Reconstruction of the distribution of sensitive data under free-will privacy

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Abstract—The local privacy mechanisms, such as k-RR, RAPPOR, and the geo-indistinguishability ones, have become quite popular thanks to the fact that the obfuscation can be effectuated at the users end, thus avoiding the need of a trusted third party. Another important advantage is that each data point is sanitized independently from the others, and therefore different users may use different levels of obfuscation depending on their privacy requirements, or they may even use entirely different mechanisms depending on the services they are trading their data for. A challenging requirement in this setting is to construct the original distribution on the users sensitive data from their noisy versions. Existing techniques can only estimate that distribution separately on each obfuscation schema and corresponding noisy data subset. But the smaller are the subsets, the more imprecise the estimations are. In this paper we study how to avoid the subsets-fractioning problem when combining local privacy mechanisms, thus recovering an optimal utility. We focus on the estimation of the original distribution, and on the two main methods to estimate it: the matrix-inversion method and the iterative Bayes update. We consider various cases of combination of local privacy mechanisms, and compare the flexibility and the performance of the two methods.

1. Introduction

Over recent years, there is a growing demand for analyzing amounts of data that are collected from a large number of users. To allow this analysis, the users need to release their data which may be sensitive and therefore put their privacy at risk. The local privacy model has been presented in the literature to solve this problem [1], [2], [3], [4]. More precisely, every user applies a privacy mechanism that obfuscates his original datum to produce a noisy version of it and then sends the latter to the data collector (instead of the original datum). The problem now is estimate statistical properties of the original users’ data from their noisy releases. In particular, we assume that there is a probability distribution on the values of a sensitive attribute and we want to estimate this distribution from the users’ noisy releases of this attribute. Existing methods to solve this problem assume that all users apply the same privacy mechanism to sanitize their original data. Based on this assumption, the authors of [3], [4] proposed the matrix inversion (INV) method in conjunction with the k-RR mechanism. A more sophisticated method is the Iterative Bayesian Update (IBU) which was proposed in [1], [2]. This method iteratively computes the maximum likelihood estimate (MLE) for the required probability distribution over the alphabet of the sensitive attribute.

While the above methods work in the setting that all users apply the same privacy mechanism to their original data, we aim in this paper to generalize this setting and assume instead that every user applies his arbitrary mechanism. Thus we consider mixtures of different mechanisms that may be used. In particular we consider k-RR [5] and RAPPOR [6] which satisfy local differential privacy. In addition we consider the geometric mechanisms [7], [8] which satisfy geo-indistinguishability. We also consider Shokri’s mechanism [9] that satisfies a particular notion of location privacy. In order to estimate the original distribution we consider GIBU that was introduced in [10] for handling mixtures of mechanisms. We show that this method is highly inefficient in its original form, and therefore we provide a more efficient algorithm that produces the same result of GIBU. In addition we consider various ways of reusing INV and IBU in our general setting that consists of different mechanisms.

We provide an experimental comparisons between the above estimation methods using both synthetic and real data. The results of this comparisons are summarized in Table 1 which shows that estimation accuracy of GIBU is superior relative to the other methods.

1.1. Contributions

• Starting with the situation that the mechanisms of the users are various, but restricted to have the same signature, we extend the classical matrix inversion method [2], known also as the empirical estimator [4], to work in this setting instead of operating under the assumption that all mechanisms are identical. We call this method \[ \text{INV}_M \].

• When all users are restricted to apply only k-RR mechanisms, but with arbitrary levels of privacy, we prove that \[ \text{INV}_M \] is consistent in the sense that its estimated distribution converges in probability to the real distribution. Furthermore, we derive an upper bound on the \( \ell_2^2 \) estimation error.

• Similarly we extend the local privacy model under Google’s RAPPOR mechanisms [6] to allow every user to set his own privacy level. Given this situation we extend the standard estimation procedure under RAPPOR. We also derive an upper bound on its \( \ell_2^2 \) error showing that the new estimator is also consistent.

• Abstracting away from the above restrictions, we consider the most general model of local privacy in which the users apply their own mechanisms, that may vary both in signatures and in privacy guarantees. In this ‘privacy-liberal’ scenario we provide a compositional and scalable algorithm GIBU that accommodates all these various mechanisms in the estimation process.

• We experimentally show that the estimation performance of GIBU is better compared to \[ \text{INV}_M \].
| TABLE 1. PERFORMANCES OF CONSTRUCTION METHODS |
|-----------------------------------------------|
| Method             | [INV]_R | [IBU]_R | [INV]_M | [IBU]_M | GIBU |
|---------------------|---------|---------|---------|---------|------|
| k-RR + k-RR         | Bad (Fig. 3) | Bad (Fig. 3) | Excellent (Fig. 7) | Good (Fig. 10) | Excellent |
| Geometric + Geometric | Bad (Fig. 5) | Bad (Fig. 5) | Bad (Fig. 8) | Bad (Fig. 11) | Excellent |
| k-RR + Geometric    | Bad (Fig. 2(b)) | Bad (Fig. 4) | Bad (Fig. 9) | Bad (Fig. 12) | Excellent |
| Shokri + Shokri     | NA | Bad (Fig. 6) | NA | Bad (Fig. 13) | Excellent |

