On sums of narrow and compact operators

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Abstract
We prove, in particular, that if $E$ is a Dedekind complete atomless Riesz space and $X$ is a Banach space then the sum of a narrow and a C-compact laterally continuous orthogonally additive operators from $E$ to $X$ is narrow. This generalizes in several directions known results on narrowness of the sum of a narrow and a compact operators for the settings of linear and orthogonally additive operators defined on Köthe function spaces and Riesz spaces.

Keywords Narrow operator · Orthogonally additive operator · Laterally continuous operator

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1 Introduction

Formally narrow operators were introduced and studied in 1990 [15,17], however some deep results on these operators were obtained earlier by Bourgain, Ghoussoub,

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Kalton, Rosenthal and other mathematicians in another terminology, see survey [18] and textbook [19]. Being a generalization of compact operators defined on atomless Köthe function spaces, narrow operators require a completely new technique and bring various new geometric implications. The most unusual thing about narrow operators is that, on the space $L_1$ the sum of two (continuous linear) narrow operators is narrow [19, Theorem 7.46], however if an r.i. space $E$ on $[0, 1]$ has an unconditional basis then every operator on $E$ is a sum of two narrow ones [19, Theorem 5.2]. Later it became clear in 2009 [8] that narrow operators have purely vector lattice nature, and the above two results are very natural. A new more general definition of a narrow operator defined on a Riesz space proposed in [8] is well agreed with the old one for operators defined on a Köthe F-space with an absolutely continuous norm. A deep result of [8] generalizing Kalton–Rosenthal’s representation theorem asserts that, under mild assumptions on Banach lattices $E$, $F$ (e.g., the result is true for order continuous Banach lattices with $E$ atomless), the sum of two narrow regular operators from $E$ to $F$ is narrow. Moreover, the set $\mathcal{N}_r(E, F)$ of all narrow regular operators from $E$ to $F$ is a band in the Riesz space $\mathcal{L}_r(E, F)$ of all regular operators from $E$ to $F$, which is orthogonal to the band of all lattice homomorphisms from $E$ to $F$. Since all continuous linear operators on $L_1$ are regular $\mathcal{L}_r(L_1) = \mathcal{L}(L_1)$, we obtain that, moreover, the set $\mathcal{N}_r(L_1)$ of all narrow operators on $L_1$ is a band in $\mathcal{L}(L_1)$ (the set of all continuous linear operators from $E$ to $X$ is denoted by $\mathcal{L}(E, X)$, and $\mathcal{L}(E) = \mathcal{L}(E, E)$). On the other hand, all examples of pairs of narrow operators with nonnarrow sum contain non-regular summands.

The peculiarity of both domain and range spaces makes a big deal in the problem on the narrowness of a sum of two narrow operators: for every Köthe Banach space $E$ on $[0, 1]$ there exist a Banach space $X$ and narrow operators from $E$ to $X$ with nonnarrow sum [14].

An exceptional Köthe Banach space for narrow operators is $L_\infty$, the norm of which is not absolutely continuous. The usual technique does not work, and there are nonnarrow continuous linear functionals on $L_\infty$. Nevertheless, questions about narrowness of the sum of two narrow operators are not less interesting. A sum of two narrow operators on $L_\infty$ need not be narrow [7]. Moreover, if $1 < p \leq \infty$ then there are regular narrow operators $T_1, T_2 \in \mathcal{L}_r(L_p, L_\infty)$ with nonnarrow sum [14]. This also shows that the order continuity assumption on the range Banach lattice $F$ is essential in the above mentioned theorem that $\mathcal{N}_r(E, F)$ is a band in $\mathcal{L}_r(E, F)$.

Let $E$, $X$ be such that a sum of two narrow operators from $E$ to $X$ need not be narrow. Then it is natural to ask of whether the sum of a narrow operator and a compact (or even finite rank) operator is narrow. The following general result is known to this extend [19, Proposition 11.2]. If $E$ is a Köthe Banach space with an absolutely continuous norm then for any Banach space $X$ the sum of a narrow operator and a hereditarily narrow operator from $E$ to $X$ is narrow (for precise definitions see below). Since the class of all hereditarily narrow operators from $E$ to $X$ includes different other classes of “small” operators (like compact, AM-compact, Dunford–Pettis operators, etc., see [19, Corollary 11.4] for more classes), as consequences, we obtain that the sum of a narrow and a compact (or other “small”) operator is narrow.

