Weak convergence rates for temporal numerical approximations of stochastic wave equations with multiplicative noise

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Abstract

In this work we establish weak convergence rates for temporal discretisations of stochastic wave equations with multiplicative noise, in particular, for the hyperbolic Anderson model. For this class of stochastic partial differential equations the weak convergence rates we obtain are indeed twice the known strong rates. To the best of our knowledge, our findings are the first in the scientific literature which provide essentially sharp weak convergence rates for temporal discretisations of stochastic wave equations with multiplicative noise. Key ideas of our proof are a sophisticated splitting of the error and applications of the recently introduced mild Itô formula. We complement our analytical findings by means of numerical simulations in Python for the decay of the weak approximation error for SPDEs for four different test functions.

Contents

1 Introduction 2
1.1 Setting . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4
2 Preliminaries 5
2.1 SDEs in Hilbert spaces . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5
2.2 Kolmogorov equations in Hilbert spaces . . . . . . . . . . . . . . . . . . . . 5
2.3 Preparatory lemmas . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7

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1 Introduction

Stochastic partial differential equations (SPDEs) are used to model various evolutionary processes subject to random forces. For example, stochastic wave equations may model the motion of a strand of DNA in a liquid or heat flow around a ring; see, e.g., [Dal09, Tho12]. In general the solution to an SPDE cannot be given explicitly, whence it is desirable to prove convergence rates for numerical approximations. Here one distinguishes strong convergence rates, i.e., with respect to the strong (mean square) error, and weak convergence rates, i.e., with respect to a suitable weak approximation error. Typically, the convergence rate for the weak error is twice the convergence rate for the strong error. However, there does not exist a straightforward way to establish this. Moreover, non-trivial exceptions to this rule exist; see, e.g., [Alf05, HJ19].

For both parabolic and hyperbolic semilinear SPDEs with coefficients depending on the state in a globally Lipschitz continuous way, strong convergence is by now well-understood. In particular, strong convergence rates for numerical approximations of stochastic wave equations have been established in, e.g., [ACLW16, CLS13, CQS16, CY07, KLL13, KLS10, QSSS06, Wal06, Wan15, WGT14]. Recently, a strong approximation scheme for a stochastic wave equation with a cubic nonlinearity has been analysed in [CHJS19].

Establishing optimal weak convergence rates for both hyperbolic and parabolic SPDEs is currently active field of research; see, e.g., [AHJK19, AJK21, AJKW17, AKL16a, AKL16b, AL16, AL19, AL17, Bre12, Bre14, Bre20, BD18, BG20, BK17, CKJ19, CH18, CH19, dB06, Deb11, DP09, GKL09, HM19, Han03, Han10, HJK16, HJ12, JdNJW21, JK21, Kop14, KLL13, KLL12, KLS15, KP14, Kru14, Lin12, LS13, Sha03, Wan15, Wan16, WG13]. Arguably, the most relevant basic SPDEs are the parabolic and hyperbolic Anderson model, i.e., the heat equation with multiplicative noise and the wave equation with multiplicative noise. However, establishing optimal weak convergence rates for SPDEs with multiplicative noise is challenging. Indeed, of the articles cited above only [BD18, CJK19, CH18, dB06, Deb11, HJK16, JdNJW21, JK21] provide weak rates for SPDEs with multiplicative noise. Roughly speaking, there are two successful approaches to obtain optimal weak convergence rates for parabolic SPDEs with multiplicative noise. One is based on regularity results for the corresponding Kolmogorov equation and Malliavin calculus; see, e.g., [BD18, Deb11]. The other is based on more elementary regularity results of the Kolmogorov equation and the mild Itô formula; see, e.g., [CJK19, HJK16, JK21].

No successful approach for proving optimal weak convergence rates has been developed yet for temporal discretisations of hyperbolic SPDEs with multiplicative noise. Indeed, the two approaches mentioned above are not applicable as they rely strongly on the smoothing effect of the semigroup. In this work we tackle this problem and develop a technique that
allows one to establish optimal weak convergence rates for hyperbolic SPDEs with multiplicative noise. A special case of our main result is presented in the following theorem.

**Theorem 1.1.** Let $T, \theta \in (0, \infty)$, $b_0, b_1 \in \mathbb{R}$, $H = L^2((0, 1); \mathbb{R})$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$ be a filtered probability space which fulfills the usual conditions, let $(W_t)_{t \in [0, T]}$ be an i.i.d. cylindrical $(\mathbb{F}_t)_{t \in [0, T]}$-Wiener process, let $A: D(A) \subseteq H \rightarrow H$ be the Dirichlet Laplacian on $H$, let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, r \in \mathbb{R})$, be a family of interpolation spaces associated to $-A$, let $\mathbf{H}_0 = H_0 \times H_{-1/2}$, $\mathbf{H}_1 = H_{1/2} \times H_0$, let $A: D(A) \subseteq \mathbf{H}_0 \rightarrow \mathbf{H}_1$ be the linear operator which satisfies that $D(A) = \mathbf{H}_1$ and $\langle v, w \rangle \in D(A): A(v, w) = (w, \theta Av)$, let $\xi \in L^6(\mathbb{P} | \mathcal{F}_0; \mathbf{H}_1)$, $\varphi \in C^4(\mathbf{H}_0, \mathbb{R})$ satisfy\(^1\) that $\sup_{k \in \{1, 2, 3, 4\}, x \in \mathbf{H}_0} \| \varphi^{(k)}(x) \|_{L^6(\mathbf{H}_0, \mathbb{R})} < \infty$, let $B: \mathbf{H}_0 \rightarrow L_2(H, \mathbf{H}_0)$ be the function which satisfies for every $(v, w) \in \mathbf{H}_0$, $u \in H_{1/2}$ that $B(v, w)u = (0, (b_0 + b_1 v) u)$, let $X: [0, T] \times \Omega \rightarrow \mathbf{H}_0$ be an $(\mathbb{F}_t)_{t \in [0, T]}$-predictable stochastic process which satisfies for every $t \in [0, T]$ that $\sup_{s \in [0, T]} \mathbb{E} \| X_s \|^2_{\mathbf{H}_0} < \infty$ and

$$X_t = e^{tA} \xi + \int_0^t e^{(t-s)A} B(X_s) \, dW_s,$$

and let $Y_N: \{0, 1, 2, \ldots, N\} \times \Omega \rightarrow \mathbf{H}_0$, $N \in \mathbb{N}$, be the stochastic processes which satisfy for all $N \in \mathbb{N}$, $n \in \{1, 2, \ldots, N\}$ that $Y_0^N = \xi$ and

$$Y_n^N = e^{(T/n)A} \left( Y_{n-1}^N + \int_{(n-1)T/N}^{nT/N} B(Y_{n-1}^N) \, dW_s \right).$$

Then it holds for all $\varepsilon \in (0, \infty)$ that $\sup_{N \in \mathbb{N}} \left( N^{1-\varepsilon} \mathbb{E} \left[ \varphi(X_T) \right] - \mathbb{E} \left[ \varphi \left( Y_N^T \right) \right] \right) < \infty$.

Note that we obtain rate of convergence $1^-$, which is indeed twice the known strong rate. Theorem 1.1 is an immediate consequence of Corollary 4.6 below. Corollary 4.6 follows from Theorem 3.11 which is the main result of this article. Indeed, Theorem 3.11 establishes an upper bound for the weak error of a temporal discretisation of a hyperbolic SPDE with multiplicative noise. Similar as in the parabolic case, a key ingredient of the proof of this upper bound is the mild Itō formula developed in \[\text{DPJR19}\]. For the convenience of the reader, we recall a suitable variant of the mild Itō formula in Proposition 3.10 below. For parabolic SPDEs the mild Itō formula is used to insert the semigroup in an appropriate place so the smoothing property can be exploited. Here, however, the mild Itō formula is used to rewrite certain terms in the error as integrals over an interval of length at most $T/N$. The use of the mild Itō formula is crucial: if one would apply the ‘classical’ Itō formula, then one would obtain a term involving an unbounded operator. Although the underlying semigroup does not does not enjoy a smoothing property as in the parabolic case, by using the mild Itō formula one can avoid the appearance of an unbounded operator and thus the roughening effect accompanied by it. Another key ingredient of the proof is an elegant decomposition of the error into terms that can be treated using this mild Itō formula approach, and terms that can be dealt with in a relatively straightforward manner; see \[\text{CJKP18}\] in the proof of Theorem 3.11. It is to be expected that this method of proof can also be applied to other types of temporal discretisations, as well as to spatial discretisations such as the finite element method. Moreover, although we consider the Hilbert space setting in this work, our approach can be extended in a straightforward way to the Banach space setting; see \[\text{HJKT16}\]. This would allow one to prove optimal weak rates for more general semilinear drift and diffusion coefficients; see \[\text{HJKT16}\] for analogous results for parabolic SPDEs. For completeness we note that optimal weak convergence rates for spatial spectral Galerkin

\[\text{Notes:}
\]

\[\text{1}\] Note that for every $r \in [0, \infty)$, $v \in H_r$ it holds that $H_r = D((-A)^r)$ and $\| v \|_{H_r} = \| (-A)^r v \|_H$.

\[\text{2}\] Observe that for every $k \in \mathbb{N}$ and every $k$-linear bounded operator $C: (\mathbf{H}_0)^k \rightarrow \mathbb{R}$ it holds that $\| C \|_{L^k(\mathbb{H}_0, \mathbb{R})} = \sup \{ |C(x_1, x_2, \ldots, x_k)| : x_1, x_2, \ldots, x_k \in \mathbf{H}_0, \| x_1 \|_{\mathbf{H}_0} = \| x_2 \|_{\mathbf{H}_0} = \ldots = \| x_k \|_{\mathbf{H}_0} = 1 \}$. 

3
approximations of stochastic wave equations have been established in \cite{JdNJW18,JdNJW21}. The approach taken in \cite{JdNJW18,JdNJW21} essentially relies on the specific structure of the spatial spectral Galerkin approximations and can thus neither be extended to temporal approximations nor to other more complicated spatial approximations such as the finite element method.

Let us comment on the optimality of the convergence rate obtained in Theorem 1.1 above. Lower bounds for strong and weak approximation errors of numerical discretisations of SPDEs have been derived in, e.g., \cite{BGJK20,CJK19,DG01,JdNJW18,JdNJW21,MGR07,MGRW08a,MGRW08b}. In particular, lower bounds for weak approximation errors of spatial spectral Galerkin approximations of stochastic wave equations can be found in \cite{JdNJW18}. Lower bounds for strong and weak approximation errors of temporal numerical approximations of stochastic wave equations remain an open problem for future research.

Nevertheless, we conjecture that the weak convergence rate for the exponential Euler scheme in Theorem 1.1 above can in general not essentially be improved.

The remainder of this article is structured as follows. In Section 2 we present some essentially well-known auxiliary results. More specifically, a standard existence and uniqueness result for semilinear SDEs in Hilbert spaces is recalled in Subsection 2.1. Regularity results for the associated Kolmogorov equations are provided in Subsection 2.2 and further preparatory lemmas are collected in Subsection 2.3. Section 3 is devoted to the weak error analysis for temporal discretisations of a class of stochastic wave equations with multiplicative noise and contains our main abstract results. A general setting for our convergence analysis is presented in Subsection 3.1, and some elementary properties of the wave semigroup are collected in Subsection 3.2. Theorem 3.11 in Subsection 3.3 establishes upper bounds for the weak errors of temporal discretisations of spatial spectral Galerkin approximations. This combined with the uniform moment bounds obtained in Subsection 3.3 and the strong convergence of the Galerkin approximations proven in Subsection 3.4 establishes the weak convergence rates of the temporal discretisations, see Corollary 3.12 below. In Section 4, we apply the weak convergence result from Corollary 3.12 to the hyperbolic Anderson model. After specifying a suitable setting in Subsection 4.1, we collect some results on multipliers on Sobolev-Slobodeckij spaces in Subsection 4.2 which we use in Subsection 4.3 to verify that Corollary 3.12 implies Corollary 4.6. Recall that Corollary 4.6 implies Theorem 1.1. Numerical simulations illustrating Corollary 4.6 are presented in Subsection 4.4.

1.1 Setting

Throughout this article we shall frequently use the following setting.

**Setting 1.2.** For every pair of $\mathbb{R}$-Hilbert spaces $(\mathcal{V}, \langle \cdot, \cdot \rangle_\mathcal{V}, \|\cdot\|_\mathcal{V})$ and $(\mathcal{W}, \langle \cdot, \cdot \rangle_\mathcal{W}, \|\cdot\|_\mathcal{W})$ let \( (L_2(\mathcal{V}, \mathcal{W}), \langle \cdot, \cdot \rangle_{L_2(\mathcal{V}, \mathcal{W})}, \|\cdot\|_{L_2(\mathcal{V}, \mathcal{W})}) \) be the $\mathbb{R}$-Hilbert space of Hilbert-Schmidt operators from $\mathcal{V}$ to $\mathcal{W}$, for every $k \in \mathbb{N}$ and every pair of $\mathbb{R}$-Banach spaces $(\mathcal{V}, \|\cdot\|_\mathcal{V})$ and $(\mathcal{W}, \|\cdot\|_\mathcal{W})$ let \( (\text{Lip}(\mathcal{V}, \mathcal{W}), \|\cdot\|_{\text{Lip}(\mathcal{V}, \mathcal{W})}) \) be the $\mathbb{R}$-Banach space of Lipschitz continuous mappings from $\mathcal{V}$ to $\mathcal{W}$, let \( (C^0_b(\mathcal{V}, \mathcal{W}), \|\cdot\|_{C^0_b(\mathcal{V}, \mathcal{W})}) \) be the $\mathbb{R}$-Banach space of $k$-times continuously Fréchet differentiable functions from $\mathcal{V}$ to $\mathcal{W}$ with globally bounded derivatives, and let \( (L^{(k)}(\mathcal{V}, \mathcal{W}), \|\cdot\|_{L^{(k)}(\mathcal{V}, \mathcal{W})}) \) be the $\mathbb{R}$-Banach space of $k$-linear bounded operators from $\mathcal{V}^k$ to $\mathcal{W}$, for every measurable space $(\Omega, \mathcal{F}, \mu)$, every measurable space $(S, \Sigma)$, and every function $f: \Omega \to S$ let \( [f]_{\mu, \Sigma} \) be the set given by

\[
[f]_{\mu, \Sigma} = \left\{ g: \Omega \to S: \left[\exists A \in \mathcal{F}: (\mu(A)=0 \quad \text{and} \quad \mu(\omega_0 \cap \mu(A) \subseteq A) \right] \right\},
\]

let $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be a separable $\mathbb{R}$-Hilbert space, let $\mathcal{U} \subseteq U$ be an orthonormal basis of $U$, let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]})$ be a filtered probability space which fulfills the usual conditions, and let $(W_t)_{t \in [0,T]}$ be an $id_U$-cylindrical $(\mathcal{F}_t)_{t \in [0,T]}$-Wiener process.
In Setting [1.2] we introduced for every pair of $\mathbb{R}$-Banach spaces $(\mathcal{V}, \|\cdot\|_\mathcal{V})$ and $(\mathcal{W}, \|\cdot\|_\mathcal{W})$ the tuples $(\text{Lip}(\mathcal{V}, \mathcal{W}), \|\cdot\|_{\text{Lip}(\mathcal{V}, \mathcal{W})})$ and $(C^k_b(\mathcal{V}, \mathcal{W}), \|\cdot\|_{C^k_b(\mathcal{V}, \mathcal{W})})$, $k \in \mathbb{N}$. Note that for every pair of $\mathbb{R}$-Banach spaces $(\mathcal{V}, \|\cdot\|_\mathcal{V})$ and $(\mathcal{W}, \|\cdot\|_\mathcal{W})$ and every $k \in \mathbb{N}$, $F \in \text{Lip}(\mathcal{V}, \mathcal{W})$, $\varphi \in C^k_b(\mathcal{V}, \mathcal{W})$ it holds that $\|F\|_{\text{Lip}(\mathcal{V}, \mathcal{W})} = \|F(0)\|_\mathcal{W} + \sup\{\|F(x) - F(y)\|_\mathcal{W} / |x - y| : x, y \in \mathcal{V}, x \neq y\} \cup \{0\}$ and $\|\varphi\|_{C^k_b(\mathcal{V}, \mathcal{W})} = \|\varphi(0)\|_\mathcal{W} + \sum_{j=1}^k \sup_{x \in \mathcal{V}} \|\varphi^{(j)}(x)\|_{L(\mathcal{V}, \mathcal{W})}$.

2 Preliminaries

2.1 Stochastic differential equations in Hilbert spaces

The existence and uniqueness result in Theorem 2.1 below is essentially well-known in the literature; cf., for example, Da Prato & Zabczyk [DPZ92] Theorem 7.4.

**Theorem 2.1.** Assume Setting [1.2] let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ be a separable $\mathbb{R}$-Hilbert space, let $S: [0, \infty) \to L(H)$ be a strongly continuous semigroup, and let $p \in [2, \infty)$, $F \in \text{Lip}(H, H)$, $B \in \text{Lip}(H, L_2(U, H))$, $\xi \in \mathcal{D}^p(\mathbb{P}|_{\mathcal{F}_0}; H)$. Then there exists an up to modifications unique $(\mathbb{F}_t)_{t \in [0, T]}$-predictable stochastic process $X: [0, T] \times \Omega \to H$ which satisfies for every $t \in [0, T]$ that $\sup_{s \in [0, T]} \mathbb{E}[\|X_s\|_H^p] < \infty$ and

$$[X_t]_{\mathbb{P}, \mathcal{B}(H)} = [S_t \xi]_{\mathbb{P}, \mathcal{B}(H)} + \int_0^t S_{t-s} F(X_s) \, ds + \int_0^t S_{t-s} B(X_s) \, dW_s. \quad (4)$$

2.2 Kolmogorov equations in Hilbert spaces

In Lemma 2.2 below we present a specific variant of a standard regularity result for backward Kolmogorov equations in Hilbert spaces that is suitable for our purpose. For the convenience of the reader we include a sketch of its elementary proof.

**Lemma 2.2.** Assume Setting [1.2] let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ be a non-trivial separable $\mathbb{R}$-Hilbert space, for every $A \in L(H)$, $F \in C^1_b(H, H)$, $B \in C^1_b(H, L_2(U, H))$, $x \in H$ let $X^{A,F,B,x}: [0, T] \times \Omega \to H$ be an $(\mathbb{F}_t)_{t \in [0, T]}$-predictable stochastic process which satisfies for every $t \in [0, T]$ that $\sup_{s \in [0, T]} \mathbb{E}[\|X_s^{A,F,B,x}\|_H^2] < \infty$ and

$$[X^{A,F,B,x}_t]_{\mathbb{P}, \mathcal{B}(H)} = [e^{tA} x]_{\mathbb{P}, \mathcal{B}(H)} + \int_0^t e^{(t-s)A} F(X^{A,F,B,x}_s) \, ds + \int_0^t e^{(t-s)A} B(X^{A,F,B,x}_s) \, dW_s, \quad (5)$$

and for every $A \in L(H)$, $F \in C^1_b(H, H)$, $B \in C^1_b(H, L_2(U, H))$, $\varphi \in C^1_b(H, \mathbb{R})$ let $v^{A,F,B,\varphi}: [0, T] \times H \to \mathbb{R}$ be the function which satisfies for every $t \in [0, T]$, $x \in H$ that $v^{A,F,B,\varphi}(t, x) = \mathbb{E}[\varphi(X^{A,F,B,x}_{T-t})]$. Then

(i) it holds for every $A \in L(H)$, $F \in C^1_b(H, H)$, $B \in C^1_b(H, L_2(U, H))$, $\varphi \in C^2_b(H, \mathbb{R})$, $t \in [0, T]$, $x \in H$ that $v^{A,F,B,\varphi} \in C^{1,2}([0, T] \times H, \mathbb{R})$ and

$$\left( \frac{\partial}{\partial t} v^{A,F,B,\varphi} \right)(t, x) + \left( \frac{\partial}{\partial x} v^{A,F,B,\varphi} \right)(t, x)(Ax + F(x))$$

$$+ \frac{1}{2} \sum_{u \in U} \left( \frac{\partial^2}{\partial u^2} v^{A,F,B,\varphi} \right)(t, x)(B(x)u, B(x)u) = 0, \quad (6)$$

(ii) it holds for every $k \in \mathbb{N}$, $A \in L(H)$, $F \in C^k_b(H, H)$, $B \in C^k_b(H, L_2(U, H))$, $\varphi \in C^k_b(H, \mathbb{R})$, $t \in [0, T]$ that $(H \ni x \mapsto v^{A,F,B,\varphi}(t, x) \in \mathbb{R}) \in C^k_b(H, \mathbb{R})$, and
(iii) it holds for every $k \in \mathbb{N}$, $c \in (0, \infty)$ that
\[
\sup_{t \in [0,T], x \in H} \left\{ \frac{(\partial^k_x e^{tA,B,\varphi})(t,x)}{\|\varphi\|_{C^k_b(H,R)}} e^{k(t,k)}(H,R) \right\} \leq \sup_{t \in [0,T], x \in H} \|\varphi\|_{C^k_b(H,R)} \leq \|\varphi\|_{C^k_b(H,R)} < \infty.
\] (7)

**Proof of Lemma 2.2.** Throughout this proof for every set $S \subseteq \{0, 1, 2, \ldots\} \cup \{\infty\}$ be the cardinality of $S$, for every set $S$ let $\mathcal{P}(S)$ be the power set of $S$, and for every $j \in \mathbb{N}$ let $\Pi_j$ be the set given by
\[
\Pi_j = \{ S \subseteq \mathcal{P}(\mathbb{N}) : \forall j \in \mathbb{N} \}
\] (the set of all partitions of $\{1, \ldots, j\}$). Observe that, e.g., Da Prato & Zabczyk [DPZ92 Theorem 9.16] (cf., for example, also Harms & Müller [HM19] item (ii) in Lemma 2.2) establishes item (ii). Moreover, note that Andersson et al. [AJKW17, item (ix) in Theorem 2.1] establishes item (i). Moreover, note that Andersson et al. [AHJK19, item (ii) in Lemma 3.2] demonstrates that for every $k \in \mathbb{N}$, $A \in L(H)$, $F \in C^k_b(H,H)$, $B \in C^k_b(H,L_2(U,H))$, $t \in [0,T]$, $p \in [1, \infty)$ it holds that
\[
(H \ni x \mapsto [X^A,F,B,x]_{\mathbb{P},B(H)} \in L^p(\mathbb{P}; H)) \subseteq C^k_b(H,L^p(\mathbb{P}; H)).
\] (9)
In addition, note that Andersson et al. [AJKW17] items (i)–(ii) and items (ix)–(x) in Theorem 2.1] (with $\alpha = \beta = \delta_1 = \ldots = \delta_k = 0$ in the notation of [AJKW17, item (ii) in Theorem 2.1) ensures that for every $k \in \mathbb{N}$, $\alpha \in (0, \infty)$, $p \in [1, \infty)$ it holds that
\[
\sup_{t \in [0,T], x \in H} \left\{ \frac{(\partial^k_x e^{tA,F,B,\varphi})(t,x)}{\|\varphi\|_{C^k_b(H,R)}} e^{k(t,k)}(H,R) \right\} \leq \sup_{t \in [0,T], x \in H} \|\varphi\|_{C^k_b(H,R)} \leq \|\varphi\|_{C^k_b(H,R)} < \infty.
\] (10)
Moreover, observe that Andersson et al. [AHJK19] item (v) in Lemma 3.2] (with $\alpha = \beta = \delta_1 = \ldots = \delta_k = 0$ in the notation of [AHJK19, item (v) in Lemma 3.2) proves that for every $k \in \mathbb{N}$, $A \in L(H)$, $F \in C^k_b(H,H)$, $B \in C^k_b(H,L_2(U,H))$, $\varphi \in C^k_b(H,R)$ it holds that
\[
\sup_{t \in [0,T], x \in H} \left\{ \frac{(\partial^k_x e^{tA,F,B,\varphi})(t,x)}{\|\varphi\|_{C^k_b(H,R)}} e^{k(t,k)}(H,R) \right\} \leq \sup_{t \in [0,T], x \in H} \|\varphi\|_{C^k_b(H,R)} \leq \|\varphi\|_{C^k_b(H,R)} < \infty.
\] (11)
Next note that for every $k \in \mathbb{N}$, $A \in L(H)$, $F \in C^k_b(H,H)$, $B \in C^k_b(H,L_2(U,H))$, $\varphi \in C^k_b(H,R)$ it holds that
\[
\sup_{x \in H} \left\{ \left( \frac{(\partial^k_x e^{A,F,B,\varphi})(t,x)}{\|\varphi\|_{C^k_b(H,R)}} e^{k(t,k)}(H,R) \right) \right\} \leq \sup_{x \in H} \|\varphi\|_{C^k_b(H,R)} \leq \|\varphi\|_{C^k_b(H,R)} < \infty.
\] (12)
Combining this, (10), and (11) establishes item (iii). The proof of Lemma 2.2 is thus completed. □
2.3 Preparatory lemmas

The next result, Lemma 2.3 below, is frequently used throughout this article.

