Handedness of complex PT-Symmetric potential barriers

Zafar Ahmed
Nuclear Physics Division, Bhabha Atomic Research Centre
Trombay, Bombay 400 085, India
zahmed@apsara.barc.ernet.in
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Abstract

Generally, when imaginary part of an optical potential is non-symmetric the reflectivity, $R(E)$, shows left/right handedness, further if it is not negative-definite the reflection and transmission, $T(E)$, coefficients become anomalous in some energy intervals and absorption is indefinite ($\pm$). We find that the complex PT-symmetric potentials could be exceptional in this regard. They may act effectively like an absorptive potential for any incident energy provided the particle enters from the preferred (absorptive) side.

When a potential is real the probability of quantal transmission, $T(E)$, and reflection, $R(E)$, are invariant with respect to the side of incidence of the particle. Recently, it has been proved [1] that when the potential is complex and non-symmetric (in space), the value of reflectivity depends on whether the particle is incident on the potential from left or right. Thus when both the symmetries: time-reversal and parity are broken the reflectivity shows handedness. For the scattering from a complex potential, we have the condition of reciprocity satisfied

$$1 - R_l(E) - A_l(E) = T(E) = 1 - R_r(E) - A_r(E)$$

(1)

and unitarity is replaced by pseudo-unitarity

$$R(E) + T(E) + A(E) = 1.$$  

(2)

Here $A$ denotes the probability of absorption and subscripts, $l$ and $r$, stand for left and right. Let the scattering co-efficient, $S(E)$, represent any of the probabilities: $R(E), T(E), A(E)$. 
For a physical process the imaginary part of the optical potential should be finite everywhere (it encloses a finite area). Phenomenologically, it is the absorption (loss of flux into unknown channels) which we try to study and model in a scattering process. We therefore require \(0 < A(E) < 1\) and this in turn requires both \(R(E)\) and \(T(E)\) to be non-anomalous \((< 1)\). Interestingly, this is achieved by choosing the imaginary part of the potential as negative-definite in the whole regime. Notice that for an optical potential \(V_c(x) = V_r(x) + iV_i(x)\) that \(A(E)\) is defined as

\[
A(E) = -\frac{2m}{\hbar^2k} \int_{-\infty}^{\infty} V_i(x)\Psi^*(x)\Psi(x) \, dx, \quad V_i(x) \leq 0, \quad V_i(\pm\infty) = 0.
\]

(3)

A deeper study of Schrödinger transmission from a complex potential reveals that when the imaginary part is not negative-definite (e.g., it oscillates or it is positive-definite), at least one of probabilities \((T(E), R(E))\) in certain intervals of energy becomes anomalous \((> 1)\) and gives rise to indefinite \((\pm)\) absorption as a function of energy. Thus, such imaginary potentials have not been considered and reported in scattering from one dimensional complex potentials.

Recently, a new scope for complex PT-symmetric \((P : x \to -x, T : i \to -i)\) potentials to have real discrete eigenvalues \([2-4]\) has been well researched. Apart from real discrete spectrum, from such potentials real energy-momentum dispersion relation have also been obtained \([5-9]\) to find that both usual and unusual energy bandstructure can exist/co-exist \([8]\). The question as to what happens when particles are scattered from complex PT-symmetric potential gains importance now. In the well explored area of barrier penetration from one-dimensional barrier potentials \([11]\), one would like to know what happens when the potential breaks time-reversal and parity symmetries individually but preserves them jointly.

PT-symmetric potentials are expressible as \(V(x) = V_e(x) + iV_o(x)\), subscript \(e\) stands for even and \(o\) stands for odd. The imaginary part of \(V(x)\) being essentially, antisymmetric, consequently the (left/right)-handedness of reflectivity \([1]\) and anomalous values of \(R(E)\) and \(T(E)\) or equivalently the indefiniteness of \(A(E)\) as a function of energy are expected. Therefore, these features also observed recently \([10]\) do not actually subscribe only to PT-symmetry or pseudo-Hermiticity of the optical potential. Complex PT-symmetric potentials are found to be pseudo-Hermitian : \(\eta H \eta^{-1} = H^\dagger\) \([12]\).