| TABLE 2. CONSTRUCTION METHODS UNDER RAPPOR MECHANISMS |
|-----------------------------------------------|
| Method             | [RAP]_R | [RAP]_M | GIBU |
|---------------------|---------|---------|------|
| High privacy        | Bad (Fig. 2(a)) | Excellent (Fig. 14(a)) | Excellent |
| Low privacy         | Bad (Fig. 2(b)) | Good (Fig. 14(b)) | Excellent |

| TABLE 3. NOTATIONS |
|-------------------|
| X                 | an alphabet of secrets of users. |
| θ[n]              | the original probability distribution on X. |
| X[i]              | random vector describing the secret of user i. |
| A[i]              | the privacy mechanism applied by user i. |
| Z'                | random vector describing the (noisy) observable of A[i]. |
| Z                 | the alphabet of Z', i.e., possible observables of A'. |
| m[i]              | marginal distribution of Z' over the alphabet Z'. |
| A[1],...,A[n]     | the average of mechanisms A[i] for i ∈ [n]. |
| q[i]              | empirical distribution for n-size noisy data. |
| θ[i]              | estimated value for the original distribution θ. |
| RAP               | the estimator under a RAPPOR mechanism |
| INV               | matrix inversion estimator |
| IBU               | iterative Bayesian update |
| [e]_R             | combining the results of estimator e |
| [e]_M             | applying estimator e to a compound mechanism. |
| GIBU              | generalized iterative Bayesian update |

- Since [INV]_M usually requires a post-processing step to obtain a valid distribution, we describe an additional method [IBU]_M that uses the average mechanism to yield an estimated distribution. However, we show that this method in some cases, e.g. with Shokri’s mechanism [9], may not converge (with large number of samples) to the real distribution.
- We compare [IBU]_M to GIBU using different mechanisms, e.g. Geometric, k-RR, and Shokri’s showing that GIBU is consistently superior.

2. Preliminaries

2.1. k-ary randomized response mechanisms

The k-ary randomized response mechanism, abbreviated as k-RR, obfuscates every datum from the alphabet X, with |X| = k, to produce a noisy observable from the same alphabet, i.e., Z = X. This mechanism was originally introduced by Warner [11] for binary alphabets, and was later extended by [5] to arbitrary k-size alphabets. Given a privacy parameter ε > 0, this mechanism applied to a datum x produces an observable z with probability

\[ P(z|x) = \frac{1}{k} + e^{-\epsilon} \left\{ \begin{array}{ll} e^\epsilon & \text{if } z = x \\ 1 & \text{if } z \neq x \end{array} \right. \]  

(1)

It follows from the definition (1) that the k-RR mechanism satisfies ε-local differential privacy.

2.2. Geometric mechanisms

Suppose that the alphabet of secrets X is a bounded linear range of integers between r_1 and r_2 inclusive, where 1 < r_2 < ∞. Then the truncated geometric mechanism [7], with parameter ε, maps every x ∈ X to an integer z in the same alphabet X with probability

\[ P(z|x) = c_z e^{-|z-x|}, \]  

(2)

where \( c_z = \frac{1}{1 + e^{-\epsilon}} \left\{ \begin{array}{ll} 1 - e^{-\epsilon} & \text{if } z \in [r_1, r_2] \\ 0 & \text{otherwise} \end{array} \right. \)  

(3)

For the planar alphabet in which the data points are the cells of a grid, we will use a planar variant of the truncated geometric mechanism which we describe in the following subsection.

2.3. Planar geometric mechanisms

Suppose that the planar space is discretized by an infinite grid of squared cells, where the side length of each cell is s. Let G be the set of centers of these cells. Then every element of G is indexed by its coordinates (i, j) ∈ Z^2. For any X ⊆ G, a planar geometric mechanism (parametrized by ε, s) reports a point z ∈ G from a real location x ∈ X according to the probability

\[ P(z|x) = \lambda e^{-\epsilon d(x,z)} \]  

where \( \lambda = 1/\sum_{(i,j) \in Z^2} e^{-\epsilon s \sqrt{i^2+j^2}} \)  

and d(·, ·) is the planar (i.e. Euclidean) distance. If X is finite we can define a truncated version of the above geometric mechanism. Basically it is obtained by drawing points in G according to the above distribution and then remapping each of them to its nearest point in X.

Since the alphabet X in this paper are bounded, we will ignore the term ‘truncated’ when we use the geometric mechanisms, and refer to them directly as linear and planar geometric mechanisms. We remark that geometric mechanisms are known to satisfy ε-geodistinguishability [12], in which the distinguishability between two points x, x’ in the space of secrets X is bounded by εd(x, x’), where d(·, ·) is the Euclidean distance. This notion differs from the ε-local differential privacy [13], [5] because in the latter notion the distinguishability between any two points is bounded by the fixed ε.
3. Local privacy model

![Diagram of local privacy model](image)

We consider $n$ users, each of them is labelled by $i \in [n]$. The sensitive datum of user $i$ is described by the random variable $X^i$. We assume that these sensitive data are drawn i.i.d from the alphabet of secrets $X$ according to a hidden distribution $\theta$. As shown in Figure 1, every user obfuscates its datum $X^i$ using an arbitrary privacy mechanism $A^i$ to yield a noisy observation $Z^i$ taking values from an alphabet of observables $Z$ which depends indeed on the applied mechanism.

