EMBEDDINGS OF $\alpha$-MODULATION SPACES

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Abstract. We show upper and lower embeddings of $\alpha$-modulation spaces in $\alpha$-modulation spaces for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, and prove partial results on the sharpness of the embeddings.

Dedicated to Professor Petar Popivanov on the occasion of his 65th birthday

0. Introduction

Let $1 \leq p, q \leq \infty$ and define the indices

$$\theta_1(p, q) = \max (0, q^{-1} - \min (p^{-1}, p'^{-1})),$$
$$\theta_2(p, q) = \min (0, q^{-1} - \max (p^{-1}, p'^{-1})).$$

Our main result is the following. For $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, $p, q \in [1, \infty]$ and $s \in \mathbb{R}$, we have the embeddings for $\alpha$-modulation spaces

$$M_{p,q}^{\alpha_2, s + d(\alpha_2 - \alpha_1)\theta_1(p,q)}(\mathbb{R}^d) \subseteq M_{p,q}^{\alpha_1, s}(\mathbb{R}^d) \subseteq M_{p,q}^{\alpha_2, s + d(\alpha_2 - \alpha_1)\theta_2(p,q)}(\mathbb{R}^d).$$

(See Theorem 2.3.) The embeddings (0.1) contain known results for embeddings of modulation spaces in Besov spaces [16] and sharpen Gröbner’s embeddings [8].

We also show the sharpness of the embeddings (0.1) in the following sense. (See Corollary 3.6.) If $p \geq \min(2, q)$ then

$$M_{p,q}^{\alpha_1,s} \subseteq M_{p,q}^{\alpha_2,t} \implies t \leq s + d(\alpha_2 - \alpha_1)\theta_2(p,q).$$

If $p \leq \max(2, q)$ then

$$M_{p,q}^{\alpha_2,t} \subseteq M_{p,q}^{\alpha_1,s} \implies t \geq s + d(\alpha_2 - \alpha_1)\theta_1(p,q).$$

For $p < \min(2, q)$ we are unable to show the implication (0.2). Nevertheless, we conjecture that the implication (0.2) holds also for $p < \min(2, q)$. By duality, this is equivalent to (0.3) for $p > \max(2, q)$.

Remark 0.1. After finalizing the proof of (0.1), we noticed the preprint [10] by Han and Wang. Their results [10, Theorems 5.1 and 5.2] generalize our Theorem 2.3 and show that the embeddings (0.1) hold for all $p, q \in (0, \infty]$, $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ and $s \in \mathbb{R}$. This paper provides an alternative proof to Han and Wang’s proof in the case $p, q \in [1, \infty]$, and

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establishes the partial sharpness of the embeddings (sharpness results are not treated in [10]).

1. Preliminaries

\( \mathbb{N}_0 \) denotes the nonnegative integers. Inclusions \( A \subseteq B \) and equalities \( A = B \) of topological spaces \( A, B \), are understood as embeddings, that is an inclusion is continuous. We use the standard notations \( \mathcal{S}(\mathbb{R}^d) \), \( \mathcal{S}'(\mathbb{R}^d) \), \( C_c^\infty(\mathbb{R}^d) \) for function and distribution spaces (see e.g. [11]). The Fourier transform of \( f \in \mathcal{S}(\mathbb{R}^d) \) is defined by

\[
\hat{f}(\xi) = \mathcal{F}f(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx.
\]

A Fourier multiplier operator is defined by \( \varphi(D)f = \mathcal{F}^{-1}(\varphi\hat{f}) \), provided \( \varphi \) and \( f \) are objects such that the expression makes sense. For \( s \in \mathbb{R} \) the Sobolev space \( H^s(\mathbb{R}^d) \) is defined as the subspace of \( f \in \mathcal{S}'(\mathbb{R}^d) \) such that \( \hat{f} \in L^2_{\text{loc}}(\mathbb{R}^d) \) and

\[
\|f\|_{H^s} = \left( \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty
\]

where \( \langle \xi \rangle = (1 + |\xi|^2)^{1/2} \).

We denote by \( |A| \) the cardinality of a finite set \( A \), and by \( \mu(A) \) the Lebesgue measure of a measurable set \( A \subseteq \mathbb{R}^d \). A closed ball in \( \mathbb{R}^d \) of center \( a \in \mathbb{R}^d \) and radius \( r \geq 0 \) is denoted \( B(a,r) = \{ x \in \mathbb{R}^d : |x - a| \leq r \} \). A closed cube in \( \mathbb{R}^d \) of center \( c \) and side length \( 2r \) is denoted \( Q(c,r) = \{ x \in \mathbb{R}^d : \max_{1 \leq j \leq d} |x_j - c_j| \leq r \} \). The conjugate exponent to \( p \in [1,\infty] \) is denoted \( p' \) and defined by \( 1/p + 1/p' = 1 \).

The notation \( X \lesssim Y \) means that \( X \leq CY \) for some constant \( C > 0 \), and \( X_i \lesssim Y_j \) for \( i \in I \) and \( j \in J \) means that the constant is uniformly bounded over the index sets \( I \) and \( J \). If \( X \lesssim Y \) and \( Y \lesssim X \) then we write \( X \asymp Y \). Coordinate reflection is denoted \( \bar{f}(x) = f(-x) \).

1.1. Besov spaces. Define

\[
D_j = \{ \xi \in \mathbb{R}^d : 2^{j-2} \leq |\xi| \leq 2^j \}, \quad j \geq 1.
\]

Let \( \{ \varphi_j \}_{j=0}^\infty \subseteq C_c^\infty(\mathbb{R}^d) \) be a sequence with the following properties [2].

\[
\begin{align*}
supp \varphi_0 &\subseteq B(0,1), \\
supp \varphi_j &\subseteq D_j, \quad j \geq 1, \\
\sum_{j=0}^\infty \varphi_j(\xi) &= 1 \quad \forall \xi \in \mathbb{R}^d.
\end{align*}
\]

Then we have for \( j \geq 0 \)

\[2^{j-1} \leq |\xi| \leq 2^j \Rightarrow \varphi_j(\xi) + \varphi_{j+1}(\xi) = 1.\]

\[\text{Note added in proof. In an updated version of their manuscript [10], Han and Wang establish the sharpness of the embeddings in all cases.}\]
The functions $\varphi_j$ for $j \geq 1$ are constructed as dilations $\varphi_j(\xi) = \varphi(2^{1-j}\xi)$ for a function $\varphi \in C^\infty_c(\mathbb{R}^d)$ supported in $D_1$ (cf. [2]). Let $p, q \in [1, \infty]$ and let $s \in \mathbb{R}$. The Besov space $B^{p,q}_s(\mathbb{R}^d)$ is defined as the space of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

\[(1.4) \quad \|f\|_{B^{p,q}_s} = \left(\sum_{j=0}^{\infty} (2^{js}\|\varphi_j(D)f\|_{L^p})^q\right)^{1/q} < \infty\]

when $q < \infty$ and with the standard modification when $q = \infty$ [2]. We abbreviate $B^{p,p}_s = B^p_s$ and $B^{p,q}_0 = B^{p,q}$.

1.2. $\alpha$-modulation spaces. We need the following definitions introduced by Feichtinger and Gröbner [4–6, 8] (cf. [3, 7]).

**Definition 1.1.** A countable set $Q$ of subsets $Q \subseteq \mathbb{R}^d$ is called an admissible covering provided

\[(1.5) \quad \sum_{Q' \in Q} |\{Q' \in Q : Q \cap Q' \neq \emptyset\}| \leq n_0 \quad \forall Q \in \mathcal{Q},\]

for some finite integer $n_0$.

