ON THE UNIVERSALITY OF THE NAZAROV-SODIN CONSTANT

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ABSTRACT. We study the number of connected components of non-Gaussian random spherical harmonics on the two dimensional sphere $S^2$. We prove that the expectation of the nodal domains count is independent of the distribution of the coefficients provided it has a finite second moment.

1. Introduction

1.1. Nodal domains of Laplace eigenfunctions. Let $(M,g)$ be a smooth, compact, connected surface and let $\Delta_g$ be the associated Laplace-Beltrami operators. We are interested in the eigenvalue problem

$$\Delta_g f_\lambda + \lambda f_\lambda = 0.$$

Since $M$ is compact, the spectrum of $-\Delta_g$ is a discrete subset of $\mathbb{R}$ with only accumulation point at $+\infty$. The eigenfunctions $f_\lambda$ are smooth and their nodal set, that is their zero set, is a smooth 1d sub-manifold outside a finite set of points [17]. In particular, it is possible to define the nodal domains counting function

$$\mathcal{N}(f_\lambda) := \text{number of connected components of } \{x \in M : f_\lambda(x) = 0\}.$$

Courant’s nodal domains theorem [16, Page 21] asserts that there exists some constant $C = C(M) > 0$ such that

$$\mathcal{N}(f_\lambda) \leq C\lambda.$$

On the other hand, it is not possible to obtain any non-trivial lower bound for $\mathcal{N}(\cdot)$. Lewis [27] showed that there exists a sequence of eigenfunction on the two dimension sphere $S^2$ with $\mathcal{N}(f_\lambda) \leq 3$ and $\lambda \to \infty$. Nevertheless, much of our physical understanding of Laplace eigenfunctions, such as the RWM proposed by Berry [6, 7, 8] and the percolation prediction of Bogomolny and Schmit [11, 12], suggests that, for “generic” Laplace eigenfunctions, we should expect

$$\mathcal{N}(f_\lambda) \geq c\lambda,$$

for some $c = c(M)$.

In order to explore this speculation, Nazarov and Sodin [28] studied the number of nodal domains of random Laplace eigenfunctions on $S^2$. The eigenvalues on $S^2$ are given by $\lambda = n(n+1)$ for any integer $n > 0$ and have multiplicity $2n + 1$, thus it is possible to define random Laplace eigenfunctions on $S^2$ as

$$f_n(x) = \frac{1}{\sqrt{2n+1}} \sum_{k=-n}^{n} a_k Y_k(x), \quad (1.1)$$

where $a_k$ are i.i.d standard Gaussian random variables and the $Y_k$’s are an orthonormal base of the spherical harmonics of degree $n$, that is Laplace eigenfunction with eigenvalue $\lambda = n(n+1)$. In this setting, Nazarov and Sodin [28] found that

$$\mathbb{E}[\mathcal{N}(f)] = c_{NS} n^2 (1 + o_{n \to \infty}(1)),$$
where $c_{NS} > 0$ is the Nazarov-Sodin constant, in agreement with the prediction of Bogo-

The purpose of this note is to explore what happens to the expected number of nodal

domains when the $a_k$’s in (1.1) are not Gaussian random variables but i.i.d. with finite second moment. In particular, we are interested in the case when $a_i$ are Bernoulli $\pm 1$ random variables. Since (normalized) Laplace eigenfunctions, such as the $Y_k$ in (1.1), are defined up to sign, the $\pm 1$ case seems to be a very natural model to study “generic” Laplace eigenfunctions.

1.2. Statement of the main result. Given some integer $n > 0$, let $\mathcal{H}_n = \mathcal{H}_n(S^2) \subset L^2(S^2)$ be the space of spherical harmonics of degree $n$ on $S^2$, that is the restriction of homogeneous harmonic polynomials of degree $n$ to $S^2$. Given an $L^2$ orthonormal basis $\{Y_k\}_{-n \leq k \leq n}$ for $\mathcal{H}_n$, we consider the function

$$f_n(x) = c_n \sum_{k=-n}^{n} a_k Y_k(x),$$

(1.2)

where $E[a_k] = 0$ and $E[|a_k|^2] = 1$ (1.3) and $c_n = (2n+1)^{-1/2}$ is a normalising constant, which has no impact on the zero set, so that $E[|f_n|^2] = 1$. With the above notation, we prove the following result:

**Theorem 1.1.** Let $f_n$ be as in (1.2). Suppose that the $a_k$’s are i.i.d. random variables such that $E[|a_0|^2] < \infty$, then

$$E[N(f_n)] = c_{NS} n^2 (1 + o_{n \to \infty}(1)).$$

One crucial difference between the Gaussian and the non-Gaussian case is that the distribution of $f_n$ as in (1.2) is not base independent (see Claim B.1 below). Therefore, most of the Gaussian tools which are pivotal to the “barrier method” in [28] are not available in the non-Gaussian case. Our proof of Theorem 1.1 takes a different approach which we will now briefly describe.

The proof of Theorem 1.1 comprises essentially of two steps:

(1) The first, and main, step is to show the universality, that is independence from the law of the $a_k$’s in (1.2), for the local nodal domains counting function

$$\mathcal{N}(F_x) = \text{number of connected components of } \{y \in B(x, O(n^{-1})): f_\lambda(y) = 0\},$$

where $B(x, O(n^{-1}))$ is a ball centered at $x \in S^2$ of radius about (we are purposely being slightly vague here) $n^{-1}$. This step requires the most care and its proof is essentially split into three parts:

(a) We show that, for most points $x \in S^2$, each of the summands in (1.2) is $o(n^{1/2})$. Here we use some quite “elementary” bounds on the $L^4$-norm of spherical harmonics.

(b) We apply a Lindeberg-type CLT to $f_n$ to deduce its (asymptotic) Gaussian behavior outside the aforementioned “bad” set of $x \in S^2$.

(c) We conclude, using the stability of the nodal set under small perturbations as in [29], that

$$\mathcal{N}(F_x) \overset{d}{\to} \mathcal{N}(F_\mu), \quad n \to \infty,$$
where $F_\mu$ are Gaussian random waves on the plane and the convergence is in distribution with respect to the product space $\Omega \times \mathbb{S}^2$ (with $\Omega$ being the probability space where the random objects are defined).

(2) In the second step, we will use, as in [29], the semi-local property of the nodal domains counting functions

$$N(f_n) = n(1 + o(1)) \int_{\mathbb{S}^2} N(F_x) d\rho(x),$$

where $\rho(\cdot)$ is the uniform measure on $\mathbb{S}^2$, to “reconstruct” $N(f_n)$ from our understanding of $N(F_x)$ and conclude the proof. This step uses some deterministic bounds on the nodal length, i.e. volume of the zero set, of $f_n$ and the Faber-Krahn inequality

We point out that the universality of the local nodal volume was observed by Kabluchko, Wigman and the author [26] in the study of the nodal volume of non-Gaussian band-limited functions, that is linear combinations of Laplace eigenfunctions. The results in [26] are based on a careful analysis of the spectral projector operator for $\Delta$ and micro-local analysis tools such as Sogge’s bound [33] on the $L^p$-norms of Laplace eigenfunctions.

On the other hand, the proof of Theorem 1.1 uses only “elementary” tools coming from the theory of orthogonal polynomials, as in [35], and the probability language of random functions, as in [9]. In fact, the exposition in this note is fully self contained a part from a result from [28], which we do not include for the sake of brevity of the article, and the well-known Faber-Krahn inequality (Lemma 2.1 below).

