NONEXISTENCE OF GLOBAL SOLUTIONS FOR A CLASS OF VISCOELASTIC WAVE EQUATIONS

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Abstract. We consider a class of nonlinear evolution equations of second order in time, linearly damped and with a memory term. Particular cases are viscoelastic wave, Kirchhoff and Petrovsky equations. They appear in the description of the motion of deformable bodies with viscoelastic material behavior. Several articles have studied the nonexistence of global solutions of these equations due to blow-up. Most of them have considered non-positive and small positive values of the initial energy and recently some authors have analyzed these equations for any positive value of the initial energy. Within an abstract functional framework we analyze this problem and we improve the results in the literature. To this end, a new positive invariance set is introduced.

1. Introduction. Let us consider the following problem. For every initial data $u_0, u_1$, find a function $t \mapsto u(t), t \geq 0$, such that

\[
\begin{cases}
Pu_{tt}(t) + Au(t) - \int_0^t g(t - \tau) Bu(\tau) d\tau + \delta Pu_t(t) = f(u(t)), & t > 0 \\
\quad u(0) = u_0, \quad u_t(0) = u_1,
\end{cases}
\]

where $u_t \equiv \frac{\partial}{\partial t} u$, $\delta > 0$ is the damping coefficient and $g(s) \geq 0$, for $s \geq 0$, is the relaxation function. The following operators, defined on Banach spaces, are linear, continuous, positive and symmetric

\[
P : V_P \to V_P', \quad A : V_A \to V_A', \quad B : V_B \to V_B'.
\]

We assume that

\[
V_A \subset V_B \subset V_P \subset H,
\]

are linear subspaces of a Hilbert space $H$ with inner product $\langle \cdot, \cdot \rangle$, norm $\| \cdot \|$, and $H', V_P', V_A', V_B'$ are the corresponding dual spaces. We identify $H = H'$, then

\[
H \subset V_P' \subset V_B' \subset V_A'.
\]

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In terms of $P$, $A$, $B$, and the corresponding duality pairs, we have the following bilinear forms
\[
\mathcal{P}(u, w) \equiv (Pu, w)_{V_P \times V_A'}, \quad u, w \in V_P,
\]
\[
\mathcal{A}(u, w) \equiv (Au, w)_{V_A \times V_A'}, \quad u, w \in V_A,
\]
\[
\mathcal{B}(u, w) \equiv (Bu, w)_{V_A \times V_B'}, \quad u, w \in V_B,
\]
and corresponding norms for $V_P, V_A, V_B$
\[
\|u\|_{V_P}^2 \equiv \mathcal{P}(u, u), \quad u \in V_P, \quad \|u\|_{V_A}^2 \equiv \mathcal{A}(u, u), \quad u \in V_A, \quad \|u\|_{V_B}^2 \equiv \mathcal{B}(u, u), \quad u \in V_B.
\]
The dynamics is studied in the following phase space
\[
\mathcal{H} \equiv V_A \times V_P,
\]
with corresponding square norm
\[
\|\!(u, v)\!\|_{\mathcal{H}}^2 \equiv \|u\|_{V_A}^2 + \|v\|_{V_P}^2.
\]
Along the paper the following hypotheses are satisfied
(i) There exist embedding constants $c > 0$, $\hat{c} > 0$, such that
\[
(H0) \quad \|u\|_{V_A}^2 \geq \hat{c}\|u\|_{V_B}^2, \quad u \in V_A, \quad \|u\|_{V_B}^2 \geq c\|u\|_{V_P}^2, \quad u \in V_B.
\]
(ii) The nonlinear source term $f : V_A \to H$, is a potential operator with potential $F : V_A \to \mathbb{R}$, that is, $f(u) = D_u F(u)$. We assume that $f(0) = 0$ and there exists a constant $r > 2$, such that
\[
(H1) \quad (f(u, u) - rF(u)) \geq 0, \quad u \in V_A.
\]
(iii) The relaxation function $g \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, satisfies the following conditions
\[
(H2) \quad g(0) > 0, \quad l \equiv 1 - \int_0^\infty g(t) \, dt > 0, \quad \dot{g}(t) \equiv \frac{d}{dt} g(t) \leq 0, \quad t \geq 0.
\]
In the dynamic analysis of an evolution equation, the first problem is to establish a functional framework and define the class of solutions to study. In such a framework, an important question is to find what initial data provides solutions that exist for any time. That is, the maximal time of existence is infinite. These solutions are called global. For functional frameworks with infinite dimensional phase space, the non-existence of global solutions is sometimes due to the blow-up of the solution in the norm of the corresponding phase space, see [1]. One of the methods to study nonexistence of global solutions of evolution equations due to blow-up is through the analysis of differential inequalities. In particular, the concavity argument, introduced by Professor Howard Levine [10, 11], has been widely used. See the book [9] and references therein, for an account of several methods to study blow-up in equations of mathematical physics.

Some mathematical models from the continuum mechanics that describe the dynamics of deformable bodies, like beams, with viscoelastic constitutive relations, led to evolution equations like the one in (P), see [25, 6]. Concrete equations of this class of abstract problems have been analyzed for several authors obtaining some qualitative properties of the solutions. In particular the blow-up problem was first studied for negative values and small positive values of the initial energy, see for instance for some early references [22, 23]. Generalizations of the concavity method have been utilized to obtain sufficient conditions to get blow-up for arbitrary positive values of the initial energy. Two notable first articles on this subject are [30, 29], see also [7, 8, 12, 13, 15, 19, 20, 27, 28], [31]-[34]. Related and interesting results have been reported in [2]-[5], [14], [16]-[18], [21, 24, 26, 35]. The purpose of this
work is to give sufficient conditions to get blow-up for the viscoelastic problem (P) with any positive value of the initial energy and improve the previous results. Our approach is to state an abstract framework in order to cover a wide class of concrete problems such as the wave, Kirchhoff, and Petrovsky equations with memory.

The analysis of problem (P) will be done for weak solutions in the following sense.

**Definition 1.1.** For every initial data

$$(u_0, u_1) \in \mathcal{H},$$

the map

$$(u_0, u_1) \mapsto (u(t), \dot{u}(t)) \in \mathcal{H}$$

is a weak local solution of problem (P), if there exists some $T > 0$, such that

$$(u, \dot{u}) \in C([0,T]; \mathcal{H})$$

with

$$u \in L^2([0,T]; V_B), \quad \dot{u} = \frac{\partial}{\partial t} u \in L^2([0,T]; V_P),$$

$$u(0) = u_0, \quad \dot{u}(0) = u_1,$$

and

$$\frac{d}{dt} P(u(t), w) + A(u(t), w) - \int_0^t g(t - \tau) B(u(\tau), w) \, d\tau + \delta P(\dot{u}(t), w) = (f(u(t)), w),$$

ea.e. $t \in (0, T)$, for every $w \in V_A$.

