ORLICZ EXTENSION OF NUMERICAL RADIUS INEQUALITIES

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Abstract. In this paper, we achieve new and improved numerical radius inequalities of operators defined on a Hilbert space by using Orlicz function and Hermite-Hadamard inequality. The upper bounds of various inequalities involving numerical radii have been obtained. Finally, we compute an upper bound of the numerical radius for block matrices of the form
\[
\begin{bmatrix}
O & P \\
Q & O
\end{bmatrix},
\]
where \(P, Q\) are any bounded linear operators on a Hilbert space.

1. Introduction

The concepts of numerical range and numerical radius of an operator play a very significant role and have been studied extensively due to their enormous applications in engineering, quantum computing, quantum mechanics, numerical analysis, differential equations, fluid dynamics, the geometry of Banach spaces, etc (see [2], [6], [19]). The study of numerical radius inequalities and its various improvements are the latest trends of research in the theory of operators on Hilbert space.

We denote \(\mathcal{B}(\mathcal{H})\) as the \(C^*\)-algebra of all bounded linear operators on a Hilbert space \(\mathcal{H}\). For any bounded linear operator \(A\) on \(\mathcal{H}\) the numerical radius, denoted by \(w(A)\), yields a norm on \(\mathcal{B}(\mathcal{H})\). The fundamental inequality between numerical radius and operator norm of any operator \(A \in \mathcal{B}(\mathcal{H})\) is as follows:

(1.1) \[ \frac{1}{2} \|A\| \leq w(A) \leq \|A\|. \]

This says that the operator norm and the norm induced by the numerical radius are equivalent on \(\mathcal{B}(\mathcal{H})\). If \(A\) is a normal operator (that is, \(A^*A = AA^*\)), then \(w(A) = \|A\|\) and if \(A^2 = O\), the zero operator on \(\mathcal{H}\), then we have \(w(A) = \frac{1}{2} \|A\|\). Many researchers later discovered sharp inequalities for a single operator.

Some preliminary results on numerical radius and its applications to stability theory of finite-difference approximations for hyperbolic initial value problems were presented by Goldberg and Tadmor [13], (see also [12]). The study of power inequalities for numerical range, and numerical radius (that is, \(w(A^n) \leq w(A)^n, n \in \mathbb{N}, A \in \mathcal{B}(\mathcal{H})\)) were studied in [3] and [26] (see also [15]), respectively. The first refinement of inequality (1.1) was obtained by Kittaneh.
We mention some results as below:

\[ w(A) \leq \frac{1}{2}(\|A\| + \|A^2\|^{1/2}). \]  

(1.2)

In 2005, Kittaneh [22] further improved the inequality (1.1) as follows: For \( A \in \mathcal{B}(\mathcal{H}) \)

\[ \frac{1}{4}\|A\|^2 + |A^*|^2 \leq w^2(A) \leq \frac{1}{2}\|A\|^2 + |A^*|^2. \]  

(1.3)

Further improvement of the upper bound of inequality (1.3) was provided by El-Haddad and Kittaneh [10] and they established the following:

For \( A \in \mathcal{B}(\mathcal{H}) \) and \( r \geq 1 \)

\[ w^r(A) \leq \frac{1}{2}\|A\|^{2r\alpha} + |A^*|^{2r(1-\alpha)} \]  

and

\[ w^{2r}(A) \leq \|\alpha|A|^{2r} + (1-\alpha)|A^*|^{2r}\|, \]

where \( 0 \leq \alpha \leq 1 \). Further refinement of the upper bounds of both the inequalities (1.2) and (1.3) was obtained by Abu-Omar and Kittaneh [1] and their result is stated as below:

\[ w^2(A) \leq \frac{1}{4}\|A\|^2 + |A^*|^2 + \frac{1}{2}w(A^2). \]  

(1.4)

A stronger version of the inequalities (1.2) and (1.3) for the upper bounds is recently obtained by Bhunia and Paul [4]:

For \( A \in \mathcal{B}(\mathcal{H}) \) and any \( r \geq 1 \)

\[ w^{2r}(A) \leq \frac{1}{4}\|A\|^{2r} + |A^*|^{2r} + \frac{1}{2}w(|A|^r|A^*|^r). \]  

(1.5)

The above results show a continuous stream of research on the numerical radius inequalities and its recent improvements and generalizations for a single operator \( A \in \mathcal{B}(\mathcal{H}) \). If \( A, B \in \mathcal{B}(\mathcal{H}) \), then Holbrook [18] proved that \( w(AB) \leq 4w(A)w(B) \). In addition, if \( A \) and \( B \) commute, then \( w(AB) \leq 2w(A)w(B) \), where the constant 2 is sharp. Also the following result is due to Fong and Holbrook [11] is worth mentioning:

\[ w(AB + BA) \leq 2\sqrt{2}w(A)\|B\|. \]

Kittaneh [22] established a generalized numerical radius inequality: Let \( A, B, C, D, S, \) and \( T \in \mathcal{B}(\mathcal{H}) \). Then

\[ w(ATB + CSD) \leq \frac{1}{2}\|A|T^*|^{2(1-\alpha)}A^* + B^*|T|^{2\alpha}B + C|S^*|^{2(1-\alpha)}C^* + D^*|S|^{2\alpha}D\| \]

for \( 0 \leq \alpha \leq 1 \). In [16], and [17], the authors computed the numerical radius of product of operators, namely, \( w(A^*XA) \), \( w(A^*XB \pm B^*YA) \) in terms of operator norm and numerical radius of a block matrix of the form \( \begin{bmatrix} O & P \\ Q & O \end{bmatrix} \). There are many recent research papers (e.g. [1], [4], [9], [28], [30] and references cited therein) on the numerical radius inequalities with its improvements and generalizations.

In this article, we attempt to unify as well as extend the above numerical radius inequalities for a large class of operators. We also establish some new numerical radius inequalities.
with improvements of many known results. Further, we provide a general upper bound for numerical radius of block matrices. The main guiding tools used in this article are the various operator inequalities, the notion of functional calculus, and the Orlicz function.

The paper is organized as follows: In Section 2, we discuss preliminaries and some important inequalities. Section 3 deals with various numerical radius inequalities of Hilbert space operators using Orlicz functions and Hermite-Hadamard inequality illustrated with some concrete examples on the Hardy space. In Section 4, we discuss the upper bound of numerical radius for the off-diagonal block matrix of operators.

2. Preliminaries

Throughout this paper, $\mathcal{H}$ denotes a separable complex Hilbert space and $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators from $\mathcal{H}$ to itself. For any $A \in \mathcal{B}(\mathcal{H})$, $A^*$ denotes the adjoint of $A$ and $|A| := (A^*A)^{1/2}$. We call an operator $A$ on a Hilbert space $\mathcal{H}$ is an isometry if $A^*A = I$.

We say an operator $A \in \mathcal{B}(\mathcal{H})$ is positive if $A$ is self adjoint and $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. The operator norm, denoted by $\|A\|$, is defined as

$$
\|A\| = \sup\{\|Ax\| : x \in \mathcal{H} \text{ with } \|x\| = 1\}.
$$

We write the cartesian decomposition of $A$ by $A = R(A) + iI(A)$, where $R(A) = \frac{1}{2}(A + A^*)$ and $I(A) = \frac{1}{2i}(A - A^*)$. For any $A \in \mathcal{B}(\mathcal{H})$, the numerical range of $A$, denoted by $W(A)$, is defined as

$$
W(A) := \{\langle Ax, x \rangle : x \in \mathcal{H}, \|x\| = 1\}.
$$

Then the numerical radius of $A$, denoted by $w(A)$, is given by

$$
w(A) = \sup\{|\langle Ax, x \rangle| : x \in \mathcal{H}, \|x\| = 1\}.
$$

Similarly, the Crawford number of $A$ is denoted by $c(A)$, and is defined as below:

