Non-Smooth Stochastic Lyapunov Functions
With Weak Extension of Viscosity Solutions

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March 1, 2019

Abstract

This paper proposes a notion of viscosity weak supersolutions to build a bridge between stochastic Lyapunov stability theory and viscosity solution theory. Different from ordinary differential equations, stochastic differential equations can have the origins being stable despite having no smooth stochastic Lyapunov functions (SLFs). The feature naturally requires that the related Lyapunov equations are illustrated via viscosity solution theory, which deals with non-smooth solutions to partial differential equations. This paper claims that stochastic Lyapunov stability theory needs a weak extension of viscosity supersolutions, and the proposed viscosity weak supersolutions describe non-smooth SLFs ensuring a large class of the origins being noisily (asymptotically) stable and ( asymptotically) stable in probability. The contribution of the non-smooth SLFs are confirmed by a few examples; especially, they ensure that all the linear-quadratic-Gaussian (LQG) controlled systems have the origins being noisily asymptotically stable for any additive noises.

1 Introduction

The development of stability analysis for stochastic dynamical systems reveals some characteristic features of the systems against deterministic dynamical systems. The features basically arise due to the system formulation being stochastic differential equations. Roughly speaking, functionals of the solutions have dynamics influenced by their Hessian terms, hence the stability analysis based on Lyapunov’s sense requires results different from the original Lyapunov stability theory for dynamical systems represented by ordinary differential equations 9, 10, 12.

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*This work was partially presented at the 11th Asian Control Conference (ASCC 2017). This work was partially supported by JSPS KAKENHI Grant Number 17H03282. Corresponding author Y. Nishimura. E-mail: yunishi@kagoshima-u.ac.jp.
The most popular stability analysis for equilibria of stochastic systems is based on asymptotic stability in probability (ASiP), and a way to ensure the property is to prove the existence of (strict) stochastic Lyapunov functions (SLFs) [9, 12]. The ASiP property and SLFs are respectively considered as stochastic versions of asymptotic stability and (strict) Lyapunov functions for equilibria of ordinary differential equations.

A characteristic feature of stochastic stability analysis appears in the discussion on ASiP at Remark 5.5 in [9], which implies that a stochastic system having smooth vector fields sometimes requires SLFs with non-smoothness at the equilibria. The situation does not occur for any deterministic system having smooth vector fields because of the necessary condition for the existence of Lyapunov functions [8]. This motivates us to relax a sufficient condition for SLFs to have non-smoothness at the target equilibria [17].

At the same time, the permission of non-smooth SLFs causes another issue of ensuring some stability for non-equilibria. For example, a stochastic system having a constant diffusion coefficient has possibility to have an SLF without smoothness at a target point, while the system has no equilibrium because of the property of the diffusion coefficient. This requires new stability definitions for the justification of the non-smoothness of an SLF [17]. The extension enables considering “asymptotic stability” for stochastic systems with additive noises such as Linear-Quadratic-Gaussian (LQG)-controlled systems. Therefore, the development of stochastic stability by non-smooth SLFs may provide fruitful discussions on characteristic features of stochastic dynamical systems.

To develop the stability analysis based on non-smooth SLFs, they should be discussed with the notions of viscosity solutions for partial differential equations [6] as with deterministic stability analysis [5, 18]. For stochastic systems, viscosity solutions are considered for stochastic optimal control problems [7] and uniform almost sure asymptotic stability (UASAS) [4]. However, the connection between non-smooth SLFs and viscosity solutions is still under construction. The works for UASAS does not achieve ASiP properties directly because the analysis is based on the (weak) invariant sets. The UASAS property requires the sublevel sets of SLFs to be invariant sets; however, the ASiP allow the sets being non-invariant sets. This implies that the construction of the bridge between viscosity solutions and our non-smooth SLFs needs another way without passing through the notion of invariant sets.

Recently, the authors provide a discussion on the relationship between viscosity solutions and a specific form of a non-smooth SLF [10] via the way of [17], which does not use the notion of invariant sets. The non-smooth SLF is firstly transformed into a \(C^2\) function for guaranteeing the existence of a global solution to the target stochastic differential equation; and then, the solution is confirmed to satisfy the property of ASiP or the stability for additive noises. After that, the \(C^2\) function is considered as a test function for a viscosity supersolution, however, the analysis concludes that the related Lyapunov equation does not achieve a viscosity supersolution due to the existence of a test function not satisfying the inequality for the supersolution. Therefore, in [10], the notion of viscosity weak supersolution is proposed for clarifying the relationship
between the notion of viscosity solutions and non-smooth SLFs.

In this paper, we generalize the analysis in [16] by considering SLFs in the class of lower semicontinuous functions as with the usual sense of viscosity solutions [4]. In the procedure, we also claim that LQG-controlled systems have the origins being “asymptotically stable” by transforming quadratic Lyapunov functions for deterministic LQ-controlled systems into lower semicontinuous SLFs.

Notations. \( \mathbb{R}^d \) is the \( d \)-dimensional Euclidean space, especially \( \mathbb{R} := \mathbb{R}^1 \). \(|x|\) denotes the Euclidean norm of \( x \in \mathbb{R}^n \). For \( a, b \in \mathbb{R} \), let \( a \wedge b \) denote the minimum of \( a \) and \( b \). Let \( \mathcal{P} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) be a filtered probability space, where \( \Omega \) is the sample space, \( \mathcal{F} \) is the \( \sigma \)-algebra that is a collection of all the events, \( \{\mathcal{F}_t\}_{t \geq 0} \) is a filtration of \( \mathcal{F} \), and \( \mathbb{P} \) is a probability measure. The probability of some event \( A \) and the expectation of some random variable \( X \) are written as \( \mathbb{P}[A] \) and \( \mathbb{E}[X] \), respectively. The function \( w := [w_1, w_2, \ldots, w_d]^T \in \mathbb{R}^d \) is a \( d \)-dimensional standard Wiener process defined on \( \mathcal{P} \). The differential form of Itô integral of a function \( \sigma_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n \) in \( w_\alpha(t) \) is denoted by \( \sigma_\alpha(x)dw_\alpha(t) \) for \( \alpha = 1, 2, \ldots \).

