MAXIMAL AMENABLE MASAS OF THE FREE GROUP FACTOR OF TWO GENERATORS ARISING FROM THE FREE PRODUCTS OF HYPERFINITE FACTORS

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Abstract. In this paper, we give examples of maximal amenable subalgebras of the free group factor of two generators. More precisely, we consider two copies of the hyperfinite factor $R_i$ of type II$_1$. From each $R_i$, we take a Haar unitary $u_i$ which generates a Cartan subalgebra of it. We show that the von Neumann subalgebra generated by the self-adjoint operator $u_1 + u_1^{-1} + u_2 + u_2^{-1}$ is maximal amenable in the free product. This provides infinitely many non-unitary conjugate maximal amenable MASAs.

1. Introduction

In operator algebra theory, maximal amenable subalgebras have fascinated many researchers because they give a good insight into ambient (non-amenable) factors. The history of the research dates back to the middle of 1960’s. In 1967, Kadison conjectured that any maximal amenable von Neumann subalgebra of any factor of type II$_1$ was isomorphic to the hyperfinite factor of type II$_1$. In the early 1980’s, Popa [12] solved this conjecture negatively. He showed that the generator subalgebra of any free group factor is maximal amenable (Even such a simple example had not been shown to be maximal amenable until his work, which tells us the difficulty of the problem). In order to show that the subalgebra is maximal amenable, he used the ultraproduct technique and looked at a property, which is called the asymptotic orthogonality property. Even now, his method has a strong influence upon the research on maximal amenability.

After Popa’s work, his result has been generalized by many researchers. Popa’s example can be seen as an amenable free component of a free product. In this direction, Boutonnet–Houdayer [2] reached a general structural theorem of maximal amenable subalgebras of (amalgamated) free products which may be of type III (See also Houdayer [8], Houdayer–Ueda [9] and Ozawa [11]). Popa’s result can also be seen as a subalgebra arising from a subgroup of a group factor. In this direction, Boutonnet–Carderi [1] gave a sufficient condition for a subalgebra of any group factor coming from a subgroup to be maximal amenable. It is also remarkable that their proof is quite concise, which relies on the study of non-normal states.

However, in order to understand the inner structure of the free group factors, it is also important to investigate subalgebras which come neither from free components nor subgroups. The reason is that there are some non-trivial presentations of the free group factors (See Dykema [3], Guionnet–Shlyakhtenko [7] for example). This means that the free group factors have certain flexibility, which makes them interesting. Hence it is important to investigate what kind of non-trivial automorphisms the free group factors admit; whether a given MASA is conjugate to the generator
subalgebras by automorphisms or not. For this purpose, it is helpful to consider whether a given MASA has similar properties to those of the generator subalgebras. In that sense, the radial MASA of the free group factors is an interesting example (For the definition, see Cameron–Fang–Ravichandran–White [4] for example). It is not known whether the radial MASA is conjugate to one of the generator MASAs by automorphisms or not; although it is not contained in any free component or subgroup in an obvious way, there is no known property which distinguishes the radial MASA from the generator MASAs. In particular, it was shown to be maximal amenable by [4] (See also Wen [10] for a simplified proof). Their strategy for proving the maximal amenability was to determine the form of a sequence which asymptotically commutes with the radial MASA by using a basis constructed by Radulescu [14]. Although their strategy itself may be simple, it could not have been carried out without their tough computing power.

Motivated by this example, we present new examples of maximal amenable von Neumann subalgebras of a free group factor. More precisely, we consider two copies of the hyperfinite factor $R_i$ ($i = 1, 2$) of type $II_1$. By Dykema [5], the free product $R_1 * R_2$ is isomorphic to the free group factor of two generators. From each $R_i$, we take a Haar unitary $u_i$ which generates a Cartan subalgebra of it. We show that the von Neumann subalgebra generated by the self-adjoint operator $u_1 + u_1^{-1} + u_2 + u_2^{-1}$ is maximal amenable in the free product.

Although our construction is similar to that of the radial MASA, it has a different aspect. Unlike the radial MASA, our construction provides “many” examples. Although we do not know whether they are really mutually non-conjugate by automorphisms or not, they are neither mutually unitary conjugate nor conjugate by automorphisms arising from the free components.

In order to show that our subalgebras are maximal amenable, we would like to develop a similar strategy to that of Cameron–Fang–Ravichandran–White [4], in which they show Popa’s asymptotic orthogonality property for the radial MASA. However, there are two problems. First, although their proof depends on combinatorics of the free groups, we cannot expect to find out such a good group-like structure in our setting. Therefore, we first consider the special case, namely, the case when the Haar unitaries come from generators of the irrational rotation $C^*$-algebras. After that, we reduce the general case to the special case. The idea of reducing the problem to that in a case when the factor comes from the irrational rotation $C^*$-algebras is conceived by Ge [6]. He embedded a system of the general case into that of the special case. However, just an imitation of Ge [6] does not work well in our setting. Our key idea for overcoming this difficulty is to embed “asymptotically” a subalgebra into another one.

Even after the problem is reduced to the special case, there arises another problem; combinatorics of the irrational rotation $C^*$-algebras is complicated. The irrational rotation $C^*$-algebras can be seen as deformations of $Z^2$. Hence it is possible to use their algebraic structures. However, since $Z^2 * Z^2$ is “less free” than $F_2$ is, its combinatorics is more complicated, which requires further computations.

This paper is organized as follows. In Section 2, we explain the main theorem of this paper. Sections 3 to 7 are devoted to showing the main theorem. In Section 3, we investigate the combinatorics for the special case and construct a good basis of $L^2$-space. In Sections 4 and 5, we do some analytical computations necessary to show the asymptotic orthogonality property. In Section 6, we reduce the problem to
the special case and show that the subalgebras have the asymptotic orthogonality property. In order to show the maximal amenability of subalgebras, besides the asymptotic orthogonality property, we need to show the singularity of them. In Section 7, we show the singularity of the subalgebras and conclude that they are maximal amenable.

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2. Main theorem

The main theorem of this paper is the following.

Theorem 1. For each \( i = 1, 2 \), let \( R_i \) be the hyperfinite factor of type \( \Pi_1 \) and \( w_i \in R_i \) be a Haar unitary of \( R_i \) which generates a Cartan subalgebra of \( R_i \). Then the von Neumann subalgebra \( B \) generated by the self-adjoint operator \( w_1 + w_1^{-1} + w_2 + w_2^{-1} \) is maximal amenable in the free product \( R_1 \ast R_2 \) with respect to the traces.

The following is an example of unitaries which satisfy the assumption of Theorem 1.

Example 2. Let \( \theta \) be an irrational number and \( A_\theta \) be the universal \( C^* \)-algebra generated by two unitaries \( u, v \) with \( \omega = e^{2\pi i \theta} vu \) (an irrational rotation \( C^* \)-algebra). Then the \( C^* \)-algebra \( A_\theta \) has a tracial state \( \tau \) defined by \( \tau(u^k v^l) = \delta_{k,0} \delta_{l,0} \) for \( k, l \in \mathbb{Z} \). Take the GNS representation of \( A_\theta \) with respect to the trace \( \tau \). Then the weak closure \( R \) of \( A_\theta \) in the GNS representation is isomorphic to the hyperfinite factor of type \( \Pi_1 \). For any \( k, l \in \mathbb{Z} \setminus \{0\} \) without any non-trivial common divisor, the unitary \( u^k v^l \) is Haar and generates a Cartan subalgebra of \( R \).

Proof. It is possible to choose \( k', l' \in \mathbb{Z} \setminus \{0\} \) with \( kl' - k'l = 1 \) because \( k \) and \( l \) do not have any non-trivial common divisor. Then the map \( u \mapsto u^k v^l \), \( v \mapsto u^{k'} v^{l'} \) extends to an automorphism of \( R \). \( \blacksquare \)

We will see that Theorem 1 has a possibility of producing many examples of maximal amenable MASAs. We see that the theorem provides many subalgebras which are mutually non-unitary conjugate. Let \( u_i, v_i \) be generators of \( R_i \) explained in the above example. Let \( w_i^{k,l} \) be a Haar unitary generating the von Neumann subalgebra \( \{u_i^{k,l} v_i^{k,l}\}'' \). Set \( B_{k,l} := \{u_i^{k,l} + (u_i^{k,l})^{-1} + w_i^{k,l} + (w_i^{k,l})^{-1}\}'' \subset R_1 \ast R_2 \). We show that \( B_{k,l} \nsubseteq B_{k',l'} \) for any \( (k, l) \neq (k', l') \). Let \( a_n \) be unitaries of \( B_{k,l} \) which converges weakly to 0. Then for any large \( M > 0 \), any small \( \epsilon > 0 \), there exists \( N > 0 \) such that for any \( n \geq N \), there exists a linear combination \( b_n \) of the words \( w \) with the following conditions.

1. We have \( \|a_n - b_n\| < \epsilon \).
2. For any linear component \( w \) of \( b_n \), the length of \( w \) is not smaller than \( M \).
3. For any linear component \( w \) of \( b_n \), the ratio of the number of \( u_i \)'s \( (i = 1, 2) \) in \( w \) and the number of \( v_i \)'s in \( w \) is \( k/l \).

Take two words \( x, y \) of \( R_1 \ast R_2 \). Then if we take \( M > 0 \) large enough compared to the lengths of \( x \) and \( y \), for any linear component \( w' \) of \( x b_n y \), the ratio of the
number of \( u_i \)'s in \( w' \) and \( v_i \)'s in \( w' \) is almost \( k/l \). Thus we have \( E_{B_{k,l}}(xb_ny) = 0 \), which implies that \( B_{k,l} \not\prec B_{k,l}' \).

In this way, it is possible to construct many non-unitary conjugate subalgebras. However, our central interest is whether they are conjugate by automorphisms or not. Thus we close this section with the following problem.

**Problem 3.** Are the maximal amenable subalgebras constructed in Theorem 4 mutually conjugate by automorphisms? Are they conjugate to the generator \( MASA \)?

Although any two Cartan subalgebras of the hyperfinite factor of type II\( _1 \) are mutually conjugate by an automorphism, it is impossible to control the position of Haar unitaries by the automorphism. Hence it is not clear whether it is possible to conjugate two of them by automorphisms arising from free components.

### 3. Radulescu type basis

In order to show the main theorem, we first consider the case when the Haar unitaries come from the irrational rotation \( C^* \)-algebras and later we reduce the general case to the special case. Hence until the end of Section 4, we always assume the following condition.

**Condition.** For each \( i = 1, 2 \), there exist Haar unitaries \( u_i, v_i \) and a complex number \( d \) of absolute value 1 satisfying the following conditions.

1. (0) For any \( n \neq 0 \), we have \( d^n \neq 1 \).
2. (1) The factor \( R_i \) is generated by \( u_i \) and \( v_i \) for each \( i \).
3. (2) We have \( u_i v_i = d v_i u_i \) for each \( i \).

Set \( M := R_1 * R_2, A := u_1 + u_1^{-1} + u_2 + u_2^{-1} \).

The purpose of this section is to construct a good basis of \( L^2 \)-space under this condition (Corollary 18), which is motivated by Radulescu [14].

#### 3.1. Decomposing \( L^2 M \) into three pieces

Let \( w = w_1 \cdots w_n \) be a word consists of the letters \( \{ u_1^\pm 1, u_2^\pm 1, v_1^\pm 1, v_2^\pm 1 \} \). The number \( n \) of letters contained in the word \( w \) is said to be the word length of \( w \) and denoted by \( |w| \). We say that the word \( w \) is reduced if the word length does not decrease by finitely many times of the transformations \( u_1^{\pm 1} u_1^{\mp 1} \leftrightarrow 1, v_1^{\pm 1} v_1^{\mp 1} \leftrightarrow 1 \) and \( u_i^t v_i^l \leftrightarrow v_i^l u_i^t \) \( (i = 1, 2, s, t = \pm 1) \). On the set of reduced words, we introduce an equivalence relation defined by \( u_i^s v_i^l \leftrightarrow v_i^l u_i^s \). Then any non-trivial word \( w \) is equivalent to a word of the following form.

\[
u_{i_1}^{k_1} v_{i_1}^{l_1} u_{i_2}^{k_2} v_{i_2}^{l_2} \cdots u_{i_n}^{k_n} v_{i_n}^{l_n},
\]

where \( n \in \mathbb{Z}_{>1}, i_1, \ldots, i_n \in \{1, 2\} \) with \( i_1 \neq i_2 \neq \cdots \neq i_n \) (This notation means that \( i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n \)), and \( |k_t| + |l_t| \geq 1 \) for \( t = 1, \ldots, n \). A word of this form is said to be a completely reduced word. For each \( l \geq 0 \), let \( \hat{W}_l \) be the set of all completely reduced words with length \( l \) consisting of letters \( u_i^{\pm 1} \) \( (i = 1, 2) \). Let \( \hat{W}_l^0 \) be the set of all completely reduced words with length \( l \) such that if we write them as \( u_{i_1}^{k_1} v_{i_1}^{l_1} u_{i_2}^{k_2} v_{i_2}^{l_2} \cdots u_{i_n}^{k_n} v_{i_n}^{l_n} \), exactly one of \( l_i \)'s is non-zero. Let \( \hat{W}_l^1 \) be the set of all completely reduced words with length \( l \) such that if we write them as \( u_{i_1}^{k_1} v_{i_1}^{l_1} u_{i_2}^{k_2} v_{i_2}^{l_2} \cdots u_{i_n}^{k_n} v_{i_n}^{l_n} \), at least two of \( l_i \)'s are non-zero. We often regard
a reduced word as a unitary element of $M$. In the following, we identify the set $\bigcup_{i \geq 0, i=0,1,2} W_i^2$ as a subset of $L^2 M$. Then it is an orthonormal basis of $L^2 M$. Set
$$W_i^l := \text{span} W_i^l$$
for each $i = 0, 1, 2$ and $l \geq 0$. Set
$$W_l := W_l^0 \oplus W_l^1 \oplus W_l^2$$
for each $l \geq 0$. Let $q_l$ be the orthogonal projection onto $W_l$.

For each $l \geq 0$, set
$$\chi_l := \sum_{w \in W_l^0} w,$$
which is a self-adjoint operator of $M$. As mentioned in p.299 of Radulescu [14], the self-adjoint operators $\{\chi_l\}_{l \geq 1}$ satisfy
$$\chi_l \chi_1 = \chi_1 \chi_l = \chi_{l+1} + 3 \chi_{l-1}$$
for $l \geq 2$ and
$$\chi_1 \chi_1 = \chi_2 + 4.$$
Notice that $A = \{\chi_1\}'$. Set
$$S_i^l := \{q_l(\chi_1 w), q_l(w \chi_1) \mid w \in W_i^k, k \leq l-1\}$$
for $i = 0, 1, 2$,
$$S_l := S_l^0 \oplus S_l^1 \oplus S_l^2.$$  
For $\gamma \in W_l^1$ and $w \in W_l^2$ with $k \leq l-1$, we have
$$\langle \gamma, q_l(\chi_1 w) \rangle = \langle \gamma, \chi_1 w \rangle = \langle \chi_1 \gamma, w \rangle.$$  
Similarly, we have $\langle \gamma, q_l(w \chi_1) \rangle = \langle \gamma \chi_1, w \rangle$. Hence $\gamma \in W_l^1 \oplus S_l^1$ means that when we expand $\chi_1 \gamma$ and $\gamma \chi_1$ by the orthonormal basis consisting of completely reduced words, no term with length less than $l$ appears. Hence we have the following decomposition.

**Lemma 4.** We have
$$W_l \oplus S_l = (W_l^0 \oplus S_l^0) \oplus (W_l^1 \oplus S_l^1) \oplus (W_l^2 \oplus S_l^2).$$

**Proof.** For $\xi \in W_l \oplus S_l$, let $\xi = \xi_0 + \xi_1 + \xi_2 \in W_l^0 \oplus W_l^1 \oplus W_l^2$ be the orthogonal decomposition. Since any term of $\chi_1 \xi_0$ is not canceled by any term of $\chi_1 (\xi_1 + \xi_2)$ and since $\xi$ is orthogonal to $S_l$, $\chi_1 \xi_0$ has no term with length less than $l$. Similarly, the vector $\xi_0 \chi_1$ has no term with length less than $l$. Hence the vector $\xi_0$ is orthogonal to $S_l^0$. Similarly, we have $\xi_1 \in (S_l^1)^\perp$ and $\xi_2 \in (S_l^2)^\perp$. \hfill $\square$

For $r, s \in \mathbb{Z}_{\geq 0}$ and $\xi \in W_l$, set
$$\xi_{r,s} := q_{l+r+s}(\chi_1 \xi \chi_s).$$
When $r < 0$ or $s < 0$, we define $\xi_{r,s}$ as $0$. We would like to see what relations there are among these vectors. The following lemma is one of them.

