ON A NEW ASYMPTOTIC PROBLEM IN THE SCATTERING SETTING

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Abstract. In recent works we considered an asymptotic problem for orthogonal polynomials when a Szegő measure on the unit circumference is perturbed by an arbitrary Blaschke sequence of point masses outside the unit disk. In the current work we consider a similar problem in the scattering setting.

The goal of this work is to consider a new asymptotic problem; the related problems in the spectral setting were solved in [4], [6].

With a given Szegő contractive function $R$ on the unite circle $\mathbb{T}$

$$|R(t)| \leq 1, \quad \log(1 - |R|) \in L^1 \quad (0.1)$$

and a positive measure $\nu$ supported on the Blaschke set

$$Z = \{\zeta_k : \sum (1 - |\zeta_k|) < \infty \}, \quad \nu(\zeta_k) = \nu_k > 0, \quad (0.2)$$

we associate the scalar product

$$\langle Df, f \rangle = \langle (I - \Gamma^* \Gamma)f, f \rangle + \sum |f(\zeta_k)|^2 \nu_k. \quad (0.3)$$

Here $\langle \cdot, \cdot \rangle$ is the inner standard product in $L^2$ on $\mathbb{T}$ with respect to the Lebesgue measure, $\Gamma$ is the Hankel operator with the symbol $R$,

$$\Gamma f = \Gamma R f = P_- (R f),$$

acting from the Hardy space $H^2$ into its orthogonal complement $H^2 = L^2 \ominus H^2$. Let us point out that we even do not require that the measure $\nu$ is finite, thus the scalar product, correspondingly the unbounded operator $D$, are defined initially on the $H^2$–functions that equal zero at all points of $Z$ except for a finite number of them. (In this place we use the Blaschke condition (0.2)).

Our generalization deals with the presence of the measure $\nu$. Without it the scalar product plays the key role in the classical now solution of the Nehari problem by Adamyan, Arov and Krein, [1], [2]. For the new point of view on this subject see [9]. The Nehari problem is also known as the generalized Schur problem. Concerning relation of the Schur problem with the Theory of Orthogonal Polynomials on the Unit Circle, CMV matrices and so on, see [7], [8].

For shortness we denote the collection of data by

$$\alpha := \{R, \nu\}$$

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and then use the notation $H^2(\alpha)$ for the closure of admissible functions $f$ from $H^2$ with respect to the metric (0.3) with $D = D(\alpha)$. The condition (0.1) guarantees that the point evaluation functional (for all $\zeta_0$ in the unit disk $D$)

$$f \mapsto f(\zeta_0)$$

is bounded in $H^2(\alpha)$. Let $k^\alpha$ be the reproducing kernel of this space with respect to the origin and let

$$K^\alpha = \frac{k^\alpha}{\|k^\alpha\|}.$$  

It is almost evident that the system

$$\{e_n(\zeta; \alpha)\}, \quad e_n(\zeta; \alpha) := \zeta^n K^{\alpha_n}(\zeta),$$

where

$$\alpha_n = \{\zeta^n R(\zeta), |\zeta|^{2n}\nu(\zeta)\}$$

forms an orthonormal basis in $H^2(\alpha)$. We claim that asymptotically this system behaves as the standard basis system in $H^2$, in particular, that

$$K^{\alpha_n}(0) \to 1, \quad n \to \infty.$$  

(0.5)

We follow the line of proof that was suggested in [5] and then improved in [10] and [3]. Actually, the general idea is very simple. There are two natural steps in approximation of the given data by “regular” ones. First, to substitute the given measure $\nu$ by a finitely supported $\nu^N$. Second, to substitute $R$ by $\rho R$ with $0 < \rho < 1$. Then the corresponding data $\alpha^{N,\rho}$ produce the metric $D(\alpha^{N,\rho})$ which is equivalent to the standard metric in $H^2$ and it is a fairly easy task to prove (0.5) for such data. Further, due to $D(\alpha^N) \leq D(\alpha) \leq D(\alpha^\rho)$ we have the evident estimations

$$K^{\alpha^N}(0) \geq K^{\alpha}(0) \geq K^{\alpha^\rho}(0).$$

And the key point is a certain duality principle, see Corollary 1.6, that will allow us to use the left or right side estimation whenever it is convenient for us.

