COMPLEX PROBABILITY THEORY AND PROGNOSTIC

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ABSTRACT

The Kolmogorov’s system of axioms can be extended to encompass the imaginary set of numbers and this by adding to the original five axioms an additional three axioms. Hence, any experiment can thus be executed in what is now the complex set C (Real set R with real probability + Imaginary set M with imaginary probability). The objective here is to evaluate the complex probabilities by considering supplementary new imaginary dimensions to the event occurring in the “real” laboratory. Whatever the probability distribution of the input random variable in R is, the corresponding probability in the whole set C is always one, so the outcome of the random experiment in C can be predicted totally. The result indicates that chance and luck in R is replaced now by total determinism in C. This new complex probability model will be applied to the concepts of degradation and the Remaining Useful Lifetime (RUL), thus to the field of prognostic.

Keywords: Complex Probability, Probability Distributions, Prognostic, Degradation, Lifetime

1. INTRODUCTION

Abou Jaoude et al. (2010); Abou Jaoude (2013; 2005; 2007); Bell (1992); Benton (1996); Boursin (1986); Chen et al. (1997); Cheney and Kincaid (2004); Dacunha-Castelle (1999); Dalmédico Dahan et al. (1992); Dalmedico Dahan and Peiffer (1986); Ekeland (1991); Feller (1968); Finney et al. (2004); Gentle (2003); Gerald and Wheatley (1999); Gleick (1997); Greene (2000; 2004) firstly, the Extended Kolmogorov’s Axioms (EKA for short) paradigm can be illustrated by the following figure (Fig. 1).

In engineering systems, the remaining useful lifetime prediction is related deeply to many factors that generally have a chaotic behavior which decreases the degree of our knowledge of the system.

As the Degree Of Our Knowledge (DOK for short) in the real universe R is unfortunately incomplete, the extension to the complex universe C includes the contributions of both the real universe R and the imaginary universe M. Consequently, this will result in a complete and perfect degree of knowledge in C = R+M (Pc = 1). In fact, in order to have a certain prediction of any event it is necessary to work in the complex universe C in which the chaotic factor is quantified and subtracted from the Degree Of Knowledge to lead to a probability in C equal to one (Pc² = DOK-Chf = 1). Thus, the study in the complex universe results in replacing the phenomena that used to be random in R by deterministic and totally predictable ones in C.

This hypothesis is verified in a previous study and paper by the mean of many examples encompassing both discrete and continuous distributions.

From the Extended Kolmogorov’s Axioms (EKA), we can deduce that if we add to an event probability in the real set R the imaginary part M (like the lifetime variables) then we can predict the exact probability of the remaining lifetime with certainty in C (Pc = 1).

We can apply this idea to prognostic analysis through the degradation evolution of a system. As a matter of fact, prognostic analysis consists in the prediction of the remaining useful lifetime of a system at any instant tₙ and during the system functioning.

Let us consider a degradation trajectory D(t) of a system where a specific instant tₙ is studied. The instant tₙ means here the time or age that can be measured also by the cycle number N.

Referring to the figure below (Fig. 2), the previous statement means that at the system age tₙ, the prognostic study must give the prediction of the failure instant tₙₙ. Therefore, the RUL predicted here at instant tₙ is the following quantity: RUL (tₙ) = tₙₙ-tₙₙ.
In fact, at the beginning ($t_0 = 0$) (point J), the failure probability $P_r = 0$ and the chaotic factor in our prediction is zero ($Chf = 0$). Therefore, $RUL(t_0 = 0) = t_N - t_0 = t_N$.

If $t_0 = t_N$ (point L) then the $RUL(t_N) = t_N - t_N = 0$ and the failure probability is one ($P_r = 1$).

If not (i.e., $0 < t_0 < t_N$) (point K), the probability of the occurrence of this instant and the prediction probability of $RUL$ are both less than one (not certain) due to non-zero chaotic factors. The degree of our knowledge is consequently less than 1. Thus, by applying here the EKA method, we can determine the system $RUL$ with certainty in $C = R + M$ where $P_c = 1$ always.

Furthermore, we need in our current study the absolute value of the chaotic factor that will give us the magnitude of the chaotic and random effects on the studied system. This new term will be denoted accordingly $MChf$ or Magnitude of the Chaotic Factor. Hence, we can deduce the following:

$$MChf(t_0) = |Chf(t_0)| \geq 0$$

$$Pc^2(t_0) = DOK(t_0) - Chf(t_0) = DOK(t_0) + |Chf(t_0)|,$$

since $-0.5 \leq Chf(t_0) \leq 0$

$$= DOK(t_0) + MChf(t_0) = 1, \quad \forall \ 0 \leq t_0 \leq t_N \Rightarrow 0 \leq MChf(t_0) \leq 0.5 \quad \text{where} \ 0.5 \leq DOK(t_0) \leq 1$$
Moreover, we can define two complementary events E and $\overline{E}$ with their respective probabilities:

$$P_{rob}(E) = p \quad \text{and} \quad P_{rob}(\overline{E}) = q = 1 - p$$

Then $P_{rob}(E)$ in terms of the instant $t_0$ is given by: $P_{rob}(E) = P_{rob}(t \leq t_0) = F(t_0)$ where $F$ is the cumulative probability distribution function of the random variable $t$.

Since $P_{rob}(E) + P_{rob}(\overline{E}) = 1$, therefore, $P_{rob}(\overline{E}) = 1 - P_{rob}(E) = 1 - P_{rob}(t \leq t_0) = P_{rob}(t > t_0)$. Let us define the two particular instants: $t_0 = 0$ assumed as the initial time of functioning (raw state) corresponding to $D = D_0 = 0$ and $t_0 = t$ is the failure instant (wear out state) corresponding to the degradation $D = 1$.

The boundary conditions are.

For $t = 0$ then $D = D_0$ (initial damage that may be zero or not) and $F(t_0) = P_{rob}(t \leq t_0) = 0$.

For $t = t_0$ then $D = 1$ and $f(t_0) = F(t_0) = P_{rob}(t \leq t_0) = 1$.

Also $F(t_0)$ is a non-decreasing function that varies between 0 and 1. In fact, $F(t_0)$ is a cumulative function (Fig. 3). In addition, since $RUL(t_0) = t_N - t_0$ and $0 \leq t_0 \leq t_N$ then $RUL(t_0)$ is a non-increasing remaining useful lifetime function (Fig. 4).

Referring to Fig. 5 below, we can infer the following:

The complex probability $Z(t_0) = P_{rob}(t_0) + P_{rob}(t_0) = P_{rob}(t_0) + i[1 - P_{rob}(t_0)]$.

