Dynamic covariate balancing:
estimating treatment effects over time

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Abstract

This paper discusses the problem of estimation and inference on the effects of time-varying treatment. We propose a method for inference on the effects treatment histories, introducing a dynamic covariate balancing method combined with penalized regression. Our approach allows for (i) treatments to be assigned based on arbitrary past information, with the propensity score being unknown; (ii) outcomes and time-varying covariates to depend on treatment trajectories; (iii) high-dimensional covariates; (iv) heterogeneity of treatment effects. We study the asymptotic properties of the estimator, and we derive the parametric $n^{-1/2}$ convergence rate of the proposed procedure. Simulations and an empirical application illustrate the advantage of the method over state-of-the-art competitors.

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1 Introduction

Understanding the effect of time-varying treatment effects is an important task for many applications in social sciences. This paper discusses the estimation and inference on the average effect of treatment trajectories (i.e., treatment history) for observational studies with $n$ independent units observed over $T$ periods. For example, we may be interested in the long-run effect of public health insurance on long-run health status (Finkelstein et al., 2012), the effect of negative political advertisements on election outcomes (Blackwell, 2013), or on the long or short-run effects of minimum wages on employment.

We focus on a setting where time-varying covariates and outcomes depend on past treatment assignments, and treatments are assigned sequentially based on arbitrary past information. Two alternative procedures can be considered in this setting. First, researchers may consider explicitly modeling how treatment effects propagate over each period through time-varying covariates and intermediate outcomes. This approach is prone to large estimation error and misspecification in high-dimensions: it requires modeling outcomes and each time-varying covariate as a function of all past covariates, outcomes, and treatment assignments. A second approach is to use inverse-probability weighting estimators for estimation and inference (Tchetgen and Shpitser, 2012; Vansteelandt et al., 2014). However, classical semi-parametric estimators are prone to instability in the estimated propensity score. There are two main reasons. First of all, the propensity score defines the joint probability of the entire treatment history and can be close to zero for moderately long treatment histories. Additionally, the propensity score can be misspecified in observational studies.

Figure 1 presents an illustrative example. The figure shows that the probability of remaining under treatment for two consecutive periods in an application from

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1Empirical examples include studying the effect of state-level change in legislation (Card and Krueger, 1993; Garthwaite et al., 2014), or studying the effect of treatments assigned at the individual level on a yearly, monthly, or daily basis (Athey and Stern, 1998; LaLonde, 1986).

2A simple example where misspecification occurs is when treatment assignments are the realization of the decisions of forward-looking agents who maximize the expected discounted future utility (Heckman and Navarro, 2007) with individual utilities unknown to the researcher.
Acemoglu et al. (2019) shifts towards zero, making inverse-probability weighting estimators unstable in finite sample.\(^3\)

Figure 1: Discussion on overlap for dynamic treatments, data from Acemoglu et al. (2019). Estimated probability of treatment for one year (left-panel) and two consecutive years (right-panel).

We overcome the problems discussed above by proposing a parsimonious and easy-to-interpret model for potential outcomes. In the same spirit of local projections (Jordà, 2005), we model the potential outcome as an (approximately) linear function of previous potential outcomes and potential covariates.\(^4\) Unlike the standard local projection framework, the model on potential outcomes allows researchers to be agnostic on the process governing treatment assignments. In particular, assignments can depend on some unknown functions of arbitrary past information.\(^5\) We allow for heterogeneity of treatment effects in possibly high dimensional covariates, with covariates that depend on treatment histories. We consider treatment dynamics

\(^3\)Estimation is performed via logistic regression with a pooled regression with year, region fixed effects and four lagged outcomes. The right panel also controls for the past treatment assignment.

\(^4\)Potential outcomes and potential covariates are indexed by the past treatment history.

\(^5\)The outcome model nests linear models with arbitrarily many auto-regressive components, without, however, required to estimate separate regressions for each time-varying covariate. It encompasses marginal structural models of Robins et al. (2000) under linearity of outcomes and covariates.
in outcomes and time-varying covariates and do not restrict (or model) how the
treatments depend on previous outcomes, covariates, or treatments.

We derive covariate balancing conditions, which circumvent the estimation prob-
lem of the propensity score by directly balancing covariates \textit{dynamically}. In addition,
we provide identification results that permit estimation of the (high-dimensional) pa-
rameters of the potential outcome model. Our method, entitled Dynamic Covariate
Balancing (DCB), builds on such results and combines high-dimensional estimators
with dynamic covariate balancing.

Balancing covariates is intuitive and commonly used in practice: in cross-sectional
studies, treatment and control units are comparable when the two groups have sim-
ilar characteristics in their covariates (Hainmueller, 2012; Imai and Ratkovic, 2014;
Li et al., 2018; Ning et al., 2017). We generalize covariate balancing of Zubizarreta
(2015) and Athey et al. (2018) to a dynamic setting. We construct weights se-
quently in time, where balancing weights in the current period depends on those
estimated in the previous period. Our balancing procedure has relevant practical
implications: (i) it allows for estimation and inference without requiring knowledge
of the propensity score; (ii) it guarantees a vanishing (and thus negligible) bias of
order faster than $n^{-1/2}$; and (iii) it solves a quadratic program to find the weights
with minimal variance and thus ensures robustness to poor overlap in a small sam-
ple. In our theoretical studies, we derive the parametric rate of convergence of the
estimator in high-dimensions, show the existence of balancing weights, and discuss
asymptotic inference on treatment histories.

Our numerical studies show the advantage of the proposed method over state-of-
the-art competitors. DCB presents correct coverage under good to moderately poor
overlap, and results are robust to increasing the dimension of covariates. Finally,
in our empirical application, we study the effect of negative advertisement on the
election outcome and the effect of democracy on GDP growth using the DCB method.

The remainder of the paper is organized as follows. In Section 2, we discuss
the framework and model in the presence of two periods. In Section 3 we discuss
balancing with two periods. In Section 4 we extend to multiple periods and discuss
theoretical guarantees. Numerical studies and the empirical application are included
in Section 5 and Section 6 respectively. Section 7 concludes.

1.1 Related Literature

Dynamic treatments have been widely discussed in several independent strands of literature. Robins (1986), Robins et al. (2000), Robins (2004), Hernán et al. (2001), Boruvka et al. (2018), Blackwell (2013), and others discuss estimation and inference on dynamic treatments. These studies mostly focus on marginal structural models, which can be sensitive to the specification of the propensity score and require, in high-dimensions, its correct specification. For a selective review, see Vansteelandt et al. (2014).

References also include Bojinov and Shephard (2019), Bojinov et al. (2020) who study inverse-probability weighting estimators and characterize their properties from a design-based perspective. Doubly robust estimators (Robins et al., 1994) for dynamic treatment assignment have been discussed in previous literature including Jiang and Li (2015); Nie et al. (2019); Tchetgen and Shpitser (2012); Zhang et al. (2013) and the recent work of Bodory et al. (2020). However, one key drawback of these methods is the instability and possible model misspecification of inverse probability weights.

Our contribution to balancing conditions for dynamic treatments is of independent interest. Differently from Zhou and Wodtke (2018), who extend the entropy balancing weights of Hainmueller (2012), we do not estimate models for each covariate given the past filtration. Instead, we only estimate models for the end-line potential outcomes, which leads to computationally efficient estimators. DCB explicitly characterizes the high-dimensional model’s bias in a dynamic setting to avoid overly conservative moment conditions, while Kallus and Santacatterina (2018) design balancing in the worst-case scenario only. We do not require estimation of the propensity model’s score function as in Yiu and Su (2018) who propose a single balancing equation. Finally, in the context of panel data, Arkhangelsky and Imbens (2019) propose practical balancing weights which, importantly, assume no dynamics (i.e., carry-overs) in treatment effects. We also note that none of the above references
address the problem of high dimensional covariates.

Our problem also connects to the literature on two-way fixed effects and multi-periods Difference-in-Differences (Abraham and Sun, 2018; Athey and Imbens, 2018; Callaway and Sant’Anna, 2019; de Chaisemartin and d’Haultfoeuille, 2019; Goodman-Bacon, 2021; Imai and Kim, 2016). The above references prohibit that individuals select into treatment and control dynamically each period, based on past outcomes and time-varying covariates.\textsuperscript{6} Here, we allow for dynamics in treatments assigned based on arbitrary past information and time-varying covariates to depend on the past treatment assignments. Also, the above references either require correct specification of the propensity score, assume that there are no high-dimensional covariates or both. Related methods also include discrete choice models and dynamic treatments using instrumental variables (Heckman et al., 2016; Heckman and Navarro, 2007), which, however, impose parametrizations on the propensity score.

A related strand of literature includes Synthetic Control (SC) methods (Abadie et al., 2010; Arkhangelsky et al., 2019; Ben-Michael et al., 2018; Doudchenko and Imbens, 2016). However, these approaches assume staggered adoption (i.e., individuals are always treated after a certain period) with an exogenous treatment time, hence prohibiting dynamics in treatment assignments. In the SC setting, Ben-Michael et al. (2018, 2019) balance covariates as in Zubizarreta (2015), fixing the time of the treatment. In their setting, staggered adoption motivates the construction of a single set of balancing weights for all post-treatment periods, hence without allowing for dynamics in treatment assignments. Here, following Robins et al. (2000)’s dynamic treatment framework, treatment assignments are time-varying and endogenously assigned based on arbitrary past information. The weights of Athey et al. (2018); Zubizarreta (2015) are a special case of our method in the absence of dynamics.

In a few studies regarding high-dimensional panel data, researchers require cor-

\textsuperscript{6}The above references impose restrictions on how potential outcomes behave conditionally on future assignments (e.g., assuming potential outcomes strong exogeneity assumptions or parallel trend conditions). Simple examples are conditions on potential outcomes conditional on the indicator of being “always under control”, i.e., on a future treatment path. However, in the presence of treatment assignments that depend on past outcomes, past potential outcomes are predictive of future assignments (e.g., whether individuals do not receive the treatment may depend dynamically on their past outcome). Our framework accommodates such a setting.
rect specification of the propensity score (Belloni et al., 2016; Bodory et al., 2020; Chernozhukov et al., 2017, 2018; Shi et al., 2018; Zhu, 2017), or impose homogeneity conditions on treatment effects (Kock and Tang, 2015; Krampe et al., 2020).

Additional references include inference in time-series analysis (Plagborg-Møller, 2019; Stock and Watson, 2018; White and Lu, 2010), which often require structural estimation for inference and impose stationarity and strong exogeneity conditions. This paper instead uses information from panel data and allows for arbitrary dependence of outcomes, covariates, and treatment assignments over time. Additional references in macroeconomics include Angrist and Kuersteiner (2011), and Angrist et al. (2018), who discuss inference using inverse probability weights estimator, without incorporating carryover effects in the construction of the weights. This difference reflects a different set of target estimands. Rambachan and Shephard (2019) discuss local projections on previous treatment assignments and characterize their properties assuming that assignments are unpredictable, i.e., independent of the past. Here, we derive novel identification results with serially correlated treatment assignments that also depend on the past outcomes.

Finally, an overview of classical tools and some recent developments in econometrics can be found in Arellano and Bonhomme (2011), Abadie and Cattaneo (2018), Abbring and Heckman (2007) and references therein.

2 Dynamics and potential projections

We first discuss the case of two time periods since it provides a simple illustration of the problem and our solution. Our focus is on ex-post evaluation, where treatment effects are evaluated after the entire history of interventions has been deployed, and the relevant outcomes under the intervention are measured in each period.

In the presence of two periods only, we observe $n$ i.i.d. copies $O_i \sim P$, $i = 1, \ldots, n$ of a random vector

$$O_i = \left( X_{i,1}, D_{i,1}, Y_{i,1}, X_{i,2}, D_{i,2}, Y_{i,2} \right)$$

\(^7\)In the above references, weights only depend on the conditional probability of the present treatment assignment, but not on the joint probability of treatment history.
where $D_1$ and $D_2$ are binary treatment assignments at time $t = 1, t = 2$, respectively. Here, $X_{i,1}$ and $X_{i,2}$ are covariates for unit $i$ observed at time $t = 1$ and $t = 2$, respectively. We observe the outcome $Y_{i,t}$ right after $D_{i,t}$, but prior to $D_{i,t+1}$. That is, at time $t = 1$, we observe $\{X_{i,1}, D_{i,1}\}$. Outcome $Y_{i,1}$ is revealed after time $t = 1$ but before time $t = 2$. At time $t = 2$ we observe $\{X_{i,2}, D_{i,2}\}$ and finally, outcome $Y_{i,2}$ is revealed. Whenever we omit the index $i$, we refer to the vector of observations for all units.

### 2.1 Estimands and potential outcomes

Potential outcomes and covariates are functions of the entire treatment history. Here,

\[
\left( Y_{i,2}(1, 1), Y_{i,2}(1, 0), Y_{i,2}(0, 1), Y_{i,2}(0, 0) \right)
\]

define the potential outcomes if individual $i$ is under treatment for two consecutive periods, under treatment for the first but not the second period, the second but not the first, and none of the periods. We define compactly $Y_{i,1}(d_1, d_2)$ and $Y_{i,2}(d_1, d_2)$ the potential outcomes in period one and two, respectively, for unit $i$, under a treatment history that assigns treatment $d_1$ in the first period and $d_2$ in the second period. Throughout our discussion, we implicitly assume that SUTVA holds (Rubin, 1990).

Treatment histories may also affect future covariates. Therefore, we denote $X_{i,2}(d_1, d_2)$, the potential covariates for a treatment history $(d_1, d_2)$. The causal effect of interest is the long-run impact of two different treatment histories $(d_1, d_2),(d_1', d_2')$ on the potential outcomes conditional on the covariates at the baseline. Let

\[
\mu(d_1, d_2) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[ Y_{i,2}(d_1, d_2) \bigg| X_{i,1} \right]
\]

denote the expectation of potential outcomes given baseline covariates. Given $\mu(\cdot)$ we can construct

\[
\text{ATE}(d_{1:2}, d_{1:2}') = \mu(d_1, d_2) - \mu(d_1', d_2').
\]

A simple example is $\text{ATE}(1, 0)$, which denotes the effect of a policy when imple-
mented on two consecutive periods against the effect of the policy when never implemented (Athey and Imbens, 2018).

The first condition we impose is the no-anticipation. This is defined below.

**Assumption 1** (No Anticipation). For $d_1 \in \{0, 1\}$, let the following hold

$$Y_{i,1}(d_1, 1) = Y_{i,1}(d_1, 0), \quad X_{i,2}(d_1, 1) = X_{i,2}(d_1, 0).$$

(2)

The no anticipation condition has two implications: (i) potential outcomes only depend on past but not future treatments; (ii) the treatment status at $t = 2$ has no contemporaneous effect on covariates. Observe that the no-anticipation allows for anticipatory effects governed by expectation, but it prohibits anticipatory effects based on the future treatment realization.\(^8\) Also, observe that the no-anticipation is not imposed on the realized treatments, and it allows potential outcomes to be correlated with the future assignments (e.g., see Equation 3).

**Example 2.1** (Observed outcomes). Consider a dynamic model of the form (omitting time-varying covariates at time $t = 2$ for expositional convenience)

$$Y_{i,2} = g_2 \left( Y_{i,1}, X_{i,1}, D_{i,1}, D_{i,2}, \varepsilon_{i,2} \right), \quad Y_{i,1} = g_1 \left( X_{i,1}, D_{i,1}, \varepsilon_{i,1} \right),$$

with $(\varepsilon_{i,2}, \varepsilon_{i,1})$ exogenous. Then we can write

$$Y_{i,2}(d_1, d_2) = g_2 \left( Y_{i,1}(d_1), X_{i,1}, d_1, d_2, \varepsilon_{i,2} \right), \quad Y_{i,1}(d_1) = g_1 \left( X_{i,1}, d_1, \varepsilon_{i,1} \right).$$

Since $g_1(\cdot)$ is not a function of $d_2$, Assumption 1 holds, for any (conditional) distribution of $(D_{i,1}, D_{i,2})$. \(\square\)

With abuse of notation, in the rest of our discussion, we index potential outcomes and covariates by past treatment history only, letting Assumption 1 implicitly hold. We define $H_{i,2} = \left[ D_{i,1}, X_{i,1}, X_{i,2}, Y_{i,1} \right]$, as the vector of past treatment assignments.

\(\text{\footnotesize\(^8\)The no-anticipation assumption on potential outcomes has been previously discussed also in Bojinov and Shephard (2019), Imai et al. (2018), Boruvka et al. (2018) to cite some.}\)
covariates, and outcomes in the previous period. We refer to

\[ H_{i,2}(d_1) = \left[ d_1, X_{i,1}, X_{i,2}(d_1), Y_{i,1}(d_1) \right] \]

as the “potential history” under treatment status \( d_1 \) in the first period. In principle, \( H_{i,2} \) can also contains interaction terms, omitted for the sake of brevity. Namely \( H_{i,2}(d_1) \) denotes the vector of potential outcomes and covariates that would be observed in the counterfactual world where the treatment at time \( t = 1 \) equals \( d_1 \).

The second condition we impose is the sequential ignorability condition.

**Assumption 2** (Sequential Ignorability). Assume that for all \((d_1, d_2) \in \{0, 1\}^2\),

\[
\begin{align*}
(A) & \quad Y_{i,2}(d_1, d_2) \perp D_{i,2} \big| D_{i,1}, X_{i,1}, X_{i,2}, Y_{i,1} \\
(B) & \quad \left( Y_{i,2}(d_1, d_2), H_{i,2}(d_1) \right) \perp D_{i,1} \big| X_{i,1}.
\end{align*}
\]

The Sequential Ignorability (Robins et al., 2000) is common in the literature on dynamic treatments. It states that treatment in the first period is randomized on baseline covariates only, while the treatment in the second period is randomized with respect to the observable characteristics in time \( t = 2 \).

**Example 2.1 Cont’d** We can equivalently write Assumption 2 as

\[ D_{i,2} = f_2\left( D_{i,1}, X_{i,1}, X_{i,2}, Y_{i,1}, \varepsilon_{D_{i,2}} \right), \quad D_{i,1} = f_1\left( X_{i,1}, \varepsilon_{D_{i,1}} \right), \quad (3) \]

where the unobservables satisfy the conditions

\[ \varepsilon_{D_{i,2}} \perp \varepsilon_{i,2} \bigg| D_{1,i}, X_{i,1}, X_{i,2}, Y_{i,1}, \quad \varepsilon_{D_{i,1}} \perp (\varepsilon_{i,1}, \varepsilon_{i,2}) \bigg| X_{i,1}, \]

and the functions \( f_2, f_1 \) are unknown.
2.2 Potential projections

Next, we discuss the model for potential outcomes. Given baseline covariates $X_{i,1}$, for a treatment history $(d_1, d_2)$, we denote

$$
\mu_1(x_1, d_1, d_2) = \mathbb{E} \left[ Y_{i,2}(d_1, d_2) \bigg| X_{i,1} = x_1 \right],
$$

$$
\mu_2(x_1, x_2, y_1, d_1, d_2) = \mathbb{E} \left[ Y_{i,2}(d_1, d_2) \bigg| X_{i,1} = x_1, X_{i,2} = x_2, Y_{i,1} = y_1, D_{i,1} = d_1 \right],
$$

respectively the conditional expectation of the potential outcome at the end-line period, given history at time $t = 1$ (base-line) and given the history at time $t = 2$.