1. **The duality**

1.1. **The space $L^2(\alpha)$**. We define the outer function $T_e$ by

$$|T_e|^2 = 1 - |R|^2, \quad T_e(0) > 0.$$  

Consider the scalar product

$$\left\langle \begin{bmatrix} 1 & \bar{R} \\ R & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\rangle.$$  

(1.1)

By $L^2_R$ we denote the closure of vectors of the form

$$\begin{bmatrix} f \\ -P_- Rf \end{bmatrix}, \quad f \in L^2.$$  

(1.2)

with respect to the above metric.

**Lemma 1.1.** We have

$$L^2_R = \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : T_e f_1 \in L^2, \ T_e f_2 \in H^2, \ Rf_1 + f_2 \in H^2 \right\}.$$  

(1.3)

Moreover, the first component $f_1$ determines the second component $f_2$ uniquely.
Proof. As usually we can find the closure as the second orthogonal complement. This proves all listed properties of the vector \( \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \).

To prove the second claim let us mention that the spaces
\[
\mathcal{H}_2 \cap \mathcal{H}_2 - \mathcal{H}_2
\]
form direct sums (due to the maximum principle in the Smirnov class: \( \frac{g_1}{g_2} \in L^2 \) with \( g_{1,2} \in H^\infty \), \( g_2 \) is outer, implies \( \frac{g_1}{g_2} \in H^2 \)). Thus the decomposition
\[
Rf_1 = -\frac{1}{T_e}(\overline{T_e}f_2) + h, \quad h \in H^2,
\]
is unique. \( \square \)

**Definition 1.2.** The space \( L^2(\alpha) \) is formed by functions \( f \) that are defined on \( T \cup Z \) and such that \( f|_T = f_1, \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in L^2_R, \) and \( f(\zeta_k) = f_k, \{ f_k \} \in L^2, \) with the norm
\[
\|f\|_{L^2(\alpha)}^2 = \left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{L^2_R}^2 + \|\{ f_k \}\|_{L^2}^2.
\]
In other words \( L^2(\alpha) = L^2_R \oplus L^2. \)

1.2. **The Hardy spaces \( \mathcal{H}^2(\alpha) \) and \( \mathcal{H}^2(\alpha) \).** The first space \( \mathcal{H}^2(\alpha) \) is defined as the closure of functions \( f \in H^2 \), that equal zero at all points of \( Z \) except for a finite number of them, in \( L^2(\alpha) \) (a function from \( H^2 \) is naturally defined on \( T \cup Z \)).

**Lemma 1.3.** If \( f \in \mathcal{H}^2(\alpha) \) then
\[
g := T_e f|_T \in H^2 \quad \text{and} \quad f(\zeta_k) = \frac{g(\zeta_k)}{T_e(\zeta_k)}, \quad (1.4)
\]

**Definition 1.4.** A function \( f \in L^2(\alpha) \) is in \( \mathcal{H}^2(\alpha) \) if conditions (1.4) hold.

Thus we have evidently \( \mathcal{H}^2(\alpha) \subset \mathcal{H}^2(\alpha) \) but they do not necessarily coincide. Of course, in a regular case they are the same, say, when topologically both spaces are equivalent to the standard \( H^2 \).

The both spaces have the reproducing kernel basis systems:
\[
\{ \zeta^n K_\alpha^*(\zeta) \}, \quad \{ \zeta^n \overline{K}_\alpha^*(\zeta) \}, \quad (1.5)
\]
where \( n \in \mathbb{Z}_+ \). Let us point out that both systems being extended to all \( n \in \mathbb{Z} \) are basises in \( L^2(\alpha) \).

1.3. **The dual \( L^2 \)–space.** Our goal is to describe the orhogonal complement of, say, \( \mathcal{H}^2(\alpha) \) in \( L^2(\alpha) \). We will see that under a certain unitary map from the given \( L^2(\alpha) \) to a similar \( L^2 \)–space such orthogonal complement transforms into \( \mathcal{H}^2 \)–subspace of the target space.

