Roy’s equations and the $\pi\pi$ experimental data

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Abstract

Roy’s equations are used to check if the scalar-isoscalar $\pi\pi$ scattering amplitudes fitted to experimental data fulfill crossing symmetry conditions. It is shown that the amplitudes describing the “down-flat” phase shift solution satisfy crossing symmetry below 1 GeV while the amplitudes fitted to the “up-flat” data do not. In this way the long standing ”up-down” ambiguity in the phenomenological determination of the scalar-isoscalar $\pi\pi$ amplitudes has been resolved confirming the independent result of the recent joint analysis of the $\pi^+\pi^-$ and $\pi^0\pi^0$ data.

1 Introduction

In 1997 a new analysis of the $\pi^-p^+ \rightarrow \pi^+\pi^-n$ reaction on a polarized target was performed in the $m_{\pi\pi}$ effective mass range from 600 to 1600 MeV [1]. For the first time the pseudoscalar ($\pi$-exchange) amplitude was separated from the pseudovector ($a_1$-exchange) amplitude in the region of the the four-momentum transfer squared from $-0.005$ to $-0.2$ (GeV/c)$^2$. Below 1000 MeV, where the $S$- and $P$-waves strongly interfere, the partial wave analysis of the $\pi^+\pi^-$ data provided us with two scalar-isoscalar solutions, called ”up” and ”down”, which differ by their intensities. Lack of information on a sign difference between the phases of the $S$- and $P$-waves near the position of the $\rho$ resonance led us to other two branches of the ”up” and ”down” amplitudes named ”steep” and ”flat”. It was shown in [2] that both “up-steep” and “down-steep” $S$-wave isoscalar amplitudes significantly violate unitarity below 1 GeV.
and should be rejected as nonphysical. Two remaining "flat" amplitudes survived the unitarity check and other tests were needed to resolve the existing "up-down" ambiguity.

In 2001 new experimental data on the $\pi^0\pi^0$ production from the E852 collaboration appeared [3] and were used in a joint analysis of the $\pi^+\pi^-$ and $\pi^0\pi^0$ data [4]. The $\pi^0\pi^0$ data are very useful to compare with the $\pi^+\pi^-$ data due to an absence of the $P$-wave in the $\pi^0\pi^0$ channel and therefore a lack of the "up-down" ambiguity. The one-pion and $a_1$-exchange model described in [4] was used to calculate the $S$-wave intensities of the $\pi^0\pi^0$ production by choosing as an input the "up-flat" or "down-flat" phase shifts. The isospin relations between the $\pi^+\pi^- \rightarrow \pi^+\pi^-$ and $\pi^+\pi^- \rightarrow \pi^0\pi^0$ amplitudes supplemented by the parameterization of the isotensor scalar amplitude taken from [1] were helpful in these calculations. It was shown that the $\pi^0\pi^0$ $S$-wave intensities determined for the "down-flat" phase shifts agree with the experimental values within the errors. However, for the "up-flat" phase shifts in the $m_{\pi\pi}$ range from 850 to 970 MeV important differences between the calculated $\pi^0\pi^0$ intensities and the corresponding experimental values occur. This fact led the authors to a conclusion that the "up-flat" data set should also be rejected.

2 Roy’s equations as a test for the $\pi\pi$ amplitudes

Another independent test of the "up-flat" and "down-flat" amplitudes consists in checking if they fulfill crossing symmetry conditions below 1 GeV. In order to achieve this task we have used Roy’s equations [5] for the scalar-isoscalar, $\ell = 0$, $I = 0$, scalar-isotensor, $\ell = 0$, $I = 2$, and the vector-isovector, $\ell = 1$, $I = 1$, $\pi\pi$ partial waves determined in a wide $m_{\pi\pi}$ range. We were especially interested in the $m_{\pi\pi}$ region between 800 and 1000 MeV where differences between phase shifts of the “up-flat” and “down-flat” data sets are largest and reach about 45° (see Fig. 4 in [2]). In a recent analysis of Roy’s equations [6] a special attention was put on the effective mass lower than 800 MeV.

As an input in Roy’s equations we have used imaginary parts of the partial waves amplitudes $f^I_\ell(s)$ related to the $\pi\pi$ phase shifts $\delta^I_\ell$ and inelasticities $\eta^I_\ell$:

$$f^I_\ell(s) = \sqrt{\frac{s}{s - 4\mu^2}} \frac{1}{2i} \left( \eta^I_\ell e^{2i\delta^I_\ell} - 1 \right),$$  (1)

where $\mu$ is the charged pion mass and $s = m_{\pi\pi}^2$.