In this Letter, we wish to report that if a particle enters from the absorptive side of the complex PT-symmetric barrier, we can still have non-anomalous and absorptive scattering
at any energy: \(0 \leq R(E), T(E) \leq 1\) and \(0 \leq A(E) \leq 1\). This we refer to as handedness of a complex PT-symmetric potential.

Several interesting features of scattering from a one-dimensional potential can be easily understood by constructing a novel rectangular potential as

\[
V(|x| < a) = 0, V(-a < x < 0) = V_0 + is_1 V_2, V(0 < x < a) = V_0 + is_2 V_2. 
\] (4)

and extracting \(T(E)\) and \(R(E)\) analytically. When \(s_1 = -1 = s_2\), it will be simple absorptive rectangular well. When \(s_1 = -1\) and \(s_2 = 1\) it will be PT-symmetric with absorptive side as on the ‘left’. When \(s_1 = 1\) and \(s_2 = -1\) again be PT-symmetric with absorptive side on the ‘right’. Non-PT-symmetric complex potentials with indefinite imaginary part can be had when \(s_1 \neq s_2\) and \(s_1 s_2 < 0\). Assuming the incidence of the particle from ‘left’ we derive the scattering co-efficients, \(R^{p,q}_l(E)\), as

\[
\left| \frac{q(k^2 - p^2) \sin pa \cos qa + p(k^2 - q^2) \cos pa \sin qa + ik(p^2 - q^2) \sin pa \sin qa}{2ikpq \cos pa \cos qa + p(k^2 + q^2) \cos pa \sin qa + q(p^2 + k^2) \sin pa \cos qa - ik(p^2 + q^2) \sin pa \sin qa} \right|^2. 
\] (5)

Here \(k = \sqrt{2mE}/\hbar, p = \sqrt{2m(V_1 + is_1 V_2)}/\hbar, q = \sqrt{2m(V_1 + is_2 V_2)}/\hbar\). Notice that \(R^{p,q}_l(E)\) is not symmetric in \(p\) and \(q\) (when \(s_1 \neq s_2\)) displaying its left/right handedness [1]. The expression for \(R^{p,q}_r(E)\) will be nothing but \(R^{q,p}_r(E)\). The transmission co-efficient, \(T(E)\) is given as

\[
\left| \frac{2ikpq}{2ikpq \cos pa \cos qa + p(k^2 + q^2) \cos pa \sin qa + q(p^2 + k^2) \sin pa \cos qa - ik(p^2 + q^2) \sin pa \sin qa} \right|^2 
\] (6)

which is symmetric under the exchange of \(p\) to \(q\) displaying the *reciprocity*: its independence on the side of the incidence of particle. This invariance is absolute whether or not \(s_1 = s_2\). Let us fix \(s_1 = -1\) and \(s_2 = 1\) so that the imaginary part is absorptive on the left and next by computing from (5), very remarkably, we find that \(R_l(E) < R_r(E)\) and that \(R_r(E)\) is anomalous but \(R_l(E)\) turns out to be physical (< 1).

Rectangular complex potential is more general than the complex PT-symmetric potentials constructed from Dirac-delta potentials [6,8,10]. Interestingly, this rectangular potential (4) like its real counter-parts (rectangular well/barrier) will have \(S(E)\) as oscillatory. An interesting study reveals [13] that the rectangular potential is the most localized potential one
can have and that other most commonly known one-dimensional potentials (e.g, Gaussian, Eckart, Lorentzian) entail \( S(E) \) as smooth function of energy. Consequent to this peculiar feature of a rectangular potential, here we do not find parameters \( V_1, V_2, a \) so that potential is absorptive for any energy excepting the situations where \( a \) is very small and the particle enters from the absorptive side.

