GEOMETRIC APPROACH TOWARDS COMPLETE LOGARITHMIC SOBOLEV INEQUALITIES

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Abstract. In this paper, we use the Carnot-Caratheodory distance from sub-Riemanian geometry to prove entropy decay estimates for all finite dimensional symmetric quantum Markov semigroups. This estimate is independent of the environment size and hence stable under tensorization. Our approach relies on the transference principle, the existence of \(t\)-designs, and the sub-Riemannian diameter of compact Lie groups and implies estimates for the spectral gap.

1. Introduction

Logarithmic Sobolev inequalities is a versatile tool in analysis and probability. It was first introduced by Gross \[Gro75b, Gro75a\], and later found rich connections to geometry, graph theory, optimal transport as well as information theory. (See e.g. \[BÉ85, OV00, BGL13\] and the overview \[Led04\] by Ledoux and by Gross \[Gro14\]). The natural framework of logarithmic Sobolev inequalities is given by Markov semigroups, i.e. a semigroups of measure preserving maps on a measure space. Barky-Emery theory \[BÉ85\], however, indicates the importance of geometric data in obtaining good estimates. In recent years, logarithmic Sobolev inequalities for quantum Markov semigroups have attracted a lot of attentions: see e.g. \[Bar17, DR20, RD19, KT13a, CM17, CM20\] for the connections to other functional and geometric inequalities; \[DR20, DB14 \] for application in quantum information theory; \[Wir18, WZ20, BGJ20b\] for infinite dimensional examples; \[CRF20, BCL+19\] for quantum Gibbs sampler on lattice spin systems. Quantum Markov semigroups model the Markovian evolution of open quantum systems, which inevitably interact with the surrounding environment. The motivation of this work is to study the entropy form of log-Sobolev inequalities, so-called modified log-Sobolev inequality, for finite dimensional quantum systems and their tensorization property.

A quantum Markov semigroup on finite dimensional quantum system is described by a Lindblad generator. Let \(M_n\) be the \(n \times n\) matrix algebra and \(\text{tr}\) be the standard matrix trace. We consider a (symmetric) Lindlabd generator (also called Lindbladian) on \(M_n\)

\[
L(x) = \sum_{j=1}^{k} a_j^2 x + x a_j^2 - 2a_j x a_j
\]

where \(a_j \in M_n\) are self-adjoint operator. It was proved by Gorini, Kossakowski and Sudarshan \[GKS76a\] and Lindblad \[Lin76\] that \(L\) generates a semigroup \(T_t = e^{-tL}\) of complete positive trace preserving maps, and conversely all such generators symmetric to the trace inner product has the form \(L\). The fixed point algebra \(N := \{x | T_t(x) = x, \forall t \geq 0\}\) is the commutant \(N = \{a_j | 1 \leq j \leq k\}\) as a subalgebra. Let \(E_N : M_n \to N\) be the conditional expectation onto \(N\), which is the projection onto the fixed point space. We say the semigroup \(T_t\) or its generator \(L\) satisfies \(\lambda\)-modified logarithmic Sobolev inequalities (\(\lambda\)-MLSI) for \(\lambda > 0\) if for all positive operators \(\rho\),

\[
\lambda \text{ tr}(\rho \log \rho - \rho \log E_N(\rho)) \leq \text{tr}(L \rho \log \rho).
\]

This inequality characterizes a strong convergence property in terms of entropy that

\[
D(T_t \rho || E_N(\rho)) \leq e^{-\lambda t} D(\rho || E_N(\rho)),
\]

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where \( D(\rho|\sigma) = \text{tr}(\rho \log \rho - \rho \log \sigma) \) is the quantum relative entropy. In contrast to classical Markov semigroups, it is crucial to allow for environment system due to potential quantum entanglement. This leads us to consider the amplified semigroup \( T_t \otimes \text{id}_{\mathcal{M}_n} \) over a noiseless (finite dimensional) auxiliary systems \( \mathcal{M}_n \), which goes beyond the ergodic case. We say the semigroup \( T_t \) satisfies \( \lambda \)-complete logarithmic Sobolev inequalities (\( \lambda \)-CLSI) if for all \( m \geq 1 \), \( T_t \otimes \text{id}_{\mathcal{M}_n} \) satisfies \( \lambda \)-MLSI. The CLSI was first introduced in \([GJL20] \) and later studied in \([BGJ20a, BGJ20b, WZ20]\). We write CLSI(\( L \)) for the optimal (largest) constant \( \lambda \) such that \( T_t = e^{-Lt} \) satisfies \( \lambda \)-CLSI. The CLSI constant governs the convergence rate independently of the size of the environment system, and more importantly, satisfies the tensorization property \( \text{CLSI}(L_1 \otimes \text{id} + \text{id} \otimes L_2) = \min\{\text{CLSI}(L_1), \text{CLSI}(L_2)\} \). The tensorization property was used in \([CRF20]\) as a key condition to obtain size independent MSLI for quantum lattice systems. Therefore, it is desired to know whether all finite dimensional quantum Markov semigroup \( T_t = e^{-Lt} \) admits CLSI(\( L \)) > 0.

It turns out that the above questions is closely related to matrix valued version of logarithmic Sobolev inequalities for classical Matrix semigroup. Indeed, let \( G \) be a compact Lie group and its Lie algebra \( \mathfrak{g} \). Given a generating family \( X = \{X_1, \cdots, X_s\} \) of the Lie algebra \( \mathfrak{g} \) via Lie bracket, \( X \) gives a hypoelliptic sub-Laplacian

\[
\Delta_X = -\sum_{j=1}^s X_j X_j
\]

Given a unitary representation \( u : G \to \mathbb{M}_n \), one can transfer the sub-Laplacian \( \Delta_X \) to a Lindblad generator \( L_X(x) = \sum_j -[a_j, [a_j, x]] \) where \( a_j \) are self-adjoint elements such that \( \pi(\exp(t a_j)) = e^{t a_j} \). Then \( L_X \) generates a quantum Markov semigroup \( T_t = e^{-tL_X} : M_n \to M_n \), and the conditional expectation onto the fixed point subalgebra \( N \) is given by

\[
E_N(x) = \int_G \pi(g)^* x \pi(g) d\mu(g).
\]

Here \( d\mu \) is the Haar measure on \( G \). \( L_X \) is called a transferred Lindbladian of \( \Delta_X \) via the representation \( u \). Conversely, it was observed in \([GJL20]\) that every finite dimensional self-adjoint Lindbladian can be realized as a transferred Lindbladian from a connected compact Lie group. (We refer to Section 3 for more information on the transference principle.)