For the case where the norm of $E$ is not absolutely continuous and a compact operator need not be narrow, a weaker question makes sense: is the sum of two narrow operators, at least one of which is compact, narrow? The strongest result in this direct-
tion, due to Mykhaylyuk [11], asserts that, if \( E \) is a Köthe F-space, \( X \) a locally convex F-space, \( T_1, T_2 \in \mathcal{L}(E, X) \) narrow operators such that \( T_2 \) maps the set of all signs to a relatively compact subset of \( X \) (which, in particular, is the case for a compact \( T_2 \)), then the sum \( T_1 + T_2 \) is narrow.

Another direction of generalization concerns nonlinear maps. The notion of a narrow operator was generalized to orthogonally additive operators in [16]. Fortunately, in some contexts when dealing with narrow linear operators the linearity is only used for orthogonal elements. This made possible to extend most of the results on narrow linear operators obtained in [8] to orthogonally additive operators. On the other hand, all necessary background for such operators has been already built [9,10]. One of the results of [16] asserts that every laterally continuous C-compact orthogonally additive operator acting from an atomless Dedekind complete Riesz space is narrow.

So, our main result, in particular, generalizes the mentioned theorem from [16]. Moreover, two papers of the second named author [3] and [5] were devoted to partial cases of our main result obtained under additional assumptions on the domain Riesz space. In the present paper we show that these assumptions are superfluous. We use here mainly a new technique partially based on ideas from the previous papers. Nevertheless, our proof looks to be even shorter.

2 Narrow orthogonally additive operators on Riesz spaces

For familiarly used notions and facts on Riesz spaces we refer the reader to [2]; on Banach spaces—to [1] and on F-spaces—to [21]. We denote the F-norm of an element \( x \) of an F-space by \( \|x\| \) (the metric distance between \( x \) and zero) and use the triangle inequality as for elements of a normed space with the exception that a scalar cannot be taken off the F-norm.

A disjoint sum in a Riesz space is denoted by \( x \sqcup y \) or \( \bigcup_{i \in I} x_i \), so \( z = x \sqcup y \) means that \( z = x + y \) and \( x \perp y \). An element \( x \) of a Riesz space \( E \) is called a fragment of \( y \in E \) (write \( x \sqsubseteq y \)) provided \( x \perp y - x \). It is a standard exercise to show that the relation \( \sqsubseteq \) is a partial order on \( E \) called the lateral order [12]. The set of all fragments of an element \( e \in E \) is denoted by \( \mathcal{F}_e \). Observe that if \( z = x \sqcup y \) then both \( x \) and \( y \) are fragments of \( z \). We say that an element \( a \neq 0 \) of a Riesz space \( E \) is an atom if the only fragments of \( a \) are 0 and \( a \) itself. A Riesz space having no atom is said to be atomless.

Let \( E \) be a Riesz space and \( X \) be a linear space. A function \( T : E \to X \) is called an orthogonally additive operator (OAO in short) if \( T(x \sqcup y) = T(x) + T(y) \) for all disjoint elements \( x, y \in E \) (we use parentheses for the argument of an OAO to draw attention to the fact that the operator need not be linear). Simple examples of OAOs are the positive, negative parts and the modules of an element: \( T_1(x) = x^+ \), \( T_2(x) = x^- \), \( T_3(x) = |x| \), \( x \in E \). For more examples of OAOs including integral Uryson operators see [9,10,16].

Let \( E \) be an atomless Riesz space and \( X \) an F-space. A function \( f : E \to X \) is said to be narrow at a point \( e \in E \) provided for every \( \varepsilon > 0 \) there exists a decomposition \( b = b' \sqcup b'' \) into disjoint fragments such that \( \|f(b') - f(b'')\| < \varepsilon \). The function \( f \) is called narrow if it is narrow at each point \( e \in E \). Observe that if \( f \) is linear then it is
narrow at a point \( e \) if and only if for every \( \varepsilon > 0 \) there exists \( x \in E \) such that \( |x| = |e| \) and \( \|f(x)\| < \varepsilon \). So, initially the latter condition was accepted as the definition of a narrow operator on a Riesz space in [8].

