Table I. Parameters of the various collective modes

| mode | mixed | $\Omega^2$ | $C_{\perp}^2$ | $C_{\parallel}^2$ |
|------|-------|------------|---------------|-----------------|
| $I_r$ | $I_r, a_r$ | $2 |\beta_2| \Delta_0^2$ | $u_L$ | $u_T + \frac{(u_T - u_L)^2}{u_T}$ |
| $a_r$ | $I_r, a_r$ | $1/2$ | $0$ | $\lambda_T^2$ |
| $a_3$ | $a_3$ | $u_T/\Delta_0$ | $\lambda_T^2$ | $0$ |
| $I_t$ | $I_t, a_t$ | $2 |\beta_2| \Delta_0^2$ | $u_T$ | $u_T - \frac{(u_T - u_L)^2}{u_T}$ |
| $a_t$ | $I_t, a_t$ | $1$ | $\lambda_T^2$ | $\lambda_T^2$ |
| $R_t$ | $R_t$ | $0$ | $u_T$ | $u_T$ |
| $\varepsilon$ | $\varepsilon$ | $1/2 |\beta| \Delta_0^2$ | $u_L$ | $u_T$ |
| $R_r$ | $R_r$ | $0$ | $u_T$ | $u_L$ |
Table II. Screening lengths $\lambda$ in directions perpendicular and parallel to the order parameter direction ($z$).

| mode | $\lambda_\perp^2$ | $\lambda_\parallel^2$ |
|------|--------------------|------------------------|
| $A_r$ | 0                  | $\lambda_T^2$          |
| $A_t$ | $\lambda_T^2$     | $\lambda_T^2$         |
| $A_3$ | $\lambda_L$       | 0                      |
| mode | perpendicular | parallel (z) |
|------|---------------|--------------|
| ε    | $\xi_\perp^2$ | $\xi_\parallel^2$ |
| $I_r$| $l_\parallel^2$ | $l_\perp^2 \left(1 + u^2/u_\perp^2\right)$ |
| $I_t$| $l_\parallel^2$ | $l_\perp^2 \left(1 - u^2/u_\perp^2\right)$ |
Collective modes, AC response and magnetic properties of the 3D Dirac semi-metal in the triplet superconducting state.

B. Rosenstein\textsuperscript{1}, B.Ya. Shapiro\textsuperscript{2} and I. Shapiro\textsuperscript{2}

\textsuperscript{1}Department of Electrohysics, National Chiao Tung University, Hsinchu, Taiwan, R.O.C. and Ariel University, Israel. and

\textsuperscript{2}Department of Physics, Institute of Superconductivity, Bar-Ilan University, 52900 Ramat-Gan, Israel.

It was recently shown that conventional phonon-electron interactions may induce a triplet pairing state in time-reversal invariant 3D Dirac semi-metals. Starting from the microscopic model of the isotropic Dirac semi-metal, the Ginzburg-Landau equations for the vector order parameter is derived using the Gor’kov technique. The collective modes including gapless Goldstone modes, and gapped Higgs modes of various polarizations are identified. They are somewhat analogous to the modes in the B phase of $He^3$, although in the present case quantitatively there is a pronounced difference between longitudinal and transverse components. The difference is caused by the vector nature of the order parameter leading to two different coherence lengths or penetration depths. The system is predicted to be highly dissipative due to the Goldstone modes. The time dependent Ginzburg-Landau model in the presence of external fields is used to investigate some optical and magnetic properties of such superconductors. The AC conductivity of a clean sample depends on the orientation of the order parameter. It is demonstrated that the difference between the penetration depths results in rotation of the polarization vector of microwave passing a slab made of this material. The upper critical magnetic field $H_{c2}$ was found. It turns out that at fields close to $H_{c2}$ the order parameter orients itself perpendicular to the field direction. In certain range of parameters the triplet superconducting phase persists at arbitrarily high magnetic field like in some p wave superconductors.

I. INTRODUCTION

Recently 3D Dirac semi-metals (DSM) like $Na_3Bi$ and $Cd_3As_2$ with electronic states described by Bloch wave functions, obeying the “pseudo-relativistic” Dirac equation (with the Fermi velocity $v_F$ replacing the velocity of light) were observed\cite{1} and attracted widespread attention. The discovery of the 3D Dirac materials makes it possible to study their physics including remarkable electronic properties. This is rich in new phenomena like giant diamagnetism that diverges logarithmically when the chemical potential approaches the 3D Dirac point, a linear-in-frequency AC conductivity that has an imaginary part\cite{2}, quantum magnetoresistance showing linear field dependence in the bulk\cite{3}. Most of the properties of these new materials were measured at relatively high temperatures. However recent experiments at low temperature on topological insulators and suspected 3D Dirac semi-metals exhibit superconductivity. Early attempts to either induce or discover superconductivity in Dirac materials were promising. The well known topological insulator $Bi_2Se_3$ doped with $Cu$, becomes superconducting at $T_c = 3.8K$\cite{4}. At present its pairing symmetry is unknown. Some experimental evidence\cite{5} point to a conventional phononic pairing mechanism. The spin independent part of the effective electron - electron interaction due to phonons was studied theoretically\cite{6}. For a conventional parabolic dispersion relation, the triplet superconducting state. Its electronic properties like specific heat, electrical resistivity, and magnetic-susceptibility indicate that $PbTaSe_2$ is a moderately coupled, type-II BCS superconductor with large electron-phonon coupling constant of $\lambda = 0.74$. It was shown theoretically to possess a very asymmetric 3D Dirac point created by strong spin-orbit coupling. If the 3D is confirmed, it might indicate that the superconductivity is conventional phonon mediated.

More recently when the $Cu$ doped $Bi_2Se_3$ was subjected to pressure\cite{7}, $T_c$ increased to 7K at 30GPa. Quasilinear temperature dependence of the upper critical field $H_{c2}$, exceeding the orbital and Pauli limits for the singlet pairing, points to the triplet superconductivity. The band structure of the superconducting compounds is apparently not very different from its parent compound $Bi_2Se_3$, so that one can keep the two band $k \cdot p$ description ($Se$ $p_z$ orbitals on the top and bottom layer of the unit cell mixed with its neighboring $Bi$ $p_z$ orbital). Electronic-structure calculations and experiments on the compounds under pressure\cite{7} reveal a single bulk three-dimensional Dirac cone like in $Bi$ with large spin-orbit coupling. Moreover very recently some pnictides were identified as exhibiting Dirac spectrum. This effort recently culminated in discovery of superconductivity in $Cd_3As_2$\cite{8}. It is claimed that the superconductivity is p-wave at least on the surface.

The case of the Dirac semi-metals is very special due to the strong spin dependence of the itinerant electrons’ effective Hamiltonian. It was pointed out\cite{9} that in this case the triplet possibility can arise although the triplet gap is smaller than that of the singlet, the difference sometimes is not large for spin independent electron - electron interactions. Very recently the spin dependent part of the phonon induced electron - electron interaction was considered\cite{10} and it
was shown that the singlet is still favored over the triplet pairing. Another essential spin dependent effective electron-electron interaction is the Stoner exchange among itinerant electrons leading to ferromagnetism in transition metals. While in the best 3D Weyl semi-metal candidates it is too small to form a ferromagnetic state, it might be important to determine the nature of the superconducting condensate. It turns out that it favors the triplet pairing\cite{11}. Also a modest concentration of magnetic impurities makes the triplet ground state stable.

In a multicomponent superconductor collective modes including gapless Goldstone modes (sound), and gapped Higgs modes of various polarizations play an important role in determining thermal and optical properties of the material\cite{12}. In addition, as mentioned above, generally the applied magnetic field is an ultimate technique to probe the superconducting state. In a growing number of experiments, in addition to magnetotransport, magnetization curves, the magnetic penetration depth and upper critical magnetic field were measured\cite{13}. It is therefore of importance to construct a Ginzburg - Landau (GL) description\cite{14} of these novel materials. This allows to study inhomogeneous order parameter configurations (junctions, boundaries, etc.), the collective modes (somewhat analogous to the modes in the B phase of $He^3$ and magnetic and optical response that typically involve inhomogeneous configurations (like vortices) not amenable to a microscopic description.

In the present paper we derive such a GL type theory for triplet superconductor from the microscopic isotropic DSM model with attractive local interaction. The order parameter in this case is a vector field and the GL theory of vector field already considered in literature\cite{15–17} in connection with putative $p$-wave superconductors have several extraordinary features, both quantitative and qualitative.

