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Particle with spin $S = 3/2$ in Riemannian space-time

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Abstract

Equations for 16-component vector-bispinor field, originated from Rarita–Schwinger Lagrangian for spin 3/2 field extended to Riemannian space-time are investigated. Additional general covariant constrains for the field are produced, which for some space-time models greatly simplify original wave equation.

Peculiarities in description of the massless spin 3/2 field are specified. In the flat Minkowski space for massless case there exist gauge invariance of the main wave equation, which reduces to possibility to produce a whole class of trivial solutions in the the form of 4-gradient of arbitrary(gauge) bispinor function, $\Psi^0(x) = \partial_\alpha \psi(x)$. Generalization of that property for Riemannian model is performed; it is shown that in general covariant case solutions of the gradient type $\Psi^0(\alpha) = (\nabla_\alpha + \Gamma_\alpha)\Psi(x)$ exist in space-time regions where the Ricci tensor obeys an identity $R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 0$.

1 Massive field, additional constraints

Starting with fundamental investigation by Pauli and Fierz [1, 2], also Rarita and Schwinger [3], the field with spin 3/2 always attracted attention:

Ginzburg [4], Davydov [5], Tamm – Davydov – Ginzburg [6, 7], Ginzburg – Smorodinsky [8, 9], Fradkin [10], Belinfante [11], Fainberg [12], Petras [13, 14], Jonson and Sudarshan [15, 16], Bender and McKoy [17], Munczek [18], Velo and Zwanziger [19, 20], Hagen and Singh [21, 22], Baisya [26], Fisk and Tait [27], Hortaçsu [28], Mathews et al [29, 30], Madore and Tait [31, 32], Hasumi, Endo and Kimura [33], Lopes – Spehler – Leite – Fleury [34, 35], Auriola et al [36], Inoue – Omote – Kobayashi [37], Loide [38, 39], Pletjuxov and Strazhev [40], Labonte [41, 42], Capri and Kobes [43], Barut and Xu [44], Darkhosh [45], Rinderi and Sivakumar [46], Cox [47], Penrose [48], Pascalutsa [49], Haberzettl [50], Deser S., Waldron A., Pascalutsa [52, 53], Kirchbach and Ahluwalia [54], Gsponer and Hurni [55], Pilling [57, 58], Kaloshin and Lomov [59, 60], Napsuciale – Kirchbach – Rodriguez [61].

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On the curved space-time background, the field of spin 3/2 is investigated much less than fields of spins 0, 1/2, 1. This circumstance is due to complexity of this object: all its 16 components are tightly linked to each other by presence of curved geometry. Let us consider some peculiarities in description of the particle with spin 3/2 in Riemannian space-time, first specifying the massive case.

Lagrangian by Rarita – Schwinger extended to generally covariant case has the form (let it be \( k = mc/\hbar \))

\[
L = \frac{1}{2} \left[ \bar{\Psi}_\alpha \gamma^\beta(x) \vec{D}_\beta \Psi^\alpha - \bar{\Psi}_\alpha \gamma^\beta(x) \vec{D}_\beta \Psi^\alpha \right] \\
+ \frac{1}{3} \left[ \bar{\Psi}_\alpha \gamma^\alpha(x) \vec{D}_\beta \Psi^\beta - \bar{\Psi}_\alpha \gamma^\alpha(x) \vec{D}_\beta \Psi^\beta \right] \\
+ \frac{1}{6} \left[ \bar{\Psi}_\alpha \gamma^\alpha(x) \gamma^\beta \vec{D}_\beta \gamma^\rho(x) \Psi_\rho - \bar{\Psi}_\alpha \gamma^\alpha(x) \gamma^\beta(x) \vec{D}_\beta \gamma^\rho(x) \Psi_\rho \right] \\
+ \kappa \bar{\Psi}_\alpha \Psi^\alpha - \frac{1}{3} \kappa \bar{\Psi}_\alpha \gamma^\alpha(x) \gamma_\beta(x) \Psi^\beta .
\]