2 Preliminary Discussions

In this section, we provide preliminary discussions on stochastic systems, viscosity solutions, target points and stochastic stability based on [4, 7, 9, 17].

2.1 System Representation and Basic Definitions

In this paper, we consider the following stochastic system:

\[
dx(t) = f(x(t))dt + \sum_{\alpha=1}^{d} \sigma_\alpha(x(t))dw_\alpha(t),
\]

where \( x \in \mathbb{R}^n \) is a state vector, \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a drift coefficient and \( \sigma_1, \ldots, \sigma_d : \mathbb{R}^n \rightarrow \mathbb{R}^n \) are diffusion coefficients; we also assume that all the coefficients \( f(x), \sigma_1(x), \ldots, \sigma_d(x) \) of (1) are locally Lipschitz.

For a function \( v : \mathbb{R}^n \rightarrow \mathbb{R} \) with \( n = 1, 2, \ldots \), we define an infinitesimal operator \( \mathcal{L} \) such that:

\[
(\mathcal{L}v)(x,v,p,X) := pf(x) + \frac{1}{2} \sum_{\alpha=1}^{d} \sigma_\alpha^T X \sigma_\alpha,
\]

where \((p, X)\) is an element of a semijet:

\[
\mathcal{J}^{-2}v(x) := \{(p, X) \in \mathbb{R}^n \times S(n) \mid \text{for } y \rightarrow x, v(y) \geq v(x) + p(y - x) + \frac{1}{2} (y - x)^T X (y - x) + o(|y - x|^2)\},
\]

where \( S(n) \) is the set of symmetric \( n \times n \) matrices. The semijet is, roughly, a set of generalized notions of the first and the second sub-derivatives of \( v(x) \) in \( x \). For \( x \in M(v) := \{x \in \mathbb{R}^n | v(x) \text{ is } C^2 \} \) with choosing \( p = (\partial v/\partial x)^T \) and \( X = \partial p/\partial x \), we employ an abridged notation: \((\mathcal{L}v)(x) = (\mathcal{L}v)(x, v, p, X)\).

3
2.2 Viscosity Supersolutions and Lyapunov Equations

In this subsection, we consider the basic definition of Lyapunov equations based on viscosity supersolutions for stochastic systems [6, 7].

Let \( x \in \mathbb{R}^n, \ v : \mathbb{R}^n \to \mathbb{R}, \ p \in \mathbb{R}^n, \ X \in \mathbb{R}^n \times \mathbb{R}^n \), and

\[
F(x, v, p, X) = 0 \quad (4)
\]

be degenerate elliptic; that is, for any \( Y \in \mathbb{R}^n \times \mathbb{R}^n \) satisfying \( Y - X \) being positive definite, \( F(x, v, p, X) \geq F(x, v, p, Y) \).

Here, we define the notion of a viscosity supersolution.

**Definition 1 (viscosity supersolution [4, 6])** A lower semicontinuous function \( v(x) \) is said to be a viscosity supersolution for (4) if

\[
F(x, v, p, X) \geq 0 \quad (5)
\]

holds for any \( x \in \mathbb{R}^n \) and for any \((p, X) \in J^{-2}v(x)\).\]

For our convenience, we put another notion of viscosity supersolution without semijet directly:

**Definition 2 (viscosity supersolution [7])** A lower semicontinuous function \( v : \mathbb{R}^n \to \mathbb{R} \) is said to be a viscosity supersolution for (4) on an open subspace \( O \subset \mathbb{R}^n \) if for any \( \phi : O \to \mathbb{R} \) being smooth,

\[
F\left(x_0, \phi(x_0), \frac{\partial \phi}{\partial x}(x_0), \frac{\partial^2 \phi}{\partial x^2}(x_0)\right) \geq 0 \quad (6)
\]

holds for any \( x_0 \in O \) satisfying \( v(x_0) = \phi(x_0) \) and \( \phi(x) \leq v(x) \) for a neighborhood of \( x_0 \).\]

The above function \( \phi(x) \) is said to be a test function. Definition [4] and Definition [7] are the same notions [7].

A viscosity supersolution is a key notion for building a bridge between viscosity solution theory and Lyapunov stability theory. Supposing \( v(x) \) being \( C^2 \) and setting \( x_0 = 0 \) and \( F = -L_v \), we obtain \((L_v)(x) \leq 0 \) for any \( x \in O \) via (6); that is, \( v(x) \) is an SLF [9, 12]. Thus,

\[
-(L_v)(x, v, p, X) - l(x) = 0 \quad (7)
\]

is said to be a Lyapunov equation, where \( l : \mathbb{R}^n \to \mathbb{R} \) is assumed to be continuous and positive semi-definite.