We shall consider that the solution in this sense is unique and satisfies the following energy equation for $T > t \geq t_0 > 0$,

$$E(u(t), \dot{u}(t)) - E(u(t_0), \dot{u}(t_0)) = -\delta \int_{t_0}^t \|\ddot{u}(s)\|_{V_B}^2 \, ds - \frac{1}{2} \int_{t_0}^t g(s) \|u(s)\|_{V_\alpha}^2 \, ds + \frac{1}{2} \int_{t_0}^t (g \circ u)(s) \, ds,$$

where

$$E(t) \equiv E(u(t), \dot{u}(t)) \equiv \frac{1}{2} \|\dot{u}(t)\|_{V_P}^2 + J(u(t)),$$

$$J(u(t)) \equiv \frac{1}{2} \|u(t)\|_{V_\alpha}^2 - F(u(t)) + \frac{1}{2} ((g \circ u)(t) - G(t) \|u(t)\|_{V_\alpha}^2),$$

and

$$(g \circ u)(t) \equiv \int_0^t g(t - \tau) \|u(t) - u(\tau)\|_{V_\alpha}^2 \, d\tau, \quad G(t) \equiv \int_0^t g(\tau) \, d\tau, \quad \xi > 0.$$

$$E(t_0) \geq E(t) = \frac{1}{2} \|u(t_0)\|_{V_\alpha}^2 + \frac{1}{2} ((g \circ u)(t) - G(t) \|u(t)\|_{V_\alpha}^2) - F(u(t)).$$

Furthermore, if the maximal time of existence $T_{\text{MAX}} < \infty$ then

$$\lim_{t \to T_{\text{MAX}}} \|(u(t), \dot{u}(t))\|_\mathcal{H} = \infty,$$

equivalently, if $\hat{c} > 1 - l$,

$$\lim_{t \to T_{\text{MAX}}} F(u(t)) = \infty,$$

since, from (H0) and (H2),

$$E(t_0) \geq \frac{1}{2} \left( 1 - \frac{1 - l}{\hat{c}} \right) \|(u(t), \dot{u}(t))\|_{V_\alpha}^2 - F(u(t)).$$
2. Main result. In this section we shall analyze the nonexistence of global solutions for the problem \((P)\), for any positive value of the initial energy. To this end, the velocity is decomposed orthogonally as follows

\[
\dot{u} = \mathcal{P}(\dot{u}, u) u + h, \quad \mathcal{P}(u, h) = 0,
\]

\[
\|\dot{u}\|_{V_P}^2 = \|h\|_{V_P}^2 + \frac{\|\mathcal{P}(\dot{u}, u)\|^2}{\|u\|_{V_P}^2} \geq \frac{\|\mathcal{P}(\dot{u}, u)\|^2}{\|u\|_{V_P}^2} \equiv Q(\dot{u}, u).
\] (1)

We assume that the constants involved in the problem \((P)\) satisfy the following inequalities

\[
(H3) \quad \hat{c} > 1 - l, \quad r > 1 + \sqrt{1 - \frac{1}{1 - 1 - l}}.
\]

We define

\[
\alpha \equiv \frac{1}{4} (r - 2), \quad \gamma \equiv 2rE_0,
\]

\[
\beta \equiv \frac{\varepsilon}{r} \{ (\hat{c} - (1 - l))(r - 1)^2 - \hat{c}\},
\]

that, according with \((H3)\), are positive constants. In particular, we emphasize that \(\beta > 0\). Also, we define the functions

\[
\psi(t) \equiv \|u(t)\|_{V_P}^2,
\]

\[
\phi(t) \equiv \left( \frac{d}{dt} \psi(t) - \frac{\delta}{\alpha} \psi^{\frac{1}{2}}(t) \right)^2 + \frac{\beta}{\alpha} \psi(t),
\]

\[
\sigma_\nu(t) \equiv \frac{1 + 2\alpha}{2} \left( \phi(t) - \frac{\beta \nu}{\alpha} \psi(t) \right),
\]

\[
\mu_\lambda(t) \equiv \frac{1 + 2\alpha}{2} \left( \phi(t) - \frac{\beta}{\alpha(1 + 2\alpha)} \psi(t) \left( \frac{\beta \psi(t)}{\alpha \phi(t)} \right)^{2\alpha} \right),
\]

for \(t \geq 0, \nu > 0, \lambda \in (0, 1), \) and

\[
\psi_0 \equiv \psi(0), \quad \phi_0 \equiv \phi(0) = \left( \frac{\psi_0 - \delta}{\alpha} \frac{\psi^{\frac{1}{2}}_0}{\psi_0} \right)^2 + \frac{\beta}{\alpha} \psi_0, \quad \dot{\psi}_0 \equiv \frac{d}{dt} \psi(0).
\] (4)

**Theorem 2.1.** Consider any solution \((u, \dot{u})\) of problem \((P)\) in the sense of Definition 1.1. Assume that hypotheses \((H0) - (H3)\) hold. If

\[
\dot{\psi}_0 > \frac{\delta}{\alpha} \psi_0 > 0,
\] (5)

is satisfied, then there exists a nonempty interval

\[
\mathcal{I} \equiv (a, b) \subset \left( 0, \frac{1 + 2\alpha}{2} \phi_0 \right),
\]

with the following consequences:

(i) If \(\gamma = 2rE_0 \in \mathcal{I}\), then \(\psi(t)\) blows-up in a finite time \(t^* > 0\), that is

\[
\lim_{t \to t^*} \psi(t) = \infty.
\]

Hence, the solution of problem \((P)\) is not global.


(ii) \( a = \sigma_{\nu}(0) \) and \( b = \mu_{\lambda}(0) \), moreover
\[
a = \frac{\beta \psi_0}{((1 + 2\alpha)\nu^*)^{\frac{1}{\alpha}}} < \frac{\beta \psi_0}{(1 + 2\alpha)^{\frac{1}{\alpha}}},
\]
\[
b = \frac{\alpha \phi_0}{\lambda^*} > \frac{1 + 2\alpha}{2} \phi_0 - \left(\frac{1 + 2\alpha}{2\alpha} - \chi(\lambda^*)\right) \beta \psi_0 > \frac{1 + 2\alpha}{2} \phi_0 - \frac{\beta}{2\alpha} \psi_0,
\]
for some \( \frac{2\alpha}{1 + 2\alpha} < \lambda^* < 1 \) and \( \nu^* > 1 + 2\alpha \), where \( 1 < \chi(\lambda^*) < \frac{1 + 2\alpha}{2\alpha} \) is a function of \( \lambda^* \).

(iii) For fixed \( \psi_0, \dot{\psi}_0 \),
\[
\delta \mapsto t^*, \text{ is strictly increasing, and}
\]
\[
\delta \mapsto |I| = b - a, \text{ is strictly decreasing.}
\]

For fixed \( \psi_0, \delta \),
\[
\dot{\psi}_0 \mapsto t^*, \text{ is strictly decreasing, and}
\]
and
\[
\dot{\psi}_0 \mapsto |I|, \text{ is strictly increasing.}
\]

We have the bounds
\[
0 < \frac{1 + 2\alpha}{2} \phi_0 - |I| < \left(\frac{1 + 2\alpha}{2\alpha} - \chi(\lambda^*) + \frac{1}{((1 + 2\alpha)\nu^*)^{\frac{1}{\alpha}}}\right) \beta \psi_0,
\]
\[
t^* \geq \left(\frac{\psi_0}{\alpha \psi_0 - \delta}\right)^{-1}.
\]

(iv) Furthermore, for \( \psi_0 \) fixed, we have the limit values as \( \dot{\psi}_0 \to \infty \),
\[
a \to 0, \quad \left| b - \frac{1 + 2\alpha}{2} \phi_0 \right| \to 0, \quad \phi_0 \to \infty, \quad t^* \to 0,
\]
\[
\nu^* \to \infty, \quad \lambda^* \to \frac{2\alpha}{1 + 2\alpha}, \quad \chi(\lambda^*) \to \frac{1 + 2\alpha}{2\alpha}.
\]

**Corollary 1.** Consider any solution \((u, \dot{u})\) of problem (P) in the sense of Definition 1.1. Assume that hypotheses of Theorem 2.1 are met. Given any number \( \xi > 0 \), we can choose initial data with \( P(u_0, u_1) \) large enough, so that the conclusions of Theorem 2.1 are satisfied for initial energy with \( 2rE_0 = \xi \).

