Hamiltonian quantization of solitons in the $\phi^4_{1+1}$ quantum field theory. I. The semiclassical mass shift.

David Stuart
Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 OWA, England
email:dmas2@cam.ac.uk

Abstract
We carry out the soliton sector quantization of the $\phi^4_{1+1}$ theory in the semiclassical limit, deriving the nonrelativistic Schrödinger equation as an equation describing the limiting soliton dynamics and proving the semiclassical mass shift formula of Dashen, Hasslacher and Neveu, which was computed by another method in [6].

MSC classification: 81T08
Keywords: soliton, $\phi^4$, kink, quantization

1 Introduction
We study of the interaction of a scalar quantum field $\phi$ with a fixed (external) electromagnetic field $A^\text{ext}_\mu dx^\mu$ in two dimensional space-time. The dynamics is determined by the action functional

$$S = \int \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} g^2 \left( \phi^2 - \frac{m^2}{g^2} \right)^2 + \lambda \epsilon_{\mu\nu} \partial_\mu A^\text{ext}_\nu \phi \right) dxdt. $$

The quartic interaction, a double well potential, supports the existence of solitons in the classical theory. The existence of the corresponding quantum theory can be proved by the methods of constructive quantum field theory. The aim is to analyze the dynamics of the soliton in this quantized theory as $g \to 0$, which corresponds to a nonrelativistic limit for the soliton, which has mass which diverges as $g^{-2}$ in this limit. In this first paper attention is focused on the case $\lambda = 0$. We develop the analytical framework for quantizing the theory, identify the appropriate degrees of freedom to describe the soliton and the $g \to 0$ limiting dynamics and provide an interpretation and proof of the Dashen-Hasslacher-Neveu semiclassical mass correction formula from [6], see (1.19)-(1.20). We refer to [14] and [5, Chapter 6] for a general physical discussion of quantum solitons and [11, §23.8] for a review of mathematical work on solitons in the context of constructive field theory, and in particular to [3] for bounds on the soliton mass.

Classical Theory. The Hamiltonian is the functional

$$H(\phi, \pi) = \int H(\phi, \pi) dx, \quad H(\phi, \pi) = \frac{1}{2} (\pi^2 + \partial_x \phi^2) + \mathcal{U}(\phi).$$

The potential function $\mathcal{U}$ is the double-well potential

$$\mathcal{U}(\phi) = \frac{m^4}{2g^2} \left( 1 - \frac{\phi^2}{m^2} \right)^2 = \frac{1}{2} g^2 \left( \phi^2 - \Phi_0^2 \right)^2.$$  

The parameters $m, g$ are assumed to be positive numbers. The functional is well-defined as a non-negative number, possibly equal to $+\infty$, on pairs $(\phi, \pi) \in H^1_{\text{loc}} \times L^2_{\text{loc}}$; the pairs for which $H(\phi, \pi) < \infty$ are the finite energy configurations. The two classical vacua are $\pm \Phi_0$, where $\Phi_0 = m/g$. Clearly the constant configuration $(\phi, \pi) = (\Phi_0, 0)$ minimizes the value of the Hamiltonian energy functional amongst all finite energy configurations which satisfy

$$\lim_{|x| \to \infty} \phi(x) = \Phi_0;$$  

(1.3)
a similar assertion holds for \((-\Phi_0,0)\). Expanding around these vacua leads to the Hamiltonians

\[
H(\pm\Phi_0 + \varphi, \pi) = \int \left[ \frac{1}{2} (\pi^2 + \partial_x \varphi^2 + 4m^2 \varphi^2) \pm 2mg\varphi^3 + \frac{1}{2} g^2 \varphi^4 \right] dx. \tag{1.4}
\]

The quadratic part of this Hamiltonian, namely

\[
H_0^{ac}(\varphi, \pi) = \int \frac{1}{2} (\pi^2 + \partial_x \varphi^2 + 4m^2 \varphi^2) dx, \tag{1.5}
\]
describes the quantum mechanics of non-interacting relativistic scalar bosons of mass 2m - these bosons are the fundamental particles of the theory. The cubic and quartic terms describe interactions between these particles, the strength being determined by the (positive) coupling constant \(g\).

The classical soliton,

\[
\Phi_S(x) = \frac{m}{g} \tanh mx, \quad \Pi_S(x) = 0, \tag{1.6}
\]
is a solution of the classical Hamiltonian equations of motion

\[
\dot{\pi} - \partial_x^2 \dot{\phi} + U'(\phi) = 0, \quad \dot{\phi} - \pi = 0. \tag{1.7}
\]
The soliton has the property that \(\Phi_S\) interpolates between the two vacua as its asymptotic boundary values, i.e.,

\[
\Phi_S(x) \to \pm\Phi_0 \quad \text{as} \quad x \to \pm\infty. \tag{1.8}
\]
The soliton minimizes the value of the Hamiltonian energy functional amongst all finite energy configurations which satisfy these boundary conditions. However, the soliton is not unique due to translation invariance: the set of energy minimizers is \(\{\Phi_S(\cdot - \xi, 0)\}_{\xi \in \mathbb{R}}\). The energy of an energy minimizer equals the minimum value of \(H\) on the set of finite energy configurations verifying (1.8); this minimum value is the classical rest mass of the soliton, given by

\[
M_{cl} = \frac{4m^3}{3g^2} = \frac{M_{cl}}{g^2}, \quad M_{cl} = \frac{4m^3}{3}. \tag{1.9}
\]

Expanding around the soliton leads to the Hamiltonian

\[
H_g^{sol}(\varphi, \pi) \equiv H(\Phi_S + \varphi, \pi) = \frac{M_{cl}}{g^2} + \int \frac{1}{2} (\pi^2 + \partial_x \varphi^2 + 4m^2 \varphi^2 - 6m^2 \tanh^2 mx \varphi^2) dx + \int (2mg\tanh mx \varphi^3 + \frac{1}{2} g^2 \varphi^4) dx. \tag{1.10}
\]

This Hamiltonian describes fluctuations around the basic soliton, centered at the origin. These fluctuations are determined infinitesimally by the linearized operator \(K = -\partial_x^2 + 4m^2 - 6m^2 \tanh^2 mx\). As discussed below, this operator has a one dimensional kernel which reflects the fact that physically the soliton is able to move along the orbit of the translation group without any “energetic cost”, i.e. dynamically the parameter \(\xi = \xi(t)\) becomes time-dependent, and one studies solutions of the form

\[
\Phi_S(x - \xi(t)) + \varphi(t, x). \tag{1.11}
\]

Now Lorentz invariance implies the existence of exact solutions of the classical equations of motion (1.7) of the form \(\Phi_S\left(\frac{x - ut + x}{\sqrt{1-u^2}}\right)\), in which the soliton moves along a straight line. More importantly for present purposes, this behaviour is actually stable generic behaviour in the low energy limit, and the dynamics can be approximated on appropriate time scales (in the \(H^1 \times L^2\) norm) by the Newtonian equation of motion for a freely moving particle of mass \(M_{cl}\), i.e.,

\[
\dot{\eta} = 0 \quad \text{where} \quad \eta = M_{cl} \xi \quad \text{momentum}. \tag{1.12}
\]

These types of problems, with some representative theorems, are surveyed in [29]. The inclusion of the mass in (1.12) is a matter of convention here, but in the presence of external potentials is unavoidable. We now discuss how this picture might be expected to be modified in the quantum case.
The quantum field theory for the Hamiltonian \( H_0 \) was constructed by the Hamiltonian method in [8]. With a spatial cut-off the theory admits a Schrödinger representation formulation with respect to a Gaussian measure \( \mu_0 \) on the space of tempered distributions (see Proposition 2.1; see [7] for a review). Moving to the soliton sector via (1.10) corresponds essentially to shifting the field by \( \Phi_S - \Phi_0 \), which is not a Cameron-Martin vector for \( \mu_0 \), and in measure theoretic terms leads to a representation supported on a measure which is singular with respect to the vacuum measure - it will be called the shifted vacuum representation. Construction of the quantum field theory using this as starting point leads to the quantum theory in the soliton, as opposed to vacuum, sector. In fact it is useful to use two different but equivalent representations in the soliton sector to reveal the physics. We now consider what the expected physics is in the limit of small \( g \).

The soliton as a nonrelativistic quantum particle. Firstly, it is to be hoped, that in the limit of small coupling \( g \) the soliton will behave as a quantum particle of mass \( M_{cl} \), as opposed to vacuum, sector. In fact it is useful to use two different but equivalent representations to focus in on the quantum fluctuations. Then, in favorable circumstances, it is to be hoped that the operator \( Q \) can be realized in the Schrödinger picture as the operator of multiplication by \( Q \) on a wave function \( \Psi = \Psi(t, Q) \) whose evolution can be approximated for small \( g \).

\[
\Phi_S(x - \xi - X) + \varphi \approx \Phi_S(x - \xi) - \Phi_S'(x - \xi)X + \varphi
\]  

where \( \xi \) is a classical c-number giving the location of the classical solution about which we quantize, while \( X \) represents an \( O(g) \) quantum fluctuation in its location. We will study quantum dynamics around nontrivial classical motions \( \xi(t) \) in a succeeding article, but for present purposes we will take \( \xi = 0 \) and rescale \( X = gQ \) to focus in on the quantum fluctuations. Then, in favorable circumstances, it is to be hoped that the operator \( Q \) can be realized in the Schrödinger picture as the operator of multiplication by \( Q \) on a wave function \( \Psi = \Psi(t, Q) \) whose evolution can be approximated for small \( g \) by the equation

\[
i \frac{\partial \Psi}{\partial t} + \frac{1}{2M_{cl}} \frac{\partial^2 \Psi}{\partial Q^2} = 0.
\]  

(1.14)

The momentum operator conjugate to \( Q \) is \( P = -i \frac{\partial}{\partial Q} \). We will make use of the Gauss-Hermite wave packet solutions to this equation, derived in [11], which are given in terms of the Hermite polynomials \( \text{He}_n(x) = (-1)^n e^{x^2} \frac{\partial^n}{\partial x^n} e^{-x^2} \) by

\[
\chi_n(t, Q; \sigma) = \frac{1}{\sqrt{n! \sqrt{2\pi}}} \int \frac{e^{-\frac{(t + i\tau)^2}{4\sigma^2(t)}}}{\sqrt{\sigma}} \frac{\exp \left[ i\frac{tQ^2}{4\sigma(t)} - \frac{Q^2}{4\sigma(t)^2} - \frac{i}{4}(2n + 1)\pi \right]}{\text{He}_n \left( \frac{Q}{\sigma(t)} \right)}
\]  

(1.15)

where \( \sigma, \tau \) are real positive constants related by \( 2\sigma^2 = \tau/M_{cl} \), and \( \sigma(t)^2 = (1 + t^2/\tau^2)\sigma^2 \) is the variance which increases with \( t \). With the mass \( M_{cl} \) given, \( \sigma \) is determined by the initial variance parameter \( \sigma \), which will only be indicated explicitly as an argument for \( \chi \) when necessary. The combinatorial factor ensures normalization \( \int |\chi_n(t, Q)|^2 dQ = 1 \) at all times \( t \), and the \( \{\chi_n(t, Q)\}_{n=0}^\infty \) form an orthonormal basis for \( L^2(dQ) \) at each fixed \( t \). The phase factor is chosen so that \( \chi_n(0, Q) \) is real. In particular \( \chi_n(0, Q) = \sigma^{-\frac{1}{2}} \chi_n^0(Q/\sigma) \) where \( \chi_n^0(y) = (2\pi)^{-\frac{1}{4}} (n!)^{-\frac{1}{2}} \exp[-y^2/4] \text{He}_n(y) \).

Bosons in the soliton background. In addition to this Schrödinger particle, the quadratic part of the Hamiltonian obtained by expanding around a kink located at the origin, namely,

\[
H_0^{sol}(\varphi, \pi) = \int \frac{1}{2} \left( \pi^2 + \partial_x \varphi^2 + 4m^2 \varphi^2 - 6m^2 \text{sech}^2 mx \varphi^2 \right) dx,
\]  

(1.16)

describes (after second quantization):
• An assembly of bosons moving in the background potential $V(x) = -6m^2\text{sech}^2mx$ created by the soliton itself, described in normal form by the Hamiltonian $\hbar(\omega_*) = \int \omega_k a_k^\dagger a_k dk$ with $\omega_k = \sqrt{4m^2 + k^2}$, which defines a self-adjoint operator on the Fock space $\mathcal{F}_0$ defined in (2.8).

• An oscillatory mode (pulsation of the soliton) of frequency $\omega_d = \sqrt{3}m$, described by harmonic oscillator Hamiltonian $\hbar(\omega_d) = \omega_d a_d^\dagger a_d$. In the Schrödinger picture this determines a self-adjoint operator acting on $L^2(\mathbb{R}, dq)$ in the usual way, see (2.35), with the Gaussian measure $\gamma((2\omega_d)^{-1}) \equiv \exp[-\omega_d q_d^2] dq_d$ of covariance $(2\omega_d)^{-1}$ arising as the square of the ground state.

This expected picture of the quantum field theory in the soliton sector - a quantum particle interacting with a quantum field - is broadly similar to that which appears on quantization of the Abraham model, see [25]. However there is a difference that in the case of the Abraham model a particle-field decomposition is given from the beginning whereas in the present case these features have to be derived using an appropriate choice of solution of the Heisenberg relation. Indeed, use of the shifted vacuum representation - there exists a unitary map $\mathcal{F}_0 \to \mathcal{F}_0$. This representation is unitarily equivalent to the Fock space generated by the modes described in the two items following (1.16); this is formulated precisely in Theorem 1.1, and explained in detail and explicitly in §2.2. This representation is unitarily equivalent to the shifted vacuum representation - there exists a unitary map $\mathcal{F}_0 \to \mathcal{F}_0$. It is constructed using the operator $K = -\partial_x^2 + 4m^2 - 6m^2\text{sech}^2mx$ which appears on linearization about the soliton, and will be referred to as the solitonic representation. All together with this choice of representation of the fields the quadratic part of the Hamiltonian takes the form

\begin{equation}
\hbar_{\text{sol}}^\dagger = \frac{1}{2M_{\text{cl}}} \left(-i \frac{\partial}{\partial Q} \right)^2 + \hbar(\omega_d) + \hbar(\omega_*)
\end{equation}

acting on the space

\begin{equation}
\mathcal{H}_0 = L^2(\mathbb{R}, dq) \otimes \mathcal{F} \quad \text{where} \quad \mathcal{F} = L^2(\mathbb{R}, \gamma((2\omega_d)^{-1})) \otimes \mathcal{F}_0.
\end{equation}

The triple colons indicate normal ordering with respect to this representation. The Hilbert space $\mathcal{F}$ is the Fock space generated by the modes described in the two items following (1.16); this is formulated precisely in Theorem 1.1 and explained in detail and explicitly in §2.2. This representation is unitarily equivalent to the shifted vacuum representation - there exists a unitary map $\mathcal{F}_0 \to \mathcal{F}_0$ such that for all Schwartz test functions $f$

\begin{equation}
\mathcal{V} \circ \exp[i\phi(f)] \circ \mathcal{V}^{-1} = \exp[i\phi(f)] \quad \mathcal{V} \circ \exp[i\pi(f)] \circ \mathcal{V}^{-1} = \exp[i\pi(f)],
\end{equation}

and it is proved in Section 3 that

\begin{equation}
\mathcal{V} \circ \hbar_{\text{sol}}^\dagger \circ \mathcal{V}^{-1} = i\hbar_{\text{sol}}^\dagger + \Delta M_{\text{cl}},
\end{equation}

where $\hbar_{\text{sol}}^\dagger$ means the normal ordered second quantization of (1.16) with respect to the shifted vacuum representation, and $\Delta M_{\text{cl}}$ is the Dashen-Hasslacher-Neveu semiclassical mass shift

\begin{equation}
\Delta M_{\text{cl}} = m \frac{2\sqrt{3}}{\pi} \frac{3m}{\pi}
\end{equation}

computed by another method in [10]. To explain our results in slightly more detail, the definition and construction of the quantum theory requires three preparatory actions:

1. Choice of solution of the Heisenberg commutation relation in both the vacuum sector and the solitonic sector; in fact in the latter case two different solutions are useful as discussed above.

2. Ultra-violet regularization of the fields in both sectors, carried out in a consistent way (see [3]1). Introduction of a spatial cut-off in the interaction terms of the Hamiltonian.

3. Subtraction of the same counter-terms (see [3]2) for both vacuum sector and solitonic sector Hamiltonians. (The counter-terms used correspond to normal ordering for the vacuum sector.)

The regularization is achieved in all representations via convolution with a smooth function, the ultra-violet cut-off being determined by a positive real number $\kappa$; as $\kappa \to +\infty$ the cut-off is removed. Regarding the third point, the counter-terms are chosen by normal ordering using the vacuum representation (2.10)-(2.17). Following this through in the vacuum sector, taking the formal Hamiltonian (1.4) as starting point, leads to a normal ordered and regularized Hamiltonian $\hbar_{\text{sol}}^\dagger$ acting on the Fock space $\mathcal{F}_0$. In the solitonic sector
we take (1.10) as the starting point. (There is actually some additional freedom, in that the expansion in (1.10) can equally well be carried out with the soliton located at an arbitrary $\xi \in \mathbb{R}$. It is necessary to take advantage of this to extend classical modulation theory to describe soliton motion in dynamically nontrivial situations, but this will not be done in this paper.) As mentioned previously, we use convolution and subtract the same counter-terms as in the vacuum sector; this corresponds to normal ordering in the solitonic sector using the shifted vacuum representation (2.20)–(2.30), see §4.2 and leads to the study of a Hamiltonian

$$\begin{align*}
\mathcal{H}_{g,\kappa}^\text{sol} &= \frac{M_{\text{cl}}}{g^2} + \int \mathcal{H}_{g,\kappa}^0 \, dx,
\mathcal{H}_{g,\kappa}^0 &= \mathcal{H}_{0,\kappa}^0 + \mathcal{H}_{1,\kappa}^0.
\end{align*}$$

(The double colon indicates normal ordering with respect to the shifted vacuum representation of the fields acting on $\mathcal{F}_0$, while the triple colon is used for normal ordering in the solitonic representation.) We study the Schrödinger evolution with initial data $\Psi_0 \in \mathcal{F}_0$. In order to obtain the simple normal form (1.17) for the quadratic part of the Hamiltonian, and hence uncover the dynamics in the limit $g \to 0$, it is necessary to move to the representation (2.40)–(2.41) on the Hilbert space $\mathcal{H}$ via the unitary transformation $\bar{V} : \mathcal{F}_0 \to \mathcal{H}$ obtained in Theorem 2.7. Even in the absence of an external field this has dynamical consequences, yielding a precise interpretation of the Dashen-Hasslacher-Neveu semiclassical mass correction formula.

**Theorem 1.1.** In the limit $\kappa \to +\infty$ the operators $\mathcal{H}_{g,\kappa}^\text{sol}$ determine a self-adjoint operator $\mathcal{H}_g^\text{sol}$ on $\mathcal{F}_0$ which is bounded below and determines a strongly continuous one-parameter unitary group via the Stone theorem. Let $[-t_1(g), t_1(g)]$ be a time interval given for each $g > 0$ which satisfies $\lim_{g \to 0} g^2 t_1(g) = 0$, then

$$\lim_{g \to 0} \sup_{|t| \leq t_1(g)} \left\| e^{i\Theta(t)} \bar{V} \exp[-i t : \mathcal{H}_g^\text{sol} :] \Psi_0(0) - \exp[-i t : \mathcal{H}_0^\text{sol} :] \bar{V} \Psi_0(0) \right\| = 0, \quad (1.21)$$

where

$$\Theta = \frac{M_{\text{cl}}}{g^2} + \Delta \mathcal{M}_{\text{scl}}.$$ 

Here $\Delta \mathcal{M}_{\text{scl}}$ is as above in (1.20).

This is proved in Section 4.

**Remark 1.2.** It should be emphasized that while the quadratic Hamiltonian $h(\omega_u)$ describing bosons in the soliton background looks the same as that for free bosons, the interactions in physical space are affected by the presence of the soliton. This shows up, for example, in the formula $b(U) = \int (2\omega_\xi)^{-\frac{1}{2}} a_\xi^\dagger U(\xi) dk$ for the operator creating a boson in state determined by a Schwartz function $U$ (in the continuous spectral subspace); see (2.33) for this and related formulæ. The notation $\bar{U}$ indicates the distorted Fourier transform $\mathcal{F}_\text{cl}$, which appears in place of the Fourier transform in the free case. Now $\bar{U}$ is constructed from the scattering analysis of the linearized operator $K$, see [14] and the Appendix, and depends on the background soliton. Insertion of such dependencies into the Hamiltonian ensures that the actual physical space dynamics of the bosons does depend on the presence of the soliton.

### 1.1 Notation

The pairing between a tempered distribution $\Phi \in \mathcal{S}'(\mathbb{R})$ and a test function $f \in \mathcal{S}(\mathbb{R})$ is written either $\langle \Phi, f \rangle$ or $(\Phi, f)$. We write the Fourier transform as $\hat{f}(k) = (2\pi)^{-1/2} \int e^{-ikx} f(x) \, dx$, and the distorted Fourier transform $U \mapsto \bar{U}$ is given in (2.33).

Given a non-negative self-adjoint operator $A$ with domain $\text{Dom} A \subset L^2(\mathbb{R})$, we write $(f, g)_A = (f, Ag)_{L^2} = (A^{\frac{1}{2}} f, A^{\frac{1}{2}} g)_{L^2}$, for the corresponding symmetric bilinear form defined on $\text{Dom} A^{\frac{1}{2}} \times \text{Dom} A^{\frac{1}{2}}$. In the particular case $A = K_0 = 4m^2 - \partial^2$ this procedure gives the Sobolev $H^1$ inner product, and fractional powers give the general $H^s$ Sobolev inner products. In particular, the case $H^{\frac{1}{2}}$ arises from the inner product

$$\langle \phi, \psi \rangle_{K_0^{\frac{1}{2}}} = \int \overline{\psi}(\xi)(4m^2 + \xi^2)^{\frac{1}{2}} \psi(\xi) d\xi, \quad (1.22)$$

which, together with its dual inner product defined as $\langle f, g \rangle_{C_0^{\infty}}$, where $C_0 = K_0^{-1}$, appears in the Schrödinger representation for the free field of mass $2m$. Given a covariance operator $C$ on a vector space we write $\gamma(C)$ for the Gaussian measure with covariance $C$; for example,
• If $s$ is a positive number then $\gamma(s^2) = (2\pi s^2)^{-1/2} \exp[-x^2/2s^2]dx$ on $\mathbb{R}$, so that in terms of the wave packets $\Phi(0,x)^2dx$ with $s^2 = \sigma^2$;
• if $C$ is a continuous and nondegenerate bilinear form on $S(\mathbb{R})$, then $\gamma(C)$ is the measure on $S'(\mathbb{R})$ with Fourier transform $S'(\mathbb{R}) \ni f \mapsto \exp(-f,Cf)/2$.

We write the quantum fields describing fluctuations around the soliton as $(\varphi, \pi)$ in the shifted vacuum representation (2.29)-(2.31), but $(\Phi, \Pi)$ in the solitonic representation (2.40)-(2.41). The two representations are unitarily related via $\mathcal{V}$ introduced in (2.2). The two Fock spaces, $\mathcal{F}$ and $\mathcal{F}$, are defined in (2.3) and (2.31), with number operators $N_0$ and $\tilde{N}$ in (2.12) and (2.31). Schrödinger representation versions of both representations, indicated by using corresponding bold face fonts $\varphi, \pi, \Phi, \Pi, \mathcal{F}, \mathcal{F}$, etc., for the fields and Fock spaces. The double colon $\mathcal{O}$ (resp. triple colon $\mathcal{O}_i$) is used to indicate an operator normal ordered with respect to the shifted vacuum (resp. solitonic) representation. $D, \mathcal{P}(\Phi), \mathcal{P}(\Phi)$ are dense subspaces defined just after (2.3) and in (2.2) and a variant $\hat{P}$ is defined in (3.9). Regularized fields $\varphi, \pi, \Phi, \Pi$, etc., are all defined by convolution with an approximate identity $\delta_0, \kappa$ introduced just after (2.8) and in §2.2, and a variant $\delta_0$ is introduced in §2.2. The two Fock spaces, $\mathcal{F}$ and $\mathcal{F}$, are defined in (2.31) and (2.46). Schrödinger representation versions of both representations, indicated by using corresponding bold face fonts $\varphi, \pi, \Phi, \Pi, \mathcal{F}, \mathcal{F}$, etc., for the fields and Fock spaces. The double colon $\mathcal{O}$ (resp. triple colon $\mathcal{O}_i$) is used to indicate an operator normal ordered with respect to the shifted vacuum (resp. solitonic) representation. $D, \mathcal{P}(\Phi), \mathcal{P}(\Phi)$ are dense subspaces defined just after (2.3) and in (2.2) and a variant $\hat{P}$ is defined in (3.9). Regularized fields $\varphi, \pi, \Phi, \Pi$, etc., are all defined by convolution with an approximate identity $\delta_0, \kappa$ introduced just after (2.8) and in §2.2, and a variant $\delta_0$ is introduced in §2.2. The two Fock spaces, $\mathcal{F}$ and $\mathcal{F}$, are defined in (2.31) and (2.46). Schrödinger representation versions of both representations, indicated by using corresponding bold face fonts $\varphi, \pi, \Phi, \Pi, \mathcal{F}, \mathcal{F}$, etc., for the fields and Fock spaces. The double colon $\mathcal{O}$ (resp. triple colon $\mathcal{O}_i$) is used to indicate an operator normal ordered with respect to the shifted vacuum (resp. solitonic) representation. $D, \mathcal{P}(\Phi), \mathcal{P}(\Phi)$ are dense subspaces defined just after (2.3) and in (2.2) and a variant $\hat{P}$ is defined in (3.9). Regularized fields $\varphi, \pi, \Phi, \Pi$, etc., are all defined by convolution with an approximate identity $\delta_0, \kappa$ introduced just after (2.8) and in §2.2, and a variant $\delta_0$ is introduced in §2.2. The two Fock spaces, $\mathcal{F}$ and $\mathcal{F}$, are defined in (2.31) and (2.46). Schrödinger representation versions of both representations, indicated by using corresponding bold face fonts $\varphi, \pi, \Phi, \Pi, \mathcal{F}, \mathcal{F}$, etc., for the fields and Fock spaces. The double colon $\mathcal{O}$ (resp. triple colon $\mathcal{O}_i$) is used to indicate an operator normal ordered with respect to the shifted vacuum (resp. solitonic) representation. $D, \mathcal{P}(\Phi), \mathcal{P}(\Phi)$ are dense subspaces defined just after (2.3) and in (2.2) and a variant $\hat{P}$ is defined in (3.9). Regularized fields $\varphi, \pi, \Phi, \Pi$, etc., are all defined by convolution with an approximate identity $\delta_0, \kappa$ introduced just after (2.8) and in §2.2, and a variant $\delta_0$ is introduced in §2.2.

