A novel Boundary Element model for solving stationary inhomogeneous heat conduction problems

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Abstract. A new formulation of BEM applied to steady heat conduction problems, governed by the inhomogeneous Laplace Equation will be presented. The thermal property is isotropic, but varies according to a function known throughout the domain. DIBEM and DST are used together to generate the numerical model. Being used in the interpolation process, two distinct RBF will be used to evaluate the formulation performance. In order to show the generality and capacity of the proposed model, the heat flux variation in a machine tool temperature sensor is calculated. The comparison of the quality of the results achieved is made using FEM.

Keywords: Boundary Element Method; Non homogeneous Laplace's problems; Heat Flow Measurements.

1. Introduction

Many problems of modern engineering have their complexity increased if the constitutive property of the medium involved is considered smoothly inhomogeneous, as occurs for example in heat transfer processes in functional materials [1]. The solution to these important and complex industrial problems is done necessarily through the use of numerical methods of solution, based on the idea of discretization. It can be argued that this important class of problems is preferably dealt with by domain discretization methods such as the Finite Element Method (FEM) and the Finite Differences Method (FDM). For some reason, such problems are not suitable for modeling with the Boundary Element Method (BEM).

Mathematically, the problems under consideration result in models in which domain integrals refer to the variation of the constitutive property in the domain. However, regarding the BEM it has evolved considerably in the treatment of these integrals due to its association with procedures based on the interpolation with Radial Base Functions (RBF) [2]. This association led to the creation of the Direct Interpolation Boundary Element Method (DIBEM) [3,4].

The DIBEM is designed to directly interpolate the entire kernel of the domain integrals, including the fundamental solution. It is important to highlight that DIBEM uses the regularization procedure to avoid singularities between source points and the usual field points. Loeffler et al. [5] applied DIBEM to Helmholtz and Poisson’s problems in homogeneous regions [3], clearly showing good results through comparisons with other methods.

Considering the DIBEM in cases where the domains present are piecewise homogeneous, was recently used a tool called Domain Superposition Technique (DST) developed by Loeffler and Mansur...
[6], in which the energy associated with any homogeneous internal region can be interpreted as an energy source and thus easily modeled in terms of coefficients of influence. Thus, some Laplace’s problems were solved by Loeffler et al. [7].

More recently, Barcelos and Loeffler [1] applied DST and DIBEM with regularization in Laplace’s problems in regions with isotropic constitutive properties, varying smoothly across the domain or sectorally surrounded by a constant \( K(x) \) property.

In this work, the DST has applied again in conjunction with the regularized DIBEM; however, the complexity of the problem associated with the Laplace equation will be greater, since the surrounding domain will contribute different constitutive properties, distributed in regions defined in the chosen geometry. Two RBF will be used during the generation of results. Linear elements will be used during numerical and analytical solutions of boundary integrals; the proposed geometry in the simulation will not be conventional, this will demonstrate the versatility of the proposed model.

2. Application of DST with BEM to piecewise problems

The proposed problem is based on a model where \( K(x) \) represents the heterogeneous constitutive property in a physical domain \( \Omega(x) \). Such a property is isotropic and smooth in the sectors that make up the entire domain. The potential \( u(x) \) is under steady-state at \( \Omega(x) \) and its field is considered irrotational. There are no sources.

Thus the government equation will be represented by:

\[
[K(X)u_i(X)]_i = 0. \tag{1}
\]

The surrounding region \( \Omega^{sur}(x) \) and an arbitrary subdomain \( \Omega^{int}(x) \) are contained in Figure 1. According to Loeffler and Mansur [6], the Figure 1 is created.