Thanks to the above transference principle, it suffices to study sub-Laplacians on compact Lie groups for matrix-valued functions. Nevertheless, many classical tools assuming ergodicity do not apply in this setting. One the technical difficulty is the fact that the generator \( L_X \) is governed by a sub-Laplacian operator \( \Delta_X \). The impressive body of work by Baudoin, Thalmaier, and Grong \([Bau14, GT19, BGKT19]\) indicates that a naive curvature identity

\[
\nabla_H \Delta_X = \Delta_X \nabla_H + R(\nabla_H)
\]

for some first order tensor \( R \) and generator \( \Delta_X \) may fail. In fact this does not appear to hold for the basic example \( G = SU(2) \) and \( L = -X^2 - Y^2 \) given by two out of three directions. This means that entropy decay estimates from quantum information theory have to go beyond the standard Bakry-Émry theory and circumvent the use of the famous Rothaus lemma, both are standard tools in the ergodic case. We refer to \([KT13a, LOZ10]\) for the Rothaus lemma in the ergodic quantum case which no longer applies with additional environment.

The main theorem of this work is a lower bound of the CLSI constant of a so-called transferred quantum Markov semigroup \( T_t = e^{-L_X t} \) via the sub-Riemannian structure of \( X = \{X_1, \cdots, X_s\} \) on \( G \).

**Theorem 1.1.** Let \( G \) be a connected compact Lie group and \( \mathfrak{g} \) be its Lie algebra. Let \( X = \{X_1, \cdots, X_s\} \) be a family of left invariant vector field generating \( \mathfrak{g} \). Suppose \( \pi : G \to \mathbb{M}_n \) is
unitary representation such that
\begin{equation}
E_N(x) := \sum_{j=1}^{m} \alpha_j \pi(g_j)^* x \pi(g_j)
\end{equation}
for a finite probability distribution $\sum_{j=1}^{m} \alpha_j = 1, \alpha_j \geq 0$. Then the CLSI constant of the transferred Lindbladian $L_X(x) = - \sum_{j=1}^{s} [a_j, [a_j, x]]$ satisfies
\[ \text{CLSI}(L_X) \geq \frac{C}{smd_X (d_X + 1)^2} , \]
where $C$ is an universal constant and $d_X$ is the diameter of $G$ in the Carnot-Carathéodory distance induced by $X$.

Here the Carnot-Carathéodory distance, also called sub-Riemannian distance, is defined as
\begin{equation}
 d_H(p, q) = \inf_{\gamma(0)=p, \gamma(1)=q} \int_0^1 \|\gamma'(t)\|_H dt
\end{equation}
where the infimum is taken over all piecewise smooth curves whose derivatives $\gamma'(t)$ are a.e. in the horizontal direction $H = \text{span}\{X_k(\gamma(t))\}$. This distance defines the same topology and hence $G$ admits finite diameter $d_X$ with respect to this new metric. The equation (1.5) is an analog of spherical design for $G = SO(n)$ and of unitary design for $G = U(n)$, which are of interest from combinatorics and quantum computing. Thanks to Carathéodory theorem (see [Wat18]), we know that the design (1.5) always exists with $m \leq n^2 + 4n + 2$. Therefore, Theorem 1.1 shows that every quantum Markov semigroup transferred from a sub-Laplacian on a compact Lie group satisfies CLSI. As a corollary, we obtain a positive solution to the existence of CLSI constants in finite dimensions.

**Corollary 1.2.** Every self-adjoint Lindbladian $L$ on a finite dimensional matrix algebra satisfies $\text{CLSI}(L) > 0$.

The above results can be extended to Lindbladians $L$ satisfying GNS-symmetry of states via the noncommutative change of measure in [JLR19]. Very recently this result has been independently obtained in [GR21] using very different techniques. These two results are complementary: while the proof presented here requires knowledge of the Carnot-Carathodory diameter of $G$ and the size of a design for the conditional expectation and implies spectral gap, the proof by Gao and Rouzé on the other hand relies on the spectral gap and the Popa-Pimnser index [PPS6] of the inclusion $N \subset M_m$. The lower bound in Theorem 1.1 does not depend much on the dimension of the representation $\pi$, and holds uniformly for sub-representations of a given tensor product representations $\pi_0^\otimes k \otimes \bar{\pi}_0^\otimes k$. In contrast, the Popa-Pimnser index for a direct sum of irreducible representations can become very large.

The rest of paper is organized as follows. Section 2 discusses the complete logarithmic Sobolev constant on the weighted interval. In Section 3, we use the interval result to prove Theorem 1.1.

2. Complete Logarithmic Sobolev Inequalities on the Interval

In this section we discuss the complete logarithmic Sobolev inequalities (CLSI) for the weighted interval. Let $[0, 1]$ be the unit interval and $\mu$ be a probability measure on $[0, 1]$. We write $L^\infty([0, 1], \mu)$ (resp. $C([0, 1])$ and $C^\infty([0, 1])$) as the space of $L^\infty$ (resp. continuous and smooth) functions. Denote $\delta = \frac{d}{dx}$ as the derivative operator. We shall first consider $\delta$ is a closable derivation on smooth functions $f$ with periodic boundary conditions $f(0) = f(1)$. In this case, the underlying space is equivalent to unit circle $\mathbb{T}$. We write $\delta^*$ as the adjoint operator on $L_2([0, 1], \mu)$ and $\Delta_\mu = \delta^* \delta$ as the weighted Laplacian operator. A matrix valued function $f \in C([0, 1], M_n)$ is positive if for every $t \in [0, 1]$, $f(t) \geq 0$ is a positive (semi-definite) matrix. We are interested in proving
the following matrix-valued modified logarithmic Sobolev inequalities that for all smooth periodic positive \( f \in C^\infty([0,1], M_n) \),