**Lemma 2.3.** Assume Setting 1.2, let \((\mathcal{V}, \|\cdot\|_{\mathcal{V}})\) and \((\mathcal{W}, \|\cdot\|_{\mathcal{W}})\) be \(\mathbb{R}\)-Banach spaces, and let \(F \in \text{Lip}(\mathcal{V}, \mathcal{W})\). Then

(i) it holds for every \(v \in \mathcal{V}\) that \(\|F(v)\|_{\mathcal{W}} \leq \|F\|_{\text{Lip}(\mathcal{V}, \mathcal{W})} \max\{1, \|v\|_{\mathcal{V}}\}\)

(ii) it holds for every \(p \in [1, \infty)\), \(\xi \in \mathcal{L}^p(\mathbb{P}; \mathcal{V})\) that

\[
\|F(\xi)\|_{\mathcal{L}^p(\mathbb{P}; \mathcal{W})} \leq \|F\|_{\text{Lip}(\mathcal{V}, \mathcal{W})} (1 + \|\|\xi\|_{\mathcal{L}^p(\mathbb{P}; \mathcal{V})}) .
\]

**Proof of Lemma 2.3.** Observe that the fact that \(\sup_{p \in [1, \infty)} \|F(\xi)\|_{\mathcal{L}^p(\mathbb{P}; \mathcal{W})} \leq \|F\|_{\text{Lip}(\mathcal{V}, \mathcal{W})} (1 + \|\|\xi\|_{\mathcal{L}^p(\mathbb{P}; \mathcal{V})})\). This proves item (i). The proof of Lemma 2.3 is thus completed.

**Lemma 2.4.** It holds

(i) that \(\sup_{\alpha \in [0, 2]} t^{\alpha} |1 - \cos(t)| = 2\) and

(ii) that \(\inf_{\alpha \in \mathbb{R}\setminus[0, 2]} \sup_{t \in (0, \infty)} t^{\alpha} |1 - \cos(t)| = \infty\).

**Proof of Lemma 2.4.** First, note that for every \(\alpha \in [0, 2]\), \(t \in (1, \infty)\) it holds that

\[
t^{-\alpha} |1 - \cos(t)| \leq 2t^{-\alpha} \leq 2 .
\]

Next observe that the fundamental theorem of calculus assures that for every \(t_1 \in (0, \infty)\), \(t_2 \in (t_1, \infty)\) it holds that

\[
(t_2)^{-\alpha} |1 - \cos(t_2)| = (t_2)^{-\alpha}(1 - \cos(t_2)) = (t_1)^{-\alpha}(1 - \cos(t_1)) + \int_{t_1}^{t_2} \left( \frac{\sin(s) s^2 - (1 - \cos(s)) 2s}{s^3} \right) ds .
\]

In addition, note that the fundamental theorem of calculus implies that for every \(s \in (0, \pi)\) it holds that

\[
\sin(s) s - (1 - \cos(s)) 2 = \int_0^s (\cos(u) u + \sin(u)) du = \int_0^s (\cos(u) u - \sin(u)) du = \int_0^s \int_0^u (-\sin(r) r + \cos(r) - \cos(r)) dr du.
\]

7
This and (17) demonstrate that the function \([0, \pi) \ni t \mapsto t^{-2}|1 - \cos(t)| \in (0, \infty)\) is monotonically decreasing, i.e., that for every \(t_1 \in (0, \pi), t_2 \in (t_1, \pi)\) it holds that
\[
(t_1)^{-2}|1 - \cos(t_1)| \geq (t_2)^{-2}|1 - \cos(t_2)|. \tag{19}
\]
Moreover, note that the fundamental theorem of calculus proves that for every \(t \in (0, \infty)\) it holds that
\[
t^{-2}|1 - \cos(t)| = t^{-2}(\cos(0) - \cos(t))
\]
\[
= -t^{-2} \left[ \int_0^t (-\sin(s)) \, ds \right] = t^{-2} \left[ \int_0^t \sin(s) \, ds \right] = t^{-2} \left[ \int_0^t \int_0^s \cos(u) \, du \, ds \right]
\]
\[
= t^{-2} \left[ \int_0^t \int_0^s 1 \, du \, ds \right] + t^{-2} \left[ \int_0^t \int_0^s (\cos(u) - \cos(0)) \, du \, ds \right]
\]
\[
= \frac{1}{2} + t^{-2} \left[ \int_0^t \int_0^s (-\sin(r)) \, dr \, du \, ds \right]. \tag{20}
\]
Hence, we obtain that for every \(t \in (0, \infty)\) it holds that
\[
|t^{-2}|1 - \cos(t)| - \frac{1}{2}| = t^{-2} \left| \int_0^t \int_0^s \int_0^u \sin(r) \, dr \, du \, ds \right|
\]
\[
\leq t^{-2} \left[ \int_0^t \int_0^s \int_0^u 1 \, dr \, du \, ds \right] = t^{-2} \left[ \frac{t^3}{3!} \right] = \frac{t}{6}.
\tag{21}
\]
Therefore, we obtain that
\[
\limsup_{t \searrow 0} \left| \frac{t^{-2}|1 - \cos(t)| - \frac{1}{2}}{t} = 0. \tag{22}\right.
\]
Combining this and (19) ensures that for every \(\alpha \in [0, 2], t \in (0, 1]\) it holds that
\[
t^{-\alpha}|1 - \cos(t)| \leq t^{-2}|1 - \cos(t)| \leq \frac{1}{2}. \tag{23}\]
In addition, note that \(\sup_{\alpha \in [0, 2], t \in (0, \infty)} (t^{-\alpha}|1 - \cos(t)|) \geq \pi^{-1}|1 - \cos(\pi)| = |1 + 1| = 2.
\] Combining this, (16), and (23) establishes item (i). Furthermore, observe that for every \(\alpha \in (-\infty, 0)\) it holds that
\[
\limsup_{t \rightarrow \infty} (t^{-\alpha}|1 - \cos(t)|) = \infty. \tag{24}\]
In addition, note that (22) shows that for every \(\alpha \in (2, \infty)\) it holds that
\[
\limsup_{t \searrow 0} \left( t^{-\alpha}|1 - \cos(t)| \right) = \limsup_{t \searrow 0} \left( t^{-(\alpha-2)} \left[ t^{-2}|1 - \cos(t)| \right] \right)
\]
\[
\geq \left[ \limsup_{t \searrow 0} t^{-(\alpha-2)} \right] \left[ \lim_{t \searrow 0} \left( t^{-2}|1 - \cos(t)| \right) \right] = \frac{1}{\alpha} \left[ \limsup_{t \searrow 0} t^{-(\alpha-2)} \right] = \infty. \tag{25}\]
Combining this and (24) establishes item (ii). The proof of Lemma 2.4 is thus completed. \(\square\)

The estimate in Lemma 2.5 below is essentially well-known and follows from the Hölder inequality for Schatten norms. We refer to Meise & Vogt [MV97, Chapter 16] for the definition and further standard properties of Schatten class operators.

**Lemma 2.5.** Let \((H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)\) and \((U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)\) be separable \(\mathbb{R}\)-Hilbert spaces, let \(U \subseteq U\) be an orthonormal basis of \(U\), for every \(p \in [1, \infty)\) let \((L_p(U, H), \|\cdot\|_{L_p(U, H)})\) be the \(\mathbb{R}\)-Banach space of Schatten-\(p\) operators from \(U\) to \(H\) and let \(r \in (0, \infty), T \in L^{(2)}(H, \mathbb{R}), A \in L_{1+r}(U, H), B \in L_{1+r}(U, H)\). Then it holds that
\[
\sum_{u \in U} |T(Au, Bu)| \leq \|T\|_{L^{(2)}(H,R)} \|A\|_{L_{1+r}(U,H)} \|B\|_{L_{1+r}(U,H)}. \tag{26}\]
Therefore, we obtain that
\[
\langle T_0x, y \rangle_H = T(x, y) \quad \text{and} \quad \|T_0\|_{L(H)} = \|T\|_{L^2(H, \mathbb{R})}.
\]
(27)

Next note that, e.g., Meise & Vogt [MV97, item 6. in Lemma 16.6 and item 2. in Lemma 16.7] ensures that \(\|B^*\|_{L_{1+r}(H, U)} = \|B\|_{L_{1+r}(U, H)}\) and \(\|T_0A\|_{L_{1+r}(U, H)} \leq \|T_0\|_{L(H)} \|A\|_{L_{1+r}(U, H)}\). Combining the Hölder inequality for Schatten norms (see, e.g., Dunford & Schwartz [DS63, item (c) in Lemma XI.9.14]) and (27) hence establishes that \(B^*T_0A \in L_1(U)\) and
\[
\|B^*T_0A\|_{L_1(U)} \leq \|B\|_{L_{1+r}(U, H)} \|T\|_{L^2(H, \mathbb{R})} \|A\|_{L_{1+r}(U, H)}.
\]
(29)

Furthermore, note that for all sequences \((x_k)_{k \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}} \subseteq U\) with \(\sum_{k \in \mathbb{N}} \|x_k\|_U \|y_k\|_U < \infty\) and \(B^*T_0A = \sum_{k \in \mathbb{N}} \langle \cdot, x_k \rangle_U y_k\) it holds that
\[
\sum_{u \in U} |\langle B^*T_0A, u \rangle_U| = \sum_{u \in U} \left| \left\langle \sum_{k \in \mathbb{N}} \langle u, x_k \rangle_U y_k, u \right\rangle \right| = \sum_{u \in U} \left| \sum_{k \in \mathbb{N}} \langle u, x_k \rangle_U \langle y_k, u \rangle_U \right| \leq \sum_{k \in \mathbb{N}} \sum_{u \in U} |\langle u, x_k \rangle_U \langle y_k, u \rangle_U| \leq \sum_{k \in \mathbb{N}} \|x_k\|_U \|y_k\|_U.
\]
(30)
The fact that
\[
\|B^*T_0A\|_{L_1(U)} = \inf \left\{ \sum_{k \in \mathbb{N}} \|x_k\|_U \|y_k\|_U : \sum_{k \in \mathbb{N}} \|x_k\|_U \|y_k\|_U < \infty \text{ and } B^*T_0A = \sum_{k \in \mathbb{N}} \langle \cdot, x_k \rangle_U y_k \right\}
\]
(31)
therefore implies that \(\sum_{u \in U} |\langle B^*T_0A, u \rangle_U| \leq \|B^*T_0A\|_{L_1(U)}\). Combining this, (28), and (29) establishes (26). The proof of Lemma 2.5 is thus completed.

3 Weak convergence rates for temporal numerical approximations of semilinear stochastic wave equations

3.1 Setting

In Setting 3.1 below we present a framework for our convergence analysis that is frequently used throughout this section. For an easier understanding we provide some informal comments in advance: To begin with, note that the spectral structure of the operator \(A\) is specified by its eigenbasis \(\mathbb{H}\) and eigenvalues \(\lambda_h, h \in \mathbb{H}\). Our standard choice for \(A\) is the Dirichlet Laplacian on the \(L^2\) space of equivalence classes of square-integrable functions on the unit interval. The interpolation spaces \(H_r, r \in \mathbb{R}\), are employed to measure smoothness in terms of powers of \(-A\), while the product spaces \(H_r, r \in \mathbb{R}\), and the operator \(A\) facilitate the interpretation of the wave equation as a system of first order equations in time. The operator \(A\) is introduced to further simplify the analysis as it allows to interpret the spaces \(H_r, r \in \mathbb{R}\), as interpolation spaces associated to \(A\); compare Lemma 3.2 in Subsection 3.2 below. Further note that the function \(\varphi\) plays the role of the test function occurring in the weak
approximation error, whereas $\xi$, $F$, and $B$ constitute the initial condition, the semilinearity in the drift term, and the diffusion coefficient of the abstract SPDE to be approximated. The parameters $\gamma$, $\beta$, and $\rho$ determine the spatial regularity of the solution; compare Lemma 3.3 in Subsection 3.3 below. Depending on the choice of the step size parameter $h$ and the subset $I$ of the eigenbasis $\mathbb{H}$, the process defined by (35) represents either an approximation of the mild solution or the mild solution itself.

**Setting 3.1.** Assume Setting [1,2], let $r \in (0, \infty)$, $\beta \in (\gamma/2, \gamma] \cap [\gamma - 1/2, \gamma]$, $\rho \in [0, 2(\gamma - \beta)]$, let $(H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H)$ be a non-trivial separable $\mathbb{R}$-Hilbert space, let $\mathbb{H} \subseteq H$ be an orthonormal basis of $H$, let $\lambda : \mathbb{H} \to \mathbb{R}$ satisfy sup$_{h \in \mathbb{H}} \lambda_h < 0$ and sup$_{h \in \mathbb{H}} |\lambda_h|^{-\beta} < \infty$, let $A : D(A) \subseteq H \to H$ satisfy $D(A) = \{ v \in H : \sum_{h \in \mathbb{H}} |\lambda_h| h \langle v, h \rangle_H^2 < \infty \}$ and $\forall v \in D(A) : Av = \sum_{h \in \mathbb{H}} \lambda_h \langle h, v \rangle_H h$, let $(\mathbb{H}_r, \langle \cdot, \cdot \rangle_{\mathbb{H}_r}, \| \cdot \|_{\mathbb{H}_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$, for every $r \in \mathbb{R}$ let $(\mathbb{H}_r, \langle \cdot, \cdot \rangle_{\mathbb{H}_r}, \| \cdot \|_{\mathbb{H}_r})$ be the $\mathbb{R}$-Hilbert space which satisfies $(\mathbb{H}_r, \langle \cdot, \cdot \rangle_{\mathbb{H}_r}, \| \cdot \|_{\mathbb{H}_r}) = (H^r \times H^{r-1/2}, \langle \cdot, \cdot \rangle_{H^r \times H^{r-1/2}}, \| \cdot \|_{H^r \times H^{r-1/2}})$, let $A : D(A) \subseteq H^r \to H^r$ satisfy $D(A) = H_1$ and $\forall (v, w) \in H^r : \Lambda(v, w) = (w, Av)$, let $\Lambda : D(\Lambda) \subseteq H^r \to H^r$ satisfy $D(\Lambda) = H_1$ and $\forall (v, w) \in H^r : \Lambda(v, w) = (w, Av)$, let $\varphi \in C_0^1(\mathbb{H}_0, \mathbb{R})$, $\xi \in L^2(\mathbb{R}^2; H^r_{\max}(\rho, \gamma - \beta))$ satisfy $E[\| \xi \|_{H^r_0}^2] < \infty$, let $F \in \text{Lip}(\mathbb{H}_{\rho, \gamma - \beta}, \mathbb{H}_0)$, $B \in C_0^1(\mathbb{H}_0, L_2(U, \mathbb{H}_0))$ satisfy $F|_{\mathbb{H}_0} \in C^1(\mathbb{H}_0, \mathbb{H}_0)$, $F|_{\mathbb{H}_0} \in \text{Lip}((\mathbb{H}_0, H_{\rho, \gamma - \beta}))$, and $B|_{\mathbb{H}_0} \in \text{Lip}(\mathbb{H}_{\rho, \rho} \times L_2(U, \mathbb{H}_0))$, let $m \in [1, \infty)$, $\iota, I \in [0, \infty)$, $\mu : \Omega \to (\mathbb{R} \setminus \{0\})$ satisfy for every $v, w \in \mathbb{H}_{\gamma - \beta}$ that

$$\max \{ \| F|_{\mathbb{H}_0}^2 C^1_{\mathbb{H}_0}(\| h \|_{\mathbb{H}_0}), \| B|_{\mathbb{H}_0}^2 C^1_{\mathbb{H}_0}(\| U \|_{L_2(U, \mathbb{H}_0)}) \} \leq m, \quad (32)$$

$$\sum_{u \in \mathbb{U}} |\mu_u|^2 \| B|_{\mathbb{H}_0}^2 \| U \|_{\mathbb{H}_{\gamma - \beta}}^2 \leq \delta^2 \max \{ 1, \| v \|_{\mathbb{H}_{\gamma - \beta}}^2 \}, \quad (33)$$

and

$$\sum_{u \in \mathbb{U}} \| (B|_{\mathbb{H}_0} - B|_{\mathbb{H}_0})(w) \|_{\mathbb{H}_0}^2 \leq \delta^2 \| v - w \|_{\mathbb{H}_{\gamma - \beta}}^2, \quad (34)$$

for every $I \subseteq \mathbb{H}$ let $P_I : \cup_{r \in \mathbb{R}} H_r \to \cup_{r \in \mathbb{R}} H_r$ and $P_I : \cup_{r \in \mathbb{R}} H_r \to \cup_{r \in \mathbb{R}} H_r$ satisfy for every $r \in \mathbb{R}$, $v \in H_r$, $w \in H_{r-1/2}$ that $P_I(v) = \sum_{h \in \mathbb{H}} |\lambda_h|^{-r} h \langle v, h \rangle_H^2 \| h \|_{\mathbb{H}_r}^{-r} h$ and $P_I(v, w) = (P_I v, P_I w)$, let $\lambda_h : [0, \infty) \to \mathbb{R}$, $h \in [0, \infty)$, satisfy for every $h \in [0, \infty)$, $x \in [0, \infty)$ that $|x|_h = \max \{ 0, 2h, 3h, \ldots \} \cap [0, x]$ and $|x|_0 = x$, and for every $I \subseteq \mathbb{H}$, $h \in [0, \infty]$ let $Y^I_{\rho, \rho} : [0, T] \times \Omega \to P_I(\mathbb{H}_0)$ be an $(\mathbb{F}^t)_{t \in [0, T]}$-predictable stochastic process which satisfies for every $t \in [0, T]$ that $\sup_{s \in [0, T]} E[|Y^I_{\rho, \rho}|_{\mathbb{H}_0}^2] < \infty$ and

$$[Y^I_{\rho, \rho}]_{\mathbb{P}^\mathbb{H}, P_I(\mathbb{H}_0)} = \left[ e^{\Lambda P_I |\xi|^2} \mathbb{P} \mathbb{E}(P_I(\mathbb{H}_0)) + \int_0^t e^{(t-s)} A P_I Y^I_{\rho, \rho}(s) ds \right] + \int_0^t e^{(t-s)} A P_I B Y^I_{\rho, \rho}(s) dW_s. \quad (35)$$

Note that the family of interpolation spaces $H_r$, $r \in \mathbb{R}$, introduced in Setting 3.1 satisfies for every $r \in [0, \infty)$, $v \in H_r$ that $H_r = D((-A)^r)$ and $\| v \|_{H_r} = \| (-A)^r v \|_H$. Furthermore, observe that Setting 3.1 ensures that for every $I \subseteq \mathbb{H}$, $t \in [0, T]$ it holds that $\mathbb{P} \left[ \int_0^t \| e^{(t-s)} A P_I F(Y^I_{\rho, \rho}(s)) \|_{\mathbb{H}_0}^2 + \| e^{(t-s)} A P_I B Y^I_{\rho, \rho}(s) \|_{L_2(U, \mathbb{H}_0)}^2 \right] ds < \infty = 1$ and

$$[Y^I_{\rho, \rho}]_{\mathbb{P}^\mathbb{H}, P_I(\mathbb{H}_0)} = \left[ e^{\Lambda P_I |\xi|^2} \mathbb{P} \mathbb{E}(P_I(\mathbb{H}_0)) + \int_0^t e^{(t-s)} A P_I Y^I_{\rho, \rho}(s) ds \right] + \int_0^t e^{(t-s)} A P_I B Y^I_{\rho, \rho}(s) dW_s; \quad (36)$$

cf., for example, Theorem 2.1 above, Lemma 3.3 below, and Jacobe de Naurois et al. [JdNJW21] Remark 3.1 for sufficient conditions which ensure the existence of such a process.
3.2 Basic results for the linear wave equation

The statement and the proof of the next result, Lemma 3.2 below, can be found in, e.g., Jacobe de Naurois et al. [JdNJW15, Lemma 2.4].

**Lemma 3.2.** Assume Setting 3.1. Then the \( \mathbb{R} \)-Hilbert spaces \( (H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|\), \( r \in \mathbb{R} \), are a family of interpolation spaces associated to \( \Lambda \).

The next result, Lemma 3.3 below, can be found in, e.g., Lindgren [Lin12, Section 5.3]. A proof of Lemma 3.3 can be found in, e.g., Lindgren [Lin12, Section 5.3].

**Lemma 3.3.** Assume Setting 3.1 and let \( S : [0, \infty) \rightarrow L(H_0) \) be the function which satisfies for every \( t \in [0, \infty) \), \( (v, w) \in H_0 \) that

\[
S_t(v, w) = \left( \cos(t(-A)^{1/2})v + (-A)^{-1/2} \sin(t(-A)^{1/2})w, \quad (-A)^{1/2} \sin(t(-A)^{1/2})v + \cos(t(-A)^{1/2})w \right).
\] (37)

Then

(i) it holds that \( S : [0, \infty) \rightarrow L(H_0) \) is a strongly continuous semigroup of bounded linear operators on \( H_0 \) and

(ii) it holds that \( A : D(A) \subseteq H_0 \rightarrow H_0 \) is the generator of \( S \).

The statement and the proof of the next result, Lemma 3.4 below, can be found in, e.g., Jacobe de Naurois et al. [JdNJW15, Lemma 2.6].

