Bloch oscillations of Bose-Einstein condensates: Breakdown and revival

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We investigate the dynamics of Bose–Einstein condensates (BEC) in a tilted one–dimensional periodic lattice within the mean–field (Gross–Pitaevskii) description. Unlike in the linear case the Bloch oscillations decay because of nonlinear dephasing. Pronounced revival phenomena are observed. These are analyzed in detail in terms of a simple integrable model constructed by an expansion in Wannier–Stark resonance states. We also briefly discuss the pulsed output of such systems for stronger static fields.

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I. INTRODUCTION

Despite its apparent simplicity, the dynamics of quantum particles in periodic structures is full of surprises, even in the one–dimensional case. For almost a century Bloch waves have been known which are delocalized states in a lattice leading to transport. If an additional static field \( F \) is introduced, these states become localized and counterintuitively transport is dramatically reduced. Instead an oscillatory motion is found, the famous Bloch oscillations of Bose-Einstein condensates: Breakdown and revival

\[
\text{where } M \text{ is the atomic mass, } g \text{ is the interaction strength.}
\]

and \( V(x) = V(x + d) \) is the periodic lattice potential.

This nonlinear system shows basically all the features found in the analysis of the linear equation, such as Bloch oscillations of the condensate \( B \). In addition, the nonlinearity introduces new effects, such as soliton–like motion, nonlinear Zener tunneling \([10, 11]\) and “classically” chaotic dynamics \([12, 13, 14]\). This system has been analyzed with various methods, see e.g. \([15, 16, 17]\).

In the present article, we will focus on Bloch oscillations and analyze the influence of the nonlinearity. The paper is organized as follows: In section II we present results from a numerical solution of the GPE and show nonlinear Bloch oscillations for relatively weak fields and different strength and signs of the nonlinear interaction. Section III introduces our main tool, a discrete representation by an expansion in Wannier–Stark resonance states and derives approximate results based on this approach. These results are used to analyze the dynamical behavior of Bloch oscillations in section IV. Finally we discuss the modification of the coherent pulse output of a Bloch oscillating condensate for stronger fields in section V. The paper closes with some concluding remarks.

II. NUMERICAL STUDY OF NONLINEAR BLOCH OSCILLATIONS

Due to the nonlinearity of the GPE analytical studies are difficult, so numerical simulations are helpful in guiding theoretical investigations.

In all numerical studies we will use a cosine potential \( V(x) = V_0 \cos(2\pi x/d) \). We furthermore use scaled units with \( d = 2\pi, V_0 = M = 1 \) (see \([4]\) for more details). It is worth noting that the scaled interaction strength is inversely proportional to the depth of the potential.

The GPE then reads

\[
\text{i} \hbar \partial_t \psi = \left[ -\frac{\hbar^2}{2M} \partial_x^2 + V(x) + Fx + g |\psi|^2 \right] \psi,
\]

where \( \psi \) is the wavefunction of the condensate.

and we will use the value \( \hbar = 3.806 \) for the scaled Planck constant adapted to the experiment of the Kasevich group \([6]\) (see also \([18]\)). The nonlinearity param-
eter is of the order \( g \leq 1 \) in this experiment. Here we will extend the analysis, however, to much stronger nonlinearities up to \( |g| = 10 \). This regime could be reached experimentally by increasing the transverse confinement or decreasing the depth of the optical lattice.

Here we are mainly interested in the dynamics of Bloch oscillations and therefore use a weak field \((F = 0.005)\) and initial states populating almost exclusively the lowest Bloch band. In this case the decay is negligible. In the linear case the band gap between the lowest and the next higher Bloch band for the field free case is \( \delta = 0.998 \) and the probability for Zener tunneling is \( \approx 10^{-12} \). This does not change noticeably in the nonlinear case as has been checked numerically, showing that nonlinear Zener tunneling \([10, 11]\) does not play a role for the given parameters. Furthermore note that the dynamical effects which are observed in the following examples (e.g. the damping of Bloch oscillations) are essentially captured by a simple integrable model which is introduced in section \([11]\). This should be compared to the breakdown of Bloch oscillations due to dynamical instability discussed in \([11, 10, 20]\).

