GENERALIZED MANY-DIMENSIONAL EXCITED RANDOM WALK IN BERNOULLI ENVIRONMENT

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Abstract. We study an extension of the generalized excited random walk (GERW) on \(\mathbb{Z}^d\) introduced in [Ann. Probab. 40 (5), 2012, [7]] by Menshikov, Popov, Ramírez and Vachkovskaia. Our extension consists in studying a version of the GERW where excitation depends on a random environment. Given \(p \in (0,1]\) (a parameter of the model) whenever the process visits a site for the first time, with probability \(p\) it gains a drift in a given direction (could be any direction of the unit sphere). Otherwise, with probability \(1-p\), it behaves as a \(d\)-martingale with zero-mean vector. Whenever the process visits an already-visited site, the process acts again as a \(d\)-martingale with zero-mean vector.

We refer to the model as a GERW in Bernoulli environment, in short \(p\)-GERW. Under the same hypothesis of [7] (bounded jumps, uniform ellipticity), we show that the \(p\)-GERW is ballistic for all \(p \in (0,1]\). Under the stronger assumptions that the increments of the regeneration times associated to the \(p\)-GERW are i.i.d. (condition which is satisfied, for example, for the excited random walk in a Bernoulli i.i.d. environment), we also obtain a Law of Large Numbers and a Central Limit Theorem.

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1. Introduction

The many-dimensional excited random walk (ERW) is a model introduced in 2003 by Benjamini and Wilson [1]. It is a discrete time non Markovian random walk in \( \mathbb{Z}^d \), with \( d \geq 2 \). It jumps as a simple random walk biased in direction \( e_1 \) (with bias \( \delta \)) every time it visits a site for the first time, where \( \{e_i : 1 \leq i \leq d\} \) denotes the canonical base of \( \mathbb{Z}^d \), otherwise it jumps as a simple symmetric random walk.

In [1], Benjamini and Wilson proved that ERW is transient in direction \( e_1 \), i.e., \( \lim_{n \to \infty} X_n \cdot e_1 = \infty \) almost surely. Furthermore, they also show that, if \( d \geq 4 \), ERW is ballistic to the right, i.e.,
\[
\lim \inf_{n \to \infty} \frac{X_n \cdot e_1}{n} > 0, \quad a.s.
\]
Later on, Kozma extended the proof of ballisticity to \( d = 3 \) in [5], and \( d = 2 \) in [6]. In 2007, Bernard and Ramirez [2] proved a Law of Large Numbers (LLN) and a Central Limit Theorem (CLT) for \( d \geq 2 \). Specifically, they prove that
\[
\lim_{n \to \infty} \frac{X_n \cdot e_1}{n} = v, \quad a.s.,
\]
for some \( v = v(\delta, d) \in \mathbb{R} \), and that
\[
\left\{ \frac{X_{\lceil nt \rceil} \cdot e_1 - \lceil nt \rceil v}{\sqrt{n}} \right\}_{t \geq 0},
\]
converges in distribution as \( n \to \infty \) (with respect to the Skorohod topology on the space of càdlàg functions) to a Brownian Motion with a finite variance depending on \( \delta \) and \( d \). Their proof relies on the introduction of an appropriate regeneration structure that was first used in the context of random walks in random environments, see for instance [10].

The proofs of directional transience in [1], the LLG and the CLT in [2], rest upon two important ingredients. A coupling between the ERW and the simple symmetric random walk (SSRW) which implies that the distance between the ERW and the SSRW at time \( n \), in the direction \( e_1 \), is non decreasing in \( n \), while for the others directions it is zero. Using this coupling, the authors provide a lower bound on the cardinality of the set of visited sites by the ERW up to time \( n \) (the range of ERW) in terms of \textit{tan points} for the SSRW, i.e., those sites \( x \in \mathbb{Z}^d \) such that \( x \) is the first site visited in the set \( \{x + ke_1 : k \geq 0\} \). A direct consequence of the coupling is that when the SSRW reaches a tan point, the ERW visits a new site and thus it is pushed in direction \( e_1 \) by a positive drift. Then in [2], using this coupling, the authors proved that the range of the ERW up to time \( n \) in dimension \( d \geq 2 \) is of order \( n^{3/4} \) with large probability. This fact alone is not enough to provide a direct proof for a linear speed of the process, however it is instrumental to
guarantee the existence of a renewal structure for the process which leads to the limit theorems.

A drawback of the technique based on tan points is that it is tailored to the basic model of ERW and it is not robust, i.e., the coupling with the SSRW would not work if for example we consider a random walk with bounded jumps, rather than nearest neighbor jumps, or even if we suppose a drift not parallel to any canonical direction. A more robust technique was developed by Menshikov, Popov, Ramirez and Vachkovskaia in [7]. The model they considered is a generalization of the ERW and is as follows: on already visited sites the process behaves like a $d$-dimensional martingale with bounded jumps and zero mean vector (rather than a SSRW) and whenever the process visits a site for the first time it behaves as follows: it has bounded jumps, satisfies a uniformly elliptic condition and a drift condition in an arbitrary direction $\ell$ of the unit sphere in $\mathbb{R}^d$. They call this model *generalized excited random walk* (GERW). They show that the GERW with a drift condition in direction $\ell$ is ballistic in that direction. Besides that, they proved a LLG and a CLT (both for dimensions $d \geq 2$) for a special case of the GERW, which they called *excited random walk in random environment*. This special model consists in an excited random walk in an i.i.d. random environment, which means that the process still has a mean drift in direction $\ell$ when it visits a site for the first time and whenever it hits an already visited site it has a zero mean drift. Along with that, the probability transitions for nearest neighbors of the process are explicit. Similarly to what was done for ERW, the first step in their proof consists in controlling the range of the process. Proposition 4.1 in [7] states that the range of the GERW is smaller than $n^{1/2+\alpha}$ with probability that decays as an stretched exponential, where $\alpha > 0$ does not depend on the parameters of the model. The proof of Proposition 4.1 completely avoids the use of the coupling and the tan points. Again, this control on the range of the process allows the construction of a regeneration structure for the GERW. If $\{X_n\}_{n \geq 0}$ denotes the GERW, the regeneration structure consists in a properly defined sequence of finite regeneration times $\tau_k$, $k \geq 1$, that correspond to those times when the process $\{X_n \cdot \ell\}_{n \geq 0}$ reaches for the first time the level $X_{\tau_k} \cdot \ell$ and never comes back below $X_{\tau_k} \cdot \ell$ after time $\tau_k$. The renewal structure considered in [7] follows the standard approach and notation presented in [2] and [10].

What makes the GERW (and the ERW) an interesting model is the self-interaction encoded in the different behavior the process has on sites visited for the first time as compared to sites already visited. It is customary to think that initially all sites have a *cookie*. Whenever the process visits a site for the first time, it eats the cookie and gains a drift in a given direction. On subsequent visits to a site, since there is no cookie left, the process has no drift (for this reason ERW are also referred to as cookie random walks). A natural question is what happens to GERW when on the first visit to a
site the random walk may or may not find/eat a cookie. Would the process still be ballistic in the direction of the drift? What about LLN and CLT?

In order to address this question, we introduce and study a model which is a variation of the GERW. Specifically, given \( p \in (0, 1] \), if at time \( n \) the process visits a site for the first time, it finds a cookie with probability \( p \) (thus gaining a drift). Otherwise, with probability \( 1 - p \), it finds no cookie (no drift) and behaves as a \( d \)-martingale with zero-mean vector. If instead the process has already visited the site, there is no cookie and the process acts again as a \( d \)-martingale with zero-mean vector. We call this model \( p \)-GERW. Note that, if \( p = 1 \) than the \( p \)-GERW reduces to GERW. Our model is well motivated since for the many-dimensional excited random walk a relevant question is if it is still possible to guarantee properties such as directional transience and ballisticity by reducing the number of cookies in the system. This goes in the same direction of well-known results in dimension one. Specifically, in dimension one for the nearest neighbor ERW with an independent cookie environment, the mean number of cookies per site should be greater than one for the system to be ballistic, see [4].

Under the same hypothesis of [7] described above, we show that the \( p \)-GERW is ballistic for all \( p \in (0, 1] \) and under the same stronger assumptions, that is, the increments of the regeneration times be i.i.d., we obtain a Law of Large Numbers and a Central Limit Theorem.

1.1. The model. We now formally introduce the \( p \)-GERW. Recall that \( d \geq 2 \) is the fixed dimension and let \( p \in (0, 1] \) be predetermined. In a broader sense our process is a random element \((X, \pi)\) of \((\mathbb{Z}^d)^{\mathbb{Z}_+} \times [0, 1]^\mathbb{Z}_d\) endowed with the product Borel \( \sigma \)-algebra. The second coordinate \( \pi = \{\pi(x)\}_{x \in \mathbb{Z}^d} \in [0, 1]^\mathbb{Z}_d \) is a random element whose marginals have uniform distribution in \([0, 1]\) and independents. We denote by \( Q \) the probability law of \( \pi \). The first coordinate \( X = \{X_n\}_{n \geq 0} \) is a \( \mathbb{Z}^d \) valued process with \( X_0 = 0 \) which is adapted to a filtration \( \mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0} \), where \( \mathcal{F}_n = \sigma(X_1, \ldots, X_n, \pi(X_1), \ldots, \pi(X_n)) \) and \( \sigma(Y) \) represents the smallest \( \sigma \)-algebra generated by a random vector \( Y \). We denote the law of \((X, \pi)\) by \( P \) and by \( E \) its expectation, we can think of \( P \) as the semi-direct measure \( Q \otimes \hat{P}_\pi \), where \( P_\pi \) is the quenched measure for \( X \), i.e., the conditional probability law of \( X \) given a realization \( \hat{\pi} \) of \( \pi \). Now fix \( \ell \in S^{d-1} \), where \( S^{d-1} \) is the unit sphere of \( \mathbb{R}^d \), and let \( ||\cdot|| \) be the euclidean norm in \( \mathbb{R}^d \). The process \( X \) is called a \( p \)-GERW in direction \( \ell \), if it satisfies the following conditions:

**Condition I** (Bounded increments). There exists a constant \( K > 0 \) such that \( \sup_{n \geq 0} ||X_{n+1} - X_n|| < K \) on every realization.

**Condition II.** If there exists \( \lambda > 0 \) such that:

- almost surely on the event \( \{X_k \neq X_n \text{ for all } k < n\} \), either
  
  \[ E[X_{n+1} - X_n | \mathcal{F}_n] \cdot \ell \geq \lambda, \text{ if } \pi(X_n) \leq p, \]
or
\[ \mathbb{E}[X_{n+1} - X_n | F_n] = 0 \], if \( \pi(X_n) > p \).

- almost surely on the event \( \{ \exists k < n \text{ such that } X_k = X_n \} \),
\[ \mathbb{E}[X_{n+1} - X_n | F_n] = 0. \]

**Condition III.** If there exist \( h, r > 0 \) such that
- Uniformly elliptic in direction \( \ell \): for all \( n \)
\[ \mathbb{P} \left[ (X_{n+1} - X_n) \cdot \ell > r | F_n \right] \geq h, \text{ a.s.} \] (UE1)

- Uniformly elliptic on the event \( \{ \mathbb{E}[X_{n+1} - X_n | F_n] = 0 \} \), for all \( \ell' \in S^{d-1} \), with \( ||\ell'|| = 1 \)
\[ \mathbb{P} \left[ (X_{n+1} - X_n) \cdot \ell' > r | F_n \right] \geq h, \text{ a.s.} \] (UE2)

**Example:** In this paragraph, we provide a concrete example of a \( p \)-GERW which may be thought of as a generalization of the classical ERW. Specifically it evolves as the classical ERW but when a site is visited for the first time, it finds a cookie (thus gaining a drift) with probability \( p \). This generalization reduces to the classical ERW when \( p = 1 \). Fix \( \delta \in (1/2, 1] \) and let \( q^{(0)}(x, e_i), x \in \mathbb{Z}^d, i = 1, \ldots, d \), be defined as
\[ p(x, e_1) = \delta/d, \quad p(x, -e_1) = (1 - \delta)/d, \]
\[ p(x, \pm e_i) = 1/2d \quad \text{for all } i = 2, \ldots, d, \]
and \( q^{(1)}(x, e_i), x \in \mathbb{Z}^d, i = 1, \ldots, d \), be the transition probabilities of a SRW in \( \mathbb{Z}^d \). Let \( \{X_n\}_{n \geq 0} \) be a process in \( \mathbb{Z}^d \) with transition probabilities
\[ P \left[ X_{n+1} = x + e_i \bigg| X_n = x, \sum_{j=0}^{n-1} 1_{\{X_j = x\}} = 0 \right] = \]
\[ = 1_{\{\pi(x) \leq p\}} q^{(0)}(x, e_i) + 1_{\{\pi(x) > p\}} q^{(1)}(x, e_i), \]
and for every \( m \in \{1, 2, \ldots, n - 1\} \) we have
\[ P \left[ X_{n+1} = x + e_i \bigg| X_n = x, \sum_{j=0}^{n-1} 1_{\{X_j = x\}} = m \right] = q^{(1)}(x, e_i). \]

The \( \{X_n\}_{n \geq 0} \) is clearly a \( p \)-GERW which we call \( p \)-ERW. See Figure 1 for a simulated realization of the \( p \)-ERW in \( \mathbb{Z}^2 \). This simulation suggests that the \( p \)-ERW is ballistic in direction \( e_1 \). Indeed, we will prove that the \( p \)-ERW satisfies a ballistic Law of Large Numbers and a Central Limit Theorem.

**1.2. Main results.** Our first result is that for every \( p \in (0, 1] \), the \( p \)-GERW is ballistic in \( \ell \) direction.

**Theorem 1.1** (Ballisticity of \( p \)-GERW). Let \( X \) be a \( p \)-GERW in direction \( \ell \in S^{d-1} \). Then
\[ \liminf_{n \to \infty} \frac{X_n \cdot \ell}{n} > 0, \text{ a.s.} \]
Figure 1. 20000 steps simulation of p-ERW for $d = 2$, $p = 0.25$, $q^{(0)}(x, e_1) = 0.375$, $q^{(0)}(x, -e_1) = 0.125$ and $q^{(0)}(x, \pm e_2) = 0.25$. The initial position of the random walk is $X_0 = (0, 0)$ and the final $X_{20000} = (397, -20)$.

The next two results are the Law of Large Numbers and the Central Limit Theorem which hold for a special case of p-GERW. Specifically, we need to introduce a forth condition (see, Condition IV in Section 2) which is related to the distribution of the increments of the regeneration times associated to the p-GERW. A p-GERW satisfying this forth condition will be called p-Strong General Excited Random Walk (p-SGERW). It can be shown (see, Corollary 2.3) that the p-ERW introduced in the Example in Section 1.1 is an example of p-SGERW.

**Theorem 1.2 (Law of Large Numbers).** Assume the process $X$ is a p-SGERW in direction $\ell$ (i.e., satisfies Conditions I, II, III and IV), then there exists $v \in \mathbb{R}^d$ such that $v \cdot \ell > 0$ and

$$\lim_{n \to \infty} \frac{X_n}{n} = v, \quad \text{a.s..} \quad (1)$$

Let $X$ be a p-SGERW in direction $\ell$ and $v \in \mathbb{R}^d$ from (1). Let us define the process

$$B^n_t = \frac{X_{\lfloor nt\rfloor} - \lfloor nt\rfloor v}{n^{1/2}}, \quad t \geq 0. \quad (2)$$

**Theorem 1.3 (Central Limit Theorem).** The process $B^n$ converges in distribution, as $n \to \infty$, to a $d$-dimensional Brownian Motion with a non-degenerate covariance matrix.

This text is organized in the following way: The renewal structure and the main structure of the proof are presented in Section 2. Our main contribution is given in Section 3 where several auxiliary results required to control the renewal structure are proved. It is in these proofs that the particular features of the p-GERW are used. Finally Appendix A, B contain some proofs which were omitted in the main text.
2. REGENERATION STRUCTURE AND PROOFS OF THE MAIN THEOREMS

We start this section defining the regeneration structure for the p-GERW which is a key element in the proofs of the theorems stated in Section 1.2. We will follow closely the regeneration structure constructed in [7]. We make small adjustments in the definition of the regeneration times that will not affect the main properties of the structure. Afterwards we outline the proofs of Theorems 1.1, 1.2 and 1.3. We do not provide the complete proofs, since they are analogous to the ones presented in [7]. Our main contribution is to establish the necessary properties on the regeneration times which is a challenging task for the p-GERW.

