Semiclassical Approach to Competing Orders in Two-leg Spin Ladder with Ring-Exchange

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We investigate the competition between different orders in the two-leg spin ladder with a ring-exchange interaction by means of a bosonic approach. The latter is defined in terms of spin-1 hardcore bosons which treat the Néel and vector chirality order parameters on an equal footing. A semiclassical approach of the resulting model describes the phases of the two-leg spin ladder with a ring-exchange. In particular, we derive the low-energy effective actions which govern the physical properties of the rung-singlet and dominant vector chirality phases. As a by-product of our approach, we reveal the mutual induction phenomenon between spin and chirality with, for instance, the emergence of a vector-chirality phase from the application of a magnetic field in bilayer systems coupled by four-spin exchange interactions.

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I. INTRODUCTION

Multiple-spin exchange interactions have attracted much interest for a long time. These interactions appear either as many-body direct exchange processes or as higher-order corrections in the strong-coupling expansion of the half-filled Hubbard model. In most cases (especially in Mott insulators), the four-spin ring (or cyclic) exchange is rather small compared with the usual Heisenberg term. However, it may play a significant role in $^3$He, where hardcore repulsion makes many-body exchanges more likely than the standard two-body one, and Mott insulators close to the metal-insulator transitions. In fact, such interactions are expected to be crucial for explaining unusual magnetic behavior in $^3$He absorbed on graphite and Wigner crystals and some Mott insulators on triangular lattices. The relevance of the four-spin cyclic exchange has also been reported in fully describing the inelastic neutron-scattering experiments for such cuprates as La$_2$CuO$_4$, La$_2$Ca$_8$Cu$_{24}$O$_{41}$, La$_4$Ca$_{10}$Cu$_{24}$O$_{41}$, CaCu$_2$O$_2$Y, and SrCu$_2$O$_3$.

A second motivation to investigate multiple-spin exchange interactions stems from the exotic physics that emerges from their competition with the Heisenberg spin exchange. For instance, certain sorts of spin nematic phases are known to be stabilized in some two-dimensional Heisenberg magnets with ring-exchange interaction. Aromatic multiple-exchanges can lead to new emerging quantum critical behaviors like the deconfined quantum criticality or the spin Bose-metal. On top of these novel phases, even more exotic topological phases have been predicted to be realized with such interactions.

A paradigmatic and minimal model that realizes these two aspects of the multiple-spin exchange interactions would be the two-leg spin ladder with a ring-exchange:

$$\mathcal{H} = J \sum_{r=1}^{r=r_{\text{rung}}} (S_{1,r} S_{1,r+1} + S_{2,r} S_{2,r+1}) + J_\perp \sum_{r} (S_{1,r} \cdot S_{2,r} + K_4 \sum_{\text{plaquettes}} (P_3 + P_4^{-1}) ,$$

where $S_{a,r}$ denotes the spin-$1/2$ operator on the chain-$a$ and the rung $r$ of the spin ladder. The parameters $J$ and $J_\perp$ respectively are the intrachain- and the interchain exchange coupling (Fig. 1(a)). The ring exchange $P_4$ is defined on each plaquette (two-rung cluster: Fig. 1(b)) and cyclically permutes the states of the four spins on the plaquette. This model is considered as relevant to describe the physical properties of ladder compounds e.g. La$_4$Ca$_{10}$Cu$_{24}$O$_{41}$ (Ref. 9) and CaCu$_2$O$_3$ (Ref. 10) and CaCu$_2$O$_3$ (Ref. 16). The model is interesting in its own right and has been studied extensively over the years and its phase diagram has been explored by both analytical and numerical approaches.

![FIG. 1. Two-leg ladder (a) and four-spin plaquette (b) on which ring-exchange $P_4$ is defined.](image-url)
The zero-temperature phase diagram is rich and six different phases have been identified: the ferromagnetic (NAF) and the VC order parameters by means of a semiclassical approach which treats VC orders. The next step of the approach is to perform a simple mean-field approximation (2) when the model (1) is invariant, i.e., self-dual, under the duality transformation (2) when \( K_4 = J/2 \). Physics in the vicinity of this point is dictated by the competition between the RS and VC phases as well.

In this section, we introduce three hardcore bosons to map the model (1) onto an effective spin-1 boson Hamiltonian. This approach will enable us to illustrate how competition among different orders (specifically, antiferromagnetic (AF) order and chiral one) is described in our framework.

### A. Dimer basis and spin-chirality rotation

First we begin with analyzing the Hilbert space of each rung which is an obvious unit of construction. In describing the states on each rung, it is convenient to move from the standard spin-1/2 \((\uparrow/\downarrow)\) basis to the singlet-triplet basis:

\[
|s\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \quad |t_x\rangle = -\frac{i}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle), \quad |t_y\rangle = \frac{i}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)
\]

These four states may be thought of as created by the following four (constrained) bosons obeying the standard bosonic commutation relations:

\[
|s\rangle = s^\dagger |0\rangle, \quad |t_\alpha\rangle = t^\dagger_\alpha |0\rangle \quad (\alpha = x, y, z)
\]

\[
s^\dagger s + \sum_{\alpha=x,y,z} t^\dagger_\alpha t_\alpha = 1 .
\]

For our purpose, however, it is more convenient to identify \(|s\rangle\) with the empty state (boson vacuum) and represent the remaining triplet states by using three hardcore bosons \(b_\alpha\)

\[
|0\rangle_{hcb} \equiv |s\rangle, \quad b^\dagger_\alpha |0\rangle_{hcb} \equiv |t_\alpha\rangle
\]

obeying the following nonstandard commutation relations:

\[
[b_{r,\alpha}, b^\dagger_{r',\beta}] = \left\{ \delta_{\alpha\beta}(1 - n^B_r) - b^\dagger_{r,\beta} b_{r',\alpha} \right\} \delta_{r,r'} ,
\]

with \(n^B_r\) being the number operator of the hardcore boson:

\[
n^B_r \equiv \sum_{\alpha=x,y,z} b^\dagger_{r,\alpha} b_{r,\alpha} = S_{1,r} \cdot S_{2,r} + 3/4 .
\]

### II. EFFECTIVE SPIN-1 HARDCORE BOSONS APPROACH

In this section, we introduce three hardcore bosons to map the model (1) onto an effective spin-1 boson Hamiltonian. This approach will enable us to illustrate how competition among different orders (specifically, antiferromagnetic (AF) order and chiral one) is described in our framework.

The relation between the RS and VC phases can be simply understood by means of a spin-chirality duality transformation. In describing the phases of the two-leg spin ladder, the lowest triplet excitation in the vector-chirality channel but rather in the vector-chirality channel.

The relation between the RS and VC phases is not the standard trimon created by the spin operators \(S_{1,r} = S_{2,r}\) but is built from the vector chirality \(S_1 \times S_2\), as we will show below. Consequently, the dominant ground state with a finite gap. In sharp contrast to the usual RS phase of the two-leg spin ladder, the lowest triplet excitation of the VC phase is not the standard trimon created by the spin operators \(S_{1,r}\) but is built from the vector chirality \(S_1 \times S_2\), as we will show below.

The next step of the approach is to perform a simple mean-field approximation (2) when the model (1) is invariant, i.e., self-dual, under the duality transformation (2) when \( K_4 = J/2 \). Physics in the vicinity of this point is dictated by the competition between the RS and the VC phases.

The sequential steps of our method, we investigate the mutual induction phenomenon between spin and chirality in Sec. II. Finally, our concluding remarks are presented in Sec. III. The paper is supplied with one appendix which provides some technical information on the semiclassical approach.
The crucial step is to realize that under the hardcore constraint
\[ n_r^B = 0,1 \]  
the boson creation operator can be represented by a complex combination of two different order parameters\[^{[25]}\]
\[ b_{r,\alpha}^\dagger = \frac{1}{2} (S_{1,r} - S_{2,r})^\alpha + i (S_{1,r} \times S_{2,r})^\alpha \quad (\alpha = x, y, z) \]  
(9)

From this equation, one sees that the competition between the AF fluctuations (carried by \((S_{1,r} - S_{2,r})\)) and the (vector) chiral ones \((S_{1,r} \times S_{2,r})\) is implemented in a bosonic object \(b\) in a unifying way. Also, it is easy to see that (spin-independent) gauge transformation \(b_{r,\alpha}^\dagger \rightarrow e^{i\phi} b_{r,\alpha}^\dagger\) translates into an SO(2) rotation (spin-chirality rotation\[^{[29]}\]) for the order-parameter doublet \(\langle S_{1,r} - S_{2,r}, S_{1,r} \times S_{2,r}\rangle\):
\[ \begin{pmatrix} S_{1} - S_{2} \\ 2S_{1} \times S_{2} \end{pmatrix} \mapsto \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} S_{1} - S_{2} \\ 2S_{1} \times S_{2} \end{pmatrix} . \]  
(10)

In particular, the spin-chirality duality transformation\[^{[2]}\] can be obtained from Eq. (10) with \(\varphi = \pi/2\).

