GENERIC BICATEGORIES

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Abstract. It is well known that to give an oplax functor of bicategories \(1 \to \mathcal{C}\) is to give a comonad in \(\mathcal{C}\). Here we generalize this fact, replacing the terminal bicategory by any bicategory \(\mathcal{A}\) for which the composition functor admits generic factorisations. We call bicategories with this property generic, and show that for generic bicategories \(\mathcal{A}\) one may express the data of an oplax functor \(\mathcal{A} \to \mathcal{C}\) much like the data of a comonad; the main advantage of this description being that it does not directly involve composition in \(\mathcal{A}\).

We then go on to apply this result to some well known bicategories, such as cartesian monoidal categories (seen as one object bicategories), bicategories of spans, and bicategories of polynomials with cartesian 2-cells.

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1. Introduction

A classical and simple fact about monads in a bicategory \(\mathcal{C}\) is that they are in bijection with lax functors \(L: 1 \to \mathcal{C}\) where \(1\) is the terminal bicategory [11]. Dually, comonads in \(\mathcal{C}\) correspond to oplax functors \(L: 1 \to \mathcal{C}\). The purpose of this paper is to provide a generalization of this dual, showing that this correspondence may be realized as a special case of a more general result.

This is done by replacing the terminal bicategory with bicategories \(\mathcal{A}\) satisfying the following special property: every functor

\[A_{X,Z}(e, - \circ -): A_{Y,Z} \times A_{X,Y} \to \text{Set}, \quad X, Y, Z, e \in A\]
is a coproduct of representables. A more informative and equivalent characterization is as follows: every composition functor
\[ \circ : \mathcal{A}_{Y,Z} \times \mathcal{A}_{X,Y} \to \mathcal{A}_{X,Z}, \quad X,Y,Z \in \mathcal{A} \]
admits generic factorisations. We will call bicategories \( \mathcal{A} \) satisfying this property generic.

Informally, this property means that the bicategory \( \mathcal{A} \) contains “diagonal” 2-cells. A simple example of this is given by taking \( \mathcal{A} \) to be a cartesian monoidal category \( (E, \times, 1) \) seen as a one-object bicategory, where we have diagonal maps \( \delta : T \to T \times T \) for each \( T \in E \). Another example is given by taking \( \mathcal{A} \) to be the bicategory of spans \( \text{Span}(E) \) in a category \( E \) with pullbacks; here our diagonal maps are morphisms \( \delta \) induced into pullbacks as in

\[
\begin{array}{cccccc}
T & \downarrow \delta & \downarrow T \times T & \downarrow \pi_1 & S & \downarrow \pi_2 \\
\downarrow s & & \downarrow & & \downarrow t & \downarrow h \\
X & & S & & Y & \downarrow \text{pb} \\
\end{array}
\]

such that \( \pi_1 \delta \) and \( \pi_2 \delta \) are identities. This can also be done for the bicategory of polynomials \( \text{Poly}_c(E) \) with cartesian 2-cells, but becomes more complicated.

Such bicategories also contain “nullary diagonals” or augmentations; these are the 2-cells into identity 1-cells, and turn out to be unique in such bicategories.

The main result of this paper is that for generic bicategories \( \mathcal{A} \), the functors \( \mathcal{A} \to \mathcal{C} \) which respect these diagonals are precisely the oplax functors. Here “respecting diagonals” means that each diagonal \( \delta \) and augmentation \( \varepsilon \) in \( \mathcal{A} \) has a corresponding comultiplication map \( \Phi_{\delta} \) and counit map \( \Lambda_{\varepsilon} \) in \( \mathcal{C} \) satisfying coherence conditions much like those for a comonad.

When the domain bicategory \( \mathcal{A} \) is generic, this description has an important advantage over the usual definition of an oplax functor: it does not involve composition in the domain bicategory. This reduction being possible since the information concerning composition in \( \mathcal{A} \) is encoded into these diagonal maps. Of course, this property is particularly useful if composition in \( \mathcal{A} \) is complicated; the bicategory of polynomials being an archetypal example.

In Section 2 we develop the theory of such bicategories \( \mathcal{A} \) and their diagonal maps, and prove the main result of this paper, Theorem 19, in which we prove the equivalence of oplax functors and functors which respect these diagonals.

In Section 3, we use this result to give a description of oplax functors out of the bicategory of spans which does not involve composition of spans (pullbacks), and then give a description of oplax functors out of the bicategory of polynomials which does not involve composition of polynomials.

These descriptions allow for a simpler proof of the universal properties of spans [2], and a much simpler proof of the universal properties of polynomials. In our next paper we will use these descriptions to give an efficient proof of these universal properties.

In Section 4 we discuss how this description of oplax functors can be seen as an instance of doctrinal Yoneda structures, seen as a consequence of the simpler Day convolution structure on generic bicategories.

2. Properties of Generic Bicategories

In this section we start off by recalling the basic theory of generic morphisms and functors which admit them. We then define generic bicategories and consider the
properties of generic morphisms in these generic bicategories. After discussing the coherence properties of these generic morphisms, we go on to give the main result of this paper; showing that the functors which respect these generic morphisms are precisely the oplax functors.

2.1. Generic morphisms and factorisations. Generic morphisms (and weaker analogues of them) have historically arisen in the characterization the analytic endofunctors of $\text{Set}$, as well as the study of qualitative domains. Characterizations of endofunctors which admit them have been studied by Weber, and this is known to be related to familial representability as studied by Diers.

In this paper we do not consider arbitrary endofunctors which admit generics, but instead composition functors which admit generics, giving us a richer structure to consider.

**Definition 1.** Given a functor $T : A \to B$ between categories $A$ and $B$, we say a morphism $\delta : B \to TA$ in $B$ (where $A \in A$ and $B \in B$) is $T$-generic if for any commutative square of the form below

![Diagram](https://example.com/diagram.png)

there exists a unique morphism $f$ in $A$ such that $T \cdot f = \delta$.

**Remark 2.** These are precisely the diagonally universal morphisms of Diers, who noted that it must follow $g \cdot \delta = h$ since both fillers below render commutative the top triangles.

**Definition 2.** We say a functor $T : A \to B$ between categories $A$ and $B$ admits generic factorisations if for any morphism $f : B \to TC$ in $B$ there exists a $T$-generic morphism $\delta : B \to TA$ in $B$ and morphism $f : A \to C$ in $A$ rendering commutative

![Diagram](https://example.com/diagram.png)

We are now ready to define generic bicategories, the structures to be considered in this paper. It will be helpful to write composition in diagrammatic order, denoted by the symbol "$;\$".

**Definition 3.** We say a bicategory $\mathcal{A}$ is generic if for every triple of objects $X, Y, Z \in \mathcal{A}$ the composition functor

$\mathcal{A}_{X,Y} \times \mathcal{A}_{Y,Z} \to \mathcal{A}_{X,Z}$

admits generic factorisations. Moreover, we simply call generic those 2-cells $\delta : c \to l;r$ which are $;\$-generic.
Remark 5. Unpacking the above definition into a more useful form, we see that a 2-cell \( \delta : c \to l; r \) is generic if and only if every commuting diagram of the form

\[
\begin{array}{ccc}
\gamma_1 \downarrow & & \gamma_2 \downarrow \\
\phi_1 \phi_2 & & \phi_1 \phi_2 \\
t_1 t_2 & & t_1 t_2 \\
\end{array}
\]

(where \( \theta_1, \theta_2, \phi_1, \phi_2 \) and \( \gamma \) are arbitrary 2-cells) admits a filler \( \gamma_1; \gamma_2 \) as displayed, such that the top triangle commutes and the bottom triangle commutes component-wise. Moreover, the pair \( \gamma_1, \gamma_2 \) must be unique such that the top triangle commutes, justifying the notation.

Remark 6. As we will see in Section 3, there are a number of well known bicategories and monoidal categories which are generic, such as:

- any cartesian monoidal category;
- finite sets and bijections with the disjoint union monoidal structure;
- the bicategory of spans;
- the bicategory of polynomials with cartesian 2-cells.

Generic bicategories may be alternatively defined in terms of familial representability, a property which is often easier to verify. This is a consequence of the following known relationship\(^1\) between functors which admit generics and the familial representability conditions of Diers [4].

**Proposition 7** (Diers). Given a functor \( T : A \to B \) between categories \( A \) and \( B \) the following are equivalent:

1. the functor \( T \) admits generic factorisations;
2. for every \( B \in B \) there exists a set \( M_B \) and function \( P(-) : M_B \to A_{ob} \) yielding isomorphisms

\[
B(B,TA) \cong \sum_{\delta \in M_B} A(P_\delta, A)
\]

natural in \( A \in A \).

**Proof.** Suppose that \( T \) admits generic factorisations. Call two generic morphisms \( \delta \) and \( \delta' \) equivalent if there exists an isomorphism \( \alpha \) rendering commutative a diagram as below:

\[
\begin{array}{ccc}
TM & \xrightarrow{T\alpha} & TM' \\
\delta' & \searrow & \delta \\
B & \swarrow & B \\
\end{array}
\]

Now take \( M_B \) to be the set of equivalence classes of generic morphisms out of \( B \), with each class labeled by a chosen representative. It follows that for any \( f : B \to TA \) we can find a representative generic morphism \( \delta_f \) and unique morphism \( f \) rendering commutative

\[
\begin{array}{ccc}
B & \xrightarrow{f} & TA \\
\delta_f & \searrow & \gamma_f \\
TM & \swarrow & TM \\
\end{array}
\]

We note also that the representative generic \( \delta_f \) is itself unique (such a generic necessarily lies in the same equivalence class). Therefore the assignment \( f \mapsto \)

\(^1\)We include the proof of this relationship due to the difficulty of finding a reference.
\((\delta_f, \mathcal{T})\) is bijective, where each \(P_{\delta_f}\) is taken as the \(M\) above. Trivially, given a map \(x : A \to A'\) the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{f} & TA \\
\downarrow{\delta_f} & & \downarrow{r_x} \\
TM & \xrightarrow{Tf} & TA'
\end{array}
\]

commutes, and by genericity \(x\mathcal{T}\) is the unique such map making the outside commute; thus showing naturality.

