Constructions and Bounds for Codes With Restricted Overlaps

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Abstract—Non-overlapping codes have been studied for almost 60 years. In such a code, no proper, non-empty prefix of any codeword is a suffix of any codeword. In this paper, we study codes in which overlaps of certain specified sizes are forbidden. We prove some general bounds and we give several constructions in the case of binary codes. Our techniques also allow us to provide an alternative, elementary proof of a lower bound on non-overlapping codes due to Levenshtein (1964).

Index Terms—Non-overlapping codes, weakly mutually uncorrelated codes, cross-bifix-free codes.

I. INTRODUCTION

Let $u$ and $v$ be (not necessarily distinct) words of length $n$ over a specified alphabet. Let $t$ be an integer such that $1 \leq t \leq n - 1$. We say that $u$ and $v$ have a $t$-overlap if the prefix of $u$ of length $t$ is identical to the suffix of $v$ of length $t$. A code $C$ is $t$-overlap-free if no codewords $u$ and $v$ in $C$ have a $t$-overlap. A code $C$ is non-overlapping if it is $t$-overlap-free for all $t$ such that $1 \leq t \leq n - 1$.

Motivated by applications including frame synchronization, non-overlapping codes have been studied by numerous authors over the years, e.g., see [2], [3], [4], [5], [9], [12], and [17].

Here we consider a less restrictive definition. Suppose that $t_1$ and $t_2$ are integers such that $1 \leq t_1 \leq t_2 \leq n - 1$. We say that a code $C$ is $(t_1, t_2)$-overlap-free if it is $t$-overlap-free for all $t$ such that $1 \leq t \leq t_2$. Two special cases of interest are codes that are $(k, n - 1)$-overlap-free (i.e., overlaps of size at least $k$ are not allowed) and codes that are $(1, k)$-overlap-free (i.e., overlaps of size at most $k$ are not allowed).

Motivated by applications in DNA-based storage systems and synchronization protocols, $(k, n - 1)$-overlap-free codes were studied in [16] and termed $k$-weakly mutually uncorrelated codes. On the other hand, $(1, k)$-overlap-free codes could be useful in a setting where we have “approximate” synchronization, i.e., if we can assume that codewords will not “drift” too much. For example, suppose (see Figure 1) that we transmit blocks 1, 2, and 3 so on, each a codeword of the same length $n$. We consider channels where a received block might be corrupted, with bits changed and up to $k$ bits inserted or deleted. We detect a loss of synchronization by checking if each block of $n$ received bits is a codeword. If we use an $(n - k, n - 1)$-overlap-free code, we are guaranteed to detect a loss of synchronization after $2n$ bits are received. If we use a $(1, k)$-overlap-free code, we are guaranteed to detect a loss of synchronization after $3n$ bits if there are inserted bits, but after only $n$ bits are received if bits have been deleted. Thus, in channels where deletions are more likely than insertions, $(1, k)$-overlap-free codes have an advantage over $(n - k, n - 1)$-overlap-free codes.

We comment that codes for synchronization is a large and thriving area, which we cannot hope to cover comprehensively here. Notable related problems are codes designed to correct bursts of insertions or deletions [6], [10], [11], [15], and variants of the non-overlapping problem in two-dimensions [1].

In general, we wish to determine the maximum number of codewords in a $(t_1, t_2)$-overlap-free code. In Section II, we prove two upper bounds, on the size of $(k, n - 1)$-overlap-free codes and $(1, k)$-overlap-free codes. Section III begins a study of constructions for $(1, k)$-overlap-free codes over a binary alphabet. Our first construction, the Doubling Construction, gives an inductive approach to the construction of these codes. Section IV introduces a graph-based interpretation of these codes. This approach is used to prove the optimality of our codes for $k \leq 6$. Section V presents an explicit construction that we term the $m$-minimum Construction, as well as the closely related Zero Block Construction. Both of these permit “good” codes to be constructed for specified values of $k$. The second of these two constructions can be analyzed by exploiting a connection with $n$-step Fibonacci numbers. We provide exact as well as asymptotic bounds; it is shown that the constructed codes are within a small constant factor of being optimal. Section VI revisits the classical problem of non-overlapping codes and discusses how our techniques apply to this problem. In particular, we provide an alternative,
elementary proof of a lower bound on non-overlapping codes due to Levenshtein [9] in 1964. Finally, Section VII is a brief discussion and summary.

II. TWO UPPER BOUNDS

Chee et al. [5] proved that if C is a non-overlapping code over an alphabet of cardinality q, then $|C| \leq q^n/(2n-1)$. This bound can be proven using a simple combinatorial argument; see Blackburn [4]. Also, a stronger bound has been proven by Levenshtein [12] using analytic combinatorics.

For $(k, n-1)$-overlap-free codes, Yazdi et al. [16] proved that such a code C satisfies the inequality $|C| \leq q^n/(n-k+1)$. The following stronger bound can be proven using the argument from [4]. Note that the special case $k = 1$ of Theorem 1 is essentially the bound proven in [4].

**Theorem 1**: If C is a $(k, n-1)$-overlap-free code over an alphabet of cardinality q, then

$$|C| \leq \frac{q^n}{2n-2k+1}.$$ 

**Proof**: Let C be a $(k, n-1)$-overlap-free code over an alphabet F of cardinality q. For $w \in F^{2n-2k+1}$ and $1 \leq i \leq 2n-2k+1$, define $w(i) = (w_i, w_{i+1}, \ldots, w_{i+n-1})$, where the subscripts are reduced modulo $2n-2k+1$. Thus, $w(i)$ is the cyclic subword of length n of w starting at $w_i$. Define

$$X = \{(w,i) : w \in F^{2n-2k+1}, 1 \leq i \leq 2n-2k+1, w(i) \in C\}.$$ 

Suppose there exists $w \in F^{2n-2k+1}$ and $i, i'$ such that $i \neq i'$ and $(w, i), (w, i') \in X$. We claim that $(w, i)$ and $(w, i')$ have an overlap of size at least k. This occurs because the overlap between $(w, i)$ and $(w, i')$ is at least

$$n + n - (2n - 2k + 1) = 2k - 1,$$

and hence the overlap at one end is at least $\lfloor (2k-1)/2 \rfloor = k$. This violates the non-overlapping properties of C. Hence, for each $w \in F^{2n-2k+1}$ there is at most one $i$ such that $(w, i) \in X$. Thus it follows that

$$|X| \leq q^{2n-2k+1}.$$ 

Also, $|X| = (2n - 2k + 1)|C|q^{n-2k+1}$, since there are $2n - 2k + 1$ choices for $i$, $|C|$ choices for $w(i)$, and $q^{n-2k+1}$ choices for the remaining entries in $w$.

Hence,

$$(2n - 2k + 1)|C|q^{n-2k+1} \leq q^{2n-2k+1},$$

which immediately yields the stated upper bound on $|C|$. □

It is natural to ask if there is a “related” upper bound for $(1, k)$-overlap-free codes.

**Theorem 2**: Let C be a $(1, k)$-overlap-free code, where $k \leq n/2$. Then

$$|C| \leq \frac{1}{2k} q^n.$$ 

**Proof**: Let $k \leq n/2$ and let C be a $(1, k)$-overlap-free code. Let $\mathcal{X}$ be the set of all codewords with the middle $n-2k$ positions removed. Clearly, $|C| \leq q^{n-2k}|\mathcal{X}|$. The elements of $\mathcal{X}$ are q-ary words of length $2k$. We have $\mathcal{X} = \mathcal{Y} \cup \mathcal{Z}$, where the elements in $\mathcal{Y}$ have (cyclic) period strictly dividing $2k$, and the elements of $\mathcal{Z}$ have period exactly $2k$.