(1.1)

Here \( \Psi_\alpha \) stands for a wave function for a particle with transformation properties of local bispinor and general covariant vector; symbols \( \rightarrow \) and \( \leftarrow \) designate operators \( D_\alpha \) acting on the right and on the left respectively

\[
\vec{D}_\alpha = \vec{\nabla}_\alpha + \Gamma_\alpha(x) - ie A_\alpha(x) , \quad \vec{D}_\alpha = \vec{\nabla}_\alpha - \Gamma_\alpha(x) + ie A_\alpha(x),
\]

\( A_\alpha(x) \) designates a 4-potential of external electromagnetic field; for shortness the combination \( e/\bar{\hbar}c \) is noted as \( e \).

From Lagrangian (1.1) it follow equations for \( \Psi(x) \) and \( \bar{\Psi}(x) \):

\[
\left[ \left[ \gamma^\alpha(x) \vec{D}_\alpha + \kappa \right] \delta^\beta_\sigma - \frac{1}{3} \left[ \gamma^\beta(x) \vec{D}_\sigma + \gamma_\sigma(x) \vec{D}_\beta \right] \right] \Psi_\beta(x) = 0 ,
\]

(1.2a)

and

\[
\bar{\Psi}_\beta(x) \left[ \left[ \gamma^\alpha(x) \vec{D}_\alpha - \kappa \right] \delta^\beta_\sigma - \frac{1}{3} \left[ \gamma^\beta(x) \vec{D}_\sigma + \gamma_\sigma(x) \vec{D}_\beta \right] \right] + \frac{1}{3} \gamma^\beta(x) \left[ \gamma^\alpha(x) \vec{D}_\alpha + \kappa \right] \gamma_\sigma(x) = 0 .
\]

(1.2b)

Below we use spinor representation for Dirac matrices, so we use identities

\[
\bar{\Psi}_\beta = \Psi_\beta^+ \gamma^0 , \quad (\gamma^\beta(x))^+ = \gamma^0 \gamma^\beta(x) , \quad (\Gamma_\beta(x))^+ \gamma^0 = -\gamma^0 \Gamma_\beta(x).
\]

(1.3)

The order in writing operators \( \gamma^\alpha(x) \) and \( \vec{D}_\beta \) (also \( \gamma^\alpha(x) \) and \( \vec{D}_\beta \)) does not matter, this quantities commute with each other; besides there exist identities

\[
\Psi_\beta = \Psi_\beta^+ \gamma^0 , \quad (\gamma^\beta(x))^+ = \gamma^0 \gamma^\beta(x) , \quad (\Gamma_\beta(x))^+ \gamma^0 = -\gamma^0 \Gamma_\beta(x).
\]
\[
\begin{align*}
\gamma^\rho(x) \Gamma_\sigma(x) & - \gamma_\sigma(x) \Gamma^\rho(x) = \nabla_\sigma \gamma^\rho(x) , \quad (1.4a) \\
\gamma^\rho(x) \overset{\rightarrow}{D}_\sigma & = \overset{\rightarrow}{D}_\sigma \gamma^\rho(x) , \quad \gamma^\rho(x) \overset{\leftarrow}{D}_\sigma = \overset{\leftarrow}{D}_\sigma \gamma^\rho(x) . \quad (1.4b)
\end{align*}
\]

Below we will use the formulas
\[
\begin{align*}
\gamma^\alpha(x) \gamma^\beta(x) + \gamma^\beta(x) \gamma^\alpha(x) & = 2 g^{\alpha\beta}(x) , \quad \gamma^\alpha \gamma_\alpha = 4 , \\
\gamma^\alpha(x) \gamma^\beta(x) & = g^{\alpha\beta}(x) + 2 \sigma^{\alpha\beta}(x) , \quad \sigma^{\alpha\beta}(x) = \sigma^{ab} e^{\alpha}_a(x) e^{\beta}_b(x) , \\
\gamma^\alpha(x) \gamma^\beta(x) \gamma^\rho(x) & = \gamma^\alpha(x) g^{\beta\rho}(x) - \gamma^\beta(x) g^{\alpha\rho}(x) + \\
& \gamma^\rho(x) g^{\alpha\beta}(x) + i \gamma^5 \epsilon^{\alpha\beta\rho\sigma}(x) \gamma^\sigma(x) ; \quad (1.5)
\end{align*}
\]

they follow from the properties of usual Dirac matrices multiplied by relevant tetrads.

