Quantal Analysis of String-Inspired Lineal Gravity with Matter Fields*

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Abstract

We show that string-inspired lineal gravity interacting with matter fields cannot be Dirac-quantized owing to the well known anomaly in energy-momentum tensor commutators.

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Since the string-inspired model for lineal gravity with matter fields has been extensively studied within various semi-classical approximations\(^1\) we are led to inquire whether the quantized theory can be analyzed exactly. Quantal results for this model without matter\(^2\)\(^3\) or with point particles\(^4\) are now available. Here we consider this gravity theory in the presence of a massless scalar field.

The model is formulated as a gauge theory\(^5\) of the extended Poincaré group\(^6\) and the analysis is carried out in a Schrödinger representation, so that issues related to functional integration do not arise. Our results are the following.

1. The quantum theory can be presented in a well-defined, unambiguous fashion, thus putting to rest the worry that “there is not a unique quantization of dilaton gravity”\(^1\). Evidently the gauge principle resolves ambiguities.

2. All dynamical variables can be separated and decoupled. The operator equations of motion can be solved.

3. There is no interaction between matter and gravity degrees of freedom, save a correlation interaction induced by diffeomorphism constraints.

4. The diffeomorphism constraints cannot be satisfied owing to a commutator anomaly, so a diffeomorphism invariant state space for the quantum theory cannot be constructed.

5. A semi-classical reduction of the fully quantized model fails to reproduce familiar results.

Detailed exposition of our investigation will be presented elsewhere\(^7\); here we give a brief description.

The gauge group is the 4-parameter extended Poincaré group. The Lie algebra possesses a central element \(I\) in the commutator of translations \(P_a\); the Lorentz generator \(J\) satisfies conventional commutation relations.

\(^1\)For a summary, see \(^1\).
\[ [P_a, P_b] = \epsilon_{ab} I \quad [P_a, J] = \epsilon_a^{\ b} P_b \]  

(1)

[Notation: tangent space is indexed by \((a, b, \ldots)\) with metric tensor \(h_{ab} = \text{diag}(1, -1)\) and \(\epsilon^{ab} = -\epsilon^{ba}, \epsilon^{01} = 1\). Light cone components will be frequently employed; they are defined by \((\pm) \equiv \frac{1}{\sqrt{2}}((0) \pm (1)).\)]

The gauge connection \(A_\mu\) is identified with metric quantities,

\[ A_\mu = e^{a}_\mu P_a + \omega_\mu J + a_\mu I \]  

(2)

where \(e^a_\mu\) is the Zweibein, determining the metric tensor of space-time, \(g_{\mu\nu} = e^a_\mu e^b_\nu h_{ab}\); \(\omega_\mu\) is the spin-connection — an independent variable at the initial stage; \(a_\mu\) is a \(U(1)\) gauge potential associated with the central element, which gives rise to a cosmological constant. A gauge group transformation \(U\), parameterized as \(U = e^{\theta^a P_a + \omega^a J + 2I}\), acts on the connections by (inhomogeneous) adjoint action,

\[ e^a_\mu \rightarrow (e^U)_a^\mu = (\Lambda^{-1})_b^a (e^b_\mu + \epsilon^b_c \theta^c e_\mu + \partial_\mu \theta^b) \]  

(3a)

\[ \omega_\mu \rightarrow (\omega^U)_\mu = \omega_\mu + \partial_\mu \alpha \]  

(3b)

\[ a_\mu \rightarrow (a^U)_\mu = a_\mu - \theta^a \epsilon_{ab} e_\mu - \frac{1}{2} \theta^a \theta^b + \frac{1}{2} \partial_\mu \theta^a \epsilon_{ab} \theta^b \]  

(3c)

where \(\Lambda_b^a\) is the Lorentz transformation matrix: \(\Lambda_b^a = \delta_b^a \cosh \alpha + \epsilon_b^a \sinh \alpha\). In the usual way, one constructs from (1) and (2) the gauge group curvature.

\[ F \equiv \frac{1}{2} e^{\mu\nu} F_{\mu\nu} = \epsilon^{\mu\nu} (\partial_\mu A_\nu + A_\mu A_\nu) = f^a P_a + f^2 J_2 + f^3 I \]

\[ = \epsilon^{\mu\nu} \left\{ (\partial_\mu e^a_\nu + e^a_b \omega_\mu e^b_\nu) P_a + \partial_\mu \omega_\nu J + (\partial_\mu a_\nu + \frac{1}{2} e^a_\mu \epsilon_{ab} e^b_\nu) I \right\} \]  

(4)

To build a gauge and diffeomorphism invariant action, a Lagrange multiplier multiplet \((\eta_1, \eta_2, \eta_3)\) is introduced, and is taken to transform in the coadjoint representation:

\[ \eta_1 \rightarrow (\eta^U)_1 = (\eta_b - \eta_3 \epsilon_{bc} \theta^c) \Lambda^b_a \]  

