On positivity and semistability of vector bundles in finite and mixed characteristics

Adrian Langer

January 18, 2013

ADDRESS:
Institute of Mathematics, Warsaw University, ul. Banacha 2, 02-097 Warszawa, Poland

Dedicated to Professor C. S. Seshadri on his 20th birthday

Abstract

We survey results concerning behavior of positivity of line bundles and possible vanishing theorems in positive characteristic. We also try to describe variation of positivity in mixed characteristic. These problems are very much related to behavior of strong semistability of vector bundles, which is another main topic of the paper.

Introduction

The main aim of this paper is to survey problems concerning positivity of line bundles and stability of vector bundles on schemes defined over finite fields or over finitely generated rings over \( \mathbb{Z} \). Note that these two topics are very much related because a degree zero vector bundle \( E \) on a curve is strongly semistable if and only if the line bundle \( \mathcal{O}_{\mathbb{P}(E)}(1) \) on the projectivization of \( E \) is nef (see, e.g., [Mr, Proposition 7.1]).

The motivating problems are the following:

In this rare case the number of years does not coincide with the number of birthdays.
• What can we say about relation between nefness, semiampleness, effectivity and pseudoeffectivity for line bundles on varieties defined over finite fields?

• What vanishing theorems can hold for suitably positive line bundles in positive characteristic (or over \( \overline{F}_p \))?

• Is there any relation between nefness in characteristic zero and in positive characteristic?

• What can we say about variation in families of positivity of line bundles and semistability of vector bundles?

The known results do not answer any of these questions. In this paper we pose and study some conjectures that try to answer all of the above questions. Some of these question are very arithmetic in nature and in fact they imply very strong properties of reductions of varieties. In some simple cases they can be recovered using known results or they give another point of view on well known conjectures from arithmetic algebraic geometry.

The paper is divided in several sections describing each of these problems and surveying known results. First we recall some notation used throughout the paper. In Section 1 we describe positivity of line bundles on varieties defined over finite fields. In Section 2 we survey known results on Kodaira type vanishing theorems in positive characteristic. In Section 3 we study vanishing theorems for general reductions from characteristic zero. In Section 4 we recall several known constructions of strictly nef line bundles in characteristic zero. This is related to Keel’s question of existence of such bundles over finite fields. In Section 5 we study variation of positivity of line bundles in mixed characteristic. In Section 6 we consider a related question concerning vector bundles. In both Sections 6 and 7 we pose several conjectures that should fully explain behavior of strong semistability in mixed characteristic.

0.1 Notation

Let \( X \) be a complete variety defined over some algebraically closed field \( k \).

Let \( N_1(X) \) (\( N^1(X) \)) be the group of 1-cycles (divisors, respectively) modulo numerical equivalence. By the Néron-Severi theorem \( N_1(X)_\mathbb{Q} = N_1(X) \otimes \mathbb{Q} \) and \( N^1(X)_\mathbb{Q} = N^1(X) \otimes \mathbb{Q} \) are finite dimensional \( \mathbb{Q} \)-vector spaces, dual to each other by the intersection pairing.
A \(\mathbb{Q}\)-divisor \(D\) is called *pseudoeffective* if its numerical class in \(N^1(X)_{\mathbb{Q}}\) is contained in the closure of the cone generated by the classes of effective divisors.

A line bundle \(L\) on \(X\) is called *semiample*, if there exists a positive integer \(n\) such that \(L^\otimes n\) is globally generated.

A line bundle \(L\) on a variety \(X\) is called *strictly nef* if it has positive degree on every curve in \(X\).

A locally free sheaf \(E\) on \(X\) is called *nef* if and only if for any \(k\)-morphism \(f : C \to X\) from a smooth projective curve \(C/k\) each quotient of \(f^*E\) has a non-negative degree. We say that \(E\) is *numerically flat* if both \(E\) and \(E^*\) are nef.

Let \(X\) be a normal projective \(k\)-variety and let \(H\) be an ample Cartier divisor on \(X\). Let \(E\) be a rank \(r\) torsion free sheaf on \(X\). Then we define the *slope* \(\mu_H(E)\) of \(E\) as quotient of the degree of \(\det E = (\bigwedge^r E)^{**}\) with respect to \(H\) by the rank \(r\).

We say that \(E\) is *slope \(H\)-semistable* if for every subsheaf \(E' \subset E\) we have \(\mu_H(E') \leq \mu_H(E)\).

If \(k\) has positive characteristic then we say that \(E\) is *strongly slope \(H\)-semistable* if all the Frobenius pull backs \((F^p_X)^*E\) of \(E\) for \(n \geq 0\) are slope \(H\)-semistable.

Let \(X\) be an algebraic \(k\)-variety. We say that a *very general point* of \(X\) satisfies some property if there exists a countable union of proper subvarieties of \(X\) such that the property is satisfied for all points outside of this union.

## 1 Nef line bundles over finite fields

The following fact (see, e.g., [Ke1, Lemma 2.16]) is standard and it follows easily from existence of the Picard scheme and the fact that an abelian variety has only finitely many rational points over a given finite field.

**Proposition 1.1.** A numerically trivial line bundle on a projective scheme defined over \(\overline{\mathbb{F}}_p\) is torsion. In particular, a nef line bundle on a projective curve over \(\overline{\mathbb{F}}_p\) is semiample.

In the surface case Artin [Ar, 2-2.11] proved the following result:

**Theorem 1.2.** A nef and big line bundle on a smooth projective surface defined over \(\overline{\mathbb{F}}_p\) is semiample.

In [Ke1, Theorem 0.2] Sean Keel gave the following criterion for semiample-

THEOREM 1.3. Let $L$ be a nef line bundle on a projective scheme $X$ defined over a field of positive characteristic. Let $L^\perp$ be the closure of the union of all subvarieties $Y \subset X$ such that $L^{\dim Y} \cdot Y = 0$, taken with the reduced scheme structure. Then $L$ is semiample if and only if its restriction to $L^\perp$ is semiample.

This theorem, combined with earlier ideas of Seshadri, occurred to be the main new ingredient in Seshadri’s new proof of Mumford’s conjecture (see [Se]).

A basic tool used in proofs of Theorems 1.2 and 1.3 is Proposition 1.1.

Keel’s theorem implies Artin’s theorem, because if $X/\bar{\mathbb{F}}_p$ is a smooth projective surface and $L$ is a nef and big line bundle on $X$ then $L^\perp$ is at most one-dimensional and hence $L|_{L^\perp}$ is numerically trivial. Thus by Proposition 1.1 $L|_{L^\perp}$ is torsion and Theorem 1.3 implies that $L$ is semiample.

Note that Keel’s theorem trivially fails in the characteristic zero case. As an example one can take, e.g., any non-torsion line bundle of degree zero on a smooth projective curve. It is more difficult to produce counterexamples to Artin’s theorem in the characteristic zero case but they also exist:

THEOREM 1.4. (see [Ke1, Theorem 3.0]) Let $C$ be a smooth projective curve of genus $g \geq 2$ over a field of characteristic zero. Let $X = C \times C$ and let $L = p_1^* \omega_C(\Delta)$, where $\Delta$ is the diagonal and $p_1$ is the projection of $X$ onto the first factor. Then $L$ is nef and big but it is not semiample.

Note that in positive characteristic the bundle $L$ in the above theorem is semiample. All these results and lack of good construction methods raised the question whether there exist any nef line bundles on varieties defined over finite fields which are not semiample. In [Ke2, Section 5] Keel gives Kollár’s example of a nef but non-semiample line bundle on a non-normal surface defined over a finite field. The example is obtained by glueing two copies of $\mathbb{P}^1 \times \mathbb{P}^1$ but the obtained line bundle is not strictly nef.

Keel’s proof of non-semiampleness in Theorem 1.4 goes via showing that the restriction of $L$ to $2\Delta$ is non-torsion. Interestingly, Totaro used a similar strategy to show the following example of a nef but non-semiample line bundle on a smooth projective surface over $\mathbb{F}_p$:

Example 1.5. Let $C$ be a smooth projective curve of genus 2 defined over $\mathbb{F}_p$. Assume that for every line bundle $L$ of order $\leq p$ the map $H^1(C, L) \to H^1(C, F^*L)$, induced by the Frobenius morphism on $C$, is injective. In [To, Lemma 6.4] Totaro showed that a general curve of genus 2 satisfies this assumption.

Then one can embedd $C$ into $\mathbb{P}^1 \times \mathbb{P}^1$ as a curve of bidegree $(2, 3)$. In this case there exists twelve $\mathbb{F}_p$-points $p_1, \ldots, p_{12}$ on $C$ such that if $X$ is the blow up of

4
$\mathbb{P}^1 \times \mathbb{P}^1$ at these points then the line bundle $L$, associated to the strict transform $\tilde{C}$ of $C$, has order $p$ after restricting to $\tilde{C}$ but the restriction of $L^{\otimes p}$ to $2\tilde{C}$ is non-trivial. In this case Totaro shows the following theorem (see [10, proof of Theorem 6.1]):

**Theorem 1.6.** The line bundle $L$ is nef but it is not semiample. In fact, we have $h^0(X, L^n) = 1$ for every positive integer $n$.