Conversely, suppose we are given such a family of isomorphisms\(^2\)

\[
\mathcal{B}(B, TA) \cong \sum_{m \in \mathcal{M}_B} \mathcal{A}(P_m, A)
\]

natural in \(A \in \mathcal{A}\), where \(B \in \mathcal{B}\) is given. We first note that by naturality, the inverse assignment is necessarily defined by

\[
m \in \mathcal{M}_B , \quad P_m \xrightarrow{\alpha} A \mapsto B \xrightarrow{\delta_m} TP_m \xrightarrow{T\alpha} TA
\]

where \(\delta_m\) is the morphism corresponding to the identity at \(P_m\). Also, this \(\delta_m\) is generic since given any commuting diagram as on the outside below

\[
\begin{array}{ccc}
B & \xrightarrow{f} & TA \\
\downarrow{\delta_m} & & \downarrow{Th} \\
TP_m & \xrightarrow{Tg} & TD
\end{array}
\]

the morphism \(Th \cdot f\) must correspond to the pair \((\delta_m, g)\) under the bijection. By naturality, \(f\) must factor through this same \(\delta_m\), and so the pair \((\delta_m, \mathcal{T})\) corresponding to \(f\) is unique such that the top triangle commutes. That \(g = h \cdot \mathcal{T}\) is also a consequence of naturality. It is implicit in the above argument that \(T\) then admits generic factorisations. \(\square\)

Taking \(T\) to be the composition functor, we have the following.

**Corollary 8.** A bicategory \(\mathcal{A}\) is generic if and only if for any triple of objects \(X, Y, Z \in \mathcal{A}\) and 1-cell \(c : X \to Z\) the functor

\[
\mathcal{A}_{X,Z}(c, - ; -) : \mathcal{A}_{X,Y} \times \mathcal{A}_{Y,Z} \to \text{Set}
\]

is a coproduct of representables, meaning that for any \((X, Y, Z, c)\) there exists a set \(\mathcal{M}_{X,Y,Z}^c\) equipped with projections

\[
(\mathcal{A}_{X,Y})_{ob} \xleftarrow{l(-)} \mathcal{M}_{X,Y,Z}^c \xrightarrow{r(-)} (\mathcal{A}_{Y,Z})_{ob}
\]

such that for all \(a : X \to Y\) and \(b : Y \to Z\) we have isomorphisms

\[
(2.1) \quad \mathcal{A}_{X,Z}(c, a ; b) \cong \sum_{m \in \mathcal{M}_{X,Y,Z}^c} \mathcal{A}_{X,Y}(l_m, a) \times \mathcal{A}_{Y,Z}(r_m, b)
\]

natural in \(a\) and \(b\).

We have defined generics as universal maps into a composite of two 1-cells; what one might call "2-generics". We might ask if there is a corresponding notion for "0-generics" into composites of zero 1-cells, that is, identity 1-cells. However, as for each \(n : X \to X\) the functor

\[
\mathcal{A}_{X,X}(n, 1_X) : 1 \to \text{Set}
\]

\(^2\)Here \(\mathcal{M}_B\) is an arbitrary set, so we do not use the suggestive notation \(\delta\) for its elements.
is trivially a coproduct of representables, there is no condition to impose on these
2-cells, and so any 2-cell \( \varepsilon : n \to 1_X \) may be regarded as a “0-generic”. Regardless,
these 2-cells still have an interesting property; they are unique.

**Proposition 9.** Suppose \( \mathcal{A} \) is a generic bicategory. Then for each \( X \in \mathcal{A} \), the
identity 1-cell \( 1_X \) is sub-terminal in \( \mathcal{A}_{X,X} \).

**Proof.** Given a morphism \( n : X \to X \) and two 2-cells \( s, t : n \to 1_X \) we have two
commuting squares

\[
\begin{array}{ccc}
    n & \xrightarrow{\delta_1} & l; n \\
    \downarrow s & \xRightarrow{\theta; \phi} & \downarrow h; s \\
    n; r & \xrightarrow{1_X; 1_X} & 1_X; 1_X
\end{array}
\quad
\begin{array}{ccc}
    n & \xrightarrow{\delta_2} & l; n \\
    \downarrow s & \xRightarrow{\theta; \phi} & \downarrow h; s \\
    n; r & \xrightarrow{1_X; 1_X} & 1_X; 1_X
\end{array}
\]

where \( \delta_1 \) and \( h : l \to 1_X \) are given by factorizing the unitor \( n \to 1_X; n \) through a
generic, and \( \delta_2 \) and \( k : r \to 1_X \) are given by factorizing the other unitor \( n \to n; 1_X \).
Now both of these squares admit a unique filler, and moreover both these fillers
must be equal as uniqueness is forced by the top left triangles; we denote this
filler \( \theta; \phi \). Equating the left components of the bottom right triangles we then find
\( s = h\theta = t \).

It will be useful to give such 2-cells a name as they still play an important role,
despite the lack of a non-trivial universal property.

**Definition 10.** We call any 2-cell of the form \( \varepsilon : n \to 1_X \) in a bicategory \( \mathcal{A} \) an
augmentation.

### 2.2. Coherence of generics

The following two lemmata show that there exists “nice” choices of generics. This will later be useful in regard to stating and checking
coherence conditions.

**Lemma 11.** Suppose \( \mathcal{A} \) is a generic bicategory. Then for any factorization of a
left unitor at a 1-cell \( c : X \to Y \) through a generic \( \delta \) as below

\[
\begin{array}{ccc}
    c & \xrightarrow{\Delta} & 1_X; c \\
    \downarrow s & \xRightarrow{\theta; \phi} & \downarrow 1_X; \phi
\end{array}
\]

the induced 2-cell \( \phi \) is invertible.

**Proof.** Define \( \phi^* : c \to r \) to be the composite

\[
c \xrightarrow{\delta} l; r \xrightarrow{\theta; \phi} 1_X; 1_X \xrightarrow{\text{unitors}} r
\]

and note that when this is post-composed by \( \phi \) we recover the identity 2-cell at \( c \),
by commutativity of the diagram 2.2 and naturality of unitors. We also note that
by naturality of unitors the diagram

\[
\begin{array}{ccc}
    c & \xrightarrow{\Delta} & 1_X; c \\
    \downarrow s & \xRightarrow{\theta; \phi} & \downarrow 1_X; \phi^*
\end{array}
\]

commutes and thus admits a filler such that both triangles commute. Moreover,
we note that as uniqueness is forced by the top triangle this filler must be \( \theta; \phi \).
Equating the second components of the bottom right triangle we have established
\( \phi \) followed by \( \phi^* \) as being the identity.

\( \square \)
Remark 12. As \( \phi \) is invertible above, composing the generic \( \delta \) with \( \phi \) still yields a generic. This shows that there exists “nice” generics \( c \to l; c \) and augmentations \( l \to 1_X \) which compose to the unitor. Moreover, it is clear this may be similarly done for right unitors.

Lemma 13. Suppose \( \mathcal{A} \) is a generic bicategory. Let \( W, X, Y, Z \) be objects in \( \mathcal{A} \), let \( T \) be the functor given by composition

\[
(\mathcal{A}_{W, X} \times \mathcal{A}_{X, Y}) \times \mathcal{A}_{Y, Z} \to \mathcal{A}_{W, Y} \times \mathcal{A}_{Y, Z} \to \mathcal{A}_{W, Z}
\]

and consider 1-cells

\[
d: W \to Z, \quad l: W \to X, \quad m: X \to Y, \quad r: Y \to Z.
\]

Then a 2-cell \( d \to (l; m); r \) in \( \mathcal{A} \) is \( T \)-generic if and only if it has the form

\[
d \xrightarrow{\delta_1} h; r \xrightarrow{\delta_2; r} (l; m); r
\]

for a pair of generics \( \delta_1 \) and \( \delta_2 \).

Proof. Suppose we are given generics \( \delta_1 \) and \( \delta_2 \) composable as in the diagram on the left below

\[
\begin{array}{c}
\delta_1 \downarrow \quad \eta_1; \eta_2 \quad \downarrow \quad (\eta_1; \eta_2) \quad \delta_2 \\
\eta_1 \quad \delta_2 \quad \eta_2 \quad r \\
(l; m); r \quad (f; g); h
\end{array}
\]

where \( \eta_1, \eta_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \) and \( \gamma \) are arbitrary 2-cells such that the outside diagram commutes. Then there exists a filler \( \eta_1; \eta_2 \) splitting the diagram into two commuting regions, by genericity of \( \delta_1 \). Moreover, there exists a filler \( \zeta_1; \zeta_2 \) for the commuting diagram on the right above as \( \delta_2 \) is generic. We thus have a diagonal filler \( (\zeta_1; \zeta_2); \gamma \) for the diagram on the left above. For uniqueness, suppose we are given another filler \( (\zeta_1'; \zeta_2' ); \gamma_2 \) and note that since \( \delta_1 \) is generic, we have \( [\eta_1; \eta_2] \circ \delta_2 \); \( \gamma_2 = \gamma_1; \gamma_2 \) component wise. Hence \( \gamma_2' = \gamma_2 \) and \( \eta_1; \eta_2 \) \( \circ \delta_2 \). Since \( \delta_2 \) is generic it follows that \( \zeta_1' = \zeta_1 \) and \( \zeta_2' = \zeta_2 \).