Let us start our discussion with a brief look at the Bloch oscillation for the linear case \( g = 0 \). As an initial state, we use a Gaussian wave packet

\[
\psi(x, t = 0) = \frac{1}{(2\pi)^{1/4}\tilde{\sigma}^{1/2}} e^{-ix_0^2/4\tilde{\sigma}^2}
\]

with width \( \tilde{\sigma} = 40\pi \) which is then projected onto the lowest Bloch band and normalized to unity. For the time propagation a split–operator method \([21]\) is used which can also be applied to the nonlinear case. In figure 1 we observe the familiar Bloch oscillation with a large amplitude because of the weak field. Let us recall that the region over which the Bloch oscillation extends can be estimated as \( \Delta / F \approx 200 \approx 32 \cdot 2\pi \) within the tilted band picture, where \( \Delta = 0.9994 \) is the width of the dispersion relation \( E(\kappa) \) in the field free case. The numerical results confirm this estimate as the top of figure 2 shows. As expected for such an initially wide distribution in coordinate space, the width of the wave packet remains practically constant varying periodically with a relative amplitude of about \( 10^{-3} \) (figure 2 bottom).

Let us now discuss the influence of a nonlinearity fixing for the moment the nonlinear parameter at \( g = 5 \) (repulsive interaction) and \( g = -5 \) (attractive interaction). From figure 3 one can observe that the Bloch oscillations continue to exist, at least for the short times up to \( t \approx 10T_B \) shown in the figure. In addition to the well known localization of \( |\psi(x, t)|^2 \) in the minima of
the cosine–potential, one observes a further filamentation which is particularly pronounced for an attractive interaction. As shown in figure 4, the amplitude of the oscillation decreases and the oscillation in the width strongly increases.

![Figure 4](image1)

**FIG. 4:** Expectation values of the position $\langle x \rangle_t$ and width $\Delta x_t$ for the wave function shown in figure 3 for a repulsive nonlinearity $g = +5$.

![Figure 5](image2)

**FIG. 5:** Same as figure 4 however for for an attractive nonlinearity $g = -5$.

Also shown in figures 4 and 5 is the time dependence of the width $\Delta x_t$ of the wave packet. In sharp contrast to the tiny oscillations of the width in the linear case (see figure 2) we find here very pronounced oscillations which are rapidly growing (as already described by Holthaus [15]). Such a phenomenon is known as breathing and is exhibited in the linear system by wave packets which are initially strongly localized in coordinate space [1, 22]. Note that the oscillations of the width $\Delta x_t$ for a repulsive and attractive nonlinearity are opposite to each other.

For a stronger nonlinearity $g = 10$, as illustrated in figures 6 and 7, the Bloch oscillations are damped more strongly. However, the oscillation does not fade completely but shows a revival with a smaller amplitude after a shrinking to approximately two lattice periods. A corresponding behavior is observed for the width, where the breathing amplitude of the wave function first grows fast up to a time of about eight Bloch periods. After this time, the width remains limited and oscillates in the interval from 31 to 35 lattice periods.

![Figure 6](image3)

**FIG. 6:** Expectation value $\langle x \rangle_t$ of the position and width $\Delta x_t$ for a repulsive nonlinearity $g = +10$. The inset shows a magnification of the time interval between 8 and 16 $T_B$.

![Figure 7](image4)

**FIG. 7:** Width $\Delta x_t$ of the wave packet shown in figure 6 for a repulsive nonlinearity $g = +10$.

Furthermore, the oscillation of $\langle x \rangle_t$ shows phase jumps that can be seen, e.g. in the inset of figure 6 at $t \approx 14 T_B$. A similar behavior has also been described in [15]. This phase jump coincides with a minimum in the amplitude. These phenomena can be understood in terms of an expansion in Wannier-Stark basis functions as explained in the next section.