**B. Generalized boson Hubbard model**

It is convenient to rewrite the model Hamiltonian\(^{[1]}\) in terms of the hardcore boson introduced above. In terms of the bosonic operators \(b_{r,\alpha}\), the (spin) Hamiltonian\(^{[1]}\) can be mapped onto the following spin-1 Bose-Hubbard model\[^{[29]}\]
\[ \mathcal{H} = t \sum_{r,\alpha} \left( b_{r,\alpha}^\dagger b_{r+1,\alpha} + b_{r+1,\alpha}^\dagger b_{r,\alpha} \right) + u \sum_{r,\alpha} \left( b_{r,\alpha}^\dagger b_{r+1,\alpha}^\dagger b_{r+1,\alpha} b_{r,\alpha} \right) + \sum_{r} \left[ J_{bl} T_r T_{r+1} + J_{bq} \left( T_r T_{r+1}\right)^2 \right] + V_c \sum_{r} n_{r}^B n_{r+1}^B - \mu \sum_{r} n_r^B \]  
(11)

The spin \(T\) of the hardcore boson \(b\) is given by
\[ T_{r,\alpha} \equiv -i \epsilon_{\alpha\beta\gamma} b_{r,\beta}^\dagger b_{r,\gamma} \]  
(12)
where the projection onto the occupied (i.e. \(n_r^B \neq 0\)) states is implied on both sides. The ring-exchange model\(^{[1]}\) is reproduced if we choose:
\[ t = \frac{1}{2} J + K_4, \quad u = \frac{1}{2} J - K_4, \quad J_{bl} = \frac{1}{2} J + K_4, \quad J_{bq} = 0, \quad V_c = 4K_4, \quad \mu = 4K_4 - J_\perp \]  
(13)

Note that the Hamiltonian\(^{[1]}\) is the most general one allowed by the requirements of (i) SU(2) symmetry, (ii) time-reversal invariance, (iii) exchange of the two chains \((1 \leftrightarrow 2)\) and (iv) short-range interactions (i.e. interactions only involve two adjacent rungs). In this respect, model\(^{[1]}\) describes two-leg spin ladder with general four-spin exchange interactions\[^{[36]}\].

The ring-exchange is special in the sense that there is no bi-quadratic exchange interaction: \(J_{bq} = 0\). It is also important to note that the U(1) gauge-symmetry for the bosons, i.e. the spin-chirality rotation\[^{[19]}\], is explicitly broken unless \(u = 0\). The spin-chirality duality symmetry\[^{[2]}\] transforms the pairing term as: \(u \rightarrow -u\). In the self-dual case, i.e. \(u = 0\), model\(^{[1]}\) is directly relevant to spinor Bose quantum gases with hyperfine spin \(F = 1\) loaded into an optical lattice\[^{[23]}\].

In the following sections, we develop a semiclassical approach to discuss the physical properties of various phases of the model\(^{[1]}\) (or the equivalent model\[^{[1]}\]). In particular, we are interested in the two non-magnetic phases\[^{[16]}\] (dubbed ‘rung singlet’ and ‘dominant vector chirality’ in Ref.\(^{[27]}\)). The effective models derived in Sec.\(^{[1]}\) provide us with a simple and natural framework of describing the competition of spin and chirality degrees of freedom.

**C. Coherent-state-construction of path-integral**

As usual, the starting point of the path-integral approach is the construction of the many-body coherent state basis for the two-leg ladder model with a ring-exchange\(^{[1]}\) and the spin-1 boson Hubbard model\(^{[1]}\). In this respect, let us consider spin-1 hardcore bosons \(b_{r,\alpha}^\dagger\) \((\alpha = x, y, z)\) on a unit of construction (e.g. a rung in the case of the two-leg ladder). As has been mentioned in Sec. II A, these boson operators are parametrized in terms of the two competing (real) order parameters \(\mathbf{q}\) and \(\mathbf{p}\):
\[ b_{r,\alpha}^\dagger = (\mathbf{q}_r)_\alpha + i (\mathbf{p}_r)_\alpha \quad (\alpha = x, y, z) , \]  
(14)
where in our ladder problem, \((\mathbf{q}_r)_\alpha\) and \((\mathbf{p}_r)_\alpha\) are respectively the staggered magnetization and the vector-chiral order parameter:
\[ \mathbf{q}_r = \frac{1}{2} (S_{1,r} - S_{2,r}) \quad \mathbf{p}_r = S_{1,r} \times S_{2,r} \]  
(15)
(see Eq.\(^{[9]}\)).

Clearly, arbitrary states on each unit can be represented as\[^{[22]}\]
\[ |\psi\rangle = s |s\rangle + \sum_{\alpha=x,y,z} t_\alpha |t_\alpha\rangle \equiv |s, t\rangle , \]  
(16)
where
\[ s^* s + t^* t = 1 \]  
(17)
is required by the normalization condition. Since the overall phase is irrelevant, we may fix the gauge in such a way that the singlet amplitude \(s\) is real and positive:
\[ s = \sqrt{1 - t^* t} \quad (t^* t \leq 1) \]  
(18)
and parametrize the complex vector \(t\) in terms of two real vectors \(\mathbf{A}\) and \(\mathbf{B}\) as:
\[ t \equiv \mathbf{A} - i \mathbf{B} \quad s = \sqrt{1 - (\mathbf{A} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B})} \]  
\[ (\mathbf{A}, \mathbf{B} \in \mathbb{R}^3 \quad \mathbf{A} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{B} \leq 1) . \]  
(19)
One of the greatest merits of using this representation is that the two order parameters $q$ and $p$, which are related to the boson by Eq. (14), are expressed simply by $A$ and $B$: \begin{align}
\langle s, t | q | s, t \rangle &= (s^* t + t^* s)/2 = \sqrt{1 - (A^2 + B^2)} A \quad \text{(20a)} \\
\langle s, t | p | s, t \rangle &= i(s^* t - t^* s)/2 = \sqrt{1 - (A^2 + B^2)} B . \quad \text{(20b)}
\end{align}

From these equations, it is obvious that the spin-chirality rotation $\hat{T}$ is equivalent to the gauge transformation of the triplet operators $t$

\[ t^* \mapsto e^{i\varphi} t^* \quad \text{(21)} \]

or the following $O(2)$ transformation for the pair $(A, B)$:

\[ \begin{pmatrix} A \\ B \end{pmatrix} \mapsto \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} . \quad \text{(22)} \]

In the case of the two-leg ladder where the triplet boson is defined on the rung, the pair of order parameters is given by Eq. (9) and the above is nothing but the spin-chirality transformation (10).\[ \text{(28c)} \]

The magnetic moment $T$, that generates the O(3) rotation of the boson triplet, is expressed as (see Eq. (12)):

\[ \langle s, t | T | s, t \rangle = -i(t^* \times t) = -2(A \times B) . \quad \text{(23)} \]

If we take a single rung as the unit in the case of the two-leg ladder, $T = S_1 + S_2$. The boson density is another important quantity and takes the following expression:

\[ \langle s, t | n^B | s, t \rangle = t^* \cdot t = (A^2 + B^2) . \quad \text{(24)} \]

The many-body coherent state basis are constructed as the tensor-product of the local coherent states:

\[ \bigotimes_r | s_r, t_r \rangle = \bigotimes_r | A_r, B_r \rangle . \quad \text{(25)} \]

With these expressions, we can write down the desired path-integral formula for many-spin systems by following the standard steps:\[ \text{(28a)} \]

\[ S = \int dt \sum_r \langle A_r \cdot \partial_t | B_r - B_r \cdot \partial_t A_r \rangle - \int dt \mathcal{H}_{cl}(\{A\}, \{B\}) , \quad \text{(26)} \]

where $\mathcal{H}_{cl}$ is given by the expectation value of the Hamiltonian with respect to the many-body coherent state $\bigotimes_r | s_r, t_r \rangle$. In obtaining $\mathcal{H}_{cl}$, the easiest way is to use the spin-1 Bose-Hubbard model (11) with the coupling constants (13) and then plug the following expressions:

\[ \begin{align}
&b^\dagger_{r, \alpha} \mapsto \sqrt{1 - (A_r^2 + B_r^2)} (A_r + i B_r) , \\
&b_{r, \alpha} \mapsto \sqrt{1 - (A_r^2 + B_r^2)} (A_r - i B_r) . \quad \text{(27)}
\end{align} \]

