Exact solution for the spin-$s$ XXZ quantum chain with non-diagonal twists

C. M. Yung and M. T. Batchelor

Department of Mathematics, School of Mathematical Sciences
Australian National University, Canberra ACT 0200, Australia

Abstract

We study integrable vertex models and quantum spin chains with toroidal boundary conditions. An interesting class of such boundaries is associated with non-diagonal twist matrices. For such models there are no trivial reference states upon which a Bethe ansatz calculation can be constructed, in contrast to the well-known case of periodic boundary conditions. In this paper we show how the transfer matrix eigenvalue expression for the spin-$s$ XXZ chain twisted by the charge-conjugation matrix can in fact be obtained. The technique used is the generalization to spin-$s$ of the functional relation method based on “pair-propagation through a vertex”. The Bethe ansatz-type equations obtained reduce, in the case of lattice size $N = 1$, to those recently found for the Hofstadter problem of Bloch electrons on a square lattice in a magnetic field.

Running title: Spin-$s$ XXZ chain with twists
1 Introduction

Vertex models and related quantum spin chains are most often studied with periodic boundary conditions [1, 2, 3]. If the Boltzmann weights underlying the models are obtained from solutions of the Yang-Baxter equation, or $R$-matrices, then commutativity of the associated transfer matrices follows and leads to integrability. A large class of $R$-matrices and therefore integrable vertex models with periodic boundaries can be constructed via the machinery of quantum groups [4, 5]. Several techniques – coordinate Bethe ansatz, algebraic Bethe ansatz, analytic Bethe ansatz, etc. – have been developed for diagonalizing the corresponding transfer matrices.

More general toroidal boundary conditions than periodic are in fact integrable; the condition being that the “twist matrix” specifying the boundary condition is a “symmetry” of the $R$-matrix [6]. Especially interesting are the cases where the twist matrix is non-diagonal. In spite of the existence of commuting transfer matrices for such cases, their diagonalization by the above mentioned Bethe ansatz techniques seems to fail. This is due to the non-existence in such cases of a trivial reference state.

In this paper we show that the spin-$s$ XXZ chain twisted by the (non-diagonal) charge-conjugation matrix can in fact be diagonalized. The technique used is a functional relation method [1] (also known as the “T-Q” method or the “method of commuting transfer matrices”) based on the “pair-propagation through a vertex” property of the relevant $R$-matrices. This generalizes to spin-$s$ a recent result for the spin-$\frac{1}{2}$ chain (or six-vertex model) [7].

One way [3, 8] to study the spin-$s$ XXZ chain with periodic boundaries is to first express the transfer matrix $t_{2s}^{(2s)}(u)$ in terms of a much simpler auxiliary transfer matrix $t_{1}^{(2s)}(u)$ using the fusion procedure [3]. This auxiliary transfer matrix is then diagonalized with the algebraic Bethe ansatz to give the eigenvalue expression and associated Bethe ansatz equations for the original spin-$s$ chain. We show that in the presence of the twist, there is an analogous procedure. The corresponding auxiliary transfer matrix is then diagonalized with the abovementioned functional relation method. As a consequence we obtain the transfer matrix eigenspectrum for the spin-$s$ XXZ chain with the charge-conjugation twist.

The paper is organized as follows: Section 2 is introductory in nature. We describe the general theory of integrable toroidal boundary conditions here and make a survey of available results for the case of the spin-$\frac{1}{2}$ XXZ Heisenberg chain. In Section 3 we study the integrable spin-$s$ generalization of the XXZ chain twisted by the charge conjugation matrix. In particular we relate the associated transfer matrix $t_{2s}^{(2s)}(u)$ to the auxiliary transfer matrix $t_{1}^{(2s)}(u)$, which is defined in terms of the $R$-matrix $R^{(2s,1)}(u)$. Section 4 is the body of the paper. In Section 4.1 we show that the property of “pair-propagation
through a vertex” for the \( R \)-matrix \( R^{(1,1)}(u) \) and related spin-\( \frac{1}{2} \) XXZ chain [1] generalizes to arbitrary spin-\( s \). This allows the \( T - Q \) relations for this auxiliary transfer matrix to be obtained in Section 4.2. For this purpose we required a conjectured property for two families of matrices \( Q_L(u) \) and \( Q_R(u) \), which is partially proved in Appendix A. In Section 4.3 we obtain the eigenvalue expression and Bethe ansatz-type relations for the auxiliary transfer matrix, leading to the corresponding expressions for the spin-\( s \) XXZ chain. In Section 5 we discuss the contents of this paper in a general context.

After this work was completed, we became aware of an interesting connection between integrable non-diagonal twists for the auxiliary transfer matrix \( t_1^{(2s)}(u) \) in the case of lattice size \( N = 1 \) and the so-called Hofstadter problem of Bloch electrons on a square lattice in a magnetic field [22]. In Appendix B we make contact with the latter work; in particular, recovering their Bethe ansatz results derived using a completely different method.

2 Integrable models with toroidal boundary conditions

Let us recall a few fundamental facts concerning the quantum inverse scattering method and the construction of integrable models [2, 3]. The principal ingredient is the Yang-Baxter equation

\[
R_{12}(u)R_{13}(u + v)R_{23}(v) = R_{23}(v)R_{13}(u + v)R_{12}(u)
\]  

which is an equation in the space \( V_1 \otimes V_2 \otimes V_3 \), with \( V_i \equiv V \simeq \mathbb{C}^n \), \( R(u) \in \text{End}(V \otimes V) \) and \( R_{12}(u) = R(u) \otimes 1, R_{23}(u) = 1 \otimes R(u) \), etc. Given an \( R \)-matrix, which by definition is a solution of (2.1) one constructs a monodromy matrix

\[
\hat{T}(u) = R_{a1}(u)R_{a2}(u) \cdots R_{aN}(u),
\]

which is to be thought of as an operator-valued matrix in the auxiliary space \( V_a \). A consequence of (2.1) is the intertwining relation

\[
R_{12}(u - v) \frac{\hat{T}(u)}{1} R_{23}(v) \frac{\hat{T}(v)}{2} = \frac{\hat{T}(v)}{1} \frac{\hat{T}(u)}{2} R_{12}(u - v)
\]

for the monodromy matrices. The indices 1 and 2 here label different auxiliary spaces and are not to be confused with those appearing in the definition (2.2) which label different quantum spaces. The important object

\[
t_P(u) = \text{tr}_a \hat{T}(u),
\]

2
has the interpretation of a row-to-row transfer matrix for an \( n \)-state vertex model with periodic boundary conditions and forms a commuting family.

We consider an extension of (2.4) to more general toroidal boundaries; namely by studying the transfer matrix

\[
 t(u) = \text{tr}_a \ T^a (u) \ F^a \tag{2.5}
\]

where \( F \) is a square matrix of dimension \( \text{dim}V_a \). The matrix \( F \) is called a “twist matrix” and the boundary conditions twisted. The following simple result shows that for certain twist matrices \( F \) commutativity of transfer matrices is retained:

**Proposition 1** [6] Let the twist matrix \( F \) be a “symmetry” of the \( R \)-matrix, i.e it obeys \( [R(u), F \otimes F] = 0 \). Then the transfer matrix \( (2.3) \) satisfies \([t(u), t(v)] = 0\) and we say \( F \) is integrable.

**Proof** From the definition, we have \( t(u) \ t(v) = \text{tr}_{12} \frac{1}{2} TT^1 (u) \frac{2}{2} T^2 (v) F \). Using the interwining relation (2.3) to change the order of \( T^1 (u) \frac{2}{2} T^2 (v) \) we obtain

\[
 \text{tr}_{12} \frac{1}{2} TT^1 (u) \frac{2}{2} T^2 (v) F = \text{tr}_{12} \frac{1}{2} F R_{12}^{-1}(u-v) \frac{2}{2} T^1 (v) \frac{1}{1} T^1 (u) R_{12}(u-v) F
\]

(symmetry) \[
 = \text{tr}_{12} R_{12}^{-1}(u-v) \frac{2}{2} T^2 (v) \frac{1}{1} T^1 (u) F \otimes R_{12}(u-v)
\]

\[
 = \text{tr}_{12} \frac{2}{2} T^2 (v) \frac{1}{1} T^1 (u) \frac{1}{1} F \ F = t(v) t(u)
\]

A quantum spin chain can be associated with the transfer matrix (2.5) in the usual way. For simplicity we assume the \( R \)-matrices are regular, with \( R(0) = cP \), where \( P \) is the exchange operator with matrix elements \( R_{a b}^c d = \delta_a^d \delta_b^c \). Given the multi-indices \( (\alpha_1, \ldots, \alpha_N) \) and \( (\beta_1, \ldots, \beta_N) \) we have from (2.5)

\[
 t(0)^{\beta_1 \ldots \beta_N}_{\alpha_1 \ldots \alpha_N} = \delta_{\alpha_1}^{\beta_1} \ldots \delta_{\alpha_{N-1}}^{\beta_{N-1}} F_{\alpha_N}^{\beta_N} c^N \tag{2.6}
\]

