GENERATION AND SYZYGIES OF THE FIRST SECANT VARIETY

PETER VERMEIRE

Abstract. Under certain effective positivity conditions, we show that the secant variety to a smooth variety satisfies $N_{3,p}$. For smooth curves, we provide the best possible effective bound on the degree $d$ of the embedding, $d \geq 2g + 3 + p$.

1. Introduction

This should not be considered a final version. Instead, I wanted to correct an error in version 2 of the posted preprint. I was able to re-establish the degree bounds found there for curves, but I have not yet completed work on the higher-dimensional case.

We work throughout over an algebraically closed field of characteristic zero. Let $X \subset \mathbb{P}^n$ be a smooth variety embedded by a line bundle $L$ and let $\Sigma_i$ denote the (complete) variety of $(i+1)$-secant $i$-planes. Though secant varieties are a very classical subject, the majority of the work done involves determining the dimensions of secant varieties to well-known varieties. Perhaps the two most well-known results in this direction are the solution by Alexander and Hirschowitz (completed in [1]) of the Waring problem for homogeneous polynomials and the classification of the Severi varieties by Zak [37].

More recently there has been great interest, e.g. related to algebraic statistics and algebraic complexity, in determining the equations defining secant varieties (e.g. [2], [4], [9], [10], [11], [12], [13], [16], [23], [28], [31], [32], [33], [34], [35], [46], [48]). In this work, we use the detailed geometric information concerning secant varieties developed by Bertram [5], Thaddeus [49], and the author [50] to lay some fundamental groundwork for studying not just the equations defining secant varieties, but the syzygies among those equations as well.

It was conjectured in [20] and it was shown in [44] that if $C$ is a smooth curve embedded by a line bundle of degree at least $4g + 2k + 3$ then $\Sigma_k$ is set theoretically defined by the $(k + 2) \times (k + 2)$ minors of a matrix of linear forms. It was further shown in [51] that if $X \subset \mathbb{P}^n$ satisfies condition $N_2$ then $\Sigma_1(v_d(X))$ is set theoretically defined by cubics for $d \geq 2$. 

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In [53] it was shown that if $C$ is a smooth curve embedded by a line bundle of degree at least $2g + 3$ then $T_{\Sigma_1}$ is 5-regular, and under the same hypothesis it was shown in [47] that $\Sigma_1$ is arithmetically Cohen-Macaulay. Together with the analogous well-known facts for the curve $C$ itself [22], [24], [39], this led to the following conjecture, extending that found in [51]:

**Conjecture 1.1.** [47] Suppose that $C \subset \mathbb{P}^n$ is a smooth linearly normal curve of degree $d \geq 2g + 2k + 1 + p$, where $p, k \geq 0$. Then

1. $\Sigma_k$ is ACM and $I_{\Sigma_k}$ has regularity $2k + 3$ unless $g = 0$, in which case the regularity is $k + 2$.
2. $\beta_{n-2k-1,n+1} = \binom{g+k}{k+1}$.
3. $\Sigma_k$ satisfies $N_{k+2,p}$.

□

**Remark 1.2.** Recall [19] that a variety $Z \subset \mathbb{P}^n$ satisfies $N_{r,p}$ if the ideal of $Z$ is generated in degree $r$ and the syzygies among the equations are linear for $p-1$ steps. Note that condition $N_{p,2}$ implies $N_{3,p}$.

By the work of Green and Lazarsfeld [24],[36], the conjecture holds for $k = 0$. Further, by [21] and by [55] it holds for $g \leq 1$, and by [47] parts (1) and (2) hold for $k = 1$. In this work, we show that part (3) holds for $k = 1$ (Theorem 4.5). More generally, we show that for an arbitrary smooth variety, $\Sigma_1$ satisfies $N_{3,p}$ for all sufficiently positive embeddings (Corollary 4.8), and we give effective results on arbitrary smooth varieties embedded by adjoint linear systems (Corollary 4.9).

Our approach combines the geometric knowledge of secant varieties mentioned above with the well-known Koszul approach of Green and Lazarsfeld. To fix notation, if $L$ is a vector bundle on a projective variety $X$, then we let $E_L = d^*(L \boxtimes O)$, where $d : \text{Bl}_{\Delta}(X \times X) \to \text{Hilb}^2 X$ is the natural double cover, and if $L$ is a globally generated line bundle on a projective variety $X$ inducing a morphism $f : X \to \mathbb{P}^n$, then we have the vector bundle $M_L = f^* \Omega_{\mathbb{P}^n}(1)$ on $X$.

2. Preliminaries

Our starting point is the familiar:

**Proposition 2.1.** Let $X \subset \mathbb{P}^n$ be a smooth variety embedded by a line bundle $L$. Then $\Sigma_1$ satisfies $N_{3,p}$ if $H^1(\Sigma_1, \wedge^a M_L(b)) = 0$, $2 \leq a \leq p + 1$, $b \geq 2$.