**Lemma 5.** For $\xi \in W_l$ with $l \geq 1$, we have
$$\chi_1 \xi_{r,s} = \xi_{r+1,s} + 3 \xi_{r-1,s}$$
for $r \geq 1$, $s \geq 0$,
$$\xi_{r,s} \chi_1 = \xi_{r,s+1} + 3 \xi_{r,s-1}$$
for $r \geq 0$, $s \geq 1$. 


Proof. This is shown by the same argument as that of the proof of Lemma 1 (a) of Radulescu [14].

3.2. Structures of the space $\left( \bigoplus_I W_I^2 \right) \oplus \left( \bigoplus_I W_I^0 \right)$. In this subsection, we present some formulas on vectors of the space $\left( \bigoplus_I W_I^2 \right) \oplus \left( \bigoplus_I W_I^0 \right)$.

**Lemma 6.** For $\xi \in (W_I^1 \oplus W_I^2) \oplus S_l$ which is orthogonal to the set $\{ u_i^{\pm 1} v_i^{\pm(l-1)} \mid i = 1, 2 \} \cup \{ v_i^{\pm 1} \}$, we have

$$\chi_1 \xi_{0,s} = \xi_{1,s}$$

for $s \geq 0$,

$$\xi_{r,0} \chi_1 = \xi_{r,1}$$

for $r \geq 0$.

**Proof.** Let

$$\xi = \sum_{w \in \tilde{W}_l} \lambda_w w$$

be the decomposition along the orthonormal basis $\tilde{W}_l$. We would like to show that $\langle \chi_1 \xi_{0,s}, w'' \rangle = 0$ for any $w'' \in \tilde{W}_k$, with $k' \leq l + s - 1$. In order to achieve this, we decompose $\xi$ into some pieces and compute the inner product of each component. The word $w \in \tilde{W}_l$ may begin with $u_i^{\pm 1}$ or $v_i^{\pm 1}$. We may consider these two kinds of terms separately. In the following, we consider the case when $\lambda_w = 0$ if $w$ begins with $u_j^{\pm 1}$ (the other case is obvious; $w \neq v_j^{\pm 1}$ implies that $q_l(\chi_1 w_{0,s}) = \chi_1 w_{0,s}$).

**Claim 1.** For any word $\tilde{w} \in \tilde{W}_{l-1}$ and $w'' \in W_{k'}$ with $k' \leq l - 1$, we have

$$\left( \sum_{x \in \tilde{W}_l^0 \text{ with } x\tilde{w} \in \tilde{W}_l} \lambda_{x\tilde{w}} q_l(\chi_1 w'') \right) = 0.$$

**Proof of Claim 1.** Notice that for different $\tilde{w}$'s, the vectors $\chi_1 x\tilde{w}$'s are mutually orthogonal. Thus no cancellation occurs among terms of $\chi_1 x\tilde{w}$ coming from different $\tilde{w}$'s. On the other hand, by assumption, we have $\langle \xi, q_l(\chi_1 w'') \rangle = 0$. Thus we get the conclusion of Claim 1. \qed

Next, we show another necessary claim.

**Claim 2.** Let $\tilde{w} \in \tilde{W}_{l-1}$ and $x \in \tilde{W}_1^0$ with $x\tilde{w} \in W_l$. Let $w'$ be a word of $W_a^0$ such that $x\tilde{w}$ does not cancel with $u'$, that is, $|xuw'| = |x| + |w| + |w'|$. Then for any $j = 1, 2$, $p = \pm 1$, if we multiply $u_j^p$ from the left, it cancel with $x\tilde{w}w'$ if and only if it cancel with $x\tilde{w}$.

**Proof of Claim 2.** Case 1. When $x\tilde{w} \neq u_j^p v_i^{\pm(l-|i|)}$, then the word $u_j^p$ cannot commute with $x\tilde{w}$. Hence for the word $u_j^p$, in order to cancel with $x\tilde{w}w'$, it should cancel with $x\tilde{w}$. Conversely, if the word $u_j^p$ cancel with $x\tilde{w}$, the length of $u_j^p x\tilde{w}w'$ should be strictly less than $1 + l + s$. Thus the word $u_j^p$ cancels with $x\tilde{w}w'$.

Case 2. When $x\tilde{w} = u_j^p v_i^{\pm(l-|i|)}$ ($s \neq 0$), then this is directly checked by using $s \neq 0$. \qed

We also need to show the following two claims.
Claim 3. Let \( w \in \tilde{W}_{l-1} \) be a word orthogonal to \( v_j^{\pm(l-1)} \) for any \( j \in \{1, 2\} \) and \( x \in \tilde{W}_1 \) with \( xw \in W_1 \). Then for any reduced word \( w' \), we have \(|xw| = |x| + |w| = |w'| \) if and only if \(|xw| = |w| + |w'| \). (In this claim, the assumption that the vector \( \xi \) is orthogonal to \( u \pm v \pm (l-1) \) is used).

Proof of Claim 3. This is shown by the same way as that of the proof of Claim 2.

Claim 3'. Let \( x \) be \( u \pm_{l-j} \) and \( \tilde{w} \) be \( v \pm l \). Then the same conclusion as that of Claim 3 holds.

Proof of Claim 3'. This is trivial.

By Claims 1, 2, 3 and 3', for any \( w' \in \tilde{W}_0 \) with \(|\tilde{w}w'| = l + s - 1\), we have

\[
\langle \sum_{x \in \tilde{W}_1 \text{ with } x\tilde{w}w' \in \tilde{W}_{l+s}} \lambda_x x\tilde{w}w', q_{l+s}(\chi_1 w'') \rangle = 0
\]

for any \( w'' \in \tilde{W}_l \) with \( k' \leq l + s - 1 \). When \( \tilde{w} \) and \( w' \) run over all possible values, the sum of the above equality is

\[
0 = \langle \sum_{w \in \tilde{W}_l} \left( \sum_{w' \in \tilde{W}_0 \text{ with } |ww'| = l + s} \lambda_w w w' \right), q_{l+s}(\chi_1 w'') \rangle
= \langle \xi_0, s, q_{l+s}(\chi_1 w'') \rangle
= \langle \chi_1 \xi_0, s, w'' \rangle
\]

for all \( w'' \in \tilde{W}_k \) with \( k' \leq l + s - 1 \). Hence \( \chi_1 \xi_0, s \) has no term with length less than \( l + s - 1 \). Thus we have \( \chi_1 \xi_0, s = q_{l+s+1}(\chi_1 \xi_0, s) = \chi_1 \xi_0, s \). The other equality is shown in the same way.

Lemma 7. For \( \xi \in \tilde{W}_l \oplus S_0 \), we have the following statements.

1. When \( l \geq 2 \), we have

\[
\chi_1 \xi_0, s = \xi_1, s
\]

for \( s \geq 0 \),

\[
\xi_{r,0} \chi_1 = \xi_{r,1}
\]

for \( r \geq 0 \).

2. When \( l = 1 \) and \( \xi \) is \( c_1(u_1 + e u_1^{-1}) + c_2(u_2 + e u_2^{-1}) \) for some \( e \in \{\pm 1\} \) and \( c_1, c_2 \in \mathbb{C} \), we have

\[
\chi_1 \xi_0, s = \xi_1, s - e \xi_0, s - 1
\]

for \( s \geq 0 \),

\[
\xi_{r,0} \chi_1 = \xi_{r,1} - e \xi_{r-1,0}
\]

for \( r \geq 0 \).

Proof. This is shown by the same argument as that of the proof of Lemma 1 (b)(c) of Radulescu [13].
3.3. Structures of the space \((\bigoplus W^3_l)\). In this subsection, we investigate the structure of the space \(\bigoplus W^3_l\). Fix \(l \in \mathbb{Z} \setminus \{0\}\). When \(l \neq \pm 1\), set
\[
\gamma_{1,l} := v''_1^{(l - \text{sgn}(l))} u''_1^l,
\]
\[
\gamma_{2,l} := v''_1^{(l - \text{sgn}(l))} (u_{3-l} + u_{3-l}^{-1}),
\]
\[
\gamma_{3,l} := (u_{3-l} + u_{3-l}^{-1}) v''_1^{(l - \text{sgn}(l))},
\]
\[
\gamma_i := \frac{2}{d^{(l - \text{sgn}(l)) - d^{-(l - \text{sgn}(l))}}} (\gamma_{1,l}^{(l,1)} - \gamma_{1,l}^{(l,-)}) - \gamma_{2,l}^{(l,1)}.
\]
and
\[
\gamma_{i,l} := d^{(l - \text{sgn}(l)) - d^{-(l - \text{sgn}(l))}} (d^{(l - \text{sgn}(l))} \gamma_{1,l}^{i} - d^{-(l - \text{sgn}(l))} \gamma_{1,l}^{i}) - \gamma_{2,l}^{(l,1)}.
\]
For \(l > 0\), we also set
\[
W^{1,1,l} := \text{span}\{u''_1^{(l - \text{sgn}(l))}, v''_1^{(l - \text{sgn}(l))} u''_1^l \mid i = 1, 2, j = 1, 2\},
\]
\[
W^{1,2,l} := W^3_l \oplus (W^{1,1,l} \oplus C derived from \(v''_1\)) + C derived from \(v''_1\)),
\]
\[
W^{1,1,1} := \text{span}\{\gamma_{1,l}^{(l,1)}, \gamma_{1,l}^{(l,1)} \mid i = 1, 2\}
\]
when \(l \neq 1\) and
\[
W^{1,1,2} := \text{span}\{(u_{3-l} - u_{3-l}^{-1}) v''_1^{(l - \text{sgn}(l))}, v''_1^{(l - \text{sgn}(l))} (u_{3-l} - u_{3-l}^{-1}) \mid i = 1, 2\}
\]
when \(l \neq 1\). When \(l = 1\), we set \(W^{1,1,2} = 0\). Obviously, the space \(W^{1,1,l}\) is generated by the completely reduced words of the form \(xv''_1 x' \in \mathcal{W}_l\) with \(x, x' \in \bigcup_{k \geq 0} W^0_k, |x| + |x'| \geq 2, k' \neq 0, i = 1, 2\).

**Lemma 8.** For any \(l \in \mathbb{Z}_{>0}\), we have
\[
W^3_l \otimes S^1_l = (W^{1,1,l} \otimes S^1_l) \oplus W^{1,1,1} \otimes W^{1,1,2} \oplus C derived from \(v''_1\) + C derived from \(v''_1\).
\]

**Proof.** First, note that we have \(W^3_l \otimes S^1_l = (W^{1,1,l} \otimes S^1_l) \oplus (W^{1,1,l} \otimes S^1_l)\). Thus in order to get the conclusion, it is enough to determine the structure of the space \(W^{1,1,l} \otimes S^1\).

**Claim.** For \(l \in \mathbb{Z} \setminus \{0, \pm 1\}\), let \(\xi \in W^3_l \otimes S^1_l\) be a linear combination of \(v''_1^{(l - \text{sgn}(l))}, v''_1^{(l - \text{sgn}(l))} \mid i, i' = 1, 2, t, t' = \pm 1\) Then \(\xi\) is a linear combination of \(\gamma_{1,l}^{(l,1)}, (u_{2} - u_{2}^{-1}) v''_1^{(l - \text{sgn}(l))}\) and \(v''_1^{(l - \text{sgn}(l))} (u_{2} - u_{2}^{-1})\).

**Proof of Claim.** For simplicity, we assume that \(l > 0\) (When \(l < 0\), Claim is shown in the same way). It is possible to write \(\xi\) as
\[
c^1_1 v''_1^{(l,1)} u_1 + c_{1,-} v''_1^{(l,1)} u^{-1}_1 + c_{2,+} v''_1^{(l,1)} u_2 + c_{2,-} v''_1^{(l,1)} u^{-1}_2 + c_{3,+} u_2 v''_1^{(l,1)} + c_{3,-} u^{-1}_2 v''_1^{(l,1)}.
\]
Then since \(\xi\) is orthogonal to \(\chi_{1,w} \) with \(|w| \leq l - 1\), we have
\[
d^{-(l-1)} c^1_{1,+} + d^{-(l-1)} c_{1,-} + c_{3,+} + c_{3,-} = 0.
\]
Since \(\xi\) is orthogonal to \(w \chi_{1,1}\) with \(|w| \leq l - 1\), we have
\[
c^1_{1,+} + c_{1,-} + c_{2,+} + c_{2,-} = 0.
\]
Obviously, the vectors \(\gamma_{1,l}^{(l,1)}, (u_{2} - u_{2}^{-1}) v''_1^{(l,1)}\) and \(v''_1^{(l,1)} (u_{2} - u_{2}^{-1})\) satisfy the above two equalities. On the other hand, the vector space defined by these two equalities is at most four dimensional. Thus \(\xi\) is a linear combination of \(\gamma_{1,l}^{(l,1)}, (u_{2} - u_{2}^{-1}) v''_1^{(l,1)}\) and \(v''_1^{(l,1)} (u_{2} - u_{2}^{-1})\). \(\square\)
Thus \( \xi \in W_l^1 \otimes S_l^1 \) which is a linear combination of \( v_j^{\pm l}, v_j^{\pm(l-1)} u_{j'}^{t'} v_j^{\pm t'} \), \( j, j', t', t'' = 1, 2, t', t'' = \pm 1 \) is in fact contained in \( W_l^{1,\alpha,1} \oplus W_l^{1,\alpha,2} \oplus C_{v_l^1} \oplus C_{v_l^2} \) . Hence we have
\[
W_l^{1,\alpha} \otimes S_l^1 = W_l^{1,\alpha,1} \oplus W_l^{1,\alpha,2} \oplus C_{v_l^1} \oplus C_{v_l^2},
\]
which implies the conclusion of the lemma. \( \square \)

**Lemma 9.** For any \( n, m \geq 0 \), \( i = 1, 2 \), we have
\[
\chi_n \gamma_{j,l} \chi_m \in \text{span}\{ (\gamma_{j,l}^r)_r,s \mid j = 1, \pm, 2, 3, r, s \geq 0 \} \cup A v_l^{-1}.
\]

**Proof.** Set
\[
V_i := \text{span}\{ (\gamma_{j,l}^r)_r,s \mid j = 1, \pm, 2, 3, r, s \geq 0 \}.
\]
Notice that we have \((v_i^{l-\text{sgn}(l)})_{0,s}, (v_i^{l-\text{sgn}(l)})_{r,0} \in V_i \cup A v_l^{-1} \). For \( s \geq 1 \), we have
\[
\chi_1 (u_i v_i^{l-\text{sgn}(l)})_{0,s} = \chi_1 (u_i v_i^{l-\text{sgn}(l)}) \left( \sum_{w \not= u_i^{-1}} w \right) u_i v_i^{l-\text{sgn}(l)} v_i = (u_i v_i^{l-\text{sgn}(l)}) \chi_s - (v_i^{l-\text{sgn}(l)} u_i^{-1})_{0,s-1},
\]
\[
\chi_1 ((u_{3-i} + u_{3-i}^{-1}) v_i^{l-\text{sgn}(l)})_{0,s} = \chi_1 ((u_{3-i} + u_{3-i}^{-1}) v_i^{l-\text{sgn}(l)}) \chi_s = (u_{3-i} + u_{3-i}^{-1}) v_i^{l-\text{sgn}(l)} \chi_s + 2 v_i^{l-\text{sgn}(l)} \chi_s
\]
and
\[
\chi_1 (v_i^{l-\text{sgn}(l)} (u_{3-i} + u_{3-i}^{-1}))_{0,s} = (v_i^{l-\text{sgn}(l)} (u_{3-i} + u_{3-i}^{-1}))_{1,s}.
\]
Thus we have \( \chi_1 (\gamma_{j,l}^r)_0,s \in V_i \cup A v_l^{-1} \). Similarly, we have \((\gamma_{j,l}^r)_r,0 \chi_1 \in V_i \cup A v_l^{-1} \).
Hence by using Lemma 5 it is possible to conclude that \( \chi_n \gamma_{j,l} \chi_m \in V_i \cup A v_l^{-1} \). \( \square \)

**3.4. Constructing a Riesz basis of \( L^2 M \otimes A \).**

**Lemma 10.** For \( \xi \in (W_l^0 \otimes S_l^0) \oplus ((W_l^{1,\alpha,2} \oplus W_l^{1,\beta}) \otimes S_l^1) \oplus (W_l^2 \otimes S_l^2) \), we have the following statements.

