Measures with infinite Lyapunov exponents for the periodic Lorentz gas

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Abstract

We study invariant measures for the Lorentz gas which are supported on the set of points with infinite Lyapunov exponents. We construct examples of such measures which are measures of maximal entropy and ones which are not.

Keywords: billiard, Lorentz gas, Lyapunov exponent, measure of maximal entropy.

1 Introduction

For a given dynamical system a measure of maximal entropy captures information about the “most chaotic” part of the dynamics. More precisely a measure of maximal entropy is a probability measure invariant under the dynamics whose metric entropy is equal to the topological entropy of the given system. For systems with finite topological entropy much is known about measures of maximal entropy. Axiom A diffeomorphisms have a unique measure of maximal entropy on each topologically transitive component and this measure is Markovian [11]. Recently in [9] it was shown that $C^{1+\alpha}$ diffeomorphisms of compact Riemannian manifolds have at most countably many ergodic measures of maximal entropy and these measures are Markovian. For systems with infinite topological
entropy nothing in this direction has been known up to now, although the main tool used
to derive the above results, Markov partitions, has been developed for some time [4].

In [6] it was shown that the billiard ball map for the periodic Lorentz gas has infinite
topological entropy. In this article we study the set of points with infinite Lyapunov
exponents. Using the cell structure developed in [4, 10] we construct an ergodic invari-
ant probability measure with infinite topological entropy supported on this set. Since
the topological entropy is infinite this is a measure of maximal entropy. From the con-
struction it is clear that there many such measures can coexist on a single component
of topological transitivity. We also construct an ergodic invariant probability measure
with finite entropy which is supported on this set showing that infinite exponents do not
necessarily lead to infinite entropy.

2 Cell structure and Lyapunov exponents

We study a periodic Lorentz gas on a plane. For simplicity, we assume that every
fundamental domain of this gas contains a single round scatterer. Let the fundamental
domain be a unit square and the scatterer be the circle of radius \( r > 0 \) centered at the
origin. Thus we get a periodic array of circles of radius \( r \) centered at sites of the integral
lattice. A pointlike particle moves freely at unit speed between the circles and reflects
elastically off them. The circles are immovable and rigid.

We assume that \( r < 1/2 \), so that the moving particle is not trapped between four
neighboring circles. Then the particle can move freely without collisions indefinitely;
such Lorentz gases are said to have no horizon.

Since the gas is periodic we can consider a fundamental domain: the configura-
tion space \( Q \) of this system is the unit torus \( 0 \leq q_1, q_2 < 1 \) without the disc \( q_1^2 + q_2^2 \leq r^2 \)
(mod 1). The phase space is \( M = Q \times S^1 \). This is a billiard system of Sinai type (with
dispersing boundary). The billiard ball map \( T \) is defined on the surface
\[
M_1 = \{(q, v) \in M : q \in \partial Q \text{ and } (v, n) \geq 0\}
\]
where \( n \) is the inward unit normal vector to the boundary \( \partial Q \) of the domain \( Q \). The map
\( T \) is simply the first return map on the surface \( M_1 \), i.e. it sends the particle at a reflection
point to its next reflection. We introduce the coordinates \((s, \varphi)\) on \( M_1 \), where \( s \) is the arc
length on the circle \( \partial Q \) and \( \varphi \) is the angle between the vector \( v \) and the above normal
vector \( n \) to the circle, \( 0 \leq s < 2\pi r \) and \(-\pi/2 \leq \varphi \leq \pi/2 \). Since \( s \) is a cyclic coordinate,
\( M_1 \) is a cylinder. The map \( T \) on \( M_1 \) preserves the measure \( d\nu = c_\varphi \cos \varphi ds \, d\varphi \), where
\( c_\varphi = (2\pi r)^{-1} \) is the normalizing factor.

Sinai [13] was first to study the properties of the map \( T \) in detail. He proved the
\( T \) is hyperbolic, i.e. has nonzero Lyapunov exponents a.e., and constructed stable and
unstable fibers at a.e. point \( x \in M_1 \). He also developed a proof of ergodicity of \( T \), which
was later improved in [3]. In addition to ergodicity, the mixing and K-property of \( T 
was established in [13, 2] and its Bernoulli property was proved in [8]. Another proof of
ergodicity, which worked for multidimensional Lorentz gas as well, was provided in [14]. Sinai [13] derived a formula for the Kolmogorov entropy of the map $T$, which was later reproved and studied in [7]. In particular, it was shown in [7] that the entropy $h(T)$ has the following asymptotics as $r \to 0$:

$$h(T) = 2 \ln(1/r) + O(1).$$

Markov partitions for the map $T$ were constructed in [4]. Those provide a symbolic representation of $T$ by topological Markov chains with countable alphabet. Based on Markov partitions, it was later shown in [6] that the topological entropy of $T$ is infinite. In particular, the natural ergodic measure $\nu$ on $M_1$ is not a measure of maximal entropy, since its entropy is finite. The statistical properties of the map $T$ were studied in [5]: a stretched exponential bound on the decay of correlations was established and the central limit theorem along with its weak invariance principle was proved. Note that the periodic Lorentz gas with no horizon, apparently, displays a ‘superdiffusive’ behavior, as was conjectured and explained in [1].

We will use the ‘cell structure’ of the surface $M_1$ described in detail in [4, 10]. The map $T$ has a countable number of domains of continuity, which accumulate at a finite number of singular points, at which the time of the first return is infinite. We will call such points supersingular. For example, four points on the circle $\partial Q$ with coordinates $(0, \pm r)$ and $(\pm r, 0)$ and with $\varphi = \pm \pi/2$ are such supersingular points (they produce eight supersingular points in $M_1$). There might be more supersingular points for small radius $r$. The domains of continuity of the map $T$ (we call them cells) form a fairly standard structure in the neighborhood of every supersingular point, independent of $r$ (the structure is the same if the scatterers are not necessarily circles but smooth convex domains on the torus). The structure of cells is shown in Fig. 1. We denote cells $A_n$, $n \geq 2$, where $n$ means that the first return time is about $n$ on the cell $A_n$. Fig. 2 shows which points are included in the cell $A_n$. The sizes of the cells are shown in Fig. 1. Here $O(n^{-a})$ means a value between $c^{-1}n^{-a}$ and $cn^{-a}$ for some $c > 1$. Since there are only a finite number of supersingular points, we can assume that the value of $c$ is the same for all of them.

The inverse map $T^{-1}$ also has a countable number of domains of continuity, which accumulate at the supersingular points. They have a symmetric form shown by dashed lines in Fig. 1. We denote them by $A'_n$, $n \geq 2$, and call ‘inverse cells’. Clearly, any cell $A_n$ is mapped by $T$ onto an inverse cell $A'_n$ with the same value of $n$ but located in the neighborhood of another supersingular point. Fig. 3 shows how $A_n$ is mapped onto $A'_n$ under $T$.

It is shown in [4, 5] that unstable directions for the map $T$ are continuous at every supersingular point and the limit $(ds^u, d\varphi^u)$ of the unstable directions is positive and finite: $0 < d\varphi^u/ds^u < \infty$. Likewise, the limit of the stable directions at every supersingular point is negative and finite: $0 > d\varphi^s/ds^s > -\infty$. Therefore, we have transversality of stable directions and increasing sides of the reverse cell $A'_n$ and transversality of unstable directions in the neighborhood of supersingular point and decreasing (long) sides of the
cells $A_n$. Thus it is clear from Fig. 3 that for all points $x \in A_n$ the one-step expansion in the unstable direction has a factor $O(n^{3/2})$, and one-step contraction in the stable direction has a factor $O(n^{3/2})$.

We will study points $x \in M_1$ such that $T^i x$ belongs to some cell $A_{n_i}$ (near some supersingular point) with large $n_i = n_i(x) > 0$ for every $i \in \mathbb{Z}$. In other words, we will study points whose trajectories stay very close to supersingular points all the time. It is clear that the positive Lyapunov exponent of any such point is

$$\Lambda_+(x) = \lim_{I \to \infty} \frac{3}{2I} \sum_{i=1}^{I} \ln n_i(x) + O(1)$$

$$= \lim_{I \to \infty} \frac{3}{2I} \sum_{i=1}^{I} \ln n_{-i}(x) + O(1) \quad (1)$$

In particular, if both limits in (1) are infinite, then $\Lambda_+(x) = \infty$. Similar formulas hold for the negative Lyapunov exponent, $\Lambda_-(x)$.

In virtue of (1), any point $x \in M_1$ such that $n_i(x) \to \infty$ as $|i| \to \infty$ has infinite Lyapunov exponents. Such points form a Cantor-like nonempty set concentrated in the vicinity of supersingular points. Obviously, this set does not support any finite invariant measure, because its every trajectory is attracted by supersingular points.

