BILINEAR FRACTIONAL INTEGRAL OPERATORS ON MORREY SPACES

QIANJUN HE  DUNYAN YAN

Abstract. We prove a plethora of boundedness property of the Adams type for bilinear fractional integral operators of the form

\[ B_\alpha(f, g)(x) = \int_{\mathbb{R}^n} \frac{f(x - y)g(x + y)}{|y|^{n-\alpha}} dy, \quad 0 < \alpha < n. \]

For \( 1 < t \leq s < \infty \), we prove the non-weighted case through the known Adams type result. And we show that these results of Adams type is optimal. For \( 0 < t < s \leq \infty \) and \( 0 < t \leq 1 \), we obtain new result of a weighted theory describing Morrey boundedness of above form operators if two weights \((v, \vec{w})\) satisfy

\[ [v, \vec{w}]^{r, as}_{t, Q/a} = \sup_{Q, Q' \in \mathcal{Q}} \min\left( \frac{|Q|}{|Q'|}, \frac{|Q|}{|Q'|^{\frac{1}{t}}} \right)^{1 - \frac{1}{t}} \left( \int_Q v^{\frac{1}{t}} \right)^{1 - \frac{1}{t}} \left( \int_{Q'} w^{\frac{1}{s}} \right)^{1 - \frac{1}{s}} < \infty, \quad 0 < t \leq 1 < s < \infty \]

and

\[ [v, \vec{w}]^{r, as}_{1/t, Q/a} := \sup_{Q, Q' \in \mathcal{Q}} \left( \frac{|Q|}{|Q'|} \right)^{\frac{1}{r}} \left( \int_Q v^{\frac{1}{t}} \right)^{\frac{1}{r}} \left( \int_{Q'} w^{\frac{1}{s}} \right)^{\frac{1}{s}} < \infty, \quad s \geq 1 \]

where \( \|v\|_{L^{t, \infty}(Q)} = \sup_Q v \) when \( t = 1, a, r, s, t \) and \( q \) satisfy proper conditions. As some applications we formulate a bilinear version of the Olsen inequality, the Fefferman-Stein type dual inequality and the Stein-Weiss inequality on Morrey spaces for fractional integrals.

1. Introduction

In the paper, we will consider the family of bilinear fractional integral operators

\[ B_\alpha(f, g)(x) := \int_{\mathbb{R}^n} \frac{f(x - y)g(x + y)}{|y|^{n-\alpha}} dy, \quad 0 < \alpha < n. \quad (1.1) \]

Such operators have a long history and were studied by Bak [2], Grafakos [5], Grafakos and Kalton [6], Hoang and Moen [9], Kenig and Stein [12], Kuk and Lee[14], Moen [16], among others.

For \( 0 < \alpha < n \), the classical fractional integral \( I_\alpha \) is given by

\[ I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy. \quad (1.2) \]

It is easily to know that \( B_\alpha(f, g) \) and \( I_\alpha f \) have following pointwise control relationship. For any pair of conjugate exponents \( 1/l + 1/l' = 1 \), Hölder’s inequality yields

\[ |B_\alpha(f, g)| \lesssim I_\alpha(|f|^{l})^{1/l} I_\alpha(|g|^{l'})^{1/l'}. \quad (1.3) \]

In [16], Moen initially introduced the fractional integral function \( M_\alpha \), given by

\[ M_\alpha(f, g)(x) = \sup_{d>0} \frac{1}{(2d)^{n-\alpha}} \int_{|y| \leq d} |f(x - y)g(x + y)| dy. \quad (1.4) \]

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A simple computation in [4] shows that for $0 < \alpha < n$,
\[ \mathcal{M}_\alpha(f, g)(x) \leq cB_\alpha(f, g)(x). \]

We first recall some standard notation. For any measurable function $f$ the average of $f$ over a set $E$ is given by
\[ \int_E f dx = \frac{1}{|E|} \int_E f dx. \]
The Euclidean norm of a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is given by $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$. We also use the $l^\infty$ norm $|x| = \max(|x_1|, \ldots, |x_n|)$. Note that $|x|_\infty \leq |x| \leq \sqrt{n} |x|_\infty$ for all $x \in \mathbb{R}^n$.

A cube with center $x_0$ and side length $d$, denoted $Q = Q(x_0, d)$, will be all points $x \in \mathbb{R}^n$ such that $|x - x_0|_\infty \leq \frac{d}{2}$. For an arbitrary cube $Q$, $c_Q$ will be its center and $l(Q)$ its side length, that is, $Q = Q(c_Q, l(Q))$. Given $\lambda > 0$ and a cube $Q$ we let $\lambda Q = Q(c_Q, \lambda l(Q))$. The set of dyadic cubes, denoted $\mathcal{D}$, is all cubes of the form $2^k(m + [0, 1]^n)$ where $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Finally for $k \in \mathbb{Z}$ we let $\mathcal{D}_k$ denote the cubes of level $2^k$, that is, $\mathcal{D}_k = \{ Q \in \mathcal{D} : l(Q) = 2^k \}$.

Morrey spaces, named after Morrey, seem to describe the boundedness property of the classical fractional integral operators $I_\alpha$ more precisely than Lebesgue spaces. We first recall the definition of the Morrey (quasi-)norms [18]. For $0 < q < p < \infty$, the Morrey norm is given by
\[ \| f \|_{\mathcal{M}^p_q} = \sup_{Q \in \mathcal{D}} |Q|^1/p \left( \int_Q |f(x)|^q dx \right)^{1/q}. \] (1.5)
Applying Hölder’s inequality to (1.5), we see that
\[ \| f \|_{\mathcal{M}^p_{q_1}} \geq \| f \|_{\mathcal{M}^p_{q_2}} \quad \text{for all } q_1 \leq q_2 > 0. \] (1.6)
This tells us that
\[ L^p = \mathcal{M}^p_1 \subset \mathcal{M}^p_{q_1} \subset \mathcal{M}^p_{q_2} \quad \text{for all } p \leq q_1 \leq q_2 > 0. \] (1.7)

Remark 1.1. In addition, we know that $L^{p, \infty}$ is contained in $\mathcal{M}^p_1$ with $1 \leq p < \infty$ (see [13, Lemma 1.7]). More precisely, $\| f \|_{\mathcal{M}^p_q} \leq C \| f \|_{L^{p, \infty}}$ with $1 \leq q < p < \infty$, here and in what follows, the letter $C$ will denote a constant, not necessarily the same in different occurrences, and let $p'$ satisfy $1/p + 1/p' = 1$ with $p > 1$.

The following result is due to Adams [1] (see also Chiarenza and Frasca [3]), which turned out sharp [17].

Proposition 1.2. Let $0 < \alpha < n$, $1 < q \leq p < \infty$ and $1 < t \leq s < \infty$. Assume $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{s} = \frac{q}{p}$. Then there exists a constant $C > 0$ such that
\[ \| I_\alpha f \|_{\mathcal{M}^s_t} \leq C \| f \|_{\mathcal{M}^p_q} \]
holds for all measurable functions $f$.

For the case $1 < t \leq s < \infty$, we prove the following theorem under the unweighted setting.
Theorem 1.3. Suppose that the parameters $p_1, q_1, p_2, q_2, s, t$ and $\alpha$ satisfy

$$1 < q_1 \leq p_1 < \infty, \quad 1 < q_2 \leq p_2 < \infty, \quad 1/q_1 + 1/q_2 < 1, \quad 1 < t \leq s < \infty, \quad 0 < \alpha < n.$$ 

Assume that $\frac{1}{s} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}$ and $\frac{1}{t} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{\alpha}{n}$. Then there exists a constant $C > 0$ such that

$$\|B_\alpha(f, g)\|_{\mathcal{M}^t_s} \leq C\|f\|_{\mathcal{M}^{p_1}_{q_1}}\|g\|_{\mathcal{M}^{p_2}_{q_2}}$$

holds for all measurable functions $f$ and $g$.

Applying inequality (1.6), we can say the following result as corollary of Theorem 1.3.

Theorem 1.4. Suppose that the parameters $p_1, q_1, p_2, q_2, s, t$ and $\alpha$ satisfy

$$1 < q_1 \leq p_1 < \infty, \quad 1 < q_2 \leq p_2 < \infty, \quad 1/q_1 + 1/q_2 < 1, \quad 1 < t \leq s < \infty, \quad 0 < \alpha < n.$$ 

Assume that $\frac{1}{s} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}$ and $\frac{1}{t} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{\alpha}{n}$. Then there exists a constant $C > 0$ such that

$$\|B_\alpha(f, g)\|_{\mathcal{M}^t_s} \leq C\|f\|_{\mathcal{M}^{p_1}_{q_1}}\|g\|_{\mathcal{M}^{p_2}_{q_2}}$$

holds for all measurable functions $f$ and $g$.

However, this is not the end of the story; we can prove even more. Here we present our full statement of the main theorem. In specially case Theorem 1.3 can be extended to a large extent.

Theorem 1.5. Suppose that $0 < \alpha < n, 1 < q_1 \leq p_1 = n/\alpha, 1 < q_2 \leq p_2 = q_2n/\alpha$ and $1/q_1 + 1/q_2 < 1$. Then there exists a constant $C > 0$ such that

$$\|B_\alpha(f, g)\|_{\mathcal{M}^{p_2}_{q_2}} \leq C\|f\|_{\mathcal{M}^{p_1}_{q_1}}\|g\|_{\mathcal{M}^{p_2}_{q_2}}$$

holds for all positive measurable functions $f$ and $g$.

2. The proofs of Theorems 1.3 - 1.5

Proof of Theorem 1.3. We take parameters

$$1 < u_1, v_1 < \infty, \quad 1 < u_2, v_2 < \infty, \quad 1 < l < q_1, \quad 1 < l' < q_2$$

such that

$$\frac{1}{u_1} = \frac{1}{p_1} - \frac{1}{l/n}, \quad \frac{1}{u_2} = \frac{1}{p_2} - \frac{1}{l'/n}, \quad \frac{v_1}{u_1} = \frac{v_2}{u_2} = \frac{t}{s}.$$

Since

$$1 < l < q_1, \quad 1 < l' < q_2 \quad \text{and} \quad \frac{1}{q_1} + \frac{1}{q_2} < 1$$

there exists a pair of conjugate of exponents $1/l + 1/l' = 1$.