1. (i) For any \( n, m \geq 0 \) and \( l \geq 2 \), we have
\[
\chi_n \xi \chi_m = \xi_{n,m} - (\xi_{n,m-2} + \xi_{n-2,m}) + \xi_{n-2,m-2}.
\]
(ii) For any \( n, m \geq 0 \) and \( l \geq 2 \), we have
\[
\xi_{n,m} = \sum_{r \leq n,s \leq m, (r,s) \text{ has the same parity as that of } (n,m)} \chi_r \xi_s.
\]
2. When \( l = 1 \) and \( \xi \) is of the form \( c_1 (u_1 + e u_1^{-1}) + c_2 (u_2 + e u_2^{-1}) \) for some \( e \in \{ \pm 1 \} \) and \( c_1, c_2 \in C \), for any \( n, m \geq 0 \), we have the following two statements.
   (i) We have
\[
\chi_n \xi \chi_m = \xi_{n,m} - (\xi_{n,m-2} + \xi_{n-2,m}) + \xi_{n-2,m-2} + \sum_{k \geq 2} (-1)^k (c \xi_{n-k-1,m-k} + \xi_{n-k+1,m-k-1} + 2 \xi_{n-k,m-k}).
\]
   (ii) We have
\[
\xi_{n,m} = \sum_{r \leq n,s \leq m, r-s \text{ has the same parity as that of } n-m} e^{n-r} \chi_r \xi_s.
\]
Proof. By using lemmas [3, 6] and [7], this is shown in the same way as that of Lemma 2 of Radulescu [14]. □

Lemma 11. For \( \xi \in (W\oplus S) \oplus ((W_{1}^{1,\alpha,2} \oplus W_{1}^{1,\beta}) \oplus S_{1}) \oplus (W_{2} \oplus S_{2}) \), \( l \geq 1 \), the projection \( p_{k} \) commutes with the projection \( q_{n} \) and the range of \( p_{k} \land q_{n} \) is the subspace of \( L^{2}(M) \) spanned by \( \{ \xi_{r,s} \mid r+s = n-l \} \).

Proof. This is shown in the same way as the fact mentioned in the paragraph preceding to Lemma 2 of Radulescu [14]. However, for readers convenience, we present a proof. Let \( r_{n} \) be a projection onto the subspace span\( \{ \xi_{r,s} \mid r+s = n-l \} \). By Lemma 11 (1) (i) (2), we have \( \xi_{r,s} \in A\xi \). Hence we have \( r_{n} \leq p_{k} \land q_{n} \). On the other hand, by Lemma 11 (1) (i) (2) (ii), we have \( p_{k} \leq \sum r_{n} \). Hence the range of \( q_{n}p_{k} \) is contained in that of \( r_{n} \). Thus we have

\[ p_{k} \land q_{n} \leq q_{n}p_{k}q_{n} \leq r_{n} \]

(The first equality holds without any assumption). Hence two projections \( p_{k} \) and \( q_{n} \) commute and we have \( p_{k} \land q_{n} = r_{n} \). □

Lemma 12. For \( \xi, \xi' \in (W_{l} \oplus S_{l}) \oplus ((W_{1}^{1,\alpha,2} \oplus W_{1}^{1,\beta}) \oplus S_{1}) \oplus (W_{2} \oplus S_{2}) \), we have the following statements.

1. When \( l \geq 2 \), \( n, m, n', m' \geq 0 \), we have

\[ \langle \xi_{n,m}, \xi'_{n',m'} \rangle = \delta_{n,n'}\delta_{m,m'}3^{n+m}\langle \xi, \xi' \rangle. \]

2. When \( l = 1 \) and \( \xi \) is of the form \( c_{1}(u_{1} + \epsilon u_{1}^{-1}) + c_{2}(u_{2} + \epsilon u_{2}^{-1}) \) for some \( \epsilon \in \{ \pm 1 \} \) and \( c_{1}, c_{2} \in C \), for any \( n, m, n', m' \), we have

\[ \langle \xi_{n,m}, \xi'_{n',m'} \rangle = \delta_{n,n'}\delta_{m,m'}3^{n+m}(\epsilon^{-|n-n'|})\langle \xi, \xi' \rangle. \]

Proof. By using lemmas [3, 6] and [7], this is shown in the same way as Lemma 3 of Radulescu [14]. □

For \( l \geq 1 \), let \( P_{l} \) be the projection onto the subspace of \( L^{2}M \) spanned by \( \{ AwA \mid w \in W_{k}, k \leq l-1 \} \).

Lemma 13. (1) The projection \( P_{l-1} \) commutes with \( q_{l} \) and the range of \( P_{l-1}q_{l} \) is exactly \( S_{l} \).

(2) For \( \xi, \xi' \in \bigcup_{l \geq 1}((W_{l} \oplus W_{l}^{1,\alpha,2} \oplus W_{l}^{1,\beta} \oplus W_{l}^{2} \oplus S_{l}) \oplus S_{l}) \) with \( \langle \xi, \xi' \rangle = 0 \), we have \( A\xiA \bot A\xi'A \).

(3) For \( \xi \in \bigcup_{l \geq 1}((W_{l} \oplus W_{l}^{1,\alpha,2} \oplus W_{l}^{1,\beta} \oplus W_{l}^{2} \oplus S_{l}) \oplus S_{l}) \), \( \xi' \in W_{l}^{1,\alpha,1} \oplus C_{l}^{1} \oplus C_{l}^{2}, \) we have \( A\xiA \bot A\xi'A \).

Proof. Although this is shown by the same argument as that of the proof of Lemma 4 of Radulescu [14], we present a proof.

(1) We show this by induction on \( l \). When \( l = 0 \), then statement (1) is obvious because we have \( P_{l-1} = 0 \). Assume that statement (1) holds for any \( k = 0, \ldots, l \). We first show the following claim.

Claim. We have \( q_{l+1}(x_{p}w_{q}x_{s}) \in S_{l+1} \) for any \( p, q \geq 0, w \in W_{k}^{1} \) with \( k \leq l \).

Proof of Claim. Let \( w = w_{1} + w_{2} \in S_{k} \oplus (W_{k}^{1} \oplus S_{k}) \) be the orthogonal decomposition. Then by the induction hypothesis, we have \( w_{1} \in \text{ran}P_{l-1} \). Hence it is a finite sum of elements of the form \( x_{p'}w'x_{q'} \) for some \( p', q' \geq 0, w' \in W_{k}^{1} \). Hence by induction, it is possible to show that \( w \) is a sum of elements of the form \( x_{r}x_{s} \) for some \( r, s \geq 0 \). Hence we may assume that
proof. When $r = 0$, the first equality follows from Lemma 5. When $r = 1$, this is shown by the direct computation. In the other cases, this is shown by counting the number of $v^l_i$, we have $A_i \vDash A'_{i}$. □

By Claim, we have
\[ \text{ran}_t P^i_t \subset S^i_{t+1} \subset \text{ran}_t P^i_t \wedge q_{t+1}. \]
The second inclusion of the above is obvious. Hence $P^i_t$ commutes with $q_{t+1}$ and the range of $P^i_t \wedge q_{t+1}$ is $S^i_{t+1}$.

(2) When $l \neq l'$, then by statement (1), we have $A_i \vDash A'_{i}$. When $l = l'$, then by Lemma 12 we have $A_i \vDash A'_{i}$.

(3) When $\xi \in W^1, \alpha, \beta$, $\xi' \in W^1, \alpha, \gamma$, this is shown by the direct computation. In the other cases, this is shown by counting the number of $v^l_i$, we have $A_i \vDash A'_{i}$. □

For each non-zero integer $l$, an integer $k$ and $r, s \geq 0$, set
\[ \xi_{i,l,k} = \frac{1}{3^{l+1}} \sum_{w \in W^0_l, \text{ending with } u^l_{i+1}} w v^l_i u^k, \quad r \geq 0 \]
\[ \xi_{i,l,k} = \frac{1}{3^{l+1}} \sum_{w \in W^0_l, \text{beginning with } u^l_{i-1}} w', \quad r = 0. \]

Set $\tilde{\chi}_1 := \chi_1 / \sqrt{3}$. Then we have the following.

Lemma 14. We have
\[ \tilde{\chi}_1 \xi_{r,s} = \begin{cases} \xi_{r+1,s} + \xi_{r-1,s} + \frac{1}{\sqrt{3}} (d^r \xi_{0,s} + d^{-r} \xi_{0,s}^{-1}) & (r \geq 1) \\
\xi_{1,s} + \frac{1}{\sqrt{3}} (d^r \xi_{0,s} + d^{-r} \xi_{0,s}^{-1}) & (r = 0) \end{cases} \]

Similarly, we have
\[ \xi_{i,l,k} \tilde{\chi}_1 = \begin{cases} \xi_{i,r+1,k} + \xi_{i,r-1,k} + \frac{1}{\sqrt{3}} (d^r \xi_{0,k} + d^{-r} \xi_{0,k}^{-1}) & (s \geq 1) \\
\xi_{i,1,k} + \frac{1}{\sqrt{3}} (d^r \xi_{0,k} + d^{-r} \xi_{0,k}^{-1}) & (s = 0) \end{cases} \]

Proof. When $r \geq 1$, the first equality follows from Lemma 5. We show the first equality when $r = 0$. We have
\[ \chi_1 \xi_{0,s} = \frac{1}{3^{s/2-1}} \chi_1 v^l_i u^k \sum_{w \in W^0_i, w_1 = u^l_{i-1}} w \]
\[ = \sqrt{3} \xi_{1,s} + \frac{1}{3^{s/2-1}} (d^r v^l_i u^{k+1} + d^{-r} v^l_i u^{k-1}) \sum_{w \in W^0_i, w_1 = u^l_{i-1}} w \]
\[ = \sqrt{3} \xi_{1,s} + d^r \xi_{0,s} + d^{-r} \xi_{0,s}^{-1}. \]

Thus we have
\[ \tilde{\chi}_1 \xi_{0,s} = \xi_{1,s} + \frac{1}{\sqrt{3}} (d^r \xi_{0,s} + d^{-r} \xi_{0,s}^{-1}). \]

The second equality is shown in the same way. □

We have the following.
Lemma 15. (1) For any \((i, l, k)\), we have
\[
\| \xi_{r,s}^i,j,k \|_2 = \begin{cases} 
2 & (r, s \geq 1) \\
\sqrt{6} & (r = 0 \text{ or } s = 0, \ (r, s) \neq (0, 0)) \\
3 & (r = s = 0). 
\end{cases}
\]

(2) When \((i, l, k, r, s) \neq (i', l', k', r', s')\), then the vector \(\xi_{r,s}^i,j,k\) is orthogonal to \(\xi_{r',s'}^{i',j,k'}\).

Proof. (1) The number of the words with length \(r\) ending with either \(u_2\) or \(u_2^{-1}\) is exactly \(2 \cdot 3^{r-1}\) when \(r \neq 0\) and 1 when \(r = 0\). Thus we have the desired equality.

(2) This is obvious. \(\square\)

Consider a finite sum
\[
\xi := \sum_{i=1,2, \ j = 1, \cdot \cdot \cdot , 2, \ 3, \ r,s \geq 0, \ t \in \mathbb{Z}} a_{r,s}^{i,j}(\gamma_j^{i,l})_{r,s}.
\]

Let
\[
\xi = \sum_{r,s,k,i=1,2} \beta_{r,s}^{i,l,k} \xi_{r,s}^i,j,k
\]

be the expansion along the orthonormal system \(\{\xi_{r,s}^{i,l,k}\}\) (this is always possible).

Lemma 16. Let \(\xi\) and \(\{\beta_{r,s}^k\}\) be as above. Then we have the following.

(1) We have
\[
\beta_{r,s}^{i,l,k} = \frac{1}{3|k|^2} (\sqrt{3} \sum_{j=0}^{[k]-1} d_{\text{sgn}(k)}^{i,l,j} \beta_{r+j,s+|k|-j-1}^{i,l,0} + \sum_{j=1}^{[k]-1} d_{\text{sgn}(k)}^{i,l,j} \beta_{r+j,s+|k|-j}^{i,l,0})
\]

for any \(i, l, k, r, s\).

(2) There exists a constant \(C\) such that if we have \(\|\xi\|_2 \leq 1\), for any \(k_0 \in \mathbb{N}\), we have
\[
\frac{1}{3} \sum_{|k| \geq k_0} \| \beta_{r,s}^{i,l,k} \|_2 \leq \frac{1}{3} \sum_{|k| \geq k_0} |\beta_{r,s}^{i,l,k}|^2 \leq \frac{C}{3 \cdot 2^{k_0}}.
\]

The constant \(C\) does not depend on \(\xi\) and \(k_0\).

Proof. Notice that \(\xi_{r,s}^{i,l,k} \perp \xi_{r',s',s'}^{i',l',k'}\), \(\gamma_j^{i,l} \perp \gamma_{j,s}^{i',l'}\) for any \((i, l) \neq (i', l')\). Thus we may assume that only one \((i, l)\) appears in the sum of the definition of \(\xi\). In the last of the proof, we denote \(a_{r,s}^{i,j} \) by \(\alpha_{r,s}^{i,j}\), \(\beta_{r,s}^{i,l,k} \) by \(\beta_{r,s}^{i,l,k}\).

(1) The components contributing to \(\xi_{r,s}^{k,k}\) are \((\gamma_2^{i,l,\text{sgn}(l)})_{r+|k|-s-1}, (\gamma_3^{i,l,\text{sgn}(l)})_{r-1,-s+|k|}\)

and \((\gamma_1^{i,l,\text{sgn}(l)})_{r+j,s+|k|-j-1} \ (j = 0, \cdot \cdot \cdot , |k| - 1)\). The coefficient coming from \((\gamma_2^{i,l,\text{sgn}(l)})_{r+|k|-s-1}\) is
\[
3 \frac{d_{\text{sgn}(k)}}{3} a_{r+|k|-s-1}^2.
\]

The coefficient coming from \((\gamma_3^{i,l,\text{sgn}(l)})_{r-1,-s+|k|}\) is
\[
3 \frac{d_{\text{sgn}(k)}}{3} a_{r-1,-s+|k|}^3.
\]

The coefficient coming from \((\gamma_1^{i,l,\text{sgn}(l)})_{r+j,s+|k|-j-1} \ (j = 0, \cdot \cdot \cdot , |k| - 1)\) is
\[
3 \frac{d_{\text{sgn}(k)}}{3} a_{r+j,-s+|k|-j-1}^1.
\]
Hence the coefficient of $\xi_{r,s}^k$ is

$$\beta_{r,s}^k = 3^{r+s} \left( d_{r+[k],s-1}^k + a_{r-1,s+|k|}^3 + \sum_{j=0}^{|k|-1} d_{s\text{sgn}(k),j}^{\text{sgn}(k)} a_{r+j,s+|k|-j-1} \right).$$

We also have

$$d_{r+[k],s-1}^k + a_{r-1,s+|k|}^3 + \sum_{j=0}^{|k|-1} d_{s\text{sgn}(k),j}^{\text{sgn}(k)} a_{r+j,s+|k|-j-1} = d_{s\text{sgn}(k),(|k|-1)}^{\text{sgn}(k)} a_{r+[k],s-1}^2 + a_{r+[k]-2,s+1}^3 + a_{r+[k]-1,1}^{1,\text{sgn}(k)}$$

$$- d_{s\text{sgn}(k),(|k|-1)}^{\text{sgn}(k)} a_{r+[k]-2,s+1}^2 + a_{r+[k]-1,1}^{2,\text{sgn}(k)}$$

$$+ d_{s\text{sgn}(k),(|k|-2)}^{\text{sgn}(k)} a_{r+[k]-1,1}^2 + a_{r+[k]-3,s+2}^3 + a_{r+[k]-2,s+1}^{1,\text{sgn}(k)}$$

$$- d_{s\text{sgn}(k),(|k|-1)}^{\text{sgn}(k)} a_{r+[k]-3,s+2}^2 + a_{r+[k]-2,s+1}^{2,\text{sgn}(k)}$$

$$+ \ldots - \ldots$$

$$+ d_{s\text{sgn}(k),(|k|-2)}^{\text{sgn}(k)} a_{r+[k]-2,s+1}^2 + a_{r+[k]-2,s+1}^{3,\text{sgn}(k)}$$

$$+ (d_{s\text{sgn}(k),(|k|-1)}^{\text{sgn}(k)} a_{r+[k]-2,s+1}^3 + a_{r+[k]-1,1}^{1,\text{sgn}(k)}).$$

Thus we gate the conclusion.

