MOMENT MAPS TO LOOP ALGEBRAS
CLASSICAL $R$–MATRIX AND INTEGRABLE SYSTEMS

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Abstract

A class of Poisson embeddings of reduced, finite dimensional symplectic vector spaces into the dual space $\tilde{\mathfrak{g}}^*$ of a loop algebra, with Lie Poisson structure determined by the classical split $R$–matrix $R = P_+ - P_-$ is introduced. These may be viewed as equivariant moment maps inducing natural Hamiltonian actions of the “dual” group $\tilde{\mathfrak{g}}_R = \tilde{\mathfrak{g}}^+ \times \tilde{\mathfrak{g}}^-$ of a loop group $\tilde{\mathfrak{g}}$ on the symplectic space. The $R$–matrix version of the Adler-Kostant-Symes theorem is used to induce commuting flows determined by isospectral equations of Lax type. The compatibility conditions determine finite dimensional classes of solutions to integrable systems of PDE’s, which can be integrated using the standard Liouville-Arnold approach. This involves an appropriately chosen “spectral Darboux” (canonical) coordinate system in which there is a complete separation of variables. As an example, the method is applied to the determination of finite dimensional quasi-periodic solutions of the sine-Gordon equation.

0. Introduction.

In [AHP, AHH1-AHH3] a unified framework was developed, describing a wide class of integrable finite dimensional Hamiltonian systems, as well as finite dimensional solutions

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to integrable systems of PDE’s. The basic idea is to represent all these systems in terms of commuting isospectral flows determined by Lax equations within finite dimensional Poisson subspaces of loop algebras consisting of orbits that are rational in the loop parameter. The commuting flows are generated by Hamiltonians that are spectral invariants, or equivalently $Ad^*$–invariants on the space $\tilde{g}^{++}$ dual to the half of the loop algebra admitting holomorphic extensions to the interior disk of a circle within the complex plane of the loop parameter. According to the Adler-Kostant-Symes (AKS) theorem, such invariants generate commuting flows with respect to the Lie Poisson structure on $\tilde{g}^{++}$ and are determined by equations of Lax type.

Finite dimensional, generically quasi-periodic solutions to integrable systems of PDE’s arise within this framework as the compatibility conditions satisfied by the matrix elements, within a given representation. The procedure developed in [AHP, AHH1-AHH3] consists of three steps; first, a suitably defined moment mapping is used to embed symplectic vector spaces quotiented by Hamiltonian group actions as Poisson subspaces of the space $\tilde{g}^{++}$. The image consists of rational coadjoint orbits with fixed poles. The AKS theorem is then used to define the Hamiltonians and determine the dynamical equations, with the ring of invariants generated by the invariant spectral curve given by the characteristic equation of the loop algebra element. The generic systems so obtained may be subjected to further reductions under both discrete and continuous Hamiltonian symmetry groups. Finally, a suitably defined Darboux (canonical) coordinate system is introduced, associated to the divisor of zeroes of the matrix representing the loop algebra element on the spectral curve. Within this “spectral Darboux” coordinate system, there is a complete separation of variables which, through application of the Liouville-Arnold integration method on the invariant Lagrangian manifolds, reduces the problem of integration to quadratures given by Abelian integrals. The resulting flows are thus seen to be linearized by the Abel map in the natural linear coordinate system on the Jacobian variety of the spectral curve, or some quotient thereof. Through the Jacobi inversion method, the matrix elements may be expressed in terms of theta functions, thereby producing, through classical Hamiltonian methods, the types of solutions typically obtained through more sophisticated algebro-geometric means [AvM, KN, D, AHH1].

This approach has been applied to a wide variety of integrable systems, both finite and infinite dimensional, such as: the cubically nonlinear Schrödinger equation and various multi-component generalizations thereof [AHP, AHH3, HW1, W1], the sine-Gordon equation [HW2, TW], the massive Thirring model [W2] the Neumann and Rosochatius systems [AHP, AHH1, AHH3] and various generalized tops [AHH2, HH].