The square of the norm of $Z(t_0)$ is:

$$[Z(t_0)]^2 = \text{DOK}(t_0) = 1 + 2P_{rob}(t_0)P_{rob}(t_0) + 2P_{rob}(t_0)$$

The Chaotic Factor and the Magnitude of the Chaotic Factor are:

$$\text{Chf}(t_0) = 2P_{rob}(t_0)[1 - P_{rob}(t_0)] = -2P_{rob}(t_0)$$

And $\text{Chf}(t_0)$ is null when $P_{rob}(t_0) = P_{rob}(0) = 0$ (point J) or when $P_{rob}(t_0) = P_{rob}(t_0) = 1$ (point L) and $\text{MChf}(t_0) = [\text{Chf}(t_0)] = 2P_{rob}(t_0)[1 - P_{rob}(t_0)] = 2P_{rob}(t_0) - 2P_{rob}^2(t_0)$ is null when $P_{rob}(t_0) = P_{rob}(0) = 0$ (point J) or when $P_{rob}(t_0) = P_{rob}(t_0) = 1$ (point L).

At any instant $t_0$ (point K), the probability expressed in the complex set C is:

$$P_{rob}(t_0) + P_{rob}(t_0)i = P_{rob}(t_0) + [1 - P_{rob}(t_0)] = 1 \text{ always}.$$  

Hence, the prediction of $RUL(t_0)$ of the system degradation in C is permanently certain.

Let us consider thereafter many probability distributions to model the function $F(t_0)$.

### 2. APPLICATION TO DIFFERENT PROBABILITY DISTRIBUTIONS

#### 2.1. The Uniform Probability Distribution

(Guillen, 1995; Gullberg, 1997; Kuhn, 1996; Liu, 2001; Mandelbrot, 1997; Montgomery and Runger, 2005; Müller, 2005; Or luc and Poirier, 2005; Poincaré, 1968; Prigogine, 1997; Prigogine and Stengers, 1992; Robert and Casella, 2010; Science et Vie, 1999; Srinivasan and Mehata, 1978; Stewart, 1996; 2002; Van Kampen, 2007; Walpole, 2002; Ducrocq and Warusfel, 2004; Weinberg, 1992)

With a probability density function:

$$f(t) = \frac{dF(t)}{dt} = \begin{cases} \frac{1}{b-a} & \text{if } a \leq t \leq b \\ 0 & \text{elsewhere} \end{cases}$$

and a cumulative distribution function:

$$F(t_0) = P_{rob}(t \leq t_0) = \int_{t_0}^{t} f(t) \, dt = \int_{t_0}^{t} f(t) \, dt = \begin{cases} \frac{t-a}{b-a} & \text{if } a \leq t_0 \leq b \\ 0 & \text{elsewhere} \end{cases}$$

With the two boundaries $a = 0$ and $b = t_N$ then:

$$F(t_0) = \begin{cases} \frac{t_0-a}{t_N-a} & \text{if } 0 \leq t_0 \leq t_N \end{cases}$$

We have taken the domain for the uniform variable $t_0 = [0, t_N = 1000]$ and $dt_0 = 0.1$ then:

$$F(t_0) = \begin{cases} \frac{t_0}{1000} & \text{if } 0 \leq t_0 \leq 1000 \end{cases}$$

#### 2.1.1. The Real Probability $P_r$:

$$P_r(t_0) = F(t_0) = \frac{t_0}{t_N} \quad \text{if } 0 \leq t_0 \leq t_N = 1000$$

We note that $P_r(t_0)$ is a non-decreasing function.

#### 2.1.2. The Complementary Probability $P_{rob}$:

$$P_{rob}(t_0) = 1 - F(t_0) = 1 - \frac{t_0}{t_N} \quad \text{if } 0 \leq t_0 \leq t_N = 1000$$
Fig. 3. Occurrence probability

Fig. 4. RUL prognostic model

Fig. 5. Degradation prognostic model
We note that $P_m(t_0)/i$ is a non-increasing function.

### 2.1.3. The Degree of Our Knowledge DOK

DOK is the measure of our certain knowledge (100% probability) about the expected event, it does not include any uncertain knowledge (with probability less than 100%):

$$DOK(t_0) = Pc(t_0) = \text{DOK}(t_0) = \text{Chf}(t_0)$$

$$= 1 - 2P_m(t_0)P_r(t_0) - \text{Chf}(t_0)$$

$$= 1 - 2P_m(t_0) - [1 - P_m(t_0)]$$

$$= 1 - 2F(t_0) - [1 - F(t_0)]$$

$$= 1 - 2\frac{t_0}{t_N} + 2\left(\frac{t_0}{t_N}\right)^2$$

Which is a parabola concave upward having a vertex (a minimum) at:

$$t_b = \frac{t_0}{2} = 500 = 0.5 \times 10^3, \quad 0.5$$

### 2.1.4. The Chaotic Factor Chf and MChf:

$$\text{Chf}(t_0) = 2P_r(t_0)P_m(t_0) - \text{Chf}(t_0)$$

$$= -2P_r(t_0)[1 - P_m(t_0)]$$

$$= -2P_r(t_0)[1 - F(t_0)]$$

$$= -2\frac{t_0}{t_N} - [1 - \frac{t_0}{t_N}]$$

$$= -2\frac{t_0}{t_N} + 2\left(\frac{t_0}{t_N}\right)^2$$

Which is a parabola concave upward having a vertex (a minimum) at:

$$t_b = \frac{t_0}{2} = 500 = 0.5 \times 10^3, \quad 0.5$$

Therefore, we can infer the magnitude of the chaotic factor MChf:

$$\text{MChf}(t_0) = \sqrt{\text{Chf}(t_0)} = \sqrt{-2P_r(t_0)[1 - P_m(t_0)]}$$

$$= 2P_r(t_0)[1 - P_m(t_0)]$$

$$= 2\frac{t_0}{t_N} - [1 - \frac{t_0}{t_N}]$$

$$= 2\frac{t_0}{t_N} - 2\left(\frac{t_0}{t_N}\right)^2$$

Which is a parabola concave downward having a vertex (a maximum) at:

$$t_b = \frac{t_0}{2} = 500 = 0.5 \times 10^3, \quad 0.5$$

### 2.1.5. Pc: The Probability in the Complex Set C:

$$Pc^2(t_0) = \text{DOK}(t_0) - \text{Chf}(t_0)$$

$$= 1 - 2\frac{t_0}{t_N} + 2\left(\frac{t_0}{t_N}\right)^2 + 2\frac{t_0}{t_N} - 2\left(\frac{t_0}{t_N}\right)^2 = 1 \Rightarrow Pc(t_0) = 1$$

Thus we deduce that in the set C, we have a complete knowledge of the random variable since $Pc = 1$.

### 2.1.6. The Intersection Point:

$$P_c(t_0) = P_m(t_0)/i \Rightarrow t_b = \frac{t_N}{2} = \frac{1000}{2} = 500$$

$$\text{Chf}(t_0) = \frac{500}{1000} = 0.5$$

$$P_m(t_0) = 0 \Rightarrow t_b = \frac{t_N}{2} = \frac{1000}{2} = 500$$

Thus we conclude that $P_r(t_0)$, $P_m(t_0)/i$, DOK and MChf all intersect at $(500, 0.5)$ (Fig. 6-9).

### 2.1.7. The EKA Parameters Analysis in the Prognostic of Degradation:

We note from the figure below that the DOK is maximum (DOK = 1) when MChf is minimum (MChf = 0) (points J & L) and that means when the magnitude of the chaotic factor (MChf) decreases our certain knowledge increases.

