On uniqueness of Heine–Stieltjes polynomials for second order finite-difference equations

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Abstract
A second order finite-difference equation has two linearly independent solutions. It is shown here that, like in the continuous case, at most one of the two can be a polynomial solution. The uniqueness in the classical continuous Heine–Stieltjes theory is shown to hold under broader hypotheses than usually presented. A difference between regularity condition and uniqueness is emphasized. Consistency in our uniqueness results is also checked against one of the Shapiro problems. An intrinsic relation between the Heine–Stieltjes problem and the discrete Bethe Ansatz equations allows one to immediately extend the uniqueness result from the former to the latter. The results have implications for nondegeneracy of polynomial solutions of physical models.

Keywords: finite-difference equations, Heine–Stieltjes polynomials, uniqueness of solutions, regularity condition, Shapiro problem, discrete Bethe Ansatz equations, $h$-analogue of Abel’s theorem

1. Introduction
Let $A(x)$ and $B(x)$ be given polynomials of degrees $m + 1$ and $m$, respectively. The subject of the classical Heine–Stieltjes theory is to determine a polynomial $V(x)$ of degree $m - 1$ such that the second-order differential equation

$$A(x)y'' + 2B(x)y' + V(x)y = 0$$

has a solution which is a polynomial of a preassigned degree $n$ [1–5]. Assume with Stieltjes [2] that $A(x)$ has real unequal roots,

1 A translation of the often quoted passage in [1] can be found in [6].
\[
A(x) = (x-a_0)(x-a_1)...(x-a_m), \quad a_0 < a_1 < \cdots < a_m, \quad (2)
\]
and
\[
B(x) = \frac{A(x)}{x-a_0} = \frac{\rho_0}{x-a_0} + \frac{\rho_1}{x-a_1} + \cdots + \frac{\rho_m}{x-a_m}, \quad \rho_\nu > 0, \quad \nu = 0, 1, 2, \ldots, m. \quad (3)
\]

This is equivalent to the assumption that the zeros of \(A(x)\) alternate with those of \(B(x)\) and that the leading order coefficients of \(A(x)\) and \(B(x)\) have the same sign. Under the above conditions, the basic properties of polynomial solutions are [2, 3, 5, 6]:

- there are exactly
  \[
  \sigma_{nm} = \binom{n + m - 1}{n}
  \]
  polynomials \(V(x)\), which are called van Vleck polynomials [6–9].
- Equation (1) cannot have two polynomial solutions linearly independent of each other.
- If \(y\) is a polynomial solution, \(y \neq 0\) at \(x = a_\nu\).
- All the zeros of \(y\) are distinct.
- The zeros of \(y\) lie in the interval \([a_0, a_m]\).

The case with \(m = 1\) corresponds to the hypergeometric differential equation, while the case with \(m = 2\) corresponds to the Heun equation [8]. For \(m \leq 3\) polynomial solutions mostly characterize quasi-exactly-solvable (QES) models [10–16], although not all such polynomial solutions are exhausted by the QES models [14]. General (extended) Heine–Stieltjes polynomials were often studied in connection with a special Lipkin–Meshkov–Glick model corresponding to the standard two-site Bose–Hubbard model [8, 9].

Compared to the continuous case of equation (1), very little is known about general properties of polynomial solutions of a linear homogeneous second-order finite-difference equation
\[
\Delta_h y(x) + r(x)\Delta_h y(x) + u(x)y(x + h) = 0, \quad (4)
\]
where the first difference quotient of \(y(x)\), or Nörlund’s operator \(\Delta_h\) [17, 18], is defined here in usual sense
\[
\Delta_h y(x) = \frac{y(x + h) - y(x)}{h}.
\]
The finite-difference equation (4) can be disguised in further equivalent forms
\[
g(x)\Delta_h^2 y(x) + [r(x) + hu(x)]\Delta_h y(x) + u(x)y(x) = g(x)y(x + 2h) + \left[ hr(x) + h^2 u(x) - 2g(x) \right] y(x + h) + [g(x) - hr(x)]y(x) = 0.
\quad (5)
\]
(The first one follows on making use of the identity \(ay(x + h) = ha\Delta_h y(x) + ay(x)\).) Last but not the least, if \(y(x) = \prod_{j=1}^{\nu} (x - x_j)\) is a polynomial solution, then equation (5) leads at any zero \(x_k\) of \(y(x)\) to a discrete Bethe Ansatz equation (see section 5 of [19])
\[
\prod_{j=1}^{\nu} \frac{(x_k - x_j + h)}{(x_k - x_j - h)} = \frac{hr(x_k - h) - g(x_k - h)}{g(x_k - h)}. \quad (6)
\]
The motivation to study polynomial solutions of finite-difference equations has got a boost after it was demonstrated that physical models with a discrete nondegenerate spectrum can be characterized in terms of orthogonal polynomials of a discrete variable and their weight function [20–25]. The latter applies to all problems where Hamiltonian operator is a self-adjoint extension of a tridiagonal Jacobi matrix of deficiency index (1, 1) [26]. For instance a displaced harmonic oscillator can be characterized in terms of the classical Charlier polynomials and the Rabi model by a norm preserving deformation of the Charlier polynomials [22, 23]. Some earlier applications of classical discrete polynomials in physics not related to Lanczos-Haydock scheme [20, 21] have been given by Lorente [27]. He showed that the respective orthonormal Kravchuk and Meixner functions are related to a quantum harmonic oscillator and the hydrogen atom of discrete variable, and that the Hahn polynomials are related to Calogero-Sutherland model on the lattice.