2 The Heisenberg Commutation Relations (CCR)

To solve the quantum field theory associated to the Hamiltonian (1.1) it is necessary to find a Hilbert space $\mathcal{H}$, such that the classical fields, $\phi$ and $\pi$, are replaced by operator-valued distributions acting on $\mathcal{H}$. These operator-valued distributions - called quantum fields, and denoted $\Phi^H$ and $\Pi^H$ - are required to verify the Heisenberg equal time commutation relation, namely, $[\Phi^H(t,x), \Pi^H(t,y)] = i\delta(x-y)$, as well as the equations of motion (1.7), appropriately interpreted. This is the quantum theory in the Heisenberg picture. In the Schrödinger picture, one instead works with time-independent quantum fields $\Phi, \Pi$ which verify the canonical commutation relation (CCR)

$$[\Phi(x), \Pi(y)] = i\delta(x-y). \quad (2.1)$$

These fields are then used to build, starting from the formal expression (1.1), a self-adjoint operator acting on $\mathcal{H}$. Once this is achieved, the theorem of Stone provides a strongly continuous one-parameter group of unitary transformations, i.e., a collection $\{S(t)\}_{t \in \mathbb{R}}$ of linear mappings

$$S(t) : \mathcal{H} \to \mathcal{H} \text{ such that } t \mapsto S(t)\Psi \text{ is continuous for all } \Psi \in \mathcal{H}, \quad (2.2)$$

$$S'(t)S(t) = S(t)S'(t) = 1, \quad S(0) = 1 \quad \text{ and } \quad (2.3)$$

$$S(t)S(s) = S(t+s). \quad (2.4)$$

This one-parameter group defines the quantum dynamics and also connects the Heisenberg and Schrödinger pictures, through the relations $\Phi^H(t,x) = S(-t)\Phi(x)S(t)$, and $\Pi^H(t,x) = S(-t)\Pi(x)S(t)$, etc. We will work in the Schrödinger picture, so that a proof of an existence theorem for the quantum dynamics consists of fixing a representation of (2.1) for time-independent fields, and then proving self-adjointness of the Hamiltonian obtained by substituting these fields into (1.1) - this latter process requires regularization and taking limits.

2.1 Quantization in the vacuum sector.

We first recall from [10] the quantization procedure in the case of the topologically trivial boundary conditions (1.2). Write the classical field as $\Phi_0 + \varphi$, where the field $\varphi$ is subject to the boundary condition $\lim_{|x| \to \infty} \varphi(x) = 0$. The classical Hamiltonian is now

$$H^{vac}(\varphi, \pi) = H(\Phi_0 + \varphi, \pi) = \int \left[ \frac{1}{2} (\pi^2 + \partial_x \varphi^2 + 4m^2 \varphi^2) + 2mg\varphi^3 + \frac{1}{2} \eta^2 \varphi^4 \right] dx, \quad (2.5)$$

$$= H_0^{vac} + H^{vac}_{fg}. \quad (2.6)$$
Here
\[ H_{0}^{\text{vac}}(\varphi, \pi) = \frac{1}{2} \int \left[ \pi^{2} + \varphi K_0 \varphi \right] \, dx \quad \text{and} \quad H_{1, g}^{\text{vac}}(\varphi) = \int 2mg\varphi^3 + \frac{1}{2}g^2\varphi^4 \, dx \]  

(2.7)

and \( K_0 = (-\partial_x^2 + 4m^2) \). Later we will also make use of the associated covariance operator
\[ C_0 = K_0^{-1} = (-\partial_x^2 + 4m^2)^{-1}, \]

and its square root. We now recall the standard solution of (2.1) for the vacuum sector fields \((\Phi, \Pi) = (\Phi_0 + \varphi, \pi)\) and the operators obtained by substituting these into the classical expressions for the Hamiltonian, giving sufficient detail for what we will need below.

**Fock Space.** Now to define a corresponding pair of quantum fields, still denoted \(\varphi, \pi\), we introduce Fock space, defined as the (complete) Hilbert direct sum of the symmetric \(n\)-fold tensor powers of \(L^2(\mathbb{R})\), defined with Lebesgue measure \(dk\), i.e.,
\[ \mathcal{F}_0 = \bigoplus_{n=0}^{\infty} \text{Sym}^n (L^2(\mathbb{R}, dk)). \]

(2.8)

For \(n = 0\) it is to be understood that \(\text{Sym}^0(L^2(\mathbb{R})) = \mathbb{C}\). A typical element, \(\Psi \in \mathcal{F}_0\), is a sequence of functions \(\{\Psi_n\}_{n=0}^{\infty}\), where \(\Psi_n \in L^2(\mathbb{R}^n)\) is symmetric with respect to interchange of any pair of coordinates:
\[ \Psi(k_1, \ldots, k_i, \ldots, k_j, \ldots, k_n) = \Psi(k_1, \ldots, k_j, \ldots, k_i, \ldots, k_n). \]

The Fock space norm is \(\|\Psi\|^2 = \sum_n \|\Psi_n\|_{L^2(\mathbb{R}^n)}^2\). The element with \(\Psi_0 = 1\) and \(\Psi_n = 0\) for \(n \geq 1\) is called the vacuum, and will be denoted \(\Omega_0\), or \(|0\rangle\).

A useful dense subspace, \(\mathcal{D}\), is the algebraic span of the finite symmetric products \(\text{Sym}^n \prod_{j=1}^{n} f_j(k_j)\) of Schwartz functions \(f_j\). For each \(k \in \mathbb{R}\) we introduce the annihilation and creation operators, given, respectively, by
\[ (a_k \Psi)_{n-1}(k_1, \ldots, k_{n-1}) = \sqrt{n} \Psi_n(k, k_1, \ldots, k_{n-1}), \quad \text{and} \]
\[ (a_k^\dag \Psi)_{n+1}(k_1, \ldots, k_{n+1}) = \sum_{j=1}^{n+1} \frac{1}{\sqrt{n+1}} \delta(k - k_j) \Psi_n(k_1, \ldots, \hat{k}_j, \ldots, k_{n+1}). \]

(2.9)

(2.10)

(The hat indicates an omitted argument.) Recall that the domain of \(a_k^\dag\) consists only of the zero vector, and properly speaking the expression above gives rise to a densely defined bilinear form on \(\mathcal{F}_0\), rather than a densely defined operator. Alternatively, these formal expressions can be regarded as defining operator-valued distributions, and it can be checked that they satisfy \([a_k, a_l^\dag] = \delta(k - l)\), interpreted appropriately, see (20).

We recall a basic estimate for Wick Operators from [7]. Given a function or distribution \(w \in \mathcal{S}'(\mathbb{R}^{m+n})\), a corresponding Wick operator on Fock space is given formally by
\[ \mathbb{W}_w = \int_{\mathbb{R}^{m+n}} a_{k_m}^\dag \cdots a_{k_1}^\dag w(k_1, \ldots, k_m, k_1', \ldots, k_n') a_{k_1} \cdots a_{k_n} \prod_{j=1}^{m} dk_j \prod_{j=1}^{n} dk_j'. \]

(2.11)

Writing
\[ N_0 = \int a_k^\dag a_k \, dk \]

(2.12)

for the number operator as usual, we have the following bounds in the case that the kernel is square integrable:
\[ \| (I + N_0)^{-m/2} \mathbb{W}_w (I + N_0)^{-n/2}\| \leq \| w \| \]

(2.13)

and, more generally, the identity (on finite particle vectors)
\[ \mathbb{W}_w (I + N_0)^\alpha = (I + N_0 + n - m)^\alpha \mathbb{W}_w \quad (\alpha \in \mathbb{R}) \]

(2.14)
implies that for \(a + b \geq m + n\),
\[
\| (\mathbb{1} + N_0)^{-a/2} \Psi_w (\mathbb{1} + N_0)^{-b/2} \| \leq (1 + |m - n|)^{(m-n-a)/2} \| w \|
\]  
(2.15)

where on the left hand side \(\| \cdot \|\) means Fock space operator norm, while on the right hand side \(\| w \|\) means the operator norm of the mapping \(\text{Sym}^n(L^2(\mathbb{R})) \to \text{Sym}^m(L^2(\mathbb{R}))\) determined by the kernel \(w\).

Introducing the dispersion relation \(\omega_k = \sqrt{k^2 + 4m^2}\), we define the fields
\[
\varphi(x) = \frac{1}{\sqrt{2\pi}} \int \frac{1}{\sqrt{2\omega_k}} (a_k e^{ikx} + a_k^\dagger e^{-ikx}) \, dk, \text{ and }
\]
(2.16)
\[
\pi(x) = \frac{1}{\sqrt{2\pi}} \int -i \frac{\omega_k}{2} (a_k e^{ikx} - a_k^\dagger e^{-ikx}) \, dk.
\]
(2.17)

Again, these expressions really define operator-valued distributions, e.g., if \(f \in \mathcal{S}(\mathbb{R})\) then \(\varphi(f)\) is the unbounded, densely defined operator given by
\[
\varphi(f) = \int \frac{1}{\sqrt{2\omega_k}} (a_k \hat{f}(-k) + a_k^\dagger \hat{f}(k)) \, dk,
\]

where \(\hat{f}(k) = (2\pi)^{-1/2} \int e^{-ikx} f(x) \, dx\) is the Fourier transform. Another useful way of expressing the above is to introduce operators, defined for each real-valued \(f \in \mathcal{S}(\mathbb{R})\), by
\[
a(f) = \int \hat{f}(-k) a_k \, dk, \quad a^\dagger(f) = \int \hat{f}(k) a_k^\dagger \, dk.
\]

With these definitions it is possible to write
\[
\varphi(f) = \frac{1}{\sqrt{2}} \left(a(K_0^{-1/4} f) + a^\dagger(K_0^{-1/4} f)\right), \quad \pi(f) = -\frac{i}{\sqrt{2}} \left(a(K_0^{1/4} f) - a^\dagger(K_0^{1/4} f)\right),
\]
(2.18)

and the Heisenberg relation is a consequence of the only non-zero commutator \([a(f), a^\dagger(g)] = \int f(x)g(x) \, dx = \int \hat{f}(-k)\hat{g}(k) \, dk\).

One can now check that the pair \((\Phi, \Pi) = (\Phi_0 + \varphi, \pi)\) solves (2.1), again interpreted appropriately. After discarding an (infinite) constant, the free Hamiltonian is \((\text{II}) \S III.1.4)\):
\[
: H_0^{\text{ac}} : = \frac{1}{2} \int : \pi^2 + \varphi K_0 \varphi : \, dx = \int \omega_k a_k^\dagger a_k \, dk.
\]
(2.19)

(In fact, to obtain the semiclassical correction to the soliton mass, we will keep track of a regularized version of the discarded constant and compare it with the corresponding quantity in the solitonic quantization.) As usual, colons indicate the normal ordered product of the field operators, obtained by moving all the annihilation operators to the right. The final expression
\[
\ln(\omega_k) = \int \omega_k a_k^\dagger a_k \, dk
\]
(2.20)
is the Hamiltonian for an assembly of noninteracting bosons with dispersion relation \(\omega_k = \sqrt{4m^2 + k^2}\). It defines a self-adjoint operator with domain
\[
\text{Dom}(H_0^{\text{ac}}) \overset{\text{def}}{=} \left\{ \Psi \in \bigoplus_{n=0}^{\infty} \text{Sym}^n(L^2(\mathbb{R})) : \sum_n \| \sum_{i=1}^{n} \omega_k_i \Psi_n(k_1, \ldots k_n) \|_{L^2}^2 < \infty \right\}.
\]
(2.21)

**The Schrödinger representation.** There is an alternative representation of the Heisenberg relations (2.1) in which the Hilbert space is a Gaussian space. To be precise, let
\[
\mu_0 = \gamma \left( \frac{1}{2\sqrt{K_0}} \right)
\]
(2.22)
be the Gaussian measure on \( S'(\mathbb{R}) \) with covariance 
\[
\frac{1}{2\sqrt{2\pi}} = \frac{1}{2} C_0^\frac{n}{2},
\]
where \( C_0^\frac{n}{2} \) is the operator with integral kernel
\[
C_0^\frac{n}{2}(x, y) = (-\Delta + 4m^2)^{-\frac{n}{4}}(x, y) = \frac{1}{2\pi} \int_\mathbb{R} \frac{e^{ik(x-y)}}{(k^2 + 4m^2)^\frac{n}{4}} \, dk = \frac{1}{2\pi} \int_\mathbb{R} \frac{e^{ik(x-y)}}{\omega_k} \, dk,
\]
and form the Gaussian Hilbert space \( L^2(S'(\mathbb{R}), \mu_0) \). Write (in boldface) \( \varphi \) for a typical point of \( S'(\mathbb{R}) \), so that the coordinate functions are the functions \( \varphi \mapsto \varphi(f) \) for \( f \in S(\mathbb{R}) \); we use the same notation to indicate the corresponding multiplication operators on \( L^2(S'(\mathbb{R}), \mu_0) \). Addition and multiplication of such coordinate functions generates the polynomials, which span a dense subspace. Recall from [15, Chapter 2] the Wiener chaos orthogonal decomposition, which yields a collection \( \{E_n\}_{n=0}^\infty \) of mutually orthogonal projection operators on \( \oplus_{n=0}^\infty \), the range of \( E_n \) being the orthogonal complement of the closed linear span of polynomials of degree \( n-1 \) within the closed linear span of polynomials of degree \( n \) (see also [11, §6.3]).

**Proposition 2.1.** There exists a unitary map \( \mathbb{I} \) taking \( F_0 \) onto \( L^2(S'(\mathbb{R}), \mu_0) \), such that \( \mathbb{I} \Omega_0 = 1 \) (i.e., the function \( S'(\mathbb{R}) \to \mathbb{C} \) which is identically equal to one), and if \( f_j \in S(\mathbb{R}) \forall j \)
\[
\mathbb{I} \varphi(f_1) \varphi(f_2) \ldots \varphi(f_N) \mathbb{I}^{-1} = \mathbb{E}_N \varphi(f_1) \varphi(f_2) \ldots \varphi(f_N).
\]

In addition \( \mathbb{I} \) induces the following action on the operators:
\[
\mathbb{I} \circ \varphi(f) \circ \mathbb{I}^{-1} = \varphi(f) \ (\text{multiplication operator}),
\]
\[
\mathbb{I} \circ \pi(f) \circ \mathbb{I}^{-1} = -iD_f + i\varphi(C_0^{-\frac{n}{2}} f),
\]
where \( D_f \) is the directional derivative operator (along \( f \in S(\mathbb{R}) \)) given by \( D_f A(\varphi) = \lim_{\epsilon \to 0} \frac{A(\varphi + f \epsilon) - A(\varphi)}{\epsilon} \) on an appropriate domain (which includes the polynomials, i.e. the algebra generated by the coordinate functions). We will on occasion use bold face \( F_0 \equiv L^2(S'(\mathbb{R}), \mu_0) \) to indicate that the Schrödinger representation description of Fock space is being used.

The Cameron-Martin space for the measure \( \mu_0 \) is \( H^\frac{n}{2} \), and so the operation of displacement of the field \( \delta_g : \varphi \mapsto \varphi + g \) (i.e. translation in the space \( S' \)) produces by push-forward an equivalent measure (i.e., \( (\delta_g)_* \mu_0 \) is mutually absolutely continuous with \( \mu_0 \)) if and only if \( g \in H^\frac{n}{2} \), see [4, Theorem 2.4.5]. In the case \( g \in H^\frac{n}{2} \) the Radon-Nikodym derivative is given by
\[
\frac{d(\delta_g)_* \mu_0}{d\mu_0} = \exp\left[ + 2\varphi(K_{0^\frac{n}{2}} g) - (g, K_{0^\frac{n}{2}} g)_{L^2} \right],
\]
where, for \( \tilde{g} = K_{0^\frac{n}{2}} g \in H^{-\frac{n}{2}} \), the function \( \varphi \mapsto \varphi(\tilde{g}) \) is well-defined in \( L^2(d\mu_0(\varphi)) \) as the measurable extension of the function \( \{ \varphi \mapsto \varphi(f) \}_{f \in S} \) (as in the discussion in the proof of Theorem 2.1).

**Self-adjointness.** We recall a result on self-adjointness from [4] and [5]; see also [10, Theorem II.3.1.3] and [7]. In the following \( H_g^{\text{vac}} \): is the self-adjoint operator discussed above with domain \( \mathcal{D}(H_g^{\text{vac}}) \in L^2(\mathbb{R}) \cap L^2(\mathbb{R}) \) the operator obtained by substitution of \( \varphi + g \) into
\[
H_{g,vac}(\varphi) = \int \left[ 2mgb(x)\varphi^3 + \frac{1}{2}g^2b(x)\varphi^4 \right] dx,
\]
normal ordering and forming \( H_g^{\text{vac}} = H_g^{\text{vac}} + H_{I,g}^{\text{vac}} \) defines an operator which is bounded below and self-adjoint on
\[
\text{Dom}(H_g^{\text{vac}}) = \text{Dom}(H_0^{\text{vac}}) \cap \text{Dom}(H_{I,g}^{\text{vac}}).
\]
2.2 Quantization in the solitonic sector.

In order to describe the quantum field theory in the solitonic sector, we take as starting point the classical Hamiltonian

\[ H_{\text{sol}}^g(\varphi, \pi) = \frac{M_{cl}}{g^2} + H_0^{\text{sol}}(\varphi, \pi) + H_{I,g}^{\text{sol}}(\varphi), \]  

where

\[ H_0^{\text{sol}}(\varphi, \pi) = \frac{1}{2} \int \left[ \pi^2 + \varphi K \varphi \right] dx \quad \text{and} \quad H_{I,g}^{\text{sol}}(\varphi) = \int \left[ 2mg \tanh mx \varphi^3 + \frac{1}{2} g^2 \varphi^4 \right] dx, \]

in place of (2.7), where

\[ K = -\partial_x^2 + 4m^2 - 6m^2 \sech^2 mx = K_0 - 6m^2 \sech^2 mx. \]  

In quantizing this Hamiltonian two natural possibilities present themselves:

(i) treat the \( \sech^2 mx \) term which appears in \( H_0^{\text{sol}} \) perturbatively, and base the quantization on the same vacuum sector solution (2.16)-(2.17) of the Heisenberg CCR (2.1);

(ii) form another soliton sector solution of the Heisenberg CCR based on the operator \( K \) in place of \( K_0 \).

The first option is based on the quantum field Hamiltonian \( H_0^{\text{vac}} + \tilde{H}_I^{\text{sol}} \), where the latter operator is obtained by substitution of (2.17) and then normal ordering the formal expression

\[ \tilde{H}_I^{\text{sol}}(\varphi) = H_{I,g}^{\text{sol}}(\varphi) - 3m^2 \int \sech^2 mx \varphi^2 dx; \]  

this is convenient for existence theory. The second option allows an explicit analysis of the semiclassical limit, so we will make use of both. (It is important that these two solutions of (2.1) are unitarily equivalent, so that both quantizations refer to the same theory - this issue is addressed below in Theorems 2.7 and 2.14.)

Next we describe the two approaches in detail.

**Soliton quantization using vacuum sector solution of CCR.** In this approach we continue to use the same solution (2.16)-(2.17) of the CCR, but shifted by the classical soliton profile \( \Phi_S \). Explicitly these are, in full,

\[ \Phi(x) = \Phi_S(x) + \varphi(x) = \Phi_S(x) + \frac{1}{\sqrt{2\pi}} \int \frac{1}{\sqrt{2\omega_k}} (a_k e^{ikx} + a_k^\dagger e^{-ikx}) dk, \]  

\[ \Pi(x) = \pi(x) = \frac{1}{\sqrt{2\pi}} \int -i \frac{\omega_k}{2} (a_k e^{ikx} - a_k^\dagger e^{-ikx}) dk, \]

acting on the Hilbert space \( \mathcal{F}_0 \) defined in (2.8). An alternative way of giving the representation, which is useful for comparison with other representations, is to pair them with a real Schwartz function

\[ \Phi(f) = \int \Phi_S(x) f(x) dx + \int \frac{1}{\sqrt{2\omega_k}} \left( a_k \hat{f}(-k) + a_k^\dagger \hat{f}(k) \right) dk, \]  

\[ \Pi(f) = -i \int \frac{\omega_k}{2} \left( a_k \hat{f}(-k) - a_k^\dagger \hat{f}(k) \right) dk. \]

**Theorem 2.3** (Self-adjointness). (i) The quadratic Hamiltonian in the solitonic sector obtained by normal ordering the classical expression \( \frac{1}{2} \int (\pi^2 + \varphi K \varphi) dx \) with respect to the representation (2.29)-(2.30), namely,

\[ :H_0^{\text{sol}}: = :H_0^{\text{vac}}: - \frac{1}{2} \int 6m^2 \sech^2 mx \varphi(x)^2 dx; \]
is well-defined on the domain \( \text{Dom} (\hat{H}^\text{vac}_0) \subset F_0 \), and is essentially self-adjoint on this domain with self-adjoint extension also written \( \hat{H}^\text{sol}_0 \). This operator verifies the lower bound \( \hat{H}^\text{sol}_0 \geq \Delta \text{mac} = \frac{b^2}{2} - \frac{6m}{x^2} \).

(ii) Let \( b \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) and assume there exists a positive constant \( \delta \) such that \( b(x) \geq \delta \text{sech}^2mx \) for all \( x \). Then the formal Hamiltonian

\[
\hat{H}^\text{sol}_0(\varphi, \pi) = \frac{M_c}{g^2} + \frac{1}{2} \int \left[ \pi^2 + \varphi K \varphi \right] \, dx + \int \left[ 2mgb(x) \text{tanh} \, mx \varphi^3 + \frac{1}{2}g^2b(x)\varphi^4 \right] \, dx
\]

(2.33)
defines, after substitution and normal ordering with (2.29)–(2.30) as in Theorem 2.2 an operator on \( F_0 \) which is bounded below and has a self-adjoint extension.

Proof. (i) The essential self-adjointness assertion is a consequence of [22, Theorem X.14] given that it follows from (2.14) that the operator \( \int 6m^2 \text{sech}^2mx \varphi(x)^2 \, dx \) is bounded on \( \text{Dom} (\hat{N}_0) \supset \text{Dom} (\hat{H}^\text{vac}_0) \), and so is a relatively bounded perturbation of \( \hat{H}^\text{vac}_0 \): and is well defined on \( \text{Dom} (\hat{H}^\text{vac}_0) \); it is also a consequence of the result in this reference that any core for \( \hat{H}^\text{vac}_0 \): is a core for \( \hat{H}^\text{sol}_0 \). The precise lower bound is proved in (3.3) together with a determination of the domain of the self-adjoint extension in Theorem 3.7.

(ii) As for Theorem 2.2 the assertion in (ii) follows from classic self-adjointness results, but applied now to the Hamiltonian \( \hat{H}^\text{vac}_0 + \hat{H}^\text{g}_{\varphi} \), see (2.28). The semi-boundedness condition on the perturbing polynomial now takes the requirement that \(-6m^2 \text{sech}^2mx \varphi^2 + 2mgb(x) \text{tanh} \, mx \varphi^3 + \frac{1}{2}g^2b(x)\varphi^4 \) be bounded below. This will be met if \( b(x) \geq \delta \text{sech}^2mx \) holds everywhere, for some positive number \( \delta \). Under this condition there is a lower bound (3.19) for the regularized interaction which is sufficient to ensure that \( \exp[-t \hat{H}^\text{sol}_0] \) is integrable (in the Schrödinger representation) and hence that the closure of the operator sum defines a self-adjoint operator, see [20, Theorem X.54], and it is bounded below by Theorem X.58 in the same reference, while its domain can be determined from [3].

\[ \bbox[white]{\text{□}} \]

Remark 2.4. Notice that the representation (2.29)–(2.30) differs from (2.16)–(2.17) by a displacement by the classical soliton profile \( \Phi_S \). This operator is not unitarily implementable because \( \Phi_S \) is not in the Cameron-Martin space, as (discussed following Proposition 2.4) is \( H^2 \).

**Soliton quantization using soliton sector solution of CCR.** The formulae (2.18) defining the free field have to be modified to take account of the different spectral properties of \( K \) as compared to \( K_0 \). The operator \( K \) is a non-negative self-adjoint operator on \( L^2(\mathbb{R}) \) with domain \( \text{Dom} (K) = H^2(\mathbb{R}) \). The spectrum of \( K \) consists of the following:

- a one-dimensional kernel \( \{ \psi_0 \} \);
- one simple discrete nonzero eigenvalue \( 3m^2 > 0 \),

\[
K \psi_1 = \omega^2 \psi_1, \quad \omega_d = \sqrt{3m}
\]

with corresponding spectral subspace \( \{ \psi_1 \} \);

- continuous spectrum \( [4m^2, +\infty) \).

In addition to the \( L^2(\mathbb{R}) \) eigenfunctions \( \psi_0 \in \mathcal{S}(\mathbb{R}) \) and \( \psi_1 \in \mathcal{S}(\mathbb{R}) \), there are generalized eigenfunctions \( E_k \in L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R}) \) which satisfy

\[
KE_k = (k^2 + 4m^2)E_k.
\]

Explicit formulae for these eigenfunctions, and the corresponding spectral resolution for \( K \), are derived and displayed in the appendix. (Overall translation invariance means that if the soliton is translated by \( \xi \in \mathbb{R} \), as explained prior to (1.9), then the corresponding eigenfunctions and generalized eigenfunctions are translated by \( \xi \) also. This means that, for each \( \xi \in \mathbb{R} \), the spectral resolution of the operator \( K(\xi) = (-\partial_x^2 + 4m^2 - 6m^2 \text{sech}^2m(x - \xi)) \) can be deduced immediately from that of the operator \( K \), on which we concentrate.) Spectral decomposition provides an integral representation for \( U \in L^2(\mathbb{R}) \), which can be given explicitly as

\[
U(x) = \langle \psi_0, U \rangle_{L^2} \psi_0(x) + \langle \psi_1, U \rangle_{L^2} \psi_1(x) + \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}} E_k(y)U(y)E_k(x) \, dy \, dk.
\]

(2.34)
It is useful to define the \textit{distorted Fourier transform} $U \mapsto \hat{U}$ by
\[
\hat{U}(k) = (2\pi)^{-1/2} \int E_k(x)U(x) \, dx,
\]
with inverse
\[
f(k) \mapsto (2\pi)^{-1/2} \int E_k(x)f(k) \, dk,
\]
which sets up a partial isometry whose initial space is $L^2(dk)$ and whose final space is the subspace $\langle \{\psi_0, \psi_1\} \rangle \subset L^2(dx)$, i.e., the $L^2$-orthogonal complement of the discrete spectral subspace.

These facts form the basis for the construction of a set of solutions of the Heisenberg relations (2.1) of the form
\[
\left( \Phi(x), \Pi(x) \right) = \left( \Phi_S(x) + \phi(x), \pi(x) \right)
\]
different to (2.20), (2.30). Before giving the full expression (2.33), (2.35), it is useful to explain how this solution is built up. It is supposed to describe a quantum mechanical particle (the kink) with momentum $P$, interacting with the oscillatory mode of frequency $\sqrt{3}m$ and the radiation modes associated to the continuous spectrum $[4m^2, +\infty)$. To describe an isolated quantum particle, we could make use of a pair of operators $(Q, P)$ which satisfy $[Q, P] = i$, and act on the space $L^2(\mathbb{R}, dQ)$ with $Q$ as the (unbounded) operator of coordinate multiplication, i.e. $Q: g(Q) \mapsto Qg(Q)$, and $P = -i\partial_Q$; slightly more generally $[Q, \eta - i\partial_Q] = i$ for any constant $\eta$. For the case at hand, we will use such a pair of operators to describe the kink; its centre being described by the operator of multiplication by $Q$, and represents quantum mechanical fluctuations around the classical location of the kink at the origin; these are small in an appropriate sense when $g$ is small. The remaining modes are described by the “fluctuation” fields around the soliton $(\phi, \pi)$, given by formulae analogous to (2.18). Define a new Fock space as the complete direct sum
\[
\mathcal{F} = \bigoplus \text{Sym}^n \mathbb{P}^+_0(L^2(\mathbb{R})) \, ,
\]
constructed this time out of the Hilbert space of square integrable fluctuations of the kink which are orthogonal to the infinitesimal translations $\psi_0$, i.e., $\mathbb{P}_0$ is the orthogonal projector onto this subspace so that
\[
\mathbb{P}^+_0(L^2(\mathbb{R})) = \langle \{\psi_0\} \rangle ^\perp \subset L^2(\mathbb{R}) ;
\]
the norm on the Fock space will be written $\| \cdot \|$, and the Fock vacuum $\Omega$. The creation/annihilation operators $b^\dagger, b$ act on $\mathcal{F}$, and the corresponding fluctuation fields are given by
\[
\hat{\phi}(f) = \frac{1}{\sqrt{2}} \left( b(K^{-1/4}f) + b^\dagger(K^{-1/4}f) \right) \quad \text{and} \quad \hat{\pi}(f) = -i\frac{1}{\sqrt{2}} \left( b(K^{1/4}f) - b^\dagger(K^{1/4}f) \right),
\]
for $f \in \mathcal{S}(\mathbb{R}) \cap \mathbb{P}^+_0(L^2(\mathbb{R}))$, in analogy to (2.18).