Similarly, assuming \( F \) is invertible, we have

\[
 \left( t(0)^{-1} t'(0) \right)^{\beta_1 \ldots \beta_N}_{\alpha_1 \ldots \alpha_N} = c^{-1} \left( R'(0)^{\beta_2 \ldots \beta_N}_{\alpha_2 \ldots \alpha_N} \delta_{\alpha_1}^{\beta_1} \ldots \delta_{\alpha_N}^{\beta_N} + \ldots + \delta_{\alpha_1}^{\beta_1} \ldots \delta_{\alpha_{N-2}}^{\beta_{N-2}} R'(0)^{\beta_{N-1} \beta_N}_{\alpha_{N-1} \alpha_N} + \delta_{\alpha_2}^{\beta_2} \ldots \delta_{\alpha_N}^{\beta_N} (F^{-1})_{\alpha_1}^{\beta_1} R'(0)_{\alpha_N}^{\beta_N} \right) \tag{2.7}
\]

From this we conclude that the quantum spin chain has Hamiltonian

\[
 H \equiv t(0)^{-1} t'(0) = \frac{1}{c} \left( \sum_{j=1}^{N-1} h_{j,j+1} + h_{N,1}^F \right), \tag{2.8}
\]
with

\[ h_{i,j} = R'_{ij}(0) \mathcal{P}_{ij}, \quad h^F_{N,1} = F^{-1} h^N_{N,1} F, \]

(2.9)

where \( F \) acts non-trivially only in the \( p \)-th slot of \( V_1 \otimes \cdots \otimes V_N \). When the twist matrix \( F \) is the identity, the spin chain with periodic b.c.'s is recovered.

**Example: Spin-\(\frac{1}{2} \) XXZ chain**

The six-vertex model \( R \)-matrix is given by

\[
R(u) = \begin{pmatrix}
\sinh(u + \lambda) & \sinh(u) & \sinh(\lambda) \\
\sinh(u) & \sinh(\lambda) & \sinh(u) \\
\sinh(u + \lambda) & \sinh(u) & \sinh(\lambda)
\end{pmatrix}
\]

and admits the following two types of integrable twist matrices:

(i) \( F = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \),

(ii) \( F = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \),

where \( \alpha, \beta \) are completely arbitrary. When \( \alpha, \beta \) are non-zero, the corresponding spin-\(\frac{1}{2} \) XXZ Hamiltonian is given by

\[
H = \frac{1}{\sinh(\lambda)} \sum_{j=1}^{N} \sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + \frac{\cosh(\lambda)}{2} \sigma_j^z \sigma_{j+1}^z,
\]

(2.10)

with boundary conditions

(i) \( \sigma^+_{N+1} = \frac{\beta}{\alpha} \sigma^+_1 \), \( \sigma^-_{N+1} = \frac{\alpha}{\beta} \sigma^-_1 \), \( \sigma^z_{N+1} = \sigma^z_1 \),

(ii) \( \sigma^+_{N+1} = \frac{\beta}{\alpha} \sigma^+_1 \), \( \sigma^-_{N+1} = \frac{\alpha}{\beta} \sigma^-_1 \), \( \sigma^z_{N+1} = -\sigma^z_1 \).

(2.11)

For boundary conditions of type (i), the total spin operator \( \sum_{j=1}^{N} \sigma_j^z \) commutes with the Hamiltonian; the trivial reference state is available and the corresponding transfer \( i^{(i)}(u) \) matrix can be diagonalized with the algebraic Bethe ansatz \[6\], with very little difference from the periodic case. The transfer matrix eigenvalue expression is given by

\[
\Lambda^{(i)}(u) = \alpha \sinh(u + \lambda)^N \prod_{j=1}^{n} \frac{\sinh(u - u_j - \lambda)}{\sinh(u - u_j)}
\]

\[ + \beta \sinh(u)^N \prod_{j=1}^{n} \frac{\sinh(u - u_j + \lambda)}{\sinh(u - u_j)}
\]

(2.12)

*When one of \( \alpha, \beta \) is allowed to be zero, the vertex model has fixed boundaries; it does not seem possible to obtain a local Hamiltonian in this case. Furthermore, the eigenvalue expression for such type (ii) twists is unknown.*
in the sector with total spin \( N - 2n \), with the Bethe ansatz equations for \( u_k \) determined from analyticity of \( \Lambda^{(i)}(u) \).

On the other hand, boundary conditions of type (ii) leave only a \( \mathbb{Z}_2 \)-invariance in the eigenspectrum. For the case \( \alpha = \beta = 1 \)† (which can be considered as anti-periodic boundary conditions on the vertex model), the problem was recently solved \[7\] using the functional equation method of Baxter \[1\]. The transfer matrix \( \tilde{t}(u) \) has eigenvalue expression

\[
\tilde{\Lambda}(u) = \sinh(u + \lambda)^N \prod_{j=1}^{N} \frac{\sinh[\frac{1}{2}(u - u_j - \lambda)]}{\sinh[\frac{1}{2}(u - u_j)]} \\
- \sinh(u)^N \prod_{j=1}^{N} \frac{\sinh[\frac{1}{2}(u - u_j + \lambda)]}{\sinh[\frac{1}{2}(u - u_j)]}.
\]

(2.13)

General boundary conditions of type (ii) are in fact very closely related. This is due to the relation

\[
\det \left( t^{(ii)}(u) - \sqrt{\alpha \beta \lambda} \right) \propto \det \left( \tilde{t}(u) - \lambda \right)
\]

(2.14)

between the characteristic polynomials for their corresponding transfer matrices, leading to the result

\[
\Lambda^{(ii)}(u) = \sqrt{\alpha \beta} \tilde{\Lambda}(u).
\]

(2.15)

Type (i) boundary conditions with \( \alpha \beta = 1 \) were studied in Ref. \[9\]: At criticality the model gives rise to a conformal field theory with central charge \( c < 1 \), with the twist parameter determining the value of \( c \). All the above toroidal boundary conditions have in fact been studied in Ref. \[10\] (by direct numerical diagonalization in the case of type (ii) twists), with conjectures given for their complete operator content. In their language, the type (ii) boundary conditions share the same physical properties because they are in the same conjugacy class of \( O(2) \), being the global symmetry of the infinite chain.

3 Spin-\( s \) XXZ quantum chains

The integrable spin-\( s \) generalization of the XXZ spin chain and related \( R \)-matrices have been studied by many authors; amongst them \[11, 12, 1, 3, 13, 14, 15, 16, 17, 8\]. In particular, the spin-1 Hamiltonian was first constructed in Ref. \[11\]. Bethe ansatz equations for the spin-\( s \) models with periodic boundary conditions were obtained and studied in Refs. \[15, 17, 8\]. For integrable boundary conditions associated with diagonal twists, Bethe ansatz equations were obtained and studied numerically in Ref. \[18\] and

†We refer to this as the charge-conjugation twist
and \( R \) is possible to establish the key relations [8].

The transfer matrix for the spin-s XXZ chain is associated with the \( R \)-matrix \( R^{(2s,2s)}(u) \) which acts in a tensor product of two spin-s representations of \( U_q(su(2)) \). We will study it via its construction by the fusion procedure. For this purpose we will require the \( R \)-matrix \( R^{(l,m)}(u) \) acting in the tensor product of the spin-\( l/2 \) and spin-\( m/2 \) representations of \( U_q(su(2)) \) for arbitrary \( l, m \). These can all be constructed iteratively from \( R^{(l,1)}(u) \), whose explicit form is given by [8]

\[
R^{(l,1)}(u) = \sinh \left( u + \frac{1}{2} \left[ l + S^z \otimes \sigma^z \right] \right) + \sinh(\lambda) \left[ S^+ \otimes \sigma^- + S^- \otimes \sigma^+ \right]. \tag{3.1}
\]

Here \( S^z \) and \( S^\pm \) are generators for \( U_q(su(2)) \):

\[
[S^z, S^\pm] = \pm 2S^\pm, \quad [S^+, S^-] = \frac{\sinh(\lambda S^z)}{\sinh \lambda}. \tag{3.2}
\]

In the spin-\( l/2 \) representation these operators take the forms \( S^z = \text{diag}(l, l-2, \ldots, -l) \) and

\[
S^+ = (S^-)^t = \frac{1}{\sinh \lambda} \begin{pmatrix}
0 & f(1) & \cdots \\
\vdots & \ddots & \ddots \\
& \cdots & f(l) \\
0 & \cdots & 0
\end{pmatrix},
\]

where \( f(j) \equiv \sqrt{\sinh(j\lambda) \sinh(\lambda(l+1-j))} \). The operators \( \sigma^z \) and \( \sigma^\pm \) are in the spin-\( 1/2 \) representation.

