From quasi-entropy to skew information

Dénes Petz\textsuperscript{1,3,4} and V.E. Sándor Szabó\textsuperscript{3,4}

\textsuperscript{3} Alfréd Rényi Institute of Mathematics, H-1364 Budapest, POB 127, Hungary

\textsuperscript{4} Department for Mathematical Analysis, BUTE, H-1521 Budapest, POB 91, Hungary

Abstract

This paper gives an overview about particular quasi-entropies, generalized quantum covariances, quantum Fisher informations, skew-informations and their relations. The point is the dependence on operator monotone functions. It is proven that a skew-information is the Hessian of a quasi-entropy. The skew-information and some inequalities are extended to a von Neumann algebra setting.

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\textsuperscript{1}E-mail: petz@math.bme.hu. Partially supported by the Hungarian Research Grant OTKA T068258.
\textsuperscript{2}E-mail: sszabo@math.bme.hu. Partially supported by the Hungarian Research Grant OTKA TS-49835.
1 Introductory preliminaries

Let $\mathcal{M}$ denote the algebra of $n \times n$ matrices with complex entries. For positive definite matrices $D_1, D_2 \in \mathcal{M}$, for $A \in \mathcal{M}$ and a function $F : \mathbb{R}^+ \to \mathbb{R}$, the quasi-entropy is defined as

$$S^A_F(D_1, D_2) := \langle AD_1^{1/2}, F(\Delta(D_2/D_1))(AD_1^{1/2}) \rangle = \text{Tr } D_1^{1/2} A^* F(\Delta(D_2/D_1))(AD_1^{1/2}), \quad (1)$$

where $\Delta(D_2/D_1) : \mathcal{M} \to \mathcal{M}$ is a linear mapping acting on matrices:

$$\Delta(D_2/D_1) A = D_2 AD_1^{-1}.$$

This concept was introduced in [21, 22], see also Chapter 7 in [20] and it is the quantum generalization of the $F$-entropy of Csiszár used in classical information theory (and statistics) [4, 16].

The concept of quasi-entropy includes some important special cases. If $D_1$ and $D_2$ are different and $A = I$, then we have a kind of relative entropy. For $F(x) = -\log x$ we have Umegaki’s relative entropy $S(D_1 \parallel D_2) = \text{Tr } D_1 (\log D_1 - \log D_2)$. More generally,

$$F(x) = \frac{1}{\alpha(1-\alpha)} (1 - x^\alpha),$$

is operator monotone decreasing for $\alpha \in (-1, 1)$. (For $\alpha = 0$, the limit is taken and it is $-\log x$.) Then the Rényi entropies are produced

$$S_\alpha(D_1 \parallel D_2) := \frac{1}{\alpha(1-\alpha)} \text{Tr } (I - D_2^{\alpha} D_1^{-\alpha}) D_1.$$

If $D_1 = D_2 = D$ and $A, B \in \mathcal{M}$ are arbitrary, then one can approach to the generalized covariance [25]. An operator monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$ will be called standard if $xf(x^{-1}) = f(x)$ and $f(1) = 1$. A standard function $f$ admits a canonical representation

$$f(t) = e^{\beta \frac{1 + t}{\sqrt{2}}} \exp \int_0^1 \frac{\lambda^2 - 1}{\lambda^2 + 1} \cdot \frac{1 + t^2}{(\lambda + t)(1 + \lambda t)} h(\lambda) \, d\lambda, \quad (2)$$

where $h : [0, 1] \to [0, 1]$ is a measurable function and $\beta$ is a real constant [11].

If $f$ is a standard function, then

$$q\text{Cov}_D^f(A, B) := \langle AD^{1/2}, f(\Delta(D/D))(BD^{1/2}) \rangle - (\text{Tr } DA^*)(\text{Tr } DB). \quad (3)$$

is a generalized covariance. The usual symmetrized covariance corresponds to the function $f(t) = (t + 1)/2$:

$$\text{Cov}_D(A, B) := \frac{1}{2} \text{Tr } (D(A^*B + BA^*)) - (\text{Tr } DA^*)(\text{Tr } DB).$$
The quantum Fisher information is similarly defined to (1), but $F(x) = 1/f(x)$ for a standard function $f : \mathbb{R}^+ \to \mathbb{R}^+$:

$$
\gamma_f^D(A, B) = \langle A D^{-1/2} \frac{1}{f}(\Delta(D/D))(BD^{-1/2}) \rangle
$$

Quantum Fisher information was characterized by the monotonicity under coarse-graining [23]. This kind of non-affine parametrization was used in [23, 25], since the relation to operator means was emphasized. Sometimes the affine parametrization is more convenient and Hansen’s canonical representation of the inverse of a standard operator monotone function can be used [12].

