The Greene-Krantz Conjecture in Dimension Two

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Abstract: We give a proof of the Greene-Krantz conjecture on convex domains in $\mathbb{C}^2$. Curiously, the proof technique depends on subelliptic estimates for the $\overline{\partial}$ problem.

1 Introduction

The last thirty-five years have seen a flourishing of the study of the automorphism groups of smoothly bounded domains in $\mathbb{C}^n$. The subject has an unusual nature, because the only smoothly bounded domain with transitive automorphism group is the unit ball $B$ (see [WON]). So we tend to instead focus our attention on the more general class of domains with noncompact automorphism group. It is a classical result of Cartan that such a domain $\Omega$ has the property that there is a point $P \in \Omega$ and a point $X \in \partial \Omega$ and automorphisms (i.e., biholomorphic selfmaps of $\Omega$) $\varphi_j$ such that $\varphi_j(P) \to X$ as $j \to \infty$. We call $X$ a boundary orbit accumulation point.

Naturally we are interested in the geometric nature of the point $X$. It is known (see [GRK1]) that $X$ must in fact be a point of pseudoconvexity. But we wish to know more about the Levi geometry of $X$. With this thought in mind, the following conjecture has been formulated (see [GRK1]):

**Greene-Krantz Conjecture:** Let $\Omega$ be a smoothly bounded domain in $\mathbb{C}^n$. Suppose that $X \in \partial \Omega$ is a boundary orbit accumulation point for the automorphism group action in the sense that there are automorphisms $\varphi_j$ and a point $P \in \Omega$ such that $\varphi_j(P) \to X$ as $j \to \infty$. Then $X$ is a point of finite type in the sense of Kohn/D’Angelo/Catlin.

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2. Key Words: pseudoconvex, domain, biholomorphic mapping, automorphism group, boundary orbit accumulation point.
This conjecture has been the object of intense study for the past twenty years or more, and there are a number of interesting partial results—see for instance [KIMS], [KIMK], [KIK1], [KIK2], [KIK3]. In the present paper we prove this conjecture for smoothly bounded convex domains in complex dimension two.

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2 Notation and Basic Ideas

We take it that the reader is familiar with complex domains and with pseudoconvexity. See [KRA1] for background and details. When the ambient space has complex dimension two, there are two notions of finite type, and they are as follows:

Definition 2.1 A first order commutator of vector fields is an expression of the form

\[ [L, M] \equiv LM - ML. \]

Note that the commutator is itself a vector field.

Inductively, an \( m \)th order commutator is the commutator of an \((m - 1)\)st order commutator and a vector field \( L \).

Definition 2.2 A holomorphic vector field is any linear combination of the expressions

\[ \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} \]

with coefficients in the ring of \( C^\infty \) functions.

A conjugate holomorphic vector field is any linear combination of the expressions

\[ \frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial \bar{z}_2} \]

with coefficients in the ring of \( C^\infty \) functions.

Definition 2.3 Let \( M \) be a vector field defined on the boundary of \( \Omega = \{ z \in \mathbb{C}^2 : \rho(z) < 0 \} \). We say that \( M \) is tangential if \( M \rho = 0 \) at each point of \( \partial \Omega \).
Now we define a gradation of vector fields which will be the basis for our
definition of analytic type. Throughout this section \( \Omega = \{ z \in \mathbb{C}^2 : \rho(z) < 0 \} \)
and \( \rho \) is \( C^\infty \) with \( \nabla \rho \neq 0 \) on \( \partial \Omega \). If \( X \in \partial \Omega \) then we may make a change of
coordinates so that \( \partial \rho/\partial z_1(X) \neq 0 \). Define the holomorphic vector field
\[
L = \frac{\partial \rho}{\partial z_2} \frac{\partial}{\partial z_1} - \frac{\partial \rho}{\partial z_1} \frac{\partial}{\partial z_2},
\]
and the conjugate holomorphic vector field
\[
\overline{L} = \frac{\partial \rho}{\partial \overline{z}_2} \frac{\partial}{\partial \overline{z}_1} - \frac{\partial \rho}{\partial \overline{z}_1} \frac{\partial}{\partial \overline{z}_2}.
\]
Both \( L \) and \( \overline{L} \) are tangent to the boundary because \( L\rho = 0 \) and \( \overline{L}\rho = 0 \).
They are both non-vanishing near \( X \) by our normalization of coordinates.

