Integrable sigma models with $\theta = \pi$

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Abstract

A fundamental result relevant to spin chains and two-dimensional disordered systems is that the sphere sigma model with instanton coupling $\theta = \pi$ has a non-trivial low-energy fixed point and a gapless spectrum. This result is extended to two series of sigma models with $\theta = \pi$: the $SU(N)/SO(N)$ sigma models flow to the $SU(N)_1$ WZW theory, while the $O(2N)/O(N) \times O(N)$ models flow to $O(2N)_1$ ($2N$ free Majorana fermions). These models are integrable, and the exact quasiparticle spectra and $S$ matrices are found. One interesting feature is that charges fractionalize when $\theta = \pi$. I compute the energy in a background field, and verify that the perturbative expansions for $\theta = 0$ and $\pi$ are the same as they must be. I discuss the flows between the two sequences of models, and also argue that the analogous sigma models with $Sp(2N)$ symmetry, the $Sp(2N)/U(N)$ models, flow to $Sp(2N)_1$.

1 Introduction

One very interesting property of field theories is that they can have critical points which are completely unseen in standard weak-coupling perturbation theory. To prove the existence of these fixed points requires using alternative perturbative methods (such as the large-$N$ or large-spin expansion), or relying on non-perturbative methods. One of the remarkable features of two-dimensional field theories is that in some cases these non-trivial critical points can be understood non-perturbatively.

A famous example arose in the study of integer and half-integer spin chains [1]. The spin-1/2 Heisenberg quantum spin chain is exactly solvable, and from Bethe’s exact solution it is known that the spectrum is gapless. The obvious guess for the field theory describing the spin chain in the continuum limit is the sphere sigma model. This sigma model is an $SU(2)$-symmetric field theory where the field takes values on a two-sphere. The two-sphere can be parametrized by three fields $(v_1, v_2, v_3)$ obeying the constraint $(v_1)^2 + (v_2)^2 + (v_3)^2 = 1$. The Euclidean action is

$$\frac{1}{g} \sum_{i=1}^{3} \int dx dy (\partial_\mu v_i)^2$$

This field theory, however, is not the correct continuum description of the spin chain, because it is also exactly solvable, and its spectrum is gapped [2]. It was proposed that that when this sigma model is modified by adding an extra term called the theta term, the spectrum can
become gapless \([1]\). The theta term is inherently non-perturbative: it does not change the beta function derived near the trivial fixed point at all. Nevertheless, the physics can be dramatically different from when \(\theta = 0\) \([3]\). In two dimensions, this can result in a non-trivial fixed point. Not only does the spectrum become gapless at the non-trivial critical point, but in addition, charge fractionalization occurs: the charges of the quasiparticles of the theory are fractions of the charge of the fields in the action.

To define the theta term, field configurations are required to go to a constant at spatial infinity, so the spatial coordinates \((x,y)\) are effectively that of a sphere. Since the field takes values on a sphere as well, the field is therefore a map from the sphere to a sphere. An important characteristic of such maps is that they can have non-trivial topology: they cannot necessarily be continuously deformed to the identity map. This is analogous to what happens when a circle is mapped to a circle (i.e. a rubber band wrapped around a pole): you can do this an integer number of times called the winding number (a negative winding number corresponds to flipping the rubber band upside down). It is the same thing for a sphere: a sphere can be wrapped around a sphere an integer number of times. An example of winding number 1 is the isomorphism from a point on the spatial sphere to the same point on the field sphere. The identity map has winding number 0: it is the map from every point on the spatial sphere to a single point on the field sphere, e.g. \((v_1(x,y), v_2(x,y), v_3(x,y)) = (1,0,0)\). Field configurations with non-zero winding number are usually called instantons. The name comes from viewing one of the directions as time (in our case, one would think of say \(x\) as space and \(y\) as Euclidean time). Since instanton configurations fall off to a constant at \(y = \pm \infty\), the instanton describes a process local in time and hence “instant”. Therefore, the field configurations in the sphere sigma model can be classified by an integer \(n\). This allows a term \(S_\theta = in\theta\) to be added to the action, where \(\theta\) is an arbitrary parameter. Since \(n\) is an integer, the physics is periodic under shifts of \(2\pi\) in \(\theta\). Haldane argued that when \(\theta = \pi\), the sphere sigma model flows to a non-trivial critical point \([1]\).

Around the same time, a similar proposal arose in some more general sigma models \([4]\). These models are most easily formulated by having the field take values in the coset space \(G/H\), where \(G\) and \(H\) are Lie groups, with \(H\) a subgroup of \(G\). In this language, the two-sphere is equivalent to \(O(3)/O(2)\): while the vector \((v_1, v_2, v_3)\) can be rotated by the \(O(3)\) symmetry group, it is invariant under the \(O(2)\) subgroup consisting of rotations around its axis. Thus the space of distinct three-dimensional fixed-length vectors (the sphere) is the coset \(O(3)/O(2)\). In order to describe two-dimensional non-interacting electrons with disorder and a strong transverse magnetic field, one takes \(G = U(2N)\) and \(H = U(N) \times U(N)\) \([4]\). For \(N = 1\), one recovers the sphere sigma model. Pruisken conjectured that in the replica limit \(N \to 0\), this sigma model has a critical point at \(\theta = \pi\). This critical point possibly describes the transitions between integer quantum Hall plateaus. Subsequently, a number of conjectures for non-trivial fixed points in sigma models have been made. For a survey of the applications of these conjectures to disordered systems, see \([5]\).

The purpose of this paper is to generalize this result and prove that several infinite hierarchies of sigma models have non-trivial fixed points when \(\theta = \pi\). A useful tool is to study the spectrum and scattering matrix of the particles in an equivalent \(1+1\)-dimensional formulation. I will show that when these sigma models have \(\theta = 0\), they are gapped, with the spectrum consisting of massive particles in the symmetric representation of \(SU(N)\) and \(O(2P)\) respectively (plus bound states in more general representations, and kinks with fractional charge in the latter case). When \(\theta = \pi\), the spectrum consists of gapless quasiparticles which are in the fundamental
representations (vector, antisymmetric tensor, . . . ) of \( SU(N) \) and \( O(2P) \) (the latter including kinks in the spinor representation).

The stable low-energy fixed points for the sphere sigma model and all the sigma models discussed in this paper are Wess-Zumino-Witten (WZW) models. The WZW model for a group \( H \) is a sigma model where the field takes values in \( H \times H/H \approx H \). It also has an extra term, called the Wess-Zumino term, which has an integer coefficient \( k \). In two dimensions, when \( k \neq 0 \), the model has a stable low-energy fixed point \( \mathcal{H}_k \). The conformal field theory describing this fixed point is called the \( \mathcal{H}_k \) WZW model.

One argument for the flow in the sphere sigma model at \( \theta = \pi \) goes as follows \([7]\). First one uses Zamolodchikov’s \( c \)-theorem, which makes precise the notion that as one follows renormalization group flows, the number of degrees of freedom goes down. Zamolodchikov shows that there is a quantity \( c \) associated with any two-dimensional unitary field theory such that \( c \) must not increase along a flow. At a critical point, \( c \) is the central charge of the corresponding conformal field theory \([8]\). At the trivial fixed point of a sigma model where the manifold is flat, the central charge is the number of coordinates of the manifold. For the sphere, this means that \( c = 2 \) at the trivial fixed point. This if the sphere sigma model flows to a non-trivial fixed point for \( \theta = \pi \), this fixed point must have \( O(3) \approx SU(2) \) symmetry and must have central charge less than 2.

The only such unitary conformal field theories are \( SU(2)_k \) for \( k < 4 \). (The central charge of \( SU(2)_k \) is \( 3k/(k + 2) \); in general, the central charge of \( H_k \) is \( k \text{dim}(H)/(k + h) \), where \( h \) is the dual Coxeter number of \( H \)). One can use the techniques of \([9]\) to show that there are relevant operators at these fixed points, and at \( k = 2 \) or 3, no symmetry of the sphere sigma model prevents these relevant operators from being added to the action \([6]\). So while it is conceivable that the sphere sigma model with \( \theta = \pi \) could flow near to these fixed points, these relevant operators would presumably appear in the action and cause a flow away. However, there is only one relevant operator (or more precisely, a multiplet corresponding to the WZW field \( w \) itself) for the \( SU(2)_1 \) theory. The sigma model has a discrete symmetry \( (v_1, v_2, v_3) \rightarrow (-v_1, -v_2, -v_3) \) when \( \theta = 0 \) or \( \pi \); the winding number \( n \) goes to \( -n \) under this symmetry, but \( \theta = \pi \) and \( \theta = -\pi \) are equivalent because of the periodicity in \( \theta \). This discrete symmetry of the sigma model turns into the symmetry \( w \rightarrow -w \) of the WZW model. While the operator \( tw \) is \( SU(2) \) invariant, it is not invariant under this discrete symmetry. Therefore, this operator is forbidden from appearing in the effective action. The operator \((tw)^2\) is irrelevant, so it is consistent for the sphere sigma model at \( \theta = \pi \) to have the \( SU(2)_1 \) WZW model as its low-energy fixed point. A variety of arguments involving the spin chain strongly support this conjecture \([1, 4]\).

An important question is therefore whether the existence of these non-perturbative fixed points in sigma models at \( \theta = \pi \) can be established definitively. The fact that the sphere sigma model has a non-trivial fixed point at \( \theta = \pi \) was proven in \([10, 11]\). This proof does not involve the spin-1/2 chain which motivated the result. Rather, it is a statement about the non-trivial fixed point in the sphere sigma model at \( \theta = \pi \). This proof utilizes the integrability of the sphere sigma model at \( \theta = 0 \) and \( \pi \). Integrability means that there are an infinite number of conserved currents which allow one to find exactly the spectrum of quasiparticles and their scattering matrix in the corresponding \( 1+1 \) dimensional field theory. The quasiparticles for \( \theta = 0 \) are gapped and form a triplet under the \( SU(2) \) symmetry, while for \( \theta = \pi \) they are gapless, and form \( SU(2) \) doublets (left- and right-moving) \([2, 10]\). This is a beautiful example of charge fractionalization: the fields \((v_1, v_2, v_3)\) form a triplet under the \( SU(2) \) symmetry, but when \( \theta = \pi \) the excitations of the system are doublets. To prove that this is the correct particle spectrum, first one computes a scattering matrix for these particles which is consistent with all the symmetries of the theory. From the exact \( S \) matrix, the \( c \) function can be computed. It was found that at high energy \( c \)
indeed is 2 as it should be at the trivial fixed point, while \( c = 1 \) as it should be at the \( SU(2) \)
low-energy fixed point [10]. As an even more detailed check, the free energy at zero temperature
in the presence of a magnetic field was computed for both \( \theta = 0 \) [12] and \( \pi \) [11]. The results can
be expanded in a series around the trivial fixed point. One can identify the ordinary perturbative
contributions to this series, and finds that they are the same for \( \theta = 0 \) and \( \pi \), even though the
particles and \( S \) matrices are completely different [11]. This is as it must be: instantons and the
\( \theta \) term are a boundary effect and hence cannot be seen in ordinary perturbation theory. One
can also identify the non-perturbative contributions to these series, and see that they differ. Far
away from the trivial fixed point, non-perturbative contributions can dominate which allow a
non-trivial fixed point to appear when \( \theta = \pi \) even though there is none at \( \theta = 0 \).

The purpose of this paper is to generalize these computations to the \( SU(N)/SO(N) \) and
\( O(2P)/O(P) \times O(P) \) sigma models. To have any hope of being able to take the replica limit
\( N \to 0 \), one needs a solution for any \( N \). The \( SU(N)/SO(N) \) sigma model reduces to the sphere
sigma model when \( N = 2 \), while \( O(2P)/O(P) \times O(P) \) reduces to two copies of the sphere when
\( P = 2 \). Thus these sigma models are the generalizations of the sphere sigma model with \( SU(N) \)
and \( O(2P) \) symmetry respectively. One difference, however, is that in general they do not allow
a continuous \( \theta \) parameter, but instead \( \theta \) can only be zero or \( \pi \). I will show that when \( \theta = \pi \), the
\( SU(N)/SO(N) \) and \( O(2P)/O(P) \times O(P) \) sigma models have stable low-energy fixed points. The
corresponding conformal field theories are the \( SU(N)_1 \) and \( O(2P)_1 \) WZW models, respectively.

In section 2 I define the sigma models. In section 3, I review \( S \) matrices in an integrable
model with a global symmetry \( G \). In section 4, I discuss the Gross-Neveu model. This is not
a sigma model, but is closely related. These results are crucial in what follows. In section 5,
I will find the particles and their scattering matrices for the sigma models. In section 6, I will
do a substantial check on this picture by using the exact \( S \) matrix to compute the energy in
the presence of a background field. In particular, I check that the perturbative contributions to
this energy are in agreement with conventional perturbation theory, and that the expansions for
\( \theta = 0 \) and \( \pi \) are the same (but the non-perturbative contributions differ). In section 7, I discuss
the low-energy fixed points at \( \theta = \pi \), and flows between them. I also present a conjecture for
the low-energy behavior of the \( Sp(2N)/U(N) \) sigma model. This last section is written to be
reasonably self-contained, so readers interested in the results but not the details (and willing to
trust me) can skip sections 3-6.