\[
\lambda \int_0^1 \text{tr}(f(x) \log f(x) - f(x) \log E_\mu f) d\mu(x) \leq \int_0^1 \text{tr}((\Delta f)(x) \log f(x)) d\mu(x).
\]

where \( E_\mu f = \int_0^1 f d\mu \) is the weighted mean. The left hand side above is the relative entropy \( D(f||E_\mu f) \) for the matrix-valued \( f \) with respect to its mean \( E_\mu f \), and the right hand side is the Fisher information \( I_{\Delta f}(f) \) (also called entropy production). We denote CLSI\(([0,1], \mu)\) (resp. MLSI\(([0,1], \mu)\)) for the optimal (largest) constant \( \lambda \) such that (2.1) is satisfied for \( n \geq 1 \) and periodic positive \( f \in C^\infty([0,1], M_n) \) (resp. for all periodic positive scalar valued function \( f \in C^\infty([0,1]) \)). We also denote CLSI\(((0,1), \mu)\) (resp. MLSI\(((0,1), \mu)\)) as the CLSI (resp. MLSI) constant for functions \( f \) without periodic boundary conditions \( f(0) = f(1) \).

We emphasize that it is the constant CLSI\(([0,1], \mu)\) (or MLSI\(([0,1], \mu)\)) that gives the exponential decay rate of relative entropy as in (1.3). On the other hand, the constants CLSI\(((0,1), \mu)\) and MLSI\(((0,1), \mu)\) are not associated with a semigroup because the derivation \( \delta = i \frac{\partial}{\partial x} \) are not closable without periodic boundary conditions. Nevertheless, the open interval constant CLSI\(((0,1), \mu)\) apply to more general functions and are more flexible to use with semigroups. It follows from the standard symmetrization and periodization argument in [BGL13] Proposition 4.5.5 & 5.7.5 that the CLSI constants of CLSI\(([0,1], \mu)\) and CLSI\(((0,1), \mu)\) are related by a factor \( \frac{1}{4} \),

\[
\frac{1}{4} \text{CLSI}((0,1), \mu) \leq \text{CLSI}([0,1], \mu) \leq \text{CLSI}((0,1), \mu).
\]

It is clear that \( \text{CLSI}([0,1], \mu) \leq \text{MLSI}([0,1], \mu) \) and \( \text{CLSI}((0,1), \mu) \leq \text{MLSI}((0,1), \mu) \) but the other direction estimate is still unknown. The constant MLSI\(((0,1), \mu)\) and MLSI\(((0,1), \mu)\) for scalar-valued functions are discussed in [BGL13] Proposition 5.7.5.

**Proposition 2.1.** Let \( n \) be a positive integer and \( d\mu(x) = \frac{1}{n} x^{n-1} dx \) be a probability on \([0,1]\). Then \( \text{CLSI}((0,1), \mu) \geq \frac{1}{4} \text{CLSI}([0,1], \mu) \geq (2e^{1/2})^{-1} \) for all \( n \geq 1 \).

**Proof.** Denote \( dx \) as the Lebesgue measure. Consider the probability measure \( d\nu^n(x) = \frac{1}{a_n} x^{n-1} e^{-\frac{x^2}{2}} dx \) on \([0,1]\) with \( a_n = \int_0^1 x^{n-1} e^{-\frac{x^2}{2}} dx \). Since for periodic boundary functions, the underlying space is circle which has zero Ricci curvature. Then the Bakry-Émery’s weighted Ricci tensor is

\[
\text{Ric}(d\nu^n) = \text{Hess}(\frac{x^2}{2} - (n-1) \ln(x)) = 1 + \frac{n-1}{x^2} \geq 1
\]

This implies \( \text{CLSI}((0,1), \nu^n) \geq 2 \) for functions with periodic condition \( f(0) = f(1) \). By comparing the two measures \( \frac{n}{a_n e^{\frac{x^2}{2}}} \leq \frac{d\nu^n(x)}{d\mu(x)} \leq \frac{n}{a_n} \) and the change of measure in [LLL20] Theorem 2.14, we have \( \text{CLSI}([0,1], \frac{1}{n} x^{n-1} dx) \geq 2e^{-1/2} \).

More generally, we have the following criterion.

**Corollary 2.2.** Let \( d\mu(x) = f(x) dx \) be a probability measure \([0,1]\) with second differentiable density function \( f \). If there exists \( k > 0 \) and \( a > 0 \) such that

\[
k f^2(x) - f''(x)f(x) - (f'(x))^2 \geq a > 0, \quad \forall x \in (0,1).
\]

Then \( \text{CLSI}((0,1), \mu) \geq \frac{1}{4} \text{CLSI}([0,1], \mu) \geq (2e^k)^{-1} \).

**Proof.** Let us consider the probability measure \( d\gamma = \frac{1}{a} f(x) e^{-kx^2} \) with \( a = \int_0^1 f(x) e^{-kx^2} dx \). The weighted Ricci tensor is

\[
\text{Ric}(d\gamma) = \text{Hess}(kx^2 - \ln(f(x))) = \frac{2k f(x)^2 - f''(x)f(x) - (f'(x))^2}{f(x)^2} \geq a > 0.
\]
The two measures \(d\mu\) and \(d\nu\) are comparable
\[
\frac{ce^{-k}}{a} \leq \frac{d\nu}{d\mu} \leq \frac{c}{a}.
\]
By the change of measure again, we have \(\text{CLSI}([0,1],d\mu) \geq 2e^{-k}\).

The next estimate, despite of giving worse constants, applies to open interval constant and just depends on the growth order.

**Proposition 2.3.** Let \(d\mu(x) = \frac{1}{a}h(x)dx\) be a probability measure on \([0,1]\) and \(a = \int_0^1 h(x)dx\). Suppose 
\[
c_1x^\alpha \leq h(x) \leq c_2 \beta x^{\beta-1}
\]
for some \(c_1, c_2 > 0\) and \(0 < \alpha < \beta \) with \(\beta \geq 1\). Then \(\text{CLSI}((0,1),d\mu) > 0\).

