The half–form $\sqrt{dx}$

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In this brief article we try to find an “interpretation” for the formalism $\sqrt{dx}$.

The problem to make sense of $\int \sqrt{dx}$ is treated in this article. In what follows we explore a solution called “the corrected” integral, formally we denote it by $\gamma \int_{[a,b]} \sqrt{dx}$.

Some inspiration for this study can be suggested by Ramanujan Ram from its convergent series. The problem is approached from the point of view of the calculus, with some relations to the fractional calculus in Ross Ro. A well definition of the symbol $\sqrt{dx}$ can be useful in geometric quantization (Woodhouse W) and different areas of mathematics. In the contest of geometric quantization the symbol $\sqrt{dx}$ denotes a section of the quantum line bundle defined on a compact, complex, symplectic manifold $M$.

I. INTEGRATION OF $\sqrt{dx}$ WITH A RIEMANNIAN INTEGRAL

In this section we start to examine the situation evaluating the following integral:

$$\int_{[a,b]} \sqrt{dx}. \tag{1}$$

Let $\mathcal{P} = \{x_0 = a, x_1, \ldots, x_n = b\}$ be a partition of the interval $[a,b]$ in $n$ subintervals of amplitude $\frac{b-a}{n}$. Thus

$$\int_{[a,b]} \sqrt{dx} = \lim_{n \to +\infty} \sum_{i=1}^{n} \sqrt{\Delta x_i}, \tag{2}$$

where $\Delta x_i = x_i - x_{i-1}$ for $i = 1, \ldots, n$. This is the Riemann integral where, instead to consider as “base of rectangles” the quantities $\Delta x_i$, we consider $\sqrt{\Delta x_i}$. The result is:

$$\int_{[a,b]} \sqrt{dx} = \lim_{n \to +\infty} \sum_{i=1}^{n} \sqrt{\frac{b-a}{n}} = \lim_{n \to +\infty} \sqrt{(b-a)n} = +\infty. \tag{3}$$

The integral diverges at $+\infty$, the result is not satisfactory at all.

II. ON A RAMANUJAN SUM

In order to find a way to make this infinity “disappear”, let us consider this result due to Ramanujan Ram:

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} = \frac{2}{3}n\sqrt{n} + \frac{1}{2\sqrt{n}} - \zeta\left(\frac{3}{2}\right) + \frac{1}{4\sqrt{n}} + o\left(\frac{1}{n^{\frac{3}{2}}}\right). \tag{4}$$

This is an asymptotic expansion of the sum of the square roots of the first $n$ natural numbers. The main term in the expansion, when $n$ goes to infinity, is $\frac{2}{3}n\sqrt{n}$. Another similar series, always due to Ramanujan is:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} = 2\sqrt{n} + \frac{1}{2\sqrt{n}} + \zeta\left(\frac{1}{2}\right) + o\left(\frac{1}{n^{\frac{1}{2}}}\right). \tag{5}$$

The idea can be to modify the sum of rectangles using the factor $\gamma(n) = \frac{1}{\sqrt{n}}$. In this case:

$$\int_{[a,b]} \sqrt{dx} = \lim_{n \to +\infty} \gamma(n) \cdot \sum_{i=1}^{n} \sqrt{\frac{b-a}{n}} = \sqrt{(b-a)} \lim_{n \to +\infty} \frac{\sqrt{n}}{\sqrt{n}} = \sqrt{(b-a)}. \tag{6}$$

This is not the Riemann integral used before but it is a variation with a normalized sum. The problem was in fact the divergence of the series of square roots. This sort of “correction” bring to a finite result that corresponds to the square root of the initial interval.

We denote this corrected integral with the notation:

$$\gamma \int_{[a,b]} \sqrt{dx} = \lim_{n \to +\infty} \gamma \sum_{i=1}^{n} \sqrt{\Delta x_i}, \tag{7}$$

where $\gamma \sum$ is the sum corrected by the factor $\gamma(n)$ and $\gamma \int$ is the “corrected” integral.

