Morse homology for the heat flow

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Abstract

We use the heat flow on the loop space of a closed Riemannian manifold to construct an algebraic chain complex. The chain groups are generated by perturbed closed geodesics. The boundary operator is defined in the spirit of Floer theory by counting, modulo time shift, heat flow trajectories that converge asymptotically to nondegenerate closed geodesics of Morse index difference one.

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1 Introduction

Let $M$ be a closed Riemannian manifold and denote by $\nabla$ the Levi-Civita connection and by $\mathcal{LM}$ the loop space, that is the space of free loops $C^\infty(S^1, M)$. For $x: S^1 \to M$ consider the action functional

$$S_V(x) = \int_0^1 \left( \frac{1}{2} |\dot{x}(t)|^2 - V(t, x(t)) \right) dt.$$ 

Here and throughout we identify $S^1 = \mathbb{R}/\mathbb{Z}$ and think of $x \in \mathcal{LM}$ as a smooth map $x: \mathbb{R} \to M$ which satisfies $x(t+1) = x(t)$. Smooth means $C^\infty$ smooth.

The potential is a smooth function $V: S^1 \times M \to \mathbb{R}$ and we set $V_t(q) := V(t, q)$.

The critical points of $S_V$ are the 1-periodic solutions of the ODE

$$\nabla_t \dot{x} = -\nabla_V(x),$$

where $\nabla V_t$ denotes the gradient and $\nabla_t \dot{x}$ denotes the covariant derivative, with respect to the Levi-Civita connection, of the vector field $\dot{x} := \frac{d}{dt}x$ along the loop $x$ in direction $\dot{x}$. By $\mathcal{P} = \mathcal{P}(V)$ we denote the set of 1-periodic solutions of (1). These solutions are called perturbed closed geodesics, since in the case $V = 0$ they are closed geodesics.

From now on we assume that $S_V$ is a Morse function on the loop space, i.e. the 1-periodic solutions of (1) are all nondegenerate. We proved in [W02] that $S_V$ is Morse for a generic potential $V_t$ and that in this case the set

$$\mathcal{P}^a(V) := \{ x \in \mathcal{P}(V) \mid S_V(x) \leq a \}$$

is finite for every real number $a$. By $E^a_x$ we denote the eigenspace corresponding to negative eigenvalues of the Hessian of $S_V$ at $x \in \mathcal{P}^a(V)$. The dimension of $E^a_x$ is finite and called the Morse index of $x$. Choose an orientation $\langle x \rangle$ of the vector space $E^a_x$. By $\nu = \nu(V, a)$ we denote a choice of orientations for all $x \in \mathcal{P}^a(V)$. Now consider the $\mathbb{Z}$-module

$$CM^a_x = CM^a_x(V, \nu) := \bigoplus_{x \in \mathcal{P}^a(V)} \mathbb{Z}\langle x \rangle.$$ 

It is graded by the Morse index.
If in addition $\mathcal{S}_V$ is Morse–Smale, then the module $\text{CM}_a^*$ carries a boundary operator $\partial = \partial(V, a, \nu)$ defined as follows. Consider the (negative) $L^2$ gradient flow lines of $\mathcal{S}_V$ on the loop space. These are solutions $u : \mathbb{R} \times S^1 \to M$ of the heat equation
\[
\partial_s u - \nabla_t \partial_t u - \nabla V_t(u) = 0 \quad (2)
\]
satisfying
\[
\lim_{s \to \pm \infty} u(s, t) = x^\pm(t), \quad \lim_{s \to \pm \infty} \partial_s u(s, t) = 0, \quad (3)
\]
where $x^\pm \in \mathcal{P}(V)$. The limits are uniform in $t$ together with the first partial $t$-derivative, i.e. in $C^1(S^1)$; see remark 1.5. By definition the moduli space $\mathcal{M}(x^-, x^+; V)$ is the space of solutions of (2) and (3). The action functional $\mathcal{S}_V$ is called Morse–Smale below level $a$ if the operator $D_u$ obtained by linearizing (2) is onto as a linear operator between appropriate Banach spaces, see (12) below, and this is true for all $u \in \mathcal{M}(x^-, x^+; V)$ and $x^\pm \in \mathcal{P}^a(V)$. Note that Morse–Smale implies Morse (use that $u := x \in \mathcal{M}(x, x; V)$). Under the Morse–Smale hypothesis the space $\mathcal{M}(x^-, x^+; V)$ is a smooth manifold whose dimension is equal to the difference of the Morse indices of the perturbed closed geodesics $x^\pm$. In the case of index difference one it follows that the quotient $\mathcal{M}(x^-, x^+; V)/\mathbb{R}$ by the (free) time shift action is a finite set. Counting these elements with appropriate signs defines the boundary operator $\partial$ on $\text{CM}_a^*$. The Morse complex $(\text{CM}_a^*, \partial)$ is called the heat flow complex and the corresponding homology groups $H^a_* (\mathcal{L}M, \mathcal{S}_V)$ are called heat flow homology.

