CORRESPONDENCE BETWEEN YOUNG WALLS AND YOUNG TABLEAUX REALIZATIONS OF CRYSTAL BASES FOR THE CLASSICAL LIE ALGEBRAS

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ABSTRACT. We give a 1-1 correspondence with the Young wall realization and the Young tableau realization of the crystal bases for the classical Lie algebras.

INTRODUCTION

Young tableaux and Young walls play important roles in the interplay, which can be explained in a beautiful manner using the crystal base theory for quantum groups, between the fields of representation theory and combinatorics. Indeed, since representation theory is known to be a vital tool in the solution of certain kinds of two-dimensional solvable lattice models in statistical mechanics, Young tableaux and Young walls are central ingredients in mathematical physics.

The quantum groups introduced by Drinfeld and Jimbo, independently are deformations of the universal enveloping algebras of Kac-Moody algebras [2, 5]. More precisely, let \( \mathfrak{g} \) be a Kac-Moody algebra and \( U(\mathfrak{g}) \) be its universal enveloping algebra. Then, for each generic parameter \( q \), we associate a Hopf algebra \( U_q(\mathfrak{g}) \), called the quantum group, whose structure tends to that of \( U(\mathfrak{g}) \) as \( q \) approaches 1.

The important feature of quantum groups is that the representation theory of \( U(\mathfrak{g}) \) is the same as that of \( U_q(\mathfrak{g}) \). More precisely, let \( M \) be a \( U(\mathfrak{g}) \)-module in the category \( \mathcal{O}_{\text{int}} \), which has a weight space decomposition \( M = \bigoplus_{\lambda \in P} M_\lambda \). Then, for each generic \( q \), there exists a \( U_q(\mathfrak{g}) \)-module \( M^q \) in the category \( \mathcal{O}^q_{\text{int}} \) with a weight space decomposition \( M^q = \bigoplus_{\lambda \in P} M^q_\lambda \) such that \( \dim_{\mathbb{C}(q)} M^q_\lambda = \dim_{\mathbb{C}} M_\lambda \) for all \( \lambda \in P \) and the structure of \( M^q \) tends to that of \( M \) as \( q \) approaches 1. Therefore, to understand the structure of representations over general quantum groups \( U_q(\mathfrak{g}) \), it is enough to understand that of representations over \( U_q(\mathfrak{g}) \) for some special parameter \( q \) which is easy to treat.

The crystal bases, introduced by Kashiwara [11, 12], can be viewed as bases at \( q = 0 \) for the integrable modules over quantum groups and they are given a structure of colored oriented graph, called the crystal graphs, which reflect the combinatorial structure of integrable modules.

For classical Lie algebras, Kashiwara and Nakashima gave an explicit realization of crystal bases for finite dimensional irreducible modules [14]. In their work, crystal bases were
characterized as the sets of semistandard Young tableaux with given shapes satisfying certain additional conditions. In [16], Littelmann gave another description of crystal bases for finite dimensional simple Lie algebras using the Lakshmibai-Seshadri monomial theory. His approach was generalized to the path model theory for all symmetrizable Kac-Moody algebras [17]. Littelmann’s theory also gives rise to colored oriented graphs, which turned out to be isomorphic to the crystal graphs [13]. Moreover, in [20], Nakashima gave a generalization of the Littlewood-Richardson rule for $U_q(\mathfrak{g})$ associated with the classical Lie algebras using the crystal base theory.

In [4] and [8], Young wall, an affine combinatorial object, was introduced. The crystal bases for basic representations (i.e., highest weight representation of level 1) for quantum affine algebras are realized as the sets of reduced proper Young walls. Motivated by the fact that a classical Lie algebra $\mathfrak{g}$ lies inside an affine Lie algebra $\hat{\mathfrak{g}}$ and any crystal graph $B(\lambda)$ for a finite dimensional irreducible $\mathfrak{g}$-module $V(\lambda)$ appears as a connected component in the crystal graph $B(\Lambda)$ of a basic representation $V(\Lambda)$ over $\hat{\mathfrak{g}}$ without 0-arrows, Kang, Kim, Lee and Shin gave a new realization of crystal bases for finite dimensional modules over classical Lie algebras in terms of Young walls corresponding to the connected component having the least number of blocks [9].

In this paper, we give a new realization of crystal bases for finite dimensional irreducible modules over classical Lie algebras derived from the Young wall realization. The basis vector is parameterized by certain tableaux with given shape which is different from generalized Young diagram given by Kashiwara and Nakashima. Moreover, motivated by the fact that the tableau realization of $B(\lambda)$ given by Kashiwara and Nakashima can be derived from the Young wall realization of the crystal graph $B(\Lambda)$ corresponding to the connected component having the largest number of blocks, we give a crystal isomorphism from our new realization to the realization of Kashiwara and Nakashima, which will be a part of insertion scheme for the crystal of the classical Lie algebras given in [15].

The contents of this paper is organized as follows: In Section 1, we review the crystal base theory and the tableaux realization of crystal bases for the classical Lie algebras given by Kashiwara and Nakashima. In Section 2, we recall the notions and properties of Young walls and the Young wall realization of crystal bases for classical Lie algebras given in [9]. In Section 3, we give a new realization of crystal bases in terms of new tableaux, which are obtained from Young wall realization. In the end of Section 3, we give a 1-1 correspondence between our new tableau realization and the tableau realization given by Kashiwara and Nakashima, which turns out to be a crystal isomorphism.

1. Quantum groups and crystal bases

Let $I$ be an index set and let $(A, \Pi, \Pi^\vee, P, P^\vee)$ be a Cartan datum, i.e.,

(a) $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix,
called the
are classified into three categories:
Moody algebras called the
affine Lie algebras
finite dimensional simple Lie algebras
and those of affine type yield infinite dimensional Kac-
more details.) The indecomposable generalized Cartan matric es of finite type give rise to
algebra and the quantum affine algebra, respectively.
Let
be a
subring of
be the set of simple roots defined by
for
or
for
1
for
1
1

To each Cartan datum, we can associate two algebras
and
, called the Kac-Moody algebra and quantized universal enveloping algebra [7, 4]. The generalized Cartan matrices are classified into three categories: finite type, affine type and indefinite type. (See [7] for more details.) The indecomposable generalized Cartan matrices of finite type give rise to finite dimensional simple Lie algebras and those of affine type yield infinite dimensional Kac-Moody algebras called the affine Lie algebras. The corresponding quantum groups will be called the quantum classical algebra and the quantum affine algebra, respectively.

Let
be the set of dominant integral weights and let
be the fundamental weight defined by
. From now on, we denote by
and
the fundamental weights for the quantum classical algebra and the quantum affine algebra, respectively.

The crystal base theory is developed for
-modules
in the category
consisting of
-modules
such that

(a) 
has a weight space decomposition,

(b) there exist a finite number of elements
such that

where
is the set of all weights of
and
be the set of

(c) all
and
are locally nilpotent on
.

By the representation theory of
, every element
in
can be written uniquely as
where
and
and
are the Kashiwara operators
and
on
are defined by

Let
be a
-module in the category
and let
be the subring of
of
that are regular at

Definition 1.1. A pair
is called a crystal base if the following conditions are satisfied:

(a) 
is a free
submodule of
such that
,

(b) 
is a
basis of
,

(c) 
is a
-lattice of

(d) 
is a free
submodule of
for all
,

(e) 
for
and only if
.
For each $b \in B$ and $i \in I$, we define
\[ \varepsilon_i(b) = \max\{k \geq 0 \mid \epsilon_i^k b \in B\}, \quad \varphi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k b \in B\} \]
Let $M_j$ be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}_\text{int}^q$, with a crystal basis $(L_j, B_j)$ ($j = 1, \cdots, N$). Fix an index $i \in I$ and consider a vector $b = b_1 \otimes \cdots \otimes b_N \in B_1 \otimes \cdots \otimes B_N$. To each $b_j \in B_j$ ($j = 1, \cdots, N$), we assign a sequence of $-$’s and $+$’s with as many $-$’s as $\varepsilon_i(b_j)$ followed by as many $+$’s as $\varphi_i(b_j)$:
\[ b = b_1 \otimes b_2 \otimes \cdots \otimes b_N \mapsto (\varepsilon_1(b_1), \varphi_1(b_1), \ldots, \varepsilon_N(b_N), \varphi_N(b_N)) \]
In this sequence, we cancel out all the $(+, -)$-pairs to obtain a sequence of $-$’s followed by $+$’s, called the $i$-signature of $b$:
\[ i\text{-}sgn(b) = (-, -, \cdots, -, +, +, \cdots, +) \]
Then the crystal bases have a nice behavior with respect to the tensor product.

**Proposition 1.2.** [11, 12] Let $M_j$ be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}_\text{int}^q$ with a crystal basis $(L_j, B_j)$ ($j = 1, \cdots, N$) and let $b = b_1 \otimes \cdots \otimes b_N$ be a vector in $B_1 \otimes \cdots \otimes B_N$. Then $\tilde{e}_i$ acts on $b_j$ corresponding to the right-most $-$ in $i\text{-}sgn(b)$ and $\tilde{f}_i$ acts on $b_k$ corresponding to the left-most $+$ in $i\text{-}sgn(b)$:
\[ \tilde{e}_i b = b_1 \otimes \cdots \tilde{e}_i b_j \otimes \cdots \otimes b_N, \quad \tilde{f}_i b = b_1 \otimes \cdots \tilde{f}_i b_k \otimes \cdots \otimes b_N. \]
Moreover, $\tilde{e}_i b = 0$ (resp. $\tilde{f}_i b = 0$) if there is no $-$ (resp. $+$) in the $i$-signature of $b$.