The paper is organized as follows. The model of the (phonon mediated or unconventional) local interactions of 3D Dirac fermion is presented and the method of its solution (in the Gorkov equations form) including the symmetry analysis of possible pairing channels and the vectorial nature of the triplet order parameter is given in Section II. In Section III the Gorkov formalism, sufficiently general to derive the GL equations, is briefly presented. The most general form of the GL energy of the triplet superconductor in magnetic field consistent with the symmetries is given in IV. The coefficient of the relevant terms are calculated from the microscopic DSM model in section V. Section VI is devoted to applications of the GL model. The ground state degeneracy, the character of its excitations and basic magnetic properties are discussed. The vector order parameter is akin to optical phonons with sharp distinction between transverse and longitudinal modes. Transverse and longitudinal coherence lengths and penetration depths are calculated and the upper critical magnetic field is discussed. Section VI includes generalizations to include Pauli paramagnetism, discussion of an experimental possibility of observation of the excitation and conclusion.

II. THE LOCAL PAIRING MODEL IN THE DIRAC SEMI-METAL.

A. Pairing Hamiltonian in the Dirac semi-metal.

Electrons in the 3D Dirac semimetal are described by field operators $\psi_{f,s}(\mathbf{r})$, where $f = L, R$ are the valley index (pseudospin) for the left/right chirality bands with spin projections taking the values $s = \uparrow, \downarrow$ with respect to, for example, the $z$ axis. To use the Dirac ("pseudo-relativistic") notations, these are combined into a four component bi-spinor creation operator, $\psi^\dagger = \left(\psi_{L\uparrow}^\dagger, \psi_{L\downarrow}^\dagger, \psi_{R\uparrow}^\dagger, \psi_{R\downarrow}^\dagger\right)$, whose index $\gamma = \{f, s\}$ takes four values. The non-interacting massless Hamiltonian with Fermi velocity $v_F$ and chemical potential $\mu$ reads\cite{18}

$$K = \int_\mathbf{r} \psi^\dagger(\mathbf{r}) \tilde{K} \psi(\mathbf{r}); \quad (1)$$

$$\tilde{K}_{\gamma\delta} = -ihv_F \nabla^i \alpha_{\gamma\delta}^i - \mu \delta_{\gamma\delta}, \quad (2)$$

where the three $4 \times 4$ matrices, $i = x, y, z$,

$$\alpha = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}$$

are presented in the block form via Pauli matrices $\sigma$. They are related to the Dirac $\gamma$ matrices (in the chiral representation, sometimes termed "spinor") by $\alpha = \beta \gamma$ with

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  \hspace{1cm} (4)

Here $\mathbf{1}$ is the $2 \times 2$ identity matrix.
We consider a special case of 3D rotational symmetry that in particular has an isotropic Fermi velocity. Moreover we assume time reversal, $\Theta \psi (r) = i \sigma_y \psi^* (r)$, and inversion symmetries although the pseudo-Lorentz symmetry will be explicitly broken by interactions. The spectrum of single particle excitations is linear. The chemical potential $\mu$ is counted from the Dirac point.

As usual in certain cases the actual interaction can be approximated by a model local one:

$$V_{eff} = -\frac{g}{2} \int \psi^+ (r) \psi^+ (r) \gamma_5 \psi (r) \psi (r).$$  \hspace{1cm} (5)

Unlike the free Hamiltonian $K$, Eq.(1), this interaction Hamiltonian does not mix different spin components.

Spin density in Dirac semi-metal has the form

$$S (r) = \frac{1}{2} \psi^+ (r) \Sigma \psi (r),$$  \hspace{1cm} (6)

where the matrices

$$\Sigma = -\alpha \gamma_5 \gamma_5 = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix},$$  \hspace{1cm} (7)

are also the rotation generators.

B. The symmetry classification of possible pairing channels.

Since we consider the local interactions as dominant, the superconducting condensate (the off-diagonal order parameter) will be local

$$O = \int \psi^+ (r) M_{\alpha\beta} \psi^+ (r),$$  \hspace{1cm} (8)

where the constant matrix $M$ should be a $4 \times 4$ antisymmetric matrix. Due to the rotation symmetry they transform covariantly under infinitesimal rotations generated by the spin $S^i$ operator, Eq.(6):

$$\int_r \int_{r'} \left[ \psi^+ (r) M_{\alpha\beta} \psi^+ (r') \psi^+ (r') \Sigma_{ij} \psi (r') \right]$$

$$= - \int_r \psi^+ (r) \left( \Sigma_{ij} M_{\delta\kappa} + M_{\gamma\delta} \Sigma_{ti} \right) \psi^+ (r).$$

Here and in what follows "t" denotes the transpose matrix. The representations of the rotation group therefore characterize various possible superconducting phases.

Out of 16 matrices of the four dimensional Clifford algebra six are antisymmetric and one finds one vector and three scalar multiplets of the rotation group. The multiplets contain:

(i) a triplet of order parameters:

$$\left\{ M^T_z, M^T_y, M^T_z \right\}$$

$$= \{-\beta \alpha_z, -i \beta \gamma_5, \beta \alpha_z \} = \{T_x, T_y, T_z\}$$

The algebra is

$$\Sigma_i T_j + T_j \Sigma_i^T = 2i \varepsilon_{ijk} T_k.$$  \hspace{1cm} (11)

Note that the three matrices $T_i$ are Hermitian.

(ii) three singlets

$$M^S_1 = i\alpha_y; \quad M^S_2 = i\Sigma_y; \quad M^S_3 = -i \beta \alpha_y \gamma_5.$$  \hspace{1cm} (12)

Which one of the condensates is realized at zero temperature is determined by the parameters of the Hamiltonian and is addressed next within the Gaussian approximation. As was shown in our previous work [11, 19], either exchange interactions or magnetic impurities make the triplet state a leading superconducting channel in these materials. Therefore we will consider in the next section only the vector channel.
III. GORKOV EQUATIONS AND THE TRIPLET PAIRING

A. Gorkov equations for Green’s functions in matrix form

Using the standard BCS formalism, the Matsubara Green’s functions ($\tau$ is the Matsubara time)

$$ G_{\alpha\beta} (\mathbf{r}, \tau; \mathbf{r}', \tau') = - \langle T_\tau \psi_\alpha (\mathbf{r}, \tau) \psi_\beta^\dagger (\mathbf{r}', \tau') \rangle; \quad (13) $$

$$ F_{\alpha\beta} (\mathbf{r}, \tau; \mathbf{r}', \tau') = \langle T_\tau \psi_\alpha (\mathbf{r}, \tau) \psi_\beta (\mathbf{r}', \tau') \rangle; \quad (14) $$

$$ F_{\alpha\beta}^+ (\mathbf{r}, \tau; \mathbf{r}', \tau') = \langle T_\tau \psi_\alpha^\dagger (\mathbf{r}, \tau) \psi_\beta (\mathbf{r}', \tau') \rangle, \quad (15) $$

obey the Gor’kov equations\[20\]:

$$ - \frac{\partial G_{\gamma\delta} (\mathbf{r}, \tau; \mathbf{r}', \tau')}{\partial \tau} - \int_{\tau''} \langle \mathbf{r} | \hat{K}_{\gamma\beta} | \mathbf{r}' \rangle G_{\beta\delta} (\mathbf{r}''; \tau; \mathbf{r}', \tau') = 0. \quad (16) $$

These equations are conveniently presented in matrix form (superscript $t$ denotes transposed and $I$ - the identity matrix):

$$ \int_{X''} \left[ \begin{array}{c} D^{-1} (X, X'') G (X'', X') - \\
- \Delta (X) F^+ (X, X') \end{array} \right] = I \delta (X - X'); \quad (17) $$

$$ \int_{X''} D^{t-1} (X, X'') F^+ (X'', X') + \Delta^{t*} (X) G (X, X') = 0. \quad (18) $$

Here $X = (\mathbf{r}, \tau), \Delta_{\alpha\beta} (X) = g F_{\beta\alpha} (X, X)$ and

$$ D_{\alpha\beta}^{-1} (X, X') = - \delta_{\alpha\beta} \frac{\partial}{\partial \tau} \delta (X - X') - \frac{\partial}{\partial \tau} \langle \mathbf{r} | \hat{K}_{\alpha\beta} | \mathbf{r}' \rangle. \quad (19) $$

In the homogeneous case the Gor’kov equations for Fourier components of the Green’s functions simplify considerably:

$$ D^{-1} (\omega, \mathbf{p}) G (\omega, \mathbf{p}) - \Delta F^+ (\omega, \mathbf{p}) = I; \quad (20) $$

$$ \tilde{D}^{-1} (\omega, \mathbf{p}) F^+ (\omega, \mathbf{p}) + \Delta^{t*} G (\omega, \mathbf{p}) = 0. \quad (21) $$

The matrix gap function can be chosen as

$$ \Delta_{\beta\gamma} = g F_{\gamma\beta} (0) = \Delta \sigma M_{\gamma\beta}, \quad (22) $$

with real constant $\Delta \sigma$. Here $D^{-1} (\omega, \mathbf{p}) = i \omega + \mu - \mathbf{\alpha} \cdot \mathbf{p}$, is the noninteracting inverse Dirac Green’s function for the Hamiltonian Eq.\[21\] and $\tilde{D}^{-1} (\omega, \mathbf{p}) = i \omega - \mu - \mathbf{\alpha}^t \cdot \mathbf{p}$, where $\omega_n = \pi T (2n + 1)$ is the fermionic Matsubara frequency.