Notice that

$$\frac{1}{u_1} + \frac{1}{u_2} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} = \frac{1}{s}.$$

It follows from this, $\frac{v_1}{u_1} = \frac{v_2}{u_2} = \frac{t}{s}$ and Hölder’s inequality that

$$\|h_1 \cdot h_2\|_{\mathcal{M}^t_s} \leq \|h_1\|_{\mathcal{M}^{p_1}_{q_1}}\|h_2\|_{\mathcal{M}^{p_2}_{q_2}}. \quad (2.1)$$
Thus, if we insert about pointwise estimate for $B_\alpha (1.3)$, use the Adams original result of Proposition 1.2 and inequality (2.1), we have
\[
\| B_\alpha (f, g) \|_{\mathcal{M}_t^1} \leq \| I_\alpha (|f|^l)^{1/l} \|_{\mathcal{M}_{t_1}^{u_1}} \| I_\alpha (|g|^{l'})^{1/l'} \|_{\mathcal{M}_{t_2}^{u_2}}.
\] (2.2)
By the condition \( \frac{1}{u_1} = \frac{1}{p_1} - \frac{\alpha}{n} \), we obtain
\[
\frac{1}{u_1/l} = \frac{1}{p_1/l} - \frac{\alpha}{n}.
\] (2.3)
Meanwhile, observing that
\[
\frac{v_1/l}{u_1/l} = \frac{v_1}{u_1} = \frac{q_1}{p_1} = \frac{q_1/l}{p_1/l} \quad \text{and} \quad \| I_\alpha (|f|^l)^{1/l} \|_{\mathcal{M}_{t_1}^{u_1}} = \| I_\alpha (|f|^l)^{1/l} \|_{\mathcal{M}_{t_1}^{u_1/l}}.
\] (2.4)
Therefore, by equations (2.3) and (2.4) we conclude that
\[
\| I_\alpha (|f|^l)^{1/l} \|_{\mathcal{M}_{t_1}^{u_1}} \lesssim \| f \|_{\mathcal{M}_{t_1}^{p_{1,l}}}.
\] (2.5)
Similary, we imply that
\[
\| I_\alpha (|g|^{l'})^{1/l'} \|_{\mathcal{M}_{t_2}^{u_2}} \lesssim \| g \|_{\mathcal{M}_{t_2}^{p_{2,l}}}.
\] (2.6)
Combining (2.2), (2.5) and (2.6), we get the following estimate
\[
\| B_\alpha (f, g) \|_{\mathcal{M}_t^1} \leq C \| f \|_{\mathcal{M}_{t_1}^{p_{1,l}}} \| g \|_{\mathcal{M}_{t_2}^{p_{2,l}}}.
\]
This completes the proof of Theorem 1.3. \( \square \)

**Proof of Theorem 1.4.** Let \( s, t_1, p_1, q_1, p_1 \) and \( q_1 \) as in Theorem 1.3, then
\[
\frac{t_1}{s} = \frac{q_1}{p_1} = \frac{q_2}{p_2} \quad \text{and} \quad \frac{1}{s} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}.
\]
It follows that
\[
\frac{1}{t_1} = \frac{p_1}{q_1} \frac{1}{s} = \frac{p_1}{q_1} \left( \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} \right) = \frac{1}{q_1} + \frac{p_1}{q_1 p_2} - \frac{p_1}{q_1} \frac{\alpha}{n} \leq \frac{1}{q_1} + \frac{1}{q_1} \frac{\alpha}{n} = \frac{1}{t},
\]
that is equivalent to
\[
t \leq t_1.
\]
Therefore, by Theorem 1.3 and the relation (1.6) with \( 1 \leq t \leq t_1 \), we obtain
\[
\| B_\alpha (f, g) \|_{\mathcal{M}_t^1} \leq \| B_\alpha (f, g) \|_{\mathcal{M}_{t_1}^{p_{1,l}}} \leq C \| f \|_{\mathcal{M}_{t_1}^{p_{1,l}}} \| g \|_{\mathcal{M}_{t_2}^{p_{2,l}}}.
\]
This finishes the proof of Theorem 1.4. \( \square \)

We invoke a bilinear estimate from [19].

**Proposition 2.1.** Let \( 0 < \alpha < n, 1 < p \leq p_0 < \infty, 1 < q \leq q_0 < \infty \) and \( 1 < r \leq r_0 < \infty \). Assume that
\[
q > r, \quad \frac{1}{p_0} > \frac{\alpha}{n}, \quad \frac{1}{q_0} \leq \frac{\alpha}{n},
\]
and
\[
\frac{1}{r_0} = \frac{1}{p_0} + \frac{1}{q_0} - \frac{\alpha}{n}, \quad \frac{r}{r_0} = \frac{p}{p_0}.
\]
Then
\[
\| g \cdot I_\alpha f \|_{\mathcal{M}_{r_0}^{p_0}} \leq C \| g \|_{\mathcal{M}_{q_0}^{p_0}} \| f \|_{\mathcal{M}_{r_0}^{p_0}},
\]
where the constant \( C \) is independent of \( f \) and \( g \).
We now prove Theorem 1.5.

**Proof of Theorem 1.5.** We first claim that we can choose parameters $1 < v \leq u < \infty$ and a pair of conjugate of exponents $l, l' > 1$ such that

$$p_1 = \frac{n}{\alpha} - \frac{\ln}{\alpha}, \quad p_2 < \frac{l'n}{\alpha}, \quad v > q_2, \quad \frac{v}{u} = \frac{q_1}{p_1}, \quad \frac{1}{u} = \frac{1}{p_1} - \frac{\alpha}{ln} = \frac{\alpha}{l'n}. \tag{2.7}$$

This is possible by assumption. In fact, let us choose $1 < v \leq u < \infty$ and $l, l' > 1$ such that

$$\frac{v}{u} = \frac{q_1}{p_1}, \quad \frac{1}{u} = \frac{1}{p_1} - \frac{\alpha}{ln}.$$

Then we have

$$\frac{1}{u} = \frac{1}{p_1} - \frac{\alpha}{ln} = \frac{\alpha}{l'n} = \frac{\alpha}{l'n} \frac{l'n}{\alpha} = \alpha - \frac{\alpha}{ln} = \alpha - \frac{\alpha}{l'n} = \alpha - \frac{\alpha}{l'n} v = \frac{q_1}{p_1} \frac{l'n}{\alpha}.$$

Therefore, if we choose $l, l'$ satisfy

$$1 < l < q_1, \quad \max(1, \frac{q_2}{q_1}) < l' < q_2 \quad \text{and} \quad l' \to q_2,$$

Then we have

$$v > q_2, \quad \frac{ln}{\alpha} < p_2 < \frac{l'n}{\alpha}.$$

Consequently, we could justify the claim that we can choose the parameters $1 < v \leq u < \infty$ and $l, l' > 1$ so that they satisfy (2.7).

By inequality (2.2) and recur to Proposition (2.1) with

$$v > q_2, \quad p_2 < \frac{l'n}{\alpha}, \quad u = \frac{l'n}{\alpha}, \quad \frac{1}{p_2} = \frac{1}{u} + \frac{\alpha}{l'n} \quad \text{and} \quad \frac{q_2}{p_2} = \frac{q_2}{p_2},$$

we have

$$\|B_\alpha(f, g)\|_{\mathcal{M}_{q_2}^{p_2}} \leq \|I_\alpha(|f|^{1/l})|l'\|_{\mathcal{M}_{q_2}^{p_2}} \leq \|I_\alpha(|f|^{1/l})|l'\|_{\mathcal{M}_{q_2}^{p_2}} = \|I_\alpha(|f|^{1/l})|l'\|_{\mathcal{M}_{q_2}^{p_2}}.$$

Since

$$\frac{v}{l'} > \frac{q_2}{l'} > 1, \quad 1 < \frac{p_2}{l'} < \frac{ln}{\alpha}, \quad \frac{u}{l'} = \frac{ln}{\alpha}, \quad \frac{1}{p_2/l'} = \frac{1}{u/l'} + \frac{1}{p_2/l'} - \frac{\alpha}{n} \quad \text{and} \quad \frac{q_2}{p_2/l'} = \frac{q_2}{p_2/l'},$$

then we have

$$\|B_\alpha(f, g)\|_{\mathcal{M}_{q_2}^{p_2}} \leq C\|I_\alpha(|f|^{1/l})|l'\|_{\mathcal{M}_{u/l'}} \|g\|_{\mathcal{M}_{q_2}^{p_2}} = C\|I_\alpha(|f|^{1/l})|l'\|_{\mathcal{M}_{u/l'}} \|g\|_{\mathcal{M}_{q_2}^{p_2}}.$$

Meanwhile, notice that

$$1 < l < q_1, \quad u = p_1l', \quad v = q_1l', \quad \frac{v/l}{u/l} = \frac{q_1/l}{p_1/l} \quad \text{and} \quad \frac{1}{u/l} = \frac{1}{p_1/l} - \alpha.$$

Hence, by Proposition 1.2, we obtain

$$\|B_\alpha(f, g)\|_{\mathcal{M}_{q_2}^{p_2}} \leq C\|f\|_{\mathcal{M}_{p_1}^{q_1}} \|g\|_{\mathcal{M}_{q_2}^{p_2}} = C\|f\|_{\mathcal{M}_{q_1}^{p_1}} \|g\|_{\mathcal{M}_{q_2}^{p_2}},$$

which gives us the desired result. \qed

**Corollary 2.2.** Let $\alpha, p_i, q_i$ be as in Theorem 1.5 and $p_i \neq q_i$ with $i = 1, 2$. Then

$$\|B_\alpha(f, g)\|_{\mathcal{M}_{q_2}^{p_2}} \leq C\|f\|_{L^{p_1}} \|g\|_{L^{p_2}}.$$
3. Sharpness of the Results

In this section we prove that
\[
\|B_\alpha(f, g)\|_{M^t} \leq C\|f\|_{M^p_1}^r\|g\|_{M^q_2}^s
\]
holds only when \(t \leq \max\left(\frac{q_1}{p_1}, \frac{q_2}{p_2}\right)\).

**Theorem 3.1.** Let \(0 < \alpha < n\), \(0 < t \leq s < \infty\) and \(0 < q_j \leq p_j < \infty\) for \(j = 1, 2\). Suppose that
\[
1 = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} \quad \text{and} \quad \frac{t}{s} > \max\left(\frac{q_1}{p_1}, \frac{q_2}{p_2}\right).
\]
Then there exists no constants \(C > 0\) such that
\[
\|B_\alpha(f, g)\|_{M^t} \leq C\|f\|_{M^p_1}^r\|g\|_{M^q_2}^s < \infty.
\]

**Proof.** We proof of this theorem based on following the equivalent definition of Morrey norm
\[
\|f\|_{M^p_1}^r \approx \sup_{Q \in \mathcal{Q}} |Q|^\frac{1}{p_1} \left(\int_Q |f(y)|^p dy\right)^\frac{1}{q_1}, \quad (3.1)
\]
where \(\mathcal{Q}\) denotes the family of all open cubes in \(\mathbb{R}^n\) with sides parallel to the coordinate axes. Without loss of generality, we may assume that
\[
1 \leq \frac{t}{s} > \max\left(\frac{q_1}{p_1}, \frac{q_2}{p_2}\right) = \frac{q_1}{p_1}
\]
and that \(q_1 < p_1\). If \(1 < \frac{t}{s}\), then the Morrey norm of a measurable function \(f\) is infinite unless \(f\) vanishes almost everywhere.

