On the covering number of symmetric groups of even degree

Eric Swartz

Abstract. If a group $G$ is the union of proper subgroups $H_1, \ldots, H_k$, we say that the collection $\{H_1, \ldots, H_k\}$ is a cover of $G$, and the size of a minimal cover (supposing one exists) is the covering number of $G$, denoted $\sigma(G)$. Maróti showed that $\sigma(S_n) = 2^{n-1}$ for $n$ odd and sufficiently large, and he also gave asymptotic bounds for $n$ even. In this paper, we give an exact value for $\sigma(S_n)$ when $n$ is divisible by 6 and show how determining the value of $\sigma(S_n)$ may be approached for other even values of $n$.

1. Introduction

Let $G$ be a group and $A = \{A_i \mid 1 \leq i \leq n\}$ a collection of proper subgroups of $G$. If $G = \bigcup_{i=1}^{n} A_i$ (as a set theoretic union), then $A$ is called a cover of $G$. A cover of size $n$ is said to be minimal if no cover of $G$ has fewer than $n$ members. The size of a minimal covering of $G$, supposing one exists, is called the covering number and is denoted by $\sigma(G)$.

Note that if $G$ is a cyclic group, then no generator of $G$ is contained in a proper subgroup, and so $G$ has no cover. On the other hand, if $G$ is not cyclic, then one could take all cyclic subgroups as a cover. Indeed, B.H. Neumann [16] showed that a group is the union of finitely many proper subgroups if and only if it has a finite noncyclic homomorphic image. In this paper we will restrict ourselves to finite groups.

The covering number $\sigma(G)$ of a finite group $G$ provides an upper bound for $\omega(G)$, which is defined to be the largest integer $m$ such that there exists a subset $S$ of $G$ of size $m$ with the property that any two distinct elements of $S$ generate $G$. There has been great interest in this topic in recent years (see [2, 3, 4, 10]), especially with regards to the application of $\sigma(G)$ as an upper bound for $\omega(G)$. For a survey regarding the covering number and related problems, see [18].

In [6], Cohn conjectures that the covering number of any (noncyclic) solvable group has the form $p^\alpha + 1$, where $p$ is a prime and $\alpha$ is a positive integer. In [19], Tomkinson confirms this conjecture, showing that the covering number of any (noncyclic) solvable group has the form $|H/K| + 1$, where $H/K$ is the smallest chief factor of $G$ having more than one complement in $G$.

Furthermore, Tomkinson suggests that it might be of interest to determine the covering number of simple groups. Along these lines, the covering number for 2-dimensional linear groups was determined by Bryce, Fedri, and Serena in [5], and the covering number for the Suzuki groups $Sz(q)$ was determined by Lucido in [13]. Holmes used innovative combinatorial and computational techniques using GAP [8] in [9] to calculate the covering number of many sporadic simple groups.

Naturally, there has been great interest in symmetric and alternating groups. Maróti made great progress on both in [15]. For alternating groups, Maróti showed that $\sigma(A_n) \geq 2^{n-2}$ for $n \neq 7, 9$ with equality if and only if $n \equiv 2 \pmod{4}$ and further proved that $\sigma(A_7) \leq 31$ and

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\( \sigma(A_6) \geq 80 \). Small values of \( n \) have been resolved elsewhere. Cohn [6] showed that \( \sigma(A_5) = 10 \); Kappe and Redden [12] showed that \( \sigma(A_7) = 31 \), \( \sigma(A_8) = 71 \), and \( 127 \leq \sigma(A_9) \leq 157 \); and recently Epstein, Magliveras, and Nikolova-Popova [7] showed that \( \sigma(A_9) = 157 \).

For symmetric groups, Maróti showed for \( n \) odd that \( \sigma(S_n) = 2^{n-1} \) unless \( n = 9 \) and showed for \( n \) even that \( \sigma(S_n) \sim \frac{1}{2} \binom{n}{n/2} \). We note that \( \sigma(S_4) = 4 \) by [19] and \( \sigma(S_6) = 13 \) by [1]. Kappe, Nikolova-Popova, and the author showed in [11] that \( \sigma(S_8) = 64 \), \( \sigma(S_9) = 256 \) (confirming that \( \sigma(S_n) = 2^{n-1} \) for all odd \( n \)), \( \sigma(S_{10}) = 221 \), and \( \sigma(S_{12}) = 761 \), establishing the upper bound of 761 for \( \sigma(S_{12}) \) that Maróti gave in [15] was in fact the exact value.

It is obvious that computational methods can only be taken so far with symmetric groups of even degree, and the goal of this paper is analyze these groups in the same spirit as [15]. We will prove the following theorem:

**Theorem 1.1.** Let \( n \equiv 0 \pmod{6} \), \( n \geq 24 \). If \( \sigma(S_n) \) denotes the subgroup covering number of \( S_n \), then \( \sigma(S_n) = \frac{1}{2} \binom{n}{n/2} + \sum_{i=0}^{n/3-1} \binom{n}{i} \). Moreover, \( \sigma(S_{18}) = 36772 = \frac{1}{2} \binom{18}{9} + \sum_{0 \leq i \leq 5, i \not\equiv 2} \binom{18}{i} \), and in each case the minimal cover using only maximal subgroups is unique.

This paper is organized as follows: we give background on subgroups of the symmetric group and establish our approach to determining \( \sigma(S_n) \) for \( n \equiv 0 \pmod{6} \) in Section 2, showing why there is a natural division between large and small values of \( n \) and why the value of \( n \) modulo 3 matters. We establish \( \sigma(S_n) \) for \( n > 30 \) and divisible by 6 in Section 3 and for the individual values of \( n = 18, 24, 30 \) in Section 4. Finally, we discuss how finding the value of \( \sigma(S_n) \) for \( n \) even but not divisible by 6 may be approached in Section 5.

### 2. Subgroups of symmetric groups and approach

It is clear from the definition that without a loss of generality one need only consider maximal subgroups when determining the covering number of a group, and the maximal subgroups of the symmetric group are known by the O’Nan-Scott Theorem, which may be stated as follows:

**Theorem 2.1 ([17]).** Let \( H \) be a maximal subgroup of \( S_n \). Then \( H \) is isomorphic to one of the following:

1. \( S_k \times S_\ell \), where \( k + \ell = n \);
2. \( S_k \wr S_\ell \), where \( k\ell = n \);
3. \( S_k \wr S_\ell \), where \( k\ell = n \) and \( k > 2 \);
4. \( \text{AGL}(d, p) \), where \( p^d = n \);
5. \( T^k, (\text{Out}(T) \times S_k) \), where \( T \) is a nonabelian simple group and \( |T|^{k-1} = n \);
6. an almost simple group.