For each $Q \in \mathcal{Q}$, let

\[(1.6) \quad r_Q = \sup\{r \in \mathbb{R} : B(c, r) \subseteq Q \text{ for some } c \in \mathbb{R}^d\},\]
\[(1.7) \quad R_Q = \inf\{R \in \mathbb{R} : Q \subseteq B(c, R) \text{ for some } c \in \mathbb{R}^d\}.\]

**Definition 1.2.** Let $\alpha \in [0, 1]$. An admissible covering $\{Q\}_{Q \in \mathcal{Q}}$ is called an $\alpha$-covering provided there exists a constant $K \geq 1$ such that

\[(1.8) \quad \mu(Q) \asymp \langle x \rangle^{\alpha d}, \quad x \in Q, \quad Q \in \mathcal{Q},\]
\[(1.9) \quad R_Q/r_Q \leq K, \quad Q \in \mathcal{Q}.\]

**Definition 1.3.** Let $\alpha \in [0, 1]$ and let $\{Q\}_{Q \in \mathcal{Q}}$ be an $\alpha$-covering of $\mathbb{R}^d$. Then $\{\psi_Q\}_{Q \in \mathcal{Q}}$ is called a bounded admissible partition of unity corresponding to $\mathcal{Q}$ (Q-BAPU) provided

\[(1.10) \quad \sup_{Q \in \mathcal{Q}} \|\mathcal{F} \psi_Q\|_{L^1} < \infty.\]

We will call a Q-BAPU an $\alpha$-BAPU when $Q$ is an $\alpha$-covering.

**Definition 1.4.** Let $\alpha \in [0, 1]$, $p, q \in [1, \infty]$, $s \in \mathbb{R}$, let $\{Q\}_{Q \in \mathcal{Q}}$ be an $\alpha$-covering of $\mathbb{R}^d$ and let $\{\psi_Q\}_{Q \in \mathcal{Q}}$ be a Q-BAPU. The weighted
\(\alpha\)-modulation space \(M_{\alpha,s}^p(\mathbb{R}^d)\) is defined as all \(f \in \mathscr{S}'(\mathbb{R}^d)\) such that
\[
\|f\|_{M_{\alpha,s}^p} = \left(\sum_{Q \in \mathcal{Q}} \langle \xi_Q \rangle^{qs} \|\psi_Q(D)f\|_{L^p}^q\right)^{1/q} < \infty
\]
where \(\xi_Q \in \mathcal{Q}\) for all \(Q \in \mathcal{Q}\), when \(q < \infty\). If \(q = \infty\) the global \(l^q\) norm in (1.11) is replaced by \(l^\infty\).

The \(\alpha\)-modulation spaces contain as extreme cases the frequency-weighted modulation spaces (cf. [4, 9]) \(M_{\alpha,s}^{p,q} = M_{\alpha,s}^p (\alpha = 0)\) and the Besov spaces \(B_{s,q}^{p,q} = M_{\alpha,s}^{\alpha,q} (\alpha = 1)\) (cf. [4]). The number \(\alpha\) thus parametrizes a scale of spaces that in some sense is intermediate between the modulation spaces and the Besov spaces. We abbreviate \(M_{\alpha,s}^p = M_{\alpha,s}^p(\alpha = 0)\), \(M_{\alpha,s}^{p,p} = M_{\alpha,s}^{p,p}\) and \(M_{\alpha,s}^{p,q} = M_{\alpha,s}^{p,q}\) (the weighted or classical modulation spaces). For \(t \geq s\) we have the embedding \(M_{\alpha,t}^{p,q} \subseteq M_{\alpha,s}^{p,q}\), \(\alpha \in [0, 1], p, q \in [1, \infty]\).

For \(\alpha\) in the interval \(0 \leq \alpha < 1\), that is, excluding the Besov spaces, we will use the following \(\alpha\)-covering and an associated \(Q\)-BAPU (cf. [3]). Set
\[
B_k = B(k|k|^\beta, r|k|^\beta), \quad k \in \mathbb{Z}^d \setminus 0,
\]
where \(\beta = \alpha/(1 - \alpha)\). Note that \(B_k = B(\xi_k, r|\xi_k|^{\alpha})\) where \(\xi_k = k|k|^\beta\).

For \(r > 0\) sufficiently large, \(\mathcal{Q} = \{B_k\}_{k \in \mathbb{Z}^d \setminus 0}\) is an \(\alpha\)-covering of \(\mathbb{R}^d\) according to [3, Theorem 2.6]. Moreover, a \(Q\)-BAPU \(\{\psi_k\}_{k \in \mathbb{Z}^d \setminus 0}\) such that \(\text{supp } \psi_k \subseteq B_k\) for all \(k \in \mathbb{Z}^d \setminus 0\) can be constructed (see [3, Proposition A.1]).

We will use Borup and Nielsen’s Banach frame construction for \(M_{\alpha,s}^{p,q}(\mathbb{R}^d)\), based on multivariate brushlet systems (cf. [3]). Let
\[
Q_k = Q(k|k|^\beta, r|k|^\beta), \quad k \in \mathbb{Z}^d \setminus 0,
\]
where again \(\beta = \alpha/(1 - \alpha)\). If \(r > 0\) is sufficiently large then \(\mathcal{Q} = \{Q_k\}_{k \in \mathbb{Z}^d \setminus 0}\) is an \(\alpha\)-covering of \(\mathbb{R}^d\). One can construct a sequence of functions
\[
(w_{n,k})_{n \in \mathbb{N}_0^d, k \in \mathbb{Z}^d \setminus 0} \subseteq \mathscr{S}(\mathbb{R}^d)
\]
such that \((w_{n,k})_{n \in \mathbb{N}_0^d, k \in \mathbb{Z}^d \setminus 0}\) is an orthonormal system, with \(\text{supp } \hat{w}_{n,k} \subseteq Q_k\), for each \(k \in \mathbb{Z}^d \setminus 0\). Each function \(w_{n,k}\) is constructed as a tensor product
\[
w_{n,k} = \bigotimes_{j=1}^d w_{n_{j,k}},
\]
where \(Q_k = \Pi_{j=1}^d I_{k_{j}}, k_{j}\), whose components are, simplifying notation to \(n = n_j, I = I_{k_{j}}\),
\[
w_{n,I}(x) = \sqrt{\mu(I)/2} e^{i\alpha x} \left( g(\mu(I)(x + e_{n,I}) + g(\mu(I)(x - e_{n,I})) \right), \quad x \in \mathbb{R},
\]
where $e_{n,I} = \pi(n + 1/2)/\mu(I)$, $a_I$ denotes the left end point of $I$, i.e. $I = [a_I, b_I]$, and $g \in \mathcal{F}C_c^\infty(\mathbb{R})$ with $\text{supp} \, \hat{g} \subseteq [0, 1]$. For more details about the sequence of functions $(w_{n,k})_{n \in \mathbb{N}_0^d, k \in \mathbb{Z}^d \setminus 0}$ we refer to \cite{3}.

Borup and Nielsen \cite{3} show that the sequence $(w_{n,k})$ is a (quasi-) Banach frame for $M_{\alpha,s}^{p,q}(\mathbb{R}^d)$ for $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. We restrict our interest to the exponents $p, q \in [1, \infty]$. Let $p, q \in [1, \infty]$, $s \in \mathbb{R}$, let $f \in M_{\alpha,s}^{p,q}(\mathbb{R}^d)$, and define the coefficient sequence

$$c_{n,k} = (f, w_{n,k})_{L^2}, \quad n \in \mathbb{N}_0^d, \quad k \in \mathbb{Z}^d \setminus 0$$

where $w_{n,k}$ is defined by (1.13). The coefficient operator is defined by $(Df)_{n,k} = c_{n,k}$, $n \in \mathbb{N}_0^d$, $k \in \mathbb{Z}^d \setminus 0$. The Banach frame property means in this case that

$$\|f\|_{M_{\alpha,s}^{p,q}} \asymp \|c\|_{m_{\alpha,s}^{p,q}},$$

where the sequence space $m_{\alpha,s}^{p,q} = m_{\alpha,s}^{p,q}(\mathbb{N}_0^d \times \mathbb{Z}^d \setminus 0)$ is defined by the norm

$$\|c\|_{m_{\alpha,s}^{p,q}} = \left( \sum_{k \in \mathbb{Z}^d \setminus 0} \left( \sum_{n \in \mathbb{N}_0^d} \left| n \right|^{-\alpha \left\langle \frac{s}{q} - \frac{s}{p} - \frac{d}{p} \right\rangle} \left| c_{n,k} \right|^p \right)^{q/p} \right)^{1/q}$$

when $p, q < \infty$ and suitably modified otherwise. Moreover, there exists a reconstruction operator $R$ defined by

$$R \, c = \sum_{k \in \mathbb{Z}^d \setminus 0, n \in \mathbb{N}_0^d} c_{n,k} \, \tilde{w}_{n,k},$$

where $(\tilde{w}_{n,k})_{k \in \mathbb{Z}^d \setminus 0, n \in \mathbb{N}_0^d}$ is a dual frame defined by $\tilde{w}_{n,k} = \psi_k(D)w_{n,k}$, $n \in \mathbb{N}_0^d$, $k \in \mathbb{Z}^d \setminus 0$. The operator $R$ is bounded as

$$\|R \, c\|_{M_{\alpha,s}^{p,q}} \lesssim \|c\|_{m_{\alpha,s}^{p,q}}, \quad c \in m_{\alpha,s}^{p,q},$$

and $RD = \text{id}_{M_{\alpha,s}^{p,q}}$. These results are proved in \cite{3} Theorem 4.3.

