ESTIMATES FOR THE INITIAL COEFFICIENTS OF BI-UNIVALENT FUNCTIONS

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ABSTRACT. A bi-univalent function is a univalent function defined on the unit disk with its inverse also univalent on the unit disk. In the present investigation, estimates for the initial coefficients are obtained for bi-univalent functions belonging to certain classes defined by subordination and relevant connections with earlier results are pointed out.

1. INTRODUCTION

Let \( \mathcal{A} \) be the class of analytic functions defined on the open unit disk \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) and normalized by the conditions \( f(0) = 0 \) and \( f'(0) = 1 \). A function \( f \in \mathcal{A} \) has Taylor’s series expansion of the form
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]
The class of all univalent functions in the open unit disk \( \mathbb{D} \) of the form (1.1) is denoted by \( \mathcal{S} \). Determination of the bounds for the coefficients \( a_n \) is an important problem in geometric function theory as they give information about the geometric properties of these functions. For example, the bound for the second coefficient \( a_2 \) of functions in \( \mathcal{S} \) gives the growth and distortion bounds as well as covering theorems. Some coefficient related problems were investigated recently in [1, 3, 8, 9, 17, 27].

Since univalent functions are one-to-one, they are invertible but their inverse functions need not be defined on the entire unit disk \( \mathbb{D} \). In fact, the famous Koebe one-quarter theorem ensures that the image of the unit disk \( \mathbb{D} \) under every function \( f \in \mathcal{S} \) contains a disk of radius \( 1/4 \). Thus, inverse of every function \( f \in \mathcal{S} \) is defined on a disk, which contains the disk \( |z| < 1/4 \). It can also be easily verified that
\[
F(w) := f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^2 - 5a_2 a_3 + a_4)w^4 + \cdots
\]
in some disk of radius at least \( 1/4 \). A function \( f \in \mathcal{A} \) is called bi-univalent in \( \mathbb{D} \) if both \( f \) and \( f^{-1} \) are univalent in \( \mathbb{D} \). In 1967, Lewin [16] introduced the class \( \sigma \) of bi-univalent analytic functions and showed that the second coefficient of every \( f \in \sigma \) satisfy the inequality \( |a_2| \leq 1.51 \). Let \( \sigma_1 \) be the class of all functions \( f = \phi \circ \psi^{-1} \) where \( \phi, \psi \) map \( \mathbb{D} \) onto a domain containing \( \mathbb{D} \) and \( \phi'(0) = \psi'(0) \). In 1969, Suffridge [26] gave a function in \( \sigma_1 \subset \sigma \), satisfying \( a_2 = 4/3 \) and conjectured that \( |a_2| \leq 4/3 \) for all functions in \( \sigma \). In 1969, Netanyahu [19] proved this conjecture for the subclass \( \sigma_1 \). Later in 1981, Styer and Wright [25] disproved the
conjecture of Suffridge [26] by showing $a_2 > 4/3$ for some function in $\sigma$. Also see [7] for an example to show $\sigma \neq \sigma_1$. For results on bi-univalent polynomial, see [14, 22]. In 1967, Brannan [4] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \sigma$. In 1985, Kedzierawski [13, Theorem 2] proved this conjecture for a special case when the function $f$ and $f^{-1}$ are starlike functions. In 1985, Tan [28] obtained the bound for $a_2$ namely $|a_2| \leq 1.485$ which is the best known estimate for functions in the class $\sigma$. For some open problems and survey, see [11, 23]. In 1985, Kedzierawski [13] proved the following:

$$
|a_2| \leq \begin{cases} 
1.5894, & f \in \mathcal{S}, f^{-1} \in \mathcal{S}; \\
\sqrt{2}, & f \in \mathcal{S}^*, f^{-1} \in \mathcal{S}^*; \\
1.507, & f \in \mathcal{S}^*, f^{-1} \in \mathcal{S}; \\
1.224, & f \in \mathcal{K}, f^{-1} \in \mathcal{S},
\end{cases}
$$

where $\mathcal{S}^*$ and $\mathcal{K}$ denote the well-known classes of starlike and convex functions in $\mathcal{S}$.

