On the Structure of Unoriented Topological Conformal Field Theories

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Abstract

We give a classification of open Klein topological conformal field theories in terms of Calabi-Yau $A_\infty$-categories endowed with an involution. Given an open Klein topological conformal field theory, there is a universal open-closed extension whose closed part is the involutive variant of the Hochschild chains of the open part.

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1 Introduction

1.1 Oriented and Klein TQFTs

The study of topological conformal field theories began with the works of Segal on conformal field theory [Seg88]. Inspired by Segal’s work, Atiyah gave a list of axioms for what he defines as a topological quantum field theory [Ati88]. It was Moore and Segal [Moo01, MS06] who first defined the concept of topological conformal field theory and suggested the importance of studying them.

For a finite set of $D$-branes $\Lambda$, let $\text{Cob}_\Lambda$ be the category whose class of objects are 1-manifolds (disjoint unions of circles and intervals) with boundary labelled by $D$-branes and with class of morphisms given by cobordism of these. Given a field $K$, a 2-dimensional topological quantum field theory (henceforth a TQFT) is a symmetric monoidal functor $F : \text{Cob}_\Lambda \to \text{Vect}_K$, where $\text{Vect}_K$ is the category of $K$-vector spaces. Depending on the boundary components, we can study open TQFTs if $\Sigma$ has only open and free boundary components; closed TQFTs if $\Sigma$ has only closed boundary components or open-closed TQFTs, where $\Sigma$ has open, closed and free boundary components.

A finite-dimensional unital associative $K$-algebra $A$ is called (commutative) Frobenius if it is equipped with a (commutative) bilinear form $\mu : A \otimes A \to K$ such that $\mu(ab,c) = \mu(a,bc)$ and is non-degenerate.

It is well known that:

1. The category of 2-dimensional open TQFTs is equivalent to the category of not-necessarily commutative Frobenius algebras (page 7 [MS06]) and

2. the category of 2-dimensional closed TQFTs is equivalent to the category of commutative Frobenius algebras (Theorem 3.3.2 [Koc04]).

If we change the morphisms in $\text{Cob}_\Lambda$ allowing not only oriented surfaces but unoriented surfaces, we get Klein topological quantum field theories. Closed Klein topological quantum field
theories have been studied and classified in terms of Frobenius algebras endowed with extra
structure coming from the extra generator one has to consider: the real projective plane \( \mathbb{RP}^2 \)
with two holes [TT06, AN06]. Open Klein topological quantum field theories are equivalent to
non-commutative Frobenius algebras endowed with an involution [Bra12]. Open-closed Klein
topological quantum field theories are completely described algebraically in terms structure
algebras [AN06].

1.2 Oriented and Klein TCFTs

If we endow the morphisms of \( \text{Cob}_\Lambda \) with a complex structure we can define a category \( \mathcal{C}(M_\Lambda) \)
with the same class of objects of \( \text{Cob}_\Lambda \) and where the arrows are given by singular chains on
moduli spaces of Riemann surfaces. In this new setting it makes sense to work at a chain
level so we can then consider functors of the form \( \mathfrak{F} : \mathcal{C}(M_\Lambda) \rightarrow \text{Comp}_K \), where \( \text{Comp}_K \) is
the category of chain complexes over \( K \). Such a functor \( \mathfrak{F} \) is called a 2-dimensional topological
conformal field theory (a TCFT henceforth). As in the topological setting, we talk about open,
closed and open-closed TCFTs depending on the boundary components of the Riemann sur-
faces we work with. Open TCFTs were classified by Costello [Cos07] in terms of \( A_\infty \)-categories
satisfying a Calabi-Yau condition. The work done by Costello relies on a ribbon graph decom-
position of the moduli space of Riemann surfaces with marked points. Costello also gives a
universal extension from open TCFTs to open-closed TCFTs and proves that the homology of
the Calabi-Yau \( A_\infty \)-category associated to the closed part of an open-closed TCFT is described
in terms of the Hochschild homology of the Calabi-Yau \( A_\infty \)-category associated to its open part.

Costello’s work was partially generalized to the unoriented setting, that is replacing Riemann
surfaces with Klein surfaces, by Braun [Bra12]. In his work, Braun gives a decomposition of the
moduli space of Klein surfaces in terms of Möbius graphs, allowing him to state the classifi-
cation of open Klein TCFTs in terms of involutive \( A_\infty \)-algebras using techniques from operads
theory.

1.3 The results of this research

By extending Costello’s techniques to the unoriented setting, the research developed here rep-
resents a completion of the picture started by Braun. The main result is:

**Theorem 1.1.**

1. There is an equivalence between open Klein TCFTs and Calabi-Yau \( A_\infty \)-categories
   endowed with an involution.

2. Given an open Klein TCFT, there exists a universal open-closed extension to open-closed Klein
   TCFTs.

3. The homology of the closed part of the above open-closed TCFT is described in terms of the invol-
   utive Hochschild homology of its open part.

The description of involutive Hochschild homology has been studied in detail in [FVG15].
Involutive Hochschild homology and “usual” Hochschild homology do not coincide unless
the algebras involved are commutative and endowed with the trivial involution.
2 A closer look to topological conformal field theories

By endowing the morphisms in $\mathcal{Cob}_\Lambda$ with a complex structure we can define a category $\mathcal{M}_\Lambda$ with the class of objects of $\mathcal{Cob}_\Lambda$ and with class of arrows given by moduli spaces of Riemann surfaces.

Let $\mathcal{C}: \text{Top} \to \text{Comp}_K$ be a functor from topological spaces to chain complexes. As Riemann surfaces form moduli spaces, applying $\mathcal{C}$ to the space of arrows of $\mathcal{M}_\Lambda$ yields a differential graded symmetric monoidal category $\mathcal{O}_\Lambda$ with $\text{Obj}(\mathcal{O}_\Lambda) = \text{Obj}(\mathcal{M}_\Lambda)$ and with class of arrows:

$$\text{Hom}_{\mathcal{O}_\Lambda}(I, J) := \mathcal{C}(\text{Hom}_{\mathcal{M}_\Lambda}(I, J)).$$

Given a set of D-branes $\Lambda$, a 2-dimensional open-closed TCFT with set of D-branes $\Lambda$ is a pair $(\Lambda, \mathfrak{F})$, where $\mathfrak{F}$ is a h-split (the map $\mathfrak{F}(I \sqcup J) \to \mathfrak{F}(I) \otimes \mathfrak{F}(J)$ is an isomorphism in homology) symmetric monoidal functor

$$\mathfrak{F}: \mathcal{O}_\Lambda \to \text{Comp}_K.$$

As in the topological setting, we can consider just open and free boundary components in order to work not with $\mathcal{O}_\Lambda$ but with a subcategory $\mathcal{O}_\Lambda$; or we can consider just closed boundary components in order to work with a subcategory $\mathcal{C}_\Lambda$. Therefore, considering h-split symmetric monoidal functors $\mathfrak{f}: \mathcal{O}_\Lambda \to \text{Comp}_K$ will yield open TCFTs whilst considering h-split symmetric monoidal functors $\mathfrak{f}: \mathcal{C}_\Lambda \to \text{Comp}_K$ will return closed TCFTs.

Costello classified open TCFTs in terms of Calabi-Yau $A_\infty$-categories. An $A_\infty$-category $\mathcal{C}$ consists of:

1. A class of objects $\text{Obj}(\mathcal{C})$;
2. for each $X_1, X_2 \in \text{Obj}(\mathcal{C})$, a $\mathbb{Z}$-graded abelian group of homomorphisms $\text{Hom}_\mathcal{C}(X_1, X_2)$;
3. for all $n \geq 1$, composition maps

$$b_n: \text{Hom}_\mathcal{C}(X_1, X_2) \otimes \cdots \otimes \text{Hom}_\mathcal{C}(X_n, X_{n+1}) \to \text{Hom}_\mathcal{C}(X_1, X_{n+1})$$

of degree $n - 2$ satisfying homotopy associativity conditions [Cos07].

Remark 2.1. If we restrict the class of objects of $\mathcal{C}$ to a set with a single object we get the concept of $A_\infty$-algebra.

A Calabi-Yau category is a $K$-algebroid $\mathcal{E}$ equipped with a trace map

$$\text{Tr}_A: \text{Hom}_{\mathcal{E}}(A, A) \to K$$

for every $A \in \text{Obj}(\mathcal{E})$ such that, for each $A, B \in \text{Obj}(\mathcal{E})$, the induced pairing

$$\text{Hom}_{\mathcal{E}}(A, B) \otimes \text{Hom}_{\mathcal{E}}(B, A) \to \text{Hom}_{\mathcal{E}}(A, A) \to K$$

is symmetric and non-degenerate.
Remark 2.2. Frobenius algebras are the result of having Calabi-Yau categories with a single object.

A Calabi-Yau $\mathcal{A}_\infty$-category is an $\mathcal{A}_\infty$-category $\mathcal{E}$ with a non-degenerate pairing of chain complexes $\langle -, - \rangle_{A,B} : \text{Hom}_\mathcal{E}(A, B) \otimes \text{Hom}_\mathcal{E}(B, A) \rightarrow \mathbb{K}$, satisfying certain conditions [Cos07].

Costello proves in Lemma 7.3.4 [Cos07] that the category of open TCFTs is quasi-equivalent to the category of unital Calabi-Yau $\mathcal{A}_\infty$-categories. The way he proves this result is heavily based on a ribbon graph decomposition for the moduli space of Riemann surfaces [Cos06], what allows one to replace $O$ with another category for which we can define a set of generators and relations.

The results obtained by Costello are all twisted by a local system of coefficients on the moduli spaces which has been ignored here. This twisting is useful and necessary to Costello due to his motivations related to Gromov-Witten theory; we ignore it for the sake of simplicity in the notations: all the results contained in this manuscript hold if we keep track of this local system.

3 Homological algebra and category theory

Braun [Bra12] gives a classification of open Klein topological conformal field theories in terms of Calabi-Yau $\mathcal{A}_\infty$-categories endowed with involution using algebras over modular operads. It will be necessary to begin with an introduction of the concepts and notations which will be used henceforth and that will be central in these notes.

3.1 DGSM categories

Let $\mathbb{K}$ be a field. A graded $\mathbb{K}$-module is a $\mathbb{K}$-module $V$ together with a decomposition indexed by integers: $V = \bigoplus_{p \in \mathbb{Z}} V^p$. We define a differential graded $\mathbb{K}$-module as a graded $\mathbb{K}$-module $V$ equipped with a map $\delta : V \rightarrow V$ of degree 1 such that $\delta^2 := \delta \circ \delta = 0$.

We define a $\mathbb{K}$-category $\mathcal{A}$ as the datum formed by:

1. A class of objects $\text{Obj}(\mathcal{A})$;
2. a $\mathbb{K}$-module $\text{Hom}_\mathcal{A}(X_1, X_2)$ for each pair $X_1, X_2 \in \text{Obj}(\mathcal{A})$;
3. a $\mathbb{K}$-linear associative composition map

$$
\mu : \text{Hom}_\mathcal{A}(X_1, X_2) \otimes \text{Hom}_\mathcal{A}(X_2, X_3) \rightarrow \text{Hom}_\mathcal{A}(X_1, X_3)
$$

$f \otimes g \mapsto g \circ f$

admitting identity maps $\text{Id}_X \in \text{Hom}_\mathcal{A}(X_1, X_1)$.

We define an involutive $\mathbb{K}$-category as a $\mathbb{K}$-category $\mathcal{A}$ endowed with a functor $* : \mathcal{A}^{\text{op}} \rightarrow \mathcal{A}$ which is the identity on objects and such that the following identity holds: $(f \circ g)^* = g^* \circ f^*$.

For involutive $\mathbb{K}$-categories $(\mathcal{A}, *)$ and $(\mathcal{B}, \dagger)$, a functor $\mathfrak{F} : (\mathcal{A}, *) \rightarrow (\mathcal{B}, \dagger)$ is a functor of the underlying categories such that $\mathfrak{F} \circ * = \dagger \circ \mathfrak{F}$. 

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**Remark 3.1.** This concept is usually known in the literature as dagger $\mathbb{K}$-category.

An involutive differential graded category (involutive DG category henceforth) is an involutive $\mathbb{K}$-category $\mathcal{A}$ whose morphism class is a DG $\mathbb{K}$-module and whose composition maps are morphisms of differential graded $\mathbb{K}$-modules.

**Example 3.2.** Differential graded categories consisting of one object may be identified with differential graded algebras, that is: graded algebras $A$ endowed with a map $\delta$ satisfying the Leibniz rule: for every $f, g \in A^p$, $\delta(f, g) = \delta(f) g + (-1)^p f \delta(g)$.

