On the trees of quantum fields

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Abstract. The solution of some equations involving functional derivatives is given as a series indexed by planar binary trees. The terms of the series are given by an explicit recursive formula. Some algebraic properties of these series are investigated. Several examples are treated in the case of quantum electrodynamics: the complete fermion and photon propagators, the two-body Green function, and the one-body Green function in the presence of an external source, the complete vacuum polarization, electron self-energy and irreducible vertex.

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1 Introduction

Renormalization theory has been recently revitalized by the discovery a Hopf algebra that transforms the dreadful combinatorics of renormalization into a mechanical application of the Hopf algebra properties of rooted trees [1,2,3].

In the companion paper [4], Butcher’s theory has been presented as an alternative way to describe the Hopf structure of the algebra of renormalization. A particularly useful aspect of Butcher’s approach is that solutions of non-linear differential equations can be written as a sum over rooted trees.

In the present paper, Butcher’s strategy is adapted to equations involving functional derivatives, that were first proposed by Schwinger [5] and will be called Schwinger equations in the rest of the paper. Schwinger equations are not commonly considered as a useful tool for computation. The purpose of this article is to show that, by using series over planary binary trees, Schwinger equations can be turned into explicit calculation methods.

The series we manipulate are indexed by planar binary trees. So we first present some basic properties of planar binary trees. Then the solution of simple Schwinger equations are written as a sum over planar binary trees, with recursively defined coefficients. To make a comparison with power series, the Schwinger equation would correspond to a differential equation for the sum of the series, whereas the formula we put forward corresponds to a recursive definition of the terms of the series: it does not contain so much information as the Schwinger equation, but it is more explicit to calculate the terms of the series.

As an example, the full fermion and photon propagators of quantum electrodynamics (QED) are written as a sum over planar binary trees. Renormalization is briefly described and applications are given for QED with an external source, the vacuum polarization, the fermion self-energy and irreducible vertex.

2 Planar binary trees

In contrast to Ref.[4], we do not use rooted trees but planar binary trees. Both can be drawn on a plane, but no permutation of vertices is allowed for planar trees.

As an example, \( \mathcal{V} \) and \( \mathcal{V}' \) are two different planar trees, although they represent the same rooted tree.

In planar trees, we distinguish two types of vertices: the leaves (which have no sons) and the remaining vertices (including the root), which we call internal vertices. In planar binary trees, internal vertices have exactly two sons.

We now follow the notation of Loday and Ronco [6]. Planar binary trees have an odd number of vertices. We denote \( Y_n \) the set of planar binary trees with \( 2n+1 \) vertices. If \( t \) belongs to \( Y_n \), \( t \) has \( n+1 \) leaves and \( n \) internal vertices. The number of elements of \( Y_n \) is \( \frac{(2n)!}{n!(n+1)!} \): the Catalan numbers, which enter many combinatorial problems [6] and should probably be called Ming numbers [8].

If \( t_1 \in Y_m \) and \( t_2 \in Y_n \), the grafting of \( t_1 \) and \( t_2 \) is the tree \( t \in Y_{m+n+1} \) obtained by putting \( t_1 \) on the left of \( t_2 \) and by joining the roots of \( t_1 \) and \( t_2 \) to a new vertex that becomes the root of \( t \). This operation is denoted by \( t = t_1 \vee t_2 \). For instance, grafting \( \mathcal{V} \) and \( \mathcal{V}' \) gives \( \mathcal{V} \vee \mathcal{V}' = \mathcal{V} \).
In the companion paper, rooted trees were graded by the number of their vertices. Here, planar binary trees have an odd number of vertices, and it is more natural to grade them differently: for each tree \( t \), we define \( |t| \) as the integer \( n \) such that \( t \in Y_n \). Thus, a tree \( t \) has \( 2|t| + 1 \) vertices.

An essential property of planar binary trees is that each tree \( t \) different from \( \bullet \) can be written in a unique way as \( t_1 \lor t_2 \), where \( t_1 \) and \( t_2 \) are called the branches of \( t \). Moreover, grafting provides a recursive definition of planar binary trees \( \mathcal{T} \) :

\[
Y_n = \bigcup_{k=0}^{n-1} Y_k \lor Y_{n-k-1}, \quad Y_0 = \{ \bullet \}. \tag{1}
\]

The notation \( Y_k \lor Y_{n-k-1} \) means that all the trees of \( Y_k \) are grafted with all the trees of \( Y_{n-k-1} \).

Planar binary trees have received much attention recently because of their relation to new algebraic structures \( \mathcal{T} \).

### 3 Schwinger equations

In this section, the solution of a linear Schwinger equation is given as a sum over planar binary trees. But we first introduce the concept of a functional derivative.

A functional \( A(\phi) \) is defined loosely as a map sending a distribution \( \phi \) to a complex number (see Ref. [11] for details). If \( \psi \) is a distribution, the functional derivative of \( A(\phi) \) in the direction \( \psi \) is defined as the limit for \( \epsilon \to 0 \) of \( (A(\phi + \epsilon \psi) - A(\phi))/\epsilon \). Finally, the functional derivative of \( A(\phi) \) with respect to \( \phi(x) \), \( \delta A(\phi)/\delta \phi(x) \), is defined as the functional derivative of \( A(\phi) \) in the direction \( \delta_x \), where \( \delta_x \) is the Dirac function \( \delta_x(y) = \delta(y-x) \).

#### 3.1 Examples of functional derivatives

A classical example is \( A(\phi) = \int dx f(x) \phi(x) \), giving easily

\[
\frac{\delta A(\phi)}{\delta \phi(x)} = f(x).
\]

A further example, that will be useful in the sequel, is \( A(\phi) = \int dx dy f(y) \phi(x) \phi(y) \). Then

\[
\frac{\delta A(\phi)}{\delta \phi(x)} = \int dy f(y) \phi(x) \phi(y) + \int dy f(y) \phi(y) \phi(x),
\]

\[
\frac{\delta^2 A(\phi)}{\delta \phi(x) \delta \phi(y)} = \frac{\delta}{\delta \phi(x)} \frac{\delta A(\phi)}{\delta \phi(y)} = f(x) + f(y).
\]

More generally, if

\[
A(\phi) = \int dx_1 \ldots dx_n f(x_1, \ldots, x_n) \phi(x_1) \ldots \phi(x_n),
\]

then

\[
\frac{\delta^n A(\phi)}{\delta \phi(x_1) \ldots \delta \phi(x_n)} = \sum_{\sigma \in S_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),
\]

where \( S_n \) is the set of permutations of \( n \) elements.

In practice, \( A(\phi) \) is often a Green function. Take the Green function defined by \( (A_0 - \phi(x)) A(\phi;x,y) = \delta(x-y) \), which can be written \( A(\phi) = (A_0 - \phi)^{-1} \). To calculate the functional derivative, we put \( A(\phi + \epsilon \psi) = (A_0 - \phi - \epsilon \psi)^{-1} \).

The operator identity \( Y^{-1} = X^{-1} + X^{-1}(X-Y)Y^{-1} \) gives us

\[
A(\phi + \epsilon \psi) = A(\phi) + \epsilon A(\phi) \psi A(\phi + \epsilon \psi).
\]

Thus, taking the limit \( \epsilon \to 0 \),

\[
\frac{\delta A(\phi;x,y)}{\delta \psi} = \int ds A(\phi;x,s) \psi(s) A(\phi;s,y).
\]

If we choose now the distribution \( \psi(s) = \delta(s-z) \) we find

\[
\frac{\delta A(\phi;x,y)}{\delta \phi(z)} = A(\phi;z) A(\phi;z,y), \tag{2}
\]

This identity will be used repeatedly in the sequel.

#### 3.2 A simple Schwinger equation

As an introduction to the method of planar binary trees we consider the Schwinger equation

\[
X = A + F(X, \frac{\delta X}{\delta v(z)}), \tag{3}
\]

where \( A \) is a functional of \( v \), \( F \) is linear in \( X \) and \( \frac{\delta X}{\delta v(z)} \), and \( z \) is a variable over which \( F \) integrates. In an equation like (3), \( F \) is called the integral operator of the equation and \( A \) is called its initial data.

A common example of such an equation is obtained when \( X = X(x,y) \), the initial data \( A(x,y) \) are the Green function \( (A_0 - v(x)) A(x,y) = \delta(x-y) \) discussed in section 3.1 and the integral operator is

\[
F(X, \frac{\delta X}{\delta v(z)})(x,y) = \int ds dz X(s,z) f(s,z) \frac{\delta X(s,y)}{\delta v(z)},
\]

for some function \( f(s,z) \).

Such a Schwinger equation summarizes an infinity of equations that can be obtained by taking successive functional derivatives of Eq.(3) with respect to \( v(z) \).

\[
X = A + F(X, \frac{\delta X}{\delta v(z)}),
\]

\[
\frac{\delta X}{\delta v(z_1)} = \frac{\delta A}{\delta v(z_1)} + F(X, \frac{\delta X}{\delta v(z)}),
\]

\[
F(X, \frac{\delta X}{\delta v(z_1)}) \frac{\delta^2 X}{\delta v(z_2)},
\]

\[
\ldots
\]

When we take the \( n \)-th functional derivative of both sides of the \((n-1)\)-th equation with respect to \( v(z_n) \), the equation gets an additional variable \( z_n \), and the chain rule is used to apply \( \frac{\delta}{\delta v(z_n)} \) to the right-hand side of the \((n-1)\)-th equation.