**Proof.** Let \(\Phi(x) = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-t^2/2}dt\) be the error function normalized. Let \(g : [0,\infty) \to [0,1]\) be a decreasing function such that 
\[
H(g(x)) = \Phi(x) \quad \text{and} \quad H(y) = \int_0^y h(x)dx.
\]
Thus \(H'(g(x))g'(x) = \sqrt{\frac{2}{\pi}} e^{-x^2/2}\) and
\[
(2.3) \quad g'(x) = \sqrt{\frac{2}{\pi}} \frac{e^{-x^2/2}}{h(H^{-1}(\Phi(x)))}.
\]
Write \(\mathbb{E}f = (\int f \, dx)1\) as the expectation to the uniform measure. Using the fact the Gaussian measure has Ricci curvature \(1\) and Lemma \(2.2\) for positive matrix-valued function \(\rho(t) = f(g(t))\) we have
\[
2D(\rho||\mathbb{E}(\rho)) \leq \sqrt{\frac{2}{\pi}} \int_0^\infty \tau(\rho'(t),J^\log_\rho(t)\rho'(t))e^{-t^2/2}dt \leq \int_0^1 \tau(f'(x),J^\log f'(x))|g'(g^{-1}(x))|^2h(x)dx.
\]
where \(J^\log_\rho(X) = \int_0^\infty \frac{1}{\sigma + r} X^{1/(\sigma + r)} dr\) is the double operator integral for log function. By the change of variable we have \(D(\rho||\mathbb{E}(\rho)) = D(f||\mathbb{E}(f))\), then it suffices to find an upper bound for \(g'(x)\). Now, our assumption \(h(x) \leq c_2 \beta x^{\beta-1}\) implies \(H(x) \leq c_2x^\beta\) and hence by (2.3)
\[
(2.4) \quad h(H^{-1}(y)) \geq c(H^{-1}(y))^{\alpha/\beta} \geq c_1 \frac{y}{c_2}^{\alpha/\beta}.
\]
Now, we use the inequality \(\Phi(x) \geq \sqrt{\frac{2}{\pi}} \frac{e^{-x^2/2}}{x(1+x^2)}\) and \(\Phi(t) \geq \Phi(1)\) for \(t \leq 1\). Together with (2.4), we obtain
\[
g'(x) \leq c^{-1}c_2^{\alpha/\beta} \left(\frac{2}{\pi}\right)^{\frac{1}{2}(1-\frac{\alpha}{\beta})} \left(x(1 + \frac{1}{x^2})\right)^{\frac{\alpha}{\beta}} e^{-\frac{x^2}{2}(1-\frac{\alpha}{\beta})}.
\]
Note that for small \(t\) we may replace \(t(1+1/t^2)\) with a constant. Thus for \(0 \leq \alpha < \beta\), this term is bounded, and hence its square is also bounded.

For \(h(x) = \frac{1}{n}x^{n-1}\), the CLSI constant from above is of the order \(n^2\). Nevertheless, Lemma \(2.3\) help us understand measures whose density functions \(h\) do not have desired smooth \((C^2)\) properties as in Proposition \(2.1\).
Remark 2.4 (CLSI constant for uniform measure). The above Proposition 2.1 gives a tight constant then [LJL20] Example 4.7 that
\[ \text{CLSI}((0, 1), dx) \geq \frac{1}{4} \text{CLSI}([0, 1], dx) \geq \frac{1}{5}, \]
which was obtained by comparing the uniform measure with a another modified Gaussian distribution. A sharper constant
\[ \text{CLSI}((0, 1), dx) \geq \frac{1}{4} \text{CLSI}([0, 1], dx) \geq \frac{\pi^2}{\ln 3}, \]
was obtained in [BGJ20a] Theorem 4.12] using heat kernel estimate and monotonicity of Fisher information. Note for the scalar case MLSI([0, 1], dx) = 4\pi^2 and MLSI((0, 1), dx) = \pi^2. The method in [BGJ20a] also applies to the weighted measure but the heat kernel estimate for the weighted Laplacian \(\Delta_\mu\) is less explicit.