Unfortunately, only the hypergeometric case $m = 1$, where the polynomial coefficients $g(x)$, $r(x)$, $u(x)$ have degrees 2, 1, 0, respectively, has been studied exhaustively within the realm of classical orthogonal polynomials of a discrete variable [28–31]. Generalized Bochner theorem for finite-difference equations has been dealt with in [32]. An important step forward has been achieved by Turobiner [10–13] within the realm of QES equations [10–15]. The latter yield a specific subclass of finite-difference equations (4) where the polynomial coefficients $g(x)$, $r(x)$, $u(x)$ have degree at most four.

The motivation of present work is to translate the properties of the classical continuous Heine–Stieltjes theory into the realm of finite-difference equations. As in the continuum case, a second order finite-difference equation (4) has two linearly independent solutions for a fixed triplet of polynomial coefficients $g(x)$, $r(x)$, $u(x)$. Here we derive the conditions under which two linearly independent polynomial solutions of equation (4) are forbidden, i.e. the polynomial solutions of general second-order finite-difference equation (4) are unique (see theorems 1 and 2). As a by product, an $h$-analogue of Abel’s theorem for the Heine–Stieltjes problem is derived, which yields an explicit analytic expression of finite difference Wronskian, or Casoratian, $W_{n}(x)$ in terms of a rational function involving products of generalized gamma function $\Gamma_{h}(x)$ in both numerator and denominator. A comparison with the classical hypergeometric equation is provided in section 2.1. Using an intrinsic relation between the Heine–Stieltjes problem problem (4) and the discrete Bethe Ansatz equations (6), the uniqueness result is extended from the former to the latter in section 2.2. The results are discussed from different perspectives in section 3. Section 3.1 shows that the uniqueness in our sense ensures uniqueness even if the regularity condition for the Hahn class of hypergeometric orthogonal polynomials (see section 2.3 of [31]) does not preclude two polynomial solutions. A comparison with one of the Shapiro problems is discussed in section 3.2. There are two basic ways how to make use of the uniqueness theorems for Heine–Stieltjes polynomials. First, they yield a straightforward proof of the nondegeneracy of QES levels which yield the so-called exceptional spectrum of physical models [10–13, 15, 21, 25], the proof of which is more involved by other means (see [16, 21, 25]). Second, they serve as no-go theorems in certain exceptional cases—see section 3.

2. Uniqueness

For each $x \in \mathbb{R}$ one can define the lattice $\Lambda_{h}(x) = \{ x + kh \mid k \in \mathbb{Z} \}$. For a given $x_{0} \in \mathbb{R}$ the second order finite-difference equation (5) is seen to connect the values of $y(x)$ at the points of $\Lambda_{h}(x_{0})$. The function constant on each $\Lambda_{h}(x_{0})$ is called $h$-periodic function. Two functions $y_{1}$ and $y_{2}$ are called linearly dependent in a finite-difference sense if there are $h$-periodic functions $C_{1}$ and $C_{2}$ such that $C_{1}(x)y_{1}(x) + C_{2}(x)y_{2}(x) \equiv 0$. Otherwise the functions $y_{1}$ and
\( y_2 \) are called linearly independent. We shall use repeatedly the following elementary argument: If \( y(x) \) is known to be a polynomial of degree not larger than \( N \) and, at the same time, to vanish in at least \( N + 1 \) different points, then \( y(x) \equiv 0 \). In what follows we shall consider the second order finite-difference equation (5) with polynomial coefficients only. Using the argument, one finds immediately that:

- **(P1)** If a polynomial \( y(x) \) solves equation (5) on an infinite subset of \( \Lambda_0(x_0) \), then \( y(x) \) solves it for all \( x_0 \in \mathbb{R} \).
- **(P2)** If a linear combination \( C_1 y_1(x) + C_2 y_2(x) \) of two polynomials vanishes on an infinite subset of \( \Lambda_0(x_0) \), then it vanishes for all \( x \in \mathbb{R} \).

The latter implies that linear dependence of two polynomial solutions \( y_1 \) and \( y_2 \) in a finite-difference sense reduces to the linear dependence in conventional sense, i.e. with \( C_1 \) and \( C_2 \) being independent of \( x \).

The following theorem, and theorem 2 below, encompass all QES equations on a uniform linear-type lattice \([10–13]\) and all classical orthogonal polynomials of a discrete variable \([28–31]\).

