CONSTRUCTION OF MAXIMAL HYPERSURFACES WITH
BORDER CONDITIONS

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Abstract

We construct maximal hypersurfaces with a Neumann boundary condition in Minkowski
space via mean curvature flow. In doing this we give general conditions for long time ex-
istence of the flow with boundary conditions with assumptions on the curvature of a the
Lorentz boundary manifold.

1. Introduction and notation

In this paper we use Mean Curvature Flow (MCF) with a Neumann boundary
condition to construct maximal hypersurfaces with boundary in Minkowski space
\( \mathbb{R}^{n+1}_1 \) for \( n \geq 2 \), which are perpendicular to a given Lorentz surface, \( \Sigma \) at their
boundary. Maximal surfaces are well known to be useful in the study of semi-
Riemannian manifolds and mathematical relativity. A famous example in which
these surfaces play a central part is the first proof of the positive mass conjecture
by Schoen–Yau \[12\]. Correspondingly the existence and properties of such surfaces
have been well studied, and we do not give a full literature review here. We mention
here Bartnik \[1\], for existence of entire maximal hypersurfaces in asymptotically
flat spacetimes, Bartnik and Simon \[2\] where solvability of the Dirichlet problem
in Minkowski space was proven, and Gerhardt \[6\] for the existence of foliations
of constant mean curvature and the solvability of the Dirichlet problem in curved
spacetimes. Ecker and Huisken \[4\] first used a parabolic prescribed mean curvature
flow to construct surfaces of prescribed mean curvature, and the assumptions on
boundary manifolds for such flows have been weakened by Gerhardt \[7\].

We require two things to construct our maximal surfaces, firstly that the flow
stays in a bounded region of Minkowski space, and secondly that that under MCF
the flowing surface remains strictly spacelike (which then implies the flow exists
for all time). The first of these may be achieved by assuming the existence of
suitable comparison solutions. The second requirement will be fulfilled by a curva-
ture assumption on the boundary manifold, which in dimension 2 is akin to mean
convexity. We remark that the flow is still interesting in the absence of some of
these assumptions, for example, we may get convergence to homothetic solutions
(see \[10\]), and that the estimates in this paper may still be of interest in some such
situations. If the flow remains in a bounded region, then for any sequence of times
we may find a subsequence \( t_i \) such that \( M_{t_i} \) converges to a minimal surface. To
obtain better convergence, for example convergence of the whole flow, we need to
assume that the maximal surface is stable under the flow, see the final section of
this paper for a discussion on this.

Suppose \( \Sigma \) is a semi-Riemannian hypersurface with a positive unit normal \( \mu \).
Let \( F : M^n \times [0,T] \to \mathbb{R}^{n+1}_1 \) be such that

\[
\begin{align*}
\frac{dF}{dt} &= H \nu \\
F(\cdot,0) &= M_0 \\
F(x,t) &\subset \Sigma \\
\langle \nu, \mu \rangle (x,t) &= 0
\end{align*}
\]

(1)
then \( F \) moves by Mean Curvature Flow with a Neumann free boundary condition \( \Sigma \) (here \( \nu(x,t) \) is the normal to \( F \) at time \( t \)). We will assume from here onwards that \( \Sigma \) is topologically a cylinder, and \( M^n \) is topologically a \( n \)-ball.

We will need various geometric quantities on various manifolds. A bar will imply quantities on \( \mathbb{R}^{n+1} \), for example \( \Delta, \nabla, \ldots \) and so on; no extra markings \( \Delta, \nabla, \ldots \) will refer to geometric quantities on \( M_t \) our flowing surface at time \( t \) and for any other manifold \( Z \) \( \Delta^Z, \nabla^Z, \ldots \) etc. will refer to the Laplacian, covariant derivatives, \( \ldots \) on \( Z \).

We state the main theorem of this paper:

**Theorem 1.** Suppose that \( \Sigma \) satisfies Conditions 1 and 2 below, such that the flowing surface \( M_t \) remains in a compact region of \( \mathbb{R}^{n+1} \). Then the flow exists for all time and is smooth with uniform bounds on all derivatives. Furthermore there exists a sequence \( t_i \to \infty \) such that \( M_{t_i} \to M_\infty \) where \( M_\infty \) is a maximal surface satisfying the boundary condition. If at the boundary of \( M_\infty \), \( A_\Sigma(\nu_\infty, \nu_\infty) > 0 \) then the whole flow converges to \( M_\infty \) in the sense that \( M_t \to M_\infty \) smoothly as \( t \to \infty \).

**Remark 1.** The assumption that the flow stays in a compact region may be attained in practice by showing the existence of suitable bounding comparison solutions. See Section 3 for further details.

The Theorem is proven as follows: In Section 2 we show that the above flow is equivalent to a quasilinear PDE, indicating that the key to obtaining the above is a suitable gradient estimate. In Section 3 we determine what constitutes a comparison solution with boundary conditions, see equation (3) and Proposition 4. In Section 4 we calculate the necessary evolution equations and boundary derivatives. In Section 5 we use an iteration argument to prove suitable estimates on the mean curvature culminating in Proposition 13. This then allows us to prove the gradient estimate, Theorem 17 which demonstrates that the above flow exists for all time and is uniformly smooth, see Corollary 19. In Section 6 we prove sequential convergence and construct comparison solutions to give conditions for stability of maximal surfaces under MCF, which are therefore conditions for convergence of the whole flow, see Corollary 24.

Clearly the geometry of \( \Sigma \) plays an important role, and it is necessary to impose some conditions. Indeed, in the absence of any assumptions we may construct the following example of singular behaviour: In \( \mathbb{R}^2 \) we parametrise a trumpet boundary manifold \( \Sigma \) graphically by \( y = \log \sinh |x| \), which has been chosen so that the Minkowski equivalent of the grim reaper solution to MCF given by \( u(x,t) = \log \cosh x + t \) is perpendicular at every point. Then starting at any negative time we obtain the solution in Figure 1. At time \( t = 0 \) we see that this solution is tangent to the light cone at infinity, and the Neumann boundary condition is no longer defined. We are able to continue the flow for \( t > 0 \) on the interior but we no longer have a boundary to speak of and the flowing manifold is no longer strictly spacelike.

One possible condition we could impose on \( \Sigma \) is convexity, and this immediately allows application of a maximum principle to get a spacelike flow, but is extremely restrictive in terms of allowed \( \Sigma \). Instead we assume the following weaker conditions:

**Condition 1 (Curvature assumptions on \( \Sigma \)).** The curvature of \( \Sigma \) is uniformly bounded and there exists a smooth timelike unit vectorfield \( V \), such that everywhere on \( \Sigma, V \) is an eigenvector of the second fundamental form of \( A_\Sigma(\cdot, \cdot) \) and \( \nabla_\mu V = 0 \). At a point \( p \in \Sigma \), let \( W_I \) for \( 1 \leq I \leq n-1 \) be the remaining (spacelike) eigenvectors of \( A_\Sigma(\cdot, \cdot) \). We assume that for \( 1 \leq I \leq n-1 \) the curvature satisfies

\[
A_\Sigma(W_I, W_I) + A_\Sigma(V, V) \geq 0.
\]
This allows significantly more varied boundary manifolds than a convexity assumption, and is similar to 2-convexity.