2.1. Regeneration times. Consider \(\{X_n\}_{n \geq 0}\) a p-GERW in direction \(\ell \in S^{d-1}\). Fix \(a > 0\) and define

\[
\rho(X_m) := \inf \left\{ n \geq m : X_n \cdot \ell \geq X_m \cdot \ell + a \right\},
\quad
\eta(X_m) := \inf \left\{ n \geq m : X_n \cdot \ell < X_m \cdot \ell \right\}.
\]

We now define the sequence of regeneration times \(\{\tau_k\}_{k \geq 0}\). First, we set \(\tau_0 \equiv 0\) and then we define \(\tau_{k+1}\) from \(\tau_k\) recursively for \(k \geq 0\). If \(\tau_k = \infty\), then \(\tau_{k+1} = \infty\). Assuming \(\tau_k < \infty\), we define

\[
\rho^{(k)}_1 := \rho(X_{\tau_k}),
\quad
\eta^{(k)}_1 := \begin{cases} \eta(X_{\rho^{(k)}_1}) & \rho^{(k)}_1 < \infty, \\ \infty & \rho^{(k)}_1 = \infty. \end{cases}
\]

where \(\rho\) and \(\eta\) are given in (3). Moreover, for every \(i \geq 2\), we recursively define

\[
\rho^{(k)}_i := \begin{cases} \inf \left\{ n \geq \eta^{(k)}_{i-1} : X_n \cdot \ell \geq \max_{k \leq \eta^{(k)}_{i-1}} X_k \cdot \ell + a \right\} & \eta^{(k)}_{i-1} < \infty, \\ \infty & \eta^{(k)}_{i-1} = \infty, \end{cases}
\]

\[
\eta^{(k)}_i := \begin{cases} \eta(X_{\rho^{(k)}_i}) & \rho^{(k)}_i < \infty, \\ \infty & \rho^{(k)}_i = \infty. \end{cases}
\]

Setting \(q_k := \inf \{ n \geq 1 : \rho^{(k)}_n < \infty, \eta^{(k)}_n = \infty \}\), we define \(\tau_{k+1} := \rho^{(k)}_{q_k}\). The time \(\tau_k\), with \(k \geq 1\), represents the \(k\)-th regeneration time. Clearly \(\tau_k\) is not a \(\mathcal{F}_n\)-stopping time for every \(k \geq 1\) and it depends on the whole future of \(\{(X_n, \pi(X_n))\}_{n \geq 1}\). For a better understanding of the sequence of the regeneration times see Figure 2.

**Remark 2.1.** The particular choice of \(a\) is irrelevant in this manuscript. Without prejudice to the proofs presented here we could have chosen \(a = 1\). The choice of \(a\) is only used to show the non-degeneracy of the covariance matrix in Theorem 1.3, the proof is not presented here since it follows from the same arguments as in Theorem 4.1 of [9].
Figure 2. A representation of the regeneration times for a $p$-ERW in $\mathbb{Z}^2$ with drift direction $e_1$ and $a = 1$. The dotted horizontal lines represents that the RW never goes below that position in direction $e_1$.

To deal with the information produced by $\{(X_n, \pi(X_n))\}_{n \geq 1}$ until time $\rho_i^{(k)}$ for each $i \geq 0$ and $k \geq 1$, we also define the $\sigma$-algebras:

$$G_0^{(k)} := \mathcal{F}_0$$

$$G_i^{(k)} = \sigma(\tau_1, \ldots, \tau_k, (X_{n \wedge \tau_k})_{n \geq 0}, (\pi(X_{n \wedge \tau_k}))_{n \geq 0}) \text{ for } k \geq 1,$$

and for all $k \geq 1$ and $i \geq 1$

$$G_i^{(k)} = \sigma(\tau_1, \ldots, \tau_k, (X_{n \wedge \rho_i^k})_{n \geq 0}, (\pi(X_{n \wedge \rho_i^k}))_{n \geq 0}) \text{ for } k \geq 1, \ i \geq 1.$$

The next result is an estimation on the tail probabilities for the increments of the regeneration times which will guarantee existence of moments needed in the proof of Theorem 1.1.

**Proposition 2.1.** Consider a $p$-GERW and $\{\tau_k\}_{k \geq 0}$ its sequence of associated regeneration times. Then there exist positive constants $C'$ and $\phi$ such that for every $n \geq 1$

$$\sup_{k \geq 0} \mathbb{P}[\tau_{k+1} - \tau_k > n | G^{(k)}_0] \leq C' e^{-n^\phi}, \ a.s..$$

Proposition 2.1 will be proved in Section 3 (with some details provided in Appendix A). As a consequence we have the two next results:

**Corollary 2.1.** For every $k \geq 1$ we have that $\tau_k < \infty \ a.s.$

**Corollary 2.2.** For every $k \geq 0$ and $m \geq 1$, we have that,

$$\mathbb{E}[\tau_{k+1}^m | G^k_0] < \infty.$$  

Below, we present a helpful result, which is a version of Proposition 2.2 in [7]. It is worth mentioning that we work on larger spaces, since in our model we have the cookie environment. However the proof follows the same steps as that in [7] and it is deferred to Appendix A. Recall that $(\mathbb{Z}^d)^{\mathbb{Z}_+}$ is the space of trajectories for $X$. 
Proposition 2.2. Suppose \( \{X_n\}_{n \geq 0} \) is a \( p \)-GERW in direction \( \ell \) and \( \{\tau_k\}_{k \geq 1} \) its sequence of associated regeneration times. Let \( X \) define the \( p \)-sequence of sites with independent cookie configurations to explore, since we have equality instead is due to the fact that the process has a totally new area of the \( p \)- GERW. 

\[ \begin{align*}
\text{(i) For every } k \geq 1, \\
\mathbb{P}[X_{\tau_k+} \in A|G^{(k)}_0] &= \sum_{n=1}^{\infty} 1_{\{\tau_k=n\}} \mathbb{P}[X_{n+} \in A|\eta(X_n) = \infty, \mathcal{F}_n], \text{ a.s..} \\
\text{(ii) For every } k, j \geq 1, \\
\mathbb{P}[X_{\rho_j^{(k)}+} \in A|G_j^{(k)}] &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} 1_{\{\tau_k=n\}} 1_{\{\rho_j^{(k)}=n+m\}} \mathbb{P}[X_{n+m+} \in A|\eta(X_n) = \infty, \mathcal{F}_{n+m}], \text{ a.s..} \\
\end{align*} \]

Where \( X_{k+} \) represents the trajectory of the process from \( k \) onwards.

Finally we have all the elements to formally introduce Condition IV and define the \( p \)-SGERW.

Condition IV. Let \( \{\tau_k\}_{k \geq 0} \) be the associated sequence of regeneration times of the \( p \)-GERW. If under \( \mathbb{P} \) we have:

\( \begin{align*}
\text{(i) the increments } \tau_{k+1} - \tau_k, \ k \geq 0, \text{ are independent, and for } k \geq 1 \text{ they are also identically distributed as } \tau_1|\eta(X_0) = \infty. \\
\text{(ii) the random variables } X_{\tau_1}, X_{\tau_{k+1}} - X_{\tau_k}, \ k \geq 1, \text{ are independent, and } \\
X_{\tau_{k+1}} - X_{\tau_k}, \ k \geq 1, \text{ are identically distributed as } X_{\tau_1}|\eta(X_0) = \infty.
\end{align*} \)

As mentioned in the Introduction, a \( p \)-GERW is called a \( p \)-SGERW if it also satisfies Condition IV.

In the next corollary we show that the \( p \)-ERW, introduced in the Example in Section 1.1, is an example of \( p \)-SGERW, that is, it satisfies Condition IV.

Corollary 2.3. The \( p \)-ERW satisfies Condition IV; hence, it is a \( p \)-SGERW.

Proof. Let \( \{X_n\}_{n \geq 0} \) be a \( p \)-ERW and \( \{\tau_k\}_{k \geq 1} \) be the associated sequence of regeneration times. Recall that \( \tau_k < \infty \) a.s. and let \( A \) be a Borel subset and \( \theta \) the canonical shift on \( (\mathbb{Z}^d)^{\mathbb{Z}^+} \), then for all \( k \geq 1 \)

\[ \mathbb{P}[X_{\tau_k+} - X_{\tau_k} \in A|G^{(k)}_0] = \mathbb{P}[X_{\tau_k+} \in A \circ \theta_{\tau_k}|G^{(k)}_0] \]

\[ = \sum_{n=1}^{\infty} 1_{\{\tau_k=n\}} \mathbb{P}[X_{n+} \in A \circ \theta_n|\eta(X_n) = \infty, \mathcal{F}_n] \]

\[ = \mathbb{P}[X \in A|\eta(X_0) = \infty], \quad (4) \]

where we used Proposition 2.2 part (i) in the second equality. The last equality instead is due to the fact that the process has a totally new area of sites with independent cookie configurations to explore, since we have \( \{\eta(X_n) = \infty\} \), then the historical of it does not matter. Besides that the \( p \)-ERW has homogeneous transition probabilities that together with
independence implies that $X_{n+1}\{\tau_k=n\}$ has the same distribution as $X_{\{\eta(X_0)=\infty\}}$, since from direction $e_1$ both processes evolve on identically distributed environments. From the previous equality (4), we have that the $p$-ERW satisfies Condition IV. \hfill \Box

As a consequence of Proposition 2.1 and Condition IV we have the important result for the $p$-SGERW and its associated sequence of regeneration times.

**Corollary 2.4.** Let $X$ be a $p$-SGERW with associated sequence of regeneration times $\{\tau_k\}_{k \geq 0}$. Then we have that

(i) $E[\tau_k^m] < \infty$ and $E[\tau_k^m|\eta(X_0) = \infty] < \infty$, for all $m \geq 1$;

(ii) $E[X_\tau_k^m] < \infty$ and $E[X_\tau_k^m|\eta(X_0) = \infty] < \infty$, for all $m \geq 1$.

**2.2. On the proofs of the main theorems.** Here we just outline the proofs of our main Theorems. As mentioned above, the proofs are analogous to those presented in [7] and they follow from the results on the regeneration times presented in Section 2.1. We point out that the main contribution of this paper is the proof of Proposition 2.1, which makes the regeneration structure work for the $p$-GERW.

The first step in the proof of Theorem 1.1 is to show that for a $p$-GERW in direction $\ell$ with an associated sequence of regeneration times $\{\tau_n\}_{n \geq 1}$ as in Section 2, there exists a constant $C > 0$ such that

$$\limsup_{n \to \infty} \frac{\tau_n}{n} < C,$$ a.s..

The proof is the same as that of Lemma 3.1 in [7]. All we need is finite fourth moment of $\tau_k$ and $X_\tau_k$, for $k \geq 1$, which we have by Corollary 2.2. Afterwards the proof of Theorem 1.1 is also the same as that of Theorem 1.1 in [7].

Concerning the proof of Theorem 1.2 for the $p$-SGERW, Condition IV allows us to follow closely the proof of Proposition 2.1 in [10], which is a Law of Large Numbers for random walks in random environment. There the proof is for nearest-neighbor jumps, but it is simple to adjust it for the case of uniformly bounded jumps. From that proof we obtain

$$\lim_{n \to \infty} \frac{X_n}{n} = \frac{E[X_{\tau_1}|\eta(X_0) = \infty]}{E[\tau_1|\eta(X_0) = \infty]} = v,$$ a.s..

At last, to show Theorem 1.3 we can follow closely the proof of Theorem 4.1 in [9] defining the covariance matrix $A$ as

$$A = \frac{E[(X_{\tau_1} - \tau_1 v)^4(X_{\tau_1} - \tau_1 v)|\eta(X_0) = \infty]}{E[\tau_1|\eta(X_0) = \infty]}.$$ 

Here, as in [9], we have to choose $a > 2\sqrt{d}$ to show that the matrix $A$ is non-degenerate.
3. Proof of Proposition 2.1

To prove Proposition 2.1 we need to state and prove several auxiliary results. First we show that the probability that a \( p \)-GERW \( X = \{X_n\}_{n \geq 0} \) visits less than \( n^{1/2 + \alpha} \) distinct sites until time \( n \) decays as a stretched exponential for all \( \alpha \in (0, 1/6) \) (see, Proposition 3.1). This result can then be used to show that \( X_n \cdot \ell \) is at least of order \( n^{1/2 + \alpha} \) with high probability (see, Proposition 3.2). This establish super-diffusive behaviour for \( X \), although still sub-ballistic, but it is all we need to obtain a key result for the regeneration structure, namely that the probability of \( \{\eta(X_0) = \infty\} \) is bounded from below by a constant whose behavior according to the choice of \( p \) can be explicitly described (see, Proposition 3.3). Finally, in Proposition 3.4 we obtain some additional estimates related to the regeneration structure and we can then prove Proposition 2.1. The above strategy is analogous to that in [7], but several novel ideas had to be implemented to deal with the randomness of the cookie environment.

In this section we state the auxiliary results mentioned above and their proofs are postponed to Section 3.1.

Given a stochastic process \( \{X_n\}_{n \geq 0} \) on the lattice \( \mathbb{Z}^d \), we denote its range at time \( n \) by

\[
\mathcal{R}_n^X := \{ x \in \mathbb{Z}^d : X_k = x \text{ for some } 0 \leq k \leq n \},
\]

i.e., the set of sites visited by the process up to time \( n \). The next result states that if \( X \) is a \( p \)-GERW, then the probability that \( |\mathcal{R}_n^X| \leq n^{1/2 + \alpha} \) decays as a stretched exponential in \( n \) for every \( \alpha \in (0, 1/6) \), where \( |A| \) denotes the number of elements of a set \( A \).

**Proposition 3.1.** Let \( X = \{X_n\}_{n \geq 0} \) be a \( p \)-GERW. Then, for all \( 0 < \alpha < 1/6 \) there exist positive constants \( \gamma_1, \gamma_2 \), which depend on \( d, K, h, \) and \( r \), such that

\[
P[|\mathcal{R}_n^X| < n^{\frac{1}{2} + \alpha}] < \exp\{-\gamma_1 n^{\gamma_2}\},
\]

for all \( n \geq 1 \).

Since the proof of Proposition 3.1 is rather lengthy, it will be deferred to Appendix B.

**Remark 3.1.** It is important to notice that \( \gamma_1 \) and \( \gamma_2 \) do not depend on \( p \). In particular, Proposition 3.1 holds true for \( p = 0 \) which is the case where the random walk is a \( d \)-martingale with zero mean vector satisfying a uniform elliptic condition. Our result also refines the proof Proposition 4.1 in [7] stated for \( p = 1 \) and some \( \alpha > 0 \), in that it quantifies the maximal value of \( \alpha \) for which the statement holds true. For \( p \in (0, 1] \) the determination of an upper bound for \( \alpha \) might not seem relevant, since the ballisticity will imply that \( \mathcal{R}_n^X \) is \( \Theta(n) \), but we do believe that our prove might be useful to discuss generalizations of the \( p \)-GERW, for instance when the probability of having a cookie on the first visit is time dependent.
Remark 3.2. Note that the event \( \{|R_n^X| < n^{\frac{1}{2} + \alpha}\} \) for all \( 0 < \alpha < 1/6 \) can be written as \( \{|R_n^X| < n^{\frac{3}{2} - \varepsilon}\} \) for all \( \varepsilon > 0 \).