**Lemma 3.4.** Assume Setting 3.1. Then \( \sup_{t \in [0, \infty)} \|e^{tA}\|_{L(H_0)} = 1 \).

The next result, Lemma 3.5 below, provides another useful elementary estimate for the semigroup \( (e^{tA})_{t \in [0, \infty)} \) generated by the operator \( A : D(A) \subseteq H_0 \rightarrow H_0 \) from Setting 3.1 cf., e.g., Kovacs et al. [KLL13, Lemma 4.4] for a similar result.

**Lemma 3.5.** Assume Setting 3.1 and let \( \alpha \in [0, 1], t \in (0, \infty) \). Then

\[
et^{-\alpha}\|A^{-\alpha}(\text{id}_{H_0} - e^{tA})\|_{L(H_0)} \leq \sqrt{2} \sup_{s \in (0, \infty)} (s^{-\alpha}|1 - e^{is}|) \leq 2^{3/2}. \] (38)

**Proof of Lemma 3.5.** First, observe that for every \( s \in (0, \infty) \) it holds that

\[
s^{-\alpha}|1 - e^{is}| = s^{-\alpha}[|1 - \cos(s)|^2 + |\sin(s)|^2]^{1/2}
\]
\[
= s^{-\alpha}[1 - 2 \cos(s) + |\cos(s)|^2 + |\sin(s)|^2]^{1/2}
\]
\[
= s^{-\alpha}[2 - 2 \cos(s)]^{1/2} = \sqrt{2}(s^{-\alpha}|1 - \cos(s)|^{1/2}). \] (39)

In addition, note that Lemma 3.3 implies that for every \( (v, w) \in H_0 \) it holds that

\[
A^{-\alpha}(\text{id}_{H_0} - e^{tA})(v, w)
\]
\[
= A^{-\alpha} \left( (\text{id}_{H} - \cos(t(-A)^{1/2}))v - (-A)^{-1/2} \sin(t(-A)^{1/2})w, \quad (-A)^{1/2} \sin(t(-A)^{1/2})v + (\text{id}_{H_{-1/2}} - \cos(t(-A)^{1/2}))w \right)
\]
\[
= \left( (-A)^{-\alpha/2}(\text{id}_{H} - \cos(t(-A)^{1/2}))v - (-A)^{(-1 + \alpha)/2} \sin(t(-A)^{1/2})w, \quad (-A)^{1 - \alpha/2} \sin(t(-A)^{1/2})v + (-A)^{-\alpha/2}(\text{id}_{H_{-1/2}} - \cos(t(-A)^{1/2}))w \right). \] (40)
Hence, we obtain that for every \((v, w) \in H_0\) it holds that

\[
t^{-\alpha} \|A^{-\alpha}(id_{H_0} - e^{tA})(v, w)\|_{H_0} \\
= t^{-\alpha} \left[ \left\|(-A)^{-\alpha/2}(id_{H} - \cos(t(-A)^{1/2})) v - (-A)^{-(1+\alpha)/2} \sin(t(-A)^{1/2}) w \right\|^2_{H} \right. \\
+ \left. \left\|(-A)^{(1-\alpha)/2} \sin(t(-A)^{1/2}) v + (-A)^{-\alpha/2}(id_{H_{-1/2}} - \cos(t(-A)^{1/2})) w \right\|^2_{H_{-1/2}} \right]^{1/2} \\
= \left[ \sum_{h \in H} t^{-2\alpha} |\lambda_h|^{-\alpha} \left( (1 - \cos(t|\lambda_h|^{1/2})) \langle h, v \rangle_H + \sin(t|\lambda_h|^{1/2}) (|\lambda_h|^{1/2} h, w)_{H_{-1/2}} \right)^2 \right. \\
+ \left. \left( \sum_{h \in H} t^{-2\alpha} |\lambda_h|^{-\alpha} \left( (1 - \cos(t|\lambda_h|^{1/2})) \langle h, v \rangle_H + (1 - \cos(t|\lambda_h|^{1/2})) (|\lambda_h|^{1/2} h, w)_{H_{-1/2}} \right)^2 \right]^{1/2} \right]^{1/2} \\
\leq \sqrt{2} \left[ \sum_{h \in H} t^{-2\alpha} |\lambda_h|^{-\alpha} \left( (1 - \cos(t|\lambda_h|^{1/2}))^2 + |\sin(t|\lambda_h|^{1/2})|^2 \right) \langle h, v \rangle_H^2 \right. \\
+ \left. \left( \sum_{h \in H} t^{-2\alpha} |\lambda_h|^{-\alpha} \left( (1 - \cos(t|\lambda_h|^{1/2}))^2 + |\sin(t|\lambda_h|^{1/2})|^2 \right) \langle |\lambda_h|^{1/2} h, w \rangle_{H_{-1/2}}^2 \right]^{1/2} \right]^{1/2} \\
= \sqrt{2} \left[ \sup_{h \in H} \left\{ t^{-2\alpha} |\lambda_h|^{-\alpha} \left( (1 - \cos(t|\lambda_h|^{1/2}))^2 + |\sin(t|\lambda_h|^{1/2})|^2 \right) \right\} \right]^{1/2} \\
\leq \sqrt{2} \left[ \sup_{h \in H} \left\{ t^{-2\alpha} |\lambda_h|^{-\alpha} \left( (1 - \cos(s|\lambda_h|^{1/2}))^2 + |\sin(s|\lambda_h|^{1/2})|^2 \right) \right\} \right]^{1/2} \| (v, w) \|_{H_0}. \\
\]

Combining this and the fact that for every \(s \in \mathbb{R}\) it holds that \(|1 - e^{is}|^2 = 2|1 - \cos(s)|\) demonstrates that for every \((v, w) \in H_0\) it holds that

\[
t^{-\alpha} \|A^{-\alpha}(id_{H_0} - e^{tA})(v, w)\|_{H_0} \\
\leq \sqrt{2} \left[ \sup_{s \in (0, \infty)} \left\{ s^{-\alpha} \left( |1 - \cos(s)|^2 + |\sin(s)|^2 \right)^{1/2} \right\} \right] \| (v, w) \|_{H_0} \quad (43) \\
= \sqrt{2} \left[ \sup_{s \in (0, \infty)} \left( s^{-\alpha} |1 - e^{is}| \right) \| (v, w) \|_{H_0} \right] = 2 \left[ \sup_{s \in (0, \infty)} \left( s^{-\alpha} |1 - \cos(s)|^{1/2} \right) \right] \| (v, w) \|_{H_0}. \\
\]

This and Lemma 3.3 establish (43). The proof of Lemma 3.5 is thus completed. \(\square\)

We end this subsection with a further auxiliary result for the semigroup \((e^{tA})_{t \in [0, \infty)}\).

**Lemma 3.6.** Assume Setting 3.1 and let \(I \subseteq H\) be finite. Then it holds for every \(t \in [0, \infty)\) that \(AP_{I|H_0} \in L(H_0)\) and

\[
e^{t(A_{P_{I|H_0}})} = e^{tA} P_{I|H_0} + P_{H \setminus I|H_0}. \quad (44)\]

**Proof of Lemma 3.6.** First, note that the finiteness of \(I \subseteq H\) ensures that for every \(x \in H_0\) it holds that \(P_{I}x \in H_1 = D(A)\) and \(AP_{I|H_0} \in L(H_0)\). This and Lemma 3.3 imply that for
every \( s, t \in [0, \infty) \), \( x \in H_0 \) it holds that
\[
(e^{sA}P_I + P_{H\setminus I})(e^{tA}P_I + P_{H\setminus I})x = e^{(s+t)A}P_I x + e^{sA}P_I P_{H\setminus I} x + e^{tA}P_I P_{H\setminus I} x + (P_{H\setminus I})^2 x \tag{45}
\]
and
\[
\lim_{h \to 0} \sup_{t \geq 0} \left\| \frac{1}{t} \left( (e^{hA}P_I + P_{H\setminus I})x - x \right) - A P_I x \right\|_{H_0} = \lim_{h \to 0} \sup_{t \geq 0} \left\| \frac{1}{t} \left( (e^{hA}P_I + P_{H\setminus I})x - (P_I + P_{H\setminus I})x \right) - A P_I x \right\|_{H_0} = \lim_{h \to 0} \sup_{t \geq 0} \left\| \frac{1}{t} (e^{hA}P_I x - P_I x) - A P_I x \right\|_{H_0} = 0. \tag{46}
\]
Moreover, observe that for every \( x \in H_0 \) it holds that \((e^{0A}P_I + P_{H\setminus I})x = x\). Combining this, \((45)\), and \((46)\) demonstrates that \((e^{tA}P_I|_{H_0} + P_{H\setminus I}|_{H_0})_{t \geq 0} \) is a strongly continuous semigroup of bounded linear operators on \( H_0 \) with generator \( AP_I|_{H_0} \in L(H_0) \). Hence, we obtain that \((e^{t(\Lambda P_I|_{H_0}))}_{t \geq 0} = (e^{tA}P_I|_{H_0} + P_{H\setminus I}|_{H_0})_{t \geq 0} \). The proof of Lemma 3.6 is thus completed. \(\square\)

3.3 A priori bounds for the numerical approximations

**Lemma 3.7.** Assume Setting 3.1. Then

(i) it holds for every \( h, t \in [0, T] \), \( I \subseteq \mathbb{H} \) that \( P(Y^{h,I}_t \in H_{\max(\rho, \gamma - \beta)}) = 1 \),

(ii) it holds that \( \sup_{h, t \in [0, T], I \subseteq \mathbb{H}} E \left[ \| Y^{h,I}_t \|^6_{H_0} \right] < \infty \), and

(iii) it holds that \( \sup_{h, t \in [0, T], I \subseteq \mathbb{H}} E \left[ \| Y^{h,I}_t \|^2_{H_{\max(\rho, \gamma - \beta)}} \right] < \infty \).

**Proof of Lemma 3.7.** First, note that
\[
\left\| F \right\|_{H_p, H_{\max(\rho, \gamma - \beta)}} \leq \left\| A^{\max(\rho, \gamma - \beta) - 2(\gamma - \beta)} \right\|_{L(H_0)} \left\| F \right\|_{H_p, H_{\gamma(\gamma - \beta)}} < \infty. \tag{47}
\]
The fact that \( E[\| \xi \|^6_{H_0} + \| \xi \|^2_{H_p}] < \infty \) and Theorem 2.1 (with \( H \in \{ H_0, H_p \} \), \( s \in \{ e^{\Lambda A}, e^{\Lambda A|_{H_p}} \} \), \( p \in \{ 6, 2 \} \), \( F \in \{ P_I F|_{H_p}, P_I F|_{H_p} \} \), \( B \in \{ P_I B, P_I B|_{H_p} \} \), \( \xi = \xi \) for \( t \in [0, \infty) \), \( I \subseteq \mathbb{H} \) in the notation of Theorem 2.1) hence imply that there exist up to modifications unique \((F^t)_{t \in [0, T]} \)-predictable stochastic processes \( X^I \colon [0, T] \times \Omega \to H_p \), \( I \subseteq \mathbb{H} \), which satisfy for every \( I \subseteq \mathbb{H} \), \( t \in [0, T] \) that \( \sup_{s \in [0, T]} E \left[ \| X^I_s \|^6_{H_0} + \| X^I_s \|^2_{H_p} \right] < \infty \) and
\[
[X^I_t]_{P, B(I, p)} = [e^{\Lambda A}P_I \xi]_{P, B(I, p)} + \int_0^t e^{(t-s)\Lambda} P_I F(X^I_s) \, ds + \int_0^t e^{(t-s)\Lambda} P_I B(X^I_s) \, dW_s. \tag{48}
\]
Combining this, \((35)\), and Theorem 2.1 ensures that for every \( I \subseteq \mathbb{H} \), \( t \in [0, T] \) it holds that \( P(X^I_t = Y^{0,I}_t) = 1 \). The fact that for every \( I \subseteq \mathbb{H} \) it holds that \( \sup_{t \in [0, T]} E \left[ \| Y^{0,I}_t \|^6_{H_0} + \| Y^{0,I}_t \|^2_{H_p} \right] < \infty \) hence implies that that for every \( I \subseteq \mathbb{H} \) it holds that
\[
\sup_{t \in [0, T]} E \left[ \| Y^{0,I}_t \|^6_{H_0} + \| Y^{0,I}_t \|^2_{H_p} \right] < \infty. \tag{49}
\]
Next note that Da Prato & Zabczyk [DPZ92, Lemma 7.7], item (ii) in Lemma 2.3, Lemma 3.4, and (35) prove that for every $\delta, \theta \in [0, \infty)$, $p \in [2, \infty)$, $h, t \in [0, T]$, $I \subseteq \mathbb{H}$ it holds that

$$
\left(\mathbb{E}\left[\|Y_{t}^{h,I}\|_{H_{h}}^{p}\right]\right)^{1/p} \\
\leq \left(\mathbb{E}\left[\left|e^{tA}\right|_{P}^{p}\right]\right)^{1/p} + \int_{0}^{t} \left(\mathbb{E}\left[\left|e^{(t-s)A}\right|_{P}^{p}\mathbb{P}_{I}F\left(Y_{s}^{h,I}\right)\right]\right)^{1/p} ds \\
+ \sqrt{p(p-1)/2} \left(\int_{0}^{t} \left(\mathbb{E}\left[\left|e^{(t-s)A}\right|_{P}^{p}\mathbb{P}_{I}B\left(Y_{s}^{h,I}\right)\right]\right)^{2/p} ds\right)^{1/2}
$$

$$
\leq \left(\mathbb{E}\left[\left|\xi\right|_{H_{h}}^{p}\right]^{1/p} + \mathbb{P}^{P_{I}}_{\xi \in \mathcal{H}_{h}} \int_{0}^{t} \left(1 + \left(\mathbb{E}\left[\left|Y_{s}^{h,I}\right|_{H_{h}}^{p}\right]\right)^{1/p}\right) ds\right)^{1/2} \\
+ \sqrt{p(p-1)/2} \left(\int_{0}^{t} \left(1 + \left(\mathbb{E}\left[\left|Y_{s}^{h,I}\right|_{H_{h}}^{p}\right]\right)^{1/p}\right)^{2/p} ds\right)^{1/2}
$$

(50)

Moreover, note that

$$
\|B\|_{H_{h}} \|\xi\|_{\mathcal{L}(P_{\xi \in \mathcal{H}_{h}}; L^{2}(U, \mathcal{H}_{h}))} \\
\leq \max\{\|B\|_{H_{h}} \|\xi\|_{\mathcal{L}(P_{\xi \in \mathcal{H}_{h}}; L^{2}(U, \mathcal{H}_{h}))}, \|A^{-\beta}\|_{L^{2}(U, \mathcal{H}_{h})} \|B\|_{H_{h}} \|\xi\|_{\mathcal{L}(P_{\xi \in \mathcal{H}_{h}}; L^{2}(U, \mathcal{H}_{h}))}\} < \infty.
$$

(51)

This, (17), (49), and (50) (with $p = 2$, $\delta = \rho$, $\theta = \max\{\rho, \gamma - \beta\}$ in the notation of (50)) ensure that for every $I \subseteq \mathbb{H}$ it holds that

$$
\sup_{t \in [0, T]} \mathbb{E}\left[\left|\left|Y_{t}^{h,I}\right|_{H_{h}}^{p}\right|_{\mathcal{H}_{h}(\rho, \gamma - \beta)}\right] < \infty.
$$

(52)

In addition, note that (50) (with $p = 6$, $\delta = 0$, $\theta = 0$ in the notation of (50)) implies that for every $h \in (0, T], I \subseteq \mathbb{H}$, $k \in \mathbb{N}_{0} \cap [0, T/h]$ it holds that

$$
\sup_{t \in (kh, (k+1)h) \cap [0, T]} \left(\mathbb{E}\left[\left|\left|Y_{t}^{h,I}\right|_{H_{h}}^{6}\right|_{\mathcal{H}_{h}}\right]\right)^{1/6} \\
\leq \left(\mathbb{E}\left[\left|\left|\xi\right|_{H_{h}}^{6}\right|_{\mathcal{H}_{h}}\right]\right)^{1/6} + \sqrt{(k+1)h} \left(1 + \sup_{j \in \{0, 1, \ldots, k\}} \left(\mathbb{E}\left[\left|\left|Y_{j}^{h,I}\right|_{H_{h}}^{6}\right|_{\mathcal{H}_{h}}\right]\right)^{1/6}\right)
$$

(53)

$$
\cdot \left(\sqrt{(k+1)h} \|F\|_{\mathcal{H}_{h}} \|\xi\|_{\mathcal{L}(P_{\xi \in \mathcal{H}_{h}}; L^{2}(U, \mathcal{H}_{h}))} + \sqrt{15} \|B\|_{\mathcal{H}_{h}} \|\xi\|_{\mathcal{L}(P_{\xi \in \mathcal{H}_{h}}; L^{2}(U, \mathcal{H}_{h}))}\right).
$$

Hence, we obtain that for every $h \in (0, T], I \subseteq \mathbb{H}$ it holds that

$$
\sup_{t \in [0, T]} \mathbb{E}\left[\left|\left|Y_{t}^{h,I}\right|_{H_{h}}^{6}\right|_{\mathcal{H}_{h}}\right] < \infty.
$$

(54)

Moreover, observe that (50) (with $p = 2$, $\delta = \rho$, $\theta = \rho$ in the notation of (50)) implies that for every $h \in (0, T], I \subseteq \mathbb{H}$, $k \in \mathbb{N}_{0} \cap [0, T/h]$ it holds that

$$
\sup_{t \in (kh, (k+1)h) \cap [0, T]} \left(\mathbb{E}\left[\left|\left|Y_{t}^{h,I}\right|_{H_{h}}^{2}\right|_{\mathcal{H}_{h}}\right]\right)^{1/2} \\
\leq \left(\mathbb{E}\left[\left|\left|\xi\right|_{H_{h}}^{2}\right|_{\mathcal{H}_{h}}\right]\right)^{1/2} + \sqrt{(k+1)h} \left(1 + \sup_{j \in \{0, 1, \ldots, k\}} \left(\mathbb{E}\left[\left|\left|Y_{j}^{h,I}\right|_{H_{h}}^{2}\right|_{\mathcal{H}_{h}}\right]\right)^{1/2}\right)
$$

(55)

$$
\cdot \left(\sqrt{(k+1)h} \|F\|_{\mathcal{H}_{h}} \|\xi\|_{\mathcal{L}(P_{\xi \in \mathcal{H}_{h}}; L^{2}(U, \mathcal{H}_{h}))} + \|B\|_{\mathcal{H}_{h}} \|\xi\|_{\mathcal{L}(P_{\xi \in \mathcal{H}_{h}}; L^{2}(U, \mathcal{H}_{h}))}\right).
$$
Hence, we obtain that for every $h \in (0, T)$, $I \subseteq \mathbb{H}$ it holds that $\sup_{t \in [0,T]} \mathbb{E}[||Y^h, I_t^J||^2_{H^p}] < \infty$. Combining this, (47), (50) (with $p = 2$, $\delta = \rho$, $\theta = \max\{\rho, \gamma - \beta\}$ in the notation of (50)), and (51) implies that for every $h \in (0, T)$, $I \subseteq \mathbb{H}$ it holds that

$$\sup_{t \in [0,T]} \mathbb{E}[||Y^h, I_t^J||^2_{H^{\max\{\rho, \gamma - \beta\}}} < \infty. \quad (56)$$

This and (52) establish item (i). Next note that (50) ensures that for every $p \in [2, \infty)$, $\delta \in [0, \infty)$, $I \subseteq \mathbb{H}$, $h,t \in [0,T]$ it holds that

$$\sup_{s \in [0,t]} \left( \mathbb{E}[||Y^h, I_s^J||^p_{H^p}] \right)^{1/p} \leq 2\left( \left( \mathbb{E}[||\xi||^p_{H^p}] \right)^{1/p} + T \left| |F|_{H^1} \right|_{\text{Lip}(H^1, H^1)} + \sqrt{p(p-1)T} \left| |B|_{H^1} \right|_{\text{Lip}(H^1, L^2(U, H^1))} \right)^2$$

$$+ 2\left( \sqrt{T} \left| |F|_{H^1} \right|_{\text{Lip}(H^1, H^1)} + \sqrt{p(p-1)} \left| |B|_{H^1} \right|_{\text{Lip}(H^1, L^2(U, H^1))} \right)^2$$

$$\cdot \int_0^t \sup_{s \in [0,t]} \left( \mathbb{E}[||Y^h, I_s^J||^p_{H^p}] \right)^{1/p} ds. \quad (57)$$

Gronwall’s inequality, (49), and (54) hence imply that

$$\sup_{h,t \in [0,T], I \subseteq \mathbb{H}} \left( \mathbb{E}[||Y^h, I_t^J||^6_{H^6}] \right)^{1/3} \leq 2\left( \left( \mathbb{E}[||\xi||^6_{H^6}] \right)^{1/3} + T \left| |F|_{H^1} \right|_{\text{Lip}(H^1, H^1)} + \sqrt{15T} \left| |B|_{H^1} \right|_{\text{Lip}(H^1, L^2(U, H^1))} \right)^2 \quad (58)$$

$$\cdot \exp\left( 2T \left( \sqrt{T} \left| |F|_{H^1} \right|_{\text{Lip}(H^1, H^1)} + \sqrt{15} \left| |B|_{H^1} \right|_{\text{Lip}(H^1, L^2(U, H^1))} \right)^2 \right) < \infty.$$ 

This establishes item (ii). In the next step we observe that Gronwall’s inequality, (47), (49), (51), (56), and (57) imply that

$$\sup_{h,t \in [0,T], I \subseteq \mathbb{H}} \mathbb{E}[||Y^h, I_t^J||^2_{H^p}] \leq 2\left( \left( \mathbb{E}[||\xi||^2_{H^1}] \right)^{1/2} + T \left| |F|_{H^1} \right|_{\text{Lip}(H^1, H^1)} + \sqrt{T} \left| |B|_{H^1} \right|_{\text{Lip}(H^1, L^2(U, H^1))} \right)^2$$

$$\cdot \exp\left( 2T \left( \sqrt{T} \left| |F|_{H^1} \right|_{\text{Lip}(H^1, H^1)} + \left| |B|_{H^1} \right|_{\text{Lip}(H^1, L^2(U, H^1))} \right)^2 \right) < \infty. \quad (59)$$

Combining this, (47), (50) (with $p = 2$, $\delta = \rho$, $\theta = \max\{\rho, \gamma - \beta\}$ in the notation of (50)), and (51) establishes item (iii). The proof of Lemma 3.7 is thus completed.

\[\square\]

3.4 Upper bounds for the strong approximation errors

The statement and the proof of the next result, Proposition 3.8 below, are minor modifications of the statement and the proof of Jacobe de Naurois et al. [JdNJW15, Lemma 3.3].

