CUP PRODUCTS ON CURVES OVER FINITE FIELDS

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Abstract. Suppose $k$ is a finite field, that $C$ is a smooth projective geometrically irreducible curve over $k$, and that $n$ is a positive integer not divisible by the characteristic of $k$. In this paper we compute cup products of elements of the étale cohomology groups $H^1(C, \mathbb{Z}/n)$ and $H^1(C, \mu_n)$. Over the algebraic closure $\overline{k}$ of $k$, such cup products are connected to values of the Weil pairing on the $n$-torsion of the Jacobian of $\overline{C} = \overline{k} \otimes_k C$ by using a fixed isomorphism between $\mathbb{Z}/n$ and $\mu_n$ over $\overline{k}$. Over $k$, such cup products are more subtle due to the fact that they take values in the group $H^2(C, \mu_n) = \text{Pic}(C)/n \cdot \text{Pic}(C)$ rather than in the group $H^2(C, \mu_n) = \mathbb{Z}/n$.

1. Introduction

Let $C$ be a smooth projective irreducible curve of positive genus over an algebraically closed field $k$. Let $n$ be a positive integer not divisible by the characteristic of $k$ and let $\mu_n$ be the group of $n$th roots of unity in $k$. The Weil pairing

$$\langle , \rangle_{\text{Weil}} : \text{Jac}(C)[n] \times \text{Jac}(C)[n] \rightarrow \mu_n$$

on the $n$-torsion of the Jacobian of $C$ is a classical topic in algebraic geometry. It arises from the principal polarization of this Jacobian, and has applications to the study of Galois actions on $\text{Jac}(C)[n]$ (see [16, §20] and [7, Dualité §3]) that $\langle , \rangle_{\text{Weil}}$ may be identified with the cup product

$$H^1(C, \mu_n) \times H^1(C, \mu_n) \rightarrow H^2(C, \mu_n^{\otimes 2}) = \mu_n.$$

Let $\mathcal{F}_n$ be $\mathbb{Z}/n$ or $\mu_n$. Since $H^1(C, \mu_n) = H^1(C, \mathbb{Z}/n) \otimes \mu_n$, we can use any choice of isomorphism $\mathbb{Z}/n \rightarrow \mu_n$ to compute the cup product

$$\cup_{\overline{C}} : H^1(C, \mathbb{Z}/n) \times H^1(C, \mathcal{F}_n) \rightarrow H^2(C, \mathcal{F}_n)$$

from the Weil pairing. Note that $H^2(C, \mu_n) = \mathbb{Z}/n$.

The object of this paper is to generalize classical formulas for the Weil pairing used to compute the pairing (1.3) by replacing $C$ by a smooth projective geometrically irreducible curve $C$ of positive genus $g(C)$ over a finite field $k$. We give expressions for the cup product and the triple product

$$H^1(C, \mathbb{Z}/n) \times H^1(C, \mu_n) \rightarrow H^2(C, \mu_n) = \text{Pic}(C)/n \cdot \text{Pic}(C)$$

and

$$H^1(C, \mathbb{Z}/n) \times H^1(C, \mathbb{Z}/n) \times H^1(C, \mu_n) \rightarrow H^3(C, \mu_n) = \mathbb{Z}/n$$

when $n$ is relatively prime to $\#k$.

In our first result, Theorem 1.1, we will provide general formulas for (1.4) and (1.5) based on work of McCallum and Sharifi in [12]. The computation of the terms arising in Theorem 1.1 is in general more difficult than that of various expressions for the Weil pairing [8, 13, 10].

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The computation of the above pairings can be reduced to the case in which \( n \) is a power of a prime \( \ell \). There are then infinitely many closed points \( O \) of \( C \) of degree \( d(O) \) prime to \( \ell \cdot (\ell - 1) \). For \( \mathcal{F}_n = \mathbb{Z}/n \) or \( \mathcal{F}_n = \mu_n \), we describe below a direct sum decomposition

\[
H^1(C, \mathcal{F}_n) = H^1(k, \mathcal{F}_n) \oplus H^1(C, \mathcal{F}_n)_O
\]

in which \( H^1(C, \mathcal{F}_n)_O \) consists of classes which are normalized at \( O \) in an appropriate sense. The pairing of \( H^1(k, \mathbb{Z}/n) \) with \( H^1(k, \mu_n) \) is trivial since \( H^2(k, \mu_n) = 0 \). We will describe below how to compute the pairing \((\text{1.4})\) when exactly one of the terms belongs to \( H^1(k, \mathcal{F}_n) \) using a new homomorphism which we call the Legendre derivative of Frobenius. Finally suppose that the terms in \((\text{1.4})\) belong to \( H^1(C, \mathbb{Z}/n)_O \) and \( H^1(C, \mu_n)_O \). We show in Theorem \((\text{1.4})\) that when \( C \) has genus 1, the pairing on normalized classes can be computed using only the Weil pairing of the restrictions of these classes to \( H^1(C, \mathcal{F}_n) \). The proof does not in fact use the arithmetic approach in Theorem \((\text{1.4})\), but is instead a delicate homological calculation. In \((\text{1.4})\) we prove that the counterpart of Theorem \((\text{1.3})\) for higher genus curves need not hold by constructing an infinite family of genus 2 counterexamples.

To state our results precisely, we will identify \( H^1(C, \mu_n) \) with the quotient of

\[
D(C) = \{ b \in k(C)^* : \text{div}_C(b) \in n \cdot \text{Div}(C) \}
\]

by the subgroup \( (k(C)^*)^n \); see Lemma \((\text{2.3})\). Let \( [b] \in H^1(C, \mu_n) \) be the cohomology class determined by \( b \in D(C) \). There is a surjection \( H^1(C, \mu_n) \rightarrow \text{Pic}(C)[n] \) which sends \([b]\) to the \( n \)-torsion divisor class \([\text{div}_C(b)/n]\) of \( \text{div}_C(b)/n \). We will view \( \text{Pic}(C)[n] \) as a subgroup of \( \text{Pic}(\overline{C})[n] = \text{Jac}(\overline{C})[n] \) when \( \overline{C} = \overline{k} \otimes_k C \) and \( \overline{k} \) is a fixed algebraic closure of \( k \). Let \( \delta = \text{div}_C(b)/n \in \text{Div}(C) \).

Suppose \( \alpha \in H^1(C, \mathbb{Z}/n) = \text{Hom}(\pi_1(C), \mathbb{Z}/n) = \text{Hom}(\text{Pic}(C), \mathbb{Z}/n) \). Let \( \pi_\alpha : C_\alpha \rightarrow C \) be the étale morphism of smooth projective curves over \( k \) associated to \( \alpha \). Define \( d_\alpha = [k(C_\alpha) : k(C)] \) and let \( \alpha' \in \text{Hom}(\text{Gal}(k(C_\alpha)/k(C)), \mathbb{Z}/n) \) be the injection associated to \( \pi_\alpha \).

**Theorem 1.1.** Suppose \( \alpha \in H^1(C, \mathbb{Z}/n) \) and \( b \in D(C) \) are as above.

(i) Let \( \sigma \in \text{Gal}(k(C_\alpha)/k(C)) \) be an element such that \( \alpha'(\sigma) \in \mathbb{Z}/n \) generates the images of \( \alpha' \) and \( \alpha \). There is an element \( \epsilon \in k(C_\alpha) \) such that \( b = \text{Norm}_{k(C_\alpha)/k(C)}(\epsilon) \). There is a divisor \( \delta \in \text{Div}(C_\alpha) \) such that

\[
\text{div}_{C_\alpha}(\epsilon) = \frac{n}{d_\alpha} \pi_\alpha^*(\alpha) + (1 - \sigma) \cdot \epsilon .
\]

Writing \([\delta]\) for the class in \( \text{Pic}(C) \) of a divisor \( \delta \), we have

\[
\alpha \cup [b] = \alpha'(\sigma) \cdot \left( \left[ \text{Norm}_{k(C_\alpha)/k(C)}(\epsilon) \right] + \frac{n}{2}[\delta] \right)
\]

in \( \text{Pic}(C)[n] \cdot \text{Pic}(C) = H^2(C, \mu_n) \).

(ii) Under the assumptions of (i), suppose \( \tau \in H^1(C, \mathbb{Z}/n) = \text{Hom}(\text{Pic}(C), \mathbb{Z}/n) \). Then

\[
\tau \cup \alpha \cup [b] = \tau \left( \alpha'(\sigma) \cdot \left( \left[ \text{Norm}_{k(C_\alpha)/k(C)}(\epsilon) \right] + \frac{n}{2}[\delta] \right) \right)
\]

in \( \mathbb{Z}/n = H^1(C, \mu_n) \).

Because cup products are bilinear, to compute them we make the following assumption for the remainder of this introduction.

**Hypothesis 1.2.** The integer \( n = \ell^z \) is a positive power of a prime \( \ell \) that is relatively prime to \( \#k \).

We next define normalized classes in \( H^1(C, \mathbb{Z}/n) \) and in \( H^1(C, \mu_n) \). These arise in Miller’s work in \((\text{1.3})\) on the efficient computation of the Weil pairing associated to \( C = \overline{k} \otimes_k C \).

Since \( C \) is geometrically irreducible over \( k \), there is a divisor of \( C \) that has degree 1 by \((\text{2.9})\) Cor. 5, \( \S VII.5 \). This implies there must be a closed point \( O \) for which the degree \( d(O) \) is prime to \( \ell \cdot (\ell - 1) \). An element \( \alpha \in \text{Hom}(\text{Pic}(C), \mathbb{Z}/n) \) will be said to be normalized at \( O \) if \( O \) splits to the curve \( C_\alpha \) from part (i) of Theorem \((\text{1.3})\). On the other hand, an element \( b \in D(C) \) will be said to be normalized at \( O \) if the Laurent expansion of \( b \) with respect to a uniformizing parameter
at \( O \) has leading coefficient in \((k(O))\)\(^n\). We will show in Lemma 3.2 that this definition does not depend on the choice of uniformizer. We denote the \( \mathbb{Z}/n \)-module of classes \( \alpha \) in \( H^1(C, \mathbb{Z}/n) \) that are normalized at \( O \) by \( H^1(C, \mathbb{Z}/n)_O \), and we denote the \( \mathbb{Z}/n \)-module of classes \( \{b\} \) represented by \( b \in D(C) \) at \( O \) by \( H^1(C, \mu_n)_O \).

We now fix a closed point \( O \) of \( C \) for which the degree \( d(O) \) is prime to \( \ell \cdot (\ell - 1) \). The structure morphism \( C \to \text{Spec}(k) \) induces homomorphisms \( H^1(k, \mathbb{Z}/n) \to H^1(C, \mathbb{Z}/n) \) and \( H^1(k, \mu_n) \to H^1(C, \mu_n) \). Here \( H^1(k, \mathbb{Z}/n) = \text{Hom}(\text{Gal}(\overline{k}/k), \mathbb{Z}/n) \) is isomorphic to \( \mathbb{Z}/n \) whereas the group \( H^1(k, \mu_n) \) is isomorphic to \( k^*/(k^*)^n \). We will show in Lemma 3.2 that there are direct sum decompositions \((1.6)\) for both \( \mathcal{F}_n = \mathbb{Z}/n \) and \( \mathcal{F}_n = \mu_n \).

We now consider the restriction of the cup product pairing \((1.4)\) to these summands. As noted earlier, the restriction of the arguments of \((1.4)\) to \( H^1(k, \mathbb{Z}/n) \) and \( H^1(k, \mu_n) \) is trivial since \( H^2(k, \mu_n) = 0 \).

Under Hypothesis 1.2, let \( T_l(C) \) be the \( \ell \)-adic Tate module of \( C \) and let \( \Phi_{k,C} \) be the arithmetic Frobenius acting on \( T_l(C) \). There exists a unique automorphism \( A \) of \( \mathbb{Q}_l \otimes \mathbb{Z}_\ell T_l(C) \) such that \( \Phi_{k,C} = 1 + n.A \). Multiplication by \( A^{-1} \) defines a homomorphism

\[
(1.7) \quad d\mathcal{L} : \text{Pic}^0(C)[n] = \frac{T_l(C) \cap AT_l(C)}{nAT_l(C)} \to \frac{T_l(C)}{nT_l(C) + nAT_l(C)} = \text{Pic}^0(C)/n \cdot \text{Pic}^0(C)
\]

which we call the Legendre derivative of Frobenius. In Theorem 5.3 we will show how the Legendre derivative \( d\mathcal{L} \) determines the value of \((1.4)\) when the first argument is restricted to \( H^1(k, \mathbb{Z}/n) \) and \( H^1(k, \mu_n) \).

Similarly, in Theorem 5.5 we will discuss the case when the second argument is restricted to \( H^1(k, \mu_n) \).

We finally consider the restriction of both arguments of \((1.4)\) to normalized classes in \( H^1(C, \mathbb{Z}/n)_O \) and \( H^1(C, \mu_n)_O \). For all \( \alpha \in H^1(C, \mathbb{Z}/n)_O \) and \( \{b\} \in H^1(C, \mu_n)_O \), we define

\[
(1.8) \quad (\alpha, \{b\})_{[O]} = \frac{1}{d(O)} \cdot \text{Pic}(C)/n \cdot \text{Pic}(C) = H^2(C, \mu_n)
\]

where \([O]\) is the class in \( \text{Pic}(C) \) of the divisor of degree \( d(O) \) defined by the point \( O \) and \( \pi \cup [\bar{b}] \in \mathbb{Z}/n \) is the value of the pairing in \((1.4)\) when \( \pi \) and \( [\bar{b}] \) are the restrictions of \( \alpha \) and \( \{b\} \) to \( H^1(C, \mathbb{Z}/n) \) and \( H^1(C, \mu_n) \), respectively.

We prove the following result for curves \( C \) of arbitrary positive genus.

**Theorem 1.3.** Suppose \( n = \ell^2 \) and that \( O \) is a closed point of degree \( d(O) \) prime to \( \ell \cdot (\ell - 1) \). The following two conditions are equivalent:

(i) For all \( \alpha \in H^1(C, \mathbb{Z}/n)_O \) and \( \{b\} \in H^1(C, \mu_n)_O \), we have \( \alpha \cup \{b\} = (\alpha, \{b\})_{[O]} \) from \((1.8)\).

(ii) The triple product on normalized classes

\[
H^1(C, \mathbb{Z}/n)_O \times H^1(C, \mathbb{Z}/n)_O \times H^1(C, \mu_n)_O \to H^3(C, \mu_n) = \mathbb{Z}/n
\]

is zero.

When the genus of \( C \) is one, we show that condition (ii) of Theorem 1.3 is satisfied. In other words, we prove the following result.

**Theorem 1.4.** Under the hypotheses of Theorem 1.3, assume further that \( g(C) = 1 \). Then for all \( \alpha \in H^1(C, \mathbb{Z}/n)_O \) and \( \{b\} \in H^1(C, \mu_n)_O \), we have

\[
\alpha \cup \{b\} = (\alpha, \{b\})_{[O]} \quad \text{from } (1.8).
\]

One key ingredient in the proof of Theorem 1.4 is to show that when \( \alpha, \{b\}, [\pi], [\bar{b}] \) are as in \((1.8)\) and \( g(C) = 1 \), then \( [\pi] \cup_{\mathbb{Z}/n} [\bar{b}] = 0 \) in \( H^2(C, \mu_n) \) implies \( \alpha \cup \{b\} = 0 \) in \( H^2(C, \mu_n) \); see Theorem 7.2.

In §9 we will give infinitely many curves \( C \) of genus 2 for which the equivalent conditions of Theorem 1.3 do not hold when \( n = \ell = 3 \). The size of the image of the cup product in an analogous number theoretic situation arising from the theory of cyclotomic fields is discussed by McCallum and Sharifi in [12].
We now outline the contents of the sections of this paper. In §2 we will set up the notation and assumptions for the remainder of the paper and we will collect some results on étale cohomology groups. Moreover, we will show Theorem 1.1. For the remainder of the article, we assume Hypothesis 1.2. In §3 we will prove the direct sum decomposition (1.6) in Lemma 3.2 and we will show in Theorem 3.6 how to reduce to the case when \( d(O) = 1 \). In §4 we will prove various results that are necessary for analyzing the formulas in Theorem 1.1. In §5 we will consider restrictions of the cup product maps (1.4) and (1.5) that are connected to the derivative of the arithmetic Frobenius. In particular, in Proposition 5.1 we will introduce the Legendre derivative of Frobenius, and we will prove Theorems 5.3 and 5.5. In §6 we will consider curves of arbitrary positive genus and we will prove Theorem 1.3 by proving that condition (ii) of Theorem 1.3 holds in this case. In §7 we will construct an infinite family of curves of genus two for which the equivalent conditions of Theorem 1.3 do not hold when \( n = \ell = 3 \); see Theorem 7.8.

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2. Étale cohomology groups

Throughout this paper we will assume that \( C \) is a smooth projective geometrically irreducible curve of genus \( g(C) \geq 1 \) over a finite field \( k \) of order \( q \). We will suppose \( n \) is a positive integer that is relatively prime to \( q \). Let \( k(C) \) be the function field of \( C \) and let \( \overline{k(C)} \) be a separable closure of \( k(C) \) containing a fixed algebraic closure \( \overline{k} \) of \( k \). Let \( \text{Div}(C) \) be the divisor group of \( C \), let \( \text{Pic}(C) \) be the Picard group of \( C \), and let \( \text{Pic}^0(C) \) be the group of divisor classes of degree 0.

Let \( \overline{C} = \overline{k} \otimes_k C \) and let \( \eta \) be a geometric point of \( \overline{C} \), which can then also be viewed as a geometric point of \( C \). We have an exact sequence

\[
1 \to \pi_1(\overline{C}, \eta) \to \pi_1(C, \eta) \to \text{Gal}(\overline{k}/k) \to 1
\]

of étale fundamental groups in which \( \text{Gal}(\overline{k}/k) \) is isomorphic to the profinite completion \( \hat{\mathbb{Z}} \) of \( \mathbb{Z} \). There are natural isomorphisms

\[
H^1(k, \mathbb{Z}/n) = \text{Hom}(\text{Gal}(\overline{k}/k), \mathbb{Z}/n)
\]

and

\[
H^1(C, \mathbb{Z}/n) = \text{Hom}(\pi_1(C, \eta), \mathbb{Z}/n) \quad \text{and} \quad H^1(\overline{C}, \mathbb{Z}/n) = \text{Hom}(\pi_1(\overline{C}, \eta), \mathbb{Z}/n).
\]

In fact, we have from [1 §2.1.2] (see also [2 §3]) the following result:

**Lemma 2.1.** (Achinger) For all \( i \geq 0 \), and all locally constant constructible sheaves \( \mathcal{F} \) of \( \mathbb{Z}/n \)-modules, the natural homomorphisms

\[
H^i(\pi_1(C, \eta), \mathcal{F}_\eta) \to H^i(C, \mathcal{F}) \quad \text{and} \quad H^i(\pi_1(\overline{C}, \eta), \mathcal{F}_\eta) \to H^i(\overline{C}, \mathcal{F})
\]

are isomorphisms when \( \mathcal{F}_\eta \) is the stalk of \( \mathcal{F} \) at \( \eta \).

Let \( \mathcal{F} \) be a constructible sheaf of finite groups of order dividing \( n \) on \( C \), and let \( \mathcal{F}_\eta \) denote its restriction to \( \overline{C} \). The following lemma results directly from the spectral sequence

\[
H^i(\text{Gal}(\overline{k}/k), H^j(\overline{C}, \mathcal{F})) \Rightarrow H^{i+j}(C, \mathcal{F})
\]

together with the fact that \( \text{Gal}(\overline{k}/k) \cong \hat{\mathbb{Z}} \) has cohomological dimension one.

**Lemma 2.2.** One has a split exact sequence of \( \mathbb{Z}/n \)-modules

\[
0 \to \text{Hom}(\text{Gal}(\overline{k}/k), \mathbb{Z}/n) \to H^1(C, \mathbb{Z}/n) \to H^1(\overline{C}, \mathbb{Z}/n)^{\text{Gal}(\overline{k}/k)} \to 0
\]
in which \(\text{Hom}(\text{Gal}(\overline{k}/k), \mathbb{Z}/n)\) is cyclic of order \(n\) and \(H^1(C, \mathbb{Z}/n)\) has order dividing \(n^{2g(C)}\). The sequence \(2.6\) is the \(\mathbb{Z}/n\)-module dual of the sequence
\[
0 \to \text{Pic}^0(C)/n \cdot \text{Pic}^0(C) \to \text{Pic}(C)/n \cdot \text{Pic}(C) \to \mathbb{Z}/n \to 0
\]
resulting from the degree map \(\text{Pic}(C) \to \mathbb{Z}\) and the Artin map \(\text{Pic}(C) \to \pi_1(C, \eta)^{ab}\).