At the beginning $P_r(t_0) = t_0/t_N = 0/t_N = 0$, the system is intact (zero damage: $D = 0$) and has zero chaotic factor before any usage, at this instant $DOK(0) = 1$ and RUL(0) = $t_N - 0 = t_N$ with $Pc(0) = 1$. Afterward, $0 < t_0 < t_N$, RUL(t0) = $t_N - t_0$ with $P_r(t_0) = t_0/t_N \neq 0$ and $Pc(t_0) = 1$ and MChf starts to increase during the functioning due to the environment and intrinsic conditions thus leading to a decrease in DOK until they both reach 0.5 at $t_b = t_N/2 = 500$ (point K) where $RUL(t_N/2) = t_N - t_N/2 = 500$. Since the real probability $P_r$ is a uniform distribution MChf will intersect with DOK at the point $(t_0/2 = 500, 0.5)$ (point K).

With the increase of the time of functioning, MChf returns to zero and the DOK returns to 1 where we reach total damage (D = 1) and hence the total failure of the system (point L). At this last point the failure here is certain, $P_r(t_N) = t_N/t_N = 1$ and RUL(t_N) = $t_N - t_N = 0$ with $Pc(t_N) = 1$, so the logical explanation of the value DOK = 1 follows (Fig. 10).
Fig. 6. EKA parameters in uniform probability distribution

Fig. 7. EKA parameters in uniform probability distribution
Fig. 8. The probabilities Pr and Pm/i in terms of t and of each other

Fig. 9. DOK and Chf in terms of t and of each other in uniform probability distribution
Fig. 10. EKA parameters and the prognostic of degradation

We note that the same logic and analysis apply concerning the degradation and the remaining useful lifetime for all the six probability distributions.

2.2. The Logarithmic Distribution:

With a probability density function:

$$f(t) = \frac{\alpha}{1 + \alpha t}$$

if $0 \leq t \leq t_N$

elsewhere

where $\alpha = (e - 1)/1000 = 0.001718281...$

and a cumulative distribution function:

$$F(t) = \int_0^t f(t) \, dt = \int_0^t \frac{\alpha}{1 + \alpha s} \, ds = \begin{cases} \ln(1 + \alpha t) & \text{if } 0 \leq t \leq t_N \\ 0 & \text{elsewhere} \end{cases}$$

We have taken the domain for the logarithmic variable $t_0 = [0, t_N = 1000]$ where $dt_0 = 0.1$.

2.2.1. The Real Probability $P_r$:

$$P_r = F(t_0) = \ln(1 + \alpha t_0) \quad \text{if } 0 \leq t_0 \leq t_N = 1000$$

We note that $P_r$ is a non-decreasing function.

2.2.2. The Complementary Probability $P_m/i$:

$$P_m/i = 1 - P_r = 1 - F(t_0) = 1 - \ln(1 + \alpha t_0)$$

if $0 \leq t_0 \leq t_N = 1000$

We note that $P_m/i$ is a non-increasing function.

2.2.3. The Degree of Our Knowledge DOK:

DOK is the measure of our certain knowledge (100% probability) about the expected event, it does not include any uncertain knowledge (with probability less than 100%):

$$DOK = 1 - 2P_r(1 - P_r) = 1 - 2P_m(E)P_{\bar{m}}(E)$$

$$= 1 - 2F(t_0)[1 - F(t_0)]$$

$$= 1 - 2\ln(1 + \alpha t_0)[1 - \ln(1 + \alpha t_0)]$$

$$= 1 - 2\ln(1 + \alpha t_0) + 2[\ln(1 + \alpha t_0)]^2$$

which is a curve concave upward having a minimum at:

$$t_0 = 377.54, \quad 0.5$$

2.2.4. The Chaotic Factor Chf and MChf:

$$Chf = -2P_r(1 - P_r) = -2P_m(E)P_{\bar{m}}(E)$$

$$= -2F(t_0)[1 - F(t_0)]$$

$$= -2\ln(1 + \alpha t_0)[1 - \ln(1 + \alpha t_0)]$$

$$= -2\ln(1 + \alpha t_0) + 2[\ln(1 + \alpha t_0)]^2$$

which is a curve concave upward having a minimum at:

$$t_0 = 377.54, \quad -0.5$$
Therefore, we can infer the magnitude of the chaotic factor MChf:

\[
MChf(t_o) = \left| \text{Chf}(t_o) - 2.8P(t_o)(1-P(t_0)) \right| \\
= 2.8P(t_o)(1-P(t_0)) + 2.8F(t_o)(1-F(t_o)) \\
= 2.8\ln(1+\alpha_o)(1-\ln(1+\alpha_o)) \\
= 2.8\ln(1+\alpha_o) - 2(\ln(1+\alpha_o))^2
\]

Which is a curve concave downward having a maximum at:

\[(t_o = 377.54, 0.5)\]

### 2.2.5. Pc: The Probability in the Complex Set C:

\[Pc^2 = DOK - \text{Chf} = 1 - 2.8\ln(1+\alpha_o) + 2.8(\ln(1+\alpha_o))^2\]

\[+2.8\ln(1+\alpha_o) - 2(\ln(1+\alpha_o))^2 = 1 \Rightarrow Pc = 1\]

Thus, we deduce that in the set C, we have complete knowledge of the random variable since Pc = 1.

### 2.2.6. The Intersection Point:

\[P_r(t_o) = P_m(t_o) / i \Leftrightarrow \ln(1+\alpha_o) = 1 - \ln(1+\alpha_o)\]

\[\Leftrightarrow 2.8\ln(1+\alpha_o) = 1 \Rightarrow \ln(1+\alpha_o) = \frac{1}{2}\]

\[\Leftrightarrow t_o = (\exp(0.5) - 1) / \alpha = 377.54\]

And \[P_r(t_o = 377.54) = \ln(1+\alpha \times 377.54) = 0.5\]

And \[P_m(t_o = 377.54) / i = 1 - \ln(1+\alpha \times 377.54)\]

\[= 1 - 0.5 = 0.5\]

So \[P_r(t_0)\] and \[P_m(t_0)/i\] intersect at \((377.54, 0.5)\).

Moreover, the minimum of DOK and the maximum of MChf occur at \((377.54, 0.5)\).

So we conclude that \([P_r(t_0), P_m(t_0)/i, DOK and MChf all intersect at (377.54, 0.5)\) (Fig. 11-13).

### 2.2.7. The EKA Parameters Analysis in the Prognostic of Degradation:

In this case, we note from the figure below that the DOK is maximum (DOK = 1) when MChf is minimum (MChf = 0) (points J & L). Afterward, MChf starts to increase with the decrease of DOK until it reaches 0.5 at \(t_0 = 377.54\) (point K). Since the real probability \(P_r\) is a logarithmic distribution (convex curve) it will intersect with DOK at the point \((377.54, 0.5)\) (point K). With the increase of \(t_0\), MChf returns to zero and the DOK returns to 1 where we reach total damage (D = 1) and hence the total certain failure (\(P_r = 1\)) of the system (point L). We note that the point K is no more at the middle of DOK since the distribution is not anymore uniform and symmetric.