![Figure 1. The DST applied in inhomogeneous medium.](image)

The function \( K^{sur}(x) \) is an surrounding property in \( \Omega^{sur}(x) \) and \( K^{int}(x) \) the property restricted to the internal sector. Given this model, the BEM is applied with its classic fundamental solution \( u^*(X) \) presented by Brebbia et al [8], obtaining the strong integral form:

\[
\int_{\Omega(x)} [K(X)u_i(X)]_i u^*(X) d\Omega(X) = 0. \tag{2}
\]

Thus, the deduction of the inverse integral equation can be done as follows and respecting Eq. (2):

\[
\int_{\Omega^{sur}(x)} [K^{sur}(X)u_i(X)]_i u^*(X) d\Omega^{sur}(X) = \int_{\Omega(x)} [K'(X)u_i(X)]_i u^*(X) d\Omega(X). \tag{3}
\]

In the previous equation \( K'(X) \) and domain \( \Omega(X) \) are respectively:

\[
K'(X) = K^{sur}(X) - K^{int}(X) ; \Omega(X) = \Omega^{sur}(X) - \Omega^{int}(X). \tag{4}
\]

Suitable algebraic manipulation results in the following final equation [1]:
It’s emphasized that:

\[ \Gamma(X) = \Gamma^{\text{sur}}(X) - \Gamma^{\text{int}}(X). \]  

In Thus, being the energy stored in the internal sector given just by the potential energy [6], Eq. (5) can be simplified so that:

\[ \begin{align*}
    c(\xi)u(\xi)K^{\text{sur}}(\xi) + \int_{\Gamma(\xi)} u(\xi)K^{\text{sur}}(\xi)q^{*}(\xi; X) d\Gamma^{\text{sur}}(X) - \\
    \int_{\Gamma(\xi)} q(\xi)K^{\text{sur}}(\xi)u^{*}(\xi; X) d\Gamma^{\text{sur}}(X) - \int_{\Omega(\xi)} u^{*}(\xi; X) d\Omega^{\text{sur}}(X) = \\
    c(\xi)u(\xi)\bar{K}(\xi) + \int_{\Gamma(\xi)} u(\xi)\bar{K}(\xi)q^{*}(\xi; X) d\bar{\Gamma}(X) - \int_{\Gamma(\xi)} q(\xi)\bar{K}(\xi)u^{*}(\xi; X) d\bar{\Gamma}(X) - \\
    \int_{\Omega(\xi)} u(\xi)\bar{K}(\xi)u^{*}(\xi; X) d\bar{\Omega}(X).
\end{align*} \]  

Brebbia et al [8] presents the fundamental solution \( u^{*}(\xi; X) \) and the normal derivative \( q^{*}(\xi; X) \) as shown below:

\[ u^{*}(\xi; X) = -\frac{1}{2\pi} \ln[r(\xi; X)]; \quad q^{*}(\xi; X) = u^{*}(\xi; X)n_{i}(X). \]

3. The DIBEM to piecewise problems

According to Loeffler et al. [5], the entire kernel of domain integrals of Eq. (7) can be approximated using linear combination of Radial Basis Functions (RBF):

\[ u(X)K^{\text{sur}}(X)u^{*}(\xi; X) = u(X) \xi \Lambda(X) \approx \xi \alpha / F^{1}(X^{\prime}; X), \]

and

\[ u(X)\bar{K}(\xi)u^{*}(\xi; X) = u(X) \xi \Pi(X) \approx \xi \beta / F^{1}(X^{\prime}; X). \]

The RBF used in this work were that of Thin Plate (\( F_{1}^{1}(X^{\prime}; X) \)) and Wendland (\( F_{2}^{1}(X^{\prime}; X) \)), by Buhmann [2].

\[ F_{1}^{1}(X^{\prime}; X) = r^{2}(X^{\prime}; X)\ln[r(X^{\prime}; X)]; \quad F_{2}^{1}(X^{\prime}; X) = \left[ 1 - \frac{r(X^{\prime}; X)}{\delta} \right]^{3}. \]

The argument of the interpolation function \( F_{1}^{1}(X^{\prime}; X) \) is composed by the Euclidean distance \( r(X^{\prime}; X) \) defined by the positions of the base point \( X^{\prime} \) with respect to the field point \( X \) on the domain \( \Omega(X) \). After the discretization process, the \( X \) field points are used to generate the nodal points in which the potential term \( u(X) \) is calculated. The terms \( \alpha \) and \( \beta \) are coefficients and the terms \( \Lambda \) and \( \Pi \) represent diagonal matrices containing the partial derivatives of the constitutive property \( K^{\text{sur}} \) and \( \bar{K}(\xi) \) respectively.

