When is Uniform Rotation an Energy Minimum?

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ABSTRACT

A simple variational calculation is presented showing that a uniformly rotating barotropic fluid in an external potential attains a true energy minimum if and only if the rotation profile is everywhere subsonic. If regions of supersonic rotation are present, fluid variations exist that could take the system to states of lower energy. In any given system, these states may or may not be dynamically accessible, but their existence is important. It means that extending the degrees of freedom available to the fluid (say by weak magnetic fields) may open a path to fluid instabilities. Whether astrophysical gaseous nebula tend toward states of uniform rotation or toward more Keplerian core-disk systems appears to be largely a matter of whether the rotation profile is transonic or not. The suggestion is made that the length scale associated with coherent molecular cloud cores is related to the requirement that the cores be stable and rotate subsonically.

Subject headings: accretion, accretion disks — hydrodynamics — instabilities
1. Introduction

It is a classical notion of hydrodynamics that a state of uniform rotation represents an energy minimum. A simple discussion of this is given by Lynden-Bell & Pringle (1974). The very same paper notes, however, that accretion disks behave in the opposite sense of seeking a state of uniform rotation. Instead, they strive to segregate matter and angular momentum, hoarding all the mass at their centers, and expelling all the angular momentum to large distances. Lynden-Bell and Pringle showed that this behavior emerges as a consequence of mass exchange in the disk, so that whatever unfavorable energy cost is incurred by the differential rotation is more than offset by the decrease in gravitational potential energy.

The question arises of when, more generally, a rotating system system will tend toward a state of constant angular velocity, and when it will accrete toward the singular “black hole” state. Do systems with mass exchange in a gravitational potential necessarily lead to a black hole final state? The problem is clearly one of widespread astrophysical interest, as it has a direct bearing on when rotating gas coalesces into cores and when it simply rotates uniformly.

The magnetorotational instability (Balbus & Hawley 1998) is a natural mechanism to drive a rotating system system to either of these states. Imagine a rotating cloud with a small angular velocity gradient. The singular state will be attained only if the instability acts continuously without turning itself off. But this is precisely what the outcome of a core collapse would be: should the core contract due to angular momentum loss, it would rotate more rapidly and increase the initial angular velocity gradient. This in turn increases the growth rate of the instability, leading to an even greater outward flow of angular momentum from the core. On the other hand, should the extracted angular momentum bring the more slowly rotating outer zones into uniform rotation with the more rapidly rotating core, the instability would cease. What then are the conditions that lead a system down one path or the other?

In this Letter we derive a very simple, general result of some astrophysical significance. Though our treatment is brief and straightforward, we are unaware of any discussion of the result in the prior astrophysical literature. We consider a uniformly rotating barotropic gas confined by an external potential. The case of self-gravitating cylinders has been investigated by Inagaki & Hachisu (1978) and Hachisu (1979), but is significantly more complex than our treatment. (A comparison of our work with the results of these authors will be presented in a later more detailed numerical and analytic study.) We show that the energy is a true minimum in a state of uniform rotation if and only if the rotation velocity is everywhere less than the sound speed. If there exists a region of supersonic rotation, then it is possible to find second-order variations that will lower the energy from the extremal state. In this case, the uniformly rotating state is not a minimum, but a saddle point. Supersonic rotation, when it is unstable, evolves not toward a uniform profile, but towards a state of central mass concentration with a surrounding Keplerian-like disk.

The plan of this Letter is as follows. In §2, we present a variational treatment of a non–self-gravitating, uniformly rotating barotropic fluid as described above. In §3, we present a brief discussion of the astrophysical significance of our results, arguing that they provide a simple way of deciding when systems evolve toward uniform rotation, and when they evolve toward a strong central mass with a surrounding disk–halo structure. We also show that our findings may be relevant to understanding the length scales over which
molecular cloud cores show coherence.