To simplify the statement of the next result, we will consider a slight generalization of the \( p \)-GERW. For a fixed set \( A \subset \mathbb{Z}^d \) we say that \( \{X_n\}_{n \geq 1} \) is a \( p \)-GERW with excitation-allowing set \( A \), if it satisfies Condition I, Condition III in Section 1.1 and the following variation of Condition II:

**Condition II**. If there exists \( \lambda > 0 \) such that:

- almost surely on the event \( \{X_k \neq X_n \text{ for all } k < n\} \), either
  \[
  \mathbb{E}[X_{n+1} - X_n|\mathcal{F}_n] \cdot \ell \geq \lambda, \quad \text{if } \pi(X_n) \leq p \text{ and } X_n \in A,
  \]
  or
  \[
  \mathbb{E}[X_{n+1} - X_n|\mathcal{F}_n] = 0, \quad \text{if } \pi(X_n) > p \text{ or } X_n \notin A.
  \]
- almost surely on the event \( \{\exists k < n \text{ such that } X_k = X_n\} \),
  \[
  \mathbb{E}[X_{n+1} - X_n|\mathcal{F}_n] = 0.
  \]

We can think of set \( A \) as the set of sites where there is the possibility of having cookies. The definition will be useful to deal with time translations of the \( p \)-GERW, where \( A \) will represent the set of sites not yet visited by the process.

Set \( H(a, b) \subset \mathbb{Z}^d \) for \( a < b \) as:

\[
H(a, b) := \{x \in \mathbb{Z}^d : x \cdot \ell \in [a, b]\},
\]
which represents the strip in direction \( \ell \) between levels \( a \) and \( b \). Roughly speaking, the next proposition states that if the number of sites outside the excitation-allowing set in a strip with length of order \( n^{\frac{1}{2} + \alpha} \), \( 0 < \alpha < 1/6 \), containing the origin, is also of order \( n^{\frac{1}{2} + \alpha} \), then \( X_n \cdot \ell \) is at least of order \( n^{\frac{1}{2} + \alpha} \) with high probability.

**Proposition 3.2.** Fix \( 0 < \alpha < 1/6 \) and suppose that \( \{X_n\}_{n \geq 1} \) is a \( p \)-GERW with excitation-allowing set \( A \subset \mathbb{Z}^d \). If for some \( n \geq 1 \)

\[
\left| (\mathbb{Z}^d \setminus A) \cap H\left(-n^{\frac{1}{2} + \alpha}, \frac{2\lambda}{3}n^{\frac{1}{2} + \alpha}\right) \right| \leq \frac{1}{3}n^{\frac{3}{2} + \alpha}, \tag{5}
\]

then, for some positive constants \( \gamma_3, \gamma_4 \) depending on \( d, K, r, \lambda, \alpha \) and \( p \), we have

\[
\mathbb{P}\left[ X_n \cdot \ell < \frac{p}{3} \lambda n^{\frac{1}{2} + \alpha} \right] < 6n \exp\{-\gamma_3 n^{\gamma_4}\}, \tag{6}
\]

where

\[
\gamma_3 = \min\left\{\gamma_1, \frac{1}{2K^2}, \frac{\lambda^2}{18K^2}, \left((1/3 - 2/3(1 - \varepsilon))\lambda p\right)^2\right\},
\]
\[
\gamma_4 = \min\left\{\gamma_2, 2\alpha, 1/2 + \alpha\right\},
\]
\( \varepsilon \in (1/2, 1) \) and \( \gamma_1, \gamma_2 \) are the same as in Proposition 3.1.
For every $\ell \in \mathbb{S}^{d-1}$, let $M_\ell$ denote the positive half-space in direction $\ell$, that is, $M_\ell = \{x \in \mathbb{Z}^d : x \cdot \ell > 0\}$. The next result provides a lower bound on the probability of $\{\eta(X_0) = \infty\}$, which will allow us to prove that regeneration times $\{\tau_k\}_{k \geq 1}$ are almost surely finite.

**Proposition 3.3.** Fix $0 < \alpha < 1/6$ and let $X$ be a $p$-GERW in direction $\ell$ with excitation-allowing set $A \subset \mathbb{Z}^d$ such that $M_\ell \subset A$. There exists $\psi > 0$ depending on $d, K, h, r, \lambda, \alpha$ and $p$ such that

$$
P[\eta(X_0) = \infty] \geq P[X_{n} \cdot \ell > 0 \text{ for all } n \geq 1] \geq \psi,
$$

where $\psi = h C \left( \frac{h}{\alpha} \right)^{\frac{1}{1-\alpha}} c c \in (0, 1), \delta = (2 - \alpha)(1/2 + \alpha)$,

$$
C > [r^{-1}] \left( \frac{K}{3} \right)^{\frac{1}{1-\alpha}} \eta,
$$

$$
\eta = \left( \frac{2 - \alpha}{\gamma_3 \varphi_1} \right)^{\frac{1}{\alpha}}, \quad \varphi_1 = \min \{\alpha, (2 - \alpha)\gamma_4\},
$$

and $\gamma_3, \gamma_4$ are as in Proposition 3.2.

We now state the last auxiliary result which will be used in the proof of Proposition 2.1. It provides bounds on probabilities associated to the regeneration times.

**Proposition 3.4.** Let $\{\tau_k\}_{k \geq 0}$ be the regeneration times for a $p$-GERW in direction $\ell$. Take $\gamma_3$ and $\gamma_4$ as in Proposition 3.2 and $\psi$ as in Proposition 3.3, then

(i) $\sup_{j,k \geq 1} P[\eta_j^{(k)} < \infty | G_j^{(k)}] < 1 - \psi$, a.s.,

(ii) $\sup_{k \geq 1} P[(X_{\tau_k + n} - X_{\tau_k}) \cdot \ell < \frac{p}{3} \lambda n^{\frac{1}{2} + \alpha} G_0^{(k)}] < \frac{e^{-\gamma_3 n^{\gamma_4}}}{\psi}$, a.s.,

(iii) $\sup_{j \geq 1, k \geq 0} P[n \leq \eta_j^{(k)} - \rho_j^{(k)} < \infty | G_0^{(k)}] < 12 e^{-\gamma_3 n^{\gamma_4}}$, a.s..

**Remark 3.3.** In the proof of (ii) in Proposition 3.4, we will need a small adaptation of Proposition 3.2. If we want to estimate $\{(X_{\tau_k + n} - X_{\tau_k}) \cdot \ell < \frac{p}{3} \lambda n^{\frac{1}{2} + \alpha}\}$, where $\tau_k$ is a regeneration time, we can still apply Proposition 3.2 to bound the probability of this event. Important to notice that conditional to be in position $X_{\tau_k}$ at time $\tau_k$, the process will not go below that position in direction $\ell$ and it has a completely unexplored environment forward of it. Then in condition (5) we only need to consider the intersection of the strip with $\{x \in \mathbb{Z}^d : x \cdot \ell \geq X_{\tau_k} \cdot \ell\}$. With this in mind, the reader just have to follow straightforwardly the proof of Proposition 3.2.

The proof of Proposition 3.4 will follow closely the proof of Proposition 4.4 in [7] (with minor adjustments due to the fact that the renewal structure is slightly different) and it is deferred to Appendix A.
Using Proposition 3.4, we can now prove Proposition 2.1 basically in the same way as in [2] and [7]. For the sake of completeness the proof is provided in Appendix A. Below we give a sketch of this proof.

**Proof sketch of Proposition 2.1.** The idea is to define the following events:

\[ G_n := \left\{ (X_{\tau_k+n} - X_{\tau_k}) \cdot \ell \leq u_n \right\}, \]
\[ B_n := \bigcap_{j=1}^{v_n} \left\{ \eta_j^{(k)} < \infty \right\} \quad \text{and} \quad F_n := \bigcup_{j=1}^{v_n} \left\{ w_n \leq \eta_j^{(k)} - \rho_j^{(k)} < \infty \right\}, \]

where, for each positive integer \( n \), \( u_n = \lfloor n^{a_1} \rfloor \), \( v_n = \lfloor n^{a_2} \rfloor \) and \( w_n = \lfloor n^{a_3} \rfloor \), with \( a_1, a_2 \) and \( a_3 \) positive real numbers such that \( a_1 < 1/2 + \alpha \), and \( a_2 + a_3 < a_1 \). We choose \( n \) large enough such that \( (K+1)v_n(w_n+1) + 2 + K \leq u_n \) and \( u_n < (p\lambda/3)n^{1/2+\alpha} \). Then we show that

\[ G_n \cap B_n \cap P_n \subset \{ \tau_{k+1} - \tau_k \leq n \}, \]

and now we will be able to control the probability of the event \( \{ \tau_{k+1} - \tau_k \leq n \} \). Hence we have

\[ \mathbb{P}[\tau_{k+1} - \tau_k > n|G_0^{(k)}] \leq \mathbb{P}[G_n|G_0^{(k)}] + \mathbb{P}[B_n|G_0^{(k)}] + \mathbb{P}[F_n|G_0^{(k)}]. \] (7)

For each sum portion in (7), we can use Proposition 3.4 part ii), i) and iii), respectively, to control those probabilities. Thus we obtain for each one:

(i) \( \mathbb{P}[B_n|G_0^{(k)}] \leq (1 - \psi)|n^{a_2}| \);

(ii) \( \mathbb{P}[G_n|G_0^{(k)}] \leq e^{-\gamma_3 n^{\gamma_4}} \psi \);

(iii) \( \mathbb{P}[F_n|G_0^{(k)}] \leq 2|n^{a_2}|e^{-\gamma_3 n^{a_3}} \).

Finally we finish the proof using the above upper bounds in (7).

### 3.1. Proof of auxiliaries results.

#### 3.1.1. Proof of Proposition 3.2.

Let us begin observing that the process \( (X_n \cdot \ell, n \geq 0) \) is a \( \mathcal{F} \)-submartingale and thus \( (-X_n \cdot \ell, n \geq 0) \) is a \( \mathcal{F} \)-supermartingale. Adaptability and integrability follows from the definitions and Condition I. Moreover we have two possible situations by Condition II,

\[ \mathbb{E}[(X_{n+1} - X_n) \cdot \ell|\mathcal{F}_n] = 0 \quad \text{or} \quad \mathbb{E}[(X_{n+1} - X_n) \cdot \ell|\mathcal{F}_n] \geq \lambda. \]

Thus, we have \( \mathbb{E}[X_{n+1} \cdot \ell|\mathcal{F}_n] \geq X_n \cdot \ell \).

As a first step we show that

\[ \mathbb{P}\left[ \max_{k \leq n} X_k \cdot \ell > \frac{2}{3} \lambda n^{\frac{1}{2}+\alpha} \right. \]
\[ \left. , \quad X_n \cdot \ell < \frac{2}{3} \lambda n^{\frac{1}{2}+\alpha} \right] \leq ne^{-C_1 n^{2\alpha}}, \] (8)
for $C_1 > 0$. Note that

$$\left\{ \max_{k \leq n} X_k \cdot \ell > \frac{2}{3} \lambda n^{\frac{1}{2} + \alpha}, \ X_n \cdot \ell < \frac{p}{3} \lambda n^{\frac{1}{2} + \alpha} \right\} \subset \bigcup_{k=1}^n \left\{ X_n \cdot \ell - X_k \cdot \ell < \left( \frac{p}{3} - \frac{2}{3} \right) \lambda n^{\frac{1}{2} + \alpha} \right\},$$

and by Azuma’s inequality for supermartingales with increments uniformly bounded by $K$ (see Lemma 1 of [11]), for every $k = \{1, \ldots, n - 1\}$ it holds that

$$\mathbb{P} \left[ X_n \cdot \ell - X_k \cdot \ell < \left( \frac{p}{3} - \frac{2}{3} \right) \lambda n^{\frac{1}{2} + \alpha} \right] \leq \exp \left( -\frac{\lambda^2 n^{2\alpha}}{18K^2} \right).$$

Then (8) follows from the usual union bound with $C_1 = (\frac{1}{3} \lambda)^2 / 2K^2$. Moreover, again using Azuma’s inequality (for supermartingales), we also have that

$$\mathbb{P} \left[ \min_{k \leq n} X_k \cdot \ell < -n^{\frac{1}{2} + \alpha} \right] \leq n \exp \left\{ -C_2 n^{2\alpha} \right\}, \quad (9)$$

for $C_2 = 1/2K^2$.

Now let $D_k = \mathbb{E}[X_{k+1} - X_k | \mathcal{F}_k]$ and $Y_n = X_n - \sum_{k=0}^{n-1} D_k$. It follows that $Y_n$ is a martingale with bounded increments. Let $G$ be the following event

$$G := \left\{ |\mathcal{R}_n^X| \geq n^{\frac{1}{2} + \alpha} \right\} \cap \left\{ X_k \in H \left( -n^{\frac{1}{2} + \alpha}, \frac{2}{3} \lambda n^{\frac{1}{2} + \alpha} \right) \right\}, \text{ for all } k \leq n \right\}.$$

Using the hypotheses (5), on $G$ we have at least $|\mathcal{R}_n^X| - \frac{1}{3} n^{\frac{1}{2} + \alpha} \geq \frac{2}{3} n^{\frac{1}{2} + \alpha}$ sites visited on the excitation-allowing $A$. Therefore, there exists a Binomial random variable $W$ with parameters $N = \frac{2}{3} n^{\frac{1}{2} + \alpha}$ and $p$ such that on $G$

$$\left( \sum_{k=0}^{n-1} D_k \right) \cdot \ell \geq \lambda W.$$

In order to prove (6), we write that probability as

$$\mathbb{P} \left[ \left\{ X_n \cdot \ell < \frac{p}{3} \lambda n^{\frac{1}{2} + \alpha} \right\} \cap G \right] + \mathbb{P} \left[ \left\{ X_n \cdot \ell < \frac{p}{3} \lambda n^{\frac{1}{2} + \alpha} \right\} \cap G^c \right] \quad (10)$$

and we control both terms separately. We start with the second term. Set

$$E = \left\{ X_n \cdot \ell < \frac{p}{3} \lambda n^{\frac{1}{2} + \alpha} \right\}, \ M = \left\{ |\mathcal{R}_n^X| < n^{\frac{1}{2} + \alpha} \right\},$$

$$J = \left\{ \min_{k \leq n} X_k \cdot \ell < -n^{\frac{1}{2} + \alpha} \right\} \quad \text{and} \quad T = \left\{ \max_{k \leq n} X_k \cdot \ell > \frac{2}{3} \lambda n^{\frac{1}{2} + \alpha} \right\}. $$
It follows that
\[
\mathbb{P}[E \cap G^c] = \mathbb{P}[(E \cap M) \cup (E \cap J) \cup (E \cap T)] \\
\leq \mathbb{P}[E \cap M] + \mathbb{P}[E \cap J] + \mathbb{P}\left[\max_{k \leq n} X_k \cdot \ell > \frac{2}{3} \lambda n^{1+\alpha},\ X_n \cdot \ell < \frac{p}{3} \lambda n^{1+\alpha}\right],
\]
and from Proposition 3.1 and (9) and (8), we obtain
\[
\mathbb{P}[E \cap G^c] \leq \mathbb{P}\left[|R_n^X| < n^{\frac{1}{2}+\alpha}\right] + \mathbb{P}\left[\min_{k \leq n} X_k \cdot \ell < -n^{\frac{1}{2}+\alpha}\right] + ne^{-C_1 n^{2\alpha}}.
\]
\begin{equation}
\leq e^{-\gamma_1 n^{\gamma_2}} + ne^{-C_2 n^{2\alpha}} + ne^{-C_1 n^{2\alpha}}.
\end{equation}

