From real materials to model Hamiltonians with density matrix downfolding

(Converting a many-electron problem to a fewer-electron one)

Hitesh J. Changlani
Florida State University

Based on
HJC, H. Zheng, L. K. Wagner J. Chem. Phys. (2015)
H.Zheng*, HJC*, K. Williams, B. Busemeyer, L. K. Wagner, Front. Phys (2018)
Quantum many body problem

**Approximate “effective” $H$**

\[
H_{\text{Hubbard}} = -t \sum_{\langle i, j \rangle, \sigma} c_{i,\sigma}^\dagger c_{j,\sigma} + U \sum_{i} n_{i,\uparrow} n_{i,\downarrow}
\]

The real thing. Deals with **all electrons** (eg. Fionn’s talk).

Hilbert space large, so **smaller systems**.

How do we know we have an “exotic” phase? (eg. spin liquid etc)

Deals with **valence electrons** only (physically insightful): eg. many models of high Tc superconductivity

Smaller Hilbert space locally, more conducive for **larger scale simulations**

**Several new diagnostics** to characterize exotic phases
Models: hard to solve exactly

\[ \tilde{H} \psi = E \psi \]

\[ \psi = \sum_{q_1, q_2 \ldots q_N} \psi^{q_1q_2\ldots q_N} |q_1 q_2 q_N \rangle \]

\( q \) is \( \bigcirc \) or \( \uparrow \) or \( \downarrow \)

(depending on model)

The Hilbert space is HUGE!

\( N=100 \) spins \( \rightarrow 10^{16} \) PB

IBM has largest storage array of 120 PB

But many recent advances

\[ \psi = \]

Quantum Monte Carlo (sign problem or trial wavefunction bias)

Ceperley, Becca, Sorella, Scalapino, Scalettar, Zhang, Alavi, Umrigar, Sandvik, Prokof’ev......

Can be made exact

Matrix Product State (MPS / DMRG)

Tensor Product/Network State (TP/NS)

Wilson, White, Ostlund, Nishino, Vidal, Xiang, Verstraete, Cirac

Others: DMFT (Kotliar, Georges,Millis), DMET (Knizia, Chan)
Example of what the “model world” is interested in
2D kagome magnet: experimental realization Herbertsmithite

Fu, Imai, Han, Y.S. Lee (Science 2015)

\[ H = \sum_{i,j} J_{ij} S_i \cdot S_j \]

Every copper surrounded by four oxygens much like the high temperature cuprates.

Coppers form a kagome lattice rather than a square lattice.

Both Mott insulators but low energy physics different.

Frustrated

Un - frustrated
Kagome phase diagram: many competing phases with similar energy

Quantum spin liquid is a phase of matter with:
(1) no symmetry breaking, no local order parameter
(2) topological properties/fractional anyonic excitations: quantum computation?
(3) very high multireference character

Many competitive phases (Marston Zeng state: 36 site unit cell)
Energy difference between candidates is 0.006 J per site
\((J=200 \text{ K}) 0.006*200 = 1.2 \text{ K} = 0.1 \text{ meV}\)

Where does the “real material” lie in the phase diagram?
Flowchart for strongly correlated models

Fit Experimental data: one personal “success” story for frustrated NaCaNi$_2$F$_7$

Fit magnetic couplings to STATIC structure factor (typically 0.1 meV to 10 meV)

EXPT 1

Plumb, HJC, et al, arXiv:1711.07509, to appear (2018)

make prediction for DYNAMIC properties (classical Monte Carlo + molecular dynamics or semiclassical spin wave theory)

EXPT 2

Indications for a quantum spin liquid (quantum model solution itself needs more work to be sure .... )

Lattice model only of magnetic ions

Calculations with classical and semiclassical model methods (pyrochlore – 3d version of kagome)

S. Zhang*, HJC*, Plumb, Moessner, Tchernyshyov, to be submitted (2018)
Flowchart for strongly correlated models

Fit Experimental data to get effective Hamiltonian – not always predictive

Lattice model only of magnetic ions (make a phase diagram in coupling space)

Fit magnetic coupling to high temperature data (assume same model for low T)

Work very hard
Be right
Make prediction

"1/3 plateau"