1.3. Discussion and Related results. Since the pioneering work of Nazarov and Sodin [29], the Nazarov-Sodin constant has been object of much attention. In [28], Nazarov and Sodin extended their work to the number of nodal domains of any ergodic Gaussian stationary field on any (compact and sufficiently smooth) manifold and in [30] they investigated the variance of $N(f_n)$ in the Gaussian setting. Beliaev, McAuley and Murihead [3, 5, 4] studied, for certain Gaussian random fields $F$, the number of connected components for the excursion sets $\{F \geq \ell\}$ and level sets $\{F = \ell\}$ for $\ell \neq 0$ and found estimates for the variance and a Central Limit Theorem. As far as nodal domains of random functions are concerned, Theorem 1.1 seems to be the first result addressing the universality of the nodal domains counting function.

The precise value on the Nazarov-Sodin constant is currently unknown. Estimates based on percolation [11, 12] suggest that

$$\frac{c_{NS}}{4\pi} = \frac{3\sqrt{3} - 5}{\pi} \approx 0.0624..., $$

however this seems slightly inconsistent with numerical simulations [2]. Pleijel’s nodal domains Theorem [31], see also [14], asserts that

$$\frac{c_{NS}}{4\pi} \leq \left( \frac{2}{j_0} \right)^2 \approx 0.691,$$

where $j_0$ is the smallest zero of the 0-th Bessel function. Using some curvature bounds [8] it is possible to show the sharper bound

$$\frac{c_{NS}}{4\pi} \leq \frac{1}{\sqrt{2\pi}} \approx 0.0225.$$
As far as lower bounds are concern, the only result we are aware of is by Ingremeau and Rivera [24],
\[
\frac{c_{NS}}{4\pi} \geq 1.39 \times 10^{-4}.
\]

Finally, we would like to discuss some deterministic lower bounds on the nodal domains counting function. Results in this direction seem few and rare. Ghosh, Reznikov and Sarnak [22, 23] gave a (non-trivial) lower bound for the number of nodal domains for Maass forms on a compact part of the modular surface. Jung and Zelditch [25] gave a lower bound for a large class of negatively curved surfaces.

We conclude by discussing the flat two dimensional torus \( T^2 = \mathbb{R}^2/\mathbb{Z}^2 \). On \( T^2 \), by Fourier expansions, every Laplace eigenfunction with eigenvalue \( 4\pi^2 n^2 \) can be expressed as a linear combination of exponentials \( \{ \exp(2\pi i \xi \cdot x) \}_{|\xi|^2 = n} \), where \( \xi \in \mathbb{Z}^2 \). In particular, it is possible to form linear combinations of eigenfunctions, with the same eigenvalue, as in (1.2). Bourgain [13], and the subsequently Buckley and Wigman [15], showed that, for “most” deterministic Laplace eigenfunctions
\[
N(f_n) = c_{NS} n^2 \left( 1 + n \to \infty \right).
\]

In particular, Bourgain’s result [13] holds for linear combination of eigenfunctions, as in (1.2), with \( a_k = \pm 1 \) coefficients. In other words, on \( T^2 \) when \( a_k = \pm 1 \), Theorem 1.1 holds deterministically.

1.4. Notation. To simplify the exposition we adopt the following standard notation: we write \( A \lesssim B \) and \( A \gtrsim B \) or \( A = O(B) \) to designate the existence of an absolute constant \( C > 0 \) such that \( A \leq CB \) and \( A \geq CB \). If the said constant \( C > 0 \) depends on some parameter, \( \beta \) (say), we write \( A \lesssim \beta B \) ecc, if no parameter is specified in the notation, then the constant is absolute. The letters \( C, c \) will be used to designate positive constants which may change from line to line. Finally, we write \( o_{\beta \to \infty}(1) \) for any function that tends to zero as \( \beta \to \infty \).

2. Tools

2.1. Harmonic-analysis tools. A very useful tool in the study of the nodal set of Laplace eigenfunctions on surfaces is the well-known Faber-Krahn inequality [16, Chapter 4], which we state here in a convenient form for our purpose:

**Lemma 2.1** (Faber-Krahn inequality). Let \( f_n \) be as in (1.2) and \( \Gamma \subset M \) be a nodal domain of \( f_n \) with inner radius \( r > 0 \), that is the radius of the largest geodesic ball inscribed in \( \Gamma \), then we have
\[
r \gtrsim n^{-1}.
\]

We will also need the following consequence of elliptic regularity [20, page 336] for harmonic functions. Before state the result, we need to introduce some notation. Recall that \( \mathcal{H}_n \) is the space of spherical harmonics of degree \( n \) and let \( B(x, r) \) be the spherical disk centered at \( x \in \mathbb{S}^2 \) of radius \( r > 0 \). We will need the following result:

**Lemma 2.2.** Let \( f \in \mathcal{H}_n \), \( R > 0 \) be some (large) parameter and \( k = 0, 1, 2 \), then
\[
\|f\|_{C^k(B(x,R/n))}^2 \lesssim (nR)^{2k+2} \int_{B(x,10R/n)} |f(y)|^2 dy,
\]
where the constant implied in the \( \lesssim \)-notation does not depend on \( n \), \( R \) or \( x \).
For the sake of completeness, we will provide a proof of Lemma 2.2 in Appendix A. We will also need the following bound on the $L^4$-norm of spherical harmonics, see [34, Theorem 2]:

**Lemma 2.3.** Let \( \{Y_k\}_{k=-n}^n \) be an orthonormal base for \( \mathcal{H}_n \), then we have
\[
\int_{S^2} |Y_k|^4 d\rho(x) \lesssim n^{2/3} \log n,
\]
where \( d\rho(\cdot) \) is the uniform measure on \( S^2 \).

Although much sharper bounds are known for the $L^4$-norm of Laplace eigenfunction, see for example the fundamental work by Sogge [33], Lemma 2.3 will suffice for our purposes and its, rather elementary, proof will be provided in Appendix A.

### 2.2. Orthogonal polynomials tools.

We will also need the following two facts about the two points function of \( f_n \) as in (1.2). Again, we need to introduce some notation. Let \( P_n(\cdot) \) be the classic Legendre polynomial
\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n(x^2 - 1)^n}{dx^n} \quad P_n(1) = 1. \tag{2.1}
\]
Moreover, let \( J_0(\cdot) \) be the 0-th Bessel function
\[
J_0(x) = \frac{1}{2\pi} \int_{S^1} e(x \cdot y) dy, \tag{2.2}
\]
where \( S^1 \subset \mathbb{R}^2 \) is the unit circle and \( e(\cdot) = \exp(2\pi i \cdot) \). We then have the following, see for example [1, Page 454]:

**Lemma 2.4** (Two point function). Let \( \{Y_k\}_{k=-n}^n \) be an orthonormal base for \( \mathcal{H}_n \) and \( x, y \in S^2 \), then we have
\[
\frac{1}{2n+1} \sum_{k=-n}^n Y_k(x)Y_k(y) = P_n(\cos \Theta(x,y)),
\]
where \( \Theta(x,y) \) is the angle between \( x, y \) and \( P_n \) is as in (2.1).

