A generalised Joyce construction for a family of nonlinear partial differential equations

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The construction of Dominic Joyce referred to in the title is for self-dual Riemannian metrics with two Killing fields \cite{Joyce}. In this note we show that this construction (in the form described by Calderbank and Pedersen in \cite{CalderbankPedersen}) can be extended to a family of nonlinear fourth order PDE in two dimensions, essentially reducing them to linear equations. One equation in this family is the affine maximal equation, and we will see that the construction in this case is almost the same as one due to Chern and Terng.

1 The main result

We consider convex functions \( u(x_i) \) defined on a domain in \( \mathbb{R}^n \) and write \( J = \det(u_{ij}) \), where \( (u_{ij}) \) is the Hessian \( \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right) \). Let \( \psi \) be any smooth, strictly convex function on the half-line \((0, \infty)\) and consider the functional

\[
\mathcal{F} = \int \psi(J) \, dx_1 \ldots dx_n.
\]  

(1)

The corresponding Euler-Lagrange equations \( \delta \mathcal{F} = 0 \) are

\[
\sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \left( J \psi'(J) u^{ij} \right) = 0,
\]  

(2)

where \( (u^{ij}) \) is the matrix inverse of \( (u_{ij}) \). This a nonlinear fourth order PDE for the function \( u \). In the case when \( \phi(J) = -J^\alpha \) for some \( \alpha \in (0, 1) \) these equations have been studied by Trudinger and Wang \cite{TrudingerWang}. The case covered by Joyce’s original construction is when \( n = 2 \) and \( \psi(J) = -\log J \), as we will discuss further in Section 3. (Of course we only consider the functional \( \mathcal{F} \) as a motivation for writing down the partial differential equations (2), and the actual convergence of the integral (1) is irrelevant.) Let us say that a point \( (x_1, x_2) \) is an “ordinary point” if the derivative \( \nabla J \) does not vanish there.

To state our result, let \( f \) be a solution of the equation

\[
f'(t) = t^{1/2} \psi''(t).
\]
Thus \( f' > 0 \) and \( f \) is a 1 \(-\) 1 map from \((0, \infty)\) to some finite or infinite interval \( I \). Let \( p \) be the inverse map raised to the power \(-1/2\), so \( J^{-1/2} = p(r) \) when \( r = f(J) \). Now consider the linear, second order PDE for a function \( \xi(H, r) \) defined on some domain in \( \mathbb{R} \times I \):

\[
\frac{\partial^2 \xi}{\partial H^2} + \frac{1}{p(r)} \frac{\partial}{\partial r} \left( p(r) \frac{\partial \xi}{\partial r} \right) = 0.
\]  \( (3) \)

**Theorem 1** Suppose \( \xi_1, \xi_2 \) are two solutions of equation (3) and \( \det \left( \frac{\partial(\xi_1, \xi_2)}{\partial(H, r)} \right) \) is positive at some point \((H_0, r_0)\). Define three 1\(-\)forms \( \epsilon_1, \epsilon_2, \epsilon \) by

\[
\epsilon_1 = p(r) \left( \frac{\partial \xi_2}{\partial r} dH - \frac{\partial \xi_2}{\partial H} dr \right) \quad \epsilon_2 = p(r) \left( -\frac{\partial \xi_1}{\partial r} dH + \frac{\partial \xi_1}{\partial H} dr \right)
\]

\[
\epsilon = \xi_1 \epsilon_1 + \xi_2 \epsilon_2.
\]

Then \( \epsilon, \epsilon_1, \epsilon_2 \) are closed 1\(-\)forms and \( \epsilon_1 \wedge \epsilon_2 \) is non\(-\)vanishing near \((H_0, r_0)\). Thus we can find functions \( u, x_1, x_2 \) with \( du = \epsilon, dx_1 = \epsilon_1, dx_2 = \epsilon_2 \) and \( x_1, x_2 \) give local co\(-\)ordinates around \((H_0, r_0)\). If we regard \( u \) as a function of \((x_1, x_2)\) it is a solution of the equation (2), on a suitable domain. All points in this domain are ordinary points. Conversely, any solution of (2) with \( n = 2 \), in the neighbourhood of an ordinary point, is obtained in this way, with solutions \( \xi_1, \xi_2 \) of (3) which are unique up to translation in the \( H \)-variable and the addition of constants.

