Influence of Electromagnetic Fields on the Evolution of Initially Homogeneous and Isotropic Universe

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Abstract—Simple exact solutions presented here describe universes whose spatial geometries are asymptotically homogeneous and isotropic near the initial singularity but whose evolution proceeds under the influence of primordial magnetic fields. In all these “deformed” Friedmann models (spatially flat, open or closed), the initial magnetic fields are concentrated near some axis of symmetry and their lines are the circles given by the lines of the azimuthal coordinate $\varphi$. Caused by the expansion of the universe, the time dependence of a magnetic field induces (in accordance with the Faraday law) the emergence of source-free electric fields. In comparison with the Friedmann models, the cosmological expansion proceeds with acceleration in the spatial directions across the magnetic field and with deceleration along the magnetic lines, so that in the flat and open models, in fluid comoving coordinates, the lengths of $\varphi$-circles of sufficiently large radius or for sufficiently late times decrease and vanish as $t \to \infty$. This means that in the flat and open models we have a partial dynamical closure of space-time at large distances from the symmetry axis, i.e., from the regions where the electromagnetic fields in our solutions are concentrated. To get simple exact solutions of the Einstein–Maxwell and perfect fluid equations, we assume a rather exotic stiff matter equation of state $\varepsilon = p$ for the perfect fluid (which supports isotropic and homogeneous “background” Friedmann geometries). However, it seems reasonable to expect that similar effects might occur in the mutual dynamics of geometry and strong electromagnetic fields in universes with more realistic matter equations of state.

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INTRODUCTION

Our physical intuition often proves to be very helpful in solving many problems of mechanics as well as theoretical and mathematical physics that can be considered in the framework of Newtonian mechanics or relativistic mechanics and electrodynamics in Minkowski space-time. However, it is much more difficult to adapt our intuition to the cases in which various physical phenomena occur in curved space-times, in strong gravitational, electromagnetic and other matter fields, when the behavior of these fields is governed by more complicated nonlinear equations of Einstein’s General Relativity, which implies that the space-time geometry, together with matter fields, is involved in their joint dynamics. In these cases, to better understand this context in general, one can consider some very simple exact solutions and the corresponding idealized models, which can be analyzed in detail, thereby providing us with useful information about the qualitative behavior of gravitational, electromagnetic and other matter fields and their interactions in an arbitrarily strong field regime.

In this paper, we construct some very simple models describing the evolution of universes under the influence of electromagnetic fields. There is a number of cosmological solutions with homogeneous time-dependent magnetic fields, which can be found in the literature of different years (e.g., in [1–5]). A rather detailed survey of exact solutions for inhomogeneous cosmological models, including some models with magnetic fields, was given later by Krasinski [6].

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The main features of the solutions presented below are that their spatial geometries at the very beginning of the cosmological expansion are asymptotically homogeneous and isotropic, but the solutions include some primordial vortex magnetic fields that are regular everywhere and whose lines of force are circles. In these solutions, the influence of electromagnetic fields on the geometry and its dynamics is negligible near the cosmological singularity; however, at later stages of the evolution, this influence becomes more “observable” and causes the development of large scale inhomogeneities. To simplify our models and to be able to construct exact solutions for a complete system of the Einstein–Maxwell and perfect fluid equations, we have assumed a rather exotic stiff matter equation of state \( \varepsilon = p \) for the perfect fluid that supports the isotropic and homogeneous “background” Friedmann geometry. However, it seems reasonable to expect that similar effects can occur in the mutual dynamics of geometry and electromagnetic fields in universes with more realistic matter equations of state.

To construct the desired solutions, we begin with the observation of Belinskii [7], who showed that for the fields with two commuting isometries, the presence of a stiff perfect fluid does not change the dynamical part of Einstein’s field equations (the projections of the Einstein equations on the Killing vectors), which completely coincides with that for vacuum. The perfect fluid parameters enter only the remaining part of the Einstein equations (“constraint equations”). Hence, to construct the solutions of this system, one can use various methods that have been developed earlier for solving the pure vacuum Einstein equations with two-dimensional symmetries. First of all, we mention here the inverse scattering approach and soliton-generating technique of Belinskii and Zakharov [8], as well as their application in the present context in [7]. It is easy to see that the same is true in the space-times with two commuting isometries in which, in addition to gravity, a stiff fluid and electromagnetic fields are present. Therefore, the symmetry-reduced Einstein–Maxwell and stiff perfect fluid equations are also integrable, and to solve this system we can apply the corresponding inverse scattering methods and the soliton-generating technique as well as the integral equation methods developed long ago for constructing electrovacuum solutions [9–11] (a short survey of various methods can be found in [12]). However, we follow here a different approach (the motivation is given in Section 7).