As a consequence of the above process, each random observation $Z^i$ has a marginal probability distribution $m^i$ that depends on the real distribution $\theta$ and the obfuscation mechanism $A^i$; namely $m^i = \theta A^i$. In particular $m_i^i$ is the probability of observing $z \in Z^i$ from the user $i$. Note that the observations $Z^i$ have in general different marginal distributions, and possibly different alphabets $Z^i$ depending on the corresponding mechanisms. This makes estimating the original distribution $\theta$ from the observations challenging, compared to the traditional case, e.g., [1], [2], [3], [4], in which all observations are produced by the same mechanism and therefore follow the same marginal distribution.

4. Estimation by combining results

Estimation methods existing in the literature of local privacy work under the assumption that all the noisy data are produced by the same mechanism. In the following we describe one approach of reusing these methods in privacy work under the assumption that all the noisy data marginal distribution.

[1], [2], [3], [4], in which all observations are produced

depe nding on the corresponding mechanisms. This makes estimating the original distribution $\theta$ from the observations challenging, compared to the traditional case, e.g., [1], [2], [3], [4], in which all observations are produced by the same mechanism and therefore follow the same marginal distribution.

5. Estimation by combining mechanisms

Instead of combining the estimates from disjoint subsets of the noisy data as in Section 4, we develop in this section estimators that rely on combining the underlying users’ mechanisms into one. Then the used estimator is applied on the compound mechanism instead of applying it individually on each underlying mechanism.

5.1. Compound-mechanism inversion [INV]M

In the following we assume that the mechanisms of the users have the same alphabet $Z$ of observables. In this case, we can construct the empirical distribution $q[n]$ where the $z$-th component $q[n]_z$ is the proportion of $z \in Z$ in the noisy data, that is

$$q[n]_z = \frac{1}{n} \sum_{i \in [n]} \mathbb{1}_{Z^i = z} \quad \forall z \in Z.$$  

Then we define the estimate of the compound-mechanism inversion estimator $[ INV ]_M$ as follows.

$$\hat{\theta}[n] = q[n] (A[n])^{-1} \text{ where } A[n] = \frac{1}{n} \sum_{i=1}^{n} A^i. \quad (7)$$

As a special case, if all users apply the same mechanism $A$, then clearly $A[n] = A$ and (7) coincides, in this case, with the matrix inversion estimator (also known as the empirical estimator) INV which was described by [2], [4].

From the definition (7) of $[ INV ]_M$ the resulting estimate $\hat{\theta}[n]$ may in general contain negative components. Therefore we post-process it using two methods to yield a valid (non-negative) distribution. The first method is projecting $\hat{\theta}[n]$ onto the probability simplex, with respect to the $\ell_2$ distance [14]. The second method is truncating the negative components of $\hat{\theta}[n]$ to 0 and then normalizing the resulting vector.

A fundamental property that describes the consistency of $[ INV ]_M$ is shown by the following proposition.

**Proposition 1.** Suppose that $A^i$, for $i \in [n]$, have the same alphabet of observables and $A[n]$ is invertible for every $n > 0$. Let $\hat{\theta}[n]$ be the result of $[ INV ]_M$ (7). Then $(\hat{\theta}[n] - \theta) A[n] \to 0$ almost surely. Furthermore if $A[n]$ converges to some $\bar{A}$, then $\hat{\theta}[n] \to \theta$ almost surely.

In particular, if all users apply the same mechanism, or they choose their mechanisms from a finite set according to some probability distribution, then $\theta[n] \to \theta$ almost surely. Moreover we will show in Section 5.2 that if all mechanisms are $k$-RR, with various levels of privacy, the expected $\ell_2^2$ error of $[ INV ]_M$ (7) converges with large $n$ to 0 and therefore $\hat{\theta}[n] \to \theta$ in probability.

5.2. [INV]M under various k-RR mechanisms

Suppose that every user $i \in [n]$ sanitizes his datum using a $k$-RR (1), with a privacy level $\epsilon_i$ of his choice. In this scenario we can express the estimate of $[ INV ]_M$ as follows. First we construct the average mechanism $A[n]$ only on a subset of noisy data. This impacts the estimation performance as shown experimentally in Section 7.
using (1). Note in this case that \( A[n] \) is also a \( k \)-RR mechanism with \( \epsilon[n] \) that satisfies
\[
\frac{1}{k-1 + \epsilon[n]} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{k-1 + \epsilon[i]}.
\]  
Then by (7) the estimate \( \hat{\theta}[n] \) of \( [\text{INV}]_{M} \) is obtained by solving the equation \( \hat{\theta}[n] A[n] = q[n] \), which yields
\[
\hat{\theta}[n] = \frac{e[n] + k - 1}{e[n] - 1} q[n] - \frac{1}{e[n] - 1} 1. \tag{9}
\]
Using the above expression of the estimate \( \hat{\theta}[n] \) we can derive an upper bound on the expected \( \ell_2^2 \) error of \( [\text{INV}]_{M} \) under various \( k \)-RR mechanisms as follows.

**Proposition 2.** Suppose that every user \( i \in [n] \) applies a \( k \)-RR mechanism with arbitrary \( \epsilon_i > 0 \). Let \( \epsilon[n] \) be defined by (8). Then the estimate \( \hat{\theta}[n] \) of \( [\text{INV}]_{M} \) satisfies
\[
\mathbb{E}[\|\hat{\theta}[n] - \theta\|^2_2] \leq \frac{1}{n} \sum_{x \in X} \frac{\theta(x)^2}{x^2} + \frac{k - 1}{n} \left( \frac{k + 2(\epsilon[n] - 1)}{(\epsilon[n] - 1)^2} \right)
\]
where the equality holds if the values \( \epsilon_i \), for all \( i \in [n] \), are equal.