The real and imaginary parts of \( L \) (equivalently of \( \overline{L} \)) generate (over the
ground field \( \mathbb{R} \)) the complex tangent space to \( \partial \Omega \) at all points near \( X \). The vector field \( L \) alone generates the space of all holomorphic tangent vector
fields and \( \overline{L} \) alone generates the space of all conjugate holomorphic tangent
vector fields.

**Definition 2.4** Let \( \mathcal{L}_1 \) denote the module, over the ring of \( C^\infty \) functions,
generated by \( L \) and \( \overline{L} \). Inductively, \( \mathcal{L}_\mu \) denotes the module generated by \( \mathcal{L}_{\mu-1} \)
and all commutators of the form \([F,G]\) where \( F \in \mathcal{L}_1 \) and \( G \in \mathcal{L}_{\mu-1} \).

Clearly \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \cdots \). Each \( \mathcal{L}_\mu \) is closed under conjugation. *It is not
generally the case that \( \bigcup_\mu \mathcal{L}_\mu \) is the entire three-dimensional tangent space at
each point of the boundary.* A counterexample is provided by
\[
\Omega = \{ z \in \mathbb{C}^2 : |z_1|^2 + 2e^{-1/|z_2|^2} < 1 \}
\]
and the point \( X = (1,0) \).

**Definition 2.5** Let \( \Omega = \{ \rho < 0 \} \) be a smoothly bounded domain in \( \mathbb{C}^2 \)
and let \( X \in \partial \Omega \). We say that \( \partial \Omega \) is of \textit{finite analytic type} \( m \) at \( X \) if
\( \langle \partial \rho(X), F(X) \rangle = 0 \) for all \( F \in \mathcal{L}_{m-1} \) while \( \langle \partial \rho(X), G(X) \rangle \neq 0 \) for some
\( G \in \mathcal{L}_m \). In this circumstance we call \( X \) a \textit{point of analytic type} \( m \).

Now we turn to a precise definition of finite geometric type. Let \( D \) denote
the unit disc in the complex plane. If \( X \) is a point in the boundary of a
smoothly bounded domain then we say that an analytic disc \( \phi : D \to \mathbb{C}^2 \) is a
\textit{non-singular disc tangent to} \( \partial \Omega \) at \( X \) if \( \phi(0) = X, \phi'(0) \neq 0 \), and \( (\rho \circ \phi)'(0) = 0 \).
Definition 2.6 Let $\Omega = \{\rho < 0\}$ be a smoothly bounded domain and $X \in \partial \Omega$. Let $m$ be a non-negative integer. We say that $\partial \Omega$ is of finite geometric type $m$ at $X$ if the following condition holds: there is a non-singular disc $\phi$ tangent to $\partial \Omega$ at $X$ such that, for small $\zeta$,

$$|\rho \circ \phi(\zeta)| \leq C|\zeta|^m$$

BUT there is no non-singular disc $\psi$ tangent to $\partial \Omega$ at $X$ such that, for small $\zeta$,

$$|\rho \circ \psi(\zeta)| \leq C|\zeta|^{(m+1)}.$$ 

In this circumstance we call $X$ a point of finite geometric type $m$.

The principal result about finite type in dimension two is the following theorem (see [KRA1, §11.5]):

Theorem 2.7 Let $\Omega = \{\rho < 0\} \subseteq \mathbb{C}^2$ be smoothly bounded and $X \in \partial \Omega$. The point $X$ is of finite geometric type $m \geq 2$ if and only if it is of finite analytic type $m$.

Now let us say a few words about subelliptic estimates. A partial differential operator $\mathcal{L}$ of order $k$ is said to satisfy elliptic estimates if, whenever $\mathcal{L}u = f$ and $f$ lies in the Sobolev space $W^s$ then $u$ lies in the Sobolev space $W^{s+k}$. The operator is said to satisfy subelliptic estimates if the index $s+k$ in the conclusion is replaced by $s+k'$ for some $0 < k' < k$. The $\overline{\partial}$-Neumann operator on a strongly pseudoconvex domain, and more generally on a finite type domain, is known to satisfy a subelliptic (but definitely not an elliptic) estimate. See [CAT1]–[CAT2], [KRA3], [FOK] for the details. It is also possible to express the subellipticity condition in terms of Lipschitz or Besov spaces rather than Sobolev spaces. We leave the details for the interested reader.

