A PRESCRIBED GAUSS-KRONECKER CURVATURE PROBLEM ON THE PRODUCT OF UNIT SPHERES

WANG ZHIZHANG

1. INTRODUCTION

Prescribed Gauss-Kronecker curvature problems are widely studied in the literature. Famous among them is the Minkowski problem. It was studied by H. Minkowski, A.D. Alexandrov, H. Lewy, A.V. Pogorelov, L. Nirenberg and last solved by S.Y. Cheng and S.T. Yau [CY]. After that, V.I. Oliker [O] researched the arbitrary hypersurface with prescribed Gauss curvature in Euclidean space. On the other hand, L.A. Caffarelli, L. Nirenberg, and J. Spruck studied the boundary-value problem of prescribed Weingarten curvature of graphs over some Euclidean domain in [CNS2], [CNS3], [CNS4]. Then B. Guan and J. Spruck [GS] studied the boundary-value problem in the case of hypersurfaces that can be represented as a radial graph over some domain on some unit sphere. But on the product unit spheres, the similar problem has not been studied systematically. The present paper tries to ask and partly solve a problem of this kind.

Let \( S^m \subset \mathbb{R}^{m+1} \), \( S^n \subset \mathbb{R}^{n+1} \), and \( S^{m+n+1} \subset \mathbb{R}^{m+1} \oplus \mathbb{R}^{n+1} \) are three unit spheres. \( \vec{\gamma}, \vec{\rho} \) are position vectors of \( S^m, S^n \) respectively, and \( u \) is a smooth function defined on \( S^m \times S^n \).

Consider a hypersurface \( M \subset S^{m+n+1} \) defined by a natural embedding \( \vec{X} : S^m \times S^n \rightarrow S^{m+n+1} \)

\[
(\vec{\gamma}, \vec{\rho}) \mapsto \frac{1}{(1 + e^{-2u})^{1/2}} \vec{\gamma} + \frac{1}{(1 + e^{2u})^{1/2}} \vec{\rho}.
\]

This map firstly appears in [H], but that paper only discusses the prescribed mean curvature problem. The fact that the map is an embedding will be proved in Section 2. Now we state explicitly the main problem: for a given positive smooth function \( K \) defined on \( \bar{\Omega} \), can we find a closed strictly convex hypersurface in \( S^{m+n+1} \) which is described by (1.1), and whose Gauss-Kronecker curvature is \( K \)? We will show that there is no global solution to this problem. So we have to restrict this problem to a subdomain of \( S^m \times S^n \). We solve this problem for some special domains defined as follows:

**Definition 1.1.** (PHC-domains) Assume \( m \neq n \). For \( m > n \), a domain \( \Omega \subset S^m \times S^n \) is called a PHC-domain if it satisfies

(i) \( \Omega \) is a product domain of the form \( \Omega = \Omega_x \times S^n \) with \( \Omega_x \subset S^m \);

(ii) \( \Omega_x \) is contained in some hemisphere;

(iii) \( \Omega_x \) is a strictly infinitesimally convex domain with smooth boundary.

For \( m < n \), we give a similar definition by changing the position of \( m \) and \( n \): let \( \Omega = S^m \times \Omega_y \), \( \Omega_y \subset S^n \), and replace \( \Omega_x \) with \( \Omega_y \) in (i),(ii),(iii) above.

We know strictly geodesically convex is equivalent to strictly locally convex, and they can be induced by strictly infinitesimally convex. For details see [S]. Our main result is the

**Theorem 1.1.** Assume that \( \Omega \subset S^m \times S^n \) is a PHC-domain. For a given smooth positive function \( K \) defined on \( \bar{\Omega} \), there is an embedding \( \vec{X} \) given by (1.1) on \( \bar{\Omega} \), giving a closed strictly convex smooth hypersurface in \( S^{m+n+1} \), whose Gauss-Kronecker curvature is \( K \).
The prescribed Gauss curvature problems always relate to some Monge-Ampère type equation. Assume \( m > n \). Consider the Dirichlet problem on a PHC-domain \( \Omega \)

\[
\begin{aligned}
\det M(u) &= K(f(u, |\nabla_x u|^2, |\nabla_y u|^2))^{\frac{m+n+2}{2}} \quad \text{in } \Omega, \\
\psi &= \psi_0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( K, \psi \in C^\infty(\bar{\Omega}), K > 0 \) on \( \bar{\Omega} \), \( M(u) \) is defined in (2.9) and the rest of the notation defined in (2.1),(2.2). This is the equation associated to our main problem. Although by the effort of much people, Monge-Ampère equations are well understood now, in our case there are some new difficulties.

The framework to obtain a solution is the classical continuity method (see [N]). So we need to give the openness part and the closedness part. For the openness part, the condition of the uniqueness of linearized equation relies on the smallness of boundary-values. Hence we consider Dirichlet problem (1.2) with sufficiently small boundary-values. For the closedness part, the openness also gives the comparison Lemma 3.1, leading to the construction of the subsolution using suitable boundary-values. As in [CNS1], the subsolution gives the initial solution, the \( C^0 \) estimate and the \( C^2 \) estimate on the boundary. The interior \( C^1 \) estimate is needed, since the manifold we consider here is a product of a domain and a unit sphere, and the sphere has no boundary. We choose a natural function (3.22) and estimate it at its maximum value point to obtain the bound. For the interior \( C^2 \) estimate, the difficulty is that we can not diagonalize the three matrices \( (M(u)_{A\bar{A}})_{(m+n)\times(m+n)}, (M(u)_{ij})_{m\times m} \) and \( (M(u_{A,B})_{n\times n}) \) at the same time (the conservation of the indices is stated in the head of Section 2). Because of this, we need to introduce a term \( \sum_A M(u)^{AA} \sum_M M(u)_{AA} \). Inspired by papers of S.T. Yau [Y] and B. Guan [G], we choose a function (4.11). Then estimating it at the maximum value point, and computing explicitly the function \( f \) in equation (1.2), we obtain the needed term and the interior \( C^2 \) estimate, in which we also generalized the idea of using a \( C^1 \) term in the paper [Y]. For the \( C^2 \) estimate on the boundary, we have the same difficulty as for the interior estimate and the difficulty that the manifold is a product manifold. Inspired by [CNS1], we use a function developed in [G] and the coordinate functions in Euclidean space to obtain the estimate. Then from Evans-Krylov theory (see [GT]), we have the \( C^{2,\alpha} \) estimate. At last, differentiate (1.2) and using Schauder theory, we have

**Proposition 1.2.** Let \( 0 < \tau < 2, K > 0 \) smooth and \( \psi \in ABF(\tau, K) \). Assume that \( u \) is the solution of problem (1.2). Then there is a constant \( C_0 \) depending on \( \psi, m, n, K, \partial \Omega, k, \alpha \) such that

\[
||u||_{C^{k,\alpha}(\bar{\Omega})} \leq C_0,
\]

where \( k \) is a positive integer and \( 0 < \alpha < 1 \).

Here \( \tau \) is defined in (2.1) and \( ABF(\tau, K) \) is defined in Definition 3.2. Now by the continuity method, we have

**Theorem 1.2.** Let \( \Omega \subset S^m \times S^n \) be a PHC-domain, and \( m > n \). For a given smooth function \( K > 0 \) and \( \psi \in ABF(\tau, K) \), problem (1.2) has a unique convex smooth solution (meaning that \( M(u) \) is positive definite).

Since the main problem leads to equation (1.2), for \( m > n \) Theorem 1.2 gives the \( m > n \) part of Theorem 1.1. For \( m \leq n \) we take \( v = -u \) and change the position of \( S^m \) and \( S^n \), so it becomes the previous case and the map \( \bar{X} \) is not changed. Then we obtain the rest of Theorem 1.1.

The present paper is organized as follows: in Section 2, we compute out the Gauss-Kroneker curvature of the hypersurface \( M \) defined by map \( \bar{X} \), and give the openness part of equation (1.2). Section 3 gives the \( C^0 \) and \( C^1 \) estimates of (1.2). And the last two sections give the \( C^2 \) estimate in the interior and on the boundary.

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2. Equation and Openness

We firstly compute the Gauss-Kronecker curvature of the hypersurface \( M \) defined by (1.1). Let \( \{e_1, \cdots, e_m\}, \{e_{m+1}, \cdots, e_{m+n}\} \) be local orthonormal coordinates of \( S^m, S^n \). Throughout our paper, Latin indices (\( i, j, \cdots \)), Greek indices (\( \alpha, \beta, \cdots \)) and capital Latin indices (\( A, B, \cdots \)) take values in the sets \( \{1, \cdots, m\}, \{m+1, \cdots, m+n\} \) and \( \{1, \cdots, m+n\} \) respectively. Now we define

\[
(2.1) \quad \tau = \frac{2(m-n)}{m+n+2}; \quad f(r, p, q) = e^\tau \left( 1 + \frac{p}{1 + e^{2r}} + \frac{e^{2r}}{1 + e^{2r}} q \right),
\]

and

\[
(2.2) \quad |\nabla_x u|^2 = \sum_i u_i^2, \quad |\nabla_y u|^2 = \sum_\alpha u_\alpha^2; \quad |\nabla u|^2 = |\nabla_x u|^2 + |\nabla_y u|^2.
\]

Obviously,

\[
(2.3) \quad \bar{\gamma}_i = e_i; \quad \bar{\gamma}_\alpha = 0; \quad \bar{\rho}_i = 0; \quad \bar{\rho}_\alpha = e_\alpha.
\]

Then the tangent vectors of \( M \) is

\[
(2.4) \quad \bar{X}_A = (1 + e^{2u})^{-\frac{3}{2}}[e^u u_A \bar{\gamma} + e^u (1 + e^{2u}) \bar{\gamma}_A + (1 + e^{2u}) \bar{\rho}_A e - e^{2u} u_A \bar{\rho}]
\]

and the induced metric \( g \) is

\[
(2.5) \quad g_{AB} = <\bar{X}_A, \bar{X}_B> = \frac{e^{2u}}{(1 + e^{2u})^2}[u_A u_B + (1 + e^{2u}) <\bar{\gamma}_A, \bar{\gamma}_B> + (1 + e^{-2u}) <\bar{\rho}_A, \bar{\rho}_B>].
\]

Here \( <\cdot, \cdot> \) is the standard inner product of \( \mathbb{R}^{m+n+2} \), and we choose special local coordinates such that \( |\nabla_x u| = u_1 \) and \( |\nabla_y u| = u_{m+1} \), then

\[
(2.6) \quad \det(g_{AB}) = \frac{e^{2mu}}{(1 + e^{2u})^{m+n}}(1 + |\nabla_x u|^2 + e^{2u} |\nabla_y u|^2).
\]