**Example 1.3.** The representation $V(\lambda_1)$ for $U_q(B_n)$, called the vector representation and denoted by $V$, has a basis $\{ [i], [\overline{i}] ; i \in A \} \cup \{ [\emptyset] \}$ ($A = \{1, \cdots, n\}$) and the action of generators of $U_q(B_n)$ is given as follows:
\[ q^h [j] = q^{e_j(h)} [j], \quad q^h [\overline{j}] = q^{-e_j(h)} [\overline{j}], \quad q^h [\emptyset] = [\emptyset], \]
\[ e_i [j] = \delta_{i+1,j} [j^\uparrow], \quad e_i [\overline{j}] = \delta_{i,j} [j^\uparrow], \quad e_i [\emptyset] = 0 \]
\[ f_i [j] = \delta_{i,j} [j^\downarrow], \quad f_i [\overline{j}] = \delta_{i+1,j} [\overline{j}^\downarrow], \quad f_i [\emptyset] = 0 \]
\[ e_n [j] = 0, \quad e_n [\overline{j}] = \delta_{n,j} [\overline{j}^\downarrow], \quad e_n [\emptyset] = [2]^n_n [\emptyset] \]
\[ f_n [j] = \delta_{n,j} [j^\uparrow], \quad f_n [\overline{j}] = 0, \quad f_n [\emptyset] = [2]^n_n [\emptyset] \]
Then the crystal base $(\mathbf{L}, \mathbf{B})$ of $V$ is given by
\[ \mathbf{L} = \bigoplus_{i=1}^n (A [i] \oplus A [\overline{i}]) \oplus A [\emptyset], \]
\[ \mathbf{B} = \{ [i], [\overline{i}] ; i \in A \} \cup \{ [\emptyset] \}. \]
Moreover, the crystal graph of $\mathbf{B} = B(\lambda_1)$ is given by
\[ \begin{array}{cccccccc}
1 & \rightarrow & 2 & \rightarrow & \cdots & \rightarrow & n-1 & \rightarrow & n \rightarrow [\emptyset] \\
\downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
\overline{1} & \leftarrow & \overline{2} & \leftarrow & \cdots & \leftarrow & \overline{n-1} & \leftarrow & \overline{n} \leftarrow [\emptyset].
\end{array} \]
We close this section with the realization of crystal bases of irreducible highest weight modules for $U_q(\mathfrak{g})$ using Young tableaux, where $\mathfrak{g} = A_n, C_n, B_n, D_n$. In particular, we will focus on the classical Lie algebra $\mathfrak{g} = B_n$. The description of other types is referred to [14].

For a sequence of half integers $l_j \in \frac{1}{2}\mathbb{Z}$ ($j = 1, \ldots, n$), such that $l_i - l_j + 1 \in \mathbb{Z}_{\geq 0}$, let $Y = (l_1, \ldots, l_n)$ be a diagram which has $n$ rows with the $j$-th row of length $|l_j|$. Moreover, if $l_j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ for all $j$, $Y$ is called the generalized Young diagram of type $B_n$. Then there is a bijection $GY$ between the set of generalized Young diagrams and the set of dominant integral weights of $B_n$ defined by $GY : (l_1, \ldots, l_n) \mapsto \sum_{j=1}^{n} l_j \epsilon_j$, where $(\epsilon_1, \ldots, \epsilon_n)$ is the orthonormal base of $\mathfrak{h}^*$ [20].

The crystal graph $B(\lambda)$ is realized as the set of tableaux of shape $GY^{-1}(\lambda)$ on $\{i, \overline{i} \mid i \in \mathcal{A}\} \cup \{0\}$ with the linear order

$$1 < 2 < \cdots < n < 0 < \pi < \cdots < \overline{\pi} < \overline{\pi}.$$ 

To describe $B(\lambda)$, we need some definitions and conditions.

(1CC) Given a column $C$ of length $N$ containing the entries (reading from top to bottom) $i_1, i_2, \ldots, i_N$, we say that $C$ satisfies the one-column condition (1CC) if for $i_p = a$ and $i_q = \overline{a}$ ($1 \leq a \leq n$), $p + (N - q + 1) \leq a$.

Let $C$ and $C'$ be adjacent columns of length $N$ and $M$ ($1 \leq M \leq N \leq n$) consisting of the entries (reading from top to bottom) $i_1, i_2, \ldots, i_N$ and $j_1, j_2, \ldots, j_M$, respectively. (Note that $C$ can be a half-column.)

**Definition 1.4.** (a) For $1 \leq a \leq b < n$, we say that $C$ and $C'$ is in the $(a, b)$-configuration if there exist $1 \leq p \leq q < r \leq s \leq M$ such that $(i_p, i_q, i_r, i_s) = (a, b, \overline{a}, \overline{b})$ or $(i_p, j_q, j_r, j_s) = (a, b, \overline{a}, \overline{b})$.

(b) For $1 \leq a < n$, we say that $C$ and $C'$ is in the $(a, n)$-configuration if there exist $1 \leq p \leq q < r = q + 1 \leq s \leq M$ such that $i_p = a, j_s = \overline{a}$ and one of the following conditions is satisfied:

(i) $i_q$ and $i_r$ are $n$ or $\overline{n}$,

(ii) $j_q$ and $j_r$ are $n$ or $\overline{n}$.

(c) We say that $C$ and $C'$ is in the $(n, n)$-configuration if there exist $1 \leq p < q \leq M$ such that $(i_p, j_q) = (n, 0), (n, \overline{n})$ or $(0, \overline{n})$.

Now, for $C$ and $C'$ in the $(a, b)$-configuration, we define $p(a, b; C, C') = (q - p) + (s - r)$. In particular, if $a = b = n$, we set $p(a, b; C, C') = 0$.

(2CC) Given adjacent two columns $C$ and $C'$, we say that $C$ and $C'$ satisfy the two-column condition (2CC) if for every $(a, b)$-configuration, $p(a, b; C, C') < b - a$.

**Proposition 1.5.** [14] For a dominant integral weight $\lambda$, the crystal graph $B(\lambda)$ is realized as the set of tableaux $T$ of shape $GY^{-1}(\lambda)$ with entries $\{i, \overline{i} \mid i \in \mathcal{A}\} \cup \{0\}$ such that

(a) the entries of $T$ weakly increase along the rows, but the element $0$ cannot appear more than once,
(b) the entries of $T$ strictly increase down the columns, but the element $0$ can appear more than once,
(c) for a half column $C$ of $T$, $i$ and $\overline{i}$ can not appear at the same time,
(d) for each column $C$ of $T$, $(1CC)$ holds,
(e) for each pair of adjacent columns $C, C'$ of $T$, $(2CC)$ holds.

2. Crystal bases in terms of Young walls for the classical Lie algebras

In this section, we will review the realization of crystal bases of irreducible highest weight modules for the classical Lie algebras $\mathfrak{g} = A_n, C_n, B_n$ and $D_n$ in terms of Young walls given in [9].

Young wall is a combinatorial object for realizing the crystal bases for quantum affine algebras. They are built of colored blocks with three different shapes. Given an affine dominant integral weight $\Lambda$ of level 1 (i.e. $\Lambda(c) = 1$ for the canonical central element $c$), we fix a frame $Y_\Lambda$ called the ground state wall of weight $\Lambda$ and on this frame, we build a wall of thickness less than or equal to one unit with the rules for building the walls. The coloring of blocks, description of ground state walls and the patterns for building the walls are given in [8].

A column in a Young wall is called a full column if its height is a multiple of the unit length and its top is of unit thickness. A Young wall is said to be proper if none of the full columns have the same height for the quantum affine algebras of type $A_{2n-1}^{(2)}, B_n^{(1)}$ and $D_n^{(1)}$ and every Young wall is defined to be proper for the quantum affine algebras of type $A_n^{(1)}$.

Definition 2.1. (a) A block of color $i$ in a proper Young wall is called a removable $i$-block if the wall remains a proper Young wall after removing the block. A column in a proper Young wall is called $i$-removable if the top of that column is a removable $i$-block.

(b) A place in a proper Young wall where one may add an $i$-block to obtain another proper Young wall is called an $i$-admissible slot. A column in a proper Young wall is called $i$-admissible if the top of that column is an $i$-admissible slot.

Let $\mathbf{F}(\Lambda)$ be the set of all proper Young walls on $Y_\Lambda$ with a affine dominant integral weight $\Lambda$ of level 1. We now define the action of Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ on $\mathbf{F}(\Lambda)$. Fix $i \in I$ and let $Y = (y_k)_{k=0}^\infty \in \mathbf{F}(\Lambda)$ be a proper Young wall.

(1) To each column $y_k$ of $Y$, we assign
\[
\begin{align*}
\text{if } y_k & \text{ is twice } i\text{-removable}, \\
\text{if } y_k & \text{ is once } i\text{-removable but not } i\text{-admissible}, \\
\text{if } y_k & \text{ is once } i\text{-removable and once } i\text{-admissible}, \\
\text{if } y_k & \text{ is once } i\text{-admissible but not } i\text{-removable}, \\
\text{if } y_k & \text{ is twice } i\text{-admissible}, \\
\text{otherwise}.
\end{align*}
\]

(2) From the (infinite) sequence of +’s and −’s, cancel out every (+, −)-pair to obtain a finite sequence of −’s followed by +’s, reading from left to right. This sequence (− · · · − + · · · +) is called the i-signature of the proper Young wall Y.

(3) We define \( \tilde{e}_i Y \) to be the proper Young wall obtained from Y by removing the i-block corresponding to the right-most − in the i-signature of Y. We define \( \tilde{e}_i Y = 0 \) if there exists no − in the i-signature of Y.

(4) We define \( \tilde{f}_i Y \) to be the proper Young wall obtained from Y by adding an i-block to the column corresponding to the left-most + in the i-signature of Y. We define \( \tilde{f}_i Y = 0 \) if there exists no + in the i-signature of Y.

We also define
\[
\begin{align*}
\text{wt}(Y) &= \Lambda - \sum_{i \in I} k_i \alpha_i \in P, \\
\varepsilon_i(Y) &= \text{the number of } −\text{’s in the } i\text{-signature of } Y, \\
\varphi_i(Y) &= \text{the number of } +\text{’s in the } i\text{-signature of } Y,
\end{align*}
\]
where \( k_i \) denotes the number of i-blocks in Y that have been added to \( Y_\Lambda \).