Solving these equations one obtains (in matrix form)

$$ G^{-1} = D^{-1} + \Delta \tilde{D} \Delta^{t*}; \quad (23) $$

$$ F^+ = - \tilde{D} \Delta^{t*} G, \quad (24) $$

with the gap function to be found from the consistency condition

$$ \Delta^{t*} = - g \sum_{\omega, \mathbf{p}} \tilde{D} \Delta^{t*} G. \quad (25) $$

Now we find solutions of this equation for each of the possible superconducting phases.
B. Homogeneous triplet solution of the gap equation.

In this phase rotational symmetry is spontaneously broken simultaneously with the electric charge $U(1)$ (global gauge invariance) symmetry. Assuming $z$ direction of the $p$ - wave condensate the order parameter matrix takes a form:

$$\Delta = \Delta_z T_z = \Delta_z \beta \alpha_z,$$

where $\Delta_z$ is a constant. The energy scale will be set by the Debye cutoff $T_D$ of the electron - phonon interactions, see below:

The spectrum of elementary excitations at zero temperature was discussed in ref. [11]. There is a saddle point with energy gap $2\Delta_z$ on the circle $p_z^2 \equiv p_x^2 + p_y^2 = \mu^2 / v_F^2$, $p_z = 0$. The gap $\Delta_z$ as a function of the dimensionless phonon-electron coupling $\lambda = gN$, where $N$ being the density of states (all spins and valleys included), increases upon reduction in $\mu$. At large $\mu >> T_D$, as in BCS, the gap becomes independent of $\mu$ and one has the relation

$$\frac{1}{g} = \frac{N}{12} \sinh^{-1} \frac{T_D}{\Delta_z}; N = \frac{2\mu^2}{\pi^2 v_F^2 \hbar^3},$$

leading to an exponential gap dependence on $\lambda$ when it is small: $\Delta_z = T_D / \sinh (12 / \lambda) \simeq 2T_D e^{-12 / \lambda}$.

The critical temperature is obtained from Eq. (20) with discret $\omega$ by substituting $\Delta_z = 0$. To utilize the orthonormality of $T_i$, $\text{Tr}(T_i T_j^*) = 4\delta_{ij}$, one multiplies the gap equation by the matrix $T_z / g$ and takes the trace:

$$\frac{1}{g} \text{Tr}(T_z T_z^*) = \frac{4}{g} = T_c B_{zz}.$$

The bubble integral is

$$B_{ij} = \sum_{\nu \mu} \text{Tr} \left( T_i \tilde{D} T_j^* D \right) = 4\delta_{ij} T_c \times$$

$$\times \sum_{\nu \mu} \frac{\omega^2_n}{\omega^4_n + (\nu^2 F^2 - \mu^2)^2 + 2\omega^2_n (\nu^2 F^2 + \mu^2)}.$$

Performing first the sum over Matsubara frequencies and then integrate over $q$ one obtains, similarly to the singlet BCS, (see Appendix A for details):

$$T_c = \frac{2\gamma_E}{\pi} T_D e^{-12 / \lambda},$$

where $\log \gamma_E = 0.577$ is the Euler constant.

IV. A GENERAL GL DESCRIPTION OF A TRIPLET SUPERCONDUCTOR IN A MAGNETIC FIELD.

In this section the effective description of the superconducting condensate in terms of the varying (on the mesoscopic scale) order complex parameter vector field $\Delta_i(\mathbf{r})$ is presented.

A. The GL description for a vector order parameter

The static phenomenological description is determined by the GL free energy functional $F[\Delta(\mathbf{r}), \mathbf{A}(\mathbf{r})]$ expanded to second order in gradients and fourth order in $\Delta$. In a magnetic field $\mathbf{B}$, as usual, space derivatives of the microscopic Hamiltonian become covariant derivatives $\nabla \to D = \nabla + i A$-$e^-2$, $\epsilon^* = 2e$ due to gauge invariance under $\Delta \to e^{i\chi(\mathbf{r})} \Delta_i, A_i \to A_i - \frac{\mathbf{e}}{\epsilon} \nabla \chi$. Naively the only modification of the GL energy is in the gradient term, Eq. (23); the most general gradient term consistent with rotation symmetry and the $U(1)$ gauge symmetry is

$$F_{\text{grad}} = N \int d\mathbf{r} \left\{ u_T \begin{pmatrix} (D_j \Delta_i)^* (D_j \Delta_i) - (D_i \Delta_j)^* (D_j \Delta_i) \\ + u_L (D_i \Delta_j)^* (D_j \Delta_i) \end{pmatrix} \right\}.$$
The factor $N$, the density of states, is customarily introduced into energy [11]. It was noted in [16], that, unlike in the usual scalar order parameter case, the longitudinal and transverse coefficients are in general different, leading to two distinct coherence lengths, see Section V. Possibilities for the local terms are [15]

$$F_{\text{loc}} = N \int \left\{ \alpha (T - T_c) \Delta_i^* \Delta_i + \frac{\beta_1}{2} (\Delta_i^* \Delta_i)^2 + \frac{\beta_2}{2} |\Delta_i| \Delta_i^* \right\}.$$  

(27)

The magnetic part, $F_{\text{mag}} = B^2/8\pi$, completes the free energy.

B. The set of time independent GL equations for triplet order parameter

The set of the GL equations corresponding to this energy are obtained by variation with respect to $\Psi_j^*$ and $A_i$. The first is:

$$- \left\{ u_T \left( \delta_{ij} D^2 - \frac{1}{2} \{ D_i, D_j \} \right) \right\} + \frac{1}{2} u_L \{ D_i, D_j \} \right\} \Delta_j + \alpha (T - T_c) \Delta_i + \beta_1 \Delta_i \Delta_j + \beta_2 \Delta^* \Delta_j = 0.$$ 

(28)

The anticommutator appears due complex conjugate terms in Eq.(26) [21]. The Maxwell equation for the supercurrent density is:

$$J_i = \frac{ie^*}{\hbar} N \left( u_T \Delta_j^* D_i \Delta_j + u \Delta_j^* D_j \Delta_i \right) + cc,$$ 

(29)

where $u = u_L - u_T$.

Having defined coefficients $u_{T,L}, \beta_{1,2}$ and $\alpha$, our aim in the next Section is to deduce them from the microscopic Dirac semi-metal model.

V. GL COEFFICIENTS FROM THE GOR’KOV EQUATIONS

For the calculation of coefficients of the local part, the homogeneous Gor’kov equation, Eq.(20) suffices, while for calculation of the gradient terms a general linearized equation, Eq.(15) is necessary.