Starting with eqs. (1.2a, b), one can derive additional constraints for components of the wave function \(\Psi_\alpha(x)\); thereby, in accordance with Pauli – Fierz approach \([1, 2]\), these constraints are deduced from the initial lagrangian (1.1)

Indeed, let us multiply eq. (1.2a) from the left by the matrix \(\gamma^\sigma(x)\):
\[
\left[ \gamma^\beta \gamma^\alpha D_\alpha + \kappa \gamma^\beta - \frac{1}{3} \gamma^\sigma \gamma^\beta D_\sigma - \frac{4}{3} D^\beta + \frac{4}{3} \gamma^\alpha \gamma^\beta D_\alpha - \frac{4}{3} \kappa \gamma^\beta \right] \Psi_\beta = 0 ,
\]
from whence it follows
\[
D_\beta \Psi_\beta = \frac{\kappa}{2} \gamma_\beta \Psi_\beta . \quad (1.6)
\]

It is a first additional constraint. Now, let us act on eq. (1.2a) from the left by operator \(D^\rho\):
\[
\left[ D^\beta \gamma^\alpha D_\alpha + \kappa D^\beta - \frac{1}{3} \gamma^\sigma D^\sigma D_\sigma - \frac{1}{3} \gamma^\sigma D_\sigma D^\beta + \\
\frac{1}{3} \gamma^\sigma \gamma^\alpha D^\sigma D_\alpha \gamma^\beta - \frac{\kappa}{3} \gamma^\sigma D_\sigma \gamma^\beta \right] \Psi_\beta(x) = 0 .
\]

Then with the use of identity
\[
D^\beta D_\alpha = D_\alpha D^\beta + D^\beta , \quad \text{where} \quad D^\beta_\alpha = D^\beta D_\alpha - D_\alpha D^\beta ,
\]
we get
\[
\gamma^\alpha D_\alpha \left( \frac{2}{3} D^\beta - \frac{\kappa}{3} \gamma^\beta \right) \Psi_\beta + \gamma^\alpha D^\beta_\alpha \Psi_\beta + \kappa D^\beta \Psi_\beta + \frac{1}{3} \sigma^{\alpha\beta} D_{\alpha\beta} \gamma^\rho \Psi_\rho = 0 .
\]

Here, the first term vanishes due to (1.6). Thus, we arrive at
\[
-D_{\alpha\beta} \gamma^\alpha \Psi_\beta + \frac{\kappa^2}{2} \gamma^\rho \Psi_\rho + \frac{1}{3} \sigma^{\alpha\beta} D_{\alpha\beta} \gamma^\rho \Psi_\rho = 0 . \quad (1.7)
\]

This second additional constraint can be transformed to the form of algebraic relationships. Indeed, let us detail operator \(D_{\alpha\beta}^\rho\):
\[ D_{\alpha\beta} = (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) + \dot{D}_{\alpha\beta} - ie \text{ } F_{\alpha\beta}, \]  

where \( F_{\alpha\beta} \) is a electromagnetic tensor; \( \dot{D}_{\alpha\beta} \) is determined by relation

\[ \dot{D}_{\alpha\beta} = \nabla_\beta \Gamma_\alpha - \nabla_\alpha \Gamma_\beta + \Gamma_\alpha \Gamma_\beta - \Gamma_\beta \Gamma_\alpha. \]