In the present paper, we consider the system of Einstein–Maxwell and stiff perfect fluid equations for the class of fields that possess only one space-like Killing vector field, i.e., for the field configurations whose metric components, electromagnetic vector potential, energy density and 4-velocity of the fluid depend on three of the four space-time coordinates, namely, on time and two spatial coordinates. The dynamical part of these equations also coincides with the dynamical equations for electrovacuum Einstein–Maxwell fields with one Killing vector. Therefore, for the Einstein–Maxwell and stiff perfect fluid equations we can use the solution-generating methods (based on the internal or “hidden” symmetries of these equations) that were developed earlier for the electrovacuum Einstein–Maxwell equations.

Although the latter system has not been found to be integrable, it is known that it can be presented in the form of generalized Ernst equations for two complex Ernst-like potentials. These equations admit some symmetry transformations that constitute the eight-parameter group \( \text{SU}(2, 1) \) [13–15]. Therefore, these are also symmetry transformations of the system of Einstein–Maxwell and stiff perfect fluid equations with one space-like Killing vector field. For our present purposes we apply a particular type of electrovacuum \( \text{SU}(2, 1) \)-symmetry transformations that were discovered earlier by Harrison [16]. These transformations allow one to use the solutions for pure gravity (with a stiff perfect fluid, as in our case) to generate the corresponding solutions that also include some electromagnetic fields.

In the subsequent sections, we describe the three-dimensional Ernst-like equations and the Harrison symmetry transformations for the Einstein–Maxwell and stiff perfect fluid equations for space-times with one Killing vector field. We also describe in detail the properties of the solutions obtained.
by applying the Harrison transformations to different types of homogeneous and isotropic Friedmann solutions with a stiff perfect fluid and with open, flat and closed spatial geometries. Some very simple solutions that arise in this way demonstrate a number of interesting phenomena that occur in the interaction of strong gravitational and electromagnetic fields and that are described below. Certain important remarks and comments are given in the concluding Section 7.

1. FRIEDMANN COSMOLOGICAL MODELS WITH STIFF MATTER

The space-times with spatially isotropic and homogeneous distributions of matter (ideal fluid) are described by one of the Friedmann models with the metric

\[ ds^2 = a^2(t)[dt^2 - d\chi^2 - S^2(\chi)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)], \]  

(1.1)

where the function \( S(\chi) \) determines one of three possible types of the geometry of these models, in which the three-dimensional physical spaces can be open, flat or closed spaces of (spatially) constant curvature (evolving, however, with time). The function \( a(t) \) determines the dynamics of the model in accordance with the Einstein equations, provided the matter equation of state is given. For a stiff matter fluid (the equation of state \( \varepsilon = p \)), the Einstein equations determine \( a(t) \), \( S(\chi) \) and the energy density \( \varepsilon(t) \):

\[
\begin{align*}
    a(t) &= a_0 \sqrt{\sin(2t)}, & S(\chi) &= \sin \chi, & k &= -1, \\
    a(t) &= a_0 \sqrt{2t}, & S(\chi) &= \chi, & k &= 0, & \text{and} & & \varepsilon &= \frac{3a_0^4}{8\pi a^5(t)}, \\
    a(t) &= a_0 \sqrt{\sin(2t)}, & S(\chi) &= \sin \chi, & k &= 1,
\end{align*}
\]

(1.2)

where \( k = -1, 0, 1 \) for open, flat and closed Friedmann models, respectively.

2. THE EINSTEIN–MAXWELL AND STIFF FLUID EQUATIONS

To construct new solutions, we consider the Einstein–Maxwell and stiff fluid equations in the absence of any charges and currents \((i, k, \ldots = 1, \ldots, 4; \gamma = 1, \epsilon = 1)\):

\[
\begin{align*}
    R_{ik} - \frac{1}{2} R g_{ik} &= 8\pi \left( T_{ik}^{(EM)} + T_{ik}^{(\varepsilon=p)} \right), \\
    \nabla_k F^k_i &= 0, & \nabla_{[j} F_{ik]} &= 0,
\end{align*}
\]

(2.1)

where \( T_{ik}^{(EM)} \) and \( T_{ik}^{(\varepsilon=p)} \) are the energy-momentum tensors of the electromagnetic field and stiff matter fluid, respectively:

\[
\begin{align*}
    T_{ik}^{(EM)} &= -\frac{1}{4\pi} \left[ F_{il} F^l_k - \frac{1}{4} F_{lm} F^{lm} g_{ik} \right], & F_{ik} &= \partial_i A_k - \partial_k A_i, \\
    T_{ik}^{(\varepsilon=p)} &= (\varepsilon + p) u_i u_k - p g_{ik}, & \varepsilon &= p, & u_k u^k &= 1.
\end{align*}
\]