Now we form a solution of the Heisenberg relations (2.1). This is achieved by the following definition of quantum fields given, as operator-valued distributions, by
\[
\Phi(f) = -\langle \psi_0, f \rangle_{L^2} \sqrt{\mathcal{M}_d} Q + \frac{1}{\sqrt{2}} \left( b(K^{-1/4}f_1) + b^\dagger(K^{-1/4}f_1) \right),
\]
\[
\pi(f) = -\frac{P}{\sqrt{\mathcal{M}_d}} \langle \psi_0, f \rangle_{L^2} - \frac{i}{\sqrt{2}} \left( b(K^{1/4}f_1) - b^\dagger(K^{1/4}f_1) \right),
\]
where $f \in \mathcal{S}(\mathbb{R})$ and $f_1 = f - \langle f, \psi_0 \rangle_{L^2}\psi_0$ is the component of $f$ in $\mathbb{P}^+_0(L^2(\mathbb{R}))$. The commutation relation reads
\[
\langle \Phi(f), \pi(g) \rangle = \left( \langle Q, P \rangle \langle f, \psi_0 \rangle_{L^2} (g, \psi_0)_{L^2} + \frac{i}{2} \left[ b(K^{-1/4}f_1), b^\dagger(K^{1/4}g_1) \right] \right)
\]
\[
- \frac{i}{2} \left[ b^\dagger(K^{1/4}f_1), b(K^{-1/4}g_1) \right]
\]
\[
= i \langle f, g \rangle_{L^2}.
\]

12
A more explicit form is obtained from (2.38)-(2.39) by extracting the test functions, leading to:

$$\phi(x) = -\sqrt{M_0} Q \psi_0(x) + \frac{1}{\sqrt{2\omega_d}} (a_d + a_d^\dagger) \psi_1(x)$$

$$+ \frac{1}{\sqrt{2\pi}} \int \frac{1}{\sqrt{2\omega_k}} (a_k E_k(x) + a_k^\dagger \tilde{E}_k(x)) \, dk,$$

$$\pi(x) = -\frac{P}{\sqrt{M_0}} \psi_0(x) - i \frac{\omega_d}{2} (a_d - a_d^\dagger) \psi_1(x)$$

$$+ \frac{1}{\sqrt{2\pi}} \int -i \frac{\omega_k}{2} (a_k E_k(x) - a_k^\dagger \tilde{E}_k(x)) \, dk.$$

The operators $a_d, a_d^\dagger$ are annihilation and creation operators for the discrete mode with frequency $\omega_d$. The $a_k, a_k^\dagger$ satisfy $[a_k, a_k^\dagger] = \delta(k-l)$ (which holds in the same sense as the corresponding relation in the vacuum sector). The operators $a_d, a_d^\dagger, a_l, a_l^\dagger$ can be related to the $b^\dagger, b$ via the formulae (which define operator-valued distributions):

$$a_d = b(\psi_1), \quad a_d^\dagger = b^\dagger(\psi_1),$$

$$a_k = b\left(\frac{1}{\sqrt{2\pi}} \tilde{E}_k\right), \quad a_l^\dagger = b^\dagger\left(\frac{1}{\sqrt{2\pi}} E_l\right).$$

The Heisenberg relation is a consequence of the completeness relation (A.10):

$$[\Phi(x), \Pi(y)] = i \psi_0(x)\psi_0(y) + i \psi_1(x)\psi_1(y) + \frac{i}{2\pi} \int_R E_k(x) \tilde{E}_k(y) \, dk$$

$$= i\delta(x-y).$$

For comparison with (2.31)-(2.32), the representation can be written, after pairing with a Schwartz test function $U$,

$$\phi(U) = -\sqrt{M_0} Q \langle \psi_0, U \rangle_{L^2} + \frac{1}{\sqrt{2\omega_d}} (a_d + a_d^\dagger) \langle \psi_1, U \rangle_{L^2}$$

$$+ \int \frac{1}{\sqrt{2\omega_k}} \left( a_k \tilde{U}(k) + a_k^\dagger \tilde{U}(k) \right) \, dk,$$

$$\pi(U) = -\frac{P}{\sqrt{M_0}} \langle \psi_0, U \rangle_{L^2} - i \frac{\omega_d}{2} (a_d - a_d^\dagger) \langle \psi_1, U \rangle_{L^2}$$

$$+ \int -i \frac{\omega_k}{2} \left( a_k \tilde{U}(k) - a_k^\dagger \tilde{U}(k) \right) \, dk.$$
Applying second quantization to this shows that $\mathcal{F}$ can be realized as a tensor product space: there is a unitary equivalence

$$\mathcal{J} : \mathcal{F} \rightarrow L^2(\mathbb{R}, \exp[-\omega_d q_d^2] dq_d) \otimes \bigoplus_{n=0}^{\infty} \text{Sym}^n L^2(\mathbb{R}, dk) = L^2(\mathbb{R}, \gamma((2\omega_d)^{-1})) \otimes \mathcal{F}_0,$$

under which

$$\mathcal{J} a_d \mathcal{J}^{-1} = \frac{1}{\sqrt{2\omega_d}} \partial_q d, \quad \mathcal{J} a_d^\dagger \mathcal{J}^{-1} = \frac{1}{\sqrt{2\omega_d}} \left(2\omega_d q_d - \partial_q d\right),$$

(2.45)

and $\Omega$ maps to the function identically equal to 1. We introduce a number operator

$$\hat{N} = a_d^\dagger a_d + \int a_k^\dagger a_k dk.$$

(2.46)

Now to describe the full quantization of the soliton, using these two ingredients, we form the total Hilbert space as in $\text{(1.18)}$.

A useful dense domain consists of the algebraic span of $g(Q)h(q_d)\text{Sym} \prod_{j=1}^{n} f_j(k_j)$ where all the $g,h,\{f_j\}$ are Schwartz functions. Restricting $g,h$ to be the Hermite functions as in Remark 2.10 gives an alternative dense set written as $\mathcal{P}(\phi)$ which essentially corresponds to the action of the polynomials in the field on the vacuum. Substituting $\text{(2.40)-(2.41)}$ into the Hamiltonian and normal ordering gives (formally)

$$\hat{H}_2 = \frac{\hat{N}^2}{2\hbar} + \hat{H}_0^{\text{sol}} + O(g)$$

with

$$\hat{H}_0^{\text{sol}} = \frac{1}{\hbar} \int \left[ \pi^2 + \phi K \Phi \right] dx.$$  

(2.47)

Lemma 2.5. Substitute $\text{(2.40)-(2.41)}$ into $\text{(2.47)}$ and interpret the resulting expression as a bilinear form valued integral, in the weak topology. Then as a bilinear form

$$\hat{H}_0^{\text{sol}} = \frac{g^2 P^2}{2M_d} + \omega_d a_d^\dagger a_d + \int \omega_k a_k^\dagger a_k dk$$

(2.48)

$$= \frac{g^2 P^2}{2M_d} + h(\omega_d) + \mathfrak{H}(\omega),$$

(2.49)

where $h(\omega_d)$ is the Hamiltonian for a one dimensional oscillator with frequency $\omega_d$ and $\mathfrak{H}(\omega)$ is as in $\text{(2.20)}$.

Proof: This is proved in the same way as the corresponding result for the vacuum representation, [7, Theorem 4.4], but making use of the properties of the eigenfunction expansion given in $\text{§A.1}$ in place of the Fourier transform.

The operator appearing the previous Lemma is quadratic and generates a unitary evolution on the space $L^2(\mathbb{R}, dQ) \otimes \mathcal{F}$ which, under the description above, can be given as

$$\exp[-it\hat{H}_0^{\text{sol}}] = \exp[-it\frac{P^2}{2M_d}] \otimes \exp[-it\omega_d] \otimes \Gamma\left(\exp[-it\omega]\right),$$

(2.50)

where the $\Gamma$ notation in the final line stands for the transformation on $\bigoplus \text{Sym}^n L^2(\mathbb{R},dk)$ induced by the map $\exp[-it\omega] : f(k) \mapsto \exp[-it\omega]f(k)$, which is unitary on $L^2(\mathbb{R},dk)$, see [24, Chapter 1].

Remark 2.6. The operator $\hat{H}_0^{\text{sol}}$ is self-adjoint. To specify its domain, decompose simultaneously with respect to the operators $\hbar(\omega_d)$ and the number operator $\hat{N}_0$, so that $\Psi$ corresponds to the sequence $\{\sum_m \Psi_m, m \geq 0\}$ where $\{f_m\}_{m \geq 0}$ are the normalized eigenfunctions of $\hbar(\omega_d)$ with $\hbar(\omega_d)f_m = m \omega_d f_m$ and each $\Psi_{m,m} = \Psi_{m,m}(Q,k_1,\ldots,k_n)$ is symmetric in $k_1,\ldots,k_n$. Then $(\hbar(\omega_d) + \mathfrak{H}(\omega))\Psi$ corresponds to the sequence $\{\sum_m (m \omega_d + \sum_{i=1}^{n} \omega_k)\Psi_{n,m} f_m \}_{n \geq 0}$ and

$$\text{Dom} \hat{H}_0^{\text{sol}} = \left\{ \Psi : \sum_{n,m} \left( \|m \omega_d + \sum_{i=1}^{n} \omega_k\|_{L^2(dQdk)}^2 + \|m \omega_d + \sum_{i=1}^{n} \omega_k\|^2 \right) \frac{d\Psi_{n,m}}{dQ} \|f_m\|_{L^2(dQdk)}^2 \\
+ \| \frac{d^2\Psi_{n,m}}{dQ^2} \|_{L^2(dQdk)}^2 < \infty \right\}. $$

14
Schrödinger representation in the solitonic sector. There are two approaches to this: in the first, the representation (2.29)–(2.30) is equivalent to a Schrödinger representation in which \( \varphi \) is the multiplication operator acting on the space \( L^2(\mathbb{R}, \mu_0) \), exactly as in the vacuum case (Proposition 2.1). The second approach is to construct a Schrödinger representation based on (2.10)–(2.11). In naive analogy to the vacuum case, the Schrödinger representation might then be expected to be based on a Gaussian measure with covariance \( \frac{1}{2} K^{-\frac{1}{2}} \), with \( K \) as in (2.27). However, \( K \) has a one dimensional kernel \( \text{Ker} K = \mathcal{P}_0(L^2(\mathbb{R})) = \langle \{ \psi_0 \} \rangle \), so this is not immediately applicable and additional arguments are needed. Introduce the operator \( K^\theta = \theta \mathcal{P}_0 + K \), which is strictly positive for \( \theta > 0 \), with inverse \( C^\theta = \theta^{-1} \mathcal{P}_0 + (\mathcal{P}_0 K \mathcal{P}_0)^{-1} \).

**Theorem 2.7.** For each \( \theta > 0 \), the Gaussian measure \( \mu(\theta) = \gamma(\frac{1}{2}(C^\theta)^{\frac{1}{2}}) \) on \( S'(\mathbb{R}) \) with covariance \( \frac{1}{2}(C^\theta)^{\frac{1}{2}} \) is equivalent (in the sense of mutual absolute continuity) with the vacuum Gaussian measure \( \mu_0 = \gamma(\frac{1}{2}C_0^{\frac{1}{2}}) \) of covariance \( \frac{1}{2}C_0^{\frac{1}{2}} \). The Radon-Nikodym derivative is

\[
\frac{d\mu(\theta)}{d\mu_0} = \det(\mathbb{I} + S)^\frac{1}{2} \exp\left[ -\left( \phi, (C_0^{\frac{1}{2}}(K^\theta)^{\frac{1}{2}} - 1)\phi \right)_{K^\theta^{\frac{1}{2}}} \right],
\]

where \( S = S(\theta) \) is given by

\[
S = C_0^{\frac{1}{2}}((K^\theta)^{\frac{1}{2}} - K_0^{\frac{1}{2}})C_0^{\frac{1}{2}}.
\]

The expression (2.30) defines an element of \( L^p(d\mu_0) \) for some \( p > 1 \). The operator \( S \) is trace-class on \( L^2(\mathbb{R}) \), or equivalently, the operator \( (C_0^{\frac{1}{2}}(K^\theta)^{\frac{1}{2}} - 1) \) is trace-class on \( H^\frac{1}{2} \), the Sobolev space determined by the inner product \( \langle \phi, \psi \rangle_{K^\theta^{\frac{1}{2}}} \) defined in (2.22).

This theorem is proved below, following some technical results on the covariance operators.

**Theorem 2.8.** If \( \theta > 0 \) the operator \( K_0^{\frac{1}{2}}((C^\theta)^{\frac{1}{2}} - C_0^{\frac{1}{2}})K_0^{\frac{1}{2}} \) is trace-class on \( L^2(\mathbb{R}) \).

**Proof.** Notice that since \( \mathcal{P}_0 \) is the spectral projection onto the kernel of \( K \), the positive square root of \( K^\theta \) is given by \( (K^\theta)^{\frac{1}{2}} = K^{\frac{1}{2}} + \theta^{\frac{1}{2}} \mathcal{P}_0 \). We also introduce for comparison the operator \( K_0^{\frac{1}{2}} + \theta^{\frac{1}{2}} \mathcal{P}_0 \), which is also a strictly positive self-adjoint operator with domain \( H^1(\mathbb{R}) \), with inverse \( (K_0^{\frac{1}{2}} + \theta^{\frac{1}{2}} \mathcal{P}_0)^{-1} \) which is bounded on \( L^2 \). Recalling that the trace-class operators form an ideal (within the algebra of bounded operators) characterized by having finite trace norm, we see that the theorem is a consequence of the following two lemmas and the triangle inequality for the trace norm.

**Lemma 2.9.** If \( \theta > 0 \) the operator \( K_0^{\frac{1}{2}}((K_0^{\frac{1}{2}} + \theta^{\frac{1}{2}} \mathcal{P}_0)^{-1} - C_0^{\frac{1}{2}})K_0^{\frac{1}{2}} \) is trace-class on \( L^2(\mathbb{R}) \).

**Lemma 2.10.** If \( \theta > 0 \) the operator \( K_0^{\frac{1}{2}}(((C^\theta)^{\frac{1}{2}} - (K_0^{\frac{1}{2}} + \theta^{\frac{1}{2}} \mathcal{P}_0)^{-1})K_0^{\frac{1}{2}} \) is trace-class on \( L^2(\mathbb{R}) \).

To prove these we will make use of the following trace-class criterion.

**Theorem 2.11** (12, Section III.10). An integral operator \( Af(x) = \int_{\mathbb{R}} A(x,y)f(y)dy \) is a trace-class operator on \( L^2(\mathbb{R}) \) if

(i) the function \( (x, y) \to A(x, y) \) is continuous,

(ii) \( (f, Af)_{L^2} \geq 0 \) for all continuous and compactly supported \( f \), and

(iii) \( \int_{\mathbb{R}} A(x, x)dx < \infty \).

**Proof of Lemma 2.9** This follows from the following explicit formulae, which will also be of use below:

\[
\left( (K_0^{\frac{1}{2}} + \theta^{\frac{1}{2}} \mathcal{P}_0)^{-1} - C_0^{\frac{1}{2}} \right) f = -\theta^{\frac{1}{2}} \frac{(C_0^{\frac{1}{2}} \psi_0, f)_{L^2} C_0^{\frac{1}{2}} \psi_0}{1 + \theta^{\frac{1}{2}}(C_0^{\frac{1}{2}} \psi_0, \psi_0)_{L^2}}.
\]
so that
\[ K_0^\frac{1}{2} \left( (K_0^\frac{1}{2} + \theta^\frac{1}{2} P_0)^{-1} - C_0^\frac{1}{2} \right) K_0^\frac{1}{2} f = -\theta^\frac{1}{2} \frac{(C_0^\frac{1}{2} \psi_0, f)_{L^2} C_0^\frac{1}{2} \psi_0}{1 + \theta^\frac{1}{2} (C_0^\frac{1}{2} \psi_0, \psi_0)_{L^2}}. \] (2.52)

Now choose an orthonormal basis \( \{e_j\}_{j=1}^{\infty} \) with \( e_1 \) proportional to \( C_0^\frac{1}{2} \psi_0 \), and recall that a self-adjoint bounded non-negative operator \( A \) on \( L^2 \) is trace-class if and only if \( \sum_j |(A e_j, e_j)_{L^2}| < \infty \) for some o.n. basis.

**Proof of Lemma 2.10.** The operator inequality \( 0 \leq K \leq K_0 \), which is evident by inspection, implies (by operator monotonicity of inversion and taking square roots) that \( 0 \leq K_0^\frac{1}{2} < (K_0)_{2} \frac{1}{2} \), and hence for any \( \theta > 0 \)

\[ (K_0^\frac{1}{2} + \theta^\frac{1}{2} P_0)^{-1} < ((K_0^\frac{1}{2} + \theta^\frac{1}{2} P_0)^{-1} = (C_0^\frac{1}{2} \right) \]

It follows that
\[ K_0^\frac{1}{2} ((C_0^\frac{1}{2} - (K_0^\frac{1}{2} + \theta^\frac{1}{2} P_0)^{-1}) K_0^\frac{1}{2} > 0. \] (2.53)

Writing
\[ K_0^\frac{1}{2} ((C_0^\frac{1}{2} - (K_0^\frac{1}{2} + \theta^\frac{1}{2} P_0)^{-1}) K_0^\frac{1}{2} = K_0^\frac{1}{2} ((C_0^\frac{1}{2} - C_0^\frac{1}{2}) K_0^\frac{1}{2} = K_0^\frac{1}{2} ((K_0^\frac{1}{2} + \theta^\frac{1}{2} P_0)^{-1} - C_0^\frac{1}{2}) K_0^\frac{1}{2}, \]

it is sufficient to show that the two operators on the right hand side satisfy the conditions (i) and (iii) in Theorem 2.11. This is clearly true of the second operator on the right, since by formula (2.52) it has a kernel proportional to \( C_0^\frac{1}{2} \psi_0 \otimes C_0^\frac{1}{2} \psi_0 \), a tensor product of a Schwartz function with itself. It therefore remains to prove the same for the operator \( A_2 = K_0^\frac{1}{2} ((C_0^\frac{1}{2} - C_0^\frac{1}{2}) K_0^\frac{1}{2}. \) We write this as \( A_2 = K_0^\frac{1}{2} A_3 K_0^\frac{1}{2}, \) with \( A_3 = ((C_0^\frac{1}{2} - C_0^\frac{1}{2}) \), and use the following two items to analyze these operators.

(a) The action of the operator \( K_0^\frac{1}{2} \) can be realized as a pseudodifferential operator acting on the kernel \( A_3(x, y) \), so that the Fourier transforms of the integral kernels of \( \{A_2, A_3\} \), i.e.,

\[ \hat{A}_3(k, l) = (2\pi)^{-1} \int \exp[-ikx - ily]A_3(x, y)dxdy \quad (j = 2, 3, \)

are related by
\[ \hat{A}_2(k, l) = (4m^2 + k^2)^\frac{1}{2} \hat{A}_3(k, l)(4m^2 + l^2)^\frac{1}{2}, \]

or, in a convenient notation,
\[ A_2(x, y) = (4m^2 - \partial_x^2)^\frac{1}{2}(4m^2 - \partial_y^2)^\frac{1}{2} A_3(x, y). \]

(b) The work in the Appendix yields an explicit formula for the integral kernel of the operator \( A_3: \)

\[ A_3(x, y) = ((C_0^\frac{1}{2} - C_0^\frac{1}{2}) (x, y) = \frac{1}{\sqrt{\psi_1(x)\psi_1(y)}} + \frac{1}{\sqrt{\psi_0(x)\psi_0(y)}} (2.54) \]

+ \[ \frac{1}{2\pi} \int_{(k^2 + m^2)(k^2 + m^2)} \frac{\mathcal{F}(k, x) \mathcal{F}(k, x) - (k^2 + m^2)(k^2 + m^2)}{(k^2 + m^2)(k^2 + m^2)} e^{ik(x-y)} dk, \]

where \( \mathcal{F}(k, x) = (-k^2 - 3imk \tanh mx + 2m^2i - 2m^2 \operatorname{sech}^2 mx) \).

The first two terms in (2.54) give no difficulty: as tensor products of Schwartz functions with themselves, even after the action of the pseudodifferential operators as in item (a) they produce smooth kernels which decrease rapidly along the diagonal, and so satisfy the requirements of (i) and (iii) in Theorem 2.11. So we concentrate on the contributions from the integral in (2.54).

Firstly, notice that away from the diagonal \( x = y \) the integral defines a smooth function since it is a well-behaved oscillatory integral. Thus it is sufficient to restrict to the positive quadrant \( \{x > 0, y > 0\} \) and the negative quadrant \( \{x < 0, y < 0\} \), in checking that this contribution to the kernel verifies (i) and (iii)
in Theorem 2.11. Write \( \tanh mx = \mp 1 + \tanh mx \pm 1 \), and similarly for \( y \), with the \( \pm \) depending on the quadrant. Expanding out, there is a cancellation of the \( k^4 \) in the numerator, leading to an expression of the form

\[
\int_{\mathbb{R}} \frac{\mathcal{F}(k, y)\mathcal{F}(k, x) - (k^2 + m^2)(k^2 + 4m^2)}{(k^2 + m^2)(k^2 + 4m^2)^{\frac{3}{2}}} e^{ik(x-y)} \, dk = \sum_{j=0}^{3} \sum_{\alpha_j=1}^{N_j} f_j^{\alpha_j}(x)g_j^{\alpha_j}(y)I_j(x-y)
\]

where each \( N_j \in \{1, 2, 3 \ldots \} \), the functions \( \{f_j^{\alpha_j}, g_j^{\alpha_j}\} \) are all either constants or smooth functions which together with their derivatives, decay exponentially at infinity, and

\[
I_j(x-y) = \int_{\mathbb{R}} \frac{k^3 e^{ik(x-y)}}{(k^2 + m^2)(k^2 + 4m^2)^{\frac{3}{2}}} \, dk.
\]  

(2.55)

For example, the \( j = 3 \) term, analysis of which is critical to the argument, is given by

\[
(3im(\tanh mx \pm 1) - 3im(\tanh my \pm 1))I_3(x-y).
\]

(As indicated above, the \( \pm \) signs should be chosen according to whether we work in the positive of negative quadrant of the plane. The \( \pm 1s \) actually cancel, and are irrelevant in bounded regions - they are only put in to ensure exponential decay when \( (x, y) \to (\pm \infty, \pm \infty) \).)

\[ \square \]

Lemma 2.12. Let \( f, g \) be Schwartz functions and \( I_j \) as in (2.55).

- For \( j \in \{0, 1, 2\} \) the integral

\[
\Gamma_{1,j}(x, y) \equiv (4m^2 - \partial_x^2)^\frac{1}{2}(4m^2 - \partial_y^2)^\frac{1}{2}(f(x)I_j(x-y)g(y))
\]

defines a continuous function which decays rapidly along the diagonal \( y = x \) so that \( \int_{\mathbb{R}} |\Gamma_{1,j}(x, x)| dx < \infty \).

- For \( j = 3 \) the integral

\[
\Gamma_2(x, y) \equiv (4m^2 - \partial_x^2)^\frac{1}{2}(4m^2 - \partial_y^2)^\frac{1}{2}\left[(f(x) - f(y))I_j(x-y)\right]
\]

defines a continuous function and \( \int_{\mathbb{R}} |\Gamma_2(x, x)| dx < \infty \).

Proof. First some heuristics: observe that the generalized integral

\[
I_{a, b}(z) = \int_{\mathbb{R}} \frac{k^a e^{ikz}}{(k^2 + m^2)(k^2 + 4m^2)^{\frac{3}{2}}} \, dk
\]  

(2.56)
is absolutely convergent and defines a continuous function of \( z \) for \( a < 1 + 2b \), so that all the integrals appearing in both assertions themselves define continuous functions of \( z \). However, the pseudodifferential operators \( (4m^2 - \partial_{x/y}^2)^\frac{1}{2} \) acting on these integrals are of order \( \frac{1}{2} \), and so their combined effect is heuristically another power of \( k \), which takes the second integral, i.e. \( \Gamma_2 \), out of the regime of absolute convergence. We discuss this case first. The point is that continuity holds, the just-mentioned lack of smoothness notwithstanding, due to the presence of the factor \( f(x) - f(y) \), which serves to restore continuity near the diagonal \( x = y \), which is the region of difficulty. To actually prove this we use a Fourier representation

\[
\Gamma_2(x, y) = (2\pi)^{-\frac{1}{2}} \iint e^{ikx + ily}(4m^2 + k^2)^{\frac{1}{2}}(4m^2 + l^2)^{\frac{1}{2}} \left[\frac{f(k + l)}{a(l)} - \frac{f(k + l)}{a(k)}\right] \, dkdl,
\]

where

\[
\frac{1}{a(k)} = \frac{k^3}{(k^2 + m^2)(k^2 + 4m^2)^{\frac{1}{2}}}.
\]
Over a common denominator the integrand is \((2\pi)^{-\frac{7}{4}}e^{i(kx+ly)}\) times
\[
\hat{f}(k + l) \left[ \frac{k^3(l^2 + m^2)(l^2 + 4m^2)\hat{f}^3 + l^3(k^2 + m^2)(k^2 + 4m^2)^\frac{3}{2}}{(l^2 + m^2)(l^2 + 4m^2)(k^2 + 4m^2)^\frac{5}{2}} \right].
\] (2.57)

To analyze this integral we change variables \((k, l) \to (u = k + l, l)\), so (2.57) becomes
\[
\hat{f}(u) \left[ \frac{(l + u)^3(l^2 + m^2)(l^2 + 4m^2)\hat{f}^3 + l^3((-l + u)^2 + m^2)((-l + u)^2 + 4m^2)^\frac{3}{2}}{((-l + u)^2 + m^2)(l^2 + m^2)((-l + u)^2 + 4m^2)^\frac{5}{2}} \right].
\] (2.58)

Now divide the domain \(\mathbb{R}^2 = \mathcal{R}_1 \cup \mathcal{R}_2\) with, given a small number \(\epsilon\),
\[
\mathcal{R}_1 = \{(u, l) : |u| \geq \epsilon |l|\} \quad \text{and} \quad \mathcal{R}_2 = \{(u, l) : |u| \leq \epsilon |l|\} \subset \{(k = u - l, l) : |k| \geq (1 - \epsilon) |l|\}.
\]

In \(\mathcal{R}_1\) the inequality \(|u| \geq \epsilon |l|\) and the rapid decay of \(\hat{f}\) (which is a consequence of the assumption that \(f \in \mathcal{S}(\mathbb{R})\)) implies that for arbitrarily large positive integers \(N_1, N_2\) there exists a constant such that
\[
|\hat{f}(u)| \leq C_5 (1 + |u|)^{-N_1} (1 + \epsilon |l|)^{-N_2}.
\]

This implies absolute integrability over \(\mathcal{R}_1\). For absolute integrability over \(\mathcal{R}_2\) it is necessary to make use of a cancellation arising from the “\(f(x) - f(y)\)” structure. The factor \(\hat{f}(u)\) in (2.58) ensures rapid decay in \(u\), so we need only consider growth in \(l\). The inequality \(|k| \geq (1 - \epsilon) |l|\) implies the denominator in (2.58), and hence (2.58), is \(\geq C_6 (1 + |l|)\), and the highest, and only dangerous, power of \(l\) in the numerator arises (after expanding out the polynomial parts) solely from the term
\[
l^5 \left((l + u)^2 + 4m^2\right)^\frac{3}{2} - (l^2 + 4m^2)^\frac{3}{2} = \frac{l^5 \left((l + u)^2 + 4m^2\right)^3 - (l^2 + 4m^2)^3}{((l + u)^2 + 4m^2)^\frac{5}{2} + (l^2 + 4m^2)^\frac{5}{2}}.
\]

The cancellation is now manifest, and the inequality \(|u| \leq \epsilon |l|\) implies that the numerator in (2.58) is \(\leq C_3 |\hat{f}(u)| (1 + |u|/|l|^7)\), ensuring absolute integrability over \(\mathcal{R}_2\). Continuity of \(\Gamma_2\) is now a consequence of the dominated convergence theorem, since the integrand in the formula for \(\Gamma_2\) is bounded by an absolutely integrable function which is independent of \(x, y\).

