An equivalent pipe network model for free surface flow in porous media and its comparison with the continuum model

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Abstract. A simple and efficient equivalent pipe network method is developed for unconfined seepage problems in porous media. Instead of the continuum-based method, the porous media is represented by an orthogonal of pipe elements rather than triangular and quadrilateral elements in which water flow occurs. The equivalent expressions of the hydraulic conductivity parameters, the continuity equation, boundary conditions and finite element analysis are reduced to a one-dimensional problem of line elements. Two typical examples are applied to investigate the validity and advantages of the proposed equivalent pipe element method in comparison with the continuum-based method. From the steady-state free surface flow in the homogeneous, the equivalent pipe network method can well reproduce the free surface locations with the numerical solutions. However, the calculated results from the fractured porous media illustrate that the performance of the proposed method is better than the continuum-based method when considering a high-hydraulic conductivity horizontal fracture in the porous media.

1. Introduction
Understanding free surface flow in porous media plays an important role in many geotechnical engineering [1-4]. Because the unknown of the free surface and the nonlinearity of the seepage face, predictions of free surface between the wet and dry domains have been a challenging issue. The porous media is generally homogenized as a continuum on the macroscopic scale, in which water can flow through the whole domain including pores and solid matrix. Based on the continuum-based theory [5], a number of numerical methods have been proposed to simulate free surface flow in porous media, which can be divided into two main categories: adaptive-mesh methods [6] and fixed-mesh methods [7-11].

Recently, increasing attractions on the equivalent pipe network model have been drawn for seepage analysis in porous media, in which the porous media is discretized into pipe networks for numerical analysis of subsurface flow [12-16]. However, the traditional pipe network model is established by using triangular mesh and water flow is delimited in each triangular edge [12-13]. As a result, the equivalent permeability parameters depend on the coordinate of the mesh nodes and need to be calculated for each triangular element, the advantage of the equivalent pipe network method should be discussed more in detail. Therefore, the primary objectives of this study are (1) to propose a simple and efficient numerical method for steady free surface flow in porous media based on the one-dimensional line
element model (ID LEM) and (2) to assess the advantages of the proposed ID LEM in comparison with the continuum-based two-dimensional quadrilateral element model (2D QEM).

2. Problem statements
A general illustration of steady-state free surface flow through a two-dimensional soil dam is shown in Fig. 1. The entire flow domain is separated by the free surface into two parts: the dry and wet domains. The subsurface flow can only occur in the wet domain below the free surface when unsaturated flow behaviors in the dry domain such as capillary flow are ignored in this study. For the free surface flow in a fixed domain, the Darcy’s law, which applied only to saturated flow in the wet domain, can be extended to the whole flow domain including the dry domain in a generalized form as

$$v = -H(\phi - z)k\nabla \phi$$

where $v$ is the flow velocity vector, $k$ is the saturated hydraulic conductivity tensor, $\phi = p + z$ is the total water head, $p$ is the pressure head, $z$ is the vertical coordinate, $H(\phi - z)$ is a Heaviside function

$$H(\phi - z) = \begin{cases} 1 & \text{if } \phi - z \geq 0 \\ 0 & \text{if } \phi - z < 0 \end{cases}$$

The water flow through the porous media should satisfy the continuity equation and following boundary conditions

(1) The water head boundary condition on AB and FG

$$\phi = \phi_1$$

(2) the flux boundary condition on AG

$$q_n = -n u^T = \overline{q}$$

where $\overline{q}$ is the prescribed flux on AG, $n$ is the outward unit normal vector of the boundary. For an impermeable boundary, it yields $\overline{q} = 0$.

(3) The Signorini’s type boundary of the seepage face on BCDEF

$$\begin{cases} \phi \leq z, q_n \leq 0 \\ (\phi - z)q_n = 0 \end{cases}$$

(4) The free surface boundary on BE

$$\begin{cases} \phi = z \\ q_n = 0 \end{cases}$$

Based on the variational principle, the continuum-based 2D QEM are given as below:

$$K\phi = q$$

$$K = \sum_{c} \int_{\Omega} H(\phi - z)B^T k B d\Omega$$
\[ q = \sum \int_N^T N_q d\Gamma \]  
\[ \phi = \{ \phi_1, \phi_2, \ldots, \phi_{N_N} \} \]

in which \( N \) is the matrix of interpolation functions and \( B \) is the gradient-fluid potential transformation matrix.