**Proposition 1** If $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a standard operator monotone function, then

$$
\frac{1}{f(t)} = \int_0^1 \frac{1 + \lambda}{2} \left( \frac{1}{t + \lambda} + \frac{1}{1 + t\lambda} \right) d\mu(\lambda),
$$

where $\mu$ is a probability measure on $[0, 1]$.

The theorem implies that the set $\{1/f : f \text{ is standard operator monotone}\}$ is convex and gives the extremal points

$$
g_\lambda(x) := \frac{1 + \lambda}{2} \left( \frac{1}{t + \lambda} + \frac{1}{1 + t\lambda} \right) \quad (0 \leq \lambda \leq 1).
$$

One can compute directly that

$$
\frac{\partial}{\partial \lambda} g_\lambda(x) = -\frac{(1 - \lambda^2)(x + 1)(x - 1)^2}{2(x + \lambda)^2(1 + x\lambda)^2}.
$$

Hence $g_\lambda$ is decreasing in the parameter $\lambda$. For $\lambda = 0$ we have the largest function $g_0(t) = (t + 1)/(2t)$ and for $\lambda = 1$ the smallest is $g_1(t) = 2/(t + 1)$. Note that this was also obtained in the setting of positive operator means [14], harmonic and arithmetic means.

Covariance and Fisher information are bilinear (or sesqui-linear) forms, in the applications they are mostly reduced to self-adjoint matrices.

The space $\mathcal{M}$ has an orthogonal decomposition

$$
\{B \in \mathcal{M} : [D, B] = 0\} \oplus \{i[D, A] : A \in \mathcal{M}\}.
$$

We denote the two subspaces by $\mathcal{M}_D$ and $\mathcal{M}_D^\perp$, respectively. If $A_2 \in \mathcal{M}_D$, then

$$
F(\Delta(D/D))(A_2 D^{\pm 1/2}) = A_2 D^{\pm 1/2}
$$

implies

$$
\text{qCov}_D^f(A_1, A_2) = \text{Tr} \, DA_1^* A_2 - (\text{Tr} \, DA_1^*)(\text{Tr} \, DA_2), \quad \gamma_D^f(A_1, A_2) = \text{Tr} \, D^{-1} A_1^* A_2
$$

independently of the function $f$. Moreover, if $A_1 \in \mathcal{M}_D^\perp$, then

$$
\gamma_D^f(A_1, A_2) = \text{qCov}_D^f(A_1, A_2) = 0.
$$

Therefore, the effect of the function $f$ and the really quantum situation are provided by the components from $\mathcal{M}_D^\perp$. 

3
2 Quasi-entropy

The quasi-entropies are monotone and jointly convex [20, 22]:

Let \( \alpha : M_0 \to M \) be a mapping between two matrix algebras. The dual \( \alpha^* : M \to M_0 \) with respect to the Hilbert-Schmidt inner product is positive if and only if \( \alpha \) is positive. Moreover, \( \alpha \) is unital if and only if \( \alpha^* \) is trace preserving. \( \alpha : M_0 \to M \) is called a **Schwarz mapping** if

\[
\alpha(B^*B) \geq \alpha(B^*)\alpha(B)
\]

for every \( B \in M_0 \).

**Proposition 2** Assume that \( F : \mathbb{R}^+ \to \mathbb{R} \) is an operator monotone function with \( F(0) \geq 0 \) and \( \alpha : M_0 \to M \) is a unital Schwarz mapping. Then

\[
S^A_F(\alpha^*(D_1), \alpha^*(D_2)) \geq S^A_F(D_1, D_2)
\]

holds for \( A \in M_0 \) and for invertible density matrices \( D_1 \) and \( D_2 \) from the matrix algebra \( M \).

