FINITELY PRESENTED SIMPLE MODULES OVER LEAVITT PATH ALGEBRAS

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ABSTRACT. Let $E$ be an arbitrary graph and $K$ be any field. We construct various classes of non-isomorphic simple modules over the Leavitt path algebra $L_K(E)$ induced by vertices which are infinite emitters, closed paths which are exclusive cycles and paths which are infinite, and call these simple modules Chen modules. It is shown that every primitive ideal of $L_K(E)$ can be realized as the annihilator ideal of some Chen module. Our main result establishes the equivalence between a graph theoretic condition and various conditions concerning the structure of simple modules over $L_K(E)$.

1. Introduction

Leavitt path algebras were introduced and initially studied in [1], [11], as algebraic analogues of graph C*-algebras and as natural generalizations of the Leavitt algebras of type $(1, n)$ built in [19]. The study of the module theory over Leavitt path algebras was initiated in [9], in connection with some questions in algebraic K-theory. Very recently, following the results of [17], Chen [16] has provided a method of constructing simple modules $V_\pi$ over a Leavitt path algebra $L_K(E)$ of an arbitrary graph $E$ by using the equivalence class $\pi$ of infinite paths tail-equivalent to a fixed infinite path $p$ in $E$. (See below for the definition of tail-equivalence for infinite paths.) He also constructed simple modules $N_w$ corresponding to various sinks $w$ in $E$. The authors have used in [13] this family of simple modules to determine the structure of the Leavitt path algebras of arbitrary graphs which have only a finite number of isoclasses of simple modules. It is as well interesting to observe that Chen’s work is related to some constructions in non-commutative algebraic geometry (see [22] and [21]).

In this paper, we introduce additional classes of simple modules using vertices which are infinite emitters and also exclusive cycles, and call all these simple modules over $L_K(E)$ Chen modules. We give a description of the annihilators of the various Chen modules and, as a consequence, we show that every primitive ideal of $L_K(E)$ can be realized as the annihilator of a Chen module. Here we take advantage of results from [20], describing the structure of the primitive ideals over a Leavitt path algebra of an arbitrary graph (see also [3], where the

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structure of primitive Leavitt path algebras of arbitrary graphs is described.) We also show, using results from [16], that the Chen modules are pairwise non-isomorphic.

Next we investigate the structure of simple modules over a Leavitt path algebra of a finite graph $E$. The structure of the simple finitely presented $L_K(E)$-modules was determined in [9] in terms of the irreducible finite-dimensional representations of the usual path algebra of the reverse graph $\overline{E}$ of $E$. A lot is known about these representations, for instance Le Bruyn and Procesi determined in [18, Section 5] the possible dimension vectors for them. So, a natural question is to determine all the finite graphs $E$ such that all simple $L_K(E)$-modules are finitely presented. One can also ask what are the connections between finitely presented simple modules and Chen simple modules. To begin with, we show that the Chen module $V_{[p]}$ corresponding to an infinite path $p$ is finitely presented if and only if $p$ is tail-equivalent to the rational infinite path $ggg \cdots$ where $g$ is some closed path.

For an algebra $A$, denote by $\hat{A}$ the set of isoclasses of simple left $A$-modules, and by $\text{Prim}(A)$ the set of primitive ideals of $A$. There is a canonical map

$$\hat{A} \rightarrow \text{Prim}(A)$$

sending $[N]$ to $\text{Ann}_A(N)$, the annihilator of $N$.

Our main result is the following:

**Theorem 1.1.** Let $E$ be a finite graph and $K$ an arbitrary field. Write $L = L_K(E)$. Then the following conditions are equivalent:

1. Every simple left $L$-module is finitely presented.
2. Every simple Chen module is finitely presented.
3. Every vertex $v$ in $E$ is the base of at most one cycle.
4. The map $\hat{L} \rightarrow \text{Prim}(L)$ is a bijection.
5. All simple left $L$-modules are Chen modules.

It is interesting to note, from the recent investigation done in [6], that the algebras $L_K(E)$ appearing in the theorem are precisely the Leavitt path algebras having finite Gelfand-Kirillov dimension.

2. Preliminaries

Before we proceed to set the basic definitions, let us remark that there is a lack of uniformity in the notation and terminology used in graph theory. Even in the setting of Leavitt path algebras or graph C*-algebras, different authors often use different conventions regarding the basic concepts from graph theory. In particular we advise the reader that the notation herewith will be the same as in [2] but will differ substantially from the one used by Chen in [16].

A (directed) graph $E = (E^0, E^1, r, s)$ consists of two sets $E^0$ and $E^1$ together with maps $r, s : E^1 \rightarrow E^0$. The elements of $E^0$ are called vertices and the elements of $E^1$ edges. We generally follow the notation, terminology and results from [2]. We outline some of the concepts and results that will be used in this paper.
A vertex $v$ is called a **sink** if it emits no edges, that is, $s^{-1}(v) = \emptyset$, the empty set. The vertex $v$ is called a **regular vertex** if $s^{-1}(v)$ is finite and non-empty and $v$ is called an **infinite emitter** if $s^{-1}(v)$ is infinite. For each $e \in E^1$, we call $e^*$ a ghost edge. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. A **finite path** $\mu$ of length $n > 0$ is a finite sequence of edges $\mu = e_1 e_2 \cdots e_n$ with $r(e_i) = s(e_{i+1})$ for all $i = 1, \ldots, n - 1$. In this case $\mu^* = e_n^* \cdots e_2^* e_1^*$ is the corresponding ghost path. The set of all vertices on the path $\mu$ is denoted by $\mu^{0}$. Any vertex $v$ is considered a path of length 0.

Given an arbitrary graph $E$ and a field $K$, the **Leavitt path $K$-algebra** $L_K(E)$ is defined to be the $K$-algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:

1. $s(e)e = e = er(e)$ for all $e \in E^1$.
2. $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.
3. (The “CK-1 relations”) For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if $e \neq f$.
4. (The “CK-2 relations”) For every regular vertex $v \in E^0$,
   \[ v = \sum_{e \in E^1, s(e) = v} ee^*. \]

A non-trivial path $\mu = e_1 \ldots e_n$ in $E$ is **closed** if $r(e_n) = s(e_1)$, in which case $\mu$ is said to be based at the vertex $s(e_1)$. A closed path $\mu$ as above is called **simple** provided it does not pass through its base more than once, i.e., $s(e_i) \neq s(e_1)$ for all $i = 2, \ldots, n$. The closed path $\mu$ is called a **cycle based at** $v$ if $s(e_1) = v$ and it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$.

When talking about a cycle we loosely understand the set of closed paths obtained by rotation of a representative of $c$. When talking about a cycle based at a vertex $v$ we understand the closed path in this family which starts at $v$.

A cycle $c$ is called an **exclusive cycle** if it is disjoint with every other cycle; equivalently, no vertex on $c$ is the base of a different cycle other than the rotate of $c$ based at $v$. These cycles were termed cycles without (K) in [20].

An **exit** for a path $\mu = e_1 \ldots e_n$ is an edge $e$ such that $s(e) = s(e_i)$ for some $i$ and $e \neq e_i$. A graph $E$ is said to satisfy **Condition (L)** if every cycle in $E$ has an exit.

If there is a path from vertex $u$ to a vertex $v$, we write $u \geq v$. A subset $D$ of vertices is said to be **downward directed** if for any $u, v \in D$, there exists a $w \in D$ such that $u \geq w$ and $v \geq w$. A subset $H$ of $E^0$ is called **hereditary** if, whenever $v \in H$ and $w \in E^0$ satisfy $v \geq w$, then $w \in H$. A hereditary set is **saturated** if, for any regular vertex $v$, $r(s^{-1}(v)) \subseteq H$ implies $v \in H$.

A subset $S$ of $E^0$ is said to have the **Countable Separation Property** (CSP) with respect to a set $C$, if $C$ is a countable subset of $E^0$ with the property that for each $u \in S$ there is a $v \in C$ such that $u \geq v$.