And (2.6) implies that \( \bar{X} \) is an embedding. Out of the two normal unit vectors of \( M \) in \( \mathbb{S}^{m+n+1} \), we choose

\[
(2.7) \quad \vec{n} = \frac{-\bar{\gamma} + e^u \bar{\rho} + \sum_i u_i \bar{\gamma}_i + e^u \sum_\alpha u_\alpha \bar{\rho}_\alpha}{(1 + e^{2u} + |\nabla_x u|^2 + e^{2u} |\nabla_y u|^2)^\frac{1}{2}},
\]

and

\[
(2.8) \quad \vec{n}_i = [\ln(1 + e^{2u} + |\nabla_x u|^2 + e^{2u} |\nabla_y u|^2)]^\frac{1}{2} \vec{n} - (1 + e^{2u} + |\nabla_x u|^2 + e^{2u} |\nabla_y u|^2)^{-\frac{1}{2}} \times \sum_j (u_{ij} - \delta_{ij}) \bar{\gamma}_j - u_i \bar{\gamma} + e^u u_i \bar{\rho} + e^u \sum_\beta (u_{i\beta} + u_{i\beta}) \bar{\rho}_\beta],
\]

\[
\vec{n}_\alpha = [\ln(1 + e^{2u} + |\nabla_x u|^2 + e^{2u} |\nabla_y u|^2)]^\frac{1}{2} \vec{n} - (1 + e^{2u} + |\nabla_x u|^2 + e^{2u} |\nabla_y u|^2)^{-\frac{1}{2}} \times \sum_j u_{j\alpha} \bar{\gamma}_j + e^u \sum_\beta (u_{\alpha\beta} + u_{\alpha\beta} + \delta_{\alpha\beta}) \bar{\rho}_\beta],
\]

where we use \( \bar{\gamma}_{ij} = \bar{\gamma}_{ij} = 0 \) and \( \bar{\rho}_{\alpha\beta} + \bar{\rho}_{\beta\alpha} = 0 \). Now we denote a symmetric matrix

\[
(2.9) \quad M(u)_{AB} = \begin{cases} 
    u_{ij} - u_{ij} - \delta_{ij} & A, B \in \{1, \cdots, m\} \\
    u_{\alpha\alpha} & A \in \{1, \cdots, m\}, B \in \{m+1, \cdots, m+n\} \\
    u_{\alpha\beta} + u_{\alpha\beta} + \delta_{\alpha\beta} & A, B \in \{m+1, \cdots, m+n\}
\end{cases}.
\]

Then the second fundamental tensor of \( M \) along \( \vec{n} \) is

\[
(2.10) \quad h_{AB} = -<\vec{n}_A, \bar{X}_B> = e^u [(1 + e^{2u})(1 + e^{2u} + |\nabla_x u|^2 + e^{2u} |\nabla_y u|^2)]^{-\frac{1}{2}} M(u)_{AB}.
\]
The linearized operator of \( F(2.11) \) then (2.15) is satisfied. And (2.16) makes sense if (2.16) holds.

**Proposition 2.3.** Assume \( m > n \) and \( M(u) \) is positive or negative definite, which implies that \((u_{ij} - u_i u_j - \delta_{ij}) > 0\) or \((u_{ij} + u_i u_j - \delta_{ij}) < 0\). In the first case, at the maximum value point of \( u \), we have \( u_{ij} \leq 0 \) and \( \nabla u = 0 \). Then \((\delta_{ij}) < (u_{ij}) \leq 0\), which is a contradiction. The second case has the same contradiction at the minimum value point of \( u \).

**Remark 2.2.** In the problem (1.2), \( u \) attains its maximum value only on the boundary of \( \Omega \). We can see this by the same argument as used in Proposition 2.1.

Now we discuss the openness part of problem (1.2). Let \( D = \{ u \in C^{2,\alpha}(\bar{\Omega});|u|_{\partial \Omega} = \psi, M(u) \text{ is positive definite} \} \). Consider the map \( F \) induced by problem (1.2)

\[
F : D \rightarrow C^{\alpha}(\bar{\Omega})
\]

\[
u \mapsto (f(u,|\nabla_x u|^2,|\nabla_y u|^2))^{-m+n+2} \det M(u).
\]

The linearized operator of \( F \) at \( u \) along function \( v \) is

\[
DF(u)v = f^{-\frac{m+n+2}{2}} \det M(u)\sum_{A,B} M(u)^{AB} v_{AB} - 2 \sum_{i,j} M(u)^{ij} u_{ij} + 2 \sum_{\alpha,\beta} M(u)^{\alpha\beta} u_{\alpha}v_{\beta}
\]

\[
-\frac{m+n+2}{2} f^{-\frac{m+n+2}{2}} \det M(u)[f_r v + 2f_p \sum_i u_i v_i + 2f_q \sum_{\alpha} u_{\alpha}v_{\alpha}],
\]

where \((M(u)^{AB})\) is the inverse matrix of \((M(u))_{AB}\). When \( f_r \geq 0 \), the linearized problem has a unique solution. Then by linear elliptic PDE theory, \( DF(u) \) is a continuous linear bijective map. Now by the implicit function theorem and the openness of positivity, we obtain the openness. Hence, we only need to find the condition guaranteeing \( f_r \geq 0 \). Since

\[
f_r = e^{\tau} \left[ \tau + \frac{\tau(1 + e^{2\tau}) - 2e^{2\tau}}{(1 + e^{2\tau})^2} p + \frac{\tau e^{2\tau} q}{1 + e^{2\tau}} + \frac{2e^{2\tau} q}{(1 + e^{2\tau})^2} \right],
\]

at \( r = u, p = |\nabla_x u|^2, q = |\nabla_y u|^2 \), we only need

\[
\tau(1 + e^{2\tau}) - 2e^{2\tau} \geq 0.
\]

Now by Remark 2.2, if we require

\[
\sup_{\partial \Omega} u = \sup_{\partial \Omega} \psi \leq \frac{1}{2} \ln \frac{\tau/2}{1 - \tau/2},
\]

then (2.15) is satisfied. And (2.16) makes sense if \( m > n \) by (2.1).

**Proposition 2.3.** Assume \( m > n \) and \( \sup_{\partial \Omega} \psi \leq \frac{1}{2} \ln \frac{\tau/2}{1 - \tau/2} \), then the openness of problem (1.2) holds.
3. $C^0$ AND $C^1$ ESTIMATES

From this section on, we always assume $m > n$ and use the Einstein convention: indices appearing twice in an expression, once as a subscript, once as a superscript implicitly summed over. The following comparison Lemma maybe known, but the author did not find an appropriate reference, so it is included here.

**Lemma 3.1.** Let $\Omega$ be a domain on the product of unit spheres. Let

\[ G(u) = \det M(u) [g(v, |\nabla_x u|^2, |\nabla_y u|^2)]^{-1}, \]

where $M(u)$ defined in (2.9) is positive definite and $g(r, p, q)$ is a positive smooth function on $\mathbb{R}^3$.

Assume that there is a constant $r_0$ such that for $r \leq r_0$, $g_r > 0$. For two smooth functions $u, v$, if $u, v \leq r_0$ and $G(u) \leq G(v)$ then one of the following holds

\[ v - \sup_{\Omega} (v - u) \leq u, \text{ or } v \leq u. \]

**Proof:** By $G(u) \leq G(v)$

\[ \frac{\det M(v)}{\det M(u)} \geq \frac{g(v, |\nabla_x v|^2, |\nabla_y v|^2)}{g(u, |\nabla_x u|^2, |\nabla_y u|^2)}. \]

Since $M(u)$ is symmetric and positive definite, we can assume $M(u) = CC^T$, where $C$ is a non-degenerate matrix. Then

\[ \frac{\det M(v)}{\det M(u)} \leq \frac{m + n + tr(C^{-1}(M(v) - M(u))(C^{-1})^T)}{m + n} \]

\[ = \left\{ 1 + \frac{1}{m + n} tr[M(u)^{-1}(M(v) - M(u))] \right\}^{m+n}, \]

where, $tr$ means taking the trace of a matrix. If (3.2) does not hold, and the function $v - u$ attains its maximum value at some point in $\Omega$, then at that point, $v - u > 0$, $\nabla v = \nabla u$, and $[(v - u), AB]_{AB}$ is non-positive definite. So

\[ \text{(3.5)} \quad tr[(M(u))^{-1}(M(v) - M(u))] = M(u)^{AB}(v - u)_{AB} \leq 0. \]

And since $u < v \leq r_0$, $\nabla u = \nabla v, g_r > 0$, we have

\[ \text{(3.6)} \quad g(v, |\nabla_x v|^2, |\nabla_y v|^2) = g(v, |\nabla_x u|^2, |\nabla_y u|^2) > g(u, |\nabla_x u|^2, |\nabla_y u|^2). \]

Hence by (3.3)-(3.6), we have a contradiction. \qed

We introduce a class of function sets,

**Definition 3.2.** (ABF-sets) Assume $\Omega$ is a PHC-domain, $m > n$ and $K$ is a smooth positive function defined on $\Omega$. We assume $\Omega = \Omega_0 \times S^n$. Let $\psi$ be a smooth function on $\Omega_0$. We call $\psi$ an **admissible boundary function** (simply, ABF) with respect to $\tau$ and $K$, if it satisfies

(i) The matrix $M(\psi)$ is positive definite,

(ii) $\sup_{\Omega} \psi \leq \frac{1}{2} \ln \frac{\tau/3}{1 - \tau/2}$, and $F(\psi) \geq K$.

The set of all ABFs with respect to $\tau$ and $K$ is denoted by $\text{ABF}(\tau, K)$.

In fact for $0 < \tau < 2, K > 0$ on a PHC-domain, the set $\text{ABF}(\tau, K)$ is always non-empty. Indeed for a PHC-domain $\Omega_0$, we can always assume that $\Omega_0$ is contained in the hemisphere $\{x \in \mathbb{R}^{m+1}; x \in S^m \text{ and } x_1 > 0\}$. So we can take a constant $E > 0$ sufficiently small, such that $x_1 - E > 0$. Then for any constant $F > 0$, let

\[ \varphi = -\ln F(x_1 - E). \]
The derivative of $\varphi$ is

$$
(3.8) \quad \varphi_i = -\frac{(x_1)_i}{x_1 - E}, \quad \varphi_{ij} = \frac{x_1 \delta_{ij}}{x_1 - E} + \frac{(x_1)_i(x_1)_j}{(x_1 - E)^2},
$$

where we use $(x_1)_j + x_1 \delta_{ij} = 0$ (cf. [CY]). Then

$$
(3.9) \quad \varphi_{ij} - \varphi_i \varphi_j - \delta_{ij} = \frac{E}{x_1 - E} \delta_{ij}.
$$

Then we take $A$ big enough such that

$$
(3.10) \quad \varphi - A \leq \frac{1}{2} \ln \frac{\tau/3}{1 - \tau/2},
$$

and

$$
(3.11) \quad F(\varphi - A) \geq e^{(m-n)(A-\varphi)} \det M(\varphi) \left(1 + |\nabla \varphi|^2\right)^{m+n+2} \geq K
$$

both holds. So let $\psi = \varphi - A$, then $\psi$ satisfies (i)(ii) of definition (3.2). Then $\text{ABF}(\tau, K)$ is always not empty.

