THE HOMOTOPY TYPE OF THE COMPLEMENT OF THE CODIMENSION-TWO COORDINATE SUBSPACE ARRANGEMENT

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A complex coordinate subspace of $\mathbb{C}^n$ is given by

$$L_\sigma = \{(z_1, \ldots, z_n) \in \mathbb{C}^n | \sigma \cap z_i = \cdot \cdot \cdot = z_{i_k} = 0\}$$

where $\sigma = \{i_1, \ldots, i_k\}$ is a subset of $[m]$. For each simplicial complex $K$ on the set $[m]$ we associate the complex coordinate subspace arrangement $CA(K) = \{L_\sigma | \sigma \notin K\}$ and its complement $U(K) = \mathbb{C}^n \setminus \bigcup_{\sigma \notin K} L_\sigma$. On the other hand, to $K$ we can associate the Davis-Januszkiewicz space $DJ(K) = \bigcup_{x \in K} BT_x \subset BT^n$, where $BT^n$ is the classifying space of $n$-dimensional torus, that is, the product of $n$ copies of infinite-dimensional projective space $\mathbb{CP}^\infty$, and $BT_x := \{(x_1, \ldots, x_n) \in BT^n | x_i = * \}$ where $i \notin \sigma$. Let $Z_K$ be the fibre of $DJ(K) \rightarrow BT^n$. By [BP, 8.9], there is an equivariant deformation retraction $U(K) \rightarrow Z_K$, and the integral cohomology of $Z_K$ has been calculated in [BP] 7.6 and 7.7.

**Theorem 1.** The complement of the codimension-two coordinate subspace arrangement in $\mathbb{C}^n$ has the homotopy type of the wedge of spheres

$$\bigvee_{k=2}^{n} (k-1) \binom{n}{k} S^{k+1}.$$  

**Proof.** Let $K$ be a disjoint union of $n$ vertices. Then $DJ(K)$ is the wedge of $n$ copies of $\mathbb{CP}^\infty$ and $U(K)$ is the complement of the set of all codimension-two coordinates subspaces $z_i = z_j = 0$ for $1 \leq i < j \leq n$ in $\mathbb{C}^n$. Therefore to prove the theorem we have to determine the homotopy fibre of the inclusion $\bigvee_{k=1}^{n} \mathbb{CP}^\infty \rightarrow \prod_{k=1}^{n} \mathbb{CP}^\infty$. This is done by applying Proposition 5 to the case $X_1 = \cdots = X_n = \mathbb{CP}^\infty$ and noting that $\Omega\mathbb{CP}^\infty \cong S^1$. \hfill $\Box$

It should be emphasized that Theorem 1 holds without suspending. Previously, decompositions were known only after some number of suspensions, the best of which was by Schaper [S] who required one suspension. To finish the proof of Theorem 1 it remains to prove Proposition 5. This was originally proved by Porter [P] by examining subspaces of contractible spaces. We present an accelerated proof based on the Cube Lemma.

We work in the category of based, connected topological spaces and continuous maps. Let $*$ denote the basepoint. For spaces $X, Y$, let $X \times Y = (X \times Y)/(*) \times Y, X \wedge Y = (X \times Y)/(X \times *),$ and $X \wedge Y = \Sigma X \vee Y$. Denote the identity map on $X$ by $\mathbb{1}$. Denote the map which sends all points to the basepoint by $\mathbb{1}$.

**Lemma 2.** Let $A, B,$ and $C$ be spaces. Define $Q$ as the homotopy pushout of the map $A \times B \rightarrow C \times B$ and the projection $A \times B \rightarrow A$. Then $Q \simeq (A \wedge B) \vee (C \times B)$.

**Proof.** Consider the diagram of iterated homotopy pushouts

$$
\begin{array}{ccc}
A \times B & \xrightarrow{i_2} & B \times C \times B \\
\downarrow \pi_2 \quad & & \downarrow s \\
A \times B & \xrightarrow{*} & A \wedge B \\
\end{array}
$$

where $\pi_2, i_2$ are the projection and inclusion respectively. Here, it is well known that the left square is a homotopy pushout, and the right homotopy pushout defines $Q$. Note that $i_2 \circ \pi_2 \simeq * \times B$. The outer rectangle in an iterated homotopy pushout diagram is itself a homotopy pushout, so $Q \simeq Q$. The right pushout then shows that the homotopy cofibre of $C \times B \rightarrow Q$ is $\Sigma B \vee (A \wedge B)$. Thus $\mathbb{1}$ has a left homotopy inverse. Further, $s \circ i_2 \simeq *$ so pinching out $B$ in the right pushout gives a homotopy cofibration $C \times B \rightarrow Q \rightarrow A \wedge B$ with $r \circ t$ homotopic to the identity map. \hfill $\Box$

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Lemma 3. Let $Y_1, \ldots, Y_n$ be spaces. Then there is a homotopy equivalence

$$\Sigma(Y_1 \times \cdots \times Y_n) \simeq \bigvee_{k=1}^n \left( \bigvee_{1 \leq i_1 < \cdots < i_k \leq n} \Sigma Y_{i_1} \wedge \cdots \wedge Y_{i_k} \right).$$

Proof. Induct on the decomposition $\Sigma(A \times B) = \Sigma A \vee \Sigma B \vee (\Sigma A \wedge B)$. □

The following was proved by Mather [M] and is known as the Cube Lemma.