Here \( \varepsilon \) and \( p \) are the energy density and pressure of the fluid, respectively, and \( u^k \) is the 4-velocity of the fluid. Equations (2.1) can also be presented in the form

\[
\begin{align*}
    R_{ik} &= 8\pi \left( T_{ik}^{(EM)} + 2\varepsilon u_i u_k \right), & \nabla_k F^k_i &= 0, & F_{ik} &= \partial_i A_k - \partial_k A_i.
\end{align*}
\]

(2.2)

**Dynamical restrictions on the fluid motion.** The dynamical equations of motion of the fluid are determined by the conservation law \( \nabla_k T^{ik} = 0 \) and can be reduced to the equations

\[
\begin{align*}
    \nabla_k (\sqrt{\varepsilon} u^k) &= 0, & u^k \nabla_k (\sqrt{\varepsilon} u^i) &= \nabla^i \sqrt{\varepsilon}.
\end{align*}
\]

(2.3)
We also assume that the motion is curl-free, i.e., there exists a potential $\phi$ such that
\[
\sqrt{\varepsilon} u^i = \nabla^i \phi \quad \Rightarrow \quad \nabla_k \nabla^k \phi = 0, \quad \varepsilon = \nabla^k \phi \nabla_k \phi,
\]
and in this case the equations of motion (2.3) are satisfied identically.

3. METRICS ADMITTING A KILLING VECTOR FIELD

Let us now assume that the space-time metric admits one time-like or space-like Killing vector field $\xi^i = \delta^i_4$. Then it can be parameterized as follows:
\[
ds^2 = \epsilon_0 H(dx^4 + \Omega_\alpha dx^\alpha)^2 - \epsilon_0 H^{-1} \gamma_{\alpha\beta} dx^\alpha dx^\beta,
\]
where $\alpha, \beta, \ldots = 1, 2, 3$ and $\epsilon_0 = \pm 1$. The value $\epsilon_0 H$ is the norm of the Killing vector field $\xi^i$, so that $\epsilon_0 = 1$ for the time-like vector and $\epsilon_0 = -1$ for the space-like one; $\gamma_{\alpha\beta}$ is a (conformal) metric on the 3-space orthogonal to $\xi^i$. The metric functions $H \geq 0$, $\Omega_\alpha$ and $\gamma_{\alpha\beta}$ depend only on $x^1$, $x^2$ and $x^3$.

Kinematic restrictions on the stiff matter configurations. In what follows, we assume that there is no any motion of a stiff matter fluid along the Killing vector field, i.e.,
\[
\xi^k u_k = 0 \quad \Rightarrow \quad \epsilon_0 = -1, \quad \xi_i \xi^i = -H.
\]
The last two equations in (3.2) follow from the first one and from the condition that $u^k$ is a time-like vector. In the coordinates in which the metric takes the form (3.1), the components $u^i$ of the 4-velocity of the fluid are
\[
u_i = \{u_\alpha, 0\}, \quad u^i = \{H \gamma^\alpha_\beta u_\beta, -H \gamma^\delta_\gamma \Omega_\gamma u_\delta\},
\]
where the matrix $\|\gamma^{\alpha\beta}\|$ is the inverse of $\|\gamma_{\alpha\beta}\|$ and the three components $u_\alpha$ remain arbitrary. However, we have assumed above that the fluid motion possesses a potential. This implies that the potential $\phi$ is independent of $x^4$ and the components of the 4-velocity and the energy density have the expressions
\[
u_i = \varepsilon^{-1/2} \{\partial_i \phi, 0\}, \quad u^i = H \varepsilon^{-1/2}\{\gamma^\alpha_\gamma \partial_\alpha \phi, -\gamma^\gamma_\delta \Omega_\gamma \partial_\delta \phi\}, \quad \varepsilon = H \gamma^\delta \partial_\gamma \phi \partial_\delta \phi.
\]

Ernst-like form of the symmetry-reduced field equations. Using $h^i_j \equiv \delta^i_j - (\epsilon_0 H)^{-1} \xi^i \xi^j$ as a projector on the 3-space orthogonal to the Killing vector, it is convenient to split equations (2.2) into three (coupled to each other) parts corresponding to the projections of the Ricci tensor
\[
R_{ij} \xi^i \xi^j, \quad R_{ij} \xi^i h^k_l, \quad R_{ij} h^i_j h^k_l.
\]
In the absence of a fluid, i.e., for electrovacuum Einstein–Maxwell equations, it is known (see Neugebauer and Kramer [13], Israel and Wilson [14] and Kinnersley [15]) that the equations corresponding to the first two projections in (3.3), together with the Maxwell equations, can be presented in the form which is a three-dimensional analogue of the well-known two-dimensional Ernst equations for space-times that admit two commuting isometries. On the other hand, in the absence of electromagnetic fields, in the two-dimensional case of gravity with stiff matter fluid, Belinskii [7] observed that the stiff fluid does not contribute to the dynamical part of the vacuum Einstein equations if there is no any motion of this fluid along the two commuting Killing vector fields. It is easy to join these two observations and show that in the three-dimensional case, in the presence of both electromagnetic
fields and a stiff matter fluid, if there is no any motion of the stiff matter fluid along the Killing vector field, the field equations corresponding to the first two projections in (3.3), together with the Maxwell equations, do not get any input from the fluid and take exactly the same form as the generalized Ernst equations [13–15] for electrovacuum space-times with one Killing vector field, while the equations corresponding to the third projection in (3.3) lead to three-dimensional tensor equations for the metric $\gamma_{\alpha\beta}$. As a result, we obtain a complete system

