ON H-SIMPLE NOT NECESSARILY ASSOCIATIVE ALGEBRAS

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Abstract. An algebra $A$ with a generalized $H$-action is a generalization of an $H$-module algebra where $H$ is just an associative algebra with 1 and a relaxed compatibility condition between the multiplication in $A$ and the $H$-action on $A$ holds. At first glance this notion may appear too general, however it enables to work with algebras endowed with various kinds of additional structures (e.g. (co)module algebras over Hopf algebras, graded algebras, algebras with an action of a (semi)group by (anti)endomorphisms). This approach proves to be especially fruitful in the theory of polynomial identities. We show that if $A$ is a finite dimensional (not necessarily associative) algebra over a field of characteristic 0 and $A$ is simple with respect to a generalized $H$-action, then there exists $\lim_{n \to \infty} \sqrt[n]{c^H_n(A)} \in \mathbb{R}_+$ where $(c^H_n(A))_{n=1}^\infty$ is the sequence of codimensions of polynomial $H$-identities of $A$. In particular, if $A$ is a finite dimensional (not necessarily group graded) graded-simple algebra, then there exists $\lim_{n \to \infty} \sqrt[n]{c_{gr}^n(A)} \in \mathbb{R}_+$ where $(c_{gr}^n(A))_{n=1}^\infty$ is the sequence of codimensions of graded polynomial identities of $A$. In addition, we study the free-forgetful adjunctions corresponding to (not necessarily group) gradings and generalized $H$-actions.

1. Introduction

Study of polynomial identities in algebras is an important aspect of study of algebras themselves. It turns out that the asymptotic behaviour of numeric characteristics of polynomial identities of an algebra is tightly related to the structure of the algebra [17, 33].

In 1980s, S. A. Amitsur conjectured that if an associative algebra $A$ over a field of characteristic 0 satisfies a nontrivial polynomial identity, then there exists an integer $PI$-exponent $\lim_{n \to \infty} \sqrt[n]{c_n(A)}$ where $c_n(A)$ is the codimension sequence of ordinary polynomial identities of $A$. (See the definition of $c_n(A)$ in Remark 3.1 below.) The original Amitsur conjecture was proved by A. Giambruno and M. V. Zaicev [16] in 1999. Its analog for finite dimensional Lie algebras was proved by M. V. Zaicev [33] in 2002. In 2011 A. Giambruno, I. P. Shestakov and M. V. Zaicev proved the analog of the conjecture for finite dimensional Jordan and alternative algebras [15].

In general, the analog of Amitsur’s conjecture for arbitrary non-associative algebras and even for infinite dimensional Lie algebras is wrong. First, the codimension growth can be overexponential [32]. Second, the exponent of the codimension growth can be non-integer [13, 25, 26]. Third, in 2014 M.V. Zaicev constructed an example of an infinite dimensional non-associative algebra $A$ for which $\lim_{n \to \infty} \sqrt[n]{c_n(A)} = 1$ and $\lim_{n \to \infty} \sqrt[n]{c_n(A)} > 1$ [34].

Algebras endowed with an additional structure, e.g. a grading, an action of a group, a Lie algebra or a Hopf algebra, find their applications in many areas of mathematics and physics. Gradings on simple Lie and associative algebras have been studied extensively [4, 5, 6, 10, 27, 28, 29]. For algebras with an additional structure, it is natural to consider the corresponding polynomial identities.

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E. Aljadeff, A. Giambruno, and D. La Mattina [11, 2] proved that if an associative PI-algebra is graded by a finite group, then the graded PI-exponent exists and it is an integer, i.e. the graded analog of Amitsur’s conjecture holds for such algebras. The same is true for finite dimensional associative and Lie algebras graded by arbitrary groups [22, Theorem 3], [21, Theorem 1]. If $H$ is a finite dimensional semisimple Hopf algebra, then the codimensions of polynomial $H$-identities of any finite dimensional $H$-module associative or Lie algebra satisfy the analog of Amitsur’s conjecture too [20, Theorem 3], [21, Theorem 7]. If an algebra is graded by a semigroup, then its graded PI-exponent can be non-integer even if the algebra itself is finite dimensional and associative [23, Theorem 5] (see also [24]).

In order to embrace the cases when an algebra is graded by a semigroup or an infinite group, or a group is acting on an algebra not only by automorphisms, but by anti-automorphisms too, it is useful to consider more general additional structures. Although polynomial identities in algebras endowed with an arbitrary set of multilinear operations were considered in the literature (see e.g. [31, Section 49.2]), it turns out that so-called generalized $H$-actions (see the definition in Section 2) where $H$ is an arbitrary associative algebra with 1 are the most adequate for our purposes. Probably, the first who used such actions and studied polynomial $H$-identities was Allan Berele [8, remark before Theorem 15] in 1996 (see also the paper of Yu. A. Bahturin and V. V. Linchenko [3]). The example constructed in [23, Theorem 5] shows that for generalized $H$-actions the exponent of the $H$-codimension growth can be non-integer even for finite dimensional $H$-simple associative algebras. Therefore, the natural question arises as to whether $H$-PI-exponent exists at all, at least in the case when the algebra is $H$-simple.

In 2012 A. Giambruno and M. V. Zaicev proved the existence of the ordinary PI-exponent for any finite dimensional simple algebra not necessarily associative [18, Theorem 3]. Recently D. Repovš and M. V. Zaicev proved the existence of the graded PI-exponent for finite dimensional graded-simple algebras graded by commutative semigroups [30, Theorem 2].

In the present article we combine A. Giambruno and M. V. Zaicev’s techniques with the techniques of generalized $H$-actions and show that for any finite dimensional $H$-simple algebra with a generalized $H$-action there exists an $H$-PI-exponent (Theorem 6.1). This enables to prove (see Corollary 6.2) the existence of the graded PI-exponent for any finite dimensional graded-simple algebra graded in a very general sense (not necessary by a semigroup, see the precise definition of such a grading in Example 2.3). Note that the notion of an $H$-simple algebra is much wider than the notion of a simple algebra since, e.g., an $H$-simple associative or Lie algebra is not even necessarily semisimple.

One of the important steps in the proof of Theorem 6.1 is Theorem 5.5 where we show that $H$-colengths of a finite dimensional algebra with a generalized $H$-action are polynomially bounded (see Corollary 5.6 for the analog in the graded case).

Polynomial $H$-identities and graded polynomial identities are elements of the algebras $F\{X \mid H\}$ and $F\{X^{T\text{-gr}}\}$ defined in Sections 3 and 4 respectively. In fact, if $H$ is an arbitrary unital associative algebra and $T$ is an arbitrary set, then neither $F\{X \mid H\}$ is an algebra with a generalized $H$-action, nor $F\{X^{T\text{-gr}}\}$ is a $T$-graded algebra (which, however, does not prevent studying polynomial $H$-identities in algebras with generalized $H$-actions and graded polynomial identities in $T$-graded algebras at all). In Section 7 we show that if we enlarge the categories of algebras in a proper way, then both $F\{X \mid H\}$ and $F\{X^{T\text{-gr}}\}$ will correspond to free-forgetful adjunctions.

2. Algebras with a generalized $H$-action

Let $H$ be an arbitrary associative algebra with 1 over a field $F$. We say that a (not necessarily associative) algebra $A$ is a algebra with a generalized $H$-action if $A$ is a left $H$-module and for every $h \in H$ there exist some $k \in \mathbb{N}$ and some $h_1^i, h_2^i, h_3^i, h_4^i \in H$, $1 \leq i \leq k$,
such that
\[ h(ab) = \sum_{i=1}^{k} \left( (h_i' a)(h_i'' b) + (h_i'' b)(h_i''' a) \right) \text{ for all } a, b \in A. \] (2.1)

Equivalently, there exist linear maps \( \Delta, \Theta : H \to H \otimes H \) (not necessarily coassociative) such that
\[ h(ab) = \sum \left( (h(1)_1 a)(h(2)_2 b) + (h(1)_2 b)(h(2)_1 a) \right) \text{ for all } a, b \in A. \]

(Here we use the notation \( \Delta(h) = \sum h(1) \otimes h(2) \) and \( \Theta(h) = \sum h(1)_1 \otimes h(2)_2 \).)

**Example 2.1.** An algebra \( A \) over a field \( F \) is a (left) \( H \)-module algebra for some Hopf algebra \( H \) if \( A \) is endowed with a structure of a (left) \( H \)-module such that \( h(ab) = (h(1)_1 a)(h(2)_2 b) \) for all \( h \in H, a, b \in A \). Here we use Sweedler’s notation \( \Delta(h) = \sum h(1) \otimes h(2) \) where \( \Delta \) is the comultiplication in \( H \) and the symbol of the sum is omitted. If \( A \) is an \( H \)-module algebra, then \( A \) is an algebra with a generalized \( H \)-action.

**Example 2.2.** Recall that if \( T \) is a semigroup, then the semigroup algebra \( FT \) over a field is the vector space with the formal basis \( (t)_{t \in T} \) and the multiplication induced by the one in \( T \). Let \( A \) be an associative algebra with an action of a semigroup \( T \) by endomorphisms and anti-endomorphisms. Then \( A \) is an algebra with a generalized \( FT \)-action.

