The Minkowski Difference for Convex Polyhedra and Some its Applications *

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Abstract

The aim of the paper is to develop a unified algebraical approach to representing the Minkowski difference for convex polyhedra. Namely, there is proposed an exact analytical formulas of the Minkowski difference for convex polyhedra with different representations. We study the cases when both operands under the Minkowski difference operation simultaneously have a vertex or a half-space representation. We also focus on the description of the Minkowski difference for a such mixed case where the first operand has the linear constraint structure and the second one is expressible as the convex hull of a finite collection of some given points. Unlike the widespread geometric approach considering mostly two-dimensional or three-dimensional spaces, we investigate the objects in finite-dimensional spaces of arbitrary dimensionality.

keywords: Minkowski difference, convex polyhedron, vertex representation, half-space representation, polyhedra, distance, projection, linear separability criterion, variational inequality
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1 Introduction

The Minkowski difference of sets having the different configuration is basic to the treatment of a large class of problems often occurring in lots of interesting applications from a variety of areas, especially in problems of engineering design [1], [2]; in data classification [3], [4].

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image analysis and processing, motion planning for robots, real-time collision detection, computer graphics, and many other front-line fields.

The Minkowski difference operation is actually useful as an investigative as well as a conceptual tool. But unfortunately, it is widely known fact that there exist serious difficulties related to implementation of the Minkowski difference for individual formulations of sets. They represent the basic impediment to making use of the Minkowski difference operation in various practical applications. For finite-dimensional spaces of arbitrary dimensionality, an exact analytical representation of the Minkowski difference for the convex polyhedra with the different as well as identical configuration is stated here for the first time as a whole.

To be adequate for a number of mathematical purposes, the different approaches (with cross-fertilization of ideas) to such a basic concept of analysis as the Minkowski difference of sets are required. The geometric viewpoint among the others has the leading role. Motivation for the development of this geometric approach has basically come from engineering design, computational geometry, collision detection, robotics and many other subjects. The nature of these subjects dictates the sufficiency of considering only two-dimensional or three-dimensional spaces (see, for instance, [7, 12, 22]). Unlike the geometric approach, we study the objects in spaces of arbitrary dimensionality and develop the algebraic approach.

Namely, we present the exact analytical representation of the Minkowski difference for convex polyhedra given by different ways. More precisely, we investigate the following cases where:

- both operands under the Minkowski difference operation are determined in the similar way as the convex hull of the finite collection of some given points,
- the operands have the different representation (namely, the first operand is given by a linear constraint system, and the second one is expressible in terms of the convex hull),
- both operands have the same representation as the intersection of the closed half-spaces.

Let us note that thanks to the obtained results, there appeared the possibility to treat the relevant problems such as:
• problem of linear separation of the convex polyhedra in the Euclidean space,
• variational inequalities problems,
• problem of finding the distance between the convex polyhedra by projecting the origin of the Euclidean space onto a convex polyhedron,
• problem of finding the nearest points of convex polyhedra.

2 Definitions and Preliminaries

This section includes the brief description of notations, definitions of all utilized in the present paper main concepts. We use standard notation that is certainly well known to all readers. Nevertheless, let us briefly describe some of notations. As usual, $\| \cdot \|$ stands for the Euclidean norm of the vector in $\mathbb{R}^n$, $\langle \cdot, \cdot \rangle$ denotes the standard scalar product of two vectors, $\text{conv}\{\cdot\}$ corresponds to the convex hull of some collection of the given vectors. Let $\text{Bd}\{\cdot\}$, $\text{int}\{\cdot\}$ denote the boundary and interior of some set $\tilde{\Phi}$, respectively.

In a context of applications for the concept of the Minkowski difference, we need further to recall some key definitions and theorems which are closely related to the linear separability property of sets. In the theory of convex sets and nonlinear programming, the study of the linear separation problem is a highly important topic with a large literature (see, for instance, [23], [24], [25], [26], [27], [28], [29], etc.).

Definition 2.1 (separating hyperplane) (see, for instance, [24], p. 198) The hyperplane
\[ \pi(c, \gamma) := \{ x \in \mathbb{R}^n : \langle c, x \rangle = \gamma \} \]
with normal vector $c \neq 0$ separates the sets $A$ and $B$ from the Euclidean space $\mathbb{R}^n$, iff $\langle c, a \rangle \geq \gamma$ for all $a \in A$ and $\langle c, b \rangle \leq \gamma$ for all $b \in B$, i.e., iff it holds that:
\[ \sup_{b \in B} \langle c, b \rangle \leq \gamma \leq \inf_{a \in A} \langle c, a \rangle. \]

Definition 2.2 (strong separability) (see [24], p. 198) Two sets $A$ and $B$ are said to be strongly separable, iff there exists some vector $c \in \mathbb{R}^n$ such that:
\[ \sup_{b \in B} \langle c, b \rangle < \inf_{a \in A} \langle c, a \rangle. \]
If it holds that $\langle c, b \rangle < \langle c, a \rangle$ for all $a \in A$, $b \in B$, then the sets $A$ and $B$ are said to be strictly separable.
The next theorem gives the rigorous justification of the fact that the problem of strong separation of the two arbitrary sets \( A, B \subset \mathbb{R}^n \) can be reformulated as the problem of strong separating the origin of \( \mathbb{R}^n \) from the Minkowski difference \( A - B = \{a - b, a \in A, b \in B\} \), and vice versa.

**Theorem 2.1 (strong separation)** ([27], p. 150) For the sets \( A \) and \( B \) to be strongly separable, it is necessary and sufficient that the origin of \( \mathbb{R}^n \) be strongly separable from the set \( A - B \).

The previous theorem immediately implies that two sets \( A \) and \( B \) are not strongly separable if and only if the set \( A - B \) is not strongly separable from the origin of \( \mathbb{R}^n \).

The analogous results are certainly well known for the problems on non-strong linear separation of the considered sets.

**Theorem 2.2 (linear separation)** ([27], p. 151) For two sets \( A \) and \( B \) to be linearly separable, it is necessary and sufficient that the origin of \( \mathbb{R}^n \) be linearly separable from the set \( A - B \).

The preceding theorem obviously implies that to say two sets \( A \) and \( B \) are linearly inseparable is to say the origin of \( \mathbb{R}^n \) is linearly inseparable from the set \( A - B \).