**Proof:** Because $L$ also induces an embedding $\Sigma_1 \subset \mathbb{P}^n$, we abuse notation and denote the associated vector bundle on $\Sigma_1$ by $M_L$. Letting $F = \oplus \Gamma(\Sigma_1, \mathcal{O}_{\Sigma_1}(n))$ and applying [18, 5.8] to $\mathcal{O}_{\Sigma_1}$ gives the exact sequence:

$$0 \to \text{Tor}_{a-1}(F, k)_{a+b} \to H^1(\Sigma_1, \wedge^a M_L(b)) \to H^1(\Sigma_1, \wedge^a \mathcal{O}_{\mathbb{P}^n}^{r+1} \otimes \mathcal{O}_{\Sigma_1}(b))$$

The vanishing in the hypothesis implies that $\text{Tor}_1(F, k)_d = 0$ for $d \geq k + 1$, and hence that the first syzygies of $\mathcal{O}_{\Sigma_1}$, which are the generators of the ideal of $\Sigma_1$, are
in degree \( \leq k \). The rest of the vanishings yield the analogous statements for higher syzygies.

The technical portion of the paper is devoted to reinterpreting the vanishings in Proposition 2.1 in terms of vanishings on the Hilbert scheme \( \text{Hilb}^2 X \), and then finally on \( X \) itself.

**Notation and Terminology 2.2.** Recall that an embedding \( X \subset \mathbb{P}^n \) **separates \( k \) points** if every subscheme of \( X \) of length \( k \) spans a \( \mathbb{P}^{k-1} \subset \mathbb{P}^n \). A very ample line bundle \( L \) is **\( k \)-very ample** if the induced embedding separates \( k + 1 \) points. It is immediate that \( k \)-very ampleness implies \( (k - 1) \)-very ampleness.

We will assume throughout that \( X \subset \mathbb{P}^n \) is a \( 3 \)-very ample embedding of a smooth variety by \( L = \mathcal{O}_X(1) \) that satisfies \( N_{2,2} \). For curves, an embedding given by a line bundle of degree at least \( 2g + 3 \) suffices [24]. As we will be interested only in the first secant variety for the remainder of the paper, we write \( \Sigma \) for \( \Sigma_1 \).

Under these hypotheses, the reader should keep in mind throughout the following morphisms [50]

\[
\begin{array}{c}
\text{Hilb}^2 X \\
\downarrow \varphi \\
\Sigma \\
\downarrow \pi \\
\Sigma
\end{array}
\]

\[
\begin{array}{c}
Z \cong \text{Bl}_\Delta (X \times X)^c \\
\downarrow \pi_2 \\
X \\
\downarrow \pi_1 = \pi|_Z \\
X^c \\
\downarrow \pi \\
\Sigma
\end{array}
\]

where

- \( \pi \) is the blow up of \( \Sigma \) along \( X \)
- \( i \) is the inclusion of the exceptional divisor of the blow-up
- \( d \) is the double cover, \( \pi_i \) are the projections
- \( \varphi \) is the morphism induced by the linear system \( |2H - E| \) which gives \( \tilde{\Sigma} \) the structure of a \( \mathbb{P}^1 \)-bundle over \( \text{Hilb}^2 X \); note in particular that \( \tilde{\Sigma} \) is smooth.

We make frequent use of the rank 2 vector bundle \( \mathcal{E}_L = \varphi_* \mathcal{O}(H) = d_*(L \boxtimes \mathcal{O}) \), and note that \( R^i \pi_* \mathcal{O}_{\tilde{\Sigma}} = H^i(X, \mathcal{O}_X) \otimes \mathcal{O}_X \) for \( i \geq 1 \) (this is shown in [53, Proposition 9] for curves, but the same proof works in the general case).

**Proposition 2.3.** If \( X \) is a smooth variety embedded by a \( 3 \)-very ample line bundle \( L \) satisfying \( N_{2,2} \), then \( \Sigma \) satisfies \( N_{3,p} \) if

\[
H^1(\tilde{\Sigma}, \pi^* \wedge^a M_L(b)) \rightarrow H^0(\Sigma, \wedge^a M_L(b) \otimes R^1 \pi_* \mathcal{O}_{\tilde{\Sigma}})
\]

is injective for \( 2 \leq a \leq p + 1, \ b \geq 2 \).
PROOF: This follows immediately from the start of the 5-term sequence associated to the Leray-Serre spectral sequence:

\[ 0 \to H^1(\Sigma, \wedge^a M_L(b)) \to H^1(\widetilde{\Sigma}, \pi^* \wedge^a M_L(b)) \to H^0(\Sigma, \wedge^a M_L(b) \otimes R^1 \pi_* O_{\Sigma}) \]

and Proposition 2.1. □

To simplify notation, we introduce a technical condition:

**Notation 2.4.** For \( p \geq 1 \), we say a line bundle \( L = \mathcal{O}_X(1) \) on \( X \subset \mathbb{P}^n \) satisfies \( N^\Sigma_p \) if

1. \( L \) is 3-very ample and satisfies \( N^\Sigma_{2,p} \).
2. \( H^i(\widetilde{\Sigma}, \mathcal{O}_\Sigma(bH - E)) = 0 \) for \( i, b \geq 1 \).