The next lemma gives a description of \(H^1(C, \mathbb{Z}/n)\) and of \(H^i(C, \mu_n)\) for \(i = 1, 2\).

**Lemma 2.3.** There are natural isomorphisms
\[
\begin{align*}
H^1(C, \mathbb{Z}/n) &= \text{Hom}(\text{Pic}(C), \mathbb{Z}/n) \quad \text{and} \\
H^2(C, \mu_n) &= \text{Pic}(C)/n \cdot \text{Pic}(C).
\end{align*}
\]
Define
\[
D(C) = \{b \in k(C)^* : \text{div}_C(b) \in n \cdot \text{Div}(C)\}.
\]
Then there is a natural isomorphism
\[
2.9 \quad H^1(C, \mu_n) = D(C)/(k(C)^*)^n.
\]
Moreover, for \(b \in D(C)\), choose an \(n\)th root \(b^{1/n}\) of \(b\) in \(k(C)\). Then the class \([b] \in D(C)/(k(C)^*)^n\) corresponds to the class of the one-cocycle \(c_{b^{1/n}} : \pi_1(C, \eta) \to \mu_n\) defined by \(c_{b^{1/n}}(\sigma) = \sigma(b^{1/n})/b^{1/n}\) for all \(\sigma \in \pi_1(C, \eta)\).

**Proof.** The group \(\text{Hom}(\text{Pic}(C), \mathbb{Z}/n)\) is identified with \(\text{Hom}(\pi_1(C, \eta), \mathbb{Z}/n)\), as in \((2.7)\), via the Artin map. Since \(H^2(C, \mathbb{G}_m) = 0\), \(\text{Pic}(C)/n \cdot \text{Pic}(C)\) is identified with \(H^2(C, \mu_n)\), as in \((2.8)\), via the Kummer sequence.

By \([14]\) p. 125, \(H^1(C, \mu_n)\) is identified with isomorphism classes of pairs \((L, \phi)\) in which \(L\) is a line bundle on \(C\) and \(\phi\) is an isomorphism \(O_C \to L^{\otimes n}\). Given \(b \in D(C)\), define \(b = \text{div}_C(b)/n\), and let \(L_b\) be the line bundle \(O_C(b)\). We then have an isomorphism \(\phi_b : O_C \to L_b^{\otimes n} = O_C(\text{div}_C(b))\) sending the global section 1 to the global section \(b^{-1}\) of \(O_C(\text{div}_C(b))\). This induces the isomorphism \(H^1(C, \mu_n) \to D(C)/(k(C)^*)^n\) in \((2.9)\).

For \(b \in D(C)\) and a choosen \(n\)th root \(b^{1/n}\) in \(k(C)\), the map \(c_{b^{1/n}}\) satisfies the one-cocycle condition. Since the one-coboundaries are the maps \(d_c : \pi_1(C, \eta) \to \mu_n\), for \(\zeta \in \mu_n\), with \(d_c(\sigma) = \sigma(\zeta)/\zeta\) for all \(\sigma \in \pi_1(C, \eta)\), they naturally correspond to \(b \in (k(C)^*)^n\).

We now prove the formula in part (i) of Theorem 1.1 by adjusting the arguments of \([12]\) Thm. 2.4] in the following way. We are concerned with the pairing
\[
H^1(C, \mathbb{Z}/n) \times H^1(C, \mu_n) \to H^2(C, \mu_n)
\]
rather than with the pairing
\[
H^1(C, \mu_n) \times H^1(C, \mu_n) \to H^2(C, \mu_n^{\otimes 2})
\]
that is the function field counterpart of the one considered in \([12]\) Thm. 2.4. As in Theorem 1.1 suppose \(\alpha \in H^1(C, \mathbb{Z}/n)\) and that \(b \in D(C)\) has class \([b] \in H^1(C, \mu_n)\). To connect the notation of \([12]\) to our case, define \(m_\sigma\) for \(\sigma \in \text{Gal}(k(C_\alpha)/k(C))\) to be the smallest non-negative integer representing the residue class \(\alpha'(\sigma) \in \mathbb{Z}/n\). The computation of \([12]\) Lemma 2.2] holds in our case when we replace \((a, -ab)_S\) in the notation of \([12]\) by
\[
(2.10) \quad \alpha \cup (-[b]) + (\alpha, \alpha)
\]
when \((\alpha, \alpha)\) is defined in the following way. The canonical exact sequence
\[
0 \to \mathbb{Z}/n \to \mathbb{Z}/n^2 \to \mathbb{Z}/n \to 0
\]
defines a class \(\epsilon \in H^2(\mathbb{Z}/n, \mathbb{Z}/n)\). Let \(\epsilon_\alpha \in H^2(C, \mathbb{Z}/n)\) be the pullback of this class via \(\alpha : \pi_1(C, \eta) \to \mathbb{Z}/n\). If \(n\) is odd, let \((\alpha, \alpha) = 0\) be the trivial element of \(H^2(C, \mu_n)\). If \(n\) is even, let \((\alpha, \alpha) \in H^2(C, \mu_n)\) be the image of \(\epsilon_\alpha\) under the composition of the map \(\mathbb{Z}/n \to \mathbb{Z}/2\) given by multiplication by \(n(n - 1)/2\) followed by the unique map \(\mathbb{Z}/2 \to \mu_n\) sending 1 mod 2 to \(-1 \in \mu_n\). (We could have replaced \(n(n - 1)/2\) here simply by \(n/2\), but we retain \(n(n - 1)/2\) to fit with \([12]\))
The computations in [12] Lemmas 2.2, 2.3 and [12] Thm. 2.4 now carry over to show part (i) of Theorem 1.1 in our case when we use (2.10) as a replacement for \((a, -ab)s\).

Part (ii) of Theorem 1.1 now results from part (i) and the next lemma.

**Lemma 2.4.** The cup product pairing

\[(2.11) \quad H^1(C, \mathbb{Z}/n) \times H^2(C, \mu_n) \to H^3(C, \mu_n) = \mathbb{Z}/n\]

is a perfect duality between the groups \(H^1(C, \mathbb{Z}/n)\) and \(H^2(C, \mu_n)\). This pairing agrees with the evaluation map

\[(2.12) \quad \text{Hom}(\text{Pic}(C), \mathbb{Z}/n) \times \frac{\text{Pic}(C)}{n \cdot \text{Pic}(C)} \to \mathbb{Z}/n\]

when we use the identifications in Lemma 2.3.

**Proof.** The first statement is shown in [15] Cor. II.3.3(b)]. The subtlety in verifying that (2.11) and (2.12) agree is to check that the natural evaluation map in (2.12) agrees with the canonical class isomorphism \(H^3(C, \mu_n) = \mathbb{Z}/n\) in (2.11).

We will use the following terminology. Suppose \(\chi \in H^1(C, \mathbb{Z}/n) = \text{Hom}(\text{Pic}(C), \mathbb{Z}/n)\) and that \(x\) is a closed point of \(C\) with class \([x] \in \text{Pic}(C)/n \cdot \text{Pic}(C)\). We will say that the pairings agree for \(\chi\) and \([x]\) if (2.11) and (2.12) agree when we make the identifications in Lemma 2.4.

The first case is when \(\chi \in \text{Hom}(\pi_1(C, \eta), \mathbb{Z}/n)\) is trivial on \(\pi_1(C, \eta)\) when \(\overline{C} = \overline{k} \otimes_k C\). Under the identification \(H^1(C, \mathbb{Z}/n) = \text{Hom}(\text{Pic}(C), \mathbb{Z}/n)\), \(\chi\) is then a multiple of the map \(\text{deg} : \text{Pic}(C) \to \mathbb{Z}/n\) induced by the degree. We may thus reduce to the case in which \(\chi = \text{deg}\). Class field theory then identifies \(\chi\) with the element of \(H^1(C, \mathbb{Z}/n) = \text{Hom}(\pi_1(C, \eta), \mathbb{Z}/n)\) sending the coset \(\Phi_{\overline{k}/k}\pi_1(C, \eta)\) to \(1 \in \mathbb{Z}/n\) when \(\Phi_{\overline{k}/k}\) is the arithmetic Frobenius of \(G = \text{Gal}(\overline{C}/C) = \text{Gal}(\overline{C}/k)\). In the proof of [13] Corollary V.2.3 the Hochschild-Serre spectral sequence produces a diagram

\[
\begin{array}{cccc}
0 & \to & H^0(\overline{C}, \mathbb{Z}/n)_G & \to & H^1(C, \mathbb{Z}/n) & \to & H^1(\overline{C}, \mathbb{Z}/n)_G & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{Hom}(H^2(\overline{C}, \mu_n)_G, \mathbb{Z}/n) & \to & \text{Hom}(H^2(C, \mu_n), \mathbb{Z}/n) & \to & \text{Hom}(H^2(C, \mu_n)_G, \mathbb{Z}/n) & \to & 0 \\
\end{array}
\]

in which the vertical arrows are isomorphisms and come from duality pairings over \(C\) and \(\overline{C}\). Here \(H^0(\overline{C}, \mathbb{Z}/n)_G = \mathbb{Z}/n\) is naturally identified with \(H^1(G, H^0(\overline{C}, \mathbb{Z}/n))\) by sending \(1 \in \mathbb{Z}/n\) to \(\chi = \text{deg}\). This reduces the proof that the pairings agree for \(\chi = \text{deg}\) and the restriction of \([x]\) to showing that the value of the pairing

\[H^0(\overline{C}, \mathbb{Z}/n)_G \times H^2(\overline{C}, \mu_n)_G \to \mathbb{Z}/n\]

on \(\text{deg} = 1\) in \(H^0(\overline{C}, \mathbb{Z}/n)_G = \mathbb{Z}/n\) and the restriction of \([x]\) to \(H^2(\overline{C}, \mu_n)_G = (\text{Pic}(\overline{C})/n \cdot \text{Pic}(\overline{C}))_G = \mathbb{Z}/n\) is \(\text{deg}(x)\). This is so because of the definition of the canonical isomorphism \(H^2(C, \mu_n) = \mathbb{Z}/n\) in the proof of [13] Theorem V.2.1(a)]. Hence the pairings agree for \(\chi\) trivial on \(\pi_1(\overline{C}, \eta)\) and all \([x]\).

Suppose now that \(\chi\) is an arbitrary element of \(H^1(C, \mathbb{Z}/n)\) and that \(x\) is a closed point such that \([k(x) : k]\) is prime to \(n\). There will then be a character \(\chi'\) that is trivial on \(\pi_1(\overline{C}, \eta)\) and has the same restriction as \(\chi\) to every decomposition group associated to \(x\) in \(\pi_1(C, \eta)\). Now \(\chi - \chi'\) has trivial restriction to all such decomposition groups. Hence the value of the pairing in (2.12) on \(\chi\) and \([x]\) agrees with the value of this pairing on \(\chi'\) and \([x]\). We will show below the same is true for the pairing in (2.11). Before proving this, we note that this will imply the pairings agree for \(\chi\) and \([x]\) because we have shown they agree for \(\chi'\) and arbitrary \([x]\).

One way to prove the above claim about (2.11) is to use the spectral sequence

\[H^p(C, R^q j_* \mu_n, \xi) \Rightarrow H^{p+q}(\xi, \mu_n, \xi)\]

and the exact sequence

\[(2.13) \quad 0 \to \mu_n, C \to j_* \mathbb{G}_m, \xi \to j_* \mathbb{G}_m, \xi \to R^1 j_* \mu_n, \xi \to 0\]
associated to the inclusion $j : \xi = \text{Spec}(k(C)) \to C$ of the generic point $\xi$ of $C$ into $C$. As in \cite{3} §5, this gives an exact sequence
\begin{equation}
H^1(\xi, \mu_{n, \xi}) \to H^0(C, R^1 j_* \mu_{n, \xi}) \xrightarrow{\omega} H^2(C, \mu_{n, C}) \to 0
\end{equation}
in which the transgression $\omega$ is a double boundary map associated to splitting \cite{24} into two exact sequences. In \cite[Cor. 5.3]{3} it is shown how \eqref{2.14} is naturally identified with the sequence
\begin{equation}
(k(C)^*/(k(C)^*)^n) \to \bigoplus_{x \in C^0} k(C)_{x}^*/T_x \xrightarrow{\omega} H^2(C, \mu_{n, C}) \to 0
\end{equation}
in which $C^0$ is the set of closed points of $C$ and $T_x$ for $x \in C^0$ is the subgroup of elements of $k(C)^*_x$ having valuation divisible by $n$. Since boundaries in spectral sequences are compatible with pairings, we see that \eqref{2.14} arises from \eqref{2.15} together with the natural pairing
$$H^1(C, \mathbb{Z}/n) \times H^0(C, R^1 j_* \mu_{n, \xi}) \to H^1(C, \mathbb{Z}/n \otimes R^1 j_* \mu_{n, \xi}) = H^1(C, R^1 j_* \mu_{n, \xi})$$
and the transgression map
\begin{equation}
H^1(C, R^1 j_* \mu_{n, \xi}) \to H^3(C, \mu_{n, C})
\end{equation}
associated to the double boundary map in cohomology associated to \eqref{2.14}. As in the last paragraph of the proof of \cite[Lemma 5.2]{3}, we have an isomorphism
$$R^1 j_* \mu_{n, \xi} \cong \bigoplus_{x \in C^0} i_x \cdot i_x^* R^1 j_* \mu_{n, \xi}$$
where $i_x : x \to C$ is the closed immersion associated to $x \in C^0$. This reduces us to considering the pairing
\begin{equation}
H^1(C, \mathbb{Z}/n) \times H^0(C, i_x \cdot i_x^* R^1 j_* \mu_{n, \xi}) \to H^1(C, \mathbb{Z}/n \otimes i_x \cdot i_x^* R^1 j_* \mu_{n, \xi}) = H^1(C, i_x \cdot i_x^* R^1 j_* \mu_{n, \xi}).
\end{equation}
Pick a decomposition group $\pi_1(C, \eta)_x$ for $x \in \pi_1(C, \eta)$. The sheaf $i_x \cdot i_x^* R^1 j_* \mu_{n, \xi}$ on $x$ is then associated to a module $M_x$ for $\pi_1(C, \eta)_x$. The $\pi_1(C, \eta)$-module associated to $i_x \cdot i_x^* R^1 j_* \mu_{n, \xi}$ is isomorphic to the module $M_x$ of $\pi_1(C, \eta)_x$ to $\pi_1(C, \eta)$. We want to show that the value of the pairing \eqref{2.11} on the pair $\chi$ and $[x]$ is the same as the value on $\chi'$ and $[x]$ because $\chi$ and $\chi'$ have the same restriction to every decomposition group associated to $x$. This follows from Shapiro’s Lemma applied to \eqref{2.17}.

We have shown that the pairings \eqref{2.11} and \eqref{2.12} agree on every pair $\chi$ and $[x]$ for which $[x] : k$ has degree prime to $n$. To finish the proof, we now show that the subgroup $T$ of $\text{Pic}(C)/n \cdot \text{Pic}(C)$ generated by classes $[x]$ defined by $x$ with $[x] : k$ prime to $n$ is all of $\text{Pic}(C)/n \cdot \text{Pic}(C)$. Let $C' \to C$ be the finite abelian unramified cover of $C$ associated to $T$. Since there are divisors of degree 1 on $C$, there are points with degrees that are prime to $n$. Hence since $C$ was assumed to be geometrically irreducible with constant field $\mathbb{F}_q$, the same is true for $C'$. By the definition of $T$, every closed point $x \in C$ with $[x] : k$ prime to $n$ splits in $C'$. Thus if $m$ is any integer prime to $n$, we have $\#(C'/\mathbb{F}_{q^m}) = d \cdot \#(C/\mathbb{F}_{q^m})$ when $d$ is the covering degree of $C'$ over $C$. Since $C$ and $C'$ are geometrically irreducible with the same constant field, the Weil conjectures force $d = 1$. Hence $T = \text{Pic}(C)/n \cdot \text{Pic}(C)$ as required.

3. Complements of classes coming from $k$

Throughout this section we will assume Hypothesis \ref{1.2} i.e. $n = \ell^z$ for a prime $\ell$. When analyzing cup product pairings it will be useful to have a complement for the image of the inflation homomorphisms $H^1(k, \mathbb{Z}/n) \to H^1(C, \mathbb{Z}/n)$ and $H^1(k, \mu_n) \to H^1(C, \mu_n)$. Let $k(O)$ be the residue field of a closed point $O$ of $C$, and let $d(O) = [k(O) : k]$.

**Definition 3.1.** Suppose $O$ is a closed point of $C$ with $d(O)$ prime to $\ell \cdot (\ell - 1)$. A class $\alpha \in H^1(C, \mathbb{Z}/n)$ will be said to be normalized at $O$ if $O$ splits in the cyclic cover $C_\alpha \to C$ associated to $\alpha$. Let $H^1(C, \mathbb{Z}/n)_O$ be the subgroup of all such $\alpha$. Let $\pi_O$ be a uniformizing parameter in the local ring $O_{C,O}$. An element $b \in D(C)$ will be said to be normalized at $O$ with respect to $\pi_O$ if the leading term in its Laurent expansion with respect to $\pi_O$ lies in $(k(O)^*)^n$. A class in $H^1(C, \mu_n)$ will
be said to be normalized at \( O \) with respect to \( \pi_O \) if it has the form \([b]\) for an element \( b \) of this kind. Let \( H^1(C, \mu_n)_O \) be the subset of all such \([b]\).

**Lemma 3.2.** Assume Hypothesis \([\ref{HypX}] \). There is a closed point \( O \) of \( C \) with \( d(O) \) prime to \( \ell \cdot (\ell - 1) \).

(i) There is a direct sum decomposition

\[
H^1(C, \mathbb{Z}/n) = H^1(C, \mathbb{Z}/n)_O \oplus H^1(k, \mathbb{Z}/n)
\]

where \( H^1(k, \mathbb{Z}/n) \cong \mathbb{Z}/n \) and the restriction map sends \( H^1(C, \mathbb{Z}/n)_O \) isomorphically to \( H^1(C, \mathbb{Z}/n)_{\text{Gal}(K/k)} \).

(ii) The subgroup \( H^1(C, \mu_n)_O \) depends on \( O \) but does not depend on the choice of uniformizer \( \pi_O \) at \( O \). There is a direct sum decomposition

\[
H^1(C, \mu_n) = H^1(C, \mu_n)_O \oplus H^1(k, \mu_n)
\]

where \( H^1(k, \mu_n) \cong k^*/(k^*)^n \) and the restriction map sends \( H^1(C, \mu_n)_O \) isomorphically to \( H^1(C, \mu_n)_{\text{Gal}(K/k)} \).

**Proof.** Since \( C \) is geometrically irreducible over \( k \), there is a divisor of \( C \) that has degree 1 by \([\ref{C}] \), Cor. 5, §VII.5), which implies there must be a point \( O \) for which \( d(O) \) is prime to \( \ell \cdot (\ell - 1) \).

To prove (i), suppose \( \alpha \in H^1(C, \mathbb{Z}/n) \). Let \( \pi_1(C, \eta)_O \) be a decomposition group associated to \( O \) in \( \pi_1(C, \eta) \). Then \( \pi_1(C, \eta)_O \) is procyclic, and there is a generator \( \Phi \) of \( \pi_1(C, \eta)_O \) that maps to \( \Phi_{\bar{k}/k} \) in \( \text{Gal}(\bar{k}/k) \). Since \( d(O) \) is prime to \( \ell \cdot (\ell - 1) \) and \( n = \ell^2 \), there is a character \( \rho \in H^1(k, \mathbb{Z}/n) = \text{Hom}(\text{Gal}(\bar{k}/k), \mathbb{Z}/n) \) such that \( \rho(\Phi_{\bar{k}/k}) = \alpha(\Phi) \). Denote by \( \rho \) also the inflation of \( \rho \) to \( H^1(C, \mathbb{Z}/n) = \text{Hom}(\pi_1(C, \eta), \mathbb{Z}/n) \) via the natural surjection \( \pi_1(C, \eta) \to \text{Gal}(\bar{k}/k) \). Then \( \hat{\alpha} = \alpha - \rho \) has the property that \( \hat{\alpha}(\Phi) = 0 \), so \( \hat{\alpha} \in H^1(C, \mathbb{Z}/n)_O \) and \( \alpha = \hat{\alpha} + \rho \). Thus \( H^1(C, \mathbb{Z}/n)_O \) and \( H^1(k, \mathbb{Z}/n) \) together generate \( H^1(C, \mathbb{Z}/n) \). If \( \alpha \in H^1(C, \mathbb{Z}/n)_O \cap H^1(k, \mathbb{Z}/n) \) then \( C_\alpha \to C \) results from a constant field extension of \( k \) of degree equal to the order of \( \alpha \) and dividing \( n = \ell^2 \), and \( O \) splits in this cover. However \( d(O) = |k(O) : k| \) has degree prime to \( n \), so this cover must be trivial and \( \alpha \) is trivial. This establishes the direct sum decomposition \( H^1(C, \mu_n) = H^1(C, \mu_n)_O \oplus H^1(k, \mu_n) \) in part (i). The last statement in part (ii) follows from the exact sequence \([\ref{E1}] \).