Suppose $y \in \mathcal{Y}$. If y has period p, where $p$ strictly divides $2k$, then $p \leq k$. Then the first and last p elements of a corresponding codeword $w \in \mathcal{X}$ agree. This codeword has a $p$-overlap with itself, which contradicts the $(1, k)$-overlap-free property. We conclude that $\mathcal{Y} = \emptyset$.

Now we claim that no pair of distinct elements in $\mathcal{Z}$ are cyclic shifts of each other. For a contradiction, suppose that $z_1, z_2$ are a pair of distinct elements from $\mathcal{Z}$ that are cyclic shifts of each other. Let $c_1, c_2$ be the corresponding codewords in C. Write $\sigma$ for the ‘cyclic shift left by one position’ operator, so

$$\sigma(a_1 a_2 \cdots a_{2k}) = a_{2k} a_1 \cdots a_{2k-1}.$$ 

Then $z_1 = \sigma^j(z_2)$ for some $j \in \{1, \ldots, 2k-1\}$. Swapping $z_1$ and $z_2$ if needed, we may assume that $j$ lies in the set $\{1, \ldots, k\}$ (as swapping replaces $j$ by $2k - j$). But now the j-prefix of $z_2$ is equal to the j-suffix of $z_1$. So the j-prefix of $c_2$ is equal to the j-suffix of $c_1$. This contradicts our assumption that C is $(1, k)$-overlap-free, and so our claim follows.

We can partition the set of all q-ary sequences of length $2k$ and period exactly $2k$ into equivalence classes under cyclic shift. Each class contains $2k$ sequences, and so there are at most $q^{2k}/2k$ classes. The previous paragraph shows that no class contains two elements of $\mathcal{Z}$, and so $|\mathcal{Z}| \leq q^{2k}/2k$. Hence

$$|C| \leq q^{n-2k}|\mathcal{X}| = q^{n-2k}(|\mathcal{Y}| + |\mathcal{Z}|) = q^{n-2k}|\mathcal{Z}| \leq \frac{q^n}{2k}. \quad \square$$

III. CONSTRUCTIONS

In this section, and the next two sections, we investigate constructions and bounds for $(1, k)$-overlap-free codes. All of our constructions will be based on the following template.
Construction 3: Let $F$ be an alphabet of size $q$ and let $n$ and $t$ be positive integers such that $n \geq 2t$. Let $\mathcal{P}$ and $\mathcal{S}$ be two sets of $t$-tuples from $F^t$. Define
\[
C(\mathcal{P}, \mathcal{S}, n, t) = \{ p \parallel x : s : p \in \mathcal{P}, s \in \mathcal{S}, x \in F^{n-2t} \}.
\]
Thus a codeword $c \in C$ has a prefix chosen from $\mathcal{P}$, a suffix chosen from $\mathcal{S}$, and the remaining $n-2t$ elements are arbitrary symbols from $F$. We also observe that $|C(\mathcal{P}, \mathcal{S}, n, t)| = |\mathcal{P}| \times |\mathcal{S}| \times q^{n-2t}$.

The following Lemma is immediate.

Lemma 4: $C(\mathcal{P}, \mathcal{S}, n, t)$ is $t$-overlap-free if and only if $\mathcal{P} \cap \mathcal{S} = \emptyset$.

Suppose $\mathcal{P}$ and $\mathcal{S}$ are two sets of $k$-tuples from $F^k$. Suppose $t$ is a positive integer such that $t < k$. Define $\mathcal{P}|_t$ to be the set of all $t$-prefixes of tuples from $\mathcal{P}$ and $\mathcal{S}|_t$ to be the set of all $t$-suffixes of tuples from $\mathcal{S}$. So
\[
\mathcal{P}|_t = \{(p_1, \ldots, p_t) : \text{there exists } (p_1, \ldots, p_t, p_{t+1}, \ldots, p_k) \in \mathcal{P} \},
\]
and
\[
\mathcal{S}|_t = \{(s_{k-t+1}, \ldots, s_k) : \text{there exists } (s_1, \ldots, s_{k-t+1}, s_{k-t+1}, \ldots, s_k) \in \mathcal{S} \}.
\]

The following is a straightforward extension of Lemma 4.

Theorem 5: $C(\mathcal{P}, \mathcal{S}, n, t)$ is a $(1,k)$-overlap-free code if and only if $\mathcal{P}|_t \cap \mathcal{S}|_t = \emptyset$ for $1 \leq t \leq k$.

A. The Doubling Construction

Theorem 5 suggests a way to build up $(1,k)$-overlap-free codes inductively. We will refer to this process as the Doubling Construction. For the rest of the paper, we consider the binary case, where $q = 2$.

We take $F = \{0,1\}$. Suppose we begin with $k = 1$. Without loss of generality, we can define $\mathcal{P} = \{0\}$ and $\mathcal{S} = \{1\}$. So $C(\mathcal{P}, \mathcal{S}, n, 1)$ would consist of all $2^{n-2}$ binary $n$-tuples that begin with a 0 and end with a 1.

Next, we consider $k = 2$. We consider extensions of the solution for $k = 1$, where we append a symbol to a tuple in $\mathcal{P}$ and we prepend a symbol to a tuple in $\mathcal{S}$:
\[
\begin{array}{|c|c|}
\hline
\mathcal{P} & \mathcal{S} \\
00 & 01 \\
01 & 11 \\
\hline
\end{array}
\]
We cannot include 01 in both $\mathcal{P}$ and $\mathcal{S}$. Without loss of generality, we include 01 in $\mathcal{P}$ but not in $\mathcal{S}$. So we obtain the following solution for $k = 2$:
\[
\mathcal{P} = \{00, 01\} \quad \text{and} \quad \mathcal{S} = \{11\}.
\]
Thus $C(\mathcal{P}, \mathcal{S}, n, 2)$ would consist of
\[
2 \times 2^{n-4}
\]
binary $n$-tuples.

We can use a similar process to proceed from $k = 2$ to $k = 3$. We append a symbol to each tuple in $\mathcal{P}$ and we prepend a symbol to each tuple in $\mathcal{S}$:

\[
\begin{array}{|c|c|}
\hline
\mathcal{P} & \mathcal{S} \\
0000 & 00011 \\
00001 & 10011 \\
00010 & 01011 \\
00011 & 11011 \\
00100 & 00111 \\
00101 & 10111 \\
01000 & 01111 \\
01001 & 11111 \\
01010 & 010111 \\
01011 & 110111 \\
\hline
\end{array}
\]
Now the 3-tuple 011 is duplicated. We retain it in $\mathcal{S}$ and delete it from $\mathcal{P}$ (this will lead to the largest code, since $3 \times 2 > 2 \times 1$). We obtain the following solution for $k = 3$: $\mathcal{P} = \{000, 001, 010\}$ and $\mathcal{S} = \{011, 111\}$. Thus $C(\mathcal{P}, \mathcal{S}, n, 3)$ consists of
\[
3 \times 2 \times 2^{n-6} = 6 \times 2^{n-6}
\]
binary $n$-tuples.

Now we proceed from $k = 3$ to $k = 4$. We get the following:
\[
\begin{array}{|c|c|}
\hline
\mathcal{P} & \mathcal{S} \\
00000 & 000111 \\
00001 & 100111 \\
00010 & 010111 \\
00011 & 110111 \\
00100 & 001111 \\
00101 & 101111 \\
01000 & 011111 \\
01001 & 111111 \\
01010 & 0101111 \\
01011 & 1101111 \\
\hline
\end{array}
\]
The 4-tuple 0011 is duplicated. Again, we retain it in $\mathcal{S}$ and delete it from $\mathcal{P}$. We obtain the following solution for $k = 4$:
\[
\mathcal{P} = \{0000, 0001, 0010, 0100, 0101\}
\]
and
\[
\mathcal{S} = \{0011, 1011, 0111, 1111\}.
\]
Thus $C(\mathcal{P}, \mathcal{S}, n, 4)$ consists of
\[
5 \times 4 \times 2^{n-8} = 20 \times 2^{n-8}
\]
binary $n$-tuples.