**Example 2.3.** Let \( A = \bigoplus_{t \in T} A^{(t)} \) be a graded algebra for some set of indices \( T \), i.e. for every \( s, t \in T \) there exists \( r \in T \) such that \( A^{(s)} A^{(t)} \subseteq A^{(r)} \). Denote this grading by \( \Gamma \). Note that \( \Gamma \) defines on \( T \) a partial operation \( \ast \) with the domain \( T_0 := \{(s, t) \mid A^{(s)} A^{(t)} \neq 0\} \) by \( s \ast t := r \). Consider the algebra \( F^T \) of all functions from \( T \) to \( F \) with pointwise operations. Then \( F^T \) acts on \( A \) naturally: \( ha = h(t)a \) for all \( a \in A^{(t)} \). Denote by \( \zeta : F^T \to \text{End}(A) \) the corresponding homomorphism. Let \( h_t(s) := \{ 1 \text{ if } s = t, 0 \text{ if } s \neq t \} \). If the support
\[ \text{supp} \Gamma := \{ t \in T \mid A^{(t)} \neq 0 \} \]
of \( \Gamma \) is finite, \( T_0 \) is finite too and we have
\[ h_r(ab) = \sum_{(s, t) \in T_0} h_s(a)h_t(b). \] (2.2)

(Since the expression is linear in \( a \) and \( b \), it is sufficient to check it only for homogeneous \( a, b \).) In this case \((h_t)_{t \in \text{supp} \Gamma} \) generates \( F^T \) modulo \( \ker \zeta \) as an \( F \)-vector space and, by the linearity, \( (2.2) \) implies \( (2.1) \) for every \( h \in F^T \). Therefore, in the case of a finite support of the grading, \( A \) is an algebra with a generalized \( F^T \)-action.

Let \( A \) be an algebra with a generalized \( H \)-action for some associative algebra \( H \) with 1 over a field \( F \). We say that a subspace \( V \subseteq A \) is invariant under the \( H \)-action if \( HV = V \), i.e. \( V \) is an \( H \)-submodule. If \( A^2 \neq 0 \) and \( A \) has no non-trivial two-sided \( H \)-invariant ideals, we say that \( A \) is \( H \)-simple.

3. Polynomial \( H \)-identities

Let \( F \) be a field and let \( Y \) be a set. Denote by \( F\{Y\} \) the absolutely free non-associative algebra on the set \( Y \), i.e. the algebra of all non-associative polynomials in variables from \( Y \) and coefficients from the field \( F \). Then \( F\{Y\} = \bigoplus_{n=1}^{\infty} F\{Y\}^{(n)} \) where \( F\{Y\}^{(n)} \) is the linear span of all monomials of total degree \( n \). Let \( H \) be an associative algebra with 1 over \( F \).

Consider the algebra
\[ F\{Y|H\} := \bigoplus_{n=1}^{\infty} H^\otimes n \otimes F\{Y\}^{(n)} \]
with the multiplication \((u_1 \otimes w_1)(u_2 \otimes w_2) := (u_1 \otimes u_2) \otimes w_1 w_2\) for all \(u_1 \in H^\otimes j, u_2 \in H^\otimes k,\) \(w_1 \in F\{Y\}^{(j)}, w_2 \in F\{Y\}^{(k)}\). We use the notation

\[
y_{i_1}^{h_1} y_{i_2}^{h_2} \cdots y_{i_n}^{h_n} := (h_1 \otimes h_2 \otimes \cdots \otimes h_n) \otimes y_{i_1} y_{i_2} \cdots y_{i_n}
\]

(the arrangements of brackets on \(y_j\) and on \(y_{j_i}^{h_i}\) are the same). Here \(h_1 \otimes h_2 \otimes \cdots \otimes h_n \in H^\otimes n,\) \(y_1, y_2, \ldots, y_n \in Y\). In addition, we identify \(Y\) with the subset \(\{y^{i_1}_{j_1} \mid y \in Y\} \subset F\{Y\}/H\).

Note that if \((\gamma_\beta)_{\beta \in \Lambda}\) is a basis in \(H\), then \(F\{Y\}/H\) is isomorphic to the absolutely free non-associative algebra over \(F\) with free formal generators \(y^{\gamma_\beta}, \beta \in \Lambda, y \in Y\). We call \(F\{Y\}/H\) the absolutely free non-associative algebra on \(Y\) with symbols from \(H\).

Below we consider \(F\{X\}/H\) where \(X := \{x_1, x_2, x_3, \ldots\}\). The elements of \(F\{X\}/H\) are called \(H\)-polynomials.

Let \(A\) be an algebra over \(F\) with a generalized \(H\)-action. Any map \(\psi: X \to A\) has a unique homomorphic extension \(\bar{\psi}: F\{X\}/H \to A\) such that \(\bar{\psi}(x_i) = h\psi(x_i)\) for all \(i \in \mathbb{N}\) and \(h \in H\). An \(H\)-polynomial \(f \in F\{X\}/H\) is a polynomial \(H\)-identity of \(A\) if \(\bar{\psi}(f) = 0\) for all maps \(\psi: X \to A\). In other words, \(f(x_1, x_2, \ldots, x_n)\) is an \(H\)-identity of \(A\) if and only if \(f(a_1, a_2, \ldots, a_n) = 0\) for any \(a_i \in A\). In this case we write \(f \equiv 0\). The set \(\text{Id}_H(A)\) of all polynomial \(H\)-identities of \(A\) is an ideal of \(F\{X\}/H\).

Denote by \(W_n^H, n \in \mathbb{N}\), the space of all multilinear non-associative \(H\)-polynomials in \(x_1, \ldots, x_n\), i.e.

\[
W_n^H = (x_{\sigma(1)}^{h_1} x_{\sigma(2)}^{h_2} \cdots x_{\sigma(n)}^{h_n} \mid h_i \in H, \sigma \in \text{S}_n) \subset F\{X\}/H.
\]

(We consider all possible arrangements of brackets.) Then the number \(c_n^H(A) := \dim\left(\frac{W_n^H}{\text{Id}_H(A)}\right)\) is called the \(n\)th \(H\)-codimension of polynomial \(H\)-identities or the \(n\)th \(H\)-codimension of \(A\). If \(f \in W_n^H\), then its image in \(\frac{W_n^H}{\text{Id}_H(A)}\) is denoted by \(\bar{f}\). The limit

\[
\text{Plexp}^H(A) := \lim_{n \to \infty} \sqrt[n]{c_n^H(A)},
\]

if it exists, is called the \(H\)-PI-exponent of \(A\).

One of the main tools in the investigation of polynomial identities is provided by the representation theory of symmetric groups. The symmetric group \(\text{S}_n\) acts on the space \(\frac{W_n^H}{\text{Id}_H(A)}\) by permuting the variables. If the characteristic of the base field \(F\) is zero, then irreducible \(FS_n\)-modules are described by partitions \(\lambda = (\lambda_1, \ldots, \lambda_t) \vdash n\) and their Young diagrams \(D_\lambda\). The character \(\chi_n^H(A)\) of the \(FS_n\)-module \(\frac{W_n^H}{\text{Id}_H(A)}\) is called the \(n\)th cocharacter of polynomial \(H\)-identities of \(A\). We can rewrite it as a sum

\[
\chi_n^H(A) = \sum_{\lambda \vdash n} m(A, H, \lambda) \chi(\lambda)
\]

of irreducible characters \(\chi(\lambda)\). The number \(\ell_n^H(A) := \sum_{\lambda \vdash n} m(A, H, \lambda)\) is called the \(n\)th colength of polynomial \(H\)-identities of \(A\). Let \(e_{T_\lambda} = a_{T_\lambda} b_{T_\lambda}\) and \(e_{T_\lambda}^* = b_{T_\lambda} a_{T_\lambda}\) where \(a_{T_\lambda} = \sum_{\pi \in \text{R}_{T_\lambda}} \pi\) and \(b_{T_\lambda} = \sum_{\sigma \in \text{C}_{T_\lambda}} (\text{sign} \sigma) \sigma\) be the Young symmetrizers corresponding to a Young tableau \(T_\lambda\). Then \(M(\lambda) = FS_n e_{T_\lambda} \cong FS_n e_{T_\lambda}^*\) is an irreducible \(FS_n\)-module corresponding to a partition \(\lambda \vdash n\). We refer the reader to [7, 9, 17] for an account of \(S_n\)-representations and their applications to polynomial identities.