HJC, Lauchli
PRB (R) (2015)

Awaga, JPSJ (2010)

Magnetization measurement
As theorists, want to minimize dependence on experiments so, the questions are…

• Is electronic structure likely to be “super accurate” to resolve small energy scales in solids (especially strongly correlated Mott insulators)?
  **Answer: Who knows the future, but currently it is not**

• Does one need to fully solve the many electron problem (obtain eigenstates) to understand the important low-energy physics of a problem?
  **Answer: Not necessarily**

• Can electronic structure tell us important physics that helps build relevant useful models (especially for solids) and helps diagnose what is right or wrong with them?
  **Answer: This is what the talk is about. I will show that QMC is quite useful for this purpose.**
What exactly do we mean by “effective Hamiltonian”? posing the problem

Classic example is the Schrieffer Wolff transformation
What is typically done to get a model?

**Density functional theory**

![Energy Diagram](image1)

Construct Wannier functions

Get hoppings “t” (integral of the kinetic energy projected in Wannier basis)
O. Andersen, R. Martin, Saha-Dasgupta, Valenti...

Hubbard U (Interactions) ? Use Post DFT method, constrained RPA
(Imada, Ariyasetiawan, Kotliar, Georges, Biermann, Casula, Werner, Valenti, Jeschke....)

How do we know whether these approximations are good or bad?
Our viewpoint for effective Hamiltonian determination

- Why treat kinetic and potential parts of Hamiltonian differently?
- Use information from accurate wavefunctions which do not care about this distinction
- Method must have internal consistency checks – is the model good or bad?
- Model is an “auxiliary system” with different electron number, so have to “match properties” instead of wavefunctions
- (several variants in other contexts: Ceperley, HJC, Henley, Wagner, White, Chan...)

These ideas motivated our first work: H.J. Changlani, H. Zheng, L.K. Wagner, J. Chem. Phys (2015)
Posing the problem precisely

Ab initio density matrix downfolding (AI-DMD)

Given a set of low energy wavefunctions (not necessarily eigenstates) how does one determine or “learn” the effective Hamiltonian parameters

Reconstruction problem

\[ \tilde{H} = C + \sum_{ij} t_{ij} c_i^\dagger c_j + \sum_{ijkl} V_{ijkl} c_i^\dagger c_j^\dagger c_l c_k \]

Finally solve

\[ \tilde{H} \psi = E \psi \]

Our criterion:

\[ \langle c_i^\dagger c_j \rangle = \langle c_i^\dagger c_j \rangle \quad \text{Model} \]

\[ \langle c_i^\dagger c_j^\dagger c_l c_k \rangle = \langle c_i^\dagger c_j^\dagger c_l c_k \rangle \quad \text{Model} \]

\[ \langle c_i^\dagger c_j^\dagger c_l c_k \rangle = \langle c_i^\dagger c_j^\dagger c_l c_k \rangle \quad \text{Ab-initio} \]
DMD scheme for effective Hamiltonians

Given a “good” one particle basis

For a single state

$$\tilde{E}_s \equiv \langle H \rangle_s = C + \sum_{ij} t_{ij} \langle c_j c_i \rangle + \sum_{ijkl} V_{ijkl} \langle c_{i} c_{j}^\dagger c_{l} c_{k}^\dagger \rangle$$

TRUE whether state is eigenstate or NOT

For many states

$$\begin{pmatrix} \tilde{E}_1 \\ \tilde{E}_2 \\ \tilde{E}_3 \\ \vdots \\ \tilde{E}_M \end{pmatrix} = \begin{pmatrix} 1 & \langle c_i c_j \rangle_1 & \ldots & \langle c_i c_j c_l c_k \rangle_1 & \ldots \\ 1 & \langle c_i c_j \rangle_2 & \ldots & \langle c_i c_j c_l c_k \rangle_2 & \ldots \\ 1 & \langle c_i c_j \rangle_3 & \ldots & \langle c_i c_j c_l c_k \rangle_3 & \ldots \\ 1 & \langle c_i c_j \rangle_4 & \ldots & \langle c_i c_j c_l c_k \rangle_4 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \ldots & \ldots & \ldots & \ldots \\ 1 & \ldots & \ldots & \ldots & \ldots \\ 1 & \langle c_i c_j \rangle_M & \ldots & \langle c_i c_j c_l c_k \rangle_M & \ldots \end{pmatrix} \begin{pmatrix} C \\ t_{ij} \\ \vdots \\ V_{ijkl} \\ \vdots \end{pmatrix}$$

