Instabilities of wave function monopoles in Bose-Einstein condensates

Th. Busch, J. R. Anglin
Institut für Theoretische Physik, Universität Innsbruck, A–6020 Innsbruck, AUSTRIA

We present analytic and numerical results for a class of monopole solutions to the two-component Gross-Pitaevski equation for a two-species Bose condensate in an effectively two-dimensional trap. We exhibit dynamical instabilities involving vortex production as one species pours through another, from which we conclude that the sub-optical sharpness of potentials exerted by matter waves makes condensates ideal tools for manipulating condensates. We also show that there are two equally valid but drastically different hydrodynamic descriptions of a two-component condensate, and illustrate how different phenomena may appear simpler in each.

PACS number(s): 03.75.Fi, 03.65 Ge

Monopoles are particle-like field configurations with which one can associate a topological charge. As perhaps the most obvious way of making a collective particle out of a condensate field, they are a natural goal for condensate state engineering. In this Letter we discuss a specific case \( D = 2 \) of a class of \( D \)-dimensional monopoles, previously introduced in general and examined for \( D = 1 \). Although the notion of topological charge (in \( D = 2 \), the ‘winding number’) is useful in identifying and classifying these particle-like configurations, in our case this charge is not guaranteed to be conserved. Indeed the structures we discuss are quite prone to dynamical instabilities. They are nevertheless also robust, in that their instabilities typically do not destroy them, but rather distort them, in ways which are both intrinsically interesting, and illustrative of some basic features of multi-component superfluid dynamics.

Our paper is organized as follows. After presenting our concept of a ‘wave function monopole’, we consider a one-dimensional ‘ring monopole’ which bears much the same useful analogy to the 2D monopole as the one-dimensional persistent current does to the vortex (and which also merits serious consideration in its own right). In this context we briefly discuss the consequences of having different scattering lengths among the two atomic species. We then present and analyze the results of numerical solutions to the two-component Gross-Pitaevski equation.

The basic concept of a monopole in \( D \) dimensions is sketched in Fig. 1 (for \( D = 2 \)). The arrows represent an order parameter having as many components as the dimensionality of the space in which the monopole exists. There is much freedom in the actual identification of this order parameter. For instance, in two dimensions its magnitude and direction could represent the modulus and phase of the macroscopic wave function of a one-species condensate; this ‘monopole’ is the well-known superfluid vortex. In the case we now consider, however, the two dimensions of the arrows instead represent the wave functions of each component of a two-species condensate, in a configuration in which both components happen to be real: \( (\psi_1, \psi_2) = f(r)(\cos \theta, \sin \theta) \), where \( r, \theta \) are the usual polar co-ordinates.

As with the vortex, the modulus of the order parameter is the density of the condensate (here, the total density of both components), so that the ‘particle-like’ core of the monopole is in fact a local minimum of density: a void or bubble, maintained by destructive interference of matter waves. As with the vortex, however, it is instructive first to avoid the core \( r \to 0 \) of the monopole, and consider only its behaviour at large \( r \), where the radial dimension becomes unimportant, leaving the effectively one-dimensional problem of a two-species condensate on a circle of fixed radius \( R \). So we replace \( f(r) \to 1 \), and examine the ‘ring monopole’ \( (\psi_1, \psi_2) \) as a stationary solution to the one-dimensional Gross-Pitaevski equation

\[
i \dot{\psi}_j = -\frac{1}{2R^2} \frac{\partial^2}{\partial \theta^2} \psi_j + g(\psi_1^2 + \psi_2^2 - \mu) \psi_j ,
\]

where \( \mu = 1 - (2gR^2)^{-1} \) provides \( \dot{\psi}_j = 0 \).