Recall also the first definition of a narrow linear operator on a Köthe F-space (to distinguish it from the given above definition for Riesz spaces, we use the term “function narrow operator”). By a Köthe F-space on a finite atomless measure space \((\Omega, \Sigma, \mu)\) we mean an F-space \( E \) which is a linear subspace of the linear space \( L_0(\mu) \) of all equivalence classes of measurable functions acting from \( \Omega \) to the scalar field \( K \in \{\mathbb{R}, \mathbb{C}\} \) possessing the following properties: \( 1_\Omega \in E \), and for every \( x \in E \) and \( y \in L_0(\mu) \) the condition \( |y| \leq |x| \) implies that \( y \in E \) and \( \|y\| \leq \|x\| \) (the characteristic function of a set \( A \in \Sigma \) is denoted by \( 1_A \), and the inequality \( u \leq v \) in \( E \) means that \( \tilde{u}(t) \leq \tilde{v}(t) \) holds for \( \mu \)-almost all \( t \in \Omega \), where \( \tilde{u} \in u \) and \( \tilde{v} \in v \) are some/any representatives of the classes \( u, v \)). If, moreover, for a Köthe F-space \( E \) one has that \( E \) is a Banach space and \( E \subseteq L_1(\mu) \) then \( E \) is called a Köthe Banach space on \((\Omega, \Sigma, \mu)\). Given a Köthe F-space \( E \) and an F-space \( X \), we say that an operator \( T \in L(E, X) \) is

- **function narrow** if for each \( A \in \Sigma \) and \( \varepsilon > 0 \) there is a decomposition \( A = B \sqcup C \) with \( B, C \in \Sigma, \mu(B) = \mu(C) \) such that \( \|T(1_B - 1_C)\| < \varepsilon \);
- **function weakly narrow** if for each \( A \in \Sigma \) and \( \varepsilon > 0 \) there is a decomposition \( A = B \sqcup C \) with \( B, C \in \Sigma \) such that \( \|T(1_B - 1_C)\| < \varepsilon \);
- **hereditarily narrow** if \( E \) is a Köthe Banach space, and for every \( A \in \Sigma \) with \( \mu(A) > 0 \) and every atomless \( \sigma \)-algebra \( \Sigma_1 \) of measurable subsets of \( A \) the restriction \( T|_{E(\Sigma_1)} \) of \( T \) to the subspace \( E(\Sigma_1) \) of all \( \Sigma_1 \)-measurable elements of \( E \) supported on \( A \), which is a Köthe Banach space itself, is function narrow.

If \( E \) is a Köthe Banach space then the definition of a weakly narrow operator is equivalent to the above given definition of a narrow operator for every Banach space \( X \) if and only if the set of all simple functions is dense in \( E \) [20]. Moreover, if \( E \) has an absolutely continuous norm, that is, \( \lim_{\mu(A) \to 0} \|x \cdot 1_A\| = 0 \) for all \( x \in E \), then all the definitions are equivalent [8], [19, Proposition 10.2]. It is still an open problem, whether every function weakly narrow operator from \( L_\infty \) is function narrow [19, Problem 10.3].

The idea of the proof of the main result is to consider the set \( \mathfrak{F}_e \) of all fragments of a fixed element of the domain Riesz space \( E \) as the main object for investigation. This becomes possible because the definitions of all notions from the main theorem could be equivalently restricted to \( \mathfrak{F}_e \). Since the set \( \mathfrak{F}_e \) is a Boolean algebra with respect to the natural operations, we come to analogous questions for functions defined on a Boolean algebra.

### 3 Almost dividing measures on Boolean algebras

Following terminology in [13], an OAO defined on a Boolean algebra is called a measure, and a narrow OAO is called an almost dividing measure. Now in more details.
Let \((u_\alpha)\) be a net in a Boolean algebra \(\mathcal{B}\). The notation \(u_\alpha \downarrow 0\) means that the net \((u_\alpha)\) decreases and \(\inf_\alpha u_\alpha = 0\). We say that a net \((x_\alpha)\) in \(\mathcal{B}\) order converges to an element \(x \in \mathcal{B}\) if there exists a net \((u_\alpha)\) in \(\mathcal{B}\) with the same index set such that \(x_\alpha \triangle x \leq u_\alpha\) for all indices \(\alpha\) and \(u_\alpha \downarrow 0\) (here and in the sequel \(x \triangle y = (x - y) \cup (y - x)\)). In this case we write \(x_\alpha \xrightarrow{o} x\) and say that \(x\) is the order limit of \((x_\alpha)\).

Let \(\mathcal{B}\) be a Boolean algebra, \(X\) an F-space or \(X = [0, +\infty]\). A function \(f : \mathcal{B} \to X\) is said to be order continuous at a point \(b \in \mathcal{B}\) if for every net \((x_\alpha)\) in \(\mathcal{B}\) the condition \(x_\alpha \xrightarrow{o} b\) implies \(f(x_\alpha) \to f(b)\). If \(f\) is order continuous at every point we say that \(f\) is order continuous.