A. Local (potential) terms in Gor'kov

Iterating once the equation Eq.(20) with help of Eq.(19) one obtains the local terms to third order in the gap function:

$$\frac{1}{g} \Delta^* + \sum_{\omega} \left\{ \bar{D}(\omega, \mathbf{p}) \Delta^* D(\omega, \mathbf{p}) - \bar{D}(\omega, \mathbf{p}) \Delta^* D(\omega, \mathbf{p}) \bar{D}(\omega, \mathbf{p}) \Delta^* D(\omega, \mathbf{p}) \bar{D}(\omega, \mathbf{p}) \right\}.$$ 

(30)

Using $\Delta^* = \Delta_i^* T_i$, multiplying by $T_i^*$ and taking the trace, one gets the linear local terms

$$N \alpha (T - T_c) \Delta_i^* = \frac{4}{g} \Delta_i^*,$$ 

(31)

where the bubble integral was given in Eq.(24). Expressing $g$ via $T_c$, see Eq.(26) allows to write the coefficient $\alpha$ of $\Delta_i^*$ in the Gorkov equation Eq.(20) is

$$\alpha (T - T_c) = \frac{8\mu^2}{3\pi^2 v_F^2 \hbar^3 N} \log \frac{T}{T_c} \approx \frac{4}{3} \frac{T - T_c}{T_c}.$$ 

(32)
The cubic terms in Eq. (30), multiplied again by $T^i_t$ and "traced" take the form
\begin{equation}
N \left( \beta_1 \Delta^*_j \Delta^*_j + \beta_2 \Delta^*_j \Delta^*_j \Delta^*_i \right)
\end{equation}

\begin{equation}
= -\Delta^*_j \Delta^*_j \sum_{\omega \mathbf{p}} \text{Tr} \left\{ T^i_t \tilde{D}^j_t \tilde{D}^*_t \tilde{D}^i_t \right\}.
\end{equation}

The calculation is given in Appendix A and results in:
\begin{equation}
\beta_1 = \frac{7 \zeta(3)}{20 \pi^2} \frac{1}{T_c^2} \quad ; \quad \beta_2 = - \frac{1}{3} \beta_1.
\end{equation}

The Riemann zeta function is $\zeta(3) = 1.2$.

**B. Linear gradient terms**

To calculate the gradient terms, one first linearizes the Gor’kov equations, Eq.(15)
\begin{equation}
\int_{X''} D^{-1}(X, X') G(X'', X')
\end{equation}

\begin{equation}
= \int_{X'} D^{t} (X - X') \Delta^t (X'') D \left( X'' - X' \right).
\end{equation}

In particular,
\begin{equation}
\frac{1}{g} \Delta^t(X) = F^+(X, X)
\end{equation}

\begin{equation}
= - \int_{X'} D^{t} (X - X') \Delta^t (X') D \left( X' - X \right).
\end{equation}

The anomalous Green’s functions are no longer space translation invariant, so that the following Fourier transform is required: The (time independent) order parameter is also represented via Fourier components $\Delta^*(P) = \sum_{P} e^{-i\mathbf{p} \cdot \mathbf{r}} \Delta^*(P)$. The linear part Gor’kov equation (this time including nonlocal parts) depending on the ”external” momentum $\mathbf{P}$ reads:
\begin{equation}
\frac{1}{g} \Delta^t(P) + \sum_{\omega \mathbf{p}} \tilde{D}(\omega, \mathbf{p}) \Delta^t(P) D(\omega, \mathbf{p} - \mathbf{P}).
\end{equation}

To find the coefficients of the gradient terms, one should consider contributions quadratic in $P$ from the expansion of both $\Delta^t(P)$ and $D(\omega, \mathbf{p} - \mathbf{P})$. In view of the gap equation Eq.(23,24), the expansions of $\Delta^t(P)$ cancel each other up to small corrections of order $T - T_c$. So that multiplying by $T^i_t$ and taking the trace
\begin{equation}
\frac{1}{2} P_k P_l \sum_{\omega \mathbf{p}} \text{Tr} \left\{ T^i_t \tilde{D}(\omega, \mathbf{p}) T^t_j \tilde{D}^t_{kl}(\omega, \mathbf{p}) \right\} \Delta^*_i,
\end{equation}

where $\tilde{D}_{kl}(\omega, \mathbf{p}) = \frac{\partial^2 D(\omega, \mathbf{p})}{\partial p_k \partial p_l}$. Comparing this with the gradient terms in the GL equation, Eq.(28), see Appendix B for details, one deduces
\begin{equation}
u_T = \frac{28 \zeta(3)}{15 \pi^2} \frac{v_F^2 \hbar^2}{T_c^2} \quad ; \quad \nu_L = \frac{1}{32} \nu_T.
\end{equation}

Note the very small longitudinal coefficient, $\nu_L << \nu_T$. As we shall see in the following section it has profound phenomenological consequences.
VI. BASIC PROPERTIES OF THE TRIPLET SUPERCONDUCTOR

A. Ground state structure and degeneracy

A ground state is characterized by three independent parameters corresponding to three Goldstone bosons. The GL energy is invariant under both the vector $O(3)$ space rotations, $\Delta_i \to R_{ij} \Delta_j$, and the superconducting phase $U(1)$, $\Delta_i \to e^{i\chi} \Delta_i$. In the superconducting state characterized by the vector order parameter $\Delta$ (where $|\Delta|$ = $\Delta$, energy gap) the $U(1)$ is broken: $U(1) \to 1$, while the $O(3)$ is only partially broken down to its $O(2)$. There are therefore three Goldstone modes. Here we explicitly parametrize these degrees of freedom by phases following ref. [15]. Generally a complex vector field can be written as

$$\Delta = \Delta (n \cos \chi + im \sin \chi),$$  \hspace{1cm} (40)

where $n$ and $m$ are arbitrary unit vectors and $0 < \chi < \pi/2$, see Fig. 1.

Using this parametrization the homogeneous part of the free-energy density, Eq.(27), takes the form

$$f_{\text{loc}} = \left\{ \frac{\alpha (T - T_c) \Delta^2 + \frac{1}{2} \beta_1 \Delta^4 +}{\beta_1 + \beta_2} \left( \cos^2 (2\chi) + (n \cdot m)^2 \sin^2 (2\chi) \right) \Delta^4 \right\}.$$  \hspace{1cm} (41)

This form allows us to make several interesting observations. The crucial sign is that of $\beta_2$. In previous studies only $\beta_2 > 0$ (so called phase A) was considered. In our case however $\beta_2 < 0$ and different ground state configurations should be considered. In phase B the minimization gives, $n = \pm m$. Note two different solutions. So that the "vacuum manifold" is

$$\Delta = \Delta_0 ne^{i\chi}.$$  \hspace{1cm} (42)

Here the range of $\chi$ was enlarged, $-\pi/2 < \chi < \pi/2$, to incorporate $n = \pm m$. The ground state energy density therefore is achieved at

$$\Delta_0^2 = \frac{\alpha (T_c - T)}{\beta_1 + \beta_2} = \frac{\alpha (T_c - T)}{\beta}.$$  \hspace{1cm} (43)

Mathematically the vacuum manifold in phase B is isomorphic to $S_2 \otimes S_1/Z_2$. This determines the thermodynamics of the superconductor very much in analogy with the scalar superconductor with $\beta = \beta_1 + \beta_2$. However the collective modes, the $AC$ conductivity and the magnetic properties are markedly different [23].

B. Collective excitation modes.

Here the response of the superconductor in phase B to an external perturbation, like boundary or magnetic field, is considered. The basic excitation modes are uncovered by the linear stability analysis very similar to the so-called Anderson - Higgs mechanism in field theory applied to (scalar order parameter) superconductivity a long time ago [24]. Two basic scales, the coherence length (scale of variations of the order parameter) and magnetic penetration depth (scale of variations of the magnetic field), are obtained from the expansion of the GL energy to second order in fluctuations around superconducting ground state at zero field. In the superconducting state the most convenient gauge is the "unitary" gauge in which the $U(1)$ phase of the order parameter is set to zero, so that we are left with three massive vector potential fields $A_1, A_2, A_3$. Chosing the ground state as $\Delta = \Delta_0 (0, 0, 1)$, see Fig.2 the order parameter in the unitary gauge can be parametrized by five real fields

$$\Delta = \Delta_0 (1 + \varepsilon) (R_1 + iI_1, R_2 + iI_2, 1).$$  \hspace{1cm} (44)

The spontaneous breaking of the space rotation symmetry $O(3)$ into its $O(2)$ subgroup of rotations within the $x - y$ plane according to the Goldstone theorem leads to two gapless modes $R_\alpha$, $\alpha = 1, 2$. Due to the residual symmetry the
Fourier components of the fluctuation fields can be generally written as combination of the radial and the tangential components:

\[
R_\alpha = (R_r k_\alpha + R_t \varepsilon \alpha \beta k_\beta) / k_\perp; \quad I_\alpha = (I_r k_\alpha + I_t \varepsilon \alpha \beta k_\beta) / k_\perp
\]

with \( k_\perp = (k_x^2 + k_y^2)^{1/2} \). The GL energy to quadratic order in eight fluctuation fields \( \eta = \{ \varepsilon, I_r, I_t, R_r, a_r, a_t, a_3 \} \),

\[
f = \frac{N \Delta_0^2}{2} \sum_k \eta_k^* M \eta_k
\]

decomposes into three independent sectors, where \( M \) is dimensionless fluctuation matrix.

