$W^\epsilon(x) = \psi(\epsilon x)(x) + \epsilon^{s-1}(1 - \psi(\epsilon x))\langle x \rangle^s,$$

which characterizes the symbol of the collision operator in the process of the limit. We emphasize that the function $\psi$ is defined in (1.23).

We begin with a technical lemma which describes the behavior of the fractional Laplace-Beltrami operator in the process of the limit. We postpone the proof to the end of Section 5.

**Lemma 4.1.** Suppose $0 < s < 1$. For any smooth function $f$ defined in $SS^2$, the following equivalence holds:

$$\|f\|_{L^2(SS^2)}^2 + \epsilon^{2s-2} \int_{\sigma^2} \frac{|f(\sigma) - f(\tau)|^2}{|\sigma - \tau|^{2s+2s}} 1_{|\sigma - \tau| \leq \epsilon} d\sigma d\tau \sim \|f\|_{L^2(SS^2)} + \|(-\Delta_{SS}^{a/2} f)\|_{L^2(SS^2)} + \epsilon^{2s-2} \|(-\Delta_{SS}^{b/2} f)\|_{L^2(SS^2)}$$

$$\sim \|f\|_{L^2(SS^2)} + \|(-\Delta_{SS}^{a/2} f)\|_{L^2(SS^2)} + \epsilon^{s-1}(-\Delta_{SS}^{s/2} f)\|_{L^2(SS^2)}.$$
where the projection operators $\mathbb{P}_{< \frac{1}{\epsilon}}$ and $\mathbb{P}_{> \frac{1}{\epsilon}}$ are defined as follows: if $f(\sigma) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m Y_l^m(\sigma)$, then

$$\left( \mathbb{P}_{\leq \frac{1}{\epsilon}} f(\sigma) \right) = \sum_{\|l\|+1 \leq \frac{1}{\epsilon}} \sum_{m=-l}^{l} f_l^m Y_l^m(\sigma),$$

$$\left( \mathbb{P}_{> \frac{1}{\epsilon}} f(\sigma) \right) = \sum_{\|l\|+1 > \frac{1}{\epsilon}} \sum_{m=-l}^{l} f_l^m Y_l^m(\sigma).$$

(4.2)

**Remark 4.1.** We remark that the projection operators $\mathbb{P}_{\leq \frac{1}{\epsilon}}$ and $\mathbb{P}_{> \frac{1}{\epsilon}}$ commutate with the fractional Laplace-Beltrami operator. Moreover, since the Laplace-Beltrami operator is a self-adjoint operator with orthogonal basis of the eigenfunctions, the spectrum theorem yields that

$$\|f\|_{L^2(S^6)}^2 + \|((-\Delta_{SSS})^{1/2})\mathbb{P}_{\leq \frac{1}{\epsilon}} + e^{2s-2}(-\Delta_{SSS})^{s/2}\mathbb{P}_{> \frac{1}{\epsilon}}\|f\|_{L^2(S^6)}^2 \sim \|W^c((-\Delta_{SSS})^{1/2})f\|_{L^2(S^6)}^2 + \|f\|_{L^2(S^6)}^2.$$

We first give the upper bounds of the operator in weighted Sobolev spaces and in the anisotropic spaces.

**Theorem 4.1.** Suppose $w_1, w_2 \in \mathbb{R}$ verifying $w_1 + w_2 = \gamma + 2$, $w_3, w_4 \in \mathbb{R}$ verifying $w_3 + w_4 = \gamma + 1$, $a_1, b_1 \in [0, 1]$ verifying $a_1 + b_1 = 1$ and $a, b \in [0, 2]$ verifying $a + b = 2$. Then for smooth functions $g, h$ and $f$, it hold

1. if $\gamma + 2 > 0$,

$$\|Q^L(\mathbb{L}, g, h)\|_{\mathcal{V}} \lesssim \|g\|_{L^{2}_{\gamma+2}(-w_1^+ + (-w_2^+)} \|h\|_{H^{a_1}_{w_1}} \|f\|_{H^{b_1}_{w_2}},$$

(4.3)

2. if $\gamma + 2 = 0$,

$$\|Q^L(\mathbb{L}, g, h)\|_{\mathcal{V}} \lesssim \|g\|_{L^{1}_{w_5}} \|h\|_{H^{a_1}_{w_5}} \|f\|_{H^{b_1}_{w_5}},$$

(4.4)

where $w_5 = \max(\delta, (-w_1^+) + (-w_2^+)$ with $\delta > 0$ which is sufficiently small,

3. if $\gamma + 2 < 0$,

$$\|Q^L(\mathbb{L}, g, h)\|_{\mathcal{V}} \lesssim \|g\|_{L^{1}_{w_5}} \|h\|_{H^{a_1}_{w_5}} \|f\|_{H^{b_1}_{w_5}},$$

(4.5)

where $w_5 = \max(-\gamma + 2, \gamma + 2(-w_1^+) + (-w_2^+),$

and

1. if $\gamma > 0$

$$\|Q^L(\mathbb{L}, g, h)\|_{\mathcal{V}} \lesssim \|g\|_{L^{1}_{\gamma+2}} \|g\|_{L^{1}_{\gamma+1}((-w_1^+)\} + (-w_4^+) + \|g\|_{L^{1}_{\gamma+2}} \left(\|((-\Delta_{SSS})^{1/2} h\|_{L^{2}_{\gamma+2}} + \|h\|_{H^{a_1}_{w_5}}\right) \times \left((-\Delta_{SSS})^{b/2} f\right)_{L^{2}_{\gamma+2}} + \|f\|_{H^{b_1}_{w_5}} + \|h\|_{H^{a_1}_{w_5}} \|f\|_{H^{b_1}_{w_5}} \right),$$

(2)

if $\gamma = 0$,

$$\|Q^L(\mathbb{L}, g, h)\|_{\mathcal{V}} \lesssim \|g\|_{L^{1}_{\gamma+2}} \|g\|_{L^{1}_{\gamma+1}((-w_1^+)\} + (-w_4^+) + \|g\|_{L^{1}_{\gamma+2}} \left(\|((-\Delta_{SSS})^{1/2} h\|_{L^{2}_{\gamma+2}} + \|h\|_{H^{a_1}_{w_5}}\right) \times \left((-\Delta_{SSS})^{b/2} f\right)_{L^{2}_{\gamma+2}} + \|f\|_{H^{b_1}_{w_5}} + \|h\|_{H^{a_1}_{w_5}} \|f\|_{H^{b_1}_{w_5}} \right),$$

(3)

if $\gamma < 0$,

$$\|Q^L(\mathbb{L}, g, h)\|_{\mathcal{V}} \lesssim \|g\|_{L^{1}_{\gamma+2}} \|g\|_{L^{1}_{\gamma+1}((-w_1^+)\} + (-w_4^+) + \|g\|_{L^{1}_{\gamma+2}} \left(\|((-\Delta_{SSS})^{1/2} h\|_{L^{2}_{\gamma+2}} + \|h\|_{H^{a_1}_{w_5}}\right) \times \left((-\Delta_{SSS})^{b/2} f\right)_{L^{2}_{\gamma+2}} + \|h\|_{H^{a_1}_{w_5}} \|f\|_{H^{b_1}_{w_5}} \right).$$


Proof. In (4.4), it is proved that for any smooth functions \(g, h\) and \(f\), there holds
\[
\lim_{\epsilon \to 0} \langle Q^\epsilon (g, h), f \rangle = \langle Q^1 (g, h), f \rangle,
\]
where \(Q^\epsilon\) is a collision operator with the cross-section \(B^\epsilon\) under the assumption (B1). Thus the bounds of the Landau operator can be reduced to the uniform bounds of the collision operator \(Q^\epsilon\) with respect to the parameter \(\epsilon\). Since \(Q^\epsilon\) is still the Boltzmann collision operator, we may copy the argument used in the proof of Theorem 1.1 and Theorem 1.4 to get the desired results.

Let us follow the same notations used in Theorem 1.1 and Theorem 1.4. In the next we want to point out the difference from the previous proofs. In order to cancel the singularity caused by the kernel and get the uniform estimates with respect to the parameter \(\epsilon\), we shall frequently make full use of the following fact:
\[
\int_0^{\pi/2} b^k (\cos \theta) \sin \theta \theta^2 \, d\theta = 1.
\]
It seems that there is no need to introduce the angular cutoff function \(\psi\) to make the decomposition for the term \(\mathcal{D}_k^1\) in the proof of Lemma 2.1, the terms \(\mathcal{E}_k^1\) and \(\mathcal{E}_k^2\) in the proof of Lemma 2.2, the term \(\mathcal{M}_{3,k,p}^3\) in the proof of Lemma 2.3 and the term \(\mathcal{E}_{l,p}\) in the proof of Lemma 3.2. Keep in mind of this and follow almost the same calculation, then we are led to the similar results stated in Lemma 2.1, Lemma 2.2 and Lemma 2.3 by replacing the index \(s\) by 1. In particular, we have
\[
|\mathcal{E}_{l,p}| \lesssim 2^{(l-p)/2} 2^k \|g\|_{L^1} \|\tilde{f}\|_{H^{1/2}} \|\tilde{f}\|_{H^{1/2}},
\]
where \(\mathcal{E}_{l,p}\) is defined in (3.4). With these in hand, following the same argument used in the proof of Theorem 1.1, will yield the desired results (4.3) and (4.4) and

Next we turn to the upper bounds of the Landau operator in the anisotropic spaces. Thanks to Lemma 4.1, we have
\[
\|
\begin{aligned}
&\epsilon^{2s-2} \int_{\sigma, \tau \in \mathcal{S}^2} \frac{|f(\sigma) - f(\tau)|^2}{|\sigma - \tau|^{2+2s}} 1_{|\sigma - \tau| \leq \epsilon} \, d\sigma \, d\tau \\
&\lesssim \|f\|_{L^2(\mathcal{S}^2)}^2 + \|(\Delta_{\mathcal{S}^2})^{1/2} f\|_{L^2(\mathcal{S}^2)}^2,
\end{aligned}
\]
which implies that for smooth functions \(g\) and \(h\), it holds
\[
\epsilon^{2s-2} \int_{\mathcal{S}^2 \times \mathcal{S}^2} \left( g(\sigma) - g(\tau) \right) h(\sigma - \tau) |\sigma - \tau|^{-(2s+2)} 1_{|\sigma - \tau| \leq \epsilon} \, d\sigma \, d\tau
\]
\[
\lesssim \sum_{l,m} g_l^m h_l^m \|(-\Delta_{\mathcal{S}^2})^{1/2} Y_l^m\|_{L^2(\mathcal{S}^2)}^2 + 1 \|(-\Delta_{\mathcal{S}^2})^{1/2} Y_l^m\|_{L^2(\mathcal{S}^2)}^2 + 1
\]
\[
\lesssim \sum_{l,m} g_l^m h_l^m (l(l+1)+1)^2
\]
\[
\lesssim \|((1-\Delta_{\mathcal{S}^2})^{a/2} g\|_{L^2(\mathcal{S}^2)}\|((1-\Delta_{\mathcal{S}^2})^{b/2} h\|_{L^2(\mathcal{S}^2)}^2,
\]
where \(a, b\) verifies \(a + b = 2\). It yields for \(a, b \in [0, 2]\) with \(a + b = 2\),
\[
|\mathcal{M}_{k,p}^{3,1}| \lesssim 2^{r-k} \int_{v, \tau} \|\mathcal{S} p g\|_{L^1} \|(1-\Delta_{\mathcal{S}^2})^{a/2} (T_v, \tilde{\mathcal{S}} p h)\|_{L^2} \|(1-\Delta_{\mathcal{S}^2})^{b/2} (T_v, \tilde{\mathcal{S}} p f)\|_{L^2} \, dv \, d\tau
\]
\[
\lesssim 2^{r-k} \|\mathcal{S} p g\|_{L^1} \|(1-\Delta_{\mathcal{S}^2})^{a/2} h\|_{L^2} + \|(1-\Delta_{\mathcal{S}^2})^{b/2} \tilde{\mathcal{S}} p f\|_{L^2} + \|\tilde{\mathcal{S}} p f\|_{H^2}.
\]
From which together with Lemma 5.8, one has
\[
\sum_p |\mathcal{M}_{k,p}^{3,1}| \lesssim 2^{r-k} \|g\|_{L^2} \|(1-\Delta_{\mathcal{S}^2})^{a/2} h\|_{L^2} + \|h\|_{H^2} \|(1-\Delta_{\mathcal{S}^2})^{b/2} f\|_{L^2} + \|f\|_{H^2}.
\]

Observing that the term \(\mathcal{M}_{k,p}^{3,2}\) enjoys the similar structure as the term \(\mathcal{E}_{l,p}\), (4.6) indicates that it is not difficult to derive
\[
|\mathcal{M}_{k,p}^{3,2}| \lesssim 2^{(r+1)k} 2^p \|g\|_{L^1} \|\tilde{\mathcal{S}} p h\|_{L^2} \|\tilde{\mathcal{S}} p f\|_{L^2},
\]
which implies

\[ \sum_p |\mathfrak{M}^{1,2}_{k,p}| < 2^{(\gamma+1)k} \|g\|_{L^1} \|h\|_{H^{\alpha/2}} \|f\|_{H^{\beta/2}}. \]

We finally arrive at for \( k \geq 0 \\
\sum_p |\mathfrak{M}^{3}_{k,p}| + \sum_l |\mathfrak{M}^{2}_{k,l}| < 2^{\gamma k} \|g\|_{L^2} (\|(-\Delta_{SS})^{n/2}h\|_{L^2} + \|h\|_{H^{\alpha}}) (\|(-\Delta_{SS})^{\beta/2}f\|_{L^2} + \|f\|_{H^\beta}) + 2^{(\gamma+1)k} \|g\|_{L^1} \|h\|_{H^{\alpha/2}} \|f\|_{H^{\beta/2}}.

Thanks to [2.1] and [2.2] we also have for \( k \geq 0 \\
\sum_{l \leq \rho-N_0} |\mathfrak{M}^{1}_{k,p,l}| + \sum_{m < \rho-N_0} |\mathfrak{M}^{4}_{k,p,m}| < 2^{(\gamma+1)k} \|g\|_{L^1} \|h\|_{H^{\alpha/2}} \|f\|_{H^{\beta/2}} + 2^{(\gamma+1)k} \|g\|_{L^1} \|h\|_{H^{\alpha/2}} \|f\|_{H^{\beta/2}},

We conclude that for for \( k \geq 0 \\
\langle Q_k^e(g, h), f \rangle \|_{\nu} \lesssim \langle g \|_{L^1} + \|g\|_{L^2} \rangle \|h\|_{H^{\alpha}} \|f\|_{H^{\beta}},

which is enough to derive the upper bounds in the anisotropic spaces by following the argument used in the proof of Theorem 1.4. This completes the proof of the theorem for the upper bounds of the operator.

The lower bounds of the Landau collision operator can be stated as:

**Theorem 4.2.** If the function \( g \) is a non-negative smooth function verifying (1.37), then for sufficiently small \( \eta \), there exist three constants \( C_1(\delta, \lambda, \eta), C_2(\lambda, \delta) \) and \( C_3(\delta, \lambda, \eta) \) such that

1. if \( \gamma + 2 > 0 \),
   \[ \langle -Q_L(g, f), f \rangle \|_{\nu} \gtrsim C_1(\delta, \lambda, \eta) \bigl( \|(-\Delta_{SS})^{1/2}f\|_{L^2} + \|f\|_{H^{1/2}}^2 \bigr) - \eta C_2(\delta, \lambda) \|f\|_{L^2} - \eta C_3(\delta, \lambda, \eta) \|f\|_{L^2}^{\gamma+2}, \]

2. if \( \gamma = -2 \),
   \[ \langle Q_L(g, f), f \rangle \|_{\nu} \gtrsim C_1(\delta, \lambda, \eta) \bigl( \|(-\Delta_{SS})^{1/2}f\|_{L^2} + \|f\|_{H^{1/2}}^2 \bigr) - \eta C_2(\delta, \lambda) \|f\|_{L^2}^2 \]
   \[ \quad - \exp \{ (C_3(\delta, \lambda, \eta) + \|g\|_{L^1})^{\alpha(w/\eta)} \} \|f\|_{L^2}^2, \]

where \( w > 0 \).

3. if \( -3 < \gamma < -2 \),
   \[ \langle Q_L(g, f), f \rangle \|_{\nu} \gtrsim C_1(\delta, \lambda, \eta) \bigl( \|(-\Delta_{SS})^{1/2}f\|_{L^2} + \|f\|_{H^{1/2}}^2 \bigr) - \eta C_2(\delta, \lambda) \|f\|_{L^2}^2 \]
   \[ \quad - C_3(\delta, \lambda, \eta) \|g\|_{L^1}^{(\gamma+5)p} \|f\|_{L^2}^2, \]

with \( p > \frac{3}{\gamma+5} \).