We note that if the sets \( A \) and \( B \) are convex, then the Minkowski difference \( A - B \) is convex as well (see, for example, [24], p. 162). It is not hard to prove that if \( A \) and \( B \) are simultaneously bounded sets, then the set \( A - B \) is also bounded. Lastly, under the condition that at least one of the sets \( A \) and \( B \) is bounded, closedness of both sets under the Minkowski difference operation implies closedness of \( A - B \). The proof of this assertion may be found, for instance, in [24], p. 201.

Another question can now be addressed. What conditions on some sets \( A, B \) ensure that they are linearly (strongly or not) separable from each other? As already noted above, we reduce the problem of linear separation of \( A \) and \( B \) to the program of separation the origin of \( \mathbb{R}^n \) from \( A - B \). Questions about the separability tests or criteria for sets can actually be answered from the platform of the Minkowski difference operation. It is widely known that the Minkowski difference is often interpreted as the translational configuration space obstacle (see, for instance, [9]). One says that \( A - B \) represents the set of translations of \( B \) that brings it into interference with \( A \). The additional nice property of the Minkowski difference consists in that
for any objects $A$ and $B$ it holds

$$\text{dist}(A, B) = \text{dist}(0, A - B),$$

where $\text{dist}(A, B) = \inf\{\|a - b\|, a \in A, b \in B\}$

denotes the distance between $A$ and $B$. Clearly, $\text{dist}(0, A - B) = \|P_{A-B}(0)\|$, where $P_{A-B}(0)$ denotes the projection of the origin of $\mathbb{R}^n$ onto $A - B$. As it is known, two convex objects collide if and only if their Minkowski difference contains the origin. For the origin of $\mathbb{R}^n$ and $A - B$, the next results of this subsection give a sort of linear separation principle.

First we recall briefly some other relevant notations and definitions about the cones of generalized strong and strict support vectors, etc. In [27], there have been defined the following sets:

$$W_{\Phi} := \{w \in \mathbb{R}^n : \inf_{x \in \Phi} \langle w, x \rangle \geq 0\}, V_{\Phi} := \{v \in \mathbb{R}^n : \inf_{x \in \Phi} \langle v, x \rangle > 0\},$$

$$Q_{\Phi} := \{q \in \mathbb{R}^n : \langle q, x \rangle > 0, \forall x \in \tilde{\Phi}\},$$

$$\Omega_{\Phi} := \{y \in \mathbb{R}^n : \langle y, x \rangle \geq \|y\|^2, \forall x \in \tilde{\Phi}\},$$

$$E(\Omega_{\Phi}) := \{x \in \mathbb{R}^n : x = \lambda y, \lambda \geq 0, y \in \Omega_{\Phi}\},$$

where $\tilde{\Phi}$ is a nonempty subset of $\mathbb{R}^n$. The set $W_{\Phi} \setminus \{0\}$ is called a cone of generalized support vectors (or, briefly, GSVs) of the set $\Phi$. The notations $V_{\Phi}$ and $Q_{\Phi}$ are used for the cones of generalized strong and strict support vectors of the set $\Phi$, respectively. The main properties of the mentioned cones and their relationship with well-known other ones have been investigated in [27]. For details, the interested reader is recommended to refer to [27]. However, recalling some main properties of GSVs seems as vital for understanding the results of the present paper:

$$W_{\Phi} = W_{cl(\Phi)} = W_{co(cl(\Phi))}, V_{\Phi} = V_{co(\Phi)} = V_{cl(co(\Phi))},$$

$$V_{\Phi} \subseteq Q_{\Phi} \subseteq W_{\Phi}, V_{\Phi} = E(\Omega_{\Phi}) \setminus \{0\},$$

$$V_{\Phi} = V_{cl(\Phi)} = V_{co(cl(\Phi))}, W_{\Phi} = W_{co(\Phi)} = W_{cl(co(\Phi))}, Q_{\Phi} = Q_{co(\Phi)}.$$
polyhedra which have the compactness property, due to the presence of
convexity and continuity of the linear function \( \langle c, x \rangle \), we can state that
\( \langle c, x \rangle \) attains its minimum on \( \tilde{\Phi} \). So, for these cases, we can rewrite
\( t_{\tilde{\Phi}}(c) := \min_{x \in \Phi} \langle c, x \rangle \). Further, we note that the following problem

\[
\max_{\|c\|=1} t_{\tilde{\Phi}}(c)
\]

is solvable. For the proof of the fact, the interested reader is directed
to [27] (see p. 149). Let the vector \( c^* \in \mathbb{R}^n \) denote an optimizer
of the problem (1). The following three theorems describe a linear
separability criterion for the pair of objects such as the origin and
some nonempty set of \( \mathbb{R}^n \). Based on the optimal value of the objective
function of (1), the above-mentioned criterion allows us to recognize
these objects as strongly separable, non-strongly linearly separable, or
inseparable.

**Theorem 2.3** (strong separability criterion) ([27], p.149) For the origin
of \( \mathbb{R}^n \) to be strongly separable from the nonempty set \( \tilde{\Phi} \subset \mathbb{R}^n \) it is necessary and sufficient to have \( t_{\tilde{\Phi}}(c^*) > 0 \).

**Theorem 2.4** (non-strong linear separability criterion) ([27], p.150) For the origin of \( \mathbb{R}^n \) to be non-strongly linearly separable from the
nonempty set \( \tilde{\Phi} \subset \mathbb{R}^n \) it is necessary and sufficient to have \( t_{\tilde{\Phi}}(c^*) = 0 \).

**Theorem 2.5** (linear inseparability criterion) ([27], p.150) For the
origin of \( \mathbb{R}^n \) to be linearly inseparable from the nonempty set \( \tilde{\Phi} \subset \mathbb{R}^n \) it is necessary and sufficient to have \( t_{\tilde{\Phi}}(c^*) < 0 \).

The following theorems allows to detect which one of the cones of
generalized support vectors \((V_{\tilde{\Phi}}, W_{\tilde{\Phi}}\{0\}, \text{ and } Q_{\tilde{\Phi}})\) is empty, and
which ones are not.

