Classical Liouville action
on the sphere with three hyperbolic singularities

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Abstract

The classical solution to the Liouville equation in the case of three hyperbolic singularities of its energy-momentum tensor is derived and analyzed. The recently proposed classical Liouville action is explicitly calculated in this case. The result agrees with the classical limit of the three point function in the DOZZ solution of the quantum Liouville theory.

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1 Introduction

In the last few years a considerable progress in the Liouville theory has been achieved [1]. The solution to the quantum theory based on the structure constants proposed by Otto and Dorn [2] and by A. and Al. Zamolodchikov [3] was recently completed by Ponsot and Teschner [4–6] and by Teschner [1, 7].

On the other hand there exists so called geometric approach originally proposed by Polyakov [8] and further developed by Takhtajan [9–12]. In contrast to the operator formulation of the DOZZ theory the correlators of primary fields with elliptic and parabolic weights are expressed in terms of a path integral over conformal class of Riemannian metrics with prescribed singularities at the punctures. Some of the predictions derived from this representation can be rigorously proved and lead to deep geometrical results [13–17].

In spite of these achievements the relation between both approaches is not fully understood [18, 19]. This in particular concerns the problem of path integral representation of the Liouville correlators with hyperbolic weights which in order requires an appropriate choice of the Liouville action functional. A step in this direction was done in our previous paper [20] where we constructed the classical Liouville action satisfying the Polyakov conjecture in the case of hyperbolic singularities.

In the present paper we calculate the classical Liouville action for three hyperbolic singularities and show that the result agrees with the classical limit of the corresponding DOZZ three point function. In the case of three singularities the monodromy problem considerably simplifies and the solution can be find in terms of hypergeometric functions [21]. It shares all the properties required by the construction of the Liouville action and the proof of the Polyakov conjecture given in [20].

The organization of the paper is as follows. In Sect.2 we briefly describe the $SL(2, \mathbb{R})$ monodromy problem for $n \geq 3$ hyperbolic singularities on the Riemann sphere. The singularity structure of the corresponding solutions to the Liouville equation is analyzed. In Sect.3 we present the essential steps of the construction of the classical Liouville action and calculate its partial derivatives with respect to the conformal weights. In Sect.4 we derive the solution to the monodromy problem in the case of three hyperbolic singularities. The corresponding solution to the Liouville equation defines a singular hyperbolic geometry on the sphere. We illustrate the structure of the closed geodesics and the singular lines of this geometry on two examples. In Sect.5 we find an explicit expression for the classical Liouville action and compare it with the classical limit of the DOZZ three point function invariant with respect to the reflections $\alpha \to Q - \alpha$.

The classical Liouville action proposed in [20] can be easily generalized to the case of an arbitrary number of elliptic, parabolic and hyperbolic singularities on the Riemann sphere. The results of the present paper provide additional justification for this construction. The path integral representation of correlation functions based on this action functional may provide a new insight into the geometric content of the quantum Liouville theory and is certainly worth further investigations. Also the classical theory itself rises some interesting questions concerning the existence and the uniqueness of regular solutions to the $SL(2, \mathbb{R})$ monodromy problem. Let us finally mention that the classical solutions with hyperbolic singularities are of interest in 3-dim gravity where they describe multi-black-hole configurations [22–26].
2 \textit{SL}(2, \mathbb{R}) \text{ monodromy problem}

The construction of the Liouville action proposed in [20] relies on solutions of a special monodromy problem for the Fuchsian equation

\[
\partial^2 \psi(z) + \frac{1}{8\beta} T(z) \psi(z) = 0
\]

where

\[
T(z) = \sum_{j=1}^{n} \left[ \frac{\Delta_j}{(z - z_j)^2} + \frac{c_j}{z - z_j} \right]
\]

is assumed to be regular at infinity and \( \Delta_j \) are hyperbolic conformal weights

\[
\Delta_j = 2\beta(1 + \lambda_j^2), \quad \lambda_j \in \mathbb{R}, \quad j = 1, \ldots, n.
\]

The problem is to adjust the accessory parameters \( c_j \) in such a way that the equation (1) admits a fundamental system of solutions with \( \text{SL}(2, \mathbb{R}) \) monodromies around all singularities. If \( \Psi(z) \) is such a system with Wronskian normalized to 1, then for each singularity \( z_j \) there exists an element \( B_j \in \text{SL}(2, \mathbb{R}) \) such that the system \( \Psi^j(z) = B_j \Psi(z) \) has the canonical form

\[
\Psi^j(z) = \begin{pmatrix} \psi^j_+(z) \\ \psi^j_-(z) \end{pmatrix} = (i\lambda_j)^{-\frac{1}{2}} \begin{pmatrix} e^{\frac{i}{2} \vartheta_j(z - z_j) \frac{1}{2}(1 + i\lambda_j)} u^j_+(z) \\ e^{-\frac{i}{2} \vartheta_j(z - z_j) \frac{1}{2}(1 - i\lambda_j)} u^j_-(z) \end{pmatrix},
\]