### III. WANNIER–STARK BASIS SET EXPANSION

An alternative approach to a direct numerical integration of the GPE is an expansion in an adequate discrete basis as for example the ground states of single potential wells [12, 23]. In this work we adopt a different approach, following [13, 17], and expand the wave function into the resonance eigenstates of the linear system, the so–called Wannier–Stark states $\Psi_{\alpha,n}(x)$ which are eigenstates of...
the linear Hamiltonian $H_0$:

$$H_0 \Psi_{\alpha,n}(x) = \mathcal{E}_{\alpha,n} \Psi_{\alpha,n}(x),$$

where $\alpha$ is the ladder index and $n$ is the site index. The energies form the Wannier–Stark ladder

$$\mathcal{E}_{\alpha,n} = \mathcal{E}_{\alpha,0} + 2\piFn.$$  

(5)

The Wannier–Stark states extend over several periods of the potential (see remark [24] and review [4] for more information). This approach has proven to be extremely convenient to describe the dynamics in tilted optical lattices in the linear case, especially for higher field strengths [4,18].

Up to section IV we will restrict the discussion to small field strengths $F$. Then one can neglect decay and Landau–Zener tunneling and use the lowest ladder $\alpha = 0$ only, henceforth the index $\alpha$ is omitted. Also neglecting decay, the imaginary part of the energy $\mathcal{E}_n$ is set to zero. Plugging the expansion $\psi(x,t) = \sum_m c_m(t) \Psi_m(x)$ into the GPE (4) leads to a set of coupled ordinary differential equations:

$$i\hbar \sum_m \dot{c}_m \Psi_m = \sum_m (\mathcal{E}_0 + 2\pi Fn) c_m \Psi_m + \sum_{klm} c_k^* c_l c_m \chi_{klm}^* \chi_{l,m}^n \Psi_l \Psi_m.$$  

(6)

The energy $\mathcal{E}_0$ only leads to a global phase factor and hence is omitted in the following. The Wannier–Stark states $\Psi_n$ are orthogonal to their left eigenstates $\Psi^L_m$ for $m \neq n$. Nevertheless, since we neglect the resonance properties of the system we can identify left and right eigenvectors, i.e. assume that $H_0$ is hermitian. So multiplying equation (6) by $\Psi_n$ and integrating yields

$$i\hbar \dot{c}_n = 2\pi Fn c_n + \sum_{klm} \chi_{klm}^* c_k^* c_l c_m,$$  

(7)

with the coupling tensor

$$\chi_{klm}^n = \int \Psi_n^*(x) \Psi_k^*(x) \Psi_l(x) \Psi_m(x) \ dx,$$  

(8)

which is symmetric under the exchange of its first and last two indices. Due to the discrete translational invariance of the Wannier–Stark states $\Psi_n(x) = \Psi_0(x - 2\pi n)$ one finds

$$i\hbar \dot{c}_n = 2\pi Fn c_n + \sum_{klm} \chi_{klm}^* c_{k+n}^* c_{l+n} c_m + \sum_{klm} \chi_{klm}^* c_{k+n} c_{l+n} c_m,$$  

(9)

defining $\chi_{klm} = \chi_{klm}^0$.

Though not suited for direct numerical calculations because of the triple infinite sum, equation (9) provides a basis for further approximations. In the following we will reduce it to a simple integrable model, which nevertheless captures important features of the dynamics. To this end we decompose the coefficients $c_n$ into phase and amplitude

$$c_n = \sqrt{\rho_n} e^{i\varphi_n}. \quad \quad \quad (10)$$

The imaginary parts of the coupling tensor $\chi_{klm}$ are negligible and so one arrives at the coupled equations

$$i\hbar \dot{\varphi}_n = -2\pi Fn - g\rho_n \sum_{klm} \chi_{klm} \left( \frac{\rho_{k+n} \rho_{l+n} \rho_{m+n}}{\rho_n^3} \right)^{1/2} \times \cos(\varphi_{l+n} + \varphi_{m+n} - \varphi_{k+n} - \varphi_n), \quad \quad \quad (11)$$

$$i\hbar \dot{\rho}_n = 2g\rho_n^2 \sum_{klm} \chi_{klm} \left( \frac{\rho_{k+n} \rho_{l+n} \rho_{m+n}}{\rho_n^3} \right)^{1/2} \times \sin(\varphi_{l+n} + \varphi_{m+n} - \varphi_{k+n} - \varphi_n). \quad \quad \quad (12)$$