In the same spirit of Jordà (2005) we model $\mu_1, \mu_2$ linearly. The model we introduce takes the following form.

**Assumption 3 (Model).** We assume that for some $\beta_{d_1, d_2}^1 \in \mathbb{R}^{p_1}, \beta_{d_1, d_2}^2 \in \mathbb{R}^{p_2}$

$$
\mu_1(x_1, d_1, d_2) = x_1 \beta_{d_1, d_2}^1, \quad \mu_2(x_1, x_2, y_1, d_1, d_2) = \left[ d_1, x_1, x_2, y_1 \right] \beta_{d_1, d_2}^2.
$$

The above models can be seen as a *local projection* model on potential outcomes, with the end-line potential outcome depending linearly on information up to and from each period. An important feature of the proposed model is that we impose it directly on potential outcomes without requiring conditions on treatment assignments. The coefficients $\beta_{d_1, d_2}^1, \beta_{d_1, d_2}^2$ are different and indexed by the treatment history, capturing the effects of $(d_1, d_2)$ and heterogeneity (note that covariates also contain intercepts).

**Example 2.2 (Linear Model).** Let $X_{i,1}, X_{i,2}$ also contain an intercept. Consider the following set of conditional expectations

$$
\mathbb{E} \left[ Y_{i,1}(d_1) \bigg| X_{i,1} \right] = X_{i,1} \alpha_{d_1}, \quad \mathbb{E} \left[ X_{i,2}(d_1) \bigg| X_{i,1} \right] = W_{d_1} X_{i,1}
$$

$$
\mathbb{E} \left[ Y_{i,2}(d_1, d_1) \bigg| X_{i,1}, X_{i,2}, Y_{i,1}, D_{i,1} = d_1 \right] = \left( X_{i,1}, X_{i,2}(d_1), Y_{i,1}(d_1) \right) \beta_{d_1, d_2}^2,
$$

for some arbitrary parameters $\alpha_{d_1} \in \mathbb{R}^{p_1}$ and $\beta_{d_1, d_2}^2 \in \mathbb{R}^{p_2}$. In the above display, $W_{d_1}, V_{d_1}$ denote unknown matrices in $\mathbb{R}^{p_2 \times p_1}$. The model satisfies Assumption 3. $\blacksquare$
Example 2.2 shows that the linearity condition imposed in Assumption 3 holds exactly whenever the potential outcomes follow a linear model and dependence between covariates is explained via an autoregressive structure. All our results hold if we relax Assumption 3 to assume only approximate linearity up to an order $O(r_p)$ that decreases as we increase the number of regressors.

As noted in Example 2.2, the local projection model has an important advantage (especially in high-dimensions): while valid under linearity of covariates and outcomes, it does not require specifying (and estimate) a structural model for each time-varying-covariate, which is cumbersome in high dimensions and prone to significant estimation error. Instead, the local projection model is parsimonious in the number of parameters. This motivates its large use in applications, dating back to Jordà (2005). Here, we revisit the model within a causal framework.

We conclude this discussion with the following identification result.

**Lemma 2.1** (Identification of the potential outcome model). Let Assumption 1, 2, 3 hold. Then\(^ {9} \)

\[
\begin{align*}
E \left[ Y_{i,2}(d_1, d_2) \mid H_{i,1}, D_{i,1} = d_1 \right] &= E \left[ Y_{i,2} \mid H_{i,2}, D_{i,2} = d_2, D_{i,1} = d_1 \right] = H_{i,2}(d_1)\beta_{d_1,d_2}^2 \\
E \left[ Y_{i,2}(d_1, d_2) \mid X_{i,1} \right] &= E \left[ E \left[ Y_{i,2}(d_1, d_2) \mid H_{i,2}, D_{i,1} = d_1 \right] \mid X_{i,1}, D_{i,1} = d_1 \right] = X_{i,1}\beta_{d_1,d_2}^1.
\end{align*}
\]

The proof is in the Appendix. The above result is new in the context of local projections. The lemma states that we can identify coefficients that capture causal effects of treatment histories using information from conditional expectations. Namely, for estimation, we can first regress the observed outcome on the information in the second period. We then regress its (estimated) conditional expectation on information in the first period (see Algorithm 2). Note that the coefficients $\beta_{d_1,d_2}^1$ would not be consistently estimated by simple linear regressions of the observed outcomes on information in the first period. This is illustrated in Remark 1.

In the following section, we characterize the balancing conditions that guarantee

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\(^9\)Here, we are implicitly assuming that the event $(D_{i,2}, D_{i,2}) = (d_1, d_2)$ has non zero probability for the conditional expectation to be well defined. This holds by discreteness of the treatment assignments. An explicit condition is imposed in Assumption 4 (ii).
that the estimation error is asymptotically negligible.

**Remark 1 (Why a model on potential outcomes?).** Assuming linearity of each conditional expectation of outcomes unconditional on the previous treatment assignment leads to model incompatibility. However, models on potential outcomes are more flexible. Suppose that covariates are time invariant and let (with $Y_{i,0} = 0$)

\[
Y_{i,t} = Y_{i,t-1} \alpha + D_{i,t} \beta + X_{i,1} \gamma + \varepsilon_{i,t} \\
\Rightarrow \mathbb{E}[Y_{i,2}|X_{i,1}, D_{i,1}] = \alpha \beta D_{i,1} + \mathbb{E}[\beta D_{i,2}|X_{i,1}, D_{i,1}] + X_{i,1}(\gamma + \alpha \gamma). \tag{4}
\]

Observe that $\mathbb{E}[Y_{i,2}|X_{i,1}, D_{i,1}]$ is not a linear function of $X_{i,1}$ unconditionally on treatment assignments, since linearity of binary allocations can be violated. Also, regressing $Y_{i,2}$ onto $(D_{i,1}, X_{i,1})$ does not return consistent estimates of the causal effects, since the regression coefficients would also capture the effect of $D_{i,1}$ mediated through $D_{i,2}$ through the component $\mathbb{E}[\beta D_{i,2}|X_{i,1}, D_{i,1}]$.\(^{10}\) This issue does not arise if we impose the model directly on the potential outcomes, as we do in the proposed potential projections, and identify the model as in Lemma 2.1. Returning to the previous example, observe in fact that

\[
\mathbb{E}[Y_{i,2}(d_1, d_2)|X_{i,1}] = \alpha \beta d_1 + \beta d_2 + X_{i,1}(\gamma + \alpha \gamma),
\]

which is linear in $X_{i,1}$, and does depend on the realized assignment $D_{i,2}$, hence satisfying Assumption 3.

\(^{10}\)It is interesting to note that this difference also relates to the causal interpretability of impulse response functions (IRF). IRF (often estimated with local projections) capture the effect of a contemporaneous treatment also mediated through future assignments if treatments are serially correlated. This can be noted from Equation (4) where a local projection on $D_{i,1}$ would also capture its effect mediated through $D_{i,2}$. Here, we are concerned with the effects of a treatment history such as $(d_1 = 1, d_2 = 0)$ as opposed to the effect of a treatment $d_1 = 1$ also mediated through future assignments. This motivates our model on potential outcomes directly.
3 Dynamic Covariate Balancing

In this section, we discuss the main algorithmic procedure. We start introducing an estimator based on doubly-robust scores.

3.1 Balancing histories for causal inference

Given the local projection model, we are interested in balancing covariates to estimate treatment effects consistently. Following previous literature on doubly-robust scores (Jiang and Li, 2015; Nie et al., 2019; Tchetgen and Shpitser, 2012; Zhang et al., 2013), we propose an estimator that exploits the modeling conditions in each of the two periods while reweighing observations to guarantee balance. Here, we adapt such an estimator to the local projection model.

Formally, we consider an estimator of $\mu(d_1, d_2)$,

$$\hat{\mu}(d_1, d_2) = \frac{1}{n} \sum_{i=1}^{n} \left( \hat{\gamma}_{i,2}(d_1, d_2)Y_i - \hat{\gamma}_{i,1}(d_1, d_2)H_{i,2} \hat{\beta}_2^{d_1,d_2} - \sum_{i=1}^{n} \left( \hat{\gamma}_{i,1}(d_1, d_2) - \frac{1}{n} \right)X_{i,1} \hat{\beta}_1^{d_1,d_2} \right),$$

(5)

where we discuss the choice of the parameters $\hat{\beta}_1^{d_1,d_2}, \hat{\beta}_2^{d_1,d_2}$ in Section 3.2.

A possible choice of the weights $\hat{\gamma}_1, \hat{\gamma}_2$ are inverse probability weights. As in the case of multi-valued treatments (Imbens, 2000), these weights can be written as follows

$$w_{i,1}(d_1, d_2) = \frac{1\{D_{i,1} = d_1\}}{nP(D_{i,1} = d_1|X_{i,1})}, \quad w_{i,2}(d_1, d_2) = \frac{w_{i,1}(d_1, d_2)1\{D_{i,2} = d_2\}}{P(D_{i,2} = d_2|Y_{i,1}, X_{i,1}, X_{i,2}, D_{i,1})}. \quad (6)$$

However, in high dimensions, IPW weights require the correct specification of the propensity score, which in practice may be unknown. Motivated by these considerations, we propose replacing the inverse-probability weights with more stable weights, which are constructed by exploiting linearity in covariates.

We start studying covariate balancing conditions induced by the local projection...
model. A simple observation is that we can write

$$\hat{\mu}(d_1, d_2) = \bar{X}_1 \beta_1 d_1, d_2 + T_1 + T_2 + T_3, \quad (7)$$

where

$$T_1 = (\hat{\gamma}_1(d_1, d_2)^\top X_1 - \bar{X}_1)(\beta_1 d_1, d_2 - \hat{\beta}_1 d_1, d_2) + (\hat{\gamma}_2(d_1, d_2)^\top H_2 - \hat{\gamma}_1(d_1, d_2)^\top H_2)(\beta_2 d_1, d_2 - \hat{\beta}_2 d_1, d_2)$$

and

$$T_2 = \hat{\gamma}_2(d_1, d_2)^\top [Y_2 - H_2 \beta_2 d_1, d_2], \quad T_3 = \hat{\gamma}_1(d_1, d_2)^\top [H_2 \beta_2 d_1, d_2 - X_1 \beta_1 d_1, d_2] .$$

The covariate balancing conditions are provided by the first component $T_1$, while the remaining two are centered around zero under regularity conditions.

**Lemma 3.1** (Covariate balancing conditions). *The following holds*

$$T_1 \leq \ell \begin{array}{l}
\| \hat{\beta}_1 d_1, d_2 - \beta_1 d_1, d_2 \|_1 \| X_1 - \hat{\gamma}_1(d_1, d_2)^\top X_1 \|_\infty + \\
\| \hat{\beta}_2 d_1, d_2 - \beta_2 d_1, d_2 \|_1 \| \hat{\gamma}_2(d_1, d_2)^\top H_2 - \hat{\gamma}_1(d_1, d_2)^\top H_2 \|_\infty,
\end{array} \quad (i)$$

Element $(i)$ is equivalent to what is discussed in Athey et al. (2018) in one period setting. Element $(ii)$ depends instead on the additional error induced by the presence of a second period. Therefore the above suggests two balancing conditions only:

$$\| \bar{X}_1 - \hat{\gamma}_1 d_1, d_2^\top X_1 \|_\infty, \quad \| \hat{\gamma}_2(d_1, d_2)^\top H_2 - \hat{\gamma}_1(d_1, d_2)^\top H_2 \|_\infty . \quad (8)$$

The first balancing condition imposes that weights in the first-period balance covariates in the first period only. The second condition requires that histories in the second period are balanced, *given* the weights in the previous period.

The remaining terms of the decomposition (7), $T_2$ and $T_3$ are mean zero under the following conditions.
Lemma 3.2 (Balancing error). Let assumptions 1 - 3 hold. Suppose that \( \hat{\gamma}_1 \) is measurable with respect to \( \sigma(X_1, D_1) \) and \( \hat{\gamma}_2 \) is measurable with respect to \( \sigma(X_1, X_2, Y_1, D_1, D_2) \). Suppose in addition that \( \hat{\gamma}_{i,1}(d_1, d_2) = 0 \) if \( D_{i,1} \neq d_1 \) and \( \hat{\gamma}_{i,2}(d_1, d_2) = 0 \) if \( (D_{i,1}, D_{i,2}) \neq (d_1, d_2) \). Then

\[
E \left[ T_2 \mid X_1, D_1, Y_1, X_2, D_2 \right] = 0, \quad E \left[ T_3 \mid X_1, D_1 \right] = 0.
\]

The proof is in the Appendix. Lemma 3.2 conveys a key insight: if we can guarantee that each component in Equation (8) is \( o_p(1) \), under mild regularity assumptions, the estimator \( \hat{\mu} \) is centered around the target estimand plus an estimation error which is asymptotically negligible. As a result, the estimation error of the linear (high-dimensional) coefficients does not affect the rate of convergence of the estimator.

Interestingly, we note that Lemma 3.2 imposes the following intuitive condition. The balancing weights in the first period are non-zero only for those units whose assignment in the first period coincide with the target assignment \( d_1 \), and this also holds in the second period with assignments \( (d_1, d_2) \). Moreover, we can only balance based on information observed before the realization of potential outcomes but not based on future information. A special case is IPW in Equation (6), for known propensity score. An illustrative example is provided in Figure 2.

3.2 Algorithm description

We can now introduce Algorithm 1. The algorithm works as follows. First, we construct weights in the first period that are nonzero only for those individuals with treatment at time \( t = 1 \) equal to the target treatment status \( d_1 \). We do the same for \( \hat{\gamma}_{i,2} \) for the desired treatment history \( (D_{i,1}, D_{i,2}) = (d_1, d_2) \). We then solve a quadratic program with linear constraints. In the first period, we balance covariates as in the one-period setting. In the second period, we balance present covariates with the same covariates, weighted by those weights obtained in the previous period. The weights sum to one, they are positive (to avoid aggressive extrapolation), and they do not assign the largest weight to few observations. We choose the weights to
minimize their small sample variance to be robust to poor overlap in small samples.

Algorithm 2 summarizes the estimation of the regression coefficients. The algorithm considers two separate model specifications which can be used. The first allows for all possible interactions of covariates and treatment assignments as in Assumption 3. The second is more parsimonious and assumes that treatment effects enter linearly in each equation, while it uses all the observations in the sample. The second specification can also contain linear interaction components, omitted for brevity. Note that the algorithm for the linear (second) specification builds predictions in the second period only for those units with $D_{i,1} = d_1$, and for all units in the first period. This is without loss of generality, since the remaining units receive a zero weight.

3.3 Existence, convergence rate and asymptotic inference

We conclude this introductory discussion by developing properties of the estimator. We first impose the following tail decay conditions.

**Assumption 4.** Let the following hold:
Algorithm 1 Dynamic covariate balancing (DCB): two periods

Require: Observations \((D_1, X_1, Y_1, D_2, X_2, Y_2)\), treatment history \((d_1, d_2)\), finite parameters \(K\), constraints \(\delta_1(n, p), \delta_2(n, p)\).

1: Estimate \(\beta_{d_1,2}^1, \beta_{d_1,2}^2\) as in Algorithm 2.
2: \(\hat{\gamma}_{i,1} = 0\), if \(D_i,1 \neq d_1, \hat{\gamma}_{i,2} = 0\) if \((D_{i,1}, D_{i,2}) \neq (d_1, d_2)\)
3: Estimate

\[
\hat{\gamma}_1 = \arg \min_{\gamma_1} \|\gamma_1\|^2, \quad \text{s.t. } \left\| X_1 - \frac{1}{n} \sum_{i=1}^{n} \gamma_{i,1} X_{i,1} \right\|_\infty \leq \delta_1(n, p),
\]

\[
1^\top \gamma_1 = 1, \gamma_1 \geq 0, \|\gamma_1\|_\infty \leq \log(n)^{-2/3}.
\] (9)

\[
\hat{\gamma}_2 = \arg \min_{\gamma_2} \|\gamma_2\|^2, \quad \text{s.t. } \left\| \frac{1}{n} \sum_{i=1}^{n} \hat{\gamma}_{i,1} H_{i,2} - \frac{1}{n} \sum_{i=1}^{n} \gamma_{i,2} H_{i,2} \right\|_\infty \leq \delta_2(n, p),
\]

\[
1^\top \gamma_2 = 1, \gamma_2 \geq 0, \|\gamma_2\|_\infty \leq K \log(n)^{-2/3}.
\]

\text{return } \hat{\mu}(d_1, d_2) \text{ as in Equation (5)}.

(i) \(H_{i,2}^{(j)}\) is subgaussian given the past history for each \(j\) and \(\|X_{i,1}\|_\infty \leq M < \infty\).

(ii) Assume that (i) \(P(D_{i,1} = 1|X_{i,1}), P(D_{i,2} = 1|D_{i,1}, X_1, X_2, Y_1) \in (\delta, 1 - \delta), \delta \in (0, 1)\).

The first condition states that histories are Sub-Gaussian and covariates are uniformly bounded. The second condition imposes overlap of the propensity score.

Theorem 3.3 (Existence of a feasible \(\hat{\gamma}_t\)). Let Assumptions 1 - 4 hold. Suppose that \(\delta_t(n, p) \geq c_0 \log^{3/2}(np)/n^{1/2}\), for a finite constant \(c_0\). Then, with probability \(\eta_n \to 1\), for each \(t \in \{1, 2\}\), for some \(N > 0, n > N\), there exists a feasible \(\hat{\gamma}_t^*\), solving the optimization in Algorithm 1, where

\[
\hat{\gamma}_{t,0} = 1/n, \quad \hat{\gamma}_{i,t}^* = \hat{\gamma}_{i,t-1}^* \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t|F_{t-1})} / \sum_{i=1}^{n} \hat{\gamma}_{i,t-1}^* \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t|F_{t-1})},
\]

and \(F_0 = \sigma(X_1), F_1 = \sigma(X_1, X_2, Y_1, D_1)\).

Theorem 3.3 has important practical implications. Inverse probability weights tend to be unstable in a small sample for moderately large periods. The algorithm
the parameter $\delta_{i,1}$ in Example 3.1.

Dimensional regression, where $\|o\|_{\gamma}$ intuition in Theorem 3.3, for some Corollary 1.

thus finds weights that minimize the small sample variance, with the IPW weights being allowed to be one of the possible solutions. We formalize this in the following corollary.

**Corollary 1.** Under the conditions in Theorem 3.3, for some $N > 0$, $n > N$, with probability $\eta_n \to 1$, $n||\hat{\gamma}_t||^2 \leq n||\hat{\gamma}_t^*||^2$

We now discuss asymptotic inference.