As the vanishing condition in the definition of \( N^\Sigma_p \) is non-trivial to understand, we explore several cases where it is satisfied in the next section.

**3. Condition \( N^\Sigma_p \)**

For curves, verification of \( N^\Sigma_p \) is straightforward.

**Proposition 3.1.** Let \( X \subset \mathbb{P}^n \) be a smooth curve satisfying \( N_p \), \( p \geq 2 \), with \( L = \mathcal{O}_X(1) \) non-special. Then \( L \) satisfies \( N^\Sigma_p \).

**Proof:** We need to show \( H^i(\widetilde{\Sigma}, \mathcal{O}_\Sigma(bH - E)) = 0 \) for \( i, b \geq 1 \).

Because \( X \) is projectively normal we have \( H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(bH-E)) = 0 \) for \( i > 0 \), \( b \geq 1 \). Thus \( H^i(\widetilde{\Sigma}, \mathcal{O}_\Sigma(bH - E)) = H^{i+1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(bH-E) \otimes I_{\Sigma}) \). By [47, 2.4(6)], we know that \( H^{i+1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(bH-E) \otimes I_{\Sigma}) = H^{i+1}(\mathbb{P}^n, I_{\Sigma}(b)) \).

Now, for \( i \geq 1 \), the arguments in [53] and in [47] go through under the stated hypotheses to give \( H^{i+1}(\mathbb{P}^n, I_{\Sigma}(b)) = 0 \) for \( b \geq 1 \). The extra hypothesis used in those papers (namely, that \( \deg(L) \geq 2g + 3 \)) is needed only to show \( H^1(\mathbb{P}^n, I_{\Sigma}(b)) = 0 \) for \( b \geq 1 \).

Verifying condition \( N^\Sigma_p \) in the general case takes somewhat more work, but the end results are reasonable. We first need a computation which will be used in both Proposition 3.3 and in Theorem 4.6.

**Lemma 3.2.** Let \( X \) be a smooth variety embedded by a 3-very ample line bundle \( L \) satisfying \( N^\Sigma_{2,2} \). Then \( d^* \wedge^2 \mathcal{E}_L = L \boxtimes L(-E_\Delta) \).

**Proof:** Consider the sequence on \( \widetilde{\Sigma} \):

\[ 0 \to \mathcal{O}_\Sigma(-E) \to \mathcal{O}_\Sigma(-E) \to 0 \]

As \( R^0 \varphi_* \mathcal{O}_\Sigma(-E) = 0 \), pushing down to \( \text{Hilb}^2 X \) we have ([47, 3.10])

\[ 0 \to \mathcal{O}_{\text{Hilb}^2 X} \to \mathcal{O}_{\text{Hilb}^2 X} \oplus M \to R^1 \varphi_* \mathcal{O}_\Sigma(-E) \to 0 \]

where \( d^* M = \mathcal{O}_Z(-E_\Delta) \).
Thus $R^1\varphi_*O_{\Sigma}(E) = M$. However, we know by [15, 5.1.2] that
\[(R^1\varphi_*O_{\Sigma}(E))^\ast = R^0\varphi_*\left(\varphi^*\omega_{\text{Hilb}\mathcal{X}} \otimes O_{\Sigma}(E)\right)\]
where $\omega_{\text{Hilb}\mathcal{X}} = \varphi^* \wedge^2 \mathcal{E}_L(-2H)$ [25, Ex.III.8.4b]. Thus we have
\[M^* = \wedge^2 \mathcal{E}_L \otimes O_{\text{Hilb}\mathcal{X}}(-1)\]
and so $\varphi^* \wedge^2 \mathcal{E}_L = O_{\Sigma}(2H - E) \otimes \varphi^* M^*$. Restricting (pulling back) this equality to $Z$ and noting ([52, 3.6]) that $O_Z(2H - E) = L \otimes L(-2E\Delta)$, we have $d^* \wedge^2 \mathcal{E}_L = L \otimes L(-E\Delta)$. We now interpret the vanishing condition in the definition of $\mathcal{N}_p^\Sigma$ in terms of $X$.

**Proposition 3.3.** Let $X \subset \mathbb{P}^n$ be a smooth variety embedded by a 3-very ample line bundle $L$ satisfying $N_{2,2}$ such that $H^i(X \times X, L^{r+s} \boxtimes L^r \otimes T^2_\Delta) = 0$ for $i, r \geq 1$, $s \geq 0$, $0 \leq q \leq 2r$. Then $H^i(\tilde{\Sigma}, O_{\Sigma}(bH - E)) = 0$ for $i, b \geq 1$.