A Boolean algebra \(\mathcal{B}\) is said to be order complete if any nonempty subset of \(\mathcal{B}\) has the supremum. By a partition of unity in a Boolean algebra \(\mathcal{B}\) we mean a maximal disjoint family \(\mathcal{A} \subseteq \mathcal{B}\), that is, \((\forall x \in \mathcal{B}) \ ((\forall a \in \mathcal{A} a \cap x = 0) \Rightarrow (x = 0))\). A disjoint union \(\bigcup \mathcal{A}\) (that is, the union of a disjoint system \(\mathcal{A} \subseteq \mathcal{B}\), if exists, is denoted by \(\bigcup \mathcal{A}\). Although in some cases an infinite union \(\bigcup \mathcal{A}\) of \(\mathcal{A} \subseteq \mathcal{B}\), which is defined to be the supremum of \(\mathcal{A}\) in \(\mathcal{B}\), does not exist, it is immediate that if \(\mathcal{A}\) is a partition of unity then \(\bigcup \mathcal{A} = 1\) exists. Conversely, if \(\bigcup \mathcal{A} = 1\) then \(\mathcal{A}\) is a partition of unity. Likewise, a partition of an element \(e \in \mathcal{B}\) is a partition of unity of the Boolean algebra \(\mathcal{B}_b = \{x \in \mathcal{B} : x \leq b\}\).

Let \(\mathcal{B}\) be a Boolean algebra, \(X\) a linear space. A function \(f : \mathcal{B} \to X\) is said to be a measure provided \(f(x \sqcup y) = f(x) + f(y)\) for every pair of disjoint elements \(x, y \in \mathcal{B}\). Obviously, \(f(0) = 0\) for a measure. A measure \(f : \mathcal{B} \to X\) is said to have finite rank if the closed linear span \([T(\mathcal{B})]\) is a finite dimensional subspace of \(X\).

Let \(\mathcal{B}\) be an atomless Boolean algebra, \(X\) an F-space. A function \(f : \mathcal{B} \to X\) is said to be almost dividing provided for every \(\varepsilon > 0\) every element \(b \in \mathcal{B}\) has a two point partition \(b = b' \sqcup b''\) with \(\|f(b') - f(b'')\| < \varepsilon\).

Let \(\mathcal{B}\) be a Boolean algebra, \(X\) an F-space. A function \(f : \mathcal{B} \to X\) is said to be compact provided its image \(f(\mathcal{B})\) is a relatively compact subset of \(X\).

An F-space \(X\) is said to have the approximation property if for every relatively compact subset \(K \subseteq X\) and every \(\varepsilon > 0\) there exists a finite rank operator \(P \in \mathcal{L}(X)\) such that \(\|x - Px\| \leq \varepsilon\) for all \(x \in K\).

Next is the main result of the section.

**Theorem 3.1** Let \(\mathcal{B}\) be an atomless order complete Boolean algebra, \(X\) an F-space. Let \(S, T : \mathcal{B} \to X\) be measures with \(S\) almost dividing and \(T\) order continuous. Assume that at least one of the following conditions holds:

(i) \(T\) is a finite rank measure;
(ii) \(T\) is compact and the closed linear span \([T(\mathcal{B})]\) of the range \(T(\mathcal{B})\) isomorphically embeds in an F-space with the approximation property;
(iii) \(T\) is compact and \(X\) is a Banach space.

Then the sum \(S + T\) is almost dividing.

In particular, every order continuous finite rank measure \(T : \mathcal{B} \to X\) is almost dividing, and every order continuous compact measure \(T : \mathcal{B} \to X\) is almost dividing under the additional assumption that the closed linear span of the range of \(T\) isomorphically embeds in an F-space with the approximation property.

For the proof, we need some lemmas.
Lemma 3.2 Let $B$ be an atomless order complete Boolean algebra, $X$ an F-space and $f : B \to X$ an order continuous measure. If $u = \bigsqcup A$ is a partition of an element $u \in B$ then $f(u) = \sum_{a \in A} f(a)$, where the series converges unconditionally in $X$. In particular, the set $\{ a \in A : f(a) \neq 0 \}$ is at most countable.

**Proof** If $A$ is finite then the assertion is trivial. Let $A$ be infinite. Denote by $\Lambda_1$ the set of all finite subsets of $A$ partially ordered by inclusion. Since $A$ is infinite, $\Lambda_1$ is a directed set. For each $\lambda \in \Lambda_1$ we set $u_\lambda = \bigsqcup \lambda$. Observe that $u_\lambda \downarrow u$. Indeed, $u_\lambda \triangle u_\lambda = u - u_\lambda = \bigsqcup (A \setminus \lambda) \downarrow 0$. By the order continuity of $f$ we obtain

$$
\sum_{a \in \lambda} f(a) = f\left(\sum_{a \in \lambda} a\right) = f(u_\lambda) \to f(u).
$$

This means that the series $\sum_{a \in A} f(a) = f(u)$ converges unconditionally in $X$. \end{proof}

By $B^+$ we mean the set of all nonzero elements of a Boolean algebra $B$.