**Proposition 2.2.** [4, 8] The set \( F(\Lambda) \) together with the maps \( \text{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i \) and \( \tilde{f}_i \) \( (i \in I) \) becomes an affine crystal.

Let \( \delta = d_0 \alpha_0 + \cdots + d_n \alpha_n \) be the null root for the quantum affine algebra \( U_q(\hat{g}) \), and set \( a_i = d_i \) if \( \hat{g} \neq D^{(2)}_{n+1} \), \( a_i = 2d_i \) if \( \hat{g} = D^{(2)}_{n+1} \). The part of a column consisting of \( a_0 \)-many 0-blocks, \( a_1 \)-many 1-blocks, \( \ldots \), \( a_n \)-many n-blocks in some cyclic order is called a \( \delta \)-column.

**Definition 2.3.** (a) A column in a proper Young wall is said to contain a removable \( \delta \) if we may remove a \( \delta \)-column from Y and still obtain a proper Young wall.

(b) A proper Young wall is said to be reduced if none of its columns contain a removable \( \delta \).

Let \( Y(\Lambda) \subset F(\Lambda) \) denote the set of all reduced proper Young walls on \( Y_\Lambda \) with an affine dominant integral weight \( \Lambda \) of level 1. Then we have:

**Proposition 2.4.** [4, 8] Let \( B(\lambda) \) be the crystal basis of the basic representation \( V(\Lambda) \) over quantum affine algebras, then there exists a crystal isomorphism
\[
Y(\Lambda) \xrightarrow{\sim} B(\Lambda) \quad \text{given by } Y_\Lambda \mapsto u_\Lambda,
\]
where \( u_\Lambda \) is the highest weight vector in \( B(\Lambda) \).
Let \( g \) be a classical Lie algebra of type \( A_n, C_n, B_n \) and \( D_n \). Then these Lie algebras lie inside an affine Lie algebra \( \hat{g} = A_n^{(1)}, A_{2n-1}^{(2)}, B_n^{(1)} \) and \( D_n^{(1)} \), respectively. (This fact can be expected because the Dynkin diagram of \( g \) can be obtained by removing the 0-node from the Dynkin diagram of \( \hat{g} \).)

Fix such a pair \( g \subset \hat{g} \) and let \( \Lambda \) be a dominant integral weight of level 1 for the affine Lie algebra \( \hat{g} \). Then by Proposition 2.4, the crystal graph \( B(\Lambda) \) is realized as the set \( Y(\Lambda) \) of all reduced proper Young walls built on the ground-state wall \( Y_\Lambda \). If we remove all 0-arrows in \( Y(\Lambda) \), then it is decomposed into a disjoint union of infinitely many connected components. Moreover, we can show that each connected component is isomorphic to the crystal graph \( B(\lambda) \) for some dominant integral weight \( \lambda \) for \( g \).

Conversely, any crystal graph \( B(\lambda) \) for \( g \) arises in this way. That is, given a dominant integral weight \( \lambda \) for \( g \), there is a dominant integral weight \( \Lambda \) of level 1 for \( \hat{g} \) such that \( B(\lambda) \) appears as a connected component in \( B(\Lambda) \) without 0-arrows.

The first step is to identify the highest weight vector \( H_\lambda \) for \( B(\lambda) \) with some reduced proper Young wall in \( Y(\Lambda) \) which is annihilated by all \( \tilde{e}_i \) for \( i = 1, \cdots, n \). However, given a dominant integral weight \( \lambda \) for \( g \), there are infinitely many such Young walls in \( Y(\Lambda) \). Equivalently, given \( \lambda \), there are infinitely many connected components of \( Y(\Lambda) \) without 0-arrows that are isomorphic to \( B(\lambda) \). Among these, they choose the characterization of \( B(\lambda) \) corresponding to the connected components having the least number of blocks [9].

From now on, we focus on the classical Lie algebra \( g = B_n \). We define the linear functionals \( \omega_i \) by

\[
\omega_i = \begin{cases} 
\lambda_i & \text{for } i = 1, \cdots, n-1, \\
2\lambda_n & \text{for } i = n.
\end{cases}
\]

Let \( F(\lambda) \subset Y(\Lambda) \) be the set of all reduced proper Young walls lying between highest weight vector \( H_\lambda \) and lowest weight vector \( L_\lambda \). (An algorithm of constructing the highest weight vector \( H_\lambda \) and lowest weight vector \( L_\lambda \) inside \( Y(\Lambda) \) was given in [9].) Set \( \lambda = \omega_{i_1} + \cdots + \omega_{i_t} + b\lambda_n \) for \( b = 0 \) or 1. For each \( Y \in F(\lambda) \), we denote by \( Y_{\omega_{i_k}} \) \( (k = 1, \cdots, t) \) (resp. \( Y_{\lambda_n} \)) the part of \( Y \) consisting of the blocks lying above \( H_{\omega_{i_k}} \) (resp. \( H_{\lambda_n} \)) and we denote by \( Y_{\overline{\omega_{i_k}}} \) (resp. \( Y_{\overline{\lambda_n}} \)) the intersection of \( Y \) and \( \overline{H_{\omega_{i_k}}} \) (resp. \( \overline{H_{\lambda_n}} \)), where \( \overline{H_{\omega_{i_k}}} \) (resp. \( \overline{H_{\lambda_n}} \)) is the Young wall consisting of \( H_{\omega_{i_k}} \) (resp. \( L_{\omega_{i_k}} \)), and \( i_k \times (t-k) \) or \( i_k \times (t-k+\frac{1}{2}) \)-many \( \delta \)-columns. Moreover, we denote by \( Y_{\omega_{i_k} + \omega_{i_{k+1}}} \) (resp. \( Y_{\omega_{i_k} + \lambda_n} \)) the union of \( Y_{\omega_{i_k}} \) (resp. \( Y_{\omega_{i_k}} \)) and \( Y_{\omega_{i_{k+1}}} \) (resp. \( Y_{\lambda_n} \)).
Example 2.5. If $g = B_4$, $\lambda = \omega_3 + \omega_4 + \lambda_4$, and let

\[
Y = \begin{array}{cccccccccc}
& & & & & & & & & 4 \\
& & & & & & & & & 3 \\
& & & & & & & & & 2 \\
& & & & & 1 & 0 & 0 & 1 & \ \\
& & & & & 2 & 2 & 2 & 2 & \ \\
& & & & & 3 & 3 & 3 & 3 & \ \\
& & & & & 4 & 4 & 4 & 4 & 4 \\
& & & & & 4 & 4 & 4 & 4 & 4 \\
& & & & & 4 & 4 & 4 & 4 & 4 \\
& & & & & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]

we have

\[
\circ \ Y_{\omega_3} = \begin{array}{cccc}
2 & 0 & 0 & 0 \\
3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 \\
1 & 0 & 0 & 1 \\
\end{array},
\circ \ Y_{\omega_4} = \begin{array}{cccc}
2 & 0 & 0 & 0 \\
3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 \\
1 & 0 & 0 & 1 \\
\end{array},
\text{ and } \circ \ Y_{\lambda_4} = \begin{array}{cccc}
2 & 0 & 0 & 0 \\
3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 \\
1 & 0 & 0 & 1 \\
\end{array}.
\]

Definition 2.6. (a) For $\overline{Y}_{\omega_{ik}}$, we define $Y^+_{\omega_{ik}}$ (resp. $Y^-_{\omega_{ik}}$) for $k = 1, \ldots, t$ by the part consisting of the blocks lying above (resp. below) the $n$-th row.

(b) For $\overline{Y}_{\omega_{ik} + \omega_{ik+1}}$ and $\overline{Y}_{\omega_{it} + \lambda_n}$ of $Y$, we define $Y^{\omega_{ik}}$ (resp. $Y^{\omega_{ik+1}}$) in $\overline{Y}_{\omega_{ik} + \omega_{ik+1}}$ is the intersection between the part consisting of blocks lying above the highest weight vector $\overline{\Pi}_{\omega_{ik}}$ (resp. $\overline{\Pi}_{\omega_{ik+1}}$) and the part consisting of blocks lying below the lowest weight vector $\overline{L}_{\omega_{ik+1}}$ (resp. $\overline{L}_{\omega_{ik}}$) reading from top (resp. right) to bottom (resp. left). Moreover, $Y^{\omega_{it}}$ and $Y^{\lambda_n}$ are defined by a similar way.