If we apply the monotonicity \([3]\) to the embedding \( \alpha(X) = X \oplus X \) of \( M \) into \( M \oplus M \) and to the densities \( D_1 = \lambda E_1 \oplus (1 - \lambda)F_1, \ D_2 = \lambda E_2 \oplus (1 - \lambda)F_2 \), then we obtain the joint concavity of the quasi entropy.

**Proposition 3** Under the conditions of Theorem 2, the joint concavity

\[
\lambda S^A_F(E_1, E_2) + (1 - \lambda)S^A_F(F_1, F_2) \leq S^A_F(\lambda E_1 + (1 - \lambda)F_1, \lambda E_2 + (1 - \lambda)F_2)
\]

holds.

The case \( F(t) = t^{\alpha} \) is the famous Lieb’s concavity theorem [15].

Our next aim is to compute

\[
\frac{\partial^2}{\partial t \partial s} S_F(D + tA, D + sB) \bigg|_{t=s=0}.
\]

We shall use the formulas

\[
\frac{d}{dt} h(D + tB) \bigg|_{t=0} = Bh'(D) \quad (B \in M_D), \quad \frac{d}{dt} h(D + ti[D, X]) \bigg|_{t=0} = i[h(D), X],
\]

see [20].

**Lemma 1** If \( A, B \in M_D \), then the derivative \([10]\) equals \(-F''(1)\text{Tr} \ D^{-1}AB\).
Proof: It is enough to check the case $F(t) = t^n$. Then
\[ S_F(D + tA, D + sB) = \text{Tr} (D + tA)^{1-n}(D + sB)^n \]  
and the derivative is $(1-n)\text{Tr} D^{-1}AB$. \[ \square \]

**Lemma 2** If $A \in \mathcal{M}_D$ and $B \in \mathcal{M}_D$, then the derivative (10) equals 0.

Proof: We compute for $F(t) = t^n$ using (11). If $B = [D, X]$, then we have the derivative
\[ \text{Tr} (1-n)D^{-n}A[D^n, X] = 0 \]
and this gives the statement. \[ \square \]

**Lemma 3** Let $X = X^* \in \mathcal{M}$ and $F : \mathbb{R}^+ \to \mathbb{R}$ be a continuously differentiable function. Then
\[ \frac{\partial^2}{\partial t \partial s} S_F(D + ti[D, X], D + si[D, X]) \bigg|_{t=s=0} = 2F(1)\text{Tr} DX^2 - 2S_F^X(D, D). \]  
Derivation of formula (11) gives
\[ 2\text{Tr} DX^2 - 2\text{Tr} XD^{1-n}XD^n \]
and this is the stated result for the particular $F$. \[ \square \]

## 3 Skew information

The Wigner-Yanase-Dyson skew information is the quantity
\[ I_p(D, A) := -\frac{1}{2} \text{Tr} [D^p, A][D^{1-p}, A] \quad (0 < p < 1). \]
Actually, the case $p = 1/2$ is due to Wigner and Yanase [29] and the extension was proposed by Dyson. The convexity of $I_p(D, A)$ in $A$ is a famous result of Lieb [15].

It was observed in [24] that the Wigner-Yanase-Dyson skew information is connected to a monoton Riemannian metric (or Fisher information) which corresponds to the function
\[ f_p(x) = p(1-p) \frac{(x-1)^2}{(x^p-1)(x^{1-p}-1)}. \]
It was proven in [24] that this is an operator monotone function, a generalization was obtained in [12] [28].
Let $f$ be a standard function and $X = X^* \in M$. The quantity
\[
I_f^D(X) := \frac{f(0)}{2} \gamma_f^D(i[D, X], i[D, X])
\]
was called skew information in [12] in this general setting. Note that the parametrization in [12] is by $c = 1/f$ which is called there Morozova-Chentsov function. The skew information is nothing else but the Fisher information restricted to $M_c^D$, but it is parametrized by the commutator. Skew information appears, for example, in uncertainty relations [1, 5, 6, 7, 13, 17, 18], see also Theorem 4. In that application, the skew information is regarded as a bilinear form.