In the subsequent section, an extended form of the moment map embedding method of
1. Moment Maps to Loop Algebras and the Classical \( R \)-Matrix.

We first establish notations for loop groups and algebras. Let \((M, \omega)\) denote the symplectic vector space whose elements are pairs \((F, G)\) of \(N \times r\) matrices

\[
M = \{(F, G) \in M^{N,r} \times M^{N,r}\},
\]

with symplectic form:

\[
\omega = \text{tr} \ (dF^T \wedge dG).
\]

Let \(\tilde{G}\) denote the group of smooth loops in \(\text{GL}(r)\) (real or complex), or some subgroup thereof, viewed as invertible matrix-valued smooth functions \(g(\lambda)\) defined on a smooth, simple closed curve \(\Gamma\) enclosing the origin of the complex \(\lambda\) plane. Let \(\tilde{G}^+, \tilde{G}^-\) be the subgroups of loops admitting holomorphic extensions, respectively, to the interior and exterior regions \(\Gamma^+, \Gamma^-\) (including \(\infty\)), such that for \(g \in \tilde{G}^-\), \(g(\infty) = I\). The Lie algebras corresponding to \(\tilde{G}, \tilde{G}^\pm\), are denoted \(\tilde{g}\) and \(\tilde{g}^\pm\) respectively. Their elements are smooth \(\text{gl}(r)\) valued functions on \(\Gamma\), with elements \(X_+ \in \tilde{g}^+\) admitting holomorphic extensions to \(\Gamma^+\) and \(X_- \in \tilde{g}^-\) to \(\Gamma^-\), the latter satisfying \(X(\infty) = 0\).

We identify \(\tilde{g}\) as a dense subspace of the dual space \(\tilde{g}^*\) through the pairing:

\[
<X, Y> := \frac{1}{2\pi i} \oint_\Gamma \text{tr} \ (X(\lambda)Y(\lambda)) \frac{d\lambda}{\lambda},
\]

\(X \in \tilde{g}^*, \ Y \in \tilde{g}\).

Using the vector space decomposition

\[
\tilde{g} = \tilde{g}^- \oplus \tilde{g}^+,
\]

this allows us to identify the dual spaces as

\[
\tilde{g}^{++} \sim \tilde{g}^-, \quad \tilde{g}^{-*} \sim \tilde{g}^+,
\]

where \(\tilde{g}^-, \tilde{g}^+\) denote, respectively, the subspaces of \(\tilde{g}\) consisting of elements admitting a holomorphic extension to \(\Gamma^-\) or \(\Gamma^+\), with \(X_+ \in \tilde{g}_+\) satisfying \(X_+(0) = 0\).
Let
\[ P_\pm : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}^\pm \]
\[ P_\pm : X \mapsto X_\pm \] (1.6)
be the projections to the subspaces \( \tilde{\mathfrak{g}}^\pm \) relative to the decomposition (1.4) and define the endomorphism \( R : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}} \) as the difference:
\[ R := P_+ - P_. \] (1.7)
Then \( R \) is a classical R-matrix \([S]\), in the sense that the bracket \([\ , \ ]_R\) defined on \( \tilde{\mathfrak{g}} \) by:
\[ [X, Y]_R := \frac{1}{2} [RX, Y] + \frac{1}{2} [X, RY] \] (1.8)
is skew symmetric and satisfies the Jacobi identity, determining a new Lie algebra structure on the same space as \( \tilde{\mathfrak{g}} \), which we denote \( \tilde{\mathfrak{g}}_R \). The corresponding group is the “dual group” \( \tilde{\mathfrak{G}}_R = \tilde{\mathfrak{G}}^- \times \tilde{\mathfrak{G}}^+ \) associated to \( \tilde{\mathfrak{G}} \). The adjoint and coadjoint actions of \( \tilde{\mathfrak{G}}_R \) on \( \tilde{\mathfrak{g}}_R \sim \tilde{\mathfrak{g}}^* \) are given by:
\[ \text{Ad}_R(g) : (X_+ + X_-) \mapsto g_+ X_+ g_+^{-1} + g_- X_- g_-^{-1} \] (1.9a)
\[ \text{Ad}^*_R(g) : (Y_+ + Y_-) \mapsto (g_- X_+ g_-^{-1})_+ + (g_+ X_- g_+^{-1})_-, \] (1.9b)
\[ X_\pm \in \tilde{\mathfrak{g}}^\pm, \quad Y_\pm \in \tilde{\mathfrak{g}}^\pm, \quad g_\pm \in \tilde{\mathfrak{G}}^\pm. \] (1.9)
The Lie Poisson bracket on \( \tilde{\mathfrak{g}}^*_R \sim \tilde{\mathfrak{g}}^* \) dual to the Lie bracket \([\ , \ ]_R\) is:
\[ \{f, g\}|_X = <[df, dg]_R, X> \] (1.10)
for smooth functions \( f, g \) on \( \tilde{\mathfrak{g}}^*_R \) (with the usual identifications, \( df|_X, \ dg|_X \in \tilde{\mathfrak{g}}^*_R \sim \tilde{\mathfrak{g}}^* \)).