**Theorem 1.** Let the second-order finite-difference equation (4) has polynomial coefficients such that \( g(x) \) and \( g(x) - hr(x) \) have real roots,

\[
g(x) = (x - b_0)(x - b_1)\ldots(x - b_m), \quad b_0 < b_1 < \ldots < b_m,
\]

\[
g(x) - hr(x) = (x - a_0)(x - a_1)\ldots(x - a_m), \quad a_0 < a_1 < \ldots < a_m.
\]

For each root \( a_j \) define a uniform lattice \( \Lambda_{a_j} := \{ a_j + kh \mid k \in \mathbb{N}_0 \} \), which extends to the right of the root \( a_j \). It is not excluded that \( a_j, \ b_j \in \Lambda_{a_j} \) for \( a_j, \ b_j > a_j \). Assume further that there is at least a single \( \Lambda_{a_j} \) which does not contain any root of \( g(x) \). Then equation (4) cannot have two polynomial solutions \( y_1 \) and \( y_2 \) linearly independent of each other.

**Proof.** For any two functions \( y_1 \) and \( y_2 \) the Leibniz’s theorem of finite-difference calculus (pp 34–35 of Milne-Thomson \([18]\)) implies

\[
\Delta_h \left[ y_1(x + h) \cdot \Delta_h y_2(x) - \Delta_h y_1(x) \cdot y_2(x + h) \right] = y_1(x + h) \cdot \Delta_h^2 y_2(x) - \Delta_h^2 y_1(x) \cdot y_2(x + h).
\]

Hence for two nontrivial solutions \( y_1 \) and \( y_2 \) of the finite-difference equation (4) we have

\[
g(x) \Delta_h \left[ y_1(x + h) \cdot \Delta_h y_2(x) - \Delta_h y_1(x) \cdot y_2(x + h) \right] + r(x) \left[ y_1(x + h) \cdot \Delta_h y_2(x) - \Delta_h y_1(x) \cdot y_2(x + h) \right] = 0.
\]

The latter is of the form

\[
\Delta X(x) = - \frac{r(x)}{g(x)} X(x) \quad \text{or} \quad X(x + h) = \frac{g(x) - hr(x)}{g(x)} X(x) = R(x) X(x),
\]

where \( X \) stands for the square bracket in equation (8), which can be identified with a finite difference Wronskian, or Casoratian, \([33]\).
The hypotheses of theorem 1 determine $R(x)$ as a rational function with zeros and poles on the real axis

$$R(x) = \frac{g(x) - hr(x)}{g(x)} = \frac{\prod_{j=0}^{m} (x - a_j)}{\prod_{j=0}^{m} (x - b_j)}.$$  

(11)

Now if $\Lambda_a$ does not contain any zero of $g(x)$, i.e. $R(x)$ is not singular on $\Lambda_a$, then, in virtue of $R(a_j) = 0$, the first-order recurrence (9) implies $X(x) \equiv 0$ for all $x \in \Lambda_{a_l + h}$. In other words, for each $x_j \in \Lambda_{a_l + h}$ there are $C_1(x_j)$ and $C_2(x_j)$ not both zero, such that $C_1(x_j) \gamma_1(x_j) + C_2(x_j) \gamma_2(x_j) = C_1(x_j) \gamma_1(x_j + h) + C_2(x_j) \gamma_2(x_j + h) = 0$. Taking $x_j = a_l + h$, the linear combination $y(x) := C_1(x_j) \gamma_1 + C_2(x_j) \gamma_2(x)$ is a solution of equation (4) on $\Lambda_{a_l + h}$ which satisfies $y(x_j) = y(x_j + h) = 0$. Considering the latter as the initial values of the *Cauchy problem* for the recursive form (5) of equation (4), one has $y(x) \equiv 0$ on $\Lambda_{a_l + h}$. Because $g(x) \neq 0$, the solutions of the Cauchy problem for equation (5) are *uniquely* determined by the initial values [33], and hence $C_1$ and $C_2$ are constants on entire $\Lambda_{a_l + h}$. In virtue of the elementary argument (P2), the linear combination $C_1 \gamma_1(x) + C_2 \gamma_2(x)$ vanishes for all $x \in \mathbb{R}$, i.e. $\gamma_1$ and $\gamma_2$ are *linearly dependent* in the conventional sense.

**Remark.** On considering equation (9) as a downward recurrence $X(x) = R^{-1}(x)X(x + h)$, an alternative version of Theorem 1 follows which guarantees the uniqueness, provided that there is at least a single $\Lambda_{b_l}$ which does not contain any root of $g(x) - hr(x)$. Here $\Lambda_{b_l}$ is defined for each root $b_l$ as a uniform lattice which extends to the left of the root $b_l$, $\Lambda_{b_l} := [b_l - kh | k \in \mathbb{N}_0]$.

**Theorem 2.** Let us consider the second-order finite-difference equation (4) with the polynomial coefficients as in theorem 1. Assume further that there is at least a single $\Lambda_{a_l}$ which contains more roots (e.g. the single root $a_l$) of $g(x) - hr(x)$ than the roots of $g(x)$ (e.g. none of the roots $b_l$ of $g(x)$). Then equation (4) cannot have two polynomial solutions $\gamma_1$ and $\gamma_2$ linearly independent of each other.