Let $\mathcal{M}_{\alpha,s}^{p,q}(\mathbb{R}^d)$ be the completion of $\mathcal{S}(\mathbb{R}^d)$ in the norm $\|\cdot\|_{M_{\alpha,s}^{p,q}(\mathbb{R}^d)}$. In the next result we collect some important properties of the $\alpha$-modulation spaces. The result is a generalization of the corresponding result for modulation spaces.

**Proposition 1.5.** Let $\alpha \in [0, 1], \, s \in \mathbb{R}$ and $p, q \in [1, \infty]$. The following holds.

(i) The space $M_{\alpha,s}^{p,q}(\mathbb{R}^d)$ is a Banach space which is independent of the sequence $\{\xi_Q\}_{Q \in \mathcal{Q}}$ as long as $\xi_Q \in Q$ for all $Q \in \mathcal{Q}$, and also independent of the $\alpha$-covering $\{Q\}_{Q \in \mathcal{Q}}$ and of the $\mathcal{Q}$-BAPU $\{\psi_Q\}_{Q \in \mathcal{Q}}$. Varying these parameters gives rise to equivalent norms.

(ii) The $L^2$-product $(\cdot, \cdot)$ on $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ extends to a continuous sesquilinear form on $M_{\alpha,s}^{p,q}(\mathbb{R}^d) \times M_{\alpha,-s}^{p,q}(\mathbb{R}^d)$.

Furthermore,

$$\|f\| = \sup |(f, g)|$$
with supremum taken over all \( g \in \mathcal{S}(\mathbb{R}^d) \) such that \( \|g\|_{M^p,q_{\alpha,s}} \leq 1 \), is a norm equivalent to \( \|f\|_{M^p,q_{R,s}} \). If \( p, q < \infty \), then the dual space of \( M^p,q_{\alpha,s} \) can be identified with \( M^{p',q'}_{\alpha,-s} \) through the form \( \langle \cdot, \cdot \rangle \).

(iii) Assume that \( 0 \leq \theta \leq 1 \), \( p, q, p_1, p_2, q_1, q_2 \in [1, \infty] \), \( s, s_1, s_2 \in \mathbb{R} \) satisfy

\[
\frac{1}{p} = \frac{1 - \theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1 - \theta}{q_1} + \frac{\theta}{q_2}, \quad s = (1 - \theta)s_1 + \theta s_2.
\]

Then complex interpolation gives

\[
\left( M^{p_1,q_1}_{\alpha,s_1}, M^{p_2,q_2}_{\alpha,s_2} \right)_{\theta} = M^{p,q}_{\alpha,s}.
\]

(iv) It holds \( M^{p,q}_{\alpha,s} \subseteq M^{p,q}_{\alpha,s} \) with equality if \( p < \infty \) and \( q < \infty \).

**Proof.** (i) See \([5, \text{Theorems 2.2, 2.3 and 3.7}]\) and \([6, \text{Theorem 4.1}]\).

(ii) The fact that the dual space of \( M^{p,q}_{\alpha,s} \), for \( 1 \leq p, q < \infty \), can be identified with \( M^{p',q'}_{\alpha,-s} \) is a consequence of \([5, \text{Theorem 2.8}]\). Let \( 1 \leq p, q \leq \infty \). From \([5, \text{Theorem 2.3}]\) it follows

\[
|\langle f, g \rangle| \lesssim \|f\|_{M^p,q_{R,s}} \|g\|_{M^{p',q'}_{R,s}}, \quad g \in \mathcal{S}(\mathbb{R}^d).
\]

For the reverse inequality we first let \( 0 \leq \alpha < 1 \). By \((1.15)\)

\[
\|f\|_{M^p,q_{R,s}} \lesssim \|c\|_{m^{p,q}_{\alpha,s}},
\]

where the sequence \( c \) is defined by \((1.14)\). The \( m^{p,q}_{\alpha,s} \)-norm of \( c \) is the mixed \( \ell^p,q \)-norm of \( \omega c \), where the weight \( \omega \) depends on \( p, \alpha, s \) as

\[
\omega_{n,k} = |k|^{-\alpha} \omega_0 \in S,
\]

\[
\omega_{n,k} = \frac{1}{|k|^{\alpha}}(1 + d(\frac{1}{2} - \frac{1}{q}))
\]

An application of \([11, \text{Lemma 3.1}]\) yields

\[
\|c\|_{m^{p,q}_{\alpha,s}} = \|\omega c\|_{\ell^{p,q} = \sup |(\omega c, d)|_{\ell^2}}
\]

with supremum taken over all sequences \( (d_{n,k}) \) of finite support and \( \|d\|_{\ell^{p',q'}} \leq 1 \). Let \( (d_{n,k}) \) be a sequence of finite support such that \( \|d\|_{\ell^{p',q'}} \leq 1 \) and

\[
\|\omega c\|_{\ell^{p,q}} \leq 2|\omega_0, d|_{\ell^2},
\]

and set

\[
g = \sum_{k \in \mathbb{Z}^d \setminus 0} \sum_{n \in \mathbb{N}^d_0} \omega_k d_{n,k} w_{n,k}.
\]

Then \( g \in \mathcal{S}(\mathbb{R}^d) \) since the sum is finite, and \( \langle f, g \rangle = (\omega c, d)|_{\ell^2} \). The following inequality follows from the proofs of \([3, \text{Lemma 3.2 and Lemma 4.2}]\). If \( p, q \in [1, \infty] \) and \( s \in \mathbb{R} \), then

\[
\left\| \sum_{k \in \mathbb{Z}^d \setminus 0} \sum_{n \in \mathbb{N}^d_0} d_{n,k} w_{n,k} \right\|_{M^{p',q'}_{\alpha,-s}} \lesssim \|d\|_{M^{p',q'}_{\alpha,-s}}.
\]

This gives

\[
\|g\|_{M^{p',q'}_{\alpha,-s}} \lesssim \|\omega d\|_{m^{p',q'}_{\alpha,-s}} = \|d\|_{\ell^{p',q'}} \leq 1.
\]
Hence we have proved that $\|f\|_{M^{p,q}_\alpha} \lesssim \|f\|$ when $0 \leq \alpha < 1$.

It remains to prove the corresponding inequality when $\alpha = 1$, in which case $M^{p,q}_\alpha = B^{p,q}_s$. Let $\{\varphi_j\}_{j=0}^\infty \subseteq C^\infty_0(\mathbb{R}^d)$ be a sequence that satisfies (1.2) and $\varphi_j(x) = \varphi(2^{j-1}x)$ for $j \geq 1$ where $\varphi \in C^\infty_0(\mathbb{R}^d)$ and $\text{supp} \varphi \subseteq D_1$. The $B^{p,q}_s$-norm defined by (1.4) is the mixed Lebesgue norm $L^{p,q}(\mathbb{R}^d \times \mathbb{N}_0)$, where $\mathbb{R}^d$ is equipped with the Lebesgue measure and $\mathbb{N}_0$ with the counting measure, of the function $F(x,j) = 2^j \varphi_j(D)f(x)$. According to [1, Lemma 3.1] we have

$$\|f\|_{B^{p,q}_s} = \sup \left\{ \sum_{j=0}^\infty 2^j \langle \varphi_j(D)f, g_j \rangle_{L^2} \right\}$$

where the supremum is taken over all sequences $(g_j)_0^\infty$ of simple functions of compact support $g_j$ such that $g_j \equiv 0$ for $j > N$ for some $N \geq 0$, and

$$\left( \sum_{j=0}^\infty \|g_j\|_{L^{p,q}}^{q'} \right)^{1/q'} \leq 1$$

if $q' < \infty$, and sup$_{0 \leq j < \infty} \|g_j\|_{L^{p,q}} \leq 1$ if $q' = \infty$. Therefore there exists $N \geq 0$ and $(g_j)_0^N \subseteq L^{p,q}(\mathbb{R}^d)$ such that

$$\|f\|_{B^{p,q}_s} \leq 2 \sum_{j=0}^N 2^j \langle \varphi_j(D)f, g_j \rangle_{L^2} = 2(f, \sum_{j=0}^N 2^j \varphi_j(D)g_j)_{L^2}$$

and

$$\left( \sum_{j=0}^N \|g_j\|_{L^{p,q}}^{q'} \right)^{1/q'} \leq 1$$

(modified as above if $q' = \infty$). Set $g = \sum_{j=0}^N 2^j \varphi_j(D)g_j \in \mathcal{S}(\mathbb{R}^d)$.