We shall be using the following concepts and results from [23] in our investigation. A **breaking vertex** of a hereditary saturated subset $H$ is an infinite emitter $w \in E^0 \setminus H$ with the property that $1 \leq |s^{-1}(w)| r^{-1}(E^0 \setminus H) | < \infty$. The set of all breaking vertices of $H$ is denoted by $B_H$. For any $v \in B_H$, $v^H$ denotes the element $v - \sum_{s(e) = v, r(e) \not\in H} ee^*$. Given a hereditary
saturated subset $H$ and a subset $S \subseteq B_H$, $(H, S)$ is called an admissible pair. The admissible pairs form a partially ordered set under the relation $(H_1, S_1) \leq (H_2, S_2)$ if and only if $H_1 \subseteq H_2$ and $S_1 \subseteq H_2 \cup S_2$. Given an admissible pair $(H, S)$, $I(H, S)$ denotes the ideal generated by $H \cup \{v^H : v \in S\}$. It was shown in [23] that the graded ideals of $L_K(E)$ are precisely the ideals of the form $I(H, S)$ for some admissible pair $(H, S)$. Moreover, $L_K(E)/I(H, S) \cong L_K(E \setminus (H, S))$. Here $E \setminus (H, S)$ is the quotient graph of $E$ in which $(E \setminus (H, S))^0 = (E^0 \setminus H) \cup \{v' : v \in B_H \setminus S\}$ and $(E \setminus (H, S))^1 = \{e \in E^1 : r(e) \notin H\} \cup \{e' : e \in E^1, r(e) \in B_H \setminus S\}$ and $r, s$ are extended to $(E \setminus (H, S))^1$ by setting $s(e') = s(e)$ and $r(e') = r(e)'$. Thus when $S = B_H$, $E \setminus (H, B_H)^0 = E^0 \setminus H$ and $E \setminus (H, B_H)^1 = \{e \in E^1 : r(e) \notin H\}$, so we can identify the graph $E \setminus (H, B_H)$ with the subgraph $E/H$ of $E$.

If $H$ is a hereditary saturated subset of $E^0$, $c$ is a cycle without exits in $E \setminus (H, B_H) = E/H$, based at $v \in E^0 \setminus H$, and $f(x)$ is a polynomial in $K[x, x^{-1}]$, we will denote by $I(H, B_H, f(c))$ the ideal of $L_K(E)$ generated by $I(H, B_H)$ and $f(c)$. Here $f(c)$ is the element of $L_K(E)$ obtained by formally substituting $x$ by $c$, $x^{-1}$ by $c^*$ and the constant term $a_0$ by $a_0 v$ in the canonical expression of $f(x)$ as a polynomial in $x, x^{-1}$.

3. Special types of simple modules

Let $E$ be an arbitrary graph and $L = L_K(E)$. In this section, we introduce three new classes of simple left modules over $L$ as an addition to the two types of simple left modules $N_w$ and $V_{[p]}$ defined by X.W. Chen ([16]) and described below. We call all these simple modules Chen modules. We first give a description of the annihilators of these Chen modules. Then we show that every primitive ideal of $L$ can be realized as the annihilator ideal of some Chen module. All the Chen modules are shown to be pairwise non-isomorphic. These results are utilized in the next section.

Given an infinite path $p = e_1e_2 \ldots e_n \ldots$ and an integer $n \geq 1$, Chen ([16]) defines $\tau_{\leq n}(p) = e_1 \cdots e_n$ and $\tau_{> n}(p) = e_{n+1}e_{n+2} \cdots$. Two infinite paths $p, q$ are said to be tail-equivalent if there exist positive integers $m, n$ such that $\tau_{> m}(p) = \tau_{> n}(q)$. This is an equivalence relation and the equivalence class of all paths tail equivalent to an infinite path $p$ is denoted by $[p]$. An infinite path $p$ is called a rational path if $p = ggg \cdots$ where $g$ is some (finite) closed path in $E$.

Given an infinite path $p$, Chen defines $V_{[p]} = \bigoplus_{q \in [p]} K q$, a $K$-vector space having $\{ q : q \in [p] \}$ as a basis. $V_{[p]}$ is made a left $L$-module by defining, for all $q \in [p]$ and all $e \in E^1$, $v \cdot q = 0$ according as $v = s(q)$ or not;

$e \cdot q = eq$ or 0 according as $r(e) = s(q)$ or not;

$e^* \cdot q = \tau_{> 1}(q)$ or 0 according as $q = eq'$ or not.

In [16], Chen shows that under the above action of $L$, $V_{[p]}$ becomes a simple left $L$-module.

Similarly, given a sink $w$ in the graph $E$, Chen defines $N_w$ to be the $K$-vector space having as its basis all the (finite) paths in $E$ that end in $w$. By defining an action of $L$ on the basis elements of $N_w$ in the same fashion as was done for $V_{[p]}$ (with the addition that $e^* \cdot w = 0$ for all $e \in E^1$), he shows that $N_w$ becomes a simple left $L$-module.

Throughout this section, we shall use the following notation.
For $v \in E^0$, define
\[ M(v) = \{ w \in E^0 : w \geq v \} \quad \text{and} \quad H(v) = E^0 \setminus M(v). \]

Similarly, if $p$ is an infinite path in $E$, we define
\[ M(p) = \{ w \in E^0 : w \geq v \text{ for some } v \in p^0 \} \quad \text{and} \quad H(p) = E^0 \setminus M(p). \]

Clearly both $M(v)$ and $M(p)$ are downward directed sets. Also, for any vertex $v$ which is a sink or an infinite emitter, and for any infinite path $p$, the sets $H(v)$ and $H(p)$ are hereditary saturated subsets of $E^0$. If $v$ is a finite emitter, it might be that $H(v)$ is not saturated, and $v$ may belong to the saturation of $H(v)$. Note also that $H(p) = H([p])$, that is, $H(p)$ does not depend on the tail-representative of $[p]$.

**Lemma 3.1.** If $w$ is a sink, then the annihilator of the simple module $N_w$ is $I(H(w), B_{H(w)})$.

**Proof.** Denote by $J$ the annihilator of $N_w$. We first prove that $I(H(w), B_{H(w)}) \subseteq J$. Since $J$ is an ideal of $L(E)$, it suffices to check that $H(w)$ and $\{ u^{H(w)} : v \in B_{H(w)} \}$ annihilate $N_w$. If $v \in H(w)$ then $v$ does not connect to $w$ and thus $v p = 0$ for every path $p$ ending at $w$. If $v \in B_{H(w)} \subseteq E^0 \setminus H(w)$, let $p$ be a path in $E$ such that $s(p) = v$ and $r(p) = w$. Let $e$ be the initial edge of $p$, say $p = ep_1$. Then $r(e) \notin H(w)$, because $r(e)$ connects to $w$. It follows that
\[ u^{H(w)} p = (v - \sum_{f \in s^{-1}(v), r(f) \notin H} ff^*)ep_1 = (e - e)p_1 = 0. \]

Therefore, we have shown the inclusion $I(H(w), B_{H(w)}) \subseteq J$. In order to show the reverse inclusion, consider the graph $F = E \setminus (H(w), B_{H(w)})$, and recall that $L(E)/I(H(w), B_{H(w)}) \cong L(F)$. We can look at $N_w$ as a simple $L(F)$-module, and we have to show that it is faithful as a $L(F)$-module. Write $\overline{J}$ for the annihilator of $N_w$ as a left $L(F)$-module. Obviously we have $\overline{J} \cap F^0 = \emptyset$, because every vertex in $F$ connects to $w$. On the other hand, $F$ satisfies condition (L) as every vertex in $F$ connects to the sink $w$ and this means, by Proposition 1 of [14], that every non-zero ideal of $L(F)$ contains a vertex. Consequently, $\overline{J} = 0$. This proves the result. \hfill \Box

Recall that a cycle $c$ is said to be an **exclusive cycle** if no vertex on $c$ is the base of a different cycle (other than the rotate of $c$ based at that vertex).

If an infinite path $p$ is tail-equivalent to the rational path $c^\infty$, where $c$ is a cycle in $E$, we say that $p$ ends in a cycle.

**Lemma 3.2.** Let $p$ be an infinite path. Then
1. If $p$ does not end in an exclusive cycle then the annihilator of $V_{[p]}$ is $I(H(p), B_{H(p)})$.
2. If $p$ ends in an exclusive cycle $c$ based at a vertex $v$, then the annihilator of $V_{[p]}$ is $I(H(p), B_{H(p)}), (c - v))$.

**Proof.** (1) The proof is similar to the proof of Lemma 3.1. We leave the details to the reader.

(2) Proceeding as in the proof of Lemma 3.1, we arrive at a simple module $V_{[p]}$ over $L(F)$, where $F = E \setminus (H(p), B_{H(p)})$, and $F$ has a unique cycle without exits, which is $c$. Moreover $F^0 \cap \overline{J} = 0$, where $\overline{J}$ is the annihilator of $V_{[p]}$ in $L(F)$ and hence a primitive ideal of $L(F)$.  


By Theorem 4-3 (iii) in [20], there exists an irreducible polynomial \( f \in K[x, x^{-1}] \) such that \( \mathcal{J} \) is the ideal generated by \( f(c) \). Since \( c - v \) annihilates \( V_{[p]} \), we conclude that \( f = x - 1 \). □

Next we wish to introduce new classes of simple modules, similar to the simple modules \( N_w \) and \( V_{[p]} \). Let \([q]\) be an infinite rational path with \( q = c^\infty \) and \( c = e_1e_2 \cdots e_n \) a cycle based at \( v \). In [16] Chen defines, for \( a \in K^\times \), a certain simple \( L(E) \)-module \( V^{a}_{[q]} \), which is the twisted module \( V^{a}_{[q]} \), where \( \sigma \) is the “gauge” automorphism of \( L(E) \) sending \( v \) to \( v \) for \( v \in E^0 \), \( e \) to \( e \) and \( e^* \) to \( e^* \) for \( e \in E^1 \) with \( e \neq e_1 \), and \( e_1 \) to \( ae_1 \) and \( e_1^* \) to \( a^{-1}e_1^* \). Denoting by \(*\) the module operation in \( V^{a}_{[q]} \), we have \( c * q = \sigma(c)c^\infty = acc^\infty = ac^\infty = aq \) and similarly \( c_i * c_i^\infty = ac_i^\infty \) for all rotates \( c_i \) of \( c \). Moreover, \( V^{a}_{[q]} \) is a simple module.

