Universal factorized formula for the cross-section of two-particle scattering

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Abstract

We analyze the process of two-particle scattering with unstable particle in an intermediate state. It was shown that the cross-section can be represented in the universal factorized form for an arbitrary set of particles. Phenomenological analysis of factorization effect is fulfilled.

PACS number(s): 11.10St, 130000

Keywords: unstable particles, factorization, cross-section.

1. Introduction

The peculiar properties of the unstable particles (UP) and resonances were being discussed during the last decades. Among them, the assumption that the decay of UP or resonance (R) proceeds independently of its production remains of interest [1] [2] [3]. Formally, this effect is expressed as the factorization of a cross-section or decay rate [3]. The processes of type \( ab \rightarrow Rx \rightarrow cdx \) were considered in Ref. [3]. It was shown, that the factorization always is valid for a scalar \( R \) and does not take place for a vector and spinor \( R \). The factorization usually is related with the narrow-width approximation (NWA) [4], which makes five critical assumptions [5].

There is another way to get factorization effect, which is connected with propagator structure [6]. The decay processes of type \( a \rightarrow Rx \rightarrow cdx \), where \( R \) is UP with a large width, were analyzed systematically in Ref. [6]. It was shown in this work, that the factorization always is valid for a scalar \( R \), while for a vector and spinor \( R \) it occurs when the propagators’
numerators are $\eta_{\mu\nu}(q) = g_{\mu\nu} - q_\mu q_\nu/q^2$ and $\hat{\eta}(q) = \hat{q} + q$, respectively, where $\hat{q} = q_i \gamma^i$ and $q = \sqrt{q_i q^i}$. Such a structure of propagators always provides the exact factorization for any tree process and is an analog of NWA, which is discussed in Section 3. These propagators were constructed in the model of UP with a smeared mass [7] and describe some effective (dressed by self-energy insertion) unstable fields. Note that the structure of the expressions $\eta_{\mu\nu}(q)$ and $\hat{\eta}(q)$ is not related with the choice of the gauge (see the third section).

In this work, we systematically analyze the processes of type $ab \rightarrow R \rightarrow cd$, where $R$ is scalar, vector or spinor UP with a large width (or resonance) and $a, b, c, d$ are the stable or long-lived particles of any kind. It was shown that the cross-section $\sigma(ab \rightarrow R \rightarrow cd)$ can be represented in the universal factorized form when the same expressions $\eta_{\mu\nu}(q) = g_{\mu\nu} - q_\mu q_\nu/q^2$ and $\hat{\eta}(q) = \hat{q} + q$ are used to describe the propagator’s numerator of vector and spinor UP, respectively. This result have been received strictly by direct calculations for all types of particles $a, b, c, d$ and $R$ (Section 2). The factorization approach is applied in Section 3 for the complicate processes of scattering with the consequent decays of the final states. In Section 4, we analyze some methodological and phenomenological aspects of factorization.

2. Universal factorized formula for the cross-section of two-particle scattering

In this section, we consider inelastic scattering of type $ab \rightarrow R \rightarrow cd$, where $R$ is the UP with a large width in $s$-channel and $a, b, c, d$ are stable (quasi-stable) particles of any kind. The vertexes are defined by the Lagrangian in the simplest standard form:

\[
L_k = g \phi \phi_1 \phi_2; \quad g \phi \bar{\psi}_1 \psi_2; \quad gV_{1\mu}V_{2\nu}; \quad gV_{\mu}(\phi_1^{\mu} \phi_2 - \phi_2^{\mu} \phi_1); \quad gV_{\mu} \bar{\psi}_1 \gamma^\mu (c_V + c_A \gamma_5) \psi_2;
\]

\[
gV_{1\mu}V_{2\nu}V_\alpha [g^{\mu\nu}(p_2 - p_1) + \alpha + g^{\mu\alpha}(2p_1 + p_2)^\nu - g^{\nu\alpha}(p_1 + 2p_2)^\mu].
\] (1)

In the expressions (1) $\phi, V$ and $\psi$ are the scalar, vector and spinor fields, respectively, $p_1$ and $p_2$ are the momenta of the particles $a$ and $b$ (or $c$ and $d$).

