Quantum Sphere via Reflection Equation Algebra

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Abstract

Quantum sphere is introduced as a quotient of the so-called Reflection Equation Algebra. This enables us to construct some line bundles on it by means of the Cayley-Hamilton identity whose a quantum version was discovered in [PS], [GPS]. A new way to introduce some elements of "braided geometry" on the quantum sphere is discussed.

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1 Introduction

Quantum sphere $S^2_q$ is the simplest example of a quantum variety. It was introduced in [P] by means of a quantum reduction procedure which in a schematic way can be represented by the formula

$$S^2_q = SU_q(2)/(SU_q(1) = SU(1)).$$

This means that the quantum sphere (by abusing the language we use the term "quantum variety" for the corresponding coordinate rings) is realized as a subspace in the algebra $k(SU_q(2))$. Other quantum varieties are usually introduced in a similar way. Nowadays, the language of the so-called Galois-Hopf extension is employed. It makes use of a couple of Hopf algebras giving rise to a "quantum coset" (cf. [S], [BM]).

1 All coordinate rings are treated in the spirit of affine algebraic geometry as quotients of a polynomial ring. The basic field $k$ is assumed to be $\mathbb{C}$ or $\mathbb{R}$. In the latter case $q$ is assumed to be real. The choice of the field is similar to the classical case. In any case $q$ is generic.
However, this approach has some defects. First, it does not allow us to control flatness of deformation of reduced objects\(^\text{2}\). It is worth noticing that the differential calculus on \(SL_q(n)\) constructed by S.Woronowicz \([W1], [W2]\) with the use of the Leibniz rule is not a flat deformation of its classical counterpart (cf. \([AAM], [Ar], [FP1], [FP2], [I]\)). However, this differential calculus plays the crucial role in all constructions (metric, bundles, connection etc.) of quantum (braided) version of differential geometry on quantum varieties. So, these objects are pointless in a sense, if we want to treat them as an approximation of their classical counterparts.

Second, the above way of introducing quantum varieties cannot be generalized to "non-quasiclassical quantum varieties", associated to non-standard quantum \(R\)-matrices constructed in \([G]\)\(^3\). Although a \(k(G_q)\) type algebra can be apparently defined for numerous non-standard \(R\)-matrices (cf. \([G]\)), it does not have any subalgebra which would give rise to a "quantum coset".

The main purpose of this note is to present another way of defining "quantum varieties" which on one hand would be valid both in the quasiclassical and non-quasiclassical case and on the other hand would enable us to control flatness of deformation. In the framework of this approach the quantum function algebras or their dual objects (quantum groups) play only the role of symmetry groups (however, we can do without these objects at all). The crucial role in our approach belongs to the so-called reflection equation (RE) algebra. We consider this algebra as a very fruitful tool of "braided geometry".

The central objects of this geometry are quantum (braided) varieties which are introduced as some quotients of the RE algebra. Otherwise stated, they are realized explicitly by means of some "braided system of equations" in the spirit of affine algebraic geometry. In virtue of \([PS], [GPS]\) there exists some polynomial (of Cayley-Hamilton type) identity for the matrix formed by the generators of this algebra. Moreover, the coefficients of the corresponding polynomial are elements of the center of the RE algebra. This property enables us to introduce quantum line bundles on some quantum varieties (in particular, quantum sphere) in terms of projective modules in the spirit of the Serre-Swan approach (cf. \([SW], [SW]\)) and to show the projectivity of the corresponding modules.

In this paper we want to demonstrate the usefulness and the power of the described approach in "braided geometry" on the example of quantum sphere. The note is organized as follows. In section 2 we introduce quantum sphere (or what is the same quantum hyperboloid if we ignore involution operator) and discuss a way to introduce differential calculus, tangent space and some other structures on it without either reduction procedure or Leibniz rule at all. Quantum group (or their dual objects) are used only as substitutes of symmetry groups. In section 3 we introduce the RE algebra and describe some its

\(^2\)We refer the reader to \([DGK]\) for a definition. However, if a classical (resp., quantum) object decomposes into a direct sum of irreducible \(U(g)\)- (resp., \(U_q(g)\)-) modules the flatness means that the both objects consist of similar components.