If the initial state is broad, populating about 20 wells, the amplitudes $\rho_n(t = 0)$ are small. Because of $\dot{\varphi}_n \approx \rho_n$ this implies that the amplitudes $\rho_n$ change only slowly in time compared to the phases $\varphi_n$ and can be assumed to be constant.

Furthermore we reduce the expression for $\dot{\varphi}_n$ to the most important contributions. Numerically examining the $\chi_{klm}$ shows that the dominating terms are $\chi_{000}$, $\chi_{kk0} = \chi_{k0k}$ and $\chi_{0kk}$, which is not unexpected considering equation (5). It can also be argued (and verified numerically) that the terms in equation (11) which have a nonzero argument of the cosine have little importance, as their contributions average out. This leaves the terms including $\chi_{000}$ and $\chi_{kk0}$ = $\chi_{k0k}$ and finally leads to:

$$i\hbar \dot{\varphi}_n = -2\pi Fn - g\gamma_n \rho_n \quad \quad \quad (13)$$

with

$$\gamma_n = \chi_{000} + 2 \sum_{k \neq 0} \chi_{k00} \frac{\rho_{k+n}}{\rho_n^3}. \quad \quad \quad (14)$$

These equations (13) are integrated to

$$\rho_n(t) = \rho_n(0), \quad \varphi_n(t) = \omega_n t \quad \quad \quad (15)$$

with

$$i\hbar \omega_n = -2\pi Fn - g\gamma_n \rho_n. \quad \quad \quad (16)$$

Note that this solution is exact for $g = 0$. Numerical calculations show that one can safely neglect the dependence of $\gamma_n$ on the index $n$ and set $\gamma_n \approx \gamma$. For the given parameters equation (14) yields $\gamma_n \geq \gamma_0 = 0.278$. However, the best fit with the results from a wave packet propagation are obtained for $\gamma = 0.15$.

This admittedly quite crude approximation shows very good agreement with an exact numerical solution. In figures 5 and 6 the approximation (15) is compared with the results obtained by a wave packet propagation using the split–operator method [21]. A normalized Gaussian initial distribution with coefficients

$$c_n \sim e^{-n^2/4\sigma^2}, \quad \sigma = \sigma_0/2\pi = 20, \quad \quad \quad (17)$$
is used which closely resembles a Gaussian wave packet projected onto the lowest Bloch band in configuration space. The dynamics for a moderate nonlinearity $g = 1$ is well described by equation (15), only the growth of the width is somewhat underestimated. For $g = 10$ the approximation (15) becomes less accurate. In particular it overestimates the revival of the Bloch oscillation and underestimates the growth of the width of the wave packet. However it still captures the important features at least qualitatively: the decay and revival of the oscillations, the phase jump around $t = 14T_B$ and the breathing of the wave function.

The systematic growth of the width of the wave packet is mainly due to a broadening of the amplitude distribution $\rho_n$ and therefore clearly not included in approximation (15). To discuss this effect we briefly reintroduce the time-dependence of the $\rho_n$. Again we reduce the triple sum to keep the calculations feasible. Note that the terms are oscillating due to the sine. Using approximation (15) we see that for $k = l + m$ the terms proportional to $F$ in the argument of the sine cancel and hence the sine oscillates slowest. Thus we approximate the dynamics of the amplitudes $\rho_n$ by

$$\hbar \dot{\rho}_n \approx 2g\rho_n^2 \sum_{l,m} \chi_{l+m,l,m} \left( \frac{\rho_{l+m+n}\rho_{l+n}\rho_{m+n}}{\rho_n^3} \right)^{1/2} \times \sin(\varphi_{l+n} + \varphi_{m+n} - \varphi_{l+m+n} - \varphi_n), \quad (18)$$

where the sum can be truncated at $|l|, |m| = 30$. Equation (18) for $\rho$ and equation (13) for $\varphi$ are now solved numerically with $\gamma = 0.15$ and the initial condition (17). The results displayed in figure 10 show that this model captures the growth of the width of the wave packet. We will, however, not go into details here and return to the approximation (15) to discuss the dynamics of Bloch oscillations.