At each instant \(t_0\), the remaining useful lifetime \(RUL(t_0)\) is certainly predicted in the complex set C with \(Pc\) maintained as equal to one through a continuous compensation between DOK and Chf. This compensation is from instant \(t_0 = 0\) where \(D(t_0) = 0\) until the failure instant \(t_N\) where \(D(t_N) = 1\) (Fig. 11).

### 2.3. The Power Probability Distribution:

With a probability density function:

\[
f(t) = \frac{dF(t)}{dt} = \begin{cases} \frac{14.1^3}{1000^{14}} & \text{if } 0 \leq t \leq t_N \\ 0 & \text{elsewhere} \end{cases}
\]

and a cumulative distribution function:

\[
F(t) = \int_0^t f(t)dt = \int_0^{t_N} \frac{14.1^3}{1000^{14}} dt = \begin{cases} \frac{t_N^{14}}{1000^{14}} & \text{if } 0 \leq t_N \leq t_N \\ 0 & \text{elsewhere} \end{cases}
\]

We have taken the domain for the power variable \(t_0 = [0, t_N = 1000]\) and \(dt_0 = 0.1\).

#### 2.3.1. The Real Probability \(P_r:\)

\[
P_r = F(t_o) = \frac{t_o^{14}}{1000^{14}} \quad \text{if } 0 \leq t_o \leq t_N = 1000
\]

We note that \(P_r\) is a non-decreasing function.

#### 2.3.2. The Complementary Probability \(P_m/i:\)

\[
P_m/i = 1 - P_r = 1 - F(t_o) = 1 - \frac{t_o^{14}}{1000^{14}} \quad \text{if } 0 \leq t_o \leq t_N = 1000
\]

We note that \(P_m/i\) is a non-increasing function.
Fig. 11. EKA parameters in logarithmic cumulative distribution

Fig. 12. EKA parameters in logarithmic cumulative distribution
2.3.3. The Degree of Our Knowledge DOK:

DOK is the measure of our certain knowledge (100% probability) about the expected event, it does not include any uncertain knowledge (with probability less than 100%):

\[ DOK = 1 - 2P_c(1 - P_c) = 1 - 2P_{rob}(E)P_{rob}(\bar{E}) \]
\[ = 1 - 2F(t_o)[1 - F(t_o)] \]
\[ = 1 - 2 \frac{t_o^{14}}{1000^{14}} \left( 1 - \frac{t_o^{14}}{1000^{14}} \right) = 1 - 2 \frac{t_o^{14}}{1000^{14}} \]
\[ + 2 \left( \frac{t_o^{14}}{1000^{14}} \right)^2 = 1 - 2 \frac{t_o^{14}}{1000^{14}} + 2 \left( \frac{t_o^{14}}{1000^{14}} \right)^2 \]

Which is a curve concave upward having a minimum at:

\[ (t_o = 951.7, \ 0.5) \]

2.3.4. The Chaotic Factor Chf and MChf:

\[ Chf = -2P_c(1 - P_c) = -2P_{rob}(E)P_{rob}(\bar{E}) \]
\[ = -2F(t_o)[1 - F(t_o)] \]
\[ = -2 \frac{t_o^{14}}{1000^{14}} \left( 1 - \frac{t_o^{14}}{1000^{14}} \right) = -2 \frac{t_o^{14}}{1000^{14}} + 2 \left( \frac{t_o^{14}}{1000^{14}} \right)^2 \]
\[ = -2 \frac{t_o^{14}}{1000^{14}} + 2 \left( \frac{t_o^{14}}{1000^{14}} \right)^2 \]

Which is a curve concave upward having a minimum at:

\[ (t_o = 951.7, \ 0.5) \]

Therefore, we can infer the magnitude of the chaotic factor MChf:

\[ MChf(t_o) = \left| Chf(t_o) \right| = \left| -2P_c(t_o)[1 - P_c(t_o)] \right| \]
\[ = 2P_c(t_o)[1 - P_c(t_o)] = 2F(t_o)[1 - F(t_o)] \]
\[ = 2 \frac{t_o^{14}}{1000^{14}} \left( 1 - \frac{t_o^{14}}{1000^{14}} \right) = 2 \frac{t_o^{14}}{1000^{14}} - 2 \frac{t_o^{14}}{1000^{14}} \]

Which is a curve concave downward having a maximum at:

\[ (t_o = 951.7, \ 0.5) \]

2.3.5. Pc: The Probability in the Complex Set C:

\[ P_c^2 = DOK - Chf = 1 - 2 \frac{t_o^{14}}{1000^{14}} + 2 \left( \frac{t_o^{14}}{1000^{14}} \right)^2 \]
\[ + 2 \frac{t_o^{14}}{1000^{14}} = 1 \Rightarrow Pc = 1 \]

Thus we deduce that in the set C, we have a complete knowledge of the random variable since Pc = 1.
2.3.6. The Intersection Point:

\[ P_r(t_0) = P_m(t_0) / i \Leftrightarrow \frac{t_0^{14}}{1000^{14}} = 1 - \frac{t_0^{14}}{1000^{14}} \]
\[ \Leftrightarrow 2 \cdot \frac{t_0^{14}}{1000^{14}} = 1 \Leftrightarrow \frac{t_0^{14}}{1000^{14}} = \frac{1}{2} \]
\[ \Leftrightarrow t_0 = \sqrt[14]{0.5 \times 1000^{14}} = 951.7 \]
\[ \text{and } P_r(t_0) = \frac{951.7^{14}}{1000^{14}} = 0.5 \]
\[ \text{and } P_m(t_0) / i = 1 - \frac{951.7^{14}}{1000^{14}} = 1 - 0.5 = 0.5 \]

So \( P_r(t_0) \) and \( P_m(t_0) / i \) intersect at \((951.7, 0.5)\).

Moreover, the minimum of DOK and the maximum of MChf occur at \((951.7, 0.5)\).

So we conclude that \( P_r(t_0), P_m(t_0) / i, \text{DOK and MChf} \) all intersect at \((951.7, 0.5)\) (Fig. 14-16).

2.3.7. The EKA Parameters Analysis in the Prognostic of Degradation:

In this case, we note from the figure below that the DOK is maximum (DOK=1) when MChf is minimum (MChf = 0) (points J & L). Afterward, MChf starts to increase with the decrease of DOK until it reaches 0.5 at \( t_0 = 951.7 \) (point K). Since the real probability \( P_r \) is a power distribution it will intersect with DOK at the point \((951.7, 0.5)\) (point K). With the increase of \( t_0 \), MChf returns to zero and the DOK returns to 1 where we reach total damage (D =1) and hence the total certain failure (\( P_r = 1 \)) of the system (point L). At this last point L the failure here is certain, \( P(t_0) = t_0/t_N = 1 \) and \( RUL(t_0) = t_N-t_0 = 0 \) with \( Pcr(t_N) = 1 \), so the logical explanation of the value DOK = 1 follows. We note that the point K is no more at the middle of DOK since the distribution is neither uniform nor symmetric (Fig. 14).