**Remark 1.** From (3)-(4) and by introducing the functions,
\[
\mathcal{F}(t) \equiv \psi^{-\alpha}(t), \quad \mathcal{G}(t) \equiv \mathcal{F}(t)e^{\delta t},
\]
we observe that (5) has the equivalent forms
\[
\phi_0 > \frac{\beta}{\alpha} \psi_0 > 0 \iff \dot{\psi}_0 > \frac{\delta}{\alpha} \psi_0 > 0 \iff P(u_0, u_1) > \frac{2\delta}{r + 2\alpha} \| u_0 \|_{V}^2 > 0
\]
\[
\iff \dot{\mathcal{F}}_0 = \frac{d}{dt} \mathcal{F}(0) < -\delta \mathcal{F}(0) \equiv -\delta \mathcal{F}_0 \iff \dot{\mathcal{G}}_0 = \frac{d}{dt} \mathcal{G}(0) < 0.
\]

The following set \( V \subset H \) is fundamental in the proof of Theorem 2.1
\[
V \equiv \{ (u, \dot{u}) : P(u, \dot{u}) > \frac{2\delta}{r + 2\alpha} \| u \|_{V}^2 \},
\]
and, from Remark 1, this is characterized by
\[ \mathcal{V} = \left\{ (u, \dot{u}) : \mathcal{G} < 0 \right\}. \]

The set \( \mathcal{V} \) is positive invariant and this property is proved next. To this end we shall need the function \( \mathcal{M}(s) \) defined by
\[ \mathcal{M}(s) = \frac{2\alpha^2}{1 + 2\alpha} \gamma s^{1 + 2\alpha} - \alpha\beta s^2 + \alpha^2 \psi_0^{-1}(1 + 2\alpha) \left( \phi_0 - \frac{2\gamma}{1 + 2\alpha} \right), \quad s \geq 0. \]

We recall that \( \mathcal{W} \subset \mathcal{H} \) is a positive invariant set, along the solution \((u, \dot{u})\), if
\[(u_0, u_1) \equiv (u(0), \dot{u}(0)) \in \mathcal{W} \Rightarrow (u(t), \dot{u}(t)) \in \mathcal{W} \text{ for any } t > 0. \]

**Lemma 2.2.** Consider any solution \((u, \dot{u})\) of problem \((P)\) in the sense of Definition 1.1. Assume that hypotheses \((H0) - (H3)\) hold. If there exists a constant \( \kappa_0^2 > 0 \) such that
\[ \mathcal{M}(s) \geq \kappa_0^2 > 0, \quad s \geq 0, \] then, along the solution \((u, \dot{u})\), the set \( \mathcal{V} \) is positive invariant.

Furthermore,
\[ \dot{\mathcal{G}}(t) \equiv \frac{d}{dt}\mathcal{G}(t) \leq -\kappa_0 < 0, \quad \text{for any } t \geq 0. \]

**Proof.** (of Lemma 2.2.) Let \((u, \dot{u})\) be a solution of problem \((P)\), such that \((u_0, u_1) \equiv (u(0), \dot{u}(0)) \in \mathcal{V}\). To show the invariance property, we shall construct a differential inequality for the function
\[ \psi(t) \equiv \|u(t)||^2_{\mathcal{V}_\alpha} \in \mathbb{R}^+, \quad t \geq 0. \]

We calculate the following derivatives
\[
\begin{align*}
\frac{d}{dt}\psi(t) &= 2\mathcal{P}(u(t), \dot{u}(t)) \\
\frac{d^2}{dt^2}\psi(t) &= 2(\|\dot{u}(t)||^2_{\mathcal{V}_\alpha} - \|u(t)||^2_{\mathcal{V}_\alpha} + (f(u(t), u(t))) \bigg) \\
&+ 2 \int_0^t g(t - \tau)\mathcal{B}(u(\tau), u(t)) \, d\tau - 2\delta\mathcal{P}(u(t), \dot{u}(t)) \tag{7}
\end{align*}
\]

We shall estimate the terms of the right hand side of the second derivative of \( \psi(t) \). First, we consider
\[ \int_0^t g(t - \tau)\mathcal{B}(u(\tau), u(t)) \, d\tau = G(t)\|u(t)||^2_{\mathcal{V}_\alpha} + \int_0^t g(t - \tau)\mathcal{B}(u(\tau) - u(t), u(t)) \, d\tau. \]

And, we get
\[ \begin{align*}
2 \int_0^t g(t - \tau)\mathcal{B}(u(\tau) - u(t), u(t)) \, d\tau &\leq 2 \int_0^t g(t - \tau)\|u(\tau) - u(t)||_{\mathcal{V}_\alpha}||u(t)||_{\mathcal{V}_\alpha} \, d\tau \\
&\leq \int_0^t g(t - \tau) \left( r\|u(\tau) - u(t)||^2_{\mathcal{V}_\alpha} + \frac{1}{r}\|u(t)||^2_{\mathcal{V}_\alpha} \right) \, d\tau \\
&= r(g \circ u)(t) + \frac{1}{r} G(t)\|u(t)||^2_{\mathcal{V}_\alpha}.
\end{align*} \]
Now, by the energy equation and hypothesis \((H1)\), we obtain the following
\[
2(\|u(t)\|_V^{2} - \|u(t)\|_V^{2}) + (f(u(t), u(t)) + G(t)\|u(t)\|_V^{2}) \\
+ 2r(E(t) - E(t)) \\
\geq (r + 2)\|\dot{u}(t)\|_V^{2} + (r - 2)(\dot{c} - G(t))\|u(t)\|_V^{2} + r(g \circ u)(t) - 2rE_0.
\]

Consequently, from \((7)\), the estimates above and \((H0), (H2)\), we get
\[
\frac{d^2}{dt^2}\psi(t) \geq (r + 2)\|\dot{u}(t)\|_V^{2} - 2\deltaE(u(t), \dot{u}(t)) \\
+ \frac{c}{r} (\dot{c}((r - 1)^2 - 1) - (1 - l)(r - 1)^2)\|u(t)\|_V^{2} - 2rE_0,
\]

since, from \((H3)\),
\[
\dot{c}((r - 1)^2 - 1) - (1 - l)(r - 1)^2 = (\dot{c} - (1 - l))(r - 1)^2 - \dot{c} > 0.
\]