Fix a positive small number \(\delta \ll 1\) and \(N = \left[\frac{q_1}{p_1} \delta^{-1}\right]\) be a large integer. We let the set of lattice points
\[
J := (0, 1)^n \cap \frac{1}{N}\mathbb{Z}^n.
\]
For each point \(j \in J\), we place a small cube \(Q_j\) centered at \(j\) with the side length \(\delta\) and set
\[
E := \bigcup_{j \in J} Q_j.
\]
Then we have
\[
|E| = N^n \delta^n \approx \delta^{\frac{nq_1}{p_1}}.
\]
Set
\[
f(x) := \delta^{-\frac{n}{p_1}} \chi_{3Q_j}(x) \quad \text{and} \quad g(x) := \delta^{-\frac{n}{p_2}} \chi_{3Q_j}(x),
\]
where \(3Q_j\) denotes the triple of \(Q_j\).

Then a simple arithmetic calculation, we claim
\[
\|f\|_{M^p_1} \leq C \quad \text{and} \quad \|g\|_{M^q_2} \leq C. \quad (3.2)
\]
In fact, we use the equivalent definition of Morrey norm \((3.1)\). When \(l(Q) \leq 3\delta\), we know that
\[
\|f\|_{M^p_1} = \sup_{Q \in \mathcal{Q}, l(Q) \leq 3\delta} |Q|^{\frac{1}{p_1} - \frac{n}{p_1}} \delta^{-\frac{n}{p_1}} \left(\int_{Q \cap 3Q_j} dy\right)^\frac{1}{q_1} \leq \sup_{Q \in \mathcal{Q}, l(Q) \leq 3\delta} |Q|^{\frac{1}{p_1}} \delta^{-\frac{n}{p_1}} \leq 3^{\frac{n}{p_1}}.
\]
When \( l(Q) > 3\delta \), we show that
\[
\|f\|_{\mathcal{M}_{\mathcal{L}}^{1}} = \sup_{Q \in \mathcal{Q}, l(Q) > 3\delta} |Q|^{\frac{1}{m} - \frac{1}{p} - \frac{m}{n}} \left( \int_{3Q} dy \right)^{\frac{1}{n}} \leq \sup_{Q \in \mathcal{Q}, l(Q) > 3\delta} 3^{\frac{m}{n}} |Q|^{\frac{1}{m} - \frac{1}{p} - \frac{m}{n}} \leq 3^{\frac{m}{n}}.
\]
Similar to computing \( \|g\|_{\mathcal{M}_{\mathcal{L}}^{p}} \), we obtain the claim.

Now, for \( x \in Q_{j} \),
\[
B_{\alpha}(f, g)(x) \approx \int_{g \in \mathbb{R}^{n}} \frac{f(x - y)g(x + y)}{|y|^{\alpha}} dy \geq \delta^{-\alpha} \int_{|y| \leq \delta} f(x - y)g(x + y) dy = \delta^{\alpha} \delta^{-\frac{m}{n} - \frac{n}{p}} = \delta^{-\frac{n}{q}},
\]
where the above estimate is based on the facts: when \( x \in Q \) and \( |y|_{\infty} \leq l(Q) \), then \((x - y, x + y) \in 3Q \times 3Q \). This tells us that
\[
\int_{(0,1)^{n}} B_{\alpha}(f, g)(x)^{t} dt \geq \int_{E} B_{\alpha}(f, g)(x)^{t} dt \leq CN_{n} \delta^{n} \delta^{-\frac{m}{n}} \approx C(\delta^{n})^{\frac{m}{n} - \frac{1}{q}}.
\]
This implies that
\[
\|B_{\alpha}(f, g)\|_{\mathcal{M}_{t}^{q}} \geq \left( \int_{(0,1)^{n}} B_{\alpha}(f, g)(x)^{t} dt \right)^{\frac{1}{t}} \geq C \left[ (\delta^{n})^{\frac{m}{n} - \frac{1}{q}} \right]^{\frac{1}{t}}.
\]
Taking \( \delta \) small enough, we have the desired result by \( \frac{q}{s} > \frac{m}{p} \) and (3.2).

4. The two-weight case for bilinear fractional integral operators

Our results are new and provide the first non-trivial weighted estimates for \( B_{\alpha} \) on Morrey spaces and the only known weighted estimates for \( B_{\alpha} \) on Morrey spaces \( \mathcal{M}_{t}^{q} \) when \( 0 < t \leq 1 \). The estimates we obtain parallel earlier results by Iida, Sato, Sawano and Tanaka [10] for the less singular bilinear fractional integral operator
\[
I_{\alpha}(f, g)(x) = \int_{\mathbb{R}^{n}} \frac{f(y)g(z)}{|y|^{\alpha} + |z|^{\alpha}} dydz.
\]
We first introduce two weight estimates for classical fractional integral operators on Morrey spaces, the following result is due to Iida, Sato, Sawano and Tanaka [10].

**Proposition 4.1.** Let \( v \) be a weight on \( \mathbb{R}^{n} \) and \( \tilde{w} = (w_{1}, w_{2}) \) be a collection of two weights on \( \mathbb{R}^{n} \).

Assume that
\[
0 < \alpha < 2n, \quad \tilde{q} = (q_{1}, q_{2}), \quad 1 < q_{1}, q_{2} < \infty, \quad 0 < q \leq p < \infty, \quad 0 < t < s < r \leq \infty
\]
and \( 1 < a < \min(r/s, q_{1}, q_{2}) \). Here, \( q \) is given by \( 1/q = 1/q_{1} + 1/q_{2} \). Suppose that
\[
\frac{1}{s} = \frac{1}{p} + \frac{1}{r} - \frac{\alpha}{n}, \quad \frac{t}{s} = \frac{q}{p}, \quad 0 < t \leq 1
\]
and the weights \( v \) and \( \tilde{w} \) satisfy the following condition:
\[
[v, \tilde{w}]_{t, q/a}^{r, as} := \sup_{Q, Q' \in \mathcal{Q}, Q' \subset Q} \left( \frac{|Q|}{|Q'|} \right)^{\frac{1}{q/a}} |Q'|^{\frac{1}{2}} \left( \frac{\int_{Q} v^{t}}{\int_{Q} v^{t}} \right)^{\frac{1}{2}} \prod_{i=1}^{2} \left( \frac{\int_{Q'} w_{i}^{-q_{1}/a} w_{i}^{q_{1}/a}}{w_{i}^{-q_{1}/a} w_{i}^{q_{1}/a}} \right)^{\frac{1}{q_{1}/a}} < \infty. \quad (4.1)
\]

Then we have
\[
\|I_{\alpha}(f, g)v\|_{\mathcal{M}_{t}^{q}} \leq C[v, \tilde{w}]_{t, q/a}^{r, as} \sup_{Q \in \mathcal{Q}} |Q|^{1/p} \left( \int_{Q} (|f|w_{1})^{q_{1}} \right)^{1/q_{1}} \left( \int_{Q} (|g|w_{2})^{q_{2}} \right)^{1/q_{2}}.
\]
For the case $0 < t \leq 1$, we have two weight inequalities of bilinear fractional integral operators.

**Theorem 4.2.** Let $v$ be a weight on $\mathbb{R}^n$ and $\tilde{w} = (w_1, w_2)$ be a collection of two weights on $\mathbb{R}^n$. Assume that

$$0 < \alpha < n, \ 1 < q_1, q_2 < \infty, \ 0 < q \leq p < \infty, \ 0 < t \leq s < \infty \ \text{and} \ \ 0 < r \leq \infty.$$ 

Here, $q$ is given by $1/q = 1/q_1 + 1/q_2$. Suppose that

$$\frac{\alpha}{n} > \frac{1}{r}, \ \frac{1}{s} = \frac{1}{p} + \frac{1}{r} - \frac{\alpha}{n}, \ \frac{t}{s} = \frac{q}{r}, \ \ 0 < t \leq 1$$

and the weights $v$ and $\tilde{w}$ satisfy the following two conditions:

(i) If $0 < s < 1$, $\frac{1}{s} < r$ and $1 < a < \min(r(1-s)/s, q_1, q_2)$

$$[v, \tilde{w}]_{t,q/a}^{r,as} : = \sup_{Q \subset \mathbb{R}^n} \left( \frac{|Q|}{|Q|'} \right)^{1-as} |Q'|^{\frac{1}{r}} \left( \int_Q v^{t-1} \right)^{\frac{1}{t}} \prod_{i=1}^{2} \left( \int_{Q_i} w_i^{-(q_i/a)^r} \right)^{1/(q_i/a)^r} < \infty \quad (4.2)$$

(ii) If $s \geq 1$ and $1 < a < \min(q_1, q_2)$

$$[v, \tilde{w}]_{t,q/a}^{r,as} : = \sup_{Q \subset \mathbb{R}^n} \left( \frac{|Q|}{|Q|'} \right)^{1-as} |Q'|^{\frac{1}{r}} \left( \int_Q v^{t-1} \right)^{\frac{1}{t}} \prod_{i=1}^{2} \left( \int_{Q_i} w_i^{-(q_i/a)^r} \right)^{1/(q_i/a)^r} < \infty, \quad (4.3)$$

where $\left( \int_Q v^{t-1} \right)^{\frac{1}{t}} = \|v\|_{L^\infty(Q)}$ when $t = 1$. Then we have

$$\|B_\alpha(f, g)v\|_{\mathcal{M}_1^t} \leq C[v, \tilde{w}]_{t,q/a}^{r,as} \sup_{Q \subset \mathbb{R}^n} |Q|^{1/p} \left( \int_Q (|f|w_1)^{q_1} \right)^{1/q_1} \left( \int_Q (|g|w_2)^{q_2} \right)^{1/q_2}.$$

**Remark 4.3.** Inequality (4.2) holds if $v$ and $\tilde{w}$ satisfy

$$[v, \tilde{w}]_{as,q/a}^{r} : = \sup_{Q \subset \mathbb{R}^n} |Q|^{\frac{1}{r}} \left( \int_Q v^{t-1} \right)^{\frac{1}{t}} \prod_{i=1}^{2} \left( \int_{Q_i} w_i^{-(q_i/a)^r} \right)^{1/(q_i/a)^r} < \infty. \quad (4.4)$$

Indeed, for any cubes $Q \subset Q'$, it immediately follows that $0 < t \leq s < 1$. Since

$$0 < t \leq s \leq 1 \implies \frac{1-s}{as} \leq \frac{1-t}{t} \implies \frac{t}{1-t} \leq \frac{as}{1-s},$$

then by using Hölder’s inequality we have

$$\left( \frac{|Q|}{|Q|'} \right)^{\frac{1-as}{as}} |Q'|^{\frac{1}{r}} \left( \int_Q v^{t-1} \right)^{\frac{1}{t}} \prod_{i=1}^{2} \left( \int_{Q_i} w_i^{-(q_i/a)^r} \right)^{1/(q_i/a)^r} \leq \left( \frac{|Q|}{|Q|'} \right)^{\frac{1-as}{as}} |Q'|^{\frac{1}{r}} \left( \int_Q v^{\frac{as}{1-as}} \right)^{\frac{1}{\frac{as}{1-as}}} \prod_{i=1}^{2} \left( \int_{Q_i} w_i^{-(q_i/a)^r} \right)^{1/(q_i/a)^r} \leq |Q'|^{\frac{1}{r}} \left( \int_{Q_i} v^{\frac{as}{1-as}} \right)^{\frac{1}{\frac{as}{1-as}}} \prod_{i=1}^{2} \left( \int_{Q_i} w_i^{-(q_i/a)^r} \right)^{1/(q_i/a)^r} \leq [v, \tilde{w}]_{as,q/a}^{r} < \infty.$$

Thus, when $s = t$, $p = q$, Theorem 4.2 recovers the two-weight results due to Moen [16].