As regards the first term in (10), let \(\varepsilon \in (1/2, 1)\), \(B = \{W \leq Np(1-\varepsilon)\}\), and write \(\mathbb{P}[E \cap G]\) as
\[
\mathbb{P}\left[\{X_n \cdot \ell < \frac{p}{3} \lambda n^{1+\alpha}\} \cap G \cap B\right] + \mathbb{P}\left[\{X_n \cdot \ell < \frac{p}{3} \lambda n^{1+\alpha}\} \cap G \cap B^c\right]
\leq \mathbb{P}[B] + \mathbb{P}\left[\{X_n \cdot \ell < \frac{p}{3} \lambda n^{1+\alpha}\} \cap G \cap B^c\right].
\]
To bound \(\mathbb{P}[B]\) we use the Chernoff bound (cf., e.g., Theorem 4.5 of [8]) to obtain
\begin{equation}
\mathbb{P}[B] \leq \exp\left(-\frac{\varepsilon^2}{2} Np\right) = \exp\left(-C_3 n^{\frac{1}{2}+\alpha} p\right),
\end{equation}
where \(C_3 = \varepsilon^2/3\). To upper bound \(\mathbb{P}\left[\{X_n \cdot \ell < \frac{p}{3} \lambda n^{1+\alpha}\} \cap G \cap B^c\right]\), we use that \(Y_n = X_n - \sum_{k=0}^{n-1} D_k\) is martingale with bounded increments and apply Azuma’s inequality (see, for example, Theorem 2.19 in [3]). Thus, denoting \(F = \{X_n \cdot \ell < \frac{p}{3} \lambda n^{1+\alpha}\} \cap G \cap B^c\) we obtain
\[
\mathbb{P}[F] \leq \mathbb{P}\left[X_n \cdot \ell - \left(\sum_{k=1}^{n-1} D_k\right) \cdot \ell < \frac{p}{3} \lambda n^{1+\alpha} - \lambda Np(1-\varepsilon)\right]
\leq \mathbb{P}\left[X_n \cdot \ell - \left(\sum_{k=1}^{n-1} D_k\right) \cdot \ell < p\lambda n^{1+\alpha}\left(\frac{1}{3} - \frac{2}{3}(1-\varepsilon)\right)\right].
\]
Hence, we have that \(C_4 := -(1/3 - 2/3(1-\varepsilon)) > 0\), since \(\varepsilon \in (1/2, 1)\). By (13) and Azuma’s inequality we obtain
\begin{equation}
\mathbb{P}[F] \leq \mathbb{P}\left[X_n \cdot \ell - \left(\sum_{k=1}^{n-1} D_k\right) \cdot \ell < -C_4 p\lambda n^{1+\alpha}\right]
\leq 2 \exp\left(-\frac{C_4 p^2 \lambda^2 n^{2\alpha}}{2K^2}\right) = 2 \exp\left(-C_5 p^2 n^{2\alpha}\right),
\end{equation}
where \(C_5 = (C_4^2 \lambda^2)/2K^2\).
Inequality (6) then follows from (11), (12) and (14) which imply that
\[
\mathbb{P}\left[X_n \cdot \ell < \frac{p}{3} \lambda n^{1+\alpha}\right] \leq e^{-\gamma_1 n^{\gamma_2}} + ne^{-C_2 n^{2\alpha}} + ne^{-C_1 n^{2\alpha}} + e^{-C_4 n^{1+\alpha} p} + 2e^{-C_5 p^2 n^{2\alpha}} \leq 6n e^{-\gamma_1 n^{\gamma_2}},
\]
where

\[ \gamma_3 = \min \left\{ \gamma_1, \frac{1}{2K^2}, \frac{\lambda^2}{18K^2}, \frac{\varepsilon^2 p}{3}, \frac{((1/3 - 2/3(1 - \varepsilon))\lambda p)^2}{2K^2} \right\} \quad \text{and} \]

\[ \gamma_4 = \min \{ \gamma_2, 2\alpha, 1/2 + \alpha \} . \]

\[ \square \]

3.1.2. Proof of Proposition 3.3. The main idea of this proof is to define an event to guarantee that the process \( X \) will advance in direction \( \ell \) in such a way that \( X_n \cdot \ell > 0 \), for all \( n \geq 1 \). Then we will obtain a lower bound for this event using Proposition 3.2 and Azuma’s inequality. As it can be noticed from the statement, one of our main concerns is to make explicit the dependency of the bound on the parameter \( p \).

\textbf{Proof of Proposition 3.3.} Since on \( \{ X_n \cdot \ell > 0 \text{ for all } n \geq 1 \} \) the process doesn’t visit the sites in \( \mathbb{Z}^d/M\ell \), it is sufficient to prove the proposition for \( A = \mathbb{Z}^d \).

Without loss of generality we consider \( r \leq 1 \) in Condition III. Define

\[ U_0 = \left\{ (X_{k+1} - X_k) \cdot \ell \geq r , \text{ for all } k = 0, 1, \ldots, \lfloor r^{-1}m \rfloor - 1 \right\} . \]

Observe that in \( U_0, X_{\lfloor r^{-1}m \rfloor} \cdot \ell \geq m \) and by (UE1) of Condition III we have

\[ \mathbb{P}[U_0] \geq h^{\lfloor r^{-1}m \rfloor} . \tag{15} \]

Consider the following time translation of the process \( X \): \( W_k = X_{\lfloor r^{-1}m \rfloor + k}, k \geq 0 \). Then \( W \) is a \( p \)-GERW with excitation-allowing set

\[ A' = \mathbb{Z}^d / \{ X_0, \ldots, X_{\lfloor r^{-1}m \rfloor - 1} \} \]

starting at \( W_0 = y_0 := X_{\lfloor r^{-1}m \rfloor} \).

Set \( \delta = (2 - \alpha)/(1/2 + \alpha) \) and

\[ m = C \left( \frac{3}{\lambda p} \right)^{\frac{1}{3\gamma_1}}, \]

where \( C > 0 \) is a constant depending on \( \alpha, K, \lambda \) and \( r \), such that

\[ C > \left( [r^{-1}]^\delta \lambda \right)^{\frac{1}{3\gamma_1}} \lor \left( [r^{-1}] \left( \frac{K}{3} \right)^{\frac{1}{3\gamma_1}} \eta \right) \]

for

\[ \eta = \left( \frac{2 - \alpha}{\gamma_3 \varphi_1} \right)^{\frac{1}{\gamma_1}} \text{ with } \varphi_1 = \min \{ \alpha, (2 - \alpha)\gamma_4 \} , \]

and \( \gamma_3, \gamma_4 \) as in the statement of Proposition 3.2. Note that for all \( \alpha \) used in Proposition 3.1, i.e., \( 0 < \alpha < 1/6 \), we have that \( \delta > 1 \).

The left-hand side of (5) with the set \( A' - y_0 \) is bounded above by \( [r^{-1}m] \).

Note that, for all \( n \geq m^{2-\alpha} ,

\[ \frac{1}{3} m^{1/2 + \alpha} \geq \frac{1}{3} m^{(2-\alpha)(1/2 + \alpha)} \geq \left( \frac{m^{\delta - 1}}{3[r^{-1}]} \right) [r^{-1}]m \geq [r^{-1}]m , \]
where the last inequality follows from \( C > \left( \left\lceil r^{-1} \right\rceil \lambda \right)^{\frac{1}{2}} \). Thus (5) with excitation-allowing set \( A' - y_0 \) is satisfied for all \( n \geq m^{2-\alpha} \).

Denote \( m_0 = 0, m_1 = m \) and, for \( k \geq 1, m_{k+1} = \frac{p}{3} \lambda m^\delta \). The sequence of \((m_k, k \geq 1)\) is increasing. The latter can be proved by induction since

\[
\frac{m_2}{m_1} = \frac{\lambda p}{3} m^{\delta - 1} = \left( \frac{C}{r^{-1}} \right)^{\delta - 1} > \frac{K \delta - 1}{3} > \frac{1}{3} \left( \frac{2 - \alpha}{\gamma^3 \varphi} \right)^{\frac{\delta - 1}{\alpha}} > 1 ,
\]

for all \( \alpha \in (0, 1/6) \), and assuming \( m_k/m_{k-1} > 1 \), we have

\[
\frac{m_{k+1}}{m_k} = \frac{\lambda p^\delta}{3} \frac{m_k^\delta}{m_{k-1}^\delta} = \left( \frac{m_k}{m_{k-1}} \right)^\delta > 1 .
\]

For every \( k \geq 1 \) consider the following events

\[
G_k = \left\{ \min_{m_{k-1}^2 < j \leq m_k^2} \left( W_j - W_{m_{k-1}^2} \right) \cdot \ell > -m_k \right\},
\]

\[
U_k = \left\{ W_{m_k^2} \cdot \ell \geq m_{k+1} \right\} .
\]

**Claim:** The following set inclusion holds:

\[
\{ X_n \cdot \ell > 0 \, \text{for all} \, n \geq 1 \} \supset \left( \bigcap_{k=1}^\infty (G_k \cap U_k) \right) \cap U_0 . \tag{16}
\]

We will postpone the proof of the Claim to the end of the proof of Proposition 3.3.

As seen in proof of Proposition 3.2, the process \((X_n \cdot \ell, n \geq 0)\), is a \( \mathcal{F} \)-submartingale, so \((W - y_0) \cdot \ell\) is also \( \mathcal{F} \)-submartingale. Write

\[
G_k^c = \bigcup_{j = m_{k-1}^2}^{m_k^2} \left\{ \left( W_j - W_{m_{k-1}^2} \right) \cdot \ell \leq -m_k \right\},
\]

and by Azuma’s inequality (for supermartingales)

\[
\mathbb{P} \left[ \left( W_j - W_{m_{k-1}^2} \right) \cdot \ell \leq -m_k \right] = \mathbb{P} \left[ \left( W_{m_{k-1}^2} - W_j \right) \cdot \ell \geq m_k \right] \\
\leq \exp \left( - \frac{m_k^2}{2K^2 (j - m_{k-1}^2)} \right) \leq \exp \left( - \frac{m_k^2}{2K^2 m_{k-1}^2} \right) \leq \exp \left( - \frac{m_k^\alpha}{2K^2} \right).
\]

Thus

\[
\mathbb{P}[G_k|U_0] \geq 1 - \left( m_{k-1}^2 - m_{k-1}^{2-\alpha} \right)e^{-m_k^\alpha/2K^2} \geq 1 - m_k^{2-\alpha} e^{-m_k^\alpha/2K^2} ,
\]

for every \( j = \{ m_{k-1}^2 \} + 1, \ldots, m_k^2 \). Since the process \( W - y_0 \) satisfies Conditions I, II, III and the set \( A' - y_0 \) fulfills (5) for all \( n \geq m^{2-\alpha} \), by Proposition 3.2, it holds that

\[
\mathbb{P}[U_k|U_0] = \mathbb{P} \left[ W_{m_k^2} \cdot \ell \geq \frac{\lambda p}{3} m_k^{2-\alpha} (\ell + 1) \right] \geq 1 - 6m_k^{2-\alpha} e^{-\gamma_3 m_k^{2-\alpha} \gamma_4} .
\]
Now, write
\[
\mathbb{P} \left[ \left( \bigcap_{k=1}^{\infty} (G_k \cap U_k) \right) \cap U_0 \right] = \mathbb{P}[U_0] \left( 1 - \sum_{k=1}^{\infty} \mathbb{P}[G_k \mid U_0] + \mathbb{P}[U_k^c \mid U_0] \right),
\]
which is bounded from below by
\[
h^{[r^{-1}]} \left( 1 - \sum_{k=1}^{\infty} \left( m_k^{2-\alpha} e^{-\frac{m_k^\alpha}{2} \eta} + 6m_k^{2-\alpha} e^{-\gamma_3 m_k^{\gamma_3} \eta} \right) \right) \geq h^{[r^{-1}]} \left( 1 - 7 \sum_{k=1}^{\infty} m_k^{2-\alpha} e^{-\gamma_3 m_k^{\gamma_3}} \right). \tag{17}
\]
Now we are going to analyze the series \( \sum_{k=1}^{\infty} m_k^{2-\alpha} e^{-\gamma_3 m_k^{\gamma_3}} \). Note that \( m \) is large enough so that the sequence \( (m_k^{2-\alpha} e^{-\gamma_3 m_k^{\gamma_3}})_k \) is decreasing.

Indeed, \( m \) is bigger than the inflection point \( \left( \frac{2-\alpha}{\gamma_3 \varphi_1} \right)^{\frac{1}{\gamma_1}} \) of the function \( z(x) = x^{2-\alpha} e^{-\gamma_3 x^{\varphi_1}} \), \( x > 0 \):
\[
m = \left( \frac{C}{r^{-1}} \right) \left( \frac{3}{\lambda p} \right)^{\frac{1}{\gamma_1}} > \left( \frac{K}{\lambda p} \right)^{\frac{1}{\gamma_1}} \eta \left( \frac{3}{\lambda p} \right)^{\frac{1}{\gamma_1}} \geq \left( \frac{K}{\lambda p} \right)^{\frac{1}{\gamma_1}} \eta \geq \left( \frac{2-\alpha}{\gamma_3 \varphi_1} \right)^{\frac{1}{\gamma_1}}.
\]
Thus we have,
\[
\sum_{k=1}^{\infty} m_k^{2-\alpha} e^{-\gamma_3 m_k^{\gamma_3}} \leq \int_{m_1}^{\infty} x^{2-\alpha} e^{-\gamma_3 x^{\varphi_1}} dx. \tag{18}
\]
By a change of variables, we write,
\[
\int_{m_1}^{\infty} x^{2-\alpha} e^{-\gamma_3 x^{\varphi_1}} dx = \varphi_1^{-1} \gamma_3^{\frac{\alpha-3}{\varphi_1}} \Gamma \left( \frac{3-\alpha}{\varphi_1}, \gamma_3 m_1^{\varphi_1} \right), \tag{19}
\]
where \( \Gamma \) is the incomplete gamma function. As mentioned above \( m \) is large enough so that the sequence \( (m_k^{2-\alpha} e^{-\gamma_3 m_k^{\gamma_3}})_k \) is decreasing. Thus, in order to obtain that (19) is smaller than \( 1/7 \), we may increase \( m \) even further by choosing a sufficiently bigger \( C \). Thus, with such a suitable chosen \( C \) we obtain
\[
\sum_{k=1}^{\infty} m_k^{2-\alpha} e^{-\gamma_3 m_k^{\gamma_3}} \leq \int_{m_1}^{\infty} x^{2-\alpha} e^{-\gamma_3 x^{\varphi_1}} dx
\]
\[
= \varphi_1^{-1} \gamma_3^{\frac{\alpha-3}{\varphi_1}} \Gamma \left( \frac{3-\alpha}{\varphi_1}, \gamma_3 m_1^{\varphi_1} \right) < \frac{1}{7}. \tag{20}
\]
\[\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt.\]
Using (20) in (17), we obtain that,
\[
\mathbb{P}\left[\bigcap_{k=1}^{\infty} (G_k \cap U_k) \cap U_0\right] \geq h^{[r^{-1}]m} \left(1 - 7 \sum_{k=1}^{\infty} m_k^{2-\alpha} e^{-\gamma_3m_k^{a_1}}\right)
\geq h^{[r^{-1}]m} \frac{c}{[\alpha/(\alpha + 1)]^{m_k^{a_1}}} c = \psi,
\]
where \(c\) is a positive constant such that \(c \in (0,1)\). Proposition 3.3 then follows from (16) which we are going to prove below.

**Proof of the Claim:** First observe that follows from (16) which we are going to prove below.

**Proof of Proposition 2.1.** First we consider the case \(k \geq 2\) we have \(X_n \cdot \ell > 0\) for all \(n \in (0, [r^{-1}]m + m_k^{a_1}]\). Indeed, suppose that in \(G_1 \cap U_0\) there exists at least a \(k \in ([r^{-1}]m, [r^{-1}]m + m_k^{a_1}]\) such that \(X_k \cdot \ell \leq 0\). Thus, we would have \((X_k - X_{[r^{-1}]m}) \cdot \ell \leq -m\), which contradicts \(G_1\).

**Appendix A. Proof of Propositions 2.1, 2.2 and 3.4**

The proof of Proposition 2.1 closely follows Proposition 1 in [2] and Proposition 2.1 in [7].

**Proof of Proposition 2.1.** First we consider the case \(k \geq 1\). Fix \(0 < \alpha < 1/6\) and let \(a_1, a_2\) and \(a_3\) be positive real numbers such that \(a_1 < 1/2 + \alpha\), and \(a_2 + a_3 < a_1\). For each positive integer \(n\), we denote \(u_n = [n^{a_1}]\), \(v_n = [n^{a_2}]\) and \(w_n = [n^{a_3}]\). Now we choose \(n\) large enough such that \((K+1)v_n(w_n+1) + 2 + K \leq u_n\) and \(u_n < (p\lambda/3)n^{1/2+\alpha}\). Let us define the following events:

\[
G_n := \{(X_{\tau_k+n} - X_{\tau_k}) \cdot \ell \leq u_n\},
\]
\[
B_n := \bigcap_{j=1}^{v_n} \{\eta_j^{(k)} < \infty\} \quad \text{and} \quad F_n := \bigcup_{j=1}^{v_n} \{w_n \leq \eta_j^{(k)} - \rho_j^{(k)} < \infty\},
\]
where, in the definition of \(F_n\), we use the convention that \(\eta_j^{(k)} - \rho_j^{(k)} = \infty\) whenever \(\eta(X_{\tau_j^{(k)}}) = \infty\).