III. THE GEOMETRIC QUANTIZATION PROGRAM AND THE DEFINITION OF $\sqrt{dx}$

An attempt to define the half–form $\sqrt{dx}$ has been done in the program of geometric quantization. The geometric quantization is a process that associate to a symplectic manifold an Hilbert space representing the space of quantum states (for references about the geometric quantization program see Woodhouse W or works of Kostant K and Souriau So).
The idea is to see the half-form \( \sqrt{dx} \) as a section of the square root of the canonical line bundle associated to the polarization adopted during the process of geometric quantization. The concept of \( \frac{1}{2} \)-form is strictly correlated to the concept of \( \frac{1}{2} \)-density. In particular we have that \( \frac{1}{2} \)-densities can be identified with \( \frac{1}{2} \)-forms. Let us consider the case of \( \mathbb{R} \). The space \( \mathbb{R} \) is a vectorial space where we can choose an orientation. If \( F(\mathbb{R}) \) represents the set of frames of \( \mathcal{R} \), we can consider the action of \( GL(1, \mathbb{R}) \) on \( F(\mathbb{R}) \) that is simply the scalar multiplication. We define the set of \( \frac{1}{2} \)-densities as:

\[
|\mathbb{R}|^{\frac{1}{2}} = \left\{ \nu : F(\mathbb{R}) \to \mathbb{C} : \nu(a \cdot v) = \nu(v) | \det a |^{\frac{1}{2}}, \right. \\
\left. \forall v \in F(\mathbb{R}), a \in GL(1, \mathbb{R}) \right\}.
\]

(8)

This set is a line bundle denoted by \( |\mathbb{R}|^{\frac{1}{2}} \to \mathbb{R} \) where the 1-dimensional fiber at \( x \in \mathbb{R} \) is \( |\mathbb{R}_x|^{\frac{1}{2}} \to \mathbb{R}_x \). Rigorous definitions of \( \frac{1}{2} \)-forms and \( \frac{1}{2} \)-distributions with its properties can be found in Guillemin and Sternberg[GS], Hall[H] and Rawnsley[Raw].

In geometric quantization the \( \frac{1}{2} \)-forms define, intrinsically, an half-form Hilbert space with an inner product and a norm. Let us consider the case of \( \mathbb{R} \). Then the symplectic manifold is \( M = T^*\mathbb{R} = \mathbb{R}^2 \). Let \( P \) be the vertical polarization of \( M \) with the orientation of \( \mathbb{R} \) so that oriented 1-forms are positive multiple of \( dx \). Let \( \sqrt{K_P} \) to be the trivial bundle with trivializing section \( \sqrt{dx} \) such that \( \sqrt{dx} \otimes \sqrt{dx} = dx \). Then the half-form \( \sigma = f(x) \sqrt{dx} \), for some real function \( f(x) \), has the norm:

\[
\|\sigma\|^2 = \int_{\mathbb{R}} |f(x)|^2 dx.
\]

(9)

IV. THE CORRECTED INTEGRAL AS APPLICATION FROM THE \( \frac{1}{2} \)-FORMS TO \( \mathbb{R} \)

Let us consider the following \( \frac{1}{2} \)-form \( \sigma = \sqrt{dx} \). In order to be precise, we must view this form as the form \( 1 \otimes \sqrt{dx} \) of the quantum line bundle \( L \otimes \sqrt{K_P} \). In this case we have that:

\[
\|\sigma\|^2 = \int_{[a,b]} dx = b - a,
\]

(10)

where we considered as a base space the interval \([a,b]\). Now the question is if exists a square root of the following equation:

\[
\sigma \cdot \sigma = b - a,
\]

(11)

where the product here is the squared norm in the Hilbert space of half-forms. In order to answer the question let us consider the corrected integral:

\[
\gamma \int_{[a,b]} \sqrt{dx} = \sqrt{b - a}.
\]

(12)

We can see that:

\[
\gamma \int_{[a,b]} \sqrt{dx} \cdot \gamma \int_{[a,b]} \sqrt{dx} = (b - a).
\]

(13)

So it is possible to apply the corrected integral in order to find the corrected result. In other terms we can see the corrected integral as a map \( \gamma : |\mathbb{R}|^{\frac{1}{2}} \to \mathbb{R} \), omitting the quantum line bundle associated.

V. INTERESTING INTEGRALS IN \( \sqrt{dx} \)

Let us consider the integral:

\[
\int_{[a,b]} x \sqrt{dx}.
\]

(14)

Without the correction the integral diverges. We can try with:

\[
\int_{[a,b]} x \sqrt{dx}.
\]

(15)

Let us consider the following partition of \([a,b]\) with \( x_i = a + (i - 1) \cdot h \) and \( x_{i+1} = a + i \cdot h \), where \( h = \frac{b-a}{n} \). Thus:

\[
\gamma \int_{[a,b]} x \sqrt{dx} = \lim_{n \to +\infty} \gamma(n) \sum_{i=1}^{n} (a + i \cdot h) \sqrt{h} =
\]

\[
= \lim_{n \to +\infty} a \cdot \sqrt{b-a} + \frac{\sqrt{(b-a)^3}}{n^2} \sum_{i=1}^{n} i =
\]

\[
= a \cdot \sqrt{b-a} + \frac{\sqrt{(b-a)^3}}{2},
\]