In sum, the local study of the nonlinear equation (2), in two dimensions, is essentially equivalent to that of the linear equation (3).

## 2 The proof

This is entirely elementary. From now on we always suppose the dimension \( n \) is 2, and we work locally so we will not specify the precise domains of definition of the various functions. Given a convex function \( u \) we consider the Riemannian metric \( g = \sum u_{ij} dx_i dx_j \), and in particular the conformal structure which this defines. Write \( \xi_i \) for the Legendre transform co\(-\)ordinates \( \xi_i = \frac{\partial u}{\partial x_i} \). Suppose now that we have some other local co\(-\)ordinates \( \lambda_1, \lambda_2 \), inducing the same orientation as \( x_1, x_2 \) (i.e. \( \det \left( \frac{\partial \lambda_i}{\partial x_i} \right) > 0 \)). We can write the metric \( g = \sum g_{ab} d\lambda_a d\lambda_b \) in these co\(-\)ordinates. Recall that the co\(-\)ordinates are called isothermal if the matrix \((g_{ab})\) is a multiple of the identity matrix at each point, or in other words

\[
g = V (d\lambda_1^2 + d\lambda_2^2),
\]

for a positive function \( V(\lambda_1, \lambda_2) \). Let \( \epsilon_{ij} \) denote the alternating tensor \( \epsilon_{12} = -\epsilon_{21} = 1, \epsilon_{11} = \epsilon_{22} = 0 \).
Observation 1 The co-ordinates $\lambda_1, \lambda_2$ are isothermal if and only if the partial derivatives $\frac{\partial x_i}{\partial \lambda_a}, \frac{\partial \xi_i}{\partial \lambda_a}$ are related by the four equations

$$\frac{\partial \xi_i}{\partial \lambda_a} = \epsilon_{ab} \epsilon_{ij} \sqrt{J} \frac{\partial x_j}{\partial \lambda_b},$$

(4)

where $J = J(x(\lambda_1, \lambda_2))$.

To see this we write

$$g = \sum_{ij} u_{ij} dx_i dx_j = \sum_j d\xi_j dx_j = \sum_{j,a,b} \frac{\partial \xi_j}{\partial \lambda_a} \frac{\partial x_j}{\partial \lambda_b} d\lambda_a d\lambda_b.$$

So the isothermal condition is

$$\sum_j \frac{\partial \xi_j}{\partial \lambda_a} \frac{\partial x_j}{\partial \lambda_b} = V \delta_{ab}.$$

In matrix notation, if $A = \left( \frac{\partial x_i}{\partial \lambda_a} \right), B = \left( \frac{\partial \xi_i}{\partial \lambda_a} \right)$, this is

$$A^T B = V 1,$$

or in other words

$$B^T = VA^{-1}. \quad (5)$$

Taking determinants we have $\det B \det A = V^2$. On the other hand the matrix $(u_{ij})$ is $BA^{-1}$, so $J = \det B \det A^{-1}$. Thus $V = \sqrt{J} \det A$ and (5) is

$$B^T = \sqrt{J} \det A \ A^{-1},$$

which is the same as (4), by the formula for the inverse of a $2 \times 2$ matrix.

Now consider any positive function $P(\lambda_1, \lambda_2)$ and the pair of second order linear PDE

$$\frac{\partial}{\partial \lambda_1} \left( P \frac{\partial \xi}{\partial \lambda_1} \right) + \frac{\partial}{\partial \lambda_2} \left( P \frac{\partial \xi}{\partial \lambda_2} \right) = 0,$$

(6)

$$\frac{\partial}{\partial \lambda_1} \left( P^{-1} \frac{\partial x}{\partial \lambda_1} \right) + \frac{\partial}{\partial \lambda_2} \left( P^{-1} \frac{\partial x}{\partial \lambda_2} \right) = 0.$$

(7)