For the ring monopole we can obtain analytically the Bogoliubov spectrum of perturbations, \( \psi_j \to \psi_j + \epsilon \phi(\theta, t) \). Working to linear order in \( \epsilon \) yields the modes

\[
\phi_{\pm 1k} = \cos(\Omega_{k+} t) \begin{pmatrix} X_{1k}^\pm \\ X_{2k}^\pm \end{pmatrix} - i \frac{2R^2 \Omega_{k+}}{k(k+2)} \sin(\Omega_{k+} t) \begin{pmatrix} Y_{1k}^\pm \\ Y_{2k}^\pm \end{pmatrix} = \begin{pmatrix} (k-2) \cos(k-1)\theta + C_{k+} \cos(k+1)\theta \\ -(k-2) \sin(k-1)\theta + C_{k+} \sin(k+1)\theta \end{pmatrix}
\]

\[
\phi_{\pm 2k} = \cos(\Omega_{k+} t) \begin{pmatrix} X_{1k}^\pm \\ X_{2k}^\pm \end{pmatrix} - i \frac{2R^2 \Omega_{k+}}{k(k+2)} \sin(\Omega_{k+} t) \begin{pmatrix} Y_{1k}^\pm \\ Y_{2k}^\pm \end{pmatrix} = \begin{pmatrix} (k+2) \cos(k+1)\theta + C_{k+} \cos(k+1)\theta \\ -(k+2) \sin(k+1)\theta + C_{k+} \sin(k+1)\theta \end{pmatrix}.
\]

The ± index distinguishes acoustic and optical branches,
in which the two species’ density perturbations are respectively in and out of phase:

$$C_{k\pm} = 2 + \frac{2k^2}{gR^2} \pm k\sqrt{1 + 8(gR^2)^{-1} + 4k^2(gR^2)^{-2}}$$

$$\frac{\Omega^2_{k\pm}}{k^2} = \frac{g}{2R^2} \frac{k^2 + 4}{4R^4} \pm \sqrt{\left(\frac{g}{2R^2} + \frac{k^2 + 4}{4R^4}\right)^2 - \frac{k^2 - 4}{4R^4} \left( g + \frac{k^2 - 4}{4R^2}\right)} \ .$$  (3)

Note that rotating these modes by $\theta \rightarrow (\theta + \pi/2)$ produces an independent set of modes, with the same frequencies; every frequency is thus two-fold degenerate.

The modes $k = 0$ (for which the $\pm$ branches co-incide) and $k \pm = 2\pi$ provide the four zero modes due to the model’s U(2) symmetry (which already includes spatial rotation). All the other modes are positive frequency excitations, except for the two $k \pm = 1\pi$ modes; and for $gR^2 > 3/4$, these become dynamical instabilities. Infinitesimal excitation of these modes then grows exponentially until it becomes finite, and the Bogoliubov theory is inadequate. To follow the evolution into this regime, we solve the Gross-Pitaevskii equation numerically, using the split operator method as described in [4]. It turns out that the finite perturbation does not mix with other modes to produce irreversible (long revival time) decay, but grows to a maximum size, then shrinks back; and the cycle repeats. This indicates that the two unstable modes form an isolated subsystem of two degrees of freedom, with a ‘Mexican hat’ potential. [8]

Before going on to examine instabilities of monopoles in two dimensions, we pause here to address an issue of generalization, which will lead to the development of some concepts useful for understanding two-dimensional phenomena. In [8] it is assumed that the three scattering lengths and two particle numbers in the problem are all equal; this is very close to the case for $^{87}$Rb [3]. In fact the simple solution we examine here can be generalized to encompass all positive scattering lengths $g_{ij}$. When separation is favoured ($\det g < 1$) [8], one can obtain solutions in which our cosine and sine are replaced by Jacobi elliptic functions. (Phase-separating monopoles in two dimensions resemble crossed domain walls, dividing two condensates each with a Josephson $\pi$-junction through the monopole core. They appear to have qualitatively similar behaviour to the simplest case with $g_{ij} = g$.) When mixing is favoured ($\det g > 1$), one can also generalize monopoles, but in a somewhat surprising way. If one merely changes basis, defining $\psi_{\pm} = \frac{1}{\sqrt{2}}(\psi_1 \pm i\psi_2)$, one discovers that our monopole can also be described as two superimposed, counter-rotating vortices (SCV). Changing scattering lengths so as to favour mixing of the two species $\psi_{\pm}$ will then stabilize this configuration.