If this is iterated to all values of \( n \), we obtain an infinite system of nonlinear integral equations. This system seems difficult to solve because the \( n \)-th differential of \( X \) depends on the \( k \)-th differentials of \( X \) for \( k = 0 \) to \( n + 1 \).
3.3 The series solution

To write the solution of Eq. (3), we must introduce some notation. Sets of arguments will be often needed, so we write \( \{z\}_{ij} = z_i, z_{i+1}, \ldots, z_j \). Furthermore, if \( f(\{z\}_{1,n}) = f(z_1, \ldots, z_n) \) is a function of \( n \) variables, then \( f_\Sigma(\{z\}_{1,n}) \) is defined as the sum of \( n \) terms, where the first variable \( z_1 \) is shifted step by step from the first to the \( n \)-th position:

\[
f_\Sigma(\{z\}_{1,n}) = f(\{z\}_{1,n}) + f(z_2, z_1, \{z\}_{3,n}) + \cdots + f(\{z\}_{2,n-1, z_1, z_n}) + f(\{z\}_{2,n, z_1}).
\]

For each planar binary tree \( t \), we define an infinite-dimensional vector \( \phi(t) \), with components \( \phi^n(t) \), where \( n \) goes from 0 to infinity. The \( n \)-th component is a function of \( n \) variables \( z_1, \ldots, z_n \). Now we define the initial data. For \( t = \bullet \), we take \( \phi^0(\bullet) = A \) and

\[
\phi^1(\bullet; z_1) = \frac{\delta A}{\delta v(z_1)}.
\]

The most natural choice is to define \( \phi^n(\bullet) \) as 1/\( n! \) times the \( n \)-th functional derivative of \( A \) with respect to \( v(z) \), but this is not always the most economical choice in practice. The only condition that we need for \( n > 1 \) is

\[
\frac{\delta \phi^{n-1}(\bullet; \{z\}_{1,n-1})}{\delta v(z_n)} = \phi^n_\Sigma(\bullet; \{z\}_{1,n}).
\]

When \( A(x, y) \) is a Green function of the kind discussed in section 3.1, it is not difficult to build such a \( \phi(\bullet) \) from given initial data \( A \). We can use Eq. (4) to show that

\[
\begin{align*}
\phi^0(\bullet) &= A(x, y), \\
\phi^1(\bullet; z_1) &= A(x, z_1)A(z_1, y), \\
\vdots \\
\phi^n(\bullet; \{z\}_{1,n}) &= A(x, z_1)A(z_1, z_2) \cdots A(z_n, y),
\end{align*}
\]

satisfies Eq. (3) and condition (3).

With this notation we can now write the solution of Eq. (3) as

\[
X = \sum_t \phi^0(t),
\]

where \( t \) spans the set of planar binary trees. Moreover

\[
\begin{align*}
\frac{\delta X}{\delta v(z_1)} &= \sum_t \phi^1(t; z_1), \\
\vdots \\
\frac{\delta^n X}{\delta v(z_1) \cdots \delta v(z_n)} &= \sum_{\sigma \in S_n} \sum_t \phi^n(t; z_{\sigma(1)}, \ldots, z_{\sigma(n)}).
\end{align*}
\]

For each planar binary tree \( t \), the vector \( \phi(t) \) is calculated as a function of the vectors \( \phi(t_1) \) and \( \phi(t_2) \), where \( t_1 \) and \( t_2 \) are the branches of \( t \). Since \( \phi(\bullet) \) is defined from Eq. (8), this defines \( \phi(t) \) recursively. The recursive definition of \( \phi(t) \) is given explicitly by

\[
\begin{align*}
\phi^n(t; \{z\}_{1,n}) &= \sum_{k=0}^n F(\phi^k(t_1; \{z\}_{1,k}), \\
\phi^n_{\Sigma+1}(t_2; z, \{z\}_{k+1,n}),
\end{align*}
\]

for \( t \neq \bullet \) and \( \phi^n(\bullet) \) is defined in Eq. (5).

In a quantum field interpretation, \( \bullet \) represents the bare fields, \( \nabla \Sigma \) represents the interaction, and the sum over trees represents all the combinations of interaction that gives the full propagator.

A proof of Eqs. (3) and (8) is given in the appendix.

3.4 Enumeration

If the initial data \( A \) is such that \( \phi^n(\bullet) \) has only one term, as in Eq. (4), the chain rule applied to the functional derivative gives a number of terms for \( \phi^n(t) \) that we denote \( |\phi^n(t)| \). Equation (8) gives us the following recurrence relation for \( |\phi^n(t)| \):

\[
|\phi^n(t)| = \sum_{k=0}^n (n - k + 1)|\phi^k(t_1)||\phi^{n-k+1}(t_2)|,
\]

\[
|\phi^n(\bullet)| = 1.
\]

Using the binomial identity

\[
\sum_{k=0}^n \binom{a+k}{b+n-k} \binom{b}{a} = \binom{a+b+n+1}{a+b+1},
\]

it can be shown that the solution of this equation is

\[
|\phi^n(t)| = \varphi(t) \left( \frac{2|t| + n}{2|t|} \right),
\]

where \( \varphi(t) \) is an integer which depends only on the tree \( t \) (not on \( n \)) and is defined recursively by

\[
\varphi(t) = \varphi(t_1)(2|t_2| + 1)\varphi(t_2),
\]

\[
\varphi(\bullet) = 1.
\]

\( t_1 \) and \( t_2 \) are the two branches of \( t \).

3.5 A compact notation

To write a compact expression for the recursive definition of \( \phi(t) \) we define the deconcatenation of \( (z_1, \ldots, z_n) \) by

\[
\Delta(z_1, \ldots, z_n) = \sum_{i=0}^n (z_1, \ldots, z_i) \otimes (z_{i+1}, \ldots, z_n).
\]

If \( z \) belongs to the vector space \( V \), the map \( \phi(t) \) acts on the tensor module (Fock space) \( T(V) = \bigoplus_{n=0}^\infty V^\otimes n \). Sometimes,
as in QED, \( \phi \) is defined on \( T(V) \times M \), where \( M \) is a fixed vector space, for instance \( M = V^2 \) for the photon propagator.

We define the operator \( d(z) \) by

\[
d(z)\phi^n(t; z_1, \ldots, z_n) = \phi_{\Sigma}^{n+1}(t; z, z_1, \ldots, z_n).
\]

The recursive definition of \( \phi \) becomes

\[
\phi(t) = F \circ (I_d \otimes d(z)) \circ (\phi(t_1) \otimes \phi(t_2)) \circ \Delta,
\]

where \( F(a \otimes b) = F(a, b) \).

It would be interesting to find a family of equations such that, for any \( \phi \) from \( T(V) \) to \( \mathbb{C} \), there is a member of the family of which \( \phi \) is a solution. This would generalize Butcher’s density theorem \(^{14}\), and would provide a general class of equations that would be satisfied by the renormalized Green functions.

### 3.6 Algebra structure

In the case of rooted trees, Butcher \(^{14}\) has defined a group structure of Runge-Kutta methods that Hairer and Wanner \(^{15}\) interpreted as a composition of Butcher series. A similar approach can be used for planar binary trees. A powerful aspect of Butcher’s approach is that algebraic operations are defined on two spaces at the same time: the space of Runge-Kutta methods, and the space of maps over trees. The same strategy will be used here, and the operations will be defined on the space of integral operators and on the space of maps over planar binary trees.

We start with the addition. If we have two Schwinger equations 
\[
X = A + F(Y + \frac{\delta S}{\delta \phi}) \quad \text{and} \quad Y = B + G(Y + \frac{\delta S}{\delta \phi}),
\]

the addition of the integral operators is
\[
H = F + G
\]

and the addition of the maps \( \phi \) (corresponding to the first equation) and \( \psi \) (for the second equation) is defined recursively by

\[
\chi^n(\bullet) = \phi^n(\bullet) + \psi^n(\bullet),
\]

for the initial data \( A+B \) and

\[
\chi^n(t; \{z\}_{1,n}) = \sum_{k=0}^{n} F(\chi^k(t_1; \{z\}_{1,k}), \chi_{\Sigma}^{n-k+1}(t_2; \{z\}_{k+1,n})) + \sum_{k=0}^{n} G(\chi^k(t_1; \{z\}_{1,k}), \chi_{\Sigma}^{n-k+1}(t_2; \{z\}_{k+1,n}))
\]

This addition defines clearly a commutative group structure for the integral operators. It defines also a commutative group structure for the space of maps, where the unit element is \( \phi(t) = 0 \) for all \( t \), and the opposite of \( \phi(t) \) is \( \psi(t) = -(\lambda^t)\phi(t) \).

Multiplication by a scalar \( \lambda \) is defined similarly. An integral operator \( F \) becomes \( \lambda F \), and the corresponding map \( \phi(t) \) becomes \( \lambda^t \phi(t) \). Notice that maps are not equivalent to integral operators since they contain also the initial data. The present definition of the multiplication by a scalar corresponds to the case where the initial data are not changed. If the initial data are also multiplied by \( \lambda \), then \( \phi(t) \) becomes \( \lambda^{t+1} \phi(t) \).

This addition is useful when a Schwinger equation is the sum of various terms. Addition and multiplication by a scalar is probably all we need to renormalize quantum electrodynamics. However, we can proceed with Butcher’s strategy and define another operation coming from a composition of solutions. If we start from two Schwinger equations

\[
X = A + F(Y + \frac{\delta S}{\delta \phi}) \quad \text{and} \quad Y = B + G(Y + \frac{\delta S}{\delta \phi}),
\]

the composition of the solutions is defined as the \( Y \) obtained with the initial data \( B = X \).