Here we show, that the cross-section $\sigma(ab \rightarrow R \rightarrow cd)$ can be expressed in a factorized universal form in terms of decay widths $\Gamma(R \rightarrow ab)$ and $\Gamma(R \rightarrow cd)$, when the expressions for propagators’ numerators $\eta_{\mu\nu}(q) = g_{\mu\nu} - q_\mu q_\nu/q^2$ and $\hat{\eta}(q) = \hat{q} + q$ are used. This expressions are constructed within the model of unstable particles with a smeared mass, which briefly considered in Appendix. The validity of these expressions have been discussed in Refs. [6, 7] and will be considered in the third section. It is convenient to employ the universal
expressions for widths $\Gamma(R \to ab)$ and $\Gamma(R \to cd)$ in a stable particle approximation [6]:

$$\Gamma_i(R \to ab) = \frac{g^2}{8\pi} \bar{\lambda}(m_a, m_b; m_R) f_i(m_a, m_b; m_R),$$  \hspace{1cm} (2)

where $m_R^2 = q^2, q^2 = (p_1 + p_2)^2$ and:

$$\bar{\lambda}(m_a, m_b; m_R) = [1 - 2\frac{m_a^2 + m_b^2}{m_R^2} + \frac{(m_a^2 - m_b^2)^2}{m_R^4}]^{1/2}. \hspace{1cm} (3)$$

The same expressions and relations are in order for the width $\Gamma(R \to cd)$. The functions $f_i(m_a, m_b; m_R)$ are defined by the corresponding vertexes. If these vertexes are described by Eqs. (1), then the functions $f_i$ (further we omit the arguments) in tree approximation are defined by the following expressions [6]:

$$\phi \to \phi_1\phi_2, \ f_1 = \frac{1}{2m_\phi}; \ \phi \to V_1V_2, \ f_2 = \frac{1}{m_\phi}[1 + \frac{(m_\phi^2 - m_1^2 - m_2^2)^2}{8m_1^2m_2^2}];$$

$$\phi \to \bar{\psi}_1\psi_2, \ f_3 = m_\phi[1 - \frac{(m_1 + m_2)^2}{m_\phi^2}], \ \phi \to \phi_1V, \ f_4 = \frac{m_\phi^3}{2m_V^2}\bar{\lambda}^2(m_1, m_V; m_\phi);$$

$$V \to \phi_1\phi_2, \ f_5 = \frac{m_V}{6}\bar{\lambda}^2(m_1, m_2; m_V); \ \rightarrow V_1\phi, \ f_6 = \frac{1}{3m_V}[1 + \frac{(m_V^2 + m_1^2 - m_\phi^2)^2}{8m_1^2m_1^2}];$$

$$V \to \bar{\psi}_1\psi_2, \ f_7 = \frac{2m_V}{3}c_+[1 - \frac{m_1^2 + m_2^2}{2m_V^2} - \frac{(m_1^2 + m_2^2)^2}{2m_V^4}] + 3c_- \frac{m_1m_2}{m_V^2};$$

$$V \to V_1V_2, \ f_8 = \frac{m_V^5}{24m_1^2m_2^2}[1 + 8(\mu_1 + \mu_2) - 2(9\mu_1^2 + 16\mu_1\mu_2 + 9\mu_2^2) + 8(\mu_1^4 - 4\mu_1^2\mu_2 + \mu_2^2 + \mu_1^3 + 8\mu_1\mu_2^2 - 18\mu_1^2\mu_2 + 8\mu_1\mu_2 + \mu_2^3)], \ \mu_{1,2} = m_{1,2}/m_V;$$

$$\psi \to \phi_1\psi_1, \ f_9 = \frac{m_\psi}{2}(1 + 2\frac{m_1}{m_\psi} + \frac{m_1^2 - m_\phi^2}{m_\psi^2});$$

$$\psi \to V\psi_1, \ f_{10} = m_\psi^2c_+[\frac{(m_\phi^2 - m_1^2)^2}{2m_\phi^2m_1^2} + \frac{m_\phi^2 + m_1^2 - 2m_\psi^2}{2m_\psi^2}] - 3c_- \frac{m_1}{m_\psi};$$

$$c_+ = c_1^2 + c_\lambda^2, \ c_- = c_1^2 - c_\lambda^2. \hspace{1cm} (4)$$

Note that the function $f_8$, given in Ref. [6], contains an error and we give here corrected expression for this function. It is convenient in the further calculations to employ the relations, which take place in the center-of-mass system:

$$p_1^0 = \frac{1}{2}q[1 + \frac{m_a^2 - m_b^2}{q^2}], \ p_2^0 = \frac{1}{2}q[1 + \frac{m_b^2 - m_a^2}{q^2}],$$