However, there are points $x \in M_1$ for which the sequence $\{n_i(x)\}$ has infinity as a limit point (both as $i \to \infty$ and $i \to -\infty$) but that sequence is ‘recurrent’, i.e. for every $n \geq 2$ there are asymptotic frequencies

$$p_n^\pm(x) = \lim_{I \to \infty} \frac{\# \{i \in [1, I] : n_{\pm i}(x) = n \}}{I} \quad (2)$$

and $\sum_n p_n^+(x) = \sum_n p_n^-(x) = 1$. If the sequences $\{p_n^+(x)\}$ and $\{p_n^-(x)\}$ decay slowly enough, then the corresponding point $x$ will have infinite Lyapunov exponent. It is clear that the following condition is sufficient for infinite Lyapunov exponents:

$$\sum_n p_n^\pm(x) \ln n = \infty \quad (3)$$

### 3 Measures with infinite Lyapunov exponents

Here we construct ergodic measures for the map $T$ with finite and infinite Kolmogorov entropy, such that a.e. point $x \in M_1$ with respect to those measures has infinite Lyapunov exponents.

The construction starts with the following observation. There are positive constants $c > 1$ and $n_* > 1$ such that every cell $A_n$ with $n > n_*$ intersects all inverse cells $A'_m$ near the same supersingular point with $c\sqrt{n} \leq m \leq c^2 n^2$ so that both longer sides of $A_n$ cross both longer sides of $A'_m$ (as shown in Fig. 3). This observation is based on Fig. 1 and 3 and was made in \[4\].
Let \( n_0, n_1, n_2, \ldots \) be a sequence of integers such that \( n_i > n_* \) for all \( i \geq 0 \) and 
\[
c\sqrt{n_i} \leq n_{i+1} \leq c^{-2} n_i^2 \quad \text{for all } i \geq 0.
\]
Then there is a sequence of cells \( A_{n_i} \) such that 
\[
A_{n_i} \cap T^{-1} A_{n_{i+1}} \neq \emptyset \quad \text{for all } i \geq 0,
\]
and the intersection 
\[
\bigcap_{i=0}^{\infty} T^{-i} A_{n_i}
\]
is a monotone curve in the cell \( A_{n_0} \) which stretches from its top (short) side to its bottom (short) side.

The same is true for inverse cells: there is a sequence of inverse cells \( A'_{n_i} \) such that the intersection 
\[
\bigcap_{i=0}^{\infty} T^i A'_{n_i}
\]
is a monotone curve stretching from the top short side of \( A'_{n_0} \) to its bottom short side.

Consequently, for any double-infinite sequence of integers \( \{n_i\}, -\infty < i < \infty \), such that 
\( n_i > n_* \) and 
\[
c\sqrt{n_i} \leq n_{i+1} \leq c^{-2} n_i^2 \quad \text{for all } i \in \mathbb{Z} \]
(note that this condition is symmetric: \( c\sqrt{n_i} \leq n_{i-1} \leq c^{-2} n_i^2 \)) there is a sequence of cells \( \{A_{n_i}\} \) such that the intersection 
\[
\bigcap_{i=-\infty}^{\infty} T^{-i} A_{n_i} \tag{4}
\]
is a single point in the cell \( A_{n_0} \).

We now fix an increasing sequence of integers, \( N_0, N_1, N_2, \ldots \), such that 
\( N_{i+1} = \lceil c^{-4} N_i^2 \rceil + 1 \) for all \( i \geq 0 \) and \( N_0 \gg n_* \). It has the following two properties:

(i) for any double-infinite sequence \( \{n_i\}, -\infty < i < \infty \) such that 
\( n_i = N_{s(i)} \) with some \( s(i) \in \mathbb{Z}^+ \) and \( |s(i) - s(i + 1)| \leq 1 \) for all \( i \in \mathbb{Z} \). Note the intersection \( [4] \) is a single point in the cell \( A_{n_0} \);

(ii) for all \( i \geq 0 \)
\[
N_i \geq c^{-4(2^i-1)} N_0^{2^i}
\]
so that if \( N_0 \) is large enough, the sequence \( \{N_i\} \) grows at the following super-exponential rate:
\[
N_i \geq 2^{2^i} \tag{5}
\]

The set of double-infinite sequences \( \{n_i\} \) described by the condition (i) above is a topological Markov chain with a countable number of states, which can be identified with \( N_1, N_2, \ldots \). The allowed transitions from every state \( N_i \) are the ones to \( N_i \) itself and to the two neighboring states, \( N_{i-1} \) and \( N_{i+1} \). The only exception is the first state, \( N_1 \), from which the transitions to itself and to \( N_2 \) are allowed.

