Knizhnik-Zamolodchikov type equations for the root system $B$ and Capelli central elements

D V Artamonov$^1$ and V A Golubeva$^2$
$^1$Faculty of Economics, Lomonosov Moscow State University
1-46 Leninskie Gory, 119991 Moscow, GSP-1, 1-46, Russia
$^2$Department of Applied Mathematics and Physics, Moscow Aviation Institute
4 Volokolamskoe Shosse, 125993 Moscow, A-80, GSP-3, Russia

Abstract. The construction of the well-known Knizhnik-Zamolodchikov equations uses the central element of the second order in the universal enveloping algebra for some Lie algebra. But in the universal enveloping algebra there exist central elements of higher orders. It seems desirable to use these elements for the construction of Knizhnik-Zamolodchikov type equations. In the present paper we give a construction of such Knizhnik-Zamolodchikov type equations for the root system $B$ associated with Capelli central elements in the universal enveloping algebra for the orthogonal algebra.

1. Introduction
The Knizhnik-Zamolodchikov equations are a system of differential equations which is satisfied by correlation functions in the WZW theory [1]. Later it turned out that these equation are related with many other areas of mathematics (quantum algebra, isomonodromic deformation).

The Knizhnik-Zamolodchikov equations are also interesting as a nontrivial example of an integrable Pfaffian system of Fuchsian type. It must be mentioned that the monodromy representation of this system is known explicitly. Thus we get a solution for the Riemann-Hilbert problem in a very particular case.

The Knizhnik-Zamolodchikov equations read

$$dy = \lambda \left( \sum_{i \neq j=1}^{n} \tau_{ij} \frac{d(z_i - z_j)}{z_i - z_j} \right) y$$

where $\lambda$ is some complex parameter, $y(z_1, \ldots, z_n)$ is a vector function that takes values in a tensor power $V^\otimes n$ of a representation space $V$ of a finite-dimensional Lie algebra $\mathfrak{g}$, and $\tau_{ij}$ is defined by the formula

$$\tau_{ij} = \sum_s 1 \otimes \cdots \otimes \rho(I_s) \otimes \cdots \otimes \rho(\omega(I_s)) \otimes \cdots \otimes 1$$  \hspace{1cm} (1.1)

Here $\{I_s\}$ is a basis of the Lie algebra $\mathfrak{g}$ and $\{\omega(I_s)\}$ is its dual basis with respect to the Killing form. The elements $I_s$ occur on tensor factors $i$ and $j$, and $\rho : \mathfrak{g} \to End(V)$ is a representation of the Lie algebra $\mathfrak{g}$. Thus $\tau_{ij}$ is the matrix of the linear operator on $V^\otimes n$. 

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The Knizhnik-Zamolodchikov equations have singularities on the planes $z_i = z_j$. It is natural to look for similar systems that have other singular locus.

Several authors constructed different Knizhnik-Zamolodchikov type equations, that have singularities on the reflection hyperplanes of different root system. Thus, in [2] a system of the Knizhnik-Zamolodchikov type equations is constructed that have as the singular set the reflection hyperplanes corresponding to an arbitrary root system. Let us present such a system.

Matsuo considers a root system $\Delta$, the set of positive roots is denoted as $\Delta^+$ and the corresponding Weyl group is denoted as $W$. The function $y$ takes values in the group algebra $\mathbb{C}[W]$ and the independent variable belong to the dual to the vector space spanned by roots.

Denote a reflection corresponding to the root $\alpha$ as $\sigma_\alpha$.

The Matsuo Knizhnik-Zamolodchikov type equations contain arbitrary parameters $\lambda_{|\alpha|}$ depending on the length of the roots $\alpha$ and one additional parameter $\lambda$. The system has the following form.

$$\frac{\partial y}{\partial \xi} = \left( \sum_{\alpha \in \Delta^+} \lambda_{|\alpha|} (\alpha, z) \frac{1}{(\alpha, u)} (\sigma_\alpha - 1) \right) y$$

Note that solutions of this system take values in the space $\mathbb{C}[W]$ and no Lie algebra is involved in it’s construction explicitly. Nevertheless if one takes in the Matsuo’s construction the root system $A$ and some special representation of the Weyl group one can obtain from Matsuo’s system an ordinary Knizhnik-Zamolodchikov system corresponding to the algebra $\mathfrak{sl}_n$ and its standard representation. Later systems of Matsuo’s type were intensively studied by numerous authors (I. Cherednik [3] and others).