$$(p_1q) = \frac{1}{2}(q^2 + m_a^2 - m_b^2), \ (p_2q) = \frac{1}{2}(q^2 + m_b^2 - m_a^2),$$

$$(p_1p_2) = \frac{1}{2}(q^2 - m_a^2 - m_b^2), \ |\tilde{p}_1| = |\tilde{p}_2| = \frac{1}{2}g\bar{\lambda}(m_a, m_b; q). \hspace{1cm} (5)$$
The analogous relations occur for the momenta $k_1$ and $k_2$ of the particles $c$ and $d$. In Eqs. (5) the symbol $q$ has different meanings in the expressions $(p_1q)$, $q = p_1 + p_2$ (q is 4-momentum) and in the expression $q[1 + f(q)]$, where $q = \sqrt{(q \cdot q)}$ is a number.

With the help of the relations (2)-5 and above discussed expressions for propagators, we have got by tedious but straightforward calculations the universal factorized cross-section for all permissible combinations of particles $(a, b, R, c, d)$:

$$
\sigma(ab \rightarrow R \rightarrow cd) = \frac{16\pi(2J_R + 1)}{(2J_a + 1)(2J_b + 1)\lambda^2(m_a, m_b; \sqrt{s})} \frac{\Gamma^{ab}_R(s)\Gamma^{cd}_R(s)}{|P_R(s)|^2}.
$$

In Eq. (6) $J_k$ is spin of the particle $(k = a, b, R)$, $s = (p_1 + p_2)^2$, $\Gamma^{ab}_R(s) = \Gamma(R(s) \rightarrow ab)$ and $P_R(s)$ is propagator’s denominator of the UP or resonance $R$. The expressions for $\Gamma^{ab}_R(s)$ and $\Gamma^{cd}_R(s)$ follow from Eqs. (2-4), when squared mass of UP is $m^2_R = q^2 = s$. The factorization of cross-section does not depend on the definition of $P_R(s)$, which can be determined in a phenomenological way, in Breit-Wigner or pole form [8, 9], etc. The expression (6) is a natural generalization of the spin-averaged Breit-Wigner (non-relativistic) cross-section, defined by the expression (37.51) in Ref. [10]. Note that the factorization is exact in our approach, while in the traditional one it occurs as an approximation.

The cross-section of exclusive process $ab \rightarrow R \rightarrow cd$, defined by Eq. (6), does not depend on $J_c$ and $J_d$. So, it can be summarized over final channels $R \rightarrow cd$:

$$
\sigma(ab \rightarrow R(s) \rightarrow all) = \frac{16\pi k_R}{k_a k_b \lambda^2(m_a, m_b; \sqrt{s})} \frac{\Gamma^{ab}_R(s)\Gamma^{tot}_R(s)}{|P_R(s)|^2}.
$$

In Eq. (7) $k_i = 2J_i + 1$ and $\Gamma^{tot}_R(s) = \sum_{cd} \Gamma^{cd}_R(s)$, where for simplicity we restrict ourselves by two-particle channels.

The factorization effect, expressed by Eq. (6), has two aspects. On the one hand it means that the decay of UP proceeds independently of its production in the approach considered. On the other hand it leads to significant simplification of calculations, in particular, in the case of the complicate processes (the scattering with chain decay of products).

### 3. Cross-section of the process $ab \rightarrow R \rightarrow R_1x \rightarrow cdx$

Here, we consider factorization effects in the case of complicate chain processes. For example, let us discuss the process of scattering $ab \rightarrow R \rightarrow R_1x$ with consequent decay $R_1 \rightarrow cd$. In this case, Eq. (6) has the form:

$$
\sigma(ab \rightarrow R(s) \rightarrow R_1x) = \frac{16\pi k_R}{k_a k_b \lambda^2(m_a, m_b; \sqrt{s})} \frac{\Gamma^{ab}_R(s)\Gamma^{R_1x}_R(s)}{|P_R(s)|^2}.
$$

In Eq. (8) $k_i = 2J_i + 1$.  

To calculate the value $\Gamma^{R_1x}(s)$ we apply convolution formula, which accounts FWE in the decay $R(s) \to R_1x$ [6]:

$$\Gamma(R(s) \to R_1x) = \int_{q_1^2}^{q_2^2} \Gamma(R(s) \to R_1(q)x) \rho_{R_1}(q) \, dq^2. \quad (9)$$

In Eq. (9) $q = p_R - p_x$, $q_{1,2}$ are defined by kinematics of the process and $\rho_{R_1}(q) = q \Gamma_{R_1}^{tot}(q)/\pi |P_R(q)|^2$ is interpreted in the model [6] as distribution function of the smeared mass of unstable particle $R_1$. Convolution structure of Eq. (9) is caused by factorization of decay rate $\Gamma(R \to R_1x \to x, all)$. This effect takes place exactly when the model propagators $\hat{\eta}(q)$ and $\eta_{\mu \nu}(q)$ are used (as in the present work).

