(Once more) In defense of Relational Quantum Mechanics: A note on ‘Relative facts do not exist. Relational quantum mechanics is incompatible with quantum mechanics’

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Abstract

This is a short note to answer Lawrence, Markiewicz and Żukowski objection [see arXiv:2108.11793] to Rovelli’s theory and concerning non-contextuality.
I. INTRODUCTION

The aim of the present note is to give a short reply to the recent J. Lawrence et al. preprint [1] concerning the Relational Quantum Mechanics interpretation (RQM) proposed by C. Rovelli [4–7] (for previous claims see also [2, 3] and see the replies by Di Biagio and Rovelli [8] and Drezet [9]).

Like in [9] I remind that in RQM the main issue concerns the interpretation of the full wavefunction $|\Psi_{SO}\rangle$ involving observer (O) and observed system (S). In RQM the fundamental object relatively to (O) is not $|\Psi_{SO}\rangle$ but the reduced density matrix $\hat{\rho}_{SO}^{(red.)} = \text{Tr}_O[\hat{\rho}_{SO}] = \text{Tr}_O[|\Psi_{SO}\rangle\langle\Psi_{SO}|]$. (1)

As it is well known $\hat{\rho}_{SO}^{(red.)}$ is independent of the basis chosen to represent the degrees of freedom for (O). In [9] I showed that it solves the dilemma discussed in [2, 3] concerning the ‘preferred basis problem’. Here I show that the same features debunk the claims of [1] concerning non-contextuality.

[1] starts with a GHZ state [10] for a system S of 3 spins $m = 1, 2, 3$:

$$|\text{GHZ}\rangle_S = \frac{1}{\sqrt{2}}[|+1\rangle_S^{(1)}, +1\rangle_S^{(1)}, +1\rangle_S^{(1)}_{s_1,s_2,s_3} + |-1\rangle_S^{(1)}, -1\rangle_S^{(1)}, -1\rangle_S^{(1)}_{s_1,s_2,s_3}] .$$

(2)

where $|p\rangle_S^{(1)}_m \equiv |\text{sign}(p)_z\rangle_S^{(1)}_m$ (with $p = \pm 1$) are spin eigenstates along the $z$ direction. We also have in different spin bases:

$$|\text{GHZ}\rangle_S = \frac{1}{2}[|+1\rangle_S^{(2)}, +1\rangle_S^{(3)}, -1\rangle_S^{(3)}_{s_1,s_2,s_3} + |-1\rangle_S^{(2)}, -1\rangle_S^{(3)}, +1\rangle_S^{(3)}_{s_1,s_2,s_3} + |+1\rangle_S^{(2)}, +1\rangle_S^{(3)}, +1\rangle_S^{(3)}_{s_1,s_2,s_3} + |-1\rangle_S^{(2)}, -1\rangle_S^{(3)}, -1\rangle_S^{(3)}_{s_1,s_2,s_3} + |+1\rangle_S^{(2)}, -1\rangle_S^{(3)}, +1\rangle_S^{(3)}_{s_1,s_2,s_3} + |-1\rangle_S^{(2)}, +1\rangle_S^{(3)}, +1\rangle_S^{(3)}_{s_1,s_2,s_3} + |+1\rangle_S^{(2)}, +1\rangle_S^{(3)}, -1\rangle_S^{(3)}_{s_1,s_2,s_3} + |-1\rangle_S^{(2)}, -1\rangle_S^{(3)}, -1\rangle_S^{(3)}_{s_1,s_2,s_3}]$$

(3)

where we used $|p\rangle_S^{(2)}_m = \frac{1}{\sqrt{2}}[|+1\rangle_S^{(1)}_s \pm -1\rangle_S^{(3)}_s] \equiv |\text{sign}(p)_x\rangle_S^{(1)}_m$, and $|p\rangle_S^{(3)}_m = \frac{1}{\sqrt{2}}[|+1\rangle_S^{(1)}_s \pm i - 1\rangle_S^{(3)}_s] \equiv |\text{sign}(p)_y\rangle_S^{(1)}_m$. This implies

$$\sigma_{x_{S_1}} \sigma_{y_{S_2}} \sigma_{y_{S_3}} |\text{GHZ}\rangle_S = -|\text{GHZ}\rangle_S$$