Observation 2 If $\xi(\lambda_1, \lambda_2)$ is a solution of (6) then there is a solution $x(\lambda_1, \lambda_2)$ of the first order system

$$\frac{\partial x}{\partial \lambda_a} = P \epsilon_{ab} \frac{\partial \xi}{\partial \lambda_b},$$

(8)

unique up to the addition of a constant, and $x(\lambda_1, \lambda_2)$ satisfies (7).
This is because the consistency condition for the first order system is
$$\frac{\partial}{\partial \lambda_2} \left( P \frac{\partial \xi}{\partial \lambda_2} \right) = \frac{\partial}{\partial \lambda_1} \left( -P \frac{\partial \xi}{\partial \lambda_1} \right),$$
which is the equation (6). In a different language, we are saying that the 1-form\( \epsilon = P^{-1} \frac{\partial \xi}{\partial \lambda_2} d\lambda_1 - P \frac{\partial \xi}{\partial \lambda_1} d\lambda_2 \) is closed, so can be written as \( dx \), for a function \( x \).

Note that, changing \( P \) to \( P^{-1} \), there is a complete symmetry between \( x \) and \( \xi \) so we can also start with a solution of (7) and construct a solution of (6).

**Observation 3** Suppose \( \xi_1, \xi_2 \) are two solutions of (6), with \( \det \left( \frac{\partial \xi_i}{\partial \lambda_a} \right) > 0 \). Let \( x_2 \) be the solution of (7) corresponding to \( \xi_1 \) by (8), and \( x_1 \) be the solution corresponding to \( -\xi_2 \). Then \( \det \left( \frac{\partial x_i}{\partial \lambda_a} \right) > 0 \), so \( x_1, x_2 \) give local co-ordinates.

Write
$$\epsilon = \xi_1 dx_1 + \xi_2 dx_2.$$

Then \( \epsilon \) is a closed 1-form and so \( \epsilon = du \) for a function \( u \). If we express \( u \) as a function of \( x_1, x_2 \) then \( \frac{\partial u}{\partial x_i} = \xi_i \).

This is all straightforward. The conditions (8) imply that
$$\det \left( \frac{\partial x_i}{\partial \lambda_a} \right) = P^2 \det \left( \frac{\partial \xi_i}{\partial \lambda_a} \right) > 0.$$

We have \( de = d\xi_1 dx_1 + d\xi_2 dx_2 \) and, writing \( \xi_{i,a} = \frac{\partial \xi_i}{\partial \lambda_a} \),
$$de = (\xi_{1,1} d\lambda_1 + \xi_{1,2} d\lambda_2)(-\xi_{2,2} d\lambda_1 + \xi_{2,1} d\lambda_2) + (\xi_{2,1} d\lambda_1 + \xi_{2,2} d\lambda_2)(\xi_{1,2} d\lambda_1 - \xi_{1,1} d\lambda_2) = 0.$$

Now return to our functions \( \psi(J), f(J) \) and the Euler-Lagrange equation (2).

**Observation 4** A convex function \( u \) satisfies equation (2) if and only if \( f(J) \) is harmonic with respect to the metric \( g = \sum u_{ij} dx_i dx_j \).

This is true in any dimension. The formula for the derivative of an inverse matrix is
$$\frac{\partial}{\partial x_k} u^{ij} = -\sum_{pq} u^{ip} u_{pq} u_{qj},$$
whereas the formula for the derivative of the determinant is
$$\frac{\partial J}{\partial x_i} = J \sum_{pq} u^{pq} u_{pq}.$$ 

These yield the identity
$$\sum_j \frac{\partial}{\partial x_j} u^{ij} = -\sum_{pqj} u^{ip} u_{jpq} u^{qj} = -\sum_p u^{ip} J^{-1} \frac{\partial J}{\partial x_p}. \quad (9)$$
Our Euler-Lagrange equation (2) is

$$\sum_i \frac{\partial v_i}{\partial x_i} = 0,$$

where

$$v_i = \sum_j \frac{\partial}{\partial x_j} (J \psi' (J) u^{ij}).$$

So

$$v_i = \sum_j \frac{\partial (J \psi' (J))}{\partial x_j} u^{ij} + J \psi' (J) \frac{\partial u^{ij}}{\partial x_j}.$$