It is shown in the appendix that, if \( \gamma \) is the map corresponding to \( Y \) (i.e. \( Y = \sum \phi(t) \)), then \( \chi(t) = \gamma(t) + \psi(t) \), where \( \phi(t) \) is the map associated to the equation for \( X \) and \( \psi^n(t) \) is given by \( \psi^n(\bullet) = 0 \) and

\[
\psi^n(t; \{z\}_{1,n}) = \sum_{k=0}^{n} G(\phi^k(t_1; \{z\}_{1,k}), \chi(t_2; \{z\}_{k+1,n}) + \psi^n(t_2; \{z\}_{k+1,n}) + \psi^n(t_2; \{z\}_{k+1,n}).
\]

This defines a product of integral operators and of maps. In the first proof of Eq.(1) given in the appendix, the integral operator corresponding to this product is constructed. Notice that this operator acts on vectors \( \langle X_{\Sigma} \rangle \).

This product has a unit element (given by \( G = 0 \)).

In the next section, the present method will be applied to the example of quantum electrodynamics (QED).

### 4 The case of QED

We work in the flat Minkowski space with a diagonal metric \( g \) (the diagonal is \( (1, -1, -1, -1) \)). The electron charge is \( e = -|e| \). Repeated indices are summed over.

In 1951, Schwinger \(^{16}\) devised coupled equations involving functional derivatives of \( S(x, y; J) \), the full fermion propagator of QED in the presence of an external electromagnetic source \( J_\mu(x) \):

\[
\square g_{\mu\nu} - (1 - \xi) \partial_\mu \partial_\nu A^\nu(x; J) = -J_\mu(x) - ie\text{tr}[\gamma_\mu S(x, x; J)],
\]

\[
\left[ i\gamma_\mu \partial_\mu - m - e\gamma_\mu A_\mu(x; J) + i\gamma_\mu \frac{\delta}{\delta J_\mu(x)} \right] S(x, y; J) = \delta(x - y).
\]

Building on a work by Polivanov \(^{10}\), Bogoliubov and Shirkov \(^{17}\) transformed this equation into a Schwinger equation coupling the full fermion propagator \( S(x, y) \) with the full photon propagator \( D_{\mu\rho}(x, y) \):

\[
\frac{\square g_{\mu\nu} - (1 - \xi) \partial_\mu \partial_\nu D_{\nu\rho}(x, y)}{\delta A_\rho(z)} = g_\mu^\mu \delta(x - y) - ie\int d^4z \text{tr} \left[ \gamma_\mu \frac{\delta S(x, x; A)}{\delta A_\rho(z)} \right] D_{\nu\rho}(x, y; A),
\]

\[
\left[ i\gamma_\mu \partial_\mu - m - e\gamma_\mu A_\mu(x) \right] S(x, y; A) = \delta(x - y) + ie\int d^4z \gamma_\mu D_{\mu\rho}(x, z; A) \frac{\delta S(x, y; A)}{\delta A_\rho(z)},
\]

\[
\frac{\square g_{\mu\nu} - (1 - \xi) \partial_\mu \partial_\nu D_{\nu\rho}(x, y)}{\delta A_\rho(z)} = g_{\mu\rho} \delta(x - y) - ie\int d^4z \text{tr} \left[ \gamma_\mu \frac{\delta S(x, x; A)}{\delta A_\rho(z)} \right] D_{\nu\rho}(x, y; A),
\]

\[
\left[ i\gamma_\mu \partial_\mu - m - e\gamma_\mu A_\mu(x) \right] S(x, y; A) = \delta(x - y) + ie\int d^4z \gamma_\mu D_{\mu\rho}(x, z; A) \frac{\delta S(x, y; A)}{\delta A_\rho(z)}.
\]
where $A(x)$ is now an external electromagnetic field. As explained in Ref.[7], Eqs. (11) and (12) are not completely equivalent to Eq. (10), they are valid in the limit $A = 0$ ($J = 0$), which is the standard case of QED.

Multiplying Eq. (11) by the bare photon propagator, $D_{\mu\nu}(x,y)$ and Eq. (13) by the bare fermion propagator in the presence of $A$, $S^0(x,y;A) = \left[\gamma^{\mu} \partial_{\mu} - m - e\gamma^\mu A_\mu\right]^{-1}$, we obtain our starting Schwinger equations:

$$D_{\mu\nu}(x,y;A) = D^0_{\mu\nu}(x,y) - i e \int d^4 zd^4 z' D^0_{\mu\lambda}(x,z) \delta S(z,z',A) D_{\nu\lambda}(z',y;A),$$

$$S(x,y;A) = S^0(x,y;A) + i e \int d^4 zd^4 z' S^0(x,z;A) \gamma^\lambda D_{\lambda\mu}(z,z';A) \delta S(z,z',A). \tag{13}$$

In principle, these equations fully determine $S(x,y;A)$ and $D_{\mu\nu}(x,y;A)$.

### 4.1 The tree solution

The method of the previous section is now used to write the solution of Eq. (13). Since $S^0(x,y;A)$ depends on the external potential $A(x)$, the small extension presented in the appendix is required. All quantities will be taken at $A(x) = 0$, so the external potential will not be mentioned (e.g. $S(x,y)$ means $S(x,y;A)$ at $A = 0$).

The notation $\{\lambda \, z\}_{i,j} = \lambda_i z_j, \lambda_{i+1} z_{j+1}, \ldots, \lambda_j z_1$ enables us to write the solution as

$$S(x,y) = \sum_{t} e^{2|t|} \phi^0(t^*;x,y), \tag{14}$$

$$\delta S(x,y) \over \delta A_{\lambda_1}(z_1) = \sum_{t} e^{2|t|+1} \phi^0(t^*;x,y;\lambda_1 z_1), \tag{15}$$

$$D_{\mu\nu}(x,y) = \sum_{t} e^{2|t|} \phi^0(t^*;x,y), \tag{16}$$

$$\delta^n S(x,y) \over \delta A_{\lambda_1}(z_1) \ldots \delta A_{\lambda_n}(z_n) = \sum_{\sigma \in S_n} \sum_{t} e^{2|t|+n} \phi^n(t^*;x,y;\{\lambda_1, z\}_{\sigma(1)}), \tag{17}$$

$$\delta^n D_{\mu\nu}(x,y) \over \delta A_{\lambda_1}(z_1) \ldots \delta A_{\lambda_n}(z_n) = \sum_{\sigma \in S_n} \sum_{t} e^{2|t|+n} \phi^n_{\mu\nu}(t^*;x,y;\{\lambda_1, z\}_{\sigma(1)}). \tag{18}$$

Another change in the notation is that trees have now two colors. The map $\phi(t)$ has a supersymmetric flavor because it has a fermion component for the full fermion propagator and a photon component for the full photon propagator. It is convenient to transfer the fermion/photon index from $\phi$ to the tree $t$. Thus, $\phi(t^*)$ is the photon component of $\phi(t)$ and $\phi(t^0)$ is its fermion component. Furthermore, $\phi(t^0)$ will be drawn with a white root and $\phi(t^*)$ with a black root. All trees are built as $t_1^0 \vee t_2^*$, where the white (photon) branch is on the left and the black (fermion) branch is on the right.

For example, the fermion trees are

$$
\text{fermion trees: }
\begin{array}{c}
  \cdot \\
  \cdot \\
  \cdot
\end{array}
$$

and the photon trees are

$$
\text{photon trees: }
\begin{array}{c}
  \cdot \\
  \cdot \\
  \cdot
\end{array}
$$

Frabetti has calculated the number of trees with $2p + 2q + 1$ vertices, $p + 1$ white leaves and $q + 1$ black leaves as

$$\epsilon_{p,q} = \frac{(p+q)! \cdot (p+q+1)!}{p!q! \cdot (p+1)!(q+1)!} \tag{19}$$

The following recursive definition gives $\phi(t)$ in terms of $\phi(t_1^0)$ and $\phi(t_2^*)$, where $t_1^0$ and $t_2^*$ are the branches of $t$:

$$\phi^n(t^*;x,y;\{\lambda, z\}_{1,n}) = i \sum_{k=0}^{n} \sum_{k'}^{n-k} \int d^4 zd^4 z' \phi^{n-k-k'}(t^0; x, z; \{\lambda, z\}_{1,n-k-k'}) \gamma^\lambda \phi^k_{\lambda\lambda}(t_1^0; z, z'; \{\lambda, z\}_{n-k-k'}), \tag{20}$$

$$\phi^n_{\mu\nu}(t^0; x, z; \{\lambda, z\}_{1,n}) = -i \sum_{k=0}^{n} \int d^4 zd^4 z' D^0_{\mu\lambda}(x,z) \delta A_{\lambda}(z') \phi^n_{\lambda\mu}(t_1^0; z', z; \lambda', z'; \{\lambda, z\}_{k+1,n}). \tag{21}$$

This recursive definition is completed by giving the components of $\phi^n(\cdot)$:

$$\phi^n_{\mu\nu}(\cdot; x,y) = D_{\mu\nu}(x,y), \tag{22}$$

$$\phi^n(\cdot; x,y) = 0 \quad \forall \; n \geq 1, \tag{23}$$

$$\phi^1(\cdot; x,y; \lambda z_1) = S^0(x, z_1) \gamma^\lambda \phi^0_{\lambda\lambda}(x, y), \tag{24}$$

$$\phi^n(\cdot; x,y; \{\lambda, z\}_{1,n}) = S^0(x, z_1) \gamma^\lambda \phi^n_{\lambda\lambda}(z_1, z_2) \gamma^\lambda \cdots \gamma^\lambda S^0(z_n, y). \tag{25}$$