A monoidal category is a category $\mathcal{B}$ with a functor $\odot : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, an object $I \in \text{Obj}(\mathcal{B})$, which will be called identity, and for objects $X, Y, Z$ in $\mathcal{B}$, natural isomorphisms

\[
\begin{align*}
(X \odot Y) \odot Z & \xrightarrow{\alpha_{X,Y,Z}} X \odot (Y \odot Z) \\
I \odot X & \xrightarrow{\lambda_X} X \\
X \odot I & \xrightarrow{\mu_X} X
\end{align*}
\]

making the diagrams below commute:

\[
\begin{array}{ccc}
A \odot (B \odot (C \odot D)) & \longrightarrow & (A \odot B) \odot (C \odot D) \\
\downarrow & & \downarrow \\
A \odot ((B \odot C) \odot D) & \longrightarrow & (A \odot (B \odot C)) \odot D
\end{array}
\]

\[
\begin{array}{ccc}
(X \odot I) \odot Y & \longrightarrow & X \odot (I \odot Y) \\
X \odot Y & \longrightarrow & \downarrow \\
\end{array}
\]

A symmetric monoidal category is a monoidal category $(\mathcal{C}, \otimes)$ such that, for $X, Y, Z \in \text{Obj}(\mathcal{C})$, there is a natural isomorphism $\sigma_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$ which makes the diagrams below commute:

\[
\begin{array}{ccc}
X \odot I & \longrightarrow & I \odot X \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y \odot X
\end{array}
\]

\[
\begin{array}{ccc}
X \otimes Y & \longrightarrow & X \otimes Y \\
\downarrow & & \downarrow \\
Y \otimes X & \longrightarrow & \downarrow \\
\end{array}
\]

\[
\begin{array}{ccc}
(X \otimes Y) \odot Z & \longrightarrow & X \otimes (Y \odot Z) \\
\downarrow & & \downarrow \\
(Y \otimes X) \odot Z & \longrightarrow & Y \otimes (Z \odot X)
\end{array}
\]

A involutive differential graded symmetric monoidal category (involutive DGSM for short) is an involutive DG category $\mathcal{D}$ which is symmetric and monoidal.
3.2 Modules over categories

For an involutive DGSM category \( \mathcal{A} \), a left \( \mathcal{A} \)-module is a monoidal functor \( \mathcal{L} : \mathcal{A} \rightarrow \text{Comp}_K \). A right \( \mathcal{A} \)-module is a monoidal functor \( \mathcal{R} : \mathcal{A}^{\text{op}} \rightarrow \text{Comp}_K \). Given an involutive DGSM category \( \mathcal{B} \) and two monoidal functors \( \mathcal{M}, \mathcal{N} : \mathcal{A} \rightarrow \mathcal{B} \), a natural transformation \( \phi : \mathcal{M} \rightarrow \mathcal{N} \) consists of a collection of maps \( \phi(a) \), where \( a \in \text{Obj}(\mathcal{A}) \), in \( \text{Hom}_\mathcal{B}(\mathcal{M}(a), \mathcal{N}(a)) \) satisfying:

1. The following diagrams commute

\[
\begin{array}{ccc}
\mathcal{M}(a_1) & \xrightarrow{\phi(a_1)} & \mathcal{N}(a_1) \\
\mathcal{M}(f) \downarrow & & \downarrow \mathcal{N}(f) \\
\mathcal{M}(a_2) & \xrightarrow{\phi(a_2)} & \mathcal{N}(a_2)
\end{array}
\]

for morphisms \( f : a_1 \rightarrow a_2 \) and \( a_1, a_2 \in \text{Obj}(\mathcal{A}) \);

2. the morphisms \( \phi(a) \) are all closed of degree 0, for \( a \in \text{Obj}(\mathcal{A}) \).

We have two categories, one of left \( \mathcal{A} \)-modules, denoted by \( \mathcal{A} \text{-Mod} \), and another one of right \( \mathcal{A} \)-modules, which will be denoted by \( \text{Mod} \text{-}\mathcal{A} \).

Given two involutive DGSM categories \( \mathcal{A} \) and \( \mathcal{B} \), we can form their tensor product category \( \mathcal{A} \otimes \mathcal{B} \). This category is formed by the following classes of objects and morphisms:

1. For the class of objects: \( \text{Obj}(\mathcal{A} \otimes \mathcal{B}) = \text{Obj}(\mathcal{A}) \times \text{Obj}(\mathcal{B}) \);

2. for the class of morphisms:

\[
\text{Hom}_{\mathcal{A} \otimes \mathcal{B}}(a_1 \times b_1, a_2 \times b_2) = \text{Hom}_{\mathcal{A}}(a_1, a_2) \otimes \text{Hom}_{\mathcal{B}}(b_1, b_2),
\]

with: \( a_1, a_2 \in \text{Obj}(\mathcal{A}) \) and \( b_1, b_2 \in \text{Obj}(\mathcal{B}) \). If a morphism \( f \) in \( \mathcal{A} \otimes \mathcal{B} \) has components \( f = (f_1, f_2) \), then the involution is:

\[
f^*(-,-) := f_1^*(-) \otimes f_2^*(-).
\]

An \( \mathcal{A} \text{-}\mathcal{B}\text{-bimodule} \) is a (monoidal) functor \( \tilde{\mathcal{F}} : \mathcal{A} \otimes \mathcal{B}^{\text{op}} \rightarrow \text{Comp}_K \).

3.3 Derived tensor products and push-forwards

Let \( \mathcal{M} \) be a \( \mathcal{B} \text{-}\mathcal{A}\text{-bimodule} \) and \( \mathcal{N} \) a left \( \mathcal{A} \)-module. The left \( \mathcal{B} \)-module \( \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} \) is given by saying that \( (\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})(b) \) is the complex with maps \( \mathcal{M}(b,a) \otimes_K \mathcal{N}(a) \rightarrow (\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})(b) \) such that make the diagram commute:

\[
\begin{array}{ccc}
\mathcal{M}(b,a) \otimes_K \text{Hom}_{\mathcal{A}}(a',a) \otimes_K \mathcal{N}(a') & \longrightarrow & \mathcal{M}(b,a) \otimes_K \mathcal{N}(a') \\
\downarrow & & \downarrow \\
\mathcal{M}(b,a') \otimes_K \mathcal{N}(a') & \longrightarrow & (\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})(b)
\end{array}
\]

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Consider a functor between involutive DGSM categories $f : \mathcal{A} \to \mathcal{B}$. Under these conditions $\text{Hom}_\mathcal{B}(-,-)$ is a $\mathcal{B}$-bimodule and becomes an $\mathcal{A}$-$\mathcal{B}$-bimodule and a $\mathcal{B}$-$\mathcal{A}$-bimodule via the functors

$\text{Hom}_\mathcal{B} : \mathcal{B} \otimes \mathcal{B}^\text{op} \to \text{Comp}_K$

$\mathcal{B}(b_1 \otimes b_2) \leadsto \text{Hom}_\mathcal{B}(b_1, b_2)$

$\mathcal{F} : \mathcal{A} \otimes \mathcal{B}^\text{op} \to \mathcal{B} \otimes \mathcal{B}^\text{op}$

$a \otimes b \leadsto f(a) \otimes b$

$\mathcal{G} : \mathcal{B} \otimes \mathcal{A}^\text{op} \to \mathcal{B} \otimes \mathcal{B}^\text{op}$

$b \otimes a \leadsto b \otimes f(a)$

We define a functor $f^+ : \mathcal{A} \to \mathcal{B}$ by setting

$f^+(\mathcal{M}) := \text{Hom}_\mathcal{B} \otimes_\mathcal{A} \mathcal{M} := \mathcal{B} \otimes_\mathcal{A} \mathcal{M}$.

We define a functor $f^* : \mathcal{B} \to \mathcal{A}$ as the composition of $\mathcal{M} : \mathcal{B} \to \text{Comp}_K$ with $f : \mathcal{A} \to \mathcal{B}$:

$f^* : \mathcal{A} \to \mathcal{B} \to \text{Comp}_K$

$a \leadsto f(a) \leadsto \mathcal{M}(f(a))$

Let us denote by $S_n$ the symmetric group on $n$ letters. For $\mathcal{A}$ an involutive DGSM category let $\text{Sym} \mathcal{A}$ be the subcategory whose objects are those of $\mathcal{A}$ and whose morphisms are the identity maps and the symmetry isomorphisms:

$a_1 \otimes a_2 \otimes \cdots \otimes a_n \cong a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$, for $\sigma \in S_n$.

We define the category $\text{Sym}_K \mathcal{A}$ as the sub-linear category of $\text{Sym} \mathcal{A}$ whose morphisms are spanned by the morphisms in $\text{Sym} \mathcal{A}$.

Following [Cos07], we denote by $\text{Comp}^\Delta_K$ the symmetric monoidal category of simplicial chain complexes. We define the realization of $C \in \text{Obj}(\text{Comp}^\Delta_K)$ as

$|C| = \bigoplus_{n \geq 0} \frac{C[n]}{C^{\text{deg}}[n]}[-n]$,

where $C^{\text{deg}}[n]$ is the image of the degeneracy maps and $[-n]$ denotes a degree shifting.

Given an $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{M}$ and a left $\mathcal{B}$-module $\mathcal{N}$, we define the left $\mathcal{A}$-module:

$\mathcal{M} \otimes^L_\mathcal{B} \mathcal{N} := \mathcal{M} \otimes_\mathcal{B} \text{Bar}_B(\mathcal{N})$,

where $\text{Bar}_B(\mathcal{N}) := |\text{Bar}^\Delta_B(\mathcal{N})|$ and $\text{Bar}^\Delta_B(\mathcal{N})$ is the following simplicial $\mathcal{A}$-module:

$(\text{Bar}^\Delta_B(\mathcal{N}))[n] := \mathcal{B} \otimes \text{Sym}_K \mathcal{B} \otimes \text{Sym}_K \mathcal{B} \cdots \otimes \text{Sym}_K \mathcal{B} \otimes \text{Sym}_K \mathcal{B} \mathcal{N}$.

The face maps come from the product maps $\mathcal{B} \otimes \text{Sym}_K \mathcal{B} \to \mathcal{B}$ whilst the degeneracy maps come from the maps $\text{Sym}_K \mathcal{B} \to \mathcal{B}$.

The definition for the derived tensor product makes sense due to the following Lemmata:
Lemma 3.3 ([Cos07], Lemma 4.3.3). The projection $\pi : \text{Bar}_B(\mathfrak{N}) \to \mathfrak{N}$ is a quasi-isomorphism.

Lemma 3.4 ([Cos07], Lemma 4.3.4). For any $B$-module $\mathfrak{N}$, $\text{Bar}_B(\mathfrak{N})$ is a flat $B$-module.

For $f : \mathcal{A} \to \mathcal{B}$ a functor between involutive DGSM categories and $\mathfrak{N}$ a left $\mathcal{A}$-module we define

$$L_f \mathfrak{N} := B \otimes_{\mathcal{A}} A \mathfrak{N}. \quad (1)$$

Remark 3.5. Let us recall that

$$L_f \mathfrak{N} := B \otimes_{\mathcal{A}} A \mathfrak{N} \simeq B \otimes_{\mathcal{A}} \text{Bar}_A A \mathfrak{N} = B \otimes_{\mathcal{A}} \left[ \text{Bar}_A A \mathfrak{N} \right]$$

and it is well known that we can write the last tensor product as the coend $\int \mathfrak{E} \mathcal{B} \otimes \mathfrak{E}$. On the other hand, for $\mathfrak{N} \in \text{Obj}(\mathcal{A} \cdot \text{-Mod})$ and $\mathfrak{F} : \mathcal{A} \to \mathcal{B}$ a functor between involutive DGSM categories, we can write (Theorem 1, chapter X, section 4 [ML98]), for each $c \in \text{Obj}(\mathcal{B})$:

$$(\text{Lan}_{\mathcal{F}} \mathfrak{N})(c) = \int_{a \in \text{Obj}(\mathcal{A})} \text{Hom}_B(\mathfrak{F}(a), c) \otimes \mathfrak{N}(a).$$

Then we can think of (1) as an example of a derived left Kan extension.

### 3.4 Quasi-isomorphisms in a category

A morphism $f : C_\bullet \to D_\bullet$ of complexes in an abelian category $\mathcal{A}$ is a quasi-isomorphism if the corresponding homology morphism

$$H_n(f) : H_n(C_\bullet) \to H_n(D_\bullet)$$

is an isomorphism for each $n \in \mathbb{Z}$.

An $\mathcal{A}$-module $\mathfrak{M}$ is flat if the functor $(-) \otimes_{\mathcal{A}} \mathfrak{M} : \text{Mod} - \mathcal{A} \to \text{Comp}_K$ is exact, that is: if it sends quasi-isomorphisms to quasi-isomorphisms. We denote by $\mathcal{A}$-flat the full subcategory of flat $\mathcal{A}$-modules and the inclusion by $i : \mathcal{A}$-flat $\hookrightarrow \mathcal{A}$-Mod.

A category $\mathcal{C}$, not necessarily abelian, has a notion of quasi-isomorphism when we are given a subset of $\text{Hom}_\mathcal{C}(-, -)$ which is closed under composition and contains all isomorphisms. Objects in $\mathcal{C}$ are said to be quasi-isomorphic if they can be connected by a chain of quasi-isomorphisms. We write $c_1 \simeq c_2$ when two objects $c_1$, $c_2$ are quasi-isomorphic.

We define a natural transformation $\phi$ between exact functors $\mathfrak{F}, \mathfrak{G}$ as a quasi-isomorphism $\phi_c : \mathfrak{F}(c) \to \mathfrak{G}(c)$ is a quasi-isomorphism for every object $c \in \text{Obj}(\mathcal{C})$.

Given categories $\mathcal{C}$ and $\mathcal{D}$ with the notion of quasi-isomorphism, we define a quasi-equivalence as a pair of functors $\mathfrak{F} : \mathcal{C} \to \mathcal{D}$ and $\mathfrak{G} : \mathcal{D} \to \mathcal{C}$ such that the following quasi-isomorphisms of functors hold: $\mathfrak{F} \circ \mathfrak{G} \simeq 1_\mathcal{D}$ and $\mathfrak{G} \circ \mathfrak{F} \simeq 1_\mathcal{C}$.
Lemma 3.6 (cf. [Cos07], Lemma 4.4.1). Given two involutive DGSM categories \( A \) and \( B \), let us assume that the homology functor \( H_\bullet(\mathcal{F}) : H_\bullet(A) \to H_\bullet(B) \) is fully faithful. Then the functor \( \mathcal{F}_* \mathcal{L} \mathcal{F}_* \) is quasi-isomorphic to \( 1_{A \text{-Mod}} \).

Theorem 3.7 (cf. [Cos07], Lemma 4.4.3). For involutive DGSM categories \( A \) and \( B \), if \( \mathcal{F} : A \to B \) is a quasi-isomorphism, then the functors \( \mathcal{L} \mathcal{F}_* \) and \( \mathcal{F}_* \) are inverse quasi-equivalences between \( A \text{-Mod} \) and \( B \text{-Mod} \).