\(^3\)In the sequel we use the term quasiclassical for the objects related to the quantum group \(U_q(sl(n))\).
properties. In section 4 we realize the quantum sphere in terms of this algebra and use this realization in order to introduce line bundles on quantum sphere via the Cayley-Hamilton identity.

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2 Differential calculus and tangent space on quantum sphere

The purpose of this section is to describe a way to introduce a quantum sphere (hyperboloid)\(^4\) and some derived objects of quantum geometry (differential calculus, tangent space) on it without any reduction procedure.

Let \( V \) be spin 1 \( U_q(sl(2)) \)-module. Then the space \( V^\otimes 2 \) being equipped with a structure of \( U_q(sl(2)) \)-module decomposes into a direct sum of three irreducible components

\[
V^\otimes 2 = V_0 \oplus V_1 \oplus V_2. \tag{2.1}
\]

where \( V_i \) stands for the spin \( i \) \( U_q(sl(2)) \)-module. (We avoid using any coordinate writing.)

Then a quantum sphere (or more precisely, its the coordinate ring) \( k(S^2_q) \) can be defined as the quotient

\[
T(V)/\{V_1, v_0 - c\}, \ c \in k \text{ is assumed to be fixed}. \tag{2.2}
\]

Here \( T(V) \) stands for the free tensor algebra of the space \( V \), \( \{I\} \) stands for the ideal in \( T(V) \) generated by a subset \( I \) and \( v_0 \) is a generator of the 1-dimensional component \( V_0 \).

We consider this quotient as a "q-commutative" algebra. Its "q-non-commutative" counterpart can be defined in a similar way if we replace \( V_1 \) in the denominator of (2.2) by \( V_1 - \hbar \alpha(V_1) \) where \( \alpha : V^\otimes 2 \to V \) is a non-trivial \( U_q(sl(2)) \)-morphism (it is unique up to a factor which in the sequel is supposed to be fixed). Thus, we get two parameter algebra denoted \( A_{\hbar,q} \) which becomes \( k(S^2_q) \) as \( \hbar = 0 \) and moreover it is a flat deformation of the classical counterpart \( A_{0,1} = k(S^2) \). Let us note that the Podles' sphere is another parametrization of this algebra equipped with an involution operator (see a discussion on an involution operator in section 3).

Remark 1 For the Lie algebras \( g = sl(n), \ n > 2 \) the following difficulty appears. Let us equip \( g \) considered as a vector space with the structure of a \( U_q(sl(n)) \)-module. Then it is

4If the coordinate ring of a sphere or a hyperboloid is defined as a quotient of the polynomial ring in the spirit of affine algebraic geometry no difference between these varieties appears. In particular, all irreducible \( sl(2) \)-modules involved in differential calculus are finite-dimensional. A difference appears while we consider the hyperboloid equipped with other functional spaces. For the same reason we do not make any difference between the QG \( U_q(sl(n)) \) and \( U_q(su(2)) \).
no clear what is a reasonable way to define a morphism analogous to that $\alpha$ above since in the space $g^\otimes 2$ the component isomorphic to $g$ itself occurs twice. By the same reason it is not clear what is a $q$-analogue of the symmetric algebra of the space $g$. This problem can be solved in terms of the so-called reflection equation algebra (see section 3).

**Remark 2** Let us remark that there exist other tensor or quasitensor categories whose fusion rings look like that of category of $sl(2)$-modules. Let $R : V^\otimes 2 \rightarrow V^\otimes 2$ be a Hecke symmetry, i.e., a solution of the Yang-Baxter (YB) equation satisfying the quadratic equation

$$R^2 = \text{id} + (q - q^{-1})R$$

such that the Poincaré series of the corresponding "skewsymmetric algebra" (cf. [G]) is of the form

$$P_-(V, t) = 1 + nt + t^2, \ n = \dim V.$$

A big family of such type Hecke symmetries was constructed in [G].

Then in the category generated by the space $V$ in the spirit of the paper [GM] there is a space $V$ whose tensor square decomposes similarly to (2.1).