Conversely, suppose we are given a 2-cell \( \delta: d \to (l; m); r \) which is \( T \)-generic. Now, we know that the \( T \)-generic \( \delta \) can be factored through a generic \( \delta_1 \) giving the triangle on the left below

\[
\begin{array}{c}
\delta \downarrow \quad \eta_1; \eta_2 \quad \downarrow \quad (\eta_1; \eta_2) \quad \delta_2 \\
\eta_1 \quad \delta_2 \quad \eta_2 \quad r \\
(l; m); r \quad (f; g); h
\end{array}
\]

and the 2-cell \( \alpha \) can be factored through a generic \( \delta_2 \) yielding the right triangle above. In particular, the components of \( (\eta_1; \eta_2); \beta \) are invertible as this is an induced isomorphism of \( T \)-generic morphisms [13, Lemma 5.7]. Hence upon taking \( \delta_1 \) to be \( \delta_1 \) pasted with \( \beta \), and \( \delta_2 \) to be \( \delta_2 \) pasted with \( \gamma_1; \gamma_2 \), we see that \( \delta \) is a pasting of generics \( \delta_1^* \) and \( \delta_2^* \).

Remark 14. The above lemma is an instance of a more general fact: if \( \delta_1: C \to SB \) is \( S \)-generic and \( \delta_2: B \to TA \) is \( T \)-generic, then

\[
C \xrightarrow{\delta_1} SB \xrightarrow{\delta_2} STA
\]

is \( ST \)-generic. Moreover, if both \( S \) and \( T \) admit generic factorisations then all \( ST \)-generics have this form.
Remark 15. Clearly, we can state and prove an analogue of the above lemma if we replace $T$ by the functor $S$ given as the composite

$$\mathcal{A}_{W,X} \times (\mathcal{A}_{X,Y} \times \mathcal{A}_{Y,Z}) \to \mathcal{A}_{W,Y} \times \mathcal{A}_{Y,Z} \to \mathcal{A}_{W,Z}$$

It is also clear that given a composite of generics

$$d \xrightarrow{\delta_1} h; r \xrightarrow{\delta_2} (l; m); r$$

which is $T$-generic, that the composite

$$d \xrightarrow{\delta_1} h; r \xrightarrow{\delta_2} (l; m); r \xrightarrow{\text{assoc}} l; (m; r)$$

is $S$-generic, and hence by the analogue of the above lemma we may write this composite as

$$d \xrightarrow{\delta_3} l; k \xrightarrow{\iota \delta_4} l; (m; r)$$

for some pair of generics $\delta_3$ and $\delta_4$.

It is sometimes advantageous to not consider all generics, but only a smaller class of generics satisfying some coherence properties outlined in the following definition.

Definition 16. Let $\mathcal{A}$ be a generic bicategory. Let $\Delta_2$ and $\Delta_0$ be given collections of generics and augmentations in $\mathcal{A}$ respectively. Denote by $\Omega_2$ the set of domains of the generics in $\Delta_2$. We say the pair $(\Delta_2, \Delta_0)$ is coherent if:

1. (completeness of generics) for every generic $\delta': c' \to l'; r'$ in $\mathcal{A}$ there exists a generic $\delta: c \to l; r$ in $\Delta_2$ and isomorphisms $\zeta_1, \zeta_2$ and $\zeta$ rendering commutative

   $$c \xrightarrow{\delta} l; r$$

   $$\zeta \downarrow \downarrow \zeta_1 \zeta_2$$

   $$c' \xrightarrow{\delta'} l'; r'$$

2. (completeness of augmentations) for every augmentation $\varepsilon': n' \to 1_X$ in $\mathcal{A}$ there exists an augmentation $\varepsilon: n \to 1_X$ in $\Delta_0$ and isomorphism $\xi: n \to n'$ rendering commutative

   $$n \xrightarrow{\varepsilon} n'$$

   $$\varepsilon \sim \xi$$

   $$1_X \xrightarrow{\varepsilon'}$$

3. (associator coherence) for all generics $\delta_1, \delta_2 \in \Delta_2$ composable as below, there exists generics $\delta_3, \delta_4 \in \Delta_2$ rendering commutative

   $$\varepsilon \downarrow \delta_3 \downarrow l; k \xrightarrow{\iota \delta_4} h; r \xrightarrow{\text{assoc}} l; (m; r) \xrightarrow{\text{assoc}} (l; m); r$$

4. (left unitor coherence) for all $c: X \to Y$ in $\Omega_2$ there exists a $\delta \in \Delta_2$ and $\varepsilon \in \Delta_0$ composable as below and rendering commutative

   $$\delta \xrightarrow{\iota \delta} l; c \xrightarrow{\text{unitor}} 1_X; c$$
(5) (right unitor coherence) for all \(c : X \to Y\) in \(\Omega_2\) there exists a \(\delta \in \Delta_2\) and \(\varepsilon \in \Delta_0\) composable as below and rendering commutative:

\[
\begin{array}{ccc}
\delta & \sim & \varepsilon \\
\downarrow & & \downarrow \\
\text{unitor} & & \text{counit} \\
\nwedge & & \nwedge \\
\end{array}
\]

**Remark 17.** If \(\mathcal{A}\) is generic, we may always take \((\Delta_2, \Delta_0)\) to be the class of all generic 2-cells and augmentations. This is a consequence of the previous two lemmata.

**Remark 18.** Informally, the conditions (3) to (5) guarantee that each 1-cell \(c \in \Omega_2\) admits the structure of an “\(\mathcal{A}\)-comonoid”; a simple example of this being that objects in cartesian monoidal categories admit the structure of a comonoid.

### 2.3. Functors which respect generics

It is well known that to give an oplax functor \(L : 1 \to \mathcal{C}\) is to give a comonad in \(\mathcal{C}\). The following theorem generalizes this fact, replacing the terminal category by any generic bicategory \(\mathcal{A}\).

At the same time, the following theorem may be seen as a coherence result; it provides a reduction in the data of an oplax functor out of such an \(\mathcal{A}\), showing that the coherence data of such an oplax functor is completely determined by the data at the diagonals.

The most important property of this result however is that it provides a description of oplax functors \(L : \mathcal{A} \to \mathcal{C}\) out of generic bicategories \(\mathcal{A}\) which does not involve composition in the domain bicategory; by this we mean expressions of the form \(L (a; b)\) or \(L (1_X)\) do not appear in our description below.

For completeness, we also give a reduced description of oplax natural transformations and icons [9] between such oplax functors.

**Theorem 19.** Let \(\mathcal{A}\) and \(\mathcal{C}\) be bicategories, and suppose \(\mathcal{A}\) is generic. Suppose we are given a coherent class \((\Delta_2, \Delta_0)\) of generics and augmentations of \(\mathcal{A}\). Then given a locally defined functor

\[
L_{X,Y} : \mathcal{A}_{X,Y} \to \mathcal{C}_{LX,LY}, \quad X, Y \in \mathcal{A}
\]

the following data are in bijection:

1. For every pair of composable 1-cells \(a\) and \(b\), a constraint 2-cell

\[
\varphi_{a,b} : L (a; b) \to L (a) ; L (b)
\]

and for every identity 1-cell \(1_X\), a constraint 2-cell

\[
\lambda_X : L (1_X) \to 1_{LX}
\]

exhibiting \(L\) as an oplax functor;

2. For every generic \(\delta : c \to l ; r \in \Delta_2\), a comultiplication 2-cell

\[
\Phi_{\delta} : L (c) \to L (l) ; L (r)
\]

and for every augmentation \(\varepsilon : n \to 1_X\) in \(\Delta_0\), a counit 2-cell

\[
\Lambda_{\varepsilon} : L (n) \to 1_{LX}
\]

satisfying the following coherence axioms:
\(\text{(a) (naturality of comultiplication)}\) for any 2-cell \(\zeta: c \to c'\) and commuting diagram as on the left below\(^3\) with \(\delta_1, \delta_2 \in \Delta_2\).

\[
\begin{array}{c}
\begin{array}{ccc}
\delta_1 & \downarrow & \delta_1 \\
 l; r & \downarrow & l; r \\
 c & \relax & c'
\end{array}
\end{array}
\quad
\begin{array}{ccc}
\Phi_{\delta_1} & \downarrow & \Phi_{\delta_1} \\
 Ll; Lr & \downarrow & Ll; Lr \\
 Lc & \relax & Lc'
\end{array}
\]

the diagram on the right above commutes;

\(\text{(b) (naturality of counits)}\) for any 2-cell \(\xi: n \to n'\) and pair of augmentations \(\varepsilon: n \to 1_X\) and \(\varepsilon': n' \to 1_X\) in \(\Delta_0\) giving a commuting diagram as on the left below.

\[
\begin{array}{ccc}
\xi & \downarrow & \xi \\
 n & \relax & n' \\
 \varepsilon & \downarrow & \varepsilon \\
 1_X & \relax & 1_X \\
 \end{array}
\quad
\begin{array}{ccc}
\Phi_{\varepsilon} & \downarrow & \Phi_{\varepsilon} \\
 Ln & \relax & Ln' \\
 \Lambda_{\varepsilon} & \downarrow & \Lambda_{\varepsilon} \\
 1_{L_\xi} & \relax & 1_{L_{\xi'}} \\
 \end{array}
\]

the diagram on the right above commutes;

\(\text{(c) (associativity of comultiplication)}\) for every \(\delta_1, \delta_2, \delta_3, \delta_4 \in \Delta_2\) yielding an equality as on the left below.