(2) For each $k$, $j = 0, \ldots, k-1$, set

$$c_j^k := \frac{\sqrt{3}}{3^{k+2}} \beta_{r+j,s+|k|-j-1}^{\text{sgn}(k)} = 3^{r+s} (d_{s\text{sgn}(k),j}^{\text{sgn}(k)} a_{r+j+1,s+|k|-j-2}^2 + a_{r+j-1,s+|k|-j+1}^3 + a_{r+j,s+|k|-j-1}^{1,\text{sgn}(k)}),$$

$$d_j^k := \frac{1}{3^{k+2}} \beta_{r+j,s+|k|-j}^0 = 3^{r+s} (a_{r+j-1,s+|k|-j+1}^3 + a_{r+j,s+|k|-j-1}).$$

Then we have

$$\sum_{j=0}^{|k|-1} |c_j^k|^2 + \sum_{j=1}^{|k|-1} |d_j^k|^2 = 3^{1-|k|} \left( \sum_{j=0}^{|k|-1} |\beta_{r+j,s+k-j-1}^1|^2 + 3^{-|k|} \sum_{j=1}^{|k|-1} |\beta_{r+j,s+k-j}^0|^2 \right)$$

$$\leq \frac{3}{4 \cdot 3^{|k|}} \left( \frac{|k|-1}{3^{|k|}} \sum_{j=0}^{|k|-1} |\beta_{r+j,s+k-j-1}^1|^2 \|\xi_{r+j,s+k-j-1}^1\|_2^2 + \sum_{j=1}^{|k|-1} |\beta_{r+j,s+k-j}^0|^2 \|\xi_{r+j,s+k-j}^0\|_2^2 \right)$$

$$= \frac{3}{4 \cdot 3^{|k|}} \sum_{j=0}^{|k|-1} |\beta_{r+j,s+k-j-1}^1|^2 \|\xi_{r+j,s+k-j-1}^1\|_2^2 + \sum_{j=1}^{|k|-1} |\beta_{r+j,s+k-j}^0|^2 \|\xi_{r+j,s+k-j}^0\|_2^2$$

$$\leq \frac{3}{4 \cdot 3^{|k|}} \sum_{r,s,k} \beta_{r,s}^k \|\xi_{r,s}^k\|_2^2$$

$$\leq \frac{3}{4 \cdot 3^{|k|}}.$$
Hence we have
\[
\sum_{|k| \geq k_0} |\beta^k_{r,s}|^2 \\
\leq \sum_{|k| \geq k_0} \left( |c^k_{k-1}| + |d^k_{k-1}| + \cdots + |c^k_1| + |d^k_0| \right)^2 \\
\leq \sum_{|k| \geq k_0} \left( (2|k| + 1)|c^k_{k-1}|^2 + |d^k_{k-1}|^2 + \cdots + |c^k_1|^2 \right) \\
\leq \sum_{|k| \geq k_0} \frac{3(2|k| + 1)}{4 \cdot 3^k} \\
\leq C \frac{2^{k_0}}{}.
\]

Set
\[
L := \text{span}\{AW^1_{l}A, Av^lA \mid i = 1, 2, l > 0\}.
\]

Summarizing the above results, we have the following.

**Lemma 17.** (See Lemma 3.2 of Cameron–Fang–Ravichandran–White [4]) There exists a sequence of orthonormal vectors \(\{\xi_n\}_{n=1}^{\infty}\) satisfying the following conditions.

1. Each vector \(\xi_n\) lies in \(W_i^{(n)}\) for some \(l(n) \geq 1\), \(i(n) = 0, 2, (1, \alpha, 2), (1, \beta)\).
2. The subspaces \(\text{span}A\xi_nA\) \((n \in \mathbb{N})\) are pairwise orthogonal in \(L^2M\).
3. We have
\[
L^2M \oplus (L^2A \oplus L) = \bigoplus \text{span}A\xi_nA.
\]
4. For any \(n\) with \(l(n) > 1\), the sequence \(\{(\xi_n)r,s/\|\xi_n\|_2\}\) is an orthonormal basis of the subspace \(\text{span}A\xi_nA\).
5. For each \(n, m > 0\), there exists a bounded invertible operator \(T_{n,m}\) from the subspace \(\text{span}A\xi^0_nA\) to \(\text{span}A\xi^0_mA\) defined by \((\xi^0_n)r,s \mapsto (\xi^0_m)r,s\). Furthermore, there exists a constant \(C_0\) satisfying \(|T_{n,m}|, |T_{n,m}^{-1}| \leq C_0\) for any \(n, m\).
6. The subspace \(L\) is contained in the subspace \(\bigoplus_{r,s \geq 1, \ i=1,2} C_{\xi_{i,l}^k} \).

**Proof.** This is shown by the same argument as that of the proof of Lemma 3.2 of Cameron–Fang–Ravichandran–White [4]. However, for readers’ convenience, we present a proof. For each \(i = 0, 2, (1, \alpha, 2), (1, \beta)\), \(l \geq 1\), choose an orthonormal basis \(\{\psi^k_{i,l}\}_{k=1}^{K_i}\) of \(W_i^l \oplus S_i^l\). Let \(\{\xi_n\}\) be a rearrangement of \(\{(\eta^k_{i,l})_{i=0,2, (1, \alpha, 2), (1, \beta)} k=1, \ldots, K_i\}\). Then by construction, the sequence \(\{\xi_n\}\) satisfies condition (1). By Lemma 13 (2), the sequence \(\{\xi_n\}\) satisfies condition (2).

We show that \(\{\xi_n\}\) satisfies condition (4). By Lemma 11, the set \(\{(\xi_n)r,s\}\) spans \(A\xi_nA\). By condition (2) and Lemma 12, the vectors \(\{(\xi_n)r,s\}_{n,r,s}\) are mutually orthogonal if \(l(n) > 1\). Thus we have condition (4).

Next, we show that the sequence \(\{\xi_n\}\) satisfies condition (3). By Lemma 13 (3), \(L\) is orthogonal to \(\sqrt{\text{span}}A\xi_nA\). By Lemma 13 (2), the subspaces \(A\xi_nA\)'s are mutually orthogonal. Thus it is enough to show that the set \(\{A\xi_nA\}_n\) really spans \(L^2M \oplus (L^2A \oplus L)\). Take an element \(\xi \in W_i \oplus (L^2A \oplus L)\) which is orthogonal to any
AξnA. Then ξ is orthogonal to the space \(W_l \oplus S_l\), which means that ξ ∈ \(S_l\). On the other hand, by the same argument as in the proof of Lemma 13 (1), any vector \(w \in W_k\) is written as a linear combination of the form \(χ_nγχ_m\) for some \(n, m \geq 0\), \(γ \in W_k' \ominus S_k'\) \((k' \leq k)\). Thus the vector ξ is orthogonal to \(AwA\) for any \(w \in W_k\), \(k \leq l - 1\). Hence by Lemma 13 (1), the vector ξ is orthogonal to \(S_l\). Thus the vector ξ is zero.

Condition (5) follows in the same way as the proof of Lemma 3.2 of Cameron–Fang–Ravichandran–White [4]. Condition (6) is trivial.

By an immediate consequence of Lemma 17, we have the following.

**Corollary 18.** We have the following.

1. The family \(\{(ξ_m)_{r,s} \mid m ∈ \mathbb{N}, r, s ≥ 0\}\) is a Riesz basis of \(L^2M \ominus (A ⊕ L)\).
2. We have \(L \subset \bigoplus_{i, l, k, r, s} \xi_{i, l, k}_{r, s}\).
3. We have \(L \oplus (L^2M \ominus (A ⊕ L)) = L^2M \ominus A\).

4. Locating the support of the sequences of \((M^\omega \ominus A^\omega) \cap A'\)

Now, we would like to explain how to show the maximal amenability of the subalgebra. In order to show the maximal amenability, we look at the following notion.

**Definition 19.** (Lemma 2.1 of Popa [12]) Let \(M\) be a factor of type \(II_1\) and \(A\) be a von Neumann subalgebra of \(M\). We say that the subalgebra \(A\) has the asymptotic orthogonality property if for any \(x_1 = (x_1^n), x_2 = (x_2^n) ∈ (M^\omega \ominus A^\omega) \cap A'\), any \(y_1, y_2 ∈ M \ominus A\), we have \(τ(\omega)(y_1^* x_1^* y_2 x_2) = 0\).

Although the definition of the asymptotic orthogonality property is rather technical, this property is crucial because of the following proposition.

**Proposition 20.** (Corollary 2.3 of Cameron–Fang–Ravichandran–White [4], See also Popa [12]) Let \(A\) be a singular maximal abelian subalgebra of a factor \(M\) of type \(II_1\) with the asymptotic orthogonality property. Then it is maximal amenable.

This is why we would like to show that the subalgebra has the asymptotic orthogonality property. In order to achieve this, we first show that any sequence of \((M^\omega \ominus A^\omega) \cap A'\) eventually get out of the space span\(\{\xi_{i,l,k}_{r,s}, (ξ_m)_{r,s} \mid r ≤ M \text{ or } s ≤ M\}\) for any \(M > 0\). Then we show that any vectors \(η_1, η_2\) orthogonal to the space and any vector \(a, b ∈ M \ominus A\), the value \(|τ(\omega)(a^* η_1^* b η_2)|\) is small if \(M\) is large enough. In this section, we show the first part.

As in Section 3, set \(\hat{χ}_1 := \frac{1}{\sqrt{3}}(u_1 + u_1^{-1} + u_2 + u_2^{-1})\).

Then we have the following lemma.

**Lemma 21.** (See Lemma 4.3 of Cameron–Fang–Ravichandran–White [4], See also Lemma 11 of Wen [16]) Let \(D ⊂ A\) be a diffuse von Neumann subalgebra. Let \(x = (x_n)\) be an \(ω\)-centralizing sequence of \(M\) commuting with \(D\), \(\|x_n\| = 1\) and
Then the complex number 

\[ \tilde{\chi}_1 \tilde{\xi}_{r,s} - \tilde{\xi}_{r,s-1} \]

is shown in a similar way to Lemma 4.1 of Cameron–Fang–Ravichandran–White [4].

By Lemma 14, we have

\[ \tilde{\chi}_1 \tilde{\xi}_{r,s} = \sum_{i,l,k} \alpha_{r,s}^{i,l,k} \tilde{s}_{r,s} \]

Then the complex number \( \beta_{r,s}^{i,l,k} \) is the following.

For \( x = \sum_{r,s,i,l,k} \alpha_{r,s}^{i,l,k} \tilde{s}_{r,s} \) with \( \|x\|_2 = 1 \), \( s' \geq s \geq 1 \), we have the following inequalities.

\[ \left( \sum_{r,s,i,l,k} |\alpha_{r,s-2,0}^{i,l,k} + \cdots + \alpha_{r,s,0}^{i,l,k}| \right)^{1/2} \leq \left( \sum_{r,s,i,l,k} |\alpha_{r,s}^{i,l,k}|^2 \right)^{1/2} \]

\[ \leq \left( \sum_{r,s,i,l,k} |\alpha_{r,s}^{i,l,k} - \alpha_{r,s+1,0}^{i,l,k} - \cdots - \alpha_{r,s+1,0}^{i,l,k}| \right)^{1/2} \]

\[ \leq 3^{s-1} C_0 \|x, \tilde{\chi}_1\|_2 \]

This is shown in a similar way to Lemma 4.1 of Cameron–Fang–Ravichandran–White [4].

Similarly, we have the following.
Lemma 23. For $x = \sum_{r \geq r', i, l, k} \alpha^i_{r,s} \xi^i_{r,s}$ with $\|x\|_2 = 1$, $r' \geq r \geq 1$, we have the following inequalities.

\[
\left( \sum_{s \geq r', i, l, k} |\alpha^i_{0,s-r} + \alpha^i_{0,s-r+2} + \cdots + \alpha^i_{0,s+r} - \frac{d^l}{\sqrt{3}} (\alpha^i_{0,s-r+1} + \cdots + \alpha^i_{0,s+r-1}) \right)^{1/2} - \left( \sum_{s \geq r', i, l, k} |\alpha^i_{l,k}|^2 \right)^{1/2} \leq \left( \sum_{s \geq r', i, l, k} |\alpha^i_{r,s} - (\alpha^i_{0,s-r} + \cdots + \alpha^i_{0,s+r})| \right)^{1/2} + \frac{d^l}{\sqrt{3}} (\alpha^i_{0,s-r+1} + \cdots + \alpha^i_{0,s-r-1}) + \frac{d^l}{\sqrt{3}} (\alpha^i_{0,s-r+1} + \cdots + \alpha^i_{0,s+r-1}) \right)^{1/2} \leq 3^{r-1} C_0 \|[x, \tilde{x}_1]\|_2.
\]

Lemma 24. For $x^n = \sum_{r \geq r', s, i, l, k} \alpha^{n,i,l,k}_{r,s} \xi^{i,l,k}_{r,s}$ with $\|x^n\|_2 = 1$ and $\|[x^n, \tilde{x}_1]\|_2 \to 0$ as $n \to \omega$, we have

\[
\lim_{n \to \omega} \sum_{r \geq r', s, i, l, k} |\alpha^{n,i,l,k}_{r,s} - \frac{1}{\sqrt{3}} (\alpha^{n,i,l,k-1}_{r+1,0} + \alpha^{n,i,l,k+1}_{r+1,0})|^2 = 0,
\]

\[
\lim_{n \to \omega} \sum_{s \geq r', i, l, k} |\alpha^{n,i,l,k}_{0,s} - \frac{1}{\sqrt{3}} (d^l \alpha^{n,i,l,k-1}_{0,s+1} + d^{-l} \alpha^{n,i,l,k+1}_{0,s+1})|^2 = 0.
\]

Proof. This is shown in a similar way to that in Lemma 4.2 of Cameron–Fang–Ravichandran–White [1].