2.3 Target Points

For considering non-smooth SLFs, we should consider the case of the origin of (1) being an non-equilibrium. In [17], the origin is categorized as an instantaneous point or an almost sure equilibrium as long as \( f(0) = 0 \). This classification is meaningful because the two notions are mutually independent; at the same time,
they are somewhat awkward because sufficient conditions for stability notions for instantaneous points include the ones for almost sure equilibria. To make the discussion on stability properties simpler, we define a new notion for covering both the instantaneous point and the almost sure equilibrium:

**Definition 3 (equilibrium)** The origin of (1) is said to be a noisy equilibrium if \( f(0) = 0 \). It is also said to be an almost sure equilibrium if \( f(0) = 0 \) and \( \sigma(0) = 0 \).

This paper mainly considers stability for noisy equilibria, and then almost sure equilibria as a special case for noisy equilibria.

### 2.4 Global Solutions

In this subsection, we summarize the results of global solutions.

The target system (1) has at least a solution in local time because all the coefficients are locally Lipschitz. The existence of a solution in global time is not ensured obviously because a linear growth condition or global boundedness of the coefficients are not guaranteed. In the basic analysis of stochastic stability [9, 12], a global solution is ensured by the existence of a smooth SLF. However, a non-smooth SLF \( V(x) \) are impossible to guarantee a global solution directly because the dynamical analysis for \( V(x) \) is broken when sample paths leaves \( M(V) \). This implies that, we need a \( C^2 \) function implicitly to derive a sufficient condition for the existence of a global solution.

**Definition 4 (FCiP [17])** The system (1) is said to be forward complete in probability (FCiP) if for each \( x_0 \in \mathbb{R}^n \), there exists a continuous function \( \psi : [0, \infty) \times (0, 1) \to [0, \infty) \) such that

\[
P \left[ \forall t \in [0, \infty), \ |x(t)| \leq \psi(t, \epsilon) \right] \geq 1 - \epsilon
\]

holds for all \( \epsilon \in (0, 1) \).

Roughly, FCiP ensures the existence of a solution to (1) for \( t \in [0, \infty) \). This implies that, the sentence “the system is FCiP” can be translated into “the system has a global solution (in forward time in probability)”. In the previous work [17], a sufficient condition for FCiP via \( C^2 \) function as follows:

**Lemma 1 ([17])** Let the system (1) be considered. If there exists a positive definite, proper and \( C^2 \) function \( y : \mathbb{R}^n \to \mathbb{R} \) such that

\[
(\mathcal{L}y)(x) \leq cy(x) + g
\]

for all \( x \in \mathbb{R}^n \) and for some constants \( c, g \in [0, \infty) \), then system (1) is FCiP. ♦
2.5 Basic Stability Analysis

In this subsection, we provide stability notions for noisy equilibria by making a small change of the definitions in [17].

Let a stopping time be defined by $\tau_0 := \inf\{t > 0 | x(t) = 0\}$, where $\infty = \inf \emptyset$.

**Definition 5 (stability)** The system (1) and the origin are respectively assumed to be FCiP and a noisy equilibrium. The origin is said to be noisily stable (NS) if $x(t)$ satisfies

\[ \forall \eta > 0, \forall \epsilon > 0, \exists \delta = \delta(\eta, \epsilon) > 0, \text{ s.t. } \sup_{|x_0| \leq \delta} \mathbb{P} \left( \sup_{0 \leq t \leq \tau_0} |x(t \wedge \tau_0)| > \eta \right) < \epsilon; \] (10)

and furthermore, the origin is an almost sure equilibrium, then the origin is said to be stable in probability (SiP).

**Definition 6 (asymptotic stability)** The system (1) and the origin are respectively assumed to be FCiP and a noisy equilibrium. The origin is said to be noisily asymptotically stable (NAS) if it is NS and satisfies

\[ \forall \eta' > 0, \forall x_0 \in \mathbb{R}^n, \mathbb{P} \left( \limsup_{t \to \infty} |x(t \wedge \tau_0)| > \eta' \right) = 0; \] (11)

and furthermore, the origin is an almost sure equilibrium, then the origin is said to be asymptotically stable in probability (ASiP).

The above definitions are the same as the ones in the previous work [17] except the two changes: one is assuming the FCiP properties, and the other is the classification of the origin. Therefore, the following sufficient conditions for the stability properties are directly obtained by the proofs for the main results in [17]:

**Theorem 1** The system (1) and the origin are respectively assumed to be FCiP and a noisy equilibrium. If there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ such that it is continuous, positive definite, proper, $C^2$ except at the origin and $(\mathcal{L}V)(x) \leq 0$ holds for all $x \in \mathbb{R}^n \setminus \{0\}$, the origin is NS. In addition, if it is an almost sure equilibrium, it is SiP.

**Theorem 2** The system (1) and the origin are respectively assumed to be FCiP and a noisy equilibrium. If there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ such that it is continuous, positive definite, proper, $C^2$ except at the origin and $(\mathcal{L}V)(x) < 0$ holds for all $x \in \mathbb{R}^n \setminus \{0\}$, the origin is NAS. In addition, if it is an almost sure equilibrium, it is ASiP.

Roughly, the function $V(x)$ is said to be a stochastic Lyapunov function (SLF) if it satisfies the conditions for Theorem 1 and a strict SLF if it satisfies the conditions for Theorem 2. Note, however, that, this paper considers more relaxed versions of SLFs and strict SLFs; the concrete definitions of them are provided in Subsection 3.3 below.
Remark 1 In the previous works [9, 17], the origin is said to be SiP if the origin is an almost sure equilibrium and \( x(t) \) satisfies

\[
\forall \eta > 0, \forall \epsilon > 0, \exists \delta = \delta(\eta, \epsilon) > 0, \text{ s.t.} \sup_{|x_0| \leq \delta} \mathbb{P} \left( \sup_{0 \leq t} |x(t)| > \eta \right) < \epsilon,
\]

which is the same conditions for the SiP in Definition 3 because of the assumption of the FCiP property. The ASiP property in Definition 6 is also the same in the previous works.