\[
\begin{array}{ccc}
\delta_1 & \downarrow & \delta_1 \\
 l; k & \relax & l; k \\
 c & \relax & c \\
 \Phi_{\delta_1} & \downarrow & \Phi_{\delta_1} \\
 Ll; Lk & \relax & Ll; Lk \\
 Lc & \relax & Lc \\
 \phi_{\delta_1} & \downarrow & \phi_{\delta_1} \\
 (l; k; r) & \relax & (l; k; r) \\
 l; (m; r) & \relax & Ll; (Lm; Lr) \\
 \langle \text{assoc} \rangle & \relax & \langle \text{assoc} \rangle \\
 (l; m); r & \relax & (l; m); r \\
 \end{array}
\]

the diagram on the right above commutes;

\(\text{(d) (left counit axiom)}\) for any 1-cell \(c: X \to Y\), generic \(\delta \in \Delta_2\) and augmentation \(\varepsilon \in \Delta_0\) yielding an equality as on the left below.

\[
\begin{array}{ccc}
\delta & \downarrow & \delta \\
 n; c & \relax & n; c \\
 c & \relax & c \\
 \Phi_{\delta} & \downarrow & \Phi_{\delta} \\
 Ln; Lc & \relax & Ln; Lc \\
 1_{L_\xi} & \relax & 1_{L_\xi} \\
 \Lambda_{\varepsilon; Lc} & \downarrow & \Lambda_{\varepsilon; Lc} \\
 1_{L_\xi; c} & \relax & 1_{L_\xi; c} \\
 \end{array}
\]

the diagram on the right above commutes;

\(\text{(e) (right counit axiom)}\) for any 1-cell \(c: X \to Y\), generic \(\delta \in \Delta_2\) and augmentation \(\varepsilon \in \Delta_0\) yielding an equality as on the left below.

\[
\begin{array}{ccc}
\delta & \downarrow & \delta \\
 c; n & \relax & c; n \\
 c & \relax & c \\
 \Phi_{\delta} & \downarrow & \Phi_{\delta} \\
 Lc; Ln & \relax & Lc; Ln \\
 1_{L_\xi} & \relax & 1_{L_\xi} \\
 \Lambda_{\varepsilon; Lc} & \downarrow & \Lambda_{\varepsilon; Lc} \\
 1_{L_\xi; c} & \relax & 1_{L_\xi; c} \\
 \end{array}
\]

the diagram on the right above commutes.

Suppose now we are given a locally defined functor \(L\) equipped with a collection \((\varphi, \lambda)\) as in (1), or equivalently equipped with a collection \((\Phi, \Lambda)\) as in (2). Denote this data by the 5-tuple \((L, \varphi, \Phi, \lambda, \Lambda)\) whilst noting the collections \((\varphi, \lambda)\) and \((\Phi, \Lambda)\) uniquely determine each other. Let \((K, \psi, \Psi, \gamma, \Gamma)\) be another such 5-tuple. Then the following data are in bijection:

1. an opplax natural transformation \(\theta: L \Rightarrow K\) of opplax functors;
(2) for every object $X \in \mathcal{A}$, a 1-cell $\vartheta_X : LX \to KX$ in $\mathcal{C}$, and for every 1-cell $f : X \to Y$ in $\mathcal{A}$, a 2-cell

$$\begin{align*}
LX & \xrightarrow{f} LY \\
\vartheta_X & \xrightarrow{\vartheta_{fX}} \vartheta_Y \\
KX & \xrightarrow{KL} KY
\end{align*}$$

natural in 1-cells $f : X \to Y$ and satisfying the following conditions:

(a) for every generic $\delta : c \to l; r$ in $\Delta_2$,

$$\begin{align*}
LX & \xrightarrow{Le} LZ \\
\vartheta_X & \xrightarrow{\vartheta_{eX}} \vartheta_Z \\
KX & \xrightarrow{KL} KZ
\end{align*} = \begin{align*}
LX & \xrightarrow{Le} LZ \\
\vartheta_X & \xrightarrow{\vartheta_{eX}} \vartheta_Z \\
KX & \xrightarrow{KL} KZ
\end{align*}$$

(b) for every augmentation $\varepsilon : n \to 1_X$ in $\Delta_0$,

$$\begin{align*}
LX & \xrightarrow{Ln} LX \\
\vartheta_X & \xrightarrow{\vartheta_{nX}} \vartheta_X \\
KX & \xrightarrow{Kn} KX
\end{align*} = \begin{align*}
LX & \xrightarrow{Ln} LX \\
\vartheta_X & \xrightarrow{\vartheta_{nX}} \vartheta_X \\
KX & \xrightarrow{Kn} KX
\end{align*}$$

When $L$ and $K$ agree on objects, this restricts to the bijection of the following data:

1. An icon between oplax functors $\vartheta : L \Longrightarrow K : \mathcal{A} \to \mathcal{C}$

2. A collection of natural transformations $\vartheta_{X,Y} : L_{X,Y} \Longrightarrow K_{X,Y} : \mathcal{A}_{X,Y} \to \mathcal{C}_{X,Y}$, $X, Y \in \mathcal{A}$ rendering commutative the diagrams

$$\begin{align*}
L(c) & \xrightarrow{\varphi} L(l) : L(r) \\
\vartheta_c & \xrightarrow{\vartheta_l ; \vartheta_r} \\
K(c) & \xrightarrow{\psi} K(l) : K(r)
\end{align*} \quad \begin{align*}
L(n) & \xrightarrow{\vartheta_n} K(n) \\
\vartheta_n & \xrightarrow{\Lambda_n \ ; \ \Gamma_n} \\
L(1_X) & \xrightarrow{\lambda_X} 1_{LX}
\end{align*}$$

Proof. We divide the proof into parts, verifying each bijection separately.

**Bijection With Oplax Functors.** We first show how to pass between the data of (1) and (2), and then verify this defines a bijection.

$(1) \implies (2)$: Suppose we are given the data $(L, \varphi, \lambda)$ of (1). We define $\Phi_\delta$ for each generic $\delta : c \to l; r$ by the composite

$$(2.3) \quad L(c) \xrightarrow{L\delta} L(l) ; L(r) \xrightarrow{\varphi_{l,r}} L(l) ; L(r)$$

and define $\Lambda_\varepsilon$ for each augmentation $\varepsilon : n \to 1_X$ by the composite

$$(2.4) \quad L(n) \xrightarrow{L\varepsilon} L(1_X) \xrightarrow{\lambda_X} 1_{LX}$$
For naturality of comultiplication, we see that given a diagram as on the left below

\[ \begin{array}{c}
  \begin{array}{ccc}
    c & \xrightarrow{\delta_1} & l; r \\
    \zeta & \zeta_1; \zeta_2 & \zeta_1; \zeta_2 \\
    c' & \xrightarrow{\delta_2} & l'; r'
  \end{array}
  \end{array} \]

the right commutes by naturality of \( \varphi \) and local functoriality of \( L \). For naturality of counits note that given a commuting diagram as on the left below

\[ \begin{array}{c}
  \begin{array}{ccc}
    n & \xrightarrow{\zeta} & 1_X \\
    n' & \xrightarrow{\zeta'} & l; r
  \end{array}
  \end{array} \]

the right trivially commutes. For associativity of comultiplication, note that given a commuting diagram

\[ \begin{array}{c}
  \begin{array}{ccc}
    c & \xrightarrow{\delta_3} & l; k \\
    l; \delta_4 & \xrightarrow{\delta_1} & h; r \\
    l; (m; r) & \xrightarrow{\text{assoc}} & (l; m); r
  \end{array}
  \end{array} \]

we have the commutativity of the diagram

\[ \begin{array}{c}
  \begin{array}{ccc}
    Lc & \xrightarrow{L; \delta_2} & L (l; k) \\
    L (l; (m; r)) & \xrightarrow{L (\text{assoc})} & (l; m); r \\
    L (l; (m; r)) & \xrightarrow{L (\text{assoc})} & (l; m); r \\
    Lc & \xrightarrow{\text{assoc}} & (L; (m; r)); r
  \end{array}
  \end{array} \]

by naturality of \( \varphi \), associativity of \( \varphi \) and local functoriality of \( L \). For the left counit axiom, suppose we are given a commuting diagram as on the left below

\[ \begin{array}{c}
  \begin{array}{ccc}
    c & \xrightarrow{\text{unitor}} & 1_X; c \\
    \delta & \delta; c & \delta; c \\
    \epsilon; c & \epsilon; c & \epsilon; c
  \end{array}
  \end{array} \]

and note the composite on the right above is the unitor by local functoriality of \( L \), naturality of \( \varphi \), and the unital axiom on \( \lambda \). The right counit axiom is similar.

(2) \( \implies \) (1): Suppose we are given the data \( (L, \Phi, \Lambda) \) for a coherent class \( (\Delta_2, \Delta_0) \). Now for any generic \( \delta' : c' \to l'; r' \) in \( \mathcal{A} \) we have a commuting diagram as on the left below with \( \zeta_1, \zeta_2, \zeta \) invertible and \( \delta \in \Delta_2 \)

\[ \begin{array}{c}
  \begin{array}{ccc}
    c & \xrightarrow{\delta} & l; r \\
    \zeta & \zeta_1; \zeta_2 & \zeta_1; \zeta_2 \\
    c' & \xrightarrow{\delta'} & l'; r'
  \end{array}
  \end{array} \]

\[ \begin{array}{c}
  \begin{array}{ccc}
    Lc & \xrightarrow{\Phi_1} & Ll; Lr \\
    L (l; r) & \xrightarrow{\Phi_1} & L (l; r) \\
    L (l; r) & \xrightarrow{\Phi_1} & L (l; r) \\
    Lc & \xrightarrow{\text{unitor}} & L (1_X; c); Lc
  \end{array}
  \end{array} \]
and so we may define $\Phi_{\varepsilon'}$ as the unique morphism making the diagram on the right above commute; this being well defined as a consequence of naturality of comultiplication.