where \( \vartheta_j \in \mathbb{R} \). \( u^j_\pm(z) \) are analytic functions on the disc \( D_j = \{ z \in X : |z - z_j| < \min_{i, i \neq j} |z_i - z_j| \} \) such that \( u^j_\pm(0) = 1 \). Each system \( \Psi^j(z) \) defines by the formula

\[
e^{-\frac{\varphi(z, \bar{z})}{2}} = \pm i \frac{\sqrt{m}}{2} \left( \psi^j_+(z) \overline{\psi^j_-(z)} - \overline{\psi^j_+(z)} \psi^j_-(z) \right)
\]

the same single-valued, real solution \( \varphi(z, \bar{z}) \) to the Liouville equation

\[
\partial \bar{\partial} \varphi = \frac{m}{2} e^{\varphi}.
\]

In the case of hyperbolic weights the singularity structure of \( \varphi \) can be described by means of the multi-valued conformal maps

\[
\rho_j(z) = \begin{pmatrix} \psi^j_+(z) \\ \psi^j_-(z) \end{pmatrix}^{1/2} = e^{\frac{\vartheta_j}{2}} \left[ z - z_j + \frac{c_j}{2\Delta_j} (z - z_j)^2 + \mathcal{O}(\left( z - z_j \right)^3) \right].
\]

Singularities of \( \rho_j(z) \) on \( X = \mathbb{C} \setminus \bigcup_{j=1}^{n} \{ z_j \} \) are branch points of infinite order located at zeros of \( \psi^j_+(z) \) or \( \psi^j_-(z) \). In spite of this complicated analytical structure one easily checks that the metric \( e^{\varphi} d^2 z \) coincides with the pull-back of the metric

\[
\frac{\lambda_j^2 d^2 \rho}{m |\rho|^2 \sin^2 (\lambda_j \log |\rho|)}
\]

by the map \( \rho_j(z) \). We shall briefly discuss the consequences of this fact.
The only singularities of the metric $e^\phi d^2z$ are closed, non-intersecting smooth lines with the following properties:

- the locations $z_j$ of the energy-momentum tensor singularities are the only limit points\(^3\) of these lines;
- there are no regions in $\hat{\mathbb{C}}$ bounded by singular lines which do not contain at least one point $z_j$;
- none of the singular lines which separate $z_j$ from all other points $z_i, i \neq j$ contains branch points of $\rho_j$.

Let $\Sigma_j$ be the “most distant from $z_j$” singular line around $z_j$, defined by the property that there are no singular lines separating $\Sigma_j$ from all the points $z_i, i \neq j$. $\Sigma_j$ is an inverse image of one of the singular lines of the metric (8) on the $\rho$ plane:

$$\rho_j (\Sigma_j) = \{ \rho \in \mathbb{C} : \lambda_j \log |\rho| = \pi l_j \}$$

for some $l_j \in \mathbb{Z}$. It follows from the behavior of the hyperbolic geometry near singular lines that there exist a closed geodesic $\Gamma_j$ separating $\Sigma_j$ from all other “most distant” singular lines $\Sigma_i, i \neq j$. Then

$$\Gamma_j = \rho_j^{-1} (\{ \rho \in \mathbb{C} : \lambda_j \log |\rho| = \pi (l_j + \frac{1}{2}) \})$$

and the map $\rho_j$ is invertible on the hole $H_j$ around $z_j$:

$$H_j = \rho_j^{-1} (\{ \rho \in \mathbb{C} : \lambda_j \log |\rho| \leq \pi (l_j + \frac{1}{2}) \})$$

The geometry on $H_j$ is therefore isomorphic to that of the metric (8) on the disc $\{ \rho \in \mathbb{C} : \lambda_j \log |\rho| \leq \pi (l_j + \frac{1}{2}) \}$.

The region “between the holes” ($H_i \cap H_j = \emptyset$ for all $i \neq j$):

$$M \equiv \hat{\mathbb{C}} \setminus \bigcup_{j=1}^n H_j$$

(9)

carries a hyperbolic geometry with geodesic boundaries. It is however not guaranteed that for $n > 3$ there are no line singularities in $M$. We say that a solution to the $SL(2,\mathbb{R})$ monodromy problem is regular if the corresponding metric $e^\phi d^2z$ is regular on $M$. Note that in the case of three singularities any solution is regular.