(c) For $\overline{Y}_{\omega_{ik} + \omega_{ik+1}}$ of $Y$, we define $L^-_{(\omega_{ik}, \omega_{ik+1})}$ (resp. $L^+_{(\omega_{ik}, \omega_{ik+1})}$) is the part of $L_{\omega_{ik}}$ (resp. $L_{\omega_{ik+1}}$) consisting of the right isosceles triangular blocks below (resp. above) the $n$-row. Moreover, $L^-_{(\omega_i, \lambda_n)}$ and $L^+_{(\omega_i, \lambda_n)}$ are defined by a similar way. Then

\[
Y^-_{(\omega_{ik}, \omega_{ik+1})} = Y \cap L^-_{(\omega_{ik}, \omega_{ik+1})} \quad \text{and} \quad Y^+_{(\omega_{ik}, \omega_{ik+1})} = Y \cap L^+_{(\omega_{ik}, \omega_{ik+1})},
\]

\[
Y^-_{(\omega_{it}, \lambda_n)} = Y \cap L^-_{(\omega_{it}, \lambda_n)} \quad \text{and} \quad Y^+_{(\omega_{it}, \lambda_n)} = Y \cap L^+_{(\omega_{it}, \lambda_n)}.\]

(d) We define $|Y^-_{\omega_{ik}}|$ by the wall obtained by reflecting $Y^-_{\omega_{ik}}$ along the $n$-row and shifting the blocks to the right as much as possible. Moreover, we define $|Y^-_{(\omega_{ik}, \omega_{ik+1})}|$ (resp. $|Y^-_{(\omega_{it}, \lambda_n)}|$) by the wall obtained by reflecting $Y^-_{(\omega_{ik}, \omega_{ik+1})}$ (resp. $Y^-_{(\omega_{it}, \lambda_n)}$) with respect to the upper $n$-row and shifting the blocks to the right as much as possible.
Example 2.7. Let $Y$ be a Young wall given in Example 2.5, then we have

\[ Y_{\omega_3}^+ = \emptyset, Y_{\omega_3}^- = \begin{array}{c} 3 \\ 2 \\ 1 \end{array}, Y_{\omega_4}^+ = \begin{array}{c} 3 \\ 2 \\ 1 \end{array} \text{ and } Y_{\omega_4}^- = \begin{array}{c} 3 \\ 2 \\ 1 \end{array} \]

\[ Y_{\omega_3} = \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \\ 4 \end{array}, \quad Y_{\omega_4} = \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array} \text{ in } Y_{\omega_3+\omega_4}, \]

\[ Y_{\omega_4}^- = \begin{array}{c} 2 \\ 3 \\ 4 \\ 1 \\ 3 \\ 4 \\ 1 \end{array}, \quad Y_{\lambda_4} = \begin{array}{c} 2 \\ 3 \\ 4 \\ 1 \\ 3 \\ 4 \\ 1 \end{array} \text{ in } Y_{\omega_4+\lambda_4}. \]

Moreover, the shaded parts represent $L_{(\omega_3,\omega_4)^-}(\omega_3,\omega_4), L_{(\omega_3,\omega_4)^+}(\omega_3,\omega_4)$ and $Y_{(\omega_3,\omega_4)^+}$, and

\[ Y_{(\omega_3,\omega_4)^-} = \begin{array}{c} 3 \\ 2 \\ 4 \end{array}, \quad Y_{(\omega_3,\omega_4)^+} = \begin{array}{c} 2 \\ 3 \\ 4 \end{array} \quad Y_{(\omega_4,\lambda_4)^-} = \begin{array}{c} 3 \\ 2 \\ 4 \end{array}, \quad Y_{(\omega_4,\lambda_4)^+} = \begin{array}{c} 3 \\ 2 \\ 4 \end{array}. \]

Now, we assume that $Y_{\omega_{ik}+\omega_{ik+1}}$ and $Y_{\omega_{it}+\lambda_n}$ satisfy the following condition: If the top of the $p$-th column of $Y_{\omega_{ik}}$ (resp. $Y_{\omega_{it}}$) from the right is $\begin{array}{c} 0 \\ a \end{array}$ and the top of the $q$-th column of $Y_{\omega_{ik+1}}$ (resp. $Y_{\lambda_{it}}$) from the right is $\begin{array}{c} a \\ b \end{array}$ with $p > q$, then it is called $Y_{\omega_{ik}+\omega_{ik+1}}$ (resp. $Y_{\omega_{it}+\lambda_n}$) satisfies (C1).

Definition 2.8. (a) We define $L_{\omega_{ik}}^+(a;p,q)$ (resp. $L_{\omega_{ik+1}}^+(a;p,q)$) to be the right isosceles triangle formed by $a$-block in the $q$-th column, $(a+p-q-1)$-block in the $q$-th column and $(a+p-q-1)$-block in the $(p-1)$-th column in $Y_{\omega_{ik}}$ (resp. $Y_{\omega_{ik+1}}$).

(b) We define $L_{\omega_{ik}}^-(a;p,q)$ (resp. $L_{\omega_{ik+1}}^-(a;p,q)$) by the wall obtained by reflecting $L_{\omega_{ik}}^+(a;p,q)$ (resp. $L_{\omega_{ik+1}}^+(a;p,q)$) with respect to the $n$-row.

(c) We also define $Y_{\omega_{ik}}^+(a;p,q), Y_{\omega_{ik}}^-(a;p,q)$ and $Y_{\omega_{ik}}^-(a;p,q)$ by

\[ Y_{\omega_{ik}}^+(a;p,q) = L_{\omega_{ik}}^+(a;p,q) \cap Y, \quad Y_{\omega_{ik+1}}^+(a;p,q) = L_{\omega_{ik+1}}^+(a;p,q) \cap Y, \]

\[ Y_{\omega_{it}}^+(a;p,q) = L_{\omega_{it}}^+(a;p,q) \cap Y, \]

(d) $|Y_{\omega_{ik}}^-(a;p,q)|$ is defined by the wall obtained by reflecting $Y_{\omega_{ik}}^-(a;p,q)$ with respect to the $n$-row and shifting the blocks to the right as much as possible.

Example 2.9. Let $Y$ be a Young wall given in Example 2.5, then we see that there exist

\[ \begin{array}{c} 2 \\ 3 \end{array} \text{ and } \begin{array}{c} 2 \\ 3 \end{array} \]

on top of the third and first column in $Y_{\omega_3}$ and $Y_{\omega_4}$, respectively. In this case, $a = 2$ and

\[ Y_{\omega_3}^+(2;3,1) = \emptyset, \quad Y_{\omega_3}^-(2;3,1) = \begin{array}{c} 3 \\ 2 \end{array}, \quad Y_{\omega_4}^+(2;3,1) = \begin{array}{c} 3 \\ 2 \end{array} \text{ and } Y_{\omega_4}^-(2;3,1) = \begin{array}{c} 3 \\ 2 \end{array}. \]
Now, we close this section with the realization theorem using Young walls for crystal bases.

**Proposition 2.10.** [9] Let $\lambda \in P^+$ be a dominant integral weight for $\mathfrak{g} = B_n$, and write
\[
\lambda = \omega_{i_1} + \cdots + \omega_{i_t} \quad (1 \leq i_1 \leq \cdots \leq i_t \leq n) \quad \text{or} \quad 
\lambda = \omega_{i_1} + \cdots + \omega_{i_t} + \lambda_n \quad (1 \leq i_1 \leq \cdots \leq i_t \leq n).
\]
We define $Y(\lambda)$ to be the set of all reduced proper Young walls satisfying the following conditions:

1. **(Y1)** For each $k = 1, \cdots, t$, we have $Y_{\omega_{i_k}}^+ \subset Y_{\omega_{i_k}}^-$;
2. **(Y2)** For each $k = 1, \cdots, t$, we have $Y_{\omega_{i_k}}^+ \subset Y_{\omega_{i_k+1}}^-$, $Y_{\omega_{i_t}}^+ \subset Y_{\lambda_n}^-;
3. **(Y3)** For each $k = 1, \cdots, t$, we have
\[
|Y_{(\omega_{i_k}, \omega_{i_k+1})}^-| \subset Y_{(\omega_{i_k}, \omega_{i_k+1})}^+, \quad |Y_{(\omega_{i_t}, \lambda_n)}^-| \subset Y_{(\omega_{i_t}, \lambda_n)}^+;
\]
4. **(Y4)** For each $k = 1, \cdots, t-1$, suppose that $Y_{\omega_{i_k}+\omega_{i_k+1}}^+$ or $Y_{\omega_{i_t}+\lambda_n}$ satisfy (C1), then we have
\[
Y_{\omega_{i_k}}^+(p, q, a) \subset |Y_{\omega_{i_k}}^-(p, q, a)|, \quad Y_{\omega_{i_k+1}}^+(p, q, a) \subset |Y_{\omega_{i_k+1}}^-(p, q, a)|,
Y_{\omega_{i_t}}^+(p, q, a) \subset |Y_{\omega_{i_t}}^-(p, q, a)|.
\]

Then there is an isomorphism of crystal graphs for $U_q(B_n)$-modules
\[
Y(\lambda) \simrightarrow B(\lambda) \quad \text{given by} \quad H_\lambda \mapsto u_\lambda,
\]
where $u_\lambda$ is the highest weight vector in $B(\lambda)$.

### 3. A new realization of crystal bases

In this section, we give a new realization of crystal bases for finite dimensional irreducible modules over classical Lie algebras derived from Young wall realization $Y(\lambda)$. The basis vectors are parameterized by certain tableaux with given shapes, which is different from generalized Young diagram given by Kashiwara and Nakashima.

We define the linear functionals $\omega_i$ by
1. $\mathfrak{g} = A_n, C_n$:
\[
\omega_i = \lambda_i \quad \text{for} \quad i = 1, \cdots, n,
\]
2. $\mathfrak{g} = D_n$:
\[
\omega_i = \begin{cases} 
\lambda_i & \text{for} \quad i = 1, \cdots, n-2, \\
\lambda_{n-1} + \lambda_n & \text{for} \quad i = n-1, \\
2\lambda_n & \text{for} \quad i = n, \\
2\lambda_{n-1} & \text{for} \quad i = n+1.
\end{cases}
\]
Then a dominant integral weight \( \lambda \) can be expressed as

\[
\lambda = \begin{cases} 
\omega_{i_1} + \cdots + \omega_{i_t} & \text{if } g = A_n, C_n, \\
\omega_{i_1} + \cdots + \omega_{i_t} + b\lambda_n & \text{if } g = B_n, \\
\omega_{i_1} + \cdots + \omega_{i_t} + b_1\lambda_{n-1} + b_2\lambda_n & \text{if } g = D_n,
\end{cases}
\tag{3.1}
\]

where \( 1 \leq i_1 \leq \cdots \leq i_t \leq n, b = 0 \) or \( 1, (b_1, b_2) = (1, 0) \) or \( (0, 1) \).

**Definition 3.1.** For a sequence of half integers \( l_j \in \frac{1}{2}\mathbb{Z} \) (\( j = 1, \cdots, n \)), such that \( l_j - l_{j+1} \in \mathbb{Z}_{\geq 0} \), let \( Y = (l_1, \cdots, l_n) \) be a diagram which has \( n \) rows with the \( j \)-th row (from bottom to top) of length \( |l_j| \). Then \( Y \) is called *generalized reverse Young diagram* of type \( A_n \) and \( C_n \) (resp. \( B_n \)) if all \( l_j \) are non-negative integers (resp. half integers). Moreover, \( Y \) is called *generalized reverse Young diagram* of type \( D_n \) if all \( l_j \) are half integers and \( l_1 \geq l_2 \geq \cdots \geq l_{n-1} \geq |l_n| \).