1. Three massive fields \( I_r, a_r, a_3 \) can mix due to kinetic terms:

\[
M_1 = \begin{pmatrix}
2 \Delta_0^2 | \beta_2 | + u_T k^2 & \frac{i \eta}{u_T} k_3 & 0 \\
\frac{i \eta}{u_T} k_3 & 1 + \lambda_T^2 k_3^2 & -\lambda_T^2 k_3 k_\perp \\
0 & -\lambda_T^2 k_3 k_\perp & u_L + \lambda_T^2 k_\perp^2
\end{pmatrix}
\]

where \( \lambda_T^2 = \hbar^2 c^2 / 8 \pi u_T e^*_T N \Delta_0^2 \).

To order \( k^2 \) the eigenvalues are,

\[
E \left( k^2 \right) / V = N \Delta_0^2 \left( \Omega^2 + C_\perp^2 k_\perp^2 + C_\parallel^2 k_\parallel^2 \right).
\]

The values of the gaps in the spectrum \( \Omega^2 \) and corresponding velocities in directions perpendicular and parallel to the vector order parameter (taken to be \( z \)), \( C_\perp, C_\parallel \) are given in Table I

2. The tangential massive, \( I_t, a_t \), sector

\[
M_2 = \begin{pmatrix}
2 | \beta_2 | \Delta_0^2 + u_T k^2 & \frac{i \eta}{u_T} k_3 \\
\frac{i \eta}{u_T} k_3 & 1 + \lambda_T^2 k_3^2
\end{pmatrix}
\]

3. The Goldstone transverse mode \( R_t \) does not mix, \( M_3 = u_T k^2 \), so that \( E_0 / V = N \Delta_0^2 u_T k^2 \).

4. Higgs and Goldstone radial collective modes \( \varepsilon, R_r \) form a \( 2 \times 2 \) matrix:

\[
M_4 = \begin{pmatrix}
u_T k^2 + u k_\perp & u k_\perp k_3 \\
u k_\perp k_3 & u_T k^2 + u k_3^2 + \Delta_0^2 \beta_2
\end{pmatrix}
\]

C. Coherence length and penetration depth for massive collective modes

The six "massive" fields, \( \varepsilon, I_\alpha \) and \( A \) with different longitudinal and transversal characteristic lengths: \( l_\perp = C_\perp / \Omega; l_\parallel = C_\parallel / \Omega \), and the same for \( \xi \) and \( \lambda \), see Table I.

This is different compared with the one component (singlet) superconductor in two respects. First the number of Higgs modes is larger since in addition to the superfluid density determined by \( \varepsilon \), there are two additional components \( I_r \) and \( I_t \). Second, as mentioned above, since the superconducting condensate is oriented, one has two different velocities. As far as (massive) photon modes are concerned, the number of modes remains the same but the anisotropy persists. Let us start with m
Here definitions of the coherence lengths (of fluctuation of the superfluid density $\varepsilon$ perpendicular and parallel to the direction of the order parameter $n$), correlation lengths of the relative weights between different components of the vector order parameter ($I_x, I_y$) and screening lengths (vector potential $A$ fluctuations):

$$
\xi_\perp^2 = \frac{2u_L}{\beta \Delta_0}; \quad \xi_\parallel^2 = \frac{u_T}{u_L} \xi_\perp^2; \quad (53)
$$

$$
\lambda_\perp^2 = \frac{\hbar^2 c^2}{8\pi u_T e^2 N \Delta_0}; \quad \lambda_\parallel^2 = \frac{u_T}{u_L} \lambda_\perp^2.
$$

In the isotropic superconductor one recovers the standard formulas since $u_T = u_L$.

Our calculation in the previous Section for the Dirac semi-metal, see Eq.(39), demonstrate that both are quite different since $u_L/u_T = 1/32 << 1$. This is obviously of great importance for large magnetic field properties of such superconductors and will be discussed below.

VII. MAGNETIC AND OPTICAL PROPERTIES

A. Strong magnetic fields: is there an upper critical field $H_{c2}$?

In strong homogeneous magnetic field $H$ (assumed to be directed along the $z$ axis) superconductivity typically (but not always, see an example of the $p$-wave superconductor that develops flux phases [23]) disappears at certain critical value $H_{c2}$. This bifurcation point is determined within the GL framework by the lowest eigenvalue of the linearized GL equations. This is an exact requirement of stability of the normal phase [14, 24]. The linearized GL equation Eq.(28) reads:

$$
\left[(a - u_T D^2) \delta_{ij} - \frac{u}{2} \{D_i, D_j\}\right] \Delta_j = 0, \quad (54)
$$

where coefficients are in Eq.(39), and $u = u_L - u_T$. We use the Landau gauge, $A_x = H_{c2} y$; $A_y = A_z = 0$. Assuming translation symmetry along the field direction, $\partial_x \Delta_z = 0$, the operators of the eigenvalue problem depend on $x$ and $y$ only.

Since we have three components of the order parameter, there are three eigenvalues. It is easily seen from Eq.(54) that the $z$- component of the order parameter $\Delta_z$ parallel to the external field direction is independent of the other two, $\Delta_x, \Delta_y$, leading to the ordinary Abrikosov value:

$$
-u_T D^2 \Delta_z = -a \Delta_z \rightarrow H_{c2}^\parallel \equiv \frac{\Phi_0}{2\pi \xi_T^2}, \quad (55)
$$

where $\xi_T^2 \equiv u_T$ (see Eq.(39)). To avoid confusion with customary notations for layered materials (like high $T_c$ cuprates), the material that is modelled here is isotropic and "parallel", "perpendicular" and refer to the relative orientation of the magnetic field to the vector order parameter rather than to a layer. The orientation of the order parameter in isotropic material considered here, due to degeneracy of the ground state, is determined by the external magnetic field as we exemplify next.

The two remaining eigenvalues involving only the order parameter components $\Delta_x$ and $\Delta_y$ perpendicular to the field (see Fig.3) are obtained from diagonalizing the "Hamiltonian":

$$
\mathcal{H} \left( \begin{array}{c} \Delta_x \\ \Delta_y \end{array} \right) = -a \left( \begin{array}{cc} \Delta_x \\ \Delta_y \end{array} \right) \\
\mathcal{H} = -\left( u_T D_y^2 + u_L D_x^2 - \frac{u}{2} \{D_x, D_y\} \right) \frac{u_T}{u_T D_x^2 + u_L D_y^2} \left( \begin{array}{c} D_x \\ D_y \end{array} \right).
$$

This nontrivial eigenvalue problem fortunately can be solved exactly, see Appendix C. The lowest eigenstate being a superposition of just two lowest even Landau levels, $|0\rangle$ and $|2\rangle$ are given. The lowest of these eigenvalues is

$$
\frac{\epsilon^* H_{c2}^\perp}{\hbar c} \left( -\sqrt{3( u_T^2 + u_L^2) - 2u_T u_L} \right) = \alpha (T_c - T). \quad (57)
$$
The corresponding critical field \( H_{c2}^\perp \) ("perpendicular" refers to the order parameter direction (Fig.3)) that can be expressed via an effective "perpendicular" coherence length,

\[
H_{c2}^\perp = \frac{\Phi_0}{2\pi \left( \frac{4}{3} (\xi_L^2 + \xi_T^2) - \sqrt{3\xi_L^2 + 3\xi_T^2 - 2\xi_L^2\xi_T^2} \right)}.
\] (58)

It is always larger than \( H_{c2}^\parallel \), and therefore is physically realized. The upper field \( H_{c2}^\perp \) becomes infinite at \( r_c = uL/uT = (13 - 4\sqrt{10})/3 \approx 0.117 \). This means that in such material superconductivity persists at any magnetic field like in some \( p \)-wave superconductors. It was found in Section IV that for the simplest Dirac semi-metal, \( r = 1/32 < r_c \) see Eq.(39). Thus there is no upper critical field in this case. Of course, different microscopic models that belong to the same universality class, might have higher \( r \). In any case the Abrikosov lattice is expected to be markedly different from the conventional one and even from the vector order parameter model studied in [15].