Therefore, we obtain
\[
\frac{d^2}{dt^2}\psi(t) + \delta\frac{d}{dt}\psi(t) - \frac{c}{r} (\dot{c}((r - 1)^2 - 1) - (1 - l)(r - 1)^2)\psi(t) \\
- \frac{1}{4} (r + 2) \left(\frac{d\psi(t)}{\psi(t)}\right)^2 + 2rE_0 \geq 0.
\]

We can simplify the notation by substituting the constants defined in \((2)\). After multiplying the differential inequality \((8)\) by \(\psi(t)\), we get
\[
\psi(t)\frac{d^2}{dt^2}\psi(t) + \delta\psi(t)\frac{d}{dt}\psi(t) - (1 + \alpha) \left(\frac{d}{dt}\psi(t)\right)^2 - \beta\psi^2(t) + \gamma\psi(t) \geq 0.
\]

If we now introduce \(F(t) \equiv \psi^{-\alpha}(t)\), this inequality becomes
\[
\frac{d^2}{dt^2}F(t) + \delta\frac{d}{dt}F(t) + \alpha\betaF(t) - \alpha\gammaF^{1+\alpha}(t) \leq 0,
\]

Next, if \(G(t) \equiv e^{\delta t}F(t)\), we get
\[
\frac{d^2}{dt^2}G(t) - \delta\frac{d}{dt}G(t) + \alpha\betaG(t) - \alpha\gammaG^{1+\alpha}(t)e^{-\delta t} \leq 0.
\]

After these estimates we proceed to show the positive invariance result. By contradiction, if the set \(\mathcal{V}\) is not positive invariant, then there exists \(\hat{t} > 0\) such that
\[
\frac{d}{dt}\mathcal{G}(\hat{t}) < 0, \quad t \in [0, \hat{t}), \quad \text{and} \quad \frac{d}{dt}\mathcal{G}(\hat{t}) = 0.
\]

Hence, we get
\[
\frac{d^2}{dt^2}\mathcal{G}(\hat{t}) + \alpha\beta\mathcal{G}(\hat{t}) - \alpha\gamma\mathcal{G}^{1+\alpha}(\hat{t}) \leq 0, \quad t \in [0, \hat{t}).
\]

Multiplying \((9)\) by \(\frac{d}{dt}\mathcal{G}(\hat{t}) < 0\), we get the following integral
\[
\left(\frac{d}{dt}\mathcal{G}(\hat{t})\right)^2 + \alpha\beta\mathcal{G}^2(\hat{t}) - \frac{2\alpha^2}{1 + 2\alpha}\gamma\mathcal{G}^{1+\alpha}(\hat{t}) \geq \mathcal{G}^2(\hat{t}) + \alpha\beta\mathcal{G}^2 - \frac{2\alpha^2}{1 + 2\alpha}\gamma\mathcal{G}^{1+\alpha} \\
= \alpha^2\psi_0^{-2(1+\alpha)} \left(\frac{\dot{\psi}_0 - \frac{\delta}{\alpha}\psi_0}{\psi_0}\right)^2 + \alpha\beta\psi_0^{-2\alpha} - \frac{2\alpha^2}{1 + 2\alpha}\gamma\psi_0^{-2(1+\alpha)} \\
= \alpha^2\psi_0^{-1+2\alpha} \left(\frac{1}{\psi_0}\left(\frac{\dot{\psi}_0 - \frac{\delta}{\alpha}\psi_0}{\psi_0}\right)^2 + \frac{\beta}{\alpha}\psi_0 - \frac{2\gamma}{1 + 2\alpha}\right).
\]
\[ \alpha^2 \psi_0^{-(1+2\alpha)} \left( \phi_0 - \frac{2\gamma}{1+2\alpha} \right), \quad t \in [0, \hat{t}], \]

and then

\[ \left( \frac{d}{dt} \mathcal{G}(t) \right)^2 \geq \mathcal{M}(\mathcal{G}(t)), \quad t \in [0, \hat{t}). \]

By our assumption (6), from (10) we obtain

\[ \frac{d}{dt} \mathcal{G}(t) \leq -\kappa_0 < 0, \quad t \in [0, \hat{t}). \]

If \( t \to \hat{t}, \) by continuity, it is necessarily true that

\[ \frac{d}{dt} \mathcal{G}(\hat{t}) \leq -\kappa_0 < 0. \]

Then, the condition that defines \( \hat{t} \) is never met. As long as the solution exits, the derivative of \( \mathcal{G}(t) \) always remains strictly negative and consequently \( \mathcal{V} \) is positive invariant.

**Proof.** (of Theorem 2.1.) Let us proceed by contradiction. If the solution is global, then

\[ t \to \psi(t) \equiv \| u(t) \|_{L^p}^2 \in \mathbb{R}^+ , \]

that is, it is well defined for any \( t \geq 0. \)

From (5) and Remark 1 the initial data are such that \((u_0, u_1) \in \mathcal{V} \). Hence, by Lemma 2.2 \((u(t), \dot{u}(t)) \in \mathcal{V} \) for any \( t > 0, \) and

\[ \frac{d}{dt} \mathcal{G}(t) \leq -\kappa_0 < 0, \quad t \geq 0. \]

Consequently,

\[ 0 < e^{\delta t} \psi^{-\alpha}(t) = \mathcal{G}(t) \leq \psi_0^{-\alpha} - t\kappa_0. \]

This inequality is not possible for all time. As \( t \to t^* \equiv (\kappa_0 \psi_0^\alpha)^{-1}, \) we conclude that \( \psi(t) \to \infty. \) That is, \( \psi(t) \) blows-up at \( t^* \), and then the solution is not global.

In order to complete the proof for the blow-up, we proceed to show that the condition (6) in Lemma 2.2 is true. To this end, we notice that \( \mathcal{M}(s) \) attains an absolute minimum at \( s_0 \equiv \left( \frac{2}{\gamma} \right)^\alpha, \) that is

\[ \mathcal{M}(s) \geq \mathcal{M}(s_0) = \alpha^2 \psi_0^{-(1+2\alpha)} \left( \phi_0 - \mathcal{N}(\gamma) \right), \]

where

\[ \mathcal{N}(\gamma) \equiv \frac{2\gamma}{1+2\alpha} + \frac{\beta}{\alpha(1+2\alpha)} \left( \frac{\beta}{\gamma} \right)^{2\alpha} \psi_0^{1+2\alpha}. \]

We define \( \kappa_0^2 \equiv \mathcal{M}(s_0) \). Then, \( \kappa_0^2 > 0 \) holds if and only if

\[ \mathcal{N}(\gamma) < \phi_0. \quad (11) \]

The function \( \mathcal{N}(\gamma) \) is such that

\[ \mathcal{N}(\gamma) \to \infty \quad \text{as either} \quad \gamma \to 0 \quad \text{or} \quad \gamma \to \infty, \]

and

\[ \left| \mathcal{N}(\gamma) - \frac{2\gamma}{1+2\alpha} \right| \to 0 \quad \text{as} \quad \gamma \to \infty. \]