In particular, we may separate these maximal subgroups into four different classes:

1. The alternating group \( A_n \);
2. Intransitive groups, i.e., those groups isomorphic to \( S_k \times S_\ell \), \( k + \ell = n \), which stabilize a breakdown of the underlying set of size \( n \) into one set of size \( k \) and one set of size \( \ell \);
3. Imprimitive groups, i.e., those groups isomorphic to \( S_k \wr S_\ell \), where \( k\ell = n \), which stabilize an imprimitive breakdown of the underlying set of size \( n \) into \( \ell \) blocks of size \( k \);
4. Primitive groups, i.e., those groups that act primitively on a set of size \( n \) and are not the alternating group \( A_n \).

This separation is especially useful thanks to the following result of Maróti:

**Lemma 2.2 ([14, Corollary 1.2]).** If \( G \) is a primitive subgroup of \( S_n \) that is not the alternating group \( A_n \) and \( n > 24 \), then \( |G| < 2^n \).
Essentially, what this result says is that when \( n \) is large, the maximal subgroups of \( S_n \) that act primitively on a set of size \( n \) (other than \( A_n \)) are really small in comparison to the alternating group \( A_n \), the intransitive groups, and the imprimitive groups isomorphic to \( S_{n/2} \) wr \( S_2 \). In fact, as we will see in Section 3, the primitive groups (other than \( A_n \)) and the imprimitive groups (other than those isomorphic to \( S_{n/2} \) wr \( S_2 \)) are too small to be in a minimal cover when \( n \) is large, and determining the exact covering number comes down to determining exactly which intransitive groups are needed in a minimal cover.

The following notation will prove useful and will be used throughout the paper. Given an element \( g \in S_n \), we say that the permutation \( g \) has cycle structure \((n_1^{a_1}, \ldots, n_k^{a_k})\) with \( n_1 < n_2 < \cdots < n_k \) if \( g \), when written as the product of disjoint cycles, contains \( a_i \) cycles of length \( n_i \) for \( 1 \leq i \leq k \). We will simply write \( n_i \) if \( a_i = 1 \), and we will suppress any fixed points unless \( g \) is the identity, in which case we say that \( g \) has cycle structure \((1)\). (Hence \( n_1 > 1 \) unless \( g \) is the identity.) For instance, regardless of the value of \( n \), we say that the permutation \((1\ 2)(3\ 4)(5\ 6\ 7\ 8\ 9)\) has cycle structure \((2^2, 5)\).

We now describe the general approach. We begin by noting that the alternating group \( A_n \) covers half the elements of the group and can easily shown to be in a minimal cover. The largest intransitive subgroups, i.e., those isomorphic to \( S_{n/2} \) wr \( S_2 \), covers half the elements of the group and can easily shown to be in a minimal cover. The elements with cycle structure \((n, r, s)\), where \( r + s = n \), can be written as the product of \((r - 1) + (s - 1) = n - 2\) transpositions, and so are in \( A_n \) and already covered. This leaves the elements \( g \) with no fixed points that are the product of at least three disjoint cycles. One of these cycles has length at most \([n/3]\), so, as Maróti noted in [15], taking all subgroups isomorphic to \( S_i \times S_{n-i} \) for \( i \leq [n/3] \) will suffice to cover the elements. We will need to determine exactly which of these subgroups are actually needed in a minimal cover. Elements with cycle structure \((i, r, s)\), where \( i \leq [n/3] \) and \( i + r + s = n \), are the elements that (potentially) will shed light on this issue, and (as we will see) especially important are elements with \( i \) close to \([n/3]\). In light of this, it is not difficult to see that the value of \( n \) modulo 3 will be important in the analysis.

### 3. Large values of \( n \) divisible by 6

In this section we will prove the following result, which holds for all large values of \( n \equiv 0 \pmod 6 \):

**Theorem 3.1.** Let \( n \equiv 0 \pmod 6 \), \( n > 30 \). If \( \sigma(S_n) \) denotes the subgroup covering number of \( S_n \), then \( \sigma(S_n) = \frac{1}{2} \binom{n}{n/2} + \sum_{i=0}^{n/3-1} \binom{n}{i} \). Moreover, the minimal cover using only maximal subgroups is unique and consists of all subgroups isomorphic to one of \( S_{n/2} \) wr \( S_2 \), \( A_n \), or \( S_i \times S_{n-i} \), where \( 1 \leq i \leq n/3 - 1 \).

To prove this we will both show that the subgroups listed form a cover and also show that there is a set of permutations all the subgroups listed to cover that and can be covered by no smaller collection of subgroups.

First, as noted by Maróti in [15], there is a subgroup covering of \( S_n \), \( n \) even, of size \( \frac{1}{2} \binom{n}{n/2} + \sum_{i=0}^{[n/3]} \binom{n}{i} \) consisting of the maximal subgroups isomorphic to \( S_{n/2} \) wr \( S_2 \), \( A_n \), or \( S_i \times S_{n-i} \) for
1 \leq i \leq \lceil n/3 \rceil$. We will refer to this collection of subgroups of $S_n$ as $C_0$. We now show that certain subgroups are unnecessary in this case and define a collection of subgroups of $S_n$ that we will eventually show is a minimal cover.

**Definition 3.2.** Let $n \equiv 0 \pmod{6}$. We define sets of maximal subgroups $H_i$ of $S_n$ as follows: we let $H_{-1}$ be the set of maximal subgroups isomorphic to $S_{n/2} \wr S_2$, we let $H_0$ be the maximal subgroup $A_n$, and for $1 \leq i \leq n/3 - 1$ we let $H_i$ be the set of maximal subgroups isomorphic to $S_i \times S_{n-i}$. For a particular $n$, we define $H := \bigcup_{i=1}^{n/3} H_i$.

**Lemma 3.3.** When $n \equiv 0 \pmod{6}$, $H$ is a cover of $S_n$, i.e., the subgroups isomorphic to $S_{n/3} \times S_{2n/3}$ are not needed in the above cover $C_0$.