**Remark 3.2.** (a) It is just a diagram obtained by reflecting generalized Young diagram to the origin.

(b) There is a bijection \( GRY \) between the set of generalized reverse Young diagrams and the set of dominant integral weights of \( g \) defined by

\[
G R Y : (l_1, \cdots, l_n) \mapsto \sum_{j=1}^{n} l_j \epsilon_j,
\]

where \( (\epsilon_1, \cdots, \epsilon_n) \) is the orthonormal base of \( \mathfrak{h}^* \).

From now on, we will focus on the classical Lie algebra \( g = B_n \). The basic data for other types are presented in the end of this section. Let \( Y \in Y(\lambda) \) be a reduced proper Young wall with a dominant integral weight \( \lambda \). Then we can associate a tableau \( T_Y \) of shape \( GRY^{-1}(\lambda) \) determined by the following steps.

**Step 1.** At first, consider \( Y_{\omega_{i_k}} = (y_{\omega_{i_k}}^1, \cdots, y_{\omega_{i_k}}^i) \) in \( Y \), where \( y_{\omega_{i_k}}^a \) (\( a = 1, \cdots, i_k \)) is the \( a \)-th column of \( Y_{\omega_{i_k}} \) from the right. Then each column \( y_{\omega_{i_k}}^a \) corresponds to a box with entry, denoted by \( T_{\omega_{i_k}}^a \), as follows:

\[
\begin{array}{c|c}
0 \text{ or } 0 & \mapsto 1 \\
\hline
1 & \mapsto n \\
\hline
2 & \mapsto 2 \\
\hline
\hline
2 \text{ or } 0 & \mapsto 3 \\
\hline
\hline
j & \mapsto j \quad (4 \leq j \leq n) \\
\hline
n & \mapsto n \\
\hline
\hline
j & \mapsto j \quad (2 \leq j \leq n - 2) \\
\hline
\end{array}
\]

\[
\begin{array}{c|c}
\hline
0 & \mapsto 0 \\
\hline
1 & \mapsto 1 \\
\hline
\hline
\end{array}
\]
Here, the blocks of left-hand side represent the top blocks in each column \( y_{\omega k}^a \) of \( Y_{\omega k} \) and the boxes with entry of right-hand side represent \( T_{\omega k}^a \). Now, a tableau \( T_{\omega k} \) corresponded to \( Y_{\omega k} \) is obtained by stacking from \( T_{\omega k}^1 \) to \( T_{\omega k}^t \).

**Example 3.3.** If \( g = B_4 \), \( \lambda = \omega_3 \) and

\[
Y = \begin{array}{c}
\hline
1 & 2 & 3 \\
2 & 3 & 4 \\
0 & 1 & 2 \\
\hline
\end{array}
\in Y(\omega_3), \quad \text{then } T_Y = \begin{array}{c}
\hline
2 \\
3 \\
0 \\
\hline
\end{array}
\]

**Step 2.** Consider \( Y_{\lambda n} = (y_{\lambda n}^1, \cdots, y_{\lambda n}^n) \), where \( y_{\lambda n}^a \) \((a = 1, \cdots, n)\) is the \( a \)-th column of \( Y_{\lambda n} \) from the right. On the one hand, if \( Y_{\lambda n} = H_{\lambda n} \), then it corresponds to a tableau

\[
\begin{array}{c}
\hline
1 \\
\hline
\end{array}
\]

Suppose that there are some added blocks above \( H_{\lambda n} \) in \( Y_{\lambda n} \). Then the columns having added blocks above \( H_{\lambda n} \) correspond to boxes with entries as follows.

\[
\begin{array}{c}
\hline
+ \rightarrow \bigcirc \\
\hline
\end{array}
\quad \begin{array}{c}
\hline
j \rightarrow \bigcirc \quad (2 \leq j \leq n - 2) \\
\hline
\end{array}
\quad \begin{array}{c}
\hline
+ \rightarrow \bigcirc \\
\hline
\end{array}
\quad \begin{array}{c}
\hline
\bigcirc \text{ or } \bigcirc \rightarrow \bigcirc \\
\hline
\end{array}
\]

Moreover, if \( k \)-many columns with added blocks above \( H_{\lambda n} \) in \( Y_{\lambda n} \) correspond to \( k \)-many boxes with entries \( i_1, \cdots, i_k \in \{1, \cdots, n\} \), then \( (n - k) \)-many columns with no added block above \( H_{\lambda n} \) correspond to \( (n - k) \)-many boxes with entry on \( \{1, \cdots, n\} \setminus \{i_1, \cdots, i_k\} \). Finally, \( Y_{\lambda n} \) corresponds to a tableau of half column of length \( n \) obtained by stacking \( T_{\lambda n}^a \) \((a = 1, \cdots, n)\) for the entries of \( T_{\lambda n}^a \) to strictly increase down the columns.

**Remark 3.4.** \( T(\lambda_n) \) is just the tableau realization of \( B_{sp} \) given by Kashiwara and Nakashima.

**Example 3.5.** Let \( g = B_3 \), \( \lambda = \lambda_3 \) and

\[
Y = (y_{\lambda_3}^1, y_{\lambda_3}^2, y_{\lambda_3}^3) = \begin{array}{c}
\hline
1 & 2 \\
3 & 3 \\
3 & 3 \\
\hline
\end{array}
\]

Since \( y_{\lambda_3}^1 \) and \( y_{\lambda_3}^2 \) correspond to \( \bigcirc \) and \( \bigcirc \), respectively, we have

\[
T_Y = \begin{array}{c}
\hline
2 \\
3 \\
\hline
\end{array}
\]

**Step 3.** Finally, a tableau \( T_Y \) corresponded to \( Y \in Y(\lambda) \) of shape \( GRY^{-1}(\lambda) \) is obtained by attaching \( T_{\omega_k} \) \((k = 1, \cdots, t - 1)\) and \( T_{\omega_t} \) to the left-hand side of \( T_{\omega_{k+1}} \) and \( T_{\lambda n} \) to be bottom-justified.
Example 3.6. If $g = B_4$, $\lambda = \omega_2 + \omega_3 + \lambda_4$ and

$$
\lambda = \omega_2 + \omega_3 + \lambda_4
$$

and

$$
Y = \begin{bmatrix}
3 & 4 & 4 & 4 \\
3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 \\
0 & 1 & 0 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 \\
\end{bmatrix}
\in Y(\lambda),
$$

then $T_Y = \begin{bmatrix}
1 & 2 & 3 \\
0 & 3 & 3 \\
3 & 0 & 3 \\
3 & 6 & 2 \\
\end{bmatrix}$.

Until now, we define a new tableau $T_Y$ of shape $GRY^{-1}(\lambda)$ associated with $Y \in Y(\lambda)$ with a dominant integral weight $\lambda$. We will characterize these new tableaux by the conditions in $Y(\lambda)$.

At first, as we have seen, if we let $Y \in Y(\lambda)$, then $\Upsilon_{\omega_{ik}} = (y^1_{\omega_{ik}}, \ldots, y^{i_k}_{\omega_{ik}})$ corresponds to one column $T_{\omega_{ik}}$ of $T_Y$. Moreover, one column $y^a_{\omega_{ik}}$ ($a = 1, \ldots, i_k$) in $\Upsilon_{\omega_{ik}}$ corresponds to one box $T^a_{\omega_{ik}}$ in $T_{\omega_{ik}}$. Then since $Y$ is proper and $T_{\omega_{ik}}$ is obtained by stacking from $T^1_{\omega_{ik}}$ to $T^{i_k}_{\omega_{ik}}$, the entries of the tableau $T_Y$ strictly increase down the columns. Moreover, we can stack $n$-block as $\begin{bmatrix} n & n \end{bmatrix}$ in $\Upsilon_{\omega_{ik}}$ by the pattern of stacking blocks because the $n$-block is a block of half-unit-height and $Y$ is proper. So, the 0 element can appear more than once in tableau $T_{\omega_{ik}}$.

On the other hand, consider the condition that $Y_{\omega_{ik}} \subset Y_{\omega_{ik+1}}$ and $Y_{\omega_{it}} \subset Y_{\lambda_n}$. Note that $y^a_{\omega_{ik}}$ (resp. $y^a_{\omega_{it}}$) and $y^a_{\omega_{ik+1}}$ (resp. $y^a_{\lambda_n}$) of $\Upsilon_{\omega_{ik}}$ (resp. $\Upsilon_{\omega_{it}}$) and $\Upsilon_{\omega_{ik+1}}$ (resp. $\Upsilon_{\lambda_n}$) correspond to the boxes with entries in the same row in $T_{\omega_{ik}} + \omega_{ik+1}$ (resp. $T_{\omega_{it}} + \lambda_n$). Therefore, this condition implies that the entries are weakly increasing in each rows of $T_Y$.