Let us recall now various definitions required in sequel. An analytic function $f$ is subordi-
age to another analytic function $g$, written $f \prec g$, if there is an analytic function $w$ with $|w(z)| \leq |z|$ such that $f = g \circ w$. If $g$ is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. Let $\varphi$ be an analytic univalent function in $\mathbb{D}$ with positive real part and $\varphi(\mathbb{D})$ be symmetric with respect to the real axis, starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. Ma and Minda [18] gave a unified presentation of various subclasses of starlike and convex functions by introducing the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ of functions $f \in \mathcal{S}$ satisfying $zf'(z)/f(z) \prec \varphi(z)$ and $1 + zf''(z)/f'(z) \prec \varphi(z)$ respectively, which includes several well-known classes as special case. For example, when $\varphi(z) = (1 + A z)/(1 + B z)$ ($-1 \leq B < A \leq 1$), the class $\mathcal{S}^*(\varphi)$ reduces to the class $\mathcal{S}^*[A, B]$ introduced by Janowski [12]. For $0 \leq \beta < 1$, the classes $\mathcal{S}^*(\beta) := \mathcal{S}^*((1 + (1 - 2\beta) z)/(1 - z))$ and $\mathcal{K}(\beta) := \mathcal{K}((1 + (1 - 2\beta) z)/(1 - z))$ are starlike and convex functions of order $\beta$. Further let $\mathcal{S}^*(0) := \mathcal{S}^*$ and $\mathcal{K}(0)$ are the classes of starlike and convex functions respectively. The class of strongly starlike functions $\mathcal{S}^*_\alpha := \mathcal{S}^*((1 + \alpha z)/(1 - z))$ of order $\alpha$, $0 < \alpha \leq 1$. Denote by $\mathcal{R}(\varphi)$ the class of all functions satisfying $f'(z) \prec \varphi(z)$ and let $\mathcal{R}(\beta) := \mathcal{R}((1 + (1 - 2\beta) z)/(1 - z))$ and $\mathcal{R}(0) := \mathcal{R}(0)$.

For $0 \leq \beta < 1$, a function $f \in \sigma$ is in the class $\mathcal{S}^*_\alpha(\beta)$ of bi-starlike function of order $\beta$, or $\mathcal{K}_\sigma(\beta)$ of bi-convex function of order $\beta$ if both $f$ and $f^{-1}$ are respectively starlike or convex functions of order $\beta$. For $0 < \alpha \leq 1$, the function $f \in \sigma$ is strongly bi-starlike function of order $\alpha$ if both the functions $f$ and $f^{-1}$ are strongly starlike functions of order $\alpha$. The class of all such functions is denoted by $\mathcal{S}^*_{\alpha, \sigma}$. These classes were introduced by Brannan and Taha [6] in 1985 (see also [5]). They obtained estimates on the initial coefficients $a_2$ and $a_3$ for functions in these classes. Recently, Ali et al. [2] extended the results of Brannan and Taha [6] by generalizing their classes using subordination. For some related results, see [10, 24, 29]. For the various applications of subordination one can refer to [11, 13, 17, 27] and the references cited therein.

Motivated by Ali et al. [2] in this paper estimates for the initial coefficient $a_2$ of bi-univalent functions belonging to the class $\mathcal{R}_\sigma(\lambda, \varphi)$ as well as estimates on $a_2$ and $a_3$ for functions in classes $\mathcal{S}^*_{\sigma}(\varphi)$ and $\mathcal{K}_\sigma(\varphi)$, defined later, are obtained. Further work of Kedzierawski [13] actuates us to derive the estimates on initial coefficients $a_2$ and $a_3$ when $f$ is in the some subclass of univalent functions and $f^{-1}$ belongs to some other subclass of univalent functions. Our results generalize several well-known results in [2, 10, 13, 24], which are pointed out here.
2. COEFFICIENT ESTIMATES

Throughout this paper, we assume that $\varphi$ is an analytic function in $\mathbb{D}$ of the form

\begin{equation}
\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots \quad \text{with } B_1 > 0, \text{ and } B_2 \text{ is any real number.}
\end{equation}

**Definition 2.1.** Let $\lambda \geq 0$. A function $f \in \sigma$ given by (1.1) is in the class $R_{\sigma}(\lambda, \varphi)$, if it satisfies

\[(1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) < \varphi(z) \quad \text{and} \quad (1 - \lambda) \frac{F(w)}{w} + \lambda F'(w) < \varphi(w).\]