In chapter 6 we explain how to perturb the Morse function $\mathcal{S}_V$ by a regular perturbation $v \in \mathcal{O}_{reg}^a$ to achieve the Morse–Smale condition without changing the set of critical points. By definition heat flow homology of $\mathcal{S}_V$ is then equal to heat flow homology of the perturbed functional. It is an open question if $\mathcal{S}_V$ is Morse–Smale for a generic potential $V$. In section 1.1 we introduce a class of abstract perturbations $V : \mathcal{L}M \to \mathbb{R}$ for which we can establish transversality. In contrast we call the potentials $V_t$ geometric perturbations.

**Theorem 1.1.** Let $V \in C^\infty(S^1 \times M)$ be a potential such that $\mathcal{S}_V$ is Morse and let $a$ be a regular value of $\mathcal{S}_V$. Take a choice of orientations $\nu = \nu(V, a)$ and fix a regular perturbation $v \in \mathcal{O}_{reg}^a$. Then $\partial = \partial(V, a, \nu, v)$ satisfies $\partial \circ \partial = 0$. Furthermore, heat flow homology defined by
\[
H^a_* (\mathcal{L}M, \mathcal{S}_V) := \frac{\ker \partial(V, a, \nu, v)}{\text{im} \partial(V, a, \nu, v)}
\]
is independent of the choice of orientations $\nu$ and the regular perturbation $v$.

The construction of the Morse complex in finite dimensions goes back to Thom [T49], Smale [Sm60, Sm61], and Milnor [M65]. It was rediscovered by Witten [Wi82] and extended to infinite dimensions by Floer [F89a, F89b]. We refer to [AM06] for an extensive historical account.
1.1 Perturbations

We introduce a class of abstract perturbations of equation (6) for which the analysis works. Later in section 6.1 we extract a countable subset and construct a separable Banach space of perturbations for which transversality works. The abstract perturbations take the form of smooth maps \( V : \mathcal{L}M \to \mathbb{R} \). For \( x \in \mathcal{L}M \) let \( \text{grad} V(x) \in \Omega^0(S^1, x^*TM) \) denote the \( L^2 \)-gradient of \( V \); it is defined by

\[
\int_0^1 \langle \text{grad} V(u), \partial_s u \rangle \, dt = \frac{d}{ds} V(u)
\]

for every smooth path \( \mathbb{R} \to \mathcal{L}M : s \mapsto u(s, \cdot) \). The covariant Hessian of \( V \) at a loop \( x : S^1 \to M \) is the operator \( H_V(x) : \Omega^0(S^1, x^*TM) \to \Omega^0(S^1, x^*TM) \) defined by

\[
H_V(u) \partial_s u := \nabla_s \text{grad} V(u)
\]

for every smooth map \( \mathbb{R} \to \mathcal{L}M : s \mapsto u(s, \cdot) \) and every \( (s, t) \in \mathbb{R} \times S^1 \). Although condition (V1) and the first part of (V2) are special cases of (V3) we state the axioms in the form below, because some of our results don’t require all the conditions to hold.

(V0) \( V \) is continuous with respect to the \( C^0 \) topology on \( \mathcal{L}M \). Moreover, there is a constant \( C = C(V) \) such that

\[
\sup_{x \in \mathcal{L}M} |V(x)| + \sup_{x \in \mathcal{L}M} \|\text{grad} V(x)\|_{L^\infty(S^1)} \leq C.
\]

(V1) There is a constant \( C = C(V) \) such that

\[
|\nabla_s \text{grad} V(u)| \leq C \left( |\partial_s u| + \|\partial_s u\|_{L^1} \right),
\]

\[
|\nabla_t \text{grad} V(u)| \leq C \left( 1 + |\partial_t u| \right)
\]

for every smooth map \( \mathbb{R} \to \mathcal{L}M : s \mapsto u(s, \cdot) \) and every \( (s, t) \in \mathbb{R} \times S^1 \).

