Tunable three-dimensional nonreciprocal transmission in a layered nonlinear elastic wave metamaterial by initial stresses

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Abstract In this work, the three-dimensional (3D) propagation behaviors in the nonlinear phononic crystal and elastic wave metamaterial with initial stresses are investigated. The analytical solutions of the fundamental wave and second harmonic with the quasi-longitudinal (qP) and quasi-shear (qS1 and qS2) modes are derived. Based on the transfer and stiffness matrices, band gaps with initial stresses are obtained by the Bloch theorem. The transmission coefficients are calculated to support the band gap property, and the tunability of the nonreciprocal transmission by the initial stress is discussed. This work is expected to provide a way to tune the nonreciprocal transmission with vector characteristics.

Key words nonlinear elastic wave metamaterial, nonreciprocal transmission, three-dimensional (3D) elastic wave, initial stress

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1 Introduction

Phononic crystals consist of two or more materials periodically, and can generate band gap of elastic waves[1–5]. Band gaps are certain frequency regions in which the elastic wave propagation is prohibited[6–7]. Elastic wave metamaterial is a new concept proposed in recent years, which brings extraordinary phenomena[8–15]. These periodic structures have the ability to control the wave propagation and vibration, which results in some advanced devices in practice.

The material nonlinearity can illustrate interesting wave phenomena and transmission behaviors, which has attracted considerable attention[16–20]. The distinguishing property of the material nonlinearity is the generation of higher-order harmonic. Liang et al.[21–22] studied an acoustic diode consisting of a linear phononic crystal and a nonlinear layer to show the nonreciprocal transmission of the acoustic wave. The nonreciprocal transmission means that waves can propagate in one direction but are prohibited in the reverse. Recently, increasing attention...
has been paid to the phenomenon in which the reciprocity theorem of the classic wave system is broken\textsuperscript{[23–30]}.

However, the above-mentioned studies mainly focused on scalar waves with only one displacement component. In recent years, the propagation of three-dimensional (3D) harmonic waves in layered structures have been reported\textsuperscript{[31–34]}. In our previous work, the nonreciprocal transmission in 3D cases in a layered nonlinear elastic wave metamaterial was discussed\textsuperscript{[35]}. Moreover, some studies have indicated that the effects of the external initial stresses on the band gap are significant\textsuperscript{[36–38]}. As a result, the initial stress can offer a new opportunity to tune the 3D nonreciprocal transmission.

In this investigation, the nonreciprocal transmission of 3D waves in a layered nonlinear elastic wave metamaterial with initial stresses is studied. Combining the band gap in the linear phononic crystal and material nonlinearity breaks the reciprocity theorem of elastic waves. According to the transfer and stiffness matrices, the band gaps and transmission coefficients of the fundamental wave and the second harmonic are obtained. The effects of the initial stresses on propagation behaviors are discussed.

2 Governing equation with initial stresses

Figure 1(a) shows a one-dimensional (1D) nonlinear phononic crystal with initial stresses and the local coordinate of each sub-cell. This structure is formed by two different nonlinear materials A and B, which consists of \( m \) unit cells. \( d_1 \) and \( d_2 \) denote the widths of the layers A and B, respectively, and the thickness of a unit cell is \( d = d_1 + d_2 \). The characters \( 2n - 1 \), \( 2n \), and \( 2n + 1 \) indicate the interfaces of the \( n \)th unit cell. The normal initial stresses \( \sigma_{11}^0 \), \( \sigma_{22}^0 \), and \( \sigma_{33}^0 \) are taken into account. For an incident elastic wave in the 3D space, the propagation direction is denoted by the polar and azimuthal angles \( \theta_1 \) and \( \theta_2 \).

As shown in Fig. 1(b), a nonlinear elastic wave metamaterial is composed of a 1D phononic crystal with layers of linear materials C and D and a nonlinear medium A. We assume that the elastic wave from the right to the left is the positive direction, and the reverse case represents the negative one. The nonreciprocal transmission can be obtained by the combination of the linearly periodic structure and nonlinear material. Then, the elastic wave can propagate in the positive direction but stop in the negative one, which shows the diode characteristic of the elastic wave.

For the 3D elastic wave propagation, the displacement components with time \( t \) can be

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Layered nonlinear phononic crystal and elastic wave metamaterial with initial stresses (color online)}
\end{figure}
written as
\[ u_i = u_i(x, y, z, t). \]  

(1)

The governing equation of the anisotropic monoclinic medium with initial stresses can be expressed as\[^{[31,39]}\]
\[ \sigma_{ij} + (\sigma_{ik} w_{ik} + \sigma_{lj} \varepsilon_{ij} - \sigma_{ik} \varepsilon_{kj}) = \rho u_{i,t}, \quad i, j, k, l = x, y, z, \]
where \( \rho \) is the mass density, the commas in the subscripts refer to the derivative with respect to time or space coordinates, and
\[ w_{ik} = \frac{u_{i,k} - u_{k,i}}{2}, \quad \varepsilon_{il} = u_{l,i}, \quad \varepsilon_{kj} = \frac{u_{k,j} + u_{j,k}}{2}. \]

(3)

The constitutive equation with material nonlinearity can be written as\[^{[40]}\]
\[ \sigma_{ij} = c_{ijkl} u_{k,l} + \frac{1}{2} m_{ijklmn} u_{k,l} u_{m,n}, \]
where \( m_{ijklmn} \) is the third-order elastic constant, \( c_{ijkl}, c_{ijlm}, c_{jklm} \), and \( c_{jilm} \) denote the second-order constants, \( \delta_{km}, \delta_{im} \), and \( \delta_{ik} \) represent the Kronecker delta, and \( u_k \) and \( u_m \) are the mechanical displacements.

The displacement components can be expressed as
\[ u_i = u_i^{(1)} + u_i^{(2)}, \]

(5)

where \( u_i^{(1)} \) and \( u_i^{(2)} \) denote the displacement components of the fundamental wave and the second harmonic, respectively.