From Eqs. (8) and (9) it follows:

$$\sigma(ab \to R \to R_1x) = \frac{16\pi k_R}{k_a k_b \lambda^2(m_a, m_b; \sqrt{s}) |P_R(s)|^2} \int_{q_1^2}^{q_2^2} \Gamma(R(s) \to R_1(q)x) \rho_{R_1}(q) \, dq^2. \quad (10)$$

Using the expression for $\rho_{R_1}(q)$, from Eq. (10) we can get the cross-section of exclusive process, for example $ab \to R \to R_1x \to cdx$. To this effect we represent $\Gamma_{R_1}^{tot}(q)$ in the form:

$$\Gamma_{R_1}^{tot}(q) = \sum_X \Gamma_{R_1}^{X}(q); \quad \Gamma_{R_1}^{cd}(q) = \Gamma(R_1(q) \to cd). \quad (11)$$

As a result, from (10) and (11) we get:

$$\sigma(ab \to R \to cdx) = \frac{16k_R}{k_a k_b \lambda^2(m_a, m_b; \sqrt{s}) |P_R(s)|^2} \int_{q_1^2}^{q_2^2} \Gamma(R(s) \to R_1(q)x) \frac{q \Gamma_{R_1}^{cd}(q)}{|P_{R_1}(q)|^2} \, dq^2. \quad (12)$$

Similar structure arises in the case $R \to R_1R_2$, i.e. when there are two UP in the final state, which have two-particle decay channels (semi-analytical approach). Thus, the model gives a convenient instrument to describe two-particle scattering accompanied by complicated decay-chain processes. However, we have checked by direct calculations only two types of processes - the decay of type $a \to Rx \to bcx$ [6] and the scattering of type $ab \to R \to cd$. The more complicated processes, such as decay $a \to R_1R_2 \to cdef$ and scattering $ab \to R \to R_1R_2 \to cdef$, will be the subject of the next paper.

4. Methodological and phenomenological analysis of the factorization effect

The model factorization of a decay width and cross-section of the processes with UP in an intermediate state was established by straightforward calculations at tree level. However,
these calculations in the effective theory of UP \[7\] account for some loop diagrams. The vertex and self-energy type corrections can be included into $\Gamma_R(s)$ and $P_R(s)$ respectively. These corrections do not breakdown a factorization, but the interaction between initial and final states does. However, such an interaction has no clear and explicit status in perturbation theory due to UP (or resonance) is not a perturbative object in the resonance neighborhood \[7, 11, 12\]. As it was noted in Ref. \[13\], such non-factorizable corrections give small contribution to the processes $e^+e^- \rightarrow ZZ, WW, 4f$ near the resonance range.

Now we consider another aspect of factorization effect, namely, the determination of dressed propagator of UP. Factorization of decay width and cross-section does not depend on the structure of propagator’s denominator $P_R(q)$, but crucially depends on the structure of its numerator in the case of vector and spinor UP. As it was verified by direct calculations, the factorization always takes place in the case of scalar UP. The expressions $\eta_{\mu\nu}(m_R) = g_{\mu\nu} - q_\mu q_\nu/m_R^2$ and $\tilde{\eta}(m_R) = \hat{q} + m_R$ for vector and spinor UP, respectively, do not lead to exact factorization. But the expressions $\eta_{\mu\nu}(q) = g_{\mu\nu} - q_\mu q_\nu/q^2$ and $\tilde{\eta}(q) = \hat{q} + q$ strictly lead to factorization for any kinds of other particles. It should be noted that the definition of the functions $\eta_{\mu\nu}(q)$ and $\tilde{\eta}(q)$ is not related with the choice of the gauge, because effective theory of UP \[7\] is not the gauge theory. The choice of $q$ instead of $m_R$ in the $\eta_{\mu\nu}$ and $\tilde{\eta}$ may seems contradict to the equation of motion for vector and spinor UP. However, this statement is valid for the stable particle with fixed mass. In the case of UP the question arises what the mass participates in equation of motion - pole mass or one of the renormalized mass \[14\]?