Proposition 3.8 ([Cos07], Lemma 4.4.4). Let us consider \( \mathcal{F}_* \) and \( \mathcal{L} \mathcal{F}_* \) the induced quasi-equivalences between \( A \text{-Mod} \times A \text{-Mod} \leftrightarrow B \text{-Mod} \times B \text{-Mod} \). Then the diagram below commutes up to quasi-isomorphism:

\[
\begin{array}{ccc}
\text{Mod-}A \times A \text{-Mod} & \xrightarrow{\otimes_A} & \text{Comp}_K \\
\downarrow \mathcal{F}_* & & \downarrow \mathcal{L} \mathcal{F}_* \\
\text{Mod-}B \times B \text{-Mod} & \xleftarrow{\otimes_B} & \\
\end{array}
\]

4 Fundamentals from graph theory

The role played by graphs is central in the theory of moduli spaces of Riemann or Klein surfaces as ribbon graphs provide orbi-cell decompositions of moduli spaces of Riemann surfaces [Cos04, Cos06]. In order to deal with Klein surfaces, ribbon graphs are not enough and one has to introduce the concept of Möbius graph. Möbius graphs provide an orbi-cell decomposition of moduli spaces of Klein surfaces. For further details we refer to the main reference of this section: [Bra12].

4.1 Ribbon graphs

A finite graph \( \gamma \) consists of:

1. A finite set of vertices \( V(\gamma) \) and half-edges \( H(\gamma) \);
2. an involution \( \iota : H(\gamma) \to H(\gamma) \) and a map \( \lambda : H(\gamma) \to V(\gamma) \).

Given a graph \( \gamma \), we say that two half-edges \( a, b \) form an edge if \( \iota(a) = b \); a half-edge \( a \) is connected to a vertex \( v \) if \( \lambda(a) = v \). A leg in \( \gamma \) is a univalent vertex, an external edge is an edge that meets a leg and an internal edge is an edge for which neither end is univalent.

Remark 4.1. We can imagine and edge \( e \) as a pair of half-edges \( e_1, e_2 \) by cutting \( e \) in half. Observe that the involution \( \iota \) swaps the half-edges. On the other hand \( \lambda \), by sending a half-edge \( e_i \) to a vertex \( v_i \), is gluing \( e_i \) with \( v_i \).

Remark 4.2. Let \( e = (e_1, e_2) \) be an external edge and let \( l \) be a leg. Let \( a \in H(\gamma) \) be a half-edge of \( l \) connected to \( v \in V(\gamma) \), we say that “\( a \) meets \( l \)” if \( \lambda(a) = v = \lambda(e_1) \).
Given two graphs $\gamma_1, \gamma_2$, a graph isomorphism $g : \gamma_1 \rightarrow \gamma_2$ is given by a pair $(g_1, g_2)$ of bijections $g_1 : V(\gamma_1) \rightarrow V(\gamma_2)$ and $g_2 : H(\gamma_1) \rightarrow H(\gamma_2)$ satisfying $\lambda \circ g_2 = g_1 \circ \lambda$ and $\iota \circ g_2 = g_1 \circ \iota$.

A ribbon graph is a finite graph equipped with a cyclic ordering of the half-edges at each vertex and a labelling of the legs, that is: the $n$ legs of the ribbon graph $\gamma$ are labelled by the elements of $\{1, \ldots, n\}$. An isomorphism of ribbon graphs is an isomorphism of graphs that preserves the cyclic ordering at each vertex and the labelling of the legs.

Given a ribbon graph $\gamma$ and an internal edge $e$ which is not a loop, we define the edge contraction $\gamma/e$ by endowing the graph $\gamma/e$ we get after contracting the edge $e$ with the obvious cyclic ordering coming from the cyclic orderings at the vertices bordering $e$.

For a ribbon graph $\gamma$ and two internal edges $e_1, e_2$ that are not loops we have the following isomorphism: $(\gamma/e_1)/e_2 \cong (\gamma/e_2)/e_1$, assuming both sides are defined.

A reduced ribbon graph is a ribbon graph where each vertex is either univalent or has valence at least 3. Given a graph with at least one vertex having valence at least 3, we can associate to it reduced graphs by repeatedly contracting an edge attached to a vertex of valence 2 until the graph is reduced.

4.2 Möbius graphs

A Möbius graph is a ribbon graph $\gamma$ with a colouring of the half-edges by two colours, which means that we have a map $c : H(\gamma) \rightarrow \mathbb{Z}_2$. An isomorphism of Möbius graphs is an isomorphism of graphs preserving the sum (modulo 2) of the labellings on each leg such that, at each vertex $v$, it can happen that either:

1. The map preserves the cyclic ordering at $v$ and the colouring of the half-edges at $v$; or
2. the map reverses the cyclic ordering at $v$ and reverses the colouring at the half-edges connected to $v$.

Given a Möbius graph $\gamma$ and an internal edge $e$ (not being a loop) where both the half-edges of $e$ have the same colour, we define the graph contraction $\gamma/e$ as we did for ribbon graphs. This is well defined on isomorphism classes and can be extended to all internal edges except loops, regardless the colouring. For a Möbius graph $\gamma$ and two internal edges $e_1, e_2$ (which are not loops) whose half-edges have the same colouring, the following isomorphism holds: $(\gamma/e_1)/e_2 \cong (\gamma/e_2)/e_1$, assuming both sides are defined.

A reduced Möbius graph is a Möbius graph where each vertex is either univalent or has valence at least 3. Given a Möbius graph with at least one vertex having valence at least 3, we can associate to it reduced Möbius graphs by repeatedly contracting an edge attached to a vertex of valence 2 until the graph is reduced.
5 Fundamentals on Klein surfaces

We revisit the concepts of Klein and nodal Klein surfaces and state equivalences of categories between Klein surfaces and Riemann surfaces with an involution following the results and techniques developed in [Bra12]. These equivalences will establish a duality that will make the forthcoming results almost a direct consequence of the results in [Cos04, Cos06, Cos07].

5.1 Klein surfaces and symmetric Riemann surfaces

Let $D \subset \mathbb{C}$ be a non-empty open subset and $f : D \to \mathbb{C}$ a smooth map. We say that $f$ is di-analytic if its restriction to each component of $D$ is either analytic or anti-analytic. If $A$ and $B$ are non-empty subsets of the complex upper half-plane $\mathbb{C}^+$, a map $g : A \to B$ is called analytic (resp. dianalytic) on $A$ if it extends to an analytic (resp. dianalytic) map $g' : U \to \mathbb{C}$ where $U$ is an open neighbourhood of $A$ in $\mathbb{C}$.

An atlas $\Xi$ on a surface $K$ is dianalytic if all the transition maps of $\Xi$ are dianalytic. A dianalytic structure on $K$ is a maximal dianalytic atlas. A Klein surface is a surface equipped with a dianalytic structure.

A morphism between Klein surfaces is a non-constant continuous map $(K_1, \partial K_1) \xrightarrow{f} (K_2, \partial K_2)$ such that for all $x \in K_1$ there are respective charts $(U_1, \phi_1)$ and $(U_2, \phi_2)$ around $x$ and $f(x)$ and an analytic map $F : \phi_1(U_1) \to \mathbb{C}$ making the diagram below commute:

$$
\begin{array}{ccc}
U_1 & \xrightarrow{f} & U_2 \\
\phi_1 \downarrow & & \phi_2 \\
\phi_1(U_1) & \xrightarrow{F} & \mathbb{C} \\
\end{array}
$$

Here $\varphi(x + iy) := x + i|y|$ is the folding map. We call $f$ dianalytic if we can choose charts where $\varphi \circ F$ is dianalytic.

**Lemma 5.1** ([Bra12], p. 59). The composition of dianalytic morphisms of Klein surfaces is dianalytic.

**Lemma 5.2** ([Bra12], p. 59). A morphism of Klein surfaces $f : K_1 \to K_2$ is dianalytic if, and only if, $f^{-1}(\partial K_2) = \partial K_1$.

A symmetric Riemann surface $(X, \iota)$ is a Riemann surface $X$ with an anti-analytic involution $\iota : X \to X$. For symmetric Riemann surfaces $(X_1, \iota_1)$ and $(X_2, \iota_2)$, a morphism between them is a non-constant continuous morphism $X_1 \xrightarrow{f} X_2$ of Riemann surfaces such that $f \circ \iota_1 = \iota_2 \circ f$.

Given a symmetric Riemann surface $(X, \iota)$, the quotient surface $K = X/\iota$ has a dianalytic structure making the quotient map $\pi : X \to X/\iota$ a morphism of Klein surfaces. We have $\pi^{-1}(\partial K) = \partial X$ if, and only if, $\pi$ is dianalytic. We call $(X, \iota)$ a dianalytic symmetric Riemann surface.
Proposition 5.3 ([Bra12], Lemma 5.2.6). Let $K$ be a Klein surface, then:

1. There is a Riemann surface $R_K$ and a dianalytic morphism $f : R_K \to K$ which is a double cover. We shall call the surface $R_K$ the orienting double of $K$;

2. if $X$ is a Riemann surface and $h : X \to K$ is a dianalytic morphism, then there is a unique analytic map $g : X \to R_K$ such that $h = f \circ g$.

3. the orienting double $R_K$ has an anti-analytic involution $\sigma$ such that $f \circ \sigma = f$;

4. any double cover $h : X \to K$ admitting such an involution and being dianalytic is universal with respect to this property;

5. the map $f$ is unramified, $\sigma$ is unique and $R_K$ is disconnected if and only if, $K$ is orientable.

Given a Klein or a symmetric Riemann surface $(X, \iota)$ whose underlying surface has $g$ handles, $0 \leq u \leq 2$ crosscaps and $h$ boundary components, we define its topological type as the triple $(g, u, h)$.

5.2 Nodal Klein and Riemann surfaces

A singular topological surface $(X, N)$ is a Hausdorff space $X$ with a discrete set $N \subset X$ of general singularities such that $X - N$ is a topological surface. Henceforth, we will consider these surfaces compact and possibly with boundary, where the boundary is defined to be the boundary of $X - N$.

Let $(X, N)$ be a singular surface. A boundary node is a singularity $z \in N$ with a neighbourhood homeomorphic to a neighbourhood $B \ni (0, 0)$, where we define $B := \{(x, y) \in (C^+)^2 \mid xy = 0\}$, such that the homeomorphism sends $z$ to $(0, 0)$. Similarly, an interior node is a singularity with a neighbourhood homeomorphic to $I \ni (0, 0)$, where $I := \{(x, y) \in C^2 \mid xy = 0\}$. If $X$ has only nodal singularities, then an atlas on $X$ is given by charts on $X - N$ together with charts at the nodes. We call a singular surface with only nodal singularities a nodal surface.

A map $f : I \to C$ is called (anti-)analytic if the compositions $f \circ g$, where $g : C \to I$ can either send $z$ to $(z, 0)$ or $(0, z)$ are (anti-)analytic. A map $f : C \to I$ is called (anti-)analytic if the composition $C \xrightarrow{f} I \xleftarrow{} C^2$ has (anti-)analytic components.

A nodal Riemann surface is a nodal surface $(X, N)$ together with a maximal analytic atlas. A nodal Klein surface is a nodal surface $(X, N)$ together with a maximal dianalytic atlas. An irreducible component of a nodal surface is a connected component of the surface obtained by pulling apart all the nodes. A nodal symmetric Riemann surface $(X, \iota)$ is a nodal Riemann surface with an anti-analytic involution $\iota : X \to X$.

An admissible symmetric Riemann surface $(X, \iota)$ is a nodal symmetric Riemann surface $(X, N)$ such that $\pi(n)$ is a boundary node for each node $n \in N$. 

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A dianalytic nodal symmetric Riemann surface is an admissible symmetric Riemann surface such that \( \pi \) is dianalytic. Observe that this imply that this kind of surface can only have boundary nodes.

A Klein surface with \( n \) marked points \((X, N)\) is a nodal Klein surface \((X, N)\) with an ordered \( n \)-tuple \( P = (p_1, \ldots, p_n) \) of distinct points on \( X - N \). A morphism \( f : (X_1, P) \to (X_2, P') \) of surfaces with \( n \) marked points is a morphism between the underlying surfaces such that \( f(p_i) = p'_i \) for each \( p_i \in P \) and \( p'_i \in P' \).

A symmetric Riemann surface \((X, \iota)\) with \((m, n)\) marked points is given by \((X, \iota, P, P')\), where \((X, \iota)\) is a nodal symmetric Riemann surface with an ordered \( 2m \)-tuple of distinct points on \( X - N \), \( P = (p_1, \ldots, p_{2m}) \), such that \( \iota(p_i) = p_{m+i} \) for \( i \in \{1, \ldots, m\} \) and an ordered \( n \)-tuple \( P' = (p'_1, \ldots, p'_n) \) of distinct points on \( X - N \) such that \( \iota(p'_j) = p'_j \) for \( j \in \{1, \ldots, n\} \). A map of marked symmetric Riemann surfaces \( f : (X_1, \iota_1, P, P') \to (X_2, \iota_2, Q, Q') \) is a morphism between the underlying symmetric Riemann surfaces such that \( f(p_i) = q_i \) and \( f(p'_j) = q'_j \). A marked symmetric Riemann surface is called admissible if the underlying symmetric Riemann surface is admissible and the points \( \pi(p_i) \) and \( \pi(p'_j) \) all lie in the boundary of \( X/\iota \); in this case all the \( p_i \) must be on the boundary.

A Klein surface with \( n \) oriented marked points is a Klein surface with marked points \((X, P)\) equipped with a choice of orientation locally on each marked point.

The category \( \mathcal{dnKlein} \) has objects Klein surfaces with only boundary nodes and oriented marked points on the boundary; its class of arrows is made of dianalytic morphisms.

The category \( \mathcal{dnSymRiemann} \) has objects dianalytic symmetric Riemann surfaces (possibly with boundary) with marked points. The arrows in \( \mathcal{dnSymRiemann} \) are given by analytic maps.

**Remark 5.4 ([Bra12], Section 5.3).** Let \((X, P)\), with \( P = (p_1, \ldots, p_n) \), be a Klein surface with \( n \) marked points. If \( f : R_X \to X \) is the orienting double we have that \( f^{-1}(p_i) \) gives two points in \( R_X \). We can make \( R_X \) a marked surface by ordering these two points using the local orientation on \( p_i \). This yields a collection of points \( P' \) which, together with the anti-analytic involution \( \sigma \) on \( R_X \), allows us to define the orienting double of \((X, P)\) as \((R_X, \sigma, P', 0)\). Observe that \((R_X, \sigma, P', 0)\) is an object in \( \mathcal{dnSymRiemann} \).