IV. ANALYSIS OF THE DYNAMICAL BEHAVIOR

Further insight can be provided by a closer look at the dynamics of the wave function in momentum space. This can be achieved with the approximate time evolution of the expansion coefficients $c_n$ derived in the previous section.

First we briefly reconsider the linear case. For $g = 0$ the equations (15) reduce to:

$$\rho_n(t) = \rho_n(0) \quad \text{and} \quad \varphi_n(t) = -2\pi Fnt/\hbar. \quad (19)$$

The Wannier–Stark functions $\Psi_n$ are related by a spatial translation $\Psi_n(x) = \Psi_0(x - 2\pi n)$. In momentum space this reads

$$\Psi_n(k) = e^{-i2\pi nk}\Psi_0(k) \quad (20)$$
and the time evolution of the wave function in momentum space is
\[
\psi(k, t) = \Psi_0(k) \sum_n \sqrt{\rho_n} \exp \left( -i 2\pi n \left( k + F t / h \right) \right)
\]
\[
\sim \Psi_0(k) \tilde{C}(k + F t / h),
\]
(21)
neglecting a global phase. Thus the wave function is the product of a time-independent function \(\Psi_0(k)\) and the discrete Fourier transformation \(\tilde{C}(k)\) of the amplitudes \(\sqrt{\rho_n}\), evaluated at the point \(k + F t / h\).

The function \(\tilde{C}(k)\) is periodic in momentum space: \(\tilde{C}(k + n) = \tilde{C}(k)\) for \(n \in \mathbb{Z}\). Thus the function \(\tilde{C}(k + F t / h)\) is periodic in time with the Bloch period \(T_B = h / F\). For a broad Gaussian distribution of the amplitudes \(\rho_n\), the discrete Fourier transform \(\tilde{C}(k)\) is a comb function with narrow peaks at \(k = n\).

So one arrives at a simple view of the dynamics in momentum space: The comb function \(\tilde{C}(k)\) moves uniformly under the envelope \(\Psi_0(k)\), as illustrated in figure 11. In coordinate space this periodic motion appears as a Bloch oscillation [1].

![Bloch oscillations in momentum space. The wave function \(\psi(k, t)\) shown for \(t = 0\) (solid line) moves uniformly under the envelope of the Wannier–Stark function \(|\Psi_0(k)|\) (dashed line).](image)

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![Time evolution of the function \(\tilde{C}(k, t)\) in equation (22) for \(g = 10\). The function \(\tilde{C}\) is scaled as \(\tilde{C}(0, 0) = 1\).](image)

The oscillations of the width \(\Delta x_t\), the breathing, can also be understood with this approach. In the linear case such breathing occurs for wave functions that are initially strongly localized in coordinate space and thus have a broad momentum distribution. As explained, the nonlinear term leads to a broadening of the function \(\tilde{C}(k, t)\) and hence to a broadening of the wave function in momentum space and breathing in coordinate space. For even longer times the coefficients dephase completely and the oscillations in both the position and the width are damped.

In order to understand the revivals and the phase jumps in the oscillations of \(\langle x \rangle_t\), we need to look at the time evolution of the function \(\tilde{C}(k, t)\). In figure 12 the function \(|\tilde{C}(k, t)|\) for \(g = 10\) is plotted at times \(t = 0\), \(t = 6 T_B\), \(t = 12 T_B\) and \(t = 18 T_B\). The dynamics of the expansion coefficients was calculated with equation (13) resp. (14). One observes that the initially narrow peaks are broadened and an oscillatory structure develops with two maxima at the edges of the band populated by \(|\tilde{C}(k, t)|\). These maxima eventually merge, leading to a revival of the Bloch oscillations. The new maximum after the merger is displaced by \(\Delta k = 0.5\) in comparison to the linear case and hence the phase of the Bloch oscillations is reversed. This maximum broadens again, leading to a periodic breakdown and revival. The phase of the Bloch oscillations is reversed after each breakdown and the amplitudes of the revivals decrease. However, these further revivals are observed only within the approximation (14) and not in a wave packet propagation.