**Proof:** Suppose $b = 2r$ is even. We know by the proof of [52, 3.6] that $O_Z(bH - E) = L^{b-1} \boxtimes L \otimes O(-2\Delta)$; thus
\[H^i(Z, O_Z(bH - rE)) = H^i(X \times X, L^r \otimes L^r \otimes T^2_\Delta) = 0\]
Because $O_{\Sigma}(bH - rE) = \varphi^* \mathcal{O}_{\text{Hilb}\mathcal{X}}(r)$, we know $d_* O_Z(bH - rE) = \mathcal{O}_{\text{Hilb}\mathcal{X}}(r) \otimes (\mathcal{O} \oplus M)$ for some line bundle $M$, and hence we know that $H^i(\text{Hilb}\mathcal{X}, \mathcal{O}_{\text{Hilb}\mathcal{X}}(r)) = 0$, but this says that $H^i(\tilde{\Sigma}, O_{\Sigma}(bH - rE)) = 0$. From the sequences
\[0 \to O_{\Sigma}(bH - (k + 1)E) \to O_{\Sigma}(bH - kE) \to O_Z(bH - kE) \to 0\]
for $k + 1 \leq r$ we see that $H^i(\tilde{\Sigma}, O_{\Sigma}(bH - E)) = 0$, as the cohomology of the rightmost terms vanishes by hypothesis since $H^i(Z, O_Z(bH - kE)) = H^i(X \times X, L^{b-k} \boxtimes L^k \otimes T^2_\Delta) = 0$.

Now, suppose that $b = 2r + 1$ is odd. As in the previous paragraph, we have $O_{\Sigma}((b-1)H - rE) = \varphi^* \mathcal{O}_{\text{Hilb}\mathcal{X}}(r)$, thus we see that $\varphi_* O_{\Sigma}(bH - rE) = \mathcal{O}_{\text{Hilb}\mathcal{X}}(r) \otimes \varphi_* O_{\Sigma}(H) = \mathcal{O}_{\text{Hilb}\mathcal{X}}(r) \otimes E$. It is therefore enough to show that $H^i(\text{Hilb}\mathcal{X}, \mathcal{O}_{\text{Hilb}\mathcal{X}}(r) \otimes E) = 0$, and then repeating the same argument as above gives $H^i(\tilde{\Sigma}, O_{\Sigma}(bH - E)) = 0$.

We have the sequence on $Z$
\[0 \to K \to d^* \mathcal{E}_L \to L \otimes \mathcal{O} \to 0\]
where $K = d^* \wedge^2 \mathcal{E}_L \otimes (L^* \boxtimes \mathcal{O}) = \mathcal{O} \boxtimes L(-E\Delta)$ by Lemma 3.2. As in the proof of Lemma 3.2, we have $d_* d^* \mathcal{E}_L = \mathcal{E}_L \oplus (\mathcal{E}_L \otimes M)$, thus
\[d_* \left( O_Z(2rH - rE) \otimes d^* (\mathcal{E}_L \otimes M^*) \right) = \mathcal{E}_L \otimes M^*(r) \oplus \mathcal{E}_L(r)\]
Thus it suffices to show $H^i(Z, O_Z(2rH - rE) \otimes d^* (\mathcal{E}_L \otimes M^*)) = 0$. However, we have
\[K \otimes O_Z(2rH - rE) \otimes d^* M^* = L^r \otimes L^{r+1}(-2rE\Delta)\]
and
\[ L \boxtimes O \otimes O_Z(2rH - rE) \otimes d^*M^s = L^{r+1} \boxtimes L^0((-2r + 1)E_\Delta) \]
and so the cohomology of each vanishes by hypothesis.

Fortunately, the vanishing in Proposition 3.3 is not too difficult to understand.

**Proposition 3.4.** Let \( X \) be a smooth variety of dimension \( d \), \( M \) a very ample line bundle. Choose \( k \) so that \( k \geq d + 3 \) and so that \( M^{k-d-1} \otimes \omega_X^* \) is big and nef. Letting \( L = M^k \), we have
\[ H^i(X \times X, L^{r+s} \boxtimes L^r \otimes T^q_\Delta) = 0 \]
for \( i, r \geq 1, s \geq 0, 0 \leq q \leq 2r \).

**Proof:** Note as above that \( H^i(Z, L^{r+s} \boxtimes L^r \otimes O(-2rE_\Delta)) \), where \( E_\Delta \rightarrow \Delta \) is the exceptional divisor of the blow-up. Note further that \( K_Z = K_X \otimes K_X \otimes O((\dim X - 1)E_\Delta) \).

Assume first that \( r \geq 2 \). Then
\[ L^{r+s} \boxtimes L^r \otimes O(-2rE_\Delta) = K_Z \otimes (L^{r+s} - K_X) \boxtimes (L^r - K_X) \otimes O((-d + 1 - q)E_\Delta) \]
but this is \( K_Z + B \) where
\[ B = [(L - K_X) \boxtimes (L - K_X)] \otimes [L^{r+s-1} \boxtimes L^{r-1} \otimes O((-d + 1 - q)E_\Delta)] \]

Because \( M^k - K_X \) is ample, \( (L - K_X) \boxtimes (L - K_X) \) is ample. We are thus left to show that
\[ M^{k(r+s-1)} \boxtimes M^{k(r-1)} \otimes O((-d + 1 - q)E_\Delta) \]
is globally generated. However, as \( k \geq d + 3 \), we have \( k(r-1) \geq d - 1 + 2r \) and so \( M^{k(r+s-1)} \boxtimes M^{k(r-1)} \otimes O((-d + 1 - 2r)E_\Delta) \) is globally generated by [6, 3.1]. Thus \( B \) is big and nef and so vanishing follows from Kawamata-Viehweg vanishing [29],[54].