3 Stability Analysis via Viscosity Weak Supersolution

This section provides the main results of this paper. It shows our non-smooth SLFs being not included in the notion of viscosity supersolutions via a simple example, followed by a new definition of viscosity weak supersolution, and then, our sufficient conditions with the construction of the bridge between stochastic Lyapunov stability analysis and viscosity solution theory.

3.1 Non-Smooth SLFs vs. Viscosity Supersolutions

Firstly, we provide a simple example that an SLF does not a viscosity supersolution. Consider

\[
dx(t) = -x(t)dt + dw(t), \quad x, w \in \mathbb{R}.
\]

The origin is NAS because it has a function \( V_1(x) = |x| \) satisfying \((-LV_1)(x) = -|x|\) except the origin. Here we calculate elements of the semijet of the function; that is, considering (3) with \( v = V_1 \), we obtain

\[
(p, X) = \begin{cases} 
(1, X) \text{ with } X \leq 0, & x > 0; \\
(p_0, X_0), & x = 0; \\
(-1, X) \text{ with } X \leq 0, & x < 0; 
\end{cases}
\]

where \( p_0 \in [-1, 1] \) and

\[
X_0 = \begin{cases} 
X \text{ with } X \leq 0 & \text{if } p_0 = \pm 1; \\
X \text{ with } X \in \mathbb{R} & \text{if } p_0 \in (-1, 1).
\end{cases}
\]

Therefore, \( V_1(x) \) satisfies

\[
-(LV_1)(x, V_1, p, X) = px - \frac{1}{2} X \geq 0
\]

for any \( x \neq 0 \), while the inequality does not hold for some combination of \( (p_0, X_0) \) for \( x = 0 \). This simply implies that \( V_1(x) \) is not a viscosity supersolution to \(- (LV_1)(x, V_1, p, X) = 0\) despite satisfying all the conditions for Theorem 2.
The above observation is also confirmed via Definition 2. Considering $\phi(x) = x^2$ as a test function in $x = 0$, we obtain

$$-(\mathcal{L}\phi)(x) = 2x^2 - 1;$$

(17)

this does not imply (6) because $-(\mathcal{L}\phi)(0) = -1 \not\geq 0$. Noticing the word “for any $\phi$” in Definition 2, we conclude that $V_1(x)$ is not a viscosity supersolution. Consequently, to build a bridge between stochastic Lyapunov stability theory and viscosity solution theory, we need a relaxed notion of viscosity supersolutions.

### 3.2 Viscosity Weak Supersolutions

Reconsider the simple example in the previous subsection. While (15) includes elements banishing $V_1(x)$ from being a viscosity supersolution, it also has elements making $V_1(x)$ being an SLF. For example, we choose $(p_0, X_0) = (0, 0)$ or $\phi(x) = x^4$, then we obtain $-(\mathcal{L}V_1)(0, V_1, p_0, X_0) = 0$ or $-(\mathcal{L}\phi)(0) = 0$. That is, we can consider the possibility for our SLFs to have “some” element in the semijet satisfying [3] or (6). This motivates us to consider the following relaxed version of viscosity supersolution:

**Definition 7 (viscosity weak supersolution)** A lower semicontinuous function $v(x)$ is said to be a viscosity weak supersolution for (4) if there exists $(p, X) \in J^{-2}V_0(x)$ such that (5) holds for any $x \in \mathbb{R}^n$. □

Of course, another version is defined as follows:

**Definition 8 (viscosity weak supersolution)** A lower semicontinuous function $v : \mathbb{R}^n \to \mathbb{R}$ is said to be a viscosity weak supersolution for (4) on an open subspace $O \subset \mathbb{R}^n$ if for some $\phi : O \to \mathbb{R}$ being smooth, (6) holds for any $x_0 \in O$ satisfying $v(x_0) = \phi(x_0)$ and $\phi(x) \leq v(x)$ for a neighborhood of $x_0$. □

The main claim of this paper is to show that the notion of viscosity weak supersolutions is useful for stability analysis of stochastic systems using non-smooth SLFs.

### 3.3 Stability Conditions via Viscosity Weak Supersolution

Here we derive sufficient conditions for stability properties using viscosity weak supersolutions. First, we define our SLFs:

**Definition 9 (SLF)** Consider the system (1). A continuous, positive definite and proper function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a stochastic Lyapunov function (SLF) if it is a viscosity weak supersolution to the Lyapunov equation (7). Furthermore, $V(x)$ is said to be a strict stochastic Lyapunov function (strict SLF) if there exists a continuous and positive definite function $l : \mathbb{R}^n \to \mathbb{R}$ such that $V(x)$ is a viscosity weak supersolution to (7). □

Then, we claim the following sufficient conditions for stability properties:
Theorem 3 The system (1) and the origin are respectively assumed to be FCiP and a noisy equilibrium. If there exists an SLF, the origin is NS. In addition, if it is an almost sure equilibrium, it is SiP.

The proof is shown in Appendix A. Then, we also derive sufficient conditions for asymptotic stability properties:

Theorem 4 The system (1) and the origin are respectively assumed to be FCiP and a noisy equilibrium. If there exists a strict SLF, the origin is NAS. In addition, if it is an almost sure equilibrium, it is ASiP.

The proof is also shown in Appendix B.