The non-quasiclassical quantum (or braided) variety looking like the quasiclassical quantum sphere and its non-commutative counterpart can be constructed in the same way as above without any reduction procedure. Also remark that there exists another quantum sphere (hyperboloid) being a deformation of the classical one. This quantum sphere corresponds to the involutary solution of the quantum YB equation arising from the triangular classical $r$-matrix $H \wedge X$.

In a similar non-coordinate manner and without any reduction procedure there can be introduced spaces of differentials on the quantum sphere. It is done in [A], [AC]. Let us describe briefly a way to introduce $\Omega^1$ suggested in these papers. Denote $V''$ the space isomorphic to $V$ as $U_q(sl(2))$-module, but generated by the differentials of the elements from $V$. Let $k(S^2_q) \otimes V''$ be free left $k(S^2_q)$-module. Let us introduce the space $\Omega^1$ as its quotient over the submodule generated by the elements $(V \otimes V')_0$. Hereafter we use the notation $(V_i \otimes V_j)_k$ for the spin $k$ component in the tensor product $V_i \otimes V_j$.

The second differential space $\Omega^2$ can be introduced similarly to $\Omega^1$. It is not difficult to describe decompositions of the $k(S^2_q)$-modules $\Omega^1$ and $\Omega^2$ and of the algebra $k(S^2_q)$ itself into direct sums of $U_q(sl(2))$-modules for a generic $q$ (cf. [AC]). They look like the decompositions of their classical counterparts into sums of $sl(2)$-modules.

Let $d$ be the differential acting in the classical differential algebra (as usual, $d$ is subject to the Leibniz rule). Since $d$ is a $U_q(sl(2))$-morphism it maps any $U_q(sl(2))$-module either to an isomorphic module or to 0. Let us define $d$ in the quantum case in a similar way by sending any $U_q(sl(2))$-module containing in $k(S^2_q)$-module $\Omega^1, \Omega^2$ or the algebra $k(S^2_q)$ itself either to an isomorphic $U_q(sl(2))$-module or to 0 similarly to the classical case (also we can require the quantum operator $d$ to be a deformation of its classical counterpart).
By this the quantum differential \( d \) is defined uniquely up to a factor on any irreducible \( \mathcal{U}_q(sl(2)) \)-component. By construction we have just the same cohomology as in the classical case:
\[
\dim H^0 = 1, \quad \dim H^1 = 0, \quad \dim H^2 = 1.
\]
Moreover, the same result will be valid if we realize this construction on any non-quasiclassical sphere mentioned in remark 2.

Thus, we have constructed our version of quantum differential calculus without Leibniz rule and without any algebraic structure in the space
\[
\Omega^* = \Omega^0 \oplus \Omega^1 \oplus \Omega^2, \quad \Omega^0 = k(S^2_q).
\]

Let us pass now to discussion of what the tangent bundle on the quantum sphere is (or, equivalently, what the phase space would be if we considered the quantum sphere as a configuration space). This problem can be split into two parts. First, we want to represent the quantum tangent space as a \( k(S^2_q) \)-module. Second, we want to assign to the elements of this space an operator meaning.

Let us begin with the classical case. The tangent space on the usual sphere can be introduced by the equation
\[
(V \otimes V')_0 = 0,
\]
where the space \( V' \) is generated not by differentials as it was the case for the quantum sphere but by the elements of the Lie algebra \( su(2) \) itself (we consider here the compact form of the Lie algebra in question). In the explicit coordinate form the above equation looks as follows
\[
xX + yY + zZ = 0
\]
where \( x, y, z \) are generators of the algebra \( k(S^2) \) and \( X, Y, Z \) are the corresponding generators of the Lie algebra \( su(2) \). As operators they can be represented by infinitesimal rotations:
\[
X = z\partial_y - y\partial_z, \quad Y = x\partial_z - z\partial_x, \quad Z = y\partial_x - x\partial_y.
\]

On the hyperboloid the relation (2.3) takes the form
\[
xY + yX + \frac{1}{2}hH = 0,
\]
where \( X, Y, H \) are "hyperbolic infinitesimal rotations".