The following is the Olsen inequality for bilinear fractional operators, which can see more in the papers [7, 8, 17].
Corollary 4.4. Let \( v \) be a weight on \( \mathbb{R}^n \) and assume that
\[
0 < \alpha < n, \quad 1 < q_1, q_2 < \infty, \quad 0 < q \leq p < \infty, \quad 0 < t \leq s < 1, \quad \frac{s}{1-s} < r \leq \infty \quad \text{and} \quad a > 1,
\]
where \( q \) is given by \( 1/q = 1/q_1 + 1/q_2 \). Suppose that
\[
\frac{\alpha}{n} > \frac{1}{r}, \quad \frac{1}{s} = \frac{1}{p} + \frac{1}{r} - \frac{\alpha}{n} \quad \text{and} \quad \frac{t}{s} = \frac{q}{p}.
\]
Then we have
\[
\|B_{\alpha}(f,g)v\|_{\mathcal{M}_t^s} \leq C\|v\|_{\mathcal{M}^{rt}_s} \sup_{Q \in \mathcal{D}} |Q|^{1/p} \left( \int_Q |f|^{q_1} \right)^{1/q_1} \left( \int_Q |g|^{q_2} \right)^{1/q_2}.
\]

Proof. This follows from Theorem 4.2 by letting \( w_1 = w_2 = 1 \) and noticing that, for every \( Q \subset Q' \),
\[
\left( \frac{|Q|}{|Q'|} \right)^{\frac{1-s}{as}} |Q'|^\frac{1}{r} = |Q|^{\frac{1-s}{as}}|Q'|^{\frac{1}{r} - \frac{1-s}{as}} \leq |Q|^\frac{1}{r}, \quad (4.5)
\]
The inequality (4.5) can be deduced from the facts that \( \frac{1}{r} - \frac{1-s}{as} < 0 \), which follow from \( \frac{s}{1-s} < r \).

The following is the Fefferman-Stein type dual inequality for bilinear fractional integral operators on Morrey spaces.

Corollary 4.5. Assume that the parameters \( 0 < s_i < 1 \) and \( \frac{s}{1-s_i} < r \leq \infty \) with \( i = 1, 2 \), satisfy
\[
\frac{1-s}{as} = \frac{1-s_1}{s_1} + \frac{1-s_2}{s_2} \quad \text{and} \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}.
\]
Then, for any collection of two weights \( w_1 \) and \( w_2 \), we have
\[
\|B_{\alpha}(f,g)w_1 w_2\|_{\mathcal{M}_t^s} \leq C \sup_{Q \in \mathcal{D}} |Q|^{1/p} \left( \int_Q |f||W_1|^{q_1} \right)^{1/q_1} \left( \int_Q |g||W_2|^{q_2} \right)^{1/q_2},
\]
where
\[
W_i(x) = \sup_{Q \in \mathcal{D}} |Q|^{1/r_i} \left( \int_Q w_i^{\frac{s_i}{1-s_i}} \right)^{\frac{1-s_i}{s_i}} \quad \text{for} \quad i = 1, 2.
\]

Proof. We need only the inequality (4.4) with \( v = w_1 w_2 \) and \( w_i = W_i \) with \( i = 1, 2 \). It follows from Hölder’s inequality that
\[
Q^{\frac{1}{r}} \left( \int_Q (w_1 w_2)^{\frac{as}{1-as}} \right)^{\frac{1-as}{as}} \leq Q^{\frac{1}{r}} \prod_{i=1}^2 \left( \int_Q w_i^{\frac{s_i}{1-s_i}} \right)^{\frac{1-s_i}{s_i}} = Q^{\frac{1}{r}} \prod_{i=1}^2 |Q|^{1/r_i} \left( \int_Q w_i^{\frac{s_i}{1-s_i}} \right)^{\frac{1-s_i}{s_i}}.
\]
Corollary 4.5 follows immediately from the inequality
\[
W_i(x) \geq |Q|^{1/r_i} \left( \int_Q w_i^{\frac{s_i}{1-s_i}} \right)^{\frac{1-s_i}{s_i}} \quad \text{for all} \quad x \in Q.
\]

For one weight inequality we take \( r = \infty \) and \( v = w_1 w_2 \) to arrive at the following theorem.

Theorem 4.6. Let \( \vec{w} = (w_1, w_2) \) be a collection of two weights on \( \mathbb{R}^n \) and assume that
\[
0 < \alpha < n, \quad \vec{q} = (q_1, q_2), \quad 1 < q_1, q_2 < \infty, \quad 0 < q \leq p < \infty, \quad 0 < t \leq s < \infty \quad \text{and} \quad a > 1,
\]
where \( q \) denotes the number determined by the Hölder relationship \( 1/q = 1/q_1 + 1/q_2 \). Suppose that
\[
\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{t}{s} = \frac{q}{p}, \quad 0 < t \leq 1
\]
and the weights \( \vec{w} \) satisfy the following two conditions:

(i) If \( 0 < s < 1 \),
\[
[w^s_t]_{t,q} := \sup_{Q \subset Q'} \left( \frac{|Q|}{|Q'|} \right)^{\frac{1-s}{s}} \left( \frac{\int_Q |f_1(x)w_1(x)w_2(x)|^t}{\int_Q |w_1(x)|^t} \right)^{\frac{1}{t}} \prod_{i=1}^{2} \left( \frac{\int_{Q'} w_{-q_i}^t}{\int_{Q'} w_{-q_i}^t} \right)^{\frac{1}{t}} < \infty
\]  
(4.6)

(ii) If \( s \geq 1 \),
\[
[w^s_t]_{t,q} := \sup_{Q \subset Q'} \left( \frac{|Q|}{|Q'|} \right)^{\frac{1-s}{s}} \left( \frac{\int_Q |f_1(x)w_1(x)w_2(x)|^t}{\int_Q |w_1(x)|^t} \right)^{\frac{1}{t}} \prod_{i=1}^{2} \left( \frac{\int_{Q'} w_{-q_i}^t}{\int_{Q'} w_{-q_i}^t} \right)^{\frac{1}{t}} < \infty,
\]  
(4.7)

where \( \left( \int_Q (w_1(x)w_2(x))^{\frac{t}{1-t}} \right)^{\frac{1}{t}} = \|w_1w_2\|_{L^\infty(Q)} \) when \( t = 1 \). Then we have
\[
\|B_\alpha (f,g)w_1w_2\|_{\mathcal{M}_t^f} \leq C[w^s_t]_{t,q} \sup_{Q \subset Q'} |Q|^{1/p} \left( \frac{\int_Q |f_1(x)w_1(x)|^{q_1}}{\int_Q |w_1(x)|^{q_1}} \right)^{1/q_1} \left( \frac{\int_Q |g_2(x)w_2(x)|^{q_2}}{\int_Q |w_2(x)|^{q_2}} \right)^{1/q_2}.
\]

Remark 4.7. In the same manner as in Remark 4.3, by using Lemma 5.6 below, the inequality (4.6) holds for \( 0 < s < 1 \) if
\[
\sup_{Q \subset Q'} \left( \frac{\int_Q (w_1(x)w_2(x))^{\frac{t}{1-t}}}{\int_Q |w_1(x)|^{t}} \prod_{i=1}^{2} \left( \frac{\int_{Q'} w_{-q_i}^t}{\int_{Q'} w_{-q_i}^t} \right)^{\frac{1}{t}} \right) < \infty.
\]  
(4.8)

Thus, when \( s = t \) and \( p = q \), Theorem 4.6 recovers the one-weight result due to Mone [16].

5. The proofs of Theorems 4.2 and 4.6

We shall state and prove a principal lemma. Our key tool is the following bilinear maximal operator.

Definition 5.1. Let \( 0 < \alpha < n \) and \( 0 < t \leq 1 \). Assume that \( v \) be a weight on \( \mathbb{R}^n \) and \((f, g)\) a couple of locally integrable functions on \( \mathbb{R}^n \). Then define a bilinear maximal operator \( \tilde{M}_\alpha^f(f,g,v)(x) \) by
\[
\tilde{M}_\alpha^f(f,g,v)(x) = \sup_{x \in Q \subset \mathcal{D}} |Q|^\frac{1}{q} \left( \int_Q |f(y)dy| \cdot \int_Q |g(y)|dy \right)^{1/q} \left( \int_Q v(y)\frac{t}{1-t}dy \right)^{1-\frac{1}{q}},
\]
where \( x \in \mathbb{R}^n \) and \( \left( \int_Q v(y)\frac{t}{1-t}dy \right)^{1-\frac{1}{q}} = \|v\|_{L^\infty(Q)} \) when \( t = 1 \).

The following is our principal lemma, which seems to be of interest on its own.

Lemma 5.2. Assume that \( v \) be a weight on \( \mathbb{R}^n \) and \((f, g)\) a couple of locally integrable functions on \( \mathbb{R}^n \). For any \( x \in Q_0 \subset \mathcal{D} \), set
\[
(f_0, g_0) = (f(\cdot)\chi_{Q_0}(x \cdot), g(\cdot)\chi_{Q_0}(\cdot - x)) \quad \text{and} \quad (f_1, g_1) = (f\chi_{3Q_0}, g\chi_{3Q_0}).
\]
Then there exists a constant \( C \) independent of \( v, f, g \) and \( Q_0 \) such that
\[
\|B_\alpha(f_0,g_0)v\|_{L^t(Q_0)} \leq C\|\tilde{M}_\alpha^f(f_1,g_1,v)\|_{L^t(Q_0)}
\]  
(5.1)
holds for $0 < \alpha < n$ and $0 < t \leq 1$.