3 The Main Argument

The result that we shall actually prove in this paper is the following:

Theorem 3.1 Let $\Omega$ be a smoothly bounded, convex domain in $\mathbb{C}^n$. Suppose that $X \in \partial \Omega$ is a boundary orbit accumulation point for the automorphism group action in the sense that there are automorphisms $\varphi_j$, a point $P \in \Omega$, and a point $X \in \Omega$ such that $\varphi_j(P) \to X$ as $j \to \infty$. Then $X$ is a point of finite type in the sense of Kohn/D’Angelo/Catlin.
Now fix a smoothly bounded domain $\Omega \subseteq \mathbb{C}^2$. Assume that $P \in \Omega$ and $X \in \partial \Omega$ and that there are automorphisms $\varphi_j$ of $\Omega$ such that $\varphi_j(P) \to X$ as $j \to \infty$. Note that, because the domain $\Omega$ is smoothly bounded and pseudoconvex, it is complete in the Bergman metric (see [OHS]).

Now consider a small Bergman metric ball $\beta$ centered at $P$. Choose $j_1$ so that $\beta_1 \equiv \varphi_{j_1}(\beta)$ is disjoint from $\beta$, and so that the Euclidean distance of $\beta_1$ to the boundary is about $2^{-1}$. Now choose $j_2$ so that $\beta_2 \equiv \varphi_{j_2}(\beta)$ is disjoint from $\beta$ and $\varphi_1(\beta)$ and so that the Euclidean distance of $\beta_2$ to the boundary is about $2^{-2}$. Keep going.

Now fix a $\partial$-closed $(0,1)$ form $\psi$ with $C^\infty_c$ coefficients that is supported in $\beta$. Define $\psi_\ell = (\varphi_{-1}\ell)^* \psi$. Thus $\psi_\ell$ is a $\overline{\partial}$-closed $(0,1)$ form with $C^\infty_c$ coefficients supported on $\beta_\ell$. Because of the derivative of $\varphi_{j_\ell}$, $\psi_\ell$ has supremum norm about $2^{-\ell}$. That will mean that the sum of the $\psi_\ell$ will have an $L^2$ or Sobolev norm that converges.

If we write $\psi_\ell = \psi^1_\ell d\bar{z}_1 + \psi^2_\ell d\bar{z}_2$, then we may note that the equation $\overline{\partial}u_\ell = \psi_\ell$ can be solved with one of the simple equations

$$u^1_\ell(z_1, z_2) = \int \int_{\zeta \in \mathbb{C}} \frac{\psi^1_\ell(\zeta, z_2)}{\zeta - z_1} dA(\zeta)$$

or

$$u^2_\ell(z_1, z_2) = \int \int_{\zeta \in \mathbb{C}} \frac{\psi^2_\ell(z_1, \zeta)}{\zeta - z_2} dA(\zeta),$$

see [KRA1, §1.1]. And it turns out that $u^1_\ell = u^2_\ell$.

It follows from standard results on fractional integration (see [STE]) that, if $\psi_\ell$ is in some Sobolev class $W^s$ then $\psi^m_\ell$ will be in a smoother Sobolev class $W^{s'}$, with $s' > s$, in the $m$th variable, $m = 1, 2$. And now a simple argument with the triangle inequality shows that $u^m_\ell$ lies in $W^{s''}$ as a function of both variables for some $s' \geq s'' > s$, $m = 1, 2$. So we see that the $\overline{\partial}$ problem satisfies a subelliptic estimate on $\psi_\ell$.

But in fact, thanks to the intervention of the automorphisms $\varphi_\ell$, the $\overline{\partial}$ problem satisfies the very same subelliptic estimate for each $\psi_\ell$. As a result, the $\overline{\partial}$ problem satisfies a subelliptic estimate on the form

$$\psi \equiv \sum \psi_\ell.$$

Now it is definitely not the case that the $\overline{\partial}$-closed $(0,1)$ forms with $C^\infty_c$ coefficients are dense in any space of forms with Sobolev coefficients. But
we shall be able to argue that they are dense in certain forms that we care about. See also the footnote below.