**Assumption 5 (Convergence rate).** Let $\delta_t(n, p)$ is such that $\delta_t(n, p) \geq c_0 \log(np)/n^{1/4}$ for a finite constant $c_0$, $\|\hat{\beta}_{d_1,2}^t - \beta_{d_1,2}^t\|_1 \delta_t(n, p) = o_p(1/\sqrt{n})$, $t \in \{1, 2\}$, $\|\hat{\beta}_{d_1,2}^t - \beta_{d_1,2}^t\|_1 = o_p(n^{-1/4})$.

The above condition states that the estimation error of the linear regressor times the parameter $\delta_t(n, p) = o(1)$ is of order $o(1/\sqrt{n})$. A simple example is an high-dimensional regression, where $\|\hat{\beta}_{d_1,2}^t - \beta_{d_1,2}^t\|_1 = O_p(\sqrt{\log(p)/n})$. We formalize this intuition in Example 3.1.

First, we define, under Assumption 3,

$$Y_{i,2}(d_1, d_2) = H_{i,2}(d_1)\beta_{d_1,2}^2 + \varepsilon_{i,2}(d_1, d_2), \quad H_{i,2}(d_1)\beta_{d_1,2}^2 = X_{i,1}(d_1)\beta_{d_1,2}^1 + \nu_{i,1}(d_1),$$

where $\nu_{i,1}(d_1) = \mathbb{E}\left[Y_{i,2}(d_1, d_2)|H_{i,2}(d_1)\right] - \mathbb{E}\left[Y_{i,2}(d_1, d_2)|X_{i,1}\right]$ denotes the difference between the two local projections over two consecutive periods.
Example 3.1 (Sufficient conditions for Lasso). Suppose that $H_2, X_1$ are uniformly bounded and $\|\beta_{d_1:2}\|_0, \|\beta_{d_1:2}\|_0 \leq s, \|\beta_{d_1:1}\|_\infty, \|\beta_{d_1:2}\|_\infty < \infty$. Suppose that $H_2, X_1$ both satisfy the restricted eigenvalue assumption, and the column normalization condition (Negahban et al., 2012).\(^\text{11}\) Suppose that $\hat{\beta}_{d_1:2}^1, \hat{\beta}_{d_1:2}^2$ are estimated with Lasso as in Algorithm 2 with a full interaction model and with penalty parameter $\lambda_n \approx s \sqrt{\log(p)/n}$. Let Assumptions 1 - 4 hold. Let $\varepsilon_2(d_{1:2})|H_2$ be subgaussian almost surely and $\nu_1(d_1)|X_1$ be sub-gaussian almost surely. Then for each $t \in \{1, 2\}$,

$$\left\|\hat{\beta}_{d_1:2}^t - \beta_{d_1:2}^t\right\|_1 = O_p\left(s^2 \sqrt{\log(p)/n}\right).$$

Therefore,

$$\|\hat{\beta}_{d_1:2}^t - \beta_{d_1:2}^t\|_1 \delta_t(n, p) = o_p(1/\sqrt{n}),$$

for $\delta_t(n, p) \approx \log(np)/n^{1/4}$ and $s^2 \log^{3/2}(np)/n^{1/4} = o(1)$. The proof is contained in the Appendix and follows similarly to Negahban et al. (2012), with minor modifications. The above result provides a set of sufficient conditions such that Assumption 5 holds for a feasible choice of $\delta_t$.\(^\text{12}\)

Assumption 6. Let the following hold:

(A) $\mathbb{E}[\varepsilon_2^4(d_1, d_2)|H_2], \mathbb{E}[\nu_1^4(d_1)|X_1] < C$ for a finite constant $C$ almost surely;

(B) $\text{Var}(\varepsilon_2(d_1, d_2)|H_{i:2}), \text{Var}(\nu_1(d_1, d_2)|X_{i:1}) > u_{\text{min}} > 0$.

The above condition states that the residuals from projections in two consecutive time periods have non-zero variance and a bounded fourth moment. We can now present the following theorem.

Theorem 3.4 (Asymptotic Inference). Let Assumptions 1 - 6 hold. Then, whenever $\log(np)/n^{1/4} \to 0$ with $n, p \to \infty$,

$$P\left(|\hat{V}_2(d_1, d_2)^{-1/2} \sqrt{n} \left(\hat{\mu}(d_1, d_2) - \mu(d_1, d_2)\right)\right| > \sqrt{\chi^2(\alpha)}) \leq \alpha,$$

\(^\text{11}\)Formally $\|X_j\|_2/\sqrt{n} \leq 1$. Sufficient conditions that guarantee that the restricted eigenvalue assumption holds are discussed in (Negahban et al., 2012).

\(^\text{12}\)Note that the example holds in the (more general) case of a fully interacted model, while the model linear in treatment assignments can be considered as a special case if correctly specified.
where

$$\hat{V}_2(d_{1:2}) = n \sum_{i=1}^{n} \hat{\gamma}_{i,2}^2(d_{1:2})(Y_i - H_{i,2}\hat{\beta}_{d_{1:2}})^2 + n \sum_{i=1}^{n} \hat{\gamma}_{i,1}^2(d_{1:2})(H_{i,2}\hat{\beta}_{d_{1:2}} - X_{i,1}\hat{\beta}_{d_{1:2}})^2$$

and $\chi^2(\alpha)$ is $1-\alpha$-quantile of a chi-squared random variable with 2 degrees of freedom.

Theorem 3.4 provides an explicit expression for constructing confidence intervals for the estimator $\hat{\mu}(d_{1:2})$ around the expectation of the potential outcome of interest. The $1-\alpha$ confidence band takes the following form

$$\text{CI}(d_1, d_2; \alpha) = \left[ \hat{\mu}(d_1, d_2) - \sqrt{\chi^2(\alpha)}\frac{\hat{V}_2(d_1, d_2)}{\sqrt{n}}, \hat{\mu}(d_1, d_2) + \sqrt{\chi^2(\alpha)}\frac{\hat{V}_2(d_1, d_2)}{\sqrt{n}} \right].$$

The confidence band also depends on the estimated variance $\hat{V}_2(d_1, d_2)$ and the critical quantile corresponding to the square-root of a chi-squared random variable.\(^{13}\)

Observe that, unlike Athey et al. (2018), here we need to take into account the joint distribution of observables and unobservables, which also depend on the random balancing weights.\(^{14}\) Tighter confidence bands can be obtained under stronger assumptions (see Remark 4).

We also study the convergence rate of the estimator. In the following theorem, we show that $\hat{V}(d_1, d_2) = O_p(1)$ and hence the estimator admits the parametric convergence rate even when $p \to \infty$.

**Theorem 3.5.** Let the conditions in Theorem 3.4 hold. Then as $n, p \to \infty$,

$$\hat{\mu}(d_{1:T}) - \mu(d_{1:T}) = O_P\left(n^{-1/2}\right).$$

Theorem 3.5 showcases that the proposed estimator guarantees parametric con-

---

\(^{13}\)For example, for a 95% confidence span, the critical quantile equals 2.45.

\(^{14}\)The reason why we do not use the critical quantile of a standard Gaussian random variable is due to the possible lack of almost sure convergence of $n||\hat{\gamma}_2||^2$ since weights characterize a triangular array of arbitrary dependent random variables. In simulations, we note that the Gaussian quantile can perform well under strong sparsity and strong overlap, but its corresponding coverage deteriorates as overlap decreases. Instead, the chi-squared critical quantile presents valid coverage throughout all the design considered.
vergence rate even in the presence of high-dimensional covariates. It implies that the estimation error due to the high-dimensionality $p$ is asymptotically negligible. Observe that the theorem does not require any restriction on the propensity score, such as the $n^{-1/4}$ rate of convergence rate commonly encountered in the doubly-robust literature (Farrell, 2015). Inference on the ATE can be implemented as in Theorem 4.5.

The proofs are contained in the Appendix.

Remark 2 (Pooled regression and limited carry-overs). In some application, we may be interested in a regression of the following form

$$Y_{i,t}(d_{1:t}) = \beta_0 + \beta_1 d_t + \beta_2 Y_{i,t-1}(d_{1:(t-1)}) + X_{i,t}(d_{1:(t-1)})\gamma + \tau_t + \varepsilon_{i,t},$$

where $\tau_t$ denotes fixed effects, and in the estimand

$$\mathbb{E}[Y_{i,t+h}(d_{1:t}, d_{t+1}, \cdots, d_{t+h})] - \mathbb{E}[Y_{i,t+h}(d_{1:t}, d'_{t+1}, \cdots, d'_{t+h})],$$

denoting the effect of changing treatment history in the past $h$ periods. In such a case, the estimation can be performed by considering each $(i, t)$ as an observation for all $t > h$ and estimate its corresponding weight.\textsuperscript{15} We obtain the corresponding variances after clustering residuals of the same individuals over different periods. \hfill \Box

4 The general case: multiple time periods

In this section we generalize our procedure to $T$ time periods. We define the estimand of interest as:

$$\text{ATE}(d_{1:T}, d'_{1:T}) = \mu_T(d_{1:T}) - \mu_T(d'_{1:T}), \quad \mu_T(d_{1:T}) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[Y_T(d_{1:T}) \big| X_{i,1}\right]. \quad (11)$$

This estimand denotes the difference in potential outcomes conditional on baseline covariates. We define $\mathcal{F}_t = \left(D_1, \cdots, D_{t-1}, X_1, \cdots, X_t, Y_1, \cdots, Y_{t-1}\right)$ the information

\textsuperscript{15}Note that here $\tau_t$ acts as an additional covariate for balancing.
at time $t$ after excluding the treatment assignment $D_t$. We denote

$$H_{i,t} = \left[ D_{i,1}, \ldots, D_{i,t-1}, X_{i,1}, \ldots, X_{i,t}, Y_{i,1}, \ldots, Y_{i,t-1} \right] \in \mathcal{H}_t$$

the vector containing information from time one to time $t$, after excluding the treatment assigned in the present period $D_t$. Interaction components may also be considered in the above vector, and they are omitted for expositional convenience only.

We let the potential history be

$$H_{i,t}(d_{1:(t-1)}) = \left[ d_{1:(t-1)}, X_{i,1:t}(d_{1:(t-1)}), Y_{i,1:(t-1)}(d_{1:(t-1)}) \right]$$

as a function of the treatment history. The following Assumption generalizes Assumptions 1-3 from the two-period setting: no-anticipation, sequential ignorability, and potential outcome models.

**Assumption 7.** For any $d_{1:T} \in \{0, 1\}^T$, and $t \leq T$,

(A) (No-anticipation) The potential history $H_{i,t}(d_{1:T})$ is constant in $d_{i:T}$;

(B) (Sequential ignorability) $\left( Y_{i,T}(d_{1:T}), H_{i,t+1}(d_{1:(t+1)}), \ldots, H_{i,T-1}(d_{1:(T-1)}) \right) \perp D_{i,t} | \mathcal{F}_i$;

(C) (Potential projections) For some $\beta_t^d_{d_{1:T}} \in \mathbb{R}^{pt}$,

$$\mathbb{E}\left[ Y_{i,T}(d_{1:T}) | D_{i,1:(t-1)} = d_{1:(t-1)}, X_{i,1:t}, Y_{i,1:(t-1)} \right] = H_{i,t}(d_{1:(t-1)}) \beta_t^d_{d_{1:T}}.$$
We construct the estimator as an analogue to the two-period setting. Once DCB weights are formed, we construct the estimator of $\mu_T(d_{1:T})$ as

$$\hat{\mu}_T(d_{1:T}) = \frac{1}{n} \sum_{i=1}^{n} \hat{\gamma}_{i,T}(d_{1:T}) Y_{i,T} - \frac{1}{n} \sum_{i=1}^{n} \sum_{t=2}^{T} \left( \hat{\gamma}_{i,t}(d_{1:T}) - \hat{\gamma}_{i,t-1}(d_{1:T}) \right) H_{i,t} \hat{\beta}_{d_{1:T}}^t$$

$$- \frac{1}{n} \sum_{i=1}^{n} \left( \hat{\gamma}_{i,1}(d_{1:T}) - \frac{1}{n} \right) X_{i,1} \hat{\beta}_{d_{1:T}}.$$ (13)

In order to introduce balancing weights for general $T$-periods, it is useful to characterize the estimation error.

**Lemma 4.1.** Suppose that $\hat{\gamma}_{i,T}(d_{1:T}) = 0$ if $D_{i,1:T} \neq d_{1:T}$. Then

$$\hat{\mu}_T(d_{1:T}) - \mu_T(d_{1:T}) = \sum_{t=1}^{T} \left( \hat{\gamma}_t(d_{1:T}) H_t - \hat{\gamma}_{t-1}(d_{1:T}) H_t \right) (\hat{\beta}_{d_{1:T}}^t - \hat{\beta}_{d_{1:T}}^{t-1}) + \gamma_T^T(d_{1:T}) \varepsilon_T$$

$$+ \sum_{t=2}^{T} \hat{\gamma}_{t-1}(d_{1:T}) \left( H_t \beta_{d_{1:T}}^t - H_{t-1} \beta_{d_{1:T}}^{t-1} \right)$$

$$+ \left( I_1 \right) + \left( I_2 \right) + \left( I_3 \right)$$

where $\varepsilon_{i,t}(d_{1:T}) = Y_{i,T}(d_{1:T}) - H_{i,t}(d_{1:(t-1)}) \hat{\beta}_{d_{1:T}}^T$.

The proof is relegated to the Appendix. Lemma 4.1 decomposes the estimation error into three main components. The first component, $\left( I_1 \right)$, depends on the estimation error of the coefficient and on balancing properties of the weights. To guarantee consistent estimation in high dimensional settings, $\left( I_1 \right)$ suggests imposing conditions on

$$\left\| \hat{\gamma}_t(d_{1:T}) H_t - \hat{\gamma}_{t-1}(d_{1:T}) H_t \right\|_\infty$$

at each point in time.

The second component characterizing the estimation error is $\left( I_2 \right) = \gamma_T^T(d_{1:T})^T \varepsilon_T$. Such an element is centered around zero, conditional on $\mathcal{F}_T$, whenever we do not use the outcome at the end-line period for estimation of the balancing weights. Finally,
the last component, \((I_3)\), characterizes the asymptotic variance. In the following lemma, we provide conditions that guarantee that \((I_3)\) is centered around zero, as in the two-period setting.

**Lemma 4.2.** Let Assumption 7 hold. Suppose that the sigma algebra \(\sigma(\hat{\gamma}_t(d_{1:T})) \subseteq \sigma(F_t, D_t)\). Suppose in addition that \(\hat{\gamma}_{i,t}(d_{1:T}) = 0\) if \(D_{i,1:t} \neq d_{1:t}\). Then

\[
\mathbb{E}\left[\hat{\gamma}_{i,t-1}(d_{1:T})H_t\beta_{d_{1:T}} - \hat{\gamma}_{i,t-1}(d_{1:T})H_{t-1}\beta_{d_{1:T}}^{-1}\bigg| F_{t-1}, D_{t-1}\right] = 0.
\]

The above condition states that weights need to be estimated using observations that match the desired treatment path up at every \(t\), and are equal to zero on the other treatment paths. The proof is presented in the Appendix.

Consequently, DCB weights can be easily constructed. Algorithm 3 contains all the details. Choice of the tuning parameters can be adaptive and details are included in Appendix D for the sake of brevity.

**Algorithm 3** Dynamic covariate balancing (DCB): multiple time periods

**Require:** Observations \(\{Y_{i,1}, X_{i,1}, D_{i,1}, \cdots, Y_{i,T}, X_{i,T}, D_{i,T}\}\), treatment history \((d_{1:T})\), finite parameters \(\{K_{1,t}, K_{2,1}, K_{2,2}, \cdots, K_{2,T}\}\), constraints \(\delta_1(n,p), \delta_2(n,p), \cdots, \delta_T(n,p)\).

1: Estimate \(\beta_{d_{1:T}}^t\) as in Algorithm D.1 in Appendix D.
2: Let \(\hat{\gamma}_{i,0} = 1/n\) and \(t = 0\);
3: for each \(t \leq T - 1\) do
4: \(\hat{\gamma}_{i,t} = 0\), if \(D_{i,1:t} \neq d_{1:t}\)
5: Estimate time \(t\) weights with

\[
\hat{\gamma}_t = \arg\min_{\gamma_t} \sum_{i=1}^{n} \gamma_{i,t}^2, \quad \text{s.t.} \quad \left\| \frac{1}{n} \sum_{i=1}^{n} \hat{\gamma}_{i,t-1}H_{i,t} - \hat{\gamma}_{i,t}H_{i,t} \right\|_{\infty} \leq K_{1,\delta}(n,p),
\]

\[
1^\top \gamma_t = 1, \gamma_t \geq 0, \|\gamma_t\|_{\infty} \leq K_{2,\delta} \log(n)n^{-2/3}.
\]

6: end for \(\triangleright\) obtain \(T\) balancing vectors

**return** Estimate of the average potential outcome as in Equation (13)

Coefficients are estimated recursively as discussed in the two periods setting (see Algorithm D.1 in the Appendix). Namely, we project the estimated outcome from
each period over the previous filtration, sequentially. We impose high-level assumptions on the coefficients, which are commonly satisfied in both high and low dimensional settings similarly to what discussed in the two-periods setting.\textsuperscript{16}

Remark 3 (Estimation error of the coefficients with many periods). The estimation error \( \| \hat{\beta}_{d_1:T}^t - \beta_{d_1:T}^t \|_1 \) can scale either linearly or exponentially with \( T \), depending on modeling assumptions. Whenever we let coefficients be different across entire different treatment histories, \( \| \hat{\beta}_{d_1:T}^t - \beta_{d_1:T}^t \|_1 \) would scale exponentially with \( T \), since we would need to run different regressions over the subsample with treatment histories \( D_{1:t} = d_{1:t} \) as in Algorithm D.1. On the other hand, additional assumptions permit to estimate \( \hat{\beta}_{d_1:T}^t \) using most or all in-sample information. A simple example, is to explicitly model the effect of the treatment history \( d_{1:T} \) on the outcome (see e.g., Remark 2).

\[ \square \]

4.1 Asymptotic properties

We provide stability as well as asymptotic normality of the proposed estimator as long as \( \log(pn)/n^{1/4} \to 0 \) while \( n, p \to \infty \). Here, \( p = \sum_t p_t \) and \( p_t \) denotes dimensionality of models as in Assumption 3. We consider a finite-time horizon and \( T < \infty \) regime.

We discuss the first regularity condition below, which mimics the analogous conditions from two periods.

Assumption 8 (Overlap and tails’ conditions). Assume that \( P(D_{i,t} = d_t | F_{t-1}, D_{t-1}) \in (\delta, 1 - \delta), \delta \in (0, 1) \) for each \( t \in \{1, \cdots, T\} \). Assume also that \( H_{i,t} \) is Sub-Gaussian given past history and \( \|X_{i,1}\|_\infty \leq M < \infty \).