Remark 3.1. If \(A\) is an ordinary algebra, then the ordinary polynomial identities and cocharacters of \(A\) can be defined as \(H\)-identities and \(H\)-cocharacters for \(H = F\) acting on \(A\) trivially:

\[
W_n := W_n^F, \quad \text{Id}(A) := \text{Id}_F(A), \quad c_n(A) := c_n^F(A), \quad \chi_n(A) := \chi_n^F(A), \quad m(A, \lambda) := m(A, F, \lambda), \quad \ell_n(A) := \ell_n^F(A).
\]

Remark 3.2. Note that here we do not consider any \(H\)-action on \(F\{Y\}/H\) itself. However \(F\{-|H\}\) can be viewed as a free functor if we enlarge the category of algebras with a generalized \(H\)-action in a proper way (see Section 7.2).
Example 4.1. Consider the multiplicative semigroup on a set $I_\mathbb{F}$ consisting of products of the variables $f \in \mathbb{F}$ of the sets $X$. Let $H$ be a field. Analogous remarks can be made in the case when $H$ is associative. One can analogously construct the free associative algebra $F\langle X \rangle$ on $X$ with symbols from $H$ (see [20, Section 3.1]) and treat polynomial $H$-identities as elements of an ideal $\text{Id}_\mathbb{H}^H(A)$ of $F\langle X \rangle$. We have $\text{Id}_\mathbb{H}^H(A) = \langle h^2 \rangle$ for any $H$-polynomial such that $f^2 = h$. If char $F = 0$, then $f^2 \in \text{Id}_\mathbb{H}^H(A)$ for all $\lambda \vdash n$, then $FS_n\overline{f}$ is a non-zero homomorphic image of the irreducible $FS_n$-module $M(\lambda)$. As a consequence, $FS_n\overline{f} \cong M(\lambda)$ too. We will use this property in Section 6.

Remark 3.3. Suppose $A$ is associative. One can analogously construct the free associative algebra $F\langle X |H \rangle$ on $X$ with symbols from $H$ (see [20, Section 3.1]) and treat polynomial $H$-identities as elements of an ideal $\text{Id}_\mathbb{H}^H(A)$ of $F\langle X |H \rangle$. However, the map $x_i^s \mapsto x_i^s$, $i \in \mathbb{N}$, $h \in H$, induces an isomorphism $F\langle X |H \rangle / \text{Id}_\mathbb{H}^H(A) \cong F\langle X |H \rangle / \text{Id}_\mathbb{H}^H(A)$ of algebras and isomorphisms $F\langle X |H \rangle / \text{Id}_\mathbb{H}^H(A) \cong P^{\mathbb{H}}_{\mathbb{H}} / \text{Id}_\mathbb{H}^H(A)$ of $FS_n$-modules where $n \in \mathbb{N}$ and $P^{\mathbb{H}}$ is the $FS_n$-module of associative $H$-polynomials multilinear in $x_1, x_2, \ldots, x_n$. In particular, the definitions of codimensions and cocharacters do not depend on whether we use $F\langle X |H \rangle$ or $F\langle X |H \rangle$. Analogous remarks can be made in the case when $A$ is a Lie algebra (see [21, Section 1.3]).

4. Graded polynomial identities

Let $T$ be a set and let $F$ be a field.

Consider the absolutely free non-associative algebra $F\{X^{T-gr}\}$ on the disjoint union $X^{T-gr} := \bigcup_{t \in T} X^{(t)}$ of the sets $X^{(t)} = \{x_1^{(t)}, x_2^{(t)}, \ldots\}$.

We say that $f = f(x_1^{(t_1)}, \ldots, x_s^{(t_s)})$ is a graded polynomial identity for a $T$-graded algebra $A = \bigoplus_{t \in T} A^{(t)}$ and write $f \equiv 0$ if $f(a_1^{(t_1)}, \ldots, a_s^{(t_s)}) = 0$ for all $a_j^{(t_j)} \in A^{(t_j)}$, $1 \leq j \leq s$. The set $\text{Id}^{T-gr}(A)$ of graded polynomial identities of $A$ is an ideal of $F\{X^{T-gr}\}$.

Example 4.1. Consider the multiplicative semigroup $T = \mathbb{Z}_2 = \{0, 1\}$ and the $T$-grading $\text{UT}_2(F) = \text{UT}_2(F)^{(0)} \oplus \text{UT}_2(F)^{(1)}$ on the algebra $\text{UT}_2(F)$ of upper triangular $2 \times 2$ matrices over a field $F$ defined by $\text{UT}_2(F)^{(1)} = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$ and $\text{UT}_2(F)^{(0)} = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. We have

$$[x^{(1)}, y^{(1)}] := x^{(1)}y^{(1)} - y^{(1)}x^{(1)} \in \text{Id}^{T-gr}(\text{UT}_2(F))$$

and $x^{(0)}y^{(0)} \in \text{Id}^{T-gr}(\text{UT}_2(F))$.

Let

$$W_n^{T-gr} := \langle x_1^{(t_1)}x_2^{(t_2)} \cdots x_s^{(t_s)} | t_i \in T, \sigma \in S_n \rangle_{F} \subset F\{X^{T-gr}\}$$
(with all possible arrangements of brackets), \( n \in \mathbb{N} \). The number
\[
c_n^{T,\text{gr}}(A) := \dim \left( \frac{W_n^{T,\text{gr}}}{W_n^{T,\text{gr}} \cap \text{Id}^{T,\text{gr}}(A)} \right)
\]
is called the \( n \text{th} \) \textit{codimension of graded polynomial identities} or the \( n \text{th} \) \textit{graded codimension} of \( A \).

The symmetric group \( S_n \) acts on the space \( \frac{W_n^{T,\text{gr}}}{W_n^{T,\text{gr}} \cap \text{Id}^{T,\text{gr}}(A)} \) by permuting the variables within each \( \chi^{(t)} \):
\[
\sigma x^{(t_1)}_i \cdots x^{(t_n)}_i := x^{(t_1)}_{\sigma(i_1)} \cdots x^{(t_n)}_{\sigma(i_n)}
\]
for \( n \in \mathbb{N}, \sigma \in S_n, 1 \leq i_k \leq n, 1 \leq k \leq n \). The character \( \chi^{T,\text{gr}}(A) \) of the \( FS_n \)-module \( \frac{W_n^{T,\text{gr}}}{W_n^{T,\text{gr}} \cap \text{Id}^{T,\text{gr}}(A)} \) is called the \( n \text{th} \) \textit{cocharacter} of graded polynomial identities of \( A \). If \( |F| = 0, \) we can rewrite it as a sum
\[
\chi^{T,\text{gr}}(A) = \sum_{\lambda \vdash n} m(A, T, \lambda, \chi) \chi(\lambda)
\]
of irreducible characters \( \chi(\lambda) \). The number \( \ell_n^{T,\text{gr}}(A) := \sum_{\lambda \vdash n} m(A, T, \lambda, \chi) \) is called the \( n \text{th} \) \textit{co-length} of graded polynomial identities of \( A \).

The proposition below provides a relation between the ordinary and the graded codimensions.

**Proposition 4.2.** Let \( A \) be a \( T \)-graded algebra over a field \( F \) for some set \( T \) not necessarily finite. Then \( c_n(A) \leq c_n^{T,\text{gr}}(A) \). If \( T \) is finite, then \( c_n^{T,\text{gr}}(A) \leq |T|^n c_n(A) \) for all \( n \in \mathbb{N} \).

**Proof.** Let \( t_1, \ldots, t_n \in T \). Denote by \( W_{t_1, \ldots, t_n} \) the vector space of multilinear non-associative polynomials in \( x^{(t_1)}_1, \ldots, x^{(t_n)}_n \). Then \( W_n^{T,\text{gr}} = \bigoplus_{t_1, \ldots, t_n \in T} W_{t_1, \ldots, t_n} \). Moreover
\[
\frac{W_n^{T,\text{gr}}}{W_n^{T,\text{gr}} \cap \text{Id}^{T,\text{gr}}(A)} \cong \bigoplus_{t_1, \ldots, t_n \in T} \frac{W_{t_1, \ldots, t_n}}{W_{t_1, \ldots, t_n} \cap \text{Id}^{T,\text{gr}}(A)} \tag{4.1}
\]
since we can pick out the component corresponding to a given \( W_{t_1, \ldots, t_n} \) in a graded polynomial identity by putting \( x^{(t)}_i = 0 \) for all \( 1 \leq i \leq n \) and \( t \neq t_i \). Note that \((4.1)\) holds over a field of any characteristic.

Let \( f_1, \ldots, f_{c_n(A)} \) be a basis in \( \frac{W_n}{W_n \cap \text{Id}^{T,\text{gr}}(A)} \) where \( f_i \in W_n \). Then for every monomial \( w = x_{\sigma(1)} \cdots x_{\sigma(n)} \) (with some arrangement of brackets), \( \sigma \in S_n \), there exist \( \alpha_{w,i} \in F \) such that
\[
x_{\sigma(1)} \cdots x_{\sigma(n)} - \sum_{i=1}^{c_n(A)} \alpha_{w,i} f_i(x_1, \ldots, x_n) \in \text{Id}(A).
\]
For every choice of \( t_1, \ldots, t_n \in T \) we replace \( x_i \) with \( x^{(t_i)}_i \) and get
\[
x^{(t_{\sigma(1)})}_{\sigma(1)} \cdots x^{(t_{\sigma(n)})}_{\sigma(n)} - \sum_{i=1}^{c_n(A)} \alpha_{w,i} f_i \left( x^{(t_1)}_1, \ldots, x^{(t_n)}_n \right) \in \text{Id}^{T,\text{gr}}(A)
\]
and
\[
\frac{W_n^{T,\text{gr}}}{W_n^{T,\text{gr}} \cap \text{Id}^{T,\text{gr}}(A)} = \left\langle f_i \left( x^{(t_1)}_1, \ldots, x^{(t_n)}_n \right) \right| 1 \leq i \leq c_n(A), t_1, \ldots, t_n \in T \right\rangle_F.
\]
This implies the upper bound for \( c_n^{T,\text{gr}}(A) \) in the case when \( T \) is finite.