In practice, the double sum of Eq. (19) is replaced by:

$$\phi^n(t^0; x, y; \{\lambda, z\}_{1,n}) = S^0(x, z_1) \gamma^\lambda \phi^{n-1}(t^0; z_1, y; \{\lambda, z\}_{2,n}) + i \sum_{k=0}^{n} \int d^4 zd^4 z' S^0(x, z) \gamma^\lambda \phi^k_{\lambda\lambda}(t_1^0; z, z'; \{\lambda, z\}_{1,k}) \phi^{n-k-k+1}(t_2^0; z, z; \lambda', z'; \{\lambda, z\}_{k+1,n}). \tag{26}$$

This is still a recursive definition, but it uses now the smaller components of $\phi$ for the same tree $t^0$. Notice that the first term of Eq. (26) is absent if $n = 0$. 
Three remarks can be useful at this point. Firstly, considering all quantities at $A = 0$, we find as in section 3.1 that $\delta S^0(x, y)/\delta A(x) = e S^0(x, y) \gamma^\lambda S^0(\lambda, y)$. Therefore, in the definition of $\phi^\lambda(t)$, a factor $e^n$ was suppressed and transferred to the solution Eqs. (14) and (15). It must be checked that this is compatible with renormalization. Secondly, it will be shown in section 6.1 that, from Furry’s theorem, the components $\phi^\lambda(t^0)$ are zero when $n$ is odd. This reduces the sums in Eqs. (16), (17) and (18) to the even components of $\phi(t^n)$. Finally, the vector space $V$ of section 3.3 has now become the space $\{0, 1, 2, 3\} \times \mathbb{R}^4$.

4.2 Diagrammatic interpretation

The recursive solution of the previous sections can be illustrated in the usual diagrammatic language:

$$
\phi(\bullet) = \begin{pmatrix}
0 \\
0 \\
\vdots
\end{pmatrix} \quad \phi(\bullet) = \begin{pmatrix}
\vdots \\
\vdots \\
\vdots
\end{pmatrix}
$$

In other words, $\phi^n(\bullet)$ is a propagator with $n$ dangling photon lines ($n$ starts at 0). More generally, all the components $\phi^n(t^0)$ or $\phi^n(t^0)$ has $n$ dangling photon lines. To see the action of the recursive equation for the photon, observe that, in Eq. (17) the fermion extremities of $t^n_1$ are closed on an additional photon line by a bare vertex (on the left), and each dangling photon line of $t^n_1$ is linked in turn to the photon extremities of $t^n_2$ (on the right). Diagrammatically:

For the fermion propagator, in Eqs. (16) and (18), each dangling photon line of $t^n_1$ is linked in turn to the right photon extremity of $t^n_2$. The left photon extremity of $t^n_2$ is linked to the left extremity of the electron propagator $t^n_2$ by an additional bare vertex.

If we neglect the first term in Eq. (18) we can write diagrammatically:

The next term is

$$
\phi(\bigtriangledown) = \begin{pmatrix}
+ \\
\vdots
\end{pmatrix}
$$

4.3 Diagram enumeration

As a warm-up exercise, we can calculate the number of diagrams in the component $\phi^n(t)$, that we denote $|\phi^n(t)|$. From Eqs. (16) and (17) we find the equations for $|\phi^n(t^0)|$ and $|\phi^n(t^0)|$.

$$
|\phi^n(t^0)| = \sum_{k=0}^{n} (n - k + 1) |\phi^k(t^0)| |\phi^{n-k+1}(t^0)|,
$$

with $\phi^n(\bullet) = 1$, $\phi^n(\bullet) = \delta_{n,0}$.

The solution of this recursive equation is

$$
|\phi^n(t^0)| = \bar{\phi}(t)^{2|t^0| + n},
$$

where $\bar{\phi}(t)$ does not depend on the colour of $t$ and was defined in section 3.4.

This can be used to calculate the total number of diagrams for the electron or photon propagators (the number is the same) contributing at $e^{2n}$. Let us define

$$
s_n = \sum_{t \in Y_n} \bar{\phi}(t) \sum_{t \in Y_n} \bar{\phi}(t_1) (2|t_2| + 1) \bar{\phi}(t_2).
$$

Using Eq. (18) for $n > 0$ we find

$$
s_n = \sum_{k=0}^{n-1} \sum_{|t_1|=k} \bar{\phi}(t_1) \sum_{|t_2|=n-k-1} (2|t_2| + 1) \bar{\phi}(t_2)
$$

$$
= \sum_{k=0}^{n-1} (2k + 1)s_{k}s_{n-k-1}.
$$

The starting value is $s_0 = 1$. For $n = 0, 1, 2, 3, 4, 5$ this gives us 1, 1, 4, 27, 248, 2830, in agreement with Refs. [18, 19]. The generating function $y(x)$ for the sequence $s_n$ satisfies the differential equation $2x^2y' + xy^2 - y + 1 = 0$ with $y(0) = 1$.

This enumeration takes into account neither symmetry nor Furry’s theorem, which says that $|\phi^n(t^0)| = 0$ if $n$ is odd. The main point of this enumeration is to show that each tree represents the sum of a large number of diagrams when $|t|$ is large. This may prove useful for practical calculations.
4.4 Fourier transform

In applications, it is often convenient to work in the $k$-space. The Fourier transform of $\phi(t; x, y; \{\lambda, z\}_1, n)$ is defined by

$$
\psi(t; q, q'; \{\lambda, p\}_1, n) = \int d^4 x d^4 y d^4 z_1 \ldots d^4 z_n e^{i(q - q' + p_1 + \ldots + p_n) \cdot x} \phi(t; x, y; \{\lambda, z\}_1, n).
$$

This corresponds to outgoing momenta $p_i$ along the dangling photon lines. If this is introduced into Eqs. (13) and (17), we find

$$
\psi(t; q, q'; \{\lambda, p\}_1, n) = (2\pi)^4 \delta(q + p_1 + \ldots + p_n - q') \tilde{\phi}(t; q; \{\lambda, p\}_1, n).
$$

The full fermion and photon propagators in Fourier space are

$$
S(q) = \sum_{\lambda} e^{2i|t|} \tilde{\phi}(t; q),
$$

$$
D_{\lambda\mu}(q) = \sum_{\lambda} e^{2i|t|} \tilde{\phi}_{\lambda\mu}(t; q).
$$

Here $\tilde{\phi}(t)$ satisfies the recursive relation

$$
\tilde{\phi}^n(\cdot; q; \{\lambda, p\}_1, n) = S^n(q) \gamma^1 \ldots \gamma^n, \quad \tilde{\phi}^{n-1}(\cdot; q; p_1; \{\lambda, p\}_2, n) +
$$

$$
i \sum_{k=0}^{n-1} \int \frac{dp}{(2\pi)^4} S^k(q) \gamma^1 \ldots \gamma^k \tilde{\phi}^n_{\lambda\mu}(t, q; \{\lambda, p\}_1, n) =\ldots
$$

$$
\tilde{\phi}^n_{\lambda\mu}(t; q; \{\lambda, p\}_1, n) = -i \sum_{k=0}^{n-1} \int \frac{dp}{(2\pi)^4} D_{\mu\lambda}(q)
$$

$$
\tilde{\phi}^k_{\lambda\mu}(t; q; p; \{\lambda, p\}_1, n) +
$$

$$
\text{tr}[(\gamma^1 \ldots \gamma^{k+1})(t; q; \{\lambda, p\}_1, n)]
$$

(19)

where we have noted $P_k = p_1 + \ldots + p_k$ and with the initial data

$$
\tilde{\phi}^0_{\mu\nu}(\cdot; q) = D_{\mu\nu}(0) = -\frac{\gamma_{\mu\nu} - (1 - \epsilon/4) \gamma_{\mu\nu}}{q^2 + \epsilon},
$$

$$
\tilde{\phi}^0_{\mu\nu}(\cdot; q) = 0 \quad \text{for} \quad n \geq 1,
$$

$$
\gamma^1_{\mu\nu}(\cdot; q; \{\lambda, p\}_1, n) = S^0(q) \gamma^1 \gamma^2 \ldots \gamma^n
$$

$$
\gamma^1_{\mu\nu}(\cdot; q; \{\lambda, p\}_1, n) = S^0(q) \gamma^1 \gamma^2 \ldots \gamma^n
$$

As for the real space case, the first term of Eq. (19) is absent for $n = 0$ and

$$
\tilde{\phi}^n_{\lambda\mu}(t; q; \{\lambda, p\}_1, n) = \tilde{\phi}^n_{\lambda\mu}(t; q; \{\lambda, p\}_1, n) +
$$

$$
\tilde{\phi}^n_{\lambda\mu}(t; q; \lambda_1, \ldots, \lambda_{n+1}, p_1, \ldots, p_n).
$$

Again, Furry’s theorem enables us to restrict the sum to the even components of $\tilde{\phi}_{\mu\nu}(t; q)$.

5 Renormalization

In Ref. [4], it has been shown that the renormalization through Hopf algebra created by Kreimer could be done also within the Butcher approach using a difference of Runge-Kutta methods (bare quantities minus counter-terms). The same method can be tried here.