The class $R_{\sigma}(\lambda, \varphi)$ includes many earlier classes, which are mentioned below:

1. $R_{\sigma}(\lambda, (1 + (1 - 2\beta)z)/(1 - z)) = R_{\sigma}(\lambda, \beta)$ ($\lambda \geq 1; 0 \leq \beta < 1$) [10, Definition 3.1]
2. $R_{\sigma}(\lambda, ((1+z)/(1-z))^\alpha) = R_{\sigma,\alpha}(\lambda)$ ($\lambda \geq 1; 0 < \alpha \leq 1$) [10, Definition 2.1]
3. $R_{\sigma}(1, \varphi) = R_{\sigma}(\varphi)$ [2, p. 345].
4. $R_{\sigma}(1, (1 + (1 - 2\beta)z)/(1 - z)) = R_{\sigma}(\beta)$ ($0 \leq \beta < 1$) [24, Definition 2]
5. $R_{\sigma}(1, ((1+z)/(1-z))^\alpha) = R_{\sigma,\alpha}$ ($0 < \alpha \leq 1$) [24, Definition 1]

Our first result provides estimate for the coefficient $a_2$ of functions $f \in R_{\sigma}(\lambda, \varphi)$.

**Theorem 2.2.** If $f \in R_{\sigma}(\lambda, \varphi)$, then

\begin{equation}
|a_2| \leq \sqrt{\frac{B_1 + B_1 - B_2}{1 + 2\lambda}}
\end{equation}

**Proof.** Since $f \in R_{\sigma}(\lambda, \varphi)$, there exist two analytic functions $r, s : \mathbb{D} \rightarrow \mathbb{D}$, with $r(0) = 0 = s(0)$, such that

\begin{equation}
(1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) = \varphi(r(z)) \quad \text{and} \quad (1 - \lambda) \frac{F(w)}{w} + \lambda F'(w) = \varphi(s(z)).
\end{equation}

Define the functions $p$ and $q$ by

\begin{equation}
p(z) = \frac{1 + r(z)}{1 - r(z)} = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots \quad \text{and} \quad q(z) = \frac{1 + s(z)}{1 - s(z)} = 1 + q_1z + q_2z^2 + q_3z^3 + \cdots,
\end{equation}

or equivalently,

\begin{equation}
r(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left( p_1z + \left( p_2 - \frac{p_1^2}{2} \right)z^2 + \left( p_3 + \frac{p_1}{2}\left( \frac{p_1^2}{2} - p_2 \right) - \frac{p_1p_2}{2} \right)z^3 + \cdots \right)
\end{equation}

and

\begin{equation}
s(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left( q_1z + \left( q_2 - \frac{q_1^2}{2} \right)z^2 + \left( q_3 + \frac{q_1}{2}\left( \frac{q_1^2}{2} - q_2 \right) - \frac{q_1q_2}{2} \right)z^3 + \cdots \right).
\end{equation}

It is clear that $p$ and $q$ are analytic in $\mathbb{D}$ and $p(0) = 1 = q(0)$. Also $p$ and $q$ have positive real part in $\mathbb{D}$, and hence $|p_i| \leq 2$ and $|q_i| \leq 2$. In the view of (2.3), (2.5) and (2.6), clearly

\begin{equation}
(1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) = \varphi \left( \frac{p(z) - 1}{p(z) + 1} \right) \quad \text{and} \quad (1 - \lambda) \frac{F(w)}{w} + \lambda F'(w) = \varphi \left( \frac{q(w) - 1}{q(w) + 1} \right).
\end{equation}
On expanding (2.11) using (2.5) and (2.6), it is evident that

\[(2.8) \ \ \phi \left( \frac{p(z) - 1}{p(z) + 1} \right) = 1 + \frac{1}{2} B_1 p_1 z + \left( \frac{1}{2} B_1 (p_2 - \frac{1}{2} p_1^2) + \frac{1}{4} B_2 p_1^2 \right) z^2 + \cdots \]

and

\[(2.9) \ \ \phi \left( \frac{q(w) - 1}{q(w) + 1} \right) = 1 + \frac{1}{2} B_1 q_1 w + \left( \frac{1}{2} B_1 (q_2 - \frac{1}{2} q_1^2) + \frac{1}{4} B_2 q_1^2 \right) w^2 + \cdots \]

