Abstract: On directed lattices, with half as many neighbours as in the usual undirected lattices, the Ising model does not seem to show a spontaneous magnetisation, at least for lower dimensions. Instead, the decay time for flipping of the magnetisation follows an Arrhenius law on the square and simple cubic lattice. On directed Barabási-Albert networks with two and seven neighbours selected by each added site, Metropolis and Glauber algorithms give similar results, while for Wolff cluster flipping the magnetisation decays exponentially with time.

Introduction
Sánchez et al some years ago showed that the Metropolis algorithm, applied to directed Watts-Strogatz (small-world) networks has a spontaneous magnetisation, even in the limit of directed random graphs[1]. More recently, Sumour and Shabat [2, 3] investigated Ising models on directed Barabási-Albert networks [4] with the usual Glauber dynamics. No spontaneous magnetisation was found, in contrast to the case of undirected Barabási-Albert networks [5, 7] where a spontaneous magnetisation was found below a critical temperature which increases logarithmically with system size. Now we simulate directed square, cubic and hypercubic lattices in two to five dimensions with heat bath dynamics in order to separate the network effects from the effects of directedness. And we compare different spin flip algorithms, including cluster flips [8], for Ising-Barabási-Albert networks. In all these cases spins were flipped according to algorithms well established to equilibrate systems described by an Ising energy, even though the directed systems are not described by such an energy [1], in contrast to an earlier assertion by one of us [3].
Figure 1: Variation of normalised magnetisation on a square lattice at a temperature which is an order of magnitude lower than the critical temperature of the usual undirected square lattice.

**Lattices**

We start with all spins up and update them by going through the lattice regularly like a typewriter, numbering them consecutively $i = 1, 2, 3 \ldots$. Helical boundary conditions were used by storing the lowest hyperplane in a buffer on top of the lattice; with the upper buffer fixed to +1 the decay times were much longer. The probability for spin $S_i = \pm 1$ to be +1 in this directed lattice is

$$p_i = 1/(1 + \exp(2E_i)/kT), \quad E_i = -JS_i \sum_{k<i} S_k$$

where the inner sum runs over all nearest neighbours $k$ of $i$ with $k < i$, on the hypercubic lattice; in the usual undirected case the restriction $k < i$ is missing. Thus on a square lattice, the spin $S_i$ is influenced only by its left and top neighbours, not by the right and bottom neighbours. This directed influence violates Newton’s law $actio = -reactio$. 

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Figure 2: Time for the magnetisation to become negative, versus inverse temperature, on large (+, lower cloud) and small (x, upper cloud) lattices. We show the median of three runs at each temperature.

For the square lattice, Fig.1 shows an example how the magnetisation jumps from nearly +1 to nearly −1, similar to [3] and different from the usual undirected Ising model. Fig.2 shows the time after which the magnetisation first becomes negative; this time fluctuates strongly but follows roughly an exponential increase as in an Arrhenius law $\propto \exp(\text{const } J/kT)$. For a lattice with hundred times less spins the times are slightly larger, suggesting that a rare nucleation event destroys the metastable state and lets the magnetisation flip. A power law seems to fit less well, Fig.3.

In higher dimensions, Fig.4, the increase of the decay time happens in a much smaller interval of $J/kT$ (hundred times smaller in 3D than in 2D) and perhaps indicates a transition to ferromagnetism. However, closer inspection of three dimensions at the lowest temperatures, Fig.4b, suggests also here an exponential Arrhenius increase instead of a divergence at a positive temperature. Perhaps the same effect happens in four and five dimensions; and
Figure 3: Same data as in Fig.2, shown on a log-log plot with a critical temperature assumed at $J/kT = 4.3$. Note that the curvature persists for both small and large lattices, suggesting that now power law corresponding to a straight line is valid.

at least in infinite dimensions (or perhaps already in four dimensions) the slope of log(flipping time) versus $J/kT$ becomes infinite, meaning a time-independent $T_c$ below which the magnetisation retains its initial sign.

Fig.5 shows a probability distribution function $P(m)$ for the magnetisation $m$ as in usual Ising models: Two wings for $T < T_c$ and one central peak for $T > T_c$. However, in usual large Ising models the magnetization does not flip below $T_c$ while in our directed square or cubic lattices it does. The temperature $T_\tau$ above which this probability distribution gets its two wings decreases logarithmically towards zero with increasing observation time; for $T < T_\tau$ the magnetisation retains its initial sign. ($T_\tau \rightarrow T_c$ for large directed Ising models.)

**Barabási-Albert Networks**

We now return to the directed scale-free networks of Barabási-Albert type
simulated by Glauber kinetics in [2, 3]. We always use half a million spins, with each site added to the network selecting \( m = 2 \) or \( 7 \) already existing sites as neighbours influencing it; the newly added spin does not influence these neighbours.

With \( J/kT = 1.0 \) and 1.7, we confirmed [2, 3] the unusual behaviour of the magnetisation, which for \( m = 7 \) stays close to 1 for a long time inspite of fluctuations. These fluctuations were smaller for Metropolis than for Glauber (heat bath) kinetics, Fig.6. For \( m = 2 \), the magnetisations are much smaller (not shown). With Swendsen-Wang cluster flips, the magnetisation scattered about zero (not shown). Only for Wolff cluster flips, a nice exponential decay
Figure 5: Probability distribution function for the magnetisation (in percent of the saturation magnetisation) for $31 \times 31 \times 31$ using ten million iterations $J/kT = 0.6$ (center) and 0.7 (wings).

towards zero is found in Fig.7.

**Conclusion**

In conclusion we found a freezing in of the magnetisation similar to [2, 3], following an Arrhenius law at least in low dimensions. This lack of a spontaneous magnetisation (in the usual sense) is consistent with the fact that if on a directed lattice a spin $S_j$ influences spin $S_i$, then spin $S_i$ in turn does not influence $S_j$. Thus there is no feedback and this hinders the stabilisation of a spontaneous magnetisation. The system has no well defined energy (in contrast to [3]) and no thermal equilibrium and thus is similar to cellular automata or asymmetric neural networks. Thus, even for the same scale-free networks, different algorithms give different results.

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Figure 6: Barabási-Albert network with half a million spins and $m = 7$ at $kT/J = 1.7$. Part a for Glauber kinetics fluctuates more than part b for Metropolis kinetics.

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Figure 7: Decay of unnormalised magnetisation in Wolff cluster flip algorithm [8] for same network as in Fig.6, $m = 7$ at $kT/J = 1.7$.

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