Now let \( r = 1 \). Then
\[ L^{1+s} \boxtimes L \otimes O(-2E_\Delta) = K_Z \otimes (M^{k+s} - K_X) \boxtimes (M^k - K_X) \otimes O((-d + 1 - q)E_\Delta) \]
but this is \( K_Z + B \) where
\[ B = [(M^{k-d-1} - K_X) \boxtimes (M^{k-d-1} - K_X)] \otimes [M^{k+s+d+1} \boxtimes M^{d+1}] \otimes O((-d+1-q)E_\Delta) \]
As above, \( B \) is big and nef. \( \square \)

**Remark 3.5.** There are numerous ways to rearrange the terms in Proposition 3.4 to produce the desired vanishing.

For example, a similar argument shows that if \( M \) is very ample, \( \omega_X \otimes M \) is big and nef, and \( B \) is nef, then letting \( L = \omega_X \otimes M^k \otimes B \) gives the vanishing for \( k \geq d + 2 \)
(Cf. [17, Theorem 1]). If, further, \( B \) is also big, then letting \( L = \omega_X \otimes M^k \otimes B \) gives the vanishing for \( k \geq d + 1 \).
Remark 3.6. In Proposition 3.4, if $\omega_X$ is big and nef (e.g. $X$ is Fano) then a slight revision of the argument shows it is enough to take $L = M^k$ for $k \geq d + 1$.

Remark 3.7. Note that the vanishing condition in Proposition 3.3 is intimately related to the surjectivity of the higher-order Gauss-Wahl maps as defined in [57].

4. Main Results

Proposition 4.1. Let $X \subset \mathbb{P}^n$ be a smooth variety embedded by a line bundle $L$ satisfying $N^i_{L^k}$ with $H^i(X, L^k) = 0$ for $i, k \geq 1$. Then $\Sigma$ satisfies $N_{3,p}$ if $H^i(\Sigma, \pi^* \wedge^{a-1+i} M_L \otimes \mathcal{O}(2H - E)) = 0$ for $2 \leq a \leq p + 1, i \geq 1$.

Proof: We use Proposition 2.3. From the sequence on $\Sigma$

$$0 \to \pi^* \wedge^a M_L(bH - E) \to \pi^* \wedge^a M_L(bH) \to \pi^* \wedge^a M_L(bH) \otimes \mathcal{O}_Z \to 0$$

we know

$$H^1(Z, \pi^* \wedge^a M_L(bH) \otimes \mathcal{O}_Z) = H^1\left(Z, \left(\wedge^a M_L \otimes L^b\right) \boxtimes \mathcal{O}_X\right)$$

$$= H^1\left(X \times X, \left(\wedge^a M_L \otimes L^b\right) \boxtimes \mathcal{O}_X\right)$$

$$= H^1(X, \mathcal{O}_X) \otimes H^0(X, \wedge^a M_L \otimes L^b).$$

The first equality follows as the restriction of $\pi^* \wedge^a M_L(bH)$ to $Z$ is $\wedge^a M_L(bH) \boxtimes \mathcal{O}_X$, the second is standard, and for the third we use the Künneth formula together with the fact that $h^1(X, \wedge^a M_L \otimes L^b) = 0$ as $X$ satisfies $N_{2,p}$.

Thus

$$h^1(\Sigma, \wedge^a M_L(b)) = \text{Rank} \left(\left.H^1(\Sigma, \pi^* \wedge^a M_L(bH - E)) \to H^1(\Sigma, \pi^* \wedge^a M_L(bH))\right)\right)$$

and so by Proposition 2.3 it is enough to show that $H^1(\Sigma, \pi^* \wedge^a M_L \otimes \mathcal{O}(bH - E)) = 0$ for $2 \leq a \leq p + 1, b \geq 2$.

From the sequence

$$0 \to \pi^* \wedge^{a+1} M_L \otimes \mathcal{O}(bH - E) \to \wedge^{a+1} \wedge^b \mathcal{O}(bH - E) \to \wedge^a M_L \otimes \mathcal{O}((b+1)H - E) \to 0$$

and the fact that $H^i(\Sigma, \mathcal{O}(bH - E)) = 0$, we see that $H^1(\Sigma, \pi^* \wedge^a M_L \otimes \mathcal{O}(bH - E)) = H^{b-2}(\Sigma, \pi^* \wedge^{a+b-2} M_L \otimes \mathcal{O}(2H - E))$ for $b \geq 2$.