When we proceed from $k = 4$ to $k = 5$, we obtain the following:
\[
\begin{array}{|c|c|}
\hline
\mathcal{P} & \mathcal{S} \\
000000 & 0001111 \\
000001 & 1001111 \\
000010 & 0101111 \\
000011 & 1101111 \\
000100 & 0011111 \\
000101 & 1011111 \\
000110 & 0111111 \\
000111 & 1111111 \\
001000 & 01011111 \\
001001 & 11011111 \\
001010 & 01111111 \\
001011 & 11111111 \\
\hline
\end{array}
\]
Now there are two duplicated 5-tuples. We will retain both 5-tuples in $\mathcal{S}$ in order to balance the sizes of $\mathcal{P}$ and $\mathcal{S}$. So we obtain the following solution for $k = 5$:
\[
\mathcal{P} = \{00000, 00001, 00010, 00100, 00101, 01001, 01010\}
\]
and
\[
\mathcal{S} = \{00011, 10011, 01011, 11011, 00111, 10111, 01111, 11111\}.
\]
TABLE I
RESULTS OBTAINED FROM THE DOUBLING CONSTRUCTION

| $n$ | $|P_k|$ | $|S_k|$ | $C(n,k)$ |
|-----|--------|--------|----------|
| 2   | 2      | 1      | $2 \times 2^{n-4}$ |
| 3   | 3      | 2      | $6 \times 2^{n-6}$ |
| 4   | 5      | 4      | $20 \times 2^{n-8}$ |
| 5   | 8      | 8      | $64 \times 2^{n-10}$ |
| 6   | 15     | 14     | $210 \times 2^{n-12}$ |
| 7   | 26     | 27     | $702 \times 2^{n-14}$ |
| 8   | 50     | 50     | $2500 \times 2^{n-16}$ |
| 9   | 94     | 94     | $8836 \times 2^{n-18}$ |
| 10  | 180    | 179    | $32220 \times 2^{n-20}$ |
| 11  | 343    | 343    | $117649 \times 2^{n-22}$ |
| 12  | 659    | 659    | $4343281 \times 2^{n-24}$ |
| 13  | 1267   | 1266   | $1640422 \times 2^{n-26}$ |
| 14  | 2444   | 2444   | $5973136 \times 2^{n-28}$ |
| 15  | 4726   | 4725   | $22330350 \times 2^{n-30}$ |
| 16  | 9157   | 9158   | $83859806 \times 2^{n-32}$ |
| 17  | 17779  | 17779  | $316929841 \times 2^{n-34}$ |
| 18  | 34575  | 34575  | $1195430625 \times 2^{n-36}$ |
| 19  | 67340  | 67339  | $4534602680 \times 2^{n-38}$ |
| 20  | 131323 | 131323 | $12745730329 \times 2^{n-40}$ |
| 21  | 256416 | 256416 | $65749165056 \times 2^{n-42}$ |
| 22  | 501208 | 501207 | $251208958056 \times 2^{n-44}$ |
| 23  | 980684 | 980684 | $961741107856 \times 2^{n-46}$ |

Thus $C(\mathcal{P}, \mathcal{S}, n, 5)$ consists of binary $n$-tuples.

We can make a few observations as to what happens when we increase $k$ by one in the Doubling Construction.

1) First, we double the size of $\mathcal{P}$ and $\mathcal{S}$ by appending 0 and 1 to every tuple in $\mathcal{P}$ and prepending 0 and 1 to every tuple in $\mathcal{S}$.

2) Then we look for duplicates in $\mathcal{P}$ and $\mathcal{S}$. Note that a duplicate occurs in the new $\mathcal{P}$ and $\mathcal{S}$ whenever there was a $k$-tuple in the old $\mathcal{P}$ whose suffix of size $k-1$ is identical to a prefix of size $k-1$ of a $k$-tuple in the old $\mathcal{S}$. For example, when $k = 4$, we see that 0001 is in $\mathcal{P}$ and 0011 is in $\mathcal{S}$. The suffix of size 3 of 0001, namely 001, is the same as the prefix of size 3 of 0011. Thus, when we append 1 to 0001 and we prepend 0 to 0011, we obtain the duplicate string 0011.

3) Finally, we eliminate one copy of each duplicate so as to balance the resulting sizes of $\mathcal{P}$ and $\mathcal{S}$ as much as possible.

The results in Table I are obtained using the Doubling Construction. Note that here and elsewhere we denote the maximum size of a $(1, k)$-overlap-free code in $\{0,1\}^n$ by $C(n, k)$.

IV. OPTIMAL SOLUTIONS—A GRAPH-BASED APPROACH

In this section, we discuss a graph-based approach that can (in principle) be used to prove that a solution is optimal. In practice, the method will only be feasible for small values of $k$. Again, we restrict our attention to the case $q = 2$ for convenience. Denote $F = \{0,1\}$ and suppose $k$ is a fixed positive integer.

We construct a bipartite graph $G_k$. The vertex set is $X \cup Y$, where $|X| = |Y| = 2^k$. We associate each vertex in $X$ with a $k$-tuple from $F^k$, and similarly each vertex in $Y$ corresponds to a $k$-tuple from $F^k$. The vertices in $X$ will be denoted by $x_p$, where $p \in F^k$, and the vertices in $Y$ will be denoted by $y_s$, where $s \in F^k$. We will join vertices $x_p$ and $y_s$ by an edge if and only if a prefix of $p$ is identical to a suffix of $s$. For example, the graph $G_2$ is depicted in Figure 2.

In general, the graph $G_k$ records incompatible prefixes and suffixes. More precisely, if $x_p y_s$ is an edge of $G_k$, then there cannot exist two $n$-tuples in a $(1, k)$-overlap-free code where $p$ is a $k$-prefix of an $n$-tuple and $s$ is a $k$-suffix of a (not necessarily distinct) $n$-tuple.

The following lemma is immediate.

Lemma 6: Suppose $C$ is a $(1, k)$-overlap-free code. Let $\mathcal{P}$ denote all the $k$-prefixes of $n$-tuples in $C$ and let $\mathcal{S}$ denote all the $k$-suffixes of $n$-tuples in $C$. Denote $X_C = \{x_p : p \in \mathcal{P}\}$ and $Y_C = \{y_s : s \in \mathcal{S}\}$. Then $X_C \cup Y_C$ is an independent set of vertices in $G_k$.

Theorem 7: Suppose $n \geq 2k$. Suppose that $X_C \cup Y_C$ is an independent set of vertices in $G_k$, where $X_C \subseteq X$ and $Y_C \subseteq Y$. Then there is a $(1, k)$-overlap-free code in $F^n$ having size

$$|X_C| \times |Y_C| \times 2^{n-2k}.$$  

Proof: Suppose $X_C \cup Y_C$ is an independent set of vertices in $G_k$. Include all $n$-tuples of the form $p \parallel x \parallel s$ where $p \in \mathcal{P}$, $s \in \mathcal{S}$, and $x \in F^{n-2k}$. This is a $(1, k)$-overlap-free code having size $|X_C| \times |Y_C| \times 2^{n-2k}$.

Theorem 8: Suppose $n \geq 2k$. Suppose that $X_C \cup Y_C$ is an independent set of vertices in $G_k$, where $X_C \subseteq X$ and $Y_C \subseteq Y$, such that $|X_C| \times |Y_C|$ is maximized. Then the maximum size of any $(1, k)$-overlap-free code in $F^n$ is exactly $|X_C| \times |Y_C| \times 2^{n-2k}$.