The \( R \)-matrix \( R^{(l,1)}(u) \) degenerates at \( u = \lambda \) and \( u = -l\lambda \):

\[
R^{(l,1)}(\lambda) = B^{(l)}(\lambda)P^{(l+1)}, \quad R^{(l,1)}(-l\lambda) = C^{(l)}P^{(l-1)}, \tag{3.3}
\]

where \( B^{(l)} \) and \( C^{(l)} \) are matrices and \( P^{(l\pm1)} \) are projection operators onto subspaces of dimension \( l + 2 \) and \( l \), respectively. Using this property and the fusion procedure [3] it is possible to establish the key relations [8]

\[
D_{12}^{(l)} R_{13}^{(l,m)}(u + \lambda) R_{23}^{(1,m)}(u) D_{12}^{-1} =
\begin{cases}
0 & \text{if } l < m, \\
\sinh \lambda R_{12}^{(l,m)}(u) & \text{if } l = m,
\end{cases}
\tag{3.4}
\]

analytically for spin-1 in Ref. [19]. General toroidal boundary conditions for the spin-1 case was also investigated by direct numerical diagonalization in Ref. [20]. In this paper, we will mainly follow the approach of Ref. [8] to construct the integrable spin-s chain.
Expressions for the matrices $B^{(l)}$, $C^{(l)}$ and $D^{(l)}$ are given in [3]. The spaces $V_{(12)}$ and $V_{<12>}$ in equation (3.4) are the symmetrized and anti-symmetrized components respectively of $V_1 \otimes V_2$. The relations (3.2) and (3.4) allow the $R$-matrix $R_{(l,l)}^{(1,l)}(u)$, through which we obtain the spin-$l/2$ XXZ chain, to be obtained recursively. A non-recursive definition is also available [3].

Thus far, all our considerations have been purely local in character; in particular, they are independent of boundary considerations. We will now specialize to the boundary conditions specified by the non-diagonal twist matrix (the charge-conjugation twist) $F^{(l)} = (F^{(l)})^{-1}$, which is the $(l+1) \times (l+1)$ matrix with 1 along the anti-diagonal and 0 elsewhere. It has the properties

$$F^{(l)} S^{\pm} F^{(l)} = S^{\mp}, \quad F^{(l)} S^{z} F^{(l)} = -S^{z}. \quad (3.5)$$

Using this it is easy to see from (3.1) that $[R^{(l,1)}(u), F^{(l)} \otimes F^{(1)}] = 0$. The recursion relations (3.4) then imply that $F^{(l)}$ is a symmetry (in the sense of Proposition 1) of $R_{(l,l)}^{(1,l)}(u)$.

Define the monodromy matrices

$$T^{(l)}(u) = R^{(1,2s)}_{a_1}(u) \cdots R^{(1,2s)}_{a_N}(u) \quad (3.6)$$

with a spin-$\frac{l}{2}$ auxiliary space and a quantum space which is an $N$-fold tensor product of spin-$s$ representations, and transfer matrices

$$t^{(2s)}_l(u) = \text{tr}_a T^{(l)}(u) F^{(l)}. \quad (3.7)$$

By Proposition 1 (or rather, the obvious generalization to the case of non-isomorphic quantum spaces), these form a commuting family: $[t^{(2s)}_l(u), t^{(2s)}_n(v)] = 0$ for all $l, n$.

With respect to the same decomposition as in (3.4) we also have

$$D^{(l)}_{12} F^{(l)} F^{(1)} D^{(l)}_{12} = \begin{pmatrix} F^{(l+1)} & 0 \\ 0 & -F^{(l-1)} \end{pmatrix}. \quad (3.8)$$

Consider now the product $t^{(2s)}_l(u + \lambda) t^{(2s)}_l(u)$: From the definition (3.7) and commuting the $R$-matrices appropriately this becomes

$$\text{tr}_{ab} F^{(l)} F^{(1)} R^{(1,2s)}_{a_1}(u + \lambda) R^{(1,2s)}_{a_1}(u) \cdots R^{(1,2s)}_{a_N}(u + \lambda) R^{(1,2s)}_{b_N}(u),$$

which can be written in the form

$$\text{tr}_{ab} \left( \begin{array}{c} D^{(l)}_{ab} F^{(l)} F^{(1)} D^{(l)}_{ab} \\ D^{(l)}_{ab} R^{(1,2s)}_{a_1}(u + \lambda) R^{(1,2s)}_{a_1}(u) \end{array} \right) \left( \begin{array}{c} D^{(l)}_{ab} R^{(1,2s)}_{a_1}(u + \lambda) R^{(1,2s)}_{b_N}(u) D^{(l)}_{ab} \\ \vdots \end{array} \right) .$$

By application of eqs (3.4) and (3.8) we arrive at the following result:
The Hamiltonian for the spin chain is then given by

\[ H = t_{l+1}^{(2s)}(u) - \sinh u^N \sinh[u + \lambda(2s + 1)]Nt_{l-1}^{(2s)}(u + 2\lambda) \]

\[ \sinh(u + 2s\lambda)^Nt_{l+1}^{(2s)}(u) - \sinh u^N \sinh[u + \lambda(2s + 1)]Nt_{l-1}^{(2s)}(u + 2\lambda) \]

\[ \sinh(u + 2s\lambda)^Nt_{l+1}^{(2s)}(u) - \sinh u^N t_{l-1}^{(2s)}(u + 2\lambda) \]

Define the shift operator \( z \) by \( z^{-1}f(u)z = f(u + \lambda) \) for all functions \( f(u) \) and the function \( d^{(m)}(u) \) by \( d^{(m)}(u) = \sinh u^N \sinh[u + \lambda(m + 1)]^N \). Then the generating function for the transfer matrices \( t_l^{(2s)}(u) \) is given by

\[
\left[ 1 - zt_1^{(2s)}(u) - z^2 d^{(2s)}(u) \right]^{-1} = \sum_{k=0}^{2s} z^k t_k^{(2s)}(u) + \sum_{l=2s+1}^{\infty} z^k \prod_{l=2s}^{k-1} \sinh(u + \lambda l)^N t_k^{(2s)}(u). \tag{3.10}
\]

This can be proven by multiplying both sides by \( \left( 1 - zt_1^{(2s)}(u) - z^2 d^{(2s)}(u) \right) \) (on the right) and using the recursion relations (3.9) to show that all coefficients of \( z^k \) for \( k \geq 1 \) vanish. The corresponding generating function [8] for periodic b.c.’s differ from (3.10) in only the sign of the coefficient of the \( z^2 \) term on the l.h.s.

The transfer matrix \( t_2^{(2s)}(u) \), out of which the Hamiltonian for the spin-s XXZ chain is constructed, can be obtained in terms of the transfer matrix \( t_1^{(2s)}(u) \) either using the recursion relations (3.9) or the generating function (3.10). It is this transfer matrix which is manageable; for instance, because the auxiliary space for \( t^{(2s)}(u) \) is two-dimensional, the functional relation method of Baxter [1] can be generalized in a straightforward manner to obtain the eigenvalue expression and associated Bethe ansatz-type equations.

At \( u = -(2s - 1)\lambda \) the \( R \)-matrix \( R^{(2s,2s)}(u) \) becomes proportional to the exchange operator \( P \), with proportionality constant \( \prod_{j=1}^{2s} \sinh(j\lambda) \). According to Eq. (2.8) the Hamiltonian for the spin chain is then given by

\[
H = t_2^{(2s)}(u)^{-1} \frac{d}{du} t_2^{(2s)}(u) \bigg|_{u=-(2s-1)\lambda} = \prod_{j=1}^{2s} \sinh(j\lambda)^{-1} \left( \sum_{j=1}^{N-1} h_{j,j+1} + h_{N,1}^F \right), \tag{3.11}
\]

with two-site interaction \( h_{j,j+1} = R_{ij}^{(2s,2s)}(-(2s - 1)\lambda)P_{ij} \). The Hamiltonian in terms of \( U_q(su(2)) \) generators is very complicated, but using Eq. (3.3) the boundary conditions can be simply written as

\[
S_{N+1}^z = -S_1^z, \quad S_{N+1}^\pm = S_1^\mp. \tag{3.12}
\]
4 Transfer matrix diagonalization

In this section we will obtain the eigenvalue expression \( \Lambda_1^{(2s)}(u) \) for the transfer matrix \( t_1^{(2s)}(u) \), from which the eigenvalue expressions for the transfer matrix \( t^{(2s)} \) and the Hamiltonian \( (3.11) \) follow. For this purpose we will require explicit expressions for the non-zero matrix elements of \( R^{(1,2s)}(u) \), which can be obtained by interchanging \( S^i \) and \( \sigma^i \) in \( (3.1) \):

\[
R^{(1,2s)}(u)_{1j}^{1j} = a_j(u), \\
R^{(1,2s)}(u)_{2j}^{2j} = a_{n+1-j}(u), \\
R^{(1,2s)}(u)_{1j+1}^{2j} = b_j(\lambda), \\
R^{(1,2s)}(u)_{2j}^{1j+1} = b_j(\lambda),
\]

(4.1)

where

\[
\begin{align*}
a_j(u) &= \sinh[u + (2s + 1 - j)\lambda], \\
b_j(\lambda) &= \sqrt{\sinh(j\lambda) \sinh[\lambda(2s + 1 - j)]}.
\end{align*}
\]

(4.2)

4.1 Pair-propagation through a vertex

A key ingredient of Baxter’s “method of commuting transfer matrices” \( \| \) is the property of “pair-propagation through a vertex” for the \( R \)-matrix; ie. the existence of vectors \( h^{\text{in}}, h^{\text{out}}, v^{\text{in}} \) and \( v^{\text{out}} \) such that:

\[
R(u)^{\mu\nu}_{\alpha\beta} v^{\text{in}}_{\beta} h^{\text{in}}_{\nu} = v^{\text{out}}_{\alpha} h^{\text{out}}_{\mu},
\]

(4.3)

for \( \alpha, \beta \in \{1, \ldots, 2s + 1\} \) and \( \mu, \nu \in \{1, 2\} \). Here \( v^{\text{in}}_{\beta} \) are entries of the vector \( v^{\text{in}} \) etc.

