ASYMPTOTIC SPREADING SPEED FOR THE WEAK COMPETITION SYSTEM WITH A FREE BOUNDARY

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(Communicated by Masaharu Taniguchi)

Abstract. This paper is concerned with a diffusive Lotka-Volterra type competition system with a free boundary in one space dimension. Such a system may be used to describe the invasion of a new species into the habitat of a native competitor, and its long-time dynamical behavior can be described by a spreading-vanishing dichotomy. The main purpose of this paper is to determine the asymptotic spreading speed of the invading species when its spreading is successful, which involves two systems of traveling wave type equations.

1. Introduction. The classical Lotka-Volterra reaction-diffusion system
\[
\begin{align*}
\frac{du}{dt} &= d_1 u_{xx} + u(a_1 - b_1 u - c_1 v), \quad x \in \mathbb{R}, \quad t > 0, \\
\frac{dv}{dt} &= d_2 v_{xx} + v(a_2 - b_2 v - c_2 u), \quad x \in \mathbb{R}, \quad t > 0
\end{align*}
\]
(1.1)
is a model frequently used to describe competitive behavior between two distinct species. Here \(u(x,t)\) and \(v(x,t)\) denote the population densities of two competing species at the position \(x\) and time \(t\); the constants \(d_i, a_i, b_i\) and \(c_i\) \((i = 1, 2)\) are the diffusion rates, intrinsic growth rates, intra-specific competition rates, and inter-specific competition rates, respectively, all of which are assumed to be positive. By setting
\[
\begin{align*}
\hat{u}(x,t) &:= \frac{b_1}{a_1} u \left( \sqrt{\frac{d_1}{a_1}} x, \frac{t}{a_1} \right), \\
\hat{v}(x,t) &:= \frac{b_2}{a_2} v \left( \sqrt{\frac{d_1}{a_1}} x, \frac{t}{a_1} \right), \\
d &:= \frac{d_2}{d_1}, \quad r := \frac{a_2}{a_1}, \quad k := \frac{a_2 c_1}{a_1 b_2}, \quad h := \frac{a_1 c_2}{a_2 b_1},
\end{align*}
\]

2010 Mathematics Subject Classification. Primary: 35B40, 35K20, 35R35; Secondary: 92B05.
Key words and phrases. Competition model, free boundary problem, spreading-vanishing dichotomy, semi-wave, asymptotic spreading speed.

The work is supported by the NSF of China (11671243, 11771262, 11572180), the Fundamental Research Funds for the Central Universities (GK201701001), and the Australian Research Council.

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and dropping the hat signs, system (1.1) becomes the following nondimensional system:

\[
\begin{align*}
  u_t &= u_{xx} + u(1 - u - kv), & x \in \mathbb{R}, \ t > 0, \\
  v_t &= dv_{xx} + rv(1 - v - hu), & x \in \mathbb{R}, \ t > 0.
\end{align*}
\]

(1.2)

It is easy to see that (1.2) has four equilibria: \((0, 0), (1, 0), (0, 1)\) and \((u^*, v^*) = \left(\frac{1-k}{1-h}, \frac{1-k}{1-h}\right)\), with \((u^*, v^*)\) meaningful only when \((1-k)(1-h) > 0\). When the entire real line \(\mathbb{R}\) is replaced by a bounded open interval in \(\mathbb{R}\), under the zero Neumann boundary conditions, the asymptotic behavior of the solution \((u(x, t), v(x, t))\) for (1.2) with initial functions \(u(x, 0), v(x, 0) > 0\) can be summarized below (see, for example [22]):

(I) if \(k < h < 1\), then \(\lim_{t \to \infty} (u(x, t), v(x, t)) = (1, 0)\);

(II) if \(h < k < 1\), then \(\lim_{t \to \infty} (u(x, t), v(x, t)) = (0, 1)\);

(III) if \(h, k < 1\), then \(\lim_{t \to \infty} (u(x, t), v(x, t)) = (u^*, v^*)\);

(IV) if \(h, k > 1\), then \(\lim_{t \to \infty} (u(x, t), v(x, t)) = (1, 0)\) or \((0, 1)\) or \((u^*, v^*)\) (depending on the initial condition).

The cases (I) and (II) are usually called the weak-strong competition case, while (III) and (IV) are known as the weak and strong competition cases, respectively.

A number of variations of (1.2) (or (1.1)) have been used to model the spreading of a new or invasive species. For example, to describe the invasion of a new species into the habitat of a native competitor, Du and Lin [7] considered the following free boundary problem

\[
\begin{align*}
  u_t &= u_{xx} + u(1 - u - kv), & 0 < x < g(t), \ t > 0, \\
  v_t &= dv_{xx} + rv(1 - v - hu), & 0 < x < \infty, \ t > 0, \\
  u_x(0, t) &= v_x(0, t) = 0, & u(x, t) = 0, \ g(t) \leq x < \infty, \ t > 0, \\
  g'(t) &= -\gamma u_x(g(t), t), & t > 0, \\
  g(0) &= g_0, & u(x, 0) = u_0(x), \ 0 \leq x \leq g_0, \\
  v(x, 0) &= v_0(x), & 0 \leq x < \infty,
\end{align*}
\]

(1.3)

where \(x = g(t)\) is usually called a free boundary, which is to be determined together with \(u\) and \(v\). The initial functions satisfy

\[
\begin{align*}
  u_0 \in C^2([0, g_0]), \ v_0'(0) &= u_0(g_0) = 0 \text{ and } u_0(x) > 0 \text{ in } [0, g_0); \\
  v_0 \in C^2([0, \infty)) \cap L^\infty(0, \infty), \ v_0'(0) = 0, \\
  \liminf_{x \to \infty} v_0(x) > 0 \text{ and } v_0(x) > 0 \text{ in } [0, \infty).
\end{align*}
\]

(1.4)

This model describes how a new species with population density \(u\) invades into the habitat of a native competitor \(v\). It is assumed that the species \(u\) exists initially in the range \(0 < x < g_0\), invades into new territory through its invading front \(x = g(t)\). The native species \(v\) undergoes diffusion and growth in the available habitat \(0 < x < \infty\). Both \(u\) and \(v\) obey a no-flux boundary condition at \(x = 0\). The equation \(g'(t) = -\gamma u_x(g(t), t)\) means that the invading speed is proportional to the gradient of the population density of \(u\) at the invading front, which coincides with the well-known Stefan free boundary condition. A detailed deduction of this free boundary condition can be found in [1], which is based on the consideration of “population loss” at the front. All parameters \(d, k, h, r, g_0\) and \(\gamma\) are assumed to be positive. See [7] for more biological background.
The weak-strong competition case of (1.3) was rather thoroughly treated in [7] and [12]. It is shown in [7] that when the invading species $u$ is the inferior competitor ($k > 1 > h$), if the resident species $v$ is already well established initially (i.e., $v_0$ satisfies the conditions in (1.4)), then $u$ can never invade deep into the underlying habitat, and it dies out before its invading front reaches a certain finite limiting position, whereas if the invading species $u$ is superior ($h > 1 > k$), a spreading-vanishing dichotomy holds for $u$ (see Theorem 4.4 in [7]). Moreover, when spreading of $u$ happens, the precise asymptotic spreading speed is determined in [12].

We would like to stress that it is in general difficult to determine the precise spreading speed for a system of equations with free boundaries such as (1.3), and [12] appears to be the first success in that direction. For systems of two species with free boundaries, one may find many interesting results in [15, 16, 24, 27, 28, 30] and the references therein, where various spreading-vanishing dichotomies have been established. However, in all these works the question of whether there is a precise asymptotic spreading speed has been left open. This is in sharp contrast to the corresponding one species models, where the precise spreading speed has been determined in many rather general situations; see, for example, [3, 4, 5, 8, 10, 11, 17, 19]. This fact has been widely used in the existing literature for competition systems with two species $u$ and $v$ and with free boundaries, to estimate the spreading speed determined by the system: The spreading speed of the species $u$ in the absence of $v$ can be obtained from the corresponding single species model, which provides an upper bound for the spreading speed of $u$ governed by the system where both $u$ and $v$ are present. Knowing the precise spreading speed determined by the system is also important from the point of view of modeling, for if this speed is always the same as the spreading speed in the absence of the competitor, then the corresponding single species model perhaps would be enough to describe the spreading of the concerned species.

In a recent paper [13], the result of [12] was applied to treat a weak-strong competition case similar to (1.3), but with the species $v$ also growing over a varying habitat $0 < x < h(t)$ with $h(t)$ determined by the Stefan condition (as in [16]). In such a case, it was shown in [13] that the precise spreading speeds of the system can often be determined by making use of [12].

The main purpose of this paper is to continue the work of [12] in a different direction, namely to determine the precise spreading speed of (1.3) for the weak competition case:

$$0 < k < 1, \ 0 < h < 1.$$ 

In this case, it is easy to show that a spreading-vanishing dichotomy holds for the invading species $u$, although in contrast to the weak-strong competition case ($h > 1 > k$) in [7], in the weak competition case here, when $u$ spreads successfully, the two populations converge to the co-existence steady state $(u^*, v^*)$ as time goes to infinity (so the native competitor $v$ always survives the invasion of $u$). The main part of this work is to determine the precise spreading speed of $u$ when the invasion is successful, which confirms that this speed is strictly less than the single species spreading speed (i.e., the spreading speed of $u$ in the absence of $v$), as one would expect.

Note that due to the different spreading behavior of $(u, v)$ as $t \to \infty$ from the situation of [12], nontrivial difficulties arise in adapting the approach of [12], and new techniques are also required (see, for example, Proposition 2.5 below). Moreover,
although we only treat (1.3) where the $v$ species does not have a free boundary, it is likely that the techniques and conclusions here are useful for treating the weak competition systems where both species have free boundaries (such cases were considered, for example, in [15, 27, 28, 30]).

Let us now describe the results of this paper more precisely. From [7] we know that (1.3) has a unique solution, which is defined for all $t > 0$. The first description of its long-time behavior is given by Theorems 1.1 and 1.2 below, which follow from relatively standard arguments. (Some detailed comparisons of these results to existing ones in [15, 27, 28, 30] can be found at the beginning of section 5.)

Theorem 1.1. Suppose that $h, k \in (0, 1)$ and $(u, v, g)$ is the solution of (1.3) with $u_0$ and $v_0$ satisfying (1.4). Then, as $t \to \infty$, the following dichotomy holds.

Either (i) the species $u$ spreads successfully:

$$\lim_{t \to \infty} g(t) = \infty \quad \text{and} \quad \lim_{t \to \infty} (u(\cdot, t), v(\cdot, t)) = (u^*, v^*) \in C^2_{\text{loc}}([0, \infty)).$$

or (ii) the species $u$ vanishes eventually:

$$\lim_{t \to \infty} g(t) < \infty \quad \text{and} \quad \lim_{t \to \infty} (u(\cdot, t), v(\cdot, t)) = (0, 1) \in C^2_{\text{loc}}([0, \infty)).$$

Theorem 1.2. Under the assumptions of Theorem 1.1, there exists $\gamma^* \in [0, \infty)$ depending on $(u_0, v_0)$ such that alternative (i) in Theorem 1.1 happens if and only if $\gamma > \gamma^*$. Moreover, $\gamma^* = 0$ (and hence $u$ always spreads successfully) if $g_0 \geq \frac{\pi}{2\sqrt{1-k}},$ and $\gamma^* > 0$ if $g_0 < \frac{\pi}{2}$.

Our main result in this paper is the following theorem, which gives the precise spreading speed of $u$ when case (i) happens in Theorem 1.1.

Theorem 1.3. If (i) happens in Theorem 1.1, then there exists $c^0 > 0$ such that

$$\lim_{t \to \infty} \frac{g(t)}{t} = c^0.$$

As usual, the positive constant $c^0$ in Theorem 1.3 is called the asymptotic spreading speed of $u$. The key in the proof of this theorem is to find a way to determine $c^0$. Similar in spirit to the weak-strong competition case considered in [12], two systems of traveling wave type equations are needed in order to determine $c^0$. The first one is obtained by looking for traveling wave solutions of (1.2), namely

$$\begin{cases}
\Phi'' - c\Phi' + (1 - \Phi - k\Psi) = 0, & \Phi' > 0, \quad -\infty < s < \infty, \\
d\Psi'' - c\Psi' + r\Psi(1 - \Psi - h\Phi) = 0, & \Psi' < 0, \quad -\infty < s < \infty, \\
(\Phi(-\infty), \Psi(-\infty)) = (0, 1), \quad (\Phi(\infty), \Psi(\infty)) = (u^*, v^*). 
\end{cases}$$

The second system is (1.6) below.

Clearly, if $(\Phi(s), \Psi(s))$ solves (1.5), then

$$(u(x, t), v(x, t)) := (\Phi(ct - x), \Psi(ct - x))$$

is a solution of (1.2), which is often called a traveling wave solution with speed $c$.

By Theorem 4.2 and Example 4.2 in [18], we have the following result on (1.5):

Proposition 1.4. Assume $h, k \in (0, 1)$. Then there exists a critical speed $c_* \geq 2\sqrt{1-k}$ such that (1.5) has a solution when $c \geq c_*$ and it has no solution when $c < c_*$. 


We can further show that \( c_s \leq 2\sqrt{\alpha} \). Making use of \( c_s \), we have the following result on the system below, which gives traveling wave type solutions to (1.3):

\[
\begin{align*}
\phi'' - c\phi' + \phi(1 - \phi - k\psi) &= 0, \quad \phi' > 0 \text{ for } 0 < s < \infty, \\
d\psi'' - c\psi' + r\psi(1 - \psi - h\phi) &= 0, \quad \psi' < 0 \text{ for } -\infty < s < \infty, \\
\phi(s) &= 0 \text{ for } s \leq 0, \quad \psi(-\infty) = 1, \quad (\phi(\infty), \psi(\infty)) = (u^*, v^*).
\end{align*}
\]

**Theorem 1.5.** Assume \( h, k \in (0, 1) \). If \( c \in [0, c_s) \), then the system (1.6) admits a unique solution \((\phi_c, \psi_c) \in [C(\mathbb{R}) \cap C^2(\mathbb{R}^+)] \times C^2(\mathbb{R})\); if \( c \geq c_s \), then (1.6) does not have a solution. Moreover, the following conclusions hold:

\( i \) \( \lim_{c \searrow c_s} (\phi_c, \psi_c) = (0, 1) \) in \( C^2_{\text{loc}}([0, \infty)) \times C^2_{\text{loc}}(\mathbb{R}) \),

\( ii \) for any \( \gamma > 0 \), there exists a unique \( c_\gamma \in (0, c_s) \) such that \( \gamma\phi'_{c_\gamma}(0) = c_\gamma \),

\( iii \) the function \( \gamma \mapsto c_\gamma \) is strictly increasing and \( \lim_{\gamma \to \infty} c_\gamma = c_s \).

We will show that the asymptotic spreading speed of \( u \) in Theorem 1.3 is given by

\[ c^0 := c_\gamma. \]

Let us note that if \((\phi, \psi, c)\) solves (1.6), then

\[ \tilde{u}(x, t) := \phi(ct - x), \quad \tilde{v}(x, t) := \psi(ct - x) \]

satisfy

\[
\begin{align*}
\tilde{u}_t &= \tilde{u}_{xx} + \tilde{u}(1 - \tilde{u} - k\tilde{v}), \quad -\infty < x < ct, \quad t \in \mathbb{R}, \\
\tilde{v}_t &= d\tilde{v}_{xx} + r\tilde{v}(1 - \tilde{v} - h\tilde{u}), \quad -\infty < x < \infty, \quad t \in \mathbb{R}, \\
\tilde{u}(x, t) &= 0, \quad ct \leq x < \infty, \quad t \in \mathbb{R}.
\end{align*}
\]

If \( c = c_\gamma \), then we have additionally

\[ (ct)' = c = -\gamma \tilde{u}_x(ct, t). \]

We call \((\phi_c, \psi_c)\) with \( c = c_\gamma \) the semi-wave associated with (1.3). This pair of functions (and its suitable variations) will play a crucial role in the proof of Theorem 1.3, and they also provide upper and lower bounds for the solution pair \((u, v)\) (see the proof of Lemmas 4.1 and 4.2 for details).

The rest of this paper is organized as follows. In section 2, we present some basic results including the existence of solutions to (1.3), and the existence of solutions to a more general system than (1.6). In section 3, we prove Theorem 1.5 based on an upper and lower solution result (Proposition 2.5) established in section 2, by following the strategy of [12]; as mentioned earlier, we have to overcome several nontrivial difficulties here. In section 4, assuming the validity of Theorem 1.1, we prove Theorem 1.3 by making use of Theorem 1.5. The proof of Theorems 1.1 and 1.2 is given in section 5. Section 6 consists of the proof of Proposition 2.5 stated in section 2.

2. Preliminary results. In this section, we collect some basic facts which will be needed in our proof of the main results. We first note that (1.3) always has a unique solution. Indeed, by Theorems 2.4 and 2.5 in [7], we have the following results on the solution of system (1.3).

**Proposition 2.1.** For any initial function \((u_0, v_0)\) satisfying (1.4), the free boundary problem (1.3) admits a unique solution

\[ (u, v, g) \in C^{1+\alpha,(1+\alpha)/2}(D_1) \times C^{1+\alpha,(1+\alpha)/2}(D_2) \times C^{1+\alpha/2}(0, \infty)), \]
where $D_1 = \{(x,t) : x \in [0,g(t)], t \in [0,\infty)\}$ and $D_2 = \{(x,t) : x \in [0,\infty), t \in [0,\infty)\}$. Furthermore, there exists a positive constant $M$ depending on $d, r, \gamma, \|u_0\|_\infty$ and $\|v_0\|_\infty$, such that

$$0 < u(x,t) \leq M, \quad 0 < g(t) \leq M \text{ for } 0 < x < g(t), \quad t > 0,$$

$$0 < v(x,t) \leq M \text{ for } 0 < x < \infty, \quad t > 0.$$

Next we recall a comparison result for the free boundary problem, which is a special case of Lemma 2.6 in [7].

**Proposition 2.2.** (Comparison principle) Assume that $T \in (0,\infty)$, $g, \overline{g} \in C^1([0,\infty), \mathbb{R})$, $u, \overline{u} \in C(D_T^+) \cap C^{2,1}(D_T^+)$ with $D_T^+ = \{(x,t) \in \mathbb{R} : x \in (0,g(t)), \ t \in (0,T)\}$, $\overline{u}, \overline{v} \in C(D_T^+) \cap C^{2,1}(D_T^+)$ with $D_T^+ = \{(x,t) \in \mathbb{R} : x \in (0,\overline{g}(t)), \ t \in (0,T)\}$, $\overline{u}, \overline{v} \in L^\infty([0,\infty) \times [0,T])$, and

$$\begin{align*}
\gamma \equiv x \in (1 - \overline{u} - h\overline{v}), & \quad 0 < x < \overline{g}(t), \quad 0 < t < T, \\
\gamma \equiv x \in (1 - \overline{u} - k\overline{v}), & \quad 0 < x < g(t), \quad 0 < t < T, \\
\gamma \equiv x \in (1 - \overline{u} - h\overline{v}), & \quad 0 < x < \infty, \quad 0 < t < T, \\
\gamma \equiv x \in (1 - \overline{u} - h\overline{v}), & \quad 0 < x < \infty, \quad 0 < t < T, \\
\gamma \equiv 0, & \quad x < g(t), \quad 0 < t < T, \\
\gamma \equiv 0, & \quad x < \overline{g}(t), \quad 0 < t < T, \\
\gamma \equiv 0, & \quad x < \infty, \quad 0 < t < T, \\
\gamma \equiv 0, & \quad x < g(t), \quad 0 < t < T, \\
\gamma \equiv 0, & \quad x < \overline{g}(t), \quad 0 < t < T, \\
\gamma \equiv 0, & \quad x < \infty, \quad 0 < t < T.
\end{align*}$$

Then the solution $(u,v,g)$ of (1.3) satisfies

$$
g(t) \leq \overline{g}(t), \quad v(x,t) \leq \overline{v}(x,t) \leq \overline{u}(x,t), \quad 0 \leq x < \infty, \quad t \in [0,T],$$

$$\overline{u}(x,t) \leq u(x,t) \text{ for } 0 \leq x < g(t), \quad u(x,t) \leq \overline{v}(x,t) \text{ for } 0 \leq x < g(t), \quad t \in [0,T].$$

**Remark 2.3.** In system (1.3), if the boundary conditions at $x = 0$ are replaced by $u(0,t) = m_1(t)$, $v(0,t) = m_2(t)$, $t > 0$, then Proposition 2.2 also holds if we replace

$$\begin{align*}
\overline{u}_x(0,t) & \equiv 0, \quad \overline{v}_x(0,t) \geq 0, \quad \overline{u}_x(0,t) \equiv 0, \quad \overline{v}_x(0,t) \leq 0 \\
\overline{u}_x(0,t) \geq 0, \quad \overline{v}_x(0,t) \leq 0, \quad \overline{u}_x(0,t) \equiv 0, \quad \overline{v}_x(0,t) \geq 0 \\
\overline{u}_x(0,t) \geq 0, \quad \overline{v}_x(0,t) \leq 0, \quad \overline{u}_x(0,t) \equiv 0, \quad \overline{v}_x(0,t) \geq 0
\end{align*}$$

for $0 < t < T$.

