DISCRIMINANTS OF CONVEX CURVES ARE HOMEOMORPHIC

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Abstract. For a given real generic curve $\gamma : S^1 \to \mathbb{P}^n$ let $D_\gamma$ denote the ruled hypersurface in $\mathbb{P}^n$ consisting of all osculating subspaces to $\gamma$ of codimension 2. In this short note we show that for any two convex real projective curves $\gamma_1 : S^1 \to \mathbb{P}^n$ and $\gamma_2 : S^1 \to \mathbb{P}^n$ the pairs $(\mathbb{P}^n, D_{\gamma_1})$ and $(\mathbb{P}^n, D_{\gamma_2})$ are homeomorphic.

§0. Preliminaries and results

Definition. A smooth curve $\gamma : S^1 \to \mathbb{P}^n$ is called nondegenerate or locally convex if the local multiplicity of its intersection with any hyperplane does not exceed $n$, i.e. in local terms $\gamma'(t), ..., \gamma^{(n)}(t)$ are linearly independent at every $t$ or its osculating complete flag is well-defined at every point. A curve $\gamma : S^1 \to \mathbb{P}^n$ is called convex if the total multiplicity of its intersection with any hyperplane does not exceed $n$.

The set $\text{Con}_n$ of all convex curves in $\mathbb{P}^n$ forms 1 connected component of the space $\mathfrak{N}\mathfrak{D}_n$ of all nondegenerate curves if $n$ is even and 2 connected components (since the osculating frame orients $\mathbb{P}^{2k+1}$) if $n = 2k + 1$, see [MSh]. Different results about convex curves show that they have the most simple properties among all curves. In this paper we prove one more result of the same nature.

Definition. A curve $\gamma : S^1 \to \mathbb{P}^n$ is called generic if at every point $\gamma(t), t \in S^1$ one has a well-defined osculating subspace of codimension 2, i.e. in local terms $\gamma'(t), ..., \gamma^{(n-1)}(t)$ are linearly independent at every $t$.

Note that any smooth curve $\gamma : S^1 \to \mathbb{P}^n$ can be made generic by a small smooth deformation of the map. The space $\mathfrak{N}\mathfrak{D}_n$ of all nondegenerate curves is the union of some connected components in the space $\mathfrak{G}\mathfrak{C}\mathfrak{M}_n$ of all generic curves. (The number of connected components in $\mathfrak{N}\mathfrak{D}_n$ equals 10 for odd $n \geq 3$ and equals 3 for even $n \geq 2$, see [MSh].)

Definition. Given a generic $\gamma : S^1 \to \mathbb{P}^n$ we call by its standard discriminant $D_\gamma \subset \mathbb{P}^n$ the hypersurface consisting of all codimension 2 osculating subspaces to $\gamma$.

In many cases (algebraic, analytic etc) the assumption of genericity in the definition of discriminant can be omitted.

The following proposition answers the question posed by V. Arnold in [Ar2], p.37.

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Main proposition. a) For any 2 convex curves $\gamma_1 : S^1 \to \mathbb{P}^n$ and $\gamma_2 : S^1 \to \mathbb{P}^n$ the pairs $(\mathbb{P}^n, D_{\gamma_1})$ and $(\mathbb{P}^n, D_{\gamma_2})$ are homeomorphic.

b) For any convex curve $\gamma$ the complement $\mathbb{P}^n \setminus D_\gamma$ consists of $\left[\frac{n}{2}\right] + 1$ components. All components are contractible to $S^1$ for $n$ even and all but one are contractible to $S^1$ for $n$ odd. The remaining component is a cell.

Now we want to place this result into a more general context of associated discriminants in the spaces of (in)complete flags.

**Notation.** Let $F_{n+1}$ denote the space of all complete flags in $\mathbb{P}^n$ (or, equivalently, in $\mathbb{R}^{n+1}$). Given a nondegenerate curve $\gamma : S^1 \to \mathbb{P}^n$ one can consider its associated curve $\tilde{\gamma} : S^1 \to F_{n+1}$ where $\tilde{\gamma}(t)$ is the complete osculating flag to $\gamma$ at $\gamma(t)$. Note that any associated curve $\tilde{\gamma} : S^1 \to F_{n+1}$ is tangent to the special distribution of $n$-dimensional cones in $F_{n+1}$ and any integral curve of this distribution is the associated curve of some nondegenerate projective curve, see e.g. [Sh1].