In order to get the lower bound for \( c_n^{T,\text{gr}}(A) \), for a given \( n \)-tuple \( (t_1, \ldots, t_n) \in T^n \) we consider the map \( \varphi_{t_1, \ldots, t_n} : W_n \to \frac{W_n^{T,\text{gr}}}{W_n^{T,\text{gr}} \cap \text{Id}^{T,\text{gr}}(A)} \) where \( \varphi_{t_1, \ldots, t_n}(f) \) is the image of
Thus \( \psi \xi \) is an arbitrary graded polynomial identity \( f = f(x_1, \ldots, x_n) \in W_n \). The multilinearity of \( f \) implies that \( f(x_1, \ldots, x_n) \equiv 0 \) is an ordinary polynomial identity if and only if

\[
f\left(x_1^{(t_1)}, \ldots, x_n^{(t_n)}\right) \equiv 0
\]

is a graded polynomial identity for every \( t_1, \ldots, t_n \in T \). In other words, \( W_n \cap \text{Id}(A) = \bigcap_{(t_1,\ldots,t_n) \in T^n} \ker \varphi_{t_1,\ldots,t_n} \). Since \( W_n \) is a finite dimensional vector space, there exists a finite subset \( \Lambda \subseteq T^n \) such that \( W_n \cap \text{Id}(A) = \bigcap_{(t_1,\ldots,t_n) \in \Lambda} \ker \varphi_{t_1,\ldots,t_n} \).

Consider the embedding

\[
W_n \hookrightarrow W_n^{T,\text{gr}} = \bigoplus_{t_1,\ldots,t_n \in T} W_{t_1,\ldots,t_n}
\]

where the image of \( f(x_1, \ldots, x_n) \in W_n \) equals \( \sum_{(t_1,\ldots,t_n) \in \Lambda} f\left(x_1^{(t_1)}, \ldots, x_n^{(t_n)}\right) \). Then (4.1) and our choice of \( \Lambda \) imply that the induced map \( \frac{W_n}{W_n \cap \text{Id}(A)} \to \frac{W_n^{T,\text{gr}}}{W_n^{T,\text{gr}} \cap \text{Id}T^{\text{gr}}(A)} \) is an embedding and the lower bound follows.

The limit \( \text{PlExp}^{T,\text{gr}}(A) := \lim_{n \to \infty} \sqrt[n]{c_n^{T,\text{gr}}(A)} \) (if it exists) is called the graded PI-exponent of \( A \).

In Example 2.3 we have shown that each \( T \)-graded algebra \( A \) with a finite support is an algebra with a generalized \( F^T \)-action. The lemma below shows that instead of studying graded codimensions and cocharacters of \( A \) we can study codimensions and cocharacters of its polynomial \( F^T \)-identities.

**Lemma 4.3.** Let \( \Gamma : A = \bigoplus_{t \in T} A^{(t)} \) be a grading on an algebra \( A \) over a field \( F \) by a set \( T \) such that \( \text{supp} \Gamma \) is finite. Then \( c_n^{T,\text{gr}}(A) = c_n^{F^T}(A) \) and \( \chi_n^{T,\text{gr}}(A) = \chi_n^{F^T}(A) \) for all \( n \in \mathbb{N} \). If, in addition, \( \text{char} F = 0 \), we have \( \ell_n^{T,\text{gr}}(A) = \ell_n^{F^T}(A) \).

**Proof.** Let

\[
\xi : F\{X|F^T\} \to F\{X^{T,\text{gr}}\}
\]

be the algebra homomorphism defined by \( \xi(x_i^h) = \sum_{t \in \text{supp} \Gamma} h(t)x_i^{(t)} \), \( i \in \mathbb{N} \), \( h \in F^T \). Let \( f \in \text{Id}^{F^T}(A) \). Consider an arbitrary homomorphism \( \psi : F\{X^{T,\text{gr}}\} \to A \) such that \( \psi(x_i^{(t)}) \in A^{(t)} \) for all \( t \in T \) and \( i \in \mathbb{N} \). Then the algebra homomorphism \( \psi \xi : F\{X|F^T\} \to A \) satisfies the condition

\[
\psi \xi(x_i^h) = \sum_{t \in \text{supp} \Gamma} h(t)\psi\left(x_i^{(t)}\right) = h\left(\sum_{t \in \text{supp} \Gamma} \psi\left(x_i^{(t)}\right)\right) = h \psi \xi(x_i).
\]

Thus \( \psi \xi(f) = 0 \) and \( \xi(f) \in \text{Id}^{T,\text{gr}}(A) \). Hence \( \xi\left(\text{Id}^{F^T}(A)\right) \subseteq \text{Id}^{T,\text{gr}}(A) \). Denote by

\[
\tilde{\xi} : F\{X|F^T\}/\text{Id}^{F^T}(A) \to F\{X^{T,\text{gr}}\}/\text{Id}^{T,\text{gr}}(A)
\]

the homomorphism induced by \( \xi \).

Let

\[
\eta : F\{X^{T,\text{gr}}\} \to F\{X|F^T\}
\]

be the algebra homomorphism defined by \( \eta(x_i^{(t)}) = x_i^{ht} \) for all \( i \in \mathbb{N} \) and \( t \in T \). Consider an arbitrary graded polynomial identity \( f \in \text{Id}^{T,\text{gr}}(A) \). Let \( \psi : F\{X|F^T\} \to A \) be a homomorphism satisfying the condition \( \psi(x_i^{(t)}) = h \psi(x_i) \) for every \( i \in \mathbb{N} \) and \( h \in F^T \). Then for
any $i \in \mathbb{N}$ and $g, t \in T$ we have
\[
h_g \psi \eta \left( x_i^{(t)} \right) = h_g \psi (x_i^{ht}) = h_g h_t \psi (x_i) = \begin{cases} 0 & \text{if } g \neq t, \\ \psi \eta (x_i^{(t)}) & \text{if } g = t. \end{cases}
\]

Thus $\psi \eta (x_i^{(t)}) \in A^{(t)}$. Therefore, $\psi \eta (f) = 0$ and $\eta (\text{Id}^{T,gr} (A)) \subseteq \text{Id}^{F^T} (A)$. Denote by $\tilde{\eta} : F\{X^{T,gr}\}/\text{Id}^{T,gr} (A) \to F\{X|F^T\}/\text{Id}^{F^T} (A)$ the induced homomorphism.

Below we use the notation $\tilde{\phi}(t) = f + \text{Id}^{F^T} (A) \in F\{X|F^T\}/\text{Id}^{F^T} (A)$ for $f \in F\{X|F^T\}$ and $\tilde{\phi} = f + \text{Id}^{T,gr} (A) \in F\{X^{T,gr}\}/\text{Id}^{T,gr} (A)$ for $f \in F\{X^{T,gr}\}$. Observe that
\[
x_i^{ht} = \sum_{t \in \text{supp} \Gamma} h(t) x_i^{ht} \in \text{Id}^{F^T} (A)
\]
for every $h \in F^T$ and $i \in \mathbb{N}$. Hence
\[
\tilde{\eta} \tilde{\xi} \left( \bar{x}_i^h \right) = \tilde{\eta} \left( \sum_{t \in \text{supp} \Gamma} h(t) \bar{x}_i^{(t)} \right) = \sum_{t \in \text{supp} \Gamma} h(t) \bar{x}_i^{ht} = \bar{x}_i^h
\]
for every $h \in F^T$ and $i \in \mathbb{N}$. Thus $\tilde{\eta} \tilde{\xi} = \text{id}_{F\{X|F^T\}/\text{Id}^{F^T} (A)}$ since $F\{X|F^T\}/\text{Id}^{F^T} (A)$ is generated by $\bar{x}_i^h$ where $h \in F^T$ and $i \in \mathbb{N}$. Moreover $\tilde{\xi} \tilde{\eta} \left( \bar{x}_i^{(t)} \right) = \tilde{\xi} \left( x_i^{ht} \right) = \bar{x}_i^t$ for every $t \in \text{supp} \Gamma$ and $i \in \mathbb{N}$. Therefore, $\tilde{\xi} \tilde{\eta} = \text{id}_{F\{X^{T,gr}\}/\text{Id}^{T,gr} (A)}$ and $F\{X^{T,gr}\}/\text{Id}^{T,gr} (A) \cong F\{X|F^T\}/\text{Id}^{F^T} (A)$ as algebras. The restriction of $\tilde{\xi}$ provides the isomorphism of the $FS_n$-modules $\frac{W^T_n}{W^T_n \cap \text{Id}^{F^T} (A)}$ and $\frac{W^{T,gr}_n}{W^{T,gr}_n \cap \text{Id}^{T,gr} (A)}$. Hence
\[
c_n^{F^T} (A) = \dim \frac{W^T_n}{W^T_n \cap \text{Id}^{F^T} (A)} = \dim \frac{W^{T,gr}_n}{W^{T,gr}_n \cap \text{Id}^{T,gr} (A)} = c_n^{T,gr} (A)
\]
and $\chi_n^{T,gr} (A) = \chi_n^{F^T} (A)$ for all $n \in \mathbb{N}$. If, in addition, $\text{char} F = 0$, we have $\ell_n^{T,gr} (A) = \ell_n^{F^T} (A)$. \hfill \Box

**Remark 4.4.** Again, analogously to Remark 3.4 in the case when $A$ is an associative or Lie algebra, one can use, respectively, free associative or Lie graded algebras, however the graded codimensions will be the same.

