A goal oriented error estimator and mesh adaptivity for sea ice simulations

Carolin Mehlmann∗ Thomas Richter†

February 12, 2020

For the first time we introduce an error estimator for the numerical approximation of the equations describing the dynamics of sea ice. The idea of the estimator is to identify different error contributions coming from spatial and temporal discretization as well as from the splitting in time of the ice momentum equations from further parts of the coupled system. Errors are measured in user specified functional outputs like the total sea ice extent or divergence. The error estimator is based on the dual weighted residual method that asks for the solution of an additional dual problem for obtaining sensitivity information. Estimated errors can be used to validate the accuracy of the solution and, more relevant, to reduce the discretization error by guiding an adaptive algorithm that optimally balances the mesh size and the time step size to increase the efficiency of the simulation.

1 Introduction

We consider the viscous-plastic (VP) sea model, that was introduced by Hibler in 1979 and which is still one of the most widely used sea ice rheologies as detailed by Stroeve et al. (2014). The model includes strong nonlinearities such that solving the sea ice dynamics at high resolutions is extremely costly and good solvers are under active research. Mostly, solutions to the VP model are approximated by iterating an elastic-viscous-plastic (EVP) modification of the model that was introduced by Hunke and Dukowicz (1997) and that allows for explicit sub-cycling. Alternatively the VP model is tackled directly with simple Picard iterations as described by Hibler (1979) or solved with Newton-like methods as described by Lemieux et al. (2010) or Mehlmann and Richter (2017b). All approaches are not satisfactory as they are extremely expensive and often are not able

∗Max-Planck-Institute of Meteorology, Bundesstrasse 53, 22176 Hamburg, Germany, caro-lin.mehlmann@mpimet.mpg.de
†Institute of Analysis and Numerics, Otto-von-Guericke University Magdeburg, Universitätsplatz 2, 39106 Magdeburg, Germany, thomas.richter@ovgu.de
to give an accurate solution in reasonable computational time. It is therefore of utmost
important to reduce the complexity of the computations, e.g. by using coarse meshes
and large time step sizes, as long as this does not deteriorate the accuracy assumptions.

We derive an error estimator that identifies the errors coming from spatial and temporal
discretization. Furthermore, the error estimator allows for a localization of the error to
each element and each time step such that local step sizes can be adjusted. This goal
oriented error estimator for the viscous-plastic sea model is an extension of the dual
weighted residual method that was introduced by Becker and Rannacher (2001). The aim
of the estimator is to identify discretization errors \( J(U) - J(U_{k,h}) \) between the unknown
exact solution \( U \) and the numerical approximation \( U_{k,h} \), where \( k \) indicates the temporal
and \( h \) the spatial discretization parameter, in functionals \( J(\cdot) \). These functionals can be
any measures of interest, e.g. the average sea ice extent in a certain time span
\[
J(U_{k,h}) = \int_{t_1}^{t_2} \int_{\Omega} A_{k,h}(x,y,t) \, d(x,y) \, dt. \tag{1}
\]

We denote by \( A_{k,h} \) the ice concentration (one component of the solution \( U_{k,h} \) which will
be introduced later), by \( \Omega \subset \mathbb{R}^2 \) the spatial domain of interest and by \([t_1,t_2]\) the time
span of interest, e.g. the summer months. The error estimator will give approximations
to \( J(U) - J(U_{k,h}) \) which can be attributed to a spatial error, a temporal error and to
a splitting error - coming from partitioning the system into momentum equation and
balance laws, which is the standard procedure in sea ice numerics, see Lemieux et al.
(2014). The estimation of errors in space and time for parabolic problems was discussed
by Schmich and Rannacher (2012). Here, we extend the application to the VP model
and additionally consider the splitting error. Lipscomb et al. (2007) pointed out that
decoupling the system in time can lead to a numerical unstable solution such that a small
time step is required to achieve a stable approximation. Lemieux et al. (2014) introduced
an implicit-explicit time integration method, which resolves this issues and allows the use
of larger time steps. The error estimator will be able to predict the accuracy implications
of this temporal splitting.

The convergence of approximations of the sea ice velocity on different spatial resolu-
tions is studied by Williams and Tremblay (2018) for a one dimensional test case. The
authors observe that the simulated velocity field depends on the spatial resolution and
found that the mean sea ice drift speed rises by 32% by increasing resolution from 40 km
to 5 km. The temporal and spatial scaling properties of the mean deformation rate and
the sea ice thickness are studied by Hutter et al. (2018).

The paper is structured as follows. In Section 2 we start by presenting the sea ice
model in strong and variational formulation which is required for the Galerkin finite
element discretization in space and time. Further give details on the partitioned solution
approach. In Section 3 we derive the goal oriented error estimator for the sea ice model
and describe its numerical realization. We numerically analyse the error estimator in
Section 4 and conclude in Section 5. For better readability we keep the mathematical
formulation as simple as possible and refer to the literature for details.
| Parameter   | Definition            | Value       |
|-------------|-----------------------|-------------|
| $\rho_{\text{ice}}$ | sea ice density       | 900 kg/m$^3$ |
| $\rho_{\text{atm}}$ | air density           | 1.3 kg/m$^3$ |
| $\rho_{\text{ocean}}$ | water density         | 1026 kg/m$^3$ |
| $C_{\text{atm}}$ | air drag coefficient  | $1.2 \cdot 10^{-3}$ |
| $C_{\text{ocean}}$ | water drag coefficient| $5.5 \cdot 10^{-3}$ |
| $f_c$      | Coriolis parameter    | $1.46 \cdot 10^{-4}$ s$^{-1}$ |
| $P^*$    | ice strength parameter| $27.5 \cdot 10^3$ N/m$^2$ |
| $C$       | ice concentration parameter| 20           |

Table 1: Physical parameters of the momentum equation.

2 Model Description and Discretization

Let $\Omega \subset \mathbb{R}^2$ be the spatial domain. We denote the time interval of interest by $I = [0, T]$.

The dynamics of sea ice is described by three variables, the sea ice concentration $A$, the sea ice thickness $H$ and the sea ice velocity $\mathbf{v}$, such that the complete solution is given by $U = (\mathbf{v}, A, H)$. The VP sea ice model as introduced by Hibler (1979) consists of the momentum equation and the balance laws

$$\rho_{\text{ice}} H \partial_t \mathbf{v} + f_c \mathbf{e}_r \times \mathbf{v} - \text{div } \sigma - \tau(\mathbf{v}) - \rho_{\text{ice}} H f_c \mathbf{e}_r \times \mathbf{v}_{\text{ocean}} = 0,$$

$$\partial_t A + \text{div } (\mathbf{v} A) = S_A$$

$$\partial_t H + \text{div } (\mathbf{v} H) = S_H$$

with $0 \leq H$ and $0 \leq A \leq 1$. The forcing term $\tau(\mathbf{v})$ models ocean and atmospheric traction

$$\tau(\mathbf{v}) = C_{\text{ocean}} \rho_{\text{ocean}} \Vert \mathbf{v}_{\text{ocean}} - \mathbf{v} \Vert^2 (\mathbf{v}_{\text{ocean}} - \mathbf{v}) + C_{\text{atm}} \rho_{\text{atm}} \Vert \mathbf{v}_{\text{atm}} \Vert^2 \mathbf{v}_{\text{atm}},$$

with the ocean velocity $\mathbf{v}_{\text{ocean}}$ and the wind velocity $\mathbf{v}_{\text{atm}}$. By $\rho_{\text{ice}}$ we denote the ice density, by $f_c$ the Coriolis parameter, by $\mathbf{e}_r$ the radial ($z$-direction) unit vector. Following Coon (1980) we have replaced surface height effects by the approximation $g \nabla \tilde{H} = -f_c \mathbf{e}_r \times \mathbf{v}_{\text{ocean}} \approx 0$. In this paper we focus on the dynamical part of the sea ice model such that we neglect thermodynamic effects and set $S_A = 0$ and $S_H = 0$.

The system of equations (2) is closed by Dirichlet conditions $\mathbf{v} = 0$ on the boundary of the domain and initial conditions $H(0) = H^0$, $A(0) = A^0$ and $\mathbf{v}(0) = \mathbf{v}^0$ for ice thickness, concentration and velocity at time $t = 0$.

Finally, we present the nonlinear viscous-plastic rheology which relates the stress $\sigma$ to the strain rate

$$\dot{\epsilon} = \frac{1}{2} \left( \nabla \mathbf{v} + \nabla \mathbf{v}^T \right), \quad \dot{\epsilon}' := \dot{\epsilon} - \frac{1}{2} \text{tr}(\dot{\epsilon}) I,$$
where $\text{tr}(\cdot)$ is the trace. The rheology is given by

$$\sigma = 2\eta \dot{\epsilon}' + \zeta \text{tr}(\dot{\epsilon}) I - \frac{P}{2} I, \quad (3)$$

with the viscosities $\eta$ and $\zeta$, given by $\eta = \frac{1}{4} \zeta$ and

$$\zeta = \frac{P}{2 \Delta(\dot{\epsilon})}, \quad \Delta(\dot{\epsilon}) := \sqrt{\frac{1}{2} \dot{\epsilon}' : \dot{\epsilon}' + \text{tr}(\dot{\epsilon})^2 + \Delta_{\text{min}}^2}. \quad (4)$$

$\Delta_{\text{min}} = 2 \cdot 10^{-9}$ is the threshold that describes the transition between the viscous and the plastic regime. The ice strength $P$ in (3) is modeled as

$$P(H, A) = P^* H \exp \left( - C (1 - A) \right), \quad (5)$$

with the constant $C = 20$. All problem parameters are collected in Table 1.