We define a smooth diffeomorphism \( F: \Omega \times \mathbb{R} \to \hat{\Sigma} \subset \mathbb{R}^{n+1} \), where \( \Omega \subset \mathbb{R}^n \) is open and bounded with smooth boundary \( \partial \Omega \), to be a \textit{spacelike foliation compatible with the boundary} if:

1. The image of \( \partial \Omega \times \mathbb{R} \) under \( F \) is \( \Sigma \).
2. If \( x^i, i = 1, \ldots, n \) are coordinates on \( \Omega \) and \( \lambda \) parametrises \( \mathbb{R} \) then \( \langle \frac{\partial F}{\partial \lambda}, \frac{\partial F}{\partial x^i} \rangle = 0 \) and that \( F(\cdot, \lambda) \) is a spacelike hypersurface with normal in the timelike direction \( \frac{\partial F}{\partial \lambda} \), where we assume there exists a constant such that \( -\left| \frac{\partial F}{\partial \lambda} \right|^2 > C_F > 0 \) uniformly.
3. If \( \gamma \) is the outward unit normal to \( \partial \Omega \) then \( \gamma^i \frac{\partial F}{\partial x^i} \) is in the direction \( \mu \).
4. All geometric quantities on these hypersurfaces, for example spacelikeness and curvature, may be uniformly bounded across all hypersurfaces.

Such a foliation is always possible when \( \Sigma \) has uniformly bounded curvature and is a topological cylinder.

Given a compatible foliation as above, one may construct the smooth time function \( \tau: \mathbb{R}^{n+1}_1 \to \mathbb{R} \) defined by \( \tau(y) = P(F^{-1}(y)) \) where \( P: \Omega \times \mathbb{R} \to \mathbb{R} \) is the standard projection. Such a \( \tau \) satisfies \( \nabla_\mu \tau = 0 \) on \( \Sigma \), and in fact \( \nabla \tau = -\left| \frac{\partial F}{\partial \lambda} \right|^{-2} \frac{\partial F}{\partial \lambda} \).

We will write the lapse function \( \psi = \sqrt{-\left| \frac{\partial F}{\partial \lambda} \right|^2} \).

For any compatible spacelike foliation, we define the normal vector field \( \hat{V} := \psi^{-1} \frac{\partial F}{\partial \lambda} \).

\textbf{Condition 2} (Existence of a compatible foliation). There exists a spacelike foliation compatible with the boundary such that there exists a constant \( C_V > 0 \) such that \( 0 < \langle V, \hat{V} \rangle \leq C_V \), where \( V \) is the unit vector field from Condition 1.

We define two notions of gradient, \( v = -\langle V, \nu \rangle \) and \( \hat{v} = -\langle \hat{V}, \nu \rangle \), where we choose a signs on \( V \) and \( \hat{V} \) such that these functions are both positive.
Remark 2. Due to the above condition, it is easy to see that there exists a $\tilde{C}_V$ depending only on $C_V$ such that

$$\frac{1}{C_V} \tilde{v} \leq \hat{v} \leq \tilde{C}_V v.$$ 

Remark 3. We observe that as in [4] we see that restricting any $p$-tensor $B$ defined on $\mathbb{R}^{n+1}_{1}$ to $M$ may be estimated via $|B| \leq \nu^p|B|$.

To obtain a good gradient estimate in settings where the flow does not stay in a bounded region, we will also consider:

**Condition 3** (Boundedness of maximum volume). The maximum volume of a spacelike hypersurface with boundary on $\Sigma$ is bounded above by $C_{\text{vol}} < \infty$.

Let $\hat{\Sigma}$ be the inside of $\Sigma$ in $\mathbb{R}^{n+1}_{1}$, that is the volume through which the flow takes place. This automatically holds in the case in which a flow stays in a bounded region, for example when we have a suitable comparison solution. However this means that for $\Sigma$ which are tangent to cones at infinity our gradient estimate gets worse as the solution moves towards spatial infinity.

Remark 4. We note that the counter example in Figure 1 violates both Conditions 1 and 3.

2. Rewriting the problem

In coordinates on $\bar{F}$ as in the previous section, writing $i$ for the $x_i$th coordinate,

$$g_{ij} = \hat{g}_{ij}(\lambda) \quad \bar{g}_{i\lambda} = 0 \quad \bar{g}_{\lambda\lambda} = -\psi^2 < 0$$

where $\hat{g}_{ij}(\lambda)$ is the metric of the hypersurface defined by $\bar{F}(\cdot, \lambda)$. We now write a hypersurface in $\hat{\Sigma}$ graphically using $\bar{F}$. Set $\bar{F}(x) = \bar{F}(x, \phi(x))$ and we calculate that

$$\frac{\partial F}{\partial x^i} = \frac{\partial \bar{F}}{\partial x^i} + \frac{\partial \bar{F}}{\partial \lambda} D_i \phi, \quad g_{ij} = \hat{g}_{ij} - \psi^2 D_i \phi D_j \phi.$$

We then have that

$$g^{ij} = \hat{g}^{ij} + \nu^2 \psi^2 D_p \hat{g}^{ik} D_q \hat{g}^{kj},$$

where $\nu^{-1} = \sqrt{1 - \psi^2 D_r \phi D_s \phi}$.

We may calculate the volume form ito be

$$\det g_{ij} = \nu^{-2} \det \hat{g}_{ij}(x, \phi(x)),$$

and note that the “upward” (that is in the same direction as $\frac{\partial F}{\partial \lambda}$) unit normal is

$$\nu = \nu \left[ \psi D_k \hat{g}^{kp} \frac{\partial \bar{F}}{\partial x^p} + \nu^{-1} \frac{\partial \bar{F}}{\partial \lambda} \right].$$

We see that $\nu = -\langle \hat{V}, \nu \rangle$, where $\hat{V}$ is as in the previous section.

Any function $f$ on $M$ may also be written as a function on $\Omega$. As such we may calculate that

$$|\nabla f|^2 = D_i \hat{g}^{ij} D_j f + \psi^2 \nu^2 (D_i \hat{g}^{ij} D_j \phi) \geq D_i \hat{g}^{ij} D_j f \geq C|Df|^2$$

where $C$ depends only on $\bar{F}$. We use this to obtain integral estimates, which are necessary since to the authors knowledge there is no equivalent of the Michael–Simon Sobolev inequality in Minkowski space. We obtain boundary and Sobolev inequalities on our flowing manifold by simply using the Euclidean equivalents on $\Omega$. Of course these estimates are not coordinate invariant and so include factors of $\nu$, but these are good enough for our purposes.
Lemma 2. Suppose $\Sigma$ satisfies Condition 3. Let $f \in C^1$ be a positive function on a spacelike hypersurface $M$ inside $\Sigma$ with $\partial M \subset \Sigma$ such that at the boundary $(\nu, \mu) = 0$. Then we may estimate

$$\left( \int_M |f|^{\frac{n+1}{n-1}} dV \right)^{\frac{n-1}{n}} \leq C_1 \sup_{x \in M} \hat{v} \int_M |\nabla f| + |f|dV$$

and if in addition $M$ satisfies the perpendicular boundary condition then

$$\int_{\partial M} f dV^0 \leq C_2 \int_M |\nabla f| + f(|A| + \hat{v})dV$$

For constants $C_1, C_2$ depending on $n$, $\Sigma$ and $\mathcal{F}$.