### 5. Upper Bound for $H$-co-lengths

Throughout Sections 3 and 4 we assume that the characteristic of the base field $F$ is 0. In [11, Theorem 1], A. Giambruno, S. P. Mishchenko, and M. V. Zaicev proved that
\[
\ell_n (A) = \sum_{\lambda \vdash n} m (A, \lambda) \leq (\dim A) (n + 1)^{(\dim A)^2 + \dim A}
\]
for all $n \in \mathbb{N}$.

It turns out that for $H$-codimensions of finite dimensional algebras with a generalized $H$-action we have the same upper bound (Theorem 5.5 below).

Let $A$ be a finite dimensional algebra with a generalized $H$-action for some associative algebra $H$ with 1.

**Lemma 5.1.** Let $C$ be a unital commutative associative algebra over $F$. Define on $A \otimes C$ the structure of an algebra with a generalized $H$-action by $h(a \otimes c) := ha \otimes c$ for $a \in A$ and $c \in C$. Then $\text{Id}^H (A \otimes C) = \text{Id}^H (A)$. 
Proof. Since $C$ is unital, $A \otimes C$ contains an $H$-invariant subalgebra isomorphic to $A$ and therefore $\text{Id}^H(A \otimes C) \subseteq \text{Id}^H(A)$. The proof of the converse inclusion is completely analogous to the case of associative algebras without an action \cite[Lemma 1.4.2]{17}. □

Let $a_1, \ldots, a_r$ be a basis in $A$. Fix a number $k \in \mathbb{N}$. Denote by $F[\xi_{ij} \mid 1 \leq i \leq s, 1 \leq j \leq k]$ the unital algebra of commutative associative polynomials in the variables $\xi_{ij}$ with coefficients from $F$. The algebra $A \otimes F[\xi_{ij} \mid 1 \leq i \leq s, 1 \leq j \leq k]$ is again an algebra with a generalized $H$-action $h(a \otimes f) := ha \otimes f$ for $a \in A$ and $f \in F[\xi_{ij} \mid 1 \leq i \leq s, 1 \leq j \leq k]$. Denote by $\tilde{A}_k$ the intersection of all $H$-invariant subalgebras of $A \otimes F[\xi_{ij} \mid 1 \leq i \leq s, 1 \leq j \leq k]$ containing the elements $\xi_j := \sum_{i=1}^s a_i \otimes \xi_{ij}$ where $1 \leq j \leq k$. (The symbols $\xi_j$ represent “generic elements” of the algebra $A$.)

**Lemma 5.2.** Let $f = f(x_1, \ldots, x_k) \in F\{X[H]\}$. Then $f \in \text{Id}^H(A)$ if and only if $f(\xi_1, \ldots, \xi_k) = 0$ in $\tilde{A}_k$.

**Proof.** Lemma 5.1 implies
\[
\text{Id}^H(A) = \text{Id}^H(A \otimes F[\xi_{ij} \mid 1 \leq i \leq s, 1 \leq j \leq k]) \subseteq \text{Id}^H(\tilde{A}_k).
\]
In particular, $f \in \text{Id}^H(A)$ implies $f(\xi_1, \ldots, \xi_k) = 0$.

Conversely, suppose $f(\xi_1, \ldots, \xi_k) = 0$. We claim that $f(b_1, \ldots, b_k) = 0$ for all $b_j \in A$. Indeed, $b_j = \sum_{i=1}^s \alpha_{ij} a_i$ for some $\alpha_{ij} \in F$. Consider the homomorphism
\[
\varphi : A \otimes F[\xi_{ij} \mid 1 \leq i \leq s, 1 \leq j \leq k] \to A
\]
of algebras and $H$-modules defined by $a \otimes \xi_{ij} \mapsto \alpha_{ij} a$ for all $a \in A$. Then
\[
f(b_1, \ldots, b_k) = f(\varphi(\xi_1), \ldots, \varphi(\xi_k)) = \varphi(f(\xi_1, \ldots, \xi_k)) = 0
\]
and $f \in \text{Id}^H(A)$. □

**Lemma 5.3.** Denote by $R_{kn}$ the linear span in $\tilde{A}_k$ of all products $(h_1 \xi_{i_1}) \cdots (h_n \xi_{i_n})$ where $h_j \in H$ and $1 \leq i_j \leq k$ for $1 \leq j \leq n$. Then $\dim R_{kn} \leq (\dim A)(n+1)^{k \dim \! A}$ for all $n \in \mathbb{N}$.

**Proof.** The space $R_{kn} \subseteq A \otimes F[\xi_{ij} \mid 1 \leq i \leq s, 1 \leq j \leq k]$ is a subspace of the linear span of elements $a_\ell \otimes \prod_{1 \leq i \leq s} \xi_{i_j}^{s_{ij}}$ where $1 \leq \ell \leq s = \dim A$, $s_{ij} \in \mathbb{Z}_+$, $\sum_{1 \leq j \leq k} s_{ij} = n$. The number of such elements does not exceed $(\dim A)(n+1)^{k \dim \! A}$, and we get the desired upper bound for $\dim R_{kn}$. □

Now we show that all irreducible $FS_n$-submodules that occur in the decomposition of $W^H_{n+1} / \text{Id}^H(A)$ with nonzero multiplicities, correspond to Young diagrams of height less than or equal to $\dim A$.

**Lemma 5.4.** Let $\lambda \vdash n$, $n \in \mathbb{N}$. Suppose $\lambda_{(\dim A)+1} > 0$. Then $m(A, H, \lambda) = 0$.

**Proof.** It is sufficient to prove that $e_{\lambda_\alpha}^* f \in \text{Id}^H(A)$ for all $f \in W^H_{n+1}$. Fix some basis of $A$. Since polynomials are multilinear, it is sufficient to substitute only basis elements. Note that $e_{\lambda_\alpha}^* = b_{\lambda_\alpha} a_{\lambda_\alpha}$ where $b_{\lambda_\alpha}$ alternates the variables of each column of $T_{\lambda_\alpha}$. Hence if we make a substitution and $e_{\lambda_\alpha}^* f$ does not vanish, this implies that different basis elements are substituted for the variables of each column. But if $\lambda_{(\dim A)+1} > 0$, then the length of the first column is greater than $\dim A$. Therefore, $e_{\lambda_\alpha}^* f \in \text{Id}^H(A)$. □

Now we can prove the main result of this section.
Theorem 5.5. Let $A$ be a finite dimensional algebra with a generalized $H$-action for some associative algebra $H$ with 1 over a field $F$ of characteristic 0. Then
\[ \ell_n^H(A) \leq (\dim A)(n+1)^{\dim A} \]
for all $n \in \mathbb{N}$.

Proof. Fix for each partition $\lambda \vdash n$ a Young tableau $T_\lambda$ of the shape $\lambda$. Then for $\lambda, \mu \vdash n$ we have $e_T^H F_{S^\mu} e_T^\mu = \begin{cases} F e_{\lambda} & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu. \end{cases}$ (See e.g. [11] Lemma 4.23 and Exercise 4.24.)