Of course, this change of basis is only a symmetry of the system when all scattering lengths are equal (and hence only then is it really true that the monopole and the SCV are the same object). Nevertheless the two bases can always be used, and so one has two very different hydrodynamic pictures which describe the same superfluid physics. This is our first principle, that with two-component superfluids one has two equally valid hydrodynamic descriptions, and that one is free to base intuition on either of them. As will be apparent below, the differences between them can be dramatic.

We now return to the case where all scattering lengths are equal, and investigate the monopole in two dimensions. We note that $(\psi_1, \psi_2 = f(r)(\cos \theta, \sin \theta)$ is a stationary solution to the Gross-Pitaevskii equation (GPE)

$$i\psi_j = -\frac{1}{2}\nabla^2 \psi_j + g(|\psi_1|^2 + |\psi_2|^2 - \mu)\psi_j + \frac{1}{2}g^2 \psi_j^3 \ , \ (4)$$

as long as $f(r)$ satisfies the same nonlinear equation as the modulus of the two-dimensional vortex in the trap:

$$f'' + \frac{1}{r}f' + \left(\frac{r^2}{2} - \frac{1}{r^2}\right)f = 2g(f^2 - \mu)f \ . \ (5)$$

We have studied such monopoles in harmonic traps numerically, using the split-operator method to solve the two-component GPE in two dimensions, in both real and imaginary time. We have examined a wide range of different parameter regimes, and compared results with grids of varying sizes; and while below we show displacements and motions which are exactly in one of our grid directions, we have also checked that the same behaviour occurs at arbitrary angles.

If a monopole is displaced from the centre of the trap, it begins to fall back towards the centre; but this does...
not mean that it is stable. By considering the linearized theory in imaginary time, one can show that there are lower energy configurations near the monopole, reached by short range perturbations near the core. While this analysis does not suffice to determine whether the instability is dynamical or merely energetic, numerical study of small amplitude oscillations of a monopole in a trap indicates that monopoles are dynamically unstable: the amplitude of small oscillations grows exponentially.

The instability, when infinitesimal, is a self-amplifying oscillation of the monopole back and forth through the centre of the trap; this is at least qualitatively similar to the instability on the ring. As the motion grows finite, vortex pairs form. As can be seen from Fig. 4, the motion of the monopole involves the pouring of one of the two species through a narrow channel formed by the other species (which is comparatively passive throughout). As shown in Fig. 5, very rapid flow develops along the sides of this channel, until vortex pairs suddenly nucleate at the maxima of the passive species’ density. Each pair then separates, into an inner vortex pinned by the passive species, and an outer vortex, which gradually drifts away. Fig. 6 shows this process as the growth of a pair of branch cuts in the condensate phase. It may be quite hard to directly detect vortex pairs formed in this manner; but their appearance vividly illustrates a potentially useful fact: that sharp-edged potentials (such as most readily generate vortices) are much more easily formed with matter waves than with light waves. If one wishes to manipulate condensates with high resolution forces, therefore, the best tools are probably other condensates.

Despite spawning vortices, an initially displaced monopole continues its fall towards the centre of the trap with little apparent impediment. As (or just before) it reaches the centre, however, its motion changes. In most cases we have investigated, it abruptly slows (though in some cases, and for reasons which are not clear to us, it seems instead to accelerate, and then decay rather violently). Thereafter, the minimum in total density at the monopole core gradually fills in, until only a slight depression remains. But even at this late stage, a double-lobed pattern of the two densities (as in Fig. 4) remains; and in each species a phase difference of \( \pi \) persists across the monopole, although the sharp jump in phase has been smoothed out. The instability appears to have saturated, leaving a ‘smeared’ monopole which appears to be dynamically (although not energetically) stable. This eventual stability is much easier to understand in the SCV basis. There, all that happens is that the superimposed counter-rotating vortices slip apart, becoming a pair of reciprocally pinned vortices: each condensate species has one vortex, and each also fills a strong potential well formed by the other species’ vortex. (This may be seen as an analogue of the SCV instability on the ring.) Once this separation has occurred, the stability of two pinned vortices is unsurprising. See Fig. 7.