By the definition of the function $f$ we have

$$\frac{\partial}{\partial x_j} (J \psi' (J)) = \sqrt{J} \frac{\partial f(J)}{\partial x_j} + \psi' (J) \frac{\partial J}{\partial x_j}.$$

Using (9) we obtain

$$v_i = \sqrt{J} \frac{\partial f(J)}{\partial x_j} u^{ij}.$$

Thus the equation (10) is the Laplace equation in the metric $g$:

$$\sum_i \frac{\partial}{\partial x_i} \left( \sqrt{J} u^{ij} \frac{\partial f(J)}{\partial x_j} \right) = 0.$$

With these four observations the main result, Theorem 1, is almost obvious. Suppose we start with an ordinary point of a solution $u$ to (2). Then $r = f(J)$ is harmonic, by Observation 4, and we can suppose that its derivative does not vanish in the region considered. There is then a conjugate harmonic function $H$, which by definition is one such that the local co-ordinates $(H, r)$ are isothermal. By Observations 1 and 2 the functions $x_i, \xi_j$ satisfy the equations (6),(7) respectively with $\lambda_1 = H, \lambda_2 = r$ and $P = p(r)$. Since $P$ does not depend on $H$ the equation (6) can be written in the form (3). By Observations 2 and 3 we can recover the original function $u$ from the two solutions $\xi_1, \xi_2$ of the linear PDE (or, equally well, the two solutions $x_1, x_2$). It is also clear that, conversely, starting with any two solutions $\xi_1, \xi_2$ to the linear equation we construct a solution to (2) by this method.

Note that the conjugate function $H$, in this situation, can be defined simply as the solution of the system

$$\frac{\partial H}{\partial x_i} = \epsilon_{ij} v_j,$$

so we can also think of $H$ as the Hamiltonian generating the area-preserving vector field $v$ in the $(x_1, x_2)$ plane. We leave this as an exercise for the reader.
3 Examples and discussion

1. Let $u^*(\xi_1, \xi_2)$ be the Legendre transform of $u(x_1, x_2)$. The Hessian $(\frac{\partial^2 u^*}{\partial \xi_i \partial \xi_j})$ at the point $\xi_i = \frac{\partial u}{\partial x_i}$ is the inverse of the Hessian of $u$ at $x_i$. Thus we can write
   \[ F = \int \psi(J^{-1}) J_* d\xi_1 d\xi_2, \]
   where $J_* = \det(\frac{\partial^2 u^*}{\partial \xi_i \partial \xi_j})$. This implies that the Legendre transform takes a solution $u$ of the equation (2) associated with $\psi$ to a solution $u^*$ of the equation associated to the function
   \[ \psi^*(t) = t \psi(t^{-1}). \]
   In our construction this just means replacing the function $p(r)$ by $p(-r)^{-1}$ and interchanging the roles of the co-ordinates $x_i, \xi_i$.

2. If $\psi(t) = -\log t$ the equation (2) is $\sum \frac{\partial^2 u^*}{\partial x_i \partial x_j} = 0$. This is the equation defining a zero scalar curvature Kahler metric in “symplectic” co-ordinates, see [1], [4]. Explicitly we introduce two further co-ordinates $\theta_1, \theta_2$ and consider the Riemannian metric, in four dimensions,

   \[ \sum u_{ij} dx_i dx_j + \sum u^{ij} d\theta_i d\theta_j. \]

   The well-known fact that, in two complex dimensions, such metrics are self-dual gives the link with Joyce’s original formulation of his construction. Under the Legendre transform we get another description corresponding to the function $\psi^*(t) = t \log t$ This leads to the equations describing zero scalar curvature metrics in “complex” co-ordinates. When $\psi(t) = -\log t$ we get $p(r) = r$ and equation (3) is the familiar equation defining axi-symmetric harmonic functions on $\mathbb{R}^3$.