For the \( R \)-matrix \( R^{(1,2s)}(u) \), the set of equations (4.3) can be written as

\[
\begin{align*}
a_j(u) v^{\text{in}}_j h^{\text{in}}_1 + b_{j-1}(\lambda) v^{\text{in}}_{j-1} h^{\text{in}}_2 &= v^{\text{out}}_j h^{\text{out}}_1, \\
a_{2s+2-j}(u) v^{\text{in}}_j h^{\text{in}}_2 + b_j(\lambda) v^{\text{in}}_{j+1} h^{\text{in}}_1 &= v^{\text{out}}_j h^{\text{out}}_2,
\end{align*}
\]

(4.4, 4.5)

where \( j \) runs from 1 to \( 2s + 1 \). The first thing to note about the system of equations (4.4), (4.5) is that it is linear in \( v^{\text{in}}_j \) and \( v^{\text{out}}_j \). Therefore, consistency of (4.4) and (4.5)
requires the vanishing of the \((4s + 2) \times (4s + 2)\) determinant

\[
\begin{vmatrix}
  a_1 h_1^{\text{in}} & b_1 h_2^{\text{in}} & \cdots & \cdots & b_{2s} h_{2}^{\text{in}} & a_{2s+1} h_1^{\text{in}} \\
  b_1 h_2^{\text{in}} & b_2 h_2^{\text{in}} & a_{2s+1} h_1^{\text{in}} & & & \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  a_{2s+1} h_2^{\text{in}} & b_1 h_1^{\text{in}} & \cdots & \cdots & b_{2s} h_{1}^{\text{in}} & a_{1} h_{2}^{\text{in}} \\
\end{vmatrix}
\]

(4.6)

An examination of small values of \(s\) leads us to the following conjecture:

**Claim 1** Define \(r_{\text{out}} = h_{2}^{\text{in,out}} / h_{1}^{\text{in,out}}\). Then the determinant (4.6) vanishes if

\[
\frac{r_{\text{out}}}{r_{\text{in}}} = e^{2\lambda \sigma},
\]

(4.7)

where \(\sigma = -s, -s + 1, \ldots, s\).

We do not have a proof for general \(s\), but this result has been important in leading to Proposition 3 below, which is a stronger result and will be proved. For that purpose, we will require a few definitions: Let \(q = e^{\lambda}\) and define the \(q\)-numbers, factorials and binomial coefficients, respectively, by

\[
\begin{align*}
[n] & = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{\sinh(n\lambda)}{\sinh(\lambda)} \\
[n]! & = [n][n-1] \cdots [1], \quad \text{with } [0]! = 1 \\
\begin{bmatrix} m \\ n \end{bmatrix} & = \frac{[m]!}{[m-n]![n]!}, \quad (m, n \in \mathbb{Z}_{\geq 0}, m \geq n).
\end{align*}
\]

The \(q\)-binomial coefficients satisfy the following relation (a \(q\)-analogue of the binomial theorem)

\[
\sum_{k=0}^{2s+1} a^k \begin{bmatrix} 2s + 1 \\ k \end{bmatrix} = \prod_{j=-s}^{s} \left(1 + qa^{2j}\right),
\]

(4.8)

which will be used later. Introduce the functions \(f^{(p)}(\sigma; u) = \sum_{k=0}^{p} f_{k}^{(p)}(\sigma; u)\), with

\[
f_{k}^{(p)}(\sigma; u) = (-e^{2\lambda \sigma})^{k} \begin{bmatrix} p \\ k \end{bmatrix} \prod_{j=1}^{k} \sinh[u + (j - 1)\lambda] \prod_{j=k+1}^{p} \sinh[u + (2s + j - p)\lambda].
\]

(4.9)

They have the useful properties summarised in the following:
Lemma 1 Let $a_j(u)$ and $b_j(\lambda)$ be defined as in (4.2). For any $\sigma \in \Phi$, the functions $f^{(p)}(\sigma; u)$ defined in (4.3) satisfy

$$f^{(p)}(\sigma; u) + e^{2\lambda a} a_{2s+2-p}(u) f^{(p-1)}(\sigma; u) = a_1(u) f^{(p-1)}(\sigma; u - \lambda), \quad (4.10)$$

$$a_{p+1}(u) f^{(p)}(\sigma; u) + e^{2\lambda b} b_p(u)^2 f^{(p-1)}(\sigma; u) = f^{(p)}(\sigma; u - \lambda)a_1(u), \quad (4.11)$$

and

$$a_1(u) a_{2s+1}(u) f^{(j-1)}(\sigma; u - \lambda) f^{(j-1)}(\sigma; u + \lambda) = a_j(u) a_{2s+2-j}(u) f^{(j-1)}(\sigma; u)^2 - b_{j-1}(\lambda)^2 f^{(j)}(\sigma; u) f^{(j-2)}(\sigma; u). \quad (4.12)$$

If $\sigma$ takes values only in the range $\{-s, -s+1, \ldots, s\}$, then we also have

$$f^{(2s)}(\sigma; u - \lambda) = e^{2\lambda a} f^{(2s)}(\sigma; u). \quad (4.13)$$

Proof These relations can be proved using the definition of $f^{(p)}(\sigma; u)$. For instance, Eq. (4.10) boils down to verifying

$$\left[ \frac{p-1}{k} \right] \sinh(u - \lambda) + \left[ \frac{p-1}{k-1} \right] \sinh(u + (p-1)\lambda) = \left[ \frac{p}{k} \right] \sinh(u + (k-1)\lambda).$$

The proof of Eq. (4.13) is more interesting; the difference between the two sides can be shown to be

$$\prod_{j=1}^{2s} \sinh[u + (j-1)\lambda] \sum_{k=0}^{2s+1} (-e^{2\lambda})^k \left[ \frac{2s+1}{k} \right].$$

Upon application of the $q$-binomial theorem (4.8), the sum above simplifies to

$$\prod_{j=-s}^{s} \left( 1 - e^{2\lambda (\sigma + j)} \right)$$

from which the result immediately follows.

We are now ready to state and prove the main result in this subsection:

Proposition 3 Let $\sigma \in \{-s, -s+1, \ldots, s\}$ and $f^{(p)}(\sigma; u)$ be defined as in (4.3). The relations

$$h_{2}^{\text{in}} = \frac{h_{2}^{\text{in}} h_{2}^{\text{out}}}{h_{1}^{\text{out}}} e^{2\lambda}, \quad (4.14)$$

$$v_{j}^{\text{out}} = a_1(u) \frac{h_{1}^{\text{in}} v_{j}^{\text{in}}}{h_{1}^{\text{out}}} \frac{f^{(j-1)}(\sigma; u - \lambda)}{f^{(j-1)}(\sigma; u)}, \quad (4.15)$$

$$v_{j+1}^{\text{in}} = \frac{1}{b_j(\lambda)} \frac{v_{j}^{\text{in}} h_{2}^{\text{out}}}{h_{1}^{\text{out}}} \frac{f^{(j)}(\sigma; u)}{f^{(j-1)}(\sigma; u)}, \quad (4.16)$$

with arbitrary $h_{1}^{\text{out}}, h_{1}^{\text{in}}$ and $v_{1}^{\text{in, out}}$ solve the equations (4.4) and (4.5). In other words, the $R$-matrix $R^{(1,2s)}(u)$ does indeed have the property of pair-propagation through a vertex.
Proof The proof is by substitution of (4.14), (4.15) and (4.16) directly into (4.4) and (4.5). For “generic” values of \( j \), these equations reduce to (4.10) and (4.11) which we have indicated above are true. For the “end values” \( j = 1 \) and \( j = 2s + 1 \) equations (4.4) and (4.5), respectively, degenerate by virtue of the vanishing of \( b_0(\lambda) \) and \( b_{2s+1}(\lambda) \). The former case can be trivially seen to be true; the latter is true by virtue of (4.13).

For the special cases of \( \sigma = \pm s \), the function \( f^{(p)}(\sigma; u) \) simplifies to

\[
f^{(p)}(\pm s; u) = e^{\mp pu} \prod_{j=1}^{p} \sinh[(2s - j + 1)\lambda],
\]

with corresponding simplifications in the expressions for \( v^\text{in}_j \) and \( v^\text{out}_j \) in Proposition 3. For the spin-\( \frac{1}{2} \) case, we recover the six-vertex model results of [1] by substituting Eq. (4.17) into Proposition 3.