2 The models

The sigma models discussed in this paper all can be written conveniently in terms of an matrix
field \( \Phi \) with action

\[
S = \frac{1}{g} \text{tr} \int d^2x \, \partial^\mu \Phi^\dagger \partial_\mu \Phi
\]

along with the constraints

\[
\Phi^\dagger \Phi = I \\
det(\Phi) = \pm 1
\]

where \( I \) is the identity matrix. In other words, \( \Phi \) is always a unitary matrix with determinant
\( \pm 1 \), with possibly additional constraints. The constraints (3) can easily be obtained from theories
without constraints by adding a potentials like \( \lambda \text{tr} (\Phi^\dagger \Phi - I)^2 \). When \( \lambda \) gets large, one recovers
the restrictions (3). In theories with interacting fermions, this often results from introducing a
The homotopy group is just the group of winding numbers of maps from the sphere to the sphere; it is the integers. The general answer is that \( \pi_2 \) is non-trivial. The second homotopy group is just the group of winding numbers of maps from the sphere to \( G/H \), so for the sphere it is the integers. The general answer is that \( \pi_2(G/H) \) is the kernel of the embedding of \( \pi_1(H) \) into \( \pi_1(G) \), where \( \pi_1(H) \) is the group of winding numbers for maps of the circle into

bosonic field to replace four-fermion interaction terms with Yukawa terms (interactions between a boson and two fermions). Integrating out the fermions then results in such potentials and hence the sigma model.

The sigma models discussed in this paper are obtained by putting additional restrictions of the matrix field \( \Phi \). The two series of models studied both require that \( \Phi \) be symmetric as well as unitary, so the matrices do not form a group: multiplying symmetric matrices does not necessarily give a symmetric matrix. Instead, these spaces are of \( G/H \) form. The \( SU(N)/SO(N) \) sigma model is obtained by requiring that \( \Phi \) be a symmetric \( N \times N \) unitary matrix with determinant 1. The \( O(2P)/O(P) \times O(P) \) sigma model is obtained by requiring that \( \Phi \) be a real, symmetric, orthogonal and traceless \( 2P \times 2P \) matrix. In general, in two dimensions a \( G/H \) sigma model has a global symmetry \( G \). The action \[ 3 \] and restriction \[ 3 \] are invariant under the symmetry

\[ \Phi \rightarrow U \Phi U^T, \]

where \( U \) is a unitary matrix with determinant one. This is the most general symmetry which keeps \( \Phi \) symmetric and unitary. In the \( O(2P)/O(P) \times O(P) \) sigma models, \( U \) must be real as well, so \( G = O(2P) \). The field \( \Phi \) in this case can diagonalized with an orthogonal matrix \( U \), so

\[ \Phi = U \Lambda U^T \quad \Phi \in O(2P)/O(P) \times O(P), \]

where \( U \) is in \( O(2P) \), and \( \Lambda \) is the matrix with \( P \) values +1 and \( P \) values −1 on the diagonal. Different \( U \) can result in the same \( \Phi \): the subgroup leaving \( \Phi \) invariant is \( H = O(P) \times O(P) \). Similarly, field configurations in the \( SU(N)/SO(N) \) sigma model can be written in the form

\[ \Phi = U U^T \quad \Phi \in SU(N)/SO(N) \]

where \( U \) is in \( SU(N) \). The subgroup \( H \) leaving \( \Phi \) invariant is \( SO(N) \). For example, \( \Phi = I \) for any real \( U \) in \( SU(N) \), i.e. if \( U \) is in the real subgroup \( SO(N) \) of \( SU(N) \). This is why \( H = SO(N) \) here.

These spaces \( G/H \) studied here are examples of symmetric spaces. A symmetric space \( G/H \) has \( H \) a maximal subgroup of \( G \) (no subgroup other than \( G \) itself contains \( H \)). The important property of a sigma model on a symmetric space is that it contains only one coupling constant \( g \) in the action. In other words, the space \( G/H \) preserves its “shape” under renormalization, with only the overall volume changing. The one and two-loop beta functions for all two-dimensional sigma models are universal and can be expressed in terms of the curvature of the field manifold \[ 3 \]. The effect of renormalization is to increase the curvature (increase \( g \)). Even though naively there is no mass scale in the theory (\( g \) is dimensionless), physical quantities depend on a scale (e.g., a lattice length or a momentum cutoff) as a result of short-distance effects. The renormalized coupling depends on this scale. At short distances, \( g \) is small, so the theory is effectively free: it is a theory of \( \text{dim}G - \text{dim}H \) free fields. At longer distances the theory is interacting: the \( \text{dim}G - \text{dim}H \) fields interact because of the constraint \[ 3 \]. In renormalization-group language, there is an unstable trivial fixed point at \( g = 0 \), and no non-trivial low-energy fixed point unless one includes a theta or WZW term.

Both sets of sigma models allow a theta term to be added to the action. This is a result long ago proven by mathematicians (for a discussion accessible to physicists, see \[ 4 \]). In mathematical language, the question is whether the second homotopy group \( \pi_2(G/H) \) is non-trivial. The second homotopy group is just the group of winding numbers of maps from the sphere to \( G/H \), so for the sphere it is the integers. The general answer is that \( \pi_2(G/H) \) is the kernel of the embedding of \( \pi_1(H) \) into \( \pi_1(G) \), where \( \pi_1(H) \) is the group of winding numbers for maps of the circle into
The rubber band on a pole example means that \( \pi_1(H) \) is the integers when \( H \) is the circle \( U(1) = SO(2) \). The only simple Lie group \( H \) for which \( \pi_1 \) is nonzero is \( SO(N) \), where \( \pi_1(SO(N)) = \mathbb{Z}_2 \) for \( N \geq 3 \) and \( \mathbb{Z} \) for \( N=2 \). Thus there are models with integer winding number, some with just winding number 0 or 1, and some with no instantons at all. Integer winding number means that \( \theta \) is continuous and periodic, while a winding number of 0 or 1 means that \( \theta \) is just 0 or \( \pi \) (just think of \( \theta \) as being the Fourier partner of \( n \)). The \( SU(N)/SO(N) \) and \( O(2P)/O(P) \times O(P) \) sigma models therefore have instantons with \( \mathbb{Z}_2 \) winding number, and so \( \theta \) can be zero or \( \pi \).

3 Generalities on exact S matrices

Even though the sigma models are originally defined in two-dimensional Euclidean space \((x, y)\), it is very convenient to continue to real time \( t = iy \). The reason is to treat the field theory as a 1 + 1 dimensional particle theory. A fundamental property of many field theories is that all states of the theory can be written in terms of particles. In other words, the space of states is combination of one-particle states, usually called a Fock space. If the theory has a stable non-trivial fixed point at low energy, then the particles should be massless: the energy is linearly related to the momentum: \( E = |P| \). If there is no fixed point, then the particles are massive: \( E = \sqrt{P^2 + M^2} \) in a Lorentz-invariant theory. In condensed-matter physics, the particles are often called “quasiparticles”, to emphasize the fact that the particles may not be the same as the underlying degrees of freedom: just because a system is made up of electrons does not mean that the collective excitations are electrons, or even resemble them. This phenomenon will occur frequently in this paper.

One finds the scattering matrix of an integrable model by utilizing a variety of constraints: unitarity, crossing symmetry, global symmetries, the consistency of bound states, and the factorization (Yang-Baxter) equations. It is convenient to write the momentum and energy of a particle in terms of its rapidity \( \theta \), defined by \( E = m \cosh \theta \), \( P = m \sinh \theta \). Lorentz invariance requires that the two-particle \( S \) matrix depends only on the rapidity difference \( \theta_1 - \theta_2 \) of the two particles.

The invariance of the \( G/H \) sigma model under the Lie-group symmetry \( G \) requires that the \( S \) matrices commute with all group elements. The \( S \) matrix can then be conveniently written in terms of projection operators. A projection operator \( \mathcal{P}_k \) maps the tensor product of two representations onto an irreducible representation labelled by \( k \). By definition, these operators satisfy \( \mathcal{P}_k \mathcal{P}_l = \delta_{kl} \mathcal{P}_k \). Requiring invariance under \( G \) means that the \( S \) matrix for a particle in the representation \( a \) with one in a representation \( b \) means that the \( S \) matrix is of the form

\[
S^{ab}(\theta) = \sum_k f_k^{ab}(\theta) \mathcal{P}_k
\]  

where \( \theta \equiv \theta_a - \theta_b \) is the difference of the rapidities, and the \( f_k^{ab} \) are as of yet unknown functions. The sum on the right-hand side is over all representations \( k \) which appear in the tensor product of \( a \) and \( b \); of course \( \sum_k \mathcal{P}_k = 1 \). In an integrable theory, the functions \( f_k^{ab}(\theta) \) are determined up to an overall function by requiring that they satisfy the Yang-Baxter equation. This stems from the requirement that the \( S \)-matrix be factorizable: the multiparticle scattering amplitudes factorize into a product of two-particle ones. There are two possible ways of factorizing the three-particle amplitude into two-particle ones; the requirement that they give the same answer is the Yang-Baxter equation. There have been hundreds of papers discussing how to solve this
equation, so I will not review this here. For a detailed discussion relevant to the sigma models here, see e.g. [15, 16, 17]. Solutions arising in the sigma models will be given below.

To obtain the overall function not given by the Yang-Baxter equation, one needs to require that the $S$ matrix be unitary, and that it obey crossing symmetry. With the standard assumption that the amplitude is real for $\theta$ imaginary, the unitarity relation $S(\theta)S(-\theta) = I$ implies

$$S(\theta)S(-\theta) = I. \text{ The latter is more useful because it is a functional relation which can be continued throughout the complex $\theta$ plane. Crossing symmetry is familiar from field theory, where}$$

rotating Feynman diagrams by $90^\circ$ relates scattering of particles $a_i$ and $b_j$ to the scattering of the antiparticle $\bar{a}_i$ with $b_j$. In $S$ matrix form, it says that

$$S^{ab}(i\pi - \theta) = C^a S^{ab}(\theta) C^a,$$

where $C^a$ is the charge-conjugation operator acting on the states in representation $a$.

Multiplying any $S$ matrix by function $F(\theta)$ which satisfies $F(\theta)F(-\theta) = 1$ and $F(i\pi - \theta) = F(\theta)$ will give an $S$ matrix still obeying the Yang-Baxter equation, crossing and unitarity (this is called the CDD ambiguity). To determine $F(\theta)$ uniquely, one ultimately needs to verify that the $S$ matrix gives the correct results for the free energy. How to do this will be described in the following sections. Before doing this calculation, one must make sure an additional criterion holds: the poles of the $S$ matrix are consistent with the bound-state spectrum of the theory. The $S$ matrices of bound states are related to those of the constituents by the bootstrap relation, which can be formulated as follows [15, 16]. Poles of $S^{ab}$ matrix elements at some value $\theta = \theta_{ab}$ with $\theta_{ab}$ imaginary and in the “physical strip” $0 < \text{Im}(\theta_{ab}) < \pi$ are usually associated with bound states. Each of the functions $f^{ab}_k(\theta)$ has a residue at this pole $R_k$. Then there are bound states in representation $k$ if $R_k \eta_k < 0$, where $\eta_k = \pm 1$ is the parity of states in representation $k$ [18] (poles which do not correspond to bound states give bound states in the process obtained by crossing). These bound states $(ab)$ have mass

$$(m_{(ab)})^2 = (m_a)^2 + (m_b)^2 + 2m_am_b \cosh \theta_{ab}$$

The bound states are not in a irreducible representation of $G$ if more than one of the residues $R_k$ is non-zero. Then the $S$ matrix for scattering the bound state $(ab)$ from another particle $c$ is given by

$$S^{(ab)c} = \left( \sum_k \frac{1}{\sqrt{|R_k|}} P_k \right) S^{ac}(\theta + i\frac{\theta_{ab}}{2}) S^{bc}(\theta - i\frac{\theta_{ab}}{2}) \left( \sum_k \frac{1}{\sqrt{|R_k|}} P_k \right),$$

where the projection operators $P_k$ act on the states in representations $a, b$. Note that the matrices in [18] are not all acting on the same spaces, so this relation is to be understood as multiplying the appropriate elements (not matrix multiplication).

All of the above considerations apply to massless particles as well, with a few modifications and generalizations [10, 19]. The theories being studied are along a flow into their stable low-energy fixed point. Even though the quasiparticles are gapless, there is still a mass scale $M$ describing the crossover: $M \to 0$ is the unstable high-energy fixed point, while $M \to \infty$ is the stable low-energy one. The rapidity variable for a massless particle is then defined via $E = me^\theta$, $P = me^\theta$ for right movers and $E = me^{-\theta}$, $P = -me^{-\theta}$ for left movers, where $m$ is not really the mass of the particle but is proportional to $M$ (different particles can and do have different values of $m$). The $S$ matrix still depends only on rapidity differences. Note in particular that the $S$ matrices for two left movers $S_{LL}$ or two right movers $S_{RR}$ depend on the ratio $E_1/E_2$ of
the two particles, and so do not depend on the mass scale $M$ at all. These are determined solely by properties of the low-energy fixed point. On the other hand, $S_{LR}$ depends on the physics of how one flows into this fixed point. Massless particles still can have bound states like massive ones. These bound states show up as poles in $S_{LL}$ and $S_{RR}$, but it does not seem possible to have poles in $S_{LR}$ in the physical strip $[10]$.