Since $B_\alpha$ is a positive operator, without loss of generality we may assume that $f, g$ are nonnegative. For simplicity, we will use the notation

$$m_Q(f, g) = \int_Q f(y)dy \cdot \int_Q g(y)dy.$$  

We begin with an auxiliary operator that will play a key role in our analysis. For $d > 0$ define,

$$B_d(f, g)(x) = \int_{|y| \leq d} f(x - y)g(x + y)dy.$$  

The operators $B_{2^k}$ are used by Kenig and Stein [12] in the analysis of $B_\alpha$. We have the following weighted estimates for $B_d$ due to [16].

Lemma 5.3. Assume that $v$ be a weight on $\mathbb{R}^n$ and $(f, g)$ a couple of locally integrable functions on $\mathbb{R}^n$. Let $0 < t \leq 1$ and $Q$ be a cube, then we have

$$\int_Q B_{t(Q)}(f, g)^t vdQ \leq C \left( \int_{3Q} f dx \cdot \int_{3Q} g dx \right)^t \left( \int_Q v^{\frac{1}{1-t}} dx \right)^{1-t},$$  

where $\left( \int_Q v^{\frac{1}{1-t}} dx \right)^{1-t} = \|v\|_{L^\infty(Q)}$ when $t = 1$.

Proof. By H"older’s inequality with $1/t$ and $(1/t)' = 1/(1 - t)$ we have

$$\int_Q B_{t(Q)}(f, g)^t vdQ \leq \left( \int_Q B_{t(Q)}(f, g)(x) dx \right)^t \left( \int_Q v^{\frac{1}{1-t}} dx \right)^{1-t} = \left( \int_Q \int_{|y| \leq l(Q)} f(x - y)g(x + y)dydx \right)^t \left( \int_Q v^{\frac{1}{1-t}} dx \right)^{1-t}.$$  

We make the change of variables $w = x + y$, $z = x - y$ in the first integral and notice that if $c_Q$ is the center of the cube, then $|x - c_Q| \leq \frac{l(Q)}{2}$ and $|t| \leq l(Q)$ imply that $(w, z) \in 3Q \times 3Q$. The lemma follows at once. \hfill \Box

Next we consider a discretization of the operator $B_\alpha$ into a dyadic model. Define the dyadic bilinear fractional integral by

$$B^\alpha_Q(f, g)(x) = \sum_{Q \in \mathcal{D}} \frac{|Q|^\alpha}{|Q|^\alpha} B_{t(Q)}(f, g)(x)\chi_Q(x).$$  

Fix a cube $Q_0 \in \mathcal{D}$. Let $\mathcal{D}(Q_0)$ be the collection of all dyadic subcubes of $Q_0$, that is, all those cubes obtained by dividing $Q_0$ into $2^n$ congruent cubes of half its side-length, dividing each of those into $2^n$ congruent cubes, and so on. By convention, $Q_0$ itself belongs to $\mathcal{D}(Q_0)$. To prove Lemma 5.2, we need the following estimate.

Lemma 5.4. For $x \in Q_0$,

$$cB^\alpha_{Q_0}(f_0, g_0)(x) \leq B_\alpha(f_0, g_0)(x) \leq CB^\alpha_{Q_0}(f_0, g_0)(x), \quad (5.2)$$  

where two constants $c$ and $C$ only depending on $\alpha$ and $n$.  

Proof. We proof of (5.2) is based on [10, 16]. We first discretize the operator \( B_\alpha(f_0, g_0) \). Notice that \(|y| \sim |y|_\infty\) and hence
\[
B_\alpha(f_0, g_0)(x) = \sum_{k \in \mathbb{Z}} \int_{Q_0 \cap \{2^{k-1} < |y| \leq 2^k\}} \frac{f(x-y)g(x+y)}{|y|^{n-\alpha}} dy \\
\leq 2^{n-\alpha} \sum_{k \in \mathbb{Z}} (2^k)^{\alpha-n} \int_{Q_0 \cap \{2^{k-1} < |y| \leq 2^k\}} \frac{f(x-y)g(x+y)}{|y|^{n-\alpha}} dy \\
\leq C \sum_{k \in \mathbb{Z}} (2^k)^{\alpha-n} \int_{Q_0 \cap |y| \leq 2^k} \frac{f(x-y)g(x+y)}{|y|^{n-\alpha}} dy \\
\leq C \sum_{x \in Q \in \mathcal{D}(Q_0)} \frac{|Q|^\alpha}{|Q|} \int_{|y| \leq l(Q)} f(x-y)g(x+y) dy.
\]
On the other hand, fix \( x \in Q_0 \) and \( \{Q_k\}_{k \in \mathbb{Z}} \) be the unique sequence of dyadic cubes with \( x \in Q_k \in \mathcal{D}(Q_0) \). Then we have
\[
\sum_{Q \in \mathcal{D}(Q_0)} \frac{|Q|^\alpha}{|Q|} B_{l(Q)}(f_0, g_0)(x) \chi_Q(x) = \sum_{k=-\infty}^{\log_2 l(Q_0)} \frac{|Q_k|^\alpha}{|Q_k|} B_{l(Q_k)}(f_0, g_0)(x) \\
= \log_2 l(Q_0) \sum_{k=-\infty}^{\log_2 l(Q_0)} \frac{|Q_k|^\alpha}{|Q_k|} \int_{2^{k-1} < |y| \leq 2^k} f(x-y)g(x+y) dy + \sum_{k=-\infty}^{\log_2 l(Q_0)} \frac{|Q_k|^\alpha}{|Q_k|} B_{l(Q_{k-1})}(f_0, g_0)(x) \\
\leq c \sum_{k=-\infty}^{\log_2 l(Q_0)} \int_{2^{k-1} < |y| \leq 2^k} \frac{f(x-y)g(x+y)}{|y|^{n-\alpha}} dy + 2^{\alpha-n} \sum_{Q \in \mathcal{D}(Q_0)} \frac{|Q|^\alpha}{|Q|} B_{l(Q)}(f_0, g_0)(x) \chi_Q(x) \\
\leq c B_{\alpha}(Q_0)(f_0, g_0)(x) + 2^{\alpha-n} \sum_{Q \in \mathcal{D}(Q_0)} \frac{|Q|^\alpha}{|Q|} B_{l(Q)}(f_0, g_0)(x) \chi_Q(x).
\]
Since \( \alpha < n \) we may rearrange the terms, then
\[
c B_{\alpha}(Q_0)(f_0, g_0)(x) \leq B_\alpha(f_0, g_0)(x).
\]
We now proceed by following [15] and observe the following.
Define
\[
M_{3\mathcal{D}}(f, g)(x) = \sup_{x \in Q \in \mathcal{D}} \int_{3Q} f dy \cdot \int_{3Q} g dy,
\]
to be the maximal function with the basis of triples of dyadic cubes. Letting \( a > 1 \) be a fixed constant to be choose later, and for \( k = 1, 2, \cdots \), we set
\[
D_k = \bigcup \{Q : Q \in \mathcal{D}(Q_0), m_{3Q}(f, g) > a^k\}.
\]
Considering the maximal cubes with respect to inclusion, we can write
\[
D_k = \bigcup_j Q_j^k.
\]
where the cubes \( \{ Q_j^k \} \subset \mathcal{D}(Q_0) \) are nonoverlapping. By the maximality of \( Q_j^k \) we can see that
\[
a^k < m_{3Q_j^k}(f, g) \leq 2^{2n} a^k. \tag{5.3}
\]
Let
\[
E_0 = Q_0 \setminus D_1 \quad \text{and} \quad E_j^k = Q_j^k \setminus D_{k+1}.
\]
We need the following properties: \( \{ E_0 \} \cup \{ E_j^k \} \) is a disjoint family of sets which decomposes \( Q_0 \) and satisfies
\[
|Q_0| \leq 2|E_0| \quad \text{and} \quad |Q_j^k| \leq 2|E_j^k|. \tag{5.4}
\]
The inequalities (5.4) can be verified as follows:

Fixed \( Q_j^k \) and by (5.3), we have that \( Q_j^k \cap D_{k+1} \subset \{ x \in Q_j^k : M_{3\mathcal{D}}(f, g)(x) > a^{k+1} \} \).

Using the operator \( M_{3\mathcal{D}} \) maps \( L^1 \times L^1 \) into \( L^{1/2, \infty} \), we have
\[
|Q_j^k \cap D_{k+1}| \leq \| \{ x \in Q_j^k : M_{3\mathcal{D}}(f, g)(x) > a^{k+1} \} \|
\leq \| \{ x \in \mathbb{R}^n : M_{3\mathcal{D}}(f \chi_{3Q_j^k}, g \chi_{3Q_j^k})(x) > a^{k+1} \} \|
\leq \left( \frac{\| M_{3\mathcal{D}} \|}{a^{k+1}} \int_{3Q_j^k} f(y)dy \cdot \int_{3Q_j^k} g(y)dy \right)^{1/2}
\leq \left( \frac{\| M_{3\mathcal{D}} \|}{a^{k+1}} \frac{1}{|3Q_j^k|^2} \int_{3Q_j^k} f(y)dy \cdot \int_{3Q_j^k} g(y)dy \right)^{1/2} |3Q_j^k|
\leq \frac{6^n \| M_{3\mathcal{D}} \|^{1/2}}{a^{1/2}} |Q_j^k|,
\]
where \( \| M_{3\mathcal{D}} \| \) be the constant from the \( L^1 \times L^1 \to L^{1/2, \infty} \) inequality for \( M_{3\mathcal{D}} \) and we have used (5.3) in the last step.

Let \( a = 6^{2n} 2^n \| M_{3\mathcal{D}} \| \), then we obtain
\[
|Q_j^k \cap D_{k+1}| \leq \frac{1}{2} |Q_j^k|. \tag{5.5}
\]
Similary, we see that
\[
|D_1| \leq \frac{1}{2} |Q_0|. \tag{5.6}
\]
Clearly, (5.5) and (5.6) imply (5.4).