Energies  Matrix of density matrices  Parameters
AI-DMD scheme for effective Hamiltonians

\[
\begin{pmatrix}
\tilde{E}_1 \\
\tilde{E}_2 \\
\tilde{E}_3 \\
\vdots \\
\tilde{E}_M
\end{pmatrix}
= \begin{pmatrix}
1 & \langle c_i^\dagger c_j \rangle_1 & \cdots & \langle c_i^\dagger c_j c_l c_k \rangle_1 & \cdots \\
1 & \langle c_i^\dagger c_j \rangle_2 & \cdots & \langle c_i^\dagger c_j c_l c_k \rangle_2 & \cdots \\
1 & \langle c_i^\dagger c_j \rangle_3 & \cdots & \langle c_i^\dagger c_j c_l c_k \rangle_3 & \cdots \\
1 & \langle c_i^\dagger c_j \rangle_4 & \cdots & \langle c_i^\dagger c_j c_l c_k \rangle_4 & \cdots \\
1 & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & \cdots & \cdots & \cdots \\
1 & \langle c_i^\dagger c_j \rangle_M & \cdots & \langle c_i^\dagger c_j c_l c_k \rangle_M & \cdots
\end{pmatrix}
\begin{pmatrix}
C \\
t_{ij} \\
\vdots \\
V_{ijkl} \\
\vdots
\end{pmatrix}
\]

$E = $ Energies $A = $ Matrix of density matrices $x = $ Parameters

\[
E = Ax
\]

\[
\mathcal{N} \equiv \|Ax - E\|^2
\]

Minimize difference

H.J. Changlani, H. Zheng, L.K. Wagner, JCP (2015)
AI-DMD scheme for effective Hamiltonians

Energy must vary with variation in density matrices to be “relevant”

\[
\begin{pmatrix}
\tilde{E}_1 \\
\tilde{E}_2 \\
\tilde{E}_3 \\
\vdots \\
\tilde{E}_M
\end{pmatrix}
= 
\begin{pmatrix}
1 & \langle c_i^\dagger c_j \rangle_1 & \langle c_i^\dagger c_j^\dagger c_l c_k \rangle_1 \\
1 & \langle c_i^\dagger c_j \rangle_2 & \langle c_i^\dagger c_j^\dagger c_l c_k \rangle_2 \\
1 & \langle c_i^\dagger c_j \rangle_3 & \langle c_i^\dagger c_j^\dagger c_l c_k \rangle_3 \\
1 & \langle c_i^\dagger c_j \rangle_4 & \langle c_i^\dagger c_j^\dagger c_l c_k \rangle_4 \\
1 & \cdots & \cdots \\
1 & \cdots & \cdots \\
1 & \langle c_i^\dagger c_j \rangle_M & \langle c_i^\dagger c_j^\dagger c_l c_k \rangle_M \\
\end{pmatrix}
\begin{pmatrix}
C \\
t_{ij} \\
\vdots \\
V_{ijkl}
\end{pmatrix}
\]

\[E = \text{Energies} \quad A = \text{Matrix of density matrices} \quad x = \text{Parameters}\]
We need a “good” (often local) one particle basis ($i$).