Proof. We write $C_n$ for any constant that depends only on $\Sigma$, $\mathcal{F}$, $n$. Using the uniform boundedness of $\det \hat{\sigma}_{ij}$, and the Sobolev inequality in the plane with boundary (e.g. [5, Lemma 1.1, Lemma 1.4]), for $f \in C^1(\Omega)$,

$$\left( \int_M |f|^{\frac{n+1}{n-1}} dV \right)^{\frac{n-1}{n}} \leq C_n \left( \int_\Omega |f|^{\frac{n+1}{n-1}} dx \right)^{\frac{n-1}{n}} \leq C_n \int_\Omega |Df| + |f|dx \leq C_n \sup_{x \in M} \hat{v} \int_M |\nabla f| + |f|dV$$

We consider the hypersurface $M$ written graphically as $\mathcal{F}(x, \phi(x))$. From properties of a compatible foliation, we have $\mu = S(x, \lambda) \gamma^k \frac{\partial \mathcal{F}}{\partial x^k}$, where $S(x, \lambda) > 0$, so the boundary condition on $\partial M$ becomes $0 = (\nu, \mu) = vS\hat{\psi}\gamma^k D_k \phi$, that is, $\gamma^k D_k \phi = 0$. Under such a condition we may see that the boundary volume form on $\partial M$ may be written as $\hat{v} \sqrt{\hat{g}_{ij}(x, \phi)}$, and so using [5, Lemma 1.4] we see, for $0 \leq f \in C^1(M)$

$$\int_{\partial M} f dV^0 \leq C_n \int_{\partial M} \frac{f}{\hat{v}} dS \leq C_n \int_\Omega \left[ |Df| + f \frac{|\hat{D} \hat{v}|}{\hat{v}} + \frac{1}{\hat{v}} \right] dC \leq C_n \int_M |\nabla f| + f(|A| + \hat{v})dC$$

where we have used that $|\nabla \hat{v}|^2 \leq C_n|A|^2 \hat{v}^2 + \hat{v}^4$. \hfill $\square$

Remark 5. By changing the constants in the above Lemma, we may exchange $\hat{v}$ for $v$ in the above, due to Condition 2.

Now we calculate the evolution of $\phi$ such that $F$ moves by mean curvature flow. We may calculate that

$$h_{ij} = -\left\langle \frac{\partial^2 F}{\partial x^i \partial x^j}, \nu \right\rangle = \hat{v} \sqrt{-\frac{\partial F}{\partial \lambda}} D_{ij}^2 \phi + \hat{b}_{ij}(x, \phi, D\phi)$$

Therefore the (reparametrised) mean curvature flow we have

$$-H = \left( \frac{d}{dt} \mathcal{F}(x, \phi(x, t)), \nu \right) = \left( \frac{\partial \mathcal{F}}{\partial \lambda} D_t \phi, \nu \right) = -\hat{v}S\hat{\psi}D_t \phi$$

and so equation (1) is equivalent (as in [13, Section 2]) to

\[
\begin{align*}
D_t \phi &= g^{ij}(x, \phi, D\phi) D_{ij}^2 \phi + b(x, \phi, D\phi) & \text{for } (x, t) \in \Omega \times [0, T] \\
\gamma^i D_i \phi &= 0 & \text{for } (x, t) \in \partial \Omega \times [0, T] \\
\phi(\cdot, 0) &= \phi_0(\cdot)
\end{align*}
\]

\[
(2)
\]
We remark that this is a quasilinear parabolic equation, and the main challenge is to show it is uniformly parabolic. From properties of \( g^{ij} \) above, this is equivalent to finding an upper bound on the quantity \( \hat{v} \), or from Remark 2 on the quantity \( v \).

3. Comparison solutions

We define a comparison solution to be a smooth mapping \( G : B^n \times \mathbb{R} \to \mathbb{R}^{n+1} \), where \( G_t = G(\cdot, t) \) is spacelike, such that if \( F \) satisfies \( \text{(1)} \) and \( F_0 \) lies on one side of \( G_0 \) then it will do for all time. Let \( \nu_G \) be the unit normal of \( G \) which points in the direction of \( F \), and \( H^G \) the mean curvature calculated with respect to \( \nu^{G} \). We aim to show that if \( G \) satisfies

\[
\begin{align*}
\langle \frac{dG}{dt}, \nu^{G} \rangle &\geq -H^G \quad \forall (x, t) \in B^n \times [0, T] \\
G(\cdot, 0) &= G_0 \\
G(x, t) &\subset \Sigma \quad \forall (x, t) \in \partial B^n \times [0, T] \\
\langle \nu^{G}, \mu \rangle (x, t) &\leq 0 \quad \forall (x, t) \in \partial B^n \times [0, T]
\end{align*}
\]

then \( G \) is a comparison solution. Furthermore we show that if \( G \) satisfies the above with equalities instead of inequalities, that is \( G \) moves by MCF, then either \( F = G \) or \( F \) lies strictly above \( G \) for \( t > 0 \). The proof of this is very similar to Stahl’s proof in the Euclidean setting [13], with some simplifications due to the geometry of Minkowski space. We require the following maximum principle:

Proposition 3 (Strong Maximum Principle). Let \( \epsilon > 0 \) be a small constant, \( \Omega \subset \mathbb{R}^n \) a compact, connected domain with smooth boundary \( \partial \Omega \) and outward pointing normal \( \gamma \). Suppose \( \phi : \Omega \times [0, T] \to \mathbb{R} \) be a continuous function of class \( C^{2,1} \) in the neighbourhood of all point \((x, t) \in M^n \times [0, T] \) with \( |\phi(x, t)| < \epsilon \) such that

\[
\begin{align*}
L(\phi) &\geq 0 \quad \forall (x, t) \in \Omega \times [0, T] \\
\gamma^i D_i \phi &\geq 0 \quad \forall (x, t) \in \partial \Omega \times [0, T] \\
\phi(\cdot, 0) &\geq 0
\end{align*}
\]

where \( a^{ij}(x, t), b^i(x, t), c(x, t) \in L^\infty(\Omega \times [0, T]) \) and

\[
L(\phi) = \frac{\partial \phi}{\partial t} - a^{ij}(x, t) D_j^2 \phi - b^i(x, t) D_i \phi - c(x, t) \phi .
\]

Then \( \phi \geq 0 \) for all \((x, t) \in \Omega \times [0, T] \). Furthermore if initially \( \phi(\cdot, 0) \) is not identically zero then \( \phi(x, t) > 0 \) for all \((x, t) \in M^n \times [0, T] \). \( \bigcup_{t \in [0, T]} \Sigma_t \times [0, T] \) which \( \nu^{G} \) points towards. If \( F_0 \) is initially in \( S \) then it will remain in \( S \). Furthermore if \( F \) is initially touching (but not entirely contained in) \( G \) then \( F \) will immediately “lift off” \( G_t \) for \( t > 0 \).

Proof. For a slightly more general maximum principle see [13, Theorem 3.1, Corollary 3.2]. \( \square \)

We have the following:

Proposition 4. Let \( G \) satisfy equation \( \text{(3)} \) and \( F \) satisfy equation \( \text{(1)} \) on a time interval \([0, T] \) such that \( F \) and \( G \) are smooth and spacelike, and let \( S \subset \Sigma \times [0, T] \) be the closure of the connected component of \( \Sigma \times [0, T] \setminus \bigcup_{t \in [0, T]} G_t \times \{t\} \) which \( \nu^{G} \) points towards. If \( F_0 \) is initially in \( S \) then it will remain in \( S \). Furthermore if \( F \) is initially touching (but not entirely contained in) \( G \) then \( F \) will immediately “lift off” \( G_t \) for \( t > 0 \).