Lemma 4.2. Let $X$ be a smooth variety embedded by a 3-very ample line bundle $L$ satisfying $N_{2,2}$ and consider the morphism $\varphi : \Sigma \to \text{Hilb}^2X \subset \mathbb{P}^s$ induced by the linear system $|2H - E|$. Then $\varphi_* \wedge^a M_L = \wedge^a M_{E_L}$, and hence $H^i(\Sigma, \pi^* \wedge^a M_L \otimes \mathcal{O}(2H - E)) = H^i(\text{Hilb}^2X, \wedge^a M_{E_L} \otimes \mathcal{O}_{\text{Hilb}^2X}(1))$.

Proof: Consider the diagram on $\Sigma$: 

\begin{center}
\begin{tikzpicture}

% Diagram here

\end{tikzpicture}
\end{center}
The vertical map in the middle is surjective as we have $\Gamma(\text{Hilb}^2 X, E_L) = \Gamma(\tilde{\Sigma}, O(H)) = \Gamma(X \times X, L \boxtimes O) = \Gamma(X, L)$. Therefore, surjectivity of the lower right horizontal map and commutativity of the diagram show that the right-hand vertical map is surjective.

Note that $R^i \varphi_* \varphi^* E_L = E_L \otimes R^i \varphi_* O_\Sigma$ by the projection formula and that the higher direct image sheaves $R^i \varphi_* O_\Sigma$ vanish as $\tilde{\Sigma}$ is a $\mathbb{P}^1$-bundle over $\text{Hilb}^2 X$. For the higher direct images, we have $R^i \varphi_* \pi^* L = 0$ as the restriction of $L$ to a fiber of $\varphi$ is $O(1)$ and hence the cohomology along the fibers vanishes. From the rightmost column, we see $R^i \varphi_* K = 0$. From the leftmost column, we have the sequence

$$0 \rightarrow \varphi^* \wedge^a M_E \rightarrow \pi^* \wedge^a M_L \rightarrow \varphi^* \wedge^{a-1} M_E \otimes K \rightarrow 0$$

but as $R^i \varphi_* (K \otimes \varphi^* \wedge^{a-1} M_E) = R^i \varphi_* K \otimes \wedge^{a-1} M_E = 0$, we have $\varphi_* \wedge^a M_L = \wedge^a M_E$.

Combining Proposition 4.1 with Lemma 4.2 yields:

**Corollary 4.3.** Let $X$ be a smooth variety embedded by a line bundle $L$ satisfying $N_{2,2}^\Sigma$ with $H^i(X, L^k) = 0$ for $i, k \geq 1$. Then $\Sigma$ satisfies $N_{3,p}$ if

$$H^i(\text{Hilb}^2 X, \wedge^{a-1+i} M_E \otimes O(1)) = 0$$

for $2 \leq a \leq p + 1$, $i \geq 1$. □

4.1. **Curves.** We need a technical lemma, completely analogous to [36, 1.4.1].

**Lemma 4.4.** Let $X \subset \mathbb{P}^n$ be a smooth curve embedded by a non-special line bundle $L$ satisfying $N_{2,2}$, let $x_1, \ldots, x_{n-2}$ be a general collection of distinct points, and let
$D = x_1 + \cdots + x_{n-2}$. Then there is an exact sequence of vector bundles on $X \times X$

$$0 \to L^{-1}(D) \boxtimes L^{-1}(D)(\Delta) \to d^* M_{\mathcal{E}_L} \to \bigoplus_i (\mathcal{O}(-x_i) \boxtimes \mathcal{O}(-x_i)) \to 0$$

**Proof:** Choose a general point $x_1 \in X$ and consider the following diagram on $X \times X$:

\[
\begin{array}{ccc}
0 & \rightarrow & d^* M_{\mathcal{E}_L(-x_1)} \\
\downarrow & & \downarrow \\
0 & \rightarrow & M_{\mathcal{E}_L(-x_1)} \boxtimes \mathcal{O} \\
\downarrow & & \downarrow \\
0 & \rightarrow & (\mathcal{O} \boxtimes L(-x_1))(-\Delta) \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\end{array}
\]

where the center column comes from [36, 1.4.1]. Following just as in that proof, we obtain

$$0 \rightarrow d^* M_{\mathcal{E}_L(-D)} \rightarrow d^* M_{\mathcal{E}_L} \rightarrow \bigoplus_i (\mathcal{O}(-x_i) \boxtimes \mathcal{O}(-x_i)) \rightarrow 0$$

from the left column. Note however that $d^* M_{\mathcal{E}_L(-D)}$ is a line bundle, and hence by Lemma 3.2 we see $d^* M_{\mathcal{E}_L(-D)} = \wedge^2 \mathcal{E}_L(-D) = L^{-1}(D) \boxtimes L^{-1}(D)(\Delta)$. \(\square\)

**Theorem 4.5.** Let $X \subset \mathbb{P}^n$ be a smooth curve embedded by a line bundle $L$ with $\deg(L) \geq 2g + p + 3$. Then $\Sigma$ satisfies $N_{3,p}$.

