On applications of the Maupertuis-Jacobi correspondence for Hamiltonians $F(x, |p|)$ in some 2-D stationary semiclassical problems

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We make use of the Maupertuis – Jacobi correspondence, well known in Classical Mechanics, to simplify 2-D asymptotic formulas based on Maslov’s canonical operator, when constructing Lagrangian manifolds invariant with respect to phase flows for Hamiltonians of the form $F(x, |p|)$. As examples we consider Hamiltonians coming from the Schrödinger equation, the 2-D Dirac equation for graphene and linear water wave theory.

1. INTRODUCTION

Maupertuis – Jacobi correspondence [1–3] allows to relate two Hamiltonians $\mathcal{H}(x, p, E)$ and $H(x, p, E)$ having in common a regular energy surface $\Sigma$; it preserves the integral curves on $\Sigma$ up to a reparametrization of time. As it was shown in [5, 6] this principle is also useful in determining the semiclassical spectral asymptotics for a selfadjoint $h$-pseudodifferential operator $\mathcal{H}(x, hD_x, E; h)$ having $\mathcal{H}(x, p, E)$ as principal semi-classical symbol. The other Hamiltonian $H(x, p, E)$ is assumed to enjoy nice properties, such as local integrability near $\Sigma$. Then we can construct some compact Lagrangian manifolds invariant by the flow of

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\( \mathcal{H}(x, p, E) \), and then determine quasimodes for \( \mathcal{H}(x, hD_x, E; h) \) microlocalized in a neighborhood of \( \Sigma \).

In this communication we want to show that Maupertuis – Jacobi correspondence allows to construct non compact Lagrangian manifolds appearing in the scattering problem for \( \mathcal{H}(x, hD_x, E; h) \), or the problem about Green function asymptotics. Here it is assumed that \( H(x, p, E) \) is a Finsler symbol, which implies the existence of special coordinates near the singular part of the Lagrangian manifold. When combining Maupertuis – Jacobi correspondence with new formulas for Maslov canonical operator [8], we show that the corresponding asymptotics can be presented in a rather explicit and simple form. We restrict ourselves to 2-D case and apply our considerations to examples from the Schrödinger (or Helmholtz) equation, the two-dimensional Dirac equation for graphene and Pseudodifferential operators from the linear water wave theory.

2. LAGRANGIAN MANIFOLDS INVARIANT WITH RESPECT TO HAMILTONIANS \( F(X, |P|) \), THE EIKONAL COORDINATES AND MAUPER TUIS – JACOBI CORRESPONDENCE

Let \( F(x, z), \ x \in \mathbb{R}^2, \ z \in [0, \infty) \) be a smooth function, and \( E \) a real parameter. Assume that the equation \( F(x, z) = E \) has the unique solution \( z = 1/C(x, E) \), where \( C(x, E) \) is a smooth positive bounded function, such that \( C(x, E) \geq c_0(E) > 0 \). Also we assume that \( \left| \frac{\partial F}{\partial z}(x, \frac{1}{C(x, E)}) \right| \geq c_1(E) > 0 \), where \( c_0(E), c_1(E) \) are some positive constants. Consider the Hamiltonians \( \mathcal{H}(x, p, E) = F(x, |p|) - E, \ H(x, p, E) = C(x, E)|p| - 1 \) in the phase space \( \mathbb{R}^4_{p,x} = T^*\mathbb{R}^2 \) together with the Hamiltonian systems

\[
\begin{align*}
(a) \quad & \frac{dp}{dt} = -\mathcal{H}_x, \quad \frac{dx}{dt} = \mathcal{H}_p; \\
(b) \quad & \frac{dp}{d\tau} = -H_x, \quad \frac{dx}{d\tau} = H_p.
\end{align*}
\] (1)

We recall that \( C(x, E)|p| \) defines a (reversible) Finsler symbol on \( T^*\mathbb{R}^2 \) [4, 7].