Lemma 3.3 (A) Let $B$ be an atomless order complete Boolean algebra, $X$ an F-space and $f : B \to X$ an order continuous at zero function with $f(0) = 0$. Then for every $b \in B^+$ and every $\varepsilon > 0$ there is a partition $b = \bigsqcup A$ such that

$$
(\forall a \in A)(\forall x \leq a) \quad \|f(x)\| < \varepsilon. \tag{3.1}
$$

(B) If, moreover, $f$ is an order continuous measure then the partition can be chosen finite.

**Proof** (A) Define a function $\tilde{f} : B \to [0, +\infty]$ by setting $\tilde{f}(x) = \sup \{\|f(y)\| : y \leq x\}$ for all $x \in B$ and prove the following properties:

1. for every $x, y \in B$ the inequality $x \leq y$ implies $\tilde{f}(x) \leq \tilde{f}(y)$;
2. $\tilde{f}$ is order continuous at zero;
3. $(\forall \varepsilon > 0)(\forall$ partition of unity $A')$, the set $\{ a \in A' : \tilde{f}(a) \geq \varepsilon \}$ is finite.

(1) is obvious.

(2) Assume $(x_\alpha)$ is a net in $B$ with $x_\alpha \overset{\alpha}{\to} 0$. Suppose on the contrary that there is $\delta > 0$ such that for every $\alpha$ there is $\beta_\alpha \geq \alpha$ with $f(x_{\beta_\alpha}) \geq \delta$. Then for every $\alpha$ we choose $y_\alpha \leq x_{\beta_\alpha}$ so that

$$
\|f(y_\alpha)\| \geq \tilde{f}(x_{\beta_\alpha}) - \frac{\delta}{2} \geq \frac{\delta}{2}.
$$

This contradicts the order continuity of $f$ at zero, because $y_\alpha \overset{\alpha}{\to} 0$. Thus, $\tilde{f}$ is order continuous at zero.

(3) Assuming (3) is not true, choose $\varepsilon > 0$ and a sequence $(a_n)_{n=1}^\infty$ of disjoint elements of $B$ such that $\tilde{f}(a_n) \geq \varepsilon$ for all $n \in \mathbb{N}$. Set $b_n = \bigsqcup_{k=n}^\infty a_k$, $n = 1, 2, \ldots$. Observe that $b_n \downarrow 0$ and by (1), $\tilde{f}(b_n) \geq \tilde{f}(a_n) \geq \varepsilon$, which contradicts (2).

Thus, properties (1)–(3) are proved.

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Fix any \( b \in \mathcal{B}^+ \) and \( \varepsilon > 0 \), and consider a maximal (with respect to the inclusion) disjoint family \( \mathcal{A} \subseteq \{ a \leq b : \tilde{f}(a) < \varepsilon \} \) (the existence of which is guaranteed by the Zorn lemma). We show that \( \mathcal{A} \) is a partition of \( b \). Assume, on the contrary, that \( b' = b - \bigsqcup \mathcal{A} > 0 \). Using atomlessness of \( \mathcal{B} \), decompose \( b' \) into an infinite disjoint sum of nonzero elements \( b' = \bigsqcup_{n=1}^{\infty} b_n, b_n > 0 \). By the maximality of \( \mathcal{A} \), one has \( \tilde{f}(b_n) \geq \varepsilon \) for all \( n \in \mathbb{N} \), which contradicts (3). Thus, \( b = \bigsqcup \mathcal{A} \). It remains to note that if \( x \leq a \in \mathcal{A} \) then \( \|f(x)\| \leq \tilde{f}(a) < \varepsilon \).