4 Case Study

This section provides some discussions on the benefits of allowing non-smooth SLFs: the validity of the notion of viscosity weak supersolutions under the assumption of the origins being almost sure equilibria, showing a particular shape of SLFs ensuring the FCiP property, and proving the LQG-controlled systems always have the origins being NAS.

4.1 ASiP vs. Viscosity Supersolutions

The simple example of NAS in Subsection 3.1 simply states the necessity of considering viscosity weak supersolutions. However, we may suspect that the notion is no need for ASiP properties. For example, another simple example of

\[ dx(t) = -\frac{1}{2} x(t)dt + x(t)dw(t), \ x, w \in \mathbb{R}, \]  

has the origin being ASiP because \( V_1(x) = |x| \) implies \( (LV_1)(x) = -(1/2)|x| < 0 \) except the origin. Because the strict SLF \( V_1(x) \) has the same semijet as \([14]\) and \([15]\), we obtain

\[-(LV_1)(x, V, p, X) = \frac{1}{2}px - \frac{1}{2}Xx^2 \begin{cases} > 0 & x \neq 0, \\ = 0 & x = 0. \end{cases} \]  

Therefore, \( V_1(x) \) is a viscosity supersolution to (7) with \( l = 0 \).

The above result states that the requirement of viscosity weak supersolutions may not be indispensable for the ASiP property. However, we claim that the employment of the notion simplifies the discovery of SLFs; let us confirm it by the following example:

\[ dx = -xdt + g_1(x)dw_1 + g_2(x)dw_2, \ x \in \mathbb{R}^3, \]  

where \( g_1 = (1, 0, x_2)^T \) and \( g_2 = (0, 1, 0)^T \). Let a candidate of a strict SLF is designed by \( V_c(x) = |x_1| + |x_2| + |x_3| \).
Letting $p = (p_i)$ and $X = (X_{ij})$ with $i, j \in 1, 2, \ldots, d$, the elements of the semijet $J_{-2}V_0$ are derived as follows: for all $x \in M(V_c)$,

$$p_i = \text{sgn}(x_i) \quad \text{and} \quad X \leq 0,$$

where $X \leq 0$ denotes that $X$ is negative semidefinite. For a point $x$ satisfying $x_k = 0$ for some $k = 1, 2, \ldots, d$ and $x_i \neq 0$ for $i \neq k$,

$$p_i = \begin{cases} \text{sgn}(x_i), & i \neq k \\ -1 \text{ or } 1, & i = k \end{cases}, \quad \text{and} \quad X \leq 0$$

or

$$\begin{cases} p_i = \text{sgn}(x_i), & \text{and} \quad (X_{ij}) \leq 0, \quad i, j \neq k \\ p_k \in (-1, 1), & \text{and} \quad X_{ik} \in \mathbb{R}. \end{cases}$$

This calculation implies that, $X$ is capable of having positively-large values at $x \in \mathbb{R}^n \setminus M(V_c)$. Therefore, $J_{-2}V_c$ generally has elements such that (5) is not satisfied. In this way, $V_c(x)$ is confirmed to be out of the notion of a viscosity super-solution; however, the semijets $J_{-2}V_c$ always has an element $(p, X)$ making $-(L V_c)(x, V_c, p, X) - l(x) \geq 0$ with some positive definite function $l(x)$; that is, $V_c(x)$ is a viscosity weak supersolution. Therefore, $V_c(x)$ is a strict SLF.

4.2 Non-Smooth SLF Ensuring FCiP Property

The SLF $V_c(x) = |x_1| + |x_2| + |x_3|$ in the previous subsection does not ensure the target system being FCiP directly because, before confirming the property, the dynamical analysis is stopped at the first exit time from $M(V_c)$. However, it has a shape ensuring the FCiP property implicitly. The following result shows that an SLF formed

$$V_0(x) = \sum_{i=1}^{n} \frac{1}{p_i} |x_i|^{p_i}, \quad p_1, \ldots, p_n > 0,$$

ensures the target system $[1]$ being FCiP:

**Theorem 5** If $V_0(x)$ is an SLF for $[1]$, the system is FCiP and the origin is NS.

The proof is shown in Appendix [C] provided that the proof for NS property is abbreviated because it is obviously ensured by the combination of the existence of an SLF and the confirmation of the FCiP property.

4.3 Stability for LQG-Controlled Systems

Here we consider the benefit of non-smooth SLFs for stability analysis of controlled systems. Let

$$dx(t) = \{Ax(t) + Bu(t)\}dt + \sum_{k=1}^{d} G_k dw_k,$$

where
where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( w \in \mathbb{R}^d \), and \( A, B \) and \( G = (G_1, \ldots, G_d) \) are \( n \times n \), \( n \times m \) and \( n \times d \)-dimensional matrices, respectively. The function \( u(t) \) is a control input vector, and is designed here by

\[
u(t) = -R^{-1}B^TPx(t), \quad (26)
\]

where \( R \) and \( P \) are positive definite \( n \times n \) matrices satisfying a Riccati equation

\[
A^TP + PA - PBR^{-1}B^TP + Q = 0,
\]

where \( Q \) is also a positive definite \( n \times n \) matrix. Then, the control input is said to be an LQG regulator because it minimizes the related cost function

\[
J(x, u) = \int_0^\infty (x(s)^TQx(s) + u^T(s)Ru(s)) \, ds,
\]

and \( V_{LQ}(x) = \inf_{u \in \mathbb{R}^m} J(x, u) \) results in \( V_{LQ}(x) = x^TPx \). If \( G = 0 \), \( V_{LQ}(x) \) is a Lyapunov function because \( V_{LQ}(x(t)) = -x^T(Q + PBR^{-1}B^TP)x \) being negative definite. However, If \( G \neq 0 \), it is generally not an SLF because

\[
(LV_{LQ})(x) = -x^T(Q + PBR^{-1}B^TP)x + \sum_{k=1}^{d} G_k^TPG_k; \quad (29)
\]

that is, the value is positive for sufficiently near the origin.