Lastly in this section, we modify some well known upper and lower solution technique to show the existence of a solution for a general cooperative system of the form

$$\begin{align*}
(d_1 \varphi'_{1} - c \varphi_{1} + f_1(\varphi) = 0, \quad s \in \mathbb{R}^+, \\
d_2 \varphi'_{2} - c \varphi_{2} + f_2(\varphi) = 0, \quad s \in \mathbb{R}, \\
\varphi_1(s) = 0, \quad s \leq 0,
\end{align*}$$

(2.1)

where $\varphi = (\varphi_1, \varphi_2), c \geq 0, d_i > 0$, and $f_i : \mathbb{R}^2 \to \mathbb{R}$ ($i = 1, 2$) satisfy the following conditions:

- (A$_1$) there is a strictly positive vector $K = (k_1, k_2) \in \mathbb{R}^2$ such that $f_i(0) = f_i(K) = 0$ for $i \in \{1, 2\}$, and $f_i(u_1, u_2) \neq 0$ for $(u_1, u_2) \in ((0,k_1] \times [0, k_2]) \setminus \{K\}, i \in \{1, 2\}$;
(A₂) the system (2.1) is a cooperative system, that is, \( f_i(u_1, u_2) \) is nondecreasing in \( u_j \) for \( i, j \in \{1, 2\}, i \neq j \), \( (u_1, u_2) \in [0, k_1] \times [0, k_2] \), and there exists a constant \( \beta_0 \geq 0 \) such that \( \beta_0 u_i + f_i(u_1, u_2) \) is nondecreasing in \( u_i \) for \( i = 1, 2 \) and \( (u_1, u_2) \in [0, k_1] \times [0, k_2] \);

(A₄) \( f_1 \) and \( f_2 \) are locally Lipschitz continuous.

We are particularly interested in solutions \( \varphi \) of (2.1) that satisfy the asymptotic boundary conditions

\[
\varphi(-\infty) = 0, \quad \varphi(\infty) = K. \tag{2.2}
\]

Indeed, solving (2.1) and (2.2) will supply the main step for solving (1.6). On the other hand, our method here to solve the more general problems (2.1) and (2.2) may have other applications.

We will write \((\tilde{\varphi}_1, \varphi_2) \leq (\varphi_1, \varphi_2)\) if \( \varphi_i \leq \tilde{\varphi}_i, i = 1, 2 \). Let \( R \subset \mathbb{R}^2 \) denote the rectangle

\[
R = [0, k_1] \times [0, k_2].
\]

It is convenient to introduce the following notations:

\[
C_R(\mathbb{R}^+, \mathbb{R}^2) \coloneqq \{ \varphi \in C(\mathbb{R}^+, \mathbb{R}^2) : \varphi(s) \in R \text{ for } s > 0 \},
\]

\[
C_R(\mathbb{R}, \mathbb{R}^2) \coloneqq \{ \varphi \in C(\mathbb{R}, \mathbb{R}^2) : \varphi(s) \in R \text{ for } s \in \mathbb{R} \}.
\]

**Definition 2.4**. Suppose that \( \varphi = (\varphi_1, \varphi_2) \in C_R(\mathbb{R}, \mathbb{R}^2), \varphi = (\varphi_1, \varphi_2) \in C_R(\mathbb{R}, \mathbb{R}^2) \), and \( \varphi_i, \varphi_j \) are twice continuously differentiable in \( \mathbb{R} \setminus \Omega_j \) for \( j = 1, 2 \), where \( \Omega_1 \subset [0, \infty) \) and \( \Omega_2 \subset \mathbb{R} \) are two finite sets, say \( \Omega_1 = \{ \xi_i \geq 0 : i = 1, 2, \cdots, m_1 \} \) and \( \Omega_2 = \{ \eta_i \in \mathbb{R} : i = 1, 2, \cdots, m_2 \} \). Moreover,

(i) the functions \( \varphi \) and \( \varphi \) satisfy the inequalities

\[
\begin{align*}
& d_1 \varphi_1'' - c \varphi_1' - \beta \varphi_1 + f_1(\varphi) \leq 0 \quad \text{for } s \in \mathbb{R}^+ \setminus \Omega_1, \\
& d_2 \varphi_2'' - c \varphi_2' - \beta \varphi_2 + f_2(\varphi) \leq 0 \quad \text{for } s \in \mathbb{R} \setminus \Omega_2, \\
& \varphi_1(s) = 0 \text{ for } s \leq 0, \quad \varphi_2(-\infty) = 0 \tag{2.3}
\end{align*}
\]

and

\[
\begin{align*}
& d_1 \varphi_1'' - c \varphi_1' - \beta \varphi_1 + f_1(\varphi) \geq 0 \quad \text{for } s \in \mathbb{R}^+ \setminus \Omega_1, \\
& d_2 \varphi_2'' - c \varphi_2' - \beta \varphi_2 + f_2(\varphi) \geq 0 \quad \text{for } s \in \mathbb{R} \setminus \Omega_2, \\
& \varphi_1(s) = 0 \text{ for } s \leq 0, \quad \varphi_2(-\infty) = 0 \tag{2.4}
\end{align*}
\]

(ii) the derivatives of \( \varphi \) and \( \varphi \) satisfy

\[
\begin{align*}
& \varphi_1'(\xi_i-) \leq \varphi_1'(\xi_i+), \quad \varphi_2'(\xi_i-) \leq \varphi_2'(\xi_i+) \text{ for } \xi_i \in \Omega_1 \setminus \{0\}, \\
& \varphi_2'(\eta_i-) \leq \varphi_2'(\eta_i+), \quad \varphi_2'(\eta_i-) \leq \varphi_2'(\eta_i+) \text{ for } \eta_i \in \Omega_2. \tag{2.5}
\end{align*}
\]

Then \( \varphi \) and \( \varphi \) are called a weak upper solution and a weak lower solution of (2.1)-(2.2) associated with \( R \), respectively.

**Proposition 2.5**. Assume that (A₁) – (A₃) hold. Suppose (2.1)-(2.2) has a pair of upper and lower solutions associated with \( R \) satisfying

\[
\varphi_1(s) \neq 0 \quad \text{and} \quad \sup_{t \leq s} \varphi(t) \leq \varphi(s) \text{ for } s \in \mathbb{R}.
\]

Then (2.1) has a monotone non-decreasing solution \( \varphi \) satisfying (2.2).
Proposition 2.5 will play an important role in the proof of Theorem 1.5 in the next section. The proof of Proposition 2.5 is based on some upper and lower solution arguments and involves the Schauder fixed point theorem. Our proof is similar in spirit to that in several works on various different traveling wave problems (see, for example, [20, 21, 25, 26, 29]), but the detailed techniques are rather different. Since the proof is long, it is postponed to Section 6 at the end of the paper.

3. Semi-wave solutions. The aim of this section is to prove Theorem 1.5. Although the strategy of [12] is followed here, the detailed techniques are very different from [12]; moreover, a number of new approaches are used in several steps (notably the proofs of Lemmas 3.5, 3.8 and 3.11 below).

Firstly, we recall some known results for the Fisher-KPP equation

\[
\begin{cases}
d\chi'' - c\chi' + a\chi(b - \chi) = 0, & 0 < s < \infty, \\
\chi(0) = 0.
\end{cases}
\] (3.1)

Lemma 3.1. Let \(a > 0, b > 0\) and \(d > 0\) be fixed constants.

(i) If \(c \in [0, 2\sqrt{abd}]\), then (3.1) has a unique solution \(\chi(s)\).

(ii) For each \(c \in [0, 2\sqrt{abd}]\), the solution \(\chi(s)\) of (3.1) is strictly increasing and has the following asymptotic behavior

\[
\chi(s) = b - [b_\chi + o(1)]e^{-\sqrt{c^2 + 4a}d}s \quad \text{as} \ s \to \infty,
\]

where \(b_\chi\) is a positive constant.

The conclusion (i) can be found in [1], and the proof of (ii) is standard (see, for example, [23]).

Lemma 3.2. ([12]) Let \(\tilde{f}, \tilde{g} \in C([0, \infty))\) with \(\tilde{g}\) nonnegative and not identically 0, and \(c, d\) be given constants with \(d > 0\). Assume that \(u_i(s) > 0\) in \((0, \infty), u_1(0) \leq u_2(0)\) and

\[
cu_1' - du_1'' - u_1[\tilde{f}(s) - \tilde{g}(s)u_1] \leq 0 \leq cu_2' - du_2'' - u_2[\tilde{f}(s) - \tilde{g}(s)u_2], 
\]

for \(0 < s < \infty\). If \(\limsup_{s \to \infty} \frac{u_1(s)}{u_2(s)} \leq 1\), then

\[
u_1(s) \leq u_2(s) \quad \text{for} \ 0 \leq s < \infty.
\]

Next, we consider the problem

\[
\begin{cases}
d\omega'' + cw' + r(hu^* - \omega)(1 - hu^* + \omega) = 0, & 0 < s < \infty, \\
\omega(0) = 0.
\end{cases}
\] (3.2)

Lemma 3.3. For any constant \(d > 0, c \geq 0\), the problem (3.2) admits a unique positive solution \(\omega(s)\) satisfying \(\omega(\infty) = hu^*\). Moreover, \(\omega(s)\) is strictly increasing in \(s\) for \(s > 0\), and there exists \(b_\omega > 0\) such that

\[
\omega(s) = hu^* - [b_\omega + o(1)]e^{-\sqrt{c^2 + 4a}d}s \quad \text{as} \ s \to \infty.
\] (3.3)

Proof. It is easily seen that

\[
\bar{\omega}(s) := 0, \quad \underline{\omega}(s) := hu^*
\] (3.4)

are a pair of lower and upper solutions for (3.2). Thus, (3.2) has at least one solution satisfying \(0 \leq \omega(s) \leq hu^*\). The strong maximum principle infers that \(\omega(s) < hu^*\) for \(s > 0\).
We thus obtain, for any fixed $s$, cooperative system by setting $e^\tilde{\omega}$. Hence, $\omega(s)$ is monotone in $(R, \infty)$ for some large $R > 0$. Otherwise $\omega(s)$ is oscillating near $s = \infty$ and hence we can find a sequence $s_n \to \infty$ as $n \to \infty$ such that $\omega'(s_n) = 0$. It follows that for any fixed $s > 0$, 

$$e^\tilde{\omega}\omega'(s) > \lim_{n \to \infty} e^\tilde{\omega}\omega'(s_n) = 0.$$ 

Thus, we have $\omega' > 0$ in $(0, \infty)$, a contradiction to the assumption. Hence, for large $R > 0$, $\omega$ is monotone in $(R, \infty)$. By (3.2) and $0 \leq \omega < hu^*$, we easily obtain $\omega(\infty) = hu^*$. Furthermore, a simple calculation indicates that the ODE system satisfied by $(\omega, \omega')$ has $(hu^*, 0)$ as a saddle point. It follows from standard ODE theory that, there exists a constant $b_\omega > 0$ such that (3.3) holds, and 

$$\omega'(s) = c + \frac{\sqrt{c^2 + 4de}}{2d}[h\omega + o(1)]e^{\frac{c - \sqrt{c^2 + 4de}}{2d}s} = o(1)e^{-\tilde{\omega}s} as s \to \infty.$$ 

We thus obtain, for any fixed $s > 0$, 

$$e^\tilde{\omega}\omega'(s) > \lim_{s \to \infty} e^\tilde{\omega}\omega'(s) = 0.$$ 

Hence $\omega' > 0$ in $(0, \infty)$ and $\omega$ is strictly increasing in $(0, \infty)$.

It remains to show the uniqueness of positive solution of (3.2) satisfying $\omega(\infty) = hu^*$. Let $\omega_1$ and $\omega_2$ be two positive solutions of (3.2) satisfying $\omega_i(\infty) = hu^*$, $i = 1, 2$. We easily see that $\omega_i \leq hu^*$ for otherwise there exists $s_i > 0$ such that 

$$\omega_i(s_i) = \max_{s \geq 0} \omega_i(s) > hu^*, \omega_i'(s_i) = 0 \geq \omega_i''(s_i),$$

which gives a contradiction to (3.2) when evaluated at $s = s_i$. The strong maximum principle then yields $\omega_i(s) < hu^*$ for $s > 0$.

Set $\chi_i(s) := 1 - hu^* + \omega_i(s)$ and we find that $\chi_i$ satisfies 

$$d\chi'' - c\chi' + r(1 - \chi)\chi = 0 for s > 0, \chi(0) = 1 - hu^*, \chi(\infty) = 1.$$ 

By Lemma 3.2 we immediately obtain $\chi_1 \equiv \chi_2$ and hence $\omega_1 \equiv \omega_2$. The uniqueness is thus proved. 

Now, we turn to consider system (1.6). For convenience, we change it to a cooperative system by setting 

$$\tilde{\phi}(s) := \phi(s), \tilde{\psi}(s) := 1 - \psi(s).$$

Clearly $(\phi, \psi)$ solves (1.6) if and only if $(\tilde{\phi}, \tilde{\psi})$ satisfies

\[
\begin{align*}
\tilde{\phi}'' - c\tilde{\phi}' + \tilde{\phi}(1 - k - \tilde{\phi} + k\tilde{\psi}) &= 0, \quad \tilde{\phi}' > 0 \quad \text{for } s \in \mathbb{R}^+, \\
\tilde{\psi}'' - c\tilde{\psi}' + r(1 - \tilde{\psi})(h\tilde{\phi} - \tilde{\psi}) &= 0, \quad \tilde{\psi}' > 0 \quad \text{for } s \in \mathbb{R}, \\
\tilde{\phi}(s) &= 0 \quad \text{for } s \leq 0, \quad \tilde{\psi}(-\infty) = 0, \quad (\tilde{\phi}(\infty), \tilde{\psi}(\infty)) = (u^*, hu^*). 
\end{align*}
\]

Let $\Sigma = \Sigma_1 \times \Sigma_2$, where

$$\Sigma_1 = \{\tilde{\phi} \in C(\mathbb{R}) \cap C^2([0, \infty)) : \tilde{\phi}(s) \equiv 0 (s \leq 0), \quad \tilde{\phi}'(s) > 0 (s > 0), \quad \tilde{\phi}(\infty) = u^*\},$$

$$\Sigma_2 = \{\tilde{\psi} \in C^2(\mathbb{R}) : \tilde{\psi}(-\infty) = 0, \quad \tilde{\psi}'(s) > 0 (s \in \mathbb{R}), \quad \tilde{\psi}(\infty) = hu^*\}.$$
We define a functional \( \theta \) on \( \Sigma \) by
\[
\theta(\bar{\phi}, \bar{\psi}) = \min \left\{ \inf_{s \in \mathbb{R}^+} \Theta_1(\bar{\phi}, \bar{\psi})(s), \inf_{s \in \mathbb{R}} \Theta_2(\bar{\phi}, \bar{\psi})(s) \right\},
\]
where
\[
\Theta_1(\bar{\phi}, \bar{\psi})(s) = \frac{\bar{\phi}''(s) + \bar{\phi}(s)(1 - k - \bar{\phi}(s) + k\bar{\psi}(s))}{\phi'(s)}, \quad s \in \mathbb{R}^+,
\]
\[
\Theta_2(\bar{\phi}, \bar{\psi})(s) = \frac{d\bar{\psi}''(s) + r(1 - \bar{\psi}(s))(h\bar{\phi}(s) - \bar{\psi}(s))}{\psi'(s)}, \quad s \in \mathbb{R}.
\]
Clearly, if \((\bar{\phi}, \bar{\psi})\) is a solution of (3.5) with \( c \geq 0 \), then \((\bar{\phi}, \bar{\psi}) \in \Sigma \) and \( \theta(\bar{\phi}, \bar{\psi}) = c \). Therefore,
\[
c \leq c_0^* := \sup_{(\bar{\phi}, \bar{\psi}) \in \Sigma} \theta(\bar{\phi}, \bar{\psi}).
\]
We will show \( c_0^* = c_* > 0 \), where \( c_* \) is given in Proposition 1.4. For the moment we assume \( c_0^* > 0 \) and prove that (3.5) has a solution for every \( c \in [0, c_0^*] \) by using an upper and lower solution argument. From Lemma 3.1, the following equation
\[
\begin{align*}
\hat{\chi}'' + \hat{\chi}(u^* - \hat{\chi}) &= 0, \quad 0 < s < \infty, \\
\hat{\chi}(0) &= 0
\end{align*}
\]
has a unique strictly increasing solution \( \hat{\chi} \) satisfying \( \hat{\chi}(\infty) = u^* \). We define
\[
\overline{\phi}(s) = \begin{cases} 
0, & -\infty < s < 0, \\
\hat{\chi}(s), & 0 \leq s < \infty,
\end{cases} \quad \overline{\psi}(s) = \begin{cases} 
hu^* - \omega(-s, 0), & -\infty < s < 0, \\
hu^*, & 0 \leq s < \infty,
\end{cases}
\]
where \( \omega(s, 0) \) stands for the unique positive solution of (3.2) with \( c = 0 \).

**Lemma 3.4.** For \( c \geq 0 \), the pair of functions \((\overline{\phi}(s), \overline{\psi}(s))\) is an upper solution of (3.5) associated with \( \mathcal{R} := [0, u^*] \times [0, hu^*] \).

**Proof.** For \( s \geq 0 \), we have
\[
\overline{\phi}'' - c\overline{\phi} + \overline{\phi}(1 - k - \overline{\phi} + k\overline{\psi}) = \hat{\chi}'' - c\hat{\chi} + \hat{\chi}(u^* - \hat{\chi}) = -c\hat{\chi}' \leq 0
\]
and
\[
d\overline{\psi}'' - c\overline{\psi}' + r(1 - \overline{\psi})(h\overline{\phi} - \overline{\psi}) = rh(1 - hu^*)(\hat{\chi} - u^*) \leq 0.
\]
For \( s < 0 \), we have
\[
\overline{\phi}'' - c\overline{\phi}' + \overline{\phi}(1 - k - \overline{\phi} + k\overline{\psi}) = 0
\]
and
\[
d\overline{\psi}'' - c\overline{\psi}' + r(1 - \overline{\psi})(h\overline{\phi} - \overline{\psi}) = -c\overline{\psi}' \leq 0.
\]
Moreover, it is easily seen that
\[
\overline{\psi}'(0-) \geq \overline{\psi}'(0+).
\]
Finally, by definition, \( \overline{\phi}(s) \equiv 0 \) for \( s \leq 0 \), \( \overline{\phi}(\infty) = u^* \), \( \overline{\psi}(\infty) = 0 \) and \( \overline{\psi}(\infty) = hu^* \). Hence \((\overline{\phi}, \overline{\psi})\) meets all the requirements for an upper solution associated with \( \mathcal{R} \) in Definition 2.4. This completes the proof. \( \square \)

**Lemma 3.5.** Assume \( h, k \in (0, 1) \), \( c_0^* > 0 \) and \( c \in [0, c_0^*] \). Then (3.5) has a solution \((\bar{\phi}, \bar{\psi})\).
Proof. For \( c \in [0, c_0^*], \) by the definition of \( c_0^* \), there exists \((\psi(s), \psi(s)) \in \Sigma\) such that \( \theta(\psi, \psi) > c \). Thus we have

\[
\begin{cases}
\phi'' - c\phi' + (1 - k - \phi + k\psi) \geq 0, & \phi' > 0, \quad s \in \mathbb{R}^+,
\psi'' - c\psi' + r(1 - \psi)(h\phi - \psi) \geq 0, & \psi' > 0, \quad s \in \mathbb{R},
\phi(s) = 0 \text{ for } s \leq 0, \quad \psi(-\infty) = 0, \quad (\phi(\infty), \psi(\infty)) = (u^*, hu^*).
\end{cases}
\]

Hence, \((\psi, \phi)\) is a lower solution of (3.5) associated with \( \mathcal{R} := [0, u^*] \times [0, hu^*] \).

Next, we show that \((\phi(s), \psi(s)) \leq (\overline{\phi}(s), \overline{\psi}(s))\) for \( s \in \mathbb{R} \), where \((\overline{\phi}, \overline{\psi})\) is the upper solution obtained in Lemma 3.4. Clearly, \( \psi(s) \leq \overline{\psi}(s) \) for \( s > 0 \) and \( \phi(s) = \overline{\phi}(s) = 0 \) for \( s < 0 \). We only need to show that \( \psi(s) \leq \overline{\psi}(s) \) for \( s \leq 0 \), and \( \phi(s) \leq \overline{\phi}(s) \) for \( s \geq 0 \). Let \( \psi_1(s) = 1 - \psi(s) \) and \( \overline{\psi}_1(s) = 1 - \overline{\psi}(s) \). In view of \( \phi(s) = \overline{\phi}(s) = 0 \) for \( s < 0 \) and (3.7), we have

\[-d\psi_1'' - c\psi_1' - r\psi_1(1 - \psi_1) \geq 0 \geq -d\psi_1'' - c\psi_1' - r\psi_1(1 - \psi_1) \text{ for } s > 0, \]

\[\psi_1(\infty) = \overline{\psi}_1(\infty) = 1, \quad \psi_1(0) > \overline{\psi}_1(0).\]

By Lemma 3.2, \( \psi_1(s) \geq \overline{\psi}_1(s) \) for \( s \geq 0 \). Hence, \( \overline{\psi}(s) \leq \overline{\psi}(s) \) for \( s \geq 0 \). Similarly, we can prove \( \phi(s) \leq \overline{\phi}(s) \) for \( s \geq 0 \). Thus

\[(\phi(s), \psi(s)) \leq (\overline{\phi}(s), \overline{\psi}(s)) \text{ for } s \in \mathbb{R}.
\]

The monotonicity of \( \phi(s) \) and \( \psi(s) \) then infers that

\[
\sup_{t \leq s} (\phi(t), \psi(t)) \leq (\overline{\phi}(s), \overline{\psi}(s)) \text{ for } s \in \mathbb{R}.
\]

Therefore we can apply Proposition 2.5 to conclude that (3.5) has a positive solution \((\phi, \psi)\) for each \( c \in [0, c_0^*]\), except that we only have \( \phi'(s) \geq 0 \) for \( s > 0 \) and \( \psi'(s) \geq 0 \) for \( s \in \mathbb{R} \).