Before giving the proof of theorem 2, it is expedient to provide an $h$-analogue of Abel’s theorem which yields an explicit analytic expression of $X(x)$ in terms of a rational function involving products of $\Gamma_h$ in both its numerator and denominator. The $h$-extension of the gamma function $\Gamma_h(x)$ is introduced through the functional equation $\Gamma_h(x + h) = x \Gamma_h(x)$ (see section 9.66 of [18]; appendix B).

**Lemma 1.** For any rational $R(x)$ of the form (11), the solution $X(x)$ of the first-order finite-difference equation (9) is either identically zero or

$$X(x) = \text{const} \times \frac{\prod_{j=0}^{m} \Gamma_h(x - a_j)}{\prod_{j=0}^{m'} \Gamma_h(x - b_j)}.$$ 

(12)

Provided that the ratio $\kappa$ of the leading polynomial coefficient of $g(x) - hr(x)$ to that of $g(x)$ is $\kappa = 1$, the rhs of equation (12) will acquire an additional multiplication factor (see
equation (A.6)) and becomes

\[ X(x) = \text{const} \times \kappa^{-t/2} \frac{\prod_{j=0}^{m} \Gamma_h(x - a_j)}{\prod_{j=0}^{m} \Gamma_h(x - b_j)}. \]  

(13)

**Proof.** First, equation (9) is recast as

\[ \Delta \ln X = \frac{1}{h} \ln \left( \frac{g(x) - hr(x)}{g(x)} \right), \]

which has the form of the first-order finite-difference equation (A.1). Its solution can be expressed in terms of Nörlund’s principal solution [17, 18], an elegant, but nowadays largely forgotten, tool of integrating finite-difference equations (see appendix A for a brief summary and definition), as

\[ X(x) = \exp \left[ \sum_{t} \frac{1}{h} \ln \left( \frac{g(t) - hr(t)}{g(t)} \right) \Delta t \right]. \]

(14)

Note in passing that use of a partial fraction decomposition (3) of the fraction in the integrand in the exponent of equation (14), as in the continuous case of Stieltjes [2] and further elaborated in section 6.81 of [5], would not bring us any further. Instead it is expedient to substitute the respective products (7) into equation (14) and use the logarithm there to split the resulting ratio into a sum of individual logarithms \( \ln(t - a_i) \) and \(-\ln(t - b_j)\) corresponding to the roots in equation (7). Each such a logarithm term integrates to a corresponding generalized gamma function \( I_h \) (see equation (B.2) of appendix B; section 9.66 of [18]). The latter recipe enables one to express (14) as in equation (13). The transition from (14) to (13) is similar to that used by Lancaster [28] in arriving from his equation (29) to his equations (30)–(33).

**Proof of theorem 2.** If \( X(x) \) of two linearly independent solutions in equation (9) is not identically zero, the hypotheses of theorem 2 imply that \( X \) is necessarily singular for some its argument value, which is impossible if \( y_1 \) and \( y_2 \) are polynomials. Indeed, the hypotheses of theorem 2 ensure that there is at least a single \( \Lambda_{a_i} \) which contains more roots (e.g. the single root \( a_i \)) of \( g(x) - hr(x) \) than the roots of \( g(x) \) (e.g. none of the roots \( b_j \) of \( g(x) \)). Unless \( X(x) \) is identically zero, lemma 1 determines the analytic form of \( X(x) \) to be either (12) or (13). Now \( \Gamma_h(x - a_i) \) has a simple pole at \( x = a_i \) (see appendix B). If there is \( a_i < a_0 \in \Lambda_{a_i} \), then also \( \Gamma_h(x - a_i) \) has a simple pole at \( x = a_i \). If there is \( b_j \in \Lambda_{a_i} \), some of the simple poles of \( \Gamma_h(x - a_i) \) and \( \Gamma_h(x - a_i) \) at \( x = a_i \) in the numerator on the rhs of equation (13) could be canceled by the simple pole of the \( \Gamma_h(x - b_j) \) at \( x = a_i \) in the denominator on the rhs of equation (13). Nevertheless, the hypotheses of theorem 2 guarantee that at least one of the simple poles of \( \Gamma_h \)'s in the numerator is not compensated by the simple pole of \( \Gamma_h(x - b_j) \) in the denominator. Then \( X(x) \) tends to infinity for \( x \to a_i \). However, as a discrete Wronskian of two polynomial solutions, \( X(x) \) cannot tend to infinity at any finite \( x \in \mathbb{R} \). Of course, the latter does not hold for general nonpolynomial solutions. Thus, as in the continuum case of section 6.81 of [5], we have a contradiction, unless, of course, \( X \equiv 0 \).
2.1. Classical hypergeometric equation

As an example, consider the classical hypergeometric equation [28–31]

\[
(ax^2 + bx + c)\Delta_h^2 y(x) + (dx + f)\Delta_h y(x) + \lambda y(x + h) = 0.
\]

(15)