Lemma 25. For $x^n = \sum_{r \geq r', s, i, l, k} \alpha^{n,i,l,k}_{r,s} \xi^{i,l,k}_{r,s} \in L$ with $\|x^n\|_2 = 1$ and $\|[x^n, \tilde{x}_1]\|_2 \to 0$ as $n \to \omega$, we have

\[
\lim_{n \to \omega} \sum_{i, l, k, s} |\alpha^{n,i,l,k}_{0,s}|^2 = 0,
\]

\[
\lim_{n \to \omega} \sum_{i, l, k, r} |\alpha^{n,i,l,k}_{r,0}|^2 = 0.
\]

Proof. By the middle $\leq$ the right of Lemma 22 and Lemma 24 for any $\epsilon > 0$ and $s \in \mathbb{N}$, there exists a natural number $N_s$ such that

\[
(*) \quad \left( \sum_{r \geq s, i, l, k} |\alpha^{n,i,l,k}_{r,s} - \alpha^{n,i,l,k}_{r+s,0}|^2 \right)^{1/2} < \epsilon
\]

for any $n \geq N_s$. Similarly, by Lemmas 23 and 24 for any $\epsilon > 0$ and $r \in \mathbb{N}$, there exists a natural number $N_r$ such that

\[
(**) \quad \left( \sum_{s \geq r, i, l, k} |\alpha^{n,i,l,k}_{r,s} - \alpha^{n,i,l,k}_{0,s+r}|^2 \right)^{1/2} < \epsilon
\]
for any \( n \geq N_r \). Fix a natural number \( s_0 \). By picking up terms over \( r \geq 2s_0 - s \) of the first inequality for \( s = 1, \cdots, s_0 \), we have

\[
\left( \sum_{r \geq 2s_0-s, i,l,k} |\alpha_{r,s}^{n,i,l,k} - \alpha_{r,s+1,0}^{n,i,l,k}|^2 \right)^{1/2} < \epsilon
\]

for \( n \geq N_1, \cdots, N_{s_0} \). By the triangle inequality, we have

\[
\left| \left( \sum_{r \geq 2s_0-s, i,l,k} |\alpha_{r,s}^{n,i,l,k}|^2 \right)^{1/2} - \left( \sum_{r \geq 2s_0, i,l,k} |\alpha_{r,s}^{n,i,l,k}|^2 \right)^{1/2} \right| < \epsilon.
\]

Here, we re-enumerated the index of the second term of the left hand side of the above inequality. Hence we have

\[
\left( \sum_{r \geq 2s_0, i,l,k} |\alpha_{r,0}^{n,i,l,k}|^2 \right)^{1/2} < \left( \sum_{r \geq 2s_0-s, i,l,k} |\alpha_{r,s}^{n,i,l,k}|^2 \right)^{1/2} + \epsilon.
\]

Taking the square of the above inequality, we have

\[
\sum_{r \geq 2s_0, i,l,k} |\alpha_{r,0}^{n,i,l,k}|^2 < \sum_{r \geq 2s_0-s, i,l,k} |\alpha_{r,s}^{n,i,l,k}|^2 + 2 \left( \sum_{r \geq 2s_0-s, i,l,k} |\alpha_{r,s}^{n,i,l,k}|^2 \right)^{1/2} \epsilon + \epsilon^2
\]

\[
\leq \sum_{r \geq 2s_0-s, i,l,k} |\alpha_{r,s}^{n,i,l,k}|^2 + 2C_0 \epsilon + \epsilon^2.
\]

Taking the average of the above inequality over \( s = 1, \cdots, s_0 \), we have

\[
\sum_{r \geq 2s_0, i,l,k} |\alpha_{r,0}^{n,i,l,k}|^2 < \frac{1}{s_0} \sum_{r \geq 2s_0-s, s=1, \cdots, s_0, i,l,k} |\alpha_{r,s}^{n,i,l,k}|^2 + 2C_0 \epsilon + \epsilon^2
\]

\[
< \frac{1}{s_0} C_0^2 + 2C_0 \epsilon + \epsilon^2.
\]

Hence we have

\[
\lim_{n \to \omega} \left( \sum_{r \geq 2s_0, i,l,k} |\alpha_{r,0}^{n,i,l,k}|^2 \right)^{1/2} \leq \frac{C_0}{\sqrt{s_0}}.
\]

Similarly, we have

\[
\lim_{n \to \omega} \left( \sum_{s \geq 2s_0, i,l,k} |\alpha_{0,s}^{n,i,l,k}|^2 \right)^{1/2} \leq \frac{C_0}{\sqrt{s_0}}.
\]

On the other hand, by Lemma 16 (2), for any \( k_0 \), we have

\[
\lim_{n \to \omega} \left( \sum_{r < 2s_0, |k| \geq k_0, i,l} |\alpha_{r,0}^{n,i,l,k}|^2 \right)^{1/2} < \frac{C}{2k_0}.
\]

Next, by looking at the partial sum over \( r = s(\geq 1) \) of inequality (*), for \( n \geq N_r \), we have

\[
\left( \sum_{i,l,k} |\alpha_{r,r}^{n,i,l,k} - \alpha_{2r,0}^{n,i,l,k}|^2 \right)^{1/2} < \epsilon.
\]
By looking at the partial sum over \( r = s \) of inequality (**), there exists a natural number \( M_r \) such that for any \( n \geq M_r \), we have

\[
\left( \sum_{i,l,k} |\alpha_{r,r}^{n,i,l,k} - \alpha_{0,2r}^{n,i,l,k}|^2 \right)^{1/2} < \epsilon.
\]

Hence for \( n \geq N_r, M_r \), we have

\[
(* *) \quad \left( \sum_{i,l,k} |\alpha_{0,2r}^{n,i,l,k} - \alpha_{2r,0}^{n,i,l,k}|^2 \right)^{1/2} < 2\epsilon.
\]

By using this and Lemma 24 and by taking \( \epsilon \) smaller, we have

\[
(***) \quad \left( \sum_{i,l,k} |\alpha_{0,2r}^{n,i,l,k} - \alpha_{2r-1,0}^{n,i,l,k}|^2 \right)^{1/2} < 2\epsilon.
\]

Suppose that there existed \( r \geq 2 \) with \( \lim_{n \to \omega} \sum_{i,l,k} |\alpha_{r,0}^{n,i,l,k}|^2 = c > 0 \). Take a small positive number \( \delta > 0 \), which depends on \( c \) and is determined later. Since \( x^0 \in L \), by Lemma 10 (1), we have

\[
\alpha_{r,s}^{n,i,l,k} = \frac{1}{3^{k/2}} (\sqrt{3} \sum_{j=0}^{\lfloor k/3 \rfloor} d_{\text{sgn}(k)}(|j|) \alpha_{r+j,s}^{n,i,l,s+|j|-1} - \sum_{j=1}^{k-1} d_{\text{sgn}(k)}(|j|) \alpha_{r+j,s}^{n,i,l,0} \alpha_{r+j,s+|j|-1}^{n,i,l,k})
\]

for any \( i, l, k, r, s \). On the other hand, by Lemma 24, we have

\[
\sum_{r \geq 0, i,l,k} |\alpha_{r,0}^{n,i,l,k} - \frac{1}{\sqrt{3}} (\alpha_{r+1,0}^{n,i,l,k-1} + \alpha_{r+1,0}^{n,i,l,k+1})|^2 < \delta
\]

for any sufficiently large \( n \). By inequalities (*), (**), (***) and (***) for any fixed \((r, s) \in \mathbb{N}^2\), we have

\[
\left( \sum_{i,l,k} |\alpha_{r,s}^{n,i,l,k} - \alpha_{r,s,0}^{n,i,l,k}|^2 \right)^{1/2} < \delta
\]

for any sufficiently large \( n \). From now, for any fixed \((r, k)\), we regard \( \{\alpha_{r,0}^{n,i,l,k}\}_{i,l} \) as a vector of \( l^2(\{1, 2\} \times \mathbb{Z}) \). Then we have

\[
\| \alpha_{r,0}^{n,i,l,k} \| = \frac{1}{3^{k/2}} (\sqrt{3} \sum_{j=0}^{\lfloor k/3 \rfloor} d_{\text{sgn}(k)}(|j|) \alpha_{r+j,|k|-1}^{n,i,l,s+|j|} - \sum_{j=1}^{k-1} d_{\text{sgn}(k)}(|j|) \alpha_{r+j,|k|-1}^{n,i,l,0} \alpha_{r+j,|k|-1+|j|,0}) < \frac{|k|}{3^{k/2}} \cdot 2\delta,
\]

\[
\| \alpha_{r+1,0}^{n,i,l,k+\text{sgn}(k)} \| = \frac{1}{3^{k/2}} (\sqrt{3} \sum_{j=0}^{\lfloor k/3 \rfloor} d_{\text{sgn}(k)}(|j|) \alpha_{r+j,|k|+1,0}^{n,i,l,s|j|} - \sum_{j=1}^{k-1} d_{\text{sgn}(k)}(|j|) \alpha_{r+j,|k|+1,0}^{n,i,l,0} \alpha_{r+j,|k|+1,0+|j|}) < \frac{|k+1|}{3^{k/2}} \cdot 2\delta,
\]

\[
\| \alpha_{r+1,0}^{n,i,l,k-\text{sgn}(k)} \| = \frac{1}{3^{k/2}} (\sqrt{3} \sum_{j=0}^{\lfloor k/3 \rfloor} d_{\text{sgn}(k)}(|j|) \alpha_{r+j,|k|-2,0}^{n,i,l,0} - \sum_{j=1}^{k-1} d_{\text{sgn}(k)}(|j|) \alpha_{r+j,|k|-2,0}^{n,i,l,0} \alpha_{r+j,|k|-2,0+|j|}) < \frac{|k-1|}{3^{k/2}} \cdot 2\delta.
\]
Thus we have
\[
\delta > \| \alpha_{r,k}^{n,i,l,k} - \frac{1}{\sqrt{3}} (\alpha_{r+1}^{n,i,l,k+1} + \alpha_{r+1}^{n,i,l,k-1}) \|_2
\]
\[
> -\frac{|k|+1}{3(\sqrt{3} - 1)} \cdot 2\delta + \frac{1}{3} \| (\sqrt{3} \sum_{j=0}^{|k|-2} d_{\text{seq}}(k) j^{n,i,l,sgn(k)} a_r^{n,i,l,k+1,j} - \sum_{j=1}^{|k|-1} d_{\text{seq}}(k) j^{n,i,l,0} a_r^{n,i,l,k+1,j}) \|_2
\]
\[
> \frac{1}{3} \| (\sqrt{3} \alpha_{r,k}^{n,i,l,k} - \alpha_{r,k}^{n,i,l,0}) \|_2
\]
\[
> (A)_{r,k} - 999\delta
\]

for any sufficiently large \( n \) (How large we should take \( n \) depends on \( r, k \) and \( \delta \)).

Thus we have \((A)_{r,k} < 1000\delta\). Now, we have \( r + |k| = (r+1) - (|k|-1) \). Hence for a fixed \( r \geq 3 \), we have
\[
\| \frac{1}{3}(\sqrt{3} \alpha_{r,k}^{n,i,l,k} - d_{\text{seq}}(k) \alpha_{r,k}^{n,i,l,0}) \|_2 \leq \| (A)_{r-|k|,|k|} - (A)_{r-|k|,|k|-1} \|_2 < 2000\delta.
\]

Thus we have
\[
\| \sqrt{3} \alpha_{r,k}^{n,i,l,k} - d_{\text{seq}}(k) \alpha_{r,k}^{n,i,l,0} \|_2 \leq 1000\delta.
\]

Hence we have
\[
\| \alpha_{r,0}^{n,i,l,1} + \alpha_{r,0}^{n,i,l,-1} - \frac{1}{\sqrt{3}} (d^t + d^{-t}) \alpha_{r+1}^{n,i,l,0} \| < 19990\delta.
\]

Thus we have
\[
\| \sqrt{3} \alpha_{r-1,0}^{n,i,l,0} - \frac{1}{\sqrt{3}} (d^t + d^{-t}) \alpha_{r+1}^{n,i,l,0} \| < 20000\delta
\]

for any sufficiently large \( n \) (depending on \( r \) and \( \delta \)). Hence if we take \( \delta \) so large that it satisfies \( \delta < c/100000 \), we have
\[
\| \alpha_{r+1}^{n,i,l,0} \|_2 \geq \frac{4}{3} c
\]
if we take a large \( n \). Take a large \( T \in \mathbb{N} \) so large that it satisfies \( T > 1/c \). Then, by induction, for any \( T \geq t > 0 \), we have
\[
\| \alpha_{r+2t-1,0}^{n,i,l,0} \|_2 \geq \frac{4t}{3} c
\]
for any large \( n \) (depending on \( r, T \) and \( \delta \)), which would contradict the fact that \( \sum_{i,k,r,s} |\alpha_{i,k,r,s}^{n,i,l,k}|^2 \leq 1 \). Hence we have \( \| \alpha_{r,0}^{n,i,l,0} \|_2 \to 0 \) for any \( r \geq 2 \). By using other inequalities, it is possible to show that \( \| \alpha_{r,0}^{n,i,l,k} \|_2 \to 0 \) for any \( r, k \).
Hence we have
\[
\lim_{n \to \infty} \sum_{i,l,k} |\alpha_{r,0}^{n,i,l,k}|^2 = \lim_{n \to \infty} \left( \sum_{r \geq 2s_0} + \sum_{r < 2s_0, |k| \geq k_0} + \sum_{r < 2s_0, |k| < k_0} \right) |\alpha_{r,0}^{n,i,l,k}|^2 \leq \frac{C^2_0}{s_0} + \frac{C}{2k_0} + 0.
\]

For any \( \epsilon > 0 \), we choose \( s_0 \) so large that we have \( \frac{C^2_0}{s_0} \leq \epsilon/2 \) and then we choose \( k_0 \) so huge that we have \( C/2k_0 < \epsilon/2 \). Then we have
\[
\sum_{r \geq 0, i,l,k} |\alpha_{0,r}^{n,i,l,k}|^2 < \epsilon.
\]

Thus we have \( \lim_{n \to \infty} \sum_{r \geq 0, i,l,k} |\alpha_{0,r}^{n,i,l,k}|^2 = 0 \). By the same argument as above, we have
\[
\lim_{n \to \infty} \sum_{i,l,k} |\alpha_{0,0}^{n,i,l,k}|^2 = 0.
\]

Thus we are done. \( \square \)

5. COUNTING THE NUMBER OF WORDS WHICH CONTRIBUTE TO THE INNER PRODUCT

In this section, we show that for any vectors \( \eta_1, \eta_2 \in \text{span}\{e_{i,r,s} \mid r \geq M \text{ or } s \geq M\} \), any vectors \( a, b \in M \otimes A \), the inner product \( |\tau(a^\ast \eta_1^r b \eta_2)| \) is small if \( M \) is large enough (Lemmas 27 and 26). This section corresponds to Section 5 and Lemma 6.1 of Cameron–Fang–Ravichandran–White [4].

Lemma 26. Let \( g, h \) be elements of \( M \otimes A \) satisfying the following conditions.

1. The vector \( g \) is of the form \( u_{i_1}^{e_1} \cdots u_{i_n}^{e_n} w_{i_1} \cdots w_{i_n} \) for some \( n \geq 1 \), \( i_1 \neq \cdots \neq i_n \), \( k_1, \cdots, k_n \in \mathbb{Z} \), where for any \( s = 1, \cdots, n \), the operator \( u_{i_s}^{k_s} w_{i_s} \) satisfies either (a) \( k_s \neq 0 \), \( w_{i_s} = 1 \) or (b) \( w_{i_s} \) is a normalizing unitary of \( \{u_{i_s}\}'' \) which is orthogonal to \( \{u_{i_s}\}'' \).

2. The vector \( h \) is of the form \( u_{i_1'}^{k_1'} x_{i_1'} \cdots u_{i_{n'}}^{k_{n'}} x_{i_{n'}} \) for some \( n' \geq 1 \), \( i'_1 \neq \cdots \neq i'_{n'} \), \( k'_1, \cdots, k'_{n'} \in \mathbb{Z} \), where for any \( t = 1, \cdots, n' \), the operator \( u_{i_t'}^{k'_t} x_{i_t'} \) satisfies either (c) \( k'_t \neq 0 \), \( x_{i_t'} = 1 \) or (d) \( x_{i_t'} \) is a normalizing unitary of \( \{u_{i_t'}\}'' \) which is orthogonal to \( \{u_{i_t'}\}'' \).

3. At least one of the operators \( u_{i_s}^{k_s} w_{i_s} \) (\( s = 1, \cdots, n \) or \( t = 1, \cdots, n' \)) satisfies the above condition (b) or (d).

Then there exists a positive constant \( C > 0 \), which depends neither on \( w_{i_s} \)'s nor \( x_{i_t}'s \), such that for any \( M > 4k := 4(|k_1| + |k_2| + \cdots + |k_n| + |k'_1| + \cdots + |k'_{n'}|) \), any vectors \( \eta_1, \eta_2 \) of the space
\[
\text{span}\left\{ (\zeta_m)_{2M+r,2M+s} \mid m \geq 0, r, s \geq 0 \right\} \cup \left( \text{span}\left\{ e_{i,l}^{e_1,l,k} \mid i = 1, 2, l \in \mathbb{Z}, k \in \mathbb{Z}, r, s \geq 0 \right\} \cap L \right),
\]
we have
\[ |\langle \eta_1 g, h \eta_2 \rangle| \leq C \frac{M^4}{3M^2} \| \eta_1 \|_2 \| \eta_2 \|_2. \]

Before proving the lemma, we have to notice that the above \( k \)'s, \( k' \)'s, \( w_i \)'s and \( x_i \)'s are not completely determined by \( h \) and \( g \). This lemma means that the above equation holds for any vectors \( h \) and \( g \) which admit the above presentation.