There exist other constructions of the Knizhnik-Zamolodchikov type equations with are closer to the original equations. One of them is the Leibman’s construction of Knizhnik-Zamolodchikov type equations associated with the root system $B$ [4] and Enriquez’s cyclotomic Knizhnik-Zamolodchikov type equations [5].

In [6] we have constructed Knizhnik-Zamolodchikov type equations for the root system $A$, but our construction was based on special central elements in the universal enveloping algebra of the orthogonal algebra, namely the Capelli elements. In the present paper we generalize our construction to the root system $B$ and show that the construction from [6] can be in fact interpreted in some sence as a very special case of the general Leibman’s construction.

2. Knizhnik-Zamolodchikov equation associated with the root system $B$

Let us explain the Leibman’s construction of Knizhnik-Zamolodchikov type equations for the root system $B$. This is a system of type

$$dy = \lambda \left( \sum_{i<j} \frac{d(z_i - z_j)}{z_i - z_j} \tau_{ij} + \sum_{i<j} \frac{d(z_i - z_j)}{z_i - z_j} \mu_{ij} + \sum_i \frac{\nu_i}{z_i} \right) y$$

were $\tau_{ij}$ is defined by (1.1). Denote by $\sigma$ an involution in the considered Lie algebra $\mathfrak{g}$. Then one has

$$\mu_{ij} = \sum_s 1 \otimes \ldots \otimes \rho(I_s) \otimes \ldots \otimes \rho(\sigma(\omega(I_s))) \otimes \ldots \otimes 1$$

where the elements $I_s$ occur on places $i, j$ and

$$\nu_i = 1/2 \sum_s 1 \otimes \ldots \otimes \rho(I_s \sigma(\omega(I_s)) + I_s^2) \otimes \ldots \otimes 1$$

(2.1)
where the elements $I_s$ occur on the place $i$.

Although the Leibman’s proof in [4] is done for the case of a simple Lie algebra and is based on calculations in the root base, in [5] there is presented a proof that is valid for an arbitrary finite-dimensional Lie algebra with a fixed central element in $U(\mathfrak{g})$ of the second order.

### 3. The Lie algebra $\mathcal{T}$

In this section we introduce a Lie algebra $\mathcal{T}$ which plays the crucial role in our construction.

Consider the space of skew-symmetric tensors with $2n$ indices. Let each index of the skew symmetric tensor take values in the set $-n, \ldots, n$.

There exists a structure of an associative algebra algebra on this space

$$\left( e_{a_1} \wedge \ldots \wedge e_{a_n} \wedge e_{b_1} \wedge \ldots \wedge e_{b_n} \right) \cdot \left( e_{-b_1} \wedge \ldots \wedge e_{-b_n} \wedge e_{c_1} \wedge \ldots \wedge e_{c_n} \right) := e_{a_1} \wedge \ldots \wedge e_{a_n} \wedge e_{c_1} \wedge \ldots \wedge e_{c_n}$$

As a corollary we have a structure of a Lie algebra, denote it as $\mathcal{T}$.