$$
\begin{align*}
(\text{Re}E + \Phi \overline{\Phi})\gamma^{\alpha\beta}\nabla_\alpha\nabla_\beta E &= \gamma^{\alpha\beta}(\nabla_\alpha E + 2\overline{\Phi}\nabla_\alpha \Phi)\nabla_\beta E, \\
(\text{Re}E + \Phi \overline{\Phi})\gamma^{\alpha\beta}\nabla_\alpha\nabla_\beta \Phi &= \gamma^{\alpha\beta}(\nabla_\alpha E + 2\overline{\Phi}\nabla_\alpha \Phi)\nabla_\beta \Phi, \\
R_{\alpha\beta}[\gamma] &= \frac{(\nabla_\alpha E + 2\overline{\Phi}\nabla_\alpha \Phi)(\nabla_\beta E + 2\Phi\nabla_\beta \Phi)}{2(\text{Re}E + \Phi \overline{\Phi})^2} - \frac{2\nabla_\alpha \Phi \nabla_\beta \overline{\Phi}}{\text{Re}E + \Phi \overline{\Phi}} + 16\pi \nabla_\alpha \phi \nabla_\beta \phi, \\
\gamma^{\alpha\beta}\nabla_\alpha\nabla_\beta \phi &= 0,
\end{align*}
$$

where the bar denotes complex conjugation, $\nabla_\alpha$ is a covariant derivative with respect to the 3-metric $\gamma_{\alpha\beta}$ and $R_{\alpha\beta}[\gamma]$ is the Ricci tensor for this metric. In contrast to the two-dimensional case, equations (3.4) are coupled: the unknown metric $\gamma_{\alpha\beta}$ enters the first two equations (3.4), while the Ernst potentials and their first derivatives enter the “source” terms on the right-hand sides of the third (tensor) equation.

The solution $\{E, \Phi, \gamma_{\alpha\beta}, \phi\}$ of equations (3.4) allows one to determine all components of the metric, electromagnetic potential, 4-velocity of the fluid and its energy density:

$$
\begin{align*}
&\{e_0, H\} = \text{Re}E + \Phi \overline{\Phi}, \\
&\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha = -e_0 H^{-2}\varepsilon_{\alpha\beta} \gamma \text{Im}\{\partial_\gamma E + 2\overline{\Phi} \partial_\gamma \Phi\}, \\
&A_1 = \text{Re}\Phi, \\
&\partial_\alpha A_\beta - \partial_\beta A_\alpha = -\Omega_\alpha \partial_\beta \text{Re}\Phi + \Omega_\beta \partial_\alpha \text{Re}\Phi - H^{-1}\varepsilon_{\alpha\beta} \gamma \partial_\gamma \text{Im}\Phi, \\
&u_i = \varepsilon^{-1/2}\{\partial_\alpha \phi, 0\}, \quad \varepsilon = H \gamma^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi.
\end{align*}
$$

4. NON-GAUGE SYMMETRIES OF THE DYNAMICAL EQUATIONS (3.4)

Just as in the electrovacuum case, the dynamical equations (3.4) can be written in a complex vector form. For this purpose, following [15], instead of the two complex potentials $E(x^\alpha)$ and $\Phi(x^\alpha)$ we introduce three unknown complex functions $u(x^\alpha)$, $v(x^\alpha)$ and $w(x^\alpha)$ so that

$$
E = \frac{u - w}{u + w}, \quad \Phi = \frac{v}{u + w}.
$$

Keeping in mind that we have the freedom to impose one more constraint on these three new variables, we can write, in complete analogy with [15], equations (3.4) in terms of a complex vector function $Y^A = \{u, v, w\}$ ($A, B, \ldots = 1, 2, 3$) in the form

$$
\begin{align*}
(Y CY^C)\gamma^{\alpha\beta}\nabla_\alpha\nabla_\beta Y^A - 2\gamma^{\alpha\beta}(\nabla_\alpha Y^C)(\nabla_\beta Y^A) &= 0, \\
R_{\alpha\beta}[\gamma] &= \frac{2\nabla_\alpha Y B \partial_\alpha Y^A \partial_\beta Y B - 2(\nabla_\alpha Y^A) \partial_\alpha Y B \partial_\beta Y B}{(Y CY^C)^2} + 16\pi \phi, \phi, \\
\gamma^{\alpha\beta}\nabla_\alpha\nabla_\beta \phi &= 0,
\end{align*}
$$