The above proposition is general in the sense that it allows the privacy level \( \epsilon_i \) for every user to be arbitrary according to his preference. As a special case when \( \epsilon_i \) for all \( i \in [n] \) are identical, i.e. all users apply the same \( k \)-RR mechanism, Proposition 2 yields the same error that was reported by [4] under this assumption.

Another important consequence of Proposition 2 is that if \( \inf_i \epsilon_i[n] > 0 \), the estimation error given by Proposition 2 converges to 0, which implies by Markov’s inequality that the \( [\text{INV}]_{M} \) estimator under \( k \)-RR mechanisms (9) is consistent, i.e. \( \hat{\theta}[n] \to \theta \) in probability.  

Finally we can see that the estimate of \( [\text{INV}]_{M} \) under \( k \)-RR mechanisms and \( \epsilon[n] \geq k \) converges in the same order as the best estimator. In fact, a lower bound for the \( \ell_2^2 \) estimation error of any estimator is \( (1 - \sum_{x \in X} \theta(x)^2)/n \) which corresponds to the maximum likelihood estimator working on non-sanitized samples [15]. From Proposition 2, and assuming that \( \epsilon[n] \geq k \), the error of \( [\text{INV}]_{M} \) is above this bound by at most \( (3 + 1/(k-1))/n \).

### 5.3. Compound-matrix iterative Bayesian update [IBU]_M

Assuming that all users apply the same mechanism \( A \), the authors of [1] proposed a procedure called iterative Bayesian update (IBU) to estimate the original distribution \( \theta \). This procedure starts with a full-support distribution \( \theta^0 \) on \( X \) and then using the empirical distribution \( q[n] \) (on \( Z \)) defined by (6), the estimate is refined iteratively as follows
\[
\theta_{t+1}^z = \frac{\theta^t_z A_{xz} \sum_{u \in X} \theta^t_u A_{uz}}{\sum_{z \in Z} \sum_{u \in X} \theta^t_u A_{uz}} \quad \forall x \in X. \tag{10}
\]

1. This means that \( \lim_{n \to \infty} P[|\hat{\theta}(n) - \theta|_2 > \delta] = 0 \) for all \( \delta > 0 \).

The procedure terminates and return \( \theta^t \) when two successive estimates are close enough. IBU coincides with GIBU (cfr. Section 6) when the users apply the same mechanism \( A \), and in this case IBU returns the MLE for \( \theta \).

In our scenario where various mechanisms are used (instead of a fixed one), one way of reusing IBU is to construct the average mechanism \( A[n] \) defined by (7) and then apply IBU to \( A[n] \) and \( q[n] \). We will call this method \([\text{IBU}]_{M}\). Note that this method (like \([\text{INV}]_{M}\)) requires that all the mechanisms have the same alphabet of observables \( Z \) to allow computing \( A[n] \).

Compared to \([\text{INV}]_{M}\), one advantage of \([\text{IBU}]_{M}\) is that \( A[n] \) is not needed to be invertible. In addition, \([\text{IBU}]_{M}\) does not require post-processing because it always returns a valid distribution. However the main limitation of \([\text{IBU}]_{M}\) is that it has no formal guarantees. In particular it does not yield the MLE unless all mechanisms are identical.

More seriously, this method may converge badly or may not converge at all when the average mechanism \( A[n] \) turns to be non-informative. Consider for example the following two mechanisms
\[
A = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}, \quad A' = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}. \tag{11}
\]
It is clear that both \( A \) and \( A' \) provide a reasonable statistical utility if only one of them is applied by all users. However, if half of the users apply \( A \) and the other half apply \( A' \), then the average mechanism \( A[n] \) is entirely non-informative because each row turns to be a uniform distribution over the observables, This of course makes estimating the original distribution using \([\text{IBU}]_{M}\) extremely inaccurate in this particular case.

### 5.4. Compound-RAPPOR estimator \([\text{RAP}]_{M}\)

The mechanism RAPPOR is built on the idea of randomized response to allow collecting statistics from end-users with differential privacy guarantees [6]. The simplest version of this mechanism, known as Basic One-Time RAPPOR, obfuscates every user’s private datum \( X_i \) (taking a value from the alphabet \( X \)) as follows. \( X' \) is encoded in a ‘one-hot’ binary vector \( B' \in \{0, 1\}^{|X|} \) such that \( B'_x = 1 \) if \( X' = x \) and \( B'_y = 0 \) otherwise. Then given a privacy parameter \( \epsilon > 0 \), every bit \( B'_i \) is obfuscated independently to a random bit \( V^i_x \) as follows.
\[
V^i_x = \begin{cases} B'_x & \text{with probability } 1 + e^{\epsilon/2} \\ 1 - B'_x & \text{with probability } 1 + e^{\epsilon/2} \end{cases}. \tag{12}
\]
Finally, the random binary vector \( V^i \in \{0, 1\}^{|X|} \) resulting from the above obfuscation scheme is reported to the server. As shown by [13], [5], the RAPPOR mechanism satisfies \( \epsilon \)-local differential privacy.