We have sup$_{j \geq 0} \|\mathcal{F}^{-1} \varphi_j\|_{L^1} \lesssim 1$. By means of (1.3) and Young’s inequality, we obtain for $k \geq 1$

$$\|\varphi_k(D)g\|_{L^{p,q}} \leq \left\| \min(N,k+1) \sum_{j=k-1}^{\infty} 2^{js} \varphi_k(D) \varphi_j(D)g_j \right\|_{L^{p,q}}$$

\[ \lesssim 2^{(k-1)s} \|g_{k-1}\|_{L^{p,q}} + 2^{ks} \|g_k\|_{L^{p,q}} + 2^{(k+1)s} \|g_{k+1}\|_{L^{p,q}}, \]

and

$$\|\varphi_0(D)g\|_{L^{p,q}} = \left\| \sum_{j=0}^{\min(N,1)} 2^{js} \varphi_0(D) \varphi_j(D)g_j \right\|_{L^{p,q}}$$

\[ \lesssim \|g_0\|_{L^{p,q}} + 2^s \|g_1\|_{L^{p,q}}. \]
which gives, by means of (1.18), \( \|g\|_{B^{p',q'}_s} \lesssim 1 \). It follows that \( \|f\|_{M^{p,q}_{s,1}} \lesssim \|f\| \).

(iii) This follows from [5, Corollary 2.4] (cf. [8, Bemerkung F.2]).

(iv) See [5, Theorem 2.2].

2. Embeddings of \( \alpha \)-modulation spaces

We need the following elementary lemma (cf. [10, Prop. 2.5] and [8]), a proof of which is provided as a service to the reader.

Lemma 2.1. If \( \alpha \in [0, 1] \) and \( s \in \mathbb{R} \) then \( M^2_{\alpha,s}(\mathbb{R}^d) = H_s(\mathbb{R}^d) \).

Proof. For the Besov space case \( (\alpha = 1) \) the result \( B^2_s(\mathbb{R}^d) = H_s(\mathbb{R}^d) \) is well known (see e.g. [2, Theorem 6.4.4]). Let \( 0 \leq \alpha < 1 \). We use the \( \alpha \)-covering (1.12) \( \{B_k\}_{k \in \mathbb{Z}^d \setminus 0} \) for \( r > 0 \) sufficiently large, and an associated BAPU \( \{\psi_k\}_{k \in \mathbb{Z}^d \setminus 0} \) such that \( 0 \leq \psi_k \leq 1 \) for all \( k \in \mathbb{Z}^d \setminus 0 \). Parseval’s formula and (1.8) yield

\[
\|f\|_{M^2_{\alpha,s}(\mathbb{R}^d)}^2 = \sum_{k \in \mathbb{Z}^d \setminus 0} \langle \xi \rangle^{2s} \int_{B_k} \psi_k(\xi) \hat{f}(\xi)^2 d\xi \\
\lesssim \sum_{k \in \mathbb{Z}^d \setminus 0} \int_{B_k} \psi_k(\xi) \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2 d\xi = \|f\|_{H_s(\mathbb{R}^d)}^2,
\]

i.e. \( H_s \subseteq M^2_{\alpha,s} \). For the opposite inclusion, we note that

\[
(2.1) \quad \sum_{k \in \mathbb{Z}^d \setminus 0} \psi_k(\xi)^2 \geq C, \quad \xi \in \mathbb{R}^d,
\]

holds for some \( C > 0 \). In fact, if this would not the case, then for any \( \varepsilon > 0 \) there exists \( \xi \in \mathbb{R}^d \) such that

\[
\sum_{k \in \mathbb{Z}^d \setminus 0} \psi_k(\xi)^2 < \varepsilon.
\]

Let \( \varepsilon < n_0^{-2} \) where \( n_0 \) is the upper bound (1.5) corresponding to the covering \( \{B_k\}_{k \in \mathbb{Z}^d \setminus 0} \), and let \( \xi \in \mathbb{R}^d \) denote the corresponding vector. Then \( \psi_k(\xi) < \sqrt{\varepsilon} \) for all \( k \in \mathbb{Z}^d \setminus 0 \). Since \( \xi \in B_j \) for some \( j \in \mathbb{Z}^d \setminus 0 \) we obtain from (1.5)

\[
\sum_{k \in \mathbb{Z}^d \setminus 0} \psi_k(\xi) = \sum_{k: B_k \cap B_j \neq \emptyset} \psi_k(\xi) < n_0 \sqrt{\varepsilon} < 1
\]

which is a contradiction. Thus (2.1) holds for some \( C > 0 \).
By means of (2.1) and again (1.8) we obtain
\[
\norm{f}_{H,\alpha}^2 \leq C^{-1} \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}^d \setminus 0} \psi_k(\xi)^2 \langle \xi \rangle^{2s} |\widehat{f}(\xi)|^2 d\xi
\]
\[
\lesssim \sum_{k \in \mathbb{Z}^d \setminus 0} \langle \xi_k \rangle^{2s} \int_{B_k} \psi_k(\xi)^2 |\widehat{f}(\xi)|^2 d\xi
\]
\[
= \norm{f}_{M^2_{\alpha,s}(\mathbb{R}^d)}^2,
\]
i.e. \(M^2_{\alpha,s} \subseteq H_s\) and the proof is complete. \(\square\)

Embeddings for \(\alpha\)-modulation spaces have been proved by Gröbner [8], Han and Wang [10], and, for the modulation space case \(\alpha = 0\), by Okoudjou [13] and the first named author of this article [15][16].

The result [16, Theorem 2.10] imply the embeddings, for \(p, q \in [1, \infty]\) and \(s \in \mathbb{R}\),
\[
B_{s+d\theta_1(p,q)}^{p,q}(\mathbb{R}^d) \subseteq M_{0,s}^{p,q}(\mathbb{R}^d) \subseteq B_{s+d\theta_2(p,q)}^{p,q}(\mathbb{R}^d).
\]
Here the indices \(\theta_1\) and \(\theta_2\) are defined by
\[
\theta_1(p, q) = \max \left(0, q^{-1} - \min(p^{-1}, p'^{-1})\right),
\]
\[
\theta_2(p, q) = \min \left(0, q^{-1} - \max(p^{-1}, p'^{-1})\right) = -\theta_1(p', q').
\]
The unweighted versions (i.e. \(s = 0\)) of these embeddings were proved in [15, Theorem 3.1]. They imply the embeddings, for \(p, q \in [1, \infty]\),
\[
B_{\theta_1(p,q)}^{p,q}(\mathbb{R}^d) \subseteq M^{p,q}(\mathbb{R}^d) \subseteq B_{\theta_2(p,q)}^{p,q}(\mathbb{R}^d),
\]
and they have been proven to be sharp. The sharpness was obtained independently by Huang and Wang [17, Theorem 1.1], and by Sugimoto and Tomita [14, Theorem 1.2], and means the following. If \(p, q \in [1, \infty]\) and \(B_{s}^{p,q}(\mathbb{R}^d) \subseteq M^{p,q}(\mathbb{R}^d)\) then \(s \geq d\theta_1(p, q)\). If \(p, q \in [1, \infty]\) and \(M^{p,q}(\mathbb{R}^d) \subseteq B_{s}^{p,q}(\mathbb{R}^d)\) then \(s \leq d\theta_2(p, q)\). (By duality, the two assertions are equivalent.) This gives the sharpness also for the weighted case (2.2), since \((D)^t\) is a homeomorphism \(B_{s}^{p,q} \mapsto B_{s-t}^{p,q}\) for any \(t, s \in \mathbb{R}\) (cf. [2]) as well as \(M_{0,s}^{p,q} \mapsto M_{0,s-t}^{p,q}\) for any \(t, s \in \mathbb{R}\) (cf. [16, Cor. 2.3]).