Based on the perturbation approach method, the governing equations with initial stresses of the fundamental wave and the second harmonic can be derived as

\[ \rho u_{p,x,t}^{(1)} - C_{p,11} u_{p,x,x}^{(1)} - 2C_{p,15} u_{p,x,z}^{(1)} - (C_{p,66} + (\sigma_{0,22}^{(0)}) - (\sigma_{11}^{(0)/2})) u_{p,x,y}^{(1)} - (C_{p,55} + (\sigma_{33}^{(0)/2})) u_{p,x,z}^{(1)} - (\sigma_{11}^{(0)/2}) u_{p,z}^{(1)} - (C_{p,55} + (\sigma_{11}^{(0)/2})) u_{p,z}^{(1)} = 0, \]

(6a)

\[ \rho u_{p,y,t}^{(1)} - (C_{p,66} + (\sigma_{0,22}^{(0)}) - (\sigma_{11}^{(0)/2})) u_{p,y,x}^{(1)} - (C_{p,25} + (\sigma_{33}^{(0)}) - (\sigma_{22}^{(0)/2})) u_{p,y,y}^{(1)} - (C_{p,46} + (\sigma_{11}^{(0)/2})) u_{p,y,z}^{(1)} - (\sigma_{22}^{(0)/2}) u_{p,y,z}^{(1)} - 2C_{p,15} u_{p,z}^{(1)} - (C_{p,25} + (\sigma_{33}^{(0)}) - (\sigma_{22}^{(0)/2})) u_{p,z}^{(1)} + (\sigma_{11}^{(0)/2}) u_{p,z}^{(1)} = 0, \]

(6b)

\[ \rho u_{p,z,t}^{(1)} - C_{p,11} u_{p,z,z}^{(1)} - 2C_{p,15} u_{p,z,z}^{(1)} - (C_{p,66} + (\sigma_{0,22}^{(0)}) - (\sigma_{11}^{(0)/2})) u_{p,x,y}^{(1)} - (C_{p,55} + (\sigma_{33}^{(0)/2})) u_{p,x,z}^{(1)} - (\sigma_{11}^{(0)/2}) u_{p,z}^{(1)} - (C_{p,55} + (\sigma_{11}^{(0)/2})) u_{p,z}^{(1)} = 0, \]

(6c)

\[ \rho u_{p,x,t}^{(2)} - C_{p,11} u_{p,x,x}^{(2)} - 2C_{p,15} u_{p,x,z}^{(2)} - (C_{p,66} + (\sigma_{0,22}^{(0)}) - (\sigma_{11}^{(0)/2})) u_{p,y,y}^{(2)} - (C_{p,55} + (\sigma_{33}^{(0)/2})) u_{p,y,z}^{(2)} - (\sigma_{22}^{(0)/2}) u_{p,y,z}^{(2)} - 2C_{p,15} u_{p,z}^{(2)} - (C_{p,25} + (\sigma_{33}^{(0)}) - (\sigma_{22}^{(0)/2})) u_{p,z}^{(2)} - (\sigma_{11}^{(0)/2}) u_{p,z}^{(2)} - (C_{p,46} + (\sigma_{33}^{(0)}) - (\sigma_{22}^{(0)/2})) u_{p,z}^{(2)} = F_1(u_p^{(1)}), \]

(6d)

\[ \rho u_{p,y,t}^{(2)} - (C_{p,66} + (\sigma_{0,22}^{(0)}) - (\sigma_{11}^{(0)/2})) u_{p,x,y}^{(2)} - (C_{p,25} + (\sigma_{33}^{(0)}) - (\sigma_{22}^{(0)/2})) u_{p,x,z}^{(2)} - (\sigma_{22}^{(0)/2}) u_{p,y,z}^{(2)} - 2C_{p,15} u_{p,z}^{(2)} - (C_{p,25} + (\sigma_{33}^{(0)}) - (\sigma_{22}^{(0)/2})) u_{p,z}^{(2)} = F_2(u_p^{(1)}), \]

(6e)
\( \rho_p u_{px,zz}^{(2)} - C_{p15} u_{px,xx}^{(2)} - (C_{p55} + C_{p14} + (\sigma_0^{0}/3)/(2) - (\sigma_1^{0}/2)) u_{px,zz}^{(2)} - C_{p46} u_{px,yy}^{(2)} - C_{p35} u_{px,zz}^{(2)} - (C_{p25} + C_{p40}) u_{py,xy}^{(2)} - (C_{p44} + C_{p23} + (\sigma_0^{0}/3)/(2) - (\sigma_2^{0}/2)) u_{py,yz}^{(2)} - (C_{p55} + (\sigma_1^{0}/2)) - (\sigma_0^{0}/3)/(2)) u_{py,zx}^{(2)} - (C_{p41} + (\sigma_2^{0}/2) - (\sigma_3^{0}/3))/(2)) u_{py,zy}^{(2)} - C_{p33} u_{py,zz}^{(2)} = F_3(u_p^{(1)}), \)  

(6f)

where the subscript \( p (p = 1, 2) \) refers to the material of each sub-cell, \( u_p^{(1)} \) and \( u_p^{(2)} \) are the displacement components of the fundamental wave and the second harmonic, respectively, \( C_{p mn} (m, n = 1, 2, \ldots, 6) \) represent the second-order elastic constants, and \( F_3(u_p^{(1)}) \) is the bulk driving force of the second harmonic generated by the interaction between the fundamental wave and material nonlinearity \( [16, 41] \). The explicit expressions of \( F_3(u_p^{(1)}) \) can be found in Ref. [35] and are not presented here for simplicity.

From Eqs. (6a)–(6c), we can see that the displacements of the fundamental wave along three directions are coupled. Accordingly, the analytical solutions depending on \( x_p, y_p, \) and \( z_p \) can be given as

\[
(u_p^{(1)}, u_p^{(1)}, u_p^{(1)}) = (U_p^{(1)}, U_p^{(1)}, U_p^{(1)}) \exp \left( \frac{i \omega}{c_p^{(1)}} (q_{p1} x_p + \alpha_p y_p + q_{p2} z_p - c_p^{(1)} t) \right),
\]

(7)

where \( U_p^{(1)}, U_p^{(1)}, \) and \( U_p^{(1)} \) are the displacement amplitudes, \( q_{p1} = \sin \theta_1 \cos \theta_2, q_{p2} = \sin \theta_1 \sin \theta_2, \) \( \omega^{(1)} = \omega/k^{(1)} \) is the phase velocity of the incident fundamental wave, \( \omega \) and \( k^{(1)} \) denote the frequency and the wave number, respectively, and \( i = \sqrt{-1} \).