Let us indicate explicitly the \( u \)- and \( \sigma \)-dependence of the vectors \( v^\text{in}_\sigma (u) \) and \( v^\text{out}_\sigma (u) \). Equation (4.16) is a recursion relation for the elements \( v^\text{in}_\sigma (u)_j \), given the initial value \( v^\text{in}_\sigma (u)_1 \). If we choose this initial value to be unity, then it is evident that

\[
v^\text{in}_\sigma (u - \lambda)_j = v^\text{in}_\sigma (u)_j \frac{f^{(j-1)}(\sigma; u - \lambda)}{f^{(j-1)}(\sigma; u)}.
\]

It can then be shown from (4.15) that \( v^\text{out}_\sigma (u) \) is simply related to \( v^\text{in}_\sigma (u) \) by

\[
v^\text{out}_\sigma (u) = a_1(u) \frac{h^\text{in}_1}{h^\text{out}_1} v^\text{in}_\sigma (u - \lambda).
\]

Proposition 3 can be interpreted in the following way: It guarantees the existence of matrices

\[
P = \begin{pmatrix} h^\text{out}_1 & * \\ h^\text{out}_2 & * \end{pmatrix}, \quad P' = \begin{pmatrix} h^\text{in}_1 & * \\ h^\text{in}_2 & * \end{pmatrix}
\]

such that the generalized similarity transformation

\[
(P \otimes 1)^{-1} R_{12}^{(1,2s)}(u)(P' \otimes 1) = \begin{pmatrix} \alpha(u) & \beta(u) \\ \gamma(u) & \delta(u) \end{pmatrix}
\]

results in a matrix which is upper block-triangular when acting on a certain vector in \( \mathfrak{g}^{2s+1} \) to be specified. To see this statement, pre-multiply (4.20) by \( P \otimes 1 \) on both sides and act on \( 1 \otimes v^\text{in}_\sigma (u) \). We end up with a 2 \( \times \) 2 matrix equation with entries in \( \mathfrak{g}^{2n+1} \). Component-wise, it reads

\[
\begin{pmatrix}
R(u)_1^{1k} v^\text{in}_\sigma (u)_k & R(u)_2^{2k} v^\text{in}_\sigma (u)_k \\
R(u)_1^{1k} v^\text{in}_\sigma (u)_k & R(u)_2^{2k} v^\text{in}_\sigma (u)_k
\end{pmatrix}
\begin{pmatrix}
h^\text{in}_1 & * \\
h^\text{in}_2 & *
\end{pmatrix}
= \begin{pmatrix}
h^\text{out}_1 & * \\
h^\text{out}_2 & *
\end{pmatrix}
\begin{pmatrix}
\alpha(u) v^\text{in}_\sigma (u)_j & \beta(u) v^\text{in}_\sigma (u)_j \\
\gamma(u) v^\text{in}_\sigma (u)_j & \delta(u) v^\text{in}_\sigma (u)_j
\end{pmatrix}.
\]
A comparison with (4.3) immediately indicates that
\[ \gamma(u)v^{in}_\sigma(u) = 0. \] (4.22)

The action of \( \alpha(u) \) and \( \delta(u) \) on the vector \( v^{in}_\sigma(u) \), whose components are determined from the recursion relation (4.16) with \( v^{in}_\sigma(u)_1 = 1 \), is determined by examination of the explicit forms for \( v^{in}_\sigma(u) \) and \( v^{out}_\sigma(u) \) given in Proposition 3: With the help of (4.18) we find that
\[ \alpha(u)v^{in}_\sigma(u) = h^{in}_1 a_1(u) v^{in}_\sigma(u - \lambda), \] (4.23)
whereas by taking determinants on both sides of (4.21), together with the result (4.12), we find that
\[ \delta(u)v^{in}_\sigma(u) = h^{out}_{2s+1}(u) v^{in}(u + \lambda). \] (4.24)

All the considerations in this subsection are local in nature. The results are directly applicable to the transfer matrix with periodic boundaries, generalizing the spin-\( \frac{1}{2} \) (six-vertex model) results of [1] and providing an alternative to the algebraic and analytic Bethe ansatz [8]. In the next subsection we will specialize to the boundary conditions specified by the twist matrices \( F^{(l)} \) defined in Section 3.

### 4.2 Diagonalization of the auxiliary transfer matrix

The main idea of the technique to diagonalize the transfer matrix \( t_1^{(2s)}(u) \) is to use Eqs. (4.20) and (4.22) – (4.24) to construct functional equations satisfied by \( t_1^{(2s)}(u) \). We note that an attempt to do a generalized algebraic Bethe ansatz calculation, along the lines of [21] for the eight-vertex model, should also probably use Eqs. (4.20) and (4.22) – (4.24) as a starting point.

Let us rewrite the auxiliary transfer matrix in the form
\[ t_1^{(2s)}(u) = \text{tr} \left( P_1^{-1} F^{(1)} P_1 \left( P_1^{-1} R_{a1}^{(1,2s)}(u) P_2 \right) \cdots \left( P_N^{-1} R_{aN}^{(1,2s)}(u) P_{N+1} \right) \right), \] (4.25)
where the matrices \( P_k \) are understood to act non-trivially only in the auxiliary space \( V_a \). These matrices are chosen to be
\[ P_{N+1} = \left( \begin{array}{c} 1 \\ r \end{array} \right), \quad P_N = \left( \begin{array}{cc} 1 \\ r e^{-2\lambda \sigma_N} \end{array} \right), \quad \ldots \]
\[ P_2 = \left( \begin{array}{ccc} 1 \\ r e^{-2\lambda \sum_{i=2}^{N} \sigma_i} \end{array} \right), \quad P_1 = \left( \begin{array}{ccc} 1 \\ r^2 e^{-2\lambda \sum_{i=1}^{N} \sigma_i} \end{array} \right), \] (4.26)
where \( \sigma_i \), for \( i = 1, \ldots, N \) is any set of variables taking values in \( \{-s, -s + 1, \ldots, s\} \). If, in addition, \( r \) is chosen to satisfy
\[ r^2 = \exp \left( 2\lambda \sum_{i=1}^{N} \sigma_i \right), \] (4.27)
then, by direct verification, we have the relation
\[
P_{N+1}^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P_1 = \begin{pmatrix} 1 & \frac{-\det(P_1)}{\det(P_{N+1})} \\ 0 & 1 \end{pmatrix}.
\] (4.28)

Let us denote \( \sigma = (\sigma_1, \ldots, \sigma_N) \). From the results (4.19) to (4.24) we deduce that
\[
t_1^{(2s)}(u) y_\sigma(u) = \frac{1}{r} a_1(u)^N y_\sigma(u - \lambda) - r a_{2s+1}(u)^N y_\sigma(u + \lambda),
\] (4.29)

where \( y_\sigma(u) = v_{\sigma_1}^{(1)}(u) \otimes \cdots \otimes v_{\sigma_N}^{(N)}(u) \) is a vector in \( (\mathbb{C}^{2s+1})^\otimes N \), with the components of \( v_{\sigma_k}^{(k)}(u) \) determined from the recursion relations
\[
v_{\sigma_k}^{(k)}(u)_1 = 1,
\]
\[
v_{\sigma_k}^{(k)}(u)_{j+1} = v_{\sigma_k}^{(k)}(u)_j \frac{r e^{-2\lambda(\sigma_1 + \cdots + \sigma_k)}}{b_j(\lambda)} \frac{f(j)(\sigma_k; u)}{f(j-1)(\sigma_k; u)}.
\] (4.30)

Note that the \( r \)'s in Eq. (4.29) are \( \sigma \)-dependent because of Eq. (4.27); they can be removed by re-scaling \( y_\sigma(u) \) by a \( u \)-dependent factor. The result can be summarised thus:

Proposition 4 Let \( t_1^{(2s)}(u) \) be the transfer matrix defined by (3.7). The vector
\[
q_\sigma(u) = e^{\sum_{i=1}^N \sigma_i u} v_{\sigma_1}^{(1)}(u) \otimes \cdots \otimes v_{\sigma_N}^{(N)}(u),
\] (4.31)

where \( v_{\sigma_k}^{(k)}(u) \) is determined by (4.30) with \( r = \exp(\lambda \sum_{i=1}^N \sigma_i) \), satisfies the equation
\[
t_1^{(2s)}(u) q_\sigma(u) = a_1(u)^N q_\sigma(u - \lambda) - a_{2s+1}(u)^N q_\sigma(u + \lambda).
\] (4.32)

The vector \( q_\sigma(u) \) in Proposition 4 is well-defined for any choice of \( \sigma = (\sigma_1, \ldots, \sigma_N) \), with \( \sigma_j \in \{-s, -s+1, \ldots, s\} \). Therefore the matrix \( Q_R(u) \), whose columns are formed from the collection of such vectors \( ((2s+1)^N \) of them altogether), satisfies the relation
\[
t_1^{(2s)}(u) Q_R(u) = a_1(u)^N Q_R(u - \lambda) - a_{2s+1}(u)^N Q_R(u + \lambda).
\] (4.33)

We will now require the following property of \( t_1^{(2s)}(u) \) which is due to crossing-symmetry of the corresponding \( R \)-matrix:

Lemma 2 The transfer matrix \( t_1^{(2s)}(u) \) satisfies the relation
\[
t_1^{(2s)}(-u - 2\lambda s) = (-1)^{N-1} t_1^{(2s)}(u).
\] (4.34)
The crucial "commutation relations" where we have defined $Q$