Proof: Suppose $C$ is a $(1, k)$-overlap-free code in $F^n$. Let $\mathcal{P}$ denote all the $k$-prefixes of $n$-tuples in $C$ and let $\mathcal{S}$ denote all the $k$-suffixes of $n$-tuples in $C$. Lemma 6 asserts that $X_C \cup Y_C$ is an independent set of vertices in $G_k$. To maximize the size of $C$, we would include all $n$-tuples of the form $p \parallel x \parallel s$ where $p \in \mathcal{P}$, $s \in \mathcal{S}$, and $x \in F^{n-2k}$. From Theorem 7, this (optimal) code has size $|X_C| \times |Y_C| \times 2^{n-2k}$.

Example 9: Suppose $k = 2$. By examining the graph $G_2$ depicted in Figure 2, it is not hard to see that the only independent sets of size 4 are $X$ and $Y$. Hence, the maximum value of $|X_C| \times |Y_C|$ is obtained when $|X_C| = 2$ and $|Y_C| = 1$ or when $|X_C| = 1$ and $|Y_C| = 2$. One optimal solution is $X_C = \{x_{00}, x_{01}\}$ and $Y_C = \{y_{11}\}$ (see the highlighted vertices in Figure 2). Therefore the maximum size of a $(1, 2)$-overlap-free code in $F^n$ is $2^n - 3$. In other words, the Doubling Construction is optimal for $k = 2$.

Remark 10: The proof of Theorem 8 uses the construction from Section III-A. In Section III-A, we inductively constructed independent sets $X_C \cup Y_C$ where we maximized $|X_C| \times |Y_C|$ at each step of the process. But it does not necessarily follow that the resulting values of $|X_C| \times |Y_C|$ are the maximum possible. In fact we will see situations where this is not the case.

The graph $G_k$ has $2^{k+1}$ vertices. If we exhaustively search for an “optimal” independent set, this approach will quickly become infeasible as $k$ increases. This can be done for a few
small values of $k$, however. The approach we take is to identify some nice structure in optimal independent sets for small $k$ and then generalize the structure to larger values of $k$.

Suppose that $X_C \cup Y_C$ is an independent set of vertices in $G_k$, where $X_C \subseteq X$ and $Y_C \subseteq Y$. If $X_C \neq \emptyset$ and $Y_C \neq \emptyset$, then we say that $X_C \cup Y_C$ is a non-trivial independent set. Now we present an upper bound on the size of a non-trivial independent set in $G_k$.

**Theorem 11:** A non-trivial independent set in $G_k$ has size at most $2^{k-1} + 1$.

**Proof:** Define $X_i = \{x : p_i = 1\}$, for $i = 0, 1$. Also, define $Y_i = \{y : s_i = 0\}$, for $i = 0, 1$. Thus $X_i$ consists of all vertices in $X$ corresponding to $k$-tuples beginning with $i$ and $Y_i$ consists of all vertices in $Y$ corresponding to $k$-tuples ending with $i$. Suppose that $X_C \cup Y_C$ is a non-trivial independent set of vertices in $G_k$; hence $X_C \neq \emptyset$ and $Y_C \neq \emptyset$. Suppose without loss of generality that there is an $x \in X_0 \cap X_C$. Then $Y_C \cap Y_0 = \emptyset$ and hence $Y_C \subseteq Y_1$. Since $Y_C \neq \emptyset$, we have $X_C \cap X_1 = \emptyset$ and hence $X_C \subseteq X_0$.

Therefore, we can restrict our attention to the subgraph $G'$ of $G$ induced by the vertices in $X_0 \cup Y_1$. $G'$ has $2^{k-1}$ vertices in each part of its partition. We show that $G'$ contains a matching $M$ of size $2^{k-1} - 1$.

First, for the $2^{k-2}$ $k$-tuples $p$ such that $p_1 = 0$ and $p_k = 1$, we match $x_p$ with $y_p$. The remaining $2^{k-2}$ $k$-tuples $p$ such that $x_p \in X_0$ have $p_1 = p_k = 0$ (call this set $P'$), and the remaining $2^{k-2}$ $k$-tuples $s$ such that $y_s \in Y_1$ have $s_1 = s_k = 1$ (call this set $S'$). We ignore the all-0 $k$-tuple in $P'$ and the all-1 $k$-tuple in $S'$; there remain $2^{k-2} - 1$ $k$-tuples in $P'$ and $2^{k-2} - 1$ $k$-tuples in $S'$.

Any $k$-tuple in $P'$ can be written uniquely in the form $p = 0 \parallel a \parallel 1 \parallel b \parallel 0$, where $a$ is a (possibly empty) sequence of 0’s and $b$ is an arbitrary binary sequence. For each such $k$-tuple, we observe that there is an edge in $G'$ from $x_p$ to $y_s$, where $s = 1 \parallel b \parallel 0 \parallel a \parallel 1$, because $p$ begins with 0 $\parallel a \parallel 1$ and $s$ ends with 0 $\parallel a \parallel 1$. This creates $2^{k-2} - 1$ additional matching edges.

We have constructed a matching of size $2^{k-1} - 1$. Since there are two unmatched vertices in $G'$, this immediately implies that the maximum size of a non-trivial independent set in $G'$ (and hence in $G_k$) is at most $2^{k-1} + 1$. □

### Remark 12:

The bound proven in Theorem 11 is tight. This can be seen by observing that $\{00 \cdots 0\} \cup Y_1$ is an independent set of size $2^{k-1} + 1$.

**Corollary 13:** For $k \geq 2$, it holds that

$$C(k, n) \leq (2^{k-2} + 1) \times 2^{k-2} \times 2^{n-k} = 2^{n-4} + 2^{n-k-2}.$$  

**Proof:** This is a straightforward application of Theorems 8 and 11. When $k \geq 2$, the value $2^{k-1} + 1$ is odd. Therefore we maximize the product $|X_C| \cdot |Y_C|$ by taking

$$|X_C| = 2^{k-2} + 1$$

and

$$|Y_C| = 2^{k-2}$$

(or vice versa).

We note that the upper bound proven in Corollary 13 is weaker than the bound proven in Theorem 2.

### A. Results for Small Values of $k$

Let $I(k)$ denote the maximum size of a non-trivial independent set in $G_k$. Table II summarizes the exact values of $I(k)$ and $C(k, n)$ for $k \leq 6$.

| $k$ | $I(k)$ | $C(k, n)$   |
|-----|-------|------------|
| 1   | 2     | $2^{n-2}$  |
| 2   | 3     | $2 \times 2^{n-4}$ |
| 3   | 5     | $6 \times 2^{n-6}$ |
| 4   | 9     | $20 \times 2^{n-8}$ |
| 5   | 16    | $64 \times 2^{n-10}$ |
| 6   | 30    | $216 \times 2^{n-12}$ |

**Remark:** The bounds for $k \leq 6$ are exact. For $k > 6$, the bounds are upper bounds only.

**Table II**

**Exact Values of $I(k)$ and $C(k, n)$ for $k \leq 6$**
This solution is in fact optimal, as was verified by an exhaustive search. Here are the 6-tuples in the sets $P$ and $S$:

$P$

000000, 000001, 000010, 000011, 000100, 000101, 000110, 000111, 001000, 001001, 001010, 001011, 001100, 001101, 001110, 001111, 010001, 010010, 010011, 010100, 010101, 010110, 010111, 011001, 011000, 011010, 011011, 011100, 011101, 011110, 011111

$S$

001101, 001111, 010111, 011011, 101011, 110011, 110110, 111001, 111011, 111100, 111101, 111111

V. The $m$-Minimum Construction

For $k \geq 7$, exhaustive searches appear to be infeasible. So we have tried various techniques to find useful lower bounds. We first describe the $m$-minimum Construction, which has enabled us to find some good solutions.