3 Classical Liouville action, $n \geq 3$

Each regular solution to the $SL(2,\mathbb{R})$ monodromy problem uniquely determines the region $M \subset \hat{\mathbb{C}}$ (9). We start from the standard Liouville action on $M$

$$S_L [M, \phi] = \frac{1}{2 \pi} \int_M d^2z \left( \partial \phi \bar{\partial} \phi + m e^\phi \right) + \frac{1}{2 \pi} \int_{\partial M} |dz| \kappa_z \phi,$$  

(10)

\(^3\)with respect to the flat metric on the complex plane.
where \( \frac{d^2 z}{d z \wedge d \bar{z}} \) and \( \kappa_{z} \) is a geodesic curvature of \( \partial M \) computed in the flat metric on the complex plane. For finite locations \( z_j \) of hyperbolic singularities \( M \) is unbounded and one has to impose an appropriate asymptotic conditions on admissible solutions. It can be done by means of the modified action

\[
S_{L}^{\infty} [M, \phi] = \lim_{R \to \infty} S_{L}^{R} [M, \phi] ,
\]

\[
S_{L}^{R} [M, \phi] = \frac{1}{2 \pi} \int_{M} d^2 z \left( \partial \phi \bar{\partial} \phi + m e^{\phi} \right) + \frac{1}{2 \pi} \int_{\partial M} |dz| \kappa_{z} \phi + \frac{1}{\pi R} \int_{|z|=R} |dz| \phi + 4 \log R ,
\]

where \( M^{R} = \{ z \in M : |z| \leq R \} \). The stationary point of this functional \( \varphi(z, \bar{z}) \) is given by the formula (5). In the case of regular solution it defines on \( M \) the hyperbolic metric \( e^{\varphi} d^2 z \) with geodesic boundaries.

On each hole \( H_{j} \) there exists a unique flat metric with the only singularity at \( z_j \) such that the boundary \( \partial H_{j} = \Gamma_{j} \) is geodesic and its length is \( 2 \pi \frac{\lambda_{j}}{\sqrt{m}} \). It can be constructed as the pull-back by \( \rho_{j}(z) \) of the metric \( \frac{\lambda_{j}^2}{m |\rho(z)|^2} d^2 \rho \). Its conformal factor is given by the formula

\[
\varphi_{j}(z, \bar{z}) = \log \left[ \frac{\lambda_{j}^2}{m |\rho_{j}(z)|^2} \left| \frac{d \rho_{j}(z)}{dz} \right|^2 \right] \]

and satisfies the sewing relations along the boundary:

\[
\varphi(z, \bar{z}) = \varphi_{j}(z, \bar{z}), \quad n^a \partial_a \varphi(z, \bar{z}) = n^a \partial_a \varphi_{j}(z, \bar{z}) \quad \text{for} \quad z \in \Gamma_{j} .
\]

Using the expansion (7) one gets

\[
\varphi_{j}(z, \bar{z}) = \log \frac{\lambda_{j}^2}{m} - \log |z - z_j|^2 + \frac{c_j}{2 \Delta_j} (z - z_j) + \frac{\bar{c}_j}{2 \Delta_j} (\bar{z} - \bar{z}_j) + \mathcal{O} \left( |z - z_j|^2 \right) .
\]

We define on \( H_{j}^{\ast} \) the regularized classical action

\[
S_{L}^{\ast} [H_{j}, \varphi_{j}] = \frac{1}{2 \pi} \int_{H_{j}^{\ast}} d^2 z \left( \partial \varphi_{j} \bar{\partial} \varphi_{j} + m e^{\varphi_{j}} \right) + \frac{1}{2 \pi} \int_{\partial H_{j}^{\ast}} |dz| \kappa_{z} \varphi_{j} + (\lambda_{j}^2 - 1) \log \epsilon .
\]

The complete classical Liouville action for hyperbolic singularities then reads

\[
S_{L} \left[ \phi \right] = 4 \beta \lim_{\epsilon \to 0} S_{L}^{\ast} [\phi] ,
\]

\[
S_{L}^{\ast} [\phi] = S_{L}^{1/\epsilon} [M, \phi] + \sum_{k=1}^{n} S_{L}^{\ast} [H_{k}, \varphi_{k}] .
\]

The action above differs from the one proposed in [20] only by a function of \( \lambda_{j} \). This can be easily verified using the identity

\[
\int_{H_{j}^{\ast}} d^2 z \ m e^{\varphi_{j}(z, \bar{z})} = 2 \pi \lambda_{j}^2 \left( \log r_j e^{\varphi_{j}} - \log \epsilon \right) = -2 \pi \lambda_{j}^2 \log \left| r_j^{-1} \frac{d \rho_{j}(z)}{dz}(z_j) \right| - 2 \pi \lambda_{j}^2 \log \epsilon
\]

valid for \( \epsilon \) small enough. The difference is due to the fact that the line integral in (15) extends over the entire boundary of \( H_{j}^{\ast} \) and not only over \( \Gamma_{j} \).
With the help of the sewing relations (13) one can rewrite the classical action $S_L^\varepsilon[\varphi]$ (16) in the form