It remains to prove \( \phi'(s) > 0 \) for \( s > 0 \) and \( \psi'(s) > 0 \) for \( s \in \mathbb{R} \). Since \( \phi'(s) \geq 0 \) for \( s \in \mathbb{R} \setminus \{0\} \) and \( \psi'(s) \geq 0 \) for \( s \in \mathbb{R} \), and none of them is identically \( 0 \), applying the strong maximum principle to the cooperative system

\[
\begin{cases}
(\phi')'' - c\phi' + (1 - k - 2\phi) + k(\phi' \psi + \phi' \psi') = 0, & \text{for } s \in \mathbb{R}^+,
(d\psi')'' - c(d\psi')' + r(1 - \psi)(h\phi - \psi) + r(1 - \psi)(h\phi - \psi') = 0, & \text{for } s \in \mathbb{R},
\phi'(s) = 0 \text{ for } s \leq 0, \quad \psi'(-\infty) = 0, \quad (\phi'(\infty), \psi'(\infty)) = (0, 0)
\end{cases}
\]

satisfied by \((\phi', \psi')\), we have \( \phi'(s) > 0 \) for \( s > 0 \) and \( \psi'(s) > 0 \) for \( s \in \mathbb{R} \). \( \Box \)

To prove uniqueness for solutions of (3.5), we need to investigate the asymptotic behavior of solutions to (3.5) as \( s \to \infty \). To this end we consider the linearized equation of (3.5) at \((u^*, hu^*)\):

\[
\begin{cases}
\phi'' - c\phi' - u^*\phi + ku^*\psi = 0,
\psi'' - c\psi' - rv^*\psi + rh \phi = 0.
\end{cases}
\]

(3.9)

If \((\phi, \psi) = (me^{\mu s}, ne^{\mu s})\) solves (3.9), then \((m, n)\) and \(\mu\) must satisfy

\[
A(\mu)(m, n)^T = (0, 0)^T,
\]

(3.10)

where

\[
A(\mu) = \begin{pmatrix}
\mu^2 - c\mu - u^* & ku^* \\
hrv^* & \mu^2 - c\mu - rv^*
\end{pmatrix}.
\]
Let
\[ P_1(\mu) := \det(A(\mu)) = (\mu^2 - c\mu - u^*)(d\mu^2 - c\mu - rv^*) - khu^*v^*. \]
Then (3.10) has a nonzero solution \((m, n)^T\) if and only if \(P_1(\mu) = 0\).

Let
\[ \mu^\pm := \frac{c \pm \sqrt{c^2 + 4u^*}}{2} \quad \text{and} \quad \mu^\pm := \frac{c \pm \sqrt{c^2 + 4u^*}}{2d} \]
be the two roots of
\[ \mu^2 - c\mu - u^* = 0 \quad \text{and} \quad d\mu^2 - c\mu - rv^* = 0, \]
respectively. Clearly
\[ P_1(0) = (1-kh)ru^*v^* > 0, \quad P_1(\pm \infty) = \infty \quad \text{and} \quad P_1(\mu^\pm) = -khu^*v^* < 0 \quad \text{for} \quad i = 1, 2. \]
Hence, for any \(c \geq 0, \ \ P_1(\mu) = 0\) has four different real roots \(\hat{\mu}_i (i = 1, 2, 3, 4)\) satisfying
\[
\hat{\mu}_1 < \min\{\mu_1^-, \mu_2^+\} \leq \max\{\mu_1^-, \mu_2^-\} < \hat{\mu}_2 < 0, \\
0 < \hat{\mu}_3 < \min\{\mu_1^+, \mu_2^-\} \leq \max\{\mu_1^+, \mu_2^+\} < \hat{\mu}_4.
\]
(3.11)

**Lemma 3.6.** Let \((\hat{\theta}(s), \hat{\psi}(s))\) be a solution of (3.5). Then there exist positive constants \(m\) and \(n\) independent of \((\hat{\theta}, \hat{\psi})\), and a positive constant \(\beta\) depending on \((\hat{\theta}, \hat{\psi})\), such that
\[
(\hat{\theta}(s), \hat{\psi}(s)) = (u^*, hu^*) - \beta e^{(m,n)(s)}[1 + o(1)] \quad \text{as} \quad s \to \infty.
\]
(3.12)

**Proof.** Let \((\hat{\theta}(s), \hat{\psi}(s))\) be an arbitrary solution of (3.5). The inequalities (3.11) imply that the first order ODE system satisfied by \((\hat{\theta}(s), \hat{\theta}'(s), \hat{\psi}(s), \hat{\psi}'(s))\) has a critical point at \((u^*, 0, hu^*, 0)\), which is a saddle point. By standard stable manifold theory (see, e.g., Theorem 4.1 and its proof in Chapter 13 of [2]), we can conclude that
\[
(u^*, hu^*) - (\hat{\theta}(s), \hat{\psi}(s)) \to (0, 0) \quad \text{exponentially as} \quad s \to \infty.
\]

Let \((\hat{\theta}, \hat{\psi}) = (u^*, hu^*) - (\hat{\theta}(s), \hat{\psi}(s))\). Then \((\hat{\theta}, \hat{\psi})\) satisfies
\[
\begin{cases}
\hat{\theta}'' - c\hat{\theta}' - u^*\hat{\theta} + ku^*\hat{\psi} + \delta_1(s)\hat{\theta} + \delta_2(s)\hat{\psi} = 0, \\
\hat{\psi}'' - c\hat{\psi}' - rv^*\hat{\psi} + hrv^*\hat{\theta} + \delta_3(s)\hat{\theta} + \delta_4(s)\hat{\psi} = 0,
\end{cases}
\]
(3.13)

where
\[
\delta_1(s) := \hat{\theta}(s), \delta_2(s) := -k\hat{\theta}(s), \delta_3(s) := -r\hat{\psi}(s), \delta_4(s) := r\hat{\psi}(s).
\]

Clearly,
\[
\delta_i(s) \to 0 \quad \text{exponentially as} \quad s \to \infty \quad \text{for} \quad i = 1, 2, 3, 4.
\]

Now we turn to consider the linear system (3.9). Recall that \(P_1(\mu) = 0\) has four different real roots satisfying \(\mu_1 < \mu_2 < 0 < \mu_3 < \mu_4\). Let \((m_i, n_i)\) be an eigenvector corresponding to \(\mu = \mu_i\) in (3.10), i.e.,
\[
(m_i, n_i) \neq (0, 0) \quad \text{and} \quad A(\mu_i)(m_i, n_i)^T = (0, 0)^T.
\]
Then (3.9) has four linearly independent solutions
\[
\Psi_i = (m_i, n_i)e^{\mu_i s}, \quad i = 1, 2, 3, 4,
\]
which form a fundamental system for (3.9). Applying Theorem 8.1 in Chapter 3 of [2] to the system (3.13), viewed as a perturbed linear system of (3.9), we conclude that (3.13) has four linearly independent solutions \( \tilde{\Upsilon}_i(s) \) to the system (3.13), satisfying
\[
\tilde{\Upsilon}_i(s) = (1 + o(1))\Upsilon(s) \quad \text{as} \quad s \to \infty, \quad i = 1, 2, 3, 4,
\]
which form a fundamental system for (3.9). So the solution \((\hat{\phi}, \hat{\psi})\) of (3.13) can be represented as
\[
(\hat{\phi}(s), \hat{\psi}(s)) = \sum_{i=1}^{4} \beta_i \tilde{\Upsilon}_i(s),
\]
where \( \beta_i \) (\( i = 1, 2, 3, 4 \)) are constants.

Since \((\hat{\phi}(\infty), \hat{\psi}(\infty)) = (0, 0), \) and \( 0 < \hat{\mu}_3 < \hat{\mu}_4, \) we necessarily have \( \beta_3 = \beta_4 = 0. \) We claim that \( \beta_2 \neq 0. \) Otherwise we necessarily have \( \beta_1 \neq 0 \) and
\[
(\hat{\phi}(s), \hat{\psi}(s)) = \beta_1 \tilde{\Upsilon}_1(s) = (1 + o(1))\beta_1(m_1, n_1)e^{\hat{\mu}_1 s} \quad \text{as} \quad s \to \infty.
\]
However, it is easily checked that all the four elements of the matrix \( A(\hat{\mu}_1) \) are positive, which implies that \( m_1 \cdot n_1 < 0 \) and so the two components of the vector \( \beta_1(m_1, n_1) \) have opposite signs. It follows that for all large \( s, \) \((\hat{\phi}(s), \hat{\psi}(s)) \) has a component which is negative, contradicting the fact that \((\hat{\phi}(s), \hat{\psi}(s)) > (0, 0) \) for all \( s > 0. \) Therefore we must have \( \beta_2 \neq 0. \) It is also easily checked that the two rows of the matrix \( A(\hat{\mu}_2) \) have opposite signs and so \( m_2 \cdot n_2 > 0. \) For definiteness, we may assume that \( m_2 \) and \( n_2 \) are positive. Moreover, due to \( \hat{\mu}_1 < \hat{\mu}_2 < 0, \) we have
\[
(\hat{\phi}(s), \hat{\psi}(s)) = \sum_{i=1}^{2} \beta_i \tilde{\Upsilon}_i(s) = (1 + o(1))\beta_2(m_2, n_2)e^{\hat{\mu}_2 s} \quad \text{as} \quad s \to \infty.
\]
Using \((\hat{\phi}(s), \hat{\psi}(s)) > (0, 0) \) for all \( s > 0 \) we further obtain that \( \beta_2 > 0, \) and hence (3.12) holds with \((m, n) := (m_2, n_2) \) and \( \beta := \beta_2. \)

**Lemma 3.7.** The solution of (3.5) is unique.

**Proof.** Let \((\hat{\phi}, \hat{\psi})\) and \((\hat{\phi}_1, \hat{\psi}_1)\) be two arbitrary solutions of (3.5). We are going to show that
\[
(\hat{\phi}(s), \hat{\psi}(s)) \geq (\hat{\phi}_1(s), \hat{\psi}_1(s)) \quad \text{for} \quad s \in \mathbb{R}. \tag{3.14}
\]
Note that if we are able to prove (3.14), then the same argument can also be used to show \((\hat{\phi}_1, \hat{\psi}_1) \geq (\hat{\phi}, \hat{\psi}). \) Hence uniqueness will follow if we can show (3.14). We use the “sliding method” of Berestycki and Nirenberg to prove this below.

For \( s \in \mathbb{R} \) and \( \xi \geq 0, \) define
\[
\hat{\phi}(s) = \hat{\phi}^\xi(s) := \hat{\phi}_1(s - \xi), \quad \hat{\psi}(s) = \hat{\psi}^\xi(s) := \hat{\psi}_1(s - \xi).
\]
We claim that there exists a constant \( \xi_0 > 0 \) such that, for every \( \xi \geq \xi_0 \)
\[
(\hat{\phi}^\xi(s), \hat{\psi}^\xi(s)) \leq (\hat{\phi}(s), \hat{\psi}(s)) \quad \text{for all} \quad s \in \mathbb{R}. \tag{3.15}
\]
Since \( \hat{\psi}_1(-\infty) = 0 < \hat{\psi}(0), \) there exists \( \xi_1 > 0 \) large enough such that \( \hat{\psi}_1(-\xi_1) \leq \hat{\psi}(0). \) Then \( \hat{\psi}^\xi(0) = \hat{\psi}_1(-\xi) \leq \hat{\psi}(0) \) for all \( \xi \geq \xi_1, \) and \( \hat{\psi}(s), \hat{\psi}^\xi(s) \) satisfy
\[
d\hat{\psi}'' + c\hat{\psi}' + r\hat{\psi}(\hat{\psi} - 1) = 0 = d\hat{\psi}'' + c\hat{\psi}' + r\hat{\psi}(\hat{\psi} - 1), \quad s \leq 0,
\]
\[
\hat{\psi}(\infty) = \hat{\psi}_1(-\infty) = 0, \quad \hat{\psi}(0) \geq \hat{\psi}(0).\]
Let \( u_1(s) := 1 - \psi(-s) \) and \( u_2(s) := 1 - \tilde{\psi}(-s) \). Then \( u_1 \) and \( u_2 \) satisfy
\[
\begin{align*}
du''_2 + cu'_2 + ru_2(1 - u_2) &= 0 = du''_1 + cu'_1 + ru_1(1 - u_1), \quad s \geq 0, \\
u_2(\infty) = u_1(\infty) &= 1, \quad u_2(0) \leq u_1(0).
\end{align*}
\]
By Lemma 3.2, we deduce that \( u_2(s) \leq u_1(s) \) for all \( s \geq 0 \). We thus obtain
\[
\psi^\xi(s) \leq \tilde{\psi}(s) \quad \text{for all } s \leq 0 \text{ and } \xi \geq \xi_1. \tag{3.16}
\]
Applying Lemma 3.6, we can find \( \xi_2 > \xi_1 \) and \( s_0 > 1 \) such that
\[
\tilde{\phi}^\xi(s) \leq \tilde{\phi}(s) \quad \text{for all } s \geq s_0.
\]
Denote \( \xi_0 = \xi_2 + s_0 \). Since \( \tilde{\phi}_1 \) is nondecreasing in \( \mathbb{R} \) and is identically 0 in \(( -\infty, 0] \), it follows that
\[
\tilde{\phi}^\xi(s) \leq \tilde{\phi}(s) \quad \text{for } s \in \mathbb{R}, \; \xi \geq \xi_0. \tag{3.17}
\]
Similarly, for \( s > 0 \), \( \psi(s) \) and \( \psi^\xi(s) \) satisfy
\[
\begin{align*}
-d\psi'' + c\psi' &= r(1 - \psi)(h\tilde{\phi} - \psi), \quad \psi > 0, \\
-d\psi'' + c\psi' &\leq r(1 - \psi)(h\tilde{\phi} - \psi) \leq \tilde{r}(1 - \psi)(h\tilde{\phi} - \psi), \quad s > 0, \\
\psi(\infty) &= \psi(\infty) = hu^*, \quad \psi(0) \geq \tilde{\psi}(0).
\end{align*}
\]
Let \( 1 - \tilde{\psi} = v_1 \) and \( 1 - \psi = v_2 \). Then \( v_1 \) and \( v_2 \) satisfy
\[
\begin{align*}
dv''_1 - cv'_1 &= -rv_1(1 - h\tilde{\phi} - v_1), \quad s > 0, \\
v''_1 - cv'_1 &\geq r(1 - \psi)(h\tilde{\phi} - \psi) = -rv_1(1 - h\tilde{\phi} - v_1), \quad s > 0, \\
v_2(0) \leq v_1(0), \quad v_2(\infty) &= v_1(\infty) = 1 - hu^*.
\end{align*}
\]
Using Lemma 3.2 again, we have \( v_2(s) \leq v_1(s) \) for all \( s \geq 0 \) and \( \xi \geq \xi_0 \), and hence
\[
\tilde{\psi}(s) \geq \psi^\xi(s) \quad \text{for all } s > 0 \text{ and } \xi \geq \xi_0. \tag{3.18}
\]
Combining (3.18), (3.16) and (3.17), we immediately obtain (3.15).

Define
\[
\check{\xi} := \inf \{ \xi_0 > 0 : (\check{\phi}(s), \check{\psi}(s)) \geq (\check{\phi}_1(s - \xi), \check{\psi}_1(s - \xi)) \text{ for } s \in \mathbb{R}, \quad \forall \xi \geq \xi_0 \}.
\]
By (3.15), \( \check{\xi} \) is well defined. Since \( \check{\phi}(0) = 0 < \check{\phi}_1(-\xi) \) for \( \xi < 0 \), we have \( \check{\xi} \geq 0 \). Clearly,
\[
(\check{\phi}(s), \check{\psi}(s)) \geq (\check{\phi}_1(s - \check{\xi}), \check{\psi}_1(s - \check{\xi})) \quad \text{for } s \in \mathbb{R}.
\]
If \( \check{\xi} = 0 \), then the above inequality already yields (3.14), and the proof is finished. Suppose \( \check{\xi} > 0 \). We are going to derive a contradiction. To simplify notations we write
\[
(\phi_\xi(s), \psi_\xi(s)) = (\check{\phi}_1(s - \check{\xi}), \check{\psi}_1(s - \check{\xi})),
\]
and set
\[
P(s) := \check{\phi}(s) - \phi_\xi(s), \quad Q(s) := \check{\psi}(s) - \psi_\xi(s).
\]
Then the nonnegative functions \( P \) and \( Q \) satisfy
\[
\begin{align*}
P'' - cP' + (1 - k - \check{\phi} - k\psi)P + k\check{\phi}Q &= 0, \quad s > \check{\xi}, \\
dQ'' - cQ' + r(\check{\psi} + \psi_\xi - 1 - h\check{\psi})Q + rh(1 - \check{\psi})P &= 0, \quad s \in \mathbb{R}, \\
P(0) &= P(\infty) = Q(\infty) = Q(\infty) = 0. \tag{3.19}
\end{align*}
\]
The strong maximum principle implies that \( P(s) > 0 \) for \( s \geq \bar{c} \) and \( Q(s) > 0 \) for \( s \in \mathbb{R} \). Rewrite (3.19) as

\[
\begin{aligned}
P''(s) - u^*P + kP(s) + h = 0, \quad s > \bar{c}, \\
dQ'(s) - rQ + rQ' + \epsilon_3(s)P + \epsilon_4(s)Q = 0, \quad s \in \mathbb{R},
\end{aligned}
\]

where

\[
\begin{aligned}
\epsilon_1 = 1 - k - \phi - \phi' + k\phi' + u^*, \quad \epsilon_2 = k\phi - ku^*, \\
\epsilon_3 = rh(1 - \psi) - rhu^*, \quad \epsilon_4 = r(\psi - \phi' - h\psi + r\psi).
\end{aligned}
\]

By Lemma 3.6, \( \epsilon_i(s) \to 0 \) exponentially as \( s \to \infty \) for \( i = 1, 2, 3, 4 \). We may now repeat the proof process of Lemma 3.6 to obtain

\[
(P(s), Q(s)) = (\tilde{C}_1 + o(1), \tilde{C}_2 + o(1))e^{\mu_2 s} \text{ as } s \to \infty,
\]

where \( \tilde{C}_1, \tilde{C}_2 \) are positive constants. By Lemma 3.6, there are positive constants \( C_*, C \) such that

\[
\begin{aligned}
\tilde{\phi}(s), \tilde{\psi}(s) = (u^*, hu^*) - C_*(m + o(1), n + o(1))e^{\bar{\mu}_2 s} \text{ as } s \to \infty, \\
(\tilde{\phi}(s), \tilde{\psi}(s)) = (u^*, hu^*) - C(m + o(1), n + o(1))e^{\bar{\mu}_2(s - \bar{c})} \text{ as } s \to \infty,
\end{aligned}
\]

which lead to

\[
\begin{aligned}
(mC e^{\bar{\mu}_2(\bar{c} - c)} - C_*)(nC e^{\bar{\mu}_2(\bar{c} - c)} - C_*) = (\tilde{C}_1, \tilde{C}_2) > (0, 0).
\end{aligned}
\]

Therefore, there exists \( \epsilon_0 > 0 \) sufficiently small so that for any \( \epsilon \in (0, 2\epsilon_0) \),

\[
\begin{aligned}
(mC e^{\bar{\mu}_2(\bar{c} - c)} - C_*)(nC e^{\bar{\mu}_2(\bar{c} - c)} - C_*) > (0, 0).
\end{aligned}
\]

It follows that, for all large \( s \), say \( s \geq M > \bar{c} \), we have

\[
\tilde{\phi}(s), \tilde{\psi}(s) \geq (\tilde{\phi}_1(s - \bar{c} + \epsilon), \tilde{\psi}_1(s - \bar{c} + \epsilon)) \text{ for } \epsilon \in (0, \epsilon_1).
\]

Since \( (P(s), Q(s)) > (0, 0) \) for \( s \in [\bar{c}, M] \), by continuity, we can find \( \epsilon_1 \in (0, \epsilon_0) \) such that, for every \( \epsilon \in (0, \epsilon_1) \),

\[
\tilde{\phi}(s), \tilde{\psi}(s) \geq (\tilde{\phi}_1(s - \bar{c} + \epsilon), \tilde{\psi}_1(s - \bar{c} + \epsilon)) \text{ for } s \in [\bar{c}, M].
\]

Hence

\[
\tilde{\phi}(s), \tilde{\psi}(s) \geq (\tilde{\phi}_1(s - \bar{c} + \epsilon), \tilde{\psi}_1(s - \bar{c} + \epsilon)) \text{ for } s \geq \bar{c}, \epsilon \in (0, \epsilon_1). \tag{3.20}
\]

Since \( \tilde{\phi}(\bar{c}) > 0 = \tilde{\phi}_1(0) \), by continuity, there exists \( \epsilon_2 \in (0, \epsilon_1] \) such that \( \tilde{\phi}(\bar{c} - \epsilon) \geq \tilde{\phi}_1(\epsilon) \) for \( \epsilon \in (0, \epsilon_2] \). It follows that

\[
\tilde{\phi}(s) \geq \tilde{\phi}_1(s - \bar{c} + \epsilon) \text{ for } s \in [\bar{c} - \epsilon, \bar{c}].
\]

Since \( \tilde{\phi}_1(s - \bar{c} + \epsilon) \equiv 0 \) for \( s \leq \bar{c} - \epsilon \), the above inequality holds for all \( s \in (-\infty, \bar{c}] \).