If $D = \text{Diag}(\lambda_1, \ldots, \lambda_n)$ is diagonal, then
\[
\gamma_f^D(i[D, X], i[D, X]) = \sum_{ij} \frac{(\lambda_i - \lambda_j)^2}{\lambda_j f(\lambda_i/\lambda_j)} |X_{ij}|^2.
\]
This implies that the identity
\[
f(0) \gamma_f^D(i[D, X], i[D, X]) = 2 \text{Cov}_D(X, X) - 2q \text{Cov}_D(\bar{f}, X)
\] (13)
holds if $\text{Tr} DX = 0$ and
\[
\bar{f}(x) := \frac{1}{2} \left( (x + 1) - (x - 1)^2 \frac{f(0)}{f(x)} \right).
\] (14)
Since the right-hand-sides of (12) and (13) are the same if $F = \bar{f}$ we have

**Theorem 1** Assume that $X = X^* \in M$ and $\text{Tr} DX = 0$. If $f$ is a standard function such that $f(0) \neq 0$, then
\[
\left. \frac{\partial^2}{\partial t \partial s} S_F(D + ti[D, X], D + si[D, X]) \right|_{t=s=0} = f(0) \gamma_f^D(i[D, X], i[D, X])
\]
for the standard function $F = \bar{f}$.

The only remaining thing to show is that if $f : \mathbb{R}^+ \to \mathbb{R}$ is a standard function, then $\bar{f}$ is standard as well. This result appeared in [8] and the proof there is not easy, even matrix convexity of functions of two variables is used. Here we give a rather elementary proof based on the fact that $1/f \mapsto \bar{f}$ is linear and on the canonical decomposition in Theorem 1.

**Lemma 4** Let $\lambda \geq 0$ and $f : \mathbb{R}^+ \to \mathbb{R}$ be a function such that
\[
\frac{1}{f(x)} := \frac{1 + \lambda}{2} \left( \frac{1}{x + \lambda} + \frac{1}{1 + x \lambda} \right) = g_\lambda(x).
\]
Then the function $\bar{f} : \mathbb{R}^+ \to \mathbb{R}$ defined in (14) is an operator monotone standard function.
Proof: From the definitions we obtain

\[ \tilde{f}(x) = \frac{x(x\lambda^2 + \lambda^2 + 2\lambda + 2x\lambda + x + 1)}{2(x + \lambda)(1 + x\lambda)} \]

and

\[ \tilde{f}'(x) = \frac{\lambda + 2x\lambda + 2\lambda^2 + x^2 \lambda + 4x\lambda^2 + 2\lambda^3 x + x^2 + \lambda^3 x^2 + \lambda^3 + \lambda^4 x^2}{2(x + \lambda)^2(1 + x\lambda)^2} \]

Hence \( \tilde{f}(0) = 0 \) and \( \tilde{f} : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \). So it is enough to prove that the holomorphic extension of \( \tilde{f} \) to the complex upper half-plane maps the upper half-plane into itself, see [3].

Let \( a, b \in \mathbb{R}, b > 0 \). Then we have

\[ \text{Im} \tilde{f}(a + ib) = \frac{b}{2((a + \lambda)^2 + b^2)((1 + \lambda a)^2 + \lambda^2 b^2)} \times ((\lambda + 2\lambda^2 + \lambda^3 + b^2 + a^2 + 2\lambda a + \lambda a^2 + 4\lambda^2 a + \lambda^2 b^2 + 2\lambda^3 a + \lambda^4 a^2 + \lambda^4 b^2). \]

Here

\[ \lambda + 2\lambda^2 + \lambda^3 + b^2 + a^2 + 2\lambda a + \lambda a^2 + 4\lambda^2 a + \lambda^2 b^2 + 2\lambda^3 a + \lambda^4 a^2 + \lambda^4 b^2 = (1 + \lambda + \lambda^4 + \lambda^3) a^2 + (4\lambda^2 + 2\lambda + 2\lambda^3) a + \lambda + 2\lambda^2 + \lambda^3 + b^2(1 + \lambda + \lambda^4 + \lambda^3). \]

The function \( g : \mathbb{R} \rightarrow \mathbb{R} \)

\[ g(a) := (1 + \lambda + \lambda^4 + \lambda^3) a^2 + (4\lambda^2 + 2\lambda + 2\lambda^3) a + \lambda + 2\lambda^2 + \lambda^3 \]

has a minimum value at

\[ a(\lambda) = -\frac{4(\lambda^2 + 2\lambda + 2\lambda^3)}{2(1 + \lambda + \lambda^4 + \lambda^3)} \]

and

\[ g(a(\lambda)) = \frac{(\lambda^2 - 1)^2 \lambda}{(\lambda - 1/2)^2 + 3/4} \geq 0. \]

Therefore the upper half-plane is mapped into itself. The properties \( x\tilde{f}(x^{-1}) = \tilde{f}(x) \) and \( \tilde{f}(1) = 1 \) are obvious. \( \Box \)

The uncertainty relation recently obtained is the following [9].