We set
\[
\mathcal{D}_0(Q_0) = \{ Q \in \mathcal{D}(Q_0) : m_{3Q}(f, g) \leq a \},
\]
\[
\mathcal{D}_j^k(Q_0) = \{ Q \in \mathcal{D}(Q_0) : Q \subset Q_j^k, a^k < m_{3Q}(f, g) \leq a^{k+1} \}.
\]
Then we obtain
\[
\mathcal{D}(Q_0) = \mathcal{D}_0(Q_0) \cup \bigcup_{k,j} \mathcal{D}_j^k(Q_0). \tag{5.7}
\]
Proof of Lemma 5.2. By Lemma 5.4 it suffices to work the dyadic operator $B_α^{2^k(Q_0)}$. Since $0 < t \leq 1$ it follows that

$$\int_{Q_0} (B_α^{2^k(Q_0)}(f_0, g_0)v)^t dx \leq c \sum_{Q \in \mathcal{D}(Q_0)} \frac{|Q|^{\frac{t}{\alpha}}}{|Q|^t} \int_Q B_t(f_0, g_0)^t v^t dx. \quad (5.8)$$

By Lemma 5.3 we have

$$\int_{Q_0} (B_α^{2^k(Q_0)}(f_0, g_0)v)^t dx \leq c \sum_{Q \in \mathcal{D}(Q_0)} \left( \frac{|Q|^{\frac{t}{\alpha}}}{|Q|} \int_{3Q} f dx \cdot \int_{3Q} g dx \right)^t \left( \int_Q v^{\frac{t}{1-t}} dx \right)^{1-t}$$

$$= c \sum_{Q \in \mathcal{D}(Q_0)} \left( |Q|^{\frac{t}{\alpha}} \int_{3Q} f dx \cdot \int_{3Q} g dx \right)^t \left( \int_Q v^{\frac{t}{1-t}} dx \right)^{1-t} \cdot |Q|. \quad (5.9)$$

First, based on (5.7) we estimate

$$\sum_{Q \in \mathcal{D}^k_j(Q_0)} \left( |Q|^{\frac{t}{\alpha}} \int_{3Q} f dx \cdot \int_{3Q} g dx \right)^t \left( \int_Q v^{\frac{t}{1-t}} dx \right)^{1-t} \cdot |Q|. \quad (5.10)$$

For every $Q \in \mathcal{D}^k_j(Q_0)$, we know $Q$ contained in a unique $Q^k_j$. Then

$$(5.10) \leq a^{(k+1)t} \sum_{Q \in \mathcal{D}^k_j(Q_0)} |Q|^{\frac{t}{\alpha}(k+1)} \left( \int_Q v^{\frac{t}{1-t}} dx \right)^{1-t} \leq a^{(k+1)t} \sum_{Q \subseteq Q^k_j} |Q|^{\frac{t}{\alpha}(k+1)} \left( \int_Q v^{\frac{t}{1-t}} dx \right)^{1-t}. \quad (5.11)$$

We now use a packing condition to handle the terms in the innermost sum of (5.11). Fix a $Q^k_j$ and consider the sum

$$\sum_{Q \subseteq Q^k_j} |Q|^{\frac{t}{\alpha}(k+1)} \left( \int_Q v^{\frac{t}{1-t}} dx \right)^{1-t}$$

$$= \sum_{i=0}^{\infty} \sum_{Q \subseteq Q^k_j \atop \ell(Q) = 2^{-i}(Q^k_j)} |Q|^{\frac{t}{\alpha}(k+1)} \left( \int_Q v^{\frac{t}{1-t}} dx \right)^{1-t}$$

$$= |Q^k_j|^{\frac{t}{\alpha}(k+1)} \sum_{i=0}^{\infty} 2^{-iat} \left( \sum_{Q \subseteq Q^k_j \atop \ell(Q) = 2^{-i}(Q^k_j)} \left( \int_Q v^{\frac{t}{1-t}} dx \right)^{1-t} \right)^t$$

$$\leq |Q^k_j|^{\frac{t}{\alpha}(k+1)} \sum_{i=0}^{\infty} 2^{-iat} \left( \sum_{Q \subseteq Q^k_j \atop \ell(Q) = 2^{-i}(Q^k_j)} \left( \int_Q v^{\frac{t}{1-t}} dx \right)^{1-t} \right)^t$$

$$= |Q^k_j|^{\frac{t}{\alpha}(k+1)} \left( \int_{Q^k_j} v^{\frac{t}{1-t}} dx \right)^{1-t} \sum_{i=0}^{\infty} 2^{-iat} = 2^{2at} \frac{|Q^k_j|^{\frac{t}{\alpha}(k+1)}}{2^at - 1} \left( \int_{Q^k_j} v^{\frac{t}{1-t}} dx \right)^{1-t} \cdot |Q^k_j|. \quad (5.12)$$
Using this inequality in (5.11) we have
\[
(5.10) \leq Ca^{(k+1)t}|Q_j^k|^{\frac{\alpha}{n}} \left( \int_{Q_j^k} v^{\frac{t}{1-t}} dx \right)^{1-t} |Q_j^k|.
\]
From (5.3), (5.4) and (5.12), we conclude that
\[
(5.10) \leq C|Q_j^k|^{\frac{\alpha}{n}} m_{3Q_j^k}(f,g)^t \left( \int_{Q_j^k} v^{\frac{t}{1-t}} dx \right)^{1-t} |Q_j^k|
= C \left[ |Q_j^k|^{\frac{\alpha}{n}} m_{3Q_j^k}(f,g) \left( \int_{Q_j^k} v^{\frac{t}{1-t}} dx \right)^{1-t} \right] |E_j^k| \leq C \int_{E_j^k} \hat{M}_\alpha^t(f_0,g_0,v)(x)^t dx.
\]
Similarly,
\[
\sum_{Q \in G_0(Q_0)} \left( |Q|^{\frac{\alpha}{n}} \int_{3Q} fdx \cdot \int_{3Q} gdx \right)^t \left( \int_{Q} v^{\frac{t}{1-t}} dx \right)^{1-t} |Q| \leq C \int_{Q_0} \hat{M}_\alpha^t(f_0,g_0,v)(x)^t dx.
\]
Summing up (5.13) and (5.14), we obtain
\[
\int_{Q_0} (B^\alpha_\alpha(Q_0)(f_0,g_0)v)^t dx \leq C \int_{Q_0} \hat{M}_\alpha^t(f_0,g_0,v)(x)^t dx.
\]
This is our desired inequality (5.1).

To prove Theorems 4.2 and 4.6, we also need two more lemmas.

Let \( 0 < \alpha < n \). For a vector \((f,g)\) of locally integrable functions and a vector \(\vec{r} = (r_1,r_2)\) of exponents, define a maximal operator
\[
M_{\alpha,\vec{r}}(f,g)(x) = \sup_{x \in Q \in \mathcal{D}} |Q|^{\frac{\alpha}{n}} \left( \int_{Q} |f(y)|^{r_1} dy \right)^{1/r_1} \left( \int_{Q} |g(y)|^{r_2} dy \right)^{1/r_2}.
\]

The following lemma concerns the maximal operator on Morrey spaces, which can found in the paper [10].

**Lemma 5.5.** Let \( 0 < \alpha < n \). Set \( \vec{q} = (q_1,q_2) \) and \( \vec{r} = (r_1,r_2) \). Assume in addition that \( 0 < r_i < q_i < \infty \), \( i = 1,2 \). If \( 0 < t \leq s < \infty \) and \( 0 < q < p < \infty \) satisfy
\[
\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n} \quad \text{and} \quad \frac{t}{s} = \frac{q}{p},
\]
where \( q \) is given by \( 1/q = 1/q_1 + 1/q_2 \), then
\[
\|M_{\alpha,\vec{r}}(f,g)\|_{M_t^s} \leq C \sup_{Q \in \mathcal{D}} |Q|^{1/p} \left( \int_{Q} |f(y)|^{q_1} dy \right)^{1/q_1} \left( \int_{Q} |g(y)|^{q_2} dy \right)^{1/q_2}.
\]

We also need the following a Characterization of a multiple weights given by Iida [11].

**Lemma 5.6.** Let \( 1 < q_1, q_2 < \infty \) and \( \hat{t} \geq q \) with \( 1/q = 1/q_1 + 1/q_2 \). Then, for two weights \( w_1, w_2 \), the inequality
\[
\sup_{Q \in \mathcal{D}} \left( \int_{Q} (w_1w_2)^{\hat{t}} \right)^{1/\hat{t}} \prod_{i=1}^{2} \left( \int_{Q} w_i^{q_i} \right)^{1/q_i} < \infty
\]
holds if and only if
\[
\begin{cases}
(w_1 w_2) \in A_{1+i(2-1/q)}, \\
w_i^{-q_i} \in A_{q_i(1/i+2-1/q)}, \quad i = 1, 2.
\end{cases}
\]

Proof of Theorem 4.2. In what follows we always assume that \(f, g\) are nonnegative and
\[
\sup_{Q \in \mathcal{Q}} |Q|^{1/p} \left( \int_Q (fw_1)^{q_1} \right)^{1/q_1} \left( \int_Q (gw_2)^{q_2} \right)^{1/q_2} = 1 \tag{5.17}
\]
by normalization. To prove this theorem we have estimate, for an arbitrary cube \(Q_0 \in Q\),
\[
|Q_0|^{1/s} \left( \int_{Q_0} (B_\alpha(f, g)v)^t \right)^{1/t}. \tag{5.18}
\]
Fix a cube \(Q_0 \in Q\) and recall that \((f_0, g_0) = (f(\cdot)\chi_{Q_0}(x - \cdot), g(\cdot)\chi_{Q_0}(\cdot - x))\). Then by a standard argument we have, for \(x \in Q_0\),
\[
\int_{Q_0} (B_\alpha(f, g)v)^t \leq \int_{Q_0} (B_0(f_0, g_0)v)^t + C_\infty, \tag{5.19}
\]
where
\[
C_\infty = \sum_{k=0}^{\infty} \int_{Q_0} \left( \int_{2^k l(Q_0) < |y| \leq 2^{k+1} l(Q_0)} \frac{f(x-y)g(x+y)}{|y|^{n-\alpha}} dy \right)^t v^t dx.
\]
Since
\[
\int_{2^k l(Q_0) < |y| \leq 2^{k+1} l(Q_0)} \frac{f(x-y)g(x+y)}{|y|^{n-\alpha}} dy \leq C \frac{|2^{k+1} Q_0|^{\alpha}}{|2^{k+1} Q_0|} \int_{|y| \leq 2^{k+1} l(Q_0)} f(x-y)g(x+y) dy,
\]
Then we have
\[
C_\infty \leq C \sum_{k=0}^{\infty} \frac{|2^{k+1} Q_0|^{\alpha}}{|2^{k+1} Q_0|^{\alpha}} \left( \int_{2^{k+1} Q_0} f dx \cdot \int_{2^{k+1} Q_0} g dx \right)^t \left( \int_{Q_0} v^{\frac{t}{1-t}} dx \right)^{1-t}. \tag{5.20}
\]
First step. Keeping in mind (5.18), (5.19) and (5.20), we now estimate for \(|Q_0|^{t/s}C_\infty\) in the Theorem 4.2. By (4.2), (4.3) and Hölder’s inequality we have
\[
c_0 = \sup_{\substack{Q \in \mathcal{Q} \setminus Q_0 \subseteq Q}} \left( \frac{|Q_0|}{|Q|} \right)^{\frac{1-s}{as}} |Q|^{\frac{1}{r}} \left( \int_{Q_0} v^{\frac{t}{1-t}} \right)^{\frac{1-t}{t}} \prod_{i=1}^{n} \left( \int_{Q_0} w_i^{-q_i} \right)^{\frac{1}{q_i}} \leq [v, w_r^{t,q/a}_{i, q/a}],
\]
and
\[
c_* = \sup_{\substack{Q \in \mathcal{Q} \setminus Q_0 \subseteq Q}} \left( \frac{|Q_0|}{|Q|} \right)^{\frac{1-s}{as}} |Q|^{\frac{1}{r}} \left( \int_{Q_0} v^{\frac{t}{1-t}} \right)^{\frac{1-t}{t}} \prod_{i=1}^{n} \left( \int_{Q_0} w_i^{-q_i} \right)^{\frac{1}{q_i}} \leq [v, w_r^{t,q/a}_{i, q/a}].
\]
From Hölder’s inequality, (5.20) and the fact that
\[
\frac{1}{s} = \frac{1}{p} + \frac{1}{r} - \frac{\alpha}{n}
\]
it follows that
\[
C_\infty \leq C \sum_{k=0}^{\infty} \left( \int_{2^{k+3}Q_0}(f w_1)^{q_1} dx \right)^{\frac{q_1}{q_1}} \left( \int_{2^{k+3}Q_0}(g w_2)^{q_2} dx \right)^{\frac{q_2}{q_2}} \left( \int_{Q_0} v^{\frac{1}{r-t}} dx \right)^{1-t} |2^{k+3}Q_0|^{\frac{aq}{n}} \\
\times \frac{|2^{k+3}Q_0|^t}{|Q_0|^t} \left( \int_{2^{k+3}Q_0} w_1^{-q_1} dx \right)^{\frac{r}{q_1}} \left( \int_{2^{k+3}Q_0} w_2^{q_2} dx \right)^{\frac{r}{q_2}} \\
= C \sum_{k=0}^{\infty} \left( \int_{2^{k+3}Q_0}(f w_1)^{q_1} dx \right)^{\frac{q_1}{q_1}} \left( \int_{2^{k+3}Q_0}(g w_2)^{q_2} dx \right)^{\frac{q_2}{q_2}} \left( \int_{Q_0} v^{\frac{1}{r-t}} dx \right)^{1-t} |2^{k+3}Q_0|^t \\
\times \frac{|2^{k+3}Q_0|^t}{|Q_0|^t} \left( \int_{2^{k+3}Q_0} w_1^{-q_1} dx \right)^{\frac{r}{q_1}} \left( \int_{2^{k+3}Q_0} w_2^{q_2} dx \right)^{\frac{r}{q_2}} |2^{k+3}Q_0|^{\frac{aq}{n} - \frac{r}{p}} \\
\leq C \sum_{k=0}^{\infty} \frac{|2^{k+3}Q_0|^t}{|Q_0|^t} \left( \int_{2^{k+3}Q_0} w_1^{-q_1} dx \right)^{\frac{r}{q_1}} \left( \int_{2^{k+3}Q_0} w_2^{q_2} dx \right)^{\frac{r}{q_2}} \left( \int_{Q_0} v^{\frac{1}{r-t}} dx \right)^{1-t}.
\]