(5a)

\[ \eta_2 \rightarrow (\eta^U)_2 = \eta_2 - \eta_a \epsilon^{a}_b \theta^b - \frac{1}{2} \eta_3 \theta^a \theta_a \]  

(5b)

\[ \eta_3 \rightarrow (\eta^U)_3 = \eta_3 \]  

(5c)

\[ \text{2Even though the group is not semi-simple, one can define an invariant, non-singular metric, so the coadjoint action can be identified with the adjoint; see Ref. 6.} \]
The gauge invariant gravitational action then reads
\[ I_g = \frac{1}{4\pi G} \int d^2x \left( \eta_a f^a + \eta_2 f^2 + \eta_3 f^3 \right) \tag{6} \]
where $G$ is “Newton’s constant”, which is dimensionless, as are the Lagrange multiplier fields. (The velocity of light and $\hbar$ are scaled to unity.) It is straightforward to show that $I_g$ is equivalent to the action
\[ \frac{1}{4\pi G} \int d^2x \sqrt{-g} \left( e^{-2\bar{\phi}} (e^{-2\bar{\phi}} R - \lambda) \right) \]
where $e^{-2\bar{\phi}}$ is proportional to $\eta_2$ with the cosmological constant $\lambda$ arising as the solution to the equation for $\eta_3$ that follows from (6). A further redefinition $\bar{g}_{\mu\nu} = e^{2\bar{\phi}} g_{\mu\nu}$ relates (6) to the string-inspired “dilation” action
\[ \frac{1}{4\pi G} \int d^2x \sqrt{-\bar{g}} e^{-2\bar{\phi}} \left( \bar{R} + 4\bar{g}^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \bar{\phi} - \lambda \right). \]

A diffeomorphism invariant action for a scalar field $\varphi$ with mass $\mu$ is of course the familiar expression
\[ I_m^0 = \frac{1}{2} \int d^2x \left( E^\mu_a E^\nu_b \partial_\mu \varphi \partial_\nu \varphi - \mu^2 \varphi^2 \right), \]
where $\epsilon_{ab} = \delta_{ab}$. However, with $\varphi$ taken to be gauge invariant, $I_m^0$ is not invariant against local (gauge) translations [see (3a)].

Construction of a Poincaré gauge invariant matter action makes use of a new field $q^a$ — the “Poincaré coordinate” — and its canonical conjugate $p_a$. Both transform as tangent-space Lorentz vectors; additionally $q^a$ shifts by a translation.

\[ q^a \rightarrow (q^U)^a = (\Lambda^{-1})^a_b \left( q^b + \epsilon^b_c \theta^c \right) \tag{7a} \]
\[ p_a \rightarrow (p^U)_a = p_b \Lambda^b_a \tag{7b} \]

As a consequence $q^a$ can be set to zero by a gauge transformation.

One verifies that the following Lagrange density for matter variables is invariant against transformations (3) and (7). ($\Pi$, $\varphi$, and $u$, $v$ are not transformed.)

\[ \mathcal{L}_m = p_a \dot{q}^a + \Pi \dot{\varphi} + e_0^a \epsilon^b_a p_b + \omega_0 p_a \epsilon^a_b q^b - u \mathcal{E} - v \mathcal{P} \tag{8} \]

The Lagrange multipliers $u$ and $v$ enforce the vanishing of the energy density $\mathcal{E}$ and momentum density $\mathcal{P}$ — these are the diffeomorphism constraints.

\[ \mathcal{E} \equiv (Dq)^a \epsilon^b_a p_b + \frac{1}{2} \left( \Pi^2 + (\varphi')^2 + (Dq)^2 \mu^2 \varphi^2 \right) \tag{9a} \]
\[ \mathcal{P} \equiv p_a (Dq)^a + \Pi \varphi' \tag{9b} \]
Here \((Dq)^a \equiv q^a + \epsilon^a_b (q^b \omega_1 - e_1^b)\), while dot/dash denote time/space differentiation.

The \(\varphi\)-field dynamics implied by (8) and (9) are the same as those arising from \(I^0_m\). This is seen by passing to the gauge \(q^a = 0\) and eliminating \(p_a\) from (8) by setting \(\mathcal{E}\) and \(\mathcal{P}\) to zero. Gauge invariance has been achieved by a Higgs-like mechanism, and vanishing \(q^a\) corresponds to a “unitary gauge” wherein the physical content of a gauge-invariant Lagrangian is exposed.