With the use of definition for the bispinor connection \( \Gamma_\alpha \), one can produce

\[ \nabla_\beta \Gamma_\alpha - \nabla_\alpha \Gamma_\beta = \frac{1}{2} \sigma^{ab} e_\alpha^{(a)} (e_{(b)\nu;\alpha;\beta} - e_{(b)\nu;\beta;\alpha}), \]

\[ + \frac{1}{2} \sigma^{ab} (e_{(a)\nu;\alpha} e_\nu^{(b);\beta} - e_{(a)\nu;\beta} e_\nu^{(b);\alpha}). \]

For the term \( (\Gamma_\alpha \Gamma_\beta - \Gamma_\beta \Gamma_\alpha) \), using the commutative relation

\[ ([\sigma^{ab}, \sigma^{mn}] = (g^{ma} \sigma^{nb} - g^{mb} \sigma^{na}) - (g^{na} \sigma^{mb} - g^{nb} \sigma^{ma}), \]

we derive the following expression

\[ \Gamma_\alpha \Gamma_\beta - \Gamma_\beta \Gamma_\alpha = -\frac{1}{2} \sigma^{ab} (e_{(a)\nu;\alpha} e_\nu^{(b);\beta} - e_{(a)\nu;\beta} e_\nu^{(b);\alpha}). \]

Summing (1.8c) and (1.8d), we get

\[ \dot{D}_{\alpha\beta} = \frac{1}{2} \sigma^{ab} e_\alpha^{(a)} (e_{(b)\nu;\beta;\alpha} - e_{(b)\nu;\alpha;\beta}), \]

\[ = \frac{1}{2} \sigma^{ab} e_\nu^{(a)} e_\mu^{(b)} R_{\mu\nu\alpha\beta}(x) = \frac{1}{2} \sigma_{\nu\mu}(x) R_{\mu\nu\alpha\beta}(x), \]

where \( R_{\mu\nu;\alpha\beta}(x) \) stands for the Riemann tensor. Substituting (1.8e) into (1.8a), we obtain

\[ D_{\alpha\beta} = (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) + \frac{1}{2} \sigma^{\mu\nu} R_{\mu\nu\alpha\beta} - ie \text{ } F_{\alpha\beta}. \]

Taking into account (1.9), now consider (1.7). For the first term in (1.7) we will obtain

\[ -\gamma^\alpha D_{\alpha\beta} \Psi^\beta = -\gamma^\alpha \left[ (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) + \frac{1}{2} \sigma^{\mu\nu} R_{\mu\nu\alpha\beta} - ie \text{ } F_{\alpha\beta} \right] \Psi^\beta; \]

note identity

\[ -\gamma^\alpha (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) \Psi^\beta = \gamma^\alpha \Psi^\nu R_{\nu\alpha}; \]

for the second term, using (1.5), one derives

\[ -\frac{1}{2} \gamma^\alpha \sigma^{\mu\nu} R_{\mu\nu\alpha\beta} \Psi^\beta = -\frac{1}{4} \left[ \gamma^\alpha g^{\mu\nu} - \gamma^\mu g^{\alpha\nu} + \gamma^\nu g^{\alpha\mu} + i \gamma^5 \epsilon_{\alpha\mu\nu\sigma}(x) \gamma_\sigma \right] R_{\mu\nu\alpha\beta} \Psi^\beta, \]

from whence, allowing for symmetry of the Riemann tensor we get \( R_{\alpha\beta} \) is the Ricci tensor:

\[ -\frac{1}{2} \gamma^\alpha \sigma^{\mu\nu} R_{\mu\nu\alpha\beta} \Psi^\beta = -\frac{1}{2} \gamma^\nu R_{\nu\beta} \Psi^\beta. \]
This, relation (1.10a) reads
\[ - \gamma^\alpha \tilde{D}_\alpha \Psi^\beta = (\frac{1}{2} R_{\alpha\beta} + ie F_{\alpha\beta}) \gamma^\alpha \Psi^\beta. \] (1.10c)