(\( \mathcal{B} \)) Fix any \( b \in \mathcal{B}^+ \) and \( \varepsilon > 0 \) and suppose on the contrary that every partition \( \mathcal{A} \) satisfying (3.1) is infinite. Choose some partition \( \mathcal{A} \) satisfying (3.1). Our goal is to construct a disjoint sequence \( (y_n) \) with \( \|f(y_n)\| \geq \varepsilon /2 \) for all \( n \in \mathbb{N} \), which would contradict (3). Since the singleton partition \( \{ b \} \) of \( b \) is finite, (3.1) does not hold true and \( x_1 \leq b \) can be chosen so that \( \|f(x_1)\| \geq \varepsilon \). By the order completeness of \( \mathcal{B} \), \( x_1 = \bigsqcup_{a \in \mathcal{A}} x_1 \cap a \). By Lemma 3.2, \( f(x_1) = \sum_{a \in \mathcal{A}} f(x_1 \cap a) \). Choose a finite subset \( A_1 \) of \( \mathcal{A} \) so that \( \|\sum_{a \in A_1} f(x_1 \cap a)\| \geq \varepsilon /2 \) and set \( y_1 = \sum_{a \in A_1} x_1 \cap a \). Now describe the second step. Since the partition \( A_1 \cup \{ b - \bigsqcup a \in A_1 \} \) of \( b \) is finite, (3.1) does not hold true and \( x_2 \leq b - \bigsqcup a \in A_1 \cap a \) can be chosen so that \( \|f(x_2)\| \geq \varepsilon \). By the order completeness of \( \mathcal{B} \), \( x_2 = \bigsqcup_{a \in \mathcal{A} \setminus A_1} x_2 \cap a \). By Lemma 3.2, \( f(x_2) = \sum_{a \in \mathcal{A} \setminus A_1} f(x_2 \cap a) \). Choose a finite subset \( A_2 \) of \( \mathcal{A} \setminus A_1 \) so that \( \|\sum_{a \in A_2} f(x_2 \cap a)\| \geq \varepsilon /2 \) and set \( y_2 = \sum_{a \in A_2} x_2 \cap a \). Continuing the recursive procedure in the obvious manner, we construct the desired sequence.

\[ \text{Lemma 3.4} \] (On rounding off coefficients) Let \( (y_k)_{k=1}^m \) be a finite collection of vectors in a finite dimensional normed space \( Y \) and \( (\lambda_k)_{k=1}^m \) be a collection of reals with \( 0 \leq \lambda_k \leq 1 \) for each \( k \). Then there exists a collection \( (\theta_k)_{k=1}^m \) of numbers \( \theta_k \in [0, 1] \) such that

\[ \left\| \sum_{k=1}^m (\lambda_k - \theta_k) y_k \right\| \leq \frac{\dim Y}{2} \max_k \|y_k\|. \quad (3.2) \]

For the proof we refer the reader to [6, p. 14].

\[ \text{Proof of Theorem 3.1} \] (i) Fix any \( b \in \mathcal{B}^+ \) and \( \varepsilon > 0 \). Let \( p \) be any norm on the finite dimensional F-space \([T(\mathcal{B})]\). Choose \( \delta > 0 \) so that

\[ \forall u \in [T(\mathcal{B})], \quad p(u) < \delta \Rightarrow \|u\| < \frac{\varepsilon}{2}. \quad (3.3) \]

Then choose \( \sigma > 0 \) so that

\[ \forall v \in [T(\mathcal{B})], \quad \|v\| < \sigma \Rightarrow p(v) < \frac{\delta}{2 \dim [T(\mathcal{B})]} \]. \quad (3.4) \]

Using Lemma 3.3, choose a finite partition \( b = \bigsqcup_{k=1}^m b_k \) such that for every \( k \in \{1, \ldots, m\} \) and every \( x \leq b_k \) one has \( \|T(x)\| < \sigma \), hence by (3.4),

\[ p(T(x)) < \frac{\delta}{2 \dim [T(\mathcal{B})]} \quad (3.5) \]
For every $k \in \{1, \ldots, m\}$, using that $S$ is almost dividing, choose a partition $b_k = b'_k \sqcup b''_k$ so that

$$
\| S(b'_k) - S(b''_k) \| < \frac{\varepsilon}{2m}.
$$

(3.6)

Set $y_k = T(b'_k) - T(b''_k)$ and $\lambda_k = 1/2$ for all $k \in \{1, \ldots, m\}$. Choose by Lemma 3.4 numbers $\theta_k \in \{0, 1\}$ to satisfy

$$
p \left( \sum_{k=1}^m 2(\lambda_k - \theta_k) y_k \right) \leq \dim [T(B)] \max \limits_k p(y_k).
$$

(3.7)

Now we set $I' = \{ k \leq m : 2(\lambda_k - \theta_k) = 1 \}$, $I'' = \{ k \leq m : 2(\lambda_k - \theta_k) = -1 \}$,

$$
b' = \bigsqcup_{k \in I'} b'_k \sqcup \bigsqcup_{k \in I''} b''_k \quad \text{and} \quad b'' = \bigsqcup_{k \in I'} b''_k \sqcup \bigsqcup_{k \in I''} b'_k.
$$