It is now possible to take the limit \(y \to x\) to find that \((2\pi)^{\frac{7}{4}} \Gamma_2(x, x)\) is equal to
\[
\int \int e^{iux} \hat{f}(u) \left[ \frac{(-l + u)^3(l^2 + m^2)(l^2 + 4m^2)^\frac{3}{2} + l^3((-l + u)^2 + m^2)((-l + u)^2 + 4m^2)^\frac{3}{2}}{((-l + u)^2 + m^2)(l^2 + m^2)((-l + u)^2 + 4m^2)^\frac{5}{2}} \right] dudl,
\] (2.59)

To establish integrability, integrate by parts to deduce that \((2\pi)^{\frac{7}{4}} (ix)^2 \Gamma_2(x, x)\) is equal to
\[
\int \int e^{iux} \left( \frac{d}{du} \right)^2 \left[ \hat{f}(u) \left( \frac{(-l + u)^3(l^2 + m^2)(l^2 + 4m^2)^\frac{3}{2} + l^3((-l + u)^2 + m^2)((-l + u)^2 + 4m^2)^\frac{3}{2}}{((-l + u)^2 + m^2)(l^2 + m^2)((-l + u)^2 + 4m^2)^\frac{5}{2}} \right) \right] dudl
\]
which can be shown to be finite exactly as previously, so that \(\int |\Gamma_2(x, x)| dx < \infty\), as claimed.

To establish the conclusions for \(\Gamma_{1,j}\) is easier. If either \(f\) or \(g\) is a constant, the preceding argument works but with \(a\) replaced by
\[
\frac{1}{a_j(k)} = \frac{k^j}{(k^2 + m^2)(k^2 + 4m^2)^\frac{3}{2}}.
\]
in the formulae. This obviates the need to observe the cancellation since the integrand is immediately of sufficiently rapid decay to apply dominated convergence. For the case that both \(f\) and \(g\) are Schwartz we use the Fourier representation
\[
\Gamma_{1,j}(x, y) = (2\pi)^{-1} \int \int e^{i(kx+ly)} (4m^2 + k^2)^\frac{3}{4} (4m^2 + l^2)^\frac{1}{4} \left( \frac{\hat{f}(k - u)\hat{g}(u + l)}{a_j(u)} \right) dkdl du,
\]
18
Now make the change variables \((u, k, l) \to (u, k', l') = (u, k + u, l + u)\), and notice that since \(j = 0, 1, 2\), absolute integrability follows easily from the bound \(|a_j(u)^{-1}| \leq \text{const.}(1 + |u|)^{j-5}\) and the fact that \(f, \hat{g}\) are Schwartz. The reminder of the argument is the same.

**Proof of Theorem 2.4.** This is an application of a theorem of Shale, in the form given in [2] Theorem 45.1 (see also [24] Theorem I.23., or [4] Chapter 6), and depends upon the trace-class property for the perturbation of the covariance. We will briefly explain the statement of the theorem in terms of measures on the space of tempered distributions being used here, since we will need to work in Proposition 2.17 with the expression for the Radon-Nikodym derivative, which however does not have an a priori meaning, and has to be defined by a limiting process.

The tempered distribution \(f \mapsto \phi(f)\) is defined as a continuous map on the space of Schwartz test functions \(f \in \mathcal{S}(\mathbb{R})\), and in its turn the map \(\phi \mapsto \phi(f)\) is continuous on \(\mathcal{S}'(\mathbb{R})\) (endowed with the weak-* topology) for all such \(f\). But the formula

\[
\|\phi(f) - \phi(g)\|_{L^2(\mu_0)}^2 = (f - g, f - g)_{L^2} + \frac{1}{2} (C_0^{\frac{1}{2}}(f-g), C_0^{\frac{1}{2}}(f-g))_{L^2}
\]

determines a unique extension of \(\phi(K_0^{\frac{1}{2}}f)\) as a measurable function of \(\phi\) in the space \(L^2(\mathcal{S}'(\mathbb{R}), \mu_0)\) for any \(f \in L^2(\mathbb{R})\), i.e. \(\phi(\chi)\) is so defined for \(\chi \in H^{-\frac{1}{2}}(\mathbb{R})\). Now if \(\{e_n\}\) is an orthonormal basis of \(L^2(\mathbb{R})\) then \(\{K_0^{\frac{1}{2}}e_n\}\) is an orthonormal basis for \(H^{-\frac{1}{2}}(\mathbb{R})\), and we can expand \(\chi = \sum \chi_n K_0^{\frac{1}{2}}e_n\), with \(\chi_n = (C_0^{\frac{1}{2}} e_n, \chi)\).

This induces a dual expansion

\[
\phi(\chi) = \sum \phi_n^* e_n = \langle \sum \phi_n C_0^{\frac{1}{2}} e_n, \chi \rangle
\]

where \(\phi_n = \phi(K_0^{\frac{1}{2}} e_n) \in L^2(\mathcal{S}'(\mathbb{R}), \mu_0)\) are well-defined for all \(n\) by the preceding discussion, and satisfy \((\phi_n, \phi_m)_{L^2(\mu_0)} = \frac{1}{2} \delta_{nm}\). With these as coordinates we identify \(\mathbb{R}^\infty\) with the infinite product probability measure \(\prod_n (\pi^{-\frac{1}{2}} \exp[-\phi_n^2] d\phi_n)\). This allows a formal interpretation of the exponential in (2.51) as \(\exp[-\sum_{m,n} \pi^{-\frac{1}{2}} \pi^{-\frac{1}{2}} \phi_n^2 (e_m, S e_n)]\), which in turn suggests choosing \(\{e_n\}\) to be an orthonormal basis of eigenfunctions of \(S\), with eigenvalues \(\lambda_n\), which satisfy \(\sum |\lambda_n| < \infty\) (under the condition that \(S\) is trace-class). We can then consider the following expression

\[
\exp[-(\phi_n (C_0^{\frac{1}{2}} (K_0^{\frac{1}{2}} - 1) \phi_n)^2)] = \lim_{N \to \infty} \prod_{n=1}^N \exp[-(\phi_n)^2 (e_n, S e_n)].
\]

This limit exists by [24] Lemma 1.24. To establish the trace-class property, note that it was proved above that the operator \(B = K_0^{\frac{1}{2}} ((C_0^{\frac{1}{2}} - C_0^{\frac{1}{2}}) K_0^{\frac{1}{2}})\) is trace-class, while \(K_0^{\frac{1}{2}} (C_0^{\frac{1}{2}} - K_0^{\frac{1}{2}}) K_0^{\frac{1}{2}} = 1 + B\) and \((1 + B)^{-1}\) are bounded by Proposition 2.13. It follows that \((1 + B)^{-1} - 1 = -B (1 + B)^{-1}\) is also trace-class, i.e. \(C_0^{\frac{1}{2}} (K_0^{\frac{1}{2}} - 1)\) is trace-class, as required. It follows that the determinant in the formula for the Radon-Nikodym derivative is well-defined and equals \(\prod_n (1 + \lambda_n)\), so that the expression (2.51) is to be interpreted as

\[
\lim_{N \to \infty} \prod_{n=1}^N \left( (1 + \lambda_n) \exp[-\lambda_n (\phi_n)^2] \right).
\]

By a result of Segal, a proof of which appears in [24], §1.6], this expression is known to converge in \(L^p(\prod_n (\pi^{-\frac{1}{2}} \exp[-\phi_n^2] d\phi_n))\) for some \(p > 1\) to give the stated formula for the Radon-Nikodym derivative.

**Remark 2.13.** To connect the preceding discussion up with the formula given in [2] Theorem 4.5] note that the term in the exponential can be rewritten

\[
-(\phi_n (C_0^{\frac{1}{2}} (K_0^{\frac{1}{2}} - 1) \phi_n)^2) = -(\phi_n (T'T - 1) \phi_n)^2,
\]

where \(T = C_0^{\frac{1}{2}} (K_0^{\frac{1}{2}})^{\frac{1}{2}}\) and ' means adjoint in the \((\ )_{K_0^{\frac{1}{2}}}\) inner product, (so that \(T' = C_0^{\frac{1}{2}} T^* K_0^{\frac{1}{2}}\) where \(T^*\) is the ordinary \(L^2\) adjoint, so that \(T'T = C_0^{\frac{1}{2}} (K_0^{\frac{1}{2}})^{\frac{1}{2}}\).)
The creation/annihilation operators are given by

\[ a^\dagger(f) = \frac{1}{\sqrt{2}} \left( \Phi((K^0) f) + i\pi((C^0) f) \right) \quad \text{and} \quad \left( a^\dagger(f) \right)^\dagger = \frac{1}{\sqrt{2}} \left( \Phi((K^0) f) - i\pi((C^0) f) \right). \]

The solutions of the Heisenberg relations \((2.1)\) are all essentially self-adjoint on the space of polynomials in the field.

**Corollary 2.15.** The solutions \((2.29)\) and \((2.40)\) of the Heisenberg relations \((2.1)\) are unitarily equivalent via a unitary isomorphism \(\mathcal{V} : \mathcal{F}_0 \rightarrow L^2(\mathbb{R}, dQ) \otimes \mathcal{F}\).

**Proof.** Now, making use of the formula for the Gaussian wave packet \((1.15)\), there is a probability measure \(\gamma(\sigma^2) = \chi_0(0, Q; \sigma)^2 dQ\) on the real line, and on the Hilbert space \(L^2(\mathbb{R}, \chi_0(0, Q; \sigma)^2 dQ)\) there is a solution of the Heisenberg relation \([Q, P] = i\) in which \(Q\) is represented by multiplication by \(Q\) and \(P\) by the operator \(f(Q) \mapsto -i\frac{d}{dQ} f(Q) + \frac{1}{2\sigma^2} Q f(Q)\); these are all unitarily equivalent to the standard Schrödinger representation via the unitary equivalence \(L^2(\mathbb{R}, dQ) \ni f \mapsto \chi_0(0, Q; \sigma)^{-1} f \in L^2(\mathbb{R}, \chi_0(0, Q; \sigma)^2 dQ)\), which yields \(-i\frac{d}{dQ}\) as the operator representing \(P\). We now include this unitary equivalence into the field theoretic situation under the identification \(\sqrt{M_{cl}} Q = -\Phi(\psi_0)\); basically \(Q\) is identified with the coordinate function on \(S'(\mathbb{R})\) connected to the zero mode \(\psi_0\). Also choose

\[ \sqrt{\theta} = 1/(2\sigma^2 M_{cl}), \]

then taking tensor products it follows that the representation determined by \((2.38)-(2.41)\) or equivalently \((2.38)-(2.39)\), on \(L^2(\mathbb{R}, dQ) \otimes \mathcal{F}\) is unitarily equivalent to the one defined on \(L^2(\mathbb{R}, \chi_0(0, Q; \sigma)^2 dQ) \otimes \mathcal{F}\) \(\cong L^2(S'(\mathbb{R}), d\mu(\theta))\), with the fields (obtained by a minor modification of \((2.38)-(2.39)\))

\[ \Phi(f) = -\psi_0 f_{L^2} \sqrt{M_{cl}} \sqrt{\theta} + \frac{1}{\sqrt{2}} \left( b((K^0) f_1) + b^\dagger((C^0) f_1) \right), \]

\[ \pi(f) = -\frac{1}{\sqrt{M_{cl}}} \left( -i \frac{d}{dQ} + \frac{i}{2\sigma^2} \theta \right) (\psi_0, f)_{L^2} - \frac{i}{\sqrt{2}} \left( b(K^{1/4} f_1) - b^\dagger(K^{1/4} f_1) \right), \]

where for \(f \in S(\mathbb{R})\) we write \(f_1 = P_0^\perp f = f - P_0 f = -\psi_0 f_{L^2} \psi_0\), so that \((K^\sigma)^a f_1 = K^\sigma f_1\) for any power \(a\); recall that \(K\) is invertible on \(P_0^\perp L^2\). In this equivalence, the function identically equal to \(1 \in L^2(S'(\mathbb{R}), d\mu(\theta))\) corresponds to \(\chi_0(0, Q; \sigma) \otimes 1 \in L^2(dQ) \otimes \mathcal{F}\), where \(\Omega\) was defined earlier as the Fock vacuum in \(\mathcal{F}\).

We next extend the definition of the \(b\) operators from \(P_0^\perp L^2\) to all of \(L^2\) with the formulae

\[ b(\psi_0) = -\sigma \frac{d}{dQ} \quad \text{and} \quad b^\dagger(\psi_0) = +\sigma \frac{d}{dQ} - \frac{1}{\sigma} Q. \]
so that, using $\sqrt{\theta} = 1/(2\sigma^2 M_{cl})$, we get
\[
\phi(f) = \frac{1}{\sqrt{2}} \left( b((C^\theta)^{1/2}P_0 f) + b^\dagger((C^\theta)^{-1/2}P_0 f) \right) + \frac{1}{\sqrt{2}} \left( b((C^\theta)^{1/2}f_1) + b^\dagger((C^\theta)^{-1/2}f_1) \right),
\]
\[
\pi(f) = \frac{-i}{\sqrt{2}} \left( b((K^\theta)^{1/2}P_0 f) - b^\dagger((K^\theta)^{-1/2}P_0 f) \right) - \frac{i}{\sqrt{2}} \left( b(K^{1/4}f_1) - b^\dagger(K^{-1/4}f_1) \right)
\]
\[
= - \frac{i}{\sqrt{2}} \left( b((K^\theta)^{1/2}f) - b^\dagger((K^\theta)^{-1/2}f) \right).
\]

From here we obtain unitary equivalence with the Schrödinger representation on $L^2(S'(\mathbb{R}), d\mu(\theta))$ by mapping vacuum to vacuum and $b, b^\dagger$ to the annihilation/creation operators defined in Theorem 2.14, as in the proof of uniqueness of Fock representations in [11, Section 6.3]. Since the covariance operator for the representations in (2.62) and (2.63)-(2.64) are the same this unitary equivalence extends to one between the fields $(\phi, \pi)$ and $(\phi', \pi')$. But this latter representation is in turn equivalent to that determined by the vacuum measure $\mu_0$, by Theorem 2.14 and the result is proved.

Another way to carry out the preceding construction of the measure $\mu(\theta)$ is to work with the subspace of tempered distributions which annihilate the zero mode, i.e.,
\[
\mathcal{S}'_0(\mathbb{R}) \overset{\text{def}}{=} \{ \phi \in \mathcal{S}'(\mathbb{R}) : \phi(\psi_0) = 0 \}.
\]

This space is itself a nuclear space, and is the dual of the quotient space $\mathcal{S}(\mathbb{R})/\langle \{ \psi_0 \} \rangle$ (also nuclear), which can be identified with $\mathcal{S}_0(\mathbb{R}) \overset{\text{def}}{=} \{ f \in \mathcal{S}(\mathbb{R}) : (f, \psi_0)_{L^2} = 0 \}$, the $L^2$-orthogonal complement of $\langle \{ \psi_0 \} \rangle$ in the space of Schwartz functions. On this latter space the operator $K$ is invertible, with inverse which we write $(P_0^\dagger C P_0^\dagger)^{1/2}$; by the Bochner-Minlos theorem there exists a Gaussian measure on $\mathcal{S}'_0(\mathbb{R})$ with covariance $\frac{1}{2}(P_0^\dagger C P_0^\dagger)^{1/2}$. This gives the space
\[
\mathcal{F} \overset{\text{def}}{=} L^2(\mathcal{S}'_0(\mathbb{R}), \gamma(\frac{1}{2}(P_0^\dagger C P_0^\dagger)^{1/2}))
\]
which is the Schrödinger representation version of the Fock space $\mathcal{F}$. Since we can write
\[
\phi(f) = \phi((\psi_0, f)_{L^2}, \psi_0 + P_0^\dagger f) = (\psi_0, f)_{L^2} \phi(\psi_0) + \phi(P_0^\dagger f)
\]
there is an isomorphism
\[
\mathcal{S}'(\mathbb{R}) = \mathbb{R} \oplus \mathcal{S}'_0(\mathbb{R})
\]
\[
\phi \mapsto (\phi(\psi_0), \phi(P_0^\dagger (\cdot)))
\]
which allows us to generate $\mu(\theta)$ as the product measure
\[
\exp(-\sqrt{\theta}\phi(\psi_0)^2 d\phi(\psi_0) \otimes \gamma(\frac{1}{2}(P_0^\dagger C P_0^\dagger)^{1/2})).
\]

**Remark 2.16.** An explicit way to write the unitary equivalence down is as follows. Let $\psi_0, \psi_1, \psi_2 \ldots$ be an orthonormal basis of $L^2(\mathbb{R})$ with $\psi_0, \psi_1$ the normalized eigenfunctions of $K$ as above. Then
\[
\nabla^{-1} : L^2(\mathbb{R}, dQ) \otimes \mathcal{F} \rightarrow \mathcal{F}_0
\]
\[
Q^{n_0} \chi_0(0, Q; \sigma) \prod_{j=1}^N \phi(\psi_j)^{n_j} \rightarrow (-\sqrt{M_{cl}})^{n_0} \varphi(\psi_0)^{n_0} \prod_{j=1}^N \varphi(\psi_j)^{n_j} \sqrt{d\mu(\theta)/d\mu_0}
\]
(2.67)

In particular, the state in which there are no bosons present and the soliton is described by a Gaussian wave packet $\chi_0(0, Q; \sigma)$ corresponds to the state $\sqrt{\frac{d\mu(\theta)}{d\mu_0}}$ in the vacuum Schrödinger representation (with $\sigma(0)$ and
\( \theta \) related as above). In the vacuum Fock representation this corresponds to a cyclic vector which we write as \( \Omega_0 \in \mathcal{F} \). We will write \( \mathcal{P}(\phi) \) for the dense subset of \( L^2(\mathbb{R}, dQ) \otimes \mathcal{F} \) obtained by taking finite linear combinations of expressions as in \( (2.67) \), essentially polynomials in the field \( \phi \). Using the description of the Hilbert space in the discussion following \( (2.15) \), this translates to the algebraic span of \( g(Q)h(q)\text{Sym}^n \prod_{j=1}^n f_j(k_j) \) with \( g(Q) = Q^{n_0} \chi_0(0, Q; \sigma) \) and \( h(q) = q^{n'} \exp \left[ -\frac{i}{\hbar} \omega_j q_j^2 \right] \) and the \( f_j \in S(\mathbb{R}) \), as in the initial definition of \( \mathcal{P}(\phi) \). These are really the same two dense subspaces, and the boldface serves to indicate when the Schrödinger representation is being used when it is necessary to emphasize this. The right hand side of \( (2.67) \) lies in \( L^p(d\mu_0) \) for some \( p > 1 \) (by the assertion about \( (2.51) \) in Theorem \( 2.7 \)), and also it is a smooth vector for \( \mathbb{N}_0 \) by \([23, \text{Theorem } 5.2]\).

The construction above has depended upon a choice of a positive real number \( \theta \), related to the variance \( \sigma \) of the Gaussian \( \chi_0(0, Q; \sigma) \) by \( \sqrt{\theta} = 1/(2\sigma^2\mathcal{M}_4) \), but as far as the unitary transformation \( \mathcal{V} \) is concerned this dependence essentially drops out, as the following result shows.

**Proposition 2.17.** The unitary transformation \( \mathcal{V} \) is independent of \( \theta \) except possibly for a multiplicative constant which is fixed by the requirement that \( \mathcal{V} \left[ \frac{d\mu(\theta)}{d\mu_0} \right] = \chi_0(0, Q; \sigma) \otimes \Omega \in L^2(dQ) \otimes \mathcal{F} \), see \( (2.67) \). More generally, if \( \sqrt{\theta'} = 1/(2(\sigma')^2\mathcal{M}_4) \)

\[
\mathcal{V} \left[ \frac{d\mu(\theta')}{d\mu_0} \right] = \frac{\det(1 + S(\theta))^\frac{1}{2}}{\det(1 + S(\theta'))^\frac{1}{2}} \chi_0(0, Q; \sigma') \otimes \Omega \in L^2(dQ) \otimes \mathcal{F} \tag{2.68}
\]

**Proof.** Working in the Schrödinger representation, and referring to \( (2.67) \), we need to check the \( \theta \) dependence in \( \chi_0(0, Q; \sigma)^{-1} \sqrt{\frac{d\mu(\theta)}{d\mu_0}} \), which seeps in via the operator \( S = S(\theta) \); explicitly:

\[
S(\theta) = S(0) + \sqrt{\theta} C_0^\dagger P_0 C_0^\dagger.
\]

This suggests that in the expression \( \exp[\sum_{m,n} \phi_m \phi_n (e_m, S e_n)] \) we can choose the \( \{ e_n \} \) to be an orthonormal set of eigenfunctions of \( S(0) \), rather than of \( S(\theta) \) as in the proof of Theorem \( 2.7 \). In particular it is easy to check that \( e_1 = K_0^\dagger \psi_0 \) is an eigenfunction with eigenvalue \( \lambda_0^0 = -1 \); let the remaining eigenvalues, none of which equal \(-1\), be written \( \lambda_0^2, \lambda_0^3, \ldots \) then with this choice the Radon-Nikodym factor becomes

\[
\sqrt{\frac{d\mu(\theta)}{d\mu_0}} = \det(1 + S(\theta))^\frac{1}{2} \exp\left[ -\frac{\sqrt{\theta}}{2} \Phi(\psi_0)^2 + \frac{1}{2} (\Phi_1)^2 \right] \lim_{N \to \infty} \prod_{n=2}^N \exp\left[ -\frac{1}{2} \lambda_0^n (\Phi_n)^2 \right],
\]

Here \( \Phi_1 = \Phi(K_0^\dagger e_1) = \Phi(K_0^\dagger \psi_0) \). This makes clear that with the relation between \( \sigma \) and \( \theta \) above, and with \( \sqrt{\mathcal{M}_4 Q} = -\Phi(\psi_0) \), the \( \theta \) dependence in the product \( \chi_0(0, Q; \sigma)^{-1} \sqrt{\frac{d\mu(\theta)}{d\mu_0}} \) remains only in the determinant factor in the Radon-Nikodym derivative (which is of course independent of \( \Phi \)) and the \( \sigma^{-\frac{1}{2}} \) factor in the wave packets \( (1.15) \). Comparing with the corresponding formula with \( \theta \) replaced by \( \theta' \), and making use of

\[
\frac{\chi_0(0, Q; \sigma)}{\chi_0(0, Q; \sigma')} = \sqrt{\frac{\sigma'}{\sigma}} \exp\left[ -\frac{Q^2}{4\sigma^2} - \frac{Q^2}{4(\sigma')^2} \right]
\]

gives \( (2.68) \). \( \square \)

**Remark 2.18.** The presence of the eigenvalue \( \lambda_0^0 = -1 \) in the spectrum of \( S(0) \) is the reason it is necessary to deform the covariance operator by the parameter \( \theta \), and arises directly from the presence of the zero mode \( \psi_0 \) in the spectral analysis of \( K \), which is itself a consequence of translation invariance, the relation being made plain by the fact that the corresponding eigenfunction of \( S(0) \) is \( e_1 = K_0^\dagger \psi_0 \).

## 3 Regularization and normal ordering of the Hamiltonian

The construction of the quantum theory involves regularization of the Hamiltonian, subtraction of counterterms followed by a careful study of the limit as the regularization is removed.
3.1 Regularization of the fields

The full Hamiltonian is constructed as a perturbation of the free Hamiltonian, and the crucial step in establishing self-adjointness is to prove a uniform bound below for a family of regularized Hamiltonians. We explain how to regularize consistently in both the vacuum and the solitonic sector, and use this to obtain a comparison of two representations, see Theorem 3.7 in particular, which leads to the semiclassical mass shift. To introduce an appropriate regularization we will make use of an approximate identity, defined as follows. Let \( \delta^{[1]} \in C^\infty_0(\mathbb{R}) \) be a non-negative, even function with \( \delta^{[1]}(x) = 0 \) for \( |x| \geq 1 \), and satisfying \( \int \delta^{[1]}(x) \, dx = 1 \). For \( \kappa > 0 \) define \( \delta^{[\kappa]}(x) = \kappa \delta^{[1]}(\kappa x) \). Then, as \( \kappa \to +\infty \), the convolution operators

\[
 f \mapsto f_\kappa := \delta^{[\kappa]} * f
\]

tend to the identity, both as operators on \( L^p \), for \( p < \infty \), and also pointwise (resp. locally uniformly) in regions of continuity (resp. uniform continuity) of the function \( f \). Now define the regularized fields at a point \( x \) by

\[
 \varphi_\kappa(x) = \varphi(\delta^{[\kappa]}(\cdot - x)), \quad \pi_\kappa(x) = \pi(\delta^{[\kappa]}(\cdot - x)).
\]

Analogous to (2.10)–(2.17), or (2.29)–(2.30), are the following equivalent formulae for the regularized fields

\[
 \varphi_\kappa(x) = \int \frac{\delta^{[1]}(k/\kappa)}{\sqrt{2\omega_k}} \left( a_k e^{ikx} + a_k^* e^{-ikx} \right) \, dk, \quad \text{and}
\]

\[
 \pi_\kappa(x) = \int -i\delta^{[1]}(k/\kappa) \sqrt{\frac{\omega_k}{2}} \left( a_k e^{ikx} - a_k^* e^{-ikx} \right) \, dk,
\]

where \( \delta^{[1]}(k) = (2\pi)^{-1/2} \int e^{-ikx} \delta^{[1]}(x) \, dx \). These latter formulae indicate that the regularization amounts to a smooth momentum cut-off at scales large compared to \( \kappa \). The regularization (3.1) determines ultraviolet regularized Hamiltonian operators in the vacuum sector, and also a regularized covariance operator:

\[
 (0|\varphi_\kappa(x)\varphi_\kappa(y)|0) = \frac{1}{2} C_{0,\kappa}^+ (x, y) = \frac{1}{4\pi} \iint_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}} \frac{\delta^{[\kappa]}(y - y') e^{ik(x - y')} \delta^{[\kappa]}(x - x') \, dx' \, dy'}{(k^2 + 4m^2)^{\frac{3}{2}}} \, dk,
\]

\[
 = \frac{1}{2} \iint_{\mathbb{R}} \frac{\delta^{[1]}(k/\kappa)}{\sqrt{(k^2 + 4m^2)^{\frac{3}{2}}}} \, dk, \quad \text{and more generally the expression}
\]

\[
 F(K_\kappa)(x, y) = \frac{1}{2\pi} \iint_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}} F(k^2 + 4m^2) \delta^{[\kappa]}(x - x') e^{ik(x - y')} \delta^{[\kappa]}(y - y') \, dx' \, dy' \, dk
\]

\[
 = \iint_{\mathbb{R}} \frac{\delta^{[1]}(k/\kappa)}{\sqrt{k^2 + 4m^2}} F((k^2 + 4m^2)^{\frac{3}{2}}) \, dk,
\]

defines the integral kernel of an operator \( F(K_\kappa) \), which is a regularization of the operator \( F(K_0) \) (for appropriate functions \( F \)). An infrared regularization is provided by inserting a factor \( b(x) \) into the nonlinear interaction.