Let \( Q = \mathbb{R} \) or \( Q = \mathbb{R}/2\pi\mathbb{Z} \) and \( Q \to T^*\mathbb{R}^2, \ \varphi \mapsto (P^0(\varphi, E), X^0(\varphi, E)) \) be a smooth embedding with image \( \Lambda^1 \) such that

\[
\mathcal{H}(X^0(\varphi, E), P^0(\varphi, E), E) = 0 \quad \text{and} \quad |P^0(\varphi, E)|C(X^0(\varphi, E)) = 1
\]

Consider the solutions \((P(t, \phi, E), \mathcal{X}(t, \phi, E)) \) and \((P(\tau, \phi, E), \mathcal{X}(\tau, \phi, E)) \) to systems (1) (a) and (b) respectively with initial data on \( \Lambda^1 \). Due to general properties of Hamiltonian
systems, we have
\[ \mathcal{H}(X(t, \phi, E), P(t, \phi, E), E) = 0 \quad \text{and} \quad \left| P(\tau, \phi, E) \right| C(X(\tau, \phi, E)) = 1 \]
and because of Maupertuis – Jacobi correspondence, trajectories \( (P(t, \phi, E), X(t, \phi, E)) \) and \( (P(\tau, \phi, E), X(\tau, \phi, E)) \) coincide modulo a reparametrization of time. Indeed one has
\[
\begin{align*}
\frac{dP}{dt} &= -\mathcal{H}_x(P, X) = -R(X)H_x(P, X) = R(X)\frac{dP}{d\tau} \\
\frac{dX}{dt} &= \mathcal{H}_p(P, X) = R(X)H_p(P, X) = R(X)\frac{dX}{d\tau}
\end{align*}
\]
where
\[
R(x) = \lim_{z \to 1/C(x, E)} \frac{F(x, z) - E}{zC(x, E) - 1} = z\frac{\partial F}{\partial z}(x, z) \bigg|_{z=1/C(x, E)}
\]
Changing time \( t \) by time \( \tau = \tau(t, \phi, E) \), by using the equation
\[
\frac{d\tau}{dt} = R(X(t, \phi, E)), \quad \tau|_{t=0} = 0,
\]
we get (inverting the equation \( \tau = \tau(t, \phi, E) \) we obtain \( t = t(\tau, \phi, E) \))
\[
\left( P(t, \phi, E), X(t, \phi, E) \right) = \left( P(\tau, \phi, E), X(\tau, \phi, E) \right) \bigg|_{\tau = \tau(t, \phi, E)} \iff \left( P(\tau, \phi, E), X(\tau, \phi, E) \right) = \left( P(t, \phi, E), X(t, \phi, E) \right) \bigg|_{t = t(\tau, \phi, E)}.
\]
In \( T^*\mathbb{R}^2 \) the solutions \( (P(\tau, \phi, E), X(\tau, \phi, E)) \) and \( (P(t, \phi, E), X(t, \phi, E)) \) define the phase flows (that we assume to be defined for all time)
\[
\Lambda^2 = \bigcup_{t \in \mathbb{R}} g_t^H \Lambda^1 = \{(p, x) = (P(t, \phi, E), X(t, \phi, E)), \phi \in Q, t \in \mathbb{R}\} = \bigcup_{t \in \mathbb{R}} g_t^H \Lambda^1 = \{(p, x) = (P(\tau, \phi, E), X(\tau, \phi, E)), \phi \in Q, \tau \in \mathbb{R}\}
\]
In particular \( \Lambda^2 \) is invariant under \( g_t^H \) and \( g_H^t \). Note that we could replace \( \mathbb{R}^2 \) by an open domain \( \Omega \subset \mathbb{R}^2 \) and consider instead the maximal classical trajectories \( g_t^H(\rho), \rho \in \Lambda^1 \) and \( t \in (T_-(\rho), T_+(\rho)) \), and similarly for \( g_H^t \). The parameters \( t \) and \( \tau \) are called \textit{proper times}. Once \( \Lambda^2 \) is a smooth manifold, it becomes an embedded Lagrangian manifold, and \( (t, \phi) \) and \( (\tau, \phi) \) are just two different coordinate systems on \( \Lambda^2 \). It is convenient to relate objects belonging to either Hamiltonians, such as eikonals or half-densities. In particular:

\textbf{Lemma 1.} The following properties hold:
1) The Jacobians of the transformation \((t, \phi) \mapsto (\tau, \phi)\) or its inverse verify
\[
\begin{align*}
\det \frac{\partial (\tau(t, \phi, E), \phi)}{\partial (t, \phi)} &= \frac{d\tau}{dt} = R(X(t, \phi, E)), \\
\det \frac{\partial (t(\tau, \phi, E), \phi)}{\partial (\tau, \phi)} &= \frac{dt}{d\tau} = 1/R(X(\tau, \phi, E))
\end{align*}
\]
and the Jacobians \(J = \det \frac{\partial X}{\partial (\tau, \phi)}\) (resp. \(J = \det \frac{\partial X}{\partial (t, \phi)}\)) in coordinates \((\tau, \phi)\) (resp. \((t, \phi)\)) are related by
\[
J(t, \phi) = R(X(\tau, \phi, E))J(\tau, \phi) \bigg|_{\tau = \tau(t)}.
\]

2) The action function (eikonal) on \(\Lambda^2\) is
\[
s(t, \phi) \equiv \int_{(0,0)}^{(t,\phi)} P(t, \phi, E) \, dX(t, \phi, E) = s_0(\phi) + \tau
\]
\[
s_0(\phi) = \int_0^\phi P^0(\phi, E) \, dX^0(\phi, E)
\]

3) The Jacobians \(J, \mathcal{J}\) satisfy to the relations:
\[
|J| = C(X(\tau, \phi))|X_\phi|, \quad |\mathcal{J}| = R(X(t, \phi))C(X(t, \phi))|X_\phi|.
\]

**Proof.** The first equalities (6), (7), (8) hold since \(\frac{\partial \phi}{\partial \tau} = \frac{\partial \phi}{\partial t} = 0\). The proof of (9) follows from chain of equalities
\[
s(t, \phi) = \int_{(0,0)}^{(t,\phi)} P(t', \phi, E) \, dX(t', \phi, E) = \int_{(0,0)}^{(\tau,\phi)} P(\tau', \phi, E) \, dX(\tau', \phi, E) = \\
\int_0^\phi P^0(\phi, E) \, dX^0(\phi, E) + \int_0^\tau P(\tau', \phi, E) \frac{dX}{d\tau'}(\tau', \phi, E) \, d\tau' = \\
\int_0^\phi P^0(\phi, E) \, dX^0(\phi, E) + \int_0^\tau |P(\tau', \phi, E)| C(X(\tau', \phi, E), E) \, d\tau' = s_0(\phi) + \tau
\]
The assertion 3) is proved in [8, 11], using that \(H(x, p, E)\) is a Finsler symbol. □

The pair \((\bar{\tau} = s_0(\phi) + \tau, \phi)\) are called **eikonal coordinates** on \(\Lambda^2\) (see [8]). There are two important examples of curves \(\Lambda^1\) in applications: \(\Lambda^1_k = \{p_1 = 0, p_2 = k, x_1 = \phi, x_2 = a, \phi \in \mathbb{R}\}\) which appears in scattering problems and \(\Lambda^1_G = \{p_1 = b \cos \phi, p_2 = b \sin \phi, x_1 = a_1, x_2 = a_2, \phi \in \mathbb{R}/2\pi \mathbb{Z}\}\) which appears in problems about the Green functions. Easy to check that in these cases \(s_0(\phi) = 0\) and \(\bar{\tau} = \tau\).
3. RELATIONSHIP WITH MASLOV CANONICAL OPERATOR

We endow the Lagrangian manifold $\Lambda^2$ with the measure $d\mu = dt \wedge d\phi$; let $A(t, \phi)$ be a smooth function on $\Lambda^2$ and

$$\psi = K^h_{\Lambda^2} A(t, \phi).$$

where $K^h_{\Lambda^2}$ is Maslov canonical operator. We want to pass in $K^h_{\Lambda^2} A(t, \phi)$ from coordinates $(t, \phi)$ to eikonal-coordinates $(\tau, \phi)$ preserving the measure $d\mu$.