$$
c(A) = \inf\{|\langle Ax, x \rangle| : x \in \mathcal{H}, \|x\| = 1\}.
$$

Before proceeding further, let us recall the notion of functional calculus given in [8] and the Orlicz function. Let $A$ be a normal operator on $\mathcal{H}$ and $\sigma(A)$ denote the spectrum of $A$. Let $B_\infty(\sigma(A))$ and $C(\sigma(A))$ be the bounded measurable complex-valued functions and the continuous complex-valued functions on $\sigma(A) \subseteq \mathbb{C}$, respectively. Clearly, $B_\infty(\sigma(A))$ is a $C^*$-algebra with the involution map defined by $f \mapsto \bar{f}$. Let $\pi : C(\sigma(A)) \to \mathcal{B}(\mathcal{H})$ be a $*$-homomorphism such that $\pi(1) = I_\mathcal{H}$, where 1 is a constant function with value one and $I_\mathcal{H}$ is an identity operator on $\mathcal{H}$. Then there exists a unique spectral measure $\mathcal{P}$ in $(\sigma(A), \mathcal{H})$ such that

$$
\pi(f) = \int_{\sigma(A)} f d\mathcal{P}.
$$

In particular, if $A \in \mathcal{B}(\mathcal{H})$ is a positive operator, then $\sigma(A)$ is a subset of $[0, \infty)$ and $f(A)$ is a positive operator for any positive continuous function whose domain contains the spectrum of $A$. 
A map $\varphi : [0, \infty] \to [0, \infty]$ is said to be an Orlicz function (non-degenerate) if it is convex, continuous, non-decreasing, with $\varphi(0) = 0$, $\varphi(u) > 0$ when $u > 0$ and $\varphi(u) \to \infty$ as $u \to \infty$ (see Definition 4.a.1., [24]). An Orlicz function is said to be degenerate if $\varphi(u) = 0$ for some $u > 0$. Throughout the paper wherever Orlicz function appears, we mean it non-degenerate. Some concrete examples of Orlicz functions are $(i)$ $\varphi(u) = u^p$, $p \geq 1$, $(ii)$ $\varphi_r(u) = e^{ur} - 1$, $1 < r < \infty$ $(iii)$ $\varphi_p(u) = \frac{u^p}{\ln(e+u)}$, $p \geq 2$ and many more. An Orlicz function can be expressed by the following integral representation:

$$\varphi(u) = \int_0^u p(t)dt,$$

where $p$ is a non-decreasing function such that $p(0) = 0$, $p(t) > 0$ for $t > 0$, $\lim_{t \to \infty} p(t) = \infty$, and is known as kernel of $\varphi$. These restrictions exclude the case only when $\varphi(u)$ is equivalent to the function $u$. The right inverse $q$ of $p$ is defined as $q(s) = \sup \{ t : p(t) \leq s \}$, $s \geq 0$. Then $q$ satisfy similar properties as $p$. Using it following complementary Orlicz function $\psi$ to $\varphi$ is defined as below:

$$\psi(v) = \int_0^v q(s)ds.$$

The pair $(\varphi, \psi)$ is called mutually complementary Orlicz functions.

The following inequalities are useful in the sequel. First, we recall the well known Hermite-Hadamard inequality [14] for a convex function $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$. Suppose that $a, b \in I$ such that $a < b$. Then Hermite-Hadamard inequality states that

$$\varphi\left(\frac{a+b}{2}\right) \leq \int_0^1 \varphi(ta + (1-t)b)dt \leq \frac{\varphi(a) + \varphi(b)}{2},$$

which is used to find the sharp inequalities.

**Lemma 2.1.** (See Theorem 1.2, [27]) Let $A$ be any self-adjoint operator on a Hilbert space $\mathcal{H}$ and $\varphi$ be a convex function such that the spectrum of $A$ contained in $[0, \infty]$. Then the operator version of Jensen inequality states that

$$\varphi(\langle Ax, x \rangle) \leq \langle \varphi(A)x, x \rangle \quad \text{(for every unit vector } x \in \mathcal{H}).$$

An easy consequence of Lemma 2.1 is the Hölder-McCarthy inequality, which is stated as below:

**Lemma 2.2.** (See Theorem 1.4, [27]) Let $A$ be any positive operator on a Hilbert space $\mathcal{H}$ and $r \geq 1$ be any real number. Then for any unit vector $x \in \mathcal{H}$,

$$\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle.$$

**Lemma 2.3.** ([23], Theorem 1) Let $A \in \mathcal{B}(\mathcal{H})$, $f, g$ be non-negative continuous functions on $[0, \infty]$ such that $f(t)g(t) = t$ for all $t \in [0, \infty]$. Then

$$|\langle Ax, y \rangle| \leq \|f(|A|)x\|\|g(|A^*|)y\|$$

holds for all $x, y \in \mathcal{H}$.

In particular, if $f(t) = t^\alpha$ and $g(t) = t^{2(1-\alpha)}$ for $0 \leq \alpha \leq 1$, then from Lemma 2.3 one gets the following special form.
Lemma 2.4. [20] Let $A$ be any bounded linear operator on a Hilbert space $\mathcal{H}$. Then for $x, y \in \mathcal{H}$
\[ |\langle Ax, y \rangle|^2 \leq \langle |A|^{2\alpha} x, x \rangle \langle |A^*|^{2(1-\alpha)} y, y \rangle, \]
where $0 \leq \alpha \leq 1$.

Lemma 2.5. [7] Suppose that $x, y, e \in \mathcal{H}$ such that $\|e\| = 1$. Then the following holds:
\[ |\langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{2}(\|x\|\|y\| + |\langle x,y \rangle|). \]

Lemma 2.6. [24] (Young’s inequality) Let $\varphi$, and $\psi$ be two complementary Orlicz functions. Then
(i) for $u, v \geq 0$, $uv \leq \varphi(u) + \psi(v)$, and
(ii) for $u \geq 0$, $up(u) = \varphi(u) + \psi(p(u))$ (equality condition).

Lemma 2.7. Let $\varphi$ be an Orlicz function and suppose that $a_i \geq 0$ for $i = 1, 2, \ldots, n$. Then
\[ \varphi\left(\sum_{i=1}^{n} \frac{a_i}{n}\right) \leq \frac{1}{n} \sum_{i=1}^{n} \varphi(a_i). \]

Proof. Let $\alpha \in [0, 1]$ and $\varphi$ be an Orlicz function. Then using non-decreasing property of the kernel function $p(t)$, we have the following:
\[ \varphi(\alpha u) = \int_{0}^{\alpha u} p(t)dt = \alpha \int_{0}^{u} p(\alpha v)dv \leq \alpha \int_{0}^{u} p(v)dv = \alpha \varphi(u), \]
which is an important and well-known property of $\varphi$ (see [24]) and we have recalled it here for the sake of completeness. Hence the desired inequality follows by choosing $\alpha$ and $u$ suitably.

Remark 2.1. (see [29]) An easy consequence of Lemma 2.7 is the Bohr’s inequality, which says that for $a_i \geq 0$ for $i = 1, 2, \ldots, n$ and $\varphi(t) = t^r$, $r \geq 1$ we have
\[ \left(\sum_{i=1}^{n} \frac{a_i}{n}\right)^r \leq \frac{1}{n} \sum_{i=1}^{n} a_i^r. \]

3. Main Results

In this section, we unify the earlier results on numerical radius inequalities as well as extend the results for a larger class of Hilbert space operators using Orlicz function and functional calculus. This also yields simple proofs of many of the earlier results. Let us begin with the following result.