For any choice of $\sigma_i = \pm 1$ and $\sigma'_i = \pm 1$, the inner product

$$
\langle q_\sigma | (-u - 2\lambda s) \cdot q_\sigma \rangle = e^{(-u-2\lambda s) \sum_{i=1}^N \sigma_i + v \sum_{i=1}^N \sigma'_i} \prod_{k=1}^N G^{(k)}_{\sigma'_k \sigma_k} (u, v),
$$

(4.39)
where
\[ G^{(k)}_{\sigma \sigma_k}(u, v) \equiv v^{(k)}_{\sigma_k}(-u - 2\lambda s) \cdot v^{(k)}(v). \]

However, from the recursion relations (4.30) together with \( r = e^{\lambda \sum_{i=1}^{N} \sigma_i} \), we have
\[ v^{(k)}_{\sigma_k}(u)_{j+1} = e^{j\lambda} \sum^k \sigma \frac{f^{(j)}(\sigma_k; u)}{b_1(\lambda) \cdots b_j(\lambda)}, \tag{4.40} \]
where \( \sum^k \sigma \) denotes \( \sum_{i=k+1}^{N} \sigma_i - \sum_{i=1}^{k} \sigma_i \), giving rise to
\[ G^{(k)}_{\sigma \sigma_k}(u, v) = 2^s \sum_{j=0}^{2s} e^{2j\lambda \sum_k^k (\sigma + \sigma')} \frac{f^{(j)}(\sigma_k; v) f^{(j)}(\sigma'_k; -u - 2s\lambda)}{(b_1(\lambda) \cdots b_j(\lambda))^2}. \tag{4.41} \]

If we restrict \( \sigma, \sigma' \) to take values only in the set \( \{ \pm s \} \) then \( G^{(k)}_{\sigma \sigma_k}(u, v) \) in Eq. (4.39) simplifies and it can be proved (see Appendix A) that the inner product (4.39) is indeed symmetric in \((u, v)\). For \( s = \frac{1}{2} \) this is, of course, the whole story \[7\]. For general \( s \) we have not been able to construct a proof of Conjecture 1 but we have confirmed the result on a computer for a range of values of \( s \) and \( N \).

We now require a technical assumption:

**Claim 2** The matrix \( Q_R(u) \) is non-singular for generic values of \( u \). In particular, it can be inverted at the point \( u = u_0 \).

Define now the matrix \( Q(u) = Q_R(u)Q_R^{-1}(u_0) \). With the property (4.37) and the above technical assumption (which is standard in the present method \[1\], but is presumably difficult to prove), we arrive at the key result that
\[ t^{(2s)}_1(u) Q(u) = Q(u) t^{(2s)}_1(u) = a_1(u)^N Q(u - \lambda) - a_{2s+1}(u)^N Q(u + \lambda), \tag{4.42} \]

together with \( Q(u)Q(v) = Q(v)Q(u) \). In other words, we have constructed a matrix \( Q(u) \) which commutes with the transfer matrix \( t^{(2s)}_1(u) \), and with \( Q(v) \) for \( v \) different from \( u \). This allows us to choose the diagonal representation for all the matrices appearing in the functional equation (4.42), which then becomes a relation for the eigenvalues of the transfer matrix.

### 4.3 Bethe ansatz-type equations

We can deduce the functional form of \( Q(u) \) in its diagonal representation (written also as \( Q(u) \), with abuse of notation) by examining Eq. (4.42) in the limits \( u \to \pm \infty \). From Eq. (3.4) the dominant term in the large \( u \) limit in each entry of \( R^{(1)}(u) \) is \( \sim e^u \). Therefore from Eq. (3.7) the dominant term in each entry of \( t^{(2s)}_1(u) \) is \( \sim e^{(N-1)u} \) (rather
than \( e^{Nu} \) because of the twist matrix \( F^{(l)} \). On general grounds the large \( u \) behaviour of \( Q(u) \) is argued to be \( \sim e^{ru} \). Substituting into Eq. (4.42) we find that \( r \) must be \( sN \). A similar argument holds for \( u \to -\infty \) and we are led to the form

\[
Q(u) \sim e^{-sNu} + \cdots + e^{sNu}
\]

\[
= e^{-sNu}(c_0 + c_1 e^{u} + \cdots + c_{2sN} e^{2sNu}),
\]

which can be parametrized alternatively as

\[
Q(u) = \prod_{j=1}^{2sN} \sinh \left( \frac{u - u_j}{2} \right),
\]

(4.43)

after taking out an irrelevant overall \( u \)-independent factor. The eigenvalue expression for the transfer matrix can then be written as

\[
\Lambda^{(2s)}_1(u) = \sinh(u + 2s\lambda)^N \prod_{j=1}^{2sN} \frac{\sinh[\frac{1}{2}(u - u_j - \lambda)]}{\sinh[\frac{1}{2}(u - u_j)]} - \sinh(u)^N \prod_{j=1}^{2sN} \frac{\sinh[\frac{1}{2}(u - u_j + \lambda)]}{\sinh[\frac{1}{2}(u - u_j)]}.
\]

(4.44)

The requirement of analyticity of \( \Lambda^{(2s)}_1(u) \) then imposes the following conditions on the “Bethe ansatz roots” \( u_j \):

\[
\left( \frac{\sinh(u_k + 2s\lambda)}{\sinh(u_k)} \right)^N = \prod_{j=1}^{2sN} \frac{\sinh[\frac{1}{2}(u_k - u_j + \lambda)]}{\sinh[\frac{1}{2}(u_k - u_j - \lambda)]}, \quad k = 1, \ldots, 2sN.
\]

(4.45)

In the spin-\( \frac{1}{2} \) case, we recover from (4.44) and (4.45) the eigenvalue expression and Bethe ansatz equations obtained in [7]. For general \( s \), substitution of (4.44) into the generating function (3.10) yields the eigenvalue expression for the transfer matrix \( \Lambda^{(2s)}_l(u) \) in terms of the Bethe ansatz roots \( u_j \), namely

\[
\Lambda^{(2s)}_l(u) = A_l(u)^N \prod_{k=1}^{2sN} \frac{\sinh[\frac{1}{2}(u - u_k - \lambda)]}{\sinh[\frac{1}{2}(u - u_k + (l - 1)\lambda)]} + (-)^{j+l} \sum_{j=1}^{l-1} A_j(u)^N \times \prod_{k=1}^{2sN} \frac{\sinh[\frac{1}{2}(u - u_k + l\lambda)] \sinh[\frac{1}{2}(u - u_k - \lambda)]}{\sinh[\frac{1}{2}(u - u_k + j\lambda)] \sinh[\frac{1}{2}(u - u_k + (j - 1)\lambda)]} + (-)^j A_0(u)^N \prod_{k=1}^{2sN} \frac{\sinh[\frac{1}{2}(u - u_k + l\lambda)]}{\sinh[\frac{1}{2}(u - u_k)]},
\]

(4.46)

where

\[
A_j(u) = \prod_{p=j}^{l-1} \sinh(u + p\lambda) \prod_{p=0}^{j-1} \sinh[u + (2s + p)\lambda].
\]

(4.47)
The eigenvalue expression for the spin-s XXZ Hamiltonian (3.11) follows from \( \Lambda_{2s}(u) \) in Eq. (4.46): We find the following expression

\[
\frac{d}{du} \log \Lambda_{2s}(u) \bigg|_{u=-(2s-1)\lambda} = N \sum_{j=1}^{2s} \coth(j\lambda) + \sum_{k=1}^{2sN} \frac{\sinh(s\lambda)}{\sinh[\frac{1}{2}(2s\lambda + u_k)] \sinh(\frac{1}{2}u_k)},
\]

with the Bethe ansatz roots \( u_k \) determined from Eq. (4.45).

We close this section with a few remarks on more general non-diagonal twists for the transfer matrix. Namely, consider

\[
\tilde{t}_1^{(2s)}(u) = \text{tr}_a R_{u_1}^{(1,2s)}(u) \cdots R_{u_N}^{(1,2s)}(u) \hat{F}^0, \tag{4.49}
\]

with

\[
\hat{F}^0 \equiv \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}.
\]

We find the relationship

\[
\det\left( \tilde{t}_1^{(2s)}(u) - \sqrt{\alpha\beta}\lambda \right) \propto \det\left( t_1^{(2s)}(u) - \lambda \right) \tag{4.50}
\]

between the characteristic polynomials for the two transfer matrices, giving the eigenvalue expression

\[
\tilde{\Lambda}_1^{(2s)}(u) = \sqrt{\alpha\beta} \Lambda_1^{(2s)}(u), \tag{4.51}
\]

with \( \Lambda_1^{(2s)}(u) \) given by Eq. (4.44). The twist matrix \( \hat{F}^0 \) generates recursively the twist matrices \( \hat{F}^{(l)} \), with the matrix elements of \( \hat{F}^{(l)} \) given by \( \hat{F}^{(l)}_{i,j} = \delta_{i+j,l+2}\alpha^{l+1-i}\beta^{i-1}, \)

through the decomposition

\[
D_{12}^{(l)} \hat{F}^{(l)} \hat{F}^0 \hat{F}^{(l)} D_{12}^{(l)-1} = \begin{pmatrix} 0 & \lambda \lambda^{-1} \\ -\lambda^{-1} \lambda & 0 \end{pmatrix}, \tag{4.52}
\]

which replaces Eq. (3.8) when \( \alpha, \beta \) are different from 1. The auxiliary transfer matrix \( \tilde{t}_1^{(2s)} \) is then relevant to the twist matrix \( \hat{F}^{(2s)} \) for the spin-s XXZ chain corresponding to boundary conditions

\[
S_{N+1}^- = -S_1^-, \quad S_{N+1}^+ = \frac{\beta}{\alpha} S_1^-, \quad S_{N+1}^- = \frac{\alpha}{\beta} S_1^+, \tag{4.53}
\]

generalizing Eq. (2.11) to arbitrary spin-s. By Eq. (4.51) the Hamiltonians corresponding to these boundary conditions share the eigenvalue expression Eq. (4.48) and Bethe ansatz equations Eq. (4.45).
5 Discussion