**Proof.** Roughly speaking, the strategy is the following. Decompose the vectors \( \eta_1 \) and \( \eta_2 \) into linear combinations of \( wyw', w''zw''' \), where \( w, w', w'', w''' \) are words with \( |w|, |w'|, |w''|, |w'''| \gg |g|, |h| \). Then we cancel each two neighboring words as possible. The vital point is that at least one of \( h \) and \( g \) is not completely canceled by condition (3). By using this fact, we show that for most \((w, w', w'')\), we have \( \tau(h^* wyw' g w'' zw''' ) = 0 \) (Claim 3).

Since the two spaces
\[ \text{span}\{ (\xi_{m}^{i})_{2M+r,2M+s} \mid m \geq 0, r, s \geq 0 \} \]
and
\[ \text{span}\{ \xi_{2M+r,2M+s}^{i,l,k} \mid i = 1, 2, l \in \mathbb{Z}, k \in \mathbb{Z}, r, s \geq 0 \} \cap L \]
are mutually orthogonal, it is enough to show the following claim.

**Claim 1.** Assume that vectors \( \eta_1 \) and \( \eta_2 \) belong to one of the above two spaces. Then we have
\[ |\langle \eta_1 g, h \eta_2 \rangle| \leq C \frac{M^4}{3M^2} \| \eta_1 \|_2 \| \eta_2 \|_2. \]

In the rest of this proof, we devote our attention to proving this claim. For simplicity, we assume that both \( \eta_1 \) and \( \eta_2 \) belong to the former subspace (We can handle the other three cases in the same way). Write \( \eta_1 \) and \( \eta_2 \) in the following way.

\[ \eta_1 = \sum_{r,s \geq 0, m \in \mathbb{N}} 3^{-\frac{4M+s+r}{2}} \lambda_{m,2M+r,2M+s} (\xi_m)_{2M+r,2M+s}, \]
\[ \eta_2 = \sum_{r',s' \geq 0, m \in \mathbb{N}} 3^{-\frac{4M+s'+r'}{2}} \mu_{m',2M+r',2M+s'} (\xi_{m'})_{2M+r',2M+s'}. \]

Then by Lemma 17(5), we have
\[ \| \eta_1 \|_2 \geq C_0^{-1} \left( \sum_{r,s,m} |\lambda_{m,2M+r,2M+s}|^2 \right)^{1/2}, \]
\[ \| \eta_2 \|_2 \geq C_0^{-1} \left( \sum_{r',s',m'} |\mu_{m',2M+r',2M+s'}|^2 \right)^{1/2}. \]

Set two vectors \( h_{j''}^{*''}, j \) and \( g_{j'}, j' \) in the following way. When \( w_{i_n} = 1 \), set
\[ h_{j''}^{*''}, j := u_{i_1}^{k_{1} - j''} w_{i_1}^{k_2} \cdots w_{i_{n-1}}^{k_{n-1}} u_{i_n}^{k_{n} - j}. \]

Here, \( j'' \) and \( j \) run over the following ranges. When \( n \geq 2 \), \( (j'', j) \) run over all \( j'' = 0, \cdots, k_1 \), \( j = 0, \cdots, k_n \). When \( n = 1 \), \( k_1 \geq 0 \), \( (j'', j) \) run over all \( j'' = 0, \cdots, k_1 \), \( j = 0, \cdots, k_1 \), \( 0 \leq j + j'' \leq k_1 \). When \( n = 1 \), \( k_1 < 0 \), \( (j'', j) \) run over all \( j'' = 0, \cdots, k_1 \), \( j = 0, \cdots, k_1 \), \( 0 \geq j + j'' \geq k_1 \).
When \( w_{i_n} \neq 1 \), set
\[
h_{j''}^{i,j} := u_{i_1}^{k_1} \cdots u_{i_n}^{k_n} w_{i_n} \text{ for } j'' = 0, \ldots, k_1, j = 0.
\]
When \( x_{i'_n} = 1 \), set
\[
g_{j',j''} := u_{i'_1}^{k'_1} x_{i'_1}^{k'_2} \cdots u_{i''_n}^{k''_n}.
\]
Here, \( j' \) and \( j'' \) run over the following ranges. When \( n' \geq 2 \), \( (j', j'') \) run over all \( j' = 0, \ldots, k'_1 \), \( j'' = 0, \ldots, k''_n \). When \( n' = 1 \), \( k'_1 \geq 0 \), \( (j', j'') \) run over all \( j' = 0, \ldots, k'_1 \), \( j'' = 0, \ldots, k''_1 \), \( 0 \leq j' \leq j'' \leq k'_1 \). When \( n' = 1 \), \( k'_1 < 0 \), \( (j', j'') \) run over all \( j' = 0, \ldots, k'_1 \), \( j'' = 0, \ldots, k''_1 \), \( 0 \geq j' \geq j'' \geq k'_1 \).

When \( x_{i''_n} \neq 1 \), set
\[
g_{j',j''} := u_{i'_1}^{k'_1} x_{i'_1}^{k'_2} \cdots u_{i''_n}^{k''_n} x_{i''_n}^{-k''_n}
\]
for \( j' = 0, \ldots, k'_1 \), \( j'' = 0 \). Here, there is an important notice:
by condition (3), either \( h_{j''}^{i,j} \) or \( g_{j',j''} \) is not 1.

Let
\[
3 \sum_{m} 2^{M+r+s-j''} \sum_{y} (\xi_m)^{M+r-j,M+s-j'} = \sum_{y} y,
\]
\[
3 \sum_{m} 2^{M+r+s-j''} \sum_{z} (\xi_m)^{M+r-j'',M+s-j'''} = \sum_{z} z
\]
be decompositions, where \( \{y\}, \{z\} \) are sets of mutually orthogonal non-zero scalar multiples of complete reduced words, respectively.

We also define a subset \( V_1(m, r, s, j, y) \) of \( \hat{W}_M^0 \) in the following way.

**Case 1.** When \( w_{i_n} = 1 \), \( V_1(m, r, s, j, y) \) is the set of all words \( w \) satisfying the following conditions.

1. The first letter of \( w \) is neither the \( (j-1) \)-st letter of \( h^* \) from the right nor the inverse of the \( j \)-th letter of \( h \) from the right.
2. The last letter of \( w \) does not cancel with the first letter of \( y \).

**Case 2.** When \( w_{i_n} \neq 1 \), \( V_1(m, r, s, j, y) \) is the set of all words whose last letters do not cancel with the first letter of \( y \).

We define a subset \( V_2(m, r, s, j', y) \) of \( \hat{W}_M^y \) in the following way.

The set \( V_2(m, r, s, j', y) \) consists of all words \( w' \) satisfying the following conditions.

1. The first letter of \( w' \) does not cancel with the last letter of \( y \).
2. The last letter of \( w' \) is neither the \( (j'-1) \)-st letter of \( y \) from the left nor the inverse of the \( j' \)-th letter of \( y \) from the left.

We define a subset \( V_3(m', r', s', j'', z) \) of \( \hat{W}_M^y \) in the following way.

**Case 1.** When \( x_{i'_n} = 1 \), \( V_3(m', r', s', j'', z) \) is the set of all words \( w'' \) satisfying the following conditions.

1. The first letter of \( w'' \) is neither the \( (j''-1) \)-st letter of \( y \) from the right nor the inverse of \( j'' \)-th letter of \( y \) from the right.
2. The last letter of \( w'' \) does not cancel with the first letter of \( z \).
Case 2. When \( x_{i,s} \neq 1 \), the set \( V_3(m', r', s', j'', z) \) consists of all words whose last letters do not cancel with the first letter of \( z \).

We define a subset \( V_4(m', r', s', j''', z) \) of \( \tilde{W}_M^0 \) in the following way.

The set \( V_4(m', r', s', j''', z) \) consists of all words \( w''' \) satisfying the following conditions.

1. The first letter of \( w''' \) does not cancel with \( z \).
2. The last letter of \( w''' \) is neither the \( (j''')-1 \)-st letter of \( h^* \) from the left nor the inverse of the \( j''' \)-th letter of \( h \) from the left.

Hereafter, if there is no danger of confusion, we sometimes abbreviate \( V_1(m, r, s, j, y) \), \( V_2(m, r, s, j', y) \), \( V_3(m', r', s', j'', z) \) and \( V_4(m', r', s', j''', z) \) to \( V_1, V_2, V_3 \) and \( V_4 \), respectively. In this setting, we have the following claim.

Claim 2. We have

\[
h^*\eta_1 g y_2 = 3^{-2M} \sum_{j, j', j'', j'''} \sum_{r, s, r', s'} 3^{j+j'+j''+j'''} w'''_{i_1} h_{j''', j} \sum_{y, z} \left( \sum_{w \in V_4(m, r, s, j, y)} w \right) \lambda_{m, 2M+r, 2M+s} \left( \sum_{w' \in V_3(m', r', s', j'', z)} w' \right) g_{j', j''} \mu_{m', 2M+r', 2M+s} \left( \sum_{w''' \in V_4(m', r', s', j''', z)} w''' \right),
\]

where \( j, j', j'', j''' \) run over the following ranges. When \( w_{i_n} = 1 \) and \( n \geq 2 \), \( j, j''' \) run over all \( j = 0, \ldots, k_n, j''' = 0, \ldots, k_1 \). When \( w_{i_n} = 1 \), \( n = 1 \) and \( k_1 \geq 0, j, j''' \) run over all \( j = 0, \ldots, k_1, 0 \leq j + j''' \leq k_1 \). When \( w_{i_n} = 1 \), \( n = 1 \) and \( k_1 < 0, j, j''' \) run over all \( j = 0, \ldots, k_1, j''' = 0, \ldots, k_1 \), \( 0 \geq j + j''' \geq k_1 \). When \( w_{i_n} \neq 1 \), \( j, j''' \) run over \( j = 0, j''' = 0, \ldots, k_1 \).

When \( w_{i_n'} = 1 \) and \( n' \geq 2 \), \( j, j'' \) run over all \( j' = 0, \ldots, k_{n'}, j'' = 0, \ldots, k_{n'}' \). When \( w_{i_n'} = 1 \), \( n' = 1 \) and \( k_{n'} \geq 0, j', j'' \) run over all \( j' = 0, \ldots, k_{n'}, j'' = 0, \ldots, k_{n'}' \), \( j' + j'' \leq k_{n'} ' \). When \( w_{i_n'} = 1 \), \( n' = 1 \) and \( k_{n'} < 0, j', j'' \) run over all \( j' = 0, \ldots, k_{n'}, j'' = 0, \ldots, k_{n'}' \). When \( w_{i_n'} \neq 1 \), \( j', j'' \) run over \( j' = 0, j'' = 0, \ldots, k_{n'} ' \).

Proof of Claim 2. Obviously, we have

\[
h^*\eta_1 g y_2 = 3^{-8M+r+s+r'+s'} h^* \sum_{m, r, s} \lambda_{m, 2M+r, 2M+s} (\xi_m)_{2M+r, 2M+s} g \sum_{m', r', s'} \mu_{m', 2M+r', 2M+s'} (\xi_{m'})_{2M+r', 2M+s'}.
\]

We would like to look at words appearing in \( h^*(\xi_m)_{2M+r, 2M+s} g(\xi_{m'})_{2M+r', 2M+s'} \) and to reduce each neighboring two words as possible. Note that in this reduction, we do temporary think that the letters \( w_i \) and \( x_i \) are free from \( u_i \). Consider the component \( h^* y' g' z' \), where \( y' \) is a linear component of \( (\xi_m)_{2M+r, 2M+s}, z' \) is a linear component of \( (\xi_{m'})_{2M+r', 2M+s'} \). When we reduce each two neighboring two blocks in this word as possible, then the word becomes a linear sum of words of the form

\[
w'''_{i_1} h_{j''', j} w w' g_{j', j''} w'' z w''',
\]
where $y$ is a component of $(\xi_{m})_{M+r-j,M+s-j'}$, $z$ is a component of $(\xi_{m'})_{M+r'-j'',M+s'-j'''}$, and $w$, $w'$, $w''$, $w'''$ are words of $\hat{W}_{\hat{M}}^{0}$ satisfying the following conditions.

1. The word $w$ does not cancel with $h_{j',j}$ or $y$.
2. The word $w'$ does not cancel with $y$ or $g_{j',j''}$.
3. The word $w''$ does not cancel with $g_{j',j''}$ or $z$.
4. The word $w'''$ does not cancel with $z$ or $h_{j''',j}$.

We have to show that the above conditions are satisfied if and only if $(w, w', w'', w''') \in V_{1} \times V_{2} \times V_{3} \times V_{4}$. For simplicity, we consider when $w_{1n} = 1$, $x_{1p} = 1$. Other cases are shown in the same way (and the argument is much easier). We show that if $w \notin V_{1}$, then the word $w$ does not satisfy condition (1). Since a word $w$ does not cancel with $h_{j',j}$, the first letter of $w$ cannot be the inverse of the last letter of $h_{j',j}$. In order to cancel the $(j-1)$st letter of $h$ from the right, there should be the inverse of that letter ahead of $w$. Hence the first letter of $w$ cannot be the $(j-1)$st letter of $h$. Of course, if the last letter of $w$ cancel with $y$, then the word $w$ cannot satisfy condition (1). Hence any word $w \notin V_{1}$ cannot contribute to the second summation. On the other hand, if $w \in V_{1}$, then the word $w$ satisfies condition (1). Similar statements hold for $w'$, $w''$ and $w'''$. Thus we get the desired expression.

**Claim 3.** Let $y$ be a linear component of $(\xi_{m})_{M+r-j,M+s-j'}$, $z$ be a linear component of $(\xi_{m'})_{M+r'-j'',M+s'-j'''}$. Consider a word $h_{j',j}^* y w g_{j',j''} z w'''$. If it satisfies $\tau(h_{j',j}^* y w g_{j',j''} z w''') \neq 0$, then at least one of the following statements holds.

1. We have $w = u_{1}^{\pm} M$.
2. We have $w' = u_{1}^{\pm} M$.
3. We have $w'' = u_{1}^{\pm} M$.
4. We have $w''' = u_{1}^{\pm} M$.

**Proof of Claim 3.** Assume that none of statements (1)–(4) holds. We would like to show that $\tau(h_{j',j}^* y w g_{j',j''} z w''') = 0$. Let

$$h_{j',j}^* = u_{1}^{k_{1}} v_{1}^{l_{1}} \ldots u_{n}^{k_{n}} v_{n}^{l_{n}},$$

$$wy = u_{1}^{p_{1}} v_{1}^{q_{1}} \ldots u_{m}^{p_{m}} v_{m}^{q_{m}},$$

$$g_{j',j''} = u_{1}^{k_{1}'} v_{1}^{l_{1}'} \ldots u_{n}^{k_{n}'} v_{n}^{l_{n}''},$$

$$w' = u_{1}^{k_{1}} v_{1}^{l_{1}} \ldots u_{m}^{k_{m}} v_{m}^{l_{m}},$$

$$w'' z w''' = u_{1}^{k_{1}'} v_{1}^{l_{1}'} \ldots u_{m}^{k_{m}'} v_{m}^{l_{m}''},$$

be the completely reduced word expressions.

**Case 1:** neither $h_{j',j}^*$ nor $g_{j',j''}$ is 1. In order to get the conclusion, it is enough to show the following.

(i) The vector $h_{j',j}^* y w g_{j',j''}$ is a linear sum of words of the form $u_{1}^{k_{1}''} v_{1}^{l_{1}''} \ldots u_{n}^{k_{n}''} v_{n}^{l_{n}''}$.

(ii) The vector $g_{j',j''} z w'''$ is a linear sum of words of the form $u_{1}^{k_{1}''} v_{1}^{l_{1}''} \ldots u_{m}^{k_{m}''} v_{m}^{l_{m}''}$.

(iii) Any pair of linear components of $h_{j',j}^* y w$ and $g_{j',j''} z w'''$ does not cancel at all.
Case 1-1: both $w_{i_n}$ and $x_{i'}$ are 1. In this case, we have statements (i)–(iii) without any assumption.