where $Y_A = \eta_{AB} Y^B$ with $\eta_{AB} = \text{diag}\{1, 1, -1\}$.
A remarkable property of equations (4.1) is that these are invariant under a linear transformation of the unknowns provided that it leaves the “metric” \( ||\eta|| \) invariant:

\[
Y^A \rightarrow U^A_b Y^B, \quad \gamma_{AB} \rightarrow \gamma_{AB}, \quad \phi \rightarrow \phi \quad \Rightarrow \quad U^C_a \eta_{CD} U^D_b = \eta_{AB},
\]

where \( ||U^A_b|| \) is a constant complex matrix. The last of the above equations means that the matrices of these transformations constitute a group that can be identified with SU(2, 1). It is known that the elements of this group can be indexed by eight independent real parameters \[15\]. Some of these parameters are pure gauge ones, i.e., the corresponding transformations change neither the space-time geometry nor the properties of the matter fields, while the others can transform a solution to a physically different one. In particular, such transformations can transform a static solution to a stationary one, a solution for waves with linear polarization to a solution that includes waves with different polarizations (the Ehlers transformation), or a pure vacuum solution to a solution with electromagnetic fields (the Harrison transformation \[16\]). The latter transformation will be of greatest interest to us.

**Harrison transformation.** As in \[15\], the Harrison transformation can be presented in the form

\[
(u + w) \rightarrow (u + w) - 2c v - c \bar{v}(u - w),
\]

\[
v \rightarrow v + c(u - w),
\]

\[
(u - w) \rightarrow (u - w)
\]

This implies the following transformation of the Ernst potentials (everywhere below, circles over symbols denote the values of potentials of the solution to which the transformation is applied):

\[
\hat{E} = \Lambda^{-1} \hat{\epsilon}, \quad \Phi = \Lambda^{-1}(\Phi + c \hat{\epsilon}), \quad \Lambda = 1 - 2c \Phi - c \bar{\epsilon} \frac{\hat{\epsilon}}{2}.
\]

The corresponding transformations of \( H \) and the energy density take the form

\[
H = \Lambda^{-1} \Lambda^{-1} \hat{H}, \quad \varepsilon = \Lambda^{-1} \Lambda^{-1} \hat{\epsilon}.
\]

However, the transformations of the metric functions \( \Omega_\alpha \) and the components of the Maxwell tensor for the electromagnetic field do not have an algebraic character; they can be determined from the general expressions (3.5).

5. **FRIEDMANN UNIVERSES DEFORMED BY ELECTROMAGNETIC FIELDS**

Our purpose now is to apply the Harrison transformation described above to the Friedmann solutions (1.1), (1.2). To this end, we first determine the metric functions, Ernst potentials and matter fields for solutions (1.1). For these solutions we choose \( \{x^1, x^2, x^3, x^4\} = \{t, \chi, \theta, \phi\} \); therefore, the Ernst potentials and the metric on the 3-dimensional sections orthogonal to the Killing vectors are

\[
\hat{\epsilon} = -a^2(t) S^2(\chi) \sin^2 \theta, \quad \hat{\Phi} = 0, \quad \gamma_{\alpha\beta} = a^4(t) S^2(\chi) \sin^2 \theta \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -S^2(\chi) \end{pmatrix}.
\]

In addition, for other metric functions and a fluid potential we have

\[
\hat{H} = a^2(t) S^2(\chi) \sin^2 \theta, \quad \hat{\Omega}_\alpha = 0, \quad \phi = \sqrt{\frac{3}{32\pi} a_0^2(t)}.
\]
The application of the Harrison transformation (4.2) to the Friedmann solutions (1.1), (1.2) leads to solutions for Friedmann universes whose metrics are deformed by electromagnetic fields:

$$ds^2 = a^2(t) \Lambda^2 \left[ dt^2 - d\chi^2 - S^2(\chi) d\theta^2 \right] - \frac{a^2(t)}{\Lambda^2} S^2(\chi) \sin^2 \theta d\varphi^2. \quad (5.1)$$

The electromagnetic fields in these solutions are described by the real vector potential $A$ or, equivalently, by a complex scalar potential $\Phi$ of the form

$$A = -2H_0 a^2(t) S^2(\chi) \cos \theta \left\{ \frac{S'(\chi)}{S(\chi)}, \frac{\dot{a}(t)}{a(t)}, 0, 0 \right\}, \quad \Phi = \frac{iH_0 a^2(t) S^2(\chi) \sin^2 \theta}{\Lambda}, \quad (5.2)$$

and the pressure and the energy density of the fluid have the expressions

$$p = \varepsilon = \frac{3a_0^4}{8\pi a^4(t) \Lambda^2}, \quad \text{where} \quad \Lambda = 1 + H_0^2 a^2(t) S^2(\chi) \sin^2 \theta. \quad (5.3)$$

Here the parameter $c$ of the Harrison transformation was chosen to be pure imaginary, $c = -iH_0$. Another choice (with $c \xi = H_0^2$) would not change the metric (5.1) but would lead to a dual transformation of the electromagnetic fields, which would lead to the appearance of a primordial electric field in the solutions.