Now fix $\tau$, $\tilde{K}$. For any $\tilde{\psi} \in \text{ABF}(\tau, \tilde{K})$ as the boundary function, we use the continuity method to solve the Dirichlet problem (1.2). For $t \in [0, 1]$, take

$$
(3.12) \quad \tilde{K}_t = (1 - t) F(\tilde{\psi}) + t \tilde{K}.
$$

Assume that $F(u_t) = \tilde{K}_t$ with $u_t|_{\partial \Omega} = \tilde{\psi}$. Obviously, at $t = 0$, we can take $u_0 = \tilde{\psi}$. By Definition 3.2, $F(\psi) \geq \tilde{K}$, so

$$
(3.13) \quad \tilde{K} \leq \tilde{K}_t \leq F(\tilde{\psi}).
$$

Let function $g$ equals $f^{(m+n+2)/2}$ in Lemma 3.1. Take $r_0 = \frac{1}{2} \ln \frac{\tau/3}{1 - \tau/2}$. When $r \leq r_0$, we have

$$
(3.14) \quad \tau (1 + e^{2r}) - 2 e^{2r} \geq \frac{\tau}{3}.
$$

By (2.14) and $0 < \tau < 2$, for $p, q > 0$,

$$
(3.15) \quad \frac{f_r(r, p, q)}{f(r, p, q)} \geq \frac{\tau + \frac{\tau}{3} p}{1 + e^{2r}} + \frac{2 e^{2r} q}{1 + e^{2r}} \geq \frac{\tau/3}{1 + e^{2r}} > 0.
$$

Since we have $\tilde{\psi} \in \text{ABF}(\tau, \tilde{K})$, by the (ii) of Definition 3.2 we have $u_t \leq r_0$. Then (3.15) implies $g_r > 0$. Moreover by (3.13), $F(u_t) \leq F(\tilde{\psi})$. Now we can use Lemma 3.1. By $u_t|_{\partial \Omega} = \tilde{\psi}$, we have

$$
(3.16) \quad \tilde{\psi} \leq u_t.
$$

Combining (3.16) and Remark 2.2, we obtain the $C^0$ estimate.

Now by the definition of derivative, (3.16) and $u_t|_{\partial \Omega} = \tilde{\psi}$, on $\partial \Omega$,

$$
(3.17) \quad \frac{\partial u_t}{\partial n} \leq \frac{\partial \tilde{\psi}}{\partial n},
$$

where $\vec{n}$ is the outer normal direction of $\Omega$. If vector $Y$ is in the tangent space of the submanifold $\partial \Omega$, then $\nabla_Y u_t \equiv \nabla_Y \tilde{\psi}$. Hence there is a constant $C_1$ depending on $\tilde{\psi}$, $m$, $n$, such that

$$
(3.18) \quad \nabla_Y u_t \leq C_1,
$$

where $Y$ is a unit vector in the tangent space of $S^m \times S^n$ supported by $\partial \Omega$, and the angle between $Y$ and $\vec{n}$ is not bigger than $\pi/2$. 
For any point \( P \in \partial \Omega_x \times S^n \), we know that \( P = (P_x, P_y) \), where \( P_x \in \partial \Omega_x \), \( P_y \in S^n \). In \( \Omega_x \) we take a geodesic curve \( l_x \) which starts at \( P_x \) with direction \(-\vec{m}(P_x)\) (the inward normal direction of \( P_x \)) and ends at point \( Q_x \in \partial \Omega_x \). Denote by \( \sigma \) the arc parameter of \( l_x \). Then at every point of \( l_x \), we choose an orthonormal frame \( \{e_1, \cdots, e_m\} \) such that \( e_m = \partial/\partial \sigma \). Then by the positivity of the matrix \( M(u) \) along geodesic curve \( l_x \),

\[
(3.19) \quad (u_t)_{\sigma, \sigma} - \frac{1}{2} (u_t)^2 - 1 > 0,
\]

where \((u_t)_{\sigma, \sigma}\) is the second order normal derivative. Take integral and by (3.18),

\[
(3.20) \quad -\frac{\partial u_t}{\partial n}(P) = (u_t)_{\sigma}(P_x, P_y) < (u_t)_{\sigma}(Q_x, Q_y) \leq C_1.
\]

Combining (3.18) (3.20), we obtain the \( C^1 \) estimate on the boundary. Namely, there is a constant \( C_2 \) depending on \( \psi, m, n, \partial \Omega \), such that on \( \partial \Omega \)

\[
(3.21) \quad |\nabla u_t| \leq C_2.
\]

Now we only need to give the interior \( C^1 \) estimate. Without loss of the generality, we only give the estimate for equation (1.2).

Consider a function

\[
(3.22) \quad \phi = |\nabla u|^2,
\]

and let the operator

\[
(3.23) \quad Lu = M(u)^{AB} v_{AB}.
\]

Assume that \( \phi \) attains its maximum value at some point \( P \) in \( \Omega \). Then at \( P \),

\[
(3.24) \quad \phi_A = 2 \sum_C u_C^2 u_{CA} = 0.
\]

By the Ricci identity on the product of unit spheres,

\[
(3.25) \quad u_{ABC} - u_{ACB} = -u_C^2 \delta_{AB} + u_B^2 \delta_{AC}.
\]

Using (3.25) and the positivity of \( M(u) \),

\[
(3.26) \quad \frac{1}{2} L \phi = \sum_C M(u)^{AB} u_C u_{CB} + \sum_C M(u)^{AB} u_C u_{CAB} \\
\geq \sum_C M(u)^{AB} u_C u_{ABC} + \sum_A M(u)^{AA} |\nabla u|^2 - M(u)^{AB} u_A u_B \sum_C M(u)^{AB} u_C u_{ABC}.
\]

Take logarithm of (1.2), and differentiate it. We get

\[
(3.27) \quad M(u)^{AB} u_{ABC} = M(u)^{AB} (u_i u_j + \delta_{ij}) C - M(u)^{AB} (u_a u_j + \delta_{a, j}) C + \frac{K_C}{K} \\
+ \frac{m + n + 2}{f} \left( \frac{f_x}{2} u_C + f_p \sum_i u_i u_C + f_q \sum_{\alpha} u_a u_{a, C} \right).
\]

Now by (3.24) and (3.26),

\[
(3.28) \quad L \phi \geq 2 \sum_C \frac{K_C u_C}{K} + \frac{m + n + 2}{f} f_x |\nabla u|^2.
\]

Then by (3.15) and the \( C^0 \) estimate, there is a positive constant \( \varepsilon_1 \) depending on \( \psi, m, n \), such that

\[
(3.29) \quad \frac{f_x(u, |\nabla_x u|^2, |\nabla_y u|^2)}{f(u, |\nabla_x u|^2, |\nabla_y u|^2)} \geq \varepsilon_1.
\]
Now by (3.29) and Schwarz inequality, at point $P$
\begin{equation}
(3.30) \quad 0 \geq L\phi \geq -C - 2\varepsilon_1 |\nabla u|^2 + (m + n + 2)\varepsilon_1 |\nabla u|^2,
\end{equation}
which implies the interior estimate. Here $C$ is a positive constant depending on $\psi, K, m, n$.
Now we have proved

**Proposition 3.3.** Let $0 < \tau < 2, \tilde{K} > 0$ and $\hat{\psi} \in AFB(\tau, \tilde{K})$. Assume that $u_\tau$ is the solution of problem (1.2) in which $K$ is $\tilde{K}_1$ defined by (3.12), and $\psi$ is $\hat{\psi}$. Then there is a constant $C_3$ depending on $\psi, m, n, \tilde{K}, \partial\Omega$ such that
\begin{equation}
(3.31) \quad \|u_\tau\|_{C^1(\Omega)} \leq C_3. \tag{3.31}
\end{equation}

\section{4. Interior $C^2$ Estimate}

Since we have a lot of positive constants, for simplicity, from this section on, we write $C$ to represent any constant of minor important. For a useful constant, we use $C$ or $C_1$ with a lower index (for example $C_1, C_2$) to represent it. These constants always relate to $\psi, K, m, n$ and the $C^1$ norm of $u$, but we do not refer to this fact everywhere. Without loss of generality, we only estimate problem (1.2) with $\psi \in AFB(\tau, \tilde{K})$, so by (3.14), (3.15) and (3.16), $u \geq \psi$. Since $\Omega$ is a PDC-domain, we can assume $\Omega_\varepsilon \subset \{x \in \mathbb{R}^{m+1}; x \in S^{m \text{ and } x_1 > 0}\}$. Then there is a positive constant $\varepsilon_2$ depending on $\partial\Omega$ such that in $\Omega$,
\begin{equation}
(4.1) \quad x_1 \geq \varepsilon_2.
\end{equation}

We let
\begin{align}
(4.2) \quad \eta &= 3 - e^{-C_4(u_\tau + 1)} - e^{-C_5\varepsilon_1}, \\
(4.3) \quad \zeta &= 3 - \eta,
\end{align}
where $C_4, C_5$ are two positive constants which will be determined in the following. Since the matrix $M(\psi)$ is positive, we can assume $M(\psi) \geq 4\varepsilon_3\text{id}$, where $\varepsilon_3$ is a positive constant depending on $\psi$. Then using (2.9),(3.23) and Proposition 3.3,
\begin{align}
(4.4) \quad L(e^{-C_4(u_\tau + 1)}) &= e^{-C_4(u_\tau + 1)} \{C_4^2 M(u)^{AB}(u - \psi)A(u - \psi)B - C_4 M(u)^AB(M(u)_{AB} - M(\psi)_{AB}) \\
&- C_4 M(u)^{ij}[u_{ij} - \psi_{ij}] - C_4 M(u)^{\alpha\beta}[-u_{\alpha\beta} + \psi_{\alpha\beta}]\} \\
&\geq e^{-C_4(u_\tau + 1)} \{C_4^2 M(u)^{AB}(u - \psi)A(u - \psi)B - (m + n)C_4 + 4\varepsilon_3 C_4 \sum_A M(u)^{AA} \\
&- CC_4 \sum_i M(u)^{ii} + C_4 M(u)^{\alpha\beta}(u - \psi)_\alpha u_\beta + C_4 M(u)^{\alpha\beta}\psi_\alpha(u - \psi)_\beta\}.
\end{align}