Given a complete flag $f \in F_{n+1}$ and some space $\mathfrak{G} = SL_{n+1}/P$ of (in)complete flags where $P$ is some parabolic subgroup one gets the Schubert cell decomposition $\mathfrak{G}ch_f$ of $\mathfrak{G}$ as follows. Each cell of $\mathfrak{G}ch_f$ consists of all flags in $\mathfrak{G}$ subspaces of which have a given set of dimensions of intersections with the subspaces of $f$. Let $\mathfrak{D}_f$ denote the union of all cells in $\mathfrak{G}ch_f$ which have codimension at least 2. (Obviously, $\text{codim } \mathfrak{D}_f = 2$.)

**Examples.** 1) If $\mathfrak{G}$ equals $\mathbb{P}^n$ then $\mathfrak{D}_f$ is the subspace of $f$ of codimension 2.

2) If $\mathfrak{G} = F_3$ then $\mathfrak{D}_f$ consists of 2 copies of $\mathbb{P}^1$ intersecting at $f$. The first $\mathbb{P}^1$ is the set of flags on $\mathbb{P}^2$ with the same point as that of $f$ and the second $\mathbb{P}^1$ consists of all flags with the same line as that of $f$.

**Definition.** For a given curve $c : S^1 \to F_{n+1}$ and a space $\mathfrak{G} = SL_{n+1}/P$ of (in)complete flags we denote by its $\mathfrak{G}$-discriminant $\mathfrak{G}D_c$ the union $\bigcup_{t \in S^1} \mathfrak{D}_{c(t)} \subset \mathfrak{G}$. (If $c$ is not a constant map then $\mathfrak{G}D_c$ is a hypersurface in $\mathfrak{G}$.)

Note that the standard discriminant $D_\gamma$ of a nondegenerate curve $\gamma : S^1 \to \mathbb{P}^n$ can be considered as the $\mathfrak{G}$-discriminant for $\mathfrak{G} = \mathbb{P}^n$.

**Definition.** Two nondegenerate curves $\gamma_1 : S^1 \to \mathbb{P}^n$ and $\gamma_2 : S^1 \to \mathbb{P}^n$ are called $\mathfrak{G}$-equivalent if the pairs $(\mathfrak{G}, 1D_{\gamma_1})$ and $(\mathfrak{G}, 1D_{\gamma_2})$ are homeomorphic. (Recall that $\tilde{\gamma}$ denotes the associated curve of $\gamma$.)

**Remark.** The notion of $\mathfrak{G}$-equivalence of nondegenerate curves is intrinsically related with the qualitative theory of linear ODE since each nondegenerate curve in $\mathbb{P}^n$ can be represented as the projectivization of the fundamental solution of some linear ODE of order $n+1$, see [Sh2]. The problem of enumeration of $\mathfrak{G}$-equivalent generic curves is apparently a very interesting and difficult question even for $n = 2$.

The following conjecture is formulated in [Sh2].

**Conjecture 1.** Any 2 convex curves $\gamma_1 : S^1 \to \mathbb{P}^n$ and $\gamma_2 : S^1 \to \mathbb{P}^n$ are $\mathfrak{G}$-equivalent for any $\mathfrak{G} = SL_{n+1}/P$.

Note that it suffices to prove this conjecture in the case of the space $F_{n+1}$ of complete flags, i.e. to the case $P = B$ where $B$ is the Borel subgroup of uppertriangular matrices.

**Conjecture 2.** For any pair of convex curves $\gamma_1 : S^1 \to \mathbb{P}^n$ and $\gamma_2 : S^1 \to \mathbb{P}^n$ the pairs $(\mathbb{P}^n, D_{\gamma_1})$ and $(\mathbb{P}^n, D_{\gamma_2})$ are diffeomorphic.