An account of uncertainty relation by smearing of mass intensifies the question. There is no unique and strict determination of dressed propagator structure for vector and spinor UP due to the specific nature of Dyson summation in these cases \[6\]. The situation is more complicated and involved in the case of hadron resonance. So, the functions $\eta_{\mu\nu}$ and $\tilde{\eta}$ have rather phenomenological (or model) than theoretical status. The model of UP \[7\] defines these functions as $\eta_{\mu\nu}(q)$ and $\tilde{\eta}(q)$, which describe the dressed propagators of UP in the resonance neighborhood.

Further, we briefly analyze the phenomenological aspect of factorization. In the low-energy experiments of type $e^+e^- \rightarrow \rho, \omega... \rightarrow \pi^+\pi^-, ...$ we can not distinguish propagators $\eta_{\mu\nu}(m_R)$ and $\eta_{\mu\nu}(q)$ even for the wide resonance. This is due to the equality $\bar{e}^-(p_1)(\hat{p}_1 + \hat{p}_2)e^-(p_2) = 0$, when the functions $\eta_{\mu\nu}$ reduce to $g_{\mu\nu}$ in both cases. In the high-energy experiments of type $e^+e^- \rightarrow Z \rightarrow \bar{f}f$, where $f$ is quark or lepton (we neglect $\gamma - Z$ interference), the transverse part of amplitude is

$$M_q \sim \bar{e}^-(p_1)\hat{q}(c_e - \gamma_5)e^-(p_2)\tilde{f}^+(k_1)\hat{q}(c_f - \gamma_5)f^+(k_2),$$ \hspace{1cm} (13)
where \( q = p_1 + p_2 = k_1 + k_2 \). From Eq. (13) with the help of the Dirac equations in momentum representation it follows

\[
M_q \sim m_\text{e} m_f \bar{e}^-(p_1) \gamma_5 e^-(p_2) \bar{f}^+(k_1) \gamma_5 f^+(k_2).
\]

As a result, we get the terms \( m_\text{e} m_f / q^2 \) and \( m_\text{e} m_f / m_Z^2 \) for \( \eta(q) \) and \( \eta(m_R) \), respectively. The difference of these values is of the order of \( (m_\text{e} m_f / m_Z^2) \cdot (m_Z - q) / m_Z \) at energy \( q^2 \sim m_Z^2 \).

Thus, the distinction between the structure of two type of the expressions \( \eta_{\mu \nu} \) is negligible in a wide range of energy.

The structure of \( \hat{\eta} \) can be studied in the process of type \( VF \rightarrow R \rightarrow V'F' \), where \( V \) and \( F \) are vector and fermion field, \( R \) is, for instance, baryon resonance with a large width. In this case, the difference between \( \hat{\eta}(m_R) \) and \( \hat{\eta}(q) \) is characterized by the value \( \sim \Gamma_R / m_R \) at peak region, and this problem demands more detailed analysis.

From this analysis it follows that method of factorization is a simple analytical analog of narrow-width approximation (NWA, which contains five critical assumptions \([5]\)) instead, we use the structure of propagators’ numerators \( \eta(q) \), which follows from usual ones under a simple transformation \( m_R \rightarrow q \), and one assumption: there is no significant interference with non-resonant processes (fifth assumption of NWA). The rest assumptions of NWA can be derived from the first our point, where some of them are not obligatory in the special cases. The method leads to factorization in two type of processes - in the decay-chain \([6]\) (universal convolution formula) and scattering ones (universal formula \([6]\)). Combining these two results, we get a simple and strict algorithm of analytical description of the complicated processes.

5. Conclusion

The factorization effect gives us a convenient phenomenological way to describe the three-particle decays and two-particle scattering processes. This effect significantly simplifies calculations and gives compact universal formulae for the decay rate and cross-section.

In this work, we have shown that the factorization always is valid when scalar UP is in the intermediate state. In the case of vector or spinor intermediate states, the factorization takes place when the specific propagators are used for these states. These propagators are derived in the model of UP with a random (smeared) mass. They negligibly differ from the traditional propagators at peak area and follow from the smearing of mass in accordance with the uncertainty relation. Our method makes it possible significantly simplify the calculation of the complicated decay-chain and scattering processes. It is some analytical analog of NWA.
and gives a simple and strict algorithm for calculations. This approach can be treated also as convenient approximation, which always is valid in the resonance range, where non-resonance contribution is small.

We have fulfilled also a short methodological and phenomenological analysis of the approach under discussion. It was shown, that in the process $e^+e^- \rightarrow f\bar{f}$ the difference between two forms of propagators is negligible in a wide range of energy. It can be significant in the processes with baryon resonance in an intermediate state, but in this case we should fulfill an additional analysis.