Similarly, for any regular variety \( M \) the tangent space \( T(M) \) has a structure of a \( k(M) \)-module and reciprocally \( k(M) \) has a \( T(M) \)-module structure since \( T(M) \) acts on \( k(M) \). These two structures are compatible in some sense giving rise to the notion of Lie-Rinehart algebra (cf. [3]).

What is a q-analogue of the space \( T(S^2) \) or otherwise stated what are q-analogues of the operators \( X, Y, H \) above which would generate the quantum tangent space? The operators \( X, H, Y \) coming from the quantum group \( U_q(sl(2)) \) do not fit for this role since
they do not satisfy any relation which would be a deformation of (2.3). As a \( k(S^2_q) \)-module the quantum tangent space can be introduced by relation (2.3) considered in the corresponding tensor or quasitensor category (the same is valid in the non-quasiclassical case). However, it is not evident in advance whether there exist operators defined on the algebra \( k(S^2_q) \) and satisfying this relation. Nevertheless, as was shown in [A] there exist some operators satisfying relations (2.3) and the defining relations of the algebra \( A_{h,q} \). We will treat these operators and all their linear combinations with coefficients from \( A_{h,q} = k(S^2_q) \) as q-analogues of vector fields on the quantum sphere (hyperboloid).

By definition, the quantum tangent space \( T(S^2_q) \) being a \( k(S^2_q) \)-module and equipped with a \( U_q(sl(2)) \)-covariant action

\[
T(S^2_q) \otimes k(S^2_q) \rightarrow k(S^2_q)
\]

is just the space of all vector fields.

Suggested in [A] (also cf. [AC]) was a method of constructing some q-analogues of metric and connection in terms of these quantum (braided) vector fields similarly to the classical way of introducing these operations (but without using any form of Leibniz rule).

### 3 Reflection equation algebra

The purpose of this section is to introduce the RE algebra and to describe its properties.

Note that this algebra in the case of a quantum R-matrix depending on a spectral parameter was introduced by Cherednik as a boundary condition in the inverse scattering problem. Later some different versions of this algebra were studied in a similar context in [KSR], [KSa]. In the framework of braided geometry it was introduced by Sh.Majid [M1] (cf. also [M2]). Also he has shown that this algebra possesses a braided Hopf algebra structure. (In the case of an involutary operator \( R \) a similar Hopf algebra was introduced earlier by one of the author, cf. [K] and references therein).

As RE algebra we call the algebra generated by \( n^2 \) elements \( l^i_j \), \( 1 \leq i, j \leq n \) subject to the following matrix relation

\[
RL_1RL_1 = L_1RL_1R \tag{3.1}
\]

where

\[
L_1 = L \otimes \text{id} \quad \text{and} \quad L = (l^i_j).
\]

This algebra will be denoted \( L_q(R) \).

If \( R \) is a Hecke symmetry and is a deformation of usual flip then the RE algebra is a flat deformation of its classical counterpart which is the symmetric algebra of the linear space span \( (l^i_j) \). This was shown in [P] (or by another method in [D]).

Let us describe some other properties of this algebra. First of all, in the quasiclassical case its product is \( U_q(sl(n)) \)-covariant. Habitually, this property is presented in a dual
form. Namely, the RE algebra is equipped with a left RTT-comodule structure so that the product in the RE algebra is covariant w.r.t. the coaction

\[ \delta : L_q(R) \to T_q(R) \otimes L_q(R), \quad \delta(l_i^j) \to t_p^i S(t_p^k) \otimes l_k^p \]  

(3.2)

(a summation upon repeated indices is assumed). Hereafter \( T_q(R) \) stands for the famous RTT algebra (cf. [FRT]) defined by the relation

\[ \text{RT}_1 T_2 = T_1 T_2 R, \]

and \( S \) is the antipode in it. (Note that this property is valid in the non-quasiclassical case if the algebra \( T_q(R) \) is well defined, cf. [G].)