(4)

and

$$p^{(2)}_{S_1} q^{(3)}_{S_2} r^{(3)}_{S_3} = -1.$$ 

(5)

Similar expressions are obtained by circular permutations:

$$p^{(3)}_{S_1} q^{(2)}_{S_2} r^{(3)}_{S_3} = -1,$$ 

(6)
and

\[ p_{S_1}^{(3)} q_{S_2}^{(3)} r_{S_3}^{(2)} = -1. \]  

We can also write:

\[
|GHZ\rangle_S = \frac{1}{2} \left[ |+1^{(2)}, +1^{(2)}, +1^{(2)}\rangle_{S_1, S_2, S_3} + | +1^{(2)}, -1^{(2)}, -1^{(2)}\rangle_{S_1, S_2, S_3} 
+ | -1^{(2)}, +1^{(2)}, -1^{(2)}\rangle_{S_1, S_2, S_3} + | -1^{(2)}, -1^{(2)}, +1^{(2)}\rangle_{S_1, S_2, S_3} \right]
\]

implying

\[
\sigma_{x_{S_1}} \sigma_{x_{S_2}} \sigma_{x_{S_3}} |GHZ\rangle_S = +|GHZ\rangle_S
\]

and thus

\[ p_{S_1}^{(2)} q_{S_2}^{(2)} r_{S_3}^{(2)} = +1. \]  

It is well known that quantum mechanics is highly contextual. If the results of spin measurements were non-contextual Eqs. 5, 6, 7 and 10 could be true together and this, clearly, is not possible: multiplying Eqs. 5, 6, 7 contradicts Eq. 10. This incompatibility also leads to a well known proof of nonlocality for hidden-variable theories [10].

II. DEBUNKING A PARADOX

In [1] the authors use the previous results. First, they consider a single observer Alice (A) (composed of 3 qubits \( A_1, A_2, A_3 \)) who measures the spins of the GHZ system \( S \) in the \( y \) bases. The GHZ states of Eq. 2 reads in the \( y \) bases:

\[
|GHZ\rangle_S = \sum_{p,q,r} C_{333}^{pq} |p^{(3)}, q^{(3)}, r^{(3)}\rangle_{S_1, S_2, S_3}
\]

where the amplitudes \( C_{333}^{pq} \) are non vanishing for each of the 8 combinations \( p, q, r \) and where \( |C_{333}^{pq}|^2 = \frac{1}{8} \). After entanglement with Alice’s qubits we have

\[
|GHZ\rangle_{SA} = \sum_{p,q,r} C_{333}^{pq} |p^{(3)}\rangle_{SA_1} |q^{(3)}\rangle_{SA_2} |r^{(3)}\rangle_{SA_3}
\]

with \( |k^{(3)}\rangle_{SA_m} := |k^{(3)}\rangle_{S_m} |k^{(3)}\rangle_{A_m} \) and \( |k^{(3)}\rangle_{A_m} \) is the state of the qubit \( A_m, m = 1, 2, 3 \) (for \( k = \pm 1 \)).
Importantly, for Alice this state shows no correlation. This is clear in RQM where the reduced density matrix reads:

$$\rho^{(\text{red.})}_{S|A} = \frac{1}{8} \sum_{p,q,r} |p(3)\rangle_{S_1} \langle p(3)|_{S_1} |q(3)\rangle_{S_2} \langle q(3)|_{S_2} |r(3)\rangle_{S_3} \langle r(3)|_{S_3}. \quad (13)$$

Due to decoherence i.e., entanglement with the environment (Alice) we have lost coherence and correlations between spins. In particular we have

$$\text{Tr}_S[\sigma_xS_1\sigma_xS_2\sigma_xS_3\hat{\rho}^{(\text{red.})}_{S|A}] = 0,$$
$$\text{Tr}_S[\sigma_xS_1\sigma_yS_2\sigma_yS_3\hat{\rho}^{(\text{red.})}_{S|A}] = 0,$$
$$\text{Tr}_S[\sigma_yS_1\sigma_xS_2\sigma_yS_3\hat{\rho}^{(\text{red.})}_{S|A}] = 0,$$
$$\text{Tr}_S[\sigma_yS_1\sigma_yS_2\sigma_xS_3\hat{\rho}^{(\text{red.})}_{S|A}] = 0,$$

which contrast with

$$\text{Tr}_S[\sigma_xS_1\sigma_xS_2\sigma_xS_3\hat{\rho}^1_S] = +1,$$
$$\text{Tr}_S[\sigma_xS_1\sigma_yS_2\sigma_yS_3\hat{\rho}^1_S] = -1,$$
$$\text{Tr}_S[\sigma_yS_1\sigma_xS_2\sigma_yS_3\hat{\rho}^1_S] = -1,$$
$$\text{Tr}_S[\sigma_yS_1\sigma_yS_2\sigma_xS_3\hat{\rho}^1_S] = -1,$$