6. Appendix

In this section, we briefly describe the model of UP with a smeared mass and construct the propagators for the vector and spinor fields. The structure of these propagators lead to the factorization effect in the processes with the participation of the UP in the intermediate state. The model field wave function, which describes UP, is represented in the form [7]:

$$\Phi_a(x) = \int \Phi_a(x, \mu) \omega(\mu) d\mu,$$

(15)

where $\Phi_a(x, \mu)$ is standard spectral component, which defines a particle with a fixed mass squared $m^2 = \mu$ in the stable particle approximation (SPA). The weight function $\omega(\mu)$ is formed by the self-energy interactions of UP with vacuum fluctuations and decay products. This function describes the smeared (fuzzed) mass-shell of UP.

The model Lagrangian, which determines a "free" (effective) unstable field $\Phi(x)$, has the convolution form:

$$L(\Phi(x)) = \int L(\Phi(x, \mu)) |\omega(\mu)|^2 d\mu.$$

(16)

In Eq.(16) $L(\Phi(x, \mu))$ is the standard Lagrangian, which describes model "free" field component $\Phi(x, \mu)$ in the stable particle approximation ($m^2 = \mu$).

From Eq.(16) and prescription $\partial \Phi(x, \mu)/\partial \Phi(x, \mu') = \delta(\mu - \mu')$ it follows the Klein-Gordon equation for the spectral component of scalar or vector field:

$$\Box \Phi_\alpha(x, \mu) = 0.$$

(17)

In analogy with (17) one can get the Dirac equation for fermion spectral component. As a result, we get the standard representation of the field function $\Phi_\alpha(x, \mu)$ with a fixed mass parameter $\mu$ (spectral component). All standard definitions, relations and frequency expansion take place for $\Phi_\alpha(k, \mu)$, however, the relation $k^0_\mu = \sqrt{k^2 + \mu}$ defines the smeared (fuzzy)
mass-shell due to a random nature of the mass parameter $\mu$. The convolution (diagonal) representation of the "free" Lagrangian (16) has an assumption (or approximation?) that the states with different $\mu$ do not interact in the approximation of the model "free" fields.

The expressions (15)–(17) define the model "free" unstable field as some effective field. As it was mentioned above, this field is formed by an interaction of "bare" UP with the vacuum fluctuations and decay products, that is includes self-energy contribution in the resonant region. Such an interaction leads to the spreading (smearing) of mass, described by the function $\omega(\mu)$ or $\rho(\mu) = |\omega(\mu)|^2$. Thus, we go from the distribution $\rho^{st}(\mu) = \delta(\mu - M^2)$ for "bare" particles to some smooth density function $\rho(\mu) = |\omega(\mu)|^2$ with mean value $\bar{\mu} \approx M^2$ and mean square deviation $\sigma_\mu \approx \Gamma$. So, the UP is characterized by the weight function $\omega(\mu)$ or probability density $\rho(\mu)$ with parameters $M$ and $\Gamma$ (or real and imaginary parts of a pole).

The commutative relations for the model operators have an additional $\delta$-function:

$$[\Phi^-_\alpha(\bar{k},\mu), \Phi^+_{\bar{\beta}}(\bar{q},\mu')]_\pm = \delta(\mu - \mu')\delta(\bar{k} - \bar{q})\delta_{\alpha\beta}, \quad (18)$$

where subscripts $\pm$ correspond to the fermion and boson fields. The presence of $\delta(\mu - \mu')$ in Eq.(18) means an assumption - the acts of creation and annihilation of the particles with various $\mu$ (the random mass squared) do not interfere. Thus, the parameter $\mu$ has the status of physically distinguishable value of a random $m^2$. This assumption is naturally related with a diagonal form of Eqs.(16) and (17) and directly follows from the interpretation of $q^2$ as a random parameter $\mu$. By integrating the both sides of Eq.(18) with weights $\omega^*(\mu)\omega(\mu')$ one can get the standard commutative relations

$$[\Phi^-_\alpha(\bar{k}), \Phi^+_{\bar{\beta}}(\bar{q})]_\pm = \delta(\bar{k} - \bar{q})\delta_{\alpha\beta}, \quad (19)$$

where $\Phi^\pm_\alpha(\bar{k})$ is the full operator field function in the momentum representation

$$\Phi^\pm_\alpha(\bar{k}) = \int \Phi^\pm_\alpha(\bar{k},\mu)\omega(\mu)d\mu. \quad (20)$$

It should be noted that Eq.(20) follows from Eq.(19) when $\int |\omega(\mu)|^2d\mu = 1$, that is $|\omega(\mu)|^2$ can be interpreted as a normalized probability density.