This and (3.20) imply

\[
\tilde{\phi}(s) \geq \tilde{\phi}_1(s - \bar{c} + \epsilon) \text{ for } s \in \mathbb{R}, \epsilon \in (0, \epsilon_2]. \tag{3.21}
\]

Denote

\[
(\psi_\epsilon(s), \psi_\epsilon(s)) := (\tilde{\phi}_1(s - \bar{c} + \epsilon), \tilde{\psi}_1(s - \bar{c} + \epsilon)).
\]

We obtain, for any fixed \( \epsilon \in (0, \epsilon_2] \),

\[
-d\psi_\epsilon'' + e\psi_\epsilon' = r(1 - \psi_\epsilon)(h\phi_\epsilon - \psi_\epsilon) \leq r(1 - \psi_\epsilon)(h\phi - \psi_\epsilon) \text{ for } s \in \mathbb{R}.
\]

Moreover, \( \psi_\epsilon(-\infty) = 0, \psi_\epsilon(\infty) = hu^* \). Hence \( u_2(s) := 1 - \psi_\epsilon(-s) \) satisfies

\[
u_2'' + cu_2' + ru_2(1 - h\phi(-s) - u_2) \leq 0 \text{ for } s \in \mathbb{R}, \quad u_2(\infty) = 1.
\]
Since \( u_1(s) := 1 - \tilde{\psi}(-s) \) satisfies
\[
du''_1 + cu'_1 + ru_1(1 - h\tilde{\psi}(-s) - u_1) = 0 \quad \text{for } s \in \mathbb{R}, \quad u_1(\infty) = 1
\]
and \( u_2(-\zeta) \leq u_2(-\tilde{\zeta}) \), we may apply Lemma 3.2 to conclude that \( u_1(s) \leq u_2(s) \) for \( s \geq -\zeta \), i.e., \( \psi(s) \geq \psi_c(s) \) for \( s \leq \zeta \). Combining this with (3.20), we obtain
\[
\psi(s) \geq \psi_c(s) = \psi_1(s - \zeta + \epsilon) \quad \text{for } s \in \mathbb{R}, \quad \epsilon \in (0, \epsilon_2).
\]

This and (3.21) clearly contradict the definition of \( \zeta \). Hence the case \( \zeta > 0 \) cannot happen, and the proof is complete.

Next we will make use of problem (1.5). Setting \( \Phi(s) := \Phi(s), \Psi(s) := 1 - \Psi(s) \), we may change (1.5) to the following cooperative system
\[
\begin{align*}
\Phi'' - c\Phi' + \Phi(1 - k - \Phi + k\Psi) &= 0, \quad \Phi' > 0, \quad s \in \mathbb{R}, \\
c\Phi'' - c\Psi' + r(1 - \Phi(\Phi - \Psi)) &= 0, \quad \Psi' > 0, \quad s \in \mathbb{R}, \\
(\Phi, \Psi)(-\infty) &= (0, 0), \\
(\Phi, \Psi)(\infty) &= (u^*, hu^*).
\end{align*}
\]
(3.22)

From Proposition 1.4, we know that there exists \( c_* \geq 2\sqrt{1 - k} \) such that (3.22) possesses a solution if and only if \( c \geq c_* \).

In what follows, we shall show \( c_0^* = c_* \) and (3.5) has no solution for \( c \geq c_* \).

Lemma 3.8. \( c_0^* \geq c_* \).

Proof. Let \( (\Phi_0, \Psi_0) \) be a solution of (3.22) with \( c = c_* \). It is easily checked that \( (0, 0, 0, 0) \) is a saddle equilibrium point of the ODE system satisfied by \( (\Phi_0, \Phi_0', \Psi_0, \Psi_0') \). It follows that
\[
\Phi_0, \Phi_0', \Psi_0, \Psi_0' \rightarrow 0 \quad \text{exponentially as } s \rightarrow -\infty.
\]

Following the idea in the proof of Lemma 3.6, we rewrite the equation satisfied by \( \Phi_0 \) as
\[
\Phi''_0 - c_* \Phi' + (1 - k)\Phi + \epsilon(s)\Phi = 0
\]
with
\[
\epsilon(s) := k\Psi_0(s) - \Phi_0(s) \rightarrow 0 \quad \text{exponentially as } s \rightarrow -\infty,
\]
and view it as a perturbed linear equation to
\[
\Phi'' - c_* \Phi' + (1 - k)\Phi = 0.
\]

Using the fundamental solutions of this latter equation we see that, as \( s \rightarrow -\infty \), the asymptotic behaviour of \( (\Phi_0, \Phi_0') \) is given by
\[
(\Phi_0(s), \Phi_0'(s)) = \begin{cases} 
(1, \alpha_1)k_0e^{\alpha_1 s}(1 + o(1)), & \text{when } c_* > 2\sqrt{1 - k}, \\
(1 + \alpha_1)k_0|s|e^{\alpha_1 s}(1 + o(1)) & \text{or } (1, \alpha_1)k_0e^{\alpha_1 s}(1 + o(1)), \quad \text{when } c_* = 2\sqrt{1 - k}.
\end{cases}
\]
(3.23)

for some \( k_0 > 0, \alpha_1 \in \left\{ \frac{1}{2} \left( c_* + \sqrt{c^2_* - 4(1 - k)} \right), \frac{1}{2} \left( c_* - \sqrt{c^2_* - 4(1 - k)} \right) \right\} \).

Fix \( \epsilon \in (0, k/\alpha_1) \) small. In view of (3.23), there exists a constant \( M_0 < 0 \) such that
\[
\frac{\Phi'_0(s)}{\Phi_0(s)} \geq \frac{\alpha_1}{2}, \quad \max\{\Phi_0(s), \Psi_0(s)\} < \min\left\{ \frac{c\alpha_1}{4k}, 1 - k \right\} \quad \text{for } s < M_0.
\]
(3.24)

Next, we prove that system (3.5) has a solution for \( c = c_* - \epsilon \). To this end, we will treat the cases \( c_* > 2\sqrt{1 - k} \) and \( c_* = 2\sqrt{1 - k} \) separately.

Case 1. \( c_* > 2\sqrt{1 - k} \).

Introduce an auxiliary function
\[ p_1(s) = 0 \text{ for } s \geq M_0, \quad p_1(s) = e^{\beta_1 s} \text{ for } s \leq M_0 - 1 \]
and for \( s \in (M_0 - 1, M_0) \), \( p_1(s) > 0, p_1'(s) \leq 0 \), where \( \beta_1 = \left( c_* - \sqrt{c_*^2 + 2d} \right)/(2d) \) < 0 and \( M_0 \) is given by (3.24). Moreover, \( p_1(s) \) is \( C^2 \) everywhere. Define
\[
\psi_1(s) := \Psi_0(s) - \epsilon_1 p_1(s),
\]
where the positive constant \( \epsilon_1 \) will be determined later.

We now calculate
\[
d\psi_1'' - (c_* - \epsilon)\psi_1' + r(1 - \psi_1)(h\Phi_0 - \psi_1) = r(1 - \psi_1)(h\Phi_0 - \psi_1) - r(1 - \psi_0)(h\Phi_0 - \Psi_0) - c_1 p_1' + \epsilon \Psi_0' + c_1 \epsilon_1 p_1'
\]
(3.25)

Hence we can fix \( \epsilon_1 > 0 \) sufficiently small so that, for \( s \in [M_0 - 1, M_0] \),
\[
d\psi_1'' - (c_* - \epsilon)\psi_1' + r(1 - \psi_1)(h\Phi_0 - \psi_1) > 0
\]
and
\[
\psi_1'(s) > 0, \quad \psi_1(s) > 0.
\]
By the definition of \( p_1(s) \) for \( s \leq M_0 - 1 \), clearly \( \psi_1'(s) > 0 \) for \( s \leq M_0 - 1 \), and \( \psi_1(s) \to -\infty \) as \( s \to -\infty \). Hence there exists a unique constant \( M_1 < M_0 - 1 \) such that \( \psi_1(M_1) = 0 \). We define
\[
\Phi(s) = \Phi_0(s) \text{ for } s \geq M_1, \quad \Psi(s) = \left\{ \begin{array}{ll} \Psi_0(s) - \epsilon_1 p_1(s), & s > M_1, \\ 0, & s \leq M_1. \end{array} \right.
\]
For \( s \in (M_1, M_0 - 1) \), by the choice of \( \Psi_0 \leq 1/4 \) in (3.24) for \( s \leq M_0 \), we have
\[
r p_1(1 + h\Phi_0 - 2\Psi_0 + c_1 p_1) - dp_1'' - \epsilon p_1' + c_1 p_1'
\]
\[
> r p_1(1 + h\Phi_0 - 2\Psi_0) - dp_1'' + c_1 p_1'
\]
\[
> - dp_1'' + c_1 p_1' + r^2 p_1 = 0.
\]
Therefore, it follows from (3.25) that
\[
d\Psi'' - (c_* - \epsilon)\Psi' + r(1 - \Psi)(h\Phi - \Psi) > 0, \quad s \in (M_1, M_0 - 1).
\]
Since \( (\Phi, \Psi) = (\Phi_0, \Psi_0) \) for \( s > M_0 \), and \( \Psi(s) = 0 \) for \( s \leq M_1 \), it is easy to verify that for any smooth extension of \( \Phi(s) \) to \( s \leq M_1 \) satisfying \( \Phi(s) \geq 0 \) in \((-\infty, M_1)\), we have
\[
d\Psi'' - (c_* - \epsilon)\Psi' + r(1 - \Psi)(h\Phi - \Psi) \geq 0
\]
for all \( s \in \mathbb{R} \). Moreover, \( \Psi'(M_1 -) = 0 \leq \Psi'(M_1 +) \).

In view of (3.24) and \( \psi_1(M_1) = 0 \), we have \( \epsilon_1 p_1(M_1) = \Psi_0(M_1) < \frac{aM}{hp_1} \). For \( s \in (M_1, M_0) \), by (3.24), we can fix \( \epsilon_1 > 0 \) sufficiently small so that
\[
\Phi_0'' - (c_* - \epsilon)\Phi_0' + \Phi_0(1 - k - \Phi_0 + k\Psi)
\]
\[
= \epsilon \Phi_0' - k \epsilon_1 p_1 \Phi_0 \geq \left( \frac{a_1}{2} - k \epsilon_1 p_1 \right) \Phi_0 \geq 0.
\]
For \( s \leq M_1 \), we choose \( \epsilon > 0 \) sufficiently small so that the quadratic equation
\[
\alpha^2 - (c_* - \epsilon)\alpha + (1 - k) = 0
\]
has a root \( \alpha_\epsilon \in (\alpha_1/2, \alpha_1) \).

Define
\[
p_2(s) = 0 \text{ for } s \geq M_1, \quad p_2(s) = e^{\alpha_\epsilon s} \text{ for } s \leq M_1 - 1
\]
and for \( s \in [M_1 - 1, M_1] \), we define \( p_2(s) \) so that \( p_2(s) > 0 \) and \( p_2(s) \) is \( C^2 \) everywhere. We define
\[
\Phi(s) = \Phi_0(s) - \epsilon_2 p_2(s),
\]
with \( \epsilon_2 > 0 \) to be determined.

Since \( \alpha_\epsilon < \alpha_1 \), by (3.23) we can find \( M_1^\prime < M_1 - 1 \) such that
\[
\Phi_0(s) > \alpha_\epsilon \Phi_0(s) \text{ for } s \leq M_1^\prime.
\]
It follows that, for \( s \leq M_1^\prime \),
\[
\Phi'(s) = \Phi_0'(s) - \epsilon_2 p_2'(s) > \alpha_\epsilon \Phi_0(s) - \epsilon_2 \alpha_\epsilon p_2(s) = \alpha_\epsilon \Phi(s).
\]
Recall that \( \alpha_\epsilon > \alpha_1/2 \). We now choose \( \epsilon_2 \) sufficiently small such that, for \( s \in [M_1^\prime, M_1] \),
\[
\Phi(s) > 0, \quad \Phi'(s) > 0
\]
and
\[
\Phi'' - (c_* - \epsilon) \Phi' + \Phi(1 - k - \Phi + k \Psi)
\]
\[
\begin{align*}
&= -k \Phi_0 \Psi_0 + c_0 \Phi_0' + c_2 [c_* p_2'' - \epsilon p_2' - p_2(1 - k - 2 \Phi_0 + \epsilon_2 p_2)] \\
&> -k \Phi_0 \Psi_0 + \epsilon \frac{\alpha_1}{2} \Phi_0 + c_2 [c_* p_2'' - \epsilon p_2' - p_2(1 - k - 2 \Phi_0 + \epsilon_2 p_2)] \\
&> \frac{\epsilon}{4} \alpha_1 \Phi_0 + c_2 [c_* p_2'' - \epsilon p_2' - p_2(1 - k - 2 \Phi_0 + \epsilon_2 p_2)] > 0.
\end{align*}
\]
Due to \( \alpha_\epsilon < \alpha_1 \) and (3.23), we easily deduce \( \lim_{s \to -\infty} \frac{\Phi_0(s)}{e^{\alpha_\epsilon s}} = 0 \). It follows that
\[
\Phi(s) = e^{\alpha_\epsilon s} \left( \frac{\Phi_0(s)}{e^{\alpha_\epsilon s}} - \epsilon_2 \right) < 0
\]
for all large negative \( s \). Since \( \Phi(M_1^\prime) > 0 \), by continuity, there exists \( M_2 < M_1^\prime \) such that
\[
\Phi(M_2) = 0, \quad \Phi(s) > 0 \text{ for } s \in (M_2, M_1^\prime).
\]
Thus for \( s \in (M_2, M_1^\prime) \), we have \(-2 \Phi_0 + \epsilon_2 p_2 < -\Phi_0 < 0 \) and
\[
\Phi'' - (c_* - \epsilon) \Phi' + \Phi(1 - k - \Phi + k \Psi)
\]
\[
\begin{align*}
&> \frac{\epsilon}{4} \alpha_1 \Phi_0 + \epsilon_2 [c_* p_2'' - \epsilon p_2' - p_2(1 - k - 2 \Phi_0 + \epsilon_2 p_2)] \\
&\geq \frac{\epsilon}{4} \alpha_1 \Phi_0 + \epsilon_2 [c_* p_2'' - \epsilon p_2' - (1 - k) p_2] = \frac{\epsilon}{4} \alpha_1 \Phi_0 > 0.
\end{align*}
\]
Define
\[
\Phi(s) = \begin{cases} 
\Phi_0(p_2(s)), & s \geq M_2, \\
0, & s \leq M_2.
\end{cases}
\]
It is easy to see that
\[
\Phi'' - (c_* - \epsilon) \Phi' + \Phi(1 - k - \Phi + k \Psi) \geq 0
\]
for all \( s \in \mathbb{R} \). Moreover, \( \Phi'(M_2^-) = 0 \leq \Phi'(M_2^+) \).

Finally, we accomplish the proof by the upper and lower solution argument. Define
\[
(\phi(s), \psi(s)) := (\Phi(s + M_2), \Psi(s + M_2)).
\]
Then \((\phi(s), \psi(s))\) is a lower solution for (3.5) with \( c = c_* - \epsilon \) associated with \( \mathcal{R} := [0, u^*] \times [0, hu^*] \). Moreover, it is easy to see that \((\overline{\phi}(s), \overline{\psi}(s))\) is an upper solution for (3.5) with \( c = c_* - \epsilon \) associated with \( \mathcal{R} \), where \((\overline{\phi}(s), \overline{\psi}(s))\) is given by (3.6).
We next check that
\[
\sup_{t \leq s} \left( \phi(t), \psi(t) \right) \leq (\overline{\phi}(s), \overline{\psi}(s)) \text{ holds for all } s \in \mathbb{R}.
\]
By the monotonicity of \( \phi \) and \( \psi \), it suffices to show
\[
(\phi(s), \psi(s)) \leq (\overline{\phi}(s), \overline{\psi}(s)) \text{ for all } s \in \mathbb{R}.
\]
(3.26)
For \( s > 0 \),
\[
\overline{\psi}(s) \leq \Psi_0(s + M_2) < hu^* = \overline{\nu}(s).
\]
Since \( M_1 - M_2 > 0 \) and \( \overline{\psi}(s) = 0 \) for \( s < M_1 - M_2 \), we thus see that
\[
\overline{\psi}(s) \leq \overline{\nu}(s) \text{ for all } s \in \mathbb{R}.
\]
Clearly \( \phi(s) = 0 = \overline{\phi}(s) \) for \( s \leq 0 \). For \( s > 0 \), due to \( \overline{\psi}(s) \leq \psi(\infty) = \Psi_0(\infty) = hu^* < 1 \), we have
\[
0 \leq \phi'' - (c_* - \epsilon)\phi' + \phi(1 - k - \phi + k\overline{\psi}) \leq \phi'' + \phi(1 - \phi).
\]
Moreover, \( \phi(\infty) = \Phi_0(\infty) = u^* = \overline{\nu}(\infty) \). Hence we can apply Lemma 3.2 to conclude that \( \phi(s) \leq \overline{\phi}(s) \) for \( s > 0 \).
We have thus proved (3.26). Clearly \( \underline{\nu}(s) \neq 0 \) and the nonlinearity functions in (3.5) satisfy (A1), (A2), (A3). We may now apply Proposition 2.5 to conclude that (3.5) with \( \overline{\nu} > 0 \) and \( \underline{\nu} > 0 \) relaxed to \( \overline{\nu}' \geq 0 \) and \( \underline{\nu}' \geq 0 \) has a solution \((\overline{\phi}, \underline{\psi})\) with \( c = c_* - \epsilon \), which would imply \( c_0^* \geq c_* - \epsilon \) if we can further prove \( \overline{\nu}' > 0 \) and \( \underline{\nu}' > 0 \). But these strict inequalities follow easily from the strong maximum principle applied to the cooperative system (3.8) satisfied by \((\overline{\nu}', \underline{\nu}')\). By the arbitrariness of \( \epsilon \), it follows \( c_0^* \geq c_* \).

**Case 2.** \( c_* = 2\sqrt{1 - \kappa} \).

It follows from Lemma 3.1 that the following problem
\[
\begin{cases}
\tilde{\chi}'' - (c_* - \epsilon)\tilde{\chi}' + \tilde{\chi}(1 - k - \tilde{\chi}) = 0, & 0 < s < \infty, \\
\tilde{\chi}(0) = 0
\end{cases}
\]
has a unique strictly increasing solution \( \tilde{\chi} \) satisfying \( \tilde{\chi}(\infty) = 1 - k \). Define
\[
\Phi(s) := \begin{cases} 0, & -\infty < s < 0, \\
\delta \tilde{\chi}(s), & 0 \leq s < \infty,
\end{cases} \quad \Psi(s) := 0,
\]
with \( \delta > 0 \) small such that \( \Phi(s) \leq \overline{\phi}(s) \) for \( s \in \mathbb{R} \), where \( \overline{\phi}(s) \) is given by (3.6). It is easy to verify that \((\Phi(s), \Psi(s))\) is a lower solution of (3.5) associated with \( \mathcal{R} \). Moreover, it is easy to see that \((\overline{\phi}(s), \overline{\psi}(s))\) is an upper solution for (3.5) with \( c = c_* - \epsilon \) associated with \( \mathcal{R} \), and \((\Phi(s), \Psi(s)) \leq (\overline{\phi}(s), \overline{\psi}(s))\) for \( s \in \mathbb{R} \). It follows from Proposition 2.5 and the strong maximum principle (applied to \( (\overline{\nu}', \underline{\nu}') \)) as in Case (i) above) that (3.5) has a solution \((\phi, \psi) \in \Sigma \) with \( c = c_* - \epsilon \), which implies \( c_0^* \geq c_* - \epsilon \). By the arbitrariness of \( \epsilon \), it follows \( c_0^* \geq c_* \).