**Theorem 2.6** (emptiness of the cone of generalized strong support
vectors) If \( \tilde{\Phi} \) is a nonempty convex and closed subset of \( \mathbb{R}^n \), then
\[
V_{\tilde{\Phi}} = \emptyset \iff 0 \in \tilde{\Phi}.
\]

The previous theorem represents the particular case of Theorem 3.3
from [29] (for the proof, see p. 703). For the convex and closed set \( \tilde{\Phi} \),
due to having \( V_{\tilde{\Phi}} = E(\Omega_{\tilde{\Phi}})\{0\} \), from Theorem 2.6, there obviously
holds the following implication: \( \Omega_{\tilde{\Phi}} = \{0\} \iff 0 \in \tilde{\Phi} \).
Theorem 2.7 (emptiness of the cone of generalized support vectors)
If \( \tilde{\Phi} \subset \mathbb{R}^n \) is a nonempty convex set, then
\[
W_{\tilde{\Phi}} = \{0\} \iff 0 \in \text{int}(\tilde{\Phi}).
\]
The preceding theorem is the particular case of Theorem 3.2 from [29]
(for the proof, see p. 703).

Theorem 2.8 (simultaneous degeneracy of the cone of generalized
strict support vectors & non-degeneracy of the cone of GSVs)
If \( \tilde{\Phi} \subset \mathbb{R}^n \) is a nonempty convex and closed set, then:
\[
W_{\tilde{\Phi}} \neq \{0\} \& Q_{\tilde{\Phi}} = \emptyset \iff 0 \in \text{Bd}(\tilde{\Phi}).
\]
According to [29], the assertion of the preceding theorem follows from
Lemmas 3.13–3.14 (see p. 702). Let us note that for the set \( \tilde{\Phi} \) hav-
ing the compactness property, it is fulfilled \( V_{\tilde{\Phi}} = Q_{\tilde{\Phi}} \). According to
Theorems 2.3, 2.6 we obviously have
\[
0 \notin \tilde{\Phi} \iff t_{\tilde{\Phi}}(c^*) > 0.
\]
From Theorems 2.5, 2.7 it immediately follows that
\[
0 \in \text{int}(\tilde{\Phi}) \iff t_{\tilde{\Phi}}(c^*) < 0.
\]
Due to Theorems 2.4, 2.8 there holds the following implication:
\[
0 \in \text{Bd}(\tilde{\Phi}) \iff t_{\tilde{\Phi}}(c^*) = 0.
\]
Thus, for the origin of \( \mathbb{R}^n \), the linear separability criterion provides a
certificate of being an exterior, interior, or boundary point of \( \tilde{\Phi} \).

3 Binary Operation of Minkowski Difference

3.1 Both Operands with a Vertex Representation

For many applications nowadays, the sets expressible as the convex hull of
finitely many points from the Euclidean space \( \mathbb{R}^n \) are especially im-
portant. They are really ubiquitous structures having a fundamental
role not only in variational analysis, computational geometry and optimization, but in data classification, image analysis and processing, motion planning for robots, collision detection, and many other frontline areas. This subsection is devoted to representing the Minkowski difference for both convex polyhedra having the above-mentioned configuration.

Further, let us be given two polyhedra of $\mathbb{R}^n$:

$L := \text{conv}\{z_i\}_{i \in I}$

where $I = \{1, 2, \cdots , l\}$, $J = \{1, 2, \cdots , m\}$, i.e.

$L = \{z \in \mathbb{R}^n : z = \sum_{i \in I} \alpha_i z_i, \sum_{i \in I} \alpha_i = 1, \alpha_i \geq 0, i \in I\}$,

$M = \{p \in \mathbb{R}^n : p = \sum_{j \in J} \beta_j p_j, \sum_{j \in J} \beta_j = 1, \beta_j \geq 0, j \in J\}$.

Obviously, $L$ and $M$ are nonempty, convex, and compact sets. Due to set algebra, the Minkowski difference of these polyhedra is defined as a set of pairwise differences of points from $L$ and $M$. Namely, as follows

$L - M = \{z - p, z \in L, p \in M\}$.

The following theorem rigorously justifies that the point set $L - M$ coincides with the convex hull of the vectors $z_i - p_j, i \in I, j \in J$.

**Theorem 3.1.** (Minkowski difference for both polyhedra given as convex hull) ([26], p. 552) $L - M = \text{conv}\{z_i - p_j\}_{i \in I, j \in J}$.

**Proof.** Part I. At first, we establish the inclusion

$\text{conv}\{z_i - p_j\}_{i \in I, j \in J} \subseteq L - M$.

By the definition of the convex hull, for any $l \in \text{conv}\{z_i - p_j\}_{i \in I, j \in J}$, there can be found the real numbers $\gamma_{ij} \geq 0, \sum_{i \in I} \sum_{j \in J} \gamma_{ij} = 1$ such that:

$l = \sum_{i \in I} \sum_{j \in J} \gamma_{ij} (z_i - p_j) = \sum_{i \in I} \left(\sum_{j \in J} \gamma_{ij}\right) z_i - \sum_{j \in J} \left(\sum_{i \in I} \gamma_{ij}\right) p_j$.

Assuming from now that $\alpha_i = \sum_{j \in J} \gamma_{ij}, \beta_j = \sum_{i \in I} \gamma_{ij}$, one can easily see that the coefficients $\alpha_i$ and $\beta_j$ satisfy the conditions $\alpha_i \geq 0, \forall i \in I, \sum_{i \in I} \alpha_i = 1, \beta_j \geq 0, \forall j \in J, \sum_{j \in J} \beta_j = 1$. Consequently, any vector $l \in \text{conv}\{z_i - p_j\}_{i \in I, j \in J}$ satisfies the following equation:
\( l = \sum_{i \in I} \alpha_i z_i - \sum_{j \in J} \beta_j p_j = z - p, \ z \in L, \ p \in M, \ \text{i.e.} \ l \in L - M. \)

Part II. Now let us justify the following backward inclusion:

\( L - M \subseteq \text{conv}\{z_i - p_j\}_{i \in I, j \in J}. \)

By the definition of the set \( L - M, \) taking some vector \( l \in L - M, \) we then have that \( l = z - p, \ z \in L, \ p \in M. \) According to the construction of the sets \( L \) and \( M, \) there will be found \( \alpha_i \geq 0 \ \forall \ i \in I, \ \beta_j \geq 0 \ \forall j \in J, \ \sum_{i \in I} \alpha_i = 1, \ \sum_{j \in J} \beta_j = 1 \) such that \( z = \sum_{i \in I} \alpha_i z_i, \ p = \sum_{j \in J} \beta_j p_j. \)