Aiming to convert the domain integral into a boundary integral, DIBEM procedure uses a primitive radial basis function \( \Psi^{1}(X^{\prime}; X) \) related to the interpolation function \( F^{1}(X^{\prime}; X) \):

\[ \xi \alpha / \int_{\Gamma(\xi)} F^{1}(X^{\prime}; X) d\Omega^{\text{sur}}(X) = \xi \alpha / \int_{\Gamma(\xi)} \eta^{1}(X^{\prime}; X) d\Gamma^{\text{sur}}(X). \]
Substituting the Eq. (12) in Eq. (7), all terms in governing equations are now composed by boundary integrals, that is:

\[
K_{\text{sur}}(\xi)u(\xi)J + \int_{\Gamma_{\text{sur}}} (X)K_{\text{sur}}(X)q^*(\xi; X) d\Gamma_{\text{sur}}(X) - \\
\int_{\Gamma_{\text{sur}}} (X)q(\xi)K_{\text{sur}}(X)u^*(\xi; X) d\Gamma_{\text{sur}}(X) - \xi \int_{\Gamma_{\text{sur}}} (X)\eta^i(X^i; X)d\Gamma_{\text{sur}}(X) \approx \\
c(\xi)u(\xi)\bar{K}(\xi) + \int_{\Gamma(X)} (X)\bar{K}(X)q^*(\xi; X) d\bar{\Gamma}(X) - \xi \int_{\Gamma(X)} \eta^i(X^i; X)d\bar{\Gamma}(X).
\]

For each source point \( \xi \), the interpolation is given by Eq. (9) and Eq. (10), corresponding to the scan of all points \( X^i \) with respect to the domain points \( X \), weighted by the coefficients \( \alpha \) and \( \beta \). The coefficients \( \alpha \) and \( \beta \) are the unknowns, which can be obtained after de discretization procedure.

4. Regularization scheme of DIBEM

Singularities arise in Eq. (9) and Eq. (10) when the source point and the field point are coincident. This singularity is avoided through a regularization procedure [4], based on the concept presented by Hadamard [9]. The procedure is applied to domain integrals of Eq. (7) as follows:

\[
c(\xi)u(\xi)K_{\text{sur}}(\xi) + \int_{\Gamma_{\text{sur}}} (X)K_{\text{sur}}(X)q^*(\xi; X) d\Gamma_{\text{sur}}(X) - \\
\int_{\Gamma_{\text{sur}}} (X)q(\xi)K_{\text{sur}}(X)u^*(\xi; X) d\Gamma_{\text{sur}}(X) - \\
\int_{\Gamma_{\text{sur}}} (X)[u(\xi) - u(\xi)]K_{\text{sur}}(X)u^i(\xi^i; X) d\Omega_{\text{sur}}(X) - \\
u(\xi)\int_{\Omega_{\text{sur}}} (X)K_{\text{sur}}(X)u^i(\xi^i; X) d\Omega_{\text{sur}}(X) = \\
c(\xi)u(\xi)\bar{K}(\xi) + \int_{\Gamma(X)} (X)\bar{K}(X)q^*(\xi; X) d\bar{\Gamma}(X) - \\
\int_{\Gamma(X)} [u(X) - u(\xi)]\bar{K}_i(X)u^i(\xi^i; X)d\bar{\Gamma}(X) - u(\xi)\int_{\Omega(X)} \bar{K}_i(X)u^i(\xi^i; X)d\bar{\Omega}(X).
\]

In Eq. (14) the DIBEM procedure is applied, approximating the kernel of the domain integrals, as follows:

\[
[u(X) - u(\xi)]K_i^j(X^j; X) \approx \xi \int_{\Gamma(X)} (X^j; X),
\]

and

\[
[u(X) - u(\xi)]\bar{K}_i(X)u^i(\xi^i; X) = \xi \beta \int_{\Gamma(X)} (X^j; X).
\]