**Lemma 3.9.** For \( c \geq c_* \), problem (3.5) has no solution.

**Proof.** Suppose on the contrary that for some \( c \geq c_* \), (3.5) has a solution \((\overline{\phi}, \underline{\psi})\). By Proposition 1.4, the system (3.22) has a solution \((\overline{\Phi}, \overline{\Psi})\) for such \( c \). We are going to derive a contradiction by making use of \((\overline{\Phi}, \overline{\Psi})\).
We note that by repeating the arguments in the proof of Lemma 3.6, the monotone increasing functions $\phi, \psi, \Phi$ and $\Psi$ can be expanded near $\infty$ in the form (3.12). In view of $\Phi'(s) > 0$ and $\Psi'(s) > 0$, there exists some $\eta_0 > 0$ such that
\[
(\Phi(s + \eta), \Psi(s + \eta)) \geq (\tilde{\phi}(s), \tilde{\psi}(s)), \quad \forall \eta \geq 0, \quad \eta \geq \eta_0.
\]
Clearly
\[
\tilde{\Phi}(s + \eta) > 0 = \tilde{\phi}(s) \text{ for } s < 0. \tag{3.27}
\]
Now we prove that
\[
\tilde{\Psi}(s + \eta) \geq \tilde{\psi}(s) \text{ for } s \in \mathbb{R} \text{ and } \eta \geq \eta_0.
\]
We only need to show this for $s < 0$. Denote, for $\eta \geq \eta_0$,
\[
\Phi_{\eta}(s) := \Phi(s + \eta), \quad \Psi_{\eta}(s) := \Psi(s + \eta),
\]
and let
\[
\tilde{\Phi}_{\eta}(s) = 1 - \tilde{\Phi}_{\eta}(-s), \quad \tilde{\Psi}_{\eta}(s) = 1 - \tilde{\Psi}_{\eta}(-s), \quad \psi(s) = 1 - \tilde{\psi}(-s).
\]
Then
\[
\begin{cases}
-c\tilde{\Psi}_{\eta}' - d\tilde{\Psi}_{\eta}'' = r\psi(1 - \tilde{\Psi}_{\eta} - h\tilde{\phi}), & s \in \mathbb{R}, \\
c\psi' - d\psi'' = r\psi(1 - \psi - h\tilde{\phi}), & s \in \mathbb{R}, \\
\tilde{\Psi}_{\eta}(\infty) = 1 = \psi(\infty), \quad \tilde{\Psi}_{\eta}(0) \leq \psi(0).
\end{cases} \tag{3.28}
\]
Using Lemma 3.2 we deduce $\tilde{\Psi}_{\eta}(s) \leq \psi(s)$ in $[0, \infty)$, and hence $\tilde{\Psi}_{\eta}(s) \geq \tilde{\psi}(s)$ in $(-\infty, 0]$.

We are now able to define
\[
\eta^* = \inf \left\{ \eta_0 \in \mathbb{R} : \quad \tilde{\Phi}_{\eta}(s) \geq \tilde{\phi}(s) \text{ in } [0, \infty), \quad \tilde{\Psi}_{\eta}(s) \geq \tilde{\psi}(s) \text{ in } \mathbb{R}, \quad \forall \eta \geq \eta_0 \right\}.
\]
We claim that $\eta^* = -\infty$. Otherwise, $\eta^*$ is a finite real number, and by continuity,
\[
\tilde{\Phi}_{\eta^*}(s) \geq \tilde{\phi}(s) \text{ in } [0, \infty), \quad \tilde{\Psi}_{\eta^*}(s) \geq \tilde{\psi}(s) \text{ in } \mathbb{R}.
\]
The first inequality of (3.28) still holds for $\eta = \eta^*$, and this inequality is strict for $s < 0$ due to (3.27). Hence $\tilde{\Psi}_{\eta^*}(s) \neq \tilde{\psi}(s)$, and by the strong maximum principle we obtain
\[
\tilde{\Psi}_{\eta^*}(s) > \tilde{\psi}(s) \text{ for } s \in \mathbb{R}.
\]
We now have
\[
c\tilde{\Phi}_{\eta^*}' - \tilde{\Phi}_{\eta^*}'' = \tilde{\Phi}_{\eta^*}(1 - k - \tilde{\phi}_{\eta^*} + k\tilde{\psi}', s \in \mathbb{R}^+, \\
c\phi' - \phi'' = \tilde{\phi}(1 - k - \phi + k\psi), s \in \mathbb{R}^+, \\
\tilde{\Phi}_{\eta^*}(0) \geq \phi(0), \quad \tilde{\Phi}_{\eta^*}(\infty) = \tilde{\phi}(\infty) = u^*.
\]
Using Lemma 3.2 and the strong maximum principle we deduce
\[
\tilde{\Phi}_{\eta^*}(s) > \tilde{\phi}(s) \text{ for } s \in [0, \infty).
\]
We may now use the expansion of $(\tilde{\Phi}_{\eta^*} - \tilde{\phi}, \tilde{\Psi}_{\eta^*} - \tilde{\psi})$ near $s = \infty$ as the proof of Lemma 3.7 to derive that
\[
\tilde{\Phi}_{\eta^* - \epsilon}(s) \geq \tilde{\phi}(s), \quad \tilde{\Psi}_{\eta^* - \epsilon}(s) > \tilde{\psi}(s) \text{ for } s \in [0, \infty) \text{ and some small } \epsilon > 0.
\]
It then follows from the monotonicity of $\tilde{\Phi}$ and $\tilde{\Psi}$ that for all $\eta \geq \eta^* - \epsilon$,
\[
\tilde{\Phi}_{\eta}(s) \geq \tilde{\phi}(s) \text{ in } [0, \infty), \quad \tilde{\Psi}_{\eta}(s) \geq \tilde{\psi}(s) \text{ in } \mathbb{R},
\]
which contradicts the definition of $\eta^*$. Hence, $\eta^* = -\infty$. The fact $\eta^* = -\infty$ implies $\Phi(s + \eta) \geq \phi(s)$ in $[0, \infty)$ for all $\eta \in \mathbb{R}$. For any fixed $s > 0$, letting $\eta \to -\infty$ and using $\Phi(-\infty) = 0$ we obtain $0 \geq \phi(s)$. This is a contradiction to the fact that $(\phi, \psi)$ is a solution of (3.5).

**Lemma 3.10.** $c_0^* = c_*$.

*Proof.* Lemmas 3.8 and 3.5 imply that (3.5) has a solution for every $c \in [0, c_0^*)$. Therefore Lemma 3.9 implies $c_* \geq c_0^*$. In view of Lemma 3.8, we must have $c_0^* = c_*$.

**Lemma 3.11.** $c_0^* \leq 2\sqrt{u^*}$.

*Proof.* Suppose on the contrary that $c_0^* > 2\sqrt{u^*}$. Then system (3.5) has a solution for some $c > 2\sqrt{u^*}$. The monotonicity of $(\phi(s), \psi(s))$ implies that $(\phi(s), \psi(s)) \leq (u^*, hu^*)$ on $\mathbb{R}^+$.

We claim that $\tilde{\phi}'(s)$ and $\tilde{\phi}''(s)$ are uniformly bounded on $\mathbb{R}^+$. Let $\beta = \max\{1 + k, r(1 + h)\}$; it follows from Lemma 6.1 and (6.7) that

$$
|\tilde{\phi}'(s)| = \left| \frac{1}{\lambda_2 - \lambda_1} \left[ \int_0^s K_{1s}(\xi, s)\tilde{\phi}(\xi)(\beta + 1 - k - \tilde{\phi}(\xi) + k\tilde{\psi}(\xi))d\xi + \int_s^\infty K_{2s}(\xi, s)\tilde{\phi}(\xi)(\beta + 1 - k - \tilde{\phi}(\xi) + k\tilde{\psi}(\xi))d\xi \right] \right| \\
\leq \frac{2(\beta + 1 + k + u^* + khu^*)u^*}{\lambda_2 - \lambda_1} := \tilde{C}_1.
$$

Using the boundedness of $\tilde{\phi}, \tilde{\psi}, \tilde{\phi}'$ and (3.5), we obtain that

$$
|\tilde{\phi}''(s)| = |c\tilde{\phi}' - \tilde{\phi}(1 - k - \tilde{\phi} + k\tilde{\psi})| \\
\leq c|\tilde{\phi}'| + |\tilde{\phi}(1 - k - \tilde{\phi} + k\tilde{\psi})| \\
\leq c\tilde{C}_1 + u^*(1 + k + u^* + khu^*) := \tilde{C}_1.
$$

Thus $|\tilde{\phi}'|, |\tilde{\phi}''| < C$ with $C := \max\{\tilde{C}_1, \tilde{C}_1\}$.

Thanks to the uniform boundedness of $\tilde{\phi}, \tilde{\psi}, \tilde{\phi}'$ and $\tilde{\phi}''$, the integrals

$$
\int_0^\infty \tilde{\phi}(s)\tilde{\psi}(s)e^{-\mu s}ds \text{ and } \int_0^\infty \tilde{\phi}(l)e^{-\mu s}ds (l = 0, 1, 2)
$$

are well defined for any $\mu > 0$. In view of $c > 2\sqrt{u^*}$, we know that

$$
\mu^2 - c\mu + u^* = 0
$$

has two positive roots, say $\tilde{\mu}_1, \tilde{\mu}_2$ with $0 < \tilde{\mu}_1 < \tilde{\mu}_2$. Now, choosing $\mu \in (\tilde{\mu}_1, \tilde{\mu}_2)$, multiplying the first equation in (3.5) by $e^{-\mu s}$ and integrating from 0 to $\infty$, we obtain

$$
\tilde{\phi}'(0) + \int_0^\infty \tilde{\phi}^2 e^{-\mu s}ds \\
= (\mu^2 - c\mu) \int_0^\infty \tilde{\phi}e^{-\mu s}ds + \int_0^\infty (1 - k + k\tilde{\psi})\tilde{\phi}e^{-\mu s}ds \\
\leq (\mu^2 - c\mu + u^*) \int_0^\infty \tilde{\phi}e^{-\mu s}ds < 0.
$$

Since $\tilde{\phi}'(0) > 0$ by the Hopf boundary Lemma, we have $\tilde{\phi}'(0) + \int_0^\infty \tilde{\phi}^2 e^{-\mu s}ds > 0$, which contradicts (3.29).
Lemma 3.12. Let \((\tilde{\phi}_c, \tilde{\psi}_c)\) denote the unique solution of \((3.5)\). Then \(0 \leq c_1 < c_2 < c_0^*\) implies
\[
\tilde{\phi}'_c(0) > \tilde{\phi}'_c(0), \quad \tilde{\phi}_c(s) > \tilde{\phi}_c(s) \text{ in } \mathbb{R}^+, \quad \tilde{\psi}_c(s) > \tilde{\psi}_c(s) \text{ in } \mathbb{R}.
\]

Proof. From the proof of Lemma 3.5 and the uniqueness of solutions to \((3.5)\), we have \((\tilde{\phi}_c(s), \tilde{\psi}_c(s)) \leq (\bar{\phi}(s), \bar{\psi}(s))\), where \((\bar{\phi}(s), \bar{\psi}(s))\) is given by \((3.6)\). Moreover, due to \(0 < c_1 < c_2\) and \(\tilde{\phi}'_c > 0\) in \(\mathbb{R}^+\) and \(\tilde{\psi}'_c > 0\) in \(\mathbb{R}\), we have
\[
\begin{aligned}
&\left\{
\begin{array}{ll}
\tilde{\phi}''_c + c_1 \tilde{\psi}'_c < \tilde{\phi}_c(1 - k - \tilde{\phi}_c + k \tilde{\psi}_c), & 0 < s < \infty, \\
-d \tilde{\psi}'_c + c_1 \tilde{\psi}'_c < r(1 - \tilde{\psi}_c)(h \tilde{\phi}_c - \tilde{\psi}_c), & -\infty < s < \infty.
\end{array}
\right.
\end{aligned}
\]

Hence, it follows from Proposition 2.5 and Lemma 3.7 that \(\tilde{\phi}_c(s) \geq \tilde{\phi}_c(s)\) in \(\mathbb{R}^+\) and \(\tilde{\psi}_c(s) \geq \tilde{\psi}_c(s)\) in \(\mathbb{R}\). Furthermore, the strong maximum principle yields \(\tilde{\phi}_c(s) > \tilde{\phi}_c(s)\) for \(s > 0\), \(\tilde{\psi}_c(s) > \tilde{\psi}_c(s)\) for \(s \in \mathbb{R}\). Let \(\phi = \tilde{\phi}_c - \tilde{\phi}_c\). Then
\[
\begin{aligned}
\tilde{\phi}''_c + c_2 \tilde{\psi}'_c &= \tilde{\phi}(1 - k - \tilde{\phi}_c + k \tilde{\psi}_c), & 0 < s > 0,
\end{aligned}
\]
\[
\tilde{\phi}(0) = 0.
\]

By the Hopf boundary lemma, we deduce \(\tilde{\phi}'(0) > 0\), that is, \(\tilde{\phi}'_c(0) > \tilde{\phi}'_c(0)\).

Lemma 3.13. Let \((\tilde{\phi}_c, \tilde{\psi}_c)\) be the unique monotone solution of \((3.5)\). Then the mapping \(c \mapsto (\tilde{\phi}_c, \tilde{\psi}_c)\) is continuous from \([0, c_0^*)\) to \(C^2_{\text{loc}}([0, \infty)) \times C^2_{\text{loc}}(\mathbb{R})\). Moreover,
\[
\lim_{c \to c_0^*} (\tilde{\phi}_c, \tilde{\psi}_c) = (0, 0) \text{ in } C^2_{\text{loc}}([0, \infty)) \times C^2_{\text{loc}}(\mathbb{R}).
\]

Proof. Suppose \(\{c_i\}\) is a sequence in \([0, c_0^*)\) such that \(c_i \to \hat{c} \in [0, c_0^*)\) as \(i \to \infty\). Let \((\tilde{\phi}_{c_i}, \tilde{\psi}_{c_i})\) be the solution of \((3.5)\) with \(c = c_i\). We claim that \((\tilde{\phi}_{c_1}, \tilde{\psi}_{c_1})\) has a subsequence that converges to \((\tilde{\phi}_c, \tilde{\psi}_c)\) in \(C^2_{\text{loc}}([0, \infty)) \times C^2_{\text{loc}}(\mathbb{R})\), which clearly implies the continuity of the mapping \(c \mapsto (\tilde{\phi}_c, \tilde{\psi}_c)\).

Firstly, we consider the case \(\hat{c} < c_0^*\). Let \(\tilde{c} \in (\hat{c}, c_0^*)\). Then \(c_i \in [\tilde{c}, \hat{c}]\) for all large \(i\), and without loss of generality we assume that this is the case for all \(i \geq 1\). For simplicity, we denote \((\tilde{\phi}_{c_i}, \tilde{\psi}_{c_i})\) by \((\tilde{\phi}_i, \tilde{\psi}_i)\). Rewrite equation \((3.5)\) in the integral form of \((6.5)\) and \((6.6)\). Noting that \(\tilde{\phi}_i\) and \(\tilde{\psi}_i\) are uniformly bounded, similar arguments as in Lemma 3.11 indicate that \(|\tilde{\phi}'_i|\) and \(|\tilde{\phi}''_i|\) are bounded for all \(i\) and \(s \in \mathbb{R}^+\). Moreover, by similar arguments as in the proof of Lemma 3.11 again, we can prove \(|\tilde{\psi}'_i|\) and \(|\tilde{\psi}''_i|\) are bounded for all \(i\) and \(s \in \mathbb{R}\). Differentiating both sides of \((3.5)\) with respect to \(s\), applying the uniform boundedness of \(\tilde{\psi}'_i\) and \(\tilde{\psi}''_i\) \((j = 0, 1, 2)\), we have \(|\tilde{\phi}'_i|\) and \(|\tilde{\phi}''_i|\) are bounded for \(s \in \mathbb{R}^+\) and \(s \in \mathbb{R}\), respectively. Hence, \(\{\tilde{\phi}'_i\}\) and \(\{\tilde{\psi}'_i\}\) \((j = 0, 1, 2)\) uniformly bounded and equi-continuous for \(s \in \mathbb{R}^+\) and \(s \in \mathbb{R}\), respectively. Using Arzelà-Ascoli’s theorem, the nested subsequence argument and Lebesgue’s dominated convergence theorem, there is a subsequence \(\{c_i\} \subset \{c_i\}\) such that \((\tilde{\phi}_{c_k}, \tilde{\psi}_{c_k}) \to (\hat{\phi}, \hat{\psi})\) uniformly as \(k \to \infty\) in \(C^2_{\text{loc}}([0, \infty)) \times C^2_{\text{loc}}(\mathbb{R})\). Moreover, \((\hat{\phi}, \hat{\psi})\) solves \((3.5)\) with \(c = \hat{c}\), except that we only have \(\phi'_i \geq 0\) and \(\psi'_i \geq 0\). Using Lemma 3.12, the required asymptotic behavior of \((\phi, \psi)\) at \(\pm \infty\) follows from
\[
(\tilde{\phi}_c, \tilde{\psi}_c) \leq (\phi, \psi) \leq (\bar{\phi}, \bar{\psi}).
\]

Applying the strong maximum principle to the system satisfied by \((\tilde{\phi}', \tilde{\psi}')\), we deduce \(\phi' > 0\) in \([0, \infty)\) and \(\psi' > 0\) in \(\mathbb{R}\). Thus \((\phi, \psi)\) is a solution of \((3.5)\) with \(c = \hat{c}\). By uniqueness, we have \((\tilde{\phi}_c, \tilde{\psi}_c) = (\hat{\phi}, \hat{\psi})\).
Proof. By Lemma 3.13 and Lemma 3.12, for fixed $\gamma$ conclude that, passing to a subsequence, 

$$ (c_{ik}, \phi_{ik}, \psi_{ik}) \rightarrow (c_0^*, \phi_*, \psi_*) \text{ in } C^2_{\text{loc}}([0, \infty)) \times C^2_{\text{loc}}(\mathbb{R}) \text{ as } k \rightarrow \infty $$

and $(\hat{\phi}_*, \hat{\psi}_*)$ solves (3.5) with $c = c_0^*$, except that we only have $\hat{\phi}'_* \geq 0$ in $[0, \infty)$ and $\hat{\psi}'_* \geq 0$ in $\mathbb{R}$. If $\hat{\phi}_* \equiv 0$, then $\hat{\psi}_*$ satisfies

$$ d\hat{\psi}''_* - c_0^* \hat{\psi}'_* = r\hat{\psi}_*(1 - \hat{\psi}_*). $$

Let $\hat{\psi}_* = 1 - \hat{\psi}_*$. Then $\hat{\psi}_*$ satisfies

$$ -d\hat{\psi}''_* \geq -d\hat{\psi}''_* + c_0^* \hat{\psi}'_* = r\hat{\psi}_*(1 - \hat{\psi}_*). $$

For large $L > 1$, assume $u_L$ is the unique positive solution of

$$ -du'' = ru(1 - u), \quad u(\pm L) = 0. $$

It is well known that $u_L \rightarrow 1$ in $C^2_{\text{loc}}(\mathbb{R})$ as $L \rightarrow \infty$. By Lemma 2.1 of [9], we have $u_L \leq \hat{\psi}_* \leq 1$ in $[-L, L]$. Letting $L \rightarrow \infty$ we obtain $\hat{\psi}_* = 1$, as we wanted. Next, assume that $\hat{\phi}_* \neq 0$. Let $(\tilde{\phi}_*, \tilde{\psi}_*)$ be a solution of (3.5) with $c = c_0^*$. Then we may repeat the proof of Lemma 3.9 to conclude $\hat{\phi}_* \leq 0$, a contradiction. \hfill $\Box$

**Lemma 3.14.** Let $(\hat{\phi}_*, \hat{\psi}_*)$ be the unique monotone solution of (3.5). For any $\gamma > 0$, there exists a unique $c = c(\gamma) \in (0, c_0^*)$ such that $\gamma \hat{\phi}'_c(0) = c$. Moreover, $\gamma \mapsto c(\gamma)$ is strictly increasing and $\lim_{\gamma \rightarrow \infty} c(\gamma) = c_0^*$. \hfill $\Box$

**Proof.** By Lemma 3.13 and Lemma 3.12, for fixed $\gamma > 0$, the function $p(c, \gamma) = \gamma \hat{\phi}'_c(0) - c$ is continuous and strictly decreasing for $c \in [0, c_0^*)$. Note that $p(0, \gamma) = \gamma \hat{\phi}'_c(0) > 0$ and $\lim_{c \rightarrow c_0^*} p(c, \gamma) = -c_0^* < 0$. Therefore, there exists a unique $c = c(\gamma) \in (0, c_0^*)$ such that $p(c, \gamma) = 0$, i.e. $\gamma \hat{\phi}'_c(0) = c$. Moreover, note that $p(c, \gamma)$ is strictly increasing in $\gamma$ for any given $c \in (0, c_0^*)$. Hence, $c(\gamma)$ is strictly increasing in $\gamma$. For any $\epsilon > 0$ and $c \in (0, c_0^* - \epsilon)$, we have $p(c, \gamma) \geq p(c_0^* - \epsilon, \gamma) \rightarrow \infty$ as $\gamma \rightarrow \infty$. It follows that $c_0^* - \epsilon < c(\gamma) < c_0^*$ for all large $\gamma$, which means that $\lim_{\gamma \rightarrow \infty} c = c_0^*$. \hfill $\Box$

Theorem 1.5 now follows directly from Lemmas 3.5–3.14.