Substituting Eq. (7) into Eqs. (6a)–(6c) yields

\[
\begin{pmatrix}
T_{p11} & T_{p12} & T_{p13} \\
T_{p21} & T_{p22} & T_{p23} \\
T_{p31} & T_{p32} & T_{p33}
\end{pmatrix}
\begin{pmatrix}
U_p^{(1)} \\
U_p^{(1)} \\
U_p^{(1)}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix},
\]

(8)

where

\[
T_{p11} = C_{p11} q_{p1}^2 + 2 C_{p15} q_{p1} q_{p2} + C_{p66} q_{p2}^2 + C_{p55} q_{p2}^2 + \frac{\sigma_0^{0} - \sigma_1^{0}/2}{\varrho_p},
\]

(9a)

\[
T_{p12} = C_{p12} q_{p1} q_{p2} + C_{p40} q_{p2}^2 + C_{p66} q_{p1} q_{p2} + C_{p25} q_{p2} q_{p1} + \frac{\sigma_0^{0} - \sigma_2^{0}/2}{\varrho_p},
\]

(9b)

\[
T_{p13} = C_{p13} q_{p1} q_{p2} + C_{p15} q_{p1}^2 + C_{p35} q_{p2}^2 + C_{p55} q_{p2} q_{p1} + \frac{\sigma_0^{0} - \sigma_3^{0}/2}{\varrho_p},
\]

(9c)

\[
T_{p21} = C_{p12} q_{p1} q_{p2} + C_{p40} q_{p2}^2 + C_{p66} q_{p1} q_{p2} + C_{p25} q_{p2} q_{p1} + \frac{\sigma_2^{0} - \sigma_1^{0}/2}{\varrho_p},
\]

(9d)

\[
T_{p22} = 2 C_{p46} q_{p1} q_{p2} + C_{p66} q_{p1}^2 + C_{p22} q_{p2}^2 + C_{p44} q_{p2}^2 + \frac{\sigma_0^{0} - \sigma_2^{0}/2}{\varrho_p},
\]

(9e)

\[
T_{p23} = C_{p46} q_{p1} q_{p2} + C_{p23} q_{p2} q_{p1} + C_{p25} q_{p2} q_{p1} + C_{p44} q_{p2} q_{p1} + \frac{\sigma_2^{0} - \sigma_3^{0}/2}{\varrho_p},
\]

(9f)

\[
T_{p31} = C_{p13} q_{p1} q_{p2} + C_{p15} q_{p1}^2 + C_{p35} q_{p2}^2 + C_{p55} q_{p2} q_{p1} + \frac{\sigma_3^{0} - \sigma_1^{0}/2}{\varrho_p},
\]

(9g)

\[
T_{p32} = C_{p46} q_{p1} q_{p2} + C_{p23} q_{p2} q_{p1} + C_{p25} q_{p2} q_{p1} + C_{p44} q_{p2} q_{p1} + \frac{\sigma_3^{0} - \sigma_2^{0}/2}{\varrho_p},
\]

(9h)

\[
T_{p33} = 2 C_{p35} q_{p1} q_{p2} + C_{p55} q_{p1}^2 + C_{p44} q_{p2}^2 + \frac{\sigma_1^{0} - \sigma_3^{0}/2}{\varrho_p} + \frac{\sigma_0^{0} - \sigma_2^{0}/2}{\varrho_p} - \rho_p (\omega^{(1)})^2.
\]

(9i)
The existence of non-trivial solutions in Eq. (8) requires the coefficient determinant being zero. Then, the characteristic equation can be derived as

$$A_6\alpha_p^6 + A_4\alpha_p^4 + A_2\alpha_p^2 + A_0 = 0,$$

(10)

where the expressions of \(A_6, A_4, A_2,\) and \(A_0\) are presented in Appendix A.

Then, three pairs of conjugate roots can be obtained by the sixth-order polynomial, which denote the coupled quasi-shear (qS2 and qS1) and quasi-longitudinal (qP) waves. We assume that \(\alpha_{p1}, \alpha_{p3},\) and \(\alpha_{p5}\) represent the transmitted qS2, qS1, and qP waves, while \(\alpha_{p2}, \alpha_{p4},\) and \(\alpha_{p6}\) mean the reflected qS2, qS1, and qP ones.

We define the amplitude ratios for the coupled waves as

$$\frac{a_{p2}(1)}{a_{p1}(1)} = \frac{t_{p1}^{(1)}}{t_{p2}^{(1)}} = \frac{T_{p13}T_{p21} - T_{p11}T_{p23}}{T_{p12}T_{p23} - T_{p13}T_{p22}},$$

(11a)

$$\frac{a_{p3}(1)}{a_{p1}(1)} = \frac{t_{p1}^{(1)}}{t_{p2}^{(1)}} = \frac{T_{p11}T_{p22} - T_{p12}T_{p21}}{T_{p12}T_{p23} - T_{p13}T_{p22}},$$

(11b)

The term \(\exp(-i\omega t)\) is ignored in the following derivation for simplicity. Then, the displacement and stress components of the coupled waves can be expressed as

$$\begin{align*}
(u_{px}^{(1)}, u_{pq}^{(1)}, u_{pq}^{(1)}) &= (1, a_{p2}^{(1)}, a_{p3}^{(1)})U_{px}^{(1)} \exp\left(i\frac{\omega}{c_{(1)}}(q_{p1}x_p + \alpha_{pq}y_p + q_{p2}z_p)\right),
\end{align*}$$

(12a)

$$\begin{align*}
(\sigma_{p12q}^{(1)}, \sigma_{p22q}^{(1)}, \sigma_{p32q}^{(1)}) &= i\frac{\omega}{c_{(1)}}(G_{p12q}^{(1)}, G_{p22q}^{(1)}, G_{p32q}^{(1)})u_{px}^{(1)} \exp\left(i\frac{\omega}{c_{(1)}}(q_{p1}x_p + \alpha_{pq}y_p + q_{p2}z_p)\right),
\end{align*}$$

(12b)

where \(q = 1, 2, \ldots, 6,\) and

$$\begin{align*}
G_{p12q}^{(1)} &= C_{p46}q_{p2}a_{p1}^{(1)} + C_{p46}\alpha_{pq}a_{p3}^{(1)} + C_{p66}\alpha_{pq} + C_{p66}q_{p1}a_{p2}^{(1)} - \frac{\sigma_{11}^0 + \sigma_{22}^0}{2} q_{p1}a_{p2}^{(1)},
G_{p22q}^{(1)} &= C_{p12q} + C_{p22}\alpha_{pq}a_{p2}^{(1)} + C_{p23}q_{p2}a_{p3}^{(1)} + C_{p25}q_{p2} + C_{p25}q_{p1}a_{p3}^{(1)} + \sigma_{22}^0 (q_{p1} + q_{p2}a_{p3}^{(1)}),
G_{p32q}^{(1)} &= C_{p44}q_{p2}a_{p1}^{(1)} + C_{p44}\alpha_{pq}a_{p2}^{(1)} + C_{p46}\alpha_{pq} + C_{p46}q_{p1}a_{p1}^{(1)} - \frac{\sigma_{22}^0 + \sigma_{33}^0}{2} q_{p2}a_{p2}^{(1)}.
\end{align*}$$