Hence the multiplicity $m(A, H, \lambda)$ of $M(\lambda) = F S^\mu e_{T_\lambda} \in W_n^H \cap \id^n(A)$ equals $\dim e_T^H F_{S^\mu} e_T^\mu \in W_n^H$. In other words, $m(A, H, \lambda) = m$ equals the maximal number of $H$-polynomials $f_1, \ldots, f_m \in W_n^H$ such that $g = \alpha_1 f_1 + \cdots + \alpha_m f_m \in \id^H(A)$ for some $\alpha_1, \ldots, \alpha_m \in F$ always implies $\alpha_1 = \cdots = \alpha_m = 0$. Denote by $k_{ij}$ the number in the $(i, j)$th box of $T_\lambda$. Then for a fixed $i$ each $e_T^H f_i$ is symmetric in the variables $x_{k_{i1}}, \ldots, x_{k_{i\lambda}}$. Applying the linearization procedure (see e.g. [17] Section 1.3), we obtain that $g$ is a polynomial $H$-identity if and only if $\tilde{g}$ is a polynomial $H$-identity, where $\tilde{g}$ is obtained from $g$ by the substitution $x_{k_{ij}} \mapsto x_i$ for all $i$ and $j$. Denote the number of rows in $T_\lambda$ by $k$. By Lemma 5.4, we may assume that $k \leq \dim A$. The $H$-polynomial $\tilde{g}$ depends on the variables $x_1, \ldots, x_k$ and Lemma 5.2 implies that $\tilde{g} \in \id^H(A)$ if and only if $\tilde{g}(\xi_1, \ldots, \xi_k) = 0$ in $A_k$. Note that $\tilde{g}(\xi_1, \ldots, \xi_k) = \alpha_1 u_1 + \cdots + \alpha_m u_m$ where $u_\ell$ is the value of $e_T^H f_\ell$ under the substitution $x_{k_{ij}} \mapsto \xi_i$ for $1 \leq i \leq k$ and $1 \leq j \leq \lambda_i$. Hence all $u_\ell \in R_k$ and if $m > (\dim A)(n+1)^{\dim A}$, then by Lemma 5.3 for any choice of $f_\ell$ the elements $u_\ell$ are linearly dependant and $\tilde{g}(\xi_1, \ldots, \xi_k) = \alpha_1 u_1 + \cdots + \alpha_m u_m = 0$ for some nontrivial $\alpha$. In particular, $\alpha_1 e_T^H f_1 + \cdots + \alpha_m e_T^H f_m \in \id^H(A)$ and $(m, A, H, \lambda) < m$. Hence for any $\lambda \vdash n$ we have $m(A, H, \lambda) \leq (\dim A)(n+1)^{\dim A} \leq (\dim A)(n+1)^{\dim A}^2$. Since the number of all partitions $\lambda \vdash n$ of height not greater than $\dim A$ does not exceed $n^{\dim A}$, we get the desired upper bound for $\ell_n^H(A)$. \hfill \Box

By Lemma 1.3 above, if a finite dimensional algebra $A$ is graded by a set $T$, then the colengths $\ell_n^{T, gr}(A)$ of graded polynomial identities of $A$ are equal to the $T$-colengths $\ell_n^T(A)$. Thus we immediately get the following corollary of Theorem 5.5.

Corollary 5.6. Let $A$ be a finite dimensional algebra over a field of characteristic 0 graded by a set $T$. Then
\[ \ell_n^{T, gr}(A) \leq (\dim A)(n+1)^{\dim A} \]
for all $n \in \mathbb{N}$.

6. Existence of the $H$-PI-exponent for $H$-simple algebras

In Theorem 6.1 below we prove that for every finite dimensional $H$-simple algebra there exists an $H$-PI-exponent.

Let $\Phi(x_1, \ldots, x_s) = \frac{1}{x_1^{x_1} \cdots x_s^{x_s}}$ for $x_1, \ldots, x_s > 0$. Since $\lim_{x \to +0} x^x = 1$, we may assume that $\Phi$ is a continuous function for $x_1, \ldots, x_s > 0$.

Theorem 6.1. Let $A$ be a finite dimensional $H$-simple algebra for some associative algebra $H$ with 1 over a field $F$ of characteristic 0, $\dim A = s$. Let
\[ d(A) := \lim_{n \to \infty} \max_{m(A, H, \lambda) \neq 0} \Phi \left( \frac{\lambda_1}{n}, \ldots, \frac{\lambda_s}{n} \right). \]

Then there exists
\[ \text{PIexp}^H(A) := \lim_{n \to \infty} \sqrt[n]{\ell_n^H(A)} = d(A). \]
Theorem 6.1 will be proved below. Again, combining Theorem 6.1 with Lemma 4.3 we get:

**Corollary 6.2.** Let \( A \) be a finite dimensional algebra over a field of characteristic 0 graded by a set \( T \) such that \( A \) does not have non-trivial graded ideals. Then there exists \( \text{Pl}exp^{F \oplus T}(A) = \lim_{n \to \infty} \sqrt[n]{c_n^{T \cdot \text{gr}}(A)} \).

First we prove that the \( H \)-codimension sequence is non-decreasing for any \( H \)-simple algebra.

**Lemma 6.3.** Let \( A \) be an \( H \)-simple algebra for some associative algebra \( H \) with 1 over any field \( F \). Then \( c_n^H(A) \leq c_{n+1}^H(A) \) for all \( n \in \mathbb{N} \).

**Proof.** Fix some \( n \in \mathbb{N} \). Let \( f_1(x_1, \ldots, x_n), \ldots, f_{c_n^H(A)}(x_1, \ldots, x_n) \) be such \( H \)-polynomials that their images form a basis in \( W_n^{H} / \text{Id}^H(A) \). Suppose the \( H \)-polynomials \( f_1(x_1, \ldots, x_n x_{n+1}), \ldots, f_{c_n^H(A)}(x_1, \ldots, x_n x_{n+1}) \) are linearly dependent modulo \( \text{Id}^H(A) \). Then there exist \( \alpha_1, \ldots, \alpha_{c_n^H(A)} \in F \) such that

\[
\alpha_1 f_1(a_1, \ldots, a_n a_{n+1}) + \cdots + \alpha_{c_n^H(A)} f_{c_n^H(A)}(a_1, \ldots, a_n a_{n+1}) = 0
\]

for all \( a_i \in A \). Since \( A \) is \( H \)-simple, \( AA = A \), and

\[
\alpha_1 f_1(a_1, \ldots, a_n) + \cdots + \alpha_{c_n^H(A)} f_{c_n^H(A)}(a_1, \ldots, a_n) = 0
\]

for all \( a_i \in A \). However, \( f_1(x_1, \ldots, x_n), \ldots, f_{c_n^H(A)}(x_1, \ldots, x_n) \) are linearly independent modulo \( \text{Id}^H(A) \). Hence \( \alpha_1 = \cdots = \alpha_{c_n^H(A)} = 0 \). Thus, \( f_1(x_1, \ldots, x_n x_{n+1}), \ldots, f_{c_n^H(A)}(x_1, \ldots, x_n x_{n+1}) \) are linearly independent modulo \( \text{Id}^H(A) \), and \( c_n^H(A) \leq c_{n+1}^H(A) \). \( \square \)

Next we prove the upper bound for \( c_n^H(A) \).

**Theorem 6.4.** Let \( A \) be a finite dimensional algebra with a generalized \( H \)-action for some associative algebra \( H \) with 1 over a field \( F \) of characteristic 0, \( \dim A = s \). Then there exist \( C > 0 \) and \( r \in \mathbb{R} \) such that

\[
c_n^H(A) \leq Cn^r \max_{\lambda \vdash n, \mu \vdash \ell, (\ell, \lambda) \neq 0} \Phi \left( \frac{\lambda_1}{n}, \ldots, \frac{\lambda_s}{n} \right)^n \text{ for all } n \in \mathbb{N}.
\]

**Proof.** Let \( \lambda \vdash n \) such that \( m(A,H,\lambda) \neq 0 \). By the hook formula, \( \dim M(\lambda) = \prod_{i,j} \frac{n!}{h_{ij}} \) where \( h_{ij} \) is the length of the hook with the edge in \( i,j \) in the Young diagram \( D_\lambda \). Hence \( \dim M(\lambda) \leq \frac{n!}{\lambda_1! \cdots \lambda_s!} \). By the Stirling formula, for all sufficiently large \( n \) we have

\[
\dim M(\lambda) \leq \frac{C_1 n^{r_1} \left( \frac{n}{e} \right)^n}{\left( \frac{\lambda_1}{e} \right)^{\lambda_1} \cdots \left( \frac{\lambda_s}{e} \right)^{\lambda_s}} = C_1 n^{r_1} \left( \frac{1}{\left( \frac{\lambda_1}{n} \right)^{\lambda_1} \cdots \left( \frac{\lambda_s}{n} \right)^{\lambda_s}} \right)^n \leq C_1 n^{r_1} \left( \Phi \left( \frac{\lambda_1}{n}, \ldots, \frac{\lambda_s}{n} \right) \right)^n
\]

for some \( C_1 > 0 \) and \( r_1 \in \mathbb{R} \) that do not depend on \( \lambda \). Together with Theorem 5.5 this yields the theorem. \( \square \)

Throughout the rest of the section we work under the assumptions of Theorem 6.1.

Let \( \lambda \vdash n, \mu \vdash m \), \( FS_n f_1 \cong M(\lambda) \) and \( FS_m f_2 \cong M(\mu) \) for some \( m,n \in \mathbb{N} \), \( f_1 \in W_n^H \) and \( f_2 \in W_m^H \). Then the image of the polynomial \( f_1(x_1, \ldots, x_n) f_2(x_{n+1}, \ldots, x_{m+n}) \) generates an \( FS_{m+n} \)-submodule of \( W_{m+n}^{H} / \text{Id}^H(A) \) which is a homomorphic image of

\[
M(\lambda) \hat{\otimes} M(\mu) := (M(\lambda) \otimes M(\mu)) \uparrow S_{m+n} := FS_{m+n} \otimes_{FS(n \times S_m)} (M(\lambda) \otimes M(\mu)).
\]
By the Littlewood—Richardson rule, all irreducible components in the decomposition of $M(\lambda) \otimes M(\mu)$ correspond to Young diagrams $D_\nu$ that are obtained from $D_{\lambda+\mu}$ by pushing some boxes down. By our assumptions, the height of $D_\nu$ cannot be greater than $s = \dim A$.