B. Dissipative dynamics

The set of the GL equations corresponding to this energy are obtained by variation with respect to \( \Delta^+ \) and \( A_i \). The first is the time dependent GL equation (the covariant derivative is replaced by partial since in a superconductor the scalar potential can be taken to be zero on the mesoscopic scale):

\[
- \Gamma \partial_t \Delta^i = \frac{\delta F}{\delta \Delta^i}.
\] (59)

This should be supplemented with the Maxwell equation including the normal metal contribution the the current \( J^0 = J - J^\parallel \), \( J_i^\parallel = \frac{\omega N}{\hbar} \left( uT \Delta^2 R^L \Delta_j + u \Delta_j \Delta^2 R^L \Delta_i \right) + cc \). This determines the dynamics of the vector potential:

\[
\frac{\sigma_n}{c} \partial_t A_i = J_i^\parallel - \frac{c}{4\pi} \left( \partial^2 \delta_{ij} - \partial_i \partial_j \right) A_j = J_i.
\] (60)

In the small fluctuations approximation the dominant role is played by the two Goldstone modes, \( R_t, R_r \), due to spontaneous breaking of the 3D rotation symmetry. The \( R_t \) mode is still isotropic, while \( R_r \) is not. Neglecting the massive excitations the dissipative dynamics of the diffusion type is governed by

\[
\partial_t R_t = D_T k^2 R_t; \quad \partial_t R_r = \left( D_T k_r^2 + D_{L} k_r^2 \right) R_r,
\] (61)

where \( D_T, L = \frac{N}{uT,L} \). The diffusion coefficient of this equation is anisotropic and is discussed in Section VIII. This would lead to increase in thermal conductivity inside the superconducting state even at low temperature.

C. The AC conductivity

In external AC field represented by (no spatial dispersion), \( A = \frac{\lambda}{2} \overline{E} (\omega) e^{i\omega t} \), one obtains in linear response

\[
J_i = - \frac{2ie^{\ast} N \Delta^2}{\omega \hbar^2} \left( uT E_i + uE_3 \delta_{3i} \right) + \sigma_n E_i.
\] (62)

Therefore the conductivity tensor reads:

\[
[\sigma(\omega)]_{ij} = \begin{pmatrix}
\sigma_n - \sigma^T_r & 0 & 0 \\
0 & \sigma_n - \sigma^T_r & 0 \\
0 & 0 & \sigma_n - \sigma^T_L
\end{pmatrix},
\] (63)

where \( \sigma^T_L(\omega) = \frac{2ie^{\ast} N \Delta^2}{\omega \hbar^2} u_{T,L} \). The AC conductivity this is different for the order parameter and the perpendicular directions. One of the interesting consequences of this phenomenon is rotation of the polarization of microwave that passes the DSM film.
D. Rotation of the polarization of the microwave

The material becomes optically active, i.e. the polarization of the electromagnetic wave rotates. The dispersion relation is:

$$-\frac{ie^2k^2}{4\pi\omega}\sigma^{-1}(\omega)\mathbf{B} = \mathbf{B}.$$  \hfill (64)

The perpendicular to the order parameters are eigenvectors with eigenvalue $-\frac{ie^2k^2}{4\pi\omega(\sigma_n - \sigma_T)}$, while the third eigenvector $(0, 0, 1)$ has the eigenvalue $-\frac{ie^2k^2}{4\pi\omega(\sigma_n - \sigma_L)}$. Assume that the incident electromagnetic wave described on the surface of DSM by the magnetic field $\mathbf{B}_0$ is perpendicular to the order parameter direction taken as $z$ (see Fig.4).

Without loss of generality it can be taken as $x$ (due to the residual $O(2)$ rotation symmetry). The Fourier component of the magnetic field in the $y - z$ plane are

$$B_y = B_0 \exp\left(ik^{(2)}x\right)$$
$$B_z = B_0 \exp\left(ik^{(3)}x\right).$$

The complex wave vectors are

$$k^{(2)} = \sqrt{\frac{1 + \sqrt{\lambda_T^4/\delta^4 + 1}}{2}} \left(-\frac{1}{\lambda_T} + i\frac{\lambda_T}{\delta^2}\right)$$
$$k^{(3)} = \sqrt{\frac{1 + \sqrt{\lambda_L^4/\delta^4 + 1}}{2}} \left(-\frac{1}{\lambda_L} + i\frac{\lambda_L}{\delta^2}\right).$$

Here the screening length is defined as usual $\delta = (c/4\pi\omega\sigma_n)^{1/2}$.

The penetration of the microwave radiation into the sample surface demonstrates differences for AC component parallel and perpendicular to the direction of the order parameter resulting in effective rotation of the incident wave vector like at Faraday effect.

VIII. DISCUSSION AND CONCLUSIONS

A. The vector nature of the order parameter

The physical properties of the triplet superconductivity appearing in 3D Dirac semi-metals were considered. Starting from the microscopic model of the isotropic Dirac semi-metal, the Ginzburg-Landau energy for this field is derived using the Gor’kov technique. The properties of the triplet superconductor phase of the Dirac semi-metal has extremely unusual features that we would like to associate qualitatively with the characteristics of the Cooper pair. The superconducting state generally is a Bose - Einstein condensate of composite bosons - Cooper pairs, classically described by the Ginzburg - Landau energy as a functional of the order parameter. In the present case the Cooper boson is described by a vector field $\Delta_i(r)$. In this respect it is reminiscent to phonon and vector mesons in particle physics[24].

Vector fields generally have both the orbital and internal degrees of freedom often called polarization. The internal degree of freedom might be connected to the "valley" degree of freedom of constituents of the composite boson. We have provided evidence that the Cooper pair in DSM has finite orbital momentum, albeit, as will be shown shortly, the spin magnetic moment is zero. Microscopically the unusual nature is related to the presence of the valley degeneracy in Dirac semi-metal. While in a single band superconductor the Pauli principle requires a triplet Cooper pair to have both odd angular momentum and spin, it is no longer the case in the Dirac semi-metal.

A massive bosonic vector field in isotropic situation (the case considered here) generally have distinct transversal and longitudinal polarizations (massless fields like photons in dielectric do not possess the longitudinal degree of freedom). The results for collective modes in triplet superconductor in DSM demonstrate pronounced disparity between dispersion of various polarizations, see Table I. In particular we have found sound velocity of the two Goldstone modes and, screening lengths of three gapped photon modes and three coherence lengths of the other (Higgs) gapped modes. These all have an impact on transport, optical and magnetic properties of these superconductors.
B. Estimates of the characteristics of the collective modes and Faraday effect in a typical Dirac semi-metal

Substituting the values of parameter of the vector GL equation found in Section IV into formulas for various coherence lengths described in Section III, one obtains,

\[ \xi_0^2 = \frac{7 \zeta(3) v_F^2 \hbar^2}{80 \pi^2 T_c}; \quad \xi_0^2 = 32 \xi_0^2; \]

\[ i_0^2 = 32 \xi_0^2; \quad l_0^2 = \xi_0^2; \]

\[ \lambda_{0T}^2 = \frac{3\pi \hbar^4 v_F}{32 e^2 \mu^2}; \quad \lambda_{0L}^2 = 32 \lambda_{0T}^2, \]

where \( \xi_0^2 (T) = \xi_0^2 / (1 - T/T_c) \) etc.

For a typical DSM one estimates the Fermi velocity and chemical potential \[ \frac{v_F}{c} = 200, \frac{\mu}{eV} = 0.2, \] and with the expected critical temperature \[ T_c = 5K, \] one obtains for \( \xi_0^2 = 230nm \) and \( \lambda_{0T} = 220nm \). This has an impact on the magnetic flux penetration into this kind of superconductors. The value of Abrikosov parameter \( \kappa = \lambda/\xi \) depends on mutual direction of the DC magnetic field. While \( \kappa_T = \lambda_T/\xi \approx 1, \kappa_L = \lambda_L/\xi \approx 30 \).

Let us estimate the characteristics of the two Goldstone modes, arising in the triplet superconducting state due to spontaneous breaking of the rotational \( O(3) \) symmetry. The isotropic transverse mode \( R_t \) defined in Section III is isotropic and its dynamics is described by Eq. (61) with diffusion constant \( D_T = N u_T / \Gamma \), while the anisotropic mode \( R_L \) involves both the transverse and the longitudinal constant that is different, \( D_L = N u_L / \Gamma \). To estimate these, let us exploit the relation \[ \frac{\Gamma}{\kappa} \] the time constant as \( \Gamma = \frac{\hbar N}{2 \pi e} \). Thus

\[ D_T = \frac{2^5 \cdot 7 \zeta(3) v_F^2 h}{15 \pi^3 T_c} = 2 \cdot 10^4 \frac{cm^2}{s}, \]

\[ D_L = \frac{7 \zeta(3) v_F^2 h}{15 \pi^3 T_c} = 620 \frac{cm^2}{s}. \]

These considerations were made for superconductor without significant pinning - disorder on the mesoscopic scale. Recently the AC response of the disordered superconductor was utilized to probe Goldstone modes \[ (23) \]. We have demonstrated that they are abundant in the triplet DSM superconductor. Therefore it would be interesting to look for effects of the Goldstone modes including damping resulting in strong sound absorption in these systems.