2.4. The Exponential Probability Distribution:

With a probability density function:

\[ f(t) = \frac{dF(t)}{dt} = \begin{cases} \frac{1}{200} \exp \left[ -\frac{(1000-t)}{200} \right] & \text{if } 0 \leq t \leq t_N \\ 0 & \text{elsewhere} \end{cases} \]

and a cumulative distribution function:

\[ F(t) = \int_0^t f(t) dt = \int_0^t f(t) dt = \begin{cases} \exp \left[ -\frac{(1000-t)}{200} \right] & \text{if } 0 \leq t \leq t_N \\ 0 & \text{elsewhere} \end{cases} \]

We have taken the domain for the logarithmic variable \( t_0 = [0, t_N = 1000] \) and \( dt_0 = 0.1 \).

2.4.1. The Real Probability \( P_r \):

\[ P_r(t) = F(t_0) = \exp \left[ -\frac{(1000-t_0)}{200} \right] \]

if \( 0 \leq t_0 \leq t_N = 1000 \)

We note that \( P_r \) is a non-decreasing function.

2.4.2. The Complementary Probability \( P_m / i \):

\[ P_m / i = 1 - P_r = 1 - F(t_0) = 1 - \exp \left[ -\frac{(1000-t_0)}{200} \right] \]

if \( 0 \leq t_0 \leq t_N = 1000 \)

We note that \( P_m / i \) is a non-increasing function.

2.4.3. The Degree of Our Knowledge DOK:

DOK is the measure of our certain knowledge (100% probability) about the expected event, it does not include any uncertain knowledge (with probability less than 100%):

\[ \text{DOK} = 1 - 2P_r(1-P_r) = 1 - 2P_m(E)P_m(E) \]
\[ = 1 - 2F(t_0)[1 - F(t_0)] \]
\[ = 1 - 2\exp \left[ -\frac{(1000-t_0)}{200} \right] \left[ 1 - \exp \left[ -\frac{(1000-t_0)}{200} \right] \right] \]
\[ = 1 - 2\exp \left[ -\frac{(1000-t_0)}{200} \right] + 2\exp \left[ -\frac{(1000-t_0)}{200} \right]^2 \]
\[ = 1 - 2\exp \left[ -\frac{(1000-t_0)}{200} \right] + 2\exp \left[ -\frac{2(1000-t_0)}{200} \right] \]

Which is a curve concave upward having a minimum at:

\( t_0 = 861.37, \ 0.5 \)
Fig. 14. EKA parameters in power probability distribution

Fig. 15. EKA parameters in power cumulative distribution
2.4.4. The Chaotic Factor Chf and MChf:

\[
\text{Chf} = -2. P_{\text{e}}(1 - P_{\text{e}}) = -2P_{\text{e}}(E)P_{\text{m}}(\overline{E}) \\
= -2F(t_0)[1 - F(t_0)] \\
= -2\exp \left[ \frac{(1000 - t_0)}{200} \right] \left( 1 - \exp \left[ \frac{(1000 - t_0)}{200} \right] \right) \\
= -2\exp \left[ \frac{(1000 - t_0)}{200} \right] + 2 \left( \exp \left[ \frac{(1000 - t_0)}{200} \right] \right)^2 \\
= -2\exp \left[ \frac{(1000 - t_0)}{200} \right] + 2\exp \left[ \frac{-2(1000 - t_0)}{200} \right] \\
\]

Which is a curve concave upward having a minimum at:

\[
(t_0 = 861.37, 0.5) \\
\]

Therefore, we can infer the magnitude of the chaotic factor MChf:

\[
\text{MChf}(t_0) = \left| \text{Chf}(t_0) \right| = -2P_{\text{e}}(t_0)[1 - P_{\text{e}}(t_0)] \\
= 2P_{\text{e}}(t_0)[1 - P_{\text{e}}(t_0)] = F(t_0)[1 - F(t_0)] \\
= 2\exp \left[ \frac{(1000 - t_0)}{200} \right] \left( 1 - \exp \left[ \frac{(1000 - t_0)}{200} \right] \right) \\
= 2\exp \left[ \frac{(1000 - t_0)}{200} \right] - 2\exp \left[ \frac{-2(1000 - t_0)}{200} \right] \\
\]

Which is a curve concave downward having a maximum at:

\[
(t_0 = 861.37, 0.5) \\
\]

2.4.5. Pc: The Probability in the Complex Set C:

\[
Pc^2 = \text{DOK} - \text{Chf} = 1 - 2\exp \left[ \frac{(1000 - t_0)}{200} \right] \\
+2 \left( \exp \left[ \frac{(1000 - t_0)}{200} \right] \right)^2 \\
+2\exp \left[ \frac{(1000 - t_0)}{200} \right] - 2 \left( \exp \left[ \frac{(1000 - t_0)}{200} \right] \right)^2 = 1 \\
\Rightarrow Pc = 1 \\
\]

Thus we deduce that in the set C, we have a complete knowledge of the random variable since \(Pc = 1\).

2.4.6. The Intersection Point:

\[
P_{\text{i}}(t_0) = \frac{P_{\text{e}}(t_0)}{1} = \exp \left[ \frac{- (1000 - t_0)}{200} \right] \\
= 1 - \exp \left[ \frac{- (1000 - t_0)}{200} \right] \Leftrightarrow 2\exp \left[ \frac{(1000 - t_0)}{200} \right] = 1 \\
\Rightarrow \exp \left[ \frac{(1000 - t_0)}{200} \right] = \frac{1}{2} \Leftrightarrow \\
t_0 = 1000 + 200\ln(0.5) = 861.37 \\
\text{And } P_{\text{i}}(t_0 = 861.37) = \exp \left[ \frac{- (1000 - 861.37)}{200} \right] = 0.5 \\
\text{and } P_{\text{e}}(t_0 = 861.37) = 1 - 0.5 = 0.5 \\
\]
Fig. 17. EKA parameters in exponential cumulative distribution

Fig. 18. EKA parameters in exponential cumulative distribution
Fig. 19. DOK and Chf in terms of $t$ and of each other in exponential cumulative distribution

So $P_r(t_0)$ and $P_m(t_0)/i$ intersect at $(861.37, 0.5)$. Moreover, the minimum of DOK and the maximum of MChf occur at $(861.37, 0.5)$. So we conclude that $P_r(t_0)$, $P_m(t_0)/i$, DOK and MChf all intersect at $(861.37, 0.5)$ (Fig. 17-19).

2.4.7. The EKA Parameters Analysis in the Prognostic of Degradation:

In this case, we note from the figure above that the DOK is maximum (DOK=1) when MChf is minimum (MChf=0) (points J & L). Afterward, the magnitude of the chaotic factor MChf starts to increase with the decrease of DOK until it reaches 0.5 at $t_0=861.37$ (point K). Since the real probability $P_r$ is an exponential distribution it will intersect with DOK at the point $(861.37, 0.5)$ (point K). With the increase of $t_0$, Chf and MChf return to zero and the DOK returns to 1 where we reach total damage ($D = 1$) and hence the total certain failure ($P_c = 1$) of the system (point L). At this last point L the failure here is certain, $P_r(t_N) = 1$ and $P_c(t_N) = 1$ and the remaining useful life ($t_N-t_N = 0$) with $P_c(t_N) = 1$, so the logical explanation of the value DOK = 1 follows. We note that the point K is no more at the middle of DOK since the exponential distribution is not symmetric (Fig. 17).