Construction 14 ($m$-Minimum Construction): Suppose $k$ is a given positive integer. For $m = 1, 2, \ldots, 2^{k-1}$, we construct a code $D_m$ as follows:

- Let $P$ consist of the first $m$ non-negative integers, represented as binary $k$-tuples (padded on the left with 0’s if necessary, i.e., in big-endian form). Define $X_C = \{x_p : p \in P\}$.
- Let $Y_C$ consist of all vertices in $Y$ that are adjacent to no vertices in $X_C$. Define $S = \{s : y_s \in Y_C\}$.
- Output the sets $P$ and $S$ for the code $D_m$ that maximizes the value of $|P| \times |S|$. The resulting $(1, k)$-overlapping-free code will have size $|P| \times |S| \times 2^{n-2k}$.

Table III summarizes results obtained from the $m$-minimum Construction. For $k \geq 6$, these are all improvements over the Doubling Construction. The optimal solution for $k = 6$ that we presented in Section IV-A is precisely the code $D_{12}$ obtained from the $m$-minimum Construction. For $k = 7$, $D_{24}$ is the code found by the $m$-minimum Construction; it has $|P| = 24$ and $|S| = 31$:

$P$

000000, 000001, 000010, 000011, 000100, 000101, 000110, 000111, 001000, 001001, 001010, 001011, 001100, 001101, 001110, 001111, 010000, 010001, 010010, 010011, 010100, 010101, 010110, 010111, 011000, 011001, 011010, 011011, 011100, 011101, 011110, 011111

$S$

011011, 011110, 011111, 010011, 010010, 010100, 010101, 011010, 011011, 011101, 011100, 011111, 010110, 010111, 011110, 011111, 011001, 011000, 011011, 011100, 011101, 011110, 011111, 011011, 011010, 011001, 011000, 011011, 011101, 011110, 011111

This yields the lower bound

$$C(7, n) \geq 744 \times 2^{n-14}.$$

A. The Zero Block Construction

We now present the Zero Block Construction, which is closely related to the $m$-minimum Construction, and is inspired by the classical construction of non-overlapping codes due to Gilbert and Levenshtein [8], [9], [12] which we discuss in Section VI.

Construction 15 (Zero Block Construction): Suppose $k$ is a given positive integer. For $z = 1, \ldots, k - 1$, we construct a code $C_z$ from a certain $X_C$ and $Y_C$ as follows:

- Let $P$ consist of the first $2^k - z$ non-negative integers, represented as binary $k$-tuples. Note that every $p \in P$ begins with a block of (at least) $z$ consecutive 0’s. Define $X_C = \{x_p : p \in P\}$.
- Let $S$ consist of all binary $k$-tuples ending with a 1 that do not contain $z$ consecutive 0’s. Define $Y_C = \{y_s : s \in S\}$.
- Output the sets $P$ and $S$ for the code $C_z$ that maximizes the value of $|P| \times |S|$. The resulting $(1, k)$-overlapping-free code will have size $|P| \times |S| \times 2^{n-2k} = |S| \times 2^{n-k-z}$.

Lemma 16: For $X_C$ and $Y_C$ as defined in Construction V-A, no vertex in $Y_C$ is adjacent to any vertex in $X_C$.

Proof: Suppose $x_p \in X_C$ and $y_s \in Y_C$. We consider two cases. If $\ell \leq z$, then the $\ell$-prefix of $p$ consists of $\ell$ 0’s. However, $s$ ends in a 1, so the $\ell$-suffix of $s$ is not the same as the $\ell$-prefix of $p$. The second case is when $\ell \geq z + 1$. Here an $\ell$-prefix of $p$ begins with $z$ 0’s. However, no $\ell$-suffix of $s$ contains $z$ consecutive 0’s, so the $\ell$-suffix of $s$ is not the same as the $\ell$-prefix of $p$. □

Thus, for any fixed value of $z$, the set $Y_C$ defined in Construction V-A is a subset of the set that would be chosen in Construction 14 (the $m$-minimum Construction). So the Zero Block Construction cannot improve on the $m$-minimum Construction; however, it is an explicit construction and potentially easier to analyze. We will consider a general bound that can be proven, as well as numerical computations for various values of $k$.
It remains to specify an appropriate value for \( z \) and to investigate the size of \( S \). It turns out that the number of binary \( \ell \)-tuples \( s \) that do not contain \( n \) consecutive 0’s is given by an \( n \)-step Fibonacci number. For a given value of \( n \geq 2 \), the \( n \)-step Fibonacci sequence is defined recursively as follows.

\[
F_{i}^{(n)} = \begin{cases} 
0 & \text{if } -n+2 \leq i \leq 0 \\
1 & \text{if } i = 1 \\
\sum_{j=1}^{n} F_{i-j}^{(n)} & \text{if } i \geq 2.
\end{cases}
\]

That is, each term in this sequence is the sum of the \( n \) previous terms. It is easy to see that

\[
F_{i}^{(n)} = 2^{i-2}
\]

for \( 2 \leq i \leq n + 1 \). Also, it is easily verified that

\[
F_{n+2}^{(n)} = 2^n - 1 \quad \text{and} \quad F_{n+3}^{(n)} = 2^{n+1} - 3.
\]

For additional information about these sequences, see [7] and [14].

The following result is well-known. We provide a proof for completeness.

**Lemma 17:** The number of binary \( \ell \)-tuples that do not contain \( z \) consecutive 0’s is \( F_{\ell}^{(z)} \).

**Proof:** Denote the number of binary \( \ell \)-tuples that do not contain \( z \) consecutive 0’s by \( g(\ell, z) \). Then it is clear that

\[
g(\ell, z) = 2^{\ell} \quad \text{if } 1 \leq \ell < z, \text{ and } g(z, z) = 2^{z-1}.
\]

Thus \( g(\ell, z) = F_{\ell}^{(z)} \) if \( 1 \leq \ell \leq z \).

Next, consider \( g(\ell, z) \) for some \( \ell > z \). We partition the set of all binary \( \ell \)-tuples that do not contain \( z \) consecutive 0’s into \( z \) disjoint subsets, denoted by \( W_i, i = 1, \ldots, z \). For \( 1 \leq i \leq z \), the set \( W_i \) consists of all the \( \ell \)-tuples that end with a 1 followed by \( i - 1 \) 0’s. It is clear that \(|W_i| = g(\ell - i, k)| \) for \( 1 \leq i \leq \ell \).

Hence,

\[
g(\ell, z) = \sum_{i=1}^{z} g(\ell - i, z)
\]

whenever \( \ell > z \). We can assume by induction that \( g(\ell - i, z) = F_{\ell-i+2}^{(z)} \) for \( 1 \leq i \leq z \). So

\[
g(\ell, z) = \sum_{i=1}^{z} F_{\ell-i+2}^{(z)} = F_{\ell+2}^{(z)}
\]

from (1), as desired.

\( \square \)

The number of choices for \( s \in S \) is exactly \( F_{k+1}^{(z)} \). Thus we have the following result.

**Theorem 18:** The size of the code obtained from the Zero Block Construction is

\[
\max \left\{ F_{k+1}^{(z)} \times 2^{n-k-z} : 1 \leq z \leq k-1 \right\}.
\]

In order to obtain an explicit closed-form bound, it is probably more convenient to work with a simple lower bound on the values \( F_{k+1}^{(z)} \).

**Lemma 19:** For \( 1 \leq z \leq k-1 \), the following bound holds:

\[
F_{k+1}^{(z)} > (1 - k^{2^{-z}}) 2^{k-1}.
\]

**Proof:** Choose a binary word \( t \) of length \( k - 1 \) randomly and uniformly and then append a 1. Let \( E_i \) be the ‘bad’ event that \( t \) contains \( 0^z \) starting at position \( i \). Note that \( t \) is of the desired form if and only if none of the events \( E_1, E_2, \ldots, E_{k-1} \) occurs. But the probability of \( E_i \) is at most \( 2^{-z} \) (indeed it is equal to this when \( i \leq k - z \), and it is 0 otherwise). So the probability that one or more of the \( E_i \)’s occurs is at most \( (k-1) 2^{-z} \). Hence the probability that none of the events \( E_1, E_2, \ldots, E_{k-1} \) occur is at least \( 1 - (k-1) 2^{-z} \).