It is clear that the observables of RAPPOR are bit-vectors drawn from \( Z = \{0, 1\}^{|X|} \) which may be too large (of size \( 2^{|X|} \)), making it impractical to obtain the inverse matrix required to apply the INV estimator. The authors of [4] described a special estimator under RAPPOR, but valid only when all users set the privacy parameter \( \epsilon \) to the same value. In the following we remove this restriction and present the compound-RAPPOR estimator which we coin as \([\text{RAP}]_{M}\).
Suppose that each user $i \in [n]$ applies RAPPOR with an arbitrary $\epsilon_i$. Then we define $\epsilon[n]$ to satisfy
\[
\frac{1}{1 + e^{\epsilon[i]/2}} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 + e^{\epsilon[i]/2}}. \tag{13}
\]
Using the vectors $V_i$ reported from individual users, we define the vector $s[n] = \frac{1}{n} \sum_{i=1}^{n} V_i$. Note that $s[n]$ has length equal to the size $X$ (the alphabet of secrets), and $s[n]_x$ is the proportion of vectors $V_i$ having the $x$-th bit set to 1. Finally we define the estimate of $[\text{RAP}]_M$ as
\[
\hat{\theta}[n] = e^{\epsilon[n]/2} + \frac{1}{e^{\epsilon[n]/2} - 1} s[n] - \frac{1}{e^{\epsilon[n]/2} - 1}. \tag{14}
\]
Since the resulting estimate $\hat{\theta}[n]$ may contain negative components, it is post-processed by projection or normalization (similar to [INV] in Section 5.1) to obtain a valid distribution. A special case of $[\text{RAP}]_M$ is obtained when all the mechanisms are identical with parameter $\epsilon$. In this case, by (13), we have $\epsilon[n] = \epsilon$, and (14) coincides with the basic estimator RAP proposed by [3], [6].

The quality of $[\text{RAP}]_M$ can be described analytically as the expected $\ell_2^2$ between the real and estimated distributions as follows.

**Proposition 3.** Suppose that every user $i \in [n]$ applies a RAPPOR mechanism with $\epsilon_i > 0$. Let $\epsilon[n]$ be defined by (13). Then the estimate $\hat{\theta}[n]$ of $[\text{RAP}]_M$ satisfies
\[
\mathbb{E}[||\hat{\theta}[n] - \theta||_2^2] \leq \frac{1}{n} \sum_{x \in X} \theta_x^2 + \frac{1}{n} \frac{k e^{\epsilon[n]/2}}{(e^{\epsilon[n]/2} - 1)^2}
\]
where the equality holds if the values $\epsilon_i$, for all $i \in [n]$, are equal.

It follows from the above result that if $\inf_i \epsilon_i = 0$, the estimation error of $[\text{RAP}]_M$ converges to 0, which implies by Markov’s inequality that this estimator is consistent in the sense that $\hat{\theta}[n] \rightarrow \theta$ in probability.

### 6. Generalized iterative Bayesian update

A rigorous approach to reconstruct the original distribution $\theta$ on $X$ is to compute the maximum likelihood estimate (MLE) of this distribution from the observed noisy data. The authors of [10] developed an iterative algorithm, called GIBU, that evaluates this MLE. Basically, in each iteration, the current estimate $\theta^t$ for the distribution over $X$ is refined to $\theta^{t+1}$ as follows.
\[
\theta^{t+1}_x = \frac{\sum_{i=1}^{n} G_{xi} \theta^t_u G_{ux}}{\sum_{u \in X} \theta^t_u G_{ux}} \forall x \in X \tag{15}
\]
where $G_{xi}$ is the conditional probability of the observation from user $i$ given that his real private datum is $x \in X$. This procedure is highly efficient since each iteration requires time of order $nt|X|^2$, and $n$ (the number of users) may be arbitrarily large. Therefore we propose a refined algorithm which is scalable. Let $A$ be the set of different mechanisms applied by the users. Then for every mechanism $A \in A$ we have the following triplet:
- the number of users $n^A$ who used the mechanism $A$,
- the alphabet of observables $Z^A$ for $A$,
- the empirical distribution over $Z^A$, written as $q^A$.

Note that the total number of users is $n = \sum_{A \in A} n^A$. Using the above triplet for every mechanism in $A$, algorithm 1 describes GIBU which returns an MLE for $\theta$.

**Algorithm 1:** GIBU working on a set $A$ of mechanisms.

**Theorem 1.** Algorithm 1 returns an MLE for $\theta$.

It can be seen that each iteration of algorithm 1 requires time in order of $\sum_{A \in A} |Z^A| |X|^2$ which is independent of $n$ and therefore shows a significant reduction in the time complexity compared to (15). We also note that every iteration is a sum of terms that can be computed in parallel since each term depends only on the triplet $(Z^A, q^A, n^A)$ of the mechanism $A$. In conclusion, algorithm 1 is scalable, allowing us to run it over large samples and compare it to other methods as we proceed in the following sections.

### 7. Evaluation of combining results

In this section we experimentally evaluate the approach of combining results described in Section 4. We consider three off-the-shelf estimators that may replace $\epsilon$ in (4), namely the standard estimator under the RAPPOR mechanism (RAP) [3], [6], INV [2], [4], and IBU [1]. We recall that each one of these methods works on noisy data produced by one mechanism as described respectively in Sections 5.4, 5.1, and 5.3.