We now take up PT-symmetric Scarf potential:

\[
V(x) = V_1 \text{sech}^2(x/a) + i V_2 \text{sech}(x/a) \tanh(x/a).
\]

This we do, in order to demonstrate in a tractable setting that for a certain choice of parameters, a complex PT-symmetric potential would act as absorptive for any energy provided the particle enters the potential from the absorptive (preferred) side. Notice that this potential is absorptive (imaginary part is negative-definite) on the left \((x \to -\infty)\) and on the right hand, it is (imaginary part is positive-definite) emissive. This potential is the first fully analytically solvable model of a complex PT-symmetric potential entailing both discrete spectrum of both the types: real when \( V_2 \leq V_1 + \Delta/4 \) and complex-conjugate pairs otherwise. Thankfully, the complex reflection/transmission amplitudes for the real(non-complex) Scarf potential have already been obtained by Khare and Sukhatme [14] in terms of complex Gamma functions assuming the incidence from the left hand side. Since the potential vanishes at \( x = \pm \infty \) and is finite everywhere, these can be extended to complex domain of the parameters. Very interestingly, we find that Gamma functions give way to simple trigonometric (circular and hyperbolic) functions in the expressions of \( R(E) \) and \( T(E) \). Thus, for the complex PT-symmetric potential (7), we find

\[
T(E) = \frac{2 \sinh^2 2\pi \kappa}{2 \cosh^2 2\pi \kappa + 4 \cosh 2\pi \kappa \cosh \pi f \cosh \pi g + \cosh 2\pi f + \cosh 2\pi g},
\]

\[
F_t(E) = \frac{1}{\sinh 2\pi \kappa} [e^{-\pi \kappa} \cosh \pi f + e^{\pi \kappa} \cosh \pi g],
\]

\[
R_t(E) = |F_t(E)|^2 T(E),
\]

\[
A_t(E) = 1 - R_t(E) - T_t(E).
\]

The parameters \( \kappa, f \) and \( g \) are defined as \( \kappa = \sqrt{E/\Delta} = ka \), \( f = \sqrt{V_1 + V_2 - \frac{1}{4}} \), \( g = \sqrt{V_1 - V_2 - \frac{1}{4}} \), where \( \Delta = \sqrt{\frac{\hbar^2}{2ma^2}} \). When \( V_1 = V_2 \), Eqs. (8a) and (8b) become quite simple.
In this tractable setting, one can readily check that $T(E)$ (8a) is invariant to the side of incidence of the particle by noticing that $T_i(E, V_2) = T_i(E, -V_2) = T_r(E, V_2)$. Equivalently, $T(E)$ is invariant if $f, g$ are inter-changed. Further, check that the transmission at any energy is non-anomalous : $0 < T(E) < 1$ as long as we have $V_1 > 0$ and $V_1 > V_2 - \Delta/4$. Generally, the imaginary part is a perturbation of smaller magnitude in many physical processes, we readily check that $R_l(E, V_2 : f, g)$ (8b) is always physical by remaining less than unity. However, $R_r(E, V_2 : f, g) = R_l(E, -V_2 : g, f)$ becomes anomalous for smaller values of energy. We also have $R_l(E) < R_r(E)$ provided $\Im(V_c(x < 0)) < 0$ (see Eq. (8c)). These results are displayed in Fig. 1.

When $V_2 > V_1 - \Delta/4$, the hyperbolic functions of $g$ become trigonometric and $T(E)$ attains anomalous values ($> 1$) at energies around the barrier height : $E \approx V_1$. In this situation, we again find that both $R_l(E) < R_r(E)$ and $R_r(E)$ is anomalous from lower (sub-barrier) energies upto energies slightly above the barrier height (see Fig. 2).

Apart from being analytically tractable, the Scarf potential is special as it is shape (form) invariant in a certain setting [12]. In order to separate out the presently claimed features from any other possible specialty of Scarf potential, we study some analytically intractable complex PT-symmetric potential models as illustrative examples. We consider two potentials :

$$V(x) = \frac{V_1 + iV_2 z}{(1 + z^2)^2}$$

and

$$V(x) = (V_1 + iV_2 z)e^{-|z|}, \ z = x/a.$$  

These potentials being analytically intractable could be more general as they do not belong to a known symmetry group. However, a common feature among the potentials (7), (9) and (10) is that these are much less localized in comparison to the rectangular potential.