6. PROPERTIES OF FRIEDMANN UNIVERSES WITH ELECTROMAGNETIC FIELDS

In solutions (5.1), the metric and electromagnetic fields possess the axial symmetry and depend on time and two spatial coordinates (for other symmetries of these solutions, see Section 7).

The initial singularity. The metric (5.1) has an initial singularity at $t = 0$ which is similar to that in the Friedmann solutions (1.1). Near this singularity, i.e., for $a(t) \to 0$, we have $\Lambda \to 1$ and the metric (5.1) becomes asymptotically homogeneous and isotropic and coincides with the Friedmann metric (1.1) in the limit.

Electromagnetic fields measured by a local observer. In our solution, the components of the Maxwell tensor have the form

$$F^{\alpha \beta} = -\frac{2H_0}{a^2(t) \Lambda^4} \begin{pmatrix} 0 & \cos \theta & -\frac{S'(\chi)}{S(\chi)} \sin \theta & 0 \\ -\cos \theta & 0 & \frac{\dot{a}(t)}{a(t)} \sin \theta & 0 \\ \frac{S'(\chi)}{S(\chi)} \sin \theta & -\frac{\dot{a}(t)}{a(t)} \sin \theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. $$

Using the metric (5.1), we introduce an orthonormal basis of one-forms

$$e_t = \Lambda a(t) dt, \quad e_\chi = \Lambda a(t) d\chi, \quad e_\theta = \Lambda a(t) S(\chi) d\theta, \quad e_\varphi = \frac{a(t)}{\Lambda} S(\chi) \sin \theta d\varphi. $$

The projections of the Maxwell tensor on this basis determine the stresses of magnetic and electric fields measured by a local fluid-comoving observer:

$$H^{\hat{t}} = 0, \quad E^{\hat{t}} = \frac{2H_0 \cos \theta}{\Lambda^2},$$

$$H^{\hat{\chi}} = 0, \quad E^{\hat{\chi}} = -\frac{2H_0 S'(\chi) \sin \theta}{\Lambda^2}, \quad (6.1)$$

$$H^{\hat{\varphi}} = \frac{2H_0 \dot{a}(t) S(\chi) \sin \theta}{a(t) \Lambda^2}, \quad E^{\hat{\varphi}} = 0.$$
The axially symmetric structure of the magnetic field in the solutions (5.1)–(6.1) is shown here on the sections orthogonal to the axis \((\theta = 0, \pi)\) on the space-like hypersurfaces \(t = \text{const}\). The arrows show the 3-vectors of the magnetic field, which have only one nonvanishing component \(H^\varphi = \frac{2a}{\sigma^2} \frac{R}{(1+R^2)^2}\) with \(R = H_{00}(t)S(\chi)\sin \theta\). The lines of force of this magnetic field are the lines of the azimuthal coordinate \(\varphi\). The length of arrows characterizes the magnitude of the magnetic field.

**The structure of primordial magnetic fields.** From these expressions, one can see that in all models the magnetic field is a vortex field, it is completely regular everywhere, and its lines coincide with the closed coordinate lines of the azimuthal coordinate \(\varphi\). The magnitude of this magnetic field vanishes on the axis of symmetry \(\theta = 0\) and \(\theta = \pi\); then it grows with the distance from this axis and reaches a maximum. At larger distances it decreases and vanishes at another pole of the 3-sphere in the closed model or vanishes asymptotically at spatial infinity in the flat and open models. The spatial structure of the magnetic field on a surface \(t = \text{const}\) is shown in the figure.

**The structure of the induced electric field.** In accordance with the Faraday law, the time dependence of the magnetic field (caused by the cosmological expansion) induces the emergence of a source-free and everywhere regular electric field, which provides the electric field flux through the magnetic field \((\varphi\text{-line})\) contours. The nonvanishing components of the electric field are \(E^\chi\) and \(E^\theta\); therefore, the spatial 3-vectors of the electric field are tangent to the two-dimensional surfaces \(\varphi = \text{const}\). To understand the structure of this field, we should find the lines of this vector field on these surfaces. The equation for electric lines of force and its solution take the form

\[
\frac{d\theta}{d\chi} = \frac{F^{\varphi t}}{F^{\chi t}} = \frac{S'(\chi)}{S(\chi)} \tan \theta \Rightarrow S(\chi) \sin \theta = \text{const}.
\]

This solution shows that in the case of the Friedmann model with a flat spatial geometry the electric field lines are the straight lines orthogonal to the \(\varphi\text{-line}\) circles and parallel to the axis \(\theta = 0\) and \(\theta = \pi\). In the case of the open model, the topology of the electric field lines is the same, while the case of the closed model is more interesting. In this case, like the lines of the magnetic field, the lines of the electric field are also closed and go along the parallels on the quasi-spherical surfaces with coordinates \((\chi, \theta)\); however, these parallels surround the axis orthogonal to the axis \(\theta = 0\) and \(\theta = \pi\) and to the magnetic field lines. These configurations of electromagnetic fields are mainly concentrated in the cylindrical region near the axis of symmetry, which resembles very much a coil whose turns are circle lines of the magnetic field, while the electric lines pass inside this coil and are parallel to its axis.