We consider an alphabet $X = \{0, 1, \ldots, |X| - 1\}$, and then we construct the real data of the users synthetically by sampling from a binomial distribution on $X$ with $\alpha = 0.5$.

In the experiments of this section we will use RAPPOR mechanisms (12) and $k$-RR mechanisms (1). We also use the truncated linear geometric mechanism [7] described in Section 2.2. We start our experiments by letting the original data be obfuscated by 10 RAPPOR mechanisms with various $\epsilon_i$, and then we apply Equations (4) and (5) with $\epsilon$ replaced by RAP. Hence we call the method in this case [RAP]R. Recall that every result of RAP requires post-processing by projection or normalization to return a valid distribution (cf. Section 5.4). Figure 2 shows the estimation performance of [RAP]R with these two post-processing methods, and also that of GIBU. The performance is measured by the earth mover’s distance.
Earth mover’s distance (EMD) between the original and estimated distributions.

![Graph](image)

Figure 2. Performance of [RAP] and GIBU for $\mathcal{X} = \{0, 1, \ldots, 19\}$. The noisy data are produced by 10 RAPPOK mechanisms having various $\epsilon_i$ in high privacy regime (a), and in low privacy regime (b) (cfr. Table 4).

We also let the original data be obfuscated by 10 $k$-RR mechanisms having $\epsilon_i$ varying between 3.0 and 8.08, and apply Equations (4) and (5) with $\epsilon$ replaced by INV and IBU. We call the methods resulting from these two substitutions of $\epsilon$ as [INV]$_R$ and [IBU]$_R$ respectively. Similar to RAP, every run of INV requires post-processing by projection or normalization to return a valid distribution (cfr. Section 5.1). Figure 3 shows the performances of [INV]$_R$ (with post-processing), [IBU]$_R$, and GIBU. We perform a similar experiment, but using a mixture of 5 truncated geometric mechanisms with $\epsilon_i$ between 0.065 and 0.869 and 5 $k$-RR mechanisms with $\epsilon_i$ between 3.0 and 4.69, and we show the results in Figure 4. Figure 5 shows the experiment results in the case of using a mixture of linear geometric mechanisms. Finally Figure 6 show the results for a mixtures of Shokri’s mechanisms which we describe later in Section 8.2.4.

It is clear from Figures 3, 4, 5, and 6 that the method of combining results shows a significantly poor estimation quality compared to GIBU. This is explained by the fact that every underlying estimator works only on a small subset of data $\mathcal{D}^2$ instead of the entire set. This implies that the estimation error of $e(A, D^3)$, for every $A$, is relatively large which makes the overall error of the final $\theta$ also large.

8. Evaluation of combining mechanisms

8.1. Performance of [INV]$_M$ relative to GIBU

In this section we experimentally compare between the estimation performance of the compound-mechanism inversion [INV]$_M$ and GIBU. We will run our experiments on two alphabets of the users’ private data. The first alphabet is linear which may represent e.g. ages, smoking rates, etc. The other alphabet is planar which represent geographic locations.

In the linear case, we define the alphabet of secrets $\mathcal{X}$ to be $\{0, 1, \ldots, 99\}$, with distance 1.0 between successive elements. In this case, we synthesize the original data of the users by sampling from $\mathcal{X}$ according to a binomial distribution with $\alpha = 0.5$.

For the planar alphabet, we consider a geographic region in San-Francisco bounded by the latitudes 37.7228,
8.1.1. Various k-RR mechanisms. We let the original data of users be obfuscated using 10 k-RR mechanisms having various values of $\epsilon_i$. In the case of linear alphabet we set these values to be between 3.0 and 8.08. In the case of the planar alphabet we set the values of $\epsilon_i$ to be between 3.05 and 8.20. The complete list of these values are shown in Table 5. Figure 7 shows the estimation performance of $[\text{INV}]_M$ with the two post-processing methods (normalization and projection), and also the performance of GIBU.

From Figure 7 we observe first that for both linear and planar alphabets, post-processing the result of $[\text{INV}]_M$ using projection exhibits significantly better estimation performance compared to normalization. We also observe that GIBU slightly outperforms $[\text{INV}]_M$ with projection.

8.1.2. Various geometric mechanisms. We let the private data of the users be sanitized by the geometric mechanisms which satisfy $\epsilon$-geo-indistinguishability. For the linear alphabet we use 10 linear geometric mechanisms, defined by (2), with $\epsilon_i$ varying from 0.020 to 0.869. For the planar alphabet we use 10 planar geometric mechanisms (described in Section 2.3) with $\epsilon_i$ between 0.190 and 3.124. The complete list of these values are shown in Table 6. Based on the noisy data, the original distribution is estimated using $[\text{INV}]_M$ and GIBU and we show their performances in Figure 8.

It is clear from Figure 8 that the performance of GIBU is significantly better than $[\text{INV}]_M$ in both cases of linear and planar alphabets.

8.1.3. Mixed geometric and k-RR mechanisms. We let now the original data of the users be sanitized by a mixture of geometric and k-RR mechanisms. Precisely, for the linear alphabet we let the original data be obfuscated by 5 linear geometric mechanisms with $\epsilon_i$ varying between 0.065 and 0.869, and 5 k-RR mechanisms with with $\epsilon_i$ varying between 3.0 and 4.69. In the case of the planar alphabet we sanitize the original data by 5 planar geometric mechanisms with $\epsilon_i$ between 0.632 and 3.124 and 5 k-RR mechanisms with $\epsilon_i$ between 3.05 and 5.67. The full list of these parameters is shown in Table 7. The results for $[\text{INV}]_M$ and GIBU, in the two cases, are shown by Figure 9 which reflects again that GIBU provides a better...
estimation performance compared to $[\text{INV}]_M$.