We now prove part (ii). If \( b \in D(C) \) then \( \text{ord}_O(b) \) is a multiple of \( n \). Therefore if one replaces \( \pi_O \) by another uniformizing parameter \( \pi_O' \) at \( O \), the leading terms in the Laurent expansions of \( b \) with respect to \( \pi_O \) and \( \pi_O' \) differ by an element of \( (k^*)^n \). Hence \( H^1(C, \mu_n)_O \) does not depend on the choice of \( \pi_O \). Since \( d = d(O) \) is prime to \( \ell \cdot (\ell - 1) \), one sees that the ratio \( \#k(O)^*/\#k^* = (q^d - 1)/(q - 1) = 1 + q + \cdots + q^{d-1} \) is prime to \( \ell \) by considering first the case in which \( q \equiv 1 \mod \ell \) and then the case in which \( q \not\equiv 1 \mod \ell \). Hence the Sylow \( \ell \)-subgroups of \( k^* \) and \( k(O)^* \) are the same. Therefore every \( b \in D(C) \) is equal to \( b \cdot s \) for some \( b \) that is normalized at \( O \) and some \( s \in k^* \). The image of \( s \) in \( H^1(k, \mu_n) = k^*/(k^*)^n = k(O)^*/(k(O)^*)^n \) is uniquely determined by the image \([b]\) of \( b \) in \( H^1(C, \mu_n) = D(C)/(k(C)^*)^n \). It follows that we have a direct sum decomposition \( H^1(C, \mu_n) = H^1(C, \mu_n)_O \oplus H^1(k, \mu_n) \). The last statement in part (ii) follows from the spectral sequence \([\ref{S}] \) with \( F = \mu_n \).

**Corollary 3.3.** Assume Hypothesis \([\ref{HypX}] \) and let \( O \) be a closed point of \( C \) with \( d(O) \) prime to \( \ell \cdot (\ell - 1) \). Every element \( \alpha \) of \( H^1(C, \mathbb{Z}/n) \) has a unique expression as a sum \( \hat{\alpha} + \rho \) with \( \hat{\alpha} \in H^1(C, \mathbb{Z}/n)_O \) and \( \rho \in H^1(k, \mathbb{Z}/n) \). Every element \( [b] \) of \( H^1(C, \mu_n) \) has a unique expression as a product \([\hat{b}] \cdot [s] \) with \( [\hat{b}] \in H^1(C, \mu_n)_O \) normalized at \( O \) and \( s \in k^* \), so \([s] \in H^1(k, \mu_n) \). Then

\[
\alpha \cup [s] = \hat{\alpha} \cup [\hat{b}] + \rho \cup [\hat{b}] + \hat{\alpha} \cup [s]
\]

**Proof.** This is clear from the fact that cup products are bilinear and anti-commutative, and \( \rho \cup [s] = 0 \) since \( \rho \cup [s] \) is the inflation of a class in \( H^2(k, \mu_n) = 0 \) to \( H^2(C, \mu_n) \).

**Remark 3.4.** Corollary \([\ref{C3}] \) reduces the computation of cup products of elements of \( H^1(C, \mathbb{Z}/n) \) and \( H^1(C, \mu_n) \) to two cases: (i) both arguments are normalized classes with respect to some choice of
closed point $O$ with $d(O)$ prime to $\ell \cdot (\ell - 1)$, or (ii) one argument is normalized and the other argument is in $H^1(k, \mu_n)$ or $H^1(k, \mathbb{Z}/n\mathbb{Z})$.

We will need the following characterizations of $H^1(C, \mathbb{Z}/n\mathbb{O})$ and $H^1(C, \mu_n)\mathbb{O}$.

**Lemma 3.5.** Let $k(C)$ be a separable closure of $k(C)$ containing $\overline{k}$, and let $M(C)$ be the maximal everywhere unramified extension of $k(C)$ inside $k(C)$. Then $\pi_1(C, \eta) = \text{Gal}(M(C)/k(C))$. Let $O$ be a closed point of $C$ with $d(O)$ prime to $\ell \cdot (\ell - 1)$, let $\tilde{O}$ be a place in $M(C)$ over $O$, and let $\pi_1(C, \eta)(\tilde{O})$ be the decomposition group of the place $\tilde{O}$ in $\pi_1(C, \eta)$. Then $\pi_1(C, \eta)(\tilde{O})$ is pro-cyclic, and there is a progenerator $\Phi$ of $\pi_1(C, \eta)(\tilde{O})$ that maps to $\Phi^d(O)_{k/k}$ in $\text{Gal}(\overline{k}/k)$.

(i) Let $\alpha \in H^1(C, \mathbb{Z}/n\mathbb{O}) = \text{Hom}(\pi_1(C, \eta), \mathbb{Z}/n\mathbb{Z})$. Then $\alpha \in H^1(C, \mathbb{Z}/n\mathbb{O})$ if and only if $\alpha(\Phi) = 0$. We have a direct sum decomposition

$$\frac{\text{Pic}(C)}{n \cdot \text{Pic}(C)} = \langle \langle O \rangle \rangle \oplus \frac{\text{Pic}^0(C)}{n \cdot \text{Pic}^0(C)}$$

where $\langle O \rangle$ is the image of the class in $\text{Pic}(C)$ of the divisor of degree $d(O)$ defined by $O$. Then $\langle \langle O \rangle \rangle \cong \mathbb{Z}/n\mathbb{Z}$, and we have that $\alpha \in H^1(C, \mathbb{Z}/n\mathbb{O}) = \text{Hom}(\text{Pic}(C), \mathbb{Z}/n\mathbb{Z})$ is normalized at $O$ if and only if $\alpha(\langle O \rangle) = 0$.

(ii) Suppose $b \in D(C)$. Then $b$ is normalized at $O$ if and only if there exists an element $b^{1/n}$ in the completion $k(C)_{\tilde{O}}$ of the function field $k(C)$ at $O$ such that $(b^{1/n})^n = b$. Let $L$ be the extension of $k(C)$ generated by all such $b^{1/n}$ as $b$ ranges over all elements of $D(C)$ that are normalized at $O$. Then $L \subset M(C)$, and there is a place $O'$ over $O$ with $L_{O'} = k(C)_{\tilde{O}}$. Let $c_{b^{1/n}} \in C^1(\pi_1(C, \eta), \mu_n)$ be the one-cocycle defined by $c_{b^{1/n}}(\sigma) = \sigma(b^{1/n})/b^{1/n}$ for $\sigma \in \pi_1(C, \eta)$. Then $c_{b^{1/n}}$ represents $[b] \in H^1(C, \mu_n)\mathbb{O}$, and $c_{b^{1/n}}$ is trivial on the decomposition group $\pi_1(C, \eta)(\tilde{O})$ of a place $\tilde{O}$ of $M(C)$ over $O'$. In particular, $c_{b^{1/n}}(\Phi) = c_{b^{1/n}}(\Phi^{-1}) = 1$.

**Proof.** The first paragraph of Lemma 3.5 is obvious.

To prove part (i), let $\pi_{\alpha} : C_{\alpha} \to C$ be the étale morphism of smooth projective curves over $k$ associated to $\alpha$. Then $O$ splits in the cyclic cover $\pi_{\alpha}$ if and only if $\alpha$ is trivial on the decomposition group $\pi_1(C, \eta)(\tilde{O})$ of any place $\tilde{O}$ in $M(C)$ over $O$. This implies the first statement of part (i) and the direct sum decomposition of $\text{Pic}(C)/n \cdot \text{Pic}(C)$. Since $d(O)$ is prime to $\ell$, $\langle \langle O \rangle \rangle \cong \mathbb{Z}/n\mathbb{Z}$, and we obtain the last statement of part (i).

To prove part (ii), let $b \in D(C)$ and choose a uniformizing parameter $\pi_{\alpha}$ in the local ring $O_{C,O}$. Then the leading term in the Laurent series expansion of $b$ with respect to $\pi_{\alpha}$ lies in $(k(O)^*)^n$ if and only if there exists an $n$th root $b^{1/n}$ of $b$ in the completion $k(C)_{\tilde{O}}$. The field $L$ described in part (ii) is then contained in $k(C)_{\tilde{O}}$, so there is a place $O'$ over $O$ in $L$ with $L_{O'} = k(C)_{\tilde{O}}$. Since $b \in D(C)$ has valuation divisible by $n$ at all places of $k(C)$ and $n$ is prime to the characteristic of $k(C)$, $L$ is contained in $M(C)$. When $\tilde{O}$ is a place of $M(C)$ over $O'$, the decomposition group $\pi_1(C, \eta)(\tilde{O})$ fixes $k(C)_{\tilde{O}} = L_{O'} \supset L$, so $c_{b^{1/n}}$ is trivial on $\pi_1(C, \eta)(\tilde{O})$ if $b \in D(C)$ is normalized at $O$.

We will need the following result later to reduce to the case in which $O$ has residue field $k$.

**Theorem 3.6.** Under the assumptions of Corollary 3.3, let $k' = k(O)$ be the residue field of $O$, so $[k' : k] = d(O)$ is relatively prime to $\ell \cdot (\ell - 1)$. Let $\pi : C' = k' \otimes_k C \to C$ be the second projection. The direct image homomorphism

$$\pi_* : H^2(C', \mu_n) = \text{Pic}(C') \otimes_{\mathbb{Z}/n} \mathbb{Z}/n \to H^2(C, \mu_n) = \text{Pic}(C) \otimes_{\mathbb{Z}/n} \mathbb{Z}/n$$

is induced by the norm $\text{Norm}_{C'/C} : \text{Pic}(C') \to \text{Pic}(C)$. Let $O'$ be a point of $C'$ over $O$. Then $O'$ is a point of $C'(k')$ and $\text{Norm}_{C'/C}(O') = O$. Let $\alpha' \in H^1(C', \mathbb{Z}/n)$ and $[b'] \in H^1(C', \mu_n)$ be the pullbacks of $\alpha \in H^1(C, \mathbb{Z}/n)$ and $[b] \in H^1(C, \mu_n)$. One has

$$\alpha \cup [b] = \frac{1}{d(O)}(\text{Norm}_{C'/C} \otimes \text{Id})(\alpha' \cup [b']).$$
where \( \text{Id} \) is the identity map on \( \mathbb{Z}/n \). If \( \alpha \in H^1(C, \mathbb{Z}/n)_O \) and \( [b] \in H^1(C, \mu_n)_O \) then \( \alpha' \in H^1(C', \mathbb{Z}/n)_{O'} \) and \( [b'] \in H^1(C', \mu_n)_{O'} \).

**Proof.** The claim about (3.19) follows from the compatibility of the Kummer sequences of \( C' \) and \( C \) with respect to \( \pi_* \). The point \( O \) splits to \( C' \) so \( \text{Norm}_{C'/C}(O') = O \). The composition \( \pi_* \circ \pi^* \) on every cohomology group \( H^i(C, \mathbb{Z}/n) \) and \( H^i(C, \mu_n) \) is multiplication by the degree \( d(O) \) of \( C' \) over \( C \). By the compatibility of cup products with pullbacks this gives

\[
d(O) \cdot (\alpha \cup [b]) = (\pi_* \circ \pi^*)(\alpha \cup [b]) = \pi_*(\alpha' \cup [b']) = (\text{Norm}_{C'/C} \otimes \text{Id})(\alpha' \cup [b'])
\]

which shows (3.20) since \( d(O) \) is prime to \( \ell \). The fact that \( \alpha' \in H^1(C', \mathbb{Z}/n)_{O'} \) if \( \alpha \in H^1(C, \mathbb{Z}/n)_O \) follows from the fact that \( O' \) and \( O \) have the same residue field. The fact that \( [b'] \in H^1(C', \mu_n)_{O'} \) if \( [b] \in H^1(C, \mu_n)_O \) follows from comparing Laurent expansions at \( O \) and at \( O' \). \( \square \)

4. **The arithmetic of covers**

In this section we will prove various results necessary for analyzing the formulas in Theorem 1.1. We will assume Hypothesis 2.2 i.e. \( n = \ell' \). Recall that \( \Phi_{k/k} \) is the arithmetic Frobenius over \( k \).

Define \( \overline{k}(C) \) to be the compositum of \( k \) and \( k(C) \) in \( \overline{k}(C) \). Define \( L(C) \) to be the maximal abelian unramified extension of \( \overline{k}(C) \) in \( k(C) \). Then \( \Phi_{E/k} \) is a progenerator of \( \text{Gal}(\overline{k}(C)/k(C)) = \text{Gal}(\overline{k}/k) \approx \hat{\mathbb{Z}} \) and \( \text{Gal}(L(C)/\overline{k}(C)) \) is isomorphic to the adelic Tate module \( T_k(C) = \prod_v T_v(C) \), where \( \ell' \) runs over all primes and \( T_v(C) \) is the \( \ell' \)-adic Tate module of \( C \). The left conjugation action on \( \text{Gal}(L(C)/\overline{k}(C)) \) of a lift \( \tilde{\Phi}_{E/k} \) of \( \Phi_{E/k} \) to \( \text{Gal}(L(C)/k(C)) \) gives an automorphism \( \Phi_{k,C} \) of \( T_k(C) \) independent of the choice of \( \tilde{\Phi}_{E/k} \). The choice of \( \tilde{\Phi}_{E/k} \) gives an isomorphism

\[
\text{Gal}(L(C)/k(C)) = T_k(C) \times \text{Gal}(\overline{k}/k).
\]

The Artin map then defines an injective homomorphism

\[
\text{art}_C : \text{Pic}(C) \to \text{Gal}(L(C)/k(C))^{ab} = T_k(C)/(1 - \Phi_{k,C})T_k(C) \times \text{Gal}(\overline{k}/k)\nonumber.
\]

The image of this homomorphism consists of the elements which project to an integral power of \( \Phi_{E/k} \) in \( \text{Gal}(\overline{k}/k) \). The restriction of the Artin map defines an isomorphism

\[
\text{art}_{C,0}^0 : \text{Pic}^0(C) \to T_k(C)/(1 - \Phi_{k,C})T_k(C) \nonumber.
\]

Here \( \text{Pic}^0(C) \) is a finite group, and on the Sylow \( \ell \)-subgroup we get an isomorphism

\[
\text{art}_{C,0}^0 : \text{Pic}^0(C)[\ell^\infty] \to T_k(C)/(1 - \Phi_{k,C})T_k(C)\nonumber.
\]

where \( T_k(C) \) is isomorphic to \( (\mathbb{Z}/q)^{g(C)} \) when \( g(C) \) is the genus of \( C \) over \( k \) since \( \ell \) is prime to \( q = \#k \).

Let \( C' \) be a smooth projective curve such that \( k(C') \) is a cyclic everywhere unramified extension of \( k(C) \) of degree \( d/n \). Let \( k' \) be the constant field of \( C' \), so \([k' : k]\) \( d \) and \( k \subset k' \subset \overline{k} \). We have \( \overline{k}(C) \subset \overline{k}(C') \) and \( L(C) \subset L(C') \). We get a commutative diagram

\[
\begin{array}{ccc}
\text{Gal}(L(C)/k(C)) & \to & T_k(C) \times \text{Gal}(\overline{k}/k) \\
\downarrow & & \downarrow \pi_* \times \text{id} \\
\text{Gal}(L(C)/k(C)) & \to & T_k(C) \times \text{Gal}(\overline{k}/k)
\end{array}
\]

in which the left vertical homomorphism results from restricting automorphisms of \( L(C') \) to \( L(C) \), \( \pi_* \) is induced by the morphism \( \pi : \overline{k} \otimes_{k'} C' = \overline{k} \to \overline{k} \otimes_k C = \overline{C} \) associated to \( \overline{k}(C) \subset \overline{k}(C') \) and \( \iota : \text{Gal}(\overline{k}/k') \subset \text{Gal}(\overline{k}/k) \) is the natural inclusion. By choosing compatible lifts of \( \Phi_{k/k'} \) and \( \Phi_{E/k/k'} \), we can assume \( \iota(\Phi_{E/k/k'}) = \Phi_{k',k} \). For \( y \in T_k(C') \) we have

\[
\pi_*(\Phi_{k',C'}(y)) = \Phi_{k',C'}(\pi_*y).
\]
Taking maximal abelian quotients of the groups in this diagram gives a commutative diagram

\[
\begin{array}{ccc}
\text{Gal}(L(C)/k(C))^{\text{ab}} & \rightarrow & \text{Gal}(L'(C)/k'(C))^{\text{ab}} \\
2 & \downarrow & 2 \\
\text{Gal}(L(C)/k(C)) & = & \frac{T_{\text{ab}}(C)}{(1 - \Phi_{k,C})T_{\text{ab}}(C)} \times \text{Gal}([k'/k]) \xrightarrow{\text{art}_{C'}} \text{Pic}(C') \\
\text{Ver} & & \pi^* \\
2 & \downarrow & 2 \\
\text{Gal}(L(C)/k(C)) & = & \frac{T_{\text{ab}}(C)}{(1 - \Phi_{k,C})T_{\text{ab}}(C)} \times \text{Gal}([k'/k]) \xrightarrow{\text{art}_{C'}} \text{Pic}(C') \\
\end{array}
\]

where $N$ is induced by the previously described homomorphism $\nu_* \times \iota$ of diagram (4.5). We then have $T_{\text{ab}}(C') = T_{\text{ab}}(C)$. With this identification, define $V : T_{\text{ab}}(C) \rightarrow T_{\text{ab}}(C') = T_{\text{ab}}(C)$ by

\[
V = \sum_{i=0}^{d-1} \Phi_i^{\text{ab}}
\]

where $d = [k(C') : k(C)] = [k' : k]$.

**Lemma 4.2.** Suppose the hypotheses of Definition 4.1. There is a commutative diagram

\[
\begin{array}{ccc}
T_{\text{ab}}(C) & \rightarrow & \text{Gal}(L(C)/k(C))^{\text{ab}} \\
\phi & \downarrow & \phi \\
T_{\text{ab}}(C') & \rightarrow & \text{Gal}(L'(C)/k'(C))^{\text{ab}} \\
\end{array}
\]

**Proof.** This is just a matter of unwinding the definition of the transfer homomorphism when $G = \text{Gal}(L(C)/k(C))$ and $H = \text{Gal}(L(C')/k'(C'))$. Since $C' = k' \otimes_k C$, we can choose the set of coset representatives for $H$ in $G$ to be $\{\phi_i^{d-1}\}_{i=0}^d$.

**Corollary 4.3.** Suppose the hypotheses of Definition 4.1. There is a commutative diagram

\[
\begin{array}{ccc}
T_{\ell}(C) & \rightarrow & T_{\ell}(C)/T_{\ell}(C) \\
\phi & \downarrow & \phi \\
T_{\ell}(C') & \rightarrow & T_{\ell}(C')/T_{\ell}(C') \\
\end{array}
\]

where $\text{art}_{C}$ (resp. $\text{art}_{C'}$) is the restriction of the Artin map $\text{art}_{C}$ (resp. $\text{art}_{C'}$) as in (4.1). We have $T_{\ell}(C') = T_{\ell}(C)$. With this identification,

\[
V(1 - \Phi_{k,C})T_{\ell}(C) = (1 - \Phi_{k',C'})T_{\ell}(C)
\]

and both $V$ and $\pi^*$ are injective.
Proof. The equality (4.10) follows from
\[ V(1 - \Phi_{k,C})T_\ell(C) = \left( \sum_{i=0}^{d-1} \Phi_{k,C}^i (1 - \Phi_{k,C})T_\ell(C) = (1 - \Phi_{k,C}^d)T_\ell(C) = (1 - \Phi_{k',C'})T_\ell(C). \]

Since \( T_\ell(C) \) is isomorphic to \( (\mathbb{Z}_\ell)^{2g(C)} \) and \( (1 - \Phi_{k',C'})T_\ell(C) \) has finite index in \( T_\ell(C) \), this equality also implies that \( V \) and \( \pi^* \) are injective. \( \Box \)

5. Arithmetic Frobenius and Legendre derivatives

In this section, we will consider restrictions of the cup product maps (1.3) and (1.5) that are connected to the derivative of the arithmetic Frobenius. We will use the notation of (2). In particular, we assume Hypothesis (1.2) i.e. \( n = \ell^2 \).

Proposition 5.1. There is a unique automorphism \( A \) of \( \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(C) \) such that \( \Phi_{k,C} = 1 + nA \).