(13a)

The fundamental wave in each nonlinear layer can generate the bulk driving force for the second harmonic as

$$\begin{align*}
F_s(u_p^{(1)}) &= \sum_{n=3,4,5,6} U_{p1i(c_m-c_n)}^{DL} \exp\left(i\frac{\omega}{c_{(1)}}(\alpha_{pm} + \alpha_{pn})y_p + \frac{2i\omega}{c_{(1)}}(q_{p1}x_p + q_{p2}z_p)\right)
+ \sum_{n=3,4,5,6} U_{p1i(c_m-c_n)}^{DT} \exp\left(i\frac{\omega}{c_{(1)}}(\alpha_{pm} + \alpha_{pn})y_p + \frac{2i\omega}{c_{(1)}}(q_{p1}x_p + q_{p2}z_p)\right)
+ \sum_{l=3,4} U_{p1i(c_l-c_n)}^{DL} \exp\left(i\frac{\omega}{c_{(1)}}(\alpha_{pl} + \alpha_{ph})y_p + \frac{2i\omega}{c_{(1)}}(q_{p1}x_p + q_{p2}z_p)\right)
+ \sum_{l=3,4} U_{p1i(c_l-c_n)}^{DT} \exp\left(i\frac{\omega}{c_{(1)}}(\alpha_{pl} + \alpha_{ph})y_p + \frac{2i\omega}{c_{(1)}}(q_{p1}x_p + q_{p2}z_p)\right)
+ \sum_{q=1}^6 U_{p1i(c_q-c_n)}^{DL} \exp\left(\frac{2i\omega}{c_{(1)}}(q_{p1}x_p + \alpha_{pq}y_p + q_{p2}z_p)\right)
+ \sum_{q=1}^6 U_{p1i(c_q-c_n)}^{DL} \exp\left(\frac{2i\omega}{c_{(1)}}(q_{p1}x_p + \alpha_{pq}y_p + q_{p2}z_p)\right).
\end{align*}$$

(14)
where $U_{p|c_n-c_n}$, $U_{DT|c_n-c_n}$, $U_{DL|c_n-c_n}$, $U_{DL|p|c_n-c_n}$, $U_{DL|p|c_n-c_n}$, and $U_{DL|p|c_n-c_n}$ are amplitudes of the bulk driving force. The superscripts DL and DT mean the components of the driving force with longitudinal and transverse waves, respectively.

The displacement components of the second harmonic can be derived by Eqs. (6d)–(6f) and (14) as
\[
(u_{pxq}^{(2)}, u_{pyq}^{(2)}, u_{pzq}^{(2)}) = \left( U_{pxq}^{(2)}, U_{pyq}^{(2)}, U_{pzq}^{(2)} \right) \exp \left( \frac{2i\omega}{c^{(2)}} (\beta_{pq} y_p + q_p z_p) \right),
\]
where $U_{pxq}^{(2)}$, $U_{pyq}^{(2)}$, and $U_{pzq}^{(2)}$ are amplitudes for the double frequency, $\beta_{pq}$ is the ratio of wave numbers for the second harmonic, and $c^{(2)} = \omega/k^{(2)}$ is the phase velocity of the second harmonic with $k^{(2)}$ denoting the wave number.

### 3 Band gap and transmission coefficient

We consider a 3D elastic wave in both the nonlinear phononic crystal and the elastic wave metamaterial. For the fundamental wave, we define the displacement and stress vectors as
\[
\begin{align*}
\mathbf{u} &= (u_{pxq}^{(1)}, u_{pyq}^{(1)}, u_{pzq}^{(1)})^T, \\
\mathbf{\sigma} &= (\sigma_{pxy}^{(1)}, \sigma_{pyz}^{(1)}, \sigma_{pzx}^{(1)})^T,
\end{align*}
\]
where
\[
\begin{align*}
u_{pxq}^{(1)} &= \sum_{q=1}^{6} U_{pxq}^{(1)} \exp \left( \frac{i\omega}{c^{(1)}} (p_{pq} y_p + q_p z_p) \right), \\
u_{pyq}^{(1)} &= \sum_{q=1}^{6} U_{pyq}^{(1)} \exp \left( \frac{i\omega}{c^{(1)}} (p_{pq} y_p + q_p z_p) \right), \\
u_{pzq}^{(1)} &= \sum_{q=1}^{6} U_{pzq}^{(1)} \exp \left( \frac{i\omega}{c^{(1)}} (p_{pq} y_p + q_p z_p) \right), \\
\sigma_{pxy}^{(1)} &= \sum_{q=1}^{6} \frac{i\omega}{c^{(1)}} U_{pxq}^{(1)} \exp \left( \frac{i\omega}{c^{(1)}} (p_{pq} y_p + q_p z_p) \right), \\
\sigma_{pyz}^{(1)} &= \sum_{q=1}^{6} \frac{i\omega}{c^{(1)}} U_{pyq}^{(1)} \exp \left( \frac{i\omega}{c^{(1)}} (p_{pq} y_p + q_p z_p) \right), \\
\sigma_{pzx}^{(1)} &= \sum_{q=1}^{6} \frac{i\omega}{c^{(1)}} U_{pzq}^{(1)} \exp \left( \frac{i\omega}{c^{(1)}} (p_{pq} y_p + q_p z_p) \right).
\end{align*}
\]

The state vectors at the left and right interfaces of each sub-cell for the nth unit cell can be written as
\[
\begin{align*}
v_{2n-1/2n+} &= T_{DL}(U_{px1}, U_{px2}, U_{px3}, U_{px4}, U_{px5}, U_{px6})^T \exp \left( \frac{i\omega}{c^{(1)}} (p_{pq} y_p + q_p z_p) \right), \\
v_{2n-1/2n+} &= T_{DL}(U_{px1}, U_{px2}, U_{px3}, U_{px4}, U_{px5}, U_{px6})^T \exp \left( \frac{i\omega}{c^{(1)}} (p_{pq} y_p + q_p z_p) \right),
\end{align*}
\]
where $v_{2n-1/2n+}$ are given in Appendix B.