Second, the RE algebra being a quadratic algebra admits a quadratic-linear counterpart which is its flat deformation and looks like the enveloping algebra \( U(gl(n)) \). More precisely, this quadratic-linear deformation (denoted \( U(gl(n)_{\hbar, q}) \)) tends to the RE algebra as \( \hbar \to 0 \) and to \( U(sl(n)_h) \) as \( q \to 1 \). Here we use the notation \( g_n \) for the Lie algebra which differs from \( g \) by the factor \( h \) introduced in the bracket of the algebra \( g \). By this we want to represent the algebra \( U(gl(n)_h) \) as a deformational object w.r.t. the symmetric algebra \( \text{Sym}(g) \).

Third, this algebra unlike the RTT algebra, has a big center. In particular, the so-called quantum trace \( \text{Tr}_q(L) \) is central (cf. [FRT]). By contrast, a similar trace in the RTT algebra is not central. Moreover, for the RE algebra some quantum analogues of the Newton relations and the Cayley-Hamilton (CH) identity hold. It was shown in [GPS] (in the quasiclassical case these relations were previously established in [PS]). Furthermore, some version of the CH identity is also valid for the algebra \( U(gl(n)_{\hbar, q}) \) (cf. [GS]). By passing to the limit \( q \to 1 \) we get a similar identity for the algebra \( U(gl(n)_h) \).

Let us make a precise. For the matrix \( L \) formed by generators of one of the above algebras \( (L_q(R), U(gl(n)_{\hbar, q}), \text{or } U(gl(n)_h)) \) the following relation is valid

\[ \sum_{i=0}^{p} (-L)^i \sigma(p - i) = 0. \]  

Here \( p \) is the rank of the Hecke symmetry \( R \) (cf. [G], in the quasiclassical case \( p = n \)) and the coefficients \( \sigma(i) \) are central elements of the algebra in question.

Following [GS] we introduce ”quantum generic orbits” as the quotients of the RE algebra over the ideal generated by the elements

\[ \{ \sigma(i) - c_i, \quad c_i \in k \}. \]

By changing \( \sigma(i) \) for the numbers \( c_i \) in relation (3.3) we get a polynomial \( \overline{P} \) with numerical coefficients. This polynomial plays a crucial role in defining line bundles over quantum varieties (see section 4).

Also observe that the operators

\[ \delta_1 : l_i^j \to l_k^i t_k^j, \quad \delta_2 : l_i^j \to l_m^i t_k^m t_k^j, \ldots \]
which map the space span \((l^j_i)\) to the RE algebra are compatible with coaction (3.2). We will call such maps *morphisms*. Otherwise stated, this property means that the maps \(L \rightarrow L^k, \ k = 2, 3, \ldots\) are morphisms.

Namely, this property allows us to ensure compatibility of identity (3.3) with coaction (3.2) (or the action of the QG \(U_q(sl(n))\) in the quasiclassical case). Note that the images of the above operators belong to the RE algebra itself but not to its tensor powers. These operators are equal to the product of the comultiplication and multiplication operators.

Remark that for the RTT algebra the maps \(T \rightarrow T^k\) do not possess similar property. For this algebra there exists some analogue of the CH identity [10] but it is not as nice as (3.3) is. First, the role of the powers \(T^k\) is played by some complicated enough and less natural expressions and second, the coefficients of the corresponding polynomial are not central. It is this fact that prevents us from introducing quantum orbits as some quotients of RTT algebra. For this algebra only reduction procedure is admitted. However, the quantum sphere can be realized in the both ways: as a quantum coset and as a restriction of the RE algebra.

Let us explain this at the quasiclassical level, i.e. by means of Poisson brackets. The quasiclassical counterpart of the RTT algebra is well known. It is the so-called Sklyanin bracket. It can be reduced to any semisimple orbit in \(sl(n)^*\). However, it is not defined on the whole \(sl(n)^*\). While the Poisson bracket being the quasiclassical counterpart of the RE algebra (denoted \(\{ , \}_RE\)) is well-defined on \(gl(n)^*\) (and \(sl(n)^*\) if we kill the trace). Moreover, this bracket is compatible with the linear Poisson-Lie bracket (denoted \(\{ , \}_PL\)), i.e. they form the Poisson pencil

\[
\{ , \}_{ab} = a\{ , \}_PL + b\{ , \}_RE, \ a, b \in k.
\]

The simultaneous quantization of this pencil is just the algebra \(U(gl(n)_{h,q})\) mentioned above.