We have calculated the AC conductivity of DSM, see Eq. (64), and applied it to describe an intriguing effect of optical activity of DSM. According to Eq. (66), the polarization vector of the incident beam rotates while passing a film of thickness \( d \) by (see Fig.4)

\[ \tan \phi = |B_z (d)/B_y (d)| = \exp \left[(\lambda_{0T}^{-1} - \lambda_{0L}^{-1})d \right], \]

under the assumption the skin depth \( \delta = (c/4\pi\sigma_0)^{1/2} \) is much larger than both \( \lambda_T \) and \( \lambda_L \). In particular for \( \lambda_L = 32 \lambda_T, \lambda_T = 220nm, \) and \( d = 1mm, \phi \rightarrow \pi/2 \) significantly deviation from the initial \( \pi/4 \) value. Let us stress that the effect is due to the difference between the two penetration depths.

C. DSM superconductor under constant magnetic field

Several new features appear when an external field is applied. The Ginzburg - Landau model was used to determine upper magnetic field \( H_{c2} \). It turns out that the lowest energy solution is when the order parameter of the texture orient itself perpendicular to the field direction. We have shown that the upper field becomes infinite for \( u_L / u_T < (13 - 4\sqrt{10}) / 3 \sim 0.117 \). In particular it is obeyed within our microscopic model. This means that in such material superconductivity persists at any magnetic field like in some \( p \)-wave superconductors. The expression for the \( H_{c2} \) for \( u_L / u_T > 0.117 \) is given by Eq. (45).

The vortex physics of strongly type II triplet superconductors of this type is very rich and some of it has already been investigated in connection with heavy fermion and other superconductors suspected to possess \( p \)-wave pairing. In particular, their magnetic vortices appear as either vector vortices or so-called skyrmions \[ (15) \] - coreless topologically nontrivial textures. The magnetic properties like the magnetization are very peculiar and even without a magnetic field the system forms a ”spontaneous flux state”. The material therefore can be called a ”ferromagnetic superconductor”.

The superconducting state develops weak ferromagnetism and a system of alternating magnetic domains \[ (22) \]. It was noted \[ (23) \] that the phase is reminiscent to the phase B of superfluid \( He_3, \) \[ (14) \] (with an obvious distinction that the order parameter in the later case is neutral rather than charged and tensorial rather than vectorial).
Since the prediction of the FFLO effect\[14] in low $T_c$ superconductors it is well known that at very high magnetic fields the direct spin - magnetic field coupling on the microscopic level might not be negligible. The singlet channel Cooper pair is effectively "broken" by the splitting since the spins of the two electrons are opposite (Pauli paramagnetic limit). It is not clear what impact it has on Dirac semi-metals. If the impact is large it could be incorporated as an additional paramagnetic term in the GL energy. In an isotropic Dirac superconductor one has only one possible term in the GL energy term linear in paramagnetic coupling and consistent with symmetries:

$$F_{par} = N\mu_p \int r \cdot (\Delta^* \times \Delta) \cdot \mathbf{B},$$

where $\mu_p$ is the effective "spin" of the Cooper pair sometimes called "Zeeman coupling"\[13, 25]. The single particle Hamiltonian in magnetic field has the Pauli term $\mu_B \mathbf{\Sigma} \cdot \mathbf{B}$, where the Bohr magneton, $\mu_B = e\hbar/2mc$, determines the strength of the coupling of the spin to magnetic field, with $m$ being the free electron mass. The direct calculation, see Appendix A, shows that $\mu_p = 0$.

Acknowledgements. We are indebted to D. Li and M. Lewkowicz for valuable discussions. Work of B.R. was supported by NSC of R.O.C. Grants No. 98-2112-M-009-014-MY3 and MOE ATU program.
IX. APPENDIX A. LOCAL TERMS IN GL FREE ENERGY

A. Critical temperature calculation

Starting from equation Eq. (23) the angle integrations result in (for $\mu >> T_D, T_c$)

$$
\frac{1}{g} = T \sum_{np} \frac{\mu^2 + \omega_n^2}{\omega_n^4 + (v_F^2 p^2 - \mu^2)^2 + 2\omega_n^2 (v_F^2 p^2 + \mu^2)}
$$

(72)

$$
= \frac{\mu^2}{12\pi^2 h^3 v_F^4} \int_{\varepsilon = -T_D}^{T_D} \frac{\tanh (\varepsilon / 2T)}{\varepsilon} \approx \frac{\mu^2}{6\pi^2 h^3 v_F^4} \log \frac{2T_D}{\pi T_c}.
$$

where $\varepsilon = v_F p - \mu$. See the last (BCS) integral in (20).

B. Cubic terms coefficients calculation

To fix the two coefficients, $\beta_1$ and $\beta_2$ in Eq. (28) we use only two components. The particular case $j = k = l = 1$ (the coefficient of $\psi_1^2 \psi_1$) gives after angle integration

$$
N (\beta_1 + \beta_2) = \frac{2T}{15\pi^2 h^3} \sum_{n} \int_{p=0}^{\infty} S(p, n)
$$

(A2)

where

$$
S(p, n) = \frac{v_F^2 p^2}{v_F^4 p^4 + 10v_F^2 p^2 (\omega_n^2 - 5\mu^2) - 15 (\mu^2 + \omega_n^2)^2}
$$

(75)

Performing finite integration (the upper bound on momentum, $\mu + T_D$, can be replaced by infinity), one obtains

$$
N (\beta_1 + \beta_2) = \frac{8\mu^2}{15\pi^4 T^2 v_F^4 h^3 s_3},
$$

(73)

where the sum is

$$
s_3 = \sum_{n=0} \frac{1}{(2n + 1)^3} = \frac{7\zeta (3)}{4}.
$$

(74)

Similarly taking $j = l = 2, k = 1$ (the coefficient of $\Delta_2^2 \Delta_1$) gives after the angle integration

$$
N \beta_2 = \frac{2T}{15\pi^2 h^3} \sum_{n} \int_{p=0}^{\infty}
$$

$$
= \frac{4\mu^2}{15\pi^4 v_F^4 h^3 T^2 s_3},
$$

resulting in Eq. (34).
C. Effect of the Pauli interaction term

The single particle Hamiltonian in magnetic field is

$$\hat{K} = -i v_F \hbar D \cdot \alpha - \mu + \mu_B \Sigma \cdot B,$$

(76)

In order to fix the coefficient of the paramagnetic term linear in both the order parameter and Pauli coupling it is enough to expand the linearized Gorkov equations Eq. (31) to the first order in the spin density. Normal Greens functions have the following corrections:

$$D_Z \approx D - \mu_B D (\Sigma \cdot B) D;$$

(77)

The Pauli term in Gor’kov equation (after multiplying by $T^i_t$ and taking the trace as usual), Eq. (28), therefore is obtained from expansion of Eq. (31),

$$\sum_\omega \text{Tr} \left\{ T^i_t \tilde{D} \Delta^* D \right\}$$

(78)

$$= -i \mu Z \varepsilon_{ijk} \Delta^j \mu_B N = \mu_B B_{ijk} \Delta^j D, \quad B_{ijk} = \sum_\omega \text{Tr} \left\{ T^i_t \tilde{D} \left( \Sigma^i \tilde{D} T^j_t - T^j_t \Sigma_k \right) D \right\},$$

The bubble sum is directly evaluated and vanishes $B_{ijk} = 0$.