Ab initio density matrix downfolding (AI-DMD)
Examples I will talk about

- A pedagogical toy model that highlights the main ideas (no QMC, everything can be exactly solved) – 3-band Hubbard model to 1-band Hubbard model

- Benzene molecule with QMC – effective one orbital per site –

- QMC on graphene

- Transition metals, as a futuristic (preliminary) application
**Example 1: (Toy) Three to one band model at half filling**

\[
H = \epsilon_p \sum_{j \in p, \eta} n_{j, \eta} + \epsilon_d \sum_{i \in d, \eta} n_{i, \eta} + t_{pd} \sum_{\langle i \in d, j \in p \rangle, \eta} \text{sgn}(p_i, d_j) \left( c_{i, \eta}^\dagger c_{j, \eta} + \text{h.c.} \right) + U_p \sum_{j \in p} n_{j, \uparrow} n_{j, \downarrow} + U_d \sum_{i \in d} n_{i, \uparrow} n_{i, \downarrow} + V_{pd} \sum_{\langle i \in p, j \in d \rangle} n_{j} n_{i},
\]

\[
\Delta \equiv \epsilon_p - \epsilon_d
\]

\[
H = E_0 - t \sum_{\langle i, j \rangle, \eta} \tilde{d}_{i, \eta}^\dagger \tilde{d}_{j, \eta} + U \sum_i \tilde{n}_{i, \uparrow} \tilde{n}_{i, \downarrow}
\]
Example 1: Three to one band model – what are the optimal orbitals?

\[
\tilde{d}_{i,\eta} = \sum_j T_{ij} c_{j,\eta}
\]

\[
\langle \tilde{d}_{i,\eta}^\dagger \tilde{d}_{j,\eta} \rangle_s = \sum_{mn} T_{im}^* \langle c_{m,\eta}^\dagger c_{n,\eta} \rangle_s T_{jn},
\]

\[
\langle \tilde{n}_{i,\uparrow} \tilde{n}_{i,\downarrow} \rangle_s = \sum_{jkmn} T_{ij}^* T_{im}^* \langle c_{j,\uparrow}^\dagger c_{m,\downarrow}^\dagger c_{n,\downarrow} c_{k,\uparrow} \rangle_s T_{in} T_{ik}
\]

\[
C = \sum_s \sum_{\eta} \left( \sum_i \langle \tilde{d}_{i,\eta}^\dagger \tilde{d}_{i,\eta} \rangle_s - N_\eta \right)^2 + \sum_{mn} \left( (TT^\dagger)_{mn} - \delta_{mn} \right)^2
\]
Example 1: Three to one band model, (renormalized) effective parameters

\[ U_d/t_{pd} = 8 \]

\[ \Delta/t_{pd} = 3 \]
Example 1: Three to one band model, Multi-scale prediction

- Obtain effective parameters for a 4 unit cell system from downfolding
- Check their transferability/predictive power on a 8 unit cell system by checking energy gaps
- Hilbert space of the 3-band 8 unit cell system is 112 million (Lanczos)
- Hilbert space of the 1-band 8 unit cell system is 4900 (exact diag)

| $\Delta/t_{pd}$ | $U_d/t_{pd}$ | $t$ (eV) | $U/t$ |
|---------------|-------------|---------|-------|
| 3.0           | 4.0         | 0.2839  | 8.698 |
| 3.0           | 8.0         | 0.3025  | 13.45 |
| 3.0           | 12.0        | 0.3155  | 16.58 |
| 5.0           | 4.0         | 0.2326  | 15.08 |
| 5.0           | 8.0         | 0.2501  | 23.75 |
| 5.0           | 12.0        | 0.2647  | 29.89 |

![Graphs](image)
Example 1: Three to one band model –
Parameters are energy window dependent

The parameters depend on the energy window of interest, much like what the renormalization group has taught us.
Quantum Monte Carlo in a nutshell

**What?**

Stochastic sampling of many electron configurations

Optimize many-body wavefunction:

\[ \psi_T(r_1, r_2, \ldots, r_N) = \mathcal{J} \sum_i d_i D_i \]

- Jastrow: Introduces electron correlation
- Determinants (HF or DFT)

Variational Monte Carlo (sample trial wf, often not chemically accurate)

Projector (Diffusion) Monte Carlo: (exact but sign problem)

Diffusion Monte Carlo with nodal constraint (systematic error, but very accurate)

**Why?**

Scalable, esp. important for solids

Accuracy improvable (by improving wavefunctions)

Can calculate observables (correlation functions, structure factor)

\[ \langle C_i \dagger C_j \rangle \langle C_i \dagger C_j C_l C_k \rangle \]

Refs: Ceperley, Alder, Umrigar, Nightingale, Mitas, Foulkes...