FIG. 3. Total density \( \rho = |\psi_1|^2 + |\psi_2|^2 \) in the centre of a harmonic trap, for a monopole initially displaced very slightly from the centre, at \( g \int |\nabla \psi_j|^2 \) = 2000. Note logarithmic vertical scale. Successive minima do not reach \( \rho = 0 \), because growing speed of the monopole involves filling in of the core.

The instability, when infinitesimal, is a self-amplifying oscillation of the monopole back and forth through the centre of the trap; this is at least qualitatively similar to the instability on the ring. As the motion grows finite, vortex pairs form. As can be seen from Fig. 4, the motion of the monopole involves the pouring of one of the two species through a narrow channel formed by the other species (which is comparatively passive throughout). As shown in Fig. 5, very rapid flow develops along the sides of this channel, until vortex pairs suddenly nucleate at the maxima of the passive species’ density. Each pair then separates, into an inner vortex pinned by the passive species, and an outer vortex, which gradually drifts away. Fig. 6 shows this process as the growth of a pair of branch cuts in the condensate phase. It may be quite hard to directly detect vortex pairs formed in this manner; but their appearance vividly illustrates a potentially useful fact: that sharp-edged potentials (such as most readily generate vortices) are much more easily formed with matter waves than with light waves. If one wishes to manipulate condensates with high resolution forces, therefore, the best tools are probably other condensates.

Despite spawning vortices, an initially displaced monopole continues its fall towards the centre of the trap with little apparent impediment. As (or just before) it reaches the centre, however, its motion changes. In most cases we have investigated, it abruptly slows (though in some cases, and for reasons which are not clear to us, it seems instead to accelerate, and then decay rather violently). Thereafter, the minimum in total density at the monopole core gradually fills in, until only a slight depression remains. But even at this late stage, a double-lobed pattern of the two densities (as in Fig. 4) remains; and in each species a phase difference of \( \pi \) persists across the monopole, although the sharp jump in phase has been smoothed out. The instability appears to have saturated, leaving a ‘smeared’ monopole which appears to be dynamically (although not energetically) stable. This eventual stability is much easier to understand in the SCV basis. There, all that happens is that the superimposed counter-rotating vortices slip apart, becoming a pair of reciprocally pinned vortices: each condensate species has one vortex, and each also fills a strong potential well formed by the other species’ vortex. (This may be seen as an analogue of the SCV instability on the ring.) Once this separation has occurred, the stability of two pinned vortices is unsurprising. See Fig. 7.

FIG. 4. \( |\psi_1|^2 \) (top) and \( |\psi_2|^2 \) for a monopole initially displaced in the direction \( \theta = \pi \), at early and late times \( (g \int |\nabla \psi_j|^2 = 2000) \). Note ‘throttle’ formed by \( |\psi_2|^2 \).

FIG. 5. Superfluid velocity component \( v_\parallel = 3(\partial \ln \psi_1) \), at various times in the same simulation illustrated in Fig. 4. Plots are ‘edge-on’ views of the velocity field \( v_\parallel(x, y) \), looking in the positive x direction: the velocity is very small everywhere except near the arc on which \( |\psi_1|^2 \) vanishes, so the edge-on view is convenient. Initial fluid velocity is negative as the monopole moves in the positive \( (\theta = 0) \) direction.