Nevertheless, we claim the following:

**Theorem 6** The origin of (25) with (26) is NAS.

The theorem is proven by designing a non-smooth strict SLF by transforming \( V_{LQ}(x) \); the detail is provided in Appendix D.

Here we consider a benefit of the above theorem to control theory. An availability of LQG controllers is that they are obtained from Riccati equations for linear-quadratic (LQ) control systems without any noise [3]; that is, the additive noises for state equations are not obstacles for optimality. As with the optimality, the above discussions imply that “for linear systems, asymptotic stability preserves against the addition of additive noises” if we recognize NAS as stochastic stability of just the counterpart for asymptotic stability of deterministic systems.

Thus, we conclude that the notion of NAS has a theoretical contribution in connecting asymptotic stability properties for deterministic and stochastic systems, and the connection is supported by non-smooth SLFs those are also viscosity weak supersolutions.

## 5 Conclusion

In this paper, we developed stability analysis for stochastic systems via non-smooth stochastic Lyapunov functions. In the procedure, we redefined the notions of stability properties including the situation of the origins losing the
equilibrium states, confirmed that the notion of viscosity supersolutions is insufficient to describe our non-smooth stochastic Lyapunov functions, proposed a new notion of viscosity weak supersolutions for building a bridge between stochastic Lyapunov stability theory and viscosity solution theory, and showed the contribution of our stability analysis to stochastic control theory.

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A Proof of Theorem 3

The existence of an SLF implies that, for any open space \( O_z \subseteq \mathbb{R}^n \) including a point \( z \in \mathbb{R}^n \), there exists a \( C^2 \) function \( \phi_z : O_z \rightarrow \mathbb{R} \) such that (6) holds with \( \phi_z(z) = V(z) \) and \( \phi_z(x) \leq V(x) \) in \( x \in O_z \). Thus, choosing \( \phi_z(x) = V(x) \) for \( x = z \in M(V) \), we design a function

\[
V'(x) = \begin{cases} 
V(x) & x \in M(V), \\
\phi_z(x) & x = z \in \mathbb{R}^n \setminus M(V)
\end{cases}
\]  

(30)

Note that, \( V'(x) \) is still non-\( C^2 \) for any \( x \in \mathbb{R}^n \setminus M(V) \) because the limitation values of the difference quotients may change by the directions of the limiting operations. However, for any fixed time \( \tau \in [0, \infty) \) and \( z = x(\tau) \in \mathbb{R}^n \), defining

\[
(LV')(z) := \begin{cases} 
(LV)(z) & z \in M(V), \\
(L\phi_z)(z) & z \in \mathbb{R}^n \setminus M(V)
\end{cases}
\]  

(31)

then we obtain \((LV')(x) \leq 0\) for any \( x \in \mathbb{R}^n \) because \( \phi_z \) ensures \( V(x) \) to be a viscosity weak supersolution to (7) with \( l = 0 \).

Then, we consider Dynkin’s formula: if \( x(t_0) \in M(V) \),

\[
\mathbb{E}[V(x(t \wedge \tau_{mv}))] - V(x(t_0)) = \mathbb{E} \left[ \int_{t_0}^{t \wedge \tau_{mv}} (LV)(x(s))ds \right]
\]

(32)

with \( \tau_{mv} := \inf\{t > 0 | x(t) \notin M(V)\} \); else if \( x(t_0) \notin M(V) \),

\[
\mathbb{E}[\phi_z(x(t \wedge \tau_z))] - \phi_z(x(t_0)) = \mathbb{E} \left[ \int_{t_0}^{t \wedge \tau_z} (L\phi_z)(x(s))ds \right]
\]

(33)
with $z = x(t_0) \in \mathbb{R}^n \setminus M(V)$ and $\tau_z := \inf\{ t > 0 | x(t) \neq z \}$. Using these formula, (31) with $(LV')(x) \leq 0$, the continuity of $V'(x)$, and the assumption of FCiP, we obtain

$$E[V'(x(t))] \leq V'(x(t_0))$$

for any $t > t_0 \geq 0$ and any $x(t_0) \in \mathbb{R}^n$.

The rest proof follows a similar way to Khasminskii [9]. Considering $\tau_\eta$ as the first exit time from $M_\eta := \{ x \in \mathbb{R}^n ||x| < \eta \}$ for $\eta > 0$ and $V'_\eta := \inf_{x \in \mathbb{R}^n \setminus M_\eta} V'(x)$, we obtain

$$V'_\eta P\left[ \sup_{t \geq 0} |x(t)| > \eta \right] \leq E[V'(x(\tau_\eta \wedge t))]$$

for all $x(t_0) \in M_\eta$. Substituting (34) into (35),

$$V'_\eta P\left[ \sup_{t \geq 0} |x(t)| > \eta \right] \leq V'(x(t_0)), \forall x(t_0) \in M_\eta$$

is derived.

In this way, we obtain (10) with $\varepsilon > \sup_{|x| \leq \delta} V'(x)/V'_\eta$; that is, the origin is NS. Furthermore, if the origin as an almost sure equilibrium, the origin is SiP.