In this paper we have studied integrable toroidal boundary conditions for vertex models and related spin chains, concentrating on boundary conditions specified by non-diagonal twists. In particular, we have applied the “T-Q” functional relation method [1] to obtain the transfer matrix eigenvalue expression and Bethe ansatz-type equations for the spin-$s$ XXZ Heisenberg chain, thereby generalizing a recent result [7] for spin-$\frac{1}{2}$. A detailed analysis of these Bethe ansatz equations should confirm various conjectures available [10, 20] for the critical properties of these models.

The primary reason why the abovementioned functional relation method is applicable is because the $R$-matrix giving rise to the transfer matrix $t_s^{(2s)}$ has the “pair-propagation through a vertex” property. Such considerations make sense in the first place because of the two-dimensional nature of the auxiliary space $V_a$ associated with this transfer matrix. For more general $R$-matrices such as those given in Refs. [4, 5], where the auxiliary space is no longer two-dimensional, the appropriate generalization is not so clear. Therefore the diagonalization of the transfer matrices for such models in the case of non-diagonal twists remains an open problem.

Acknowledgements

We thank R. J. Baxter for a collaboration on the six-vertex model which led to this work. We are also grateful to V. Rittenberg for helpful comments and suggestions. This work is supported by the Australian Research Council.

Appendices

A Partial proof of Conjecture 1

In this appendix we study the inner product $\langle t_q^{\sigma'}(-u-2\lambda s) \cdot q_\sigma(v) \rangle$ of Conjecture 1, for the case when $\sigma, \sigma'$ are allowed to take only the values $\pm s$. In this case, the function $f^{(p)}(\sigma; u)$ simplifies according to (4.17), accompanied by a simplification of $G_{\sigma_k \sigma_k}^{(k)}(u, v)$ in (4.41) to

$$G_{\sigma_k \sigma_k}^{(k)}(u, v) = \sum_{j=0}^{2s} \left[ \frac{2s}{j} \right] e^{i\lambda \sum_{k} (\sigma_k + \sigma_k')} e^{i(2\sigma_k' \lambda - \frac{s_k v}{s} + \frac{s_k u}{s})}. \quad (A.1)$$
Here \[ \binom{2s}{j} \] is the \( q \)-binomial coefficient defined in the paragraph preceding Lemma 1. By an application of the \( q \)-binomial theorem \((A.8)\) we obtain
\[
G_{\sigma\sigma'}^{(k)}(u, v) = \prod_{j=-s+1/2}^{s-1/2} \left( 1 + e^{2j\lambda+\lambda} \sum_{k=1}^{k} (\sigma + \sigma') e^{2\sigma_k' \lambda - \frac{\sigma_k' u}{2} + \frac{\sigma_k' v}{2}} \right), \quad (A.2)
\]
and therefore
\[
t^q \sigma'(-u - 2\lambda s) \cdot q \sigma(v) = \prod_{k=1}^{N} \prod_{j=-s+1/2}^{s-1/2} \left( e^{\frac{\sigma_k' u}{2} - \frac{\sigma_k' v}{2} - \lambda \sigma_k'} + e^{-\frac{\sigma_k' u}{2} + \frac{\sigma_k' v}{2} + \lambda \sigma_k' + 2j\lambda + \lambda \sum_{i=j+1}^{k} (\sigma_i + \sigma_i')} \right), \quad (A.3)
\]
Suppose now there exists a pair \( \sigma_p, \sigma'_p \) such that \( \sigma_p + \sigma'_p = 0 \). Then the terms in Eq. \((A.3)\) which involve \( \sigma_p \) can be seen to be symmetric in \((u, v)\), whereas the terms which do not involve \( \sigma_p \) can be seen (after some relabelling) to be expressible in the form \((A.3)\) with \( N \to N - 1 \). Hence we need only consider the case where \( \sigma_i = \sigma'_i \) for all \( i \), which simplifies to
\[
G_N(\sigma_1, \ldots, \sigma_N) \equiv t^q \sigma(-u - 2\lambda s) \cdot q \sigma(v) = e^{-2\lambda s \sum_{k=1}^{N} \sigma_k} \prod_{k=1}^{N} \prod_{j=-s+1/2}^{s-1/2} \left( e^{\frac{\sigma_k' u}{2} - \frac{\sigma_k' v}{2} - \lambda \sigma_k'} + e^{-\frac{\sigma_k' u}{2} + \frac{\sigma_k' v}{2} + \lambda \sigma_k' + 2j\lambda + \lambda \sum_{i=j+1}^{k} (\sigma_i + \sigma_i')} \right). \quad (A.4)
\]
If \( \sigma_k + \sigma_{k+1} = 0 \) then from Eq. \((A.4)\) we can write \( G_N(\sigma_1, \ldots, \sigma_k, -\sigma_k, \ldots, \sigma_N) = G_{N-2}(\sigma_1, \ldots, \hat{\sigma}_k, \sigma_{k+1}, \ldots, \sigma_N) \) multiplied by a function symmetric in \((u, v)\). This is true for \( 1 \leq k < N \). The only remaining case is therefore \( \sigma_1 = \cdots = \sigma_N \). But from Eq. \((A.4)\) we again have \( G_N(\sigma_1, \ldots, \sigma_{N-1}, \sigma_1) = G_{N-2}(\sigma_2, \ldots, \sigma_{N-1}) \) up to a function symmetric in \((u, v)\). We now check explicitly that \( G_1(\sigma_1) \) and \( G_2(\sigma_1, \sigma_2) \) are symmetric functions in \((u, v)\) for all choices of \((\sigma_1, \sigma_2)\). By induction on \( N \), it follows that Eq. \((A.4)\) is symmetric in \((u, v)\) for all choices of \( \sigma_i \), thereby proving that the inner product \((A.38)\) is symmetric in \((u, v)\) for all choices of \( \sigma_i, \sigma'_i \in \{\pm s\} \), as claimed in the text.

**B Relation to the Hofstadter problem**

The Bethe ansatz equations \((A.45)\) for the case when \( N = 1 \) have already made a surprising appearance \([22]\) in a quantum mechanical context. This is in relation to the so-called Hofstadter problem of Bloch electrons on a square lattice in a magnetic field. More specifically, in the one-electron problem the Hamiltonian can be expressed in terms of the generators of the group of magnetic translations, which in turn can be related to \( U_q(su(2))\)-generators at \( q = e^{i\pi P/Q} \) (with \( P, Q \) mutually prime integers) in
the $Q = (2s + 1)$-dimensional representation. The eigenvalue expression and associated Bethe ansatz equations were obtained in [22] using the functional Bethe ansatz. In this Appendix we recover these equations from our general $N$ results. On the one hand, it is a check on our results. On the other, it serves to further elucidate the connection found in Ref. [22].

Define then the “Hamiltonian” $\mathcal{H}$ by

$$\mathcal{H} = 2i \text{tr}_a F^{(1,2s)}(u - s\lambda) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (B.1)$$

Comparison with Eq. (3.7) reveals immediately that

$$\mathcal{H} = 2i t^{(2s)}_1 (u - s\lambda), \quad (B.2)$$

where the auxiliary transfer matrix on the rhs is for lattice size $N = 1$. On the other hand, it is clear from Eqs. (3.1) and (B.1) that $\mathcal{H}$ takes the form

$$\mathcal{H} = 2i \sinh(\lambda) \left(S^+ + S^-\right) \quad (B.3)$$

in terms of $U_q(su(2))$ generators. From the eigenvalue expression Eq. (4.44) for the auxiliary transfer matrix, we find the eigenvalue $E$ of $\mathcal{H}$ to be (after a shift in $u_k$)

$$E = 2i \sinh(u + s\lambda) \prod_{j=1}^{2s} \frac{\sinh[\frac{1}{2}(u - u_j - \lambda)]}{\sinh[\frac{1}{2}(u - u_j)]}$$

$$- 2i \sinh(u - s\lambda) \prod_{j=1}^{2s} \frac{\sinh[\frac{1}{2}(u - u_j + \lambda)]}{\sinh[\frac{1}{2}(u - u_j)]}, \quad (B.4)$$

together with the Bethe ansatz equations

$$\frac{\sinh(u_k + s\lambda)}{\sinh(u_k - s\lambda)} = \prod_{j=1}^{2s} \frac{\sinh[\frac{1}{2}(u_k - u_j + \lambda)]}{\sinh[\frac{1}{2}(u_k - u_j - \lambda)]}. \quad (B.5)$$