**Theorem.** The following equalities hold:

$$\psi = K^h_{\Lambda^2} \left[ A(t(\tau, \phi), \phi) \sqrt{\det \frac{\partial (\tau, \phi)}{\partial (t, \phi)}} \right] = K^h_{\Lambda^2} \left[ \frac{A(t(\tau, \phi), \phi)}{\sqrt{R(X(\tau, \phi))}} \right] = \frac{1}{\sqrt{R(x)}} K^h_{\Lambda^2} \left[ A(t(\tau, \phi), \phi) \right] (1 + O(h)).$$

**Proof.** It follows easily from (6) and the commutation formula between the pseudodifferential operator $\hat{Q} = Q(x, hD_x)$ and Maslov canonical operator [9, 10]: $\hat{Q} K^h_{\Lambda^2} [A(t, \phi)] = K^h_{\Lambda^2} [Q(x, p)|_{\Lambda^2} A(t, \phi)] (1 + O(h))$. □

Recall that the canonical operator has different representations in the neighborhood of regular points (where $J = \det \frac{\partial X}{\partial (\tau, \phi)} \neq 0$) and in the neighborhood of singular (focal) points (where $J = \det \frac{\partial X}{\partial (\tau, \phi)} = 0$). According to (10) the point $(P(\tau, \phi), X(\tau, \phi)) \in \Lambda^2$ is singular (focal) if $X_\phi(\tau, \phi) = 0$. It was proved in [8] that under existence of the eikonal coordinates, $\det(P, P_\phi) \neq 0$ in the neighborhood of the focal points. Here $(P, P_\phi)$ is the $2 \times 2$ matrix constructed from vector columns $P$ and $P_\phi$. Thus the Lagrangian manifold could be covered by regular charts $\Omega^\text{reg}_j$ with $X_\phi(\tau, \phi) \neq 0$ and singular charts $\Omega^\text{sing}_j$ with $\det(P, P_\phi) \neq 0$. Let $\{e_j(\tau, \phi)\}$ be a (finite) partition of unity subordinated to the charts $\Omega^\text{reg}_j, \Omega^\text{sing}_j$. Then due to Lemma 1 the contribution of a regular chart to the canonical operator is

$$\psi_j = \left. \frac{e^{-i\frac{\pi}{2} m_j}}{\sqrt{R(x)C(x)|_{X_\phi}}} e^{i\frac{\pi}{2} A(\tau, \phi)} e_j(\tau, \phi) \right|_{(\tau, \phi) = (\tau_j(x), \phi_j(x))}$$

where $(\tau_j(x), \phi_j(x))$ is the solution to the (vector) equation $X(\tau, \phi) = x$ in the chart $\Omega^\text{reg}_j$ and $m_j$ is the Maslov index of $\Omega^\text{reg}_j$ (see below). The contribution of a singular chart is [8]

$$\psi_j = \frac{e^{-i\frac{\pi}{2} m_j^s} e^{i\frac{\pi}{2} \int R(x)}}{\sqrt{hR(x)}} \int_\mathbb{R} e^{i\frac{\pi}{2} \sqrt{|\det(P, P_\phi)|} A(\tau, \phi)} e_j(\tau, \phi) d\phi$$

where $\tau_j(x, \phi)$ is the solution to the scalar equation $\langle P(\tau, \phi), x - X(\tau, \phi) \rangle = 0$ in the chart $\Omega^\text{sing}_j$ and $m_j^s$ is the Maslov index of $\Omega^\text{sing}_j$.  