Considering that the fundamental solution of the Poisson’s problems is used [8], one has:

\[
-u(\xi)\int_{\Gamma_{\text{sur}}} (X)q^*(\xi; X) d\Gamma_{\text{sur}}(X) + \int_{\Gamma_{\text{sur}}} (X)u(\xi)K_{\text{sur}}(X)q^*(\xi; X) d\Gamma_{\text{sur}}(X) - \\
\int_{\Gamma_{\text{sur}}} (X)q(\xi)K_{\text{sur}}(X)u^*(\xi; X) d\Gamma_{\text{sur}}(X) - \xi \int_{\Gamma_{\text{sur}}} (X)\eta^i(X^i; X)d\Gamma_{\text{sur}}(X) = \\
-u(\xi)\int_{\Gamma(X)} \bar{K}(X)q^*(\xi; X) d\bar{\Gamma}(X) + \int_{\Gamma(X)} u(X)\bar{K}(X)q^*(\xi; X) d\bar{\Gamma}(X) - \\
\xi \beta \int_{\Gamma(X)} \eta^i(X^i; X)d\bar{\Gamma}(X).
\]
5. Discretization procedure

Equation (17) can be written in a following discrete form:

\[
\begin{align*}
-(H_{12} + \cdots + H_{1n})u_1 + H_{12}u_2 + \cdots + H_{1n}u_n - G_{11}q_1 - \cdots - G_{1n}q_n \\
= 1\alpha^1 N_i^{sur} + \cdots + 1\alpha^n N_n^{sur} - 1\beta^i \tilde{N}_i - \cdots - 1\beta^m \tilde{N}_m \\
- (H_{21} + H_{23} + \cdots + H_{2n})u_2 + \cdots + H_{2n}u_n - G_{21}q_1 - \cdots - G_{2n}q_n \\
= 2\alpha^1 N_i^{sur} + \cdots + 2\alpha^n N_n^{sur} - 2\beta^i \tilde{N}_i - \cdots - 2\beta^m \tilde{N}_m \\
\vdots \\
H_{n1}u_1 + H_{n2}u_2 + \cdots - (H_{n1} + \cdots + H_{n(n-1)})u_n - G_{n1}q_1 - \cdots - G_{nn}q_n \\
= n\alpha^1 N_i^{sur} + \cdots + n\alpha^n N_n^{sur} - n\beta^i \tilde{N}_i - \cdots - n\beta^m \tilde{N}_m.
\end{align*}
\]  

(18)

Both terms \(i\) and \(m\) comprise all interpolation points of the internal sectors of \(\tilde{\Omega}(X)\).

Using the matrix form the Eq. (18) is given by:

\[
\begin{bmatrix}
H_{11} & \cdots & H_{1n} \\
\vdots & \ddots & \vdots \\
H_{n1} & \cdots & H_{nn}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
\vdots \\
u_n
\end{bmatrix}
- \begin{bmatrix}
G_{11} & \cdots & G_{1n} \\
\vdots & \ddots & \vdots \\
G_{n1} & \cdots & G_{nn}
\end{bmatrix}
\begin{bmatrix}
q_1 \\
\vdots \\
q_n
\end{bmatrix} = 
\begin{bmatrix}
\frac{1}{\alpha^1} & \cdots & \frac{1}{\alpha^n} \\
\vdots & \ddots & \vdots \\
\frac{n\alpha^1}{\alpha^n} & \cdots & \frac{n\alpha^1}{\alpha^n}
\end{bmatrix}
\begin{bmatrix}
N_1^{sur} \\
\vdots \\
N_n^{sur}
\end{bmatrix}
- \begin{bmatrix}
\frac{1}{\beta^i} & \cdots & \frac{1}{\beta^m} \\
\vdots & \ddots & \vdots \\
\frac{n\beta^i}{\beta^m} & \cdots & \frac{n\beta^i}{\beta^m}
\end{bmatrix}
\begin{bmatrix}
\tilde{N}_1 \\
\vdots \\
\tilde{N}_m
\end{bmatrix}.
\]  

(19)