## B Proof of Theorem 4

Consider $V'(x)$ defined in Appendix A and $V(x)$ being a strict SLF, which ensures the existence of $l(x)$ being positive definite in the Lyapunov equation (7). Thus, as with the previous section, the use of (31) and Dynkin's formula yield

$$(LV')(x) \leq -l(x), \forall x \in \mathbb{R}^n.$$  

(37)

The rest proof for NAS is provided via the same way to Theorems 2.3 and 2.4 in Section 4 of [12], provided that the origin is a noisy equilibrium. The proof for ASiP is adding the condition of the origin being an almost sure equilibrium into the proof for NAS.

## C Proof of Theorem 5

Let us first prove that (1) is FCiP by transforming $V_0$ into $C^2$ for all $x \in \mathbb{R}^n$. We should firstly note that the locally Lipschitz condition for all coefficients $f, \sigma_1, \ldots, \sigma_d$ ensures the existence of a solution to (1) for a time interval $[0, t_i]$ with some $t_i > 0$. Set

$$V'_0(x) = \sum_{i=1}^{n} V_i,$$

$$V_i = \begin{cases} \frac{1}{p_i} |x_i|^{p_i}, & |x_i| \geq b_i, \\ c_i(x_i), & |x_i| \in [a_i, b_i), \\ v_i(x_i), & |x_i| \in (0, a_i), \end{cases}$$

(38)
Figure 1: An example of $c_i(x_i) = \alpha_5|x_i|^5 + \alpha_4|x_i|^4 + \alpha_3|x_i|^3 + \alpha_2|x_i|^2 + \alpha_1|x_i| + \alpha_0$ and $v_i(x_i)$ with $a_i = 0.05$, $b_i = 0.4$ and $p_i = 4$. In this case, $\alpha_5 \approx 257$, $\alpha_4 \approx -301$, $\alpha_3 \approx 114$, $\alpha_2 \approx -13$, $\alpha_1 \approx 0.58$ and $\alpha_0 \approx -0.0091$.

where $0 < a_i < b_i$ for all $i = 1, 2 \ldots, n$ satisfying $(b_i)^{p_i}/p_i > v_i(\pm a_i)$ and $c_1, \ldots, c_n : \mathbb{R} \rightarrow \mathbb{R}$ satisfy

$$c_i(\pm a_i) = v_i(\pm a_i), \quad c_i(\pm b_i) = \frac{1}{p_i} b_i^{p_i},$$

$$\frac{\partial c_i}{\partial x_i}(\pm a_i) = \frac{\partial v_i}{\partial x_i}(\pm a_i), \quad \frac{\partial c_i}{\partial x_i}(\pm b_i) = \text{sgn}(b_i) b_i^{p_i - 1},$$

$$\frac{\partial^2 c_i}{\partial x_i^2}(\pm a_i) = \frac{\partial^2 v_i}{\partial x_i^2}(\pm a_i), \quad \frac{\partial^2 c_i}{\partial x_i^2}(\pm b_i) = (p_i - 1) b_i^{p_i - 2},$$

and $C^2$ in $a_i < |x_i| < b_i$ without any extremum value in this region. In short, $c_1, \ldots, c_n$ are curves connecting $v_i$ and $|x_i|^{p_i}/p_i$ smooth with keeping positive definiteness and properness. An example of $V_i$ is described in Fig. 1. Therefore, $V'_0$ is a candidate of $y$ in Lemma 1.

Case (a)

If $x \in M^b := \{x \in \mathbb{R}^n|\forall i = 1, 2 \ldots, n, |x_i| \geq b_i\}$, $(\mathcal{L}V'_0)(x) < 0$ because $V_0(x) = V'_0(x)$; this implies that \[ holds for $y(x) = V'_0(x)$ and $c = g = 0$ in $x \in M^b$.

Case (b)

If $x \in M_a := \{x \in \mathbb{R}^n|\forall i = 1, 2 \ldots, n, |x_i| \leq a_i\}$, the continuity of $f$, $\sigma_1, \ldots, \sigma_d$ and the $C^2$ property of $V_i$ for all $i = 1, 2 \ldots, n$ imply that $(\mathcal{L}V'_0)(x)$ is also continuous with $V'_0(0) = 0$ in $M_a$ Because of the continuity of $(\mathcal{L}V'_0)(x)$ and
the boundedness of $M_a$, there exists $a \in \mathbb{R}$ such that

$$\sup_{x \in M_a} (\mathcal{L}V'_0)(x) = a.$$  \hfill (42)

Therefore, (9) holds for $y(x) = V'_0(x)$ and $c = 0$, $g = \max(0, a)$ in $x \in M_a$, that is, around the origin.

Case (c)

Here we analyze $V'_0(x)$ on the rest region $M_r := \mathbb{R}^n \setminus (M^b \cup M_a)$. Let us consider $\tilde{x} \in M^1_r = \{ x \in \mathbb{R}^n | |x_1| < a_1, \ |x_j| > b_j, \ j = 2, 3, \ldots, n \} \subset M_r$; that is,

$$V'(\tilde{x}) = v_1(x_1) + \sum_{j=2}^{n} \frac{1}{p_j} |x_j|^{p_j}.$$  \hfill (43)