Since

\[
1 - (k-1) 2^{-z} > 1 - k 2^{-z},
\]

the stated bound follows.

\( \square \)

Now, using equation (2) from Theorem 18, for a given value of \( z \), we obtain a code of size at least

\[
(1 - k 2^{-z}) \times 2^{k-1} \times 2^{n-k-z} = (1 - k 2^{-z}) \times 2^{n-z-1} = (2^{-z} (1 - k 2^{-z})) 2^{n-1}.
\]

The function \( f(z) = 2^{-z} (1 - k 2^{-z}) \) is maximized when \( z = \log_2 2k \). Sadly, this is not always an integer. However, taking \( z_0 = \lfloor \log_2 2k \rfloor \) (i.e., rounding \( \log_2 2k \) to the nearest integer), we have

\[
\log_2 2k - 1/2 < z_0 < \log_2 2k + 1/2,
\]

so \( 2^{z_0} \in [\sqrt{2k}, 2\sqrt{2k}] \). It then follows that

\[
f(z_0) \geq \max\{f(\log_2 2k - 1/2), f(\log_2 2k + 1/2)\}.
\]

We have

\[
f(\log_2 2k - 1/2) = \frac{1}{\sqrt{2k}} \left( 1 - \frac{1}{\sqrt{2}} \right) \approx \frac{1}{4.83k}
\]

and

\[
f(\log_2 2k + 1/2) = \frac{1}{2\sqrt{2k}} \left( 1 - \frac{1}{\sqrt{2}} \right) \approx \frac{1}{4.38k}.
\]

Hence, \( f(z_0) \geq 1/(4.83k) \). Since the size of the resulting code is \( f(z_0) \times 2^{n-1} \), we have the following theorem.

**Theorem 20:** There exists \( z \) such that

\[
|C_z| > (1/9.67k) 2^n;
\]

hence

\[
C(k, n) \geq (1/9.67k) 2^n.
\]

We now incorporate two tweaks to improve Theorem 20. The first is to define the events \( E_1, E_2, \ldots \) used in the proof of Lemma 19 a bit more carefully.

**Lemma 21:** For \( 1 \leq z \leq k-1 \), the following bound holds:

\[
F_{k+1}^{(z)} \geq (1 - k 2^{-z-1}) 2^{k-1}.
\]

**Proof:** As before, choose a binary word \( t \) of length \( k - 1 \) randomly and uniformly and then append a 1. We define \( E_i \) as before. However, for \( 2 \leq i \leq k - z \), we now define \( E_i \) to be the event that there is a 1 in position \( i - 1 \), followed by \( z \) 0’s. It is not hard to see that if \( t \) contains \( z \) consecutive zeroes, then one of the events \( E_1, \ldots, E_{k-z-1} \) occurs. This is because the first occurrence of \( z \) consecutive 0’s must immediately follow a 1, except when the first \( z \) positions are all 0’s.
We have \( \Pr[E_1] = 2^{-z} \) and \( \Pr[E_i] = 2^{-z-1} \) for \( 2 \leq i \leq k-z \). Hence,

\[
\Pr[E_1 \lor \cdots \lor E_{k-z}] \leq 2^{-z} + (k-z-1)2^{-z-1} = 2^{-z-1}(2 + k - z - 1) \leq k 2^{-z-1},
\]
since \( z \geq 1 \). Hence,

\[
\Pr\left[ E_1 \wedge \cdots \wedge E_{k-z} \right] \geq 1 - k 2^{-z-1}.
\]
The stated bound follows. \( \square \)

Using equation (2) from Theorem 18, for a given value of \( z \), we obtain a code of size at least

\[
(2^{-z}(1 - k 2^{-z-1}))2^{n-1}.
\]

In order to maximize the size of the code, we choose \( z \) to maximize the function

\[
g(z) = 2^{-z}(1 - k 2^{-z-1}).
\]

The maximum occurs when \( z = \log_2 k \), which of course might not be an integer. We could consider an interval of length 1 whose centre is at \( \log_2 k \) (similar to our argument above), but we can do slightly better by considering a different interval (this is our second tweak).

We choose \( z \) to be an integer in the interval \([\log_2 \frac{3k}{4}, \log_2 \frac{3k}{2}]\). Notice that this is again an interval of length 1. We obtain a slightly better bound because \( g\left( \log_2 \frac{3k}{4} \right) = g\left( \log_2 \frac{3k}{2} \right) \).

In fact,

\[
g\left( \log_2 \frac{3k}{4} \right) = g\left( \log_2 \frac{3k}{2} \right) = \frac{4}{9k}.
\]

We immediately obtain the following theorem, which improves Theorem 20.

**Theorem 22:** There exists \( z \) such that

\[
|C_z| > (2/9k)2^n;
\]
hence \( C(k, n) \geq (2/9k)2^n \).

When \( k \) is a power of 2, the function \( g(z) \) is maximized at the integral value \( z = \log_2 k \). We obtain an improved result in this case.

**Theorem 23:** If \( k = 2^i \) for a positive integer \( i \), then

\[
|C| > (1/4k)2^n;
\]
hence \( C(k, n) \geq (1/4k)2^n \) for these values of \( k \).

We note that the upper bound from Theorem 2 is \( C(k, n) \leq (1/2k)2^n \), which is roughly a factor of two greater than the lower bound from Theorem 23 (when \( k \) is a power of two).

It is also possible to obtain asymptotic bounds which are stronger than the explicit general bounds discussed above. We pursue this now.

Let \( \ell \) and \( z \) be integers, with \( 1 \leq z < \ell \). For an integer \( k \) with \( 0 \leq k < \ell \), define \( \phi(\ell, k, z) \) to be the number of binary sequences of length \( \ell \) and weight \( k \) such that any two cyclically consecutive ones are separated by at least \( z \) zeros. The following lemma gives bounds for \( \phi(\ell, k, z) \) that are good when \( k \) and \( z \) are small compared to \( \ell \).

**Lemma 24:** Define \( \ell, z, k \) and \( \phi(\ell, k, z) \) as above. Then

\[
\binom{\ell}{k} - k \binom{\ell}{k-1} \leq \phi(\ell, k, z) \leq \binom{\ell}{k}.
\]

**Proof:** The lemma follows trivially in the case when \( k \leq 1 \), since \( \phi(\ell, 0, z) = 1 \) and \( \phi(\ell, 1, z) = \ell \). So we may assume that \( k \geq 2 \).