**Proof.** Let $H$ and $C_0$ be as defined above. We will show that $H$ is in fact a cover of $S_n$. Suppose, on the contrary, that it is not a cover. Then there must exist an element $g \in S_n$ that has not been covered. We examine the possible cycle structures of $g$. If $g$ is an $n$-cycle, then it is covered by the subgroups isomorphic to $S_{n/2} \wr S_2$, and if $g$ fixes any elements of the underlying $n$-element set, then it is contained in a subgroup isomorphic to $S_{n-1}(\times S_i)$. If the cycle structure of $g$ contains a cycle of length $j$ less than $n/3$, then it is contained in a subgroup isomorphic to $S_j \times S_{n-j}$. On the other hand, since $g$ is an element that fixes no point of the underlying $n$-set and not an $n$-cycle, $g$ is the product of at least two disjoint cycles, and, if $g$ consists of exactly two disjoint cycles of lengths $j$ and $n-j$, then $g$ can be written as the product of $(j - 1) + (n-j - 1) = n - 2$ transpositions. Since $n$ is even, so is $n - 2$, and thus if $g$ is an element without fixed points on the underlying set that is the product of exactly two disjoint cycles, it is in $A_n$. Hence $g$ must consist of at least three disjoint cycles, each of which has length at least $n/3$, and so $g$ is the product of exactly three disjoint $n/3$-cycles. Moreover, $n$ is even, and so is $n/3$. Thus three disjoint $n/3$-cycles preserve some partition of $n$-elements into two sets of size $n/2$, i.e., $g$ is in some subgroup isomorphic to $S_{n/2} \wr S_2$. Therefore, every element of $g$ is in some subgroup in the collection $H$, and so $H$ is a cover of $S_n$.

Next, we will show that the cover $H$ defined above is in fact minimal. To do this, we will show that at least the number of groups listed are necessary to cover a certain subset of the permutations of $S_n$. We will define sets of permutations of $S_n$, $n$ divisible by 6, as follows.

**Definition 3.4.** Let $\Pi_{-1}$ be the set of all $n$-cycles, and let $\Pi_0$ be the set of all elements with cycle structure $(n/2 - 1, n/2 + 1)$ if $n/2$ is even or the set of all elements with cycle structure $(n/2 - 2, n/2 + 2)$ if $n/2$ is odd. For $1 \leq i \leq n/3 - 1$, if $i > 1$ is odd, we let $\Pi_i$ be the set of all elements with cycle structure $(i, \lceil (n-i)/2 \rceil, \lceil (n-i)/2 \rceil)$; if $i = 1$, then let $\Pi_1$ be the set of all elements with cycle structure $(1, n/2 - 2, n/2 + 1)$; if $i$ is even and $(n-i)/2$ is odd, we let $\Pi_i$ be the set of all elements with cycle structure $(i, (n-i)/2, (n-i)/2)$; if $i$ and $(n-i)/2$ are both even, then when $i > 2$ we let $\Pi_i$ be the set of all elements with cycle structure $(i, (n-i)/2 - 1, (n-i)/2 + 1)$, and when $i = 2$ and $(n-2)/2$ is even, we let $\Pi_2$ be the set of all elements with cycle structure $(2, n/2 - 4, n/2 + 2)$. Finally, we define $\Pi := \bigcup_{i=-1}^{n/3} \Pi_i$.

**Lemma 3.5.** If $H$ and $\Pi$ are defined as above for a particular $S_n$, then $\Pi \subseteq \bigcup_{H \in \Pi} H$.

**Proof.** This is clear from the definition of $H$ and the cycle structure of each element of $\Pi$.

This next lemma shows that there is no redundancy among the $H_i$’s in covering $\Pi$.

**Lemma 3.6.** Let $n > 30$, $n \equiv 0 \pmod{6}$. Suppose that $-1 \leq i < j \leq n/3 - 1$ and that $H_i \in H_i$ and $H_j \in H_j$. Then $\Pi_i \cap H_j = \Pi_j \cap H_i = \emptyset$.

**Proof.** First, the result is clear if $i = -1$ since $n$-cycles are odd permutations that are transitive on the underlying $n$-element set, and, by Definition by 3.4, the elements of $\Pi_j$ for
$j \geq 0$ each have cycle structure containing a cycle of odd length and do not preserve a partition of the underlying $n$-set into two sets of size $n/2$. Thus we may assume that $i \geq 0$. If $i = 0$, then, while the elements of $\Pi_0$ preserve an intransitive decomposition of the underlying $n$-element set into two sets, neither of these sets has size at most $n/3 - 1$. Moreover, any $\Pi_j$ for $j \geq 1$ contains elements with cycle structure $(j, r, s)$, where $j + r + s = n$ and hence are the product of $(j - 1) + (r - 1) + (s - 1) = n - 3$ transpositions and not in $A_n$. Thus $i \geq 1$. In every case when $i \geq 2$, if $\Pi_i$ is nonempty, then $\Pi_i$ consists of elements that fix no points in the underlying set of size $n$ and are the product of three disjoint cycles, at least one of which has odd length. This means that these elements are not contained in the subgroups of $\mathcal{H}_{-1}$, $\mathcal{H}_0$, and $\mathcal{H}_1$, respectively. In every case, when $k \geq 1$ the elements of $\Pi_k$ have cycle structure $(k, r, s)$, where $k \leq n/3 - 1 < r \leq s$ (we are assuming that $n > 30$ here, and this condition is necessary for the $(n/2 - 4)$-cycle in elements with cycle structure $(2, n/2 - 4, n/2 + 2)$ to have length at least $n/3$). Thus if an element of $\Pi_i$, $1 \leq i \leq n/3 - 1$, is contained in another intransitive maximal subgroup of the form $S_m \times S_{n-m}$, both $n$ and $m$ must be larger than $n/3 - 1$, and the result holds.

**Lemma 3.7.** Let $\mathcal{H}$ and $\Pi$ be defined for a particular $n > 30$, $n \equiv 0 \pmod{6}$, as above. If $H \neq H' \in \mathcal{H}$, then $\Pi \cap H \cap H' = \emptyset$, i.e., the subgroups in $\mathcal{H}$ partition $\Pi$.

**Proof.** If $H$ and $H'$ are in different $\mathcal{H}_j$, then this follows from Lemma 3.6. Thus we may assume that $H, H'$ are in the same $\mathcal{H}_j$. If $j = -1$, then, since each $n$-cycle preserves a unique imprimitive decomposition of an $n$-element set into two sets of size $n/2$, we have that $\Pi \cap H \cap H' = \emptyset$. Since $\mathcal{H}_0$ contains only $A_n$, we may assume that $j \geq 1$. If $j \geq 1$, then $H$ and $H'$ each preserve different decompositions of the underlying $n$-element set into sets of size $j$ and $n - j$, and each element of $\Pi_j$ preserves a unique such decomposition. Hence $\Pi \cap H \cap H' = \emptyset$, as desired.