The sharpness of (2.2) reads:
\[
B_{t}^{p,q}(\mathbb{R}^d) \subseteq M_{0,s}^{p,q}(\mathbb{R}^d) \implies t \geq s + d\theta_1(p, q), \quad p, q \in [1, \infty],
\]
\[
M_{0,s}^{p,q}(\mathbb{R}^d) \subseteq B_{t}^{p,q}(\mathbb{R}^d) \implies t \leq s + d\theta_2(p, q), \quad p, q \in [1, \infty].
\]

Note that the embeddings (2.2) and (2.4) are restricted to upper and lower embeddings of 0-modulation spaces in 1-modulation spaces, and give no information on upper and lower embeddings of \(M_{\alpha_1,s}^{p,q}\) in \(M_{\alpha_2,t}^{p_q}\) for general \(\alpha_1, \alpha_2 \in [0, 1]\).

Gröbner’s embeddings [8, Theorems F.6, F.7 and pp. 66–68] reads (2.5)
\[
M_{\alpha_2,t}^{p,q}(\mathbb{R}^d) \subseteq M_{\alpha_1,t}^{p,q}(\mathbb{R}^d) \subseteq M_{\alpha_2,s}^{p,q}(\mathbb{R}^d),
\]
for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, $p, q \in [1, \infty]$ and $s \in \mathbb{R}$, where the indices $\nu_1$ and $\nu_2$ are defined by

$$
\nu_1(p, q) = \theta_1(p, q) + \max \left(0, q^{-1} - \max(p^{-1}, p'^{-1})\right),
$$

$$
\nu_2(p, q) = \theta_2(p, q) + \min \left(0, q^{-1} - \min(p^{-1}, p'^{-1})\right) = -\nu_1(p', q').
$$

Since $\nu_1(p, q) \geq \theta_1(p, q)$ and $\nu_2(p, q) \leq \theta_2(p, q)$, the embeddings (2.2) improve Gröbner’s embeddings (2.5) when $\alpha_1 = 0$ and $\alpha_2 = 1$.

We are now in a position to present our main embedding theorem, which is both a sharpening of (2.5) and a generalization of (2.2) to general $\alpha$-modulation spaces. In the proof of the theorem we need the following lemma.

**Lemma 2.2.** Suppose $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, $\{Q_j\}_{j \in J}$ is an $\alpha_1$-covering, $\{P_i\}_{i \in I}$ is an $\alpha_2$-covering, and let $\eta_j \in Q_j$ for all $j \in J$, and $\xi_i \in P_i$ for all $i \in I$. If

$$
\Omega_i = \{ j \in J ; Q_j \cap P_i \neq \emptyset \}, \quad i \in I,
$$

$$
\Lambda_j = \{ i \in I ; Q_j \cap P_i \neq \emptyset \}, \quad j \in J,
$$

then

$$
|\Omega_i| \lesssim \langle \xi_i \rangle^{d(\alpha_2 - \alpha_1)}, \quad i \in I,
$$

$$
|\Lambda_j| \lesssim 1, \quad j \in J,
$$

and $\langle \eta_j \rangle$ for $j \in \Omega_i$ for all $i \in I$, and for $i \in \Lambda_j$ for all $j \in J$.

**Proof.** By the “disjointization lemma” [5, Lemma 2.9], for any admissible covering $\{Q_j\}_{j \in J}$ we can split the index set as $J = \bigcup_{k=1}^{n_0} J_k$, where $n_0$ is finite, $\{J_k\}$ are pairwise disjoint, and $j, j' \in J_k, j \neq j'$ imply $Q_j \cap Q_{j'} = \emptyset$ for $1 \leq k \leq n_0$.

Let $i \in I$. By (1.6) we have $\mu(Q_j) \sim \langle \xi_i \rangle^{d\alpha_1}$ for all $j \in \Omega_i$. By (1.7) and (1.9) we have $P_i \subseteq B(c_i, 2R_2)$ where $R_2^d \lesssim \mu(P_i)$, for some $c_i \in \mathbb{R}^d$. Let $j \in \Omega_i$ and $x_j \in Q_j \cap P_i$. Again (1.7), (1.8), (1.9) give $Q_j \subseteq B(b_j, 2R_1)$ where $R_1^d \lesssim \langle x_j \rangle^{d\alpha_1} \lesssim \langle x_j \rangle^{d\alpha_2} \lesssim \mu(P_i) \lesssim R_2^d$, for some $b_j \in \mathbb{R}^d$. It follows that $Q_j \subseteq B(c_i, CR_2)$ for some $C > 0$. Combining these observations, we obtain for $1 \leq k \leq n_0$

$$
\langle \xi_i \rangle^{d\alpha_1} |\Omega_i \cap J_k| \asymp \sum_{j \in \Omega_i \cap J_k} \mu(Q_j) \lesssim \mu(B(c_i, CR_2)) \lesssim \langle \xi_i \rangle^{d\alpha_2},
$$

whereupon (2.7) follows from the disjointization lemma. The proof of (2.8) is similar. The final statement of the lemma is a direct consequence of (1.8). $\square$

**Theorem 2.3.** Let $p, q \in [1, \infty]$, $s \in \mathbb{R}$ and $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Then

$$
M_{\alpha_2, s + d(\alpha_2 - \alpha_1)\theta_1(p, q)}^{p, q}(\mathbb{R}^d) \subseteq M_{\alpha_1, s}^{p, q}(\mathbb{R}^d) \subseteq M_{\alpha_2, s + d(\alpha_2 - \alpha_1)\theta_2(p, q)}^{p, q}(\mathbb{R}^d),
$$

and, for some constant $C > 0$, it holds for $f \in \mathcal{F}(\mathbb{R}^d)$

$$
C^{-1} \|f\|_{M_{\alpha_2, s + d(\alpha_2 - \alpha_1)\theta_1(p, q)}^{p, q}(\mathbb{R}^d)} \leq \|f\|_{M_{\alpha_1, s}^{p, q}(\mathbb{R}^d)} \leq C\|f\|_{M_{\alpha_2, s + d(\alpha_2 - \alpha_1)\theta_2(p, q)}^{p, q}(\mathbb{R}^d)}.
$$
By duality it suffices to prove the right hand side embedding. Let \( s \in \mathbb{R} \), let \( \{ \varphi_j \} \) be an \( \alpha_1 \)-BAPU such that \( \varphi_j \geq 0 \) for all \( j \), let \( \{ \psi_i \} \) be an \( \alpha_2 \)-BAPU such that \( \psi_i \geq 0 \) for all \( i \), let \( \eta_j \in \text{supp} \varphi_j \) for all \( j \), and let \( \xi_i \in \text{supp} \psi_i \) for all \( i \).

\[
\Omega_i = \{ j ; \text{supp} \varphi_j \cap \text{supp} \psi_i \neq \emptyset \} \\
\Lambda_j = \{ i ; \text{supp} \varphi_j \cap \text{supp} \psi_i \neq \emptyset \}
\]

(2.10)

then by Lemma 2.2

\[
|\Omega_i| \lesssim (\xi_i)^{d(\alpha_2-\alpha_1)} \quad \text{for all } i, \\
|\Lambda_j| \lesssim 1 \quad \text{for all } j,
\]

and \( \langle \xi_i \rangle \asymp \langle \eta_j \rangle \) for \( j \in \Omega_i \) for all \( i \), and for \( i \in \Lambda_j \) for all \( j \). This gives, using (2.11),

\[
\|\psi_i(D) f\|_{L^2(\xi_i)^{2s-d(\alpha_2-\alpha_1)}}^2 = \|\psi_i \hat{f}\|_{L^2(\xi_i)^{2s-d(\alpha_2-\alpha_1)}}^2 \lesssim \sum_{j \in \Omega_i} \int \varphi_j^2(\xi) \psi_j^2(\xi) |\hat{f}(\xi)|^2 d\xi (\xi_i)^{2s-d(\alpha_2-\alpha_1)} \\
\lesssim \sum_{j \in \Omega_i} \int \varphi_j^2(\xi) |\hat{f}(\xi)|^2 d\xi (\xi_i)^{2s-d(\alpha_2-\alpha_1)} \\
\lesssim (\xi_i)^{d(\alpha_2-\alpha_1)} \sup_{j \in \Omega_i} \|\varphi_j \hat{f}\|_{L^2(\xi_i)^{2s-d(\alpha_2-\alpha_1)}}^2 \\
= \sup_{j \in \Omega_i} \|\varphi_j(D) f\|_{L^2(\xi_i)^{2s}}^2.
\]