$$
S_L^\varepsilon[\varphi] = \frac{1}{2\pi} \int_M d^2z \left( \partial \varphi \partial \varphi + m e^\varphi \right) + \frac{1}{2\pi} \sum_{k=1}^{n} \int_{H_k^*} d^2z \left( \partial \varphi_k \partial \varphi_k + m e^{\varphi_k} \right) 
$$

$$
+ \frac{1}{2\pi} \sum_{k=1}^{n} \int_{|z-z_k|=\varepsilon} |dz| \ k_z \varphi_k + \frac{\varepsilon}{\pi} \int_{|z|=\varepsilon^{-1}} |dz| \varphi - \left( 4 + \sum_{k=1}^{n} (1-\lambda_k^2) \right) \log \varepsilon .
$$

(17)

It follows from (5) and (12) that

$$
\varphi = - \log m + \ldots , \quad \varphi_k = - \log m + \ldots ,
$$

(18)

where the dots denote the $m-$independent terms. This implies that the integrals in the first line of (17) do not depend on $m$. The only source of $m$ dependence of the classical action are therefore the line integrals in (17) and consequently the $m-$dependent term in $S_L[\varphi]$ is

$$
- \frac{2\beta}{\pi} \log m \left( \sum_{k=1}^{n} \int_{|z-z_k|=\varepsilon} |dz| \ k_z + 2\varepsilon \int_{|z|=\varepsilon^{-1}} |dz| \right) = 4\beta(n-2) \log m .
$$

(19)

As was shown in [20] the classical action (17) satisfies the Polyakov conjecture

$$
\frac{\partial S_L[\varphi]}{\partial z_j} = -c_j .
$$

(20)

Once the accessory parameters $c_j$ are known the relations (20) allow to compute the classical Liouville action up to the $z_j-$independent terms.

We conclude this section calculating the partial derivatives of the classical action (17) with respect to the parameters $\lambda_j$. Using the equations of motion

$$
\partial \bar{\varphi} = \frac{m}{2} e^\varphi , \quad \partial \bar{\varphi}_k = 0 ,
$$

(21)

the sewing relations (13), and integrating by parts one gets

$$
\frac{\partial}{\partial \lambda_j} S_L^\varepsilon[\varphi] = \frac{1}{2\pi} \sum_{k=1}^{n} \int_{H_k^*} d^2z \frac{\partial}{\partial \lambda_j} m e^{\varphi_k} + \frac{i}{4\pi} \sum_{k=1}^{n} \int_{H_k^*} \frac{\partial \varphi_k}{\partial \lambda_j} (\partial \varphi_k d\bar{z} - \partial \bar{\varphi}_k dz)
$$

$$
+ \frac{1}{2\pi} \sum_{k=1}^{n} \int_{|z-z_k|=\varepsilon} |dz| \ k_z \frac{\partial \varphi_k}{\partial \lambda_j} + 2\lambda_j \log \varepsilon .
$$

(22)

It follows from the expansion (14) that the second and the third term in (22) cancel in the limit. Changing the variables $z = \rho_k^{-1}(\rho)$ in the first one and using (12) one gets

$$
\frac{1}{2\pi} \int_{H_k^*} d^2z \frac{\partial}{\partial \lambda_j} m e^{\varphi_k(z,\bar{z})} = \frac{1}{2\pi} \int_{\rho_k^{-1}(H_k^*)} d^2\rho \frac{\partial}{\partial \lambda_j} \left( \frac{\lambda_k^2}{|\rho|^2} \right)
$$

$$
= 2\delta_{jk} \left( \pi (l_j + 1/2) - \vartheta_j - \lambda_j \log \varepsilon \right) ,
$$

(23)
where \( \rho_k^{-1}(H^*_k) = \{ \rho \in \mathbb{C} : e^{\rho \Phi_k} \leq |\rho| \leq e^{\Phi_k(t_j+1/2)} \} \) for \( \epsilon \) small enough. This finally gives

\[
\frac{\partial}{\partial \lambda_j} S_L[\varphi] = 4\beta \lim_{\epsilon \to 0} \frac{\partial}{\partial \lambda_j} S_L^* \varphi] = -8\beta \left( \vartheta_j - \pi(l_j + 1/2) \right). \tag{24}
\]

Let us note that the Liouville action \( S_L[\varphi] \) does not depend on the signs of \( \lambda_j \). In consequence the l.h.s of (24) is an odd function of \( \lambda_j \). It follows from (4) that one can always choose \( \vartheta_j \) to be an odd function of \( \lambda_j \) as well. Then \( l_j \) changes to \( -l_j - 1 \) with the change of sign of \( \lambda_j \).