We will now show that nearly all other subgroups of $S_n$ have smaller intersection with $\Pi$ than the ones we have chosen. As noted in Section 2, any maximal subgroup $M$ of $S_n$ that does not contain $A_n$ is primitive, stabilizes an imprimitive decomposition, or stabilizes an intransitive decomposition.

**Lemma 3.8.** For any $H \in \mathcal{H}$, $|H \cap \Pi| \geq (n/2 - 1)! \cdot (n/2)!$.

**Proof.** The alternating group $A_n$ contains $n! / ((n/2 - 1)!(n/2 + 1)!)$ different elements of $\Pi$ when $n/2$ is even and $n! / ((n/2 - 2)!(n/2 + 2)!)$ different elements of $\Pi$ when $n/2$ is odd. For $i \geq 1$ if $H_i \in \mathcal{H}$ and $\Pi_i$ contains elements with cycle structure $(i, r, s)$ where $i < n/3 \leq r \leq s$, then $H_i$ contains $(i - 1)! \cdot \left(\binom{n-i}{r} \cdot (r-1)! \cdot (s-1)!\right)$ elements of $\Pi$ if $r \neq s$ and $\frac{1}{2} \cdot (i-1)! \cdot \left(\binom{n-i}{r} \cdot (r-1)^2\right)$ elements of $\Pi$ if $r = s$. In these cases, $|H_i \cap \Pi|$ is at least $(n/3 - 2)! \cdot \left(\frac{2^{n/3+1}}{n/3}\right) \cdot (n/3 - 1)! \cdot (n/3)!$. Finally, if $H$ is in $\mathcal{H}_{-1}$, then $|H \cap \Pi| = (n/2 - 1)! \cdot (n/2)!$. The result follows.

**Lemma 3.9.** Let $M$ be a primitive maximal subgroup of $S_n$ that is not the alternating group $A_n$. Then $|M \cap \Pi| < |H \cap \Pi|$ for any $H \in \mathcal{H}$.

**Proof.** Note that this argument essentially appears in the proof of [15, Claim 3.4]. Using Lemmas 2.2 and 3.8 and noting that $n > 30$, for any $H \in \mathcal{H}$ we have:

$$|M \cap \Pi| < |M| < 2^n < (n/2)^{n-1}/e^n \leq (n/2)^2 \cdot 2/n \leq |H \cap \Pi|,$$

as desired.

**Lemma 3.10.** Let $M$ be an imprimitive maximal subgroup of $S_n$ that is not in $\mathcal{H}$. Then $|M \cap \Pi| < |H \cap \Pi|$ for any $H \in \mathcal{H}$. 
PROOF. Note that this argument essentially appears in the proof of [15, Claim 3.4]. Using Lemma 3.8, we see that:

$$|M \cap \Pi| \leq |M| \leq (n/3)!^3 \cdot 3! < (n/2)!^2 \cdot 2/n \leq |H \cap \Pi|,$$

as desired. \hfill \Box

**Lemma 3.11.** Any minimal cover of $S_n$, $n$ even, must contain all subgroups isomorphic to $S_{n/2}$ wr $S_2$.

**Proof.** Assume that $G$ is a minimal cover of $S_n$ but does not contain $m$ subgroups isomorphic to $S_{n/2}$ wr $S_2$. This means that there are $m \cdot (n/2)!((n/2) - 1)!$ elements of $\Pi$ ($n$-cycles) that need to be covered by other groups. No intransitive group contains an $n$-cycle; hence the groups of $G$ that replace the $m$ subgroups must be primitive or imprimitive maximal subgroups. However, these groups contain fewer than $(n/2)!((n/2) - 1)!$ elements in total, let alone in $\Pi$ (indeed, far fewer: at most $6 \cdot (n/3)!^3$; see Lemmas 3.9 and 3.10 above). Therefore, no collection of fewer than $m$ subgroups outside of $H$ can possibly replace these elements and additionally cover the elements of any other member of $H$. Therefore, any minimal subgroup cover of $S_n$ must use contain all subgroups isomorphic to $S_{n/2}$ wr $S_2$. \hfill \Box

**Lemma 3.12.** Any minimal cover of $S_n$, $n$ even, must contain $A_n$.

**Proof.** Assume that $G$ is a minimal cover of $S_n$ that does not contain $A_n$. By Lemma 3.11, $G$ must contain all $1/2 \cdot \binom{n}{n/2}$ subgroups isomorphic to $S_{n/2}$ wr $S_2$. Thus the remaining subgroups of $G$ must cover the $n!/((n/2 - 1)(n/2 + 1))$ or $n!/((n/2 - 2)(n/2 + 2))$ different elements of $\Pi$ that are in $A_n$ (depending on the parity of $n/2$). No other maximal subgroup contains more than $(n/2)!((n/2) - 2)!$ of these elements, so at least $\binom{n}{n/2} - 1$ subgroups are needed as replacements. On the other hand, $1/2 \cdot \binom{n}{n/2} + \binom{n}{n/2 - 1}$ is already far larger than the upper bounds proved above and by Maróti in [15]. Therefore, the alternating group $A_n$ is in any minimal subgroup cover of $S_n$. \hfill \Box

**Lemma 3.13.** Let $n > 30$ such that $n \equiv 0 \pmod{6}$, and let $M$ be an intransitive subgroup of $S_n$ that is not contained in $H$. Then $M$ intersects $\Pi_i$ for at most five different values of $i$ unless $M \cong S_{n/2-2} \times S_{n/2+2}$ and $n/2$ is odd, in which case $M$ intersects $\Pi_i$ for exactly six values of $i$.

**Proof.** Let $M$ be isomorphic to $S_k \times S_{n-k}$, where $n/3 \leq k \leq n/2$. Note that unless $k = n/2 - 1$ or $k = n/2 - 2$, $M$ does not intersect $\Pi_0$, and $M$ never intersects $\Pi_{i-1}$ regardless of the value of $k$. If $M$ intersects $\Pi_i$ for some $i$, and the elements of $\Pi_i$ have cycle structure $(i, r, s)$, where $i < n/3 \leq r \leq s$, then $k$ is either $r$ or $s$ unless $i = 1$ or $i = 2$. Furthermore, unless $i = 1$ or $i = 2$, $(n - i)/2 - 1 \leq r \leq s \leq (n - i)/2 + 1$. Hence if $j - i \geq 5$, $i > 2$, the elements of $\Pi_j$ have cycle structure $(i, r_i, s_i)$ and the elements of $\Pi_j$ have cycle structure $(j, r_j, s_j)$, then $(n - i) - (n - j) \geq 5$, which implies

$$r_j \leq s_j \leq (n - j)/2 + 1 < (n - j)/2 + 3/2 \leq (n - i)/2 - 1 \leq r_i \leq s_i.$$