According to the interfacial condition, we have the following relations:
\[
v_{2n-1} = T_{p} v_{2n-1}, \quad v_{2n+1} = T_{2} v_{2n+},
\]
where $T_{p}$ is the transfer matrix of each sub-cell with the following form:
\[
T_{p} = T_{pR} T_{pL}^{-1}.
\]
Based on the constitutive equation, the stiffness matrices can be expressed as\(^{42-43}\)

\[
\begin{pmatrix}
\sigma_{2n-1} \\
\sigma_{2n}
\end{pmatrix}
= K_1
\begin{pmatrix}
u_{2n-1} \\
u_{2n}
\end{pmatrix}, \quad (21a)
\]

\[
\begin{pmatrix}
\sigma_{2n} \\
\sigma_{2n+1}
\end{pmatrix}
= K_2
\begin{pmatrix}
u_{2n} \\
u_{2n+1}
\end{pmatrix}, \quad (21b)
\]

where \(K_p\) \((p = 1, 2)\) denotes the sub-cell stiffness matrix.

Then, the stiffness matrix can be derived from Eqs. (19)–(21) as

\[
K_p (6 \times 6) = \begin{pmatrix}
-T_{pb}^{-1}T_{pa} & T_{pb}^{-1} \\
T_{pb} - T_{pa}T_{pb}^{-1}T_{pa} & T_{pa}T_{pb}^{-1}
\end{pmatrix}, \quad (22)
\]

where \(T_{p\zeta}\) \((\zeta = a, b, d)\) is the \(3 \times 3\) sub-matrix of \(T_p\).

Eliminating the mechanical quantities at the \(2n\)th interface, the cell stiffness matrix can be derived as

\[
\begin{pmatrix}
\sigma_{2n-1} \\
\sigma_{2n+1}
\end{pmatrix}
= K
\begin{pmatrix}
u_{2n-1} \\
u_{2n+1}
\end{pmatrix}, \quad (23)
\]

where

\[
K(6 \times 6) = \begin{pmatrix}
K_{1a} + K_{1b}(K_{2a} - K_{1d})^{-1}K_{1c} & -K_{1b}(K_{2a} - K_{1d})^{-1}K_{2b} \\
K_{2c}(K_{2a} - K_{1d})^{-1}K_{1c} & K_{2d} - K_{2c}(K_{2a} - K_{1d})^{-1}K_{2b}
\end{pmatrix}, \quad (24)
\]

and \(K_{p\zeta}\) \((\zeta = a, b, c, d)\) is the \(3 \times 3\) sub-stiffness matrix of \(K_p\).

Then, the wave propagation in periodic structures satisfies the Bloch theorem as

\[
u_{2n+1} = e^{ikd}\nu_{2n-1}, \quad (25)
\]

where \(k\) refers to the wave number.

Based on Eqs. (23)–(25), the eigenvalue equation is expressed as

\[|T - e^{ikd}I| = 0, \quad (26)\]

where

\[
T(6 \times 6) = \begin{pmatrix}
-K_{c}^{-1}K_{a} & K_{b}^{-1} \\
K_{c} - K_{d}K_{b}^{-1}K_{a} & K_{d}K_{b}^{-1}
\end{pmatrix}, \quad (27)
\]

and \(K_{\zeta}\) \((\zeta = a, b, c, d)\) is the \(3 \times 3\) sub-stiffness matrix of \(K\).

As a result, the band gap of the fundamental wave can be obtained by Eq. (26). As a result, the transmission coefficients of the fundamental wave can be calculated to support the band gap property. For the incident qS\(_2\) wave, the displacements at the incident boundary consist of one incident wave and three reflected waves as

\[
u_{x(1)} = U_{I1}r_{11} + U_{R2}r_{12} + U_{R4}r_{14} + U_{R6}r_{16}, \quad (28a)
\]

\[
u_{y(1)} = U_{Ir_{11}} + U_{R2}r_{2} + U_{R4}r_{4} + U_{R6}r_{6}, \quad (28b)
\]

\[
u_{z(1)} = U_{I1}r_{11} + U_{R2}r_{2} + U_{R4}r_{4} + U_{R6}r_{6}, \quad (28c)
\]

where the subscript \((1)\) refers to the incident boundary, \(U_I\) is the amplitude of the incident
wave, \( U_{R2}, U_{R4}, \) and \( U_{R6} \) are the amplitudes of the reflected waves, and
\[
\begin{align*}
    r_{11} & = \exp \left( \frac{\omega}{c} (q_{1} x_{p} + \alpha_{1} y_{p} + q_{2} z_{p}) \right), \\
    r_{12} & = \exp \left( \frac{\omega}{c} (q_{1} x_{p} + \alpha_{1} y_{p} + q_{2} z_{p}) \right), \\
    r_{14} & = \exp \left( \frac{\omega}{c} (q_{1} x_{p} + \alpha_{1} y_{p} + q_{2} z_{p}) \right), \\
    r_{16} & = \exp \left( \frac{\omega}{c} (q_{1} x_{p} + \alpha_{1} y_{p} + q_{2} z_{p}) \right).
\end{align*}
\] (29a, b, c, d)

The corresponding stresses can be expressed as
\[
\begin{align*}
    \sigma_{12(1)} & = \frac{\omega}{c} (U_{T1} G^{(1)}_{p1} r_{11} + U_{R2} G^{(1)}_{p1} r_{12} + U_{R4} G^{(1)}_{p1} r_{14} + U_{R6} G^{(1)}_{p1} r_{16}), \\
    \sigma_{22(1)} & = \frac{\omega}{c} (U_{T1} G^{(1)}_{p2} r_{11} + U_{R2} G^{(1)}_{p2} r_{12} + U_{R4} G^{(1)}_{p2} r_{14} + U_{R6} G^{(1)}_{p2} r_{16}), \\
    \sigma_{23(1)} & = \frac{\omega}{c} (U_{T1} G^{(1)}_{p3} r_{11} + U_{R2} G^{(1)}_{p3} r_{12} + U_{R4} G^{(1)}_{p3} r_{14} + U_{R6} G^{(1)}_{p3} r_{16}).
\end{align*}
\] (30a, b, c)