4 Classical solution, \( n = 3 \)

In this section we shall analyze the solution of the \( SL(2, \mathbb{R}) \) monodromy problem for three hyperbolic singularities on the Riemann sphere. The energy-momentum tensor with singularities located at \( z = 0, 1 \) and \( \infty \) has the form

\[
T(z) = \frac{\Delta_1}{z^2} + \frac{\Delta_2}{(1 - z)^2} + \frac{\Delta_1 + \Delta_2 - \Delta_3}{z(1 - z)}. \tag{25}
\]

Let us consider the systems of solutions of the corresponding Fuchsian equation (1)

\[
\Psi^1(z) = \begin{pmatrix} \psi_+^1(z) \\ \psi_-^1(z) \end{pmatrix} = \begin{pmatrix} (i\lambda_1)^{-\frac{1}{2}} e^{\frac{z}{2}\vartheta_1} \psi(i\lambda_1, i\lambda_2, i\lambda_3; z) \\ (i\lambda_1)^{-\frac{1}{2}} e^{-\frac{z}{2}\vartheta_1} \psi(-i\lambda_1, -i\lambda_2, -i\lambda_3; z) \end{pmatrix}, \tag{26}
\]

where

\[
\psi(\lambda, \mu, \rho; z) = z^{\frac{1+\lambda + \frac{1}{2}}{2}} (1 - z)^{\frac{1-\mu}{2}} 2F_1 \left( \frac{1+\lambda - \mu + \rho, 1+\lambda - \mu - \rho}{2}, 1+\lambda; z \right)
\]

and \( 2F_1(a, b, c; z) \) is the hypergeometric function. This system has Wronskian normalized to 1 and its monodromy around \( z = 0 \) is diagonal:

\[
\Psi^1(e^{2\pi i}z) = \begin{pmatrix} -e^{-\pi \lambda_1} \\ 0 \end{pmatrix} \Psi^1(z).
\]

We shall show that for arbitrary hyperbolic weights we can choose a real number \( \vartheta_1(\lambda_1, \lambda_2, \lambda_3) \) that appears in (26) in such a way that the monodromy matrix \( M^1_2 \) at \( z = 1 \),

\[
\Psi^1(e^{2\pi i}(1 - z)) = M^1_2 \Psi^1(1 - z),
\]

belongs to \( SL(2, \mathbb{R}) \). To this end let us consider an auxiliary normalized system

\[
\tilde{\Psi}^2(z) = \begin{pmatrix} \tilde{\psi}_+^2(z) \\ \tilde{\psi}_-^2(z) \end{pmatrix} = \begin{pmatrix} (i\lambda_2)^{-\frac{1}{2}} \psi(i\lambda_2, i\lambda_1, -i\lambda_3; 1 - z) \\ (i\lambda_2)^{-\frac{1}{2}} \psi(-i\lambda_2, -i\lambda_1, i\lambda_3; 1 - z) \end{pmatrix},
\]

with a diagonal monodromy around \( z = 1 \),

\[
\tilde{\Psi}^2(e^{2\pi i}(1 - z)) = \begin{pmatrix} -e^{-\pi \lambda_2} \\ 0 \end{pmatrix} \tilde{\Psi}^2(1 - z).
\]

Using the method of [21] one can calculate the transition matrix

\[
\Psi^1(z) = S^{12} \tilde{\Psi}^2(z).
\]
In the case of hyperbolic singularities $\lambda_i \in \mathbb{R}$ it can be written in the form

$$S^{12} = i\sqrt{\lambda_1 \lambda_2} \begin{pmatrix} e^{i\frac{1}{2}g_-} & e^{i\frac{1}{2}g_+} \\ -e^{-i\frac{1}{2}g_+} & -e^{-i\frac{1}{2}g_-} \end{pmatrix}$$  \hspace{1cm} (27)$$

where

$$g_\pm = \frac{\Gamma(i\lambda_1)\Gamma(\pm i\lambda_2)}{\Gamma(\frac{1+i\lambda_1+i\lambda_2+i\lambda_3}{2})\Gamma(\frac{1+i\lambda_1+i\lambda_2-i\lambda_3}{2})}.$$  

With the formula (27) one can easily calculate the monodromy $M_2^1$ of $\Psi(z)$ at $z = 1$:

$$M_2^1 = S^{12} \begin{pmatrix} -e^{-\pi\lambda_2} & 0 \\ 0 & -e^{\pi\lambda_2} \end{pmatrix} (S^{12})^{-1} = \lambda_1 \lambda_2 \begin{pmatrix} e^{-\pi\lambda_2}|g_-|^2 - e^{\pi\lambda_2}|g_+|^2 & -(e^{\pi\lambda_2} - e^{-\pi\lambda_2}) e^{i\theta_1} g_+ g_- \\ (e^{\pi\lambda_2} - e^{-\pi\lambda_2}) e^{-i\theta_1} \frac{g_+ g_-}{g_+ g_-} & e^{\pi\lambda_2}|g_-|^2 - e^{-\pi\lambda_2}|g_+|^2 \end{pmatrix}.$$  