(V2) There is a constant \( C = C(V) \) such that

\[
|\nabla_s \nabla_s \text{grad} V(u)| \leq C \left( |\nabla_s \partial_s u| + \|\nabla_s \partial_s u\|_{L^1} + (|\partial_s u| + \|\partial_s u\|_{L^2})^2 \right),
\]

\[
|\nabla_t \nabla_s \text{grad} V(u)| \leq C \left( |\nabla_t \partial_s u| + (1 + |\partial_t u|) (|\partial_s u| + \|\partial_s u\|_{L^1}) \right),
\]

and

\[
|\nabla_s \nabla_s \text{grad} V(u) - H_V(u) \nabla_s \partial_s u| \leq C \left( |\partial_s u| + \|\partial_s u\|_{L^2} \right)^2
\]

for every smooth map \( \mathbb{R} \to \mathcal{L}M : s \mapsto u(s, \cdot) \) and every \( (s, t) \in \mathbb{R} \times S^1 \).
For any two integers \( k > 0 \) and \( \ell \geq 0 \) there is a constant \( C = C(k, \ell, V) \) such that

\[
|\nabla^k_t \nabla^\ell_s \nabla V(u)| \leq C \sum_{k_j, \ell_j > 0} \left( \prod_{k_j > 0} \left| \nabla^{k_j}_t \nabla^{\ell_j}_s |u| \right| \right) \prod_{\ell_j = 0} \left( |\nabla^{k_j}_s |u| + \| \nabla^{k_j}_s |u| \|_{L^p} \right)
\]

for every smooth map \( \mathbb{R} \to \mathcal{L}M : s \mapsto u(s, \cdot) \) and every \((s, t) \in \mathbb{R} \times S^1\); here \( p_j \geq 1 \) and \( \sum \ell_j = 1 \); the sum runs over all partitions \( k_1 + \cdots + k_m = k \) and \( \ell_1 + \cdots + \ell_m = \ell \) such that \( k_j + \ell_j \geq 1 \) for all \( j \). For \( k = 0 \) the same inequality holds with an additional summand \( C \) on the right.

Remark 1.2. In (V0) the \( L^\infty \) bound for \( \nabla V \) is imposed, since occasionally we need \( L^p \) bounds for fixed but arbitrary \( p \). Continuity of \( V \) with respect to the \( C^0 \) topology is used to prove [SW03, lem. 10.2] and proposition 3.14.

Remark 1.3. Each geometric potential \( V \) provides an abstract perturbation \( \mathcal{V} \) such that for smooth loops \( x \) and smooth vector fields \( \xi \) along \( x \) we have

\[
\mathcal{V}(x) := \int_0^1 V_t(x(t)) \, dt, \quad \nabla \mathcal{V}(x) = \nabla V_t(x), \quad \mathcal{H}_V(x) \xi = \nabla_\xi \nabla V_t(x).
\]

Remark 1.4. To prove transversality in section 6 we use perturbations\(^1\) of the form

\[
\mathcal{V}(x) := \rho \left( \|x - x_0\|_{L^2}^2 \right) \int_0^1 V_t(x(t)) \, dt,
\]

where \( \rho : \mathbb{R} \to [0, 1] \) is a smooth cutoff function and \( x_0 : S^1 \to M \) is a smooth loop. Any such perturbation satisfies (V0)-(V3). Here compactness of \( M \) is crucial, in particular, finiteness of the diameter of \( M \).

1.2 Main results

There are two main purposes of this text. One is to construct the Morse chain complex for the action functional on the loop space. The other one is to provide proofs of the results announced and used in [SW03] to calculate the adiabatic limit of the Floer complex of the cotangent bundle. More precisely, in [SW03] we proved in joint work with D. Salamon that the connecting orbits of the heat flow are the adiabatic limit of Floer connecting orbits in the cotangent bundle \( T^*M \) with respect to the Hamiltonian given by kinetic plus potential energy. The key idea is to appropriately rescale the Riemannian metric on \( M \). Both purposes are achieved simultaneously by theorems 1.6–1.14.

\(^1\)Here and throughout the difference \( x - x_0 \) of two loops denotes the difference in some ambient Euclidean space into which \( M \) is (isometrically) embedded. Note that cutting off with respect to the \( L^2 \) norm – as opposed to the \( L^\infty \) norm – prevents us from expressing the difference in terms of the exponential map.