Let \( A \in \mathfrak{gl}(N) \) be a fixed \( N \times N \) matrix having no eigenvalues on \( \Gamma \), and define the following action of \( \tilde{\mathfrak{G}}_R \) on \( M \)
\[ \tilde{\mathfrak{G}}_R : M \to M \]
\[ g(\lambda) : (F, G) \to (F_g, G_g) \] (1.11)
where \( (F_g, G_g) \) are defined by:
\[ F_g := F - \frac{1}{2\pi i} \oint_\Gamma (A - \lambda I)^{-1} F (g_+^{-1}(\lambda) - g_-^{-1}(\lambda)) d\lambda \] (1.12a)
\[ G_g := G - \frac{1}{2\pi i} \oint_\Gamma (A^T - \lambda I)^{-1} G (g_+^T(\lambda) - g_-^T(\lambda)) d\lambda. \] (1.12b)
It is straightforward to verify, using the identity
\[(A - \lambda I)^{-1}(A - \sigma I)^{-1} = \frac{(A - \lambda I)^{-1} - (A - \sigma I)^{-1}}{\lambda - \sigma}\] (1.13)
and residue calculus, that the \(\widetilde{\mathfrak{g}}_R\) composition rule is satisfied, so (1.11), (1.12a,b) does, indeed define a \(\widetilde{\mathfrak{g}}_R\)–action on \(M\). A similar calculation shows that this action preserves the symplectic form (1.2) and is, in fact, generated as a Hamiltonian action by the equivariant moment map:
\[
\tilde{J}_{A,Y} : M \rightarrow \widetilde{\mathfrak{g}}_R^* \\
\tilde{J}_{A,Y}(F,G) = \lambda Y + \lambda G^T (A - \lambda I)^{-1} F,
\] (1.14)
where \(Y \in \mathfrak{g}(r)\) is any \(r \times r\) matrix. The constant term \(\lambda Y\) may be included without affecting the equivariance of the moment map (1.14) since \(\lambda Y\) is an infinitesimal character for the Lie algebra \(\widetilde{\mathfrak{g}}_R\):
\[
<\lambda Y, [X,Y]_R> = 0, \quad \forall X,Y \in \widetilde{\mathfrak{g}}_R.
\] (1.15)
The splitting \(\tilde{J}_{A,Y}(F,G) = \tilde{J}_+(F,G) + \tilde{J}_-(F,G),\) \(\tilde{J}_\pm(F,G) \in \widetilde{\mathfrak{g}}_\pm\) is determined by the pole structure, with the \(\lambda Y\) term plus the poles at eigenvalues of \(A\) in \(\Gamma^-\) included in the \((\widetilde{\mathfrak{g}}^-)^* \sim \widetilde{\mathfrak{g}}_+\) part and the poles at eigenvalues in \(\Gamma^+\) in the \((\widetilde{\mathfrak{g}}^+)^* \sim \widetilde{\mathfrak{g}}_-\) part.