Now
\begin{align}
(4.5) \quad M(u)^{AB}(-\eta)_{AB} + 2M(u)^{ij}u_{ij} - 2M(u)^{\alpha\beta}u_{\alpha\beta} \\
&\geq e^{-C_4(u_\tau + 1)} \{C_4^2 M(u)^{AB}(u - \psi)_\alpha u_\beta \geq (m + n)C_4 + 4\varepsilon_3 C_4 \sum_A M(u)^{AA} \\
&- CC_4 \sum_i M(u)^{ii} + 3C_4 M(u)^{\alpha\beta}(u - \psi)_\alpha u_\beta + C_4 M(u)^{\alpha\beta}\psi_\alpha(u - \psi)_\beta\} \\
&+ e^{-C_5\varepsilon_1}[C_4^2 M(u)^{ij}(x_1)_i(x_1)_j + 2C_5 M(u)^{ij}u_{ij} \geq (m + n)C_4 + 4\varepsilon_3 C_4 \sum_A M(u)^{AA} \\
&- CC_4 \sum_i M(u)^{ii} + 3C_4 M(u)^{\alpha\beta}\psi_\alpha(u - \psi)_\beta + C_4 M(u)^{\alpha\beta}(u - \psi)_\alpha u_\beta \\
&- 3C_4 M(u)^{\alpha\beta}(u - \psi)_\alpha - C_4 M(u)^{ij}(u - \psi)_{ij}\} \\
&+ e^{-C_5\varepsilon_1}[C_4^2 M(u)^{ij}(x_1)_i(x_1)_j + 2C_5 M(u)^{ij}u_{ij}(x_1)_i - C_5 M(u)^{ij}(x_1)_i].
\end{align}
By Schwarz inequality and because the \( C^1 \) norms of \( \psi \) and \( u \) are bounded, we have
\[
|3C_4 M(u)^{AB}(u - \psi)_{A u\beta}| \leq \frac{C_4^2}{2} M(u)^{AB}(u - \psi)_{A}(u - \psi)_{B} + \frac{9}{2} M(u)^{\alpha\beta} u_{\alpha} u_{\beta}
\]
\[
\leq \frac{C_4^2}{2} M(u)^{AB}(u - \psi)_{A}(u - \psi)_{B} + C \sum_{A} M(u)^{\alpha\alpha},
\]
and we also have a similar inequality for the term \( C_4 M(u)^{AB}(u - \psi)_{A} \) and the inequality \( |2C_5 M(u)^{ij} u_{i(x_1)_j}| \leq C_5^2 M(u)^{ij} (x_1)_i (x_1)_j + M(u)^{ij} u_{ij} \). Then
\[
M(u)^{AB} (-\eta)_{A B} + 2 M(u)^{ij} u_{i\eta j} - 2 M(u)^{\alpha\beta} u_{\alpha} u_{\beta}
\]
\[
\geq e^{-C_4(u-\psi+1)} \{-\hat{C}_1 C_4 + (4\varepsilon_3 C_4 - \hat{C}_2) \sum_{A} M(u)^{AA} - \hat{C}_3 C_4 \sum_{i} M(u)^{ii} \}
\]
\[
-2\hat{C}_4 C_4 \sum_{i,\alpha} |M(u)^{ai}| + e^{-C_4 x_1} [-M(u)^{ij} u_{ij} - \hat{C}_5 M(u)^{ij} (x_1)_i j],
\]
where \( \hat{C}_1, \cdots, \hat{C}_4 \) are four positive constants. Obviously by the positivity of \( M(u) \),
\[
2|M(u)^{ai}| \leq 2|M(u)^{\alpha\alpha} M(u)^{ii}|^{1/2} \leq \frac{\varepsilon_3}{m \hat{C}_4} M(u)^{\alpha\alpha} + \frac{m \hat{C}_4}{\varepsilon_3} M(u)^{ii}.
\]
Now by (4.7) and the fact \((x_1)_i + x_1 \delta_{ij} = 0\),
\[
M(u)^{AB} (-\eta)_{A B} + 2 M(u)^{ij} u_{i\eta j} - 2 M(u)^{\alpha\beta} u_{\alpha} u_{\beta}
\]
\[
\geq e^{-C_4(u-\psi+1)} \{-\hat{C}_1 C_4 + (3\varepsilon_3 C_4 - \hat{C}_2) \sum_{A} M(u)^{AA} - \hat{C}_3 C_4 \sum_{i} M(u)^{ii} \}
\]
\[
+e^{-C_4 x_1} (\hat{C}_5 x_1 - \hat{C}_4^2) \sum_{i} M(u)^{ii},
\]
where \( \hat{C}_5 \) is also a positive constant depending on \( \hat{C}_4, \varepsilon_3, m \) and \( n \). Now we take \( C_5 = 2C_4^2 / \varepsilon_2 \), and \( \hat{C}_4 = \max\{\hat{C}_2 / \varepsilon_3, 2(\hat{C}_5 e^{C_4} / C_4^3)\} \). Now since \( x_1 \leq 1, \varepsilon \geq \psi \) and (4.1), we have
\[
M(u)^{AB} (-\eta)_{A B} + 2 M(u)^{ij} u_{i\eta j} - 2 M(u)^{\alpha\beta} u_{\alpha} u_{\beta} \geq 4 \varepsilon_4 \sum_{A} M(u)^{AA} - C,
\]
where \( \varepsilon_4 = e^{-C_4(1+\max_0 |u-\psi|)} \hat{C}_2 / 2 \). Now define a function
\[
\phi = e^{-C_4 u} \Delta u + \zeta,
\]
where \( C_4 \) is a positive constant which will be determined in the following, and \( \Delta \) is the Laplace operator of \( S^n \times S^n \). This type of function is well known (see \( Y \)), but we modify it and use the idea of \( B \) to handle the extra term \( \sum_{A} M(u)^{AA} \sum_{A} M(u)_{AA} \) which will appear in the following. Assume \( \phi \) attains its maximum value at point \( P \in \Omega \). Then at \( P \),
\[
\phi_A = -C_6 \eta_{A} e^{-C_4 u} \Delta u + e^{-C_4 u} \sum_{C} u_{CC A} + \zeta_A = 0.
\]
Then
\[
L(\phi)
= e^{-C_4 u} [C_6^2 M(u)^{AB} \eta_{AB} \Delta u - C_6 M(u)^{AB} \eta_{AB} \Delta u - 2C_6 M(u)^{AB} \eta_{A} \sum_{C} u_{CC B} 
\]
\[
+ \sum_{C} M(u)^{AB} u_{CC AB} ] + M(u)^{AB} \zeta_{AB}.
\]
Use the notation \( R_{BCD}^{AB} \) to denote the Riemannian curvature (see \( C \) appendix A.6 and the Ricci identities using in the following also see this book). Firstly by the Ricci identity and
(2.9),

\begin{equation}
M(u)^{AB} \eta_{A} u_{CCB} = M(u)^{AB} \eta_{A} u_{BCC} - M(u)^{AB} \eta_{A} u_{E} R_{BCC}^{E} \\
\leq M(u)^{AB} \eta_{A} M(u)_{BCC} + M(u)^{Ai} \eta_{A} (u_{i} u_{j} + \delta_{ij}) - M(u)^{A} \eta_{A} (u_{i} u_{j} + \delta_{ij}) \\
+ C \sum_{A} M(u)^{AA}
\end{equation}

\begin{equation}
= M(u)^{AB} \eta_{A} M(u)_{BCC} + M(u)^{Ai} \eta_{A} u_{i} \sum_{j} M(u)_{ij} + M(u)^{A} \eta_{A} u_{i} \sum_{j} (u_{j}^{2} + 1) \\
-M(u)^{A} \eta_{A} u_{i} \sum_{\beta} M(u)_{i\beta} + M(u)^{A} \eta_{A} u_{i} \sum_{\beta} (u_{i}^{2} + 1) + M(u)^{A} \eta_{A} u_{i} u_{j} \\
-M(u)^{A} \eta_{A} u_{i} u_{j} + C \sum_{A} M(u)^{AA}
\end{equation}

\begin{equation}
\leq M(u)^{AB} \eta_{A} M(u)_{BCC} + (|M(u)^{Ai} \eta_{A} u_{i}| + |M(u)^{A} \eta_{A} u_{i}|) \sum_{A} M(u)^{AA} \\
+ M(u)^{A} \eta_{A} u_{i} u_{j} - M(u)^{A} \eta_{A} u_{i} u_{j} + C \sum_{A} M(u)^{AA}.
\end{equation}

Since

\begin{equation}
|M(u)^{Ai} \eta_{A} u_{i}| \leq \frac{\varepsilon_{A}}{2} \sum_{i} M(u)^{ii} + CM(u)^{AB} \eta_{A} \eta_{B},
\end{equation}

and we have a similar inequality for the term \( |M(u)^{A} \eta_{A} u_{i}| \). By Schwarz inequality and (2.9)

\begin{equation}
M(u)^{Ai} \eta_{A} u_{i} u_{j} \leq M(u)^{Ai} \eta_{A} (M(u)_{i} u_{j}) + C \sum_{A} M(u)^{AA}
\end{equation}

\begin{equation}
\leq (M(u)^{AB} \eta_{A} \eta_{B})^{1/2} (M(u)^{ij} M(u)_{ik} u_{k} M(u)_{ij} u_{i})^{1/2} + C \sum_{A} M(u)^{AA},
\end{equation}

then diagonalizing the matrix \( (M(u)^{ij} M(u)_{ik} u_{k}) \) at point \( P \) and using the positivity of \( M(u) \), we have

\begin{equation}
M(u)^{ij} M(u)_{ik} u_{k} M(u)_{ij} u_{i} \leq C \sum_{i} M(u)^{ii} (\sum_{i} M(u)^{ii})^{2}.
\end{equation}

By the above two inequalities,

\begin{equation}
M(u)^{Ai} \eta_{A} u_{i} u_{j} \leq C(M(u)^{AB} \eta_{A} \eta_{B})^{1/2} (\sum_{i} M(u)^{ii})^{1/2} \sum_{i} M(u)_{ii} + C \sum_{A} M(u)^{AA}
\end{equation}

\begin{equation}
\leq \frac{\varepsilon_{A}}{2} \sum_{A} M(u)^{AA} \sum_{i} M(u)_{ii} + CM(u)^{AB} \eta_{A} \eta_{B} \sum_{A} M(u)^{AA} + C \sum_{A} M(u)^{AA},
\end{equation}

Similarly, we have an inequality for the term \( -M(u)^{A} \eta_{A} u_{i} u_{j} \). Now combining (4.14), (4.15) and (4.18),

\begin{equation}
\sum_{C} M(u)^{AB} \eta_{A} u_{CBB} \leq \sum_{C} M(u)^{AB} \eta_{A} M(u)_{BCC} + \varepsilon_{C} \sum_{A} M(u)^{AA} \sum_{A} M(u)^{AA}
\end{equation}

\begin{equation}
+ CM(u)^{AB} \eta_{A} \eta_{B} \sum_{A} M(u)_{AA} + C \sum_{A} M(u)^{AA}.
\end{equation}

It will be used in the later. By the Ricci identity, we get

\begin{equation}
u_{CAB} = u_{ABCC} - u_{EB} R_{ACC}^{E} - 2u_{EC} R_{BCC}^{E} - u_{AE} R_{BCC}^{E}.
\end{equation}
Then by (2.9),

\[(4.21) \quad \sum_C M(u)^{AB} u_{CCAB} \]

\[= \sum_C M(u)^{AB} u_{ABCC} - 2 \sum_C M(u)^{AB} u_{ECR^E_{BCA}} - 2 \sum_C M(u)^{AB} u_{AE} R^E_{BCC} \]

\[\leq \sum_C M(u)^{AB} u_{ABCC} - C \sum_A M(u)^{AA} \sum_A M(u)_{AA} - C \sum_A M(u)^{AA} - C \]

\[= \sum_C M(u)^{AB} M(u)_{ABCC} + 2 \sum_C M(u)^{ij} u_{iC} u_{jC} + 2 \sum_C M(u)^{ij} u_{ijCC} \]

\[-2 \sum_C M(u)^{ij} u_{iC} u_{jC} - 2 \sum_C M(u)^{ij} u_{ijCC} - C \sum_A M(u)^{AA} \sum_A M(u)_{AA} \]

\[-C \sum_A M(u)^{AA} - C, \]

and

\[(4.22) \quad \sum_C M(u)^{ij} u_{iC} u_{jC} - \sum_C M(u)^{ij} u_{iC} u_{jC} \]

\[= \sum_C M(u)^{ij} u_{iC} u_{jC} - \sum_C M(u)^{ij} u_{iC} u_{jC} \]

\[= \sum_C M(u)^{AB} u_{AC} M(u)_{BC} + \sum_j M(u)^{ij} (u_{ij} + \delta_{ij}) u_{Aj} \]