Considering

$$\sum_{i=1}^{n} \frac{1}{p_i} \mathcal{L}|x_i|^{p_i} < 0, \ \forall x \in M(V_0) \setminus \{0\},$$  \hfill (44)

we obtain

$$(\mathcal{L}V'_0)(\tilde{x}) = (\mathcal{L}v_1)(x_1) + \sum_{j=2}^{n} \frac{1}{p_j} (\mathcal{L}|x_j|^{p_j})(x)
\leq (\mathcal{L}v_1)(x_1) - \frac{1}{p_1} (\mathcal{L}|x_1|^{p_1})(x)$$  \hfill (45)

for all $\tilde{x} \in M^1_r \cap M(V_0) \setminus \{0\}$. Recalling $\tilde{x}_1 \in (-a_1, a_1)$, (45) implies that there exists a finite value $\tilde{a}$ such that $(\mathcal{L}V'_0)(\tilde{x}) < \tilde{a}$ for all $\tilde{x} \in M^1_r \cap M(V_0) \setminus \{0\}$. Furthermore, the above results and the assumption that $V'_0(x)$ is a viscosity weak supersolution imply that there exists $\tilde{a}$ for $\tilde{x} \in M^1_r \cap \{ x \in \mathbb{R}^n | x_1 = 0 \}$.

The above discussion also holds for any $x \in M_r$; there exists $\tilde{a}$ for any $x \in M_r$. For example, for $\tilde{x} \in \{ x \in \mathbb{R}^n | |x_1| \in [a_1, b_1) \}$, the above discussion holds with replacing $v_1(x_1)$ by $c_1(x_1)$ and without the discussion on $x_1 = 0$. Of course, for more complicated case such as $\tilde{x} \in \{ x \in \mathbb{R}^n | x_2 \in [a_2, a_2), \ |x_3| \in (-a_3, a_3) \}$, there also exists $\tilde{a}$ via the similar way to the above discussion. Therefore, (9) holds for $y(x) = V'_0(x)$ and $c = 0$, $g = \max(0, \tilde{a})$ in $x \in M_r$.

Conclusion of the Proof

Combining Cases (a)-(c), we obtain (9) with $y(x) = V'_0(x)$, $c = 0$ and $g = \max(0, \tilde{a}, \tilde{a})$. 

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D Proof of Theorem 6

First, we summarize the LQG-controlled system as follows:

\[ dx(t) = A_{LQ}x(t)dt + \sum_{k=1}^{d} G_k dw_k, \]

(46)

where \( A_{LQ} = A - BR^{-1}B^TP \).

Reminding that the positive definite and symmetric matrix \( P \) of \( V_{LQ}(x) = x^TPx \) has only real valued orthogonal eigenvectors \( t_1, \ldots, t_n \in \mathbb{R}^n \), a matrix \( T = (t_1, \ldots, t_n) \) is an orthogonal matrix; that is, \( T^{-1} = TT^T \). Transforming \( x = Tz \) with \( z \in \mathbb{R}^n \) being a new state variable, we obtain the followings:

\[ dz(t) = T^{-1}dx(t) \]

\[ = T^{-1}A_{LQ}Tz(t)dt + T^{-1}\sum_{k=1}^{d} G_k dw_k \]

\[ =: \tilde{A}z(t)dt + \sum_{k=1}^{d} \tilde{G}_k dw_k; \]

(47)

\[ V_{LQ}(x) = z^T T^TPz =: z^T\tilde{P}z = \tilde{V}_{LQ}(z), \]

(48)

where \( \tilde{P} = \text{diag}(\tilde{p}_1, \ldots, \tilde{p}_n) \) is derived by using \( \tilde{p}_1, \ldots, \tilde{p}_n > 0 \), which are the eigenvalues of \( P \), because \( PT = TP \) holds.

Here we design a candidate of a non-smooth SLF

\[ \tilde{V}(z) = \sum_{i=1}^{n} \tilde{p}_i |z_i|, \]

(49)

which has

\[ \frac{\partial \tilde{V}}{\partial z}(z) = \text{sgn}(z)^T \tilde{P}, \]

(50)

\[ \frac{\partial^2 \tilde{V}}{\partial z^2}(z) = 0, \]

(51)

for \( z \in M(\tilde{V}) \), where

\[ \text{sgn}(z) = (\text{sgn}(z_1), \ldots, \text{sgn}(z_n))^T. \]

(52)

Further considering \( z = Z^{1/2}y(z) \) with

\[ Z^{1/2} = \text{diag}(|z_1|^{1/2}, \ldots, |z_n|^{1/2}), \]

(53)

\[ y(z) = Z^{1/2}\text{sgn}(z), \]

(54)

\[ 17 \]
we obtain

\[
(\mathcal{L}\bar{V})(z) = \text{sgn}(z)^T \bar{P} \bar{A} z \\
= \text{sgn}(z)^T \bar{P} \bar{A} z^{1/2} y(z) \\
= \frac{1}{2} y^T(z)(\bar{P} \bar{A} + \bar{A}^T \bar{P}) y(z) \\
= \frac{1}{2} y^T(z) T^T (PA_{LQ} + A_{LQ}^T P) T y(z) \\
= -\frac{1}{2} y^T(z) T^T (Q + PB R^{-1} B^T) T y(z) \tag{55}
\]

for \( z \in M(\bar{V}) \); that is, \((\mathcal{L}\bar{V})(z) < 0 \) for any \( z \in M(\bar{V}) \). For \( z \in \mathbb{R}^n \setminus M(\bar{V}) \), choosing a test function as with \( V'_0(x) \) in (38), \( \bar{V}(z) \) is confirmed to be a viscosity weak supersolution to \(- (\mathcal{L}\bar{V})(z) - l(z) = 0 \) with a positive definite function \( l : \mathbb{R}^n \rightarrow \mathbb{R} \). Thus, we conclude that \( \bar{V}(z) \) is a strict SLF for (47). This completes the proof.