Theorem 3.1. Suppose that $A \in \mathcal{B}(\mathcal{H})$. Then for $0 \leq \alpha, t \leq 1$ and any Orlicz function $\varphi$
\[ \varphi(w(A)) \leq \left\| \int_{0}^{1} \varphi(t|A|^{2\alpha} + (1 - t)|A^*|^{2(1-\alpha)})dt \right\| \leq \frac{1}{2}\|\varphi(|A|^{2\alpha}) + \varphi(|A^*|^{2(1-\alpha)})\|. \]
Proof. Choose \( x \in \mathcal{H} \) such that \( \|x\| = 1 \). Given \( \alpha, t \in [0, 1] \). By using Lemma 2.1, Lemma 2.3 AM-GM inequality, and inequality (2.1) we have the following:

\[
\varphi(|\langle Ax, x \rangle|) \leq \varphi(\langle |A|^{2\alpha} x, x \rangle) \leq \frac{1}{2} \left( \langle |A|^{2\alpha} x, x \rangle + \langle |A^*|^{2(1-\alpha)} x, x \rangle \right)
\]

\[
\leq \int_0^1 \varphi(t \langle |A|^{2\alpha} x, x \rangle + (1 - t) \langle |A^*|^{2(1-\alpha)} x, x \rangle) \, dt
\]

\[
= \int_0^1 \varphi(\langle (t|A|^{2\alpha} + (1 - t)|A^*|^{2(1-\alpha)}) x, x \rangle) \, dt
\]

\[
\leq \int_0^1 \langle (t|A|^{2\alpha} + (1 - t)|A^*|^{2(1-\alpha)}) x, x \rangle \, dt
\]

\[
= \left\langle \int_0^1 (\varphi(t|A|^{2\alpha} + (1 - t)|A^*|^{2(1-\alpha)}) \right. \, dt, x, x \rangle.
\]

Since \( \varphi \) is convex, so we have \( \varphi(t|A|^{2\alpha} + (1 - t)|A^*|^{2(1-\alpha)}) \leq t\varphi(|A|^{2\alpha}) + (1 - t)\varphi(|A^*|^{2(1-\alpha)}) \). Therefore

\[
\varphi(|\langle Ax, x \rangle|) \leq \left\langle \int_0^1 (t|A|^{2\alpha} + (1 - t)|A^*|^{2(1-\alpha)}) \, dt, x, x \rangle \leq \frac{1}{2} \left( \|\varphi(|A|^{2\alpha}) + \varphi(|A^*|^{2(1-\alpha)}) \right) x, x \rangle.
\]

Taking the supremum over \( \|x\| = 1 \), we get the desired inequality. \( \blacksquare \)

An immediate consequence of the above result is an improvement of the Haddad and Kittaneh inequality [10].

**Corollary 3.2.** Suppose \( A \in \mathcal{B}(\mathcal{H}) \), \( \varphi(t) = t^r \) for \( t \geq 0 \) and \( r \geq 1 \). Then for \( 0 \leq \alpha \leq 1 \)

\[
w^r(A) \leq \left\| \int_0^1 \left( t|A|^{2\alpha} + (1 - t)|A^*|^{2(1-\alpha)} \right)^r \, dt \right\| \leq \frac{1}{2} \| |A|^{2\alpha} + |A^*|^{2(1-\alpha)} \|.
\]

**Remark 3.3.** The above findings are stronger versions of celebrated Kittaneh inequalities (see [10], [21], and [22]) as well as results of Sababheh and Moradi [28].

**Remark 3.4.** Suppose that \( A \) is an isometry on a Hilbert space \( \mathcal{H} \) and \( \varphi(t) = t^r \), \( r \geq 1 \). Then from the above result we have \( 1 = \varphi(1) \leq \frac{1}{2} \| I + P \| \leq 1 \), that is \( \| I + P \| = 2 \), where \( P \) is the orthogonal projection on the range of \( A \).

**Remark 3.5.** Let \( H^2(\mathbb{D}) \) be the Hardy space over the unit disc \( \mathbb{D} \) and \( \theta \) be any inner function on \( \mathbb{D} \). Then \( T_\theta \) is an analytic Toeplitz operator on \( H^2(\mathbb{D}) \). Now we have from our result \( w(T_\theta) = 1 \) (see [25]). Then for \( \varphi(t) = t^r \), \( r \geq 1 \), we get \( \| I + P_{H^2(\mathbb{D})} \| = 2 \), where \( P_{H^2(\mathbb{D})} \) is the orthogonal projection onto the range of \( T_\theta \), that is \( \theta H^2(\mathbb{D}) \). In a similar way, in case of Crawford number, we have \( \varphi(c(A)) \leq \frac{1}{2} \| |A|^{2\alpha} + |A^*|^{2(1-\alpha)} \| \), \( 0 \leq \alpha \leq 1 \). In this case, \( c(T_\theta) = 0 \) (see [25]), hence \( \varphi(c(T_\theta)) = 0 \), for any Orlicz function. Then for \( \varphi(t) = t^r \), \( r \geq 1 \), we get \( 0 \leq \| I + P_{H^2(\mathbb{D})} \| \leq 2 \).

**Theorem 3.6.** Let \( A \in \mathcal{B}(\mathcal{H}) \) and \( \varphi \) be an Orlicz function. Then

\[
w^2(A) \leq \| \varphi(|A|) + \psi(|A^*|) \|,
\]
where \( \psi \) is a complementary Orlicz function to \( \varphi \).

**Proof.** Suppose that \( A \in \mathcal{B}(\mathcal{H}) \). Then for any \( x \in \mathcal{H} \) with \( \|x\| = 1 \), we have

\[
w^2(A) = \sup_{\|x\|=1} |\langle Ax, x \rangle|^2 \leq \sup_{\|x\|=1} \langle |A|x, x \rangle \rangle (\text{Lemma 2.4}) 
\]

\[
\leq \sup_{\|x\|=1} \{ \varphi(\langle |A|x, x \rangle) + \psi(\langle |A^*|x, x \rangle) \} (\text{Lemma 2.6}) 
\]

\[
\leq \sup_{\|x\|=1} \langle \{ \varphi(|A|) + \psi(|A^*|) \}x, x \rangle (\text{Lemma 2.1}) 
\]

\[
= w(\varphi(|A|)) + \psi(|A^*|)) 
\]

\[
\leq \|\varphi(|A|) + \psi(|A^*|)\|. 
\]

\[
\square
\]

Now we have the following inequalities in particular choices of Orlicz function \( \varphi \).

**Corollary 3.7.** Let \( A \in \mathcal{B}(\mathcal{H}) \) and \( \varphi(t) = \frac{t^p}{p} \) for \( p > 1 \) and \( t \geq 0 \). Then the corresponding complementary Orlicz function is \( \psi(t) = \frac{t^q}{q} \), where \( \frac{1}{p} + \frac{1}{q} = 1 \), and

(i) \( w^2(A) \leq w(\frac{|A|^p}{p} + \frac{|A^*|^q}{q}) \leq \|\frac{|A|^p}{p} + \frac{|A^*|^q}{q}\|, \)

(ii) \( w^2(A) \leq \|\frac{|A|^p}{2} + \frac{|A^*|^q}{2}\| \), when \( p = 2 \) (Kittaneh [22]).

**Remark 3.8.** Suppose that \( A \) is an isometry on a Hilbert space \( \mathcal{H} \). Then we have \( w(A) = 1 \) (see [25]). Also \( |A| = \sqrt{A^*A} = I \) and \( |A^*| = \sqrt{AA^*} = P \), where \( P \) is the orthogonal projection on the range of \( A \). Then from the above result with \( \varphi(t) = \frac{t^p}{p} \), \( \psi(t) = \frac{t^q}{q} \) for \( p, q > 1 \), we get

\[
\|\frac{1}{p}I + \frac{1}{q}P\| = 1.
\]

In our next result, we obtain an upper bound for numerical radius as below.

**Theorem 3.9.** Let \( A, B, X \in \mathcal{B}(\mathcal{H}) \), and \( \varphi, \psi \) be two mutually complementary Orlicz functions. Then

(i) \( w^r(A^*XB) \leq \|X\|^r w(\varphi(|A|^r) + \psi(|B|^r)) \) for \( r \geq 2 \), and

(ii) \( w(A^*XB) \leq \varphi(\sqrt{w(B^*X^{2\alpha}B)}) + \psi(\sqrt{w(A^*X^{2(1-\alpha)}A)}) \) for \( 0 \leq \alpha \leq 1 \).