Remark 2.5 (Extension to piecewise smooth functions). Here we discuss some subtlety about the domain of \(\Delta_\mu\) and of the modified log-Sobolev inequality. On one hand, the semigroup \(T_t = e^{-t\Delta_\mu}\) is defined for all functions \(f \in L_1((0, 1), \mu)\). MLSI((0, 1), \mu) \geq \lambda is equivalent to that for any density function \(\rho \in L_1((0, 1), \mu), \rho \geq 0\) and \(\int \rho d\mu = 1\),
\[ D(T_t\rho\|1) \leq e^{-\lambda t} D(\rho\|1). \]
where \(D(f\|1) = \int \log f d\mu\) is the entropy functional. For smooth \(\rho\), we can take derivative at \(t = 0\) and obtain the modified log-Sobolev inequality
\[ \lambda D(\rho\|1) \leq I_{\Delta_\mu}(\rho) := \int \Delta_\mu \rho \log \rho d\mu. \]
This inequality can be extended to piecewise smooth \(\rho\) where the Fisher information \(I_{\Delta_\mu}(\rho)\) has to be interpreted as Dirichelet form
\[ I_{\Delta_\mu}(\rho) = \lim_{n \to \infty} E((\rho, f_n)\rho) := \lim_{n \to \infty} \int \frac{d\rho}{dx}(\rho) \frac{d\rho}{dx}(f_n(\rho)) d\mu \]
where \(f_n(t) = \max\{\min\{\log(t), n\}, -n\}\) is the truncated logarithmic function (see [Wir18] Definition 5.17)). Suppose \(\rho : [0, 1] \to \mathbb{R}\) is continuous piecewise smooth and its derivative \(\rho'\) is defined and continuous except for finite points in \([0, 1]\). For our purpose, it suffices to consider \(\rho\) is bounded from below (also bounded from above by continuity). Thus \(I(\rho) = E(\rho, f_m(\rho))\) for some finite \(m\). Let \(\epsilon_n\) be an approximation identity of smoothing kernels. Take \(\rho_n = \rho * \epsilon_n\) by convolution, and \(\rho_n' = \rho' * \epsilon_n\) be the derivative of \(\rho_n\). It is readily to see that
\[ \|\rho_n - \rho\|_2 \leq \|\rho_n - \rho\|_\infty \leq \|\rho_n' - \rho\|_1 \leq \|\rho_n' - \rho\|_2 \to 0, \]
which means that both \(\rho_n \to \rho\) and \(\rho_n' \to \rho'\) in \(L_2\). Hence \(\rho \in \text{dom}(\frac{d}{dx}) = \text{dom}(\Delta_\mu/2)\) by closable extension and by the Leibniz rule \(f_m(\rho)\) also in \(\text{dom}(\frac{d}{dx})\). Thus the Fisher information \(I_{\Delta_\mu}(\rho)\) is also well-defined. Note that by data processing inequality \(\rho \mapsto \rho * \epsilon_n\) and lower-semicontinuity of relative entropy,
\[ \limsup_n D(\rho_n\|1) \leq D(\rho\|1) \leq \liminf_n D(\rho_n\|1). \]
For the Fisher information,
\[ I_{\Delta_\mu}(\rho) = E(\rho, f_m(\rho)) = \lim_n E(\rho_n, f_m(\rho_n)) = \lim_n I_{\Delta_\mu}(\rho_n). \]
Here we use [DPWS02] Corollary 7.5 for the continuity \(\|\delta(f_m(\rho_n)) - \delta(f_m(\rho))\|_2 \to 0\). Thus for continuous, piecewise smooth and strictly positive \(\rho\),
\[ \lambda D(\rho\|E(\rho)) \leq I_{\Delta_\mu}(\rho). \]
The same argument works for the matrix-valued functions. For smooth matrix-valued density \( \rho : [0,1] \to \mathbb{M}_n \),

\[
I_{\Delta_\mu}(\rho) = \int_0^1 \text{tr} \left( (\Delta_\mu \otimes \text{id}_n)(x) \log \rho(x) \right) dx = \int_0^1 \text{tr} \left( \rho'(x) J_{\rho(x)}^\text{ln} \rho'(x) \right) dx .
\]

where \( J_\sigma(X) = \int_0^\infty \frac{1}{\sigma + r} X \frac{1}{\sigma + r} dr \) is the double operator integral for \( f(x, y) = \frac{\log x - \log y}{x - y} \). Indeed, this is clear for smooth \( \rho_n = \rho \ast (\epsilon_n 1_{M_n}) \) (entry-wise mollification) and by limit

\[
I_{\Delta_\mu}(\rho) = \lim_n I_{\Delta_\mu}(\rho_n) = \lim_n \int_0^1 \text{tr} \rho'^n(x) J_{\rho_n(x)}^\text{ln} \rho'^n(x) dx = \int_0^1 \text{tr} \rho'(x) J_{\rho(x)}^\text{ln} \rho'(x) dx .
\]

Here we use the fact \( \rho' \to \rho' \) in \( L_2([0,1], M_n) \) and \( \rho_n \to \rho \) in \( \| \cdot \|_\infty \) (because \( \rho \) is continuous). To sum up, our discussion above justifies that modified log-Sobolev inequalities

\[
\lambda D(\rho||E_{\rho\mu}) \leq I_{\Delta_\mu}(\rho) = \int_0^1 \tau(\rho'(x), J_{\rho(x)}^\text{ln} \rho'(x)) dx
\]

extends to piecewise smooth, strictly positive matrix-valued density function.

3. Complete Logarithmic Sobolev Inequalities On Matrix Algebras

In this section we prove that every symmetric quantum Markov semigroup \( T_t = e^{-tL} : \mathbb{M}_n \to \mathbb{M}_n \) on Matrix algebra satisfies complete logarithmic Sobolev inequality. A quantum Markov semigroup \( T_t : \mathbb{M}_n \to \mathbb{M}_n \) is a continuous family of maps satisfying

i) for each \( t \geq 0 \), \( T_t \) is completely positive and unital \( T_t(1) = 1 \).
i) for any \( t, s \geq 0 \), \( T_t \circ T_s = T_{t+s} \) and \( T_0 = \text{id}_{\mathbb{M}_n} \).

where \( \text{id}_{\mathbb{M}_n} \) is the identity map on \( \mathbb{M}_n \). We denote \( L_2(M_n) \) as the Hilbert-Schmidt space equipped with the inner product \( (a,k) = \text{tr}(a^*k) \). We say a semigroup \( T_t \) is symmetric if for each \( t \geq 0 \), \( T_t \) is a self-adjoint map on \( L_2(\mathbb{M}_n) \). Namely, for any \( x, y \in \mathbb{M}_n \),

\[
\text{tr}(T_t(x)^*y) = \text{tr}(x^*T_t(y)) .
\]

The generator of the semigroup (also called Lindbladian) is a operator on \( L_2(\mathbb{M}_n) \) defined as

\[
Lx = \lim_{t \to 0} \frac{1}{t}(x - T_t(x)) , \quad T_t = e^{-tL} ,
\]

where \( L \) is a operator on \( L_2(\mathbb{M}_n) \). In most of our discussion, we restrict ourselves to the symmetric cases. Thanks to [GKS76b, Lin76], the generator of symmetric semigroups is given by

\[
L(x) = -\sum_{k=1}^s [a_k, [a_k, x]] = \sum_{k=1}^s (a_k^2 x + x a_k^2 - 2a_k x a_k) .
\]

where \( a_k \in M_n \) are some self-adjoint operators. Then, \( L \) admits a \(*\)-preserving derivation given by

\[
\delta : \mathbb{M}_n \to \bigoplus_{j=1}^s \mathbb{M}_n , \quad \delta(x) = i[ a_1, x] \oplus i[ a_2, x] \oplus \cdots \oplus i[ a_s, x] .
\]

Recall that \( \delta \) is called a derivation because it satisfies the Leibniz rule \( \delta(xy) = \delta(x)y + x\delta(y) \). In particular, \( L = \delta^*\delta \). The fixed-point algebra is

\[
N := \{ x \in \mathbb{M}_n \mid T_t(x) = x , \forall t \geq 0 \} = \{ x \in \mathbb{M}_n \mid Lx = 0 \} = \{ a_1, \ldots, a_s \}' .
\]

We denote by \( E_N \) be the conditional expectation onto \( N_{\text{fix}} \).