**The electromagnetic contribution to the energy density.** The expressions given above also show that near the singularity the main contribution to the electromagnetic energy density is made by the only nonvanishing component of the magnetic field, which goes to infinity as \(1/t\), while the electric field input is much less because its components remain finite when \(t \to 0\). This allows...
us to say that the role of the primordial field in our solution is played by the magnetic field, and the electric field arises due to Faraday induction. On the other hand, even the infinitely growing magnetic field contributes to the energy density much less than the energy density of the fluid: the latter grows as $t^{-3}$ for $t \to 0$, while the value of the magnetic field input in the energy density is of order $t^{-2}$. This allows the universe to be initially (for small $t$) homogeneous and isotropic due to the homogeneous and isotropic initial distribution of the stiff fluid.

**Behavior of the space-time geometry in spatial directions.** On the axis of symmetry ($\theta = 0, \pi$) the metrics (5.1) at any time coincide with the corresponding metrics of Friedmann solutions without electromagnetic fields. An important deviation of the geometry (5.1) from the Friedmann one occurs in the flat and open models at spatial infinity in those directions in which $S(\chi) \sin \theta \to \infty$. In the regions far from the axis $\theta = 0, \pi$, the deviations in the flat and open models from Friedmann universes are crucial: the spatial geometry becomes asymptotically closed in these directions. This closure of the spatial metric manifests itself in nonmonotonous behavior of the lengths of circles that are the coordinate lines of $\phi$ when their “radius” $\chi$ grows (the length of these lines is determined by the value of $\sqrt{-g_{\phi\phi}}$ in (5.1)). Indeed, for sufficiently small values of $\chi$ (or, equivalently, of $S(\chi)$) the lengths of these lines increase when $\chi$ increases. However, with the further increase in $\chi$ (with nonvanishing $\sin \theta$) the value of $\Lambda$ becomes proportional to $S(\chi)^2 \sin^2 \theta$; therefore, $g_{\phi\phi}$ is then proportional to $S(\chi)^{-2} \sin^{-2} \theta$ and vanishes asymptotically at spatial infinity in these directions.

**Spatial closure of the Melvin and Bertotti–Robinson static magnetic universes.** Partial spatial closure of the geometry (5.1) described above resembles very much the spatial closure of the static Melvin magnetic universe in spatial directions orthogonal to its axis of symmetry. In the Melvin universe, even though the value of the magnetic field (parallel to the axis of symmetry and constant along this axis) decreases with distance from this axis, the anisotropic electromagnetic stresses in this “bundle of magnetic lines” are so strong that these cause the closure of the spatial geometry in the directions orthogonal to the symmetry axis. Another example of such closure is the static Bertotti–Robinson electromagnetic universe. The physical space of this universe is filled with a completely homogeneous (spatially constant) magnetic/electric field. In this case, the closure of spatial geometry occurs at a finite distance from the axis of symmetry rather than at spatial infinity. As a result, this model possess the AdS$^2 \times S^2$ space-time geometry (where AdS$^2$ stands for the two-dimensional anti-de Sitter space-time), so that the internal geometry on the sections ($t = \text{const}$, $z = \text{const}$) is homogeneous, isotropic and isometric to the geometry on the 2-spheres of constant radius which is inversely proportional to the stress of a magnetic/electric field.

**Acceleration of cosmological expansion and dynamical closure of the flat and open models.** As mentioned above, on the symmetry axis ($\theta = 0, \pi$) the metrics (5.1) at any time $t$ coincide with the corresponding metrics of Friedmann solutions. Due to the presence of the factor $\Lambda^2$ in all metric components except $g_{\phi\phi}$, the cosmological expansion outside the axis $\theta \neq 0, \pi$ proceeds faster than in the corresponding Friedmann models in all spatial directions across the magnetic field.