A necessary and sufficient condition for the existence of a polynomial solution of equation (15) is that a characteristic polynomial,

\[
\theta(z) := az(z - 1) + dz + \lambda,
\]

has a nonnegative integer root (see the \( n = 2 \) case of theorem 2 of [28]). If there is a polynomial solution of degree \( n \), then \( \theta(n) = 0 \). The latter is equivalent to

\[
\lambda + nd + n(n - 1)a = 0, \quad \text{or} \quad \lambda_n = -n(n - 1)a - nd, \quad n = 0, 1, 2, \ldots.
\]

Equation (15) is a special case of the eigenvalue problems for the Hahn class of orthogonal polynomials [29, 31]. In the latter case the regularity condition says that all eigenspaces of the hypergeometric eigenvalue problem are one dimensional if and only if \( \lambda_n \neq \lambda_l \) for \( l \neq n \) in the set of numbers \( \{\lambda_n\}_{n=0}^\infty \) defined by equation (15), or if and only if \( q[n] + d \neq 0 \) (see section 2.3 of [31]). Here \([-1] = -1/q, \quad [0] = 0, \quad [n] = \sum_{k=0}^{n-1}q^k, \quad n \geq 1, \) and \( q \in \mathbb{R} \setminus \{-1, 0\} \) is the Hahn parameter (for the uniform linear lattice in our case \( q = 1 \) and \( [n] = n \)). However, the regularity condition does not exclude the corresponding eigenspace to be, for instance, two dimensional for \( \lambda_n = \lambda_l \) with \( l \neq n \). The latter is precluded by the following Corollary.

Corollary 1. Polynomial solutions of the second-order finite-difference hypergeometric equation equation (15) are nondegenerate, i.e., for a given eigenvalue \( \lambda \) there is at most a single solution to equation (15).

If \( d = ka \), then \( \lambda_n = -n(n - k - 1)a \) and \( \lambda_n \) may equal \( \lambda_l \) for some \( l \neq n \). For instance, \( \lambda_l = \lambda_k = k \) for \( n = 1, k \). If \( k > 1 \) there is thus, under the hypotheses of theorems 1 and 2, no polynomial solution of degree \( k \).

2.2. Discrete Bethe Ansatz equations

Using an intrinsic relation between the Heine–Stieltjes theory and the discrete Bethe Ansatz equations one can immediately arrive at the following result.

Theorem 3. Provided that the Heine–Stieltjes problem has unique polynomial solution, the corresponding discrete Bethe Ansatz equations (6) have also a unique polynomial solution up to permutations of zeros \( x_k \)'s.

Proof. A solution \( y(x) = \sum_{i=0}^{n}y_i x^i \) to the discrete Bethe Ansatz equations (6) implies that the second-order difference equation (4) is satisfied at the \( n \) points \( x_1, x_2, \ldots, x_n \). The necessary condition that \( y(x) = \sum_{i=0}^{n}y_i x^i \) solves equation (4) is the vanishing of the leading \( n \)th degree. The latter requires that the sum of the coefficients of the leading degree of the polynomials \( g(x), \quad [hr(x) + h^2u(x) - 2g(x)], \) and \([g(x) - hr(x)]\) in the recurrence form (5) of equation (4) vanishes. In the hypergeometric case this is the condition (16). If the polynomial coefficients of equation (4) are assumed to satisfy the necessary condition, the lhs of equation (4) becomes a polynomial of one less, i.e. \( (n - 1) \)th degree. By the elementary
argument, if a polynomial in $x$ of degree $n - 1$, that can vanish only at $n - 1$ different points, vanishes at the $n$ distinct points $x_1, x_2, \ldots, x_n$, then it must vanish identically. Thus the lhs of equation (4) vanishes identically. This leads to a second-order difference equation whose polynomial solutions are unique.

Ismail et al have earlier shown that the solution to the discrete Bethe Ansatz equations (6) with the right-hand side derived from the Meixner $M_n(x; \beta, c)$ and the Hahn polynomials $Q_n(x; \alpha, \beta, N)$ are unique up to permutations (see section 5 of [19]). Theorem 3 extends the results of Ismail et al (see section 5 of [19]) to the general case. Some special cases when the uniqueness may break down are discussed in section 3.2.

3. Discussion

Our uniqueness theorems encompass all QES equations on a uniform linear-type lattice [10–13] and all classical orthogonal polynomials of a discrete variable [28–31]. The hypotheses of our uniqueness theorems look rather different from those in the classical continuous Heine–Stieltjes theory [1–5]. In the finite-difference case, the respective $g(x)$ and $g(x) - hr(x)$ can be identified as the coefficients of $y(x + 2h)$ and $y(x)$ of (4) recast in recurrence form (5) of equation (4). Unlike the continuous case of [2, 5] (i.e. with $\Delta$ in equation (4) replaced with ordinary derivatives (see equation (1)), one does not assume that the zeros of $g(x)$ alternate with those of $r(x)$ (see equations (2) and (3)). The hypotheses of theorems 1 and 2 are also silent about relative degrees of the polynomial coefficients $g(x), r(x), u(x)$.