Another remark is that, in the process of pushing boxes down, the value of $\Phi$ is non-decreasing since the function $\frac{1}{x^{\epsilon}(x-1)^s}$ is increasing as $x \in (0, \frac{\epsilon}{2})$ for fixed $0 < \epsilon < 1$.

**Lemma 6.5.** There exists a constant $N \in \mathbb{N}$ such that for every $\epsilon > 0$ there exist a number $q \in \mathbb{N}$, natural numbers $n_1 < n_2 < n_3 < \ldots$ such that $n_{i+1} - n_i \leq N + q$, and partitions $\lambda^{(i)} \vdash n_i$, $\mu$ such that $\Phi \left( \frac{\lambda^{(i)}}{n_i}, \ldots, \frac{\lambda^{(i)}}{n_i} \right) \geq d(A) - \epsilon$ for all $i \in \mathbb{N}$.

**Proof.** Denote by $B$ the subalgebra of the associative algebra $\text{End}_F(A)$ generated by all operators of left and right multiplication by elements of $A$ as well as the images of all elements of $H$ considered as operators on $A$. Since $A$ is an $H$-simple algebra, $A$ is an irreducible left $B$-module. Denote by $\mathcal{B}$ the set of all operators $u \in B$ of the following form: there exist $a_1, \ldots, a_m, a_1', \ldots, a_m', \tilde{a}_1, \ldots, \tilde{a}_m \in A$, $m, \tilde{m} \in \mathbb{Z}_+$, $h \in H$ and some arrangement of brackets such that $ua_i = a_i \cdot \tilde{a}_i h \tilde{a}_i \tilde{a}_i' \tilde{a}_i' h$ for all $a_i \in A$. Using (2.1), we can incorporate operators of the $H$-action in the operators of left and right multiplication by elements of $A$ and present each operator from $\mathcal{B}$ as a linear combination of operators from $\mathcal{B}$. Since $\text{End}_F(A)$ is finite dimensional, we can choose a finite basis $u_1, \ldots, u_{\dim B}$ in $\mathcal{B}$ of the vector space $B$. Denote by $N$ the maximal number $2(m + \tilde{m})$ among all $u_i$.

Since $A$ is $H$-simple, $A^2 \neq 0$ and for every $a, b \in A$, $a \neq 0$, $b \neq 0$, we have $A = Ba = Bb$ and $(Ba)(Bb) \neq 0$. The choice of $N$ implies that there exist some $a_1, \ldots, a_m, \tilde{a}_1, \ldots, \tilde{a}_m, b_1, \ldots, b_k \in A$, $k, \tilde{k}, m, \tilde{m} \in \mathbb{Z}_+$, $h_1, h_2 \in H$, such that

$$(a_1 \cdots a_m(h_1 a_1 \tilde{a}_1 \cdots \tilde{a}_m)(b_1 \cdots b_k(h_2 b_1 \cdots b_k)) \neq 0$$

(for some arrangements of brackets on the multipliers) and $k + \tilde{k} + m + \tilde{m} \leq N$.

Now we choose $q \in \mathbb{N}$ such that $\Phi \left( \frac{\mu_1}{q}, \ldots, \frac{\mu_2}{q} \right) \geq d(A) - \epsilon/2$ and $m(A, H, \mu) \neq 0$ for some $\mu \vdash q$. Recall that $\Phi$ is continuous on $[0; 1]^s$ and therefore uniformly continuous on $[0; 1]^s$ since $[0; 1]^s$ is compact. Since we can take $q$ arbitrarily large, we may assume also that

$$\Phi \left( \frac{\mu_1 + \sum_{j=1}^i d_j}{iq + \sum_{j=1}^i d_j}, \frac{i\mu_2}{iq + \sum_{j=1}^i d_j}, \ldots, \frac{i\mu_s}{iq + \sum_{j=1}^i d_j} \right) \geq d(A) - \epsilon$$

(6.2)

for all $i \in \mathbb{N}$ and all $0 \leq d_i \leq N$.

Choose $\tilde{f} \in W_H \setminus 1d^H(A)$ such that $FS_q \tilde{f} \cong M(\mu)$. Remarks made in the beginning of the proof imply that for some arrangements of brackets, some $h_1, h_2 \in H$, and some $k, \tilde{k}, m, \tilde{m} \geq 0$ such that $d_i := k + \tilde{k} + m + \tilde{m} \leq N$, we have

$$f_1 := \left( y_1 \cdots y_k \tilde{f} h_1(x_1, \ldots, x_q) \tilde{y}_1 \cdots \tilde{y}_k \right) \left( z_1 \cdots z_m \tilde{f} h_2(\tilde{x}_1, \ldots, \tilde{x}_q) \tilde{z}_1 \cdots \tilde{z}_m \right) \notin 1d^H(A).$$

(The notation $f^h$ was introduced in Remark 3.3 above.)

Consider the $FS_{q+k+k}$-submodule $M$ of $W_{q+k+k}^H(A)$ generated by the image of $y_1 \cdots y_k \tilde{f} h_1(x_1, \ldots, x_q) \tilde{y}_1 \cdots \tilde{y}_k$. By Remark 3.3, the image of $\tilde{f} h_1(x_1, \ldots, x_q)$ generates an $FS_q$-submodule isomorphic to $M(\mu)$ and therefore $M$ is a homomorphic image of

$$M(\mu) \otimes FS_{k+k} := (M(\mu) \otimes FS_{k+k}) \uparrow FS_{q+k+k}.$$
Since all partitions of \(k + \tilde{k}\) are obtained from the row of length \(k + \tilde{k}\) by pushing some boxes down, by the Littlewood — Richardson rule, all the partitions in the decomposition of \(M\) are obtained from \((\mu_1 + k + \tilde{k}, \mu_2, \ldots, \mu_s)\) by pushing some boxes down. The same arguments can be applied to \(z_1 \cdots z_m f^h(x_1, \ldots, x_q) \tilde{z}_1 \cdots \tilde{z}_m\).

Let \(n_1 := 2q + d_1\) and let \(\lambda^{(1)}\) be one of the partitions corresponding to the irreducible components in the decomposition of \(FS_{n_1} f_1\). Then by \((6.2)\), the remarks above and the remark before the lemma, we have \(\Phi \left( \frac{\lambda^{(1)}_{n_1}}{n_1}, \ldots, \frac{\lambda^{(1)}_n}{n_n} \right) \geq d(A) - \varepsilon\).

Again,

\[
f_2 := (y_1 \cdots y_k f_1^h(x_1, \ldots, x_q) \tilde{y}_1 \cdots \tilde{y}_k) \left( z_1 \cdots z_m f^h(x_1, \ldots, x_q) \tilde{z}_1 \cdots \tilde{z}_m \right) \notin \text{Id}^H(A)
\]

for some arrangements of brackets, some \(h_1, h_2 \in \text{H}\), and some \(k, \tilde{k}, m, \tilde{m} \geq 0, d_2 := k + \tilde{k} + m + \tilde{m} \leq N\) (maybe different from those for \(f_1\)). Again, we define \(n_2 := 3q + d_1 + d_2\). Denote by \(\lambda^{(2)}\) one of the partitions corresponding to the irreducible components in the decomposition of \(FS_{n_2} f_2\). We continue this procedure and prove the lemma.

**Proof of Theorem \((6.1)\)** Fix some \(\varepsilon > 0\). Consider \(n_i \in \mathbb{N}\) and \(\lambda^{(i)} \vdash n_i\) from Lemma \((6.5)\). We have

\[
c^H_{n_i}(A) \geq \dim M(\lambda^{(i)}) = \frac{n_i!}{\prod_{j,k \in \text{H}} h_{jk}} \geq \frac{n_i!}{n_i^{s(s-1)} \lambda^{(i)}_1 \cdots \lambda^{(i)}_s!} \geq \frac{C_1 n_i^{r_i}}{\left( \frac{\lambda^{(i)}_{n_i}}{n_i} \right)^{n_i}} = C_1 n_i^{r_i} \left( \Phi \left( \frac{\lambda^{(i)}_{n_i}}{n_i}, \ldots, \frac{\lambda^{(i)}_n}{n_n} \right) \right)^{n_i}
\]

(6.3)

for some \(C_1 > 0\) and \(r_i \leq 0\) which do not depend on \(i\).