Proof. We consider \( F \) and \( G \) in coordinates \( \tilde{F} \) inside \( \Sigma \) as in the previous section, and we write them as (smooth) graphs \( u(x, t) \) and \( w(x, t) \) respectively. Since initially \( F_0 \) lies on one side of \( G_0 \), without loss of generality we may assume that
$u \geq w$ initially and that $\nu^G$ is an upwards pointing unit vector field. As in the calculations in the previous section we see that

$$u_t = g^{ij}(x, u, Du)D_{ij}^2 u + b(x, u, Du)$$
$$w_t \leq g^{ij}(x, w, Dw)D_{ij}^2 w + b(x, w, Dw)$$

while at the boundary,

$$\gamma^i D_i u = 0, \quad \gamma^i D_i w \leq 0$$

Writing $\phi = u - w$ then by standard methods we may write

$$\phi_t \geq a^{ij}(x, t)D_{ij}^2 u + b^i(x, t)D_i \phi + c(x, t)\phi, \quad \gamma^i D_i \phi \geq 0$$

where $a^{ij}(x, t), b^i(x, t), c(x, t) \in L^\infty(\Omega \times [0, T]).$ Since $\phi(\cdot, 0) \geq 0$, when $\phi$ is small (i.e. when $F$ and $G$ are close together or touching) we may now apply the above strong maximum principle to obtain the proposition. □

4. Evolution equations and boundary identities

In this section we collect the necessary evolution equations and boundary identities. Firstly, we need standard evolution equations for evolution of the metric and normal:

**Lemma 5.** On the interior of $M$ we have that

$$\frac{d\nu}{dt} = \nabla H,$$
$$\frac{dg_{ij}}{dt} = 2Hh_{ij}.$$

**Proof.** See [4, Proposition 3.1]. □

**Proposition 6.** The mean curvature evolves by

$$\left(\frac{d}{dt} - \Delta\right)H = -H|A|^2,$$

**Proof.** See [4, Proposition 3.3]. □

From the spatial and time derivatives of the boundary condition we have:

**Lemma 7.** For $p \in \partial M \times [0, T)$ and $W \in T_pM_t \cap T_p\Sigma$ then

$$A(\mu, W) = -A^\Sigma(\nu, W).$$

and also

$$\nabla_\mu H = -H A^\Sigma(\nu, \nu).$$

**Proof.** See [10, Lemma 5.2, Lemma 5.4]. □

Importantly we will also need the evolution equation for $v = -\langle V, \nu \rangle$.

**Lemma 8.** On the interior of the flowing manifold,

$$\left(\frac{d}{dt} - \Delta\right)v = -v|A|^2 + 2g^{ij}A(\nabla_i V, j) + g^{ij}\langle \nabla_i^2 V, \nu \rangle$$

holds.

**Proof.** We calculate from Lemma 5

$$\frac{dv}{dt} = -\nabla_v^\top H - H \langle \nabla_v V, \nu \rangle$$
and

\[ \Delta v = -g^{ij} \left( \left( \nabla_i \nabla_j V, \nu \right) + 2A(i, (\nabla_j V)^\top) + \nabla V \cdot h_{ij} + h_{ij} g^{kl} h_{kl} \langle \nu, V \rangle - \left( \nabla_{\nabla_i \nabla_j V, \nu} \right) \right) + 2v g^{ij} A(i, (\nabla_j V)^\top) + v |A|^2 - \nabla V \cdot H - H \langle \nabla_V V, \nu \rangle, \]

where we used the Codazzi–Mainardi and Weingarten formulae.

Lemma 9. We define the function \( u : M \to \mathbb{R} \) by \( u = \tau(F(x,t)) \), then

\[ \left( \frac{d}{dt} - \Delta \right) u = -g^{ij} \nabla_i^2 \tau \]

and we furthermore remark that

\[ |\nabla u|^2 = \psi^{-2}(\dot{v}^2 - 1) \]

Proof. We calculate for a general ambient function \( u \)

\[ \frac{d u}{d t} = H \nabla_{\nu} u, \quad \Delta u = g^{ij} \nabla_i^2 u + H \nabla_{\nu} u. \]

Now since \( \nabla u \) is stricly timelike, we calculate

\[ \nabla u = \nabla_i \tau g^{ij} \frac{\partial}{\partial x^j} = (\nabla \tau)^\top = \nabla \tau - \psi^{-1} \dot{v} \nu \]

and so

\[ |\nabla u|^2 = \psi^{-2}(\dot{v}^2 - 1) \]

as claimed.

Lemma 10. For any \( f \in C^1(M \times [0,T]) \) we have

\[ \frac{d}{dt} \int_M f dV = \int_M \frac{d f}{d t} + H^2 f dV \]

Proof. Since at the boundary \( \frac{d F}{d t} \perp \mu \), we do not need to concern ourselves with the manifold flowing “out” of \( \Sigma \). Therefore as is standard we may calculate using Lemma 5

\[ \frac{d}{dt} \int_M f dV = \frac{d}{dt} \int_{M^\infty} f \sqrt{\det g_{ij}} d x = \int_M \frac{d f}{d t} + H^2 f dV \]

We also require the boundary derivative

Lemma 11. At the boundary if \( V \) is a (strictly) timelike eigenvalue of the second fundamental form such that \( \nabla_\mu V = 0 \) we have

\[ \nabla_\mu v = -v [A^\Sigma(\nu, \nu) - A^\Sigma(V, V)] \]

Proof. Using Lemma 7 we calculate that

\[ \nabla_\mu v = -A(\mu, V^\top) = A^\Sigma(\nu, V^\top) = A^\Sigma(\nu, V - \nu) = -v A^\Sigma(\nu, \nu) + v A^\Sigma(V, V) \]

because an eigen vector has the property, \( A^\Sigma(V,i)g_{ij} \frac{\partial}{\partial x^j} = \lambda V \) and so \( \lambda = -A^\Sigma(V, V) \). Therefore \( A(V, \nu) = \lambda (V, \nu) = v A^\Sigma(V, V) \).
5. Gradient estimates

Throughout this section we assume Conditions 1, 2, and 3 on Σ, at least on a time interval \([0,T]\), to obtain the key estimate required for long time existence of the flow, namely the gradient estimate. Firstly we use Condition 1 to establish signs on the boundary derivatives of \(v\) and \(H\). We observe that since \(\sum_{I=1}^{n-1} (\langle W_i, \nu \rangle)^2 = v^2 - 1\), Condition 1 implies

\[
A^2(\nu, \nu) - A^2(V, V) = \sum_I A(W_i, W_i)(\langle W_i, \nu \rangle)^2 + A^2(V, V)(v^2 - 1) \geq -A^2(V, V) \sum_I (\langle W_i, \nu \rangle)^2 + A^2(V, V)(v^2 - 1) = 0.
\]

This then gives us the useful boundary properties that

\[
\nabla \mu v \leq 0, \quad \nabla \mu H^2 = -H^2 A^2(\nu, \nu) \leq -H^2 A^2(V, V).
\]

**Remark 6.** If instead of the curvature Condition 1 on Σ we assume that Σ has merely bounded curvature, the best estimates we may get on the boundary derivatives of \(v\) and \(H\) are (for some \(C(\Sigma)\))

\[
\nabla \mu v \leq Cv^3, \quad \nabla \mu H^2 \leq CH^2 v^2.
\]

This extra factor of \(v^2\) adds significant technical problems, with the boundary terms overpowering the evolution equation terms.