Now we consider the time dependence of the expectation values of position and width. These quantities can be evaluated analytically in the linear case \(g = 0\) using

\[
\varphi_n(t) = \omega_n t \quad \text{with} \quad \hbar \omega_n = -2\pi F n - g\gamma n,
\]
(23)
and the amplitudes are assumed to be Gaussian
\[
\rho_n(t) = \rho_n \sim e^{-n^2 / 2\sigma^2}.
\]
(24)
As in the linear case, the static field term \(-2\pi F n\) in equation (24) for the frequency leads to a uniform motion of the function \(\tilde{C}(k, t)\) in momentum space. The nonlinear term \(-g\gamma \rho_n\) leads to a dephasing of the coefficients \(c_n\) and broadens the Fourier transform \(\tilde{C}(k, t)\). This dephasing causes a damping of the Bloch oscillations in coordinate space.
a tight–binding approximation [1, 22]. In this approximation the expectation value of the position oscillates harmonically with the Bloch frequency $\omega_B$,

$$\langle x \rangle_t = \bar{x} + A \cos(\omega_B t)$$

(25)

and amplitude

$$A = \frac{\Delta}{2F} e^{-2\pi^2 \Delta p^2 / \hbar^2},$$

(26)

where $\Delta p$ is the width in momentum space and $\Delta$ is the bandwidth of the dispersion relation $E(\kappa)$ in the field–free case. For a small nonlinearity the broadening of the wave function in momentum space due to the nonlinearity happens slowly compared to the Bloch oscillations. Thus we can assume that $\langle x \rangle_t$ still executes damped harmonic oscillations with the amplitude (26), where the damping is determined by the slowly increasing momentum width $\Delta p_t$.

According to equation (22) we can estimate the momentum width $\Delta p_t$ by the width of the peaks of the function $\tilde{C}(k,t)$. For a broad distribution of the coefficients $c_n$, as assumed throughout this paper, the sum in equation (22) can be replaced by an integral:

$$\tilde{C}(k,t) = \int_{-\infty}^{+\infty} \sqrt{\rho_n} \exp(-i(2\pi nk - \varphi_n(t))) \, dn. \quad (27)$$

This expression is valid for $|k| < 0.5$, otherwise $\tilde{C}(k,t)$ is determined by its periodicity. The amplitudes $\rho_n$ and phases $\varphi_n(t)$ are approximated by equation (15), where the amplitudes $\rho_n$ are normalized as

$$\rho_n = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-n^2/2\sigma^2}. \quad (28)$$

We note that $\tilde{C}(k,t)$ depends on the momentum $k$ only through the expression $k = k + Ft/\hbar$, reflecting the uniform motion in momentum space due to the static field:

$$\tilde{C}(\tilde{k}) = \frac{1}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{+\infty} \exp\left(-\frac{n^2}{4\sigma^2} - i(2\pi nk + \beta e^{-n^2/2\sigma^2})\right) \, dn$$

(29)

with $\beta = \gamma g t / (\sqrt{2\pi} \sigma \hbar)$.