The complete Lagrange density for the gravity-matter system that we study follows from (6), (8) and (9).

\[
\mathcal{L}_{g+m} = \frac{1}{4\pi G} (\eta_a \dot{e}_1^a + \eta_2 \dot{\omega}_1 + \eta_3 \dot{a}_1) + p_a q^a + \Pi \dot{\varphi} + \epsilon^a_0 G_a + \omega_0 G_2 + a_0 G_3 - u E - v P
\]  

(Spatial, but not temporal, integration by parts is carried out freely.) The symplectic structure identifies the canonical coordinates as \((e^a_1, \omega_1, a_1), q^a\) and \(\varphi\), while their respective canonical momenta are \(\frac{1}{4\pi G} (\eta_0, \eta_2, \eta_3), p_a\) and \(\Pi\). The Hamiltonian is a superposition of the diffeomorphism constraints and the gauge constraints; \((e^a_0, \omega_0, a_0)\) enforce the vanishing of the gauge generators \((G_a, G_2, G_3)\).

\[
G_a = \frac{1}{4\pi G} \left( \eta_a' + \epsilon_a^b \eta_b \omega_1 + \eta_3 \epsilon_a^b e_1^b \right) + \epsilon_a^b p_b
\]

\[
G_2 = \frac{1}{4\pi G} \left( \eta_2' + \eta_a \epsilon_a^b e_1^b \right) - q^a \epsilon_a^b p_b
\]

\[
G_3 = \frac{1}{4\pi G} \eta_3'
\]

Using the Poisson brackets implied by the symplectic structure, one verifies that the algebra of constraints closes; they are first-class. The gauge generators commute with \(\mathcal{E}\) and \(\mathcal{P}\), while among themselves they follow the Lie algebra.

\[
[G_a(x), G_b(y)]_{PB} = \epsilon_{ab} G_3(x) \delta(x - y)
\]

\[
[G_a(x), G_2(y)]_{PB} = \epsilon_a^b G_b(x) \delta(x - y)
\]

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3Our construction of the gauge invariant matter action follows the method given by G. Grignani and G. Nardelli (Ref. 8). The haphazard-appearing expressions (8), (9) can in fact be systematically derived within an explicitly gauge covariant formalism, see Ref. 7.

4In two dimensions, non-minimal coupling to the gravitational gauge potentials is also possible, see Ref. [6], but we do not make use of it here.
The diffeomorphism constraints satisfy

\[ [\mathcal{E}(x), \mathcal{E}(y)]_{PB} = [\mathcal{P}(x), \mathcal{P}(y)]_{PB} = (\mathcal{P}(x) + \mathcal{P}(y)) \delta'(x - y) \]  
(13a)

\[ [\mathcal{E}(x), \mathcal{P}(y)]_{PB} = (\mathcal{E}(x) + \mathcal{E}(y)) \delta'(x - y) \]  
(13b)

Eqs. (13a,b) may be decoupled by defining \( T_{\pm} = \frac{1}{2}(\mathcal{E} \pm \mathcal{P}) \)

\[ [T_{\pm}(x), T_{\pm}(y)]_{PB} = \pm (T_{\pm}(x) + T_{\pm}(y)) \delta'(x - y) \]  
(13c)

\[ [T_{\pm}(x), T_{\mp}(x)]_{PB} = 0 \]  
(13d)

[A common time argument in the above quantities has been suppressed.] Note that (13) coincides with the algebra of a conformally invariant theory, even though the field is massive.

Quantization consists of replacing Poisson brackets by commutators. The question of the quantum nature of the above constraint algebra will not be addressed as yet. Rather we set ourselves the task of satisfying the requirement of vanishing constraints by attempting to solve the corresponding functional differential equations that a quantum mechanical wave functional \( \Psi \), in the Schrödinger representation, must satisfy. Moreover, we do not well-order operators in the constraints at intermediate steps of the calculation; the ordering is stipulated only at the end, when a constraint is taken to act on the wave functional. We work in “momentum” space for the metric variables and in “position” space for the matter variables \( i.e., \Psi \) depends on \((\eta_1, \eta_2, \eta_3)\), \( q^a \) and \( \varphi \) while \((e_1^a, \omega_1, a_1)\) are realized as the functional derivatives \( 4\pi Gi \left( \frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \eta_2}, \frac{\partial}{\partial \eta_3} \right) \) and similarly \( p_a \) and \( \Pi \) as \( \frac{1}{i} \frac{\delta}{\delta q^a} \) and \( \frac{1}{i} \frac{\delta}{\delta \varphi} \).

Before proceeding with the task of solving the quantum constraints, we record the general classical solution to the equations that follow from (9), (10), (11). Henceforth, to simplify the problem and to make contact with previous work, we set \( \mu \), the mass of \( \varphi \), to zero. The general solution to the equations of motion is the following.

The equations for the metric variables do not involve the matter variables; they require the vanishing of \( F \), hence \( A_{\mu} \) is a pure gauge and explicitly is given by

\[ e_{\mu}^\pm = \exp(\pm \alpha) \partial_\mu \theta^\pm \]  
(14a)

\[ \omega_\mu = \partial_\mu \alpha \]  
(14b)

\[ a_\mu = \partial_\mu \beta + \frac{1}{2} \epsilon_{ab} \partial_\mu \theta^a \theta^b \]  
(14c)
where $\theta^a$, $\alpha$, $\beta$ are arbitrary functions of space-time $(t, x)$ and specify an arbitrary gauge transformation.