QWALK software: Wagner, Bajdich, Mitas, JCP (2009)
Example 2: Benzene molecule

One particle orbitals

Occupancy 0.99  
Occupancy 0.99  
Occupancy 0.5

\[
H = -t \sum_{ij} c_{i,\sigma}^\dagger c_{j,\sigma} + \text{h.c.} + U \sum_i n_{i,\uparrow} n_{i,\downarrow}
\]

\[
H = - \sum_{ij} t_{ij} c_{i,\sigma}^\dagger c_{j,\sigma} + \text{h.c.} + U \sum_i n_{i,\uparrow} n_{i,\downarrow} + \sum_{ij} V_{ij} n_i n_j
\]

Hubbard  
Extended Hubbard (PPP)

H.J. Changlani, FSU
Example 2: Benzene molecule

Testing the one particle basis

Full ab-initio calculation: 30 electrons (eg. 15 u, 15 d)

“Effective” Lattice model: 6 electrons (eg. 3 u, 3 d)

| Spin | DFT   | SJ-VMC         | SJ-DMC         | CISDJ-VMC     | CISDJ-DMC     | $N_\uparrow, N_\downarrow$ | Used in Fit? |
|------|-------|----------------|----------------|---------------|---------------|----------------------------|--------------|
| 0    | -37.6303 | -37.6229(6)       | -37.7213(9)       | -37.6352(6)  | -37.7259(9)  | 2.96, 2.96                  | Yes          |
| 1    | -37.4634 | -37.4546(6)     | -37.5555(7)     | -37.4814(6)  | -37.5707(7)  | 3.94, 1.98                  | Yes          |
|      |         |                |                | -37.4561(6)  | -37.5479(6)  | 3.94, 1.98                  | Yes          |
|      |         |                |                | -37.4531(6)  | -37.5470(6)  | 3.94, 1.98                  | Yes          |
| 2    | -37.3203 | -37.2987(6)     | -37.3974(7)     | -37.3141(6)  | -37.4020(7)  | 4.92, 1.00                  | Yes          |
| 3    | -37.0378 | -37.0116(4)     | -37.1074(7)     | -37.0118(4)  | -37.1083(7)  | 4.88, 0.02                  | No           |

Energy Units: Ha
Example 2: Benzene molecule
Hubbard model, double occupancy

\[ H_{\text{Hubbard}} = -t \sum_{i,j,\sigma} c_{i,\sigma}^\dagger c_{j,\sigma} + U \sum_i n_{i,\uparrow} n_{i,\downarrow} \]

**Why is U hard to get?**

U depends very sensitively on the level of correlation in the wavefunction.

**SJ = Slater-Jastrow**
(1 determinant)

**CISD-J = Multidet - Jastrow**

Half filled ground state (S=0)
Example 2: Benzene molecule
Comparison of model vs ab-initio correlation functions in ground state
Example 2: Benzene molecule

Hubbard model

\[ E = A x \]

\[ t_{01} = 3.04(4) \]
\[ U = 5.2(2) \]

\[ \Delta E_{\text{max}} = 2.24 \]
\[ \Delta E_{\text{rms}} = 0.49 \]

Outliers
Example 2: Benzene molecule

Hubbard and extended Hubbard (PPP) model

Hubbard only

\[ H = -t \sum_{ij} c_{i,\sigma}^\dagger c_{j,\sigma} + \text{h.c.} + U \sum_i n_{i,\uparrow} n_{i,\downarrow} \]

Extended Hubbard

\[ H = - \sum_{ij} t_{ij} c_{i,\sigma}^\dagger c_{j,\sigma} + \text{h.c.} + U \sum_i n_{i,\uparrow} n_{i,\downarrow} + \sum_{ij} V_{ij} n_i n_j \]
Example 2: Benzene molecule

Reconstructing eigenstates by solving lattice model of 6 electrons + comparison to experiment

Comparable to previous semi-empirical fits + We DO NOT use experimental data!

Some subtleties:
Need to calculate parameters from variational and diffusion MC, and extrapolate
Example 3: Graphene (periodic solid)
What is the “effective” U?