If $X = A + F(X, \frac{\partial X}{\partial x}(z))$ is the equation to renormalize, and if $R$ represents a renormalization prescription (see Ref. [4]), then the renormalized field $X_R$ is given by

$$
X_R = \sum_t \phi_R(t),
$$

where $\phi_R(t)$ is defined recursively from the integral operator $F - R \circ F$ by $\phi^0_R(\bullet) = \phi^0(\bullet)$ and

$$
\phi^n_R(\bullet; \{z\}_1, n) = \sum_{k=0}^n F(\phi^k_R(\bullet; \{z\}_1, k), \phi^{n-k+1}_R(t_1, \ldots, t_k; \{z\}_k, \ldots, \{z\}_n)) - R \left[ F(\phi^k_R(\bullet; \{z\}_1, k), \phi^{n-k+1}_R(t_1, \ldots, t_k; \{z\}_k, \ldots, \{z\}_n)) \right].
$$

Within this approach, renormalization looks rather simple. At each stage of a calculation, use the renormalized $\phi$ instead of the bare ones, and subtract the singularity. This rule seems promising for the case of QED, where $\phi^n(\bullet)$ is regular and needs not be renormalized, and where the recursive operations described by Eqs. (13) and (17) involve only a single loop (a fermion loop for the photon propagator and a photon loop for the fermion propagator). Since $\phi^k_R(t_1)$ and $\phi^{n-k+1}(t_2)$ are regular, there is no subdivergency, and thus no overlapping divergence problem.

Following Kreimer [4], the renormalization group equations are a consequence of the fact that $F - R(F) = F'(F) - R(F) + F - F'(F)$ is independent of the parameters of $R'$.

Clearly, this presentation of renormalization is far too sketchy and further investigation is required to include renormalization prescriptions and calculation of counter-terms, and to check that the results are the same as in the standard theory.

6 Applications

In this section, the previous results are applied to Furry’s theorem and the Ward-Takahashi identities. Finally, a sum over trees is defined for the two-particle Green function.

6.1 Furry’s theorem

Within our approach, Furry’s theorem implies $\phi^n(\t^\circ) = 0$ for odd $n$. To show this, remark that, in Eq. (20), the integral

$$
\int \frac{d^4p}{(2\pi)^4} \text{tr}[(\gamma^1 \ldots \gamma^{k+1})(t; q; \{\lambda, p\}_1, n)]
$$

is zero. This concludes the proof of Furry’s theorem.
is a fermion loop with \( n - k + 2 \) external photon lines. According to Furry’s theorem, this loop is zero when \( n - k \) is odd. From its explicit definition, \( \phi^n(0) = 0 \) is zero if \( n \) is odd (in fact, if \( n \geq 1 \)). We reason recursively on the number of vertices of \( t^\circ \). If \( \phi^n(t^\circ) = 0 \) for odd \( n \) and \( t^\circ \) with up to \( 2N - 1 \) vertices, take \( t^\circ \) with \( 2N + 1 \) vertices. In Eq. (21), the integral over \( p \) is zero if \( n - k \) is odd (Furry’s theorem) and \( \phi^n(t^\circ) \) is zero if \( k \) is odd (because \( t^\circ_0 \) has less vertices than \( t \)). Thus, for the left-hand side of Eq. (21) to be eventually non-zero, \( k \) and \( n - k \) must be even, so \( n \) must be even. Therefore, \( \phi^n(t^\circ) = 0 \) for odd \( n \).

### 6.2 Ward-Takahashi identities

Within the present approach, the Ward-Takahashi identities take the form

\[
\sum_{\mu} k_\mu \partial^n(t^\circ; q; \mu, k, \{\lambda, p\})_{1,n} = \partial^n(t^\circ; q; \{\lambda, p\})_{1,n} - \partial^n(t^\circ; q + \phi; \{\lambda, p\})_{1,n}. \tag{21}
\]

Again, these identities will be proved recursively over the number of vertices of \( t^\circ \). An elementary calculation shows that Eq. (21) is true for \( t^\circ = \bullet \) (Ref. 201, p.238). Using Eq. (20), we can write Eq. (21) as

\[
\sum_{\mu} k_\mu \frac{\partial^n(t^\circ; q; \{\lambda, p\})_{1,n}}{\partial A_\mu(k)} = \partial^n(t^\circ; q; \{\lambda, p\})_{1,n} - \partial^n(t^\circ; q + \phi; \{\lambda, p\})_{1,n}. \tag{22}
\]

The last equation can be proved by applying the differential operator \( \sum_{\mu} k_\mu \frac{\partial}{\partial A_\mu(k)} \) to both sides of Eq. (19). Consider first the second term of Eq. (19). The differential operator acting on \( \phi^n_{\Sigma}(t^\circ_0) \) adds a photon line to \( \phi^n_{\Sigma}(t^\circ_0) \), which gives zero by Furry’s theorem. The differential operator acting on \( \phi^n_{\Sigma - 1}(t^\circ_0) \) will transform according to Eq. (22), which is valid because \( t^\circ_2 \) has less vertices than \( t^\circ \). Carrying out the sum over \( k \), we see that Eq. (22) is true for the second term of the right-hand side of Eq. (19).

Consider now the first term of Eq. (19). For \( n = 0 \) it does not exist, so Eq. (22) is proved for \( n = 0 \) because it is true for the second term. The recursion runs now over \( n \), the functional derivative operates either on \( S^0(q) \) or on \( \phi^{n-1}(t^\circ; q + p_1; \{\lambda, p\})_{2,n} \). Since Eq. (22) is true for both, and for the second term, it is true for \( \phi^n(t^\circ; q; \{\lambda, p\})_{1,n} \).

### 6.3 Homology and Ward-Takahashi identities

The previous Ward-Takahashi identities induce a boundary operator on \( \phi(t) \). In that case, we need to start from the fully symmetrized form of \( \phi(t) \):

\[
\phi^n_{\Sigma}(t; q; \{\lambda, z\})_{1,n} = \sum_{\sigma \in S_n} \phi^n(t; q; \{\lambda, z\})_{\sigma(1), \sigma(n)}.
\]

We define \( \partial \) by

\[
\partial \phi^n_{\Sigma}(t; q; \{\lambda, z\})_{1,n} = \sum_{\mu=0}^3 k_\mu \phi^n_{\Sigma}(t; q; \{\lambda, z\})_{1,n} = \partial^n_{\Sigma - 1}(t; q; \{\lambda, z\})_{1,n} - \partial^n_{\Sigma - 1}(t; k; \{\lambda, z\})_{1,n} - \partial^n_{\Sigma - 1}(t; + k; \{\lambda, z\})_{1,n} - \partial^n_{\Sigma - 1}(t; + k; \{\lambda, z\})_{1,n},
\]

where the last line is the Ward-Takahashi identity. If we write the usual expression for the boundary operator

\[
\partial \phi^n(t; q; \{\lambda, z\})_{1,n} = \sum_{i=1}^n (-1)^i \partial_i \phi^n(t; q; \{\lambda, z\})_{1,n},
\]

we find by explicit calculation that \( \partial^2 = 0 \). This was proved for the fermion component, but \( \partial^2 = 0 \) for the photon component too, because of Furry’s theorem. Notice that, for \( n = 0 \) we have put \( \partial \phi(0) = 0 \).

### 6.4 Two-particle Green function

In Ref. 4 Schwinger proceeds by giving an equation for the full two-particle fermion Green function:

\[
\left[ i \gamma^\mu \partial_{x^\mu} - m - e A_\mu(x_1) \right] S(x_1, x_2; y_1, y_2; J) = \delta(x_1 - y_1)S(x_2, y_2; J) - \delta(x_1 - y_2)S(x_2, y_1; J).
\]

From the identity [3]

\[
\frac{\delta}{\delta J^\mu(x)} = - \int dy G_{\lambda \mu}(y, x) \frac{\delta}{\delta A_\lambda(y)},
\]

we go from the \( J \) to the \( A \) variable (in the limit \( A = 0 \))

\[
\left[ i \gamma^\mu \partial_{x^\mu} - m - e A_\mu(x_1) \right] S(x_1, x_2; y_1, y_2; A) = \delta(x_1 - y_1)S(x_2, y_2; A) - \delta(x_1 - y_2)S(x_2, y_1; A) + \int d^4z \gamma^\mu \frac{\delta S(x_1, x_2; y_1, y_2; A)}{\delta A_\lambda(z)} G_{\lambda \mu}(z, x_1).
\]

It remains to multiply by \( S^0 \) and to take \( A = 0 \). This yields

\[
S(x_1, x_2; y_1, y_2) = S^0(x_1, y_1)S(x_2, y_2) - S^0(x_1, y_2)S(x_2, y_1) + \int d^4z \gamma^\mu \frac{\delta S^0(x_1, x_2; y_1, y_2)}{\delta A_\lambda(z')} G_{\lambda \mu}(z', z).
\]

In Eq. (23), \( S(x, y) \) plays the role of initial data, whereas it is the solution of Eq. (13). Therefore, we are making a composition of solutions, which was considered in section 3.6. The situation is not exactly the same as that of section 5.6, but the proof is similar (only more cumbersome) and we obtain an expression for the two-particle Green function as a sum over planar binary trees

\[
S(x_1, x_2; y_1, y_2) = \sum_{t} e^{2\text{il}(t)} \chi^0(t; x_1, x_2; y_1, y_2).
\]
According to the rule of composition, \( \chi \) is the sum of two terms:

\[
\chi^n(t; x_1, x_2; y_1, y_2; \{\lambda, z\}_{1,n}) = S^0(x_1, y_1)\phi^0(x^*, x_2, y_2; \{\lambda, z\}_{1,n}) - S^0(x_1, y_1)\phi^0(t^*, x_2, y_1; \{\lambda, z\}_{1,n}) + \\
\psi^0(t; x_1, x_2; y_1, y_2; \{\lambda, z\}_{1,n}),
\]

and \( \psi \) itself is given by

\[
\psi^n(t; x_1, x_2; y_1, y_2; \{\lambda, z\}_{1,n}) = S^0(x_1, z_1)\gamma^\lambda
\]

\[
\chi^{n-1}(t; x_1, x_2; y_1, y_2; \{\lambda, z\}_{2,n}) + \\
\int d^2z d^2z' S^0(x, z)\gamma^n\phi^n_{\lambda z}(t^*_n; z', z; \{\lambda, z\}_{1,k})
\]

\[
\chi^{n-k+1}_{\Sigma}(t_2; x, x_2; y_1, y_2; \lambda, z', \{\lambda, z\}_{k+1,n}).
\]

Equation (24) is not very simple, but it provides a way to calculate recursively all orders of the perturbation expansion for the two-particle Green function of QED. In that sense, it is not so complicated.