The algebra $\mathcal{T}$ has a representation on the space of skew-symmetric tensors with $n$ indices defined by the formula

$$ e_{a_1} \wedge \ldots \wedge e_{a_n} \wedge e_{b_1} \wedge \ldots \wedge e_{b_n} \left( e_{-b_1} \wedge \ldots \wedge e_{-b_n} \right) := e_{a_1} \wedge \ldots \wedge e_{a_n} $$

(3.1)

The algebra has an involution $\omega$ defined as follows

$$ \omega(e_{a_1} \wedge \ldots \wedge e_{a_n} \wedge e_{b_1} \wedge \ldots \wedge e_{b_n}) = e_{-a_1} \wedge \ldots \wedge e_{-a_n} \wedge e_{-b_1} \wedge \ldots \wedge e_{-b_n} $$

In $U(\mathcal{T})$ there is a central element of the second order, namely the element

$$ C = \sum_{a_1, \ldots, a_n, b_1, \ldots, b_n} \left( e_{a_1} \wedge \ldots \wedge e_{a_n} \wedge e_{b_1} \wedge \ldots \wedge e_{b_n} \right) \cdot \left( e_{-a_1} \wedge \ldots \wedge e_{-a_n} \wedge e_{-b_1} \wedge \ldots \wedge e_{-b_n} \right) $$

where $\cdot$ denotes the multiplication in $U(\mathcal{T})$. The proof of this fact is essentially contained in [7].

Using general constructions described in Section (2) one can construct the Knizhnik-Zamolodchikov type equations associated with the root system $B$ based with coefficients in the Lie algebra $\mathcal{T}$.

In the next section we give an interpretation of these equations as the Knizhnik-Zamolodchikov type equations whose construction is based on some certain higher order central elements in $U(\mathfrak{o}_{2n+1})$.

### 4. Capelli elements and noncommutative Pfaffians

Let us define some certain central elements in the universal enveloping algebra of the orthogonal algebra.

#### 4.1. The split realization of the orthogonal algebra

We use the split realization of the orthogonal algebra. This means that we define the orthogonal algebra as the algebra that preserves the quadratic form with the matrix

$$ G = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix} $$

The row and columns are indexed by $i, j = -n, -n+1, \ldots, n-1, n$. The zero is skipped in the case $N = 2n$ and is included in the case $N = 2n + 1$. The algebra $\mathfrak{o}_N$ is generated by matrices

$$ F_{ij} = E_{ij} - E_{-j-i} $$

The commutation relations between these generators are the following

$$ [F_{ij}, F_{kl}] = \delta_{kj} F_{il} - \delta_{il} F_{kj} + \delta_{i-k} F_{jl} + \delta_{-i-j} F_{k-i} $$
4.2. Noncommutative Pfaffians and Capelli elements

Now describe some special higher order central elements in the universal enveloping for the orthogonal algebra.

Let \( \Phi = (\Phi_{ij}) \) be a \( k \times k \) matrix, where \( k \) is even, whose elements belong to some noncommutative ring. The noncommutative Pfaffian is defined by

\[
PfF \Phi = 2^{-k/2}/(k/2)! \sum_{\sigma \in S_k} (-1)^\sigma \Phi_{\sigma(1)\sigma(2)} \cdots \Phi_{\sigma(k-1)\sigma(k)}
\]

For a subset \( I \subset \{-n, \ldots, n\} \) define a submatrix \( F_I = (F_{ij})_{i,j \in I} \). For this subset put

\[
PfF F_I := PfF(F_{-i,j})_{-i,j \in I}
\]

**Definition 4.1.** Put

\[
C_k = \sum_{I \subset \{1, \ldots, N\}, |I| = k} PfF F_I PfF F_I
\]

The elements \( C_k \) are are called the Capelli elements.

**Theorem 4.2** ([7]). For odd \( N \) the elements \( C_k \) are algebraically independent and generate the center, for even \( N \) the same is true if one takes instead the highest Capelli element \( C_N = (PfF F)^2 \). Let \( \Delta \) be the standard comultiplication in the universal enveloping algebra.

Below we need two formulas. There proofs can be found in [6].

**Lemma 4.3.** We have

\[
PfF F_I = (p/2)! (q/2)! / (k/2)! \sum_{I' \cup I'' = I} (-1)^{|I'|} PfF F_{I'} PfF F_{I''}
\]

where \((-1)^{|I'|}\) is a sign of a permutation of the set \( I = \{i_1, \ldots, i_k\} \) that places first the subset \( I' \subset I \) and then the subset \( I'' \subset I \). The numbers \( p, q \) are even fixed numbers, satisfying \( p + q = k = |I| \).