Proposition 3.8. Assume Setting 3.1 and let $h \in [0,T)$, $I, J \subseteq \mathbb{H}$. Then it holds that

$$\sup_{t \in [0,T]} \|Y^h, I_t^J - Y^h, I_t^J\|_{\mathcal{L}^2(P; H^p)} \leq \sqrt{2} \exp\left( T \|P_{I \cap J} F|_{H^p, H^p} \|_{\text{Lip}(H^p, H^p)} + \sqrt{T} \|P_{I \cap J} B\|_{\text{Lip}(H^p, H^p)} \right)^2$$

$$\cdot \left[ \sup_{t \in [0,T]} \|P_{I \setminus J} Y^h, I_t^J - P_{J \setminus I} Y^h, I_t^J\|_{\mathcal{L}^2(P; H^p)} \right]. \quad (60)$$
Proof of Proposition 3.8. Note that item [3] in Lemma 3.7 implies that

\[
\sup_{s \in [0,T]} \|Y_s^{h,I} - Y_s^{\ast,I}\|_{L^2(P; H_0)} \leq \sup_{s \in [0,T]} \|Y_s^{h,I}\|_{L^2(P; H_0)} + \sup_{s \in [0,T]} \|Y_s^{\ast,I}\|_{L^2(P; H_0)} < \infty. \tag{61}
\]

Moreover, observe that Lemma 3.4 ensures that for every \( t \in (0, T) \), \( s \in (0, t) \) it holds that

\[
\|e^{(t-|s|)h}A P_{I \cap J}(F(Y_{|s|;h}^{h,I}) - F(Y_{|s|;h}^{\ast,I}))\|_{L^2(P; H_0)} \leq \|P_{I \cap J} F|_{\text{Lip}(H_0, H_0)} \left[ \sup_{u \in [0,s]} \|Y_u^{h,I} - Y_u^{\ast,I}\|_{L^2(P; H_0)} \right] \tag{62}
\]

and

\[
\|e^{(t-|s|)h}A P_{I \cap J}(B(Y_{|s|;h}^{h,I}) - B(Y_{|s|;h}^{\ast,I}))\|_{L^2(P; L^2(U, H_X))} \leq \|P_{I \cap J} B|_{\text{Lip}(H_0, L^2(U, H_X))} \left[ \sup_{u \in [0,s]} \|Y_u^{h,I} - Y_u^{\ast,I}\|_{L^2(P; H_0)} \right] \tag{63}
\]

Combining (32), (35), (61), and Jentzen & Kurniawan [JK21] Corollary 3.1 (with \( H = H_0 \), \( p = 2 \), \( \vartheta = 0 \), \( y = \|P_{I \cap J} F\|_{\text{Lip}(H_0, H_0)} \), \( z = \|P_{I \cap J} B\|_{\text{Lip}(H_0, L^2(U, H_X))} \), \( X_t = Y_t^{h,I}, X_t = Y_t^{\ast,I} \), \( Y_t^{h,I}, Y_t^{\ast,I} = e^{(t-|s|)h}A P_{I \cap J} F(Y_{|s|;h}^{h,I}) \), \( Z_t^{h,I} = e^{(t-|s|)h}A P_{I \cap J} B(Y_{|s|;h}^{h,I}) \) for \( s \in [0, t] \), \( t \in [0, T] \) in the notation of [JK21] Corollary 3.1) therefore establishes that

\[
\sup_{t \in [0,T]} \|Y_t^{h,I} - Y_t^{\ast,I}\|_{L^2(P; H_0)} \leq \sqrt{2} \exp \left( (T \|P_{I \cap J} F|_{\text{Lip}(H_0, H_0)} + \sqrt{T} \|P_{I \cap J} B\|_{\text{Lip}(H_0, L^2(U, H_X))})^2 \right)
\]

\[
\times \left[ \sup_{t \in [0,t]} \|Y_t^{h,I}\|_{P, B(H_0)} - \left( \int_0^t e^{(t-|s|)h}A P_{I \cap J} F(Y_{|s|;h}^{h,I}) \, ds \right) + \int_0^t e^{(t-|s|)h}A P_{I \cap J} B(Y_{|s|;h}^{h,I}) \, dW_s \right] \left( \int_0^t e^{(t-|s|)h}A P_{I \cap J} F(Y_{|s|;h}^{\ast,I}) \, ds \right) + \int_0^t e^{(t-|s|)h}A P_{I \cap J} B(Y_{|s|;h}^{\ast,I}) \, dW_s \right] \left( \int_0^t e^{(t-|s|)h}A P_{I \cap J} F(Y_{|s|;h}^{h,I}) \, ds \right) + \int_0^t e^{(t-|s|)h}A P_{I \cap J} B(Y_{|s|;h}^{h,I}) \, dW_s \right]
\]

\[
= \sqrt{2} \exp \left( (T \|P_{I \cap J} F|_{\text{Lip}(H_0, H_0)} + \sqrt{T} \|P_{I \cap J} B\|_{\text{Lip}(H_0, L^2(U, H_X))})^2 \right)
\]

\[
\times \left[ \sup_{t \in [0,t]} \|Y_t^{h,I}\|_{P, B(H_0)} - \left( \int_0^t e^{(t-|s|)h}A P_{I \cap J} F(Y_{|s|;h}^{h,I}) \, ds \right) + \int_0^t e^{(t-|s|)h}A P_{I \cap J} B(Y_{|s|;h}^{h,I}) \, dW_s \right] \left( \int_0^t e^{(t-|s|)h}A P_{I \cap J} F(Y_{|s|;h}^{\ast,I}) \, ds \right) + \int_0^t e^{(t-|s|)h}A P_{I \cap J} B(Y_{|s|;h}^{\ast,I}) \, dW_s \right]
\]

\[
= \sqrt{2} \exp \left( (T \|P_{I \cap J} F|_{\text{Lip}(H_0, H_0)} + \sqrt{T} \|P_{I \cap J} B\|_{\text{Lip}(H_0, L^2(U, H_X))})^2 \right)
\]

\[
\times \left[ \sup_{t \in [0,t]} \|Y_t^{h,I} - Y_t^{\ast,I}\|_{L^2(P; H_0)} \right].
\]

The proof of Proposition 3.8 is thus completed.
The proof of the next result, Corollary 3.9 below, is a minor modification of the third step in the proof of Jacobe de Naurois et al. [JdNMW21 Theorem 3.6].

**Corollary 3.9.** Assume Setting 3.1, let \( h \in [0, T] \), and let \( I_n \subseteq \mathbb{H} \), \( n \in \mathbb{N} \), satisfy that \( \cup_{n \in \mathbb{N}} I_n = \mathbb{H} \) and \( \forall n \in \mathbb{N} : I_n \subseteq I_{n+1} \). Then

\[
\limsup_{n \to \infty} \sup_{t \in [0,T]} \mathbb{E} \left [ \| Y_{t}^{h,I_n} - Y_{t}^{h,I_n} \|_{H_0}^2 \right ] = 0. \tag{65}
\]

**Proof of Corollary 3.9.** Observe that Proposition 3.8, (35), Lemma 3.4, Minkowski’s integral inequality, and Itô’s isometry imply that for every \( n \in \mathbb{N} \) it holds that

\[
\sup_{t \in [0,T]} \| Y_{t}^{h,I_n} - Y_{t}^{h,I_n} \|_{L^2(P;H_0)} \leq \sqrt{2} \exp \left ( T \| \mathbb{P}_{I_n} F \|_{\text{Lip}(H_0,H_0)} + \sqrt{T} \| \mathbb{P}_{I_n} B \|_{\text{Lip}(H_0,L^2(U,H_0))} \right ) ^2
\]
\[
\cdot \left [ \sup_{t \in [0,T]} \| \mathbb{P}_{H \setminus I_n} Y_{t}^{h,I_n} \|_{L^2(P;H_0)} \right ]
\]
\[
\leq \sqrt{2} \exp \left ( T \| \mathbb{P}_{I_n} F \|_{\text{Lip}(H_0,H_0)} + \sqrt{T} \| \mathbb{P}_{I_n} B \|_{\text{Lip}(H_0,L^2(U,H_0))} \right ) ^2
\]
\[
\cdot \left [ \| \mathbb{P}_{H \setminus I_n} \xi \|_{L^2(P;H_0)} + \int_0^T \| \mathbb{P}_{H \setminus I_n} F(Y_{[s]}^{h,I_n}) \|_{L^2(P;H_0)} ds \\
+ \left ( \int_0^T \| \mathbb{P}_{H \setminus I_n} B(Y_{[s]}^{h,I_n}) \|_{L^2(P;L^2(U,H_0))} ds \right ) ^{1/2} \right ].
\]

This, item (ii) in Lemma 2.3, item (ii) in Lemma 3.7, and Lebesgue’s theorem of dominated convergence ensure that

\[
\limsup_{n \to \infty} \sup_{t \in [0,T]} \| Y_{t}^{h,I_n} - Y_{t}^{h,I_n} \|_{L^2(P;H_0)} = 0. \tag{66}
\]

The proof of Corollary 3.9 is thus completed. \( \square \)

### 3.5 Upper bounds for the weak approximation errors

Theorem 3.11 below constitutes our main result as it establishes suitable upper bounds for weak approximation errors related to temporal discretisations of hyperbolic SPDEs with multiplicative noise. In particular, it allows us to derive weak convergence rates for temporal discretisations of hyperbolic SPDEs with multiplicative noise. In particular, it allows us to derive weak convergence rates for temporal discretisations of hyperbolic SPDEs with multiplicative noise. The proof of Theorem 3.11 relies on Proposition 3.10 below, which states a special case of the mild Itô formula in Da Prato et al. [DPJR19 Corollary 1].

**Proposition 3.10.** Assume Setting 3.2, let \((H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H)\) be a separable \(R\)-Hilbert space, let \(S : [0, \infty) \to L(H)\) be a strongly continuous semigroup, let \(\varphi \in C^2(H, \mathbb{R})\), \(\tau_0, \tau_1 \in [0, T]\) satisfy \(\tau_0 < \tau_1\), let \(X : [\tau_0, \tau_1] \times \Omega \to H, Y : [\tau_0, \tau_1] \times \Omega \to H, Z : [\tau_0, \tau_1] \times \Omega \to L_2(U, H)\) be \((\mathbb{F}_t)_{t \in [\tau_0, \tau_1]}\)-predictable stochastic processes which satisfy for every \(t \in [\tau_0, \tau_1]\) that \(\mathbb{P} \left [ \int_{\tau_0}^t \left ( \| S_{t-s} Y_s \|_H + \| S_{t-s} Z_s \|_{L_2(U, H)}^2 \right ) ds < \infty \right ] = 1\) and

\[
[X_t]_{\mathbb{P}, B(H)} = [S_{t-s} X_0]_{\mathbb{P}, B(H)} + \int_{\tau_0}^t S_{t-s} Y_s ds + \int_{\tau_0}^t S_{t-s} Z_s dW_s. \tag{67}
\]

Then it holds for every \(t \in [\tau_0, \tau_1]\) that

\[
\mathbb{P} \left [ \int_{\tau_0}^t \left ( \| \varphi'(S_{t-s} X_s) S_{t-s} Y_s \| + \| \varphi'(S_{t-s} X_s) S_{t-s} Z_s \|_{L_2(U, R)}^2 \right ) ds < \infty \right ] = 1, \tag{68}
\]

\[
\mathbb{P} \left [ \int_{\tau_0}^t \left ( \| \varphi'(S_{t-s} X_s) S_{t-s} Y_s \| + \| \varphi'(S_{t-s} X_s) S_{t-s} Z_s \|_{L_2(U, R)}^2 \right ) ds < \infty \right ] = 1, \tag{69}
\]
\[
\mathbb{P} \left[ \int_{t_0}^t \left( \| \varphi''(S_{t-s}X_s) \|_{L^2(U,H)} \| S_{t-s}Z_s \|_{L^2(U,H)}^2 \right) ds < \infty \right] = 1, \tag{70}
\]

and
\[
[\varphi(X_t)]_{\mathcal{P}, \mathcal{B}(\mathbb{R})} = [\varphi(S_{t_0}X_0)]_{\mathcal{P}, \mathcal{B}(\mathbb{R})} + \int_{t_0}^t \left( \varphi'(S_{t-s}X_s) S_{t-s}Y_s + \frac{1}{2} \sum_{u \in \mathcal{U}} \varphi''(S_{t-s}X_s) (S_{t-s}Z_s u, S_{t-s}Z_s u) \right) ds + \int_{t_0}^t \varphi'(S_{t-s}X_s) S_{t-s}Z_s dW_s. \tag{71}
\]

The key idea in the proof of the weak error estimate in Theorem 3.11 below is a specific decomposition of the error into terms that can be estimated using the mild Itô formula in Proposition 3.10; see (86)–(88) below.

**Theorem 3.11.** Assume Setting 3.1, let \( h \in (0, T] \), for every finite \( I \subseteq \mathbb{H} \) and every \( x \in \mathcal{P}_I(\mathcal{H}_0) \) let \( X_t^{I,x} : [0,T] \times \Omega \to \mathcal{P}_I(\mathcal{H}_0) \) be an \( (\mathcal{F}_t)_{t \in [0,T]} \)-predictable stochastic process which satisfies for every \( t \in [0,T] \) that \( \sup_{s \in [0,T]} \mathbb{E}[\| X_s^{I,x} \|^2_{\mathcal{H}_0}] < \infty \) and

\[
[X_t^{I,x}]_{\mathcal{P}, \mathcal{B}(\mathcal{P}_I(\mathcal{H}_0))} = [e^{tA}x]_{\mathcal{P}, \mathcal{B}(\mathcal{P}_I(\mathcal{H}_0))} + \int_0^t e^{(t-s)A} \mathcal{P}_I F(X_s^{I,x}) ds
\]

\[
+ \int_0^t e^{(t-s)A} \mathcal{P}_I B(X_s^{I,x}) dW_s,
\]

and for every finite \( I \subseteq \mathbb{H} \) let \( v^I : [0,T] \times \mathcal{P}_I(\mathcal{H}_0) \to \mathbb{R} \) be the function which satisfies for every \( t \in [0,T] \), \( x \in \mathcal{P}_I(\mathcal{H}_0) \) that \( v^I(t, x) = \mathbb{E}[\varphi(X_t^{I,x})] \). Then

(i) it holds for every \( t \in [0,T] \) and every finite \( I \subseteq \mathbb{H} \) that \( \mathbb{P}_I(\mathcal{H}_0) \ni x \mapsto v^I(t, x) \in \mathbb{R} \) \( \in C^4(\mathcal{P}_I(\mathcal{H}_0), \mathbb{R}) \),

(ii) it holds that

\[
\sup_{I \text{ is finite}} \max_{k \in \{1,2,3,4\}} \sup_{t \in [0,T]} \sup_{x \in \mathcal{P}_I(\mathcal{H}_0)} \| \frac{\partial^k}{\partial x^k} v^I \|_{L^2(\mathcal{P}_I(\mathcal{H}_0), \mathbb{R})} < \infty, \tag{73}
\]

and

(iii) it holds for every finite \( I \subseteq \mathbb{H} \) that

\[
\mathbb{E} \left[ \varphi(Y_t^{0,I}) \right] - \mathbb{E} \left[ \varphi(Y_t^{h,I}) \right] \leq 6 \max \{ T, T^{2-2(\gamma-\beta)} \} h^{2(\gamma-\beta)}
\]

\[
\cdot \left[ \max_{k \in \{1,2,3,4\}} \sup_{t \in [0,T]} \sup_{x \in \mathcal{P}_I(\mathcal{H}_0)} \| \frac{\partial^k}{\partial x^k} v^I \|_{L^2(\mathcal{P}_I(\mathcal{H}_0), \mathbb{R})} \right]
\]

\[
\cdot \| F \|_{H_p} \| \text{Lip}(H_p, H_{2(\gamma-\beta)}) + 3 m^4 + \| F \|_{\text{Lip}(H_{4-\gamma}, H_0)}
\]

\[
+ 2 \left[ \sum_{h \in \mathbb{H}} |\lambda_h|^{-\beta} \| B |_{H_p} \|_{H_{2(\gamma-\beta)}}^2 + \| F \|_{\text{Lip}(H_{4-\gamma}, H_0)} \right]
\]

\[
\cdot \max \left\{ \| A^{\rho-\max\{\rho, \gamma-\beta\}} \|_{L(\mathcal{H}_0)}, \| A^{\gamma-\max\{\rho, \gamma-\beta\}} \|_{L(\mathcal{H}_0)} \right\}
\]

\[
\cdot \left[ 1 + \sup_{t \in [0,T]} \mathbb{E} \left[ \| Y_t^{h,I} \|_{H_{\max\{\rho, \gamma-\beta\}}}^2 + \| Y_t^{h,I} \|_{\mathcal{H}_0} \right] \right] < \infty. \tag{74}
\]
Proof of Theorem 3.11. Throughout this proof let $\delta: [0, \infty) \to [0, h]$ be the function which satisfies for every $x \in [0, \infty)$ that $\delta(x) = x - \lfloor x \rfloor_h$, for every $p \in [1, \infty)$ and every $\mathbb{R}$-Hilbert space $(W, \langle \cdot, \cdot \rangle_W, \| \cdot \|_W)$ let $(L_p(U, W), \| \cdot \|_{L_p(U, W)})$ be the $\mathbb{R}$-Banach space of Schatten-$p$ operators from $U$ to $W$ and let $(L_p(W), \| \cdot \|_{L_p(W)})$ be the $\mathbb{R}$-Banach space of Schatten-$p$ operators from $W$ to $W$. For every finite $I \subseteq \mathbb{H}$ let $v^I_{1,0} : [0, T] \times P_I(H_0) \to \mathbb{R}$ be the function which satisfies for every $t \in [0, T]$, $x \in P_I(H_0)$ that $v^I_{1,0}(t, x) = (\frac{\partial}{\partial t} \delta)^I(v^I(t, x))$, for every finite $I \subseteq \mathbb{H}$ and every $i \in \{1, 2, 3, 4\}$ let $v^I_{1,i} : [0, T] \times P_I(H_0) \to \mathbb{R}$ be the function which satisfies for every $t \in [0, T]$, $x_0, x_1, \ldots, x_i \in P_I(H_0)$ that
\[
v^I_{1,i}(t, x_0)(x_1 \ldots, x_i) = (\frac{\partial}{\partial t} \delta)^I(v^I(t, x_0))(x_1, \ldots, x_i),
\]
for every finite $I \subseteq \mathbb{H}$ let $\varphi^I, \psi^I : [0, T] \times P_I(H_0) \to \mathbb{R}$ be the functions which satisfy for every $t \in [0, T]$, $x \in P_I(H_0)$ that $\varphi^I(t, x) = v^I_{0,1}(t, x)(P_IF(x))$ and $\psi^I(t, x) = \sum_{u \in \mathbb{U}} v^I_{0,2}(t, x)(P_IB(x)u, P_IB(x)u)$, and for every finite $I \subseteq \mathbb{H}$ and every $i \in \{1, 2\}$ let $\varphi^I_{0,i}, \psi^I_{0,i} : [0, T] \times P_I(H_0) \to L^i(H_0, \mathbb{R})$ be the functions which satisfy for every $t \in [0, T]$, $x \in P_I(H_0)$ that $\varphi^I_{0,i}(t, x) = (\frac{\partial}{\partial t} \delta)^I(\varphi^I(t, x))$ and $\psi^I_{0,i}(t, x) = (\frac{\partial}{\partial t} \delta)^I(\psi^I(t, x))$. Observe that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0, \infty)$ it holds that $\text{AP}_{P_I[H_0]} \subseteq L(H_0)$ and
\[
e^\iota(\text{AP}_{P_I[H_0]}) = e^\iota(\text{AP}_{P_I[H_0]}) + P_{H \setminus P_I[H_0]}.
\]
Lemma 3.4 and (32) therefore ensure that
\[
sup_{I \subseteq \mathbb{H}} \sup_{t \in [0, T]} \| e^{t(\text{AP}_{P_I[H_0]})} \|_{L(H_0)} + \| P_IF[H_0] \|_{C^0_{t}(H_0, H_0)} + \| P_IB[H_0] \|_{C^0_t(H_0, L^2(U, U))} < \infty.
\]
Combining this, (72), (76), item (ii) in Lemma 2.2 (with $H = H_0$, $k = 4$, $A = \text{AP}_{P_I[H_0]}$, $F = P_IF[H_0]$, $B = P_IB$, $\varphi = \varphi^I$, $\psi^A, F, B, \varphi(t, x) = \psi^I(t, x)$ for $t \in [0, T]$, $x \in P_I(H_0)$, $I \in \{J \subseteq \mathbb{H} : J$ is a finite set$\}$ in the notation of item (ii) in Lemma 2.2, and item (iii) in Lemma 2.2 (with $H = H_0$ in the notation of item (iii) in Lemma 2.2) establishes items (i) and (ii). It thus remains to prove item (iii). For this, observe that for every finite $I \subseteq \mathbb{H}$ it holds that $E[\varphi(Y^h_{T,I})] = E[\psi^I(T, Y^h_{T,I})]$ and
\[
E[\varphi^I(Y^h_{T,I})] = E[E(\varphi^I(Y^h_{T,I})|F_0)] = E[\psi^I(0, P_IF_0)] = E[\psi^I(0, Y^h_{T,I})].
\]
Moreover, note that for every finite $I \subseteq \mathbb{H}$ it holds that $(e^{tA}|_{P_I(H_0)})_{t \in [0, \infty)} \subseteq L(P_I(H_0))$ is a strongly continuous semigroup with generator $A|_{P_I(H_0)} \subseteq L(P_I(H_0))$. This and (35) imply that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0, T]$ it holds that
\[
[Y^h_{T,I}]_{P,B(P_I(H_0))} = [P_IF_0]_{P,B(P_I(H_0))} + \int_0^t (A[Y^h_{T,I}]_{s} + e^{\delta(s)A}[P_IF(Y^h_{T,I})]_{s}) ds
\]
\[
+ \int_0^t e^{\delta(s)A} P_IB(Y^h_{T,I})_s ds.
\]
In addition, observe that item (i) in Lemma 2.2 ensures that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0, T]$, $x \in P_I(H_0)$ it holds that
\[
v^I \in C^{1,2}([0, T] \times P_I(H_0), \mathbb{R})
\]
and
\[
v^I_{1,0}(t, x) = -v^I_{0,1}(t, x)(Ax + P_IF(x)) - \frac{1}{2} \sum_{u \in \mathbb{U}} v^I_{0,2}(t, x)(P_IB(x)u, P_IB(x)u).
\]
Combining item (ii), the fact that for every finite $I \subseteq \mathbb{H}$ it holds that $A|_{P_f(H_0)} \in L(P_f(H_0))$, item (ii) in Lemma 2.3, Lemma 3.4, and item (iii) in Lemma 3.7 therefore proves that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\mathbb{E}\left[ \int_0^T |v_{l,0}^I(t, Y_{t}^{h,I})| \, dt + \int_0^T |v_{l,1}^I(t, Y_{t}^{h,I}) (AY_{t}^{h,I} + e^{\delta(t)A}P_f(Y_{t}^{h,I}))| \, dt \\
+ \frac{1}{2} \sum_{u \in U} \int_0^T |v_{l,2}^I(t, Y_{t}^{h,I}) (e^{\delta(t)A}P_f(Y_{t}^{h,I})u, e^{\delta(t)A}P_f(Y_{t}^{h,I})u)| \, dt \right] < \infty.
$$

(82)

Furthermore, note that Itô’s isometry, item (ii), item (iii) in Lemma 2.3, Lemma 3.4, and item (iii) in Lemma 3.7 imply that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\mathbb{E}\left[ \int_0^T v_{l,0}^I(t, Y_{t}^{h,I}) e^{\delta(t)A}P_f(Y_{t}^{h,I}) \, dW_t \right]^2 < \infty.
$$

(83)

Hence, we obtain that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\mathbb{E}\left[ \int_0^T v_{l,0}^I(t, Y_{t}^{h,I}) e^{\delta(t)A}P_f(Y_{t}^{h,I}) \, dW_t \right] = 0.
$$