\[-\sum_{\beta} M(u)^{ij} (u_{ij} + \delta_{ij}) u_{\beta C} - 2 \sum_{\gamma} M(u)^{ij} (u_{ij} + \delta_{ij}) u_{\beta \gamma} \]

\[= \Delta u + 2 \sum_j M(u)^{ik} (u_{ij} + \delta_{ij}) u_{ik} + 2 \sum_{\gamma} M(u)^{ij} (u_{ij} + \delta_{ij}) u_{ij} \]

\[-\sum_{\beta} M(u)^{ij} (u_{ij} + \delta_{ij}) u_{\beta A} - 2 \sum_{\alpha} M(u)^{ij} (u_{ij} + \delta_{ij}) u_{\alpha A} \]

\[\geq \frac{\varepsilon_5}{2} \sum_{A,B} u_{AB}^2 + 2 \sum_j M(u)^{ik} M(u)_{kl} (u_{ij} + \delta_{ij}) \]

\[+ 2 \sum_{\gamma} M(u)^{ij} M(u)_{ij} (u_{ij} + \delta_{ij}) - C \sum_A M(u)^{AA} - C, \]

where \(\varepsilon_5 = \frac{(m + n + 2)e^{-2C_3}}{6(1 + e^{2C_5})^2(1 + C_3^2)^2} \) and we have used the inequality

\[(4.23) \quad -\sum_j M(u)^{ij} (u_{ij} + \delta_{ij}) u_{Aj} \]

\[= -\sum_j M(u)^{ij} (u_{ij} + \delta_{ij}) M(u)_{Aj} - \sum_j M(u)^{ij} (u_{ij} + \delta_{ij}) (u_{ik} u_{ij} + \delta_{ij}) \]

\[\geq -C \sum_A M(u)^{AA} - C, \]
and a similar inequality for term $-\sum_\beta M(u)\eta^\beta (u_\alpha u_\beta + \delta_{\alpha\beta})u_{A\beta}$. Now by (4.22),

\[
(4.24) \quad 2\left(\sum_C M(u)^{ij} u_i C u_{jC} - \sum_C M(u)^{\alpha\beta} u_{\alpha C} u_{\beta C}\right) \\
\geq -\varepsilon_3 \sum_{A,B} u_{AB}^2 - C \sum_A M(u)^{AA} \sum_A M(u)_{AA} - C \sum_A M(u)^{AA} - C.
\]

Now the term $\sum_A M(u)^{AA} \sum_A M(u)_{AA}$ appears, which is one of our main difficulties. By the Ricci identity, (4.21) becomes

\[
(4.25) \quad \sum_C M(u)^{AB} u_{CCAB} \\
\geq \sum_C M(u)^{AB} M(u)_{ABCC} + 2\left(\sum_C M(u)^{ij} u_i u_{CC}\right) - \sum_C M(u)^{\alpha\beta} u_{\alpha} u_{\beta} u_{CC} \\
-\varepsilon_3 \sum_{A,B} u_{AB}^2 - C \sum_A M(u)^{AA} \sum_A M(u)_{AA} - C \sum_A M(u)^{AA} - C.
\]

At point $P$, by (4.3),(4.10),(4.12),(4.13),(4.25) we have

\[
(4.26) \quad L(\phi) \\
\geq e^{-C_6 |\nabla u|^2} M(u)^{AB} \eta_{AB} \Delta u - C_6 M(u)^{AB} \eta_{AB} \Delta u - 2C_6 M(u)^{AB} \eta_A \sum_C u_{CCB} \\
+ \sum_C M(u)^{AB} M(u)_{ABCC} + 2C_6 M(u)^{ij} u_i \eta_j \Delta u - 2C_6 M(u)^{\alpha\beta} u_{\alpha} \eta_{\beta} \Delta u \\
-\varepsilon_3 \sum_{A,B} u_{AB}^2 - C \sum_A M(u)^{AA} \sum_A M(u)_{AA} - C \sum_A M(u)^{AA} - C \\
+ M(u)^{AB} \zeta_{AB} - 2M(u)^{ij} u_i \zeta_j + 2M(u)^{\alpha\beta} u_{\alpha} \zeta_{\beta} \\
= e^{-C_6 |\nabla u|^2} M(u)^{AB} \eta_{AB} \Delta u - 2C_6 M(u)^{AB} \eta_A \sum_C u_{CCB} + \sum_C M(u)^{AB} M(u)_{ABCC} \\
+ C_6 [M(u)^{AB} (-\eta, AB) + 2M(u)^{ij} u_i \eta_j - 2M(u)^{\alpha\beta} u_{\alpha} \eta_{\beta}] \sum_A M(u)_{AA} \\
+ C_6 [M(u)^{AB} (\eta, AB) + 2M(u)^{ij} u_i \eta_j - 2M(u)^{\alpha\beta} u_{\alpha} \eta_{\beta}) (|\nabla x|^2 - |\nabla y|^2 + m - n) \\
-\varepsilon_3 \sum_{A,B} u_{AB}^2 - C \sum_A M(u)^{AA} \sum_A M(u)_{AA} - C \sum_A M(u)^{AA} - C \\
+ M(u)^{AB} \zeta_{AB} - 2M(u)^{ij} u_i \zeta_j + 2M(u)^{\alpha\beta} u_{\alpha} \zeta_{\beta}
\]

Now using the first equality of (4.4), the bounds on $\psi$, on $x_1$, and on the $C^1$ norm of $u$, we have

\[
(4.27) \quad M(u)^{AB} (-\eta, AB) + 2M(u)^{ij} u_i \eta_j - 2M(u)^{\alpha\beta} u_{\alpha} \eta_{\beta} \leq C \sum_A M(u)^{AA} + C.
\]

Combining the above two inequalities and (4.10), we have

\[
(4.28) \quad L(\phi) \\
\geq e^{-C_6 |\nabla u|^2} M(u)^{AB} \eta_{AB} \Delta u - 2C_6 M(u)^{AB} \eta_A \sum_C u_{CCB} + \sum_C M(u)^{AB} M(u)_{ABCC} \\
+ (4\varepsilon_4 C_6 - C) \sum_A M(u)^{AA} \sum_A M(u)_{AA} - \varepsilon_5 \sum_{A,B} u_{AB}^2 - (C_6 + C) \sum_A M(u)^{AA} \\
- C(C_6) \} + 4\varepsilon_4 \sum_A M(u)^{AA} - C.
\]
Here $C(C_0)$ is a positive constant depending on $\psi$, $m$, $n$, the $C^1$ norm of $u$ and also $C_0$. Now we use equation. Take logarithm of (1.2) and differentiate it:

\begin{equation}
M(u)^{AB} M(u)_{AB} = [\ln(K)]_C + \frac{m+n+2}{2} [\ln(f)]_C,
\end{equation}

\begin{equation}
M(u)^{AB} M(u)_{ABCC} = M(u)^{A'A'} M(u)^{BB'} M(u)_{ABC} M(u)_{A'B'C} + [\ln(K)]_{CC} + \frac{m+n+2}{2} [\ln(f)]_{CC}.
\end{equation}

We choose a local orthonormal frame at $P$ such that the matrix $M(u)$ is diagonal at $P$. Then by (4.30),

\begin{equation}
\sum_C M(u)^{AB} M(u)_{ABCC} \geq \sum_{A,B,C} \frac{M(u)^2_{AB}}{M(u)_{AA} M(u)_{BB}} \frac{m+n+2}{2} \sum_C [\ln(f)]_C^2 + \frac{m+n+2}{2} \sum_C \frac{(f)_{CC}}{f} - C,
\end{equation}

where we used the bound of $K$. Obviously,

\begin{equation}
\sum_{A,B,C} \frac{M(u)^2_{AB}}{M(u)_{AA} M(u)_{BB}} \geq \sum_{B\neq C} \frac{M(u)^2_{BC}}{M(u)_{CC} M(u)_{BB}} + \sum_{A\neq C} \frac{M(u)^2_{AC}}{M(u)_{AA} M(u)_{CC}} + \sum_{A,C} \frac{M(u)^2_{AC}}{M(u)^2_{AA}}.
\end{equation}

Then by (4.29) and Schwarz inequality,

\begin{equation}
\frac{m+n+2}{4} \sum_C [\ln(f)]_C^2 = \frac{1}{m+n+2} \sum_A \sum_C \frac{M(u)^2_{AC}}{M(u)_{AA}} - [\ln(K)]_C^2 \leq \frac{1}{m+n+2} (1 + \frac{1}{m+n+1}) \sum_A \left( \sum_C \frac{M(u)^2_{AC}}{M(u)_{AA}} \right)^2 + C
\end{equation}

\begin{equation}
\leq \frac{m+n+2}{m+n+1} \sum_A \left( \frac{M(u)^2_{AC}}{M(u)_{AA}} \right)^2 + C.
\end{equation}

Now by (2.1)

\begin{equation}
(f)_C = f_p u_C + 2[f_p \sum_i u_i u_{iC} + f_q \sum_\alpha u_\alpha u_{\alpha C}],
\end{equation}

then by the Schwarz inequality and $f_p, f_q > 0$,

\begin{equation}
\frac{m+n+2}{4} \sum_C (f)_C^2 \leq C + \varepsilon_5 \sum_{AB} u_{AB}^2 + \frac{m+n+2}{f^2} \sum_C [f_p \sum_i u_i u_{iC} + f_q \sum_\alpha u_\alpha u_{\alpha C}]^2
\end{equation}

\begin{equation}
\leq C + \varepsilon_5 \sum_{AB} u_{AB}^2 + \frac{m+n+2}{f^2} \sum_C \sum_i u_i^2 + f_q \sum_\alpha u_\alpha^2 \sum_{i,C} u_i^2 + f_q \sum_\alpha u_\alpha^2 \sum_{i,C} u_\alpha^2.
\end{equation}

Now by (2.1) and Proposition 3.3,

\begin{equation}
\frac{m+n+2}{f^2} \sum_i u_i^2 - f_q \sum_\alpha u_\alpha^2 \sum_{i,C} u_i^2 + f_q \sum_{i,C} u_{iC}^2
\end{equation}

\begin{equation}
= \frac{m+n+2}{(1 + e^{2u} + |\nabla_x u|^2 + e^{2u} |\nabla_y u|^2)^2} (1 + e^{2u}) \sum_{i,C} u_i^2 + e^{2u} (1 + e^{2u}) \sum_{\alpha,C} u_\alpha^2
\end{equation}

\begin{equation}
\geq \frac{(m+n+2)e^{-2c_3}}{(1 + e^{2c_3})^2(1 + C_0^2)} \sum_{A,B} u_{AB}^2 \geq 6\varepsilon_5 \sum_{A,B} u_{AB}^2.
\end{equation}
Now (4.35) becomes

\[ m + n + 2 \sum_C \frac{[\ln(f)]^2}{f} \leq C - 5\varepsilon_5 \sum_{AB} u_{AB}^2 + \frac{m + n + 2}{f} [f_p \sum_{i,C} u_{iC}^2 + f_q \sum_{\alpha,C} u_{\alpha C}^2]. \]