The following is [35, Theorem 8.21.6]:

**Lemma 2.5.** Let \( P_n \) be as in (2.1) and \( J_0 \) be as in (2.2). There exists some absolute constant \( C > 0 \) such that
\[
P_n(\cos(\theta)) = \frac{\theta}{\sin(\theta)} J_0 \left( (n + \frac{1}{2}) \theta \right) + F,
\]
where
\[
F \lesssim \begin{cases} 
\theta^{1/2} n^{-3/2} & C/n \leq \theta \leq \pi/2 \\
\theta^2 & 0 \leq \theta \leq C/n
\end{cases}
\]

### 2.3. Probabilistic tools.

We will need a multi-dimensional version of Lindeberg-CLT, see for example [21, Proposition 6.2]:

**Lemma 2.6** (CLT). Let \( d > 0 \) be a positive integer and let \( \{V_{n,k}\}_{n,k} \) be a triangular array whose \( n \)-th row consists of independent \( \mathbb{R}^d \)-valued random vectors with mean zero. That is, for any fixed \( n,k \), \( V_{n,k} = (V_{n,k}^i)_{i=1}^d \) is a \( d \)-dimension vector. Moreover, we normalize the \( V_{n,k}^i \) by writing \( (s_n^i)^2 = \sum_k \mathbb{E}[(V_{n,k}^i)^2] \) and \( \hat{V}_{n,k}^i = (s_n^i)^{-1} V_{n,k}^i \). Assume the following two conditions:
(1) Let \((\Sigma_n)_{ij} = \mathbb{E}[\tilde{V}_{n,i} \tilde{V}_{n,j}]\), be the covariance matrix of the \(n\)-th row of \(\{\tilde{V}_{n,k}\}_{n,k}\), then
\[
\lim_{n \to \infty} \Sigma_n = \Sigma_0,
\]
for some positive define \(d \times d\) positive matrix.

(2) We have
\[
\max_{i=1,\ldots,d} \frac{1}{s_n^2} \sum_k \mathbb{E} \left[ (V_{n,k}^i)^2 1_{V_{n,k}^i > s_n^i} \right] \to 0, \quad n \to \infty,
\]
for any positive \(\varepsilon > 0\), where \(1\) is the indicator function.

Then, we have
\[
\sum_k \tilde{V}_{n,k} \overset{d}{\to} N(0, \Sigma_0) \quad n \to \infty,
\]
where the convergence is in distribution and the rate of convergence depends on the rates of convergence in (1) and (2) only.

We will also need the Continuous Mapping Theorem in the following form, see for example [9, Theorem 2.1.]:

**Lemma 2.7 (Continuous mapping).** Let \(P_n, P\) be probability measures on a space \((S, \mathcal{S})\), where \(\mathcal{S}\) is the Borel \(\sigma\)-field. Suppose that
\[
P_n \overset{d}{\to} P \quad n \to \infty,
\]
where the convergence is in distribution, or, in other words, with respect to the weak* topology. Moreover, let \(h : S \to S\) be a map and let \(D_h\) be the set of discontinuity points of \(h\). If \(P(D_h) = 0\), then
\[
P_n h^{-1} \overset{d}{\to} P h^{-1} \quad n \to \infty.
\]

Finally, we need the following standard fact about uniform convergence, see [10, Theorem 3.5]

**Lemma 2.8.** Let \(X_n\) be a sequence of random variables such that
\[
X_n \overset{d}{\to} X,
\]
where the convergence is in distribution, for some random variable \(X\). Suppose that there exists some constants \(C, \alpha > 0\), independent of \(n\) such that \(\mathbb{E}[X_n^{1+\alpha}] \leq C\), then
\[
\mathbb{E}[X_n] \to \mathbb{E}[X].
\]

### 2.4. Gaussian fields tools

We will need the main result from [29]. Before stating it, we introduce some definitions which will be useful throughout the script. Let \(\Omega\) be an abstract probability space, with probability measure \(\mathbb{P}(\cdot)\) and expectation \(\mathbb{E}[\cdot]\). A (real-valued) Gaussian field \(F\) is a map \(F : \mathbb{R}^2 \times \Omega \to \mathbb{R}\) such that all finite dimensional distributions \((F(x_1, \cdot), \ldots, F(x_n, \cdot))\) are multivariate Gaussian vectors and \(F(x, \cdot)\) is continuous in \(x\). We say that \(F\) is centered if \(\mathbb{E}[F] = 0\) and stationary if its law is invariant under translations \(x \to x + \tau\) for \(\tau \in \mathbb{R}^2\). The covariance function of \(F\) is
\[
\mathbb{E}[F(x) \cdot F(y)] = \mathbb{E}[F(x - y) \cdot F(0)].
\]
Since the covariance is positive definite, by Bochner’s theorem, it is the Fourier transform of some measure \( \mu \) on \( \mathbb{R}^2 \). So we have
\[
\mathbb{E}[F(x)F(y)] = \int_{\mathbb{R}^2} e(\langle x - y, s \rangle) \, d\mu(s).
\]
The measure \( \mu \) is called the spectral measure of \( F \). Since \( F \) is real-valued, \( \mu \) is symmetric, that is \( \mu(-A) = \mu(A) \) for any (measurable) subset \( A \subset \mathbb{R}^2 \). By Kolmogorov’s theorem, \( \mu \) fully determines \( F \). Thus, from now on, we will simply write \( F = F_\mu \) for the centered, stationary Gaussian field with spectral measure \( \mu \). For the rest of the script \( \mu \) will always denote the uniform measure on \( S^1 \) and also let \( \mathcal{N}(F_\mu, R) \) be the number of nodal domains of \( F_\mu \) fully contained, that is not intersecting the boundary, in \( B(R) \), the ball centered at zero of radius \( R > 0 \). We are finally ready to state the main result from [29] which we will need:

**Theorem 2.9** (Nazarov-Sodin). Let \( \mu \), \( F_\mu \) and \( \mathcal{N}(F_\mu, R) \) be as above. Then, there exists some constant \( c_{NS} > 0 \) such that
\[
\mathbb{E}[\mathcal{N}(F_\mu, R)] = c_{NS} \pi R^2 + o(R^2).
\]

We will also need the following standard lemma, see for example [29, Lemma 6]:

**Lemma 2.10** (Bulinskaya’s lemma). Let \( F = F_\mu \), with \( \mu \) be the Lebesgue measure on the unit circle \( S^1 \). Then \( F \in C^1_s(W) \) almost surely, where is \( C^1_s(W) \) as in Lemma 2.12 (below).

### 2.5. Analysis tools.

We will need the following well-known fact about the nodal length, that is the volume of the zero set, of a function \( f \) as in (1.2).

**Lemma 2.11.** Let \( f_n \) be as in (1.2) and let \( \mathcal{L}(f_n) = \text{Vol}(x \in \mathbb{S}^2 : f(x) = 0) \), then we have
\[
\mathcal{L}(f_n) \lesssim n.
\]

Although Lemma 2.11 follows from the much more general result of Donnelly-Fefferman [18], we will give an “elementary” proof in Appendix A. We will also need the following consequence of Thom’s isotopy Theorem, see for example [19, Theorem 3.1] and also [28, Claim 4.3]. We state it in a way convenient for our purposes and will provide a proof, for completeness, in Appendix A:

**Lemma 2.12.** Let \( W > 0 \) be some parameter, \( B(W) \) be the ball of radius \( W \) centered at the origin and \( \partial B(W) \) its boundary. Let us write
\[
C^1_s(W) := \{ g \in C^1(B(2W)) | |g(x)| + |\nabla g(x)| > 0 \text{ for all } x \in B(W) \}
\]
and let \( \mathcal{N}(g, W) \) be the number of nodal domain of \( g \) fully contained in \( B(W) \), that is \( \Gamma \cap B(W) = \emptyset \) for all nodal domains of \( g \). Then, \( \mathcal{N}(\cdot, W) \) is a continuous functional on \( C^1_s(W) \).