(84)

The Itô formula, (78)–(80), and (82) therefore imply that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\mathbb{E}\left[ \varphi(Y_{t}^{h,I}) - \varphi(Y_{t}^{0,I}) \right] = \mathbb{E}\left[ \int_0^T v_{l,0}^I(T, Y_{t}^{h,I}) - v_{l,0}^I(0, Y_{0}^{h,I}) \right] = \mathbb{E}\left[ \int_0^T v_{l,0}^I(t, Y_{t}^{h,I}) \, dt + \int_0^T v_{l,1}^I(t, Y_{t}^{h,I}) (AY_{t}^{h,I} + e^{\delta(t)A}P_f(Y_{t}^{h,I})) \, dt \\
+ \frac{1}{2} \sum_{u \in U} \int_0^T v_{l,2}^I(t, Y_{t}^{h,I}) (e^{\delta(t)A}P_f(Y_{t}^{h,I})u, e^{\delta(t)A}P_f(Y_{t}^{h,I})u) \, dt \right].
$$

(85)

Combining this and (81) demonstrates that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\mathbb{E}\left[ \varphi(Y_{t}^{h,I}) - \varphi(Y_{t}^{0,I}) \right] = \mathbb{E}\left[ \int_0^T v_{l,0}^I(t, Y_{t}^{h,I}) (e^{\delta(t)A}P_f(Y_{t}^{h,I}) - P_f(Y_{t}^{h,I})) \, dt \\
+ \frac{1}{2} \sum_{u \in U} \int_0^T v_{l,2}^I(t, Y_{t}^{h,I}) (e^{\delta(t)A}P_f(Y_{t}^{h,I})u, e^{\delta(t)A}P_f(Y_{t}^{h,I})u) \, dt \\
- \frac{1}{2} \sum_{u \in U} \int_0^T v_{l,2}^I(t, Y_{t}^{h,I}) (P_f(Y_{t}^{h,I})u, P_f(Y_{t}^{h,I})u) \, dt \right].
$$

(86)

Moreover, observe that the triangle inequality ensures that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\mathbb{E}\left[ \int_0^T v_{l,0}^I(t, Y_{t}^{h,I}) (e^{\delta(t)A}P_f(Y_{t}^{h,I}) - P_f(Y_{t}^{h,I})) \, dt \right] \\
\leq \mathbb{E}\left[ \int_0^T v_{l,0}^I(t, Y_{t}^{h,I}) ((e^{\delta(t)A} - \text{id}_{H_0})P_f(Y_{t}^{h,I})) \, dt \right] \\
+ \mathbb{E}\left[ \int_0^T (v_{l,0}^I(t, Y_{t}^{h,I}) (P_f(Y_{t}^{h,I}) - v_{l,0}^I(t, e^{\delta(t)A}Y_{t}^{h,I}) (P_f(Y_{t}^{h,I}))) \, dt \right] \\
+ \mathbb{E}\left[ \int_0^T v_{l,0}^I(t, e^{\delta(t)A}Y_{t}^{h,I}) (P_f(Y_{t}^{h,I}) - P_f(e^{\delta(t)A}Y_{t}^{h,I})) \, dt \right].
$$

(87)
Furthermore, note that the triangle inequality implies that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\left| \mathbb{E} \left[ \sum_{u \in \mathcal{U}} \int_0^T v_{0,2}^I(t, Y_t^h, I) \left( e^{\delta(t)A} \mathbf{P}_I B(Y_t^h, I) u, e^{\delta(t)A} \mathbf{P}_I B(Y_t^h, I) u \right) dt \right] \right| 
$$

$$
\leq T \left[ \sup_{t \in [0,T]} \left( \|v^I_{0,1}(t, x)\|_{L(\mathbf{P}_I(H_0), \mathbb{R})} \right) \right] \left[ \|e^{\delta(t)A} - \text{id}_{H_0}\|_{L(H_0)} \right] 
$$

$$
\cdot \|F\|_{H_p} \|\text{Lip}(H_p, H_{2(\gamma-\beta)})\| \left[ \sup_{t \in [0,T]} \mathbb{E} \left[ \max\{1, \|Y_t^h, I\|_{H_p}\} \right] \right] 
$$

$$
\leq 2^{\gamma/2} \|e^{\delta(t)A} - \text{id}_{H_0}\|_{L(H_0)} \|F\|_{H_p} \|\text{Lip}(H_p, H_{2(\gamma-\beta)})\| \left[ \sup_{t \in [0,T]} \mathbb{E} \left[ \max\{1, \|Y_t^h, I\|_{H_p}\} \right] \right]. 
$$

In the next step we estimate the second term on the right-hand side of (87). Note that item (i) in Lemma 2.3 and Lemma 3.5 imply that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\left| \mathbb{E} \left[ \int_0^T v_{0,1}^I(t, Y_t^h, I) \left( e^{\delta(t)A} - \text{id}_{H_0}\right) \mathbf{P}_I F(Y_t^h, I) dt \right] \right| 
$$

$$
\leq T \left[ \sup_{t \in [0,T]} \left( \|v^I_{0,1}(t, x)\|_{L(\mathbf{P}_I(H_0), \mathbb{R})} \right) \right] \left[ \|e^{\delta(t)A} - \text{id}_{H_0}\|_{L(H_0)} \right] 
$$

$$
\cdot \|F\|_{H_p} \|\text{Lip}(H_p, H_{2(\gamma-\beta)})\| \left[ \sup_{t \in [0,T]} \mathbb{E} \left[ \max\{1, \|Y_t^h, I\|_{H_p}\} \right] \right]. 
$$

In the next step we estimate the second term on the right-hand side of (87). Note that item (ii), (35), and Proposition 3.10 (with $H = (\mathbf{P}_I(H_0))^2$, $S_r = [(\mathbf{P}_I(H_0))^2]$ hold for $(x_1, x_2) \mapsto$
\( \varphi = [(P_I(H_0))^2 \ni (x_1, x_2) \mapsto v_{0,1}^I(t, x_1, x_2) \in \mathbb{R}], \tau_0 = [t]_h, \tau_1 = t, X_s = (Y_s^{h,I}, P_I F(Y_{\{t\}_h}^{h,I}), Y_s = (e^{\delta(s)A} P_I F(Y_{\{t\}_h}^{h,I}, 0), Z_su = (e^{\delta(s)A} P_I B(Y_{\{t\}_h}^{h,I})u, 0) \) \\
for \( r \in [0, \infty), s \in [t, [t]], t \in (0, T] \setminus \{2h, 3h, \ldots\}, I \in \{J \subseteq \mathbb{H}: J \text{ is a finite set}\}, u \in U \) \\
in the notation of Proposition 3.10, imply that for every finite \( I \subseteq \mathbb{H} \) and every \( t \in [0, T] \) it holds that \\
\[
\begin{align*}
&\left[ v_{0,1}^I(t, Y_{\{t\}_h}^{h,I})(P_I F(Y_{\{t\}_h}^{h,I})) - v_{0,1}^I(t, e^{\delta(t)A} Y_{\{t\}_h}^{h,I})(P_I F(Y_{\{t\}_h}^{h,I})) \right]_{P_I, \mathcal{B}(\mathbb{R})} \\
= \int_t^T v_{0,2}^I(t, e^{(\tau-t)A} Y_{\{t\}_h}^{h,I})(P_I F(Y_{\{t\}_h}^{h,I}), e^{\delta(t)A} P_I F(Y_{\{t\}_h}^{h,I})) ds \\
+ \int_t^T v_{0,2}^I(t, e^{(\tau-t)A} Y_{\{t\}_h}^{h,I})(P_I F(Y_{\{t\}_h}^{h,I}), e^{\delta(t)A} P_I B(Y_{\{t\}_h}^{h,I})) dW_s (90)\\n+ \frac{1}{2} \sum_{u \in U} \int_t^T v_{0,3}^I(t, e^{(\tau-t)A} Y_{\{t\}_h}^{h,I}) (P_I F(Y_{\{t\}_h}^{h,I}), e^{\delta(t)A} P_I B(Y_{\{t\}_h}^{h,I}) u) ds.
\end{align*}
\]
Moreover, observe that Itô’s isometry, item [ii] in Lemma 2.3, Lemma 3.4, and item [iii] in Lemma 3.7 ensure that for every finite \( I \subseteq \mathbb{H} \) and every \( t \in [0, T] \) it holds that \\
\[
\mathbb{E}\left[ \left| \int_t^T v_{0,2}^I(t, e^{(\tau-t)A} Y_{\{t\}_h}^{h,I})(P_I F(Y_{\{t\}_h}^{h,I}), e^{\delta(t)A} P_I B(Y_{\{t\}_h}^{h,I}) dW_s \right|^2 \right] < \infty. (91)
\]
This, item [ii] in Lemma 2.3, Lemma 3.4, (32), and (90) imply that for every finite \( I \subseteq \mathbb{H} \) it holds that \\
\[
\begin{align*}
\left| \mathbb{E}\left[ \int_0^T \left(v_{0,1}^I(t, Y_{\{t\}_h}^{h,I})(P_I F(Y_{\{t\}_h}^{h,I})) - v_{0,1}^I(t, e^{\delta(t)A} Y_{\{t\}_h}^{h,I})(P_I F(Y_{\{t\}_h}^{h,I})) \right) dt \right] \right| \\
&\leq hT \sup \left\{ \|v_{0,k}(t, x)\|_{L^2(I_P(I_H, \mathbb{R}))}: x \in \mathbb{P}_{I_H}(H_0), \right\} \|F\|_{\mathbb{P}_{I_H}(H_0)} \|\mathcal{L}_{\text{lip}}(\mathbb{P}_{I_H}(H_0)) \| \\
&\cdot \left( \|F\|_{\mathbb{P}_{I_H}(H_0)} \|\mathcal{L}_{\text{lip}}(\mathbb{P}_{I_H}(H_0)) + \frac{1}{2} \|B\|_{\mathbb{P}_{I_H}(H_0)} \|^2 \right) \left( \sup_{t \in [0, T]} \mathbb{E}\left[ \max \{1, |Y_{\{t\}_h}^{h,I}|^3 \}_{H_0} \right] \right) \\
&\leq \frac{3}{2} hT m^3 \sup \left\{ \|v_{0,k}(t, x)\|_{L^2(I_P(I_H, \mathbb{R}))}: x \in \mathbb{P}_{I_H}(H_0), \right\} \\
&\cdot \left( \sup_{t \in [0, T]} \mathbb{E}\left[ \max \{1, |Y_{\{t\}_h}^{h,I}|^3 \}_{H_0} \right] \right). (92)
\end{align*}
\]
In the next step we estimate the third term on the right-hand side of (87). Note that Lemma 3.5 implies that for every finite \( I \subseteq \mathbb{H} \) it holds that \\
\[
\begin{align*}
&\left| \mathbb{E}\left[ \int_0^T v_{0,3}^I(t, e^{\delta(t)A} Y_{\{t\}_h}^{h,I})(P_I F(Y_{\{t\}_h}^{h,I}), e^{\delta(t)A} Y_{\{t\}_h}^{h,I}) dt \right] \right| \\
&\leq \left( \sup_{t \in [0, T], x \in \mathbb{P}_{I_H}(H_0)} \|v_{0,1}^I(t, x)\|_{L^2(\mathbb{P}_{I_H}(H_0), \mathbb{R}))} \|F\|_{\mathcal{L}_{\text{lip}}(\mathbb{P}_{I_H}(H_0))} \right. \\
&\cdot \left. \int_0^T \mathbb{E}\left[ \|A^{\gamma-\gamma} (\text{id}_{H_0} - e^{\delta(t)A} Y_{\{t\}_h}^{h,I}) H_{h_0} \|_{H_0} \right] dt \right) \\
&\leq 2^{3/2} h^{2(\gamma-\beta)} T \|F\|_{\mathcal{L}_{\text{lip}}(H_{\beta-\gamma}, H_0)} \left( \sup_{t \in [0, T], x \in \mathbb{P}_{I_H}(H_0)} \|v_{0,1}^I(t, x)\|_{L^2(\mathbb{P}_{I_H}(H_0), \mathbb{R}))} \right. \\
&\cdot \left. \int_0^T \mathbb{E}\left[ \|Y_{\{t\}_h}^{h,I}\|_{H_{-\gamma}} \right] dt \right). (93)
\end{align*}
\]
In the next step we estimate the fourth term on the right-hand side of (97). Note that item (ii) in Lemma 3.7 implies that for every finite \( I \subseteq \mathbb{H} \) and every \( t \in [0,T] \) it holds that

\[
(P_I(H_0) \ni x \mapsto \varphi^I(t, x) \in \mathbb{R}) \in C^2(P_I(H_0), \mathbb{R}).
\] (94)

In addition, observe that item (ii) in Lemma 2.3, Lemma 3.4, item (ii) in Lemma 3.7 and (97) assure that for every finite \( I \subseteq \mathbb{H} \) and every \( t \in [0,T] \), \( x, v_1, v_2 \in P_I(H_0) \) it holds that

\[
\varphi^I_{0.1}(t, x)(v_1) = v^I_{0.2}(t, x)(P_IF(x), v_1) + v^I_{0.1}(t, x)(P_IF'(x)(v_1))
\] (95)

and

\[
\varphi^I_{0.2}(t, x)(v_1, v_2) = v^I_{0.3}(t, x)(P_IF(x), v_1, v_2) + v^I_{0.2}(t, x)(P_IF'(x)(v_2), v_1)
\]

\[
+ v^I_{0.1}(t, x)(P_IF''(x)(v_1, v_2))
\] (96)

This and item (iii) in Lemma 2.3 imply that for every finite \( I \subseteq \mathbb{H} \) and every \( t \in [0,T] \), \( x \in P_I(H_0) \) it holds that

\[
\|\varphi^I_{0.1}(t, x)\|_{L^2(P_I(H_0), \mathbb{R})} \leq 2 \sup \left\{ \|v^I_{0,k}(s, y)\|_{L^2(P_I(H_0), \mathbb{R})} : s \in [0,T], k \in \{1,2,3\} \right\}
\]

\[
\cdot \|F|_{P_I(H_0)}\|_{C^2(P_I(H_0), \mathbb{R})} \max\{1, \|x\|_{H_0}\},
\] (97)

and

\[
\|\varphi^I_{0.2}(t, x)\|_{L^2(P_I(H_0), \mathbb{R})} \leq 4 \sup \left\{ \|v^I_{0,k}(s, y)\|_{L^2(P_I(H_0), \mathbb{R})} : s \in [0,T], k \in \{1,2,3\} \right\}
\]

\[
\cdot \|F|_{P_I(H_0)}\|_{C^2(P_I(H_0), \mathbb{R})} \max\{1, \|x\|_{H_0}\}.
\] (98)

Next observe that (33), (94), and Proposition 3.10 (with \( H = P_I(H_0) \), \( S_r = e^{rA}|_{P_I(H_0)} \), \( \varphi = \varphi^I(t, \cdot) \), \( \tau = \tau_I \), \( T = T_I \), \( X_t = Y_{[t,b]}^{h,I} \), \( Y_s = e^{\delta(s)A}P_IF(Y_{[t,b]}^{h,I}) \), \( Z_s = e^{\delta(s)A}P_IB(Y_{[t,b]}^{h,I}) \) for \( r \in [0,\infty) \), \( s \in [\tau_I, T] \) \( t \in (0,T) \setminus \{h, 2h, 3h, \ldots\} \), \( J \subseteq \mathbb{H} \), \( J \) is a finite set) in the notation of Proposition 3.10 imply that for every finite \( I \subseteq \mathbb{H} \) and every \( t \in [0,T] \) it holds that

\[
\left[ v^I_{0.1}(t, e^{(t-s)A}Y_{[t,b]}^{h,I})(P_IF(\cdot)) - v^I_{0.1}(t, Y_{[t,b]}^{h,I})(P_IF(Y_{[t,b]}^{h,I})) \right]_{P,B(R)}
\]

\[
= - \int_{[t,b]} \varphi^I_{0.1}(t, e^{(t-s)A}Y_{[t,b]}^{h,I})(e^{\delta(t)A}P_IB(Y_{[t,b]}^{h,I})) \, ds
\]

\[
- \int_{[t,b]} \varphi^I_{0.1}(t, e^{(t-s)A}Y_{[t,b]}^{h,I})(e^{\delta(t)A}P_IB(Y_{[t,b]}^{h,I})) \, dW_s
\]

\[
- \frac{1}{2} \sum_{s \in U} \int_{[t,b]} \varphi^I_{0.2}(t, e^{(t-s)A}Y_{[t,b]}^{h,I})(e^{\delta(t)A}P_IB(Y_{[t,b]}^{h,I})u)(e^{\delta(t)A}P_IB(Y_{[t,b]}^{h,I})u) \, dW_s
\] (99)

Moreover, note that Itô’s isometry, item (iii), item (ii) in Lemma 2.3, Lemma 3.4, item (iii) in Lemma 3.7 and (97) assure that for every finite \( I \subseteq \mathbb{H} \) and every \( t \in [0,T] \) it holds that

\[
\mathbb{E}\left[ \left| \int_{[t,b]} \varphi^I_{0.1}(t, e^{(t-s)A}Y_{[t,b]}^{h,I})(e^{\delta(t)A}P_IB(Y_{[t,b]}^{h,I})) \, dW_s \right|^2 \right] < \infty.
\] (100)

Combining this, item (iii) in Lemma 2.3, Lemma 3.4, (92), (97), (98), and (99) implies that
for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\left| \mathbb{E} \left[ \int_0^T \left( v_{0.1}^{I} (t, e^{\delta(t)A} Y_{[t]^h}) (P_I F(e^{\delta(t)A} Y_{[t]^h})) - v_{0.1}^{I} (t, Y_{[t]^h}) (P_I F(Y_{[t]^h})) \right) dt \right] \right|
\leq 4hT \sup_{t \in [0, T], k \in \{1, 2, 3\}} \left\{ \left\| |v_{0.1}^{I} (t, x)|_{L^2([0, T])} \right\|_{P_I (H_0, R)} \right\} + \frac{1}{2} \left\| B |P_I (H_0)|_{\text{Lip}(P_I (H_0, L_2(U, H_0))} \right\| \left( \sup_{t \in [0, T]} \mathbb{E} \left[ \max \left\{ 1, \| Y_{[t]}^{h,I} \|_{H_0}^3 \right\} \right] \right)
$$

(101)

In the next step we estimate the first term on the right-hand side of (89). Note that Lemma 2.3 and Lemma 3.4 imply that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\left| \mathbb{E} \left[ \sum_{u \in U} \int_0^T v_{0.2}^{I} (t, Y_{[t]^h}) \left( (e^{\delta(t)A} - \text{id}_{H_0}) P_I B (Y_{[t]^h}) u, (e^{\delta(t)A} + \text{id}_{H_0}) P_I B (Y_{[t]^h}) u \right) dt \right] \right|
\leq \mathbb{E} \left[ \int_0^T \left\| |v_{0.2}^{I} (t, x)|_{L^2([0, T])} \right\|_{P_I (H_0, R)} \right\} \left( \sup_{t \in [0, T]} \mathbb{E} \left[ \max \left\{ 1, \| Y_{[t]}^{h,I} \|_{H_0}^3 \right\} \right] \right)
$$

(102)

Moreover, observe that $A$ and $P_I$ commute. This, item (i) in Lemma 2.3 and Lemma 3.4 imply that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0, T]$ it holds that

$$
\left\| (\text{id}_{H_0} + e^{\delta(t)A}) P_I B (Y_{[t]^h}) \right\|_{L_{2\beta/\gamma} (U, H_0)}
\leq \left\| \text{id}_{H_0} + e^{\delta(t)A} \right\|_{L_1 (H_0)} \left\| A^{-\gamma} \right\|_{L_{2\beta/\gamma} (H_0)} \left\| A^\gamma P_I B (Y_{[t]^h}) \right\|_{L_1 (H_0)}
\leq 2 \left\| A^{-2\beta} \right\|_{L_{1} (H_0)} \left\| B \right\|_{H_0} \left\| \text{Lip}_{H_0} (U, H_0) \right\| \left( \| Y_{[t]}^{h,I} \|_{H_0} \right) \left( \max \left\{ 1, \| Y_{[t]}^{h,I} \|_{H_0}^2 \right\} \right)
$$

(103)

In addition, note that item (i) in Lemma 2.3 and Lemma 3.5 imply that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0, T]$ it holds that

$$
\left\| (\text{id}_{H_0} - e^{\delta(t)A}) P_I B (Y_{[t]^h}) \right\|_{L_{2\beta/(2\beta-\gamma)} (U, H_0)}
\leq \left\| A^{2(\beta-\gamma)} (\text{id}_{H_0} - e^{\delta(t)A}) \right\|_{L_1 (H_0)} \left\| A^{\gamma-2\beta} \right\|_{L_{2\beta/(2\beta-\gamma)} (H_0)} \left\| A^\gamma P_I B (Y_{[t]^h}) \right\|_{L_1 (H_0)}
\leq 2^{\gamma/2} \left\| A^{-2\beta} \right\|_{L_{1} (H_0)} \left\| B \right\|_{H_0} \left\| \text{Lip}_{H_0} (U, H_0) \right\| \left( \| Y_{[t]}^{h,I} \|_{H_0} \right) \left( \max \left\{ 1, \| Y_{[t]}^{h,I} \|_{H_0}^2 \right\} \right)
$$

(104)

Combining (102), (103), and (104) ensures that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\left| \mathbb{E} \left[ \sum_{u \in U} \int_0^T v_{0.2}^{I} (t, Y_{[t]^h}) \left( (e^{\delta(t)A} - \text{id}_{H_0}) P_I B (Y_{[t]^h}) u, (e^{\delta(t)A} + \text{id}_{H_0}) P_I B (Y_{[t]^h}) u \right) dt \right] \right|
\leq 2^{\gamma/2} \left( \gamma \right) T \left\| A^{-2\beta} \right\|_{L_1 (H_0)} \left\| B \right\|_{H_0} \left\| \text{Lip}_{H_0} (U, H_0) \right\| \left( \sup_{t \in [0, T]} \mathbb{E} \left[ \max \left\{ 1, \| Y_{[t]}^{h,I} \|_{H_0}^2 \right\} \right] \right)
$$