**Proof.** We first focus on the lower bound to the functional \( \langle -Q^e(g, f), f \rangle \) where \( g \) satisfies the condition (1.37). Observe that

\[ \langle -Q^e(g, f), f \rangle = \frac{1}{2} \mathcal{L}^e_g(f) - \frac{1}{2} \mathcal{L}^e_k(f), \]
where
\[ \mathcal{E}_g^\gamma(f) \overset{\text{def}}{=} \int_{\mathbb{S}^2} |v - v_*|^\gamma g_* b^\epsilon(\cos \theta)(f^\epsilon - f)^2 d\sigma d v_* d v, \]
\[ \mathcal{L}_g^\epsilon(f) \overset{\text{def}}{=} \int_{\mathbb{R}^3} d v_* d v \int_{\mathbb{S}^2} B^\epsilon(|v - v_*|, \sigma) g_*(f^\epsilon - f)^2 d\sigma. \]

By the cancellation lemma, it holds
\[ |\mathcal{L}_g^\epsilon(f)| \lesssim \mathcal{R}, \]
where we have the control on the right-hand side thanks to Lemma 3.1. Therefore we have
\[ \langle -Q^\epsilon(g, f), f \rangle \geq \frac{1}{2} \mathcal{E}_g^\gamma(f) - C \mathcal{R}. \]

To give the lower bound of \( \mathcal{E}_g^\gamma(f) \), we begin with the typical case \( \gamma = 0 \). Due to the geometric decomposition, we have
\[ \mathcal{E}_g^{0,\epsilon}(f) \geq \mathcal{E}_{1,g}^{0,\epsilon}(f) - \mathcal{E}_{2,g}^{0,\epsilon}(f), \]
where
\[ \mathcal{E}_{1,g}^{0,\epsilon}(f) \overset{\text{def}}{=} \frac{1}{2} \int_{u,v,\sigma} g_* b^\epsilon(\cos \theta)(T_{v_*} f)(r_c) - (T_{v_*} f)(r_\tau))^2 d\sigma d v_* d u, \]
\[ \mathcal{E}_{2,g}^{0,\epsilon}(f) \overset{\text{def}}{=} 2 \int_{u,v,\sigma} g_* b^\epsilon(\cos \theta)(f(v_* + u^\epsilon) - f(v_* + |u| u^\epsilon)) d\sigma d v_* d u. \]

**Step 1: Estimate of \( \mathcal{E}_{1,g}^{0,\epsilon}(f) \):** As a direct consequence of Lemma 4.1, we first derive the lower bound of \( \mathcal{E}_{1,g}^{0,\epsilon}(f) \), that is, if \( g \geq 0 \), then
\[ \mathcal{E}_{1,g}^{0,\epsilon}(f) + \|g\|_{L^1} \|f\|_{L^2}^2 \lesssim \int_{\mathbb{R}^3} g_* \|W^\epsilon((-\Delta_{\mathbb{S}^2})^{1/2}) T_{v_*} f\|_{L^2}^2 d v_* + \|g\|_{L^1} \|f\|_{L^2}^2. \]

**Step 2: Estimate of \( \mathcal{E}_{2,g}^{0,\epsilon}(f) \):** Recalling the estimate of \( \mathcal{E}_\gamma^0 \) (defined in (3.3)), we make a slight modification of the proof to get
\[ |\mathcal{E}_{2,g}^{0,\epsilon}(f)| \lesssim \|g\|_{L^1} \|f\|_{H^1}^2. \]

We remark that the difference lies in the observation
\[ \int_0^{\pi/2} b^\epsilon(\cos \theta) \sin \theta \theta^2 d\theta \sim 1, \]
which means there is no need to split \( \mathcal{E}_{2,g}^{0,\epsilon}(f) \) into two parts: cut off and non cut off parts.

We finally arrive at
\[ \mathcal{E}_g^{0,\epsilon}(f) + \|g\|_{L^1} \|f\|_{L^2}^2 \geq \int_{\mathbb{R}^3} g_* \|W^\epsilon((-\Delta_{\mathbb{S}^2})^{1/2}) T_{v_*} f\|_{L^2}^2 d v_* - \|g\|_{L^1} (\eta^{-1} \|f\|_{H^1} + \eta \|f\|_{L^2}^2) + \|g\|_{L^1} \|f\|_{H^1}^2 \]
\[ \geq \int_{\mathbb{R}^3} g_* \|W^\epsilon((-\Delta_{\mathbb{S}^2})^{1/2}) T_{v_*} f\|_{L^2}^2 d v_* - \|g\|_{L^1} (\eta^{-1} \|\psi(e D) f\|_{H^1}^2 \]
\[ + \eta^{-1} \|\psi(e D) f\|_{L^2}^2 + \|g\|_{L^2}^2 \]

Thanks to the condition (1.37), we also have
\[ \mathcal{E}_g^{0,\epsilon}(f) + \|g\|_{L^1} \|f\|_{L^2}^2 \geq C(\lambda, \delta) \|W^\epsilon(D) f\|_{L^2}^2, \]
which was proven in [14]. Combining these two inequalities, we obtain
\[ \mathcal{E}_g^{0,\epsilon}(f) + \|g\|_{L^1} (\eta^{-1} \|\psi(e D) f\|_{H^1}^2 + \eta^{-1} \|f\|_{L^2}^2 \]
\[ \geq C(\lambda, \delta, \eta) \|W^\epsilon(D) f\|_{L^2}^2 + \int_{\mathbb{R}^3} g_* \|W^\epsilon((-\Delta_{\mathbb{S}^2})^{1/2}) T_{v_*} f\|_{L^2}^2 d v_* \].
By Lemma\textsuperscript{3.4} and the condition (1.37), we have
\begin{equation}
C(\lambda, \delta, \eta)\left(\int_{\mathbb{R}^3} u \ast W_{\mathbb{R}^3}((\Delta_{\mathbb{S}^2})^{1/2}) T_{\psi} W_{\gamma/2} f \left\| \gamma/2 \right\|_{L^2}^2 d v_{\ast} + \| W_{\mathbb{R}^3} (D) \left\| \gamma/2 \right\|_{L^2}^2 \right)
\end{equation}
(4.8) \begin{equation}
\lesssim C(\delta, \lambda)\left(\varepsilon_g \varepsilon_f (f) + \left\| f \right\|^2_{\gamma/2} + C_{\eta} (\left\| f \right\|^2_{\gamma/2} + \left\| (1 - \psi (\varepsilon D)) W_{\gamma/2} f \right\|^2_{H^1} + \eta \left\| f \right\|^2_{\gamma/2}) \right).
\end{equation}

Noticing that for smooth function \( f \), we have
\begin{equation}
\langle -Q_l (g, f), f \rangle_{\nu} = \lim_{\epsilon \to 0} \langle -Q^\epsilon (g, f), f \rangle \geq \lim_{\epsilon \to 0} \frac{1}{2} \varepsilon \psi (\varepsilon (\psi D))^2 (f) - C \mathcal{R},
\end{equation}
from which together with Fatou Lemma, (4.8) and Lemma\textsuperscript{3.1} we arrive at the desired results. \( \square \)

5. Toolbox: Weighted Sobolev Spaces, Interpolation Theory and the Profile of the Laplace-Beltrami Operator

In this section, we first list some preliminaries and then give a new profile of the weighted Sobolev Spaces. We remark that these theories are the key points to give the new sharp upper and lower bounds for the collision operator in weighted Sobolev spaces. Then we will state a new version of interpolation theory which slightly relaxes the assumption that operators need to be commutated with each other. Then in the next we will list some basic properties of the real spherical harmonics and introduce the definition of the Laplace-Beltrami operator. After giving the profile of the fractional Laplace-Beltrami operator on the unit sphere, we will give a detailed proof to (1.30) and (1.31) which are crucial to capture the anisotropic structure of the collision operator. We address that results in this section have independent interest.

5.1. Weighted Sobolev spaces. Before stating the results, we list some basic facts which will be used in the proof of the new profile of the weighted Sobolev spaces.

Lemma 5.1 (Bernstein inequalities). There exists a constant \( C \) independent of \( j \) and \( f \) such that
1) For any \( s \in \mathbb{R} \) and \( j \geq 0 \),
\begin{equation}
C^{-1} 2^{j s} \| \mathcal{F} f \|_{L^2(\mathbb{R}^3)} \leq \| \mathcal{F} f \|_{H^s(\mathbb{R}^3)} \leq C 2^{j s} \| \mathcal{F} f \|_{L^2(\mathbb{R}^3)}.
\end{equation}
2) For integers \( j, k \geq 0 \) and \( p, q \in [1, \infty] \), the Bernstein's inequality is shown as
\begin{equation}
\sup_{|\alpha|=k} \| \partial^\alpha \mathcal{F} f \|_{L^q(\mathbb{R}^3)} \lesssim 2^{j k} 2^{j (1-q-1)/p} \| \mathcal{F} f \|_{L^p(\mathbb{R}^3)},
\end{equation}
(5.2) \begin{equation}
2^{j k} \| \mathcal{F} f \|_{L^p(\mathbb{R}^3)} \lesssim \sup_{|\alpha|=k} \| \partial^\alpha \mathcal{F} f \|_{L^p(\mathbb{R}^3)} \lesssim 2^{j k} \| \mathcal{F} f \|_{L^p(\mathbb{R}^3)}.
\end{equation}
3) For any \( f \in H^s \), it holds that
\begin{equation}
\| f \|_{H^s(\mathbb{R}^3)} \sim \sum_{k=-1}^{\infty} 2^{ks} \| \mathcal{F}_k f \|_{L^2(\mathbb{R}^3)}.
\end{equation}

Lemma 5.2. (see \textsuperscript{[15]}) Let \( s, r \in \mathbb{R} \) and \( a(v), b(v) \in C^\infty \) satisfy for any \( \alpha \in \mathbb{Z}^3_+ \),
\begin{equation}
|D^\alpha_x a(v)| \leq C_{1,\alpha} \langle v \rangle^{r-|\alpha|}, |D^\alpha_x b(\xi)| \leq C_{2,\alpha} \langle \xi \rangle^{s-|\alpha|}
\end{equation}
for constants \( C_{1,\alpha}, C_{2,\alpha} \). Then there exists a constant \( C \) depending only on \( s, r \) and finite numbers of \( C_{1,\alpha}, C_{2,\alpha} \) such that for any \( f \in \Sigma(\mathbb{R}^3), \)
\begin{equation}
\| a(v) b(D) f \|_{L^2} \leq C \| \langle D \rangle^{s} \langle v \rangle^{r} f \|_{L^2},
\end{equation}
\begin{equation}
\| b(D) a(v) f \|_{L^2} \leq C \| \langle v \rangle^{r} \langle D \rangle^{s} f \|_{L^2}.
\end{equation}

Remark 5.1. As a direct consequence, we get
\begin{equation}
\| \langle D \rangle^m \langle v \rangle^{r} f \|_{L^2} \sim \| \langle v \rangle^{r} \langle D \rangle^m f \|_{L^2} \sim \| f \|_{H^m}.\end{equation}
**Definition 5.1.** A smooth function $a(v, \xi)$ is said to a symbol of type $S_{1,0}^m$ if $a(v, \xi)$ verifies for any multi-indices $\alpha$ and $\beta$,

$$|\partial^\beta_v \partial^\alpha_\xi a(v, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m - |\alpha|},$$

where $C_{\alpha, \beta}$ is a constant depending only on $\alpha$ and $\beta$.

**Lemma 5.3.** Let $l, s \in \mathbb{R}$, $M(\zeta) \in S_{1,0}^l$ and $\Phi(v) \in S_{1,0}^l$. Then there exists a constant $C$ such that

$$\| [M(D_v), \Phi(v)] f \|_{H^s} \leq C \| f \|_{H^s_{1,1}}.$$  

**Proof.** We prove it in the spirit of [15]. Thanks to expansion of the pseudo-differential operator, there holds for any $N \in \mathbb{N}$,

$$M(D_v)\Phi = \Phi M(D_v) + \sum_{1 \leq |\alpha| < N} \frac{1}{\alpha!} \Phi_{\alpha} M^\alpha(D_v) + r_N(v, D_v)$$

where $\Phi_{\alpha}(v) = \partial^\alpha_v \Phi$, $M^\alpha(\xi) = \partial^\alpha_\xi M(\xi)$ and

$$r_N(v, \xi) = N \sum_{|\alpha| = N} \int_0^1 \frac{(1 - \tau)^{N-1}}{\alpha!} r_{N,\tau,\alpha}(v, \xi) d\tau.$$

Here

$$r_{N,\tau,\alpha}(v, \xi) = \int \left[(1 - \Delta_y)^n \Phi_{\alpha}(v + y)\right] I(\xi; y)^{-2m} d y$$

with $2m > N - l + 3$, $2n > N - r + 3$ and

$$I(\xi, y) = \frac{1}{(2\pi)^3} \int e^{-i\eta y} (1 - \Delta_\eta)^m \left[ \langle \eta \rangle^{-2n} M^\alpha(\xi + \tau \eta) \right] d\eta.$$  

It is not difficult to check that for there holds uniformly with respect to $\tau \in [0, 1],

$$|D^\beta_v \partial^\alpha_\xi r_{N,\tau,\alpha}(v, \xi)| \leq C_{\beta, \alpha} \langle \xi \rangle^{-N - |\beta|} \langle v \rangle^{l - N - |\alpha|}.$$  

Then (5.4) and Lemma 5.2 lead us to the lemma with $s = 0$. The case $s \neq 0$ can be treated similarly and we skip the proof here. \qed

Now we are in a position to give the new profile of the weighted Sobolev spaces $H^m_l(\mathbb{R}^3)$.

**Theorem 5.1.** Let $m, l \in \mathbb{R}$. Then for $f \in H^m_l(\mathbb{R}^3)$, we have

$$\sum_{k = -1}^\infty 2^{2k} \| \mathcal{P}_k f \|_{H^m}^2 \sim \| f \|_{H^m_l}^2.$$  

**Proof.** We first observe that $2^{k(l+1)} \varphi(\frac{v}{2^k})$ verifies the condition in Lemma 5.2. Then we have

$$2^{2k} \| \mathcal{P}_k f \|_{H^m}^2 = 2^{-2k} \| 2^{k(l+1)} \mathcal{P}_k f \|_{H^m}^2 \leq 2^{-2k} \| (D_v)^m \langle v \rangle^{l+1} \mathcal{P}_k f \|_{L^2}^2 \leq 2^{-2k} \left[ \| \langle v \rangle^{l+1} \mathcal{P}_k (D_v)^m f \|_{L^2}^2 + \| f \|_{H^m_{l-1}}^2 \right],$$

where we use Lemma 5.3 in the last step. This implies

$$\sum_{k = -1}^\infty 2^{2k} \| \mathcal{P}_k f \|_{H^m}^2 \leq \sum_{k = -1}^\infty \| \langle v \rangle^{l+1} \mathcal{P}_k (D_v)^m f \|_{L^2}^2 + \| f \|_{H^m_{l-1}}^2 \leq \| f \|_{H^m_l}^2.$$  

We are now ready to prove Proposition 4.8:

**Proof of Proposition 4.8.** We first note that by the definition of $S_{1,0}^l$, $\rho(\xi)$ verify the condition in Lemma 5.2. Then we have

$$\| \langle v \rangle^{l+1} \mathcal{P}_k (D_v)^m f \|_{L^2} \leq C \| f \|_{H^m_{l-1}}.$$  

Additionally, we have

$$\| \langle v \rangle^{l+1} \mathcal{P}_k (D_v)^m f \|_{L^2} \leq C \| f \|_{H^m_{l-1}}.$$  

where $C$ is a constant depending only on $l$ and $m$. Then we have

$$\sum_{k = -1}^\infty \| \langle v \rangle^{l+1} \mathcal{P}_k (D_v)^m f \|_{L^2}^2 + \| f \|_{H^m_{l-1}}^2 \leq \| f \|_{H^m_l}^2.$$  

Finally, we have

$$\sum_{k = -1}^\infty 2^{2k} \| \mathcal{P}_k f \|_{H^m}^2 \sim \| f \|_{H^m_l}^2$$

as desired. \qed
To prove the inverse inequality, we first treat with the case \( m \geq 0 \). Thanks to Remark 5.1, we have
\[
\|f\|^2_{H^m} \sim \sum_{k=-1}^\infty \|\mathcal{D}_k \langle \cdot \rangle^k \langle D \rangle^m f\|^2_{L^2} \sim \sum_{k=-1}^\infty 2^{2kl} \|\mathcal{D}_k \langle D \rangle^m f\|^2_{L^2} 
\]
\[
\lesssim \sum_{k=-1}^\infty \left( 2^{2kl} \|\mathcal{D}_k f\|^2_{H^m} + 2^{-k} \|\langle D \rangle^m f\|^2_{L^2} \right) 
\]
\[
\lesssim \sum_{k=-1}^\infty 2^{2kl} \|\mathcal{D}_k f\|^2_{H^m} + \|f\|^2_{H^{-1/2}},
\]
where we use Lemma 5.3 in the last two steps. Then by iterated argument, we may arrive at for any \( N \in \mathbb{R}^+ \),
\[
\|f\|^2_{H^m} \lesssim \sum_{k=-1}^\infty 2^{2kl} \|\mathcal{D}_k f\|^2_{H^m} + \|f\|^2_{H^{-N/2}}.
\]
Thanks to the fact that for \( m \geq 0 \), there holds
\[
\sum_{k=-1}^\infty 2^{2kl} \|\mathcal{D}_k f\|^2_{H^m} \gtrsim \|f\|^2_{L^2},
\]
choose \( N \) sufficiently large, then we may get
\[
\|f\|^2_{H^m} \lesssim \sum_{k=-1}^\infty 2^{2kl} \|\mathcal{D}_k f\|^2_{H^m},
\]
which gives the proof to desired result with \( m \geq 0 \).