The integral (29) can be evaluated using the stationary phase approximation. However, there exists only a finite $k$–interval for which stationary points exist. For $|k| > |k_c|$ with

$$k_c = \frac{\beta}{2\pi \sigma e^{1/2}} \quad (30)$$

the integral vanishes in the simple stationary phase approximation [27]. For $|k| < |k_c|$ one obtains

$$\tilde{C}(k,t) \approx \left(\frac{2\pi \sigma^2}{\beta^2}\right)^{1/4} \left(\frac{e^{i\tilde{k}+} - e^{i\tilde{k}-}}{\sqrt{1 - z_+}} + \frac{e^{i\tilde{k}+} - e^{i\tilde{k}-}}{\sqrt{1 - z_-}}\right), \quad (31)$$

where $z_\pm$ are the two solutions of the equation

$$ze^{-z} = \left(\frac{2\pi \sigma \tilde{k}}{\beta}\right)^2 \quad (32)$$

and the abbreviations $\zeta_\pm = -2\pi \sigma \tilde{k} (z_\pm^{1/2} + z_\pm^{-1/2})$ were used. As an example the function $|\tilde{C}(k,t)|$ is plotted in figure 13 for $gt = 90 T_B$ and $\gamma = 0.15$.

![figure 13](image)

**FIG. 13**: Function $|\tilde{C}(k,t)|$ for $gt = 90 T_B$ and $\gamma = 0.15$. The integral was evaluated numerically (solid line) resp. using the stationary phase method (dashed line).

Estimating the momentum width as $\Delta p \approx \hbar |k_c|$ one arrives at

$$\Delta p_t \approx \frac{|\gamma g|}{(2\pi)^{3/2}\pi \sigma^2 e^{1/2}} t. \quad (33)$$

Thus the damped Bloch oscillations in coordinate space are described by

$$\langle x \rangle_t \approx \bar{x} + \frac{\Delta}{2F} \exp\left(-\frac{\gamma^2 g^2 t^2}{4\pi \hbar^2 \sigma^2}\right) \cos(\omega_B t) \quad (34)$$

according to (26). The amplitude decreases exponentially with $-\gamma^2 t^2$ in agreement with the estimate given in [12]. Furthermore we can calculate approximately the time up to the first rephasing of the coefficients and thus the first revival of the Bloch oscillations. This revival occurs if the outer peaks of $\tilde{C}(k,t)$ meet at $k = n + 1/2$, $n \in \mathbb{Z}$, as illustrated in figure 12. Therefore the first rephasing and revival occurs for $k_c = 0.5$ which yields a revival time

$$t_{rev} \approx \frac{(2\pi)^{3/2} e^{1/2} \hbar \sigma^2}{2 |\gamma g|}. \quad (35)$$

For $g = 10$ and $\gamma = 0.15$ one obtains

$$t_{rev} \approx 17 T_B \quad (36)$$

in reasonable agreement with the revival of Bloch oscillations observed numerically for the wave packet propagation shown in figure 3.
incoherent sum of the basis states. Assuming that the amplitudes $\rho_n$ are constant in time according to equation (13) one has

$$\langle x \rangle = \sum_n \rho_n \langle \Psi_n | x | \Psi_n \rangle. \quad (37)$$

Using the translational properties of the Wannier–Stark states (cf. 4) one arrives at

$$\langle x \rangle_{\infty} \approx \langle \Psi_0 | x | \Psi_0 \rangle + 2\pi \sum_n n \rho_n. \quad (38)$$

The amplitudes of the initial state are symmetric around $n = 0$ and hence this approximation yields $\langle x \rangle_{\infty} \approx \langle \Psi_0 | x | \Psi_0 \rangle = -10.5 \cdot 2\pi$. This estimation fairly agrees with the numerical results displayed in figure 6. As argued above, the systematic growth of the width of the wave packet is mainly due to a broadening of the amplitude distribution $\rho_n$ and hence cannot be explained using the simple model discussed here.

V. STRONG STATIC FIELD AND DECAY

An expansion into Wannier–Stark resonances is also very helpful in order to understand the dynamics and decay in strong static fields. In the following we will discuss the dynamics for the parameters $\hbar = 3.3806$ and $F = 0.0661$, corresponding to the experiment of the Kasevich group 6. A detailed discussion of this experiment in terms of Wannier–Stark resonances, however neglecting the nonlinearity, can be found in 18. Thus we will only briefly discuss the influence of the nonlinearity on the pulse shape.