### 2.1 Variational formulation and discretization

The dual weighted residual estimator by Becker and Rannacher (2001) relies on a variational formulation of the system of partial differential equations in space and time. Galerkin discretizations are based on approximating the differential equation by restricting the search space of solutions to a finite dimensional space. In contrast to difference methods, where the discretization is based on an approximation of the differential operators, the equation itself is not changed. While the finite element method is a well established Galerkin method for the spatial discretization, time Galerkin discretizations are not widely used. Before describing the time discretization in detail we want to clarify that the Galerkin approach has to be considered mostly as a mathematical concept. The resulting discretization scheme is nearly identical to the backward Euler method and this is how we actually solve the problem. Nevertheless, the reformulation into a Galerkin approach is essential to derive the error estimate.

To start with, we multiply the three equations (2) with test functions $\phi$, $\psi_A$ and $\psi_H$ and integrate over the temporal interval $[0, T]$ and the spatial domain $\Omega$. To shorten the notation we introduce the $L^2$-inner products in space and in the space-time domain

$$(f, g) := \int_\Omega f(x) \cdot g(x) \, dx,$$

$$(f, g) := \int_0^T \int_\Omega f(x,t) \cdot g(x,t) \, dx \, dt.$$ 

Then, the variational formulation of the system of sea ice equations is given by the relation

$$\langle \rho_{\text{ice}} H \partial_t \mathbf{v} + f_c \mathbf{e}_r \times \mathbf{v} - \rho_{\text{ice}} H f_c \mathbf{e}_r \times \mathbf{v}_{\text{ocean}}, \phi \rangle + \langle \mathbf{\tau}(\mathbf{v}), \phi \rangle + \langle \sigma, \nabla \phi \rangle$$

$$+ \langle \partial_t A + \text{div} (\mathbf{v} A), \psi_A \rangle + \langle \partial_t H + \text{div} (\mathbf{v} H), \psi_H \rangle = 0 \quad (6)$$
for all test functions $\phi \in \mathcal{V}$, $\psi_A \in \mathcal{V}^A$ and $\psi_H \in \mathcal{V}^H$, where $\mathcal{V}, \mathcal{V}^A, \mathcal{V}^H$ are suitable function spaces.

The finite element discretization of (6) would consist of replacing the function spaces by discrete subspaces of finite element functions, e.g. $\mathbf{v}_h \in \mathcal{V}_h \subset \mathcal{V}$, where $\mathcal{V}_h$ is the space of piecewise linear functions defined on a triangulation $\Omega_h$ of the domain $\Omega$.

In time, we proceed in a similar way and replace the solution $U = (\mathbf{v}, A, H)$ and the test function $\Phi = (\phi, \psi_A, \psi_H)$ by functions that are piecewise constant on the time partitioning

$$0 = t_0 < t_1 < \cdots < t_N = T, \quad I_n := (t_n, t_{n-1}).$$

The time step size is denoted by $k_n := t_n - t_{n-1}$. For simplicity we assume that $k = k_n$ is constant for all the time steps. Finally, the velocity is found in the space-time Galerkin space

$$\mathbf{v}_{k,h} := \{\mathbf{v}_{k,h} \in \mathcal{V}, \mathbf{v}_{k,h} |_{I_n} \in \mathcal{V}_h\}.$$ 

On each subinterval $I_n = (t_{n-1}, t_n]$ the velocity

$$\mathbf{v}_{n,h} := \mathbf{v}_{k,h} |_{I_n}$$

is a constant function in time with values in the discrete finite element space $\mathcal{V}_h$. Similar constructions are done for the ice concentration $A_{k,h} \in \mathcal{V}_{k,h}$ and the ice thickness $H_{k,h} \in \mathcal{V}_{k,h}^H$. Since these discrete spaces are discontinuous in time (but the real solution $U = (\mathbf{v}, A, H)$ is continuous in time), they are no subspaces of $\mathcal{V}, \mathcal{V}_A, \mathcal{V}_H$. The method is therefore called **discontinuous Galerkin time discretization**, in short: $dG(0)$, see [Schmich and Vexler (2008)](Schmich2008).

Similar to the **finite volume method**, which can be seen as the spatial counterpart to the $dG(0)$ discretization we must add numerical fluxes in time to guarantee continuity of the solution in the limit $k \to 0$. Here, we add **jump terms** at each discrete time steps

$$\sum_{n=1}^{N} \left\{ (\rho_{\text{ice}} H(t_{n-1})^+[\mathbf{v}]_{n-1}, \phi(t_{n-1})^+) 
\quad + ([A]_{n-1}, \psi_A(t_{n-1})^+) + ([H]_{n-1}, \psi_H(t_{n-1})^+) \right\}, \quad (8)$$

where we denote by $\phi(t_{n-1})^+$ the right-sided value of the (possibly) discontinuous function $\phi$, by $\phi(t_{n-1})^-$ the value from the left and by $[\mathbf{v}]_{n-1} = \mathbf{v}(t_{n-1})^+ - \mathbf{v}(t_{n-1})^-$ the jump at time $t = t_{n-1}$. The real solution is continuous and it holds $[\mathbf{v}]_n = 0$ and $[A]_n = [H]_n = 0$. If the jump is not zero, the terms in (8) can be regarded as penalty terms.

To shorten notation we combine $U = (\mathbf{v}, h)$ with $h = (A, H)$ and $\Phi = (\phi, \psi)$ with $\psi = (\psi_A, \psi_H)$ and we assume that these functions come from function spaces $U \in \mathcal{X} := \mathcal{V} \times \mathcal{V}^A \times \mathcal{V}^H$ and $\Phi \in \mathcal{X}$. The exact notation of all function spaces is introduced in [Mehlmann](Mehlmann).
and Richter (2017a). Then, the variational formulation (6) plus the additional jump terms (8) can be written in an abstract notation by introducing the form \( B(U)(\Phi) \) which simply collects all the integrals from (6) and the jumps from (8):

\[
U \in \mathcal{X}, \quad B(U)(\Phi) = 0 \quad \forall \Phi \in \mathcal{X}.
\]  

The analytical solution \( U \) is described by this variational formulation. All jumps vanish such that the solution is continuous in time. The discrete solution \( U_{k,h} \) is given by restricting (9) to the finite dimensional discrete space \( X_{k,h} \):

\[
U_{k,h} \in X_{k,h}, \quad B(U_{k,h})(\Phi_{k,h}) = 0 \quad \forall \Phi_{k,h} \in X_{k,h}.
\]  

In time, these discrete functions are piecewise constant, as defined in (7). In space, the discretization is quickly described: We define a conforming finite element space \( V_h \). In our implementation we use the space of piecewise bi-linear functions defined on a quadrilateral mesh \( \Omega_h \) of the domain \( \Omega \). Danilov et.al. (2015) consider standard linear finite elements on triangular meshes.

The goal oriented error estimator relies heavily on the concept of Galerkin discretizations: the discrete solution \( U_{k,h} \) is found by limiting (9) to the finite dimensional space \( X_{k,h} \) while the variational form itself is unchanged. This is in contrast to finite difference schemes that are based on discrete approximations of the differential operators. The most important tool in the analysis of Galerkin discretizations is the Galerkin orthogonality: the discretization error \( U - U_{k,h} \) is orthogonal on the Galerkin space, i.e.

\[
B(U)(\Phi_{k,h}) - B(U_{k,h})(\Phi_{k,h}) = 0 \quad \forall \Phi_{k,h} \in \mathcal{X}_h,
\]  

which directly follows from subtracting (10) from (9), since \( X_{k,h} \subset \mathcal{X} \) is a subspace and the real solution \( U \in \mathcal{X} \) satisfies (9) also, if tested with discrete functions \( \Phi_{k,h} \in X_{k,h} \).