Case 1-2: neither $w_{i_n}$ nor $x_{i'}$ is 1. In Claim 1, we temporarily thought that the letters $w_i$ and $x_i$ were free from $u_i$. However, in this claim, we do not. This may cause $h_{j''''}^{*}w_j$ and $w$ to cancel. Nonetheless, the assumption that $w \neq u_{i_n}^{\pm M}$ ensures that if we reduce $h_{j''''}^{*}w_jw'$ as possible, it is either of the form
\[
\begin{align*}
&u_{i_1}^{k_1}w_{i_1} \cdots u_{i_n}^{k_n}(w_{i_n}u_{i_1}^{p_1}w_{i_1}^{s_1})w_{i_n}u_{i_2}^{p_2}w_{i_2}^{s_2} \cdots u_{i_m}^{p_m}w_{i_m}^{s_m} \\
&\quad (i_n \neq j_2) \text{ or} \\
&u_{i_1}^{k_1}w_{i_1} \cdots u_{i_n}^{k_n}w_{i_n}u_{j_1}^{p_1}w_{j_1}^{s_1} \cdots u_{j_m}^{p_m}w_{j_m}^{s_m} \\
&\quad (i_n \neq j_1). 
\end{align*}
\]
Since $w_{i_n}$ is orthogonal to the subalgebra $\{u_{i_n}\}$ and normalizes it, by freeness, the vector $h_{j''''}^{*}w_jw'$ satisfies condition (i). Similarly, since $w'' \neq u^{\pm M}_{i_n}$, the vector $g_{j',j''''}w''zw'''$ satisfies condition (ii). Since $w' \neq u^{\pm M}_{i_n}$, if $k'_1 = 0$, then condition (iii) is shown by the same argument. If $k'_1 \neq 0$, then $w''$ and $g_{j',j''''}$ do not cancel at all. Hence condition (iii) is trivial.

Case 1-3: exactly one of $w_{i_n}$ and $x_{i'}$ is 1. This case is treated by the combination of the arguments in Cases 1-1 and 1-2.

Case 2: exactly one of $h_{j''''}^{*}w_j$ and $g_{j',j''''}$ is 1. For simplicity, we assume $g_{j',j''''} = 1$ (the other case is handled in the same way). We have
\[
\tau(h_{j''''}^{*}w_jw'_jg_{j',j''''}w''zw''') = \tau(w'_j \cdot w''zw'''h_{j''''}^{*}w_j).
\]
Since any component of $h_{j''''}^{*}w_j$ contains $v_j^{\pm 1}$, for $\tau(w'_j \cdot w''zw'''h_{j''''}^{*}w_j)$, in order to be non-zero, at least one of $z$ and $y$ contains $v_j^{\pm 1}$. Since neither $w''$ nor $w$ is $u^{\pm M}_{i_n}$, by the same argument as that of Case 1, any component of $w''zw'''h_{j''''}^{*}w_j$ contains at least one $v_j^{\pm 1}$ if we reduce it as possible. Thus we have $\tau(w'_j \cdot w''zw'''h_{j''''}^{*}w_j) = 0$. 

Set
\[
\begin{align*}
\eta_{1,j',j'''} &:= 3^{-M+\frac{i'+i''}{2}} \sum_{r,s,m} \sum_{y} \left( \sum_{w \in V_1} w \right) \lambda_{m,2M+r,2M+s} \left( \sum_{w' \in V_2} w' \right), \\
\eta'_{1,j,j''} &:= 3^{-M+\frac{i'+i''}{2}} \sum_{r,s,m} \sum_{y} \left( \sum_{u_i^M \in V_1} u_i^M \right) \lambda_{m,2M+r,2M+s} \left( \sum_{w' \in V_2} w' \right), \\
\eta_{1,j,j'''} &:= 3^{-M+\frac{i'+i''}{2}} \sum_{r,s,m} \sum_{y} \left( \sum_{u_i^M \in V_1} u_i^M \right) \lambda_{m,2M+r,2M+s} \left( \sum_{w' \in V_2} w' \right), \\
\eta_{2,j',j'''} &:= 3^{-M+\frac{i'+i''}{2}} \sum_{r,s',m'} \sum_{z} \left( \sum_{w'' \in V_3} w'' \right) \mu_{m',2M+r',2M+s'} \left( \sum_{w''' \in V_4} w''' \right), \\
\eta_{2,j,j''} &:= 3^{-M+\frac{i'+i''}{2}} \sum_{r,s',m'} \sum_{z} \left( \sum_{u_i^M \in V_3} u_i^M \right) \mu_{m',2M+r',2M+s'} \left( \sum_{w''' \in V_4} w''' \right), \\
\eta_{2,j,j'''} &:= 3^{-M+\frac{i'+i''}{2}} \sum_{r,s',m'} \sum_{z} \left( \sum_{u_i^M \in V_3} u_i^M \right) \mu_{m',2M+r',2M+s'} \left( \sum_{w''' \in V_4} w''' \right),
\end{align*}
\]
\[ \eta_{2,j,j'}^{\mu} := 3^{-M + \frac{1}{2} + \frac{1}{2}} \sum_{z} \left( \sum_{w'' \in V_3, \ w'' \neq u_i^M} w'' \right) \mu_{m',2M+r',2M+s} \left( \sum_{i=1/2, \ s \in \{ \pm 1 \}, \ u_i^M \in V_4} u_i^M \right). \]

Then we have
\[
\| h_{j,j',j} \|_2 = \| \eta_{1,j,j'} \|_2 \leq 4 \cdot 3^{-M + \frac{1}{2} + \frac{1}{2}} \sum_{r,s,m} \lambda_{m,2M+r,2M+s} \left( \sum_{w \in V_1} w \right) \| y \|_2 \| y \|_2 
\leq 4 \cdot 3^{-M + \frac{1}{2} + \frac{1}{2}} \sum_{r,s,m} \lambda_{m,2M+r,2M+s} \left( \sum_{w \in V_1} w \right) \| y \|_2 
= 4 \cdot 3^{-M + \frac{1}{2} + \frac{1}{2}} \sum_{r,s,m} \lambda_{m,2M+r,2M+s} (|\xi_m|_{M+r-j,2M+s-j'} + 2^{M-j}) 
\leq 4 \cdot C_0 3^{-M + \frac{1}{2} + \frac{1}{2}} \left( \sum_{r,s,m} \lambda_{m,2M+r,2M+s} \right)^{1/2} \leq 4 \cdot C_0 3^{-M + \frac{1}{2} + \frac{1}{2}} \| \eta_1 \|_2.
\]

Similar statements hold for other three vectors. We also have
\[
\| h_{j,j',j}^{\mu} - (\eta_{1,j,j'} + \eta_{2,j,j'}) \|_2 
\leq 3^{-M} \sum_{r,s,m} \lambda_{m,2M+r,2M+s} \left( \sum_{w \in V_1} w \right) \| y \|_2 
\leq 3^{-M} \cdot 3^M \cdot C_0 \left( \sum_{r,s,m} \lambda_{m,2M+r,2M+s} \right)^{1/2} \leq C_0 \| \eta_1 \|_2.
\]

Hence we have
\[
|\langle \eta_1 g, h_{2} \rangle| = |\tau(h^{*} \eta_1 g_{2}^{*})| 
\leq \sum_{j,j',j''} \left( |\tau(h_{j,j',j''}^{\mu}(\eta_{1,j,j'} + \eta_{2,j,j'}) g_{j,j'',j'''} \eta_{2,j,j''})| + |\tau(h_{j,j',j''}^{\mu}(\eta_{1,j,j'} + \eta_{2,j,j'}) g_{j,j'',j'''}(\eta_{2,j',j''} + \eta_{2,j',j'''}))| + |\tau(h_{j,j',j''}^{\mu}(\eta_{1,j,j'} + \eta_{2,j,j'}) g_{j,j'',j'''}(\eta_{2,j',j''} + \eta_{2,j',j'''}))| \right).
\]

By Claim 3, the third term in the above sum is zero. Hence the right hand side is not greater than
\[
M^4 \left( 4 \cdot 4 \cdot \frac{C_0^4}{3^{M/2}} \| \eta_1 \|_2 \| \eta_2 \|_2 + 0 \right) \leq 16 \cdot C_0^4 \cdot \frac{M^4}{3^{M/2}} \| \eta_1 \|_2 \| \eta_2 \|_2.
\]

Thus we are done. \( \square \)

We also have the following.
Lemma 27. Let $g_1, g_2$ be words of $W^0_l$ for some non-negative integer $l \geq 0$. Let $h$ be a vector satisfying the following condition.

The vector $h$ is of the form $\sum u_{i_1}^{k_1} x_{i_1} \cdots u_{i_n}^{k_n} x_{i_n}$, for some $n' \geq 1$, $i'_1 \neq \cdots \neq i'_n$, $k_1', \cdots, k_n' \in \mathbb{Z}$, where for any $t = 1, \cdots, n'$, the operator $u_{i'_t}^{k'_t} x_{i'_t}$ satisfies either (c) $k'_t \neq 0$, $x_{i'_t} = 1$ or (d) $x_{i'_t}$ is a normalizing unitary of $\{u_{i'_t}^{k'_t}\}$ in $R_{i'_t}$ which is orthogonal to $\{u_{i'_t}^{k'_t}\}$.

Set $k := |k'_1| + \cdots + |k'_n|$ (Actually, this depends on the presentation of $h$. However, we fix the presentation or take the minimum). Then there exists a constant $C > 0$, which does not depend either on none of $x_{i_t}$'s, such that for any $M > 2 \max\{l, k\}$, any vectors $\eta_1, \eta_2$ of

$$\text{span}\{(\xi_m)_{M+r, M+s} \mid m \geq 0, r, s \geq 0\} \cup \left(\text{span}\{\xi_m^{i, l, k} \mid i = 1, 2, l \in \mathbb{Z}\setminus\{0\}, k \in \mathbb{Z}, r, s \geq 0\} \cap L\right),$$

we have

$$\langle \eta_1 (g_1 - g_2), h \eta_2 \rangle \leq CM^{4} 3^{-M/2} \|\eta_1\|_2 \|\eta_2\|_2.$$

Proof. When $h \in W^0_l$, then the lemma is shown by the same argument as that of the proof of Lemmas 5.1, 5.2, 5.3 and 6.1 of Cameron–Fang–Ravichandran–White [4]. When $h \notin W^0_l$, then the lemma is shown by Lemma 27.

6. ASYMPTOTIC ORTHOGONALITY PROPERTY OF THE SUBALGEBRA

In this section, by using the results of Sections 4 and 5, we show that the subalgebra has the asymptotic orthogonality property. In order to achieve this, we have to reduce the general cases to the special case, that is, the case when the Haar unitaries come from generators of the irrational rotation $C^*$-algebras.

Lemma 28. Let $\alpha$ be a free action of $\mathbb{Z}/m\mathbb{Z}$ on a diffuse separable abelian von Neumann algebra $C$ and $\beta$ be an ergodic action of $\mathbb{Z}$ on a diffuse separable abelian von Neumann algebra $D$. Let $u, v$ be generating Haar unitaries of $C$ and $D$, respectively. Then for any positive number $\epsilon > 0$, there exists a normal injective $^*$-homomorphism $\theta$ from $C \rtimes_{\alpha} \mathbb{Z}/m\mathbb{Z}$ into $D \rtimes_{\beta} \mathbb{Z}$ satisfying $\theta(C) = D$, $\|\theta(u) - v\|_2 < \epsilon$.

Proof. Since both $u$ and $v$ are generating Haar unitary, the map $u^k \mapsto v^k$ extends to a $^*$-isomorphism $\theta_0$ from $C$ onto $D$ satisfying $\theta_0(u) = v$. Since $\theta_0 \circ \alpha \circ \theta_0^{-1}$ is a free action of $\mathbb{Z}/(m\mathbb{Z})$ on $D$, it is possible to find a partition $\{p_i\}_{i=1}^m$ of unity by projections in $D$ with

$$\theta_0 \circ \alpha \circ \theta_0^{-1}(p_i) = p_{i+1}$$

for any $i = 1, \cdots, m$, where $p_{m+1} = p_1$. Then there exists a partition $\{q_l\}_{l=1}^L$ of $p_1$ by projections in $D$ such that there exist complex numbers $\{\lambda(i, l)\}_{i=1, \cdots, m; l=1, \cdots, L} \subset C$ with

$$\| \sum_{i=1}^m \sum_{l=1}^L \lambda(i, l) \theta_0 \circ \alpha^{i-1} \circ \theta_0^{-1}(q_l) - v \|_2 < \frac{\epsilon}{2}.$$

Set

$$v_0 := \sum_{i=1}^m \sum_{l=1}^L \lambda(i, l) \theta_0 \circ \alpha^{i-1} \circ \theta_0^{-1}(q_l).$$
Since the action $\beta$ of $\mathbb{Z}$ on $D$ is ergodic, there exists a unitary $w$ of $D \rtimes \beta \mathbb{Z}$ satisfying the following two conditions.

(1) The unitary $w$ normalizes $D$.
(2) We have $w^{i+1} q_l w^{-i} = \theta_0 \circ \alpha^{-1} \circ \theta_0^{-1}(q_l)$ for any $l = 1, \ldots, L, i = 1, \ldots, m+1$.
(3) We have $w^m = 1$.

For $x \in C$, set
\[
\theta(x) := \sum_{i=1}^{m} \theta_0 \circ \alpha^{-i} \circ \theta_0^{-1}(p_1) w^{i+1} \left( \theta_0 \circ \alpha^{-i+1}(x) \right) w^{-i+1}.
\]

Claim 1. The map $\theta$ is a $*$-isomorphism from $C$ onto $D$.

Proof of Claim 1. Notice that the map $\theta$ maps each $\alpha^{-1} \circ \theta_0^{-1}(q_l)$ to $\theta_0 \circ \alpha^{-1} \circ \theta_0^{-1}(q_l)$ because we have
\[
\theta(\alpha^{-1} \circ \theta_0^{-1}(q_l)) = \sum_{i} \theta_0 \circ \alpha^{-i} \circ \theta_0^{-1}(p_1) w^{i+1} \left( \theta_0 \circ \alpha^{-i+1}(\alpha^i \circ \theta_0^{-1}(q_l)) \right) w^{-i+1}
\]
\[
= \sum_{i} \theta_0 \circ \alpha^{-i} \circ \theta_0^{-1}(p_1) w^{i+1} (\theta_0 \circ \alpha^{-i+1}(\theta_0 \circ \alpha^{-1}(q_l))) w^{-i+1}
\]
\[
= \sum_{i} \theta_0 \circ \alpha^{-i} \circ \theta_0^{-1}(p_1) w^{i+1} (w^{i+1} \theta_0 \circ \alpha^{-i+1}(\theta_0 \circ \alpha^{-1}(q_l))) w^{-i+1}
\]
\[
= \sum_{i} \theta_0 \circ \alpha^{-i} \circ \theta_0^{-1}(p_1) \theta_0 \circ \alpha^{-i} \circ \theta_0^{-1}(\theta_0 \circ \alpha^{-1}(q_l))
\]
\[
= \theta_0 \circ \alpha^{-i} \circ \theta_0^{-1}(q_l).
\]

Note that the third equality of the above computation follows from conditions (2) and (3). On the other hand, the map $\text{Ad} w^{-1} \circ \theta_0 \circ \alpha^{-1}$ is a $*$-isomorphism from $C$ onto $D$. Hence so is its restriction to $C_{\alpha^{-1} \circ \theta_0^{-1}(q_l)}$. Thus $\theta$ is a $*$-isomorphism from $C$ onto $D$.

Claim 2. We have $\theta \circ \alpha = \left( \text{Ad} w|_D \right) \circ \theta$.