**Proposition 5.5 ([Bra12], Proposition 5.3.11).** The orienting double sets an equivalence of categories \( \mathcal{dnKlein} \to \mathcal{dnSymRiemann} \).

A Klein or Riemann surface with \( n \) marked points, possibly oriented, is *stable* if it has only finitely many automorphisms.

### 5.3 Moduli spaces of Klein surfaces

Let \( \mathcal{K}_{g,u,h,n} \) be the moduli space of stable Klein surfaces in \( \mathcal{dnKlein} \) with topological type \((g, u, h)\) and \( n \) marked points on the boundary. Let us consider the subspace \( \mathcal{K}_{g,u,h,n} \subset \mathcal{K}_{g,u,h,n} \) of non-
singular Klein surfaces. These moduli spaces are not empty except for the cases:
\[(g, u, h, n) \in \{(0, 0, 1, 0), (0, 0, 1, 1), (0, 0, 1, 2), (0, 0, 2, 0), (0, 1, 1, 0)\}.
\]
We denote by \(\tilde{D}_{g,u,h,n} \subset \overline{R}_{g,u,h,n}\) the subspace consisting of those Klein surfaces whose irreducible components are all discs.

**Lemma 5.6 ([Bra12], Proposition 5.5.2).** The spaces \(\tilde{D}_{g,u,h,n}\) admit a decomposition into orbi-cells labelled by Möbius graphs.

**Sketch of the proof.** The proof given by Braun can be divided into two different parts. In a first instance it is shown that a Klein surface in \(\tilde{D}_{g,u,h,n} \subset \overline{R}_{g,u,h,n}\) can be labelled with a Möbius graph. This association is done by giving a vertex for each irreducible component of the surface, an edge for each node and a leg for each marked point. In order to specify the Möbius structure we have to choose a ribbon structure and a colouring of the half-edges by a colour \(\{0, 1\}\); this is done by using the orienting double. The orienting double is also used in order to check the well-definedness of the choice: in order to choose orientations and colourings we have to choose between one of the two pre-images yielded by the orienting cover (as it is a double cover); choosing the other pre-image would reverse orientations, cyclic orderings and colourings.

![Figure 1: The association between graphs and surfaces.](image)

The second part of the proof is focused on proving that, for any given Möbius graph \(\gamma\), the space of surfaces \(K \in \tilde{D}_{g,u,h,n}\) which are labelled by \(\gamma\) is an orbi-cell; this goes as follows: a disc with an analytic structure is holomorphic to the unit disc in \(\mathbb{C}^2\), whose automorphism group is \(PSL_2(\mathbb{R})\). This means that the space of \(n \geq 3\) marked points (which correspond to legs in a reduced Möbius graph) on the unit disc is the configuration space of marked points in \(S^1\) modulo \(PSL_2(\mathbb{R})\), let's call this space \(C\). As the automorphisms of the unit disc preserve the cyclic ordering of marked points, \(C\) decomposes into cells labelled by ribbon corollas.

According to Lemma 5.4.5 [Bra12], the moduli space of Klein discs can be identified with the moduli space of marked unit discs coloured by \(\{0, 1\}\) modulo the action of the anti-analytic map reversing the cyclic ordering of the marked points. We have shown that this space decomposes into cells and reversing of the cyclic ordering of the marked points sends cells to cells. Therefore we have a cell decomposition of the moduli space of Klein discs.

Summarizing, we associate each vertex \(v\) of a Möbius graph \(\gamma\) with a Klein disc, equivalently with a Möbius corolla, which labels a cell \(X(v)\) (actually we associate it to a ribbon corolla, but
as we are dealing with colours, it turns out to be a Möbius corolla). Finally, if $\text{Aut}(\gamma)$ denotes the group of automorphisms of $\gamma$ preserving labelings and colourings, one can identify the orbi-space of surfaces associated to $\gamma$ with the orbicell: $\left( \prod_{v \in \gamma} X(v) \right) / \text{Aut}(\gamma)$. 

**Lemma 5.7 ([Bra12], Lemma 5.5.4).** A stable Klein surface with $n > 0$ oriented marked points has no non-trivial automorphisms.

**Sketch of the proof.** One needs to show that the orienting double has no non-trivial automorphisms. The arguments of this result follow the steps taken in Lemma 3.0.11 [Cos06].

An orbi-space is defined as a quotient by group acting with finite stabilizers, in this case $\text{Aut}(\gamma)$, and a stable Klein surface has no non-trivial automorphisms, whence we have the following:

**Corollary 5.8 ([Bra12], Corollary 5.5.5).** If $n > 0$ then $\tilde{D}_{g,u,h,n}$ is an ordinary space and decomposes into a cell complex.

**Lemma 5.9 ([Bra12], Lemma 5.5.10).** The inclusion $\partial \bar{K}_{g,u,h,0} \hookrightarrow \bar{K}_{g,u,h,0}$ defines a homotopy equivalence.

**Sketch of the proof.** The main point of this proof is to construct a deformation retract of $\bar{K}_{g,u,h,0}$ onto $\partial \bar{K}_{g,u,h,0}$. Using the equivalence given in Proposition 5.5, Braun is allowed to use the arguments used by Costello ([Cos06], Lemmata 3.0.7, 3.0.8 and 3.0.9): given a Klein surface $K \in \bar{K}_{g,u,h,0}$ we flow $\partial K$ into $K$ until $K$ becomes singular. Costello’s arguments and the orienting double prove that the singularities exist and are all nodes. This permits the definition of a map

$$\phi : K_{g,u,h,0} \times [0,1] \to \bar{K}_{g,u,h,0}$$

which is extended to a map $\phi' : \bar{K}_{g,u,h,0} \times [0,1] \to \bar{K}_{g,u,h,0}$ which turns out to be a deformation retract of the inclusion.

**Lemma 5.10 ([Bra12], Lemma 5.5.11).** The inclusion $\partial \bar{K}_{g,u,h,n} \hookrightarrow \bar{K}_{g,u,h,n}$ defines a homotopy equivalence.

**Sketch of the proof.** In this case Braun uses a Klein analogue for the map used by Costello ([Cos06], Lemma 3.0.5); that is: a map $\bar{K}_{g,u,h,n+1} \to \bar{K}_{g,u,h,n}$ which forgets the last marked point and contracts any resulting unstable component. Then the orienting double is used in order to conclude that the map is a locally trivial fibration.

**Proposition 5.11 ([Bra12], Proposition 5.5.9).** The inclusion $\tilde{D}_{g,u,h,n} \hookrightarrow \bar{K}_{g,u,h,n}$ defines a homotopy equivalence.
Proof. Let us assume the statement for lower dimensions. For \( k \geq 1 \) one defines \( \partial_k \tilde{\mathcal{K}}_{g,u,h,n} \) as the space of surfaces \( \Sigma \in \partial \tilde{\mathcal{K}}_{g,u,h,n} \) endowed with a map \( \{1,2,\ldots,k\} \to \{ \text{Nodes on } \Sigma \} \). We can see that the spaces \( \partial_k \tilde{\mathcal{K}}_{g,u,h,n} \) are \( k-1 \) simplices of a simplicial space whose realization \( |\partial_k \tilde{\mathcal{K}}_{g,u,h,n}| \) is weakly equivalent to \( \tilde{\mathcal{K}}_{g,u,h,n} \). Similarly, one defines the space \( \partial_k \tilde{\mathcal{D}}_{g,u,h,n} \) as the space of surfaces \( \Sigma' \in \tilde{\mathcal{D}}_{g,u,h,n} \) endowed with maps \( \{1,2,\ldots,k\} \to \{ \text{Nodes on } \Sigma' \} \). We can assure that \( \partial_k \tilde{\mathcal{D}}_{g,u,h,n} \) form a simplicial space whose realization is weakly equivalent to \( \tilde{\mathcal{D}}_{g,u,h,n} \).

By induction, we know that for each \( k \geq 1 \) there is an equivalence \( \partial_k \tilde{\mathcal{D}}_{g,u,h,n} \cong \partial_k \tilde{\mathcal{K}}_{g,u,h,n} \), therefore \( |\partial_k \tilde{\mathcal{D}}_{g,u,h,n}| \cong |\partial_k \tilde{\mathcal{K}}_{g,u,h,n}| \). Finally, the diagram below commutes:

\[
\begin{array}{ccc}
|\partial_k \tilde{\mathcal{D}}_{g,u,h,n}| & \cong & |\partial_k \tilde{\mathcal{K}}_{g,u,h,n}|\\
\downarrow & & \downarrow \\
\tilde{\mathcal{D}}_{g,u,h,n} & \cong & \tilde{\mathcal{K}}_{g,u,h,n}
\end{array}
\]

what allows us to conclude that \( \tilde{\mathcal{D}}_{g,u,h,n} \cong \tilde{\mathcal{K}}_{g,u,h,n} \). \( \square \)

## 6 The definition of an open-closed Klein TCFT

Let \( \Lambda \) be a set of D-branes. We define a topological category \( \mathcal{W}_\Lambda \) where:

1. The class of objects \( \text{Obj}(\mathcal{W}_\Lambda) \) is given by quadruples \( \alpha := ([O],[C],s,t) \), with \( O,C \in \mathbb{N} \), where \( [O] = \{0,\ldots,O-1\} \) and \( [C] = \{0,\ldots,C-1\} \), and maps \( s,t : [O] \to \Lambda \);

2. the class of morphisms \( \mathcal{W}_\Lambda(\alpha,\beta) \) is given by the moduli spaces of Klein surfaces \( \Sigma \) with \( \alpha \) incoming boundary components and \( \beta \) outgoing boundary components. The open boundary components are disjoint intervals labelled in \( [O] \); the closed boundary components are parameterised circles labelled in \( [C] \). An open interval in \( \partial \Sigma \) has associated an ordered pair \( \{s(i),t(i)\} \) of D-branes indicating where the interval begins and where it ends. Surfaces in \( \mathcal{W}_\Lambda(\alpha,\beta) \) have free boundary components, which can be either intervals or circles. Free boundary components are the remaining components of \( \partial \Sigma \) after removing from it both open and closed components and must be labelled by D-branes in a way compatible with the labelling \( \{s(i),t(i)\} \).

Let \( \mathcal{W}_{\Lambda,\text{open}} \subset \mathcal{W}_\Lambda \) be the full subcategory whose objects are of the form \( \alpha = ([O],\emptyset,s,t) \).

Composition of morphisms is given by gluing Klein surfaces: we glue together incoming open (resp. closed) boundary components with outgoing open (resp. closed) boundary components. Open boundary components can only be glued together if their D-brane labelling agree. Disjoint union makes \( \mathcal{W}_\Lambda \) into a symmetric monoidal category.

We require the positive boundary condition: Klein surfaces are required to have at least one incoming closed boundary component on each connected component.
Remark 6.1. We allow the following exceptional surfaces: the disc, the annulus and the Möbius strip with no open or closed boundary components and only free boundary components; these surfaces are unstable and so we define their associated moduli space to be a point.

![Figure 2: Components 1 and 2 are open; components 3 and 4 are free and component 5 is closed.](image)

Let us consider the functor $\mathcal{C}_\bullet : \text{Top} \to \text{Comp}_K$ which computes singular homology groups. The functor $\mathcal{C}_\bullet$ yields a DG category $\mathcal{O}_\Lambda = \mathcal{C}_\bullet(W_\Lambda)$ whose objects are finite sets $a = ([O], [C], s, t)$ and where the space of morphisms is $\text{Hom}_{\mathcal{O}_\Lambda}(a, \beta) := \mathcal{C}_\bullet(W_\Lambda(a, \beta))$. Let $\mathcal{O}_\Lambda$ be the full subcategory whose objects are of the form $([O], [C], s, t)$. Similarly, let $\mathcal{C}_\Lambda$ be the full subcategory whose objects are of the form $(\emptyset, [C], s, t)$.

A split functor between DGSM categories $\mathfrak{F} : (\mathcal{A}, \sqcup) \to (\mathcal{B}, \otimes)$ is a symmetric monoidal functor satisfying $\mathfrak{F}(a \sqcup b) \cong \mathfrak{F}(a) \otimes \mathfrak{F}(b)$ for objects $a, b \in \text{Obj}(\mathcal{A})$. If we have quasi-isomorphisms instead, we talk about a $h$-split functor.

An open-closed Klein topological conformal field theory (henceforth an open-closed KTCFT) is a pair $(\Lambda, \mathfrak{F})$ where $\Lambda$ is finite set of D-branes and $\mathfrak{F}$ is a $h$-split symmetric monoidal functor $\mathfrak{F} : \mathcal{O}_\Lambda \to \text{Comp}_K$; a morphism of open-closed KTCFTs $(\Lambda_1, \mathfrak{F}_1) \to (\Lambda_2, \mathfrak{F}_2)$ is given by a map $\Lambda_1 \to \Lambda_2$ and a morphism $\mathfrak{F} \to L^* \mathfrak{F}_2$, where $L : \mathcal{O}_\Lambda \to \mathcal{O}_\Lambda$ is the functor induced by the map $\Lambda_1 \to \Lambda_2$; an open KTCFT is a $h$-split symmetric monoidal functor $\mathfrak{F} : \mathcal{O}_\Lambda \to \text{Comp}_K$; a closed KTCFT is defined as a $h$-split symmetric monoidal functor $\mathfrak{F} : \mathcal{C}_\Lambda \to \text{Comp}_K$.

Morphisms between open (resp. closed) KTCFTs are defined the same way we defined a morphism between open-closed KTCFTs.