### 2.2 Equivalence to the backward Euler method

The Galerkin formulation (10) appears rather abstract. We will show that it coincides with the backward Euler method that is most often used for solving the sea ice dynamics. On each interval \( I_n = (t_{n-1}, t_n] \) the solution \( U_{k,h}|_{I_n} =: U_{n,h} \in V_h \) is constant in time, as defined in (7). As it is has discontinuities at \( t_{n-1} \) and \( t_n \) there is no natural coupling to the adjacent intervals. The only coupling between intervals is introduced by the jump terms (8), where \( U_{n,h} \) couples to \( U_{n-1,h} \). These jumps \( [v_{k,h}]_{n-1} = v_{n,h} - v_{n-1,h} \) take the role of the time derivative. There is no coupling of \( I_n \) to \( I_{n+1} \) or the previous intervals \( I_1, \ldots, I_{n-2} \), as shown in Figure 1. Since \( U_{n,h} \) is constant on \( (t_{n-1}, t_n] \) it holds that \( \partial_t U_{n,h} = 0 \). Further, all temporal integrals can be exactly evaluated with the box rule such that the space-time Galerkin formulation corresponds to the time-stepping scheme.
Figure 1: Visualization of the temporal jump of piecewise constant functions at time point $t = t_n$.

for $n = 1, 2, \ldots, N$

$$(v_n - v_{n-1}, \phi_h) + k(f_e \mathbf{e}_r \times v_n - \rho_{ice} H_n f_e \mathbf{e}_r \times v_{ocean}, \phi_h) + k(\mathbf{\tau}(v_n), \phi_h) + (\boldsymbol{\sigma}(v_n, A_n, H_n), \nabla \phi_h) = 0$$

(12)

Division by the step size $k$ reveals the classical backward Euler scheme which is standard in sea ice dynamics as described by [Lemieux et al. (2014)](https://doi.org/10.1002/mma.2859). The Galerkin formulation (10) has to be considered as a mere concept of notation. Using the Galerkin formulation be essential for derivation of the error estimator.

**Remark 2.1** The transport equations for $A$ and $H$ are under the constraints $0 \leq H \leq 0$ and $0 \leq A \leq 1$. Without proper modeling of the thermodynamics there is no natural effect to guarantee these constraints. We will therefore apply a simple modification of the discrete system to take care of $A \leq 1$. We introduce the right hand side

$$(A_n - A_{n-1}, \phi_A) + k(\text{div}(v_n A_n), \psi_A) = -\min\{0, 1 - A_n\}, \phi_A)$$

that only gets active if $A_n > 1$ and that will then force $A_n$ below one.

2.3 Partitioned solution approach

The discrete formulation (10) naturally splits into time-steps $t_{n-1} \rightarrow t_n$ as shown in (12). The three components velocity $v$, ice concentration $A$ and ice thickness $H$ however are coupled. It is standard to apply a partitioned solution approach in every time step, either by first solving the momentum equation for the sea ice velocity followed by the balance laws, or vice versa, see [Lemieux et al. (2014)](https://doi.org/10.1002/mma.2859). We start with the momentum equation
and replace the dependency of the ice mass $\rho_{\text{ice}} H$ and the ice strength $P(H, A)$ given in (5) by the ice concentration and thickness from the last time step. To realize this decoupling within a Galerkin method we introduce the projection operator $\mathcal{S} : \mathcal{X} \rightarrow \mathcal{X}$ that projects $A \in \mathcal{V}^A$ (or $H$, respectively) on the interval $I_n = (t_{n-1}, t_n]$ onto the value $H(t_{n-1}) \cdot$ For discrete functions this corresponds to $\mathcal{S}(H_n) = H_{n-1}$ and $\mathcal{S}(A_n) = A_{n-1}$.

In terms of the backward Euler scheme, which is used for simulation, this means replacing $H_n$ and $A_n$ in the momentum equation (terms tested with $\phi_h$) by the values $H_{n-1}$ and $A_{n-1}$ from the previous step. For the Galerkin formulation this calls for a slight modification of the variational formulation $B(U)(\Phi) = 0$, namely the introduction of the projection operator $\mathcal{S}$ in the momentum part

$$
B_s(U)(\Phi) = \sum_{n=1}^{N} (\rho_{\text{ice}} \mathcal{S}(H)(t_{n-1}) \cdot [v]_{n-1}, \phi(t_{n-1}) \cdot \\
+ \langle\rho_{\text{ice}} \mathcal{S}(H) \partial_t v + f e \times v - \rho_{\text{ice}} \mathcal{S}(H) f e \times v_{\text{ocean}}, \phi \rangle \\
+ \langle\tau(v), \phi \rangle + \langle\sigma(v, \mathcal{S}(A), \mathcal{S}(H)), \nabla \phi \rangle \\
+ \sum_{n=1}^{N} ([A]_{n-1}, \phi_A(t_{n-1}) \cdot \\
+ \sum_{n=1}^{N} ([H]_{n-1}, \phi_H(t_{n-1}) \cdot \\
+ \langle\partial_t A + \text{div} (v A), \psi_A \rangle \\
+ \langle\partial_t H + \text{div} (v H), \psi_H \rangle \\
+ \langle\min\{0, 1 - A\}, \phi_A \rangle. 
$$

Once again, we indicate this variational form only for the formulation of the error estimator, the solution itself is computed by the backward Euler scheme (12) by replacing $A_n, H_n$ in the momentum equations by the previous approximations $A_{n-1}, H_{n-1}$.

For the following we define the splitting error that is introduced by partitioning the sea ice system in time as the difference between both variational formulations

$$
\beta(U)(\Phi) := B_s(U)(\Phi) - B(U)(\Phi). 
$$

3 Goal oriented error estimation

In this section, we derive a goal oriented error estimator for partitioned solution approaches. The new error estimator will be based on concepts of the dual weighted residual (DWR) method introduced by Becker and Rannacher (2001). The DWR estimator can be easily applied to all problems given in a variational Galerkin formulation and it has been applied to various problems for error estimation in space and time such as fluid dynamics (Schmich and Rannacher (2012)) or fluid-structure interactions (Richter, 2017, Chapter 8). The concept can also be extended to the estimation of modeling errors, see Braack and Ern (2003), an approach that has similarities to the treatment of partitioned solution schemes considered within this work.

Before deriving the error estimator for decoupled systems, we briefly recapitulate the idea of the dual weighted residual method.
3.1 The dual weighted residual method

To illustrate the idea of the DWR estimator we consider the heat equation \( \partial_t u - \Delta u = f \) in a space-time variational formulation \( A(u, \phi) = F(\phi) \) with

\[
A(u, \phi) = \sum_{n=1}^{N} ([u]_{n-1}, \phi(t_{n-1}^+)) + \langle \partial_t u, \phi \rangle + \langle \nabla u, \nabla \phi \rangle
\]

\[
F(\phi) = \langle f, \phi \rangle.
\]

(15)

Discretization is by means of the \( dG(0) \) method in time (which corresponds to the implicit Euler scheme) and finite elements in space such that we have to deal with the two problems

\[
u \in \mathcal{V}
\]
\[
A(u, \phi) = F(\phi) \quad \forall \phi \in \mathcal{V}
\]
\[
u_{k,h} \in \mathcal{V}_{k,h}
\]
\[
A(u_{k,h}, \phi_{k,h}) = F(\phi_{k,h}) \quad \forall \phi_{k,h} \in \mathcal{V}_{k,h}
\]

with the Galerkin subspace \( \mathcal{V}_{k,h} \subset \mathcal{V} \).

We start by defining an error measure, e.g. the average temperature at final time \( T \) described by the error functional

\[
J(u) = \int_{\Omega} u(T, x) \, dx.
\]

Key to error estimation with the dual weighted residual method is the introduction of the dual solution \( z \in \mathcal{V} \), which is given as solution to the dual problem

\[
z \in \mathcal{V}
\]
\[
A(\phi, z) = J(\phi) \quad \forall \phi \in \mathcal{V}
\]

(16)

and its discretization

\[
z_{k,h} \in \mathcal{V}_{k,h}
\]
\[
A(\phi_{k,h}, z_{k,h}) = J(\phi_{k,h}) \quad \forall \phi_{k,h} \in \mathcal{V}_{k,h}.
\]

Computing all integrals, the dual variational formulation reveals the implicit Euler method of a problem that runs backward in time.

\[
(z_N, \phi) + k(\nabla z_N, \nabla \phi) = k(1, \phi)
\]

(17)

\[
(z_n - z_{n+1}, \phi) + k(\nabla z_n, \nabla \phi) = 0 \quad n = N - 1, \ldots, 1.
\]

(18)

Instead of having an “initial value” at time \( T \) we solve an initial problem (17) for \( z_N \).

With the dual solutions \( z \) and \( z_{k,h} \) the error in the functional is given by

\[
J(u - u_{k,h}) = A(u - u_{k,h}, z),
\]

which directly follows by inserting \( \phi := u - u_{k,h} \in \mathcal{V} \) in (16). This is allowed since \( u_{k,h} \in \mathcal{V}_{k,h} \subset \mathcal{V} \) comes from a Galerkin subspace. The error estimator is then completed by using Galerkin orthogonality (11) to introduce an interpolation \( I_{k,h} z \in \mathcal{V}_{k,h} \) of the dual solution and using that \( A(u, z - I_{k,h} z) = F(z - I_{k,h} z) + A(u_{k,h}, z - I_{k,h} z) = F(z - I_{k,h} z) - A(u_{k,h}, z - I_{k,h} z). \)
This relation is called error identity. It cannot be exactly evaluated in practical application since it involves the unknown dual solution $u \in V$. However, there are well established strategies for approximating the local interpolation error $z - I_{k,h} z$ and $u - I_{k,h} u$ by reconstruction techniques, see Richter and Wick (2015). These interpolation errors act as weights that focus the residuals on areas with high sensitivity to the error functional.