To solve the matter-field equations, we choose two functions of space-time, $X^a(t, x)$, and parameterize $u$ and $v$ as

$$v \pm u = \dot{X}^\pm/X'^\pm$$  \hspace{1cm} (15)

Also we introduce six arbitrary mode functions of the single variable $X^+$ or $X^-$: $N_\pm(X^\pm)$, $Q_\pm(X^\pm)$, $\phi_\pm(X^\pm)$. The general solution, which satisfies all the equations, save the diffeomorphism constraints, may be presented as:

$$q^\pm = e^{\mp \alpha}(Q^\pm(X^\pm) \mp \theta^\pm)$$ \hspace{1cm} (16a)

$$p_\pm = \mp e^{\mp \alpha} \frac{\partial}{\partial x} N_\pm(X^\pm)$$ \hspace{1cm} (16b)

$$\eta_3 = \lambda$$ \hspace{1cm} (17a)

$$\frac{1}{4\pi G} \eta_\pm = e^{\mp \alpha}\left(N_\pm(X^\pm) \mp \lambda \frac{\theta^\mp}{4\pi G}\right)$$ \hspace{1cm} (17b)

$$\frac{1}{4\pi G} \eta_2 = -\int^{X^+} dz \ N_+(z)Q^+(z) - \int^{X^-} dz \ N_-(z)Q^-(z)$$
$$+ N_+(X^+)\theta^+ - N_-(X^-)\theta^- - \frac{\lambda}{4\pi G} \theta^+\theta^-$$ \hspace{1cm} (17c)

$$\varphi = \phi_+(X^+) + \phi_-(X^-)$$ \hspace{1cm} (18a)

$$\Pi = \frac{\partial}{\partial x} \left(\phi_+(X^+) - \phi_-(X^-)\right)$$ \hspace{1cm} (18b)

Finally the diffeomorphism constraints that $\mathcal{E}$ and $\mathcal{P}$ vanish are satisfied by identifying $\phi_\pm$ with $N_\pm$, $Q_\pm$,

$$\left(\phi'_\pm\right)^2 = N'_\pm Q'^\pm$$ \hspace{1cm} (19)

[In (17) and (19) the dash on the mode functions signifies differentiation with respect to argument; viz. $\frac{\partial}{\partial x} \phi_\pm(X^\pm) = X'^\pm \phi'_\pm(X^\pm)$, etc.]
The form of the general solution illustrates well the flexibility inherent in a gauge theory of gravity. With all functions arbitrary, neither the geometry nor the field motion are specified; indeed upon setting the gauge parameters \((\theta^a, \alpha, \beta)\) to zero, \((e^a_\mu, \omega_\mu, a_\mu)\) vanish and no geometry can be constructed. The “unitary gauge” of vanishing \(q^a\) is achieved by choosing \(\theta^a = -\varepsilon^a_b Q^b\). Metric variables are no longer zero; they describe vanishing curvature, without selecting a specific coordinate system on which the embedding functions \(X^a\) are defined. To select coordinates it is natural to set \(e^a_\mu = \delta^a_\mu\), \(\omega_\mu = 0\), \(\theta^a = x^a\), \(\alpha = 0\) and also \(\beta\) is taken to vanish so that \(a_\mu = \frac{1}{2}\varepsilon_{\mu\nu} x^\nu\). It follows that \(X^\pm\) depends only on \(x^\pm\) and coincides with the function inverse to \(Q^\pm\), but its specific form is still open. A simple choice is \(X^a = x^a\), so that \(u = 1\), \(v = 0\). Eq. (18) then shows that the matter field \(\phi\) consists of a superposition of left-moving and right-moving waves, \(\phi_\pm\), while (19) expresses \(N_\pm\) — the remaining mode function — in terms of \(\phi_\pm\).

We now return to the problem of solving the quantum mechanical theory; viz. solving the gauge and diffeomorphism constraints. We begin by recording the known solution in the pure gravity case, with the matter variables \((p_a, q^a, \Pi, \varphi)\) omitted. Only the gauge constraints act, and they are solved by

\[
\Psi(\eta) \bigg|_{\text{pure gravity}} = \delta(\eta_3') \delta(M') e^{i\Omega}\Psi(M, \lambda) \bigg|_{\text{pure gravity}}
\]

Here \(M\) is the gauge-invariant combination

\[
M = \eta_a \eta^a - 2\eta_2 \eta_3
\]

and \(\Omega\) is the Kirillov-Kostant 1-form on the coadjoint orbit of the extended Poincaré group.