Since \(f \in \sigma\) has the Maclaurin series given by (1.1), a computation shows that its inverse \(F = f^{-1}\) has the expansion given by (1.2). It follows from (2.7), (2.8) and (2.9) that

\[(1 + \lambda)a_2 = \frac{1}{2} B_1 p_1, \]

\[(2.10) \ \ (1 + 2\lambda)a_3 = \frac{1}{2} B_1 \left( p_2 - \frac{1}{2} p_1^2 \right) + \frac{1}{4} B_2 p_1^2, \]

\[-(1 + \lambda)a_2 = \frac{1}{2} B_1 q_1, \]

\[(2.11) \ \ (1 + 2\lambda)(2a_2^2 - a_3) = \frac{1}{2} B_1 \left( q_2 - \frac{1}{2} q_1^2 \right) + \frac{1}{4} B_2 q_1^2. \]

Now (2.10) and (2.11) yield

\[(2.12) \ \ 8(1 + 2\lambda)a_2^2 = 2(p_2 + q_2)B_1 + (B_2 - B_1)(p_1^2 + q_1^2). \]

Finally an application of the known results, \(|p_i| \leq 2\) and \(|q_i| \leq 2\) in (2.12) yields the desired estimate of \(a_2\) given by (2.2).

\[\square\]

**Remark 2.3.** Let \(\phi(z) = (1 + (1 - 2\beta)z)/(1 - z), 0 \leq \beta < 1\). So \(B_1 = B_2 = 2(1 - \beta)\). When \(\lambda = 1\), Theorem 2.2 gives the estimate \(|a_2| \leq \sqrt{2(1 - \beta)/3}\) for functions in the class \(R_\sigma(\beta)\) which coincides with the result [29, Corollary 2] of Xu et al. In particular if \(\beta = 0\), then above estimate becomes \(|a_2| \leq \sqrt{2/3} \approx 0.816\) for functions \(f \in R_\sigma(0)\). Since the estimate on \(|a_2|\) for \(f \in R_\sigma(0)\) is improved over the conjectured estimate \(|a_2| \leq \sqrt{2} \approx 1.414\) for \(f \in \sigma\), the functions in \(R_\sigma(0)\) are not the candidate for the sharpness of the estimate in the class \(\sigma\).

**Definition 2.4.** A function \(f \in \sigma\) is in the class \(J^*(\phi)\), if it satisfies

\[\frac{zf'(z)}{f(z)} < \phi(z) \quad \text{and} \quad \frac{wF'(w)}{F(w)} < \phi(w).\]

Note that for a suitable choice of \(\phi\), the class \(J^*(\phi)\), reduces to the following well-known classes:

1. \(J^*(1 + (1 - 2\beta)z)/(1 - z) = J^*(\beta) \quad (0 \leq \beta < 1)\).
2. \(J^*(((1 + z)/(1 - z))^\alpha) = J_{\sigma,\alpha} \quad (0 < \alpha \leq 1)\).
Theorem 2.5. If \( f \in \mathcal{S}_\sigma^*(\varphi) \), then

\[
|a_2| \leq \min \left\{ \sqrt{B_1 + |B_2 - B_1|}, \frac{\sqrt{B_1^2 + B_1 + |B_2 - B_1|}}{2}, \frac{B_1\sqrt{B_1}}{\sqrt{B_1^2 + |B_1 - B_2|}} \right\}
\]

and

\[
|a_3| \leq \min \left\{ B_1 + |B_2 - B_1|, \frac{B_1^2 + B_1 + |B_2 - B_1|}{2}, R \right\},
\]

where

\[
R := \frac{1}{4} \left( B_1 + 3B_1 \max \left\{ 1; \left| \frac{B_1 - 4B_2}{3B_1} \right| \right\} \right).
\]

Proof. Since \( f \in \mathcal{S}_\sigma^*(\varphi) \), there are analytic functions \( r, s : \mathbb{D} \rightarrow \mathbb{D} \), with \( r(0) = 0 = s(0) \), such that

\[
\frac{zf'(z)}{f(z)} = \varphi(r(z)) \quad \text{and} \quad \frac{wF'(w)}{F(w)} = \varphi(s(z)).
\]