Now we define formally this dual space. We define the Blaschke product
\[
B(\zeta) = \prod_{\zeta_k \in Z} \frac{\zeta_k - \zeta}{1 - \overline{\zeta_k} \zeta_k}
\]
and the function \( T = \frac{T_e}{T_e} \).
Put

\[ R^\tau(i) = \frac{R(t)}{T_c(i)} T(t), \quad t \in \mathbb{T}. \]  

(1.6)

Note that

\[
\begin{bmatrix} 1 & R^\tau \cr R & 1 \end{bmatrix} (i) = \begin{bmatrix} T & 0 \\
0 & T_c \end{bmatrix} (t) \begin{bmatrix} 1 & R \cr R & 1 \end{bmatrix}^{-1} (t) \begin{bmatrix} T & 0 \\
0 & T_c \end{bmatrix} (t),
\]

(1.7)

and, also, due to \(|R^\tau(i)| = |R(t)|\), we have \(T^\tau_c(t) = T_c(t)\).

We define the measure \(\nu^\tau\) supported on \(\mathbb{Z}^\tau := \overline{\mathbb{Z}}\) by

\[ \nu^\tau(\zeta_k) = \left| \frac{1}{T} \right|^2 \langle \zeta_k \rangle \nu_k. \]

(1.8)

We have \(B^\tau(t) = B(t)\), and thus \(T^\tau(t) = T(t)\).

Finally, the map \(\tau : L^2(\alpha) \to L^2(\alpha^\tau)\) is defined by

\[ \begin{bmatrix} 1 \\
R \\
1 \end{bmatrix} (t) \begin{bmatrix} f_1 \\
f_2 \end{bmatrix} (t) = \begin{bmatrix} \overline{T} & 0 \\
0 & T_c \end{bmatrix} (t) T \begin{bmatrix} f_1^\tau \\
f_2^\tau \end{bmatrix} (t) \]

(1.9)

on \(L^2_R\) component, and by

\[ f^\tau(\zeta_k) = -\frac{1}{T} \langle \zeta_k \rangle f(\zeta_k) \nu_k, \]

(1.10)

on \(L^2_\nu\), so that

\[ \tau f|_\mathbb{Z} = f_1^\tau, \quad (\tau f)(\zeta_k) = f^\tau(\zeta_k). \]

(1.11)

It is an easy task to check correctness of the definition, as well as the fact that the map is an involution.

1.4. The duality Theorem.

**Theorem 1.5.** \(\tau\) maps unitary \(L^2(\alpha) \subseteq \hat{H}^2(\alpha)\) onto \(\hat{H}^2(\alpha^\tau)\).

**Proof.** Let \(f\), with the components

\[ \begin{bmatrix} f_1 \\
f_2 \end{bmatrix} \in L^2_R, \quad \{f_k\} \in L^2_\nu, \]

be a vector from the orthogonal complement to \(\hat{H}^2(\alpha)\). By orthogonality to the vectors of the form \(Bh, \ h \in H^2\), we get

\[ \langle f, Bh \rangle_{L^2(\alpha)} = \left\langle \frac{T e \tilde{f}_1^\tau(i) \tilde{f}_2^\tau(i)}{T e \tilde{f}_2^\tau(i)}, \begin{bmatrix} B & Bh \\
- P R Bh \end{bmatrix} \right\rangle = \langle \overline{T e \tilde{f}_1^\tau(i)}, h \rangle = 0. \]

(1.12)

That is \(T^\tau_c f^\tau_1 \in H^2\).

Substituting in the scalar product the function \(\frac{B(h)}{\tau - \zeta_k}\) we get

\[ \left\langle \frac{\tilde{T e f_1^\tau(i)} \tilde{T e f_2^\tau(i)}}{\tau - \zeta_k} + f_k \frac{B^\tau(\zeta_k) \nu_k}{\tau - \zeta_k} \right\rangle = 0, \]

or,

\[ (T^\tau_c f^\tau_1)(\zeta_k) + f_k \frac{B^\tau(\zeta_k) \nu_k}{\tau - \zeta_k} = 0. \]

By Definition 1.4 the target vector \(\tau f\) is in \(\hat{H}^2(\alpha^\tau)\).