7 Vacuum polarization, self-energy

To calculate vacuum polarization and self-energy, we introduce a further operation on planar binary trees. Let

\[
P(t) = \sum_{i=1}^{n(t)} u_i \otimes v_i,
\]

where \( n(t) \) is an integer recursively defined by

\[
n(\bullet) = 0,
\]

\[
n(t) = 0 \text{ if } t = t_1 \lor \bullet,
\]

\[
n(t) = 1 + n(t_2) \text{ if } t = t_1 \lor t_2, \quad t_2 \neq \bullet,
\]

and the planar binary trees \( u_i \) and \( v_i \) are determined by

\[
P(\bullet) = 0,
\]

\[
P(t) = 0 \text{ if } t = t_1 \lor \bullet,
\]

\[
P(t) = (t_1 \lor \bullet) \otimes t_2 + \sum_{i=1}^{n(t_2)} (t_1 \lor u_i) \otimes v_i
\]

\[
\text{if } t = t_1 \lor t_2, \quad t_2 \neq \bullet.
\]

The trees \( u_i \) and \( v_i \) in Eq. (25) are generated by Eq. (24) for \( t = t_2 \).

As a more graphical definition, for a tree \( t \), we consider the path starting from the root and climbing up the tree by taking, at each vertex, the right branch. This path terminates at the extreme right leaf of the tree and goes through \( n(t) + 2 \) vertices (including the root and the leaf). For each vertex \( s_i \) along that path, excluding the root and the leaf, we cut \( t \) into two trees \( u_i \) and \( v_i \), where \( v_i \) is the subtree of \( t \) that has \( s_i \) as a root, and \( u_i \) the subtree of \( t \) that has \( s_i \) as a leaf. For example:

\[
P\left(\begin{array}{c}
\bigcirc \\
\end{array}\right) = \bigcirc \otimes \bigcirc.
\]

This operation will help calculating the vacuum polarization and the self-energy. If we define \( Y \) by

\[
X = \sum_t \phi^0(t) = \frac{1}{1/\phi^0(\bullet) - Y},
\]

it is proved in the appendix that \( Y = \sum_t \psi^0(t) \), where

\[
\psi^0(\bullet) = 0,
\]

\[
\psi^0(t) = \frac{1}{\phi^0(\bullet)}\left(\phi^0(t) - \frac{1}{\phi^0(\bullet)} \sum_{i=1}^{n(t)} \phi^0(u_i)\psi^0(v_i)\right).
\]

In this equation, \( u_i, v_i \) and \( n(t) \) are determined from \( t \) by Eq. (24).

7.1 Vacuum polarization

From the general formula, we deduce that the vacuum polarization is given, in terms of the map \( \phi \) for the full photon propagator, by

\[
\Pi_{\lambda \mu}(q) = \sum_t e^{2|t|} \psi^0_{\lambda \mu}(t^*; q),
\]

with \( \psi^0(\bullet; q) = 0 \) and

\[
\psi^0_{\lambda \mu}(t^*; q) = (\gamma^\alpha q_\alpha - m)\phi^0_{\lambda \mu}(t^*; q)(\gamma^\beta q_\beta - m) - \\
(\gamma^\alpha q_\alpha - m)\sum_{i=1}^{n(t)} \phi^0(u_i^*; q) g^{\rho \nu'} \psi^0_{\mu \nu'}(v_i; q).
\]

Incidentally, it can be noted that the full vacuum polarization \( \Pi^{\lambda \mu}(q) \) is related to the fermion loop by (Ref. 23 p.477):

\[
\Pi^{\lambda \mu}(q) = -i \sum_t e^{2|t| + 2} \int \frac{d^4p}{(2\pi)^4} \text{tr}[\gamma^\lambda \phi^4(t^*; p; \mu, -q)].
\]

7.2 Self-energy

Similarly, the self-energy is given, in terms of the map \( \phi \) for the full fermion propagator, by

\[
\Sigma(q) = \sum_t e^{2|t|} \psi^0(t^*; q),
\]

with \( \psi^0(\bullet; q) = 0 \) and

\[
\psi^0(t^*; q) = (\gamma^\alpha q_\alpha - m)\phi^0(t^*; q)(\gamma^\beta q_\beta - m) - \\
(\gamma^\alpha q_\alpha - m)\sum_{i=1}^{n(t)} \phi^0(u_i^*; q)\psi^0(v_i^*; q).
\]
From the formula for the self-energy, we can deduce the complete one-particle irreducible three-point function \( \Gamma''(p, p') \) \[21\], p.335. We reintroduce the external potential to write
\[
\Sigma(x, y; A) = \sum_t (i\gamma^\lambda \partial_{x^\lambda} - m - e\gamma^\lambda A(x)) \phi(t^*; x, y; A) \langle -i\gamma^\mu \bar{\partial}_{y^\mu} - m - e\gamma^\mu A(y) \rangle - \sum_t (i\gamma^\lambda \partial_{x^\lambda} - m - e\gamma^\lambda A(x)) \]
\[
\frac{n(t)}{CQ} \int dz' \phi(u^*_i; x, z'; A) \psi(v^*_i; z', y; A),
\]
where \( \bar{\partial}_{y^\mu} \) acts on the left. In the real space, \( \Gamma''(x, y; z) \) is given
\[
\Gamma''(x, y; z) = \sum_t e^{(|t|+1)} \phi^0(t^*; x, y; \nu, z),
\]
with
\[
\psi^1(t^*; x; y; \nu, z) = -\gamma^\nu \delta(z - x) - \phi^0(t^*; x, y)(-i\gamma^\mu \bar{\partial}_{y^\mu} - m) - (i\gamma^\lambda \partial_{x^\lambda} - m) \phi^0(t^*; x, y) \gamma^\nu \delta(z - y) + (i\gamma^\lambda \partial_{x^\lambda} - m) \psi^1(t^*; x, y; \nu, z)(-i\gamma^\mu \bar{\partial}_{y^\mu} - m) + \gamma^\nu \delta(z - x) \sum_i \int dz' \phi^0(u^*_i; x, z'; \nu, z) \psi^1(v^*_i; z', y) - (i\gamma^\lambda \partial_{x^\lambda} - m) \sum_i \int dz' \phi^0(u^*_i; x, z'; \nu, z) \psi^1(v^*_i; z', y) - (i\gamma^\lambda \partial_{x^\lambda} - m) \sum_i \int dz' \phi^0(u^*_i; x, z'; \nu, z) \psi^1(v^*_i; z', y).
\]
In Fourier space, this gives us
\[
\Gamma''(q, q + p) = \sum_t e^{(|t|+1)} \psi^1(t^*; q; \nu, p),
\]
with
\[
\psi^1(t^*; q; \nu, p) = -\gamma^\nu \phi^0(t^*; q) (i\gamma^\mu (\nu + p) - m) - (i\gamma^\lambda \partial_{q^\lambda} - m) \phi^0(t^*; q) \gamma^\nu + (i\gamma^\lambda \partial_{q^\lambda} - m) \phi^0(t^*; q; \nu, p) (i\gamma^\mu (\nu + p) - m) + \gamma^\nu \sum_i \phi^0(u^*_i; q + p) \psi^0(v^*_i; q + p) - (i\gamma^\lambda \partial_{q^\lambda} - m) \sum_i \phi^0(u^*_i; q) \psi^0(v^*_i; q + p) - (i\gamma^\lambda \partial_{q^\lambda} - m) \sum_i \phi^0(u^*_i; q; \nu, p) \psi^0(v^*_i; q + p).
\]

8 Interaction with an external field

In this section we come back to the original Schwinger equation, because the presence of an external source is a convenient way to represent the nuclei in the QED of matter.

Starting from Eq.\([14]\), we multiply by the corresponding bare Green functions and we introduce \( A''(x; J) \) into the second equation to obtain
\[
S(x, y; J) = S^0(x, y) - ie \int d^4zd^4\bar{z}' S^0(x, z) \gamma^\mu D_{\mu\nu}(z, z')
\]
\[
J''(z') S(z, y; J) - ie^2 \int d^4zd^4\bar{z}' S^0(x, z) \gamma^\mu
\]
\[
D_{\mu\nu}(z, z') \text{tr}[\gamma^\nu S(z', \bar{z}; J)] S(z, y; J) -
\]
\[
ie \int d^4z S^0(x, z) \gamma^\mu \frac{\delta S(x, y; J)}{\delta J^\mu(z)}.
\]

Here, the Schwinger equation is a sum of three terms. The first term is simply the classical interaction with the external source \( J''(z_2) \), it can be solved by defining a bare propagator in the presence of this source
\[
S^0(x, y; J)^{-1} = i\gamma^\mu \partial_{\mu} - m + e\gamma^\mu \int d^4z D_{\mu\nu}(x, z) J''(z).
\]

Equation \([27]\) becomes now
\[
S(x, y; J) = S^0(x, y; J) - ie^2 \int d^4zd^4\bar{z}' S^0(x, z; J) \gamma^\mu
\]
\[
D_{\mu\nu}(z, z') \text{tr}[\gamma^\nu S(z', \bar{z}; J)] S(z, y; J) -
\]
\[
ie \int d^4z S^0(x, z) \gamma^\mu \frac{\delta S(x, y; J)}{\delta J^\mu(z)}.
\]

This equation is solved by the usual methods, and the recursive definition of \( \phi(t) \) is
\[
\phi^n(\bullet; x; y; \{\lambda, z\}_{1,n}) = (-1)^n \int d^4s_1 \ldots d^4s_n S^0(x, s_1; J)
\]
\[
\sum_{k=0}^{n-k} \int d^4zd^4\bar{z}' \phi^{n-k-k'}(\bullet; x, z; \{\lambda, z\}_{1,n-k-k'}) D_{\mu\nu}(z, z')
\]
\[
\text{tr}[\gamma^\nu \phi^{n-k-k'}(t_1; \bar{z}', \bar{z}; \{\lambda, z\}_{n-k-k' +1,n})]
\]
\[
\phi^{n-k}(t_2; z, y; \{\lambda, z\}_{n-k+1,n}) -
\]
\[
i \int d^4z \phi^{n-k}(\bullet; x, z; \{\lambda, z\}_{1,k})
\]
\[
\gamma^{\mu} \frac{\delta S^0_{\mu\nu}(x, z; \{\lambda, z\}_{1,k})}{\delta J^\nu(z)}
\]
where the last term is non zero only if \( t \) has the special shape \( t = \bullet \lor t_2 \).