Then

\[
l = \sum_{i \in I} \alpha_i z_i - \sum_{j \in J} \beta_j p_j = \sum_{j \in J} \left( \sum_{i \in I} \alpha_i z_i \right) - \sum_{i \in I} \alpha_i \left( \sum_{j \in J} \beta_j p_j \right) = \sum_{i \in I} \sum_{j \in J} \alpha_i \beta_j (z_i - p_j) = \sum_{i \in I} \sum_{j \in J} \gamma_{ij} (z_i - p_j),
\]

where \( \gamma_{ij} \geq 0, \ \sum_{i \in I} \sum_{j \in J} \gamma_{ij} = 1. \) Thus, there is true the following inclusion: \( L - M \subseteq \text{conv}\{z_i - p_j\}_{i \in I, j \in J}. \) The latter should be compared to the earlier proved inclusion \( \text{conv}\{z_i - p_j\}_{i \in I, j \in J} \subseteq L - M. \) This comparison allows us to complete the proof. \( \square \)

Since due to Theorem 3.1, \( L - M \) has a representation as a convex hull of finite number \((l \cdot m)\) of points \( z_i - p_j \) for all \( i \in I, \ j \in J, \) it is characterized as a nonempty, compact, and convex point set. Consequently, for the case, the operation of the Minkowski difference thereby preserves the compactness and convexity.

For practical applications, there has the extremely importance a question consisting of how to decrease the number of points forming \( L - M. \) For the low enough dimension of \( \mathbb{R}^n, \) we have a possibility to make this number of points as small as possible by means of using some software package. In the case of the two-dimensional or three-dimensional space, the function Convhull, for instance, in MATLAB not only computes and returns the convex hull of the given collection of points, but provides the option of removing vertices that do not contribute to the area or volume of the convex hull. Moreover, this package allows to visualize the output of Convhull with the help of the function Plot in 2-D. The function Trisurf or Trimesh provides the possibility of plotting the output of Convhull in 3-D. Let us note that in four or more dimensions, there can efficiently be used, for instance, the proposed in [30] Quickhull algorithm for computing the convex hulls. This method is realized in MATLAB by means of the function.
This function returns the indices of input points that form the faces of the convex hull. Consequently, to compute, for instance, the distance between $L$ and $M$ or to linearly separate these sets, one first should select those of the $(l \cdot m)$ points of the type $z_i - p_j, \ i \in I, j \in J$ that are really formed the faces of $L - M$. Luckily, the inner points of the Minkowski difference $L - M$ may be ignored. As a result of taking into account of the only points that are vertices of the convex hull, the collection of points $z_i - p_j, \ i \in I, j \in J$ may considerably be reduced. Therefore, it is more easier to deal with such a reduced family of input points. The expected time complexity of the Quickhull algorithm depends on the different parameters such as dimension of the space, the number of input and processed points, the maximum number of facets (for details, see [30], [31]).

### 3.2 Binary Mixture of Sets with Different Constraint Structure

In this subsection, we focus on a precise representation of the Minkowski difference for a common case where the first point set from the sets pair under the operation has the general constraint (i.e. not necessarily linear constraint) structure, and the second operand is identified by the abstract constraint. Such sets identification is exhibited as too crucial for applications to be considered below. Furthermore, the abstract constraint specification is a really useable form since it does not restrict the variations on how the corresponding point set might be defined. The presence of the abstract constraint characterizes our approach as very flexible [29], [32], since constraints might not even be present. The purpose of considering in this subsection such a case of the more general settings is twofold - to recall some fundamental results from the previous research and to explain how they can be applied to a topic of our interest.

In a wide range of practical applications, the set of feasible solutions $\Phi$ is representable by a system of inequality constraints in the general form:

$$\Phi = \{ x \in X : f_k(x) \leq b_k, \ k \in K \}, \ K = \{ 1, 2, \ldots, r \},$$

(2)

where $f_k(x), \ k \in K$ are arbitrary real-scaled quasi-convex functions defined on a convex set $X \subseteq \mathbb{R}^n$. We recall that a function $f(x)$ is said to be a quasi-convex on a convex set $X$ if and only if $[S^d, f]^L_X$
is a convex set for all $d \in \mathbb{R}^1$, where

$$[S^d, f]_{X}^{L_0} := \{x \in X : f(x) \leq d\}.$$ 

Therefore, as an intersection of the convex sets $[S^{b_k}, f_k]_{X}^{L_0}$, $k \in K$, the set $\Phi$ is convex, too.

**Theorem 3.2** (closedness of the lower level set) Let $X$ be a closed set of $\mathbb{R}^n$, then $f(x)$ is a lower semicontinuous function over $X$ if

and only if the lower level set $[S^d, f]_{X}^{L_0}$ is closed for all $d \in \mathbb{R}^1$.

The interested reader can find the proof of the theorem, for instance, in [24] (see p. 81).

The previous theorem implies that if $f_k(x)$, $k \in K$ are lower semicontinuous functions over a closed set $X$, then the lower level sets $[S^{b_k}, f_k]_{X}^{L_0}$ are closed for all $b_k$, $k \in K$. Consequently, being an intersection of closed sets $[S^{b_k}, f_k]_{X}^{L_0}$, $k \in K$, the set $\Phi$ is closed as well.

Next, we present the analytical description of the Minkowski difference of two sets $\Phi$ and $\Psi$, when $\Phi$ is given by (2), and $\Psi$ is an arbitrarily defined set.

**Theorem 3.3** (Minkowski difference when the two sets under the operation are given by a constraints system and an abstract constraint, respectively) (24, p. 716) Let be given an arbitrary nonempty set $\Psi \subseteq \mathbb{R}^n$, the set $\Phi \neq \emptyset$ be defined by (2), $X = \mathbb{R}^n$, then $\Phi - \Psi = \Phi_1$, where

$$\Phi_1 = \{x \in \mathbb{R}^n : f_k(x + y) \leq b_k, k \in K, y \in \Psi\},$$

$$K = \{1, 2, \ldots, r\}, \Phi - \Psi = \{z \in \mathbb{R}^n : z = x - y, x \in \Phi, y \in \Psi\}.$$

**Proof.** Part I. First we select arbitrarily some fixed point $\tilde{x}$ from $\Phi$. We further consider $\tilde{x}(y) = \tilde{x} - y$ for $\forall y \in \Psi$. It is clear that $f_k(\tilde{x}(y) + y) = f_k(\tilde{x}) \leq b_k$, $\forall k \in K$, $\forall y \in \Psi$, i.e. $\tilde{x}(y) \in \Phi_1$. Therefore, we have

$$\tilde{x} \in \Phi \Rightarrow \tilde{x}(y) = \tilde{x} - y \in \Phi_1, \forall y \in \Psi.$$ 

Through the fact that $\tilde{x} \in \Phi$ was chosen arbitrarily, there was proved the following inclusion: $\Phi - \Psi \subseteq \Phi_1$.