Using elementary properties of the gamma function one checks that $\det S^{12} = 1$. Thus the monodromy matrix $M_1^1$ belongs to $SL(2, \mathbb{R})$ if and only if $g_+ g_- e^{i\theta_1}$ is a real number. This yields the condition:

$$e^{2i\theta_1(\lambda_1, \lambda_2, \lambda_3)} = \frac{g_+ g_-}{g_+ g_-} = \frac{\Gamma^2(-i\lambda_1)}{\Gamma^2(i\lambda_1)} \gamma \left( 1+i\lambda_1+i\lambda_2+i\lambda_3 \right) \gamma \left( 1-i\lambda_1-i\lambda_2+i\lambda_3 \right) \frac{\Gamma(1-x)}{\Gamma(x)} = \frac{\Gamma(x)}{\Gamma(1-x)}.$$  \hspace{1cm} (28)$$

where

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}.$$  

It follows that $\Psi(z)$ with $\theta_1(\lambda_1, \lambda_2, \lambda_3)$ determined by (28) is a normalized solution to the $SL(2, \mathbb{R})$ monodromy problem with singularities at $z = 0, 1, \infty$ and with a diagonal monodromy at $z = 0$. Using the transformation properties of the the Fuchsian equation (1) and the conformal map

$$w(z) = \begin{pmatrix} z_2 - z_3 \\ z_2 - z_1 \end{pmatrix} = \frac{z - z_1}{z - z_3} \hspace{1cm} (29)$$

one can obtain a normalized solution $\Psi(z)$ for arbitrary locations $z = z_1, z_2, z_3$ of singularities with a diagonal monodromy at $z_1$:

$$\Psi(z) = \left( \frac{dw(z)}{dz} \right)^{-\frac{1}{2}} \Psi^1(w(z)).$$  \hspace{1cm} (30)$$

With an appropriate choice of a conformal map $w(z)$ and a permutation of the conformal weights one can use this formula to obtain solutions diagonal at other locations.

In particular, for the weights $\Delta_1, \Delta_2, \Delta_3$ at $0, 1, \infty$ the solution $\Psi^2(z)$ with a diagonal monodromy at $z = 1$ reads

$$\psi^2_\pm(z) = (i\lambda_2)^{-\frac{1}{4}} e^{\pm \frac{1}{4}i\theta_1(\lambda_2, \lambda_1, \lambda_3)} (-1)^{\frac{1}{2}} \psi(\pm i\lambda_2, \pm i\lambda_1, \pm i\lambda_3, 1 - z)$$  \hspace{1cm} (31)$$

and the solution $\Psi^3(z)$ with a diagonal monodromy at $z = \infty$ is given by

$$\psi^3_\pm(z) = (i\lambda_3)^{-\frac{1}{4}} e^{\pm \frac{1}{4}i\theta_1(\lambda_3, \lambda_2, \lambda_1)} (-z^2)^{\frac{1}{2}} \psi(\pm i\lambda_3, \pm i\lambda_2, \pm i\lambda_1, 1 - \frac{1}{z})$$  \hspace{1cm} (32)$$
All solutions $\Psi^1(z), \Psi^2(z)$ and $\Psi^3(z)$ define the same singular solution to the Liouville equation on the punctured sphere. The closed geodesic and the closed line singularities of the corresponding hyperbolic geometry $e^\varphi d^2 z$ can be found as appropriate levels of the functions $\rho_j$ in regions where they are invertible. Two examples of this structure are presented on Fig.1.

5 Classical Liouville action, $n = 3$

For arbitrary location of singularities of $T(z)$ the accessory parameters can be determined from the regularity of the energy–momentum tensor at infinity or, equivalently, from (25) and the transformation properties of $T(z)$ applied in the case of the global conformal map (29). They have the form

$$c_i = \frac{\Delta_i + \Delta_j - \Delta_k}{z_j - z_i} + \frac{\Delta_i + \Delta_k - \Delta_j}{z_k - z_i}$$

with $i, j, k = 1, 2, 3, i \neq j \neq k$. Integrating the equations (20), and their complex conjugate counterparts one then obtains:

$$S_L[\varphi] = (\Delta_1 + \Delta_2 - \Delta_3) \log |z_1 - z_2|^2 + (\Delta_2 + \Delta_3 - \Delta_1) \log |z_2 - z_3|^2$$

$$+ (\Delta_3 + \Delta_1 - \Delta_2) \log |z_3 - z_1|^2 + R(\lambda_1, \lambda_2, \lambda_3).$$