Now, for the third term in (1.7) we derive
\[ \frac{1}{3} (\sigma^{\alpha\beta} D_{\alpha\beta}) \gamma^\rho \Psi_\rho = \frac{1}{3} \sigma^{\alpha\beta} \left[ (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) + \frac{1}{2} \sigma^{\mu\nu} R_{\mu\nu\alpha\beta} - ie F_{\alpha\beta} \right] \gamma^\rho \Psi_\rho. \]

Here the first term vanish identically (let it be \( \gamma^\sigma \Psi_\sigma = \Phi(x) \)):
\[ (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) \Phi(x) = \partial/\partial x^\alpha (\partial \Phi/\partial x^\beta) - \partial/\partial x^\beta (\partial \Phi/\partial x^\alpha) + \Gamma^\sigma_{\alpha\beta} (\partial \Phi/\partial x^\sigma) \equiv 0. \]

The second term \((R)\) is the Ricci scalar reads
\[ \frac{1}{6} \sigma^{\alpha\beta} \sigma^{\mu\nu} R_{\mu\nu\alpha\beta} \gamma^\rho \Psi_\rho = \frac{1}{24} \gamma^\alpha (\gamma^\beta \gamma^\mu) R_{\mu\nu\alpha\beta} \gamma^\rho \Psi_\rho \]
\[ = -\frac{1}{12} \gamma^\alpha \gamma^\beta R_{\alpha\beta} \gamma^\rho \Psi_\rho = -\frac{1}{12} \left( g^{\alpha\beta} + 2 \sigma^{\alpha\beta} \right) R_{\alpha\beta} \gamma^\rho \Psi_\rho = -\frac{1}{12} R \gamma^\rho \Psi_\rho. \]

Therefore, the third term in (1.7) reduces to
\[ \frac{1}{3} (\sigma^{\alpha\beta} D_{\alpha\beta}) \gamma^\rho \Psi_\rho = -\frac{1}{12} R \gamma^\rho \Psi_\rho + \frac{ie}{3} \sigma^{\alpha\beta} F_{\alpha\beta} \gamma^\rho \Psi_\rho. \]

Thus, the second additional constraint (1.7) is equivalent to the algebraic relationship
\[ \left( \frac{1}{2} R_{\alpha\beta} + ie F_{\alpha\beta} \right) \gamma^\alpha \Psi^\beta + \left[ \frac{1}{2} \kappa^2 - \frac{1}{3} \left( \frac{1}{4} R + ie F_{\alpha\beta} \sigma^{\alpha\beta} \right) \right] \gamma^\rho \Psi_\rho = 0; \] (1.11a)

for convenience let us written down the first condition as well
\[ D_\beta \Psi^\beta = \frac{\kappa}{2} \gamma_\beta \Psi^\beta. \] (1.11b)

Sometime, these two relations permit us to greatly simplify the initial wave equation (1.2a). For instance, for a free particle in Minkowski space-time, in Cartesian coordinates and tetrad, eqs. (1.11, b) give
\[ \gamma^a \Psi_a(x) = 0, \quad \partial^a \Psi_a(x) = 0, \] (1.12a)
so that eq. (1.2a) assumes the form of four separate Dirac equations
\[ (\gamma^a \partial_a + \kappa) \Psi_a(x) = 0. \] (1.12b)

Analogous situation arises in any curved space-time with vanishing Ricci tensor. Indeed, let
\[ R_{\alpha\beta}(x) = 0, \quad F_{\alpha\beta}(x) = 0, \] (1.13a)
then the full systems of equations determining the particle with spin 3/2 is
\[
\gamma^\beta(x) \Psi_\beta(x) = 0 , \quad (\nabla_\beta + \Gamma_\beta(x)) \Psi^\beta(x) = 0 , \\
\left[ \gamma^\alpha(x) \left( \nabla_\alpha + \Gamma_\alpha(x) \right) + \kappa \right] \Psi^\beta(x) = 0 . \tag{1.13b}
\]
It should be noted that because \(\Psi^\beta(x)\) stands for a general covariant vector, and \(\nabla_\alpha\) stands for a covariant derivative, the karsr equation in (1.13b) is not equivalent to four independent Dirac-like equations.