The fibres of the Poisson map \(\tilde{J}_{A,Y}\) are the orbits of the stability subgroup \(G_A := \text{Stab}(A) \subset \text{Gl}(N)\) under the Hamiltonian \(\text{Gl}(N)\) action defined by
\[
g : M \rightarrow M \\
g : (F,G) \mapsto (gF, (g^T)^{-1} G).
\] (1.16)
On a suitably defined open, dense set, this action is free, allowing us to define the quotient Poisson space \(M/G_A\). Since the map \(\tilde{J}_{A,Y} : M \rightarrow \widetilde{\mathfrak{g}}_R^*\) passes to the quotient, defining a 1–1 Poisson map \(\tilde{J}_{A,Y} : M/G_A \rightarrow \widetilde{\mathfrak{g}}_R^*\), we may identify \(M/G_A\) with the image Poisson space \(\mathfrak{g}_A^Y \subset \widetilde{\mathfrak{g}}_R^*\) consisting of elements \(\mathcal{N}(\lambda)\) of the form
\[
\mathcal{N}(\lambda) = \lambda Y + \lambda \sum_{i=1}^{n} \sum_{l_i=1}^{n_i} \frac{N_{i,l_i}}{(\lambda - \alpha_i)^{l_i}},
\] (1.17)
where \(\{\alpha_i\}_{i=1,...,n}\) are the eigenvalues of \(A\), \(\{l_i\}_{i=1,...,n}\) are the dimensions of the corresponding Jordan blocks of generalized eigenspaces, and the ranks and Jordan structures of the \(r \times r\) matrices \(N_{i,l_i}\) are determined by the multiplicities of the eigenvalues of \(A\) and its Jordan structure.
The Hamiltonian equations on the phase space $g^Y_A \sim M/G_A$ are generated by elements of the ring $T^Y_A$ consisting of $Ad^*$-invariant polynomials on $\tilde{g}^*$, restricted to the Poisson subspace $g^Y_A \subset \tilde{g}^*_R$. According to the Adler-Kostant-Symes (AKS) theorem, in its $R$-matrix form $[S]$, the elements of this ring Poisson commute, generating commuting Hamiltonian flows, and Hamilton’s equations for $\phi \in T^Y_A$ have the Lax form:

$$\frac{dN(\lambda)}{dt} = [(d\phi)_+,N] = -[(d\phi)_-,N],$$

$$(d\phi)_+ \in \tilde{g}^+, \quad d\phi = (d\phi)_+ + (d\phi)_-.$$

This implies that the spectral curve $S$, with affine part defined by the characteristic equation

$$\det(L(\lambda) - zI) := P(\lambda,z) = 0,$$

where

$$L(\lambda) := \frac{a(\lambda)}{\lambda}N(\lambda), \quad a(\lambda) := \prod_{i=1}^{n}(\lambda - \alpha_i)^{n_i}$$

is invariant under these flows, and its coefficients generate the ring $T^Y_A$.

Integrable systems of PDE’s then arise as the equations satisfied by the matrix elements of $N$ implied by the compatibility conditions

$$\frac{d(d\phi)_+}{dx} - \frac{d(d\psi)_+}{dt} + [d(\phi)_+,d(\psi)_+] = 0, \quad \phi, \psi \in T^Y_A,$$

where $t, x$ are the respective flow parameters for the Hamiltonians $\phi$ and $\psi$. The flows may be integrated through a standard algebro-geometric construction that leads to a linearizing map to the Jacobi variety of the spectral curve $S$ ([KN, D, AHH1]). To obtain interesting examples, the generic systems so obtained must usually be further reduced by certain continuous or discrete symmetry groups. The easiest way to arrive at the linearization involves the introduction of a suitably defined “spectral Darboux” (canonical) coordinate system associated to the divisor of zeroes of the eigenvectors of $L(\lambda)$ on the spectral curve. The general construction is detailed in [AHH3]. Its application to the particular problem of determining finite dimensional quasi-periodic solutions to the sine-Gordon equation is described in the next section. Full details for this case may be found in [HW2].
2. Quasiperiodic Solutions of the Sine-Gordon Equation.