We denote the collection of the above double-infinite sequences \( \{n_i\} \) by \( \Omega_1 \). In virtue of the property (i) every sequence \( \omega = \{n_i\} \in \Omega_1 \) corresponds to a point \( x = x(\omega) \in M_1 \) defined by the intersection \( [4] \). The set of points 
\[
\Omega_{1,M} = \{x(\omega) : \omega \in \Omega_1\}
\]
is a closed Cantor-like subset of \( M_1 \) invariant under \( T \), i.e. \( T \Omega_{1,M} = T^{-1} \Omega_{1,M} = \Omega_{1,M} \).

Let \( \mu_1 \) be a Markov measure on the symbolic space \( \Omega_1 \) defined by the following transition probabilities: \( \pi_{i+1,i} = 1/3 \) for all \( i \geq 1 \), \( \pi_{i-1,i} = 2/3 \) for all \( i \geq 2 \) and \( \pi_{1,1} = 2/3 \).
(Here $\pi_{i,j}$ stands for the probability of transition from $N_j$ to $N_i$). This Markov measure is ergodic and mixing, its stationary distribution is $p(N_i) = 1/2^i$ for $i \geq 1$.

The measure $\mu_1$ projected from $\Omega_1$ down to $M$ generates an ergodic measure $\nu_1$ for the map $T$, which is concentrated on $\Omega_{1,M}$. By the ergodic theorem, for $\nu_1$-almost every point $x \in \Omega_{1,M}$ the asymptotic frequencies $p_n^\pm(x)$ defined by (2) exist and are equal to $p_n^\pm(x) = 1/2^i$ if $n = N_i$ for some $i \geq 1$ and zero otherwise. It is then a simple calculation based on (3) and (5) that the Lyapunov exponents are infinite a.e. in $M$ with respect to the ergodic measure $\nu_1$:

$$\sum_n p_n^\pm(x) \ln n = \sum_i p_{N_i}^\pm \ln N_i \geq \sum_i 2^{-i} \ln 2^{2^i} = \infty$$

The measure $\nu_1$ constructed above has a finite entropy. Indeed, it is a Markov measure, and so its entropy, see, e.g., [11], is given by

$$h = - \sum_i p_i \sum_j \pi_{ij} \log \pi_{ij}$$

(6)

where $\pi_{ij}$ are the transition probabilities and $p_i$ is the stationary distribution. In fact, since for every state $i$ only two transition probabilities $\pi_{ij}$ are positive, as defined above, we have $h \leq \log 2$ for the measure $\mu_1$.

The above construction of the Markov measure $\mu_1$ can be modified so that its entropy will be infinite and Lyapunov exponents will be still infinite a.e. We outline the construction below.

We now consider all the double-infinite sequences $\{n_i\}$, $-\infty < i < \infty$, satisfying $n_i > n_*$ and $c\sqrt{n_i} \leq n_{i+1} \leq c^{-2}n_i$ for all $i \in \mathbb{Z}$, as defined above. We denote the set of these sequences by $\Omega_2$. Obviously, $\Omega_2$ is a topological Markov chain with a countable number of states which can be identified with $n_* + 1, n_* + 2, \ldots$. We will number these states by $1, 2, \ldots$ so that the $i$th state is identified with $n_* + i$. Every sequence $\omega = \{n_i\} \in \Omega_2$ corresponds to a point $x = x(\omega) \in M_1$ defined by the intersection (4). The set of points

$$\Omega_{2,M} = \{x(\omega) : \omega \in \Omega_2\}$$

is a closed Cantor-like subset of $M_1$ invariant under $T$, i.e. $T\Omega_{2,M} = T^{-1}\Omega_{2,M} = \Omega_{2,M}$.

We are going to find an ergodic and mixing Markov measure $\mu_2$ on $\Omega_2$ with transition probabilities $\pi_{ij}$ and with stationary distribution $p_i$ satisfying two conditions:

(iii) its entropy given by (6) is infinite;

(iv) one has

$$\sum_i p_i \ln(n_* + i) = \infty$$

so that, by the condition (3), the projection of the measure $\mu_2$ on $M_1$ will have infinite Lyapunov exponents a.e.

The existence of Markov measures satisfying (iii) and (iv) is not based on the dynamics of the Lorentz gas or billiards, and we only sketch a proof. It is clear that (iv) is always
satisfied if the probabilities \( p_i \) decay slowly enough, for example, if \( p_i > \text{const} \cdot (i \ln^2 i)^{-1} \).