We can extend the system (1.13b) to the class of space-time model with more general structure of the Ricci tensor
\[
R_{\alpha\beta}(x) = \frac{1}{4} R(x) g_{\alpha\beta}(x) . \tag{1.14a}
\]
In this case, additional constraints reduce to
\[
D^\beta(x) \Psi_\beta(x) = \frac{1}{2} \kappa \gamma^\beta(x) \Psi_\beta(x) , \\
\left( \frac{1}{12} R(x) - \frac{m^2 c^2}{h^2} \right) \left[ \gamma^\beta(x) \Psi_\beta(x) \right] = 0 . \tag{1.14b}
\]
Simplest examples of such models are de Sitter and anti de Sitter spaces.

## 2 Massless field

Now let us specify the massless case. It is known that in Minkowski space-time, equation for massless field with spin 3/2 can be transformed to a special form when it become evident existence of trivial solutions in the form of 4-gradient of arbitrary bispinor
\[
i\gamma^5 \epsilon^{abcd}_a \gamma_d \partial_b \tilde{\Psi}_c(x) = 0 , \quad \tilde{\Psi}_c(x) = \partial_c \psi(x) . \tag{2.1}
\]
This property proves gauge invariance of massless wave equation, which give possibility to remove redundant degrees of freedom.

Let us consider analogous problem in the case of a curved space-time. It is convenient to start with the following matrix form of eq. (1.2a)
\[
\left[ \alpha^\nu(x) D_\nu + \kappa \beta(x) \right] \Psi(x) = 0 , \tag{2.2a}
\]
\[
\Psi(x) = (\Psi_\sigma(x)) , \quad (\beta)^\sigma_\rho = \delta^\sigma_\rho - \frac{1}{3} \gamma^\rho(x) \gamma^\sigma(x) , \\
(\alpha^\nu)^\sigma_\rho = \gamma^\nu(x) \delta^\sigma_\rho - \frac{1}{3} \gamma^\sigma(x) \delta^\nu_\rho \\
- \frac{1}{3} \gamma_\rho(x) g^{\sigma\sigma}(x) + \frac{1}{3} \gamma_\rho(x) \gamma^\nu(x) \gamma^\sigma(x) . \tag{2.2b}
\]
Let us perform two successive transformation over eq. (2.2a). Furs, multiply it from the left by a matrix \(C\), ant then translate equation to a new representation with the help of other matrix \(S\):
\[ \beta', \alpha' \Rightarrow \beta = C \beta', \alpha' = C \alpha' \Rightarrow \]

\[ \tilde{\beta} = S \beta' S^{-1}, \tilde{\alpha}' = S \alpha' S^{-1}, \bar{\Psi} = S \Psi. \]

(2.3)

The relevant matrices are taken in the form

\[ C_{\alpha}^{\beta} = \delta_\alpha^\beta + c \gamma_\alpha(x) \gamma^\beta(x), \quad S_{\alpha}^{\beta} = \delta_\alpha^\beta + a \gamma_\alpha(x) \gamma^\beta(x), \]

\[ (S^{-1})^{\beta}_{\alpha} = \delta_\alpha^\beta + b \gamma_\alpha(x) \gamma^\beta(x), \quad a + b + 4 ab = 0. \]

(2.4)