Let \( p \) and \( q \) be defined as in (2.4), then it is clear from (2.13), (2.5) and (2.6) that

\[
\frac{zf'(z)}{f(z)} = \varphi \left( p(z) - \frac{1}{p(z) + 1} \right) \quad \text{and} \quad \frac{wF'(w)}{F(w)} = \varphi \left( q(z) - \frac{1}{q(z) + 1} \right).
\]

It follows from (2.14), (2.8) and (2.9) that

\[
a_2 = \frac{1}{2} B_1 p_1,
\]

\[
2a_3 = \frac{B_1 p_1}{2} a_2 + \frac{1}{2} B_1 \left( p_2 - \frac{1}{2} p_1^2 \right) + \frac{1}{4} B_2 p_1^2,
\]

\[
-a_2 = \frac{1}{2} B_1 q_1
\]

and

\[
4a_2^2 - 2a_3 = -\frac{B_1 q_1}{2} a_2 + \frac{1}{2} B_1 \left( q_2 - \frac{1}{2} q_1^2 \right) + \frac{1}{4} B_2 q_1^2.
\]

The equations (2.15) and (2.17) yield

\[
p_1 = -q_1,
\]

\[
8a_2^2 = (p_1^2 + q_1^2) B_1^2
\]

and

\[
2a_2 = \frac{B_1 (p_1 - q_1)}{2}.
\]

From (2.15), (2.18) and (2.21), it follows that

\[
8a_2^2 = 2B_1 (p_2 + q_2) + (B_2 - B_1)(p_1^2 + q_1^2).
\]

Further a computation using (2.16), (2.18), (2.15) and (2.19) gives

\[
16a_2^2 = 2B_1^2 q_1^2 + 2B_1 (p_2 + q_2) + (B_2 - B_1)(p_1^2 + q_1^2).
\]
Similarly a computation using (2.16), (2.18), (2.21) and (2.20) yields
\[4(B_1^2 - B_2 + B_1)a_2^2 = B_1^3(p_2 + q_2).\]
Now (2.22), (2.23) and (2.24) yield the desired estimate on \(a_2\) as asserted in the theorem. To find estimate for \(a_3\) subtract (2.16) from (2.18), to get
\[-4a_3 = -4a_2^2 + \frac{B_1(q_2 - p_2)}{2}.\]
Now a computation using (2.23) and (2.25) leads to
\[16a_3 = 2B_1^2q_1^2 + 4B_2p_2 + (B_1 - B_2)(p_1^2 + q_1^2).\]
From (2.15), (2.16), (2.17) and (2.18), it follows that
\[4a_3 = \frac{B_1}{2}(3p_2 + q_2) + (B_2 - B_1)p_1^2\]
\[= \frac{B_1q_2}{2} + \frac{3B_1}{2} \left( p_2 - \frac{2(B_1 - B_2)}{3B_1}p_1^2 \right).\]
On applying the result of Keogh and Merkes \[15\] (see also \[20\]), that is for any complex number \(v\), \(|p_2 - vp_1^2| \leq 2\max\{1; |2v - 1|\}\), along with \(|q_2| \leq 2\) in (2.28), we obtain
\[4|a_3| \leq B_1 + 3B_1 \max \left\{ 1; \left| \frac{B_1 - 4B_2}{3B_1} \right| \right\}.\]
Now the desired estimate on \(a_3\) follows from (2.26), (2.27) and (2.29) at once. \(\square\)

**Remark 2.6.** If \(f \in \mathcal{S}_\sigma(\beta)\) \((0 \leq \beta < 1)\), then from Theorem 2.5 it is evident that
\[|a_2| \leq \min \left\{ \sqrt{2(1 - \beta)}, \sqrt{(1 - \beta)(3 - 2\beta)} \right\} = \left\{ \begin{array}{ll}
\sqrt{2(1 - \beta)}, & 0 \leq \beta \leq 1/2; \\
(1 - \beta)(3 - 2\beta), & 1/2 \leq \beta < 1. 
\end{array} \right.\]
Recall Brannan and Taha’s \[5\] Theorem 3.1 coefficient estimate, \(|a_2| \leq \sqrt{2(1 - \beta)}\) for functions \(f \in \mathcal{S}_\sigma(\beta)\), who claimed that their estimate is better than the estimate \(|a_2| \leq 2(1 - \beta)\), given by Robertson \[21\]. But their claim is true only when \(0 \leq \beta \leq 1/2\). Also it may noted that our estimate for \(a_2\) given in (2.30) improves the estimate given by Brannan and Taha \[5\] Theorem 3.1.