Part II. Conversely, we take now an arbitrary fixed point $\tilde{t} \in \Phi_1$ and, for all $y \in \Psi$, check whether the points $\tilde{t}(y) = \tilde{t} + y$ belong to $\Phi$. Then there can be observed that $f_k(\tilde{t}(y)) = f_k(\tilde{t} + y) \leq b_k$, $\forall k \in K$, $\forall y \in \Psi$, i.e. $\tilde{t}(y) \in \Phi$. In other words, it holds $\tilde{t} \in \Phi - \Psi$, $\forall y \in \Psi$. Thanks to the arbitrary choice of $\tilde{t} \in \Phi_1$, we get that $\Phi_1 \subseteq \Phi - \Psi$. 

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We can now finalize our proof of the theorem. Since, taking into account the forward and backward inclusions, we have the claimed equality: $\Phi_1 = \Phi - \Psi$. □

An abstract constraint $y \in \Psi$ is very convenient as well for representing the Minkowski difference in the case of a more complicated nature of the second point set from the sets pair under the operation.

The previous theorem was presented for the first time in [29], but it is brought out here along with the prime role it plays in our further research.

In actual practice, strict inequalities are rarely seen in constraints, however, if the nonempty set $\Phi$ happens to be described by the strict inequality constraints, then $\Phi_1$ should be also expressed by the system of strict inequalities.

For the proof of the previous theorem, no matter how the set $\Psi$ was expressed analytically or alternatively in other ways. For instance, $\Psi$ may be specified in a similar way as the set $\Phi$:

$$\Psi := \{x \in \mathbb{R}^n : g_s(x) \leq d_s, s \in S\}, S = \{1, 2, \ldots, t\}.$$ 

In particular, $\Psi$ can consist of a single point as in the conditions of the next lemma.

**Lemma 3.1** (Minkowski difference for a set with constraint structure and a singleton) Let be given an arbitrary vector $p \in \mathbb{R}^n$, $\Phi$ be defined by (2), $\Phi \neq \emptyset$, $\Phi_1 = \{x \in \mathbb{R}^n : f_k(x + p) \leq b_k, k \in K\}, K = \{1, 2, \ldots, r\}$,

then $\Phi - p = \Phi_1$, where $\Phi - p = \{z \in \mathbb{R}^n : z = x - p, x \in \Phi\}$.

The result of the previous lemma evidently follows from Theorem 3.3 under the assumption that $\Psi$ is a singleton, i.e. $\Psi = \{p\}$. Lemma 3.1 has in turn the following quite obvious corollaries.

**Corollary 3.1** (Minkowski difference for the nonnegative orthant and a singleton) Let be given an arbitrary vector $p = (p^1, \cdots, p^n)$, $\Phi$ be described as the nonnegative orthant

$$\Phi = \mathbb{R}^n_+ = \{x = (x^1, \cdots, x^n) : x^j \geq 0, j = 1, \ldots, n\},$$

then $\Phi - p = \{x = (x^1, \cdots, x^n) : x^j \geq -p^j, j = 1, \ldots, n\}$. 

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Corollary 3.2 (Minkowski difference for a set with linear constraint structure and a singleton) Let be given an arbitrary vector \( p \in \mathbb{R}^n \), \( \Phi \neq \emptyset \),

\[
\Phi = \{ x \in \mathbb{R}^n : \langle a_k, x \rangle \leq b_k, a_k \in \mathbb{R}^n, b_k \in \mathbb{R}^1, k \in K \},
\]

\( K = \{1, 2, \cdots , r\} \), (3)

then \( \Phi - p = \{ x \in \mathbb{R}^n : \langle a_k, x \rangle \leq \tilde{b}_k, \tilde{b}_k = b_k - \langle a_k, p \rangle, k \in K \} \).

Corollary 3.3 (Minkowski difference for a set with box constraints structure and a singleton) Let be given an arbitrary vector \( p \in \mathbb{R}^n \), \( \Phi \) be specified by the box constraints on \( x \) of the form

\[
\Phi = \{ x \in \mathbb{R}^n : l \leq x \leq u, \ l, u \in \mathbb{R}^n \}, \ \Phi \neq \emptyset,
\]

then \( \Phi - p = \{ x \in \mathbb{R}^n : l - p \leq x \leq u - p \} \).

Corollary 3.4 (Minkowski difference for a closed ball and a singleton) Let be given an arbitrary vector \( p \in \mathbb{R}^n \), \( \Phi \) be given as a closed ball of radius \( q \) around some point of \( o \), i.e.

\[
\Phi = \{ x \in \mathbb{R}^n : \| x - o \|^2 \leq q^2, \ o \in \mathbb{R}^n, q \in \mathbb{R}^1 \},
\]

then \( \Phi - p = \{ x \in \mathbb{R}^n : \| x - \bar{o} \|^2 \leq q^2, \ \bar{o} = o - p \} \).

3.3 Operands: Convex Polyhedra with Different Representation

Based on the foregoing results, we introduce in this subsection the representation of the Minkowski difference for some binary mixture of convex polyhedra having differently defined shapes. In more detail, we focus now on the mixed case where the first convex polyhedron has the linear constraint structure and the second one is expressible as the convex hull of a finite collection of points from \( \mathbb{R}^n \). In this case, for the space of arbitrary dimensionality, there is successfully reached an exact representation of the Minkowski difference without the necessity of any transition to higher dimensions. Furthermore, it will be shown below that a number of linear constraints describing the Minkowski difference of sets is exactly the same as it is for the first operand.