The main motivation of this paper was an attempt to formalize the idea that any 2 convex curves as qualitatively equivalent in any natural sense. It is difficult to overestimate the role of my visit to the Max-Planck Institute during the summer 1996 where the main bulk of this paper was written.
Some generalities on convex curves.

Definition. For any $t \in S^1$ and $1 \leq k \leq n - 1$ let $L_t^k$ denote the osculating subspace to $\gamma$ at $\gamma(t)$ of dimension $k$.

1.1. Theorem (criterion of convexity). A curve $\gamma : S^1 \to \mathbb{P}^n$ is convex if and only if for any $r$-tuple of positive integers $k_1, ..., k_r$ such that $\sum k_i = n$ and any $r$-tuple of pairwise different moments $t_1, ..., t_r$ the intersection $L_{t_1}^{n-k_1} \cap ... \cap L_{t_r}^{n-k_r}$ is a point.

Proof. In order to save the space we refer the interested reader to [Co],[Ma] and further references.

\[\square\]

Definition. Given a nondegenerate curve $\gamma : S^1 \to \mathbb{P}^n$ we call by its dual $\gamma : S^1 \to (\mathbb{P}^n)^*$ the curve consisting of all osculating hyperplanes to $\gamma$.

Remark. If $\gamma$ is convex then $\gamma^*$ is also convex, see [Ar1], [Ar2].

Notation. If a point $p$ lies on the osculating hyperplane $H_\tau$ to $\gamma$ we say that the order of tangency $\sharp_p(\gamma(\tau))$ of $p$ at $\gamma(\tau)$ equals to $i$ if $p$ belongs to the osculating subspaces at $\gamma(\tau)$ of the codimension at most $i$. (For example, for every point $p$ on a line $l$ tangent to a circle $c$ at $c(1)$ on $\mathbb{P}^2$ except for the tangency point $c(1)$ one has $\sharp_p(c(1)) = 1$. On the other side, $\sharp_{c(1)}(c(1)) = 2$.)

Given a nondegenerate $\gamma : S^1 \to \mathbb{P}^n$ and a point $p \in \mathbb{P}^n$ we call by the number of roots $\sharp_p(\gamma)$ of $p$ the sum of the orders of tangency $\sharp_p(\gamma(t))$ taken over all osculating hyperplanes $H_t^i$ through $p$.

(The term ‘number of roots’ comes from the example when $\gamma$ is the rational normal curve in $\mathbb{P}^n$. In this case all points in $\mathbb{P}^n$ can be interpreted as homogeneous polynomials in 2 variables of degree $n$ with real coefficients (considered up to a constant factor) and $\gamma$ is the family of polynomials of the form $(ax_1 + bx_2)^n$, $a^2 + b^2 \neq 0$. In this situation $\sharp_p(\gamma)$ coincides with the total number of real roots of such a polynomial on $\mathbb{P}^1$ counted with multiplicities.)

Observation. The number of roots $\sharp_p(\gamma)$ coincides with the total multiplicity (i.e. sum of local multiplicities) of the intersection of $H_p$ with $\gamma^*$. Here $H_p$ denotes the hyperplane in $(\mathbb{P}^n)^*$ corresponding to the point $p \in \mathbb{P}^n$.

1.2. Corollary. A curve $\gamma$ is convex if and only if for any $p \in \mathbb{P}^n$ one has $\sharp_p(\gamma) \leq n$.

Projection.

Given a convex curve $\gamma : S^1 \to \mathbb{P}^n$ and its osculating hyperplane $H_\tau$ at the point $\gamma(\tau)$ let us denote by $\gamma^\tau : S^1 \to H_\tau$ the curve obtained by projection of $\gamma$ onto $H_\tau$ along the pencil of tangent lines to $\gamma$, i.e. for any $t \in S^1$ one has $\gamma^\tau(t) = H_\tau \cap l_t$ where $l_t$ is the tangent line to $\gamma$ at $\gamma(t)$.