For two states \( \rho \) and \( \sigma \) with \( \text{tr}(\rho) = \text{tr}(\sigma) \), the relative entropy is defined as

\[
D(\rho||\sigma) = \begin{cases} 
\text{tr}(\rho \ln \rho - \rho \ln \sigma), & \text{if supp}(\rho) \subseteq \text{supp}(\sigma) \\
+\infty, & \text{otherwise},
\end{cases}
\]
where \( \text{supp}(\rho) \) (resp. \( \text{supp}(\sigma) \)) is the support projection of \( \rho \) (resp. \( \sigma \)). The Fisher information (also called entropy production) is \( I(\rho) = \text{tr}(L\rho \log \rho) \).

**Definition 3.1.** We say \( T_t \) satisfies \( \lambda \)-modified logarithmic Sobolev inequalities (\( \lambda \)-MLSI) for \( \lambda > 0 \) if for all state \( \rho \)

\[
2\lambda \Delta \rho \|E_N(\rho)\| \leq I(\rho) .
\]

We say \( T_t \) satisfies \( \lambda \)-complete logarithmic Sobolev inequalities (\( \lambda \)-CLSI) for \( \lambda > 0 \) if for all \( m \geq 1 \), \( T_t \otimes \text{id}_{\mathbb{M}_m} \) satisfies \( \lambda \)-MLSI.

**Remark 3.2.** As a matter of simplicity the results in this work is stated for environments given by matrix algebras. As the proof will show the auxiliary matrix algebra \( \mathbb{M}_m \) can be replaced by any finite von Neumann algebra \( \mathcal{M} \) with a specified trace.

It was proved in [GJL20] that a symmetric quantum Markov semigroup on matrix algebra is always a transference of a classical Markov semigroup on a compact Lie group with sub-Laplacian as the generator. Recall that for a Riemannian manifold \( M \), a Hörmander system is a finite family of vector fields \( X = \{X_1, \ldots, X_s\} \) such that for some global constant \( l_X \), the set of iterated commutators (no commutator if \( k = 1 \))

\[
\bigcup_{1 \leq k \leq l_X} \{ [X_{j_1}, [X_{j_2}, \ldots, [X_{j_{k-1}}, X_{j_k}]]] | 1 \leq j_1, \ldots, j_k \leq s \}
\]

spans the tangent space \( T_pM \) at every point \( p \in M \). We denote \( \Delta_X = \sum_{j=1}^s X_j^* X_j \) as the sub-Laplacian where \( X_j^* \) is the adjoint operator of \( X_j \) with respect to \( L_2(M, \mu) \) and \( \mu \) is the volume form of \( M \).

**Lemma 3.3** ([GJL20]). Let \( T_t : \mathbb{M}_n \to \mathbb{M}_n^\prime \) be a symmetric quantum Markov semigroup. There exists a connected compact Lie group \( G \), a unitary representation \( u : G \to \mathbb{M}_n \) and a Hörmander system \( X = \{X_1, \ldots, X_d\} \) of a compact connected Lie group \( G \) such that the following diagram commute

\[
\begin{array}{ccc}
\mathbb{M}_n & \xrightarrow{T_t \otimes \text{id}_{\mathbb{M}_n}} & \mathbb{M}_n^\prime \\
\uparrow \pi & & \uparrow \pi \\
L_\infty(G, \mu; \mathbb{M}_n) & \xrightarrow{S_t \otimes \text{id}_{\mathbb{M}_n}} & L_\infty(G, \mu; \mathbb{M}_n^\prime)
\end{array}
\]

where \( L_\infty(G, \mu; \mathbb{M}_n) \) denotes the matrix-valued function on \( G \) and \( \pi : \mathbb{M}_n \to L_\infty(G, \mu; \mathbb{M}_n) \) denotes the transference map \( \pi(x)(g) = u(g)^* xu(g) \).

We briefly describe the construction, as it will be used in later discussion (See [GJL20] Lemma 4.10, 5.1 for detailed proof). Let \( \{a_1, \ldots, a_r\} \) be the self-adjoint elements in the \( \mathbb{M}_n \). Denote \( u_m = i(\mathbb{M}_n)_{a,m} \) as the Lie algebra of the unitary group \( U(\mathbb{M}_m) \). Then \( X = \{ia_1, \ldots, ia_r\} \) generates a Lie subalgebra \( \mathfrak{g} \) of \( u_m \) which by basically Lie’s second theorem (see also [GJL20] Lemma 4.10)) is the Lie algebra of connected compact Lie group \( G \). Let \( u \) be the unitary representation induced by the Lie algebra embedding \( \mathfrak{g} \subset u_m \). One can show that the \(*\)-homomorphism \( \pi : \mathbb{M}_n \to L_\infty(G, \mu; \mathbb{M}_n) \) satisfies that

\[
X_j \otimes \text{id}_{\mathbb{M}_n}(\pi(x)) = -i\pi([a_j, x]), \Delta_X \otimes \text{id}_{\mathbb{M}_m}(\pi(\rho)) = \pi(L(\rho)),
\]

which yields the intertwining relation of the semigroups \( (3.2) \).