We comment that the condition \( |\nabla g - \frac{x \cdot \nabla g}{|x|^2} x| > 0 \) assures that the nodal set does not touch the boundary of \( B(W) \) tangentially at one point. In particular, it implies that all the nodal domains that do not intersect \( \partial B(W) \) are fully contained in \( B(W) \) (and well separated from the boundaries).
3. Convergence in distribution

We are finally ready to begin the proof of Theorem 1.1. In order to state the main result of this section, we need to introduce some notation. Let \( \exp_x : T_xS^2 \cong \mathbb{R}^2 \to S^2 \) be the exponential map, let use define
\[
F_{x,R}(y) = F_x(y) := f(\exp_x(Ry/n)),
\]
where \( R > 0 \) is some parameter and \( y \in B(0,1) \), the ball centered at zero of radius 1. In (3.1), we tacitly assume that \( n \) is much larger than \( R \) so that the exponential map is a diffeomorphism. Now, we write \( N(F_x) \) for the number of nodal domains of \( F_x \) fully contained in \( B(0,1) \). Recalling that \( \Omega \) is the abstract probability space where all the objects in the script are defined, we can think of \( N(F_x) \) as a random variable \( \Omega \times S^2 \to \mathbb{R} \), where \( \Omega \times S^2 \) is equipped with the probability measure
\[
d\sigma = dP \otimes d\rho,
\]
where \( d\rho \) is the uniform probability measure on the unit sphere.

The main result of this section is the following:

**Proposition 3.1.** Let \( \mu, F_\mu, N(F_\mu, R) \) be as in section 2.4 and \( R > 0 \) some fixed parameter. Then, we have
\[
N(F_\mu) \xrightarrow{d} N(F_\mu, R) \quad n \to \infty,
\]
where the convergence happens in the cross space \( \Omega \times S^2 \) equipped with the measure \( d\sigma \) in (3.2) and the speed of convergence depends on \( R \).

The proof of Proposition 3.1 relies the observation that \( F_x \) is suitable close \( F_\mu \), as a random function from \( (\Omega \times S^2, d\sigma) \) into \( C^1(B(0,1)) \), continuously differentiable functions on \( B(0,1) \). We rigorously express this claim in the next section after having introduced the relevant notation and background.

3.1. Convergence of random functions. Before stating the main result of this section, we give a brief digression on the convergence of random functions. Let \( C^s(V) \) be the space of \( s \)-times, \( s \geq 0 \) integer, continuously differentiable functions on \( V \), a compact subset of \( \mathbb{R}^2 \). Since \( C^s(V) \) is a separable metric space, Prokhorov’s Theorem, see [9, Chapters 5 and 6], implies that \( \mathcal{P}(C^s(V)) \), the space of probability measures on \( C^s(V) \), is metrizable via the Lévy–Prokhorov metric. This is defined as follows: for a (measurable) subset \( A \subset C^s(V) \), denote by \( A_+ \) the \( \varepsilon \)-neighborhood of \( A \), that is
\[
A_+ := \{ p \in C^s(V) \mid \exists q \in A, \ ||p - q|| < \varepsilon \} = \bigcup_{p \in A} B(p, \varepsilon),
\]
where \( || \cdot || \) is the \( C^s \)-norm and \( B(p, \varepsilon) \) is the (open) ball centered at \( p \) of radius \( \varepsilon > 0 \). The Lévy–Prokhorov metric \( d_P : \mathcal{P}(C^s(V)) \times \mathcal{P}(C^s(V)) \to [0, +\infty) \) is defined for two probability measures \( \mu \) and \( \nu \) as:
\[
d_P(\mu, \nu) := \inf \{ \varepsilon > 0 : \mu(A_+) \leq \nu(A_+ \varepsilon) + \varepsilon, \ \nu(A) \leq \mu(A_+ \varepsilon) + \varepsilon \ \forall A \subset C^s(V) \}.
\]

Given an integer \( s \geq 1 \), \( F_x(y) \), as in (3.1), induces a probability measure on \( C^s(B(1)) \) via the push-forward measure
\[
(F_x)_* \operatorname{Vol}(A) = \operatorname{Vol}(\{ x \in S^2 : F_x(\cdot) \in A \}),
\]
where $A \subset C^s(B(1))$ is a measurable subset. Similarly, the push-forward of $F_\mu$ defines a probability measure on $C^s(B(1))$ which we denote by $(F_\mu)_*\mathbb{P}$. We can now measure the distance between $F^B_x$ and $F_\mu$ as the distance between their push-forward measures in $\mathcal{P}(C^s(B(1)))$, the space of probability measures on $C^s(B(1))$, equipped with the Lévy–Prokhorov metric. Therefore, to shorten notation, we will write

$$d_P(F_x, F_\mu) := d_P((F_x)_*d\sigma, (F_\mu)_*\mathbb{P}).$$

With the above notation, we have the following result:

**Proposition 3.2.** Let $R > 0$ be some parameter and $F_x(\cdot)$ be as in (3.1), $F_\mu$ be as in section 2.4. Then, we have

$$d_p(F_x, F_\mu) \to 0 \quad n \to \infty,$$

where $d_p$ is defined with respect to the $C^1$-norm.

Since the proof of Proposition 3.2 is somehow long and technical, we postpone it to section 4 below. In the rest of the section we will show how Proposition 3.1 follows from Proposition 3.2.

**3.2. Proof of Proposition 3.1.** In this section we prove Proposition 3.1 assuming Proposition 3.2.

**Proof of Proposition 3.1 given Proposition 3.2.** In order to prove Proposition 3.1 we apply Lemma 2.7 to $P_n = (F_x)_*d\sigma$ and $P = (F_\mu)_*d\mathbb{P}$ with $h = N(\cdot)$. By Proposition 3.2, we have

$$P_n \xrightarrow{d} P \quad n \to \infty.$$

Thanks to Lemma 2.10, applied with $W = 10R$ (say), we may assume that $F_\mu(R\cdot) \in C^1(B(10))$, where $C^1$ is as in Lemma 2.12. Therefore Lemma 2.12 implies that $P(D_h) = 0$. Hence, the assumptions of Lemma 2.7 are satisfied and Proposition 3.1 follows. □

**4. Proof of Proposition 3.2**

This section is entirely dedicated to the proof of Proposition 3.2. The main tool in the proof will be Lemma 2.6. However, in order to apply the Lindeberg-CLT, we need all the summands in (1.2) to have size, before normalization, $o(n^{1/2})$. This is not always the case as there exists spherical harmonics, as $g(\theta, \psi) = \sqrt{2n + 1}P_n(\cos(\theta))$, with satisfy $\max_{S^2} |g|^2 \gtrsim n$. In order to circumvent this difficulty, we show that the portion of space where spherical harmonics are large is small. This will be the content of the next section.

**4.1. Getting rid of large values.** This section is dedicated to the proof of the following lemma:

**Lemma 4.1.** Let $\{Y_k\}_{k=-n}^n$ be an orthonormal base for $\mathcal{H}_n$, spherical harmonics of degree $n$, and let $R, K > 0$ be some (large) parameters. Then there exists a subset $\mathcal{B} \subset S^2$ of volume at most $O(K^4R^{10}n^{-1/3}\log n)$ such that

1. Uniformly for all $x \notin \mathcal{B}$, we have

$$\max_{-n \leq k \leq n} \sup_{B(x, R/n)} |Y_k(x)| \lesssim K^{-1} n^{1/2}$$
Uniformly for all \( x \notin B \), we have
\[
\max_{-n \leq k \leq n} \sup_{B(x,R/n)} |n^{-1} \nabla Y_k(x)| \lesssim K^{-1} n^{1/2}.
\]
Moreover, the constants implied in the \( O \) and \( \lesssim \)-notation are independent of \( K, R, x, n \).