Proceeding to the limit $z_1 \to 0, z_2 \to 1, z_3 \to \infty$ in (24) one gets the equations

$$\frac{\partial}{\partial \lambda_j} R(\lambda_1, \lambda_2, \lambda_3) = -8\beta \left( \vartheta_j(\lambda_1, \lambda_2, \lambda_3) - \pi \left( l_j + \frac{1}{2} \right) \right)$$

where $\vartheta_1$ is given by (28), and $\vartheta_2$ and $\vartheta_3$ are determined by (31) and (32), respectively.
In order to calculate the constants $l_j$ one can consider the limit of small positive $\lambda_j$. It follows from (28) that $\lim_{\lambda_j \to 0^+} \vartheta_j(\lambda_1, \lambda_2, \lambda_3) = \pi$. Thus, for $\lambda_j \to 0^+$,

$$\rho_j(z) = e^{a + \frac{i}{u_j}} (z - z_j) + \mathcal{O}(\lambda_j)(z - z_j) + \mathcal{O}((z - z_j)^2),$$

where $a$ does not depend on $\lambda_j$. The geodesics around $z_j$ are then described by the equations

$$|z - z_j| \approx e^{-a - \frac{\pi}{2} \lambda_j} e^{\frac{\pi}{2} (l_j + 1)}.$$

For $\lambda_j$ small enough the “most distant from $z_j$” closed geodesic should be still close to $z_j$. On the other hand the next to the “most distant” singular line should stay away from $z_j$ in this limit. Both conditions are met only for $l_j = 0$.

Integrating (35) and taking into account (19) one finally gets

$$R(\lambda_1, \lambda_2, \lambda_3) = 8\beta \sum_{\sigma_1, \sigma_2 = \pm} F \left( \frac{1 + i\lambda_1}{2} + \sigma_1 \frac{i\lambda_2}{2} + \sigma_2 \frac{i\lambda_3}{2} \right) + 8\beta \sum_{j=1}^{3} \left( H(i\lambda_j) + \frac{\pi}{2} |\lambda_j| \right) + 4\beta \log m + \text{const.}$$

with

$$F(x) = \int_{\frac{x}{2}}^{x} dy \log \gamma(y), \quad H(x) = \int_{0}^{x} dy \log \frac{\Gamma(-y)}{\Gamma(y)}.$$

The parameterization of the conformal weights used in [3] reads

$$\Delta_\alpha = \alpha(Q - \alpha)$$

where $\alpha \in (0, Q/2] \subset \mathbb{R}$ or $\alpha \in Q/2 + i\mathbb{R}^+$. The “background charge” $Q$ is related to the Liouville coupling constant $b$ as

$$Q = b + \frac{1}{b}.$$

Comparing the formulae for the central charge obtained in the DOZZ and in the geometric approach [9, 18]

$$c_L = 1 + 6Q^2 = 1 + 48\beta$$

as well as the expressions for the conformal weights (3) and (38) one obtains the relations

$$\beta = \frac{Q^2}{8}, \quad \alpha = \frac{Q}{2} (1 + i\lambda),$$

where $\lambda \in \mathbb{R}$ in the case of hyperbolic weights. The semi-classical regime of the Liouville theory corresponds to $c_L \gg 1$, which can be achieved by taking $b \to 0$. Consequently the relations (39) turn into

$$\beta = \frac{1}{8b^2}, \quad \alpha = \frac{1}{2b} (1 + i\lambda).$$

The DOZZ three-point function has the form [3]

$$C(\alpha_1, \alpha_2, \alpha_3) = \left[ \pi \mu \gamma(b^2) b^{2-2b} \right]^{(Q - \sum \alpha_i)/b} \times$$

$$\frac{\Upsilon_0 \Upsilon(2\alpha_1) \Upsilon(2\alpha_2) \Upsilon(2\alpha_3)}{\Upsilon(\sum \alpha_i - Q) \Upsilon(Q + \alpha_1 - \alpha_2 - \alpha_3) \Upsilon(\alpha_1 - \alpha_2 + \alpha_3) \Upsilon(\alpha_1 + \alpha_2 - \alpha_3) \Upsilon(\alpha_1 - \alpha_2 + \alpha_3)}.$$
where for $x$ in the strip $0 < \Re(x) < Q$ the function $\Upsilon(x)$ can be defined through the integral representation

$$
\log \Upsilon(x) = \int_0^\infty \frac{dt}{t} \left[ (\frac{Q}{2} - x)^2 e^{-t} - \frac{\sinh^2 \left( \frac{Q}{2} - x \right)}{\sinh \frac{t}{2} \sinh \frac{Q}{2}} \right] \tag{42}
$$