(105)
In the next step we estimate the second term on the right-hand side of (88). Note that item (i), (35), and Proposition 3.10 (with \( H = (P_I(H_0))^3 \), \( S_r = [(P_I(H_0))^3 \supseteq \langle x_1, x_2, x_3 \rangle \mapsto (e^{A,x_1, x_2, x_3}) \in (P_I(H_0))^3] \), \( \tau_0 = \{t\} \), \( \tau_1 = t \), \( X_s = (Y_{s,h}, P_I B(Y_{t,h}) u, P_I B(Y_{t,h}) u) \), \( Y_s = (e^{(s)} P_I F(Y_{t,h}), 0, 0) \), \( Z_s u' = (e^{(s)} P_I B(Y_{t,h}) u', 0, 0) \) for \( r \in [0, \infty) \), \( s \in [\{t\}, t] \), \( t \in (0, T) \setminus \{h, 2h, 3h, \ldots\} \), \( I \in \{J \subseteq \mathbb{H} : J \) is a finite set\}, \( u, u' \in U \) in the notation of Proposition 3.10) ensure that for every finite \( I \subseteq \mathbb{H} \) and every \( t \in [0, T] \), \( u \in U \) it holds that

\[
\begin{align*}
&\left[ v_{0,2}(t, Y_{t,h}^I) (P_I B(Y_{t,h}^I) u, P_I B(Y_{t,h}^I) u) \\
&- v_{0,2}(t, e^{(s)} A Y_{t,h}^I) (P_I B(Y_{t,h}^I) u, P_I B(Y_{t,h}^I) u) \right] \mathbb{P}(dR) \\
&= \int_{[t,h]} v_{0,3}(t, e^{(s)} A Y_{h}^I) (P_I B(Y_{h}^I) u, P_I B(Y_{h}^I) u, e^{(t)} A P_I F(Y_{t,h}^I)) ds \\
&+ \int_{[t,h]} v_{0,3}(t, e^{(s)} A Y_{h}^I) (P_I B(Y_{h}^I) u, P_I B(Y_{h}^I) u, e^{(t)} A P_I B(Y_{h}^I) u') dW_s \\
&+ \frac{1}{2} \sum_{u' \in U} \int_{[t,h]} v_{0,4}(t, e^{(s)} A Y_{s}^I) (P_I B(Y_{s}^I) u, P_I B(Y_{s}^I) u, e^{(t)} A P_I B(Y_{t,h}^I) u') ds.
\end{align*}
\]  

Moreover, observe that Itô’s isometry, item (ii), item (i) in Lemma 2.3, Lemma 3.4, and item (iii) in Lemma 3.7 ensure that for every finite \( I \subseteq \mathbb{H} \) and every \( t \in [0, T] \), \( u \in U \) it holds that

\[
\mathbb{E} \left[ \left| \int_{[t,h]} v_{0,3}(t, e^{(s)} A Y_{s}^I) (P_I B(Y_{s}^I) u, P_I B(Y_{s}^I) u, e^{(t)} A P_I B(Y_{t,h}^I) u') dW_s \right|^2 \right] < \infty. \tag{107}
\]

Combining this, item (i) in Lemma 2.3, Lemma 3.4, (32), and (106) implies that for every finite \( I \subseteq \mathbb{H} \) it holds that

\[
\begin{align*}
&\mathbb{E} \left[ \left| \sum_{u \in U} \int_0^T \left( v_{0,2}(t, Y_{t,h}^I) (P_I B(Y_{t,h}^I) u, P_I B(Y_{t,h}^I) u) \\
- v_{0,2}(t, e^{(t)} A Y_{t,h}^I) (P_I B(Y_{t,h}^I) u, P_I B(Y_{t,h}^I) u) \right) dt \right| \right] \\
&= \left| \sum_{u \in U} \int_0^T \mathbb{E} \left[ v_{0,2}(t, Y_{t,h}^I) (P_I B(Y_{t,h}^I) u, P_I B(Y_{t,h}^I) u) \\
- v_{0,2}(t, e^{(t)} A Y_{t,h}^I) (P_I B(Y_{t,h}^I) u, P_I B(Y_{t,h}^I) u) \right] dt \right| \\
&\leq hT \sup \left\{ \|v_{0,k}(t, x)\|_{L^1(|P_I(H_0)|)} : x \in P_I(H_0), t \in [0, T], k \in \{3, 4\} \right\} \|B|_{P_I(H_0)} \|_{L^1(|P_I(H_0)|)}^2 \\
\times \left( \left( \|F|_{P_I(H_0)} \|_{L^1(|P_I(H_0)|)} + \frac{1}{2} \|B|_{P_I(H_0)} \|_{L^1(|P_I(H_0)|)}^2 \right)^{1/2} \right) \\
\times \left[ \sup_{t \in [0, T]} \mathbb{E} \left[ \max\{1, \|Y_{t,h}^I\|_{H_0}^4 \} \right] \right] \\
&\leq \frac{3}{2} hT m^4 \sup \left\{ \|v_{0,k}(t, x)\|_{L^1(|P_I(H_0)|)} : x \in P_I(H_0), t \in [0, T], k \in \{3, 4\} \right\}
\end{align*}
\]
In the next step we estimate the third term on the right-hand side of (88). Note that the Cauchy-Schwarz inequality, Lemma 3.5, (33), and (34) imply that for every finite \( I \subset \mathbb{H} \) it holds that

\[
\mathbb{E} \left[ \sum_{u \in U} \int_0^T v_{0,2}^J(t, e^{\delta(t)A}Y_{[t,J]}^I) \right. \\
\left. \left( \mathbf{P}_I B(Y_{[t,J]}^I) - \mathbf{P}_I B(e^{\delta(t)A}Y_{[t,J]}^I) \right) u, \left[ \mathbf{P}_I B(Y_{[t,J]}^I) + \mathbf{P}_I B(e^{\delta(t)A}Y_{[t,J]}^I) \right] u \right) dt \right]
\]

\[
\leq \left[ \sup_{t \in [0,T], x \in \mathbf{P}_I(H_0)} \| v_{0,2}^J(t, x) \|_{L^2(\mathbf{P}_I(H_0), \mathcal{R})} \right]
\cdot \mathbb{E} \left[ \int_0^T \left( \sum_{u \in U} |\mu_u|^{-2} \| B(Y_{[t,J]}^I) - B(e^{\delta(t)A}Y_{[t,J]}^I) \|_{H_0}^2 \right)^{1/2} \\
\cdot \left( \sum_{u \in U} |\mu_u|^2 \| B(Y_{[t,J]}^I) + B(e^{\delta(t)A}Y_{[t,J]}^I) \|_{H_0}^2 \right)^{1/2} dt \right] 
\]

\[
\leq 2 \mathfrak{c} \left[ \sup_{t \in [0,T], x \in \mathbf{P}_I(H_0)} \| v_{0,2}^J(t, x) \|_{L^2(\mathbf{P}_I(H_0), \mathcal{R})} \right]
\cdot \int_0^T \| A^{2(\beta-\gamma)}(id_{H_0} - e^{\delta(t)A}) \|_{L(H_0)} \mathbb{E} \left[ \max\{1, \| Y_{[t,J]}^I \|_{H_{\gamma-\beta}}^2 \} \right] dt 
\]

\[
\leq 2^{3/2} h^{2(\gamma-\beta)} T \mathfrak{c} \left[ \sup_{t \in [0,T], x \in \mathbf{P}_I(H_0)} \| v_{0,2}^J(t, x) \|_{L^2(\mathbf{P}_I(H_0), \mathcal{R})} \right]
\cdot \mathbb{E} \left[ \max\{1, \| Y_{[t,J]}^I \|_{H_{\gamma-\beta}}^2 \} \right]. 
\]

In the next step we estimate the final term on the right-hand side of (88). Note that item \( i \) implies that for every finite \( I \subset \mathbb{H} \) and every \( t \in [0, T] \) it holds that

\[
(\mathbf{P}_I(H_0) \ni x \mapsto \psi^I(t, x) \in \mathbb{R}) \in C^2(\mathbf{P}_I(H_0), \mathbb{R}). 
\]

In addition, observe that item \( i \) ensures that for every finite \( I \subset \mathbb{H} \) and every \( t \in [0, T], x, v_1, v_2 \in \mathbf{P}_I(H_0) \) it holds that

\[
\psi_{0,1}^I(t, x)(v_1) = \sum_{u \in U} v_{0,3}^J(t, x) (\mathbf{P}_I B(x) u, \mathbf{P}_I B(x) u, v_1) + 2 \sum_{u \in U} v_{0,2}^J(t, x) (\mathbf{P}_I B'(x)(v_1) u, \mathbf{P}_I B(x) u), 
\]

\[
\psi_{0,2}^I(t, x)(v_1, v_2) = \sum_{u \in U} v_{0,4}^J(t, x) (\mathbf{P}_I B(x) u, \mathbf{P}_I B(x) u, v_1, v_2) + 2 \sum_{u \in U} v_{0,3}^J(t, x) (\mathbf{P}_I B'(x)(v_2) u, \mathbf{P}_I B(x) u, v_1) \\
+ 2 \sum_{u \in U} v_{0,3}^J(t, x) (\mathbf{P}_I B'(x)(v_1) u, \mathbf{P}_I B(x) u, v_2) + 2 \sum_{u \in U} v_{0,2}^J(t, x) (\mathbf{P}_I B''(x)(v_1, v_2) u, \mathbf{P}_I B(x) u) 
\]

\[
\text{(111)} 
\]

\[
\text{(112)} 
\]
+ 2 \sum_{u \in \mathcal{U}} v_{0,2}(t, x) \left( \mathbf{P}_I B(x)(v_1 u), \mathbf{P}_I B(x)(v_2 u) \right) .

Combining this and item (i) in Lemma 2.3 shows that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0, T]$, $x \in \mathbf{P}_I(\mathcal{H}_0)$ it holds that

$$
\| \psi_{0,1}^I(t, x) \|_{L^p(\mathbf{P}_I(\mathcal{H}_0), R)} \leq 3 \sup \left\{ \| \psi_{0,k}^I(s, y) \|_{L^p(\mathbf{P}_I(\mathcal{H}_0), R)} : s \in [0, T], k \in \{2, 3\} \right\} \cdot \| \mathbf{B}\|_{C_0^0(\mathbf{P}_I(\mathcal{H}_0), L^2(U, \mathcal{H}_0))} \max \left\{ 1, \| x \|_{\mathcal{H}_0}^2 \right\} \tag{113}
$$

and

$$
\| \psi_{0,2}^I(t, x) \|_{L^p(\mathbf{P}_I(\mathcal{H}_0), R)} \leq 9 \sup \left\{ \| \psi_{0,k}^I(s, y) \|_{L^p(\mathbf{P}_I(\mathcal{H}_0), R)} : s \in [0, T], k \in \{2, 3, 4\} \right\} \cdot \| \mathbf{B}\|_{C_0^0(\mathbf{P}_I(\mathcal{H}_0), L^2(U, \mathcal{H}_0))} \max \left\{ 1, \| x \|_{\mathcal{H}_0}^2 \right\} . \tag{114}
$$

In addition, observe that (35), (10), and Proposition 3.10 (with $H = \mathbf{P}_I(\mathcal{H}_0)$, $S_r = e^{r A}\mathbf{P}_I(\mathcal{H}_0)$, $\varphi = \psi^I(t, \cdot)$, $\tau_0 = [t]_h$, $\tau_1 = t$, $X_s = Y_{s,t}^h$, $Z_s = e^{(s,t)A} \mathbf{P}_I B(Y_{s,t}^h)$ for $r \in [0, \infty)$, $s \in ([t]_h, t]$), $t \in (0, T) \setminus \{h, 2h, 3h, \ldots\}$, $I \in \{J \subseteq \mathbb{H} : J$ is a finite set $\}$ in the notation of Proposition 3.10 imply that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0, T]$ it holds that

$$
\left[ \psi^I(t, Y_{t,h}^h) - \psi^I(t, e^{(s,t)A} Y_{s,t}^h) \right]_{P,B}(R) = \int_{[t]_h}^t \psi_{0,1}^I(t, e^{(t-s)A} Y_{s,t}^h) \left( e^{(s,t)A} F(Y_{s,t}^h) \right) ds + \int_{[t]_h}^t \psi_{0,1}^I(t, e^{(t-s)A} Y_{s,t}^h) \left( e^{(s,t)A} \mathbf{P}_I B(Y_{s,t}^h) \right) dW_s + \frac{1}{2} \sum_{u \in \mathcal{U}} \int_{[t]_h}^t \psi_{0,2}^I(t, e^{(t-s)A} Y_{s,t}^h) \left( e^{(s,t)A} \mathbf{P}_I B(Y_{s,t}^h) \right) u, e^{(s,t)A} \mathbf{P}_I B(Y_{s,t}^h) u \right) ds . \tag{115}
$$

Next note that Itô’s isometry, item (iii) in Lemma 2.3, Lemma 3.4, item (ii) in Lemma 3.7 and (113) prove that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0, T]$ it holds that

$$
\mathbb{E} \left[ \left( \int_{[t]_h}^t \psi_{0,1}^I(t, e^{(t-s)A} Y_{s,t}^h) \left( e^{(s,t)A} \mathbf{P}_I B(Y_{s,t}^h) \right) dW_s \right)^2 \right] < \infty . \tag{116}
$$

Combining this, item (iii) in Lemma 2.3, Lemma 3.4, (32), (113), (114), and (115) ensures that for every finite $I \subseteq \mathbb{H}$ and every $t \in [0, T]$ it holds that

$$
\| \psi^I(t, Y_{t,h}^h) - \psi^I(t, e^{(s,t)A} Y_{s,t}^h) \|_{L^p(\mathbf{P}_I(\mathcal{H}_0), R)} \leq h \sup \left\{ \| \psi_{0,k}^I(s, x) \|_{L^p(\mathbf{P}_I(\mathcal{H}_0), R)} : x \in \mathbf{P}_I(\mathcal{H}_0), s \in [0, T], k \in \{2, 3, 4\} \right\} \cdot \| \mathbf{B}\|_{C_0^0(\mathbf{P}_I(\mathcal{H}_0), L^2(U, \mathcal{H}_0))} \max \left\{ 1, \| Y_{s,t}^h \|_{\mathcal{H}_0}^4 \right\} \tag{117}
$$

This implies that for every finite $I \subseteq \mathbb{H}$ it holds that

$$
\mathbb{E} \left[ \sum_{u \in \mathcal{U}} \int_0^T v_{0,2}(t, x) \left( \mathbf{P}_I B(x)(v_1 u), \mathbf{P}_I B(x)(v_2 u) \right) dt \right] \leq 15 h m^4 \sup \left\{ \| \psi_{0,k}^I(s, x) \|_{L^p(\mathbf{P}_I(\mathcal{H}_0), R)} : x \in \mathbf{P}_I(\mathcal{H}_0), s \in [0, T], k \in \{2, 3, 4\} \right\} \cdot \max \left\{ 1, \| Y_{s,t}^h \|_{\mathcal{H}_0}^4 \right\} .
$$
\[
- \sum_{u \in U} \int_0^T \left( v_{0,2}(t, Y_t^{h,I}) \langle P_I B(Y_t^{h,I}) u, P_I B(Y_t^{h,I}) u \rangle \right) dt \\
\]

\[
= \left| \int_0^T \mathbb{E} \left[ \psi^I(t, Y_t^{h,I}) - \psi^I(t, e^{\delta(t) A} Y_t^{h,I}) \right] dt \right| \\
\leq \frac{15}{2} h T m^4 \sup \left\{ \| v_{0,2}(t, x) \|_{L^2(U; P_I(H_0), R^d)} : t \in [0, T], k \in \{1, 2, 3, 4\} \right\} \\
\times \left[ \sup_{t \in [0, T]} \mathbb{E} \left[ \max \left\{ 1, \| Y_t^{h,I} \|_{H_0}^4 \right\} \right] \right].
\]

Combining (86), (87), (88), (89), (92), (93), (101), (105), (108), (109), and (118) ensures that for every finite \( I \subseteq \mathbb{H} \) it holds that

\[
\mathbb{E} \left[ \varphi(Y_{T}^{0,H}) - \varphi(Y_{T}^{h,H}) \right] \leq \max \{ h, h^{2(\gamma-\beta)} \} T \\
\times \sup \left\{ \| v_{0,2}(t, x) \|_{L^2(U; P_I(H_0), R^d)} : t \in [0, T], k \in \{1, 2, 3, 4\} \right\} \\
\times \left( 2^{y/2} \| F \|_{L^2(I; H_0)} \sup_{t \in [0, T]} \mathbb{E} \left[ \max \left\{ 1, \| Y_t^{h,I} \|_{H_0}^4 \right\} \right] \right) \\
+ 17 m^4 \sup_{t \in [0, T]} \mathbb{E} \left[ \max \left\{ 1, \| Y_t^{h,I} \|_{H_0}^4 \right\} \right] \\
+ 2^{y/2} \| F \|_{L^2(H_{\beta-\gamma}; H_0)} \sup_{t \in [0, T]} \mathbb{E} \left[ \| Y_t^{h,I} \|_{H_{\beta-\gamma}}^2 \right] \\
+ 2^{y/2} \| A^{-2\beta} \|_{L^2(I; H_0)} \| B \|_{L^2(I; H_0)} \sup_{t \in [0, T]} \mathbb{E} \left[ \max \left\{ 1, \| Y_t^{h,I} \|_{H_{\beta}}^2 \right\} \right] \\
+ \sup_{t \in [0, T]} \mathbb{E} \left[ \max \left\{ 1, \| Y_t^{h,I} \|_{H_{\beta-\gamma}}^2 \right\} \right].
\]

Combining this, the estimates \( \max \{ h, h^{2(\gamma-\beta)} \} \leq \max \{ T^{1-2(\gamma-\beta)}, 1, h^{2(\gamma-\beta)} \} \) and \( 2^{y/2} \leq 6 \), the fact that \( \| A^{-2\beta} \|_{L^2(I; H_0)} = 2 \| \sum_{h \in H} |\lambda_h|^{-\beta} \|_2 \) and items (ii)-(iii) in Lemma \( 3.7 \) establishes item (iii). The proof of Theorem \( 3.11 \) is thus completed. \( \square \)

**Corollary 3.12.** Assume Setting \( \Box \). Then

\[
\sup_{h \in (0, T]} \left( h^{2(\beta-\gamma)} \left| \mathbb{E} \left[ \varphi(Y_{T}^{0,H}) \right] - \mathbb{E} \left[ \varphi(Y_{T}^{h,H}) \right] \right| \right) < \infty.
\]

**Proof of Corollary 3.12.** Throughout this proof let \( I_n \subseteq \mathbb{H} \), \( n \in \mathbb{N} \), be a non-decreasing sequence of finite sets which satisfies that \( \bigcup_{n \in \mathbb{N}} I_n = \mathbb{H} \). Observe that the triangle inequality implies that for every \( n \in \mathbb{N} \), \( h \in (0, T] \) it holds that

\[
\mathbb{E} \left[ \varphi(Y_{T}^{0,H}) \right] - \mathbb{E} \left[ \varphi(Y_{T}^{h,H}) \right] \\
\leq \mathbb{E} \left[ \varphi(Y_{T}^{0,I_n}) \right] - \mathbb{E} \left[ \varphi(Y_{T}^{I_n}) \right] + \mathbb{E} \left[ \varphi(Y_{T}^{I_n}) \right] - \mathbb{E} \left[ \varphi(Y_{T}^{0,H}) \right] \\
+ \mathbb{E} \left[ \varphi(Y_{T}^{0,H}) \right] - \mathbb{E} \left[ \varphi(Y_{T}^{I_n}) \right] \\
\leq \left( \sup_{x \in H_0} \| \varphi' \|_{L^2(H_0; R^d)} \right) \left( \| Y_{T}^{0,H} - Y_{T}^{0,I_n} \|_{L^2(P; H_0)} + \| Y_{T}^{h,H} - Y_{T}^{h,I_n} \|_{L^2(P; H_0)} \right) \\
+ \mathbb{E} \left[ \varphi(Y_{T}^{0,I_n}) \right] - \mathbb{E} \left[ \varphi(Y_{T}^{h,I_n}) \right].
\]

Furthermore, note that Corollary \( 3.9 \) ensures that for every \( h \in (0, T] \) it holds that

\[
\lim_{n \to \infty} \left( \| Y_{T}^{0,H} - Y_{T}^{0,I_n} \|_{L^2(P; H_0)} + \| Y_{T}^{h,H} - Y_{T}^{h,I_n} \|_{L^2(P; H_0)} \right) = 0.
\]

(122)
Moreover, observe that items (ii)–(iii) in Lemma 3.7 and items (ii)–(iii) in Theorem 3.11 imply that

\[
\sup_{h \in (0, T]} \left( h^{2(\beta - \gamma)} \left[ \limsup_{n \to \infty} \| \varphi(Y_{T}^{0, I_n}) - \varphi(Y_{T}^h, I_n) \| \right] \right) < \infty.
\]

(123)

This, (121), and (122) imply (120). The proof of Corollary 3.12 is thus completed. \( \square \)

4 Weak convergence rates for temporal numerical approximations of the hyperbolic Anderson model

4.1 Setting

Throughout this section we shall frequently use the following setting.

**Setting 4.1.** For every measure space \((\Omega, \mathcal{F}, \mu)\), every measurable space \((S, \Sigma)\), every set \(O\), and every function \(f: O \to S\) let \([f]_{\mu, \Sigma}\) be the set given by

\[
[f]_{\mu, \Sigma} = \left\{ g: \Omega \to S : \left[ \exists A \in \mathcal{F} : (\mu(A) = 0 \text{ and } \{\omega \in \Omega : f(\omega) \not\in g(\omega)\} \subseteq A) \right] \right\},
\]

(124)

let \(\lambda: \mathcal{B}((0, 1)) \to [0, 1]\) be the Lebesgue-Borel measure on \((0, 1)\), for every \(r \in [0, \infty)\), \(p \in (1, \infty)\) let \((W^{r,p}((0, 1), \mathbb{R}), \|\cdot\|_{W^{r,p}((0, 1), \mathbb{R})})\) be the Sobolev-Slobodetskii space with smoothness parameter \(r\) and integrability parameter \(p\) of equivalence classes of \(\mathcal{B}((0, 1))/\mathcal{B}(\mathbb{R})\)-measurable functions, for every \(r \in [0, 2)\), \(p \in (1, \infty)\) let \((W^{r,p}((0, 1), \mathbb{R}), \|\cdot\|_{W^{r,p}((0, 1), \mathbb{R})})\) be the \(\mathbb{R}\)-Banach space which satisfies that

\[
W^{r,p}((0, 1), \mathbb{R}) = \left\{ f \in W^{r,p}((0, 1), \mathbb{R}) : \left[ \exists g \in C([0, 1], \mathbb{R}) : (g|_{[0, 1])} f \right] \right\} : r \leq \frac{1}{p}
\]

and \(\forall v \in W^{r,p}((0, 1), \mathbb{R}) : \|v\|_{W^{r,p}((0, 1), \mathbb{R})} = \|v\|_{W^{r,p}((0, 1), \mathbb{R})}\), for every \(p \in (1, \infty)\) let \((L^p(\lambda; \mathbb{R}), \|\cdot\|_{L^p(\lambda; \mathbb{R})})\) be the \(\mathbb{R}\)-Banach space which satisfies that \((L^p(\lambda; \mathbb{R}), \|\cdot\|_{L^p(\lambda; \mathbb{R})}) = (W^{0,p}((0, 1), \mathbb{R}), \|\cdot\|_{W^{r,p}((0, 1), \mathbb{R})})\) and let \(A_p: D(A_p) \subseteq L^p(\lambda; \mathbb{R}) \to L^p(\lambda; \mathbb{R})\) be the linear operator which satisfies that \(D(A_p) = \mathcal{V}^{2,p}((0, 1), \mathbb{R})\) and \(\forall h \in D(A_p) : A_p(h) = \Delta h\), for every \(p \in (1, \infty)\) let \((V_{r,p}, \|\cdot\|_{V_{r,p}})\), \(r \in \mathbb{R}\), be a family of interpolation spaces associated to \(-A_p\) and for every \(\delta \in (0, 1)\) let \((C^\delta([0, 1], \mathbb{R}), \|\cdot\|_{C^\delta([0, 1], \mathbb{R})})\) be the space of \(\delta\)-Hölder continuous functions from \([0, 1]\) to \(\mathbb{R}\).