We have the following lemma:

**Lemma 3.2** If the boundary orbit accumulation point $X$ is of infinite type, then for each $\epsilon > 0$ there is a $\overline{\partial}$-closed $(0,1)$ form $f$ on $\Omega$ with $L^2$ coefficients so that the equation $\overline{\partial}u = f$ does not have any solution in the Besov space of order $\epsilon > 0$.

**Proof:** The idea for the proof goes back to an old result of Kerzman (see [KER]) and is reasonably well known. See also [KRA1, §10.3]. We sketch the idea here.

We may assume that $X = (1,0) \in \partial \Omega$ and that the complex normal direction at $X$ is $\langle 1,0 \rangle$. With these normalizations, we define

$$f = \frac{d\overline{z}_2}{\log(1 - z_1)}.$$ 

By the convexity of $\Omega$, it is clear that the principal branch of the logarithm is well defined and that $f$ has bounded coefficients.

Now any solution of the equation $\overline{\partial}u = f$ will have the form

$$u(z) = \frac{\overline{z}_2}{\log(1 - z_1)} + h(z_1, z_2),$$

where $h$ is some holomorphic function on $\Omega$.

Since $X$ is a point of infinite type then we know that, for any positive integer $m$, there is a nonsingular complex curve $\mu_m : D \rightarrow \mathbb{C}^2$ that is tangent to order $2m$ with $\partial \Omega$ at $X$. Let $\nu_X$ denote the Euclidean outward unit normal vector to $\partial \Omega$ at $X$. Then, for $\delta > 0$ small, the analytic disc

$$\left\{ \mu_m(\zeta) - \delta \nu_X : |\zeta| < C\delta^{1/(2m)}, \zeta \in D \right\}$$

lies in $\Omega$ (see [KRA2] for the elementary calculations needed to justify this assertion). Thus

$$\theta_\delta : t \mapsto \mu_m(C\delta^{1/(2m)}e^{it}) - \delta \nu_X, \quad 0 \leq t < 2\pi,$$

describes the boundary of an analytic disc in $\Omega$. 

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With $(1, 0) \in \partial \Omega$ as our point of infinite type, let $m$ be a positive integer as above and (by the well-known semicontinuity of type—see [KRA1, §11.5]) choose a neighborhood $W$ of $(1, 0)$ so that boundary points in $W$ are of finite type at least $2m$. We may assume that $W$ is a Euclidean ball, and that it lies in a tubular neighborhood of $\partial \Omega$. Pick $\delta > 0$ small (small enough so the $\delta^{1/(2m)}$ is much less than the radius of $W$) and define

$$\tilde{\Omega} = (W \cap \Omega) \bigcup \left\{ z \in \Omega : \text{dist}(z, \partial \Omega) > \frac{\delta}{12} \right\}.$$  

We examine the complex line integral

$$F(\delta, \zeta) = \oint_{\sigma_2} u(\zeta_1 - 2\delta, \zeta_2 + z_2) - u(\zeta_1 - \delta, \zeta_2 + z_2) \, dz_2$$  

for $\zeta \in \tilde{\Omega}$. We note that the curves

$$t \mapsto -\delta \nu_X + \mu(C' \delta^{1/(2m)} e^{it}) \quad \text{and} \quad t \mapsto -2\delta \nu_X + \mu(C' \delta^{1/(2m)} e^{it})$$

both lie in $\Omega$ precisely because $X$ is a point of infinite type (more precisely, a point of type at least $2m$).

Seeking a contradiction, if $u$ satisfies a Besov condition of order $\epsilon$, then we may straightforwardly estimate that

$$\|F(\delta)\|_{L^2(\zeta)} \leq \int_{\tilde{\Omega}} \int_{\sigma_2} \left| u(\zeta_1 - 2\delta, \zeta_2 + z_2) - u(\zeta_1 - \delta, \zeta_2 + z_2) \right|^2 dV(\zeta)/12$$

$$\leq \int_{\sigma_2} \int_{\tilde{\Omega}} \left| u(\zeta_1 - 2\delta, \zeta_2 + z_2) - u(\zeta_1 - \delta, \zeta_2 + z_2) \right|^2 dV(\zeta)^{1/2} d|z_2|$$