At the transmitted boundary, the displacements are composed of three coupled waves along the forward direction as
\[
\begin{align*}
    u_{x(2)} & = U_{T1} r_{T1} + U_{T3} r_{T3} + U_{T5} r_{T5}, \\
    u_{y(2)} & = U_{T1} a_{p1} r_{T1} + U_{T3} a_{p3} r_{T3} + U_{T5} a_{p5} r_{T5}, \\
    u_{z(2)} & = U_{T1} a_{p3} r_{T1} + U_{T3} a_{p5} r_{T3} + U_{T5} a_{p5} r_{T5},
\end{align*}
\] (31a, b, c)

where the subscript \((2)\) denotes the transmitted boundary, \( U_{T1}, U_{T3}, \) and \( U_{T5} \) represent the amplitudes of transmitted waves, and
\[
\begin{align*}
    r_{T1} & = \exp \left( \frac{\omega}{c} (q_{1} x_{p} + \alpha_{2} y_{p} + q_{2} z_{p}) \right), \\
    r_{T3} & = \exp \left( \frac{\omega}{c} (q_{1} x_{p} + \alpha_{2} y_{p} + q_{2} z_{p}) \right), \\
    r_{T5} & = \exp \left( \frac{\omega}{c} (q_{1} x_{p} + \alpha_{2} y_{p} + q_{2} z_{p}) \right).
\end{align*}
\] (32a, b, c)

The stresses at the transmitted boundary can be given by
\[
\begin{align*}
    \sigma_{12(2)} & = \frac{\omega}{c} (U_{T1} G^{(1)}_{p1} r_{T1} + U_{T3} G^{(1)}_{p3} r_{T3} + U_{T5} G^{(1)}_{p5} r_{T5}), \\
    \sigma_{22(2)} & = \frac{\omega}{c} (U_{T1} G^{(1)}_{p2} r_{T1} + U_{T3} G^{(1)}_{p3} r_{T3} + U_{T5} G^{(1)}_{p5} r_{T5}), \\
    \sigma_{23(2)} & = \frac{\omega}{c} (U_{T1} G^{(1)}_{p3} r_{T1} + U_{T3} G^{(1)}_{p3} r_{T3} + U_{T5} G^{(1)}_{p5} r_{T5}).
\end{align*}
\] (33a, b, c)

The global stiffness matrix represents the relation of stress and displacement components at the incident and transmitted boundaries. The periodic structures contain \( m \) unit cells, and therefore we can derive the following relation\(^{[42–43]}\):
\[
\begin{align*}
    K^{m} (6 \times 6) & = \begin{pmatrix}
    K_{1}^{m-1} + K_{2}^{m-1}(K_{1}^{M} - K_{1}^{m-1})^{-1}K_{1}^{m-1} & -K_{2}^{m-1}(K_{1}^{M} - K_{1}^{m-1})^{-1}K_{2}^{M} \\
    K_{3}^{m-1}(K_{1}^{M} - K_{1}^{m-1})^{-1}K_{1}^{m-1} & K_{1}^{M} - K_{1}^{M}(K_{1}^{M} - K_{1}^{m-1})^{-1}K_{1}^{m-1}
    \end{pmatrix}, \quad (34)
\end{align*}
\]

where \( K^{m} \) is the global stiffness matrix with \( m \) unit cells, \( K_{1}^{m-1} \) denotes the sub-stiffness matrix with \( m - 1 \) unit cells, and \( K_{1}^{M} \) refers to the sub-stiffness matrix of the \( m \)th unit cell.
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Then, combination of Eqs. (28)–(34) yields
\[
\begin{pmatrix}
\sigma_{12(1)} \\
\sigma_{22(1)} \\
\sigma_{23(1)} \\
\sigma_{12(2)} \\
\sigma_{22(2)} \\
\sigma_{23(2)}
\end{pmatrix} = K^m
\begin{pmatrix}
u_{x(1)} \\
u_{y(1)} \\
u_{z(1)} \\
u_{x(2)} \\
u_{y(2)} \\
u_{z(2)}
\end{pmatrix},
\]
(35)

The reflection \( F_{RT2T2} = \frac{U_{T2}}{U_{T1}} \), \( F_{RT2T1} = \frac{U_{T1}}{U_{T1}} \), \( F_{RT2L} = \frac{U_{T2}}{U_{T1}} \) and transmission \( F_{DT2T2} = \frac{U_{T2}}{U_{T2}} \), \( F_{DT2T1} = \frac{U_{T1}}{U_{T2}} \), \( F_{DT2L} = \frac{U_{T2}}{U_{T2}} \) coefficients can be derived as
\[
\eta = (K^m M_2 - M_1)^{-1}(N_1 - K^m N_2),
\]
where \( \eta = (F_{RT2T2}, F_{RT2T1}, F_{RT2L}, F_{DT2T2}, F_{DT2T1}, F_{DT2L}) \), and the elements of matrices \( M_1, M_2, N_1, \) and \( N_2 \) are presented in Appendix C. The results of the second harmonic can also be obtained by the previous derivation. The transmission coefficients of the second harmonic are denoted as \( H_{DT2T2}, H_{DT2T1}, \) and \( H_{DT2L} \) for the incident \( qS_2 \) wave and \( H_{DT1T2}, H_{DT1T1}, \) and \( H_{DT1L} \) for the incident \( qS_1 \) wave.

4 Numerical simulation and discussion

In this section, the numerical results of band gaps and transmission coefficients with initial stresses are presented. In Fig. 1(a), each unit cell is composed of nonlinear materials A and B whose material parameters were presented in Refs. [31] and [44]. The phononic crystal in the nonlinear elastic wave metamaterial consists of two different linear materials C and D[45]. These materials as the monoclinic media have 13 independent elastic constants[35].

Figures 2 and 3 show band gaps and transmission coefficients of the fundamental wave and the second harmonic in the nonlinear phononic crystal. The polar and azimuthal angles \( \theta_1 = 26^\circ \) and \( \theta_2 = 25^\circ \), the thickness ratio \( d_1 : d_2 = 2 : 1 \), and the phase velocities \( c(1) = 1.510 \text{ m/s} \) are considered. Figures 2(a)–2(c) show the effects of the normal initial stresses \( \sigma_{11}, \sigma_{22}, \) and \( \sigma_{33} \) on the band gaps of the fundamental wave. We can see that the central frequency of band gaps increases with \( \sigma_{11} \) and \( \sigma_{22} \), but \( \sigma_{33} \) gives the opposite influence. The transmission coefficients with \( \sigma_{11}^0 = \sigma_{22}^0 = \sigma_{33}^0 = 1.5 \text{ GPa} \) are calculated to support the band gap property in Figs. 2(d)–2(f). It is clear that the frequency regions with zero transmission coefficients correspond to the band gaps.

Figures 3(a)–3(c) illustrate the effects of the normal initial stresses in the 3D space on the band gaps of the second harmonic. It can be seen that the band gaps have a similar change by the initial stresses to the case of the fundamental wave. We can find that the initial stresses make the band gaps change evidently. Thus, the initial stress can be used as a tunable way for the band gaps of both the fundamental wave and the second harmonic. In Figs. 3(d)–3(i), the transmission coefficients with \( \sigma_{11}^0 = \sigma_{22}^0 = \sigma_{33}^0 = 1.5 \text{ GPa} \) for the incident \( qS_1 \) and \( qS_2 \) waves are presented, which agree well with the band gaps.