Further if we take \(\phi(z) = ((1 + z)/(1 - z))^\alpha, 0 < \alpha \leq 1\) in Theorem 2.5, we have \(B_1 = 2\alpha\) and \(B_2 = 2\alpha^2\). Then we obtain the estimate on \(a_2\) for functions \(f \in \mathcal{S}_\sigma(\beta, \alpha)\) as:
\[|a_2| \leq \min \left\{ \sqrt{4\alpha - 2\alpha^2}, \sqrt{\alpha^2 + 2\alpha}, \frac{2\alpha}{\sqrt{1 + \alpha}} \right\} = \frac{2\alpha}{\sqrt{1 + \alpha}}.\]
Note that Brannan and Taha \[5\] Theorem 2.1 gave the same estimate \(|a_2| \leq 2\alpha/\sqrt{1 + \alpha}\) for functions \(f \in \mathcal{S}_\sigma(\beta, \alpha)\).

**Definition 2.7.** A function \(f\) given by (1.1) is said to be in the class \(K_\sigma(\phi)\), if \(f\) and \(F\) satisfy the subordinations
\[1 + \frac{zf''(z)}{f'(z)} < \phi(z)\] and \[1 + \frac{wF''(w)}{F'(w)} < \phi(w).\]
Note that $K_{\varphi}(\frac{1 + (1 - 2\beta)z}{1 - z}) =: K_{\varphi}(\beta)$ ($0 \leq \beta < 1$).

**Theorem 2.8.** If $f \in K_{\varphi}(\varphi)$, then

$$|a_2| \leq \min \left\{ \sqrt{\frac{B_1^2 + B_1 + |B_2 - B_1|}{6}}, \frac{B_1}{2} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{B_1^2 + B_1 + |B_2 - B_1|}{6}, \frac{B_1(3B_1 + 2)}{12} \right\}.$$

**Proof.** Since $f \in K_{\varphi}(\varphi)$, there are analytic functions $r, s : \mathbb{D} \to \mathbb{D}$, with $r(0) = 0 = s(0)$, satisfying

$$1 + \frac{zf''(z)}{f'(z)} = \varphi(r(z)) \text{ and } 1 + \frac{wF''(w)}{F'(w)} = \varphi(s(z)).$$

Let $p$ and $q$ be defined as in (2.4), then it is clear from (2.31), (2.5) and (2.6) that

$$1 + \frac{zf''(z)}{f'(z)} = \varphi \left( \frac{p(z) - 1}{p(z) + 1} \right) \text{ and } 1 + \frac{wF''(w)}{F'(w)} = \varphi \left( \frac{q(z) - 1}{q(z) + 1} \right).$$

It follows from (2.32), (2.8) and (2.9) that

$$2a_2 = \frac{1}{2}B_1 p_1,$$

$$6a_3 = B_1 p_1 a_2 + \frac{1}{2}B_1 \left( p_2 - \frac{1}{2}p_1^2 \right) + \frac{1}{4}B_2 p_1^2,$$

$$-2a_2 = \frac{1}{2}B_1 q_1$$

and

$$6(2a_2^2 - a_3) = -B_1 q_1 a_2 + \frac{1}{2}B_1 \left( q_2 - \frac{1}{2}q_1^2 \right) + \frac{1}{4}B_2 q_1^2.$$  

Now (2.33) and (2.35) yield

$$p_1 = -q_1$$

and

$$4a_2 = \frac{B_1(p_1 - q_1)}{2}.$$  

From (2.34), (2.36), (2.37) and (2.33), it follows that

$$48a_2^2 = 2B_1^2 p_1^2 + 2B_1(p_2 + q_2) + (B_2 - B_1)(p_1^2 + q_1^2).$$

In view of $|p_1| \leq 2$ and $|q_1| \leq 2$ together with (2.38) and (2.39) yield the desired estimate on $a_2$ as asserted in the theorem. In order to find $a_3$, we subtract (2.34) from (2.36) and use (2.37) to obtain

$$-12a_3 = -12a_2^2 + \frac{B_1(q_2 - p_2)}{2}.$$
Now a computation using (2.39) and (2.40) leads to
\[(2.41)\]
\[-48a_3 = 2B_1^2p_1^2 - 4B_2p_2 + (B_1 - B_2)(p_1^2 + q_1^2).\]