**Proof:** We verify the condition in Corollary 4.3. Pulling the sequence on $\text{Hilb}^2 X$

$$0 \to M_{\mathcal{E}_L} \to \Gamma(\text{Hilb}^2 X, \mathcal{E}_L) \to \mathcal{E}_L \to 0$$

back to $Z = X \times X$ yields the diagram
As in Lemma 3.2, we have $d_*\mathcal{O}_Z = \mathcal{O}_{\text{Hilb}^2 X} \oplus M$ where $d_* M = \mathcal{O}(-\Delta)$, $d_* K = \mathcal{E}_L \otimes M$, and $K = d^* \wedge^2 \mathcal{E}_L \otimes (L^* \boxtimes \mathcal{O}) = \mathcal{O} \boxtimes L(-\Delta)$. From the left vertical sequence we have

$$0 \to \wedge^a d^* \mathcal{E}_L \to \wedge^a M_{\text{L} \boxtimes \mathcal{O}} \to \wedge^{a-1} d^* \mathcal{E}_L \otimes K \to 0$$

and pushing down to $\text{Hilb}^2 X$ yields

$$0 \to \wedge^a M_{\mathcal{E}_L} \oplus (\wedge^a M_{\mathcal{E}_L} \otimes M) \to d_* \wedge^a M_{\text{L} \boxtimes \mathcal{O}} \to \wedge^{a-1} M_{\mathcal{E}_L} \otimes \mathcal{E}_L \otimes M \to 0$$

Twisting this sequence by $\mathcal{O}_{\text{Hilb}^2 X}(1) \otimes M^*$ gives

$$0 \to \wedge^a M_{\mathcal{E}_L}(1) \otimes M^* \oplus \wedge^a M_{\mathcal{E}_L}(1) \to \mathcal{O}_{\text{Hilb}^2 X}(1) \otimes M^* \otimes d_* \wedge^a M_{\text{L} \boxtimes \mathcal{O}} \to \wedge^{a-1} M_{\mathcal{E}_L}(1) \otimes \mathcal{E}_L \to 0$$

Since $d^* \mathcal{O}_{\text{Hilb}^2 X}(1) \otimes M^* = L \boxtimes L \otimes \mathcal{O}(-\Delta)$, it suffices (as in Proposition 3.3) to show that

$$H^i(Z, \wedge^{a-1+i} d^* M_{\mathcal{E}_L} \otimes L \boxtimes L \otimes \mathcal{O}(-\Delta)) = 0$$

for $2 \leq a \leq p + 1$, $i = 1, 2$.

Now, by Lemma 4.4 we have exact sequences

$$0 \to \wedge^{r-1} Q \otimes \mathcal{O}(D) \boxtimes \mathcal{O}(D) \to \wedge^r d^* M_{\mathcal{E}_L} \otimes L \boxtimes L \otimes \mathcal{O}(-\Delta) \to \wedge^r Q \otimes L \boxtimes L \otimes \mathcal{O}(-\Delta) \to 0$$

where $Q = \bigoplus_i \mathcal{O}(-x_i) \boxtimes \mathcal{O}(-x_i)$.

On the right, we have a direct sum of vector bundles of the form $F \boxtimes F(-\Delta)$ where $F$ is a line bundle of degree $\deg(L) - r$. Thus $H^1$ and $H^2$ of the right side will vanish when $\deg(L) - r \geq 2g + 1$.

On the left, we have a direct sum of vector bundles of the form $F \boxtimes F$ where $F$ is a line bundle of degree $n - 2 - (r - 1) = \deg(L) - g - r - 1$. Because $x_1, \ldots, x_{n-2}$ are general, $H^1$ and $H^2$ of the left side will vanish when $\deg(L) - g - r - 1 \geq g$. 


Combining these, we see that \( H^i(Z, \wedge^{a-1+i} d^* M_{\mathcal{E}_L} \otimes L \boxtimes L \otimes O(-E_\Delta)) = 0 \) for \( 2 \leq a \leq p+1, i = 1, 2 \) as long as \( \deg(L) \geq 2g + p + 3 \).

\[ \square \]

4.2. Higher Dimensions.

**Theorem 4.6.** Let \( X \subset \mathbb{P}^n \) be a smooth variety embedded by a line bundle \( L \) satisfying \( N_p^\Sigma \), \( p \geq 1 \), with \( H^i(X, L^k) = 0 \) for \( i, k \geq 1 \). If \( H^i(X, N_{X/\mathbb{P}^n}^* \otimes \wedge^{a-1+i} M_L \otimes L^2) = 0 \) for \( 2 \leq a \leq p+1 \) and for \( i \geq 1 \), then \( \Sigma \) satisfies \( N_{3,p} \).