Similarly, for any augmentation $\varepsilon': n' \to 1_X$ in $\mathcal{A}$ there exists an augmentation $\varepsilon: n \to 1_X$ in $\Delta_0$ and isomorphism $\xi: n \to n'$ rendering commutative the left diagram below

and so we may define $\Lambda_{\varepsilon'}$ as the unique morphism making the right diagram above commute; similarly well defined by naturality of counits.

We have now extended the definition of $\Phi$ and $\Lambda$ to all generic morphisms and augmentations. Moreover, the naturality properties now hold with respect to all generics $\delta$ and augmentations $\varepsilon$. Indeed, given any generics $\delta$ and $\delta'$ in $\mathcal{A}$ and a diagram as on the left below (not assuming $\zeta_1, \zeta_2$ or $\zeta$ are invertible)

we can factor as on the right, where $\tilde{\delta}$ and $\tilde{\delta}'$ are in $\Delta_2$ and $\theta, \theta_1, \theta_2, \gamma, \gamma_1$ and $\gamma_2$ are invertible. Applying the naturality condition to the three squares on the right then gives the naturality condition for the left diagram. A similar calculation may be done concerning augmentations.

To show that one may recover an oplax functor $L: \mathcal{A} \to \mathcal{C}$ we note we may define a general oplax constraint cell $\varphi_{a,b}: L(a; b) \to La; Lb$ by taking a diagram as on the left below with $\delta$ generic and then defining the right diagram to commute.

Note that this is well defined since given two diagrams as on the left above, we have a commuting diagram as on the left below

composing to the identity, and this implies the right diagram commutes by naturality of comultiplication (with $\zeta$ taken to be the identity). Trivially, we take each unit $\lambda_X: L(1_X) \to 1_X$ to be the component of $\Lambda$ at $\text{id}_{1_X}$.

To see that the family $\varphi$ satisfies naturality of the constraints suppose that we are given a diagram as on the left below with the horizontal paths composing to

and so we may define $\Phi_{\varepsilon'}$ as the unique morphism making the diagram on the right above commute; this being well defined as a consequence of naturality of comultiplication.
identities
\[ a; b \xrightarrow{\delta} l; r \xrightarrow{s_1; s_2} a; b \]
\[ L(a; b) \xrightarrow{\Phi_\delta} Ll; Lr \xrightarrow{Ls_1; Ls_2} La; Lb \]
\[ a'; b' \xrightarrow{\alpha; \beta} l'; r' \xrightarrow{s'_1; s'_2} a'; b' \]
\[ L(a'; b') \xrightarrow{\Phi_\delta'} Ll'; Lr' \xrightarrow{Ls'_1; Ls'_2} La'; Lb' \]
and note that the right diagram commutes by naturality of comultiplication.

Before checking associativity we first note that given any generics \( \delta_1', \delta_2', \delta_4' \) and \( \delta_4' \) in \( \mathcal{A} \) such that (1) commutes below,

we can construct regions (2) and (3) as on the right above, where \( \delta_1 \) and \( \delta_2 \) lie in \( \Delta_2 \). By naturality of the associator (4) commutes. Then since our given class of generics is coherent, we can find a \( \delta_3 \) and \( \delta_4 \) in \( \Delta_2 \) such that the outside diagram commutes above. By genericity of \( \delta_3 \) we then have induced 2-cells \( \alpha \) and \( \beta \) such that (5) and (6) commute (invertible as \( \delta_3' \) is also generic). Now, by associativity of comultiplication the commutativity of the outside diagram is respected by the transformation \( \delta \mapsto \Phi_\delta \), and this is equivalent to the commutativity of (1) being respected as the pasting with (2),(3),(4),(5) and (6) may be undone.

Now, to see that the family \( \varphi \) satisfies associativity of the constraints consider the outside diagram of

where the appropriate horizontal composites are identity 2-cells. We first factor \( \delta_5 \cdot s_1 \) through a generic \( \delta_2 \) to recover 2-cells \( \xi_1 \) and \( \xi_2 \) and the commuting region (1). Similarly, we create the region (2). Now take \( \delta_3 \) and \( \delta_4 \) to be generics such that region (3) commutes, which exist by Lemma 13. We then note that region (4) commutes by naturality of the associator in \( \mathcal{A} \). Finally, note that we have an induced \( (\gamma_1; \gamma_2) \) by genericity of \( \delta_3 \), and thus \( \delta_7 \gamma_2 \) yields an induced \( (\alpha; \beta) \) through the generic \( \delta_4 \).
We have now constructed the above diagram and shown each region commutes; all that remains is to notice in the corresponding diagram below

naturality of comultiplication implies (1), (2), (5) and (6) commute; associativity of comultiplication implies (3) commutes; naturality of the associators in $\mathcal{C}$ implies (4) commutes, and (7) commutes as $L$ is locally a functor.

Before checking the unitary axioms on $\lambda$ we note that given a generic $\delta'$ and augmentation $\varepsilon'$ composable as in the middle diagram below

we have an isomorphism $\zeta : c \to c'$ by axiom (1) of a coherent class. By axiom (5) we then have a $\delta$ and $\varepsilon$ in the coherent class such that the outside diagram commutes. It follows from genericity of $\delta$ that we have an induced isomorphism $u_1; u_2$ such that the above diagram commutes. As the commutativity of the outside diagram is respected by assumption, and the commutativity of the left and right regions is respected by naturality of comultiplication and augmentations respectively (and the pasting with these regions can be undone), it follows that the commutativity of the middle diagram is respected.

Now, to see the left unit axiom on $\lambda$ is satisfied note that given any commuting diagram as on the left below

we get a commuting diagram as on the right above by naturality of comultiplication, the left counit axiom, and naturality of counits (the bottom composite in this diagram is a $\varphi$ followed by a $\lambda$). The right unitary axiom is similar.

Finally, note that the composite assignment

$$ (1) \mapsto (2) \mapsto (1) $$
is the identity, since with $\Phi$ defined as in (2.3), the oplax constraint cells as recovered by (2.5), given by the family of constraints

$$L (a; b) \overset{L \delta}{\longrightarrow} L (l; r) \overset{\varphi_{l,r}}{\longrightarrow} L l; L r \overset{L s_1; L s_2}{\longrightarrow} L a; L b$$

are clearly equal to $\varphi_{a,b}$ by naturality. Moreover, the composite assignment

$$(2) \mapsto (1) \mapsto (2)$$

is the identity, since with $\varphi$ defined as by (2.5), the comultiplication cells $\Phi$ at an arbitrary generic $\tilde{\delta} \in \Delta_2$ are given by the composite in the top line on the left below

(2.7)

$$Lc \overset{L \tilde{\delta}}{\longrightarrow} L \left( \tilde{l}; \tilde{r} \right) \overset{\varphi_{l,r}}{\longrightarrow} L l; L r \overset{L s_1; L s_2}{\longrightarrow} L \tilde{l}; L \tilde{r}$$

where $\delta \in \Delta_2$ is a generic and the right diagram commutes. Then we note that

we have an induced $\tilde{\delta}_1; \tilde{\delta}_2$ rendering commutative the left diagram above by genericity of $\tilde{\delta}$, the middle diagram shows that the induced diagonal is necessarily a pair of identities (by component-wise commutativity of the bottom triangle), and whiskering the left diagram with $s_1; s_2$ gives the right diagram, where as we have noted the induced diagonal making the diagram commute is a pair of identities. Consequently, $s_1 \tilde{\delta}_1$ and $s_2 \tilde{\delta}_2$ are identities. We then note that in diagram 2.7 the region (1) commutes by naturality of comultiplication, and applying local functoriality of $L$ we then see the given composite is $\Phi_{\tilde{\delta}}$ as required.

The bijection of the nullary data may be similarly proven using the respective naturality properties, and so we omit the details.

**Bijection With Oplax Natural Transformations.** As the the data of (1) and (2) is the same, we need only check that the coherence conditions correspond.

$(1) \iff (2):$ Suppose we are given an oplax natural transformation $\vartheta : L \Rightarrow K$ in the usual sense. Then by the definition of $\Phi$ at a $\delta \in \Delta_2$ we have
which by compatibility with composition is

\[
\begin{array}{ccc}
L_X & \xrightarrow{L_{(c)}} & L_Z \\
\phi & \phi_{L \delta} & \\
K_X & \xrightarrow{K_{(c)}} & K_Z \\
\end{array}
= \\
\begin{array}{ccc}
L_X & \xrightarrow{L_{(r)}} & L_Z \\
\phi & \phi_z & \\
K_X & \xrightarrow{K_{(r)}} & K_Z \\
\end{array}
\]

and by definition of \( \Psi \) this gives the required coherence condition. We omit the nullary version.

(2) \( \implies \) (1): Suppose we are given the data of (2) subject to the coherence conditions of (2). Then by the definition of the constraint data \( \varphi \) we have

\[
\begin{array}{ccc}
L_X & \xrightarrow{L_{(f;g)}} & L_Z \\
\phi_f & \phi_{\varphi_f;g} & \\
K_X & \xrightarrow{K_{(f;g)}} & K_Z \\
\end{array}
= \\
\begin{array}{ccc}
L_X & \xrightarrow{L_{(g;h)}} & L_Z \\
\phi_g & \phi_{\varphi_g;h} & \\
K_X & \xrightarrow{K_{(g;h)}} & K_Z \\
\end{array}
\]

and so applying naturality of \( \varphi \), this is equal to the left below

\[
\begin{array}{ccc}
L_X & \xrightarrow{L_{(f;g)}} & L_Z \\
\phi_f & \phi_{\varphi_f;g} & \\
K_X & \xrightarrow{K_{(f;g)}} & K_Z \\
\end{array}
= \\
\begin{array}{ccc}
L_X & \xrightarrow{K_{(f;g)}} & L_Z \\
\phi_g & \phi_{\varphi_g;h} & \\
K_X & \xrightarrow{K_{(g;h)}} & K_Z \\
\end{array}
\]

which by the assumed coherence axiom is the right above. Applying the definition of \( \psi \), we recover the compatibility of an oplax natural transformation with composition. Again, we will omit the analogous nullary condition.

