WIGNER’S PHOTONS

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Abstract

If Einstein’s photon is \( E = cp = \hbar \omega \), Wigner’s photon is its helicity which is a Lorentz-invariant concept coming from the \( E(2) \)-like little group for massless particles. In addition, the \( E(2) \)-like little group has two translation-like degrees of freedom. What happens to them? They are associated with the gauge degree of freedom. Since the physics of polarized light waves can be formulated within the framework of the Lorentz group, it is now possible to use polarization experiments to study the \( E(2) \)-like little group in terms of quantities that can be measured in laboratories.

I. INTRODUCTION

The purpose of this meeting is to discuss photons, and there are many papers based on Wigner functions. The oscillator-based Wigner function is the natural language for the \( Sp(2) \) and \( Sp(4) \) groups [1]. These two groups also happened to be the natural languages for one- and two-mode squeezed states respectively. Since they are are isomorphic to \( O(2,1) \) and \( O(3,2) \) Lorentz groups respectively, the squeezed state is not only the physics of coherent photon states, but also the physics of Lorentz transformations. Let us note also that Lorentz boosts are squeeze transformations [2].

Wigner’s influence is not limited to Wigner functions and squeezed states. In his 1931 book entitled Gruppentheorie und ihre Anwendung auf Quantenmechanik der Atomspektren [3], Wigner gives a complete group theoretical description of angular-momentum states. Of course, transitions between quantized atomic states are mediated by photons carrying their angular momenta. It is therefore safe to say that this book was the first book on the quantum theory of photons, as well as on applications of group theory in physics. This book was translated from German into English and from the left-handed coordinate system to the right-handed coordinate system by Griffin in 1959 [4].

The main purpose of the present report is to discuss the plane-wave solution of Maxwell’s equations within the framework of Wigner’s little groups. In his 1939 paper on representations of the Poincaré group [7], Wigner formulated the internal space-time symmetries of relativistic particles in terms of the little groups. There he observed that the internal

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space-time symmetry of massless particles is dictated by a subgroup of the Lorentz group isomorphic to $E(2)$ or the two-dimensional Euclidean group. But Wigner did not address the question of how his formalism can accommodate Maxwell’s equations, and left this as a home-work problem for younger generations. This homework problem has now been solved by various authors, and Wigner’s 1939 paper provides a place for photons and Maxwell’s equations.

In 1949, Wigner published a paper on localization of relativistic systems [6]. This paper led to the problem known today as the photon localization [7]. As in the case of the little group, this localization problem with a single photon. However, the Lorentz group continues to play important roles in many-photon systems. As is well demonstrated at this conference, the squeezed state of light is the physics of the $O(2,1)$ and $O(3,2)$ Lorentz groups. In addition, it has been found recently that the Lorentz group is the natural language for the theory of polarized lights which is almost 150 years old [8].

Indeed, Wigner cannot be separated from classical and modern optics. In Sec. II, we establish a connection between Wigner’s representation theory and Maxwell’s equations. In Sec. III we discuss other areas of the science of photons where Wigner made fundamental contributions.

II. MAXWELL AND WIGNER

In 1939, Wigner observed that internal space-time symmetries of relativistic particles are dictated by their respective little groups [5]. The little group is the maximal subgroup of the Lorentz group which leaves the four-momentum of the particle invariant. The Lorentz group is generated by three rotation generators $J_i$ and three boost generators $K_i$, which satisfy the commutation relations:

\[
\begin{align*}
[J_i, J_j] &= i\epsilon_{ijk}J_k, \\
[J_i, K_j] &= i\epsilon_{ijk}K_k, \\
[K_i, K_j] &= -i\epsilon_{ijk}J_k.
\end{align*}
\]

(2.1)

If a massive particle is at rest, its momentum is invariant under three-dimensional rotations. Thus, its little group is generated by $J_1$, $J_2$, and $J_3$, and its spin orientation is changed under the little group transformation. If we use the metric convention $(x, y, z, t)$, the transformation matrix applicable to spin-1 particles is

\[
\begin{pmatrix}
  r_{11} & r_{12} & r_{13} & 0 \\
  r_{21} & r_{22} & r_{23} & 0 \\
  r_{31} & r_{32} & r_{33} & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix},
\]

(2.2)

where the three-by-three matrix consisting of the first three rows and columns constitute a rotation matrix in the three-dimensional space. This matrix leaves the four-momentum $(0,0,0,m)$ invariant, but can change the direction of the angular momentum.