After a large algebraic and matrix treatment, it’s possible to write the coefficients \(\alpha\) and \(\beta\) as a function of the \(u(X)\) according to Eq. (20):

\[
\begin{bmatrix}
H_{11} & \cdots & H_{1n} \\
\vdots & \ddots & \vdots \\
H_{n1} & \cdots & H_{nn}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
\vdots \\
u_n
\end{bmatrix}
- \begin{bmatrix}
G_{11} & \cdots & G_{1n} \\
\vdots & \ddots & \vdots \\
G_{n1} & \cdots & G_{nn}
\end{bmatrix}
\begin{bmatrix}
q_1 \\
\vdots \\
q_n
\end{bmatrix} = 
\begin{bmatrix}
D_{11} & \cdots & D_{1n}^{sur} \\
\vdots & \ddots & \vdots \\
D_{n1} & \cdots & D_{nn}
\end{bmatrix}
\begin{bmatrix}
\tilde{D}_{11} & \cdots & \tilde{D}_{1n} \\
\vdots & \ddots & \vdots \\
\tilde{D}_{n1} & \cdots & \tilde{D}_{nn}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
\vdots \\
u_n
\end{bmatrix}.
\]  

(20)

6. Numerical Simulation: Heat diffusion in a machine tool

This example evaluates the normal temperature gradient variation at \(x_2=0\) by contacting a machining tool with surfaces to be machined. Figure 2 outlines the problem in question and presents the boundary conditions. The dashed outline indicates thermal insulation, i.e. the normal temperature gradient condition is null. The relative error for the temperature will be taken at the external boundary of sector 1. As already mentioned, the temperature gradient calculation was done at \(x_2 = 0\).

In the left corner of Figure 2 we have the established domain and its boundary conditions, in the right side we observe the logic assembled to apply to DST according to Figure 1. In this way \(K^{sur}(x_1,x_2)\) was chosen as the constitutive property of \(\Omega^{sur}(X)\). The boundaries of this problem contain Dirichlet type boundary conditions in terms of temperature \(T(X)\) with unit in Celsius \(^\circ\)C and Neumann type for the normal temperature gradient \(\frac{dT(X)}{dn}\) in \(^\circ\)C/cm. These units of measurement were taken for convenience.
Figure 2. The sector 1 with $K^{int1}(x_1,x_2)$ receives the heat flow during the part application process for this example.

The percent relative error "$pre\%$" will be measured at the boundary of interest through Eq. (21):

$$pre\% = 100 \left( \frac{1}{n} \sum_{j=1}^{n} \left| \frac{V_j^{FEM}}{V_j^{DIBEM}} - 1 \right| \right) \%.$$  

The term $n$ is number of total points, $V_{FEM}$ corresponds to the largest reference value calculated by the FEM at this boundary, $V_j^{FEM}$ the FEM value calculated at point $j$ and $V_j^{DIBEM}$ the DIBEM value also calculated at this same point $j$.

Mesh generation to represent the DIBEM and FEM discretization process was done through the Delaunay Triangulation [10]. Poles are considered to be all internal points outside the boundary of the mesh used by DIBEM.

Figure 3 presents the functions of the model’s constitutive properties, as well as the new functions after application of the DST.

| Constitutive property of model | Constitutive property applying DST | Constitutive property gradient |
|-------------------------------|-----------------------------------|--------------------------------|
| $K^{intw}(x_1,x_2) = K_0 \frac{W}{m^2C}$ | $K^{intw}(x_1,x_2) = K_0 \frac{W}{m^2C}$ | $\frac{\partial K^{intw}(x_1,x_2)}{\partial x_1} = 0 \frac{W}{m^2C}$, $\frac{\partial K^{intw}(x_1,x_2)}{\partial x_2} = 0 \frac{W}{m^2C}$ |

$K^{int1}(x_1,x_2) = K_0 e^{(1+0.01811x_2)} \frac{W}{m^2C}$  

$K^{int2}(x_1,x_2) = K_0 x_2 \frac{W}{m^2C}$

$K^{int1}(x_1,x_2) = K_0 e^{(1+0.01811x_2)} \frac{W}{m^2C}$  

Figure 3. Constitutive properties with DST.

During the discretization process, the mesh used for the FEM contains 4405 nodes with 8464 triangular elements. Figure 4 (a) shows a FEM mesh similar to that used, but with little discretization and the FEM numerical solution is represented by Figure 4 (c). The construction of DIBEM meshes.
was made paying attention to a uniform distribution of the internal points according to Figure 4 (b). Figure 5 and Table 1 present the results found for this problem.