3. The equivalent pipe network model

In this section, the continuous porous media is divided into an orthogonal network of pipe elements and the related one-dimensional line finite element model is developed. Alternatively, the flow domain is replaced by an orthogonal network of line elements with uniform scale in the horizontal and vertical directions, as illustrated in Fig. 2. Based on the principle of flux equivalence between the homogeneous pore-scale micromodel and pipe network model, the equivalent hydraulic conductivities in the horizontal and vertical direction through the porous media and the pipe network model are given as

\[ k_{px} w_x = k_x B_x \]  
\[ k_{py} w_y = k_y B_y \]

where \( k_{px} \) and \( k_{py} \) are the equivalent permeabilities, \( w_x \) and \( w_y \) are the widths, \( B_x \) and \( B_y \) are the spacings of vertical and horizontal pipe elements, respectively.

For the steady-state analysis in the orthogonal network of line elements, the water flow through an arbitrary line element \( ij \) under a local coordinate system \( l \) is constant, which can be written as

\[ \frac{\partial q_y}{\partial t} - \frac{\partial}{\partial l} \left[ H (\phi - z) k_y w_y \frac{\partial \phi}{\partial l} \right] = 0 \]  

In order to solve Eq.(14) in the orthogonal network of line elements, the boundary conditions in the line element model can be processed equivalently as follows:

1. The water head boundary condition on AB and FG

\[ \phi_i = \bar{\phi} \quad (\text{node } i \in \text{AB+FG}) \]  

2. The flux boundary condition on AG

\[ q_y = k_y w_y \frac{\partial \phi}{\partial l} = \bar{q} \quad (\text{node } i \in \text{AG}) \]  

3. The Signorini’s type boundary of the seepage face on BCDEF

\[ \begin{cases} \phi_i \leq z_i, q_y \leq 0 \\ (\phi_i - z_i) q_y = 0 \end{cases} \quad (\text{node } i \in \text{BCDEF}) \]  

4. The free surface boundary on BE

![Fig. 2 The equivalent pipe network model](image)
By employing the continuous Heaviside function \( H_z(\phi - z) \), the discrete 1D LEM of Eq.(14) can be obtained as

\[
K\phi^{n+1} = q^n
\]  

in which

\[
K = \sum_{ij} B^T_{ij} k_{ij} w_{ij} B_{ij} dl
\]  

\[
q^n = K_x^\phi
\]

\[
K_x = \sum_{i\alpha} K_{i\alpha}^x
\]

\[
K_{i\alpha}^x = \int_{S_i} \left[1 - H_z(\phi^n - z)\right] B^T_{i\alpha} k_{i\alpha} w_{i\alpha} B_{i\alpha} dl
\]

\[
H_z(\phi - z) = \begin{cases} 
1 & \text{if } \phi - z \geq 0 \\
\frac{(\lambda + \phi - z)}{\lambda} & \text{if } 0 > \phi - z > -\lambda \\
0 & \text{if } \phi - z \leq -\lambda
\end{cases}
\]  

The convergence criterion for the two finite element procedures is given as follows:

\[
\left\|\phi^{n+1} - \phi^n\right\| \leq \delta \left\|\phi^n\right\|
\]

where \( \delta \) denotes a user-specified error tolerance and take the value of 0.001 in this study.

4. Examples simulations and comparisons

4.1. Steady flow in a homogeneous dam

In the first example, the case of steady-state free surface seepage in a 2D dam is considered as shown in Fig. 3, in which the dam is isotropic and homogeneous with a height of 10m and a width of 5m. The saturated hydraulic conductivity is equal to 1m/d. The upstream and downstream water levels are 10m and 2m, respectively. The bottom boundary is specified to be impermeable. In the 2D QEM, this example is meshed by quadrilateral elements. In contrast, the porous medium is transformed into an orthogonal network in the 1D LEM, which is formed by two set of line elements with constant equivalent saturated hydraulic conductivity, width and spacing.