Our first step will be to show that
\[
G_n^c \cap B_n^c \cap F_n^c \subset \{\tau_{k+1} - \tau_k \leq n\},
\quad (21)
\]
Afterwards we will estimate separately the probabilities of \( G_n, B_n \) and \( F_n \) to finish the proof.

On the event \( B_n^c \), we can define
\[
M = \inf \{ 1 \leq j \leq v_n : \eta_j^{(k)} = \infty \},
\]
and from definition we have \( \tau_{k+1} = \rho_j^{(k)} \). Hence, we shall prove that \( \{ \rho^{(k)}_M - \tau_k \leq n \} \) always happens in \( G_n^c \cap B_n^c \cap F_n^c \), then we have (21).

For each natural \( m \geq \tau_k \), define
\[
r_n = \max \{ (X_j - X_{\tau_k}) \cdot \ell : \tau_k \leq j \leq m \}.
\] (22)

By the definition of \( M \) we have \( \eta_{M-1}^{(k)} < \infty \). Set \( \eta_{0}^{(k)} = \tau_k \) and write
\[
\sum_{j=1}^{M-1} \left( (r_{\eta_{j}^{(k)}} - r_{\rho_{j}^{(k)}}) + (r_{\rho_{j}^{(k)}} - r_{\eta_{j-1}^{(k)}}) \right) = r_{\eta_{M-1}^{(k)}} - r_{\rho_{0}^{(k)}} = r_{\eta_{M-1}^{(k)}}.
\] (23)

We are going to analyze separately each term on the right hand side of the of (23), that is, \( (r_{\eta_{j}^{(k)}} - r_{\rho_{j}^{(k)}}) \) and \( (r_{\rho_{j}^{(k)}} - r_{\eta_{j-1}^{(k)}}) \). By the definition of \( \rho_j^{(k)} \) we get directly
\[
r_{\rho_{j}^{(k)}} = (X_{\rho_{j}^{(k)}} - X_{\tau_k}) \cdot \ell.
\] (24)

By Condition I, we have that each jump of the process has a maximum range \( K \), thus for each \( 1 \leq j \leq M - 1 \), we have that \( (r_{\eta_{j}^{(k)}} - r_{\rho_{j}^{(k)}}) \leq K(\eta_{j}^{(k)} - \rho_{j}^{(k)}) \). Now in the event \( F_n^c \) we have \( \eta_{j}^{(k)} - \rho_{j}^{(k)} < w_n \) for \( 1 \leq j \leq M - 1 \), thus \( (r_{\eta_{j}^{(k)}} - r_{\rho_{j}^{(k)}}) \leq Kw_n \). For \( (r_{\rho_{j}^{(k)}} - r_{\eta_{j-1}^{(k)}}) \), from what we saw in (24), Condition I and by the definition of \( \rho_j^{(k)} \), we have \( (r_{\rho_{j}^{(k)}} - r_{\eta_{j-1}^{(k)}}) \leq K + 1 \) for each \( 1 \leq j \leq M - 1 \). From those inequalities and (23) we obtain
\[
r_{\eta_{M-1}^{(k)}} \leq \sum_{j=1}^{M-1} ((Kw_n) + (K + 1)) \leq v_n(K + 1)(w_n + 1).
\]

Since we are considering \( n \) large enough such that \( v_n(K+1)(w_n+1)+2+K \leq u_n \), then \( r_{\eta_{M-1}^{(k)}} + 2 + K \leq u_n \) on \( B_n^c \cap F_n^c \).

Now on the event \( G_n^c \cap B_n^c \cap F_n^c \), we have \( (X_{\tau_k+j} - X_{\tau_k}) \cdot \ell > r_{\eta_{M-1}^{(k)}} + 2 + K \). Set \( i = \min \{ j \leq n : (X_{\tau_k+j} - X_{\tau_k}) \cdot \ell > r_{\eta_{M-1}^{(k)}} + 1 \} \). Since \( \tau_{k+1} = \rho_{M}^{(k)} \) and \( \rho_{M}^{(k)} \) is the first time the process move forward at least one position in direction \( \ell \) from \( \eta_{M-1}^{(k)} \), i.e. the maximum the walk reaches in direction \( \ell \) until time \( \eta_{M-1}^{(k)} \), we have that \( \rho_{M}^{(k)} - \tau_k = i \leq n \) which gives (21).

Using (21), we have that
\[
\mathbb{P}[\tau_{k+1} - \tau_k > n|G_0^{(k)}] \leq \mathbb{P}[G_n|G_0^{(k)}] + \mathbb{P}[B_n|G_0^{(k)}] + \mathbb{P}[F_n|G_0^{(k)}]. \quad (25)
\]
Now we will bound these three probabilities. By Proposition 3.4 part (ii)
\[
P[G_n|G_0^{(k)}] = P[(X_{n+n} - X_{n}) \cdot \ell \leq u_n|G_0^{(k)}] \leq P[(X_{n+n} - X_{n}) \cdot \ell \leq \frac{P}{3}\lambda n^{2+\alpha}|G_0^{(k)}] \leq \frac{e^{-\gamma n^\beta}}{\psi}.
\] (26)

By Proposition 3.4 part (iii) we have
\[
P[F_n|G_0^{(k)}] = P\left[\bigcup_{j=1}^{n} \{w_n \leq \eta_j^{(k)} - \rho_j^{(k)} < \infty\} \big| G_0^{(k)}\right] \leq \sum_{j=1}^{n} \frac{P}{2} \left\{w_n \leq \eta_j^{(k)} - \rho_j^{(k)} < \infty\right\} G_0^{(k)} \leq \sum_{j=1}^{n} 2e^{-\gamma w_n^\beta} = 2n^\alpha e^{-\gamma n^{\alpha}}. \] (27)

Finally using Proposition 3.4 part (i) we have
\[
P[B_n|G_0^{(k)}] = P\left[\bigcap_{j=1}^{n} \{\eta_j^{(k)} < \infty\} \big| G_0^{(k)}\right] = P\left[\eta_0^{(k)} < \infty\big| G_0^{(k)}, \bigcap_{j=1}^{n} \{\eta_j^{(k)} < \infty\}\right] P\left[\bigcap_{j=1}^{n} \{\eta_j^{(k)} < \infty\} \big| G_0^{(k)}\right].
\] (28)

From the definition of the regeneration times
\[
\{\eta_{v_n}^{(k)} < \infty\} = \{\eta_{v_n}^{(k)} < \infty\} \cap \{\rho_{v_n}^{(k)} < \infty\} = \emptyset,
\]
then we have
\[
P\left[\eta_0^{(k)} < \infty\big| G_0^{(k)}, \bigcap_{j=1}^{n} \{\eta_j^{(k)} < \infty\}\right] = P\left[\rho_{v_n}^{(k)} < \infty\big| G_0^{(k)}, \bigcap_{j=1}^{n} \{\eta_j^{(k)} < \infty\}\right] \times P\left[\{\eta_0^{(k)} < \infty\} \big| G_0^{(k)}, \{\rho_{v_n}^{(k)} < \infty\}, \bigcap_{j=1}^{n} \{\eta_j^{(k)} < \infty\}\right] \leq P\left[\{\eta_{v_n}^{(k)} < \infty\} \big| G_0^{(k)}\right].
\]

Using the last inequality and Proposition 3.4 part (i) in (28), we obtain
\[
P[B_n|G_0^{(k)}] \leq P\left[\{\eta_{v_n}^{(k)} < \infty\} \big| G_0^{(k)}\right] P\left[\bigcap_{j=1}^{n} \{\eta_j^{(k)} < \infty\} \big| G_0^{(k)}\right] \leq \prod_{j=1}^{n} P\left[\eta_j^{(k)} < \infty\big| G_0^{(k)}\right] \leq \prod_{j=1}^{n} (1 - \psi) = (1 - \psi)^{v_n} = (1 - \psi)^{n^{\alpha+\beta}}.
\] (29)
where we have used induction in second inequality. Next, using (26), (27) and (29) in (25) we obtain

$$\mathbb{P}[\tau_{k+1} - \tau_k > n|G_0^{(k)}] \leq \mathbb{P}[G_n|G_0^{(k)}] + \mathbb{P}[B_n|G_0^{(k)}] + \mathbb{P}[F_n|G_0^{(k)}]$$

$$\leq e^{-\gamma_n \psi} + 12[n^{a_2}]e^{-\gamma_3[n^{a_2}]} + (1 - \psi)[n^{a_2}]$$

$$\leq e^{-\gamma_n \psi} + 12[n^{a_2}]e^{-\gamma_3[n^{a_2}]} + e^{-\psi[n^{a_2}]} ,$$

finishing the proof for $k \geq 1$.

It remains to prove the result for $k = 0$, i.e., there exist positive constants $\tilde{C}$ and $\zeta$ such that for some $n \geq 1$

$$\mathbb{P}[\tau_1 > n] \leq \tilde{C}e^{-n^\zeta} .$$  \hspace{1cm} (30)

The proof of (30) is analogous as that for $k \geq 1$, the only difference is in events. To prove (30) we consider

$$G_n = \{X_n \cdot \ell \leq u_n\},$$

$$B_n = \bigcap_{j=1}^{v_n} \{\eta_j^{(0)} < \infty\} \quad \text{and} \quad F_n = \bigcup_{j=1}^{v_n} \{w_n \leq \eta_j^{(0)} - \rho_j^{(0)} < \infty\} .$$

Then we can conclude the proof. \hspace{1cm} \square

The proof of Proposition 2.2 follows closely that of Proposition 2.2 in [7].

**Proof of Proposition 2.2.** (i): We first prove the case $k = 1$. For each $n \geq 1$, we construct a set of trajectories $\Lambda_n$ of the form $\{x_0, \ldots, x_n\}$ such that $x_n \cdot \ell > \sup_{0 \leq i \leq n-1} x_i \cdot \ell$. For a element $\gamma \in (\mathbb{Z}^d)^N$, we will denote $\gamma_n$ as a projection to the $n$ first coordinates and $\pi(\gamma_n)$ as the $n$ first Bernoulli’s trials.

Let $b \in \Lambda_n$, then we have that $G_0^{(1)}$ is generated by the disjoint collection of the form $\{\tau_1 = n\} \cap \{\gamma_n = b\} \cap \{\pi(\gamma_n) = \pi(b)\}$. Thus we obtain,

$$\mathbb{P}[X_{\tau_1+} \in A|G_0^{(1)}] =$$

$$= \sum_{n=1}^{\infty} 1_{\{\tau_1 = n\}}(\omega) \sum_{\gamma_n \in \Lambda_n} 1_{\{\omega_n = \gamma_n\}} \mathbb{P}[X_{\tau_1+} \in A|\tau_1 = n, \omega_n = \gamma_n, \pi(\omega_n) = \pi(\gamma_n)]$$

$$= \sum_{n=1}^{\infty} 1_{\{\tau_1 = n\}}(\omega) \sum_{\gamma_n \in \Lambda_n} 1_{\{\omega_n = \gamma_n\}} \mathbb{P}[X_{\tau_1+} \in A|\eta(X_n) = \infty, \omega_n = \gamma_n, \pi(\omega_n) = \pi(\gamma_n)]$$

$$= \sum_{n=1}^{\infty} 1_{\{\tau_1 = n\}}(\omega) \mathbb{P}[X_{\tau_1+} \in A|\eta(X_n) = \infty, F_n] .$$  \hspace{1cm} (31)

The third equality in (31) follows from observing that in the event $\{\tau_1 = n\}$,

$$\{\omega_n = \gamma_n, \pi(\omega_n) = \pi(\gamma_n)\} = \{\eta(X_n) = \infty, \omega_n = \gamma_n, \pi(\omega_n) = \pi(\gamma_n)\} .$$
The fourth equality in (31) follows from noticing that \(1_{\{\tau_1 = n\}}(\omega)1_{\{\omega_n = \gamma_n\}} = 0\), if \(\gamma_n \not\in \Lambda_n\). As a matter of fact, if \(\gamma_n \not\in \Lambda_n\), by the property of the regeneration times, it is not possible the trajectory \(\gamma\) has \(\tau_1 = n\), since in the first \(n\) coordinates it must exists a \(0 \leq j \leq n - 1\) such that \(x_j \cdot \ell > x_n \cdot \ell\).

The case \(k \geq 2\) can be proved in a similar way. Indeed, the sequence of regeneration time is increasing and in the event \(\{\tau_k = n\}\) we always have \(\{\eta(X_n) = \infty\}\).

(ii): This proof is similar to Proposition 2.2 part (ii) in [7]. For a element \(\gamma \in (\mathbb{Z}^d)^N\), we will denote by \(\gamma_n\) as a projection to the \(n\) first coordinates and \(\pi(\gamma_n)\) as the \(n\) first Bernoulli’s trials. First we consider the case \(k = 1\) and \(j = 1\).

For an \(u > 0\), let us denote by \(T_u\) the first time the process reaches or exceeds this position in direction \(\ell\), i.e., \(T_u = \inf\{k \geq 1, X_k \cdot \ell \geq u\}\). For each \(n \geq 1\) and \(m \geq 1\), we construct a set of trajectories \(\Lambda_{n,m}\) of the form \(\{x_0, x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}\}\) satisfying the following properties:

(i) \(x_n \cdot \ell > \sup_{0 \leq i \leq n-1} x_i \cdot \ell\).

(ii) For each \(0 \leq i \leq n - 1\), we have,

\[
\min_{T_{x_i}, t \leq i \leq n-1} x_t \cdot \ell \leq x_i \cdot \ell.
\]

(iii) \(x_{n+m} \cdot \ell \geq x_n \cdot \ell + 1 > \sup_{0 \leq i \leq n+m-1} x_i \cdot \ell\)

One can see that, for this set of trajectories \(\Lambda_{n,m}\), if \(\eta(x_n) = \infty\) then \(\rho_1^{(1)} = n + m\) and \(\tau_1 = n\). Let \(b \in \Lambda_{n,m}\), thus \(G_1^{(1)}\) is generated by the disjoint collection of the form \(\{\tau_1 = n\} \cap \{\gamma_{n+m} = b\} \cap \{\pi(\gamma_{n+m}) = \pi(b)\}\).