However, the above differences are mostly only apparent, until one realizes that already in the classical continuous Heine–Stieltjes theory the assumptions that (i) the zeros of $A(x)$ alternate with those of $B(x)$ and that (ii) the leading order coefficients of $A(x)$ and $B(x)$ have the same sign, are not necessary for the uniqueness of solutions. Indeed, one can multiply both the numerator and denominator in $B(x)/A(x)$ on the lhs of equation (3) with the same polynomial factor $(x - \gamma)^n, n_l \geq 1$, without changing the rhs of equation (3), and hence the reasoning leading to the uniqueness. It is also not necessary that all $\rho_i > 0$ as in equation (3). (The latter has been recognized as late as 2000 by Dimitrov and Van Assche [34].)

A broader sufficient condition for the Wronskian $W \{y_1, y_2 \}$ to diverge to infinity is that there is merely at least one $\nu$ such that $\rho_\nu > 0$ and $a_\nu$ is different from all other $b_\nu$’s. The latter points could be illustrated for a continuous hypergeometric analogue of equation (15),

$$\left( ax^2 + bx + c \right) y''(x) + (dx + f)y'(x) + \lambda y(x) = 0. \quad (17)$$

3.1. Regularity condition versus uniqueness

The regularity condition of the eigenvalue problems for the Hahn class of orthogonal polynomials does not answer what happen if $\lambda_n = \lambda_l$ for $l \neq n$ in the set of numbers $\{\lambda_n\}_{n=0}^\infty$ defined by equation (15). Will the eigenspace corresponding to $\lambda_n = \lambda_l$ be zero-, one-, or two-dimensional? The question of uniqueness and existence of the polynomial solutions of the hypergeometric equation (17) reduces to solving Lesky’s downward TTRR (see equation (3) in [35])

$$(n - k)[(n + k - 1)a + d]a_{nk} = (k + 1)[(k + 2)c a_{n,k+2} + (kb + f)a_{n,k+1}] \quad (18)$$
for the coefficients $a_{nk}$ of the polynomial solution of the $n$th degree,

$$y(x) = a_{nn} x^n + a_{n,n-1} x^{n-1} + \cdots + a_{n0}.$$  \hfill (19)

The TTRR runs downward for $k = n - 1, n - 2, \ldots, 0$, with the initial condition $a_{n,n+1} \equiv 0$. Without any loss of generality one can assume $a_{nn} = 1$. With the initial conditions on $a_{n,n+1}$ and $a_{nn}$ being fixed, any other not linearly dependent solution has to have $a_{n,n+1} = 0$ for $W_k(y)$ (see equation (10)) of the Cauchy problem for the TTRR (18) to be nonzero. This is impossible for a polynomial solution of the $n$th degree, which implies uniqueness of the polynomial solution of the $n$th degree, provided it exists (i.e. $(n + k - 1)a + d \neq 0$).

The condition (16) is valid both in the continuous and discrete cases. Thus for $d = -ka$ some of $\lambda_k$ may equal $\lambda_l$ also in the continuous case (e.g. $\lambda_k = \lambda_k = k$ for $n = 1, k$). Let $[x]$ denotes the smallest integer not less than $x$, or the ceiling function. Then, unless some additional conditions are satisfied, Lesky’s TTRR (18) does not have any solution for $d = -ka$ and $n \in \{ \lfloor (k + 2)/2 \rfloor, k + 1 \}$. Obviously the uniqueness of polynomial solutions persists even though the above assumption (ii) is not satisfied.

3.2. Shapiro problem

The additional conditions under which Lesky’s TTRR (18) has a solution for any degree $n$ and when the uniqueness of polynomial solutions breaks down are formulated separately for $k$ even and odd. The latter is related to the problem of describing when a linear ordinary differential equation with polynomial coefficients admits at least 2 polynomial solutions, which is the first of five open problems listed by Shapiro [6]. An exhaustive answer in the special case $2B(x) = -A'(x)$ has been obtained by Eremenko and Gabrielov [36]. The following discussion is limited to the hypergeometric equation (17) but is not constraint to $2B(x) = -A'(x)$.

For $d = -ka$ and even $k = 2t > 0$ (Lesky’s special case 2), uniqueness persists unless $f = -tb$. Then $df + f = -2tax - tb$, or $2B(x) = df + f = -tA'(x)$ in the notation of equation (1), and hence all the residues $\rho_j$ of the ratio $B(x)/A(x) = -tA'(x)/[2A(x)]$ in equation (3) are necessarily negative. For odd $k = 2t - 1 > 0$ (Lesky’s special case 3), the ratio $B(x)/A(x) = -t/(x - a_0) - (t - 1)/(x - a_1)$, i.e. none of the residues $\rho_j$ of the ratio $B(x)/A(x)$ is positive. Thus not just any algebraic dependence of $A(x)$ and $B(x)$ but only a particular one [6] leads to that the uniqueness of polynomial solutions ceases to hold and there are possible two linearly independent solutions of the continuous equation (17) for the same value of $\lambda$.