As a slight modification of the above construction, let \( f(x) = 1 + a_1x + \cdots + a_nx^n, n \geq 1 \), be an irreducible polynomial in \( K[x, x^{-1}] \) and let \( c = e_1e_2 \cdots e_m \) be an exclusive cycle. Set \( q = c^\infty \). We are going to define a new module \( V^{f}_{[q]} \). Let \( K' = K[x, x^{-1}]/(f(x)) \), which is a field because \( f(x) \) is irreducible. Define a \( L_{K'}(E) \)-module by \( M = V_{[q]}^{\sigma} \). Observe that \( M \) is well-defined because \( \sigma \) is invertible in \( K' \), and that \( M \) is a simple \( L_{K'}(E) \)-module. We denote by \( V^{f}_{[q]} \) the \( L_{K}(E) \)-module obtained by restricting scalars on \( M \) from \( L_{K'}(E) \) to \( L_{K}(E) \).

**Lemma 3.3.** The \( L_K(E) \)-module \( V^{f}_{[q]} \) is simple.

**Proof.** Let \( U \) be a nonzero \( L_K(E) \)-submodule of \( V^{f}_{[q]} \). We first claim that, given two nonzero scalars \( \lambda, \mu \) in \( K' \) such that \( \lambda q \in U \), then also \( \mu q \in U \). For this it clearly suffices to prove that \( (\sigma^\infty)(\lambda q) \in U \). We have

\[
\epsilon * (\lambda q) = \sigma(c)(\lambda c^\infty) = (\sigma\lambda)q.
\]

which shows the claim. Next, we follow the proof of [16 Theorem 3.3(1)]. Let \( u = \sum_{i=1}^{l} \lambda_i p_i \) be a nonzero element in \( U \), with \( \lambda_i \in K' \setminus \{0\} \) and \( p_i \) distinct paths in \([q]\). We can uniquely write \( p_i = p'_i c^\infty \), where \( p'_i \) is a finite path (possibly of length \( 0 \)) which does not involve \( e_1 \). We can assume that the length of \( p'_i \) is larger than or equal to the maximum of the lengths of all the other paths \( p'_i, i \geq 2 \). We get

\[
[(p'_i)^*] * u = \lambda_1 c^\infty
\]

because the path \( p'_i \) does not involve the edge \( e_1 \) and has maximum length. Therefore \( \lambda_1 q \in U \).

Now let \( \mu \) be an arbitrary nonzero element of \( K' \), and \( p_0 \in [q] \). We have \( p_0 = p'_0 q \), with \( p'_0 \) a finite path not involving \( e_1 \). By the claim we have \( \mu q \in U \), and so

\[
\mu p_0 = p'_0 (\mu q) \in U.
\]

It follows that \( U = V^{f}_{[q]} \), showing the result. □

The module \( V^{f}_{[q]} \) above has the appropriate annihilator, as follows.

**Lemma 3.4.** Let \( f(x) = 1 + a_1x + \cdots + a_nx^n, n \geq 1 \), be an irreducible polynomial in \( K[x, x^{-1}] \) and let \( c = e_1e_2 \cdots e_m \) be an exclusive cycle. Set \( q = c^\infty \). Then the annihilator of \( V^{f}_{[q]} \) is \( I(H(q), B_{H(q)}, f(c)) \).
Proof. As in the proof of Lemma 3.2(2), we only have to show that the annihilator \( \mathfrak{J} \) of \( V_f \) in \( L(F) \) contains \( f(c) \), where \( F = E \setminus (H(q), B_{H(q)}) \). But this follows from

\[
f(c) \ast c^\infty = \sigma(f(c))c^\infty = f(\sigma(c))c^\infty = f(\overline{x})c^\infty = 0.
\]

□

Let \( v \) be an infinite emitter such that \( v \in B_{H(v)} \).

Then we can build the primitive ideal \( P = I(H(v), B_{H(v)} \setminus \{v\}) \) (see [20]) and the factor ring

\[
L(E)/P \cong L(F)
\]

where \( F = E \setminus (H(v), B_{H(v)} \setminus \{v\}) \). Here \( F^0 = (E^0 \setminus H(v)) \cup \{v\} \),

\[
F^1 = \{ e \in E^1 : r(e) \notin H(v) \} \cup \{ e' : e \in E^1, r(e) = v \}
\]

and \( r \) and \( s \) are extended to \( F \) by setting \( s(e') = s(e) \) and \( r(e') = v' \) for all \( e \in E^1 \) with \( r(e) = v \). Note that \( v' \) is a sink in \( F \). We claim that \( M_F(v') = F^0 \). Since \( M_E(v) = E^0 \setminus H(v) \), it suffices to show that every vertex in \( M_E(v) \) connects to \( v' \). Now, since \( v \in B_{H(v)} \), there is \( e \in E^1 \) such that \( s(e) = v \) and \( r(e) = M_E(v) \). If \( r(e) = v \) then \( e' \in F^1 \) and \( s(e') = v, r(e') = v' \). If \( r(e) \neq v \), then, since every vertex in \( M_E(v) \) connects to \( v \), there exists a path \( p = f_1 \cdots f_m \) in \( E \) such that \( s(p) = r(e) \) and \( r(p) = v \). Now \( q := ef_1f_2 \cdots f_{m-1}f_m' \) is a path in \( F \) such that \( s(q) = v \) and \( r(q) = v' \), as claimed.

Accordingly we may consider the simple module \( N_{v'} \) of \( L(F) \) introduced by Chen corresponding to the sink \( v' \) in \( F \). Now \( N_{v'} \) is a simple faithful \( L(F) \)-module by Lemma 3.1 because \( M_F(v') = F^0 \). Using the quotient map \( L(E) \to L(F) \), we may view \( N_{v'} \) as a simple module over \( L(E) \). We shall denote this \( L(E) \)-module by \( N_{v'}^{B_{H(v)}} \).

**Lemma 3.5.** Assume that \( v \) is an infinite emitter and that \( v \in B_{H(v)} \). Then the annihilator of \( N_{v'}^{B_{H(v)}} \) is precisely \( I(H(v), B_{H(v)} \setminus \{v\}) \).

**Proof.** This follows from the fact that the simple \( L(F) \)-module \( N_{v'} \) is faithful. □

Next, suppose \( v \) is an infinite emitter such that \( r(s^{-1}(v)) \subseteq H(v) \).

Then \( v \) is the unique sink of \( E \setminus (H(v), B_{H(v)}) \). Let \( N_{v} \) be the corresponding simple \( L_K(E \setminus (H(v), B_{H(v)})) \)-module introduced by Chen. It is clear that \( N_{v} \) is a faithful simple \( L_K(E \setminus (H(v), B_{H(v)})) \)-module. Consider \( N_{v} \) as a simple \( L_K(E) \)-module through the quotient map \( L_K(E) \to L_K(E \setminus (H(v), B_{H(v)})) \). We denote this as \( N_{v}^{H(v)} \). The next lemma follows immediately.

**Lemma 3.6.** Let \( v \) be an infinite emitter and suppose that \( r(s^{-1}(v)) \subseteq H(v) \). Then the annihilator of the simple module \( N_{v}^{H(v)} \) is \( I(H(v), B_{H(v)}) \).

**Definition 3.7.** Let \( E \) be an arbitrary graph and \( K \) an arbitrary field. By a *Chen module* we mean a simple left \( L_K(E) \)-module of one of the following types:

1. \( N_w \), where \( w \) is a sink in \( E \);
2. \( N_{v}^{B_{H(v)}} \), where \( v \) is an infinite emitter such that \( v \in B_{H(v)} \);
(3) $N_{v}^{p}$, where $v$ is an infinite emitter such that $r(s^{-1}(v)) \subseteq H(v)$;
(4) $V_{[p]}$, where $p$ is an infinite path on $E$;
(5) $V_{[q]}^{s}$, where $q = c^{\infty}$, $c$ is an exclusive cycle, and $f(x)$ is an irreducible polynomial in $K[x, x^{-1}]$, with $f(x) \neq x - 1$.

Proposition 3.8. All simple modules listed in Definition 3.7 are pairwise non-isomorphic.

Proof. This follows from the computation of the annihilators of these modules and from [16, Theorems 3.3(2), 3.7(3)]. □

We can now state the main result of this section.