### 8.2. Performance of $[\text{IBU}]_M$ relative to GIBU

In this section we compare between the performance of $[\text{IBU}]_M$ described in Section 5.3 and that of GIBU. We will use the same experimental setup of Section 8.1. In particular, we base our comparison on private data drawn from linear and planar alphabets. In the linear case, the alphabet is $\{0, 1, \ldots, 99\}$ and the private data of the users are synthesized using a binomial distribution. In the planar case, the alphabet is the grid cells of San Francisco and the private data are obtained from the Gowalla dataset.

#### 8.2.1. Various $k$-RR mechanisms

We let the original data of users be sanitized by 10 $k$-RR mechanisms with various values of $\epsilon_i$. In the case of linear alphabet, the values of $\epsilon_i$ are between 3.0 to 8.08, and in the case of planar alphabet, they are between 3.05 and 8.20 (cfr. Table 5). In these two cases we estimate the original distribution on the alphabet using $[\text{IBU}]_M$ and GIBU and show the results in Figure 10.

It is clear that GIBU outperforms $[\text{IBU}]_M$ in both cases of linear and planar data. Note that although each one of these methods maximizes a likelihood function based on reported data, only GIBU yields the true MLE since it takes into account the various marginal distributions induced by individual mechanisms. $[\text{IBU}]_M$ on the other hand returns the distribution under a ‘fake’ hypothesis that all observations are produced by the average mechanism $A[n]$, hence returning a worse estimate.

This also explains why the performance of $[\text{IBU}]_M$ is worse than the performance of $[\text{INV}]_M$ under $k$-RR mechanisms (cfr. Figure 7) although both methods use the average mechanism. In fact, unlike the case of $[\text{IBU}]_M$, the definition of $[\text{INV}]_M$ is independent of the above hypothesis that observations are produced by $A[n]$.

#### 8.2.2. Various geometric mechanisms

Now we let the original data be sanitized by 10 different geometric geometric mechanisms. For the linear alphabet we use linear geometric mechanisms with $\epsilon_i$ between 0.02 and 0.869, whereas for the planar alphabet we use the planar variants of geometric mechanisms with $\epsilon_i$ between 0.19 and 3.124 (cfr. Table 6). Figure 11 shows the estimation performances of $[\text{IBU}]_M$ and GIBU in these two cases. It is clear from this figure that GIBU is superior.

#### 8.2.3. Mixed geometric and $k$-RR mechanisms

We consider now the scenario when both geometric and $k$-RR mechanisms are used by different users to obfuscate their data. In the case of the linear alphabet, we use 5 linear geometric mechanism with various $\epsilon_i$ between 0.065 and 0.869 together with 5 $k$-RR mechanisms with $\epsilon_i$ between 3.0 and 4.69. In the planar case we use 5 planar geometric mechanisms with $\epsilon_i$ between 0.632 and...
3.124 with 5 \( k \)-RR mechanisms having \( \epsilon_i \) between 3.05 and 5.67 (cfr. Table 7). The performances of [IBU]_M and GIBU are shown by Figure 12 from which it is clear that GIBU is superior.

8.2.4. Privacy mechanisms of Shokri et al. . In the following we use the privacy mechanism of Shokri et al. [9] to obfuscate the original data of the users and again we compare the estimation performances of [IBU]_M and GIBU.

This mechanism is defined so that the privacy, quantified by the adversary’s expected loss, is maximized while the expected loss of quality, experienced by the user, is maintained under a given threshold \( Q \). Suppose that the user’s real datum is \( x \in \mathcal{X} \). Then for any \( z \in \mathcal{X} \) the function \( \ell(x, z) \) quantifies the user’s loss of quality when the mechanism reports \( z \) instead of \( x \). We let \( \ell(x, z) \) describe also the adversary’s loss when his guess is \( z \). The user is assumed to have a personal profile modeled as a distribution \( \pi \) over the alphabet \( \mathcal{X} \). The adversary is assumed to know \( \pi \) which he uses, in addition to the mechanism \( A \), to make his guess so that his expected loss is minimized. It is shown in [9] that the adversary’s best guess, given an observation \( y \), is

\[
\arg\min_{z \in \mathcal{X}} \sum_{x \in \mathcal{X}} \pi_x A_{xy} \ell(x, z).
\]

Hence the mechanism \( A \) is defined to be the solution of the following optimization problem.

\[
\begin{align*}
\text{Maximize} & \quad \sum_{y \in \mathcal{X}} \min_{x \in \mathcal{X}} \sum_{x \in \mathcal{X}} \pi_x A_{xy} \ell(x, z), \\
\text{subject to} & \quad \sum_{x \in \mathcal{X}} \pi_x \sum_{x \in \mathcal{X}} A_{xz} \ell(x, z) \leq Q, \\
& \quad \sum_{z \in \mathcal{X}} A_{xz} = 1 \quad \forall x \in \mathcal{X}, \\
& \quad A_{xz} \geq 0 \quad \forall x, z \in \mathcal{X}.
\end{align*}
\]

Note that the objective function above is the expected adversary’s loss using his best strategy, and the first constraint restricts the user’s expected loss of quality to be below the threshold \( Q \).