According to [8] Maslov index $m_j$ coincides with Morse index of the trajectory starting from the point $(P, X)$ with coordinates $(\tau = 0^+, \phi)$ and coming to the point $(P(\tau, \phi), X(\tau, \phi)) \in \Omega_j^{\text{reg}}$: it equals to a number of zeroes of Jacobian $J = \det \frac{\partial X}{\partial (\tau', \phi)}$ (or the function $X_\phi(\tau', \phi)$) when $\tau'$ runs from $0^+$ to $\tau$. To find the index $m_j^s$ of a singular chart $\Omega_j^{\text{sing}}$ one need to take an arbitrary regular point $(P(\tau, \phi), X(\tau, \phi)) \in \Omega_j^{\text{sing}}$ and compare the signs of $J = \det \frac{\partial X}{\partial (\tau', \phi)}$ and $\det(P, P_\phi)$. Then $m_j^s$ equals Morse index of $(P(\tau, \phi), X(\tau, \phi))$ if they coincide, and Morse index plus 1 otherwise. Finally to construct the canonical operator one should patch all $\psi_j$ together (see [9, 10]). At last note that integral (14) could be expressed in the form of Airy or Pearcey functions (see explicit formulas in [8]) under the assumption that the certain subset of Lagrangian singularities $\{(P(\tau, \phi), X(\tau, \phi))|_{X_\phi=0}\}$ are in the so-called general position ([1, 9]).

We consider the following example. The Lagrangian manifold presented in Fig. 1 has 2 caustics (red lines). Under the area in configuration space between edges of a caustic the Lagrangian manifold is folded into 3 leaves. So in this area 3 functions of the form (13) are to be patched together. Under the area “outside caustics” there is only one leave of the manifold, equation $X(\tau, \phi) = x$ has a unique solution and the canonical operator takes the form of (13) with a single function. In the vicinity of caustic edges canonical operator is a sum of a regular (13) and singular (14) parts.

4. EXAMPLES

Let us present several examples of application of the Maupertuis – Jacobi correspondence for the construction of Maslov canonical operator. We do not discuss here further applications to Partial Differential Equations.

Example 1 (from the Schrödinger equation, see [12, 13]). Let $U(x)$ be a smooth bounded function, $U(x) < E$. Consider the classical Hamiltonian $H(x, p) = F(x, |p|) = \frac{p^2}{2} + U(x)$. Then

$$C(x, E) = \frac{1}{\sqrt{2(E - U(x))}}, \quad R(x) = z^2 \bigg|_{z = 1/C(x)} = 2(E - U(x))$$

(15)

and

$$\psi(x) = \frac{1}{\sqrt{2(E - U(x))}} K^h \left[ A(t(\tau, \phi), \phi) \right].$$

(16)

Example 2 (from the two-dimensional Dirac equation for graphene, [14]). Let $U(x), m(x)$ be smooth bounded functions. Consider the effective Hamiltonians $H^\pm(x, p) = F(x, |p|) =$
$U(x) \pm \sqrt{p^2 + m(x)^2}$. Then

$$C(x, E) = \frac{1}{\sqrt{(E - U)^2 - m^2}}, \quad R = \pm \frac{z^2}{\sqrt{z^2 + m(x)^2}} \bigg|_{z=1/c} = \frac{(E - U(x))^2 - m^2(x)}{E - U(x)},$$

(17)

and

$$\psi = \frac{\sqrt{E - U(x)}}{\sqrt{(E - U(x))^2 - m(x)^2}} K_{\Lambda^2}^2 \left[ A(t(\tau, \phi), \phi) \right].$$

(18)

**Example 3** (from the water waves theory, [6, 15, 16]). Let $D(x) > 0$ be the smooth function, representing the depth of a basin, and consider the effective Hamiltonian $H(x, p) = F(x, |p|) = \sqrt{|p|} \tanh(|p|D(x)) - E$. It is easy to see that there exists a unique smooth positive solution $y = Y(\mathcal{E}(x))$ to the equation $\sqrt{y \tanh(y)} = \mathcal{E}(x) = \sqrt{D(x)}E$ and

$$C(x, E) = \frac{D(x)}{y(E \sqrt{D(x)})}, \quad R = \frac{D(x)z / \cosh^2(zD(x)) + \tanh(zD(x))}{2\sqrt{z \tanh(zD(x))}} \bigg|_{z=\lambda} = \frac{(y^2 - y^2 \tanh^2(y) + y \tanh(y))}{2\sqrt{D(x)E} \tanh(y)} \bigg|_{y=Y(\sqrt{D(x)})} = \frac{y^2 - D(x)^2E^2 + D(x)E^2}{2D(x)E} \bigg|_{y=Y(\sqrt{D(x)})E}.$$