The quantities \( a, b, c \) are unknown numerical parameters; relationship between \( a \) and \( b \) ensures identity \( S S^{-1} = I \). In accordance with (2.3) and (2.4), we find \( \beta', \tilde{\beta} \) and \( \alpha', \tilde{\alpha}' \):

\[ (\beta')^\rho_\sigma = (\delta^\rho_\sigma - \frac{c + 1}{3} \gamma_\rho \gamma^\sigma), \]

\[ (\tilde{\beta})^\rho_\sigma = \{\delta^\rho_\sigma + [b + (4c + 1) (a - (4a + 1) \frac{c + 1}{3})] \gamma_\rho \gamma^\sigma\}, \]

\[ (\alpha'^\nu_\sigma)^\rho = [\gamma^\nu \delta^\rho_\sigma - \frac{1}{3} \gamma^\nu \delta^\rho_\sigma + (2c - \frac{1}{3}) \gamma_\rho g^{\nu\sigma} + \frac{1}{3} \gamma_\rho \gamma^\nu \gamma^\sigma], \]

\[ (\tilde{\alpha}^\nu_\sigma)^\rho = \gamma^\nu \delta^\rho_\sigma \{1 - \left[b + \frac{1}{3} + b \left(\frac{2c - 1}{3} (1 + 4a) + 2a\right)\right]\} \]

\[ + \gamma^\nu \delta^\rho_\sigma \left\{\frac{2b - 1}{3} + \left[b + \frac{1}{3} + b \left(\frac{2c - 1}{3} (1 + 4a) + 2a\right)\right]\} \]

\[ + \gamma_\rho g^{\nu\sigma} \left\{[(2c - 1) \frac{1}{3} + 2a] + \left[b + \frac{1}{3} + b ((2c - 1) \frac{1}{3} + 4a) + 2a]\right]\} \]

\[ + i \gamma^5 \epsilon^{\nu\sigma\mu} \gamma_\mu \left[b + \frac{1}{3} + b ((2c - 1) \frac{1}{3} + 4a) + 2a]\right]. \]

(2.5b)

Let us try to choose \((a, b, c)\) so that in expression for \(\tilde{\alpha}'\) all terms excluding one containing Levi-Civita tensor vanish. To this end, we must impose restrictions

\[ a + b + 4 ab = 0, \quad 1 - \left[b + \frac{1}{3} + b ( (2c - 1) \frac{1}{3} + 4a) + 2a\right] = 0, \]

\[ \frac{2b + 1}{3} + \left[b + \frac{1}{3} + b ( (2c - 1) \frac{1}{3} + 4a) + 2a\right] = 0, \]

\[ (1 + 4a) \frac{2c - 1}{3} + 2a + \left[b + \frac{1}{3} + b ( (2c - 1) \frac{1}{3} + 4a) + 2a\right] = 0. \]

Solution of the system is

\[ a = -\frac{1}{3}, \quad b = -1, \quad c = +2 \]

(2.6a)

Thus, the transformation \( S \) is
\[ S_\alpha^\beta = \delta_\alpha^\beta - \frac{1}{3} \gamma_\alpha(x) \gamma^\beta(x) , \quad \tilde{\Psi}_\alpha = S_\alpha^\beta \Psi_\beta \]

and correspondingly in new representation the wave equation is determined by the matrices

\[
\begin{align*}
(\tilde{\beta})_\rho^\sigma &= \delta_\rho^\sigma - \gamma_\rho(x) \gamma^\sigma(x) , \\
(\tilde{\alpha}_\nu^\rho)_\sigma &= +i \gamma^5 \epsilon_\rho^{\nu\sigma\mu}(x) \gamma_\mu(x) .
\end{align*}
\]  

Expression for \( \tilde{\beta} \) in (2.6b) can be rewritten as differently

\[
(\tilde{\beta})_\rho^\sigma = -2 \sigma_\rho^\sigma(x)
\]

and further, with the use of identity

\[
2 \sigma_\rho^\sigma(x) = 2 \left( \frac{1}{4} \gamma_\mu(x) \right) \left[ \gamma^\mu(x) \sigma_\rho^\sigma(x) \right] = \sigma_\rho^\sigma(x) + \frac{i}{4} \gamma_\mu(x) \gamma^5 \epsilon_\rho^{\sigma\nu} \gamma_\nu(x)
\]

for the matrix \( \tilde{\beta} \) we get

\[
(\tilde{\beta})_\rho^\sigma = \frac{i}{2} \gamma^5 \epsilon_\rho^{\nu\sigma\mu}(x) \gamma_\mu(x) \gamma_\nu(x) .
\]  