Let \(n \geq n_1\). Then \(n_i \leq n < n_{i+1}\) for some \(i \in \mathbb{N}\). Taking into account (6.3), Lemma \((6.3)\) and the fact that \(\Phi(x_1, x_2, \ldots, x_s) \geq 1\) as \(0 \leq x_1, \ldots, x_s \leq 1\), we get

\[
\sqrt[n]{c^H_{n_i}(A)} \geq \sqrt[n]{c^H_{n_i}(A)} \geq \sqrt[n]{C_1 n_i^{r_i} \left( \Phi \left( \frac{\lambda^{(i)}_{n_i}}{n_i}, \ldots, \frac{\lambda^{(i)}_n}{n_n} \right) \right)^{n_i}}
\]

\[
\geq \sqrt[n]{C_1 n_i^{r_i} \left( \Phi \left( \frac{\lambda^{(i)}_{n_i}}{n_i}, \ldots, \frac{\lambda^{(i)}_n}{n_n} \right) \right)^{n_i}} \geq \sqrt[n]{C_1 n_i^{r_i}} (d(A) - \varepsilon)^{\frac{n-n_q}{n}}.
\]

Hence \(\lim_{n \to \infty} \sqrt[n]{c^H_{n_i}(A)} \geq d(A) - \varepsilon\). Since \(\varepsilon > 0\) is arbitrary, we get \(\lim_{n \to \infty} \sqrt[n]{c^H_{n_i}(A)} \geq d(A)\). Now Theorem \((6.4)\) yields \(\lim_{n \to \infty} \sqrt[n]{c^H_{n_i}(A)} = d(A)\).

**7. Free-forgetful adjunctions corresponding to gradings and generalized \(H\)-actions**

In this section we analyze the free constructions from Sections \((3)\) and \((4)\) from the categorical point of view. Here we consider the categories of not necessarily associative algebras, though the analogous adjunctions, of course, exist in the case of associative and Lie algebras too.
7.1. Gradings. Let $T$ be a set and let $F$ be a field. Denote by $\text{Vect}_{F}^{T,gr}$ the category where the objects are all $T$-graded vector spaces over $F$, i.e. vector spaces $V$ with a fixed decomposition $V = \bigoplus_{t \in T} V^{(t)}$, and the sets $\text{Vect}_{F}^{T,gr}(V,W)$ of morphisms between $V = \bigoplus_{t \in T} V^{(t)}$ and $W = \bigoplus_{t \in T} W^{(t)}$ consist of all linear maps $\varphi: V \to W$ such that $\varphi (V^{(t)}) \subseteq W^{(t)}$ for all $t \in T$.

Denote by $\text{NAAlg}_{F}^{T,pgr}$ (“not necessarily associative partially $T$-graded algebras”) the category where the objects are all not necessarily associative algebras $A$ over $F$ with distinguished subspaces $\bigoplus_{t \in T} A^{(t)} \subseteq A$ (the inclusion can be proper) graded by $T$ and if $A \supseteq \bigoplus_{t \in T} A^{(t)}$ and $B \supseteq \bigoplus_{t \in T} B^{(t)}$ are two such objects then, by the definition, the set $\text{NAAlg}_{F}^{T,pgr}(A,B)$ of morphisms $A \to B$ consists of all algebra homomorphisms $\varphi: A \to B$ such that $\varphi(A^{(t)}) \subseteq B^{(t)}$ for every $t \in T$.

Denote by $U: \text{NAAlg}_{F}^{T,pgr} \to \text{Vect}_{F}^{T,gr}$ the forgetful functor that assigns to each object $A \supseteq \bigoplus_{t \in T} A^{(t)}$ the $T$-graded vector space $\bigoplus_{t \in T} A^{(t)}$ and restricts homomorphisms to the distinguished subspaces.

Let $V = \bigoplus_{t \in T} V^{(t)}$ be a $T$-graded space. Let $V^{(t)}$ be bases in $V^{(t)}$. Denote by $KV$ the absolutely free non-associative algebra $F\{Y\}$ on the basis $Y = \bigsqcup_{t \in T} Y^{(t)}$. In the basis invariant form,

$$KV = \bigoplus_{n=1}^{\infty} \bigoplus \text{all possible arrangements of brackets } V \otimes \cdots \otimes V$$

and the multiplication is defined by $vw = v \otimes w$ (the arrangement of brackets in both sides is the same). We identify $V$ with the corresponding subspace in $KV$ and treat $KV \supseteq V = \bigoplus_{t \in T} V^{(t)}$ as an object of $\text{NAAlg}_{F}^{T,pgr}$.

For each $\varphi \in \text{Vect}_{F}^{T,gr}(V,W)$ there exists a unique algebra homomorphism $K\varphi: KV \to KW$ such that $(K\varphi)_{V} = \varphi$.

**Proposition 7.1.** The functor $K: \text{Vect}_{F}^{T,gr} \to \text{NAAlg}_{F}^{T,pgr}$ is the left adjoint to $U: \text{NAAlg}_{F}^{T,pgr} \to \text{Vect}_{F}^{T,gr}$.

**Proof.** If $V \in \text{Vect}_{F}^{T,gr}$ and $A \in \text{NAAlg}_{F}^{T,pgr}$, then each morphism $KV \to A$ is uniquely determined by its restriction to $V$. Hence we obtain a natural bijection $\text{NAAlg}_{F}^{T,pgr}(KV,A) \to \text{Vect}_{F}^{T,gr}(V,UA)$. \hfill \Box

Suppose now that $V = \bigoplus_{t \in T} V^{(t)}$ where $V^{(t)}$ are the vector spaces with the formal bases $(x^{(t)}_{i})_{i \in \mathbb{N}}$. Then $KV$ can be identified with $F\{X^{T,gr}\}$ from Section 4. Every $T$-graded algebra $A$ can be treated as an object of $\text{NAAlg}_{F}^{T,pgr}$ where the subspace $\bigoplus_{t \in T} A^{(t)}$ coincides with $A$. In this case we have a bijection $\text{NAAlg}_{F}^{T,pgr}(KV,A) \to \text{Vect}_{F}^{T,gr}(V,UA)$ which means that every map $\psi: X^{T,gr} \to A$, such that $\psi (X^{(t)}) \subseteq A^{(t)}$ for each $t \in T$, can be uniquely extended to an algebra homomorphism $\tilde{\psi}: KV \to A$ such that $\tilde{\psi} (X^{(t)}) \subseteq A^{(t)}$.

7.2. Generalized $H$-actions. Let $H$ be a unital associative algebra over a field $F$. Denote by $hM$ the category of left $H$-modules and by $h\text{NAAlgSubMod}$ (“not necessarily associative algebras with subspaces that are $H$-modules”) the category where the objects are all not necessarily associative algebras $A$ over $F$ with distinguished subspaces $A_{0} \subseteq A$ (the inclusion can be proper), which are left $H$-modules, and for objects $A \supseteq A_{0}$ and $B \supseteq B_{0}$ the set $h\text{NAAlgSubMod}(A,B)$ of morphisms consists of all algebra homomorphisms $\varphi: A \to B$ where $\varphi(A) \subseteq B_{0}$ and $\varphi|_{A_{0}}$ is a homomorphism of $H$-modules. Here we again have an obvious forgetful functor $U: h\text{NAAlgSubMod} \to hM$ where $UA := A_{0}$ and $U\varphi := \varphi|_{A_{0}}$. 

Let \( K \) be a functor \( \mathcal{H} \mathcal{M} \to \mathcal{H} \text{NAAlgSubMod} \) that assigns to each left \( H \)-module \( V \) the absolutely free non-associative algebra \( KV := F\{Z\} \) where \( Z \) is a basis in \( V \). In other words,

\[
KV = \bigoplus_{n=1}^{\infty} \bigoplus_{\text{all possible arrangements of brackets}} V \otimes \cdots \otimes V
\]

and the multiplication is defined by \( vw = v \otimes w \) (the arrangement of brackets in both sides is the same). We identify \( V \) with the corresponding subspace in \( KV \) and treat \( KV \supseteq V \) as an object of \( \mathcal{H} \text{NAAlgSubMod} \). For each \( \varphi \in \mathcal{H} \mathcal{M}(V,W) \) there exists a unique algebra homomorphism \( K\varphi: KV \to KW \) such that \( (K\varphi)|_V = \varphi \).

**Proposition 7.2.** The functor \( K: \mathcal{H} \mathcal{M} \to \mathcal{H} \text{NAAlgSubMod} \) is the left adjoint to \( U: \mathcal{H} \text{NAAlgSubMod} \to \mathcal{H} \mathcal{M} \).

**Proof.** If \( V \in \mathcal{H} \mathcal{M} \) and \( A \in \mathcal{H} \text{NAAlgSubMod} \), then each morphism \( KV \to A \) is uniquely determined by its restriction to \( V \). Hence we obtain a natural bijection \( \mathcal{H} \text{NAAlgSubMod}(KV,A) \to \mathcal{H} \mathcal{M}(V,UA) \).

Suppose now that \( V \) is a free left \( H \)-module with a formal \( H \)-basis \( Y \), i.e. \( V = \bigoplus_{y \in Y} Hy \). Then \( KV \) can be identified with \( F\{Y|H\} \) from Section[3]. Every algebra \( A \) with a generalized \( H \)-action can be treated as an object of \( \mathcal{H} \text{NAAlgSubMod} \) where the \( H \)-module \( A_0 \) coincides with \( A \). In this case we have a bijection \( \mathcal{H} \text{NAAlgSubMod}(KV,A) \to \mathcal{H} \mathcal{M}(V,UA) \) which means that every map \( \psi: Y \to A \) can be uniquely extended to an algebra homomorphism \( \bar{\psi}: KV \to A \) such that \( \bar{\psi}(hy) = h \bar{\psi}(y) \) for every \( y \in Y \).

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