1.3. Lemma. For any $\tau \in S^1$ the curve $\gamma^\tau$ is a convex curve in $H_\tau$. Osculating hyperplanes to $\gamma^\tau$ and its discriminant $D_{\gamma^\tau}$ are obtained by intersection of the osculating hyperplanes and $D_\gamma$ with $H_\tau$.

Proof. The argument splits into 2 principal parts. At first we show that $\gamma^\tau$ is nondegenerate, i.e. $(\gamma^\tau)'(t), (\gamma^\tau)'(n-1)(t)$ are linearly independent at any $t \in S^1$. Then we prove that $\gamma^\tau$ is convex.
convex, i.e. its total multiplicity of intersection with any hyperplane in \( H_\tau \) does not exceed \( n - 1 \). Observe that \( \gamma \) has the only intersection point with \( H_\tau \), namely \( \gamma(\tau) \). Assume first that \( t \neq \tau \). In this case one can choose a system of affine coordinates \( x_1, \ldots, x_n \) in \( \mathbb{P}^n \) such that \( H_\tau \) coincides with the hyperplane \( \{ x_n = 0 \} \); \( \gamma(t) \) is the point with coordinates \( (0, \ldots, 0, 1) \) and the tangent line \( l_t \) to \( \gamma \) at \( \gamma(t) \) is the \( x_n \)-axis. In these coordinates the curve \( \gamma^\tau \) has the form

\[
\gamma^\tau(t) = \gamma(t) - \frac{\gamma_n(t)}{\gamma'_n(t)} \gamma'(t)
\]

where \( \gamma_n \) is the last coordinate of \( \gamma \). (Under our assumptions \( \gamma'_n(t) \neq 0 \).) Therefore

\[
(\gamma^\tau)^{(i)}(t) = (-1)^i \frac{\gamma_n(t)}{\gamma'_n(t)} \gamma^{(i)}(t) + \ldots
\]

where \( \ldots \) denotes the terms containing derivatives of \( \gamma \) of order lower than \( i \). By the above assumptions \( \gamma'_n(t) \neq 0 \) and since \( \gamma', \ldots, \gamma^{(n)}(t) \) are linearly independent one gets that the derivatives \( (\gamma^\tau)^{(i)}(t), i = 1, \ldots, n - 1 \) are linearly independent as well.

The alternative geometric argument is as follows. Since the point \( \gamma(t) \) does not lie on \( H^\tau \) one has that the osculating complete flag \( f(t) \) to \( \gamma \) at \( \gamma(t) \) is transversal to \( H_\tau \) and the same holds for all \( t' \) close to \( t \). Therefore the complete flags obtained by intersection of \( f(t') \) with \( H_\tau \) are well defined. But in their turn these flags coincide with the osculating flags to \( \gamma^\tau \) which are therefore well-defined in some neighborhood of \( t \).

It is left to show that \( \gamma^\tau \) is nondegenerate at \( t = \tau \). This follows from the local calculation given below. In this case we can choose a system of coordinates such that in the neighborhood of \( \tau \) (we assume \( \tau = 0 \)) the curve \( \gamma \) has the form

\[
x_1 = t + \ldots, \quad x_2 = t^2 + \ldots, \quad x_n = t^n + \ldots
\]

The osculating hyperplane \( H^0 \) at \( \tau = 0 \) is given by \( \{ x_n = 0 \} \). The projected curve \( \gamma^0(t) \) is given by

\[
\gamma^0(t) = \gamma(t) - \frac{\gamma_n(t)}{\gamma'_n(t)} \gamma'(t) = (t + \ldots, t^2 + \ldots, t^n + \ldots) - \frac{t^n + \ldots}{nt^{n-1} + \ldots}(1 + \ldots, 2t + \ldots, nt^{(n-1)} + \ldots)
\]

\[
= \frac{1}{n}((n - 1)t + \ldots, (n - 2)t^2 + \ldots, t^{(n-1)} + \ldots, 0)
\]

which shows that \( \gamma^0 \) is nondegenerate at \( t = 0 \).