From (2.38) and (2.40), it follows that
\[(2.42)\]
\[-12a_3 = \frac{B_1(q_2 - p_2)}{2} - \frac{3(p_1 - q_1)^2B_1^2}{16}.\]

Now (2.41) and (2.42) yield the desired estimate on \(a_3\) as asserted in the theorem. \(\square\)

**Remark 2.9.** If \(f \in K_\sigma(\beta)\) \((0 \leq \beta < 1)\), then theorem 2.8 gives
\[
|a_2| \leq \min \left\{ \sqrt{\frac{(1-\beta)(3-2\beta)}{3}}, 1-\beta \right\} = 1-\beta
\]
and
\[
|a_3| \leq \min \left\{ \frac{(1-\beta)(3-2\beta)}{3}, \frac{(1-\beta)(4-3\beta)}{3} \right\} = \frac{(1-\beta)(3-2\beta)}{3},
\]
which improves the Brannan and Taha’s [5, Theorem 4.1] estimates 
\[
|a_2| \leq \sqrt{1-\beta} \quad \text{and} \quad |a_3| \leq 1-\beta
\]
for functions \(f \in K_\sigma(\beta)\).

**Theorem 2.10.** Let \(f \in \sigma\) be given by (1.1). If \(f \in K(\varphi)\) and \(F \in R(\varphi)\), then
\[
|a_2| \leq \sqrt{\frac{3|B_1| + |B_2 - B_1|}{8}}
\]
and
\[
|a_3| \leq \frac{5|B_1| + |B_2 - B_1|}{12}.
\]

**Proof.** Since \(f \in K(\varphi)\) and \(F \in R(\varphi)\), there exist two analytic functions \(r,s : \mathbb{D} \to \mathbb{D}\), with \(r(0) = 0 = s(0)\), such that
\[(2.43)\]
\[1 + \frac{zf''(z)}{f'(z)} = \varphi(r(z)) \quad \text{and} \quad F'(w) = \varphi(s(z)).\]

Let the functions \(p\) and \(q\) are defined by (2.4). It is clear that \(p\) and \(q\) are analytic in \(\mathbb{D}\) and \(p(0) = 1 = q(0)\). Also \(p\) and \(q\) have positive real part in \(\mathbb{D}\), and hence \(|p_i| \leq 2\) and \(|q_i| \leq 2\).
Proceeding as in the proof of Theorem 2.2 it follow from (2.43), (2.8) and (2.9) that
\[
2a_2 = \frac{1}{2}B_1p_1,
\]
\[(2.44)\]
\[6a_3 - 4a_2^2 = \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2p_1^2,
\]
and
\[
-2a_2 = \frac{1}{2}B_1q_1
\]
\[(2.45)\]
\[3(2a_2^2 - a_3) = \frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2q_1^2.
\]
A computation using (2.44) and (2.45), leads to
\[
a_2^2 = \frac{2(p_2 + 2q_2)B_1 + (p_1^2 + 2q_1^2)(B_2 - B_1)}{32},
\]
and
\[
a_3 = \frac{2(3p_2 + 2q_2)B_1 + (3p_1^2 + 2q_1^2)(B_2 - B_1)}{48}.
\]
Now the desired estimates on \(a_2\) and \(a_3\), follow from (2.46) and (2.47) respectively.

Remark 2.11. If \(f \in \mathcal{K}(\beta)\) and \(F \in \mathcal{R}(\beta)\), then from Theorem 2.10 we see that
\[|a_2| \leq \sqrt{3(1 - \beta)/2} \quad \text{and} \quad |a_3| \leq 5(1 - \beta)/6.\]
In particular if \(f \in \mathcal{K}\) and \(F \in \mathcal{R}\), then \(|a_2| \leq \sqrt{3}/2 \approx 0.867\) and \(|a_3| \leq 5/6 \approx 0.833\).