**Proof.** (i) Suppose that \( A, B, X \in \mathcal{B}(\mathcal{H}) \). Let \( x \in \mathcal{H} \) with \( \|x\| = 1 \) and \( r \geq 2 \). Then we compute the following expression

\[
|\langle A^*XBx, x \rangle|^r = |\langle XBx, Ax \rangle|^r 
\]

\[
\leq \|X\|^r \|Ax\|^r \|Bx\|^r 
\]

\[
= \|X\|^r \langle Ax, x \rangle^r \langle Bx, x \rangle^r 
\]

\[
\leq \|X\|^r \langle |A|^r x, x \rangle \langle |B|^r x, x \rangle (\text{Lemma 2.2}) 
\]

\[
\leq \|X\|^r (\varphi(\langle |A|^r x, x \rangle) + \psi(\langle |B|^r x, x \rangle)) 
\]

\[
\square
\]
Further, applying Lemma 2.4 we have
\[ \|X\|^{\alpha} (\langle |A|^{\alpha} x, x \rangle + \langle |B|^{\alpha} x, x \rangle) \] (Lemma 2.1)
\[ \leq \|X\|^{\alpha} w(\varphi(|A|^{\alpha}) + \psi(|B|^{\alpha})). \]

Hence the result follows.

(ii) Further, applying Lemma 2.4 we have
\[ |\langle A^*XB, x \rangle| = |\langle XB, Ax \rangle| \]
\[ \leq \langle |X|^{2\alpha} Bx, Bx \rangle^{\frac{1}{2}} \langle |X|^{2(1-\alpha)} Ax, Ax \rangle^{\frac{1}{2}} \]
\[ \leq \varphi(\langle B^*|X|^{2\alpha} Bx, x \rangle^{\frac{1}{2}}) + \psi(\langle A^*|X|^{2(1-\alpha)} Ax, x \rangle^{\frac{1}{2}}), \]
where \( 0 \leq \alpha \leq 1 \). This proves the desired inequality.

Now we establish upper bounds for \( \varphi(w^2(A)) \).

**Theorem 3.10.** For any Orlicz function \( \varphi \), and \( A \in \mathcal{B}(\mathcal{H}) \) the following inequalities hold:

(i) \( \varphi(w^2(A)) \leq \|\alpha \varphi(|A|^{\frac{1}{\alpha}}) + (1 - \alpha) \varphi(|A^*|^{\frac{1}{1-\alpha}}) \| \) for all \( \alpha \in (0, 1) \), and

(ii) \( \varphi(w^2(A)) \leq \|\alpha \varphi(|A|^2) + (1 - \alpha) \varphi(|A^*|^2) \| \) for all \( \alpha \in (0, 1) \).

**Proof.** (i) Let \( A \in \mathcal{B}(\mathcal{H}) \), \( \alpha \in (0, 1) \) and choose \( x \in \mathcal{H} \) such that \( \|x\| = 1 \). Then by Lemma 2.4 and Young’s inequality, we get
\[
\varphi(\langle |Ax, x| \rangle) \leq \varphi(\langle |A|^{\frac{1}{\alpha}} x, x \rangle) \]
\[
\leq \alpha \varphi(\langle |A|^{\frac{1}{\alpha}} x, x \rangle) + (1 - \alpha) \varphi(\langle |A^*|^{\frac{1}{1-\alpha}} x, x \rangle) \]
\[
\leq \alpha \varphi(\langle |A|^{\frac{1}{\alpha}} x, x \rangle) + (1 - \alpha) \varphi(\langle |A^*|^{\frac{1}{1-\alpha}} x, x \rangle) \quad \text{(convexity of } \varphi \text{ and Lemma 2.1)}
\]
which proves the desired inequality.

(ii) Using convexity of \( \varphi \), Lemma 2.1, Lemma 2.2, and Young’s inequality it follows that:
\[
\varphi(\langle |Ax, x| \rangle) \leq \alpha \varphi(\langle |A|^{2\alpha} x, x \rangle) + (1 - \alpha) \varphi(\langle |A^*|^{2(1-\alpha)} x, x \rangle) \quad \text{(Lemma 2.1)}
\]
\[
\leq \alpha \varphi(\langle |A|^{2\alpha} x, x \rangle) + (1 - \alpha) \varphi(\langle |A^*|^{2(1-\alpha)} x, x \rangle) \]
\[
\leq \alpha \varphi(\langle |A|^2 x, x \rangle) + (1 - \alpha) \varphi(\langle |A^*|^2 x, x \rangle) \]
\[
= \langle \{\alpha \varphi(\langle |A|^2 x, x \rangle) + (1 - \alpha) \varphi(\langle |A^*|^2 x, x \rangle) \} x, x \rangle,
\]
which gives the desired inequality.

Now we obtain a general form of inequality (1.3) established by Abu-Omar and Kittaneh (see [1]). In fact we have the following theorem.

**Theorem 3.11.** Let \( \varphi \) be an Orlicz function and \( A \in \mathcal{B}(\mathcal{H}) \). Then
\[
\varphi(w^2(A)) \leq \left\| \int_0^1 \varphi(t|A|^2 + (1-t)|A^*|^2) dt \right\| + \frac{1}{2} \varphi(w(A^2)) \leq \frac{1}{4} \|\varphi(|A|^2) + \varphi(|A^*|^2)\| + \frac{1}{2} \varphi(w(A^2)).
\]
Proof. Suppose that $A \in B(\mathcal{H})$ and $x \in \mathcal{H}$ with $\|x\| = 1$. Then Lemma 2.4, Lemma 2.5, AM-GM inequality, and convexity of $\varphi$ implies that

$$\varphi(|\langle Ax, x \rangle|^2)$$
$$= \varphi(|\langle Ax, x \rangle \langle x, A^* x \rangle|)$$
$$\leq \frac{1}{2} \varphi(\|Ax\|\|A^* x\| + |\langle Ax, A^* x \rangle|)$$
$$\leq \frac{1}{2} \varphi(\langle Ax, Ax \rangle^{\frac{1}{2}} \langle A^* x, A^* x \rangle^{\frac{1}{2}}) + \frac{1}{2} \varphi(|\langle A^2 x, x \rangle|)$$

(3.1)

Using the Hermite-Hadamard inequality (2.1) in (3.1), we get

$$\varphi(|\langle Ax, x \rangle|^2)$$
$$\leq \frac{1}{4} \left( \varphi(\|A\|^2 x, x) + \varphi(\|A^*\|^2 x, x) \right) + \frac{1}{2} \varphi(|\langle A^2 x, x \rangle|)$$

(Lemma 2.1).

This proves the desired inequality. 

Remark 3.12. Let $T_z$ be the shift operator on the $E$-valued Hardy space $H^2_E(D)$ over the unit disc $D$. Then $w^2(T_z) = 1$ (see [25]). Therefore by choosing $\varphi(t) = t^r, r \geq 1$ in the above result, we have $1 \leq \frac{1}{4} \|I + P\|^2 + \frac{1}{2}$, that is $\|I + P\| = 2$, where $P$ is the orthogonal projection onto $zH^2_E(D)$.

Sometimes it is very difficult to obtain the complementary Orlicz function $\psi$ to $\varphi$ explicitly. So here we obtain an upper bound of numerical radius, which involves only $\varphi$.

Theorem 3.13. Let $A, B, X \in B(\mathcal{H})$ and $\varphi$ be any Orlicz function. Then

$$\varphi(w(A^* X B)) \leq \left\| \int_0^1 \varphi(t \|X\| |A|^2 + (1 - t) \|X\| |B|^2) dt \right\| \leq \frac{1}{2} w(\varphi(\|X\| |A|^2) + \varphi(\|X\| |B|^2))$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Since $\varphi$ is convex so by applying Lemma 2.1 one gets

$$\varphi(|\langle A^* X B x, x \rangle|) = \varphi(|\langle X B x, A x \rangle|)$$
$$\leq \varphi(\|X\| |A x\| |B x\|)$$
$$= \varphi(\|X\| |\langle A^2 x, x \rangle + (\langle B^2 x, x \rangle)^{\frac{1}{2}}\| |\langle A^2 x, x \rangle + (\langle B^2 x, x \rangle)^{\frac{1}{2}}\|)$$
$$\leq \varphi \left( \frac{\|X\| |\langle A^2 x, x \rangle + (\langle B^2 x, x \rangle)^{\frac{1}{2}}\|}{2} \right).$$
Using the convexity of \( \varphi \), and Lemma 2.1 we get

\[
\varphi(|\langle A^* X B x, x \rangle|) \\
\leq \left\{ \int_0^1 \{ \varphi(t\|X\|A^2) + (1-t)\|X\|B^2 \} dt \right\} x, x \\\n\leq \frac{1}{2} (\varphi(\|X\|A^2) + \varphi(\|X\|B^2)) x, x .
\]

The result is immediately follows by taking supremum over \( \|x\| = 1 \).