The expressions (15), (16) and (18) are the principal elements of the model. The weight function $\omega(\mu)$ (or $\rho(\mu)$) is full characteristic of UP in the framework of the model. The relations (18) define the structure of the model amplitude and transition probability.

Here, we consider the model amplitude for the simplest processes with UP in an initial or final state and get the convolution formula as a direct consequence of the model. The expression for a scalar operator field [7] is

$$\phi^\pm(x) = \frac{1}{(2\pi)^{3/2}} \int \omega(\mu)d\mu \int \frac{a^\pm(\bar{q},\mu)}{\sqrt{2q_0^\mu}}e^{\pm iqx}d\bar{q}, \quad (21)$$
where \( q_\mu^0 = \sqrt{q^2 + \mu} \) and \( a^\pm(\bar{q}, \mu) \) are the creation or annihilation operators of UP with the momentum \( q \) and mass squared \( m^2 = \mu \). Taking into account Eq.(18) one can get:

\[
[a^-(\bar{k}, \mu), \phi^+(x)]_\pm; [\phi^-(x), a^+(\bar{k}, \mu)]_\pm = \frac{\omega(\mu)}{(2\pi)^{3/2} \sqrt{2k_\mu}} e^{\pm ikx},
\]

(22)

where \( k_\mu^0 = \sqrt{k^2 + \mu} \). The expressions (22) differ from the standard ones by the factor \( \omega(\mu) \) only. From this result it follows that, if \( \hat{a}^+(k, \mu)|0\rangle \) and \( \langle 0|\hat{a}^-(k, \mu) \) define UP with the mass \( m = \sqrt{\mu} \) and momentum \( k \) in the initial or final states, then the amplitude for the transition \( \Phi \to \phi \phi_1 \) is

\[
A(k, \mu) = \omega(\mu)A^{st}(k, \mu),
\]

(23)

where \( A^{st}(k, \mu) \) is the amplitude in the stable particle approximation. This amplitude is calculated in the standard way and can include the higher corrections. Moreover, it can be an effective amplitude for the processes with hadron participation. From Eq.(23) it follows that the differential (on \( \mu \)) probability of transition is

\[
dP(k, \mu) = \rho(\mu)|A(k, \mu)|^2 d\mu.
\]

To define the transition probability of the process \( \Phi \to \phi \phi_1 \), where \( \phi \) is UP with a large width, we should take into account the status of the parameter \( \mu \) as a physically distinguishable value, which follows from Eq.(18). Thus, the differential (on \( k \)) probability is

\[
d\Gamma(k) = \int d\Gamma^{st}(k, \mu) \rho(\mu) d\mu.
\]

(24)

In Eq.(24) the differential probability \( d\Gamma^{st}(k, \mu) \) is defined in the standard way (the stable particle approximation):

\[
d\Gamma^{st}(k, \mu) = \frac{1}{2\pi} \delta(k_\Phi - k_\phi - k_1)|A^{st}(k, \mu)|^2 d\bar{k}_\phi d\bar{k}_1,
\]

(25)

where \( k = (k_\Phi, k_\phi, k_1) \) denotes the momenta of particles. From Eqs.(24) and (25) it directly follows the well-known convolution formula for a decay rate

\[
\Gamma(m_\Phi, m_1) = \int_{\mu_1}^{\mu_2} \Gamma^{st}(m_\Phi, m_1; \mu) \rho(\mu) d\mu,
\]

(26)

where \( \rho(\mu) = |\omega(\mu)|^2 \), \( \mu_1 \) and \( \mu_2 \) are the threshold and maximal invariant mass squared of an unstable particle \( \phi \).