Taking the supremum over \( i \) we obtain

\[
\|f\|_{M^{2,\infty}_{\alpha_2,s-d(\alpha_2-\alpha_1)/2}(\mathbb{R}^d)} \lesssim \|f\|_{M^{2,\infty}_{\alpha_1,s}(\mathbb{R}^d)},
\]

which proves the embedding

(2.11) 

\[
M^{2,\infty}_{\alpha_1,s}(\mathbb{R}^d) \subseteq M^{2,\infty}_{\alpha_2,s-d(\alpha_2-\alpha_1)/2}(\mathbb{R}^d).
\]

Next we observe that Young’s inequality and (1.10) for \( \{ \psi_i \} \) gives, for all \( i \) and any \( p \in [1, \infty] \),

(2.12) 

\[
\|\psi_i(D) f\|_{L^p} = \left\| \sum_{j \in \Omega_i} \mathcal{F}^{-1} \left( \psi_i \varphi_j \hat{f} \right) \right\|_{L^p} \lesssim \sum_{j \in \Omega_i} \|\varphi_j(D) f\|_{L^p}.
\]

This gives

\[
\|f\|_{M^{1}_{\alpha_2,s}} = \sum_i (\xi_i)^s \|\psi_i(D) f\|_{L^1} \lesssim \sum_i (\xi_i)^s \|\varphi_j(D) f\|_{L^1} \\
\times \sum_j (\eta_j)^s \|\varphi_j(D) f\|_{L^1} = \sum_i \sum_j (\xi_i)^s (\eta_j)^s \|\varphi_j(D) f\|_{L^1} \\
\lesssim \|f\|_{M^{1}_{\alpha_1,s}},
\]

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which proves the embedding
\[(2.13) \quad M_{\alpha_1,s}^1(\mathbb{R}^d) \subseteq M_{\alpha_2,s}^1(\mathbb{R}^d).\]
We also obtain from \((2.12)\)
\[
\|f\|_{M_{\alpha_2,s-d(\alpha_2-\alpha_1)}^1} = \sup_i \langle \xi_i \rangle^{s-d(\alpha_2-\alpha_1)} \|\psi_i(D) f\|_{L^1} 
\lesssim \sup_i \sum_{j \in \Omega_i} \langle \xi_i \rangle^{-d(\alpha_2-\alpha_1)} \langle \eta_j \rangle^s \|\varphi_j(D) f\|_{L^1} \lesssim \|f\|_{M_{\alpha_1,s}^1};
\]
which proves the embedding
\[(2.14) \quad M_{\alpha_1,s}^{\infty}(\mathbb{R}^d) \subseteq M_{\alpha_2,s-d(\alpha_2-\alpha_1)}^{\infty}(\mathbb{R}^d).\]
Again \((2.12)\) gives
\[
\|f\|_{M_{\alpha_2,s}^{\infty,1}} = \sum_i \langle \xi_i \rangle^s \|\psi_i(D) f\|_{L^\infty} \lesssim \sum_i \sum_{j \in \Omega_i} \langle \eta_j \rangle^s \|\varphi_j(D) f\|_{L^\infty} 
= \sum_j \sum_{i \in \Lambda_j} \langle \eta_j \rangle^s \|\varphi_j(D) f\|_{L^\infty} \lesssim \|f\|_{M_{\alpha_1,s}^{\infty,1}},
\]
which proves the embedding
\[(2.15) \quad M_{\alpha_1,s}^{\infty,1}(\mathbb{R}^d) \subseteq M_{\alpha_2,s}^{\infty,1}(\mathbb{R}^d).\]
Finally \((2.12)\) gives
\[
\|f\|_{M_{\alpha_2,s-d(\alpha_2-\alpha_1)}^{\infty}} = \sup_i \langle \xi_i \rangle^{s-d(\alpha_2-\alpha_1)} \|\psi_i(D) f\|_{L^\infty} 
\lesssim \sup_i \sum_{j \in \Omega_i} \langle \xi_i \rangle^{-d(\alpha_2-\alpha_1)} \langle \eta_j \rangle^s \|\varphi_j(D) f\|_{L^\infty} 
\lesssim \|f\|_{M_{\alpha_1,s}^{\infty}};
\]
which proves the embedding
\[(2.16) \quad M_{\alpha_1,s}^{\infty}(\mathbb{R}^d) \subseteq M_{\alpha_2,s-d(\alpha_2-\alpha_1)}^{\infty}(\mathbb{R}^d).\]
By Lemma \(2.1\) we have
\[(2.17) \quad M_{\alpha_1,s}^2(\mathbb{R}^d) = M_{\alpha_2,s}^2(\mathbb{R}^d).\]
The result now follows from interpolation between \((2.11), (2.13), (2.14), (2.15), (2.16)\) and \((2.17)\), and duality.

3. Sharpness of the embeddings

The notion of \(\alpha\)-covering is connected with the metric calculus presented in [12, Section 18.4]. Let \(0 \leq \alpha \leq 1\), and let \(g\) be the Riemannian metric
\[
g_\eta(\xi) = \frac{|\xi|^2}{\langle \eta \rangle^{2\alpha}}.
\]
If \(0 < r < 1\) then it follows by straightforward considerations that
\[
g_\eta(\xi - \eta) \leq r^2 \quad \implies \quad C^{-1} g_\eta(\zeta) \leq g_\xi(\zeta) \leq C g_\eta(\zeta), \quad \zeta \in \mathbb{R}^d,
\]
for some constant \( C \) which depends on \( r \) only. Hence \( g \) is a slowly varying metric in the sense of \cite[Def. 18.4.1]{embedding}, and (18.4.2) in \cite{embedding} is satisfied with \( c = r^2 \). The results in \cite{embedding} gives the following proposition.

**Proposition 3.1.** Let \( 0 \leq \alpha \leq 1 \) and \( 0 < r < 1 \). The following holds.

(i) For some sequence \( \{\xi_i\}_{i \in I} \subseteq \mathbb{R}^d \), the balls \( B_i = B(\xi_i, r\langle \xi_i \rangle^\alpha/2) \) constitute an \( \alpha \)-covering.

(ii) There are functions \( \psi_i \in C_c^\infty(\mathbb{R}^d) \), \( i \in I \), such that \( \text{supp} \psi_i \subseteq B_i \), \( 0 \leq \psi_i \leq 1 \), \( \sum_{i \in I} \psi_i = 1 \), and for every multiindex \( \beta \), there is a finite constant \( C_\beta > 0 \) such that

\[
\sup_{i \in I} \left( \langle \xi_i \rangle^{\alpha|\beta|} \|\partial^\beta \psi_i\|_{L^\infty} \right) \leq C_\beta.
\]

(iii) If \( Q = \{B_i\}_{i \in I} \) then \( \{\psi_i\}_{i \in I} \) is a \( Q \)-BAPU.

**Proof.** (i) and (ii) follow immediately from \cite[Lemma 18.4.4]{embedding} with \( \varepsilon < 1/8 \). Therefore, in order to prove (iii) it suffices to show

\[
\sup_{i \in I} \|\mathcal{F} \psi_i\|_{L^1} < \infty,
\]

which is a special case of the following Lemma \ref{lemma:3.2}.