An interesting phenomenon can be observed in these space-times if we consider the character of the cosmological expansion along the magnetic line circles. In contrast with the directions across the magnetic field, the expansion along the magnetic field lines proceeds more slowly than in the Friedmann models due to the presence of the factor $\Lambda^{-2}$ in $g_{\phi\phi}$. At the early stage of the cosmological expansion, the lengths of magnetic line circles, considered as functions of time for finite comoving “radius” $\chi$, grow with time. However, later, even in the flat and open models, the decelerated expansion along the magnetic field lines ($\phi$-lines with $\chi = \text{const}$) changes to a contraction: the lengths of $\phi$-circles with given comoving “radius” $\chi$ pass through a maximum and then begin to decrease with time and vanish asymptotically as $t \to \infty$, while the cosmological expansion in other spatial directions in the flat and open models continues until $t = \infty$ and the actual “radius” of these
circles continues to grow with acceleration. This means that under the influence of electromagnetic fields (6.1) on the dynamics of these universes, their space-times suffer partial dynamical closure.

Speculatively, it seems interesting to note that in our models the acceleration of the cosmological expansion occurs in almost all radial directions from the observer located somewhere in the region where the electromagnetic fields are concentrated, and it can be measured by some “ordinary” methods, while the time behavior of the lengths of circles of very large radius can be much more difficult to measure and this may escape the observer’s attention.

7. CONCLUDING REMARKS

1. The metrics (1.1) depend on three coordinates $t, \chi$ and $\theta$ and obviously admit the Killing vector field $\xi_\varphi = \partial/\partial \varphi$. Precisely with this Killing vector field we have associated the Ernst potentials and their Harrison transformation used in our construction. However, the Friedmann metrics admit other isometries and, in particular, another Killing field, say $\xi_\psi$, commuting with $\xi_\varphi$. For the convenience of the reader, we recall here a known coordinate transformation (see, for example, [7]) leading to forms of Friedmann universes that depend on two coordinates only. Namely, the coordinate transformation $\{\chi, \theta\} \rightarrow \{x, \psi\}$ determined by the expressions

$$
\begin{align*}
S(\chi) \sin \theta &= X(x), \\
S(\chi) \cos \theta &= X'(x)Y(\psi),
\end{align*}
$$

where

$$
\begin{align*}
\{\sinh x, \sinh \psi\}, & \quad k = -1, \\
\{x, \psi\}, & \quad k = 0, \\
\{\sin x, \sin \psi\}, & \quad k = 1,
\end{align*}
$$

leads from the metrics (1.1) to the metrics that depend only on $t$ and $x$,

$$
\begin{align*}
ds^2 &= a^2(t)[dt^2 - dx^2 - X^2(x)\,d\varphi^2 - X'^2(x)\,d\psi^2],
\end{align*}
$$

and obviously admit two commuting isometries $\xi_\varphi = \partial/\partial \varphi$ and $\xi_\psi = \partial/\partial \psi$. Using the formalism described above, we can associate the Ernst potentials and their Harrison transformation not with $\xi_\varphi$ only (as was done in the paper) but with an arbitrary linear combination of these two Killing vector fields. This will lead to solutions that describe the Friedmann universes deformed by electromagnetic fields of more complicated and, probably, richer and more interesting structures.

2. As we have seen in the paper, even a simple scalar field (which was alternatively interpreted here as the potential for the stiff fluid motion) can play an important role in the evolution of the universe. In the modern literature, various gravity models (e.g., string gravity and supergravity models) are considered in four and higher dimensions. The symmetry reductions of these models include a large number of “moduli fields”: the scalar fields of different structures and couplings as well as various gauge fields. These (symmetry-reduced) models admit a large number of “hidden” symmetries. Many of these symmetries (including the Harrison-type transformations) were used by different authors for constructing various charged black hole solutions in these models (“charging transformations”). Our present construction shows that similar transformations can give rise to simple exact solutions which describe the evolution of various cosmological models in these theories under the influence of various (spatially nonsingular) scalar and gauge fields.

3. To conclude our discussion of the solutions constructed in this paper, we should mention that it is difficult to offer an immediate astrophysical interpretation of these solutions as relevant to our Universe. This is mainly caused by two important circumstances: the “exotic” character of the used equation of state $p = \varepsilon$ and the existence of tremendous observational restrictions on the possible anisotropy of our Universe. Concerning the former, as it was already mentioned in the Introduction, this equation of state was used for a pure technical reason, namely, to obtain field equations that possess “hidden” symmetries, including the Harrison transformations, but in the presence of an ideal fluid that “supports” a Friedmann cosmological background. However, one
may expect that the presence of electromagnetic fields in the universes with more realistic matter
equations of state may give rise to similar physical phenomena caused by a nonlinear interaction of
gravitational and electromagnetic fields, which can be studied in our solutions in great detail.

As regards the observationally restricted possibility for our Universe to be anisotropic due to the
existence of primordial magnetic fields, it seems to be the most interesting property of the solutions
constructed here that the influence of initially presented magnetic fields (even sufficiently strong
ones) can be almost negligible at the very beginning of the evolution of such a universe, which allows
the initial geometry of the universe to be homogeneous and isotropic, but this influence can become
much more significant at later stages of its evolution.

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