**Lemma 4.4.** We have

\[
\Delta PfF F_I = \sum_{I' \cup I'' = I} (-1)^{|I'|} PfF F_{I'} \otimes PfF F_{I''}
\]

where \((-1)^{|I'|}\) is a sign of a permutation of the set \( I = \{i_1, \ldots, i_k\} \) that places first the subset \( I' \subset I \) and then places the subset \( I'' \subset I \).

4.3. The action of Pfaffians on tensors

Let us describe the action of Pfaffians in the tensor representations.

**Proposition 4.5.** On the base vectors \( e_{-2}, e_{-1}, e_0, e_1, e_2 \) of the standard representation of \( \mathfrak{s}_5 \), the Pfaffians \( PfF F_I \), where \(|I| = 4\), act as zero operators.

**Proof.** The proposition is proved by direct calculation using the formulae, where \( a \star b = \frac{1}{2}(ab + ba) \):

\[
PfF F_{-2} = F_{0-1} \star F_{-21} = F_{-1-1} \star F_{-20} + F_{-2-1} \star F_{-10}
\]

\[
PfF F_{-1} = F_{0-2} \star F_{-21} = F_{-1-2} \star F_{-20} + F_{-2-2} \star F_{-10}
\]

\[
PfF F_0 = F_{1-2} \star F_{-21} = F_{-1-2} \star F_{-2-1} + F_{-2-2} \star F_{-1-1}
\]

\[
PfF F_1 = F_{1-2} \star F_{-20} = F_{0-2} \star F_{-2-1} + F_{-2-2} \star F_{0-1}
\]

\[
PfF F_2 = F_{1-2} \star F_{-10} = F_{0-2} \star F_{-1-1} + F_{-1-2} \star F_{0-1}
\]
Now prove an analogue of the previous statement in an arbitrary dimension.

**Proposition 4.6.** On the base vectors $e_{-n}, \ldots, e_n$ of the standard representation of $\mathfrak{so}_N$ the Pfaffians $Pf F_I$ for $|I| > 2$ act as zero operators.

**Proof.** Put $q = 4$, $p = k - 4$ in Lemma (4.3). One has

$$Pf F_I e_j = \sum_{I' \cup I'' = I, |I'| = k - 4, |I''| = 4} (p/2)! (q/2)! / (k/2)! (-1)^{|I'|} Pf F_{I'} Pf F_{I''} e_j$$

If $j \notin I''$, then obviously $Pf F_{I''} e_j = 0$. If $j \in I''$, then using Proposition (4.5) one also obtains $Pf F_{I''} e_j = 0$.

Let us find an action of a Pfaffian of the order $k$ on a tensor product of $< \frac{k}{2}$ vectors, that is on a tensor product $e_{r_2} \otimes e_{r_4} \otimes \ldots \otimes e_{r_t}$, where $t < k$.

**Proposition 4.7.** We have

$$Pf F_I e_{r_2} \otimes e_{r_4} \otimes \ldots \otimes e_{r_t} = 0, \quad t < k$$

**Proof.** The following formulae takes place (see Lemma (4.4)):

$$\Delta Pf F_I = \sum_{I' \cup I'' = I} (-1)^{|I'|} Pf F_{I'} \otimes Pf F_{I''}$$

By definition one has

$$Pf F_I e_{r_2} \otimes e_{r_4} \otimes \ldots \otimes e_{r_k} = (\Delta^k Pf F_I) e_{r_2} \otimes e_{r_4} \otimes \ldots \otimes e_{r_k}$$

Since $t < k$, the comultiplication $\Delta^k Pf F_I$ contains only summands in which on some place the Pfaffian stands whose indexing set $I$ satisfies $|I| \geq 4$ (Lemma (4.4)). From Proposition (4.5) it follows that every such a summand acts as a zero operator.

Find an action of a Pfaffian of the order $k$ on a tensor product of $k/2$ vector, that is on the tensor product $e_{r_2} \otimes e_{r_4} \otimes \ldots \otimes e_{r_k}$.