4. Asymptotic spreading speed. We prove Theorem 1.3 in this section, assuming the validity of Theorem 1.1.

**Lemma 4.1.** Suppose spreading occurs, i.e., alternative (i) happens in Theorem 1.1. Then

$$ \liminf_{t \rightarrow \infty} \frac{q(t)}{t} \geq c_\gamma. $$

**Proof.** Let $V(t)$ be the unique solution of

$$ V' = rV(1 - V), \quad V(0) = \|v_0\|_\infty. $$

Then a simple comparison consideration yields $v(x, t) \leq V(t)$ for $x \geq 0$ and $t > 0$. Since $\lim_{t \rightarrow \infty} V(t) = 1$, we can find $T_0' > 0$ such that

$$ v(x, t) < 1 + \delta \quad \text{for } x \geq 0, \quad t \geq T_0'. \quad (4.1) $$

Theorem 1.3 now follows directly from Lemma 4.1.
We now consider the auxiliary problem
\[ \begin{align*}
\phi_0'' - c\phi_0' + (1 - 2\delta - \phi_0 - k\phi_0) &= 0, \quad \phi_0' > 0 \quad \text{for } s > 0, \\
\delta v_0'' - cv_0' + r\psi_0(1 + 2\delta - \psi_0 - h\phi_0) &= 0, \quad \psi_0' < 0 \quad \text{for } -\infty < s < \infty, \\
\phi_0(s) &= 0 \quad \text{for } s \leq 0, \quad \gamma \phi_0'(0) = c, \quad \psi_0(-\infty) = 1 + 2\delta, \\
(\phi_0(\infty), \psi_0(\infty)) &= (u_0^*, v_0^*),
\end{align*} \tag{4.2} \]
where \( \delta > 0 \) is small. We claim that there exists a unique \( c_0^* > 0 \) such that (4.2) has a unique solution \((\phi_0, \psi_0)\) when \( c = c_0^* \); moreover,
\[ \lim_{\delta \to 0} c_0^* = c_0. \]
Indeed, if we define
\[ \sigma_{\pm \delta} = \sqrt{1 \pm 2\delta}, \quad \phi_0(s) = \sigma_{-\delta}^2 \tilde{\phi}_0(\sigma_{-\delta}s), \quad \psi_0(s) = \sigma_{+\delta}^2 \tilde{\psi}_0(\sigma_{-\delta}s), \]
and
\[ \tilde{c} = c/\sigma_{-\delta}, \quad \tilde{\gamma} = \gamma/\sigma_{-\delta}, \quad \tilde{r}_0 = r \left( \frac{\sigma_{+\delta}}{\sigma_{-\delta}} \right)^2, \quad \tilde{h}_0 = h \left( \frac{\sigma_{+\delta}}{\sigma_{-\delta}} \right)^2, \quad \tilde{k}_0 = k \left( \frac{\sigma_{+\delta}}{\sigma_{-\delta}} \right)^2, \]
then a direct calculation shows that \((c, \phi_0, \psi_0)\) solves (4.2) if and only if \((\tilde{c}, \tilde{\phi}_0, \tilde{\psi}_0)\) satisfies (1.6) \[ \tilde{c}\tilde{\phi}_0'(0) = \tilde{c} \quad \text{when } (r, h, k, u^*, v^*) \quad \text{in } (1.6) \quad \text{is replaced by } (\tilde{r}_0, \tilde{h}_0, \tilde{k}_0, \ u_0^*, \ v_0^*). \]
So the claim follows directly from Theorem 1.5 and Lemma 3.14, and the continuous dependence of the unique solution on the parameters.

Since \( \psi_0(-\infty) = 1 + 2\delta, \psi_0(+\infty) = v_0^* > v^* \) and \( v_0^* < 0 \), there exists \( L > 1 \) such that
\[ \psi_0(x) > 1 + \delta \quad \text{for } x \leq -L, \quad \psi_0(x) > v_0^* > v^* \quad \text{for } x \geq L. \tag{4.3} \]
Similarly it follows from \( \phi_0(+\infty) = u_0^* < u^* \) and \( \phi_0' > 0 \) that
\[ \phi_0(x) < u_0^* < u^* \quad \text{for } x \geq 0. \]

By the spreading assumption, we have
\[ \lim_{t \to \infty} g(t) = \infty, \quad \lim_{t \to \infty} \left( u(x,t), v(x,t) \right) = (u^*, v^*) \quad \text{in any compact subset of } [0, \infty). \]
Hence, in view of \( u^* > u_0^* \) and \( v^* < v_0^* \), there exists \( T_0 > T_0' \) large such that for \( x \in [0, L + 1] \) and \( t \geq T_0, \)
\[ g(t) > 1, \quad u(x, t) > u_0^*, \quad v(x, t) < v_0^*. \]

We now define
\[ g(t) = c_0^*(t - T_0) + 1, \quad u(x, t) = \phi_0(g(t) - x), \quad v(x, t) = \psi_0(g(t) - x). \]
Then \( g(T_0) = 1 < g(T_0) \),
\[ u(x, T_0) = \phi_0(1 - x) < u_0^* < u(x, T_0) \quad \text{for } x \in [0, 1] = [0, g(T_0)], \]
and in view of (4.3) and (4.1), we also have
\[ v(x, T_0) = \psi_0(1 - x) > v_0^* > v(x, T_0) \quad \text{for } x \in [0, L + 1], \]
\[ v(x, T_0) = \psi_0(1 - x) > 1 + \delta > v(x, T_0) \quad \text{for } x > L + 1. \]
Let us also note that
\[ u(0, t) = \phi_0(g(t)) < u_0^* < u(0, t) \quad \text{for } t \geq T_0, \]
\[ v(0, t) = \psi_0(g(t)) > v_0^* > v(0, t) \quad \text{for } t \geq T_0, \]
and moreover,
\[ g'(t) = c^\delta_{\gamma} = \gamma \phi'_\delta(0) = -\gamma w_*(g(t), t) \] for \( t \geq T_0 \).

Furthermore,
\[ u_t - u_{xx} = c^\delta_{\gamma} \phi'_\delta - \phi''_\delta = \phi_\delta(1 - 2\delta - k\psi_\delta) \leq \phi_\delta(1 - \phi_\delta - k\psi_\delta) = u(1 - u - k\tau), \]
\[ \tau_t - d\tau_{xx} = c^\delta_{\gamma} \psi'_\delta - d\psi''_\delta = r\psi_\delta(1 + 2\delta - \psi_\delta - h\phi_\delta) \geq r\psi_\delta(1 - \psi_\delta - h\phi_\delta) = r\tau(1 - \tau - h\tau). \]

Hence, we can use Proposition 2.2 and Remark 2.3 to conclude that
\[ g(t) \geq g(t) \] for \( t \geq T_0 \).

It follows that \( \lim\inf_{t \to \infty} \frac{g(t)}{t} \geq c^\delta_{\gamma} \), which yields the required inequality by letting \( \delta \to 0 \).

**Lemma 4.2.** Under the assumptions of Lemma 4.1 we have
\[ \limsup_{t \to \infty} \frac{g(t)}{t} \leq c_{\gamma}. \]

**Proof.** For small \( \tau > 0 \) we consider the auxiliary problem
\[
\begin{aligned}
\phi''_\tau - c\phi'_\tau + \phi_\tau(1 + 2\tau - \phi_\tau - k\psi_\tau) &= 0, \quad \phi'_\tau > 0 \text{ for } 0 < s < \infty, \\
\psi''_\tau - c\psi'_\tau + r\psi_\tau(1 - 2\tau - \psi_\tau - h\phi_\tau) &= 0, \quad \psi'_\tau < 0 \text{ for } -\infty < s < \infty, \\
\phi_\tau(s) &\equiv 0 \text{ for } s \leq 0, \quad \gamma \phi'_\tau(0) = c, \quad \psi_\tau(-\infty) = 1 - 2\tau, \\
(\phi_\tau, \psi_\tau)(\infty) &= (u^*_\tau, v^*_\tau) := \left( \frac{1+2\tau-k(1+2\tau)}{1-hk}, \frac{1-2\tau-h(1+2\tau)}{1-hk} \right).
\end{aligned}
\]

As in the proof of Lemma 4.1 we can use a change of variable trick to reduce (4.4) to (1.6), and then apply Theorem 1.5 and Lemma 3.14 to conclude that there exists a unique \( c^\tau_{\gamma} > 0 \) such that (4.4) has a unique solution \((\phi_\tau, \psi_\tau)\) when \( c = c^\tau_{\gamma} \), and moreover,
\[ \lim_{\tau \to 0} c^\tau_{\gamma} = c_{\gamma}. \]

Let us also observe that
\[ u^* < u^*_\tau, \quad v^* > v^*_\tau, \quad 0 < \phi_\tau(x) < u^*_\tau \text{ for } x > 0, \quad v^*_\tau < \psi_\tau(x) < 1 - 2\tau \text{ for } x \in (-\infty, \infty). \]

For clarity we divide the analysis below into three steps.

**Step 1.** We prove that for any small \( \tau > 0 \), we can find \( T'_0 > 0 \) such that for each \( T \geq T'_0 \), there exists \( L(T) > 0 \) having the following property:
\[ v(x, T) \geq 1 - \tau \text{ for } x \geq L(T). \]

By (1.4) we have
\[ \tilde{v}_0 := \inf_{x \geq 0} v_0(x) > 0. \]

Consider the auxiliary problem
\[
\begin{aligned}
w_t - dw_{xx} &= rw(1 - w), \quad x > 0, \quad t > 0, \\
w(0, t) &= 0, \quad t > 0, \\
w(x, 0) &= \tilde{v}_0, \quad x > 0.
\end{aligned}
\]

It is well known that the solution of (4.5) satisfies
\[ \lim_{t \to \infty} w(x, t) = w_*(x) \text{ locally uniformly for } x \in [0, \infty), \]

and
\[ u_t - u_{xx} = c^\delta_{\gamma} \phi'_\delta - \phi''_\delta = \phi_\delta(1 - 2\delta - k\psi_\delta) \leq \phi_\delta(1 - \phi_\delta - k\psi_\delta) = u(1 - u - k\tau), \]
\[ \tau_t - d\tau_{xx} = c^\delta_{\gamma} \psi'_\delta - d\psi''_\delta = r\psi_\delta(1 + 2\delta - \psi_\delta - h\phi_\delta) \geq r\psi_\delta(1 - \psi_\delta - h\phi_\delta) = r\tau(1 - \tau - h\tau). \]
where \( w_* \) is the unique solution of
\[
-w''_* = r w_*(1 - w_*) \quad \text{for} \quad x \in [0, \infty), \quad w_*(0) = 0.
\]
Moreover, \( w_* \) has the property that \( w'_* > 0 \) and \( w_* (\infty) = 1 \). Therefore, there exist positive constants \( L_1, T'_0 \) large enough such that
\[
w(L_1, t) \geq w_*(L_1) - \tau/2 \geq 1 - \tau \quad \text{for} \quad t \geq T'_0.
\]
Applying the maximum principle to the equation satisfied by \( w_* (x, t) \), we deduce \( w_* (x, t) \geq 0 \) for \( x > 0 \) and \( t > 0 \). It follows that
\[
w(x, t) \geq 1 - \tau \quad \text{for} \quad x \geq L_1 \quad \text{and} \quad t \geq T'_0.
\]
Fix \( T \geq T'_0 \) and note that \( v \) satisfies
\[
\begin{cases}
  v_t - rv_{xx} = rv(1 - v), & x > g(T), \quad 0 < t \leq T, \\
  v(x, 0) \geq \bar{v}_0, & x \geq g(T).
\end{cases}
\]
Set \( \tilde{w} (x, t) := w(x - g(T), t) \). Then \( \tilde{w} (x, t) \) satisfies
\[
\tilde{w}_t - r \tilde{w}_{xx} = r\tilde{w}(1 - \tilde{w}) \quad \text{for} \quad x > g(T) \quad \text{and} \quad 0 < t \leq T.
\]
Since
\[
\tilde{w}(g(T), t) = 0 < v(g(T), t) \quad \text{for} \quad t \in (0, T], \quad \tilde{w}(x, 0) = \bar{v}_0 \leq v(x, 0) \quad \text{for} \quad x > g(T),
\]
we can use the comparison principle to deduce
\[
v(x, t) \geq \tilde{w}(x, t) = w(x - g(T), t) \quad \text{for} \quad x > g(T) \quad \text{and} \quad 0 < t \leq T.
\]
Thus we obtain
\[
v(x, T) \geq w(x - g(T), T) \geq w(L_1, T) \geq 1 - \tau \quad \text{for} \quad x \geq L(T) := L_1 + g(T).
\]
This completes the proof of Step 1.

**Step 2.** We prove that for any small \( \tau > 0 \), there exists \( T'_1 > 0 \) such that
\[
u(x, t) \leq u^*_{\tau/2}, \quad v(x, t) \geq v^*_{\tau/2} \quad \text{for} \quad x \geq 0, \quad t \geq T'_1.
\]

We prove the claimed inequalities in (4.6) by a comparison argument involving the following ODE system
\[
\begin{cases}
  \tilde{u}'(t) = \tilde{u}(1 - \tilde{u} - k \tilde{v}), & t > 0, \\
  \tilde{v}'(t) = r \tilde{v}(1 - \tilde{v} - h \tilde{u}), & t > 0, \\
  (\tilde{u}(0), \tilde{v}(0)) = (\|u_0\|_{\infty}, \bar{v}_0).
\end{cases}
\]
Indeed, by the comparison principle for cooperative system we easily obtain
\[
u(x, t) \leq \tilde{u}(t), \quad v(x, t) \geq \tilde{v}(t) \quad \text{for} \quad x \geq 0, \quad t > 0.
\]
But it is well known (for example, see [14]) that
\[
\lim_{t \to \infty} (\tilde{u}(t), \tilde{v}(t)) = (u^*, v^*).
\]
The inequalities in (4.6) thus follow directly once we recall \( u^* < u^*_{\tau/2} < u^*_{\tau} \) and \( v^* > v^*_{\tau/2} > v^*_{\tau} \).

**Step 3.** We complete the proof of the lemma by constructing a suitable comparison function triple \((\mathcal{P}(x, t), \mathcal{V}(x, t), \mathcal{G}(t))\), and applying the comparison principle.

We fix \( T_0 := \max \{ T'_0, T'_1 \} \). Then by the conclusions in Steps 1 and 2 we obtain
\[
u(x, T_0) \leq u^*_{\tau/2} < u^*_{\tau} \quad \text{for} \quad x \geq 0, \quad v(x, T_0) \geq v^*_{\tau/2} > v^*_{\tau} \quad \text{for} \quad x \geq 0,
\]
Moreover, spreading speed of $u$ and $v$ in Theorem 1.2 is presented here.

The conclusion there is weaker than our Theorem 1.1. For these reasons, a complete system when spreading of $u$ case. Theorem 5.4 in [30] gives a description for the long-time behavior of the general heterogeneous system was considered, which contains (1.3) as a special systems can be found in [15, 27, 28], but in these papers, both species have free boundaries, and so the precise conclusions are rather different. In [30], a more spreading-vanishing dichotomies for various weak competition

5. The spreading-vanishing dichotomy. In this section, we prove Theorems 1.1 and 1.2. Similar spreading-vanishing dichotomies for various weak competition systems can be found in [15, 27, 28], but in these papers, both species have free boundaries, and so the precise conclusions are rather different. In [30], a more general heterogeneous system was considered, which contains (1.3) as a special case. Theorem 5.4 in [30] gives a description for the long-time behavior of the system when spreading of $u$ happens, but when applied to the special case here, the conclusion there is weaker than our Theorem 1.1. For these reasons, a complete proof for our Theorems 1.1 and 1.2 is presented here.
Let us recall that for the problem
\[
\begin{align*}
  u_t &= du_{xx} + u(a - bu), & 0 < x < h(t), & t > 0, \\
  u_x(0,t) &= 0, & u(x,t) &\equiv 0, & h(t) \leq x, & t > 0, \\
  h'(t) &= -\nu u_x(h(t),t), & \nu > 0, & t > 0, \\
  h(0) &= h_0 > 0, & u(x,0) &= u_0(x), & 0 \leq x \leq h_0,
\end{align*}
\]
the following result holds.

**Lemma 5.1.** ([6]) If \( h_0 \geq \frac{\pi}{2}\sqrt{\frac{a}{b}} \), then spreading always happens. If \( h_0 < \frac{\pi}{2}\sqrt{\frac{a}{b}} \), then there exists \( \nu^* > 0 \) depending on \( u_0 \) such that vanishing happens when \( \nu \leq \nu^* \), and spreading happens when \( \nu > \nu^* \).

**Lemma 5.2.** Let \((u,v,g)\) be the solution of (1.3). If \( \lim_{t \to \infty} g(t) = g_\infty < \infty \), then the solution of equation (1.3) satisfies
\[
\lim_{t \to \infty} \|u(\cdot,t)\|_{C([0,g(t)])} = 0, \quad \lim_{t \to \infty} v(\cdot,t) = 1 \text{ in } C_{\text{loc}}([0,\infty)).
\]

**Proof.** The proof is similar to the proof of Lemma 4.6 in [7]. For readers’ convenience, we give the details here. Define
\[
s := \frac{g_0 x}{g(t)}, \quad \hat{u}(s,t) := u(x,t), \quad \hat{v}(s,t) := v(x,t).
\]

By direct calculation,
\[
\hat{u}_t = \frac{g'(t)}{g(t)} s \hat{u}_s, \quad \hat{u}_x = \frac{g_0}{g(t)} \hat{u}_s, \quad \hat{u}_{xx} = \frac{g_0^2}{g^2(t)} \hat{u}_{ss}.
\]

Hence \( \hat{u} \) satisfies
\[
\begin{align*}
  \hat{u}_t - \frac{g_0^2}{g^2(t)} \hat{u}_{ss} - \frac{g'(t)}{g(t)} s \hat{u}_s &= \hat{u}(1 - \hat{u} - k\hat{v}), & 0 < s < g_0, & t > 0, \\
  \hat{u}_x(0,t) &= \hat{u}(g_0,t) = 0, & t > 0, \\
  \hat{u}(s,0) &= u_0(s), & 0 \leq s \leq g_0.
\end{align*}
\]

By Proposition 2.1, there exists \( M > 0 \) such that
\[
\|1 - \hat{u} - k\hat{v}\|_{L^\infty} \leq 1 + (1+k)M, \quad \left\| \frac{g'(t)}{g(t)} \right\|_{L^\infty} \leq \frac{M}{g_0}.
\]

Since \( g_0 \leq g(t) < g_\infty < \infty \), the differential operator is uniformly parabolic. Therefore we can apply standard \( L^p \) theory to obtain, for any \( p > 1 \),
\[
\|\hat{u}\|_{W^{2,1}_p([0,\infty) \times [0,2])} \leq C_1,
\]
where \( C_1 \) is a constant depending on \( p, g_0, M \) and \( \|u_0\|_{C^{1,\alpha}[0,\infty)} \). For each \( T \geq 1 \), we can apply the partial interior-boundary estimate over \([0, g_0] \times [T, T+2]\) to obtain
\[
\|\hat{u}\|_{W^{2,1}_p([0,\infty) \times [T+1/2, T+2])} \leq C_2 \text{ for some constant } C_2 \text{ depending on } \alpha, g_0, M \text{ and } \|u_0\|_{C^{1,\alpha}[0,\infty)}, \text{ but independent of } T.
\]

Therefore, we can use the Sobolev imbedding theorem to obtain, for any \( \alpha \in (0,1) \),
\[
\|\hat{u}\|_{C^{1+\alpha,1/2+\alpha/2}([0,\infty) \times [0,\infty)})} \leq C_3,
\]
where \( C_3 \) is a constant depending on \( \alpha, g_0, M \) and \( \|u_0\|_{C^{1,\alpha}[0,\infty)} \). Similarly we may use interior estimates to the equation of \( \hat{v} \) to obtain
\[
\|\hat{v}\|_{C^{1+\alpha,1/2+\alpha/2}([0,\infty) \times [0,\infty)})} \leq C_4,
\]
where \( C_4 \) is a constant depending on \( \alpha, g_0, M \) and \( \|v_0\|_{C^{1,\alpha}[0,\infty+1]} \).
Since
\[ g'(t) = -\gamma u_x(g(t), t) = -\gamma \frac{g_0}{g(t)} \hat{u}_x(g_0, t), \]
it follows that there exists a constant \( \hat{C} \) depending on \( \alpha, \gamma, g_0, \|u_0, v_0\|_{C^{1+\alpha}[0, g_0]} \) and \( g_\infty \) such that
\[ \|g\|_{C^{1+\alpha/2}(\mathbb{R}^n \times [0, +\infty])} \leq \hat{C}. \]  
(5.3)

For contradiction, we assume that
\[ \limsup_{t \to +\infty} \|u(\cdot, t)\|_{C([0, g(t)])} = \delta > 0. \]
Then there exists a sequence \( (x_k, t_k) \) with \( 0 \leq x_k < g(t_k), \) \( 1 < t_k < +\infty \) such that \( u(x_k, t_k) \geq \frac{\delta}{2} > 0 \) for all \( k \in \mathbb{N}, \) and \( t_k \to +\infty \) as \( k \to +\infty. \) By (5.3), we know \( |u_x(g(t), t)| \) is uniformly bounded for \( t \in [0, +\infty), \) and there exists \( \sigma > 0 \) such that \( x_k \leq g(t_k) - \sigma \) for all \( k \geq 1. \) Therefore there exists a subsequence of \( \{x_k\} \) that converges to some \( x_0 \in \mathbb{R}^n \) with \( g_\infty - \sigma \). Without loss of generality, we may assume \( x_k \to x_0 \) as \( k \to +\infty, \) which leads to \( s_k = \frac{g(x_k)}{g(t_k)} \to s_0 = \frac{g(x_0)}{g_\infty} < g_0. \)