Hence we obtain,

\[
\mathbb{P}[X_{\rho_1^{(1)}} \in A|G_1^{(1)}] = \sum_{n=1}^{\infty} 1_{\{\tau_1=n\}}(\omega) \sum_{m=1}^{\infty} 1_{\{\rho_1^{(1)}=n+m\}}(\omega) \sum_{\gamma_{n+m} \in \Lambda_{n,m}} 1_{\{\omega_{n+m} = \gamma_{n+m}\}} \times \mathbb{P}[X_{n+m+} \in A|\tau_1 = n, \omega_{n+m} = \gamma_{n+m}, \pi(\omega_{n+m}) = \pi(\gamma_{n+m})].
\]

Thus, in the event \(\{\tau_1 = n\}\), we have \(\{\omega_{n+m} = \gamma_{n+m}, \pi(\omega_{n+m}) = \pi(\gamma_{n+m})\} = \{\eta(X_n) = \infty, \omega_{n+m} = \gamma_{n+m}, \pi(\omega_{n+m}) = \pi(\gamma_{n+m})\}\) = \(\{\eta(X_n) = \infty, \omega_{n+m} = \gamma_{n+m}, \pi(\omega_{n+m}) = \pi(\gamma_{n+m})\}\). Now observe that,

\[
1_{\{\tau_1=n\}}(\omega)1_{\{\rho_1^{(1)}=n+m\}}(\omega)1_{\{\omega_{n+m} = \gamma_{n+m}\}} = 0,
\]

if \(\gamma_{n+m} \not\in \Lambda_{n,m}\). One can see that if \(\gamma_{n+m} \not\in \Lambda_{n,m}\), by the property of the regeneration times, it is not possible the trajectory \(\gamma\) has \(\tau_1 = n\), by the fact that in the first \(n\) coordinates we have a \(0 \leq j \leq n - 1\) such that \(x_j \cdot \ell > x_n \cdot \ell\). Besides that, if we have a \(n + 1 \leq j \leq n + m - 1\), such that \(x_j \cdot \ell \geq x_n \cdot \ell + 1\),
then \( \rho_{1}^{(1)} = j \) by the definition of \( \rho_{1}^{(1)} \). Going back to (32), we have

\[
\mathbb{P}[X_{\rho_{1}^{(1)}+} \in A | G]^{(1)}] = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} 1_{\{\tau_{1}=n\}}(\omega) 1_{\{\rho_{1}^{(1)}=n+m\}}(\omega) 1_{\{\omega_{n+m}=\gamma_{n+m}\}} \times \mathbb{P}[X_{n+m+} \in A | \eta(X_{n}) = \infty, \mathcal{F}_{n+m}]
\]

\[
= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} 1_{\{\tau_{1}=n\}}(\omega) 1_{\{\rho_{1}^{(1)}=n+m\}}(\omega) \mathbb{P}[X_{n+m+} \in A | \eta(X_{n}) = \infty, \mathcal{F}_{n+m}].
\]

The case that \( k = 1 \) and \( j > 1 \) is similar. We just need to make the proper adjustments in \( \Lambda_{n,m} \). For \( k > 1 \), we also prove in the similar way using the fact that, for every natural \( n \) and \( k > 1 \), the event \( \{\eta(X_{n}) = \infty\} \cap \{n < \tau_{k}\} \) is \( G_{0}^{(k)} \) measurable.

The proof of Proposition 3.4 closely follows the proof of Proposition 4.4 in [7].

**Proof of Proposition 3.4.** Proof of (i): From the definition of \( \eta_{j}^{(k)} \) we have that,

\[
\mathbb{P}[\eta_{j}^{(k)} < \infty | G]^{(k)} = \mathbb{P}[\eta(X_{\rho_{j}^{(k)}}) < \infty | G]^{(k)} = 1 - \mathbb{P}[\eta(X_{\rho_{j}^{(k)}}) = \infty | G]^{(k)}. \quad (33)
\]

The event \( \{\eta(X_{\rho_{j}^{(k)}}) = \infty\} \) means that the process does not come back in direction \( \ell \) to the position \( X_{\rho_{j}^{(k)}} \), hence \( X_{\rho_{j}^{(k)} \cdot i} > X_{\rho_{j}^{(k)} \cdot \ell} \) for all \( i \in \mathbb{N} \).

Let \( B \) be a Borel set of \((\mathbb{Z}^{d})^{\mathbb{N}}\) such that \( X_{\rho_{j}^{(k)} \cdot i} > X_{\rho_{j}^{(k)} \cdot \ell} \) for all \( i \in \mathbb{N} \), then by part (ii) of Proposition 2.2 we have,

\[
\mathbb{P}[\eta(X_{\rho_{j}^{(k)}}) = \infty | G]^{(k)} = \mathbb{P}[X_{\rho_{j}^{(k)}}+ \in B | G]^{(k)} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} 1_{\{\tau_{n}=n\}}(\omega) 1_{\{\rho_{j}^{(k)}=n+m\}}(\omega) \mathbb{P}[X_{n+m+} \in B | \eta(X_{n}) = \infty, \mathcal{F}_{n+m}]
\]

\[
= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} 1_{\{\tau_{n}=n\}}(\omega) 1_{\{\rho_{j}^{(k)}=n+m\}}(\omega) \mathbb{P}[\eta(X_{n+m}) = \infty | \eta(X_{n}) = \infty, \mathcal{F}_{n+m}]. \quad (34)
\]

Since we have that \( \{\eta(X_{n+m}) = \infty\} \subset \{\eta(X_{n}) = \infty\} \) then,

\[
\mathbb{P}[\eta(X_{n+m}) = \infty | \eta(X_{n}) = \infty, \mathcal{F}_{n+m}] = \frac{\mathbb{P}[\{\eta(X_{n+m}) = \infty\} | \mathcal{F}_{n+m}]}{\mathbb{P}[\{\eta(X_{n}) = \infty\} | \mathcal{F}_{n+m}]} \geq \mathbb{P}[\eta(X_{n+m}) = \infty | \mathcal{F}_{n+m}] \geq \psi. \quad (35)
\]

The last inequality in (35) follows from Proposition 3.3 and the fact that \( m+n \) is a maximum point in \( \ell \) direction, so like in the origin the process
has a completely new environment to explore forward in $\ell$ direction. Then using (35) in (34) we get

$$
P \left[ \eta(X_{\ell}^{(k)}) = \infty | G_j^{(k)} \right] \geq \psi \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} 1_{\{n = m\}}(\omega) 1_{\{\rho_j^{(k)} = n + m\}}(\omega) \geq \psi . \quad (36)$$

Then, by (33) and (36) we conclude that $P \left[ \eta_j^{(k)} < \infty | G_j^{(k)} \right] < (1 - \psi)$.

**Proof of (ii):** Use Proposition 2.2 part (i) to write

$$
P \left[ (X_n - X_{\ell}) \cdot \ell < \frac{p}{3} \lambda n^{\frac{3}{2} + \alpha} | G_0^{(k)} \right] =
$$

$$
= \sum_{m=1}^{\infty} 1_{\{\tau_m = m\}}(\omega) P \left[ (X_{n+m} - X_m) \cdot \ell < \frac{p}{3} \lambda n^{\frac{3}{2} + \alpha} | \eta(X_m) = \infty, F_m \right]
$$

$$
\leq \sum_{m=1}^{\infty} 1_{\{\tau_m = m\}}(\omega) \frac{P \left[ (X_{n+m} - X_m) \cdot \ell < \frac{p}{3} \lambda n^{\frac{3}{2} + \alpha} | F_m \right]}{P[\eta(X_m) = \infty | F_m]},
$$

which, by Propositions 3.2, Remark 3.3 and Proposition 3.3, is bounded from above by

$$
\sum_{m=1}^{\infty} 1_{\{\tau_m = m\}}(\omega) e^{-\gamma_3 n^{\gamma_4}} \frac{e^{-\gamma_4 n^{\gamma_4}}}{P[\eta(X_m) = \infty | F_m]} \leq \frac{e^{-\gamma_4 n^{\gamma_4}}}{\psi}.
$$

**Proof of (iii):** By definition of $\eta_j^{(k)}$, we have that,

$$
P \left[ n \leq \eta_j^{(k)} - \rho_j^{(k)} < \infty | G_0^{(k)} \right] = P \left[ n \leq \eta(X_{\rho_j^{(k)}}) - \rho_j^{(k)} < \infty | G_0^{(k)} \right]
$$

$$
= \sum_{i=n}^{\infty} P \left[ \eta(X_{\rho_j^{(k)}}) - \rho_j^{(k)} = i | G_0^{(k)} \right] = \sum_{i=n}^{\infty} P [\eta(X_i) = i] \leq \sum_{i=n}^{\infty} P [X_i \cdot \ell < 0].
$$

The third equality in (37) holds since the system is invariant and by the definition of $\rho_j^{(k)}$ we know the process in direction $\ell$ sees a new environment. Then to go below that position in direction $\ell$, it is the same to go under $X_0$ in direction $\ell$. Hence, using Proposition 3.2, we obtain,

$$
P \left[ n \leq \eta_j^{(k)} - \rho_j^{(k)} < \infty | G_0^{(k)} \right] \leq \sum_{i=n}^{\infty} P [X_i \cdot \ell < 0] \leq \sum_{i=n}^{\infty} P \left[ X_i \cdot \ell < \frac{p}{3} \lambda n^{\frac{3}{2} + \alpha} \right]
$$

$$
\leq 6 \sum_{i=n}^{\infty} i e^{-\gamma_3 i^{\gamma_4}} \leq 6 \left( ne^{-\gamma_3 n^{\gamma_4}} + \int_{\gamma_3 n^{\gamma_4}}^{\infty} x e^{-x} dx \right) \leq 12 e^{-\gamma_3 n^{\gamma_4}} .
$$
APPENDIX B. PROOF OF PROPOSITION 3.1

This section is devoted to prove Proposition 3.1. We will start by stating some auxiliaries lemmas whose proofs are deferred to the end of this Appendix.

Let $X$ be a $p$-GERW. We denote by $L_n(m)$ its local time up to time $n$ on the $m$-th strip in direction $\ell$ which is defined as

$$L_n(m) := \sum_{j=0}^{n} 1\{X_j, \ell \in [m,m+1)\}, \ m \in \mathbb{Z}.$$ 

The next result provides a control on the tail of the local times $L_n(m)$.

Lemma B.1. Let $X$ be a $p$-GERW, for any $\delta > 0$ there exists a constant $\gamma'_1$ depending on $K$, $h$ and $r$ such that for all $m$ we have,

$$\mathbb{P}\left[L_n(m) \geq n^{\frac{1}{2}+2\delta}\right] \leq \exp\left(-\gamma'_1 n^{\delta}\right).$$

B.1. Some auxiliary results for $d$-dimensional martingales. In this brief section we lists a few results for $d$-dimensional martingales which will be used in the proof of Proposition 3.1.

Lemma B.2. Let $Y$ be a $d$-dimensional martingale with uniformly $K$-bounded increments, i.e., $\sup_{n \geq 1} \|Y_{n+1} - Y_n\| \leq K$. Then, for all $b \geq 0$ and for all $n \geq 1$, it holds that

$$\mathbb{E}\left(\|Y_{n+1}\|^b | \mathcal{F}_n\right) \geq \|Y_n\|^b 1\{|Y_n| > K/(\sqrt{2}-1)\}.$$

The next result is a bound on the number of visits of a $d$-dimensional martingale to any site in $\mathbb{Z}^d$.

Lemma B.3. Let $Y$ be a $d$-dimensional martingale which satisfies Condition I and (UE2) in Condition III and suppose that $Y_0 = x_0$. Then, for any $\delta > 0$ and $\phi > 0$, there exists a constant $\gamma'_3 > 0$ depending on $K$, $\phi$, $r$ and $h$ such that for all $x_0, y_0 \in \mathbb{Z}^d$ and for all $n$, we have,

$$\mathbb{P}\left[\sum_{j=1}^{n} 1\{Y_j = y_0\} > n^{\phi + \delta}\right] \leq \exp\left(-\gamma'_3 n^{\delta}\right).$$

Let us denote by $\tau^X_B$ the hitting time to a set $B$ for a process $X$, i.e.,

$$\tau^X_B := \min\{n \geq 0 : X_n \in B\},$$

and denote by $B(x,q)$ a discrete ball centered in $x \in \mathbb{Z}^d$ and with radius $q$, i.e., $B(x,q) := \{y \in \mathbb{Z}^d : \|x - y\| \leq q\}$. The next result implies that a $d$-dimensional martingale $Y$ hits with high probability sets that contains enough points close to the origin of the process.
Lemma B.4. Let $Y$ be a $d$-dimensional martingale which satisfies Condition 1 and (UE2) in Condition III. Assume that $Y_0 = x$. Consider an arbitrary $\delta > 0$, a set $U$ and suppose that $|B(x, m^{1/2}) \setminus U| \leq m^{1-\phi-2\delta}$ for some $m$ and $0 < \phi < 1$. Then there exists a constant $\gamma_d' > 0$ depending on $d$, $K$, $\phi$, $h$ and $r$ such that
\[ \mathbb{P} \left[ \tau_{U_\delta} \geq m^{1-\delta} \right] \leq \exp \left\{ -\gamma_d' m^\delta \right\}. \]

B.2. Proof of Proposition 3.1. Using Lemma B.1 and Lemma B.4 we now prove Proposition 3.1\textsuperscript{2}.

Proof of Proposition 3.1. Consider $b \in (0, 1)$ and $\varepsilon > 0$. Set $e_w$ the strip width exponent and define
\[ H_j^n := H(2(j-1)n^{e_w}, 2(j+1)n^{e_w}), \quad n \geq 1, \ j \geq 1, \]
so that $H_j^n$ is strip of width $4n^{e_w}$ in direction $\ell$. The strip $H_j^n$ will be called a trap if $|R^n_X \cap H_j^n| \geq n^{\varepsilon t}$, where $e_t = 2e_w(1 - (b/2)) - 2\varepsilon$ is the trap exponent. Set
\[ G = \{ |R^n_X| \geq n^{\frac{1}{2} + e_w(1-b)-4\varepsilon} \}. \]

We are going to prove that
\[ \mathbb{P}[G] \geq 1 - \left( (2Kn + 1) e^{-\gamma_1' n^{\frac{1}{2}}} + \frac{n^{1-2e_w+\varepsilon}}{2} e^{-\gamma_1' n^{\varepsilon}} \right) \tag{38} \]
for every $\varepsilon > 0$ sufficiently small. This establishes Proposition 3.1 since, as we will see, for every $0 < \alpha < 1/6$, we can choose $e_w$, $b$ and $\varepsilon$ such that $\alpha < e_w(1-b)-4\varepsilon$ (i.e., $\{ |R^n_X| < n^{\frac{1}{2} + \alpha} \} \subset G^c$) and (38) holds.

Thus we have to prove (38). Let us first introduce the event
\[ G_1 = \{ L_n(k) \leq n^{\frac{1}{2} + \varepsilon} \text{ for all } k \in [-Kn, Kn] \}. \]

By Lemma B.1, it holds that
\[ \mathbb{P}[G_1] \geq 1 - (2Kn + 1) \exp \left\{ -\gamma_1' n^{\frac{1}{2}} \right\}. \tag{39} \]

Now, let us define $\sigma_0 = 0$ and inductively
\[ \sigma_{k+1} = \min \{ j \geq \sigma_k + n^{2e_w-\varepsilon} : |R^n_X \cap B(X_j, n^{e_w})| \leq n^{\varepsilon t} \}, \tag{40} \]
(formally, if such $j$ does not exist, we put $\sigma_{k+1} = \infty$). Consider the event
\[ G_2 = \left\{ \text{at least one new point is hit on each of the time intervals} \right\}
\[ [\sigma_{j-1}, \sigma_j), \ j = 1, \ldots, \frac{1}{2} n^{1-2e_w+\varepsilon} \}, \]
where hitting a new point means to visit a not-yet-visited site. Note that on $G_2^c$, the process does not hit a new point in time interval $[\sigma_{j-1}, \sigma_j)$ for some $j = 1, \ldots, \frac{1}{2} n^{1-2e_w+\varepsilon}$. When this happens, the process $X$ evolves as a $d$-dimensional martingale during time interval $[\sigma_{j-1}, \sigma_j)$ and for $Y = X_{\sigma_{j-1}+}$.

\textsuperscript{2}Let us point out that Lemma B.2 is used to prove Lemma B.3 which, in turn, is used to prove Lemma B.4.
This allows us to write

\[ \gamma_{(R_{\tilde{\delta}_j-1})}^Y \geq \sigma_j - \sigma_{j-1} \geq n^{2e_w - \varepsilon}. \]

To control the probability of \( G^c \), we will apply Lemma B.4. For this sake we point out that \( 2e_w \geq e_t \) and introduce \( \tilde{\delta} \) in \((0, 1)\) such that

\[ e_t = 2e_w(1 - \tilde{\delta}). \]

This allows us to write

\[ \frac{\tilde{\delta}}{2} = \frac{1}{2} - \frac{e_t}{4e_w} = \frac{1}{2} - \frac{1}{2} \left( 1 - \frac{b}{2} \right) - \frac{\varepsilon}{2e_w} = \frac{b}{4} - \frac{\varepsilon}{2e_w}. \]

Setting \( \delta = \frac{\varepsilon}{2e_w} \), for every \( b \in (0, 1) \), we can choose \( \varepsilon \) sufficiently small such that

\[ \frac{\tilde{\delta}}{2} > \delta > 0. \]

Then, we are ready to apply Lemma B.4 with the following choice of parameters: \( \delta = \frac{\varepsilon}{2e_w} \) (as above), \( m = n^{2e_w} \), \( \phi = \tilde{\delta} - 2\delta \), and choosing \( U = (R_{\tilde{\delta}_j-1})^c \).