**Remark 7.** The gradient estimate we give below depends on a Stampaccia iteration argument to get an estimate on \(H\). We note that it is also possible to obtain a gradient estimate without estimating \(H\) using purely maximum principle arguments as in [7]. However in an unbounded situation, the methods below give a much better exponent in \(u\).

As is common with Minkowski space problems [1][3][4] we will estimate \(v^{-2} |\nabla v|^2\) in terms of \(|A|^2\) and \(H^2\), allowing us to obtain a sign on the evolution of \(v\). For this to work, we also need to be able to estimate the extra \(H^2\) term by a sufficiently small power of \(v\). Unfortunately the boundary derivative of \(H^2\) may be positive (when \(A^2(V, V) < 0\)) and so a direct application of maximum principle does not work. We instead use a Stampaccia iteration technique, and to apply this we need Condition 3. An immediate corollary of this assumption is the following:

**Lemma 12.** Given Condition 3 there exists a finite constant \(C(\Sigma)\) which depends on the maximum area of the flowing manifold, but is independant of \(T\), such that

\[
\int_0^T \int_M H^2 dV dt \leq C.
\]

**Proof.** We have that

\[
\frac{d}{dt} \int_M dV = \int_M H^2 dV,
\]

and so by integrating we obtain

\[
\int_0^T \int_M H^2 dV dt = \int_{M_T} dV - \int_{M_0} dV \leq \int_{M_T} dV \leq C(\Sigma)
\]

by Condition 3.

We aim to prove:

**Proposition 13.**

\[
\sup_{(x,t) \in M \times [0,T]} |H| \leq C_1 + C_2 \sup_{(x,t) \in M \times [0,T]} v^p,
\]

where \(C_1, C_2, p > 0\) are constants depending only on \(n, \Sigma\) and \(M_0\) and \(p < 1\).
We introduce the notation $\mathcal{H} = \sup_{(x,t) \in M \times [0,T]} |H|$ and $v = \sup_{(x,t) \in M \times [0,T]} v$. The key to showing the Proposition is the following which may be interpreted as an estimate of the $L^p$ norm of $|H|$ in terms of $v^2$.

**Lemma 14.** For $k, \gamma > 0$ where $k \in \mathbb{Z}$ and $p = n + 2k + \gamma$, there exists a constant $C_1(n, p, \gamma, \Sigma, M_0), C_2(n, p, \gamma, \Sigma, M_0) > 0$ such that
\[
\int_0^T \int_M |H|^p dV dt \leq C_1 v^{k-1} + C_2 v^k \mathcal{H}^{n+\gamma-2} .
\]

**Proof.** Suppose $p > n$ and let $C_n$ be any constant depending on $n, p, \Sigma$ which may change from line to line. By Proposition 6 and Lemmas 7 and 10,
\[
\frac{d}{dt} \int_M |H|^p dV = \int_{\partial M} -p |H|^p A^2(V, V) dV^3 \\
+ \int_M -p H^p |A|^2 - p(p - 1) H^{p-2} |\nabla H|^2 + H^{p+2} dV .
\]

By Lemma 2, we have that
\[
\int_{\partial M} -p |H|^p A^2(V, V) dV^3 \leq C_n \int_{\partial M} |H|^p dV^3 \\
\leq C_n \int_M |H|^{p-1} |\nabla H| + |H|^p (v + |A|) dV
\]
and so using Young's inequality and $|A|^2 \geq \frac{1}{n} H^2$ then
\[
\frac{d}{dt} \int_M |H|^p dV \leq \int_M |H|^{p-2} \left[ -(p-n) H^2 |A|^2 - p(p-1) |\nabla H|^2 \right. \\
+ C_n |H| |\nabla H| + C_n H^2 (v + |A|) \big] dV
\]
and so integrating,
\[
\int_0^T \int_M |H|^{p+2} dV dt \leq C_n v \int_0^T \int_M |H|^p dV dt + \int_{M_0} |H|^p dV .
\]

Iterating this estimate, we see that for $p$ as described in the statement of the Lemma
\[
\int_0^T \int_M |H|^{p+2} dV dt \leq C_1 v^k \int_0^T \int_M |H|^{p+\gamma} dV dt + C_2 v^{k-1}
\]
which completes the proof in light of Lemma 12.

We will consider $f_k = (H^2 - k)_+$, the cutoffs of the function $f = H^2$. We define the time dependent set $A(k) = \{ x \in M_t : f_k > 0 \}$, and look to estimate a measure of this set,
\[
\|A(k)\| = \int_0^T \int_{A(k)} dV dt .
\]

**Lemma 15.** For any $k > 0$, there exists a constant $C(k, \Sigma)$ independant of $T$ such that
\[
\|A(k)\| \leq C
\]

**Proof.**
\[
\|A(k)\| = \int_0^T \int_{A(k)} dV dt \leq \frac{2}{k} \int_0^T \int_{A(k)} H^2 dV dt \leq \frac{2C}{k}
\]
where the constant is from Lemma 12.

We will also need the following iteration Lemma:
Lemma 16. Suppose \( \phi : (k_0, \infty) \to \mathbb{R} \) is a non-negative non-increasing function such that for all \( h > k \geq k_0 \) then
\[
\phi(h) \leq \frac{C}{(h-k)^\alpha} (\phi(k))^\beta
\]
where \( C, \alpha \) and \( \beta \) are positive constants. Then if \( \beta > 1 \) then \( \phi(k_0 + d) = 0 \) for
\[
d^\alpha = C[\phi(k_0)]^{\beta-1}2^\alpha \pi^{\frac{\beta}{\gamma}}.
\]

Proof. See [14, Lemma 4.1 i]. \( \square \)

We now prove the Proposition:

Proof of Proposition 13. We look at the evolution of \( f^p_k \) for some large \( p > \frac{n}{2} \). From Proposition 6 and [11],
\[
\left( \frac{d}{dt} - \Delta \right) f^p_k = pf^{p-1}_k \left[ -2H^2|A|^2 - 2|\nabla H|^2 \right] - p(p-1)f^{p-2}_k 4H^2|\nabla H|^2
\]
\[
- \nabla_{\mu} f_k = -pf^{p-1}_k H^2 A^\nu(V,V) \leq C_n f^{p-1}_k H^2.
\]

Therefore using Lemma 2 we have:
\[
\int_{\partial M} |\nabla f^p_k|^\theta dV^0 \leq C_n \int_{\partial M} f^{p-1}_k H^2 dV^\beta
\]
\[
\leq C_n \int_M f^{p-2}_k |H|^3|\nabla H| + f^{p-1}_k |H||\nabla H| + f^{p-1}_k H^2(v + |A|)dV .
\]

Estimating similarly to in Lemma 14 (and using that \( 2p > n \))
\[
\frac{d}{dt} \int_M f^p_k dV \leq \int_M pf^{p-1}_k \left[ -2H^2|A|^2 - 2|\nabla H|^2 + C_n |H||\nabla H| + C_n H^2(v + |A|) \right]
\]
\[
+ p(p-1)f^{p-2}_k \left( -4H^2|\nabla H|^2 + C_n |H|^3|\nabla H| \right) + H^2 f^p_k dV
\]
\[
\leq \int_M pf^{p-1}_k \left[ C_n H^2 - |\nabla H|^2 \right]
\]
\[
+ p(p-1)f^{p-2}_k \left(-2H^2|\nabla H|^2 + C_n |H|^4 \right) dV .
\]