To obtain the sine-Gordon equation

\[ \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \sin(u) \]  

we shall need commuting isospectral flows in the twisted loop algebra \( \tilde{\mathfrak{su}}(2) := \tilde{\mathfrak{su}}^{(1)}(2) \). This is the subalgebra of \( \tilde{\mathfrak{gl}}(2) \) consisting of elements \( X(\lambda) \) that are invariant under the three involutive endomorphisms:

\[ \sigma_1 : X(\lambda) \mapsto JX^T(\lambda)J \] 
\[ \sigma_2 : X(\lambda) \mapsto -X^\dagger(\bar{\lambda}) \] 
\[ \sigma_3 : X(\lambda) \mapsto \tau X(-\lambda)\tau \]

where

\[ \tau := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \] 

The first of these implies that \( X \) is traceless, the second that it is in \( \tilde{\mathfrak{su}}(2) \) and the third that it is in the “twisted” subalgebra \( \tilde{\mathfrak{su}}(2) \).

We choose the matrix \( Y \) as

\[ Y = J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \] 

\( \Gamma \) as a circle centred at the origin of the \( \lambda \)-plane, and \( A \) as a diagonal \( 2N \times 2N \) dimensional matrix, having distinct eigenvalues, all inside \( \Gamma^+ \), of the form

\[ A = \text{diag} \left( \alpha_1, \alpha_1, \ldots, \alpha_p, \alpha_p, -\alpha_1, \ldots, -\alpha_p, i\beta_{2p+1}, -i\beta_{2p+1}, \ldots, i\beta_N, -i\beta_N \right), \] 

where \( \{\alpha_j\}_{j=1,\ldots,p} \) have nonvanishing real and imaginary parts and \( \{\beta_j\}_{j=2p+1,\ldots,N} \) are real.

The space \( M = \{(F,G) \in M^{2N,2} \times M^{2N,2} \} \) must correspondingly be reduced by the endomorphisms

\[ \Sigma_1 : (F,G) \mapsto (GJ, -FJ) \] 
\[ \Sigma_2 : (F,G) \mapsto (-JG, JF) \] 
\[ \Sigma_3 : (F,G) \mapsto (-iKF\tau, iKG\tau) \]

where

\[ J = \text{diag} \left( J, J, \ldots, J \right). \]
is the block diagonal $2N \times 2N$ matrix with $2 \times 2$ blocks equal to $J$ and

$$\mathcal{K} = \begin{pmatrix} 0 & I_{2p} \\
-I_{2p} & 0 \\
& \ddots \\
& & J \end{pmatrix} \quad (2.8)$$

is the $2N \times 2N$ matrix consisting of a $4p \times 4p$ block formed from $I_{2p}$, the $2p \times 2p$ identity matrix and its negative, and $N - 2p$ diagonal $2 \times 2$ blocks $J$.

The moment map $\hat{J}_A^Y$ of eq. (1.14) then intertwines the automorphism groups generated by $\Sigma_1, \Sigma_2, \Sigma_3$ and by $\sigma_1, \sigma_2, \sigma_3$. Denoting by $\left(\begin{pmatrix} F_{2j-1}^\perp \\ F_{2j}^\perp \end{pmatrix}, \begin{pmatrix} G_{2j-1}^\perp \\ G_{2j}^\perp \end{pmatrix}\right)$ the consecutive pairs of $2 \times 2$ blocks in $(F, G)$, the fixed point set $M_{\Sigma} \subset M$ under $\Sigma_1, \Sigma_2, \Sigma_3$ consists of $(F, G)$ with $2 \times 2$ blocks of the form