Next, we will take care of the condition (iii).

Let a probability distribution, \( ||p_i|| \), be given. We will show how to find a transition matrix, \( \Pi = ||\pi_{ij}|| \), preserving the distribution \( ||p_i|| \), so that for every state \( i \geq 1 \) all the positive transition probabilities \( \pi_{ij} > 0 \) satisfy

\[
c\sqrt{n_* + i} \leq n_* + j \leq c^{-2}(n_* + i)^2
\]

in which case the Markov measure defined by \( ||p_i|| \) and \( ||\pi_{ij}|| \) will be concentrated on \( \Omega_2 \).

For any \( k \geq 1 \) denote

\[
q_k = p_k - p_{k+1} + p_{k+2} - p_{k+3} + \cdots
\]

We assume that the probabilities \( p_i \) decrease monotonically, \( p_1 > p_2 > \cdots \), and that

\[
\frac{1}{2} p_k \leq q_k \leq \frac{3}{4} p_k
\]

for every \( k \geq 1 \). To assure this, it is enough to assume that

\[
\frac{1}{3} \leq \frac{p_{k+2} - p_{k+1}}{p_{k+1} - p_k} \leq 1
\]

for all \( k \geq 1 \), which simply means that \( \{p_k\} \) decays without abrupt drops.

Now, for any \( k \geq 1 \) define a matrix of transition probabilities, \( \Pi(k) = ||\pi_{ij}(k)|| \), by \( \pi_{ii} = 1 \) for all \( 1 \leq i < k \), \( \pi_{kk} = q_k/p_k \), \( \pi_{i,i+1} = q_{i+1}/p_i \) for all \( i \geq k \) and \( \pi_{i,i-1} = q_i/p_i \) for all \( i \geq k + 1 \). It is easy to check that all \( \Pi(k) \) preserve the same distribution \( ||p_i|| \). Now, let \( 1 = k_1 < k_2 < k_3 < \cdots \) be a sequence of numbers such that \( k_m = \lceil \bar{c} m \rceil \) for some sufficiently large \( \bar{c} > 2 \). We then define the matrix of transition probabilities \( \Pi = ||\pi_{ij}|| \)

by

\[
\Pi = \Pi(k_1) \cdot \Pi(k_2) \cdot \Pi(k_3) \cdots
\]

(here every entry \( \pi_{ij} \) requires only a finite number of multiplications). If \( \bar{c} \) and \( n_* \) are large enough, this matrix clearly satisfies (7). Now, due to (8) all the positive entries of the matrices \( \Pi(k), k \geq 1 \), are not smaller than \( 1/4 \). In that case, the entropy of the conditional distribution,

\[
h(i) = -\sum_j \pi_{ij} \log \pi_{ij}
\]

increases to infinity as \( i \to \infty \). Moreover, it is bounded below by some increasing sequence \( h(i) \geq H_i \), \( H_i \to \infty \) as \( i \to \infty \), independent on the stationary distribution \( ||p_i|| \). (The sequence \( H_i \) depends only on the value of \( \bar{c} \) which is only determined by \( c \) and \( n_* \) in (4). Finally, we can find a probability distribution \( ||p_i|| \) satisfying (8) and such that \( \sum_i p_i H_i = \infty \). (Obviously, such a distribution always exists, whatever the increasing sequence \( \{H_i\} \)). Hence the condition (iii) is provided.

The Markov measure \( \mu_2 \) satisfying the conditions (iii) and (iv) projected from \( \Omega_2 \) down to \( M_1 \) induces a measure, \( \nu_2 \), concentrated on \( \Omega_{2,M} \). A.e. point with respect to
\( \nu_2 \) has infinite Lyapunov exponents. This measure has infinite entropy, and so it is a measure of maximal entropy. Note that for every mixing subshift of finite type with finite entropy \([12]\) the measure of maximal entropy is a Markov one, and its transition probabilities are positive for all (topologically) allowed transitions between states. The measure of maximal entropy that we have constructed here is also a Markov one, but some of its transition probabilities are zero even for topologically allowed transitions. This means that we did not use all of the available topological richness of the chain \( \Omega_2 \) defined above. The transitions that we have used were enough to make the entropy of the Markov measure infinite.

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Figure captions.

1. Fig. 1. A supersingular point $S$ and a cell $A_n$ near it.
2. Fig. 2. The outgoing vectors form the cell $A_n$.
3. Fig. 3. The image of $A_n$ under the map $T$ is a reverse cell, $A_n'$. 