The standard feedback-approach for running adaptive simulations based on the DWR method is as follows:

**Algorithm 3.1** Let $0 = t_0 < t_1 < \cdots < t_N = T$ be the initial time mesh, $\Omega_h$ the initial spatial mesh. Let $V_{k,h}$ be the resulting space-time function space.

1. Solve the primal problem $u_{k,h} \in V_{k,h}$
2. Solve the dual problem $z_{k,h} \in V_{k,h}$
3. Approximate the weights $u - I_{k,h} u$, $z - I_{k,h} z$
4. Evaluate the error identity (19)
5. Stop, if $|J(u) - J(u_{k,h})|$ is sufficiently small
6. Otherwise use the error estimator to adaptively refine the spatial and temporal discretization and restart with a finer function space $V'_{k,h}$ on the refined mesh $\Omega_{h'}$.

In Step 3, the approximation of the weights is the delicate part of the error estimator. Here, we must replace the unknown exact solutions $u$ and $z$. For heuristic approaches we refer to Becker and Rannacher (2001), or in particular Schmich and Vexler (2008) or Meidner and Richter (2014) for time discretizations.

Application of the DWR method will always require some numerical overhead, mainly by the computation of the auxiliary dual problem. It turns out that the dual problem is always a linear problem, also in the case of the fully nonlinear sea ice problem. In the following sections we describe the steps that are required for applying the DWR estimator to the sea ice model in a partitioned solution framework. We refer to Mehlmann (2019) for the full derivation.

### 3.1.1 Extension to the nonlinear case

For linear problems, the dual problem is just the transposed of the primal problem, compare (16). The general nonlinear setting requires a more involved approach. Here we cast the error estimation problem into the framework of optimization problems. We minimize the functional value $J(U)$ under the constraint that $U$ satisfies the sea ice problem, $B(U)(\Phi) = 0$. We introduce the Lagrangian

$$L(U, Z) = J(U) - B(U)(Z),$$

where $Z$ takes the role of the Lagrange multiplier. We then obtain the nonlinear error identity (since $B(U)(Z) = 0$ and $B(U_{k,h})(Z_{k,h}) = 0$)

$$J(U) - J(U_{k,h}) = L(U, Z) - L(U_{k,h}, Z_{k,h}).$$

10
The right hand side is written as an integral

\[ J(U) - J(U_{k,h}) = \int_0^1 \frac{d}{ds}L(U_{k,h} + s(U - U_{k,h}), Z_{k,h} + s(Z - Z_{k,h})) \, ds, \]

and this integral is approximated with the trapezoidal rule

\[ J(U) - J(U_{k,h}) = \frac{1}{2} L'(U, Z)(U - U_{k,h}, Z - Z_{k,h}) + \frac{1}{2} L'(U_{k,h}, Z_{k,h})(U - U_{k,h}, Z - Z_{k,h}) + R(x_{k,h}, e), \]

\[ R(x_{k,h}, e) = \int_0^1 L'''(x_{k,h} + \lambda e)(e, e, e) \cdot \lambda(1 - \lambda) d\lambda, \]

with \( x_{k,h} = (U_{k,h}, Z_{k,h}) \) and \( e = (U - U_{k,h}, Z - Z_{k,h}) \). The third order remainder is omitted in practical application. The derivatives \( L' \) appearing in (22) are the directional derivatives and if we consider the definition of the Lagrangian (20) they take the form

\[ L'(U, Z)(\delta U, \delta Z) = J'(U)(\delta U) - B'(U)(\delta U, Z) - B(U, \delta Z). \]

Details are given in Section 3.2. To proceed with the nonlinear error identity (23) we now define \( Z \in X \) and \( Z_{k,h} \in X_{k,h} \) as the solutions to the linearized dual problems

\[ B'(U_{k,h})(\Psi, Z_{k,h}) = J'(U_{k,h})(\Psi) \]

with \( \Psi \in X \) and \( \Psi \in X_{k,h} \).

The specific form for the sea ice problem will be given in Section 3.1.2. Galerkin orthogonality (11) also holds for this linearized dual problem. Hence, the first term \( L'(U, Z)(\cdot, \cdot) \) in (23) is zero and the complete nonlinear error estimator is given by

\[ J(U) - J(U_{k,h}) = R(x_{k,h}, e) - \frac{1}{2} B(U_{k,h})(Z - I_{k,h}Z) - \frac{1}{2} \left\{ J'(U_{k,h})(U - U_{k,h}) - B'(U_{k,h})(U - I_{k,h}U, Z_{k,h}) \right\}, \]

where we used primal and dual Galerkin orthogonality (11) to replace the errors \( U - U_{k,h} \) and \( Z - Z_{k,h} \) by local interpolation errors. The first part is the primal residual and it is multiplied with the dual weights \( Z - I_{k,h}Z \), the second part is the dual residual which is multiplied with the primal weights.

For details and many examples on the DWR method we refer to Becker and Ranacher (2001) or Schmich and Vexler (2008) who focus on the time dependent part. Algorithm 3.1 must not be changed for the general nonlinear case. Only the dual problem is constructed in a more complex setting. It is, however, still a linear problem.

3.1.2 The goal oriented error estimator for partitioned solution approaches

The partitioned solution approach is realized as a non-consisting discretization of the sea ice equation: for discretization we replace the variational formulation \( B(U)(\Phi) \) by
the splitting form \( B_s(U_{k,h})(\Phi_{k,h}) \). This means that the Galerkin orthogonality \( (11) \) is perturbed and it only holds

\[
B(U)(\Phi_{k,h}) - B_s(U_{k,h})(\Phi_{k,h}) = 0,
\]

or, using the splitting error \( [14] \)

\[
B(U)(\Phi_{k,h}) - B(U_{k,h})(\Phi_{k,h}) = \beta(U_{k,h})(\Phi_{k,h}).
\]

This requires subtle modifications in the derivation of the error estimate. Again, we only sketch the derivation and refer to [Mehmann 2019] for details.

Most important, the nonlinear error identity \( (21) \) must be based on two separate Lagrangians

\[
J(U) - J(U_{k,h}) = L(U, Z) - L_s(U_{k,h})(Z_{k,h}),
\]

where

\[
L_s(U_{k,h})(Z_{k,h}) := J(U_{k,h}) - B_s(U_{k,h})(Z_{k,h})
\]  

(26)

is based on the projection operator \( S \). Then, the DWR method is derived by dividing the error identity into the Galerkin error and into a splitting error by introducing \( \pm L(U_{k,h})(Z_{k,h}) = 0 \)

\[
J(U) - J(U_{k,h}) = \underbrace{L(U, Z) - L(U_{k,h})(Z_{k,h})}_\text{Galerkin} + \underbrace{L(U_{k,h})(Z_{k,h}) - L_s(U_{k,h})(Z_{k,h})}_\text{splitting}.
\]

While the Galerkin part can be estimated as outlined in Section 3.1.1, the splitting part is quickly derived by using the definitions of the Lagrangians in \( (20) \), \( (26) \) and \( (14) \)

\[
L(U_{k,h})(Z_{k,h}) - L_s(U_{k,h})(Z_{k,h}) = \beta(U_{k,h})(Z_{k,h}).
\]

This error contribution can be evaluated since it only depends on quantities that are available, namely the primal and dual discrete solutions. We summarize:

**Theorem 3.2 (DWR estimator for partitioned solution schemes)** Let \( U, Z \in X \) and \( U_{k,h}, X_{k,h} \in X_{k,h} \) be primal and dual solutions to

\[
\begin{align*}
B(U)(\Phi) &= 0 \quad \forall \Phi \in X \\
B'(U)(\Psi, Z) &= J'(U)(\Psi) \quad \forall \Psi \in X \\
B_s(U_{k,h})(\Phi_{k,h}) &= 0 \quad \forall \Phi_{k,h} \in X_{k,h} \\
B'_s(U_{k,h})(\Psi_{k,h}, Z_{k,h}) &= J'(U_{k,h})(\Psi_{k,h}) \quad \forall \Psi_{k,h} \in X_{k,h}.
\end{align*}
\]

Then, it holds that

\[
J(U) - J(U_{k,h}) = R(x_{k,h}, e) - \frac{1}{2} B_s(U_{k,h})(Z - I_{k,h}Z) \\
+ \frac{1}{2} \left\{ J'(U_{k,h})(U - I_{k,h}U) - B'_s(U_{k,h})(U - I_{k,h}U, Z_{k,h}) \right\} \\
+ \frac{1}{2} \left\{ \beta(U)(Z + Z_k) + \beta'(U)(U, Z_k) \right\}
\]  

(27)
where $R(x_{k,h}, e)$ is given in (23) and with the primal and dual splitting errors

$$\beta(U)(\Phi) := B_s(U)(\Phi) - B(U)(\Phi),$$

$$\beta'(U)(\Psi, Z) := B'_s(U)(\Psi, Z) - B'(U)(\Psi, Z).$$

By $I_{k,h} : X \to X_{k,h}$ we denote an interpolation to the space-time domain, by $U_k$ and $Z_k$ we denote semidiscrete solutions which are discretized in time only.