This Poisson pencil can be restricted on any semisimple orbit (as well as on any other type of orbit, cf. [D2]) in \(gl(n)^*\) (or \(sl(n)^*\)). So, on such an orbit we have the reduced Sklyanin bracket and the restricted Poisson pencil \(\{ , \}_{ab}\). In general, the reduced Sklyanin bracket has nothing in common with this pencil but on a symmetric orbit it becomes a particular case of this Poisson pencil. This is the reason why the quantum sphere (hyperboloid) can be realized in the both ways: its classical counterpart is a symmetric orbit!

Note that all these structures are quasiclassical counterparts of \(U_q(sl(n))\)-covariant algebras. However, the family of Poisson structures possessing this property is much bigger. We refer the reader to [DCS] where such Poisson brackets are classified. (In fact on the sphere the bracket \(\{ , \}_RE\) becomes the so-called R-matrix bracket classified earlier in [CP].)

Let us complete this section with a discussion of the representation theory of the RE algebra in the quasiclassical case. Since this algebra is \(U_q(sl(n))\)-covariant it is natural
to look for representations of the RE algebra which would be $U_q(sl(n))$-morphisms. Let a linear space $U$ be a $U_q(sl(n))$-module. Then the space $\text{End}(U)$ can be equipped with the structure of a $U_q(sl(n))$-module as well. Moreover, the product in this algebra is $U_q(sl(n))$-covariant. We say that a map

$$\rho : L_q(R) \to \text{End}(U)$$

is a representation if it is true in the category of associative algebras and if $\rho$ is a $U_q(sl(n))$-morphism.

Note that this definition looks like the definition of representations in a super-category.

We say that a representation $\rho$ of the algebra $L_q(R)$ is \textit{generic} if

$$\rho(\text{Tr}_q(L)) = a \text{id}, \quad a \neq 0.$$ 

On the contrary, if $a = 0$ we say that the representation is \textit{exceptional}.

We can say nothing about exceptional representations. However, the theory of generic finite-dimensional representations can be constructed as follows.

Consider the map

$$\gamma_h : L_q(R) \to k, \quad \gamma_h(l_i^j) = h\delta_i^j, \quad h \in k.$$ 

It is a $U_q(sl(n))$-morphism. Let us replace $L$ in the defining relations of the RE algebra by $L - \gamma_h(L)$ (i.e., $l_i^j$ by $l_i^j - \gamma_h(l_i^j)$). Then these relations turn into

$$RL_1RL_1 - L_1RL_1R = h(RL_1 - L_1R), \quad h = h(q - q^{-1}). \quad (3.4)$$

The algebra defined by (3.4) is just the algebra $U(gl(n)_{h,q})$ mentioned above.

By imposing the condition $\text{Tr}_q(L) = 0$ we get a $q$-analogue of the algebra $U(sl(n)_h)$. Denote this two parameter algebra $U(sl(n)_{h,q})$. Since this algebra also is $U_q(sl(n))$-covariant we consider its representations in the same sense as above.

It is evident from the construction that the representations of the RE algebra such that

$$\rho(\text{Tr}_q(L)) = -\gamma_h(\text{Tr}_q(L)) \text{id}$$

and those of the algebra $U(sl(n)_{h,q})$ are in the one-to-one correspondence. (Let us note that $\gamma_h(\text{Tr}_q(L)) \neq 0$ for a generic $q$.)

The representations of the latter algebra can be constructed in the quasiclassical case via the embedding

$$U(sl(n)_{h,q}) \hookrightarrow U_q(sl(n)) \quad (3.5)$$

realized in [LS]. More precisely, embedding (3.5) differs from that of [LS] by a factor which is the Casimir element but it becomes a number if the QG $U_q(sl(n))$ is represented in an irreducible module. So, by rescaling the generators of the algebra $U(sl(n)_{h,q})$ we can get embedding (3.5) represented in a module. Thus, any irreducible representation of the QG $U_q(sl(n))$ gives rise to that of $U(sl(n)_{h,q})$. 