Consider the condition (Y1) $Y_{\omega_{ik}}^+ \subset |Y_{\omega_{ik}}^-|$ $(1 \leq i_k \leq n)$ in Proposition 2.9. Then this condition implies that the following can not exist in $T_{\omega_{ik}}$ as a subtableau :

$$
\begin{array}{ccc}
\begin{array}{|c|}
\hline
1 \\
\hline
\end{array} & \begin{array}{|c|}
\hline
2 \\
\hline
\end{array} & \begin{array}{|c|}
\hline
\vdots \\
\hline
\end{array} & \begin{array}{|c|}
\hline
a \\
\hline
\end{array} & \begin{array}{|c|}
\hline
\vdots \\
\hline
\end{array} & \begin{array}{|c|}
\hline
i_{k-1} \\
\hline
\end{array} & \begin{array}{|c|}
\hline
i_k - (a+1) \\
\hline
\end{array} & \begin{array}{|c|}
\hline
i_{k-2} \geq \\
\hline
\end{array} & \begin{array}{|c|}
\hline
i_{k-3} \geq \\
\hline
\end{array} & \begin{array}{|c|}
\hline
T \\
\hline
\end{array}
\end{array}
\end{array}
$$

Therefore, if $T_{\omega_{ik}}$ has $a$-box and $\bar{a}$-box, then the number of rows between $a$-box and $\bar{a}$-box is larger than or equals to $i_k - a$. That is, if $a$-box and $\bar{a}$-box lie in $p$-th and $q$-th rows of $T_{\omega_{ik}}$ from bottom to top with $p > q$, we have

$$
p - q - 1 \geq i_k - a. \quad (3.2)
$$
Remark 3.7. (a) In fact, we know that the following also can not exist in \( T_{\omega_{i_k}} \) as a subtableau.

\[
\begin{array}{c}
\begin{array}{ccccccc}
\text{a} & \text{a} & \text{a} & \text{a} & \text{a} & \text{a} & \text{a} \\
\text{a} & \text{a} & \text{a} & \text{a} & \text{a} & \text{a} & \text{a} \\
\text{a} & \text{a} & \text{a} & \text{a} & \text{a} & \text{a} & \text{a} \\
\text{a} & \text{a} & \text{a} & \text{a} & \text{a} & \text{a} & \text{a} \\
\text{a} & \text{a} & \text{a} & \text{a} & \text{a} & \text{a} & \text{a} \\
\end{array}
\end{array}
\]

(1 \leq a < i_k).

But, it is evident that this condition is subject to above condition by the pattern of stacking the blocks in Young walls and proper condition of \( Y \). Anyway, if \( T_{\omega_{i_k}} \) has \( a \)-box and \( \bar{b} \)-box, then the number of rows between \( a \)-box and \( \bar{b} \)-box is larger than or equals to \( i_k - \max(a, b) \).

That is, if \( a \)-box and \( \bar{b} \)-box lie in \( p \)-th and \( q \)-th rows of \( T_{\omega_{i_k}} \) from bottom to top with \( p > q \), we have

\[
p - q - 1 \geq i_k - \max(a, b). \tag{3.3}
\]

(b) It is just the one column condition (1CC) of Kashiwara-Nakashima tableaux. We will also call it the one column condition and denote by (1CC).

Consider the condition (Y3) in Proposition 2.9. That is,

\[
|Y_{\omega_{i_k}}^-| \subset Y_{\omega_{i_k+1}}^+ \quad \text{and} \quad |Y_{\omega_{i_t}}^-| \subset Y_{\lambda_n}^+.
\]

This condition says that the following can not exist in \( T_{\omega_{i_k} + \omega_{i_{k+1}}} \) \((k = 1, \cdots, t - 1)\) and \( T_{\omega_{i_t} + \lambda_n} \) as a subtableau:

\[
\begin{array}{c}
\begin{array}{ccccccc}
\text{n} & \text{n} & \text{n} & \text{n} & \text{n} & \text{n} & \text{n} \\
\text{n} & \text{n} & \text{n} & \text{n} & \text{n} & \text{n} & \text{n} \\
\text{n} & \text{n} & \text{n} & \text{n} & \text{n} & \text{n} & \text{n} \\
\text{n} & \text{n} & \text{n} & \text{n} & \text{n} & \text{n} & \text{n} \\
\text{n} & \text{n} & \text{n} & \text{n} & \text{n} & \text{n} & \text{n} \\
\end{array}
\end{array}
\]

Here, \( n - i_k + 2 \leq a \leq n \) and \( n - i_t + 2 \leq a \leq n \), respectively. Therefore, if \( T_{\omega_{i_k}} \) (resp. \( T_{\omega_{i_t}} \)) and \( T_{\omega_{i_{k+1}}} \) (resp. \( T_{\lambda_n} \)) have \( a \) and \( \bar{a} \), respectively, then the number of rows between \( a \)-box and \( \bar{a} \)-box is smaller than \( n - a \). That is, if \( a \)-box and \( \bar{a} \)-box lie in \( p \)-th and \( q \)-th rows of \( T_{\omega_{i_k}} \) (resp. \( T_{\omega_{i_t}} \)) and \( T_{\omega_{i_{k+1}}} \) (resp. \( T_{\lambda_n} \)) from bottom to top with \( p > q \), we have \( p - q - 1 < n - a \) and so

\[
p - q \leq n - a. \tag{3.4}
\]
Remark 3.8. (a) In fact, we know that the following also can not exist in $T_{\omega_{i_k} + \omega_{i_{k+1}}}$ as a subtableau.

\[
\begin{array}{ccc}
\begin{array}{cccc}
 a & \cdots & a \\
 \pi & 0 & \cdots & 0 \\
\end{array}
\end{array}
\begin{array}{ccc}
\begin{array}{cccc}
 a & \cdots & a \\
 \pi & 0 & \cdots & 0 \\
\end{array}
\end{array}
\begin{array}{ccc}
\begin{array}{cccc}
 a & \cdots & a \\
 \pi & 0 & \cdots & 0 \\
\end{array}
\end{array}
\begin{array}{ccc}
\begin{array}{cccc}
 a & \cdots & a \\
 \pi & 0 & \cdots & 0 \\
\end{array}
\end{array}
\begin{array}{ccc}
\begin{array}{cccc}
 a & \cdots & a \\
 \pi & 0 & \cdots & 0 \\
\end{array}
\end{array}
\end{array}
\]  

where $n - i_k + 2 \leq a \leq n$. Moreover, the following can not exist in $T_{\omega_{i_k} + \omega_{i_{k+1}}}$ as a subtableau.

\[
\begin{array}{c}
\begin{array}{ccc}
 0 & 1 \\
 0 & \pi & 0 \\
 0 & \pi & 0 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
 0 & 1 \\
 0 & \pi & 0 \\
 0 & \pi & 0 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
 0 & 1 \\
 0 & \pi & 0 \\
 0 & \pi & 0 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
 0 & 1 \\
 0 & \pi & 0 \\
 0 & \pi & 0 \\
\end{array}
\end{array}
\end{array}
\]  

But, it is clear that this condition is subject to above condition (1.7). Anyway, if $T_{\omega_{i_k}}$ and $T_{\omega_{i_{k+1}}}$ have $a$ and $\bar{b}$, respectively, the number of rows between $a$-box and $\bar{b}$-box is smaller than $n - \min(a, b)$. That is, if $a$-box and $\bar{b}$-box lie in $p$-th and $q$-th rows of $T_{\omega_{i_k}}$ and $T_{\omega_{i_{k+1}}}$ from bottom to top with $p > q$, we have $p - q - 1 < n - \min(a, b)$ and so

$$p - q \leq n - \min(a, b).$$

(3.5)

Similarly, it also holds for $T_{\omega_{i_l} + \lambda_n}$.

(b) This condition contains the condition very similar to the two column condition (2CC) for the tableau of Kashiwara and Nakashima in the $(a, n)$, $(n, n)$-configuration. We will call it the first two column condition and denote by (2CC-1).

Consider the condition (Y4) in Proposition 2.9. That is,

$$Y^+_{\omega_{i_k}}(p, q, a) \subset Y^-_{\omega_{i_k}}(p, q, a), \quad Y^+_{\omega_{i_{k+1}}}(p, q, a) \subset Y^-_{\omega_{i_{k+1}}}(p, q, a) \quad (1 \leq k \leq t - 1),$$

$$Y^+_{\omega_{i_l}}(p, q, a) \subset Y^-_{\omega_{i_l}}(p, q, a).$$

At first, we consider the assumptions of the condition (Y4). The top of $\mathbf{Y}_{\omega_{i_k}}$ (resp. $\mathbf{Y}_{\omega_{i_l}}$) at $p$-th column from the right is $\frac{a_{p+1}}{a_{p}}$ and the top of $\mathbf{Y}_{\omega_{i_{k+1}}}$ (resp. $\mathbf{Y}_{\lambda_n}$) at $q$-th column from the right is $\frac{a}{a_{q+1}}$ with $p > q$. Then these assumptions can be represented in the tableau $T_{\omega_{i_k} + \omega_{i_{k+1}}}$ and $T_{\omega_{i_l} + \lambda_n}$ as follows.

\[
\begin{array}{c}
\begin{array}{ccc}
 p-th & a \\
 \pi & q-th \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
 p-th & a \\
 \pi & q-th \\
\end{array}
\end{array}
\]  

Now, consider the subtableau $T_{\omega_{i_k}+\omega_{i_{k+1}}-2}$ and $T_{\omega_{i_j}+\lambda}$ consisting of boxes lying between $q$-th row and $p$-th row in $T_{\omega_{i_k}+\omega_{i_{k+1}}-2}$ and $T_{\omega_{i_j}+\lambda}$. Then the condition $Y_{\omega_{i_k}}^+(p,q,a) \subset |Y_{\omega_{i_k}}^-(p,q,a)|$ (resp. $Y_{\omega_{i_j}}^+(p,q,a) \subset |Y_{\omega_{i_k}}^-(p,q,a)|$) imply that the following can not exist in $T_{\omega_{i_k}+\omega_{i_{k+1}}-2} \cap T_{\omega_{i_j}+\lambda}$ and $T_{\omega_{i_j}+\lambda}$ as a subtableau.