From the above intertwining relation, we can view \( T_t \) as a sub-semigroup for the matrix valued semigroup \( S_t \otimes \text{id}_{\mathbb{M}_n} \). In particular, when \( t \to \infty \), we have the commutation relation for the conditional expectations

\[
(\mathbb{E}_G \otimes \text{id}_{\mathbb{M}_n}) \circ \pi = \pi \circ \mathbb{E}_N,
\]
where $\mathbb{E}_G f = (\int_G f d\mu) 1_G$ is the expectation on $G$ and $\mu$ be the normalized Haar measure over $G$. In particular, the conditional expectation onto the fixed-point subalgebra $N = u(G)'$ is given by

$$E_N(\rho) = \int_G u(g)^* \rho u(g) d\mu(g).$$

By Carathéodory's Theorem (see e.g. Proposition 4.9 in [Wat18]), there exist finitely many elements $\{g_j\}_{j=1}^m \subset G$ such that for every $\rho \in \mathbb{M}_n$,

$$(3.3) \quad E_N(\rho) = \sum_{j=1}^m \alpha_j u(g_j)^* \rho u(g_j),$$

where $\{\alpha_j\}$ is a finite probability distribution s.t. $\sum \alpha_j = 1$ and $\alpha_j \geq 0$. Furthermore, we have $\text{CLSI}(T_t) \geq \text{CLSI}(S_t)$. This transference does not apply to MLSI because $T_t$ is a restriction of the matrix-valued amplification $S_t \otimes \text{id}_n$. We are now ready to prove the main theorem of this paper.

**Theorem 3.4.** Let $T_t = e^{-tL} : \mathbb{M}_n \to \mathbb{M}_n$ be symmetric quantum Markov semigroup. Suppose $L$ is a transferred Lindbladian of a sub-Laplacian $\Delta_X$ on a connected compact Lie group $G$ given by the Hörmander system on $X = \{X_1, \ldots, X_s\}$. Then

$$\text{CLSI}(L_X) \geq \frac{C}{\text{smd}_X(d_X + 1)^2}.$$  

where $C_X$ is some constant depending on $X$ and $d_X$ is the diameter of $G$ in the Carnot-Caratheodory distance induced by $X$.

**Proof.** Recall $d_H$ be the horizontal distance defined in (1.6) and denote $d_X := \sup_{g_1, g_2 \in G} d_H(g_1, g_2)$ as the horizontal diameter. Without loss of generality, we can always assume that $X = \{X_1, \ldots, X_s\}$ forms orthonormal set with respect to the Riemannian metric. Namely, for any point $g \in G$ and $\lambda_k \in \mathbb{R}$,

$$|\sum_k \lambda_k g_{k,g}|^2 = \sum_k |\lambda_k|^2.$$  

Let $\{\alpha_j\}$ be the probability given in (3.3). Define recursively $\beta_1 = \alpha_1 + d_{\text{CC}}(g_1, g_2)$ and for $1 \leq j \leq m-1$, $\beta_{j+1} = \beta_j + \alpha_j + d_H(g_j, g_{j+1})$. ($m+1$ is viewed as 1.) Then

$$\beta_m \leq \sum_{j=1}^m \alpha_j + \sum_{j=1}^m d_H(g_j, g_{j+1}) \leq 1 + m d_X.$$  

We split the interval

$$I_j = [\beta_j, \beta_{j+1}] = [\beta_j, \beta_j + \alpha_j] \cup [\beta_j + \alpha_j, \beta_{j+1}] = I_j(1) \cup I_j(2)$$

into intervals of length $|I_j(1)| = \alpha_j$ and $|I_j(2)| = d_H(g_j, g_{j+1})$. Consider the new transference map $\pi : M_n \to \ell^\infty_\infty(M_n)$ defined by

$$\pi(\rho)(j) = u(g_j)^* \rho u(g_j).$$

Let $\mathbb{E}_\mu(f) = \sum_{j=1}^m \alpha_j f(j)$ be the expectation on $\ell^\infty_\infty$. Then we have

$$E_N(\rho) = \mathbb{E}_\mu(\pi(\rho)),$$

$$D(\rho||E_N(\rho)) = \text{tr}(\rho \log \rho) - \text{tr}(\rho \log \mathbb{E}_\mu(\pi(\rho))) = D(\pi(\rho)||\mathbb{E}_\mu(\pi(\rho))).$$

Let $\gamma_j : [0, d_H(g_j, g_{j+1})] \to G$ be a piecewise smooth horizontal path such that

$$\gamma_j(0) = g_j, \quad \gamma_j(d_H(g_j, g_{j+1})) = g_{j+1}, \quad \gamma_j'(t) \in T_{\gamma_j(t)}H,$$
for a.e. \( t \in (0, d_H(g_j, g_{j+1})) \) and \( |\gamma'(t)| = 1 \) of unit length in the Riemannian metric. Then we define the piecewise smooth matrix-valued function \( \tilde{\rho} : M_m \to L_\infty([0, \beta_m], M_n) \),

\[
\tilde{\rho}(t) = \begin{cases} 
    u(g_j)^* \rho u(g_j), & \text{if } t \in I_j(1) \\
    u(\gamma_j(t - \beta_j - \alpha_j))^* \rho u(\gamma_j(t - \beta_j - \alpha_j)), & \text{if } t \in I_j(2).
\end{cases}
\]

Denote \( \mathbb{E}\rho = (\int \rho dt)1 \) as the mean and take \( \tilde{\sigma} := : \mathbb{E}\tilde{\rho} \). By the chain rule [JLR19, Lemma 3.4] and the non-negativity of the relative entropy

\[
D(\pi(\rho)\|\tilde{\sigma}) = D(\pi(\rho)\|\mathbb{E}\pi(\rho)) + D(\mathbb{E}\pi(\rho)\|\tilde{\sigma}) \geq D(\pi(\rho)\|\mathbb{E}\pi(\rho))
\]

Moreover

\[
D(\pi(\rho)\|\tilde{\sigma}) = \sum_{j=1}^m \alpha_j D(u(g_j)^* \rho u(g_j))\|\tilde{\sigma}) \leq D(\tilde{\rho}\|\tilde{\sigma}).
\]