Furthermore, \( \mathcal{N}(\gamma) \) has an absolute minimum at \( \gamma_0 \equiv \beta \psi_0, \) that is

\[ \mathcal{N}(\gamma) \geq \mathcal{N}(\gamma_0) = \frac{\beta}{\alpha} \psi_0, \quad \gamma > 0. \]
From (5) and Remark 1, \( \phi_0 > \frac{\beta}{\alpha} \psi_0 \). This condition implies that there exist exactly two different roots \( \gamma^* \), denoted by \( a \) and \( b \), of
\[
\mathcal{N}(\gamma^*) = \phi_0,
\]
such that
\[
0 < a < \gamma_0 < b < \frac{1 + 2 \alpha}{2} \psi_0.
\]
Then (11) holds equivalently (6) is satisfied if and only if
\[
\gamma \in \mathcal{I} \equiv (a, b).
\]
Let \( \gamma \in \mathcal{I} \), which will be called the blow-up interval, if \( \gamma \neq \gamma_0 \) then
\[
\frac{\beta}{\alpha} \psi_0 < \mathcal{N}(\gamma) < \phi_0.
\]
The strict monotonicity of \( \mathcal{N}(\gamma) \) for \( \gamma < \gamma_0 \) and \( \gamma > \gamma_0 \), implies that, for fixed \( \psi_0 \), the blow-up interval \( \mathcal{I} \) grows as \( \dot{\psi}_0 \) grows. That is,
\[
\lim_{\dot{\psi}_0 \to \infty} \left| \frac{1 + 2 \alpha}{2} \phi_0 - b \right| = 0 = \lim_{\dot{\psi}_0 \to \infty} a, \quad \lim_{\dot{\psi}_0 \to \infty} \phi_0 = \infty.
\]
Since \( \mathcal{N}(\gamma) \geq \mathcal{N}(\gamma_0) \) and from the definition of \( \kappa_0 \), we verify that the initial data are compatible with the conclusion of Lemma 2.2, that is, at \( t = 0 \) the following is satisfied
\[
-\dot{G}_0 = -\frac{d}{dt} G(0) = \alpha \psi_0^{-(\alpha + 1)} \left( \psi_0 - \frac{\delta}{\alpha} \psi_0 \right)
\]
\[
= \alpha \psi_0^{-\frac{1 + 2 \alpha}{\alpha}} (\phi_0 - \mathcal{N}(\gamma_0))^{\frac{1}{2}} \geq \alpha \psi_0^{-\frac{1 + 2 \alpha}{\alpha}} (\phi_0 - \mathcal{N}(\gamma))^{\frac{1}{2}} = M^{\frac{1}{2}}(s_0) \equiv \kappa_0.
\]
The same lower bound of \( \mathcal{N}(\gamma) \) implies the following lower bound of \( t^* \),
\[
t^* \equiv (\kappa_0 \psi_0)^{-1} = \alpha^{-1} \psi_0^{\frac{1}{\alpha}} (\phi_0 - \mathcal{N}(\gamma))^{-\frac{1}{2}} \geq \alpha^{-1} \psi_0^{\frac{1}{\alpha}} \left( \phi_0 - \frac{\beta}{\alpha} \psi_0 \right)^{-\frac{1}{2}} \geq \alpha^{-1} \psi_0^{\frac{1}{\alpha}} \left( \frac{\psi_0}{\psi_0^{\frac{1}{2}}} - \frac{\delta}{\alpha} \psi_0^{\frac{1}{2}} \right)^{-1} = \left( \alpha \psi_0^{\frac{1}{2}} - \delta \right)^{-1}.
\]
We observe that this lower bound is finite because \( \alpha \psi_0^{\frac{1}{2}} - \delta > 0 \) is just the condition (5). The properties of the blow-up time \( t^* \) and the limit value as \( \psi_0 \) goes to infinity follow from its definition which depends on \( \phi_0 \).

We shall employ the functions \( \sigma_\nu \) and \( \mu_\lambda \), defined in (3), in order to find the values of \( a \) and \( b \), respectively. They are defined as the roots of \( \mathcal{N}(\gamma) = \phi_0 \). The equation to find \( a \) is the following
\[
\mathcal{N}(\sigma_\nu) = \phi_0, \quad \sigma_\nu \equiv \sigma_\nu(0).
\]
This equations is satisfied if and only if,
\[
m(\nu) \equiv \frac{\nu}{\alpha} + \left( \frac{2}{(1 + 2 \alpha)^{\frac{1 + 2 \alpha}{2 \alpha}}} \right) \frac{1}{\nu^{\frac{2}{\alpha}}} = \frac{\phi_0}{\beta \psi_0}.
\]
From the definition of \( \phi_0 \),
\[
\frac{\phi_0}{\beta \psi_0} > \frac{1}{\alpha}.
\]
Furthermore,

\[ m(\nu) \to \infty, \quad \text{as either } \nu \to 0 \quad \text{or} \quad \nu \to \infty, \]

and

\[ m(\nu) \geq m(1 + 2\alpha) = \frac{1}{\alpha}, \quad \nu > 0. \]

Consequently, equation (13) equivalently (12), has two roots and only one is such that \( \nu^* > 1 + 2\alpha. \) Hence,

\[ a = \sigma_{\nu^*} = \frac{\beta \psi_0}{(1 + 2\alpha \nu^*)^{\frac{1}{\alpha}}} < \frac{\beta \psi_0}{(1 + 2\alpha)^{\frac{1}{\alpha}}}, \]

and, for fixed \( \psi_0, \) we get \( \lim_{\nu^* \to \infty} \nu^* = \infty. \)

Now, we shall calculate \( b. \) To this end, we have to solve the equation

\[ \mathcal{N}(\mu \lambda) = \phi_0, \quad \mu \lambda \equiv \mu \lambda(0). \] (14)

For \( \lambda \in [\lambda_0, 1], \lambda_0 = 2\alpha(1 + 2\alpha)^{-1}. \) The equation (14) holds if and only if

\[ n(\lambda) \equiv \frac{1}{1 + 2\alpha} \left( \frac{\lambda \beta \psi_0}{\alpha \psi_0} \right)^{1 + 2\alpha} = \lambda - \frac{2\alpha}{1 + 2\alpha} \equiv l(\lambda). \] (15)

We observe that \( n(\lambda) \) and \( l(\lambda), \) are strictly monotone increasing and, from to the definition of \( \phi_0, \)

\[ n(\lambda_0) > l(\lambda_0) = 0, \quad n(1) < l(1) = \frac{1}{1 + 2\alpha}. \]

Consequently, there exists one and only one number \( \lambda^* \in (2\alpha(1 + 2\alpha)^{-1}, 1) \) where \( n(\lambda^*) = l(\lambda^*). \) That is, only one root \( \lambda^* \) of equation (15), and consequently (14) is satisfied. Hence,

\[ b = \mu_{\lambda^*} = \frac{\alpha \phi_0}{\lambda^*}, \]

and, for fixed \( \psi_0, \) we get

\[ \lim_{\psi_0 \to \infty} \lambda^* = \frac{2\alpha}{1 + 2\alpha}. \]

Next, we will show the following lower bound for \( b, \)

\[ b > \frac{1 + 2\alpha}{2} \phi_0 - \frac{1}{2\alpha} \beta \psi_0 = \frac{1 + 2\alpha}{2\psi_0} \left( \psi_0 - \frac{\delta}{\alpha} \psi_0 \right)^2 + \beta \psi_0, \]