Totaro used the above theorem to show the first example of nef and big line bundle on a smooth projective threefold, which is not semiample. This shows that Artin’s theorem does not generalize to higher dimensions. These examples do not answer the following question of Keel (see [Ke2, Question 0.9]), which we provocatively formulate as a conjecture:

**Conjecture 1.7.** Let $L$ be a strictly nef line bundle on a smooth projective surface $X$ defined over $\bar{\mathbb{F}}_p$. Then $L$ is ample.

By the Nakai-Moishezon criterion (see [Ht2, Chapter V, Theorem 1.10]), or by Theorem 1.2 this conjecture is equivalent to non-existence of strictly nef line bundles $L$ on $X$ with $L^2 = 0$. In fact, in view of Totaro’s example, one can pose an even stronger conjecture:

**Conjecture 1.8.** Let $L$ be a nef line bundle on a smooth projective surface $X$ over $\bar{\mathbb{F}}_p$. Then the Iitaka dimension $\kappa(L)$ of $L$ is non-negative. Equivalently, we can find some positive integer $m$ such that $L^{\otimes m}$ has a section.

If $L$ is nef and $L^2 > 0$ then $\kappa(L) = 2$, so in the above conjecture we can assume that $L^2 = 0$. We can also try to relax the nefness assumption and pose the following conjecture:

**Conjecture 1.9.** Let $D$ be a pseudoeffective $\mathbb{Q}$-divisor on a smooth projective surface $X$ over $\bar{\mathbb{F}}_p$. Then $D$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor.

Conjecture 1.9 is equivalent to non-existence of a nef line bundle $L$ with Iitaka dimension $\kappa(L) = -\infty$ and the numerical Iitaka dimension $\nu(X) = 1$. Obviously, all of the above conjectures can be also considered in higher dimensions but similarly to the surface case no answer seems to be known up to date. In fact, in higher dimensions Conjecture 1.9 can be generalized into two different ways: either as asking wether the cone of curves $\text{NE}(X) \subset N_1(X)_{\mathbb{Q}}$ is closed or as asking wether the cone of effective divisors is closed.

The assertion of Conjecture 1.9 seems to be much stronger than the one of Conjecture 1.8 but in fact we have the following lemma:
Lemma 1.10. Conjectures 1.8 and 1.9 are equivalent.

Proof. We only need to check that Conjecture 1.8 implies Conjecture 1.9.

Let $D$ be a pseudoeffective $\mathbb{Q}$-divisor. Then there exists a decomposition (so-called Zariski decomposition) $D = P + N$, where $P$ is a nef $\mathbb{Q}$-divisor and $N$ is a (negative) effective $\mathbb{Q}$-divisor $N$ such that $P \cdot N = 0$. By our assumption we know that some positive multiple of $P$, and therefore also of $D$, has a section.

2 Killing cohomology by finite morphisms

If $L$ is an ample line bundle on a smooth variety $X$ defined over a field of characteristic zero then Kodaira’s vanishing theorem says that $H^i(X, L^{-1})$ vanishes for $i < \dim X$. Kawamata–Viehweg vanishing theorem says that the same vanishing holds if $L$ is only nef and big. However, Raynaud in [Ra] constructed an example showing that already Kodaira’s vanishing theorem fails in positive characteristic. In this section we do not try to recover Kodaira’s vanishing theorem adding additional assumptions on the base variety as was done by Deligne and Illusie in [DI]. Instead try to kill cohomology on all varieties but using finite morphisms:

Theorem 2.1. Let $X$ be a proper variety over a field of positive characteristic and let $L$ be a semiample line bundle on $X$.

1. For any $i > 0$ there exists a finite surjective morphism $\pi : Y \to X$ such that the induced map $H^i(X, L) \to H^i(Y, \pi^*L)$ is zero.

2. If $L$ is big then for any $i < \dim X$ there exists a finite surjective morphism $\pi : Y \to X$ such that the induced map $H^i(X, L^{-1}) \to H^i(Y, \pi^*L^{-1})$ is zero.

This theorem was proven by Hochster and Huneke [HH, Theorem 1.2] in case $L$ is a tensor power of a very ample line bundle (see also [Sm, Theorem 2.1] and its erratum for the case when $L$ is a tensor power of an ample line bundle), and by Bhatt [Bh, Propositions 7.2 and 7.3] in general. Note that in case $L$ is a tensor power of an ample line bundle, the only non-trivial case is when $L = \mathcal{O}_X$. In the remaining cases, it is sufficient to pass to the normalization and use Serre’s vanishing theorem (see [Hi2, Chapter III, Theorem 5.2]) and Serre’s duality (see [Hi2, Chapter III, Corollary 7.7]) in the dual case.

One can ask whether Theorem 2.1 works under weaker assumptions on $L$, possibly after restricting the base field to the algebraic closure of a finite field (this is the most interesting case, as it is the only case that arises when reducing
from characteristic zero). By Proposition 1.1, Theorem 2.1.1 holds for nef line bundles on curves over $\overline{\mathbb{F}}_p$ but it fails for nef line bundles on smooth projective surfaces over $\overline{\mathbb{F}}_p$. More precisely, one can prove that in Example 1.5 we have the following non-vanishing theorem (see [La2, Theorem 3.1]):

**Theorem 2.2.** Let $M = L - p^{-1}$ or $M = L^p - 1$. Then for any complete $\overline{\mathbb{F}}_p$-surface $Y$ and any generically finite surjective morphism $\pi : Y \to X$ the induced map $H^1(X, M) \to H^1(Y, \pi^* M)$ is non-zero.

Similarly, Theorem 1.2 implies that Theorem 2.1.1 holds for nef and big line bundles on smooth projective surfaces over $\overline{\mathbb{F}}_p$ but one can show that it fails for nef and big line bundles on smooth projective threefolds over $\overline{\mathbb{F}}_p$ (see [La2, Proposition 4.1]).

In analogy to the Kawamata–Viehweg vanishing theorem, it is more natural to generalize Theorem 2.1.2 to nef and big line bundles on smooth projective varieties. In fact, in low dimensions one can show an even stronger theorem:

**Theorem 2.3.** Let $L$ be a nef and big line bundle on a normal projective variety over field of positive characteristic. Fix an integer $0 \leq i < \min(\dim X, 2)$. Then for sufficiently large $m$ the map

$$H^i(X, L^{-1}) \to H^i(X, L^{-p^m})$$

induced by the $m$-th Frobenius pull back is zero.

Unfortunately, the vanishing holds for trivial reasons because under the above assumptions one has $H^i(X, L^{-n}) = 0$ for $n \gg 0$ (see [Fu, Theorem 10]; see also [La1, Theorem 2.22 and Corollary 2.27] for effective versions of this theorem).

The only known examples of nef and big line bundle $L$ on a smooth projective variety $X$ of dimension $> 2$ such that $H^2(X, L^{-n}) \neq 0$ for all $n \gg 0$ were constructed by Fujita (see [Fu, pp. 526–527]). He used Raynaud’s counterexample to Kodaira’s vanishing theorem in positive characteristic (see [Ra]). By construction, in Fujita’s example the map induced by the $m$-th Frobenius pull back on $H^2(X, L^{-1})$ vanishes for all $m \gg 0$. This leaves open the following question:

**Question 2.4.** Let $L$ be a nef and big line bundle on a smooth projective variety defined over an algebraically closed field of positive characteristic. Fix an integer $0 \leq i < \dim X$. Is the map $H^i(X, L^{-1}) \to H^i(X, L^{-p^m})$ induced by the $m$-th Frobenius pull back zero for $m \gg 0$?
Note that [La2, Example 5.4] shows that the answer to this question is negative if one allows singular varieties. But for smooth varieties an answer to the above question is not known even if $L$ is semiample and big.

One can also try to weaken conditions on $L$ in Theorem 2.3 still hoping that we can kill cohomology using the Frobenius morphism. This works in some cases as shown by the following theorem proven in [La2, Theorem 6.1]:

**Theorem 2.5.** Let $X$ be a smooth projective surface defined over an algebraic closure of some finite field. Let $L$ be a nef line bundle on $X$ such that $\kappa(X, L) = -\infty$ (i.e., no power of $L$ has any sections). Then for large $n$ the map $H^1(X, L^{-1}) \to H^1(X, (F^n_X)^*L^{-1})$ induced by the $n$-th Frobenius morphism $F^n_X$ is zero.

Note that if in Example 1.5 we take $M = L^{p+1}$ then we get a nef line bundle with $M^2 = 0$ on a smooth projective surface over $\bar{\mathbb{F}}_p$ such that $H^1(X, M^{-1}) \to H^1(X, (F^n_X)^*M^{-1})$ induced by the $n$-th Frobenius pull back is always non-zero (see Theorem 2.2).

The above theorem is consistent with Conjecture 1.8 saying that there does not exist a nef line bundle $L$ on a smooth projective surface defined over $\bar{\mathbb{F}}_p$ such that $\kappa(L) = -\infty$ (cf. Corollary 3.4).

An interesting point in proof of Theorem 2.5 is that we use the higher rank case of Proposition 1.1 which follows from boundedness of the family of semistable vector bundles with trivial Chern classes.