7 Categories via generators and relations

The steps taken by Braun in [Bra12] suggest that one can adapt the techniques developed by Costello in [Cos07] in order to give a classification of open KTCFTs. This is what we do in this section: we give a description, in terms of generators and relations, of the moduli space of Klein surfaces in $dnKlein$.
whose irreducible components are either a disc or an annulus. This moduli space yields a category $\tilde{\mathcal{D}}_\Lambda$ which turns out to be quasi-equivalent to the category $\tilde{\mathcal{O}C}_\Lambda$. The arguments of Costello can be applied in the unoriented setting with a little subtlety: in order to reflect the involution which appears when we apply the equivalence of categories stated in Proposition 5.5, we have to introduce further generators and relations.

### 7.1 Moduli spaces and categories

We define the moduli space $\overline{K}_\Lambda(\alpha, \beta)$ of Klein surfaces in $dnKlein$ (so we allow nodes) as follows: its elements are stable Klein surfaces with $\alpha$ incoming boundary components labelled by $[O]$; we assume there are no closed incoming boundary components. Surfaces have $\beta$ outgoing boundary components labelled in a similar way: $O$ open boundary components and $C$ closed boundary components labelled by $[O]$ and $[C]$ respectively. Closed boundary components have exactly one marked point on them, whilst open marked points are distributed all along the boundary components of the surfaces. Klein surfaces in $\overline{K}_\Lambda(\alpha, \beta)$ have free boundary components, which are the intervals between open marked points and those components with no marked points on them; free boundary components must be labelled by D-branes in $\Lambda$ in a way compatible with the maps $s, t : [O] \to \Lambda$. Let us remark that, although surfaces in $\overline{K}_\Lambda(\alpha, \beta)$ are asked to be stable, we allow the following exceptional surfaces: the disc with zero, one or two open marked points, the annulus with no open or closed points and the Möbius strip with no open or closed points. Let $\mathcal{K}_\Lambda(\alpha, \beta) \subset \overline{K}_\Lambda(\alpha, \beta)$ be the subspace of non-singular Klein surfaces.

![Figure 3: A surface in $\overline{K}_\Lambda(\alpha, \beta)$.

Let us define $\tilde{\mathcal{G}}_\Lambda(\alpha, \beta) \subset \overline{K}_\Lambda(\alpha, \beta)$ as the subspace consisting of Klein surfaces whose irreducible components are either a disc or an annulus of modulus one. Annuli are required to have one of their sides labelled as an outgoing boundary component. Observe that $\tilde{\mathcal{G}}_\Lambda(\alpha, \beta)$ contains the exceptional surfaces.

**Proposition 7.1.** The inclusion $\tilde{\mathcal{G}}_\Lambda(\alpha, \beta) \hookrightarrow \overline{K}_\Lambda(\alpha, \beta)$ is a weak homotopy equivalence of orbi-spaces.

**Proof.** This result follows from Proposition 5.11 if one observes that the weak homotopy equivalence $i : \tilde{\mathcal{D}}_{g,u,h,n} \rightarrow \tilde{\mathcal{K}}_{g,u,h,n}$ holds if we replace points on the interior of each surface in $\tilde{\mathcal{D}}_{g,u,h,n}$ and their images by $i$ in $\overline{K}_{g,u,h,n}$ with boundary components; we replace at most one point in the same disc. The equivalence holds if we include one marked point in each new boundary component. 

\[\square\]
Let us denote by $\mathcal{K}_\Lambda$ the category with the objects of $W_\Lambda$ and arrows given by $\mathcal{K}_\Lambda(\alpha, \beta)$; there is a full subcategory $\mathcal{K}_\Lambda,\text{open}$ whose objects $\alpha$ are those of $W_\Lambda$ which have no closed part. Gluing surfaces comes from maps

$$\mathcal{K}_\Lambda(\alpha, \beta) \times \mathcal{K}_\Lambda(\beta, \gamma) \to \mathcal{K}_\Lambda(\alpha, \gamma)$$

that glue outgoing open boundary components in $\mathcal{K}_\Lambda(\alpha, \beta)$ to incoming open boundary components in $\mathcal{K}_\Lambda(\beta, \gamma)$. The exceptional surfaces are glued as follows: gluing the disc with two outgoing marked points, one incoming and one outgoing, or both incoming, to a surface $\Sigma$ corresponds to gluing the points of $\Sigma$ together. Gluing the disc with one marked point to a marked point of $\Sigma$ corresponds to forgetting the marked point.

The inclusion in Proposition 7.1 leads to a subcategory $\tilde{G}_\Lambda,\text{open} \subset \mathcal{K}_\Lambda,\text{open}$. Observe that disjoint union gives $\mathcal{K}_\Lambda,\text{open}$ and $\tilde{G}_\Lambda,\text{open}$ the structure of symmetric monoidal categories. The following result is the unoriented analogue of Proposition 6.1.5 [Cos07]:

**Proposition 7.2.** The DGSM category $\mathcal{C}_\bullet(\mathcal{K}_\Lambda,\text{open})$ is quasi-isomorphic to $\tilde{O}_\Lambda$. Under the induced quasi-equivalence between $\text{Obj}(\tilde{O}_\Lambda)$-$\mathcal{C}_\bullet$-$\mathcal{B}$-bimodules and $\text{Obj}(\tilde{O}_\Lambda)$-$\tilde{O}_\Lambda$-bimodules, $\mathcal{C}_\bullet(\mathcal{K}_\Lambda)$ is quasi-isomorphic to $\tilde{O}_\Lambda$.

**Sketch of the proof.** The proof for this result is akin to the proof for Proposition 6.1.5 [Cos07]. Let us remind the main points:

For a pair of objects $\alpha, \beta \in \text{Obj}(W_\Lambda)$, let us denote $W'_{\Lambda,\text{open}}(\alpha, \beta)$ the moduli space of Klein surfaces in $\text{dnKlein}$ (like $\mathcal{K}_\Lambda(\alpha, \beta)$) where the marked open boundaries have been replaced by parameterized intervals (like $W_\Lambda(\alpha, \beta)$). We do not allow these intervals to intersect each other or the nodes on the boundary of the surfaces. By associating each outgoing open boundary interval with a number $t \in [0, 1/2]$, we can define gluing maps

$$W'_{\Lambda,\text{open}}(\alpha, \beta) \times W'_{\Lambda,\text{open}}(\beta, \gamma) \to W'_{\Lambda,\text{open}}(\alpha, \gamma),$$

making $W'_{\Lambda,\text{open}}$ into a category.

Inclusions $W'_{\Lambda,\text{open}}(\alpha, \beta) \to W'_{\Lambda,\text{open}}(\alpha, \beta)$ and $\mathcal{K}_\Lambda(\alpha, \beta) \to W'_{\Lambda,\text{open}}(\alpha, \beta)$ mapping 0 and 1/2 to open boundaries respectively define homotopy equivalences on the spaces of morphisms.

We follow [Cos07] to give $\tilde{G}_\Lambda(\alpha, \beta)$ an orbi-cell decomposition. Let $\Sigma \in \tilde{G}_\Lambda(\alpha, \beta)$ and assume $A \subset \Sigma$ is an irreducible component which is an annulus with a closed boundary component. We write $A_{\text{open}}$ and $A_{\text{closed}}$ for the open and closed boundary components of $A$, respectively.

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Let $p \in A_{\text{closed}}$ be the unique marked point in the closed boundary component. We can identify $A$ with the cylinder $S^1 \times [0, 1]$ in a way that identifies $p$ with the point $(1, 0)$. This identification allows us to cut $A$ from $p$ to a point of $A_{\text{open}}$. We declare the 0-cells as the marked points, the nodes and the intersection points between the cut and $A_{\text{open}}$; 1-cells are defined to be the boundary components $A_{\text{open}}, A_{\text{closed}}$ and the cut itself; finally we declare the 2-cell to be $\Sigma$.

This process yields a stratification of $\tilde{G}_\Lambda(\alpha, \beta)$ by saying that two surfaces are in the same level if the corresponding marked 2-cell complexes are isomorphic.

Let $C_{\text{cell}}$ be the functor taking a finite cell complex to an object in $\text{Comp}_K$ (see Appendix A [Cos07]); by applying $C_{\text{cell}}$ to $\tilde{G}_\Lambda(\alpha, \beta)$ we define a category $\tilde{D}_{\Lambda, \text{open}}(\alpha, \beta)$ as $C_{\text{cell}}(\tilde{G}_\Lambda(\alpha, \beta))$ which, due to the quasi-isomorphism $C_{\text{cell}}(X) \cong C_*(X)$ (for $X$ an orbi-cell complex), leads to the following result, which is the unoriented analogue of Lemma 6.1.7 [Cos07]:

**Proposition 7.3.** The following assertions hold:

1. There is a quasi-isomorphism of DGSM categories $\tilde{D}_{\Lambda, \text{open}} \cong \tilde{\mathcal{O}}_\Lambda$, where we define $\tilde{D}_{\Lambda, \text{open}}(\alpha, \beta)$ as $C_{\text{cell}}(\tilde{G}_{\Lambda, \text{open}}(\alpha, \beta))$;
2. there is a quasi-isomorphism of DGSM categories $\tilde{D}_\Lambda \cong \tilde{\mathcal{O}}_\Lambda$.

### 7.2 Generators and relations

Using the equivalences of categories stated in Proposition 5.5, we can move some of the results in [Cos07] into the Klein setting. This implies the definition of several categories, analogous to those appearing in [Cos07], which will simplify the problem of understanding KTCFTs in terms of involutive $A_\infty$-categories.

A DG category $\mathcal{A}$ is generated by some set of arrows $A$ if $\text{Hom}_{\mathcal{A}}(\cdot, \cdot)$ has $A$ as a generating set; $\mathcal{A}$ has $R$ as a set of relations if $\text{Hom}_{\mathcal{A}}(\cdot, \cdot)$ is given by the quotient $A/R$. We say that $\mathcal{A}$ is generated as a symmetric monoidal category by $A$ modulo $R$ if $\text{Hom}_{\mathcal{A}}(\cdot, \cdot)$ is of the form $A/R$ and the axioms of symmetric monoidal categories are satisfied.

Let $\tilde{D}_{\Lambda, \text{open}}^+ \subset \tilde{D}_{\Lambda, \text{open}}$ be the subcategory with the same objects but where a morphism is given by a disjoint union of discs, with each connected component having exactly one outgoing boundary marked point. For an ordered set $\lambda_0, \ldots, \lambda_{n-1}$ of D-branes, with $n \geq 1$, let $\{\lambda_0, \ldots, \lambda_{n-1}\} \in \text{Obj}(\tilde{\mathcal{O}}_\Lambda)$ with $O = n$, $s(i) = \lambda_i, t(i) = \lambda_{i+1}$ for $0 \leq i \leq O - 1$; we use the notation $\{\lambda_0, \ldots, \lambda_{n-1}\}^c := \{\lambda_1, \ldots, \lambda_{n-1}, \lambda_0\}$. Let us define $D^+(\lambda_0, \ldots, \lambda_{n-1})$ as the disc.
with \( n \) marked points and D-brane labelling given by the different \( \lambda_i \), where all the boundary marked points are incoming except for that between \( \lambda_{n-1} \) and \( \lambda_0 \), which is outgoing. The boundary components of the discs are compatibly oriented. There is an exceptional morphism in \( \tilde{\mathcal{D}}_{\Lambda,\text{open}}^+ \) given by a disc \( D^\tau(\lambda_0, \lambda_1) \), which will be called a twisted disc. The particularity of this disc is that, contrary to the discs \( D^+(\lambda_0, \ldots, \lambda_{n-1}) \), it has boundary components oriented incomplibly.

![Figure 4: The twisted disc.](image)

Let \( \tilde{\mathcal{C}} \subset \tilde{\mathcal{D}}_{\Lambda,\text{open}}^+ \) be the subcategory with \( \text{Obj} \left( \tilde{\mathcal{C}} \right) = \text{Obj} \left( \tilde{\mathcal{D}}_{\Lambda,\text{open}}^+ \right) \) but whose arrows are not allowed to have connected components which are the disk with at most 1 open marked point, or the disc with two open marked incoming points or the annulus with neither open nor closed marked points. The morphisms in \( \tilde{\mathcal{C}} \) are assumed to be not complexes but graded vector spaces.

**Proposition 7.4.** Let \( D(\lambda_0, \ldots, \lambda_{n-1}) \) be the disc in \( \tilde{\mathcal{D}}_{\Lambda,\text{open}}^+ \) whose marked points are all incoming. The subcategory \( \tilde{\mathcal{C}} \) is freely generated, as a symmetric monoidal involutive category over \( \text{Obj} \left( \tilde{\mathcal{D}}_{\Lambda,\text{open}}^+ \right) \), by the discs \( D(\lambda_0, \ldots, \lambda_{n-1}) \), for \( n \geq 3 \), the discs with two outgoing marked points, subject to the relation that \( D(\lambda_0, \ldots, \lambda_{n-1}) \) is cyclically symmetric (that is \( D(\lambda_0, \ldots, \lambda_{n-1}) = \pm D(\lambda_1, \ldots, \lambda_{n-1}, \lambda_0) \)), and the twisted disc.

**Proof.** The proof for this result follows the steps of Proposition 6.2.1 [Cos07]. If we denote by \( \tilde{\mathcal{E}} \) a category with the same sets of generators and relations as \( \tilde{\mathcal{C}} \), we can construct a fully faithful functor \( \tilde{\mathcal{E}} \to \tilde{\mathcal{C}} \), indeed: to prove that the functor is full we observe every surface in \( \text{Hom}_{\tilde{\mathcal{C}}}(\alpha, \beta) \) can be built using disjoint unions of surfaces in \( \tilde{\mathcal{E}} \) and gluing discs. Observe that the twisted disc, as remarked above, allows us to change the orientations of the marked points, whilst the disc with two outgoing marked points turns incoming boundaries into outgoing boundaries.