Note that for every \(p \in (1, \infty)\) it holds that \(A_p\) is the Dirichlet Laplacian on \(L^p(\lambda; \mathbb{R})\). The relationship between the spaces \(W^{2r,p}((0, 1), \mathbb{R})\) and \(V_{r,p}\), \(r \in (0, 1), p \in (1, \infty)\), is discussed in Lemma 4.2 below.

4.2 Preparatory lemmas

Various results closely related to Lemmas 4.2–4.5 below are available in the literature; see, e.g., Lemarie-Rieusset & Gala [LRG06] Lemma 1 for a result closely related to Lemmas 4.3 and 4.4 below. We provide these lemmas in the exact form that we need.

**Lemma 4.2.** Assume Setting 4.1. Then

(i) it holds for every \(r \in (0, 1)\) \(\backslash\{1/4\}\) that \(V_{r,2} \subseteq W^{2r,2}((0, 1), \mathbb{R})\) continuously,

(ii) it holds for every \(r \in (0, 1)\) \(\backslash\{1/4\}\) that \(W^{2r,2}((0, 1), \mathbb{R}) \subseteq V_{r,2}\) continuously, and
(iii) it holds for every $p \in (1, \infty)$, $r \in (0, 1)$, $s \in [0, r)$ that $V_{r,p} \subseteq W^{2s,p}((0,1),\mathbb{R})$ continuously.

**Proof of Lemma 4.2.** First, note that, e.g., Triebel [Tri78] Theorem 1.15.3, Definition 2.3.1/1, item (d) in Theorem 2.3.2, Definition 4.2.1/1, Definition 4.3.3/2, equation (7) in Theorem 4.3.3, item (b) in Theorem 4.9.1, and item (b) in Theorem 5.5.1 [with $k = 1$, $B_1 = \text{id}_{C^1(0,1)}$, $m_1 = 0$, $m = 2$, $p = 2$, $\theta = r$ for $r \in (0, 1) \setminus \{1/4\}$ in the notation of Triebel Definition 4.3.3/2 and equation (7) in Theorem 4.3.3] implies that for every $r \in (0, 1) \setminus \{1/4\}$ it holds that $V_{r,2} \subseteq W^{2r,2}((0,1),\mathbb{R}) \subseteq V_{r,2}$ continuously (cf. Triebel [Tri78] Definition 2.3.1/1 and Definition 4.2.1/1) for a definition of $W^{r,p}((0,1),\mathbb{C})$, $r \in (0, \infty)$, $p \in (1, \infty)$, and cf. Triebel [Tri78] Section 4.2.4, Remark 2 in Section 4.4.1, and Remark 2 in Section 4.4.2 for equivalent definitions of $W^{r,p}((0,1),\mathbb{C})$, $r \in (0, \infty)$, $p \in (1, \infty)$. This proves items (i) and (ii). Next observe that, e.g., Triebel [Tri78] Theorem 1.15.3, Definition 4.2.1/1, Definition 4.3.3/2, equation (7) in Theorem 4.3.3, items (a)–(b) in Theorem 4.6.1, item (b) in Theorem 4.9.1, item (c) in Theorem 5.4.4/1, and item (b) in Theorem 5.5.1 [with $k = 1$, $B_1 = \text{id}_{C^1(0,1)}$, $m_1 = 0$, $m = 2$, $p = p$, $\theta = r - (\varepsilon/2)\{1/(2p)\}(r)$ for $r \in (1, \infty)$, $\varepsilon \in (0, r)$, $r \in (0, 1)$ in the notation of Triebel Definition 4.3.3/2 and equation (7) in Theorem 4.3.3] ensures that for every $r \in (1, \infty)$, $r \in (0, 1)$, $\varepsilon \in (0, r)$ it holds that $V_{r,p} \subseteq W^{2r(\varepsilon-r)\{1/(2p)\}+\{1/(2p)\}(r),\mathbb{R})$ continuously. This establishes item (iii). The proof of Lemma 4.2 is thus completed. \qed

**Lemma 4.3.** Assume Setting 4.1 and let $r \in [0, 1/2) \setminus \{1/4\}$, $\delta \in (2r, 1)$. Then

(i) it holds for every $f \in V_{r,2}$, $v \in C^3([0, 1],\mathbb{R})$ that $fv \in V_{r,2}$ and

(ii) it holds that

\[
\sup \left\{ \frac{\|fv\|_{V_{r,2}}}{\|fv\|_{C^3(0,1),\mathbb{R}}}; \quad f \in V_{r,2}\setminus \{0\}, \quad v \in C^3([0, 1],\mathbb{R})\setminus \{0\} \right\} \leq \frac{\sqrt{3}}{\sqrt{\delta - 2r}} \sup \left\{ \frac{\|w\|_{W^{2r,2}(0,1),\mathbb{R}}}{\|w\|_{W^{r,2}(0,1),\mathbb{R}}} \right\} \left( \sup \left\{ \frac{\|w\|_{W^{2r,2}(0,1),\mathbb{R}}}{\|w\|_{W^{2r,2}(0,1),\mathbb{R}}} \right\} \right) < \infty. \tag{126}
\]

**Proof of Lemma 4.3.** To prove items (i) and (ii) we distinguish between the case $r = 0$ and the case $r > 0$. We first prove items (i) and (ii) in the case $r = 0$. Observe that the fact that for every $w \in C([0, 1],\mathbb{R})$ it holds that

\[
\|w\|_{C^3([0, 1],\mathbb{R})} = \sup_{x \in [0, 1]} |w(x)| + \sup_{x, y \in [0, 1], x \neq y} \left( \frac{|w(x) - w(y)|}{|x - y|^3} \right)
\]

establishes items (i) and (ii) in the case $r = 0$. Next we prove items (i) and (ii) in the case $r > 0$. Note that item (i) in Lemma 4.2 and (23) in Jentzen & Röckner [JR12] imply that for every $f \in V_{r,2}$, $v \in C^3([0, 1],\mathbb{R})$ it holds that $f \in W^{2r,2}((0, 1),\mathbb{R})$ and $fv \in W^{2r,2}((0, 1),\mathbb{R})$. Combining this and item (ii) in Lemma 4.2 establishes item (i) in the case $r > 0$. Moreover, observe that (23) in Jentzen & Röckner [JR12] assures that

\[
\sup \left\{ \frac{\|fv\|_{W^{2r,2}(0,1),\mathbb{R}}}{\|fv\|_{C^3(0,1),\mathbb{R}}}; \quad f \in W^{2r,2}(0,1),\mathbb{R})\setminus \{0\}, \quad v \in C^3([0, 1],\mathbb{R})\setminus \{0\} \right\} \leq \frac{\sqrt{3}}{\sqrt{\delta - 2r}}. \tag{128}
\]

Combining this and items (i)–(ii) in Lemma 4.2 establishes item (ii) in the case $r > 0$. The proof of Lemma 4.3 is thus completed. \qed

**Lemma 4.4.** Assume Setting 4.1 and let $r \in (0, 1/2) \setminus \{1/4\}$, $\delta \in (2r, 1)$. Then

\[
\sup \left\{ \frac{\|fv\|_{V_{r,2}}}{\|fv\|_{C^3(0,1),\mathbb{R}}}; \quad f \in L^2(\lambda,\mathbb{R})\setminus \{0\}, \quad v \in C^3([0, 1],\mathbb{R})\setminus \{0\} \right\} \leq \frac{\sqrt{3}}{\sqrt{\delta - 2r}} \sup \left\{ \frac{\|w\|_{V_{r,2}}}{\|w\|_{W^{2r,2}(0,1),\mathbb{R}}} \right\} \left( \sup \left\{ \frac{\|w\|_{V_{r,2}}}{\|w\|_{W^{2r,2}(0,1),\mathbb{R}}} \right\} \right) < \infty. \tag{129}
\]
Proof of Lemma 4.4. First, note that Lemma 4.3 proves that for every $u \in L^2(\lambda; \mathbb{R})$, $v \in C^\delta([0, 1], \mathbb{R})$ it holds that $v(-A_2)^{-\tau}u \in V_{\gamma, 2}$. The fact that for every $f \in L^2(\lambda; \mathbb{R})$, $v \in C^\delta([0, 1], \mathbb{R})$ it holds that $fv \in L^2(\lambda; \mathbb{R})$ and the self-adjointness of $L^2(\lambda; \mathbb{R}) \ni v \mapsto (-A_2)^{-\tau}v \in L^2(\lambda; \mathbb{R})$ therefore imply that for every $f \in L^2(\lambda; \mathbb{R})$, $v \in C^\delta([0, 1], \mathbb{R})$ it holds that

$$
\|fv\|_{V_{\gamma, 2}} = \|(-A_2)^{-\tau}(fv)\|_{L^2(\lambda; \mathbb{R})} = \sup_{u \in L^2(\lambda; \mathbb{R}) \setminus \{0\}} \frac{\langle (A_2)^{-\tau}u, fv \rangle_{L^2(\lambda; \mathbb{R})}}{\|u\|_{L^2(\lambda; \mathbb{R})}} 
= \sup_{u \in L^2(\lambda; \mathbb{R}) \setminus \{0\}} \frac{\langle (A_2)^{-\tau}(v(A_2)^{-\tau}u), (-A_2)^{-\tau}f \rangle_{L^2(\lambda; \mathbb{R})}}{\|u\|_{L^2(\lambda; \mathbb{R})}} 
\leq \left[ \sup_{u \in L^2(\lambda; \mathbb{R}) \setminus \{0\}} \frac{\|v(A_2)^{-\tau}u\|_{V_{\gamma, 2}}}{\|u\|_{L^2(\lambda; \mathbb{R})}} \right] \|f\|_{V_{\gamma, 2}}. 
$$

(130)

Combining this and Lemma 4.3 (with $v = v$, $f = (-A_2)^{-\tau}u$ in the notation of Lemma 4.3) establishes (129). The proof of Lemma 4.4 is thus completed.

Lemma 4.5. Assume Setting 4.3 for every $\mathbb{R}$-Hilbert space $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ let $(L_2(H), \langle \cdot, \cdot \rangle_{L_2(H)}, \|\cdot\|_{L_2(H)})$ be the $\mathbb{R}$-Hilbert space of Hilbert-Schmidt operators from $H$ to $H$, let $\tau \in (-1/4, 1/4)$, and for every $m \in V_{\max(0, r), 2}$ let $M_m: D(M_m) \to L^2(\lambda; \mathbb{R})$ be the linear operator which satisfies that $D(M_m) = \{h \in L^2(\lambda; \mathbb{R}) : mh \in L^2(\lambda; \mathbb{R})\}$ and $\forall h \in D(M_m): M_mh = mh$. Then

(i) it holds for every $m \in V_{\max(0, r), 2}$, $h \in L^2(\lambda; \mathbb{R})$ that $(-A_2)^{-1/2}h \in D(M_m)$ and $M_m(-A_2)^{-1/2}h \in V_{\max(0, r), 2}$ and

(ii) it holds that $\sup_{m \in V_{\max(0, r), 2}\setminus\{0\}} \left[ \frac{\|(-A_2)^{\tau}M_m(-A_2)^{-1/2}\|_{L^2(\lambda; \mathbb{R})}}{\|m\|_{V_{\gamma, 2}}} \right] < \infty.$

Proof of Lemma 4.5. Throughout this proof let $g = \max\{0, r\}$, $\varepsilon \in (0, \frac{1}{2} - g)$, let $e_n: [0, 1] \to \mathbb{R}$, $n \in \mathbb{N}$, be the functions which satisfy for every $n \in \mathbb{N}$, $x \in [0, 1]$ that $e_n(x) = \sqrt{2} \sin(n\pi x)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\gamma_n: \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, be independent standard Gaussian random variables, and let $K_p \in [1, \infty)$, $p \in [1, \infty)$, be the extended real numbers which satisfy for every $p \in [1, \infty)$ that

$$
K_p = \sup \left\{ \left( \mathbb{E} \left[ \sum_{k=1}^n |\gamma_k|^p \right] \right)^{1/p} : n \in \mathbb{N}, x_1, \ldots, x_n \in \mathbb{R} \setminus \{0\} \right\}. 
$$

(131)

Observe that the Khintchine inequalities imply that for every $p \in [1, \infty)$ it holds that $K_p < \infty$. Moreover, note that item [1] in Lemma 4.2 and the fractional Sobolev inequalities prove that for every $h \in L^2(\lambda; \mathbb{R})$, $\delta \in (0, 1/2)$ it holds that there exists a $v \in C^\delta([0, 1], \mathbb{R})$ such that $(-A_2)^{-1/2}h = [v]_{\lambda, B(\mathbb{R})}$. Lemma 4.3 and the fact that for every $f \in L^2(\lambda; \mathbb{R})$, $v \in C^\delta([0, 1], \mathbb{R})$, $v_1, v_2 \in [v]_{\lambda, B(\mathbb{R})}$ it holds that

$$
[v_1, f]_{\lambda, B(\mathbb{R})} = [v_2 f]_{\lambda, B(\mathbb{R})} 
$$

(132)

hence imply that for every $h \in L^2(\lambda; \mathbb{R})$, $m \in V_{\gamma, 2}$ it holds that $M_m(-A_2)^{-1/2}h \in V_{\gamma, 2}$. This establishes item [6]. Furthermore, observe that for every $p \in (1, \infty)$ it holds that

$$
\sup_{n \in \mathbb{N}} \left( \mathbb{E} \left[ \left( \sum_{k=1}^n (k\pi)^{-1} |\gamma_k e_k|^2 \right)^{p/2} \middle| C^\delta_{p+\varepsilon}([0, 1], \mathbb{R}) \right] \right)^{1/2} \leq \sup \left\{ \frac{\|v\|_{L^2\left(\lambda; C^{p+\varepsilon}([0, 1], \mathbb{R})\right)}}{\|v\|_{L^2\left(\lambda; \mathbb{R}\right)}} : v \in C^\delta([0, 1], \mathbb{R}) \setminus \{0\} \right\}
\cdot \sup_{n \in \mathbb{N}} \left( \mathbb{E} \left[ \left( \sum_{k=1}^n (k\pi)^{-1+2(p+\varepsilon)} |\gamma_k e_k|^2 \right)^{p/2} \right] \right)^{1/2}. 
$$

(133)
Moreover, note that Lemma 4.3 and the fractional Sobolev inequalities demonstrate that for every $p \in (\varepsilon^{-1}, \infty)$ it holds that

$$
\sup \left\{ \sup_{v \in C^2([0,1], \mathbb{R}) \setminus \{0\}} \left[ \|v\|_{C^{2,\varepsilon}([0,1], \mathbb{R})} : v(0) = v(1) = 0 \right] \right\} < \infty.
$$

(134)

Moreover, note that Hölder’s inequality, Fubini’s theorem, and (131) imply that for every $p \in (\varepsilon^{-1}, \infty)$ it holds that

$$
\sup_{n \in \mathbb{N}} \left( \mathbb{E} \left[ \left( \sum_{k=1}^{n} (k\pi)^{-1+2(\varepsilon+\varepsilon)} \gamma_k e_k \right)^2 \right] \right)^{1/2}
$$

$$
= \sup_{n \in \mathbb{N}} \left( \mathbb{E} \left[ \frac{1}{n} \sum_{k=1}^{n} (k\pi)^{-1+2(\varepsilon+\varepsilon)} \gamma_k e_k(x) \right] \right)^{1/2}
$$

$$
\leq \sup_{n \in \mathbb{N}} \left( \mathbb{E} \left[ \left( \sum_{k=1}^{n} (k\pi)^{-1+2(\varepsilon+\varepsilon)} \gamma_k e_k(x) \right)^2 \right] \right)^{1/2}
$$

$$
= \sup_{n \in \mathbb{N}} \left( \frac{1}{n} \mathbb{E} \left[ \left( \sum_{k=1}^{n} (k\pi)^{-1+2(\varepsilon+\varepsilon)} \gamma_k e_k(x) \right)^2 \right] \right)^{1/2}
$$

$$
\leq K_p \sup_{n \in \mathbb{N}} \left( \frac{1}{n} \mathbb{E} \left[ \left( \sum_{k=1}^{n} (k\pi)^{-2+4(\varepsilon+\varepsilon)} |e_k(x)| \right)^2 \right] \right)^{1/2}
$$

$$
\leq \sqrt{2} K_p \sup_{n \in \mathbb{N}} \left( \sum_{k=1}^{n} (k\pi)^{-2+4(\varepsilon+\varepsilon)} \right)^{1/2} = \sqrt{2} K_p \left( \sum_{k=1}^{\infty} (k\pi)^{-2+4(\varepsilon+\varepsilon)} \right)^{1/2} < \infty.
$$

(135)

Next observe that for every $m \in V_{\varepsilon,2}$ it holds that

$$
\|(-A_2)^{\gamma} M_m (-A_2)^{-1/2}\|_{L^2(L^2(\lambda, \mathbb{R}))}^2 = \sup_{n \in \mathbb{N}} \left( \sum_{k=1}^{n} \| M_m (-A_2)^{-1/2} [\lambda, B(\mathbb{R})] \|_{V_{\varepsilon,2}}^2 \right)
$$

$$
= \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \| m \sum_{k=1}^{n} (k^{\pi})^{1-\gamma_k e_k} \|_{V_{\varepsilon,2}}^2 \right] (136)
$$

Moreover, note that Lemma 4.3, Lemma 4.4, (133), (134), and (135) imply that

$$
\sup_{m \in V_{\varepsilon,2}\setminus\{0\}} \left\{ \sup_{n \in \mathbb{N}} \left( \mathbb{E} \left[ \left( \sum_{k=1}^{n} (k\pi)^{-1} \gamma_k e_k \right)^2 \right] \right)^{1/2} \right\}
$$

$$
\leq \sqrt{2} \left( \sup_{w \in V_{\varepsilon,2}\setminus\{0\}} \left\| \frac{\|w\|_{V_{\varepsilon,2}}}{\|w\|_{W^{2,\varepsilon}([0,1], \mathbb{R})}} \right\| \sup_{w \in V_{\varepsilon,2}\setminus\{0\}} \left( \mathbb{E} \left[ \left( \sum_{k=1}^{n} (k^{\pi})^{-1} \gamma_k e_k \right)^{2} \right] \right)^{1/2} \right)
$$

$$
< \infty.
$$

(137)

Combining this and (136) establishes item (ii). The proof of Lemma 4.5 is thus completed.

□
4.3 The hyperbolic Anderson model

**Corollary 4.6.** For every pair of $\mathbb{R}$-Hilbert spaces $(V, \langle \cdot, \cdot \rangle_V, \|\cdot\|_V)$ and $(W, \langle \cdot, \cdot \rangle_W, \|\cdot\|_W)$ let $(L_2(V, W), \langle \cdot, \cdot \rangle_{L_2(V, W)}, \|\cdot\|_{L_2(V, W)})$ be the $\mathbb{R}$-Hilbert space of Hilbert-Schmidt operators from $V$ to $W$, for every measure space $(\Omega, \mathcal{F}, \mu)$, every measurable space $(S, \Sigma)$, every set $\mathcal{O}$, and every function $f : \mathcal{O} \to S$ let $[f]_{\mu, \Sigma}$ be the set given by

$$[f]_{\mu, \Sigma} = \{ g : \Omega \to S : \left[ \exists A \in \mathcal{F} : (\mu(A) = 0 \text{ and } \{ \omega \in \Omega : f(\omega) \# g(\omega) \subseteq A \}) \right] \},$$

(138)

let $T, \varrho \in (0, \infty)$, $b_0, b_1 \in \mathbb{R}$, let $\lambda : \mathcal{B}(0, 1) \to [0, 1]$ be the Lebesgue-Borel measure on $(0, 1)$, let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ be the $\mathbb{R}$-Hilbert space given by $(H, \langle \cdot, \cdot \rangle_H) = (L^2(\lambda; \mathbb{R}), \langle \cdot, \cdot \rangle_{L^2(\lambda; \mathbb{R})}, \|\cdot\|_{L^2(\lambda; \mathbb{R})})$, let $(\Omega, \mathcal{F}, \mu, (\mathcal{F}_t)_{t \in [0, T])}$ be a filtered probability space which fulfills the usual conditions, let $(W_t)_{t \in [0, T]}$ be an id$_H$-cylindrical $(\mathcal{F}_t)_{t \in [0, T]}$-Wiener process, for every $n \in \mathbb{N}$ let $e_n \in H$ satisfy $e_n = [(\sqrt{2} \sin(n \pi x)]_{x \in (0, 1]} \lambda (\mathbb{R})$, let $A : D(A) \subseteq H \to H$ satisfy $D(A) = \{ h \in H : \sum_{n=1}^{\infty} [(n \pi)^2 \langle e_n, h \rangle_H]^2 < \infty \}$ and $\forall h \in D(A) : Ah = \sum_{n=1}^{\infty} \langle n \pi^2 \langle e_n, h \rangle_H^2 e_n \}$, let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$, for every $r \in \mathbb{R}$ let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$ be the $\mathbb{R}$-Hilbert space which satisfies $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r}) = (H_{r/2} \times H_{r/2 \downarrow} \times \cdots \times H_{r/2 \downarrow_{2}}, \lambda_{r/2 \downarrow})$, let $A : D(A) \subseteq H_0 \to H_0$ satisfy $D(A) = H_1$ and $\forall (v, w) \in H_1 : A(v, w) = (v, \varrho Av)$, let $\varphi \in C^4(\mathbb{H}_0, \mathbb{R})$ satisfy for every $k \in \{1, 2, 3, 4\}$ that $\sup_{x \in \mathbb{H}_0} \|\varphi^{(k)}(x)\|_{L^0(\mathbb{H}_0, \mathbb{R})} < \infty$, let $\xi \in C^6([F_{\mathcal{F}_H}; \mathbb{H}_0])$, $B : \mathbb{H}_0 \to L_2(\mathbb{H}_0, \mathbb{H}_0)$ satisfy for every $(v, w) \in \mathbb{H}_0$, $u \in \mathbb{H}$ that $B(v, w)u = (0, B(v)u)$, let $X : [0, T] \times \Omega \to \mathbb{H}_0$ be an $(\mathcal{F}_t)_{t \in [0, T]}$-predictable stochastic process which satisfies for every $t \in [0, T]$ that $\sup_{s \in [0, T]} \mathbb{E}[\|X_s\|_{\mathbb{H}_0}^2] < \infty$ and

$$[X_t]_{\mathcal{P}, \mathcal{B}(\mathbb{H}_0)} = [e^{A_h} \xi]_{\mathcal{P}, \mathcal{B}(\mathbb{H}_0)} + \int_0^t e^{(t-s)A}B(X_s) dW_s,$$

(139)

for every $h \in (0, T]$ let $[\cdot]_h : [0, \infty) \to \mathbb{R}$ satisfy for every $x \in [0, \infty)$ that

$$[x]_h = \max \{ \{0, h, 2h, 3h, \ldots \} \cap [0, x] \},$$

(140)

and for every $h \in (0, T]$ let $Y^h : [0, T] \times \Omega \to \mathbb{H}_0$ be an $(\mathcal{F}_t)_{t \in [0, T]}$-predictable stochastic processes which satisfies for every $t \in [0, T]$ that $\sup_{s \in [0, T]} \mathbb{E}[\|Y^h_s\|_{\mathbb{H}_0}^2] < \infty$ and

$$[Y^h_t]_{\mathcal{P}, \mathcal{B}(\mathbb{H}_0)} = [e^{A_h} \xi]_{\mathcal{P}, \mathcal{B}(\mathbb{H}_0)} + \int_0^t e^{(t-[s]_h)A}B(Y^h_{[s]_h}) dW_s.$$