2.5. The Normal Probability Distribution:

With a probability density function:

$$f(t) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{1}{2}\left(\frac{t-\bar{T}}{\sigma}\right)^2\right),$$

for $-\infty < t < \infty$

and a cumulative distribution function:

$$F(t) = \int_{-\infty}^{t} f(t)dt = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{1}{2}\left(\frac{t-\bar{T}}{\sigma}\right)^2\right)dt,$$

for $0 \leq t \leq \bar{T}$

We have taken the domain for the normal variable $t_0 = [0, \bar{T}_N = 1000]$, $d_{tu} = 0.1$, $\bar{T} = 500$ (Mean) and $\sigma = 150$ (Standard deviation).

Note that:

$$\int \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{1}{2}\left(\frac{t-\bar{T}}{\sigma}\right)^2\right)dt = 1$$

2.5.1. The Real Probability $P_r$:

$$P_r(t_0) = F(t_0) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{1}{2}\left(\frac{t_0-\bar{T}}{\sigma}\right)^2\right)dt$$

if $0 \leq t_0 \leq \bar{T}_N = 1000$

We note that $P_r(t_0)$ is a non-decreasing function.
2.5.2. The Complementary Probability $P_m/i$: 

\[ P_m(t_0) / i = 1 - P_i(t_0) = 1 - F(t_0) \]

\[ = 1 - \frac{1}{\sqrt{2\pi\sigma_i}} \exp \left[ -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right] \cdot dt \]

\[ = \frac{1}{\sqrt{2\pi\sigma_i}} \exp \left[ -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right] \cdot dt \]

if $0 \leq t_0 \leq t_{i0} = 1000$

We note that $P_m(t_0)/i$ is a non-increasing function.

2.5.3. The Degree of Our Knowledge DOK:

DOK is the measure of our certain knowledge (100% probability) about the expected event, it does not include any uncertain knowledge (with probability less than 100%).

\[ \text{DOK}(t_i) = P_c(t_i) + 2P_i(t_i)P_m(t_i) \]

\[ = 1 - 2P_m(E)P_m(E) = 1 - 2P_i(t_i) [1 - P_i(t_i)] \]

\[ = 1 - 2P_i(t_i) [1 - F(t_i)] = 1 - 2 \frac{1}{\sqrt{2\pi\sigma_i}} \exp \left[ -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right] \cdot dt \]

\[ = 1 - 2 \frac{1}{\sqrt{2\pi\sigma_i}} \exp \left[ -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right] \cdot dt \]

\[ + 2 \left( \frac{1}{\sqrt{2\pi\sigma_i}} \exp \left[ -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right] \right)^2 \]

Which is a curve concave upward having a minimum at:

\[ \int_0^\infty \frac{1}{\sqrt{2\pi\sigma_i}} \exp \left[ -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right] \cdot dt \]

\[ = 0.5 \Leftrightarrow \text{At } (t_i = \bar{T} = 500, \ 0.5) \]

Since the normal distribution is symmetric about the mean which is $\bar{T} = 500$.

2.5.4. The Chaotic Factor Chf and MChf:

\[ \text{Chf}(t_i) = 2P_i(t_i)P_m(t_i) = -2P_m(E)P_m(E) \]

\[ = -2P_i(t_i) [1 - F(t_i)] = -2P_i(t_i) [1 - F(t_i)] \]

\[ = -2 \frac{1}{\sqrt{2\pi\sigma_i}} \exp \left[ -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right] \cdot dt \]

\[ \times \frac{1}{\sqrt{2\pi\sigma_i}} \exp \left[ -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right] \cdot dt \]

\[ = -2 \frac{1}{\sqrt{2\pi\sigma_i}} \exp \left[ -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right] \cdot dt \]

\[ + 2 \left( \frac{1}{\sqrt{2\pi\sigma_i}} \exp \left[ -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right] \right)^2 \]

Which is a curve concave downward having a maximum at:

\[ \int_0^\infty \frac{1}{\sqrt{2\pi\sigma_i}} \exp \left[ -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right] \cdot dt \]

\[ = 0.5 \Leftrightarrow \text{At } (t_i = \bar{T} = 500, \ 0.5) \]

Since the normal distribution is symmetric about the mean which is $\bar{T} = 500$. 

Since the normal distribution is symmetric about the mean which is $\bar{T} = 500$. 

Therefore, we can infer the magnitude of the chaotic factor MChf:

\[ \text{MChf}(t_i) = \left| \text{Chf}(t_i) \right| = \left| -2P_i(t_i) [1 - F(t_i)] \right| \]

\[ = 2P_i(t_i) [1 - P_i(t_i)] = 2P_i(t_i) [1 - F(t_i)] \]

\[ = 2 \frac{1}{\sqrt{2\pi\sigma_i}} \exp \left[ -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right] \cdot dt \]

\[ \times \frac{1}{\sqrt{2\pi\sigma_i}} \exp \left[ -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right] \cdot dt \]

\[ = 2 \frac{1}{\sqrt{2\pi\sigma_i}} \exp \left[ -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right] \cdot dt \]

\[ -2 \left( \frac{1}{\sqrt{2\pi\sigma_i}} \exp \left[ -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right] \right)^2 \]

Which is a curve concave downward having a maximum at:

\[ \int_0^\infty \frac{1}{\sqrt{2\pi\sigma_i}} \exp \left[ -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right] \cdot dt \]

\[ = 0.5 \Leftrightarrow \text{At } (t_i = \bar{T} = 500, \ 0.5) \]
2.5.5. 

**Pc:** The Probability in the Complex Set C:

\[
Pc(t_0) = DOK(t_0) - \text{Chf}(t_0)
\]

\[
= 1 - 2 \int_0^t \left\{ \frac{1}{\sqrt{2\pi} \sigma_i} \exp \left( -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right) \right\} \, dt
\]

\[
\times \left( \frac{1}{\sqrt{2\pi} \sigma_i} \exp \left( -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right) \right) \, dt
\]

\[
+ 2 \int_0^t \left\{ \frac{1}{\sqrt{2\pi} \sigma_i} \exp \left( -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right) \right\} \, dt = 1
\]

\[
\Rightarrow Pc(t_0) = 1
\]

Thus we deduce that in the set C, we have a complete knowledge of the random variable since \( Pc = 1 \).

2.5.6. The Intersection Point:

\[
P_r(t_0) = F_m(t_0) / i \equiv \frac{1}{\sqrt{2\pi} \sigma_i} \exp \left( -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right) \, dt
\]

\[
= 1 - \int_0^t \left\{ \frac{1}{\sqrt{2\pi} \sigma_i} \exp \left( -\frac{1}{2} \left( \frac{t - \bar{T}}{\sigma_i} \right)^2 \right) \right\} \, dt
\]

\[
\Rightarrow t_0 = \bar{T} = 500
\]

and \( P_r(t_0) =\bar{T} = 500 \), \( i = 1 \), \( 0.5 = 0.5 \).