**Lemma 3.2.** Let \( 0 \leq \alpha \leq 1 \) and suppose \( \{\psi_i\}_{i \in I} \subseteq C_c^\infty(\mathbb{R}^d) \) is a family of functions such that \( \text{supp} \psi_i \subseteq B(\xi_i, r\langle \xi_i \rangle^\alpha) \), \( i \in I \), for some sequence \( \{\xi_i\}_{i \in I} \subseteq \mathbb{R}^d \) and some \( r > 0 \), and for any multiindex \( \beta \) there is \( C_\beta > 0 \) such that

\[
\sup_{i \in I} \left( \langle \xi_i \rangle^{\alpha|\beta|} \|\partial^\beta \psi_i\|_{L^\infty} \right) \leq C_\beta.
\]

Then for \( p \in [1, \infty] \) there is a constant \( C_p > 0 \) such that

\[
\sup_{i \in I} \langle \xi_i \rangle^{-d\alpha/p'} \|\mathcal{F} \psi_i\|_{L^p} \leq C_p.
\]

**Proof.** Set

\[
\varphi_i(\xi) = \psi_i(\langle \xi \rangle^\alpha \xi + \xi), \quad i \in I.
\]

Then \( \text{supp} \varphi_i \subseteq B(0, r) \) for all \( i \in I \), and \cite{embedding} gives \( \|\partial^\beta \varphi_i\|_{L^\infty} \leq C_\beta \) for all \( i \in I \). If \( p < \infty \) and \( n > d/(2p) \) is an integer then integration by parts gives, for some constants \( c_\beta \),

\[
\|\mathcal{F} \varphi_i\|_{L^p}^p = (2\pi)^{-dp/2} \int_{\mathbb{R}^d} \langle x \rangle^{-2np} \left| \int_{\mathbb{R}^d} \varphi_i(\xi) \langle x \rangle^{2n} e^{-ix\xi} d\xi \right|^p dx
\]

\[
= (2\pi)^{-dp/2} \int_{\mathbb{R}^d} \langle x \rangle^{-2np} \left| \sum_{|\beta| \leq 2n} c_\beta \int_{\mathbb{R}^d} \partial^\beta \varphi_i(\xi) e^{-ix\xi} d\xi \right|^p dx
\]

\[
\lesssim \int_{\mathbb{R}^d} \langle x \rangle^{-2np} \left( \sum_{|\beta| \leq 2n} \|\partial^\beta \varphi_i\|_{L^1} \right)^p dx \lesssim 1
\]
for all \( i \in I \). If \( p = \infty \) the observations above give \( \| \mathcal{F} \varphi_i \|_{L^\infty} \leq (2\pi)^{-d/2}\| \varphi_i \|_{L^1} \lesssim 1 \) for all \( i \in I \). The result now follows from \( \| \mathcal{F} \varphi_i \|_{L^p} = \langle \xi \rangle^{\alpha / \beta} \| \mathcal{F} \varphi_i \|_{L^p} \).

Given an \( \alpha \)-covering and an \( \alpha \)-BAPU according to Proposition 3.1, the next lemma says that we may adjoin a sequence of balls to the covering, and modify the BAPU accordingly, without destroying the \( \alpha \)-covering and the \( \alpha \)-BAPU properties. A function indexed by the new index set equals one on a ball of radius proportional to \( \langle \xi_j \rangle^\alpha \) where \( \xi_j \) is the center of the support of the function. This will be useful in the proofs of the forthcoming sharpness results Propositions 3.4 and 3.5.

**Lemma 3.3.** Let \( 0 \leq \alpha \leq 1 \), \( 0 < r < 1 \), and let \( \{ B_i \}_{i \in I} \) and \( \{ \psi_i \}_{i \in I} \) be as in Proposition 3.1. Let \( J \) be a countable index set such that \( I \cap J = \emptyset \), and let \( \{ B_j \}_{j \in J} \) be balls such that \( B_j = B(\xi_j, r \langle \xi_j \rangle^\alpha / 2) \) where \( \xi_j \in \mathbb{R}^d \) for \( j \in J \), and \( B_j \cap B_k = \emptyset \), when \( j, k \in J \) and \( j \neq k \).

Then there are functions \( \varphi_i \in C_c^\infty(\mathbb{R}^d) \), \( i \in I \cup J \), such that the following is true:

(i) \( 0 \leq \varphi_i \leq 1 \), \( \text{supp} \varphi_i \subseteq B_i \) when \( i \in I \cup J \);

(ii) \( \varphi_j = 1 \) on \( B(\xi_j, r \langle \xi_j \rangle^\alpha / 4) \) for \( j \in J \);

(iii) \( \{ \varphi_i \}_{i \in I \cup J} \) is an \( \alpha \)-BAPU, and for each multiindex \( \beta \) there exists \( C_\beta > 0 \) such that

\[
\sup_{i \in I \cup J} \left( \langle \xi_i \rangle^{\alpha \beta} \| \partial^\beta \varphi_i \|_{L^\infty} \right) \leq C_\beta. \tag{3.3}
\]

**Proof.** Let \( \varphi \in C_c^\infty(\mathbb{R}^d) \), \( 0 \leq \varphi \leq 1 \), \( \text{supp} \varphi \subseteq B(0, r/2) \) and \( \varphi(\xi) = 1 \) for \( \xi \in B(0, r/4) \). We set

\[
\varphi_j(\xi) = \varphi(\langle \xi_j \rangle^{-\alpha}(\xi - \xi_j)) \quad \text{for} \quad j \in J
\]

and

\[
\varphi_i(\xi) = \psi_i(\xi) \prod_{j \in J} (1 - \varphi_j(\xi)) \quad \text{for} \quad i \in I.
\]

Then properties (i) and (ii) are satisfied. The estimate \( \sup_{j \in J} \langle \xi_j \rangle^{\alpha \beta} \| \partial^\beta \varphi_j \|_{L^\infty} \leq C_\beta \) for any multiindex \( \beta \) follows immediately. These estimates combined with (3.1) and straightforward considerations give \( \sup_{i \in I} \langle \xi_i \rangle^{\alpha \beta} \| \partial^\beta \varphi_i \|_{L^\infty} \leq C_\beta \) for all multindices \( \beta \). Thus (3.3) holds for all multindices \( \beta \). Likewise one can easily verify

\[
\sum_{i \in I \cup J} \varphi_i(\xi) = 1 \quad \forall \xi \in \mathbb{R}^d,
\]

as well as the fact that \( \{ B_i, B_j \}_{i \in I, j \in J} \) is an admissible \( \alpha \)-covering. To prove (iii) it thus suffices to observe that \( \sup_{j \in J} \| \mathcal{F} \varphi_j \|_{L^1} < \infty \) follows from \( \| \mathcal{F} \varphi_j \|_{L^1} = \| \mathcal{F} \varphi \|_{L^1} \), and that \( \sup_{i \in I} \| \mathcal{F} \varphi_i \|_{L^1} < \infty \) follows from (3.3) and Lemma 3.2. \( \square \)
We are now in a position to prove two results which show that the embeddings (2.9) in Theorem 2.3 are optimal, in most cases. This is a consequence of the following Propositions 3.4 and 3.5.

**Proposition 3.4.** If \( p, q \in [1, \infty], 0 \leq \alpha_1 \leq \alpha_2 \leq 1 \) and \( t, s \in \mathbb{R} \) then

\[
M_{\alpha_1, s}^{p, q} \subseteq M_{\alpha_2, t}^{p, q} \quad \implies \quad t \leq s + d(\alpha_2 - \alpha_1)\left(\frac{1}{q} - \frac{1}{p'}\right).
\]

**Proof.** We prove the result by showing that the assumption

\[
\varepsilon := t - s - d(\alpha_2 - \alpha_1)(1/q - 1/p') > 0
\]

implies that

\[
M_{\alpha_1, s}^{p, q} \subseteq M_{\alpha_2, t}^{p, q}
\]

cannot hold.

Let \( \{\varphi_j\}_{j \in J} \) be an \( \alpha_1 \)-BAPU constructed according to Proposition 3.1 and let \( \{\psi_i\} \) be an \( \alpha_2 \)-BAPU constructed according to Proposition 3.1 and modified according to Lemma 3.3. Then there exists an infinite index set \( I \) such that the following is true for some \( r > 0 \):

(i) If \( i_1, i_2 \in I \) and \( i_1 \neq i_2 \), then \( \text{supp} \; \psi_{i_1} \cap \text{supp} \; \psi_{i_2} = \emptyset \);

(ii) \( \psi_i(\xi) = 1 \) on \( B_i = B(\xi_i, r(\xi_i)^{\alpha_2}), \; \xi_i \in \mathbb{R}^d, \; i \in I \).