4 The Gross-Neveu models

Before discussing the particle content of the sigma models, it is useful to first discuss the Gross-Neveu models. These are not sigma models, but are closely related: they are asymptotically-free field theories with a global symmetry $G$. For groups $O(N)$ and $U(N)$ they were originally formulated in terms of fermions with a four-fermion interaction. For $O(N)$, the action was first written in terms of $N$ Majorana (real) fermions with a four-fermion interaction: $[20]$

$$S = \int \sum_{j=1}^{N} \left( \psi_{L}^{j} i \partial_{L} \psi_{L}^{j} + \psi_{R}^{j} i \partial_{R} \psi_{R}^{j} \right) + g \left( \sum_{j=1}^{N} \psi_{L}^{j} \psi_{R}^{j} \right)^{2}. \quad (8)$$

For $SU(N)$ the action can also be written in terms of fermions (this is sometimes known as the chiral Gross-Neveu model $[22]$), but for general $G$ the Gross-Neveu models are most easily defined in terms of the $G$ Wess-Zumino-Witten model at level 1. The WZW model has a local symmetry $G_{L} \times G_{R}$ generated by chiral currents $j_{a}^{L}$ and $j_{a}^{R}$ (the explicit definitions are given in section 7 below). The action of the $G$ Gross-Neveu model is then a perturbation of the $G_{1}$ WZW action:

$$S = S(G_{1}) + g \sum_{a} \int j_{a}^{L} j_{a}^{R} \quad (9)$$

The coupling $g$ is naively dimensionless, but it has lowest-order beta function

$$\beta(g) \propto g^{2},$$

where the constant of proportionality is negative. This means that for positive $g$ the coupling is marginally relevant $[20]$. In other words, the WZW fixed point in (1) is unstable and (8) defines a massive field theory called the Gross-Neveu model. For negative $g$, the coupling is marginally irrelevant and the WZW fixed point is stable, a fact which will become important for sigma models with $\theta = \pi$.

As one would expect, there are particles in the $O(N)$ Gross-Neveu model with the same quantum numbers as the fermions in (8). These are in the defining ($N$-dimensional vector) representation of $O(N)$. For general $G$, this is also true: there are particles in the defining representation of $G$. However, there is more. For $O(N)$, there are kinks as well, as follows from a semi-classical analysis of the action (8). These particles are in the spinor representations of $O(N)$ $[23]$. Thus the $O(N)$ Gross-Neveu model exhibits charge fractionalization. For even $N$, there are two spinor representations, of dimension $2^{N/2-1}$; for odd $N$ there is one spinor representation, of dimension $2^{(N-1)/2}$. However, there still more quasiparticles, which are bound states of the fermions. As will follow from studying the bound-state structure of the exact $S$ matrix, there are particles in all of the fundamental representations. The fundamental representations for any Lie group are the representations with highest weight vectors $\mu^{i}$ whose inner product with the simple root vectors $\alpha^{j}$ obeys $\mu^{i} \cdot \alpha^{j} = \delta^{ij} \quad [24]$. There is one fundamental representation corresponding
notations are explicitly to a finite number of particles. For SU(groups, the Gross-Neveu bootstrap closes if \( \Delta = \frac{1}{\hbar} \). Gross-Neveu model is \([22]\) one does not generate an infinite number of particles: the poles in all the strip. This means that the rest of this paper, \( \Delta = \frac{1}{\hbar} \) and the bootstrap together means that the pole must be at \( \theta \). Thus the projection operators on the two-index symmetric (S) and antisymmetric (A) representations are explicitly

\[
P_S(a_i(\theta_a)b_j(\theta_b)) = \frac{1}{2} (a_i(\theta_a)b_j(\theta_b) + a_j(\theta_a)b_i(\theta_b))
\]

\[
P_A(a_i(\theta_a)b_j(\theta_b)) = \frac{1}{2} (a_i(\theta_a)b_j(\theta_b) - a_j(\theta_a)b_i(\theta_b)).
\]

The subscripts in \( a_i \) and \( b_j \) represent the \( i^{th} \) and \( j^{th} \) particles in the vector multiplets, so the S matrix in \([9]\) is indeed a matrix.

Requiring that the S matrix obey the Yang-Baxter equation means that

\[
\frac{f^{VV}_S}{f^{VV}_A} = \frac{\theta + 2\pi i \Delta}{\theta - 2\pi i \Delta}
\]

where \( \Delta \) will be determined shortly. The simplest solution for the overall function consistent with crossing symmetry and unitarity for particles in the vector representation of SU(N) is \( f^{VV}_S = S^{VV}_{min} \)

\[
S^{VV}_{min} = \frac{\Gamma(1 - \frac{\theta}{2\pi i}) \Gamma\left(\frac{\theta}{2\pi i} + \Delta\right)}{\Gamma(1 + \frac{\theta}{2\pi i}) \Gamma\left(-\frac{\theta}{2\pi i} + \Delta\right)}
\]

This is called the minimal solution, because the resulting S matrix has no poles in the physical strip: a zero in \([13]\) cancels the pole in \([12]\). It is not the unique solution, because of the CDD ambiguity.

The Gross-Neveu model has bound states and so the S matrix must have poles in the physical strip. This means that \( f^{VV}_S = S^{VV}_{min} X(\theta) \), where \( X(\theta) \) contains the poles. If there are to be bound state particles with in the antisymmetric representation of SU(N), but not in the symmetric representation, then there must be a pole in \( f^{VV}_A \) but not in \( f^{VV}_S \). The relation \([12]\) means the pole must be at \( \theta = 2\pi i \Delta \) and so particles in the antisymmetric representation have mass \( m_A/m_V = 2 \cos \pi \Delta \). To determine \( \Delta \), one must ensure that the bootstrap closes. This means that one does not generate an infinite number of particles: the poles in all the S matrices correspond to a finite number of particles. For SU(N), the bootstrap closes if \( \Delta = 1/N \). For general groups, the Gross-Neveu bootstrap closes if \( \Delta = 1/h \), where \( h \) is the dual Coxeter number. In the rest of this paper, \( \Delta = 1/h \), with \( h \) appropriate for the case at hand. Putting \([11][12][13]\) and the bootstrap together means that the S matrix for the scattering of vector particles in the Gross-Neveu model is \([22]\)

\[
S^{VV}_{GN}(\theta) = X(\theta)S^{VV}_{min}(\theta) \left( P_S + \frac{\theta + 2i\pi \Delta}{\theta - 2i\pi \Delta} P_A \right)
\]
where
\[
X(\theta) = \frac{\sinh((\theta + 2\pi i\Delta)/2)}{\sinh((\theta - 2\pi i\Delta)/2)}.
\]

The extra factor \(X(\theta)\) indeed reinstates the pole at \(\theta = 2\pi i\Delta\) canceled by \(S_{\text{min}}\). Using this bootstrap to compute the \(S\) matrices of the bound states, one finds for example that the scattering matrix \(S^{AV}\) for a particle in the vector representation with one in the antisymmetric representation has a pole at \(\theta = 3\pi i\Delta\), leading to particles with mass \(m_3/m_V = \sinh(3\Delta\pi)/\sinh(\Delta\pi)\). These particles are in the representation with highest weight \(\mu_3\) (the Young tableau with 3 boxes and one column, or equivalently the three-index antisymmetric tensor).

One keeps repeating this bootstrap procedure and finds particles in all the antisymmetric representations, ending up with the particles in the conjugate representation arising from the bound state of \(N - 1\) vector particles. This provides a substantial check on the \(S\) matrix, because the crossing relation (6) relates \(S_{VV}\) to \(S_{\bar{V}\bar{V}}\). One can indeed check that the \(S\) matrices built up from (14) satisfy this relation. For the \(SU(N)\) Gross-Neveu model, there is one copy of each fundamental representation [21]. The dimension of the antisymmetric tensor with \(j\) indices (\(j\) boxes in the one-column Young tableau) is \(N!/(j!(N-j)!))\), so there are \(2^N - 2\) particles in all. Each multiplet of particles has mass \(M \sin \pi j/N\), where \(M\) is an overall scale related to the coupling constant \(g\). The particles in the \(N - j\) representation are the antiparticles of those in the \(j\) representation. One can check that there are no additional particles by computing the energy in a background field as done in subsequent sections [25], or by computing the free energy at non-zero temperature [26]. Moreover, the \(SU(N)\) Gross-Neveu model can be solved directly using the Bethe ansatz [22]; the results agree with those above.

The \(S\) matrices for the \(O(N)\) Gross Neveu model are derived in the same manner as those with \(SU(N)\) symmetry, but there are a number of additional complications.

The scattering of particles both in the vector representation of \(O(N)\) is of the form
\[
S_{VV} = F_{SV}^{VV} P_S + F_{AV}^{VV} P_A + F_0^{VV} P_0,
\]
where the projection operators on the symmetric (\(S\)), antisymmetric (\(A\)) and singlet (0) representations are explicitly
\[
\begin{align*}
P_S(a_ib_j) &= \frac{1}{2} (a_ib_j + a_jb_i) - \frac{1}{N} \delta_{ij} \sum_{k=1}^{N} a_kb_k \\
P_A(a_ib_j) &= \frac{1}{2} (a_ib_j - a_jb_i) \\
P_0(a_ib_j) &= \frac{1}{N} \delta_{ij} \sum_{k=1}^{N} a_kb_k
\end{align*}
\]
(17)

where I have suppressed the \(\theta\) dependence. For example, the scattering process \(a_1b_1 \rightarrow a_1b_1\) has \(S\) matrix element \((N-2)F_S^{VV} + N F_A^{VV} + 2F_0^{VV})/2N\), while the process \(a_1b_1 \rightarrow a_2b_2\) has element \((F_0^{VV} - F_S^{VV})/N\). The extra term in (16) as compared to (13) stems from the fact that the trace \(\delta_{ij}a_ib_j\) is an \(SO(N)\) invariant.

Requiring that the \(S\) matrix obey the Yang-Baxter equation means that
\[
\frac{F_A^{VV}}{F_S^{VV}} = \frac{\theta + 2\pi i\Delta}{\theta - 2\pi i\Delta}.
\]
(18)
The minimal solution (no poles in the physical strip) for the overall function for $O(N)$ with particles in the vector is $F_{SV}^{VV} = S_{min}^{VV}$, where

$$
S_{min}^{VV}(\theta) = \frac{\Gamma \left(1 - \frac{\theta}{2\pi} + \Delta\right) \Gamma \left(\frac{\theta}{2\pi} + \Delta\right)}{\Gamma \left(1 + \frac{\theta}{2\pi} - \Delta\right) \Gamma \left(-\frac{\theta}{2\pi} + \Delta\right) \Gamma \left(\frac{\theta}{2\pi} + \Delta\right) \Gamma \left(\frac{\theta}{2\pi} - \Delta\right)}
$$

The $S$ matrix with $F_{SV}^{VV} = S_{min}^{VV}$ defined by $[10, 11, 19, 20]$ is the $S$ matrix for the $O(N)/O(N-1)$ sigma model $[2]$. The $S$ matrix does not contain any poles because the $O(N)/O(N-1)$ model has no particles in any representations other than the vector, and hence no bound states.

The $O(N)$ Gross-Neveu model has bound states. Just like in $SU(N)$, there are particles in all the antisymmetric representations, whose $S$ matrices follow from the bootstrap. These representations are self-conjugate: there is no $\bar{N}$ representation. That means that both the bound-state ($s$-channel) pole at $\theta = 2\pi i \Delta$ and the crossed pole ($t$-channel) channel at $\theta = i\pi - 2\pi i \Delta$ appear in $S_{VN}^{VV}$, yielding

$$
S_{VN}^{VV}(\theta) = X(\theta) X(\pi i - \theta) S_{min}^{VV}(\theta) \left( P_S + \frac{\theta + 2\pi i \Delta}{\theta - 2\pi i \Delta} P_A + \frac{\theta + 2\pi i \Delta - i\pi}{\theta - 2\pi i \Delta} P_0 \right)
$$

where $X(\theta)$ is as in $[13]$ with $\Delta = 1/h$, with the dual Coxeter number here $h = N - 2$. This value of $\Delta$ ensures the bootstrap closes. The particles obtained by fusing the vector particle $j$ times therefore have mass $m_j/m_V = 2 \sin(j\pi \Delta)/\sin(\pi \Delta)$. One difference between the $O(N)$ and $SU(N)$ Gross-Neveu models is apparent in $[21]$: if there is a pole in the antisymmetric channel at $\theta = 2\pi i \Delta$, there is also one in the singlet channel. This means that there are extra bound states in the $O(N)$ Gross-Neveu model. For example, there are $N(N - 1)/2 + 1$ particles with mass $m_2$: $N(N - 1)/2$ of them transform in the two-index antisymmetric representation of $O(N)$, while one is a singlet under $O(N)$. In general, at mass $m_j$ there are particles in all the $k$-index completely antisymmetric representations $k = 0, 2, \ldots j$ for even $j$ and $k = 1, 3, \ldots j$ for odd $j$ $[13]$. Despite this additional complication, one can in principle obtain the $S$ matrices for all of these states by using fusion. This was done explicitly for $k = 2$ in $[17]$.