As a corollary to Theorem 2.5 we get the following theorem analogous to Theorem 2.3:

**Corollary 2.6.** Let $X$ be a smooth projective variety of dimension $d \geq 2$ defined over an algebraic closure of some finite field. Let $L$ be a strictly nef line bundle on $X$. Then for large $n$ the map $H^1(X, L^{-1}) \to H^1(X, (F^n_X)^*L^{-1})$ induced by the $n$-th Frobenius morphism $F^n_X$ is zero.

### 3 Vanishing theorems in mixed characteristic

Let $R$ be a domain which contains $\mathbb{Z}$ and which, as a ring, is finitely generated over $\mathbb{Z}$. Let $\mathcal{X}$ be a projective $R$-scheme and let $\mathcal{L}$ be an invertible sheaf of $\mathcal{O}_{\mathcal{X}}$-modules. Let $\mathcal{X}_s$ denote the fibre over $s \in S$ and let $\mathcal{L}_s$ be the restriction (i.e., pull-back) of $\mathcal{L}$ to $\mathcal{X}_s$.

Let $R \subset K$ be an algebraic closure of the field of quotients of $R$. By assumption $K$ is of characteristic zero, so we can think of $\mathcal{X} \to S = \text{Spec } R$ as a model of the generic geometric fibre $\mathcal{X}_K$ with polarization $\mathcal{L}_K$. 


The following theorem (see [Sm, 3.5]), conjectured by Huneke and K. Smith in [HS, 3.9], was proven (in more general setting of rational singularities) by N. Hara in [Ha, Theorem 4.7] and later by V. Mehta and V. Srinivas in [MSr, Theorem 1.1].

**Theorem 3.1.** Let us assume that $X_K$ is smooth and $L_K$ is ample. Then there exists a non-empty Zariski open subset $U \subset S$ such that for every closed point $s \in U$ the natural map

$$H^i (\mathcal{X}_s, L_s^{-1}) \to H^i (\mathcal{X}_s, F^* L_s^{-1}),$$

induced by the Frobenius morphism on the fiber $\mathcal{X}_s$, is injective for all $i \geq 0$.

Note that for $i < \dim \mathcal{X}_K$ Kodaira’s vanishing theorem says that $H^i (\mathcal{X}_K, L_K^{-1}) = 0$ so by semicontinuity of cohomology (see [H2, III, Theorem 12.8]) we have $H^i (\mathcal{X}_s, L_s^{-1}) = 0$ for $s$ from some open subset of $S$. So the above theorem is non-trivial only in case $i = \dim X$. On the other hand, one can ask if similar theorems hold in other cases when we do not have vanishing of cohomology at the generic fibre. Here is one such example in the surface case:

**Proposition 3.2.** Let us assume that $\mathcal{X}_K$ is a smooth surface and $L_K$ is a line bundle with $\kappa (L_K) = -\infty$. Assume also that there exists an ample line bundle $A_K$ on $\mathcal{X}_K$ such that $c_1 L_K \cdot c_1 A_K > 0$. Then there exists a non-empty Zariski open subset $U \subset S$ such that for every closed point $s \in U$ and every positive integer $n$ the natural map

$$H^1 (\mathcal{X}_s, L_s^{-1}) \to H^1 (\mathcal{X}_s, (F^n)^* L_s^{-1}),$$

induced by composition of $n$ absolute Frobenius morphisms on the fiber $\mathcal{X}_s$, is injective.

**Proof.** Let $B^1_{\mathcal{X}_s}$ be the sheaf of exact 1-forms. By definition we have an exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}_s} \to F_\ast \mathcal{O}_{\mathcal{X}_s} \to F_\ast B^1_{\mathcal{X}_s} \to 0.$$

Therefore to check that

$$H^1 (\mathcal{X}_s, L_s^{-1}) \to H^1 (\mathcal{X}_s, F_\ast \mathcal{O}_{\mathcal{X}_s} \otimes L_s^{-1}) = H^1 (\mathcal{X}_s, F_\ast L_s^{-1})$$

is injective, it is sufficient to prove that $H^0 (\mathcal{X}_s, F_\ast B^1_{\mathcal{X}_s} \otimes L_s^{-1}) = 0$. But $F_\ast B^1_{\mathcal{X}_s}$ is a subsheaf of $F_\ast \Omega^1_{\mathcal{X}_s}$, so by the projection formula we have

$$H^0 (F_\ast B^1_{\mathcal{X}_s} \otimes L_s^{-1}) \subset H^0 (F_\ast \Omega_{\mathcal{X}_s} \otimes L_s^{-1}) = H^0 (\Omega_{\mathcal{X}_s} \otimes F_\ast L_s^{-1}).$$
So it is sufficient to show that there exists an open subset $U \subset S$ such that for every closed point $s \in U$ the sheaf $\Omega_{\mathcal{X}_s} \otimes F^* \mathcal{L}_s^{-1}$ has no sections. Similarly, to check that $$H^1(\mathcal{X}_s, (F^{n-1})^* \mathcal{L}_s^{-1}) \to H^1(\mathcal{X}_s, (F^n)^* \mathcal{L}_s^{-1})$$ is injective it is sufficient to prove that $\Omega_{\mathcal{X}_s} \otimes (F^n)^* \mathcal{L}_s^{-1}$ has no sections.

We can find a Zariski open subset $V \subset S$ and a line bundle $\mathcal{A}$ extending $\mathcal{A}_K$. Since ampleness is an open property, shrinking $V$ if necessary, we can assume that $\mathcal{A}$ on $\mathcal{X}_V \to V$ is relatively ample. Existence of the relative Harder-Narasimhan filtration of $\Omega_{\mathcal{X}_V}/V$ (see [HL, Theorem 2.3.2]) implies that further shrinking $V$ we can assume that for all closed points $s \in V$ we have

$$\mu_{\max, \mathcal{A}_s}(\Omega_{\mathcal{X}_s}) = \mu_{\max, \mathcal{H}}(\Omega_{\mathcal{X}_K}).$$

Since $c_1 \mathcal{L}_s \cdot c_1 \mathcal{A}_s = c_1 \mathcal{L}_K \cdot c_1 \mathcal{A}_K > 0$, we see that if the characteristic $p$ at a closed point $s \in V$ is larger than $\mu_{\max, \mathcal{A}_s}(\Omega_{\mathcal{X}_s})/(c_1 \mathcal{L}_K \cdot c_1 \mathcal{A}_K)$, then for every positive integer $n$

$$\mu_{\max, \mathcal{A}_s}(\Omega_{\mathcal{X}_s} \otimes (F^n)^* \mathcal{L}_s^{-1}) = \mu_{\max, \mathcal{A}_K}(\Omega_{\mathcal{X}_K}) - p^n(c_1 \mathcal{L}_K \cdot c_1 \mathcal{A}_K) < 0.$$ But existence of sections of $\Omega_{\mathcal{X}_s} \otimes (F^n)^* \mathcal{L}_s^{-1}$ would contradict this inequality.

\[\Box\]

**Lemma 3.3.** Let $C$ be a $\mathbb{Q}$-divisor on a smooth projective surface $X$. If $C^2 \geq 0$ and $CP > 0$ for some nef divisor $P$ then $C$ is pseudoeffective.

**Proof.** If $CA < 0$ for some ample divisor $A$ then taking appropriate combination $H = aA + bP$ for some $a, b > 0$ we have $CH = 0$. Since $H$ is ample and $C$ is numerically non-trivial, the Hodge index theorem (see [H2, Chapter V, Theorem 1.9]) gives $C^2 < 0$. \[\Box\]

**Corollary 3.4.** Let $L$ be a pseudoeffective line bundle on a smooth projective surface defined over a field of characteristic zero. Let us assume that $L^2 \geq 0$ and $H^1(X, L^{-1})$ is non-zero. Then for almost all primes $p$ the reduction of $L$ modulo $p$ has a non-negative Iitaka dimension.
Proof. If \( L \) is pseudoeffective then and \( L^2 \geq 0 \) then by Lemma 3.3 almost all reductions of \( L \) are pseudoeffective. Let \( L = P + N \) be the Zariski decomposition (see proof of Lemma 1.10). If \( L \) is not nef then \( P^2 = L^2 - N^2 > 0 \) (since \( N \) is non-zero we have \( N^2 < 0 \) as follows from \( PN = 0 \) by the Hodge index theorem). Hence \( P \) is big, which implies that \( L \) is also big. The same argument shows that if we take a reduction of \( L \) which is pseudoeffective but not nef then it is big. So we can assume that a reduction of \( L \) is nef. In this case the assertion follows from Proposition 3.2 and Theorem 2.5. \( \square \)

Remarks 3.5. 1. In the above corollary, instead of assuming that \( H^1(X, L^{-1}) \) is non-zero it is sufficient to assume that there exist a smooth projective surface \( Y \) and a generically finite morphism \( \pi : Y \to X \) such that \( H^1(Y, \pi^*L^{-1}) \) is non-zero.