**Remark 3.14.** Suppose \( X \in \mathcal{B}(\mathcal{H}) \) is a contraction, that is, \( \|X\| \leq 1 \). Then using the property \( \varphi(\alpha t) \leq \alpha \varphi(t) \) for \( 0 < \alpha \leq 1, t \geq 0 \), we get \( \varphi(\|X\|A^2) \leq \|X\|\varphi(|A|^2) \), and similarly \( \varphi(\|X\|B^2) \leq \|X\|\varphi(|B|^2) \). Therefore, the above result yields the following inequality for a contraction \( X \):

\[
\varphi(w(A^* XB)) \leq \|X\| \int_0^1 \varphi(t|A|^2 + (1-t)|B|^2) dt \leq \frac{\|X\|}{2} (\varphi(|A|^2) + \varphi(|B|^2) ) .
\]

For \( X \in \mathcal{B}(\mathcal{H}) \), and two positive operators \( A, B \in \mathcal{B}(\mathcal{H}) \) the operator \( A^alpha XB^{1-alpha} \) for \( 0 \leq \alpha \leq 1 \) has its special importance in operator theory. Here we establish an upper bound of \( \varphi(w(A^alpha XB^{1-alpha})) \) for Orlicz function \( \varphi \). Subsequently, we compute \( w^r(A^alpha XB^{1-alpha}) \) for \( r \geq 1 \) and many others by choosing \( \varphi \) suitably.

**Theorem 3.15.** Let \( A, B, X \in \mathcal{B}(\mathcal{H}) \) such that \( A, B \geq 0 \).

(i) For any Orlicz function \( \varphi \) and \( \alpha \in (0, \frac{1}{2}] \)

\[
\varphi(w(A^alpha XB^{1-alpha})) \leq w(\alpha \varphi(\|X\|A) + (1-\alpha) \varphi(\|X\|B)) .
\]

(ii) Further, if \( \alpha \in [0, 1] \) then

\[
\varphi(w(A^alpha XB^{1-alpha})) \leq \frac{1}{2} w(\varphi(\|X\|A^{2alpha}) + \varphi(\|X\|B^{2(1-alpha)})) .
\]

**Proof.** (i) Let \( x \in \mathcal{H} \) with \( \|x\| = 1 \), and \( A, B, X \in \mathcal{B}(\mathcal{H}) \) such that \( A, B \geq 0 \). Using Lemma 2.4, Young’s inequality and Lemma 2.2 we get

\[
\varphi(|\langle A^alpha XB^{1-alpha} x, x \rangle|) = \varphi(|\langle XB^{1-alpha} x, A^alpha x \rangle|) \\
\leq \varphi(\|X\|A^alpha x\|B^{1-alpha} x\|) \\
= \varphi(\|X\|\langle A^{2alpha} x, x \rangle^{\frac{1}{2}} \langle B^{2(1-alpha)} x, x \rangle^{\frac{1}{2}}) \\
\leq \varphi(\langle X\|\langle \alpha A^{2alpha} x, x \rangle^{\frac{1}{2}} + (1-\alpha)\langle B^{2(1-alpha)} x, x \rangle^{\frac{1}{2}} \rangle) \\
\leq \varphi(\|X\|\langle \alpha A x, x \rangle + (1-\alpha)\langle B x, x \rangle) \) (since \( \varphi \) is convex).
\]
This proves the desired inequality.

(ii) Proceeding as in the case (i), we get the following inequality
\[
\varphi(|\langle A^\alpha X B^{1-\alpha} x, x \rangle|) = \varphi(|\| X \| \langle A^{2\alpha} x, x \rangle^{\frac{1}{2}} (B^{2(1-\alpha)} x, x)^{\frac{1}{2}} |) \text{ (Lemma 2.4)}
\leq \varphi(|\| X \| \frac{1}{2} \langle A^{2\alpha} x, x \rangle + \frac{1}{2} \langle B^{2(1-\alpha)} x, x \rangle|)
\leq \frac{1}{2} (\varphi(|\| X \| A^{2\alpha}) + \varphi(|\| X \| B^{2(1-\alpha)} x, x, x). \text{ (Lemma 2.1)}
\]

This proves the desired result.

\textbf{Remark 3.16.} In addition, if \( X \in B(H) \) as a contraction, then

(i) For \( \alpha \in (0, \frac{1}{2}] \):
\[
\varphi(w(A^\alpha X B^{1-\alpha})) \leq \| X \| w(\alpha \varphi(A) + (1 - \alpha) \varphi(B)).
\]

(ii) For \( \alpha \in [0, 1] \):
\[
\varphi(w(A^\alpha X B^{1-\alpha})) \leq \frac{\| X \|}{2} w(\varphi(A^{2\alpha}) + \varphi(B^{2(1-\alpha)})).
\]

In the following we establish an upper bound of the numerical radius of finite sum of operators of the form \( \sum_{i=1}^{n} A_i^* X_i B_i \).

\textbf{Theorem 3.17.} Let \( A_k, B_k, X_k \in B(H) \) for \( k = 1, 2, \ldots, n \) and \( \varphi \) be an Orlicz function. Suppose that \( f, g : [0, \infty) \to [0, \infty) \) be two continuous functions such that \( f(t)g(t) = t \) for all \( t \in [0, \infty) \). Then the following inequality holds true:
\[
\varphi(w(\sum_{k=1}^{n} A_k^* X_k B_k)) \leq \frac{1}{2n} \sum_{k=1}^{n} \| \varphi(nB_k^* f^2(|X_k|) B_k) + \varphi(nA_k^* g^2(|X_k^*|) A_k) \|.
\]

\textbf{Proof.} Let \( x \in H \) with \( \| x \| = 1 \). Then we get
\[
\varphi(|\langle \sum_{k=1}^{n} (A_k^* X_k B_k) x, x \rangle|) \leq \varphi(\sum_{k=1}^{n} |\langle X_k B_k x, A_k x \rangle|) \leq \varphi(\sum_{k=1}^{n} \| f(|X_k|) B_k x \| \| g(|X_k^*|) A_k x \|) \text{ (Lemma 2.3)}
\leq \varphi(\sum_{k=1}^{n} \langle f^2(|X_k|) B_k x, B_k x \rangle^{\frac{1}{2}} \langle g^2(|X_k^*|) A_k x, A_k x \rangle^{\frac{1}{2}})\]
Theorem 3.18. Let \( A_k, B_k, X_k \in \mathcal{B}(\mathcal{H}) \) for \( k = 1, 2, \ldots, n \). Suppose that \( f, g : [0, \infty) \to [0, \infty) \) be two continuous functions such that \( f(t)g(t) = t \) for all \( t \in [0, \infty) \). Then for an Orlicz function \( \phi \) the following inequality holds:

\[
\phi \left( \left\| \sum_{k=1}^{n} (A_k^* X_k B_k) \right\| \right) \leq \frac{1}{2n} \left( \phi \left( \sum_{k=1}^{n} (nB_k^2 f(|X_k|) B_k) \right) + \phi \left( nA_k^2 g^2(|X_k^*|) A_k \right) \right) \]

Proof. Let \( x \in \mathcal{H} \) with \( \|x\| = 1 \), and \( A_k, B_k, X_k \in \mathcal{B}(\mathcal{H}) \) for \( k = 1, 2, \ldots, n \). Using the similar steps used in previous Theorem 3.17 we obtain the following expression:

\[
\phi \left( \left\| \sum_{k=1}^{n} (A_k^* X_k B_k) x, x \right\| \right) \leq \frac{1}{2n} \sum_{k=1}^{n} \left( \phi(nB_k^2 f(|X_k|) B_k) x, x \right) + \left( \phi(nA_k^2 g^2(|X_k^*|) A_k) x, x \right) \]

\[
\leq \frac{1}{\sqrt{2n}} \left| \sum_{k=1}^{n} \left( \phi(nB_k^2 f(|X_k|) B_k) x, x \right) + i \sum_{k=1}^{n} \left( \phi(nA_k^2 g^2(|X_k^*|) A_k) x, x \right) \right| \]

(since \( |p + q| \leq \sqrt{2}|p + iq| \) for all \( p, q \in \mathbb{R} \))

\[
= \frac{1}{\sqrt{2n}} \left| \left\{ \phi(nB_k^2 f(|X_k|) B_k) x, x \right\} + i \phi(nA_k^2 g^2(|X_k^*|) A_k) x, x \right| \]

\[
\leq \frac{1}{\sqrt{2n}} \phi \left( \left\| \sum_{k=1}^{n} (nB_k^2 f(|X_k|) B_k) + i \phi(nA_k^2 g^2(|X_k^*|) A_k) \right\| \right). \]
This proves the desired inequality.