This yields for 0 < s < 1
\[
|Q_0|^{t/s}C_\infty \leq c_0 \sum_{k=0}^{\infty} \left( \frac{|Q_0|}{|2^{k+3}Q_0|} \right)^{(1-1/a)(1/s-1)} = Cc_0
\]
and for s ≥ 1
\[
|Q_0|^tC_\infty \leq c_0 \sum_{k=0}^{\infty} \left( \frac{|Q_0|}{|2^{k+3}Q_0|} \right)^{(1-1/a)} = Cc_s,
\]
where we have used 1 − 1/a > 0.

Second step. For 0 < t ≤ 1, we shall estimate
\[
|Q_0|^{1/s} \left( \int_{Q_0} (B_\alpha(f_0, g_0)v)^t \right)^{1/t}.
\]
By (4.2) and (4.3) we have
\[
c_1 = \sup_{Q \in \mathcal{G}} |Q|^{1/r} \left( \int_Q v^{\frac{1}{r-t}} \right)^{1-t} \prod_{i=1}^2 \left( \int_Q w_i^{-(q_i/a)'})^{1/(q_i/a)'} \leq [v, w_i]_{r, q_i/a}^{r, a}.
\]
To apply Lemma (5.2) we now compute, for any Q ∈ \mathcal{G},
\[
\left( \int_Q |f(y)| dy \cdot \int_Q |g(y)| dy \right) \left( \int_Q v^{\frac{1}{r-t}} \right)^{1-t} \\
\leq \left( \int_Q (f w_1)^{q_1/a} \right)^{aq_1} \left( \int_Q (g w_2)^{q_2/a} \right)^{aq_2} \left( \int_Q v^{\frac{1}{r-t}} \right)^{1-t} \prod_{i=1}^2 \left( \int_Q w_i^{-(q_i/a)'} \right)^{1/(q_i/a)'} \\
\leq c_1 |Q|^{-1/r} \left( \int_Q (f w_1)^{q_1/a} \right)^{aq_1} \left( \int_Q (g w_2)^{q_2/a} \right)^{aq_2}.
\]
This implies, for x ∈ Q_0,
\[
\tilde{M}_\alpha(f_0, g_0, v)(x) \leq c_1 M_{\alpha-n/r, q_i/a}(f, g)(x).
\]
Inequality (5.21), Lemma 5.2 and Lemma 5.5 yield
\[
|Q_0|^{1/s} \left( \int_{Q_0} (B_\alpha(f, g)v)^t \right)^{1/t} \leq Cc_1 \|M_{\alpha-n/r, q_i/a}(f, g)\|_{M^t_t} \leq Cc_1,
\]
where we have used the assumption
\[
\frac{1}{s} = \frac{1}{p} - \frac{\alpha - n/r}{n} \quad \text{and} \quad \frac{t}{s} = \frac{q}{p}
\]
and (5.17). This completes proof of Theorem 4.2.

\textbf{Proof of Theorem 4.6.} Keeping in mind (5.18), (5.19) and let \( v = w_1 w_2 \), we only need estimate the following inequality
\[
\left( \int_{Q_0} (B_\alpha(f, g)w_1 w_2) \right)^t \leq \int_{Q_0} (B_0(f_0, g_0)w_1 w_2)^t + c_\infty, \tag{5.22}
\]
where
\[
c_\infty = \sum_{k=0}^\infty \int_{Q_0} \left( \int_{|y|^{\alpha - n} < 2^k + 1} \frac{f(x - y)g(x + y)}{|y|} \right)^t w_1(x)^t w_2(x)^t \, dx.
\]

Similar to the estimate for \( C_\infty \) we have
\[
|Q_0|^{t/s} c_\infty \leq C|\vec{w}|_{t, \vec{q}}^{as}.
\]
Next, we will estimate, for Theorem 4.6 in the conditions (4.6) and (4.7),
\[
|Q_0|^{1/s} \left( \int_{Q_0} (B_\alpha(f_0, g_0)w_1 w_2)^t \right)^{1/t} .
\]
For \( 0 < t \leq 1 \), by assumption we have
\[
c_2 = \sup_{Q \in \mathcal{D}} \left( \int_Q (w_1 w_2)^{\frac{1}{1-t}} \right)^{\frac{1-t}{t}} \prod_{i=1}^2 \left( \int_Q w_i^{-q_i} \right)^{\frac{1}{q_i}} < \infty.
\]
Then we can deduce from Lemma 5.6 and the reverse Hölder’s inequality that there a constant \( \theta \in (1, \min(q_1, q_2)) \) such that, for any cube \( Q \in \mathcal{D} \),
\[
\left( \int_Q (w_1 w_2)^{\frac{1}{1-t}} \right)^{\frac{1-t}{t}} \leq C \left( \int_Q (w_1 w_2)^{\frac{1}{1-t}} \right)^{\frac{1-t}{t}} \tag{5.23}
\]
and
\[
\left( \int_Q w_i^{-(q_i/\theta)^y} \right)^{1/(q_i/\theta)^y} \leq C \left( \int_Q w_i^{-q_i} \right)^{1/q_i}, \quad \text{for each} \ i = 1, 2. \tag{5.24}
\]
Combining (5.23) and (5.24) with the weight conditions in Theorem 4.6, we obtain
\[
[v, \vec{w}]_{t, \vec{q}/a}^{\infty, as} \leq [\vec{w}]_{t, \vec{q}}^{as} < \infty.
\]
Going through a similar argument in Theorem 4.2, we have
\[
|Q_0|^{1/s} \left( \int_{Q_0} (B_\alpha(f_0, g_0)w_1 w_2)^t \right)^{1/t} \leq C.
\]
Consequently, Theorem 4.6 is proved. \( \square \)
6. Examples and necessary conditions

7.1. A bilinear Stein-Weiss inequality. Given $0 < \alpha < n$ let $T_\alpha$ be define by

$$T_\alpha f(x) = I_{n-\alpha} f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$ 

Stein and Weiss\cite{stein1975} proved the following weighted inequality for $T_\alpha$:

$$\left( \int_{\mathbb{R}^n} \left( \frac{T_\alpha f(x)}{|x|^{\beta}} \right)^t \, dx \right)^{1/t} \leq \left( \int_{\mathbb{R}^n} (f(x)|x|^{\gamma})^q \, dx \right)^{1/q},$$

where

$$\beta < \frac{n}{t}, \quad \gamma < \frac{n}{q'}, \quad \alpha + \beta + \gamma = n + \frac{n}{t} - \frac{n}{q}, \quad \beta + \gamma \geq 0.$$ (6.1)

Conditions (6.1), (6.2) and (6.3) are actually sharp. Condition (6.1) ensures that $|x|^{-\beta q}$ and $|x|^{-\gamma p'}$ are locally integrable. Condition (6.2) follows from a scaling argument and condition (6.3) is a necessary condition for the weights to satisfy a general two weight inequality \cite{stein1975}.