The final result reads as follows:

\[ \mathcal{H}_{cl} = \mathcal{H}_{\text{magnetic}} + \mathcal{H}_{\text{hopping}} + \mathcal{H}_{\text{charge}} \quad \text{(28a)} \]

\[ \mathcal{H}_{\text{magnetic}} \equiv (2J + 4K_4) \sum_r \left\{ (A_r \cdot A_{r+1} + B_r \cdot B_{r+1}) - (A_r \cdot B_{r+1} + B_r \cdot A_{r+1}) \right\} , \quad \text{(28b)} \]

\[ \mathcal{H}_{\text{hopping}} \equiv \sum_r 4 \sqrt{1 - (A_r^2 + B_r^2)} \sqrt{1 - (A_{r+1}^2 + B_{r+1}^2)} \times \left\{ \frac{J}{2} A_r \cdot A_{r+1} + K_4 B_r \cdot B_{r+1} \right\} , \quad \text{(28c)} \]

\[ \mathcal{H}_{\text{charge}} \equiv (J_+ + 2K_4) \sum_r \left\{ (A_r^2 + B_r^2) - \frac{3}{4} \right\} + 4K_4 \sum_r \left\{ (A_{r+1}^2 + B_{r+1}^2) - \frac{3}{4} \right\} . \quad \text{(28d)} \]

The magnetic- and the charge part are invariant under the spin-chirality U(1) transformation (10) and the competition between the $(\pi, \pi)$ antiferromagnetic correlation (A) and the chirality correlation (B) is controlled solely by the two coupling constants in $\mathcal{H}_{\text{hopping}}$:

\[ J_A \equiv \frac{t + u}{2} = \frac{J}{2} , \quad J_B \equiv \frac{t - u}{2} = K_4 . \quad \text{(29)} \]

### III. MEAN-FIELD PHASE DIAGRAM

In this section, we determine the classical ground state of the model (10) or its bosonic equivalent (11), on the basis of which we develop the effective field theories in Sec. (14).

#### A. General properties

Before presenting the results, we describe the relationship between the obtained $\{A, B\}$-configurations and the physical phases. By construction, it is obvious that our calculation is nothing but the mean-field approximation for the spin-1 boson using the following product state:

\[ | \Psi \rangle_{\{\eta_r, a_r, b_r\} \rangle} = \bigotimes_r \left\{ \cos \frac{\eta_r}{2} | \rangle + \sin \frac{\eta_r}{2} \sum_{\alpha=x,y,z} (a - i b)_{r, \alpha} | 1 \rangle \right\} , \quad \text{(30)} \]

where we have introduced

\[ s_r = \cos \frac{\eta_r}{2} , \quad t_r = A_r + i B_r = \sin \frac{\eta_r}{2} (a_r + i b_r) . \quad \text{(31)} \]
with $a^2 + b^2 = 1$. The parameter $\eta_r (0 \leq \eta_r \leq \pi)$ controls the local boson density $n_B$ through the relation

$$\langle \Psi | r_r^B | \Psi \rangle_{(\eta_r, a_r, b_r)} = \sin^2 \frac{\eta_r}{2}.$$  \hspace{1cm} (32)

As has been mentioned in the previous section, the spin-1 (hardcore) boson operators are parametrized in terms of the two competing (real) order parameters $q$ and $\tilde{p}$ (see Eq. \[14\]). One can then compute the expectation values of these operators in the coherent state \[30\]:

$$\langle \Psi | q_r | \Psi \rangle_{(\eta_r, a_r, b_r)} = \left\langle \frac{1}{2} (S_1 - S_2)^\alpha \right\rangle = \frac{1}{2} \sin \eta_r (a_r) \alpha ,$$

$$\langle \Psi | \tilde{p}_r | \Psi \rangle_{(\eta_r, a_r, b_r)} = \left\langle (S_1 \times S_2)^\alpha \right\rangle = \frac{1}{2} \sin \eta_r (b_r) \alpha .$$  \hspace{1cm} (33)

Therefore, the phases with $\eta_r \neq 0, \pi$ in general correspond to superfluids in that $\langle b_r \rangle \neq 0$.

Information on magnetism may be obtained by the spin-1 magnetic moment \[12\] on each rung:

$$\langle \Psi | T_r | \Psi \rangle_{(\eta_r, a_r, b_r)} = \langle S_1 + S_2 \rangle = -2 \sin^2 \frac{\eta_r}{2} a_r \times b_r .$$  \hspace{1cm} (34)

From Eqs. \[33\] and \[34\], it is obvious that if, for some reasons, the system chooses the state with $a = 0, b \neq 0$ ($\eta \neq 0, \pi$), a moment free (i.e. $\langle S_1 + S_2 \rangle = \langle S_1 - S_2 \rangle = 0$) chiral phase $\langle S_1 \times S_2 \rangle \neq 0$ ($p$-type spin-nematic) is realized. Similarly, $a \neq 0, b = 0$ implies another type of phases with collinear spin order (either $(\pi, \pi)$ or $(0, \pi)$).

Since we are dealing with spin-1 bosons, we may expect (typically for large enough biquadratic interaction $J_{bb}$) a phase characterized by the following rank-2 tensor (spin-nematic phase) to occur:

$$Q_r^{\alpha \beta} = \frac{1}{2} (T_r^{\alpha} T_r^{\beta} + T_r^{\beta} T_r^{\alpha}) - \frac{1}{3} \delta^{\alpha \beta} T_r.$$  \hspace{1cm} (35)

In the mean-field state \[30\], the above tensor order parameter takes the value:

$$\langle \Psi | Q_r^{\alpha \beta} | \Psi \rangle_{(\eta_r, a_r, b_r)}$$

$$= - \sin^2 \frac{\eta_r}{2} \left\{ (a_r)_{\alpha} (a_r)_{\beta} + (b_r)_{\alpha} (b_r)_{\beta} - \frac{1}{3} \delta_{\alpha \beta} \right\} .$$  \hspace{1cm} (36)

When $a$ and $b$ are parallel to each other, the spin sector is in general spin-nematic with vanishing magnetic moment $\langle T \rangle = 0$. Note that the two $\langle T \rangle = 0$ phases described above have finite $\langle Q^{\alpha \beta} \rangle$ and that, in a sense, they may be thought of as spin-nematic. The only difference between spin-1 nematic and the other two states comes from the filling-dependent overall factors $\sin \eta$ and $\sin^2 (\eta/2)$: the former can exist even in the spin-1 limit $\eta = \pi$ while the latter are not.

The charge part (Eq. \[28d\]) dictates the charge density distribution. For instance, for sufficiently large (positive) $\mu$, the density saturates $n_B = 1$ and the system reduces to the (localized) spin-1 chain. For large enough $V_c$, on the other hand, the system may develop inhomogeneity, i.e. form a charge-density wave with alternating $n_B = 0$ and $n_B = 1$.

The general mean-field phase diagram, which results from these equations, will be presented elsewhere. Here, we only stress that the set of coupling constants $(t, u)$ is crucial for the competition between antiferromagnetism and chirality. To see this, we first calculate the kinetic energy by using the mean-field ansatz \[30\]:

$$\langle t \sum_{r, \alpha} \left( b_{r, \alpha}^+ b_{r+1, \alpha} + b_{r+1, \alpha}^+ b_{r, \alpha} \right) \rangle$$

$$+ \langle u \sum_{r, \alpha} \left( b_{r, \alpha}^+ b_{r+1, \alpha} + b_{r+1, \alpha}^+ b_{r, \alpha} \right) \rangle$$

$$= \sum_r \sin \eta_r \sin \eta_{r+1} \left\{ J_A a_r a_{r+1} + J_B b_r b_{r+1} \right\} ,$$  \hspace{1cm} (37)

where the two couplings, that characterize the anisotropy in the spin-chirality space, are given by Eq. \[25\]. For $u \neq 0$ and $\eta \neq 0, \pi$, an anisotropic superfluid forms; $a_r (\propto \text{Re}(t_r^*)$) is dominant when $|J_A| > |J_B|$, while $b_r (\propto \text{Im}(t_r^*))$ is dominant when $|J_A| < |J_B|$.