**Remark 3.3 (Weights)** The error estimator (27) depends on the unknown solutions $U$ and $Z$ but also on semidiscrete solutions $U_k$ and $Z_k$, which are still continuous in space. All these terms must be approximated by suitable reconstruction techniques in order to evaluate the error estimator. In general, reconstruction is be a postprocessing mechanism: In time, we combine two intervals and reconstruct a linear function by connecting $U_{n-1}$ in $t_{n-1}$ with $U_n$ at $t_n$, i.e.

$$i_{2k}^{(1)} U_{k,h} \bigg|_{[t_{n-1}, t_n]} = U_{n-1} + \frac{t - t_{n-1}}{k}(U_n - U_{n-1}).$$

Then, we approximate $U_k \approx i_{2k}^{(1)} U_{k,h}$. In space a similar procedure is done by combining the piecewise linear function $U_{k,h}$ on four adjacent quadrilaterals to one quadratic function. We refer to Richter and Wick (2015) and Mehlmann (2019) for details. The notation $i_{2k}$ means: interpolation to linear, (1), functions on a mesh with double, (2k) spacing in time. Correspondingly, $i_{2h}^{(2)}$ stands for the interpolation to the space of quadratic, (2), functions on the space with double spatial mesh spacing, (2h).

### 3.1.3 Decomposing the error estimator

One application of the error estimator is to identify different contributions to the overall error, namely the error coming from discretization in space $\eta_{\rho h}$, from the discretization in time $\eta_{\rho k}$ and from the splitting $\eta_{\beta}$. This information can help to optimally balance the discretization, e.g. by avoiding excessive refinement (in space or time) or by avoiding (or by applying) a more costly IMEX (see Lemieux et. al. (2014)) approach to avoid the splitting error.

The structure of the error estimator (27) consists of residuals weighted by primal or dual interpolation errors (first and second line of (27)) and by two terms, $\beta$ and $\beta'$, which measure the splitting error $\eta_{\beta}$. The residual terms refer to the discretization error in space $\eta_{\rho h}$ and in time $\eta_{\rho k}$. An allocation of this combined space-time error can be achieved by introducing intermediate interpolations, which we discuss for the primal residual term (the first term in (27)). Here we introduce $\pm I_k Z$, an interpolation into the space of functions that are piecewise constant in time but still non-discrete in space

$$B_s(U_{k,h})(Z - I_{k,h} Z) = B_s(U_{k,h})(Z - I_k Z) + B_s(U_{k,h})(I_k Z - I_{k,h} Z).$$

We can split the residual into two separate parts, as the dependency on the weight (which takes the role of the test function) is always linear.
Naturally, the interpolation $I_k Z$ is not available. However, we can approximate the interpolation errors by the reconstruction operator that have been mentioned in Remark 3.3. To be precise, the two terms in (29) are approximated by

$$B_s(U_{k,h})(Z - I_{k,h} Z)B_s(U_{k,h})(i_{2h}^{(1)} Z_{k,h} - Z_{k,h}) + B_s(U_{k,h})(i_{2h}^{(2)} Z_{k,h} - Z_{k,h}).$$

The philosophy is simple: for estimating the error in time, we compare the discrete solution $Z_{k,h}$ with its higher order reconstruction in time $i^{(1)}_{2h} Z_{k,h}$, the spatial error is estimated by considering the spatial reconstruction operator only. All further residual terms in (27) are handled in the same way.

### 3.2 Realization for sea ice dynamics

In order to apply the adaptive feedback loop presented in Algorithm 3.1, we will first indicate the exact discrete formulations for solving the primal and dual problems. These are not written as $dG(0)$ formulation but in the form of the Euler method. Afterwards we give some further remarks on the evaluation of the error estimator.

**Algorithm 3.4 (Primal sea ice problem)** Let $v_0$ and $h_0 = (A_0, H_0)$ be the initial solutions at time $t = 0$. Iterate for $n = 1, ..., N$

1. Solve for the velocity $v_n \in V_h$

$$k^{-1}(\rho_{ice} H_{n-1}(v_n - v_{n-1}), \phi)_{\Omega}$$

$$+ \langle f_c \vec{e}_r \times v_n + \tau(v_n) - \rho_{ice} H_{n-1} f_c \vec{e}_r \times v_{ocean}, \phi \rangle_{\Omega}$$

$$+ \langle \sigma(v_n, H_{n-1}, A_{n-1}), \nabla \phi \rangle_{\Omega} = 0 \quad \forall \phi \in V_h$$

2. Solve the transport equations for $A_n \in V_h^A$ and $H_n \in V_h^H$

$$k^{-1}(A_n - A_{n-1}, \psi_A)_{\Omega} + \langle \text{div}(v_n A_n), \psi_A \rangle_{\Omega} = -(\min\{0, 1 - A_n\}, \phi_A) \quad \forall \psi_A \in V_h^A,$$

$$k^{-1}(H_n - H_{n-1}, \psi_H)_{\Omega} + \langle \text{div}(v_n H_n), \psi_H \rangle_{\Omega} = 0 \quad \forall \psi_H \in V_h^H$$

To derive the dual sea ice model defined in Theorem 3.2, we must differentiate the form $B_s(U)(\Phi)$, which is described in (13), in the direction of the solution $U = (v, h)$. The dual solution $Z = (w, q)$ replaces the test function and the new test function $\Psi = (\psi, \psi_H, \psi_A)$ is the argument of the directional derivative. The complete derivative of the variational formulation is presented in Mehlmann (2019).

Most characteristic for the dual problem is the reversal of the time direction, the problem runs *backward in time*. This property is most easily seen if we go back to the linear heat equation discussed in Section 3.1. Since the variational formulation (15) is linear, the derivative is simply the form itself and the dual form is derived by switching the role of test function and solution. We only consider the term $\langle \partial_t u, \phi \rangle$ which turns into

$$\int_0^T (\partial_t \phi, z) \, dt = -\int_0^T (\phi, \partial_t z) \, dt + \phi(T) z(T) - \phi(0) z(0),$$

14
where we used partial integration. Up to boundary terms we obtain the dual time derivative \((-\partial_t z, \phi)\).

This reversal of direction also carries over to the splitting scheme. While the primal iteration first solves the momentum equation, the dual problem naturally results in first solving the (dual) transport problems, followed by the (dual) momentum equation.

**Algorithm 3.5 (Partitioned solution approach for the dual system)** Let \(v_n\) and \(h_n\) for \(n = 0, \ldots, N\), be the discrete solution of the primal problem. We set \(v_{N+1} := 0\) and \(z_{N+1} := 0\) and iterate backward in time from \(n = N\) to \(n = 1\)

1. Solve the dual transport equations for \(q_n = (q_{A,n}, q_{H,n}) \in V_h^2\)

\[
k^{-1}(q_{A,n} - q_{A,n+1}, \psi_A)\Omega - (v_n \cdot \nabla q_{A,n}, \psi_A)\Omega + \sigma'_A(v_{n+1}, H_n, A_n)(\psi_A, z_{n+1}) = J'_A(U_n)(\psi_A) - (\chi_{A>1}, \psi_A)\Omega
\]

and

\[
k^{-1}(q_{H,n} - q_{H,n+1}, \psi_H)\Omega - (v_n \cdot \nabla q_{H,n}, \psi_H)\Omega + (f_c\rho_{A,1}e_r \times (v_{n+1} - v_{\text{ocean}}(t_{n+1})), \psi_H)\Omega + \sigma'_A(v_{n+1}, H_n, A_n)(\psi_A, z_{n+1}) + k^{-1}((v_{n+1} - v_n)z_{n+1}, \psi_H) = J'_H(U_n)(\psi_H)
\]

for all \(\psi_A, \psi_H \in V_h\).

2. Solve the dual momentum equation for \(z_n \in V_h^2\)

\[
k^{-1}(\rho_{A,1}z_n - H_n z_{n+1}, \phi)\Omega + (f_c\rho_{A,1}e_r \times \phi, z_n)\Omega + (\tau'(v_n)(\phi), z_n)\Omega + (\sigma'(v_n, H_{n-1}, A_{n-1})(\phi, z_n))\Omega - (H_n \nabla q_{H,n}, \phi)\Omega - (A_{n} \nabla q_{A,n}, \phi)\Omega = J'_\phi(U_n)(\phi)
\]

for all \(\phi \in V_h^2\). By \(\chi_{A>1}(x)\) we denote the characteristic function satisfying \(\chi_{A>1}(x) = 1\) for \(A(x) \geq 1\) and \(\chi_{A>1}(x) = 0\) for \(A(x) < 1\). By \(\sigma'\) we denote the derivatives of the stress tensor with respect to \(A\), \(H\) or \(v\), by \(\tau'\) the derivative of the forcing and by \(J'\) that of the functional. These terms are detailed in Mehlmann (2019).