By (4.34),

\[ \frac{m + n + 2}{2} \sum_C \frac{(f)_{CC}}{f} = \frac{m + n + 2}{2} \sum_C (f_p u_{CC})_C + \frac{m + n + 2}{2} \sum_i (f_p u_{iC})_C + \frac{m + n + 2}{2} \sum_{\alpha,C} (f_q u_{\alpha C})_C + \frac{m + n + 2}{f} [f_p \sum_{i,C} u_{iC}^2 + f_q \sum_{\alpha,C} u_{\alpha C}^2] \]

\[ \leq -5\varepsilon_5 \sum_{AB} u_{AB}^2 - C + \frac{m + n + 2}{f} [f_p \sum_{i,C} u_{iC}^2 + f_q \sum_{\alpha,C} u_{\alpha C}^2] + \frac{m + n + 2}{f} [f_p \sum_{i,C} u_{iCC} + f_q \sum_{\alpha,C} u_{\alpha CC}]. \]

By (4.12),

\[ \frac{m + n + 2}{f} [f_p \sum_{i,C} u_{iCC} + f_q \sum_{\alpha,C} u_{\alpha CC}] \geq -C(C_6) - \varepsilon_5 \sum_{AB} u_{AB}^2, \]

where \( C(C_6) \) is a positive constant depending on an undetermined constant \( C_6 \). Then by (4.33), (4.37), (4.38), (4.39),

\[ -\frac{m + n + 2}{2} \sum_C \frac{[\ln(f)]^2}{f} + \frac{m + n + 2}{2} \sum_C \frac{(f)_{CC}}{f} \leq -\frac{m + n + 2}{m + n + 1} \sum_{A,C} \frac{(M(u))_{AA}^2}{M(u)_{CC}} + 3\varepsilon_5 \sum_{A,B} u_{AB}^2 - C(C_6). \]

Now by (4.31), (4.32), (4.40) and (4.1), \( \eta < 3 \),

\[ \sum_C M(u)^{AB} M(u)_{ABCC} \]

\[ \leq 2 \sum_{A \neq C} \frac{M(u)_{ACC}^2}{M(u)_{AA} M(u)_{CC}} + \frac{1}{m + n + 1} \sum_{A,C} \frac{(M(u)_{AA}^2)}{M(u)_{CC}} + 3\varepsilon_5 \sum_{A,B} u_{AB}^2 - C(C_6). \]

Then by (4.19), (4.28), (4.41), and Ricci identity, we have

\[ L(\phi) \geq e^{-C_6 \eta} \{ C_6 (C_6 - \hat{C}_6) M(u)^{AB} \eta_{AB} \sum_A M(u)_{AA} - 2C_6 \sum_C M(u)_{AA} \eta_{AM(u)_{BBC}} + (2C_6 \varepsilon_4 - C) \sum_A M(u)^{AA} \sum_A M(u)_{AA} + 2 \sum_{A \neq C} \frac{M(u)_{ACC}^2}{M(u)_{AA} M(u)_{CC}} \]

\[ + \frac{1}{m + n + 1} \sum_{A,C} \frac{M(u)_{AAC}^2}{M(u)_{AA}^2} + 2\varepsilon_5 \sum_{A,B} u_{AB}^2 - (CC_6^2 + CC_6 + C) \sum_A M(u)^{AA} \]

\[ -C(C_6) \} + 4\varepsilon_4 \sum_A M(u)^{AA} - C. \]
Now take $C_6 > \hat{C}_6$. Since
\begin{align}
(4.43) \quad 2C_6 \sum_{A \neq C} \frac{\eta_A M(u) M(u)_{AA}}{M(u)_{AA}} &= C_6 \sum_{A \neq C} \frac{2(M(u)_{CC} \eta_A) M(u)_{AA}}{M(u)_{AA} M(u)_{CC}} \\
&\leq C_6 \sum_{A \neq C} \frac{(C_6 - \hat{C}_6) M(u)_{CC} \eta_A^2 + \frac{M(u)_{CC}^2}{C_6 - \hat{C}_6}}{M(u)_{AA} M(u)_{CC}} \\
&\leq C_6 (C_6 - \hat{C}_6) \sum_{A \neq C} \frac{\eta_A^2 M(u)_{CC}}{M(u)_{AA}} + \frac{C_6}{C_6 - \hat{C}_6} \sum_{A \neq C} \frac{M(u)_{CC}^2}{M(u)_{AA} M(u)_{CC}},
\end{align}
and
\begin{align}
(4.44) \quad 2C_6 \sum_{A} \frac{\eta_A M(u)_{AAA}}{M(u)_{AA}} &\leq \frac{1}{m + n + 1} \sum_{A} \frac{M(u)_{AAA}^2}{M(u)_{AA}^2} + (m + n + 1)C_6^2 \sum_{A} \eta_A^2,
\end{align}
then
\begin{align}
(4.45) \quad C_6 (C_6 - \hat{C}_6) M(u)_{AB} \eta_A \eta_B \sum_{A} M(u)_{AA} - 2C_6 \sum_{C} M(u)_{AB} \eta_A M(u)_{BC} \\
&\quad + 2 \sum_{A \neq C} \frac{M(u)_{CC}^2}{M(u)_{AA} M(u)_{CC}} + \frac{1}{m + n + 1} \sum_{A \neq C} \frac{M(u)_{CC}^2}{M(u)_{AA}^2} \\
&\geq (2 - \frac{C_6}{C_6 - \hat{C}_6}) \sum_{A \neq C} \frac{M(u)_{CC}^2}{M(u)_{AA} M(u)_{CC}} - CC_6^2.
\end{align}
Now by (4.1), (4.45), $x_1 \leq 1$, we have
\begin{align}
(4.46) \quad L(\phi) &\geq e^{-C_6 \eta} \left\{ (2 - \frac{C_6}{C_6 - \hat{C}_6}) \sum_{A \neq C} \frac{M(u)_{CC}^2}{M(u)_{AA} M(u)_{CC}} - (\hat{C}_6 C_6^2 + \hat{C}_6) \sum_{A} M(u)_{AA} \\
&\quad + (2C_6 \varepsilon_4 - \hat{C}_7) \sum_{A} M(u)_{AA} + \varepsilon_5 \sum_{A,B} u_{AB}^2 \right\} 4\varepsilon_4 \sum_{A,B} M(u)_{AA} - C(C_6).
\end{align}
We take $C_6$ big enough to satisfy
\begin{align}
(4.47) \quad 2 - \frac{C_6}{C_6 - \hat{C}_6} &> 0, \quad 2C_6 \varepsilon_4 - \hat{C}_7 > 0, \quad \text{and} \quad 4\varepsilon_4 - e^{-C_6} (\hat{C}_6 C_6^2 + \hat{C}_6) > 0
\end{align}
Now by (4.47) and $\eta \geq 1$, (4.46) implies that $\sum_{A,B} u_{AB}^2$ is bounded at point $P$. So the function $\phi$ has a uniform upper bound, and by the positivity of matrix $M(u)$, there is a constant $C_7$ depending on $K, \psi, m, n, \partial\Omega$ such that
\begin{align}
(4.48) \quad \sum_{A,B} |u_{AB}|^2 &\leq C_7.
\end{align}
This gives the interior $C^2$ estimate. Here we generalize the idea of [Y] to deal with the $C^3$ term. In order to obtain the $C^2$ estimate, now we only need the estimate on the boundary.

5. $C^2$ Estimate on the Boundary
Let $P$ be on the boundary $\partial\Omega_\delta \times S^n$, $P = (P_x, P_\delta)$, and $\Omega^\delta(P) = \Omega^\delta_x(P_x) \times B^\delta_\delta(P_\delta)$. Here $\Omega^\delta_x(P_x) = \Omega_x \cap B^\delta_x(P_x)$, and $B^\delta_\delta(P_\delta), B^\delta_x(P_x)$ are $\delta$ geodesic sphere neighborhoods of $S^n$ and $S^n$ centered at $P_x$ and $P_\delta$, respectively. Since $\Omega$ is a PHC-domain, for sufficiently small $\delta$, we can find a frame $\{e_1, \cdots, e_{m+1}\}$ on $\tilde{\Omega}^\delta(P)$ such that: $e_m$ is the outer normal direction on $\partial\Omega^\delta_x(P_\delta)$; the previous $m - 1$ ones are tangent vectors of $\partial\Omega_{\delta_x}$; the last $n$ ones are tangent vectors of $S^n$. By compactness of $\partial\Omega_{\delta_x}$, we can take $\delta$ independent from boundary points.
(The proof is similar to Lebesgue’s Covering Lemma.) Taking $\delta < 1$ sufficiently small, we consider a local function on $\Omega^0(P)$,

$$
\phi = (u - \psi)_C + v,
$$

where $\psi \in AB(\tau, K)$, and $v$ is an undetermined function which we will give explicitly in the following. If $\phi$ attains its maximum value at some point $Q$ in $\Omega^0(P)$. Then at point $Q$,

$$
(u - \psi)_{C_A} + v_A = 0.
$$

By (3.23) and Ricci identity,

$$
L \phi = M(u)^{AB}(u - \psi)_{CAB} + M(u)^{AB}v_{AB}
$$

\begin{align*}
\geq & \quad M(u)^{AB}u_{ABC} + M(u)^{AB}v_{AB} - C \sum_A M(u)^{AA}.
\end{align*}

For any function $\xi$ define an operator

$$
\hat{L}(\xi) = M(u)^{AB}\xi_{AB} - 2M(u)^{ij}u_i\xi_j + 2M(u)^{\alpha\beta}u_{\alpha\beta}
$$

\begin{align*}
& \quad - m + n + 2 f [f_p \sum_i u_i \xi_i + f_q \sum_\alpha u_{\alpha} \xi_\alpha].
\end{align*}

Then by (4.29), (5.2) and (5.4), (5.3) becomes

$$
L \phi
$$

\begin{align*}
\geq & \quad 2M(u)^{ij}u_iu_{jC} - 2M(u)^{\alpha\beta}u_{\alpha\beta} + m + n + 2 f [f_p \sum_i u_i u_{jC} + f_q \sum_\alpha u_{\alpha} u_{\alpha C}]
\end{align*}

\begin{align*}
+ & \quad M(u)^{AB}v_{AB} - C \sum_A M(u)^{AA}
\end{align*}

\begin{align*}
\geq & \quad \hat{L} v - C \sum_A M(u)^{AA}.
\end{align*}

Now we define a vector field in $\mathbb{R}^{m+1}$. For $0 < \epsilon < 1/4$ and $P_x \in \partial \Omega_x$, let

$$
\chi(P_x) = -e_m(P_x) + \epsilon \tilde{\gamma}(P_x).
$$

Here we use $\tilde{\gamma}(\cdot)$ and $e_m(\cdot)$ to denote the vectors at point “™” in $\mathbb{R}^{m+1}$, and $\tilde{\gamma}$ is defined in (1.1). We choose a coordinate system for $\mathbb{R}^{m+1}$ with first coordinate axis given by $\chi(P_x)$. Denote by $< \cdot, \cdot >_m$ and $| \cdot |_m$ the inner product and corresponding norm of $\mathbb{R}^{m+1}$. Let $H = \{ \gamma \in S^m; < -e_m(P_x), \gamma >_m = 0 \}$ be a totally geodesic submanifold of $S^m$. Since $\Omega_x$ is a strictly infinitesimally convex domain, it is a strictly locally convex domain (see [SI]). Since the exponential map of $S^m$ takes the subspace $T_{P_x}(\partial \Omega_x)$ onto $H$, we have $H \cap \Omega_x = \{ P_x \}$. (If there is another point $P_x \in H \cap \Omega_x$, the minimal geodesic curve connected $P_x$ and $\tilde{P}_x$ is contained in $H \cap \Omega_x$ which contradicts the strictly locally convexity of $\partial \Omega_x$.) This means that for any $Q_x' \in \Omega_x^0$,