In order to check that \( \tilde{\mathcal{E}} \to \tilde{\mathcal{C}} \) is faithful, we construct an inverse functor \( \tilde{\mathcal{C}} \to \tilde{\mathcal{E}} \), which is the identity \( 1_{\tilde{\mathcal{C}}} \) on objects. Let us consider \( \Sigma \in \tilde{\mathcal{C}}(\alpha, \beta) \), then we can write \( \Sigma = \Sigma' \circ Y \), where both \( \Sigma', Y \) are surfaces in \( \tilde{\mathcal{E}} \). The surface \( \Sigma' \) is composed by disjoint unions of identity maps, discs with all incoming boundaries and twisted discs; the surface \( Y \) is composed by disjoint unions of identity maps, discs with two outgoing boundaries and twisted discs. This decomposition allows us to write a map \( \tilde{\mathcal{C}}(\alpha, \beta) \to \tilde{\mathcal{E}}(\alpha, \beta) \). We conclude that the functor \( \tilde{\mathcal{E}} \to \tilde{\mathcal{C}} \) is faithful.

**Corollary 7.5.** The category \( \tilde{\mathcal{D}}_{\Lambda,\text{open}}^+ \) is freely generated as a symmetric monoidal involutive DG category over \( \text{Obj} \left( \tilde{\mathcal{D}}_{\Lambda,\text{open}}^+ \right) \), by the discs \( D^+(\lambda_0, \ldots, \lambda_{n-1}) \) and \( D^\tau(\lambda_0, \lambda_1) \), modulo the relations:

1. For \( n = 2 \) : \( D^\tau(\lambda_0, \lambda_1) \circ D^\tau(\lambda_0, \lambda_1) = \text{Id}_{\{\lambda_0, \lambda_1\}} \);
2. for $n = 3$ we have: $D^+(\lambda_0, \lambda_0, \lambda_1) \circ D^+(\lambda_0) = \text{Id}_{(\lambda_0, \lambda_1)} = D^+(\lambda_0, \lambda_1, \lambda_1) \circ D^+(\lambda_1)$;

3. for $n \geq 3$, gluing twisted discs to each of the incoming boundary components of $D^+(\lambda_0, \ldots, \lambda_{n-1})$ is equivalent to gluing a twisted disc to the outgoing boundary component of $D^+(\lambda_0, \ldots, \lambda_{n-1})$;

4. for $n \geq 4$: $D^+(\lambda_0, \ldots, \lambda_i, \lambda_i, \ldots, \lambda_{n-1}) \circ D^+(\lambda_i) = 0$.

Remark 7.6. Relation 3 will be needed in order to guarantee the Calabi-Yau condition in the forthcoming involutive $A_\infty$-categories.

Theorem 7.7. The category $\tilde{D}_{\Lambda, \text{open}}$ is freely generated, as a symmetric monoidal DG involutive category over $\text{Obj}(\tilde{D}_{\Lambda, \text{open}})$, by $\tilde{D}_{\Lambda, \text{open}}^+$ and the discs with two incoming or two outgoing boundary components (denoted by $D_{\text{in}}(\lambda_0, \lambda_1)$ and $D_{\text{out}}(\lambda_0, \lambda_1)$ respectively) modulo the following relations:

1. Gluing a disc with two outgoing boundary components to a disc with two incoming boundary components yields the identity;

2. the disc $D(\lambda_0, \ldots, \lambda_{n-1})$, whose marked points are all incoming, is cyclically symmetric:

$$D(\lambda_0, \ldots, \lambda_{n-1}) = \pm D(\lambda_1, \ldots, \lambda_{n-1}, \lambda_0)$$

under the existing permutation isomorphism $\{\lambda_0, \ldots, \lambda_{n-1}\}^c \cong \{\lambda_1, \ldots, \lambda_{n-1}, \lambda_0\}^c$.

Proof. The proof follows the arguments used in Proposition 7.4.

Let $A(\lambda_0, \ldots, \lambda_{n-1})$ be the annulus with $n \geq 1$ marked points and the intervals between them labelled with D-branes with the inner boundary component labelled as closed. As in the case of the discs $D^+(\lambda_0, \ldots, \lambda_{n-1})$, the boundary components of the annuli $A(\lambda_0, \ldots, \lambda_{n-1})$ are compatibly oriented.
**Theorem 7.8.** The annuli $A(\lambda_0, \ldots, \lambda_{n-1})$, the twisted disc and the identity in $\tilde{D}_\Lambda,_{\text{open}}(\alpha, \alpha)$ freely generate $\tilde{D}_\Lambda$ as an Obj($\tilde{O}\mathcal{C}_\Lambda$)-$\tilde{D}_\Lambda,_{\text{open}}$-bimodule, modulo the following relations:

1. Gluing the disc with one marked point $D(\lambda_i)$ to $A(\lambda_0, \ldots, \lambda_{n-1})$ in any of the boundary marked points except that between $\lambda_{n-1}$ and $\lambda_0$ yields zero;

2. the disjoint union of the identity element on $\alpha$ with that on $\beta$ is the identity on $\alpha \sqcup \beta$.

*Proof.* This result follows from Proposition 7.4.

Let $\tilde{D}_\Lambda^+$ be the Obj($\tilde{O}\mathcal{C}_\Lambda$)-$\tilde{D}_\Lambda,_{\text{open}}^+$-bimodule with the generators and relations stated above.

### 7.3 The differential in $\tilde{D}_\Lambda$

The definition of the differential for $D_\Lambda$ given in [Cos07] can be used in our context. The complexes $\tilde{D}_\Lambda$ admit a differential $d$ which is defined on discs as follows: if $\ast$ denotes the gluing of the open marked points between $\lambda_i$ and $\lambda_j$:

$$d(D(\lambda_0, \ldots, \lambda_{n-1})) = \sum_{0 \leq i < j \leq n-1 \atop 2 \leq j-i} \pm D(\lambda_i, \ldots, \lambda_j) \ast D(\lambda_j, \ldots, \lambda_i).$$

For annuli, the differential is:

$$d(A(\lambda_0, \ldots, \lambda_{n-1})) = \sum_{0 \leq i < j \leq n-1 \atop 2 \leq |i-j|} \pm A(\lambda_0, \ldots, \lambda_j, \lambda_i, \ldots, \lambda_{n-1}) \ast D(\lambda_i, \ldots, \lambda_j)$$

$$+ \sum_{0 \leq i < j \leq n-1 \atop (j,i) \neq (0,n-1)} \pm A(\lambda_j, \ldots, \lambda_i) \ast D(\lambda_i, \ldots, 0, 1, \ldots, \lambda_j).$$

**Remark 7.9.** The signs in the previous formula for the differential are not important for our purposes; nevertheless, we point out that they depend on the orientation chosen for the cells in $\tilde{G}_\Lambda$ of marked points on discs and annuli.

**Lemma 7.10 (cf. [Cos07], Lemma 6.3.1).** The assertions below hold:

1. The Obj($\tilde{O}\mathcal{C}_\Lambda$)-$\tilde{D}_\Lambda,_{\text{open}}$-bimodule $\tilde{D}_\Lambda$ is $\tilde{D}_\Lambda,_{\text{open}}$-flat.

2. If $M$ is a h-split $\tilde{D}_\Lambda,_{\text{open}}$-module, then $\tilde{D}_\Lambda \otimes_{\tilde{D}_\Lambda,_{\text{open}}} M$ is a h-split Obj($\tilde{O}\mathcal{C}_\Lambda$)-module.

These results also hold if one considers $\tilde{D}_\Lambda^+$ and $\tilde{D}_\Lambda^+$ instead of $\tilde{D}_\Lambda,_{\text{open}}$ and $\tilde{D}_\Lambda$.

*Proof.* The Obj($\tilde{O}\mathcal{C}_\Lambda$)-$\tilde{D}_\Lambda,_{\text{open}}$-bimodule $\tilde{D}_\Lambda$ is generated, for $\alpha \in \text{Obj}(\tilde{D}_\Lambda,_{\text{open}})$, by the identity elements in $\tilde{D}_\Lambda,_{\text{open}}(\alpha, \alpha)$, the twisted disc $D^\tau(\lambda_0, \lambda_1)$ and the annuli $A(\lambda_0, \ldots, \lambda_{n-1})$.

We filter $\tilde{D}_\Lambda$ as a bimodule with a filtration on the generators by saying that the identity element in $\tilde{D}_\Lambda(\alpha, \alpha)$ and the twisted disc are in $F^0$ and each annulus $A(\lambda_0, \ldots, \lambda_{n-1})$ is in $F^n$. 

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In order to show the first point of the Lemma, we have to prove that the functor $\tilde{D}_\Lambda \otimes \tilde{D}_{\text{open}}(-)$ is exact, that is: given a quasi-isomorphism $M_1 \to M_2$ of h-split $\tilde{D}_{\text{open}}$-modules we must prove that the map below is also a quasi-isomorphism:

$$\tilde{D}_\Lambda(\beta, -) \otimes \tilde{D}_{\text{open}} M_1(-) \to \tilde{D}_\Lambda(\beta, -) \otimes \tilde{D}_{\text{open}} M_2(-).$$

Giving both sides the filtration induced by $\tilde{D}_\Lambda(\beta, -)$, it is enough to show the statement on the associated graded complexes.

Let $\alpha \in \text{Obj}(\tilde{D}_{\text{open}})$ and $\text{Obj}(\tilde{D}_{\Lambda}) \ni \beta = C \cup \alpha$ for $C \in \mathbb{N}$; observe that we are adding $C$ closed states to $\alpha$. We will show the result for $C = 1$. Let $M$ be a h-split $\tilde{D}_{\text{open}}$-module. In degree $n$, by sending the generators of $\tilde{D}_\Lambda$ which are the identity in $\tilde{D}_{\text{open}}$ to $\alpha$ and the annulus $A(\lambda_0, \ldots, \lambda_{n-1}) := a \to \{\lambda_0, \ldots, \lambda_{n-1}\}^c$, we get that $\tilde{D}_\Lambda(\alpha \cup 1, -) \otimes \tilde{D}_{\text{open}} M(-)$ is spanned by the spaces of the form $a \otimes K M(\alpha \cup \{\lambda_0, \ldots, \lambda_{n-1}\}^c)$.

There is just one relation to be considered in $\tilde{D}_\Lambda$: gluing a disc with one boundary marked point to any of the marked points of $\alpha$, but that between $\lambda_{n-1}$ and $\lambda_0$, is zero. This is the same as saying that the following composition is zero:

$$a \otimes K M(\alpha \cup \{\lambda_0, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{n-1}\}^c)$$

$$\begin{array}{c}
\downarrow (1) \\
\text{deg}^n(\tilde{D}_\Lambda(\alpha \cup 1, -) \otimes \tilde{D}_{\text{open}} M(-))
\end{array}$$

$$a \otimes K M(\alpha \cup \{\lambda_0, \ldots, \lambda_{i-1}, \lambda_i, \lambda_i, \lambda_{i+1}, \ldots, \lambda_{n-1}\}^c)$$

The map (1) corresponds to the element

$$\tilde{D}_{\text{open}}(\alpha \cup \{\lambda_0, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{n-1}\}^c, \alpha \cup \{\lambda_0, \ldots, \lambda_{i-1}, \lambda_i, \lambda_i, \lambda_{i+1}, \ldots, \lambda_{n-1}\}^c)$$

obtained from the tensor product of $\text{Id}_\alpha$ and $\text{Id}_{\{\lambda_0, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{n-1}\}}$ with the map corresponding to the disc with a single marked point. The map (2) corresponds to gluing the disc with one marked point (as $\tilde{D}_\Lambda(\alpha \cup 1, -) \otimes \tilde{D}_{\text{open}} M(-)$ is spanned by $a \otimes K M(\alpha \cup \{\lambda_0, \ldots, \lambda_{n-1}\}^c)$).
As (1) is always injective (because we can find an splitting coming from the disc with one marked point), taking quotient is an exact operation and hence
\[ \tilde{D}_\Lambda(\alpha \cup 1, -) \otimes \tilde{D}_\Lambda,_{\text{open}} M(-) \]
is an exact functor. The same argument applies for any $C \in \mathbb{Z}$ by observing that, as each annulus $A(\lambda_0, \ldots, \lambda_{n-1})$ has a closed boundary component, each integer $C$ corresponds to an annulus, which contributes with an element of the form $\{\lambda_0, \ldots, \lambda_{n-1}\}^c$. Therefore the first part of the Lemma is proved.

The second part is proved similarly. Let $N := \tilde{D}_\Lambda \otimes \tilde{D}_\Lambda,_{\text{open}} M$ and, for simplicity, assume $\beta = \alpha \cup 1$. In order to show that $N(\beta) \otimes N(\beta') \rightarrow N(\beta \cup \beta')$ is a quasi-isomorphism, we take the filtration induced by $\tilde{D}_\Lambda$ and check the result for the associated graded complexes. Roughly speaking we have: for $\lambda_\bullet, \lambda'_\bullet \in \text{Obj}(\tilde{D}_\Lambda,_{\text{open}})$, $N(\beta)$ is spanned by $a \otimes_k M(\alpha \cup \{\lambda_0, \ldots, \lambda_{n-1}\})^c$, similarly $N(\beta')$ is spanned by $a' \otimes_k M(\alpha' \cup \{\lambda'_0, \ldots, \lambda'_{m-1}\})^c$. The tensor product $N(\beta) \otimes N(\beta')$ is spanned by
\[ a \otimes_k a' \otimes_k M(\alpha \cup \{\lambda_0, \ldots, \lambda_{n-1}\})^c \otimes_k M(\alpha' \cup \{\lambda'_0, \ldots, \lambda'_{m-1}\})^c \]
which is quasi-isomorphic to
\[ a \otimes_k a' \otimes_k M(\{\alpha \cup \{\lambda_0, \ldots, \lambda_{n-1}\}\}^c \cup (\alpha' \cup \{\lambda'_0, \ldots, \lambda'_{m-1}\})) \]
because we are assuming $M$ to be h-split. These elements span $\tilde{D}_\Lambda(\beta \cup \beta', -) \otimes \tilde{D}_\Lambda,_{\text{open}} M(-)$.