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From now on we replace the potential $V$ by an abstract perturbation $\mathcal{V}$ satisfying (V0)–(V3). In this case the action is given by

$$S_\mathcal{V}(x) = \frac{1}{2} \int_0^1 |\dot{x}(t)|^2 \, dt - \mathcal{V}(x)$$

(5)

for smooth loops $x : S^1 \to M$ and the heat equation has the form

$$\partial_s u - \nabla_t \partial_t u - \text{grad} \mathcal{V}(u) = 0$$

(6)

for smooth maps $u : \mathbb{R} \times S^1 \to M, (s, t) \mapsto u(s, t)$. Here grad$\mathcal{V}(u)$ denotes the value of grad$\mathcal{V}$ on the loop $u_s : t \mapsto u(s, t)$. The relevant set $\mathcal{P}(\mathcal{V})$ of critical points of $S_\mathcal{V}$ consists of the (smooth) loops $x : S^1 \to M$ that satisfy the ODE

$$\nabla_t \dot{x} = -\text{grad} \mathcal{V}(x).$$

(7)

The subset $\mathcal{P}_a(\mathcal{V})$ consists of all critical points $x$ with $S_\mathcal{V}(x) \leq a$. For two nondegenerate critical points $x^\pm \in \mathcal{P}(\mathcal{V})$ we denote by $\mathcal{M}(x^-, x^+; \mathcal{V})$ the set of all solutions $u$ of (6) such that

$$\lim_{s \to \pm \infty} u(s, t) = x^\pm(t), \quad \lim_{s \to \pm \infty} \partial_s u(s, t) = 0.$$  

(8)

The limits are uniform in $t$ together with the first partial $t$-derivative. These solutions are called connecting orbits. The energy of such a solution is given by

$$E(u) = \int_{-\infty}^{\infty} \int_0^1 |\partial_s u|^2 \, dt \, ds = S_\mathcal{V}(x^-) - S_\mathcal{V}(x^+).$$

(9)

**Remark 1.5** (Asymptotic limits). In (3) and (8) we require convergence in $C^1(S^1)$ as opposed to $C^0(S^1)$ which is standard in elliptic Floer theory. We need the stronger assumption in theorem 2.10 to establish exponential decay. Actually $W^{1,2}(S^1)$ convergence already works. Compare [SW03] where the asymptotic $C^0$ limits of $(u, \nu)$ and $(\partial_s u, \nabla_t \partial_t u)$ are required to be $(x^\pm, \partial_s x^\pm)$ and zero, respectively. Now $\nu$ corresponds to $\partial_t u$ in the adiabatic limit studied in [SW03].

**Theorem 1.6** (Regularity). Fix a constant $p > 2$ and a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V3). Let $u : \mathbb{R} \times S^1 \to M$ be a continuous function of class $W^{1,p}_{1,\text{loc}}$, that is $u, \partial_t u, \nabla_t \partial_t u, \partial_s u$ are locally $L^p$ integrable. Assume further that $u$ solves the heat equation (6) almost everywhere. Then $u$ is smooth.

**Remark 1.7.** It seems unlikely that the assumption $u \in W^{1,p}_{1,\text{loc}}$ can be weakened to $u \in W^{1,p}_{\text{loc}}$, as announced in [SW03], unless we also weaken $p > 2$ to $p > 3$; see [W09, rmk. 5.2]. However, the stronger assumption $u \in W^{1,p}_{\text{loc}}$ is satisfied in our applications of theorem 1.6. These are [SW03, proof of lemma 10.2], the Banach bundle setup introduced in chapter 4, step 1 of the proof of theorem 1.13, and the proof of proposition 6.7 on surjectivity of the universal section.
Theorem 1.8 (Apriori estimates). Fix a perturbation $V : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V1) and a constant $c_0$. Then there is a positive constant $C = C(c_0, V)$ such that the following holds. If $u : \mathbb{R} \times S^1 \to M$ is a smooth solution of (6) such that $\mathcal{S}_V(u(s, \cdot)) \leq c_0$ for every $s \in \mathbb{R}$ then

$$\|\partial_t u\|_\infty + \|\nabla \partial_t u\|_\infty + \|\partial_s u\|_\infty + \|\nabla \partial_s u\|_\infty + \|\nabla \partial_s u\|_\infty \leq C.$$

Theorem 1.9 (Exponential decay). Fix a perturbation $V : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V3) and assume $\mathcal{S}_V$ is Morse.