2. Corollary 3.4 implies that if a line bundle \( L \) is strictly nef with non-vanishing \( H^1(X, L^{-m}) \) for some positive integer \( m \), then its reduction to positive characteristic is almost never strictly nef. This happens, e.g., in Mumford’s example (see Example 4.1). In fact, in this case Biswas and Subramanian (see [BS, Theorem 1.1]) proved that strictly nef line bundles on ruled surfaces over \( \mathbb{F}_p \), are always ample.

4 Examples of strictly nef line bundles

Note that if \( L \) is a strictly nef line bundle on a proper variety \( X \) and \( f : Y \to X \) is a finite morphism then \( f^*L \) is also strictly nef. This gives a lot of examples of strictly nef line bundles once we have constructed some such bundles. In this section we review known constructions of strictly nef line bundles on smooth projective surfaces that do not come from this construction.

Example 4.1. The most famous example of a strictly nef line bundle is due to Mumford (see [H1, I, Example 10.6]). Namely, let \( C \) be a smooth complex projective curve of genus \( \geq 2 \). Then on \( C \) there exists a rank 2 stable vector bundle \( E \) with trivial determinant and such that all symmetric powers \( S^i E \) are also stable. Let \( \pi : X = \mathbb{P}(E) \to C \) be the projectivization of \( E \) and let \( L = \mathcal{O}_{\mathbb{P}(E)}(1) \). Then \( L \) is a strictly nef line bundle on \( X \) with \( L^2 = 0 \). Note that in this example \( H^1(X, L^{-2}) \) is non-zero. More precisely, let us not that the relative Euler exact sequence

\[
0 \to \Omega_{X/C} \to \pi^*E \otimes L^{-1} \to \mathcal{O}_X \to 0
\]
is non split, as it is non-split after restricting to the fibers of $\pi$. After tensoring this sequence by $L$ and using $\det E \otimes \mathcal{O}_C$ we get the sequence

$$0 \to L^{-1} \to \pi^* E \to L \to 0,$$

which gives a non-zero element in $\text{Ext}^1(L,L^{-1}) = H^1(X,L^{-2})$.

For generalization of Mumford’s example to higher dimensions see S. Subramanian’s paper [Su]. For uncountable fields of positive characteristic a similar example was considered by V. Mehta and S. Subramanian [MSu]. The next example shows existence of strictly nef line bundles even over countable fields of positive characteristic, provided they have sufficiently large transcendental degree over its prime field.

**Example 4.2.** Consider the projective plane $\mathbb{P}^2$ over some field $k$ and let us take $r = s^2$, where $s > 3$, $k$-rational points $p_1, \ldots, p_r \in \mathbb{P}^2(k)$. Let $\rho : X \to \mathbb{P}^2$ be the blow up at these points and let us take $L = \rho^* \mathcal{O}_{\mathbb{P}^2}(s) \otimes \mathcal{O}(-E)$, where $E$ is the exceptional divisor of $\rho$. Clearly, we have $L^2 = 0$. If all the chosen points lie on a geometrically irreducible degree $s$ curve $C \subset \mathbb{P}^2$ defined over $k$ then $L$ is nef. This follows from the fact that the strict transform $\tilde{C}$ gives an element of the linear system $|L|$ and hence for every irreducible curve $D \subset Y$ we have $D \cdot L = D \cdot \tilde{C} \geq 0$ with equality if and only if $D = \tilde{C}$. This is also the main idea behind Totaro’s construction of a nef non-semiample line bundle, except that to obtain an example where $C$ has genus 2 he blow ups $\mathbb{P}^1 \times \mathbb{P}^1$ instead of $\mathbb{P}^2$. Obviously, the bundle $L$ obtained in this way is not strictly nef as $L \cdot \tilde{C} = 0$. However, Nagata proved the following theorem:

**Theorem 4.3.** Assume that the points $p_1, \ldots, p_r$ are very general. Then $L$ is strictly nef.

**Proof.** Let $D$ be any reduced curve on the blow up $X$ and let $C \in |\mathcal{O}_{\mathbb{P}^2}(d)|$ be its image. Let $m_1, \ldots, m_r$ be the multiplicities of $C$ at the points $P_1, \ldots, P_r$, respectively. Then $LD = sd - \sum_{i=1}^r m_i$. But by [Na, Chapter 3, Proposition 1] we have $sd - \sum_{i=1}^r m_i > 0$. □

Unfortunately, this theorem does not say anything for varieties defined over $\overline{\mathbb{F}}_p$.

Note that a similar construction can be used also in different cases: we can blow up some points $p_1, \ldots, p_r$ (where $r$ can be arbitrary) on a smooth projective surface $X$ and take the pull back of an ample line bundle on $X$ twisted by a suitable
negative combination of exceptional divisors, arranging this so that the obtained line bundle has self intersection 0. If the number \( r \) of points is sufficiently large and the points are in a very general position then the obtained line bundle should be strictly nef. This type of construction was used, e.g., in \([LR, Example 3.3]\) but it seems that the proof of strict nefness of the obtained divisor is incorrect.

Example 4.4. Let \( F \) be a real quadratic field and let \( D \) be a totally indefinite quaternion \( F \)-algebra. Let us recall that a quaternion algebra over \( F \) is an \( F \)-algebra
\[ D = F + Fi + Fj + Fi \]
given by \( i^2 = a, j^2 = b \) and \( ij = -ji \), where \( a, b \in F \) are some non-zero elements. \( D \) is totally indefinite, if for both embeddings \( F \hookrightarrow \mathbb{R} \) we have \( \mathbb{R} \otimes_F D \simeq M_2(\mathbb{R}) \). In this case we get two inequivalent real representations \( \rho_i : D \to M_2(\mathbb{R}), i = 1, 2 \). On the algebra \( D \) we can introduce a norm \( N : D \to F \) by
\[ N(x_0 + x_1i + x_2j + x_3ij) = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2 \]
for \( x_i \in F \). Let \( \hat{G} \) be the group of elements of norm 1 in a fixed maximal order \( R \) in \( D \) and let \( G = \hat{G}/\langle \pm 1 \rangle \). Let \( \mathbb{H} \) be the complex upper half plane. The group \( G \) acts on the product \( \mathbb{H} \times \mathbb{H} \) by
\[ \lambda(z_1, z_2) = (\rho_1(\lambda)z_1, \rho_2(\lambda)z_2). \]
In case \( D \) is a division algebra, the quotient surface \( X = \mathbb{H} \times \mathbb{H}/G \) is compact. Let us also assume that \( X \) is smooth (all these assumptions are satisfied in some cases). Let \( p_1, p_2 : \tilde{X} = \mathbb{H} \times \mathbb{H} \to \mathbb{H} \) be the two projections. Then \( \Omega^1_{\tilde{X}} \simeq p_1^*\Omega^1_{\mathbb{H}} \oplus p_2^*\Omega^1_{\mathbb{H}} \) as \( G \)-linearized bundles. So by descent we have \( \Omega^1_X \simeq L \oplus M \) for some line bundles \( L \) and \( M \). Then we have the following lemma:

**Lemma 4.5.** ([\text{SB1} Lemma 3]) The line bundles \( L \) and \( M \) are strictly nef with \( L^2 = M^2 = 0 \).

**Proof.** Let \( C \) be a reduced and irreducible curve in \( X \) and let \( \tilde{C} \) be an irreducible component of its pre-image in \( \tilde{X} \). The line bundle \( L|_C \) is represented by a form whose pull-back to \( \tilde{C} \) is the pull-back of a positive form from \( \mathbb{H} \). Therefore \( CL = \deg L|_C > 0 \). This shows that \( L \) is strictly nef and in particular \( L^2 \geq 0 \). If \( L^2 > 0 \) then \( L \) is ample by the Nakai–Moishezon criterion (see \([\text{Ha}, V, Theorem 1.10]\)). But by Bogomolov’s vanishing theorem \( \Omega^1_X \) does not contain any ample subbundles. Therefore \( L^2 = 0 \). The same proof works also for \( M \). \( \square \)

[\text{SB1}] contains a more general example of the same type but we will need this particular case later on (see Example 5.6).
5 Variation of positivity of line bundles

It is known that ampleness is an open condition in families (not necessarily flat). More precisely, let $S$ be an irreducible noetherian scheme and let $\pi : \mathcal{X} \to S$ be a proper morphism. Let $L$ be a line bundle on $\mathcal{X}$.

**Theorem 5.1.** (see [Gr, III, Theorem 4.7.1]) If $L_{s_0}$ is ample on $\mathcal{X}_{s_0}$ for some point $s_0 \in S$ then $L_s$ is ample for a general point of $S$, i.e., there exists an open neighborhood $U \subset S$ of $s_0$ such that $L_s$ is ample on $\mathcal{X}_s$ for all $s \in U$.

**Corollary 5.2.** If $L_{s_0}$ is nef on $\mathcal{X}_{s_0}$ for some geometric point $s_0 \in S$ then $L_s$ is nef for a very general point of $S$, i.e., there exist countably many open and dense subsets $U_m \subset S$ such that $L_s$ is nef for every geometric point $s \in \bigcap U_m$.

**Proof.** Using Chow’s lemma we can reduce to the case where $\pi$ is projective. Let $\mathcal{O}_{\mathcal{X}}(1)$ be a $\pi$-ample line bundle on $\mathcal{X}$. By Theorem 5.1 we know that for every positive integer $m$ the set $U_m$ of points for which $(L^m \otimes \mathcal{O}_{\mathcal{X}}(1))_s$ is ample is open and dense in $S$. It is easy to see that these sets satisfy the required assertion.