Hence unless $k = n/2 - 1$ or $i = 1$ or $i = 2$, $M$ intersects $\Pi_i$ for at most five different values of $i$. On the other hand, when $n/2$ is odd, $M \cong S_{n/2-2} \times S_{n/2+2}$ intersects $\Pi_i$ for $1 \leq i \leq 6$ (and this $M$ never intersects any other $\Pi_i$), and when $n/2$ is even, $M = S_{n/2-1} \times S_{n/2+1}$ intersects $\Pi_i$ for $0 \leq i \leq 4$ (and this $M$ never intersects any other $\Pi_i$). These are the two worst-case possibilities, and so the lemma is proved. \hfill \Box

**Lemma 3.14.** Let $H \in H$ be an intransitive group and $M$ any maximal subgroup of $S_n$ that is not contained in $H$. Unless $H \cong S_{n/3-1} \times S_{2n/3+1}$ and $M \cong S_{n/3+1} \times S_{2n/3-1}$, $|H \cap \Pi| > |M \cap \Pi|$. 

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PROOF. It should be clear that the values of $|H \cap \Pi|$ and $|M \cap \Pi|$ are close only when $H$ and $M$ each stabilize an intransitive decomposition of $n$ into sets of sizes close to $n/3$ and $2n/3$. Furthermore, $|H \cap \Pi|$ is smallest when $H \cong S_{n/3-1} \times S_{2n/3+1}$. (We will fix $H_{n/3-1} \cong S_{n/3-1} \times S_{2n/3+1}$ for the remainder of this proof.) Based on our definition of the $\Pi_i$, subgroups isomorphic to $S_{n/3} \times S_{2n/3}$ only intersect $\Pi_i$ when $i = n/3 - 1$, and in this case we have

$$|M \cap \Pi| = (n/3 - 1)! \left(\frac{2n/3}{n/3 - 1}\right) (n/3 - 2)! (n/3)!$$

$$= \frac{(n/3 - 1)! (2n/3)!}{(n/3 - 1)(n/3 + 1)!} < \frac{(n/3 - 2)! (2n/3 + 1)!}{(n/3)(n/3 + 1)!}$$

$$= (n/3 - 2)! \left(\frac{2n/3 + 1}{n/3}\right) (n/3 - 1)! (n/3)!$$

$$= |H_{n/3-1} \cap \Pi|.$$  

For $M \cong S_{n/3+2} \times S_{2n/3-2}$, we have

$$|M \cap \Pi| = (n/3 + 1)! \left(\frac{2n/3 - 2}{n/3 - 3}\right) (n/3 - 4)! (n/3)!$$

$$+ (n/3 + 1)! \left(\frac{2n/3 - 2}{n/3 - 5}\right) (n/3 - 6)! (n/3 + 2)!$$

$$= \frac{(n/3 + 1)! (2n/3 - 2)!}{(n/3 - 3)(n/3 + 1)!} + \frac{(n/3 + 1)! (2n/3 - 2)!}{(n/3 - 5)(n/3 + 3)!}$$

$$< \frac{(n/3 - 2)! (2n/3 + 1)!}{(n/3)(n/3 + 1)!}$$

$$= (n/3 - 2)! \left(\frac{2n/3 + 1}{n/3}\right) (n/3 - 1)! (n/3)!$$

$$= |H_{n/3-1} \cap \Pi|.$$  

Finally, based on Lemma 3.13 above, since the only subgroup $M$ that possibly intersects $\Pi_i$ for six different values of $i$ stabilizes a breakdown $(n/2 - 1, n/2 + 1)$ and contains far fewer elements than our chosen $H$, the case when $M \cong S_{n/3+3} \times S_{2n/3-3}$ has $|M \cap \Pi|$ as large as possible among the remaining cases. However, calculations similar to the above show that $|M \cap \Pi| < |H_{n/3-1} \cap \Pi|$ in this case as well, and the only remaining possibility is that $M \cong S_{n/3+1} \times S_{2n/3-1}$. In this case, for any $H \in \mathcal{H}_i$, where $i \leq n/3 - 2$ has $|H \cap \Pi| > |M \cap \Pi|$ by a calculation similar to those above, and the result is proved. $\square$

Unfortunately, when $H \cong S_{n/3-1} \times S_{2n/3+1}$ and $M \cong S_{n/3+1} \times S_{2n/3-1}$, $|H \cap \Pi| < |M \cap \Pi|$, and so this situation requires more care. Indeed, such subgroups $M$ cover elements with cycle structures $(n/3 - 1, n/3, n/3 + 1)$, $(n/3 - 2, n/3 + 1, n/3 + 1)$, $(n/3 - 3, n/3 + 1, n/3 + 2)$, and $(n/3 - 4, n/3 + 1, n/3 + 3)$, respectively, and when $n > 30$,

$$|M \cap \Pi| = \frac{n!}{3!} \left(\frac{2n}{3} - 1\right)! \left(\frac{1}{(n/3)(n/3 - 1)} + \frac{1}{(n/3 + 1)(n/3 - 2)} + \frac{1}{(n/3 + 2)(n/3 - 3)}\right)$$

$$+ \frac{1}{(n/3 + 3)(n/3 - 4)}.$$
(When \( n = 30 \), as we will see in Section 4, the elements with cycle structure \((2, 11, 17)\) must be included as well, so the cases for \( n \leq 30 \) will be handled separately.)

**Lemma 3.15.** Suppose that some nonempty collection \( C' \) of subgroups each isomorphic to \( S_{n/3+1} \times S_{2n/3-1} \) replaces a collection of subgroups \( C \) from \( \mathcal{H} \) in a minimum cover of \( S_n \). Then \( C \) must contain subgroups from each of \( \mathcal{H}_{n/3-4}, \mathcal{H}_{n/3-3}, \mathcal{H}_{n/3-2}, \) and \( \mathcal{H}_{n/3-1} \).