**Remark 3.6 (Dual problem)** The complexity of the dual system appears immense. However, the dual equation is linear such that the solution of each time step is very simple in comparison to the standard sea ice equation that often requires many (up to hundreds) of Picard iterations in each time step, see Lemieux and Tremblay (2009).

If the sea ice system is linearized by a Newton method, it turns out that the dual system matrix is just the transposed of the Jacobian. It is hence not necessary to implement the rather complicated form of the equations in Algorithm 3.5. Instead, it is sufficient to assemble the Newton Jacobian and take its transpose.
We have written primal and dual problem in the form of an Euler discretization. To evaluate the error estimator we must compute various residuals, compare (27). Considering the $dG(0)$ discretization all functions are constant in time such that the temporal integrals of the variational formulation can be exactly computed by the box rule (which finally results in the Euler form). The error estimator however additionally depends on evaluations of residuals that are weighted with higher order reconstructions, e.g. terms like

$$\langle \tau(v_{k,h}), i_{2k}^{(1)} z_{k,h} - z_{k,h} \rangle = \sum_{n=1}^{N} \int_{I_n} \tau(v_n) : \left( i_{2k}^{(1)} z_{k,h} - z_{k,h} \right) |_{I_n} \, dt$$

where $i_{2k}^{(1)} z_{k,h}$ is linear in time. Since $\tau(v_n)$ is constant in time, these integrals containing the higher order reconstructions are exactly evaluated with the midpoint rule, compare (28) for the exact definition of the reconstruction:

$$\langle \tau(v_{k,h}), i_{2k}^{(1)} z_{k,h} - z_{k,h} \rangle = \frac{k}{2} \sum_{n=1}^{N} \tau(v_n) : \left( z_{n-1} - z_n \right).$$

4 Numerical examples

Usually, a posteriori error estimators are used for two objectives: to compute an approximation with a certain accuracy as stopping criteria for the simulation, and, to adaptively control the discretization parameters, namely the mesh size and the time step size. The first goal is not realistic in sea ice simulations. Uncertainties from measurement and from model inaccuracies are so large that quantitative error measures are not available. Furthermore, large scale simulations are computationally extremely challenging. Mostly there is little room for using finer and finer meshes. However, the described analysis of the different error contributions is of great computational importance as it allows to optimally balance all error contributions to avoid excessive over-refinement in space or in time. The estimator can help to steer the simulation such that a given error rate can be obtained with the smallest effort.

Fully adaptive simulations, possibly even using dynamic meshes that change from time step to time step, call for an enormous effort in terms of implementation that is usually only given in academic software codes (such as Gascoigne 3d, Becker et al. (2019), which is used in this work). Global climate models do not offer this flexibility. However, some models like FESOM (Danilov et al. (2015)), MPAS (Ringler et al. (2013)) or ICON (Korn (2017)) offer the possibility for regional refinement in different zones. The error estimator can be used for automatically selecting the proper refinement level of all zones to reach the best accuracy on a discretization that is as coarse as possible.

We start by describing a benchmark problem that has been introduced by Mehlmann and Richter (2017b). While keeping simplicity (e.g. square domain) it features typical characteristics in terms of the forcing and the parameters. Then, we present different numerical studies on the error estimator. First testing its accuracy and effectivity in terms of adaptive mesh control, then focusing on possible cases for an integration of such techniques in productive models.
4.1 Definition of a benchmark problem

We consider the quadratic domain $\Omega = (0, 500 \text{ km})^2$. A circular steady ocean current is described by

$$v_{\text{ocean}}(x, y) = 0.01 \text{ m s}^{-1} \left( \frac{y}{250 \text{ km}} - 1 \right) \left( 1 - \frac{x}{250 \text{ km}} \right),$$

The wind field mimics a cyclone that is diagonally passing from the midpoint to the corner of the computational domain

$$v_{\text{atm}}(x, y, t) = 15 \text{ m s}^{-1} \omega(x, y) R(\alpha) \left( \frac{x - m_x(t)}{y - m_y(t)} \right),$$

with the rotation matrix

$$R(\alpha) := \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$ 

The center of the cyclone is given by $m_x(t) = m_y(t) = 250 \text{ km} + 50 t \text{ km/day}$ and the convergence angle is set to $\alpha = 90^\circ - 18^\circ$. To reduce the wind strength away from the center, we choose $\omega(x, y)$ as

$$\omega(x, y) = \frac{1}{50} \exp \left( -\frac{r(x, y)}{100 \text{ km}} \right),$$

$$r(x, y) = \sqrt{(x - m_x(t))^2 + (y - m_y(t))^2}.$$ 

The time interval of interest $I = [0, T]$ will be specified in the different test cases.

Solution of the nonlinear and linear systems The nonlinear problems resulting in each time step of the forward simulation are solved with a modified Newton scheme that is described in [Mehlmann and Richter (2017b)]. The linear systems within the Newton iteration and the linear problems of the dual system are solved with a GMRES method, preconditioned by a geometric multigrid solver as introduced in [Mehlmann and Richter (2017a)]. The model is implemented in the software library Gascoigne 3d, see [Becker et. al.] (2019).

4.2 Sharpness of the error estimate

As functional of interest, we evaluate the average sea ice extent within a subset $\Omega_2 = (375 \text{ km}, 500 \text{ km})^2 \subset \Omega$ over the first day of simulation $I_J = [0, 1 \text{ day}]$

$$J_A(A) = \int_{I_J} \int_{\Omega_2} A(x, y, t) d(x, y) \, dt.$$ 

(30)

Similar measures are considered for sea ice model evaluations or model intercomparisons, see [Stroeve et. al.] (2014) or [Kwok and Rothrock] (2009).
Table 2: On a sequence of spatially and temporally refined meshes we compare the functional output \( J_A(U_{k,h}^s) \) to the reference value \( \tilde{J}_A \approx 1.49907 \) and indicate the error \( \tilde{J}_A - J_A(U_{k,h}^s) \), the total error estimator \( \eta_{k,h} = \frac{1}{2}(\eta_h + \eta_k + \eta_\beta) \) and its contributions attributing the spatial discretization error \( \eta_h \), the temporal discretization error \( \eta_k \) and the splitting error \( \eta_\beta \). We observe that the complete error estimate \( \eta_{k,h} \) is very close to the real error and that the temporal error is dominating on fine spatial meshes.

At initial time, the ice is at rest, \( \mathbf{v}_0 = 0 \), the ice concentration is constant \( A = 1 \) and the ice height is a spatial variation around a thickness of \( H = 0.3 \text{ m} \)

\[
H^0(x, y) = 0.3 \text{ m} + 0.005 \text{ m} \left( \cos \left( \frac{x}{25 \text{ km}} \right) + \cos \left( \frac{y}{50 \text{ km}} \right) \right).
\]

To obtain an accurate reference value \( \tilde{J}_A \) we run the simulation on \( I_x = [0, 1 \text{ day}] \) on fine mesh with \( h_{ref} = 1 \text{ km} \) with step size \( k_{ref} = 0.125 \text{ h} \). Here, we obtain the value

\[
\tilde{J}_A := J_A(U_{ref,ref}^s) = 1.49907 \pm 10^{-5}.
\]  

In Table 2, we evaluate the functional error \( |\tilde{J}_A - J_A(U_{k,h}^s)| \) and the error estimator given by (31), which we denote by \( \eta_{k,h} \). The functional error shows linear convergence in time, whereas no obvious spatial convergence order can be identified. This is due to the dominance of the temporal error in this test case. To validate the accuracy of the error estimator we introduce the effectivity index

\[
\text{eff}_{k,h} := \frac{\tilde{J}_A - J(U_{k,h}^s)}{\eta_{k,h}},
\]

which measures the sharpness of the estimate. For \( \text{eff}_{k,h} \to 1 \) \((k, h \to 0)\) the error estimator would be exact, values below 1 indicate an overestimation of the error, values