Note that we can assume that the sequence $\{U_m\}_{m \in \mathbb{N}}$ is descending, i.e., $U_{m+1} \subset U_m$ for all $m$ and one can ask if such a sequence must stabilize. In general, this is too much to hope for but $\bigcap U_m$ contains the generic geometric point of $S$ so we can ask if it contains any closed points. This is interesting only if $S$ has only countably many points as only then the set of closed geometric points $s \in S$ for which $L_s$ is nef can be empty. Indeed, this can really happen as shown by the following example due to Monsky [Mo1], Brenner [Br2] and Trivedi [Tr]:

**Example 5.3.** Let us start with recalling the following result of Monsky [Mo1, Theorem]:

**Theorem 5.4.** Let $R_t = K_t[x, y, z]/(P_t)$, where $K_t$ is an algebraic closure of $\mathbb{F}_2(t)$ and set

$$P_t = z^4 + xyz^2 + x^3 z + y^3 z + tx^2 y^2.$$  

Then the Hilbert-Kunz multiplicity of $R_t$ is equal to $3 + 4^{-m(t)}$, where

$$m(t) = \begin{cases} 
\text{degree of } \lambda \text{ over } \mathbb{F}_2, & \text{if } t = \lambda^2 + \lambda \text{ is algebraic over } \mathbb{F}_2, \\
\infty, & \text{if } t \text{ is transcendental over } \mathbb{F}_2.
\end{cases}$$
Now let $k = \overline{\mathbb{F}}_2$ and let us set $S = \mathbb{A}_k^1$ with coordinate $t$ and $\mathbb{P}_k^2$ with homogeneous coordinates $[x : y : z]$. Let $Y \subset \mathbb{P}_k^2 \times_k S$ be given by

$$z^4 + xyz^2 + x^3z + y^3z + tx^2y^2 = 0$$

and let $\mathcal{E} = p_1^* \Omega_{\mathbb{P}_k^2}$, where $p_1 : Y \to \mathbb{P}_k^2$ is the canonical projection. Consider the projection $p_2 : Y \to S$. Then $\mathcal{E}_s$ is not strongly semistable for every closed point $s \in S$ (even on the singular fiber over $0 \in S$) but $\mathcal{E}_\eta$ is strongly semistable for the generic point $\eta \in S$. This follows from Monsky’s theorem and the computation of the Hilbert-Kunz multiplicity of $R_t$ in terms of strong Harder–Narasimhan filtration of bundles $\mathcal{E}_s$ for $s \in S$ due to Brenner [Br2, Theorem 1] and Trivedi [Tr, Theorem 5.3]. This computation implies that $\mathcal{E}_s$ is strongly semistable for $s : \text{Spec } K_t \to S$ if and only if the Hilbert-Kunz multiplicity of $R_t$ is equal to 3.

Let $\mathscr{X}$ be the projectivization of $\mathcal{F} = p_1^*(S^2 \Omega_{\mathbb{P}_k^2}(1))$ over $Y$. Let $\mathcal{L} = \mathcal{O}_{\mathscr{X}}(1)$ and let $\pi : \mathscr{X} \to S$ be the composition of the projections $\mathcal{X} \to Y$ and $p_2 : Y \to S$. Then $\mathcal{L}_\mathcal{X}$ is nef for a generic geometric point $\mathcal{T} \in S$ but $\mathcal{L}_s$ is not nef for every closed geometric point $s \in S$.

One can also show a similar example in equal characteristic 3 (see [Mo2]).

Note that in the above example $S$ was defined over an algebraic closure of a finite field. It seems to be unknown if similar examples can occur for $S$ defined over a countable field of positive characteristic containing transcendental elements over its prime field, or even in case $S$ is defined over $\overline{\mathbb{Q}}$. One might expect that the strange behavior of variation of nefness in positive equal characteristic cannot occur in mixed characteristic:

**Conjecture 5.5.** Let $R$ be a finitely generated integral domain over $\mathbb{Z}$, containing $\mathbb{Z}$. Let $\pi : \mathcal{X} \to S = \text{Spec } R$ be a smooth proper morphism. Let $\mathcal{L}$ be an invertible sheaf of $\mathcal{O}_{\mathcal{X}}$-modules and assume that the restriction of $\mathcal{L}$ to the generic geometric fibre of $\pi$ is nef. Then the set $T$ of closed points $s \in S$ such that $\mathcal{L}_s$ is semiample is dense in $S$.

Totaro’s Example [1.3] comes from characteristic zero by reduction modulo $p$. The above conjecture suggests that such examples are rather rare and almost all reductions of a fixed nef line bundle are semiample.

Conjecture 5.5 generalizes [Mi, Problem 5.4] which considers the same question in case $\mathcal{X}$ is a projectivization of a rank 2 vector bundle over a curve (in this case if $s \in S$ is a closed point then nefness of $\mathcal{L}_s$ implies its semiampleness by the Lange–Stuhler theorem; see Proposition [7.1].

15
Note that one can show examples in which the set $T$ is not open in the set of closed points of $S$. The first such examples come from an unpublished work [EST] of Ekedahl, Shepherd–Barron and Taylor:

**Example 5.6.** Consider $X$ from Example [4.4] The line subbundle $L^{-1} \subset T_X \simeq L^{-1} \oplus M^{-1}$ defines a foliation. If we reduce $X$ modulo some prime of characteristic $p$ then the $p$-curvature map $L_p \otimes (L_p)^{-1} \rightarrow T_{X_p}/L_p^{-1} = M_p^{-1}$, given by taking the $p$-th power of a derivation, is $\mathcal{O}_{X_p}$-linear. If $p$ is inert in $F$ then this map is non-zero (see [EST, p. 23]). In this case we get a section of $L_p \otimes M_p^{-1}$ and, similarly, we get a section of $M_p \otimes L_p^{-1}$. Note that $L_p$ is not nef (and hence it is not semiample). Otherwise, we would have $-LM = L_p(pL_p - M_p) \geq 0$, whereas $LM > 0$. Since $L_p$ is pseudoeffective and $L_p^2 = 0$, existence of the Zariski decomposition of $L_p$ implies that $L_p$ is big (see proof of Corollary [3.4]). Let us recall that by Chebotarev’s density theorem the number of rational primes $p$ which remain inert in $F$ is infinite (of Dirichlet density $1/2$). So in this case we have a strictly nef line bundle $L$ for which infinitely many reductions are not semiample.

In fact, it is not clear how to prove that in the remaining cases the reduction of $L$ is semiample (possibly apart from finitely many primes).

Other examples of a similar type were obtained by Brenner [Br1] in case $\mathcal{X}$ is a projectivization of a rank 2 vector bundle over a curve (note that these examples did not solve Miyaoka’s problem [Mi, Problem 5.4]).

### 6 Variation of semistability of vector bundles

Let $X$ be a smooth complex projective variety and let $\mathcal{O}_X(1)$ be an ample line bundle on $X$. Let $E$ be a slope semistable (with respect to $\mathcal{O}_X(1)$) locally free $\mathcal{O}_X$-module.

We are interested in behavior of $E$ when taking reduction modulo $p$. More precisely, all of the above data can be described by a finite number of equations. Therefore there exist a subring $R \subset \mathbb{C}$, finitely generated as an algebra over $\mathbb{Z}$, and a triple $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1), \mathcal{E})$ consisting of a smooth projective $R$-scheme $\pi : \mathcal{X} \to S = \text{Spec} R$, an $R$-ample line bundle $\mathcal{O}_{\mathcal{X}}(1)$ and a family $\mathcal{E}$ of locally free slope semistable sheaves on the fibers of $\pi$, such that on the fiber over the generic geometric point $\text{Spec} \mathbb{C} \to S$ we recover the triple $(X, \mathcal{O}_X(1), E)$. Note that we have implicitly used openness of slope semistability in flat families of sheaves.
Let us recall that for every maximal ideal \( m \subset R \) the residue field \( k = R/m \) is finite of characteristic \( p > 0 \). Now we would like to relate various properties of \( E \) to the behavior of its reductions modulo \( p \). We pose a series of conjectures that should completely describe the behavior of strong semistability in mixed characteristic. The first conjecture is motivated by [SB2], where it was proven in the rank 2 case:

**Conjecture 6.1.** Let \( \Sigma^{\text{nss}} \) be the set of closed points \( s \in S \) such that \( \mathcal{E}_s \) is not strongly slope semistable. If \( \Sigma^{\text{nss}} \) is infinite then \( \text{End } E \) is a numerically flat vector bundle. Moreover, \( \text{End } E \) is not étale trivializable.

**Lemma 6.2.** If \( \Sigma^{\text{nss}} \) is infinite then \( \text{End } E \) is not étale trivializable. In particular, Conjecture 6.1 is true in the curve case.