**Proof.** By Lemma 2.2, we have
\[
\sup_{B(x,R/n)} |Y_k(x)|^2 \lesssim (nR)^2 \int_{B(x,10R/n)} |Y_k(y)|^2 dy,
\]
Thus, in order to prove Lemma 4.1, it is enough to provide upper an bound on \( \int |Y_k|^2 \).
To this end, we first observe that
\[
\int_{B(x,10R/n)} |Y_k(y)|^2 dy \lesssim R^2 n^{-2} \int_{B(0,10)} |Y_{k,x}(y)|^2 dy,
\]
where \( Y_{k,x}(y) = Y_k(\exp_x(Ry/n)) \). Moreover, using Lemma 2.3 and switching the order of integration, we have
\[
\int_{S^2} \int_{B(0,10)} |Y_{k,x}(y)|^4 d\rho(x) \lesssim n^3 \log n,
\]
where \( \rho(\cdot) \) is the uniform measure on \( S^2 \). Furthermore, we observe that convexity of the \( L^2 \)-norm (Jensen’s inequality) implies
\[
\int_{S^2} \left( \int_{B(x,10)} |Y_{k,x}(y)|^2 d\rho(y) \right)^2 d\rho(x) \lesssim \int_{S^2} |Y_{k,x}(y)|^4 d\rho(x) \lesssim n^2 \log n.
\]
Thus, combining Chebyshev’s bound, (4.1) and (4.2), we have
\[
\rho \left( \sup_{B(x,R/n)} |Y_k(x)| > K^{-1} n^{1/2} \right) \lesssim K^4 n^{-2} \int_{S^2} \sup_{B(x,R/n)} |Y_k(x)|^4 dx \lesssim K^4 R^8 n^{-4/3} \log n \quad (4.3)
\]
Hence, Lemma 4.1 follows by taking the union bound over the \( O(n) \) choices of \( k \) in (4.3).

We are now ready to begin the proof of Proposition 3.2. We begin with a standard reduction step, which shows that it is enough to consider finite dimensional distributions to prove convergence of random functions (in our case).

**4.2. Step I: Reduction.** In order to prove Proposition 3.2 we will first make a reduction step which relays on the following, well-known, result about tightness of sequences of measures on \( \mathcal{P}(C^1(V)) \). First, a sequence of probability measures \( \{\nu_k\}_{k=0}^\infty \) on some topological space \( X \) is **tight** if for every \( \epsilon > 0 \), there exists a compact set \( K = K(\epsilon) \subset X \) such that
\[
\nu_k(X \setminus K) \leq \epsilon,
\]
uniformly for all \( k \geq 0 \). We will need the following lemma, borrowed from [32, Lemma 1], see also [10, Chapter 6 and 7]:
Lemma 4.2 (Tightness). Let $V$ be a compact subset of $\mathbb{R}^n$, and $\{\nu_n\}$ a sequence of probability measures on the space $C^1(V)$ of continuously differentiable functions on $V$. Then $\{\nu_n\}$ is tight if the following conditions hold:

1. For every $|\alpha| \leq 1$, there exists some $y \in V$ such that for every $\varepsilon > 0$ there exists $K > 0$ with

$$\nu_n(g \in C^2(V) : |D^\alpha g(y)| > K) \leq \varepsilon.$$ 

2. For every $|\alpha| \leq 1$ and $\varepsilon > 0$, we have

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \nu_n \left( g \in C^2(V) : \sup_{|y-y'| \leq \delta} |D^\alpha g(y) - D^\alpha g(y')| > \varepsilon \right) = 0.$$

As a consequence of Lemma 4.2 we have the following:

Lemma 4.3. Let $R > 0$ be some parameter and $F_x(\cdot)$ be as in (3.1) and let $\nu_n = (F_x)_* d\sigma$, where $d\sigma$ is the product measure on $\Omega \times S^2$ and $(F_x)_*$ is the push-forward measure. Then, the sequence $\nu_n$ is tight.

Proof. In order to check condition (1) in Lemma 4.2, we observe that, by Lemma 2.4, we have

$$\mathbb{E}[|D^\alpha F_x(0)|^2] \lesssim 1.$$ 

Thus, by Chebichev’s bound, we have

$$\sigma(|D^\alpha F_x(0)| > K) \leq \mathbb{P}(|D^\alpha F_x(0)| > K) \lesssim K^{-2},$$ 

which implies condition (1) in Lemma 4.2.

In order to check condition (2) in Lemma 4.2, we first observe that

$$\sup_{|y-y'| \leq \delta} |D^\alpha F_x(y) - D^\alpha F_x(y')| \lesssim \sup_{B(0,1)} |\nabla D^\alpha F_x| \delta. \quad (4.4)$$

By Lemma 2.2, we have

$$\sup_{B(0,1)} |\nabla D^\alpha F_x|^2 \lesssim_R \int_{B(0,2)} |F_x(y)|^2 dy,$$

thus, by Lemma 2.4, we deduce

$$\mathbb{E}[\sup_{B(0,1)} |\nabla D^\alpha F_x|^2] \lesssim_R 1.$$ 

Again by Chebichev’s bound and (4.4), we conclude that

$$\sigma\left( \sup_{|y-y'| \leq \delta} |D^\alpha F_x(y) - D^\alpha F_x(y')| \gtrsim K \delta \right) \lesssim K^{-2},$$

which, taking $K = \delta^{-1/2}$ say, implies condition (2) in Lemma 4.2. \qed

In light of Lemma 4.3, in order to prove Proposition 3.2 it is enough to prove convergence of the final-dimensional distributions, that is the following lemma:

Lemma 4.4. Let $R > 0$ be some parameter and $F_x(\cdot)$ be as in (3.1), moreover let $F_\mu$ be as in section 2.4. Fix some integer $d > 0$ and let $y_1, \ldots, y_d$ be $d$ fixed points in $B(0,1)$, then

$$(F_x(y_1), \ldots, F_x(y_d)) \overset{d}{\to} (F_\mu(y_1), \ldots, F_\mu(y_d)) \quad n \to \infty,$$
and for any multi-index $|\alpha| \leq 1$, we also have

$$(D^n F_x(y_1), ..., D^n F_x(y_d)) \overset{d}{\to} (D^n F_\mu(y_1), ..., D^n F_\mu(y_d)) \quad n \to \infty.$$ 

4.3. Step II: Convergence of final dimensional distributions. In light of the reduction step, we are left with proving Lemma 4.4, this is the content of this section.

Proof of Lemma 4.4. We are first going to focus on the first claim in Lemma 4.4. By Portmanteau Theorem [9, Theorem 2.1] and [9, Theorem 2.6], we have can fix some bounded continuous function $g : \mathbb{R}^k \to \mathbb{R}$ (say) and we have to show that

$$\int_{\Omega \times S^2} g((F_x(y_1), ..., F_x(y_d)))d\sigma \overset{n \to \infty}{\to} \int_{\Omega} g(F_\mu(Ry_1), ..., F_\mu(Ry_d))d\mathbb{P}. \quad (4.5)$$

In order to prove (4.5), let us write

$$\int_{\Omega \times S^2} g((F_x(y_1), ..., F_x(y_d)))d\sigma = \int_{S^2} d\rho(x) \int_{\Omega} g((F_x(Ry_1), ..., F_x(Ry_d)))d\mathbb{P},$$

we wish to show that the inner integral converges to the r.h.s. of (4.5). This would follow from Lemma 2.6 provided, we can check its assumptions, which we are going to do next.