Note that by the definition of \( \sigma_{j-1} \), we have that \( |R_{\sigma_{j-1}} \cap B(X_{\sigma_{j-1}}, n^{e_w})| \leq n^{e_t} \), which for our choice of parameters implies

\[
|B(X_{\sigma_{j-1}}, n^{e_w}) \setminus (R_{\sigma_{j-1}})^c| = |B(X_{\sigma_{j-1}}, (n^{2e_w})^{1/2}) \setminus (R_{\sigma_{j-1}})^c| \leq n^{e_t} = (n^{2e_w})^{e_t/2e_w} = (n^{2e_w})^{e_t - \tilde{\delta}} = (n^{2e_w})^{1 - \phi - 2\delta},
\]

and thus we can use Lemma B.4 to conclude that

\[ \mathbb{P}[\gamma_{(R_{\tilde{\delta}_j-1})}^Y \geq \sigma_j - \sigma_{j-1} \geq n^{2e_w - \varepsilon}] \leq e^{-\gamma'n^\varepsilon}, \]

for every \( j = 1, \ldots, \frac{1}{2}n^{1-2e_w+\varepsilon} \). Thus

\[ \mathbb{P}[G_2] \geq 1 - \frac{1}{2} n^{1-2e_w+\varepsilon} e^{-\gamma'n^\varepsilon}. \tag{41} \]

Next, assuming that \( n \) is large enough so that \( 8n^{1-\varepsilon} < n/2 \), we will show that \( (G_1 \cap G_2) \subset G \). Suppose that both \( G_1 \) and \( G_2 \) occur, but \( |R_n^X| < n ^{\frac{1}{2} + e_w (1-b) - 4\varepsilon} \). Let us denote by \( \hat{L}_j \) the total number of visits to \( H_j^n \) up to time \( n \), i.e.,

\[ \hat{L}_j = \sum_{k=2}^{2(j+1)n^{e_w}-1} L_n(k), \]

On \( \{|R_n^X| < n ^{\frac{1}{2} + e_w (1-b) - 4\varepsilon} \} \) we have

\[ n ^{\frac{1}{2} + e_w (1-b) - 4\varepsilon} > |R_n^X| \geq n^{e_t} |\{ j : H_j^n \text{ is a trap} \}|, \]

thus the number of traps is at most

\[ 2n ^{\frac{1}{2} + e_w (1-b) - 4\varepsilon - e_t} = 2n^{\frac{1}{2} - e_w - 2\varepsilon}. \]
On $G_1$, we have,
\[
\sum_{j \in \mathbb{Z}} \hat{L}_j 1_{\{H_j \text{ is a trap}\}} \leq 4n^{e_w} \times 2n^{\frac{1}{2} - e_w - 2\epsilon} \times n^{\frac{1}{2} + \epsilon} = 8n^{1-\epsilon}.
\]
Now observe that, since for $j \leq n$ we have $R_j^X \subset R_{n}^X$, if $|R_j^X \cap B(X_j, n^{e_w})| > n^{\epsilon\epsilon}$ then $X_j$ must be in a trap. Since $n$ is such that $8n^{1-\epsilon} < n/2$, we obtain that, on the event
\[
\left\{ \sum_{j \in \mathbb{Z}} \hat{L}_j 1_{\{H_j \text{ is a trap}\}} \leq 8n^{1-\epsilon} \right\},
\]
the total time (up to time $n$) spent in non-traps is at least $n - 8n^{1-\epsilon} > n/2$. From the definition (40), the latter implies that $\sigma_{\frac{n^{1-2e_w+\epsilon}}{2}} < n$.

Indeed, up to time $\sigma_{\frac{n^{1-2e_w+\epsilon}}{2}}$ we can have at most $n/2$ instances $j$ such that $|R_j^X \cap B(X_j, n^{e_w})| \geq n^{\epsilon\epsilon}$. Therefore, on the event $G_2$ we have that $|R_j^X| \geq \frac{1}{2} n^{1-2e_w+\epsilon}$. Recall that we assumed that $G_1$ and $G_2$ occur, but $|R_j^X| < n^{\frac{1}{2} + e_w(1-b)-4\epsilon}$. Since for $e_w < \frac{1}{6}$ (and $n$ sufficiently large) it holds that $\frac{1}{2} n^{1-2e_w+\epsilon} > n^{\frac{1}{2} + e_w(1-b)-4\epsilon}$, for every $b \in (0, 1)$ and $\epsilon > 0$, we obtain a contradiction. Then, indeed, $(G_1 \cap G_2) \subset G$, and (38) follows from (39) and (41),
\[
P[G] \geq P[G_1 \cap G_2] \geq 1 - (P[G_1] + P[G_2])
\geq 1 - \left( (2Kn + 1) e^{-\gamma_1 n^{\frac{\epsilon}{2}}} + \frac{n^{1-2e_w+\epsilon}}{2} e^{-\gamma_4 n^\epsilon} \right).
\]
To conclude the proof of Proposition 3.1, just note that for every $\alpha < 1/6$, we can find $e_w < 1/6$ and $b \in (0, 1)$ and $\epsilon$ (sufficiently small), such that $\alpha < e_w(1-b) - 4\epsilon$.

\[\square\]

B.3. Proof of the Lemmas. In this section we provide the proof of the Lemmas used to prove Proposition 3.1.

The proof of Lemma B.1 follows closely that of Lemma 5.1 from [7].

Proof of Lemma B.1. Without loss of generality we can consider $m = -1$. As we will see in this proof, it will be only require the uniform elliptic condition, then we can choose any $m$ and the proof will use the same techniques. We denote $\hat{t}_0 = 0$ and
\[
\hat{t}_{k+1} = \min\{j > \hat{t}_k : X_j \cdot \ell \in [-1, 0)\}.
\]
With this notation we have $L_n(-1) = \max\{k : \hat{t}_k \leq n\}$.

By Condition III there exist a positive constant $C_1$ and a natural number $i_0 \geq 1$ such that for any stopping time $T$ we have
\[
P[(X_{T+i_0} - X_T) \cdot \ell \geq 2|\mathcal{F}_T] \geq C_1.
\] (42)
As we saw in the proof of Proposition 3.2 the process \( \{X_n \cdot \ell\}_{n \geq 0} \) is a \( \mathcal{F}_n \)-submartingale. By the optional stopping theorem for any positive integer \( j \) and \( x \in \mathbb{Z}^d \) such that \( x \cdot \ell \geq 1 \), we have

\[
\mathbb{E}[(X_{T_n \wedge T_0} \cdot \ell | \mathcal{F}_{t_j + i_0}, X_{t_j + i_0} = x)] \geq x \cdot \ell \geq 1,
\]

where \( T_n = \tau_{H(n^{1/2+\delta},+\infty)} \circ \theta_{t_j + i_0} \) and \( T_0 = \tau_{H(-\infty,0)} \circ \theta_{t_j + i_0} \) and \( \theta \) is the canonical time shift on the space of trajectories. Now we obtain

\[
\mathbb{E}[(X_{T_n \wedge T_0} \cdot \ell | \mathcal{F}_{t_j + i_0}, X_{t_j + i_0} = x)] = \mathbb{E}[(X_{T_n} \cdot \ell)1_{\{T_n < T_0\}} + (X_{T_0} \cdot \ell)1_{\{T_0 < T_n\}} | \mathcal{F}_{t_j + i_0}, X_{t_j + i_0} = x] \leq (n^{1/2+\delta} + K)\mathbb{P}[T_n < T_0 | \mathcal{F}_{t_j + i_0}, X_{t_j + i_0} = x].
\]

With (43) and (44) we have

\[
\mathbb{P}[T_n < T_0 | \mathcal{F}_{t_j + i_0}, X_{t_j + i_0} = x] \geq \frac{1}{n^{1/2+\delta} + K}.
\]

We set the events \( E = \{(X_{t_j + i_0} - X_{t_j}) \cdot \ell \geq 2\} \) and \( F_y = \{X_{t_j + i_0} = y\} \) where \( y \in \mathbb{Z}^d \). Then we obtain

\[
\mathbb{P}[X_{t_j + l} \cdot \ell > 0 \forall i_0 \leq l \leq \tau_{H(n^{1/2+\delta},+\infty)} \circ \theta_{t_j | \mathcal{F}_{t_j}}] \geq \mathbb{P}[(X_{t_j + i_0} - X_{t_j}) \cdot \ell \geq 2, X_{t_j + l} \cdot \ell > 0 \forall i_0 \leq l \leq i_0 + T_n | \mathcal{F}_{t_j}] \geq \sum_{y \in \mathbb{Z}^d} \mathbb{P}[E \cap F_y | \mathcal{F}_{t_j}] \mathbb{P}[X_{t_j + l} \cdot \ell > 0 \forall i_0 \leq l \leq i_0 + T_n | \{E \cap F_y\}, \mathcal{F}_{t_j}],
\]

where the last inequality above holds since \( \mathbb{P}[E \cap F_y | \mathcal{F}_{t_j}] > 0 \) only if \( y \cdot \ell \geq 1 \). Thus (43) holds, and using (45) in (46) we obtain

\[
\mathbb{P}[X_{t_j + l} \cdot \ell > 0 \forall i_0 \leq l \leq \tau_{H(n^{1/2+\delta},+\infty)} \circ \theta_{t_j | \mathcal{F}_{t_j}}] \geq \frac{1}{n^{1/2+\delta} + K} \sum_{y \in \mathbb{Z}^d} \mathbb{P}[E \cap F_y | \mathcal{F}_{t_j}] \geq \frac{C_1}{n^{1/2+\delta} + K} \geq C_2 n^{-\frac{1}{2}-\delta},
\]

where in the last inequality we use (42).

Set

\[
B = \{X_{t_j + l} \cdot \ell > 0, \forall i_0 \leq l \leq \tau_{H(n^{1/2+\delta},+\infty)} \circ \theta_{t_j}\}.
\]

We will find a lower bound on the probability of the event that the process spends more than \( n \) steps outside the strip \([-1,0]\) in direction \( \ell \).

\[
\mathbb{P}[\hat{t}_{j+1} - \hat{t}_j > n | \mathcal{F}_{t_j}] \geq \mathbb{P}[\{\hat{t}_{j+1} - \hat{t}_j > n\} \cap B | \mathcal{F}_{t_j}] \geq \mathbb{P}[B | \mathcal{F}_{t_j}] \mathbb{P}[\hat{t}_{j+1} - \hat{t}_j > n | B, \mathcal{F}_{t_j}] \geq \mathbb{P}[B | \mathcal{F}_{t_j}] (1 - \mathbb{P}[\hat{t}_{j+1} - \hat{t}_j \leq n | B, \mathcal{F}_{t_j}]) \geq \mathbb{P}[B | \mathcal{F}_{t_j}] (1 - \mathbb{P}[(X_n + X_k) \cdot \ell \geq n^{1/2+\delta}]) \geq C_2 n^{-\frac{1}{2}-\delta} (1 - e^{-\frac{25}{2K^2}}) \geq C_3 n^{-\frac{1}{2}-\delta}.
\]
In the fourth inequality in (48), since we have the event \( B \), we know that the process at time \( k < n \) will be in \( H(\frac{n}{2} + \delta, +\infty) \). Then in the last inequality in (48) we use (47) and Azuma’s inequality for super-martingales.

Finally we have,

\[
P[L_n(-1) \geq n^{\frac{1}{2} + 2\delta}] = P[\max\{k: \hat{t}_k \leq n\} \geq n^{\frac{1}{2} + 2\delta}]
\leq P[\bigcap_{j=0}^{n^{\frac{1}{2} + 2\delta} - 1} \{\hat{t}_{j+1} - \hat{t}_j \leq n\}]
\leq P[\hat{t}_1 \leq n \prod_{j=1}^{n^{\frac{1}{2} + 2\delta} - 1} P[\hat{t}_{j+1} - \hat{t}_j \leq n\{\hat{t}_1 - \hat{t}_0 \leq n\}, \ldots, \hat{t}_j - \hat{t}_{j-1} \leq n\}]
\leq \left(1 - \frac{C_3}{n^{\frac{1}{2} + \delta}}\right)n^{\frac{1}{2} + 2\delta} \leq \left((1 - \frac{C_3}{n^{\frac{1}{2} + \delta}})^{\frac{1}{2} + \delta}\right) n^\delta \leq e^{-C_3 n^\delta},
\]

which is bounded above by

\[
P[\hat{t}_1 \leq n] \prod_{j=1}^{n^{\frac{1}{2} + 2\delta} - 1} P[\hat{t}_{j+1} - \hat{t}_j \leq n\{\hat{t}_1 - \hat{t}_0 \leq n\}, \ldots, \hat{t}_j - \hat{t}_{j-1} \leq n\}]
\leq \left(1 - \frac{C_3}{n^{\frac{1}{2} + \delta}}\right)n^{\frac{1}{2} + 2\delta} \leq \left((1 - \frac{C_3}{n^{\frac{1}{2} + \delta}})^{\frac{1}{2} + \delta}\right) n^\delta \leq e^{-C_3 n^\delta},
\]

finishing the proof.

Now we prove Lemma B.2. This Lemma is similar to Lemma 5.2 in [7] but it is more general in that our statement holds true for every \( b \geq 0 \) (rather than claiming the existence of a \( b \in (0, 1) \) as in [7]). For the proof of Lemma B.2 we use Taylor expansion of first order, some standard inequalities and the fact that \( Y \) is a \( d \)-dimensional martingale with \( K \)-bounded increments.

**Proof of Lemma B.2.** Note that for \( b \geq 1 \) and \( b = 0 \) the proof is straightforward. For \( b \in (0, 1) \), let us begin observing that for all \( y \in \mathbb{R}^d \) such that \( \|y\| > K/(\sqrt{2} - 1) \) it holds that

\[
\left|\frac{2y \cdot z + \|z\|^2}{\|y\|^2}\right| < 1, \forall z \in \mathbb{R}^d \text{ with } \|z\| \leq K.
\]

Let \( b \in (0, 2) \). For any real number \( u \neq 0 \) such that \( |u| < 1 \), there exists \( \theta \in (0, 1) \) such that if \( \xi = (1 - \theta)u \), it holds that

\[
(1 + u)^{b/2} = 1 + \frac{b}{2}(1 + \xi)^{b/2 - 1}u \geq 1 + \frac{b}{2}e^{-(1-\theta)(1-b/2)}u.
\]

Due to the fact that \( (1 - b/2) > 0, 1 - \theta > 0 \) and the assumption that \( |u| < 1 \), we obtain that

\[
(1 + u)^{b/2} \geq 1 + \frac{b}{2}e^{-(1-\theta)(1-b/2)}u \geq 1 + \frac{b}{2}e^{-(1-\theta)(1-b/2)}u.
\]
If we denote $P_{y,y+z} = \mathbb{P}(Y_{n+1} - Y_n = z | F_n, Y_n = y)$, we can write

$$
\mathbb{E} \left( \|Y_{n+1}\|_b^b - \|Y_n\|_b^b | F_n, Y_n = y \right) = \sum_z P_{y,y+z} \left( \|y+z\|_b^b - \|y\|_b^b \right)
$$

$$
= \|y\|_b^b \sum_z P_{y,y+z} \left( \frac{\|y+z\|_b^b}{\|y\|_b^b} - 1 \right) = \|y\|_b^b \sum_z P_{y,y+z} \left( \left( \frac{\|y+z\|_b^b}{\|y\|_b^b} \right)^{b/2} - 1 \right)
$$

$$
= \|y\|_b^b \sum_z P_{y,y+z} \left( 1 + 2y \cdot z + \|z\|_2^2 \|y\|_2^2 \right)^{b/2} \left( \frac{1}{\|y\|_2^2} - 1 \right).
$$

Note that, since $Y$ has uniformly $K$-bounded increments, the summation only runs over $z$ such that $\|z\| \leq K$. Therefore, for all $y$ such that $\|y\| > K/(\sqrt{2} - 1)$, we have that $\left| \frac{2y \cdot z + \|z\|_2^2}{\|y\|_2^2} \right| < 1$, for all $z$ in the summation. Thus, we obtain that

$$
\mathbb{E} \left( \|Y_{n+1}\|_b^b - \|Y_n\|_b^b | F_n, Y_n = y \right) \geq \|y\|_b^b \sum_z P_{y,y+z} \left( \frac{b}{2} e^{-(1-\theta)(1-b/2)} \frac{2y \cdot z + \|z\|_2^2}{\|y\|_2^2} - 1 \right)
$$

$$
= \|y\|_b^b \sum_z P_{y,y+z} \left( \frac{b}{2} e^{-(1-\theta)(1-b/2)} \frac{2y \cdot z + \|z\|_2^2}{\|y\|_2^2} \right)
$$

$$
= \|y\|_b^b \sum_z P_{y,y+z} \left( \frac{b}{2} e^{-(1-\theta)(1-b/2)} \frac{\|z\|_2^2}{\|y\|_2^2} \right) \geq 0.
$$

where, in the last equality, we used that $\sum_z z P_{y,y+z} = 0$, which holds since $Y$ is a $d$-dimensional martingale. \(\square\)

The proof of Lemma B.3 closely follows that of Lemma 5.3 from [7]. Here we will use similar techniques that we apply in the proof of Lemma B.1 and B.2.