In operator theory, and its allied areas it is required to determine the bounds of numerical radius of operator like \( ATB + CSD \) for \( A, B, C, D, S, T \in \mathcal{B}(\mathcal{H}) \). The upper bound of \( w(ATB + CSD) \) was established by Kittaneh \[22\):

\[
(3.3) \quad w(ATB + CSD) \leq \frac{1}{2} \|A|T^*|^{2(1-\alpha)} A^* + B^*|T|^{2\alpha} B + C|S^*|^{2(1-\alpha)} C^* + D^*|S|^{2\alpha} D\|
\]

where \( \alpha \in [0, 1] \). An attempt has been taken to establish a general version of inequality (3.3), which includes power type numerical radius inequality as well as provides new inequalities for a particular \( \varphi \). Let us begin with the following statement:

**Theorem 3.19.** Suppose that \( A, B, C, D, S, T \in \mathcal{B}(\mathcal{H}) \) and \( \varphi \) be an Orlicz function. Moreover, let \( f, g : [0, \infty) \to [0, \infty) \) be two continuous functions such that \( f(t)g(t) = t \) for all \( t \in [0, \infty) \). Then we have

\[
\varphi(w(ATB + CSD))
\leq \left\| \int_0^1 \varphi(t(B^*f^2(|T|)B + Ag^2(|T^*|)A^*) + (1 - t)(D^*f^2(|S|)D + Cg^2(|S^*|)C^*))dt \right\|
\leq \frac{1}{2}\|\varphi(Ag^2(|T^*|)A^* + B^*f^2(|T|)B + \varphi(Cg^2(|S^*|)C^* + D^*f^2(|S|)D))\|.
\]

**Proof.** Let \( x \in \mathcal{H} \) with \( \|x\| = 1 \). Applying the AM-GM inequality and convexity of \( \varphi \), we get

\[
\varphi(|<(ATB + CSD)x, x>|) \\
\leq \varphi(|<TBx, A^*x>| + |<SDx, C^*x>|) \\
\leq \varphi(f^2(|T|)Bx, Bx)^\frac{1}{2} (g^2(|T^*|)A^*x, A^*x)^\frac{1}{2} \\
\quad + f^2(|S|)Dx, Dx)^\frac{1}{2} (g^2(|S^*|)C^*x, C^*x)^\frac{1}{2}) \quad \text{(Lemma 2.3)}
\]

Using inequality (2.1), we get

\[
\varphi(|<(ATB + CSD)x, x>|) \\
\leq \int_0^1 \varphi(t\{Ag^2(|T^*|)A^* + B^*f^2(|T|)B\}x, x) + (1 - t)\{Cg^2(|S^*|)C^* + D^*f^2(|S|)D\}x, x)dt
\leq \frac{1}{2}\left(\{\varphi(B^*f^2(|T|)B + Ag^2(|T^*|)A^*) + \varphi(D^*f^2(|S|)D + Cg^2(|S^*|)C^*)\}x, x\right).
\]

Taking supremum over \( \|x\| = 1 \) on both sides of the above, one gets the desired inequality. This proves the theorem.

**Corollary 3.20.** Choose \( f(t) = t^\alpha \) and \( g(t) = t^{1-\alpha} \) for \( t \geq 0 \) and \( \alpha \in [0, 1] \). Suppose that \( \varphi(t) = t \) for \( t \geq 0 \). Then we have Kittaneh inequality \[22\] as below:

\[
w(ATB + CSD) \leq \frac{1}{2} \|A|T^*|^{2(1-\alpha)} A^* + B^*|T|^{2\alpha} B + C|S^*|^{2(1-\alpha)} C^* + D^*|S|^{2\alpha} D\|.
\]
Now it is planned to establish a new numerical radius inequality in a more general setting. Consequently, we derive several recent improved numerical radius inequalities in one frame.

**Theorem 3.21.** Let \( A \in B(\mathcal{H}) \), and \( \varphi \) be an Orlicz function. Then

\[
\varphi(w^2(A)) \leq \alpha \left\| \frac{1}{4}(\varphi(\|A\|^2) + \varphi(\|A^*\|^2)) + \frac{1}{2}\varphi(|\Re\|A\||A^*||)\right\| + \frac{(1-\alpha)}{2} \left\| \int_0^1 \varphi(t\|A\|^2 + (1-t)\|A^*\|^2)dt \right\| \]

\[
+ \frac{1-\alpha}{2} \varphi(w(A^2)) \]

\[
\leq \frac{1}{4}\|\varphi(\|A\|^2) + \varphi(\|A^*\|^2)\| + \frac{\alpha}{2}\|\varphi(|\Re\|A\||A^*||)\| + \frac{1-\alpha}{2} \varphi(w(A^2)),
\]

where \( \alpha \in [0, 1] \).

**Proof.** Let \( A \in B(\mathcal{H}) \) and \( x \in \mathcal{H} \) with \( \|x\| = 1 \). Then we have

\[
\varphi(\langle Ax, x \rangle^2) = \varphi(\alpha\langle Ax, x \rangle^2 + (1 - \alpha)\langle Ax, x \rangle^2)
\]

\[
\leq \alpha \varphi(\langle |A| \langle |A^*| \rangle x, x \rangle^{\frac{1}{2}}(\langle |A^*| x, x \rangle^{\frac{1}{2}})^2) + (1 - \alpha)\varphi(\langle Ax, x \rangle^2).
\]

Now we have the following

\[
\alpha \varphi(\langle |A| \langle |A^*| \rangle x, x \rangle^{\frac{1}{2}}(\langle |A^*| x, x \rangle^{\frac{1}{2}})^2)
\]

\[
\leq \alpha \varphi\left(\langle \frac{|A| + |A^*|}{4} x, x \rangle \right)
\]

\[
\leq \alpha \varphi\left(\langle \frac{|A|^2 + |A^*|^2 + 2|\Re\|A\||A^*||)}{4} x, x \rangle \right)
\]

\[
\leq \alpha \left(\langle \frac{1}{4} \varphi(|A|^2) + \frac{1}{4} \varphi(|A^*|^2) + \frac{1}{2} \varphi(|\Re\|A\||A^*||) \rangle (Lemma \[2.1\]).
\]

On the other hand, by Theorem \[3.1\] we get

\[
(1 - \alpha)\varphi(\langle Ax, x \rangle^2)
\]

\[
\leq \frac{(1-\alpha)}{2} \left\| \int_0^1 \varphi(t\|A\|^2 + (1-t)\|A^*\|^2)dt \right\| \langle Ax, x \rangle \right) + \frac{(1-\alpha)}{2} \varphi(\langle A^2 x, x \rangle) \]

\[
\leq \frac{(1-\alpha)}{4} (\varphi(|A|^2) x, x) + \langle \varphi(|A^*|^2) x, x \rangle) + \frac{(1-\alpha)}{2} \varphi(\langle A^2 x, x \rangle) \) (Lemma \[2.1\]).
Therefore, we have
\[ \varphi(|\langle A x, x \rangle|^2) \]
\[ \leq \alpha \left( \left( \frac{1}{4} \varphi(|A|^2) + \frac{1}{4} \varphi(|A^*|^2) + \frac{1}{2} \varphi(\langle |A||A^*| \rangle) \right) x, x \right) \]
\[ + \frac{(1 - \alpha)}{2} \left( \int_0^1 \varphi(t|A|^2 + (1 - t)|A^*|^2) dt \right) x, x \right) + \frac{(1 - \alpha)}{2} \varphi(|\langle A^2 x, x \rangle|) \]
\[ \leq \frac{1}{4} (\varphi(|A|^2) x, x) + \varphi(|A^*|^2) x, x) + \frac{\alpha}{2} (\varphi(\langle |A||A^*| \rangle) x, x) + \frac{1 - \alpha}{2} \varphi(|\langle A^2 x, x \rangle|). \]

This proves the desired inequality.