9
Note that in a non-quasiclassical case the above method is not valid. Another way to develop representation theory of (non-commutative) quantum hyperboloid was suggested in [DGR]. That method is valid in all cases mentioned in remark 2. It is based on the fact that if \( \text{End}(U) \) is multiplicity free the map \( V \to \text{End}(U) \) being a morphism is unique up to a factor. If \( U \) is the fundamental module \( (U = V \) in the terms of remark 2) \( V \) can be identified with the traceless component of \( \text{End}(U) \).

This observation allows us to consider the problem of defining an involution operator in the algebra \( A_{h,q} \) from a new viewpoint. The traditional approach begins with definition of this operator on the algebra \( k(G_q) \) (and on its dual object) by means of some compatibility condition with the coproduct. However, in our realization of the quantum sphere any Hopf structures (habitual or braided) are irrelevant. From the point of view of above realization of the quantum varieties, the problem can be reformulated as follows: what are desired properties of an involution operator in the space \( \text{End}(V) \), \( V \) being an object of a tensor (or quasitensor) category? (Note that in a super-category the classical condition \( (ab)^* = b^*a^* \) is replaced by its super-analogue.) A case of an involutive \( R \) was considered in [GRZ]. In [DGR] some compatibility condition with a q-analogue of Lie bracket was suggested.

### 4 Quantum bundles on quantum sphere

In this section we consider a particular case of the RE algebra related to the QG \( U_q(sl(2)) \). Thus, we get a new realization of the quantum sphere.

In this case the algebra \( \mathcal{L}_q(R) \) is generated by four elements \( l_1^1, l_2^1, l_1^2, l_2^2 \). The space span \( (l_i^j) \) is a direct sum of the trivial component generated by \( \text{Tr}_q(L) \) and \( V = \ker \gamma_h \). By imposing the relation \( \text{Tr}_q(L) = 0 \) on the RE algebra we get an algebra generated by the space \( V \). Introducing one more constraint \( \det_q(L) = c \neq 0 \) where \( \det_q(L) \) is the quantum determinant we get just another realization of the quantum sphere (hyperboloid). Therefore the above numerical polynomial \( \mathcal{P} \) becomes

\[
\mathcal{P} = L^2 + c_2 \text{id}
\]

with an appropriate non-trivial factor \( c_2 \).

Thus, we have realized the quantum sphere as a quotient of the RE algebra. In a similar way the non-commutative analogue \( A_{h,q} \) of the quantum sphere can be realized as a quotient of the algebra \( U(sl(n)_{h,q}) \). The corresponding polynomial \( \mathcal{P} \) has the same form (4.6) (cf. [GS]).

This realization of the quantum sphere is very useful for definition of line bundles on it. Let us explain this.

First, we will construct the line bundles related to the fundamental \( U_q(sl(2)) \)-module \( V = V_{1/2} \) (we call these line bundles basic). Let

\[
M = V \otimes A_{h,q}
\]
be a free right $A_{h,q}$-module. It is a $U_q(sl(2))$-module too. Consider its submodule $M_\nu$ generated by the coordinates of the vector

$$v \triangleleft L - \nu v, \quad v = (v_1, v_2) \in V.$$  \hspace{1cm} (4.7)

By $v \triangleleft L$ we mean the vector whose the $j$-th coordinate is $v_i \otimes l^i_j$, $1 \leq i, j \leq 2$.

**Proposition 1** [GS] The quotient-module $\overline{M}_\nu = M/M_\nu$ is not trivial iff $\nu$ is a root of the equation $P(\nu) = 0$. Let $\nu_1$ and $\nu_2$ be the roots. Then the operators

$$P_1 = \frac{(L - \nu_1 \text{id})}{(\nu_2 - \nu_1)}, \quad P_2 = \frac{(L - \nu_2 \text{id})}{(\nu_1 - \nu_2)}$$

are projectors. By definition these operators act on the $A_{h,q}$-module $M = V \otimes A_{h,q}$ as follows

$$(v_i \otimes f^i) \triangleleft L = (v \triangleleft L)_i \otimes f^i, \quad \forall f^i \in A_{h,q}.$$  \hspace{1cm} (4.8)

Moreover, $P_1 + P_2 = \text{id}$ and the quotient $\overline{M}_{\nu_i}$ can be identified with $\text{Im} P_i$, $i = 1, 2$. That is the modules $\overline{M}_{\nu_i}$, $i = 1, 2$ are projective.