Next we will use the duality argument to deal with the case \( m < 0 \). Notice that
\[
\int_{\mathbb{R}^3} fg \, dv = \sum_{k=-1}^\infty \int_{\mathbb{R}^3} \mathcal{D}_k f \mathcal{D}_k g \, dv \lesssim \sum_{k=-1}^\infty \|\mathcal{D}_k f\|_{H^m} \|\mathcal{D}_k g\|_{H^{-m}} 
\]
\[
\lesssim \left( \sum_{k=-1}^\infty 2^{2kl} \|\mathcal{D}_k f\|^2_{H^m} \right)^\frac{1}{2} \|g\|_{H^{-m}},
\]
Then for any Schwartz function \( g \),
\[
|\int_{\mathbb{R}^3} \langle \cdot \rangle^k \langle D \rangle^m f \rangle \langle D \rangle^m g \, dv| \lesssim \left( \sum_{k=-1}^\infty 2^{2kl} \|\mathcal{D}_k f\|^2_{H^m} \right)^\frac{1}{2} \|\langle D \rangle^m f \rangle \langle D \rangle^m g\|_{L^1} \|
\]
\[
\lesssim \left( \sum_{k=-1}^\infty 2^{2kl} \|\mathcal{D}_k f\|^2_{H^m} \right)^\frac{1}{2} \|g\|_{L^2},
\]
which implies
\[
\|f\|^2_{H^m} \lesssim \sum_{k=-1}^\infty 2^{2kl} \|\mathcal{D}_k f\|^2_{H^m}.
\]
We complete the proof to the lemma. \( \square \)

5.2. **Interpolation theory.** The couple of Banach spaces \((X, Y)\) is said to be an interpolation couple if both \( X \) and \( Y \) are continuously embedding in a Hausdorff topological vector space. Let \((X, Y)\) be a real interpolation couple, then the real interpolation space \((X, Y)_{\theta, p}\) with \( \theta \in (0, 1) \) and \( p \in [1, \infty) \) is defined as follows:
\[
(X, Y)_{\theta, p} \overset{\text{def}}{=} \left\{ x \in X + Y \bigg| \|x\|_{\theta, p} \overset{\text{def}}{=} \left\| t^{-\theta} K(t, x) \right\|_{L^p_\tau(0, \infty)} < \infty \right\},
\]
where \( K(t, x) = \inf_{x=a+b, a \in X, b \in Y} (\|a\|_X + t\|b\|_Y) \) and \( L^p_\tau(0, \infty) \) is a Lebesgue space \( L^p \) with respect to the measure \( dt/t \).
Let $X$ be a real Banach space with norm $\| \cdot \|$. Let $T$ be a closed operator: $\mathcal{D}(T) \subset X \rightarrow X$ satisfying there exists a constant $M$ such that for any $\lambda > 0$

$$
(0, \infty) \subset \rho(T), \| \lambda R(\lambda, T) \|_{L(X)} \leq M.
$$

Then $\mathcal{D}(T)$ is a Banach space with the graph norm $\| x \|_{\mathcal{D}(T)} = \| x \| + \| Tx \|$ for $x \in \mathcal{D}(T)$.

**Proposition 5.1** (see [17]). Let $A$ satisfy (5.5). If we set $\mathcal{D}_A(\theta, p) \overset{\text{def}}{=} (X, \mathcal{D}(A))_{\theta, p}$, then

$$
\mathcal{D}_A(\theta, p) = \left\{ x \in X \left\| \lambda^\theta \| A R(\lambda, A) x \| \right\|_{L^p(0, \infty)} < \infty \right\}.
$$

Let $A, B$ are two closed operators satisfying (5.5). Set $[A, B] \overset{\text{def}}{=} AB - BA$. In general, if $[A, B] \neq 0$, it is difficult to derive

$$
(X, \mathcal{D}(A) \cap \mathcal{D}(B))_{\theta, 2} = \mathcal{D}_A(\theta, 2) \cap \mathcal{D}_B(\theta, 2).
$$

The aim of this subsection is to show that under some special conditions on the operators $A$ and $B$, the interpolation space $(X, \mathcal{D}(A) \cap \mathcal{D}(B))_{\theta, 2}$ still verifies (5.6). We will use this fact to prove (1.31).

Let us give the typical examples of the operators which verify the condition (5.5). Let $\Omega_{ij} = x_i \partial_i - x_j \partial_j$ with $1 \leq i < j \leq 3$ and the domain of the operator is defined as

$$
\mathcal{D}(\Omega_{ij}) = \left\{ f \in L^2(\mathbb{R}^3) \exists g \in L^2(\mathbb{R}^3), \forall h \in C_c^\infty(\mathbb{R}^3), \int_{\mathbb{R}^3} (\Omega_{ij} h) f dx = - \int_{\mathbb{R}^3} h g dx \right\}.
$$

From which, we give the definition such that

$$
g \overset{\text{def}}{=} \Omega_{ij} f.
$$

Then $\Omega_{ij}$ is a closed operator and verifies the condition (5.5). Another example is the partial derivative operator $\partial_k$ with $1 \leq k \leq 3$. We mention that in this case the domain of the operator $\partial_k$ is defined by

$$
\mathcal{D}(\partial_k) = \left\{ f \in L^2(\mathbb{R}^3) \exists g \in L^2(\mathbb{R}^3), \forall h \in C_c^\infty(\mathbb{R}^3), \int_{\mathbb{R}^3} (\partial_k h) f dx = - \int_{\mathbb{R}^3} h g dx \right\}.
$$

The new interpolation theory can be stated as follows:

**Theorem 5.2.** Let $A, B_1, B_2$ and $B_3$ are closed operators satisfying the condition (5.5) and

$$
[B_i, B_j] = 0, [A, B_1] = -B_2, [A, B_2] = -B_1, [A, B_3] = 0.
$$

If we set $\mathcal{D}(B) = \bigcap_{i=1}^3 \mathcal{D}(B_i)$ and $\| x \|_{\mathcal{D}(B)} = \| x \| + \sum_{i=1}^3 \| B_i x \|$. Then

$$
(X, D)_{\theta, 2} = \mathcal{D}_A(\theta, 2) \cap \mathcal{D}_B(\theta, 2)
$$

where $D = \mathcal{D}(A) \cap \mathcal{D}(B)$.

**Proof.** By the definition of the real interpolation space, it is easy to check

$$
(X, D)_{\theta, 2} \subset \mathcal{D}_A(\theta, 2) \cap \mathcal{D}_B(\theta, 2).
$$

Therefore we only need to prove the inverse conclusion. In other words, we only need to prove $\mathcal{D}_A(\theta, 2) \cap \mathcal{D}_B(\theta, 2) \subset (X, D)_{\theta, 2}$.

By the definition of real interpolation space $(X, D)_{\theta, 2}$, it is reduced to prove that for $f \in \mathcal{D}_A(\theta, 2) \cap \mathcal{D}_B(\theta, 2)$,

$$
\| t^{-\theta} K(t, f) \|_{L^2(0, \infty)} < \infty,
$$

where $K(t, f) = \inf_{a+b, a \in X, b \in D} (\| a \| + t \| b \|)$ and $\| b \|_D \overset{\text{def}}{=} \| b \| + \| Ab \| + \sum_{i=1}^3 \| B_i b \|$. 


Thanks to the condition \([B_i, B_j] = 0\), by Proposition 3.1 in [17], one has
\[
\mathcal{D}_B(\theta, 2) = \bigcap_{i=1}^{3} \mathcal{D}_{B_i}(\theta, 2).
\]
Then if \(f \in \mathcal{D}_A(\theta, 2) \cap \mathcal{D}_B(\theta, 2)\), by Proposition 5.1, we have for \(\lambda > 0\),
\[
(5.9) \quad \lambda^\theta \|\lambda A R(\lambda, A) f\|, \lambda^\theta \|B_1 R(\lambda, B_1) f\| \in L^2(0, \infty).
\]
Set
\[
V(\lambda) = \lambda^\theta R(\lambda, B_3)[R(\lambda, B_1) R(\lambda, B_2)]^2 R(\lambda, A) R(\lambda, B_1) R(\lambda, B_2) f.
\]
Then we have the following decomposition
\[
f = f - V(\lambda) + V(\lambda).
\]
From the fact
\[
(5.10) \quad \lambda A R(\lambda, T) = I + T R(\lambda, T),
\]
it follows
\[
V(\lambda) - f \quad = -f + \lambda R(\lambda, B_3) R(\lambda, B_1) R(\lambda, B_2) R(\lambda, A) R(\lambda, B_1) f
\]
\[
+ \lambda^2 R(\lambda, B_3) R(\lambda, B_1) R(\lambda, B_2) R(\lambda, A) R(\lambda, B_1) f.
\]
Using the condition (5.5), we infer
\[
\|\lambda^2 R(\lambda, B_3) R(\lambda, B_1) R(\lambda, B_2) R(\lambda, A) R(\lambda, B_1) f\| \lesssim \|B_2 R(\lambda, B_2) f\|.
\]
It gives
\[
\|V(\lambda) - f\| \lesssim \| - f + \lambda^2 R(\lambda, B_3) R(\lambda, B_1) R(\lambda, B_2) R(\lambda, A) R(\lambda, B_1) f\|
\]
\[
+ \|B_2 R(\lambda, B_2) f\|.
\]
Using (5.10) again and following the similar argument, we derive
\[
\| - f + \lambda^2 R(\lambda, B_3) R(\lambda, B_1) R(\lambda, B_2) R(\lambda, A) R(\lambda, B_1) f\|
\]
\[
\lesssim \| - f + \lambda^2 R(\lambda, B_3) R(\lambda, B_1) R(\lambda, B_2) R(\lambda, A) f\| + \|B_1 R(\lambda, B_1) f\|.
\]
Then by the inductive method, we are led to
\[
(5.11) \quad \|V(\lambda) - f\| \lesssim \|AR(\lambda, A) f\| + \sum_{i=1}^{3} \|B_i R(\lambda, B_i) f\|.
\]
Now we turn to give the bound for \(\|V(\lambda)\|_D\). Thanks to the condition (5.5), we have
\[
\|V(\lambda)\| \lesssim \|f\|.
\]
Observe that if \([T_i, T_j] = 0\), one has
\[
R(\lambda, T_1) R(\lambda, T_j) = R(\lambda, T_i) R(\lambda, T_j), T_i R(\lambda, T_j) = R(\lambda, T_j) T_i.
\]
From which together with the condition (5.5) and \([B_i, B_3] = [A, B_3] = 0\), we deduce that
\[
\|B_3 V(\lambda)\| \lesssim \|B_3 R(\lambda, B_3) f\|.
By standard computation for the resolvent and the condition \([5.7]\), we have the following three facts:

\[(5.12)\]
\[\begin{align*}
[R(\lambda, B_1)R(\lambda, B_2), R(\lambda, A)] & = R(\lambda, A)R(\lambda, B_1)R(\lambda, B_2)[\lambda(B_1 + B_2) - B_1^2 - B_2^2] \\
& \quad \times R(\lambda, B_1)R(\lambda, B_2)R(\lambda, A),
\end{align*}\]
\[(5.13)\]
\[\begin{align*}
[R(\lambda, B_1)R(\lambda, B_2)^2, R(\lambda, A)] & = R(\lambda, A)R(\lambda, B_1)R(\lambda, B_2)^2(-\lambda^2(B_2 + 2B_1) + 2\lambda(B_1B_2 + B_1^2 + B_2^2) \\
& \quad + B_1^2 - 2B_1^2B_2]R(\lambda, B_1)R(\lambda, B_2)^2R(\lambda, A),
\end{align*}\]
and
\[(5.14)\]
\[\begin{align*}
[(R(\lambda, B_1)R(\lambda, B_2))^2, R(\lambda, A)] & = R(\lambda, A)[R(\lambda, B_1)R(\lambda, B_2)]^2[2\lambda^2(B_1 + B_2) - 4\lambda^2(B_1B_2 + B_1^2 + B_2^2) \\
& \quad + 2\lambda(B_1^3 + B_2^3 + 2B_1^2B_3 + 2B_2^2B_1 - 2B_1^3 - 2B_2^3)]R(\lambda, B_1)R(\lambda, B_2)]^2R(\lambda, A).
\end{align*}\]

Now we begin with the estimates for \(\|AV(\lambda)\|\) and \(\|B_1V(\lambda)\|\). It is easy to check

\[AV(\lambda) = \lambda^8 AR(\lambda, B_3)[R(\lambda, B_1)R(\lambda, B_2)]^2R(\lambda, A)R(\lambda, B_1)R(\lambda, B_2)f\]
\[\begin{align*}
& = \lambda^8 AR(\lambda, A)[R(\lambda, B_1)R(\lambda, B_2)]^3R(\lambda, B_3)f \\
& \quad + \lambda^8 A[(R(\lambda, B_1)R(\lambda, B_2))^2, R(\lambda, A)]R(\lambda, B_1)R(\lambda, B_2)R(\lambda, B_3)f \\
& \quad \overset{\text{def}}{=} R_1 + R_2.
\end{align*}\]

By \((5.10)\) and the condition \((5.5)\), we get

\[(5.15)\]
\[\|TR(\lambda, T)\|_{L(X)} \lesssim 1.\]

From which together with \((5.14)\) and the condition \((5.5)\), we obtain
\[\|R_2\| \lesssim \|f\|.\]

Notice that

\[R_1 = \lambda^8(-I + AR(\lambda, A))[R(\lambda, B_1)R(\lambda, B_2)]^3R(\lambda, B_3)f\]
\[\begin{align*}
& = \lambda^8[R(\lambda, B_1)R(\lambda, B_2)]^3R(\lambda, B_3)(-I)f \\
& \quad + \lambda^8[R(\lambda, A), (R(\lambda, B_1)R(\lambda, B_2))]^2R(\lambda, B_1)R(\lambda, B_2)R(\lambda, B_3)f \\
& \quad + \lambda^8[R(\lambda, B_1)R(\lambda, B_2)]^2[R(\lambda, A), R(\lambda, B_1)f[R(\lambda, B_2)]R(\lambda, B_3)f \\
& \quad + \lambda^8[R(\lambda, B_1)R(\lambda, B_2)]^3R(\lambda, B_3)R(\lambda, A)f \\
& = \lambda^8[R(\lambda, B_1)R(\lambda, B_2)]^3R(\lambda, B_3)AR(\lambda, A)f \\
& \quad + \lambda^8[R(\lambda, A), (R(\lambda, B_1)R(\lambda, B_2))]^2R(\lambda, B_1)R(\lambda, B_2)R(\lambda, B_3)f \\
& \quad + \lambda^8[R(\lambda, B_1)R(\lambda, B_2)]^2[R(\lambda, A), R(\lambda, B_1)R(\lambda, B_2)]R(\lambda, B_3)f \\
& \quad \overset{\text{def}}{=} R_3 + R_4 + R_5.
\]

Thanks to \((5.5)\), \((5.15)\), \((5.12)\) and \((5.14)\), we are led to
\[\|R_3\| \lesssim \lambda\|AR(\lambda, A)f\|, \|R_4\| + \|R_5\| \lesssim \|f\|,\]
which imply that
\[\|AV(\lambda)\| \lesssim \|f\| + \lambda\|AR(\lambda, A)f\|.\]
Similarly we have
\[ B_1 V(\lambda) = \lambda^8(-I + AR(\lambda, B_1))R(\lambda, B_1)R(\lambda, B_2)^2R(\lambda, A)R(\lambda, B_1)R(\lambda, B_2)R(\lambda, B_3) \]
\[ = \lambda^8R(\lambda, A)R(\lambda, B_2)^3R(\lambda, B_1)^2R(\lambda, B_3)B_1R(\lambda, B_1) \]
\[ - \lambda^8[R(\lambda, B_1)R(\lambda, B_2)^2, R(\lambda, A)]R(\lambda, B_1)R(\lambda, B_2)R(\lambda, B_3) \]
\[ + \lambda^8[R(\lambda, B_1)^2R(\lambda, B_2)^2, R(\lambda, A)]R(\lambda, B_1)R(\lambda, B_2)R(\lambda, B_3). \]

Thanks to (5.5), (5.15), (5.13) and (5.14), we are led to
\[ \|B_1 V(\lambda)\| \lesssim \|f\| + \lambda\|B_1 R(\lambda, B_1)f\|. \]

By the same argument, we can get
\[ \|B_2 V(\lambda)\| \lesssim \|f\| + \lambda\|B_2 R(\lambda, B_2)f\|. \]

Patching together all the estimates, we finally get
\[ (5.16) \quad \|V(\lambda)\|_D \lesssim \|f\| + \lambda\|AR(\lambda, A)f\| + \sum_{i=1}^3 \lambda\|B_i R(\lambda, B_i)f\|. \]

Then for \( \lambda \geq 1 \), one has
\[ \lambda^\theta(\|V(\lambda) - f\| + \lambda^{-1}\|V(\lambda)\|_D) \]
\[ \lesssim \lambda^{\theta}(\|AR(\lambda, A)f\| + \sum_{i=1}^3 \lambda\|B_i R(\lambda, B_i)f\|) + \lambda^{\theta-1}\|f\|. \]

From which together with (5.9), we get
\[ \left\| \lambda^\theta(\|V(\lambda) - f\| + \lambda^{-1}\|V(\lambda)\|_D) \right\|_{L^2_{t,\theta}(1,\infty)} \lesssim \|f\| + \|f\|_{\mathcal{D}_A(\theta, 2)} + \|f\|_{\mathcal{D}_B(\theta, 2)}. \]

In other words, we obtain
\[ \|t^{-\theta} K(t, f)\|_{L^2_{t,\theta}(0,1)} \lesssim \|t^{-\theta}(\|V(t^{-1}) - f\| + t\|V(t^{-1})\|_D)\|_{L^2_{t,\theta}(0,1)} \lesssim \|f\| + \|f\|_{\mathcal{D}_A(\theta, 2)} + \|f\|_{\mathcal{D}_B(\theta, 2)}. \]

For \( t \geq 1 \), we have for \( f \in \mathcal{D}_A(\theta, 2) \cap \mathcal{D}_B(0, 2) \),
\[ K(t, f) \leq \|f\|, \]
which yields
\[ \|t^{-\theta} K(t, f)\|_{L^2_{t,\theta}(1,\infty)} \lesssim \|f\|. \]

We complete the proof to (5.8) and then ends the proof of the theorem. \( \square \)

5.3. **Spherical harmonics.** In this subsection, we give the introduction of the definition and basic properties of the real spherical harmonics.