We have that \( |\nabla f^p_k| \leq f^{p-1}_k \left[ \frac{C_n}{\varepsilon} H^2 + \epsilon |\nabla H|^2 \right] \), and so
\[
\frac{d}{dt} \int_M f^p_k dV \leq \int_M C_n f^{p-2}_k H^4 dV - C_n \int_M |\nabla f^p_k| + f^p_k dV
\]
\[
\leq C_n \varphi \int_{A(k)} H^2 p dV - C_n \frac{\varphi}{\nu} \left( \int_M f^{\frac{n}{2}+1}_k \right)^{\frac{\alpha}{2}} .
\]

We now set \( k > k_0 = \sup_{x \in M_0} H^2 \) and integrate to get
\[
\sup_{t \in [0,T]} \int_M f^p_k dV + \frac{C_n}{\varphi} \int_0^T \left( \int_M f^{\frac{n}{2}+1}_k \right)^{\frac{\alpha}{2}} \frac{\varphi}{\nu} dt \leq C_n \varphi \int_{A(k)} H^2 p dV .
\]

By standard methods,
\[
\sup_{t \in [0,T]} \int_M f^p_k dV + \frac{C_n}{\varphi} \int_0^T \left( \int_M f^{\frac{n}{2}+1}_k \right)^{\frac{\alpha}{2}} \frac{\varphi}{\nu} dt \geq C_n \frac{\varphi}{\nu} \frac{T}{\|A(k)\|^{\frac{\alpha}{2}}}
\]
\[
\sum_{t \in [0,T]} \int_M f^p_k dV dt
\]
\[
\sum_{t \in [0,T]} \int_M \left( \int_M f^{\frac{n}{2}+1}_k \right)^\frac{\alpha}{2} \frac{\varphi}{\nu} dt
\]
\[
\sum_{t \in [0,T]} \int_M f^{\frac{n}{2}+1}_k \frac{\varphi}{\nu} dt .
\]
and so by Hölder’s inequality,

\[ |h - k| p^n A(h) | \leq \int_0^T \int_{A(k)} f_k^p dV dt \]

\[ \leq C_n v^{1 + \frac{n}{p^n}} \left( \int_0^T \int_M H^2 \right)^{\epsilon} \]

We now set \( \epsilon = \frac{1}{p(n+1)} \), let \( j \in \mathbb{Z} \) be so large that \( p > 2 \) where \( 2p = n + 2j \). By Lemma \[14\]

\[ |h - k| p^n A(h) | \leq C_n v^{1 + \frac{n}{p^n}} (v^{j-1} + v^j)^n \]

Therefore from Lemma \[16\] Lemma \[15\] we see that \( \|A(k_0 + 1 + d)\| = 0 \) for particular \( d \) depending on \( v \) and \( \beta \). Explicitly, we may estimate:

\[ f^n \leq k_0 + 1 + C_n v^{\frac{1}{p^n} + \frac{n}{p^n}} (v^{j-1} + v^j)^n \]

\[ \leq k_0 + 1 + C_n v^{\frac{n+2j}{p^n}} (1 + C_n v^{\frac{2j}{p^n}}) . \]

The Proposition is now proved by making \( j \) very large. \( \square \)

We may now use standard methods to obtain a gradient estimate which is exponential in a height function \( u \).

**Theorem 17.** There exist constants \( C_1, C_2 \) depending on \( n, \Sigma \) and \( M_0 \) such that

\[ v \leq C_1 e^{C_2 \text{osc} u} \sup_{x \in M_0} v \]

**Proof.** We consider the function \( f = ve^{\lambda u} \). Using Lemma \[9\],

\[ \left( \frac{d}{dt} - \Delta \right) e^{\lambda u} = e^{\lambda u} \left( -2\lambda g^{ij} \nabla_j u - 2\lambda^2 |\nabla u|^2 \right) \]

We estimate

\[ \frac{|\nabla v|^2}{v^2} \leq (1 + \epsilon_1) A \left( \frac{V^T}{v}, i \right) g^{ij} A \left( \frac{V^T}{v}, j \right) + C_n (1 + \frac{1}{\epsilon_1}) v^2 \]

Therefore since \( |V^T|^2 \leq \sqrt{v^2 - 1} \), we may estimate as in \[11\] Theorem 3.1

\[ |A|^2 \geq \left( 1 + \frac{1}{\epsilon_1} \right) \left( \frac{1}{1 + \epsilon_1} \frac{|\nabla v|^2}{v^2} - \frac{C}{\epsilon_1} v^2 \right) - H^2 \]

We use these inequalities and Lemma \[8\] to obtain that for \( \epsilon_1 \) and \( \epsilon_2 \) small,

\[ \left( \frac{d}{dt} - \Delta \right) f = ve^{\lambda u} \left( -|A|^2 + 2v^{-1} g^{ij} A(\nabla_i V, j) + v^{-1} g^{ij} \langle \nabla_i V, j \rangle \right) \]

\[ -2\lambda \left( \frac{\nabla v}{v}, \nabla u \right) - 2\lambda^2 |\nabla u|^2 \]

\[ \leq ve^{\lambda u} \left( -(1 - \epsilon_2) |A|^2 + C_n \left( 1 + \lambda + \frac{1}{\epsilon_2} \right) v^2 - 2\lambda \left( \frac{\nabla v}{v}, \nabla u \right) - \lambda^2 |\nabla u|^2 \right) \]

\[ \leq ve^{\lambda u} \left( -(1 - \epsilon_2) \left( 1 + \frac{1}{\epsilon_2} \right) \frac{|\nabla v|^2}{v^2} + H^2 + C_n \left( 1 + \lambda + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} \right) v^2 \right) \]

\[ - 2\lambda \left( \frac{\nabla v}{v}, \nabla u \right) - \lambda^2 |\nabla u|^2 \]

\[ \leq ve^{\lambda u} \left( H^2 + C_n \left( 1 + \lambda + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} \right) v^2 - \lambda^2 \left( 1 - \frac{1}{(1 - \epsilon_2)(1 + \frac{1}{\epsilon_2})} |\nabla u|^2 \right) \right) . \]
Choosing, for example, $\epsilon_1 = \frac{1}{16}$ and $\epsilon_2$ so that \((1 - \epsilon_2)(1 + \frac{1}{n}) = 1 + \frac{1}{2n}\), then using Proposition 13 and Lemma 9,
\[
\left( \frac{d}{dt} - \Delta \right) f \leq ve^{\lambda u} \left( \frac{\lambda^2}{4n + 2} \left| \nabla u \right|^2 + C_n v^2 + C_n \lambda v^2 \right) \leq ve^{\lambda u} \left( \frac{\lambda^2}{4n + 2} \left| \nabla u \right|^2 + C_n \lambda v^2 \right).
\]
Therefore due to the uniform lower bound on $\psi$, and the equivalence of $v$ and $\tilde{v}$, when $\nu > 2C_V$ (where $C_V$ is the constant from Remark 2), we may choose $\lambda$ sufficiently large, to obtain on the interior of $M_t$
\[
\left( \frac{d}{dt} - \Delta \right) f \leq ve^{\lambda u} (v^{2p} - v^2)
\]
while meanwhile at the boundary, due to Condition 1 and Lemma 11
\[
\nabla \nu f \leq 0.
\]
We now apply a maximum principle argument to remove the possibility of large increasing maxima of $f$ when $\nu > 2C_V$.