(F) Let $u : [0, \infty) \times S^1 \to M$ be a smooth solution of (6). Then there are positive constants $\rho$ and $c_0, c_1, c_2, \ldots$ such that

$$\|\partial_s u\|_{C^0([T, \infty) \times S^1)} \leq c_k e^{-\rho T}$$

for every $T \geq 1$. Moreover, there is a periodic orbit $x \in \mathcal{P}(V)$ such that $u(s, \cdot)$ converges to $x$ in $C^2(S^1)$ as $s \to \infty$.

(B) Let $u : (-\infty, 0] \times S^1 \to M$ be a smooth solution of (6) with finite energy. Then there are positive constants $\rho$ and $c_0, c_1, c_2, \ldots$ such that

$$\|\partial_s u\|_{C^0((-\infty, -T] \times S^1)} \leq c_k e^{-\rho T}$$

for every $T \geq 1$. Moreover, there is a periodic orbit $x \in \mathcal{P}(V)$ such that $u(s, \cdot)$ converges to $x$ in $C^2(S^1)$ as $s \to -\infty$.

The covariant Hessian of $\mathcal{S}_V$ at a loop $x : S^1 \to M$ is the linear operator $A_x : W^{2,2}(S^1, x^*\mathcal{T}M) \to L^2(S^1, x^*\mathcal{T}M)$ given by

$$A_x \xi := -\nabla \nabla \xi - R(\xi, \dot{x}) \dot{x} - \mathcal{H}_V(x) \xi$$

(10)

where $R$ denotes the Riemannian curvature tensor and the Hessian $\mathcal{H}_V$ is defined by (4). This operator is self-adjoint with respect to the standard $L^2$ inner product. The number of negative eigenvalues is finite. It is denoted by $\text{ind}_V(A_x)$ and called the Morse index of $A_x$. If $x$ is a critical point of $\mathcal{S}_V$ we define its Morse index by $\text{ind}_V(x) := \text{ind}_V(A_x)$ and we call $x$ nondegenerate if $A_x$ is bijective. In this notation the linearized operator $\mathcal{D}_u : W^{2,1}_u \to \mathcal{L}^p_u$ is given by

$$\mathcal{D}_u \xi := \nabla \xi + A_u \xi$$

(11)

where $u(t) := u(s, t)$ and the spaces $W_u = W^{2,1}_u$ and $\mathcal{L}_u = \mathcal{L}^p_u$ are defined as the completions of the space of smooth compactly supported sections of the pullback tangent bundle $u^*\mathcal{T}M \to \mathbb{R} \times S^1$ with respect to the norms

$$\|\xi\|_{\mathcal{L}} = \left( \int_{-\infty}^{\infty} \int_0^1 |\xi|^p \, dt \, ds \right)^{1/p},$$

$$\|\xi\|_{W} = \left( \int_{-\infty}^{\infty} \int_0^1 |\xi|^p + |\nabla \xi|^p + |\nabla^2 \xi|^p \, dt \, ds \right)^{1/p}.$$
Theorem 1.10 (Fredholm). Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V3), a constant $p > 1$, and two nondegenerate critical points $x^\pm \in \mathcal{P}(\mathcal{V})$. Assume $u : \mathbb{R} \times S^1 \to M$ is a smooth map such that $\|\nabla_t \nabla_t \partial_s u_s\|_2$ is bounded, uniformly in $s \in \mathbb{R}$, and

$$u_s = \exp_{x^\pm}(\eta^\pm_s), \quad \|\eta^\pm_s\|_{W^{2,2}} \to 0, \quad \|\partial_s u_s\|_{W^{1,2}} \to 0, \quad as \ s \to \pm \infty.$$  

Then the operator $D_u : \mathcal{W}^{1,p}_u \to \mathcal{L}^p_u$ is Fredholm and

$$\text{index } D_u = \text{ind}_\mathcal{V}(x^-) - \text{ind}_\mathcal{V}(x^+).$$

Moreover, the formal adjoint operator $D_u^* = -\nabla_s + A_{u^*} : \mathcal{W}^{1,p}_u \to \mathcal{L}^p_u$ is Fredholm with index $D_u^* = -\text{index } D_u$.