Now we show that \( \gamma^\tau \) is convex. By Corollary 1.2. one has to prove that for any \( p \in H_\tau \) the number of roots \( \sharp_p(\gamma^\tau) \) is less or equal \( n - 1 \). This follows from the equality

\[
\sharp_p(\gamma^\tau) + 1 = \sharp_p(\gamma)
\]

which together with convexity of \( \gamma \) gives the required result. Indeed, assume that some \( p \in H_\tau \) lies in the intersection \( L_{t_1}^{k_1}(\gamma) \cap L_{t_2}^{k_2}(\gamma) \cap \ldots \cap L_{t_r}^{k_r}(\gamma) \) where \( t_1 = \tau \). Since each subspace \( L_{t_i}^{k_i} \) for \( i \neq 1 \) is transversal to \( H_\tau \) (see criterion of convexity) one has that \( p \) lies in the intersection \( L_{t_1}^{k_1}(\gamma^\tau) \cap L_{t_2}^{k_2-1}(\gamma^\tau) \cap \ldots \cap L_{t_r}^{k_r-1}(\gamma^\tau) \). Therefore, by definition of the number of roots, one gets the above equality.

\[ \square \]

For any \( k \)-tuple of moments \( (t_1, \ldots, t_k) \), \( t_i \in S^1 \) let \( H_{t_1} \cap \ldots \cap H_{t_k} \) denote the intersection of the osculating hyperplanes \( H_{t_i}, i = 1, \ldots, k \). In what follows we use the following convention. If some of the moments \( t_{j_1}, t_{j_2}, \ldots, t_{j_r} \) coincide we define the intersection \( H_{t_{j_1}} \cap H_{t_{j_2}} \cap \ldots \cap H_{t_{j_r}} \) as the osculating subspace to \( \gamma \) at \( \gamma(t_{j_1}) \) of codimension \( r \). Under this convention one has that \( H_{t_1} \cap \ldots \cap H_{t_k} \) always has codimension \( k \), see 1.1.
1.4. Corollary. The projection $\gamma^{t_1, \ldots, t_k}$ of $\gamma$ onto any intersection of osculating hyperplanes $H_{t_1} \cap \ldots \cap H_{t_k}$ by a pencil of $k$-dimensional osculating subspaces to $\gamma$ is a convex curve. For any point $p \in H_{t_1} \cap \ldots \cap H_{t_k}$ one has

$$\sharp_p(\gamma^{t_1, \ldots, t_k}) + k = \sharp_p(\gamma).$$

Proof. Apply the above lemma several times.

\[ \Box \]

Elliptic hull of $\gamma$ and root filtration of $\mathbb{P}^n$.

Definition. For a convex $\gamma : S^1 \to \mathbb{P}^n$ we denote by its elliptic hull $\operatorname{Ell}_\gamma$ the set of all $p \in \mathbb{P}^n$ with

$$\begin{cases} \sharp_p(\gamma) = 0, & \text{if } n \text{ is even} \\ \sharp_p(\gamma) = 1, & \text{if } n \text{ is odd} \end{cases}$$

1.5. Lemma. a) If $n$ is even then $\operatorname{Ell}_\gamma$ is a nonempty convex set in some affine chart of $\mathbb{P}^n$, comp. [ShS];

b) If $n$ is odd then $\operatorname{Ell}_\gamma$ is a disjoint union of $\bigcup_{\tau \in S^1} \operatorname{Ell}_\gamma^\tau$ and, therefore, is fibered over $\gamma$ with a contractible fiber.

Proof. a) Note that if $\gamma : S^1 \to \mathbb{P}^{2k}$ is convex then $\gamma$ lies in some affine chart in $\mathbb{P}^{2k}$. Indeed, take some osculating hyperplane $H_\tau$. The curve $\gamma$ is tangent to $H_\tau$ only at $\gamma(\tau)$ with the multiplicity $2k$. Locally $\gamma$ lies on one side w.r.t. $H_\tau$. Therefore, one can make a small shift of $H_\tau$ in order to get rid of the intersection points with $\gamma$ near $\gamma(\tau)$. But no new intersection can appear for a sufficiently small shift since the only intersection point of $\gamma$ and $H_\tau$ is $\gamma(\tau)$.