2. Analysis

2.1. Equilibrium First Order Variation

Consider a rotating barotropic gas confined by an external potential $\Phi$. The energy of the system is

$$ E = \int \left( \frac{1}{2} \rho v^2 + \varepsilon + \rho \Phi \right) dV $$

(1)

where $\rho$ is the mass density, $v$ the velocity, and $\varepsilon$ is the internal energy density. Introduce first order Eulerian variations into the fluid quantities (e.g. $\rho \to \rho + \delta \rho$, etc.) subject to the constraint that the total mass and the total angular momentum remain constant. We use Lagrangian multipliers $-\zeta$ and $-\Omega$ to ensure these conditions (the minus signs are for later convenience), and vary the quantity

$$ I = E - \zeta \int \rho dV - \Omega \int \rho R v dV, $$

(2)

where $R$ is the cylindrical radius about the axis of rotation. If the energy is an extremum, then $\delta I = 0$, or

$$ 0 = \int \left[ \delta \rho \left( -\zeta - \Omega R v + \frac{\delta (\text{int})}{\delta \rho} \right) + \delta v \rho (v - \Omega R) \right] dV, $$

(3)

where the variational derivative of the integrand of equation (1) is

$$ \frac{\delta (\text{int})}{\delta \rho} = \frac{v^2}{2} + \frac{d\varepsilon}{d\rho} + \Phi. $$

(4)

We require the coefficients of $\delta v$ and $\delta \rho$ to vanish independently. The first condition leads to solid body rotation,

$$ v = \Omega R, $$

(5)

whence we may identify the Lagrange multiplier $\Omega$ with the angular velocity. To understand the consequences of the second condition (vanishing of the $\delta \rho$ coefficient), we need to interpret the $d\varepsilon/d\rho$ derivative. If our $\delta$ variations are nondissipative, this is an adiabatic derivative, and the first law of the thermodynamics gives

$$ \frac{d\varepsilon}{d\rho} = \frac{1}{\rho} (\varepsilon + P) = H, $$

(6)

where $P$ is the gas pressure, and $H$ the enthalpy function. The vanishing of the $\delta \rho$ coefficient then yields

$$ -\frac{v^2}{2} + H + \Phi = \zeta, $$

(7)
whence we may identify the Lagrange multiplier $\zeta$ with the conserved Bernoulli constant. Equation (7) is simply the integrated form of the hydrostatic equilibrium equation for a uniformly rotating system

$$0 = \frac{1}{\rho} \nabla P + \nabla \left( \Phi - \frac{R^2 \Omega^2}{2} \right)$$

(recall $dH = (1/\rho)dP$), showing that the equilibrium solution is an energy extremum. We do not know at this point that the extremum corresponds to a minimum. For that, we must go to second order in the energy variation.

### 2.2. Second Order Variations

The second-order variation in the integrand of $E$ is

$$\delta^2 E = \int \frac{1}{2} \left[ \rho (\delta v)^2 + 2v \delta \rho \delta v + (\delta \rho)^2 \frac{dH}{d\rho} \right] dV.$$  (9)

Define the volume specific angular momentum $j = \rho R v$. Then

$$\rho (\delta v)^2 + 2v \delta \rho \delta v = \left( \frac{\delta j}{\rho R} \right)^2 - \rho v^2 \left( \frac{\delta \rho}{\rho} \right)^2.$$  (10)

Note additionally that

$$\frac{dH}{d\rho} = \frac{1}{\rho} \frac{dP}{d\rho} = \frac{a^2}{\rho},$$  (11)

where $a$ is the adiabatic sound speed, so that upon substituting for the cross term $2v \delta \rho \delta v$ in (9), we are brought to

$$\delta^2 E = \int \frac{\rho}{2} \left[ \left( \frac{\delta j}{\rho R} \right)^2 + (a^2 - v^2) \left( \frac{\delta \rho}{\rho} \right)^2 \right] dV.$$  (12)

Equation (12) is the main result of this Letter. If the rotation is subsonic everywhere, then $\delta^2 E > 0$, and the original equilibrium is a true energy minimum. But if the rotation is supersonic at some point, it is possible to find variations (say with $\delta j = 0$) for which $\delta^2 E < 0$. In this case, the energy is not a local minimum, and the possibility of instability is present. Whether a given system is unstable or not depends upon whether it is possible to reach the states of lower energy. Weak magnetic fields give fluids new internal degrees of freedom, and may provide the pathway to these lower energy states. Subsonic and supersonic rotation should have very different consequences for the host system.

### 3. Discussion

We are now in a position to answer the question posed in the Letter title. Subsonic uniform rotation is a true energy minimum, whereas supersonic rotation carries no such
guarantee. Note that our procedure does not ensure instability in the latter case, it simply shows that neighboring states of lower energy exist. If a weak magnetic field is present, however, these states should be accessible. Thus, we would expect the MRI to eliminate a small angular velocity gradient and in subsonic rotation, but to accentuate it in supersonic rotation.