Often quoted Hubbard U/t = 3.8

Wannier function for graphene

Sorella et al (Nature Scientific Reports 2012)

Full ab-initio calculation on 3x3 cell: 72 electrons (eg. 36 u, 36 d)
“Effective” Lattice model on 3x3 cell: 18 electrons (eg. 9 u, 9 d)
Example 3: Graphene (periodic solid)

What is the “effective” U?

Tight binding model

Hubbard model

Our result $U/t \approx 1.3 \pm 0.2$, cRPA result $1.6 \pm 0.2$ (PRL 2013)

Graphene is well in semi-metallic phase
Example 4: Transition metal atoms
rotationally symmetric case: Hund's coupling $J$

Antiferromagnetism and correlation of electrons in transition metals

A. M. Oleś

Max-Planck—Institut für Festkörperforschung, D-7000 Stuttgart 80, Federal Republic of Germany
and Institute of Physics, Jagellonian University, PL-30-059 Kraków, Poland

(Received 12 October 1982)

\[
H = \sum_{m,n,i,\sigma} t_{mn} a_{mi\sigma}^\dagger a_{ni\sigma} + (U + 2J) \sum_{m,i} n_{mi\uparrow} n_{mi\downarrow} + U \sum_{m,i,j,\sigma} n_{mi\sigma} n_{mj,-\sigma} + (U - J) \sum_{m,i,j,\sigma} n_{mi\sigma} n_{mj\sigma}
- J \sum_{m,i,j,\sigma} a_{mi\sigma}^\dagger a_{mi,-\sigma} a_{mj,-\sigma} a_{mj\sigma} + J \sum_{m,i,j} a_{mi\uparrow}^\dagger a_{mi\downarrow}^\dagger a_{mj\downarrow} a_{mj\uparrow}
\]

Hubbard-like Hamiltonians for interacting electrons in $s$, $p$, and $d$ orbitals

M. E. A. Coury,\textsuperscript{1,2,*} S. L. Dudarev,\textsuperscript{2} W. M. C. Foulkes,\textsuperscript{1} A. P. Horsfield,\textsuperscript{1,1} Pui-Wai Ma,\textsuperscript{2} and J. S. Spencer\textsuperscript{1}

\textsuperscript{1}Imperial College London, London SW7 2AZ, United Kingdom
\textsuperscript{2}CCFE, Culham Science Centre, Abingdon, Oxfordshire OX14 3DB, United Kingdom

(Received 20 July 2015; revised manuscript received 2 November 2015; published 1 February 2016)

Others: Ohno, Kanamori, Dworin-Narath, Parisier-Pople-Parr, Georges, Kotliar, Imada...
Example 4: Transition metal atoms
rotationally symmetric case: Hund's coupling $J$

$$H = \sum_{m,n,i,\sigma} t_{mn}a_{mi\sigma}^{+}a_{ni\sigma} + (U+2J)\sum_{m,i} n_{mi \uparrow}n_{mi \downarrow} + U\sum_{m,i,j,\sigma} n_{mi\sigma}n_{mj,-\sigma} + (U-J)\sum_{m,i,j,\sigma} n_{mi\sigma}n_{mj\sigma}$$

$$-J\sum_{m,i,j,\sigma} a_{mi\sigma}^{+}a_{mi,\sigma}^{\downarrow}a_{mj,-\sigma}^{\uparrow}a_{mj,\sigma}^{\downarrow} + J\sum_{m,i,j} a_{mi\downarrow}^{+}a_{mi\uparrow}a_{mj\uparrow}a_{mj\downarrow}$$

Mn

- **d 4**
  - J = 0.95 eV

- **d 5**
  - J = 1.05 eV

V

- **d 3**
  - J = 0.60 eV

- **d 4**
  - J = 0.65 eV
Summary of DMD

Model Hamiltonians using wavefunction data
- main idea is to relate “energies” and “reduced density matrices”

Look beyond Hubbard-only models, eg. Kanamori form

Promising but also challenging when many energy scales (FeSe, cuprates)

Ab-initio community can help those studying strongly correlated models
Acknowledgements

- David Ceperley
- Cyrus Umrigar
- (late) Christopher Henley
- Garnet Chan
- Bryan Clark
- Norm Tubman

Thanks for listening!