and

$$
\Upsilon_0 = \left. \frac{d \Upsilon(x)}{dx} \right|_{x=0}. \tag{43}
$$

From (42) one gets the relations

$$
\Upsilon \left( \frac{Q}{2} \right) = 1, \tag{44}
$$

$$
\Upsilon(x + b) = \gamma(bx) b^{1-2bx} \Upsilon(x). \tag{45}
$$

It follows from (45) that for $b \to 0$

$$
\frac{d}{dx} \log \Upsilon(x) \approx \frac{1}{b} \left[ \log \gamma(bx) + (1 - 2bx) \log b \right] \tag{46}
$$

and therefore

$$
\log \Upsilon(x) \approx \frac{1}{b^2} \left[ F(bx) + \left( bx(1 - bx) - \frac{1}{4} \right) \log b \right] \tag{47}
$$

with the integration constant determined from (44). One also checks that for $b \to 0$

$$
\log \Upsilon_0 \approx \frac{1}{b^2} \left[ F(0) - \frac{1}{4} \log b \right]. \tag{48}
$$

Using (47) and the identity ($\lambda \in \mathbb{R}$):

$$
F(1 + i\lambda) = F(0) - \frac{1}{2} \pi \left| \lambda \right| - H(i\lambda) + i\lambda (\log \left| \lambda \right| - 1),
$$

one obtains

$$
\log \Upsilon \left( \frac{1 + i\lambda}{2b} \right) \approx \frac{1}{b^2} \left[ F \left( \frac{1 + i\lambda}{2} \right) + \frac{\lambda^2}{4} \log b \right], \tag{49}
$$

$$
\log \Upsilon \left( \frac{1 + i\lambda}{b} \right) \approx \frac{1}{b^2} \left[ F(0) - H(i\lambda) - \frac{1}{2} \pi \left| \lambda \right| + \left( \lambda^2 - \frac{1}{4} \right) \log b \right] + \frac{i\lambda}{b^2} \left[ \log \left| \frac{\lambda}{b} - 1 \right] \right]. \tag{50}
$$

The classical limit of the three-point function [3] can then be written in the following form

$$
\log C \approx -\frac{1}{b^2} \left[ -4F(0) + \sum_{\sigma_1, \sigma_2 = \pm} F \left( \frac{1 + i\lambda_1}{2} + \sigma_1 \frac{i\lambda_2}{2} + \sigma_2 \frac{i\lambda_3}{2} \right) \right.
$$

$$
\left. + \sum_{j=1}^3 \left( H(i\lambda_j) + \frac{1}{2} \pi \left| \lambda_j \right| \right) + \frac{1}{2} \log (\pi \mu b^2) \right] \tag{51}
$$

$$
- \frac{i}{b^2} \sum_{j=1}^3 \lambda_j \left[ 1 - \log \left| \lambda_j \right| + \frac{1}{2} \log (\pi \mu b^2) \right].
$$
Under the reflection $\alpha \to Q - \alpha$, $\lambda \to -\lambda$ the DOZZ three point function changes accordingly to the formula [3]

$$S(i\alpha_1 - iQ/2) = \frac{C(Q - \alpha_1, \alpha_2, \alpha_3)}{C(\alpha_1, \alpha_2, \alpha_3)} , \quad (52)$$

where $S$ is the so called reflection amplitude

$$S(P) = \left(\pi \mu \gamma (b^2)\right)^{-2iP/b} \frac{\Gamma(1 + 2iP/b) \Gamma(1 + 2iPb)}{\Gamma(1 - 2iP/b) \Gamma(1 - 2iPb)} . \quad (53)$$

On the other hand the classical Liouville action is by construction symmetric with respect to this reflection. This discrepancy can be overcome if one considers instead of $C(\alpha_1, \alpha_2, \alpha_3)$ the symmetric three-point function $\tilde{C}(\alpha_1, \alpha_2, \alpha_3)$:

$$\tilde{C}(\alpha_1, \alpha_2, \alpha_3) = \left(\prod_{j=1}^3 \sqrt{S(i\alpha_j - i\frac{Q}{2})}\right) C(\alpha_1, \alpha_2, \alpha_3) . \quad (54)$$

Taking into account the classical limit of the reflection amplitude for real $\lambda$

$$\log S \left( -\frac{\lambda}{2b} \right) \approx \frac{2i}{b^2} \lambda \left[ 1 - \log |\lambda| + \frac{1}{2} \log (\pi \mu b^2) \right] . \quad (55)$$

and the formulae (34), (19), and (51) one easily verifies that in the limit $b \to 0$

$$\log \tilde{C} \approx -\frac{1}{b^2} S_L [\varphi] + \text{const.}$$

where the relation $m = \pi \mu b^2$ is assumed and the constant on the r.h.s. is independent of $z_j$, $\lambda_j$ and $m$.

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