Theorem 3.9. Let $E$ be an arbitrary graph and $K$ an arbitrary field, and let $P$ be any primitive ideal of $L_{K}(E)$. Then there exists a Chen simple module $S$ such that the annihilator of $S$ is $P$.

Proof. Let $P$ be a primitive ideal of $L(E)$, and set $H = P \cap E^{0}$. By [20, Theorem 4.3], $P$ satisfies one of the following:

(i) $P = I(H, B_{H}, f(c))$, where $c$ is a exclusive cycle based at a vertex $u$, $E^{0} \setminus H = M(u)$, and $f(x)$ is an irreducible polynomial in $K[x, x^{-1}]$.
(ii) $P$ is a graded ideal of the form $I(H, B_{H}, \{u\})$, where $u \in B_{H}$, and $M(u) = E^{0} \setminus H$;
(iii) $P$ is a graded ideal of the form $I(H, B_{H})$, and $E \setminus (H, B_{H})$ is downward directed, satisfies the condition (L) and the countable separation property.

Suppose that (i) holds. Write $q = c^{\infty}$. Then $H = H^{p}(q) = E^{0} \setminus M(u)$ and so, by Lemma 3.4, the annihilator of $V_{[q]}^{s}$ is precisely $I(H, B_{H}, f(c)) = P$.

Next assume that (ii) holds. Now $u$ is an infinite emitter with $u \in B_{H(u)}$ and, since $M(u) = E^{0} \setminus H$, we must have $H(u) = H$. Thus the annihilator of $N_{u}^{p}$ is $P = I(H, B_{H}, \{u\})$, by Lemma 3.5.

Finally, suppose that (iii) holds. Let $S$ be a countable (finite or infinite) subset of $E^{0} \setminus H$ such that every vertex of $E \setminus (H, B_{H})$ connects to some vertex of $S$. We claim that either there is a (unique) sink in $E \setminus (H, B_{H})$ to which all the vertices in $E \setminus (H, B_{H})$ connect, or else there exists an infinite path $p$ on $E \setminus (H, B_{H})$ such that each vertex in $E \setminus (H, B_{H})$ connects to a vertex in $p^{0}$. To see this, set $F := E \setminus (H, B_{H})$.

If $F$ has a sink $v$, then since $F^{0}$ is downward directed, $v$ is a unique sink, $v \in S$ and $F^{0} = M(v)$.

Assume that $F$ does not have any sink. If $S$ is infinite, let $v_{1}, v_{2}, v_{3}, \ldots$ be an enumeration of the elements of $S$. If $S = \{v_{1}, \ldots, v_{k}\}$ is finite, we consider the infinite sequence $v_{1}, \ldots, v_{k}, v_{1}, \ldots, v_{k}, v_{1}, \ldots, v_{k}, \ldots$ obtained by repeating the finite sequence $v_{1}, \ldots, v_{k}$ infinitely many times, and we write $v_{kr+i} = v_{i}$. Proceeding as in the proof of Theorem 3.5 in [3], we may inductively define a sequence $\lambda_{1}, \lambda_{2}, \ldots$, of paths in $F$ such that

(1) $\lambda_{i}$ is an initial segment of $\lambda_{j}$ whenever $i \leq j$,
(2) The length of $\lambda_{i}$ is $\geq i$ for all $i$, and
(3) $v_{i} \geq r(\lambda_{i})$ for all $i$. 


Now, we may use the paths $\lambda_i$ to build an infinite path $p$ such that each vertex of $S$ connects to a vertex in $p^0$.

Therefore every vertex of $F$ connects to a vertex in $p^0$, as claimed.

We also note that when the path $p$ constructed above is a rational infinite path $ggg \cdots$, where $g$ is a closed path, Condition (L) on $F$ together with the condition that every vertex connects to $g^0$ imply that $g$ cannot be an exclusive cycle.

If there is a unique sink $v$ in $E \setminus (H, B_H)$ to which all the vertices in $E \setminus (H, B_H)$ connect, then $H = H(v)$, and there are two possibilities: either $v$ is a sink in $E$, or else $v$ is an infinite emitter such that $r(s^{-1}(v)) \subseteq H(v)$. In either case, by Lemmas 3.1 and 3.6, the annihilator of $N_v$, respectively of $N_v^{H(v)}$, is precisely $P = I(H(v), B_{H(v)})$.

If there is an infinite path $p$ on $E \setminus (H, B_H)$ such that $(E \setminus (H, B_H))^0 = M(p)$, consider $p$ as an infinite path in the graph $E$. Then $H = H(p)$ and, since $E \setminus (H, B_H)$ has Condition (L), $p$ cannot end in an exclusive cycle in $E$. Hence, it follows from Lemma 3.2(1) that the annihilator of $V_{[p]}$ is precisely $P = I(H, B_H)$.

This concludes the proof of the theorem. \hfill \Box

From the proof of the above theorem dealing with condition (iii), we get the following sharper reformulation of [3, Theorem 5.7], characterizing primitive Leavitt path algebras.

**Theorem 3.10.** Let $E$ be an arbitrary graph and $K$ be any field. Then the Leavitt path algebra $L_K(E)$ is primitive if and only if $E$ contains either a sink $w$ or an infinite path $p$ that does not end in a cycle without exits, such that $E^0 = M(w)$ or $M(p)$.

Given a graph $E$ and a hereditary subset $H$, one can construct the restricted graph $E_H$ as follows:

$$(E_H)^0 = H \text{ and } (E_H)^1 = \{e \in E^1 : s(e) \in H\}.$$ 

We now state a result on Chen modules that will be useful in next section.

**Lemma 3.11.** Let $E$ be an arbitrary graph and let $H$ be a hereditary subset of $E^0$. Consider the idempotent $e = \sum_{w \in H} w$ in the multiplier algebra $M(L_K(E))$ of $L_K(E)$. Then $L_K(E_H)$ is isomorphic to $eL_K(E)e$. Moreover, if $M$ is a Chen simple $L_K(E_H)$-module of one of the types (1)-(5) described in Definition 3.7, then $L_K(E)e \otimes_{eL_K(E)e} M$ is also a Chen simple $L_K(E)$-module of the same type.

**Proof.** The algebra $eL_K(E)e$ is linearly generated by all the paths $pq^*$, where $p, q$ are paths in $E$ such that $s(p), s(q) \in H$. Using this, it is easy to show that there is a graded isomorphism $L_K(E_H) \rightarrow eL_K(E)e$, which sends the vertices in $E_H^0 = H$ to the same vertices in $E^0$ and the edges in $(E_H)^1$ to the corresponding edges in $E$.

Write $R = L_K(E)$ and $S = L_K(E_H)$. If $M = V^{E_H}_{[p]}$ is a simple $L_K(E_H)$-module given by an infinite path $p$ on $E_H$, then we have an isomorphism

$$Re \otimes_{eRe} V^{E_H}_{[p]} \cong V_{[p]}$$
sending $\gamma \eta^* \otimes q$ to $\gamma \eta^* q$, where $\gamma$ and $\eta$ are finite paths in $E$ such that $s(\eta) \in H$, and $q$ is an infinite path tail-equivalent to $p$ starting at a vertex of $H$. It is easily checked that this map is surjective. To see that it is injective, observe that we can write any element in $Re \otimes eRe V_{[p]}^{EH}$ in the form $\sum_{i=1}^{r} \gamma_i \otimes a_i$, where $a_i \in V_{[p]}^{EH}$, and $\gamma_i$ are distinct paths in $E$ such that $r(\gamma_i) \in H$ and all the other vertices of the path $\gamma_i$ do not belong to $H$. If $\sum_{i=1}^{r} \gamma_i a_i = 0$, then we get $a_i = 0$ for all $i$, and so $\sum_{i=1}^{r} \gamma_i \otimes a_i = 0$.

It is a simple matter to show that the same holds for all types of Chen simple modules considered in Definition 3.7.

4. When every simple module is finitely presented

Let $E$ be a finite graph. We first show that the simple module $V_{[p]}$ is finitely presented if and only if $p$ is tail-equivalent to the rational infinite path $\gamma$. We then show that every (Chen) simple left $L := L_K(E)$-module is finitely presented if and only if every vertex in the graph $E$ is the base of at most one cycle in $E$ (Theorem 4.5). Finally, we obtain a proof of our main result, Theorem 1.1.

The next proposition gives necessary and sufficient conditions under which the simple left module $V_{[p]}$ (where $p$ is an infinite path) is finitely presented. In its proof, we use the notation $P[E]$ to denote the path algebra over the field $K$ of the reverse graph $\overline{E}$ of $E$, with $\overline{E}^0 = E^0$ and $\overline{E}^1 = \{e^* : e \in E^1\}$.

A closed path $c$ is said to be primitive in case $c \neq d^r$ for any closed path $d$ and any $r \geq 2$. We remark that these paths are called simple closed paths in [16]. Observe that, if $c = e_1 e_2 \cdots e_n$ is primitive, then all the rotates $c_i = e_e e_i e_i+1 \cdots e_n e_1 \cdots e_i-1$, $i = 1, \ldots, n$, are different.