**Proposition 4.8.** If $\{r_2, r_4, \ldots, r_k\}$ is not contained in $I$, then

$$Pf F_I e_{r_2} \otimes e_{r_4} \otimes \ldots \otimes e_{r_k} = 0$$

Otherwise, take a permutation $\gamma$ of $I$, such that

$$(\gamma(i_1), \gamma(i_2), \ldots, \gamma(i_k)) = (r_1, r_2, r_3, \ldots, r_{k-1}, r_k)$$

Then

$$Pf F_I e_{r_2} \otimes \ldots \otimes e_{r_k} = (-1)^{\gamma} (-1)^{k(k-1)/2} \times$$

$$\times \sum_{\delta \in Aut(r_1, r_3, \ldots, r_{k-1})} (-1)^{\delta} e_{-\delta(r_1)} \otimes e_{-\delta(r_3)} \otimes \ldots \otimes e_{-\delta(r_{k-1})} \otimes e_{-\delta(r_k)}$$
Proof. By definition one has

$$P_f F_1 e_{r_2} \otimes e_{r_4} \otimes \ldots \otimes e_{r_k} = (\Delta^k P_f F_1) e_{r_2} \otimes e_{r_4} \otimes \ldots \otimes e_{r_k}$$

Applying many times the formulae for comultiplication one obtains

$$\Delta^k P_f F_1 = \sum_{I^1 \cup \ldots \cup I^k} (-1)^{(I^1 \ldots I^k)} P_f F_{I^1} \otimes \ldots \otimes P_f F_{I^k}$$

Using Proposition (4.7) one gets that, only the summands for which $|I^j| = 2, j = 1, \ldots, k$ are nonzero operators. Hence, the summation over divisions can be written in the following way:

$$P_f F_1 e_{r_2} \otimes e_{r_4} \otimes \ldots \otimes e_{r_k} = 1/2^{k/2} \sum_{\sigma \in S_k} (-1)^{\sigma} F_{-\sigma(i_1)} e_{r_2} \otimes \ldots \otimes F_{-\sigma(i_k)} e_{r_k}$$

$$= 1/2^{k/2} \sum_{\sigma \in S_k} (-1)^{\sigma} F_{-\sigma(i_1)} e_{r_2} \otimes \ldots \otimes F_{-\sigma(i_k)} e_{r_k}$$

Consider the expression $F_{-\sigma(i_1)} e_{r_2}$. This is $e_{-\sigma(i_1)}$ if $\sigma(i_2) = r_2$, this is $-e_{-\sigma(i_2)}$ if $\sigma(i_1) = r_2$ and zero otherwise. Thus the summand is nonzero only if the permutation $\sigma$ satisfies the following condition. In each pair $(\sigma(i_{2t-1}), \sigma(i_{2t}))$ either $\sigma(i_{2t-1}) = r_{2t}$ or $\sigma(i_{2t}) = r_{2t}$. Show that one can consider only the permutations $\sigma$ such that $\sigma(i_{2t}) = r_{2t}$, that is the permutations of type $(\sigma(i_1), r_2, \sigma(i_2), r_3, \ldots, \sigma(i_{k-1}), r_k)$. But when only summands corresponding to such permutations are considered one must multiply the resulting sum on $2^{k/2}$. It is enough to prove that the permutations

$$\sigma = (\sigma(i_1), \sigma(i_2) = r_2, \sigma(i_3), \ldots, \sigma(r_k)), \quad \sigma' = (\sigma(i_2) = r_2, \sigma(i_1), \sigma(i_3), \ldots, \sigma(r_k))$$

give the same input. Recall that the input for $\sigma$ is

$$(-1)^{\sigma} F_{-\sigma(i_1)} e_{r_2} \otimes \ldots \otimes F_{-\sigma(i_k)} e_{r_k}$$

One has from one hand that

$$F_{-\sigma(i_1)} e_{r_2} = e_{-\sigma(i_1)}$$

and from the other hand that

$$F_{-\sigma'(i_1)} e_{r_2} = -e_{-\sigma'(i_2)} = -e_{-\sigma(i_1)}$$

Also one has $(-1)^{\sigma} = (-1)^{\sigma'}$. Thus the inputs corresponding to $\sigma$ and $\sigma'$ are the same. Hence one can consider the only the permutations $\sigma$ of type $(\sigma(i_1), r_2, \sigma(i_2), r_3, \ldots, \sigma(i_{k-1}), r_k)$ but multiplying the resulting sum on $2^{k/2}$.