Therefore, if $T_{\omega_{i_k}}$ and $T_{\omega_{i_{k+1}}-2}$ has $b$-box and $b$-box between $p$-th column and $q$-th column, then the number of rows of between $b$-box and $b$-box in $T_{\omega_{i_k}}$ or $T_{\omega_{i_{k+1}}-2}$ is larger than to $p-q-(b-a+1)$. That is, if $b$-box and $b$-box lie in $r$-th and $s$-th rows of $T_{\omega_{i_k}}$ or $T_{\omega_{i_{k+1}}-2}$ from bottom to top with $r > s$, we have $r-s-1 > p-q-(b-a+1)$ and so

$$(p-r) + (s-q) < b-a. \quad (3.6)$$

**Remark 3.9.** (a) In fact, we know that the following also can not exist in $T_{\omega_{i_k}+\omega_{i_{k+1}}-2} \cap T_{\omega_{i_j}+\lambda}$ and $T_{\omega_{i_j}+\lambda}$ as a subtableau.

where $0 \leq k \leq p-q-a+1$. Therefore, if $T_{\omega_{i_k}}$ or $T_{\omega_{i_{k+1}}-2}$ has $b$-box and $b$-box between $p$-th column and $q$-th column, then the number of rows of between $b$-box and $b$-box is larger than $p-q-(\max(b,c)-a+1)$. That is, if $b$-box and $b$-box lie in $r$-th and $s$-th rows of $T_{\omega_{i_k}}$ from bottom to top with $r > s$, we have $r-s-1 > p-q-(\max(b,c)-a+1)$ and so

$$(p-r) + (s-q) < \max(b,c) - a. \quad (3.7)$$

(b) This condition is very similar to the two column condition (2CC) for the tableau of Kashiwara and Nakashima in the $(a,b)$-configuration. We will call it the second two column condition and denote by (2CC-2).

(c) The first two column condition (2CC-1) covers a lot of parts of this condition (2CC-2).

Now, we are ready to give an another explicit realization of crystal graph $B(\lambda)$ over $B_n$.

For a dominant integral weight $\lambda$, we define $T(\lambda)$ for $g = B_n$ by the set of tableaux of shape $GRY^{-1}(\lambda)$ with entries $\{i, \bar{i} | 1 \leq i \leq n \} \cup \{0\}$ such that

1. the entries of $T$ weakly increase along the rows, but the element 0 cannot appear more than once,
2. the entries of $T$ strictly increase down the columns, but the element 0 can appear more than once,
(3) for a half column $C$ of $T$, $i$ and $\bar{i}$ cannot appear at the same time,
(4) for each column $C$ of $T$, (1CC) holds,
(5) for each pair of adjacent columns $C, C'$ of $T$, (2CC-1) and (2CC-2) holds,
and we identify a tableau $T$ of shape $GRY^{-1}(\lambda)$ with the vector
\[ x_1 \otimes \cdots \otimes x_t \in B^t \quad \text{or} \quad v_{sp} \otimes x_1 \otimes \cdots \otimes x_t \in B_{sp} \otimes B^t, \]
where $x_i$ is an entry of $T$ by reading from top to bottom and from right to left. Then we have:

**Theorem 3.10.** For a dominant integral weight $\lambda$, there exists a crystal isomorphism for $U_q(B_n)$-modules $\varphi: Y(\lambda) \rightarrow T(Y)$ sending $Y$ to $T_Y$.

**Proof.** It is clear that $\varphi$ is a bijection. So, it suffices to show that it is a crystal morphism. Suppose that $\tilde{f}_i Y$ ($1 \leq i < n$) is obtained by stacking the block $\begin{array}{c} 1 \end{array}$ on top of some column $y_a$ of $Y$. Then these columns of $Y$ and $\tilde{f}_i Y$ correspond to the boxes $\begin{array}{c} i \end{array}$ and $\begin{array}{c} i+1 \end{array}$ or $\begin{array}{c} i+1 \end{array}$ and $\begin{array}{c} i \end{array}$ of $T_Y$ and $T_{\tilde{f}_i Y}$. Moreover, it is easy to see that the $i$-signatures of each column $y_k$ of $Y$ and a corresponded box $\varphi(y_k)$ of $T_Y$ are the same. Therefore, by the tensor product rule of Kashiwara operators, it is easy to see that $\tilde{f}_i T_Y$ is obtained by acting on $\varphi(y_a)$ of $T_Y$ and so $\varphi(\tilde{f}_i Y) = \tilde{f}_i \varphi(Y)$. For $i = n$, it is proved by the same method. Moreover, for $\tilde{e}_i$, it is also proved by the similar argument. □

By the same algorithm, we can obtain new tableau realization of crystal bases of irreducible highest weight modules over other classical Lie algebras. The basic data in **Step 1** and **Step 2** over other classical Lie algebras are as follows:

**Step 1.**
1) $A_n$ ($n \geq 1$)
\[ \begin{array}{c} j \end{array} \longrightarrow \begin{array}{c} j+1 \end{array} \quad (0 \leq j \leq n), \]

2) $C_n$ ($n \geq 3$)
\[ \begin{array}{c} 0 \end{array} \text{ or } \begin{array}{c} \bar{\bar{0}} \end{array} \longrightarrow \begin{array}{c} 1 \end{array} \]
\[ \begin{array}{c} 0 \end{array} \text{ or } \begin{array}{c} \bar{\bar{0}} \end{array} \longrightarrow \begin{array}{c} 2 \end{array} \]
\[ \begin{array}{c} 0 \end{array} \text{ or } \begin{array}{c} \bar{\bar{0}} \end{array} \longrightarrow \begin{array}{c} 3 \end{array} \]
\[ \begin{array}{c} n \end{array} \quad \text{ or } \begin{array}{c} \bar{n} \end{array} \longrightarrow \begin{array}{c} n \end{array} \]
\[ \begin{array}{c} j \end{array} \quad \text{ or } \begin{array}{c} \bar{j} \end{array} \longrightarrow \begin{array}{c} j \end{array} \quad (2 \leq j \leq n - 1) \]
\[ \begin{array}{c} j \end{array} \quad \text{ or } \begin{array}{c} \bar{j} \end{array} \longrightarrow \begin{array}{c} j \end{array} \quad (4 \leq j \leq n) \]
\[ \begin{array}{c} 0 \end{array} \text{ or } \begin{array}{c} \bar{\bar{0}} \end{array} \longrightarrow \begin{array}{c} 1 \end{array} \]

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3) \( D_n \) \((n \geq 4)\)

\[
\begin{array}{c}
\emptyset \text{ or } \begin{array}{c}
\_
\end{array} \rightarrow 1 \\
\begin{array}{c}
\emptyset
\end{array} \text{ or } \begin{array}{c}
\cdot
\end{array} \rightarrow 2 \\
\begin{array}{c}
\cdot
\end{array} \text{ or } \begin{array}{c}
\cdot
\end{array} \rightarrow 3 \\
\begin{array}{c}
\cdot
\end{array} \text{ or } \begin{array}{c}
\cdot
\end{array} \rightarrow n \\
\begin{array}{c}
\cdot
\end{array} \text{ or } \begin{array}{c}
\cdot
\end{array} \rightarrow n
\end{array}
\]

Step 2. Now, consider the \( \Upsilon_{\lambda_{n-1}} \) and \( \Upsilon_{\lambda_n} \) for \( g = D_n \). If \( \Upsilon_{\lambda_{n-1}} = H_{\lambda_{n-1}} = \ldots \begin{array}{c}
\cdot
\end{array} \), \( \Upsilon_{\lambda_n} = H_{\lambda_n} = \ldots \begin{array}{c}
\cdot
\end{array} \), they correspond to the tableaux

\[
\begin{array}{c}
\begin{array}{c}
\cdot
\end{array} \\
\begin{array}{c}
\cdot
\end{array}
\end{array}
\quad \text{ and } \quad
\begin{array}{c}
\begin{array}{c}
\cdot
\end{array} \\
\begin{array}{c}
\cdot
\end{array}
\end{array}
\]

respectively.

Suppose that there are some added blocks above \( H_{\lambda_{n-1}} \) and \( H_{\lambda_n} \) in \( \Upsilon_{\lambda_{n-1}} \) and \( \Upsilon_{\lambda_n} \). Then the columns having added blocks above \( H_{\lambda_{n-1}} \) or \( H_{\lambda_n} \) correspond to boxes with entries as follows.

\[
\begin{array}{c}
\begin{array}{c}
\cdot
\end{array} \quad \text{ or } \quad \begin{array}{c}
\cdot
\end{array} \rightarrow \begin{array}{c}
\cdot
\end{array} \\
\begin{array}{c}
\cdot
\end{array} \quad \text{ or } \quad \begin{array}{c}
\cdot
\end{array} \rightarrow \begin{array}{c}
\cdot
\end{array} \\
\begin{array}{c}
\cdot
\end{array} \quad \text{ or } \quad \begin{array}{c}
\cdot
\end{array} \rightarrow \begin{array}{c}
\cdot
\end{array} \\
\begin{array}{c}
\cdot
\end{array} \quad \text{ or } \quad \begin{array}{c}
\cdot
\end{array} \rightarrow \begin{array}{c}
\cdot
\end{array} \\
\begin{array}{c}
\cdot
\end{array} \quad \text{ or } \quad \begin{array}{c}
\cdot
\end{array} \rightarrow \begin{array}{c}
\cdot
\end{array} \\
\begin{array}{c}
\cdot
\end{array} \quad \text{ or } \quad \begin{array}{c}
\cdot
\end{array} \rightarrow \begin{array}{c}
\cdot
\end{array} \\
\begin{array}{c}
\cdot
\end{array} \quad \text{ or } \quad \begin{array}{c}
\cdot
\end{array} \rightarrow \begin{array}{c}
\cdot
\end{array} \\
\begin{array}{c}
\cdot
\end{array} \quad \text{ or } \quad \begin{array}{c}
\cdot
\end{array} \rightarrow \begin{array}{c}
\cdot
\end{array} \\
\begin{array}{c}
\cdot
\end{array} \quad \text{ or } \quad \begin{array}{c}
\cdot
\end{array} \rightarrow \begin{array}{c}
\cdot
\end{array} \\
\begin{array}{c}
\cdot
\end{array} \quad \text{ or } \quad \begin{array}{c}
\cdot
\end{array} \rightarrow \begin{array}{c}
\cdot
\end{array}
\end{array}
\]