**Proof.** Suppose \( C \) does not contain subgroups from each of these four classes. Since none of the subgroups of (at least) one class \( \mathcal{H}_{n/3-k}, 1 \leq k \leq 4 \), are in \( C \), we may ignore the elements of \( M \cap \Pi_{n/3-k} \) for \( M \in C' \) (they are already covered by the subgroups in \( \mathcal{H}_{n/3-k} \)). However, if any of these classes of elements is removed, calculations similar to those in the proof of Lemma 3.14 show that the subgroups of \( \mathcal{H} \) now all contain more elements of \( \Pi \) than \( M \) contains on the remaining three classes \( \Pi_{n/3-j}, j \neq k \). Hence covering the elements of \( \bigcup_{H \in C}(H \cap \Pi) \) by subgroups isomorphic to \( S_{n/3+1} \times S_{2n/3-1} \) takes at least \( |C| + 1 \) subgroups, a contradiction to minimality. Thus the cover that uses \( C' \) cannot in fact be a minimum cover, and so \( C \) must contain subgroups from each of the four classes, as desired. \( \square \)

**Proof of Theorem 3.1.** By Lemmas 3.11 and 3.12, any minimal cover contains \( \mathcal{H}_{-1} \) and \( \mathcal{H}_0 \). Moreover, Lemma 3.14 shows that the only possible way to get a smaller subgroup cover than \( \mathcal{H} \) would be to replace subgroups in \( \mathcal{H} \) by subgroups isomorphic to \( S_{n/3+1} \times S_{2n/3-1} \). Assume that some collection of subgroups \( C \) is removed from \( \mathcal{H} \) and is replaced by a collection of subgroups \( C' \) isomorphic to \( S_{n/3+1} \times S_{2n/3-1} \). Let \( c := |C| \) and let \( c_i \) be the number of subgroups in \( C \cap \mathcal{H}_{n/3-i} \). Thus \( c_1 + c_2 + c_3 + c_4 = c \) and \( |C'| \leq (c - 1) \). Moreover, by Lemma 3.15, each of \( c_1, c_2, c_3, c_4 \) is at least 1.

In order to cover just the \( c_1 \) subgroups from \( \mathcal{H}_{n/3-1} \), it requires at least

\[
c_1 \cdot \frac{(n/3 - 2)! (2n/3 + 1)!}{(n/3)(n/3 + 1)} \cdot \frac{(n/3)(n/3 - 1)}{(n/3)! (2n/3 - 1)!} = c_1 \cdot \frac{4n/3 + 2}{n/3 + 1}
\]

subgroups isomorphic to \( S_{n/3+1} \times S_{2n/3-1} \) since the elements with cycle structure \((n/3 - 1, n/3, n/3 + 1)\) are partitioned among the subgroups in both isomorphism classes of maximal subgroups. This implies that \( c - 1 = |C'| \geq c_1 \cdot \frac{4n/3 + 2}{n/3 + 1} \), which in turn implies that

\[
c_1 \leq \frac{n/3 + 1}{4n/3 + 2}(c - 1).
\]

Since \( c_3 \) and \( c_4 \) are at least one, this means that \( C' \) must cover at least

\[
\frac{n/3 + 1}{4n/3 + 2}(c - 1) \cdot \frac{(n/3 - 2)! (2n/3 + 1)!}{(n/3)(n/3 + 1)} + \frac{n + 1}{4n/3 + 2}(c - 1 - 2) \cdot \frac{(n/3 - 3)! (2n/3 + 2)!}{2(n/3 + 1)(n/3 + 1)} + \frac{n/3 - 4)! (2n/3 + 3)!}{(n/3)! (2n/3 + 4)!} + \frac{n/3 + 2}{n/3 + 1} + \frac{n/3 + 1}{n/3 + 1} \cdot \frac{(n/3 - 5)! (2n/3 + 3)!}{(n/3 + 1)(n/3 + 1)} \geq (c - 1) \cdot \frac{n/3 + 1}{4n/3 + 2} \cdot \frac{(n/3 - 2)! (2n/3 + 1)!}{(n/3)(n/3 + 1)} + \frac{n + 1}{4n/3 + 2} \cdot \frac{(n/3 - 3)! (2n/3 + 2)!}{2(n/3 + 1)(n/3 + 1)}
\]
elements of $\Pi$; on the other hand, we know that $C'$ covers at most
\[ (c-1) \cdot \frac{n}{3}! \left( \frac{2n}{3} - 1 \right)! \left( \frac{1}{(n/3)(n/3-1)} + \frac{1}{(n/3+1)(n/3-2)} + \frac{1}{(n/3+2)(n/3-3)} \right) \]
\[ + \frac{1}{(n/3+3)(n/3-4)} \]
elements of $\Pi$. However, for $n > 30$, we see that in fact
\[ \frac{n}{3}! \left( \frac{2n}{3} - 1 \right)! \left( \frac{1}{(n/3)(n/3-1)} + \frac{1}{(n/3+1)(n/3-2)} \right) \]
\[ + \frac{1}{(n/3+2)(n/3-3)} + \frac{1}{(n/3+3)(n/3-4)} \]
\[ < \frac{n+1}{4n/3+2} \left( \frac{n/3-2}{(n/3)(n/3+1)} \right) + \frac{n+1}{4n/3+2} \cdot \frac{(n/3-3)! (2n/3+2)!}{2(n/3+1)(n/3+1)}. \]

Therefore, no such collection $C$ exists, and in fact $\mathcal{H}$ is the unique minimum cover of $\Pi$ (using only maximal subgroups). However, since $\mathcal{H}$ also covers $S_n$, the result follows. \qed

4. Small values of $n$ divisible by 6

In this section, we calculate $\sigma(S_n)$ for $n = 18, 24, 30$. While the arguments are quite similar to those in Section 3 (especially for $n = 24, 30$), they differ slightly and require individual treatment.

**Lemma 4.1.** The covering number of $S_{18}$ is 36772, and the unique minimal cover containing only maximal subgroups uses all subgroups isomorphic to one of $S_9 \wr S_2$, $A_{18}$, $(S_1 \times) S_{17}$, $S_3 \times S_{15}$, $S_4 \times S_{14}$, or $S_5 \times S_{13}$.

**Proof.** Let $C$ be the cover consisting of the subgroups listed. We first show that $C$ is in fact a cover. First, all 18-cycles are covered by the subgroups isomorphic to $S_9 \wr S_2$, and any permutation that fixes a point in the underlying 18-element set is covered by a subgroup isomorphic to $(S_1 \times) S_{17}$. Any element with cycle structure $(i, 18 - i)$, where $1 \leq i \leq 9$, is covered by $A_{18}$. Hence any element not covered by $C$ must consist of at least three disjoint cycles and have no fixed points. If any one of these cycles has size 3, 4, 5, then it is covered by an element of $C$. If all cycles have size at least 6, then it is an element with cycle structure $(6, 6, 6)$ and is covered by a subgroup isomorphic to $S_9 \wr S_2$, as it preserves a partition of the underlying set of size 18 into two sets of size 9. Any element not covered then has cycle structure $(2, i_1, i_2, \ldots, i_k)$, where $\sum_{j=1}^k i_j = 16$. If some $i_j = 2$, then the element is in a subgroup isomorphic to $S_4 \times S_{14}$, and so we may assume that each $i_j \geq 6$. This implies that $k = 2$ and the element has cycle structure $(2, 6, 10), (2, 7, 9)$, or $(2, 8, 8)$. However, all elements with these cycle structures stabilize an imprimitive breakdown of the underlying 18-element set into two sets of size 9 and are hence contained in a subgroup isomorphic to $S_9 \wr S_2$. Therefore, $C$ is a cover of $S_{18}$.