Conversely, an arbitrary vector of this form is orthogonal to \(Bh, \ h \in H^2\) and \(\frac{B(h)}{\tau - \zeta_k}\), \(\forall k\), and such vectors are complete in \(\hat{H}^2(\alpha)\).

\(\square\)
Note that all definitions (1.6) ... (1.11) were given to suite the proof of this theorem.

**Corollary 1.6.** In the above notations
\[ T(0)K^{\alpha-1}(0) \tilde{K}^{\alpha'}(0) = 1. \]  
(1.13)

**Proof.** By the theorem we conclude that
\[ \tau(\zeta^{-1}\tilde{K}^{\alpha^{-1}}(\zeta)) = \tilde{K}^{\alpha'}(\zeta). \]  
(1.14)

Since $\tilde{K}^{\alpha^{-1}}(\zeta)\tilde{K}^{\alpha^{-1}}(0)$ is the reproducing kernel of $\hat{H}^2(\alpha^{-1})$, we have
\[ \langle \zeta^{-1}\tilde{K}^{\alpha^{-1}}(\zeta), \zeta^{-1}B(\zeta) \rangle_{L^2(\alpha)} = \langle \tilde{K}^{\alpha^{-1}}(\zeta), B(\zeta) \rangle_{L^2(\alpha^ {-1})} = \frac{B(0)}{K^{\alpha^{-1}}(0)}. \]  
(1.15)

On the other hand, using (1.14), in the same way as in (1.12), we get for this scalar product
\[ \langle \bar{T}\bar{t}K^{\alpha'}(\bar{t}), iB \rangle = T(0)K^{\alpha'}(0). \]

In combination with (1.15) it gives us (1.13). \qed

2. **Asymptotics**

**Proof of (0.5).** We have
\[ \tilde{K}^{\alpha_n}(0) \leq \tilde{K}^{\alpha^N_n}(0) = \frac{B^{(N)}(0)}{T_e(0)} \frac{1}{K^{\alpha^{-1}_{n-1}}(0)} \leq \frac{B^{(N)}(0)}{T_e(0)} \frac{1}{K^{\alpha_{n-1}^{\rho}}(0)} \tilde{K}^{\alpha^N_{n-1}^{\rho}}(0). \]  
(2.1)

And from the other side
\[ \tilde{K}^{\alpha_n}(0) \geq \tilde{K}^{\alpha^N_n}(0) = \frac{B(0)}{T_e(0)} \frac{1}{K^{\alpha^{-1}_{n-1}}(0)} \geq \frac{B(0)}{T_e(0)} \frac{1}{K^{\alpha_{n-1}^{\rho}}(0)} \tilde{K}^{\alpha^N_{n-1}^{\rho}}(0). \]  
(2.2)

Passing to the limit in (2.1) and (2.2) and using arbitrriness of $\rho$ and $N$ we get (0.5). \qed

**Remark 2.1.** Assume that a space $\mathcal{H}$, $\hat{H}^2(\alpha) \subset \mathcal{H} \subset \hat{H}^2(\alpha)$ has the following shift invariant property: $\hat{H}^2(\alpha_n) \subset \mathcal{H}^{(\alpha_n)} \subset \hat{H}^2(\alpha_n)$ for the defined inductively
\[ \mathcal{H}^{(\alpha_n)} = \zeta \mathcal{H}^{(\alpha_{n+1})}, \quad \mathcal{H}^{(\alpha_n)}_0 := \{ f \in \mathcal{H}^{(\alpha_n)} : f(0) = 0 \}, \]  
(2.3)

with initial $\mathcal{H}^{(0)} = \mathcal{H}$. Then, naturally, there exists the reproducing kernel based orthonormal basis in $\mathcal{H}$ which also has property (0.5).

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