FIG. 6. Phase of \( \psi_1(x, y) \) before and after appearance of vortex pairs. The branch cuts are indeed jumps of \( 2\pi \).

3
FIG. 7. Monopole instability in the SCV basis. Top figures show total density $\rho = |\psi_1|^2 + |\psi_2|^2$ at early and late times; bottom figures show $|\psi_1|^2$ ($|\psi_2|^2$ being the mirror image in the x-axis.) No new vortices form; instead, the initially superimposed vortices slip apart, and each is pinned.

On the other hand, the monopole basis can be more illuminating of other phenomena. Two directly aligned monopoles perform a curious dance: they attract each other, merging into a single monopole of winding number two; this then separates again, but in the perpendicular direction; and the cycle repeats. In the SCV basis, this is a rather confusing system of orbiting vortices, and pinning peaks that form and dissolve; but in the monopole basis, it is quite simple (see Fig. 8). And the peculiar scattering of monopoles at right angles in a head-on collision is actually what is expected for monopoles in gauge theories as well [9]. We find analogous dances, with $\pi/n$ scattering, for monopoles of higher winding number $n$.

We conclude our overture to monopoles in Bose condensates with three remarks on experimental possibilities. First, despite their instability, creation of structures like these, in spherical or toroidal traps, may nevertheless be possible using the adiabatic passage technique of Dum et al. [10]. See Fig. 9. Second, unlike vortices, detection of monopoles should not pose a problem: since the two species can be imaged separately, the pattern of density lobes is large and obvious, and clearly indicates the presence of a monopole core. And finally, the two principles of alternative hydrodynamic pictures, and using condensates to manipulate condensates, may be of practical value in a wide range of future experiments on multi-component condensates.

FIG. 8. Two aligned monopoles scattering in a harmonic trap. Top figures show total density; bottom figures show $|\psi_1|^2$ and $|\psi_2|^2$ separately. Since $|\psi_2|^2$ essentially does not change, it is merely indicated as uniformly shaded lobes.

We gratefully acknowledge valuable discussions with Ignacio Cirac and Peter Zoller, as well as the support of the European Union under the TMR Network ERBFMRX-CT96-0002.

FIG. 9. Adiabatic transfer of population (see Ref. [10]), from an internal atomic ground state and the motional ground state in the trap, to the monopole configuration of two excited internal states. We take $\lambda_0 k_L = 0.1, g \int d^2x |\psi_j|^2 = 1000$, detuning swept linearly from -1 to +5 trap frequencies. Further parameter optimization may allow total transfer, but there may also be advantages, for stabilization and observation, to leaving a small ground state population in the core.

We gratefully acknowledge valuable discussions with Ignacio Cirac and Peter Zoller, as well as the support of the European Union under the TMR Network ERBFMRX-CT96-0002.

[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, E. A. Cornell Science, 269, 198 (1995).
[2] C. C. Bradley, C. A. Sackett, J. J. Tollett, R. G. Hulet, Phys. Rev. Lett. 75, 1687 (1995).
[3] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995).
[4] Th. Busch and J.R. Anglin, cond-mat/9809408.
[5] C.G. Myatt, E.A. Burt, R.W. Ghrist, E.A. Cornell, C.E. Weiman, Phys. Rev. Lett. 78, 586 (1997).
[6] Our structures could be called ‘wave function monopoles’ to distinguish them from ‘spin monopoles’ in which the order parameter making the canonical pattern is the atomic spin; see J.J.Garcia-Ripoll et al., in preparation.
[7] Since mean field theory soon breaks down in the presence of a dynamical instability, one should actually quantize these two unstable modes.
[8] T.-L. Ho, Phys. Rev. Lett. 81, 742 (1998).
[9] M. Atiyah and N. Hitchin, The Geometry and Dynamics of Magnetic Monopoles, Princeton (1988).
[10] R. Dum, J.I. Cirac, M. Lewenstein, P. Zoller, Phys. Rev. Lett. 80, 2972 (1998).