Let \( \sigma = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \in S^2 \) with \( \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi] \), the real spherical harmonics \( Y^m_l(\sigma) \) with \( l \in \mathbb{N}, -l \leq m \leq l \), are defined as \( Y^m_0(\sigma) = (4\pi)^{-1/2} \) and for any \( l \geq 1 \),
\[ Y^m_l(\sigma) = \begin{cases} 
\frac{2l+1}{4\pi} P_l(\cos \theta), & \text{if } m = 0, \\
\frac{(2l+1)(l-m)!}{2\pi (l+m)!} P^m_l(\cos \theta) \cos \phi, & \text{if } m = 1, \ldots, l, \\
\frac{(2l+1)(l-m)!}{2\pi (l+m)!} P^{-m}_l(\cos \theta) \sin \phi, & \text{if } m = -l, \ldots, -1, \end{cases} \]
where \( P_l \) denotes the \( l \)-th Legendre polynomial and \( P^m_l \) the associated Legendre functions of the first kind of order \( l \) and degree \( m \). It is well-known that
\[ (-\Delta_{SS^2}) Y^m_l = (l+1) Y^m_l. \]
We remark that the family \( (Y^{m}_{l,m})_{l,m} \) is an orthonormal basis of the space \( L^2(SS^2, d\sigma) \) with \( d\sigma \) being the surface measure on \( SS^2 \). Thus if \( f \in L^2(SS^2) \), then we have

\[
f(\sigma) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f^{m}_{l} Y^{m}_{l}(\sigma),
\]

where \( f^{m}_{l} = \int_{SS^2} f(\sigma) Y^{m}_{l}(\sigma) d\sigma \). Then the fractional Laplace-Beltrami operator \( (-\Delta_{SS^2})^{s/2} \) is defined by

\[
( (-\Delta_{SS^2})^{s/2} f)(\sigma) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (l(l+1))^{s/2} f^{m}_{l} Y^{m}_{l}(\sigma),
\]

with \( s \in \mathbb{R}^{+} \). Similarly we have

\[
(1 - \Delta_{SS^2})^{s/2} f)(\sigma) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (1 + l(l+1))^{s/2} f^{m}_{l} Y^{m}_{l}(\sigma),
\]

with \( s \in \mathbb{R}^{+} \).

Next we denote \( \mathcal{A}_{l} \) the space of solid spherical harmonics of degree \( l \), that is, the set of all homogeneous polynomials of degree \( l \) on \( \mathbb{R}^{3} \) that are harmonic. Let \( \mathcal{H}_{l} \) be the space of spherical harmonics of degree \( l \). Then we define \( \mathcal{D}_{l} \) to be a space of all linear combinations of functions of the form \( f(r) P(x) \), where \( f \) ranges over the radical functions and \( P \) over the solid spherical harmonics of degree \( l \), in such a way that \( f(r) P(x) \) belongs to \( L^2(\mathbb{R}^{3}) \). We have

**Theorem 5.3.**

\[
L^2(\mathbb{R}^{3}) = \bigoplus_{l=0}^{\infty} \mathcal{D}_{l}.
\]

Moreover, for \( Y \in \mathcal{H}_{l} \), there exists a function \( \Psi \) defined on the \([0, \infty)\) such that for \( w \geq 0 \)

\[
\int_{SS^2} e^{-2\pi i w \cdot \tau} Y(\sigma) d\sigma = \Psi(w) Y(\tau),
\]

which means that the Fourier transform maps \( \mathcal{D}_{l} \) into itself.

Suppose \( f \) is a Schwartz function. Thanks to **Theorem 5.3** we have

\[
f(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^{m}_{l}(\sigma) f^{m}_{l}(r),
\]

with \( x = r\sigma \) and \( \sigma \in SS^2 \). Then for \( s > 0 \),

\[
( (-\Delta_{SS^2})^{s/2} f)(x) \equiv \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (l(l+1))^{s/2} Y^{m}_{l}(\sigma) f^{m}_{l}(r).
\]

Similarly we have

\[
(1 - \Delta_{SS^2})^{s/2} f)(x) \equiv \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (1 + l(l+1))^{s/2} Y^{m}_{l}(\sigma) f^{m}_{l}(r).
\]

We recall the statement of the addition theorem:

**Theorem 5.4 (Addition Theorem).** Consider two unit vectors \( \sigma \) and \( \tau \), then

\[
P_{l}(\sigma \cdot \tau) = \frac{1}{2l+1} \sum_{m=-l}^{l} Y^{m}_{l}(\sigma) Y^{m}_{l}(\tau).
\]

Now we want to prove
Lemma 5.4. Suppose $H(x) \in L^2([-1,1])$. Then we have
\[
\int_{SS^2 \times SS^2} (g(\sigma) - g(\tau)) h(\sigma) H(\sigma \cdot \tau) d\sigma d\tau
= \sum_{l,m} g_l^m h_l^m \int_{SS^2 \times SS^2} (Y_l^m(\sigma) - Y_l^m(\tau)) Y_l^m(\sigma) H(\sigma \cdot \tau) d\sigma d\tau.
\]
Here we use the notation: $f_l^m \overset{\text{def}}{=} \int_{SS^2} f(\sigma) Y_l^m(\sigma) d\sigma$.

Proof. Thanks to the fact the family $\{P_n\}_{n \geq 0}$ is an orthogonal basis of the space $L^2[-1,1]$, we have
\[
H(x) = \sum_{n \geq 0} a_n P_n(x),
\]
where $a_n(x) = (n + \frac{1}{2}) \int_{-1}^1 H(x) P_n(x) dx$. In particular, it gives
\[
\int_{SS^2 \times SS^2} g(\tau) h(\sigma) H(\sigma \cdot \tau) d\sigma d\tau
= \sum_n a_n \int_{SS^2 \times SS^2} g(\tau) h(\sigma) P_n(\sigma \cdot \tau) d\sigma d\tau
= \sum_n \sum_{q=-n}^n a_n \frac{1}{2n+1} \int_{SS^2 \times SS^2} g(\tau) h(\sigma) Y_n^q(\sigma) Y_n^q(\tau) d\sigma d\tau
= \sum_n \sum_{q=-n}^n g_l^m h_l^m a_n \frac{1}{2n+1}
= \sum_{l,m} g_l^m h_l^m \int_{SS^2 \times SS^2} Y_l^m(\tau) Y_l^m(\sigma) H(\sigma \cdot \tau) d\sigma d\tau,
\]
where we use Lemma 5.4 in the second and the last equalities. On the other hand,
\[
\int_{SS^2 \times SS^2} g(\sigma) h(\sigma) H(\sigma \cdot \tau) d\sigma d\tau
= \int_{SS^2} g(\sigma) h(\sigma) d\sigma \int_{SS^2} H(\sigma \cdot \tau) d\tau
= (\sum_{l,m} g_l^m h_l^m) \int_{\theta,\phi} H(\cos \theta) \sin \theta d\theta d\phi
= \sum_{l,m} g_l^m h_l^m \int_{SS^2 \times SS^2} Y_l^m(\sigma) H(\sigma \cdot \tau) d\sigma d\tau.
\]
Combine these two equalities, then we get the desired result. \qed

5.4. $L^2$ profile of the fractional Laplace-Beltrami operator. In this subsection, we will show some equivalence of the $L^2$ norm of the fractional Laplace-Beltrami operator in the unit sphere. Then we will generalize this kind of the equivalence to the function defined in the whole space and show that in this case the fractional Laplace-Beltrami operator has strong connection to the rotation vector fields.

We first have

Lemma 5.5. Suppose that $f$ is a smooth function defined in $SS^2$. Then if $0 < s < 1$, it holds
\[
\|f\|_{L^2(SS^2)}^2 + \int_{\sigma, \tau \in SS^2} \frac{|f(\sigma) - f(\tau)|^2}{|\sigma - \tau|^{2+2s}} d\sigma d\tau \sim \|f\|_{L^2(SS^2)}^2 \int_{SS^2} (-\Delta_{SS^2})^{s/2} f d\sigma.
\]

Proof. Let $\omega_1, \omega_2, \omega_3 \in C_c^{\infty}(\mathbb{R})$. Assume that $\omega_1(x) = 1$ in the Ball $B_{\frac{3}{4}}$ with compact support in the Ball $B_1$, $\omega_2(x) = 1$ in the Ball $B_{\frac{3}{4}}$ with compact support in the Ball $B_2$ and $\omega_3(x) = 1$ in the Ball $B_{\frac{3}{4}}$ with compact support in the Ball $B_5$. Let $\chi$ be a smooth function verifying
\[
\chi(x) = \begin{cases} 
1, & \text{if } x \geq 0; \\
0, & \text{if } x < -\frac{1}{10}.
\end{cases}
\]
Suppose $u = (u_1, u_2, u_3) \in \mathbb{R}^3$. Then it is easy to check for $k \in \mathbb{N}$ with $1 \leq k \leq 3$ and $u \neq 0$,

$$\sum_{i=1}^{3} \omega_k \left( \sum_{j \neq i} \frac{u_j^2}{|u_i|^2} \right) \geq 1.$$ 

Then we set for $m \in \mathbb{N}$ with $1 \leq m \leq 3$,

$$\theta_{km+}(u) = \frac{\omega_k \left( \sum_{j \neq m} \frac{u_j^2}{|u_m|^2} \right)}{\sum_{i=1}^{3} \omega_k \left( \sum_{j \neq i} \frac{u_j^2}{|u_i|^2} \right)} \chi(u_m) \quad \text{and} \quad \theta_{km-}(u) = \frac{\omega_k \left( \sum_{j \neq m} \frac{u_j^2}{|u_m|^2} \right)}{\sum_{i=1}^{3} \omega_k \left( \sum_{j \neq i} \frac{u_j^2}{|u_i|^2} \right)} \chi(-u_m).$$

We have for $u \in SS^2$,

$$\sum_{m=1}^{3} \left[ \theta_{km+}(u) + \theta_{km-}(u) \right] = 1. \tag{5.22}$$

Observe that

$$\int_{\sigma, \tau \in SS^2} \frac{|f(\sigma) - f(\tau)|^2}{|\sigma - \tau|^{2+2s}} d\sigma d\tau \sim \sum_{m=1}^{3} \left[ \int_{\sigma, \tau \in SS^2} \frac{|f(\sigma) - f(\tau)|^2}{|\sigma - \tau|^{2+2s}} [\theta_{1m+}(\sigma) + \theta_{1m-}(\sigma)] d\sigma d\tau. \right. \tag{5.23}$$

Therefore by the symmetric structure, we only need to focus on the estimate of

$$\int_{\sigma, \tau \in SS^2} \frac{|f(\sigma) - f(\tau)|^2}{|\sigma - \tau|^{2+2s}} \theta_{13+}(\sigma) d\sigma d\tau.$$

Notice that

$$\int_{\sigma, \tau \in SS^2} \frac{|f(\sigma) - f(\tau)|^2}{|\sigma - \tau|^{2+2s}} \theta_{13+}(\sigma) d\sigma d\tau = \int_{\sigma, \tau \in SS^2} \frac{|f(\sigma) - f(\tau)|^2}{|\sigma - \tau|^{2+2s}} [\theta_{13+}(\sigma) + \theta_{23+}(\sigma)(1 - \theta_{33+}(\tau))] d\sigma d\tau.$$

From which together with the fact that $|\sigma - \tau| \geq \frac{1}{2} - \frac{1}{\sqrt{5}}$ if $\sigma \in \text{Supp} \theta_{13+}$ and $\tau \in \text{Supp} (1 - \theta_{33+})$, we deduce that

$$\int_{\sigma, \tau \in SS^2} \frac{|f(\sigma) - f(\tau)|^2}{|\sigma - \tau|^{2+2s}} \theta_{13+}(\sigma) d\sigma d\tau + \|f\|_{L^2(SS^2)}^2 \sim \int_{\sigma, \tau \in SS^2} \frac{|f(\sigma) - f(\tau)|^2}{|\sigma - \tau|^{2+2s}} \theta_{23+}(\sigma) \theta_{33+}(\sigma) \theta_{33+}(\tau) d\sigma d\tau + \|f\|_{L^2(SS^2)}^2$$

$$\sim \int_{\sigma, \tau \in SS^2} \frac{|(\theta_{13} + f)(\sigma) - (\theta_{13} + f)(\tau)|^2}{|\sigma - \tau|^{2+2s}} \theta_{33+}(\sigma) \theta_{33+}(\tau) d\sigma d\tau + \|f\|_{L^2(SS^2)}^2.$$
Suppose \( \sigma = (x_1, x_2, x_3) \in SS^2 \) and set \( x = (x_1, x_2) \). Let \( F_{13}^+(x) \overset{\text{def}}{=} (\theta_{13}+f)(x_1, x_2, \sqrt{1-x_1^2-x_2^2}) \) and \( \Theta_{33}^+(x) = \theta_{33}+(x_1, x_2, \sqrt{1-x_1^2-x_2^2}) \). Then by change of variables, we have

\[
\int_{\sigma, \tau \in SS^2} \left| \frac{\theta_{13}+(\sigma) - \theta_{13}+(\tau)}{\sigma - \tau} \right|^2 \theta_{33}+(\sigma) \theta_{33}+(\tau) d\sigma d\tau
\]

\[
= \int_{|x|,|y| \leq \sqrt{\frac{7}{5}}} \frac{|F_{13}^+(x) - F_{13}^+(y)|^2}{(|x| - |y|)^{2+2s}} \cdot \Theta_{33}^+(x) \Theta_{33}^+(y) \sqrt{\frac{1}{1-x_1^2-x_2^2} - \frac{1}{1-y_1^2-y_2^2}} dxdy
\]

\[
\gtrsim \int_{|x|,|y| \leq \sqrt{\frac{7}{5}}} \frac{|F_{13}^+(x) - F_{13}^+(y)|^2}{|x| - |y|} dxdy,
\]

which yields

\[(5.24) \quad \int_{\sigma, \tau \in SS^2} \left| \frac{f(\sigma) - f(\tau)}{\sigma - \tau} \right|^2 \theta_{13}^+(\sigma) d\sigma d\tau + \|f\|^2_{L^2(SS^2)} \gtrsim \|F_{13}^+\|^2_{H^s(B_{\frac{7}{5}})}.
\]

On the other hand,

\[
\int_{\sigma, \tau \in SS^2} \left| \frac{\theta_{13}+(\sigma) - \theta_{13}+(\tau)}{\sigma - \tau} \right|^2 \theta_{33}+(\sigma) \theta_{33}+(\tau) d\sigma d\tau
\]

\[
\lesssim \int_{|x-y| \leq \eta} \frac{|F_{13}^+(x) - F_{13}^+(y)|^2}{|x| - |y|} \Theta_{33}^+(x) \Theta_{33}^+(y) dxdy + C_{\eta} \|F_{13}^+\|^2_{L^2(B_{\frac{7}{5}})}.
\]

Choose \( \eta \) sufficiently small such that

\[ F_{13}^+(x)^2 \Theta_{33}^+(x) \Theta_{33}^+(y) = F_{13}^+(x)^2 \Theta_{23}^+(x) \Theta_{23}^+(y), \]

and

\[ F_{13}^+(x) F_{13}^+(y) \Theta_{33}^+(x) \Theta_{33}^+(y) = F_{13}^+(x) F_{13}^+(y) \Theta_{23}^+(x) \Theta_{23}^+(y). \]

Then we get

\[
\int_{\sigma, \tau \in SS^2} \left| \frac{\theta_{13}+(\sigma) - \theta_{13}+(\tau)}{\sigma - \tau} \right|^2 \theta_{33}+(\sigma) \theta_{33}+(\tau) d\sigma d\tau
\]

\[
\lesssim \int_{|x-y| \leq \sqrt{\frac{7}{5}}} \frac{|F_{13}^+(x) - F_{13}^+(y)|^2}{|x| - |y|} \Theta_{23}^+(x) \Theta_{23}^+(y) dxdy + C_{\eta} \|F_{23}^+\|^2_{L^2(B_{\frac{7}{5}})}
\]

\[
\lesssim \|F_{13}^+\|^2_{H^s(B_{\frac{7}{5}})},
\]

which implies

\[
\int_{\sigma, \tau \in SS^2} \left| \frac{f(\sigma) - f(\tau)}{\sigma - \tau} \right|^2 \theta_{13}^+(\sigma) d\sigma d\tau \lesssim \|f\|^2_{L^2(SS^2)} + \|F_{13}^+\|^2_{H^s(B_{\frac{7}{5}})}.
\]

From which together with (5.24), we have

\[(5.25) \quad \int_{\sigma, \tau \in SS^2} \left| \frac{f(\sigma) - f(\tau)}{\sigma - \tau} \right|^2 \theta_{13}^+(\sigma) d\sigma d\tau + \|f\|^2_{L^2(SS^2)} \sim \|F_{13}^+\|^2_{H^s(B_{\frac{7}{5}})} + \|f\|^2_{L^2(SS^2)}.
\]

Observe

\[ \|(-\Delta_{SS^2})^{\frac{1}{2}} (\theta_{13}+f)\|^2_{L^2(SS^2)} = \int_{\sigma, \tau \in SS^2} \left( (-\Delta_{SS^2})(\theta_{13}+f) \right)(\sigma)(\theta_{13}+f)(\sigma) d\sigma. \]
Thanks to the fact \((-\Delta_{SS} f)(\sigma) = -\sum_{1 \leq i < j \leq 3} (\Omega_{ij}^2 f)(x_1, x_2, x_3)\) with \(\sigma = (x_1, x_2, x_3)\) and \(\Omega_{ij} = x_i \partial_j - x_j \partial_i\), by change of variables, we obtain
\[
\|(-\Delta_{SS})^{1/2} (\theta_{13} + f)\|^2_{L^2(\mathbb{S}^2)} = \int_{|x| \leq \sqrt{\frac{5}{2}}} \sum_{1 \leq i < j \leq 3} \left( \Omega_{ij}^2 (\theta_{13} + f) \right)(x, \sqrt{1 - |x|^2}) \theta_{13} + f(x, \sqrt{1 - |x|^2}) \frac{1}{\sqrt{1 - |x|^2}} \, dx.
\]

It is easy to see for \(i = 1, 2\),
\[
\partial_i F_{13}^+(x_1, x_2) = \frac{1}{\sqrt{1 - |x|^2}} \left( -\Omega_{13} (\theta_{13} + f) \right)(x, \sqrt{1 - |x|^2}),
\]
which implies that
\[
\left( \Omega_{13}^2 (\theta_{13} + f) \right)(x, \sqrt{1 - |x|^2}) = \left( (\sqrt{1 - |x|^2} \partial_i) F_{13}^+ \right)(x).
\]

Then by direct calculation, it yields
\[
\begin{align*}
\|(-\Delta_{SS})^{1/2} (\theta_{13} + f)\|^2_{L^2(\mathbb{S}^2)} &= -\int_{|x| \leq \sqrt{\frac{5}{2}}} (\partial_1 (\sqrt{1 - |x|^2} \partial_1) F_{13}^+) F_{13}^+ \, dx - \int_{|x| \leq \sqrt{\frac{5}{2}}} (\partial_2 (\sqrt{1 - |x|^2} \partial_2) F_{13}^+) F_{13}^+ \, dx \\
& \quad - \int_{|x| \leq \sqrt{\frac{5}{2}}} (\Omega_{12})^2 F_{13}^+ F_{13}^+ \frac{1}{\sqrt{1 - |x|^2}} \, dx.
\end{align*}
\]

Thus we have
\[
\begin{align*}
\|(-\Delta_{SS})^{1/2} (\theta_{13} + f)\|^2_{L^2(\mathbb{S}^2)} + \|\theta_{13} + f\|^2_{L^2(\mathbb{S}^2)} &= \|F_{13}^+\|^2_{H^1(\mathbb{B}_{\frac{5}{2}})} + \|\Omega_{12} F_{13}^+\|^2_{L^2(\mathbb{B}_{\frac{5}{2}})} \\
& \sim \|F_{13}^+\|^2_{H^1(\mathbb{B}_{\frac{5}{2}})},
\end{align*}
\]
where we use the fact \(\|\theta_{13} + f\|_{L^2(\mathbb{S}^2)} \sim \|F_{13}^+\|_{L^2(\mathbb{B}_{\frac{5}{2}})}\).