We conclude by observing that the same proof works for $\tilde{D}_\Lambda^+,_{\text{open}}$ and $\tilde{D}_\Lambda^+$. \hfill \Box

8 Involutive Calabi-Yau categories and KTCFTs

8.1 Involutive $A_\infty$-categories

For an involutive graded space $A$, let $SA := A[1]$. We define an involutive $A_\infty$-algebra as an involutive graded space $A$ endowed with involution-preserving maps of degree $n - 2$
\[ m_n : (SA)^{\otimes n} \rightarrow SA, \ n \geq 1, \]
such that the identity below holds:
\[ \sum_{i+j+l=n} m_{i+j+l} \circ (\text{Id}^{\otimes i} \otimes m_j \otimes \text{Id}^{\otimes l}) = 0, \ \forall n \geq 1. \quad (2) \]

An involutive $A_\infty$-category $C$ consists of:

1. A class of objects $\text{Obj}(C)$;

2. for all $X_1, X_2 \in \text{Obj}(C)$, a $\mathbb{Z}$-graded abelian group of morphisms $\text{Hom}_C(X_1, X_2)$;
3. A functor $* : C^{op} \to C$ which is the identity on objects and such that $(f \circ g)^* = g^* \circ f^*$ for morphisms $f, g \in \text{Hom}_C(-,-)$;

4. For all $n \geq 1$, composition maps

$$m_n : \text{Hom}_C(X_1, X_2) \otimes \cdots \otimes \text{Hom}_C(X_n, X_{n+1}) \to \text{Hom}_C(X_1, X_{n+1})$$

of degree $n-2$ satisfying (2);

5. Given morphisms $f_1, \ldots, f_n \in \text{Hom}_C$, the maps $m_n$ are required to satisfy the following identity:

$$(m_n(f_1 \otimes \cdots \otimes f_n))^* = m_n(f_1^* \otimes \cdots \otimes f_n^*).$$

8.2 Involutive Calabi-Yau $A_\infty$-categories

We define an involutive $\mathbb{K}$-algebroid as a category $C$ such that each morphism set has the structure of a $\mathbb{K}$-module in a way that the composition is $\mathbb{K}$-bilinear; the category $C$ is endowed with a functor $* : C^{op} \to C$ which is the identity on objects and such that for morphisms $f, g \in \text{Hom}_C$ satisfies: $(f \circ g)^* = g^* \circ f^*$.

An involutive Calabi-Yau category over a field $\mathbb{K}$ is an involutive $\mathbb{K}$-algebroid $C$ equipped with a trace map

$$\text{Tr}_A : \text{Hom}_C(A, A) \to \mathbb{K},$$

for every $A \in \text{Obj}(C)$, which we require to be involution-preserving. The associated pairing

$$\langle -, - \rangle_{A,B} : \text{Hom}_C(A, B) \otimes \text{Hom}_C(B, A) \to \mathbb{K}$$

is required to be non-degenerate, symmetric and must satisfy the following identity, for a map $f \in \text{Hom}_C(A, B)$:

$$\langle f, g \rangle_{A,B} = \langle g^*, f^* \rangle_{A,B}$$

(3)

Observe that we have endowed our field $\mathbb{K}$ with the trivial grading and the trivial involution.

An involutive Calabi-Yau $A_\infty$-category is an involutive $A_\infty$-category $\mathcal{E}$ with symmetric and non-degenerate on homology pairing of chain complexes

$$\langle -, - \rangle_{A,B} : \text{Hom}_\mathcal{E}(A, B) \otimes \text{Hom}_\mathcal{E}(B, A) \to \mathbb{K},$$

for each $A, B \in \text{Obj}(\mathcal{E})$, required to satisfy condition (3) and the following identity:

$$\langle m_n(a_0 \otimes \cdots \otimes a_{n-2}), a_{n-1} \rangle = (-1)^{(n+1)+|a_0| \sum_{i=1}^{n-1} |a_i|} \langle m_{n-1}(a_1 \otimes \cdots \otimes a_{n-2}), a_0 \rangle.$$

8.3 Open KTCFTs and involutive Calabi-Yau $A_\infty$-categories

The main result of this chapter states that the category of open KTCFTs is quasi-isomorphic to the category of Calabi-Yau $A_\infty$-categories endowed with involution. We will get products $m_n$ from the generators of the categories defined in the previous sections and, by using the twisted
disc $D^\tau(\lambda_0, \lambda_1)$, we will equip all our $A_\infty$-categories with an involution.

Let $\mathfrak{F} : \tilde{D}_{\Lambda, \text{open}}^+ \to \text{Comp}_K$ be a split symmetric monoidal functor. For each $O \in \mathbb{N}$ and D-brane labelling given by $\{s(i), t(i)\}$, with $0 \leq i \leq O - 1$, the following isomorphism holds:

$$\mathfrak{F}(O, s, t) \cong \bigotimes_{i=0}^{O-1} \mathfrak{F}(\{s(i), t(i)\}). \quad (4)$$

Let the pair $\{s(i), t(i)\}$ correspond to the pair of D-branes $\{\lambda_i, \lambda_{i+1}\}$. We can define a category $\mathcal{B}$ with $\text{Obj}(\mathcal{B}) = \Lambda$ and $\text{Hom}_\mathcal{B}(\lambda_i, \lambda_{i+1}) := \mathfrak{F}(\{s(i), t(i)\})$. Composition of morphisms in $\mathcal{B}$ makes sense as $\mathfrak{F}$ is split. Observe that we are just associating each open boundary component (i.e. each interval and later on each open marked point) to the space $\text{Hom}_\mathcal{B}(\lambda_i, \lambda_{i+1})$.

A unital extended involutive Calabi-Yau $A_\infty$-category with objects in $\Lambda$ is a h-split symmetric monoidal functor $\mathfrak{F} : \tilde{D}_{\Lambda, \text{open}}^+ \to \text{Comp}_K$. By considering $\tilde{D}_{\Lambda, \text{open}}^+$ instead of $\tilde{D}_{\Lambda, \text{open}}$ we get the concept of unital extended involutive $A_\infty$-category. If we consider split functors instead of h-split functors, we obtain the concept of unital Calabi-Yau involutive $A_\infty$-category and the concept of unital involutive $A_\infty$-category respectively. These definitions make sense due to the following Lemmata:

**Lemma 8.1.** A split symmetric monoidal functor $\mathfrak{F} : \tilde{D}_{\Lambda, \text{open}}^+ \to \text{Comp}_K$ is the same as a unital involutive $A_\infty$-category $\mathcal{B}$ with set of objects $\Lambda$.

**Proof.** The proof follows from the isomorphism (4) above. Let us observe that:

1. The twisted disc $D^\tau(\lambda_0, \lambda_1)$ yields the involution
   
   $$(-)^* : \text{Hom}_\mathcal{B}(\lambda_0, \lambda_1) \to \text{Hom}_\mathcal{B}(\lambda_1, \lambda_0);$$

2. the discs $D^+(\lambda_0, \ldots, \lambda_{n-1})$ yield the products
   
   $$m_{n-1} : \text{Hom}_\mathcal{B}(\lambda_0, \lambda_1) \otimes \cdots \otimes \text{Hom}_\mathcal{B}(\lambda_{n-2}, \lambda_{n-1}) \to \text{Hom}_\mathcal{B}(\lambda_0, \lambda_{n-1});$$

3. the differential $d$ gives the $A_\infty$-relations between the $m_n$;

4. for $n = 2$, $D^+(\lambda_0, \lambda_1)$ yields the identity $\text{Hom}_\mathcal{B}(\lambda_0, \lambda_1) \to \text{Hom}_\mathcal{B}(\lambda_0, \lambda_1)$;

5. for $n = 1$, $D^+(\lambda)$ yields the unit $K \to \text{Hom}_\mathcal{B}(\lambda, \lambda)$.

Observe that relation 3 in Corollary 7.5 proves that the products $m_n$ preserve the involution. □

**Lemma 8.2.** A split symmetric monoidal functor $\mathfrak{F} : \tilde{D}_{\Lambda, \text{open}}^+ \to \text{Comp}_K$ is the same as a unital involutive Calabi-Yau $A_\infty$-category $\mathcal{B}$ with set of objects $\Lambda$. 

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Proof. The proof follows the same arguments of Lemma 8.1 but now we have two more generators (see Theorem 7.7): the discs with two incoming and two outgoing marked points, which yield the map
\[ \text{Hom}_B(\lambda_0, \lambda_1) \otimes \text{Hom}_B(\lambda_1, \lambda_0) \rightarrow K \]
and its inverse. The extra relations on \( \tilde{D}_{\Lambda,\text{open}} \) correspond to the cyclic symmetry condition. As in the previous result, the anti-analytic involution on the Riemann surfaces is transferred to the involutive Calabi-Yau \( A_\infty \)-category through the disc \( D^\tau(\lambda_0, \lambda_1) \). Observe that we can deduce the identity \( \langle f, \hat{g} \rangle_{A,B} = \langle \hat{g}^*, f^* \rangle_{A,B} \) from relation 3 of Corollary 7.5.

The following result is clear from the above results, and almost proves the first part of our main theorem:

**Proposition 8.3.** The category of unital extended involutive Calabi-Yau \( A_\infty \)-categories with set of objects \( \Lambda \) is quasi-equivalent to the category of open KTCFTs.

**Proof.** Let us recall that an open KTCFT is an h-split monoidal functor
\[ \mathcal{R} : \tilde{O}_\Lambda \rightarrow \text{Comp}_K. \]

The result follows from Lemma 8.2 and the quasi-isomorphism between \( \tilde{O}_\Lambda \) and \( \tilde{D}_{\Lambda,\text{open}} \) in Proposition 7.3.

**Proposition 8.4.** The following categories, each one with set of objects \( \Lambda \), are quasi-equivalent:

1. The category of unital extended involutive \( A_\infty \)-categories;
2. the category of unital involutive \( A_\infty \)-categories and
3. the category of unital involutive DG categories.

**Proof.** Let \( \alpha, \beta \in \text{Obj}(\tilde{D}^+_\Lambda,\text{open}) \). The space \( \tilde{D}^+_\Lambda,\text{open}(\alpha, \beta) \) is contractible as it is given by the chains on the moduli spaces of discs with \( \alpha \) incoming marked points and \( \beta \) outgoing marked points, hence for \( n \neq 0 \) we have:
\[ H_n(\tilde{D}^+_\Lambda,\text{open}(\alpha, \beta)) = 0. \]

This implies that \( \tilde{D}^+_\Lambda,\text{open}(\alpha, \beta) \) is quasi-isomorphic to its homology and in particular it is quasi-isomorphic to \( H_0(\tilde{D}^+_\Lambda,\text{open}(\alpha, \beta)) \).

With the notation introduced at the beginning of this section, giving a split functor
\[ \tilde{\mathcal{F}} : H_0(\tilde{D}^+_\Lambda,\text{open}(\alpha, \beta)) \rightarrow \text{Comp}_K \]
is the same as giving a unital DG category \( \mathcal{B} \) with set of objects \( \Lambda \). Observe that
\[ H_0(\tilde{D}^+_\Lambda,\text{open}([\lambda_0, \ldots, \lambda_n], \{\lambda_0, \lambda_n\})) \]
corresponds to an “alien pair of pants” given by a disc with \(n\) marked points with the point
between \(\lambda_n\) and \(\lambda_0\) is outgoing. This corresponds to the product
\[
\text{Hom}_B(\lambda_0, \lambda_1) \otimes \cdots \otimes \text{Hom}_B(\lambda_{n-1}, \lambda_n) \to \text{Hom}_B(\lambda_0, \lambda_n),
\]
which is associative as \(\text{H}_0(\tilde{\mathcal{D}}^+_{\Lambda, \text{open}}(\{\lambda_0, \ldots, \lambda_n\}, \{\lambda_0, \lambda_n\}))\) has dimension one.

We show that there is a quasi-equivalence between unital extended involutive \(A_\infty\)-categories and unital extended involutive DG categories. For that purpose we will use the equivalences obtained in Lemmata 8.1 and 8.2. We define a unital extended involutive DG category as a h-split functor of the form \(\mathcal{F} : \text{H}_0(\tilde{\mathcal{D}}^+_{\Lambda, \text{open}}) \to \text{Comp}_K\). Due to the quasi-isomorphism
\[
\tilde{\mathcal{D}}^+_{\Lambda, \text{open}} \cong \text{H}_0(\tilde{\mathcal{D}}^+_{\Lambda, \text{open}})
\]
there is a quasi-equivalence between unital extended involutive \(A_\infty\)-categories and unital extended involutive DG categories given by

\[
\begin{array}{ccc}
\tilde{\mathcal{D}}^+_{\Lambda, \text{open}} & \cong & \text{H}_0(\tilde{\mathcal{D}}^+_{\Lambda, \text{open}}) \\
\downarrow & & \downarrow \mathcal{F} \\
\text{Comp}_K & \leftarrow & \text{H}_0(\tilde{\mathcal{D}}^+_{\Lambda, \text{open}})
\end{array}
\]

Our next step is to show that there exist a quasi-equivalence between unital extended involutive \(A_\infty\)-categories and involutive \(A_\infty\)-categories. It goes as follows:

In \(\tilde{\mathcal{D}}^+_{\Lambda, \text{open}}\) the following isomorphism holds:
\[
\bigotimes_{i=1}^n \tilde{\mathcal{D}}^+_{\Lambda, \text{open}}(a_i, \{\lambda_i, \lambda'_i\}) \to \tilde{\mathcal{D}}^+_{\Lambda, \text{open}}(\bigsqcup_{i=1}^n a_i, \bigsqcup_{i=1}^n \{\lambda_i, \lambda'_i\}).
\] (6)

Let us consider a unital extended involutive \(A_\infty\)-category given by a h-split functor
\[
\mathfrak{F} : \tilde{\mathcal{D}}^+_{\Lambda, \text{open}} \to \text{Comp}_K
\]
and define a unital involutive \(A_\infty\)-category given by a split functor \(F_{\mathfrak{F}} : \tilde{\mathcal{D}}^+_{\Lambda, \text{open}} \to \text{Comp}_K\) by stating:
\[
F_{\mathfrak{F}}(O, s, t) := \bigotimes_{i=0}^{O-1} \mathfrak{F}(\{s(i), t(i)\}).
\]

This definition together with the isomorphism (6) assure the existence of maps \(F_{\mathfrak{F}}(a) \to \mathfrak{F}(a)\) which, composing with the action of \(\tilde{\mathcal{D}}^+_{\Lambda, \text{open}}\) yields maps
\[
F_{\mathfrak{F}}(a) \otimes \tilde{\mathcal{D}}^+_{\Lambda, \text{open}}(\alpha, \{\lambda_0, \lambda_1\}) \to F_{\mathfrak{F}}(\{\lambda_0, \lambda_1\}),
\]
indeed:

\[
\begin{align*}
F_\delta(a) & \rightarrow \mathfrak{F}(a) \\
F_\delta(a) \otimes \tilde{D}_\Lambda^{\text{open}}(a, \{\lambda_0, \lambda_1\}) & \rightarrow \mathfrak{F}(a) \otimes \tilde{D}_\Lambda^{\text{open}}(a, \{\lambda_0, \lambda_1\}) \\
F_\delta(a) \otimes \tilde{D}_\Lambda^{\text{open}}(a, \{\lambda_0, \lambda_1\}) & \rightarrow \mathfrak{F}(\{\lambda_0, \lambda_1\}) = F_\delta(\{\lambda_0, \lambda_1\})
\end{align*}
\]

Due to the isomorphisms (6) we get that \( F_\delta \) is monoidal, what leads us to conclude that \( F_\delta \) is a \( \tilde{D}_\Lambda^{\text{open}} \)-module.