**Proposition 4.1.** Let $E$ be a finite graph and $K$ be any field. Let $p$ be an infinite path in $E$. Then the simple left $L_K(E)$-module $V_{[p]}$ is finitely presented if and only if $p$ is tail-equivalent to the (rational) infinite path $\gamma$ where $c$ is some closed path in $E$.

**Proof.** Suppose $V_{[p]}$ is a finitely presented left $L_K(E)$-module. First observe that the simple $L_K(E)$-module $V_{[p]}$ can not be projective. This is because, since the graph $E$ is finite, the infinite path $p$ can not be tail equivalent to a path containing a line point. Since $V_{[p]}$ is simple, it is of finite length and contains no projective submodule and so by Proposition 7.2 (1) of [9], $V_{[p]}$ is a finitely generated Blanchfield module over the path algebra $P[E]$. Also, by Proposition 7.2 (2) of [9], $V_{[p]} = L_K(E) \otimes_{P[E]} N$ for some $P[E]$-module $N$ having finite $K$-dimension. Indeed we can assume $V_{[p]} = P[E] N$ for some $P[E]$-module $N$ which is finite dimensional over $K$. In particular, there are infinite paths $q_1, \ldots, q_r \in [p]$ such that every element $a$ in $V_{[p]}$ can be written as $a = \sum_{i=1}^{n} a_i q_i$ for some $a_i \in P[E]$. Since each $q_i \sim p$, we can assume that $q_i = \tau_{>n_i}(p)$ for some positive integer $n_i$. Choose an integer $m$ larger than $\max\{n_1, \ldots, n_r\}$. By hypothesis,

$$\tau_{>m}(p) = \sum_{i=1}^{r} a_i \tau_{>n_i}(p)$$
where \( a_i \in P[E] \). Write

\[
p = e_1e_2e_3\cdots,
\]

where \( e_i \) are the edges of the infinite path \( p \). Observe that

\[
\tau_{>m+1}(p) = e_{m+1}^*\tau_{>m}(p) = \sum_{i=1}^r e_{m+1}^*a_i\tau_{>n_i}(p).
\]

In this way, we can reduce the length of the paths in each of the terms \( a_i \), and also possibly obtain some terms of the form \( \tau_{>n_i+1}(p) \) for some \( i \). Repeating this process a finite number of times we arrive at the equation

\[
\tau_{>k}(p) = \sum_{i=1}^r \lambda_i\tau_{>k_i}(p)
\]

where \( \lambda_i \in K \) and \( k > \max\{k_1, \cdots, k_r\} \). This implies that \( \tau_{>k}(p) = \tau_{>l}(p) \) for some integer \( l < k \). Thus we get the equality of the infinite paths

\[
e_{k+1}e_{k+2}\cdots = e_{l+1}e_{l+2}\cdots
\]

and so \( e_{k+1} = e_{l+1}, e_{k+2} = e_{l+2}\cdots, e_{k-l} = e_k, e_{k-l+1} = e_{k+1}, e_{k-l+2} = e_{k+2} = e_{l+2}\cdots \). We conclude that

\[
p = (e_1e_2\cdots e_l)c\cdots
\]

where \( c = e_{l+1}e_{l+2}\cdots e_k \) so that \( p \) is tail-equivalent to \( c^\infty = cccc\cdots \), as desired.

Conversely, suppose \( p \) is tail-equivalent to the infinite path \( c^\infty = cccc\cdots \) for some primitive path \( c \) of length \( n \), say \( c = e_1\cdots e_n \). Let \( c_1 = c \) and for each \( i = 2, \ldots, n \), let \( c_i = e_je_{i+1}\cdots e_{i-1} \) be the \( i \)-th rotate of \( c \). For each \( i \), let \( p_i = c_i^\infty = c_1c_i \cdots \) be an infinite path. Then the finite dimensional \( K \)-vector space \( N = Kp_1 \oplus \cdots \oplus Kp_n \) is actually a \( P[E] \)-module and, by Proposition 2.2 of \([9]\), \( N \) is a finitely presented \( P[E] \)-module. Now \( V_{[p]} = L_K(E)N \) and for any sink \( u \in E^0 \), \( u \cdot V_{[p]} = 0 \). Then by Proposition 7.2 of \([9]\), \( V_{[p]} \) is finitely presented as a left \( L_K(E) \)-module.

**Notation 4.2.** If \( E \) is a graph and \( v \in E^0 \) is a source then \( E\backslash v \) denotes the ”source elimination graph” where \((E\backslash v)^0 = E^0\backslash \{v\}, (E\backslash v)^1 = E^1\backslash s^{-1}(v), s_{E\backslash v} = s|(E\backslash v)^1 \) and \( r_{E\backslash v} = r|(E\backslash v)^1 \).

The following lemma was proved in \([10]\) Lemma 1.4] under the assumption that \( L_K(E) \) is simple and it can also be derived from \([10]\) Lemma 6.1]. We give a direct proof for completeness.

**Lemma 4.3.** Let \( E \) be a finite graph. If \( v \) is a source and not a sink, then \( L_K(E) \) is Morita equivalent to \( L_K(E\backslash v) \).

**Proof.** First, observe that the hypothesis that \( v \) is a source but not a sink implies that \( |E^0| \geq 2 \).

Note that \( E\backslash v \) is a complete subgraph of \( E \). Hence, the \( K \)-algebra map \( \theta : L_K(E\backslash v) \rightarrow L_K(E) \) given, for all \( w \in (E\backslash v)^0, e \in (E\backslash v)^1 \), by \( \theta(w) = w, \theta(e) = e \) and \( \theta(e^*) = e^* \) is a non-zero graded homomorphism. Since \( \theta \) is non-zero at all the vertices of \( E\backslash v \), it is then a monomorphism.
Let $\epsilon = \theta(1_{L_K(E\setminus v)}) = \sum_{w \in E^0, w \neq v} w$. We claim that $\theta(L_K(E\setminus v)) = \epsilon L_K(E)\epsilon$. Clearly $\theta(L_K(E\setminus v)) \subseteq \epsilon L_K(E)\epsilon$. To prove the other inclusion, note that $\epsilon L_K(E)\epsilon$ is linearly spanned by elements $pq^* \in \epsilon L_K(E)\epsilon$ such that $s(p) \neq v$ and $s(q) \neq v$. Moreover, since $v$ is a source $p$ as well as $q$ cannot pass through $v$. Hence both $p$ and $q$ are paths in $E\setminus v$, consequently $pq^* = \theta(pq^*) \in \theta(L_K(E\setminus v))$, thus proving our claim.

To show the Morita equivalence, we need also to show that $L_K(E)\epsilon L_K(E) = L_K(E)$. It is enough to show that $v$ is in $L_K(E)\epsilon L_K(E)$. Let $\{e_1, \cdots, e_n\} = s^{-1}(v) \neq \emptyset$. Since $r(e_i)$ belongs to the ideal $L_K(E)\epsilon L_K(E)$, the edge $e_i$ belongs to $L_K(E)\epsilon L_K(E)$, for all $i = 1, \cdots, n$.

Then $v = \sum_{i=1}^n e_ie_i^* \in L_K(E)\epsilon L_K(E)$. This proves that $L_K(E)\epsilon L_K(E) = L(E)$. Hence $L_K(E)$ is Morita equivalent to $L_K(E\setminus v)$.

**Lemma 4.4.** Let $E$ be a finite graph. Let $c$ be a cycle in $E$ without entries, that is, such that $|r^{-1}(v)| = 1$ for all $v \in c^\circ$. Then a finite graph $F$ can be constructed from $E$ in which the cycle $c$ is replaced by a loop such that $L_K(F)$ is Morita equivalent to $L_K(E)$.

**Proof.** Write $c = e_1 \cdots e_r$, with $s(e_i) = v_i$ for all $i$. We define a graph $F$ as follows:

Let $F^0 = (E^0 \setminus c^0) \cup \{v\}$ where $v$ is a new vertex.

To define $F^1$, note that by our hypothesis, $r(e) \notin c^0$ for all $e \in E^1$ such that $s(e) \notin c^0$, that is, $E^0 \setminus c^0$ is a hereditary set. So, we define $s^{-1}_F(w) = s^{-1}_E(w)$ if $w \in E^0 \setminus c^0$.

Corresponding to an edge $f$ with $s(f) \in c^0$ and $r(f) \in E^0 \setminus c^0$, define an edge $f'$ in $F^1$ with $s_F(f') = v$ and $r_F(f') = r(f)$. Finally, we define a loop $e'$ at $v$ so that $s_F(e') = v = r_F(e')$.