**Proposition 4** Assume that \( f, g : \mathbb{R}^+ \rightarrow \mathbb{R} \) are standard functions and \( D \) is a positive definite matrix. Then for self-adjoint matrices \( A_1, A_2, \ldots, A_m \) the determinant inequality

\[ \det \left( q\text{Cov}_D^g(A_i, A_j) \right)_{i,j=1}^m \]

\[ \geq \det \left( \left[ f(0)g(0)(\text{Cov}_D(A_i, A_j) - q\text{Cov}_D^\tilde{f}(A_i, A_j)) \right]_{i,j=1}^m \right) \]

holds.

Note that the right-hand-side contains skew informations, cf. [13].
4 The setting of von Neumann algebras

Let $\mathcal{M}$ be a von Neumann algebra. Assume that it is in standard form, it acts on a Hilbert space $\mathcal{H}$, $\mathcal{P} \subset \mathcal{H}$ is the positive cone and $J : \mathcal{H} \rightarrow \mathcal{H}$ is the modular conjugation \cite{10, 20, 27}. Let $\varphi$ and $\omega$ be normal states with representing vectors $\Phi$ and $\Omega$ in the positive cone. For the sake of simplicity, assume that $\varphi$ and $\omega$ are faithful. This means that $\Phi$ and $\Omega$ are cyclic and separating vectors. The closure of the unbounded operator $A\Omega \mapsto A^*\Phi$ has a polar decomposition $J\Delta(\varphi, \omega)^{1/2}$ and $\Delta(\varphi, \omega)$ is called relative modular operator. $A\Omega$ is in the domain of $\Delta(\varphi, \omega)^{1/2}$ for every $A \in \mathcal{M}$.

For $A \in \mathcal{M}$ and $F : \mathbb{R}^+ \rightarrow \mathbb{R}$, the quasi-entropy

$$S_F^A(\omega, \varphi) := \langle A\Omega, F(\Delta(\varphi, \omega))A\Omega \rangle$$

was introduced in \cite{21}, see also Chapter 7 in \cite{20}. (The right-hand-side can be understood via the spectral decomposition of the positive operator $\Delta(\varphi, \omega)$.) For $F(t) = -\log t$ and $A = I$ the relative entropy of Araki is obtained \cite{2} and this was the motivation of the generalization.

**Theorem 2** Assume that $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is an operator monotone function with $F(0) \geq 0$ and $\alpha : \mathcal{M}_0 \rightarrow \mathcal{M}$ is a unital normal Schwarz mapping. Then

$$S_F^A(\omega \circ \alpha, \varphi \circ \alpha) \geq S_F^A(\omega, \varphi)$$

holds for $A \in \mathcal{M}_0$ and for normal states $\omega$ and $\varphi$ of the von Neumann algebra $\mathcal{M}$.

We sketch the proof based on inequalities for operator monotone and operator concave functions. (The details are clarified in \cite{21}.) First note that

$$S_{F+c}^A(\omega \circ \alpha, \varphi \circ \alpha) = S_F^A(\omega \circ \alpha, \varphi \circ \alpha) + c\omega(\alpha(A^*A))$$

and

$$S_{F+c}^A(\omega, \varphi) = S_F^A(\omega, \varphi) + c\omega(\alpha(A^*A))$$

for a positive constant $c$. Due to the Schwarz inequality \cite{7}, we may assume that $F(0) = 0$.

Let $\Omega_0$ be the representing vector for $\omega \circ \alpha$ and $\Delta := \Delta(\varphi, \omega)$, $\Delta_0 := \Delta(\varphi \circ \alpha, \omega \circ \alpha)$. The operator

$$V x\Omega_0 = \alpha(x)\Omega \quad (x \in \mathcal{M}_0)$$

is a contraction:

$$\|\alpha(x)\Omega\|^2 = \omega(\alpha(x)^*\alpha(x)) \leq \omega(\alpha(x^*x)) = \|x\Omega_0\|^2$$

since the Schwarz inequality is applicable to $\alpha$. A similar simple computation gives that

$$V^*\Delta V \leq \Delta_0.$$ (18)
Since $F$ is operator monotone, we have $F(\Delta_0) \geq F(V^*\Delta V)$. Recall that $F$ is operator concave, therefore $F(V^*\Delta V) \geq V^*F(\Delta)V$ and we conclude

$$F(\Delta_0) \geq V^*F(\Delta)V.$$ (19)

Application to the vector $A\Omega_0$ gives the inequality.