Concerning the funny assumption on $\nabla_t \nabla_t \partial_s u_s$ see the footnote in section 2.4.

Theorem 1.11 (Implicit function theorem). Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V3). Assume $x^\pm$ are nondegenerate critical points of $\mathcal{S}_\mathcal{V}$ and $D_u$ is onto for every $u \in \mathcal{M}(x^-, x^+; \mathcal{V})$. Then $\mathcal{M}(x^-, x^+; \mathcal{V})$ is a smooth manifold of dimension $\text{ind}_\mathcal{V}(x^-) - \text{ind}_\mathcal{V}(x^+)$.

Proposition 1.12 (Finite set). Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V3) and assume $\mathcal{S}_\mathcal{V}$ is Morse–Smale below level $a$ in the sense that every $u \in \mathcal{M}(x^-, x^+; \mathcal{V})$ is regular (i.e. the Fredholm operator $D_u$ is surjective), for every pair $x^\pm \in \mathcal{P}^a(\mathcal{V})$. Then the quotient space

$$\hat{\mathcal{M}}(x^-, x^+; \mathcal{V}) := \mathcal{M}(x^-, x^+; \mathcal{V}) / \mathbb{R}$$

is a finite set for every such pair of Morse index difference one. Here the (free) action of $\mathbb{R}$ is given by time shift $(\sigma, u) \mapsto u(\sigma + \cdot)$.

Theorem 1.13 (Refined implicit function theorem). Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V3) and a pair of nondegenerate critical points $x^\pm \in \mathcal{P}(\mathcal{V})$ with $\mathcal{S}_\mathcal{V}(x^+) < \mathcal{S}_\mathcal{V}(x^-)$ and Morse index difference one. Then, for every $p > 2$ and every large constant $c_0 > 1$, there are positive constants $\delta_0$ and $C$ such that the following holds. Assume $\mathcal{S}_\mathcal{V}$ is Morse–Smale below level $2c_0$. Assume further that $u : \mathbb{R} \times S^1 \to M$ is a smooth map such that $u(s, \cdot)$ converges in $W^{1,2}(S^1)$ to $x^\pm$, as $s \to \pm \infty$, and such that

$$|\partial_s u(s, t)| \leq \frac{c_0}{1 + s^2}, \quad |\partial_t u(s, t)| \leq c_0, \quad |\nabla_t \partial_s u(s, t)| \leq c_0,$$

for all $(s, t) \in \mathbb{R} \times S^1$ and

$$\|\partial_s u - \nabla_t \partial_t u - \text{grad} \mathcal{V}(u)\|_p \leq \delta_0.$$  

Then there exist elements $u_* \in \mathcal{M}(x^-, x^+; \mathcal{V})$ and $\xi \in \text{im } D_{u_*}^* \cap \mathcal{W}$ satisfying

$$u = \exp_{u_*}(\xi), \quad \|\xi\|_\mathcal{W} \leq c \|\partial_s u - \nabla_t \partial_t u - \text{grad} \mathcal{V}(u)\|_p.$$
In the previous theorem “$c_0$ large” means that the constant $c_0$ should be larger than the constant $C_0$ in axiom (V0). Recall that a subset of a complete metric space is called \textbf{residual} if it contains a countable intersection of open and dense sets. By Baire’s category theorem a residual subset is dense.

\textbf{Theorem 1.14} (Transversality). \textit{Fix a perturbation $V : LM \to \mathbb{R}$ that satisfies (V0)-(V3) and assume $S_V$ is Morse. Then for every regular value $a$ there is a complete metric space $O^a = O^a(V)$ of perturbations supported away from $P^a(V)$ and satisfying (V0)-(V3) such that the following is true. If $v \in O^a$, then $P^a(V) = P^a(V + v)$, $H_*(\{S_V \leq a\}) \cong H_*(\{S_{V+v} \leq a\})$.

Moreover, there is a residual subset $O^a_{reg} \subset O^a$ such that for each $v \in O^a_{reg}$ the perturbed functional $S_{V+v}$ is Morse–Smale below level $a$.

\textbf{Outlook}

The obvious next step is to relate heat flow homology defined in theorem 1.1 to singular homology of the loop space. In our forthcoming paper [W10] we establish the following result. Throughout singular homology $H_*$ is meant with integer coefficients.