If $q = 1$ these modules correspond in the framework of the Serre-Swan scheme to the ”basic line bundles” on the sphere. We want to stress that in comparison with a construction of these quantum line bundles in [BM], [HM] our approach does not use of any description of the quantum sphere via the QG $k(SL_q(2))$ (or what is the same $k(SU_q(2))$). Moreover, our construction has an evident generalization to orbits related to other QG $U_q(sl(n))$, $n > 2$ and is valid in non-quasiclassical cases as well (cf. [GS]). The passage to other modules (called derived) can be realized by an extension of the matrix $L$ to other $U_q(sl(2))$-modules (hereafter only the algebra $k(S^2_q)$ is considered).

The problem splits into two parts. The first question is: what is a reasonable way of constructing such an extension? In the classical case the extension to the module $V \otimes k \otimes k(S^2_q)$ can be introduced by means of the Leibniz rule

$$(u \otimes v) \triangleleft L = u \otimes (v \triangleleft L) + (u \triangleleft L) \otimes v \quad u, v \in V$$

if $k = 2$ and similarly for $k > 2$. The second part of the problem is to find the CH identity for such an extension of the matrix $L$. In the classical case the corresponding numerical polynomials can be found directly from $P$ (cf. [GS]).

However, in the quantum case we meet the following difficulty. If we extended the matrix $L$ by the Leibniz rule as above we would be unable to find the CH identity for the extended matrix (unless the Hecke symmetry is involutary). An explanation of this fact is presented in [GS].

In this paper we suggest another way to construct an extension of the matrix $L$. Define the matrix

$$L_+^{(k)} = P_+^{(k)} L_1 P_+^{(k)}$$  \hspace{1cm} (4.8)
where
\[ P_+^{(k)} : V^\otimes k \to \text{Sym}^k(V) \]
is the full \( q \)-symmetrizer of the \( k \)-th tensor power of \( V \). Its explicit form can be found in [GPS].

The action of thus introduced matrix on the module \( V^\otimes k \otimes k(S_q^2) \) is defined as follows. For any vector
\[ v_{i_1} \otimes \ldots \otimes v_{i_k} \otimes f_j \]
of this module, the first projector \( P_+^{(k)} \) in (4.8) extracts its \( q \)-symmetric component. Then the matrix \( L_1 \) acts on the first element only of each term of this symmetric component in accordance with the definition of the above Proposition 1. And, at last, the action of the second projector \( P_+^{(k)} \) in (4.8) restores the full \( q \)-symmetry. That is we have an action
\[ L_+^{(k)} : v^\otimes k \otimes k(S_q^2) \to \text{Sym}^k(V) \otimes k(S_q^2). \]

This method is motivated by the fact that in the classical case such an extension of the matrix \( L \) and that arising from the Leibniz rule are equivalent (up to a non-essential factor) upon restriction to the symmetric component. Note that for the construction of line bundles only symmetric component is relevant. This way to extend the matrix \( L \) enables us to find the CH identity for the matrix \( L_+^{(k)} \) if \( k = 2 \) in the case related to the quantum sphere (if \( k > 2 \) the problem is still open). More precisely we have the following.

**Proposition 2** [GS] If the CH identity for the matrix \( L \) is of the form
\[ L^2 - aL + b \text{id} = 0 \quad a = \nu_1 + \nu_2, \ b = \nu_1 \nu_2. \]
then the matrix \( L_+ \) defined by (4.8) \((k = 2)\) obeys the CH identity of the form:
\[ L_+^3 - a(1 + q^{-1} \frac{2q}{2q - 1}) L_+^2 + (a^2 q^{-1} \frac{2q}{2q - 1} - b) L_+ + ab \frac{q^{-1}}{2q} \text{id} = 0. \] (4.9)

Remark that if \( a = 0 \) (in particular, this case corresponds to the algebra \( k(S_q^2) \)) relation (L9) becomes
\[ L_+^3 - b L_+ = 0 \]
and it does not depend on \( q \).

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