The $S$ matrix for the kinks in the spinor representation in the $O(N)$ Gross-Neveu model is given in $[13, 10, 27]$. It can be written out explicitly, but since the answer is somewhat complicated I will not give it here. It is derived by using the fact that kink-kink bound states contain the particles in the antisymmetric representations as bound states; in fact is a sort of reverse bootstrap. The mass of the spinor states is given by $m_{\text{spinor}}/m_v = 1/\sin(\pi \Delta)$.

In general, the particles of the $G$ Gross-Neveu theory are in fundamental representations of $G$. Their masses and multiplicities are known from considering the bound-state properties of the exact $S$ matrix.

5 The particles and their scattering matrices

In this section I give the particle content and exact scattering matrices of the $SU(N)/SO(N)$ and $O(2P)/O(P) \times O(P)$ sigma models. In the next section, I will use the exact $S$ matrix to compute the energy in a background field.

The representations of the global symmetries of sigma models are somewhat more elaborate than that in the Gross-Neveu model. The results are different for each model, and also change dramatically if certain extra terms are added to the action $[3]$. 

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5.1  $SU(N)/SO(N)$ sigma model

For $SU(N)/SO(N)$, the action is written in terms of a symmetric unitary matrix field $\Phi$. Thus as opposed to the Gross-Neveu model, one expects there to be particles forming the symmetric representation of $SU(N)$, which has dimension $N(N+1)/2$. Indeed, it was long ago established that for $N = 2$ (the sphere sigma model), the particles are in the symmetric (triplet) representation of $SU(2)$ \cite{2}. For any $N$, there are non-trivial non-local conserved currents in this sigma model \cite{28}. These conserved currents are consistent with particles in the symmetric representation, and also with bound states in all representations of $SU(N)$ with Young tableau with two columns and rectangular (i.e. the same number of boxes in each of the columns). In group-theory language, these are representations with highest weight 2$\mu$ (the fundamental representations arising in the Gross-Neveu model have highest weight $\mu$).

Building the bound-state $S$ matrices with the bootstrap relation \cite{3} is one example of a general procedure called fusion \cite{29}. Fusion exploits the fact that the sum of projection operators to find new solutions of the Yang-Baxter equation. It is useful for things other than just the bound states in a given theory. The solution of the Yang-Baxter equation $SSS$ for particles are in the symmetric representation can also be obtained from $SVV$. The solution is obtained from \cite{3} like before, using instead the pole at $\theta = -2\pi i\Delta$. The tensor product of two symmetric representations contains three representations with Young tableaux

\[
4\mu_1 = \begin{array}{l}
\end{array}, \quad 2\mu_1 + \mu_2 = \begin{array}{l}
\end{array}, \quad 2\mu_2 = \begin{array}{l}
\end{array} \quad \text{(22)}
\]

It is convenient to label particles in the symmetric representation with two indices and the constraint $a_{ij} = a_{ji}$. The projection operators are then explicitly

\[
\mathcal{P}_{4\mu_1}(a_{ij}b_{kl}) = \frac{1}{6}(a_{i}b_{kl} + a_{ik}b_{jl} + a_{il}b_{jk} + a_{kl}b_{ij} + a_{kj}b_{il} + a_{jl}b_{ik})
\]

\[
\mathcal{P}_{2\mu_1+\mu_2}(a_{ij}b_{kl}) = \frac{1}{2}(a_{ij}b_{kl} - a_{kl}b_{ij})
\]

\[
\mathcal{P}_{2\mu_2}(a_{ij}b_{kl}) = \frac{1}{6}(2a_{ij}b_{kl} + 2a_{ik}b_{jl} - a_{il}b_{jk} - a_{jl}b_{ik} - a_{jk}b_{il} - a_{jl}b_{ik})
\]

This means, for example, that the $S$ matrix element for scattering an initial state of $a_{12}$ and $b_{13}$ to a final state of $a_{11}$ and $b_{23}$ is $(2\mathcal{S}_{4\mu_1} + 3\mathcal{S}_{2\mu_1+\mu_2} + \mathcal{S}_{2\mu_2})/6$, while the element for scattering the same two particles and getting $a_{11}$ and $b_{23}$ in the final state is $(\mathcal{S}_{4\mu_1} - 2\mathcal{S}_{2\mu_2})/6$.

This enables us to find the $S$ matrix for the particles in the $SU(N)/SO(N)$ sigma model at $\theta = 0$ in the symmetric representation:

\[
SSS = X(\theta)SS_{min}(\theta) \left( \mathcal{P}_{4\mu_1} + \frac{\theta + 4\pi i\Delta}{\theta - 4\pi i\Delta} \mathcal{P}_{2\mu_1+\mu_2} + \frac{\theta + 2\pi i\Delta}{\theta - 2\pi i\Delta} \mathcal{P}_{2\mu_2} \right), \quad \text{(23)}
\]

where $X(\theta)$ is the same as in the Gross-Neveu model. The overall function $SS_{min}$ is determined by using unitarity, crossing and the bootstrap. The result is that

\[
SS_{min}(\theta) = \frac{\theta - 2\pi i\Delta}{\theta + 2\pi i\Delta} \frac{\Gamma(1 - \frac{\theta}{2\pi i}) \Gamma \left( \frac{\theta}{2\pi i} + 2\Delta \right)}{\Gamma(1 + \frac{\theta}{2\pi i}) \Gamma \left( -\frac{\theta}{2\pi i} + 2\Delta \right)}. \quad \text{(24)}
\]

The symmetric representation has highest weight $2\mu_1$. The pole in the $S$ matrix at $\theta = 2\pi i\Delta$ results in a bound state transforming in the representation with highest weight $2\mu_2$, with mass.
\[ m_{2\mu_j}/m_S = 2 \cos \pi \Delta. \] In fact, the factor \( X \) ensures that the spectrum of bound states is just like that in the \( SU(N) \) Gross-Neveu model. Building the \( S \) matrices for the bound states in the \( SU(N)/SO(N) \) sigma model proceeds just like the \( SU(N) \) Gross-Neveu model. For example, the prefactor in (23) ensures that \( S^{S\bar{S}} \) determined by the bootstrap is the same as that following from crossing. To close the bootstrap, \( \Delta = 1/h \) as before, with \( h = N \) for \( SU(N) \). There are particles in all representations with highest weight \( 2\mu_j \). Each representation appears just once with mass \( M \sin \pi j/N \). For \( N = 2 \), this model reduces to the \( O(3)/O(2) \) model (1), and the \( S \) matrix (23) becomes that of a massive \( O(3) \) triplet, familiar from (2).

The \( SU(N)/SO(N) \) sigma model with no \( \theta \) term is therefore an integrable generalization of the sphere sigma model to all \( SU(N) \). I will now show that the behavior at \( \theta = \pi \) also generalizes.

To find the particles and scattering matrices when \( \theta = \pi \), it is first necessary to understand the non-trivial fixed point. Generalizing from \( N=2 \), it is natural to assume that it is \( SU(N)_1 \). In fact, the argument from (4), (5) can be adapted to show that this is completely consistent with the symmetries of the problem. To apply the consistency argument here, it is first useful to study the symmetries of the \( SU(N) \) Gross-Neveu model. To avoid confusion with the sigma model, I denote the field in the Gross-Neveu model as \( w \). With the definition (1) as a perturbed WZW model, \( w \) must be an \( SU(N) \) matrix. Therefore, it transforms under the \( SU(N) \) symmetry as \( w \rightarrow U w U^\dagger \), where \( U \) is an element of \( SU(N) \). The currents also transform as \( j \rightarrow U j U^\dagger \). In other words, \( w \) and \( j \) are in the adjoint representation of \( SU(N) \). The action (13) is manifestly symmetric under the symmetry \( SU(N)/Z_N \). The \( Z_N \) is the center of \( SU(N) \), consisting of \( I \)-matrices \( \Omega I \), where \( \Omega \) is a \( N \)-th root of unity, and \( I \) is the identity matrix. The matrices \( \Omega I \) commute with all elements of \( SU(N) \). The reason this discrete subgroup is divided out is that if \( U = \Omega I \), \( w \) and \( j \) are left invariant. Thus the symmetry acting manifestly on the action is \( SU(N)/Z_N \), not \( SU(N) \). However, the full symmetry of the Gross-Neveu model is larger. This can be seen in several ways. First of all, note that there are particles in the model in the vector representation. The full \( SU(N) \) group acts on the vector representation non-trivially. Indeed, one can assign an extra \( Z_N \) charge to the particles of the model, defined so that the particles in the \( j \)-index antisymmetric representation have charge \( \Omega^j \). Another way of seeing this extra charge is by examining terms which do not appear in the action (13). The \( SU(N)/Z_N \)-symmetric operators in the \( SU(N)_1 \) conformal field theory are \( \text{tr} w^j \), where \( j = 1 \ldots N - 1 \). Some of these are relevant operators, and if added to the action would change the physics considerably. However, they are forbidden from the action if the model is required to be invariant under the symmetry \( w \rightarrow \Omega w \).

This discrete symmetry is not part of the \( SU(N) \) acting on the \( w \) field, but rather makes the full symmetry of the model \( Z_N \times SU(N)/Z_N \). This symmetry of the action and the full \( SU(N) \) of the particle description are completely consistent with each other if the particles are kinks in the \( w \) field. (They are kinks in the \( 1 + 1 \)-dimensional picture, vortices in the \( 2 + 0 \)-dimensional picture.) The presence of kinks is easy to see. The field values \( w = I \) and \( w = \Omega \) are not related by a continuous symmetry, so field configurations with \( w(x=-\infty,t) = I \) and \( w(x=\infty,t) = \Omega I \) are topologically stable. These are the kinks. The extra \( Z_N \) symmetry is then indeed the discrete kink charge. This sort of symmetry should be familiar from the sine-Gordon model, where the soliton charge is not an explicit symmetry of the action.

The symmetries of the \( SU(N)/SO(N) \) sigma model at \( \theta = \pi \) are similar to those in the Gross-Neveu model. The \( Z_N \) center acts non-trivially on the \( SU(N)/SO(N) \) matrix \( \Phi \), as seen in (13). Thus the symmetry of the sigma model action is the full \( SU(N) \) (or to be precise, for even \( N \) it is \( Z_2 \times SU(N)/Z_2 \)). Therefore it is consistent for the low-energy fixed point of the \( SU(N)/SO(N) \) sigma model to be \( SU(N)_1 \). The reason is the same as in the \( SU(N) \) Gross-Neveu model: the
full $SU(N)$ symmetry of the action in terms of $\Phi$ implies a $\mathbb{Z}_N \times SU(N)/\mathbb{Z}_N$ symmetry of the low-energy effective action involving $w$. This forbids the relevant operators from being added to the $SU(N)_1$ action. This argument shows that the effective action near the low-energy fixed point contains only $SU(N)$-invariant irrelevant operators. The operator $j_L j_R$ is $SU(N)$ invariant and of dimension 2. Thus the effective action here is like the Gross-Neveu action (9), except here the coupling $g$ is negative so that the perturbing operator is irrelevant. The coupling $g$ is related to the scale $M$; when $M \to \infty$, $g \to 0$ from below, and the model reaches the low-energy fixed point.

The form of the effective action near the low-energy fixed point gives the $S$ matrices $S_{LL}$ and $S_{RR}$ immediately, because as explained above they are independent of $M$ and therefore follow solely from $SU(N)_1$. They must be in fact those of the massless limit of the Gross-Neveu model. Written as functions of rapidity they are

$$S^{ab}_{LL}(\theta) = S^{ab}_{RR}(\theta) = S^{ab}_{GN}(\theta).$$

(25)

Thus the spectrum of these gapless particles is the same as that of the massive particles of the Gross-Neveu model: there are particles in any antisymmetric representation, a left-moving and right-moving set for each representation. Even though the particles here are gapless, the mass ratios of the Gross-Neveu model still appear in the definition of rapidity. For example, a right mover in the representation $\mu^j$ has energy $E = m_j e^\theta$. The charge has fractionalized, as compared to $\theta = 0$.