For the permutation $\sigma$ of type $(\sigma(i_1), r_2, \sigma(i_2), r_3, \ldots, \sigma(i_{k-1}), r_k)$ using the definition of $\gamma$ one gets

$$(-1)^{\sigma} F_{-\sigma(i_1)} e_{r_2} \otimes \ldots \otimes F_{-\sigma(i_k)} e_{r_k} = (-1)^{\sigma(i_1) r_2 \ldots \sigma(i_{k-1}) r_k} e_{-\sigma(i_1)} \otimes e_{-\sigma(i_2)} \otimes \ldots \otimes e_{-\sigma(i_k)}$$

$$= (-1)^{k(k-1)/2} (-1)^\delta (-1)^\gamma = (-1)^{\sigma}$$

where $\delta$ is a permutation of the set $\{r_1, r_3, \ldots, r_{k-3}, r_k\}$ and the equality

$$(-1)^{k(k-1)/2} (-1)^\delta (-1)^\gamma = (-1)^{\sigma}$$
was used. Performing summation over all permutations $\delta$, one gets
\[
Pf F_1 e_{r_2} \otimes e_{r_4} \otimes \ldots \otimes e_{r_k} = (-1)^{k(k-1)/2}(-1)^{\gamma} \sum_{\delta \in Aut(r_1, \ldots, r_{k-1})} (-1)^{\delta} e_{-\delta(r_1)} \otimes \ldots \otimes e_{-\delta(r_{k-1})}
\]
Finally, from the formula
\[
Pf F_1 e_{r_2} \otimes e_{r_4} \otimes \ldots \otimes e_{r_t} = (\Delta^t Pf F_1) e_{r_2} \otimes e_{r_4} \otimes \ldots \otimes e_{r_t}
\]
as in the proof of Proposition (4.8), one gets the formulae of the action on an arbitrary tensor $e_{r_2} \otimes \ldots \otimes e_{r_t}$.

\textbf{Proposition 4.9.} We have
\[
Pf F_1 e_{r_2} \otimes e_{r_4} \otimes \ldots \otimes e_{r_t} = \sum_{\{j_2, j_4, \ldots, j_k\} \subset \{2, 4, \ldots, t\}} Pf^{j_2, j_4, \ldots, j_k} F_1 e_{r_2} \otimes e_{r_4} \otimes \ldots \otimes e_{r_t}
\]
where $Pf^{j_2, j_4, \ldots, j_k} F_1$ acts on the tensor multiples with numbers $j_2, j_4, \ldots, j_k$. It’s action is described by Proposition (4.8)

4.4. Pfaffians and representation of the algebra $\mathcal{T}$

Let us give a relation between the representation of the Lie algebra $\mathcal{T}$ defined by the formula (3.1) through the action of noncommutative Pfaffians.

As a corollary of Proposition (4.9) we get the following proposition

\textbf{Proposition 4.10.} In the case $\mathfrak{o}_{2n+1}$, if
\[
I \subset J, \quad |J| = 2n, \quad v = e_{i_1} \otimes \ldots \otimes e_{i_n} \quad \text{then} \quad Pf F_I Pf F_J v = CPf F_{I''} v
\]
where the constant $C$ depends only on $n$.