The natural extension of the covariance (from probability theory) is

$$q\text{Cov}_\omega^f(A, B) = \langle \sqrt{f(\Delta(\omega, \omega))} A\Omega, \sqrt{f(\Delta(\omega, \omega))} B\Omega \rangle - \omega(A)\omega(B),$$ (20)

where $\Delta(\omega, \omega)$ is actually the modular operator. Motivated by the application, we always assume that the function $f$ is standard. For such a function $f$, the inequalities

$$\frac{2x}{x+1} \leq f(x) \leq \frac{1+x}{2}$$

holds. Therefore $A\Omega$ is in the domain of $\sqrt{f(\Delta(\omega, \omega))}$ and the covariance $q\text{Cov}_\omega^f(A, B)$ is a well-defined sesquilinear form.

For a standard function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and for a normal unital Schwarz mapping $\beta : \mathcal{N} \to \mathcal{M}$ the inequality

$$q\text{Cov}_\omega^f(\beta(X), \beta(X)) \leq q\text{Cov}_\omega^f(X, X) \quad (X \in \mathcal{N})$$ (21)

is a particular case of Theorem 2 and it is the monotonicity of the generalized covariance under coarse-graining [25].

Following [12], the skew information (as a bilinear form) can be defined as

$$I^f_\omega(X, Y) := \text{Cov}_\omega(X, Y) - q\text{Cov}_\omega^f(X, Y)$$ (22)

if $\omega(X) = \omega(Y) = 0$. (Then $I^f_\omega(X) = I^f_\omega(X, X)$.)

**Theorem 3** Assume that $f, g : \mathbb{R}^+ \to \mathbb{R}$ are standard functions and $\omega$ is a faithful normal state on a von Neumann algebra $\mathcal{M}$. Let $A_1, A_2, \ldots, A_m \in \mathcal{M}$ be self-adjoint operators such that $\omega(A_1) = \omega(A_2) = \ldots = \omega(A_m) = 0$. Then the determinant inequality

$$\text{Det} \left( [q\text{Cov}_D^g(A_i, A_j)]_{i,j=1}^m \right) \geq \text{Det} \left( [2g(0)I^f_\omega(A_i, A_j)]_{i,j=1}^m \right)$$ (23)

holds.

**Proof:** Let $E(\cdot)$ be the spectral measure of $\Delta(\omega, \omega)$. Then for $m = 1$ the inequality is

$$\int g(\lambda) \, d\mu(\lambda) \leq g(0) \left( \int \frac{1+\lambda}{2} \, d\mu(\lambda) - \int \bar{f}(\lambda) \, d\mu(\lambda) \right),$$

where $d\mu(\lambda) = d(\Lambda\Omega, E(\lambda)\Lambda\Omega)$. Since the inequality

$$f(x)g(x) \geq f(0)g(0)(x-1)^2$$ (24)
holds for standard functions [9], we have
\[
    g(\lambda) \geq g(0) \left( \frac{1 + \lambda}{2} - f(0) \tilde{f}(\lambda) \right)
\]
and this implies the integral inequality.

Consider the finite dimensional subspace \( \mathcal{N} \) generated by the operators \( A_1, A_2, \ldots, A_m \). On \( \mathcal{N} \) we have the inner products

\[
    \langle \langle A, B \rangle \rangle := \text{Cov}^0_ω(A, B)
\]

and

\[
    \langle A, B \rangle := 2g(0)I^f_ω(A, B).
\]

Since \( \langle A, A \rangle \leq \langle \langle A, A \rangle \rangle \), the determinant inequality holds (see Lemma 2 in [9]). □

This theorem is interpreted as quantum uncertainty principle [1, 6, 8, 13]. In the earlier works the function \( g \) from the left-hand-side was \((x + 1)/2\) and the proofs were more complicated. The general \( g \) appeared in [9].

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