Figures 4 and 5 show the effects of the azimuthal angle \( \theta_2 \) on the band gaps of the fundamental wave and the second harmonic for \( \sigma_{11}^0 = 1.5 \text{ GPa} \). In Fig. 4(a), the surface for the transmission coefficient \( F_{DT2T2} \) varying with the frequency and azimuthal angle is illustrated. As shown in Fig. 4(b), its contour clearly illustrates the relation of the azimuthal angle and the wave frequency. Figures 5(a) and 5(b) present the whole part and the contour of the transmission coefficient \( H_{DT2T1} \) for different azimuthal angles and wave frequencies. We can see that the locations of the band gaps shift towards the high frequency regions as the azimuthal angle increases for both the fundamental wave and the second harmonic. We can also see that the width of the second band gap changes slightly with the azimuthal angle. As a result, the central frequency of the band gap can be tuned by the initial stresses and the azimuthal angle.
Then, our attention is focused on achieving the 3D nonreciprocal transmission in the layered nonlinear elastic wave metamaterial. We consider the polar and azimuthal angles $\theta_1 = 26^\circ$ and $\theta_2 = 20^\circ$, the thickness ratio $d_1 : d_2 = 1 : 1$, and $c^{(1)} = c^{(2)} = 1350 \text{ m/s}$. Figures 6(a) and 6(b) present the effects of the initial stress $\sigma_{11}^0$ on the band gaps of the fundamental wave and the second harmonic in the nonlinear elastic wave metamaterial. It can be seen that the central frequencies of band gaps for both elastic waves decrease with $\sigma_{11}^0$. It can also be seen from Fig. 6(b) that the influence of $\sigma_{11}^0$ on the band gaps in high frequency regions is more obvious.

Figures 7(a) and 7(b) show the band gaps of the fundamental wave and the second harmonic with $\sigma_{11}^0 = 3 \text{ GPa}$. It can be seen that the frequency regions of band gaps are $0.38 \text{ MHz} - 0.58 \text{ MHz}$ for the fundamental wave, as well as $0.19 \text{ MHz} - 0.29 \text{ MHz}$ and $0.46 \text{ MHz} - 0.52 \text{ MHz}$ for the second harmonic. As a result, the frequency region of the nonreciprocal transmission is $0.38 \text{ MHz} - 0.46 \text{ MHz}$, i.e., the fundamental wave is located at the stop band, while the second harmonic falls in the pass band.

Figure 8 illustrates the nonreciprocal transmission of the 3D waves for the incident $qS_2$ wave in the positive direction. Figure 8(a) shows the transmission coefficients of the fundamental wave, while Figs. 8(b) and 8(c) illustrate the transmission coefficients of the second harmonic.
Fig. 3 Effects of the initial stresses on the band gaps of the second harmonic. (a) $\sigma_{11}^0 = 0$ GPa, 1.5 GPa, 3 GPa, (b) $\sigma_{22}^0 = 0$ GPa, 1.5 GPa, 3 GPa, and (c) $\sigma_{33}^0 = 0$ GPa, 1.5 GPa, 3 GPa, the transmission coefficients of the second harmonic for the incident $qS_2$ wave with (d) $\sigma_{11}^0 = 1.5$ GPa, (e) $\sigma_{22}^0 = 1.5$ GPa, and (f) $\sigma_{33}^0 = 1.5$ GPa, and the transmission coefficients of the second harmonic in the nonlinear phononic crystal for the incident $qS_1$ wave with (g) $\sigma_{11}^0 = 1.5$ GPa, (h) $\sigma_{22}^0 = 1.5$ GPa, and (i) $\sigma_{33}^0 = 1.5$ GPa (color online).
Fig. 4 Transmission coefficient of the fundamental wave $F_{DT2T2}$ in the nonlinear phononic crystal with $\sigma_{11}^0 = 1.5$ GPa for different azimuthal angles $\theta_2$: (a) the transmission coefficient $F_{DT2T2}$ varying with both the azimuthal angle and the frequency and (b) its contour in the azimuthal angle versus the frequency plane (color online).

Fig. 5 Transmission coefficient of the second harmonic $H_{DT2T1}$ in the nonlinear phononic crystal with $\sigma_{11}^0 = 1.5$ GPa for different azimuthal angles $\theta_2$: (a) the transmission coefficient $H_{DT2T1}$ varying with both the azimuthal angle and the frequency and (b) its contour in the azimuthal angle versus the frequency plane (color online).

Fig. 6 Transmission coefficients of (a) the fundamental wave ($F_{DT2T2}$) and (b) the second harmonic ($H_{DT2T2}$) changing with both $\sigma_{11}^0$ and the frequency in the nonlinear elastic wave metamaterial (color online).

We can see that the transmission coefficients of the fundamental wave remain zero but the results of the second harmonic are not zero in the region 0.38 MHz–0.46 MHz. The frequency region of the nonreciprocal transmission without initial stresses is 0.41 MHz–0.47 MHz. We find that the initial stress $\sigma_{11}^0$ makes the central frequency of the nonreciprocal transmission change and the gap width increase about 0.2 MHz.

For the incident $qS_2$ wave in the negative direction, the transmission coefficient of the second harmonic is zero in the nonreciprocal frequency region. It is mainly because the second harmonic cannot be generated in the linear materials and the fundamental wave is located at the stop band. Thus, the transmission coefficients with initial stresses for both the fundamental wave
Fig. 7 Band gaps of the fundamental wave and the second harmonic in the nonlinear elastic wave metamaterial with $\sigma_{11}^0 = 3$ GPa: (a) fundamental wave and (b) second harmonic (color online)

Fig. 8 Nonreciprocal transmission of the incident elastic wave along the positive direction with $\sigma_{11}^0 = 3$ GPa: (a) the fundamental wave and the transmission coefficients of the second harmonic for the incident (b) $q_S^2$ and (c) $q_S^1$ wave modes (color online)

and the second harmonic are zero in the frequency region 0.38 MHz–0.46 MHz. As a result, the nonreciprocal phenomenon in the 3D space can be achieved, which permits the wave propagation only along the positive direction. Then, we can tune the location and width of the frequency region for the 3D nonreciprocal transmission by initial stresses.