$$
< \tilde{\gamma}(Q_x') - \tilde{\gamma}(P_x), -e_m(P_x) >_m > 0.
$$

Now define $S_{\delta'}$ = $\{ Q_x' \in S^m; |\tilde{\gamma}(Q_x') - \tilde{\gamma}(P_x)|_m = \delta' \}$. For $Q_x' \in \Omega_x^0(P_x) \cap S_{\delta'}$, where $\delta' < 2 \sin(\delta/2)$ (guaranteeing that $\Omega_x^0 \cap S_{\delta'}$ is non-empty), we have

$$
< \tilde{\gamma}(Q_x') - \tilde{\gamma}(P_x), \tilde{\gamma}(P_x) >_m = - (\delta')^2 / 2,
$$

where we used $|\tilde{\gamma}(Q_x') - \tilde{\gamma}(P_x)|_m = (\delta')^2$ and $|\tilde{\gamma}(Q_x')|_m = |\tilde{\gamma}(P_x)|_m = 1$. Now denote $a_x(Q_x') = < \tilde{\gamma}(Q_x') - \tilde{\gamma}(P_x), e_m(P_x) >_m$. Then by (5.8), $\sum^m a_x(Q_x') + (\delta')^4 / 4 = (\delta')^2$. By (5.7), we have $-a_m(Q_x') > 0$. Now further assume $Q_x'$ is the minimum value point of function $-a_m$. Then $a_m(Q_x') < (\delta')^2 - (\delta')^4 / 4$ for $i \neq m$. We can take a vector $\tilde{b}$ in $S_{\delta'}$ defined by $\tilde{b} = \sum_i b_i e_i(P_x) + [1 - (\delta')^2 / 2] \tilde{\gamma}(P_x)$ such that for $i \neq m$, $|b_i| > |a_i(Q_x')|$; and $-b_m < -a_m(Q_x')$. By the openness of $\Omega_x^0(P_x)$, the point corresponding to $\tilde{b}$ is in $\Omega_x^0(P_x)$ if we further require that $|b - a(Q_x')|$ is sufficiently small. This is a contradiction. Hence the
minimal value of function \(-a_m\) in \(\Omega^+_{1}(P_x) \cap S_{1}^\epsilon\) only occurs in \(\partial \Omega^+_{1}(P_x) \cap S_{1}^\epsilon\). So for any \(Q'_x \in \Omega^+_{1}(P_x) \cap S_{1}^\epsilon\), by (5.8) hold for any point in \(S_{1}^\epsilon\) and (5.6),

\[
\langle \nu(Q'_x) - \nu(P_x), \nu(P_x) \rangle > m \overset{\text{inf}}{=} \inf_{Q''_x \in \partial \Omega^+_{1}(P_x) \cap S_{1}^\epsilon} \langle \nu(Q''_x) - \nu(P_x), \nu(P_x) \rangle > m.
\]

Let \(\nabla, \tilde{\nabla}\) be the Levi-Civita connections of \(S^m\) and \(\mathbb{R}^{m+1}\). Since \(\partial \Omega_x\) is a strictly infinitesimally convex hypersurface in \(S^m\), its second fundamental tensor is positive definite everywhere. Then for \(i \neq m\), on \(\partial \Omega^+_{1}(P_x)\) the order \(m-1\) matrix

\[
< \nabla_{\epsilon_i} \nabla_{\epsilon_j} \tilde{\nu}, -e_m > > 0 = < \nabla_{\epsilon_i} \epsilon_j, -e_m > > 0.
\]

has a positive uniform (independent from the choice of boundary points) lower bound, by the compactness of \(\partial \Omega_x\). By (5.6), we can take a uniformly sufficiently small \(\epsilon\) such that the matrix \(< \nabla_{\epsilon_i} \nabla_{\epsilon_j} \tilde{\nu}, -e_m > > 0\) also has a uniform positive lower bound. Now by the Taylor expansion of \(\partial \Omega_x\) near \(P_x\) with the frame \(e_1(P_x), \ldots, e_{m-1}(P_x)\), we find that the right hand side of (5.9) is non-negative for sufficiently small \(\delta\). Moreover the choice of \(\delta\) is independent from the boundary point, as follow from: (i) there is a uniform \(\delta\) such that \(\partial \Omega^+_{1}(P_x)\) can be parameterized, and the tangent vectors along the parameterized curves at \(P_x\) are \(e_1(P_x), \ldots, e_{m-1}(P_x)\); (ii) for the function \(< \tilde{\nu}(Q''_x) - \tilde{\nu}(P_x), \nu(P_x) > > m\), where \(Q''_x \in \partial \Omega^+_{1}(P_x)\), the second order term at \(P_x\) has a uniform lower bound, and higher than second order terms at \(P_x\) have uniform upper bound. By the arbitrary choice of \(\delta\), we have \(< \tilde{\nu}(Q'_x) - \tilde{\nu}(P_x), \nu(P_x) > > m\) for \(Q'_x \in \Omega^+_{1}(P_x)\) which implies in \(\Omega^+_{1}(P_x)\) that

\[
x_1 \overset{\geq}{=} x_1(P_x) = \frac{\epsilon}{(1 + \epsilon^2)^{1/2}} > 0.
\]

Denote \(\theta = x_1(P_x)\), which is a constant only depending on \(\partial \Omega_x\). Now obviously for \(i \neq m\),

\[
(x_1)_i(P_x) = 0, \text{ and } |(x_1)_m(P_x)| \overset{\geq}{=} \frac{3}{4}.
\]

Now we let

\[
w = e^{-C_8(u^{\theta})} + e^{-C_8C_9(x_1 - \theta)}.
\]

Here \(C_8, C_9\) are two constants which we will determine in the following. Using the similar tricks of (4.4) to (4.9), we have

\[
\tilde{L}(e^{-C_8(u^{\theta})}) \overset{\geq}{=} e^{-C_8(u^{\theta})}\left[-\tilde{C}_{10}C_8 + (3\tilde{C}_8 - \tilde{C}_{11}) \sum_A M(u)^{AA} - \tilde{C}_{12}C_8 \sum_im\right],
\]

and in \(\Omega^+_{1}(P_x)\),

\[
\tilde{L}(e^{-C_8(u^{\theta})}) \overset{\geq}{=} e^{-C_8(u^{\theta})}\left\{\left(C_9C_8\right)^2M(u)^{ij}(x_1)_i(x_1)_j - C_9C_8M(u)^{ij}(x_1)_i\right\}
+ 2C_9C_8M(u)^{ij}(x_1)_i\right\}
+ \frac{(m + n + 2)C_9C_8}{f}\sum_iu_i(x_1)_i\right\}
\]

\[
\geq e^{-C_8(u^{\theta})}\left\{\left(C_9C_8\right)^2M(u)^{ij}(x_1)_i(x_1)_j + (C_9C_8 - C)\sum_iM(u)^{ij} - CC_9C_8\right\}
\]

where we used \((x_1)_i + x_1 \delta_{ij} = 0\), \(x_1 \geq \theta\) and the inequality

\[
2C_9C_8M(u)^{ij}u_i(x_1)_j \leq \frac{(C_9C_8)^2}{2}M(u)^{ij}(x_1)_i(x_1)_j + C\sum_iM(u)^{ij}.
\]

Then for any point \(Q' \in \Omega^+_{1}(P), Q' = (Q'_x, Q'_y)\), we have

\[
\left|(x_1)_i(P_x) - (x_1)_i(Q'_x)\right| \leq |\nabla(x_1)| \text{dist}_e(P_x, Q'_x) \leq \tilde{C}_{13}\delta,
\]
where \( \hat{C}_{13} \) is an absolute positive constant and \( \text{dist}_x(\cdot, \cdot) \) is the distance function of \( S^m \).

Assume \( C_9, C_8 > 1 \). Then we choose \( \delta \) such that

\[
\delta \leq \frac{1}{6\hat{C}_{13}(C_9C_8)^2}.
\]

By (5.12), (5.17), we have for

\[
(5.21)
\]

By (5.14), (5.21), (5.22), and (5.23),

\[
(5.22)
\]

Then in \( \Omega^\delta(P) \), by the positivity of \( M(u) \) and (5.16),

\[
\begin{align*}
M(u)^{ij}(x_1)_i(x_1)_j & = M(u)^{mm}(x_1)_m + \sum_{i,j \neq m} M(u)^{ij}(x_1)_i(x_1)_j + 2M(u)^{mi}(x_1)_i(x_1)_m \\
& \geq \frac{1}{4} M(u)^{mm} - \frac{1}{6(C_9C_8)^2} \sum_{i} M(u)^{ii} - \sum_{i} (M(u)^{mm} + M(u)^{ii})[(x_1)_i][(x_1)_m] \\
& \geq \frac{1}{4} M(u)^{mm} - \frac{C}{(C_9C_8)^2} \sum_{i} M(u)^{ii}.
\end{align*}
\]

Now (5.12) becomes

\[
\begin{align*}
\tilde{L}(e^{-C_9C_8(x_1 - \theta)}) & \geq e^{-C_9C_8(x_1 - \theta)} \left[ \frac{(C_9C_8)^2}{8} M(u)^{mm} + (C_9C_8 \theta - \hat{C}_{14}) \sum_i M(u)^{ii} - \hat{C}_{15}C_9C_8 \right].
\end{align*}
\]

Since at point \( P, u - \psi - C_9(x_1 - \theta) = 0 \), we know that in \( \Omega^\delta(P) \),

\[
\begin{align*}
|u - \psi - C_9(x_1 - \theta)| & \leq (|\nabla(u - \psi)| + C_9|\nabla(x_1)|) \delta \leq \hat{C}_{16}C_9 \delta,
\end{align*}
\]

where \( \hat{C}_{16} \) is a positive constant depending on \( \psi \) and \( C_9 \). We further require

\[
\delta \leq \frac{1}{C_9C_8 \hat{C}_{16}}.
\]

We take \( C_9 = \max \{1, (e\hat{C}_{12} + \hat{C}_{14})/\theta\} \), and

\[
\begin{align*}
C_8 & \geq \frac{\hat{C}_{11}}{\hat{C}_3}.
\end{align*}
\]