**Remark 20.** Notice that in Theorem 19, giving binary oplax constraint cells

\[
\varphi_{l,r}: L (l; r) \rightarrow Ll; Lr
\]

for generics \( \delta: c \rightarrow l; r \) in \( \Delta_2 \) completely determines arbitrary oplax constraint cells

\[
\varphi_{a,b}: L (a; b) \rightarrow La; Lb
\]

This is since these \( \varphi_{l,r} \) suffice to construct each \( \Phi_\delta \). Hence this theorem provides a reduction in the data of an oplax functor when the domain bicategory \( A \) is generic.

**Remark 21.** Given a family of hom-categories \( A_{X,Y} \), sets \( M_{X,Y,Z} \), and natural isomorphisms

\[
A_{X,Z} (c; a; b) \cong \sum_{m \in M_{Y,Z}} A_{X,Y} (l_m; a) \times A_{Y,Z} (r_m; b)
\]
for all \(X, Y, Z\) and \(c\), the formal composite \(a; b\) is essentially uniquely determined (by essential uniqueness of representing objects).

Given a complete class of generics \(\Delta_2\) equipped with their universal properties, one may recover the above by taking \(\mathbb{M}_{XYZ}\) to be the set of equivalence classes of generics \(\delta: c \to l; r\). It follows that composition in the bicategory is essentially uniquely determined by the generics.

### 3. Consequences and examples

In this section we discuss some of the main examples of Theorem 19. Viewing monoidal categories as one-object bicategories, we first consider the case where \(\mathcal{A}\) is a cartesian monoidal category, giving a simple and informative example of this situation. We then go on to consider more complicated examples, namely where \(\mathcal{A}\) is the bicategory of spans or the bicategory of polynomials with cartesian 2-cells.

For completeness, we also discuss the case where \(\mathcal{A}\) is the category of finite sets and bijections with the disjoint union monoidal structure, but will omit some details as this is a rather trivial example.

#### 3.1. Cartesian monoidal categories.

Given a category \(\mathcal{E}\) with finite products, one may construct the cartesian monoidal category \((\mathcal{E}, \times, 1)\) where the tensor product is the cartesian product and the unit is the terminal object. Clearly this monoidal category is generic, as

\[ \mathcal{E} (T, - \times -): \mathcal{E} \times \mathcal{E} \to \text{Set} \]

is representable (no coproducts are necessary). Now, seen as a one object bicategory, the generics are the diagonal morphisms \(\delta_T\) in \(\mathcal{E}\) of the form

\[
\begin{array}{ccc}
T & \xleftarrow{\delta_T} & T \\
\downarrow{id} & & \downarrow{id} \\
T \times T & \xrightarrow{s_1} & T \\
\end{array}
\]

and so we take \(\Delta_2\) to be the class of diagonals \(\delta_T: T \to T \times T\) for each \(T \in \mathcal{E}\). Trivially, we take the augmentations as the unique maps into the terminal object from each object \(T \in \mathcal{E}\). Applying Theorem 19 in this case then makes it clear why we may say the data of this theorem is analogous to the data of a comonad; indeed, we have the following.

**Corollary 22.** Let \(\mathcal{E}\) be a category with finite products and let \((\mathcal{C}, \odot, I)\) be a monoidal category. Denote by \((\mathcal{E}, \times, 1)\) the category \(\mathcal{E}\) equipped with the cartesian monoidal structure. Then to give an oplax monoidal functor

\[ L: (\mathcal{E}, \times, 1) \to (\mathcal{C}, \odot, I) \]

is to give a functor \(L: \mathcal{E} \to \mathcal{C}\) with comultiplication and counit maps

\[ \Phi_T: L(T) \to L(T) \odot L(T), \quad \Lambda_T: L(T) \to I \]

for every \(T \in \mathcal{E}\), such that for every \(T \in \mathcal{E}\) the diagrams

\[
\begin{array}{ccc}
LT & \xrightarrow{\Phi_T \odot LT} & LT \odot I \\
\downarrow{\text{unitor}} & & \downarrow{\text{unitor}} \\
LT & \xrightarrow{LT; \Lambda_T} & I \odot LT
\end{array}
\]

\[
\begin{array}{ccc}
LT & \xleftarrow{LT; \Lambda_T} & LT \odot I \\
\downarrow{\Phi_T \odot LT} & & \downarrow{\Phi_T \odot LT} \\
LT & \xrightarrow{\text{unitor}} & I \odot LT
\end{array}
\]
commute, the diagrams

\[
\begin{array}{cc}
LT & LT \\
\Phi_T \downarrow & \Phi_T \\
LT \otimes LT & LT \otimes LT \\
LT \otimes LT \otimes LT & LT \otimes LT \otimes LT \\
\end{array}
\]

commute, and all morphisms \( f : T \to T' \) in \( E \) render commutative

\[
\begin{array}{cc}
L(T) & L(T') \\
\Lambda_T \downarrow & \Lambda_{T'} \\
1_X & 1_Y \\
\end{array}
\]

The unitary and associativity conditions above ask that \( L \) sends each \( T \in E \) to a comonoid \((LT, \Phi_T, \Lambda_T)\) in \((C, \otimes, I)\), and the last two conditions ask that morphisms in \( E \) are sent to morphisms of comonoids. Hence this may be simply stated as follows.

**Corollary 23.** Let \( \text{Comon} (C, \otimes, I) \) be the category of comonoids in the monoidal category \((C, \otimes, I)\). Then oplax monoidal functors \((E, \times, 1) \to (C, \otimes, I)\) are in bijection with functors \( E \to \text{Comon} (C, \otimes, I)\).

### 3.2. Bicategories of spans

Given a category \( E \) with pullbacks, one may form the bicategory of spans in \( E \) denoted \( \text{Span}(E) \) with objects those of \( E \), 1-cells given by spans

\[
\begin{array}{cc}
X & \overset{T}{\rightarrow} & Z \\
\downarrow & & \downarrow \\
\overset{s}{\rightarrow} & & \overset{t}{\rightarrow} \\
\end{array}
\]

denoted \((s, t)\), 2-cells given by morphisms \( f \) rendering commutative diagrams as on the left below

and composition of 1-cells given by forming the pullback as on the right above [1].

The reader will then notice that by the universal property of pullback, giving a morphism of spans \((s, t) \to (u, v) ; (p, q)\) as on the left below

\[
\begin{array}{cc}
X & \overset{R}{\rightarrow} & Y & \overset{S}{\rightarrow} & Z \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\overset{u}{\rightarrow} & \overset{v}{\rightarrow} & \overset{p}{\rightarrow} & \overset{q}{\rightarrow} & \overset{s}{\rightarrow} \overset{t}{\rightarrow} \\
\end{array}
\]

is to give a morphism \( h : T \to Y \) as well as pair of morphisms of spans as on the right above such that each region in the diagram commutes. Therefore

\[
\text{Span}(E)_{X,Z} ((s,t) , (u,v); (p,q))
\]

is isomorphic to

\[
\sum_{h : H \to Y} \text{Span}(E)_{X,Y} ((s,h), (u,v)) \times \text{Span}(E)_{Y,Z} ((h,t), (p,q))
\]
and so the bicategory of spans is generic. Our class of generics \( \Delta_2 \) consists of, for each diagram

\[
\begin{array}{ccc}
T & \rightarrow & X \\
\downarrow & & \downarrow \\
Y & \rightarrow & Z
\end{array}
\]

in \( E \), the morphisms of spans \( \delta_{s,h,t} : (s,t) \rightarrow (s,h) ; (h,t) \) corresponding to

under this bijection. Our augmentations are the morphisms of spans as below for each morphism \( h \) in \( E \)

and will be denoted by \( \varepsilon_h \). Thus, applying Theorem 19 we have the following.

**Corollary 24.** Let \( E \) be a category with pullbacks and denote by \( \text{Span}(E) \) the bicategory of spans in \( E \). Let \( C \) be a bicategory. Then to give an oplax functor

\[
L : \text{Span}(E) \rightarrow C
\]

is to give a locally defined functor

\[
L_{X,Y} : \text{Span}(E)_{X,Y} \rightarrow C_{LX,LY}, \quad X,Y \in E
\]

with comultiplication and counit maps

\[
\Phi_{s,h,t} : L(s,t) \rightarrow L(s,h) ; L(h,t), \quad \Lambda_h : L(h,h) \rightarrow 1_{LX}
\]

for every respective diagram in \( E \)

such that:

1. for any triple of morphisms of spans as below

\[
\begin{array}{ccc}
X & \rightarrow & Z \\
\downarrow & & \downarrow \\
T & \rightarrow & T
\end{array}
\]

we have the commuting diagram

\[
\begin{array}{ccc}
L(u,v) & \xrightarrow{\Phi_{u,k,v}} & L(u,k) ; L(k,v) \\
Lf & & Lf ; Lf \\
L(s,t) & \xrightarrow{\Phi_{s,h,t}} & L(s,h) ; L(h,t)
\end{array}
\]
(2) for any morphism of spans as on the left below

\[
\begin{array}{c}
\xymatrix{
X \ar[r]^q & N \ar[r]^q & X \\
\downarrow^p & & \downarrow^p \\
M \ar[r]_p & & 
}
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{
L(p,p) \ar[r]^{Lf} & L(q,q) \\
\downarrow^\Lambda_p & & \downarrow^\Lambda_q \\
1_{LX} \ar[r]_{\Lambda_X} & 
}
\end{array}
\]

the diagram on the right above commutes;