For $N=2$, the $S$ matrix $S_{LR}(\theta)$ as a function of rapidity is the same as $S_{LL}(\theta)$ 10. This $S$ matrix satisfies the Yang-Baxter equation, and has the appropriate $SU(2)$ (not $SU(2)_L \times SU(2)_R$) symmetry. However, the situation is not quite as simple for general $N$, because there are poles in $S_{LL}$ and $S_{RR}$ in the physical strip, resulting in the bound states. These poles are forbidden in $S_{LR}$. However, it is easy to remove these and still have a sensible $S$ matrix. The bound-state $S$ matrices $S^{ab}_{GN}$ are labeled with $a, b = 1 \ldots N - 1$ corresponding to the antisymmetric representations with $a$ and $b$ indices. For example $S^{11} = S^{VV}$ and $S^{21} = S^{AV}$. The unwanted poles in $S^{ab}$ all arise from the factor $X^{11}(\theta) \equiv X(\theta)$ and its fusions. The matrix $S^{ab}_{GN}$ contains the overall function

$$X^{ab}(\theta) \equiv \prod_{i=1}^a \prod_{j=1}^b X(\theta + [2(i + j - 1) - a - b] \pi i \Delta).$$

(26)

For example, $X^{12} = X(\theta + \pi i \Delta) X(\theta - \pi i \Delta)$. The prefactor $X^{ab}$ contains all the poles in the physical strip in $S^{ab}_{GN}$. It is easy to check that $X^{ab}(\theta)X^{ab}(-\theta) = 1$ and that $X^{ab}(\theta) = X^{ab}(i \pi - \theta)$. Therefore

$$S^{ab}_{LR}(\theta) = S^{ab}_{GN}(\theta) / X^{ab}(\theta)$$

(27)

satisfies crossing, unitarity, the Yang-Baxter equation, and has no poles in the physical strip. Thus this is the $S$ matrix for left-right scattering in the $SU(N)/SO(N)$ sigma model at $\theta = \pi$.

For $N = 2$, $X = 1$, and the result reduces to that in 10.

To prove that this picture is correct, in the next section I will show that these $S$ matrices give a model which is $SU(N)_1$ in the low-energy limit, but is the $SU(N)/SO(N)$ sigma model at high energy. Moreover, in 32 the corresponding $c$ function is calculated, and indeed flows from the $c = (N - 1)(N + 2)/2$ (the value at the trivial high-energy fixed point in $SU(N)/SO(N)$ to $c = N - 1$ (the central charge for $SU(N)_1$).
5.2 \( O(2P)/O(P) \times O(P) \) sigma model

As with the Gross-Neveu models, the \( O(2P)/O(P) \times O(P) \) sigma model is a slightly more complicated version of its \( SU(N) \) analog, the \( SU(N)/SO(N) \) sigma model. The field \( \Phi \) is symmetric and unitary (and also real and traceless), so again one expects particles in the symmetric representation of \( O(2P) \) (which is \( P(2P + 1) - 1 \) dimensional), and its various bound states. This is what happens for \( P = 2 \), where the model reduces to two copies of the sphere sigma model: there are six particles: three in each of the symmetric representations. Here, for \( P > 2 \), the non-local conserved currents have not been found, although some interesting results for the local currents in the classical model (\( g \) small) have been found [12].

The \( S \) matrices for the \( O(2P)/O(P) \times O(P) \) model can be constructed by fusing the \( O(2P) \) Gross-Neveu results. The \( S \) matrix for particles in the symmetric representation of \( O(2P) \) (highest weight \( 2\mu_1 \)) is of the form [17]

\[
S^{SS} = R(\theta) \left( \mathcal{P}_{4\mu_1} + \frac{\theta + 4\pi i \Delta}{\theta - 4\pi i \Delta} \left( \mathcal{P}_{2\mu_1+\mu_2} + \frac{\theta + i\pi + 2\pi i \Delta}{\theta - i\pi - 2\pi i \Delta} \mathcal{P}_{2\mu_2} \right) \right)
\]

where \( \Delta = 1/h \) as always, with \( h = 2P - 2 \) here. The minimal solution (no poles in the physical strip) for the prefactor is

\[
S^{SS}_{\text{min}}(\theta) = \frac{\theta - 2\pi i \Delta}{\theta + 2\pi i \Delta} \frac{\Gamma \left( 1 - \frac{\theta}{2\pi i} \right) \Gamma \left( \frac{\theta}{2\pi i} + 2\Delta \right)}{\Gamma \left( 1 + \frac{\theta}{2\pi i} \right) \Gamma \left( \frac{\theta}{2\pi i} - 2\Delta \right)} \frac{\Gamma \left( \frac{1}{2} - \frac{\theta}{2\pi i} + 2\Delta \right)}{\Gamma \left( \frac{1}{2} - \frac{\theta}{2\pi i} - 2\Delta \right)} \frac{\Gamma \left( \frac{1}{2} + \frac{\theta}{2\pi i} + 2\Delta \right)}{\Gamma \left( \frac{1}{2} + \frac{\theta}{2\pi i} - 2\Delta \right)}.
\]

Like the \( O(2P) \) Gross-Neveu model, to get bound states there must be poles at \( \theta = 2\pi i \Delta \) and \( \theta = \pi i(1 - 2\Delta) \). This means that the prefactor of \( S^{SS} \) is

\[
R(\theta) = X(\theta)X(i\pi - \theta)S^{SS}_{\text{min}}(\theta).
\]

Because the factor \( X(\theta)X(i\pi - \theta) \) is the same, the bootstrap for the \( O(2P)/O(P) \times O(P) \) model gives bound states with the same spectrum as the \( O(2P) \) Gross-Neveu model. Also like the \( O(2P) \) Gross-Neveu model, there are particles which do not follow obviously from the action. These are in “double-spinor” representations, formed by the symmetric product of two spinor representations. In group theory language they have highest weight \( 2\mu_s \), where \( \mu_s \) is the highest-weight of a spinor representation; they are \( (2P - 1)!/P!(P - 1)! \) dimensional. These particles presumably are kinks like in the Gross-Neveu model, but it is not clear how to extract this information directly from the action. The \( S \) matrix for these kinks is quite complicated, since many representations appear in the tensor product of two double-spinor representations. It can presumably be obtained by fusion of the spinor \( S \) matrices, or by the reverse bootstrap like the Gross-Neveu kinks. The presence of these particles is confirmed by studying the free energy at non-zero temperature [31].

Therefore, in both series of sigma models there are particles in all representations with highest weight \( 2\mu \). However, here there are even more degeneracies and more representations appearing. Since there is a pole in [29] at \( \theta = 2\pi i \Delta \), then there must be bound states not only in the representation with highest weight \( 2\mu_2 \), but also in the singlet and in the antisymmetric representation (highest weight \( \mu_2 \)). Thus charge is fractionalized even at \( \theta = 0 \). For example, in \( O(8) \), the vector and spinor representations are 8-dimensional, the antisymmetric representation has dimension 28, the symmetric and double-spinors dimension 35, and the representation with
highest weight $2\mu_2$ has dimension 300. Thus in the $O(8)$ Gross-Neveu model, there are 8 particles in the vector (mass $2M \sin \pi/6 = M$) 8 particles in each of the spinor representations (mass $M$), and $28 + 1$ particles of mass $2M \sin \pi/3 = \sqrt{3}M$. Thus there are 53 stable particles in the $O(8)$ Gross-Neveu model. In the $O(8)/O(4) \times O(4)$ sigma model, there are 35 particles in each of the vector and two double-spinor representations, all of mass $M$. There are $300 + 28 + 1$ particles of mass $\sqrt{3}M$, giving 434 particles in all.

The behavior when $\theta = \pi$ in the $O(2P)/O(P) \times O(P)$ model is also reminiscent of the $SU(N)/SO(N)$ model at $\theta = \pi$. The arguments follow in the same fashion. The model has the $O(2P)_1$ WZW model as a stable low-energy fixed point. The $O(2P)_1$ WZW model is equivalent to $2P$ free Majorana fermions, or equivalently $2P$ decoupled Ising models. The word “free” is slightly deceptive, because just as in a single 2d Ising model, one can study correlators of the magnetization or “twist” operator, which are highly non-trivial. The consistency argument is simpler here: the only relevant $O(2P)$ symmetric operators are the fermion mass and the magnetization operator; neither is invariant under the symmetry $\Phi \to -\Phi$ of the sigma model. The $c$-function is computed in \cite{31}, and indeed flows from $c = P^2$ to $c = P$ as it must.

The $S$ matrix for the quasiparticles follows from the Gross-Neveu model. The same arguments applied above to the $SU(N)/SO(N)$ sigma model at $\theta = \pi$ show here that the massless left- and right-moving particles have the same spectrum as the $O(2P)$ Gross-Neveu model. Charges fractionalize: the left- and right-moving particles are in the fundamental representations of $O(2P)$. The $S$ matrices for the particles in antisymmetric representations are

$$S_{LL}^{ab}(\theta) = S_{RR}^{ab}(\theta) = S_{GN}^{ab}(\theta)$$

$$S_{LR}^{ab}(\theta) = S_{GN}^{ab}(\theta)/(X^{ab}(\theta)X^{ab}(i\pi - \theta))$$

For particles in the spinor representations, the poles can easily be removed as well; see \cite{16} for the definition of $X^{ab}$ for the spinor representations. Even though the field theory at the low-energy fixed point is free fermions, the $S$ matrix is factorizable away from the fixed point only in the basis related to the Gross-Neveu model.

6 Matching perturbative expansions

In this section I compute the energy of these sigma models at zero temperature in a background magnetic field. This is very useful for several reasons. First of all, it allows a direct comparison of the $S$ matrix to perturbative results. This in particular ensures that the $S$ matrices are correct as written in the last section. For example, it eliminates the possibility of extra CDD factors and/or extra bound states. Second, because the effect of a $\theta$ term is non-perturbative, the perturbative expansions for the sigma models at $\theta = 0$ and $\pi$ must be the same. Thus even though the $S$ matrices for $\theta = 0$ and $\pi$ are very different, the energy must have the same perturbative expansion. I verify that this is true for the above $S$ matrices.

The abelian subgroup of the group $G$ is $U(1)^r$, where $r$ is the rank of the group. Thus a model with a global symmetry $G$ has $r$ conserved charges. These charges can be coupled to a background field, which is constant in spacetime. In the sigma models $O(2P)/O(P) \times O(P)$ and $SU(N)/O(N)$ where $\Phi$ is a symmetric matrix, this means that the Euclidean action (2) is modified to

$$S = \frac{1}{g} \text{tr} \int d^2x \left( \partial^\mu \Phi^\dagger - \bar{A}^T \Phi^\dagger - \Phi^\dagger \bar{A}^T \right) \left( \partial_\mu \Phi + \bar{A} \Phi + \Phi \bar{A} \right)$$

(33)
where $\tilde{A}$ is a matrix in the Cartan subalgebra of $G$ (the Cartan subalgebra is comprised of the generators of the abelian subgroup of $G$). For $SU(N)/SO(N)$, $\tilde{A}$ is diagonal with entries $(A_1, A_2, \ldots A_N)$ and the constraint $\sum_i A_i = 0$. For $O(2P)/O(P) \times O(P)$, the matrix can be written in the form $\tilde{A}_{jk} = \sum_{l=1}^{P} A_l (\delta_{j,2l-1} \delta_{k,2l} - \delta_{j,2l} \delta_{k,2l-1})$.

Because $\tilde{A}$ has dimensions of mass, the energy depends on the dimensionless parameters $A_i/M$ or $A_i/m$. The strength of the background field controls the position of the theory on its renormalization group trajectory and, in particular, in the limit of large field the theory is driven to the ultraviolet fixed point. To be near the UV fixed point, all non-zero $A_i/M$ must be large in magnitude, so let $\Lambda$ be one of the non-zero components. The perturbative expansion around this fixed point therefore is an expansion for large $A/M$. The computation of this expansion is a fairly standard exercise in Feynman diagrams; for details closely related to the cases at hand, see [25]. The only effect of the sigma model interactions resulting from the non-linear constraints (3) to the one-loop energy comes through the running of the coupling constant. Specifically, the two-loop beta function is of the form

$$\beta(g) = \mu \frac{\partial}{\partial \mu} g(\mu) = -\beta_1 g^2(\mu) - \beta_2 g^3(\mu) - \ldots$$

where $\beta_1$ and $\beta_2$ are model-dependent, but universal for these sigma models. Solving this equation means that

$$\frac{1}{g(A)} = \beta_1 \ln A/\Lambda + \frac{\beta_2}{\beta_1} \ln(\ln A/\Lambda)) + O(1/\ln(A))$$

(34)

where the scale $\Lambda$ depends on the perturbative scheme used. The zero-temperature energy for $SU(N)/SO(N)$ through one loop is

$$E(A) - E(0) = -\frac{4}{g(A)} \sum_j (A_j)^2 - \frac{1}{2\pi} \sum_{i<j} (A_i - A_j)^2 \left( \ln |A_i - A_j| - \frac{1}{2} \right) + O(g),$$

with a similar formula for $O(2P)/O(P) \times O(P)$. For both cases, this means that at large $A/M$, the energy is of the form

$$E(A) - E(0) \propto A^2 (\ln(A/M) + \frac{\beta_2}{(\beta_1)^2} \ln(\ln(A/M)) + \ldots)$$

(35)

In particular, note that the ratio of the first two terms is universal. These logarithms are characteristic of an asymptotically free theory, where the perturbation of the UV fixed point is marginally relevant.