The physical content of our result may be understood as follows. The most unstable disturbances (i.e., “steepest descent” in the $\delta \rho \delta j$ plane) in a supersonically rotating gas cloud are associated with $\delta j = 0$ disturbances. While any particular system may or may not be able to follow the route of steepest descent, it is of interest to understand the reason that $\delta j = 0$ corresponds to such a path. At a given radial distance from the axis, the vanishing of $\delta j$ would mean that either (1) the density rises as the angular velocity drops; or (2) that the density drops as the angular velocity rises. The first describes efficient extraction of mass specific angular momentum and the development of a pressure-supported slowly rotating core, the second describes the development of an extended low density disk and the acquisition of mass specific angular momentum.

Typical disk flow will of course rotate rapidly near the core and slowly in the outer regions. But the $\delta j = 0$ states infintesimally close to the uniformly rotating equilibrium solution should not be confused with some kind of quasi-Keplerian disk profiles. Rather, they represent the virtual state of the disk after it is subject to variations but before it has relaxed to another equilibrium. However, this behavior is also recognizable as the evolutionary strivings of real systems, with slowly rotating pressure-supported dense cores and diffuse rotationally-supported surroundings.

Self-gravity represents a significant mathematical complication to our analysis, but not necessarily a physical one. One expects the bifurcation condition between supersonic and subsonic behavior to be reasonably accurate even in self-gravitating systems, since it is the outer regions that are prone to instability. Here, the direct effects of self-gravity tend to be locally less important. A numerical simulation is being undertaken by the authors to investigate this point in detail.

One practical application of these ideas is to be found in the scales of rotating molecular cores internal to larger turbulent cloud complexes. Barranco & Goodman (1998) and Goodman et al. (1998) present the results of an observational study of coherence in molecular cloud cores. On length scales smaller than $R_{\text{coh}} \sim 0.1$ pc, molecular linewidths are constant with scale, while on larger length scales than $R_{\text{coh}}$ the linewidths are scale-dependent. It is suggested that $R = R_{\text{coh}}$ marks the transition between coherent and noncoherent (i.e. turbulent) behavior, and is, as such, an inner scale to a turbulent cascade. The interesting question of what sets the value of $R_{\text{coh}}$ remains open, with these authors favoring an interpretation in which $R_{\text{coh}}$ is a dissipation scale for MHD processes.

The results presented in this Letter may be relevant to understanding the formation of these uniformly rotating cloud cores if the existence of lower energy states for supersonic rotation follows the scenario we have presented above. The velocity gradient $\Omega \sin i$ (where $i$ is the inclination of the rotation axis to the line of sight) of several cloud cores is presented in Table (4) of Barranco & Goodman (1998). A representative value for $\Omega$ from this tabulation is about $1 \text{ km s}^{-1} \text{ pc}^{-1}$. At $T = 10$ K, the adiabatic sound speed for molecular gas of cosmic abundances is $0.22 \text{ km s}^{-1}$. This means that the radius of the cores should all be less than the sonic rotation radius $R_s = 0.22$ pc, and such is indeed the case. Moreover, since the cores have likely coalesced, their inferred scale of 0.1 pc is quite consistent with this larger initial radius of core detachment. The above argument ignores magnetic and
turbulent support in the cores, but such support appears to be subsonic for the $H_2$ gas in these regions (Barranco & Goodman 1998). In this interpretation, molecular cores do in fact mark the scale at which a transition to coherence occurs, but this scale is present for reasons that are essentially dynamical, not dissipative, in character.

A somewhat more theoretical implication of our results concerns the edge behavior of uniformly rotating cores. A polytrope has a temperature scaling as $T \sim \rho^{\gamma - 1}$ ($\gamma$ is the usual adiabatic index), so that any finite rotation speed for sufficiently low density will become supersonic. Edge cooling will have a similar effect, more generally. In essence, uniform rotation may not be maintained across the radial extent of a gaseous body with a cold outer surface layer. Provided there is a pathway to the local lower-lying energy states, the supersonic surface layers will soak up angular momentum, and form an outer disk. This process will continue on a time scale given by the cooling time of the core’s surface layers. Keplerian-like disks with uniformly rotating cores would thereby result.

A more definitive exploration of the evolutionary differences between supersonic and subsonic rotation, including the effects of self-gravity, is best carried out by way of numerical simulation techniques. These are currently in progress, and will be reported upon in a forthcoming journal article.

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