For a massless particle, it is not possible to find a Lorentz frame in which the particle is at rest. We can however assume that its momentum is in the $z$ direction. Then the momentum is invariant under the subgroup of the Lorentz group generated by

\[
J_3, \quad N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1.
\]

(2.3)
Wigner noted in his 1939 paper that these generators satisfy the same set of commutation relations as those for the two-dimensional Euclidean group consisting of one rotation and translations in two different directions. With these generators, we can construct the following transformation matrices generated by $J_3$.

$$
\begin{pmatrix}
\cos \phi & 0 & -\sin \phi & 0 \\
0 & 1 & 0 & 0 \\
\sin \phi & 0 & \cos \phi & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
$$
\hspace{1cm} (2.4)

which performs rotations around the momentum. It is not difficult to associate this matrix with the helicity of the photon, and it is an invariant concept \[10–12\]. In addition, they generate transformation matrices of the form

$$
\begin{pmatrix}
1 & 0 & \frac{-u}{2} & \frac{v}{2} \\
0 & 1 & \frac{-u}{2} & \frac{v}{2} \\
u & v & 1 - \frac{(u^2 + v^2)}{2} & \frac{(u^2 + v^2)}{2} \\
u & v & -\frac{(u^2 + v^2)}{2} & 1 + \frac{(u^2 + v^2)}{2} \\
\end{pmatrix}.
$$
\hspace{1cm} (2.5)

This expression is given in Wigner’s original paper, and this matrix leaves the four-momentum

$$
(0, 0, \omega, \omega)
$$

invariant. But its strange appearance kept physicists away from the matrix for many years.

One quick solution to this problem was to avoid this expression. For this reason, for many years, there was a tendency to restrict representations which are invariant under this transformation. In 1964 \[13\], Weinberg systematically constructed representations which are invariant under this transformation, and ended up with the gauge-invariant electromagnetic tensor. This leads to a suspicion that the not-so-nice-looking matrix of Eq.\((2.5)\) could be a gauge transformation. What else could it be?

However, our imagination did not reach that quickly. This observation was made by several independent research groups after 1970 \[14,15\]. These days, we have an easy way to see this \[16\]. A plane wave propagating along the $z$ direction can have a four-potential \(A_1, A_2, A_3, A_0\), with the Lorentz condition \(A_3 = A_0\). If the third and the fourth components are equal, the four-by-four matrix of Eq.\((2.5)\) becomes

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
u & v & 1 & 0 \\
u & v & 0 & 1 \\
\end{pmatrix}.
$$
\hspace{1cm} (2.7)

Furthermore, its application to the four-potential leads to

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
u & v & 1 & 0 \\
u & v & 0 & 1 \\
\end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_0 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ uA_1 + vA_2 \\ uA_1 + vA_2 \end{pmatrix}.
$$
\hspace{1cm} (2.8)
This means the application of the matrix of Eq. (2.3) to the four-potential leads to the addition of a four-vector proportional to the four-momentum of Eq. (2.6). This is a gauge transformation!

We thus can give a complete interpretation of Wigner’s E(2)-like little group for photons in terms of Maxwell’s equations. In 1964, Weinberg started his analysis starting from the two-by-two representations of the $SL(2, C)$ group generated by

$$J_i = \frac{1}{2} \sigma_i, \quad K_i = \frac{i}{2} \sigma_i,$$

(2.9)

which satisfy the closed set of commutation relations given in Eq. (2.4). The commutation relations remain invariant under the sign change of the boost generators. Thus the representations of the $SL(2, C)$ group are twin representations, and there are four components in the spinor as in the case of the Dirac spinor. This leads to sixteen independent components of the tensor, and the systematic reduction leads to the four-potential and the Maxwell tensor $\mathbb{I}^7$, $\mathbb{I}^9$.