Assume now that $\mathbb{R}^{2k} \subset \mathbb{P}^{2k}$ is the affine chart containing $\gamma$. We claim that $\operatorname{Ell}_\gamma$ coincides with the intersection $\bigcap_{\tau \in S^1} \operatorname{Half}_{\tau}$. Here $\operatorname{Half}_{\tau}$ is the open halfspace in $\mathbb{R}^{2k}$ containing $\gamma$ and bounded by the osculating hyperplane $H_\tau$. First of all, $\bigcap_{\tau \in S^1} \operatorname{Half}_{\tau}$ is nonempty since it is an open convex set containing the interior of the convex hull of $\gamma$ in $\mathbb{R}^{2k}$. Then $\bigcap_{\tau \in S^1} \operatorname{Half}_{\tau}$ is contained in $\operatorname{Ell}_\gamma$. Indeed, every hyperplane through a point $p \in \bigcap_{\tau \in S^1} \operatorname{Half}_{\tau}$ is transversal to any osculating hyperplane $H_\tau$ since $p \notin H_\tau$. On the other side, $\operatorname{Ell}_\gamma \subseteq \bigcap_{\tau \in S^1} \operatorname{Half}_{\tau}$. Indeed, for every $p \notin \operatorname{Half}_{\tau}$, $\tau \in S^1$ there exists a hyperplane $L_p$ through $p$ not intersecting $\gamma$ at all. Take the affine chart $\mathbb{P}^n \setminus L_p$ containing $\gamma$ and some pencil $\mathcal{L}_p$ of ‘parallel’ hyperplanes through $p$. Since $\gamma$ is a closed curve in $\mathbb{P}^n \setminus L_p$ one gets that some hyperplane in $\mathcal{L}_p$ does not intersect $\gamma$. Therefore, there exists a hyperplane in $\mathcal{L}_p$ tangent to $\gamma$ at some $\gamma(t_p)$. But this exactly means that the osculating hyperplane $H_{t_p}$ contains $p$.

b) Take a 1-parameter family of osculating hyperplanes. According to the proof of lemma 1.3. for any $\tau \in S^1$ the curve $\gamma^\tau$ is convex in $H_\tau$ and one has $\sharp_p(\gamma^\tau) + 1 = \sharp_p(\gamma)$. Therefore the elliptic domain $\operatorname{Ell}_{\gamma^\tau}$ of every curve $\gamma^\tau$ belongs to $\operatorname{Ell}_\gamma$, i.e. $\bigcup_{\tau \in S^1} \operatorname{Ell}_{\gamma^\tau} \subset \operatorname{Ell}_\gamma$. (Note that the union $\bigcup_{\tau \in S^1} \operatorname{Ell}_{\gamma^\tau}$ is disjoint.) Conversely, by definition, for odd $n$ every point $p$ in $\operatorname{Ell}_\gamma$ has exactly one tangent hyperplane to $\gamma$ and thus $p$ belongs exactly to one osculating $H_\tau$. By the equality $\sharp_p(\gamma^\tau) + 1 = \sharp_p(\gamma)$ the point $p$ lies in the elliptic hull of $\gamma^\tau$. Moreover, by the first part of this proof, $\operatorname{Ell}_{\gamma^\tau}$ is a convex domain in $H_\tau$ and, therefore, is contractible which gives the necessary result.

\[ \Box \]

Definition. By the root filtration of $\mathbb{P}^n$ w.r.t. a convex curve $\gamma : S^1 \to \mathbb{P}^n$

$$\operatorname{P}_0(\gamma) \subset \ldots \subset \operatorname{P}_{\sharp_p(\gamma) - 1}(\gamma) = \mathbb{P}^n$$
we denote the filtration where each $\mathbb{P}_i(\gamma)$ consists of all $p \in \mathbb{P}^n$ for which the number of roots $\sharp_p(\gamma)$ is greater or equal $n-2i$.