Theorem 2.12. Let \(f \in \sigma\) be given by (1.1). If \(f \in \mathcal{K}^*(\phi)\) and \(F \in \mathcal{R}(\phi)\), then
\[|a_2| \leq \frac{\sqrt{5[B_1 + |B_2 - B_1|]}}{3}, \quad \text{and} \quad |a_3| \leq \frac{7[B_1 + |B_2 - B_1|]}{9}.\]

Proof. Since \(f \in \mathcal{K}^*(\phi)\) and \(F \in \mathcal{R}(\phi)\), there exist two analytic functions \(r,s : \mathbb{D} \to \mathbb{D}\), with \(r(0) = 0 = s(0)\), such that
\[
z f'(z) = \phi(r(z)) \quad \text{and} \quad F'(w) = \phi(s(z)).
\]
Let the functions \(p\) and \(q\) be defined as in (2.4). Then
\[
z f'(z) = \phi\left(\frac{p(z) - 1}{p(z) + 1}\right) \quad \text{and} \quad F'(w) = \phi\left(\frac{q(w) - 1}{q(w) + 1}\right).
\]
It follow from (2.49), (2.8) and (2.9) that
\[a_2 = \frac{1}{2}B_1p_1,
\]
\[2a_3 - a_2^2 = \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2p_1^2,
\]
\[-2a_2 = \frac{1}{2}B_1q_1,
\]
\[3(2a_2^2 - a_3) = \frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2q_1^2.
\]
A computation using (2.50) and (2.51) leads to
\[
a_2^2 = \frac{2(3p_2 + 2q_2)B_1 + (3p_1^2 + 2q_1^2)(B_2 - B_1)}{36},
\]
and
\[
a_3 = \frac{2(6p_2 + 2q_2)B_1 + (6p_1^2 + 2q_1^2)(B_2 - B_1)}{36}.
\]
Now the bounds for \(a_2\) and \(a_3\) are obtained from (2.52) and (2.53) respectively using the fact that \(|p_1| \leq 2\) and \(|q_1| \leq 2\).
Remark 2.13. If $f \in S^*(\beta)$ and $F \in \mathcal{R}(\beta)$, then from Theorem 2.12 it is easy to see that

$$|a_2| \leq \sqrt{10(1 - \beta)}/3 \quad \text{and} \quad |a_3| \leq 14(1 - \beta)/9.$$ 

In particular if $f \in S^*$ and $F \in \mathcal{R}$, then $|a_2| \leq \sqrt{10}/3 \approx 1.054$ and $|a_3| \leq 14/9 \approx 1.56$.

**Theorem 2.14.** Let $f \in \sigma$ given by (1.1). If $f \in S^*(\varphi)$ and $F \in \mathcal{K}(\varphi)$, then

$$|a_2| \leq \sqrt{B_1 + |B_2 - B_1|}/2$$

and

$$|a_3| \leq \frac{B_1 + |B_2 - B_1|}{2}.$$ 

**Proof.** Assuming $f \in S^*(\varphi)$ and $F \in \mathcal{K}(\varphi)$ and proceeding in the similar way as in the proof of Theorem 2.10 it is easy to see that

$$a_2 = \frac{1}{2}B_1p_1,$$

(2.54) 

$$3a_3 - a_2^2 = \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2p_1^2,$$

$$-2a_2 = \frac{1}{2}B_1q_1,$$

(2.55) 

$$8a_2^2 - 6a_3 = \frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2q_1^2.$$ 

A computation using (2.54) and (2.55) leads to

(2.56) 

$$a_2 = \frac{2(2p_2 + q_2)B_1 + (2p_1^2 + q_1^2)(B_2 - B_1)}{24}$$

and

(2.57) 

$$a_3 = \frac{2(8p_2 + q_2)B_1 + (8p_1^2 + q_1^2)(B_2 - B_1)}{72}.$$ 

Now using the result $|p_i| \leq 2$ and $|q_i| \leq 2$, the estimates on $a_2$ and $a_3$ follow from (2.56) and (2.57) respectively. \hfill \Box

Remark 2.15. Let $f \in S^*(\beta)$ and $F \in \mathcal{K}(\beta)$, $0 \leq \beta < 1$. Then from Theorem 2.14 it is easy to see that

$$|a_2| \leq \sqrt{1 - \beta} \quad \text{and} \quad |a_3| \leq 1 - \beta.$$ 

In particular if $f \in S^*$ and $F \in \mathcal{K}$, then $|a_2| \leq 1$ and $|a_3| \leq 1$.

**Acknowledgements.** The research is supported by a grant from University of Delhi.
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