Writing $q = e^\lambda$, the Bethe ansatz equations (B.5) can be written in the alternative form

$$\frac{e^{2u_k} - q^{-2s}}{e^{2u_k} - q^{-2s} - 1} = \prod_{j=1}^{2s} \frac{e^{u_k} - e^{u_j} q - e^{u_j} q^{-1}}{e^{u_k} - e^{u_j} q}, \quad k = 1, \ldots, 2s. \quad (B.6)$$

It is clear from Eq. (B.3) that $\mathcal{H}$ is independent of $u$. Evaluating $E$ at $u \to \infty$ we obtain the expression

$$E = -i(q - q^{-1}) \sum_{j=1}^{2s} e^{u_j}. \quad (B.7)$$
The Hamiltonian $H$ has the following interpretation: There exists a $q$-difference operator realization of the $(2s+1)$-dimensional representation of $U_q(su(2))$ given by

\begin{align}
q^{\pm s} \Psi(z) &= q^{-s} \Psi(qz) \quad \text{(B.8)} \\
q^{-\frac{1}{2} s} \Psi(z) &= q^s \Psi(q^{-1}z) \quad \text{(B.9)} \\
S^+ \Psi(z) &= z(q - q^{-1})^{-1} \left( q^{2s} \Psi(q^{-1}z) - q^{-2s} \Psi(qz) \right) \quad \text{(B.10)} \\
S^- \Psi(z) &= -z^{-1} (q - q^{-1})^{-1} \left( \Psi(q^{-1}z) - \Psi(qz) \right), \quad \text{(B.11)}
\end{align}

with $\Psi(z)$ belonging to the space of polynomials of degree $2s$. In the $q \to 1$ limit, this reduces to the familiar differential operator realization of $su(2)$. In the $q$-difference operator realization, the eigenvalue equation for the Hamiltonian $H$ can be written as

\begin{align}
&i(zq^{2s} - z^{-1}) \Psi(q^{-1}z) + i(z^{-1} - q^{-2s}) \Psi(qz) = E \Psi(z). \quad \text{(B.12)}
\end{align}

Let $q = e^{i\pi P/Q}$ with $Q = 2s + 1$ and consider first the case when $P$ is odd. We have $q^{2s} = q^{-1}$ and the difference equation \(\text{(B.12)}\) becomes

\begin{align}
&-i(zq^{-1} + z^{-1}) \Psi(q^{-1}z) + i(z^{-1} + zq) \Psi(qz) = E \Psi(z), \quad \text{(B.13)}
\end{align}

while the Bethe ansatz equations determining the eigenvalue $E$ become

\begin{align}
\frac{z_k^2 + q}{z_k^2 q + 1} = -\prod_{j=1}^{Q-1} \frac{z_k q - z_j}{z_k - z_j q}, \quad k = 1, \ldots, Q - 1, \quad \text{(B.14)}
\end{align}

with $z_j \equiv e^{i\pi j}$. When $P$ is even we have $q^{2s} = q^{-1}$ and we will again arrive at the difference equation \(\text{(B.13)}\) if we replace $H$ in Eq. \(\text{(B.3)}\) by $\tilde{H} = 2i \sinh(\lambda) (S^- - S^+)$. This corresponds, in Eq. \(\text{(B.1)}\), to replacing the twist matrix \(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) by \(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\), which is also an integrable twist. By Eq. \(\text{(B.10)}\), we find that the eigenvalue expression is $i$ times Eq. \(\text{(B.7)}\) with the same set of Bethe ansatz equations \(\text{(B.6)}\). The Bethe ansatz equations simplify, upon setting $e^{i\pi j} \equiv iz_j$, to the equations \(\text{(B.14)}\) without the negative sign in front of the product on the rhs. The odd and even $P$ Bethe ansatz equations can therefore be unified as

\begin{align}
\frac{z_k^2 + q}{z_k^2 q + 1} = q^Q \prod_{j=1}^{Q-1} \frac{z_k q - z_j}{z_k - z_j q}, \quad k = 1, \ldots, Q - 1, \quad \text{(B.15)}
\end{align}

with the eigenvalue expression becoming

\begin{align}
E = -i(q - q^{-1}) \sum_{j=1}^{Q-1} z_j. \quad \text{(B.16)}
\end{align}

The difference equation \(\text{(B.13)}\) is one form of Harper’s equation, which appears in the Hofstadter problem with magnetic flux $\Phi = 2\pi P/Q$ per plaquette. Apart from a sign factor\footnote{We are in agreement with Ref. \[23\] on the sign discrepancy.} in the Bethe ansatz equations, the solution \(\text{(B.16)}\) and \(\text{(B.13)}\) for this problem was first obtained in \[22\] using the functional Bethe ansatz.
References

[1] Baxter, R.J.: Exactly solved models in Statistical Mechanics. Academic Press, 1982

[2] Faddeev, L.D.: Completely integrable quantum models of field theory. Sov. Sci. Reviews 1, (1979).

[3] Kulish, P.P., Sklyanin, E.K.: Quantum spectral transform method: Recent developments. In: J. Hietarinta and C. Montonen, (eds.) Lecture Notes in Physics vol 151. Springer-Verlag, 1982, pp 61–119

[4] Jimbo, M.: Quantum $R$ matrix for the generalized Toda system. Commun. Math. Phys. 102, 537–547 (1986)

[5] Bazhanov, V.V.: Integrable quantum systems and classical Lie algebras. Commun. Math. Phys. 113, 471–503 (1987)

[6] de Vega, H.J.: Families of commuting transfer matrices and integrable models with disorder. Nucl. Phys. B 240, 495–513 (1984)

[7] Batchelor, M.T., Baxter, R.J., O’Rourke, M.J., Yung, C.M.: Exact solution and interfacial tension of the six-vertex model with anti-periodic boundary conditions, ANU preprint MRR 012-95

[8] Kirillov, A.N., Reshetikhin, N.Yu.: Exact solution of the integrable XXZ Heisenberg model with arbitrary spin: I. The ground state and the excitation spectrum. J. Phys. A 20, 1565–1585 (1987)

[9] Alcaraz, F.C., Barber, M.N., Batchelor, M.T.: Conformal invariance, the XXZ chain and the operator content of two-dimensional critical systems. Ann. Phys. (N.Y.) 182, 280–343 (1988)

[10] Alcaraz, F.C., Baake, M., Grimm, U., Rittenberg, V.: Operator content of the XXZ chain. J. Phys. A 21, L117–120 (1988)

[11] Zamolodchikov, A.B., Fateev, V.A.: A model factorized S-matrix and an integrable spin-1 Heisenberg ferromagnet. Sov. J. Nucl. Phys. 32, 298–303 (1980)

[12] Kulish, P.P., Reshetikhin, N. Yu., Sklyanin, E.K.: Yang-Baxter equation and representation theory I. Lett. Math. Phys. 5, 393–403 (1981)

[13] Kulish, P.P., Reshetikhin, N. Yu.: Quantum linear problem for the sine-Gordon equation and higher representations. J. Sov. Math. 23, 2435–2441 (1983)

[14] Sogo, K., Akutsu, Y., Abe, T.: New factorized $S$-matrix and its applications to exactly solvable $q$-state model. I. Prog. Theor. Phys. 70, 730–738 (1983)

[15] Sogo, K.: Ground state and low-lying excitations in the Heisenberg XXZ chain of arbitrary spin $s$. Phys. Lett. A 104, 51–54 (1984)
[16] Jimbo, M.: A $q$-difference analogue of $U_q(g)$ and the Yang-Baxter equation. Lett. Math. Phys. 10, 63–69 (1985)

[17] Babujian, H.M., Tsvelick, A.M.: Heisenberg magnet with an arbitrary spin $s$ and anisotropic chiral field. Nucl. Phys. B 265, 24–44 (1985)

[18] Alcaraz, F.C., Martins, M.J.: The spin-$s$ XXZ quantum chain with general toroidal boundary conditions. J. Phys. A 23, 1439–1476 (1990)

[19] Klümper, A., Batchelor, M.T., Pearce, P.A.: Central charges of the 6- and 19-vertex models with twisted boundary conditions. J. Phys. A 24, 3111–3133 (1991)

[20] Baranowski, D., Rittenberg, V.: The operator content of the ferromagnetic and anti-ferromagnetic spin one Zamolodchikov-Fateev quantum chain. J. Phys. A 23, 1029–1033 (1990)

[21] Takhtadzhian, L.A., Faddeev, L.D.: The quantum method of the inverse problem and the Heisenberg XYZ model. Russian Math. Surveys 34, 11–68 (1979)

[22] Wiegmann, P.B., Zabrodin, A.V.: Bethe-Ansatz for the Bloch Electron in Magnetic Field. Phys. Rev. Lett. 72, 1890–1893 (1994)

[23] Faddeev, L.D., Kashaev, R.M.: Generalized Bethe Ansatz Equations for Hofstadter Problem. Preprint HU-TFT-93-63, [hep-th/9312133](https://arxiv.org/abs/hep-th/9312133)