The upper bound follows since \( \binom{\ell}{k} \) is the number of weight \( k \) binary sequences of length \( \ell \). The lower bound follows if we can show that there are at most \( k \binom{\ell}{k-1} \) weight \( k \) binary sequences of length \( \ell \) that have a zero run of length less than \( z \). But all such sequences can be obtained (possibly more than once) in the following three-stage process. In Stage 1, choose a set of \( k-1 \) positions in the sequence to be equal to 1. In Stage 2, choose one of these \( k-1 \) positions, say position \( i \). In Stage 3, choose a position \( i + a \mod \ell \) where \( 1 \leq a \leq z \) and set this position equal to 1; set the remaining positions to be zero. There are at most \( \binom{\ell}{k-1} \) choices in the first stage, there are \( k-1 \) choices in the second stage and at most \( z \) choices in the third stage. So

\[
\phi(\ell, k, z) \geq \binom{\ell}{k} - k \binom{\ell}{k-1} (k-1)z \geq \binom{\ell}{k} - k z \binom{\ell}{k-1},
\]
as required. \( \square \)

**Corollary 25:** Define \( \ell, z, k \) and \( \phi(\ell, k, z) \) as above. Then

\[
\frac{1}{k!} - \frac{2kz}{\ell} \leq \frac{\phi(\ell, k, z)}{\ell^k} \leq \frac{1}{k!}.
\]

**Proof:** The upper bound follows from the upper bound of Lemma 24 and the inequality

\[
\binom{\ell}{k} \leq \ell^k / k!.
\]

For the lower bound, we use the lower bound of Lemma 24 and the same bound on a binomial coefficient to see that

\[
\frac{\phi(\ell, k, z)}{\ell^k} \geq \frac{\ell^k}{k!} \frac{kz}{k} \binom{\ell}{k-1} \frac{\ell}{k-1} = \frac{\ell^k}{k!} - \frac{kz}{k-1} \frac{\ell}{k}.
\]

The lower bound now follows since

\[
\ell^k \geq (\ell-1)^k \geq \ell^k - k^k \frac{k^k}{k!} \geq \ell^k - k z \ell^{k-1}. \quad \Box
\]

**Theorem 26:** For a positive integer \( a \), define \( \ell = 2^a \) and \( z = a-1 \) (so \( 2^{a+1} = \ell \)). Let \( \nu_a \) be the number of binary sequences of length \( \ell \) that do not contain any cyclic runs of \( z \) or more consecutive zeros. Then \( \lim_{a \to \infty} \nu_a/2^a = 1/e \) (where \( e \) is the base of the natural logarithm).

**Proof:** Let \( X_i \) be the set of sequences

\[
s = (s_0, s_1, \ldots, s_\ell)
\]
such that \( s_i = 1 \) and \( s_{i+1} = s_{i+2} = \cdots = s_{i+z} = 0 \). (Here we take subscripts modulo \( \ell \).)

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Note that \( s \) has no cyclic runs of \( z \) or more zeros if and only if \( s \) is non-zero and \( s \not\in X_i \) for \( i \in L = \{0, 1, 2, \ldots, \ell - 1\} \). Hence
\[
\nu_a = \left| \bigcup_{i \in L} X_i \right| - 1. \tag{3}
\]

By the principle of inclusion-exclusion,
\[
\left| \bigcup_{i \in L} X_i \right| = \sum_{k=0}^{\ell - 1} (-1)^k \sum_{I \subseteq \{1, \ldots, \ell\} : |I| = k} \left| \bigcap_{i \in I} X_i \right|, \tag{4}
\]
where the partial sums involving \( k \) on the right hand side are successively upper and lower bounds for the left hand side (this follows from the Bonferroni inequalities).

For a subset \( I \subseteq L \), let
\[
t_I = (t_{i_0}, t_{i_1}, \ldots, t_{i_{\ell - 1}})
\]
be the indicator binary sequence for \( I \), so
\[
t_i = \begin{cases} 
1 & \text{if } i \in I, \\
0 & \text{otherwise.}
\end{cases}
\]
When \( k \geq 1 \) we see that
\[
\bigcap_{i \in I} X_i = 2^{\ell - (z + 1)k},
\]
when any two cyclically consecutive ones in \( t_I \) are separated by at least \( z \) zeros, and is 0 otherwise. So using the notation above Lemma 24, we may simplify (4) as
\[
\left| \bigcup_{i \in L} X_i \right| = \sum_{k=0}^{\ell - 1} (-1)^k \phi(\ell, k, z) 2^{\ell - (z + 1)k}
\]
\[
= 2^\ell \left( \sum_{k=0}^{\ell - 1} (-1)^k \phi(\ell, k, z) \frac{2^k}{k!} \right), \tag{5}
\]
since \( 2^{z+1} = \ell \) by our choice of \( \ell \) and \( z \).

Define \( b = \lceil \ell^{1/4} \rceil \). We noted above that successive partial sums in the right hand side of (5) are upper and lower bounds for the left hand side, so truncating this sum after \( b + 1 \) terms, we see that
\[
\left| \bigcup_{i \in L} X_i \right| - 2^\ell \left( \sum_{k=0}^{b} (-1)^k \phi(\ell, k, z) \frac{2^k}{k!} \right)
\]
\[
\leq 2^\ell \left( \sum_{k=0}^{b} (-1)^k \phi(\ell, b + 1, z) \frac{2^k}{k^b+1} \right) \leq 2^\ell \left( \frac{2\ell}{(b + 1)!} \right), \tag{6}
\]
the final inequality following by the upper bound of Corollary 25. Now, \( z < b \) when \( a \) is sufficiently large, since \( z = a - 1 \) and \( b \geq 2^{a/4} - 1 \). So, using the bounds in Corollary 25,
\[
2^\ell \left( \sum_{k=0}^{b} (-1)^k \phi(\ell, k, z) \frac{2^k}{k!} \right) - 2^\ell \left( \sum_{k=0}^{b} (-1)^k \right)
\]
\[
\leq 2^\ell \sum_{k=0}^{b} \frac{2k}{k!} \leq 2^\ell \sum_{k=1}^{b} \frac{2k^2}{k!}
\]
\[
= 2^\ell \frac{2b^2}{e}, \tag{7}
\]
whenever \( a \) is sufficiently large. But the usual power series expansion for \( 1/e \) shows that
\[
2^\ell \left( \frac{1}{e} - \sum_{k=0}^{b} \frac{(-1)^k}{k!} \right) \leq \frac{2^\ell}{(b + 1)!}. \tag{9}
\]
Combining equations (3), (6), (7) and (9) we see that \( \nu_a = 2^\ell (1/e + \varepsilon) \), where
\[
|\varepsilon| \leq \frac{1}{2^\ell} + \frac{2}{(b + 1)!} + \frac{2}{e2^{b/4}}
\]
whenever \( a \) is sufficiently large. In particular \( \varepsilon \) tends to zero as \( a \to \infty \), and so the theorem follows. \( \square \)

We remark that Schoeny et al. [15, Subsection V.B] prove a bound on the number of binary sequences with no (zero or one) runs of length \( \log(2n) \), using a probabilistic construction. We wonder whether their bounds could be improved using techniques similar to those in the proof of Theorem 26.

**Corollary 27:** For a positive integer \( a \), define \( \ell = 2^a \) and \( z = a - 1 \). Then
\[
\lim_{a \to \infty} \frac{F_{\ell+2}^{(z)}}{2^\ell} = \frac{1}{e},
\]
(where \( e \) is the base of the natural logarithm).

**Proof:** Recall that \( \nu_a \) is the number of binary sequences of length \( \ell \) that do not contain any cyclic runs of \( z \) or more consecutive zeros. Also, \( F_{\ell+2}^{(z)} \) is the number of binary sequences of length \( \ell \) that do not contain any runs of \( z \) or more consecutive zeros. Clearly
\[
\nu_a \leq F_{\ell+2}^{(z)}.
\]
A sequence with no (non-cyclic) runs of \( z \) or more consecutive zeros, but which contains a cyclic run of \( z \) or more zeros, must either start or end with at least \( \lfloor a/2 \rfloor \) zeros. Hence
\[
F_{\ell+2}^{(z)} - \nu_a \leq 2^{2^\ell - \lfloor a/2 \rfloor}.
\]
Hence
\[
2^{-\ell} |\nu_a - F_{\ell+2}^{(z)}| \leq 2^{1 - \lfloor a/2 \rfloor}.
\]
Since the right hand side of this inequality tends to 0 as \( a \to \infty \), the corollary follows by Theorem 26. \( \square \)

We can now prove the following asymptotic lower bound on \( C(k, n) \).