Let us first check the convergence of the relative covariance matrix. First, we observe that, for all $y_1, y_2 \in B(0, 1)$ and $x \in S^2$ and fixed $R > 0$, we have

$$\Theta(\exp_x(Ry_1/n), \exp_x(Ry_2/n)) = R \frac{|y_1 - y_2|}{n} (1 + o_{n \to \infty}(1)),$$

where $\Theta(a, b)$ is the angle between $a, b \in S^2$. Therefore, Lemma 2.6 together with Lemma 2.4 and a straightforward differentiation gives that, uniformly for all $x \in S^2$, we have

$$E[D^n F_x(y_1)D^n F_x(y_j)] \to E[D^n F_\mu(Ry_1)D^n F_\mu(Ry_j)] \quad n \to \infty, \quad (4.6)$$

for all $i, j \in \{1, 2, ..., d\}$ and all multi-indices $|\alpha| \leq 1$. Thus, we have show that the first assumption of Lemma 2.6 holds.

Now, in light of the observation at the beginning of section 4.1, the second assumption of Lemma 2.6 is not satisfied uniformly for all $x \in S^2$. Thus, we will use Lemma 4.1 to get rid of a “bad” set of $x \in S^2$, as follows. Let $K = K(n) = \log n$ (say), by Lemma 4.1, applied with such $K$ and $R > 0$, there exists some set $B = B(n)$ such that

$$\rho(B) \lesssim_R n^{-1/3}(\log n)^{10},$$

and for all $x \not\in B$ the conclusion of Lemma 4.1 holds. Thus, we may re-write the l.h.s. of (4.5) as

$$\int_{S^2} d\rho(x) \int_{\Omega} g((F_x(Ry_1), ..., F_x(Ry_d)))d\mathbb{P} = \int_{S^2 \setminus B} d\rho(x) \int_{\Omega} g((F_x(Ry_1), ..., F_x(Ry_d)))d\mathbb{P} + o_{g,R}(1),$$

where the error term tends to zero as $n \to \infty$. Thus, it is enough to prove that

$$\int_{S^2 \setminus B} d\rho(x) \int_{\Omega} g((F_x(Ry_1), ..., F_x(Ry_d)))d\mathbb{P} \to \int_{\Omega} g(F_\mu(Ry_1), ..., F_\mu(Ry_d))d\mathbb{P}. \quad (4.7)$$

Hence, it is enough to check the second assumption in Lemma 2.6 holds under the conclusion of Lemma 4.1. This is what we are going to show next. Re-writing the second
assumption in Lemma 2.6, we have to show that
\[ \max_{i \in \{0, \ldots, d\}} \frac{1}{2n+1} \sum_{k=-n}^{n} \mathbb{E}[|a_i Y_k(\exp_x(R y_i)) \mathbb{1}_{|Y_k(\exp_x(R y_i))| > \epsilon(2n+1)^{1/2}}|] \rightarrow 0 \quad n \rightarrow \infty, \quad (4.8) \]
Since there are 2n + 1 summands in (4.8) it enough to prove that,
\[ \max_{i \in \{0, \ldots, d\}} \max_k \mathbb{E}[|a_i Y_k(\exp_x(R y_i)) \mathbb{1}_{|a_i Y_k(\exp_x(R y_i))| > \epsilon(2n+1)^{1/2}}|] \rightarrow 0 \quad n \rightarrow \infty, \]
for all \( \epsilon > 0 \) Since, as discussed above, it is enough to check that the second assumption in Lemma 2.6 holds under the conclusion of Lemma 4.1, we may assume\(^1\) that
\[ \max_{i \in \{0, \ldots, d\}} |Y_k(\exp_x(R y_i))| < n^{1/2}(\log n)^{-1}. \]
Therefore, we have
\[ \mathbb{E}[|a_i Y_k(\exp_x(R y_i)) \mathbb{1}_{|a_i Y_k(\exp_x(R y_i))| > \epsilon(2n+1)^{1/2}}|] \leq \mathbb{E}[|a_i \mathbb{1}_{|a_i| > 10 \epsilon \log n}|^2]. \]
Since the \( a_i \)'s have finite second moment their probability distribution decays at infinity. In order words, we may write
\[ \mathbb{E}[|a_i \mathbb{1}_{|a_i| > 10 \epsilon \log n}|^2] = \int_\Omega |a_i|^2 \mathbb{1}_{|a_i| > 10 \epsilon \log n} d\mathbb{P}(\omega) = \int_0^\infty t^2 \mathbb{1}_{t > 10 \epsilon \log n} d\mathbb{P}(|a_i| > t). \quad (4.9) \]
Since the \( a_i \)'s have finite second moment, by the Dominated Convergence Theorem, we may take the limit \( n \rightarrow \infty \) inside the integral in (4.9), to see that
\[ \max_i \mathbb{E}[|a_i \mathbb{1}_{|a_i| > 10 \epsilon \log n}|^2] \rightarrow 0 \quad n \rightarrow \infty, \]
for all \( \epsilon > 0 \) (we can actually take \( \epsilon = (\log n)^{-1/2} \), say). This proves (4.8). Hence, in light of (4.6) and (4.8), Lemma 2.6 implies (4.7), which, in turn, implies (4.5), concluding the proof of the first claim in Lemma 4.4. In light of the second claim in Lemma 4.1, Lemma 2.4 and Lemma 2.5, the second claim of Lemma 4.4 follows form an identical argument and it is therefore omitted. \( \square \)

5. PROOF OF THEOREM 1.1

We are finally ready to conclude the proof of Theorem 1.1:

Proof of Theorem 1.1. During the proof, we write \( \mathcal{N}(F_x) = \mathcal{N}(F_x, 1) \) and \( \mathcal{N}(F_\mu) = \mathcal{N}(F_\mu, R) \). First, observe that, by Lemma 2.11, we have
\[ \mathcal{L}(f_n) \lesssim n \]
Therefore, the number of nodal domains with diameter, that is the largest distance between two points on the said domain, larger than \( R/n \) is at most \( C_1 n^2/R \) for some \( C_1 > 0 \). Now suppose that \( \Gamma \) is a nodal domain with \( \text{Diam}(\Gamma) \leq R/n \), then the volume of \( x \in \mathbb{S}^2 \) such that \( \Gamma \cap B(x, R/n) \neq \emptyset \) but \( \Gamma \) is not fully contained in \( B(x, R/n) \) is at most \( O(\mathcal{L}(\Gamma) \cdot R/n) \), that is the length of \( \Gamma \) times the boundary length of \( B(x, R/n) \). Therefore, in light of Lemma 2.11, we obtain
\[ \sum_{\text{Diam}(\Gamma) \leq R/n} \int_{\Gamma \cap B(x, R/n)} dx - \pi R^2 n^2 \sum_{\text{Diam}(\Gamma) \leq R/n} 1 \lesssim \frac{R}{n} \sum_{\Gamma} \mathcal{L}(\Gamma) \lesssim R, \]
\(^1\)Note that here, if necessary, we use the conclusion of Lemma 4.1 with 2R in place of R so that the image of the exponential map is contained in \( B(x, R/n) \subset \mathbb{S}^2 \).
where \( \mathbf{1} \) is the indicator function, \( \Gamma \subset B(x, R/n) \) means that \( \Gamma \cap \partial B(x, R/n) = \emptyset \) and we tacitly assumed that \( R > 100 \) (say). Thus, bearing in mind that the number of nodal domains with diameter larger than \( R/n \) is at most \( C_1 n^2 / R \), exchanging the order of summation, we have

\[
\frac{n^2}{\pi R^2} \int N(F_x) d\rho(x) = \sum_{\text{Diam}(\Gamma) \leq R/n} \frac{n^2}{\pi R^2} \int \mathbf{1}_{\Gamma \in B(x, R/n)} dx = \sum_{\Gamma} 1 + O\left(\frac{n^2}{R} + R\right) = N(f_n) + O\left(\frac{n^2}{R}\right),
\]

as \( R \) is assumed to be much smaller than \( n \).