III. OTHER ASPECTS OF WIGNER’S PHOTONS

In order to complete the particle picture of photons, we have to second-quantize the Maxwell fields. This eventually leads to quantum electrodynamics. However, unlike Schrödinger wave functions, quantum fields do not carry probability interpretation. This point was first observed in Wigner’s 1949 paper on localization of relativistic systems $[3]$. There has been some progress in recent years on this question in terms of wavelets, but this fundamental problem still remains unsolved $[7]$.

On the other hand, there has been a concrete progress in recent years on applications of the Lorentz group in polarization optics. This subject deals with the Jones matrix and the Stokes parameters which are constantly used in laboratories where polarized photons and light waves are observed. If a light wave propagates along the $z$ direction, the electric field vector can be written as

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} A \exp \{i(kz - \omega t + \phi_1)\} \\ B \exp \{i(kz - \omega t + \phi_2)\} \end{pmatrix},$$

(3.1)

where $A$ and $B$ are real and positive numbers, and $\phi_1$ and $\phi_2$ are the phases of the $x$ and $y$ components respectively. This column matrix is called the Jones vector.

The transformation takes place when the light beam goes through an optical filter whose transmission properties are not isotropic. The absorption coefficient in one transverse direction could be different from the coefficient along the other direction. Thus, there is the “polarization” coordinate in which the absorption can be described by

$$\begin{pmatrix} e^{-\eta_1} & 0 \\ 0 & e^{-\eta_2} \end{pmatrix} = e^{-(\eta_1+\eta_2)/2} \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix},$$

(3.2)

with $\eta = \eta_2 - \eta_1$. This attenuation matrix tells us that the electric fields are attenuated at two different rates. The exponential factor $e^{-(\eta_1+\eta_2)/2}$ reduces both components at the same rate and does not affect the degree of polarization. The effect of polarization is solely determined by the squeeze matrix.
This type of mathematical operation is quite familiar to us from squeezed states of light [1]. Another basic element is the optical filter with two different values of the index of refraction along the two orthogonal directions. The effect of this filter can be written as

\[
\begin{pmatrix}
  e^{i\lambda_1} & 0 \\
  0 & e^{i\lambda_2}
\end{pmatrix}
= e^{-i(\lambda_1+\lambda_2)/2}
\begin{pmatrix}
  e^{-i\lambda_2/2} & 0 \\
  0 & e^{i\lambda_2/2}
\end{pmatrix},
\]

(3.4)

with \( \lambda = \lambda_2 - \lambda_1 \). In measurement processes, the overall phase factor \( e^{-i(\lambda_1+\lambda_2)/2} \) cannot be detected, and can therefore be deleted. The polarization effect of the filter is solely determined by the matrix

\[
P(0, \lambda) = \begin{pmatrix}
  e^{-i\lambda/2} & 0 \\
  0 & e^{i\lambda/2}
\end{pmatrix}.
\]

(3.5)

If the polarization coordinate is the same as the \( xy \) coordinate where the electric field components take the form of Eq.(3.1), the above attenuator is directly applicable to the column matrix of Eq.(3.1). If the polarization coordinate is rotated by an angle \( \theta/2 \), or by the matrix

\[
R(\theta) = \begin{pmatrix}
  \cos(\theta/2) & -\sin(\theta/2) \\
  \sin(\theta/2) & \cos(\theta/2)
\end{pmatrix}.
\]

(3.6)

Repeated applications of squeeze matrices of Eq.(3.3) and phase shifters of the form given in Eq.(3.6) together with rotation matrices of the above form will give a six-parameter transformation matrix of the form \[20\]

\[
\begin{pmatrix}
  \alpha & \beta \\
  \gamma & \delta
\end{pmatrix},
\]

(3.7)

generated by the two-by-two matrices given in Eq.(2.9). This is how the Jones-matrix formalism is framed into the Lorentz group. As we mentioned at the end of Sec. II, it is possible to construct a four-vector from the four \( SL(2, \mathbb{C}) \) spinors. The element of this four-vector are the Stokes parameters \[21\].

The Lorentz group indeed gives an elegant formalism for polarization optics. Then what new physics does this generate? This group has an \( E(2) \)-like little group as we discussed in Sec. II, and optical filters performing \( E(2) \)-like transformations may be manufactured in laboratories \[22\]. In this case, those \( E(2) \)-like filters will perform observable operations on light waves which are mathematically equivalent to gauge transformations.

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