Fig. 3 indicates the comparisons of the free surface locations simulated by the 2D QEM to those of the 1D LEM with \( B_x = B_y = 0.05 \)m and other numerical solutions. The predictions of the free surface locations by the 2D QEM and 1D LEM are mostly overlapped and present reasonable agreements with those from Lacy and Prevosti [17] and Bardet and Tobita [18], except for Oden and Kikuchi [8] and Borja and Kishnani [19], in which the free surfaces flow out at the point D rather than the seepage point E as a result of the singularity of the seepage face [11].

4.2. A fractured porous medium

In this example, the advantage of the proposed 1D LEM for modeling the free surface flow in the fractured porous media is investigated, where a horizontal fracture is imbedded in the center of a rectangular porous model as shown in Fig. 4. The size of the whole domain was 5m in length and 10m in height. The location of fracture ranges from point \((x=2.5, y=5)\) to point \((x=5, y=5)\). The prescribed water heads on the upstream and downstream boundaries are 10m and 2m, respectively. The base is specified as the impermeable boundary condition. In the 2D QEM, the horizontal fracture is modeled as an extremely thin rectangular domain and meshed into quadrilateral elements with its aperture size. On the contrary, the proposed 1D LEM uses 1D zero-thickness line elements to represent the fracture.
For 2D QEM and 1D LEM, the same mesh size $B_x = B_y = 0.25$ m are applied to the porous media and the horizontal fracture during the seepage flow analysis in the whole domain.

The saturated hydraulic conductivity of the porous media is specified as $4.87 \times 10^{-8}$ m/s and the horizontal fracture with two different aperture values $1 \times 10^{-4}$ m and $1 \times 10^{-5}$ m are used to investigate the effect of fracture hydraulic conductivity on the location of free surface. Based on the cubic law, the hydraulic conductivity ratios $k_f/k_m$ between the fracture and matrix are $1.67 \times 10^5$ and $1.67 \times 10^3$ respectively, where $k_f$ is the fracture hydraulic conductivity and $k_m$ is the matrix hydraulic conductivity.

The predictions of free surfaces calculated by 2D QEM and 1D LEM are compared in Fig. 4. For this problem, the numerical solutions of the free surface provided by Li and Li [20] are presented as well. It is found that the free surface near the interface has a more pronounced drop with increasing fracture hydraulic conductivity. The results demonstrated that the performance of the 1D LEM is much better than the 2D QEM and have good agreements with the numerical solutions of Li and Li [20], especially for large hydraulic conductivity ratio between the fracture and matrix.

Note that in the discrete fracture network model, the permeability of matrix is generally weak and water flow is dominated by the fracture networks. Thus, the fracture network can also be treated as one-dimensional line element in the finite element analysis and applied to model the free surface flow in porous media based on the flux equivalence [21]. However, the proposed method is based on the geometry similarity between the homogeneous pores and pipes as shown in Fig. 2, which is easy to understand and has concise physical meaning.

5. Conclusions

Concerning the free surface flow in porous media, a simple one-dimensional line element model is proposed to simulate steady-state free surface flow in porous media and compared with the traditional continuum-based 2D QEM. In the 1D LEM, the whole flow domain is discretized into an orthogonal network of line elements instead of the rectangular element in the continuum-based 2D QEM and water flow is assumed to only occur through the line elements, rather than the solid elements in the 2D QEM. In addition, equivalent hydraulic parameters, continuity equation, boundary conditions and finite element analysis are formulated in a form of one dimension. Therefore, the 2D free surface flow problem is reduced to a 1D problem by the 1D LEM. The calculation of the permeability matrix can be accurately integrated with 2-node linear element for the 1D LEM, while an approximate Gaussian integration of 4-node bilinear quadratic element is used in the 2D QEM. In order to evaluate the validity and computational efficiency of the proposed 1D LEM, two typical examples have been
applied to predict the free surface in porous media. From the numerical analysis in the homogeneous rectangular dams and he fractured porous media, the comparisons show that the two models yield identical results of the free surface locations and the 1D LEM gives a better performance and agrees well with the numerical solutions available in the literature.

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