Multiplication by \( A^{-1} \) defines a homomorphism
\[ d\mathcal{L} : \text{Pic}^0(C)[n] = \frac{T_\ell(C) \cap AT_\ell(C)}{nAT_\ell(C)} \rightarrow \frac{T_\ell(C)}{nT_\ell(C) + nAT_\ell(C)} = \text{Pic}^0(C), \]
which we will call the Legendre derivative of Frobenius.

Proof. By (4.4) we have an isomorphism
\[ \text{art}^0_C : \text{Pic}^0(C)[\ell] \rightarrow \frac{T_\ell(C)}{(1 - \Phi_{k,C})T_\ell(C)} = \frac{T_\ell(C)}{nAT_\ell(C)}. \]

Since this group is finite and \( T_\ell(C) \) is a free \( \mathbb{Z}_\ell \)-submodule of \( \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(C) \) of rank \( 2g(C) \), this implies \( A \) is an automorphism of \( \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(C) \). We have isomorphisms
\[ \frac{\text{Pic}^0(C)}{n \cdot \text{Pic}^0(C)} = \frac{\text{Pic}^0(C)[\ell]}{n \cdot \text{Pic}^0(C)[\ell]} = \frac{T_\ell(C)}{nT_\ell(C) + nAT_\ell(C)} \]
and
\[ \frac{\text{Pic}^0(C)[n]}{n \cdot \text{Pic}^0(C)} = \frac{T_\ell(C) \cap AT_\ell(C)}{nAT_\ell(C)} \]
from which the proposition is clear. \( \Box \)

Remark 5.2. The reason for the terminology is that in the classical theory of convexity, the derivative of the Legendre transform of a differentiable function is the inverse of the derivative of the function; see (19). Here \( \Phi_{k,C} - 1 \) is a formal derivative of the arithmetic Frobenius. It would be interesting to develop a counterpart of this theory over finite fields.

The groups \( \text{Pic}^0(C)[n] \) and \( \text{Pic}^0(C)/n \cdot \text{Pic}^0(C) \) have the same order, but \( d\mathcal{L} \) is not in general an isomorphism. For example, suppose \( C \) is an elliptic curve with affine equation \( y^2 = x^3 - 3 \) over \( k = \mathbb{Z}/7 \). Then \( C \) has an automorphism \( \zeta \) of order \( 3 \) over \( k \) fixing \( y \) and sending \( x \) to \( 2x \), so \( C \) has complex multiplication by \( \mathbb{Z}[\zeta] \). One finds \#\( C(k) = 3 \). This implies that if \( \ell = n = 3, T_\ell(C) \) must be a free rank one \( \mathbb{Z}_\ell[\zeta] \)-module, and \( \Phi_{k,C} - 1 \) acts as multiplication by a uniformizing parameter in \( \mathbb{Z}_\ell[\zeta] \). Since \( \mathbb{Z}_\ell[\zeta] \) is quadratically ramified over \( \mathbb{Z}_\ell \), it follows that \( A^{-1} \) acts by multiplication by a uniformizer in \( \mathbb{Z}_\ell[\zeta] \). This forces \( d\mathcal{L} \) to be the zero homomorphism. On the other hand, if \( C \) is any curve such that \( \text{Pic}^0(C)[n] \) is isomorphic to \( (\mathbb{Z}/n)^{2g(C)} \), then \( A \) defines an endomorphism of \( T_\ell(C) \), and \( d\mathcal{L} \) is an isomorphism.

Theorem 5.3. Under Hypothesis (1.2) the restrictions of (1.10) (resp. (1.15)) in which the first (resp. second) argument lies in \( H^1(k, \mathbb{Z}/n) \) can be computed in the following way. Suppose \( \tau, \alpha \) and \( b \) are as in parts (i) and (ii) of Theorem (1.1) and \( \alpha \in H^1(k, \mathbb{Z}/n) = \text{Hom}(\text{Gal}(\bar{k}/k), \mathbb{Z}/n) \). Let \( [b] \) be the class in \( \text{Pic}^0(C)[n] \) of the divisor \( b = \text{div}_C(b)/n \). Let \( \Phi_{\tau/k} \in \text{Gal}(\bar{k}/k) \) be the Frobenius automorphism. One has
\[ \alpha \cup [b] = \alpha(\Phi_{\tau/k}) \cdot d\mathcal{L}([b]) \in \text{Pic}(C)/n \cdot \text{Pic}(C) = H^2(C, \mu_n). \]
The cup product $\tau \cup \alpha \cup [b]$ depends only on the restriction of $\tau$ to $\text{Pic}^0(C)$, and

\begin{equation}
\tau \cup \alpha \cup [b] = \alpha(\Phi_{T/k}) \cdot \tau(d\mathcal{L}([b])) \in \mathbb{Z}/n = H^3(C, \mu_n).
\end{equation}

Remark 5.4. In Theorem 5.3 it is important that the automorphism $\Phi_{k,C}$ of $T_k(C)$ used to define $d\mathcal{L}$ is the one arising from the arithmetic Frobenius $\Phi_{T/k} \otimes 1_{C/k}$ of $T = k \otimes_k C$. The geometric Frobenius endomorphism $1_T \otimes F_{C/k}$ of $\mathcal{C} = \mathbb{C} \otimes C$ over $\mathbb{C}$ is the identity map on the underlying topological space of $C$ and that is the $q^r$th power map on $O_C$. In particular, $1_T \otimes F_{C/k}$ acts on $\mathcal{C}(k)$ by raising the coordinates of any point to the $q^r$th power (see [14] p. 186 and p. 291-292). The action of $1_T \otimes F_{C/k}$ on $T_k(C)$ is the inverse of the action of $\Phi_{k,C}$. In particular, if one writes the action of $\Phi_{k,C}$ on $T_k(C)$ as $1 + nA$ as in Proposition 5.1 then $1_T \otimes F_{C/k}$ acts as $(1 + nA)^{-1}$.

Proof of Theorem 5.3 We can reduce to the case in which $\alpha$ is the generator of $H^1(k, \mathbb{Z}/n)$ such that $\alpha(\Phi_{T/k}) = 1$. By parts (i) and (ii) of Theorem 1.1 formulas for the cup products in (5.2) and (5.3) are obtained as follows. Let $k'$ be the cyclic extension of $k$ of degree $n$. Then $C' = k' \otimes_k C$ is a cyclic unramified cover of degree $n$, and $\alpha$ defines an isomorphism from $\text{Gal}(C'/C) = \text{Gal}(k'/k)$ to $\mathbb{Z}/n$ by sending $\Phi_{k'/k}$ to $1$.

We have defined $b = \text{div}_C((b)/n \in \text{Div}(C)$. There is an element $c \in k(C')$ such that $b = \text{Norm}_{k(C')/k(C)}(c)$ since $k(C') = k'(C') = k' \otimes_k C$ is a cyclic unramified (constant field) extension of $k(C)$. Let $\psi_k$ be the automorphism of $k(C') = k' \otimes_k C$ which is the identity on $1 \otimes C$ and which is the Frobenius automorphism $\Phi_{k'/k}$ on $k' \otimes 1 = k'$. Let $\sigma : C' \rightarrow C$ be the morphism associated with the inclusion $k(C) \subset C'(C)$. There is a divisor $e \in \text{Div}(C')$ such that

\begin{equation}
\text{div}_{C'}(e) = \pi^*b + (1 - \psi_k) \cdot e
\end{equation}

since $C' \rightarrow C$ is cyclic and unramified and the norm of the divisor $\text{div}_{C'}(e) - \pi^*b$ is trivial.

In the notation of Theorem 1.1 we have $C_\alpha = C'$, $\sigma = \psi_k$, and $\alpha'(\sigma) = 1$ because $\alpha(\Phi_{T/k}) = 1$. We obtain that the cup product is given by

\begin{equation}
\alpha \cup [b] = [\text{Norm}_{k(C')/k(C)}(c)] + \frac{n}{2}[b] \in \text{Pic}(C)/n \cdot \text{Pic}(C) = H^2(C, \mu_n)
\end{equation}

where $[b]$ is the class in $\text{Pic}(C)$ of a divisor $b$. For $\tau \in H^1(C, \mathbb{Z}/n) = \text{Hom}(\text{Pic}(C), \mathbb{Z}/n)$, the triple product is given by

\begin{equation}
\tau \cup \alpha \cup [b] = \tau \left([\text{Norm}_{k(C')/k(C)}(c)] + \frac{n}{2}[b]\right) \in \mathbb{Z}/n = H^3(C, \mu_n).
\end{equation}

Since the class $[b]$ lies in $\text{Pic}^0(C)[n]$, (5.4) shows

\begin{equation}
\pi^*[b] = (\psi_k - 1)[c] \in \text{Pic}^0(C')[n].
\end{equation}

We now use diagram (4.1) of Corollary 4.3. Here, since $C' = k' \otimes_k C$ and $[k' : k] = n$, $\Phi_{k,C}$ is the automorphism of the Tate module $T_k(C) = T_k(C')$ induced by $\psi_k = 1_C \otimes \Phi_{k/A}$, and $\Phi_{k',C'} = \Phi_{k,C}^n$. Write $\Phi_{k,C} = 1 + nA$ as in Proposition 4.1. The endomorphism $V : T_k(C) \rightarrow T_k(C)$ is

\begin{equation}
V = \sum_{i=0}^{n-1} \Phi_{k,C} = \sum_{i=0}^{n-1} (1 + nA)^i.
\end{equation}

From (4.3) we have isomorphisms

\begin{equation}
\text{art}_C^0 : \text{Pic}^0(C)[\ell^\infty] \xrightarrow{\sim} \frac{T_k(C)}{(1 - \Phi_{k,C})T_k(C)} = \frac{T_k(C)}{nAT_k(C)}
\end{equation}

and

\begin{equation}
\text{art}_C^0 : \text{Pic}^0(C)[n] \xrightarrow{\sim} \frac{T_k(C) \cap AT_k(C)}{nAT_k(C)}.
\end{equation}
Hence \( \text{art}^0_{C'}([b]) = [AE] \) in \( (T_C(\ell) \cap AT_C(\ell))/nAT_C(\ell) \) for some \( E \in T_C(\ell) \) such that \( AE \in T_C(\ell) \). Therefore (5.10) shows

\[
\text{art}^0_{C'}(\pi^*[b]) = [VAE] \quad \text{in} \quad \frac{T_C(\ell)}{(1 - \Phi_{k',C'})T_C(\ell)} = \frac{T_C(\ell)}{(1 - (1 + nA)^n)T_C(\ell)}.
\]

Here

\[
VA = AV = A \sum_{i=0}^{n-1} (1 + nA)^i = A \sum_{i=0}^{n-1} \sum_{j=0}^{i} \binom{i}{j} (nA)^j = nAU
\]

where

\[
U = 1 + A \sum_{i=1}^{n-1} \sum_{j=1}^{i} \binom{i}{j} (nA)^{j-1}.
\]

The element \( D = UE \) of \( \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_C(\ell) \) lies in \( T_C(\ell) \) since \( E \) and \( AE \) are both in \( T_C(\ell) \). We have

\[
\text{art}^0_{C'}(\pi^*[b]) = [VAE] = [nAU] = [nAD] = (\Phi_{k,C} - 1)[D]
\]

where on the right \([D]\) means the class of \( D \) in \( T_C(\ell)/(1 - \Phi_{k',C'})T_C(\ell) = \text{Pic}^0(C')[\ell^\infty] \).

The action of \( \Phi_{k,C} \) on \( T_C(\ell) \) gives the action of \( \psi_k \) on \( \text{Pic}^0(C')[\ell^\infty] \). Equations (5.7) and (5.10) now show

\[
[D] - [D] \in \text{Pic}^0(C')^G
\]

when \( G = \text{Gal}(C'/C) \) is the cyclic group of order \( n \) generated by \( \psi_k \).

We have an exact sequence

\[
1 \to (k')^* \to k(C')^* \to \text{Div}(C') \to \text{Pic}(C') \to 1
\]

in which \( k' \) is the field of constants of \( k(C') \) and the map \( k(C')^* \to \text{Div}(C') \) is induced by taking divisors. Splitting this into two short exact sequences and using that \( H^1(G,k(C')^*) = 0 \) by Hilbert’s theorem 90, we get exact sequences

\[
\text{Div}(C')^G \to \text{Pic}(C')^G \to H^1(G,k(C')^*/(k')^*) \to 0 = H^1(G,\text{Div}(C'))
\]

and

\[
0 \to H^2(G,k(C')^*/(k')^*) \to H^2(G,(k')^*) \to H^2(G,k(C')^*)
\]

Since in (5.14), \( k'/k \) is a finite Galois extension with cyclic Galois group \( G \), we have \( H^2(G,(k')^*) = H^0(G,(k')^*) = 0 \). We conclude from (5.13) that the map \( \text{Div}(C')^G \to \text{Pic}(C')^G \) is surjective. However, \( C' \to C \) is a cyclic unramified \( G \)-extension, so \( \text{Div}(C')^G = \pi^*\text{Div}(C) \). Using this in (5.11) shows that there is a divisor \( \epsilon \) on \( C \) such that

\[
[D] = [\epsilon] + \pi^*[\epsilon].
\]

We now have

\[
\text{Norm}_{k(C')/k(C)}[\epsilon] = \text{Norm}_{k(C')/k(C)}[D] + \text{Norm}_{k(C')/k(C)}[\pi^*[\epsilon]]
\]

\[
= \text{Norm}_{k(C')/k(C)}[D] + n \cdot [\epsilon].
\]

Let \( [D]_1 \) be the image of \( D \in T_C(\ell) \) in \( \text{Pic}^0(C')[\ell^\infty] = T_C(\ell)/(1 - \Phi_{k,C})T_C(\ell) = T_C(\ell)/nAT_C(\ell) \).

By diagram (4.10) and equation (5.10), we have

\[
\text{Norm}_{C'/C}[D] = [D]_1.
\]

Formula (5.3) together with (5.10) and (5.17) now give

\[
\alpha \cup [b] = \text{Norm}_{k(C')/k(C)}[\epsilon] + \frac{n}{2}[b] = [D]_1 + \frac{n}{2}[b]
\]

since \( n \cdot [\epsilon] \) is trivial in \( \text{Pic}(C)/n\text{Pic}(C) \). We need to show that

\[
[D]_1 + \frac{n}{2}[b] = d\mathcal{L}(\langle\text{[b]}\rangle) \quad \text{in} \quad \frac{T_C(\ell)}{nT_C(\ell) + nAT_C(\ell)} = \frac{\text{Pic}^0(C)}{n \cdot \text{Pic}^0(C)}.
\]
Recall that we chose \( E \in T_\ell(C) \) so that \( AE \in T_\ell(C) \) and
\[
\text{art}_C^\ell([b]) = [AE].
\]
The definition of \( dL \) then shows that
\[
dL([b]) = [E] \quad \text{in} \quad \frac{T_\ell(C)}{nT_\ell(C) + nAT_\ell(C)} = \frac{\text{Pic}^0(C)}{n \cdot \text{Pic}^0(C)}.
\]
Concerning the left side of (5.19), \( D = UE \) when \( U \) is as in (5.9). If \( \ell > 2 \) then \( \frac{2}{n}[b] = 0 \) in \( \text{Pic}^0(C)/n \cdot \text{Pic}^0(C) \) since 2 is then invertible mod \( n \). So (5.19) for \( \ell > 2 \) is equivalent to
\[
[U E] - [E] = 0 \quad \text{in} \quad \frac{T_\ell(C)}{nT_\ell(C) + nAT_\ell(C)} \quad \text{when} \quad \ell > 2.
\]
We have
\[
U - 1 = A \sum_{i=1}^{n-1} \sum_{j=1}^i \left( \begin{array}{c} i \\ j \end{array} \right) (nA)^{j-1} = A \frac{n(n-1)}{2} + \left( \sum_{i=1}^{n-1} \sum_{j=2}^i (nA)^{j-1} \right) A.
\]
So
\[
[U E] - [E] = \frac{n(n-1)}{2} [AE] + \left( \sum_{i=1}^{n-1} \sum_{j=2}^i (nA)^{j-1} \right) [AE] = 0 \quad \text{in} \quad \frac{T_\ell(C)}{nT_\ell(C) + nAT_\ell(C)} \quad \text{when} \quad \ell > 2
\]
as required since \( 2 | (n-1) \), \( AE \in T_\ell(C) \), \( E \in T_\ell(C) \) and \( nA \) is an endomorphism of \( T_\ell(C) \). Suppose now that \( \ell = 2 \). Then (5.19) is equivalent to
\[
[U E] + \frac{n}{2} [AE] - [E] = 0 \quad \text{in} \quad \frac{T_\ell(C)}{nT_\ell(C) + nAT_\ell(C)} \quad \text{when} \quad \ell = 2.
\]
We have
\[
\frac{n}{2} A + U - 1 = \frac{n}{2} A + A \frac{n(n-1)}{2} + \left( \sum_{i=1}^{n-1} \sum_{j=2}^i (nA)^{j-1} \right) A = \frac{n^2}{2} A + \left( \sum_{i=1}^{n-1} \sum_{j=2}^i (nA)^{j-1} \right) A \quad \text{when} \quad \ell = 2
\]
from which (5.22) is clear because \( E \in T_\ell(C) \).

\textbf{Theorem 5.5.} Under Hypothesis \[1.2\] the restrictions of (1.3) (resp. (1.5)) in which the second (resp. third) argument lies in \( H^1(k, \mu_n) \) can be computed in the following way. Suppose \( \tau, \alpha \) and \( b \) are as in parts (i) and (ii) of Theorem 1.1 and \( \beta \in H^1(k, \mu_n) = k^*/(k^*)^n \). Let \( d = \#H^1(k, \mu_n) \). Then \( d|n \) and there is a unique homomorphism \( c \in \text{Hom}(\text{Gal}(\bar{k}/k), \mu_d) = H^1(k, \mu_d) \) such that \( c(\Phi_{\bar{k}/k}) = \zeta \) is in \( \mu_d(k) \subset k^* \) and \( [b] \) is represented by the one-cocycle \( c \). Let \( [a] \in H^1(C, \mu_n) \) be the image of \( \alpha \) under the homomorphism \( H^1(C, \mathbb{Z}/n) \to H^1(C, \mu_n) \) induced by the unique homomorphism of sheaves \( \mathbb{Z}/n \to \mu_n \) that sends the global section 1 to the global section \( \zeta \). Let \( [a] \in \text{Pic}^0(C)[n] \) be the image of \( [a] \) under the map \( H^1(C, \mu_n) \to \text{Pic}^0(C)[n] \) produced by the Kummer sequence. Then
\[
\alpha \cup [b] = -dL([a]) \in \text{Pic}(C)/n \cdot \text{Pic}(C) = H^2(C, \mu_n).
\]
and
\[
\tau \cup \alpha \cup [b] = -\tau(dL([a])) \in \mathbb{Z}/n = H^3(C, \mu_n).
\]
\textbf{Proof.} Define \( \beta \in H^1(k, \mathbb{Z}/n) = \text{Hom}(\text{Gal}(\bar{k}/k), \mathbb{Z}/n) \) to be the homomorphism that sends \( \Phi_{\bar{k}/k} \) to 1. In view of Theorem 5.3 it will suffice to show
\[
\alpha \cup [b] = -\beta \cup [a] \quad \text{and} \quad \tau \cup \alpha \cup [b] = -\tau \cup \beta \cup [a].
\]
The group \( H^1(k, \mu_n) = k^*/(k^*)^n \) has order \( d = \gcd(n, \#(k^*)) \), so \( k^* \) contains a primitive \( d^{th} \) root of unity. Let \( \phi_j : H^1(k, \mu_d) \to H^1(k, \mu_n) \) and \( \psi_j : H^1(C, \mu_d) \to H^1(C, \mu_n) \) be the homomorphisms
induced by the natural injection \( \mu_d \to \mu_n \). Moreover, let \( \lambda_j : H^j(C, \mathbb{Z}/n) \to H^j(C, \mathbb{Z}/d) \) be the homomorphisms induced by the natural surjection \( \mathbb{Z}/n \to \mathbb{Z}/d \).