\[
\Omega = \frac{1}{8\pi G \lambda} \int e^{ab} \eta_a d\eta_b
\]
\[ \rho^a \equiv q^a + \eta^a / \lambda \]  

responds only to Lorentz gauge transformations, while it is translation and \(U(1)\) invariant. This is identical to the transformation law for \(p_a\), which is now taken as conjugate to \(\rho^a\). Also \(\eta_2\) is shifted by \(\eta_a \eta^a / 2\lambda\), so that \(-2\lambda \eta_2\) is replaced by the gauge invariant variable \(M\). Finally, \(\frac{1}{8\pi G} \omega_1\), the coordinate conjugate to \(\eta_2\), is renamed \(2\lambda \Pi_M\). With these redefinitions, the \(G_a\) constraint requires that there be no further dependence on \(\eta_a\). Thus the wave-functional satisfying \(G_3\) and \(G_a\) takes the form

\[ \Psi = \delta(\eta_a') e^{i\Omega} \tilde{\Psi}(\rho^a, M, \varphi) \]  

and the remaining gauge constraint \(\tilde{G}_2\) in terms of new variables is

\[ -\tilde{G}_2 = \frac{1}{8\pi G \lambda} M' + \rho^a \epsilon^b_a p_b \]  

while the diffeomorphism constraints read,

\[ \tilde{\mathcal{E}} = \rho^a \epsilon^b_a p_b - 8\pi G \lambda \Pi_M \rho^a p_a - \frac{4\pi G}{\lambda} p_a \rho^a + \frac{1}{2} \left( \Pi^2 + \varphi'^2 \right) \]  

\[ \tilde{\mathcal{P}} = p_a \rho^a - 8\pi G \lambda \Pi_M \rho^a \epsilon^b_a p_b + \Pi \varphi' \]  

Effectively \(\tilde{\Psi}\) is governed by the Lagrange density

\[ \tilde{\mathcal{L}}_{g+m} = p_a \dot{\rho}^a + \Pi_M \dot{M} + \Pi \dot{\varphi} + \omega_0 \tilde{G}_2 - u \tilde{\mathcal{E}} - v \tilde{\mathcal{P}}. \]  

One verifies that \(\tilde{\mathcal{L}}_{g+m}\) is obtained from \(\mathcal{L}_{g+m}\) in (10) by adding a total time derivative, solving the \(G_a\) and \(G_3\) constraints, and redefining variables as indicated above.

To unravel the \(\tilde{G}_2\) constraint, we extract from the wave functional an \(M\)-dependent phase factor. Also we shift the \(M\) dependence in the remaining functional by a term proportional to \(\rho^a \rho_a\), which is equivalent to a canonical transformation from \(M\) and \(p_a\) to \(m = M - \frac{\lambda^2}{2} \rho^a \rho_a\) and \(\Pi_a = p_a + \lambda^2 \rho_a \Pi_M\) (\(\rho^a\) and \(\Pi_M = \Pi_m\) are unaffected).

\[ \tilde{\Psi}(\rho^a, M, \varphi) = e^{i\tilde{\Omega}} \Psi(\rho^a, M, \varphi) \]  

\[ \tilde{\Omega} = \frac{1}{8\pi G \lambda} \int d\rho^a \epsilon_{ab} \dot{\rho}^b M \]  

\[ \dot{\rho}^a \equiv \rho^a / \rho \quad \rho \equiv \sqrt{\rho^a \rho_a} \]
The constraints on $\tilde{\Psi}$ now become
\[ -G_2 = \rho^a \epsilon_a^b \Pi_b \equiv j \quad (30) \]

\[
\tilde{\mathcal{E}} = -\frac{4\pi G}{\lambda} \rho^2 - \frac{\lambda}{16\pi G} \rho^2 + 4\pi G \lambda^2 \Pi^2 m \rho^2 + \frac{(m' - 8\pi G \lambda j)^2}{16\pi G \lambda^2} + \frac{1}{2} \left( \Pi^2 + \varphi^2 \right) \quad (31a) \\
\tilde{\mathcal{P}} = \Pi \rho' + \Pi_m (m' - 8\pi G \lambda j) + \Pi \varphi' \quad (31b) 
\]

where $\Pi = \hat{\rho}^a \Pi_a$. The $G_2$ constraint now simply requires that $\tilde{\Psi}$ depends on $\rho^a$ only through its magnitude $\rho$, and $j$ disappears from $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{P}}$.