An Orlicz function \( \varphi \) is said to be sub-multiplicative if \( \varphi(uv) \leq \varphi(u)\varphi(v) \) for \( u, v \geq 0 \), holds. In our next result, we find an upper bound by using the sub-multiplicative property of \( \varphi \).

**Theorem 3.22.** Let \( A \in B(\mathcal{H}) \), and Orlicz function \( \varphi \) is sub-multiplicative. Then for \( \alpha \in [0, 1] \)
\[ \varphi(w^2(A)) \leq \frac{1}{4} \|\varphi(|A|^2) + \varphi(|A^*|^2)\| + \frac{\alpha}{2} \|\varphi(|A||A^*|)\| + \frac{1 - \alpha}{2} \varphi(w(A^2)). \]

**Proof.** Let \( A \in B(\mathcal{H}) \) and \( x \in \mathcal{H} \). We follow the similar steps as in Theorem 3.21. Then we have
\[ \varphi(|\langle A x, x \rangle|^2) \]
\[ \leq \alpha \varphi \left( \left( \frac{\|A\| x, x} + \frac{\|A^*\| x, x} \right)^2 \right) \]
\[ + \frac{1 - \alpha}{4} (\varphi(|A|^2) x, x) + \varphi(|A^*|^2) x, x) + \frac{1 - \alpha}{2} \varphi(|\langle A^2 x, x \rangle|). \]

Now applying the convexity and sub-multiplicative property of \( \varphi \), the first term of RHS of inequality (3.6) gives the following:
\[ \alpha \varphi \left( \left( \frac{\|A\| x, x} + \frac{\|A^*\| x, x} \right)^2 \right) \]
\[ \leq \frac{\alpha}{4} \varphi \left( \langle |A| x, x \rangle^2 \right) + \frac{\alpha}{4} \varphi \left( \langle |A^*| x, x \rangle^2 \right) + \frac{\alpha}{2} \varphi \left( \langle |A| x, x \rangle \langle |A^*| x, x \rangle \right) \]
\[ \leq \frac{\alpha}{4} \left( \varphi(|A|^2) x, x \right) + \varphi(|A^*|^2) x, x \right) + \frac{\alpha}{2} \varphi \left( \langle |A| x, x \rangle \varphi \left( \langle |A^*| x, x \rangle \right) \right) \]
\[ \leq \frac{\alpha}{4} \left( \varphi(|A|^2) x, x \right) + \varphi(|A^*|^2) x, x \right) + \frac{\alpha}{2} \langle |A| x, x \rangle \varphi \left( \langle |A^*| x, x \rangle \right). \]

Hence by plugging this upper bound in inequality (3.6), and then taking supremum over \( \|x\| = 1 \), we get the desired inequality.

In the following result, we have obtained a generalized version of Theorem 3.21.
Theorem 3.23. Let $A_i \in \mathcal{B}(\mathcal{H})$ for $i = 1, 2, \ldots, n$. Then for Orlicz function $\varphi$ the following inequality holds:

\[
\varphi\left(\left\|(\sum_{i=1}^{n} A_i) x, x\right\|^2\right) = \varphi\left(\sum_{i=1}^{n} \left\langle A_i x, x \right\rangle \right) \leq \varphi\left(\sum_{i=1}^{n} \left|\left\langle A_i x, x \right\rangle\right|^2\right)
\]

\[
\leq \varphi\left(n \sum_{i=1}^{n} \left|\left\langle A_i x, x \right\rangle\right|^2\right) \leq \frac{\alpha}{n} \sum_{i=1}^{n} \left\langle \left(\frac{1}{4} \varphi(|nA_i|^2) + \frac{1}{4} \varphi(|nA_i^*|^2) + \frac{1}{2} \varphi(\Re(n^2|A_i||A_i^*|))\right) x, x \right\rangle
\]

\[
+ \frac{(1 - \alpha)}{2n} \sum_{i=1}^{n} \left\langle \int_{0}^{1} \varphi(t|nA_i|^2 + (1 - t)|nA_i^*|^2) dt \right\rangle x, x \right\rangle + \frac{(1 - \alpha)}{2n} \sum_{i=1}^{n} \varphi(n^2|A_i^2 x, x|)
\]

\[
\leq \frac{1}{4n} \left\langle \sum_{i=1}^{n} \left(\varphi(|nA_i|^2) x, x\right) + \varphi(|nA_i^*|^2) x, x \right\rangle + \frac{(1 - \alpha)}{2n} \sum_{i=1}^{n} \varphi(n^2|A_i^2 x, x|)
\]

\[
+ \frac{(1 - \alpha)}{2n} \sum_{i=1}^{n} \varphi(n^2|A_i^2 x, x|)
\]

Hence the desired inequality.

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$, and $A_i \in \mathcal{B}(\mathcal{H})$ for $i = 1, 2, \ldots, n$. Then using the steps applied in Theorem 3.21 and Lemma 2.7, we have

In the next result, another numerical radius inequality is obtained by using Hermite-Hadamard inequality (2.1) and Orlicz function $\varphi$. 
Theorem 3.24. Let \( A \in \mathcal{B}(\mathcal{H}) \) and \( \varphi \) be an Orlicz function. Then the following inequality holds:

\[
\varphi(w^2(A)) \\
\leq \frac{1}{2} \left\| \int_0^1 \varphi(t|A|^2 + (1 - t)|A^*|^2) dt \right\| + \frac{1}{2} \varphi(w(|A||A^*|)) \\
\leq \frac{1}{4} \| (\varphi(|A|^2) + \varphi(|A^*|^2)) \| + \frac{1}{2} \varphi(w(|A||A^*|)).
\]

Proof. Suppose that \( x \in \mathcal{H} \) with \( \|x\| = 1 \). Then one gets

\[
\varphi(|\langle Ax, x \rangle|^2) \\
\leq \varphi(|\langle Ax, x \rangle| |A^*| x, x) \\
= \varphi(|\langle A^*| x, x \rangle| |A| x) \\
\leq \varphi\left(\frac{1}{2}(\| |A^*| x \| |A| x \| + |\langle |A^*| x, |A| x \rangle|)\right) \\
\leq \frac{1}{2} \varphi\left(|\langle A^*| x, |A| x \rangle| ^{\frac{1}{2}} |\langle |A| x, |A| x \rangle| ^{\frac{1}{2}} \right) + \frac{1}{2} \varphi\left(|\langle |A| A^*| x, x \rangle|\right) \\
\leq \frac{1}{2} \varphi\left(\frac{|\langle A^*| ^2 x, x \rangle + |\langle A| ^2 x, x \rangle|}{2}\right) + \frac{1}{2} \varphi\left(|\langle |A| A^*| x, x \rangle|\right).
\]

Applying the Hermite-Hadamard inequality (2.1) in above, one immediately gets

\[
\varphi(|\langle Ax, x \rangle|^2) \\
\leq \frac{1}{2} \left\{ \int_0^1 \{ \varphi(t|A|^2 + (1 - t)|A^*|^2) dt \} x, x \right\} + \frac{1}{2} \varphi\left(|\langle |A| A^*| x, x \rangle|\right) \\
\leq \frac{1}{4} (\langle \varphi(|A^*|^2)x, x \rangle + \langle \varphi(|A|^2)x, x \rangle) + \frac{1}{2} \varphi\left(|\langle |A| A^*| x, x \rangle|\right).
\]

This proves the desired inequality.