According to Schwinger \([8]\), the full photon Green function in the presence of an external current \( J \) is given by
the functional derivative of $A(x; J)$ with respect to $J(y)$. Therefore
\[
D_{\lambda\mu}(x, y; J) = -\frac{\delta A_\lambda(x; J)}{\delta J^\mu(y)} = D_{\lambda\mu}^{0}(x, y) + ie \int dz D_{\lambda\mu}^{0}(x, z) \text{tr} \left[ \gamma^\nu \frac{\delta S(z, z; J)}{\delta J^\mu(y)} \right] = D_{\lambda\mu}^{0}(x, y) + ie^2 \sum_t \int dz D_{\lambda\mu}^{0}(x, z) \text{tr} \left[ \gamma^\nu \phi^1(t; z, z; \mu, y) \right].
\]

It is also possible to start directly from Eq. (27) and to write a tree solution of this equation using bare fermion Green functions. Here, the strong field case was treated because it is probably more interesting for applications to solid-state physics.

9 Conclusion

A method was presented to write the solution of some Schwinger equations as a series over planar binary trees. In quantum field theory, it is common to expand over the number of loops or to use integral equations relating, for instance, the full propagator to the full vertex. The first method gives explicit results but becomes very complex, and hundreds of diagrams must be built and calculated after the first few terms of the perturbation expansion. The second method is formally powerful but not very explicit because an $n$-body Green function is expressed in terms of an unknown $(n+1)$-body Green function. The present approach is a way to mix these two methods to obtain an explicit recursive formula for the propagators and its functional derivatives.

The main point of the method is that explicit recursive expressions can be given for the solution of Schwinger equations. Because of the recursive structure, the results obtained at each step can be reused for the next step.

The present paper is only a first exploration of the method of series indexed by planar binary trees, and much work remains to be done to explore its algebraic properties and its applications.

Two kinds of applications were presented here. On the one hand, a series indexed by planar binary trees was given for various physical quantities (full fermion and photon propagators, full two-body Green function), and a method was given to deduce from this a series for vacuum polarization, fermion self-energy and irreducible vertex function. Although the formulas may be a bit cumbersome, they are derived and proved easily. On the other hand, the recursive nature of the terms of the series is well suited to prove properties to all orders of perturbation theory. This was done in the case of the Ward-Takahashi identities.

The present work can be expanded in various directions. Other field theories can be investigated, as well as many-particle Green functions. The present treatment was restricted to classical electromagnetic sources, it could be worthwhile to study the case of anticommuting fermion sources. Furthermore, planar binary trees could be used to solve the Hedin equation [22] and investigate the GW-approximation [23] of solid-state physics.

As a last point, it can be noticed that previous articles have presented general planar trees as the structure adapted to quantum field theory [24]. In fact, planar trees and planar binary trees are equivalent for that purpose. Since the number of planar trees with $n$ vertices is equal to the number of planar binary trees with $2n - 1$ vertices [1], there is a bijection $\Psi$ between planar trees and planar binary trees. If $t$ is a planar tree, the bijection can be realized recursively by, for example, $\Psi(\bullet) = \bullet$ and
\[
\Psi(B_+(t)) = \Psi(t) \vee \bullet, \quad \Psi(B_+(t_1, t_2, \ldots, t_k)) = \Psi(t_1) \vee \Psi(B_+(t_2, \ldots, t_k)),
\]
where $B_+$ is the grafting operator defined by Kreimer [1]. to build rooted trees. This bijection is also apparent in the existence of two methods for the numerical solution of differential equations on Lie groups: one based on planar trees [25], the other on planar binary trees [26]. Planar binary trees were chosen here because the recursive formulas look simpler and because of the mathematical results of Loday, Frabetti and collaborators.

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11 Appendix

This appendix contains proofs of some of the statements contained in the text.

11.1 Proof of Eq. (3)

Equation (3) will be proved in two steps. Firstly, it will be shown that if $\phi^n(t)$ satisfies Eq. (3), then $\frac{\delta \phi^n(t)}{\delta \nu(y)} = \phi^{n+1}_S(t)$, secondly, that the sum over trees is a solution of Eq. (3).

The first step is to show that
\[
\frac{\delta \phi^n(t; \{z\}_{1,n})}{\delta \nu(y)} = \phi^{n+1}_S(t; y, \{z\}_{1,n}). \tag{29}
\]
This will be proved inductively. According to the construction of $\phi^n(\bullet)$ from the initial data $A$, the relation is true for $t = \bullet$. Now we assume that the relation is true up to trees with $2N - 1$ vertices. Let $t$ be a tree with $2N + 1$
vertices. The functional derivative of Eq. (8) gives us
\[
\frac{\delta \phi^n(t; \{ z \}_1,n)}{\delta \nu(y)} = \sum_{k=0}^{n} F^{(\delta \phi^k(t_1; \{ z \}_1, k),} \phi^{n-k+1}_{\Sigma}(t_2; z, \{ z \}_{k+1,n}) + \\
\frac{\delta \phi^{n-1}_{\Sigma}(t_2; z, \{ z \}_{k+1,n})}{\delta \nu(y)} \nonumber
\]

Since \( t_1 \) and \( t_2 \) have less vertices than \( t \), relation (24) is true for them and we obtain
\[
\frac{\delta \phi^n(t; \{ z \}_1,n)}{\delta \nu(y)} = \sum_{k=0}^{n} F^{(\delta \phi^k(t_1; \{ z \}_1, k),} \phi^{n-k+1}_{\Sigma}(t_2; z, \{ z \}_{k+1,n}) + \\
\frac{\delta \phi^{n-1}_{\Sigma}(t_2; z, \{ z \}_{k+1,n})}{\delta \nu(y)} \nonumber
\]

where \( \phi^k_{\Sigma}(t; \{ z \}_1,n) \) distributes the variables \( z_1 \) and \( z_2 \) over the \( k \) positions \( z_1, \ldots, z_k \) without changing the order of \( z_3, \ldots, z_k \). All ways to take two number among \( k \) are used, so it is clear that \( \phi^{n}_{\Sigma}(t; z_1, z_2, \ldots, z_k) \) is symmetric in \( z_1, z_2 \).

On the other hand, we know from Eq. (6) that
\[
\phi^{n+1}(t; y, \{ z \}_1,n) = F^{(\phi^0(t), \phi^{n+2}_{\Sigma}(t_2; y, \{ z \}_1,n)) + F^{(\phi^{n+2}_{\Sigma}(t_1; y, \{ z \}_1,n), \phi^1(t_2; z)) + \\
\sum_{k=1}^{n} F^{(\phi^k(t_1; y, \{ z \}_1,k-1), \phi^{n-k+2}_{\Sigma}(t_2; z, \{ z \}_{k+1,n}) + \\
\sum_{k=1}^{n} F^{(\phi^k(t_1; \{ z \}_1,k), \phi^{n-k+2}_{\Sigma}(t_2; y, \{ z \}_{k+1,n}) + 
\]

If we symmetrize \( y \) in \( \phi^{n+1} \), we obtain
\[
\phi^{n+1}_{\Sigma}(t; y, \{ z \}_1,n) = F^{(\phi^0(t), \phi^{n+2}_{\Sigma}(t_2; z, y, \{ z \}_1,n)) + F^{(\phi^{n+2}_{\Sigma}(t_1; y, \{ z \}_1,n), \phi^1(t_2; z)) + \\
\sum_{k=1}^{n} F^{(\phi^k(t_1; y, \{ z \}_1,k-1), \phi^{n-k+2}_{\Sigma}(t_2; z, \{ z \}_{k+1,n}) + \\
\sum_{k=1}^{n} F^{(\phi^k(t_1; \{ z \}_1,k), \phi^{n-k+2}_{\Sigma}(t_2; y, \{ z \}_{k+1,n}) + 
\]

Comparing this with Eq. (30), we see that the two expressions are identical, and the property is proved for \( t \).