Let $\mathcal{T}^j = (S^1)^j$ denote the $j$-dimensional torus and let $\mathcal{T}^j / \mathbb{S}_j$ be its quotient mod the natural action of the symmetric group $\mathbb{S}_j$ by permutation of copies of $S^1$.

1.6. Lemma. a) For any $n$ and $0 \leq i < [\frac{n}{2}]$ the set $\mathbb{P}_i(\gamma) \setminus \mathbb{P}_{i-1}(\gamma)$ is naturally fibered over $\mathcal{T}^{n-2i} / \mathbb{S}_{n-2i}$ with a contractible fiber. (For $n = 2k$ and $i = k$ the set $\mathbb{P}_k(\gamma) \setminus \mathbb{P}_{k-1}(\gamma)$ is contractible, see 1.5.a);

b) This fibration is trivial.

Proof. a) Every point $p \in \mathbb{P}_i(\gamma) \setminus \mathbb{P}_{i-1}(\gamma)$ can be described as follows. There exists and unique $(n-2i)$-tuple of osculating hyperplanes $H_{t_1}, \ldots, H_{t_{n-2i}}$ to $\gamma$ (with probably coinciding moments $t_1, \ldots, t_{n-2i}$ in which case we use the same convention as above) such that $p$ belongs to the intersection $H_{t_1} \cap H_{t_2} \cap \cdots \cap H_{t_{n-2i}}$ and, moreover, lies in the elliptic hull of the curve $\gamma_{t_1, \ldots, t_{n-2i}}$. (Here $\gamma_{t_1, \ldots, t_{n-2i}}$ is the projection of $\gamma$ onto $H_{t_1} \cap H_{t_2} \cap \cdots \cap H_{t_{n-2i}}$ by the pencil of osculating subspaces of dimension $n-2i$.) Indeed, we have that $\sharp_p(\gamma_{t_1, \ldots, t_{n-2i}})+2i = \sharp_p(\gamma)$, see 1.2. Therefore $p$ must lie in the elliptic hull of $\gamma_{t_1, \ldots, t_{n-2i}}$. On the other side, any intersection $H_{t_1} \cap H_{t_2} \cap \cdots \cap H_{t_{n-2i}}$ has codimension $n-2i$, see 1.1. and any curve $\gamma_{t_1, \ldots, t_{n-2i}}$ is convex. Therefore applying 1.5. we get the necessary result.

b) The fibration of elliptic components $Ell_{\gamma_{t_1, \ldots, t_{n-2i}}}$ of the curves $\gamma_{t_1, \ldots, t_{n-2i}}$ over the set of moments $(t_1, \ldots, t_{n-2i}) \in \mathcal{T}^{n-2i} / \mathbb{S}_{n-2i}$ depends continuously on $\gamma \in \mathcal{COn}_n$. Since $\mathcal{COn}_n$ consists of 1 connected component (up to orientation for odd $n$) it suffices to show that the fibration sending $Ell_{\gamma_{t_1, \ldots, t_{n-2i}}}$ to $(t_1, \ldots, t_{n-2i})$ is trivial for some $\gamma \in \mathcal{COn}_n$.

The simplest example showing triviality is the case when $\gamma$ is the rational normal curve. Indeed, in this case the space under consideration is the fibration of the space $\Pi_n(i)$ all homogeneous forms of degree $n$ in 2 variables (up to a scalar multiple) which have exactly $n-2i$ real zeros counted with multiplicities over the space $\mathcal{T}^{n-2i} / \mathbb{S}_{n-2i}$ of their real zeros. But $\Pi_n(i)$ has the obvious structure of the product of the space of degree $n-2i$ polynomials with all real zeros (considered up to a scalar multiple) and the space of degree 2i polynomials with no real zeros (considered up to a scalar multiple). This shows that the fibration $\Pi_n(i) \to \mathcal{T}^{n-2i} / \mathbb{S}_{n-2i}$ is trivial.