| \( h \) | \( k \) | \( J_A(U_{k,h}^s) \) | \( \tilde{J}_A - J_A(U_{k,h}^s) \) | \( \eta_{k,h} \) | \( \eta_h \) | \( \eta_k \) | \( \eta_\beta \) |
|---|---|---|---|---|---|---|---|
| 64 km 8 h | 1.49763 | 1.44 \cdot 10^{-3} | 2.01 \cdot 10^{-3} | 1.20 \cdot 10^{-3} | 2.65 \cdot 10^{-3} | 1.58 \cdot 10^{-4} |
| 32 km 8 h | 1.49788 | 1.19 \cdot 10^{-3} | 1.38 \cdot 10^{-3} | 1.21 \cdot 10^{-4} | 2.19 \cdot 10^{-3} | 4.40 \cdot 10^{-4} |
| 16 km 8 h | 1.49797 | 1.10 \cdot 10^{-3} | 1.30 \cdot 10^{-3} | 6.72 \cdot 10^{-5} | 2.10 \cdot 10^{-3} | 4.38 \cdot 10^{-4} |
| 8 km 8 h | 1.49802 | 1.05 \cdot 10^{-3} | 1.25 \cdot 10^{-3} | 4.11 \cdot 10^{-5} | 2.03 \cdot 10^{-3} | 4.21 \cdot 10^{-4} |
| 64 km 4 h | 1.49833 | 7.43 \cdot 10^{-4} | 9.52 \cdot 10^{-4} | 6.28 \cdot 10^{-4} | 1.21 \cdot 10^{-3} | 6.12 \cdot 10^{-5} |
| 32 km 4 h | 1.49849 | 5.80 \cdot 10^{-4} | 6.16 \cdot 10^{-4} | 8.53 \cdot 10^{-5} | 1.02 \cdot 10^{-3} | 1.25 \cdot 10^{-4} |
| 16 km 4 h | 1.49856 | 5.15 \cdot 10^{-4} | 5.72 \cdot 10^{-4} | 4.41 \cdot 10^{-5} | 9.70 \cdot 10^{-4} | 1.30 \cdot 10^{-4} |
| 8 km 4 h | 1.49858 | 4.87 \cdot 10^{-4} | 5.51 \cdot 10^{-4} | 2.47 \cdot 10^{-5} | 9.44 \cdot 10^{-4} | 1.32 \cdot 10^{-4} |
| 64 km 2 h | 1.49863 | 4.39 \cdot 10^{-4} | 5.67 \cdot 10^{-4} | 5.48 \cdot 10^{-4} | 5.67 \cdot 10^{-4} | 2.04 \cdot 10^{-5} |
| 32 km 2 h | 1.49876 | 3.10 \cdot 10^{-4} | 2.94 \cdot 10^{-4} | 7.01 \cdot 10^{-5} | 4.82 \cdot 10^{-4} | 3.67 \cdot 10^{-5} |
| 16 km 2 h | 1.49881 | 2.59 \cdot 10^{-4} | 2.67 \cdot 10^{-4} | 3.58 \cdot 10^{-5} | 4.59 \cdot 10^{-4} | 3.95 \cdot 10^{-5} |
| 8 km 2 h | 1.49883 | 2.37 \cdot 10^{-4} | 2.54 \cdot 10^{-4} | 1.93 \cdot 10^{-5} | 4.47 \cdot 10^{-4} | 4.19 \cdot 10^{-5} |
above 1 an underestimation. In Figure 2 we plot this effectivity index and observe highly accurate estimator values for increasing spatial and temporal resolutions.

The last three columns of Table 2 show the splitting of the error estimator into spatial, temporal and splitting part as described in Section 3.1.3. These values show a dominance of the temporal error over the spatial error and to lesser degree also over the splitting error. In space, the estimator values $\eta_h$ also clearly demonstrate linear convergence in $h$, which is expected for linear finite elements. We do not observe this convergence order in the overall error, as it is dominated by the other two parts. Different configurations presented in [Mehlmann 2019] also showed a dominance of the spatial error. The dominating temporal residual error might stem from the short simulation time of $T = 1$ day. Our findings coincide with the analysis of [Lemieux et al. 2014] where the temporal error also dominates the splitting error in a one day simulation.

### 4.3 Balancing error contributions

A simple application of the decomposition of the error estimator into spatial error, temporal error and splitting error is to balance the different error contributions by the following algorithm:

**Algorithm 4.1 (Balancing errors)** Given an initial time step size $k$ and mesh size $h$.

Iterate:

1. Solve the sea ice problem $u_{k,h} \in V_{k,h}$
2. Estimate the error according to Algorithm 3.1
3. Split the error estimate $\eta_{k,h} := \frac{1}{2}(\eta_h + \eta_k + \eta_\beta)$
4. If $\eta_k + \eta_\beta > 2\eta_h$ refine time step $k \mapsto \frac{k}{2}$
   If $\eta_h > 2(\eta_k + \eta_\beta)$ refine spatial mesh $h \mapsto \frac{h}{2}$
Otherwise refine in space and time \( k, h \mapsto \frac{k}{2}, \frac{h}{2} \).

Here, we have attributed the splitting error \( \eta_\beta \) to the temporal error. We refine only spatially (or temporally) if this error contribution is twice as large as the other part. If the errors are already close to each other, we refine in space and in time. This strategy can be extended to include further error contributions. In coupled ice-ocean simulations one could balance the errors of the ocean component and the ice component.

We virtually perform a simulation based on Algorithm 4.1 by processing the results from Table 2. Starting with \( h = 64 \text{ km} \) and \( k = 8 \text{ h} \) it holds \( \eta_k + \eta_\beta = 2.71 \cdot 10^{-3} > 2\eta_h = 2.40 \cdot 10^{-3} \) (compare the first line of Table 2). Hence, we refine in time only and proceed with \( h = 64 \text{ km} \) and \( k = 4 \text{ h} \). Again, it holds \( \eta_k + \eta_\beta = 1.27 \cdot 10^{-3} > 2\eta_h = 1.26 \cdot 10^{-3} \) such that we once more refine in time only, resulting in \( h = 64 \text{ km} \) and \( k = 2 \text{ h} \). This third simulation yields \( \eta_k + \eta_\beta = 5.87 \cdot 10^{-4} \approx \eta_h = 5.48 \cdot 10^{-4} \) and we would continue by refining both in time and space.

The natural alternative to this procedure would be a uniform refinement in space and in time whenever the accuracy is not sufficient. To compare the complexity of both approaches we assume that the algorithm scales optimally, i.e. linear in the number of time steps \( \mathcal{O}(k^{-1}) \) and linear in the number of mesh elements given by \( \mathcal{O}(h^{-2}) \). Altogether we use the simple model \( E(k, h) = Ck^{-1}h^{-2} \) to measure the effort of one simulation. For simplicity, the constant is set to \( C = 64^2 \cdot 8 \). Three steps of uniform refinement result in the effort

\[
E(8, 64) + E(4, 32) + E(2, 16) = 32768 \left( \frac{1}{64^2 \cdot 8} + \frac{1}{32^2 \cdot 4} + \frac{1}{16^2 \cdot 2} \right) = 73
\]

whereas the balancing algorithm yields

\[
E(8, 64) + E(4, 64) + E(2, 64) = 32768 \left( \frac{1}{64^2 \cdot 8} + \frac{1}{64^2 \cdot 4} + \frac{1}{64^2 \cdot 2} \right) = 7,
\]

which is only 10% of the effort for the uniform standard approach. On the final mesh, the balancing algorithm yields the error \( 4.39 \cdot 10^{-4} \) compared to \( 2.59 \cdot 10^{-4} \) that would be obtained by using uniform refinement in space and time (at 10 times the cost).

### 4.4 Adaptive mesh control and steering of regional predictions

We repeat the simulations from Section 4.2 and focus on the spatial convergence of the finite element framework. The time step size is fixed to \( k = 0.5 \text{ h} \). We consider the contribution of the spatial discretization error \( \eta_h \) only and use Algorithm 3.1 for identifying optimal finite element meshes to yield small errors on meshes that are as coarse as possible. Two different strategies are investigated. First, we use a fully local

---

3Linear complexity w.r.t. spatial refinement is in principal possible by using multigrid methods for the solution of the linear systems, see [Mehlmann and Richter (2017a)](#). Due to the increasing impact of the nonlinearity on highly resolved simulations, the assumption of linearity turns out to be too optimistic. The savings from adaptivity by using smaller meshes would even be more drastic if a realistic estimate of the effort would be available.
Figure 3: Adaptive meshes resulting from four iterations of Algorithm 3.1 using local mesh refinement on an element-level. Meshes are optimized to compute the average sea ice extent subdomain $\Omega_2 = (375 \text{ km}, 500 \text{ km})^2$ in the upper right corner.

Figure 4: Adaptive meshes resulting from four iterations of Algorithm 3.1 using mesh refinement based on $4 \times 4$ predefined local regions.

The adaptive mesh concept, where mesh elements $K \in \Omega_h$ are refined into four smaller quadrilaterals, if the local error contribution $\eta_K$ is larger than the average error, i.e.

$$\eta_K > \frac{1}{|\Omega_h|} \sum_{K' \in \Omega_h} \eta_{K',h} \Rightarrow \text{refine } K.$$  \hspace{1cm} (33)

The results are shown in Fig. 3 (we do not show the coarse mesh without local refinement). The meshes created by this procedure show the typical characteristics: refinement is focused on regions where the solution shows strong patterns (e.g. cracks in the ice) and in regions where the dual problem indicates strong sensitivity of the error functions $J_A$, which, in this case, is the average ice extent in the subdomain $\Omega_2 = (375 \text{ km}, 500 \text{ km})^2$. It is also typical that the resulting meshes look non-intuitive as they appear rather tattered, non-symmetric and as the complete subdomain $\Omega_2$ is not chosen for refinement. We refer also to Becker and Rannacher (2001) with several examples showing that meshes obtained by a posteriori error estimators are superior to manually adjusted refinements.