(19)

The Hamiltonian system with the Hamiltonian $H(x, p, E) = C(x, E)|p|$ has the form

$$\frac{dp}{d\tau} = -|p| \frac{\partial}{\partial x} \left( \frac{D(x)}{Y(E \sqrt{D(x)})} \right), \quad \frac{dx}{d\tau} = \frac{p}{|p|} \frac{D(x)}{Y(E \sqrt{D(x)}}.$$

(20)

It contains function $Y(\mathcal{E}(x))$ and its derivative $Y'(\mathcal{E}(x))$ which makes difficult to finding its inverse. Let us show how to rewrite this system in a form without the function $Y(\mathcal{E}(x))$.

Along the trajectories $(P, X)$ the following equalities hold:

$$H(X, p, E) = 0 \quad \Leftrightarrow \quad C(X, E)|P| = 1 \quad \Rightarrow \quad Y(E \sqrt{D(X)}) = \frac{D(X)}{C(X, E)} = D(X)|P|.$$  

(21)

Then differentiating the equation for $Y(\mathcal{E}(x))$ we have

$$Y'(\tanh Y + Y(1 - \tanh^2 Y)) = 2\sqrt{Y \tanh Y}.$$  

(22)

This gives for the solution $(P, X)$

$$Y'(E \sqrt{D(X)}) = \frac{2YE}{Y^2 + E^2 - E^4} \bigg|_{(P, X)} = \frac{2|P| \sqrt{D(X)}E}{D(X)|P|^2 + E^2 - D(X)E^4}.$$  

(23)

Inserting these equalities (21) and (23) into the Hamiltonian system we finally have

$$\frac{dp}{d\tau} = -\frac{p^2 - E^4}{D(x)p^2 + E^2 - D(x)E^4} \cdot \frac{\partial D(x)}{\partial x}, \quad \frac{dx}{d\tau} = \frac{p}{p^2}.$$  

(24)
To write out the canonical operator we also insert $Y|_{(P,X)}$ into the expression (19) for $R$:

$$R|_{(P,X)} \equiv R(X, P, E) = \frac{D(X)P^2 - D(X)E^4 + E^2}{2E}$$

Taking into account the last expression for $R$ and that $C(X) = 1/|P|$ we obtain the formula (13) for canonical operator in a regular point in this case

$$\psi_j(x) = \frac{e^{i\pi x}e^{-i\frac{\pi}{m}j}}{\sqrt{|x_\phi(\tau, \phi)|}} \sqrt{\frac{2E|P(\tau, \phi)|}{D(X(\tau, \phi))P^2 - D(X(\tau, \phi))E^4 + E^2}} A(\tau, \phi)e_j(\tau, \phi)\Big|_{(\tau, \phi)=(\tau_j(x), \phi_j(x))}$$

As $R$ depends on $p$, it is not convenient to factor out $1/\sqrt{R(x, hD_x, E)}$ from the canonical operator. Near the focal point we write instead

$$\psi_j(x) = e^{-i\frac{\pi}{m}j}e^{i\frac{\pi}{2}} \sqrt{2E} \sqrt{\frac{\sqrt{\det (P(\tau, \phi), P_\phi(\tau, \phi))}}{D(X(\tau, \phi))P^2 - D(X(\tau, \phi))E^4 + E^2}} A(\tau, \phi)e_j(\tau, \phi)\Big|_{\tau=\tau_j(x, \phi)}$$

**Example 4** (from the water waves theory with surface tension, [6, 15, 16]). We modify Hamiltonian in Example 3 according to $\mathcal{H}(x, p) = F(x, |p|) - E = \sqrt{|p|} \tanh(|p|D(x))(1 + \mu(x)|p|^2) - E$, $x \in \mathbb{R}^2$, where $\mu(x) > 0$ is a smooth function, representing the surface tension of the fluid. Let also $\nu(x) = E(\mu(x))^{1/4}$, $\mathcal{E}(x) = E(D(x))^{1/2}$.