$$\begin{pmatrix} F_{2j-1} \\ F_{2j} \end{pmatrix} = \begin{pmatrix} \varphi_j & \bar{\gamma}_j \\ \bar{\gamma}_j & -\bar{\varphi}_j \end{pmatrix}, \quad \begin{pmatrix} G_{2j-1} \\ G_{2j} \end{pmatrix} = \begin{pmatrix} -\bar{\gamma}_j & \varphi_j \\ \bar{\varphi}_j & \gamma_j \end{pmatrix}, \quad j = 1, \ldots, p \quad (2.9a)$$

$$\begin{pmatrix} F_{2j}+2p-1 \\ F_{2j+2p} \end{pmatrix} = \begin{pmatrix} i\varphi_j & -i\bar{\gamma}_j \\ i\bar{\gamma}_j & i\bar{\varphi}_j \end{pmatrix}, \quad \begin{pmatrix} G_{2j}+2p-1 \\ G_{2j+2p} \end{pmatrix} = \begin{pmatrix} -i\bar{\gamma}_j & i\varphi_j \\ -i\bar{\varphi}_j & i\gamma_j \end{pmatrix}, \quad j = 1, \ldots, p \quad (2.9b)$$

$$\begin{pmatrix} F_{2j-1} \\ F_{2j} \end{pmatrix} = \begin{pmatrix} i\bar{\gamma}_j & \bar{\gamma}_j \\ \gamma_j & i\bar{\varphi}_j \end{pmatrix}, \quad \begin{pmatrix} G_{2j-1} \\ G_{2j} \end{pmatrix} = \begin{pmatrix} -\bar{\gamma}_j & i\gamma_j \\ -i\bar{\varphi}_j & \bar{\gamma}_j \end{pmatrix}, \quad j = 2p + 1, \ldots, N. \quad (2.9c)$$

We have

$$\Sigma_1^*\omega = \omega, \quad \Sigma_2^*\omega = \bar{\omega}, \quad \Sigma_3^*\omega = \omega \quad (2.10)$$

so $M_{\Sigma}$ is a real symplectic space, with symplectic form:

$$\bar{\omega} = 4 \sum_{j=1}^{p} (d\gamma_j \wedge d\varphi_j + d\bar{\gamma}_j \wedge d\bar{\varphi}_j) + 4i \sum_{j=2p+1}^{N} d\bar{\gamma}_j \wedge d\gamma_j. \quad (2.11)$$

The restriction of $\hat{J}_A^Y$ to $M_{\sigma}$, which we denote $\hat{J}$, is given by

$$\hat{J}(F, G) := \mathcal{N}(\lambda) = \lambda \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + 2\lambda \begin{pmatrix} b(\lambda) & -c(\lambda) \\ c(\lambda) & -b(\lambda) \end{pmatrix} \in \mathfrak{su}(2). \quad (2.12)$$

where

$$b(\lambda) = \lambda \sum_{j=1}^{p} \left( \frac{-\varphi_j \bar{\gamma}_j}{\alpha_j - \lambda^2} + \frac{\bar{\varphi}_j \gamma_j}{\alpha_j^* - \lambda^2} \right) + i\lambda \sum_{j=2p+1}^{N} \frac{|\gamma_j|^2}{\beta_j^2 + \lambda^2} \quad (2.13a)$$

$$c(\lambda) = \sum_{j=1}^{p} \left( \frac{\alpha_j \bar{\gamma}_j^2}{\alpha_j^2 - \lambda^2} + \frac{\bar{\alpha}_j \varphi_j^2}{\bar{\alpha}_j^2 - \lambda^2} \right) - i \sum_{j=2p+1}^{N} \frac{\beta_j \bar{\gamma}_j^2}{\beta_j^2 + \lambda^2}. \quad (2.13b)$$
Now, choose Hamiltonians $H_\xi, H_\eta \in \mathcal{I}_A^Y$ as:

$$H_\xi(X) = -\frac{1}{4\pi i} \oint_{\Gamma} \text{tr} \left( \frac{a(\lambda)}{\lambda^2} (X(\lambda))^2 \right) d\lambda$$