I now explain how to calculate the ground-state energy $E(A) - E(0)$ directly from the $S$-matrix, giving the promised check. In this picture, the model is treated as a 1 + 1 dimensional particle theory at zero temperature. Turning on the background field has the effect of changing the one-dimensional quantum ground state. If one were working in field theory or in a lattice model, this would change the Dirac or Fermi sea. In the exact $S$-matrix description, a similar thing happens: the ground state no longer is the empty state — it has a sea of real particles. For example, a particle $a_i$ in the vector representation of $SU(N)$ has its energy shifted by $A_i$ in a magnetic field. If the total energy is negative, then it is possible for such a particle to appear in the ground state. Because the technicalities are slightly different, I treat the cases of massive and massless particles separately.
6.1 Massive particles

For simplicity, I first treat only the case where the magnetic field is chosen so that only one kind of massive particle of mass $m$ and charge $q$ appears in the ground state. In integrable models obeying the Yang-Baxter equation, the particles fill levels like fermions: only one can occupy a given level. Then the ground state is made up of particles with rapidities

$$-B < \theta_1 < \theta_2 < \ldots \theta_N < B$$

for some maximum rapidity $B$. When the length $L$ of the system is large, these levels are very close together, so the density of particles $\rho(\theta)$ is defined so that $\rho(\theta)d\theta$ is the number of particles with rapidities between $\theta$ and $\theta + d\theta$. The energy density of the ground state is then

$$E(A) - E(0) = \frac{1}{L} \sum_{\alpha} \int_{-B}^{B} d\theta \rho(\theta)(-A + m \cosh \theta).$$

(36)

I have normalized $A$ so that the charge $q = 1$. The exact $S$-matrix allows us to derive equations for the $\rho(\theta)$ and $B$. Imposing periodic boundary conditions on the box of length $L$ requires that the rapidities $\theta_i$ of the particles in the ground state all satisfy the quantization conditions

$$e^{im \sinh \theta_i L} \prod_{j \neq i} S(\theta_i - \theta_j) = 1$$

(37)

for all $i$. This is an interacting-model generalization of the one-particle relation $m \sinh \theta_k = p_k = 2\pi n_k/L$ to the case where the particles in the ground state scatter elastically from each other with $S$ matrix element $S(\theta)$. In the large $L$ limit, taking the derivative of the log of (37) gives an integral equation for the density $\rho(\theta)$:

$$\rho(\theta) = \frac{mL}{2\pi} \cosh \theta + \int_{-B}^{B} d\theta' \rho(\theta') \phi(\theta - \theta'),$$

(38)

where

$$\phi(\theta) = -\frac{i}{2\pi} \frac{\partial \ln S(\theta)}{\partial \theta}.$$

This equation is valid for $|\theta| < B$; for $|\theta| > B$, $\rho(\theta) = 0$. The maximum rapidity $B$ is determined by minimizing the energy equation (36) with respect to $B$ subject to the constraint (38). If the particles were non-interacting, the density would be $mL \cosh \theta$ and $m \cosh B = A$, but the effect of the interactions is quite substantial.

These equations can be put in a convenient form by defining the “dressed” particle energies $\epsilon(\theta)$ as

$$\epsilon(\theta) = A - m \cosh \theta + \int_{-B}^{B} d\theta' \phi(\theta - \theta') \epsilon(\theta').$$

(39)

Substituting this into (36) and using (38) yields

$$E(A) - E(0) = -\frac{m}{2\pi} \int_{-B}^{B} d\theta \cosh \theta \epsilon(\theta).$$

(40)

In this formulation, $B$ is a function of $A/m$ determined by the boundary condition $\epsilon(\pm B) = 0$.

Consider the $O(2P)/O(P) \times O(P)$ sigma model, and choose the magnetic field to be $A_1 = A$, $A_i = 0$ for $i \neq 1$. The massive particles used above ($a_{ij} = a_{ji}$) are not eigenstates of the
matrix magnetic field $\tilde{A}$, but one can easily change basis to particles which are. This merely requires taking linear combinations of particles of the same mass. The state which has the largest eigenvalue of this magnetic field is $d \equiv a_{11} - a_{22} + 2ia_{12}$. This basis also has the advantage that scattering of $d$ particles amongst themselves is diagonal (as opposed to the scattering of $a_{11}$ with $b_{11}$, which does not necessarily yield $a_{11}$ and $b_{11}$ in the final state). The $S$ matrix element for scattering $d(\theta_1)d(\theta_2)$ to $d(\theta_1)d(\theta_2)$ is $R(\theta_1 - \theta_2)$ as given in (31). The state $d$ has the largest eigenvalue of the magnetic field, and is the only state with this eigenvalue. I therefore make the assumption that it is the only kind of particle which appears in the ground state. This assumption is standard in these computations; it can be justified by a careful treatment of the zero-temperature limit of the finite-temperature density equations discussed in (31). The kernel $\phi$ of the integral equation (39) is then proportional to the derivative of $\log R(\theta)$. It is convenient to write this in Fourier space as

$$\phi(\theta) = \int_{-\infty}^{\infty} d\omega \ e^{i\omega \theta} (1 - K(\omega))$$

$$K(\omega) = e^{-\pi \Delta |\omega|} \frac{2 \cosh((1 - 2\Delta)\pi \omega/2) \sinh(2\Delta |\omega|)}{\cosh(\pi \omega/2)}$$

where $\Delta$ is as always $1/h$, with the dual Coxeter number $h = 2P - 2$ for $O(2P)$.

For the $SU(N)/SO(N)$ sigma model, the states $a_{ij}$ are eigenstates of the magnetic field operator, with eigenvalue $A_i + A_j$. If we chose the magnetic field to be $A_1 = A$, $A_j = -A/(N-1)$ for $j > 1$, then the particle $a_{11}$ has maximum charge. Again making the assumption that this is the only particle in the ground state, the Fourier transform of the resulting kernel is

$$2e^{-\pi \Delta |\omega|} \frac{\sinh((1 - \Delta)\pi |\omega|) \sinh(2\Delta |\omega|)}{\sinh(\pi \omega)}$$

However, it is convenient here to instead choose the fields to be $A_1 = -A_2 = A$, $A_j = 0$ for $j > 2$ (this choice of field was useful also in the supersymmetric $CP^n$ model (33)). Then there are two particles with largest eigenvalue: $a_{11}$ and the antiparticle (in the $\bar{N}$ representation) $\bar{a}_{22}$. Thus they both appear in the ground state. These two particles scatter diagonally among themselves and each other. The $S$ matrix element $S_1$ for $a_{11}a_{11} \rightarrow a_{11}a_{11}$ and $\bar{a}_{22}\bar{a}_{22} \rightarrow \bar{a}_{22}\bar{a}_{22}$ is

$$S_1(\theta) = S_{\min}^{SS}(\theta) X(\theta).$$

The element $S_2$ for $a_{11}\bar{a}_{22} \rightarrow a_{11}\bar{a}_{22}$ is found easily from (34) by using crossing. It is

$$S_2(\theta) = X(i\pi - \theta)S_{\min}^{SS}(i\pi - \theta) \frac{(i\pi - \theta)(i\pi - \theta + \mu)}{(i\pi - \theta - 2\mu)(i\pi - \theta - \mu)}$$

Because of the symmetry between the two kinds of particles, their ground-state densities must be the same. The ground-state energy then follows from a simple generalization of the above analysis: the equations (39) and (40) still apply, with

$$\phi(\theta) \equiv -\frac{i}{2\pi} \frac{\theta}{\partial \theta}(\ln S_1(\theta) + \ln S_2(\theta))$$

here. Plugging in the explicit expressions of the functions yields the useful fact that $S_1(\theta)S_2(\theta) = R(\theta)$, where $R(\theta)$ is the function appearing in (31). Thus with these choices of magnetic field, the $O(2P)/O(P) \times O(P)$ and the $SU(N)/SO(N)$ sigma models can be treated by using the kernel (42): the only difference is the $\Delta = 1/(2P - 2)$ in the former and $\Delta = 1/N$ in the latter.
The linear integral equation (39) cannot be solved in closed form. However, there is a generalized Weiner-Hopf technique which allows the perturbative (and non-perturbative) expansion for large $A/M$ to be obtained [35]. I discuss this technique in the next subsection. In order to compare the energy in the $\theta = 0$ sigma model with the perturbative computation (35), I can rely on the results of [25]. There the first few terms in the large $A$ expansion for models with kernels like (42) are given. The technique requires that the kernel $K(\omega)$ be factorized into

$$ K(\omega) = \frac{1}{K_+(\omega)K_-(\omega)} \quad \text{(43)} $$

where $K_+ (\omega)$ has no poles or zeroes (and is bounded) in the upper half plane $\text{Im} (\omega) > 0$, while $K_- (\omega) = K_+ (-\omega)$ has no poles or zeroes and is bounded in the upper half plane. Then

$$ K_+(\omega) = \sqrt{\frac{2i\Delta}{\pi}} e^{i\omega \Delta \ln(i\omega) + i\mu \omega} \frac{\Gamma(-2i\Delta \omega)\Gamma\left(\frac{1}{2} - i\left(\frac{1}{2} - \Delta\right)\omega\right)}{\Gamma\left(\frac{1}{2} - i\left(\frac{1}{2} + \frac{1}{2}\right)\omega\right)}, \quad \text{(44)} $$

where

$$ \mu = 2\Delta \ln(2\Delta) + \left(\frac{1}{2} - \Delta\right) \ln\left(\frac{1}{2} - \Delta\right) + \frac{1}{2} \ln 2 - \Delta. $$

The factor $e^{i\omega \mu}$ ensures that $K_+$ is bounded appropriately as $|\omega| \to \infty$ in the upper half plane. When for small $\xi$, $K_+(i\xi)$ goes as

$$ K_+(i\xi) = \frac{k}{\sqrt{\xi}} e^{\xi \ln \xi (1 + \ldots)}, $$

the energy at large $A$ is [25]

$$ E(H) - E(0) = -\frac{k^2}{4} A^2 \left[ \ln \left(\frac{A}{m}\right) + (s + \frac{1}{2}) \ln \left(\ln \left(\frac{A}{m}\right)\right) + \ldots \right] \quad \text{(45)} $$

This expansion is of the same form as (35). The two are equal if

$$ \frac{\beta_2}{(\beta_1)^2} = s + \frac{1}{2}. \quad \text{(46)} $$

The explicit kernel (42) yields

$$ s = \Delta, $$

while perturbative computations (see e.g. [34] and references within) give for $SU(N)/SO(N)$

$$ \frac{\beta_2}{(\beta_1)^2} = \frac{N + N^2/2}{N^2} $$

while for $O(2P)/O(P) \times O(P)$ the ratio is

$$ \frac{\beta_2}{(\beta_1)^2} = \frac{2P^2 - 2P}{(2P - 2)^2} $$

Using the relations $\Delta = 1/N$ and $\Delta = 1/(2P - 2)$ respectively, one indeed sees that the condition (46) holds. This is a substantial check, and coupled with the fact that $S$ matrix also gives the correct central charge, leads me to be completely convinced that the $S$ matrices discussed above are indeed the correct sigma model $S$ matrices.
6.2 Comparing the perturbative expansions at $\theta = 0$ and $\pi$

In this subsection I analyze the equations for the ground-state energy in a magnetic field in more detail. The final result will be that the entire large-$A$ perturbative expansions for $\theta = 0$ and for $\theta = \pi$ are identical, but that non-perturbative contributions differ. This effectively confirms the $S$ matrices and spectrum above, and the identity of the low-energy fixed points.

The generalized Weiner-Hopf technique is discussed in detail in the appendix of [35]. The equation to be solved is of the form

$$\epsilon(\theta) - \int_{-B}^{B} \phi(\theta - \theta')\epsilon(\theta') = g(\theta).$$

where $\epsilon(\theta)$ and $g(\theta)$ are both vanishing for $|\theta| \geq B$, and $\phi(\theta) = \phi(-\theta)$. Defining the Fourier transforms $\tilde{\epsilon}(\omega)$, $\tilde{g}(\omega)$ and $\tilde{K}(\omega)$ (the latter related to $\phi$ as in (41)) gives

$$\int_{-\infty}^{\infty} e^{-i\omega\theta} \{\tilde{\epsilon}(\omega)\tilde{K}(\omega) - \tilde{g}(\omega)\} = 0.$$

This equation is valid for $|\theta| < B$. Fourier-transforming this gives

$$\tilde{\epsilon}(\omega)\tilde{K}(\omega) - \tilde{g}(\omega) = X_+(\omega)e^{i\omega B} + X_-e^{-i\omega B}.$$  (47)

The functions $X_\pm$ arise, roughly speaking, from the analytic continuation of $\epsilon(\theta)$ to $|\theta| > B$. The extra factors $e^{i\omega B}$ ensure that $X_+(\omega)$ is analytic in the upper half plane, while $X_-(-\omega) = X_+(\omega)$ is analytic in the lower half plane. The relation (47) can be split into two equations, one involving poles in the upper half plane and the other the lower. To split a given function, one uses

$$f(\omega) = [f(\omega)]_+ + [f(\omega)]_-$$

where

$$[f(\omega)]_\pm = \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\omega)}{\omega' - \omega \mp i\delta} d\omega'$$

where $\delta$ is a positive real number tending to zero. The functions $[f]_\pm$ are analytic in the upper ($+$) and lower ($-$) half planes. Because $\epsilon(\theta)$ and $g(\theta)$ are zero for $|\theta| \geq B$, the functions

$$\epsilon_\pm(\omega) = \tilde{\epsilon}(\omega)e^{i\omega B} \quad g_\pm(\omega) = \tilde{g}(\omega)e^{i\omega B}$$

are similarly analytic in the upper half and lower half planes. Equations for the functions $X_\pm$ can be derived by exploiting these analyticity properties and the factorization (43). Namely, (47) implies the equations

$$\frac{\epsilon_\pm(\omega)}{K_\pm(\omega)} = [g_\pm(\omega)K_\pm(\omega)]_\pm + [X_\pm(\omega)K_\pm(\omega)e^{\pm 2i\omega B}]_\pm,$$  (48)

while the $X_\pm$ are given by

$$X_\pm(\omega)K_\pm(\omega) + [X_\pm(\omega)K_\pm(\omega)e^{\mp 2i\omega B}]_\pm = -[g_\pm(\omega)K_\pm(\omega)]_\pm$$  (49)