From the exact sequence of sheaves

\[
1 \to \mu_d \to \mu_n \to \mu_n/d \to 1
\]

we get an exact sequence

\[
H^0(k, \mu_{n/d}) \to H^1(k, \mu_d) \to H^1(k, \mu_n) \to H^1(k, \mu_{n/d}).
\]

Since \( \gcd(n/d, \#(k^*)) = 1 \), we have \( H^0(k, \mu_{n/d}) = \mu_{n/d}(k) = \{1\} \) and \( H^1(k, \mu_{n/d}) = k^*/(k^*)^{n/d} = \{1\} \). So \( \phi_1 : H^1(k, \mu_d) = \text{Hom}(\text{Gal}(\overline{k}/k), \mu_d) \to H^1(k, \mu_n) \) is an isomorphism. Let \( [b]' = \phi_1^{-1}([b]) \), let \( \alpha' = \lambda_1(\alpha) \) and \( \tau' = \lambda_1(\tau) \). The natural maps of sheaves \( \mathbb{Z}/n \otimes \mu_d \to \mu_n \) and \( \mathbb{Z}/n \otimes \mathbb{Z}/n \otimes \mu_d \to \mu_n \) imply that

\[
(5.26) \quad \psi_2(\alpha' \cup [b]') = \alpha \cup [b] \quad \text{and} \quad \psi_3(\tau' \cup \alpha' \cup [b]') = \tau \cup \alpha \cup [b].
\]

Hence to prove Theorem 5.25, we can reduce to the case in which \( d = n \).

Now \( d = n \) implies \( k^* \) contains a primitive \( n^{\text{th}} \) root of unity \( \gamma \). There is a unique isomorphism of sheaves \( \mathbb{Z}/n \to \mu_n \) sending the global section 1 to the global section \( \gamma \) and a unique homomorphism of sheaves \( \mu_n \to \mu_n^{\otimes 2} \) sending \( \gamma' \) to \( (\gamma \otimes \gamma)' \). Let \( \nu_j : H^j(C, \mathbb{Z}/n) \to H^j(C, \mu_n) \) and \( \xi_j : H^j(C, \mu_n) = H^j(C, \mu_n^{\otimes 2}) \) be the resulting isomorphisms. Then

\[
\nu_1(\alpha) \cup [b] = \xi_2(\alpha \cup [b]) \in H^2(C, \mu_n^{\otimes 2}).
\]

However, we can regard \( [b] \in H^1(k, \mu_n) = \text{Hom}(\text{Gal}(\overline{k}/k), \mu_n) \) as an element of \( H^1(C, \mu_n) = \text{Hom}(\pi_1(C), \mu_n) \). So since the cup product on \( H^1(C, \mu_n) \) is anticommutative, we have

\[
\nu_1(\alpha) \cup [b] = -[b] \cup \nu_1(\alpha) \quad \text{in} \quad H^2(C, \mu_n^{\otimes 2}).
\]

Now by considering cocycle representatives we see

\[
\xi_2(\beta \cup [a]) = [b] \cup \nu_1(\alpha).
\]

Therefore

\[
\alpha \cup [b] = \xi_2^{-1}(\nu_1(\alpha) \cup [b]) = -\xi_2^{-1}([b] \cup \nu_1(\alpha)) = -\beta \cup [a]
\]

and this implies (5.25) and completes the proof.

\[\square\]

### 6. Cup products of normalized classes on curves of arbitrary positive genus

Throughout this section we will assume Hypothesis 1.2 i.e \( n = \ell^z \). We let \( O \) be a fixed closed point of \( C \) with degree \( d(O) \) prime to \( \ell \cdot (\ell - 1) \), which exists by Lemma 3.2. The goal of this section is to prove Theorem 1.3. In §9 we will focus on the case when the genus \( g(C) = 1 \) and we will prove Theorem 1.3 by proving that condition (ii) of Theorem 1.3 holds in this case. In §11 we will give an infinite family of curves of genus 2 for which the equivalent conditions of Theorem 1.3 do not hold when \( n = \ell = 3 \).

**Proof of Theorem 1.3.** We consider the natural non-degenerate pairings from Lemma 2.4

\[\text{Hom}(	ext{Pic}(C), \mathbb{Z}/n) \times \text{Pic}(C) \to \mathbb{Z}/n\]

where the pairing in the top row is given by the cup product and in the bottom row by the evaluation map.

The formula in part (i) of Theorem 1.3 implies that classes in the cup product

\[H^1(C, \mathbb{Z}/n) \cup H^1(C, \mu_n) \subseteq H^2(C, \mu_n) = \text{Pic}(C)/n \cdot \text{Pic}(C)\]
are multiples of the image of $[O]$ in $\Pic(C)/n \cdot \Pic(C)$. On the other hand, by part (i) of Lemma 3.5, elements $\alpha \in H^1(C, \mathcal{O}) \subseteq H^1(C, \mathbb{Z}/n) = \Hom(\Pic(C), \mathbb{Z}/n)$ satisfy $\alpha([O]) = 0$. Therefore, part (i) implies part (ii).

Part (ii) of Theorem 1.3 implies that elements in the cup product of normalized classes in $\Pic(C)$ are orthogonal to $H^1(C, \mathbb{Z}/n)$ under the non-degenerate pairing $(6.1)$. In fact, the direct sum decomposition of $\Pic(C)/n \cdot \Pic(C)$ in part (i) of Lemma 3.5 shows that the orthogonal complement of $\Pic(C)/\mathbb{Z}/n$ in $\Pic(C)/n \cdot \Pic(C)$ is equal to the cyclic subgroup of order $n$ generated by the image of $[O]$. This means that every element of the cup product in $\Pic(\mathbb{C})$ of normalized classes is a multiple of the image of $[O]$ in $\Pic(C)/n \cdot \Pic(C)$. We can then read off this multiple by restricting to the algebraic closure and using that $d(O)$ is prime to $\ell \cdot (\ell - 1)$, leading to the formula in part (i).

\section{The genus one case}

In this section, we will focus on the case in which $C$ has genus 1. We make the assumptions and use the notation from §1. In particular, $n = \ell^2$ and $O$ is a closed point of $C$ of degree $d(O)$ prime to $\ell \cdot (\ell - 1)$.

Our goal is to prove Theorem 1.4. By Theorem 3.6, it is enough to consider the case in which $k(O) = k$, so that $d(O) = 1$. Hence we make the following assumptions throughout this section.

\textbf{Hypothesis 7.1.} The genus of $C$ is $g(C) = 1$, $n = \ell^2$ as in Hypothesis 1.2 and $O$ is a closed point of $C$ of degree $d(O) = 1$.

A key ingredient in the proof of Theorem 1.4 is the following result.

\textbf{Theorem 7.2.} Under Hypothesis 7.1 let $\mathcal{F}_n$ be either the constant sheaf $\mathbb{Z}/n$ or the locally constant sheaf $\mu_n$. Let $\alpha \in H^1(C, \mathbb{Z}/n)$ and $\beta \in H^1(C, \mathcal{F}_n)$, and let $\overline{\alpha}$ and $\overline{\beta}$ be the restrictions of $\alpha$ and $\beta$ to $H^1(C, \mathbb{Z}/n)$ and $H^1(C, \mathcal{F}_n)$, respectively. If the cup product

$$\overline{\alpha} \cup \overline{\beta} = 0 \quad \text{in} \quad H^2(C, \mathcal{F}_n),$$

then the cup product

$$\alpha \cup \beta = 0 \quad \text{in} \quad H^2(C, \mathcal{F}_n).$$

We will defer the proof of Theorem 1.4 to §3 since we will use continuous group cohomology, which requires some additional notational set up.

Another key ingredient in the proof of Theorem 1.4 is the following result about bilinear pairings on finite modules over $\mathbb{Z}/n$ that are generated by at most two elements.

\textbf{Lemma 7.3.} Suppose $M_1, M_2$ are finite modules over $\mathbb{Z}/n$ that are both generated by at most 2 elements. Let

$$\langle \ , \rangle : \ M_1 \times M_2 \to \mathbb{Z}/n$$

be a bilinear pairing. Then one of the following three conditions holds:

(a) $M_1$ is cyclic,

(b) $M_2$ is cyclic, say it is generated by $m_2$, $M_1$ is not cyclic, and there are generators $m_{11}, m_{12}$ of $M_1$ such that $\langle m_{12}, m_2 \rangle = 0$,

(c) $M_1$ and $M_2$ are both not cyclic, and there exist generators $m_{11}, m_{12}$ of $M_1$ and $m_{21}, m_{22}$ of $M_2$ such that $\langle m_{11}, m_{2i} \rangle = 0$ if $i \neq j$ in $\{1, 2\}$.

\textbf{Proof.} Suppose this is false, and that the pair $(M_1, M_2)$ is a counterexample that minimizes $(\#M_1) \cdot (\#M_2)$. Then $M_1$ is not cyclic. If $M_2$ were cyclic and generated by $m_2$, then there would be generators $m_{11}, m_{12}$ of $M_1$ such that $\langle m_{12}, m_2 \rangle = 0$, contradicting that property (b) is not satisfied. Hence $M_2$ is not cyclic.

Suppose first that there exists an element $y \in M_2$ of order $\ell$ such that $\langle M_1, y \rangle = 0$. Then if we define $M'_2 = M_2/\mathbb{Z}y$, the pairing $(\ , \ )$ induces a bilinear pairing

$$\langle \ , \ \rangle' : M_1 \times M'_2 \to \mathbb{Z}/n.$$
By the minimality of \((M_1, M_2)\), the pairing \(\langle \cdot , \cdot \rangle\) must satisfy one of the properties (a) - (c). We know that \(M_1\) and \(M_2\) are both not cyclic. Note that \(M'_2\) cannot be the zero module, since otherwise \(\langle \cdot , \cdot \rangle = 0\), which would imply that \(\langle \cdot , \cdot \rangle\) must satisfy property (c), a contradiction. Suppose next that \(M'_2\) is cyclic, which means that \(M'_2 = \mathbb{Z}m_{21} \oplus \mathbb{Z}m_{22}\) with \(y = m_{22}\) and \(\ell m_{22} = 0\). Since \(\langle \cdot , \cdot \rangle\) must then satisfy property (b), there are generators \(m_{11}, m_{12}\) of \(M_1\) such that \(\langle m_{12}, m_{21} \rangle = 0\). Since \(\langle M_1, y \rangle = 0\) and \(m_{22} = y\), we also have that \(\langle m_{11}, m_{22} \rangle = 0\). Hence \(\langle \cdot , \cdot \rangle\) again satisfies property (c), a contradiction. Therefore, \(M'_2\) is not cyclic. This means that \(\langle \cdot , \cdot \rangle\) must satisfy property (c) for some choice of generators \(m_{11}, m_{12}\) of \(M_1\) and \(m'_{21}, m'_{22}\) of \(M'_2\). Lifting \(m'_{21}, m'_{22}\) to generators \(m_{21}, m_{22}\) of \(M_2\), we see that then \(\langle \cdot , \cdot \rangle\) again satisfies property (c), a contradiction. Hence we conclude that \(\langle M_1, y \rangle \neq 0\) for all \(y \in M_2\) of order \(\ell\). This implies that the map

\[
M_2 \rightarrow \text{Hom}(M_1, \mathbb{Z}/n)
\]

is injective. In particular, \(#M_2 \leq \#\text{Hom}(M_1, \mathbb{Z}/n) = \#M_1\).

In a similar way, one can prove that there is no \(x \in M_1\) of order \(\ell\) such that \(\langle x, M_2 \rangle = 0\). Therefore, the map

\[
M_1 \rightarrow \text{Hom}(M_2, \mathbb{Z}/n)
\]

is injective, and \(#M_1 \leq \#\text{Hom}(M_2, \mathbb{Z}/n) = \#M_2\).

Since we also have \(#M_2 \leq \#M_1\), we conclude that \(M_1\) maps isomorphically to \(\text{Hom}(M_2, \mathbb{Z}/n) = M'_2\). Because \(M_2\) is not cyclic, we can write

\[
M_2 = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \quad \text{and} \quad M'_2 = \mathbb{Z}b'_1 \oplus \mathbb{Z}b'_2,
\]

where \(\langle b'_1, b_2 \rangle = b'_1(b_2) = 0\) and \(\langle b'_2, b_1 \rangle = b'_2(b_1) = 0\). But this means that \(\langle \cdot , \cdot \rangle\) satisfies property (c), a contradiction. In other words, no minimal counterexample \((M_1, M_2)\) exists, which completes the proof of Lemma \[1.3\] \(\square\)

**Proof of Theorem 1.4.** We use that the Weil pairing is alternating (see, for example, [8, Theorem 1]), which implies that the cup product \(\cap \overline{\cdot}\) in \([1.3]\) is alternating when \(\mathcal{F}_n = \mathbb{Z}/n\). Hence it follows by Theorem \[7.2\] that

\[
(7.1) \quad \alpha \cap \alpha = 0 \quad \text{for all } \alpha \in H^1(C, \mathbb{Z}/n)_O.
\]

We apply Lemma \[7.3\] to \(M_1 = H^1(C, \mathbb{Z}/n)_O, M_2 = H^1(C, \mu_n)_O\) and the bilinear pairing \(\langle \cdot , \cdot \rangle\) that is defined as follows. Let \(\alpha \in H^1(C, \mathbb{Z}/n)_O\) and \([b] \in H^1(C, \mu_n)_O\), and define

\[
\langle \alpha, [b] \rangle = \overline{\alpha} \cup \overline{b} \quad \text{in } \mathbb{Z}/n = H^2(T, \mu_n)
\]

to be the value of the pairing in \([1.3]\) when \(\mathcal{F}_n = \mu_n\) and \(\overline{\alpha}\) and \(\overline{[b]}\) are the restrictions of \(\alpha\) and \([b]\) to \(H^1(C, \mathbb{Z}/n)\) and \(H^1(T, \mu_n)\), respectively. By Lemma \[7.3\] we need to analyze the following three cases:

(a) \(H^1(C, \mathbb{Z}/n)_O\) is cyclic. In this case \(H^1(C, \mathbb{Z}/n)_O \cup H^1(C, \mathbb{Z}/n)_O\) is trivial by \((7.1)\), so condition (ii) of Theorem \[1.3\] holds.

(b) \(H^1(C, \mu_n)_O\) is cyclic, say it is generated by \([b]\), \(H^1(C, \mathbb{Z}/n)_O\) is not cyclic, and there are generators \(\alpha_1, \alpha_2\) of \(H^1(C, \mathbb{Z}/n)_O\) such that \(\overline{\alpha_2} \cup \overline{[b]} = 0\). By Theorem \[7.2\] we then have \(\alpha_2 \cup [b] = 0\) in \(H^2(C, \mu_n)\). Since \(\alpha_1\) and \(\alpha_2\) generate \(H^1(C, \mathbb{Z}/n)_O\), it follows from \((7.1)\) that the cup product \(\alpha_1 \cup \alpha_2\) generates \(H^1(C, \mathbb{Z}/n)_O \cup H^1(C, \mathbb{Z}/n)_O\). But then

\[
(\alpha_1 \cup \alpha_2) \cup [b] = \alpha_1 \cup (\alpha_2 \cup [b]) = 0
\]

generates \(H^1(C, \mathbb{Z}/n)_O \cup H^1(C, \mathbb{Z}/n)_O \cup H^1(C, \mu_n)_O\), which implies condition (ii) of Theorem \[1.3\].
(c) $H^1(C,\mathbb{Z}/n\mathbb{O})$ and $H^1(C,\mu_n\mathbb{O})$ are both not cyclic, and there exist generators $\alpha_1, \alpha_2$ of $H^1(C,\mathbb{Z}/n\mathbb{O})$ and $[b_1], [b_2]$ of $H^1(C,\mu_n\mathbb{O})$ such that $\overline{\alpha_i} \cup \overline{b_j} = 0$ if $i \neq j$ in $\{1, 2\}$. By Theorem 7.2, we then have $\alpha_1 \cup [b_j] = 0$ in $H^2(C,\mu_n\mathbb{O})$ if $i \neq j$ in $\{1, 2\}$. As in case (b), it follows that $\alpha_1 \cup \alpha_2$ generates $H^1(C,\mathbb{Z}/n\mathbb{O}) \cup H^1(C,\mathbb{Z}/n\mathbb{O})$. Since $\alpha_1 \cup \alpha_2 = -\alpha_2 \cup \alpha_1$, we obtain

$$(\alpha_1 \cup \alpha_2) \cup [b_1] = \alpha_1 \cup (\alpha_2 \cup [b_1]) = 0,$$

$$(\alpha_1 \cup \alpha_2) \cup [b_2] = (-\alpha_2 \cup \alpha_1) \cup [b_2] = -\alpha_2 \cup (\alpha_1 \cup [b_2]) = 0.$$

Hence condition (ii) of Theorem 1.3 holds.

Therefore, Theorem 1.4 follows from Theorem 1.3 in all cases (a) - (c).

8. Cohomological calculations in the genus one case

In this section we prove Theorem 7.2 using arguments from continuous group cohomology. We assume Hypothesis 7.1. We need the following notation.

Notation 8.1. Let $\overline{k(C)}$ be a fixed separable closure of $k(C)$ containing $\mathbb{F}$, and let $M(C)$ be the maximal everywhere unramified extension of $k(C)$ inside $\overline{k(C)}$. Then $\pi_1(C, \eta) = \text{Gal}(M(C)/k(C))$. We have a short exact sequence

$$0 \to \pi_1(C, \eta) \to \pi_1(C, \eta) \to \text{Gal}(\overline{k}/k) \to 1.$$ 

The arithmetic Frobenius $\Phi_{\mathbb{F}/k}$ on $\mathbb{F}$, which is the $q$th power map, progenerates $\text{Gal}(\overline{k}/k)$. Moreover, $\pi_1(C, \eta)$ is isomorphic to the adelic Tate module $T_{\alpha}(C) = \prod_{\ell} T_{\alpha}(C)$, where $\ell$ runs over all primes and $T_{\alpha}(C)$ is the $\ell$th-adic Tate module of $C$. Defining

$$\Gamma = \frac{\pi_1(C, \eta)}{\prod_{\ell \neq \alpha} T_{\alpha}(C)}$$

we obtain a short exact sequence

$$0 \to T_{\ell}(C) \to \Gamma \to \text{Gal}(\overline{k}/k) \to 1$$

and an isomorphism

$$\Gamma \cong T_{\ell}(C) \times \text{Gal}(\overline{k}/k).$$

Let $\mathcal{O}$ be a place over $O$ in $M(C)$ with the properties in part (ii) of Lemma 3.6. Let $\mathcal{O}_{\mathcal{O}}$ be the image in $\Gamma$ of the decomposition group of $\mathcal{O}$ in $\pi_1(C, \eta) = \text{Gal}(M(C)/k(C))$. Let $\Phi$ be the Frobenius progenerator of $\Gamma_{\mathcal{O}}$, so that $\Phi$ is the unique element mapping to $\Phi_{\mathbb{F}/k}$ in $\text{Gal}(\overline{k}/k)$. Let $a_1$ and $a_2$ be progenerators of $T_{\ell}(C)$. We will write $T_{\ell}(C)$ multiplicatively, i.e. each element can be expressed as

$$a_1^{t_1} \cdot a_2^{t_2}$$

for unique $t_1, t_2 \in \mathbb{Z}_{\ell}$. We have that $\Gamma$ is progenerated by $a_1, a_2$ and $\Phi$.

Remark 8.2. Let $G$ be a profinite group, and consider the standard resolution $B_{\mathbb{Z}}$ of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module, where

$$B_{\mathbb{Z}} = \cdots \to \mathbb{Z}[G \times G \times G] \to \mathbb{Z}[G \times G] \to \mathbb{Z}[G]$$

is concentrated in degrees $\geq 0$ and $g_0, g_1, \ldots$ are arbitrary elements of $G$ (see \cite[§6.2]{17}). Recall that the corresponding bar resolution is obtained by introducing the “bar notation”

$$[g_1, g_2, \ldots, g_m] = (1, g_1, g_1g_2, \ldots, g_1g_2\ldots g_m)$$

for all $g_1, g_2, \ldots, g_m$. 




Note that we have a contracting homotopy \( H_{G,m} : B_{G,m} = \hat{\mathbb{Z}}[G^{m+1}] \to B_{G,m+1} = \hat{\mathbb{Z}}[G^{m+2}] \), given by \( H_{G,m}(g_0, g_1, \ldots, g_m) = (1, g_0, g_1, \ldots, g_m) \) (see [3, §1.5]), which satisfies
\[
(\partial_{G,m+1} \circ H_{G,m})(w) = w \quad \text{for all } w \in \text{Image}(\partial_{G,m+1}) \subseteq B_m.
\]

Let \( M \) and \( N \) be two discrete \( \hat{\mathbb{Z}}[G] \)-modules, and let \([\phi_M] \in H^1(G, M) \) and \([\phi_N] \in H^1(G, N)\) be classes that are represented by continuous one-cocycles \( \phi_M \in C^1(G, M) \) and \( \phi_N \in C^1(G, N) \). Then the cup product \([\phi_M] \cup [\phi_N] \in H^2(G, M \otimes \mathbb{Z}_2) \) is represented by the continuous two-cycle \( \phi_M \cup \phi_N \) given by
\[
(\phi_M \cup \phi_N)([g_1] [g_2]) = -\phi_M([g_1]) \otimes g_1 \phi_N([g_2])
\]
for all \( g_1, g_2 \in G \) (see [6, §V.3]).