(2.14a)

$$H_\eta(X) = \frac{1}{4\pi i} \oint_{\Gamma} \text{tr} \left( \frac{a(\lambda)}{\lambda^{2N}} (X(\lambda))^2 \right) d\lambda,$$

(2.14b)

where

$$a(\lambda) = \prod_{j=1}^{p} [(\lambda^2 - \alpha_j^2)(\lambda^2 - \bar{\alpha}_j^2)] \prod_{k=2p+1}^{N} (\lambda^2 + \beta_k^2).$$

(2.15)

Then the Lax form of Hamilton’s equations may be written as

$$\frac{dN}{d\xi} = -[dH_\xi(N)_-, N]$$

(2.16a)

$$\frac{dN}{d\eta} = [dH_\eta(N)_+, N],$$

(2.16b)

where, setting

$$L(\lambda) = \frac{a(\lambda)}{\lambda} N(\lambda) = \lambda^{2N-1} L_0 + \lambda^{2N-2} L_1 + \cdots + L_{2N-1} + a(\lambda) Y,$$

(2.17)

we have

$$dH_\xi(N)_- = -\frac{1}{\lambda}(L_{2N-1} + a(0) Y)$$

(2.18a)

$$dH_\eta(N)_+ = L_0 + \lambda Y.$$

(2.18b)

The spectral curve is of the form

$$\det(L(\lambda) - z I) = P(\lambda, z) = z^2 + a(\lambda) P(\lambda) = 0$$

(2.19a)

$$P(\lambda) = P_0 + \lambda^2 P_1 + \cdots + \lambda^{2N-2} P_{N-1} + \lambda^{2N},$$

(2.19b)

with all coefficients $P_i$ in $\mathcal{I}_A^Y$ and, in particular

$$H_\xi = P_0, \quad H_\eta = -P_{N-1}.$$  

(2.20)

Choosing the invariant level set

$$a(0) P_0 = \det(L_{2N-1} + a(0) Y) = \frac{1}{16},$$

(2.21)
implies that
\[ \mathcal{L}_{2N-1} + a(0)Y = \frac{1}{4} \begin{pmatrix} 0 & e^{iu} \\ -e^{-iu} & 0 \end{pmatrix}, \] (2.22)
where \( u \) is real. Setting
\[ \xi = x + t, \quad \eta = x - t, \] (2.23)
the compatibility equations for (2.16a,b) then imply that \( u \) satisfies the sine-Gordon equation (2.1).

Since the spectral curve \( \mathcal{C} \) determined by (2.19a,b) is invariant under the involution \((z, \lambda) \mapsto (z, -\lambda)\), it is a two-sheeted covering of the hyperelliptic curve with genus \( N - 1 \) whose affine part given is by
\[ z^2 + \tilde{a}(E)\tilde{P}(E) = 0, \] (2.24)
where
\[ \tilde{P}(\lambda^2) := P(\lambda), \quad \tilde{a}(\lambda^2) := a(\lambda), \quad \lambda^2 = E. \] (2.25)
We also define functions \( \tilde{b}, \tilde{c} \) by
\[ \tilde{b}(\lambda^2) = b(\lambda), \quad \tilde{c}(\lambda^2) = c(\lambda). \] (2.26)
Setting \( \tilde{z} := z\lambda \), we obtain the genus \( N \) hyperelliptic curve \( \tilde{\mathcal{C}} \) with affine part given by
\[ \tilde{z}^2 + E\tilde{a}(E)\tilde{P}(E) = 0, \] (2.27)
which has two additional branch points at \( E = 0, \infty \).