A set \( \Phi \subset \mathbb{R}^n \) is said to be a polyhedral set if it can be specified as the intersection of a finite family of closed half-spaces, or equivalently,
can be expressed by finitely many linear constraints of form:

$$\Phi = \{x \in \mathbb{R}^n : Ax \leq b\},$$  \hspace{1cm} (4)

where $A$ is the given nonvacuous matrix in $\mathbb{R}^{r \times n}$ with components $a_{ik}$, $b$ is the vector in $\mathbb{R}^r$ with components $b_i$. It is well known that a polyhedral set is a convex polyhedron. Let $M$ be the set of all convex combinations of some vectors $p_j, j \in J$, i.e. $M$ be specified in the similar manner as it was defined in Subsection 2.1. To construct a refined representation of the Minkowski difference for $\Phi$ and $M$, we utilize first Theorem 3.3 as follows:

$$\Phi - M = \{x \in \mathbb{R}^n : A(x + y) \leq b, y \in M\}.$$  

For the further analysis, we will need to construct the following supplementary set:

$$\Psi_1 = \{x \in \mathbb{R}^n : Ax \leq b - Ap_j, p_j \in \mathbb{R}^n, j \in J\}.$$  \hspace{1cm} (5)

Theorem 3.4 (Representation of Minkowski difference for the mixed case before refinement) Let be given an arbitrary collection of vectors $p_j \in \mathbb{R}^n, j \in J$, $\Phi$ be defined by (4), $\Phi \neq \emptyset$, $M = \text{conv}\{p_j\}_{j \in J}$, then

$$\Phi - M = \Psi_1.$$  

Proof. Part I. Without loss of generality, we fix some arbitrary point $\bar{x}$ from $\Psi_1$. By our construction of $\Psi_1$, it then holds $A\bar{x} \leq b - Ap_j$. Multiplying this system through by the nonnegative coefficients $\beta_j, j \in J$ (giving $\sum_{j \in J} \beta_j = 1$), summing, and rearranging, we obtain

$$A\bar{x} \leq b - A \sum_{j \in J} \beta_j p_j.$$  

In other words, it holds $A(\bar{x} + \sum_{j \in J} \beta_j p_j) \leq b$ subject to $\beta_j \geq 0, j \in J$, $\sum_{j \in J} \beta_j = 1$. By definition, the set $M$ as the convex hull of the vectors $p_j, j \in J$ consists of all their convex combinations, so it is convex. More precisely, all the points $y \in \Psi$ are specified by some possible convex combinations $\sum_{j \in J} \beta_j p_j$. This allows us to conclude that the following implication holds

$$A(\bar{x} + y) \leq b, \forall y \in \Psi \Rightarrow \bar{x} \in \Phi - M.$$  

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Taking into account the arbitrary selection of $\bar{x} \in \Psi_1$, we thereby get the desired inclusion $\Psi_1 \subseteq \Phi - M$.

Part II. Now, there is no loss of generality in taking some point $\bar{x}$ arbitrarily from $\Phi - M$. We aim at showing that $\bar{x} \in \Psi_1$. In this case, the proof is trivial. Indeed, by our assumption, there is fulfilled $A(\bar{x} + y) \leq b$, $\forall y \in \Psi$. Since $p_j \in \Psi$, $\forall j \in J$, we obviously have $A(\bar{x} + p_j) \leq b$ for all $j \in J$ and given points $p_j \in \mathbb{R}^n$, i.e. $\bar{x} \in \Psi_1$. Due to our arbitrary choice of $\bar{x} \in \Phi - M$, this therefore implies that $\Phi - M \subseteq \Psi_1$. □

The right-hand side for a system of inequalities in (5), take $n$ with all possible choices $j \in J$, characterized this system as overdetermined.

In what follows, having reduced the number of inequality constraints in (5), we will come latter to the refined representation of the Minkowski difference $\Phi - M$.

For all $j \in J$, let us use the following notation: $A_{p_j} = \tilde{b}_j$. Then we have

$$b^i - \tilde{b}^i = b^i - \max_{j \in J} \tilde{b}_j^i \leq b^i - \tilde{b}_j^i, \forall i \in K, \forall j \in J.$$ 

For the system from (5) can then be formulated some subsystem of the form $Ax \leq s$, where 

$s \in \mathbb{R}^r$, $s^T = b^T - \tilde{b}^T = b^T - (\max_{j \in J} \tilde{b}_j^1, \max_{j \in J} \tilde{b}_j^2, \ldots, \max_{j \in J} \tilde{b}_j^r)$.

Define the set with the following constraint representation

$$\Psi_2 = \{x \in \mathbb{R}^n : Ax \leq s\}.$$ 

**Theorem 3.5** (Refined Minkowski difference for convex polyhedra having different representations) Let be given an arbitrary collection of vectors $p_j \in \mathbb{R}^n$, $j \in J$, $\Phi$ be defined by (4), $\Phi \neq \emptyset$, $M = \text{conv}\{p_j\}_{j \in J}$, then

$$\Phi - M = \Psi_2.$$ 

**Proof.** The core of the proof consists in showing that $\Psi_1 = \Psi_2$.

Part I. There is no loss of generality in selecting some fixed point $\bar{x}$ from $\Psi_2$. Writing

$$A\bar{x} \leq s = b - \tilde{b} \leq b - \tilde{b} = b - Ap_j, \forall j \in J,$$

we see that $\bar{x}$ lies in $\Psi_1$. The arbitrariness of choosing $\bar{x} \in \Psi_2$ confirms that there is true the following inclusion: $\Psi_2 \subseteq \Psi_1$.

Part II. To prove the converse inclusion, take some point $\bar{x}$ arbitrary.
ily from $\Psi_1$. Due to the special construction of the right-hand side of constraint system from the description of $\Psi_1$, for each $i \in K$, it is not hard to see that the following subsystem of linear constraints

$$
\sum_{k=1}^{n} a_{ik} \bar{x}^k \leq b^i - \sum_{k=1}^{n} a_{ik} p_j^k,
$$

\hspace{1cm} \ldots \ldots \ldots \ldots \ldots

$$
\sum_{k=1}^{n} a_{ik} \bar{x}^k \leq b^i - \sum_{k=1}^{n} a_{ik} p_m^k,
$$

can be replaced by a unique inequality

$$
\sum_{k=1}^{n} a_{ik} \bar{x}^k \leq b^i - \max_{j \in J} \sum_{k=1}^{n} a_{ik} p_j^k.
$$

The truth of the previous assertion is quite obvious since it holds

$$
\min_{j \in J} (b^i - \sum_{k=1}^{n} a_{ik} p_j^k) = b^i - \max_{j \in J} \sum_{k=1}^{n} a_{ik} p_j^k.
$$