To this end, we prove first the following inequality

\[ \frac{\alpha}{\psi_0 \lambda^*} \left( \left( \psi_0 - \frac{\delta}{\alpha} \psi_0 \right)^2 + \frac{\beta}{\alpha} \psi_0^2 \right) = \frac{\alpha \phi_0}{\lambda^*} = b > \frac{1 + 2\alpha}{2\psi_0} \left( \psi_0 - \frac{\delta}{\alpha} \psi_0 \right)^2. \]

If \( s \equiv \left( \psi_0 - \frac{\delta}{\alpha} \psi_0 \right)^2 > 0, \) this inequality is equivalent to

\[ q(s) \equiv \left( \lambda^* \left( \frac{1 + 2\alpha}{2\alpha} \right) - 1 \right) \left( \psi_0 - \frac{\delta}{\alpha} \psi_0 \right)^2 < \frac{\beta}{\alpha} \psi_0^2, \]
where $\frac{2\alpha}{1+2\alpha} < \lambda^* < 1$. If we take into account that $\lambda^*$ is a function of $s$, we obtain the following limits.

$$\lim_{s \to \infty} \lambda^* = \frac{2\alpha}{1 + 2\alpha},$$

$$\lim_{s \to \infty} q(s) = \frac{1}{2\alpha} \lim_{s \to \infty} s \left( \lambda^* \frac{\beta \psi_0}{\alpha \phi_0} \right)^{1+2\alpha} = \frac{1}{2\alpha} \lim_{s \to \infty} s \left( \lambda^* \frac{\beta \psi_0^2}{\alpha \psi_0^2 + s} \right)^{1+2\alpha} = 0,$$

$$\lim_{s \to 0} \lambda^* = 1, \quad \lim_{s \to 0} q(s) = 0.$$

Necessarily, there exists $s^* \in (0, \infty)$, such that $q(s^*) = \max_{s \in (0, \infty)} q(s)$. We can find after some calculations that

$$s^* = \psi_0^2 \left( \frac{2\beta(1 - \lambda^*)}{(1 + 2\lambda)\lambda^* - 2\alpha(1 - \lambda^*)} \right),$$

$$q(s^*) = \frac{\beta}{\alpha} \psi_0^2 \left( \frac{(1 + 2\alpha)\lambda^* - 2\alpha)(1 - \lambda^*)}{(1 + 2\alpha)\lambda^* - 2\alpha(1 - \lambda^*)} \right),$$

and then

$$q(s) \leq q(s^*) = \frac{\beta}{\alpha} \psi_0^2 \eta(\lambda^*),$$

where

$$\eta(\lambda^*) \equiv \frac{(1 + 2\alpha)\lambda^* - 2\alpha)(1 - \lambda^*)}{(1 + 2\alpha)\lambda^* - 2\alpha(1 - \lambda^*)} < 1.$$

We observe that, if $\psi_0$ is fixed, then

$$\lim_{\psi_0 \to \infty} \lambda^* = \frac{2\alpha}{1 + 2\alpha},$$

and hence

$$\lim_{\psi_0 \to \infty} \eta(\lambda^*) = 0,$$

which implies

$$\lim_{\psi_0 \to \infty} q(s) = 0,$$

according to the limit calculated before.

In order to get the lower bound for $b$, we notice that the inequality $q(s) \leq q(s^*)$ is equivalent to

$$b > \left( \frac{1 + 2\alpha}{2\psi_0} \right) \left( \psi_0 - \frac{\delta}{\alpha} \psi_0 \right)^2 + \chi(\lambda^*)\beta \psi_0 = \frac{1 + 2\alpha}{2} \phi_0 - \left( \frac{1 + 2\alpha}{2}\phi_0 - \chi(\lambda^*) \right) \beta \psi_0,$$

where

$$\frac{1 + 2\alpha}{2\alpha} > \chi(\lambda^*) \equiv \frac{\lambda^*(1 + 2\alpha)}{\lambda^*(1 + 2\alpha) - 2\alpha(1 - \lambda^*)} > 1,$$

since

$$\frac{2\alpha}{1 + 2\alpha} < \lambda^* < 1.$$

And the following limit follows from the corresponding limit for $\lambda^*$

$$\lim_{\psi_0 \to \infty} \chi(\lambda^*) = \frac{1 + 2\alpha}{2\alpha}.$$
The properties of $|I|$ follow from its dependence on $\phi_0$. Finally, from the bounds for $a, b$, we have that

$$0 < \frac{1 + 2\alpha}{2} \phi_0 - |I| < \left(1 + \frac{2\alpha}{2\alpha} - \chi(\lambda^*) + \frac{1}{((1 + 2\alpha)\nu^*)^\frac{1}{\nu}}\right) \beta \psi_0.$$

\[\square\]

**Proof.** (of Corollary 1.) Since $\dot{\psi}_0 \to \infty \Rightarrow a \to 0$ and $b \to \infty$ then, for every $\xi > 0$ there exists $\eta > 0$, such that $\dot{\psi}_0 > \eta \Rightarrow \xi \in I = (a, b)$. Hence, any solution with $\gamma \equiv 2rE_0 = \xi$ blows-up in finite time. \[\square\]

**Remark 2.** The positive invariance of the set $V$ in Lemma 2.2 holds if (6) is fulfilled and, according to the proof of Theorem 2.1, this is characterized by the blow-up condition in terms of the initial energy: $\gamma \equiv 2rE_0 \in I$. Furthermore, the existence of the blow-up interval $I \neq \emptyset$ is a consequence of the condition on the initial data (5). With these two conditions the blow-up of $\psi(t)$ at $t^*$ follows and the solution of (P) is not global. In a previous article [5] a similar result was proved for the problem (P) with $\delta$ small and $g(t) = 0$. In that paper the positive invariance of the set $V$ was shown implicitly within the proof of the main result. Here, we make explicit such a set and its property.

**Remark 3.** From ($H3$),

$$r > 1 + \frac{1}{\sqrt{1 - \frac{1 - l}{\hat{c}}}} > 2.$$  

The blow-up property is reached for a larger set of values of $r$ as long as the quotient $0 < \frac{1 - l}{\hat{c}} < 1$ decreases. The blow-up is obtained for a larger set of relaxation functions as long as $\hat{c}$ is larger. Furthermore, if $B = A$, then $\hat{c} = 1$, the set of relaxation functions is only restricted by hypothesis ($H2$) and

$$r > 1 + \frac{1}{\sqrt{l}} > 2,$$

that is, the set of values of $r$ to get blow-up is larger as long as $l$ is close to one, that is, almost without the memory term. This is the condition on $r$ that appears in various articles that study qualitative properties of particular equations of the problem (P) with $A = B$, see [8, 15, 19, 20, 22, 23].

In the absence of the memory term, $g(t) = 0$, we have that $l = 1$ and the embedding constant becomes $\hat{c} = 1$ and $r > 2$. This problem has been studied in [5] and references therein.