In RQM Eq. 15 actually describes the correlations available to A before the interaction occurred, i.e., when the full state of SA is still factorized. It represents a catalog of knowledge or potentiality in the sense of Heisenberg. The actualization of measurements in RQM is a debatable issue and we will not consider this problem here (see e.g. [9]).

In the next step the authors of [1] consider a second observer Bob (B) (also composed of 3 qubits $B_1, B_2, B_3$) who measures the spins of the entangled GHZ system SA in the x bases of the joint system. In analogy with Eq. 8 we write after entanglement with Bob qubits:

$$|\text{GHZ}\rangle_{SAB} = \frac{1}{2} \sum_{p,q,r} |p(2)\rangle_{SAB_1} |q(2)\rangle_{SAB_2} |r(2)\rangle_{SAB_3} \quad (16)$$

with $|k^{(2)}\rangle_{SAB_m} := |k^{(2)}\rangle_{SA_m} |k^{(2)}\rangle_{B_m}$ and $|k^{(2)}\rangle_{B_m}$ is the state of the qubit $B_m$, $m = 1, 2, 3$ (for $k = \pm 1$) and where we introduced the entangled ‘x’ states for the SA system: $|k^{(2)}\rangle_{SA_m} = \ldots$
\[ \frac{1}{\sqrt{2}} \left[ | +^{(3)} \rangle_{SA_m} \pm i | -^{(3)} \rangle_{SA_m} \right] \equiv | \text{sign}(p)x \rangle_{SA_m} \. \] Crucially, the numbers \( p, q, r = \pm 1 \) in Eq. 16 must obey the GHZ-constraint:

\[ p^{(2)}_{SAB} q^{(2)}_{SAB} r^{(2)}_{SAB} = +1. \] (17)

Once more, in RQM we need to consider the reduced density matrix \( \hat{\rho}_{SA|B}^{(\text{red.})} = \text{Tr}_B[\hat{\rho}_{SAB}] \):

\[ \hat{\rho}_{SA|B}^{(\text{red.})} = \frac{1}{4} \sum_{p,q,r} | p^{(2)} \rangle_{SA_1} | p^{(2)} \rangle_{S2} | q^{(2)} \rangle_{S2} | q^{(2)} \rangle_{S3} | r^{(2)} \rangle_{S3} \] (18)

with again \( pqr = +1 \). This defines the information available to B in RQM and this density matrix shows partial coherence since we have

\[ \text{Tr}_{SA}[\sigma_{xSA_1} \sigma_{xSA_2} \sigma_{xSA_3} \hat{\rho}_{SA|B}^{(\text{red.})}] = +1, \]

\[ \text{Tr}_{SA}[\sigma_{xSA_1} \sigma_{ySA_2} \sigma_{ySA_3} \hat{\rho}_{SA|B}^{(\text{red.})}] = 0, \]

\[ \text{Tr}_{SA}[\sigma_{ySA_1} \sigma_{xSA_2} \sigma_{ySA_3} \hat{\rho}_{SA|B}^{(\text{red.})}] = 0, \]

\[ \text{Tr}_{SA}[\sigma_{ySA_1} \sigma_{ySA_2} \sigma_{xSA_3} \hat{\rho}_{SA|B}^{(\text{red.})}] = 0. \] (19)

The first line of Eq. 19 is of course reminiscent of Eq. 17 and shows that there is a preferred pointer basis defined by the specific measurement protocol. This is associated with a specific interaction Hamiltonian \( H_{SA,B} \) leading to the state given by Eq. 16.