At an increasing maximum $(p, s)$ of $f$, where $f(p, s) = \sup_{(x, t) \in M^n \times [0, s]} f(x, t)$, then for $\nu(s) = \sup_{(x, t) \in M^n \times [0, s]} v(x, t)$ we have
\[
0 \leq v^{2p} - v^2.
\]
For $(x, t) \in M^n \times [0, T]$ write $m \leq \epsilon^{\lambda u} \leq M$, then for any $(q, r) \in M^n \times [0, s]$ such that $v(q, r) \geq \nu(s) \geq m^\frac{1}{2}$, then
\[
\frac{m}{2} \nu(s) \leq \epsilon^{\lambda u(q, r)} v(q, r) \leq (\epsilon^{\lambda u} v)(p, s) \leq M \nu^p(s),
\]
therefore $\nu(s) \leq (\frac{2M}{m})^\frac{1}{1-p}$, and $f \leq (\frac{2M}{m})^\frac{1}{1-p} M$. Therefore an increasing interior maximum is bounded by exponents of $u$.

At the boundary if $v^2 \geq v^{2p}$ then we may apply the elliptic Hopf lemma (see for example [3, Lemma 3.4, p34]) to disallow an increasing boundary maximum. Otherwise we obtain exactly the situation above.

Therefore we have $f \leq \max \left\{ \sup_{M^n} ve^{\lambda u}, \left( \frac{2M}{m} \right)^\frac{1}{1-p} M, C_V M \right\}$. We observe that adding a constant function to $u$ changes nothing above, and so without loss of generality we may assume that $m = 1$. The estimate on $f$ implies the theorem. □

**Corollary 18.** If Conditions 1, 3, and 2 hold on the time interval $[0, T]$ with finite constants $C_v(T), C_V(T)$ and $C_F(T)$, and also there exists comparison solutions such that $\underline{C_u}(T) \leq u \leq \overline{C_u}(T)$. Then while $C_v(T), C_V(T), C_F(T)$ and $\overline{C_u}(T)$ are bounded a solution to (7) exists up to time $T$, which is smooth for $t > 0$ and $C^{2,\alpha}$ up to $t = 0$.

**Proof.** The above shows that equation (2) is a uniformly parabolic quasilinear equation with with a linear boundary condition. Therefore by standard quasilinear parabolic theory, for example [11], we have existence of a smooth solution for all time. The bounds on the flow and its derivatives depend on the bounds on $C_v(T), C_V(T), C_F(T), \underline{C_u}(T)$ and $\overline{C_u}(T)$. □

**Corollary 19.** If the flow is as above, but $C_v$, $C_V$, $C_F$ and $\underline{C_u}$ are uniformly bounded, then a solution to (7) exists for $T = \infty$, which is smooth for $t > 0$ and $C^{2,\alpha}$ up to $t = 0$. For any $\epsilon > 0$ the derivatives of the flow are uniformly bounded (depending on $\epsilon$) for all times $t > \epsilon > 0$. 
6. Convergence and Stability

We now look into questions of convergence when $F$ stays in a bounded region.

**Lemma 20.** If $\Sigma$ is as in Corollary 19, then there exists a sequence of times $t_k \to \infty$ such that $M_{t_k}$ tends towards $M_\infty$ in the $C^\infty$ topology where $M_\infty$ is a minimal surface satisfying the boundary condition.

*Proof.* This is as in [4, Proof of Theorem 4.2].

Convergence of the whole flow is not so straightforward and is related to stability of the maximal surfaces towards which the flow converges. This stability depends on the geometry of $\Sigma$ close to the maximal surface. To illustrate this we consider rotationally symmetric $\Sigma$.

**Lemma 21.** Let $\Sigma \subset \mathbb{R}^3_1$ be a rotationally symmetric boundary manifold, parametrised by $E(r, \theta) = f(z)(r + ze_3)$ where $r = \cos \theta e_1 + \sin \theta e_2$ such that $|f'(z)| < 1$. $\Sigma$ satisfies Condition 1 if and only if

$$\frac{f''}{1 - (f')^2} - \frac{1}{f} \leq 0. \quad (5)$$

*Proof.* We may calculate in these coordinates

$$\mu = \frac{1}{\sqrt{1 - (f')^2}}(r + f'e_3), \quad h^\Sigma_{zz} = -\frac{f''}{\sqrt{1 - (f')^2}}, \quad h^\Sigma_{z\theta} = h^\Sigma_{\theta z} = 0, \quad h^\Sigma_{\theta\theta} = \frac{f}{\sqrt{1 - (f')^2}}.$$

Therefore the principle directions are $V = \frac{f'r + e_3}{\sqrt{1 - (f')^2}}$ and $W = r_\theta$ therefore we have

$$A^\Sigma(V, V) = -\frac{f''}{(1 - (f')^2)^{3/2}}, \quad A^\Sigma(W, W) = \frac{1}{f\sqrt{1 - (f')^2}}$$

and Condition 1 becomes equation (5). □

We may obtain Conditions 3 and 2 on such a rotational $\Sigma$ by, for example, assuming $f$, $f'$ and $f''$ are uniformly bounded and smooth.

**Example 1.** In the extreme case of the above, where $\frac{f''}{1 - (f')^2} - \frac{1}{f} = 0$ everywhere, then we may integrate to get for arbitrary $A, B$

$$f(x) = \sqrt{A^2 + (z + B)^2}$$

or we obtain the pseudo-sphere in $\mathbb{R}^3_1$, i.e. the set of points $x \in \mathbb{R}^3_1$ such that $|x - Be_3|^2 = A^2$. We remark that in this case, comparison solutions move off towards infinity, and so we do not necessarily expect convergence to a maximal surface. However, we are still able to apply Corollary 18 to obtain long time existence of the flow.

**Lemma 22.** If $\Sigma$ is rotationally symmetric and satisfies (3), then $\Sigma$ admits a foliation of $\hat{\Sigma}$ of constant mean curvature surfaces, where each leaf is a plane or a hyperbolic plane which satisfies the perpendicular boundary condition.

*Proof.* We aim to do this by constructing constant mean curvature foliation of planes and hyperbolic planes of $\hat{\Sigma}$. A general hyperbolic plane may be written $P(l, \theta) = R(\sinh lr + \cosh le_3) + Je_3$. We suppose this perpendicularly intersects a rotational surface $\Sigma$, that is $\cosh lr + \sinh le_3 = \mu$. This gives that if $f' \neq 0$,

$$R = \frac{f}{f'} \sqrt{1 - (f')^2}, \quad J = z - \frac{f}{f'}$$

and so

$$P(l, \theta) = \frac{f}{f'} \sqrt{1 - (f')^2}(\sinh lr + \cosh le_3) + (z - \frac{f}{f'})e_3.$$
This represents a foliation if the leaves of the foliation do not cross, and since these are rotationally symmetric, this is equivalent to not crossing at \( l = 0 \). Therefore we have a foliation if \( \frac{\partial g}{\partial z} > 0 \) where

\[
g(z) = -\langle P(0, s), e_3 \rangle = z - \frac{f}{f'}(1 - \sqrt{1 - (f')^2}).
\]

We calculate

\[
g' = \sqrt{1 - (f')^2} \left[ 1 - \frac{f''f}{(1 + \sqrt{1 - (f')^2})(1 - (f')^2)} \right].
\]

From equation (5), \( ff'' \leq 1 - (f')^2 \) and so

\[
1 - \frac{f''f}{(1 + \sqrt{1 - (f')^2})(1 - (f')^2)} \geq 1 - \frac{1}{(1 + \sqrt{1 - (f')^2})} > 0.
\]

Therefore, we may always obtain a foliation of CMC surfaces if we have Condition 1 and \( f' > 0 \). When \( f' \to 0 \), \( g' \geq \frac{1}{2} > 0 \), and so the leaves do not cross. In this case the above parametrisation becomes degenerate, but the hyperbolic planes converge to a maximal plane.