(16)

where we used the fact that \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \). A similar result is the following:

\[
\gamma \int_{[a,b]} x^2 \sqrt{dx} = \frac{a^2}{2} \sqrt{b-a} +
\]

\[
+ (b-a) \sqrt{b-a} + \frac{1}{3} (b-a)^2 \sqrt{b-a}.
\]

(17)

The calculations are similar to the previous case where the formula used now is \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \).
VI. $\sqrt{dx}$ FROM FRACTIONAL CALCULUS

In this section we see as our definition of corrected fractional integral is in perfect agreement with the definition from the fractional calculus\(^{Ro}\). We start recalling the formula for the $\frac{1}{2}$-integral between $a$ and $b$, this is given by the formula:

$$D_{[a,b]}^{-\frac{1}{2}}f(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_a^b (b-t)^{-\frac{1}{2}} f(t) dt \quad (18)$$

Now we observe that if we compute the $\frac{1}{2}$-integral for the constant function $f(t) = 1$ we find that:

$$D_{[a,b]}^{-\frac{1}{2}}1 = \frac{2}{\Gamma\left(\frac{1}{2}\right)} \sqrt{b-a}. \quad (19)$$

We observe that we have considered the fractional integral over the interval $[a,b]$. Usually the integral (18) is considered on the interval $[0,x]$ and, for general calculations of (18), we need to use the beta integral:

$$\int_0^x (x-y)^d y^b dy = \frac{\Gamma(d+1)\Gamma(b+1)}{\Gamma(b+d+2)} x^{b+d+1}. \quad (20)$$

We can compare the result from the theory of fractional calculus $\frac{2}{\Gamma\left(\frac{1}{2}\right)} \sqrt{b-a}$ with our definition using the corrected integral (12) that gives $\sqrt{b-a}$ (only a constant factor of difference!). In fact:

$$\int_{[a,b]}^{x} 1 \sqrt{dx} = \frac{\Gamma\left(\frac{1}{2}\right)}{2} D_{[a,b]}^{-\frac{1}{2}}1. \quad (21)$$

VII. OBSERVATIONS AND CONCLUSION

The main observation is that we have realized the following relation:

$$\int_{[a,b]}^{x} 1 \sqrt{dx} = \sqrt{\int_{[a,b]}^{x} f(x) dx}. \quad (22)$$

The relation seems not true in general $\gamma \int_{[a,b]}^{x} f(x) \sqrt{dx} \neq \sqrt{\int_{[a,b]}^{x} f(x) dx}$. The fact is clear observing that for $f(x) = x$ the previous relation is false. We can try to define an integral function $F(x) = \gamma \int_{[0,x]}^{x} 1 \sqrt{ds}$, in this case:

$$F(x) = \frac{1}{\gamma} \sqrt{\int_{[0,x]}^{x} 1 ds} = \sqrt{x}. \quad (23)$$

Other questions are open, it is possible a similar definition for $\sqrt{dx}$? (for $n = 3, 4, \ldots$). Another important discussion theme concerns the geometrical meaning of this correction.

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\(^{GS}\) V.Guillemin, S.Sternberg, “Geometric Asymptotics”, Amer.Math.Soc.Surveys, 14 (revised edition 1990), pg, 251–258.

\(^{H}\) B.C.Hall, “Quantum theory for mathematicians”, Graduate Texts in Mathematics book series, vol. 267, Springer-Verlag New York (2013).

\(^{K}\) B.Kostant, “Quantization and unitary representations”, Taam C.T. (eds) Lectures in Modern Analysis and Applications III. Lecture Notes in Mathematics, vol 170. Springer, Berlin, Heidelberg.

\(^{Rav}\) S.Ramanujan, “On the sum of the square roots of the first $n$ natural numbers”, J. Indian Math. Soc., V II, (1915), 173–175.

\(^{Ro}\) J.H.Rawnsley, “Some properties of half-forms”, Bleuler K., Reetz A., Petry H.R. (eds) Differential Geometrical Methods in Mathematical Physics II. Lecture Notes in Mathematics, vol 676. Springer, Berlin, Heidelberg (1978).

\(^{So}\) J.M.Souriau, “Structure des systèmes dynamiques”, Dunod, Paris 1970.

\(^{W}\) N.M.J.Woodhouse, “Geometric quantization”, Clarendon Press Oxford, second edition, 1991.