Now the authors of [1] didn’t understand the structure of RQM and the meaning of relative facts. They claim that we can find relations between facts or information available to Bob and facts or information available to Alice. This we show below is actually a mistake. More precisely, they consider that Bob only measures one of the 3 qubits belonging to SA. Here we consider the case \( m = 1 \). Only the system \( SA_1 \) will interact with \( B_1 \). This requires to let the two other qubits of Bob \( B_2 \) and \( B_3 \) in their respective ground states. As they show we get the new state

\[ | GHZ' \rangle_{SAB} = \frac{1}{2} \sum_{p,q,r} | p^{(2)} \rangle_{SAB} | q^{(3)} \rangle_{SA_2} | in \rangle_{B_2} | r^{(3)} \rangle_{SA_3} | in \rangle_{B_2} \] (20)

with now the constraint:

\[ p^{(2)}_{SAB} q^{(3)}_{SA_2} r^{(3)}_{SA_3} = -1. \] (21)
Of course, we could develop 2 similar procedures acting only on the qubit \(B_2\) or alternatively the qubit \(B_3\). We will obtain different 2 states \(|GHZ''\rangle_{SAB}\) and \(|GHZ'''\rangle_{SAB}\) leading to the relations

\[
p^{(3)}_{SA_1}q^{(2)}_{SAB_2}r^{(3)}_{SA_3} = -1,
\]  

(22)

and

\[
p^{(3)}_{SA_1}q^{(3)}_{S_{A_2}}r^{(2)}_{SAB_3} = -1.
\]  

(23)

With these mathematical properties the deduction of [1] goes as follows:

i) It is visible from Eq. 21 (and similarly for Eqs. 22,23 by cyclic permutation) that the number \(p^{(2)}_{SAB_1}\) characterizes the entangled system \(SAB_1\) whereas \(q^{(3)}_{S_{A_2}}\) and \(r^{(3)}_{S_{A_3}}\) characterize \(SA_2\) and \(SA_3\). Therefore it is tempting to call \(p^{(2)}_{SAB_1}\) a relative fact for Bob and \(q^{(3)}_{S_{A_2}}\) and \(r^{(3)}_{S_{A_3}}\) relative facts for Alice. This idea, which is central for their paper, is clearly summarized by the analysis surrounding their equation 17 in [1]. They call \(B_m\) the number \(k^{(2)}_{SAB_m}\) defined in our Eqs. 17, 21-23 and similarly they call \(A_m\) the number \(k^{(3)}_{SA_m}\). Assuming this we go to the next step of their ‘no-go’ deduction.

ii) The 4 relations Eq. 17, 21-23. are clearly incompatible. If we multiply Eq. 21 by Eqs. 22 and 23 we obtain

\[
p^{(2)}_{SAB_1}q^{(2)}_{SAB_2}r^{(2)}_{SAB_3}(p^{(3)}_{SA_1}q^{(3)}_{S_{A_2}}r^{(3)}_{S_{A_3}})^2 = p^{(2)}_{SAB_1}q^{(2)}_{SAB_2}r^{(2)}_{SAB_3} = -1,
\]  

(24)

which clearly contradicts Eq. 17. Now as they emphasize it clearly: ‘Note that, most importantly for the sequel, the three [unitary] transformations [acting on \(m = 1, 2, 3\)] mutually commute, and thus their order of application is immaterial.’. In other words: Since the 3 operations leading to \(|GHZ''\rangle_{SAB}, |GHZ'''\rangle_{SAB}, |GHZ''''\rangle_{SAB}\) and thus Eqs. 21-23 are acting ‘locally’ only on one of the sub systems \(SA_m\) their meaning should be non contextual and absolute. This following [1] justifies why we should apriori compare these states with \(|GHZ\rangle_{SAB}\) and Eq. 17. This noncontextual reading is what they believe is contained in RQM.

If we accept this reasoning then RQM contradicts quantum mechanics. Relative facts for Alice and Bob are contradictory.