By standard real interpolation method, we obtain that for \(0 \leq s \leq 1\),
\[
\|(-\Delta_{SS})^{s/2} (\theta_{13} + f)\|_{L^2(\mathbb{S}^2)} + \|\theta_{13} + f\|_{L^2(\mathbb{S}^2)} \sim \|F_{13}^+\|_{H^1(\mathbb{B}_{\frac{5}{2}})}.
\]

Next we claim that for \(0 \leq s \leq 2\),
\[
\|(-\Delta_{SS})^{s/2} (\theta_{1m} + f)\|_{L^2(\mathbb{S}^2)} + \|\theta_{1m} + f\|_{L^2(\mathbb{S}^2)} \lesssim \|(-\Delta_{SS})^{s/2} f\|_{L^2(\mathbb{S}^2)} + \|f\|_{L^2(\mathbb{S}^2)}.
\]

This is easily followed by the real interpolation method since
\[
\|(-\Delta_{SS})^s (\theta_{1m} + f)\|_{L^2(\mathbb{S}^2)} \lesssim \|(-\Delta_{SS})^s f\|_{L^2(\mathbb{S}^2)} + \|f\|_{L^2(\mathbb{S}^2)},
\]
and \(\|\theta_{1m} + f\|_{L^2(\mathbb{S}^2)} \lesssim \|f\|_{L^2(\mathbb{S}^2)}\).

Thanks to (5.22) and (5.23), we are led to for \(0 \leq s \leq 2\),
\[
\sum_{m=1}^3 \left[ \|(-\Delta_{SS})^{s/2} (\theta_{1m} + f)\|_{L^2(\mathbb{S}^2)} + \|(-\Delta_{SS})^{s/2} (\theta_{1m} - f)\|_{L^2(\mathbb{S}^2)} \right] \sim \|(-\Delta_{SS})^{s/2} f\|_{L^2(\mathbb{S}^2)} + \|f\|_{L^2(\mathbb{S}^2)}.
\]

Then (5.25), (5.27) and (5.29) imply the lemma. □

As the consequence of Lemma 5.4 and Lemma 5.5, we get the following estimate:
Corollary 5.1. Let $g$ and $h$ are smooth functions in the sphere. Then for $a, b \in \mathbb{R}$ verifying $a + b = 2s$,

$$\left| \int_{S^2 \times S^2} (g(\sigma) - g(\tau)) h(\sigma \cdot \tau) H(\sigma \cdot \tau) d\sigma d\tau \right| \lesssim \|(1 - \Delta_{S^2})^{a/2} g\|_{L^2(S^2)} \|(1 - \Delta_{S^2})^{b/2} h\|_{L^2(S^2)},$$

where $H(\sigma \cdot \tau) = |\sigma - \tau|^{-(2+2s)}$.

Proof. Let $\lambda > 0$. Then by Lemma 5.4, we have

$$\int_{S^2 \times S^2} (g(\sigma) - g(\tau)) h(\sigma \cdot \tau) H(\sigma \cdot \tau) d\sigma d\tau = \lim_{\lambda \to 0} \int_{S^2 \times S^2} (g(\sigma) - g(\tau)) h(\sigma \cdot \tau) H(\sigma \cdot \tau) 1_{|\sigma - \tau| \geq \lambda} d\sigma d\tau$$

where we use the symmetric structure of the integral in the last step. Applying the Cauchy-Schwarz inequality and Lemma 5.5, we obtain

$$\int_{S^2 \times S^2} (g(\sigma) - g(\tau)) h(\sigma \cdot \tau) H(\sigma \cdot \tau) d\sigma d\tau \lesssim \sum_{l,m} g_l^m h_l^m \|(1 - \Delta_{S^2})^{s/2} Y_l^m\|_{L^2(S^2)} + 1 \|(1 - \Delta_{S^2})^{s/2} Y_l^m\|_{L^2(S^2)} + 1$$

which completes the proof of the lemma.

Next we generalize the estimates in the previous lemma to the functions defined in the whole space.

Lemma 5.6. Let $f$ be a smooth function in $\mathbb{R}^3$. Suppose $f(u) = f(u_1, u_2, u_3)$ with $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ and $\Omega_{ij} f \overset{\text{def}}{=} u_i \partial_{u_j} f - u_j \partial_{u_i} f$. Then if $0 < s < 1$, it hold

$$\int_{\sigma, \tau \in S^2, r \in \mathbb{R}} \frac{|f(\sigma) - f(\tau)|^2}{|\sigma - \tau|^{2+2s}} r^2 d\sigma d\tau dr + \|f\|_{L^2(\mathbb{R}^3)}^2 \sim \|(-\Delta_{S^2})^{s/2} f\|_{L^2(S^2)}^2 + \|f\|_{L^2(S^2)}^2 \sum_{1 \leq i < j \leq 3} \|f\|_{\mathcal{D}_{ij}^{(s,2)}}^2,$$

Moreover, we have for $s \in [0, 2]$,

(5.30) \[ \|(-\Delta_{S^2})^{s/2} f\|_{L^2(S^2)} + \|f\|_{L^2(S^2)} \sim \sum_{1 \leq i < j \leq 3} \|f\|_{\mathcal{D}_{ij}^{(s/2,2)}}^2, \]

Here we use notations: $\mathcal{D}_{ij}^{(s,2)} = (L^2, \mathcal{D}(\Omega_{ij}))_{s,2}$ and $\mathcal{D}_{ij}^{(s/2,2)} = (L^2, \mathcal{D}(\Omega_{ij}^2))_{s/2,2}$.

Proof. For $r > 0$ and $x = (x_1, x_2)$, we set

$$\tilde{F}_{13}^+(r, x) \overset{\text{def}}{=} r \partial_{13+} (x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}) f(r x_1, r x_2, r \sqrt{1 - x_1^2 - x_2^2})$$
Thus if take $f$,

We use the notation

Thanks to (5.27), one has

Then by interpolation, we obtain

Let $T_1 : L^2((0,\infty) \times B_{\frac{1}{\sqrt{s}}}) \rightarrow L^2((0,\infty) \times \mathbb{R}^2)$ be a linear operator defined by

Then we have

Then by interpolation, we obtain

By the definition of $\tilde{F}_1$, we have $\text{Supp} \tilde{F}_1(r, x) \in (0, \infty) \times B_{\frac{1}{\sqrt{s}}}$. Thus if take $f = \tilde{F}_1$, then we get

Let $T_2 : L^2((0,\infty) \times \mathbb{R}^2) \rightarrow L^2((0,\infty) \times B_{\frac{1}{\sqrt{s}}})$ be a linear operator defined by

Then by the similar argument mentioned before, we may obtain

Thus if take $f = \tilde{F}_2$, then we get

Therefore, we are led to

which implies that

We use the notation $\parallel \tilde{F}_1 \parallel_{L^2 H^s} \overset{\text{def}}{=} \parallel \tilde{F}_1 \parallel_{L^2((0,\infty); H^s(\mathbb{R}^2))}$. It is easy to see

and

$$
\tilde{F}_1(r, x) \overset{\text{def}}{=} \begin{cases} 
  r(\theta_{13} + f)(r, x_1, r, x_2, r\sqrt{1 - x_1^2 - x_2^2}), & \text{if } |x| \leq \sqrt{\frac{4}{5}}; \\
  0, & \text{if } |x| \geq \sqrt{\frac{4}{5}}.
\end{cases} 
$$
Notice that
\[ \|F_{13}^+\|_{L^2_\Omega(H_{1,1}^s)} \sim \|\Omega_{12}^{(\theta_{13}, f)}\|_{L^2(\mathbb{R}^3)} + \|\theta_{13} + f\|_{L^2(\mathbb{R}^3)}, \]
\[ \|F_{13}^+\|_{L^2_\Omega(H_{1,1}^s)} \sim \|\theta_{13} + f\|_{L^2(\mathbb{R}^3)}. \]

By the real interpolation method, one has
\[ \|F_{13}^+\|_{L^2_\Omega(H_{1,1}^s)} \sim \|\theta_{13} + f\|_{\mathcal{G}_{2,2}(s, 2)}. \]

Similarly
\[ \|F_{13}^+\|_{L^2_\Omega(H_{1,1}^s)} \sim \|\theta_{13} + f\|_{\mathcal{G}_{2,2}(s, 2)}. \]

From which together with (5.33), we have
\[ \|(-\Delta_{SS}^{1/2})^{s/2}(\theta_{13} + f)\|_{L^2(\mathbb{R}^3)} + \|\theta_{13} + f\|_{L^2(\mathbb{R}^3)} \]
\[ \sim \|F_{13}^+\|_{L^2((0, \infty); H^s(\mathbb{R}^3))} \sim \|\theta_{13} + f\|_{\mathcal{G}_{2,2}(s, 2)} + \|\theta_{13} + f\|_{\mathcal{G}_{2,2}(s, 2)}. \]

Observe that
\[ (\Omega_{12}^{(\theta_{13}, f)})(u) = (x_1 \partial_{x_2} - x_2 \partial_{x_1})(r^{-1}F_{13}^+(r, x)), \]
where \( u = (r x_1, r x_2, r\sqrt{1 - |x|^2}) \). It yields
\[ \|\Omega_{12}^{(\theta_{13}, f)}\|_{L^2(\mathbb{R}^3)} + \|\theta_{13} + f\|_{L^2(\mathbb{R}^3)} \lesssim \|F_{13}^+\|_{L^2_\Omega(H_{1,1}^s)} \]
Therefore by interpolation, we deduce that
\[ \|\theta_{13} + f\|_{\mathcal{G}_{2,2}(s, 2)} \lesssim \|F_{13}^+\|_{L^2_\Omega(H_{1,1}^s)} \lesssim \|\theta_{13} + f\|_{\mathcal{G}_{2,2}(s, 2)} + \|\theta_{13} + f\|_{\mathcal{G}_{2,2}(s, 2)} \]
which yields
\[ \|(-\Delta_{SS}^{1/2})^{s/2}(\theta_{13} + f)\|_{L^2(\mathbb{R}^3)} + \|\theta_{13} + f\|_{L^2(\mathbb{R}^3)} \sim \sum_{1 \leq i < j \leq 3} \|\theta_{13} + f\|_{\mathcal{G}_{2,2}(s, 2)}. \]

With the help of (5.29), we are led to
\[ \sum_{m=1}^3 \sum_{1 \leq i < j \leq 3} \|\theta_{13} + f\|_{\mathcal{G}_{2,2}(s, 2)} + \|\theta_{13} + f\|_{\mathcal{G}_{2,2}(s, 2)} \]
\[ \sim \|(-\Delta_{SS}^{1/2})^{s/2} f\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^2(\mathbb{R}^3)}. \]

Due to the fact
\[ \|\Omega_{ij}^{(\theta_{13}, f)}\|_{L^2(\mathbb{R}^3)} \lesssim \|\Omega_{ij}^{(\theta_{13}, f)}\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^2(\mathbb{R}^3)}, \]
we have
\[ \|\theta_{13} + f\|_{\mathcal{G}_{2,2}(s, 2)} \lesssim \|f\|_{\mathcal{G}_{2,2}(s, 2)}. \]

From which together with (5.36), we derive that
\[ \sum_{1 \leq i < j \leq 3} \|f\|_{\mathcal{G}_{2,2}(s, 2)} \sim \|(-\Delta_{SS}^{1/2})^{s/2} f\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^2(\mathbb{R}^3)}. \]

We complete the proof to the first equivalence.

The standard interpolation theory indicates
\[ \sum_{1 \leq i < j \leq 3} \|f\|_{\mathcal{G}_{2,2}(s, 2)} \sim \|f\|_{\mathcal{G}_{2,2}(s/2, 2),}, \]
which implies the second equivalence in the case of \( 0 \leq s \leq 1 \). Next we will prove it holds for \( 1 < s \leq 2 \).

We first show
\[ \|(-\Delta_{SS}^{1/2})^{s/2}(\theta_{13} + f)\|_{L^2(\mathbb{R}^3)}^2 + \|\theta_{13} + f\|_{L^2(\mathbb{R}^3)}^2 \]
\[ \sim \|F_{13}^+\|_{L^2(\mathbb{R}^3)}. \]
It derives from the fact that

\[(5.38) \quad (-\Delta_{SS^2})(\theta_{13}+f) = LF_{13}^+,\]

where \(L = -(1-x_1^2)\partial_2^2-(1-x_2^2)\partial_3^2 + 2x_1\partial_1 + 2x_2\partial_2\). Since \(L\) is a uniformly elliptic in \(B_{\frac{r}{\sqrt{3}}}\) and \(F_{13}^+\) vanishes in the boundary of \(B_{\frac{r}{\sqrt{3}}}\), the standard elliptic estimate implies that

\[
\|F_{13}^+\|_{H^2(B_{\frac{r}{\sqrt{3}}})} \lesssim \|F_{13}^+\|_{L^2(B_{\frac{r}{\sqrt{3}}})} + \|\mathbf{L}F_{13}^+\|_{L^2(B_{\frac{r}{\sqrt{3}}})} \lesssim \|(-\Delta_{SS^2})(\theta_{13}+f)\|_{L^2(SS^2)} + \|\theta_{13}+f\|_{L^2(SS^2)},
\]

which gives the proof to \((5.37)\) since the inverse inequality is obviously true recalling the definition of \(\Delta_{SS^2}\). By real interpolation, \((5.37)\) yields that \((5.33)\) holds for \(0 \leq s \leq 2\).

Due to the fact

\[
(\sqrt{1-|x|^2}\partial_x)^2(r^{-1}F_{13}^+)(r, x_1, r x_2, r \sqrt{1-|x|^2}) = (\Omega_{13}^2(\theta_{13}+f))(u),
\]

where \(u = (r x_1, r x_2, r \sqrt{1-|x|^2})\), we derive

\[
\|\tilde{F}_{13}^+\|_{L^2_{\Omega_{13}^2} H^s_{ss}} \sim \|\theta_{13}+f\|_{\mathcal{G}_{\Omega_{13}^2}^s(s/2,2)}.
\]

By real interpolation, it follows for \(0 \leq s \leq 2\)

\[
\|\tilde{F}_{13}^+\|_{L^2_{\Omega_{13}^2} H^s_{ss}} \sim \|\theta_{13}+f\|_{\mathcal{G}_{\Omega_{13}^2}^s(s/2,2)}.
\]

Similarly it holds for \(0 \leq s \leq 2\),

\[
\|\tilde{F}_{13}^+\|_{L^2_{\Omega_{13}^2} H^s_{ss}} \sim \|\theta_{13}+f\|_{\mathcal{G}_{\Omega_{13}^2}^s(s/2,2)}.
\]

From which together with \((5.33)\), we have for \(0 \leq s \leq 2\),

\[(5.39) \quad \|(-\Delta_{SS^2})^{s/2}(\theta_{13}+f)\|_{L^2(\mathbb{R}^3)} + \|\theta_{13}+f\|_{L^2(\mathbb{R}^3)} \sim \|F_{13}^+\|_{L^2((0,\infty);H^s(\mathbb{R}^3))} \sim \|\theta_{13}+f\|_{\mathcal{G}_{\Omega_{13}^2}^s(s/2,2)} + \|\theta_{13}+f\|_{\mathcal{G}_{\Omega_{13}^2}^s(s/2,2)}.
\]

Observe that

\[
(\Omega_{12}(\theta_{13}+f))(u) = (x_1\partial_{x_2} - x_2\partial_{x_1})(r^{-1}F_{13}^+(r, x)),
\]

where \(u = (r x_1, r x_2, r \sqrt{1-|x|^2})\). It yields

\[
\|\Omega_{12}^2(\theta_{13}+f)\|_{L^2(\mathbb{R}^3)} + \|\theta_{13}+f\|_{L^2(\mathbb{R}^3)} \lesssim \|\tilde{F}_{13}^+\|_{L^2_{\Omega_{13}^2} H^s_{ss}}.
\]

Therefore by interpolation, we deduce that

\[
\|\theta_{13}+f\|_{\mathcal{G}_{\Omega_{13}^2}^s(s/2,2)} \lesssim \|\tilde{F}_{13}^+\|_{L^2_{\Omega_{13}^2} H^s_{ss}} \lesssim \|\theta_{13}+f\|_{\mathcal{G}_{\Omega_{13}^2}^s(s/2,2)} + \|\theta_{13}+f\|_{\mathcal{G}_{\Omega_{13}^2}^s(s/2,2)}
\]

which yields

\[
\|(-\Delta_{SS^2})^{s/2}(\theta_{13}+f)\|_{L^2(\mathbb{R}^3)} + \|\theta_{13}+f\|_{L^2(\mathbb{R}^3)} \sim \sum_{1 \leq i < j \leq 3} \|\theta_{13}+f\|_{\mathcal{G}_{\Omega_{ij}^2}^s(s/2,2)}.
\]

From which together with \((5.29)\), we get the equivalence \((5.30)\). \(\square\)
5.5. **Commutator estimates for fractional Laplace-Beltrami operator.** Based on the abstract interpolation theory established in Theorem 5.2 in this subsection, we will give the estimates to the commutator estimates between the fractional Laplace-Beltrami operator and the translation operator. We emphasize that it is the key point to obtain the sharp bounds for the operator in the anisotropic spaces.