**Proof.** If \( \text{End } E \) is étale trivializable then \( \text{End } E \) is étale trivializable over \( \mathcal{X}_U \) for some open subset \( U \subset S \). In particular, \( \text{End } \mathcal{E}_s \) is strongly semistable for \( s \in U \). We claim that \( \mathcal{E}_s \) is also strongly semistable. If \( \mathcal{E}_s \) is not strongly semistable then there exists some \( n \) such that the \( n \)th Frobenius pull back of \( \mathcal{E}_s \) is destabilized by some subsheaf \( \mathcal{E}' \). But then

\[
\mu(E' \otimes (F^n)^* \mathcal{E}_s^*) = \mu(E') + \mu((F^n)^* \mathcal{E}_s^*) > \mu((F^n)^* \mathcal{E}_s) + \mu((F^n)^* \mathcal{E}_s^*) = 0
\]

and hence \( E' \otimes (F^n)^* \mathcal{E}_s^* \) destabilizes \( (F^n)^*(\text{End } \mathcal{E}_s) \), a contradiction. This implies that \( \Sigma^{\text{nss}} \) is contained in the set of closed points of \( S - U \), and therefore \( \Sigma^{\text{nss}} \) is finite.

If \( X \) is a curve then for every semistable \( E \) the bundle \( \text{End } E \) is semistable of degree 0, so it is numerically flat and the conjecture follows from the first part of the lemma.

This shows that Conjecture 6.1 is of interest only in the surface case and the only non-trivial part of the conjecture is that \( \text{End } E \) is numerically flat. Indeed, the higher dimensional case can be easily reduced to the surface case by means of restriction theorems. More precisely, if \( X \) has dimension \( d \) greater than 2 and \( E \) is a vector bundle for which \( \Sigma^{\text{nss}} \) is infinite then the restriction of \( E \) to a general complete intersection surface \( Y \subset X \) is semistable and it satisfies the assumptions of the conjecture. So if we know the conjecture for \( E|_Y \) then \( \text{End } E|_Y \) is a numerically flat vector bundle. But then \( \text{End } E \) is also numerically flat because it is semistable with respect to some ample polarization \( H \) such that \( c_1(E)H^{d-1} = c_2(E)H^{d-2} = 0 \) (cf. [Si, Theorem 2]).
7 Arithmetic of numerically flat vector bundles

Conjecture 6.1 implies that to study strong semistability of reductions of a complex vector bundle, it is sufficient to study reductions of numerically flat vector bundles. The following subsection recalls a special role of such vector bundles and their relation to representations of the fundamental group.

7.1 Flat bundles

Let $X$ be a smooth complex projective variety. Giving a representation of the topological fundamental group $\pi_1(X,x)$ on a complex vector space $V$, is equivalent to giving a complex local system $V$ (a sheaf of complex vector spaces locally isomorphic to the constant sheaf $\mathbb{C}^n$, $n \in \mathbb{N}$). Given a local system we can recover the corresponding representation as the monodromy representation.

Given $V$ we can construct a holomorphic vector bundle $\mathcal{O}_X \otimes \mathbb{C}V$ with (holomorphic) integrable connection $\nabla$ such that $\nabla(fv) = df \cdot v$, where $f$ is a local section of $\mathcal{O}_X$ and $v$ is a local section of $V$. On the other hand, given a holomorphic vector bundle $\mathcal{E}$ with integrable connection $\nabla$ we can recover a local system $V$ as a sheaf of local sections $v$ of $\mathcal{E}$ for which $\nabla(v) = 0$. This constructions provide functors giving an equivalence of categories of complex local systems and holomorphic vector bundles with integrable connection.

In [Si, Corollary 3.10] Simpson proved that these categories are equivalent to the category of (Higgs) semistable Higgs bundles $(E, \theta)$ with vanishing (rational) Chern classes. This category contains the category of semistable vector bundles with vanishing Chern classes. If a representation of $\pi_1(X,x)$ is an extension of unitary representations, then the corresponding Higgs bundle is an extension of stable vector bundles and the equivalence preserves the holomorphic structure. In particular, every semistable vector bundle with vanishing Chern classes has a holomorphic flat structure which is an extension of unitary flat bundles. Finally, let us recall that a vector bundle is semistable with vanishing Chern classes if and only if it is numerically flat.

We also need to recall a few basic results about étale trivializable bundles.

7.2 Étale trivializable bundles

Let $X$ be a smooth projective variety over an algebraically closed field $k$. A rank $r$ locally free sheaf $E$ on $X$ is called étale trivializable if there exists a finite étale
covering \( \pi : Y \to X \) such that \( \pi^* E \cong \mathcal{O}_Y \). Over finite fields étale trivializable bundles are characterized as Frobenius periodic bundles:

**Proposition 7.1.** (see [LS]) Assume that \( k = \overline{\mathbb{F}_p} \) and let \( F : X \to X \) be the Frobenius morphism. A locally free sheaf \( E \) is étale trivializable if and only if there exists an isomorphism \((F^n_X)^* E \cong E\) for some positive integer \( n \).

It is easy to see that every étale trivializable bundle is numerically flat. So we can try to characterize such bundles for \( k = \mathbb{C} \) in terms of their monodromy representation. If we have a representation \( \rho : \pi_1(X, x) \to \text{GL}_r(\mathbb{C}) \) whose image \( G \) is a finite group then by Weyl’s trick \( G \) is a unitary subgroup of \( \text{GL}_r(\mathbb{C}) \). Since every complex representation of a finite group is a direct sum of irreducible representations, the corresponding Higgs bundle \((E, \theta)\) is a direct sum of stable vector bundles. Passing to the étale covering defined by the quotient \( \pi_1(X, x) \to G \) we see that each direct summand is étale trivializable and the Higgs field \( \theta = 0 \).

On the other hand, if a bundle is étale trivializable then it is étale trivializable by a finite Galois covering and hence the corresponding monodromy representation has finite image.

### 7.3 Étale trivializability of reductions of numerically flat bundles

We keep the notation from Section 6 but now we restrict to the case where \( E \) is a numerically flat vector bundle.

**Conjecture 7.2.** The set \( \Sigma^{\text{et}} \) of closed points \( s \in S \) such that \( \mathcal{E}_s \) is étale trivializable, is infinite.

The following example shows that this conjecture is interesting even for very simple semistable vector bundles:

**Example 7.3.** Let \( X \) be a smooth complex projective variety with \( h^1(X, \mathcal{O}_X) > 0 \). Let us consider vector bundle \( E \) corresponding to the extension

\[
0 \to \mathcal{O}_X \to E \to H^1(\mathcal{O}_X) \otimes \mathcal{O}_X \to 0
\]

defined by the identity \( \text{id}_{H^1(\mathcal{O}_X)} \subseteq \text{End}(H^1(\mathcal{O}_X)) = \text{Ext}^1(H^1(\mathcal{O}_X) \otimes \mathcal{O}_X, \mathcal{O}_X) \). This is clearly a numerically flat vector bundle.

For every finite étale morphism \( \pi : Y \to X \) the map \( \pi^* : H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_Y) \) is injective as it can be split by the trace map. Let \( E_Y \) be the extension corresponding
to \( \text{id}_{H^1(\mathcal{O}_Y)} \in \text{End} (H^1(\mathcal{O}_Y)) = \text{Ext}^1(H^1(\mathcal{O}_Y) \otimes \mathcal{O}_Y, \mathcal{O}_Y) \) and consider the commutative diagram

\[
\begin{array}{c}
0 \rightarrow \mathcal{O}_Y \rightarrow \pi^*E \rightarrow H^1(\mathcal{O}_X) \otimes \mathcal{O}_Y \rightarrow 0 \\
0 \rightarrow \mathcal{O}_Y \rightarrow E_Y \rightarrow H^1(\mathcal{O}_Y) \otimes \mathcal{O}_Y \rightarrow 0.
\end{array}
\]

If \( \pi^*E \) is trivial then it injects into \( E_Y \) and hence \( E_Y \) has at least \( \text{rk} E = h^1(\mathcal{O}_X) + 1 \) linearly independent global sections. But by the definition of \( E_Y \) the connecting map \( H^0(Y, H^1(\mathcal{O}_Y) \otimes \mathcal{O}_Y) \rightarrow H^1(Y, \mathcal{O}_Y) \) is an isomorphism and hence \( h^0(E_Y) = 1 \). Therefore \( E \) is not étale trivializable.

Let \( \mathcal{X} \rightarrow S \) be a model of \( X \) as in the beginning of Section 6.

**Lemma 7.4.** There exists a non-empty open subset \( U \subset S \) such that the reduction \( \mathcal{E}_s \) of \( E \) for a closed point \( s \in U \) is étale trivializable if and only if the Frobenius morphism \( F = F_{\mathcal{X}_s} \) acts on \( H^1(\mathcal{O}_{\mathcal{X}_s}) \) bijectively.

**Proof.** If \( F^* \) acts on \( V = H^1(\mathcal{O}_{\mathcal{X}_s}) \) bijectively then the diagram

\[
\begin{array}{c}
0 \rightarrow \mathcal{O}_{\mathcal{X}_s} \rightarrow F^*\mathcal{E}_s \rightarrow V \otimes \mathcal{O}_{\mathcal{X}_s} \rightarrow 0 \\
0 \rightarrow \mathcal{O}_{\mathcal{X}_s} \rightarrow \mathcal{E}_s \rightarrow V \otimes \mathcal{O}_{\mathcal{X}_s} \rightarrow 0
\end{array}
\]

shows that \( F^*\mathcal{E}_s \cong \mathcal{E}_s \) and hence \( \mathcal{E}_s \) is étale trivializable by the Lange–Stuhler theorem (see Proposition 7.1).