This concludes the proof of the equivalence \((1) \iff (2)\). Similarly we prove that unital extended involutive DG categories are quasi-equivalent to unital involutive DG categories. Let UEI stand for “unital extended involutive”, the diagram (5) connects the latter quasi-equivalences in the sense below:

\[
\begin{array}{ccc}
\text{UEI } A_\infty\text{-categories} & \xrightarrow{(5)} & \text{UEI DG categories} \\
\approx & & \approx \\
\text{Unital } A_\infty\text{-categories} & \xrightarrow{(1)} & \text{Unital DG categories}
\end{array}
\]

Observe that (1) denotes the quasi-equivalence \((2) \iff (3)\) and that the quasi-equivalence \((3) \iff (1)\) is straightforward from the diagram above.

This concludes the proof of part (1) of Theorem 1.1. Observe that, as we have shown that there are quasi-isomorphisms \( D_{\Lambda^{\text{open}}} \cong \tilde{O}_\Lambda \text{ and } D_{\Lambda} \cong \tilde{O}_\Lambda \text{ (this is Proposition 7.3)}, \) by Proposition 3.8 we have, for a left \( \tilde{D}_{\Lambda^{\text{open}}} \)-module \( \mathfrak{M}_1 \) and its associated left \( \tilde{O}_\Lambda \)-module \( \mathfrak{M}_2 \):

\[
\tilde{O}_\Lambda(-, \beta) \otimes^L_{\tilde{O}_\Lambda} \mathfrak{M}_2 \cong \tilde{D}_{\Lambda} \otimes_{\tilde{D}_{\Lambda^{\text{open}}}} \mathfrak{M}_1.
\]

This shows that, if \( \mathfrak{M}_2 \) is h-split, so it is \( \mathfrak{M}(\beta) \). Therefore \( \mathfrak{M} \) defines nothing but an open-closed Klein topological conformal field theory, which is the universal open-closed KTCFT associated to \( \mathfrak{M}_2 \). This proves part (2) of Theorem 1.1. Our next objective is to prove part (3), concluding the proof of Theorem 1.1; this is the purpose of the next section.

### 9 Open-closed KTCFTs and involutive Hochschild homology

For an involutive DG category \( \mathcal{A} \), we define its involutive Hochschild chain complex as

\[
\mathcal{C}^{\text{inv}}(\mathcal{A}) = \bigoplus_n \left( \bigoplus_{a_0, \ldots, a_{n-1}} \text{Hom}_\mathcal{A}(a_0, a_1) \otimes \cdots \otimes \text{Hom}_\mathcal{A}(a_{n-1}, a_0) \right) [1 - n] / \sim,
\]

where \( \sim \) denotes the relation \( f_0^* \otimes g = f_0 \otimes g^* \), with \( g = (f_1, \ldots, f_{n-1}) \). The involution is given by: \( (f_0 \otimes \cdots \otimes f_{n-1})^* = f_{n-1}^* \otimes \cdots \otimes f_0^* \).
The differential for $C_*^{\text{inv}}(\mathcal{A})$ is given, for maps $f_i \in \text{Hom}_\Lambda(\alpha_i, \alpha_{i+1})$, by:

$$d(f_0 \otimes \cdots \otimes f_{n-1}) = \sum_{i=0}^{n-1} (-1)^i (f_0 \otimes \cdots \otimes df_i \otimes \cdots \otimes f_{n-1}) + \sum_{i=0}^{n-2} (-1)^i (f_0 \otimes \cdots \otimes (f_{i+1} \circ f_i) \otimes \cdots \otimes f_{n-1}) + (-1)^{n-1}((f_0 \circ f_{n-1}) \otimes \cdots \otimes f_{n-2})$$

**Lemma 9.1.** The differential $d$ preserves involutions.

*Proof.* It is a direct computation:

$$d(f_0 \otimes \cdots \otimes f_{n-1})^* = \sum_{i=0}^{n-1} (-1)^i (f_0 \otimes \cdots \otimes df_i \otimes \cdots \otimes f_{n-1})^* + \sum_{i=0}^{n-2} (-1)^i (f_0 \otimes \cdots \otimes (f_{i+1} \circ f_i) \otimes \cdots \otimes f_{n-1})^* + (-1)^{n-1}((f_0 \circ f_{n-1}) \otimes \cdots \otimes f_{n-2})^* = d(f_{n-1}^* \otimes \cdots \otimes f_0^*)$$

If $\mathcal{A}$ is unital, the normalized involutive Hochschild chain complex $\overline{C}^{\text{inv}}_*(\mathcal{A})$ is the quotient of $C_*^{\text{inv}}(\mathcal{A})$ by the sub-complex spanned by $f_0 \otimes \cdots \otimes f_{n-1}$, where at least one of the maps $f_i$ (for $i > 0$) is the identity. We have the following result:

**Lemma 9.2 (cf. [Cos07], Lemma 7.4.1).** The functor $\mathcal{A} \rightarrow \overline{C}^{\text{inv}}_*(\mathcal{A})$, from the category of involutive DG categories with set of objects $\Lambda$ to the category of complexes, is exact.

For an extended involutive Calabi-Yau $A_\infty$-category $\phi$ there is an underlying extended involutive $A_\infty$-category given by restricting to $\widetilde{D}^+_{\Lambda,\text{open}}$; indeed: let $\phi : \widetilde{D}^+_{\Lambda,\text{open}} \rightarrow \text{Comp}_K$ be an extended involutive Calabi-Yau $A_\infty$-category (see Lemma 8.2); if we consider the subcategory $\widetilde{D}^+_{\Lambda,\text{open}} \subset \widetilde{D}_{\Lambda,\text{open}}$ and take the restriction $\phi|_{\widetilde{D}^+_{\Lambda,\text{open}}}$, we get a functor $\widetilde{D}^+_{\Lambda,\text{open}} \rightarrow \text{Comp}_K$ which, by Lemma 8.2, is an extended involutive Calabi-Yau $A_\infty$-category; this is the underlying category we are talking about. The Hochschild homology of $\phi$ is defined to be the homology of the associated underlying $A_\infty$-category.

**Proposition 9.3.** For a unital involutive extended Calabi-Yau $A_\infty$-category $\phi$ the following equality holds:

$$\widetilde{D}_{\Lambda}(-,1) \otimes \widetilde{D}_{\Lambda,\text{open}} \phi = \overline{C}^{\text{inv}}_*(\phi).$$

*Proof.* We will introduce some notations that will make the proof much easier to read: we will write $\Lambda_0 := (\lambda_0, \ldots, \lambda_{n-1})$ and $\Lambda' := (\lambda_0, \ldots, \lambda_i, \lambda_{i+1}, \ldots, \lambda_{n-1})$ for $i > 0$. The proof for this result is based on the generators and relations stated in Lemma 7.5, Theorem 7.7 and Lemma 7.8. We can state the following equality, as we have defined the generators of $\overline{D}_{\Lambda}$ as those of $\overline{D}_{\Lambda}$:

$$\overline{D}_{\Lambda}(-,1) \otimes \overline{D}_{\Lambda,\text{open}} \phi = \overline{D}_{\Lambda}(-,1) \otimes \overline{D}_{\Lambda,\text{open}} \phi.$$
From Lemma 7.10 we know that the functor \( \phi \mapsto \tilde{\mathcal{D}}_\Lambda^+(-, 1) \otimes_{\mathcal{D}_{\Lambda, \text{open}}^+} \mathcal{D}_{\Lambda, \text{open}}^+ \) is exact, so we only have to check that the equality below holds for the DG category \( \mathcal{B} \) associated to \( \phi \), thought of as a left \( \mathcal{D}_{\Lambda, \text{open}}^+ \)-module:

\[
\tilde{\mathcal{D}}_\Lambda^+(-, 1) \otimes_{\mathcal{D}_{\Lambda, \text{open}}^+} \mathcal{B} = C_{\text{inv}}^\bullet(\mathcal{B}).
\]

**Remark 9.4.** The association between \( \phi : \tilde{\mathcal{D}}_\Lambda^+ \rightarrow \text{Comp}_K \) and the involutive DG category \( \mathcal{B} \) follows from the quasi-equivalence stated in Proposition 8.4.

We will proceed as in Lemma 7.10: in degree \( n \), the associated complex to the \( \text{Obj}(\tilde{OC}_\Lambda) \)-module \( \tilde{\mathcal{D}}_\Lambda^+(-, 1) \otimes_{\mathcal{D}_{\Lambda, \text{open}}^+} \mathcal{B}(\cdot) \) is spanned by \( A(\lambda)^{\otimes} \mathcal{B}(\cdot) \), which is associated, through the disc \( D(\lambda) \), to the product:

\[
\text{Hom}_\mathcal{B}(\lambda_0, \lambda_1) \otimes \cdots \otimes \text{Hom}_\mathcal{B}(\lambda_{n-1}, \lambda_0),
\]

modulo the subspace spanned by the elements of the form \( \phi_0 \otimes \cdots \otimes \phi_{n-1} \), where at least one of the \( \phi_i \) (for \( i > 0 \)) is the identity. This quotient comes from the construction of the tensor product \( \tilde{\mathcal{D}}_\Lambda^+(-, 1) \otimes_{\mathcal{D}_{\Lambda, \text{open}}^+} \mathcal{B} \), indeed: let us recall that the tensor product is characterized by the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{\mathcal{D}}_\Lambda^+(m, 1) \otimes_K \tilde{\mathcal{D}}_{\Lambda, \text{open}}^+(n, m) \otimes_K \mathcal{B}(n) & \xrightarrow{(1)} & \tilde{\mathcal{D}}_\Lambda^+(m, 1) \otimes_K \mathcal{B}(m) \\
\downarrow{(2)} & & \downarrow \\
\tilde{\mathcal{D}}_\Lambda^+(n, 1) \otimes_K \mathcal{B}(n) & \rightarrow & \tilde{\mathcal{D}}_\Lambda^+(-, 1) \otimes_{\mathcal{D}_{\Lambda, \text{open}}^+} \mathcal{B}(-)
\end{array}
\]

**Remark 9.5.** Mind the abuse of notation in the diagram: we write \( \mathcal{B}(m) \) for \( \mathcal{B}(\{\lambda_0, \ldots, \lambda_{m-1}\}^c) \).

Action (1) corresponds to gluing the surface in \( \tilde{\mathcal{D}}_{\Lambda, \text{open}}^+(n, m) \) to a disc depicting \( \mathcal{B}(n) \) whilst action (2) corresponds to gluing the same surface in \( \tilde{\mathcal{D}}_{\Lambda, \text{open}}^+(n, m) \) to an annulus representing \( \tilde{\mathcal{D}}_\Lambda^+(m, 1) \). Algebraically, action (1) corresponds to the map:

\[
\text{Hom}_\mathcal{B}(\lambda_0, \lambda_1) \otimes \cdots \otimes \text{Hom}_\mathcal{B}(\lambda_{m-1}, \lambda_0) \rightarrow \\
\text{Hom}_\mathcal{B}(\lambda_0, \lambda_1) \otimes \cdots \otimes \text{Hom}_\mathcal{B}(\lambda_i, \lambda_i) \otimes \cdots \otimes \text{Hom}_\mathcal{B}(\lambda_{m-1}, \lambda_0)
\]

defined by: \( f_0 \otimes \cdots \otimes f_{m-1} \mapsto f_0 \otimes \cdots \otimes \text{Id}_{\lambda_{i-1} \lambda_i} \otimes \cdots \otimes f_{m-1} \). On the other hand, action (2) is zero as stated in Theorem 7.8. Keeping in mind that the diagram commutes, we get the relation that the tensor product of maps where at least one is the identity (for \( i > 0 \)) yields zero, and this is what leads to the quotient space above. The following picture intends to make this reasoning clearer:
There is a further relation given by:

This relation corresponds to the following: for \( f_0 \in \text{Hom}_B(\lambda_0, \lambda_1), f_1 \in \text{Hom}_B(\lambda_1, \lambda_2) \) and \( f_2 \in \text{Hom}_B(\lambda_2, \lambda_0) \) we have: \( f_0^* \otimes f_1 \otimes f_2 = f_0 \otimes f_2^* \otimes f_1^* \).

This shows that \( \tilde{D}_\Lambda^{+}(-,1) \otimes \tilde{B}_{\lambda, \text{open}}^{+} \mathcal{B} \) is isomorphic, as a vector space, to the quotient of

by the relation \( f_0^* \otimes g = f_0 \otimes g^* \), modulo the subspace spanned by the elements of the form \( f_0 \otimes \cdots \otimes f_{n-1} \) above. This is precisely the definition given for the normalized involutive Hochschild complex \( \mathcal{C}^{\text{inv}}_{B} (\mathcal{B}) \), ignoring the differential \( d \) momentarily. The compatibility with \( d \) follows from Proposition 7.4.3 [Cos07].

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