4. Conclusions

We have established sufficient conditions (theorems 1 and 2) for the uniqueness of polynomial solutions of second order finite-difference equations. They encompass all classical orthogonal polynomials of a discrete variable [28–31] and all QES equations on a uniform linear-type lattice [11–13]. An $h$-analogue of Abel’s theorem for the Heine–Stieltjes problem was derived, which yields an explicit analytic expression of finite difference Wronskian, or Casoratian, $W_0(x)$ in terms of a rational function involving products of generalized gamma function $\Gamma_h(x)$ in both numerator and denominator. The latter was facilitated by Nörlund’s principal solution $\sum_{j}^{\infty}$ [17, 18]. It suffices to know Nörlund’s principal solution $\sum_{j}^{\infty}$ only for a constant (see equation (A.6)) and a logarithm (see equation (B.2)) to deal with a large set of finite difference problems (e.g. [28]). Using an intrinsic relation between the
Heine–Stieltjes problem (4) and the discrete Bethe Ansatz equations (6), theorem 3 extended the uniqueness of polynomial solutions of the discrete Bethe Ansatz equations of Ismail et al (see section 5 of [19]) to the general case. The uniqueness in the classical continuous Heine–Stieltjes theory was shown to hold under broader hypotheses than usually presented [2, 3, 5]. A difference between the regularity condition and uniqueness was emphasized. An extension of the results to a general lattice and a second-order finite-difference equation (4) with $\Delta$ being replaced by the more general Hahn operator [29, 31] is dealt with in a forthcoming publication [37]. An open question remains if it is possible to translate also the remaining properties of the classical continuous Heine–Stieltjes theory into the realm of finite-difference equations.

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Appendix A. Nörlund’s principal solution

That particular solutions of the given equation

$$\Delta u(x) = \phi(x), \quad (A.1)$$

always exist is seen (in the case of the real variable) by considering that $u(x)$ being arbitrarily defined at every point of the interval $0 \leq x < h$, the equation defines $u(x)$ for every point exterior to this interval. The expression

$$f(x) = A - h[\phi(x) + \phi(x + h) + \phi(x + 2h) + \phi(x + 3h) + \cdots]$$

$$= A - h \sum_{s=0}^{\infty} \phi(x + sh),$$

where $A$ is constant, is a formal solution of the difference equation, since

$$f(x + h) = A - h[\phi(x + h) + \phi(x + 2h) + \phi(x + 3h) + \cdots],$$

and therefore $f(x + h) - f(x) = h\phi(x)$. However, such solutions are in general not analytic.

Nörlund [17] has succeeded in defining a principal solution which has specially simple and definite properties. In particular, when $\phi(x)$ is a polynomial so is the principal solution. If for $A$ we write $\int_{c}^{\infty} \phi(t) \, dt$, and if this infinite integral and the infinite series both converge, Nörlund defines the principal solution of the difference equation, or sum of the function $\phi(x)$, as

$$F(x) = \sum_{c} \phi(z) \Delta z = \int_{c}^{\infty} \phi(t) \, dt - h \sum_{s=0}^{\infty} \phi(x + sh). \quad (A.2)$$

The principal solution thus defined depends on an arbitrary constant $c$. As an example, consider [17, 18]
\[ \Delta u(x) = e^{-x}, \]

\( x \) and \( h \) being real and positive. Here

\[
F(x) = \sum_{c}^{x} e^{-z} \frac{\Delta z}{h} = \int_{c}^{\infty} e^{-t} \, dt - h \sum_{s=0}^{\infty} e^{-s \cdot sh} = e^{-c} - \frac{he^{-x}}{1 - e^{-h}}, \tag{A.3}
\]

after evaluating the integral, and summing the geometrical progression.

The necessary and sufficient conditions for the existence of the sum \( F(x) \) as defined above are the convergence of the integral and of the series. In general, neither of these conditions is satisfied and the definition fails. In order to extend the definition of the sum, Nörlund adopts an ingenious and powerful recipe. This consists in a regularization of \( \phi(x) \) with a parameter \( \mu \) \( (> 0) \), say \( \phi(x, \mu) \), which is so chosen that (see chapter III of [17]; see also chapter VIII of [18])

- (i) \( \lim_{\mu \to 0} \phi(x, \mu) = \phi(x) \);
- (ii) \( \int_{c}^{\infty} \phi(t) \, dt \) and \( \sum_{s=0}^{\infty} \phi(x + sh) \) both converge.

For this function \( \phi(x, \mu) \), the difference equation

\[ \Delta u(x) = \phi(x, \mu), \tag{A.4} \]

has a principal solution, given by the definition (A.2),

\[
F(x, \mu) = \int_{c}^{\infty} \phi(t, \mu) \, dt - h \sum_{s=0}^{\infty} \phi(x + sh, \mu). \]

If in this relation we let \( \mu \to 0 \), the difference equation (A.4) becomes the difference equation (A.1) and the principal solution of the latter is defined by

\[
F(x) = \lim_{\mu \to 0} F(x, \mu),
\]

provided that this limit exists uniformly and, subject to conditions (i) and (ii), is independent of the particular choice of \( \phi(x, \mu) \). When the limit exists \( \phi(x) \) is said to be summable.