However, this is wrong and we now debunk the contradiction. First ,consider point i): It is clear that \(p^{(2)}_{SAB_1}\) must be a relative fact for Bob. However, there is no reason to consider that
\(q_{SA_3}^{(3)}\) and \(r_{SA_3}^{(3)}\) should be relative facts for Alice. Actually, the quantum state Eq. 20 contains quantum numbers \(p, q, r\) but in RQM the fundamental description for Bob is the reduced density matrix \(\hat{\rho}_{SA|B}^{(red.)} = \text{Tr}_B[\hat{\rho}_{SAB}]\) obtained by using \(\hat{\rho}_{SAB} = |GHZ\rangle_{SABSAB}\langle GHZ|\). We have:

\[
\hat{\rho}_{SA|B}^{(red.)} = \frac{1}{2} [\hat{1}^{(2)}_{SA_1SA_2} \hat{1}^{(2)}_{SA_3} \hat{1}^{(2)}_{SA_3} \hat{1}^{(2)}_{SA_3} - \hat{1}^{(2)}_{SA_1SA_2} \hat{1}^{(2)}_{SA_3} \hat{1}^{(2)}_{SA_3} - \hat{1}^{(2)}_{SA_1SA_2} \hat{1}^{(2)}_{SA_3} \hat{1}^{(2)}_{SA_3} - \hat{1}^{(2)}_{SA_1SA_2} \hat{1}^{(2)}_{SA_3} \hat{1}^{(2)}_{SA_3}]
\]

(25)

where

\[
|\Phi\rangle_{SA_2SA_3} = \frac{1}{\sqrt{2}} [\hat{1}^{(3)}_{SA_2} \hat{1}^{(3)}_{SA_3} + \hat{1}^{(3)}_{SA_2} \hat{1}^{(3)}_{SA_3}]
\]

\[
|\Psi\rangle_{SA_2SA_3} = \frac{1}{\sqrt{2}} [\hat{1}^{(3)}_{SA_2} \hat{1}^{(3)}_{SA_3} + \hat{1}^{(3)}_{SA_2} \hat{1}^{(3)}_{SA_3}]
\]

(26)

are two Bell states. The fact that EPR-like Bell states are present show that there is a coherence preserved in the description of the system \(SA\) by Bob. In particular we deduce

\[
\text{Tr}_SA[\hat{\sigma}_{xSA_1} \hat{\sigma}_{xSA_2} \hat{\sigma}_{xSA_3} \hat{\rho}_{SA|B}^{(red.)}] = +1,
\]

\[
\text{Tr}_SA[\hat{\sigma}_{xSA_1} \hat{\sigma}_{ySA_2} \hat{\sigma}_{ySA_3} \hat{\rho}_{SA|B}^{(red.)}] = -1,
\]

\[
\text{Tr}_SA[\hat{\sigma}_{ySA_1} \hat{\sigma}_{xSA_2} \hat{\sigma}_{ySA_3} \hat{\rho}_{SA|B}^{(red.)}] = 0,
\]

\[
\text{Tr}_SA[\hat{\sigma}_{ySA_1} \hat{\sigma}_{ySA_2} \hat{\sigma}_{xSA_3} \hat{\rho}_{SA|B}^{(red.)}] = 0.
\]

(27)

The second line is of course reminiscent of Eq. 21 but the meaning is here different from what is claimed in [1]. Indeed, this a relation for Bob measurements not for Alice. The idea to have a mixture of information for Bob and Alice in the same Eq. 21 is thus unjustified and is not a part of RQM but instead of the reading of RQM made by the authors of [1]. The first line is also very interesting: It shows coherence associated with the \(|\Phi\rangle_{SA_2SA_3}\) and \(|\Psi\rangle_{SA_2SA_3}\) Bell states. Moreover this result can be compared with Eq. 19 for \(\hat{\rho}_{SA|B}^{(red.)}\). The difference clearly stresses that the measurement procedures are more invasive that claimed in [1]. In RQM like in the Copenhagen interpretation the experimental context and the choice of the interaction Hamiltonian is key. This allows us us to answer to point ii) concerning non-contextuality. Indeed, the non contextuality supposed by the authors of [1] is not a part of the RQM. For RQM (like in the orthodox interpretation) the different contexts are not compatible and we have no right to compare Eqs. 17 and Eqs. 21-23 as claimed. This would
otherwise contradict the central axioms of RQM and therefore the demonstration discussed in [1] is a non sequitur fallacious reasoning.

I conclude by adding that the RQM is an interesting interpretation. The formalism is perfectly agreeing with standard quantum mechanics and recovers the orthodox interpretation at the limit were observers are essentially macroscopic without the problems concerning the definition of agents in Qbism. Several issues concerning interactions can still be debated but we should not focus on wrong problems. Often objections are done without carefully considering the philosophy of ROM but by adding some prejudices or preconceptions concerning the interpretation (i.e., [1–3]). I hope this note will contribute to clarify a bit this issue.

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