Let \( \overline{\pi} \in H^1(\overline{C}, \mathbb{Z}/n) = \text{Hom}(T_k(C), \mathbb{Z}/n) \) and let \( \overline{\beta} \in H^1(\overline{C}, \mathcal{F}_n) = \text{Hom}(T_k(C), \hat{\mathcal{F}}_n) \), where the sheaf \( \mathcal{F}_n \) is \( \mathbb{Z}/n \) or \( \tilde{\mu}_n \), and the group \( \hat{\mathcal{F}}_n \) is \( \mathbb{Z}/n \) or \( \tilde{\mu}_n \), respectively. We want to use a free resolution \( W_* \to \hat{\mathbb{Z}} \to 0 \) of \( \hat{\mathbb{Z}}[T_k(C)] \)-module that is different from the standard resolution \( B_{T_k(C), \bullet} \to \hat{\mathbb{Z}} \) to provide a different presentation of the cup product \( \overline{\pi} \cup \overline{\beta} \in H^2(\overline{C}, \mathcal{F}_n) \). We define \( W_* \) to be the (Koszul) complex
\[
W_* = \cdots \to \hat{\mathbb{Z}}[T_k(C)] h \xrightarrow{\delta_3} \hat{\mathbb{Z}}[T_k(C)] f_1 \oplus \hat{\mathbb{Z}}[T_k(C)] f_2 \xrightarrow{\delta_2} \hat{\mathbb{Z}}[T_k(C)] e
\]
\[
\delta_1\ x \xrightarrow{\delta_0} \hat{\mathbb{Z}}[T_k(C)] h \quad \text{where}
\]
which only has nonzero terms in degrees 0, 1 and 2. Note that the basis elements \( h, f_1, f_2, e \) are only used to more readily distinguish between the different nonzero terms of \( W_* \).

Writing \( B_* = B_{T_k(C), \bullet} \) and \( \partial_m = \partial_{T_k(C), m} \), we obtain a morphism of complexes
\[
\xymatrix{ W_* \ar[r]^-{\lambda_0} & 0 \ar[r]^-{0} & \hat{\mathbb{Z}}[T_k(C)] h \ar[r]^-{\delta_3} & \hat{\mathbb{Z}}[T_k(C)] f_1 \oplus \hat{\mathbb{Z}}[T_k(C)] f_2 \ar[r]^-{\delta_2} & \hat{\mathbb{Z}}[T_k(C)] e }
\]
\[
\xymatrix{ B_* \ar[r]^-{\lambda} & \cdots \ar[r]^-{\delta_3} & \hat{\mathbb{Z}}[T_k(C) \times T_k(C) \times T_k(C)] \ar[r]^-{\partial_2} & \hat{\mathbb{Z}}[T_k(C) \times T_k(C)] \ar[r]^-{\partial_1} & \hat{\mathbb{Z}}[T_k(C)] }
\]
where
\[
\lambda_0(e) = 1,
\]
\[
\lambda_1(f_1 + 0) = (1, a_1),
\]
\[
\lambda_1(0 + f_2) = (1, a_2),
\]
\[
\lambda_2(h) = (a_1, a_2, a_1) - (1, 1, a_1) + (1, 1, a_2) - (1, a_1, a_1 a_2).
\]

These equalities, and in particular [8.3], can be found using property [8.3] of the contracting homotopy \( H_{T_k(C), m} \).

**Lemma 8.3.** Let \( \mathcal{F}_n \) be the sheaf \( \mathbb{Z}/n \) or \( \tilde{\mu}_n \), and let \( \hat{\mathcal{F}}_n \) be the group \( \mathbb{Z}/n \) or \( \tilde{\mu}_n \), respectively. We have an isomorphism
\[
H^2(\overline{C}, \mathcal{F}_n) = \text{Hom}_{\hat{\mathbb{Z}}[T_k(C)]}(\hat{\mathbb{Z}}[T_k(C)] h, \hat{\mathcal{F}}_n).
\]
If \( \overline{\pi} \in H^1(\overline{C}, \mathbb{Z}/n) = \text{Hom}(T_k(C), \mathbb{Z}/n) \) and \( \overline{\beta} \in H^1(\overline{C}, \mathcal{F}_n) = \text{Hom}(T_k(C), \hat{\mathcal{F}}_n) \) then \( \overline{\pi} \cup \overline{\beta} \in H^2(\overline{C}, \mathcal{F}_n) \) is given by
\[
(\overline{\pi} \cup \overline{\beta})(h) = \overline{\pi}(a_1) \overline{\beta}(a_2) - \overline{\pi}(a_2) \overline{\beta}(a_1) \quad \text{if } \mathcal{F}_n = \mathbb{Z}/n
\]
and by
\[
(\overline{\pi} \cup \overline{\beta})(h) = \overline{\beta}(a_2) \overline{\pi}(a_1) - \overline{\pi}(a_2) \overline{\beta}(a_1) \quad \text{if } \mathcal{F}_n = \tilde{\mu}_n.
We want to use the induced complex $\mathbb{V}$ cone of the morphism
\[\Phi: T(C) \to T(C),\]
where the right vertical morphism $\Phi$ provides a free resolution $P_i$ of $\mathcal{F}_n$.

The first statement now follows from the isomorphism
\[H^2(C, \mathcal{F}_n) \cong \frac{\text{Hom}_{\mathbb{Z}[T(C)]}(\mathbb{Z}[T(C)], h, \mathcal{F}_n)}{\text{Image}(\delta_2^*)}.

Suppose first that $\mathcal{F}_n = \mathbb{Z}/n$. We use the natural isomorphism $\mathbb{Z}/n \otimes \mathbb{Z}/n = \mathbb{Z}/n$ given by multiplication. Since $T(C)$ acts trivially on $\mathbb{Z}/n$, we have, by [3.4], that $\tau \cup \beta$ is given by the two-cocycle $c_{\tau, \beta} \in C^2(T(C), \mathbb{Z}/n)$ satisfying for all $g_1, g_2 \in T(C)$,
\[c_{\tau, \beta}(g_1, g_2) = -\tau(g_1)\beta(g_2).

Using $[g_1, g_2] = (1, g_1, g_2)$ and [3.4], we obtain that, as an element of $\text{Hom}_{\mathbb{Z}[T(C)]}(\mathbb{Z}[T(C)], h, \mathcal{F}_n)$, $\tau \cup \beta$ is given by
\[(\tau \cup \beta)(h) = -\tau(a_2)\beta(a_1) + \tau(1)\beta(a_1) - \tau(1)\beta(a_2) + \tau(a_1)\beta(a_2),

which implies [8.10]. The case [8.11] when $\mathcal{F}_n = \mu_n$ follows by using the natural isomorphism $\mathbb{Z}/n \otimes \mu_n \to \mu_n$ by exponentiation and that $T(C)$ also acts trivially on $\mu_n$.

Let now $\alpha \in H^1(C, \mathbb{Z}/n)_O$ and $\beta \in H^1(C, \mathcal{F}_n)_O$, where $\Gamma$ is defined as in [8.1] and $\mathcal{F}_n$ is as above. We want to use the induced complex $V_* = \text{Ind}_{T(C)}^\Gamma W_*$ to construct a particular free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\Gamma]$-module in order to provide a presentation of the cup product $\alpha \cup \beta \in H^2(C, \mathcal{F}_n)$.

We consider the following commutative diagram of induced complexes from $T(C)$ to $\Gamma$
\[
\begin{array}{ccc}
V_* &=& \text{Ind}_{T(C)}^\Gamma W_* \\
\Phi^{-1} &\downarrow & \Phi^{-1} \\
V_* &=& \text{Ind}_{T(C)}^\Gamma W_* \\
\end{array}
\]
where the right vertical morphism $\Phi - 1$ is injective with cokernel $\mathbb{Z}$. It follows that the mapping cone of the morphism
\[V_* = \text{Ind}_{T(C)}^\Gamma W_* \xrightarrow{\Phi^{-1}} V_* = \text{Ind}_{T(C)}^\Gamma W_*
\]
provides a free resolution $P_* \to \mathbb{Z}$ of $\mathbb{Z}$ as a $\mathbb{Z}[\Gamma]$-module. By [32.6], we have $P_i = V_{i-1} \oplus V_i$ with $i^{th}$ differential $\delta_{P,i} : P_i \to P_{i-1}$ given by
\[\delta_{P,i} = \begin{pmatrix}
-\delta_{V,i-1} & 0 \\
-(\Phi - 1) & \delta_{V,i}
\end{pmatrix}.

More precisely, we have that
\[P_* = \cdots \to P_3 = V_2 \xrightarrow{\delta_{P,3}} P_2 = V_1 \oplus V_2 \xrightarrow{\delta_{P,2}} P_1 = V_0 \oplus V_1 \xrightarrow{\delta_{P,1}} P_0 = V_0
\]
is concentrated in degrees 0, 1, 2 and 3 with differentials
\[
P_1 = \mathbb{Z}[\Gamma]e \oplus \mathbb{Z}[\Gamma]f_1 \oplus \mathbb{Z}[\Gamma]f_2 \xrightarrow{\delta_{P,1}} \mathbb{Z}[\Gamma]e \to (1 - \Phi)e,
\]
\[0 \oplus f_1 \oplus 0 \mapsto (a_1 - 1)e,
\]
\[0 \oplus 0 \oplus f_2 \mapsto (a_2 - 1)e.
\]
Lemma 8.4. These equalities, and in particular (8.17) - (8.19), can be found using again property (8.3) of the contracting homotopy $H_{r,m}$. Writing $A_* = B_{r,*}$ and $\partial_{A,m} = \partial_{r,m}$, we obtain a morphism of complexes

$$\begin{align*}
P_2 & = \hat{Z}[\Gamma]/f_1 \oplus \hat{Z}[\Gamma]/f_2 \oplus \hat{Z}[\Gamma]/h \xrightarrow{\delta_{p,2}} \hat{Z}[\Gamma]/e \oplus \hat{Z}[\Gamma]/f_1 \oplus \hat{Z}[\Gamma]/f_2 = P_1 \\
& \text{where} \\
& f_1 \oplus 0 \oplus 0 \implies (1 - a_1)e \oplus (1 - \Phi)f_1 \oplus 0 \\
& 0 \oplus f_2 \oplus 0 \implies (1 - a_2)e \oplus 0 \oplus (1 - \Phi)f_2 \\
& 0 \oplus 0 \oplus h \implies 0 \oplus (a_2 - 1)f_1 \oplus (1 - a_1)f_2,
\end{align*}$$

and

$$\begin{align*}
P_3 & = \hat{Z}[\Gamma]/h \xrightarrow{\delta_{p,3}} \hat{Z}[\Gamma]/f_1 \oplus \hat{Z}[\Gamma]/f_2 \oplus \hat{Z}[\Gamma]/h = P_2 \\
& \text{Writing} A_* = B_{r,*} \text{ and } \partial_{A,m} = \partial_{r,m}, \text{ we obtain a morphism of complexes}
\end{align*}$$

These equalities, and in particular (8.17) - (8.19), can be found using again property (8.3) of the contracting homotopy $H_{r,m}$.

Lemma 8.4. Let $F_n$ and $\hat{F}_n$ be as in Lemma 8.3. We have an isomorphism

$$\begin{align*}
H^2(C, F_n) & = \frac{\text{Hom}_{2[\Gamma]}(\hat{Z}[\Gamma]/f_1 \oplus \hat{Z}[\Gamma]/f_2 \oplus \hat{Z}[\Gamma]/h, \hat{F}_n)}{\text{Image}(\delta_{p,2})}
\end{align*}$$

where

$$\delta_{p,2} : \text{Hom}_{2[\Gamma]}(\hat{Z}[\Gamma]/e \oplus \hat{Z}[\Gamma]/f_1 \oplus \hat{Z}[\Gamma]/f_2, \hat{F}_n) \to \text{Hom}_{2[\Gamma]}(\hat{Z}[\Gamma]/f_1 \oplus \hat{Z}[\Gamma]/f_2 \oplus \hat{Z}[\Gamma]/h, \hat{F}_n)$$

is given by precomposition with $\delta_{p,2}$. Suppose $\alpha \in H^1(C, Z/n) = \text{Hom}(\Gamma, Z/n)$ and $\beta \in H^1(C, F_n)$ is given by

$$\begin{align*}
\begin{cases}
\beta \in \text{Hom}(\Gamma, Z/n) & \text{if } F_n = Z/n, \\
[c_b] \in H^1(\Gamma, \hat{\mu}_n) & \text{if } F_n = \mu_n,
\end{cases}
\end{align*}$$

where $b \in D(C)$, $b^{1/n}$ is a chosen $n^{th}$ root of $b$ in $k(\Gamma)$, and $c_b \in C^1(\Gamma, \hat{\mu}_n)$ is the one-cocycle defined by $c_b(\gamma) = \gamma(b^{1/n})/b^{1/n}$ for all $\gamma \in \Gamma$. Then $\alpha \cup \beta \in H^2(C, F_n)$ is represented by the homomorphism $\xi_{\alpha, \beta} \in \text{Hom}_{2[\Gamma]}(\hat{Z}[\Gamma]/f_1 \oplus \hat{Z}[\Gamma]/f_2 \oplus \hat{Z}[\Gamma]/h, \hat{F}_n)$ given by

$$\begin{align*}
\begin{cases}
\xi_{\alpha, \beta}(f_1 \oplus 0 \oplus 0) = \alpha(\Phi) \beta(a_1) - \alpha(a_1) \beta(\Phi), \\
\xi_{\alpha, \beta}(0 \oplus f_2 \oplus 0) = \alpha(\Phi) \beta(a_2) - \alpha(a_2) \beta(\Phi), \\
\xi_{\alpha, \beta}(0 \oplus 0 \oplus h) = \alpha(a_1) \beta(a_2) - \alpha(a_2) \beta(a_1),
\end{cases}
\end{align*}$$

and by

$$\begin{align*}
\begin{cases}
\xi_{\alpha, \beta}(f_1 \oplus 0 \oplus 0) = c_b(a_1)q^{\alpha(\Phi)} \cdot c_b(\Phi^{-1})q^{\alpha(a_1)}, \\
\xi_{\alpha, \beta}(0 \oplus f_2 \oplus 0) = c_b(a_2)q^{\alpha(\Phi)} \cdot c_b(\Phi^{-1})q^{\alpha(a_2)}, \\
\xi_{\alpha, \beta}(0 \oplus 0 \oplus h) = c_b(a_1)a_1 \cdot c_b(a_1) - a_2),
\end{cases}
\end{align*}$$
Proof. Since $n = \ell^k$, we have $H^2(C, \mathcal{F}_n) = H^2(\Gamma, \mathcal{F}_n)$ by Lemma 8.21 which implies the isomorphism $\mathbf{(8.20)}$. If $\mathcal{F}_n = \mu_n$, then $H^1(C, \mu_n) = D(C)/(k(C)^*)^n$. By Lemma 8.3 if $\beta \in H^1(C, \mu_n)$ then there exists an element $b \in D(C)$ such that $\beta = [c_b]$ for the one-cocycle $c_b \in C^2(\Gamma, \mu_n)$ given in the statement of Lemma 8.3.

Suppose first that $\mathcal{F}_n = \mathbb{Z}/n$. As above, we use the natural isomorphism $\mathbb{Z}/n \otimes \mathbb{Z}/n = \mathbb{Z}/n$ given by multiplication. Since $T_2(C)$ and $\Phi$ act trivially on $\mathbb{Z}/n$, so does $\Gamma$. Therefore, we have, by $\mathbf{(8.4)}$, that $\alpha \cup \beta \in H^2(C, \mathbb{Z}/n)$ is represented by the two-cocycle $c_{\alpha,\beta} \in C^2(\Gamma, \mathbb{Z}/n)$ satisfying for all $\gamma_1, \gamma_2 \in \Gamma$,

$$c_{\alpha,\beta}([\gamma_1 | \gamma_2]) = -\alpha(\gamma_1)\beta(\gamma_2).$$

Using $[\gamma_1 | \gamma_2] = (1, \gamma_1, \gamma_1\gamma_2)$ and $\mathbf{(8.17)} - \mathbf{(8.19)}$, we obtain that $\alpha \cup \beta$ is represented by the homomorphism $\xi_{\alpha,\beta} \in \text{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}[\Gamma] f_1 \oplus \mathbb{Z}[\Gamma] f_2 \oplus \mathbb{Z}[\Gamma] h, \mathbb{Z}/n)$ given by

\begin{align}
(8.24) \quad \xi_{\alpha,\beta}(f_1 + 0 + 0) &= -\alpha(\Phi)\beta(\Phi^{-1}) + (a_1\Phi)\beta(\Phi^{-1}) - (1)\beta(a_1) + \alpha(\Phi)\beta(a_1) \\
&= \alpha(\Phi)\beta(\Phi) - (a_1)\beta(a_1) - \alpha(\Phi)\beta(\Phi) + (a_1)\beta(a_1), \\
(8.25) \quad \xi_{\alpha,\beta}(0 + f_2 + 0) &= -\alpha(\Phi)\beta(\Phi^{-1}) + (a_2\Phi)\beta(\Phi^{-1}) - (1)\beta(a_2) + \alpha(\Phi)\beta(a_2) \\
&= \alpha(\Phi)\beta(\Phi) - (a_2)\beta(a_2) - \alpha(\Phi)\beta(a_2) + (a_2)\beta(a_2), \\
(8.26) \quad \xi_{\alpha,\beta}(0 + 0 + h) &= -\alpha(\Phi)\beta(a_1) + (1)\beta(a_1) - (a_1)\beta(a_2) + (a_1)\beta(a_2) \\
&= -\alpha(\Phi)\beta(a_1) + (a_1)\beta(a_2),
\end{align}

where the second equalities in $\mathbf{(8.24)} - \mathbf{(8.26)}$ follow since $\alpha$ and $\beta$ are group homomorphisms. Therefore we obtain $\mathbf{(8.22)}$.

Suppose next that $\mathcal{F}_n = \mu_n$. As above, we use the natural isomorphism $\mathbb{Z}/n \otimes \mathbb{Z}/n = \mathbb{Z}/n$ given by exponentiation. We have that $T_2(C)$ acts trivially on $\mu_n$, and that $\Phi$ acts on $\mu_n$ as the $q$th power map. Therefore, we have, by $\mathbf{(8.4)}$, that $\alpha \cup \beta \in H^2(C, \mu_n)$ is represented by the two-cocycle $c_{\alpha,\beta} \in C^2(\Gamma, \mu_n)$ satisfying for all $\gamma_1, \gamma_2 \in \Gamma$,

$$c_{\alpha,\beta}([\gamma_1 | \gamma_2]) = \gamma_1 c_b(\gamma_2)^{-\alpha(\gamma_1)}.$$

Using $[\gamma_1 | \gamma_2] = (1, \gamma_1, \gamma_1\gamma_2)$ and $\mathbf{(8.17)} - \mathbf{(8.19)}$, we obtain that $\alpha \cup \beta$ is represented by the homomorphism $\xi_{\alpha,\beta} \in \text{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}[\Gamma] f_1 \oplus \mathbb{Z}[\Gamma] f_2 \oplus \mathbb{Z}[\Gamma] h, \mu_n)$ given by

\begin{align}
(8.27) \quad \xi_{\alpha,\beta}(f_1 + 0 + 0) &= (\Phi c_b(\Phi^{-1}) - \alpha(\Phi)) \cdot (a_1 \Phi c_b(\Phi^{-1}))^{\alpha(a_1)\Phi} \cdot (c_b(a_1))^{\alpha(1)\Phi} \\
&= (\Phi c_b(\Phi^{-1}) - \alpha(\Phi)) \cdot (\Phi c_b(\Phi^{-1}))^{\alpha(a_1)\Phi} \cdot (\Phi c_b(\Phi^{-1}))^{\alpha(1)\Phi} \\
&= c_b(\Phi^{-1})^{\alpha(a_1)} \cdot c_b(a_1)^{\alpha(1)}, \\
(8.28) \quad \xi_{\alpha,\beta}(0 + f_2 + 0) &= (\Phi c_b(\Phi^{-1}) - \alpha(\Phi)) \cdot (a_2 \Phi c_b(\Phi^{-1}))^{\alpha(a_2)\Phi} \cdot (c_b(a_2))^{\alpha(1)\Phi} \\
&= (\Phi c_b(\Phi^{-1}) - \alpha(\Phi)) \cdot (\Phi c_b(\Phi^{-1}))^{\alpha(a_2)\Phi} \cdot (\Phi c_b(\Phi^{-1}))^{\alpha(1)\Phi} \\
&= c_b(\Phi^{-1})^{\alpha(a_2)} \cdot c_b(a_2)^{\alpha(1)}, \\
(8.29) \quad \xi_{\alpha,\beta}(0 + 0 + h) &= (a_2 c_b(\Phi^{-1}))^{\alpha(a_2)} \cdot (c_b(a_2))^{\alpha(1)} \\
&= (a_2 c_b(\Phi^{-1}))^{\alpha(a_2)} \cdot (c_b(a_2))^{\alpha(1)},
\end{align}

where the second equalities in $\mathbf{(8.27)} - \mathbf{(8.29)}$ follow since $\alpha$ is a group homomorphism and since $a_1$ and $a_2$ act trivially on $\mu_n$. Therefore we obtain $\mathbf{(8.23)}$.