We begin with a lemma that the $L^2$ norm of the fractional Laplace-Beltrami operator can be bounded by the weighted Sobolev norm. It explains where the additional weight $\langle v \rangle^{2s}$ comes from in the upper bounds of the collision operator in the weighted Sobolev spaces (see Theorem 1.1).

**Lemma 5.7.** Suppose $f \in H^s_2(\mathbb{R}^3)$ with $s \geq 0$. Then there holds
\[
\| (-\Delta_{SS})^{s/2} f \|_{L^2} \lesssim \| f \|_{H^s}.
\]

*Proof.* We will use Lemma 5.1, the profile of the weighted Sobolev space, and the interpolation theory to prove the lemma. Suppose $0 \leq s \leq 2m$ with $m \in \mathbb{N}$. Then we have
\[
\| (-\Delta_{SS})^m f \|_{L^2}^2 = \sum_{k \geq -1} \| (-\Delta_{SS})^m \mathcal{P}_k f \|_{L^2}^2
\]
\[
\lesssim \sum_{k \geq -1} 2^{4mk} \| \mathcal{P}_k f \|_{H^m}^2.
\]
Since it holds
\[
\| (-\Delta_{SS})^m \tilde{\mathcal{P}}_k f \|_{L^2} \lesssim 2^{2mk} \| f \|_{H^m}, \| \tilde{\mathcal{P}}_k f \|_{L^2} \lesssim \| f \|_{L^2},
\]
by the real interpolation, we get
\[
\| (-\Delta_{SS})^{s/2} \tilde{\mathcal{P}}_k f \|_{L^2} \lesssim 2^{ks} \| f \|_{H^s}.
\]
In particular, it yields
\[
\| (-\Delta_{SS})^{s/2} \mathcal{P}_k f \|_{L^2} \lesssim 2^{ks} \| \mathcal{P}_k f \|_{H^s}.
\]
We finally derive
\[
\| (-\Delta_{SS})^{s/2} f \|_{L^2}^2 = \sum_{k \geq -1} \| (-\Delta_{SS})^{s/2} \mathcal{P}_k f \|_{L^2}^2
\]
\[
\lesssim \sum_{k \geq -1} 2^{2ks} \| \mathcal{P}_k f \|_{H^s}^2 \lesssim \| f \|_{H^s}^2.
\]
This completes the proof of the Lemma. □

Next we give several commutator estimates between the Laplace-Beltrami operator and the standard derivatives.

**Lemma 5.8.** Suppose $a, b \in \mathbb{R}$ and $\phi$ to be a radial function, then we have
\[
\mathcal{F}(-\Delta_{SS})^{a/2} = (-\Delta_{SS})^{a/2} \mathcal{F}
\]
and
\[
\phi(|D|)(-\Delta_{SS})^{a/2} = (-\Delta_{SS})^{a/2} \phi(|D|).
\]
In particular, it holds
\[
\| (1 - \Delta_{SS})^{a/2} f \|_{H^b}^2 \sim \sum_{p \geq -1} 2^{2pb} \| (1 - \Delta_{SS})^{a/2} \tilde{\mathcal{P}}_p f \|_{L^2}^2.
\]
Proof. Suppose $f$ is the Schwartz function. Thanks to Theorem 5.3, we have
\[
((-\Delta_{SS})^{a/2} f)(x) = \sum_{l,m} (l(l+1))^{a/2} Y_l^m(\sigma) f_l^m(r),
\]
with $x = r\sigma$ and $\sigma \in SS^2$. Then if $\xi = \rho\tau$ with $\tau \in SS^2$, then
\[
\mathcal{F}((-\Delta_{SS})^{a/2} f)(\xi) = \sum_{l,m} (l(l+1))^{a/2} \mathcal{F}(Y_l^m)(\xi) f_l^m(\rho) = \sum_{l,m} (l(l+1))^{a/2} Y_l^m(\tau) W_l^m(\rho).
\]
where we use (5.19) to assume that $\mathcal{F}(Y_l^m f_m)(\xi) = Y_l^m(\tau) W_l^m(\rho)$.
On the other hand, using the same notation, we have
\[
(\mathcal{F} f)(\xi) = \sum_{l,m} Y_l^m(\tau) W_l^m(\rho),
\]
which implies
\[
((-\Delta_{SS})^{a/2} (\mathcal{F} f))(\xi) = \sum_{l,m} (l(l+1))^{a/2} Y_l^m(\tau) W_l^m(\rho) = \mathcal{F}((-\Delta_{SS})^{a/2} f)(\xi).
\]
This gives the first equality.
Observe that
\[
\mathcal{F} \phi(|D|)(-\Delta_{SS})^{a/2} = \phi \mathcal{F} (-\Delta_{SS})^{a/2} = \phi(-\Delta_{SS})^{a/2} \mathcal{F} = (-\Delta_{SS})^{a/2} \phi \mathcal{F},
\]
where we use the fact that $\phi$ is a radical function in the last equality.
On the other hand, we have
\[
\mathcal{F}(-\Delta_{SS})^{a/2} \phi(|D|) = (-\Delta_{SS})^{a/2} \mathcal{F} \phi(|D|) = (-\Delta_{SS})^{a/2} \phi \mathcal{F} = \mathcal{F}(\phi(-\Delta_{SS})^{a/2} |D|)
\]
which is enough to yield the second equality in the lemma.
Finally we give the proof to the last equivalence. It follows from
\[
\| (1 - \Delta_{SS})^{a/2} f \|_{L^2}^2 \sim \sum_{p \geq -1} 2^{2p} \| \mathcal{F}_p (1 - \Delta_{SS})^{a/2} f \|_{L^2}^2
\]
and
\[
\mathcal{F} \mathcal{F}_p (1 - \Delta_{SS})^{a/2} = \varphi(2^{-p} : \mathcal{F} (1 - \Delta_{SS})^{a/2} = \varphi(2^{-p}) (1 - \Delta_{SS})^{a/2} \mathcal{F}
\]
\[
= (1 - \Delta_{SS})^{a/2} \varphi(2^{-p} : \mathcal{F} = (1 - \Delta_{SS})^{a/2} \mathcal{F} \mathcal{F}_p = \mathcal{F} (1 - \Delta_{SS})^{a/2} \mathcal{F}_p,
\]
where $\varphi(2^{-p}) \xi$ is the multiplier of the operator $\mathcal{F}_p$. We complete the proof of the lemma. \( \square \)

**Lemma 5.9.** Suppose $a, b \geq 0, m \in \mathbb{N}$ and $f$ to be a Schwartz function. Then it hold
\[
\sum_{1 \leq i \neq j \leq 3} \| \Omega_{ij} f \|_{H^m} \sim \| (1 - \Delta_{SS})^{1/2} f \|_{H^m},
\]
(5.40)
\[
\sum_{1 \leq i < j \leq 3} \| (-\Delta_{SS})^{a/2} \Omega_{ij} f \|_{H^m} + \| f \|_{H^m} \sim \| (1 - \Delta_{SS})^{(a+1)/2} f \|_{H^m},
\]
(5.41)
and
\[
\| (1 - \Delta_{SS})^{a/2} f \|_{H^m} \sim \sum_{|a| \leq m} \| (1 - \Delta_{SS})^{a/2} \partial^a f \|_{L^2}.
\]
(5.42)
Moreover, it holds
\[
\| (-\Delta_{SS})^{a/2} f \|_{H^b} \lesssim \| (1 - \Delta_{SS})^{(a+b)/2} f \|_{L^2} + \| f \|_{H^{a+b}}.
\]
Proof. (i) We first give the proof to the last inequality. Thanks to Theorem \[5.3\] we have for \( f \in L^2 \),

\[
f(x) = \sum_{l,m} (\mathcal{B}_l^m f)(x) \overset{\text{def}}{=} \sum_{l,m} f_l^m(r) Y_l^m(\sigma),
\]

where \( x = r\sigma \) with \( r \geq 0 \) and \( \sigma \in \mathbb{S}^2 \). Then it follows

\[
\|(-\Delta_{SS})^{a/2} f\|_{H^b}^2 \sim \sum_{k \geq 1} 2^{2kb} \|\mathcal{F}_k(-\Delta_{SS})^{a/2} f\|_{L^2}^2 \\
\sim \sum_{k \geq 1} 2^{2kb} \|(-\Delta_{SS})^{a/2} \mathcal{F}_k f\|_{L^2}^2 \\
\sim \sum_{k \geq 1} \sum_{l,m} 2^{2kb}(l(l+1))a^a \|\mathcal{B}_l^m(\mathcal{F}_k f)\|_{L^2}^2 \\
\lesssim \sum_{k \geq 1} \sum_{l,m} (2^{2kb} + (l(l+1))a^a) \|\mathcal{B}_l^m(\mathcal{F}_k f)\|_{L^2}^2 \\
\lesssim \|(-\Delta_{SS})^{a+b/2} f\|_{L^2}^2 + \|f\|_{H^{a+b}}^2.
\]

(ii). Now we turn to prove the equivalences in the lemma. Thanks to the fact \( \mathcal{F}\Omega_{ij} = \Omega_{ij}\mathcal{F} \), we are led to

\[\mathcal{F}_k \Omega_{ij} = \Omega_{ij} \mathcal{F}_k.\]

Then we have

\[
\sum_{1 \leq i < j \leq 3} \|\Omega_{ij} f\|_{H^0(\mathbb{R}^3)}^2 \sim \sum_{k \geq 1} \sum_{1 \leq i < j \leq 3} 2^{2ka} \|\Omega_{ij} \mathcal{F}_k f\|_{L^2}^2,
\]

which yields

\[
\sum_{1 \leq i < j \leq 3} \|\Omega_{ij} f\|_{H^0(\mathbb{R}^3)}^2 \sim \sum_{k \geq 1} 2^{2ka} \|(-\Delta_{SS})^{1/2} \mathcal{F}_k f\|_{L^2}^2 \\
\sim \sum_{k \geq 1} 2^{2ka} \|\mathcal{F}_k(-\Delta_{SS})^{1/2} f\|_{L^2}^2 \sim \|(-\Delta_{SS})^{1/2} f\|_{H^0}^2.
\]

This gives \([5.40]\).

(iii). Next we give the proof to the second equivalence. We divide the proof into two steps.

**Step 1:** \( m = 0 \). We want to prove

\[
(5.43) \quad \sum_{1 \leq i < j \leq 3} \|(-\Delta_{SS})^{a/2} \Omega_{ij} f\|_{L^2} + \|f\|_{L^2} \sim \|(1-\Delta_{SS})^{(a+1)/2} f\|_{L^2}
\]

We begin with the case \( 0 \leq a \leq 1 \). Observing that

\[
\langle \Omega_{mn} \Omega_{ij} f, \Omega_{mn} \Omega_{ij} f \rangle - \langle \Omega_{mn} \Omega_{mn} f, \Omega_{ij} \Omega_{ij} f \rangle \\
= \langle \Omega_{mn} \Omega_{ij} f, \Omega_{mn} \Omega_{ij} f \rangle + \langle \Omega_{ij} \Omega_{mn} f, \Omega_{mn} \Omega_{ij} f \rangle - \langle \Omega_{ij} \Omega_{ij} f, \Omega_{mn} \Omega_{mn} f \rangle.
\]

and

\[
[\Omega_{mn}, \Omega_{ij}] f = \delta_{ni} \Omega_{mj} + \delta_{nj} \Omega_{im} - \delta_{jm} \Omega_{in} - \delta_{mi} \Omega_{nj},
\]

we deduce that

\[
(5.45) \quad \sum_{1 \leq m < n \leq 3} \sum_{1 \leq i < j \leq 3} \|\Omega_{mn} \Omega_{ij} f\|_{L^2} \lesssim \|(-\Delta_{SS})^{1/2} f\|_{L^2} + \|(-\Delta_{SS}) f\|_{L^2}.
\]

By the fact

\[
\|(-\Delta_{SS})^{1/2} f\|_{L^2} \sim \sum_{1 \leq m < n \leq 3} \|\Omega_{mn} \Omega_{ij} f\|_{L^2},
\]

...
we get
\begin{equation}
(5.46) \quad \sum_{1 \leq i < j \leq 3} \| (-\Delta_{SS}^{3})^{1/2} \Omega_{ij} f \|_{L^2} \lesssim \| \Delta_{SS}^{3} f \|_{L^2} + \| f \|_{L^2}.
\end{equation}

On the other hand, by (5.44), we obtain
\begin{equation*}
\| (-\Delta_{SS}^{3}) f \|_{L^2} \lesssim \sum_{1 \leq m < n \leq 3} \sum_{1 \leq i < j \leq 3} \| \Omega_{mn} \Omega_{ij} f \|_{L^2} + \| (-\Delta_{SS}^{3})^{1/2} f \|_{L^2} \lesssim \sum_{1 \leq i < j \leq 3} \| (-\Delta_{SS}^{3})^{1/2} \Omega_{ij} f \|_{L^2} + \eta \| (-\Delta_{SS}^{3}) f \|_{L^2} + C_\eta \| f \|_{L^2},
\end{equation*}
where \( \eta \) is a small constant. From which together with (5.46), we obtain (5.43) with \( a = 1 \).

We turn to the case \( 0 < a < 1 \). Due to Lemma 5.6 for smooth functions \( g \) and \( f \), we have
\begin{equation*}
\| (-\Delta_{SS}^{3})^{1/2} (f g) \|_{L^2(\Omega^{3})} \lesssim (\| \nabla_{SS}^{3} g \|_{L^\infty(\Omega^{3})} + \| g \|_{L^\infty(\Omega^{3})}) \| (1 - \Delta_{SS}^{3})^{1/2} f \|_{L^2(\Omega^{3})}.
\end{equation*}

It implies
\begin{equation}
(5.47) \quad \| (-\Delta_{SS}^{3})^{a/2} \Omega_{ij} f \|_{L^2} = \| (-\Delta_{SS}^{3})^{a/2} \Omega_{ij} (\theta_{13} f) - (\Omega_{ij} \theta_{13} + f) \|_{L^2} \gtrsim \| (-\Delta_{SS}^{3})^{a/2} \Omega_{ij} (\theta_{13} + f) \|_{L^2} - \| (1 - \Delta_{SS}^{3})^{a/2} f \|_{L^2}.
\end{equation}

Using the notations introduced in Lemma 5.6, we derive
\begin{equation*}
\Omega_{ij} (\theta_{13} + f) (u) = r^{-1} \mathcal{A}_k F_{13}^+ (r, x_1, x_2),
\end{equation*}
with \( u = (r x_1, r x_2, r \sqrt{1 - |x|^2}) \) and \( 1 \leq k \leq 3 \). Here the operator \( \mathcal{A}_k \) is defined by
\begin{equation*}
\mathcal{A}_1 \defeq \sqrt{1 - |x|^2} \partial_{x_1}, \mathcal{A}_2 \defeq \sqrt{1 - |x|^2} \partial_{x_2}, \mathcal{A}_3 \defeq x_1 \partial_{x_2} - x_2 \partial_{x_1}.
\end{equation*}

Therefore, by (5.33), we obtain
\begin{equation*}
\sum_{1 \leq i < j \leq 3} \| (-\Delta_{SS}^{3})^{a/2} \Omega_{ij} (\theta_{13} + f) \|_{L^2} + \| (1 - \Delta_{SS}^{3})^{1/2} (\theta_{13} + f) \|_{L^2} \sim \sum_{1 \leq i < j \leq 3} \| \mathcal{A}_k F_{13}^+ \|_{H_1^2} L_i^2 H_i^1 + \| F_{13}^+ \|_{L_i^2 H_i^1} \sim \| F_{13}^+ \|_{L_i^2 H_i^1} \sim \| (1 - \Delta_{SS}^{3})^{(a + 1)/2} (\theta_{13} + f) \|_{L^2},
\end{equation*}
where (5.39) is used in the last equivalence. From which together with (5.47) and
\begin{equation*}
\| \Omega_{ij} f \|_{L^2} + \| (-\Delta_{SS}^{3})^{a/2} \Omega_{ij} f \|_{L^2} \sim \frac{3}{2} \left( \| (-\Delta_{SS}^{3})^{a/2} \theta_{1m} \Omega_{ij} f \|_{L^2} + \| (-\Delta_{SS}^{3})^{a/2} \theta_{1m - \Omega_{ij} f} \|_{L^2} + \| \Omega_{ij} f \|_{L^2} \right),
\end{equation*}
we have
\begin{equation*}
\sum_{1 \leq i < j \leq 3} \| (-\Delta_{SS}^{3})^{a/2} \Omega_{ij} f \|_{L^2} + \| (1 - \Delta_{SS}^{3})^{1/2} f \|_{L^2} \sim \| (1 - \Delta_{SS}^{3})^{(a + 1)/2} f \|_{L^2},
\end{equation*}
which implies (5.43) with \( 0 < a < 1 \). It completes the proof to (5.43) for \( a \in [0, 1] \).