We now define a map $\theta : L_K(F) \longrightarrow L_K(E)$ as follows: $\theta(w) = w$ for all $w \in E^0 \setminus c^0$ and $\theta(v) = v_1$ (where $v_1 = s(e_1)$). As for edges, $\theta(e) = e$ for all $e$ with $s(e) \in E^0 \setminus c^0$.

We set $\theta(f') = e_1 \cdots e_{i-1}f$ if the edge $f$ corresponding to $f'$ satisfies $s(f) = v_i$ and $r(f) \in E^0 \setminus c^0$.

Finally, set $\theta(e') = e_1 \cdots e_r$.

It can be verified that the CK-relations are preserved under this map and so $\theta$ extends to a well-defined algebra homomorphism from $L_K(F)$ onto the corner $\epsilon L_K(E)\epsilon$ where $\epsilon = \theta(1_{L_K(F)})$. (Note that $1_{L_K(F)} = v + \sum_{u \in E^0 \setminus c^0} u$.) The most tricky part of this verification is to
show the preservation of (CK2) at vertex \( v \). To show this, observe that

\[
\sum_{\alpha \in s_E^{-1}(v)} \theta(\alpha \alpha^*) = e_1 \cdots e_r e_r^* \cdots e_1^* + \sum_{i=1}^{r} \sum_{g \in s_E^{-1}(v_i) \setminus \{e_i\}} e_1 \cdots e_{i-1} gg^* e_{i-1}^* \cdots e_1^*
\]

\[
= e_1 \cdots e_{r-1} \left( e_r e_r^* + \sum_{g \in s_E^{-1}(v_r) \setminus \{e_r\}} gg^* \right) e_{r-1}^* \cdots e_1^*
\]

\[
+ \sum_{i=1}^{r-1} \sum_{g \in s_E^{-1}(v_i) \setminus \{e_i\}} e_1 \cdots e_{i-1} gg^* e_{i-1}^* \cdots e_1^*
\]

\[
= e_1 \cdots e_{r-1} e_{r-1}^* \cdots e_1^* + \sum_{i=1}^{r-1} \sum_{g \in s_E^{-1}(v_i) \setminus \{e_i\}} e_1 \cdots e_{i-1} gg^* e_{i-1}^* \cdots e_1^*
\]

\[
= \cdots \cdots \cdots \cdots
\]

\[
= e_1 e_1^* + \sum_{g \in s_E^{-1}(v_1) \setminus \{e_1\}} gg^* = v_1.
\]

Since \( \theta(v) = v_1 \), this shows that relation (CK2) at \( v \) is preserved.

To show that the map \( \theta \) is injective, we show that \( \theta \) sends, in a one-to-one way, a basis of \( L_K(F) \) to a subset of a basis of \( L_K(E) \). We make use of the basis defined in [6 Section 3] (see also [2 Chapter 1]). For each \( w \in E^0 \setminus c^0 \) which is not a sink, choose an edge \( \gamma(w) \) in \( s_E^{-1}(w) \). For each \( i = 1, \ldots, r \), set \( \gamma(v_i) = e_i \in s_E^{-1}(v_i) \). Refer to these edges as special. By [6 Theorem 1], a basis \( B_E \) of \( L_K(E) \) is given by the following elements (i) \( w \), where \( w \in E^0 \setminus c^0 \), (ii) \( p, p^* \), where \( p \) is a path in \( E \), (iii) \( pq^* \), where \( p = e_1 \cdots e_n, q = f_1 \cdots f_m \) are paths that end at the same vertex \( r(e_n) = r(f_m) \), with \( n, m \geq 1 \), with no restriction when \( e_n \neq f_m \), but with the restriction that \( e_n \) must not be special when \( e_n = f_m \). In other words we avoid terms of the form \( e_1 e_2 \cdots e_n e_n^* f_n^* f_{n-1} \cdots f_1^* \), with \( e_n \) special.

Consider a corresponding basis \( B_F \) for \( L_K(F) \) by declaring as special the same edges \( \gamma(w) \) as before for \( w \in E^0 \setminus c^0 \), and declaring \( \gamma(v) = e' \). Then it is clear that \( \theta \) restricts to an injective mapping from the basis of \( L_K(F) \) into the basis of \( L_K(E) \).

We now check that the image of \( \theta \) is exactly \( \epsilon L_K(E) \). Indeed, it is easy to check, using the hypothesis that \( c \) has no entries, that the subset \( \theta(B_F) \) of the basis \( B_E \) of \( L_K(E) \) is a linear basis for \( \epsilon L_K(E) \), and so \( \theta(L_K(F)) = \epsilon L_K(E) \).

To show that \( L_K(E) \epsilon L_K(E) = L_K(E) \), it is enough to observe that \( c^0 \subseteq L_K(E) \epsilon L_K(E) \). This follows from the fact that \( v_1 \in L_K(E) \epsilon L_K(E) \) and the equality \( v_i = e_{i-1}^* \cdots e_1^* v_i e_1 \cdots e_{i-1} \) for \( i = 2, \ldots, r \).

We have shown that \( L_K(F) \) is isomorphic to a full corner \( \epsilon L_K(E) \) of \( L_K(E) \), and so \( L_K(E) \) is Morita equivalent to \( L_K(F) \). \( \square \)

We introduce a pre-order \( \leq \) on the set of cycles of a directed graph \( E \), as follows. If \( c_1 \) and \( c_2 \) are two cycles in \( E \), set \( c_1 \leq c_2 \) in case there is a path from a vertex of \( c_2 \) to a vertex of
c_1$. Note that this is indeed a partial order in case $E$ satisfies the graph-theoretic condition (ii) in Theorem 4.5. We say that a cycle $c$ of $E$ is a maximal cycle in case, for any cycle $c'$ in $E$, $c \leq c'$ implies $c' \leq c$.

**Theorem 4.5.** Let $E$ be a finite graph and $K$ be any field. Then the following conditions are equivalent for the Leavitt path algebra $L = L_K(E)$:

(i) Every simple left $L$-module is finitely presented;
(ii) Every Chen simple module is finitely presented.
(iii) Every vertex $v$ in $E$ is the base of at most one cycle.

**Proof.** (i) $\implies$ (ii) is immediate.

Assume (ii) so that every simple Chen $L$-module is finitely presented. Assume, by way of contradiction, that there is a $v \in E^0$ which is the base of two different cycles $g, h$. Consider the infinite path

$$p = gh^2gh^3 \cdots gh^ngh^{n+1} \cdots .$$

This path $p$ cannot be tail-equivalent to the rational path $c^{\infty} = ccc \cdots$ for any closed path $c$ in $E$. Hence, by Proposition 4.1, the Chen simple module $V_p$ is not finitely presented, a contradiction. Thus every vertex in $E$ is the base of at most one cycle.

Assume (iii) so that every vertex in $E$ is the base of at most one cycle. We need to show that every simple left $L$-module is finitely presented.

Let $n$ be the number of distinct cycles in $E$. We apply induction on $n$ to show that every simple left $L$-module is finitely presented.

The base case is the case $n = 0$, that is, the case where $E$ contains no cycles. In that case, $L$ is a semi-simple artinian ring and all its left/right simple modules are projective and hence finitely presented.

Suppose $n \geq 1$ and that the result is true for graphs containing less than $n$ cycles.

Using Lemma 4.3 a finite number of times, we get a finite graph without sources $F$, containing the same closed paths as $E$, and such that $L_K(E)$ is Morita equivalent to $L_K(F) \times K^t$ for some $t \geq 0$. Since Morita equivalence between module categories (over unital rings) preserves simple modules and finite presentation (see [5]), we can therefore assume that the graph $E$ has no sources. Then all paths in $E$ can be seen as portions of paths coming from a cycle in $E$.

Let $c$ be a maximal cycle. Since we are assuming that $E$ does not have sources, the cycle $c$ has no entries. So, using Lemma 4.4 we may assume that $c$ is a loop, with $s(c) = v = r(c)$. Since $c$ is a maximal cycle and $E$ has no sources, $E^0 \{v\}$ is a hereditary saturated set. Let $M$ be the (graded) ideal of $L$ generated by $E^0 \{v\}$. Clearly $L/M \cong K[x, x^{-1}]$.

Consider an arbitrary simple left $L$-module $S$.