5 Conclusions

In this work, the effects of the initial stresses on the wave propagation of 3D waves in both nonlinear phononic crystal and elastic wave metamaterial are studied. The analytical solutions for the coupled $q_P$, $q_S^1$, and $q_S^2$ waves with the initial stresses are derived. The band gaps
and transmission coefficients of the fundamental wave and second harmonic are obtained by the transfer and stiffness matrices. The nonreciprocal transmission for an incident 3D wave can be achieved in the nonlinear elastic wave metamaterial with initial stresses. We can find that the initial stresses can tune the frequency regions of the band gap and nonreciprocal transmission in layered periodic structures. This work is expected to be helpful for designing devices of the nonreciprocal transmission with vector and tunable properties.

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\[ A_6 = C_{p22}C_{p44}S_{21} - C_{p22}C_{p66}S_{32} + C_{p22}C_{p44}C_{p66} - C_{p22}C_{p46} - C_{p22}S_{21}S_{32}, \]  
\[ A_4 = \rho_0 S_{21} S_{22} (c^{(1)})^2 - C_{p22} \rho_0 S_{21} (c^{(1)})^2 - S_{21} C_{p44} \rho_0 (c^{(1)})^2 + S_{21} C_{p22} C_{p33} q_{p32} - S_{21} S_{31} C_{p22} q_{p33}, \]  
\[ A_2 = S_{21} S_{31} (\rho_0 q_{p33} (c^{(1)})^2 - 2 C_{p23} - 2 C_{p44} q_{p31} q_{p32} - 2 C_{p55}), \]
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\[ A_0 = \sum \left( (c^{(1)})^2 \rho_1 q_2 (S_1 + C_{p33} + C_{p35}) + q_4 (C_{p11} - S_1 + C_{p55} + 2q_1 q_2 (C_{p15} + C_{p46}) + C_{p44} q_1 + C_{p55} q_1) \right) \]

\[ + \sum (c^{(1)})^2 \rho_2 q_2 (S_1 + C_{p33} + C_{p55} + q_2 (C_{p11} - S_1 + C_{p55}) + 2q_1 q_2 (C_{p15} + C_{p46}) + C_{p44} q_1 + C_{p55} q_1) \]

\[ - C_{p11} C_{p44} - 2C_{p15} C_{p46} q_1 + 2q_1 q_2 (C_{p13} C_{p46} - C_{p15} C_{p44} + 2C_{p46} C_{p55}) + C_{p33} C_{p44} q_1^2 + 2q_1 q_2 (C_{p33} C_{p46} + C_{p35} C_{p44}) + 6C_{p35} C_{p46} q_1 q_2 + S_{31} S_{12} (\rho_1 q_2 (C_{p15} + C_{p46}) - 2C_{p12} q_1 q_2 - 2C_{p13} C_{p46} q_1 q_2 - 4C_{p66} q_1 q_2) + S_{31} (c^{(1)})^2 (C_{p22} (q_1^2 - q_2^2) - C_{p44} q_1^2 q_2) + C_{p66} q_1 q_2 + 2q_1 q_2 (C_{p11} C_{p22} + C_{p12} C_{p66} + C_{p12} q_1^2 + 2q_1 q_2 (C_{p12} (C_{p25} + C_{p46}) + C_{p15} C_{p22}) + q_4^2 (C_{p22} C_{p33} - C_{p22} - 2C_{p23} C_{p44}) + 2q_1 q_2 (C_{p22} C_{p35} - C_{p23} (C_{p25} + C_{p46}) - C_{p25} C_{p44}) + 2C_{p25} C_{p66} q_1 q_2) + S_{32} (c^{(1)})^2 \rho_1 (C_{p11} + C_{p66}) q_1^2 + 2q_1 q_2 (C_{p15} + C_{p46}) + C_{p55} q_2.

(A3)
Appendix B

The elements of the coefficient matrices $T_{pl}$ and $T_{pr}$ in Eq. (18) are

$$T_{pl}(1, q) = 1, \quad T_{pl}(2, q) = a_{pqy}^{(1)}, \quad T_{pl}(3, q) = a_{pqy}^{(1)}, \quad T_{pl}(6, q) = \text{i}wG_{pqy}^{(1)} \frac{\omega}{c(1)},$$

$$T_{pr}(1, q) = 0, \quad T_{pr}(2, q) = \text{i}wG_{pqy}^{(1)} \frac{\omega}{c(1)} \exp \left( \frac{\text{i}w\alpha_{pqy}d_p}{c(1)} \right),$$

$$T_{pr}(3, q) = 0, \quad T_{pr}(4, q) = \text{i}wG_{pqy}^{(1)} \frac{\omega}{c(1)} \exp \left( \frac{\text{i}w\alpha_{pqy}d_p}{c(1)} \right),$$

$$T_{pr}(5, q) = \text{i}wG_{pqy}^{(1)} \frac{\omega}{c(1)} \exp \left( \frac{\text{i}w\alpha_{pqy}d_p}{c(1)} \right), \quad T_{pr}(6, q) = \text{i}wG_{pqy}^{(1)} \frac{\omega}{c(1)} \exp \left( \frac{\text{i}w\alpha_{pqy}d_p}{c(1)} \right).$$

Appendix C

The elements of the coefficient matrices $M_1$, $M_2$, $N_1$, and $N_2$ in Eq. (36) are

$$M_1 = \begin{pmatrix}
G_{p12}^{(1)} & G_{p14}^{(1)} & G_{p16}^{(1)} & 0 & 0 & 0 \\
G_{p22}^{(1)} & G_{p24}^{(1)} & G_{p26}^{(1)} & 0 & 0 & 0 \\
G_{p32}^{(1)} & G_{p34}^{(1)} & G_{p36}^{(1)} & 0 & 0 & 0 \\
0 & 0 & 0 & G_{p11}^{(1)} & G_{p13}^{(1)} & G_{p15}^{(1)} \\
0 & 0 & 0 & G_{p21}^{(1)} & G_{p23}^{(1)} & G_{p25}^{(1)} \\
0 & 0 & 0 & G_{p31}^{(1)} & G_{p33}^{(1)} & G_{p35}^{(1)}
\end{pmatrix},$$

$$M_2 = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}^T,$$

$$N_1 = \begin{pmatrix}
\text{i}wG_{p11}^{(1)} & \text{i}wG_{p21}^{(1)} & \text{i}wG_{p31}^{(1)} & 0 & 0 & 0
\end{pmatrix}^T,$$

$$N_2 = \begin{pmatrix}
a_{p21}^{(1)} & a_{p31}^{(1)} & 0 & 0 & 0 & 0
\end{pmatrix}^T.$$