If we denote by \( X \) the (formal) sum \( X = \sum_t \phi^0(t) \), then from Eq. (29) we obtain
\[
\frac{\delta X}{\delta \nu(z)} = \sum_t \phi^0(t; z). \quad \text{Using Eq. (9) we can write}
\]
\[
X = \sum_t \phi^0(t) = \phi^0(\bullet) + \sum_{t \neq \bullet} F(\phi^0(t_1), \phi^1(t_2; z)). 
\]

At this point intervenes the essential property Eq. (1) that each tree different from \( \bullet \) is generated in a unique way by the grafting of two trees \( t_1 \) and \( t_2 \). The sum over \( t_1 \) and \( t_2 \), which are the branches of \( t \), can be replaced by an unrestricted sum over all trees \( t_1 \) and \( t_2 \). Thus
\[
X = \phi^0(\bullet) + \sum_{t_1, t_2} F(\phi^0(t_1), \phi^1(t_2; z)) = \phi^0(\bullet) + F\left(\sum_{t_1} \phi^0(t_1), \sum_{t_2} \phi^1(t_2; z)\right) = A + F(X, \delta X).
\]

A small extension of the previous result is necessary to treat the case of QED. The Schwinger equation is now \( X = A + AF(X, \delta X) \), and the solution is \( X = \sum_t \phi(t) \), where \( \phi(\bullet) \) is the usual initial data and the recurrence relation becomes
\[
\phi^n(t; \{ z \}_1,n) = \sum_{k=0}^{n} \sum_{k'=0}^{n-k} \phi^k(\bullet; \{ z \}_1,k) F(\phi^{k'}(t_1; \{ z \}_{k+1,k+k'}, \phi^{n-k-k'}_{\Sigma}(t_2; z, \{ z \}_{k+k'+1,n})).
\]

The reasoning is the same as for the simple Schwinger equation. We start by proving that \( \frac{\delta \phi^n(t)}{\delta \nu(z)} = \phi^0_{\Sigma}(t) \). This is done by using the recursive definition of \( \phi \) to write both sides in terms of \( \phi(t_1) \) and \( \phi(t_2) \). Then we write \( X = \sum_t \phi^0(t) \) and we show that, because of the recurrence relation for \( \phi \), it satisfies \( X = A + AF(X, \delta X) \).

In this paper, the treatment was restricted to linear integral operators \( F \). However, the method can treat other cases. For instance, if \( F \) depends on \( X^2 \), this square can be, in some sense, “linearized” by the planar trees. If \( X = \sum_t \phi^0(t) \), then
\[
X^2 = \left( \sum_t \phi^0(t) \right)^2 = \sum_{t_1} \sum_{t_2} \phi^0(t_1) \phi^0(t_2) = \sum_t \psi^0(t),
\]

where \( \psi^0(t) \) (and more generally \( \psi^n(t) \)) is defined by
\[
\psi^0(\bullet) = 0, \quad \psi^n(t) = \sum_{k=0}^{n} \phi^k(t_1) \phi^{n-k}(t_2).
\]

11.2 Proof of Eq. (9)

Equation (9) will be proved by two methods. Both are fast and easy. In the first method, we define a Schwinger equation
\[
\left( \begin{array}{c}
X \\
Z
\end{array} \right) = \left( \begin{array}{c}
A \\
0
\end{array} \right) + H \left( \begin{array}{c}
X \\
Y
\end{array} \right),
\]

where \( H = \Delta_{\Sigma} + A^\dagger A = \sum_{\Sigma} \Delta_{\Sigma} + \sum_{\Sigma} A^\dagger A \).
where
\[ H(X,Z) = \left( F(X, \frac{\delta X}{\delta v(z)}) - G(X + Z, \frac{\delta X}{\delta v(z)} + \frac{\delta Z}{\delta v(z)}) \right). \]

This is a Schwinger equation whose solution is given by Eq.\((\text{23})\), where the map is
\[ \left( \phi \psi \right)(t) = \sum_{k=0}^{n} \left( F(\phi^k(t_1), \phi^{n-k+1}(t_2)) \right) G(\phi^k(t_1) + \psi(t_1), \phi^{n-k+1}(t_2) + \psi^n(t_2)), \]
which is Eq.\((\text{24})\).

The upper component of Eq.\((\text{33})\) is the first Schwinger equation \( X = A + F(X, \frac{\delta X}{\delta v(z)}) \). If we write \( Y = X + Z \), the lower component is \( Y = X + G(Y, \frac{\delta Y}{\delta v(z)}) \). Therefore, \( X + Z \) is the solution of the composition of equations. The map corresponding to this solution is \( \chi(t) = \phi(t) + \psi(t) \).

The second proof can also be useful. It starts by adding a parameter \( s \) to the Schwinger equations: \( X(s) = A + sF(X(s), \frac{\delta X(s)}{\delta v(z)}) \) and \( Y(s) = X(s) + sG(Y(s), \frac{\delta Y(s)}{\delta v(z)}) \). We take the \( n \)-th derivative of \( Y \) with respect to \( s \), and we write \( Y^{(n)} \) for its value at \( s = 0 \). The chain rule gives
\[ Y^{(n)}(s) = X^{(n)} + n \sum_{k=0}^{n} \binom{n}{k} \frac{\delta Y^{(n-k-1)}}{\delta v(z)}(t), \]
It can be shown recursively that
\[ Y^{(n)} = n! \sum_{|t|=n} (\phi^0(t) + \psi^0(t)), \]
where \( \psi \) satisfies Eq.\((\text{24})\). The result follows by expanding \( X(s) \) and \( Y(s) \) as power series of \( s \) and taking its value for \( s = 1 \).

11.3 Proof of Eq.\((\text{26})\)

Let \( X(s) = 1 + \sum_{t\neq \bullet} s^{||t||} \phi(t) \). Let us show that the series for \( Y(s) = 1 - 1/X(s) \) is given by \( Y(s) = \sum_{t\neq \bullet} s^{||t||} \psi(t) \), where \( \psi(t) \) is defined by Eq.\((\text{26})\).

The first step is to prove that, if \( P(t) \) is defined by Eq.\((\text{23})\), then, with an abuse of notation,
\[ P(Y_n) = \sum_{k=1}^{n-1} Y_k \otimes Y_{n-k}, \]
(32)
or more precisely
\[ \sum_{|t|=n} P(t) = \sum_{k=1}^{n-1} \left( \sum_{|u|=k} u \otimes \left( \sum_{|v|=n-k} v \right) \right). \]

As usually, this will be proved recursively. The property is true for \( n = 2 \), because
\[ P(Y_2) = 0, \quad P(Y_1) = Y \otimes Y, \]
If this is true up to \( n \), then, from Eq.\((\text{11})\)
\[ \sum_{|t|=n+1} P(t) = \sum_{k=0}^{n} \sum_{|t_1|=k \ |t_2|=n-k} P(t_1 \lor t_2). \]
By definition \((\text{25})\),
\[ \sum_{|t|=n+1} P(t) = \sum_{k=1}^{n} \sum_{|t_1|=k \ |t_2|=n-k} (t_1 \lor \bullet) \otimes t_2 + \sum_{n-1}^{n} \sum_{m=1}^{n} \sum_{k=m+1}^{n} \sum_{t_1 \lor Y_k = m} \otimes Y_m. \]
We use property \((\text{32})\) in the right-hand side:
\[ \sum_{|t|=n+1} P(t) = \sum_{k=1}^{n} \sum_{|t_1|=k \ |t_2|=n-k} (t_1 \lor \bullet) \otimes t_2 + \sum_{n-1}^{n} \sum_{m=1}^{n} \sum_{k=m+1}^{n} (t_1 \lor Y_k = m) \otimes Y_m. \]
In the second term, we use Eq.\((\text{11})\) to sum over \( k \) by
\[ \sum_{k=m+1}^{n} Y_{n-k} \lor Y_k = Y_{n-m+1} - Y_{n-m} \lor \bullet. \]
Therefore
\[ \sum_{|t|=n+1} P(t) = \sum_{k=0}^{n} \sum_{|t_1|=k \ |t_2|=n-k} (t_1 \lor \bullet) \otimes t_2 + \sum_{n-1}^{n} \sum_{m=1}^{n} \sum_{|t_1|=n-m+1 \ |t_2|=m} (t_1 \lor \bullet) \otimes t_2 - \sum_{n-1}^{n} \sum_{m=1}^{n} \sum_{|t_1|=n-m} (t_1 \lor \bullet) \otimes t_2. \]
To complete the proof, we start from \( X(s) = 1 + \sum_{t\neq \bullet} s^{||t||} \phi(t) \) and we define \( Y(s) = \sum_{t\neq \bullet} s^{||t||} \psi(t) \), where \( \psi(t) \) is given by \( \psi(\bullet) = 0, \psi(t) = \phi(t) - \sum \phi(u_i) \otimes \psi(v_i) \).

Thus
\[ \sum_{|t|=1}^{n} \sum_{|t|=1}^{n} \sum_{|t|=1}^{n} \sum_{i=1}^{n} (t \lor u_i) \otimes \psi(v_i). \]
From Eq.\((\text{32})\), we see that \(|t|=|u_i|+|v_i|\) and
\[ \sum_{|t|=1}^{n} \sum_{|t|=1}^{n} \sum_{i=1}^{n} (t \lor u_i) \otimes \psi(v_i). \]

\[ \left( \sum_{|u|=0}^{n} s^{||u||} \phi(u) \right) \left( \sum_{|v|=0}^{n} s^{||v||} \psi(v) \right). \]
From the definition of $X(s)$ and $Y(s)$ we deduce

$$Y(s) - \psi(t) = X(s) - \phi(t) - 1 - (X(s) - 1)Y(s).$$

From the definition of $\psi(t)$ we get $\psi(t) = \phi(t)$, and $Y(s)$ satisfies the equation

$$X(s) - X(s)Y(s) = 1,$$

or $X(s) = 1/(1-Y(s))$. To simplify the notation, we have assumed $\phi(\bullet) = 1$. The general case is proved similarly and leads to Eq. (26).

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