**Remark 3.1.** The regularized fields are not actually bounded, but by (2.13) both \( \varphi_\kappa(N_0 + 1)^{-\frac{1}{2}} \) and \( \pi_\kappa(N_0 + 1)^{-\frac{1}{2}} \) are bounded. Insertion of the regularized fields above into the free Hamiltonian (1.5) and normal ordering leads to the regularized free Hamiltonian \( H_{0,\kappa}^{rl} \), which can be computed directly to be \( \Pi(\omega_{k,\kappa}) \), in which the regularized dispersion relation is \( \omega_{k,\kappa} = 2\pi|\delta^{[1]}(k/\kappa)|^2 \omega_k \). This can be checked by applying the expression directly to \( \Psi_\kappa \in \text{Sym}^n L^2(\mathbb{R}, dk) \), and using \( \int e^{ik(z - z_0)} \, dz = 2\pi \delta(z - z_0) \) (as an \( S^1 \)-valued integral).

In order to compute the energy of the soliton it is necessary to use a comparable regularization of the fields in the solitonic sector. This would lead us to consider the regularized fields as \( \phi(\delta^{[\kappa]}(\cdot - x)) \) and
π(δ[κ](−x)) . This leads to the following definition of regularized versions of the fields:

\[
\phi_κ(x) = -\sqrt{M_{cl}}Q\psi_0 * δ^{[κ]}(x) + \frac{1}{\sqrt{2ω_d}}(a_d + a_d^\dagger)\psi_1 * δ^{[κ]}(x)
\]

\[
+ \frac{1}{\sqrt{2π}} \int \int \frac{δ^{[κ]}(x - x')}{\sqrt{2ω_k}} (a_k E_k(x') + a_k^\dagger \overline{E_k}(x')) \, dx' \, dk,
\]

\[
π_κ(x) = -\frac{P}{\sqrt{M_{cl}}}\psi_0 * δ^{[κ]}(x) - i\frac{ω_d}{2}(a_d - a_d^\dagger)\psi_1 * δ^{[κ]}(x)
\]

\[
+ \frac{1}{\sqrt{2π}} \int \int -i\frac{ω_d}{2} δ^{[κ]}(x - x') (a_k E_k(x') - a_k^\dagger \overline{E_k}(x')) \, dx' \, dk .
\]

We will also use

\[
(K\phi)_κ(x) = \sqrt{\frac{ω_d^2}{2}} (a_d + a_d^\dagger)\psi_1 * δ^{[κ]}(x) + \frac{1}{\sqrt{2π}} \int \int \sqrt{\frac{ω_d^2}{2}} δ^{[κ]}(x - x') (a_k E_k(x') + a_k^\dagger \overline{E_k}(x')) \, dx' \, dk .
\]

Remark 3.2. Since ψ_0, ψ_1 are Schwartz functions there is nothing to be gained from applying the regularization procedure to them, except for consistency, and the corresponding formulae with ψ_0 * δ^{[κ]} replaced by ψ_0 will give the same results in the limit κ → +∞.

Notice that, as a consequence of the fact that linearization about the soliton breaks translation invariance, the formulae analogous to (3.5)-(3.9) actually define different regularizations which we write as

\[
\phi^{0lt}_κ(x) = -\sqrt{M_{cl}}Q\psi_0(x) + \frac{1}{\sqrt{2ω_d}}(a_d + a_d^\dagger)\psi_1(x)
\]

\[
+ \int \frac{δ^{[κ]}(k/κ)}{\sqrt{2ω_k}} (a_k E_k(x) + a_k^\dagger \overline{E_k}(x)) \, dk ,
\]

\[
π^{0lt}_κ(x) = -\frac{P}{\sqrt{M_{cl}}}\psi_0(x) - i\frac{ω_d}{2}(a_d - a_d^\dagger)\psi_1(x)
\]

\[
+ \int -i\frac{ω_d}{2} δ^{[κ]}(k/κ) (a_k E_k(x) - a_k^\dagger \overline{E_k}(x)) \, dk .
\]

Use of these in place of (3.5)-(3.6) is generally not permissible: for example, only (3.5) - (3.6) give rise to the correct Dashen-Hasslacher-Neveu semiclassical mass shift (as computed by taking the limit of the problem in a sequence of increasing intervals in [6]). On the other hand (3.8)-(3.9) can be useful as an intermediate approximation in the analysis of Wick polynomials in the field, see §5.3.

3.2 Counter-terms

The appropriate counter-terms are determined by normal ordering of the Hamiltonian with respect to the (regularized) covariance. As emphasized in [5] Chapter 6, §4.2, it is important that the same subtractions are made for both the vacuum and solitonic sectors - otherwise it would not be possible to make any meaningful statements about the mass of the kink. (Here the “same subtractions” means same in terms of the original field φ in (1.1); to express them in terms of ϕ it is necessary to take account of the different shifts of the field used in defining the theory via (1.4) and (1.10) in the vacuum and soliton sectors.) So we start with the Hamiltonian in the form (2.6), and consider the effect of normal ordering with respect to the covariance \( \frac{1}{2} C_0^2 \). Defining the number (independent of x)

\[
γ_κ = \frac{1}{2} C_0^2(x, x) = \frac{1}{4π} \int \int_{R × R} \frac{δ^{[κ]}(x - y) e^{ik(x' - y')} \delta^{[κ]}(x - x')}{(k^2 + 4m^2)^{\frac{3}{2}}} \, dx' \, dy' \, dk = \frac{1}{2} \int_R \frac{|\delta^{[κ]}(k/κ)|^2}{(k^2 + 4m^2)^{\frac{3}{2}}} \, dk .
\]

We recall the formulae

\[
\varphi_κ^2 = \varphi_κ^2 - γ_κ , \quad \varphi_κ^3 = \varphi_κ^3 - 3γ_κ \varphi_κ \quad \text{and}
\]

\[
\varphi_κ^4 = \varphi_κ^4 - 6γ_κ \varphi_κ^2 + 3\gamma_κ^2 .
\]

(3.10)

(3.11)
In the vacuum sector the total regularized Hamiltonian is:

$$\mathcal{H}^{vac}_{0,\kappa} : = \mathcal{H}^{vac}_{0,\kappa} : + : \mathcal{H}^{vac}_{1,\kappa, g} : = \int : \mathcal{H}^{vac}_{0,\kappa} : + : \mathcal{H}^{vac}_{1,\kappa, g} : dx ,$$

where

$$\mathcal{H}^{vac}_{0,\kappa} : = \frac{1}{2} \left[ \pi_\kappa^2 + \varphi_\kappa K_0 \varphi_\kappa - K_{0,\kappa}^\dagger (x, x) \right] = \frac{1}{2} (\pi_\kappa^2 + \varphi_\kappa K_0 \varphi_\kappa) : ,$$

and

$$\mathcal{H}^{vac}_{1,\kappa, g} : = \left[ 2mg (\varphi_\kappa^3 - 3\gamma_\kappa \varphi_\kappa) + \frac{1}{2} g^2 (\varphi_\kappa^4 - 6\gamma_\kappa \varphi_\kappa^2 + 3\gamma_\kappa^2) \right]$$

$$= 2mg : \varphi_\kappa^3 : + \frac{1}{2} g^2 : \varphi_\kappa^4 : .$$

Thus we have introduced the following counter-terms (or subtractions) in the definition of the regularized Hamiltonian density:

$$\mathcal{H}^{vac}_{c.t.}(\varphi_\kappa) = -3g^2 \gamma_\kappa \varphi_\kappa^2 - 6mg \gamma_\kappa \varphi_\kappa - \frac{1}{2} K_{0,\kappa}^\dagger (x, x) + \frac{3g^2}{2} \gamma_\kappa^2 .$$

In order to derive a comparable Hamiltonian in the solitonic sector we have to take into account the shift \( \varphi_\kappa \rightarrow - \Phi_0 + \Phi_S + \varphi_\kappa \) to obtain the corresponding counter-terms (using the representation (V.2.25)-(V.2.30)):

$$\mathcal{H}^{sol}_{c.t.}(\varphi_\kappa) = -3g^2 \gamma_\kappa (-\Phi_0 + \Phi_S + \varphi_\kappa)^2 - 6mg \gamma_\kappa (-\Phi_0 + \Phi_S + \varphi_\kappa) - \frac{1}{2} K_{0,\kappa}^\dagger (x, x) + \frac{3g^2}{2} \gamma_\kappa^2 .$$

(Notice that the quadratic “mass renormalization” counter-term induces counter-terms of \( O(g^0) \) in the soliton sector Hamiltonian due to the \( g \) dependence of \( \Phi_0 \) and \( \Phi_S \).) All together this leads to the following regularized Hamiltonian density:

$$\mathcal{H}^{sol}_{c.t.}(\varphi_\kappa) = \left[ \frac{1}{2} \left[ \pi_\kappa^2 + \varphi_\kappa K \varphi_\kappa - 6m^2 \gamma_\kappa \left( \tanh^2 mx - 1 \right) - K_{0,\kappa}^\dagger (x, x) \right] - K_{0,\kappa}^\dagger (x, x) \right] + : \mathcal{H}^{sol}_{1,\kappa, g} :$$

$$= \frac{1}{2} \left[ \pi_\kappa^2 + \varphi_\kappa (K \varphi_\kappa) + 6m^2 \gamma_\kappa \text{sech}^2 mx - K_{0,\kappa}^\dagger (x, x) + \varphi_\kappa \left( K \varphi_\kappa - (K \varphi_\kappa) \right) \right] + : \mathcal{H}^{sol}_{1,\kappa, g} : ,$$

where

$$\mathcal{H}^{sol}_{1,\kappa, g} : = 2mg \tanh mx \left( \varphi_\kappa^3 - 3\gamma_\kappa \varphi_\kappa \right) + \frac{1}{2} g^2 \left( \varphi_\kappa^4 - 6\gamma_\kappa \varphi_\kappa^2 + 3\gamma_\kappa^2 \right)$$

$$= 2mg \tanh mx : \varphi_\kappa^3 : + \frac{1}{2} g^2 : \varphi_\kappa^4 : .$$

This form for the Hamiltonian will lead to the formula for the mass shift in 3.4. Notice that the subtractions here, being induced from those made in the vacuum sector involve the same covariance operator, and so the normal ordering symbol has the same meaning in both vacuum and soliton sectors.

For the determination of the relation between the free Hamiltonians in the vacuum and solitonic sectors, it is necessary to make a precise definition of the regularized free Hamiltonian, and take a limit. As indicated in (3.14), it does make a difference for finite \( \kappa \) whether we just regularize the field \( \varphi \) and then apply the operator \( K \), or regularize \( K \varphi \) (which appears in the expression for the Hamiltonian in terms of fields):

$$K \varphi_\kappa(x) - (K \varphi)_\kappa(x) = -6m^2 \text{sech}^2 mx \varphi_\kappa(x) + 6m^2 \int g[\kappa](x-x') \text{sech}^2 mx' \varphi(x') dx'$$

It turns out to be convenient to use \( \mathcal{H}^{sol}_{0,\kappa} : = \frac{1}{2} : (\pi_\kappa^2 + \varphi_\kappa (K \varphi)_\kappa) : \) as the definition of normal ordered free
regularized Hamiltonian. To check that other choices lead to the same answer in the limit, we first compute

\[ :\mathcal{H}_{0,\kappa}^{\text{sol}}: = \frac{1}{2} \left[ \pi_{\kappa}^2 + \varphi_{\kappa}(K\varphi)_{\kappa} \right] : \]

\[ = \frac{1}{2} \left[ \pi_{\kappa}^2 + \varphi_{\kappa}(K_0\varphi)_{\kappa} - 6m^2 \int \varphi_{\kappa}(x)\delta^{[\kappa]}(x-x')\text{sech}^2mx'\varphi(x')dx' : \right] : \]

\[ = \frac{1}{2} \left[ \pi_{\kappa}^2 + \varphi_{\kappa}(K\varphi)_{\kappa} - K_{0,\kappa}^2(x,x) + 6m^2 \int \varphi_{\kappa}(x)(\text{sech}^2mx'\varphi(x'))\delta^{[\kappa]}(x-x')dx' | 0 \right] : \]

\[ = \frac{1}{2} \left[ \pi_{\kappa}^2 + \varphi_{\kappa}(K\varphi)_{\kappa} - K_{0,\kappa}^2(x,x) + 6m^2\gamma_{\kappa}\text{sech}^2mx \right] \]

\[ - \left[ 3m^2\gamma_{\kappa}\text{sech}^2mx - 3m^2 \int \int \int \frac{e^{ik(x'-y)}}{2(k^2+4m^2)^{\frac{1}{2}}} \delta^{[\kappa]}(x-y)\delta^{[\kappa]}(x'-x)\text{sech}^2mx'dx'dydk \right] . \]

This indicates that not only (3.16), but also the consequent normal ordering adjustments arising from the choice in (3.17) potentially affect the regularized Hamiltonian for finite \( \kappa \). Nevertheless, the following lemmas indicate that these effects vanish in the limit \( \kappa \to +\infty \). We first show that the final term in (3.14) vanishes in an appropriate sense. In this connection, recall from [7, Theorem 4.4] that the Fock space form for the \( \kappa \) choice in (3.17) potentially affect the regularized Hamiltonian for finite \( \kappa \) in the sense of convergence as a bilinear form on \( D \times D \). We therefore show below that the final term in (3.14) has vanishing contribution to the Hamiltonian in the limit \( \kappa \to +\infty \) in this sense.

**Lemma 3.3.**

\[ \lim_{\kappa \to +\infty} \int \varphi_{\kappa}(K\varphi_{\kappa} - (K\varphi)_{\kappa})dx = 0 \]

in the sense of convergence as a bilinear form on \( D \times D \).

This is proved in Appendix [B]. A closely related calculation, also given in Appendix [B], gives the following result.

**Lemma 3.4. (a) In Fock space operator norm**

\[ \lim_{\kappa \to +\infty} \left\| (N_0 + 1)^{-\frac{1}{2}} \left( \int \text{sech}^2mx\varphi_{\kappa}(x)^2dx - \int :\varphi_{\kappa}(x)\delta^{[\kappa]}(x-x')\text{sech}^2mx'\varphi(x')dx'dx' \right)(N_0 + 1)^{-\frac{1}{2}} \right\| = 0 . \]

**Lemma 3.5. (b) Both** \( \int \text{sech}^2mx\varphi_{\kappa}(x)^2dx \) and

\[ \int \int :\varphi_{\kappa}(x)\delta^{[\kappa]}(x-x')\text{sech}^2mx'\varphi(x')dx'dx \]

converge to \( \int \text{sech}^2mx\varphi(x)^2dx \) in \( L^p(d\mu_0) \) for every \( p < \infty \).

The next result, proved in the same appendix, deals with the error in the zero point energy correction in (3.17).

**Lemma 3.6.** In the limit \( \kappa \to +\infty \)

\[ \int \left[ 6m^2\gamma_{\kappa}\text{sech}^2mx - 6m^2 \int \int \frac{e^{ik(x'-y)}}{2(k^2+4m^2)^{\frac{1}{2}}} \delta^{[\kappa]}(x-y)\delta^{[\kappa]}(x'-x)\text{sech}^2mx'dx'dydk \right] dx = O\left( \frac{\ln\kappa}{\kappa} \right) . \]

To obtain the existence theory we use the Hamiltonian in the form in (2.28), which leads us to consider the corresponding regularized spatially cut-off Hamiltonian density

\[ :\tilde{\mathcal{H}}_{0,\nu,\kappa}^{\text{sol}}: = \left[ -3m^2\text{sech}^2mx : \varphi_{\kappa}^2 : + 2mb(x) \tanh mx : \varphi_{\kappa}^3 : + \frac{1}{2}g^2b(x) : \varphi_{\kappa}^4 : \right] , \]
Lemma 3.6. Let $\mathcal{F} \in \mathcal{S}(\mathbb{R})$ be the vacuum in the Fock space $\mathcal{F}$, and $F \in \mathcal{S}(\mathbb{R})$, then

$$
(H(0)_{\omega} \mathcal{F}(Q)) = 1_{2} \mathcal{F} \mathcal{L}^{2} \mathcal{L}^{2} + \frac{1}{2} K_{0,\omega}(x, x) - \frac{1}{2} K_{0,\omega}(x, x)
$$

This indicates that the final three terms on the right hand side give the infimum of the quadratic part of the energy. In the limit $\kappa \to +\infty$ we can replace the final term by the expression in Lemma 3.5 and thence compute that the sum of these three terms has a nonzero limit $\Delta M_{\text{scl}}$:

$$
\Delta M_{\text{scl}} = \frac{1}{2} \kappa \to +\infty \int \left( K_{0,\omega}(x, x) - K_{0,\omega}(x, x) + 3m^{2} \text{sech}^{2} mx C_{0,\omega}(x, x) \right) dx = -m \left( \frac{3}{\pi} - \frac{1}{2\sqrt{3}} \right).
$$

3.3 Change of representation - quadratic terms

The aim next is to exploit the fact that the representation $\mathcal{K}_{a,\omega}$ diagonalizes the quadratic part of the Hamiltonian in the solitonic sector to obtain precise information about $\mathcal{H}_{I,\omega}^{\text{sol}}$: which is not evident in the representation $\mathcal{K}_{a,\omega}$. So consider the effect on the quadratic part of the Hamiltonian of the change (2.40) to (2.41), as described in Corollary 2.15. There is a unitary representation $\mathcal{K}_{a,\omega}$, so consider the effect on the quadratic part of the Hamiltonian of the change $\mathcal{K}_{a,\omega}$ to $\mathcal{K}_{a,\omega}$, as in (2.13). Now let $\Omega$ be the vacuum in the Fock space $\mathcal{F}$, and $F \in \mathcal{S}(\mathbb{R})$, then

$$
\langle F(0) \Omega | \mathcal{H}_{0,\omega}^{\text{sol}} | F(Q) \Omega \rangle = \frac{1}{2} \mathcal{F} \mathcal{L}^{2} \mathcal{L}^{2} + \frac{1}{2} K_{0,\omega}(x, x) - \frac{1}{2} K_{0,\omega}(x, x)
$$

where the first term on the right means the expression

$$
\frac{1}{2} \int \left[ \pi_{\omega}^{2} + \Phi_{\omega} K \Phi_{\omega} \right] dx,
$$

normal ordered with respect to the representation $\mathcal{K}_{a,\omega}$, as in (2.13); this normal ordering produces the second term on the right hand side of (3.20), with $K_{a,\omega}$ as in (2.13). Now let $\Omega$ be the vacuum in the Fock space $\mathcal{F}$, and $F \in \mathcal{S}(\mathbb{R})$, then

$$
\langle F(0) \Omega | \mathcal{H}_{0,\omega}^{\text{sol}} | F(Q) \Omega \rangle = \frac{1}{2} \mathcal{F} \mathcal{L}^{2} \mathcal{L}^{2} + \frac{1}{2} K_{0,\omega}(x, x) - \frac{1}{2} K_{0,\omega}(x, x)
$$

This indicates that the final three terms on the right hand side give the infimum of the quadratic part of the energy. In the limit $\kappa \to +\infty$ we can replace the final term by the expression in Lemma 3.5 and thence compute that the sum of these three terms has a nonzero limit $\Delta M_{\text{scl}}$:

$$
\Delta M_{\text{scl}} \overset{\text{def}}{=} \frac{1}{2} \lim_{\kappa \to +\infty} \int \left( K_{0,\omega}(x, x) - K_{0,\omega}(x, x) + 3m^{2} \text{sech}^{2} mx C_{0,\omega}(x, x) \right) dx = -m \left( \frac{3}{\pi} - \frac{1}{2\sqrt{3}} \right).
$$
Proof. It is to be understood here that the Lemma is asserting the existence of the limit in the definition of $\Delta M_{\text{sol}}$. The proof is in §3.4.

We now turn to the convergence of $iM_{\text{sol}} = \int P(x)dx + \kappa \to +\infty$, to the operator $iM_{\text{sol}}$ in (2.44). This latter operator is itself self-adjoint with domain $\text{Dom}(iM_{\text{sol}})$ given in Remark 2.4. This leads to a precise definition of the self-adjoint operator $iM_{\text{sol}}$; referenced in Theorem 2.3.

**Theorem 3.7.** In the limit $\kappa \to +\infty$ the regularized quadratic part of the Hamiltonian in the solitonic representation $V \circ iM_{\text{sol}} \circ V^{-1}$ converges, as a bilinear form on $\mathcal{P}(\phi) \times \mathcal{P}(\phi)$, to $\Delta M_{\text{sol}} + iM_{\text{sol}}$. This is a closed quadratic form whose closure is the form associated to the self-adjoint operator $iM_{\text{sol}}$, and whose form domain is $\text{Dom}(iM_{\text{sol}}^{\text{reg}})$. The quadratic form defined by the limit of $iM_{\text{sol}}$ is closable and defines a self-adjoint operator $iM_{\text{sol}}$; with domain $V^{-1}V^{-1} \text{Dom}(iM_{\text{sol}})$, which equals $\text{Dom}(iM_{\text{sol}})$, which is a domain of essential self-adjointness. The operator $iM_{\text{sol}}$ so defined is the self-adjoint operator referred to in Theorem 2.3 and $V \circ iM_{\text{sol}} \circ V^{-1} = iM_{\text{sol}} + \Delta M_{\text{sol}}$.

**Proof.** Step One. Recall from Lemma 2.4 that if the expression obtained by substitution of (2.40)-(2.41) into (2.44) is interpreted as a weak bilinear form valued integral, then it equals the bilinear form defined by the expression (1.17), on the domain $\mathcal{P}(\phi) \times \mathcal{P}(\phi)$.

Step Two. The next ingredient is the convergence, as $\kappa \to +\infty$, of $iM_{\text{sol}}$ to $iM_{\text{sol}}$ in the sense of weak bilinear form on the domain $\mathcal{P}(\phi) \times \mathcal{P}(\phi)$. Substitution of (3.5)-(3.6) into (3.21) leads to a sum of terms, of which we consider as representative that involving two annihilation operators, namely

$$
\frac{1}{4\pi} \int a_k a_l \left( \frac{\omega_k (\omega_k - \omega_l)}{\sqrt{\omega_k \omega_l}} \right) \delta^{(1)}(x - x') \delta^{(1)}(x - x'' ) E_k(x') E_l(x'') dx' dx'' d\kappa d\lambda.
$$

The integral is to be understood as a weak bilinear form valued integral, and we want to show that this converges in the weak sense, as $\kappa \to +\infty$, to

$$
\frac{1}{4\pi} \int F(k, l) \left( \frac{\omega_k (\omega_k - \omega_l)}{\sqrt{\omega_k \omega_l}} \right) \delta^{(1)}(x - x') \delta^{(1)}(x - x'' ) E_k(x') E_l(x'') dx' dx'' d\kappa d\lambda.
$$

The weak interpretation above means taking the matrix element of the integrand between two elements of $\mathcal{P}(\phi)$, which will reduce the above integral to one of the form

$$
\frac{1}{4\pi} \int F(k, l) \left( \frac{\omega_k (\omega_k - \omega_l)}{\sqrt{\omega_k \omega_l}} \right) \delta^{(1)}(x - x') \delta^{(1)}(x - x'' ) E_k(x') E_l(x'') dx' dx'' d\kappa d\lambda.
$$

with $F$ a Schwartz function. We recall the fact that $\delta^{(1)} + U(x) \to U(x)$ at points of continuity of $U$, (and in fact uniformly on intervals of uniform continuity of $U$), and apply the dominated convergence theorem. Restricting the integral to bounded intervals of $x$ this gives convergence immediately, so that

$$
\lim_{\kappa \to +\infty} \frac{1}{4\pi} \int 1_{|x| \leq 10} \left( x \right) F(k, l) \left( \frac{\omega_k (\omega_k - \omega_l)}{\sqrt{\omega_k \omega_l}} \right) \delta^{(1)}(x - x') \delta^{(1)}(x - x'' ) E_k(x') E_l(x'') dx' dx'' d\kappa d\lambda = \frac{1}{4\pi} \int 1_{|x| \leq 10} \left( x \right) F(k, l) \left( \frac{\omega_k (\omega_k - \omega_l)}{\sqrt{\omega_k \omega_l}} \right) E_k(x') E_l(x'') dx' dx'' d\kappa d\lambda.
$$

However, for infinite intervals integration by parts arguments are needed also. Referring to (A.4), we see that it is sufficient to consider the case that $E_k(x')$ is replaced by $g(x') h(k) e^{ikx'}$ and $E_l(x'')$ is replaced by $\hat{g}(x'') \hat{h}(l) e^{ltx''}$ with $h(k)$ a polynomial in $k$ divided by $\sqrt{(k^2 + m^2)(k^2 + 4m^2)}$, and similarly for $\hat{h}(l)$, and $g, \hat{g}$ either identically equal to 1, or otherwise one of the functions sech$^2 m(\cdot)$ or tanh$^2 m(\cdot)$. It follows that $G(k, l) = h(k) \hat{h}(l) F(k, l)$ is a Schwartz function, and that it is sufficient to establish that for such $g, \hat{g}, G$

$$
\lim_{\kappa \to +\infty} \int_{|x| \geq 10} 1_{|x| \geq 10} \left( x \right) G(k, l) \left( x - x' \right) \delta^{(1)}(x - x', x'' ) g(x') \hat{g}(x'') e^{ikx' + ltx''} dx' dx'' d\kappa d\lambda = \int_{|x| \geq 10} 1_{|x| \geq 10} \left( x \right) G(k, l) g(x) \hat{g}(x) e^{i(k + l)x} dx d\kappa d\lambda.
$$

(3.24)
After two integration by parts (in \( k \) and \( l \)), the right hand side can be written as

\[
\int_{\mathbb{R}^3} \mathbf{1}_{\{|x| \geq 10\}}(x)G(k, l)g(x)\hat{g}(x)e^{i(k+l)x}dkdldx = \int_{\mathbb{R}^3} \mathbf{1}_{\{|x| \geq 10\}}(x) \frac{\partial^2_{k,l} G(k, l)}{(ix)^2} g(x)\hat{g}(x)e^{i(k+l)x}dkdldx
\]

(3.26)

Carrying out the same integration by parts on the left hand side leads to

\[
\int_{\mathbb{R}^3} \mathbf{1}_{\{|x| \geq 10\}}(x) \left[ \int_{\mathbb{R}^2} \frac{G(k, l)}{(ix')^2} \delta^{[c]}(x-x') \delta^{[c]}(x-x'')g(x')\hat{g}(x'')e^{ikx'+ix''} dx' dx'' \right]dkdldx
\]

(3.27)

Noting that for \( \kappa \) large the function \( \delta^{[c]}(x-x') \) vanishes unless \( |x-x'| \leq \kappa^{-1} < 1 \), the integrand over the outer \( \mathbb{R}^3 \) integral can be bounded by

\[
\text{const.} \mathbf{1}_{\{|x| \geq 10\}}(x) \int \frac{|\partial^2_{k,l} G(k, l)| \delta^{[c]}(x-x') \delta^{[c]}(x-x'')|}{1 + x^2} dx' dx'' \leq \text{const.} \mathbf{1}_{\{|x| \geq 10\}}(x) \frac{|\partial^2_{k,l} G(k, l)|}{1 + x^2}
\]

which is integrable and independent of \( \kappa \). Hence by the dominated convergence theorem the limit of (3.27) exists and equals (3.26), establishing (3.24). Combining this with the argument for \( |x| \leq 10 \), we have proved that

\[
\lim_{\kappa \to +\infty} \frac{1}{4\pi} \int F(k, l) \left( \frac{\omega_k (\omega_k - \omega_l)}{\sqrt{\omega_k \omega_l}} \right) \delta^{[c]}(x-x') \delta^{[c]}(x-x'') E_k(x') E_l(x'') dx' dx'' dkdldx
\]

\[
= \frac{1}{4\pi} \int F(k, l) \left( \frac{\omega_k (\omega_k - \omega_l)}{\sqrt{\omega_k \omega_l}} \right) E_k(x) E_l(x) dkdldx
\]

(which is actually zero by the orthogonality relations for the \( E_k \)). The proof for the other terms is similar.