Now let $N = 2n + 1$ and $|I| = 2n$. Then a subset $I$ which consists of $2n$ elements is of type $\{1, \ldots, N\} \setminus i$. Put $Pf F_I := Pf F_{i}$. Then $Pf F_{Pf F_{J''}} = 0$ for $I'' > 2$ and
\[
[Pf F_i, Pf F_{j}] = 1/n \sum_{k \neq i, j} (-1)^{(I''')} Pf F_{ikj} Pf F_{kj}
\]
where the sign $(-1)^{(I''')}$ is defined as follows. For an index $s$ denote as $s$ either $s$ for $s < i$ or $s - 1$ for $s > i$. Then $(-1)^{(I''')} = (-1)^{n-\overline{j}+(n-1)-\overline{k}} = (-1)^{\overline{j}+\overline{k}-1}$ in the case $j < k$ and $(-1)^{\overline{j}+\overline{k}}$ in the case $j > k$. By denoting this sign as $s_{jk}$, one has $s_{jk} = -s_{kj}$. Using the theorem (4.10) one obtains that for a vector $v$ from a representation of $\mathfrak{o}_{2n+1}$ with the highest weight $(1, \ldots, 1)$ the following holds.

\textbf{Lemma 4.11.} One has
\[
[Pf F_i, Pf F_j]v = C \sum_{k \neq i, j} s_{jk} Pf F_{k}v
\]
where $C$ is some constant.

The following theorem is proved.

\textbf{Theorem 4.12.} The representation of the algebra $\mathcal{T}$ on the space of skew-symmetric tensors with $n$ indices, given by the formula (3.1) is given also by the formula
\[
e_{a_1} \wedge \ldots \wedge e_{b_n} \mapsto 1/\sqrt{C} \ Pf F_{\{a_1, \ldots, b_n\}}
\]
where the constant is taken from Lemma (4.11) and the Pfaffian is considered as an operator acting on the space of skew-symmetric tensors with $n$ indices.
5. Knizhnik-Zamolodchikov equations and Capelli elements

Now let us give an interpretation of the Knizhnik-Zamolodchikov type equations associated with the root system $B$ constructed for the algebra $\mathcal{T}$ and its representation (3.1) as a Knizhnik-Zamolodchikov type equations constructed for higher order Capelli central elements. This fact is an immediate corollary of Theorem (4.12).

Introduce the elements

$$\tau_{ij} := \sum_{I, |I| = 2n} 1 \otimes \cdots \otimes \rho(Pf F_I) \otimes \cdots \otimes \rho(Pf F_{-I}) \otimes \cdots \otimes 1 \quad (5.1)$$

where the Pfaffians occur on places $i, j$, and $\rho$ is a representation of $\mathfrak{o}_{2n+1}$ on the space of skew-symmetric tensors with $n$ indices,

$$\mu_{ij} := \sum_{s} 1 \otimes \cdots \otimes \rho(Pf F_I) \otimes \cdots \otimes \rho(Pf F_I) \otimes \cdots \otimes 1 \quad (5.2a)$$

$$\nu_i := 1/2 \sum_{s} 1 \otimes \cdots \otimes \rho(Pf F_I P f F_{-I} + P f F_I P f F_I) \otimes \cdots \otimes 1 \quad (5.2b)$$

**Theorem 5.1.** The action of elements (1.1)-(2.1) and (5.1)-(5.2b) on the space of skew-symmetric tensors with $n$ indices coincide.

As a corollary we get

**Theorem 5.2.** The elements (5.1)-(5.2b) satisfy the commutation relation for the coefficients of the Knizhnik-Zamolodchikov type equations associated with the root system $B$.

References

[1] Knizhnik V G and Zamolodchikov A B 1984 Nucl. Phys. B 247 83
[2] Matsuo A 1992 Invent. Math. 110 95
[3] Cherednik I 2005 *Double Affine Hecke algebras* (Cambridge Univ. Press)
[4] Leibman A 1994 Comm. Math. Phys. 164 293
[5] Enriquez B 2008 Selecta Math. 13 391
[6] Artamonov D V and Golubeva V A 2012 Mat. Sb. 203 5
[7] Molev A 2007 *Yangians and Classical Lie algebras* Math. Surveys Monog. 143 (Providence, RI: AMS)