Moreover, if a column whose right column is a full column is \( \begin{array}{c}
\cdot
\end{array} \), a part of ground-state wall, then it corresponds to \( \begin{array}{c}
\cdot
\end{array} \). Finally, if \( k \)-many columns with blocks above \( H_{\lambda_{n-1}} \) or \( H_{\lambda_n} \) in \( \Upsilon_{\lambda_{n-1}} \) and \( \Upsilon_{\lambda_n} \) correspond to \( k \)-many boxes with entries \( i_1, \ldots, i_k \in \{\bar{n}, \ldots, \bar{T}\} \), then \( (n - k) \)-many columns with no block above \( H_{\lambda_{n-1}} \) or \( H_{\lambda_n} \) correspond to boxes with entries on \( \{1, \ldots, n\} \setminus \{\bar{i}_1, \ldots, \bar{i}_k\} \). Now, \( \Upsilon_{\lambda_{n-1}} \) and \( \Upsilon_{\lambda_n} \) correspond to tableaux of half column of length \( n \) obtained by stacking \( T_{\lambda_{n-1}} \) and \( T_{\lambda_n} \) \((a = 1, \ldots, n)\) for the entries of \( T_{\lambda_{n-1}} \) and \( T_{\lambda_n} \) to strictly increase down the columns, respectively.
Example 3.11. Let \( g = D_4 \), \( \lambda = \lambda_3 \) and
\[
Y = (y^1_{\lambda_3}, y^2_{\lambda_3}, y^3_{\lambda_3}, y^4_{\lambda_3}) = \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}
\]
Then \( y^1_{\lambda_3} \) and \( y^2_{\lambda_3} \) correspond to \( \begin{array}{c}
1 \\
2 \\
\end{array} \) and \( \begin{array}{c}
3 \\
4 \\
\end{array} \), respectively. Moreover, since \( y^3_{\lambda_3} \) is \( \begin{array}{c}
4 \\
\end{array} \) and the its right column \( y^4_{\lambda_3} \) is a full column, it corresponds to \( \begin{array}{c}
3 \\
\end{array} \). Therefore, we have
\[
T_Y = \begin{array}{c}
3 \\
\end{array}
\]
Then we have the following realization of crystal bases for the classical Lie algebras \( g = A_n, C_n \) and \( D_n \).

At first, we define \( T(\lambda) \) for \( g = A_n \) and \( C_n \) as follows:
1) \( g = A_n \) : the set of tableaux of shape \( GRY^{-1}(\lambda) \) with entries \( \{1, \cdots, n\} \) such that
   (a) the entries of \( T \) weakly increase along the rows,
   (b) the entries of \( T \) strictly increase down the columns,
2) \( g = C_n \) : the set of tableaux of shape \( GRY^{-1}(\lambda) \) with entries \( \{i, j | i = 1, \cdots, n\} \) such that
   (a) the entries of \( T \) weakly increase along the rows,
   (b) the entries of \( T \) strictly increase down the columns,
   (c) for each column \( C \) of \( T \), (1CC) holds,
   (d) for each pair of adjacent columns \( C, C' \) of \( T \), (2CC-1) and (2CC-2) holds.

Theorem 3.12. For a dominant integral weight \( \lambda \), there exists a crystal isomorphism for \( U_q(g) \)-modules \( \varphi : Y(\lambda) \to T(\lambda) \) sending \( Y \) to \( T_Y \), where \( g = A_n \) and \( C_n \).

Proof. It is similar to the proof of Theorem Theorem 3.10, we omit it. \( \square \)

For the type of \( g = D_n \), we have another conditions. Let \( C \) and \( C' \) be adjacent columns of length \( N \) and \( M \) (\( 1 \leq M \leq N \leq n \)) consisting of the entries (reading from bottom to top) \( i_1, i_2, \cdots, i_M \) and \( j_1, j_2, \cdots, j_N \), respectively. (Note that \( C \) can be a half-column.)

Definition 3.13. For \( 1 \leq a < n \), we say that \( C \) and \( C' \) is in the \( a \)-odd-configuration (resp. \( a \)-even-configuration) if there exist \( M \geq p \geq q > r \geq s \geq 1 \) such that \( q - r + 1 \) is odd (resp. even) and
\[
(i_p, j_q, i_r, j_s) = (a, n, \overline{n}, \overline{a}) \quad \text{or} \quad (a, \overline{n}, n, \overline{a})
\]
(resp. \( (i_p, j_q, i_r, j_s) = (a, n, n, \overline{a}) \) or \( (a, \overline{n}, n, \overline{a}) \)).

Now, for \( C \) and \( C' \) in the \( a \)-odd-configuration or \( a \)-even-configuration, we define
\[
q(a; C, C') = p - s.
\]
Given adjacent two columns $C$ and $C'$, we say that $C$ and $C'$ satisfy the third two-column condition (2CC-3) if for every $a$-odd-configuration or $a$-even-configuration, $q(a; C, C') < n - a$.

Now, we can describe the crystal graph $B(\lambda)$ for $\mathfrak{g} = D_n$ as follows. We define $T(\lambda)$ for $\mathfrak{g} = D_n$ by the set of tableaux of shape $GRY^{-1}(\lambda)$ with entries $\{i, \overline{i} | i = 1, \cdots, n\}$ such that

1. the entries of $T$ weakly increase along the rows,
2. the entries of $T$ strictly increase down the columns, but the element $\overline{n}$ (resp. $n$) can appear below the element $n$ (resp. $\overline{n}$),
3. for a half column $C$ of $T$, $i$ and $\overline{i}$ cannot appear at the same time,
4. for each column $C$ of $T$, (1CC) holds,
5. for each pair of adjacent columns $C, C'$ of $T$, (2CC-1), (2CC-2) and (2CC-3) holds.

**Theorem 3.14.** For a dominant integral weight $\lambda$, there is a crystal isomorphism for $U_q(D_n)$-modules $\varphi : Y(\lambda) \to T(\lambda)$ sending $Y$ to $T_Y$.

**Proof.** It is similar to the argument of the proof of Theorem 3.10, we will omit it. \qed

**Example 3.15.** If $\mathfrak{g} = D_4$, $\lambda = \omega_4 + \lambda_4 = 3\lambda_4$ and

\[
Y = \begin{array}{cccc}
2 & 3 & 4 & 5 \\
2 & 2 & 2 & 2 \\
0 & 1 & 0 & 1 \\
2 & 2 & 2 & 2 \\
4 & 3 & 4 & 3 \\
4 & 3 & 4 & 3 \\
4 & 3 & 4 & 3 \\
4 & 3 & 4 & 3 \\
4 & 3 & 4 & 3 \\
4 & 3 & 4 & 3 \\
\end{array}
\in Y(\lambda), \quad \text{then } \varphi(Y) = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7 \\
5 & 6 & 7 & 8 \\
6 & 7 & 8 & 9 \\
7 & 8 & 9 & 10 \\
8 & 9 & 10 & 11 \\
9 & 10 & 11 & 12 \\
10 & 11 & 12 & 13 \\
11 & 12 & 13 & 14 \\
\end{array}.
\]

**Remark 3.16.** We know that the crystal graph $B(\lambda)$ for the quantum classical algebra $U_q(\mathfrak{g})$ appears as a connected component in the crystal graph $B(\Lambda)$ of basic representation for the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ without 0-arrows. Moreover, there are many connected components of $B(\Lambda)$ without 0-arrows that are isomorphic to $B(\lambda)$. Therefore, we can obtain a lot of Young wall realizations of the crystal graph $B(\lambda)$. In particular, if we choose the Young wall realization of $B(\lambda)$ corresponding to the connected components having the largest number of blocks, then applying a similar map to the map $\varphi$, we can obtain the tableau realization of $B(\lambda)$ given by Kashiwara-Nakashima (We shall not describe this Young wall realization and the map similar to $\varphi$ in this paper).

Finally, we will close this section to give a 1-1 correspondence between our new realization and the realization of Kashiwara and Nakashima for $\mathfrak{g} = A_n, C_n, B_n$ and $D_n$. Moreover, we will prove that it is just a crystal isomorphism for $U_q(\mathfrak{g})$. 
Theorem 3.17. Let $U_q(g) (g = A_n, C_n, B_n, D_n)$ be a quantum classical Lie algebra and let $\lambda$ be a dominant integral weight. Then there is a crystal isomorphism $\psi : T(\lambda) \to B(\lambda)$ for $U_q(g)$-modules.

Proof. Let $T$ be a tableau of $T(\lambda)$ consisting of columns $T_1, T_2, \cdots, T_p$ reading from right to left. We define $\psi(T)$ by

$$T_1 \leftarrow T_2 \leftarrow \cdots \leftarrow T_p.$$ 

Here, the notation $T \leftarrow U$ for the tableaux $T, U$ consisting of one column is referred to [15]. Then by Theorem 4.14 of [15], it is clear that the map $\psi$ is a crystal isomorphism. □

Example 3.18. (a) Let $g = B_4$, $\lambda = \omega_1 + \omega_2 + \omega_3$ and

$$T = \begin{array}{c}
2 \\
1 & 3 \\
2 & 3 & 7 
\end{array}.$$ 

Then

$$\psi(T) = \begin{array}{c}
2 \\
3 \\
1 & 9 
\end{array} \leftarrow \begin{array}{c}
1 \\
2 \\
2 & 2 & 3 
\end{array} = \begin{array}{c}
1 \\
1 & 7 \\
2 & 2 \\
3 
\end{array}.$$ 

(b) Let $g = D_4$, $\lambda = \omega_1 + \omega_2 + \lambda_4$ and

$$T = \begin{array}{c}
2 \\
3 \\
1 & 7 \\
1 & 7 & 7 
\end{array}.$$ 

Then

$$\psi(T) = \begin{array}{c}
2 \\
8 \\
7 \\
1 & 1 & 7 
\end{array} \leftarrow \begin{array}{c}
7 \\
1 & 7 \\
2 & 2 & 3 
\end{array} = \begin{array}{c}
1 \\
2 & 7 \\
2 & 2 \\
3 & 7 
\end{array}.$$ 

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