Allowing for (2.6b, c), equation for the particle with spin 3/2 can be presented as follows

\[
\gamma^5 \epsilon_\rho^{\nu\sigma\mu}(x) \gamma_\mu(x) \left[ i D_\nu - \frac{mc}{2h} \gamma_\nu(x) \right] \tilde{\Psi}_\sigma(x) = 0 .
\]  

At \( m = 0 \) we obtain an equation (compare it with (2.1)) for massless field

\[
i \gamma^5 \epsilon_\rho^{\nu\sigma\mu}(x) \gamma_\mu(x) \left[ \nabla_\nu + \Gamma_\nu(x) \right] \tilde{\Psi}_\sigma(x) = 0 .
\]  

Not let us investigate the problem of possible existence of solutions in the form of 4-gradient of arbitrary bispinor field. Substituting the function \( \tilde{\Psi}_\sigma^0(x) \) of the form

\[
\tilde{\Psi}_\beta^0(x) = \left[ \nabla_\beta + \Gamma_\beta(x) \right] \Psi(x) ,
\]  

into eq. (2.7b), we get

\[
\frac{i}{2} \gamma^5 \epsilon_\rho^{\nu\sigma\mu}(x) \gamma_\mu(x) \left[ D_\nu , D_\sigma \right] - \Psi(x) = 0 .
\]

Taking into account expression (1.9) for the commutator \( [D_\nu , D_\sigma] \) when \( F_{\nu\mu} = 0 \), and also allowing for that the bispinor \( \Psi \) is a scalar in general covariant sense, we get

\[
\frac{i}{4} \gamma^5 \epsilon_\rho^{\nu\sigma\mu}(x) \gamma_\mu(x) \left[ \sigma_{\alpha\beta}(x) R_{\alpha\beta\nu\sigma}(x) \right] \Psi(x) = 0 .
\]

Further, we obtain

\[
\frac{i}{4} \gamma^5 \epsilon_\rho^{\nu\sigma\mu}(x) \left[ \gamma^\beta(x) R_{\mu\beta\nu\sigma}(x) + \frac{i}{2} \gamma^5 \epsilon_\mu^{\alpha\beta\sigma\nu}(x) \gamma_\sigma(x) R_{\alpha\beta\nu\sigma}(x) \right] \Psi(x) = 0 ;
\]

therefore arrive at
\( R_{\alpha\beta\nu\sigma}(x) \left[ \epsilon_{\rho}^{\nu\gamma\mu}(x) \epsilon_{\mu}^{\alpha\beta\gamma}(x) \right] \left[ \gamma_{s}(x) \Psi(x) \right] = 0 \).

Using the known formula

\[
\epsilon_{\rho}^{\nu\sigma\mu}(x) \epsilon_{\mu}^{\alpha\beta\gamma}(x) = \det \begin{vmatrix}
\delta_{\rho}^{\gamma}
& \delta_{\rho}^{\beta}
& \delta_{\rho}^{s}

\delta_{\rho}^{\gamma}
& \delta_{\rho}^{\beta}
& \delta_{\rho}^{s}

\delta_{\rho}^{\gamma}
& \delta_{\rho}^{\beta}
& \delta_{\rho}^{s}
\end{vmatrix},
\]

from (2.8b) we derive relation needed

\[
\left[ R_{\alpha\beta}(x) - \frac{1}{2} R(x) g_{\alpha\beta}(x) \right] \gamma^{\beta}(x) \Psi(x) = 0. \tag{2.8c}
\]

Thus, we conclude that in the region where

\[
R_{\alpha\beta}(x) - \frac{1}{2} R(x) g_{\alpha\beta}(x) = 0,
\]

the massless particle with spin 3/2 possess a gauge symmetry and thereby in such regions it is a correctly defined massless object; otherwise it is not clear how one can determine a massless field.

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