The first condition is the overlap condition as in the case of two periods. The second condition is a tail restriction. In the following theorem, we characterize the existence of a solution to the optimization program.

Theorem 4.3. Let Assumptions 7, 8 hold. Consider \( \delta_t(n, p) \geq c_0 n^{-1/2} \log^{3/2}(pn) \) for a finite constant \( c_0 \), and \( K_{2,t} = 2K_{2,t-1}b_t \) for some constant \( b_t < \infty \). Then, with

\[ \text{See Appendix B.1.2.} \]
probability $\eta_n \to 1$, for each $t \in \{1, \cdots, T\}, T < \infty$, for some $N > 0$, $n > N$, there exists a feasible $\hat{\gamma}_t^*$, solving the optimization in Algorithm 3, where

$$\hat{\gamma}_{i,0}^* = 1/n, \quad \hat{\gamma}_{i,t}^* = \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t|\mathcal{F}_{t-1}, D_{t-1})} \sum_{i=1}^n \hat{\gamma}_{i,t-1}^* \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t|\mathcal{F}_{t-1}, D_{t-1})}.$$  

The above theorem shows existence of a feasible solution which encompasses stabilized inverse probability weights. Next, we characterize asymptotic properties of the estimator.

**Assumption 9.** Let the following hold: for every $t \in \{1, \cdots, T\}$, $d_{1:T} \in \{0, 1\}^T$,

(i) $\max_t ||\hat{\beta}_{d_{1:T}}^t - \beta_{d_{1:T}}^t||_1 \delta_t(n, p) = o_p(1/\sqrt{n})$, $\delta_t(n, p) \geq c_0, t n^{-1/4} \log(2pn)$ for a finite constant $c_0, t$,

(ii) $E[\varepsilon_{i,t}^4|H_t] < C$ almost surely for a finite constant $C$, with $\varepsilon_{i,t} = Y_{i,t} - H_{i,t}\beta_{d_{1:T}}^T$; suppose in addition that $E[(H_{i,t}\beta_{d_{1:T}}^t - H_{i,t-1}\beta_{d_{1:T}}^{t-1})^4|H_{i,t-1}] < C$ for a finite constant $C$ almost surely;

(iii) $\text{Var}(\varepsilon_{i,t}|H_{i,t}), \text{Var}(H_{i,t}\beta_{d_{1:T}}^t - H_{i,t-1}\beta_{d_{1:T}}^{t-1}|H_{i,t-1}) > u_{min} > 0$, for some constant $u_{min}$.

Assumption 9 imposes the consistency in estimation of the outcome models. Condition (i) is attained for many high-dimensional estimators, such as the lasso method, under regularity assumptions; see e.g., Bühlmann and Van De Geer (2011). A discussion is included in Lemma 3.1 which is valid recursively for any finite $T$ (see Appendix B.1.2). The remaining conditions impose moment assumptions similarly to the two periods setting.

**Theorem 4.4 (Asymptotic Inference).** Let Assumptions 7 - 9 hold. Then, whenever $\log(np)/n^{1/4} \to 0$ with $n, p \to \infty$,

$$P\left(\left|\frac{\sqrt{n}(\hat{\mu}(d_{1:T}) - \mu_T(d_{1:T}))}{\hat{V}_T(d_{1:T})^{1/2}}\right| > \sqrt{\chi_T(\alpha)}\right) \leq \alpha, \quad \hat{\mu}_T(d_{1:T}) - \mu_T(d_{1:T}) = O_p(n^{-1/2}),$$  

(16)
where
\[
\hat{V}_T(d_{1:T}) = n \sum_{i=1}^{n} \hat{\gamma}^2_{i,T}(d_{1:T})(Y_i - H_i \beta_{d_{1:T}})^2 + \sum_{t=1}^{T-1} n \sum_{i=1}^{n} \hat{\gamma}^2_{i,t}(d_{1:t})(H_{i,t+1} \beta_{d_{1:T}} - H_{i,t} \beta_{d_{1:T}})^2
\]
and \(\chi_T(\alpha)\) is \((1 - \alpha)\)-quantile of a chi-squared random variable with \(T\) degrees of freedom.

The proofs of the above two theorems are contained in the Appendix. The theorem shows that the estimator converges to a Gaussian distribution at the optimal rate of \(\sqrt{n}\), even when the number of variables greatly exceeds the sample size. Interestingly, the confidence interval increases with \(T\) due to higher variance and larger critical quantile.

**Theorem 4.5** (Inference on ATE). Let the conditions in Theorem 4.4 hold. Let \(d_1 \neq d'_1\). Then, whenever \(\log(np)/n^{1/4} \to 0\) with \(n, p \to \infty\),
\[
P\left( \mid (\hat{V}_T(d_{1:T}) + \hat{V}_T(d'_{1:T}))^{1/2} \sqrt{n} \left( \hat{\mu}(d_{1:T}) - \hat{\mu}(d'_{1:T}) - \text{ATE}(d_{1:T}, d'_{1:T}) \right) \right) > \sqrt{\chi^2_T(\alpha)} \leq \alpha.
\]

The proof is in the Appendix. The above theorem permits inference on the ATE.

**Remark 4** (Tighter confidence bands under more restrictive conditions). Appendix C.2 shows that under more restrictive assumptions, we can show that
\[
(\hat{V}_T(d_{1:T}) + \hat{V}_T(d'_{1:T}))^{1/2} \sqrt{n} \left( \hat{\mu}(d_{1:T}) - \hat{\mu}(d'_{1:T}) - \text{ATE}(d_{1:T}, d'_{1:T}) \right) \to_d \mathcal{N}(0, 1)
\]
and hence, tighter confidence bands can be constructed. We note, however, that the assumptions require that \(n|\hat{\gamma}_t|^2/2\) converge almost surely to a finite constant. This condition implicitly imposes restrictions on the degree of dependence of such weights.
5 Numerical Experiments

This section collects results from numerical experiments. We estimate in two and three periods

\[ E[Y_{i,T}(1) - Y_{i,T}(0)], \quad T \in \{2, 3\}. \]

We let the baseline covariates \( X_{i,1} \) be drawn from as i.i.d. \( \mathcal{N}(0, \Sigma) \) with \( \Sigma^{(i,j)} = 0.5^{|i-j|} \). Covariates in the subsequent period are generated according to an autoregressive model

\[ \{X_{i,t}\}_j = 0.5\{X_{i,t-1}\}_j + \mathcal{N}(0, 1), \quad j = 1, \cdots, p_t. \]

Treatments are drawn from a logistic model that depends on all previous treatments as well as previous covariates. Namely, \( D_{i,t} \sim \text{Bern}(1 + e^{\theta_{i,t}})^{-1} \) with

\[ \theta_{i,t} = \eta \sum_{s=1}^{t} X_{i,s} \phi + \sum_{s=1}^{t-1} \delta_s (D_{i,s} - \bar{D}_s) + \xi_{i,t}, \quad \bar{D}_s = n^{-1} \sum_{i=1}^{n} D_{i,s} \quad (17) \]

and \( \xi_{i,t} \sim \mathcal{N}(0, 1) \), for \( t \in \{1, 2, 3\} \). Here, \( \eta, \delta \) controls the association between covariates and treatment assignments. We consider values of \( \eta \in \{0.1, 0.3, 0.5\} \), \( \delta_1 = 0.5, \delta_2 = 0.25 \). We let \( \phi \propto 1/j \), with \( \|\phi\|_2^2 = 1 \), similarly to what discussed in Athey et al. (2018). Table 1 illustrates the behavior of the propensity score as a function of \( \eta \). The larger the value of \( \eta \), the weaker the overlap.

We generate the outcome according to the following equations:

\[ Y_{i,t}(d_{1:t}) = \sum_{s=1}^{t} \left( X_{i,s} \beta + \lambda_{s,t} Y_{i,s-1} + \tau d_s \right) + \varepsilon_{i,t}(d_{1:t}), \quad t = 1, 2, 3, \]

where elements of \( \varepsilon_{i,t}(d_{1:t}) \) are i.i.d. \( \mathcal{N}(0, 1) \) and \( \lambda_{1,2} = 1, \lambda_{1,3}, \lambda_{2,3} = 0.5 \). We consider three different settings: Sparse with \( \beta(j) \propto 1\{j \leq 10\} \), Moderate with moderately sparse \( \beta(j) \propto 1/j^2 \) and the Harmonic setting with \( \beta(j) \propto 1/j \). We ensure \( \|\beta\|_2 = 1 \). Throughout our simulations we set \( \tau = 1 \). In Appendix E we collect results in the presence of non-linear (misspecified) outcome models.
Table 1: Summary statistics of the distribution of the propensity score in two and three periods in a sparse setting with dim($X$) = 300.

|       | $\eta = 0.1$ |       |       | $\eta = 0.3$ |       |       | $\eta = 0.5$ |       |
|-------|--------------|-------|-------|--------------|-------|-------|--------------|-------|
|       | T=2          | T=3   | T=2   | T=3          | T=2   | T=3   | T=2          | T=3   |
| Min   | 0.012        | 0.003 | 0.004 | 0.0002       | 0.001 | 0.00000 | 0.001        | 0.00000 |
| 1st Quantile | 0.126 | 0.049 | 0.105 | 0.031       | 0.079 | 0.018 | 0.079        | 0.018 |
| Median | 0.218        | 0.097 | 0.216 | 0.097       | 0.216 | 0.094 | 0.216        | 0.094 |
| 3rd Quantile | 0.248 | 0.126 | 0.259 | 0.153      | 0.277 | 0.183 | 0.277        | 0.183 |
| Max   | 0.352        | 0.175 | 0.377 | 0.226       | 0.429 | 0.286 | 0.429        | 0.286 |

5.1 Methods

While we note that dynamic treatments are often of research interest, there is little discussion on methods that permit estimation with dynamic treatments and high-dimensional covariates. We consider the following competing methodologies. **Augmented IPW**, with *known* propensity score and with *estimated* propensity score. The method replaces the balancing weights in Equation (5) with the (estimated or known) propensity score. Estimation of the propensity score is performed using a logistic regression (denoted as aIPWl) and a penalized logistic regression (denoted as aIPWh).\footnote{See for example Nie et al. (2019) and the recent work of Bodory et al. (2020) for a discussion on doubly-robust estimators.} For both AIPW and IPW we consider stabilized inverse probability weights. We also compare to existing balancing procedures for dynamic treatments. Namely, we consider Marginal Structural Model (MSM) with balancing weights computed using the method in Yiu and Su (2018, 2020). The method consists of estimating **Covariate-Association Balancing weights CAEW (MSM)** as in Yiu and Su (2018, 2020), which consists in balancing covariates reweighted by marginal probabilities of treatments (estimated with a logistic regression), and use such weights to estimate marginal structural model of the outcome linear in past treatment assign-
ments. We follow Section 3 in Yiu and Su (2020) for its implementation.\footnote{Estimation consists in projecting the outcome on the two or three past assignments, use the CAEW for reweighting. The reader can also refer to Blackwell (2013) for references on marginal structural models.} We also consider “Dynamic” Double Lasso that estimates the effect of each treatment assignment separately, after conditioning on the present covariate and past history for each period using the double lasso discussed in one period setting in Belloni et al. (2014). The overall treatment effect is then estimated by summing over each effect.\footnote{The method, was not discussed in previous literature for dynamic treatments and is an adaptation of the Double Lasso of Belloni et al. (2014) in the one-period setting. We follow the following approach. First, we run a regression of $Y_3$ after conditioning on $X_1$ and $D_1$ only. We obtain an estimate of the treatment effect for $D_1$ on $Y_3$ from this expression. Such an effect is estimated via the Double Lasso. We then repeat the procedure by regressing $Y_3$ onto $(X_1, X_2, D_1, D_2)$ and obtain the effect of the treatment in the second period on the end-line outcome using the Double Lasso. Finally, we repeat with the third period. We obtain the final effect by summing over these three effects.} Naive Lasso runs a regression controlling for covariates and treatment assignments only. Finally, Sequential Estimation estimates the conditional mean in each time period sequentially using the lasso method, and it predicts end-line potential outcomes as a function of the estimated potential outcomes in previous periods.\footnote{A related procedure can be found in Zhou et al. (2019).} For Dynamic Covariate Balancing, DCB choice of tuning parameters is data adaptive, and it uses a grid-search method discussed in Appendix D.\footnote{The grid-search procedure consists of finding the smallest feasible constraint-value through grid search, while choosing more stringent constants for those variables whose estimated coefficients are non-zero.} We estimate coefficients as in Algorithm 2 for DCB and (a)IPW, with a linear model in treatment assignments. Estimation of the penalty for the lasso methods is performed via cross-validation.

\subsection*{5.2 Results}

We consider $\dim(\beta) = \dim(\phi) = 100$ and set the sample size to be $n = 400$. Under such design, the regression in the first period contains $p_1 = 101$ covariates, in the second period $p_2 = 203$ covariates, and in the third $p_3 = 305$ covariates.

In Table 2 we collect results for the average mean squared error in two and three
periods. Throughout all simulations, the proposed method significantly outperforms any other competitor for $T = 3$ across all designs, with one single exception for $T = 2$, good overlap and harmonic design. It also outperforms the case of known propensity score, consistently with our findings in Theorem 3.3. Improvements are particularly significant when (i) overlap deteriorates; (ii) the number of periods increases from two to three. This can also be observed in the panel at the bottom of Figure 3, where we report the decrease in MSE (in logarithmic scale) when using our procedure for $T = 3$. In Appendix E we collect additional results with misspecified models.

In the top panel of Figure 3 we report the length of the confidence interval and the point estimates for the harmonic and moderate design for estimating the ATE. The length increases with number of periods. Point estimates are more accurate for a larger degree of sparsity due to the consistency of the penalized regression procedure.

Figure 3: Top panels collect the point estimate (crosses), minus the true effect of the treatment, and confidence intervals of DCB for $p = 100$ across the three different designs. The bottom panel reports the decrease in MSE (in logarithmic scale) of the proposed method compared to the best competitor (excluding the one with known propensity score) for $T = 3$.

Finally, we report finite sample coverage of the proposed method, DCB in Table 3 for estimating $\mu(1, 1)$ and $\mu(1, 1) - \mu(0, 0)$ in the first two panel with $\eta = 0.5$.\footnote{Results for $\eta \in \{0.1, 0.3\}$ present over-coverage of the chi-squared method, and correct or under-coverage (albeit less severe than $\eta = 0.5$) when considering a Gaussian critical quantile.}
Table 2: Mean Squared Error (MSE) of Dynamic Covariate Balancing (DCB) across 200 repetitions with sample size 400 and 101 variables in time period 1. This implies that the number of variables in time period 2 and 3 are 203 and 304. Oracle Estimator is denoted with aIPW* whereas aIPWh(l) denote AIPW with high(low)-dimensional estimated propensity. CAEW (MSM) corresponds to the method in Yiu and Su (2020), D.Lasso is adaptation of Double Lasso (Belloni et al., 2014).

|                | \(\eta = 0.1\) | \(\eta = 0.3\) | \(\eta = 0.5\) |
|----------------|-----------------|-----------------|-----------------|
|                | sparse mod harm | sparse mod harm | sparse mod harm |
| aIPW*          | 0.069 0.092 0.071 | 0.102 0.104 0.118 | 0.131 0.127 0.132 |
| DCB            | 0.060 0.077 0.075 | 0.092 0.076 0.084 | 0.099 0.077 0.085 |
| aIPWh          | 0.064 0.091 0.070 | 0.180 0.204 0.218 | 0.265 0.312 0.368 |
| aIPWhl         | 0.260 0.229 0.212 | 0.157 0.201 0.165 | 0.214 0.234 0.213 |
| IPWh           | 2.37 1.78 2.80   | 10.19 6.49 11.72 | 15.25 8.09 16.67 |
| Seq.Est.       | 0.932 1.333 0.692 | 1.388 1.787 1.152 | 1.759 1.795 1.664 |
| Lasso          | 0.247 0.410 0.132 | 0.509 0.710 0.298 | 0.762 0.948 0.560 |
| CAEW (MSM)     | 0.432 0.444 0.517 | 1.934 1.274 1.974 | 3.376 2.168 4.423 |
| Dyn.D.Lasso    | 0.124 0.118 0.256 | 0.208 0.147 0.430 | 0.218 0.153 0.554 |
|                | \(T = 3\)       |                 |                 |
| aIPW*          | 0.226 0.296 0.261 | 0.403 0.251 0.339 | 0.472 0.496 0.562 |
| DCB            | 0.155 0.208 0.199 | 0.257 0.217 0.329 | 0.294 0.267 0.455 |
| aIPWh          | 0.201 0.273 0.280 | 0.595 0.747 0.835 | 0.999 1.328 1.607 |
| aIPWhl         | 0.823 0.625 0.829 | 0.623 0.704 0.638 | 1.078 1.396 1.234 |
| IPWh           | 11.03 8.09 12.84 | 34.65 20.34 39.37 | 47.65 23.30 45.47 |
| Seq.Est.       | 2.608 4.016 2.316 | 3.722 5.269 3.818 | 5.279 6.829 5.467 |
| Lasso          | 0.409 0.492 0.514 | 0.559 0.732 0.507 | 1.290 1.315 1.174 |
| CAEW (MSM)     | 3.580 2.446 4.279 | 18.50 12.07 22.85 | 30.07 18.71 33.01 |
| Dyn.D.Lasso    | 0.471 0.344 0.679 | 0.694 0.378 1.182 | 0.964 0.383 1.594 |

former is of interest when the effect under control is more precise and its variance is asymptotically negligible compared to the estimated effect under treatment (e.g., many more individuals are not exposed to any treatment). The latter is of interest when both \(\mu(1,1)\) and \(\mu(0,0)\) are estimated from approximately a proportional sample. In the third panel, we report coverage when instead a Gaussian critical quantile (instead of the square root of a chi-squared quantile discussed in our theorems) is
used. We observe that our procedure can lead to correct (over) coverage, while the Gaussian critical quantile leads to under-coverage in the presence of poor overlap and many variables, but correct coverage with fewer variables and two periods only.\textsuperscript{23}

Table 3: Conditional average Coverage Probability of Dynamic Covariate Balancing (DCB) over 200 repetitions, with $\eta = 0.5$ (poor overlap). Here, $n = 400$ and $p = 100$; implying that the number of variables at time 2 and time 3 are $2p$ and $3p$, respectively. Homoskedastic and heteroskedastic estimators of the variance are denoted with Ho and He, respectively. The first two panels use the square-root of the chi-squared critical quantiles as discussed in Theorems 4.4, 4.5 and the last panel uses instead critical quantiles from the standard normal table (see Remark 4).

|                | $T = 2$ |                | $T = 3$ |
|----------------|---------|----------------|---------|
|                | Sparse  | Moderate       | Harmonic| Sparse  | Moderate       | Harmonic|
|                | Ho      | He             | Ho      | He      | Ho             | He      |
| $\mu(1,1)$: 95% Coverage Probability |         |                |         |         |                |         |
| $p=100$        | 1.00    | 0.98           | 1.00    | 0.99    | 0.99 0.96      |         |
| $p=200$        | 0.99    | 0.99           | 0.99    | 0.98    | 0.97 0.95      |         |
| $p=300$        | 1.00    | 0.99           | 0.99    | 0.99    | 0.96 0.94      |         |
| $\mu(1,1)-\mu(0,0)$: 95% Coverage Probability |         |                |         |         |                |         |
| $p=100$        | 1.00    | 0.99           | 1.00    | 1.00    | 1.00 1.00      | 0.99    |
| $p=200$        | 1.00    | 1.00           | 0.99    | 0.99    | 1.00 0.99      | 0.97    |
| $p=300$        | 1.00    | 1.00           | 1.00    | 1.00    | 0.98 0.97      | 0.98    |
| $\mu(1,1)-\mu(0,0)$: 95% Coverage Probability with Gaussian quantile |         |                |         |         |                |         |
| $p=100$        | 0.96    | 0.94           | 0.98    | 0.96    | 0.98 0.94      | 0.97    |
| $p=200$        | 0.97    | 0.94           | 0.98    | 0.92    | 0.91 0.85      | 0.98    |
| $p=300$        | 0.99    | 0.96           | 0.99    | 0.95    | 0.89 0.84      | 0.92    |

\textsuperscript{23}In practice, the Gaussian quantile may lead to less conservative statements. However, we also warn the reader that this may also lead to under-coverage compared to the proposed chi-squared quantile.
6 Empirical applications

6.1 The effect of negative advertisement on election outcome

Here, we study the effect of negative advertisements on the election outcome of democratic candidates. We use data from Blackwell (2013) who collects information on advertisement weeks before elections held in 2000, 2002, 2004, 2006.\(^\text{24}\) There were 176 races during this period. We select a subsample of 148 races, removing the non-competitive races as in Blackwell (2013). Each race is associated with a different democratic candidate and a set of baseline and time-varying covariates. Negative advertisement is indicated by a binary variable as discussed in Blackwell (2013).\(^\text{25}\)

As shown in Figure 4 (left-panel), each week, races may or may not “go negative” with treatment assignments exhibiting correlation in time. Hence, controlling for time-varying covariates and past assignments is crucial to avoid confounding. In a first model (Case 1), we control for the share of undecided voters in the previous week, whether the candidate is incumbent, the democratic polls, and whether the

\(^{24}\)Data is available at https://dataverse.harvard.edu/dataset.xhtml?persistentId=hdl:1902.1/19801.