In the above chain of the formulas, $\bar{x}^k$, $p_j^k$ are the $k$–th components of $\bar{x}$ and $p_j$, respectively. Besides, $a_{ik}$ denotes the $k$–th element in the row indexed by $i$ of the matrix $A$, $b^i$ corresponds to the $i$–th item of the vector $b$. In consequence, the system $A\bar{x} \leq b - Ap_j$, $j \in J$ can be reduced to the system $A\bar{x} \leq s$ with the much fewer number of constraints but with the same dimension, i.e. $\bar{x} \in \Psi_2$. Due to an arbitrary manner of selecting $\bar{x}$ from $\Psi_1$, this justifies the inclusion $\Psi_1 \subseteq \Psi_2$. Through the forward and backward inclusions, we have $\Psi_1 = \Psi_2$. Consequently, from Theorem 3.4 it follows that $\Phi - M = \Psi_2$. \hfill \Box

In light of these assertions, inter alia, we can now apply the formula of the Minkowski difference $\Phi - M$ in solving the separation problem for $\Phi$ and $M$. To linearly separate the sets $\Phi$ and $M$, for a beginning, we thereby need to solve the problem (1). Of course, in the setting of (1), we take the set $\Phi - M$ instead of $\tilde{\Phi}$. After that, we are now in the position of being able to analyze the values of $t_{\Phi-M}(c^*)$ and $c^*$. This analysis allows us to characterize $\Phi$ and $M$ as linearly separable or inseparable. In the separable case, $c^*$ represents the normal vector of the best linear separator for $\Phi$ and $M$ with the maximal thickness. In the case of the sets inseparability, $c^*$ corresponds to the best pseudo-separator with the minimal thickness. In other words, for any case, the special setting of (1) provides an
opportunity for obtaining the optimal thickness of the margin between the supporting hyperplanes to the sets. For the details relatively the terms of separator, pseudo-separator, their thickness, and not only, the interested reader is again directed to [27] (see p. 161). Notice that if we need only inspect the issue of whether the Minkowski difference $\Phi - M$ fails to be strongly separable from the origin of $\mathbb{R}^n$, then we can simply test whether or not the origin satisfies the system of constraints describing $\Phi - M$. Another question may now arise. How we can reveal that the sets $\Phi$ and $M$ are non-strongly linearly separable from each other? Due to constraint structure of $\Phi - M$, it is also not hard to check the question of whether the origin of $\mathbb{R}^n$ is the boundary point of $\Phi - M$. Indeed, if some constraints is fulfilled at the origin as equations, whereas all the others hold as the strict inequalities, obviously this means then that the origin belongs to the boundary of $\Phi - M$.

Let us note that the problem of computing the distance between $\Phi$ and $M$ can be solved by means of projecting the origin of $\mathbb{R}^n$ onto $\Phi - M$. It will especially be expedient to apply such a reduction of the problems in the case when all of the right-hand side components $s_k$ for the system identifying the refined version of the Minkowski difference $\Phi - M$ are negative. Since in this event, as it was proved in [33], a projection problem for convex polyhedron given by a system of linear constraints can be solved by reduction to the problem of projecting the origin onto the polyhedron described by a finite collection of points from $\mathbb{R}^n$. In its own turn, this projection problem may be solved utilizing one of the various problem settings presented in [34]. The proposed reduction makes wider a range of suitable optimization tools which can effectively be operated for solving the projection problem in consideration.

### 3.4 Both Operands with a Half-space Representation

The purpose of this subsection is to obtain the description of the Minkowski difference for the case where both operands have the same setting in the form of the half-spaces intersection (or, briefly, have a so-called half-space representation).

Let us be given the two following polyhedra described as the inter-
section of closed half-spaces of \( \mathbb{R}^n \):

\[ \Phi = \{ x \in \mathbb{R}^n : A_1 x \leq b_1 \}, \]
\[ \Psi = \{ x \in \mathbb{R}^n : A_2 x \leq b_2 \}, \]

where \( A_1 \in \mathbb{R}^{r_1 \times n} \), \( A_2 \in \mathbb{R}^{r_2 \times n} \) are some nonvacuous matrices, \( b_1 \in \mathbb{R}^{r_1} \), \( b_2 \in \mathbb{R}^{r_2} \). By the same arguments already applied in Subsection 2.3, there can be shown that \( \Phi \) and \( \Psi \) are closed and convex sets. According to Theorem 3.3 the Minkowski difference of sets \( \Phi \) and \( \Psi \) can be expressed as follows

\[ \Phi - \Psi = \{ x \in \mathbb{R}^n : A_1 (x + y) \leq b_1, \ y \in \Psi \}. \]

If we replace further the abstract constraint by the system defining \( \Psi \), then we immediately obtain the following system of linear constraints:

\[ A_1 x + A_1 y \leq b_1, \]
\[ A_2 y \leq b_2. \]

This system can be equivalently rewritten in matrix form as follows:

\[ \Phi - \Psi = \{ z \in \mathbb{R}^{2n} : Dz \leq b \}, \]

where \( D \in \mathbb{R}^{(r_1 + r_2) \times 2n} \) is a block-structured matrix,

\[ D = \begin{pmatrix} A_1 & A_1 \\ \Theta & A_2 \end{pmatrix}. \]

Besides, the right-hand side of the system and the vector of variables have also the block structure, i.e. \( b \in \mathbb{R}^{r_1 + r_2} \) and \( b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \), \( z \in \mathbb{R}^{2n} \) and \( z = (x \ | \ y) \). Here, \( \Theta \in \mathbb{R}^{r_2 \times n} \) denotes the null matrix. Being the intersection of the closed half-spaces in \( \mathbb{R}^{2n} \), the Minkowski difference \( \Phi - \Psi \) is the closed and convex point set.

For computing the Euclidean distance between the sets \( \Phi \) and \( \Psi \), we can formulate and solve the following problem of minimizing the strongly convex quadratic function subject to the linear inequalities:

\[ \min \| z \|^2 \quad (6) \]
\[ Dz \leq b, \quad (7) \]

where the number of constraints is equal to \( r_1 + r_2 \), the amount of variables equals \( 2n \). For solving the same problem of measuring the
distance between $\Phi$ and $\Psi$, in [25], they dealt with the following optimization problem:

$$\min \| x - y \|^2$$  \hspace{1cm} (8) \\
$$A_1 x \leq b_1,$$  \hspace{1cm} (9) \\
$$A_2 y \leq b_2.$$  \hspace{1cm} (10)

Observe that this problem has the exactly similar number of constraints and variables as the problem (6)–(7). Nevertheless, the objective function of (8)–(10) does not have the strong convexity property. For this reason, the program (6)–(7) compares favorably with (8)–(10).