![Figure 4. Meshes and the reference numerical solution: (a) FEM, (b) DIBEM e (c) FEM solution.](image)

**Figure 4.** Meshes and the reference numerical solution: (a) FEM, (b) DIBEM e (c) FEM solution.

![Figure 5.](image)

**Figure 5.** (a) Error curve related to temperature values, (b) Error curve related to temperature gradient values and (c) Temperature gradient profile at $x_2=0$ in $\Omega(X)$.

![Table 1](image)

**Table 1.** Percentage relative error values for temperature gradient

| Total number of poles | Mesh BEM (a)-(b)-(c) | Percent relative error (Thin plate RBF) | Percent relative error (Wendland RBF) |
|-----------------------|----------------------|----------------------------------------|---------------------------------------|
| 0                     | BEM                  | 5.622658%                             | 2.729247%                             |
| 29                    | 172/40/108           | 0.341815%                             | 0.512791%                             |
| 41                    |                      | 0.196732%                             | 0.365725%                             |
| 176                   |                      | 0.047528%                             | 0.096317%                             |
| 196                   |                      | 0.047524%                             | 0.096317%                             |

(a) = Number of boundary elements of surrounding region.
(b) = Number of boundary elements of sector 1.
(c) = Number of boundary elements of sector 2.

For the first mesh refinement, it’s noteworthy that the Thin Plate RBF lost precision during the calculation of the temperature gradient distribution near $x_1 = 0$ (Figure 5(c)), however, Table 1 allows us to conclude that the refinement has subtly improved its performance compared to Wendland RBF.
7. Conclusions
Considering the comparison done with the FEM results, the example presented in this work can be considered satisfactorily solved. The combination between the DST and the DIBEM, two recent techniques, was successful since the error can be reduced successively with the mesh refinement reaching very acceptable engineering values.

The application of DST must precede the DIBEM with regularization because this favours the process of transforming enveloping internal domain integrals with non-self-joining kernels into boundary integrals. The Thin Plate and Wendland RBF collaborated during mesh refinement, concluding that both were important for the method convergence.

Given the good performance found, a great evolution of this work follows the search for the solution of the Helmholtz’s problem in inhomogeneous medium, analyzing the response problem and eigenvalue problem, as presented by Butkov [11].

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References
[1] Barcelos H M and Loeffler C F 2019 The direct interpolation boundary element method applied to smoothly inhomogeneous Laplace’s problems Engineering Analysis with Boundary Elements. 105 155-164.
[2] Buhmann M D 2003 Radial Basis Function: Theory and Implementations (Cambridge University Press).
[3] Loeffler C F, Cruz A L and Bulcão A 2015 Direct Use of radial basis interpolation functions for modelling source terms with the boundary element method Engineering Analysis with Boundary Elements. 50 97-108.
[4] Loeffler C F and Mansur W J 2017 A Regularization scheme applied to the direct interpolation boundary element technique with radial basis functions for solving eigenvalue problem Engineering Analysis with Boundary Elements. 74 14-18.
[5] Loeffler C F, Barcelos H M, Mansur W J and Bulcão A 2015 Solving Helmholtz Problems with the Boundary Element Method Using Direct Radial Basis Function Interpolation Engineering Analysis with Boundary Elements. 61 218-225.
[6] Loeffler C F, Mansur W J 2016 Sub-regions without subdomain partition with boundary elements Engineering Analysis with Boundary Elements. 74 14-18.
[7] Loeffler C F, Barbosa J P and Barcelos H M 2018. Performance of BEM superposition technique for solving sectorially heterogeneous Laplace’s problems with non-regular geometry Engineering Analysis with Boundary Elements. 93 105-111.
[8] Brebbia C A, Telles J C F and Wrobel L C 1984 Boundary Element Techniques. (Springer-Verlag).
[9] Pessolani R V 2002. An hp-adaptive hierarchical formulation for the boundary element method applied to elasticity in two dimensions Journal of the Brazilian Society Mechanical Sciences. 24 23-45.
[10] Delaunay B N 1934 Sur la sphère vide. Izvestia Akademil Nauk SSSR Otdelenie Matematicheskikh i Estestvennykh. 7 793-800.
[11] Butkov E 1973 Mathematical Physics (Addison-Wesley publishing company).