In the final step, we present $\tilde{\Psi}$ as a Fourier transform with respect to $m'$, and call the conjugate variable $\frac{1}{8\pi G \lambda} \gamma$. (This leaves the constant part of $m$ undetermined.)

\[ \tilde{\Psi}(\rho^a, m, \varphi) = \int D\gamma \exp \left( \frac{i}{8\pi G \lambda} \int m d\gamma \right) \Phi(\rho, \gamma, \varphi) \quad (32) \]

Acting on $\phi$, $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{P}}$ become

\[
\mathcal{E} = -\frac{4\pi G}{\lambda} \left( \frac{\Pi^2}{\rho^2} - \frac{1}{\rho^2} \Pi^2 \right) - \frac{\lambda}{16\pi G} \left( \rho^2 - \rho^2 \gamma^2 \right) + \frac{1}{2} \left( \Pi^2 + \varphi^2 \right) \quad (33a) \\
\mathcal{P} = \Pi \rho' + \Pi_\gamma \gamma' + \Pi \varphi' \quad (33b) 
\]

where $\Pi_\gamma = \frac{1}{i} \frac{\delta}{\delta \gamma}$. But now we see that the variables $(\rho, \gamma)$ and $(\Pi_\rho, \Pi_\gamma)$ can be interpreted as the radial and “angular” coordinates according to the decomposition

\[ r^a = (\rho \cosh \gamma, \rho \sinh \gamma) \quad (34) \]

Thus the final effective Lagrange density is

\[ \mathcal{L} = \pi_a \dot{r}^a + \Pi \dot{\varphi} - u \mathcal{E} - v \mathcal{P} \quad (35) \]

with

\[
\mathcal{E} = -\frac{1}{2} \left( g^{\pi a} \pi_a + \frac{1}{g} r^{a} r_a^j \right) + \frac{1}{2} \left( \Pi^2 + \varphi^2 \right) \quad g = 8\pi G / \lambda \quad (36a) \\
\mathcal{P} = \pi_a r^a + \Pi \varphi' \quad . \quad (36b) 
\]
The wave functional $\Phi(r^a, \varphi)$ must satisfy the energy and momentum constraints which are completely decoupled and separated, and there is no matter-gravity interaction. Nevertheless the constraints give rise to correlations between the variables, which are otherwise non-interacting.

In fact the final formulas (35), (36) may be gotten by another, shorter route. In the above approach, we have not chosen a gauge, but arrived at the final expressions by solving the gauge constraints. Alternatively, we may return to (25)–(28) and fix the Lorentz gauge freedom by setting $\frac{1}{8\pi G}\omega_1 = \Pi_M$ to zero. Since the bracket $[G_2, \Pi_M]$ is a c-number, no compensating terms are needed. The effective Lagrangian in this gauge reads

$$\mathcal{L}_{g+m}\bigg|_{\Pi_M=0} = p_a\dot{r}^a + \Pi\dot{\varphi} + \omega_0\dot{G}_2 - \left(u\tilde{E} + v\tilde{P}\right)\bigg|_{\Pi_M=0}$$

(37)

$$\tilde{E}\bigg|_{\Pi_M=0} = \rho^a\epsilon^b_a p_b - \frac{4\pi G}{\lambda}p^2 + \frac{1}{2}\left(\Pi^2 + \varphi'^2\right)$$

(38a)

$$\tilde{P}\bigg|_{\Pi_M=0} = p_a\rho^a + \Pi\varphi'$$

(38b)

and with the redefinitions

$$\pi_a = p_a + \frac{\lambda}{8\pi G}\epsilon_{ab}\rho^b, \quad r^a = \rho^a$$

(39)

we again arrive at (35), (36), with the two Lagrangians differing by a total time derivative (equivalent to factoring from the wave functional the phase $e^{i\frac{16G}{8\pi}\int \rho^a \epsilon_{ab} d\rho^b}$). Also in (37), there still remains the $\bar{G}_2$ constraint (25), now transformed, which requires that $\left(M - \frac{\lambda^2}{2} \rho^a \rho_a\right)' + j$ vanishes; but this merely serves to specify the $M$-dependence of the wave-functional, and does not interfere with the diffeomorphism constraints. We now proceed to analyze them.

The momentum constraint in (36b), enforced by $v$, is easily satisfied. It requires that the wave functional be invariant against arbitrary reparametrization of the line coordinate $x$, with the fields $\varphi, r^a$ transforming as scalars. This is achieved when the dependence on the fields is contained in integrals of 1-forms, e.g. $\int \mathcal{F}(\varphi, r^a) \varphi' dx = \int \mathcal{F}(\varphi, r^a) d\varphi$.

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5 Absence of matter-gravity interactions in the string-inspired model has also been claimed in [9]. Also we have shown [4] that point particles do not experience gravitational forces.
But we need not concern ourselves with satisfying this constraint: the momentum density arises when energy densities are commuted [see (13a)], so we expect that once the energy constraint is satisfied, the momentum constraint will also be met.

However, the remaining energy constraint cannot be solved for the following reason. The energy density consists of three identical free-field terms (for \( r^0 \), \( r^1 \), and \( \varphi \)) with the two gravitational contributions entering with opposite signs, regardless of sign \( (8\pi G/\lambda) \), while the matter contribution is positive. [The reversal of signs in the gravity terms is also understood from a degrees-of-freedom count: metric gravity in two space-time dimensions has a single negative degree of freedom, while the dilaton carries a positive degree of freedom.] The three fields form an \( O(2, 1) \) invariant array.