Unfortunately, it is not possible to solve the equations for $X_\pm$ explicitly. However, this form does allow the large-$A$ expansion to be systematically developed.
To study the large \( A \) expansion, it turns out to be much easier to consider the more general kernel

\[
K(\omega) = \frac{\cosh(\gamma \pi \Delta \omega)}{\sinh((\gamma + 1) \pi \Delta \omega)} \frac{2 \cosh((1 - 2 \Delta) \pi \omega / 2) \sinh(2 \Delta \pi \omega)}{\cosh(\pi \omega / 2)}.
\] (50)

In the limit \( \gamma \to \infty \), \( K \) goes back to the sigma model kernel \( K \). In the \( N=2 \) case, this deformation corresponds to deforming the sphere sigma model into the sausage sigma model \([11]\). This kernel factorizes as \( K = 1/(K_+ K_-) \), giving

\[
K_+(\omega) = \sqrt{\frac{2}{\gamma + 1}} e^{i\omega} \Gamma \left( \frac{1}{2} - i \Delta \gamma \omega \right) \Gamma \left( \frac{1}{2} - i \left( \frac{1}{2} + \Delta \right) \omega \right) \Gamma \left( -2i \Delta \omega \right) \Gamma \left( -i(\gamma + 1) \Delta \omega \right) \Gamma \left( \frac{1}{2} - i \frac{\omega}{2} \right).
\] (51)

where

\[\nu = \mu + \gamma \Delta \ln(\gamma/(\gamma + 1)) - \Delta \ln \Delta + \Delta.\]

The equations (49) can be written as a single one by exploiting the relation \( X_+(\omega) = X_-(-\omega) \). Defining

\[v(\omega) \equiv \frac{X_+(\omega)}{K_+(\omega)} - i \frac{qK_+(0)}{\omega + i\delta} A - i \frac{meB}{2} \frac{K_+(i)}{\omega - i},\]

they become

\[v(\omega) = -\frac{iAK_+(0)}{\omega + i\delta} + \frac{imB}{2} \frac{K_+(i)}{\omega - i} + \int_{C_+} \frac{e^{2\omega iB}}{\omega + \omega' + i\delta} \alpha(\omega') v(\omega') d\omega'.\] (52)

where \( \alpha(\omega) \) is defined as

\[\alpha(\omega) \equiv \frac{K_-(-\omega)}{K_+(\omega)}\]

and the integration contour circles all singularities on the positive imaginary axis. The boundary condition \( \epsilon(\pm B) = 0 \) becomes

\[iAK_-(-i) - \frac{imB}{2} K_-(-i) = \int_{C_+} e^{2\omega iB} \alpha(\omega) v(\omega') \frac{d\omega}{2\pi i}\] (53)

while the energy (41) becomes

\[E(A) - E(0) = \frac{meB}{2} K_-(-i) \left[ AK_+(0) - \frac{meB}{4} K_+(i) \int_{C_+} \frac{e^{2\omega iB}}{\omega - i} \alpha(\omega) v(\omega) \frac{d\omega}{2\pi i} \right].\] (54)

The contour here includes the double pole at \( \omega = i \), and the poles in \( \alpha(\omega) \). The function \( v(\omega) \) has no poles in the upper half plane except the explicit one at \( \omega = i \).

These equations are convenient for deriving the expansion for large \( A \). The reason is that when \( A \) is large, the range of rapidities allowed in the ground state is large, so \( B \) is large (recall that if the particles were free, \( A = m \cosh B \)). In this limit the integrals in (52,53) are small corrections, and their effect can be treated iteratively. This iterative expansion is worked out in detail in \([11]\). The first correction to \( v(\omega) \) comes from approximating the integral using the leading pieces of \( v(\omega) \) in the integrand. The boundary condition (53) relates \( B \) to \( A \). Thus the contributions to this integral come from the pole in \( v(\omega) \) at \( \omega = i \) and the poles in \( \alpha(\omega) \). The poles of \( \alpha \) are at \( \omega = i(2n - 1)/(2\gamma \Delta) \), \( \omega = (2n - 1)\hbar/(2\Delta + 1) \) and \( \omega = in/(2\Delta) \), for \( n \) a positive
integer. Using the iterative procedure it is straightforward to see the form of the expansion for large $A$: it is a power series of the form

$$E(A) = A^2 \left( \sum_{j=0}^{\infty} \frac{\alpha_j}{m} \frac{A^j}{\gamma \Delta} \right) \left( 1 + O \left( \frac{A^2}{m^2} \right) \right).$$

(55)

The first series is a result of the first series of poles in $\alpha$; the coefficients $\alpha_j$ depend on the residues at these poles. The order $A^2$ corrections are a result of the pole at $\omega = i$ and the other poles in $\alpha$. Recall that $\gamma$ is large, so these contributions are much smaller. The double pole at $\omega = i$ results in an $A$-independent piece, which is identified as $E(0)$.

The sigma model result of interest is recovered in the limit $\gamma \to \infty$. This complicates matters considerably for the power series in (55), since the exponent is going to zero. This is precisely what happens for example in the anisotropic Kondo problem as the anisotropy is tuned away. Note from (34) that $1/g \propto \ln(A)$, so the series depends on powers of $g$, while the order $A^2$ corrections depend on $e^{-\text{const}/g}$. Thus the first series consists of the perturbative contributions to the free energy, while the second part is non-perturbative: the latter will never be seen in standard sigma model perturbation theory in $g$. The exact form of the perturbative expansion cannot be displayed in closed form, but one can build it piece by piece; the first pieces are displayed in (45).

The analogous equations for $E(A)$ for the sigma models with $\theta = \pi$ follow from their $S$ matrices. I will show that the large $A/M$ expansion is of the form (55), with the identical $\alpha_n$. This effectively proves that these $S$ matrices are those of the sigma model at $\theta = \pi$. The flow to the WZW model then follows immediately, because the $S$ matrix manifestly becomes that of the appropriate WZW model in the large-$M$ limit.

The integral equations for massless particles in a background field are similar to the ones for massive particles. The massless particles are left and right moving, and because they are gapless, they begin filling the sea for arbitrarily field $A$. Therefore, right moving particles with rapidities $-\infty < \theta < B$ and left moving particles with rapidities $-B < \theta < \infty$ fill the sea. For one species of right mover, and one species of left mover, the analysis at the beginning of this section can be repeated to derive the equations

$$\epsilon_R(\theta) = A - \frac{M}{2} e^\theta + \int_{-\infty}^{B} \phi_1(\theta - \theta') \epsilon_R(\theta') d\theta' + \int_{-B}^{\infty} \phi_2(\theta - \theta') \epsilon_L(\theta') d\theta'$$

(56)

$$\epsilon_L(\theta) = A - \frac{M}{2} e^{-\theta} + \int_{-\infty}^{B} \phi_2(\theta - \theta') \epsilon_R(\theta') d\theta' + \int_{-B}^{\infty} \phi_1(\theta - \theta') \epsilon_L(\theta') d\theta'$$

(57)

valid for $-\infty < \theta < B$ and $-B < \theta < \infty$ respectively, with the boundary conditions

$$\epsilon_R(B) = \epsilon_L(-B) = 0.$$

The kernels are defined as

$$\phi_1(\theta) = -\frac{i}{2\pi} \frac{\partial}{\partial \theta} \ln S_{LL}(\theta) \quad \phi_2(\theta) = -\frac{i}{2\pi} \frac{\partial}{\partial \theta} \ln S_{LR}(\theta)$$

The energy is

$$E^{(\pi)}(A) - E^{(\pi)}(0) = -\frac{M}{2\pi} \int_{-\infty}^{B} e^{\theta} \epsilon_R(\theta) d\theta'$$

(58)
where I have used the symmetry $\epsilon_L(\theta) = \epsilon_R(-\theta)$.

The two equations (56,57) can be made into one by exploiting the left-right symmetry. In terms of Fourier transforms, $\epsilon_R(\omega) = \tilde{\epsilon}_L(-\omega)$, giving

$$
\bar{\epsilon}_R(\omega)k_1(\omega) - \bar{\epsilon}_R(-\omega)k_2(\omega) - \bar{g}_R(\omega) = Y_+(\omega)e^{i\omega B} + Y_-(\omega)e^{-i\omega B}k_2(\omega)
$$

where the symmetry $k_2(-\omega) = k_2(\omega)$ is used, and $g_R(\theta) = A - me^{\theta}/2$. Note only one term is need on the right-hand side, because the integral in $\theta$ space runs all the way to $-\infty$. Getting rid of the $\epsilon_R(-\omega)$ gives

$$
\bar{\epsilon}_R(\omega) \left( k_1(\omega) - \frac{(k_2(\omega))^2}{k_1(\omega)} \right) - \bar{g}_R(\omega) = Y_+(\omega)e^{i\omega B} + Y_-(\omega)e^{-i\omega B}k_2(\omega)
$$

where as always $Y_-(\omega) = Y_+(\omega)$. This is now an equation of the form (17), and the same generalized Weiner-Hopf analysis can be applied. Defining

$$
\tilde{k}(\omega) \equiv k_1(\omega) - \frac{(k_2(\omega))^2}{k_1(\omega)}
$$

and factorizing it as $k(\omega) = 1/(k_+(\omega)k_-(\omega))$ in the usual way, one finds an equation just like (52), namely

$$
v(\omega) = -\frac{iAk_+(0)k_1(0)}{\omega + i\delta}k_2(0) + \frac{imeB}{2}K_+(i)k_1(i) + \int_{C_+} \frac{e^{2i\omega B}}{\omega + \omega' + i\delta} \beta(\omega')v(\omega') \frac{d\omega'}{2\pi i}
$$

where

$$
\beta(\omega) = \frac{k_+(\omega)k_1(\omega)}{k_+(\omega)k_2(\omega)}.
$$

The boundary conditions and energy follows with the same substitutions.

Now I can show that the sigma models have the same perturbative expansions at $\theta = 0$ and $\pi$. Using the magnetic field described in the last subsection, the kernels $\phi_1$ and $\phi_2$ follow simply from the Gross-Neveu $S$ matrix and the relations (25,27,31,32). As before, it is assumed that only particles with largest eigenvalue of this magnetic field occupy this state. For the $O(2P)/O(P) \times O(P)$ models at $\theta = \pi$, there is only one kind of particle (left and right moving) in the ground state. The Fourier transforms of the kernels are

$$
k_1(\omega) = 1 - \tilde{\phi}_1(\omega) = \frac{\sinh((\gamma + 1)\pi\Delta\omega)}{\sinh(\gamma\pi\Delta)} \frac{\cosh((1 - 2\Delta)\pi\omega/2)}{\cosh(\pi\omega/2)}
$$

$$
k_2(\omega) = \tilde{\phi}_2(\omega) = \frac{\sinh((\gamma - 1)\pi\Delta\omega)}{\sinh(\gamma\pi\Delta)} \frac{\cosh((1 - 2\Delta)\pi\omega/2)}{\cosh(\pi\omega/2)}
$$

where I have again included an extra parameter $\gamma$ to simplify the analysis. The sigma model kernels are recovered in the limit $\gamma \to \infty$, giving the exponential factors $e^{\pi\Delta|\omega|}$ and $e^{-\pi\Delta|\omega|}$ respectively. Just like the massive case, the $SU(N)/O(N)$ model has two particles in the ground state, and ends up with the same kernels.

From these explicit forms, one finds remarkably enough that $k(\omega)$ in the massless case is identical to $K(\omega)$ in (50) in the massive case. The only difference between the equations and (52) and (60) is the extra function

$$
\frac{k_1(\omega)}{k_2(\omega)} = \frac{\sinh((\gamma + 1)\Delta\pi\omega)}{\sinh((\gamma - 1)\Delta\pi\omega)}
$$

24
in all three terms. This indeed means that the energy at $\theta = \pi$ is not the same as $\theta = 0$. However, this extra piece has no effect on the perturbative contributions, so the energy is still given by (55). This is because the extra piece introduces no new poles in the integrand in (60) (the poles in (63) are canceled by zeros in $k_-/k_+$). Moreover, the residues at the poles at $\omega = i(2n + 1)/(2\Delta \gamma)$ are the same in the massive and massless cases (up to an overall sign), because

$$k_1(i(2n + 1)/(2\Delta \gamma)) k_2(i(2n + 1)/(2\Delta \gamma)) = -1.$$ 

These residues are what determine the coefficients $\alpha_n$ in (55), so indeed the perturbative expansions at $\theta = 0$ and $\theta = \pi$ are completely identical. This was established for the sphere sigma model in [11]; here I have extended this proof to two infinite hierarchies of models.