So far, we have presented the mean-field description of the phases of the spin-1 boson model. However, in one dimension, strong quantum fluctuations may destroy the ordered states predicted by the mean-field theory. In fact, as we will show in the next section by using low-energy effective theories, some of the ordered phases are replaced by gapped short-range phases which do not break rotational symmetry.

### B. Two-leg ladder

The semiclassical ground state of the model \[1\] (or equivalently, \[11\]) is obtained by minimizing the energy functional $\mathcal{H}_0\{\{A_r\}, \{B_r\}\}$ in \[28a\] with respect to the variational parameters $\{A_r\}$ and $\{B_r\}$. As we have already seen, this is nothing but the mean-field treatment using \[30\]. The resulting phase diagram contains six phases: (i) ‘NAF-dominant’, (ii) ‘chirality-dominant’, (iii) ‘partial-AF’, (iv) ‘F-nematic’, (v) ‘ferromagnetic’ and (vi) ‘singlet-product’. It is convenient to parametrize the coupling constants as:

$$J = \cos \theta , \quad K_4 = \sin \theta (\pi < \theta \leq \pi)$$  \hspace{1cm} (38)

and map out the phase diagram as a function of $\theta$ (see Fig. \[2\]).

Let us describe the nature of the six phases.

(i) NAF-dominant: This phase is described by $A_r \neq 0$ and $B_r = 0$. The resulting phase is characterized by vanishing (total) magnetic moment on each rung $\langle S_1 + S_2 \rangle = -2A \times B$ and anti-parallel ordering of $A$ (local chirality $B = 0$). At the mean-field level, the symmetry SU(2)$\times\mathbb{Z}_2$ of the model \[1\] is broken down to SO(2) (rotation around the ordered $A$).

Since $A$ corresponds to $S_1 - S_2$, the staggered order of $A$ implies the standard $(\pi, \pi)$ Néel-ordered phase.

(ii) chirality dominant: The second phase is the dual ($A \leftrightarrow B$) of the former with $A_r = 0$ and $B_r \neq 0$. This phase has the same symmetry as the first phase and is non-magnetic (i.e. $\langle S_1 \rangle = \langle S_2 \rangle = 0$). However, the long-range order occurs in the chirality channel $B \sim S_1 \times S_2$ in a staggered manner. The transition from the first phase occurs at the so-called self-dual
point $\theta = \theta_{ad} \equiv \tan^{-1}(1/2) \approx 0.1476\pi$ where $J_A = J_B$ (see Eq. (29)).

(iii) partial-AF: At $\theta = \pi/2$, the system begins to have a small local magnetization $\langle S_1 + S_2 \rangle$ on each rung and this partial magnetization orders in an AF manner. Note that both A and B take finite values in this phase. Within the semiclassical treatment, the energy of the partial-AF phase is very close to that of the chirality-dominant phase suggesting the instability of the former against quantum fluctuations.

(iv) F-nematic: At $\theta = \cos^{-1}\left(-2/\sqrt{5}\right) \approx 0.8524\pi$ ($t = J_{bl} = J/2 + K_4 = 0$), the system enters a new non-magnetic phase which is similar to the first one (‘NAF-dominant’) except that now the A fields align in a parallel (ferromagnetic) manner. If we regard the triplet state on each rung as an effective spin-1 state, this is nothing but the spin-nematic state (one can easily check $(T) = 0, Q_{\alpha\beta} \neq 0$) where A plays the role of the director. This is why the name ‘F-nematic’ (‘F’ denotes ferro) is used here.

(v) ferromagnetic: For $\theta > \cos^{-1}\left(-\sqrt{32(60+\sqrt{7})}/1289\right) \approx 0.9354\pi$, the system is fully-occupied by the spin-1 states (i.e. $\nu_B = A^2 + B^2 = 1$) and these spin-1s form a polarized ferromagnetic state as a whole.

(vi) singlet-product: At $\theta = -\cos^{-1}(2/\sqrt{13}) \approx -0.3128\pi$ (where $3J + 2K_4 = 0$), the system becomes non-magnetic again through a first-order transition. Contrary to the ferromagnetic phase, all rungs are occupied by the singlet (i.e. $\nu_{bl} = A^2 + B^2 = 0$) in the new phase. In terms of the original spin-1/2 ladder model, this is nothing but the singlet-product state (rung-singlet). The original symmetry $SU(2) \times Z_2$ is not broken at all. In fact, a simple fluctuation analysis shows that the low-energy excitation is the gapped triplet. The singlet-product phase persists until the system enters the ‘NAF-dominant’ phase at $\theta = -\tan^{-1}(1/4) \approx -0.0780\pi$ (i.e. $J + 4K_4 = 0$), where the triplet gap vanishes.

The emergence of spin-nematic phase (‘F-nematic’) in the absence of biquadratic interaction $J_{bl}$ (see Eq. (13)) may look surprising. However, this can be understood within the simple mean-field argument presented here. First we note that $J_{bl} = J/2 + K_4 > 0 (< 0)$ for $\theta < \pi - \theta_{ad}$ ($\theta > \pi - \theta_{ad}$). In the region where we have ‘F-nematic’, the magnetic coupling $J_{bl} = J/2 + K_4 (< 0)$ is small and the chirality coupling $J_B = K_4 = \sin\theta$ is always positive (hence anti-parallel configuration of B is favored), while the NAF-coupling $J_A = J/2 = \cos\theta$ is negative. In the ‘partial-AF’ phase, the system tends to develop weak local moments T which align in an AF-manner to optimize the positive magnetic coupling $J_{bl}$; the combination $J_A < 0, J_B > 0$ stabilizes the parallel A and the anti-parallel B as the optimal configuration.

In the ‘F-nematic’ phase, on the other hand, the negative $J_{bl}$ naively favors ferromagnetic alignment of the local moments T. However, to realize it, configurations with (A-parallel, B-parallel) or (A-antiparallel, B-antiparallel) are needed and they are inconsistent with the signs of $J_A$ and $J_B$ (hence frustrated). Since $-J_A > J_B (> 0)$ and $|J_{bl}| < 1$, the system lowers the energy by choosing a non-magnetic state (‘F-nematic’) with ferromagnetic ordering of A ($B = 0$).

If we further increase $\theta$, $J_{bl}$ gets larger while the chiral-coupling $J_B$ becomes negligibly small. Then, the energy gain by forming a ferromagnetic (i.e. parallel T) configuration overcomes the energy cost coming from the frustration in the B-channel (i.e. parallel-B for positive $J_B$) and hence the ferromagnetic state is stabilized.

So far, we have presented the mean-field description of the phases of the spin-1 boson model and found that in some of these phases, rotational symmetry is spontaneously broken at the mean-field level. However, in one dimension, strong quantum fluctuations may destroy the ordered states predicted by the mean-field theory. In fact, as we will show in the next section by using effective field theories, some of the ordered phases are replaced by gapped short-range phases which do not break rotational symmetry.

IV. LOW-ENERGY EFFECTIVE ACTIONS

Having established the mean-field phase diagram, we now take into account quantum fluctuations in a semiclassical fashion. From the viewpoint of the competition between spin and chirality, it is interesting to develop the continuum descriptions of the two phases – the NAF-dominant phase and the chirality-dominant phase.

A. Low-Energy Fluctuations

In constructing the continuum limit, it is important to find out the relevant low-energy degrees of freedom. To this end, let us calculate the low-energy spectrum in the semiclassical approximation. First we parametrize the semi-classical configuration as:

\[ A_r^{(0)} = (0, 0, (-1)^r \rho_0), \quad B_r^{(0)} = (0, 0, 0) \]

for \( -\tan^{-1}(1/4) \leq \theta < \theta_{ad} = 0.1476\pi \) (39a)
or
\[ A_r^{(0)} = (0, 0, 0), \quad B_r^{(0)} = (0, 0, (-1)^\nu \rho_0) \]
for \( \theta_{ad} < \theta < \pi/2 \),

where \( \rho_0 \equiv \sqrt{\rho_0} (0 \leq \rho_0 \leq 1) \) and is given by (see Appendix A.2):
\[
\rho_0 = \begin{cases} 
\frac{1}{2} \sqrt{\frac{J + 4K_4}{J + 2K_4}} & \text{for } K_4 < J/2 \\
\frac{1}{2} \sqrt{\frac{-J + 8K_4}{K_4}} & \text{for } K_4 > J/2 .
\end{cases}
\]
The value of \( \rho_0 \) is plotted in the left panel of Fig. 3

It is easy to verify that the transverse modes \( (\omega_{x,y}(k)) \) have a linear dispersion near the zone boundary \( k = \pi \):
\[
\omega_{x,y}(k) \approx \frac{J \sqrt{(J + K_4)(3J + 4K_4)}}{\sqrt{2}(J + 2K_4)} |k - \pi| ,
\]
while the longitudinal one \( (\omega_z) \) is gapped (unless \( K_4 = J/2 \)). These linearly dispersive modes roughly describe transverse fluctuations of the director vectors \( A \) and similar to the usual AF magnons. The spectrum for the \( B \)-ordered phase (‘chirality-dominant’ phase), which is realized when \( K_4 \gg J \), can be obtained in a similar fashion.