It is well known that the quantum commutator of energy density with momentum density for free fields possesses an anomalous, non-canonical c-number contribution proportional to \( \delta'''(x-y) \). This Schwinger term cancels in the gravitational contribution, owing to the opposite signs in the energy densities of \( r^0 \) and \( r^1 \). But the matter contribution survives, producing an obstruction to solving the energy constraint: the corresponding (functional) differential equation is not integrable.

Various proofs have been given for the necessary presence of a triple derivative Schwinger term in the energy-density – momentum-density commutator. One approach uses energy-momentum conservation and positivity of the Hilbert space. Alternatively one may normal order the field bilinears with respect to a pre-selected Fock vacuum. Perhaps neither argument is appropriate in a gravity context, where concepts of energy and momentum are elusive and the vacuum state may be very unconventional, with no relation to a normalizable Fock vacuum. It is therefore fortunate, for our argument, that an alternative proof can be devised, which makes no reference to a vacuum state, but relies on manipulations in the functional space of the Schrödinger representation. The argument, which has been presented elsewhere [10], proceeds as follows.

Consider a canonical free field \((\Pi, \varphi)\) and form the combination \( \chi = \frac{1}{\sqrt{2}} (\Pi + \varphi') \). The sum of the energy and momentum densities is given by \( \chi^2 \) and one wants to determine the commutator of \( \chi^2(x) \) with \( \chi^2(y) \); or better, the commutator \( T_{f_1} \) with \( T_{f_2} \) where

\[
T_f = \frac{1}{2} \int dx \, \chi(x) \, f(x) \, \chi(x)
\]  

(40)

Owing to the necessarily singular nature of the operator product of \( \chi \) with itself, (40) requires
regularization and subtraction, so that the regulated and subtracted quantity possesses a limit as the regulator is removed, but an anomaly in the commutator may emerge. To regulate, we replace $T_f$ by $T_F$

$$T_F = \frac{1}{2} \int dx \, dy \, \chi(x) \, F(x, y) \, \chi(y)$$

and inquire what should be subtracted from $T_F$ so that the limit

$$F(x, y) \to \frac{1}{2} (f(x) + f(y)) \, \delta(x - y)$$

can be taken.

To determine the subtraction, $c_F$, we first compute in the Schrödinger representation the (regulated) unitary operator that implements the finite transformation generated by $T_F$.

$$U_F(\varphi_1, \varphi_2) = \langle \varphi_1 | e^{-iT_F} | \varphi_2 \rangle$$

[As is well known, the exponential of an operator is better behaved than the operator itself: $\langle \varphi_1 | T_F | \varphi_2 \rangle$ involves a functional $\delta$-function, $U_F(\varphi_1, \varphi_2)$ is an ordinary functional.] Since $T_F$ is quadratic in the fields, $U_F$ can be explicitly constructed and one can explicitly study the limit (42). It turns out that $U_F$ does not possess a well defined limit, but the singularities can be identified and isolated. It is found that they are contained in a $c$-number phase $e^{-icF}$, and $c_F$ is the required subtraction in the generator $T_F$. Explicitly one finds

$$c_F = -\frac{1}{4\pi} \, P \int dx \, dy \, \frac{F(x, y)}{(x - y)^2}$$

and this then has the consequence that

$$[\tilde{T}_{f_1}, \tilde{T}_{f_2}] = i \tilde{T}_{(f_1, f_2)} - \frac{i}{48\pi} \, \int dx \, (f_1 f_2''' - f_2 f_1''')$$

where the tilda denotes well-defined, subtracted generators and $(f_1, f_2)$ is the Lie bracket of $(f_1, f_2)$. The last term is of course the unavoidable Schwinger term; it is a quantum anomaly in the commutator (13b,c) and changes the constraints from first to second class.

Another perspective on the obstruction is given by the operator solution to the theory, which may be explicitly constructed. Returning to the formulation in (37–39), we see from (16–18) that the classical solution is (with $\alpha = 0$, so that $\omega_1 = 0$)
\[ \varphi = \phi_+(X^+) + \phi_-(X^-) , \quad \Pi = \frac{d}{dx} \left( \phi_+(X^+) - \phi_-(X^-) \right) \]
\[ r^\pm = Q^\pm(X^+) + \frac{g}{2} N_\mp(X^\mp) , \quad \pi_\pm = \mp \frac{1}{2} \frac{d}{dx} \left( N_\pm(X^\pm) - \frac{2}{g} Q^\mp(X^\mp) \right) \quad (46) \]

The quantities in (46) are promoted to operators, and the canonical commutators of momenta with coordinates are reproduced provided we postulate

\[ [\phi_\pm(\xi), \phi_\pm(\xi')] = -\frac{i}{4} \epsilon(\xi - \xi') \quad (47) \]
\[ [Q^\pm(\xi), N_\pm(\xi')] = \frac{i}{2} \epsilon(\xi - \xi') \quad (48) \]

with all other commutators vanishing. [It is assumed that \( X^+(X^-) \) is an increasing (decreasing) function of \( x \).] The mode operators are chiral fields. It remains to solve the diffeomorphism constraints. According to (19), this would be achieved by enforcing the equality

\[ (\phi'_\pm)^2 = \frac{1}{2} \left( N'_\pm Q'^\pm + Q'^\pm N'_\mp \right) \]

at least weakly, i.e. when acting on states. But such an equality cannot hold: the left side — the matter energy momentum density — possesses a Schwinger term in its commutator, the right side does not.