\(^{25}\)This indicates whether at least ten percent of the negative advertisement.
democratic went negative in the previous week. Each of these variables (including
treatment assignments) enters linearly in the regression. In Figure 5 we compare
imbalance in covariates between the IPW weights estimated via logistic regression
and the DCB weights for Case 1. We observe that imbalance is substantially smaller
with the proposed weights, particularly for the share of undecided voters and the
polls. The only exception is the second covariate in the left-bottom panel where
imbalance, however, is approximately zero for both methods (the magnitude is $10^{-4}$
for this case). In Table 4 we collect results that demonstrate the negative effects of
going negative for two consecutive periods. We also observe negative effects, albeit
of smaller magnitude, when implementing a second specification (Case 2), which
controls for a larger set of covariates.\footnote{These are campaign length, the baseline number of undecided voters, baseline share of demo-
cratic voters, assignments two periods before, the type of office, and year fixed effects.}
When comparing to AIPW, we observe that DCB has a standard error twice as small as AIPW and larger point estimates in
magnitude. The standard error of simple IPW is, instead, much larger than the
AIPW and DCB.

Table 4: The first row corresponds to Case 1, while the second row to Case 2. (A)-
IPW refers to (Augmented)-inverse probability weights with stabilized weights. In
parenthesis, the standard errors.

|              | ATE DCB | ATE AIPW | ATE IPW |
|--------------|---------|----------|---------|
| Case 1       | −1.767  | −0.57    | 1.01    |
|              | (0.70)  | (1.45)   | (19.97) |
| Case 2       | −0.493  | −0.22    | 0.58    |
|              | (0.764) | (1.33)   | (19.58) |

6.2 Effect of democracy on economic growth

Here, we revisit the study of Acemoglu et al. (2019) on the effects of democracy on
economic growth.\footnote{In the past, Mulligan et al. (2004) find no significant effect, while Giavazzi and Tabellini (2005)
and Acemoglu et al. (2019) find positive and significant effects.} The data consist of an extensive collection of countries observed
Figure 5: Effect of negative advertisement on election outcome: imbalance plot. Covariates are the share of undecided voters, whether the democratic candidate is incumbent, the democratic polls, and the treatment in the previous period. At the top, we report the imbalance on the treated and on the controls at the bottom. On the left panel, we illustrate the imbalance in the first period and on the right in the second period.

between 1960 and 2010.\textsuperscript{28} We consider observations starting from 1989. After removing missing values, we run regressions with 141 countries. The outcome is the log-GDP in country $i$ in period $t$ as discussed in Acemoglu et al. (2019). Following Acemoglu et al. (2019) we capture democracy with a binary treatment based on international ranking. Studying the long-run impact of democracy has two challenges: (i) GDP growth depends on a long treatment history; (ii) unconfoundeness might hold only when conditioning on a large set of covariates and past outcomes. A graphical illustration of the causal model is Figure 7.

For each country, we condition on lag outcomes in the past four years, following Acemoglu et al. (2019), past four treatment assignments which enter linearly in the regression. We consider a pooled regression (see Remark 2) and two alternative specifications. The first is parsimonious and include dummies for different regions and different intercepts for different periods.\textsuperscript{29} A second one includes a larger set of covariates (in total 235 covariates). Coefficients are estimated with a penalized linear

\textsuperscript{28}Data available at https://www.journals.uchicago.edu/doi/suppl/10.1086/700936.

\textsuperscript{29}Country specific fixed effects are not included.
regression as described in Algorithm 2 (with \texttt{model = linear}).\textsuperscript{30} Tuning parameters for balancing weights are chosen as described in the Appendix.

The estimand of interest is the $t$-long run effect of democracy.\textsuperscript{31} It represents the effect of the past $t$ consecutive years of democracy. In Figure 6 (left-panel) we collect our results, for endline outcomes pooled across 1989 to 2010. Democracy has a statistically insignificant effect on the first years of GDP growth but a statistically significant positive impact on long-run GDP growth. The two specifications present similar results, showing the robustness of the results. We report the point estimates of the DCB method and the AIPW method, with a 90% confidence band for DCB (light-gray area). Figure 6 illustrates the flexibility of the method in capturing effects of policies that are possibly non-linear in the exposure length.

\textsuperscript{30}In practice, we may also want to penalize only some of the coefficients and not others, which is also allowed in our framework.

\textsuperscript{31}Formally $\mathbb{E}\left[Y_{i,t}(1s, D_i,(t-s);(-\infty))\right] - \mathbb{E}\left[Y_{i,t}(0s, D_i,(t-s);(-\infty))\right]$.
7 Discussion

This paper discusses the problem of inference on dynamic treatments via covariate balancing. We allow for high-dimensional covariates, and we introduce novel balancing conditions that allow for the optimal $\sqrt{n}$-consistent estimation. The proposed method relies on computationally efficient estimators. Simulations and empirical applications illustrate its advantages over state-of-the-art methodologies.

Several questions remain open. First, the asymptotic properties crucially rely on cross-sectional independence while allowing for general dependence over time. A natural extension is where clusters occur, which can be accommodated by our method with minor modifications. However, future work should address more general extensions where cross-sectional i.i.d.-ness does not necessarily hold. Second, our asymptotic results assume a fixed period. This is an extension for future research, where the period is allowed to grow with the sample size. Third, our derivations impose a weak form of overlap when constructing balancing weights. A natural avenue for future research is whether conditions on overlap might be replaced by alternative (weaker) assumptions.

Finally, derivation of general balancing conditions which do not rely on a partic-
ular model specification remains an open research question.

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**Supplementary Materials**

A Definitions

Throughout our discussion, we denote $y \lesssim x$ if the left-hand side is less or equal to the right-hand side up to a multiplicative constant term. We will refer to $\beta^t$ as $\beta^t_{d_{i,T}}$ whenever clear from the context. Recall that when we omit the script $i$, we refer to the vector of all observations. We define

\[
\nu_{i,t} = H_{i,t+1} \beta_{d_{i,T}}^{t+1} - H_{i,t} \beta_{d_{i,T}}^t, \quad \varepsilon_{i,T} = Y_{i,T}(d_{1:T}) - H_{i,T} \beta_{d_{i,T}}^T.
\]

and $\hat{\nu}_{i,t}$ for estimated coefficients (omitting the argument $(d_{1:T})$ for notational convenience). In addition, we define $V_{i,t} = (Y_{i,1}, \cdots, Y_{i,t-1}, X_{i,1}, \cdots, X_{i,t})$ the vector of observations without including the treatment assignments.

B Lemmas

B.1 Lemmas in Section 3 and 4

B.1.1 Proof of Lemma 2.1

The first equation is a direct consequence of condition (A) in Assumption 2, and the linear model assumption. Consider the second equation. By condition (B) in Assumption 2, we have

\[
E[Y_{i,2}(d_1, d_2) | X_{i,1}] = E[E[Y_{i,2}(d_1, d_2) | H_{i,2}, D_{i,1} = d_1] | X_{i,1}, D_{i,1} = d_1].
\]

Using the law of iterated expectations (since $X_{i,1} \subseteq H_{i,2}$)

\[
E[Y_{i,2}(d_1, d_2) | X_{i,1}, D_{i,1} = d_1] = E[E[Y_{i,2}(d_1, d_2) | H_{i,2}, D_{i,1} = d_1] | X_{i,1}, D_{i,1} = d_1].
\]
Using condition (A) in Assumption 2, we have

\[ \mathbb{E}[Y_{i,2}(d_1,d_2)|H_{i,2},D_{i,1} = d_1] = \mathbb{E}[Y_{i,2}(d_1,d_2)|H_{i,2},D_{i,1} = d_1,D_{i,2} = d_2] \]

completing the proof.

**B.1.2 Proof of Lemma 3.1**

The result for \( \bigg| \beta_{d_1:2}^2 - \beta_{d_1:2}^2 \bigg|_1 = O_p \left( s \sqrt{\log(p)/n} \right) \)

follows verbatim from Negahban et al. (2012) Corollary 2. For the result for \( \hat{\beta}_{d_1:2}^1 \) it suffices to notice, following the same argument from Negahban et al. (2012) (Corollary 2), that

\[ \bigg| \hat{\beta}_{d_1:2}^1 - \beta_{d_1:2}^1 \bigg|_1 = O(s \lambda_n), \text{ for } \lambda_n \geq \bigg| \frac{1}{n} X_1^\top \nu_1 \bigg|_\infty, \]

since here we used the estimated outcome \( H_2 \hat{\beta}_{d_1:T}^2 \) as the outcome of interest in our estimated regression instead of the true outcome. The upper bound as a function of \( \lambda_n \) follows directly from Theorem 1 in Negahban et al. (2012).\(^{32}\) The estimation error (and distribution of \( \nu_1 \)) affects concentration of \( \bigg| \frac{1}{n} X_1^\top \nu_1 \bigg|_\infty \). We note that we can write

\[
\bigg| \frac{1}{n} X_1^\top \nu_1 \bigg|_\infty \leq \bigg| \frac{1}{n} X_1^\top \nu_1 \bigg|_\infty + \bigg| \frac{1}{n} X_1^\top (\nu_1 - \hat{\nu}_1) \bigg|_\infty = \bigg| \frac{1}{n} X_1^\top \nu_1 \bigg|_\infty + \bigg| \frac{1}{n} X_1^\top H_2 (\beta_{d_1:2}^2 - \hat{\beta}_{d_1:2}^2) \bigg|_\infty \leq \bigg| \frac{1}{n} X_1^\top \nu_1 \bigg|_\infty + ||X_1||_\infty ||H_2||_\infty ||\beta_{d_1:2}^2 - \hat{\beta}_{d_1:2}^2||_1.
\]

\(^{32}\)Note that Theorem 1 in Negahban et al. (2012) does not depend on the distribution of the data and is a deterministic statement which holds under strong convexity at the true regression parameter. For a linear model, strong convexity is satisfied under the restricted eigenvalue assumption which does not depend on the regression parameter.
We now study each component separately. By sub-gaussianity, since $\mathbb{E}[\nu_1 | X_1] = 0$ by Assumption 3, we have for all $t > 0$, by Hoeffding inequality and the union bound,

$$P\left( \left\| \frac{1}{n} X_1^T \nu_1 \right\|_\infty > t \right| X_1 \right) \leq p \exp \left( - M \frac{t^2}{s} \right)$$

for a finite constant $M$. This result follows since $\nu_1 \leq ||\beta_1||_1 ||X_1^{(j)}||_\infty \leq Ms$. It implies that

$$\left\| \frac{1}{n} X_1^T \nu_1 \right\|_\infty = O_p(\sqrt{s \log(p)/n})$$

The second component instead is $O_p(s \sqrt{\log(p)/n})$ by the bound on $||\beta_{d_1:2}^2 - \hat{\beta}_{d_1:2}^2||_1$. This complete the proof. Finally, observe also that the same argument follows recursively for any finite $T$

### B.1.3 Proof of Lemma 4.1

Since the Lemmas in Section 3 are a special case of those in Section 4 we directly prove the results for multiple periods.

Throughout the proof we omit the argument $d_{1:T}$ of $\hat{\gamma}_i(d_{1:T})$ for notational convenience. Recall that $\hat{\gamma}_i = 0$ if $D_{i,1:T} \neq d_{1:T}$. Therefore, by consistency of potential outcomes:

$$\hat{\gamma}_i Y_i = \hat{\gamma}_i Y_i(d_{1:T}) = \hat{\gamma}_i (H_i \beta_{d_{1:T}}^T + \varepsilon_{i,T}).$$

Then we can write

$$\frac{1}{n} \sum_{i=1}^n \left( \hat{\gamma}_i Y_i - \sum_{t=2}^T (\hat{\gamma}_i - \hat{\gamma}_{i-1}) H_i \hat{\beta}_{d_{1:T}}^t - (\hat{\gamma}_{i-1} - \frac{1}{n}) X_i \hat{\beta}_{d_{1:T}}^1 \right)$$

$$= \frac{1}{n} \sum_{i=1}^n \left( \hat{\gamma}_i H_i \beta_{d_{1:T}}^T - \sum_{t=2}^T (\hat{\gamma}_i - \hat{\gamma}_{i-1}) H_i \hat{\beta}_{d_{1:T}}^t - (\hat{\gamma}_{i-1} - \frac{1}{n}) X_i \hat{\beta}_{d_{1:T}}^1 \right) + \hat{\gamma}_T^T \varepsilon_T.$$
Consider first the term
\[
\frac{1}{n} \sum_{i=1}^{n} (\hat{\gamma}_{i,T} H_{i,T} \beta_{d_1,T}^{T} - (\hat{\gamma}_{i,T} - \hat{\gamma}_{i,T-1}) H_{i,T} \beta_{d_1,T}^{T})
\]
\[
= (\hat{\gamma}_{T} H_{T} - \hat{\gamma}_{T-1} H_{T}) (\beta_{d_1,T}^{T} - \beta_{d_1,T}^{T}) + \hat{\gamma}_{T-1} H_{T} \beta_{d_1,T}^{T}.
\]

Notice now that for any \( s > 1 \),
\[
\frac{1}{n} \sum_{i=1}^{n} (\hat{\gamma}_{i,s} - \hat{\gamma}_{i,s-1}) H_{i,s} \beta_{d_1,T}^{s} = (\hat{\gamma}_{s} H_{s} - \hat{\gamma}_{s-1} H_{s}) (\beta_{d_1,T}^{s} - \beta_{d_1,T}^{s}) + \hat{\gamma}_{s} H_{s} \beta_{d_1,s}^{s} - \hat{\gamma}_{s-1} H_{s} \beta_{d_1,s}^{s}.
\]

For \( s = 1 \) we have instead
\[
\frac{1}{n} \sum_{i=1}^{n} (\hat{\gamma}_{i,1} - \frac{1}{n}) X_{i,1} \beta_{d_1,T}^{1} = (\hat{\gamma}_{1} X_{1} - \bar{X}_{1}) (\beta_{d_1,T}^{1} - \beta_{d_1,T}^{1}) + \hat{\gamma}_{1} X_{1} \beta_{d_1,s}^{1} - \bar{X}_{1} \beta_{d_1,s}^{1}.
\]

Therefore, we can write
\[
\frac{1}{n} \sum_{i=1}^{n} \left( \hat{\gamma}_{i,T} Y_{i,T} - \sum_{t=2}^{T} (\hat{\gamma}_{i,t} - \hat{\gamma}_{i,t-1}) H_{i,t} \beta_{d_1,T}^{t} - (\hat{\gamma}_{i,1} - \frac{1}{n}) X_{i,1} \beta_{d_1,T}^{1} \right)
\]
\[
= (\hat{\gamma}_{T} H_{T} - \hat{\gamma}_{T-1} H_{T}) (\beta_{d_1,T}^{T} - \beta_{d_1,T}^{T}) + \sum_{s=2}^{T-1} (\hat{\gamma}_{s} H_{s} - \hat{\gamma}_{s-1} H_{s}) (\beta_{d_1,T}^{s} - \beta_{d_1,T}^{s}) + (\hat{\gamma}_{1} X_{1} - \bar{X}_{1}) (\beta_{d_1,T}^{1} - \beta_{d_1,T}^{1})
\]
\[
+ \hat{\gamma}_{T} H_{T} \beta_{d_1,T}^{T} + \gamma_{T} \varepsilon_{T} - \left[ \sum_{s=2}^{T-1} \hat{\gamma}_{s} H_{s} \beta_{d_1,s}^{s} - \hat{\gamma}_{s} H_{s} \beta_{d_1,s}^{s} \right] - \hat{\gamma}_{1} X_{1} \beta_{d_1,s}^{1} + \bar{X}_{1} \beta_{d_1,s}^{1}.
\]

The proof completes collecting the desired terms.