(3) for all diagrams of the form

\[
\begin{array}{c}
\xymatrix{
W \ar[r]^s & X \ar[r]^{T} & Y \ar[r]^t & Z \\
\downarrow^h & \downarrow^k & & & \downarrow^k \\
\downarrow^f & \downarrow^g & & & \downarrow^h \\
X \ar[r]^X & Y \ar[r]^Y & Z
}
\end{array}
\]

in \(E\), we have the commuting diagram

\[
\begin{array}{c}
\xymatrix{
L(s,t) \ar[r] & L(s,t) \\
\downarrow^{\Phi_{s,t}} & \downarrow^{\Phi_{s,t}} \\
L(s,h) \cdot L(h,t) & L(s,k) \cdot L(k,t) \\
\downarrow^{L(s,h) \cdot \Phi_{h,k,t}} & \downarrow^{\Phi_{s,k,t}} \\
L(s,h) \cdot (L(h,k) \cdot L(k,t)) \ar[r]^{\text{assoc}} & (L(s,h) \cdot (L(h,k))) \cdot L(k,t)
}
\end{array}
\]

(4) for all spans \((s,t)\) we have the commuting diagrams

\[
\begin{array}{c}
\xymatrix{
L(s,s) \cdot L(s,t) \ar[r] & L(s,t) \\
\downarrow^{\Phi_{s,s,t}} & \downarrow^{\Phi_{s,s,t}} \\
L(s,t) \cdot L(t,t) \ar[r]^{\text{unitor}} & L(s,t) \cdot 1_{LY} \\
\downarrow^{\Delta_{s,t}} & \downarrow^{\text{unitor}} \\
L(s,s) \cdot L(s,t) \ar[r]^{1_{LX} \cdot L(s,t)} & L(s,t) \cdot L(s,t) \cdot 1_{LY}
}
\end{array}
\]

Remark 25. Note that this description of an oplax functor out of the bicategory of spans does not involve pullbacks, thus allowing for a simpler proof of the universal properties of the span construction [2].

3.3. Bicategories of polynomials. Given a locally cartesian closed category \(E\), one may form the bicategory of polynomials in \(E\) with cartesian 2-cells [14, 6]. This bicategory we denote by \(\text{Pol}_c(E)\) and has objects those of \(E\), 1-cells given by diagrams

\[
\begin{array}{c}
\xymatrix{
E \ar[r]^B & Z \\
X \ar[u]^s & \ar[l]_t 
}
\end{array}
\]

in \(E\) called polynomials and denoted by \((s,p,t)\), and 2-cells given by commuting diagrams as below

\[
\begin{array}{c}
\xymatrix{
K \ar[r]^i & I \\
X \ar[r]^f & Y \ar[u]^g & \ar[l]_b \ar[l]_a \\
R \ar[r]^j & Y
}
\end{array}
\]

where the middle square is a pullback. Composition of 1-cells is more complicated and so will be omitted; especially as it is not necessary to describe oplax functors out of \(\text{Pol}_c(E)\) once we know the generics.

The reader need only know the following corollary of [14, Prop. 3.1.6], a description of polynomial composition due to Weber.
**Corollary 26.** Consider two polynomials in $E$ as below:

$$K \xrightarrow{a} I \xleftarrow{b} Y \quad \text{and} \quad \begin{array}{c}
R \xrightarrow{u} J \xleftarrow{v} Z
\end{array}$$

Then to give a cartesian 2-cell $(s, p, t) \to (a, i, b); (u, j, v)$ is to give a factorization $p = p_1; p_2$ through an object $T$, a morphism $h: T \to Y$, and a pair of cartesian morphisms $(s, p_1, h) \to (a, i, b)$ and $(h, p_2, t) \to (u, j, v)$ such that the above diagram commutes. Here we identify a septuple $(p_1, h, p_2, w, x, y, z)$ as above with another septuple $(p_1', h', p_2', w', x', y', z')$ if $w = w'$, $z = z'$ and there exists an invertible $\alpha: T \to T'$ rendering commutative the diagrams.

It follows that

$$\text{Poly}_c(E)_{X, Z}((s, p, t), (a, i, b); (u, j, v))$$

is isomorphic to

$$\sum_{p = p_1; p_2, h: T \to Y} \text{Poly}_c(E)_{X, Y}((s, p_1, h), (a, i, b)) \times \text{Poly}_c(E)_{Y, Z}((h, p_2, t), (u, j, v))$$

where the equivalence relation “∼” indicates the sum is taken over representatives of equivalence classes of triples $(p_1, h, p_2)$ (where two such triples are seen as equivalent if there is an isomorphism $\alpha$ rendering commutative the left and right diagrams as in Figure 3.2). We have thus exhibited the bicategory of polynomials with cartesian 2-cells as a generic bicategory.

Here our class of generics $\Delta_2$ consists of, for each diagram

$$E \xrightarrow{p_1} T \xleftarrow{p_2} B$$

in $E$ where $p = p_1; p_2$, the cartesian morphisms of polynomials

$$\delta_{s, p_1, h, p_2, t}: (s, p, t) \to (s, p_1, h); (h, p_2, t)$$

It is clear that if the middle diagram commutes then the rightmost diagram also does. Also, such an isomorphism $\alpha$ making the left diagram commute must be unique.
corresponding to

\[
\begin{array}{ccc}
E & \xrightarrow{p_1} & T \\
\downarrow^s & \downarrow & \downarrow^h \\
X & \xrightarrow{T} & Y \\
\downarrow^h & \downarrow & \downarrow \\
B & \xrightarrow{p_2} & B \\
\end{array}
\]

under this bijection. We take as our augmentations the cartesian morphisms

\[
\begin{array}{ccc}
X & \xrightarrow{id} & Y \\
\downarrow & \downarrow & \downarrow \\
X & \xrightarrow{id} & Y \\
\end{array}
\]

and denote these by \(\varepsilon_h\). There are more general morphisms into identity polynomials where the middle map is invertible; but using those would lead to unnecessary complexity.

Remark 27. Note that our class of generics \(\Delta_2\) does not involve representatives of equivalence classes, unlike the summation formula given.

Now, applying Theorem 19 we have the following.

**Corollary 28.** Let \(\mathcal{E}\) be a locally cartesian closed category and denote by \(\text{Poly}_c(\mathcal{E})\) the bicategory of polynomials in \(\mathcal{E}\) with cartesian 2-cells. Let \(\mathcal{C}\) be a bicategory. Then to give an oplax functor

\[L: \text{Poly}_c(\mathcal{E}) \to \mathcal{C}\]

is to give a locally defined functor

\[L_{X,Y}: \text{Poly}_c(\mathcal{E})_{X,Y} \to \mathcal{C}_{LX,LY}, \quad X,Y \in \mathcal{E}\]

with comultiplication and counit maps

\[\Phi_{s,p_1,p_2,t}: L(s,p,t) \to L(s,p_1,h) \cdot L(h,p_2,t), \quad \Lambda_h: L(h,1,h) \to 1_{LX}\]

for every respective diagram in \(\mathcal{E}\)

\[
\begin{array}{ccc}
E & \xrightarrow{p_1} & T \\
\downarrow^s & \downarrow^h & \downarrow^h \\
X & \xrightarrow{T} & Y \\
\downarrow & \downarrow & \downarrow \\
B & \xrightarrow{p_2} & B \\
\end{array}
\]

where we assert \(p = p_1;p_2\) on the left, such that:

1. for any morphisms of polynomials as below
we have the commuting diagram

\[
\begin{array}{c}
\Phi_{u,q_1,k,q_2,v} \\
L_{(u,q,v)} \downarrow \\
L(f,g) \\
L(s,p,t) \downarrow \\
\Phi_{s,p_1,h,p_2,t}
\end{array}
\quad
\begin{array}{c}
L_{(u,q_1,k,q_2,v)} \\
\downarrow \\
L_{(f,c)} \downarrow \\
L_{(s,p_1,h,p_2,t)} \\
\end{array}
\]

(2) for any morphism of polynomials as on the left below

\[
\begin{array}{c}
R \\
\downarrow \\
X \\
\downarrow \\
T \\
\downarrow \\
J
\end{array}
\quad
\begin{array}{c}
L_{(s,1,s)} \downarrow \\
L_{(f,f)} \\
\downarrow \\
L_{(s,p,t)} \\
\downarrow \\
1_{LY}
\end{array}
\]

the diagram on the right above commutes;

(3) for all diagrams of the form

\[
\begin{array}{c}
F \\
\downarrow \\
W \\
\downarrow \\
X \\
\downarrow \\
Y \\
\downarrow \\
Z
\end{array}
\quad
\begin{array}{c}
G \\
\downarrow \\
H \\
\downarrow \\
K
\end{array}
\]

in \( \mathcal{E} \), we have the commuting diagram

\[
\begin{array}{c}
L_{(s,a; b;c, t)} \\
\downarrow \\
L_{(s,a; h; b;c,t)} \\
\downarrow \\
L_{(s,a; h; b,k; c,t)} \\
\downarrow \\
\Phi_{s,a; h; b,k; c,t}
\end{array}
\quad
\begin{array}{c}
L_{(s,a; b;c, t)} \\
\downarrow \\
L_{(s,a; b,k; c,t)} \\
\downarrow \\
L_{(s,a; b,k; h; c,t)} \\
\downarrow \\
\Phi_{s,a; b,k; h; c,t}
\end{array}
\]