Proof of Claim 2. For $x \in C$, we have
\[
w \theta(x) w^{-1}
\]
\[
= w \left( \sum_{i=1}^{m} \theta_0 \circ \alpha^{-i} \circ \theta_0^{-1}(p_1) w^{i+1} \left( \theta_0 \circ \alpha^{-i+1}(x) \right) w^{-i+1} \right) w^{-1}
\]
\[
= \sum_{i=1}^{m} \left( w^i p_1 w^{-i} \right) (w^i \theta_0 \circ \alpha^{-i+1}(x)) w^{-i}
\]
\[
= \sum_{i=1}^{m} \theta_0 \circ \alpha^i \circ \theta_0^{-1}(p_1) w^i (\theta_0 \circ \alpha^{-i}(\alpha(x))) w^{-i}
\]
\[
= \theta(\alpha(x)).
\]

Claim 3. We have $\theta(\theta_0^{-1}(v_0)) = v_0$.
Proof of Claim 3. As we have see in the proof of Claim 1, we have \( \theta(\alpha^{i-1} \circ \theta_0^{-1}(q)) = \theta_0 \circ \alpha^{i-1} \circ \theta_0^{-1}(q) \). Since \( v_0 \) is a linear combination of \( \theta_0 \circ \alpha^{i-1} \circ \theta_0^{-1}(q) \), we have \( \theta(\theta_0^{-1}(v_0)) = v_0 \). \( \square \)

By Claim 2, the *-isomorphism \( \theta \) extends to a *-isomorphism form \( C \rtimes \alpha (\mathbb{Z})/(m\mathbb{Z}) \) into \( D \rtimes \beta \mathbb{Z} \) satisfying \( \theta(C) = D, \theta(\lambda_1^q) = w \). By Claim 3, we also have
\[
\|\theta(u) - v\|_2 \leq \|\theta(u - \theta_0^{-1}(v_0))\| + \|v_0 - v\|_2
\]
\[
= \|u - \theta_0^{-1}(v_0)\|_2 + \|v_0 - v\|_2
\]
\[
= \|\theta_0(u) - v_0\|_2 + \|v_0 - v\|_2
\]
\[
= 2\|v_0 - v\|_2
\]
\[
< \epsilon.
\]
\( \square \)

Let \( N \) be the free product of two hyperfinite factor \( R_i \) \( (i = 1, 2) \) of type II \( 1 \) with respect to their traces. For each \( i = 1, 2 \), choose a Haar unitary \( w_i \) of \( R_i \) which generates a Cartan subalgebra of \( R_i \). Set
\[
B := \{w_1 + w_1^{-1} + w_2 + w_2^{-1}\}'' \subset N.
\]
For each non-negative integer \( l \geq 0 \), let \( \chi^B_l \) be the sum of all reduced words of \( \{w_i^{\pm 1}\}_{i=1,2} \) with length \( l \). For each \( i = 1, 2 \), think of \( R_i \) as an increasing union \( \{B_i \rtimes \alpha_i, G_k\}_{k=1}^\infty \) of von Neumann algebras of type I, where \( B_i := \{w_i''\}, \{G_k\}_{k=1}^\infty \) is a decreasing sequence of finite abelian groups and \( \alpha_i \) be a free ergodic probability measure preserving action of \( \bigcup_k G_k \) on \( B_i \). Set
\[
N_k := (B_1 \rtimes \alpha_1 G_k) \ast (B_2 \rtimes \alpha_2 G_k),
\]
which is a von Neumann subalgebra of \( N \).

Proposition 29. Let \( B \subset N \) be as above. For \( j = 1, \cdots, J \), choose \( (x_j^i) \in (N^{\omega} \ominus B^\omega) \cap B' \). Assume that for each \( n \), there exists a positive integer \( k_n \) \( > 0 \) with \( x_j^i \in N_{k_n} \) for all \( j = 1, \cdots, J \).

Then there exists a family of weakly continuous injective *-homomorphisms \( \{\theta_n : N_{k_n} \rightarrow M\}_{n=1}^\infty \) satisfying the following conditions.

1. For each \( i = 1, 2 \), we have \( \theta_n(w_i) \rightarrow u_i \) as \( n \rightarrow \omega \).
2. For any \( n \), any normalizing unitary \( x \in B_i \rtimes \alpha_i G_{k_n} \) of \( B_i \) which is orthogonal to \( B_i \), the unitary \( \theta_n(x) \) normalizes \( \{u_i\}'' \) and is orthogonal to \( \{u_i\}'' \).
3. We have \( (\theta_n(x_j^i)) \in (M^{\omega} \ominus A^\omega) \cap A' \).

Proof. By the previous lemma, there exists a *-homomorphism \( \overline{\theta_k} \) from \( B_i \rtimes \alpha_i G_k \) into \( \{u_i, v_i\}'' \) satisfying \( \overline{\theta_k}(B_i) = \{u_i''\}, \|\overline{\theta_k}(w_i) - u_i\|_2 < 2^{-k} \). Set \( \overline{\theta} := \overline{\theta_k} \ast \overline{\theta_k}' \), which is a *-homomorphism from \( N_k \) into \( M \).

Claim. There exists an increasing sequence \( \{k_n'\} \) of natural numbers with the following conditions.

1. For any \( n \), we have \( k_n' > k_n \).
2. For any \( n \), we have \( \|E_A((\theta_{k_n'}(x_j^i))_n)\|_2 \leq \|x_j^i\|_2/2^n \).
3. For any \( n \), we have \( \|\overline{\theta_{k_n'}}(w_i) - u_i\|_2 < 1/2^n \).
Proof of Claim. For each $n$ and $j$, since $x_n^j$ is orthogonal to $B$, $x_n^j$ can be approximated by a linear sum of vectors of the following forms.

1. The vectors $b_1 - b_2$, where $b_1$ and $b_2$ are words of $w_n^{j+1}$ of the same length.
2. The words of $\lambda_n^j$, $w_n^{j+1}$ for some $i, j = 1, 2$, $g \in (\bigcup_k G_k) \setminus \{0\}$ with at least one $\lambda_n^j$.

If $g_t \neq 0$ for some $t$, then we have

\[ E_A(\overline{\theta}(u_i^1 \lambda_{y_1}^1 \cdots u_i^m \lambda_{y_m}^m)) = 0, \]

which implies that the image of any vector of the second form by $E_A \circ \overline{\theta}$ converges to 0 as $k \to \infty$. Also we have $\overline{\theta}(u_i^1 \cdots w_i^{k_m}) \to u_i^1 \cdots u_i^{k_m}$, which implies that any vector of the first form converges to a vector orthogonal to $A$. Thus we have $E_A(\overline{\theta}(x_n^j)) \to 0$ as $k \to \infty$. We also have $\overline{\theta}(w_i) \to u_i$ as $k \to \infty$. Thus Claim holds.

Set $\theta_n := \overline{\theta}_{k_n}$.

Then the family $\{\theta_n : N_{k_n} \to M\}$ is a family of weakly continuous injective *-homomorphisms satisfying conditions (1) and (2). We show condition (3). Since we have $\|E_A(\theta_n(x_n^j))\| < 1/2^n \to 0$ as $n \to \omega$, we have $\{\theta_n(x_n^j)\} \in M^\omega \cap A^\omega$. Next, we show that $\theta(x)$ commutes with $A$. Take $y \in A$. Since we have $\|\theta_n^{-1}(u_i) - u_i\|_2 = \|u_i - \theta_n(w_i)\|_2 \to 0$ as $n \to \omega$, the sequence $\{\theta_n^{-1}(y)\}$ converges to an operator $z$ of $B$ in the strong operator topology. Thus we have $x_n\theta_n^{-1}(y) - \theta_n^{-1}(y)x_n$ converges to 0 as $n$ tends to $\omega$. Hence $\theta(x)$ commutes with $A$. □

Remark 30. For each $x \in N$, set $\theta(x) := (\theta_n(E_{k_n}(x)))_n \in M^\omega$. Then the map $\theta$ is a weakly continuous injective *-homomorphism from $N$ into $M^\omega$.

Theorem 31. The subalgebra $B$ has the asymptotic orthogonality property.

Proof. Choose $x^1 = (x_n^1)$, $x^2 = (x_n^2) \in (N^\omega \cap B^\omega) \cap B'$ and $y^1, y^2 \in N \cap B$. Let $\theta_n : N_{k_n} \to M$, $\theta : N \to M^\omega$ be injective *-homomorphisms chosen in the previous proposition corresponding to $(x_n^1)$, $(x_n^2)$. Then we have $\theta(x^1), \theta(x^2) \in (M^\omega \cap A^\omega) \cap A^\omega$. Hence by Lemmas 27 and 20 we may assume that the vectors $\theta_n(x_n^1)$ and $\theta_n(x_n^2)$ lie in

\[
\text{span}\{\xi_{m}2M_{n+r,2M_{n+s}} \mid m \geq 0, r, s \geq 0\} \cup \left(\text{span}\{\xi_{k}^{2M_{n+r,2M_{n+s}}} \mid k \in \mathbb{Z}, r, s \geq 0\} \cap L\right),
\]

where $M_n \to \omega$ as $n \to \omega$. On the other hand, when we regard $N$ as $\bigcup_k N_k^{\text{weak}}$, the Hilbert subspace $N \cap B$ is linearly spanned by the following things.

1. The vectors $b_1 - b_2$, where $b_1$ and $b_2$ are words of $w_n^{j+1}$ of the same length.
2. The words of $\lambda_n^j$, $w_n^{j+1}$ for some $i, j = 1, 2$, $g \in (\bigcup_k G_k) \setminus \{0\}$ with at least one $\lambda_n^j$.

Hence we may assume that $y_1$ and $y_2$ are of the above forms. Then by Proposition 20 (1) (2), for each $n$, the vectors $\theta_n(y_1)$ and $\theta_n(y_2)$ satisfy assumptions of Lemmas 27 and 26. Hence by Lemmas 27 and 26 we have $\tau(y_n^*(x_n^1)y_2x_n^2) \to 0$ as $n \to \omega$. □
7. The subalgebra is maximal amenable.

In this section, we show that the subalgebra $B$ is maximal amenable. In order to achieve this, we use Proposition 20. In the previous section, we have already shown that the subalgebra $B$ has the asymptotic orthogonality property. Hence in order to show that the subalgebra is maximal amenable, it is enough to show that it is singular. In order to achieve this, we show the mixing property.

**Definition 32.** (Definition 3.1 of Cameron–Fang–Mukherjee [3]) Let $A$ be a diffuse abelian von Neumann subalgebra of a factor $M$ of type $\text{II}_1$. The subalgebra $A$ is said to be mixing if for any $a, b \in M \otimes A$, any sequence $\{u_n\}$ of unitaries of $A$ which converges to 0 weakly, we have $\|E_A(au_n b)\|_2 \to 0$.

It is known that if a subalgebra is mixing, then it is singular (Proposition 1.1 of Jolissaint–Stalder [10]).

**Lemma 33.** Let $w$ be a reduced word of $\{w_1^{\pm 1}, w_2^{\pm 1}\}$ with $|w| = M > 0$ and $g, h$ be words of the forms

$$g = \lambda_{g_1}^{i_1} w_{i_1}^{k_1} \cdots \lambda_{g_n}^{i_n} w_{i_n}^{k_n},$$

$$h = w_{j_1}^{l_1} \lambda_{h_1}^{l_1} \cdots w_{j_m}^{l_m} \lambda_{h_m}^{l_m},$$

respectively. Suppose that at least one of $g_i$'s is non-zero and that one of $h_i$'s is non-zero. Assume that they satisfy the following conditions.

1. The word $w$ is not $w_i^{\pm M}$.
2. The words $w_i^{k_n}$ and $w$ do not cancel, that is, we have $|w_i^{k_n}w| = |w_i^{k_n}||w|$.
3. The words $w$ and $w_i^{l_1}$ do not cancel.

Then we have $E_B(gw_h) = 0$.

**Proof.** This is shown by the same argument as that of Claim 3 of the proof of Lemma 20.

**Lemma 34.** Let $v = \sum_{p \geq M} \lambda_p w_p / \|w_p\|_2$ be a vector of $L^2 B$ with $\|v\|_2 = 1$, where $w_p$ is the sum of all words of $w_i^{\pm 1}$ with length $p$. Then there exists a positive constant $C_{g,h} > 0$ with $\|E_B(gw_{\ell}h)\|_2 \leq C_{g,h} 3^{M/2}$.

**Proof.** If either $g$ or $h$ is in $W_1^0$, then there is nothing to show (Here, we use the mixing property of the radial MASA implicitly. See Sinclair–Smith [15] Theorems 3.1 and 5.1. Although the statements are slightly different, they essentially show the mixing property of the radial MASA). Write $g$ and $h$ as the following way.

$$g = \lambda_{g_1}^{i_1} w_{i_1}^{k_1} \cdots \lambda_{g_n}^{i_n} w_{i_n}^{k_n},$$

$$h = w_{j_1}^{l_1} \lambda_{h_1}^{l_1} \cdots w_{j_m}^{l_m} \lambda_{h_m}^{l_m}.$$  

Set $|g| := |k_n|, |h| := |l_1|$. By the previous lemma, we have $\|E_B(gw_{\ell}h)\|_2 \leq 4|g||h|$. Thus we have

$$\|E_B(gw_{\ell}h)\|_2 \leq 4|g||h| \left( \sum_{p \geq M} |\lambda_p|^{1/2} \right) \leq 4|g||h| \left( \sum_{p \geq M} \frac{1}{3p} \right)^{1/2} \leq C_{g,h} \frac{1}{3^{M/2}}.$$
Proposition 35. The subalgebra $B$ is mixing in $N$. In particular, it is singular in $N$.

Proof. Notice that if $u^k \to 0$ weakly, then $u^k$ is approximated by operators $v$ of the form $\sum_{p \geq M_k} \lambda_p w_p/\|w_p\|^2$ in the strong operator topology, where $M_k \to \infty$. Hence this is obvious by the previous lemma. \hfill \Box

Theorem 36. The subalgebra $B$ is maximal amenable in $N$.

Proof. By Theorem 31, the subalgebra $B$ has the asymptotic orthogonality property. By Lemma 35, the subalgebra $B$ is a singular abelian von Neumann subalgebra of $N$. Thus by Proposition 20, the subalgebra $B$ is maximal amenable in $N$. \hfill \Box

References

[1] R. Boutonnet and A. Carderi, Maximal amenable von Neumann subalgebras arising from maximal amenable subgroups, Geom. Funct. Anal. 25 (2015) 1688–1705.
[2] R. Boutonnet and C. Houdayer, Amenable absorption in amalgamated free product von Neumann algebras, preprint, [arXiv:1606.00808](http://arxiv.org/abs/1606.00808).
[3] J. Cameron, J. Fang and K. Mukherjee, Mixing subalgebras of finite von Neumann algebras, New York J. Math. 19 (2013), 343–366.
[4] J. Cameron, J. Fang, M. Ravichandran and S. White, The radial masa in a free group factor is maximal injective, J. London Math. Soc. (2) 82 (2010), 787–809.
[5] K. Dykema, Free products of hyperfinite von Neumann algebras and free dimension, Duke Math. J. 69 (1993), 97–119.
[6] L. Ge, On maximal injective subalgebras of factors, Adv. Math. 118 no. 1 (1996), 34–70.
[7] A. Guionnet and D. Shlyakhtenko, Free monotone transport, Invent. Math. 197 no. 3 (2014), 613–661.
[8] C. Houdayer, Gamma stability in free product von Neumann algebras, Comm. Math. Phys. 336 (2015), 831–851.
[9] C. Houdayer and Y. Ueda, Asymptotic structure of free product von Neumann algebras, preprint [arXiv:1503.02490](http://arxiv.org/abs/1503.02490) to appear in Math. Proc. Cambridge Philos. Soc.
[10] P. Jolissaint, Y. Stalder, Strongly singular MASAs and mixing actions in finite von Neumann algebras, Ergodic Theory Dynam. Systems 28 no. 6 (2008), 1861–1878.
[11] N. Ozawa, A remark on amenable von Neumann subalgebras in a tracial free product, Proc. Japan Acad. Ser. A Math. Sci. 91 no. 7 (2015), 104.
[12] S. Popa, Maximal injective subalgebras in factors associated with free groups, Adv. Math. 50 no. 1 (1983), 27–48.
[13] S. Popa, Orthogonal pairs of *subalgebras in finite von Neumann algebras, J. Oper. Theo. 9 no. 2 (1983), 253–268.
[14] R. Radulescu, Singularity of the radial subalgebra of $L(F_N)$ and the Pukanszky invariant. Pacific. J. Math. 151 no. 2 (1991), 297–306.
[15] A. Sinclair and R. Smith, The Laplacian MASA in a free group factor, Trans. Amer. Math. Soc. 355 no. 2 (2003), 465–475.
[16] C. Wen, Maximal amenability and disjointness for the radial masa, J. Funct. Anal. 270 no. 2 (2016), 787–801.

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