Below, we prove a bilinear Stein-Weiss inequality on Morrey spaces. For $0 < \alpha < n$ let $BT_\alpha$ be the bilinear operator defined by

$$BT_\alpha(f, g)(x) = B_{n-\alpha}(f, g)(x) = \int_{\mathbb{R}^n} \frac{f(x-y)g(x+y)}{|y|^{\alpha}} dy.$$ 

**Theorem 6.1.** Assume that $1 < q_1 \leq p_1 < \infty$, $1 < q_2 \leq p_2 < \infty$, $0 < t \leq s < 1$, $\frac{n}{n-\alpha} < r \leq \infty$ and $1 < a < \min(q_1, q_2)$. Here, $p$ and $q$ are given by

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}. $$

Suppose that

$$\frac{1}{s} = \frac{1}{p} + \frac{1}{r} - \frac{n-\alpha}{n}, \quad \frac{1}{t} = \frac{1}{q} + \frac{1}{r} - \frac{n-\alpha}{n}$$

and $\alpha, \beta, \gamma_1, \gamma_2$ satisfy the conditions

$$\left\{ \begin{array}{l}
\beta < n(\frac{1}{s} - 1), \\
\gamma_1 < \frac{n}{q_1}, \\
\gamma_2 < \frac{n}{q_2}, \\
\alpha + \beta + \gamma_1 + \gamma_2 = n + \frac{n}{t} - \frac{n}{q_1} - \frac{n}{q_2}, \\
\beta + \gamma_1 + \gamma_2 \geq 0.
\end{array} \right. $$ (6.4)

Then the following inequality holds for all $f, g \geq 0$

$$\sup_{Q \in \mathcal{G}} |Q|^{\frac{1}{2}} \left( \int_{Q} \left( \frac{BT_\alpha(f, g)(x)}{|x|^{\beta}} \right)^t \, dx \right)^{\frac{1}{t}} \leq C \sup_{Q \in \mathcal{G}} |Q|^{\frac{1}{p_1}} \left( \int_{Q} (f(x)|x|^{\gamma_1})^{q_1} \, dx \right)^{\frac{1}{q_1}} \sup_{Q \in \mathcal{G}} |Q|^{\frac{1}{p_2}} \left( \int_{Q} (g(x)|x|^{\gamma_2})^{q_2} \, dx \right)^{\frac{1}{q_2}}.$$
Proof. We suppose that the parameters satisfy
\[ 0 < t_0 \leq s < 1 \quad \text{and} \quad \frac{t_0}{s} = \frac{q_1}{p_1} = \frac{q_2}{p_2}. \]

Similar to the estimate for Theorem 1.4, we have
\[
\sup_{Q \in \mathcal{Q}} |Q|^\frac{1}{s} \left( \int_Q \left( \frac{BT_{\alpha} (f,g)(x)}{|x|^\beta} \right)^t \, dx \right)^{\frac{1}{t}} \leq \sup_{Q \in \mathcal{Q}} |Q|^\frac{1}{s} \left( \int_Q \left( \frac{BT_{\alpha} (f,g)(x)}{|x|^\beta} \right)^{t_0} \, dx \right)^{\frac{1}{t_0}}. \tag{6.5}
\]

By Theorem 4.2, Remark 4.3 and (6.5) we only need to prove that \( a > 1 \) and a constant \( C \) such that
\[
|Q|^{\frac{1}{s}} \left( \int_Q |x|^{-\frac{a\alpha}{1-t}} \, dx \right)^{\frac{1-a}{a}} \prod_{i=1}^2 \left( \int_Q |x|^{-\gamma_i(q_i/a)'} \, dx \right)^{\frac{1}{(q_i/a)'^q}} \leq C
\]
for all cubes \( Q \). From here we follow the standard estimates for power weights. Let \( a > 1 \) be such that \( a\beta < n \left( \frac{1}{s} - 1 \right), \gamma_1 < \frac{n}{(q_1/a)'} \) and \( \gamma_2 < \frac{n}{(q_2/a)'} \). Given a cube \( Q \) let \( Q_0 = Q(0,l(Q)) \). Then either \( 2Q_0 \cap Q = \emptyset \) or \( 2Q_0 \cap Q \neq \emptyset \). In the case \( 2Q_0 \cap Q = \emptyset \) we have \( |c_Q|_\infty \geq l(Q) \) and \( |x| \sim |x|_\infty \sim |c_Q|_\infty \neq 0 \) for all \( x \in Q \). Using this fact we have
\[
|Q|^{1-\frac{a\beta}{1-t} - \frac{a}{s}} \left( \int_Q |x|^{-\frac{a\alpha}{1-t}} \, dx \right)^{\frac{1-a}{a}} \prod_{i=1}^2 \left( \int_Q |x|^{-\gamma_i(q_i/a)'} \, dx \right)^{\frac{1}{(q_i/a)'^q}}
\]
\[
= l(Q)^{\beta + \gamma_1 + \gamma_2} \left( \int_Q |x|^{-\frac{a\alpha}{1-t}} \, dx \right)^{\frac{1-a}{a}} \prod_{i=1}^2 \left( \int_Q |x|^{-\gamma_i(q_i/a)'} \, dx \right)^{\frac{1}{(q_i/a)'^q}}
\]
\[
\leq Cl(Q)^{\beta + \gamma_1 + \gamma_2} |c_Q|_\infty^{-\gamma_1 - \gamma_2} \leq C,
\]
where in the first line we have used the second equality in (6.4) and in the last estimate we have used the third inequality in (6.4). When \( 2Q_0 \cap Q \neq \emptyset \) we have that \( Q \subseteq B = B(0,5l(Q)) \), the Euclidean ball of radius \( 5l(Q) \) about the origin. Thus,
\[
|Q|^{1-\frac{a\beta}{1-t} - \frac{a}{s}} \left( \int_Q |x|^{-\frac{a\alpha}{1-t}} \, dx \right)^{\frac{1-a}{a}} \prod_{i=1}^2 \left( \int_Q |x|^{-\gamma_i(q_i/a)'} \, dx \right)^{\frac{1}{(q_i/a)'^q}}
\]
\[
\leq l(Q)^{\beta + \gamma_1 + \gamma_2} \left( \int_B |x|^{-\frac{a\alpha}{1-t}} \, dx \right)^{\frac{1-a}{a}} \prod_{i=1}^2 \left( \int_B |x|^{-\gamma_i(q_i/a)'} \, dx \right)^{\frac{1}{(q_i/a)'^q}}
\]
\[
\leq C.
\]

\[ \square \]

7.2. Necessary conditions. Apparently our techniques do not address the case \( 1 < t \leq s < \infty \). That is, other than the trivial conditions mentioned in the introduction, we do not know of sufficient conditions on weights \((v, w_1, w_2)\) that imply
\[
\|B_{\alpha} (f,g)v\|_{M_t^s} \leq C \sup_{Q \in \mathcal{Q}} |Q|^{1/p} \left( \int_Q (|f| w_1)^{q_1} \right)^{1/q_1} \left( \int_Q (|g| w_2)^{q_2} \right)^{1/q_2}
\]
when \( 1 < t \leq s < \infty \). Here we present a necessary condition for the two weight inequality for \( M_\alpha \), which in turn is necessary for \( B_\alpha \) when \( 0 < \alpha < n \).
Theorem 6.2. Let \( v \) be a weight on \( \mathbb{R}^n \) and \( \bar{v} = (w_1, w_2) \) be a collection of two weights on \( \mathbb{R}^n \). Assume that

\[
0 \leq \alpha < n, \quad q = (q_1, q_2), \quad 1 < q_1, q_2 < \infty, \quad 0 < q \leq p < \infty \quad \text{and} \quad 1 \leq t \leq s < \infty.
\]

Here, \( q \) is given by \( 1/q = 1/q_1 + 1/q_2 \). Suppose that

\[
\frac{\alpha}{n} \geq \frac{1}{s} \geq 0, \quad \frac{1}{p} = \frac{1}{r} + \frac{\alpha}{n} \quad \text{and} \quad \frac{t}{s} = \frac{q}{p}.
\]

Then, for every \( Q \in \mathcal{D} \), the weighted inequality

\[
|Q|^{1/s} \left( \int_Q (\mathcal{M}_\alpha(f, g)v)^t \right)^{1/t} \leq C \sup_{Q' \subset Q} \left| Q' \right|^{1/p} \left( \int_{Q'} (|f|w_1)^{q_1} \right)^{1/q_1} \left( \int_{Q'} (|g|w_2)^{q_2} \right)^{1/q_2}. \tag{6.6}
\]

Then there exists a constant \( C \) such that

\[
\sup_{Q \in \mathcal{D}} |Q|^{1/t} (\inf_Q v) \left( \int_Q w_1^{-q_1^i} \right)^{1/q_1} \left( \int_Q w_2^{-q_2^i} \right)^{1/q_2} \leq C. \tag{6.7}
\]

Proof. We assume to the contrary that

\[
\sup_{Q \in \mathcal{D}} |Q|^{1/t} (\inf_Q v) \left( \int_Q w_1^{-q_1^i} \right)^{1/q_1} \left( \int_Q w_2^{-q_2^i} \right)^{1/q_2} = \infty. \tag{6.8}
\]

By (6.8) we can select a cube \( Q \) such that for any large \( M \),

\[
|Q|^{1/t} (\inf_Q v) \left( \int_Q w_1^{-q_1} \right)^{1/q_1} \left( \int_Q w_2^{-q_2} \right)^{1/q_2} > M. \tag{6.9}
\]

Selecting a smaller cube \( Q \) in (6.9), without loss of generality we may assume that \( Q \) in minimal in the sense that

\[
\sup_{R \in \mathcal{D}} \int_R w_i^{-q_i^i} = \int_Q w_i^{-q_i^i}, \quad \text{for } i = 1, 2. \tag{6.10}
\]

Thanks to the fact that \( 1/p - 1/q \leq 0 \), equality (6.10) yields

\[
\sup_{R \in \mathcal{D}} |R|^{1/p} \prod_{i=1}^2 \left( \int_R \chi_Q w_i^{-q_i^i} \right)^{1/q_i^i} = |Q|^{1/p} \prod_{i=1}^2 \left( \int_Q w_i^{-q_i^i} \right)^{1/q_i^i}. \tag{6.11}
\]

We also need the following estimate due to [16],

\[
|Q|^{\frac{1}{s}} \left( \sup_Q v \right) \left( \int_Q f(y)dy \right) \left( \int_Q g(y)dy \right) \leq C \left( \int_Q (\mathcal{M}_\alpha(f, g)v)^t dx \right)^{1/s}. \tag{6.12}
\]

It follows by applying (6.6), (6.11) and (6.12) with \( f = \chi_Q w_1^{-q_1^i}, g = \chi_Q w_2^{-q_2^i} \) that

\[
|Q|^{1/t} (\inf_Q v) \left( \int_Q w_1^{-q_1} \right)^{1/q_1} \left( \int_Q w_2^{-q_2} \right)^{1/q_2} \leq C|Q|^{-1/p}|Q|^{1/s} \left( \int_Q (\mathcal{M}_\alpha(\chi_Q w_1^{-q_1^i}, \chi_Q w_2^{-q_2^i})v)^t dx \right)^{1/t}.
\]

\[
\leq C|Q|^{-\frac{1}{s}} \sup_{R \in \mathcal{D}} |Q|^{1/p} \left( \int_R \chi_Q w_1^{-q_1^i} \right)^{1/q_1} \left( \int_R \chi_Q w_2^{-q_2^i} \right)^{1/q_2}
\]

\[
= C \left( \int_Q w_1^{-q_1^i} \right)^{1/q_1} \left( \int_Q w_2^{-q_2^i} \right)^{1/q_2}.
\]
This yields a contradiction

\[ M < |Q|^{\frac{1}{r}} (\inf_Q v) \left( \int_Q w_1^{-q_1'} \right)^{1/q_1'} \left( \int_Q w_2^{-q_2'} \right)^{1/q_2'} \leq C. \]

This finishes the proof of Theorem 6.2. \(\square\)

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