4. Applications to block matrices

Let \( P, Q \in \mathcal{B}(\mathcal{H}) \). Then we compute numerical radius of some \( 2 \times 2 \) off-diagonal operator matrix in this section. In fact, we establish an generalized upper bound for \( \varphi(w(A)) \), where \( A = \begin{bmatrix} O & P \\ Q & O \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \). Our result is stated as below.
Theorem 4.1. Let $P, Q \in \mathcal{B}(\mathcal{H})$, and $\varphi$ be an Orlicz function. If $A = \begin{bmatrix} O & P \\ Q & O \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, then

$$
\varphi(w^2(A)) \leq \frac{1}{4} \left\| \begin{bmatrix} \varphi(||Q||^2) + \varphi(||P^*||^2) & \varphi(||P||^2) \\ \varphi(||P||^2) + \varphi(||Q^*||^2) \\ O & O \end{bmatrix} \right\| \\
+ \frac{\alpha}{2} \left\| \begin{bmatrix} \varphi(\mathfrak{R}(||Q||P^*||)) & \varphi(\mathfrak{R}(||P||Q^*||)) \\ \mathfrak{R}(||Q||P^*||) & \mathfrak{R}(||P||Q^*||) \end{bmatrix} \right\| \\
+ \frac{1 - \alpha}{2} \varphi\left(w\left(\begin{bmatrix} PQ \\ O \\ QP \end{bmatrix}\right)\right).
$$

Proof. Suppose that $\varphi$ is an Orlicz function and $A = \begin{bmatrix} O & P \\ Q & O \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Then we have

$$
\varphi(||A||^2) = \varphi(A^*A) = \begin{bmatrix} \varphi(||Q||^2) & \varphi(||P||^2) \\ O & O \end{bmatrix} \text{ and } \varphi(||A^*||^2) = \varphi(AA^*) = \begin{bmatrix} \varphi(||P^*||^2) & O \\ O & \varphi(||Q^*||^2) \end{bmatrix}.
$$

Further, since

$$
\mathfrak{R}(||A||A^*||) = \begin{bmatrix} \mathfrak{R}(||Q||P^*||) & \mathfrak{R}(||P||Q^*||) \\ \mathfrak{R}(||Q||P^*||) & \mathfrak{R}(||P||Q^*||) \end{bmatrix},
$$

we get $\varphi(\mathfrak{R}(||A||A^*||)) = \begin{bmatrix} \varphi(\mathfrak{R}(||Q||P^*||)) & \varphi(\mathfrak{R}(||P||Q^*||)) \\ \mathfrak{R}(||Q||P^*||) & \mathfrak{R}(||P||Q^*||) \end{bmatrix}$.

Therefore, for the operator $A = \begin{bmatrix} O & P \\ Q & O \end{bmatrix}$, the desired inequality follows immediately from Theorem 3.21.

We now have the following remark:

Remark 4.2. (see Remark 2.6, [5]) Let $P, Q \in \mathcal{B}(\mathcal{H})$ and $A = \begin{bmatrix} O & P \\ Q & O \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Then for $\varphi(t) = t$ for $t \geq 0$, and $\alpha = 1$, one gets

$$
w^2(A) \leq \frac{1}{4} \max \left\{ ||Q||^2 + ||P^*||^2, ||P||^2 + ||Q^*||^2 \right\} + \frac{1}{2} \max \left\{ ||\mathfrak{R}(||Q||P^*||)||, ||\mathfrak{R}(||P||Q^*||)|| \right\}.
$$

5. Conclusions

In this paper we have expanded the improved numerical radius inequalities for class of operators defined on a Hilbert space by using the Orlicz function and Hermite-Hadamard inequality. We have also been able to generalize and extend many known results in this important area of research.
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REFERENCES

[1] A. Abu-Omar and F. Kittaneh, Numerical radius inequalities for \( n \times n \) operator matrices, Linear Algebra Appl., 468(1) (2015), 18-26.
[2] O. Axelsson, H. Lu, B. Polman, On Numerical Radius of Matrices and Its Application for Iterative Solution Methods (2nd ed.), Report 9308, Dept. of Mathematics, Catholic Univ. Nijmegen, 1993.
[3] C. A. Berger, On the numerical range of powers of an operator, Notices of Amer. Math. Soc., 12 (No. 590) (1965).
[4] P. Bhunia and K. Paul, New upper bounds for the numerical radius of Hilbert space operators, Bull. Sci. Math., 167 (2021), No. 102959.
[5] P. Bhunia and K. Paul, Improvement of numerical radius inequalities, Preprint, https://arxiv.org/pdf/2110.02505.pdf.
[6] F. F. Bonsall, J. Duncan, Numerical Ranges II, Cambridge University Press, ISBN 978-0-521-20227-5, 1971.
[7] M. L. Buzano, Generalizzazione della diseguaglianza di Cauchy-Schwartz (Italian), Rend Sem Mat Univ E Politech Torino, 1974 31 (1971/73), 405-409.
[8] J. B. Conway, A Course in Functional Analysis, 2nd edition, Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990.
[9] S. S. Dragomir, Inequalities for the numerical radius of linear operators in Hilbert space, Springer Briefs in Mathematics, Springer, 2013.
[10] M. El-Haddad and F. Kittaneh, Numerical radius inequalities for Hilbert space operators II, Studia Math., 182 (2) (2007), 133-140.
[11] C. K. Fong and J. A. R. Holbrook, Unitarily invariant operators norms, Canad. J. Math., 35 (1983), 274-299.
[12] M. Goldberg, E. Tadmor and G. Zwas, Numerical radius of positive matrices, Linear Algebra Appl., 12 (1975), 209-214.
[13] M. Goldberg and E. Tadmor, On the numerical radius and its applications, Linear Algebra Appl., 42 (1982), 263-284.
[14] J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d’une fonction considérée par Riemann, J. Math. Pures et Appl., 58 (1893), 171-215.
[15] P. R. Halmos, A Hilbert space problem book, Van Nostrand, New York, 1967.
[16] O. Hirzallah, F. Kittaneh and K. Shebrawi, Numerical radius inequalities for certain \( 2 \times 2 \) operator matrices, Integral Equ. Oper. Theory, 71 (2011), 129-147.
[17] O. Hirzallah, F. Kittaneh and K. Shebrawi, Numerical radius inequalities for commutators of Hilbert space operators, Numer. Funct. Anal. Optim., 32(7) (2011), 739-749.
[18] J. A. R. Holbrook, Multiplicative properties of the numerical radius in operator theory, J. Reine Angew Math., 237 (1969), 166-174.
[19] R. A. Horn; C. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, 1991.
[20] T. Kato, Notes on some inequalities for linear operators, Math. Ann., 125 (1952), 208-212.
[21] F. Kittaneh, Numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, Studia Math., 158(1) (2003), 11-17.
[22] F. Kittaneh, Numerical radius inequalities for Hilbert space operators, Studia Math., 168(1) (2005), 73-80.
[23] F. Kittaneh, Notes on some inequalities for Hilbert space operators, Publ. RIMS Kyoto Univ., 24 (1988), 283-293.
[24] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces I and II, Sequence spaces, Function spaces, Springer-Verlag Berlin Heidelberg, Printed in Germany, 1996.
[25] S. Majee, A. Maji, and A. Manna, Numerical radius and Berezin number inequality, J. Math. Anal. Appl., 517(1) (2023), 126566.
[26] C. Pearcy, *An elementary proof of the power inequality for the numerical radius*, Michigan Math. J., 13(3) (1966), 289-291.

[27] J. Pečarić, T. Furuta, J. M. Hot and Y. Seo, *Mond-Pečarić method in operator inequalities*, *Inequalities for bounded self-adjoint operators on a Hilbert space*, Monograph in inequalities I, 2nd Edition, Element, Zagreb, 2005.

[28] M. Sababheh and H. R. Moradi, *More accurate numerical radius inequalities (I)*, Linear Multilinear Algebra, 69(10) (2021), 1964-1973.

[29] M. P. Vasić and D. J. Kečkić, *Some inequalities for complex numbers*, Math. Balkanica, 1 (1971), 282-286.

[30] T. Yamazaki, *On upper and lower bounds for the numerical radius and an equality condition*, Studia Math., 178(1) (2007), 83-89.

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