Suppose $MS = S$. Let $e = \sum_{w \in E^0 \{v\}} w$. Then $e$ is a full idempotent in $M$, that is $LeL = MeM = M$, and so $M$ is Morita equivalent to $eLe = L(E_H)$, where $H := E^0 \{v\}$, and $E_H$ denotes the restriction of $E$ to $H$, that is, the graph with $(E_H)^0 = H$ and $(E_H)^1 = \{e \in E^1 \mid s(e) \in H\}$. Note that we have a surjective Morita context given by $(Le, eL)$, that
is, we have surjective bimodule homomorphisms
\[ eL \otimes_L Le \to eLe, \quad Le \otimes_{eLe} eL \to LeL = M. \]

Now \( E_H \) contains \( n - 1 \) cycles and so, by induction hypothesis, all simple \( eLe \)-modules are finitely presented. Since \( M \) is also a Leavitt path algebra (see [12, Lemma 1.2]), it has local units, and so we can apply [7, Theorem 2.2] to deduce that there is an equivalence of categories between \( M \text{-Mod} \) and \( eLe \text{-Mod} \), induced by the functors \( eL \otimes_M - \) and \( Le \otimes_{eLe} - \). Therefore there is a simple \( eLe \)-module \( S' \) such that \( Le \otimes_{eLe} S' \cong S \). Since \( S' \) is finitely presented, so is \( S \). Indeed if
\[
(eLe)^n \to (eLe)^m \to S' \to 0
\]
is an exact sequence of \( eLe \)-modules, then
\[
(Le)^n \to (Le)^m \to Le \otimes_{eLe} S' \to 0
\]
is an exact sequence of \( L \)-modules, which shows that \( S \) is finitely presented.

Suppose now that \( MS = 0 \). Then \( S \) is a simple \( K[x,x^{-1}] \)-module and so there is a polynomial \( f(x) = 1 + a_1 x + \cdots + a_n x^n \), with \( a_n \neq 0 \) such that \( S \cong K[x,x^{-1}]/(K[x,x^{-1}]f(x)) \). Consequently, \( S \cong L/(Lf(c) + M) \).

Set \( s^{-1}(v) \setminus \{c\} = \{e_1, \ldots, e_k\} \).

Let \( N \) be the (finitely generated) left ideal of \( L \) generated by \( H = E^0 \setminus \{v\} \) and by all the paths of the form \( e_i^*(c^*)^j \), with \( 1 \leq i \leq k, 0 \leq j \leq n - 1 \). Observe that \( e_i^* \in M \), so that \( N \subseteq M \), as \( M \) is an ideal of \( L \). We claim that \( Lf(c) + M = Lf(c) + N \). This will show that \( S \) is finitely presented. Note that \( M \) is linearly spanned by the elements of the form \( pq^* \), where \( p, q \) are paths in \( E \) such that \( r(p) = r(q) \in H \). If \( r(pq^*) = s(q) \in H \) then \( pq^* \in Le(s(q)) \subseteq N \). Therefore we can assume that \( s(q) = v \). In this case observe that \( q = c^i e_i q_1 \) for some \( 1 \leq i \leq k \) and \( j \geq 0 \), and some path \( q_1 \). Therefore \( pq^* \in Le(c^i)^j \). It thus suffices to show that \( e_i^*(c^*)^j \) belongs to \( Lf(c) + N \) for all \( i, j \). If \( 0 \leq j \leq n - 1 \), this follows from the definition. Suppose that \( e_i^*(c^*)^j \in Lf(c) + N \) for all \( 0 \leq t \leq r \), where \( r \geq n - 1 \). Then multiplying \( f(c) \) on the left by \( e_i^*(c^*)^{r+1} \) we obtain
\[
e_i^*(c^*)^{r+1} = e_i^*(c^*)^{r+1} f(c) - a_1 e_i^*(c^*)^r - \cdots - a_n e_i^*(c^*)^{r+n} \in Lf(c) + N.
\]
Thus \( S = L/(Lf(c) + N) \) is finitely presented, as desired.

For the class of graphs \( E \) appearing in Theorem 4.5 we can indeed classify all the simple left \( L_K(E) \)-modules. Specifically, we show that in this case every simple \( L_K(E) \)-module determines and is determined by a unique primitive ideal of \( L_K(E) \).

**Corollary 4.6.** Let \( E \) be a finite graph such that every vertex in \( E \) is the base of at most one cycle. Then every simple \( L_K(E) \)-module is a Chen module. Indeed, for any primitive ideal \( P \) of \( L_K(E) \) there exists a unique simple \( L_K(E) \)-module \( S \) (which is a Chen module) such that the annihilator of \( S \) is \( P \).

**Proof.** The proof uses the same kind of induction as in Theorem 4.5. Let \( n \) be the number of distinct cycles in \( E \). If \( n = 0 \), then \( E \) is semisimple artinian, and the simple modules are in bijective correspondence with the sinks of \( E \). So all of them are of the form \( N_w \)
for a sink $w$, and distinct simple modules have distinct annihilators. Assume the result is true for graphs with less than $n$ distinct cycles, and let $E$ be a finite graph with $n$ cycles satisfying the required hypothesis. Since, by Theorem 3.9 we can realize every primitive ideal as the annihilator of at least one Chen simple module, and since, for two Morita-equivalent unital rings $R$ and $S$, there is a bijective correspondence between the isomorphism classes of simple $R$-modules and the isomorphism classes of simple $S$-modules, and also a bijective correspondence between primitive ideals of $R$ and primitive ideals of $S$ which is compatible with the above in the sense that respects annihilators of simple modules, we may use Lemma 4.3 a finite number of times and reduce to the case where $E$ does not have sources.

Let $c$ be a maximal cycle in $E$. Using Lemma 4.4 we can further assume that $c$ is a loop, based at $v$. Let $M = I(H)$, where $H = E^0 \setminus \{v\}$.

Set $L := L_K(E)$. If $S$ is a simple $L$-module such that $MS = S$, then as in the proof of Theorem 4.5 we have that $S \cong Le \otimes_e Le S'$, where $S'$ is a simple $L_K(E_H)$-module. By the inductive hypothesis, $S'$ is a Chen $L_K(E_H)$-module, and therefore $S$ is a Chen $L_K(E)$-module by Lemma 3.11. Moreover if $S_1$ and $S_2$ are two simple $L$-modules such that $MS_i = S_i$ for $i = 1, 2$, and $S_1 \not\cong S_2$, then $S'_1 \not\cong S'_2$ and so by induction hypothesis, $\text{Ann}_{L(E_H)}(S'_1) \neq \text{Ann}_{L(E_H)}(S'_2)$, which implies that

$$\text{Ann}_M(S_1) = Le \otimes_e Le \text{Ann}_{L(E_H)}(S'_1) \otimes_e Le cL$$

$$\neq Le \otimes_e Le \text{Ann}_{L(E_H)}(S'_2) \otimes_e Le cL = \text{Ann}_M(S_2)$$

and so $\text{Ann}_L(S_1) \neq \text{Ann}_L(S_2)$. If $MS_1 = S_1$ and $MS_2 = 0$ then $M \not\subseteq \text{Ann}_L(S_1)$ and $M \subseteq \text{Ann}_L(S_2)$, so that $S_1$ and $S_2$ have different annihilators.

Finally suppose that $MS_1 = 0 = MS_2$ and that $S_1 \not\cong S_2$. Then there exist distinct irreducible polynomials $f(x)$ and $g(x)$ in $K[x, x^{-1}]$ such that $S_1 \cong L/(M + Lf(c))$ and $S_2 \cong L/(M + Lg(c))$. Therefore $\text{Ann}_L(S_1) = M + Lf(c) \neq M + Lg(c) = \text{Ann}_L(S_2)$. Also it can be easily verified that $S_1 \cong V^{f(q)}_1$, where $q = c^\infty$, so that $S_1$ is a Chen simple module.

This completes the proof.

**Example 4.7.** Let $E$ be the graph with $E^0 = \{v, w\}$ and $E^1 = \{e, f\}$ such that $s(e) = r(e) = v = s(f)$ and $r(f) = w$. Then the Leavitt path algebra $L_K(E)$ is the Jacobson algebra $S_1 = K\langle x, y \mid yx = 1 \rangle$. This algebra is non-noetherian but all the simple modules are finitely presented by Theorem 3.5. The structure of the simple $L_K(E)$-modules is well-known (see [15] Lemma 3.1 or [9, 5.10(3)]). The Chen module $N_w$ corresponding to the sink $w$ is the simple module $K[x]$ (cf. [15]). The other simple modules are the Chen modules $V^{f(q)}_1$, where $q = c^\infty$ and $f(x)$ is an irreducible polynomial in $K[x, x^{-1}]$. In [15], Bavula finds all the simples modules over the algebras $S_n := S_1 \otimes (S_1) \cdots \otimes S_1$, for all $n \geq 1$. It would be interesting to know whether similar results can be obtained for tensor products of Leavitt path algebras of the form $L_K(E)$, where $E$ is a graph such that every vertex is the basis of at most one cycle.

We are now ready to prove our main result.
Proof of Theorem [1.7]: (1) $\iff$ (2) $\iff$ (3) is Theorem [4.5] and (3) $\implies$ (4) is shown in Corollary [4.6]

(4) $\implies$ (5). If $M$ is a simple $L$-module, then, by Theorem [3.9] there exists a Chen module $N$ such that Ann$_L(M) = Ann_L(N)$. Now (4) gives $M \cong N$. Therefore, every simple $L$-module is a Chen module.

(5) $\implies$ (3). Suppose that $v \in E^0$ is the base of two different cycles $g$ and $h$. We will build a simple finitely presented left $L$-module which is not a Chen module.