Gravity without matter as well as the semiclassical limit are obtained \[11\] by expanding the phase of the wave functional in powers of the gravitational constant \( G \); for us the expansion parameter will be \( g \equiv 8\pi G/\lambda \): \( \Phi = \exp \left( g^{-1} S_g + S_0 + g S_1 + \ldots \right) \). We then operate on \( \Phi \) with the energy constraint (36a) and equate to zero powers of \( g \). The first few orders give the equations

\[ g^{-2} : \quad \frac{\delta S_g}{\delta \varphi} = 0 \quad (49a) \]
\[ g^{-1} : \quad \frac{\delta S_g}{\delta r^a} \frac{\delta S_g}{\delta r_a} + r^a r'_a = 0 \quad (49b) \]
\[ g^0 : \quad \left( -\frac{1}{2} \frac{\delta^2}{\delta \varphi \delta \varphi} + \frac{1}{2} \varphi'^2 \right) e^{iS_0} = -i \frac{\delta S_g}{\delta r^a} \frac{\delta e^{iS_0}}{\delta r_a} - \frac{i}{2} \frac{\delta^2 S_g}{\delta r^a \delta r_a} e^{iS_0} \quad (49c) \]

Eq. (49a) indicates that \( S_g \) is all that survives when matter is absent and is determined by Eq. (49b) to be

\[ \Phi(r^a) \bigg|_{\text{pure gravity}} = e^{\frac{1}{16} \int dr^a r^a r_a} , \quad \frac{\delta^2 S_g}{\delta r^a \delta r_a} = 0 \quad (50) \]
which is also invariant against $x$-reparametrization — as anticipated, the momentum constraint is met automatically. This explicitly shows the absence of an obstruction — the Schwinger terms cancel owing to the opposite signs in the gravitational energy. [When (50) is used in (32) with (34) and the relation between $M$ and $m$ is recalled, we regain the pure gravity wave functional (20).]

But the addition of matter adds positive expressions to the energy density, with non-canceling Schwinger terms, and a solution cannot be found.\footnote{If one has an even number of matter fields, $\varphi_n$, and is willing to replace half of them by $i\varphi_n$, then a functional that is annihilated pairwise by the matter energy densities can be constructed. From (50) one sees that the functional involves $e^{\pm\frac{i}{2}\int \varphi_n d\varphi_{n+1}}$. But this is a real exponential, rather than a phase, and diverges for large fields. We view such a “solution” to be unacceptable. It was proposed in Ref. \cite{12}.

The last topic that we address concerns a possible semi-classical limit of the quantum theory. The semi-classical approach consists of setting

$$\Phi = e^{\frac{i}{2}\int dr^a \epsilon_{ab} r^b \Phi_{\text{matter}} }$$

with \( \Phi_{\text{matter}} = e^{i(S_0 + gS_1 + \ldots)} \) satisfying equation (49c), which is exact as $g \to 0$,

$$\left(-\frac{1}{2} \frac{\delta^2}{\delta \varphi \delta \varphi} + \frac{1}{2} \varphi'^2\right) \Phi_{\text{matter}} = r^a \epsilon_a^b \frac{1}{i} \frac{\delta \Phi_{\text{matter}}}{\delta r^b}$$

Next one supposes that the $r^a$ dependence in $\Phi_{\text{matter}}$ is such that the right side of (51b) may be written as $i\frac{\delta}{\delta \tau(x)} \Phi_{\text{matter}}$, where $\tau(x)$ is an emergent local time coordinate that allows viewing (51b) as a Schwinger-Tomonaga-type equation. However here this is not possible, again because of an obstruction: $\frac{\delta}{\delta \tau(x)}$ obviously commutes with itself at all points on the line, but $r^a(x) \epsilon_a^b \frac{\delta}{\delta r^b(x)}$ does not. Thus a sensible semi-classical description, which still retains a limiting form of the diffeomorphism constraints, cannot be given.

Evidently lineal gravity with matter is an anomalous theory. We leave it for future investigations to decide whether any of the familiar devices for relaxing incompatible quantum constraints can be employed here to define an acceptable lineal quantum gravity. But even should this be possible, it is doubtful that the semiclassical black hole puzzles have a quantum analog in this model.

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