Set
\[ \hat{u}_k(s, t) = \hat{u}(s, t_k + t), \quad \hat{v}_k(s, t) = \hat{v}(s, t_k + t) \]
for \( (s, t) \in [0, g_0] \times [-1, 1]. \) It follows from (5.1) and (5.2) that \( \{\hat{u}_k, \hat{v}_k\} \) has a subsequence \( \{\bar{u}_k, \bar{v}_k\} \) such that
\[ \|\bar{u}_k(t) - \bar{u}(t, t_k)\|_{C^{1+\alpha', (1+\alpha')/2}([0, g_0] \times [-1, 1])} \to 0 \text{ as } i \to +\infty, \]
where \( \alpha' \in (0, \alpha). \) Since \( \|g\|_{C^{1+\alpha/2}([0, +\infty])} \leq \tilde{C}, \) \( g'(t) > 0 \) and \( g(t) \leq g_\infty, \) we necessarily have \( g'(t) \to 0 \) as \( t \to +\infty. \) Hence, \( (\bar{u}^*, \bar{v}^*) \) satisfies
\[ \begin{aligned}
\bar{u}^*_t - (\frac{m_{0, \infty}}{g_\infty})^2 \bar{u}^s & \bar{u}^s = \bar{u}^*(1 - \bar{u}^* - k \bar{v}^*), \quad 0 \leq s < g_0, \quad t \in (-1, 1), \\
\bar{u}^*_0(t) & = \bar{u}^*(g_0, t) = 0, \quad t \in [-1, 1].
\end{aligned} \]
Clearly, \( \bar{u}_k(s_0) = u(x_k, t_k) \geq \frac{\delta}{2}. \) Hence, we have \( \bar{u}^*(s_0, 0) \geq \frac{\delta}{2}. \) By the maximum principle, \( \bar{u}^* > 0 \) in \( [0, g_0] \times (-1, 1). \) Thus we can apply the Hopf boundary lemma to conclude that \( \theta_0 := \bar{u}^*(g_0, 0) < 0. \) It follows that \( u_x(g(t_k), t_k) = \partial_x \bar{u}_k(g_0, 0) \frac{g_0}{g(t_k)} \leq \frac{g_0}{g_\infty} < 0 \) for all large \( i, \) and hence
\[ g'(t_k) = -\gamma u_x(g(t_k), t_k) \geq -\gamma \theta_0/2 > 0 \]
for all large \( i. \) On the other hand, recalling that \( g'(t) \to 0 \) as \( t \to +\infty, \) we obtain a contradiction. Hence we must have
\[ \lim_{t \to +\infty} \|u(\cdot, t)\|_{C([0, g(t)])} = 0. \]

Using this fact and a simple comparison argument we easily deduce \( \lim_{t \to -\infty} v(\cdot, t) = 1 \) uniformly in any compact subset of \( [0, \infty). \)

**Lemma 5.3.** Let \( (u, v, g) \) be the solution of (1.3) and suppose \( g_\infty = \infty. \) Then
\[ (u(x, t), v(x, t)) \to (u^*, v^*) \text{ as } t \to +\infty \]
uniformly for \( x \) in any compact subset of \( [0, \infty). \)

**Proof.** We define
\[ \mathfrak{u}_1 = \mathfrak{v}_1 = 1, \quad \mathfrak{u}_1 = 1 - h, \quad \mathfrak{v}_1 = 1 - k. \]
Then define inductively for \( n \geq 1, \)
\[ \begin{aligned}
\mathfrak{u}_{n+1} & = 1 - k \mathfrak{u}_n, \quad \mathfrak{v}_{n+1} = 1 - h \mathfrak{v}_n, \quad \mathfrak{u}_{n+1} = 1 - k \mathfrak{v}_n, \quad \mathfrak{v}_{n+1} = 1 - h \mathfrak{u}_n.
\end{aligned} \]
We claim that, for every \( n \), Step 1 induction argument. directly from (5.5) and (5.4). So it suffices to prove (5.5). We do that by a uniform in any compact subset of \([0, \infty)\). The conclusion of the lemma follows directly from (5.5) and (5.4). So it suffices to prove (5.5). We do that by an induction argument.

**Step 1.** (5.5) holds for \( n = 1 \).

It follows from the comparison principle that \( u(x, t) \leq \hat{u}_1(t) \) for \( t > 0 \) and \( x \in [0, g(t)] \), where \( \hat{u}_1(t) \) satisfies

\[
\begin{aligned}
\frac{d\hat{u}_1}{dt} &= \hat{u}_1(1 - \hat{u}_1), & \quad t > 0, \\
\hat{u}_1(0) &= \|u_0\|_{\infty}.
\end{aligned}
\]

Clearly, \( \lim_{t \to \infty} \hat{u}_1(t) = 1 \). Hence,

\[
\limsup_{t \to \infty} u(x, t) \leq 1 = \pi_1 \text{ uniformly for } x \in [0, \infty),
\]

By the same argument as above, one gets

\[
\limsup_{t \to \infty} v(x, t) \leq 1 = \pi_1 \text{ uniformly for } x \in [0, \infty).
\]

For any given \( l > \max \left\{ g_0, \frac{\pi}{2\sqrt{1 - k}}, \frac{\pi}{2\sqrt{\frac{d}{r(1 - k)}}} \right\} \). In view of (5.6), (5.7) and \( g_\infty = \infty \), for any small \( \epsilon > 0 \), there exists \( t_1 > 0 \) such that \( g(t) > l \) for \( t \geq t_1 \) and \( u(x, t) < \pi_1 + \epsilon, v(x, t) < \pi_1 + \epsilon \) for \( x \in [0, l], t > t_1 \). It follows that

\[
\begin{aligned}
v_t &\geq dv_{xx} + rv(1 - v - h(1 + \epsilon)), & \quad 0 < x < l, \quad t > t_1, \\
v_x(0, t) = 0, \quad v(l, t) > 0, & \quad 0 \leq x < l, \quad t > t_1,
\end{aligned}
\]

which implies that \( v \) is an upper solution to the problem

\[
\begin{aligned}
\hat{v}_t &= d\hat{v}_{xx} + r\hat{v}(1 - \hat{v} - h(1 + \epsilon)), & \quad 0 < x < l, \quad t > t_1, \\
\hat{v}_x(0, t) = 0, \quad \hat{v}(l, t) = 0, & \quad t > t_1, \\
\hat{v}(x, t_1) &= v(x, t_1), & \quad 0 \leq x \leq l.
\end{aligned}
\]

Hence

\[
v(x, t) \geq \hat{v}(x, t) \quad \text{for } x \in [0, l] \text{ and } t > t_1.
\]

In view of \( l > \frac{\pi}{2\sqrt{\frac{d}{r(1 - k)}}} \), it is well known that \( \lim_{t \to \infty} \hat{v}(x, t) = \hat{v}^*(x) \), where \( \hat{v}^*(x) \) is the unique positive solution of

\[
\begin{aligned}
d\hat{v}_{xx}^* + r\hat{v}^*(1 - \hat{v}^* - h(1 + \epsilon)) &= 0, & \quad 0 < x < l, \\
\hat{v}^*_x(0) = 0, & \quad \hat{v}^*(l) = 0.
\end{aligned}
\]

On the other hand, \( \hat{v}^* \to 1 - h(1 + \epsilon) \) uniformly in any compact subset of \([0, \infty)\) as \( l \to \infty \) (see, for example, Lemma 2.2 in [9]). Thanks to the arbitrariness of \( l \) and \( \epsilon \), we thus obtain from \( v(x, t) \geq \hat{v}(x, t) \) in \([0, l] \times (t_1, \infty)\) that

\[
\liminf_{t \to \infty} v(x, t) \geq 1 - h = \pi_1 \text{ uniformly in any compact subset of } [0, \infty).
\]
Similarly, we have
\begin{equation}
\begin{cases}
  u_t - u_{xx} \geq u(1 - u - k(1 + \epsilon)), & 0 < x < l, \ t > t_1, \\
  u_x(0, t) = 0, \ u(l, t) > 0, & t > t_1,
\end{cases}
\end{equation}
which leads to
\[
\liminf_{t \to \infty} u(x, t) \geq 1 - k = u_1 \ 	ext{uniformly in any compact subset of } [0, \infty).
\]
This completes the proof of Step 1.

**Step 2.** If \((5.5)\) holds for \(n = j \geq 1\), then it holds for \(n = j + 1\).

Since \((5.5)\) holds for \(n = j\), for any small \(\epsilon > 0\) and large \(l > \max \left\{ g_0, \frac{\pi}{2} \sqrt{1 - k}, \frac{\pi}{2} \sqrt{d/(l-\pi)} \right\}\), there is \(t_2 > 0\) such that
\[ g(t) > l, \ u(x, t) \in [u_j - \epsilon, u_j + \epsilon], \ v(x, t) \in [v_j - \epsilon, v_j + \epsilon] \text{ for } x \in [0, l], \ t > t_2. \]
It follows from the comparison principle that \(u(x, t) \leq \pi(x, t)\) for \(x \in [0, l]\) and \(t > t_2\), where \(\pi(x, t)\) satisfies
\[
\begin{cases}
  \bar{u}_t - \bar{u}_{xx} = \bar{u}(1 - \bar{u} - k(\bar{u}_j - \epsilon)), & x \in (0, l), \ t > t_2, \\
  \pi_x(0, t) = 0, \ \pi(l, t) = \pi_j + \epsilon, & t > t_2, \\
  \pi(x, t_2) = u(x, t_2), & x \in [0, l].
\end{cases}
\]
It is well known that this problem has a unique positive steady-state solution \(\hat{u}^*(x)\) and \(\lim_{t \to \infty} \hat{u}(x, t) = \hat{u}^*(x)\) uniformly for \(x \in [0, l]\). Moreover,
\[
\lim_{t \to \infty} \hat{u}^*(x) = 1 - k(v_j - \epsilon) \text{ locally uniformly in } [0, \infty).
\]
It follows, since \(\epsilon > 0\) can be arbitrarily small, that
\[
\limsup_{t \to \infty} u(x, t) \leq 1 - k v_j = \pi_{j+1} \text{ locally uniformly in } [0, \infty).
\]
Analogously, from the comparison principle we obtain \(u(x, t) \geq \underline{u}(x, t)\) for \(x \in [0, l]\) and \(t > t_2\), where \(\underline{u}(x, t)\) satisfies
\[
\begin{cases}
  u_t - u_{xx} = \underline{u}(1 - u - k(\overline{u}_j + \epsilon)), & x \in (0, l), \ t > t_2, \\
  u_x(0, t) = 0, \ \underline{u}(l, t) = \underline{u}_j - \epsilon, & t > t_2, \\
  u(x, t_2) = u(x, t_2), & x \in [0, l],
\end{cases}
\]
from which we can deduce
\[
\liminf_{t \to \infty} u(x, t) \geq 1 - k \underline{u}_j = \underline{u}_{j+1} \text{ locally uniformly in } [0, \infty).
\]
The proof for
\[
\limsup_{t \to \infty} v(x, t) \leq \pi_{j+1}, \ \liminf_{t \to \infty} v(x, t) \geq \underline{v}_{j+1} \text{ locally uniformly in } [0, \infty)
\]
is similar, and we omit the details. \(\Box\)

Theorem 1.1 now follows directly from Lemmas 5.2 and 5.3.

**Lemma 5.4.** If \(g_\infty < +\infty\), then \(g_\infty \leq \frac{\pi}{2 \sqrt{1 - k}}\). Hence \(g_0 \geq \frac{\pi}{2 \sqrt{1 - k}}\) implies \(g_\infty = +\infty\).
Proof. Assume for contradiction that \( \frac{\pi}{2\sqrt{1-k(1+\varepsilon)}} < g_\infty < \infty \). Then there exists \( T_1 > 0 \) such that \( g(T_1) > \frac{\pi}{2\sqrt{1-k(1+\varepsilon)}} \) for \( \varepsilon \) sufficiently small. By a simple comparison consideration, there exists \( T > T_1 > 0 \) such that

\[
v(x, t) \leq 1 + \varepsilon \quad \text{for} \quad x \in [0, \infty), \quad t > T.
\]

Hence \((u, g)\) satisfies

\[
\begin{align*}
u_t &\geq u_{xx} + u(1 - u - k(1 + \varepsilon)), & 0 < x < g(t), \quad t > T, \\
u_x(0, t) &= u(g(t), t) = 0, & t > T, \\
g'(t) &= -\gamma u_x(g(t), t), & t > T, \\
u(x, T) &> 0, & 0 < x < g(T),
\end{align*}
\]

which implies that \((u, g)\) is an upper solution to the problem

\[
\begin{align*}
u_t &= w_{xx} + w(1 - w - k(1 + \varepsilon)), & 0 < x < g(t), \quad t > T, \\
w_x(0, t) &= w(g(t), t) = 0, & t > T, \\
g'(t) &= -\gamma w_x(g(t), t), & t > T, \\
w(x, T) &= u(x, T), \quad g(T) = g(T), & 0 < x < g(T).
\end{align*}
\]

Thus, \( g(t) \geq g(T) \) for \( t > T \). Since \( g(T) = g(T) > g(T_1) > \frac{\pi}{2\sqrt{1-k(1+\varepsilon)}} \), it follows from Lemma 5.1 that \( g(t) \to \infty \) and hence \( g_\infty = \infty \). This contradiction leads to \( g_\infty < \frac{\pi}{2\sqrt{1-k}} \). \( \square \)

**Lemma 5.5.** If \( g_0 < \frac{\pi}{2\sqrt{1-k}} \), then there exists \( \gamma \geq 0 \) depending on \( u_0 \) and \( v_0 \) such that spreading happens when \( \gamma > \gamma^* \).

**Proof.** Since \( \limsup_{t \to \infty} v(x, t) \leq 1 \) uniformly for \( x \in [0, \infty) \), there exists \( t_3 > 0 \), which is independent of \( \gamma \), such that \( v(x, t) \leq 1 + \varepsilon \) for \( x \in [0, \infty), \quad t \geq t_3 \). Thus \((u, g)\) satisfies

\[
\begin{align*}
u_t - u_{xx} &\geq u(1 - u - k(1 + \varepsilon)), & 0 < x < g(t), \quad t > t_3, \\
u_x(0, t) &= 0, \quad u(g(t), t) = 0, & t > t_3, \\
g'(t) &= -\gamma u_x(g(t), t), & t > t_3, \\
u(x, t_3) &> 0, & 0 \leq x < g(t_3).
\end{align*}
\]

Hence \((u, g)\) is an upper solution to the problem

\[
\begin{align*}
\hat{u}_t - \hat{u}_{xx} &= \hat{u}(1 - \hat{u} - k(1 + \varepsilon)), & 0 < x < \hat{g}(t), \quad t > t_3, \\
\hat{u}_x(0, t) &= 0, \quad \hat{u}((\hat{g}(t), t) = 0, & t > t_3, \\
\hat{g}'(t) &= -\gamma \hat{u}_x(\hat{g}(t), t), & t > t_3, \\
\hat{u}(x, t_3) &= u(x, t_3), \quad \hat{g}(t_3) = g(t_3), & 0 \leq x < g(t_3).
\end{align*}
\]

The comparison principle infers \( g(t) \geq \hat{g}(t) \) for \( t > t_3 \). Applying Lemma 5.1 to (5.8) we see that there exists \( \gamma \geq 0 \) depending on \( g(t_3) \) and \( u(x, t_3) \) (which are uniquely determined by \( u_0 \) and \( v_0 \)) such that spreading happens for (5.8) when \( \gamma > \gamma^*_0 \). Thus \( \lim_{t \to \infty} g(t) = \infty \) when \( \gamma > \gamma^*_0 \), and by Lemma 5.3, spreading happens to (1.3) for such \( \gamma \). \( \square \)

**Lemma 5.6.** There exists \( \gamma^* \geq 0 \), depending on \( u_0 \) and \( v_0 \), such that \( g_\infty < \infty \) if \( \gamma \leq \gamma^* \), and \( g_\infty = \infty \) if \( \gamma > \gamma^* \).
γ > γ. Let be the two roots of equation γ = γ. Hence we can find T > 0 such that g(T) > 2π2√1−k. To emphasize the dependence of the solution of (1.3) on γ, we denote it by (uγ, vγ, gγ) instead of (u, v, g), and so gγ(T) > 2π2√1−k. By the continuous dependence of (uγ, vγ, gγ) on γ, we can find ε > 0 small so that gγ(T) > 2π2√1−k for γ ∈ [γ∗ − ε, γ∗ + ε]. It then follows from Lemma 5.4 that for all such γ, limγ→∞ gγ(t) = ∞. This implies that [γ∗ − ε, γ∗ + ε] ⊂ Λ, and inf Λ < γ∗ − ε, a contradiction to the definition of γ∗.

Lemma 5.7. If g0 < 2π, then there exists γ > 0 depending on u0 such that g∞ < +∞ if γ ≤ γ.

Proof. Clearly, (u, g) satisfies
\[
\begin{align*}
\frac{\partial u}{\partial t} - u_{xx} & \leq u(1 - u), \quad 0 < x < g(t), \quad t > 0, \\
u_x(0, t) & = 0, \quad u(g(t), t) = 0, \quad t > 0, \\
g'(t) & = -γu_x(g(t), t), \quad t > 0, \\
u(x, 0) & = u_0(x), \quad 0 \leq x \leq g_0.
\end{align*}
\]
That is, (u, g) is a lower solution to the problem
\[
\begin{align*}
\frac{\partial u}{\partial t} - u_{xx} & = u(1 - u), \quad 0 < x < g(t), \quad t > 0, \\
u_x(0, t) & = 0, \quad u(g(t), t) = 0, \quad t > 0, \\
g'(t) & = -γu_x(g(t), t), \quad t > 0, \\
u(x, 0) & = u_0(x), \quad 0 \leq x \leq g_0.
\end{align*}
\]
It follows from the comparison principle that g(t) ≤ g′(t). Since g0 < π/2, by Lemma 5.1 there exists γ > 0 depending on u0 such that g∞ < ∞ if γ ≤ γ. Hence, g∞ < +∞ if γ ≤ γ.

Theorem 1.2 now follows directly from Lemmas 5.4–5.7.

6. Proof of Proposition 2.5. Since the first equation of (2.1) is only satisfied for s > 0, our proof of Proposition 2.5 is significantly different from those in the literature for similar equations. We break the rather long proof into several lemmas.

We start with a second order ODE of the following form
\[
\begin{align*}
d_1y'' - cy' - βy + f(s) & = 0, \quad s > 0, \\
y(0) & = 0,
\end{align*}
\]
where the constants c and β are positive, and the function f is specified below.

Let
\[
\begin{align*}
λ_1 & = \frac{c - \sqrt{c^2 + 4βd_1}}{2d_1}, \\
λ_2 & = \frac{c + \sqrt{c^2 + 4βd_1}}{2d_1}
\end{align*}
\]
be the two roots of equation d1λ2 − cλ − β = 0. Then we have the following result.

Lemma 6.1. Assume f : [0, ∞) → R is piecewise continuous and |f(s)| ≤ Aeαs for all s ≥ 0 and some constants A > 0, α ∈ (0, min{−λ1, λ2}). Then (6.1) has a
unique solution satisfying $y(s) = O(e^{\alpha s})$ as $s \to \infty$, and it is given by

$$y(s) = \frac{1}{d_1(\lambda_2 - \lambda_1)} \left[ \int_0^s K_1(\xi, s) f(\xi) d\xi + \int_s^\infty K_2(\xi, s) f(\xi) d\xi \right], \quad (6.2)$$

where

$$K_1(\xi, s) = e^{\lambda_1 s} (e^{-\lambda_2 \xi} - e^{-\lambda_2 s}), \quad K_2(\xi, s) = (e^{\lambda_2 s} - e^{\lambda_1 s}) e^{-\lambda_2 \xi}.$$  

Proof. By the variation of constants formula, the solutions of (6.1) are given by

$$y(s) = \gamma (e^{\lambda_1 s} - e^{\lambda_2 s}) + \frac{1}{d_1(\lambda_2 - \lambda_1)} \left[ \int_0^s (e^{\lambda_1 (s - \xi)} - e^{\lambda_2 (s - \xi)}) f(\xi) d\xi \right], \quad \gamma \in \mathbb{R}, \quad (6.3)$$

Multiplying both sides of (6.3) by $e^{-\lambda_2 s}$, we get

$$y(s)e^{-\lambda_2 s} = \gamma (e^{(\lambda_1 - \lambda_2) s} - 1) + \frac{1}{d_1(\lambda_2 - \lambda_1)} \left[ \int_0^s (e^{\lambda_1 (s - \xi)} - e^{\lambda_2 (s - \xi)}) f(\xi) d\xi \right].$$

If $y(s) = O(e^{\alpha s})$ as $s \to \infty$, then due to $\lambda_1 < \lambda_2$ and $|\lambda_1| < \lambda_2$, we obtain

$$y(s)e^{-\lambda_2 s} \to 0, \quad e^{(\lambda_1 - \lambda_2) s} \to 0 \quad \text{and} \quad \frac{1}{d_1(\lambda_2 - \lambda_1)} \int_0^\infty e^{\lambda_1 (s - \xi)} f(\xi) d\xi \to 0$$
as $s \to \infty$. Therefore,

$$\gamma = -\frac{1}{d_1(\lambda_2 - \lambda_1)} \int_0^\infty e^{-\lambda_2 \xi} f(\xi) d\xi. \quad (6.4)$$

Substituting (6.4) into (6.3), we obtain (6.2).