(141)

Then it holds for every $\varepsilon \in (0, \infty)$ that

$$\sup_{h \in (0, T]} \left( h^{\varepsilon-1} \left| \mathbb{E}[\varphi(X_T)] - \mathbb{E}[\varphi(Y^h_T)] \right| \right) < \infty.$$  

(142)

**Proof of Corollary 4.6.** Throughout this proof for every $\mathbb{R}$-Hilbert space $(V, \langle \cdot, \cdot \rangle_V, \|\cdot\|_V)$ let $(L_2(V), \langle \cdot, \cdot \rangle_{L_2(V)}, \|\cdot\|_{L_2(V)})$ be the $\mathbb{R}$-Hilbert space of Hilbert-Schmidt operators from $V$ to $V$, for every pair of $\mathbb{R}$-Banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ let $(\text{Lip}(V, W), \|\cdot\|_{\text{Lip}(V,W)})$ be the $\mathbb{R}$-Banach space of Lipschitz continuous mappings from $V$ to $W$, for every $\ell \in \mathbb{N}$ and every pair of $\mathbb{R}$-Banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ let $(C^\ell_b(V, W), \|\cdot\|_{C^\ell_b(V,W)})$ be the $\mathbb{R}$-Banach space of $\ell$-times continuously Fréchet differentiable functions from $V$ to $W$ with globally bounded derivatives, for every $v \in H$ let $M_v : D(M_v) \subseteq H \to H$ be the linear operator which satisfies that $D(M_v) = \{ h \in H : \varrho h \in H \}$ and $[\varrho h \in D(M_v) : M_v h = \varrho h)$, and let $\varepsilon \in (0, 2/3]$. Observe that the fact that for every $\rho \in [0, 1/4)$ it holds that $B \in \text{Lip}(H, L_2(H, H_{\rho-1/2}))$
implies that for every \( \rho \in [0, 1/4) \) it holds that \( B \in \text{Lip}(H_0, L_2(H, H_{2\rho})) \). Hence, we obtain that \( B \in \text{Lip}(H_0, L_2(H, H_{1/2-\varepsilon/4})) \) and
\[
B|_{H_{1/2-\varepsilon/4}} \in \text{Lip}(H_{1/2-\varepsilon/4}, L_2(H, H_{1/2-\varepsilon/4})).
\] (143)

Moreover, note that de Naurois et al. [JdNW21] (3.74)–(3.75) in Section 3.3 and Hölder’s inequality ensure that for every \( \rho \in (0, 1/4) \), \( v, w \in H_\rho \), \( u \in H_1 \) it holds that \( B(v), B(w) \in L(H, H_{\rho-1/4}) \) and
\[
\| (B(v) - B(w))u \|_{H_{\rho-1/4}} = \sup_{\psi \in H_1 \setminus \{0\}} \frac{|(\psi, (B(v) - B(w))u)_H|}{\|\psi\|_{H^{(1/4)-\rho}_1}} = \sup_{\psi \in H_1 \setminus \{0\}} \frac{\|\psi b_1(v - w)\|_{L^1(\lambda \mathbb{R})}}{\|\psi\|_{H^{(1/4)-\rho}_1}} 
\leq \sup_{\psi \in H_1 \setminus \{0\}} \frac{\|\psi\|_{L^{1/(2\rho)}(\lambda \mathbb{R})}}{\|\psi\|_{H^{(1/4)-\rho}_1}} \sup_{\zeta \in H_\rho \setminus \{0\}} \frac{\|\zeta\|_{L^{2/(1-4\rho)}(\lambda \mathbb{R})}}{\|\zeta\|_{H_{\rho}}} \| v - w \|_{H_{\rho}} < \infty.
\] (144)

This assures that for every \( \rho \in (0, 1/4) \) it holds that \( B|_{H_\rho} \in \text{Lip}(H_\rho, L(H, H_{\rho-1/4})) \) and \( B|_{H_{2\rho}} \in \text{Lip}(H_{2\rho}, L(H, H_{2\rho+1/2})) \). Therefore, we obtain that
\[
B|_{H_{1/2-\varepsilon/4}} \in \text{Lip}(H_{1/2-\varepsilon/4}, L(H, H_{1-\varepsilon/4})).
\] (145)

Furthermore, observe that for every \((v_1, v_2), (w_1, w_2) \in H_0 \), \( u \in H_1 \) it holds that
\[
B(v_1 + w_1, v_2 + w_2)u = (0, B(v_1 + w_1)u) = (0, (b_0 + b_1(v_1 + w_1))u) = (0, (b_0 + b_1v_1)u + (0, b_1w_1)u) = (0, b_1w_1)u.
\] (146)

Combining this, the fact that for every \((v_1, v_2) \in H_0 \) it holds that \( B(v_1, v_2) \in L_2(H, H_0) \), and the fact that \( H_1 \) is a dense subset of \( H \) implies that for every \((v_1, v_2), (w_1, w_2) \in H_0 \), \( u \in H_1 \) it holds that \( B \in C^1(H_0, L_2(H, H_0)) \) and
\[
\left[ (B^{(1)}(v_1, v_2))(w_1, w_2) \right] u = (0, b_1w_1)u.
\] (147)

Hence, we obtain that for every \( k \in \mathbb{N} \) it holds that
\[
B \in C^k(H_0, L_2(H, H_0)).
\] (148)

Next observe that for every \((v_1, v_2) \in H_0 \) it holds that
\[
\sum_{n \in \mathbb{N}} (n\pi)^{1-\varepsilon} \|B(v_1, v_2)e_n\|_{H_0}^2 = \sum_{n \in \mathbb{N}} (n\pi)^{1-\varepsilon} \|(-A)^{-1/2}((b_0 + b_1v_1)e_n)\|_H^2 
= \sum_{n \in \mathbb{N}} (n\pi)^{1-\varepsilon} \|b_0(-A)^{-1/2}e_n + b_1(-A)^{-1/2}M_{v_1}e_n\|_H^2 
= \sum_{n \in \mathbb{N}} \|b_0(-A)^{-1+\varepsilon/4}e_n + b_1(-A)^{-1/2}M_{v_1}(-A)^{(1-\varepsilon)/4}e_n\|_H^2 
\leq 2|b_0|^2 \|(-A)^{-1+\varepsilon/4}\|_{L_2(H)}^2 + 2|b_1|^2 \left[ \sum_{n \in \mathbb{N}} \||-A)^{-1/2}M_{v_1}(-A)^{(1-\varepsilon)/4}e_n\|_H^2 \right].
\] (149)
Moreover, note that for every \((v_1, v_2), (w_1, w_2) \in \mathbf{H}_0\) it holds that
\[
\sum_{n \in \mathbb{N}} (n \pi)^{\varepsilon - 1} \| B(v_1, v_2) - B(w_1, w_2) e_n \|_{\mathbf{H}_0}^2
= \sum_{n \in \mathbb{N}} (n \pi)^{\varepsilon - 1} \| (-A)^{-1/2} (b_1 (v_1 - w_1) e_n) \|_{H}^2
= |b_1|^2 \left( \sum_{n \in \mathbb{N}} \| (-A)^{-1/2} M_{v_1 - w_1} (-A)^{(\varepsilon - 1)/4} e_n \|_{H}^2 \right).
\]
(150)

In addition, observe that item \(\[\]\) in Lemma 4.5 ensures that for every \(r \in (-1/4, 1/4)\), \(v \in H_{\max(0, r)}\), \(n \in \mathbb{N}\) it holds that \(M_v (-A)^{-1/2} e_n \in H_{\max(0, r)}\). This and the fact that for every \(v \in H\) it holds that \(M_v : D(M_v) \subseteq H \to H\) is a symmetric linear operator imply that for every \(r \in (-1/4, 1/4)\), \(v \in H_{\max(0, r)}\) it holds that
\[
\sum_{n \in \mathbb{N}} \| (-A)^{-1/2} M_v (-A)^r e_n \|_{H}^2 = \sum_{m, n \in \mathbb{N}} \left\| (-A)^{-1/2} M_v (-A)^r e_n, e_m \right\|_{H}^2
= \sum_{m, n \in \mathbb{N}} \left\| (-A)^{-1/2} M_v (-A)^r e_n, e_m \right\|_{H}^2 \sum_{n \in \mathbb{N}} \| (-A)^{1/2} M_v (-A)^{-1/2} e_m \|_{H}^2.
\]
(151)

Lemma 4.5, (149), (150), and the fact that for every \(r \in (1/4, \infty)\) it holds that \(\| A^{-r} \|_{L_2(H)} < \infty\) therefore ensure that
\[
\sup \left\{ \frac{\sum_{n \in \mathbb{N}} (n \pi)^{\varepsilon - 1} \| B(v_1, v_2) e_n \|_{H_{\max(0, r)}}^2}{\max\{1, \| (v_1, v_2) \|_{H_{\max(0, r)}}^2\}} : (v_1, v_2) \in \mathbf{H}_{(1-\varepsilon)/2} \right\}
\leq \sup \left\{ \frac{2|b_1|^2 \| (-A)^{-1/4} M_{v_1 - w_1} (-A)^{-1/2} e_n \|_{L_2(H)}^2}{\max\{1, \| (v_1, v_2) \|_{H_{\max(0, r)}}^2\}} : (v_1, v_2) \in \mathbf{H}_{(1-\varepsilon)/2} \right\}
\leq \infty.
\]
(152)

and
\[
\sup \left\{ \frac{\sum_{n \in \mathbb{N}} (n \pi)^{\varepsilon - 1} \| B(v_1, v_2) - B(w_1, w_2) e_n \|_{H_{\max(0, r)}}^2}{\| (v_1, v_2) - (w_1, w_2) \|_{\mathbf{H}_{(1-\varepsilon)/2}}} : (v_1, v_2), (w_1, w_2) \in \mathbf{H}_{(1-\varepsilon)/2}, (v_1, v_2) \neq (w_1, w_2) \right\}
= \sup \left\{ \frac{|b_1|^2 \| (-A)^{(\varepsilon - 1)/4} M_{v_1 - w_1} (-A)^{-1/2} e_n \|_{L_2(H)}^2}{\| (v_1, v_2) - (w_1, w_2) \|_{\mathbf{H}_{(1-\varepsilon)/2}}} : (v_1, v_2), (w_1, w_2) \in \mathbf{H}_{(1-\varepsilon)/2}, (v_1, v_2) \neq (w_1, w_2) \right\}
< \infty.
\]
(153)

Combining this, (143), (145), (148), and Corollary 3.12 (with \(U = H, \cup = \{e_n\}_{n \in \mathbb{N}}, T = T, (W_t)_{t \in [0, T]} = (W_t)_{t \in [0, T]}\)) it holds that \(\gamma = 1 - \varepsilon / 4, \beta = 1/2 + \varepsilon / 4, \rho = 1/2 - \varepsilon / 4, H = H, \mathbb{H} = \{e_n\}_{n \in \mathbb{N}}, \forall n \in \mathbb{N} : \lambda_n = -\vartheta (n \pi)^2, A = \vartheta A, \forall r \in \mathbb{R} : \| H_r \| = \vartheta r \| H_r \|, A = A, \varphi = \varphi, \xi = \xi, F = 0, B = B, \forall n \in \mathbb{N} : \mu_n = (n \pi)^{1-\varepsilon / 2}, m = \max\{\vartheta^{-\varepsilon / 2}, 1\} \| B \|_{\mathbf{C}^1_h(H_0, L_2(H, \mathbf{H}_0))} + 1, c^2 = \max\{\vartheta^{\varepsilon / 2}, \vartheta^{1-\varepsilon / 2}\} \sup \left\{ \sum_{n \in \mathbb{N}} (n \pi)^{\varepsilon - 1} \| B(v_1, v_2) e_n \|_{H_{(1-\varepsilon)/2}}^2 : (v_1, v_2) \in \mathbf{H}_{(1-\varepsilon)/2} \right\}, c^2 = \max\{\vartheta^{-1+\varepsilon / 2}, \vartheta^{1-\varepsilon / 2}\} \sup \left\{ \sum_{n \in \mathbb{N}} (n \pi)^{\varepsilon - 1} \| B(v_1, v_2) - B(w_1, w_2) e_n \|_{H_{(1-\varepsilon)/2}}^2 : (v_1, v_2), (w_1, w_2) \in \mathbf{H}_{(1-\varepsilon)/2} \right\}, \forall h \in (0, T]: Y^h, H = Y^{h, H}, Y^{0, H} = X \in the notation of Corollary 3.12 establishes (142). The proof of Corollary 4.6 is thus completed.

\[\square\]

In Corollary 4.6 the multiplicative noise term appearing in the hyperbolic SPDE (139) is specified via the mapping \(B : H \to L_2(H, H_{1/2})\), which is defined as a combination of a multiplication operator and a Nemyskii operator induced by an affine transformation. We expect that the approach to weak error analysis presented in this article can be extended to hyperbolic SPDEs formulated in a more general Banach space framework involving Banach space-valued stochastic integrals, so that more general classes of nonlinear Nemyskii operators can potentially be handled as well; compare, e.g., [CJKT15, Corollary 1], [HJK16].
4.4 Numerical simulations

Here we illustrate Corollary 4.6 with some numerical experiments. To this end, assume the setting specified in Corollary 4.6: assume that \( T = 2, \vartheta = 1, b_0 = 1, b_1 = 1 \), assume that for every \( \omega \in \Omega \) it holds that \( \xi(\omega) = (e_1, 0) \), let \( J, M \in \mathbb{N}, N \in 2\mathbb{N}, P: H \to H \) satisfy for every \( v \in H \) that \( P(v) = \sum_{j=1}^{J} \langle ej, v \rangle_H e_j \), let \( \mathfrak{M}^N = \Omega \to P(H), N, n, m \in \mathbb{N} \), be independent Gaussian random variables which satisfy for every \( N, n, m \in \mathbb{N}, v, w \in P(H) \) that \( \mathbb{E}[\mathfrak{M}^N_0] = 0 \) and \( \mathbb{E}[\langle v, \mathfrak{M}^N_0 \rangle_H \langle w, \mathfrak{M}^N_0 \rangle_H] = (T/N) \langle v, w \rangle_H \), and for every \( N, m \in \mathbb{N} \) let \( Y^N,m = (Y^N,m,1, Y^N,m,2): \{0, 1, \ldots, N\} \times \Omega \to P(H) \times P(H) \) be the stochastic process which satisfies for every \( n \in \{1, 2, \ldots, N\} \) that \( Y^N,m_n = \xi \) and

\[
Y^N,m_n = e^{(T/N)A}\left(Y^N,m_{n-1}, Y^N,m_{n-1} + P[B(Y^N,m_{n-1}) 2\mathfrak{M}^N_n]\right). \tag{154}
\]

For every \( N \in \{1, 2, \ldots, N/2\} \) we employ the random variable

\[
\left[ \frac{1}{M} \sum_{m=1}^{M} \varphi(Y^N,m) \right] - \left[ \frac{1}{M} \sum_{m=1}^{M} \varphi(Y^N,m) \right]
\]

as a Monte Carlo approximation of the weak error \( \mathbb{E}[\varphi(X_T)] - \mathbb{E}[\varphi(Y^{T/N}_T)] \); compare (142) in Corollary 4.6 above. Note that this Monte Carlo approximation involves a spatial spectral Galerkin approximation using \( J \) basis functions. For the implementation of the scheme (154), we apply the explicit representation of the propagator \( e^{(T/N)A} \) provided by Lemma 3.3. Moreover, in order to approximately compute the multiplicative noise term \( P[B(Y^N,m_{n-1}) 2\mathfrak{M}^N_n] \) in (154), we use the fact that for every \( v, u \in C([0,1], \mathbb{R}) \) it holds that

\[
P[B([v]_{\lambda,B(R)})[u]_{\lambda,B(R)}] = \sum_{j=1}^{J} \langle ej, B([v]_{\lambda,B(R)})[u]_{\lambda,B(R)} \rangle_H e_j
\]

\[
= \sum_{j=1}^{J} \int_{0}^{1} \sqrt{2} \sin(j \pi x) u(x) \, dx \, e_j \tag{156}
\]

and we employ for every \( j \in \{1, 2, \ldots, J\}, v, u \in C([0,1], \mathbb{R}) \) the approximation

\[
\int_{0}^{1} \sqrt{2} \sin(j \pi x) u(x) \, dx \approx \frac{\sum_{k=0}^{J} \sqrt{2} \sin(j \pi \frac{k}{J+1}) u(\frac{k}{J+1})}{J+1}. \tag{157}
\]

Observe that for every \( v, u \in C([0,1], \mathbb{R}) \) with \([v]_{\lambda,B(R)}, [u]_{\lambda,B(R)} \in P(H) \) the approximations (157), \( j \in \{1, 2, \ldots, J\} \), are readily implemented by means of the discrete sine transform; see the PYTHON code in Listing 1.

In Figure 1 we present approximate simulations of the weak error estimator (155) for different test functions \( \varphi \) plotted against the used numbers of time steps \( N \in \{2^3, 2^4, \ldots, 2^{11}\} \), employing \( N = 2^{12} \) time steps for the reference value, \( J = 16 \) spatial basis functions, and \( M = 5 \cdot 10^5 \) Monte Carlo runs. By using a relatively large number of Monte Carlo runs we take into account the fact that the Monte Carlo error tends to dominate the weak error caused by the temporal and spatial approximations. As the focus lies on temporal approximations, we content ourselves with a relatively small number of spatial basis functions. Note that the simulation results presented in Figure 1 are in accordance with Corollary 4.6. In particular, the results in the case \( \varphi(x) = (x(2), \pi e_1)_H_{1/2} \) and \( x = (x(1), x(2)) \in H_0 \), suggest that the weak order 1+ established in Corollary 4.6 is sharp. Moreover, the visible fluctuations of simulation values seem to be due to a dominance of the Monte Carlo error in the corresponding ranges. The PYTHON code used to obtain the simulations is presented in Listing 1.

36
estimated weak error

\[ \varphi(x) = \langle x^{(1)}, e_1 \rangle \]

\[ \varphi(x) = \langle x^{(2)}, \pi e_1 \rangle \]

\[ \varphi(x) = \frac{1}{1 + \|x^{(1)}\|^2} \]

\[ \varphi(x) = \frac{1}{1 + \|x^{(2)}\|^2} \]

order 1

Figure 1: Approximate simulations of the weak error estimator \([155]\) for different test functions \(\varphi(x), x = (x^{(1)}, x^{(2)}) \in H_0\), plotted against the used numbers of time steps \(N \in \{2^3, 2^4, \ldots, 2^{11}\}\), employing \(N = 2^{12}\) time steps for the reference value, \(J = 16\) spatial basis functions, and \(M = 5 \cdot 10^5\) Monte Carlo runs (see the text and Listing 1 for details)

Listing 1: PYTHON code used to create Figure 1

```python
import numpy as np
from scipy.fft import dst
import matplotlib.pyplot as plt

# endpoint of time interval and numbers of time steps
T= 2.; N_list = [2 ** n for n in range (3 , 13) ]

# number of spatial basis functions , number of Monte Carlo runs , and seed
J = 16; M = 5e5; np.random.seed(123)

# initial condition , given in terms of coefficients w.r.t. basis functions
xi = np.array([1] + (2 * J - 1) * [0], dtype = float).reshape(2 , J)

# arrays of eigenvalues of \((-A)^{1/2}\) and \((-A)^{-1/2}\)
sqrt_ev = np.array([j * np.pi for j in range (1 , J +1) ])
sqrt_ev_inv = sqrt_ev ** (-1)

# one-step propagation depending on step size h and array of coefficients
# w.r.t. spatial basis functions of shape (2 , J)
def propagate(h, coeff):
    C = np.cos(h * sqrt_ev ); S = np.sin(h * sqrt_ev )
    new_coeff = np.zeros([2 , J])
    new_coeff[0 ,:] = C * coeff[0 ,:] + sqrt_ev_inv * S * coeff[1 ,:]
    new_coeff[1 ,:] = - sqrt_ev * S * coeff[0 ,:] + C * coeff[1 ,:]
    return new_coeff

# transformation of coefficients w.r.t. basis functions into function values
# w.r.t. spatial basis functions of shape (2 , J)
def coeff_to_fn(coeff):
    return 1 / np.sqrt(2) * dst(coeff, type = 1, norm = 'backward')
```
# transformation of function values into coefficients w.r.t. basis functions
def fn_to_coeff(values):
    return 1 / (np.sqrt(2) * (len(values) + 1))
    * dst(values, type = 1, norm = 'backward')

# simulation of approximate solution of the stochastic wave equation
# with \( N \) time steps and \( J \) spatial basis functions
def simulate_SWE(N):
    Y = np.zeros([N, 2, J])
    Y[0, :, :] = xi
    h = T / N
    for n in range(1, N):
        coeff = Y[n-1, :, :]
        y_1 = coeff_to_fn(coeff[0, :])
        dw = coeff_to_fn(np.sqrt(h) * np.random.normal(0, 1, size = J))
        BdW = fn_to_coeff(y_1 * dw)
        coeff[1, :] += BdW
        Y[n, :, :] = propagate(h, coeff)
    return Y

# Monte Carlo approximation of \( \mathbb{E} \varphi (Y^h_T) \) with step size \( h = T / N \)
# using \( M \) samples and \( J \) spatial basis functions
def Monte_Carlo_EphiY(M, N):
    EphiY = np.zeros(4)
    for m in range(M):
        Y = simulate_SWE(N)
        Y_T = Y[-1, :, :]
        normsq = np.array([np.sum(Y_T[0, :] ** 2),
                           np.sum((sqrt_ev_inv * Y_T[1, :]) ** 2)])
        EphiY += np.array([Y_T[0,0], Y_T[1,0], (1 + normsq[0]) ** (-1),
                           (1 + normsq[1]) ** (-1)])
    EphiY *= 1/M * np.array([1, 1/np.pi, 1, 1])
    return EphiY, Y

# compute weak error estimates for different numbers of time steps
EphiY = np.zeros([len(N_list), 4])
error = np.zeros([len(N_list)-1, 4])
for i in range(len(N_list)):
    EphiY[i, :] = Monte_Carlo_EphiY(M, N_list[i])
for i in range(len(N_list)-1):
    error[i, :] = np.abs(EphiY[-1, :] - EphiY[i, :])

# plot weak error estimates
plt.rcParams['text.usetex'] = True; plt.figure()
plt.xlabel('$N$'); plt.ylabel('estimated weak error')
plt.yscale('log'); plt.xscale('log')
order_line = 1.8 * error[0, 0] * np.array([N_list[0] / N_list[i]
for i in range(0, len(N_list)-1)])
for i in range(4):
    plt.plot(N_list[: -1], error[:, i], '-o', label = labels[i])
    plt.plot(N_list[: -1], order_line, linestyle = (0 ,(1 ,3) ), color = '0.5 ', label = 'order 1')
plt.legend(prop={'size': 9})
plt.savefig('plot.pdf', bbox_inches = 'tight')

38
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