### 3.5 Applications in Variational Inequalities Problems Related to a Concept of Linear Separability

Naturally, the analytical representation of the Minkowski difference of sets already has its own utility and independent significant applications in various fields of mathematical sciences. Likewise, the Minkowski difference operation can take its place now as a mathematical tool ready for new applications. In regard the topic of interest, it is quite clear that the Minkowski difference is a tool suitable for dealing with solving the variational inequalities that are closely relevant to a concept of the linear separability of sets.

Let us consider the following variational inequalities, which consist in determining a nonzero vector $c \in \mathbb{R}^n$ such that

$$\langle c, x - y \rangle \geq \Delta, \quad x \in A, \ y \in B, \ \Delta > 0,$$  \hspace{1cm} (11) \\
$$\langle c, x - y - c \rangle \geq 0, \quad x \in A, \ y \in B,$$  \hspace{1cm} (12) \\
$$\langle c, x - y \rangle \geq 0, \quad x \in A, \ y \in B.$$  \hspace{1cm} (13)

Note that the first two of these variational inequalities correspond to the strong linear separability term in a sense that the inequalities are solvable in the case when the sets $A$ and $B$ are strongly linearly separable. The third inequality is closely connected with the potentially non-strong linear separability of the considered sets. Clearly, using the Minkowski difference $A - B$, for the above-mentioned inequalities (11)–(13), a reduction can be made to the following variational
inequalities, respectively:

\[ \langle c, z \rangle \geq \Delta, \quad x \in A - B, \quad \Delta > 0, \tag{14} \]

\[ \langle c, z - c \rangle \geq 0, \quad z \in A - B, \tag{15} \]

\[ \langle c, z \rangle \geq 0, \quad z \in A - B. \tag{16} \]

Of course, the goal is to find the nonzero vector \( c \in \mathbb{R}^n \) satisfying (14)-(16). A set of possible solutions for (16) coincides with \( W_{A-B} \setminus \{0\} \).

For the more general than convex polyhedral setting, the proof of this fact was represented in [29] (see p. 712). Moreover, it was proved that the inequality (16) has nonzero solutions if and only if \( W_{A-B} \neq \{0\} \) or, equivalently, \( 0 \notin \text{int}(A - B) \). For the convex sets \( A \) and \( B \) with nonempty interiors, due to Lemma 3.16 from [29], it holds

\[ \text{int}(A - B) = \text{int}(A) - \text{int}(B). \]

This means that (16) (in tandem with (13)) is solvable if and only if the convex polyhedra \( A \) and \( B \) with nonempty interiors have no the common interior points. For convex and closed sets \( A \) and \( B \), a solution set of (14) coincides with \( V_{A-B} \). In [22], it was justified that the variational inequality (14) together with (11) has solutions if and only if \( V_{A-B} \neq \emptyset \) or, equivalently, \( 0 \notin A - B \). Thus, the absence of any common points for \( A \) and \( B \) is the condition for the solvability of (14) and (11). Formally, the fulfillment of this condition can be verified with the help of projecting the origin of \( \mathbb{R}^n \) onto \( A - B \). If as the result of projecting we obtain \( P_{A-B}(0) \neq 0 \), then \( c = P_{A-B}(0) \) represents the solution for (14). Otherwise, the inequalities (13) and (11) have no solutions. For the variational inequality which consists in determining a vector \( c \in \mathbb{R}^n \setminus \{0\} \) satisfying (15), a solution set coincides with \( \Omega_{A-B} \). Let us remind that the considered pair of convex polyhedra \( A \) and \( B \) are closed. Then the boundedness of one of these sets implies the closedness of \( A - B \). By the closedness and convexity properties of \( A - B \), due to Theorem 3.3 from [27], we immediately obtain \( P_{A-B}(0) \in Bd(\Omega_{A-B}) \). Having obtained \( P_{A-B}(0) = 0 \), we can conclude that \( \Omega_{A-B} = \{0\} \), since Theorem 3.2 from [27] yields

\[ \min_{x \in A - B} \|x\|^2 = \max_{y \in \Omega_{A-B}} \|y\|^2. \]

The fulfillment of \( \Omega_{A-B} = \{0\} \) is equivalent to having \( 0 \in A - B \),
since \( E(\Omega_{A-B}) \setminus \{0\} = V_{A-B} \).

For the Minkowski difference operation, we underline the next application which consists in finding the nearest points of sets \( A \) and \( B \) by solving the system

\[
\begin{align*}
\langle c, x - \bar{x} \rangle & \geq 0, \quad x \in A, \\
\langle c, \bar{y} - y \rangle & \geq 0, \quad y \in B,
\end{align*}
\]

where \( c = P_{A-B}(0) \) is the projection of the origin of \( \mathbb{R}^n \) onto \( A - B \). Let us note that the projection problem can be reduced to the maximin problem of the type (1). To solve the maximin problem, we can apply some software package after a simple transition to the minimax problem as follows

\[
\max_{w \in \chi} \inf_{x \in A-B} \langle x, w \rangle = -\min_{w \in \chi} \left( -\inf_{x \in A-B} \langle x, w \rangle \right) = -\min_{w \in -\chi} \sup_{x \in A-B} \langle x, w \rangle = -\min_{w \in \chi} \sup_{x \in A-B} \langle x, w \rangle,
\]

where

\[
\chi := \{c \in \mathbb{R}^n : \|c\| = 1\}, \quad -\chi := \{-x, \; x \in \chi\}.
\]

We notice that, for instance, a package Optimization Toolbox in MATLAB contains the function Fminimax which is usable for solving the minimax constraint problem.

4 Conclusions

We have presented the novel analytical representation of the Minkowski difference for convex polyhedra given by different ways. In particular, we have considered the following cases where:

- both operands under the Minkowski difference operation are similarly determined as convex hulls of finite collections of points,

- the operands have the different performance (or more precisely, the first operand is expressible by a linear constraint system, and the second one is given in terms of convex hull),

- both operands have the same representation as the intersection of the closed half-spaces.

It can be concluded that thanks to the obtained results, there appeared the possibility to investigate the relevant problems such as: problem of linear separation of the convex polyhedra in the Euclidean space,
variational inequalities problems, problem of finding the distance between the convex polyhedra by projecting the origin of the Euclidean space onto a convex polyhedron, the problem of determining the closest points of convex polyhedra.

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