Now assume that \( \mathcal{E}_s \) is étale trivializable. Let us consider the unique decomposition \( V = V_s \oplus V_n \) such that the Frobenius morphism \( F^* \) acts on \( V_s \) as an automorphism and it is nilpotent on \( V_n \). Let \( G \) be the bundle obtained as the extension of \( V_n \otimes \mathcal{O}_{\mathcal{X}_s} \) by \( \mathcal{O}_{\mathcal{X}_s} \) defined by the canonical inclusion \( (V_n \hookrightarrow V) \in \text{Hom}(V_n, V) = \text{Ext}^1(V_n \otimes \mathcal{O}_{\mathcal{X}_s}, \mathcal{O}_{\mathcal{X}_s}) \). Then we have the diagram

\[
\begin{array}{c}
0 \rightarrow \mathcal{O}_{\mathcal{X}_s} \rightarrow G \rightarrow V_n \otimes \mathcal{O}_{\mathcal{X}_s} \rightarrow 0 \\
0 \rightarrow \mathcal{O}_{\mathcal{X}_s} \rightarrow \mathcal{E}_s \rightarrow V \otimes \mathcal{O}_{\mathcal{X}_s} \rightarrow 0,
\end{array}
\]

which shows that \( G \hookrightarrow \mathcal{E}_s \). By the definition of \( G \) there exists some \( m_0 \) such that \( (F^{m_0})^*G \) is trivial. Let \( r = \dim V_n \). This shows that for every \( m \geq m_0 \) we have

\[
h^0((F^m)^*\mathcal{E}_s) \geq h^0((F^m)^*G) = r + 1.
\]
By the Lange–Stuhler theorem we know that for some \( m \geq m_0 \) we have \((F^m)^*E_s \simeq E_s\) and hence \( h^0(E_s) \geq r + 1 \). By the definition of \( E \) we know that the connecting map \( \delta: H^0(\mathcal{O}_X) \otimes \mathcal{O}_{X_s} \to H^1(\mathcal{O}_X) \) is an isomorphism and hence \( h^0(E) = 1 \).

Using semicontinuity of cohomology, we see that there exists an open subset \( U \subset S \) such that \( h^0(E_s) = 1 \) for every \( s \in U \). This implies that for any closed \( s \in U \) we have \( r = 0 \) and \( V = V_s \).

Therefore Conjecture 7.2 for vector bundle \( E \) is equivalent to the assertion that there are infinitely many closed points \( s \in S \) for which the Frobenius acts on \( H^1(\mathcal{O}_{X_s}) \) bijectively. In the curve case this is equivalent to saying that there are infinitely many places of ordinary reduction. This is known in case of genus \( g \leq 2 \) but it is still an open problem in general.

Remark 7.5. Note that if the reduction of \( E_s \) is étale trivializable by \( \pi: Y \to X_s \) then the degree of \( \pi \) is divisible by the characteristic \( p \) of the residue field \( k(s) \).

Indeed, if the characteristic \( p \) does not divide the degree of \( \pi \) then \( \frac{1}{\deg \pi} \text{Tr}_{X_s/Y}: \pi_*\mathcal{O}_Y \to \mathcal{O}_{X_s} \) splits the injection \( \mathcal{O}_{X_s} \to \pi_*\mathcal{O}_Y \). Then the same argument as in the characteristic zero case gives a contradiction.

### 7.4 Analogue of the Grothendieck-Katz \( p \)-curvature conjecture

In this subsection we try to relate étale trivializability of reductions of a vector bundle to finiteness of the image of its monodromy representation. Before formulating the corresponding conjecture we provide its original motivation: the global case of the Grothendieck–Katz conjecture.

Let \( X \) be a smooth variety defined over a field of characteristic \( p > 0 \) and let \( \nabla: E \to \Omega_X \otimes E \) be an integrable \( k \)-connection on a locally free \( \mathcal{O}_X \)-module \( E \). In characteristic \( p \), the \( p \)-th power \( D^p \) of a derivation \( D \) is again a derivation so we can consider \( \nabla(D^p) - \nabla(D)^p \). When this is zero for all local derivations \( D \) then we say that \( \nabla \) has zero \( p \)-curvature. If \( F_g: X \to X^{(1)} \) is the geometric Frobenius morphism then \( (E, \nabla) \) is equivalent to giving a locally free \( \mathcal{O}_{X^{(1)}} \)-module \( G \). The sheaf \( G \) can be recovered from \( (E, \nabla) \) as a sheaf of local sections \( v \) of \( E \) for which \( \nabla(v) = 0 \). On the other hand, giving \( G \) we can construct a canonical connection on \( E = F_g^*G \) by differentiating along the fibers of \( F_g \), i.e., we set \( \nabla(f \otimes g) = df \otimes g \).

**Conjecture 7.6.** (Grothendieck–Katz, see [Ka]) Let \( \nabla \) be a holomorphic vector bundle with an integrable connection on a complex manifold \( X \). Then \( (E, \nabla) \) has a finite monodromy group if and only if almost all its reductions to positive characteristic have vanishing \( p \)-curvature.
Note that if $X$ projective then $(E, \nabla)$ with finite monodromy group corresponds via Simpson’s correspondence described in Subsection 7.1 to an étale trivializable bundle (with zero Higgs field). So we can try to describe representations of the fundamental group with finite image on the Higgs bundle side in the following way:

**CONJECTURE 7.7.** In the notation of Section 6 assume that $E$ is not étale trivializable. Then the set $\Sigma^{net}$ of closed points $s \in S$ such that $\mathcal{E}_s$ is not étale trivializable, is infinite.

In case of bundles described in Example 7.3, the conjecture can be reformulated as saying that for a given smooth complex projective variety $X$ with $h^1(X, \mathcal{O}_X) > 0$, there are infinitely many points $s \in S$ for which the nilpotent part of the Frobenius action on $H^1(\mathcal{O}_s)$ is non-trivial. In particular, if $X$ is a complex elliptic curve then this is equivalent to saying that there are infinitely many primes for which the reduction of $X$ is supersingular. In case of elliptic curves defined over $\mathbb{Q}$ (and also in some other cases) this is a celebrated Elkies’ result [El].

**Example 7.8.** Let $A$ be an abelian variety over a number field $K$ and let $\mathcal{L}$ be a line bundle on some model $\mathfrak{A} \to S = \text{Spec } R$ of $A$ for a finitely generated subring $R \subset K$. Note that by Theorem 7.1 a line bundle $L$ on a smooth projective variety over $\overline{\mathbb{F}}_p$ is étale trivializable if and only if there exists some $n \in \mathbb{N}$ such that $(F^n)^*L \simeq L$. Therefore Conjecture 7.7 predicts that in the above case if for almost all closed points $s \in S$ there exists $n_s \in \mathbb{N}$ such that $(F^{n_s})^*\mathcal{L}_s \simeq \mathcal{L}_s$ then $\mathcal{L}_K$ is étale trivializable on $A$.

In this case a slightly weaker result is known. Namely, assume that there exists some $n \in \mathbb{N}$ such that for almost all closed points $s \in S$ we have $(F^n)^*\mathcal{L}_s \simeq \mathcal{L}_s$ (so $n_s$ in the above reformulation is independent of $s$). Then $\mathcal{L}_K$ is étale trivializable on $A$. This is just a dual version of [El, Theorem 5.3] and it implies that Conjecture 7.7 reduces to existence of a uniform bound on all $n_s$.

Note that if $\mathcal{L}_K$ is étale trivializable then there exists a positive integer $m$ such that $\mathcal{L}_s^m \simeq \mathcal{O}_s$ for all $s$ from some non-empty open subset $U \subset S$. Since for every (rational) prime $p$ not dividing $m$ the number $p^{nm!} - 1$ is divisible by $m$ we see that $(F^{nm})^*\mathcal{L}_s \simeq \mathcal{L}_s$ for all closed points $s$ from some smaller non-empty open subset $V \subset U$. This provides us with the converse to Pink’s theorem.

Using the same methods as in proof of [An, Théorème 7.2.2] and [EL, Theorem 5.1] one can show that an analogue of Conjecture 7.7 holds in case of equal characteristic zero:
THEOREM 7.9. Let $f : \mathcal{X} \to S$ be a smooth projective morphism of varieties defined over an algebraically closed field $k$ of characteristic 0. Let $\bar{\eta}$ be the generic geometric point of $S$ and let $\mathcal{E}$ be a locally free sheaf on $\mathcal{X}$. Let us assume that there exists a dense subset $U \subset S(k)$ such that for every $s$ in $U$ the bundle $\mathcal{E}_s$ is étale trivializable. Then we have the following:

1) There exists a finite Galois étale covering $\pi : Y \to \mathcal{X}_{\bar{\eta}}$ such that $\pi^* \mathcal{E}_{\bar{\eta}}$ is a direct sum of line bundles.