Following the general method of [AHH3], we define on \( \tilde{\mathcal{C}} \) the divisor of degree \( N \) with coordinates \((E_\mu, \zeta_\mu)_{\mu=1,\ldots,N}\) determined by solving the equation
\[ 2\tilde{c}(E_\mu) + 1 = 0 \] (2.28)
for \( \{E_\mu\}_{\mu=1,\ldots,N} \) and substituting in
\[ \zeta_\mu = \sqrt{-\frac{\tilde{P}(E_\mu)}{E_\mu\tilde{a}(E_\mu)}} = \frac{2\tilde{b}(E_\mu)}{\sqrt{E_\mu}}. \] (2.29)
This defines the zeroes of the eigenvectors of \( \mathcal{L}(\lambda) \) on the spectral curve \( \tilde{\mathcal{C}} \). In terms of these, we have:
\[ u = -i\ln \left( 2 \prod_{\mu=1}^{N} E_\mu \right) + \epsilon \pi \] (2.30)
where \( \epsilon = 1, 0 \) for \( N \) even or odd, respectively. The \( \{E_\mu, \zeta_\mu\}_{\mu=1,\ldots,N} \) may be viewed as coordinate functions on the coadjoint orbit through \( \mathcal{N} = \hat{J} \in \hat{su}_R^*(2) \), and the following result is key to the linearization of the flows (cf. [HW2] for details):
**Proposition.** The functions \((E_\mu, \zeta_\mu)_{\mu=1,\ldots,N}\) form a Darboux coordinate system on the coadjoint orbit passing through \(N(\lambda)\). The corresponding orbital symplectic form is

\[
\omega = \sum_{\mu=1}^{N} dE_\mu \wedge d\zeta_\mu = -d\theta.
\] (2.31)

This may be seen either by an explicit coordinate transformation from (2.11), by verifying the following expression for the canonical 1–form

\[
\theta = 4 \sum_{i=1}^{p} (\varphi_i d\bar{\gamma}_i - \gamma_i d\bar{\varphi}_i) + 4i \sum_{j=2p+1}^{N} \gamma_j d\bar{\gamma}_j = \sum_{\mu=1}^{N} \zeta_\mu dE_\mu,
\] (2.32)

or as an application of the general “spectral Darboux coordinates” theorem of [AHH3].

It now follows directly from the definition of the coordinates \((E_\mu, \zeta_\mu)_{\mu=1,\ldots,N}\) that on the invariant level sets (Lagrangian manifolds) determined by \(\{P_i = c_i\}_{i=0}^{N-1}\), the 1–form \(\theta|_{\{P_i = c_i\}} = dS\) may be integrated to yield the Liouville generating function

\[
S(P_i, E_\mu) = \sum_{\mu=1}^{N} \int_{E_0}^{E_\mu} \sqrt{-E\tilde{a}(E)P(E)} dE.
\] (2.33)

Within the new canonical coordinate system \(\{Q_i, P_i\}_{i=0,\ldots,N-1}\) defined by

\[
Q_i = \frac{\partial S}{\partial P_i} = -\frac{1}{2} \sum_{\mu=1}^{N} \int_{E_0}^{E_\mu} \frac{E^i}{\sqrt{-E\tilde{a}(E)P(E)}} dE,
\] (2.34)

the flows for all the Hamiltonians in the spectral ring generated by the \(P_i\)’s are then linear. In particular, integrating Hamilton’s equations for \(H_\xi, H_\eta\) gives

\[
\sum_{\mu=1}^{N} \int_{E_0}^{E_\mu} \frac{E^i}{\sqrt{-E\tilde{a}(E)P(E)}} dE = C_i + 2\delta_{i,0}\xi - 2\delta_{i,N-1}\eta.
\] (2.35)

This may be interpreted as the linearizing Abel map to the Jacobi variety of \(\tilde{C}\). By a standard inversion procedure (cf. [HW2], the function \(u\) may be explicitly expressed in terms of quotients of the associated theta functions:

\[
u = -2i\ln \frac{\Theta(A(0) - U\eta - V\xi - B - \kappa)}{\Theta(A(\infty) - U\eta - V\xi - B - \kappa)} + c,
\] (2.36)

where the constant vectors \(A, U, V, B \in \mathbb{C}^N\) are obtained from those defined by the slopes and integration constants on the RHS of eq. (2.35) by applying the matrix that transforms the differentials appearing in the integrands on the LHS to a normalized canonical basis of abelian differentials.
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