Below 970 MeV the following Padé representation of both the “up-flat” and “down-flat” phase shifts has been taken:

$$\tan \delta^0_0(s) = \frac{\sum_{i=0}^{4} \alpha_{2i+1} k^{2i+1}}{\prod_{i=1}^{3} (k^2/\alpha_{2i} - 1)},$$  (2)

where $k = \frac{1}{2} \sqrt{s - 4\mu^2}$ is the pion momentum and $\alpha_j$ ($j = 1, \ldots, 7, 9$) are constant parameters. Above 970 MeV up to 2 GeV our coupled channel model [7] amplitude $A,$
We have used the CERN MINUIT program with the $\chi^2$ fits to data and to Roy’s equations separately for the “up-flat” and “down-flat” data.

The parameterization of isovector wave using a rank-two separable potential model has been described in [9] where detailed analysis of the present study is presented.

For the $P$-wave, from the $\pi\pi$ threshold till 970 MeV, we have used an extended Schenk parameterization [6]:

$$\tan \delta^p_I(s) = \sqrt{1 - \frac{4\mu^2}{s}} k^2 \left( A + Bk^2 + Ck^4 + Dk^6 \right) \left(\frac{4\mu^2 - s_\rho}{s - s_\rho}\right),$$

(3)

where $A$ is the $P$-wave scattering length and $s_\rho$ is equal to the $\rho$-mass squared. Above 970 MeV we took the $K$-matrix parameterization of Hyams et al. [10]. The parameters $C$ and $D$ were chosen to join smoothly the phase shifts given by both parameterizations around 970 MeV.

The contributions to Roy’s equations from high energies ($m_{\pi\pi} > 2$ GeV) and from higher partial waves ($l > 1$) are called driving terms. They are composed of contributions from the $f_2(1270)$ and $\rho_3(1690)$ resonances and from the Regge amplitudes for the Pomeron, $\rho$- and $f$-exchanges. The Breit-Wigner parameterization with masses, widths and $\pi\pi$ branching ratios taken from [11] were used for $f_2(1270)$ and $\rho_3(1690)$.

For the Regge parts we have used formulae of [6] without the $u$-crossed terms. We have found that the $f_2(1270)$ resonance dominates in the scalar isoscalar wave and that the introduction of the $\rho_3(1690)$ has a significant influence on the isotensor and isovector waves. In the isoscalar wave the Regge contributions are more than 10 times smaller than the resonance contributions but for the isospin 1 and 2 they are of the same order.

The thirteen constants, six for the scalar-isoscalar wave in (2), four for the isotensor wave and three for the isovector wave in (3), were calculated from the simultaneous fits to data and to Roy’s equations separately for the “up-flat” and “down-flat” data. We have used the CERN MINUIT program with the $\chi^2$ test function defined by

$$\chi^2 = \sum_{I=0,1,2} \left\{ \sum_{j=1}^{N_I} \left[ \frac{\sin(\delta^I_j(s_i) - \varphi^I_j(s_i))}{\Delta \varphi^I_i(s_i)} \right]^2 + \sum_{j=1}^{12} \left[ \frac{\text{Re} f^{I}_{\text{out}}(s_j) - \text{Re} f^{I}_{\text{in}}(s_j)}{\Delta f} \right]^2 \right\},$$

(4)

where $\varphi^I_j(s_i)$ and $\Delta \varphi^I_i(s_i)$ represent the experimental phase shifts and their errors, respectively, $s_j = [4j + 0.001]\mu^2$ for $j = 1, \ldots, 11$ and $s_{12} = 46.001\mu^2$. The real parts $\text{Re} f^{I}_{\text{in}}$ have been calculated from (1) under an assumption that the inelasticity $\eta^I_j$ is equal to 1 and the phase shifts $\delta^I_j$ are equal to $\phi^I_j(s_j)$. Other real parts, denoted by $\text{Re} f^{I}_{\text{out}}$, constitute the output values calculated from Roy’s equations. We take a $\Delta f$ value of $0.5 \times 10^{-2}$ to obtain acceptable fits to Roy’s equations. 18 experimental values of the “up-flat” or “down-flat” data between 600 and 950 MeV were used in addition to six data taken from [8].
In Fig. 1a and 1b we present results of fits to the “up-flat” and “down-flat” phase shifts and to Roy’s equations (solid lines). In both cases differences |Re $f_{out}^I$ - Re $f_{in}^I$| were of the order of $10^{-3}$ in all three partial waves. The $\chi^2$ for 18 points between 600 and 970 MeV was 16.6 in the “down-flat” case and as large as 46.4 in the “up-flat” one. We see in Fig. 1a that the solid line lies distinctly below the “up-flat” data points between 800 and 970 MeV. In contrary, the corresponding line for the “down-flat” case in Fig. 1b is very close to experimental data in the same range of $m_{\pi\pi}$. In order to improve a fit to the “up-flat” data we have used constraints given by the good fit to the “down-flat” data. Two parameters were fixed by choosing the values of the scattering length and the slope parameter and two others by the values of phase shifts calculated from this fit at 500 and 550 MeV. A new fit with these constrains gave an improved value of $\chi^2 = 13$ for 18 “up-flat” data points, corresponding to the first part of $\chi^2$ in (4) but provided us with an enormous value of $\chi^2 = 1.2 \times 10^4$ for the second part related to Roy’s equations. The phase shifts for this amplitude are presented in Fig. 1a by the dotted line. It is clear that a simultaneous good fit to the “up-flat” data and to Roy’s equations is impossible.