From such a foliation we may obtain comparison solutions, by simply solving an ordinary differential inequality to obtain a solution (3).

**Definition 1.** A solution to mean curvature flow \( F \) is said to be stable under the flow if for any sufficiently small perturbation \( \tilde{F}_0 \) of the initial conditions \( F_0 \), the perturbed flow will converge uniformly to \( F \) as \( t \to \infty \).

In Figure 2 we see three examples of possible stabitily behavior of planar maximal surfaces. The left picture shows one completely stable plane at the widest point of the sine wave, and two unstable planes at the thinnest points. We remark that since the plane is a maximal surface, and therefore a comparison solution Proposition 4 implies that MCF starting at a onesided perturbation of one of the the unstable maximal surfaces will move away towards the stable maximal surfaces. The right

![Figure 2. Two examples of foliations by CMC surfaces, demonstrating stability and instability of maximal planes.](image-url)
hand picture shows examples with one sided stability – perturbations on the lower side will flow back towards the maximal surface while flowing a onesided upwards perturbation will move away towards a higher maximal surface.

It is also easy to see that despite the existence of a comparison solution moving away from the the unstable maximal surfaces in the left picture, there exist solutions to MCF which must intersect this maximal surface for all time. For example if we were to perturb by a two sided perturbation, rotationally symmetric around the $y$-axis the solution must always intersect the unstable plane due to preservation of symmetry by the flow. If there are no other maximal surfaces nearby, a subsequence of the flow must converge to the unstable maximal surface.

Remark 8. Variational stability of a maximal surface does not imply stability under the flow. We may observe this by taking a convex cylindrical $\Sigma$ and considering graphical MCF where, as in [9], the flow then converges to planes given graphically by $u = \text{const}$. The condition for variational stability (where we assume perturbations also satisfy the boundary condition) becomes

$$2 \int_{\partial M} \phi^2 A^2(\nu, \nu) dV^0 + 2 \int_M |\nabla \phi|^2 + \phi^2 |A|^2 dV \geq 0$$

for any function $\phi$ such that $\nabla \mu \phi^2 = -2\phi^2 A^2(\nu, \nu)$, which is trivially true for constant graphs inside a convex cylinder, $\Sigma$. But from Proposition 3 a one sided perturbation of such a maximal surface will converge to a different maximal surface, and so we do not have stability under MCF.

Stability of the flow does imply variational stability. For sufficiently small variations of a maximal hypersurface which is stable under the flow, MCF will move the surface back to the maximal hypersurface. We therefore see that any small perturbation cannot have $H \equiv 0$ everywhere, and so by Lemma 10 the volume of the flowing surface strictly increases under the flow, and the maximal surface is variationally stable.

We give a condition for stability under the flow.

**Lemma 23.** Let $M$ be a smooth compact uniformly spacelike maximal surface with boundary $\partial M$, where $\partial M$ satisfies the boundary condition. Suppose there exists an $\epsilon > 0$ and a $\phi : M \rightarrow \mathbb{R}$, $\phi \geq \epsilon$ such that

$$\begin{cases} 
\Delta \phi - \phi |A|^2 \leq -\epsilon \\
\nabla \mu \phi \geq -\phi A^2(\nu, \nu) 
\end{cases}$$

then $M$ is stable from above.

**Proof.** We construct comparison solutions. We do this by verifying there exists a positive perturbation of $M$ gives a foliation where each manifold has a strict sign on the mean curvature. We take $\nu$ to be the upwards unit vector of the perturbation, and so we look for a positive perturbation with $H < 0$ and $\langle \nu, \mu \rangle \geq 0$: Once we have such a perturbation, solving an ODE will give local existence of a comparison solution. Let $F : \Omega \rightarrow \mathbb{R}^{n+1}$ parametrise $M$, we add a $G : \Omega \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^{n+1}$, such that $F = F + G$ is as in Section 1 locally, except we weaken the requirement at the boundary to $\langle \nu(x, \lambda), \mu \rangle \geq 0$. We additionally stipulate that $G|_{\epsilon=0} = 0$, and $\partial G^2|_{\epsilon=0} = \phi \nu$ for some smooth $\phi$.

Identically to the proof of Proposition 3 we calculate

$$\frac{\partial}{\partial \epsilon}|_{\epsilon=0} g_{ij} = 2\phi h_{ij}, \quad \frac{\partial \nu}{\partial \epsilon}|_{\epsilon=0} = \nabla \phi.$$ 

We also have

$$\frac{\partial}{\partial \epsilon}|_{\epsilon=0} h_{ij} = -\frac{\partial}{\partial \epsilon}|_{\epsilon=0} \left( \nu, \frac{\partial^2 (F + G)}{\partial x^i \partial x^j} \right) = \nabla^2_{ij} \phi + \phi h_{ik}^i h_{kj},$$
and so\[ \frac{\partial H}{\partial \epsilon} |_{\epsilon=0} = -R^a_{\nu} \frac{\partial g_{\nu \mu}}{\partial \epsilon} |_{\epsilon=0} + g^{ij} \frac{\partial h_{ij}}{\partial \epsilon} |_{\epsilon=0} = \Delta \phi - \phi |A|^2. \]

At the boundary we require \( \frac{\partial G}{\partial \epsilon} |_{\epsilon=0} = \langle \phi \nu, \mu \rangle |_{\epsilon=0} = 0 \) so that the boundary stays in \( \Sigma \) for a short time, and additionally that \( \langle \nu, \phi \rangle \geq 0 \) for larger \( \epsilon \). Therefore we require\[ 0 \leq \frac{\partial}{\partial \epsilon} \langle \nu, \mu \rangle |_{\epsilon=0} = \nabla_\mu \phi + A^2(\nu, \nu) \phi. \]

Therefore when \( \text{(6)} \) is satisfied, for small \( \delta > 0 \) the surfaces \( M_\delta \) given by \( F(\cdot, \delta) \) have \( H < 0 \) and \( \langle \nu \sigma, \mu \rangle \geq 0 \). Now solving an ODE gives a comparison solution satisfying \( \text{(5)} \).

**Corollary 24.** If \( M \) is as in the previous Lemma and additionally at the boundary \( A^2(\nu, \nu) > 0 \), then \( M \) is stable.

**Proof.** Pick a point \( a \in \mathbb{R}^{n+1}_1 \) and consider the function \( f = R - |x - a|^2 \) which we will show satisfies \( \text{(6)} \). We may easily see that \( \nabla_j f = -2 \nu_j \) and so, since \( M \) is maximal \( \Delta f = -2n \). At the boundary we have\[ \nabla_\mu f = -2 \langle x - a, \mu \rangle. \]

By compactness of \( \partial M \), \( A^2(\nu, \nu) > \delta \), for some \( \delta > 0 \), and similarly (by uniform spacelikeness of \( M \)) \( \langle \mu, x - a \rangle \) is bounded above. Therefore there exists a \( R_0 > 0 \) such that for all \( R \geq R_0 \), \( 2 \langle x - a, \mu \rangle \leq A^2(\nu, \nu)(R - |x - a|^2) \). Setting \( R = \max_{M} \{ R_0, \sup_{|x - a|^2} \} \) then \( \text{(6)} \) holds for \( \phi = f \). \( \square \)

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