So \( P_r(t_0) \) and \( F_m(t_0) / i \) intersect at \( (500, 0.5) \).

Moreover, the minimum of DOK and the maximum of MChf occur at \( (500, 0.5) \).

So we conclude that \( P_r(t_0) \), \( F_m(t_0) / i \), DOK and MChf all intersect at \( (500, 0.5) \) (Fig. 20-22).

2.5.7. The EKA Parameters Analysis in the Prognostic of Degradation:

In this case, we note from the figure that the DOK is maximum (DOK = 1) when MChf is minimum (MChf = 0) (points J & L). Afterward, the magnitude of the chaotic factor MChf starts to increase with the decrease of DOK until it reaches 0.5 at \( t_0 = 500 \) (point K). Since the real probability \( P_r \) is a normal distribution it will intersect with DOK at the point \( (500, 0.5) \) (point K). With the increase of \( t_0 \), Chf and MChf return to zero and the DOK returns to 1 where we reach total damage (D = 1) and hence the total certain failure (\( P_r = 1 \)) of the system (point L). At this last point L the failure here is certain, \( P_r(t_0) = t_0/t_N = 1 \) and RUL\((t_0) = t_N-t_0 = 0 \) with \( Pc(t_0) = 1 \), so the logical explanation of the value DOK=1 follows. We note that the point K is at the middle of DOK since the normal distribution is symmetric (Fig. 20).

2.6. The Lognormal Probability Distribution:

With a probability density function:

\[
f(t) = \frac{1}{\sqrt{2\pi} \sigma_i} \exp \left( -\frac{1}{2} \left( \frac{\ln(t) - \bar{T}}{\sigma_i} \right)^2 \right)
\]

for \( 0 < t < \infty \)

and a cumulative distribution function:

\[
F(t) = \int_0^t \frac{1}{\sqrt{2\pi} \sigma_i} \exp \left( -\frac{1}{2} \left( \frac{\ln(t) - \bar{T}}{\sigma_i} \right)^2 \right) \, dt
\]

for \( 0 \leq t_0 \leq t_N \).

We have taken the domain for the lognormal variable \( t_0 = [0, t_N = 1000] \), \( dt_0 = 0.1 \), \( \bar{T} = 5.3 \) (mean) and \( \sigma_i = 0.7 \) (standard deviation).

Note that:

\[
\int_0^t \frac{1}{\sqrt{2\pi} \sigma_i} \exp \left( -\frac{1}{2} \left( \frac{\ln(t) - \bar{T}}{\sigma_i} \right)^2 \right) \, dt = 1
\]

2.6.1. The Real Probability \( P_r \):

\[
P_r(t_0) = F(t_0) = \int_0^t \frac{1}{\sqrt{2\pi} \sigma_i} \exp \left( -\frac{1}{2} \left( \frac{\ln(t) - \bar{T}}{\sigma_i} \right)^2 \right) \, dt
\]

if \( 0 \leq t_0 \leq t_N = 1000 \)

We note that \( P_r(t_0) \) is a non-decreasing function.
Fig. 20. EKA parameters in normal probability distribution

Fig. 21. EKA parameters in normal probability distribution
2.6.2. The Complementary Probability $P_m / i$:

$$P_m(t_0) / i = 1 - P_{i}(t_0) = 1 - F(t_0)$$

$$= 1 - \int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma_t} \exp \left[ -\frac{1}{2} \left( \frac{\ln(t) - \bar{T}}{\sigma_t} \right)^2 \right] \, dt$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_t} \exp \left[ -\frac{1}{2} \left( \frac{\ln(t) - \bar{T}}{\sigma_t} \right)^2 \right] \, dt$$

if $0 \leq t_0 \leq t_\infty = 1000$

We note that $P_m(t_0) / i$ is a non-increasing function.

2.6.3. The Degree of Our Knowledge DOK:

DOK is the measure of our certain knowledge (100% probability) about the expected event, it does not include any uncertain knowledge (with probability less than 100%):

$$\text{DOK} = P_n(t) = P_{c}(t_0) + 2iP_{i}(t)P_{n}\left(t_0, t\right) = P_{c}(t_0) + 2i\left[ P_{n}\left(t_0, t\right) \right]$$

$$= 1 - 2P_{n}\left(t_0, t\right) = 1 - 2\left[ P_{n}\left(t_0, t\right) \right]$$

$$= 1 - 2\left[ 1 - P_{n}\left(t_0, t\right) \right]$$

$$= 1 - 2\left[ 1 - F(t_0) \right]$$

$$= 1 - 2\left[ 1 - \int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma_t} \exp \left[ -\frac{1}{2} \left( \frac{\ln(t) - \bar{T}}{\sigma_t} \right)^2 \right] \, dt \right]$$

$$= 1 - 2\left[ 1 - \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_t} \exp \left[ -\frac{1}{2} \left( \frac{\ln(t) - \bar{T}}{\sigma_t} \right)^2 \right] \, dt \right]$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_t} \exp \left[ -\frac{1}{2} \left( \frac{\ln(t) - \bar{T}}{\sigma_t} \right)^2 \right] \, dt$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_t} \exp \left[ -\frac{1}{2} \left( \frac{\ln(t) - \bar{T}}{\sigma_t} \right)^2 \right] \, dt$$

$$= 0.5 \Leftrightarrow t_0 = 200.34, \quad 0.5$$

Notice that $\ln(200.34) = 5.3 = \bar{T}$ which is the mean of the distribution and equivalently $\exp(\bar{T} = 5.3) = 200.34$.

This is conformed to the lognormal distribution.

2.6.4. The Chaotic Factor Chf and MChf:

$$\text{Chf} = 2iP(t_0)P_{n}\left(t_0, t\right) = -2P_{n}\left(t_0\right)\left[1 - P_{n}\left(t_0\right)\right] = -2F(t_0)\left[1 - F(t_0)\right]$$

$$= -2\left[ \int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma_t} \exp \left[ -\frac{1}{2} \left( \frac{\ln(t) - \bar{T}}{\sigma_t} \right)^2 \right] \, dt \right]$$

$$= -2\left[ \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_t} \exp \left[ -\frac{1}{2} \left( \frac{\ln(t) - \bar{T}}{\sigma_t} \right)^2 \right] \, dt \right]$$

$$= 0.5 \Leftrightarrow t_0 = 200.34, \quad 0.5$$

$$= -2\left[ \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_t} \exp \left[ -\frac{1}{2} \left( \frac{\ln(t) - \bar{T}}{\sigma_t} \right)^2 \right] \, dt \right]$$

$$= -2\left[ \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_t} \exp \left[ -\frac{1}{2} \left( \frac{\ln(t) - \bar{T}}{\sigma_t} \right)^2 \right] \, dt \right]$$

$$= 0.5 \Leftrightarrow t_0 = 200.34, \quad 0.5$$

$$= 0.5 \Leftrightarrow t_0 = 200.34, \quad 0.5$$

Notice that $\ln(200.34) = 5.3 = \bar{T}$ which is the mean of the distribution and equivalently $\exp(\bar{T} = 5.3) = 200.34$.