**Step Three.** Since the limiting expression defines the easy to understand self-adjoint operator \( \hat{H}^{\text{sol}}_{0:} \) in (2.41), we can use the limit of (3.27) to define a self-adjoint operator \( O \) on \( \mathcal{F}_0 \), whose domain is \( \mathcal{V}^{-1}\text{Dom}(\hat{H}^{\text{sol}}_{0:}) \) and such that \( \mathcal{V}O \mathcal{V}^{-1} \) equals the right hand side of (1.19). It remains to relate \( O \) to the operator \( \hat{H}^{\text{sol}}_{0:} \) (as defined in the vacuum representation (2.10)-(2.17)) on \( \text{Dom}(\hat{H}^{\text{vac}}_{0:}) \). For this purpose it is useful to work at the level of quadratic forms, interchangeably using the Fock and Schrödinger solitonic representations, indicating the latter with boldface, and writing \( \mathcal{P} \) for the dense set generated by the polynomials in the field in either case, see Remark 2.16.

The self-adjoint operator \( \hat{H}^{\text{sol}}_{0:} \) is related to the closed quadratic form \( (\Psi, \hat{H}^{\text{sol}}_{0:} \Psi) \), which converts under the unitary transformation \( \mathcal{V} \) into the quadratic form \( (\hat{\Psi}, (\hat{H}^{\text{sol}}_{0:}) + \Delta M_{\text{sol}}) \hat{\Psi} ) \), with \( \hat{\Psi} = \mathcal{V} \Psi \). (Here \( \Delta M_{\text{sol}} \) is the quantity appearing in (3.29) with limit \( \Delta M_{\text{sol}} \)). The relation between \( \Psi \) and \( \hat{\Psi} \) is as described in the proof of Corollary 2.14 and the Radon-Nikodym derivative (2.31) which appears there itself lies in \( L^p(d\mu_0) \) for some \( p > 1 \). Now we have the formula

\[
\hat{H}^{\text{sol}}_{0:} = \hat{H}^{\text{vac}}_{0:} + 3 \int \text{sech}^2mx : \varphi_\kappa(x)^2 : dx = \mathcal{H}(\omega_\kappa) - 3 \int \text{sech}^2mx : \varphi_\kappa(x)^2 : dx
\]

in which the regularized dispersion relation is as in Remark 3.1. Recall (from (2.20)) that \( \hat{H}^{\text{vac}}_{0:} = \mathcal{H}(\omega_\kappa) \) is self-adjoint with domain defined in (2.21), while for finite \( \kappa \) the corresponding regularized operator \( \hat{H}^{\text{sol}}_{0:} = \mathcal{H}(\omega_\kappa) \) is bounded on \( \text{Dom}(\mathbb{N}_0) \). Writing \( v(\varphi) = 3 \int \text{sech}^2mx : \varphi(x)^2 : dx \) we get

\[
(\hat{\Psi}, \mathcal{H}(\omega_\kappa) \hat{\Psi}) = (\hat{\Psi}, (\hat{H}^{\text{sol}}_{0:} + \Delta M_{\text{sol}}) \hat{\Psi}) + (\hat{\Psi}, v(\varphi_\kappa) \hat{\Psi})
\]

Now consider the limits of the three terms in the above equation.

- We have already noted that the first term on the right side converges for \( \hat{\Psi} \in \mathcal{P} \) to the quadratic form \( (\hat{\Psi}, (\hat{H}^{\text{sol}}_{0:} + \Delta M_{\text{sol}}) \hat{\Psi}) \).
Lemma 3.4 implies that \( \int f \varphi(x) \delta^{[\sigma]}(x-x') \sec^2 m x' \varphi(x') \, dx' dx \) converges to \( \int \sec^2 m x \varphi(x)^2 \, dx \) as \( \kappa \to +\infty \) in every \( L^p(d\mu_0) \), \( p < \infty \), see [7] Section 5. Therefore the second term on the right also converges to \( \langle \Psi, v(\varphi) \Psi \rangle \) as long as \( \Psi \in L^p \) for some \( p \in (2, \infty) \). But \( \mathcal{V}^{-1} \mathcal{P} \subset L^p \) for some \( p > 2 \) by Theorem 2.14 and so convergence holds for \( \Psi \in \mathcal{V}^{-1} \mathcal{P} \).

Finally, referring to Remark 3.1 and noting that \( \omega_k, \kappa \nearrow \omega_k \) monotonically as \( \kappa \nearrow +\infty \), we deduce by the monotone convergence theorem that the left hand side converges as \( \kappa \) goes to infinity:

\[
\lim_{\kappa \to +\infty} (\langle \Psi, :H_0^{\text{vac}}:\Psi \rangle_{\mathcal{F}_0} = (\Psi, :H_0^{\text{vac}}:\Psi)_{\mathcal{F}_0},
\]

(actually for any \( \Psi \in \mathcal{F}_0 \), with a finite limit occurring precisely when \( \Psi \in \text{Dom}(H_0^{\text{vac}}) \)). Positive closable quadratic forms determine self-adjoint operators uniquely, so the operator \( :H_0^{\text{vac}}: \) is fixed by the above.

**Step Four.** To conclude we have established that

\[
(\Psi, :H_0^{\text{vac}}:\Psi) = (\hat{\Psi}, :H_0^{\text{sol}}:\hat{\Psi}) + \Delta \text{M}_{\text{scl}} + (\Psi, v(\varphi)\Psi)
\]

(3.28)

for \( \hat{\Psi} \in \mathcal{P} \), or \( \Psi \in \mathcal{V}^{-1} \mathcal{P} \). Polarizing (3.28) with \( \hat{\chi} = \chi \) yields

\[
\| :H_0^{\text{vac}}:\Psi \| = \sup_{\chi \in \mathcal{V}^{-1} \mathcal{P}, \| \chi \| = 1} (\chi, :H_0^{\text{vac}}:\Psi) = \sup_{\chi \in \mathcal{V}^{-1} \mathcal{P}, \| \chi \| = 1} \left( (\hat{\chi}, :H_0^{\text{sol}}:\hat{\chi}) + \Delta \text{M}_{\text{scl}} + (\chi, v(\varphi)\Psi) \right)
\]

which is finite by the aforementioned \( L^p \) properties of \( v(\varphi) \) and \( \Psi \in \mathcal{V}^{-1} \mathcal{P} \). Therefore

\[
\mathcal{V}^{-1} \mathcal{P} \subset \text{Dom}(H_0^{\text{vac}}).
\]

Now \( \text{Dom}(H_0^{\text{vac}}) \) is dense in \( \mathcal{F}_0 \) and so \( T := H_0^{\text{vac}}|_{\mathcal{V}^{-1} \mathcal{P}} \) is a symmetric operator expressible as a direct sum of operators of multiplication by \( \sum \omega_k \kappa \), in standard Fock space form, and \( T \) has a closed extension, namely the self-adjoint operator \( H_0^{\text{vac}} \) with domain \( \mathcal{F}_{21} \); furthermore, using this multiplication operator form, the closure of the graph of \( T \), namely \( \Gamma(T) \), is easily seen to be \( \Gamma(H_0^{\text{vac}}) \). But since \( T \) is closable, results in [22], p. 250 imply that its closure \( \overline{T} \) has graph \( \Gamma(\overline{T}) = \Gamma(T) \), and hence \( \overline{T} = H_0^{\text{vac}} \), which establishes that \( \mathcal{V}^{-1} \mathcal{P} \) is a core for \( H_0^{\text{vac}} \). Therefore, since \( v(\varphi) \) is well-defined on \( \text{Dom}(H_0^{\text{vac}}) \) it is a consequence of Wüst’s Theorem [22] Theorem X.14 that \( H_0^{\text{sol}} := H_0^{\text{vac}} + v(\varphi) \) is essentially self-adjoint on \( \mathcal{V}^{-1} \mathcal{P} \) (since this subspace is a core for \( H_0^{\text{vac}} \)). Furthermore, (1.19) holds thus identifying the operator \( O \) defined above as \( H_0^{\text{sol}} \).

A slight strengthening of this result which will be useful can be read off as a corollary of the proof. Let \( \hat{\mathcal{P}}(\varphi) \) be defined as the space of finite complex linear combinations of functions \( g(Q)b(h Q) \text{Sym}^n \prod_{j=1}^n f_j(k_j) \in L^2(dQ) \otimes \mathcal{F} \) where all the \( h, \{ f_j \} \) are Hermite and Schwartz functions exactly as before, but \( g \) is now allowed to run through functions of the form

\[
g_\alpha(Q; \sigma) = \exp \left[ -\frac{\alpha Q^2}{4\sigma^2} - \frac{Q^2}{4\sigma^2} \right] H_0(\frac{Q}{\sigma})
\]

for all \( \sigma > 0 \) and real \( \alpha \). This is useful because it is invariant under the action of the unitary group \( \exp[-itH_0^{\text{sol}}] \) - see formula (1.6). Also, by Proposition 2.17, \( \mathcal{V}^{-1} F \in L^p(d\mu_0) \) for any such \( F \in \hat{\mathcal{P}}(\varphi) \) and by the argument in the preceding proof \( \mathcal{V}^{-1} \mathcal{P} \subset \text{Dom}(H_0^{\text{vac}}) \subset \text{Dom}(H_0^{\text{sol}}) \). It then follows from [22] Theorem VIII.11 that \( \mathcal{V}^{-1} \hat{\mathcal{P}} \) is a core for \( H_0^{\text{sol}} \). To summarize

**Corollary 3.8.** The space \( \hat{\mathcal{P}}(\varphi) \) is invariant under the unitary evolution generated by \( iH_0^{\text{sol}} \) and \( \mathcal{V}^{-1} \hat{\mathcal{P}} \) is a core for \( H_0^{\text{sol}} \).
3.4 Computation of the mass shift - proof of Lemma [3.6]

For the main calculation we ignore the factor $\frac{1}{2}$ and will reinsert it at the end. From (3.13) we have the formula:

$$K^\frac{1}{2}(x,y) = \sqrt{3} m \psi_1(x) \psi_1(y)$$  \hspace{1cm} (3.30)

$$+ \frac{1}{2\pi} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \left[(k^2 - 3imk \tanh my' + 2m^2 - 3m^2 \tanh^2 my') \delta(k, y') e^{ik(x' - y')} \times \delta(x - x') \left(\frac{-k^2 - 3imk \tanh mx' + 2m^2 - 3m^2 \tanh^2 mx'}{(k^2 + m^2)(k^2 + 4m^2)^\frac{3}{2}}\right)\right] \, dk \, dx \, dy \, dk,$$  \hspace{1cm} (3.31)

$$C_{0,\kappa}^\frac{1}{2}(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \left(k^2 + 4m^2)^\frac{3}{2} \delta(k, y') e^{ik(x' - y')} \delta(x - x') \, dx' \, dy' \, dk \right. \hspace{1cm} \text{and} \hspace{1cm}$$  \hspace{1cm} (3.32)

When the regularization is removed, i.e., when $\kappa = +\infty$, the first two integrals are quadratically divergent, while the third is logarithmically divergent. The fact that the final answer, (3.29), is finite is due to cancellations. It is necessary to handle these carefully, because the actual limit is not the naive $\kappa = +\infty$ limit which is defined by combining the three integrals (3.30), (3.31) and (3.32) into one and then replacing $\delta(k, y')$ by the delta function $\delta$ and performing cancellations. Doing this leads to

$$\Delta M_{\text{vac}}^\text{naive} = \sqrt{3} m + \frac{1}{2\pi} \int_{\mathbb{R}^4} \frac{9m^4 \tanh^2 mx (\tanh^2 mx - 1)}{(k^2 + m^2)(k^2 + 4m^2)^\frac{3}{2}} \, dk \, dx = \frac{m}{\sqrt{3}}. \hspace{1cm} (3.33)$$

The difference of the first two terms in the integrand (3.29) can be written

$$\left(K^\frac{1}{2} - K_{0,\kappa}^\frac{1}{2}\right)(x,y) = \sqrt{3} m (\psi_1(x))^2 + \frac{1}{2\pi} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} F(k, x, y') \delta(x - x') \delta(y - y') \, dx' \, dy' \, dk,$$

where

$$F(k, x, y) = \left[(k^2 - 3imk \tanh my + 2m^2 - 3m^2 \tanh^2 my)(-k^2 - 3imk \tanh mx + 2m^2 - 3m^2 \tanh^2 mx) \right. \hspace{1cm}$$

$$\left. \quad \quad - (k^2 + m^2)(k^2 + 4m^2)^\frac{3}{2}\right].$$

(Notice the cancellation of the $k^4$ term for all $x, y$ and also the $k^3$ term when $y = x$.) The limit $\kappa \to +\infty$ can be taken through the integral rather directly for $j = 0, 1$, but for $j = 2, 3$ it is necessary to look more carefully.

For $j = 0$: define new integration variables $\xi = \kappa(x' - x)$ and $\eta = \kappa(y' - x)$ in place of $x', y'$. This leads to the integrand

$$\frac{1}{2\pi} \left(9m^4 \tanh^2 m(x + \xi/\kappa) \tanh^2 m(x + \eta/\kappa) - 6m^4 (\tanh^2 m(x + \xi/\kappa) + \tanh^2 m(x + \eta/\kappa)) \right) \times \delta^{[1]}(\xi) e^{ik(\xi - \eta)/\kappa} \delta^{[1]}(\eta) \left((k^2 + m^2)(k^2 + 4m^2)^\frac{3}{2}\right).$$

Since $\delta^{[1]}$ is a non-negative, smooth function which is supported inside $[-1, 1]$, it is easy to see, by considering the cases $|x| \geq 2/\kappa$ and $|x| \leq 2/\kappa$, that this integrand is dominated by

$$\text{const} e^{-m|x|/2} \delta^{[1]}(\xi) \delta^{[1]}(\eta)(m^2 + k^2)^{-3/2} \in L^1(dx d\xi d\eta d\kappa)$$
with \( \text{const.} \) a fixed number which is independent of \( \kappa > 1 \). It follows that the limit \( \kappa \to +\infty \) through the integral can be taken directly by the dominated convergence theorem, leading to

\[
\frac{1}{2\pi} \int_{R \times R} \frac{(9\sech^4 mx - 12\sech^2 mx)}{(k^2 + m^2)(k^2 + 4m^2)^{\frac{3}{2}}} \, dx \, dk = -\frac{6m^3}{\pi}.
\]

To this should be added the contribution \( \sqrt{3}m \) from the discrete mode, and also from the term in \( C_{0,k}(x,x) \) corresponding to \( j = 0 \), leading to the answer

\[
\sqrt{3}m + \frac{1}{2\pi} \int_{R \times R} \frac{(9m^3 \sech^4 mx - 12m^4 \sech^2 mx)}{(k^2 + m^2)(k^2 + 4m^2)^{\frac{3}{2}}} \, dx \, dk + \frac{1}{2\pi} \int_{R \times R} \frac{3m^4 \sech^2 mx}{(k^2 + m^2)(k^2 + 4m^2)^{\frac{3}{2}}} \, dx \, dk = \frac{m}{\sqrt{3}}.
\]

This is precisely the naive answer (3.33). The correct answer (3.23) comes from a careful evaluation of the limiting values of the remaining integrals, whose naive limits are all zero.

For \( j = 1 \): the same change of variables leads to the integrand

\[
\frac{1}{2\pi} \left( \tanh m(x + \eta/\kappa)(2m^2 - 3m^2 \sech^2 m(x + \xi/\kappa)) - \tanh m(x + \xi/\kappa)(2m^2 - 3m^2 \sech^2 m(x + \eta/\kappa)) \right) \times \frac{3ikm\delta [1](\xi)e^{ik(\xi^2 - 2\eta)(\xi/\kappa)}\delta [1](\eta)}{(k^2 + m^2)(k^2 + 4m^2)^{\frac{3}{2}}}.
\]

The only difference with the \( j = 0 \) case is that it is necessary to write

\[
\tanh m(x + \xi/\kappa) - \tanh m(x + \eta/\kappa) = \int_0^1 m \sech^2 m(x + \theta \xi/\kappa + (1 - \theta)\eta/\kappa) \, d\theta,
\]

to conclude similarly that the integrand is dominated for \( \kappa > 1 \) by

\[
\text{const.} \cdot e^{-m|x|/2\delta [1](\xi)\delta [1](\eta)(m^2 + k^2)^{-1}} \in L^1(dx \, d\xi \, d\eta \, dk)
\]

so that the limit through the integral can be taken directly, and this limiting value is zero.

For \( j = 3 \): the integrand is equal to \( \frac{1}{\sqrt{3}} \) times

\[
(3im^3 \tanh mx' - 3im^3 \tanh my') \delta [1](x - y') \delta [1](x - x') e^{ik(x' - y')}/(k^2 + m^2)(k^2 + 4m^2)^{\frac{3}{2}}
\]

so that the integral \( dk \) is naively linearly divergent. However, writing \( e^{ik(x' - y')} = \frac{1}{\pi(x' - y')} \frac{d}{dk} e^{ik(x' - y')} \) and using the change of variables above, we can write

\[
\frac{3im^3}{(k^2 + m^2)(k^2 + 4m^2)^{\frac{3}{2}}} \int_0^1 m \sech^2 m(x + \theta \xi/\kappa + (1 - \theta)\eta/\kappa) \, d\theta \, \delta [1](\xi)\delta [1](\eta) \frac{d}{dk} e^{ik(\xi - \eta)/\kappa} \, d\xi \, d\eta.
\]

The integral \( d\xi \, d\eta \) is essentially a two dimensional Fourier transform of a smooth compactly supported function of \( \xi, \eta \), and as such decays rapidly as \( k \to \infty \) for any fixed \( \kappa > 0 \). Therefore, it is permissible to integrate by parts in \( k \), leading to the integrand

\[
\left( -\frac{d}{dk} \frac{3im^3}{(k^2 + m^2)(k^2 + 4m^2)^{\frac{3}{2}}} \right) \delta [1](\xi)\delta [1](\eta) e^{ik(\xi - \eta)/\kappa} \int_0^1 m \sech^2 m(x + \theta \xi/\kappa + (1 - \theta)\eta/\kappa) \, d\theta.
\]

The limit, as \( \kappa \to \infty \), of this integrated over \( x, k, \xi, \eta \in \mathbb{R}^4 \) is what is needed. It is easy to check that the integrand is dominated by a function of the same form as in the cases above, so the limit can be taken through the integral. The value of the limit is therefore

\[
-\int_{\mathbb{R}} m \sech^2 mx \times \left[ \frac{3im^3}{(k^2 + m^2)(k^2 + 4m^2)^{\frac{3}{2}}} \right]_{-\infty}^{+\infty} \, dx = -6m^2 \int \sech^2 mx \, dx.
\]
Reintroducing the $1/(2\pi)$ factor gives the overall contribution $-\frac{6m^2}{\pi^2}\sech^2 mx$, in place of the naive value of zero from the $j = 3$ term. Performing the integral over $x$ leads to the value $-6m/\pi$ which is the required correction to the naive value to give the correct mass shift (3.23). It remains to show that the remaining terms with $j = 2$ do not contribute further corrections.

For $j = 2$: it is necessary to combine the integral involving $F_2$ with the corresponding naively logarithmically divergent term $C_{0,\kappa}^2(x, x)$. All together this leads to

$$
\int_{\mathbb{R} \times \mathbb{R}} \left[ 3m^2(\sech^2 mx + \sech^2 my) - 9m^2\sech mx \sech my \cosh (x' - y') + 3m^2\sech^2 mx \right] \times \delta^{[\kappa]}(x - y') e^{ik(x' - y')} \delta^{[\kappa]}(x - x') \, dx' \, dy' \quad (3.35)
$$

all multiplied by $\frac{k^2}{(k^2 + m^2)(k^2 + 4m^2)^{1/2}}$, and integrated over $(k, x) \in \mathbb{R} \times \mathbb{R}$. Notice that the naive value of this integral, obtained by everywhere replacing $\delta^{[\kappa]}$ by the delta function $\delta$, is zero; we must prove that the limit as $\kappa \to \infty$ of the integral really is zero. Write the quantity in the square brackets in (3.35) as

$$
\left[ \right] = 9m^2 \sech mx \sech my \left( 1 - \cosh m(x' - y') \right) + 3m^2 \left( \frac{3}{2} \left( \sech mx' - \sech my' \right)^2 + \frac{\sech^2 mx - \sech^2 mx'}{2} + \frac{\sech^2 mx - \sech^2 my'}{2} \right) \quad (3.36)
$$

The first two terms in (3.36) are handled as in (3.34), with the conclusion that their contribution is zero (due to the quadratic order of vanishing of $1 - \cosh m(x' - y')$ and $(\sech mx' - \sech my')^2$ at $x' = y'$). To handle the final two terms, write $e^{ik(x' - y')} = e^{ik(x' - x)} e^{ik(x - y')}$, and then

$$
3m^2(\sech^2 mx - \sech^2 mx') \delta^{[\kappa]}(x - y') \delta^{[\kappa]}(x - x') e^{ik(x' - x)} = 3m^2(\sech^2 mx - \sech^2 mx') \delta^{[\kappa]}(x - y') \delta^{[\kappa]}(x - x') \frac{d}{dk} e^{ik(x' - x)} ,
$$

and similarly with $x'$ replaced by $y'$. Now define, as above, $\xi = \kappa(x' - x)$ and $\eta = \kappa(y' - x)$, so that

$$
\sech^2 m(x') - \sech^2 mx = \frac{2m\xi}{\kappa} \rho_{x,\kappa}(\xi) , \quad \rho_{x,\kappa}(\xi) = \int_0^1 \sech^2 m(x + \theta \xi / \kappa) \tanh m(x + \theta \xi / \kappa) \, d\theta ,
$$

and similarly with $x'$ replaced by $y'$. The final two terms in (3.36) then contribute

$$
\int_{\mathbb{R} \times \mathbb{R}} \frac{3m^2}{i} \left( \frac{k^2}{(k^2 + m^2)(k^2 + 4m^2)^{1/2}} \right) \times \int_{\mathbb{R} \times \mathbb{R}} \delta^{[1]}(\xi) \delta^{[1]}(\eta) \left[ e^{-ik\eta/\kappa} \rho_{x,\kappa}(\xi) \frac{d}{dk} e^{ik\xi/\kappa} - e^{ik\xi/\kappa} \rho_{x,\kappa}(\eta) \frac{d}{dk} e^{-ik\eta/\kappa} \right] \, d\xi \, d\eta \, dk \, dx
$$

$$
= \frac{3m^2}{i} \int_{\mathbb{R} \times \mathbb{R}} \frac{d}{dk} \left( \frac{\sqrt{2\pi} \delta^{[1]}(k/\kappa) k^2}{(k^2 + m^2)(k^2 + 4m^2)^{1/2}} \right) \int_{\mathbb{R} \times \mathbb{R}} \delta^{[1]}(\xi) \left( \rho_{x,\kappa}(\xi) - \rho_{x,\kappa}(-\xi) \right) e^{ik\xi/\kappa} \, d\xi \, dk \, dx .
$$

where we have used the assumption that $\delta^{[1]}$ is even, and have relabelled the dummy variable $\eta$ as $\xi$ in the second term, to show that the integrand has pointwise limit zero as $\kappa \to +\infty$. To see that this integral has limit zero we apply the product rule to get

$$
\frac{d}{dk} \left( \frac{\delta^{[1]}(k/\kappa) k^2}{(k^2 + m^2)(k^2 + 4m^2)^{1/2}} \right) = \frac{k^2}{(k^2 + m^2)(k^2 + 4m^2)^{1/2}} \frac{d}{dk} \delta^{[1]}(k/\kappa) + \delta^{[1]}(k/\kappa) \frac{d}{dk} \left( \frac{k^2}{(k^2 + m^2)(k^2 + 4m^2)^{1/2}} \right)
$$

and consider the resulting two integrals separately. For the first integral, estimate

$$
\left| \frac{k^2}{(k^2 + m^2)(k^2 + 4m^2)^{1/2}} \right|_{L^1(dk)} = O(\kappa^{-1}) ,
$$

33
as \( \kappa \to +\infty \), to start with. Next, observe that
\[
|\delta^{[1]}(\xi)(\rho_{x,\kappa}(\xi) - \rho_{x,\kappa}(-\xi))e^{ik/k}| \leq \text{const.} |\delta^{[1]}(\xi)| e^{-m|x|/2} \in L^1(dx d\xi),
\]
uniformly in \( k, \kappa > 1 \). It follows that the the first integral is \( O(\kappa^{-1}) \). For the second integral, observe that
\[
\left| \frac{\delta^{[1]}(k/\kappa)}{\kappa} \frac{d}{dk} \left( \frac{k^2}{(k^2 + m^2)(k^2 + 4m^2)^{1/2}} \right) \right| \leq \text{const.} \frac{1}{(k^2 + m^2)} \in L^1(dk)
\]
with \( \text{const.} \) independent of \( \kappa \). But then, since by inspection
\[
\lim_{\kappa \to \infty} |\delta^{[1]}(\xi)(\rho_{x,\kappa}(\xi) - \rho_{x,\kappa}(-\xi))e^{ik/k}| = 0,
\]
it follows from the dominated convergence theorem that the limit as \( \kappa \to +\infty \) of the second integral is also zero.

The conclusion of all the above is that the naive limit of the logarithmically divergent \( j = 2 \) term is equal to the true limit, but this is not so for the linearly divergent \( j = 3 \) term, whose true limit is equal to \(-\frac{6m^2}{2\pi} \text{sech}^2 mx \). Reinserting the factor of \( \frac{1}{2} \) leads to the final answer
\[
\Delta M_{\text{act}} = \Delta M_{\text{act}}^{\text{naive}} + \int_{\mathbb{R}} -\frac{3m^2}{2\pi} \text{sech}^2 mx \, dx = \frac{m}{2\sqrt{3}} - \frac{3m}{\pi},
\]
as claimed in (3.20).

### 3.5 Change of representation - interaction terms

We now compute the effect of the change of representation from (2.16)-(2.17) to (2.40)-2.41 on the interaction Hamiltonian (3.15). As shorthand write \( Y = -\sqrt{M_s}Q\psi_0(x) \) and
\[
\hat{\phi}_\kappa(x) = +\frac{1}{\sqrt{2\omega_d}} (a_d + a^\dagger_d) \psi_1(x)
\]
\begin{align*}
&+ \frac{1}{\sqrt{2\pi}} \int \int \frac{\delta^{[\kappa]}(x - x')}{\sqrt{2\omega_k}} \left(a_k E_k(x') + a_k^\dagger E_k^*(x') \right) \, dx' \, dk,
\end{align*}
so that \( \phi_\kappa(x) = Y + \hat{\phi}_\kappa(x) \). We write \( \hat{\phi}(x) \) for the corresponding unregularized expression, which is to be interpreted as an operator valued distribution. For comparison with (3.4) we note the formula
\[
(0 | \hat{\phi}_\kappa(x) \hat{\phi}_\kappa(y) | 0) = \frac{1}{4\pi} \int \int \frac{\delta^{[\kappa]}(x - x') E_k(x') E_k^*(y') \delta^{[\kappa]}(y - y')}{(k^2 + 4m^2)^{1/2}} \, dx' \, dy' \, dk.
\]
Define \( \gamma_\kappa(x) = \langle 0 | \hat{\phi}_\kappa(x) \hat{\phi}_\kappa(x) | 0 \rangle \) and note the fact that \( \delta \gamma_\kappa = \gamma_\kappa(x) - \gamma_\kappa \) is uniformly bounded as \( \kappa \to +\infty \), by an easier version of the calculations in (3.4).