Then $b = b' \sqcup b''$ and

$$
T(b') - T(b'') = \sum_{k \in I'} T(b'_k) + \sum_{k \in I''} T(b''_k) - \sum_{k \in I'} T(b''_k) - \sum_{k \in I''} T(b'_k)
$$

$$
= \sum_{k \in I'} (T(b'_k) - T(b''_k)) - \sum_{k \in I''} (T(b'_k) - T(b''_k))
$$

$$
= \sum_{k=1}^m 2(\lambda_k - \theta_k) y_k.
$$

Since $b'_k, b''_k \leq b_k$, by (3.5),

$$
p(y_k) \leq p(T(b'_k)) + p(T(b''_k)) < \frac{\delta}{\dim [T(B)]},
$$

and hence,

$$
p(T(b') - T(b'')) \overset{(3.7)}{\leq} \dim [T(B)] \max \limits_k p(y_k)
$$

$$
< \dim [T(B)] \frac{\delta}{\dim [T(B)]} = \delta.
$$

(3.8)

Then by (3.3),

$$
\| T(b') - T(b'') \| < \frac{\varepsilon}{2}.
$$

(3.9)
On the other hand,

\[
\|S(b') - S(b'')\| = \left\| \sum_{k \in I'} S(b'_k) + \sum_{k \in I''} S(b''_k) - \sum_{k \in I'} S(b''_k) - \sum_{k \in I''} S(b'_k) \right\|
\]

\[
= \left\| \sum_{k \in I'} (S(b'_k) - S(b''_k)) - \sum_{k \in I''} (S(b'_k) - S(b''_k)) \right\|
\]

\[
\leq \sum_{k=1}^m \|S(b'_k) - S(b''_k)\| \quad (3.6) < m \cdot \frac{\varepsilon}{2m} = \frac{\varepsilon}{2}.
\]

This together with (3.9) gives

\[
\|(S + T)(b') - (S + T)(b'')\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

(ii) Fix any \( b \in B^+ \) and \( \varepsilon > 0 \). Let \( J : [T(B)] \to Y \) be an isomorphic embedding, where \( Y \) is an F-space with the approximation property. Choose \( \delta > 0 \) so that

\[
\forall y \in Y, \quad \|y\| \leq \delta \Rightarrow \|J^{-1}y\| \leq \frac{\varepsilon}{4}. \quad (3.10)
\]

Since \( T \) is compact, \( (J \circ T)(B) \) is a relatively compact subset of \( Y \). Choose a finite rank operator \( P \in \mathcal{L}((J \circ T)(B)) \) so that

\[
\|y - Py\| \leq \delta \quad \text{for every} \quad y \in (J \circ T)(B), \quad (3.11)
\]

Define an operator \( T_1 : B \to X \) by \( T_1 = J^{-1} \circ P \circ J \circ T \). Notice that \( T_1 \) is an order continuous finite rank measure. By (i), the sum \( S + T_1 \) is almost dividing. Choose a decomposition \( b = b' \sqcup b'' \) so that

\[
\|(S + T_1)(b') - (S + T_1)(b'')\| < \frac{\varepsilon}{2}. \quad (3.12)
\]

Since \( (J \circ T)(b') \in (J \circ T)(B) \), by (3.11), \( \|(J \circ T)(b') - (P \circ J \circ T)(b')\| \leq \delta \). Hence, by (3.10),

\[
\|T(b') - T_1(b')\| = \|T(b') - (J^{-1} \circ P \circ J \circ T)(b')\| \leq \frac{\varepsilon}{4}. \quad (3.13)
\]

Analogously,

\[
\|T(b'') - T_1(b'')\| \leq \frac{\varepsilon}{4}. \quad (3.14)
\]

Finally, we obtain by (3.12), (3.13) and (3.14)

\[
\|(S + T)(b') - (S + T)(b'')\| \leq \|(S + T_1)(b') - (S + T_1)(b'')\|
\]

\[
+ \|T(b') - T_1(b')\| + \|T(b'') - T_1(b'')\|
\]

\[
< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\]
(iii) follows from (ii) because every Banach space \( X \) isometrically embeds in a Banach space with the approximation property, for example by means of the following canonical isometrical isomorphisms: \( X \hookrightarrow X^{**} \hookrightarrow \ell_\infty(B_{X^*}) = \ell_\infty(\Gamma) \).

\[ \square \]

**Remark 3.5** (1) Concerning arbitrary F-spaces, one of the sufficient conditions on an F-space \( Y \) to have the approximation property is that \( Y \) has a transfinite basis, see [11, Proposition 5.1].

(2) A large class of F-spaces that cannot be embedded in an F-space with the approximation property if the class of F-spaces with trivial dual \( X^* = \{0\} \), including \( L_p(\mu) \)-spaces with \( 0 \leq p < 1 \) on atomless measure spaces.