The success of the method of definition just described depends on the difference of the infinite integral and the infinite series having a limit when \( \mu \to 0 \). Each separately may diverge when \( \mu = 0 \) and the choice of \( \phi(x, \mu) \) has to be so made that when we take the difference of the integral and the series the divergent part disappears. It has been shown that, for a wide class of summation methods, the result is independent of the method adopted. A convenient practical choice is [17, 18]

\[
F(x) = \sum_{c}^{x} \phi(z) \frac{\Delta z}{h}
\]

\[
= \lim_{\mu \to 0} \left\{ \int_{c}^{\infty} \phi(t) e^{-\mu \lambda(t)} \, dt - h \sum_{s=0}^{\infty} \phi(x + sh) e^{-\mu \lambda(x+sh)} \right\}, \tag{A.5}
\]

where \( p \geq 1, q \geq 0 \), such that for \( \lambda(x) = x^p (\ln x)^q \) this limit exists. Nörlund’s recipe (A.5) can be seen as a two-parameter extension of the single-parameter Lindelöf and Mittag-Leffler methods of summing divergent series [38]. The latter belongs to the so-called analytic and regular summability methods [38, 39]. If applied to a power series (i) it yields the value equal
to that obtained by an analytic continuation of the series beyond the radius of convergence anytime the limit exists, (ii) provided that the sum converges for \( \mu = 0 \), the limit \( \mu \to 0 \) yields the very same sum [38, 39].

As a simple illustration, consider

\[
\Delta u(x) = a,
\]

where \( a \) is constant. The series \( a + a + a + \cdots \) obviously diverges, but for \( \mu > 0 \)

\[
\int \limits_{c}^{\infty} ae^{-\mu t} \, dt, \quad \sum \limits_{s=0}^{\infty} ae^{-\mu (x + sh)}
\]

both converge if \( h \) is a positive real number, so that we can take \( \lambda(x) = x \), i.e. \( p = 1, q = 0 \).

Hence

\[
\Delta z = \lim \limits_{\mu \to 0} \left\{ \int \limits_{c}^{\infty} ae^{-\mu t} \, dt - h \sum \limits_{s=0}^{\infty} ae^{-\mu (x + sh)} \right\}
= \lim \limits_{\mu \to 0} \left( \frac{ae^{-\mu c}}{\mu} - \frac{ahe^{-\mu x}}{1 - e^{-\mu h}} \right)
= \lim \limits_{\mu \to 0} ae^{-\mu c} \left[ \frac{1 - e^{-\mu h} - \mu he^{-\mu (x-c)}}{\mu (1 - e^{-\mu h})} \right]
= \lim \limits_{\mu \to 0} \frac{ae^{-\mu c} \left( \mu h - \frac{(\mu h)^2}{2} + \cdots + \mu h + \mu^2 h(x - c) - \cdots \right)}{\mu \left[ \mu h - \frac{(\mu h)^2}{2} + \cdots \right]}
= a \left( x - c - \frac{h}{2} \right),
\]

(A.6)

which is the principal solution. It should be noted that both the integral and the series diverge when \( \mu = 0 \).

**Appendix B. The generalized Gamma function**

Following section 9.66 of [18], if we define the function \( \Gamma_h(x) \) by the relation

\[
h \ln \Gamma_h(x) = \sum \limits_{0}^{x} \ln \Delta z + h \ln \sqrt{2\pi/h},
\]

(B.1)

we have by differencing

\[
h \Delta \ln \Gamma_h(x) := \ln \frac{\Gamma_h(x + h)}{\Gamma_h(x)} = \ln x,
\]

(B.2)

and hence

\[
\Gamma_h(x + h) = x \Gamma_h(x).
\]

(B.3)

Thus, if \( n \) be a positive integer, \( \Gamma_h(nh + h) = h^n n! \Gamma_h(h) \).
\( \Gamma_h(x) \) can be related to the conventional \( \Gamma(x) \) through

\[
\ln \Gamma_h(x) = \ln \Gamma(x/h) + \frac{1}{h} (x - h) \ln h,
\]
or

\[
\Gamma_h(x) = \Gamma(x/h) \exp \left( \frac{x - h}{h} \ln h \right).
\]

Using the above relation one finds \( \Gamma_h(h) = 1 \), and for any positive integer \( n > 0 \)

\[
\Gamma_h(nh + h) = h^n n!.
\]

The formula (section 9.66 of [18])

\[
\frac{1}{\Gamma_h(x)} = e^{-\gamma \ln h} x \prod_{i=1}^{\infty} \left( \frac{x}{sh} + 1 \right) e^{-\frac{x}{sh}},
\]

where \( \gamma \approx 0.5772 \) is the Euler–Mascheroni constant, shows that \( 1/\Gamma_h(x) \) is an integral transcendent function, with simple zeros at the points \( 0, -h, -2h, -3h, \ldots \), and therefore that \( \Gamma_h(x) \) is a meromorphic function of \( x \) with simple poles at the same points.

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