2) If $U$ is open in $S(k)$ then $\mathcal{E}_{\bar{\eta}}$ is étale trivializable.

Note that, similarly as in other cases, an analogue of this theorem is false for families defined over an algebraic closure of a finite field:

Example 7.10. In [EL, Corollary 4.3] the authors used Laszlo’s example [Ls, Section 3]) to construct a locally free sheaf $\mathcal{E}$ on $\mathcal{X} = X \times_k S \to S$, where $X$ is a smooth projective curve, $S$ is a smooth curve, both defined over $k = \mathbb{F}_2$ and such that for every closed point $s \in S$ the bundle $\mathcal{E}_s$ is étale trivializable but $\mathcal{E}_{\bar{\eta}}$ is not étale trivializable for the generic geometric point $\bar{\eta}$ of $S$.

The above example can occur only because the monodromy groups of $\mathcal{E}_s$ have orders divisible by the characteristic of $k(s)$. For positive results in other cases see [EL, Theorem 5.1].

Acknowledgements.

The author would like to thank D. Rössler and H. Esnault for useful conversations related to Section 7.

References

[An] Y. André, Sur la conjecture des $p$-courbures de Grothendieck-Katz et un problème de Dwork, Geometric aspects of Dwork theory, Vol. I, II, 55–112, Walter de Gruyter GmbH & Co. KG, Berlin, 2004.

[Ar] M. Artin, Some numerical criteria for contractability of curves on algebraic surfaces, Amer. J. Math. 84 (1962), 485–496.

[BS] M. Beltrametti, A. Sommese, Remarks on numerically positive and big line bundles, Projective geometry with applications, 9–18, Lecture Notes in Pure and Appl. Math. 166, Dekker, New York, 1994.
[Bh] B. Bhatt, Derived splinters in positive characteristic, *Compositio Math.* **148** (2012), 1757–1786.

[BS] I. Biswas, S. Subramanian, On a question of Sean Keel, *J. Pure Appl. Algebra* **215** (2011), 2600–2602.

[Br1] H. Brenner, On a problem of Miyaoka, Number fields and function fieldstwo parallel worlds, 51–59, *Progr. Math.* **239**, Birkhäuser Boston, Boston, MA, 2005.

[Br2] H. Brenner, The rationality of the Hilbert-Kunz multiplicity in graded dimension two, *Math. Ann.* **334** (2006), 91–110.

[DI] P. Deligne, L. Illusie, Relévements modulo $p^2$ et décomposition du complexe de de Rham, *Invent. Math.* **89** (1987), 247–270.

[EST] T. Ekedahl, N. Shepherd-Barron, R. Taylor, A conjecture on the existence of compact leaves of algebraic foliations, preprint, available at https://www.dpmms.cam.ac.uk/~nisb/

[El] N. Elkies, The existence of infinitely many supersingular primes for every elliptic curve over $\mathbb{Q}$, *Invent. Math.* **89** (1987), 561–567.

[EL] H. Esnault, A. Langer, On a positive equicharacteristic variant of the p-curvature conjecture, preprint, arXiv:1108.0103

[Fu] T. Fujita, Vanishing theorems for semipositive line bundles, Algebraic geometry (Tokyo/Kyoto, 1982), 519–528, *Lecture Notes in Math.* **1016**, Springer, Berlin, 1983.

[Gr] A. Grothendieck, Eléments de géométrie algébrique III. Étude cohomologique des faisceaux cohérents. 1, *Inst. Hautes Études Sci. Publ. Math.* **11** (1961), 167 pp.

[Ha] N. Hara, A characterization of rational singularities in terms of injectivity of Frobenius maps, *Amer. J. Math.* **120** (1998), 981–996.

[Ht1] R. Hartshorne, Ample subvarieties of algebraic varieties, *Lecture Notes in Mathematics* Vol. **156**, Springer-Verlag, Berlin-New York 1970.

[Ht2] R. Hartshorne, Algebraic geometry, *Graduate Texts in Mathematics* **52**, Springer-Verlag, New York-Heidelberg, 1977.
[HH] M. Hochster, C. Huneke, Infinite integral extensions and big Cohen-
Macaulay algebras, *Ann. of Math.* **135** (1992), 53–89.

[HS] C. Huneke, K. Smith, Tight closure and the Kodaira vanishing theorem,
*J. Reine Angew. Math.* **484** (1997), 127–152.

[HL] D. Huybrechts, M. Lehn, The geometry of moduli spaces of sheaves,
*Aspects of Mathematics* **E31**, Friedr. Vieweg & Sohn, Braunschweig, 1997.

[Ka] N. Katz, Algebraic solutions of differential equations (p-curvature and
the Hodge filtration), *Invent. Math.* **18** (1972), 1–118.

[Ke1] S. Keel, Basepoint freeness for nef and big line bundles in positive characteristic,
*Ann. of Math.* **149** (1999), 253–286.

[Ke2] S. Keel, Polarized pushouts over finite fields, Special issue in honor of
Steven L. Kleiman, *Comm. Algebra* **31** (2003), 3955–3982.

[LS] H. Lange, U. Stuhler, Vektorbündel auf Kurven und Darstellungen der
algebraischen Fundamentalgruppe, *Math. Z.* **156** (1977), 73–83.

[La1] A. Langer, Moduli spaces of sheaves and principal G-bundles, Algebraic
geometry–Seattle 2005. Part 1, 273–308, *Proc. Sympos. Pure Math.* **80**, Part 1, Amer. Math. Soc., Providence, RI, 2009.

[La2] A. Langer, Nef line bundles over finite fields, *Int. Math. Res. Not.* (2012),
doi:10.1093/imrn/rns152.

[LR] A. Lanteri, B. Rondena, Numerically positive divisors on algebraic surfaces,
*Geom. Dedicata* **53** (1994), 145–154.

[LS] Y. Laszlo, A non-trivial family of bundles fixed by the square of Frobe-
nius, *C. R. Acad. Sci. Paris Sér. I Math.* **333** (2001), 651–656.

[Lz] R. Lazarsfeld, Positivity in algebraic geometry I, Ergebnisse der Mathe-
matik und ihrer Grenzgebiete **48**, Springer-Verlag, Berlin, 2004.

[MSr] V. Mehta, V. Srinivas, A characterization of rational singularities, *Asian
J. Math.* **1** (1997), 249–271.
[MSu] V. Mehta, S. Subramanian, Nef line bundles which are not ample, *Math. Z.* **219** (1995), 235–244.

[Mi] Y. Miyaoka, The Chern classes and Kodaira dimension of a minimal variety, Algebraic geometry, Sendai, 1985, 449–476, *Adv. Stud. Pure Math.* **10**, North–Holland, Amsterdam, 1987.

[Mo1] P. Monsky, Hilbert-Kunz functions in a family: point-S4 quartics, *J. Algebra* **208** (1998), 343–358.

[Mo2] P. Monsky, On the Hilbert-Kunz function of $z^D - p_4(x, y)$, *J. Algebra* **291** (2005), 350–372.

[Mr] A. Moriwaki, Relative Bogomolov’s inequality and the cone of positive divisors on the moduli space of stable curves, *J. Amer. Math. Soc.* **11** (1998), 569–600.

[Na] M. Nagata, Lectures on the fourteenth problem of Hilbert, Tata Institute of Fundamental Research, Bombay 1965.

[Pi] R. Pink, On the order of the reduction of a point on an abelian variety, *Math. Ann.* **330** (2004), 275–291.

[Ra] M. Raynaud, Contre-exemple au ”vanishing theorem” en caract’eristique $p > 0$, C. P. Ramanujam–a tribute, pp. 273–278, *Tata Inst. Fund. Res. Studies in Math.* **8**, Springer, Berlin-New York, 1978.

[Se] C. S. Seshadri, Geometric reductivity (Mumford’s conjecture)-revisited, Commutative algebra and algebraic geometry, 137–145, *Contemp. Math.* **390**, Amer. Math. Soc., Providence, RI, 2005.

[SB1] N. Shepherd-Barron, Infinite generation for rings of symmetric tensors, *Math. Res. Lett.* **2** (1995), 125–128.

[SB2] N. Shepherd-Barron, Semi-stability and reduction mod $p$, *Topology* **37** (1998), 659–664.

[Si] C. Simpson, Higgs bundles and local systems, *Inst. Hautes Études Sci. Publ. Math.* No. **75** (1992), 5–95.
[Sm] K. Smith, Vanishing, singularities and effective bounds via prime characteristic local algebra, Algebraic geometry – Santa Cruz 1995, 289–325, Proc. Sympos. Pure Math. 62, Part 1, Amer. Math. Soc., Providence, RI, 1997; Erratum available at http://www.math.lsa.umich.edu/~kesmith

[Su] S. Subramanian, Mumford’s example and a general construction, Proc. Indian Acad. Sci. Math. Sci. 99 (1989), 197–208.

[To] B. Totaro, Moving codimension-one subvarieties over finite fields, Amer. J. Math. 131 (2009), 1815–1833.

[Tr] V. Trivedi, Semistability and Hilbert-Kunz multiplicities for curves, J. Algebra 284 (2005), 627–644.