Now that we have determined the classical (or mean-field) ground state, we are at the point of calculating the spectrum of the low-energy excitations. Let us consider the following small deviation \( \delta A, \delta B \) from the classical ground state:
\[
A_r = A_r^{(0)} + \delta A_r, \quad B_r = B_r^{(0)} + \delta B_r .
\]
If we plug the above expressions into the classical equation of motion and retain terms up to 1st-order in \( \delta A \) and \( \delta B \), we obtain the following ‘spin-wave’ spectra:
\[
\omega_x(k) = \omega_y(k) = \frac{J \cos \left( \frac{k}{2} \right) \sqrt{(3J + 4K_4)(3J + 4K_4 - J \cos(k))}}{\sqrt{2}(J + 2K_4)} ,
\]
\[
\omega_z(k) = \sqrt{\frac{\cos(k)J^3 + 4(J + 2K_4)J^2 + 2K_4 (J^2 - 8K_4J - 16K_4^2) \cos^2(k)}{J + 2K_4}} .
\]

The above analysis suggests that we should keep the low-energy order parameter field around \( k = \pi \) in constructing the continuum limit for the dominant-NAF phase. Specifically, we use the following two-sublattice (plaquette-wise, actually) ansatz to parametrize the \( A \) and \( B \) fields:
\[
A_r + i B_r = \sqrt{\rho_0^2 + a l_\phi(x)} (-1)^\nu e^{i\phi(x)} \times \left\{ \sqrt{1 - a^2 b_{l_\phi(x)}^* \Omega(x)} + ia(\Omega \times \mathbf{l}_{l_\phi}) \right\} ,
\]
where the order parameter field (director) satisfies \( |\Omega(x)| = 1 \) and \( \Omega \cdot \mathbf{l}_{l_\phi} = 0 \). Physically, the two small fields \( \mathbf{l}_{l_\phi} \) and \( l_\phi \) are respectively the generators of spin-SU(2) and charge-U(1):
\[
T_r^a = -i \epsilon_{\alpha\beta\gamma} b_{l_\phi}^\dagger b^\dagger_{l_\phi} b_\gamma = 2 \rho_0^2 a (l_\phi)^a + \cdots ,
\]
\[
y_{l_\phi} = \sum_\alpha b_{l_\phi}^\dagger b_\alpha = \rho_0^2 + a l_\phi(x) + \cdots .
\]
The vector order parameter \( \Omega \) represents spin and chirality in a unifying manner. When the phase \( \phi \) is dominantly around \( \phi = 0 \) or \( \pi \), the order parameter \( \Omega \) describes the transverse fluctuations of the \((\pi, \pi)\) antiferromagnetic order represented by \( A \):
\[
\Omega \sim (S_1 - S_2)_{\theta = \pi} .
\]
When \( \phi \) fluctuates around \( \pm \pi/2 \), on the other hand, \( \Omega \) constitutes the imaginary part of the quantity \( A + iB \) and describes the fluctuations of (the staggered component of) the vector chirality \( B \):
\[
\Omega \sim (S_1 \times \mathbf{S}_2)_{\theta = \pi} .
\]
As in the usual Haldane mapping\cite{Haldane81}, in order to write down the effective action, we assume that the two fields $l_\Omega$ and $l_\phi$ are small compared with $\Omega$ and retain terms up to second-order in the calculation. The condensate amplitude $\rho_0$ and the phase angle $\phi_0$ of the ground state can be determined by minimizing the constant part of the effective action:

$$
E_0 = \rho_0^2 \left\{ \rho_0^2 (J + 6 K_4) - 6 K_4 \right\}
- \rho_0^2 (1 - \rho_0^2) (J - 2K_4) \cos(2\phi_0) .
$$

(48)

Depending on the sign of $(J - 2K_4)$, either $\phi_0 = 0, \pi$ or $\phi_0 = \pi/2, 3\pi/2$ is chosen.

After some algebra, we obtain (see Appendix A 2 for details):

$$
S_{\text{eff}}[\Omega] = \frac{1}{2} \chi_s \int dx dt (\partial_x \Omega)^2
- \frac{1}{2} \rho_s \int dx dt (\partial_x \Omega)^2
+ \frac{1}{2} \chi_c \int dx dt (\partial_x \phi)^2
- \frac{1}{2} \rho_c \int dx dt (\partial_x \phi)^2
+ g(J, K_4) \int dx dt \cos(2\phi(x,t)) ,
$$

where the explicit expressions for the five coupling constants (spin/charge susceptibility $\chi_{s,c}$, the spin/charge stiffness $\rho_{s,c}$ and the sine-Gordon coupling $g(J, K_4)$) are given in Appendix (Eq. A13). Note that a similar effective action (without the cosine-term) has been derived in the context of $F = 1$ spinor Bose-Einstein condensate\cite{Eguchi15}. The spin part of the effective action is nothing but the O(3) non-linear sigma model in (1+1)-dimensions, where the long-range-ordered ground state with the gapless $k$-linear Goldstone modes is replaced by a quantum-disordered one with the gapped triplet excitations (triplon)\cite{Eguchi16}. Since near $\theta = 0$ and $\pi$, $\Omega$ may be thought of as the order parameter for the standard ($\pi, \pi$) AF ordering, the spin part describes the short-range antiferromagnetic fluctuations for $J > K_4$. Since $A$ and $B$ play the role of the director in the spin-nematic, one may think that the order parameter manifold is RP\textsuperscript{2}. However, in contrast to the case of the usual spin nematic, the directors ($A$ and $B$) themselves are directly related to the physical observables (i.e. $S_1 - S_2$ and $S_1 \times S_2$, respectively) and the order parameter in fact is defined on the two-dimensional sphere $S^2$. In principle, there can be a topological term associated with $H_2(S^2)$ in the above effective action. However, the above direct mapping suggests that there is no such topological term in the effective action.

On top of the spin part, there is the spin-chirality (or, in terms of spin-1 bosons, ‘charge’) part which describes the dynamics of the Bose phase $\phi$; the spin-chirality part of the effective action is given by the sine-Gordon model with the $(J, K_4)$-dependent couplings. Depending on the sign of $g(J, K_4)$, the cosine term pins the ground-state phase either at $\phi = 0, \pi$ (for $g(J, K_4) > 0$) or at $\phi = \pi/2, 3\pi/2$ (for $g(J, K_4) < 0$). Since $\phi \rightarrow \phi + \pi$ amounts to the 1-site translation $r \rightarrow r + 1$ (see Eq. (44)), one can see that in both cases there appear two-fold degenerate ground states related by 1-site translation; the sine-Gordon soliton connects the two degenerate ground states. At the self-dual point $K_4 = J/2$, the sine-Gordon coupling disappears and the charge (i.e. spin-chirality) part reduces to the Tomonaga-Luttinger model as is expected from the previous results\cite{Ueda02}.

Now we proceed to a more interesting case of the chirality-dominant phase realized in the large-$K_4$ region. In this phase, $\phi_0$ is locked at $\phi = \pm \pi/2$ and the effective action (45) describes the short-range (staggered) fluctuations in the chirality channel; the staggered component of the B-field plays a role of the order-parameter field: $\Omega \sim (S_1 \times S_2)_{\phi = \pi}$. As before, the ground state is quantum-disordered with a gapped triplet excitation, which is created not by the spin operators themselves but by the staggered component of the vector chirality $(S_1 \times S_2)_{\phi = \pi}$. This agrees with the results of numerical simulations\cite{Ishikawa15}. Since the duality transformation ($\phi \rightarrow \phi + \pi/2$ in Eq. (22) or $\phi \leftrightarrow \phi + \pi/2$ in Eq. (44)) interchanges two vector fields $A$ (‘spin’) and $B$ (‘chirality’), these two effective actions (49) clearly exhibit the ‘dual’ nature of spin and chirality in the model (1).