An account of higher corrections in the amplitude (23) keeps the convolution form of Eq.(26). This form can be destroyed by the interaction between the products of UP (\( \phi \)) decay and initial \( \Phi \) or final \( \phi_1 \) states. The calculation in this case can be performed in the standard way, but UP in the intermediate state is described by the model propagator. However, the calculation within the framework of perturbative theory (PT) can not be applicable to the
UP with a large width, that is to the short-living particle. In any case, the applicability of the PT, of the model approach or convolution method to the decays considered should be justified by an experiment. The validity of the CM was demonstrated for many processes, but this problem needs in more detailed investigation. If there are two UP with large widths in a final state $\Phi \rightarrow \phi_1 \phi_2$, then in analogy with the previous case one can get the double convolution formula:

$$\Gamma(m_\Phi) = \int \int \Gamma^{st}(m_\Phi; \mu_1, \mu_2)\rho_1(\mu_1)\rho_2(\mu_2)d\mu_1d\mu_2.$$  \hspace{1cm} (27)

The derivation of CF for the cases when there is a vector or spinor UP in the final state can be done in analogy with the case of scalar UP. However, in Eqs. (21), (22) and (23) one should take into account the polarization vector $e_m(q)$ or spinor $u_\nu^{\mu,\pm}(q)$, where momentum $q$ is on fuzzy mass-shell. As a result, we get the polarization matrix with $m^2 = \mu$. In the case of vector UP in the final state we have

$$\sum e_m(q)e^*_n(q) = -g_{mn} + q_mq_n/\mu,$$  \hspace{1cm} (28)

and in the case of spinor UP in the final state:

$$\sum u_\nu^{\mu,\pm}(q)\bar{u}_\beta^{\nu,\mp}(q) = \frac{1}{2q^0_\mu}(\hat{q} \mp \sqrt{\mu})_{\alpha\beta},$$  \hspace{1cm} (29)

where the summation over polarization is implied and $q^0_\mu = \sqrt{\hat{q}^2 + \mu}$. The same relations take place for the initial states, however one have to average over the polarizations. The formulae (26) and (27) describe FWE in full analogy with the phenomenological convolution method. Similar method, called the semi-analytical approach, was applied in calculations of cross-section of the processes $e^+e^- \rightarrow ZZ, WW$ at LEP2 energy [14], where the phase space of the final states was integrated in analogy with (27). This approach gives a simple expressions for the cross-sections, which are equivalent to the inclusive cross-section of the fourth-fermion reactions. The calculation of this cross-section in the framework of standard PT is very complicated and usually carried out with the help of the Monte-Carlo simulation.

The model under consideration gives a quantum field basis for CM, which takes into account the fundamental uncertainty relation, provides a simple expressions for decay rates and is in a good agreement with the experimental data on some processes. To evaluate FWE in the case, when UP is in an initial state, we have to take into consideration the process of UP production. If UP is in an intermediate state, then the description of FWE is equivalent to the traditional one, but the propagators are determined by the model.

Now, we consider the the structure of the model propagators. With the help of the traditional method, one can get from Eqs. (15), (18) and (20) the expression for the unstable
scalar Green function [7]:
\[\langle 0| T(\phi(x), \phi(y)) |0 \rangle \equiv D(x - y) = \int D(x - y, \mu) \rho(\mu) d\mu. \quad (30)\]

In Eq. (30) \(D(x, \mu)\) is a standard scalar Green function with \(m^2 = \mu\), which describes UP in an intermediate state:
\[D(x, \mu) = \frac{i}{(2\pi)^4} \int \frac{e^{-ikx}}{k^2 - \mu + i\epsilon} dk. \quad (31)\]

The right-hand side of Eq. (30) is the Lehmann-like spectral \((on \mu)\) representation of the scalar Green function. Taking into account the relation between scalar and vector Green functions, we can get the Green function of the vector unstable field in the form:
\[D_{mn}(x, \mu) = -(g_{mn} + \frac{1}{\mu} \frac{\partial^2}{\partial x^n \partial x^m})D(x, \mu)\]
\[= \frac{-i}{(2\pi)^4} \int \frac{g_{mn} - k_m k_n / \mu}{k^2 - \mu + i\epsilon} e^{-ikx} dk. \quad (32)\]

Analogously, the Green function of the spinor unstable field is
\[\hat{D}(x, \mu) = (i\hat{\partial} + \sqrt{\mu})D(x, \mu) = \frac{i}{(2\pi)^4} \int \frac{\hat{k} + \sqrt{\mu}}{k^2 - \mu + i\epsilon} e^{-ikx} dk; \quad (33)\]
where \(\hat{k} = k_i \gamma^i\). These Green functions in momentum representation have a convolution form:
\[D_{mn}(k) = \int D_{mn}(k, \mu) \rho(\mu) d\mu, \quad \hat{D}(k) = \int \hat{D}(k, \mu) \rho(\mu) d\mu. \quad (34)\]

The expression (32-34) for the propagators of vector and spinor fields leads to the effect of factorization, which in turn, gives the convolution formula.

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