$$\leq \int_{\sigma_2} \epsilon^4 d|z_2|$$

$$\approx \frac{\delta^\epsilon}{\delta^{1/(2m)}}.$$  

On the other hand,

$$F(\delta, \zeta) = \int_{\sigma_2} \frac{\overline{z}_2 + \zeta_2}{\delta^{2/(2m)}} - \frac{\overline{z}_2 + \zeta_2}{\delta^{2/(2m)}} \, dz$$

$$= \frac{\log(1 - \zeta_1 + 2\delta)}{\delta^{2/(2m)}} - \frac{\log(1 - \zeta_1 + \delta)}{\delta^{2/(2m)}}$$

$$\approx C \cdot \frac{\delta^{1/m}}{\log^2(-\delta)}.$$  

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As a result,
\[
\int_\Omega |F(\delta)|^2 dV(\zeta)^{1/2} \approx \frac{\delta^{1/m}}{\log^2(-\delta)}.
\]

Comparing our two estimates, we find that
\[
\frac{\delta^{1/m}}{\log^2(-\delta)} \leq C \cdot \delta^{\epsilon + 1/(2m)}
\]
or
\[
\frac{\delta^{1/(2m)}}{\log^2(-\delta)} \leq C \cdot \delta^\epsilon.
\]
This is false as soon as \( m \in \mathbb{N} \) is large enough.

The lemma tells us that, in the Besov space topology, the \( \overline{\partial} \) problem does not satisfy a subelliptic estimate. But it is not difficult to see that the form
\[
f(z) = \frac{d\overline{z}_2}{\log(1 - z_1)}
\]
is the limit of forms with compact support.\(^3\) For let \( \rho_2 \) be a \( C_\infty \) function that approximates \( 1/\log(1 - z_1) \) in the \( L^2 \) topology. Now the formula
\[
v(z_1, z_2) = \int \int \frac{\rho_2(z_1, \zeta)}{\zeta - z_2} d\zeta
\]
satisfies
\[
\frac{\partial}{\partial \overline{z}_2} v = \rho_2.
\]
Note that (see [KRA1, §1.1]) \( v \in C_\infty_\infty(\Omega) \). Hence
\[
\rho_1(z) \equiv \frac{\partial}{\partial z_1} v
\]
\(^3\)And notice that, if \( \psi_0 \) is a \( \overline{\partial} \)-closed \((0, 1)\) form with \( C_\infty \) coefficients on \( \beta \), then we can consider the form \( \psi \) on the union of \( \beta \), \( \varphi_{j_1}(\beta) \), etc. as described above and we can also consider the “shifted” form \( \tau \) given by \( (\varphi_{j_1}^{-1})^* \psi \) on \( \varphi_{j_1}(\beta) \), \( (\varphi_{j_2}^{-1})^* \psi \) on \( \varphi_{j_2}(\beta) \) (with intervening automorphism \( \varphi_{j_2} \circ \varphi_{j_1}^{-1} \)), and so forth. Then the difference of these two forms is a \( C_\infty \) form supported on \( \beta \) alone. So our arguments and estimates also apply to forms that have compact support and are smooth.
will give a form

\[ R = \rho_1 d\bar{z}_1 + \rho_2 d\bar{z}_2 \]

that is \( \overline{\partial} \)-closed with \( C^\infty_c \) coefficients. And of course \( R \) will approximate \( f \) in the \( L^2 \) topology.

This approximation implies of course that the problem \( \overline{\partial}u = f \), with \( f \) as in the lemma, satisfies a subelliptic estimate in the Sobolev topology. But that implies that it satisfies a subelliptic estimate in the Besov topology. And we have established in the lemma that that is impossible.

We have proved that the boundary orbit accumulation point \( X \) cannot be of infinite type.

Remark 3.3 It is worth noting that the construction presented here—of the ball \( \beta \) and subsequent target balls \( \varphi_{j_1}(\beta), \varphi_{j_2}(\beta), \) etc., does not work when the automorphism group is compact. For, when the automorphism group is compact, then these balls will no longer be pairwise disjoint. Also the norms of the \( (\varphi^{-1})^* \psi \) will no longer vanish rapidly, so that the series which is obtained by adding the forms supported on the different balls will no longer converge.

4 Concluding Remarks

In this paper we certainly have not proved the full Greene-Krantz conjecture. But we have proved a notable and interesting special case.

There is certainly interest in developing techniques for attacking the full conjecture, and we intend to attack that problem in future papers.
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