\textbf{Theorem 1.15}. \textit{Assume $S_V$ is Morse and $a$ is either a regular value of $S_V$ or equal to infinity. Then there is a natural isomorphism

$HM^*_a(\mathcal{LM}, S_V) \cong H_*(\mathcal{L}^a M)$, \hspace{1cm} $\mathcal{L}^a M := \{\gamma \in \mathcal{LM} \mid S_V(\gamma) \leq a\}$.

If $M$ is not simply connected, then there is a separate isomorphism for each component of the loop space. The isomorphism commutes with the homomorphisms $HM^*_a(\mathcal{LM}, S_V) \to HM^b(\mathcal{LM}, S_V)$ and $H_*(\mathcal{L}^a M) \to H_*(\mathcal{L}^b M)$ for $a < b$.

For a $C^1$ gradient flow on a Banach manifold, where the Morse functional is bounded below and its critical points are of finite Morse index, Abbondandolo and Majer [AM06] proved the existence of a natural isomorphism between singular and Morse homology. The geometric idea is that the unstable manifolds carry the homologically relevant information. A major point is to construct a cellular filtration of $\mathcal{L}^a M$ by open forward flow invariant subsets $F_0 \subset F_1 \subset \ldots \subset F_N \subset \mathcal{L}^a M$ such that $F_k$ contains all critical points up to Morse index $k$ and relative singular homology $H_*(F_k, F_{k-1})$ is isomorphic to the free abelian group generated over $\mathbb{Z}$ by the critical points of index $k$. Let $F_0$ be the union of disjoint, open, and forward flow invariant neighborhoods of the critical points of index zero. Then they fix small neighborhoods of the index one critical points and consider the set exhausted by the forward flow. Now they take the union of this set with $F_0$ to obtain $F_1$. Clearly $F_1$ is forward flow invariant. Moreover, it is open, because the time-$t$-map of the flow is an open map. Next continue with the index two critical points and so on.

Unfortunately the time-$t$-map for the semiflow generated by the heat equation does not take open sets to open sets due to the extremely strong regularizing
nature of the heat flow. Hence new ideas are required. Firstly, find the right notion of Conley index pairs of isolated invariant sets in the infinite dimensional situation at hand. Secondly, solve the forward time Cauchy problem for the heat equation (6) for initial values in the Hilbert manifold $\Lambda = W^{1,2}(S^1, M)$ to establish existence of a continuous semiflow $\phi : [0, \infty) \times \Lambda^a M \to \Lambda^a M$. Now use continuity of the time-$t$-map to conclude that the preimage $\varphi_T^{-1}(F_0)$ is an open subset of $\Lambda^a M$. Here $F_0$ consists locally of (strict) sublevel sets near the local minimima and $T > 0$ is chosen sufficiently large such that the time-$T$-map $\varphi_T$ maps the exit set $L_1$ of the Conley index pair $(N, L_1)$ associated to the index one critical points into $F_0$. Then the set $F_1 := N_1 \cup \varphi_T^{-1}(F_0)$ is open and semiflow invariant. Next include the index two points and so on. Full details will be provided in [W10].

1.3 Overview

In appendix A we recall for convenience of the reader facts proved in [W09] concerning parabolic regularity. These are used extensively in the present text. More precisely, we recall the fundamental $L^p$ estimate and local regularity for the linear heat operator $\partial_s u - \partial_t \partial_t u$ acting on real-valued maps $u$ defined on the lower half plane $\mathbb{H}^-$ or on cylindrical sets. Moreover, we introduce relevant parabolic spaces $W^{k,p}$ and $C^{k,p}$ and recall the product estimate lemma A.5 crucial to prove the quadratic estimate in proposition 4.2.

In chapter 2 we study the solutions to the linearized version of the heat equation (6), in other words, the kernel of the operator $D_u$ given by (11). In theorem 2.1 we show that these solutions are smooth. In fact, even weak solutions are smooth. In section 2.2 we derive pointwise bounds in terms of the $L^2$ norm. Section 2.3 then establishes exponential decay of these $L^2$ norms. The combination of these results is used in section 2.4 to prove that the operator $D_u$ is Fredholm for a rather general class of smooth cylinders $u$ in $M$ with nondegenerate asymptotic limits $x^\pm \in \mathcal{P}(V)$. The main result is theorem 1.10.