Now by (5.14), (5.21), (5.22) and (5.23),

\[
\begin{align*}
\tilde{L}(e^{-C_9(u - \psi)} + e^{-C_9C_8(x_1 - \theta)}) & \geq e^{-C_9(u - \psi)} \left\{ -\hat{C}_{12}C_8 + (3\varepsilon_3C_8 - \hat{C}_{11}) \sum_{A} M(u)^{AA} - \hat{C}_{12}C_8 \sum_{i} M(u)^{ii} \\
& \quad + e^{-C_9(u - \psi)}[u - C_9(x_1 - \theta)][(C_9C_8)^2/8 M(u)^{mm} + (C_9C_8 \theta - \hat{C}_{14}) \sum_i M(u)^{ii} - \hat{C}_{15}C_9C_8] \right\} \\
& \geq e^{-C_9(u - \psi)}[\hat{C}_{10}C_8 + (3\varepsilon_3C_8 - \hat{C}_{11}) \sum_{A} M(u)^{AA} - \hat{C}_{12}C_8 \sum_{i} M(u)^{ii} \\
& \quad + e^{-1}(C_9C_8)^2/8 M(u)^{mm} + e^{-1}(C_9C_8 \theta - \hat{C}_{14}) \sum_i M(u)^{ii} - e\hat{C}_{15}C_9C_8].
\end{align*}
\]
Since $C_8 > 1$ and by (1.2),(5.13),(5.24) and (5.25),
(5.26) \[ \bar{L}(w) \geq e^{-C_8(u-\psi)}[2\varepsilon_3C_8 \sum_A M(u)^{AA} + \varepsilon_3C_8 \sum_A \lambda_A + e^{-1}(C_9C_8)^2 \lambda_1 - (\hat{C}_{10} + e\hat{C}_{15}C_9)C_8] \]
\[ \geq e^{-C_8(u-\psi)}[\varepsilon_3C_8 \sum_A M(u)^{AA} + \varepsilon_3C_8 \sum_A \lambda_A + e^{-1}(C_9C_8)^2 \lambda_1 - (\hat{C}_{10} + e\hat{C}_{15}C_9)C_8] \]
\[ \geq e^{-C_8(u-\psi)}[\varepsilon_3C_8 \sum_A M(u)^{AA} + (C_8)\sum_{m+n+1} \hat{\lambda}_A (\hat{\lambda}_A)^{m+n} - (\hat{C}_{10} + e\hat{C}_{15}C_9)C_8] \]
\[ \geq e^{-C_8(u-\psi)}[\varepsilon_3C_8 \sum_A M(u)^{AA} + (C_8)\sum_{m+n+1} \hat{\lambda}_A (\hat{\lambda}_A)^{m+n} - (\hat{C}_{10} + e\hat{C}_{15}C_9)C_8], \]
where we assume $\lambda_1 \leq \lambda_2 \ldots \leq \lambda_{m+n}$ to be the positive eigenvalues of the matrix $M(u)^{-1}$, and $\hat{C}_{17}$ is a positive constant depending on $\varepsilon_3, C_9, C_8$. Now by (5.24), we take
(5.27) \[ C_8 = \max\{1, \frac{\hat{C}_{11}(\hat{C}_{10} + e\hat{C}_{15}C_9)^{m+n}}{C_{17}}\}. \]
Now we choose $\delta$ sufficiently small and satisfying (5.18),(5.23), then in $\Omega^\delta(P)$ by (5.26),(5.27),
(5.28) \[ \bar{L}(w) \geq \hat{C}_{19} \sum_A M(u)^{AA}. \]
For $Q' \in \Omega^\delta(P)$, we let
(5.29) \[ v(Q') = C_{10}w(Q') - C_{11}d^2(Q'), \]
where $d(\cdot) = \text{dist}(P, \cdot)$, $\text{dist}(\cdot, \cdot)$ is the distance function of $S^n \times S^n$, and $C_{10}, C_{11}$ are two positive constants which will be determined in the following. Then by (5.1) and (5.29), we know that on $\partial\Omega \cap \bar{\Omega}^\delta(P) \setminus \{P\}$ (where $w < 2$ and $u|_{\partial\Omega} = \psi$), we have
(5.30) \[ \phi = C_{10}w - C_{11}d^2 < 2C_{10}. \]
Obviously $\phi(P) = 2C_{10}$. Moreover, on $\partial\Omega^\delta \cap \bar{\Omega}$,
(5.31) \[ \phi \leq (u-\psi)C + 2C_{10} - C_{11}d^2 \leq \hat{C}_{19} + 2C_{10} - C_{11}\delta^2, \]
where $\hat{C}_{19}$ is a positive constant depending on $C_5, \psi$. Now we take $C_{11} = (\hat{C}_{19} + 1)/\delta^2$. By (5.30) and (5.31) on $\partial\Omega^\delta(P) \setminus \{P\}$,
(5.32) \[ \phi < \phi(P) = 2C_{10}. \]
We notice that the derivative of the smooth function $\text{dist}^2(P, \cdot)$ has a uniform bound which does not depend on the point $P$. Then by (5.5),(5.28) and (5.29), we have
(5.33) \[ L\phi \geq C_{10}\hat{C}_{19} \sum_A M(u)^{AA} - \hat{C}_{20}(1 + \sum_A M(u)^{AA}), \]
where $\hat{C}_{20}$ is a constant. So by (1.2) and Proposition 3.3, we only need to take $C_{10}$ big enough, then $L\phi > 0$ in $\Omega^\delta(P)$. This means that the maximum value of function $\phi$ is attained on the boundary. Then in $\Omega^\delta(P)$, (5.32) gives
(5.34) \[ (u-\psi)C \leq C_{11}d^2 + C_{10}[(1 - e^{-C_8(u-\psi)}) + (1 - e^{-C_9C_8(x_1 - \theta)})]. \]
Both sides of the above inequality are 0 at point $P$. Now we obtain the uniform lower bound of term $(u-\psi)c_m$. And letting
(5.35) \[ \phi = -(u-\psi)c + \nu, \]
we can similarly obtain the upper bound. So for $C \neq m$, there is a positive constant $C_{12}$ depending on $\psi, K, m, n, \partial\Omega$ such that on $\partial\Omega \times S^n$
(5.36) \[ |(u-\psi)c_m| \leq C_{12}. \]
By the choice of frame we made in the head of this section, and by the equality on the boundary \( u = \psi \), we know that for \( A, B \neq m \), on \( \partial \Omega^f(P) \)

\[
(5.37) \quad u_{AB} = \psi_{AB} - h_{AB}(u - \psi)_m,
\]

where \( h_{AB} \) is the second fundamental tensor along the outward normal direction \( e_m \) of \( \partial \Omega \). Obviously, if one of \( A, B \) takes value in \( m + 1, \cdots, m + n \), then \( h_{AB} = 0 \). Moreover, as \( \psi \in ABF(\tau, K, 0, m) \), \( \psi \) only depends on \( S_m \). So for \( i, j \neq m \),

\[
(5.38) \quad u_{ij} = \psi_{ij} - h_{ij}(u - \psi)_m,
\]

and \( u_{AB} = 0 \) for the other cases. Since \( M(u) \) is positive definite, we now only need the upper bound on \( u_{mm} \). We use the same argument as in the papers [T] and [G]. For \( P \in \partial \Omega \), define a function

\[
(5.39) \quad \lambda(P) = \min_{|\xi| = 1, \xi \in T_P(\partial \Omega_\epsilon)} [\nabla_\xi u(P) - (\nabla_\xi u(P))^2 - 1],
\]

where \( |\cdot| \) is the standard norm of \( S^m \times S^n \). Assume that at \( P_0 \in \partial \Omega \) and \( \xi = e_1(P_0) \in T_{P_0}(\partial \Omega_\epsilon) \), \( \lambda \) attains its minimum value. Then \( \lambda(P) \geq \lambda(P_0) \). By (5.37), (5.39) and \( u|_{\partial \Omega} = \psi \),

\[
(5.40) \quad h_{11}(P)(u - \psi)_m(P) \leq (\psi_{11}(P) - \psi_{11}^2(P)) - (\psi_{11}(P_0) - \psi_{11}^2(P_0)) + h_{11}(P_0)(u - \psi)_m(P_0).
\]

By the compactness of \( \partial \Omega_\epsilon \), there is a uniform sufficiently small \( \delta \) such that on \( \bar{\Omega}^f(P_0) \), the smooth function \( h_{11} = \nabla_\epsilon e_1, e_m \) has a uniform negative upper bound. Now on \( \bar{\Omega}^f(P_0) \), let

\[
(5.41) \quad \Psi(Q) = h_{11}(Q)^{-1}[\psi_{11}(Q) - \psi_{11}^2(Q)] - (\psi_{11}(P_0) - \psi_{11}^2(P_0)) + h_{11}(P_0)(u - \psi)_m(P_0),
\]

where \( Q \in \bar{\Omega}^f(P_0) \). By (5.40), for \( P \in \bar{\Omega}^f(P_0) \cap \partial \Omega \), obviously

\[
(5.42) \quad (u - \psi)_m(P) \geq \Psi(P), \quad \text{and} \quad (u - \psi)_m(P_0) = \Psi(P_0).
\]

Then on \( \bar{\Omega}^f(P_0) \), let

\[
(5.43) \quad \phi = \Psi - (u - \psi)_m + v.
\]

By (5.42), and a similar argument as in (5.1) to (5.5) and (5.28) to (5.33), \( \phi \) attains its maximum value at point \( P_0 \) for a choice of suitable constants of \( v \). Then \( \phi_m(P_0) \geq 0 \) which implies that \( u_{mm}(P_0) \) has a upper bound. So at \( P_0 \) by (5.38), all eigenvalues of \( M(u) \) have a upper bound. By equation (1.2), the minimum eigenvalue of \( M(u) \) at \( P_0 \) has a lower bound, which implies that \( \lambda(P_0) \) has a lower bound. Since \( P_0 \) is the minimum value point of \( \lambda \), for any boundary point \( P \), and any unit vector \( \xi \in T_P(\partial \Omega_\epsilon), \nabla_\xi u(P) - (\nabla_\xi u(P))^2 - 1 \) has a uniform lower bound. By the sentence after (5.38) and Definition 3.2,

\[
(5.44) \quad M(u)_{mm}^* = (u_{ij} - u_iu_j - \delta_{ij})(m-1) \times (m-1),
\]

where \( M(u)_{mm}^* \) is the cofactor matrix of \( M(u)_{mm} \), and the right hand side is a order \( m - 1 \) matrix with \( i, j \neq m \). So on the tangent space of \( \partial \Omega_\epsilon \), diagonalizing the matrix \( M(u)_{mm}^* \), we find that \( M(u)_{mm}^* \) has a positive uniform lower bound. With the same argument as in [CNS1], \( u_{mm} \) has a uniform upper bound. Now by (5.36) and (5.37), there is a positive constant \( C_{13} \) depending on \( \psi, K, m, n, \partial \Omega \) such that on \( \partial \Omega_\epsilon \times S^n \),

\[
(5.45) \quad |u_{AB}| \leq C_{13}.
\]

Now we have the \( C^2 \) estimate on the boundary, and combining this with the interior \( C^2 \) estimate and proposition 3.3, we obtain \( C^2 \) estimate. Then using Evans-Krylov theory (see[GT]), we have the \( C^{2,\alpha} \) estimate. Then differentiate equation (1.2) and using Schauder theory, we obtain Proposition 1.2. This gives the existence part of Theorem 1.2. For the uniqueness part, we let \( g \) equals \( f^{(m+n+2)/2} \) in Lemma 3.1. If \( u, v \) are both solutions of Problem (1.2), then \( G(u) = G(v) \). Then similar as the argument of (3.14), (3.15), we can use Lemma 3.1, and for the equality of boundary-values, we have \( u = v \).
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Wang ZhiZhang
Fudan University, Institute of Mathematics Science, Shanghai, 200433, China
E-mail: youxiang163wang@163.com