**B.1.4 Proof of Lemma 4.2**

Since \( \hat{\gamma}_{i,t}(d_{1,t}) \) is equal to zero if \( D_{i,1:t} \neq d_{1:t} \) we can focus to the case where \( D_{i,1:t} = d_{1:t} \). Since weights at time \( t - 1 \) are measurable with respect to \( F_{t-1}, D_{t-1} \), we only need to show that
\[
\mathbb{E}[\hat{\gamma}_{i,t-1}(d_{1:T}) H_{i,t} \beta_{d_1,T}^{t} | F_{t-1}, D_{t-1}, D_{i,(1:t-1)}(d_{1:t-1})] = \hat{\gamma}_{i,t}(d_{1:T}) H_{t-1} \beta_{d_1,T}^{t-1},
\]
where, recall, weights \( \hat{\gamma}_{i,t-1}(d_{1:T}) \) are a measurable function of \( F_{t-1}, D_{t-1} \). On the event that \( D_{i,(1:t-1)} \neq d_{1:t-1} \) the expression is zero on both sides and the result
trivially holds. Therefore, we can implicitly assume that $D_{i,(t-1)} = d_{1,(t-1)}$ since otherwise the result trivially holds. Under Assumption 7 we can write

$$
\mathbb{E}[\hat{\gamma}_{i,t-1}(d_{1:T})H_t\beta_{d_{1:T}}|\mathcal{F}_{t-1}, D_t] = \mathbb{E}[\hat{\gamma}_{i,t-1}(d_{1:T})|\mathcal{F}_{t-1}, D_t]
$$

by the tower property of the expectation. Now notice that under Assumption 7, $\mathbb{E}[Y_{i,T}(d_{1:T})|\mathcal{F}_{t-1}, D_{t-1}] = \mathbb{E}[Y_{i,T}(d_{1:T})|\mathcal{F}_{t-1}]$. Therefore

$$
\hat{\gamma}_{i,t-1}(d_{1:T})\mathbb{E}[Y_{i,T}(d_{1:T})|\mathcal{F}_{t-1}] = \hat{\gamma}_{i,t-1}(d_{1:T})H_t\beta_{d_{1,(t-1)}}
$$

which follows since $\hat{\gamma}_{i,t-1}(d_{1:T}) = 0$ if $D_{1:t-1} \neq d_{1:t-1}$.

**Corollary 2.** Lemma 3.2 holds.

**Proof.** It follows directly choosing $t \in \{1, 2\}$ from Lemma 4.2.

---

**B.2 Additional auxiliary Lemmas**

**Lemma B.1.** *(Existence of Feasible $\hat{\gamma}_i$)* Suppose that $|X_{i,1}^{(j)}| \leq M < \infty$, $X_{i,1} \in \mathbb{R}^p$. Suppose that for $d_1 \in \{0, 1\}$, $P(D_{i,1} = d_1|X_{i,1}) \in (\delta, 1-\delta)$. Then with probability $1 - 5/n$, for $\log(2np)/n \leq c_0$ for a constant $0 < c_0 < \infty$, where $\delta_1(n,p) \geq C M a \sqrt{2 \log(2np)/n}$, for a constant $0 < C < \infty$, there exist a feasible $\hat{\gamma}_1$. In addition,

$$
\lim_{n \to \infty} P\left(n||\hat{\gamma}_1||_2^2 \leq \mathbb{E}\left[\frac{1}{P(D_{i,1} = d_1|X_{i,1})}\right]\right) = 1.
$$

**Proof of Lemma B.1.** This proof follows in the same spirit of one-period setting (Athey et al., 2018). To prove existence of a feasible weight, we use a feasible guess. We prove the claim for a general $d_1 \in \{0, 1\}$. Consider first

$$
\hat{\gamma}_{i,1}^* = \frac{1\{D_{i,1} = d_1\}}{nP(D_{i,1} = d_1|X_{i,1})}/\left(\frac{1}{n} \sum_{i=1}^{n} 1\{D_{i,1} = d_1\}/P(D_{i,1} = d_1|X_{i,1})\right).
$$

(B.4)
For such weight to be well-defined, we need that the denominator is bounded away from zero. We now provide bounds on the denominator. Since 

\[ P(D_{i,1} = d_1 | X_{i,1}) \in (\delta, 1 - \delta) \]

by Hoeffding inequality

\[
P\left( \left| \frac{1}{n} \sum_{i=1}^{n} \frac{1\{D_{i,1} = d_1\}}{P(D_{i,1} = d_1 | X_{i,1})} - 1 \right| > t \right) \leq 2 \exp \left( - \frac{nt^2}{2a^2} \right),
\]

for a finite constant \( a \). Therefore with probability \( 1 - 1/n \),

\[
\frac{1}{n} \sum_{i=1}^{n} \frac{1\{D_{i,1} = d_1\}}{P(D_{i,1} = d_1 | X_{i,1})} > 1 - \sqrt{2a^2 \log(2n)/n}.
\]  \hspace{1cm} (B.5)

Therefore for \( n \) large enough such that \( \sqrt{2a^2 \log(2n)/n} < 1 \), weights are finite with high probability. In addition, they sum up to one and they satisfy the requirement with probability \( 1 - 1/n \)

\[
\frac{1\{D_{i,1} = d_1\}}{nP(D_{i,1} = d_1 | X_{i,1})} \lesssim n^{-2/3} \Rightarrow \gamma_{i,1}^* \leq K_{2,1} n^{-2/3}
\]

for a constant \( K_{2,1} \), where the first inequality follows by the overlap assumption and the second by Equation (B.5). We are left to show that the first constraint is satisfied. First notice that under Assumption 7

\[
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{1\{D_{i,1} = d_1\} X_{i,1}^{(j)}}{P(D_{i,1} = d_1 | X_{i,1})} | X_1 \right] = \bar{X}_1^{(j)}.
\]

In addition, since \( X_{i,1} \) is uniformly bounded, by sub-gaussianity of \( 1/P(D_{i,1} = d_1 | X_{i,1}) \), and the union bound

\[
P\left( \left\| X_1 - \frac{1}{n} \sum_{i=1}^{n} \frac{1\{D_{i,1} = d_1\}}{P(D_{i,1} = d_1 | X_{i,1})} X_{i,1} \right\|_\infty > t | X_1 \right) \leq p2 \exp \left( - \frac{nt^2}{2a^2 M^2} \right)
\]

51
for a finite constant $a^2$. With trivial rearrangement, with probability $1 - 1/n$,

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \frac{1\{D_{i,1} = d_1\}}{P(D_{i,1} = d_1|X_{i,1})} X_{i,1} \right\|_{\infty} \leq aM \sqrt{2 \log(2np)/n} \quad (B.6)$$

Consider now the denominator. We have shown that the denominator concentrates around one at exponential rate, namely that with probability $1 - 1/n$,

$$\left| \left| \frac{1}{n} \sum_{i=1}^{n} \frac{1\{D_{i,1} = d_1\}}{P(D_{i,1} = d_1|X_{i,1})} - 1 \right| \leq 2a \sqrt{\log(2n)/n} \right. \quad (B.7)$$

Therefore, with probability $1 - 2/n$,

$$\left| \left| \frac{\bar{X}_1 - \frac{1}{n} \sum_{i=1}^{n} \frac{1\{D_{i,1} = d_1\}}{P(D_{i,1} = d_1|X_{i,1})} X_{i,1}}{\frac{1}{n} \sum_{i=1}^{n} \frac{1\{D_{i,1} = d_1\}}{P(D_{i,1} = d_1|X_{i,1})}} \right|_{\infty} \right. \leq \left| \left| \bar{X}_1 - \frac{1}{n} \sum_{i=1}^{n} \frac{1\{D_{i,1} = d_1\}}{P(D_{i,1} = d_1|X_{i,1})} X_{i,1} \right|_{\infty} \right. \right| + \frac{M^2 2a \sqrt{\log(2n)/n}}{\frac{1}{n} \sum_{i=1}^{n} \frac{1\{D_{i,1} = d_1\}}{P(D_{i,1} = d_1|X_{i,1})}}$$

$$\leq \frac{Ma \sqrt{2 \log(2np)/n} + M^2 a \sqrt{\log(2n)/n}}{\frac{1}{n} \sum_{i=1}^{n} \frac{1\{D_{i,1} = d_1\}}{P(D_{i,1} = d_1|X_{i,1})}} \quad (B.8)$$

where the first inequality follows by the triangular inequality and by concentration of the term $\frac{1}{n} \sum_{i=1}^{n} \frac{1\{D_{i,1} = d_1\}}{P(D_{i,1} = d_1|X_{i,1})}$ around one at exponential rate as in Equation (B.7). The second inequality follows by concentration of the numerator as in Equation (B.6).

With probability $1 - 1/n$, the denominator is bounded away from zero. Therefore for a universal constant $C < \infty$,$^{33}$

$$P\left( \left| \left| \bar{X}_1 - \frac{1}{n} \sum_{i=1}^{n} \frac{1\{D_{i,1} = d_1\}}{P(D_{i,1} = d_1|X_{i,1})} X_{i,1} \right|_{\infty} \right| \leq CMa \sqrt{2 \log(2np)/n} \right) \geq 1 - 3/n. \quad (B.9)$$

We are left to provide bounds on $||\hat{\gamma}_1||_2^2$. For $n$ large enough, with probability at least $1 - 5/n$, $||\hat{\gamma}_1||_2^2 \leq ||\gamma_1^*||_2^2$ since $\gamma_1^*$ is a feasible solution. By overlap, the fourth moment of $1/P(D_{i,1} = d_1|X_{i,1})$ is bounded. By the strong law of large numbers and

$^{33}$Here $3/n$ follows from the union bound.
Slutsky theorem,
\[ n ||\hat{\gamma}^*_t||_2^2 = \sum_{i=1}^{n} \frac{1\{D_{i,1} = d_1\}}{nP(D_{i,1} = d_1|X_{i,1})^2} \left( \sum_{i=1}^{n} \frac{1\{D_{i,1} = d_1\}}{nP(D_{i,1} = d_1|X_{i,1})^2} \right)^2 \to_{as} \frac{\mathbb{E}[\frac{1\{D_{i,1} = d_1\}}{nP(D_{i,1} = d_1|X_{i,1})^2}]^2}{\mathbb{E}[\frac{1\{D_{i,1} = d_1\}}{nP(D_{i,1} = d_1|X_{i,1})}]} < \infty. \] (B.10)

which completes the proof.

\[ \square \]

**Lemma B.2. (Existence of a feasible \( \hat{\gamma}_t \))** Let

\[ Z_{i,t}(d_t) = \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t|Y_{i,1}, ..., Y_{i,t-1}, X_{i,1}, ..., X_{i,t-1}, D_{i,1}, ..., D_{i,t-1})}. \]

Assume that for \( d_t \in \{0, 1\} \). Assume that \( H_{i,t}^{(j)}|H_{i,t-1} \) is sub-gaussian for all \( j \in \{1, ..., p\} \) almost surely. Let Assumption 8 hold and let for a finite constant \( c_0 \),

\[ \delta_t(n, p) \geq c_0 \frac{\log^{3/2}(pn)}{n^{1/2}}, \text{ and } K_{2,t} = 2K_{2,t-1}\tilde{c}, \text{ for some finite constant } \tilde{c}. \]

Then with probability \( \eta_n \to 1 \), for some \( N > 0 \), \( n \geq N \), there exists a feasible \( \hat{\gamma}^*_t \) solving the optimization in Algorithm 3, where

\[ \hat{\gamma}^*_i,t = \hat{\gamma}_{i,t-1}Z_{i,t}(d_t) / \sum_{i=1}^{n} \hat{\gamma}_{i,t-1}Z_{i,t}(d_t) \]

In addition,

\[ \lim_{n \to \infty} P\left(n ||\hat{\gamma}_t||_2^2 \leq C_t \right) = 1 \] (B.11)

for a constant \( 1 \leq C_t < \infty \) independent of \((p, n)\).

**Proof of Lemma B.2.** The proof follows by induction. By Lemma B.1 we know that there exist a feasible \( \hat{\gamma}_1 \), with \( \lim_{n \to \infty} P(n ||\hat{\gamma}_1||_2^2 \leq C') = 1 \). Suppose now that there exist feasible \( \hat{\gamma}_1, ..., \hat{\gamma}_{t-1} \), such that

\[ \lim_{n \to \infty} P(n ||\hat{\gamma}_s||_2^2 \leq C_s) = 1 \] (B.12)

for some finite constant \( C_s \) which only depends on \( s \), and for all \( s < t \). We want to
show that the statement holds for $\hat{\gamma}_t$. We find $\gamma_t^*$ that satisfies the constraint, with
\begin{equation}
\hat{\gamma}_{i,t} = \hat{\gamma}_{i,t} - 1 \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t | H_{i,t})} / \left( \sum_{i=1}^n \hat{\gamma}_{i,t} \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t | H_{i,t})} \right).
\end{equation}

We break the proof into several steps.

**Finite and Bounded Weights** To show that such weights are finite, with high probability, we need to impose bounds on the numerator and the denominator. We want to bound for a universal constant $\bar{C} < \infty$,
\begin{equation}
P\left( \max_{i \in \{1, \ldots, n\}} \hat{\gamma}_{i,t} - 1 \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t | H_{i,t})} > \tilde{C} n^{-2/3} K_{2,t-1} \right)
\end{equation}

for a finite constant $\tilde{C}$. We now provide bounds on the denominator. Since $\sigma(H_{t-1}) \subseteq \sigma(H_t)$
\begin{align*}
E\left[ \sum_{i=1}^n \hat{\gamma}_{i,t} - 1 \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t | H_{i,t})} \right] &= E\left[ E\left[ \sum_{i=1}^n \hat{\gamma}_{i,t} - 1 \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t | H_{i,t})} | H_{t-1} \right] \right] \\
&= E\left[ \sum_{i=1}^n \hat{\gamma}_{i,t} - 1 E\left[ \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t | H_{i,t})} | H_{t-1} \right] \right] = \sum_{i=1}^n \hat{\gamma}_{i,t-1} = 1.
\end{align*}

We show concentration of the denominator around its expectation to show that the denominator is bounded away from zero with high probability. Let $C_{t-1}$ be the upper limit on $n||\hat{\gamma}_{t-1}||_2^2$, and let
\begin{equation}
c := 1/C_{t-1} \quad \eta_{n,t} := P(||\hat{\gamma}_{t-1}||_2^2 \leq 1/(cn)),
\end{equation}

54
for some constant $c$, which only depends on $t - 1$ (the dependence with $t - 1$ is suppressed for expositional convenience). Observe in addition that $\eta_{n,t} \to 1$ by the induction argument (see Equation (B.12)). We write for a finite constant $a$

\[
P\left(\left| \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t|H_{i,t})} - 1 \right| > h \right) \\
\leq P\left(\left| \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t|H_{i,t})} - 1 \right| > h \left| \frac{||\hat{\gamma}_{t-1}||^2}{2} \leq 1/(cn) \right) \eta_{n,t} + (1 - \eta_{n,t}) \sum_{i=1}^{n} \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t|H_{i,t})} - 1 \right| \\
\leq 2 \exp\left( - \frac{ah^2}{2||\hat{\gamma}_{t-1}||^2} \left| \frac{||\hat{\gamma}_{t-1}||^2}{2} \leq 1/(cn) \right) \eta_{n,t} + (1 - \eta_{n,t}) \right)
\]

(B.15)

The third inequality follows from the fact that $\hat{\gamma}_{t-1}$ is measurable with respect to $H_{t-1}$ and $\frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t|H_{i,t})}$ is sub-gaussian conditional on $H_{i,t-1}$ (since uniformly bounded). Therefore with probability at least $1 - \delta$,

\[
\left| \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t|H_{i,t})} - 1 \right| \leq \sqrt{2 \log\left( \frac{2}{\delta + \eta_{n,t} - 1} \right)}/(acn). \quad (B.16)
\]

By setting $\delta = \eta_{n,t}/n + (1 - \eta_{n,t})$, with probability at least $1 - \eta_{n,t}/n + (1 - \eta_{n,t})$,

\[
\left| \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t|H_{i,t})} - 1 \right| \leq \sqrt{2 \log(2n)}/acn,
\]

and hence the denominator is bounded away from zero for $n$ large enough (recall that $\eta_{n,t} \to 1$).

**First Constraint**  We now show that the proposed weights satisfy the first constraint in Algorithm 3. The second trivially holds, while the third has been discussed
in the first part of the proof. We write

$$
\mathbb{E} \left[ \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} H_{i,t}^{(j)} - \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t|H_{i,t})} H_{i,t}^{(j)} \right] = 0.
$$

We want to show concentration. First, we break the probability into two components:

$$
P\left( \left\| \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} H_{i,t} - \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t|H_{i,t})} H_{i,t} \right\|_{\infty} > h \right)
\leq P\left( \left\| \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} H_{i,t} - \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t|H_{i,t})} H_{i,t} \right\|_{\infty} > h \left\| \|\hat{\gamma}_{t-1}\|_2^2 \leq 1/cn \right\| \eta_{n,t} + (1 - \eta_{n,t}), \right.
\right)
$$

where $\eta_{n,t} = P(\|\hat{\gamma}_{t-1}\|_2^2 \leq 1/cn)$ for some constant $c$. We study $(I)$, whereas, by the induction argument $(II) \rightarrow 0$ (Equation (B.12)). For a constant $\bar{c} < \infty$, sub-gaussianity of $H_{i,t}|H_{t-1}$ and overlap, we can write for any $\lambda > 0$,

$$(I) \leq \sum_{j=1}^{p} \mathbb{E} \left[ \mathbb{E} \left[ \exp \left( \lambda \bar{c} \left\| \hat{\gamma}_{t-1} \right\|_2^2 - \lambda h \right) |H_{t-1}, \left\| \hat{\gamma}_{t-1} \right\|_2^2 \leq 1/cn \right] \left\| \|\hat{\gamma}_{t-1}\|_2^2 \leq 1/cn \right\| \eta_{n,t}. \right.
\right)
$$

Since $\hat{\gamma}_{t-1}$ is measurable with respect to $H_{t-1}$, we can write

$$(B.17) \leq \eta_{n,p} \exp \left( \lambda^2/(cn) - \lambda h \right). \quad (B.18)$$

Choosing $\lambda = hcn/2$ we obtain that the above equation converges to zero as $\log(p)/n = o(1)$. After trivial rearrangement, with probability at least $1 - (1 - \eta_n) - 1/n$ (recall that $\eta_n \rightarrow 1$),

$$
\left\| \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} H_{i,t} - \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t|H_{i,t})} H_{i,t} \right\|_{\infty} \lesssim \sqrt{\log(np)/n}. \quad (B.19)
$$
As a result, we can write

$$
\left\| \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} H_{i,t} - \frac{1}{\sum_{i=1}^{n} \hat{\gamma}_{i,t-1} P(D_{i,t} = d_{t}|H_{i,t})} \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} \frac{1(D_{i,t} = d_{t})}{P(D_{i,t} = d_{t}|H_{i,t})} H_{i,t} \right\|_{\infty}
$$

\[= \left\| \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} H_{i,t} \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} \frac{1(D_{i,t} = d_{t})}{P(D_{i,t} = d_{t}|H_{i,t})} - \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} \frac{1(D_{i,t} = d_{t})}{P(D_{i,t} = d_{t}|H_{i,t})} H_{i,t} \right\|_{\infty}
\]

\[\leq \left\| \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} H_{i,t} \left(1 - \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} \frac{1(D_{i,t} = d_{t})}{P(D_{i,t} = d_{t}|H_{i,t})} \right) \right\|_{\infty} + \left\| \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} H_{i,t} \left(1 - \frac{1(D_{i,t} = d_{t})}{P(D_{i,t} = d_{t}|H_{i,t})} \right) \right\|_{\infty}.
\]

Observe now that the denominators of the above expressions are bounded away from zero with high probability as discussed in Equation (B.16). The numerator of (ii) is bounded by Equation (B.19). We are left with the numerator of (i). Note first that

$$
E \left[ \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} \frac{1(D_{i,t} = d_{t})}{P(D_{i,t} = d_{t}|H_{i,t})} \right] = 1.
$$

We can write

$$
\left\| \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} H_{i,t} \left(1 - \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} \frac{1(D_{i,t} = d_{t})}{P(D_{i,t} = d_{t}|H_{i,t})} \right) \right\|_{\infty} \leq \max_{j} \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} H_{i,t}^{(j)} \left| 1 - \sum_{i=1}^{n} \hat{\gamma}_{i,t-1} \frac{1(D_{i,t} = d_{t})}{P(D_{i,t} = d_{t}|H_{i,t})} \right|.
$$

Here \((jj)\) is bounded as in Equation (B.16), with probability \(1 - 1/n\) at a rate \(\sqrt{\log(n)/n}\). The component \((j)\) instead is bounded as

$$
(j) \leq \max_{j,i} |H_{i,t}^{(j)}| \lesssim \log(pn)
$$

with probability \(1 - 1/n\) using subgaussianity of \(H_{i,t}^{(j)}\). As a result, all constraints are satisfied.
Finite Norm  We now need to show that Equation (B.11) holds. With probability converging to one,

\[ n\|\hat{\gamma}_t\|^2 \leq n\|\hat{\gamma}^*_t\|^2 = \sum_{i=1}^{n} n\hat{\gamma}^*_i,1^2 \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t|H_{i,t})^2} / \left( \sum_{i=1}^{n} \hat{\gamma}^*_i,1 \frac{1\{D_{i,t} = d_t\}}{P(D_{i,t} = d_t|H_{i,t})} \right)^2. \]

The denominator converges in probability to one by Equation (B.16). The numerator can instead be bounded by \( n\|\gamma^*_t\|^2 \) up-to a finite multiplicative constant by Assumption 8. By the recursive argument \( n\|\gamma^*_t\|^2 = O_p(1) \).