**Remark 4.** The sufficient conditions on the initial data in Theorem 2.1 are, see Remark 1,

$$\mathcal{P}(u_0, u_1) > \frac{2\delta}{r - 2} \|u_0\|_{V^p}^2 > 0.$$  

Furthermore, if $\delta = 0$, these become

$$\mathcal{P}(u_0, u_1) > 0, \quad \|u_0\|_{V^p}^2 > 0.$$  

Several authors have proved the blow-up property for particular undamped problems under a set of conditions that include these. Our main result for the damped problem with memory and any positive value of the initial energy $E_0$ is new in the literature and generalizes the one presented in [5] that analyzes the case without viscoelastic term, $g(t) = 0$, and small damping coefficient $\delta$. 
Remark 5. The length of the blow-up interval \( I \) decreases as \( \delta \) increases. Therefore, as the damping coefficient grows, then the set of initial energies that produce global non existence becomes smaller. Finally, a notable property that should be highlighted is that the blow-up time approaches zero while \( \psi_0 \) goes to infinity.

3. Applications. For concrete problems of the type \((P)\) some authors have studied the blow-up for arbitrary positive values of the initial energy, see [7, 8, 12, 13, 15, 19, 20, 27]-[34]. We next present and comment the application of our main result to some concrete problems, such as the wave, Kirchhoff, and Petrovsky equations with memory. One contribution of this work is that the upper bound, \( b \), of \( 2rE_0 \) given by Theorem 2.1 is larger than the ones showed in previous articles. This implies that given any positive value of \( E_0 \), the set of initial data that produce the blow-up property given by Theorem 2.1 is larger than the ones published in the literature.

3.1. Viscoelastic wave and Petrovsky equations.

\[
\begin{align*}
\text{(VW)} & \quad \begin{cases}
  u_t(t) - \Delta u(t) + \int_0^t g(t - \tau) \Delta u(\tau) \, d\tau + \delta u(t) = f(u(t)), & x \in \Omega, \quad t > 0, \\
  u(0,x) = u_0(x), & x \in \Omega, \\
  u(x,t) = 0, & x \in \partial \Omega, \quad t > 0,
\end{cases} \\
\text{(VP)} & \quad \begin{cases}
  u_t(t) + \Delta^2 u(t) - \int_0^t g(t - \tau) \Delta^2 u(\tau) \, d\tau + \delta u(t) = f(u(t)), & x \in \Omega, \quad t > 0, \\
  u(0,x) = u_0(x), & x \in \Omega, \\
  u(x,t) = 0 = \partial_\nu u(x,t), & x \in \partial \Omega, \quad t > 0,
\end{cases} \\
\text{(VW*)} & \quad \begin{cases}
  u_t(t) - \Delta u(t) - \int_0^t g(t - \tau) u(\tau) \, d\tau + \delta u(t) = f(u(t)), & x \in \Omega, \quad t > 0, \\
  u(0,x) = u_0(x), & x \in \Omega, \\
  u(x,t) = 0 = \partial_\nu u(x,t), & x \in \partial \Omega, \quad t > 0,
\end{cases}
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^N \), is a bounded domain with smooth boundary and \( \nu \) is the normal vector to \( \partial \Omega \).

In \((VW)\) we have \( A = B = -\Delta \), defined on \( V_A = V_B = H^2_0(\Omega) \). In \((VP)\), \( A = B = \Delta^2 \), defined on \( V_A = V_B = H^3_0(\Omega) \). In both problems, \( P = I = \text{identity operator and } H = V_P = L^2(\Omega) \). Then, \( c = 1 \) and \( r \) satisfies (16). In \((VW*)\), \( A = -\Delta \), defined on \( V_A = H^2_0(\Omega) \), \( B = P = I = \text{identity operator and } H = V_P = V_B = L^2(\Omega) \). Then, \( c = 1 \), \( \hat{c} \) is the best Sobolev’s constant of \( L^2(\Omega) \subset H^1_0(\Omega) \). The relaxation function and the source term satisfy \((H2)\) and \((H3)\), respectively.

For any of these equations, our main result implies that the solution blows-up if (5) holds for any positive value of the initial energy such that \( 2rE_0 \in I \). And, the size of the blow-up interval \( I \) is as big as we wish if \( (u_0, u_1) > 0 \) is large enough.

Blow-up has been studied for the problem \((VW)\) with nonlinear damping in [22, 23]. In [22] this result was showed for negative initial energy and in [23] the blow-up was proved by means of the potential well method if \( E_0 < d \) and

\[
I(u_0) \equiv \| \nabla u_0 \|^2_{V_A} - (f(u_0), u_0) < 0,
\]

where

\[
d \equiv \inf_{u \in \mathcal{N}} \hat{J}(u) \equiv \inf_{u \in \mathcal{N}} \left\{ \frac{1}{2} \| \nabla u \|^2_{V_A} - F(u) \right\} = \inf_{u \in \mathcal{N}} \{ (f(u), u) - 2F(u) \},
\]

is the potential depth or the mountain pass level of the corresponding elliptic problem without memory, and

\[
\mathcal{N} \equiv \{ u \neq 0 : I(u) = 0 \},
\]
is the corresponding Nehari manifold. By means of this number, the potential well method has been used to characterize the qualitative behavior for solutions of some equations of the type (P), whenever $E_0 < d$. Indeed, typically a solution is global if either $I(u_0) > 0$ or $u_0 = 0$, and blows up in finite time if $I(u_0) < 0$.

For small positive energy and under similar conditions to those in [23], the blow-up property was showed in [30] for the undamped equation, that is with $\delta = 0$. In [15] the authors reached the same conclusions for the equation with strong linear damping following the same method and under the same hypotheses as in [30].

The problem with large positive values of the initial energy was analyzed in [8, 15, 27, 29, 30]. Indeed, under the assumptions

$$\mathcal{P}(u_0, u_1) > 0, \ I(u_0) < 0, \ E_0 < C \|u_0\|_{H^P}^2,$$  

(17)

where $C > 0$ is a constant independent of the initial data, the blow-up was proved in [29]. See also [15] for a similar result. For the equation without damping, $\delta = 0$, the same conclusion was reached in [30] under the conditions

$$\mathcal{P}(u_0, u_1) > 0, \ E_0 < C_1 \mathcal{P}(u_0, u_1) - C_2 \|u_0\|_{H^P}^2,$$  

(18)

for some constants $C_j > 0, \ j = 1, 2$. Moreover, for a nonlinear damping the blow-up was concluded in [27] if

$$\mathcal{P}(u_0, u_1) > 0, \ E_0 < C \mathcal{P}(u_0, u_1),$$  

(19)

for some $C > 0$. These upper bounds of $E_0$ are smaller than the one given in Theorem 1. Furthermore, the condition $I(u_0) < 0$ is not required in Theorem 1. In our previous work [5] we studied the problem (P) without memory, that is with $g(t) = 0$, we made a discussion of this point and we showed that the sign of $I(u_0)$ has no relevance for the blow-up of solutions when $E_0 \geq d$. On the other hand, by the potential well method, the negative sign of $I(u_0)$ is essential for $E_0 < d$.