With this notation we write \( \hat{H}_{I,\kappa}^{\text{sol}} = \int \hat{H}_{I,\kappa}^{\text{sol}} \, dx \) where
\[
\hat{H}_{I,\kappa}^{\text{sol}} \equiv \mathcal{V} \circ \hat{H}_{I,\kappa,\varphi}^{\text{sol}} \circ \varphi^{-1} = \left[ 2mg \tanh mx \mathcal{V} \circ \varphi_\kappa \circ \varphi^{-1} + \frac{1}{2} \varphi^2 \mathcal{V} \circ \varphi^4 \circ \varphi^{-1} \right],
\]
where
\[
\mathcal{V} \circ \varphi_\kappa \circ \varphi^{-1} = Y^3 + 3Y^2 \hat{\phi}_\kappa + 3Y \hat{\phi}_\kappa^2 + \hat{\phi}_\kappa^3 + 3 \delta \gamma_\kappa \hat{\phi}_\kappa + 3Y \delta \gamma_\kappa,
\]
and
\[
\mathcal{V} \circ \varphi^4 \circ \varphi^{-1} = Y^4 + 4Y^3 \hat{\phi}_\kappa + 6Y^2 \hat{\phi}_\kappa^2 + 4Y \hat{\phi}_\kappa^3 + \hat{\phi}_\kappa^4 + 6Y^2 \delta \gamma_\kappa + 12Y \delta \gamma_\kappa \hat{\phi}_\kappa - 6 \delta \gamma_\kappa \hat{\phi}_\kappa^2 + 3 \delta \gamma_\kappa^2.
\]
Integration of these densities gives generalizations of the Wick monomials in (2.4) which can be estimated by the following lemma.
Lemma 3.9. Let \( b \in L^2(\mathbb{R}) \), then both \( (\int b(x) \hat{\Phi}_\kappa^n(x) : dx)(1 + \hat{N})^{-n/2} \) and the corresponding Wick monomial formed from \( \hat{\Phi}_\kappa \) define bounded operators on \( \mathcal{F} \) and, in operator norm,

\[
\lim_{\kappa \to +\infty} \left( \int b(x) : \hat{\Phi}_\kappa^n(x) : dx - \int b(x) : \hat{\Phi}_\kappa^n(x) : dx \right)(1 + \hat{N})^{-n/2} = 0.
\]

Proof. Consider the monomials formed by inserting the expression for \( \hat{\Phi} \) into \( (\int b(x) : \hat{\Phi}_\kappa^n(x) : dx)(1 + \hat{N})^{-n/2} \). Writing (with reference to (A.4))

\[
E_k(x) = e^{ikx} y(x; k), \quad y(x; k) = y_0(k) + y_1(k) \tanh mx + y_2(k) \text{sech}^2 mx
\]

where

\[
\sup_k (|y_0(k)| + \omega_k |y_1(k)| + \omega_k^2 |y_2(k)|) < \infty,
\]

and observing that \( mx b(x) \) and \( \text{sech}^2 mx b(x) \) are both in \( L^2(dx) \), the proof of the boundedness assertion reduces to the standard case (2.13)-(2.15) treated in [7, Section 5].

Next we prove the approximation result. Using the alternative regularization \( \hat{\Phi}_\kappa^{alt} \) as defined by the final integral in (3.8), the corresponding assertion

\[
\lim_{\kappa \to +\infty} \left( \int b(x) : (\hat{\Phi}_\kappa^{alt})^n(x) : dx - \int b(x) : \hat{\Phi}_\kappa^n(x) : dx \right)(1 + \hat{N})^{-n/2} = 0.
\]

is an essentially immediate consequence of (2.15) via a minor modification of the calculation in [7, Proposition 5.8]. In order to establish this result for \( \hat{\Phi}_\kappa \) we consider the effect of this change of regularization on a typical kernel for one of the Wick operators (2.11) which appear on substitution of the field into \( \int b(x) : (\hat{\Phi}_\kappa)^n(x) : dx \).

A typical kernel in the resultant sum of Wick operators is proportional to

\[
\int b(x) \prod \frac{\delta^{[|\kappa|]} * E_{k_j}(x)}{\sqrt{2\pi \omega_{k_j}}},
\]

while for \( \int b(x) : (\hat{\Phi}_\kappa^{alt})^n(x) : dx \) the corresponding kernel is

\[
\int b(x) \prod \frac{\delta^{[1]}(k_j/\kappa) E_{k_j}(x)}{\sqrt{\omega_{k_j}}}.
\]

The difference between an individual pair of factors is proportional to \( 1/\sqrt{\omega_{k_j}} \) times

\[
g(k_j; \kappa) \overset{\text{def}}{=} \delta^{[|\kappa|]} * E_{k_j}(x) - \sqrt{2\pi} \delta^{[1]}(k_j/\kappa) E_{k_j}(x) = \int \delta^{[1]}(u) e^{ik_j(x-u/\kappa)} (y(x-u/\kappa; k_j) - y(x; k_j)) du.
\]

Next, refer again to (3.10) and observe that both functions \( \tanh mx \) and \( \text{sech}^2 mx \) have derivatives bounded by \( \text{const.} e^{-m|x|/2} / (\kappa \omega_k) \) and hence that

\[
\int b(x) \prod \frac{\delta^{[|\kappa|]} * E_{k_j}(x)}{\sqrt{2\pi \omega_{k_j}}} dx - \int b(x) \prod \frac{\delta^{[1]}(k_j/\kappa) E_{k_j}(x)}{\sqrt{\omega_{k_j}}} dx
\]

\[
= \int b(x) \prod \frac{\delta^{[1]}(k_j/\kappa) E_{k_j}(x) + g(k_j; \kappa)}{\sqrt{\omega_{k_j}}} - \int b(x) \prod \frac{\delta^{[1]}(k_j/\kappa) E_{k_j}(x)}{\sqrt{\omega_{k_j}}} dx
\]

can be bounded in \( L^2(\mathbb{R}^n; \prod dk_j) \) by \( \text{const.} \| b \|_{L^2(1 + \ln \kappa)^{n-1}/\kappa} \) which completes the proof.
4 Dynamics

In this section an analysis of the dynamics generated by the quantization of the Hamiltonian (1.10) in the limit $g \to 0^+$ is given. We first consider the vacuum case (1.4) which is very simple but worth stating for purposes of comparison with the solitonic case. The framework used is that of the standard representation of the Heisenberg relations (2.10)-(2.17), acting on the Hilbert space $\mathcal{F}_0$, and leads to the expected limiting dynamics, namely a free relativistic field describing an assembly of bosons of mass $2m$ governed by the quadratic Hamiltonian $H_0^\text{vac}$.

**Theorem 4.1.** In the limit $\kappa \to +\infty$ the operator $H_{g,\kappa}^\text{vac}$ determines a self-adjoint operator $H_g^\text{vac}$ which is bounded below and determines a strongly continuous one-parameter unitary group via the Stone theorem. As the coupling constant $g$ tends to zero, this one-parameter group satisfies

$$\exp[-it H_{g}^\text{vac}] \to \exp[-it H_0^\text{vac}] \quad \text{as } g \to 0^+$$

in the sense of strong pointwise convergence, uniformly for time $|t| \leq t_0(g)$ with $\lim_{g \to 0^+} gt_0(g) = 0$.

**Proof.** The Duhamel formula

$$\exp[-it H_g^\text{vac}] - \exp[-it H_0^\text{vac}] = -i \int_0^t \exp[-i(t-s) H_g^\text{vac}] H_{1,g}^\text{vac} \exp[-is H_0^\text{vac}] \, ds$$

together with unitarity, implies that for any finite particle vector $F \in \mathcal{F}_0$ there holds

$$\|\exp[-it H_g^\text{vac}] F - \exp[-it H_0^\text{vac}] F\| \leq \int_0^t \|H_{1,g}^\text{vac} \exp[-is H_0^\text{vac}] F\| \, ds.$$ 

Now if $F$ is of the form $\prod_{j=1}^M a^\dagger(\chi_j)\Omega_0$, then since $\Omega_0$ is invariant,

$$\exp[-is H_0^\text{vac}] F = \prod_{j=1}^M a^\dagger(e^{-is\chi_j})\Omega_0,$$

so that using (2.15) we can bound

$$\|H_{1,g}^\text{vac} \exp[-is H_0^\text{vac}] F\| = \|H_{1,g}^\text{vac} (1+N_0)^{-2} (1+N_0)^2 \exp[-is H_0^\text{vac}] F\| \leq g \text{const.}(1+M)^2 \sqrt{\prod_{j=1}^M |\chi_j|},$$

for all $s$. This implies immediately that

$$\|\exp[-it H_g^\text{vac}] F - \exp[-it H_0^\text{vac}] F\| \leq \text{const.}|t|g \sqrt{\prod_{j=1}^M |\chi_j|}$$

for $F$ as above, and hence to (4.1) by the density of the finite particle vectors and the fact that (by unitarity)

$$\|\exp[-it H_g^\text{vac}] F_1 - \exp[-it H_0^\text{vac}] F_1\| \leq \|\exp[-it H_g^\text{vac}] F - \exp[-it H_0^\text{vac}] F\| + 2\|F_1 - F\|. \quad \square$$

In order to prove an analogous result in the solitonic case, consider first applying the previous argument using the representation (2.29)-(2.30). The difficulty arises in the use of the analogy to (4.2), which introduces factors which are growing in time into the estimate, due to the presence of the zero mode in the spectral decomposition of the operator $H_0^\text{sol}$; such an explicitly growing solution to the linear equation is given in $\S A.3$ see Remark $A.4$ in particular. On the time intervals of interest, these factors become arbitrarily large as $g \to 0$ (since each creation operator will potentially produce a factor) and so it is essential to find an alternative approach. A method to carry out the generalization successfully is to employ the representation (2.30)-(2.41). This leads to a description of the limiting $g = 0$ dynamics in terms of the nonrelativistic Schrödinger equation for the soliton, in addition to the assembly of relativistic bosons and a pulsation mode for the soliton, as in Theorem (1.1).
Proof of Theorem 1.1. Consider
\[ \Psi_g(t) = \exp[-itH_{g_{0}}^{\text{sol}}] \Psi_g(0), \]
and transform into the representation determined by (2.40) - (2.41), and then show that it is possible to obtain comparison estimates with the evolution generated by \( iH_{g_{0}}^{\text{sol}} \). So define \( \hat{\Psi}_g(t) \equiv \nabla \exp[-itH_{g_{0}}^{\text{sol}}] \Psi_g(0) \), which is a solution of the equation
\[ \frac{\partial}{\partial t} \hat{\Psi}_g = \nabla \circ H_{g_{0}}^{\text{sol}} : \circ \nabla^{-1} \hat{\Psi}_g, \]
with initial data \( \hat{\Psi}_g(0) = \nabla \Psi_g(0) \). Referring to §3.3 and in particular Theorem 3.7 we have
\[ \nabla \circ H_{g_{0}}^{\text{sol}} : \circ \nabla^{-1} = \left[ \frac{M_{1} + \Lambda M_{\text{scl}}}{g_{0}} + iH_{0}^{\text{sol}} : + i\hat{H}_{1, g_{0}}^{\text{sol}} \right], \]
(4.3) where \( i\hat{H}_{1, g_{0}}^{\text{sol}} \) was defined in §3.5. The operator \( \nabla \circ H_{g_{0}}^{\text{sol}} : \circ \nabla^{-1} \) is self-adjoint and generates a unitary semigroup \( \{ U(t) \}_{t \in \mathbb{R}} \) on the space \( \mathcal{D} \) defined in (1.18). Recall from the proof of Theorem 3.7 that \( \nabla^{-1} \mathcal{P} \subset H_{0}^{\text{vac}} \) and \( \nabla^{-1} \mathcal{P} \subset L^{p}(d \mu) \). It follows that \( \nabla^{-1} \mathcal{P} \subset \text{Dom}(iH_{g_{0}}^{\text{sol}}) \) and hence that \( \mathcal{P} \subset \text{Dom}(\nabla \circ H_{g_{0}}^{\text{sol}} : \circ \nabla^{-1}) \).

The Duhamel formula reads
\[ e^{i\Theta(t)} \hat{\Psi}_g(t) - \exp[-itH_{0}^{\text{sol}}] \nabla \Psi_g(0) = -i \int_{0}^{t} U(t - s) \hat{H}_{1, g_{0}}^{\text{sol}} \exp[-isH_{0}^{\text{sol}}] \hat{\Psi}_g(0) \, ds. \]
(4.4) applied to an initial state in \( \mathcal{P}(\phi) \). In fact we can consider a tensor product of a wave packet (1.15) describing the location of the kink with a finite particle state describing the bosons:
\[ \hat{\Psi}_g(0) = \chi_{n}(Q, \sigma)(a_{j}^{\dagger})^{M} \prod_{j=1}^{M} a_{j}(f_{j})\Omega, \]
(4.5) (where \( \Omega \) is the Fock vacuum in \( F \)) since states in \( \mathcal{P}(\phi) \) are finite linear combinations of such vectors. Referring to (1.15) and Appendix A.3 we see that
\[ \exp[-isH_{0}^{\text{sol}}] \chi_{n}(0, Q, \sigma)(a_{j}^{\dagger})^{M} \prod_{j=1}^{M} a_{j}^{\dagger}(f_{j}) \Omega = \chi_{n}(s, Q, \sigma)(e^{is\omega_{a}s}a_{j}^{\dagger})^{M} \prod_{j=1}^{M} a_{j}^{\dagger}(e^{is\omega_{a}s}f_{j}) \Omega. \]
(4.6) This shows that both \( \exp[-isH_{0}^{\text{sol}}] \hat{\Psi}_g(0) \) and \( \hat{H}_{1, g_{0}}^{\text{sol}} \exp[-isH_{0}^{\text{sol}}] \hat{\Psi}_g(0) \) lie in \( \hat{\mathcal{P}}(\phi) \subset \text{Dom}(\nabla \circ H_{g_{0}}^{\text{sol}} : \circ \nabla^{-1}) \). The subspace \( \nabla^{-1} \mathcal{P} \) is invariant under \( \exp[-isH_{g_{0}}^{\text{sol}}] \) and is a core for \( iH_{g_{0}}^{\text{sol}} \), see Corollary 4.6 and hence (4.4) can be proved in the usual way by application of the fundamental theorem of calculus to
\[ U(t - s) \exp[-isH_{0}^{\text{sol}}] \exp[-i\Theta(s)] \nabla \Psi_g(0). \]

By unitarity of all the operators \( \{ U(t - s) \} \) we have
\[ \|e^{i\Theta(t)} \hat{\Psi}_g(t) - \exp[-itH_{0}^{\text{sol}}] \nabla \Psi_g(0)\| \leq \int_{0}^{t} \| \hat{H}_{1, g_{0}}^{\text{sol}} \exp[-isH_{0}^{\text{sol}}] \nabla \Psi_g(0) \| \, ds. \]
(4.7) In the following \( a^{\dagger} \), with \( \iota \in \{ +1, -1 \} \), means either \( a \) if \( \iota = -1 \), or \( a^{\dagger} \) if \( \iota = 1 \). Then referring to (2.14) we have the identity
\[ (\hat{N} + 1)^{2} \prod_{j=1}^{M} a_{j}(g_{j}) = \prod_{j=1}^{M} ((\hat{N} + 1)^{2} - \hat{a}_{j}(g_{j})) \left( \hat{N} + 1 + \sum_{1 \leq k \leq M} \hat{t}_{k} \right)^{2} \prod_{j=1}^{M} \left( \hat{N} + 1 + \sum_{j \leq k \leq M} \hat{t}_{k} \right). \]
In what follows we use the operator norm bound \( \| (\hat{N} + 1)^{\frac{1}{2}} a^{\dagger}(f) \| \leq \| f \| \), and the fact that, which follows from (1.15) by observation, that
\[ \| Q^{*} \chi_{n}(t, Q, \sigma) \|^{2} dQ = \sigma(t)^{2r} |\hat{Q}^{*} \chi_{n}^{0} (\hat{Q})|^{2} d\hat{Q} \]
37
with $\hat{Q} = Q/\sigma(t)$, so that, referring to the discussion following (1.15),

$$\int |X_n(t, Q; \sigma)|^2 |Q|^{2r}dQ = c_{n,r}2^{r}\sigma^{2r}(1+t^2/\tau^2)^r$$

where $c_{n,r}$ is a number arising from the integral involving $X_n^0$, whose precise value is not needed for present purposes. In the choice of wave packets (1.15), recall that the number $\tau > 0$ is arbitrary, while $2\sigma^2 = \tau/\text{Mcl}$. Referring to the form of the interaction term $H^{\text{int}}_I, g$ in §3.5 we see that the right hand side of (1.7) can be bounded by

$$\text{const.} \left( g t_1(g) \left( 1 + \left( \frac{\tau}{\text{Mcl}} + \frac{t_1(g)^2}{\tau \text{Mcl}} \right) \right) + g^2 t_1(g) \left( 1 + \left( \frac{\tau}{\text{Mcl}} + \frac{t_1(g)^2}{\tau \text{Mcl}} \right) \right)^2 \right),$$

with the constant depending upon $n, r, M, m$, but independent of $t, g$. (Here we are using the fact that the contribution of the $Y^3$ term in (3.39) vanishes by parity.) Now choose $\tau = t_1(g)$ to deduce the result for initial data as in (1.9).

It follows from the density of the subspace spanned by initial conditions of the form (4.5) that, given $\hat{\Psi}_r(0)$, there exists a sequence of such initial conditions $\{\hat{\Psi}^\nu_r(0)\}_r$ converging to $\hat{\Psi}_r(0)$, and furthermore by unitarity $\|\hat{\Psi}^\nu_r(t) - \hat{\Psi}_r(t)\| = \|\hat{\Psi}^\nu_r(0) - \hat{\Psi}_r(0)\|$ for all times $t$. Now the corresponding solutions $\hat{\Psi}^\nu_r$ satisfy

$$\lim_{g \to 0} \sup_{|t| \leq t_1(g)} \|\hat{\Psi}^\nu_r(t) - \hat{\Psi}_r(t)e^{-i\Theta(t)}\| = 0,$$

where $\hat{\Psi}^\nu_r(t) = \exp[-itH^{\text{int}}_I]\hat{\Psi}^\nu_r(0)$. Again $\|\hat{\Psi}_r(0) - \hat{\Psi}_r(t)\| = \|\hat{\Psi}_0(0) - \hat{\Psi}_0(0)\| = \|\hat{\Psi}_0(0) - \hat{\Psi}_0(0)\|$, and so the result follows by unitarity and the triangle inequality.

$$\|\hat{\Psi}_r(t) - \hat{\Psi}_0(t)e^{-i\Theta(t)}\| \leq \|\hat{\Psi}^\nu_r(t) - \hat{\Psi}^\nu_r(t)e^{-i\Theta(t)}\| + \|\hat{\Psi}^\nu_r(t) - \hat{\Psi}_r(t)e^{-i\Theta(t)}\| + \|\hat{\Psi}^\nu_r(t) - \hat{\Psi}_r(t)e^{-i\Theta(t)}\| + 2\|\hat{\Psi}^\nu_r(0) - \hat{\Psi}_r(0)\|.$$

### A Appendix: Quantum Mechanics in the Kink Background

The analysis in this article is based on spectral representations for the linear operators which arise on linearization around the kink. The linearized one-particle Hamiltonian is the Schrödinger operator $K = (-\partial_x^2 + 4m^2 - 6m^2\text{sech}^2mx)$. This operator is one of a ladder of differential operators whose eigenfunctions can be written explicitly as follows. Starting with the operator $-\partial_x^2 + m^2$, where $m^2 > 0$, we notice the factorization

$$-\partial_x^2 + m^2 = A A^\dagger, \quad \text{where} \quad A = \partial_x + m \tanh mx \quad \text{and} \quad A^\dagger = -\partial_x + m \tanh mx. \quad (A.1)$$

Paired with $AA^\dagger$ is the operator

$$-\partial_x^2 + m^2 - 2m^2\text{sech}^2mx = A^\dagger A. \quad (A.2)$$

This process repeats: define

$$B = \partial_x + 2m \tanh mx \quad \text{and} \quad B^\dagger = -\partial_x + 2m \tanh mx, \quad (A.2)$$

then compute that

$$B^\dagger B = -\partial_x^2 + 4m^2 - 6m^2\text{sech}^2mx \quad \text{and} \quad BB^\dagger = A^\dagger A + 3m^2. \quad (A.3)$$

A.1 Spectral Resolution and Covariance Operators

It follows from the ladder structure just introduced that if \( AA^\dagger \phi = \epsilon \phi \), then \( A^\dagger AA^\dagger \phi = (\epsilon + 1)A^\dagger \phi \), and hence that

\[
A^\dagger e^{ikx} = (\tanh mx - ik)e^{ikx}
\]

is a generalized eigenfunction of \( A^\dagger A \). In addition, there is a normalizable eigenfunction, \( \text{sech} \, mx \), which lies in the kernel of \( A^\dagger A \).

In the same way, it follows that, for any \( k \in \mathbb{R} \), the function

\[
B^\dagger A^\dagger e^{ikx} = (-k^2 - 3imk \tanh mx + 2m^2 - 3m^2 \text{sech}^2 mx) e^{ikx}
\]

is an eigenfunction for \( B^\dagger B \). It is a consequence of (A.1), (A.2) and (A.3) that \( A B^\dagger A^\dagger = (-\partial_x^2 + m^2)(-\partial_x^2 + 4m^2) \). It is convenient to introduce phase factors

\[
e^{\pm i\delta_k} = \frac{[ -k^2 \pm 3imk + 2m^2 ]}{\sqrt{(k^2 + m^2)(k^2 + 4m^2)}},
\]

and then to normalize the generalized eigenfunctions as follows:

\[
E_k(x) = \begin{cases} 
\frac{[-k^2-3imk \tanh mx + 2m^2 - 3m^2 \text{sech}^2 mx]}{\sqrt{(k^2 + m^2)(k^2 + 4m^2)}} e^{i\delta_k} e^{ikx} & k \geq 0, \\
\frac{[-k^2-3imk \tanh mx + 2m^2 - 3m^2 \text{sech}^2 mx]}{\sqrt{(k^2 + m^2)(k^2 + 4m^2)}} e^{-i\delta_k} e^{ikx} & k < 0.
\end{cases} \tag{A.4}
\]

These obey \((-\partial_x^2 + 4m^2 - 6m^2 \text{sech}^2 mx) E_k(x) = (k^2 + 4m^2) E_k(x)\), as a consequence of the above algebraic structure, and are normalized so that for \( k > 0 \) there holds \( E_k(x) = e^{ikx} + O(e^{-m|x|}) \) as \( x \to -\infty \), while for \( k < 0 \) there holds \( E_k(x) = e^{ikx} + O(e^{-m|x|}) \) as \( x \to +\infty \). This normalization is natural for scattering, as will become clear.

In addition, there is a pair of square-integrable eigenfunctions, given in normalized form as:

\[
\psi_0(x) = \sqrt{\frac{3m}{4}} \text{sech}^2 mx, \quad \psi_1(x) = \sqrt{\frac{3m}{2}} \tanh mx \text{ sech} mx. \tag{A.5}
\]

We write \( P_0, P_1 \) for the corresponding orthogonal projection operators, defined by the integral kernels \( P_a(x,y) = \psi_a(x)\psi_a(y) \) for \( a \in \{0, 1\} \). These obey \((-\partial_x^2 + 4m^2 - 6m^2 \text{sech}^2 mx) \psi_0(x) = 0 \) and \((-\partial_x^2 + 4m^2 - 6m^2 \text{sech}^2 mx) \psi_1 = 3m^2 \psi_1 \).

These definitions are chosen so that the following orthonormality relations hold:

\[
\int_{\mathbb{R}} E_l(x) E_k(x) \, dx = 2\pi \delta(k-l), \quad \text{for all } k, l \in \mathbb{R}, \tag{A.7}
\]

\[
\int_{\mathbb{R}} \psi_a(x) \psi_b(x) \, dx = \delta_{ab}, \quad \text{for all } a, b \in \{0, 1\}, \tag{A.8}
\]

\[
\int_{\mathbb{R}} \psi_a(x) E_k(x) \, dx = 0, \quad \text{for all } a \in \{0, 1\} \text{ and } k \in \mathbb{R}. \tag{A.9}
\]

(In the first of these, and in related formulae, the integral is of course to be understood as being an \( S' \)-valued integral, i.e., the relation when holds when paired with Schwartz function inside the integral.) The completeness relation takes the form:

\[
\frac{1}{2\pi} \int E_k(y) E_k(x) \, dk + P_0 + P_1 = \delta(x-y). \tag{A.10}
\]
Writing $K = B^T B$, the functional calculus gives the following formula for the integral kernel of the operator $F(K)$:

$$F(K)(x, y) = F(0)\psi_0(x)\psi_0(y) + F(3m^2)\psi_1(x)\psi_1(y) + \frac{1}{2\pi} \int_{\mathbb{R}} \left[ F(k^2 + 4m^2)\overline{E_k(y)}E_k(x) \right] dk$$

$$= F(0)\psi_0(x)\psi_0(y) + F(3m^2)\psi_1(x)\psi_1(y) + \frac{1}{2\pi} \int_{\mathbb{R}} \left[ F(k^2 + 4m^2)(-k^2 + 3imk \tanh my + 2m^2 - 3m^2 \text{sech}^2 my) e^{ik(x-y)} \right] \frac{(-k^2 - 3imk \tanh mx + 2m^2 - 3m^2 \text{sech}^2 mx)}{(k^2 + m^2)(k^2 + 4m^2)} dk.$$  \hspace{1cm} (A.11)

**Proposition A.1.** For any $s \in \mathbb{R}, r \geq 0$, the operator $K^{\frac{r}{2}}$ is bounded as an operator $H^s \to H^{s-r}$. For any $s, r \in \mathbb{R}$, and $\theta > 0$, the operator $(K^\theta)^\frac{r}{2}$, where $K^\theta$ was defined just prior to Theorem 2.7, is bounded as an operator $H^s \to H^{s-r}$.