Lemma B.3. The weights solving the optimization problem in Algorithm 3 are such that

\[ \|\hat{\gamma}_t\|^2 \geq 1/n. \]

Proof. Observe that for either algorithms, weights sum to one. The minimum under this constraint only is obtained at \( \hat{\gamma}_{i,t} = 1/n \) for all \( i \) concluding the proof.

C   Proofs of the Main Theorems

Proof of Theorem 4.3

By Lemmas B.1 and B.2, Theorem 4.3 and Theorem 3.3 directly hold.

Proof of Theorem 4.4

Throughout the proof we will be omitting the script \( d_{1:T} \) in the weights and coefficients whenever clear from the context. Note that Theorem 3.4 and 3.5 are a direct corollary of Theorem 4.4.

Weights do not diverge to infinity  First notice that by Lemmas B.1, B.2, there exist a \( \hat{\gamma}^*_t \) such that for \( N \) large enough, with probability converging to one, for some constant \( C \), and \( n > N \)

\[ n\|\hat{\gamma}_t\|^2 \leq n\|\hat{\gamma}^*_t\|^2 = O_p(1). \]  (C.1)
Similar reasoning also applies to \( n \sum_{i=1}^{n} \gamma_{i,t}^2 \text{Var}(\nu_{i,t} | H_{i,t}) \) and \( n \sum_{i=1}^{n} \gamma_{i,T}^2 \text{Var}(\varepsilon_{i,T} | F_T) \) since the conditional variances are uniformly bounded by the finite third moment condition.

**Error Decomposition** We denote \( \bar{\sigma}^2 \) the lower bound on the conditional variances and \( \sigma_{\text{up}}^2 \) the upper bound on the variances. Recall \( \nu_{i,t} = H_{i,t+1} \beta_{d_1,t}^{t+1} - H_{i,t} \beta_{d_1,t}^t \) and \( \hat{\nu}_{i,t} \) for estimated coefficients, \( \hat{\nu}_{i,t} = H_{i,t+1} \hat{\beta}_{d_1,t}^{t+1} - H_{i,t} \hat{\beta}_{d_1,t}^t \). First we write the expression as

\[
\frac{\hat{\mu}(d_{1:T}) - \bar{X}_1 \beta_{d_1,T}}{\sqrt{\hat{V}_T(d_{1:T})}} = \frac{\hat{\mu}(d_{1:T}) - \bar{X}_1 \beta_{d_1,T}}{\sqrt{\sum_{i=1}^{n} \gamma_{i,T}^2 \text{Var}(\varepsilon_{i,T} | H_{i,T-1}) + \sum_{i=1}^{n} \sum_{t=1}^{T-1} \gamma_{i,t}^2 \text{Var}(\nu_{i,t} | H_{i,t})}} \times (I)
\]

\[
\times \frac{\sqrt{\sum_{i=1}^{n} \gamma_{i,T}^2 \text{Var}(\varepsilon_{i,T} | H_{i,T}) + \sum_{i=1}^{n} \sum_{t=1}^{T-1} \gamma_{i,t}^2 \text{Var}(\nu_{i,t} | H_{i,t})}}}{\sqrt{\sum_{i=1}^{n} \gamma_{i,T}^2 (Y_{i,T} - H_{i,T} \beta_{d_1,T}^T)^2 + \sum_{i=1}^{T-1} \gamma_{i,t}^2 \hat{\nu}_{i,t}^2}} . (II)
\]

**Term (I)** We consider the term (I). By Lemma 4.1, we have

\[
(I) = \frac{\sum_{t=1}^{T} (\beta^t - \hat{\beta}^t) \top (\hat{\gamma}_t - \hat{\gamma}_{t-1} H_t)}{\sqrt{\sum_{i=1}^{n} \gamma_{i,T}^2 \text{Var}(\varepsilon_{i,T} | H_{i,T-1}) + \sum_{i=1}^{n} \sum_{t=1}^{T-1} \gamma_{i,t}^2 \text{Var}(\nu_{i,t} | H_{i,t})}}
\]

\[
+ \frac{\sum_{i=1}^{n} \hat{\gamma}_i \varepsilon_{i,T} + \sum_{i=1}^{T-1} \hat{\gamma}_t \nu_{i,t}}{\sqrt{\sum_{i=1}^{n} \gamma_{i,T}^2 \text{Var}(\varepsilon_{i,T} | H_{i,T-1}) + \sum_{i=1}^{n} \sum_{t=1}^{T-1} \gamma_{i,t}^2 \text{Var}(\nu_{i,t} | H_{i,t})}} . (j)
\]

We start from (j). Notice since \( \sum_{i=1}^{n} \hat{\gamma}_{i,t} = 1 \) and the variances are bounded from below (and Lemma B.3), it follows that

\[
\sum_{i=1}^{n} \gamma_{i,T}^2 \text{Var}(\varepsilon_{i,T} | H_{i,T-1}) + \sum_{i=1}^{n} \sum_{t=1}^{T-1} \gamma_{i,t}^2 \text{Var}(\nu_{i,t} | H_{i,t}) \geq T \bar{\sigma}^2 \sum_{i=1}^{n} \frac{1}{n^2} = T \bar{\sigma}^2 / n.
\]

59
Therefore, since the denominator is bounded from below by $\sigma \sqrt{T/n}$, and since, by Holder’s inequality

$$\sum_{t=1}^{T} (\beta^t - \hat{\beta}^t)^\top (\hat{\gamma}_t H_t - \hat{\gamma}_{t-1} H_t) \lesssim T \|\beta^t - \hat{\beta}^t\|_1 \left\| \hat{\gamma}_t H_t - \hat{\gamma}_{t-1} H_t \right\|_{\infty}$$

we have

$$(j) \lesssim T \max_t \delta_t(n,p) \|\beta^t - \hat{\beta}^t\|_1 \to_p 0 \quad \text{(C.3)}$$

under Assumption 9 and the fact that $T$ is fixed. We can now write

$$(I) = o_p(1) + \frac{\sum_{i=1}^{n} \hat{\gamma}_{i,T} \varepsilon_{i,T}}{\sqrt{\sum_{i=1}^{n} \hat{\gamma}_{i,T}^2 \text{Var}(\varepsilon_{i,T}|H_{i,T-1})}} \times \left( \frac{\sum_{i=1}^{n} \hat{\gamma}_{i,T}^2 \text{Var}(\varepsilon_{i,T}|H_{i,T-1})}{\sum_{i=1}^{n} \hat{\gamma}_{i,T}^2 \text{Var}(\varepsilon_{i,T}|H_{i,T-1}) + \sum_{i=1}^{n} \sum_{t=1}^{T-1} \hat{\gamma}_{i,t}^2 \text{Var}(\nu_{i,t}|H_{i,t})} \right)$$

$$+ \sum_{t=1}^{T-1} \frac{\sum_{i=1}^{n} \hat{\gamma}_{i,T} \nu_{i,t}}{\sqrt{\sum_{i=1}^{n} \text{Var}(\nu_{i,t}|H_{i,t}) \hat{\gamma}_{i,t}^2}} \times \frac{\sqrt{\sum_{i=1}^{n} \text{Var}(\nu_{i,t}|H_{i,t}) \hat{\gamma}_{i,t}^2}}{\sqrt{\sum_{i=1}^{n} \text{Var}(\varepsilon_{i,T}|H_{i,T-1}) + \sum_{i=1}^{n} \sum_{t=1}^{T-1} \hat{\gamma}_{i,t}^2 \text{Var}(\nu_{i,t}|H_{i,t})}}$$

First, notice that $\sigma(\hat{\gamma}_T) \subseteq \sigma(D_T, F_T)$, and by Assumption 7 $\varepsilon_T \perp D_T|F_T$. Therefore,

$$\mathbb{E}[\hat{\gamma}_{i,T} \varepsilon_{i,T} | F_T, D_T] = 0, \quad \sigma^2 \|\hat{\gamma}_T\|_2^2 \leq \text{Var} \left( \sum_{i=1}^{n} \hat{\gamma}_{i,T} \varepsilon_{i,T} | F_T, D_T \right) \leq \|\hat{\gamma}_T\|_2^2 \sigma^2_\varepsilon,$$

where the first statement follows directly from 4.2 and the second statement holds for a finite constant $\sigma^2_\varepsilon$ by the third moment condition in Assumption 9. By the third moment conditions in Assumption 9 and independence of $\varepsilon_{i,T}$ of $D_T$ given $F_T$ in Assumption 7, for a constant $0 < C < \infty$,

$$\mathbb{E} \left[ \left( \sum_{i=1}^{n} \hat{\gamma}_{i,T} \varepsilon_{i,T} \right)^3 \right] | F_T, D_T = \sum_{i=1}^{n} \hat{\gamma}_{i,T}^3 \mathbb{E}[\varepsilon_{i,T}^3 | F_T] \leq C \sum_{i=1}^{n} \hat{\gamma}_{i,T}^3 \leq C \|\hat{\gamma}_T\|_2^3 \max_i \hat{\gamma}_{i,T} \lesssim \log(n) n^{-2/3} \|\hat{\gamma}_T\|_2.$$
Thus,

\[
E \left[ \sum_{i=1}^{n} \tilde{\gamma}_{i,T}^3 \epsilon_{i,T} \bigg| \mathcal{F}_T, D_T \right] / \text{Var} \left( \sum_{i=1}^{n} \tilde{\gamma}_{i,T} \epsilon_{i,T} \bigg| \mathcal{F}_T, D_T \right)^{3/2} = O(\log(n)n^{-2/3}||\tilde{\gamma}_T||^1_2) = o(1).
\]

By Liapunov theorem, we have

\[
\frac{\sum_{i=1}^{n} \tilde{\gamma}_{i,T} \epsilon_{i,T}}{\sum_{i=1}^{n} \tilde{\gamma}_{i,T}^2 \text{Var}(\epsilon_{i,T} | \mathcal{F}_T)} \rightarrow_d \mathcal{N}(0, \sigma^2).
\]

Consider now (iii) for a generic time \( t \). We study the behaviour of \( \sum_{i=1}^{n} \tilde{\gamma}_{i,t} \nu_{i,t} \) conditional on \( \sigma(\mathcal{F}_t, D_t) \). Since \( \sigma(\tilde{\gamma}_t) \subseteq \sigma(\mathcal{F}_t, D_t) \), \( \tilde{\gamma}_t \) is deterministic given \( \sigma(\mathcal{F}_t, D_t) \).

By Lemma 4.2, \( E[\tilde{\gamma}_{i,t} \nu_{i,t} | \mathcal{F}_t, D_t] = 0 \). We now study the second moment. First notice that

\[
\bar{\sigma}^2 ||\tilde{\gamma}_t||^2_2 \leq \text{Var} \left( \sum_{i=1}^{n} \tilde{\gamma}_{i,t} \nu_{i,t} | \mathcal{F}_t, D_t \right) = \sum_{i=1}^{n} \tilde{\gamma}_{i,t}^2 \text{Var}(\nu_{i,t} | \mathcal{F}_t, D_t) \leq \sum_{i=1}^{n} \tilde{\gamma}_{i,t}^2 \sigma_{ub}^2.
\]

Finally, we consider the third moment. Under Assumption 9,

\[
E \left[ \sum_{i=1}^{n} \tilde{\gamma}_{i,t}^3 \nu_{i,t}^3 \bigg| X_1, D_1 \right] = \sum_{i=1}^{n} \tilde{\gamma}_{i,t}^3 E[\nu_{i,t}^3 | \mathcal{F}_t, D_t] \leq \sum_{i=1}^{n} \tilde{\gamma}_{i,t}^3 u_{max}^3 \lesssim \log(n)n^{-2/3}||\tilde{\gamma}_t||^2_2.
\]

Since \( ||\tilde{\gamma}_t||^2 \geq 1/\sqrt{n} \) by Lemma B.3 and since \( \text{Var}(\nu_{i,t} | \mathcal{F}_t, D_t) > u_{min} \),

\[
E \left[ \sum_{i=1}^{n} \tilde{\gamma}_{i,t}^3 \nu_{i,t}^3 \bigg| \mathcal{F}_t, D_t \right] / \text{Var} \left( \sum_{i=1}^{n} \tilde{\gamma}_{i,t} \nu_{i,t} \bigg| \mathcal{F}_t, D_t \right)^{3/2} = O(\log(n)n^{-2/3}||\tilde{\gamma}_t||^1_2) = o(1).
\]

\[
\Rightarrow \frac{\sum_{i=1}^{n} \tilde{\gamma}_{i,t} \nu_{i,t}}{\sqrt{\sum_{i=1}^{n} \tilde{\gamma}_{i,t}^2 \text{Var}(\nu_{i,t} | \mathcal{F}_t, D_t)}} \rightarrow_d \mathcal{N}(0, 1).
\]
Collecting our results it follows that

\[
\frac{\sum_{i=1}^{n} \hat{\gamma}_{i,T} \varepsilon_{i,T}}{\sqrt{\sum_{i=1}^{n} \text{Var}(\varepsilon_{i,T}|H_{i,T}) \hat{\gamma}_{i,T}^2}} \quad | \quad \sigma(F_T, D_T) \rightarrow_d \mathcal{N}(0, 1)
\]

\[
\frac{\sum_{i=1}^{n} \hat{\gamma}_{i,t} \nu_{i,t}}{\sqrt{\sum_{i=1}^{n} \hat{\gamma}_{i,t}^2 \text{Var}(\nu_{i,t}|F_t, D_t)}} \quad | \quad \sigma(F_t, D_t) \rightarrow_d \mathcal{N}(0, 1), \quad \forall t \in \{1, \ldots, T-1\}
\]

(C.4)

Notice now that \(\sigma(F_t, D_t)\) consistute a filtration and that

\[
\mathbb{E}[\hat{\gamma}_{i,t} \varepsilon_{i,T} \hat{\gamma}_{i,t} \nu_{i,t}|F_T, D_T] = 0
\]

\[
\mathbb{E}[\hat{\gamma}_{i,t} \hat{\gamma}_{i,s} \nu_{i,s} \hat{\gamma}_{i,t} \nu_{i,t}|F_{\max\{s,t\}}, D_{\max\{s,t\}}] = 0.
\]

(C.5)

Since each component at time \(t\) converges conditionally on the filtration \(\sigma(F_t, D_t)\) and each component is measurable with respect to \(\sigma(F_{t+1}, D_{t+1})\), it follows the joint convergence result

\[
\begin{bmatrix} Z_1, \cdots, Z_T \end{bmatrix}^\top \rightarrow_d \mathcal{N}(0, I),
\]

\[
Z_t = \frac{\sum_{i=1}^{n} \hat{\gamma}_{i,t} \nu_{i,t}}{\sqrt{\sum_{i=1}^{n} \hat{\gamma}_{i,t}^2 \text{Var}(\nu_{i,t}|F_t, D_t)}}, \quad t \in \{1, \ldots, T-1\},
\]

\[
Z_T = \frac{\sum_{i=1}^{n} \hat{\gamma}_{i,T} \varepsilon_{i,T}}{\sqrt{\sum_{i=1}^{n} \text{Var}(\varepsilon_{i,T}|H_{i,T}) \hat{\gamma}_{i,T}^2}}.
\]

We are left to consider the components \((ii), (iv)\). Define

\[
W_T = \sqrt{\frac{\sum_{i=1}^{n} \hat{\gamma}_{i,T}^2 \text{Var}(\varepsilon_{i,T}|H_{i,T-1})}{\sum_{i=1}^{n} \hat{\gamma}_{i,T}^2 \text{Var}(\varepsilon_{i,T}|H_{i,T-1}) + \sum_{i=1}^{n} \sum_{t=1}^{T-1} \hat{\gamma}_{i,t}^2 \text{Var}(\nu_{i,t}|H_{i,t})}}
\]

\[
W_t = \frac{\sum_{i=1}^{n} \hat{\gamma}_{i,t}^2 \text{Var}(\nu_{i,t}|H_{i,t})}{\sqrt{\sum_{i=1}^{n} \hat{\gamma}_{i,T}^2 \text{Var}(\varepsilon_{i,T}|H_{i,T-1}) + \sum_{i=1}^{n} \sum_{t=1}^{T-1} \hat{\gamma}_{i,t}^2 \text{Var}(\nu_{i,t}|H_{i,t})}}, \quad t \in \{1, \ldots, T-1\}.
\]

Note that \(||W||_2 = 1\). Note also that we can write the expression \((I)\) as \(\sum_{t=1}^{T} Z_t W_t\).
Therefore we write for any \( t \geq 0 \),

\[
P\left( \left| \sum_{t=1}^{T} W_t Z_t \right| > t \right) \leq P\left( \left\| W \right\|_2 \sqrt{\sum_{t=1}^{T} Z_t^2} > t \right) = P\left( \sum_{t=1}^{T} Z_t^2 > t^2 \right),
\]

where the last equality follows from the fact that \( \left\| W \right\|_2 = 1 \). Note now that since \( Z_t \) are independent standard normal, \( \sum_{t=1}^{T} Z_t^2 \) is chisquared with \( T \) degrees of freedom.