Next we prove (5.43) holds for \( 1 < a \leq 2 \). Suppose \( a = 1 + s \) with \( 0 < s \leq 1 \). Thanks to (5.43) with \( 0 \leq a \leq 1 \), we have
\begin{equation*}
\sum_{1 \leq i < j \leq 3} \| (-\Delta_{SS}^{3})^{a/2} \Omega_{ij} f \|_{L^2} + \| f \|_{L^2} \sim \sum_{1 \leq i < j \leq 3} \| (-\Delta_{SS}^{3})^{s/2} \Omega_{ij} f \|_{L^2} + \| f \|_{L^2} \sim \sum_{1 \leq i < j \leq 3} \left( \sum_{1 \leq m < n \leq 3} \| (-\Delta_{SS}^{3})^{s/2} \Omega_{mn} \Omega_{ij} f \|_{L^2} + \| \Omega_{ij} f \|_{L^2} \right) + \| f \|_{L^2} \sim \sum_{1 \leq i < j \leq 3} \sum_{1 \leq m < n \leq 3} \sum_{p=1}^3 \| (-\Delta_{SS}^{3})^{s/2} \left( \theta_{1p} + \Omega_{mn} \Omega_{ij} f \right) \|_{L^2} + \| (1 - \Delta_{SS}^{3})^{1/2} f \|_{L^2}.
\end{equation*}
Notice that
\[
(-\Delta_{SS})^{s/2}(\theta_{13} + \Omega_{mn} \Omega_{ij} f) = (-\Delta_{SS})^{s/2} \Omega_{mn} \Omega_{ij} (\theta_{13} f) - (-\Delta_{SS})^{s/2} (\Omega_{mn} \theta_{13} f) - (-\Delta_{SS})^{s/2} (\Omega_{ij} \theta_{13} f) - (-\Delta_{SS})^{s/2} (\Omega_{mn} \Omega_{ij} \theta_{13} f),
\]
\[\|(-\Delta_{SS})^{s/2} (\Omega_{mn} \Omega_{ij} \theta_{13} f)\|_{L^2} \lesssim \|1 - \Delta_{SS}\|^{s/2} \Omega_{ij} f\|_{L^2} \lesssim \|1 - \Delta_{SS}\|^{a/2} f\|_{L^2}\]
and
\[\|(-\Delta_{SS})^{s/2} (\Omega_{mn} \Omega_{ij} \theta_{13} f)\|_{L^2} \lesssim \|1 - \Delta_{SS}\|^{s/2} f\|_{L^2},\]
then we have
\[\|(-\Delta_{SS})^{s/2} (\theta_{13} + \Omega_{mn} \Omega_{ij} f)\|_{L^2} + \|1 - \Delta_{SS}\|^{a/2} f\|_{L^2} \sim \|(-\Delta_{SS})^{s/2} (\Omega_{mn} \Omega_{ij} \theta_{13} f)\|_{L^2} + \|1 - \Delta_{SS}\|^{a/2} f\|_{L^2}\]
Thus we have
\[\sum_{1 \leq i < j \leq 3} \|\|(-\Delta_{SS})^{a/2} \Omega_{ij} f\|_{L^2} + \|f\|_{L^2} + \|1 - \Delta_{SS}\|^{a/2} f\|_{L^2}\]
\[\sim \sum_{1 \leq i < j \leq 3} \sum_{1 \leq m < n \leq 3} \sum_{p=1}^{3} \|\|(-\Delta_{SS})^{s/2} (\Omega_{mn} \Omega_{ij} \theta_{13} f)\|_{L^2} + \|\Omega_{mn} \Omega_{ij} \theta_{13} f\|_{L^2} + \|(-\Delta_{SS})^{s/2} (\Omega_{mn} \Omega_{ij} \theta_{13} f)\|_{L^2} + \|1 - \Delta_{SS}\|^{a/2} f\|_{L^2}\]
If we show
\[(5.48) \sum_{1 \leq i < j \leq 3} \sum_{1 \leq m < n \leq 3} \|\|(-\Delta_{SS})^{s/2} \Omega_{mn} \Omega_{ij} (\theta_{13} f)\|_{L^2} + \|\Omega_{mn} \Omega_{ij} (\theta_{13} f)\|_{L^2} \sim \|1 - \Delta_{SS}\|^{(s+2)/2} (\theta_{13} f)\|_{L^2},\]
then by Young inequality,
\[\|1 - \Delta_{SS}\|^{a/2} f\|_{L^2} \leq \eta \|1 - \Delta_{SS}\|^{(a+1)/2} f\|_{L^2} + C_{\eta} \|f\|_{L^2},\]
we conclude the equivalence (5.43) with \(1 \leq a \leq 2\). It remains to prove (5.48). On one hand, we observe that
\[\sum_{1 \leq i < j \leq 3} \sum_{1 \leq m < n \leq 3} \sum_{k=1}^{3} \sum_{p=1}^{3} \|\|a_{k} \psi \tilde{F}_{13} + \|\tilde{F}_{13} + \|\tilde{F}_{13} + \|\tilde{F}_{13}\|_{L^2 H^2}\]
\[\sim \|\tilde{F}_{13} + \|\tilde{F}_{13} + \|\tilde{F}_{13} H^{2+s},\]
On the other hand, it is easy to check that
\[\|(-\Delta_{SS})^{s+2}/2 (\theta_{13} f)\|_{L^2} + \|\theta_{13} f\|_{L^2}\]
\[\sim \|(-\Delta_{SS})^{s/2} (\theta_{13} f)\|_{L^2} + \|\theta_{13} f\|_{L^2}\]
\[\sim \|\tilde{F}_{13} + \|\tilde{F}_{13} + \|\tilde{F}_{13}\|_{L^2 H^2}\]
where we use (5.37) and (5.39). We end the proof to (5.48) by these two equivalences.
To complete the proof, we first use inductive method to show that for \(a \geq 0\),
\[(5.49) \|\|(-\Delta_{SS})^{s/2} \Omega_{ij} f\|_{L^2} \lesssim \|1 - \Delta_{SS}\|^{s/2} (\theta_{13} f)\|_{L^2}\]

Since \(5.49\) holds for \(0 \leq a \leq 2\), we assume \(5.49\) holds for \(a \leq m\) with \(m \geq 2\). Suppose \(a \in [m, m+1]\), then we have
\[
\|(-\Delta^a_{SS})^{a/2} \Omega_{ij} f\|_{L^2} \leq \|(-\Delta^a_{SS})^{a/2-1} \sum_{1 \leq m < n \leq 3} \Omega^2_{mn} \Omega_{ij} f\|_{L^2}
\]
\[
\lesssim \|(-\Delta^a_{SS})^{a/2-1} \Omega_{ij} (-\Delta^a_{SS}) f\|_{L^2} + \|(-\Delta^a_{SS})^{a/2-1} \sum_{1 \leq m < n \leq 3} \Omega^2_{mn} \Omega_{ij} f\|_{L^2}
\]
\[
\lesssim \|(-\Delta^a_{SS})^{a/2-1} \Omega_{ij} (-\Delta^a_{SS}) f\|_{L^2} + \sum_{1 \leq m < n \leq 3} \sum_{1 \leq i < j \leq 3} \|(-\Delta^a_{SS})^{a/2-1} \Omega_{mn} \Omega_{ij} f\|_{L^2}
\]
\[
\lesssim \|(1 - \Delta^a_{SS})^{a/2} (-\Delta^a_{SS}) f\|_{L^2},
\]
where we use the inductive assumption in the last inequality and the fact
\[
[ \sum_{1 \leq m < n \leq 3} \Omega^2_{mn} \Omega_{ij}] = \sum_{1 \leq m < n \leq 3} (\Omega_{mn} \Omega_{ij} + \Omega_{mn} \Omega_{ij} \Omega_{mn}).
\]
We complete the inductive argument to derive \(5.49\).

Now we are ready to prove \(5.43\). We may assume that \(5.43\) holds for \(a \leq m\) with \(m \geq 1\). Suppose \(a \in [m, m+1]\). We have
\[
\sum_{1 \leq i < j \leq 3} \|(-\Delta^a_{SS})^{a/2} \Omega_{ij} f\|_{L^2} + \|(-\Delta^a_{SS})^{a/2} f\|_{L^2}
\]
\[
\leq \sum_{1 \leq i < j \leq 3} \|(-\Delta^a_{SS})^{a/2} \Omega_{ij} f\|_{L^2} + \|(-\Delta^a_{SS})^{a/2} f\|_{L^2}
\]
\[
\lesssim \|(1 - \Delta^a_{SS})^{a/2} (-\Delta^a_{SS}) f\|_{L^2} \leq \|(1 - \Delta^a_{SS})^{a+1/2} f\|_{L^2},
\]
where we use \(5.49\) in the last inequality, we finally derive the desired result and ends the inductive argument for the equivalence \(5.43\).

**Step 2**: \(m \in \mathbb{N}\). By Lemma 5.8 and \(5.43\), we have
\[
\|(-\Delta^a_{SS})^{a/2} f\|_{H^m} \sim \sum_{k \geq -1} 2^{2mk} \|(-\Delta^a_{SS})^{a/2} \mathcal{F}_k f\|_{L^2}^2
\]
\[
\sim \sum_{k \geq -1} 2^{2mk} \left( \sum_{1 \leq i < j \leq 3} \|(-\Delta^a_{SS})^{a/2} \Omega_{ij} \mathcal{F}_k f\|_{L^2}^2 + \|\mathcal{F}_k f\|_{L^2}^2 \right)
\]
\[
\sim \sum_{k \geq -1} 2^{2mk} \left( \sum_{1 \leq i < j \leq 3} \|(-\Delta^a_{SS})^{a/2} \mathcal{F}_k \Omega_{ij} f\|_{L^2}^2 + \|\mathcal{F}_k f\|_{L^2}^2 \right)
\]
\[
\sim \sum_{1 \leq i < j \leq 3} \|(-\Delta^a_{SS})^{a/2} \Omega_{ij} \mathcal{F}_k f\|_{H^m}^2 + \| f\|_{H^m}^2,
\]
where we use the fact \(\Omega_{ij} \mathcal{F} = -\mathcal{F} \Omega_{ij}\) if \(i \neq j\). We complete the proof to \(5.41\).

(iv) We first claim that for \(0 \leq a \leq 1\), it holds
\[
\sum_{1 \leq i \leq 3} \|(-\Delta^a_{SS})^{a/2} \partial_i f\|_{L^2} + \|\partial_i f\|_{L^2} + \|(-\Delta^a_{SS})^{a/2} f\|_{L^2} \sim \|(1 - \Delta^a_{SS})^{a/2} (D) f\|_{L^2}.
\]
In other words, we have
\[
\|(-\Delta^a_{SS})^{a/2} f\|_{H^1} \sim \sum_{|a| \leq 1} \|(-\Delta^a_{SS})^{a/2} \partial^a f\|_{L^2}.
\]
Obviously \(5.50\) holds for \(a = 0\). Then we separate the proof of \(5.50\) into two cases.
Case 1: $0 < a < 1$. Indeed, by Plancherel's equality and Lemma 5.8, we have
\[
\sum_{1 \leq l \leq 3} \left( \|(-\Delta_{SS})^{a/2} \partial_l f \|_{L^2} + \|\partial_l f \|_{L^2} \right) = \sum_{1 \leq l \leq 3} \left( \|(-\Delta_{SS})^{a/2} \xi_l \mathcal{F} f \|_{L^2} + \|\xi_l \mathcal{F} f \|_{L^2} \right).
\]
Due to Lemma 5.6 and by setting $\xi = r\sigma = (r\sigma_1, r\sigma_2, r\sigma_3)$, we have
\[
\sum_{1 \leq l \leq 3} \|(-\Delta_{SS})^{a/2} \partial_l f \|_{L^2} + \|f \|_{H^1}^2 \sim \sum_{1 \leq l \leq 3} \int_{\sigma, \tau \in SS^2, r \in \mathbb{R}} \left| r\sigma_l (\mathcal{F} f)(r\sigma) - r\tau_l (\mathcal{F} f)(r\tau) \right|^2 r^2 d\sigma d\tau dr + \|f \|_{H^1}^2.
\]
Thanks to the observation
\[
\left| r\tau_l \right|^2 \left| r(\mathcal{F} f)(r\sigma) - r(\mathcal{F} f)(r\tau) \right|^2 - 2 \left| r\sigma_l \right|^2 \left| r(\mathcal{F} f)(r\sigma) \right|^2 \leq \left| r\sigma_l \right|^2 \left| r(\mathcal{F} f)(r\sigma) - r\tau_l (\mathcal{F} f)(r\tau) \right|^2 \leq \left| r\tau_l \right|^2 \left| r(\mathcal{F} f)(r\sigma) - r\tau_l (\mathcal{F} f)(r\tau) \right|^2 + \left| r\sigma_l \right|^2 \left| r(\mathcal{F} f)(r\sigma) \right|^2,
\]
we deduce that
\[
\sum_{1 \leq l \leq 3} \|(-\Delta_{SS})^{a/2} \partial_l f \|_{L^2} + \|f \|_{H^1} \sim \|(-\Delta_{SS})^{a/2} (D f) \|_{L^2} + \|f \|_{H^1}.
\]
This is enough to get (5.50) for $0 < a < 1$.

Case 2: $a = 1$. It is not difficult to check
\[
\sum_{|a| \leq 1} \left( \|(-\Delta_{SS})^{1/2} \partial_a f \|_{L^2} + \|f \|_{H^1} \right) \sim \sum_{|a| \leq 1} \|\Omega_{mn} \partial_a f \|_{L^2} + \|f \|_{H^1} \sim \sum_{1 \leq m < n \leq 3} \|\Delta \Omega_{mn} f \|_{L^2} + \|f \|_{H^1} \sim \sum_{1 \leq m < n \leq 3} \|\Omega_{mn} f \|_{H^1} + \|f \|_{H^1} \sim \|1 - \Delta_{SS} \|^{1/2} f \|_{H^1},
\]
where we use (5.40) in the last equivalence. We complete the proof to (5.50).

Now we are ready to prove (5.42) holds for $a \in [0, 1]$ by inductive method. Thanks to (5.50), we assume (5.40) holds for $N - 1$. Recall that
\[
\sum_{|a| \leq N} \|(-\Delta_{SS})^{a/2} \partial_a f \|_{L^2} \sim \sum_{|a| \leq N-1} \|(-\Delta_{SS})^{a/2} \partial_a f \|_{L^2} + \sum_{|a| = N} \|(-\Delta_{SS})^{a/2} \partial_a f \|_{L^2} + \|f \|_{H^N}.
\]
Thanks to (5.51), we have
\[
\sum_{|a| = N} \|(-\Delta_{SS})^{a/2} \partial_a f \|_{L^2} + \|f \|_{H^N} \sim \sum_{|\beta| = N-1} \sum_{1 \leq l \leq 3} \|(-\Delta_{SS})^{a/2} \partial_l \partial_\beta f \|_{L^2} + \|f \|_{H^N} \sim \sum_{|\beta| = N-1} \|(-\Delta_{SS})^{a/2} \partial_\beta (D f) \|_{L^2} + \|f \|_{H^N},
\]
which implies
\[ \sum_{|a|=N} \|(-\Delta_{SS})^{a/2} \partial^a f\|_{L^2} + \|f\|_{H^N} \sim \|(-\Delta_{SS})^{a/2} \partial^a f\|_{L^2} + \|f\|_{H^N}, \]
by the assumption that (5.40) holds for \( N - 1 \). We deduce that
\[ \sum_{|a|=N} \|(-\Delta_{SS})^{a/2} \partial^a f\|_{L^2} \sim \|(-\Delta_{SS})^{a/2} \partial^a f\|_{H^{N-1}} + \|(-\Delta_{SS})^{a/2} |D|^N f\|_{L^2} + \|f\|_{H^N} \sim \|(-\Delta_{SS})^{a/2} \partial^a f\|_{H^N}, \]
which completes the inductive argument to derive (5.42) with \( a \in [0, 1] \).

Finally we are in a position to prove (5.42). We still use the inductive method. Suppose (5.42) holds for \( a \leq N \) with \( N \geq 1 \). Suppose now \( a \in [N, N+1] \). Due to (5.41) and the inductive assumption, we have
\[ \|(-\Delta_{SS})^{a/2} \partial^a f\|_{H^m} \sim \sum_{1 \leq i < j \leq 3} \|(-\Delta_{SS})^{a/2} \partial^a \Omega_{ij} f\|_{H^m} + \|f\|_{H^m} \sim \sum_{1 \leq i < j \leq 3} \|(-\Delta_{SS})^{a/2} \partial^a \Omega_{ij} f\|_{L^2} + \|f\|_{H^m} \]
Using (5.41) and the inductive assumption again, we have
\[ \sum_{|a|=m} \sum_{1 \leq i < j \leq 3} \|(-\Delta_{SS})^{a/2} \partial^a \Omega_{ij} f\|_{L^2} \quad \sim \sum_{|a|=m} \|(-\Delta_{SS})^{a/2} \partial^a f\|_{L^2}. \]
Notice the fact
\[ \sum_{|a|=m} \sum_{1 \leq i < j \leq 3} \|(-\Delta_{SS})^{a/2} \partial^a \Omega_{ij} f\|_{L^2} \leq \sum_{|\beta| \leq m} \|(-\Delta_{SS})^{a/2} \partial^\beta f\|_{L^2} \]
\[ \leq \sum_{|a|=m} \sum_{1 \leq i < j \leq 3} \|(-\Delta_{SS})^{a/2} \partial^a \Omega_{ij} f\|_{L^2} \]
we derive that
\[ \sum_{|a|=m} \|(-\Delta_{SS})^{a/2} \partial^a f\|_{L^2} \leq C_1 \|(1 - \Delta_{SS})^{a/2} f\|_{H^m} \]
\[ \leq \|(-\Delta_{SS})^{a/2} f\|_{H^m} \]
\[ \leq \sum_{|a|=m} \|(-\Delta_{SS})^{a/2} \partial^a f\|_{L^2} + C_2 \|(1 - \Delta_{SS})^{a/2} f\|_{H^m} \]
Observe that
\[ \|(1 - \Delta_{SS})^{a/2} f\|_{H^m} \sim \|(1 - \Delta_{SS})^{a/2} D)^m f\|_{L^2} \]
\[ \leq \eta \|(1 - \Delta_{SS})^{a/2} (D)^m f\|_{L^2} + C_\eta \|D)^m f\|_{L^2} \]
\[
\sim \eta \|(1 - \Delta_{SS})^{a/2} f\|_{H^m} + C_\eta \|f\|_{H^m}.
\]
(5.52) yields the desired result and we complete the inductive argument to (5.42). We end the proof of the lemma. \( \square \)

**Lemma 5.10.** If \( T_h f (v) \overset{\text{def}}{=} f (v + h) \). Then for \( s \geq 0 \), it holds
\[ \|(1 - \Delta_{SS})^{a/2} T_h f\|_{L^2 (\mathbb{R}^2)} \leq (h)^s \|(1 - \Delta_{SS})^{a/2} f\|_{L^2 (\mathbb{R}^2)} \]
Proof. We begin with the case \(0 \leq s \leq 1\). Thanks to Lemma \[5.6\] we have
\[
\|(-\Delta_{SS})^{s/2} T_h f\|_{L^2(\mathbb{R}^3)} \lesssim \sum_{1 \leq i < j \leq 3} \|T_h f\|_{\mathcal{A}_{ij}(s,2)}.
\]
Since
\[
\|\Omega_{ij} T_h f\|_{L^2} \lesssim \langle h \rangle (\|f\|_{H^1} + \|\Omega_{ij} f\|_{L^2}), \quad \|T_h f\|_{L^2} = \|f\|_{L^2},
\]
then applying Lemma \[5.2\] with \(A = \Omega_{ij}\) and \(B_k = \partial_k\), we get
\[
(5.53) \quad \|T_h f\|_{\mathcal{A}_{ij}(s,2)} \lesssim \langle h \rangle^s (\|f\|_{H^1} + \|f\|_{\mathcal{A}_{ij}(s,2)}).
\]
From which together with Lemma \[5.6\] we obtain the desired result.