We now show that $C$ is actually a minimal cover. We let $\Pi'$ denote the collection of all elements of $S_{18}$ with cycle structure one of $(18), (7, 11), (1, 7, 10), (3, 7, 8), (4, 7, 7)$, or $(5, 6, 7)$. Note that these elements are partitioned among the subgroups of $C$, showing the necessity of all subgroups of $C$ in any cover using such subgroups. Now, the primitive subgroups (that are not $A_{18}$) and the imprimitive subgroups only contain 18-cycles from $\Pi'$, and these are the only maximal subgroups of $S_{18}$ containing 18-cycles. It follows that any minimal cover of $\Pi'$ uses all subgroups isomorphic to $S_9 \wr S_2$ to cover the 18-cycles and no other imprimitive subgroups or primitive subgroups (that are not $A_{18}$). Noting that the subgroups isomorphic to $S_2 \times S_{16}$ do
not contain any elements of $\Pi'$ and that the subgroups isomorphic to $S_8 \times S_{10}$ contain fewer elements of $\Pi'$ than any subgroup in $C$, we need only examine subgroups isomorphic to $S_7 \times S_{11}$ as possible replacements for members of $C$.

Assume first that $A_{18}$ is not in a minimal cover. Then all 31824 subgroups isomorphic to $S_7 \times S_{11}$ are required to cover the elements with cycle structure $(7, 11)$, and, combined with all subgroups isomorphic to $S_9 \wr S_2$, this new cover already contains more subgroups than $C$. Therefore, $A_{18}$ is contained in any minimal cover. Similarly, if any subgroup isomorphic to $(S_1 \times S_{17})$ is missing (for instance, the one fixing the element in the underlying set labeled “18”), then all $\binom{17}{3} = 19448$ subgroups isomorphic to $S_7 \times S_{11}$ with this element “18” in the 11-set are needed. As above, with subgroups isomorphic to $S_9 \wr S_2$ we already have more subgroups than are in $C$. Hence all subgroups isomorphic to $(S_1 \times S_{17})$ are in any minimal cover. This only leaves the subgroups isomorphic to $S_7 \times S_{11}$ as possible replacements for subgroups isomorphic to one of $S_3 \times S_{15}, S_4 \times S_{14}, S_5 \times S_{13}$ on the set $\Pi''$ consisting of elements with cycle structure one of $(3, 7, 8), (4, 7, 7), (5, 6, 7)$. However, each subgroup isomorphic to $S_9 \wr S_2$ already contains exactly 3181939200 elements of $\Pi''$, whereas each subgroup isomorphic to one of $S_3 \times S_{15}, S_4 \times S_{14}$ contains at least 3358297600 elements of $\Pi''$. Therefore, no collection of subgroups isomorphic to $S_7 \times S_{11}$ can possibly be more efficient, and $C$ is the unique minimal cover of $S_{18}$ (using only maximal subgroups), as desired.

**Lemma 4.2.** The covering number of $S_{24}$ is 1888233, and the unique minimal cover $\mathcal{H}$ using only maximal subgroups of $S_{24}$ is as described in Definition 3.2.

**Proof.** We construct $\Pi$ as described in Definition 3.4 for $S_n$ with $n > 30$. All subgroups isomorphic to $S_{12} \wr S_2$ and $A_{24}$ are necessary for the same reasons as for the groups $S_n$ with $n > 30$, and the other imprimitive subgroups and primitive subgroups isomorphic to $\text{PGL}(2, 23)$ can be ignored for the same reasons as for the groups $S_n$ with $n > 30$. Moreover, the subgroups isomorphic to $S_9 \times S_{16}, S_{10} \times S_{14}, S_{11} \times S_{13}$ all contain fewer elements of $\Pi$ than any subgroup they could possibly replace. This leaves only subgroups isomorphic to $S_9 \times S_{15}$ as compared to subgroups in $\mathcal{H}$. However, we now proceed exactly as in the proof of Theorem 3.1 to see that $\mathcal{H}$ is in fact the unique minimal cover using only maximal subgroups, as desired.

**Lemma 4.3.** The covering number of $S_{30}$ is 100522847, and the unique minimal cover $\mathcal{H}$ using only maximal subgroups of $S_{30}$ is as described in Definition 3.2.

**Proof.** We construct $\Pi$ as described in Definition 3.4 for $S_n$ with $n > 30$. All subgroups isomorphic to $S_{15} \wr S_2$ and $A_{30}$ are necessary for the same reasons as for the groups $S_n$ with $n > 30$, and the other imprimitive subgroups and primitive subgroups isomorphic to $\text{PGL}(2, 29)$ can be ignored for the same reasons as for the groups $S_n$ with $n > 30$. Moreover, the subgroups isomorphic to $S_{10} \times S_{20}, S_{12} \times S_{18}, S_{13} \times S_{17}, S_{14} \times S_{16}$ all contain fewer elements of $\Pi$ than any subgroup they could possibly replace. This leaves only subgroups isomorphic to $S_{11} \times S_{19}$ as compared to subgroups in $\mathcal{H}$. Note, however, in this case that these subgroups contain elements with cycle structure $(2, 11, 17)$ in addition to elements with cycle structure $(6, 11, 13), (7, 11, 12), (8, 11, 11),$ and $(9, 10, 11)$.

Let $\mathcal{C}'$ be the collection of subgroups isomorphic to $S_{11} \times S_{19}$ that we use instead of members of $\mathcal{H}$. First, we note that if two subgroups isomorphic to $S_2 \times S_{28}$ are removed from $\mathcal{H}$, it requires at least $2 \cdot \binom{28}{10} - \binom{10}{5} = 29910465$ subgroups isomorphic to $S_{11} \times S_{19}$ to replace them. There are only 227900854 subgroups altogether isomorphic to one of $S_2 \times S_{28}, S_6 \times S_{24}, S_7 \times S_{23}, S_8 \times S_{22},$ or $S_9 \times S_{21},$ and so this new cover cannot possibly be more efficient. Hence there is at most one subgroup isomorphic to $S_2 \times S_{28}$ replaced by subgroups isomorphic to $S_{11} \times S_{19}$. Effectively ignoring this subgroup, we now proceed exactly as in the proof of Theorem 3.1 with the assumption that $c_1 + c_2 + c_3 + c_4 = c$ and $|\mathcal{C}'| \leq c$ (instead of $|\mathcal{C}'| \leq c - 1$). We again reach
a contradiction to see that \( H \) is in fact the unique minimal cover using only maximal subgroups, as desired.