To complete the claim, we are only left to show that \((II) \rightarrow_p 1\) to then invoke Slutsky theorem.

**Term (II)** We can write

\[
| (II)^2 - 1 | = \left| \sum_{i=1}^{n} \tilde{\gamma}_{i,T}^2 (Y_{i,T} - H_{i,T} \beta_{T}^T)^2 + \sum_{t=1}^{T-1} \frac{\tilde{\gamma}_{i,t}^2 \nu_{i,t}^2}{\sum_{i=1}^{n} \tilde{\gamma}_{i,T}^2 \text{Var}(\epsilon_{i,T} | H_{i,T-1}) + \sum_{i=1}^{n} \sum_{t=1}^{T-1} \tilde{\gamma}_{i,t}^2 \text{Var}(\nu_{i,t} | H_{i,t})} - 1 \right|
\]

\[
\leq \left| n \sum_{i=1}^{n} \tilde{\gamma}_{i,T}^2 \text{Var}(\epsilon_{i,T} | H_{i,T-1}) + n \sum_{i=1}^{n} \sum_{t=1}^{T-1} \tilde{\gamma}_{i,t}^2 \text{Var}(\nu_{i,t} | H_{i,t}) - 1 \right|
\]

\[ (A) \]

\[
\leq \left| n \sum_{i=1}^{n} \tilde{\gamma}_{i,T}^2 \left[ (Y_{i,T} - H_{i,T} \beta_{T}^T)^2 - (Y_{i,T} - H_{i,T} \beta_{T}^T)^2 \right] \right|
\]

\[ (B) \]

\[
+ \left| \sum_{t=1}^{T-1} \left[ (H_{i,t+1} \beta_{t+1} - H_{i,t} \beta_{t+1}^T)^2 - (H_{i,t+1} \beta_{t+1} - H_{i,t} \beta_{t+1}^T)^2 \right] \right|.
\]

\[ (C) \]

To show that \((A)\) converges it suffices to note that the denominator is bounded from below by a finite positive constant by Lemmas B.1, B.2 and the fact that each variance component is bounded away from zero under Assumption 9. The conditional variance of each component in the numerator reads as follows (recall by the above
lemma that \( n||\hat{\gamma}_t||^2 = O_p(1) \)
\[
\text{Var}\left( n \sum_{i=1}^{n} \gamma_{i,T}^2 \epsilon_{i,T} | H_T \right) \leq n^2 \tilde{C} \|\gamma_T\|_2^4 \leq \log^2(n) n^2 \tilde{C} n^{-4/3} \|\gamma_T\|_2^2 = O_p(1) \log^2(n) nn^{-4/3} = o_p(1),
\]
\[
\text{Var}\left( n \sum_{i=1}^{n} \gamma_{i,T}^2 \nu_{i,t} | H_t \right) \leq \tilde{C} n^2 \|\gamma_T\|_2^4 \leq n^2 \log^2(n) \tilde{C} n^{-4/3} \|\hat{\gamma}_t||_2^2 = O_p(1) \log^2(n) nn^{-4/3} = o_p(1)
\]
and hence (A) converges to zero by the continuous mapping theorem. For the term (B), the denominator is bounded from below away from zero as discussed for (A).

The numerator is
\[
n \sum_{i=1}^{n} \hat{\gamma}_{i,T}^2 \left[ (Y_{i,T} - H_{i,T} \hat{\beta}^T)^2 - (Y_{i,T} - H_{i,T} \beta^T)^2 \right] \leq n \sum_{i=1}^{n} \hat{\gamma}_{i,T}^2 \left( H_{i,T} (\hat{\beta}^T - \beta^T) \right)^2 \]
(C.7)

We can now write
\[
n \sum_{i=1}^{n} \hat{\gamma}_{i,T}^2 \left( H_{i,T} (\hat{\beta}^T - \beta^T) \right)^2 \leq ||\hat{\beta}^T - \beta^T||_2^2 n ||\hat{\gamma}_T||^2 \max_i |H_{i,T}||^2_{\infty}.
\]

Notice now that by sub-gaussianity, with probability 1\(-1/n\), we have \( \max_i |H_{i,T}||_{\infty} = O(\log(np))^{34} \) and \( ||\hat{\beta}^T - \beta^T||_1 = o_p(n^{-1/4}) \), \( n ||\hat{\gamma}_T||^2 = O_p(1) \) and \( \log(np)/n^{1/4} = o(1) \) the above expression is \( o_p(1) \). Consider now
\[
n \sum_{i=1}^{n} \hat{\gamma}_{i,t}^2 \left[ (H_{i,t+1} \beta^{t+1} - H_{i,t} \hat{\beta}^{t+1})^2 - (H_{i,t+1} \beta^t - H_{i,t} \beta^t)^2 \right] \leq n \sum_{i=1}^{n} \hat{\gamma}_{i,t}^2 \left( H_{i,t} (\beta^t - \hat{\beta}^t) \right)^2
\]
which is \( o_p(1) \) similarly to the term in Equation (C.7).

**Rate of convergence** is \( n^{-1/2} \). To study the rate of convergences it suffices to show that (for fixed \( T \))
\[
n \left[ \sum_{i=1}^{n} \hat{\gamma}_{i,T}^2 \text{Var}(\epsilon_{i,T} | H_{i,T-1}) + \sum_{i=1}^{n} \sum_{t=1}^{T-1} \hat{\gamma}_{i,t}^2 \text{Var}(\nu_{i,t} | H_{i,t}) \right] = O(1).
\]

\(^{34}\)To not this, we can write \( P(\max_{i,j} |H_{i,T}^{(j)}| > t) \leq npP(|H_{i,T}^{(j)}| > t) \leq npe^{-t^2v} \) for some finite constant \( v \). Setting \( npe^{-t^2v} = 1/n \) the claim holds.

64
This follows directly from Lemma B.2, B.1 and the bounded conditional third moment assumption in Assumption 9.

C.1 Proof of Theorem 4.5

The proof of the corollary follows similarly to the proof of Theorem 4.4. In particular, note that we can write

\[
\sqrt{\sum_{d \in \{d_{1:T}, d'_{1:T}\}} \sum_{i=1}^{n} \hat{\gamma}^2_{i,T}(d)(Y_{i,T} - H_{i,T}\beta_d^T)^2 + \sum_{t=1}^{T-1} \hat{\gamma}^2_{i,t}(d)\hat{\nu}^2_{i,t}(d)}
\]

\[
= \frac{\hat{\mu}(d_{1:T}) - \hat{\mu}(d'_{1:T}) - \hat{\mu}(d'_{1:T}) - \hat{\mu}(d_{1:T}) - \hat{\beta}_{d_{1:T}} - \hat{\beta}_{d'_{1:T}}}{\sqrt{\sum_{d \in \{d_{1:T}, d'_{1:T}\}} \sum_{i=1}^{n} \hat{\gamma}^2_{i,T}(d)(Y_{i,T} - H_{i,T}\beta_d^T)^2 + \sum_{t=1}^{T-1} \hat{\gamma}^2_{i,t}(d)\hat{\nu}^2_{i,t}(d)}}
\]

\[
\times \frac{\sqrt{\sum_{d \in \{d_{1:T}, d'_{1:T}\}} \sum_{i=1}^{n} \hat{\gamma}^2_{i,T}(d)(Y_{i,T} - H_{i,T}\beta_d^T)^2 + \sum_{t=1}^{T-1} \hat{\gamma}^2_{i,t}(d)\hat{\nu}^2_{i,t}(d)}}{\sum_{d \in \{d_{1:T}, d'_{1:T}\}} \sum_{i=1}^{n} \hat{\gamma}^2_{i,T}(d)(Y_{i,T} - H_{i,T}\beta_d^T)^2 + \sum_{t=1}^{T-1} \hat{\gamma}^2_{i,t}(d)\hat{\nu}^2_{i,t}(d)}
\]

(C.8)

The component (II) converges in probability to one as discussed in the proof of Theorem 4.4. The component (I) behaves similarly to the component (I) in Theorem 4.4 following verbatim the same argument with a single modification: here (I) can be written as

\[
Z_t(d_{1:T})W_t + Z_t(d'_{1:T})W'_t,
\]

where

\[
Z_t(d_{1:T}) = \frac{\sum_{i=1}^{n} \hat{\gamma}_{i,t}(d_{1:T})\nu_{i,t}(d_{1:T})}{\sqrt{\sum_{i=1}^{n} \hat{\gamma}^2_{i,t}(d_{1:T})\nu^2_{i,t}(d_{1:T})|F_t, D_t}}, \quad t \leq T - 1,
\]

\[
Z_T(d_{1:T}) = \frac{\sum_{i=1}^{n} \hat{\gamma}_{i,T}(d_{1:T})\epsilon_{i,T}}{\sqrt{\sum_{i=1}^{n} \nu^2_{i,T}(d_{1:T})|H_T, \hat{\gamma}^2_{i,T}}},
\]

65
Proof of Theorem C.1.
for constants \( \{c_t \} \), similarly \( W_T \) of Lemma B.2, the asymptotic limits correspond that the conditions in Theorem 4.5 hold. Suppose in addition that for all \( t \in \{1, \cdots, T-1 \} \), \( \sum_{i=1}^{n} \gamma_{i,t}^2 \text{Var}(\varepsilon_{i,t} | F_{t-1}) \to_{as} c_t \), \( \sum_{i=1}^{n} \gamma_{i,t}^2 \text{Var}(\nu_{i,t} | F_{t-1}) \to_{as} C_T \) for constants \( \{c_t \} \). Then, whenever \( \log(np)/n^{1/4} \to 0 \) with \( n,p \to \infty \),
\[
(\hat{V}_T(d_{1:T}) + \hat{V}_T(d'_{1:T}))^{-1/2} \sqrt{n} \left( \hat{\mu}(d_{1:T}) - \hat{\mu}(d'_{1:T}) - \text{ATE}(d_{1:T}, d'_{1:T}) \right) \to_d \mathcal{N}(0,1). \tag{C.9}
\]

Proof of Theorem C.1. The proof follows verbatim from the proof of Theorem 4.5, while here the components \( W_t \to_{a.s.} c_t, W'_t \to_{a.s.} c'_t \) for constants \( c_t, c'_t \). Note that by Lemma B.2, the asymptotic limits \( c_t \) must be finite since
\[
\sum_{i=1}^{n} \gamma_{i,t}^2 \text{Var}(\nu_{i,t} | F_{t-1}) \leq \bar{n} \| \gamma_t \|^2 = O_p(1),
\]
where \( \bar{u} \) is a finite constant by Assumption 9 (ii). Following the same argument as in the proof of Theorem 4.5, we obtain that the left-hand side of Equation (C.9) converges to

\[
\sum_{t=1}^{T} c_t Z_t - \sum_{t=1}^{T} c'_t Z'_t, \quad (Z_1, \cdots, Z_T, Z'_1, \cdots, Z'_T) \sim N(0, I).
\]

The variance is therefore \( \sum_{t=1}^{T} c_t^2 + \sum_{t=1}^{T} c'_t^2 = 1 \), since \( ||(W, -W)||^2 = 1 \) as discussed in the proof of Theorem 4.5.

\[ \square \]

## D Additional Algorithms

Algorithm D.1 presents estimation of the coefficients for multiple periods. Its extensions for a linear model on the treatment assignments (hence using all in-sample information) follows similarly to Algorithm 2. Algorithm D.2 presents the choice of the tuning parameters. The algorithm imposes stricter tuning on those covariates whose coefficients are non-zero. Whenever many coefficients (more than one-third) are non-zero, we impose a stricter balancing on those with the largest size.\(^{35}\)

### Algorithm D.1 Coefficients estimation with multiple periods

**Require:** Observations, history \((d_{1:t},)\), model \(\in\{\text{full interactions, linear}\}\).

1. if model = full interactions then
2. Estimate \(\hat{\beta}_{d_{1:T}}\) by regressing \(Y_{i,T}\) onto \(H_{i,T}\) for \(i\) with \(D_{1:T} = d_{1:T}\).
3. for \(t \in \{T - 1, \cdots, 1\}\) do
4. \hspace{1em} Estimate \(\hat{\beta}_{d_{1:T}}\) by regressing \(H_{i,t+1}\hat{\beta}_{d_{1:T}}\) onto \(H_{i,t}\) for \(i\) that has the treatment history \((d_{1:t})\).
5. \hspace{1em} end for
6. else
7. \hspace{1em} Estimate \(\hat{\beta}\) by regressing \(Y_{i,T}\) onto \((H_{i,T}, D_{i,T})\) for all \(i\) (without penalizing \((D_{i,1:T})\)) and define \(H_{i,T}\hat{\beta}_{d_{1:T}} = (H_{i,T}, d_T)\hat{\beta}\) for all \(i : D_{i,1:(T-1)} = d_{1:(T-1)}\);
8. \hspace{1em} Repeat sequentially as in Algorithm 2
9. end if

\(^{35}\)E.g., 60 coefficients are prioritized for \(T = 2, p = 100\) design, since the dimension is \(Tp\).
Algorithm D.2 Tuning Parameters for DCB

Require: Observations \( \{Y_{i,1}, X_{i,1}, D_{i,1}, ..., Y_{i,T}, X_{i,T}, D_{i,T}\} \), \( \delta_t(n,p) \), treatment history \((d_{1:T}), L_t, U_t\), grid length \(G\), number of grids \(R\).

1: Estiamte coefficients as in Algorithm D.1 and let \( \hat{\gamma}_{i,0} = 1/n \);
2: Define \(R\) grids of length \(G\), denoted as \(G_1, ..., G_R\), equally between \(L_t\) an \(U_t\).
3: Define \(S_1 = \{j : |\hat{\beta}_t(j)| \neq 0\}, \quad S_2 = \{j : |\hat{\beta}_t(j)| = 0\}\).
4: (Non-sparse regression): if \(|S_1|\) is too large (i.e., \(> \dim(\hat{\beta}_t)/3\)), select \(S_1\) the set of the 1/3rd largest coefficients in absolute value and \(S_2 = S_1^c\).
5: \textbf{for each} \(s_1 \in 1 : G\) \textbf{do}
6: \quad \textbf{for each} \(K_{1,t}^a \in G_{s_1}\) \textbf{do}
7: \quad \quad \textbf{for each} \(K_{1,t}^b \in G_{s_1}\) \textbf{do}
8: \quad \quad \quad \text{Let } \hat{\gamma}_{i,t} = 0, \text{ if } D_{i,1:t} \neq d_{1:t} \text{ and define } \hat{\gamma}_t := \arg\min_{\gamma_t} \sum_{i=1}^n \gamma_{i,t}^2
\quad \text{s.t. } \left| \frac{1}{n} \sum_{i=1}^n \hat{\gamma}_{i,t-1} H_{i,t}^{(j)} - \gamma_{i,t} H_{i,t}^{(j)} \right| \leq K_{1,t}^a \delta_t(n,p), \quad \forall j : \hat{\beta}_t(j) \in S_1
\quad \quad \left| \frac{1}{n} \sum_{i=1}^n \hat{\gamma}_{i,t-1} H_{i,t}^{(j)} - \gamma_{i,t} H_{i,t}^{(j)} \right| \leq K_{1,t}^b \delta_t(n,p) \quad \forall j : \hat{\beta}_t(j) \in S_2 \tag{D.1}
\quad \quad \sum_{i=1}^n \gamma_{i,t} = 1, \quad ||\gamma_t||_{\infty} \leq \log(n) n^{-2/3}, \gamma_{i,t} \geq 0.
9: \quad \textbf{Stop if: } a \text{ feasible solution exists.}
10: \quad \textbf{end for}
11: \textbf{end for}
12: \textbf{end for}
13: \textbf{return } \hat{\mu}_{T}(d_{1:T})

E Simulations under misspecification

We simulate the outcome model over each period using non-linear dependence between the outcome, covariates, and past outcomes. The function that we choose for the dependence of the outcome with the past outcome and covariates follows similarly to Athey et al. (2018), where, differently, here, such dependence structure is applied not only to the first covariate only (while keeping a linear dependence with
the remaining ones) but to all covariates, making the scenarios more challenging for
the DCB method. Formally, the DGP is the following:

\[
Y_2(d_1, d_2) = \log(1 + \exp(-2 - 2X_1\beta_{d_1,d_2})) + \log(1 + \exp(-2 - 2X_2\beta_{d_1,d_2})) \\
+ \log(1 + \exp(-2 - 2Y_1)) + d_1 + d_2 + \varepsilon_2,
\]

and similarly for \(Y_3(d_1, d_2, d_3)\), with also including covariates and outcomes in period
\(T = 2\). Coefficients \(\beta\) are obtained from the sparse model formulation discussed in
the main text. Results are collected in Table E.2. Interestingly, we observe that
DCB performs relatively well under the misspecified model, even if our method does
not use any information on the propensity score. We also note that our adaptation
of the double lasso to dynamic setting performs comparable or better in the presence
of two periods only or a sparse structure. However, as the number of periods increase
or sparsity decreases Double Lasso’s performance deteriorates.

Table E.1: MSE under misspecified model in a sparse setting.

|       | \(T = 2\)  | \(T = 3\)  |
|-------|------------|------------|
|       | \(\eta = 0.3\) | \(\eta = 0.5\) | \(\eta = 0.3\) | \(\eta = 0.5\) |
| DCB   | 0.238      | 0.354      | 0.751      | 0.402      |
| aIPW* | 0.434      | 0.802      | 1.363      | 1.622      |
| aIPWh | 0.863      | 1.363      | 1.882      | 2.464      |
| CAEW (MSM) | 0.815 | 1.364 | 7.889 | 8.675 |
| D. Lasso | 0.121    | 0.142    | 0.689    | 0.503    |
| Seq.Est | 0.811    | 0.346    | 2.288    | 2.031    |
Table E.2: MSE under misspecified model in a moderately sparse setting.

|                | $T = 2$ |        | $T = 3$ |        |
|----------------|---------|--------|---------|--------|
|                | $\eta = 0.3$ | $\eta = 0.5$ | $\eta = 0.3$ | $\eta = 0.5$ |
| DCB            | 0.212   | 0.256  | 0.326   | 0.384  |
| aIPW*          | 0.428   | 0.789  | 1.364   | 1.616  |
| aIPWh          | 0.826   | 1.313  | 1.857   | 2.434  |
| CAEW (MSM)     | 0.781   | 1.317  | 7.833   | 8.616  |
| D. Lasso       | 0.115   | 0.133  | 0.675   | 0.494  |
| Seq.Est        | 0.847   | 0.366  | 2.316   | 2.058  |