This completes the identification of the massless $S$ matrices with the sigma models at $\theta = \pi$. They give the identical perturbative expansion as the sigma model at $\theta = 0$, but the non-perturbative pieces differ. As an additional check, I have also computed the c-function by computing the free energy with no magnetic field but at non-zero temperature [31]. This gives the correct behavior, thus completely confirming this identification.

7 Flows when $\theta = \pi$

The main point of this paper is that the stable fixed point in the sphere sigma model at $\theta = \pi$ is not an isolated instance. There are at least two infinite hierarchies of models which have this behavior. Thus current approaches to the problem of disordered electrons in two dimensions in both the replica approach and the supersymmetric approach (see [5] and [37] and references therein) are on sound footing.

One obvious question is such flows happen in other sigma models. In this section I will show how to obtain the $Sp(2N)/U(N)$ sigma model by perturbing the $O(4N)/O(2N) \times O(2N)$ model. I will use this to make a conjecture that at $\theta = \pi$ the former has the $Sp(2N)_1$ WZW model as its low-energy fixed point.

First, I will establish a flow between the two integrable hierarchies. The field $\Phi$ in the $O(2N)/O(N) \times O(N)$ sigma model is an $2N \times 2N$ traceless real symmetric unitary matrix. It is simple to see that the field configurations of the $SU(N)/SO(N)$ models are a subspace of these. The configurations of the former can be written as

$$\begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

(64)

where $A$, $B$, and $D$ are real $N \times N$ matrices. $A$ and $D$ are symmetric, and $\text{tr} A + \text{tr} D = 0$. To ensure unitarity the matrices must satisfy

$$AA + BB = DD + BB = I$$

$$AB^T + BD = 0$$

where $I$ is the $N \times N$ identity. The subspace $U(N)/SO(N)$ is obtained by requiring that $A = -D$ and $B = B^T$. The matrix $A + iB$ is indeed a symmetric unitary $N \times N$ matrix, as can be verified from the preceding matrix relations. To get the $SU(N)/SO(N)$ sigma model, one must in addition require $\det(A + iB) = 1$. Thus one can flow from the $O(2N)/O(N) \times O(N)$ sigma model to the $SU(N)/SO(N)$ model adding the potential

$$\lambda \int (\text{tr} [(B - B^T)^2 + (A + D)^2] + |\det(A + iB)|^2)$$

(65)
and making $\lambda$ large. If $\theta = \pi$ in the former model, then $\theta = \pi$ in the latter. This flow is illustrated in Fig. 1.

This flow between the low-energy fixed points can also be seen explicitly. This is the flow along the $x$-axis in figure 1. The $O(2N)_1$ fixed point is equivalent to $2N$ free massless Majorana (real) fermions. The right-moving fermions are denoted $\psi^R_\alpha$, while the left movers are denoted $\psi^L_\alpha$, where $\alpha = 1 \ldots 2N$. The $O(2N)$ symmetry currents are then

$$j^{\alpha\beta}_R = 2i \psi^\alpha_R \psi^\beta_R$$

and likewise for $j^{\alpha\beta}_L$. Because of Fermi statistics, $j^{\alpha\beta} = -j^{\beta\alpha}$, so there are $N(N-1)/2$ different right-moving currents. (I suppress the $L,R$ subscripts: equations without them are meant to apply to both $L$ and $R$.) These currents generate the $O(2N)$ symmetry, which at the critical point is enhanced to a chiral $O(2N)_L \times O(2N)_R$ symmetry. The theory of $2N$ free Majorana fermions is of course equivalent to a theory of $N$ Dirac fermions $\Psi^a$, defined by

$$\Psi^a = \psi^a + i\psi^{a+N}.$$  

for $a = 1 \ldots N$. From the Dirac fermions, one can form the $U(N)$ symmetry currents

$$j^{ab} = \overline{\Psi}^a \Psi^b.$$  

These $N^2$ right-moving and $N^2$ left-moving currents generate a $U(N)_L \times U(N)_R$ subgroup of the full symmetry.

In the terms of WZW models, the equivalence between $N$ Dirac fermions and $2N$ Majorana fermions is written as

$$O(2N)_1 = SU(N)_1 \oplus U(1).$$  

(66)
The $U(1)$ symmetry in $U(N)$ is generated by

$$j^0 \equiv \sum_{a=1}^{N} j^{aa}.$$  

This can be split off because the generator $j^0$ commutes with all the generators $j^{ab}$, so the corresponding WZW models are independent. For $N = 2$, this splitting is the simplest example of spin-charge separation: the $U(1)$ is the charge mode, while the $SU(2)_1$ are the spin modes. The equivalence (66) is a simple example of what is called a conformal embedding [38]. Yet another name often used is non-abelian bosonization: the free fermions of $O(2N)_1$ are written in terms of the bosonic fields of the WZW model. The simplest example of a conformal embedding is $SU(2)_1 = U(1)$. This means that the $SU(2)_1$ can be written in terms of a single boson $\phi$: the $SU(2)$ currents $j^x, j^y$ and $j^z$ are $\cos \phi, \sin \phi$ and $\partial \phi$ respectively.

With the conformal embedding/spin-charge separation/non-abelian bosonization (66), it is now easy to see how to flow from $O(2N)_1$ to $SU(N)_1$. One needs to add a perturbation which gives a gap to the $U(1)$ charge mode but leaves the $SU(N)_1$ untouched. If we define the $U(1)$ charge boson $\phi$ by $j^0 = \partial \phi$, the perturbation can be written as

$$S = S_{O(2N)_1} + \lambda \int \cos(\phi_L + \phi_R),$$

Since the $SU(N)_1$ currents commute with $j^0$, they are untouched by this perturbation. Therefore, in the limit $\lambda \to \infty$, the perturbed theory reaches the $SU(N)_1$ model. Thus the flow between the two sigma models can be seen in the WZW models as well: essentially all one needs to do is break the original symmetry appropriately.

This is only one example of a conformal embedding. A complete list of all such embeddings is given in [39]. An embedding of particular interest here is

$$SO(4N)_1 = Sp(2N)_1 \oplus SU(2)_N \quad (67)$$

To realize this embedding explicitly, it is convenient to write the $SO(4N)$ currents in terms of real antisymmetric $4N \times 4N$ matrices $T^{\alpha\beta}_{kl}$ where

$$j^{\alpha\beta} = i\psi^k T^{\alpha\beta}_{kl} \psi^l.$$  

Explicitly, $T^{\alpha\beta}_{kl} = \delta^\alpha_k \delta^\beta_l - \delta^\alpha_l \delta^\beta_k$. Then the $SU(2)$ subalgebra is given by the matrices

$$T^z = i\sigma^y \otimes I$$
$$T^x = \sigma^x \otimes Z$$
$$T^y = \sigma^z \otimes Z$$

where $I$ is the $2N \times 2N$ identity, the $\sigma^a$ are the Pauli matrices, and $Z$ is a fixed real antisymmetric $2N \times 2N$ matrix which obeys $Z^2 = I$. Since $T^x, T^y$ and $T^z$ are $4N \times 4N$ real antisymmetric matrices, they are linear combinations of the $T^{\alpha\beta}_{kl}$, so they do indeed form an $SU(2)$ subalgebra of $O(4N)$. The subalgebra which commutes with the $SU(2)$ subalgebra consists of matrices of the form

$$I \otimes C$$
$$i\sigma^y \otimes D$$

27
where $I$ is the $2 \times 2$ identity and $C$ and $D$ are respectively antisymmetric and symmetric real $2N \times 2N$ matrices. For these to commute with $T^x$, $T^y$ and $T^z$, $C$ and $D$ must satisfy

$$[C, Z] = 0 \quad \{D, Z\} = 0$$

The subalgebra consisting of all matrices $C$ and $D$ obeying these constraints is precisely $Sp(2N)$; by exponentiating $C$ and $D$ into some matrix $P \equiv e^C + D$, one finds $PTZP = Z$, which is the defining relation of the group $Sp(2N)$. Note that because $C$ is real, $P$ is not necessarily unitary. A unitary matrix can be obtained by exponentiating $iC$ and $D$, giving the embedding of $Sp(2N)$ in $SU(2N)$.

Therefore, in the embedding $[57]$, the currents $J^a = i\psi^k(T^a)_{kl}\psi^l$ ($a = x, y$ or $z$) form the $SU(2)_N$ current algebra, while the currents $\psi^k(I \otimes C)_{kl}\psi^l$ and $\psi^k(i\sigma^y \otimes D)_{kl}\psi^l$ form the $Sp(2N)_1$ current algebra. Thus a flow from the $SO(4N)_1$ fixed point to the $Sp(2N)_1$ fixed point occurs if one gives a gap to the $SU(2)_N$ modes. This can be done with the perturbation

$$S = S_{O(4N)_1} + \lambda \int \text{tr} (J^x_L J^x_R + J^y_L J^y_R + J^z_L J^z_R)$$

When $\lambda$ goes to $\infty$, the flow reaches the $Sp(2N)_1$ fixed point. As a tangential comment, note that one can obtain a fermionic realization of the $Sp(2N)$ Gross-Neveu model by perturbing $O(4N)_1$ by the $Sp(2N)_1$ currents instead of the $SU(2)_N$ ones.

The question now is if this flow implies a low-energy fixed point in any sigma model at $\theta = \pi$. My conjecture is that it does. Namely, consider the $Sp(2N)/U(N)$ sigma model. Field configurations in this coset space can be realized as a subspace of the $O(4N)/O(2N) \times O(2N)$ configurations. Matrices in the latter are of the form $[54]$, where $A$, $B$ and $D$ are now symmetric $2N \times 2N$ matrices. To get $Sp(2N)/U(N)$ requires the restrictions $A = -D$ and $B = B^T$ as before, plus the additional restriction

$$VZV = Z, \tag{68}$$

where $V \equiv A + iB$. In other words, the $Sp(2N)/U(N)$ subspace of $O(4N)/O(2N) \times O(2N)$ consists of matrices of the form

$$\frac{1}{2} \begin{pmatrix} V + V^* & i(V - V^*) \\ i(V - V^*) & -V - V^* \end{pmatrix}$$

where $V$ is a $2N \times 2N$ symmetric unitary matrix obeying the condition $[58]$. One can therefore obtain this model by a perturbation like $[55]$. Here, the perturbation breaks the $O(4N)$ global symmetry down to $Sp(2N)$. It is giving a large gap to the modes outside the $Sp(2N)$ subgroup, effectively removing them from the theory. This is just how the flow described above goes from the $SO(4N)_1$ WZW model to the $Sp(2N)_1$ model. It is thus very plausible that the general flows look like those in Fig. 2.

This is why I have conjectured that the low-energy fixed point of the $Sp(2N)/U(N)$ sigma model at $\theta = \pi$ is the $Sp(2N)_1$ WZW model. This is certainly true at $N = 1$, because $Sp(2) = SU(2)$ and this becomes the flow in the sphere sigma model. For $N > 1$, any non-trivial fixed point in $Sp(2N)/U(N)$ should be a perturbation of the $\theta = \pi$ fixed point of $O(4N)/O(2N) \times O(2N)$, namely $O(4N)_1$, which has central charge $2N$. If there is a non-trivial critical point of $Sp(2N)/U(N)$ when $\theta = \pi$, it must have central charge less than $2N$, which leaves only $Sp(2N)_1$.

Further evidence for this conjecture comes from the replica limit $N \to 0$. The resulting model describes the spin quantum Hall effect $[10]$, and can also be studied in a supersymmetric formulation. It is found in $[12]$ that certain correlators are equivalent to those in classical percolation.
Figure 2: The conjectured phase diagram of the $O(4N)/O(2N) \times O(2N)$ sigma model at $\theta = \pi$ with symmetry broken to $Sp(2N)$.

From this one can extract the density of states exponent in the disordered model, and indeed it agrees with the result from taking $N \to 0$ for the corresponding exponent in $Sp(2N)$ (see also [44]). There are many potential subtleties in taking the replica limit, so this is hardly a proof, but I take it as a good piece of evidence that the conjecture is true.

8 Conclusions

I have shown that several hierarchies of sigma models flow to a non-trivial low-energy fixed point when $\theta = \pi$. This provides $SU(N)$ and $O(2N)$-symmetric generalizations of the $SU(2)$ results of [1, 10]. In another paper, I will calculate the $c$-functions for these models [31].

This result is a useful check on the currently-popular approach to disordered models in two dimensions. Most recent study has been based on the assumption that that the picture of [4] is fairly general, and this paper shows that the picture holds for the $SU(N)/O(N)$ models (class CII) and the $O(2N)/O(N) \times O(N)$ models (the GSE class). All the models discussed in this paper have WZW models as their non-trivial fixed points. However, if the model believed to apply to the integer quantum Hall plateau phase transition (the $N \to 0$ limit of $U(2N)/U(N) \times U(N)$ model at $\theta = \pi$ [4]) has a fixed point, it does not seem likely that it is of WZW type, as argued in [30]. Thus there is still a great deal of interesting physics yet to be uncovered in sigma models with $\theta = \pi$.

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