3. For any function $p(r)$ there is an obvious solution $\xi = H$ to (3). Thus we get a special family of solutions to (2) with $\xi_1 = H$ and $\xi_2$ some other solution of (3). There is another special family, which corresponds to this under the Legendre transform, when $\xi_1$ is a function of $r$ only, so $x_2 = H$. These correspond to second order equations of Monge-Ampère type which give special solutions of (2). For example, in the zero scalar curvature case above we have the special solutions where $\xi_1 = \log r$. The function $u$ satisfies the equation $\det \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right) = e^{-\xi_1/2}$ and the corresponding four dimensional Riemannian metric is Ricci-flat.

4. For the Trudinger-Wang equations, when $\psi(t) = -t^\alpha$ with $0 < \alpha < 1/2$, we get $p(r) = r^{1/(1-2\alpha)}$ (up to factor which is irrelevant because it drops out of the equation (3)). If $\alpha = 1/2$ we get $p(r) = e^{r/2}$. 

6
5. Consider the graph of a function $u(x_1, x_2)$ as a surface in $\mathbb{R}^3$. The Gauss curvature is

$$K = \frac{\det(u_{ij})}{(1 + |\nabla u|^2)^2} = \frac{J}{(1 + |\nabla u|^2)^2}$$

and the induced area form is

$$dA = (1 + |\nabla u|^2)^{1/2}dx_1dx_2.$$

Thus

$$K^{1/4}dA = J^{1/4}dx_1dx_2.$$  

The left hand side is invariant under Euclidean transformations of $\mathbb{R}^3$ while the right hand side is invariant under unimodular affine transformations of $\mathbb{R}^2$. Since these two groups generate all unimodular affine transformations of $\mathbb{R}^3$ we see that this 2-form is an affine invariant of the surface. The graphs of the solutions of equation (2) when $\psi(t) = t^{1/4}$ are "affine maximal" surfaces in $\mathbb{R}^3$. According to Chern and Terng [3] these surfaces can be described locally as follows. Let $F_1, F_2, F_3$ be harmonic functions of variables $\lambda_1, \lambda_2$ and write $F = F(\lambda_1, \lambda_2)$ for the corresponding vector-valued function. Then the condition $\frac{\partial^2 F}{\partial \lambda_1^2} + \frac{\partial^2 F}{\partial \lambda_2^2} = 0$ implies the consistency of the first order system

$$\frac{\partial Z}{\partial \lambda_1} = F \times \frac{\partial F}{\partial \lambda_2} \quad \frac{\partial Z}{\partial \lambda_2} = -F \times \frac{\partial F}{\partial \lambda_1}$$  \hspace{1cm} (12)

so there is a solution $Z$ which, under appropriate non-degeneracy conditions, parametrises a surface in $\mathbb{R}^3$. Chern and Terng show that these surfaces are precisely the affine maximal surfaces. We want to relate this description to ours. Notice first that our equations can be written in a similar form. Given a pair of solutions $\xi_1, \xi_2$ of (3) we define a vector valued function $\Xi$ with components $\xi_1, \xi_2, 1$. Then if $Z = (x_1, x_2, u)$ our description is the first order system

$$\frac{\partial Z}{\partial H} = p(r) \Xi \times \frac{\partial \xi}{\partial r} \quad \frac{\partial Z}{\partial r} = -p(r) \Xi \times \frac{\partial \xi}{\partial H}.$$  \hspace{1cm} (13)

In the case when $\psi(t) = t^{1/4}$ we get $p(r) = r^2$ so the equation (3) is not the ordinary Laplace equation, in the variables $(H, r)$. However given a function $F(H, r)$ we consider the first order system

$$p(r) \frac{\partial \xi}{\partial H} = r \frac{\partial F}{\partial H} \quad p(r) \frac{\partial \xi}{\partial r} = r \frac{\partial F}{\partial r} - F.$$  \hspace{1cm} (14)

If $p(r) = r^2$ one readily checks that this system is consistent, so there is a solution $\xi(H, r)$. The equation (3) for $\xi$ is equivalent to the ordinary Laplace equation for $F$. Now take two harmonic functions $F_1, F_2$ and set $F_3(H, r) = r$. Then a few lines of calculation show that the system (12) is identical to the system (13), when in the latter we use the functions $\xi_1, \xi_2$ corresponding to $F_1, F_2$ by solving (14).
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