If $y(s)$ is given by (6.2), then it is easy to check that $y(s)$ satisfies (6.1) and $y(s) = O(e^{\alpha s})$ as $s \to \infty$. \hfill \Box

Define the operators $H_1 : C_R(\mathbb{R}^+, \mathbb{R}^2) \to C(\mathbb{R}^+, \mathbb{R})$ and $H_2 : C_R(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R})$ by

$$H_1(\varphi)(s) := \beta \varphi_1(s) + f_1(\varphi(s)), \quad H_2(\varphi)(s) := \beta \varphi_2(s) + f_2(\varphi(s)), \text{ where the positive constant } \beta \text{ is large enough such that } H_i(\varphi) \text{ is nondecreasing with respect to } \varphi_1 \text{ and } \varphi_2, \text{ for } (\varphi_1(s), \varphi_2(s)) \in \mathcal{R} = [0, k_1] \times [0, k_2].$$

Let $F_1 : C_R(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$ be given by

$$F_1(\varphi)(s) := \begin{cases} \frac{1}{d_1(\lambda_2 - \lambda_1)} \left[ \int_0^s K_1(\xi, s) H_1(\varphi)(\xi) d\xi \right], & s > 0, \\ 0, & s \leq 0, \end{cases} \quad (6.5)$$

where $K_i(\xi, s)$ is given by (6.3). By Lemma 6.1, it is easy to see that the operator $F_1$ is well defined and

$$\begin{cases} \left( d_1(F_1(\varphi))''(s) - c(F_1(\varphi))'(s) - \beta F_1(\varphi)(s) + H_1(\varphi)(s) \right) = 0, & s > 0, \\ F_1(\varphi)(s) = 0, & s \leq 0. \end{cases}$$

Let

$$\mu_1 = \frac{c - \sqrt{c^2 + 4\beta d_2}}{2d_2}, \quad \mu_2 = \frac{c + \sqrt{c^2 + 4\beta d_2}}{2d_2}$$

be the two roots of

$$d_2 \mu^2 - c \mu - \beta = 0.$$  

Define $F_2 : C_R(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$ by

$$F_2(\varphi)(s) = \frac{1}{d_2(\mu_2 - \mu_1)} \left[ \int_{-\infty}^s e^{\mu_1(s - \xi)} H_2(\varphi)(\xi) d\xi + \int_s^\infty e^{\mu_2(s - \xi)} H_2(\varphi)(\xi) d\xi \right]. \quad (6.6)$$
It is easy to show that the operator $F_2$ is well defined and satisfies
\[ d_2(F_2(\phi))''(s) - c(F_2(\phi))'(s) - \beta F_2(\phi)(s) + H_2(\phi)(s) = 0. \]

We now define $F : C_{\mathcal{R}}(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)$ by
\[ F(\phi) := (F_1(\phi), F_2(\phi)). \]

Clearly, $\phi$ is a fixed point of the operator $F$ in $C_{\mathcal{R}}(\mathbb{R}, \mathbb{R}^2)$ if and only if it is a solution of (2.1) in $C_{\mathcal{R}}(\mathbb{R}, \mathbb{R}^2)$.

Next, we introduce a Banach space with exponential decay norm. Fix $\sigma \in (0, \min\{|\lambda_1|, |\lambda_2|, |\mu_1|, |\mu_2|\})$. It is easy to see that
\[ B_\sigma(\mathbb{R}, \mathbb{R}^2) := \left\{ \phi \in C(\mathbb{R}, \mathbb{R}^2) : \sup_{s \in \mathbb{R}} |\phi(s)|e^{-\sigma|s|} < \infty \right\} \]
equipped with the norm
\[ ||\phi||_\sigma := \sup_{s \in \mathbb{R}} |\phi(s)|e^{-\sigma|s|} \]
is a Banach space.

Let $\overline{\phi}(s)$ and $\underline{\phi}(s)$ be the upper and lower solutions given in the statement of Proposition 2.5. Consider the set
\[ \Gamma = \left\{ \phi = (\phi_1, \phi_2) \in B_\sigma(\mathbb{R}, \mathbb{R}^2) : \underline{\phi} \leq \phi \leq \overline{\phi}, \phi_i \text{ is nondecreasing in} \ \mathbb{R}, i = 1, 2 \right\}. \]

Clearly $\Gamma$ is a nonempty, bounded, closed, convex subset of the Banach space $B_\sigma(\mathbb{R}, \mathbb{R}^2)$.

We are going to show that $F$ maps $\Gamma$ into itself, and is completely continuous. Then the Schauder fixed point theorem will yield a fixed point of $F$ in $\Gamma$, and we will then show that it satisfies (2.1) and (2.2).

Lemma 6.2. (i) $F_1(\phi)(s) \leq F(\phi)(s)$ for $s \in \mathbb{R}$ if $\underline{\phi} \leq \phi \leq \overline{\phi} \leq \overline{\phi}$;
(ii) $F_1(\phi)(s)$ and $F_2(\phi)(s)$ are nondecreasing in $s \in \mathbb{R}$ for any $\phi \in \Gamma$.

Proof. We show that $F_1$ satisfies (i) and (ii) stated in the lemma.

Since $F_1(\phi)(s) = 0$ for $s \leq 0$, we only need to consider the case of $s > 0$. In view of $\lambda_1 < 0 < \lambda_2$, it is easy to see that $K_1(\xi, s) > 0$ for $0 < \xi < s$ and $K_2(\xi, s) > 0$ for $s < \xi$. Thus, by (6.5) and the hypothesis $A_2$ we conclude that $F_1(\phi)(s) \leq F_1(\phi)(s)$ for $s \in \mathbb{R}$ if $\underline{\phi} \leq \phi \leq \overline{\phi}$. This proves (i) for $F_1$.

We next consider (ii). For $\phi = (\phi_1, \phi_2) \in \Gamma$, the hypothesis $A_2$ implies that $H_1(\phi)$ is nondecreasing in $\phi_i$. Since $\phi_1$ and $\phi_2$ are nondecreasing in $\mathbb{R}$, we have $\phi_i(s + \theta) \geq \phi_i(s)$ for $\theta > 0$ and $i = 1, 2$. This leads to $H_1(\phi)(s + \theta) - H_1(\phi)(s) \geq 0$.

A direct computation gives
\[ F_1(\phi)(s + \theta) - F_1(\phi)(s) = \frac{1}{d_1(\lambda_2 - \lambda_1)} \left[ \int_0^{s+\theta} K_1(\xi, s + \theta)H_1(\phi)(\xi)d\xi + \int_{s+\theta}^{\infty} K_2(\xi, s + \theta)H_1(\phi)(\xi)d\xi - \int_0^{s} K_1(\xi, s)H_1(\phi)(\xi)d\xi + \int_{s}^{\infty} K_2(\xi, s)H_1(\phi)(\xi)d\xi \right] \]
\[ = \frac{1}{d_1(\lambda_2 - \lambda_1)} \left[ \int_0^{s+\theta} e^{\lambda_1(s+\theta-\xi)}H_1(\phi)(\xi)d\xi + \int_{s+\theta}^{\infty} e^{\lambda_2(s+\theta-\xi)}H_1(\phi)(\xi)d\xi \right] \]
\[
- \int_0^s e^{\lambda_1(s-\xi)} H_1(\varphi)(\xi) d\xi - \int_s^\infty e^{\lambda_2(s-\xi)} H_1(\varphi)(\xi) d\xi \\
+ \frac{1}{d_1(\lambda_2 - \lambda_1)} \left[ \int_0^s e^{\lambda_1 s} e^{-\lambda_2 \xi} H_1(\varphi)(\xi) d\xi + \int_0^\infty e^{\lambda_1 s} e^{-\lambda_2 \xi} H_1(\varphi)(\xi) d\xi \\
- \int_s^{s+\theta} e^{\lambda_1(s+\theta)} e^{-\lambda_2 \xi} H_1(\varphi)(\xi) d\xi - \int_s^\infty e^{\lambda_1(s+\theta)} e^{-\lambda_2 \xi} H_1(\varphi)(\xi) d\xi \right] \\
= \frac{1}{d_1(\lambda_2 - \lambda_1)} \left[ \int_0^{s+\theta} e^{\lambda_1(s+\theta-\xi)} H_1(\varphi)(\xi) d\xi + \int_{s+\theta}^\infty e^{\lambda_2(s+\theta-\xi)} H_1(\varphi)(\xi) d\xi \\
- \int_0^s e^{\lambda_1(s-\xi)} H_1(\varphi)(\xi) d\xi - \int_s^\infty e^{\lambda_2(s-\xi)} H_1(\varphi)(\xi) d\xi \right] \\
+ \frac{1}{d_1(\lambda_2 - \lambda_1)} e^{\lambda_1 s} (1 - e^{\lambda_1 \theta}) \int_0^\infty e^{-\lambda_2 \xi} H_1(\varphi)(\xi) d\xi \\
\geq \frac{1}{d_1(\lambda_2 - \lambda_1)} \left[ \int_0^{s+\theta} e^{\lambda_1(s+\theta-\xi)} H_1(\varphi)(\xi) d\xi + \int_{s+\theta}^\infty e^{\lambda_2(s+\theta-\xi)} H_1(\varphi)(\xi) d\xi \\
- \int_0^s e^{\lambda_1(s-\xi)} H_1(\varphi)(\xi) d\xi - \int_s^\infty e^{\lambda_2(s-\xi)} H_1(\varphi)(\xi) d\xi \right] \\
= \frac{1}{d_1(\lambda_2 - \lambda_1)} \left\{ \int_0^\theta e^{\lambda_1(s+\theta-\xi)} H_1(\varphi)(\xi) d\xi + \int_0^\infty e^{\lambda_1(s-\xi)} (H_1(\varphi)(\xi + \theta) \\
- H_1(\varphi)(\xi)) d\xi + \int_s^\infty e^{\lambda_2(s-\xi)} (H_1(\varphi)(\xi + \theta) - H_1(\varphi)(\xi)) d\xi \right\} \geq 0.
\]

So \(F_1\) satisfies (ii).

Similarly, we can prove \(F_2\) satisfies (i) and (ii).

\[\square\]

**Lemma 6.3.** \(F(\Gamma) \subseteq \Gamma\).

**Proof.** Due to Lemma 6.2, it suffices to show that, for all \(s \in \mathbb{R}\),

\[
\varphi(s) \leq F(\varphi)(s), \quad F(\overline{\varphi})(s) \leq \overline{\varphi}(s).
\]

We firstly show

\[
\varphi_1(s) \leq F_1(\varphi)(s), \quad \forall s \in \mathbb{R}.
\]

Since \(\varphi_1(s) = F_1(\varphi)(s) = 0\) for \(s \leq 0\), we only need to consider the case of \(s > 0\). Without loss of generality, we denote \(\xi_0 = 0, \xi_{m_1+1} = \infty\) and assume \(\xi_i < \xi_{i+1}\) for \(i = 0, 1, 2, \ldots, m_1\). Here \(\xi_i, i \in \{0, \ldots, m_1\}\), are points in \(\Omega\) so that \(\varphi_1\) satisfies the first inequality (2.4) in \(\mathbb{R}^+ \setminus \Omega\). According to the definition of \(F_1(\varphi)\) and Definition 2.4, for any \(s \in (\xi_i, \xi_{i+1})\), we have,

\[
F_1(\varphi)(s) = \frac{1}{d_1(\lambda_2 - \lambda_1)} \left[ \int_0^s K_1(\xi, s) H_1(\varphi)(\xi) d\xi + \int_s^\infty K_2(\xi, s) H_1(\varphi)(\xi) d\xi \right] \\
\geq \frac{1}{d_1(\lambda_2 - \lambda_1)} \left[ \int_0^s K_1(\xi, s) \left( \beta_{\varphi_1}(\xi) - d_1 \varphi_1''(\xi) + c_0 \varphi_1'(\xi) \right) d\xi \right. \\
\left. + \int_s^\infty K_2(\xi, s) \left( \beta_{\varphi_1}(\xi) - d_1 \varphi_1''(\xi) + c_0 \varphi_1'(\xi) \right) d\xi \right]
\]
By a direct calculation, we have
\[ F = \varphi_1(s) + \frac{1}{\lambda_2 - \lambda_1} \left[ \sum_{j=1}^{i} K_1(\xi_j, s) \left( \varphi'_1(\xi_j^+) - \varphi'_1(\xi_j^-) \right) \right. \]
\[ + \sum_{j=i+1}^{m_1} K_2(\xi_j, s) \left( \varphi'_1(\xi_j^+) - \varphi'_1(\xi_j^-) \right] \geq \varphi_1(s). \]

The continuity of \( \varphi(s) \) and \( F_1(\varphi)(s) \) implies that \( F_1(\varphi)(s) \geq \varphi_1(s) \) for any \( s \in \mathbb{R}^+ \).

The proofs of \( F_1(\varphi)(s) \leq \varphi_1(s) \), \( F_2(\varphi)(s) \leq \varphi_2(s) \), \( F_2(\varphi)(s) \leq \varphi_2(s) \) for \( s \in \mathbb{R} \) are similar, and we omit the details.

**Lemma 6.4.** \( F : \Gamma \to \Gamma \) is continuous.

**Proof.** From the hypothesis (A_4), it is easy to see that, for some \( L > 0 \) and all \( \tilde{\varphi}, \hat{\varphi} \in \Gamma \),
\[ |H_1(\tilde{\varphi}) - H_1(\hat{\varphi})|_\sigma \leq (L + \beta)|\tilde{\varphi} - \hat{\varphi}|_\sigma. \]

By a direct calculation, we have
\[ |F_1(\tilde{\varphi}) - F_1(\hat{\varphi})|_\sigma \]
\[ = \frac{1}{d_1(\lambda_2 - \lambda_1)} \left[ \int_0^s K_1(\xi, s)(H_1(\tilde{\varphi})(\xi) - H_1(\hat{\varphi})(\xi))d\xi \right. \]
\[ + \int_s^\infty K_2(\xi, s)(H_1(\tilde{\varphi})(\xi) - H_1(\hat{\varphi})(\xi))d\xi \right] 
\[ \leq \frac{1}{d_1(\lambda_2 - \lambda_1)} \left[ \int_0^s K_1(\xi, s)H_1(\tilde{\varphi})(\xi) - H_1(\hat{\varphi})(\xi))d\xi \right. \]
\[ + \int_s^\infty K_2(\xi, s)H_1(\tilde{\varphi})(\xi) - H_1(\hat{\varphi})(\xi))d\xi \right] 
\[ \leq \frac{L + \beta}{d_1(\lambda_2 - \lambda_1)}|\tilde{\varphi} - \hat{\varphi}|_\sigma \sup_{s \in \mathbb{R}^+} \left[ \int_0^s K_1(\xi, s)e^{\sigma(\xi - s)}d\xi + \int_s^\infty K_2(\xi, s)e^{\sigma(\xi - s)}d\xi \right] 
\[ = \frac{L + \beta}{d_1(\lambda_2 - \lambda_1)}|\tilde{\varphi} - \hat{\varphi}|_\sigma \sup_{s \in \mathbb{R}^+} \left( 1 - e^{(\lambda_1 - \sigma)s} \right) \frac{\lambda_2 - \lambda_1}{(\lambda_1 - \sigma)(\lambda_2 - \lambda_1)} \]
\[ \leq \frac{L + \beta}{d_1(\lambda_2 - \sigma)(\lambda_2 - \lambda_1)}|\tilde{\varphi} - \hat{\varphi}|_\sigma, \]

which clearly implies \( F_1 : \Gamma \to B_\sigma(\mathbb{R}, \mathbb{R}^2) \) is continuous.

Similarly we can show \( F_2 : \Gamma \to B_\sigma(\mathbb{R}, \mathbb{R}^2) \) is continuous. Hence \( F \) is continuous on \( \Gamma \).

**Lemma 6.5.** \( F : \Gamma \to \Gamma \) is compact.

**Proof.** Since \( F \) is continuous on \( \Gamma \) by Lemma 6.4, and \( \Gamma \) is a bounded set in \( B_\sigma(\mathbb{R}, \mathbb{R}^2) \), it suffices to show that \( F(\Gamma) \) is a relatively compact set. To this end, let
\[ \rho := \sup \{|H_i(\varphi)| : \varphi \in \Gamma, i = 1, 2 \}. \]

In view of \( F_1(\varphi)(s) = 0 \) for \( s < 0 \), we get \( (F_1(\varphi))^'(s) = 0 \) when \( s < 0 \). Moreover, for any \( \varphi \in \Gamma \) and \( s > 0 \),
\[ (F_1(\varphi))^'(s) = \frac{1}{d_1(\lambda_2 - \lambda_1)} \left[ \int_0^s K_{1,s}(\xi, s)H_1(\varphi)(\xi)d\xi + \int_s^\infty K_{2,s}(\xi, s)H_1(\varphi)(\xi)d\xi \right]. \]
Hence,

$$
| (F_1(\varphi))'(s) | \leq \frac{\rho}{d_1(\lambda_2 - \lambda_1)} \left[ \int_0^s |K_1(\xi,s)| d\xi + \int_s^\infty |K_2(\xi,s)| d\xi \right]
$$

(6.7)

We thus see that \( s \to F_1(\varphi)(s) \) is Lipschitz continuous with Lipschitz constant \( L_1 := \frac{\rho}{d_1(\lambda_2 - \lambda_1)} \) independent of \( \varphi \in \Gamma \).

Similarly, from (6.6) we have

$$
(F_2(\varphi))'(s) = \frac{1}{d_2(\mu_2 - \mu_1)} \left[ \mu_1 \int_{-\infty}^s e^{\mu_1(s-\xi)} H_2(\varphi)(\xi) d\xi + \mu_2 \int_s^\infty e^{\mu_2(s-\xi)} H_2(\varphi)(\xi) d\xi \right],
$$

and

$$
| (F_2(\varphi))'(s) | \leq \frac{\rho}{d_2(\mu_2 - \mu_1)} \left[ \mu_1 \int_{-\infty}^s e^{\mu_1(s-\xi)} d\xi + \mu_2 \int_s^\infty e^{\mu_2(s-\xi)} d\xi \right]
$$

\[
\leq \frac{2\rho}{d_2(\mu_2 - \mu_1)}.
\]

Thus \( \{ F(\varphi)(s) : \varphi \in \Gamma \} \) is a family of equi-continuous functions of \( s \in \mathbb{R} \).

Let \( \Phi_j \) be a sequence of \( \Gamma \) and \( v_j = F(\Phi_j) \). Then the sequence \( v_j \) is equi-continuous. It follows from Lemma 6.2(ii) that \( v_j(s) \) is nondecreasing in \( s \in \mathbb{R} \). Noting that \( \Gamma \) is bounded in \( L^\infty(\mathbb{R}, \mathbb{R}^2) \), by the Arzela-Ascoli theorem, we conclude that for any \( R > 0 \), there exists a convergent subsequence of \( v_j|[-R,R] \) in \( C([-R,R], \mathbb{R}^2) \). Using a standard diagonal selection scheme, we can extract a subsequence \( v_{j_k} \) that converges in \( C([-R,R], \mathbb{R}^2) \) for every \( R > 0 \). Without loss of generality, we assume that the sequence \( v_j \) itself converges in each \( C([-R,R], \mathbb{R}^2) \).

From this, it follows easily that \( v_j \) is Cauchy in \( B_\sigma(\mathbb{R}, \mathbb{R}^2) \), and hence it is convergent. This proves the precompactness of \( F(\Gamma) \). \( \square \)

Since \( \Gamma \) is a bounded closed convex set of \( B_\sigma(\mathbb{R}, \mathbb{R}^2) \), by Lemmas 6.3, 6.4 and 6.5, we can apply Schauder’s fixed point theorem to conclude that \( F \) has a fixed point \( \varphi \) in \( \Gamma \), which is a non-decreasing solution of (2.1). To complete the proof of Proposition 2.5, it remains to prove the following result.

**Lemma 6.6.** The fixed point \( \varphi \) obtained above satisfies (2.2).

**Proof.** From \( \varphi(s) \leq \varphi(s) \leq \varphi(s) \) \( \varphi(s) = \varphi_1(s) = 0 \) for \( s \leq 0 \), \( \varphi_2(-\infty) = \varphi_2(-\infty) = 0 \), we obtain \( \varphi(\infty) = (0,0) \). Moreover, due to \( 0 \leq \varphi_1(s) = 0 \) for \( s \in \mathbb{R} \), we have \( 0 \leq \varphi_1(s) \). It then follows from the monotonicity of \( \varphi_1(s) \) that \( \varphi_1(\infty) \in (0,k_1) \). Using (2.1), it is well known that (cf. Lemma 2.2 in [29]) \( f_1(\varphi(\infty)) = f_2(\varphi(\infty)) = 0 \). Thus we may use (A1) to conclude that \( \varphi(\infty) = K \). Hence (2.2) holds. \( \square \)

**Acknowledgments.** The research in this work was supported by the Natural Science Foundation of China(11671243, 1171262, 61672021), the Shaanxi New-star Plan of Science and Technology(2015KJXX-21), the Natural Science Foundation of Shaanxi Province(2014JM1003, 2018JM1020), the Fundamental Research Funds for the Central Universities(GK201701001, GK201302005), and the Australian Research Council.

The authors thank the anonymous referees for their valuable suggestions leading to an improvement of the manuscript.
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Received August 2018; revised January 2019.

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