□

Proof of the main proposition.

a) We will construct the homeomorphism of pairs $(\mathbb{P}_n, D_{\gamma_1})$ and $(\mathbb{P}_n, D_{\gamma_2})$ in $[\frac{n}{2}] + 1$ steps. On the $i$th step, $i = 0, \ldots, [\frac{n}{2}]$ we obtain the partial homeomorphism $h_i$ of the terms $\mathbb{P}_i(\gamma_1)$ and $\mathbb{P}_i(\gamma_2)$ of the above filtration.

The initial step. We construct the homeomorphism $h_0 : \mathbb{P}_0(\gamma_1) \to \mathbb{P}_0(\gamma_2)$. Indeed, each of $\mathbb{P}_0(\gamma_1)$ and $\mathbb{P}_0(\gamma_2)$ is homeomorphic to $\mathcal{T}^n / \mathbb{S}_n$ as follows. Every element in $\mathcal{T}^n / \mathbb{S}_n$ is a pair $(t_1, \ldots, t_r) \in (\mathcal{T}^r \setminus \text{Diag}) / \mathbb{S}_r$, $r \leq n$ and $(k_1, \ldots, k_r)$, $\sum k_i = n$. We map such a pair $(t_1, \ldots, t_r), (k_1, \ldots, k_r)$ onto the intersection point $L_{t_1}^{n-k_1}(\gamma_{j_1}) \cap \cdots \cap L_{t_r}^{n-k_r}(\gamma_{j_2}), j = 1, 2$. This identification provides the homeomorphism $h_0 : \mathbb{P}_0(\gamma_1) \to \mathbb{P}_0(\gamma_2)$ by 1.1.

The typical step. Each point in $\mathbb{P}_i(\gamma_1) \setminus \mathbb{P}_{i-1}(\gamma_1)$ lies in the elliptic hull of the unique curve $\gamma_{t_1, \ldots, t_{n-2i}} \subset H_{t_1} \cap \cdots \cap H_{t_{n-2i}}$, i.e. the set of (nonnecessarily pairwise different) moments $(t_1, \ldots, t_{n-2i}) \in \mathcal{T}^{n-2i} / \mathbb{S}_{n-2i}$ is uniquely defined. For each individual intersection $H_{t_1} \cap \cdots \cap H_{t_{n-2i}}$ the homeomorphism $h_{i-1}$ is already defined on the complement to the elliptic hulls of the curves $\gamma_{t_1, \ldots, t_{n-2i}}$ and $\gamma_{t_1, \ldots, t_{n-2i}}$.

Since the elliptic hulls are convex domains and the fibrations of the elliptic hulls over $\mathcal{T}^{n-2i} / \mathbb{S}_{n-2i}$ are trivial we can extend $h_{i-1}$ fiberwise to $\mathbb{P}_i(\gamma_1) \setminus \mathbb{P}_{i-1}(\gamma_1)$ to get the desired partial homeomorphism $h_{ij}$. Since the partial homeomorphisms $h_{ij}$ are compatible we can define $h_i$ on $\mathbb{P}_i(\gamma_1)$ and $\mathbb{P}_i(\gamma_2)$.

□
by identifying the points of the elliptic hull of $\gamma _{t_1,\ldots,t_{n-2i}}$ with points of the elliptic hull of $\gamma _{t_1,\ldots,t_{n-2i}}$ for all tuples $(t_1,\ldots,t_{n-2i}) \in \mathcal{T}^{n-2i}/\mathcal{S}_{n-2i}$.

b) The corresponding component $\text{Comp}_i$ of $\mathbb{P}^n \setminus D_\gamma$ contained in $\mathbb{P}_i \setminus \mathbb{P}_{i-1}$ is fibered over $(\mathcal{T}^{n-2i} \setminus \text{Diag})/\mathcal{S}_{n-2i}$ with the contractible fiber. Since $(\mathcal{T}^{n-2i} \setminus \text{Diag})/\mathcal{S}_{n-2i}$ is contractible to $S^1$ for any $n-2i > 0$ one gets that $\text{Comp}_i$ is contractible to $S^1$ for all $n$ and $i \leq \left\lceil \frac{n}{2} \right\rceil$ except for $\text{Ell}_\gamma$ for even $n$ which is contractible to a point.

□

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