### 4 The sum of a narrow and a finite rank laterally continuous OAOs in Riesz spaces

A subset \( A \subseteq E \) is said to be **laterally bounded** if \( A \subseteq \mathfrak{F}_e \) for some \( e \in E \). A net \( (x_\alpha) \) in \( E \) **order converges** to an element \( x \in E \) if there exists a net \( (u_\alpha) \) in \( E \) with the same index set such that \( |x_\alpha - x| \leq u_\alpha \) for all indices \( \alpha \) and \( u_\alpha \downarrow 0 \). In this case we write \( x_\alpha \xrightarrow{o} x \), and say that \( x \) is the **order limit** of \( (x_\alpha) \). If \( x_\alpha \xrightarrow{o} x \) and, moreover, there is an index \( \alpha_0 \) such that the tail \( (x_\alpha)_{\alpha \geq \alpha_0} \) is laterally bounded, we say that \( (x_\alpha) \) **laterally converges** to \( x \), write \( x_\alpha \xrightarrow{\alpha} x \), and say that \( x \) is the **lateral limit** of \( (x_\alpha) \). We remark that in some papers the authors additionally assume the lateral increase of the net in the definition of the lateral convergence, that is, \( x_\alpha \subseteq x_\beta \) if \( \alpha < \beta \). So, our definition is more general, however the lateral continuity is more restrictive.

Let \( E \) be a Riesz space and \( X \) an F-space. We say that a function \( f : E \to X \) is **laterally continuous at a point** \( e \in E \) provided for every net \( (x_\alpha) \) in \( E \) the condition \( x_\alpha \xrightarrow{\alpha} x \) implies \( f(x_\alpha) \to f(x) \). A function which is laterally continuous at each point \( e \in E \) is called **laterally continuous**. In the case where \( X \) is a Riesz space (or a Banach lattice), to distinguish different types of convergence in \( X \), we use the terms “laterally continuous” (for the lateral convergence in \( X \)), “laterally-to-order continuous” and “laterally-to-norm continuous” respectively with the obvious meaning.

It is natural (but not so obvious as in the case of linear operators) that the lateral continuity of an OAO at zero implies its lateral continuity [4]. However, the lateral continuity of an OAO at a fixed nonzero point does not imply its lateral continuity [12].

Let \( E \) be a Riesz space and \( X \) an F-space. An OAO \( T : E \to X \) is said to be **C-compact** if \( T \) sends laterally bounded sets to relatively compact sets.

**Theorem 4.1** Let \( E \) be an atomless Dedekind complete Riesz space, \( X \) an F-space, \( S, T : E \to X \) OAOs with \( S \) narrow and \( T \) finite rank laterally continuous. Then the sum \( S + T \) is narrow.

**Proof** Fix any \( e \in E^+ \) and \( \varepsilon > 0 \). It is well known that the set of all fragments \( \mathfrak{F}_e \) of \( e \) is a Boolean algebra with zero \( 0 \), unity \( e \) and with respect to the lateral partial order \( \sqsubseteq \), which is order complete by the Dedekind completeness of \( E \) [2, Theorem 3.15] (see...
also [12] for details). \( \mathcal{F}_e \) is an atomless Boolean algebra by the atomlessness of \( E \).

Then an immediate application of Theorem 3.1 to the restrictions \( S|\mathcal{F}_e \) and \( T|\mathcal{F}_e \) yields the existence of a decomposition \( e = e' \sqcup e'' \) with \( \| (S + T)(e') - (S + T)(e'') \| < \varepsilon \).

\[ \square \]

One can prove similarly the next applications of Theorem 3.1.

**Theorem 4.2** Let \( E \) be an atomless Dedekind complete Riesz space, \( X \) an F-space, \( S, T : E \to X \) OAOs with \( S \) narrow and \( T \) laterally continuous C-compact. Assume that, for every \( e \in E \), the closed linear span \([ T(\mathcal{F}_e) ]\) of the range \( T(\mathcal{F}_e) \) isomorphically embeds in an F-space with the approximation property. Then the sum \( S + T \) is narrow.

**Theorem 4.3** Let \( E \) be an atomless Dedekind complete Riesz space, \( X \) a Banach space, \( S, T : E \to X \) OAOs with \( S \) narrow and \( T \) laterally continuous C-compact. Then the sum \( S + T \) is narrow.

To this concern, one problem still remains open.

**Problem 1** [11] Let \( 0 < p < 1 \). Is a sum of two narrow operators from \( \mathcal{L}(L_\infty, L_p) \), at least one of which is compact, is narrow?

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