To summarize, we have obtained the O(3) non-linear sigma model for the spin dynamics in the NAF/chirality-dominant phases; deep inside these phases, where the spin-chirality ($\theta$) fluctuations have a large gap and are well separated from the transverse spin fluctuations (see the inset of Fig. 3), we may expect that the pure sigma model provides us with a good description of the low-energy physics far away from the self-dual point $K_4/J = 1/2$. Since a similar non-linear sigma model is obtained in the standard field-theory treatment\cite{Eguchi16} of the two-leg ladder with $K_4 = 0$, one may identify the NAF-dominant phase with the rung-singlet (RS) phase. However, as we will see below, this is not the case. It is important to note that the above derivation of the effective action applies to higher dimensional cases as well with due modification of their coupling constants $\rho_{s,c}$ and $\chi_{s,c}$.

When the self-dual point $K_4/J = 1/2$ is approached, on the other hand, the gap of the $\omega_2$-branch (i.e. the $\phi$-channel) decreases and eventually becomes smaller than the dynamically-generated spin gap; exactly at the point, the longitudinal mode exhibits the gapless $k$-linear behavior\cite{Ueda02} around $k = \pi$ which we may identify with the gapless spin-chirality (i.e. the U(1)-phase of the spin-1 hardcore boson) fluctuations represented by the Tomonaga-Luttinger model also found in the bosonization analysis\cite{Ueda02}. As the gap in the spin sector (which is described by the non-linear sigma model) is always finite, the quantum phase transition between NAF-dominant phase and the chirality-dominant one is described by the $c = 1$ Tomonaga-Luttinger liquid\cite{Ueda02}. In fact, around the self-dual point, the spin-singlet phase (‘charge’) fluctuations play the primary role.

Away from the self-dual point, the anomalous hopping term generates the $\cos 2\phi$-term which pins the ground state at $\phi = 0, \pi$ or at $\phi = \pi/2, 3\pi/2$ to stabilize the two-fold degenerate ground states of the model (1) as far as the condensate amplitude $\rho_0$ is finite; the two-fold degeneracy corresponds to breaking of the $Z_2$-symmetry ($A, B \rightarrow (-A, -B)$) (i.e. the interchange of the two chains 1 $\leftrightarrow$ 2) or the 1-site translation $r \rightarrow r + 1$. In 1D, the rotation symmetry is restored and the only broken symmetry is the $Z_2$. This is what we have in the staggered-dimer phase and the staggered-scalar-chirality
In the case of the two-leg ladder, the spin excitations are always the gapped triplons in the three main phases (singlet-product, NAF/chirality-dominant) in the $J > 0$ region (see Fig. 2) and only the presence/absence of the $\mathbb{Z}_2$-symmetry distinguishes between the first- and the latter two phases. In this respect, the NAF-dominant- and the chirality-dominant phase may be identified respectively with the staggered dimer phase and the staggered scalar chiral phase found in Ref. [27]. Since our semiclassical treatment underestimates the effect of quantum fluctuations brought about by the pairing term, the $\mathbb{Z}_2$-broken ordered states (NAF/chirality-dominant phases) are more stabilized than in the actual two-leg ladder [27].

Last, we briefly touch upon the effective action in the F-nematic phase. A similar fluctuation analysis shows that the gapless $k$-linear modes appear at $k = 0$ in this case. The calculation is essentially the same as in the previous two cases except that we adopt the following parametrization reflecting the existence of an alternating spin-chirality $U(1)$-symmetry [49]. As in the above cases, we carry out gradient expansion and the subsequent Gaussian integration over $l_{1}$, $l_{0}$ to obtain the non-linear sigma model [49] with different coupling constants (see Eq. (A16)). At the transition point $(J + 2K_{A} = 0)$ between partial-AF and F-nematic, the sine-Gordon interaction vanishes and the transition is described by the Tomonaga-Luttinger liquid. Again, quantum fluctuations open a gap in the spin sector and the ground state is characterized by the short-range correlation of the collinear order parameter $\Omega \sim S_{1} - S_{2}$ at $k = 0$, which is consistent with the numerical observation [52].

The six coupling constants are expressed in terms of the six bosonic ones as:

$$g_{1} = J_{k} - 4J_{bq}/2, \ g_{2} = (t + u)/2, \ g_{3} = J_{bq}/2, \ g_{4} = (t - u)/2, \ g_{5} = 2J_{bq} - \mu + 3V_{c}/2, \ g_{6} = V_{c}.$$  

Under the spin-chirality duality symmetry (2), the coupling constants of model (51) transform as: $g_{2} \leftrightarrow g_{4}$, while all the others remain invariant.

### A. Mean-field analysis

Let us consider the situation where the magnetic interaction $J_{K}$ of model (11) (i.e. $g_{1}$ in the ladder model (51)) is dominant and the interaction $g_{2}(> 0)$ in the spin-spin channel is stronger than that in the chirality channel $g_{4}$. The idea is as follows. When a sufficiently strong magnetic field (say, in the $z$-direction) induces uniform magnetization, the magnetic moments $M = S_{1} + S_{2} = -2A \times B$ (see Eq. (52f)) assume (at least in the semiclassical sense) the canted configuration as is shown in Fig. 4. A simple mean-field argument given above concludes that the anti-parallel ordering of $A \propto S_{1} - S_{2}$ forces the canting of the chirality vector $B \propto S_{1} \times S_{2}$ leading to the appearance of finite vector chirality in the field direction (52) $(S_{1} \times S_{2})^{z} \neq 0$ (see Fig. 4). Note that the mean-field (or variational) energy is invariant under the global $\mathbb{Z}_{2}$-symmetry: $A \mapsto -A, \ B \mapsto -B$ corresponding to the interchange of the two chains $1 \leftrightarrow 2$. Therefore, the appearance of uniform vector chirality in the low-field phase is accompanied by the $\mathbb{Z}_{2}$-symmetry breaking.

According to the spin-chirality duality (2), the same argument applies to the case where the chirality channel $g_{4}(> 0)$ is dominant: in this case, the ordering in the chirality channel induces finite $(S_{1} - S_{2})^{z}$, that is, magnetization is induced asymmetrically in the upper- and the lower chain. It is important to note that, when the interaction in the spin-spin channel

### V. EFFECTS OF MAGNETIC FIELD AND MUTUAL INDUCTION PHENOMENA

In this section, we investigate the effect of a magnetic field on the two-leg spin ladder with a ring-exchange by means of our semiclassical approach. We show, as a by-product, that the spin-chirality duality leads to an interesting phenomenon—mutual induction between spin and chirality degrees of freedom.

To this end, it is convenient to enlarge the parameter space of model (1) and consider a two-leg ladder with a general four-spin exchange interaction [50]:

$$\mathcal{H} = g_{1}\mathcal{H}_{1} + g_{2}\mathcal{H}_{2} + g_{3}\mathcal{H}_{3} + g_{4}\mathcal{H}_{4} + g_{5}\mathcal{H}_{5} + g_{6}\mathcal{H}_{6}, \ (51)$$
is dominant, the chirality is induced and vice versa. As is clear from the above argument, this is not restricted to one dimension; our semiclassical approach immediately predicts that in two dimensions too. One of the most important conclusions of our bosonic effective theory is that this mutual induction is an observable consequence of the spin-chirality duality.

It would be interesting to check if the simplest ring-exchange ladder (1) really supports this phase or not. Again we can use mean-field analysis to map out the phase diagram in the presence of magnetic field. The result is summarized in Fig. 5. Except for the trivial saturated phase which covers the entire ferromagnetic phase as well as the high-field region, the 1/2-plateau phase, which is characterized by spin-polarized spin-1 bosons sitting every other site, occupies the large portion of the phase diagram. This is consistent with the previous results obtained by numerical simulations and strong-coupling expansions.

Magnetization increases smoothly in the two collinear phases (collinear-I and collinear-II). In the collinear-I phase, both A and B align in an AF manner and are perpendicular to the field (hence the local moments $\propto A \times B$ are parallel to the field. See Fig. 6(a)). It exists typically in the low-field region of ‘NAF-dominant’ phase and in the high-field region near saturation. In fact, it is easy to show, by finding the exact single-magnon wave function, that it is a generalizations of (the simplest version of) the triplon condensate at momentum $k = \pi$ which is used frequently in analyzing the magnetization process of dimer systems; here not only the spin component parallel (i.e. $T^z = +1$) to the field but also the anti-parallel (i.e. $T^z = -1$) one are taken into account. The collinear-II phase (Fig. 6(b)) is similar to the collinear-I except that here both A and B align ferromagnetically. In this sense, this is thought of as the classical counterpart of the triplon condensate at $k = 0$. Typically, it appears when the field is applied in the ‘F-nematic’ phase.