Next we turn to the case \(1 < s \leq 2\). Suppose \(s = 1 + a\). Then by Lemma \[5.9\] we have
\[
\|(-\Delta_{SS})^{s/2} f\|_{L^2(\mathbb{R}^3)} \lesssim \sum_{1 \leq i < j \leq 3} (\|f\|_{L^2} + \|(-\Delta_{SS})^{a/2} \Omega_{ij} f\|_{L^2}),
\]
\[
\lesssim \sum_{1 \leq i < j \leq 3} (\|f\|_{L^2} + \|\Omega_{ij} f\|_{\mathcal{A}_{ij}(a,2)}).
\]
Therefore,
\[
\|(-\Delta_{SS})^{s/2} T_h f\|_{L^2(\mathbb{R}^3)} \lesssim \|f\| + \sum_{1 \leq i < j \leq 3} \|\Omega_{ij} T_h f\|_{\mathcal{A}_{ij}(a,2)}
\]
\[
\lesssim \|f\| + \sum_{1 \leq i < j \leq 3} (\|T_h \Omega_{ij} f\|_{\mathcal{A}_{ij}(a,2)} + \langle h \rangle (\|T_h \partial_i f\|_{\mathcal{A}_{ij}(a,2)} + \|T_h \partial_j f\|_{\mathcal{A}_{ij}(a,2)})).
\]
Thanks to \[5.53\] and Lemma \[5.6\] we are led to
\[
\|(-\Delta_{SS})^{s/2} T_h f\|_{L^2(\mathbb{R}^3)} \lesssim \langle h \rangle^s \sum_{1 \leq i < j \leq 3} (\|(-\Delta_{SS})^{a/2} \Omega_{ij} f\|_{L^2(\mathbb{R}^3)} + \|\Omega_{ij} f\|_{H^a(\mathbb{R}^3)} + \|f\|_{H^0})
\]
\[
+ \|(-\Delta_{SS})^{a/2} \partial_i f\|_{L^2(\mathbb{R}^3)} + \|(-\Delta_{SS})^{a/2} \partial_j f\|_{L^2(\mathbb{R}^3)}).
\]
Due to Lemma \[5.9\] we deduce
\[
\|(-\Delta_{SS})^{s/2} T_h f\|_{L^2(\mathbb{R}^3)} \lesssim \langle h \rangle^s (\|(-\Delta_{SS})^{s/2} f\|_{L^2(\mathbb{R}^3)} + \|f\|_{H^0}).
\]
Finally we use the inductive method to handle the case \(s > 2\). We assume the result holds for \(m \leq 2N\). Suppose \(s \in (2N, 2N + 2]\). Then
\[
\|(-\Delta_{SS})^{s/2} T_h f\|_{L^2(\mathbb{R}^3)} = \sum_{1 \leq i < j \leq 3} \|(-\Delta_{SS})^{s/2 - 1} \Omega_{ij}^2 T_h f\|_{L^2(\mathbb{R}^3)}.
\]
It is easy to check that
\[
\Omega_{ij}^2 T_h f = T_h (\Omega_{ij} f) - h_i T_h (\partial_j \Omega_{ij} f) + h_j T_h (\partial_i \Omega_{ij} f) - h_i T_h (\Omega_{ij} \partial_j f)
\]
\[
+ h_i^2 T_h (\partial_j^2 f) - h_j^2 T_h (\partial_i^2 f) + h_i T_h (\Omega_{ij} \partial_i f).
\]
Then by the inductive assumption and Lemma \[5.3\], we are led to
\[
\|(-\Delta_{SS})^{s/2} T_h f\|_{L^2(\mathbb{R}^3)} \lesssim \langle h \rangle^s \sum_{1 \leq i < j \leq 3} \|(-\Delta_{SS})^{s/2 - 1} \Omega_{ij}^2 f\|_{L^2(\mathbb{R}^3)} + \|\Omega_{ij}^2 f\|_{H^{s-2}}
\]
\[
+ \|(-\Delta_{SS})^{s/2 - 1} \partial_\Omega_{ij} f\|_{L^2(\mathbb{R}^3)} + \|\partial_\Omega_{ij} f\|_{H^{s-2}}
\]
\[
+ \|(-\Delta_{SS})^{s/2 - 1} \partial_i^2 f\|_{L^2(\mathbb{R}^3)} + \|\partial_i^2 f\|_{H^{s-2}}
\]
\[
\lesssim \langle h \rangle^s (\|(-\Delta_{SS})^{s/2} f\|_{L^2(\mathbb{R}^3)} + \|f\|_{H^s}).
\]
which completes the inductive argument to get the result.

Finally we will give the proof to Lemma 4.1.

Proof. We follow the idea used in Lemma 5.5. Thanks to Lemma 5.4, we have

\[ \epsilon^{2s-2} \int_{|\sigma - \tau| \leq \epsilon} \frac{|f(\sigma) - f(\tau)|^2}{|\sigma - \tau|^{2+2s}} \, d\sigma d\tau = \epsilon^{2s-2} \sum_{l} |f_l|^2 \int_{|\sigma - \tau| \leq \epsilon} \frac{|Y_l^m(\sigma) - Y_l^m(\tau)|^2}{|\sigma - \tau|^{2+2s}} \, d\sigma d\tau. \]

Then the result in the lemma is reduced to prove

\[ \epsilon^{2s-2} \int_{|\sigma - \tau| \leq \epsilon} \frac{|f(\sigma) - f(\tau)|^2}{|\sigma - \tau|^{2+2s}} \, d\sigma d\tau + \|f\|_{L^2(S^2)}^2. \]

We divide the proof into three cases.

**Case 1: l is small.** We first claim that

\[ \|(-\Delta_{S^2})^{1/2} f\|_{L^2(S^2)}^2 + \|f\|_{L^2(S^2)}^2 - c^2 \|(-\Delta_{S^2}) f\|_{L^2(S^2)}^2 \]

\[ \lesssim \epsilon^{2s-2} \int_{|\sigma - \tau| \leq \epsilon} \frac{|f(\sigma) - f(\tau)|^2}{|\sigma - \tau|^{2+2s}} \, d\sigma d\tau + \|f\|_{L^2(S^2)}^2. \]

In particular, if we choose \( f = Y_l^m \), then there exists a universal constant \( c_1 \) and \( c_2 \) such that

\[ (1 - c_1 [l(l+1)] \epsilon^2) [l(l+1)] + 1 \]

\[ \lesssim II \equiv \epsilon^{2s-2} \int_{|\sigma - \tau| \leq \epsilon} \frac{|Y_l^m(\sigma) - Y_l^m(\tau)|^2}{|\sigma - \tau|^{2+2s}} \, d\sigma d\tau + 1 \]

\[ \lesssim (1 + c_2 [l(l+1)] \epsilon^2) [l(l+1)] + 1. \]

Then we arrive at that if \( [l(l+1)] \leq (2c_1)^{-1/2} \epsilon^{-1} \),

(5.54)

\[ II \sim l(l+1) + 1. \]

To prove the claim, we set

\[ I \equiv \epsilon^{2s-2} \int_{|\sigma - \tau| \leq \epsilon} \frac{|f(\sigma) - f(\tau)|^2}{|\sigma - \tau|^{2+2s}} \, d\sigma d\tau + \|f\|_{L^2(S^2)}^2. \]

Then it is easy to check

\[ I \sim \epsilon^{2s-2} \int_{|\sigma - \tau| \leq \epsilon} \frac{|(\overline{\partial}_{13} f)(\sigma) - (\overline{\partial}_{13} f)(\tau)|^2}{|\sigma - \tau|^{2+2s}} \, d\sigma d\tau + \|f\|_{L^2(S^2)}^2. \]
Suppose \( \sigma = (x_1, x_2, x_3) \in \mathbb{S}^2 \) and set \( x = (x_1, x_2) \). Let \( F_{13}^+(x) \overset{\text{def}}{=} (\theta_{13+}, f)(x_1, x_2, \sqrt{1-x_1^2-x_2^2}) \) and \( \Theta_{33}^+(x) = \theta_{33+}(x_1, x_2, \sqrt{1-x_1^2-x_2^2}) \). Then by change of variables, we have

\[
\epsilon^{2s-2} \int_{|x| \leq \sqrt{\frac{1}{2}}} \frac{|(\theta_{13+}, f)(\sigma) - (\theta_{13+}, f)(r)|^2}{|\sigma - r|^{2s+2}} \Theta_{33+}(\sigma) \Theta_{33+}(r) \, d\sigma \, dr \\
\geq \epsilon^{2s-2} \int_{|x| \leq \sqrt{\frac{1}{2}}} \frac{|F_{13}^+(x) - F_{13}^+(y)|^2}{|x-y|^{2s+2}} \, dx \, dy \\
\times \Theta_{33}^+(x) \Theta_{33}^+(y) \\
\approx \epsilon^{2s-2} \int_{|x| \leq \sqrt{\frac{1}{2}}} \frac{|F_{13}^+(x) - F_{13}^+(y)|^2}{|x-y|^{2s+2}} \, dx \, dy.
\]

Thanks to the Taylor expansion, it yields that

\[
I \geq \epsilon^{2s-2} \int_{|x| \leq \sqrt{\frac{1}{2}}} \frac{|\nabla F_{13}^+(x) \cdot (y-x)|^2}{|x-y|^{2s+2}} \, dx \, dy \\
- \epsilon^{2s-2} \int_0^1 \int_{|x| \leq \sqrt{\frac{1}{2}}} \frac{|\nabla^2 F_{13}^+(x + \kappa(y-x))| |x-y|^{4}}{|x-y|^{2s+2}} \, dx \, dy \, d\kappa.
\]

Note that

\[
\epsilon^{2s-2} \int_{|x| \leq \sqrt{\frac{1}{2}}} \frac{|\nabla F_{13}^+(x) \cdot (y-x)|^2}{|x-y|^{2s+2}} \, dx \, dy \\
= \int_{|x| \leq \sqrt{\frac{1}{2}}} \frac{|\nabla F_{13}^+(x)|^2}{|h|^{2s}} \int_{|h| \leq \epsilon} \frac{|\nabla F_{13}^+(x) \cdot h|^2}{|h|^{2s}} \, dh \, dx
\]

and

\[
\epsilon^{2s-2} \int_0^1 \int_{|x| \leq \sqrt{\frac{1}{2}}} \frac{|\nabla^2 F_{13}^+(x + \kappa(y-x))| |x-y|^{4}}{|x-y|^{2s+2}} \, dx \, dy \, d\kappa \\
\lesssim \epsilon^2 \|F_{13}^+\|_{H^2(B_{\sqrt{2}}^2)}^2.
\]

Then we derive that

\[
(5.56) \quad I \gtrsim \|F_{13}^+\|_{H^1(B_{\sqrt{2}}^2)}^2 \epsilon^2 \|F_{13}^+\|_{H^2(B_{\sqrt{2}}^2)}^2.
\]

On the other hand, following the similar argument, we may infer

\[
(5.57) \quad I \lesssim \|F_{13}^+\|_{H^1(B_{\sqrt{2}}^2)}^2 + \epsilon^2 \|F_{13}^+\|_{H^2(B_{\sqrt{2}}^2)}^2.
\]

Thanks to the facts \(5.26, 5.37, 5.56\) and \(5.57\) imply that

\[
\|(-\Delta_{SS^2})^{1/2}(\theta_{13+}, f)\|_{L^2(S^2)}^2 + \|\theta_{13+}, f\|_{L^2(S^2)}^2 \lesssim \epsilon^2 \|(-\Delta_{SS^2})(\theta_{13+}, f)\|_{L^2(S^2)}^2
\]

\[
\lesssim \epsilon^2 \lesssim \|(-\Delta_{SS^2})^{1/2}(\theta_{13+}, f)\|_{L^2(S^2)}^2 + \|\theta_{13+}, f\|_{L^2(S^2)}^2 + \epsilon^2 \|(-\Delta_{SS^2})(\theta_{13+}, f)\|_{L^2(S^2)}^2.
\]

Due to the decomposition \(5.22\), we finally conclude the claim.
Case 2: \( l \) is sufficiently large. Observe that

\[
II = e^{2s-2} \int_{\sigma, \tau \in \mathbb{R}^2} \frac{|Y_l^m(\sigma) - Y_l^m(\tau)|^2}{|\sigma - \tau|^{2+2s}} d\sigma d\tau 
- e^{2s-2} \int_{\sigma, \tau \in \mathbb{R}^2} \frac{|Y_l^m(\sigma) - Y_l^m(\tau)|^2}{|\sigma - \tau|^{2+2s}} 1_{|\sigma - \tau| \geq \epsilon} d\sigma d\tau + 1.
\]

Thanks to Lemma 5.5, there exits a universal constants \( c_3 \) and \( c_4 \) such that

\[
e^{2s-2} \left( \frac{1}{l(l+1)^s} + 1 - c_3 e^{-2s}\right) II \lesssim e^{2s-2} \frac{1}{l(l+1)^s} + 1 + c_4 e^{-2s}.
\]

It implies that if \([l(l+1)]^{1/2} \geq 2\epsilon_3^{1/2}\epsilon^{-1},\)

\[(5.58)\]

\[II \sim e^{2s-2} \frac{1}{l(l+1)^s} + 1.\]

Case 3: \([l(l+1)]^{1/2} \sim \epsilon^{-1}.\) We claim that in this case, \( II \sim l(l+1) + 1.\) Observe that for any \( N \in \mathbb{N},\)

\[
II \geq N^{2s-2} (\epsilon / N) e^{2s-2} \int_{\sigma, \tau \in \mathbb{R}^2} \frac{|Y_l^m(\sigma) - Y_l^m(\tau)|^2}{|\sigma - \tau|^{2+2s}} 1_{|\sigma - \tau| \leq \epsilon / l} N d\sigma d\tau + 1
\]

\[
\geq N^{2s-2} (\epsilon / N) e^{2s-2} \int_{\sigma, \tau \in \mathbb{R}^2} \frac{|Y_l^m(\sigma) - Y_l^m(\tau)|^2}{|\sigma - \tau|^{2+2s}} 1_{|\sigma - \tau| \leq \epsilon / l} N d\sigma d\tau + 1.
\]

From which together with (5.54) and (5.55), we derive that if \([l(l+1)]^{1/2} \leq (2\epsilon_1)^{-1/2} N \epsilon^{-1},\)

\[(5.59)\]

\[N^{2s-2} \left( l(l+1) + 1 \right) \lesssim II \lesssim \left( l(l+1) + 1 \right).\]

Now choose \( N \geq 2\sqrt{2} c_4^{1/(2s)} c_1^{1/2},\) then (5.55), (5.58) and (5.59) yield the claim.

Now we are in a position to prove the lemma. It is easily followed by the estimates (5.55), (5.58) and (5.59). We complete the proof to the lemma. \( \square \)

6. Conclusions and Perspectives

In this paper, by making full use of two types of decompositions performed in Phase and Frequency spaces and the geometric decomposition, we successfully establish several lower and upper bounds for Boltzmann collision operator in weighted Sobolev spaces and in the anisotropic spaces. By comparing with the behavior of the linearized operator, we show that all the bounds are sharp. We further show that the strategy of the proof is so robust that we can apply it to the rescaled Boltzmann collision operator (see assumption (B1)). Finally we obtain sharp bounds for the Landau collision operator by so-called grazing collisions limit.

It is very interesting to see whether our method used here can be applied or not to capture the exact behavior of the operator under the assumption (I.7) or (I.8). In Section 4, we make an attempt to analyze the Boltzmann collision operator in the process of the grazing collisions limit (see Lemma 4.1). We conjecture that if \( \mathcal{L}_{B}^\epsilon \) is the linearized Boltzmann collision operator with the rescaled cross-section under the assumption (B1), then

\[
\langle \mathcal{L}_{B}^\epsilon f, f \rangle _v + \| f \|^2 _{L^2_{T_1/2}} \sim \| W^\epsilon (D) f \|^2 _{L^2_{T_1/2}} + \| W^\epsilon ((-\Delta_{SS})^{1/2}) f \|^2 _{L^2_{T_1/2}} + \| W^\epsilon f \|^2 _{L^2_{T_1/2}},
\]

where the symbol function \( W^\epsilon \) is defined in (4.1). Based on the conjecture, we may ask:

1. How to establish a unified framework to solve the Boltzmann and Landau equations in the perturbation regime and prove the asymptotic formula (L11);
2. How to describe the behavior of the spectrum of the operator \( \mathcal{L}_{B}^\epsilon \) in the limit \( \epsilon \to 0 \) for \( \gamma \in [-2, -2s] \); we recall that the spectrum gap exists for \( \mathcal{L}_{B}^\epsilon \) if and only if \( \gamma \geq -2s \) while it exists for \( \mathcal{L}_{L} \) if and only if \( \gamma \geq -2. \)
The similar conjecture can be questioned on the operator with the assumption \((1.7)\) or with the Coulomb potential. Then the asymptotics of the Boltzmann equation from short-range interactions to long-range interactions and the Landau approximation for Coulomb potential can be investigated.

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