We begin by building a simple, finite-dimensional, $P[E]$-module. We write $g = \alpha_1 \cdots \alpha_n$, with $v = v_1 = s(g) = r(g)$ and $v_i = s(\alpha_i) = r(\alpha_{i-1})$ for $i = 2, \ldots, n$. Similarly, set $h = \beta_1 \cdots \beta_m$, with $w_1 = v = s(h) = r(h)$ and $w_j = s(\beta_j) = r(\beta_{j-1})$ for $j = 2, \ldots, m$. For $i = 1, \ldots, n$, put $M_{v_i} = z_i K$, a 1-dimensional vector space. For $w_j = h^0 \setminus g^0$, set $M_{w_j} = t_j K$, a 1-dimensional vector space. If $w_j = v_i \in g^0 \cap h^0$, set $t_j = z_i$. Set also $M_w = 0$ if $w \notin g^0 \cup h^0$.

This defines a family $(M_w)_{w \in E^0}$ of finite-dimensional vector spaces. Now, for $i = 1, \ldots, n$, define a linear map $\Phi_{\alpha^*_i} : M_{v_{i+1}} \to M_v$ by $\Phi_{\alpha^*_i}(z_{i+1}) = z_i$ (where $v_{i+1} = v_1$). Similarly, if $j = 1, \ldots, m$ let $\Phi_{\beta^*_j} : M_{w_{j+1}} \to M_w$ be the linear such that $\Phi_{\beta^*_j}(t_{j+1}) = t_j$. If $\alpha \notin g^1 \cup h^1$, define $\Phi_{\alpha^*} = 0$. In this way, we have defined a family $(M_w)_{w \in E^0}$, $(\Phi_{\alpha^*})_{\alpha \in E^1}$, which gives rise to a $P[E]$-module $M$, with underlying vector space $M = \bigoplus_{w \in E^0} M_w$. Observe that dim$_K(M) = |g^0 \cup h^0|$.

We claim that $M$ is a simple $P[E]$-module. To see this, let

$$a = \sum_{i=1}^n \lambda_i z_i + \sum_{j \in J} \mu_j t_j$$

be a nonzero element in $M$, where $J = \{ j \in \{ 1, \ldots, m \} : w_j \notin g^0 \}$. Assume that $\lambda_{i_0} \neq 0$. Then $\alpha_{i_0}^* a = \lambda_{i_0} z_{i_0-1} \neq 0$ (where $i_0 - 1$ is computed mod $n$). Similarly, if $\mu_{j_0} \neq 0$ for some $j \in J$, then $\beta_{j_0}^* a = \mu_{j_0} w_{j_0-1} \neq 0$. In either case we obtain that $z_1 = t_1 \in P[M]a$, which implies the simplicity of $M$.

By [9] Lemma 5.7, $\tilde{M} := L \otimes_{P[E]} M$ is a finitely presented simple $L$-module. Assume, by way of contradiction, that $\tilde{M}$ is a Chen module. It is easy to show, by using the arguments in Lemma [3.2](1) that the annihilator of $\tilde{M}$ is $I(H(g^\infty)) = I(H(h^\infty))$, and so $\tilde{M}$ cannot be a Chen module of type (1) or (5). Hence, there is an infinite path $p$ such that $\tilde{M} \cong V[p]$. Now, it follows from Proposition [4.1] that $p$ must be tail-equivalent to a rational path of the form $q^\infty$, where $q$ is a primitive closed path in $E$, so that $\tilde{M} \cong V[q^\infty]$. Write $q = e_i \cdots e_r$, where $e_i \in E^1$. For $i = 1, \ldots, r$, let $q_i = e_i \cdots e_{i-1}$ be the $i$-th rotate of $q$. Then $N = \bigoplus_{i=1}^r q_i^\infty K$ is a simple, finite-dimensional $P[E]$-module, and $P[E]N = V[q^\infty]$. This implies that $N$ is the smallest lattice of $V[q^\infty]$, see [9] Proposition 7.2(3)]. Since the minimal lattice of $M$ is $M$, we obtain a $P[E]$-isomorphism $\phi : M \to N$. In particular $r = \dim_K(N) = |g^0 \cup h^0|$. Moreover $\dim_K(wN) = 1$ for all $w \in g^0 \cup h^0$, which implies that $q$ must be a cycle, because it cannot pass through the same vertex twice. Let $i$ be the smallest positive integer such that
\( \alpha_i \neq \beta_i \). Then either \( \alpha_i^* q_i^\infty = 0 \) or \( \beta_i^* q_i^\infty = 0 \). Assume, for convenience, that \( \alpha_i^* q_i^\infty = 0 \). Then, since \( q \) is a cycle, we must have \( \alpha_i^* q_j^\infty = 0 \) for all \( j = 1, \ldots, n \), and thus \( \alpha_i^* N = 0 \). Hence \( 0 = \alpha_i^* M \neq 0 \) and we have arrived to a contradiction. Therefore, we conclude that \( \tilde{M} \) is not a Chen module.

\( \square \)

**Remark 4.8.** From Theorem 5 of [6], it is interesting to observe that, for a finite graph \( E \), the equivalent conditions of Theorem 1.1 are also equivalent to the condition that \( L_K(E) \) has finite Gelfand-Kirillov dimension.

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**References**

[1] G. Abrams, G. Aranda Pino, *The Leavitt path algebra of a graph*, J. Algebra 293 (2005), 319–334.
[2] G. Abrams, P. Ara, M. Siles Molina, *Leavitt path algebras. A primer and handbook*, Springer. (To appear)
[3] G. Abrams, J. Bell, K. M. Rangaswamy, *On prime non-primitive von Neumann regular algebras*, Trans. Amer. Math. Soc. (To appear).
[4] G. Abrams, A. Louly, E. Pardo, C. Smith, *Flow invariants in the classification of Leavitt path algebras*, J. Algebra 333 (2011), 202–231.
[5] F.W. Anderson, K. Fuller, *Rings and categories of modules*, Graduate Texts in Math. 13, Springer-Verlag, 1974.
[6] A. Alahmadi, H. Alsulami, S.K. Jain, E. Zelmanov, *Leavitt path algebras of finite Gelfand-Kirillov dimension*, J. Algebra and Applications 11 1250225 (2012) [6 pages].
[7] P. N. Anh, L. Marki, *Morita equivalence for rings without identity*, Tsukuha J. Math. 11 (1987), 1–16.
[8] P. Ara, *Finitely presented modules over Leavitt algebras*, J. Algebra 191 (2004), 1–21.
[9] P. Ara, M. Brustenga, *Module theory over Leavitt path algebras and K-theory*, J. Pure Appl. Algebra 214 (2010), 1131–1151.
[10] P. Ara, M. Brustenga, G. Cortiñas, *K-theory of Leavitt path algebras*, Münster J. of Math. 2 (2009), 5–34.
[11] P. Ara, M.A. Moreno, E. Pardo, *Non-stable K-theory for graph algebras*, Algebra Represent. Theory 10 (2007), 157-178.
[12] P. Ara, E. Pardo, *Stable rank for Leavitt path algebras*, Proc. Amer. Math. Soc. 136 (2008), 2375–2386.
[13] P. Ara, K. M. Rangaswamy, *Leavitt path algebras of finite representation type*, arXiv:1309.7917 [math.RA].
[14] G. Aranda Pino, D. Martín Barquero, C. Martín Gonzalez, and M. Siles Molina, *Socle theory for Leavitt path algebras of arbitrary graphs*, Rev. Mat. Iberoamericana 26 (2010), 611–638.
[15] V. V. Bavula, *The algebra of one-sided inverses of a polynomial algebra*, J. Pure Appl. Algebra 214 (2010), 1874–1897.
[16] X. W. Chen, *Irreducible representations of Leavitt path algebras*, Forum Math 20 (2012), DOI 10.1515.
[17] D. Goncalves, D. Royer, *On the representations of Leavitt path algebras*, J. Algebra 333 (2011), 258–272.
[18] L. Le Bruyn, C. Procesi, *Semisimple representations of quivers*, Trans. Amer. Math. Soc. 317 (1990), 585–598.
[19] W. G. Leavitt, *The module type of a ring*, Trans. Amer. Math. Soc. 103 (1962), 113–130.
[20] K. M. Rangaswamy, *The theory of prime ideals of Leavitt path algebras over arbitrary graphs*, J. Algebra 375 (2013), 73–96.
[21] S. P. Smith, *Category equivalences involving graded modules over path algebras of quivers*, Adv. Math. 230 (2012), 1780–1810.

[22] S. P. Smith, *The space of Penrose tilings and the non-commutative curve with homogeneous coordinate ring $K\langle x, y\rangle/(y^2)$*, J. Noncommutative Geometry (to appear); arXiv:1104.3811v2 [math.RA].

[23] M. Tomforde, *Uniqueness theorems and ideal structure of Leavitt path algebras*, J. Algebra 318 (2007), 270–299.

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