Proof of Theorem 7.2. Let $\alpha \in H^1(C, \mathbb{Z}/n)_0$ and let $\beta \in H^1(C, \mathcal{F}_n)_0$. We view $\alpha \in \text{Hom}(\Gamma, \mathbb{Z}/n)$. By Lemma 8.3(i), we have $\alpha(\Phi) = \alpha(\Phi^{-1}) = 0$.

Suppose first that $\mathcal{F}_n = \mathbb{Z}/n$. In this case, we use again Lemma 8.3(i) to view $\beta \in \text{Hom}(\Gamma, \mathbb{Z}/n)$ with $\beta(\Phi) = \beta(\Phi^{-1}) = 0$. By Lemma 8.4 $\alpha \cup \beta \in H^2(C, \mathbb{Z}/n)$ is represented by the homomorphism
Let $\xi_{\alpha,\beta} \in \text{Hom}_{\mathbb{Z}[\Gamma]}(\hat{\mathbb{Z}}[\Gamma]/f_1 + \hat{\mathbb{Z}}[\Gamma]/f_2 + \hat{\mathbb{Z}}[\Gamma]/h, \mathbb{Z}/n)$ be given by (8.22). Since $\alpha(\Phi) = \beta(\Phi^{-1}) = 0$ and since $\bar{\alpha}$ and $\bar{\beta}$ are the restrictions of $\alpha$ and $\beta$, respectively, to $T_\ell(C)$, we obtain

$$\begin{align*}
\xi_{\alpha,\beta}(f_1 + 0 + 0) &= 0, \\
\xi_{\alpha,\beta}(0 + f_2 + 0) &= 0, \\
\xi_{\alpha,\beta}(0 + 0 + h) &= \bar{\alpha}(a_1)\bar{\beta}(a_2) - \bar{\alpha}(a_2)\bar{\beta}(a_1).
\end{align*}$$

Therefore, (8.11) implies that if $\bar{\alpha}, \bar{\beta} : C \to \mathbb{Z}/n$ is trivial in $H^1(\mathbb{C}, \mathbb{Z}/n)$ then $\xi_{\alpha,\beta}$ is also trivial. This implies Theorem 7.2 when $F_n = \mathbb{Z}/n$.

Suppose next that $F_n = \mathbb{Z}/n$. By Lemma 8.3(ii), there exists an element $b \in D(C)$ that is normalized at $O$ and an $n$th root $b^{1/n}$ of $b$ in the completion $k(C)_O$ at $O$ such that $\beta$ is represented by the one-cocycle $c_b \in C^1(\Gamma, \hat{\mu}_n)$ given by $c_b(\gamma) = \gamma(b^{1/n})/b^{1/n}$ for all $\gamma \in \Gamma$. Moreover, $c_b(\Phi) = c_b(\Phi^{-1}) = 1$. Since $T_\ell(C)$ acts trivially on $\hat{\mu}_n$, it follows that the restriction of $c_b$ to $T_\ell(C)$ defines a homomorphism $T_\ell(C) \to \hat{\mu}_n$ that is independent of the choice of $n$th root of $b$ and that is therefore equal to the restriction $\beta$ of $\hat{\beta}$ to $H^1(\mathbb{C}, \mu_n)$.

By Lemma 8.3, $\alpha \cup \beta \in H^2(C, \mu_n)$ is represented by the homomorphism $\xi_{\alpha,\beta} \in \text{Hom}_{\mathbb{Z}[\Gamma]}(\hat{\mathbb{Z}}[\Gamma]/f_1 + \hat{\mathbb{Z}}[\Gamma]/f_2 + \hat{\mathbb{Z}}[\Gamma]/h, \hat{\mu}_n)$ given by (8.23). Since $\alpha(\Phi) = 0$ and $c_b(\Phi^{-1}) = 1$, and since $T_\ell(C)$ acts trivially on $\hat{\mu}_n$, we obtain

$$\begin{align*}
\xi_{\alpha,\beta}(f_1 + 0 + 0) &= 1, \\
\xi_{\alpha,\beta}(0 + f_2 + 0) &= 1, \\
\xi_{\alpha,\beta}(0 + 0 + h) &= \bar{\beta}(a_2)\bar{\alpha}(a_1) - \bar{\beta}(a_1)\bar{\alpha}(a_2).
\end{align*}$$

Therefore, (8.11) implies that if $\bar{\alpha}, \bar{\beta} : C \to \mathbb{Z}/n$ is trivial in $H^1(\mathbb{C}, \mu_n)$ then $\xi_{\alpha,\beta}$ is also trivial. This implies Theorem 7.2 when $F_n = \mathbb{Z}/n$.

Remark 8.5. Lemma 8.4 can also be used to get different presentations for the cup product when one of the arguments lies in $H^1(k, \mathbb{Z}/n)$ or $H^1(k, \mathbb{Z}/n)$, respectively.

(i) Suppose $\alpha \in H^1(k, \mathbb{Z}/n)$ and $\beta \in H^1(C, \mathbb{Z}/n)$. Then $\alpha(a_1) = \alpha(a_2) = 0$. Therefore, the homomorphism $\xi_{\alpha,\beta} \in \text{Hom}_{\mathbb{Z}[\Gamma]}(\hat{\mathbb{Z}}[\Gamma]/f_1 + \hat{\mathbb{Z}}[\Gamma]/f_2 + \hat{\mathbb{Z}}[\Gamma]/h, \hat{\mu}_n)$ given by (8.23) satisfies

$$\begin{align*}
\xi_{\alpha,\beta}(f_1 + 0 + 0) &= \bar{\beta}(a_1)^{q(\beta)}(\alpha(\Phi)), \\
\xi_{\alpha,\beta}(0 + f_2 + 0) &= \bar{\beta}(a_2)^{q(\beta)}(\alpha(\Phi)), \\
\xi_{\alpha,\beta}(0 + 0 + h) &= 1.
\end{align*}$$

Here the first two equalities follow, since the restriction of $c_b$ to $T_\ell(C)$ defines a homomorphism $T_\ell(C) \to \hat{\mu}_n$ that is independent of the choice of $n$th root of $b$ and that is therefore equal to the restriction $\beta$ of $\hat{\beta}$ to $H^1(C, \mu_n) = \text{Hom}(T_\ell(C), \hat{\mu}_n)$.

(ii) Suppose $\alpha \in H^1(C, \mathbb{Z}/n)$ and $\beta \in H^1(k, \mathbb{Z}/n)$. Then $\beta$ is represented by a one-cocycle $c_b$ associated to an element $b \in k$, which means that any choice of $n$th root of $b$ in $k(C)$ lies in $\overline{k}$. Hence $c_b(a_1) = c_b(a_2) = 1$. Therefore, the homomorphism $\xi_{\alpha,\beta} \in \text{Hom}_{\mathbb{Z}[\Gamma]}(\hat{\mathbb{Z}}[\Gamma]/f_1 + \hat{\mathbb{Z}}[\Gamma]/f_2 + \hat{\mathbb{Z}}[\Gamma]/h, \hat{\mu}_n)$ given by (8.23) satisfies

$$\begin{align*}
\xi_{\alpha,\beta}(f_1 + 0 + 0) &= c_b(\Phi^{-1})^q(\beta(a_1)), \\
\xi_{\alpha,\beta}(0 + f_2 + 0) &= c_b(\Phi^{-1})^q(\beta(a_2)), \\
\xi_{\alpha,\beta}(0 + 0 + h) &= 1.
\end{align*}$$

Here the first two equalities follow, since the restriction of $\alpha$ to $T_\ell(C)$ is equal to $\bar{\beta}$.

In both cases (i) and (ii), $\xi_{\alpha,\beta}$ induces a homomorphism in $\text{Hom}_{\mathbb{Z}[\Gamma]}(\hat{\mathbb{Z}}[\Gamma]/f_1 + \hat{\mathbb{Z}}[\Gamma]/f_2, \hat{\mu}_n)$. Since $T_\ell(C)$ acts trivially on $\hat{\mu}_n$, it follows that the class of $\xi_{\alpha,\beta}$ in the quotient (8.24) defines an element in the quotient group

$$\frac{\text{Hom}_{\mathbb{Z}[\Gamma]}(\hat{\mathbb{Z}}[T_\ell(C)]/f_1 + \hat{\mathbb{Z}}[T_\ell(C)]/f_2, \hat{\mu}_n)}{(\Phi - 1)\text{Hom}_{\mathbb{Z}[\Gamma]}(\hat{\mathbb{Z}}[T_\ell(C)]/f_1 + \hat{\mathbb{Z}}[T_\ell(C)]/f_2, \hat{\mu}_n)} = \frac{H^1(\mathbb{C}, \mu_n)}{(\Phi - 1)H^1(\mathbb{C}, \mu_n)}.$$
which equals $H^1(\text{Gal}(\overline{k}/k), H^1(\overline{C}, \mu_n))$. On the other hand, at least one of $\alpha$ or $\beta$ has trivial restriction to $\overline{C}$, so $\alpha \cup \beta$ restricts to 0 in $H^2(\overline{C}, \mu_n)$. Thus the above expression for $\xi_{\alpha, \beta}$ is consistent with the split exact sequence

$$0 \to H^1(\text{Gal}(\overline{k}/k), H^1(\overline{C}, \mu_n)) \to H^2(C, \mu_n) \to H^0(\text{Gal}(\overline{k}/k), H^2(\overline{C}, \mu_n)) \to 0$$

that results from the degeneration of the spectral sequence

$$H^p(\text{Gal}(\overline{k}/k), H^q(\overline{C}, \mu_n)) \Rightarrow H^{p+q}(C, \mu_n).$$

This gives another approach to computing $\alpha \cup \beta$ than the Legendre transform method in Theorems 5.3 and 5.5.

**Remark 8.6.** Since the point $O$ has degree $d(O)$ prime to $\ell \cdot (\ell - 1)$, there is a canonical splitting of the sequence (8.30) sending $d(O) \in \mathbb{Z}/n = H^2(\overline{C}, \mu_n)_{\text{Gal}(\overline{k}/k)}$ to the class $[O]$ of $O$ in $H^2(C, \mu_n) = \text{Pic}(C)/n \cdot \text{Pic}(C)$. Theorem 4.4 shows that if $\alpha \in H^1(C, \mathbb{Z}/n)O$ and $\beta \in H^1(C, \mu_n)O$, then $\alpha \cup \beta$ lies in the image of this splitting.

### 9. An infinite family of genus two examples

The goal of this section is to construct an infinite family of curves of genus 2 for which the equivalent conditions of Theorem 4.3 do not hold when $n = \ell = 3$. We introduce the following notation.

**Notation 9.1.** Let $\zeta \in \mathbb{C}$ be a primitive cube root of unity, and let $F = \mathbb{Q}(\zeta) = \mathbb{Q}(\sqrt{-3})$.

(a) Let $E$ be the elliptic curve over $F$ defined by the affine equation $y^2 = x^3 - 3$, and let $\pi : E \to \mathbb{P}^1_F$ be the morphism associated to the inclusion of function fields $F(x) \subset F(E) = F(x)[y]/(y^2 - x^3 + 3)$.

(b) Let $Y$ be the smooth projective curve with function field $F(E)(x^{1/2})$, and let $\theta : Y \to E$ be the natural morphism associated to the containment $F(E) \subset F(Y)$.

(c) Define $\tilde{y} = y/x^{3/2}$ and $w = x^{-1}$. Let $E'$ be the elliptic curve over $F$ defined by $\tilde{y}^2 = 1 - 3w^3$, and let $\pi' : E' \to \mathbb{P}^1_F$ be the morphism associated to the inclusion of function fields $F(x) = F(w) \subset F(E') = F(w)[\tilde{y}]/(\tilde{y}^2 - 1 + 3w^3)$. Moreover, let $\theta' : Y \to E'$ be the natural morphism associated to the containment $F(E') \subset F(Y)$.

(d) If $N$ is a finite extension of $F$, we define $Y_N = N \otimes_F Y$, $E_N = N \otimes_F E$ and $E'_N = N \otimes_F E'$.

We collect some obvious properties of $Y$, $E$ and $E'$ in the following remark.

**Remark 9.2.** Both $\theta$ and $\theta'$ are covers of degree 2. It follows that $F(Y)$ is a biquadratic extension of $F(x) = F(w)$ with intermediate fields $F(E)$, $F(x^{1/2})$ and $F(E')$. Moreover, $\theta$ is ramified over the points $Q_1 = (0, \sqrt{-3})$ and $Q_2 = (0, -\sqrt{-3})$ in $(x, y)$ coordinates on $E$, so $Y$ has genus 2 by the Hurwitz formula. The point $x = \infty$ on $\mathbb{P}^1_F$ is a branch point of $\pi : E \to \mathbb{P}^1_F$ and splits under $\pi' : E' \to \mathbb{P}^1_F$. So there are two points $0_Y$ and $0'_Y$ of $Y$ over $x = \infty$ on $\mathbb{P}^1_F$.

**Lemma 9.3.** Let $N$ be a finite extension of $F$. The direct image homomorphisms $\theta_* : \text{Pic}^0(Y_N) \to \text{Pic}^0(E_N)$ and $\theta'_* : \text{Pic}^0(Y'_N) \to \text{Pic}^0(E'_N)$ give a homomorphism

$$\theta_* \times \theta'_* : \text{Pic}^0(Y_N) \to \text{Pic}^0(E_N) \times \text{Pic}^0(E'_N)$$

whose kernel and cokernel are finite groups annihilated by 2.

**Proof.** We have pullback maps $\theta^* : \text{Pic}^0(E_N) \to \text{Pic}^0(Y_N)$ and $\theta'^* : \text{Pic}^0(E'_N) \to \text{Pic}^0(Y'_N)$ such that $\theta_* \circ \theta^*$ and $\theta'_* \circ \theta'^*$ are multiplication by 2. Furthermore, $\theta_* \circ \theta^* = \pi_* \circ \pi_*$ and $\theta'_* \circ \theta'^* = \pi'_* \circ \pi'_*$ are trivial since $\text{Pic}^0(\mathbb{P}^1_N)$ is trivial. Hence the composition

$$(\theta_* \times \theta'_*) \circ (\theta^* \times \theta'^*) : \text{Pic}^0(E_N) \times \text{Pic}^0(E'_N) \to \text{Pic}^0(E_N) \times \text{Pic}^0(E'_N)$$

is multiplication by 2. Since all the groups appearing in (9.1) are finitely generated abelian groups, it will suffice to show that the kernel $\mathcal{K}$ of $\theta_* \times \theta'_*$ is annihilated by 2. View $Y_N$ as a Galois cover of $\mathbb{P}^1_N$ with Galois group $G$ a Klein four group. The three intermediate quadratic covers are $E_N$, $E'_N$.
$E'_N$ and the projective line over $N$ with function field $N(x^{1/2})$. Since the latter projective line has trivial Jacobian, we see $K$ is annihilated by the group ring element $1 + \sigma$ for each non-trivial $\sigma \in G$. The sum of these three group ring elements is $2 + \text{Trace}_G$. Hence $2$ annihilates $K$ because the action of $\text{Trace}_G$ on $\text{Pic}^0(Y_N)$ factors through $\text{Pic}^0(\mathbb{P}^1_N) = 0$.

The following lemma is clear from the functorial properties of Jacobians.

**Lemma 9.4.** Let $X$ be a geometrically integral curve of genus 1 over a perfect field $K$, and suppose $0_X$ is an arbitrary point of $X(K)$. Then $X$ becomes an elliptic curve with origin $0_X$ via the morphism from $X$ to its Jacobian sending $0_X$ to the origin.

(i) Suppose $1 \leq n \in \mathbb{Z}$. The $n$-torsion of $\text{Pic}^0(X)$ has order $n^2$ if and only if the set of elements $P \in X(K)$ such that $nP = 0_X$ with respect to the group law on $X$ has order $n^2$.

(ii) Suppose $P_1, P_2 \in X(K)$. There is a class $\theta \in \text{Pic}^0(X)$ such that $n\theta = [P_1] - [P_2]$ in $\text{Pic}^0(X)$ if and only if there is a point $P \in X(K)$ such that $nP = P_1 - P_2$ with respect to the group law on $X$.

We now return to our curves $E$, $E'$ and $Y$ from Notation 9.1

**Lemma 9.5.** Let $N$ be a finite extension of $F$.

(i) The 3-torsion of $\text{Pic}^0(E_N)$ has order 9 if and only if $F(12^{1/3}) \subset N$.

(ii) The 3-torsion of $\text{Pic}^0(E'_N)$ has order 9 if and only if $F((4/3)^{1/3}) \subset N$.

(iii) Let $0_Y$ and $0'_Y$ be the two points of $Y$ defined in Remark 9.2. Then $\{0_Y, 0'_Y\}$ is the inverse image under $\theta : Y \to E$ of the point $0_E$ at infinity on the curve $E : y^2 = x^3 - 3$. We can label these points so that under $\theta' : Y \to E'$ one has $\theta'(0_Y) = (0, 1) = P_1$ and $\theta'(0'_Y) = (0, -1) = P_2$ relative to the $(w, y)$ coordinates of the curve $E' : \tilde{y}^2 = 1 - 3w^3$. The divisor class $[P_1] - [P_2]$ in $\text{Pic}^0(E'_N)$ lies in $3 \cdot \text{Pic}^0(E'_N)$ if and only if $F((4/3)^{1/3}, \zeta^{1/3}) \subset N$.

**Proof.** The complex multiplication of $\zeta \in F$ on $E : y^2 = x^3 - 3$ is defined by $\zeta \cdot (x, y) = (\zeta x, y)$, while $-(x, y) = (x, -y)$ in the group law of $E$. Thus $(x, y) = (0, \sqrt[3]{-3})$ and $(x, y) = (0, -\sqrt[3]{-3})$ are fixed by $\zeta$ and therefore sent to 0 in the group law of $E$ by $(\zeta - 1)$. Since $(\zeta^2 - 1)(\zeta - 1) = 3$ in $F$, these points are 3-torsion points of $E$ over $\mathbb{Q}(\sqrt[3]{-3}) = F$. To find the remaining 3-torsion points over $\overline{F}$, we just need a point $(x, y) \in E(\overline{F})$ such that $(\zeta^2 - 1)(x, y) = (0, \sqrt[3]{-3})$. Here $\zeta^2(x, y) = (\zeta^2 x, y), -(x, y) = (x, -y)$ and $-(0, \sqrt[3]{-3}) = (0, -\sqrt[3]{-3})$. So we are looking for a line of the form $cx + dy - e = 0$ containing three points of the form $(\zeta^2 x, y), (x, -y)$ and $(0, -\sqrt[3]{-3})$. Furthermore, one should not have $x = 0$, since such a solution leads to the previous points $(0, \sqrt[3]{-3})$ and $(0, -\sqrt[3]{-3})$. Writing out the constraints on $c, d$ and $e$ we find $d(-\sqrt[3]{-3}) = e$ and that there is a matrix equation

$$
\begin{pmatrix}
\zeta^2 x & y + \sqrt[3]{-3} \\
x & -y + \sqrt[3]{-3}
\end{pmatrix} \cdot \begin{pmatrix}
c \\
d
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
$$

Since $c$ and $d$ are not both 0 we find from this that

$$
0 = \det \left( \begin{array}{cc}
\zeta^2 x & y + \sqrt[3]{-3} \\
x & -y + \sqrt[3]{-3}
\end{array} \right) = y(-\zeta^2 x - x) + \sqrt[3]{-3}(\zeta^2 - 1)x.
$$

Since $x \neq 0$ and $\zeta^2 + 1 = -\zeta$ we can divide by $x$ to have

$$
y = -\zeta^2 \sqrt[3]{-3}(\zeta^2 - 1).
$$

On squaring we get

$$
x^3 - 3 = y^2 = -3\zeta(\zeta^2 - 1)^2 = -3\zeta(\zeta^4 - 2\zeta^2 + 1) = -3(\zeta^5 - 2\zeta^3 + \zeta) = -3(\zeta^2 - 2 + \zeta) = 9.
$$

Thus

$$
x^3 = 12 \quad \text{and} \quad y = \pm 3
$$

which shows part (i) of Lemma 9.5.