Let \( \vartheta \in C_c^\infty(\mathbb{R}^d) \) satisfy \( 0 \leq \vartheta \leq 1 \), \( \text{supp} \; \vartheta \subseteq B(0, r) \) and \( \vartheta(\xi) = 1 \) when \( \xi \in B(0, r/2) \), and define \( \vartheta_i(\xi) = \vartheta(\langle \xi_i \rangle^{-\alpha_2}(\xi - \xi_i)) \). Then \( \psi_i = 1 \) in \( \text{supp} \; \vartheta_i \). Let \( I' \subseteq I \) be any finite subset, let \( \{t_i\}_{i \in I'} \) be a sequence of nonnegative numbers, and set

\[
\hat{f}(\xi) = \sum_{i \in I'} t_i \vartheta_i(\xi) \in C_c^\infty(\mathbb{R}^d).
\]

Let \( q < \infty \). It follows from our choice of \( \vartheta_i \) that

\[
\|f\|_{M_{\alpha_2, t}^{p, q}} \geq \left( \sum_{i \in I'} (\langle \xi_i \rangle^t \|\vartheta_i(D)f\|_{L^p})^q \right)^{1/q}
= \left( \sum_{i \in I'} (\langle \xi_i \rangle^t \|\vartheta_i\|_{L^p})^q \right)^{1/q} \asymp \left( \sum_{i \in I'} (t_i \langle \xi_i \rangle^{t + d\alpha_2/p'})^q \right)^{1/q}.
\]

Next we estimate \( \|f\|_{M_{\alpha_1, s}^{p, q}} \). Set

\[
J_i = \{ j \in J ; \; \text{supp} \; \varphi_j \cap B_i \neq \emptyset \}, \quad i \in I',
\]

\[
I_j' = \{ i \in I' ; \; \text{supp} \; \varphi_j \cap B_i \neq \emptyset \}, \quad j \in J.
\]

By Lemma 2.2

\[
|J_i| \lesssim \langle \xi_i \rangle^{d(\alpha_2 - \alpha_1)}, \quad i \in I',
\]

\[
|I_j'| \lesssim 1, \quad j \in J.
\]

Denoting the center of the ball in which \( \varphi_j \) is supported by \( \eta_j \in \mathbb{R}^d \), this gives, using Hölder’s and Young’s inequalities, Lemma 2.2 and Lemma ...
3.2

\[
\|f\|_{M_{p,q}^{\alpha_1,s}} = \left( \sum_{j \in J} \langle \eta_j \rangle^{sq} \left\| \sum_{i \in I_j^t} t_i (\varphi_j \vartheta_i) \right\|_{L^p}^q \right)^{1/q} \\
\lesssim \left( \sum_{j \in J} \langle \eta_j \rangle^{sq} \sum_{i \in I_j^t} t_i^q \| F^{-1} (\varphi_j \vartheta_i) \|_{L^q}^q \right)^{1/q} \\
\lesssim \left( \sum_{i \in I} \sum_{j \in J} \langle \eta_j \rangle^{sq} t_i^q \| F^{-1} \partial_i \|_{L^q} \| \varphi_j \|_{L^q}^q \right)^{1/q} \\
\lesssim \left( \sum_{i \in I} \sum_{j \in J} \langle \xi_i \rangle^{sq + d \alpha_1 q/p'} t_i^q \right)^{1/q} \\
\lesssim \left( \sum_{i \in I} \left( t_i \langle \xi_i \rangle^{s + d(\alpha_2 - \alpha_1)/q + d \alpha_1/p'} \right)^{1/q} \right)^{1/q}.
\]

(3.6)

We may assume that \( I = N_0 \). Since \(|\xi_i| \to \infty\) as \( i \to \infty \), we may assume that \( \langle \xi_i \rangle \geq \langle i \rangle^{\frac{2}{q'}} \), by passing to a subsequence if necessary. If we set

\[
t_i := \langle i \rangle^{-\frac{2}{q'}} \langle \xi_i \rangle^{-s - d(\alpha_2 - \alpha_1)/q - d \alpha_1/p'}
\]

then (3.5) and (3.6) give a contradiction to (3.4), as \( |I'| \) is made arbitrarily large. This proves the result when \( q < \infty \). The case \( q = \infty \) is settled with slight modifications of the same proof. \( \square \)

**Proposition 3.5.** If \( p, q \in [1, \infty] \), \( 0 \leq \alpha_1 \leq \alpha_2 \leq 1 \) and \( t, s \in \mathbb{R} \) then

\[
M_{p,q}^{\alpha_1,s} \subseteq M_{p,q}^{\alpha_2,t} \implies t \leq s.
\]

**Proof.** We show that \( t > s \) implies that (3.4) does not hold.

Let \( \{\varphi_j\}_{j \in J}, \{\psi_i\} \) and \( I \) be as in the proof of Proposition 3.4 and let \( \vartheta_i = \vartheta(\xi - \xi_i) \in C_c^\infty(\mathbb{R}^d) \), where \( \vartheta \in C_c^\infty(\mathbb{R}^d) \), supp \( \vartheta \subseteq B(0, r) \) is the same as in the proof of Proposition 3.4. Let \( f \) be given by

\[
\hat{f}(\xi) = \sum_{i \in I'} t_i \vartheta_i(\xi) \in C_c^\infty(\mathbb{R}^d)
\]
for some suitable sequence \( \{ t_i \}_{i \in I'} \) where \( I' \subseteq I \) is finite. Let \( q < \infty \). We have

\[(3.7) \quad \| f \|_{M_{\alpha_2,t}^{p,q}} \geq \left( \sum_{i \in I'} \left( \| \xi_i \| \| \psi_i (D) f \|_{L^p} \right)^q \right)^{1/q} \]

\[
= \left( \sum_{i \in I'} \left( \langle \xi_i \rangle t_i \| \hat{\varphi_i} \|_{L^p} \right)^q \right)^{1/q} \times \left( \sum_{i \in I'} \left( \langle \xi_i \rangle t_i \right)^q \right)^{1/q}.
\]

In order to estimate \( \| f \|_{M_{\alpha_1,s}^{p,q}} \) we set

\[
J_i = \{ j \in J ; \text{ supp } \varphi_j \cap B(\xi_i,r) \neq \emptyset, \quad i \in I' \},
\]

\[
I'_j = \{ i \in I' ; \text{ supp } \varphi_j \cap B(\xi_i,r) \neq \emptyset, \quad j \in J. \}
\]

As in the proof of Lemma 2.2 it follows that

\[
\sup_{i \in I'} |J_i| < \infty, \quad \sup_{j \in J} |I'_j| < \infty, \quad \text{and } \langle \xi_i \rangle \asymp \langle \eta_j \rangle \text{ when } j \in J_i.
\]

As in the estimate (3.6) this gives, again using Hölder’s and Young’s inequalities and Lemma 3.2

\[(3.8) \quad \| f \|_{M_{\alpha_1,s}^{p,q}} = \left( \sum_{j \in J} \langle \eta_j \rangle^{q} \sum_{i \in I'_j} t_i \| \mathcal{F}^{-1} (\varphi_j \hat{\varphi}_i) \|_{L^p} \right)^{1/q} \]

\[
\lesssim \left( \sum_{j \in J} \langle \eta_j \rangle^{q} \sum_{i \in I'_j} t_i^{q} \| \mathcal{F}^{-1 -\theta} \|_{L^p} \| \mathcal{F}^{-1} \|_{L^1} \right)^{1/q} \]

\[
\lesssim \left( \sum_{j \in J} \sum_{i \in I', j \in J} \langle \xi_i \rangle \sum_{j \in J} \langle \eta_j \rangle^{q} t_i^{q} \| \mathcal{F}^{-1} \|_{L^p} \| \mathcal{F}^{-1} \|_{L^1} \right)^{1/q} \]

As before (3.7) and (3.8) give a contradiction to (3.4). The case \( q = \infty \) follows in the same manner. \( \square \)

A combination of (2.3), Propositions 3.4 and 3.5 and duality give the earlier mentioned optimality result concerning Theorem 2.3.

**Corollary 3.6.** Let \( p, q \in [1, \infty], \quad s \in \mathbb{R} \) and \( 0 \leq \alpha_1 \leq \alpha_2 \leq 1 \). If \( 1/p \leq \max(1/2, 1/q) \) then

\[
M_{\alpha_1,s}^{p,q} \subseteq M_{\alpha_2,t}^{p,q} \quad \Rightarrow \quad t \leq s + d(\alpha_2 - \alpha_1) \theta_2(p,q).
\]

If \( 1/p \geq \min(1/2, 1/q) \) then

\[
M_{\alpha_2,t}^{p,q} \subseteq M_{\alpha_1,s}^{p,q} \quad \Rightarrow \quad t \geq s + d(\alpha_2 - \alpha_1) \theta_1(p,q).
\]
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