Assuming $2m = 1 = \hbar$ and also $a = 1$, we have computed the scattering co-efficients by integrating the Schrödinger equation numerically on both the sides of $x = 0$ for the models (7), (9), and (10). It may be important to mention that the results (8a) and (8c) have also been checked against our numerical integration method for the Scarf potential (7). Various values of $V_1$ and $V_2$ (see Figs. 1-4) taken in arbitrary units are in no way special excepting that we plan to address to two situations : $V_2 < V_{2\text{critical}}$ and $V_2 > V_{2\text{critical}}$. We
find that there exists a characteristic critical value of $V_2$: $V_2^{\text{critical}} = f(V_1)$ above which even the transmission probability becomes unphysical at energies around the barrier height, $V_1$, (see Fig. 2) for a given complex PT-symmetric potential. For instance, for the tractable Scarf potential ($7$), we have $V_2^{\text{critical}} = V_1 - \Delta/4$. The figures 1, 3 and 4 pictorially display the claimed feature of physical scattering at least from one (absorptive) side of a complex PT-symmetric potential. For a given complex PT-symmetric potential, we find that $V_2^{\text{critical}}$ may also be much higher or much lower than $V_1$. See Figs. 3 and 4 for the potential models (9) and (10) respectively, when $V_1 = 5.0$ and $V_2 = 4.0$, the scattering from left is physical. We also find that when $V_1 = 4.0$ and $V_2 = 7.5$ the scattering from left remains physical for the models (9) and (10). For these models, when $V_2 = 8.0$, $T(E)$ attains anomalous values at energies around the barrier height, $V_1$, (as in Fig. 2). The absorption in this case becomes indefinite ($\pm$) in certain intervals of energies for the entrance of the particle from any side.

Complex PT-symmetric potentials surprise one readily as being non-Hermitian and yet possessing a real discrete spectrum. Hence, equally dramatical speculation that scattering from such potentials may yield no-absorption would not have been very surprising. We find that this situation of no absorption does not arise in the penetration through complex PT-symmetric potential barriers.

The other essence of a PT-symmetric potential governed by its parameters lies in two kinds of results it yields: usual and unusual. For instance PT-symmetric potentials give rise to real spectrum with two branches [4], real spectrum with complex-conjugate pairs of eigenvalues and usual [7,8] bandstructure containing discontinuous band gaps with unusual [5-9] bandstructure also containing rounded bands. In this regard, the present work reveals the handedness of a PT-symmetric optical potential barrier wherein the scattering/tunneling will be physical (usual) only if the particle is incident from absorptive (sink) side of the potential. If the particle enters from the other side, the scattering will be anomalous (unusual) in certain intervals of energy. Let us conclude by putting this feature in an amusing way: the complex PT-symmetric potential barriers may mimic a ‘spy-glass’ fitted in the windows of a room to view out-side without being viewed from there. Model independent explanation of the presently reported features may be very interesting.
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FIG. 1. Scattering co-efficients for the PT-symmetric complex Scarf potential (7), notice that
$R_r(E) > R_l(E) < 1$ and $0 < T(E) < 1$. Hence the scattering from left of the potential is physical,
i.e., $0 < A_l(E) < 1$.

FIG. 2. When $V_2 > V_2^{\text{critical}} = 3.75$ in (7), $T(E)$ becomes anomalous around the barrier height.
Thus, scattering from either side will yield an indefinite ($\pm$) absorption as a function of energy.
FIG. 3. For the complex PT-symmetric potential (9), for given parameters, the scattering from left is physical. Notice that $0 < T(E) < 1$ and $R_r(E) > R_l(E) < 1$.

FIG. 4. The same as for Fig. 3 excepting that the potential is given by (10).