**Theorem 28:**
\[
\lim_{k \to \infty} \frac{C(k, n)}{2^n} \geq \frac{1}{ek}.
\]

**Proof:** Take \( k = \ell + 1 = 2^a + 1 \) in equation (2) from Theorem 18. Since \( z = a - 1 \), we now have \( 2^{z+1} = k - 1 = \ell \). Then Theorem 18 says that
\[
\lim_{k \to \infty} \frac{C(k, n)}{2^n}
\]
\[
\geq \lim_{k \to \infty} \frac{F_{k+1}^{(z)}}{2^{k+1}}
\]
\[
= \lim_{\ell \to \infty} \frac{F_{\ell+2}^{(z)}}{2^{\ell+1}}
\]
\[
= \frac{1}{e2^{z+1}} \text{ from Corollary 27}
\]
in the third column of [5, Table 1] are computed using the formula (10).

We can use the techniques developed in Section V-A to give an explicit, non-asymptotic lower bound on $S(n)$.

\textbf{Lemma 32:}

$$F_{n-z}^{(z)} > (1 - n 2^{-z-1})2^{n-z-2}.$$  

\textbf{Proof:} If we take $k = n-z - 1$ in Lemma 21, we obtain

$$F_{n-z}^{(z)} \geq (1 - (n - z - 1) 2^{-z-1})2^{n-z-2}.$$  

Clearly, 

$$1 - (n - z - 1) 2^{-z-1} > 1 - n 2^{-z-1},$$

so the stated bound follows.  \hfill \square

We now choose $z$ to maximize the function $h(z) = (1 - n 2^{-z-1})2^{-z-2}$. The maximum occurs when $z = \log_2 k$, which of course might not be an integer. Choose $z$ to be an integer in the interval $[\log_2 \frac{3n}{4}, \log_2 \frac{3n}{2}]$. Then we have

$$h \left( \log_2 \frac{3n}{4} \right) = h \left( \log_2 \frac{3n}{2} \right) = \frac{1}{9n},$$

and we obtain the following theorem.

\textbf{Theorem 33:} $S(n) > (1/9n)2^n$.

When $n$ is a power of 2, the maximum value of $h(z)$ occurs when $z = \log_2 n$, and so we do slightly better: 

\textbf{Theorem 34:} If $n$ is a power of two, then $S(n) > (1/8n)2^n$.

These bounds improve previous explicit bounds. In Bilotta, Pergola and Pinzani [3], an explicit construction based on Dyck paths was given. However, it was observed by Chee et al. [5] that this construction does not yield a lower bound of the form $S(n) > (c/n)2^n$ for any constant $c > 0$. Also, Blackburn [4] proved that $S(n) > (3/64n)2^n$; our lower bound from Theorem 33 is stronger.

As far as asymptotic bounds are concerned, Levenshtein [9] proved that

$$\limsup_{n \to \infty} S(n) \geq (1/2en)2^n \approx (1/5.436n)2^n.$$  

Levenshtein’s asymptotic bound also follows easily from Corollary 27 and Theorem 31, as we now demonstrate.

\textbf{Theorem 35:} $\limsup_{n \to \infty} S(n) \geq (1/2en)2^n$.

\textbf{Proof:} We prove that

$$\limsup_{n \to \infty} \frac{2nS(n)}{2^n} \geq \frac{1}{e}.$$  

From Theorem 31, we have

$$\frac{2nS(n)}{2^n} \geq \frac{2nF_{n-z}^{(z)}}{2^n}$$

for any $z$ such that $1 \leq z \leq n - 1$. Let $\ell = 2^a$, $z = a - 1$ and $n = \ell + a + 1$ for a positive integer $a$. Then $n - z = \ell + 2$. For these values of $n$ and $z$, we compute

$$\frac{2nF_{n-z}^{(z)}}{2^n} = \frac{2(\ell + a + 1)}{2^{a+1}} \times \frac{F_{\ell + 2}^{(a-1)}}{2^\ell} = \frac{\ell + a + 1}{\ell} \times \frac{F_{\ell + 2}^{(a-1)}}{2^\ell}.$$  

Finally, it is perhaps also of interest to compute the exact size of the codes obtained from the Zero Block Construction for “small” values of $k$. We use the formula (2) from Theorem 18. For a fixed “small” value of $k$, we choose $z$ to maximize $F_{k+1}' \times 2^{-z}$. This is easily done by iterating through the possible values of $z$ to see which one gives the largest result. The exact values $F_{k+1}'$ are computed very quickly from the recurrence relation (1).

We present some data in Table IV comparing the Zero Block Construction to the $m$-minimum Construction. For the Zero Block Construction, we also include the optimal value of $z$. Table V provides a summary of the constructions and bounds in this paper. It is interesting to observe that the Zero Block Construction performs almost as well as the $m$-minimum Construction in all cases, and it gives the same result in many cases. However, the computations of the bounds for the Zero Block Construction are amazingly fast. For example, it is almost instantaneous to compute the lower bound

$$5745596237141382$$

$$785608786499535716424326$$

$$792561835200479232 \times 2^{n-200}$$

\section{VI. Non-Overlapping Codes}

We can apply the techniques of Section V-A to the construction of “classic” non-overlapping codes. Again, we restrict our attention to the binary case. The following construction is due to Gilbert and Levenshtein; it has been re-discovered several times, and is used in many applications. See [5], [8], [9], [12], and [13].

\textbf{Construction 29 (Gilbert–Levenshtein Construction):}

Suppose $n$ is a given positive integer. For $z = 1, \ldots, k - 1$, we construct a code $L_z$ as follows:

- each codeword $c = (c_1, \ldots, c_n) \in L_z$ begins with a block of $z$ consecutive 0’s,
- $c_{z+1} = c_n = 1$, and
- the sequence $(c_{z+1}, \ldots, c_{n-1})$ does not contain $z$ consecutive 0’s.

It is clear that $|L_z|$ equals the number of binary sequences of length $n-z-2$ that do not contain $z$ consecutive 0’s. Hence, from Lemma 17, we have the following.

\textbf{Lemma 30:} $|L_z| = F_{n-z}^{(z)}$.

Of course we would choose $z$ to maximize $|L_z|$. Let $S(n)$ denote the size of the code obtained from the Gilbert–Levenshtein Construction. The following result is immediate.

\textbf{Theorem 31:}

$$S(n) = \max \left\{ F_{n-z}^{(z)} : 1 \leq z \leq n - 1 \right\}. \quad (10)$$

We note that the connection between the Gilbert–Levenshtein Construction and the $n$-step Fibonacci numbers was pointed out by Chee et al. [5]. In fact, the entries
It is clear that \((\ell + a + 1)/\ell\) approaches 1 as \(a \to \infty\), because \(\ell = 2^a\). Also, from Corollary 27, \(F_{a+1}^{(a-1)} / 2^\ell\) approaches \(1/e\) as \(a \to \infty\). The desired result follows. \(\square\)

VII. DISCUSSION AND SUMMARY

In this paper, we have mainly concentrated on \((1,k)\)-overlap-free codes over a binary alphabet. Our constructions and bounds are actually quite close. There are very possible avenues for future research, including studying variable-length analogs, studying codes over non-binary alphabets, or investigating codes with other forbidden overlaps. One direction that might be fruitful for applications is the investigation of codes which are simultaneously \((1,k)\)-overlap-free and \((n-k,n-1)\)-overlap-free, where \(k < \frac{n}{2} \).

The Zero Block Construction is inspired by a classical construction of non-overlapping codes due to Gilbert and Levenshtein. It is surprising to us that the \(m\)-construction of non-overlapping codes due to Gilbert and Levenshtein works for infinitely many values of \(n\), whereas our \(m\)-minimum and Zero Block Construction give the same bound for infinitely many values of \(n\). Finally, we note that the constructions in Section VI are most effective when \(n\) is close to a power of two. We ask if there are constructions that are asymptotically better when \(n\) is not of this form, for example when \(n = \lceil 2^{a+1}/2 \rceil \) as \(a \to \infty\).

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