On top of them, yet another collinear phase (dubbed SDW for spin density wave) appears in the ‘partial-AF’ phase when the field is very weak (a tiny region below the 1/2-plateau). In the SDW phase, the local moments $\langle \hat{T} \rangle$ form an up-down pattern along the field direction while the lengths of the moments on different sublattices are not equal (Fig. 6(c)).

Our mean-field results basically agree with those of density-matrix-renormalization-group (DMRG) simulations; the exceptions are the ‘SDW’ phase and ‘collinear-II’ near saturation. In fact, in the region where we found ‘SDW’, DMRG simulations observed a phase with the dominant vector-chiral correlation (in the transverse direction). As we have remarked above, already at the mean-field level, the energies of ‘chirality-dominant’ phase and ‘SDW’ are very close to each other and the correct treatment of quantum fluctuations might stabilize the chirality over ‘SDW’ state. ‘Collinear-II’ state in the classical limit translates in the quantum description to the single-boson (either triplons over the rung singlet or singlet-rungs in the spin-polarized state) condensate at $k = 0$. In the low-field region, this is consistent with the phase diagram (Fig. 1 and 5 of Ref. 54) obtained numerically. However, what has been observed near saturation is the phase characterized by the condensation of magnon bound states. The failure of our mean-field theory in describing this magnon-pair phase is not surprising since our wave function is tailored to describe the single-boson condensate.

Clearly, the mean-field $(\theta, H)$-phase diagram (Fig. 5) of the ring-exchange ladder (1) shows that the spin-canted state illustrated in Fig. 4 is not realized at least in the simplest ladder. In fact, after this phenomenon of magnetization-induced vector chirality has been predicted first in one dimension by means of bosonization, extensive DMRG study has been carried out to show that the pure ring-exchange model (1) in a magnetic field is not sufficient to support the predicted phase with finite uniform (vector) chirality. This is consistent with our mean-field study.

To seek for the possibility of the new phase with uniform chirality, we explored the enlarged parameter space and extensively carried out the variational analysis. It turned out that it is not easy to realize the canted state (Fig. 4) at intermediate fillings (i.e., $0 < n_B < 1$); it is quite often masked by the above three collinear states or the 1/2-plateau. However, careful choices of parameters can stabilize the canted state. For instance, we show the magnetization process of the generalized ladder with $J_{d1} = 2.0$, $J_{eq} = 0.0$, $J_A = (t + u)/2 = 1.3$, $J_B = (t - u)/2 = 0.01$, $V_z = 0.6$, $\mu = -0.5$ in Fig. 7. One can clearly see that uniform $z$-component of vector chirality is induced in the low-field region. As has been mentioned above, the appearance of uniform vector chirality in the low-field phase implies the breaking of the chain-exchange $Z_2$-symmetry.

![FIG. 4. (color online) A naively expected high-field configuration of $A \propto S_1 - S_2$, $B \propto S_1 \times S_2$, and $\langle \hat{T} \rangle \propto S_1 + S_2$. Spin $\langle \hat{T} \rangle$ canting induces a uniform $z$-component $\langle S_1^z \rangle$. If a similar spin-canted state is realized for large enough $g_s$, we may expect a state where local magnetization is induced asymmetrically in the two chains $\langle S_1^z \rangle \neq \langle S_2^z \rangle$.](image-url)
Numerical results

In order to investigate the possible appearance of vector chirality, we now turn to numerical investigation using the DMRG algorithm. Since the pure ring-exchange model does not exhibit such a symmetry breaking, and because the most general spin ladder Hamiltonian contains too many parameters, we have used the previous variational analysis as a guide.

Using the following parameters $J_{bl} = 2.0$, $J_{bq} = 0.0$, $(t - u)/2 = 0.01$, $V_z = 0.6$, and $\mu = -0.5$, vector chirality was found to occur for $(t + u)/2 = 1.2$ in the variational approach (see Fig. 7). In Fig. 8, we plot the vector chirality correlation

$$C(x) = \langle (\mathbf{S}_{1,r} \times \mathbf{S}_{2,r})^z (\mathbf{S}_{1,r+x} \times \mathbf{S}_{2,r+x})^z \rangle$$

obtained numerically with DMRG. We have studied $2 \times L$ ladders with general four-spin exchange (see Eq. (51)) corresponding to the above bosonic parameters, and we use open boundary conditions. Correlations are averaged with $L/4 \leq r \leq 3L/4$. We keep up to 1200 states which is sufficient to get a discarded weight smaller than $10^{-12}$.

As can be read from the plot, vector chirality correlations do exhibit a slight increase when $(t + u)/2 \sim 1.2$, corresponding to the optimal parameter found in the variational analysis, but this only indicates quasi long-range order, i.e. an algebraic decay.

We have also tried to vary some of these parameters, or look at other magnetization values, but we have been unable so far to detect any evidence of long-range order.
VI. CONCLUDING REMARKS

In the present paper, we have investigated the phase diagram of the two-leg spin ladder with a ring-exchange interaction. In this respect, we have developed a unifying approach which treats AF fluctuations and VC (or $p$-nematic) ones on the same footing. This approach is based on the description in terms of a 1D hardcore boson operators that create the triplet states out of the singlet on each rung. The spin-chirality transformation, originally defined on the lattice spin operators,\(^{28,29}\) has a simple physical meaning here through the standard U(1) global gauge symmetry of the bosons.

Using a simple mean-field calculation on the boson operators, we have mapped out the zero-temperature phase diagram of the two-leg spin ladder with a ring-exchange interaction. This bosonic approach enables one to take into account quantum fluctuations in a semiclassical fashion. In particular, we reproduce the main results of previous intensive numerical studies\(^{27,28}\) and a low-energy description of the chirality-long-range ordering, although a mean-field analysis of Ref. 54. In this respect, we have given conditions which support this exotic phase in full agreement with the DMRG calculations are called for to fully investigate the six-dimensional systems made of two-spin clusters (spin dimers) like $S=1/2$ bilayer systems. However, the range of its applicability is much broader. In fact, as we will show elsewhere\(^{25}\) even in the two-dimensional cases without such special structures, one can construct effective bosonic degrees of freedom which encode the competition between spin and chirality degrees of freedom in close parallel to the 1D case. Then, we can apply a semiclassical field-theory approach similar to what is described in this paper to such higher-dimensional systems as the spin-$1/2$ Heisenberg model on the square lattice with a ring-exchange interaction to capture the global structure of the phase diagram.

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Appendix A: Dimer coherent state path integral

In this Appendix, we present some technical details on the derivation of the low-energy effective actions for the RS and VC phases of the two-leg spin ladder with a ring-exchange interaction.\(^{11}\)

1. Berry phase term

The quantum dynamics is generated by the Berry phase term which can be derived from the following overlap:

$$\langle s(t + \delta t), t(t + \delta t)|s(t), t(t)\rangle \approx 1 + \{ s \partial_t s^* + t \partial_t t^* \} \delta t ,$$

(A1)

where $t$ and $t + \delta t$ denote two infinitesimally separated times. Plugging (19) into the above, we obtain

$$\langle s(t + \delta t), t(t + \delta t)|s(t), t(t)\rangle = 1 + i(A \cdot \partial_t t - B \cdot \partial_t t^*) \delta t ,$$

(A2)

from which we can read off the Berry phase contribution to the (single-site) action:

$$S_{Berry} = \int dt \frac{i}{2} (t^* \cdot \partial_t t - t \cdot \partial_t t^*) = \int dt (A \cdot \partial_t t - B \cdot \partial_t t^*) .$$

(A3)

This Berry phase term is the first contribution of the general action (26). As is easily seen by rescaling: $t \mapsto \sqrt{\rho} t$ ($|t| = 1$), this term generates the dynamics in the transverse direction, i.e. there is no time-derivative for the $p$-field which describe the fluctuations in the longitudinal direction.
2. Continuum limit