Apart of the fits to the “up-flat” and “down-flat” experimental points we have also performed fits to points shifted upwards and downwards by their errors. In these fits the same four constraints described above were used below 600 MeV. Up to 937 MeV in the “down-flat” case in Fig. 1d the curves labeled higher ”in” and lower ”in” form a band including inside a band delimited by the lines higher ”out” and lower ”out”. All the curves lying inside these bands correspond to the amplitudes fulfilling the crossing symmetry so the “down-flat” data can be accepted as physical ones. In the “up-flat” case in Fig. 1c the output band lies outside of the input band from 840 to 970 MeV. It means that in this case the crossing symmetry is violated by the amplitudes fitted to the “up-flat” data.

In Fig. 2 we have presented the output results for the isotensor and isovector waves in the “down-flat” case only since in the “up-flat” case the curves are very similar. The ”in” curves were not plotted because they are almost indistinguishable from the ”out” ones.

### 3 Conclusions

We have used Roy’s equations as a tool to test if the amplitudes fitted to the “up-flat” and “down-flat” phase shifts extracted from the $\pi^-p_\uparrow \rightarrow \pi^+\pi^-n$ data fulfill crossing symmetry conditions. We have found that only the $S$-wave isoscalar amplitude corresponding to the “down-flat” data set can be accepted. The amplitude constrained to the “up-flat” data does not satisfy Roy’s equations and should be rejected as non-physical. This conclusion is in agreement with the independent results obtained from a joint analysis of the $\pi^+\pi^-$ and the $\pi^0\pi^0$ production data [4]. In this way a long standing ”up-down” ambiguity in the $\pi\pi$ experimental data has been resolved in favour of the “down-flat” data set.
Figure 1: a) and c) correspond to the “up-flat” case, b) and d) correspond to the “down-flat” case. Fits to the scalar-isoscalar phase shifts of [1] and to Roy’s equations are denoted by solid lines in a) and b). Dotted line in a) between two dashed lines represents fit with constraints described in the text. Dashed lines in a) and b) represent fits to phase shifts moved upwards and downwards by their errors; the corresponding lines in c) and d) are called higher and lower, respectively. Lines in c) and d) correspond to real parts of input amplitudes (“in”) and real parts calculated from Roy’s equations (“out”), all multiplied by $2k\lambda^{-1/2}$. Diamonds represent the $K_{e4}$ data [8].
Figure 2: Real parts of isotensor a) and isovector b) $\pi\pi$ amplitudes (multiplied by $2ks^{-1/2}$) fitted to the “down-flat” data. Triangles in a) denote data of [12]. Crosses in b) are the pseudo-data calculated from the $K$-matrix fit of [10].

References

[1] Kamiński R., Leśniak L. and Rybicki K., Z. Phys. C 74, 79 (1997).
[2] Kamiński R., Leśniak L. and Rybicki K., Acta Phys. Pol. B31, 895 (2000).
[3] Gunter J. et al. (E852 Collaboration), Phys. Rev. D 64, 072003 (2001).
[4] Kamiński R., Leśniak L. and Rybicki K., Eur. Phys. J. direct C4, 1 (2002).
[5] Roy S. M., Phys. Lett. B 36, 353 (1971); Roy S. M., Helv. Phys. Acta 63, 627 (1990).
[6] Ananthanarayan B., Colangelo G., Gasser J. and Leutwyler H., Phys. Rep. 353, 207 (2001).
[7] Kamiński R., Leśniak L., B. Loiseau, Phys. Lett. B 413, 13 (1997).
[8] Pislak S. et al., (E865 Coll.), Phys. Rev. Lett. 87, 221801 (2001).
[9] Kamiński R., Leśniak L. and B. Loiseau, Phys. Lett. B 551, 241 (2003).
[10] Hyams B. et al., Nucl. Phys. B 64, 134 (1973).
[11] K. Hagiwara et al. (Particle Data Group), Phys. Rev. D66, 010001 (2002).
[12] Hoogland W. et al., Nucl. Phys. B126, 109 (1977).