We may now finally prove the main result.

**Proof of Theorem 1.1.** The result follows immediately from the results of Theorem 3.1 and Lemmas 4.1, 4.2, and 4.3.

\[ \square \]

5. Other even values of \( n \) and open problems

Most of the proofs listed here also apply when \( n \) is even but not divisible by 6; however, the situation of elements with cycle structure \((i, r, s)\) with \( i = \lfloor n/3 \rfloor \) is much trickier. We will describe each situation in detail.

5.1. \( S_n, n \equiv 2 \pmod{6} \). Suppose that \( n \equiv 2 \pmod{6} \). Then for some even \( t \in \mathbb{N} \), we may write \( n = 3t + 2 \). In this case, elements with cycle structure \((t, t + 1, t + 1)\) are problematic, since subgroups isomorphic to \( S_t \times S_{2t+2} \) and \( S_{t+1} \times S_{2t+1} \) both contain the same number of elements with this cycle structure. Because of this, the arguments used in the proof of Theorem 3.1 will not apply. On the other hand, as is the case with \( S_n \) for \( n \equiv 0 \pmod{6} \), it should more or less boil down to the groups isomorphic to \( S_t \times S_{n-1} \) not being as efficient at covering the elements of the set \( \Pi \) (defined as in Definition 3.4) as intransitive groups of the form \( S_t \times S_{n-i} \), where \( i \leq t \). For this reason, the following conjecture seems likely:

**Conjecture 5.1.** For sufficiently large \( n \equiv 2 \pmod{6} \),

\[
\sigma(S_n) = \frac{1}{2} \binom{n}{n/2} + \sum_{i=0}^{\lfloor n/3 \rfloor} \binom{n}{i}.
\]

5.2. \( S_n, n \equiv 4 \pmod{6} \). Suppose that \( n \equiv 4 \pmod{6} \). Then for some odd \( t \in \mathbb{N} \), we may write \( n = 3t + 1 \). In this case, elements with cycle structure \((t, t, t + 1)\) or \((t, t + 1, t + 1)\) are problematic, since not all subgroups isomorphic to \( S_t \times S_{2t+1} \) will be necessary to cover them. (In [11], the Erdős-Ko-Rado Theorem was used to determine how many subgroups isomorphic to \( S_3 \times S_3 \) were necessary in a minimal cover of \( S_{10} \).) The elements with this cycle structure make it likely that some subgroups isomorphic to \( S_t \times S_{2t+1} \) are necessary in a minimal cover; on the other hand, since elements with cycle structure \((t - 1, t + 1, t + 1)\) are contained in the subgroups isomorphic to \( S_{n/2} \wr S_2 \), if all subgroups isomorphic to \( S_t \times S_{2t+1} \) are included in the cover, then the subgroups isomorphic to \( S_{t-1} \times S_{2t+2} \) are unnecessary. Indeed, we have the following:

**Lemma 5.2.** When \( n \equiv 4 \pmod{6} \), the subgroups isomorphic to \( S_{\lfloor n/3 \rfloor - 1} \times S_{n - \lfloor n/3 \rfloor + 1} \) are not needed in the cover \( C_0 \), as defined at the beginning of Section 3.

**Proof.** Let \( C \) be the collection of subgroups obtained by removing the subgroups isomorphic to \( S_{\lfloor n/3 \rfloor - 1} \times S_{n - \lfloor n/3 \rfloor + 1} \) from \( C_0 \). We will show that \( C \) is in fact a cover of \( S_n \). We proceed as in the proof of Lemma 3.3 and suppose that \( g \) is an element of \( S_n \) that is not covered by \( C \). As in the proof of Lemma 3.3, \( g \) fixes no elements of the underlying set of \( n \) elements and has cycle structure consisting of at least three disjoint cycles, each of which has length at least \( \lfloor n/3 \rfloor - 1 \). Since \( n \equiv 1 \pmod{3} \), \( n = 3 \lfloor n/3 \rfloor + 1 = (\lfloor n/3 \rfloor - 1) + 2(\lfloor n/3 \rfloor + 1) \).

In particular, both \( (\lfloor n/3 \rfloor - 1) \) and \( (\lfloor n/3 \rfloor + 1) \) are even, so elements with cycle structure \((\lfloor n/3 \rfloor - 1, \lfloor n/3 \rfloor + 1, \lfloor n/3 \rfloor + 1)\) preserve a partition of the underlying \( n \)-element set into two sets of size \( n/2 \), and each element with this cycle structure is contained in a subgroup isomorphic to \( S_{n/2} \wr S_2 \). Similarly, all elements with cycle structure \((\lfloor n/3 \rfloor - 1, \lfloor n/3 \rfloor - 1, \lfloor n/3 \rfloor + 3)\) are covered by the subgroups isomorphic to \( S_{n/2} \wr S_2 \). This means that the cycle structure of \( g \) must be \((\lfloor n/3 \rfloor - 1, \lfloor n/3 \rfloor, \lfloor n/3 \rfloor + 2)\); however, elements with this cycle structure are contained in the subgroups isomorphic to \( S_{\lfloor n/3 \rfloor} \times S_{n - \lfloor n/3 \rfloor + 1} \). Therefore, \( C \) is a cover of \( S_n \), as desired. \( \square \)
Indeed, this shows that
\[ \sigma(S_n) \leq \frac{1}{2} \left( \frac{n}{n/2} \right) + \sum_{i=0}^{\lfloor n/3 \rfloor - 2} \binom{n}{i} + \binom{n}{\lfloor n/3 \rfloor}. \]

For this reason, we conjecture the following, although (if true) it will likely be more difficult to prove than Conjecture 5.1:

**CONJECTURE 5.3.** For sufficiently large \( n \equiv 4 \pmod{6} \),
\[ \sigma(S_n) = \frac{1}{2} \left( \frac{n}{n/2} \right) + \sum_{i=0}^{\lfloor n/3 \rfloor - 2} \binom{n}{i} + \binom{n}{\lfloor n/3 \rfloor}. \]

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