As discussed before usual large scale climate models do not allow for fully adaptive meshes that call for a large technical overhead in terms of implementation, in particular when it comes to efficient realizations on parallel computers. However, several models allow for selecting local regions of higher resolution. These are usually hand-picked. Here we discuss a second possible use of the a posteriori error estimator for optimally tuning the mesh sizes in predefined local regions. We split the domain $\Omega = (0, 500 \text{ km})^2$ into $16 = 4 \times 4$ uniform local regions. Then, we proceed similar to (33) but first sum all
error indicators $\eta_{K;h}$ that belong to each of the 16 regions. Refinement is not carried out element-by-element, but for the complete region that has been selected. In Fig. 4 we show the resulting meshes. We observe that it is not sufficient to only refine the area of interest (the upper right sixteenth). Neighboring areas must be refined to give the optimal error balance.

Finally, we show in Fig. 5 the resulting spatial discretization errors for the different refinement strategies: uniform refinement of all elements, fully adaptive refinement and adaptive refinement based on predefined regions. Local refinement (both strategies) strongly reduced the required number of unknowns to reach a certain error level. The effect for a fully adaptive simulation is stronger, but also selective refinement based on predefined regions significantly decreases the problem size. Here, about 30,000 unknowns give the same error as about 130,000 unknowns on uniform meshes. The benefit of adaptivity is the omitting of unnecessary refinements.

5 Discussion and Conclusion

In this paper we introduced the first error estimator for the standard model describing the sea ice dynamics. The error estimator is derived for a general class of coupled non-stationary partial differential equations that are solved with a partitioned solution approach. It is based on the concept of the dual weighted residual method that has been introduced by Becker and Rannacher (2001). The error estimator consists mainly of two parts, the primal and dual residual error that arise in the framework of the dual weighted residual method, and here, an additional splitting error which stems from the application of the partitioned solution approach. In order to derive the error estimator for the sea ice model, we reinterpret the usual implicit Euler formulation as a variational space-time Galerkin approach.
We numerically evaluated this new error estimator on an idealized test case and measured the sea ice extent in a subdomain of interest. The temporal discretization error dominates the overall numerical error on all considered mesh resolutions. This might be due to the short simulation time and it coincides with the findings of [Lemieux et al. (2014)].

The error estimator is highly accurate as we observe an efficiency index close to 1. Despite the very strong nonlinearity of the sea ice model this means that the DWR estimator is a useful measure in sea ice simulations.

We discussed several approaches how this error estimator can be used to speedup the sea ice component in global climate models. First, the error estimator can be applied for a balancing of different error contributions, namely the spatial and the temporal discretization error as well as the error that comes form partitioning the coupled system. This approach can be extended to include further fields, like a coupled ocean-ice simulation. Second, we demonstrate how the error estimator can be used to control the mesh size of models that allow for a regional sampling at higher resolution. An automatic feedback approach guides the simulation to an optimally balanced mesh and allows for significant savings in terms of computational time.

The main technical difficulty for realizing the error estimator is the implementation of the dual problem, a problem that runs backward in time and that has a reversed partitioning structure. The concept of the dual weighted residual estimator is very flexible, with the main prerequisite of casting the problem and discretization into a variational Galerkin formulation. We have considered one typical error functional measuring the average ice extent, but further error measures are easily realized.

Acknowledgment. The work of Carolin Mehlmann has been supported by the Deutsche Bundesstiftung Umwelt. The work of both authors is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - 314838170, GRK 2297 MathCoRe.

References

[Becker et al. 2019] Becker, R.; Braack, M.; Meidner, D.; Richter, T.; Vexler, B.: The finite element toolkit GASCOIGNE. 2019. – http://www.uni-kiel.de/gascoigne/

[Becker and Rannacher 2001] Becker, R.; Rannacher, R.: An Optimal Control Approach to A Posteriori Error Estimation in Finite Element Methods. In: Iserles, A. (Hrsg.): Acta Numerica 2001 Bd. 37. Cambridge University Press, 2001, pp. 1–225

[Braack and Ern 2003] Braack, M.; Ern, A.: A posteriori control of modeling errors and discretization errors. In: Multiscale Model. Simul. 1 (2003), Nr. 2, pp. 221–238

[Coon 1980] Coon, M.D.: A review of AIDJEX modeling. In: Sea Ice Processes and Models: Symposium Proceedings, Univ. of Wash. Press, Seattle., 1980, pp. 12–27
[Danilov et al. 2015] Danilov, S.; Wang, Q.; Timmermann, R.; Iakovlev, N.; Sidorenko, D.; Kimritz, M.; Jung, T.; Schröter, J.: Finite-Element Sea Ice Model (FESIM), version 2. In: Geosci. Model Dev. 8 (2015), pp. 1747–1761

[Hibler 1979] Hibler, W.D.: A dynamic thermodynamic sea ice model. In: J. Phys. Oceanogr. 9 (1979), pp. 815–846

[Hunke and Dukowicz 1997] Hunke, E.C.; Dukowicz, J.K.: An elastic-viscous-plastic model for sea ice dynamics. In: J. Phys. Oceanogr. 27 (1997), pp. 1849–1867

[Hutter et al. 2018] Hutter, N.; Losch, M.; Menemenlis, D.: Scaling properties of Arctic sea ice deformation in a high-resolution viscous-plastic sea ice model and in satellite observations. In: Journal of Geophysical Research 170 (2018), pp. 18–38

[Korn 2017] Korn, P.: Formulation of an unstructured grid model for global ocean dynamics. In: J. Comp. Phys. 339 (2017), pp. 525–552

[Kwok and Rothrock 2009] Kwok, R.; Rothrock, D.A.: Decline in Arctic sea ice thickness from submarine and ICESat records: 1958–2008. In: Geophysical Research Letters 36 (2009), Nr. 15

[Lemieux et al. 2014] Lemieux, J.F.; Knoll, D.; Losch, M.; Girard, C.: A second-order accurate in time IMplicit–EXplicit (IMEX) integration scheme for sea ice dynamics. In: J. Comp. Phys. 263 (2014), pp. 375–392

[Lemieux and Tremblay 2009] Lemieux, J.F.; Tremblay, B.: Numerical convergence of viscous-plastic sea ice models. In: J. Geophys. Res. 114 (2009), Nr. C5

[Lemieux et al. 2010] Lemieux, J.F.; Tremblay, B.; Sedláček, J.; Tupper, P.; Thomas, S.; Huard, D.; Auclair, J.P.: Improving the Numerical Convergence of Viscous-plastic Sea Ice Models with the Jacobian-free Newton-Krylov Method. In: J. Comp. Phys. 229 (2010), pp. 2840–2852

[Lipscomb et al. 2007] Lipscomb, W. H.; Hunke, E. C.; Maslowski, W.; Jakacki, J.: Ridging, strength and stability in high-resolution sea ice models. In: Journal of Geophysical Research 112 (2007)

[Mehlmann 2019] Mehlmann, C.: Efficient numerical methods to solve the viscous-plastic sea ice model at high spatial resolutions, Otto-von-Guericke Universität Magdeburg, Diss., 2019

[Mehlmann and Richter 2017a] Mehlmann, C.; Richter, T.: A finite element multigrid-framework to solve the sea ice momentum equation. In: J. Comp. Phys. 348 (2017), pp. 847–861

[Mehlmann and Richter 2017b] Mehlmann, C.; Richter, T.: A modified global Newton solver for viscous-plastic sea ice models. In: Ocean Modeling 116 (2017), pp. 96–107
[Meidner and Richter 2014] Meidner, D.; Richter, T.: Goal-Oriented Error Estimation for the Fractional Step Theta Scheme. In: *Comp. Meth. Appl. Math.* 14 (2014), pp. 203–230

[Richter 2017] Richter, T.: *Fluid-structure Interactions.* Springer International Publishing, 2017

[Richter and Wick 2015] Richter, T.; Wick, T.: Variational Localizations of the Dual Weighted Residual Method. In: *Journal of Computational and Applied Mathematics* (2015), pp. 192–208

[Ringler et al. 2013] Ringler, T.; Petersen, M.; Higdon, R.; Jacobsen, D.; Maltrud, M.; Jones, P.: A multi-resolution approach to global ocean modelling. In: *Ocean Modelling* 69 (2013), pp. 211–232

[Schmich and Rannacher 2012] Schmich, M.; Rannacher, R.: Goal-oriented space–time adaptivity in the finite element Galerkin method for the computation of nonstationary incompressible flow. In: *Int. J. Numer. Meth. Fluids* 70 (2012), Nr. 1, pp. 1139–1166

[Schmich and Vexler 2008] Schmich, M.; Vexler, B.: Adaptivity with dynamic meshes for space–time finite element discretizations of parabolic equations. In: *SIAM Journal on Scientific Computing* 30 (2008), Nr. 1, pp. 369–393

[Stroeve et al. 2014] Stroeve, J.; Barrett, A.; Serreze, M.; Schweiger, A.: Using records from submarine, aircraft and satellites to evaluate climate model. In: *The Cryosphere* 8 (2014), pp. 1839–1854

[Williams and Tremblay 2018] Williams, J.; Tremblay, B.: The dependence of energy dissipation on spatial resolution in a viscous-plastic sea-ice model. In: *Ocean Modelling* 130 (2018), pp. 40 – 47