Now, by Lemma 2.1, we have

\[
\int_{S^2 \times \Omega} N(F_x)^2 d\sigma \lesssim R^2.
\]

Therefore, in light of Proposition 3.1, we may apply Lemma 2.8 with \( X_n = N(F_x) \), \( X = N(F_\mu(R \cdot)) \), where \( \mu \) is the uniform measure on \( S^1 \), to see that

\[
\int_{S^2 \times \Omega} N(F_x) d\sigma = \mathbb{E}[N(F_\mu)](1 + o_{n \to \infty}(1)).
\]

Combining (5.1) and (5.2), we deduce that

\[
\mathbb{E}[N(f_n)] = \frac{n^2}{\pi R^2} \mathbb{E}[N(F_\mu)](1 + o_{n \to \infty}(1)) + O\left(\frac{n^2}{R^2}\right).
\]

Hence, thanks to Theorem 2.9, we have

\[
\mathbb{E}[N(f_n)] = c_{NS} n^2 (1 + o_{n \to \infty}(1))(1 + o_{R \to \infty}(1)) + O\left(\frac{n^2}{R^2}\right),
\]

and Theorem 1.1 follows by taking \( R \to \infty \) sufficiently slowly compared to \( n \). \( \square \)

**Appendix A. Proof of auxiliary lemmas**

We will now prove Lemma 2.2.

**Proof of Lemma 2.2.** Let us consider the “harmonic lift” \( h(x, t) = \exp(-n(n+1)^{1/2}t) f_n(x) : S^2 \times [-2, 2] \to \mathbb{R} \). Note that, \( \Delta h = 0 \) on (say) any ball contained in the product space \( S^2 \times [-2, 2] \). Thus, by the mean value property of harmonic functions, for all \( y \in S^2 \), we have

\[
f_n(y) = h(y, 0) \lesssim \text{Vol}(\hat{B}((y, 0), n^{-1})) \int_{\hat{B}((y, 0), n^{-1})} h,
\]

where \( \hat{B}() \) denotes a ball in the product space. Using Cauchy-Schwartz and integrating over the auxiliary variable \( (t) \), we obtain

\[
|f_n(y)| \lesssim n \left( \int_{B(y, n^{-1})} |f_n|^2 \right)^{1/2}.
\]

Applying (A.1) (squared) to every point in \( y \in B(x, R/n) \), we conclude

\[
\sup_{B(x, R/n)} |f_n|^2 \lesssim (nR)^2 \int_{B(x, 10R/n)} |f_n|^2.
\]
We now consider bounds on the the derivatives of $f_n$. Since $h$ (as above) is harmonic, we also have $D^\alpha h$ is harmonic for all multi-indices $|\alpha| \leq 1$ say. Therefore, by the mean-value property of harmonic functions and the Divergence Theorem, for all points $w \in \mathbb{S}^2 \times [-1, 1]$ and a ball $\hat{B}(W) = \hat{B}(w, W)$ with $W > 0$, writing $D^\alpha = D$, we find

$$Dh(w) = \text{Vol}(\hat{B}(W))^{-1} \int_{\hat{B}(W)} Dh = \text{Vol}(\hat{B}(W))^{-1} \int_{\partial \hat{B}(W)} h \cdot n,$$

where $n$ is the outward pointing unit-norm vector. In particular, we have

$$Dh(w) \lesssim W^{-1} \sup_{\hat{B}(2W)} |h|.$$

Taking $w = (y, 0), W = 1/(2n)$ and using (A.1), we obtain

$$|Df_n(y)|^2 \lesssim n^4 \sup_{B(y,1/n)} |f_n|^2.$$

Another covering argument as above concludes the proof of Lemma 2.2 for the $C^1$-norm. To see Lemma 2.2 for the $C^2$-norm we simply use the mean-value property and the Divergence Theorem with $D^\beta (D^\alpha h)$, for $|\alpha| \leq 1$ and $|\beta| \leq 1$, to obtain

$$D^\beta (D^\alpha h)(y, 1) \lesssim n^2 \sup_{B(y,1,n^{-1})} |D^\alpha h| \lesssim n^6 \sup_{B(y,1/n)} |f_n|^2,$$

and repeat the covering argument. \qed

We will now prove Lemma 2.3 following [34, Theorem 2], we claim no originality.

**Proof of Lemma 2.3.** Let us suppose that $u$ is a function which maximizes $\int_{\mathbb{S}^2} u^4$ among all functions $u \in \mathcal{H}_n$ with $\|u\|_{L^2} = 1$ (note that $u$ exists since $\mathcal{H}_n$ is a finite dimensional vector space). Now, let us consider the integral kernel of the spectral projector operator $\pi_n : L^2(\mathbb{S}^2) \to \mathcal{H}$ which, in light of Lemma 2.4, is given by

$$\sum_{k=-n}^n Y_k(x)Y_k(y) = (2n+1) P_n(\langle x, y \rangle) := \varphi_n(\langle x, y \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the standard inner-product on $\mathbb{S}^2$ so that $\cos(\Theta(x, y)) = \langle \cdot, \cdot \rangle$. Then, by our choice of $u$, we claim that

$$\pi_n(u^3)(y) := \varphi_n * u^3(y) = \int_{\mathbb{S}^2} \varphi_n(\langle x, y \rangle) u(y)^3 d\rho(y) = cu(x), \quad (A.2)$$

for some constant $c > 0$. Indeed, for $f \in L^2(\mathbb{S}^2)$, we have

$$\int_{\mathbb{S}^2} f \cdot u = \int_{\mathbb{S}^2} \pi_n(f) \cdot u + \int_{\mathbb{S}^2} (f - \pi_n(f)) \cdot u = \int_{\mathbb{S}^2} \pi_n(f) \cdot u, \quad (A.3)$$

as $u \in \mathcal{H}_n$ and $(f - \pi_n(f))$ is orthogonal to it. Thus, since the integral on the r.h.s. of (A.3) is maximized for $\pi_n(f) = cu$ and $u$ by definition maximizes the integral on the l.h.s. of (A.3), we conclude (A.2).

In order to find $c > 0$ in (A.2), we multiply both sides by $u(x)$ and integrate with respect to $x \in \mathbb{S}^2$. Thus, bearing in mind that $\|u\|_{L^2} = 1$, we find

$$c = \|u\|^4_{L^4}.$$

Therefore, (A.2) and Hölder inequality (for the $L^1$-norm) implies

$$\|u\|_{L^1} \|u\|_{L^\infty} = \|\varphi_n * u^3\|_{L^\infty} \leq \|\varphi_n\|_{L^4} \|u^3\|_{L^{3/4}} = \|\varphi_n\|_{L^4} \|u\|^3_{L^4}.$$
In particular, we obtain the pair of bounds
\[ ||u||_{L^4} ||u||_{L^\infty} \leq ||\varphi_n||_{L^4} \quad ||u||_{L^4}^4 \leq ||u||_{L^\infty}^4. \]  
(A.4)
Hence, if \( ||u||_{L^\infty} \leq n^{1/3} \), then (A.4) implies the bound \( ||u||_{L^4}^4 \leq n^{2/3} \). If \( ||u||_{L^\infty} > n^{1/3} \), since a straightforward computation using Lemma 2.5 implies \( ||\varphi_n||_{L^4} \leq n^{1/2} \log n \), (A.4) implies \( ||u||_{L^4}^4 \leq n^{2/3} \log n \), as required.