X. APPENDIX B. CALCULATION OF GRADIENT TERMS IN THE GL

Rotational invariance allows to represent the sum in Eq (38) terms of coefficients $u_T$ and $u_L$:

$$-N (u_T (P^2 \delta_{mj} - P_m P_j) + u_L P_m P_j)$$

$$= P_k P_l \sum_{\omega q} \text{Tr} \left\{ T^i_t \tilde{D} (\omega, q) T^j_s \tilde{D}^\prime_{kl} (\omega, q) \right\},$$

where

$$D''_{ij} = \frac{2}{\left(q^2 - (i \omega + \mu)^2\right)^2}$$

$$\{q_j \alpha_i + q_i \alpha_j + \delta_{ij} (i \omega + \mu + \alpha \cdot q) + 2 q_i q_j D\}.$$ 

In particular

$$N u_L = - \sum_{\omega p} \text{Tr} \left\{ T^i_t \tilde{D} (\omega, p) T^*_{jz} D''_{zz} (\omega, p) \right\}$$

(81)

$$= \frac{\mu^2}{15 \pi^4 T^2 v_F \hbar^3}$$

$$= \frac{7 \zeta (3)}{60 \pi^4} \frac{\mu^2}{T^2 v_F \hbar} = \frac{7 \zeta (3) (v_F \hbar)^2 N}{120 \pi^2 T^2}$$

and

$$N u_T = - \sum_{\omega p} \text{Tr} \left\{ T^i_t \tilde{D} (\omega, p) T^*_{jz} D''_{zz} (\omega, p) \right\} = 32 u_L.$$ 

(82)
XI. APPENDIX C. COLLECTIVE MODES

Here details of the calculation of free energy in harmonic approximation are given. The expansion of order parameter Eq.(44) to quadratic order is

\[ \Delta / \Delta_0 = (1 + \varepsilon) (\delta_1, \delta_2, 1) = (0, 0, 1) + (\delta_1, \delta_2, \varepsilon) + (\delta_1 \varepsilon, \delta_2 \varepsilon, 0). \]  

(83)

The gradient terms that do not involve the vector potential \( A \) are:

\[ F^{(1)}_{\text{grad}}(\Delta^2 / \Delta_0^2) = u_T \left( \partial_j R_\alpha \partial_j R_\alpha + \partial_j I_\alpha \partial_j I_\alpha + (\partial_j \varepsilon)^2 \right) \]

\[ + u \left\{ \frac{\partial_1 R_1 \partial_1 R_1 + \partial_1 I_1 \partial_1 I_1 + \partial_2 R_2 \partial_2 R_2 + \partial_2 I_2 \partial_2 I_2 + (\partial_3 \varepsilon)^2}{(\partial_2 R_1)(\partial_1 R_2) + (\partial_3 I_1)(\partial_1 I_2) + (\partial_3 \varepsilon)(\partial_3 R_\alpha)} \right\} \]

(84)

The terms involving \( A \) read

\[ F^{(2)}_{\text{grad}}(\Delta^2 / \Delta_0^2) = 2 \frac{e^* u}{\hbar c} (\partial_3 I_\alpha) A_\alpha + \left( \frac{e^*}{\hbar c} \right)^2 (u_T A^2 + u A_3^2) \]

(85)

Potential terms result in

\[ F_{\text{pot}}(\Delta^2 / \Delta_0^2) = \Delta_0^2 \left\{ \frac{\beta_1 + \beta_2 \varepsilon^2}{2} - 2 \beta_2 (I_1^2 + I_2^2) \right\}, \]

(86)

while the magnetic energy is:

\[ F_{\text{mag}}(\Delta^2 / \Delta_0^2) = \frac{\hbar^2 e^2}{8\pi e^* N\Delta_0^2} A_1^* \left( k^2 \delta_{ij} - k_ik_j \right) A_j \]

(87)

The fluctuation matrix \( M \) in Eq.(48) thus was constructed.

XII. APPENDIX D. EXACT SOLUTION FOR UPPER CRITICAL MAGNETIC FIELD

In this Appendix the matrix \( H \) defined in Eq.(56) determining the perpendicular upper critical field is diagonalized variationally.

A. Creation and annihilation operators

Using Landau creation and annihilation operators in units of magnetic length \( \frac{eB}{e} = l^{-2} \) for the state with \( k_x = 0 \) (independent of \( x \)), so that covariant derivatives are

\[ D_x = \partial_x + iy = iy = \frac{i}{\sqrt{2}} (a + a^+) \];

\[ D_y = \partial_y = \frac{1}{\sqrt{2}} (a - a^+) \].

(88)

In terms of these operators the matrix operator \( \mathcal{H} \) takes a form:

\[ \mathcal{H} = u_T + \frac{u}{2} + \mathcal{V}; \]

\[ \mathcal{V}_{11} = 2u_T a^+ a + \frac{u}{2} (a^2 + a^{+2} + 2a^+ a) ; \]

\[ \mathcal{V}_{12} = \mathcal{V}_{21} = \frac{iu}{2} (a^{+2} - a^2) ; \]

\[ \mathcal{V}_{22} = 2u_T a^+ a - \frac{u}{2} (a^2 + a^{+2} - 2a^+ a) . \]
The exact lowest eigenvalue is a combination of two lowest Landau levels. Indeed applying the operator \( \mathcal{V} \) on a general vector on the subspace gives

\[
\mathcal{V} \begin{pmatrix} \alpha |0\rangle + \beta |2\rangle \\ \gamma |0\rangle + \delta |2\rangle \end{pmatrix} = \begin{pmatrix} \frac{u}{\sqrt{2}} (i\delta - \beta) |0\rangle \\ - \left( u \left( \frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{2}} + 2\beta \right) + 4u_T\beta \right) |2\rangle \\ + \frac{u}{3} (-\beta - i\delta) |4\rangle \\ + \frac{u}{\sqrt{2}} (i\beta + \delta) |0\rangle \\ + \left( u \left( -\frac{i\alpha}{\sqrt{2}} + \frac{i\beta}{\sqrt{2}} - 2\delta \right) - 4u_T\delta \right) |2\rangle \\ + \frac{u}{2} (-i\beta + \delta) |4\rangle \end{pmatrix}.
\] (90)

For \( \delta = i\beta \), higher Landau levels decuple and one gets eigenvalue equations

\[
\begin{vmatrix} -v & -\sqrt{2}u & 0 \\ -\frac{u}{\sqrt{2}} & -4u_T - 2u - v - \frac{iu}{\sqrt{2}} & 0 \\ 0 & u\sqrt{2} & -v \end{vmatrix} = 0,
\] (91)

resulting in three eigenvalues of \( \mathcal{H} \)

\[
h^{(1)} = u_T + u/2, h^{(\pm)}
\] (92)

\[
= 3u_T \pm \frac{3}{2} u \pm \sqrt{4u_T^2 + 4u_Tu + 3u^2}.
\]
(1997); M. Kato and K. Maki, Prog. Theor. Phys. 107 941 (2002); J. X. Zhu, Physica C 340 230 (2000); B. Rosenstein, I. Shapiro, B. Ya. Shapiro, J. Low Temp. Phys. 173 289 (2013).

[22] G. Bel, B. Rosenstein, B.Ya. Shapiro and I. Shapiro, Europhys. Lett., 64 503 (2003); B. Rosenstein, I. Shapiro, B.Ya. Shapiro, and G. Bel, Phys. Rev. B 67 224507 (2003); E. Pechenik, B. Rosenstein, B.Ya. Shapiro, I. Shapiro, Phys. Rev. B 65 214522 (2002).

[23] K-H Bennemann and J.B. Ketterson The Physics of Superconductors: Vol. II: Superconductivity in Nanostructures, High Tc, and Novel superconductors, Organic superconductors. Springer-Verlag Berlin, Heidelberg (2004).

[24] S. Weinberg, Prog. Theor. Phys. Suppl., 86, 43 (1986).

[25] S. Alama and L. Bronsard Rev. Math Phys., 16, 147 (2009)

[26] T. Cea, D. Bucheli, G. Seibold, L. Benfatto, J. Lorenzana, and C. Castellani Phys. Rev. B 89 174506 (2014).
Figures Captions

Fig. 1. A complex vector field can be written as $\Delta = \Delta (n \cos \chi + i m \sin \chi)$, where $n$ and $m$ are arbitrary unit vectors and $0 < \chi < \pi/2$.

Fig. 2. Fluctuations of the order parameter in the unitary gauge can be parametrized by the five real fields $\Delta = \Delta_0 (1 + \varepsilon) (R_1 + i I_1, R_2 + i I_2, 1)$.

Fig. 3. Superconductivity arises from the normal state when the order parameter is formed in direction perpendicular to the magnetic field.

Fig. 4. The polarization vector of the incident beam rotates while passing a film of thickness $d$ by angle $\phi \tan \phi = |B_z (d) / B_y (d)| = \exp \left[ (\lambda_T^{-1} - \lambda_L^{-1}) d \right]$.