Let \( C := \text{CLSI}([0, \beta_m], dx) \) be the CLSI constant on the interval \([0, \beta_m]\). We obtain that

\[
D(\tilde{\rho}\|\tilde{\sigma}) = D(\tilde{\rho}\|\mathbb{E}\tilde{\rho}) \leq \frac{1}{C} I(\tilde{\rho}) = \frac{1}{C} \int_0^{\beta_m} \text{tr} \left( \tilde{\rho}'(t) J_{\tilde{\rho}(t)}^{\text{log}} (\tilde{\rho}'(t)) \right) \, dt
\]

Let \( X_t = \gamma_j'(t - \beta_j) \in T_{\gamma_j(t-\beta_j)} H \) and \( g_t = \gamma_j(t - \beta_j) \). Denote \( \phi_u : g \to i(\mathbb{M}_n)_{s.a.} \) be the Lie algebra homomorphism induced by \( u \). For \( t \in I_j(2) \), we have

\[
\tilde{\rho}'(t) = \frac{d}{ds} (u(g_t+s)^* \rho u(g_t+s))|_{s=0} = u(g_t)^* (-\phi_u(X_t)\rho + \rho\phi_u(X_t))u(g_t)
\]

\[
= u(g_t)^* (-iY_t\rho + i\rho Y_t)u(g_t)
\]

\[
= -u(g_t)^* (i[Y_t, \rho])u(g_t).
\]

where \( Y_t = i\phi_u(X_t) \) are self-adjoint operators. Then we observe that

\[
I(\tilde{\rho}) = \int_{I_j(2)} \text{tr} \left( \tilde{\rho}'(t) J_{\tilde{\rho}(t)}^{\text{log}} (\tilde{\rho}'(t)) \right) \, dt
\]

and for each \( j \) and \( t \in I_j(2) \),

\[
\text{tr} \left( \tilde{\rho}'(t) J_{\tilde{\rho}(t)}^{\text{log}} \tilde{\rho}'(t) \right) = \int_0^\infty \text{tr} \left( \gamma_j(t - \beta_j)^* i[Y_t, \rho] \gamma_j(t - \beta_j) (\gamma_j(t - \beta_j)^* \rho \gamma_j(t - \beta_j) + r)^{-1}
\]

\[
\gamma_j(t - \beta_j)^* i[Y_t, \rho] \gamma_j(t - \beta_j) (\gamma_j(t - \beta_j)^* \rho \gamma_j(t - \beta_j) + r)^{-1} \right) \, dr,
\]

\[
= \int_0^\infty \text{tr} \left( i[Y_t, \rho] (\rho + r)^{-1} i[Y_t, \rho] (\rho + r)^{-1} \right) \, dr
\]

Note that \( \phi_u(X_k) = a_k \). Suppose that for each \( t \in I_j(2) \), \( Y(t) = \sum_{k=1}^s \lambda_k(t) a_k \) and hence \( X_t = \sum_{k=1}^s \lambda_k(t) X_k |_{\gamma_j(t-\beta_j)} \). The horizontal path \( \gamma_j \) has constant unit speed \( |X_t|_{\gamma_j(t)} \leq 1 \) and hence \( \sum_k \lambda_k(t)^2 \leq 1 \). Take \( \omega_k^j = (\rho + r)^{-\frac{1}{2}} i[a_k, \rho]/(\rho + r)^{-\frac{1}{2}} \). We have

\[
I(\tilde{\rho}) = \sum_{j=1}^m \int_{I_j(2)} \int_0^\infty \text{tr} \left( i[Y_t, \rho] (\rho + r)^{-1} i[Y_t, \rho] (\rho + r)^{-1} \right) \, drdt
\]

\[
= \sum_{j=1}^m \int_{I_j(2)} \int_0^\infty \sum_{k,l=1}^s \lambda_k(t) \lambda_l(t) \text{tr} \left( i[a_k, \rho] (\rho + r)^{-1} i[a_l, \rho] (\rho + r)^{-1} \right) \, drdt
\]

\[
= \sum_{j=1}^m \int_{I_j(2)} \int_0^\infty \sum_{k,l=1}^s \lambda_k(t) \lambda_l(t) \tau(\omega_k^j, \omega_l^j) \, drdt
\]
Combining the estimates above, we have
\[
D(\rho||E(\rho)) = D(\pi(\rho)||\pi(\rho)) \leq D(\pi(\rho)||E(\tilde{\rho})) \leq D(\tilde{\rho}||E(\tilde{\rho})) \leq \frac{1}{C} I(\tilde{\rho}) \leq \frac{smd_X}{C} I_L(\rho).
\]
Note that the constant \( C = \text{CLSI}(0, \beta_m, dt) = \beta_m^{-2} \text{CLSI}(0, 1, dt) \). Thus we prove the CLSI constant \( \text{CLSI}(L_X) = \frac{\text{CLSI}(0,1,dt)}{smd_X(d_X+1)} \). Finally, for non-orthonormal linearly independent \( X_k \)'s, the change of basis adds another multiplicative constant and completes the proof.

Using the noncommutative change of measure in \cite{JLR19} Theorem 4.1, we obtain the existence of CLSI constant for all finite dimensional quantum Markov semigroup satisfies detailed balance condition (see e.g. \cite{JLR19} for detailed definition.).

**Corollary 3.5.** Let \( T_t = e^{-Lt} : M_n \to M_n \) be a quantum Markov semigroup GNS-symmetric with respect to a full rank state \( \sigma \). Then \( T_t \) satisfies the \( \lambda \)-CLSI constant for some \( \lambda > 0 \).

**Proof.** By \cite{JLR19} Theorem 4.1, we know the optimal CLSI constant of every GNS-symmetric semigroup \( T_t \) is comparable to the optimal CLSI constant of a trace symmetric semigroup \( \tilde{T}_t \).

The argument in this section applies to complete Beckner inequalities \( C_p \text{SI} \), see \cite{Li20} for definitions of \( C_p \text{SI} \).

**Corollary 3.6.** Let \( T_t = e^{-Lt} : M_n \to M_n \) be a quantum Markov semigroup GNS-symmetric with respect to a full rank state \( \sigma \). Then \( T_t \) satisfies the \( \lambda \)-\( C_p \text{SI} \) constant for some \( \lambda > 0 \).

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