This is conformed to the lognormal distribution.
which is a curve concave upward having a minimum at:

$$\int_0^\infty \frac{1}{\sqrt{2\pi} \sigma_t} \exp \left[ -\frac{1}{2} \frac{(\ln(t) - \bar{T})^2}{\sigma_t^2} \right] dt = 0.5$$

$$\Rightarrow \text{At } \left( t_0 = \exp(\bar{T} = 5.3) = 200.34, -0.5 \right)$$

Therefore, we can infer the magnitude of the chaotic factor MChf:

$$\text{MChf}(t_0) = \text{Chf}(t_0) \Rightarrow [2\mathcal{P}(t_0)(1 - \mathcal{P}(t_0))]$$

$$= 2 \mathcal{P}(t_0)[1 - \mathcal{P}(t_0)] = 2 \mathcal{F}(t_0)[1 - \mathcal{F}(t_0)]$$

$$= 2 \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma_t} \exp \left[ -\frac{1}{2} \frac{(\ln(t) - \bar{T})^2}{\sigma_t^2} \right] dt$$

$$\times \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma_t} \exp \left[ -\frac{1}{2} \frac{(\ln(t) - \bar{T})^2}{\sigma_t^2} \right] dt$$

$$= 2 \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma_t} \exp \left[ -\frac{1}{2} \frac{(\ln(t) - \bar{T})^2}{\sigma_t^2} \right] dt$$

$$\times 2 \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma_t} \exp \left[ -\frac{1}{2} \frac{(\ln(t) - \bar{T})^2}{\sigma_t^2} \right] dt$$

which is a curve concave downward having a maximum at:

$$\int_0^\infty \frac{1}{\sqrt{2\pi} \sigma_t} \exp \left[ -\frac{1}{2} \frac{(\ln(t) - \bar{T})^2}{\sigma_t^2} \right] dt = 0.5$$

$$\Rightarrow \text{At } \left( t_0 = \exp(\bar{T} = 5.3) = 200.34, 0.5 \right)$$

2.6.5. **Pc: Probability in the Complex Set C:**

$$\mathcal{P}_c(t_0) = \mathcal{DOK}(t_0) - \text{Chf}(t_0)$$

$$= 1 - 2 \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma_t} \exp \left[ -\frac{1}{2} \frac{(\ln(t) - \bar{T})^2}{\sigma_t^2} \right] dt$$

$$\times \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma_t} \exp \left[ -\frac{1}{2} \frac{(\ln(t) - \bar{T})^2}{\sigma_t^2} \right] dt$$

$$+ 2 \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma_t} \exp \left[ -\frac{1}{2} \frac{(\ln(t) - \bar{T})^2}{\sigma_t^2} \right] dt$$

$$\Rightarrow \text{At } t_0 = 200.34$$

Thus we deduce that in the set C, we have a complete knowledge of the random variable since Pc = 1.

2.6.6. **The Intersection Point:**

$$\mathcal{P}_r(t_0) = \mathcal{P}_m(t_0)/i \Rightarrow \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma_t} \exp \left[ -\frac{1}{2} \frac{(\ln(t) - \bar{T})^2}{\sigma_t^2} \right] dt$$

$$= 1 - 2 \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma_t} \exp \left[ -\frac{1}{2} \frac{(\ln(t) - \bar{T})^2}{\sigma_t^2} \right] dt$$

$$\Rightarrow \mathcal{P}_r(t_0) = \mathcal{P}_m(t_0)/i \Rightarrow t_0 = 200.34$$

So $\mathcal{P}_r(t_0)$ and $\mathcal{P}_m(t_0)/i$ intersect at (200.34, 0.5).

Moreover, the minimum of DOK and the maximum of MChf occur at (200.34, 0.5).

So we conclude that $\mathcal{P}_r(t_0)$, $\mathcal{P}_m(t_0)/i$, DOK and MChf all intersect at (200.34, 0.5) (Fig. 23-25).

2.6.7. **The EKA Parameters Analysis in the Prognostic of Degradation:**

In this case, we note from the figure below that the DOK is maximum (DOK=1) when MChf is minimum (DOK=0) (points J & L). Afterward, the magnitude of the chaotic factor MChf starts to increase with the decrease of DOK until it reaches 0.5 at $t_0 = 200.34$ (point K). Since the real probability $\mathcal{P}_r$ is a lognormal distribution it will intersect with DOK at the point (200.34, 0.5) (point K). With the increase of $t_0$, Chf and MChf return to zero and the DOK returns to 1 where we reach total damage ($\mathcal{D} = 1$) and hence the total certain failure ($\mathcal{P}_r = 0$) of the system (point L). We note that the point K is no more at the middle of DOK since the lognormal distribution is not symmetric.

At each instant $t_0$, the remaining useful lifetime $\text{RUL}(t_0)$ is certainly predicted in the complex set C with Pc maintained as equal to one through a continuous compensation between DOK and Chf. This compensation is from instant $t_0 = 0$ where $\mathcal{D}(t_0) = 0$ until the failure instant $t_0$ where $\mathcal{D}(t_0) = 1$ (Fig. 23).
Fig. 23. EKA parameters in log-normal probability distribution

Fig. 24. EKA parameters in log-normal probability distribution
3. CONCLUSION

In this study I applied the theory of Extended Kolmogorov Axioms to different probability distributions: the uniform, the logarithmic, the power, the exponential, the normal and the lognormal cumulative probability distributions. In addition, I established a tight link between the new theory and degradation or the remaining useful lifetime. Hence, I developed the theory of “Complex Probability” beyond the scope of the previous first and second paper on this topic. As it was proved and illustrated, when the degradation index is 0 or 1 and correspondingly the RUL is $t_N$ or 0 then the Degree of Our Knowledge (DOK) is one and the chaotic factor (Chf and MChf) is 0 since the state of the system is totally known. During the process of degradation (0<D<1) we have: 0.5<DOK<1, -0.5<Chf<0 and 0<MChf<0.5. Notice that during the whole process of degradation we have $P_c = DOK - Chf = DOK + MChf = 1$, that means that the phenomenon which seems to be random and stochastic in R is now deterministic and certain in $C = R + M$ and this after adding to R the contributions of M and hence after subtracting the chaotic factor from the degree of our knowledge. Moreover, for each value of an instant $t_0$, I have determined its corresponding probability of survival or of the remaining useful lifetime $RUL(t_0) = t_N - t_0$. In other words, at each instant $t_0$, $RUL(t_0)$ is certainly predicted in the complex set C with $P_c$ maintained as equal to one through a continuous compensation between DOK and Chf. This compensation is from instant $t_0 = 0$ where $D(t_0) = 0$ until the failure instant $t_N$ where $D(t_N) = 1$. Furthermore, using all these graphs illustrated throughout the whole paper, we can visualize and quantify both the system chaos (Chf and MChf) and the system certain knowledge (DOK and $P_c$). This is certainly very interesting and fruitful and shows once again the benefits of extending Kolmogorov’s axioms and thus the originality and usefulness of this new field in mathematics that can be called verily: “The Complex Probability and Statistics Paradigm”.

4. REFERENCES

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