In [8] the blow-up for (WA) is proved under the condition,

$$0 < E_0 < \frac{1}{8} \frac{(2\mathcal{P}(u_0, u_1) - \frac{\delta}{\alpha} \|u_0\|_{H^P}^2)^2}{\|u_0\|_{H^P}^2}.$$  

We observe that the upper bound $b$ of $E_0$ given in Theorem 2.1 is larger than the one in [8], since the lower bound of $b$ proved in our main result is such that

$$b > \alpha \phi_0 + \frac{1}{2} \left( \frac{\psi_0}{\psi_0^2} - \frac{\delta}{\alpha} \frac{1}{\alpha} \right)^2 > \frac{1}{2} \frac{(2\mathcal{P}(u_0, u_1) - \frac{\delta}{\alpha} \|u_0\|_{H^P}^2)^2}{\|u_0\|_{H^P}^2}.$$  

By the potential well method the blow-up of solutions for the problem (VP) with $\delta = 0$ and $E_0 < d$ was showed in [33] and for any positive value of $E_0$ if the conditions given in (17) hold. See also [28] for the same problem and similar conditions and conclusions. In [19], under the assumption (19), the blow-up is showed for the problem (VP) with a nonlinear damping. The same result was proved in [20] if the following holds,

$$\mathcal{P}(u_0, u_1) > 0, \ E_0 < C \left( \mathcal{P}(u_0, u_1) + \|u_0\|_{H^P}^2 \right),$$  

(20)

where $C > 0$. As before, this upper bound for $E_0$ is smaller than the one given in Theorem 1.

To the best author’s knowledge the last example (VW)* has not been reported in the literature. There, $A \neq B$, $V_A \subsetneq V_B$. A consequence of this is that the embedding constant $\tilde{c}$, the number $r$ that measures the nonlinearity of the source term, and the number $l$ that is a measure of the relaxation function, must satisfy
the hypothesis \((H3)\). Furthermore, if the initial data satisfy \((5)\) then the blow-up of the solution is guaranteed whenever the initial energy is in the blow-up interval, that is \(2rE_0 \in \mathcal{I}\).

3.2. Dispersive wave and Petrovsky equations.

\[
\begin{align*}
\text{(DW)} \quad & \begin{cases}
    u_{tt}(t) - \Delta u_{tt}(t) - \Delta u(t) + \int_0^t g(t - \tau)\Delta u(\tau) \, d\tau \\
    + \delta u_t(t) - \delta \Delta u_t(t) = f(u(t)), & x \in \Omega, \ t > 0, \\
    u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), & x \in \Omega, \\
    u(x, t) = 0, & x \in \partial \Omega, \ t > 0,
\end{cases} \\
\text{(DP)} \quad & \begin{cases}
    u_{tt}(t) - \Delta u_{tt}(t) + \Delta^2 u(t) - \int_0^t g(t - \tau)\Delta^2 u(\tau) \, d\tau \\
    + \delta u_t(t) - \delta \Delta u_t(t) = f(u(t)), & x \in \Omega, \ t > 0, \\
    u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), & x \in \Omega, \\
    u(x, t) = 0 = \partial_\nu u(x, t), & x \in \partial \Omega, \ t > 0,
\end{cases}
\end{align*}
\]

where \(\Omega \subset \mathbb{R}^N\), is a bounded domain with smooth boundary and \(\nu\) is the normal vector to \(\partial \Omega\).

In \((\text{DW})\) we have \(A = B = -\Delta\) defined on \(V_A = V_B = H_0^1(\Omega)\). In \((\text{DP})\) we have \(A = B = \Delta^2\) defined on \(V_A = V_B = H_0^2(\Omega)\). In both problems \(P = I - \Delta\) defined on \(V_P = H_0^1(\Omega)\). Furthermore, \(H = L^2(\Omega)\), \(\hat{c} = 1\) and \(r\) satisfies \((16)\). The relaxation function and the source term satisfy \((H2)\) and \((H3)\), respectively.

Applying Theorem 2.1 to any of these problems, we conclude that any solution that satisfies \((5)\) blows-up if the initial energy is such that \(2rE_0 \in \mathcal{I}\), and Corollary 1 implies that if \((u_0, u_1) > 0\) is large enough, there exist initial data that imply blow-up.

Few authors have analyzed these problems. In [32] the problem \((\text{DW})\) was studied, where global and non-global solutions with \(E_0 < d\) are analyzed by means of the potential well method and for arbitrary positive initial energy the blow-up property was showed if \((17)\) holds. In [13] the blow-up property of the problem \((\text{DP})\) was proved if \(E_0 < d\) and \(I(u_0) < 0\). Hence, Theorem (2.1) give us a better understanding of the dynamics of these problems.

3.3. Kirchhoff equation.

\[
\begin{align*}
\text{(KE)} \quad & \begin{cases}
    u_{tt} - \varphi(\|u\|_2^2)\Delta u + \int_0^t g(t - \tau)\Delta u(\tau) \, d\tau + \delta u_t = g(u), & x \in \Omega, \ t > 0, \\
    u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), & x \in \Omega, \\
    u(x, t) = 0, & x \in \partial \Omega, \ t > 0,
\end{cases}
\end{align*}
\]

where \(\Omega \subset \mathbb{R}^N\), is a bounded domain with smooth boundary and

\[
\varphi(\|u\|_2^2) = 1 + \|u\|_2^{2p}, \quad p \geq 1.
\]

The nonlinear source \(g\) is such that

\[
(g(u), u) - \rho G(u) \geq 0, \quad \rho > 2,
\]

where \(G\) is the potential of \(g\). Then,

\[
f(u) = g(u) + \|\nabla u\|_2^{2p}\Delta u,
\]
has the potential
\[ F(u) = G(u) - \frac{1}{2(p+1)}\|\nabla u\|^{2(p+1)}_2. \]
Hence, \((H1)\) is satisfied if
\[ \rho \geq r \geq 2(p+1). \]
That is, the nonlinearity of the source term \(g\) is stronger than the one of \(\varphi\).
This is an important condition to get the blow-up. See \([5]\) and references therein
where the same problem without the viscoelastic term is studied and this example
is analyzed. When the source term \(g\) is dominated by the nonlinearity of \(\varphi\) then
all the solutions are global and are attracted by the set of all the equilibria, that is
solutions of the corresponding elliptic equation.
In our framework we have \(A = B = -\Delta\), defined on \(V_A = V_B = H^1_0(\Omega)\).
Also, \(P = I = \) identity operator and \(H = V_P = L^2(\Omega)\). Hence, \(c = 1\) and \(r\) satisfies,
besides the inequality above, \((16)\). Finally, the relaxation function and the source
term satisfy \((H2)\) and \((H3)\), respectively.
This problem has been studied in \([7, 12, 31, 34]\) within the framework of strong
solutions. That is, the initial data are in the following space
\[ (u_0, u_1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega) \subset H \equiv H^1_0(\Omega) \times L^2(\Omega). \]
In \([31]\) the blow-up property the problem \((KE)\) with nonlinear damping is ana-
lyzed for \(E_0 < d\) and \(I(u_0) < 0\). For high initial energies the blow-up of solutions
is proved in \([12]\) and \([7]\) under the conditions given in \((17)\), and the same property
is showed in \([34]\) if \((19)\) is satisfied. Consequently, the same remarks made in
the previous examples regarding these conditions and our contribution apply to this
problem.

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