For a field strength of $F = 0.0661$ decay cannot be neglected any longer. One has to take into account that the resonance states eventually diverge exponentially for $x$ or $k \to -\infty$. Hence, a wave function of the form is not normalizable. Nevertheless, the restriction to the ground Wannier–Stark ladder is still sufficient.

As described in 18 (see also remark 24) one can solve the problem of normalization by introducing truncated resonance states defined by

$$\Psi^K_n(k) = \Theta(k + K) \Psi_n(k). \quad (39)$$

The Heaviside–function $\Theta(k + K)$ truncates the resonances at $-K$. Provided that $|K|$ is large enough, the time evolution of these states is given by

$$\Psi^K_n(k, t) = \Theta(k + K + Ft/\hbar) \Psi_n(k, t). \quad (40)$$

If the support of the initial wave function is bounded in momentum space by $|k| < |K|$, we can expand it into a basis of truncated resonances. The dynamics of this state is then given by

$$\psi(k, t) = \Theta(k + K + Ft/\hbar) \Psi_0(k, t) \tilde{C}(k, t), \quad (41)$$

with $\Psi_0(k, t) = \exp(-iE_0 t/\hbar) \Psi_0(k)$ instead of equation 22.

For a coherent initial distribution of a sufficient width $c_n \sim \exp(-n^2/(2\sigma^2))$ with $\sigma \gg 1$, the function $\tilde{C}(k, t)$ is a comb function in the linear case, leading to a pulsed output. The pulse shape given by the function $\tilde{C}$ broadens and deforms under the influence of the nonlinearity as described in the previous section (cf. figure 12). This deformation is directly observable in the pulsed output.

![FIG. 14: Pulsed output for different nonlinearities $g = 0, -5, 5$ and 10 (from top to bottom). The wave function $|\psi(k, t)|^2$ is displayed for $t = 9.1T_B$.](image)

This is demonstrated in figure 14 for a coherent initial distribution $c_n \sim \exp(-n^2/4\sigma^2)$ with a width of $\sigma = 15/2$. The time evolution was again calculated using a split–operator method. The wave function $|\psi(k, t)|^2$ is plotted at a time $t = 9.1T_B$ for four different values of the nonlinearity $g = 0, -5, 5$ and 10. The broadening of the peaks with increasing $|g|$ and the characteristic double–peak structure can clearly be seen.

The resulting wave function in coordinate space is a sequence of pulses at the points

$$x = x_0 - \frac{F}{2}(t + jT_B)^2, \quad (42)$$

with $x_0 = E_0/F$ and $j \in \mathbb{Z}$. These pulses are accelerated just like classical particles in a static field, as observed in the experiment 6. The pulse shape is approximately described by the discrete Fourier transform of $\tilde{C}$. Thus one also finds a characteristic deformation of the pulses in coordinate space.

VI. CONCLUSIONS

In this article we first investigated Bloch oscillations of BECs by numerical solutions of the Gross–Pitaevskii equation (GPE) and demonstrated a revival of Bloch oscillations after an initial breakdown. These findings
have been further analyzed via discretising the GPE in a Wannier–Stark basis set expansion. Using these resonance states one can easily compare the linear and nonlinear case. This comparison leads to a better understanding of the nonlinear features of BECs in optical lattices. It allows us to derive a simple integrable model which can explain the nonlinear phenomena of breakdown and revival of the Bloch oscillations. This approach, unlike the tight–binding approximation, works as well for strong Stark fields.

Many interesting questions are left open and deserve future studies, as for example the following: (a) The effects induced by the nonlinearity for Bloch oscillations in two–dimensional lattices, where recently novel effects concerning the extreme sensitivity on the field direction with respect to the lattice have been found. (b) For BECs in tilted optical lattices a classical chaotic behavior has been reported. These interesting findings deserve further studies, for example the correspondence of the emergence of chaos with the loss of stability of the GPE solutions. (c) It is an entirely open question how the nonlinearity influences the behavior of a driven Wannier–Stark system, e.g. the stabilizing phenomena found for an additional harmonic driving.

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