The relation $\mathcal{H}(x, p) = 0$ can be rewritten in a functional form as $f(y, \mathcal{E}, \nu) = 0$, where $f(y, \mathcal{E}, \nu) = y \tanh(y) - \mathcal{E}^2(1 + \nu^2/\mathcal{E}^4)^{-1}$ is smooth on $\mathbb{R}^2_+$, and because $\partial f/\partial y(y, \mathcal{E}, \nu) > 0$, the implicit functions theorem gives $y = Y(\mathcal{E}, \nu)$ where $Y$ is smooth in $(\mathcal{E}, \nu) \in \mathbb{R}^2_+$. Since $\mu$ and $D$ are smooth functions, it follows also from the implicit function theorem in Fréchet space $C^\infty(\mathbb{R}^2)$ that $y = Y(\mathcal{E}, \nu) \in C^\infty(\mathbb{R}^2)$. As above, the equations of motion with Hamiltonian $C(x, E)|p|$ have the form

$$\frac{dp}{d\tau} = -|p| \frac{\partial}{\partial x} \left( \frac{D(x)}{Y(E\sqrt{D(x)}, E\mu(x)^{1/4})} \right), \quad \frac{dx}{d\tau} = \frac{p}{|p|} \frac{D(x)}{Y(E\sqrt{D(x)}, E\mu(x)^{1/4})}.$$  

As above they simplify on $H(x, p, E) = C(x, E)|p| - 1 = 0$ to

$$\frac{dp}{d\tau} = -|p| \left( \frac{1}{y} \frac{\partial D}{\partial x} - \frac{D(x)}{y^2} \frac{dy}{dx} \right)|_{y=D(x)|p|}, \quad \frac{dx}{d\tau} = \frac{p}{p^2}.$$  

Here $dY/dx$ is found by differentiating the equation $y = Y(\mathcal{E}(x), \nu(x))$:

$$\frac{dY}{dx} = -\left( \frac{\partial f}{\partial y} \right)^{-1} \left( \frac{\partial f}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial x} + \frac{\partial f}{\partial \nu} \frac{\partial \nu}{\partial x} \right),$$

$$\frac{\partial f}{\partial y} = \tanh(y) + y(1 - \tanh^2(y)) + 2y\mathcal{E}^{-2}\nu^4(1 + y^2/\mathcal{E}^4)^{-2} > 0.$$
Substituting this derivatives into (29) with $y(E\sqrt{D(X)}) = |P|D(X)$ we get a system similar to (24), and can we obtain also an expression for $R$ as in (25). So we are able to get a representation of Maslov canonical operator as in (26) and (27).

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Figure 1. Lagrangian manifold, characteristics (blue lines) and caustic (red lines): in the phase space (in coordinates \((x_1, x_2, p_1) \subset \mathbb{R}_x^4\)) and in projection to the configuration space \(\mathbb{R}_x^2\). The Lagrangian manifold \(\Lambda^2 = \bigcup g'_H \Lambda^1\) corresponds to a scattering problem with initial curve

\[\Lambda^1 = \{p = (0, 2), x = (\phi, 0), \phi \in \mathbb{R}\},\]

the Hamiltonian is \(H(x, p) = |p|/(E - U(x))\), where

\[E = 2, U(x) = e(x)e^{-(x_1-5)^2-(x_2-3)^2}\]

and \(e(x)\) is a cut-off function

\[e(x) = 0, x_2 \leq 0, e(x) = 1, x_2 \geq 1.\]