For part (ii) we proceed similarly using the model $E' : \tilde{y}^2 = 1 - 3w^3$, with complex multiplication defined by $\zeta(w, \tilde{y}) = (\zeta w, \tilde{y})$ and $-(w, \tilde{y}) = (w, -\tilde{y})$. As the origin of $E'$ we will use the point $0_{E'}$
at infinity relative to the above affine \((w, \tilde{y})\) model of \(E'\). Note that because of Lemma 9.4 this does not make a difference as far as part (ii) of Lemma 9.5 is concerned. The points \((w, \tilde{y}) = (0,1)\) and \((0,-1)\) are fixed by \(\zeta\) and hence annihilated by \(\zeta - 1\), so they are 3-torsion points. To find the remaining 3-torsion points, we look for a \((w_0, \tilde{y}_0)\) with \((\zeta^2 - 1) \cdot (w_0, \tilde{y}_0) = (0,1)\). Thus we need a line \(cw_0 + d\tilde{y}_0 - e = 0\) containing the points \((\zeta^2w_0, \tilde{y}_0), (w_0, -\tilde{y}_0)\) and \(-(0,1) = (0,-1)\). This implies \(-d - e = 0\) so \(-e = d\) and
\[
\begin{pmatrix}
\zeta^2 w_0 & \tilde{y}_0 + 1 \\
w_0 & -\tilde{y}_0 + 1
\end{pmatrix}
\cdot
\begin{pmatrix}
c \\
d
\end{pmatrix}
= \begin{pmatrix}0 \\
0
\end{pmatrix}.
\]
Since \(c\) and \(d\) are not both 0, this gives
\[
0 = \det
\begin{pmatrix}
\zeta^2 w_0 & \tilde{y}_0 + 1 \\
w_0 & -\tilde{y}_0 + 1
\end{pmatrix}
= \tilde{y}_0(-\zeta^2w_0 - w_0) + (\zeta^2 - 1)w_0.
\]
The solution we are looking for does not have \(w_0 = 0\), so we can divide by \(w_0\) and then square to find
\[
1 - 3w_0^3 = \tilde{y}_0^2 = \left(\frac{\zeta^2 - 1}{\zeta^2 + 1}\right)^2 = \zeta^{-2}(\zeta^4 - 2\zeta^2 + 1) = (\zeta^2 - 2 + \zeta) = -3.
\]
Thus
\[
w_0^3 = 4/3 \quad \text{and} \quad \tilde{y}_0 = \pm \frac{\zeta^2 - 1}{\zeta^2 + 1} = \mp \sqrt{-3} \in F
\]
where we set \(\sqrt{-3} = 2\zeta + 1\), which leads to part (ii) of Lemma 9.5.

Finally, for part (iii), note that we have shown \(P_1 = (0,1)\) and \(P_2 = (0,-1)\) are 3-torsion points on \(E'\) with \(P_2 = -P_1\), so \(P_1 - P_2 = 2P_1\) and \(P_1 = 2 \cdot (2P_1)\). So in view of Lemma 9.4 it will suffice to determine the extension of \(F\) generated by the coordinates of a point \(Q\) with \(3Q = P_1 = (0,1)\) when the group law on \(E'\) is the one coming from the map to the Jacobian that sends the point \(0_{E'}\) at infinity on \(E'\) to the origin. We have found above a point \((w_0, \tilde{y}_0)\) with \((\zeta^2 - 1)(w_0, \tilde{y}_0) = (0,1)\). This point has \(w_0^3 = 4/3\) and \(\tilde{y}_0\) a particular square root of \(-3\) depending on \(\zeta\). So we now look for a point \(Q = (w_1, \tilde{y}_1)\) with \((\zeta - 1)(w_1, \tilde{y}_1) = (w_0, \tilde{y}_0)\); then \(3Q = (\zeta^2 - 1)(\zeta - 1)(w_1, \tilde{y}_1) = (0,1)\). Here \(\zeta(w_1, \tilde{y}_1) = (\zeta w_1, \tilde{y}_1)\) and \(-(w_1, \tilde{y}_1) = (w_1, -\tilde{y}_1)\) and \(-(w_0, \tilde{y}_0) = (w_0, -\tilde{y}_0)\). Hence we are looking for a line
\[
(c(w - w_0) + d(\tilde{y} + \tilde{y}_0) - e = 0
\]
containing \((\zeta w_1, \tilde{y}_1), (w_1, -\tilde{y}_1)\) and \((w_0, -\tilde{y}_0)\). Here if \(w_1 = w_0\) then \((w_1, \tilde{y}_1) = \pm (w_0, \tilde{y}_0)\) and
\[
(\zeta - 1)(w_1, \tilde{y}_1) = \pm (\zeta - 1)(w_0, \tilde{y}_0) = \mp (\zeta^2 - 1)(w_0, \tilde{y}_0) = \mp (0,1) = (0, \mp 1) \neq (w_0, \tilde{y}_0).
\]
Thus we can assume \(w_1 \neq w_0\), and similarly we can assume \(\zeta w_1 \neq w_0\), so the three points \((\zeta w_1, \tilde{y}_1), (w_1, -\tilde{y}_1)\) and \((w_0, -\tilde{y}_0)\) are distinct. In order for the point \((w_0, \tilde{y}_0)\) to be on the line
\[
(w_1 - w_0, \tilde{y}_1 + \tilde{y}_0), \quad (w_1 - w_0, -\tilde{y}_1 + \tilde{y}_0)
\]
we must have \(-e = 0\). The remaining two points \((\zeta w_1, \tilde{y}_1), (w_1, -\tilde{y}_1)\) are on the line if and only if
\[
\begin{pmatrix}
\zeta w_1 - w_0 & \tilde{y}_1 + \tilde{y}_0 \\
w_1 - w_0 & -\tilde{y}_1 + \tilde{y}_0
\end{pmatrix}
\cdot
\begin{pmatrix}
c \\
d
\end{pmatrix}
= \begin{pmatrix}0 \\
0
\end{pmatrix}.
\]
Since \(c\) and \(d\) are not both 0, we conclude
\[
0 = \det
\begin{pmatrix}
\zeta w_1 - w_0 & \tilde{y}_1 + \tilde{y}_0 \\
w_1 - w_0 & -\tilde{y}_1 + \tilde{y}_0
\end{pmatrix}
= (\zeta w_1 - w_0) \cdot (\tilde{y}_1 + \tilde{y}_0) - (\tilde{y}_1 + \tilde{y}_0) \cdot (w_1 - w_0)
= \tilde{y}_1((-\zeta - 1)w_1 + 2w_0) + (\zeta - 1)w_1\tilde{y}_0.
\]
Since \(-\zeta - 1 = \zeta^2\) we get
\[
(9.3) \quad \tilde{y}_1(\zeta^2 w_1 + 2w_0) = (1 - \zeta)w_1\tilde{y}_0
\]
so on squaring this we find
\[
(1 - 3w_0^3)(\zeta^2 w_1 + 2w_0)^2 = (1 - \zeta)^2 w_1^2(-3)
\]
Writing \((\zeta^2 w_1 + 2w_0)^2 = \zeta^4 w_1^2 + 4\zeta^2w_0w_1 + 4w_0^2\) we end up with an equality
\[
-3(\zeta^4 w_1^2 + 4\zeta^2w_0w_1 + 4w_0^2) + (\zeta^4 + 3(1 - \zeta)^2)w_1^2 + 4\zeta^2w_0w_1 + 4w_0^2 = 0.
\]
We know \( w_1 \neq w_0 \) and \( w_2 \neq \zeta^{-1}w_0 = \zeta^2w_0 \), so we divide the left hand side by \((w_1 - w_0)(w_1 - \zeta^2w_0) = w_1^2 - (1 + \zeta^4)w_0w_1 + \zeta^2w_0^2 = w_1^2 + \zeta w_0w_1 + \zeta^2w_0^2 \). This gives
\[-3\zeta w_1^3 - 9\zeta^2w_0w_1^2 + 4\zeta = 0.\]

Dividing by \(-3\zeta\) we obtain
\[(9.4) \quad w_1^3 + 3\zeta w_0w_1^2 - 4/3 = 0.\]

We now write down the roots of \((9.4)\) using Cardano’s formulas. Define
\[a = 3\zeta w_0, \quad b = 0, \quad c = -4/3,\]
and write
\[
\begin{align*}
r &= \frac{1}{3}(3b - a^2) = \frac{1}{3}(-9\zeta^2w_0^2) = -3\zeta^2w_0^2 = -3\zeta^2(4/3)^{2/3}, \\
s &= \frac{1}{27}(2a^3 - 9ab + 27c) = 2w_0^3 - 4/3 = 2(4/3) - 4/3 = 4/3, \\
D &= -4r^3 - 27s^2 = -4(-27w_0^6) - 27(4/3)^2 = 4 \cdot 27 \cdot (4/3)^2 - 27 \cdot (4/3)^2 = 4^23^2, \\
A^3 &= -\frac{27s}{2} + \frac{3}{2}\sqrt{3D} = -\frac{27}{4} + \frac{3}{2}\sqrt{-3342} = 36\zeta, \\
B^3 &= -\frac{27s}{2} - \frac{3}{2}\sqrt{3D} = 36\zeta^2.
\end{align*}
\]

The roots \( w_1 \) of \((9.4)\) are then
\[
\frac{A + B}{3}; \quad \frac{-\frac{\zeta^2A + \zeta B}{3}}{3}; \quad \frac{\zeta A + \zeta^2 B}{3}.
\]

We have from \((9.3)\) that
\[
\hat{y}_1/w_1 = (1 - \zeta)\hat{y}_0/(\zeta^2w_1 + 2w_0).
\]

Since \(\hat{y}_0 = \pm \sqrt{-3} \in F\), we conclude that \(w_1 \) and \(\hat{y}_1\) generate the same extension over \(F\) as \(w_1\) and \(w_0\) do. So the extension of \(F\) generated by \(w_1\) and \(\hat{y}_1\) is \(F((4/3)^{1/3}, \zeta^{1/3})\). This completes the proof of part (iii) of Lemma 9.3.\]

**Corollary 9.6.** Let \(E, E'\) and \(Y\) be the minimal projective models over \(\mathbb{Z}\) of the curves over \(\mathbb{Z}\) associated to the affine equations used to define \(E, E'\) and \(Y\). We assume \(q\) is a rational prime such that for \(C = E, E'\) or \(Y\), \(C\) has good reduction \(C_q\) modulo \(q\). Suppose \(q\) splits in the field \(N = F(4^{1/3}, 3^{1/3})\) but does not split in the extension \(N(\zeta^{1/3})\).

(i) The 3-torsion subgroup of \(\text{Pic}^0(Y_q)\) is isomorphic to \((\mathbb{Z}/3)^4 = (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/3)\).

(ii) The point \(0_{E_q}\) at infinity associated to the affine model \(E_q : \hat{y}^2 = x^3 - 3\) splits into two points \(0_{Y_q}\) and \(0'_{Y_q}\) via the morphism \(Y_q \to E_q \to \text{Pic}^0(Y_q)\) associated to the field embedding \((\mathbb{Z}/q)(E_q) \subset (\mathbb{Z}/q)(Y_q)\).

(iii) The divisor class \([0_{Y_q}] - [0'_{Y_q}]\) in \(\text{Pic}^0(Y_q)\) does not lie in \(3 \cdot \text{Pic}(Y_q)\).

**Proof.** By Lemma 9.5 the 3-torsion on the general fibers of \(E\) and \(E'\) is defined over \(N\), so these torsion points define sections of the natural morphisms \(E_N = \text{Spec}(O_N) \otimes \mathbb{Z} \to \text{Spec}(O_N)\) and \(E'_N = \text{Spec}(O_N) \otimes \mathbb{Z} \to \text{Spec}(O_N)\). Since \(q\) must be larger than 3, these sections specialize to distinct 3-torsion points of the fibers of \(E_N\) and \(E'_N\) over a prime of \(O_N\) over \(q\). Since \(q\) splits in \(O_N\), these fibers are isomorphic to \(E_q\) and \(E'_q\), respectively. This shows part (i) because the same arguments used in the proof of Lemma 9.5 show the natural direct image homomorphism \(\text{Pic}^0(Y_q) \to \text{Pic}^0(E_q) \times \text{Pic}^0(E'_q)\) has kernel and cokernel equal to finite abelian groups annihilated by 2. Moreover, part (ii) follows from the corresponding fact on the generic fibers of \(Y\) and \(E\).

Finally, for (iii), it suffices to show that the image \(\partial\) of \([0_{Y_q}] - [0'_{Y_q}]\) in \(\text{Pic}^0(E_q)\) does not lie in \(3 \cdot \text{Pic}^0(E_q)\). Setting \(w = x^{-1}\) defines the affine model \(E'_q : \hat{y}^2 = 1 - 3w^3\) when \(\hat{y} = y/x^{3/2}\). Let \(0_{E'_q}\) be the point at infinity for this model. Then \(\partial = [P_{1,q}] - [P_{2,q}]\) when \(P_{1,q} = (0, 1)\) and \(P_{2,q} = (0, -1)\) in \((u, \hat{y})\) coordinates. Here \([P_{2,q}] = -[P_{1,q}]\) in \(\text{Pic}^0(E'_q)\) and \((\zeta - 1)[P_{1,q}] = 0\) relative to the complex multiplication action of \(\mathbb{Z}[\zeta] = O_F\) on \(E'_q\) defined by \(\zeta(w, \hat{y}) = (\zeta w, \hat{y})\). So \([P_{1,q}]\) is a 3-torsion point.
when we use 0_{E'_q} as the origin of the group law of E'_q. Lemma 9.4 shows that d = [P_{1,q}] - [P_{2,q}] lies in 3 · Pic(E'_q) if and only if P_{1,q} - P_{2,q} = 2P_{1,q} lies in 3 · E'_q(Z/q) relative to the group law of E'_q(Z/q).

Since P_{1,q} is a 3-torsion point, this will be true if and only if P_{1,q} lies in 3 · E'_q(Z/q). However, P_{1,q} is the intersection of the fiber of E'_N over a chosen prime q of O_N over q with the corresponding point P_1 on the general fiber of E'_N, i.e., the point P_1 with (w, y) coordinates (0, 1). Multiplication by 3 defines an étale morphism of abelian schemes Z[\frac{1}{3}] \otimes E'_N \rightarrow Z[\frac{1}{3}] \otimes E'_N. The pullback of the section defined by P_1 is a divisor \mathcal{P} on Z[\frac{1}{3}] which is étale over this section. Therefore \mathcal{P} must be a disjoint union of divisors of the form Spec(O) in which O is the integral closure of Z[\frac{1}{3}] in the residue field L of a closed point P' of E'_N such that 3 · P' = P_1. By Lemma 9.5, L must contain N(\zeta^{1/3}), and by construction q is a prime of O_N inert to N(\zeta^{1/3}) and q is prime to 6. Any point of the fiber of E'_N over q which when multiplied by 3 gives P_{1,q} must lie on the intersection of \mathcal{P} with this fiber. Because N(\zeta^{1/3}) \subset L, there is no point of this intersection with residue field O_N/q = Z/q. This implies (iii).

\[\Box\]

Remark 9.7. By the Cebotarev density theorem, the set of rational primes q that have the properties in Corollary 9.6 has Dirichlet density 1/18 - 1/54 = 1/27. The prime q = 439 is an example.

**Theorem 9.8.** The equivalent conditions of Theorem 1.3 need not hold if C is allowed to have genus greater than 1. More specifically, let n = \ell = 3. With the notation of Corollary 9.6, let C be the curve Y_q and let \mathcal{O} be either one of the points 0_{C_q} or 0'_C. Let \overline{E}_q = \overline{k} \otimes_k E_q and \overline{C} = \overline{k} \otimes_k C. There are normalized classes \alpha \in H^1(E_q, \mathbb{Z}/3)_{\text{et}} and [b] \in H^1(E_q, \mu_3)_{\text{et}} such that the restrictions \overline{\alpha} and \overline{[b]} to \overline{E}_q satisfy \overline{\alpha} \cup \overline{[b]} \neq 0 with respect to the pairing \cup_{\overline{E}_q} in (1.3) on \overline{E}_q. Let \theta_q : C = Y_q \rightarrow E_q be the morphism associated to the construction of \mathcal{Y}_q. The pullbacks \theta_q^*(\alpha) \in H^1(C, \mathbb{Z}/3) and \theta_q^*[b] \in H^1(C, \mu_3) are normalized classes at \mathcal{O}, but

\[\theta_q^*(\alpha) \cup \theta_q^*[b] \neq \frac{1}{d(O)} \cdot \left(\theta_q^*(\alpha) \cup \theta_q^*[b]\right) \cdot [O] \quad \text{in} \quad \text{Pic}(C)/3 \cdot \text{Pic}(C) = H^2(C, \mu_3).

Proof. Let k = \mathbb{F}_q, and let \overline{k} be a fixed algebraic closure of k. By Corollary 1.7, the 3-torsion of \text{Pic}(E_q) is isomorphic to \mathbb{Z}/3 \times \mathbb{Z}/3. In particular, \text{Pic}(E_q)[3] = \text{Pic}(E_q)[3], which implies \mu_3 \subset k^*.

Therefore, H^1(E_q, \mu_3) = H^1(E_q, \mathbb{Z}/3) \otimes \mu_3. Since the Weil pairing (1.3) on \overline{E}_q is non-degenerate, we can use (1.2) and (1.3), together with Lemmas 9.2 and 9.5, to find normalized classes \alpha \in H^1(E_q, \mathbb{Z}/3)_{\text{et}} and [b] \in H^1(E_q, \mu_3)_{\text{et}} such that the restrictions \overline{\alpha} and \overline{[b]} to \overline{E}_q satisfy \overline{\alpha} \cup \overline{[b]} \neq 0 in \mathbb{Z}/3, with respect to the pairing \cup_{\overline{E}_q} in (1.3) on \overline{E}_q.

Cup products respect restrictions and pullbacks by \theta_q, so we find that

(9.5) \[\theta_q^*(\alpha) \cup \theta_q^*[b] = \theta_q^*(\alpha) \cup \theta_q^*[b]\]

when \theta_q^* on the right is the pullback map

(9.6) \[\theta_q^* : H^2(E_q, \mu_3) \rightarrow H^2(C, \mu_3).

When we identify the domain and range of \theta_q^* in (9.6) with \mathbb{Z}/3, the map \theta_q^* becomes multiplication by 2, since \theta_q : C \rightarrow E_q is a degree two map of curves with constant field k. This and (9.5) imply \theta_q^*(\alpha) \cup \theta_q^*[b] \neq 0 in \mathbb{Z}/3. Suppose now that in fact,

(9.7) \[\theta_q^*(\alpha) \cup \theta_q^*[b] = \frac{1}{d(O)} \cdot \left(\theta_q^*(\alpha) \cup \theta_q^*[b]\right) \cdot [O] \quad \text{in} \quad \text{Pic}(C)/3 \cdot \text{Pic}(C) = H^2(C, \mu_3)

where d(O) = 1. Let \sigma be the non-trivial automorphism of C over E_q. Then the action of \sigma fixes \theta_q^*(\alpha) and \theta_q^*[b] and it is equivariant with respect to cup products. Moreover, \sigma acts on H^2(C, \mu_3) = \text{Pic}(C)/3 \cdot \text{Pic}(C) via the automorphism of \text{Pic}(C) induced by \sigma. Hence (9.7) would
imply
\[ \theta_q^* (\alpha) \cup \theta_q^* (\beta) = \sigma \left( \theta_q^* (\alpha) \cup \theta_q^* (\beta) \right) = \sigma \left( \theta_q^* (\alpha) \cup \theta_q^* (\beta) \right) = \sigma \left( \theta_q^* (\alpha) \cup \theta_q^* (\beta) \right) \cdot [O] \]
(9.8)
Subtracting the right side of (9.8) from the right side of (9.7), using \( d(O) = 1 \), we get
\[ 0 = \left( \theta_q^* (\alpha) \cup \theta_q^* (\beta) \right) \cdot ([O] - [\sigma(O)]) \] in \( \text{Pic}(C)/3 \cdot \text{Pic}(C) \).
Since \( \theta_q^* (\alpha) \cup \theta_q^* (\beta) \neq 0 \) in \( \mathbb{Z}/3 \), this would force \( [O] - [\sigma(O)] \) to lie in \( 3 \cdot \text{Pic}(C) \). However, \( \{O, \sigma(O)\} = \{0_{Y_q}, 0'_{Y_q}\} \) and we have shown in Corollary 9.6 that the difference \( 0_{Y_q} - 0'_{Y_q} \) is not in \( 3 \cdot \text{Pic}(C) \). So the contradiction shows that \( 9.6 \) cannot be true. \( \square \)

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