**Proof:**

As in Theorem 4.5, we have

\[ 0 \to \wedge^a d^* M_{\mathcal{E}_L} \to \wedge^a M_{\mathcal{E}_L} \otimes O \to \wedge^{a-1} d^* M_{\mathcal{E}_L} \otimes K \to 0 \]

and pushing down to \( \text{Hilb}^2 X \) yields

\[ 0 \to \wedge^a M_{\mathcal{E}_L} \oplus (\wedge^a M_{\mathcal{E}_L} \otimes M) \to \wedge^a M_{\mathcal{E}_L} \oplus (F_\alpha^a \otimes M) \to \wedge^{a-1} M_{\mathcal{E}_L} \otimes \mathcal{E}_L \otimes M \to 0 \]

where \( F_\alpha^a \) comes from the standard filtration

\[ 0 \subset \wedge^a M_{\mathcal{E}_L} \subset F_\alpha^2 \subset \wedge^a \Gamma(\text{Hilb}^2 X, \mathcal{E}_L) \]

of \( \wedge^a \Gamma \) associated to \( 0 \to M_{\mathcal{E}_L} \to \Gamma(\text{Hilb}^2 X, \mathcal{E}_L) \to \mathcal{E}_L \to 0 \) where

\[ F_\alpha^2 / \wedge^a M_{\mathcal{E}_L} = \wedge^{a-1} M_{\mathcal{E}_L} \otimes \mathcal{E}_L; \wedge^a \Gamma / F_\alpha^2 = \wedge^{a-2} M_{\mathcal{E}_L} \otimes \wedge^2 \mathcal{E}_L. \]

Twisting by \( O_{\text{Hilb}^2 X}(1) \), we see it is enough to show that

\[ H^i(Z, \wedge^{a-1+i} d^* M_L \otimes L \boxtimes L \otimes O(-2E_\Delta)) = 0 \]

for \( 2 \leq a \leq p+1, i \geq 1 \). However, it is well-known that for \( L \) very ample we have

\[ H^i(Z, \wedge^{a-1+i} d^* M_L \otimes L \boxtimes L \otimes O(-2E_\Delta)) = H^i(X, N_{X/\mathbb{P}^n}^* \otimes \wedge^{a-1+i} M_L \otimes L^2). \]

\[ \square \]

To verify the new vanishing condition we have:

**Proposition 4.7.** Let \( X \) be a smooth variety of dimension \( d \), \( M \) a very ample line bundle. Choose \( k \in \mathbb{N} \) so that \( M^{k-\text{ad}a-1} \otimes \omega_X^* \) is big and nef. Letting \( L = M^k \), we have

\[ H^i(X, N_{X/\mathbb{P}^n}^* \otimes \wedge^a M_L \otimes L^2) = 0 \]

for \( i \geq 1 \).

**Proof:** Consider the product of \( a+2 \) factors \( X \times X \times \cdots \times X \). Then (cf. [27])

\[ (\pi_1)_* L \boxtimes L \boxtimes \cdots \boxtimes L \otimes \mathcal{T}_{\Delta_{1,2}} \otimes \mathcal{T}_{\Delta_{1,3}} \otimes \cdots \otimes \mathcal{T}_{\Delta_{1,a+2}} = M_{L}^{\otimes a} \otimes N^*(2) \]

Arguing as in Proposition 3.4, we obtain \( H^1(X, M_{L}^{\otimes a} \otimes N^*(2)) = 0 \). However, as we are working in characteristic 0, \( \wedge^a M_L \otimes N^*(2) \) is a summand of \( M_{L}^{\otimes a} \otimes N^*(2) \).

\[ \square \]

**Corollary 4.8.** Let \( X \) be a smooth variety, \( M \) an ample line bundle, and embed \( X \) by \( M^k \). Then for all \( k >> 0 \), \( \Sigma \) satisfies \( N_{3,p} \).
Proof: Letting \( L = M^k \) for \( k \gg 0 \), we know \([24],[27]\) \( L \) satisfies \( N_p^\Sigma \). Finally, by Proposition \( 4.7 \), we have \( H^i(X, N_X^{*} \otimes \wedge^{a-1+i} M_L \otimes L^2) = 0 \) for \( 2 \leq a \leq p + 1 \) and for \( i \geq 1 \). \( \Box \)

Corollary 4.9. Let \( X \neq \mathbb{P}^d \) be a smooth projective variety of dimension \( d \), \( M \) a very ample line bundle such that \( K_X \otimes M \) is ample. Embedding \( X \) by \( L = K_X \otimes M^{(p+2)d+1} \), \( p \geq 1 \), we have \( \Sigma \) satisfies \( N_{3,p} \).

Proof: In \([17, 3.1]\) it is shown that \( L = K_X \otimes M^{d+p+2} \) satisfies \( N_{p+2} \). The result now follows as in Remark 3.5. \( \Box \)

We conjecture what we believe to be the best possible result in general:

Conjecture 4.10. Let \( X^d \subset \mathbb{P}^n \) be a smooth projective variety with \( H^i(X, \mathcal{O}(k)) = 0 \) for \( i, k \geq 1 \) satisfying \( N_{p+d+1}^\Sigma \). Then \( \Sigma \) satisfies \( N_{3,p} \). \( \Box \)

Remark 4.11. We can show that under the hypotheses of the Conjecture that \( \Sigma \) satisfies \( N_{3+d,p} \). \( \Box \)

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DEPARTMENT OF MATHEMATICS, 214 PEARCE, CENTRAL MICHIGAN UNIVERSITY, MOUNT PLEASANT MI 48859

E-mail address: vermelpj@cmich.edu