On the basis of the fluctuation analysis in Sec. [IV A], we can derive a low-energy effective actions for the NAF-dominant- and the chirality-dominant phase. Since the \( A \)-field develops in the former phase, we may make the following ansatz Eq. [A4] to parametrize the \( A \) and \( B \) fields:

\[
A_r + i B_r = \sqrt{\rho_0^2 + a l_\phi(x)} (1 - i)^r e^{i\phi(x)} \times \left\{ \frac{1}{2} - a^4 l_\Omega(x) + 2 a \Omega(\Omega \times l_\Omega) \right\} .
\]  

(A4)

Clearly, \( \Omega \) and \( \phi \) respectively describe the transverse- and the phase fluctuations of the spin-1 boson. To guarantee the normalization condition \( \langle \eta_B^2 \rangle = \rho_0^2 \) we impose the following constraints:

\[
|\Omega| = 1 \quad \Omega \cdot l_\Omega = 0 .
\]  

(A5)

The idea underlying the decomposition [A4] may be seen as follows. First we note that Eq. (A4) is a close parallel of the expression (27). Suppose that the vector field \( A \) has a finite length \( \rho_0^2 \) (as in NAF-dominant phase) and that it plays a role of the order parameter i.e. \( \Omega \sim A \). The imaginary part \( B \) may be obtained by noting that the SU(2)-generator is given by

\[
L_r = -2 A_r \times B_r
\]  

(A6)

(see Eq. [25]). As is suggested by counting the number of independent degrees of freedom, once the overall phase \( \phi \) is singled out, the angle between \( A_r \) and \( B_r \) cannot be arbitrary; a convenient choice would be to take: \( A_r \cdot B_r = 0 \). When \( A_r \sim \Omega_r, B_r \) is expressed simply as:

\[
\Omega_r \times L_r = -2 \left\{ \Omega_r \cdot B_r \right\} = -2 B_r \left( \Omega_r \right) .
\]  

(A7)

With the identification: \( l_\Omega \sim L/2 \), one can see that the vectorial part of [A4] correctly reproduces the transformation properties of \( A + i B \). The phase part \( \sqrt{\rho_0^2 + a l_\phi(x)} e^{i\phi(x)} \) is easily guessed from the well-known result in the single-component Bose liquid.

Now let us plug the ansatz [A4] into the action (25) and keep terms up to second order in the small fields \( l_{\Omega} \) and \( l_{\phi} \). The constant part [49]

\[
E_0 = \rho_0^2 \left\{ \rho_0^2(J + 6K_4) - 6K_4 \right\} - \rho_0^2(1 - \rho_0^2)(J - 2K_4)\cos(2\phi_0)
\]  

(A8)

is minimized to give \( \rho_0 \) and \( \phi_0 \):

\[
\rho_0, \phi_0 = \left\{ \begin{array}{ll}
\frac{1}{2} \sqrt{\frac{J+4K_4}{J+2K_4}}, & 0(\text{or } \pi) \quad \text{for} \quad -1/4 < K_4/J < 1/2 \\
\frac{1}{2} \sqrt{-\frac{J+4K_4}{K_4}}, & \frac{\pi}{2}(\text{or } \frac{3\pi}{2}) \quad \text{for} \quad K_4/J > 1/2 .
\end{array} \right.
\]  

(A9)

Dropping the alternating sums and changing the sum over rungs to integrals

\[
\sum_r \rightarrow \frac{1}{a} \int dx \quad \text{,}
\]  

we obtain the following results:

\[
\mathcal{S}_{\text{Berry}} = \int dx dt \left\{ l_{\phi}(\partial_t \phi) + 2 \rho_0^2 l_\Omega \right\} + \frac{\rho_0^2}{a} \left( \partial_t \phi \right),
\]  

(A11a)

\[
\mathcal{S}_{\text{magnetic}} + \mathcal{S}_{\text{charge}}
\]  

\[
\approx -(2J + 4K_4) \rho_0^2 a \int dx dt \left( l_{\Omega} \right)^2 - 4K_4 a \int dx dt l_{\phi}^2 ,
\]  

(A11b)

and

\[
\mathcal{S}_{\text{hopping}}
\]  

\[
\approx - \left\{ \left( \frac{J}{2} + K_4 \right) \pm \left( \frac{J}{2} - K_4 \right) \right\} a
\]  

\[
\times \rho_0^2 (1 - \rho_0^2) \int dx dt \left\{ \left( \partial_x \Omega \right)^2 + \left( \partial_x \phi \right)^2 \right\}
\]  

\[
+ 4a \rho_0^2 (1 - \rho_0^2) \left( \frac{J}{2} - K_4 \right) \int dx dt \left( l_{\Omega} \right)^2 ,
\]  

(A11c)

where the upper (lower) sign is chosen when \( J > 2K_4 \) (\( J < 2K_4 \)). The Gaussian integration in \( l_{\Omega} \) and \( l_{\phi} \) yields the following result:

\[
\mathcal{S}_{\text{eff}}[\Omega] = \frac{1}{2} \chi_s \int dx dt \left( \partial_t \Omega \right)^2 - \frac{1}{2} \rho_c \int dx dt \left( \partial_x \Omega \right)^2
\]  

\[
+ \frac{1}{2} \chi_c \int dx dt \left( \partial_t \phi \right)^2 - \frac{1}{2} \rho_c \int dx dt \left( \partial_x \phi \right)^2
\]  

\[
+ g(J, K_4) \int dx dt \cos(2\phi) .
\]  

(A12)

When \( J > 2K_4 \), the effective action \( \mathcal{S}_{\text{eff}}[\Omega] \) describes the \((\pi, \pi)\) AF fluctuations and the phase fluctuations in the spin-chirality space. The five coupling constants \( \chi_{s,c}, \rho_{s,c} \) and \( g(J, K_4) \) are given by:

\[
\chi_s = \frac{J + 4K_4}{4aJ(J+K_4)} \quad \rho_s = \frac{J + 4K_4 (3J + 4K_4)}{8(J + 2K_4)^2} a ,
\]  

(A13a)

\[
\chi_c = \frac{1}{4a(J + 2K_4)} \quad \rho_c = \frac{J + 4K_4 (3J + 4K_4)}{8(J + 2K_4)^2} a ,
\]  

(A13b)

\[
g(J, K_4) = \frac{(J - 2K_4)(J + 4K_4)}{16 (J + 2K_4)^2 a} \quad (> 0).
\]  

(A13c)

Note that precisely at the self-dual point \( J = 2K_4 \), \( g(J, K_4) = 0 \) and the spin-chirality fluctuations becomes gapless (i.e. Tomonoga-Luttinger like).

In the chirality-dominant phase \( K_4 > J/2 \) (or \( \theta_{ad} < \theta < \pi/2 \)), \( \phi \) is locked at \( \phi = \pi/2, 3\pi/2 \) and \( \Omega \sim (S_1 \times S_2)_{k=\pi} \).
This short-range vector-chiral fluctuations at \( k = \pi \) are governed by the non-linear sigma model \((A12)\) with

\[
\chi_\sigma = \frac{8K_4 - J}{2a(4K_4-J)(4K_4+J)} \cdot \rho_\sigma = \frac{(8K_4-J)(8K_4+J)a}{64K_4},
\]

\[
\chi_c = \frac{1}{16aK_4}, \quad \rho_c = \frac{(8K_4-J)(8K_4+J)a}{64K_4},
\]

\[
g(J, K_4) = -\frac{(2K_4-J)(8K_4-J)(8K_4+J)}{256K_4^2a} \quad (< 0).
\]

In deriving the effective model for the F-nematic phase realized for \( J + 2K_4 < 0 \), we use another parametrization \((50)\) which takes into account the transverse soft modes at \( k = 0 \) and the staggered U(1)-symmetry for \( J + 2K_4 = 0 \):

\[
A_r + i B_r = \sqrt{\rho_0^2 + a l_0(x)} e^{(-1)\cdot \phi(x)}
\times \left\{ \sqrt{1 - a^2\Omega_1(x) \Omega(x) + ia(\Omega \times l_0)} \right\},
\]

where the two phenomenological parameters are now given by:

\[
\chi_s = \frac{3J - 4K_4}{2a(J^2 + 4K_4J - 8K_4^2)},
\]

\[
\rho_s = \frac{J(J - 4K_4)(3J - 4K_4)a}{8(J - 2K_4)^2},
\]

\[
\chi_c = \frac{1}{4a(2K_4 - J)}, \quad \rho_c = \frac{a(-J)(4K_4 - J)(4K_4 - 3J)}{8(J - 2K_4)^2},
\]

\[
g(J, K_4) = -\frac{(J + 2K_4)(4K_4 - J)(4K_4 - 3J)}{16a(J - 2K_4)^2} \quad (> 0).
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