We will now prove Lemma 2.11:

**Proof of Lemma 2.11.** First, since spherical harmonics are restrictions of homogeneous polynomials to the sphere, by passing to polar coordinates, we may identify \( f_n \) with a bi-variate trigonometric polynomial \( g_n \) (say) so that
\[ \text{Vol}(x \in S^2 : f_n(x) = 0) \asymp \text{Vol}(x \in [-1/2, 1/2]^2 : g_n(x) = 0), \]
Therefore, it is enough to prove
\[ \text{Vol}(x \in [-1/2, 1/2]^2 : g_n(x) = 0) \lesssim n. \]  
(A.5)
Now we claim that there exists either an horizontal line \( \ell_h \) (say) or a vertical line \( \ell_v \) (say) such that
\[ \text{Vol}(x \in [-1/2, 1/2]^2 : g_n(x) = 0) \lesssim |\{x \in \mathcal{L}_h : g(x) = 0\}| + |\{x \in \mathcal{L}_v : g(x) = 0\}| \]  
(A.6)
Since the zero set of \( g_n \) is an union of smooth curves, it is, in particular, rectifiable (approximable by line segments). So it is enough to show that for any line segment \( \mathcal{C} \) with length \( \ell \) claim A.6 holds. Indeed, let us write \( N_1(x_1) \) for the number of intersections of \( \mathcal{C} \) with the horizontal line going through \( x_1 \in [-1/2, 1/2] \) (the lower side of the square \([-1/2, 1/2]^2\)) and, similarly, \( N_1(x_2) \) for the number of intersections of \( \mathcal{C} \) with the vertical line going through \( x_2 \in [-1/2, 1/2] \) (the left side of the square \([-1/2, 1/2]^2\)). Then, we have
\[ \int N(x_1)dx_1 + \int N(x_2)dx_2 \geq P_1(\mathcal{C}) + P_2(\mathcal{C}) \geq \frac{\ell}{10}, \]
where \( P_1, P_2 \) are the length of the projection of \( \mathcal{C} \) on the X and Y-axis respectively, and Claim A.6 follows. Since \( g_n \) is a bi-variate polynomial of degree at most \( n \), its restriction to any vertical or horizontal line is an un-variante polynomial of degree \( n \), thus, by (A.6), we have
\[ \text{Vol}(x \in [-1/2, 1/2]^2 : g_n(x) = 0) \lesssim n, \]
as required. \( \square \)

We are now going to prove Lemma 2.12, the proof follows [29, Claim 4.2], we claim no originality.

**Proof of Lemma 2.12.** Let \( \{g_k\}_{k=1}^\infty \in C^1_t \) be a sequence of functions converging to some \( g \in C^1_t(W) \) with respect to the \( C^1 \)-topology. Since \( g \in C^1_t \), \( g \) has finitely many nodal domains in \( B(2W) \). In particular, it has finitely many nodal domains which do not touch \( \partial B(W) \) and, in light of the definition of \( C^1_t \), there exists some \( a > 0 \) such that
\[ \min \text{Dist}(\Gamma, \partial B(W)) > a, \]
where the minimum is take over all nodal domains \( \Gamma \) of \( g \) contained in \( B(W) \). Moreover, there exists some \( b \) such that
\[ |g| + |\nabla g| > b. \]
Now, we claim that each connected component \( \Gamma(t) \) of the level set \( |g| \leq t \) contains precisely one nodal domain \( \Gamma \) and the \( \Gamma(t) \)’s are disjoint, provided we choose \( t = t(a, b) \).
sufficiently small. Let’s start by showing that the $\Gamma(t)$’s are disjoint. If they were not, they would meet at a point $x$ (say) where $|\nabla g(x)| = 0$, since $x \in B(2W)$, this implies that $|g(x)| > b$ and thus it does not belong to $\Gamma(b/2)$.

Let us now show that each connected component $\Gamma(t)$ of the level set $|g| \leq t$ contains precisely one nodal domain $\Gamma$, provided we choose $t = t(a, b)$ sufficiently small. Indeed, since $g$ is continuous, taking $t$ sufficiently small depending on $a$, we may assure that

$$\min \text{Dist}(\Gamma(t), \partial B(W)) > a/2.$$ 

So all $\Gamma(t)$ as sufficiently well separated from the boundaries. Now, let $x \in \partial \Gamma(t)$, then $|g(x)| = t$ and, taking $t < b/2$, we see that $|\nabla g(x)| > b/2$, thus moving in the direction of $\nabla g(x)$ if $g(x) = -t$ and in the direction $\nabla g(x)$ if $g(x) = t$, we will find a point (within $\Gamma(t)$ for appropriately small $t$ depending on $b$) such that $g(x) = 0$. Since $|g| + |\nabla g| > b$, there are exists some $c = c(b) > 0$ such that

$$\min \text{Dist}(\Gamma_i, \Gamma_j) > c.$$ 

By continuity of $g$, we may choose $t$ sufficiently small such that $\Gamma(t) \subset \Gamma_{+(c/2)}$, the $(c/2)$-neighborhood of $\Gamma$ (and all this neighborhood are well within $B(W)$, provided with also choose $c$ small compared to $a$). Thus, $\Gamma(t)$ contains precisely one nodal domain.

To conclude the proof we take $k$ sufficiently large so that each nodal domains of $g_k$ is contained in $\Gamma(t)$, for some $\Gamma(t)$, which implies, as $\mathcal{N}(\cdot)$ is integer values, that

$$\mathcal{N}(g_k, W) = \mathcal{N}(g, W),$$

as required. $\square$

**Appendix B. Not base-independent**

We prove that the distribution of $f_n$ as a random function on $C^0(S^2)$ (say) is not base independent:

**Claim B.1.** Given and orthogonal $\{Y_k\}$ for $\mathcal{H}_n$ let $v_n = (f_n)_* \mathbb{P}$ be as in section 3.1 with

$$f_n(x) = c_n \sum_{k=-n}^{n} a_k Y_k,$$

where the $a_k$’s are i.i.d. Bernoulli ±1. There exists two (different) orthonormal basis $\{Y_k\}_{k=-n}^{n}$ and $\{\tilde{Y}_k\}_{k=-n}^{n}$ such that their associated pushfoward measures, $v_n$ and $\tilde{v}_n$ (say), have different distributions.

**Proof.** Suppose, by contradiction that $v_n = \tilde{v}_n$, in the sense of distributions. Then, their finite dimensional distributions also agree, in particular, taking $x = (0, 0)$ to be the north pole, in polar coordinates on the sphere, we should have

$$\mathbb{P}(f_n(x) \leq t) = \mathbb{P}(\tilde{f}_n(x) \leq t),$$

for all $t \in \mathbb{R}$. Let us now take $Y_k(\theta, \psi) = \exp(ik\psi) P_n^k(\cos \theta)$, where $(\theta, \psi)$ are polar coordinates on $S^2$ and

$$P_n^k(x) = \frac{n(n-k)!}{(n+k)!} \left( \frac{-1}{2} \right)^k \frac{1}{2^k k!} (1 - x^2)^{k/2} d^{n+k} x^{n+k} (x^2 - 1)^n,$$

are the (normalized) associated Legendre polynomials. Moreover, let us take

$$\tilde{Y}_k = Y_k \quad k \neq 0, 1, \quad \tilde{Y}_0 = \frac{1}{\sqrt{2}} (Y_0 + Y_1), \quad \tilde{Y}_1 = \frac{1}{\sqrt{2}} (Y_0 - Y_1).$$
By definition, at the north pole $Y_k(x) = 0$ for all $k \neq 0$ and $Y_0(x) = \sqrt{2n+1}$, thus, bearing in mind the normalization constant in the definition of $f_n$, we obtain

$$
P(f_n(x) \leq t) = P(a_0 \leq t),$$

and

$$
P(f_n(x) \leq t) = P(2^{-1/2}(a_0 + a_1) \leq t),$$

which contradicts (B.1).

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