Now for our experiment we let the original data of the users be obfuscated by 10 Shokri’s mechanisms constructed with \( \pi \) set to the uniform distribution on \( \mathcal{X} \), the loss function \( \ell(\cdot, \cdot) \) set to be the Euclidean distance between the elements of \( \mathcal{X} \), and finally with various \( Q_i \) as in Table 8. The performances of [IBU]_M and GIBU for the linear and planar alphabets are shown in Figure 13 which reflects the superiority of GIBU in this experiment.

8.3. Performance of [RAP]_M relative to GIBU

We suppose that original private data take values from a linear alphabet \( \mathcal{X} = \{0, 1, \ldots, 19\} \) in which the distance
between any two successive elements is 1. The original data are synthetically generated according to a binomial distribution on \( \mathcal{X} \) with \( \alpha = 0.5 \). Then the data are obfuscated using 10 RAPPOR mechanisms.

We perform our evaluation in two settings. In the first one, which we refer to as the high privacy regime, we set \( \epsilon_i \) of the mechanisms to vary between 0.1 and 1.0. In the other setting, which we call low privacy regime, we set \( \epsilon_i \) to vary between 1.0 and 10.0. In these two settings, Figure 14 shows the performance of \([\text{RAP}]_M\) with the two post-processing methods (normalization and projection), and also the performance of GIBU.

It is clear that the performances of \([\text{RAP}]_M\) and GIBU are similar in the high privacy regime, while the latter is clearly better in the low privacy regime. A related result obtained by [4] states that RAPPOR estimator (which assumes that all users apply the same mechanism) is ‘order’ optimal when \( \epsilon < 1 \); while being not optimal for larger values of \( \epsilon \). Our evaluation extends this result to \([\text{RAP}]_M\) which accepts various mechanisms.

9. Conclusion

In this paper we consider the situation when every user applies his own privacy mechanism and privacy level to sanitize his sensitive data. Of course this situation involves many mechanisms that may differ in the level of privacy, in their signature, or in both. In this case we require to construct the probability distribution of the users’ original data on the alphabet of the secrets. We have presented several methods for this construction and presented experimental comparisons between them. These comparisons were based on both synthetic and real datasets. We find that methods that are based on composing the mechanisms are more accurate than the methods based on combining results. Furthermore, we find that GIBU has a superior accuracy compared to other methods.

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Appendix A.

TABLE 4. RAPPOR MECHANISMS USED IN FIGURES 2 AND 14

|          | high privacy | low privacy |
|----------|--------------|-------------|
|          | $e_i$ below 0.1 | $e_i$ below 1.0 |
| [RAP]_m + Normalization | 0.1 0.2 0.3 | 0.6 0.7 0.8 0.9 1.0 |
| [RAP]_c + Projection | 1.0 2.0 3.0 4.0 5.0 | 6.0 7.0 8.0 9.0 10.0 |
| GIBU     |              |             |

TABLE 5. MECHANISMS USED IN FIGURES 3, 7, AND 10

|          | linear | planar |
|----------|--------|--------|
|          | $k$-RR mechanisms with $e_i$ below | $k$-RR mechanisms with $e_i$ below |
|          | 0.020 0.025 0.031 0.039 0.050 | 0.190 0.244 0.310 0.390 0.493 |
|          | 0.065 0.088 0.131 0.236 0.869 | 0.632 0.835 1.159 1.762 3.124 |
|          | 3.00 3.54 3.96 4.34 4.69 | 6.05 6.44 6.87 7.40 8.20 |
|          | 5.06 5.46 5.93 6.60 8.08 |             |
|          | 3.05 4.19 4.81 5.27 5.67 |             |

TABLE 6. MECHANISMS USED IN FIGURES 8, 11, AND 5

|          | linear | planar |
|----------|--------|--------|
|          | truncated geometric mechanisms with $e_i$ below | truncated geometric mechanisms with $e_i$ below |
|          | 0.065 0.088 0.131 0.236 0.869 | 0.190 0.244 0.310 0.390 0.493 |
|          | 0.632 0.835 1.159 1.762 3.124 |             |
|          | 3.00 3.54 3.96 4.34 4.69 |             |
|          | 3.05 4.19 4.81 5.27 5.67 |             |

TABLE 7. MECHANISMS USED IN FIGURES 4, 9 AND 12

|          | linear | planar |
|----------|--------|--------|
|          | 5 truncated geometric mechanisms with $e_i$ below | 5 truncated planar mechanisms with $e_i$ below |
|          | 0.065 0.088 0.131 0.236 0.869 | 0.190 0.244 0.310 0.390 0.493 |
|          | 0.632 0.835 1.159 1.762 3.124 |             |
|          | 3.00 3.54 3.96 4.34 4.69 |             |
|          | 3.05 4.19 4.81 5.27 5.67 |             |

TABLE 8. MECHANISMS USED IN FIGURES 13

|          | linear | planar |
|----------|--------|--------|
|          | 10 Shokri’s mechanisms with $Q_i$ below | 10 Shokri’s mechanisms with $Q_i$ below |
|          | 0.3 0.6 0.9 1.2 1.5 | 0.3 0.6 0.9 1.2 1.5 |
|          | 1.8 2.1 2.4 2.7 3.0 |             |