**Proof.** Consider the second statement of the proposition. We remark that if $f \in C^\infty(\mathbb{R})$ is a smooth function all of whose derivatives are bounded then the operator $u \mapsto fu$ is bounded on every Sobolev space $H^s$, i.e. $\|fu\|_{H^s} \leq \text{const.}\|u\|_{H^s}$. [This is an immediate consequence of the product rule for the case $s \in \{1, 2, 3 \ldots\}$, and follows in the negative integral case by duality and in the general case by interpolation.] Making use of the formula for the kernel

$$(K^\theta)^\frac{r}{2}(x, y) = \theta^{\frac{r}{2}}\psi_0(x)\psi_0(y) + (3m^2)^\frac{r}{2}\psi_1(x)\psi_1(y) + \frac{1}{2\pi} \int_{\mathbb{R}} \left[ (-k^2 + 3imk \tanh my + 2m^2 - 3m^2 \text{sech}^2 my) e^{ik(x-y)} \right] \frac{(-k^2 - 3imk \tanh mx + 2m^2 - 3m^2 \text{sech}^2 mx)}{(k^2 + m^2)(k^2 + 4m^2)^{1-\frac{r}{2}}} dk, \hspace{1cm} (A.12)$$

it is only necessary to consider the final integral, by the preceding remark. By observation, this integral can be put in the form $\sum_{j=0}^4 \sum_{x=1}^{N_j} f^{\alpha_j}(x)g^{\alpha_j}(y)I_{j,1-\frac{r}{2}}(x-y)$, where each $N_j \in \{1, 2, 3 \ldots\}$, the functions $\{f^{\alpha_j}, g^{\alpha_j}\}$ are all smooth bounded functions, whose derivatives are in fact Schwartz functions and the $I_{j,1-\frac{r}{2}}(z)$ are as defined in (3.10) with $(a, b) = (j, 1 - \frac{r}{2})$. Again making use of the remark above, the result is consequence of the fact that for each $j \in \{0, 1, 2, 3, 4\}$, the pseudo-differential operator $(-i\partial^j)(m^2 - \partial^2)^{-1}(4m^2 - \partial^2)^{-\frac{r}{2}}$, whose integral kernel is $I_{j,1-\frac{r}{2}}$ is bounded $H^s \to H^{s-r}$. The first statement of the proposition is proved similarly, but requires $r \geq 0$ because $K$ has a kernel.

The regularization induced by smoothing of the field operators as in (3.1) leads to the following regularization of functions of the operator $K$:

$$F(K)_\kappa(x, y) = F(0)\psi_0 \ast \delta^{[\kappa]}(x)\psi_0 \ast \delta^{[\kappa]}(y) + F(3m^2)\psi_1 \ast \delta^{[\kappa]}(x)\psi_1 \ast \delta^{[\kappa]}(y) + \frac{1}{2\pi} \iint_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}} F(k^2 + 4m^2)\delta^{[\kappa]}(x-x')E_k(x')\overline{E_k(y)}\delta^{[\kappa]}(y-y') dk \, dx' \, dy'.$$  \hspace{1cm} (A.13)

**A.2 Wave Operators**

We now summarize the scattering theory for the operator $K = -\partial^2/4 + 4m^2 \text{sech}^2 mx$. Theoretically this falls under the framework for short range scattering developed in [22] §XI.4, and problem 44] or [13] Chapter XIV. The differential operators $K$ and $K_0 = (-\partial^2 + 4m^2)$ extend to define unbounded self-adjoint operators on $L^2(\mathbb{R})$, and there exist partial isometries $\mathbb{W}_\pm$ such that

$$\mathbb{W}_\pm u = \lim_{t \to \pm \infty} e^{itK} e^{-itK_0} u$$

and

$$\mathbb{W}_\pm e^{isK_0} u = e^{i\kappa} \mathbb{W}_\pm u,$$
for all \( u \in L^2(\mathbb{R}) \). These are the wave operators and are isometric from \( L^2(\mathbb{R}) \) onto the absolutely continuous subspace of \( K \), which is the orthogonal complement of the linear span of the two discrete eigenfunctions \( \psi_0 \) and \( \psi_1 \). The wave operators can be represented explicitly using the distorted Fourier transform, a representation which we will now derive.

Introduce \( V(x) = -6m^2 \text{sech}^2 mx \) and \( R_0(z) = (-\partial_x^2 - z)^{-1} \) as, respectively, notation for the potential induced by the kink, and for the free resolvent. The resolvent is well-defined on the complement of the non-negative real axis, and has the integral kernel

\[
R_0(z)(x, y) = \frac{i}{2\sqrt{z}} \exp[i\sqrt{z}|x - y|] \tag{A.14}
\]

where \( \sqrt{z} \) means the square root with \( \text{Im} \sqrt{z} > 0 \), so that in a neighbourhood of the positive real axis \( z = k^2 > 0 \) there holds

\[
\sqrt{k^2 \pm i\epsilon} = \pm (|k| \pm \frac{i\epsilon}{2|k|} + O(\epsilon^2)).
\]

Consider now the formulae for the generalized eigenfunctions \( E_k(x) \) which were derived in the preceding section. The fact that only \( e^{ikx} \) appears is a consequence of the fact that the potential \( V(x) \) is a reflectionless potential. The wave operators are completely determined by the phase factors \( e^{\pm i\delta_k} \).

Taking the limit \( \epsilon \downarrow 0 \) leads to the introduction of boundary values \( R^\pm_0(k^2) \) of the free resolvent on the upper and lower sides of the positive axis and thence, by a calculation similar to that in [13, Example 14.6.10], we obtain the following result.

**Lemma A.2.** For \( k \in \mathbb{R} \) there holds

\[
(1 + R^+_0(k^2)V)E_k(x) = e^{ikx},
\]

while for \( k \geq 0 \) there holds, respectively,

\[
(1 + R^-_0(k^2)V)E_k(x) = \pm e^{\pm 2i\delta_k} e^{ikx}.
\]

As in the same reference we can now read off the following formulæ for the adjoints of the wave operators:

\[
\mathcal{W}^+ \left( \int_{\mathbb{R}} f(k)E_k(x) \, dk \right) = \int_{-\infty}^{+\infty} f(k)e^{ikx} \, dk \tag{A.15}
\]

\[
\mathcal{W}^- \left( \int_{\mathbb{R}} f(k)E_k(x) \, dk \right) = \int_{-\infty}^{0} f(k)e^{+2i\delta_k + ikx} \, dk + \int_{0}^{+\infty} f(k)e^{-2i\delta_k + ikx} \, dk. \tag{A.16}
\]

The scattering operator \( \hat{S} \overset{\text{def}}{=} \mathcal{W}^- \circ \mathcal{W}^+ \) has the effect:

\[
\int_{0}^{+\infty} f(k)e^{-i\delta_k + ikx} \, dk + \int_{-\infty}^{0} f(k)e^{+i\delta_k + ikx} \, dk \rightarrow \int_{0}^{+\infty} f(k)e^{+i\delta_k + ikx} \, dk + \int_{-\infty}^{0} f(k)e^{-i\delta_k + ikx} \, dk,
\]

which may be written in the alternative form

\[
\hat{S} \left( \int_{\mathbb{R}} g(k)e^{ikx} \, dk \right) = \int_{0}^{+\infty} g(k)e^{+2i\delta_k + ikx} \, dk + \int_{-\infty}^{0} g(k)e^{-2i\delta_k + ikx} \, dk.
\]

### A.3 Time Evolution

The classical equation \( \dot{y} + Ky = 0 \) can be written in first order form as

\[
\frac{\partial}{\partial t} \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix} \begin{pmatrix} y \\ \dot{y} \end{pmatrix}. \tag{A.17}
\]

This generates a one parameter group of operators

\[
T_t \overset{\text{def}}{=} \exp \left[ t \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix} \right]
\]

41
which are continuous on $H^{s+1}(\mathbb{R}) \times H^{s}(\mathbb{R})$, for any $s \in \mathbb{R}$, and also on $S(\mathbb{R}) \times S(\mathbb{R})$. In terms of the eigenfunction expansion \((a)\), the general solution of this equation is
\[
y(t, x) = (y_0 + y_1 t)\psi_0(x) + (y_d e^{-it\omega_d} + \mathcal{F} e^{it\omega_d}) \psi_1(x) + \frac{1}{\sqrt{2\omega_k}} \int \frac{1}{\sqrt{2\omega_k}} (y_k e^{-it\omega_k} E_k(x) + \mathcal{F} e^{it\omega_k} \overline{E_k}(x)) \, dk.
\]

This transfers to give the time-dependent Heisenberg field in the soliton representation:
\[
\phi^H(t, x) = -\sqrt{M_{cl}}(X + vt)\psi_0(x) + \frac{1}{\sqrt{2\omega_d}} \left( a_d(t) + a_d^\dagger(t) \right) \psi_1(x) + \frac{1}{\sqrt{2\pi}} \int \frac{1}{\sqrt{2\omega_k}} \left( a_k(t) E_k(x) + a_k^\dagger(t) \overline{E_k}(x) \right) \, dk.
\]

Thus the time evolution of the creation and annihilation operators in the Heisenberg picture, which are written in boldface, is
\[
\{V_k \phi^H \}_{k \in \mathbb{R}} = \{a_k(t) \psi_1 \}_{k \in \mathbb{R}} = \{a_k(t) \psi_1 \}_{k \in \mathbb{R}} + \{a_k(t) \overline{\psi}_1 \}_{k \in \mathbb{R}}
\]

We can now give a description of the evolution determined by the semiclassical Hamiltonian \(\mathcal{H}_{sod}\), of states of the form \(f(Q) a_j^m \prod_j a(\chi_j) \Omega\) where \(f\) and \(\chi_j\) are Schwartz. (The \(\chi_j\) can be thought of as distorted Fourier transforms (see \((2.33)\)) of Schwartz functions \(\chi_j \in \{\psi_0, \psi_1\}^\perp\).) Then
\[
\exp[-it\mathcal{H}_{sod}] = \psi(t, Q) e^{it\omega_d} \psi(0, Q) = f(Q).
\]

For the purposes of quantization in the vacuum representation, it is useful to introduce \(\alpha = 2^{-\frac{1}{2}} (K_0^+ y + i K_0^- y)\) and its complex conjugate \(\pi\), in terms of which the evolution equation can be written
\[
\frac{d\eta}{dt} = \begin{pmatrix} \eta & -\frac{1}{2} \pi \end{pmatrix} \begin{pmatrix} 0 & K_0^- \frac{1}{2} V K_0^+ \ 
K_0^- \frac{1}{2} V K_0^+ & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \pi \end{pmatrix}.
\]

The \(\{u(t)\}_{t \in \mathbb{R}}\) constitute a strongly continuous one parameter group of operators on \(L^2(\mathbb{R}; \mathbb{C})^2\) which have operator norm \(\|u(t)\|_{L^2 \to L^2} \leq \|e^{-L|t|}\|_{L^2 \to L^2} \leq e^{L|t|}\) for some \(L > 0\), and satisfy the usual differentiability properties for quite general \(V\) (see [21, Theorem XI.104]). For the case at hand, the presence of the zero mode \(K_0^+ \psi_0 = 0\) shows that the bounds cannot be time-independent: there is a solution \(\eta_Z(t) = (u(t) i K_0^- \psi_0 = t K_0^- \psi_0 + i K_0^- \psi_0 growing in time (as well as the constant solution \(K_0^+ \psi_0\)). See also the remark below for further comments.

The group of operators \(\{u(t)\}\) induces an evolution of the quantum fields in the Heisenberg picture: the Heisenberg field at time \(t\) can be written
\[
\phi^H(f, t) = \frac{1}{\sqrt{2}} \left( a(K_0^+ f, t) + a^\dagger(K_0^- f, t) \right), \quad \pi^H(f, t) = \frac{-i}{\sqrt{2}} \left( a(K_0^+ f, t) - a^\dagger(K_0^+ f, t) \right)
\]

where \(f \in S(\mathbb{R})\) and the evolution of the creation and annihilation operators is given in terms of \(u(t)\):
\[
a(f, t) = a(u_{11}(t)^T f) + a^\dagger(u_{12}(t)^T f),
\]
\[
a^\dagger(f, t) = a(u_{12}(t)^T f) + a^\dagger(u_{11}(t)^T f).
\]
Theorem A.3. The evolution of the field operators determined by \( \text{(A.20)} \) is unitarily implementable on Fock space, i.e. there exists a family of unitary operators \( \{ \exp[-it:H_0^{\text{tot}}]\} \) defined on \( \mathcal{F}_0 \), which induce the above actions and map the finite particle space, in particular the Fock vacuum \( \Omega_0 \), into the subspace \( \bigcap_{s=1,2,...} \text{Dom}(\mathcal{N}_0^s) \subset \mathcal{F}_0 \) of smooth vectors for the number operator \( \mathcal{N}_0 \).

Proof. This is essentially a consequence of basic results on unitary implementability explained in [2, Chapter 4] and [17, 16], and [18]. To obtain the precise statement we need on the smoothness of the transformed vacuum with respect to the number operator we take as starting point the discussion in [23] and [21 §11.15]. In particular, see pages 313-314 of the latter reference for the unitary implementability of the finite time classical evolution operator for the charged case and [23] for a treatment of the neutral case appropriate to a real scalar field. The verification of the Hilbert-Schmidt hypothesis on \( u_{11}^T u_{12} \) amounts to checking that the integral
\[
\int_{\mathbb{R}^2} |\hat{V}(k-l)|^2 (k^2 + 4m^2)^{-\frac{1}{4}} (l^2 + 4m^2)^{-\frac{1}{4}} \, dkdl
\]
is finite, which certainly holds since \( \hat{V} \in \mathcal{S}(\mathbb{R}) \). The statement that the quantum flow maps the Fock vacuum into a smooth vector of \( \mathcal{N}_0 \) is proved by consideration of the formula
\[
\exp\left[-\frac{1}{2} \Lambda(a^\dagger, a^\dagger)\right] \Omega_0.
\]
Here \( \Lambda \) is the bilinear form associated with the operator
\[
-(u_{22}^T u_{21}) = (u_{22}^T u_{21})^T.
\]
The fact that this operator is equal to its transpose is a consequence of the pseudo-unitarity property above, see [23]. It is shown on p. 122 of this reference that if \( c_n = 2^{-k_n(n!)^{-2}} \| (\Lambda(a^\dagger, a^\dagger))^n \Omega_0 \|^2 \) then the radius of convergence of \( \sum c_n z^n \) is greater than one, so that \( c_n \leq C^2 e^{-2r n} \) for some positive \( C, r \). This implies that the formula \( \text{(A.22)} \) defines a smooth vector for \( \mathcal{N}_0 \) since \( \mathcal{N}_0^0(\Lambda(a^\dagger, a^\dagger))^n \Omega_0 = (2n)^n (\Lambda(a^\dagger, a^\dagger))^n \Omega_0 \) whose square norm is bounded by \( C^2 |2n|^{2r} e^{-2r n} \) which is summable.

Remark A.4. The presence of the zero mode shows itself in the possibility of growth in the norm of operators such as \( a^\dagger(f,t)(\mathcal{N}_0 + 1)^{-\frac{1}{2}} \), where
\[
a^\dagger(f,t) = \exp[+it:H_0^{\text{tot}}] a^\dagger(f,0) \exp[-it:H_0^{\text{tot}}],
\]
in contrast to the boundedness in the vacuum case which is an immediate consequence of \( \text{(4.1)} \) and \( \text{(2.13)} \). Indeed taking the inner product of \( \text{(A.17)} \) with \( (0,\psi_0)^T \) and \( (\psi_0, (t_0-t)\psi_0)^T \) leads to the identities
\[
\pi^H(\psi_0, t_0) = \pi^H(\psi_0, 0) \quad \text{and} \quad \Phi^H(\psi_0, t_0) = \Phi^H(\psi_0, 0) + t_0 \pi^H(\psi_0, 0)
\]
and hence
\[
a^\dagger(K_0^{-\frac{1}{2}} \psi_0, t_0) \Omega_0 = a^\dagger(K_0^{-\frac{1}{2}} \psi_0, 0) \Omega_0 + it_0 a^\dagger(K_0^{+\frac{1}{2}} \psi_0, 0) \Omega_0.
\]
From this it follows that
\[
\lim_{t_0 \to -\infty} \frac{||a^\dagger(K_0^{-\frac{1}{2}} \psi_0, t_0) \Omega_0||}{t_0} = ||a^\dagger(K_0^{+\frac{1}{2}} \psi_0, 0) \Omega_0||.
\]
Appendix: proofs of results on regularization

Proof of Lemma 3.4 We start with the formula
\[
(K\varphi)_{\kappa} - K\varphi_{\kappa} = \frac{1}{2\pi} \int \frac{-6m^2}{\sqrt{2\omega_k}} \int \delta^{[\kappa]}(z)(\text{sech}^2 m(x - z) - \text{sech}^2 m\pi) (a_k e^{ik(x-z)} + a_k^* e^{-ik(x-z)}) \, dz \, dk \\
= -2m/\kappa \frac{6m^2}{\sqrt{2\omega_k}} \int \int \int \delta^{[\kappa]}(z) \int \text{sech}^2 m(x - z'\theta/\kappa) \\
\times \tan h m(x - z'\theta/\kappa)(a_k e^{ik(x-z'/\kappa)} + a_k^* e^{-ik(x-z'/\kappa)}) \, d\theta \, dz' \, dk.
\]

(In deriving this, various terms do drop out due to the fact that differentiation does commute with convolution, so only the final \(-6m^2\text{sech}^2 m\pi\) in \((2.27)\) causes error terms.)

A typical term in \((K\varphi)_{\kappa} - K\varphi_{\kappa}\) \(dx\) is that involving two annihilation operators; it can be written
\[
\frac{1}{2\pi} \int \int \int \frac{6m^3}{\kappa \sqrt{\omega_k \omega_l}} a_l a_k z' \delta^{[\kappa]}(z') \delta^{[\kappa]}(w') \int \text{sech}^2 m(x - \theta z'/\kappa) \tan h m(x - \theta z'/\kappa) \, d\theta \\
\times \exp[ik(x - z'/\kappa)] \exp[i\ell(x - w'/\kappa)] \, dw' \, dz' \, dk \, dl \, dx.
\]

Now to show that has limit zero as a bilinear form on \(\mathcal{D} \times \mathcal{D}\), take a matrix element between two vectors in \(\mathcal{D}\), leading to an integral
\[
\frac{1}{2\pi} \int \int \int \frac{6m^3}{\kappa \sqrt{\omega_k \omega_l}} F(k, l) z' \delta^{[\kappa]}(z') \delta^{[\kappa]}(w') \int \text{sech}^2 m(x - \theta z'/\kappa) \tan h m(x - \theta z'/\kappa) \, d\theta \\
\times \exp[ik(x - z'/\kappa)] \exp[i\ell(x - w'/\kappa)] \, dw' \, dz' \, dk \, dl \, dx,
\]

with \(F \in \mathcal{S}(\mathbb{R}^2)\), which is \(O(\kappa^{-1})\).

The same holds for all the terms arising except for the one involving \(a_l a_k^*\), which is not normal ordered. This term can be normal ordered, and then the previous argument does apply but at the expense of an additional c-number term, which is given by
\[
\text{Err} = \frac{1}{2\pi} \int \int \int \frac{6m^3}{\kappa \sqrt{\omega_k \omega_l}} z' \delta^{[\kappa]}(z') \delta^{[\kappa]}(w') \int \text{sech}^2 m(x - \theta z'/\kappa) \tan h m(x - \theta z'/\kappa) \, d\theta \\
\times \exp[-ik(x - z'/\kappa)] \exp[i\ell(x - w'/\kappa)] \, dw' \, dz' \, dk \, dx.
\]

We bound the inner \(w'\) integral as
\[
\sup_{x, z, k} \left| \int \delta^{[\kappa]}(w') \exp[ik(x - w'/\kappa)] \, dw' \right| \leq \text{const.} \frac{\int (|\delta^{[\kappa]}| + |\delta^{[\kappa]}|') \, dw'}{(1 + |k|/\kappa)},
\]

and thence bound
\[
|\text{Err}| \leq C' \frac{1}{\omega_k} \int \frac{1}{1 + |k|/\kappa} \, dk \leq C' \int_0^\infty \frac{1}{(2m + k)(\kappa + k)} \, dk = C' \frac{\ln \kappa/(2m)}{\kappa - 2m} = O\left(\frac{\ln \kappa}{\kappa}\right).
\]

Proof of Lemma 3.3 The first assertion of (b) is proved in \([11]\) Section 5. The second can be proved by a modification of that argument as follows. After a change of variables \(2\pi \int \int \varphi_{\kappa}(x) \delta^{[\kappa]}(x-x') \text{sech}^2 m\pi \varphi(x') \, dx' \, dx\) can be written as
\[
\int \int \int \int \frac{\delta^{[\kappa]}(w) \delta^{[\kappa]}(z)}{2\sqrt{\omega_k \omega_l}} e^{i(k+l)z - ilw - ikz} \text{sech}^2 m(x - z) : (a_l + a_{l-}) (a_k + a_{k-}) dzdwdx : (a_l + a_{l-}) (a_k + a_{k-}) \, dkdl.
\]

Thus the inner \(dzdwdx\) integral determines the kernel whose \(L^2\) properties determine the required \(L^p(d\mu_0)\) properties according to \([11]\) Section 5. The \(dw\) integral just gives the Fourier transform \(\delta^{[\kappa]}(l/\kappa)\). The \(z\)
integral is a convolution of the function $\delta^{[\epsilon]}(z)e^{-ikz}$ with the function $h(x) \overset{df}{=} \text{sech}^2 mx$. Noting that the Fourier transform of the former function is just $\hat{\delta}^{[\epsilon]}((\cdot + k)/\kappa)$, the convolution theorem implies that the $x, z$ integral gives $\hat{h}(-(k + l))\hat{\delta}^{[\epsilon]}(-l/\kappa)$. Thus all together we are left with

$$\int\int \hat{\delta}^{[\epsilon]}(l/\kappa) \hat{h}(-(k + l))\hat{\delta}^{[\epsilon]}(-l/\kappa) \frac{1}{2\sqrt{\omega_k \omega_l}} \cdot ((a_l + a_{-l}^\dagger)(ak + a_{-k}^\dagger)dz dw dx \cdot ((a_l + a_{-l}^\dagger)(ak + a_{-k}^\dagger): dk dl.$$ 

Since the function $\frac{\hat{h}(-(k + l))\hat{\delta}^{[\epsilon]}(-l/\kappa)}{2\sqrt{\omega_k \omega_l}}$ is square integrable, the dominated convergence theorem implies that this kernel converges to $\frac{\hat{h}(-(k + l))\hat{\delta}^{[\epsilon]}(-l/\kappa)}{2\sqrt{\omega_k \omega_l}}$ in $L^2$ as $\kappa \to +\infty$, and hence the results follows by [7, Theorem 5.7]. The same calculation applied in the Fock space implies statement (a), via Theorem 4.2 in the same reference.

Proof of Lemma 3.5

We compute

$$\int \left[ 6m^2 \gamma_\kappa \text{sech}^2 mx - 6m^2 \int \int \int \frac{e^{ik(x'-y)}}{(k^2 + 4m^2)^\frac{3}{2}} \delta^{[\epsilon]}(x'-x) \text{sech}^2 mx' dx' dy dk \right] dx$$

$$= 3m^2 \int \left[ \int \int \frac{e^{ik(x'-y)}}{(k^2 + 4m^2)^\frac{3}{2}} \delta^{[\epsilon]}(x'-x) \text{sech}^2 mx - \text{sech}^2 mx' dx' dy dk \right] dx$$

$$= 3m^2 \int \left[ \int \int \frac{e^{ik(w-z)/\kappa}}{(k^2 + 4m^2)^\frac{3}{2}} \delta^{[\epsilon]}(z) \text{sech}^2 mx - \text{sech}^2 m(x + w/\kappa) dzdw dk dx$$

$$\leq \frac{\text{const.}}{\kappa} \int \frac{\left| \delta^{[\epsilon]}(k/\kappa) \right|}{(k^2 + 4m^2)^\frac{3}{2}} dk = O\left(\frac{\ln \kappa}{\kappa}\right).$$

Acknowledgements

This work has been partially supported by STFC consolidated grant ST/P000681/1 and St John’s College, Cambridge. It was completed while visiting the Institute for Analysis, Leibniz University, Hannover and the author thanks Elmar Schrohe for hospitality.

References

[1] Mark Andrews, The evolution of free wave packets, American Journal of Physics 76 (2008), no. 12, 1102-1107, DOI 10.1119/1.2982628.
[2] Baez, John C., Segal, Irving E., and Zhou, Zheng-Fang, Introduction to algebraic and constructive quantum field theory, Princeton Series in Physics, Princeton University Press, Princeton, NJ, 1992. MR1178936
[3] J. Bélliard, J. Fröhlich, and B. Gidas, Soliton mass and surface tension in the $(\lambda \mid \phi \mid ^4)_2$ quantum field model, Comm. Math. Phys. 60 (1978), no. 1, 37–72. MR0496006
[4] Bogachev, Vladimir I., Gaussian measures, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998. MR1642391
[5] S. Coleman, Aspects of symmetry, Cambridge University Press, Cambridge, 1985.
[6] R.F. Dashen, B. Hasslacher and A. Neveu, Nonperturbative methods and extended hadron models in field theory, parts i-iii, Phys.Rev. D10 (1974), 4114–4142.
[7] Glimm, J. and Jaffe, A., Boson quantum field models, 1972, pp. 77–143. MR0674511
[8] Glimm, James and Jaffe, Arthur, A $\lambda \phi^4$ quantum field without cutoffs. i, Phys. Rev. (2) 176 (1968), 1945–1951. MR0247845
[9] , Singular perturbations of selfadjoint operators, Comm. Pure Appl. Math. 22 (1969), 401–414. MR0282243
[10] , Quantum field theory and statistical mechanics, Birkhäuser Boston, Inc., Boston, MA, 1985. Expositions; Reprint of articles published 1969–1977. MR810217

45
[11] ______, Quantum physics, 2nd ed., Springer-Verlag, New York, 1987. A functional integral point of view. MR887102
[12] Gohberg, I. C. and Krein, M. G., Introduction to the theory of linear nonselfadjoint operators, Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18, American Mathematical Society, Providence, R.I., 1969. MR0246142
[13] Hörmander, Lars, The analysis of linear partial differential operators. ii, Classics in Mathematics, Springer-Verlag, Berlin, 2005. Differential operators with constant coefficients; Reprint of the 1983 original. MR2108588
[14] R. Jackiw, Quantum meaning of classical field theory, Rev. Modern Phys. 49 (1977), no. 3, 681–706, DOI 10.1103/RevMod-Phys.49.681. MR0503137
[15] Janson, Svante, Gaussian hilbert spaces, Cambridge Tracts in Mathematics, vol. 129, Cambridge University Press, Cambridge, 1997. MR1474726
[16] Kristensen, P., Mejlbo, L., and Poulsen, E. Thue, Tempered distributions in infinitely many dimensions. ii. displacement operators, Math. Scand. 14 (1964), 129–150. MR0180856
[17] ______, Tempered distributions in infinitely many dimensions. i. canonical field operators, Commun. Math. Phys. 1 (1965), 175–214. MR0180855
[18] ______, Tempered distributions in infinitely many dimensions. iii. linear transformations of field operators, Comm. Math. Phys. 6 (1967), 29–48. MR0218081
[19] Messiah, Albert, Quantum mechanics. vol. i, Translated from the French by G. M. Temmer, North-Holland Publishing Co., Amsterdam; Interscience Publishers Inc., New York, 1961. MR0129790
[20] Reed, Michael and Simon, Barry, Methods of modern mathematical physics. ii. fourier analysis, self-adjointness, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975. MR0493420
[21] ______, Methods of modern mathematical physics. iii, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1979. Scattering theory. MR529429
[22] ______, Methods of modern mathematical physics. i, 2nd ed., Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1980. Functional analysis. MR751959
[23] Ruijsenaars, S. N. M., On bogoliubov transformations. ii. the general case, Ann. Physics 116 (1978), no. 1, 105–134. MR516713
[24] Simon, Barry, The p(\phi)^2 euclidean (quantum) field theory, Princeton University Press, Princeton, N.J., 1974. Princeton Series in Physics. MR0489552
[25] Herbert Spohn, Dynamics of charged particles and their radiation field, Cambridge University Press, Cambridge, 2004. MR2097788
[26] Stuart, David M. A., Analysis of the adiabatic limit for solitons in classical field theory, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 463 (2007), no. 2087, 2753–2781. MR2360179