Lemma 4. Suppose there is a diagram of spaces and maps

\begin{equation}
\begin{array}{cccc}
E & \rightarrow & F \\
\downarrow & & \downarrow \\
G & \rightarrow & H \\
\downarrow & & \downarrow \\
A & \rightarrow & B & \rightarrow & C & \rightarrow & D
\end{array}
\end{equation}

where the bottom face is a homotopy pushout and the four sides are obtained by pulling back with $H \rightarrow D$. Then the top face is a homotopy pushout. □

Proposition 5. Let $X_1, \ldots, X_n$ be spaces. Consider the homotopy fibration

$$F_n \rightarrow X_1 \vee \cdots \vee X_n \rightarrow X_1 \times \cdots \times X_n$$

obtained by including the wedge into the product. Then there is a homotopy decomposition

$$F_n \simeq \bigvee_{k=2}^n \left( \bigvee_{1 \leq i_1 < \cdots < i_k \leq n} (k-1) (\Sigma \Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_k}) \right).$$

Proof. We induct on $n$. When $n = 2$ it is well known that $F_2 \simeq \Sigma \Omega X_1 \wedge \Omega X_2$. Let $n \geq 3$ and assume the Proposition holds for $F_{n-1}$. Let $M_k = X_1 \vee \cdots \vee X_k$ and $N_k = X_1 \times \cdots \times X_k$. Observe that $M_n$ is the pushout of $M_{n-1}$ and $X_n$ over a point. Composing each vertex of the pushout into $N_n$ we obtain homotopy fibrations $\Omega N_n \rightarrow \ast \rightarrow N_n, \Omega N_{n-1} \rightarrow X_n \rightarrow N_n, F_{n-1} \times \Omega X_n \rightarrow M_{n-1} \rightarrow N_n$, and $F_n \rightarrow M_n \rightarrow N_n$. Write $N_n$ as $N_{n-1} \times X_n$. Then Lemma 4 implies that there is a homotopy pushout

$$\xymatrix{\Omega N_{n-1} \times \Omega X_n \ar[r]^h \ar[d]_g & F_{n-1} \times \Omega X_n \ar[d] \\
\Omega N_{n-1} \ar[r] & F_n}$$

where $g$ is easily identified as the projection and $h$ is the connecting map for the homotopy fibration $F_{n-1} \times \Omega X_n \rightarrow M_{n-1} \times \ast \rightarrow N_{n-1} \times X_n$. So $h \simeq \partial h_{n-1} \times \Omega X_n$ where $\partial h_{n-1}$ is the connecting map of the fibration $F_{n-1} \rightarrow M_{n-1} \rightarrow N_{n-1}$. But $\partial h_{n-1} \simeq \ast$ as $\Omega M_{n-1} \rightarrow \Omega N_{n-1}$ has a right homotopy inverse. Thus $h \simeq \ast \times \Omega X_n$. By Lemma 4 $F_n \simeq (\Omega N_{n-1} \ast \Omega X_n) \vee (F_{n-1} \times \Omega X_n)$. Since $F_{n-1}$ is a suspension, $F_{n-1} \times \Omega X_n \simeq F_{n-1} \vee (F_{n-1} \wedge \Omega X_n)$. Combining the decomposition of $\Omega \Sigma N_n \simeq \Sigma (\Omega X_1 \times \cdots \times \Omega X_n)$ in Lemma 4 with the inductive decomposition of $F_{n-1}$ and collecting like terms, the asserted wedge decomposition of $F_n$ follows. □

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References

[BP] V.M. Buchstaber and T.E. Panov, Torus actions and their applications in topology and combinatorics, University Lecture Series 24, American Mathematical Society, (2002).

[M] M. Mather, Pull-backs in homotopy theory, Canad. J. Math. 28 (1976), 225-263.

[P] T. Porter, The homotopy groups of wedges of suspensions, Amer. J. Math. 88 (1966), 655-663.

[S] Ch. Schaper, Suspensions of affine arrangements, Math. Ann. 309 (1997), 463-473.

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