(4) for all polynomials \( (s,p,t) \) the diagrams

\[
\begin{array}{c}
L_{(s,1,s); L_{(s,p,t)}} \\
\downarrow \\
L_{(s,p,t)} \quad \text{unitor}
\end{array}
\quad
\begin{array}{c}
L_{(t,1,t)} \\
\downarrow \\
L_{(s,p,t); \Lambda_t}
\end{array}
\]

\[
\begin{array}{c}
L_{(s,p,t)} \\
\downarrow \\
L_{(s,p,t); \Lambda_t}
\end{array}
\quad
\begin{array}{c}
L_{(s,p,t)} \\
\downarrow \\
L_{(s,p,t); \Lambda_t}
\end{array}
\]

\[
\begin{array}{c}
L_{(s,p,t)} \\
\downarrow \\
L_{(s,p,t); 1_{LY}}
\end{array}
\quad
\begin{array}{c}
L_{(s,p,t)} \\
\downarrow \\
\Phi_{s,p,t; t}
\end{array}
\]

\[
\begin{array}{c}
\text{commute.}
\end{array}
\]

Remark 29. As the above description of oplax functors out of the bicategory of polynomials does not rely on polynomial composition, it may be used for an efficient proof of the universal properties of polynomials. Indeed, this allows us to avoid the large coherence diagrams which would arise in a direct proof. We will discuss this in detail in our next paper.
3.4. **Finite sets and bijections.** We give this example for completeness, but will omit some details as Theorem 19 becomes rather trivial in this case (due to all generic morphisms being invertible). Here we take \( \mathcal{A} \) to be the category of finite sets and bijections with the disjoint union monoidal structure, denoted \((\mathbb{P}, \sqcup, \emptyset)\). This monoidal category is generic since we have isomorphisms

\[
\mathbb{P}(C, A \sqcup B) \cong \sum_{C=L\sqcup R} \mathbb{P}(L, A) \times \mathbb{P}(R, B)
\]

natural in finite sets \( A \) and \( B \), where the sum is taken over decompositions of \( C \) into the disjoint union of two sets. Here we choose our class of generics \( \Delta_2 \) to contain

\[
\delta_{n_1, n_2} : [n_1 + n_2] \to [n_1] \sqcup [n_2], \quad n \mapsto \begin{cases} (1, n), & n \leq n_1 \\ (2, n), & n > n_1 \end{cases}
\]

for each pair of non-negative integers \( n_1 \) and \( n_2 \). Trivially, the only augmentation is the identity map on the empty set. Taking \((C, \otimes, I)\) to be a monoidal category, it follows from Theorem 19 that oplax monoidal functors \( L: (\mathbb{P}, \sqcup, \emptyset) \to (C, \otimes, I) \) may be specified by giving comultiplication and counit maps

\[
\Phi_{n_1, n_2} : L [n_1 + n_2] \to L [n_1] \otimes [n_2], \quad \Lambda : L (\emptyset) \to I
\]

Of course, this may more easily be seen by simply taking the skeleton.

4. **Convolution structures and Yoneda structures**

By results of Day [3], given a bicategory \( \mathcal{A} \) with locally small hom-categories one may consider the local cocompletion of \( \mathcal{A} \), a new bicategory \( \hat{\mathcal{A}} \) with objects those of \( \mathcal{A} \), hom-categories given by

\[
\hat{\mathcal{A}}_{X,Y} := \mathcal{A}_{X,Y}^{\text{op}}, \quad X, Y \in \mathcal{A}_{\text{ob}}
\]

and a composite of two presheaves

\[
F: \mathcal{A}_{X,Y}^{\text{op}} \to \text{Set}, \quad G: \mathcal{A}_{Y,Z}^{\text{op}} \to \text{Set}
\]

given by Day’s convolution formula

\[
GF: \mathcal{A}_{X,Z}^{\text{op}} \to \text{Set}, \quad GF(c) = \int_{a,b} \mathcal{A}_{X,Z} (c, a; b) \times Fa \times Gb
\]

With this definition, the family of Yoneda embeddings on the hom-categories defines a pseudofunctor \( y_{\mathcal{A}}: \mathcal{A} \to \hat{\mathcal{A}} \). This is of interest since in the case of generic bicategories \( \mathcal{A} \), this convolution structure has an especially nice form.

**Proposition 30.** Suppose \( \mathcal{A} \) is a generic bicategory. Then for any pair of presheaves

\[
F: \mathcal{A}_{X,Y}^{\text{op}} \to \text{Set}, \quad G: \mathcal{A}_{Y,Z}^{\text{op}} \to \text{Set}
\]

there exists isomorphisms as below

\[
\int_{a,b} \mathcal{A}_{X,Z} (c, a; b) \times Fa \times Gb \cong \sum_{m \in \Delta_2} F\alpha_m \times G\alpha_m
\]

thus reducing the Day convolution structure to a simpler formula.
Proof. We have

\[
\text{LHS} = \int^{a,b} \mathcal{A}_{X,Z}(c,a;b) \times F_a \times G_b
\]

\[
\cong \int^{a,b} \sum_{m \in \mathcal{M}_{X,Y,Z}} \mathcal{A}_{X,Y}(l_m, a) \times \mathcal{A}_{Y,Z}(r_m, b) \times F_a \times G_b
\]

\[
\cong \sum_{m \in \mathcal{M}_{X,Y,Z}} \int^{a,b} \mathcal{A}_{X,Y}(l_m, a) \times F_a \times \mathcal{A}_{Y,Z}(r_m, b) \times G_b
\]

\[
\cong \sum_{m \in \mathcal{M}_{X,Y,Z}} \left( \int^{a} \mathcal{A}_{X,Y}(l_m, a) \times F_a \right) \times \left( \int^{b} \mathcal{A}_{Y,Z}(r_m, b) \times G_b \right)
\]

\[
\cong \sum_{m \in \mathcal{M}_{X,Y,Z}} F_{l_m} \times G_{r_m}
\]

\[
= \text{RHS}
\]

as required. \qed

Remark 31. Unfortunately, the above formula has some disadvantages. Indeed, as \(\mathcal{M}_{X,Y,Z}^c\) is isomorphic to the set of equivalence classes of generics out of \(c\), it follows that writing down \(\mathcal{M}_{X,Y,Z}^c\) will involve a choice of representatives for each equivalence class. This is problematic since choices of representatives do not nicely behave with respect to composition.

As a consequence of this proposition and the formulas (3.1) and (3.3) given in the previous section, we have the following.

Corollary 32. The Day convolution of two presheaves of spans

\[ F: \text{Span}(\mathcal{E})^{\text{op}}_{X,Y} \to \text{Set}, \quad G: \text{Span}(\mathcal{E})^{\text{op}}_{Y,Z} \to \text{Set} \]

is given by

\[ GF: \text{Span}(\mathcal{E})^{\text{op}}_{X,Z} \to \text{Set}, \quad GF(s,t) \cong \sum_{h: T \to Y} F(s,h) \times G(h,t) \]

and the Day convolution of two presheaves of polynomials

\[ F: \text{Poly}_c(\mathcal{E})^{\text{op}}_{X,Y} \to \text{Set}, \quad G: \text{Poly}_c(\mathcal{E})^{\text{op}}_{Y,Z} \to \text{Set} \]

is given by the formula

\[ GF: \text{Poly}_c(\mathcal{E})^{\text{op}}_{X,Z} \to \text{Set}, \quad GF(s,p,t) \cong \sum_{p=p_1,p_2} \sim F(s,p_1,h) \times G(h,p_2,t) \]

The purpose of the following is to describe how Theorem 19 may be seen as an instance of a more general result. Indeed, as a special case of [12, Theorem 76] we have the following corollary.

Corollary 33 (Doctrinal Yoneda Structures). Let \(\mathcal{A}\) and \(\mathcal{C}\) be bicategories with locally small hom-categories. Let \(\mathcal{A}\) be the free small local cocompletion of \(\mathcal{A}\). Then for any locally defined identity on objects functor \(L: \mathcal{A} \to \mathcal{C}\), with the corresponding
locally defined identity on objects functor \( R = \mathcal{C}(L-, -) \) as below

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{R} & \mathcal{A} \\
\downarrow & & \downarrow \varphi \\
L & \xrightarrow{\rho} & \mathcal{A}
\end{array}
\]

the structure of an oplax functor on \( L \) is in bijection with the structure of a lax functor on \( R \).

Supposing that \( \mathcal{A} \) is generic, and hence that composition on \( \mathcal{A} \) has the reduced form given by Proposition 30, one sees that for a given locally defined functor \( L: \mathcal{A} \to \mathcal{C} \), giving an oplax functor \( (L, \varphi, \lambda): \mathcal{A} \to \mathcal{C} \) with constraint cells

\[\varphi_{a,b}: L(a;b) \to La;Lb, \quad \lambda_X: L1_X \to 1_X\]

is to give a lax functor \( (R, \phi, \omega): \mathcal{C} \to \mathcal{A} \) with constraints

\[\phi_{a,b}: Ra;Rb \to R(a;b), \quad \omega_X: 1_X \to R1_X\]

These binary constraints are functions for each \( c: X \to Z \)

\[
\sum_{m \in \mathfrak{M}^{X,Y,Z}_c} \mathcal{C}_{X,Y}(Ll_m,a) \times \mathcal{C}_{Y,Z}(Lr_m,b) \to \mathcal{C}_{X,Z}(Lc,a;b)
\]

natural in \( a, b \) and \( c \). By naturality, to give such a function is to give an assignment on the identity pair (we may call the result \( \Phi_{c,m} \))

\[
(id: Ll_m \to Ll_m, id: Lr_m \to Lr_m) \mapsto \Phi_{c,m}: Lc \to Ll_m;Lr_m
\]

A similar calculation may be done with the nullary constraints \( \Lambda \).

Remark 34. It is this observation which is the motivation for Theorem 19. However, this approach does not give an efficient proof of this theorem for a number of technical reasons. In particular, we wish to avoid considering equivalence classes of generic morphisms (such as the set \( \mathfrak{M}^{X,Y,Z}_c \)) to avoid technicalities involving choices of representatives.

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