**Proof of Lemma B.3.** Since the process $Y$ is a $d$-dimensional martingale without loss of generality we may assume $x_0 = 0$. Moreover, we may also assume $y_0 = x_0 = 0$.

Lemma B.3 is trivial for $\phi \geq 1$. Moreover, we can assume $\phi < 1/2$.

Set $\gamma' = K/(\sqrt{2} - 1)$ and define $\tilde{\tau} := \tau_{\gamma'Y}^{Z_h/B(0,\gamma' + 1)}$ the first time the process $Y$ exits the ball of radius $\gamma' + 1$ centered in the origin. Define $\bar{V} := \{Y_m \neq 0, \text{ for all } 1 \leq m \leq \tilde{\tau} \}$. By Condition III, there exists $C_1 > 0$ depending on $r$ and $h$ such that

$$
\mathbb{P}[\bar{V}] > C_1.
$$

(49)

Let $C_2$ be a large constant to be chosen later. We have that $\|Y_n\|^{2\phi}$ is a supermartingale (since we assumed $\phi < 1/2$), so by optional stopping
The above inequality together with Lemma \(B.2\) imply that
\[
\mathbb{E} \left[ \|Y_{\bar{\tau}^Y} \|_{2d}/B(0,C_2n^{1/2}) \circ \theta_{\bar{\tau}^Y} \circ \theta_{\bar{\tau}^{B(0,\gamma_2')}} \|^{2\phi}_{\mathcal{F}_{\bar{\tau}^Y}, \tilde{V}} \right] \leq \|Y_{\bar{\tau}^Y}\|^{2\phi}
\]
\[
\leq \|Y_{\bar{\tau}^Y}\|^{2\phi} 1_{\{\|Y_{\bar{\tau}^Y}\|>\gamma_2'\}} + \sum_{n=0}^{\infty} 1_{\{\|Y_{\bar{\tau}^Y}\|\leq \gamma_2'\}}.
\]

The above inequality together with Lemma \(B.2\) imply that
\[
\mathbb{E} \left[ \|Y_{\bar{\tau}^Y} \|_{2d}/B(0,C_2n^{1/2}) \circ \theta_{\bar{\tau}^Y} \circ \theta_{\bar{\tau}^{B(0,\gamma_2')}} \|^{2\phi}_{\mathcal{F}_{\bar{\tau}^Y}, \tilde{V}} \right] = \|Y_{\bar{\tau}^Y}\|^{2\phi} 1_{\{\|Y_{\bar{\tau}^Y}\|>\gamma_2'\}}.
\]

Denote \(T_1 = \tau^{Y}_{Z_d/B(0,C_2n^{1/2})} \circ \theta_{\bar{\tau}^Y} \) and \(T_2 = \tau^{Y}_{B(0,\gamma_2')} \circ \theta_{\bar{\tau}^Y} \), then we obtain
\[
\mathbb{E} \left[ \|Y_{T_1}\|^{2\phi} 1_{\{T_1<T_2\}} | \mathcal{F}_{\bar{\tau}^Y}, \tilde{V} \right] + \mathbb{E} \left[ \|Y_{T_2}\|^{2\phi} 1_{\{T_2<T_1\}} | \mathcal{F}_{\bar{\tau}^Y}, \tilde{V} \right] = \|Y_{\bar{\tau}^Y}\|^{2\phi} 1_{\{\|Y_{\bar{\tau}^Y}\|>\gamma_2'\}}
\]
\[
\implies \mathbb{E} \left[ \|Y_{T_1}\|^{2\phi} 1_{\{T_1<T_2\}} | \mathcal{F}_{\bar{\tau}^Y}, \tilde{V} \right] + (\gamma_2')^{2\phi} \geq (\gamma_2' + 1)^{2\phi}.
\]

Ergo by (50) we have
\[
(C_2n^{1/2} + K)^{2\phi} \mathbb{P} \left[ \tau^{Y}_{Z_d/B(0,C_2n^{1/2})} \circ \theta_{\bar{\tau}^Y} < \tau^{Y}_{B(0,\gamma_2')} \circ \theta_{\bar{\tau}^Y} | \mathcal{F}_{\bar{\tau}^Y}, \tilde{V} \right] + (\gamma_2')^{2\phi} \geq (\gamma_2' + 1)^{2\phi}.
\]
This implies
\[
\mathbb{P} \left[ \tau^{Y}_{Z_d/B(0,C_2n^{1/2})} \circ \theta_{\bar{\tau}^Y} < \tau^{Y}_{B(0,\gamma_2')} \circ \theta_{\bar{\tau}^Y} | \mathcal{F}_{\bar{\tau}^Y}, \tilde{V} \right] \geq (\gamma_2' + 1)^{2\phi} - (\gamma_2')^{2\phi} \left( C_2n^{1/2} + K \right)^{2\phi}.
\]

Now, for any stopping time \(T\) and \(y \in \mathbb{Z}_d/B(0,C_2n^{1/2})\), we have,
\[
\mathbb{P} \left[ n \mathbb{E} \left[ Y_T | \tau_{B(0,\gamma_2')} \circ \theta_{\bar{\tau}^Y} \right] = y \right] \geq \mathbb{P} \left[ |Y_n - Y_0| \cdot \ell' < C_2n^{1/2} \right]
\]
\[
\geq 1 - 2 \exp \left( - \frac{(C_2n^{1/2})^2}{2nK} \right) \geq 1 - 2 \exp \left( - \frac{C_2^2}{2K} \right) \geq \frac{1}{2},
\]
where \(\ell' \in \mathbb{S}_d \) in some fixed direction. The first inequality in (52) follows from noticing that if the martingale \(Y\) is initially in position \(y\) and it can not reach 0 in \(n\) steps, then there exists a direction \(\ell' \in \mathbb{S}_d \) along which we must have a distance at most \(C_2n^{1/2}\) units between the initial position \(Y_0\) and \(Y_n\). In the second inequality we used Azuma’s inequality (see, for example, Theorem 2.19 in [3]). Finally, the last inequality in (52) follows from choosing \(C_2 \geq K^{1/2} \sqrt{2} (\ln (1/4))^{1/2} \); for example we can choose \(C_2 = 2K\).
Setting $C_2 = 2K$, we can rewrite (51) as follows:

$$\mathbb{P}\left[ \frac{\tau^Y_{\{B(0, C_2 n^{1/2})\} \circ \theta_{\hat{\tau}} < \tau^{SB(0, \gamma_2' \hat{\tau})} \circ \theta_{\hat{\tau}} | \mathcal{F}_{\hat{\tau}}, \hat{V}} \right] \geq \frac{(\gamma_2' + 1)^{2\phi} - (\gamma_2')^{2\phi}}{(2K n^{1/2} + K)^{2\phi}} \geq \frac{1}{(2n^{1/2} + 1)^{2\phi}} \geq \frac{C_{K,b}}{n^{\phi}},$$

(53)

where $C_{K,\phi} = \left( (K + \sqrt{2} - 1)^{2\phi} - K^{2\phi} \right) / \left( 4K^{2\phi} \left( \sqrt{2} - 1 \right)^{2\phi} \right)$.

Recall that $T_1 = \tau^Y_{\{B(0, C_2 n^{1/2})\} \circ \theta_{\hat{\tau}}, T_2 = \tau^{SB(0, \gamma_2 \hat{\tau})} \circ \theta_{\hat{\tau}}$, with $\hat{\tau} = \tau^Y_{\{B(0, \gamma_2 \hat{\tau})\}}$, and $\hat{V} := \{Y_m \neq 0, \text{ for all } 1 \leq m \leq \hat{\tau}\}$. By (49), (53) and (52), we have,

$$\mathbb{P} [Y_m \neq 0, \text{ for all } m = 1, \ldots, n] \geq \mathbb{P} \left[ \hat{V} \cap \{T_1 < T_2\} \cap \{\tau^Y_0 \circ \theta_{T_1} > n\} \right] \geq \frac{C_{K,\phi} C_1}{2n^{\phi}}.$$

Let $L_n(0) = \sum_{j=1}^{n} 1_{\{Y_j = 0\}}, \hat{t}_0 = 0$ and $\hat{t}_{k+1} = \min \{j > \hat{t}_k : Y_j = 0\}$, thus we have $L_n(0) = \max \{k : \hat{t}_k \leq n\}$. We will apply the same technique used at the end of the proof of the Lemma B.1.

$$\mathbb{P}[\hat{t}_{k+1} - \hat{t}_k > n | \mathcal{F}_{\hat{t}_k}] = \mathbb{P}[Y_m \neq 0, \text{ for all } m = 1, \ldots, n] \geq \frac{C_{K,\phi} C_1}{2n^{\phi}}.$$

Thus,

$$\mathbb{P}[\text{there exist a } k \leq n^{\phi+\delta} - 1 \text{ such that } \hat{t}_{k+1} - \hat{t}_k > n] = 1 - \mathbb{P} \left[ \bigcap _{k=0}^{n^{\phi+\delta} - 1} \{\hat{t}_{k+1} - \hat{t}_k \leq n\} \right]$$

$$= 1 - \prod _{k=0}^{n^{\phi+\delta} - 1} \mathbb{P}[\hat{t}_{k+1} - \hat{t}_k \leq n | \{\hat{t}_1 - \hat{t}_0 \leq n\}, \ldots, \{\hat{t}_k - \hat{t}_{k-1} \leq n\}]$$

$$\geq 1 - \prod _{k=0}^{n^{\phi+\delta} - 1} \left( 1 - \frac{C_{K,\phi} C_1}{2n^{\phi}} \right) \geq 1 - \left( 1 - \frac{C_{K,\phi} C_1}{2n^{\phi}} \right)^{n^{\phi}} \geq 1 - e^{-C_3 n^{\delta}},$$

(54)

where $C_3 = C_{K,\phi} C_1 / 2$. Finally, we obtain

$$\mathbb{P} \left[ L_n(0) > n^{\phi+\delta} \right] = \mathbb{P} \left[ \max \{k : \hat{t}_k \leq n\} > n^{\phi+\delta} \right] \leq 1 - \mathbb{P} \left[ \text{there exist a } k \leq n^{\phi+\delta} - 1 \text{ such that } \hat{t}_{k+1} - \hat{t}_k > n \right] \leq 1 - \left( 1 - e^{-C_3 n^{\delta}} \right) \leq e^{-C_3 n^{\delta}}.$$

The proof of Lemma B.4 follows closely that of Lemma 5.4 from [7].
Proof of Lemma B.4. First begin providing a lower bound to the probability of the following event

\[ F = \{ Y_j \in B(x, m^{1/2}) \text{ for all } j \leq m^{1-\delta} \} . \]

On the event \( F^c \), there must exist a direction \( \ell' \in S^{d-1} \) and a time \( j \leq m^{1-\delta} \) such that \( |Y_j - Y_0| \cdot \ell' \geq m^{\frac{\delta}{2}} \). Thus,

\[ \mathbb{P}[F^c] \leq \mathbb{P} \left( \bigcup_{j=1}^{m^{1-\delta}} \{ |Y_j - Y_0| \cdot \ell' \geq m^{\frac{\delta}{2}} \} \right) \leq \sum_{j=1}^{m^{1-\delta}} \mathbb{P} \left( |Y_j - Y_0| \cdot \ell' \geq m^{\frac{\delta}{2}} \right) . \]

Using Azuma’s inequality for martingales with bounded increments (see, for example, Theorem 2.19 in [3]), we obtain that

\[ \mathbb{P}[F] \geq 1 - \sum_{j=1}^{m^{1-\delta}} 2 \exp \left( - \frac{m}{2jK^2} \right) \geq 1 - 2m^{1-\delta} \exp \left( - \frac{m}{2m^{1-\delta}K^2} \right) \geq 1 - 2m^{1-\delta} \exp \left( - \frac{m^\delta}{2K^2} \right) . \] (55)

Next, we use Lemma B.3 to estimate a lower bound to the probability of all \( y \in B(x, m^{1/2}) \) be visited less than \( m^{\phi+\delta} + \delta \) times up until time \( m^{1-\delta} \). Let us denote the event \( B = \{ \text{for all } y \in B(x, m^{1/2}), \text{ we have } \sum_{j=1}^{m^{1-\delta}} 1\{Y_j=y\} < m^{\phi+\delta} \} \). Thus we have,

\[ \mathbb{P}[B] = \mathbb{P} \left[ \bigcap_{y \in B(x, m^{1/2})} \left\{ \sum_{j=1}^{m^{1-\delta}} 1\{Y_j=y\} < m^{\phi+\delta} \right\} \right] \geq 1 - \sum_{y \in B(x, m^{1/2})} \mathbb{P} \left[ \sum_{j=1}^{m^{1-\delta}} 1\{Y_j=y\} \geq m^{\phi+\delta} \right] \geq 1 - C_1 \mathbb{P} \left[ \sum_{j=1}^{m^{1-\delta}} 1\{Y_j=y\} \geq m^{\phi+\delta} \right] \geq 1 - C_1 e^{-\gamma' m^\delta} , \] (56)

where \( C_1 = |B(x, m^{1/2})| \). In the third inequality above, we used the fact that \( \{ \sum_{j=1}^{m^{1-\delta}} 1\{Y_j=y\} \geq m^{\phi+\delta} \} \subset \{ \sum_{j=1}^{m} 1\{Y_j=y\} \geq m^{\phi+\delta} \} \), while the last inequality follows from Lemma B.3.

Now, on the event \( F \cap B \), we necessarily have that

\[ |\mathcal{R}_{m^{1-\delta}}^Y| > \frac{m^{1-\delta}}{m^{\phi+\delta}} = m^{1-\phi-2\delta} . \]
Therefore, by (55) and (56) we have
\[
\mathbb{P}\left[|\mathcal{R}^{Y}_{m^1-\delta}| > m^{1-\phi-2\delta}\right] \geq \mathbb{P}[F \cap B] \geq 1 - (\mathbb{P}[F^c] + \mathbb{P}[B^c]) \geq 1 - \left(2m^{1-\delta} \exp\left(-\frac{m^\delta}{2K^2}\right) + C_1 e^{-\gamma_3 m^\delta}\right).
\]

Now, since by hypotheses the set \(U\) is such that \(|B(x, m^{1/2})/U| \leq m^{1-\phi-2\delta}\), on the event \(G = \{\mathcal{B}(x, m^{1/2}) \cap \mathcal{R}^{Y}_{m^1-\delta} > m^{1-\phi-2\delta}\}\), we have \(\{Y_1, ..., Y_{m^1-\delta}\} \cap U \neq \emptyset\), that is \(\tau^{Y}_{U} \leq m^{1-\delta}\). With this observation we can finish the proof using (57),
\[
\mathbb{P}\left[\tau^{Y}_{U} \leq m^{1-\delta}\right] \geq \mathbb{P}[G] \geq \mathbb{P}[F \cap B] \geq 1 - \left(2m^{1-\delta} \exp\left(-\frac{m^\delta}{2K^2}\right) + C_1 e^{-\gamma_3 m^\delta}\right).
\]

\[\square\]

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