In chapter 3 we study the solutions $u$ to the (nonlinear) heat equation (6). Since $\partial_s u$ solves the linearized equation the results of chapter 2 apply. In section 3.1 we prove smoothness of $W^{1,p}_{loc}$ solutions and a compactness result for sequences of uniformly bounded gradient with respect to appropriate norms. In sections 3.2–3.4 boundedness of the action is a crucial assumption. Fix a positive constant $c_0$. Then all solutions $u$ of (6) with $\sup_{s \in \mathbb{R}} \mathcal{S}_\mathcal{V}(u_s) \leq c_0$ admit a uniform apriori estimate for $\|\partial_t u\|_\infty$ (theorem 3.5), uniform energy bounds (lemma 3.8), uniform gradient bounds (theorem 3.9), and uniform $L^2$ exponential decay (theorem 3.10). In section 3.5 we study compactness of the moduli spaces $\mathcal{M}(x^-, x^+; V)$ in the case that $\mathcal{S}_\mathcal{V} : \mathcal{L} M \to \mathbb{R}$ is a Morse function.

Chapter 4 deals with implicit function type theorems. Here, in addition to the Morse condition, the Morse–Smale condition enters: To prove that the moduli spaces are smooth manifolds we not only need nondegeneracy of the
asymptotic boundary data, that is the critical points $x^\pm$, but in addition sur-
jectivity of the linearized operators. Under these assumptions proposition 1.12
asserts that modulo time shift there are only finitely many heat flow lines from
$x^-$ to $x^+$ whenever the Morse index difference is one. Here the compactness
results of section 3.5 enter. Furthermore, we prove the refined implicit function
theorem 1.13, a major technical tool in [SW03]. Here the product estimate pro-
vided by lemma A.5 is the crucial ingredient to obtain the required quadratic
estimates. Furthermore, the choice of the sublevel set on which $\mathcal{S}_V$ needs to
be Morse–Smale requires care. The reason is that one starts out only with an approximate
solution $u$ along which the action is not necessarily decreasing. However, the assumptions guarantee that all loops $u_s$ are contained in the
sublevel set $\{S_V \leq 2c_0^2\}$.

In chapter 5 we prove unique continuation for the heat equation (6) and its
linearization. The proof is based on an extension of a result by Agmon and
Nirenberg. In contrast to forward unique continuation the result on backward
unique continuation is surprising at first sight. Of course, there is an assumption.
Namely, the action along the two semi-infinite backward trajectories $u, v$ which
coincide at time $s = 0$ must be bounded. In this case we obtain that $u = v$.

In chapter 6 we construct a separable Banach space $Y$ of abstract pertur-
bations that satisfy axioms (V0)–(V3). Assume $\mathcal{S}_V$ is Morse and $a$ is a regular
value. Then we define a Banach submanifold $\mathcal{O}^a(V)$ of admissible perturbations $v$. These have the property that $\mathcal{S}_V$ and $\mathcal{S}_{V+v}$ do have the same critical points
on their respective sublevel set with respect to $a$ and, moreover, both sublevel
sets are homologically equivalent. The proof that there is a residual subset
$\mathcal{O}^a_{reg}(V)$ of regular perturbations for which $\mathcal{S}_{V+v}$ is Morse-Smale below level $a$
requires unique continuation for the linearized heat equation and the fact that
the action is strictly decreasing along nonconstant heat flow trajectories.

In chapter 7 we define Morse homology for the heat flow. In section 7.1
we define the unstable manifold of a critical point $x$ of the action functional
$\mathcal{S}_V : LM \to \mathbb{R}$ as the set of endpoints at time zero of all backward halfcylinders
solving the heat equation (6) and emanating from $x$ at $-\infty$. The main result is
theorem 7.1 saying that if the critical point $x$ is nondegenerate, then this is a
contractible submanifold of the loop space and its dimension equals the Morse
index of $x$. Here we use unique continuation for the linear and the nonlinear
heat equation. In section 7.2 we put together everything to define the Morse
complex for the negative $L^2$ gradient of the action functional on the loop space.

Despite the title of this text the fact that the heat equation gives rise to
a forward semiflow is nowhere used. Existence of this semiflow will be proved and used in our forthcoming paper [W10] to construct a natural isomorphism
to singular homology of the loop space.

Notation. If $f = f(s, t)$ is a map, then $f_s$ abbreviates the map $f(s, \cdot) : t \mapsto f(s, t)$. In contrast partial derivatives are denoted by $\partial_s f$ and $\partial_t f$.

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2 The linearized heat equation

Fix a smooth function $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V3) and a smooth map $u : \mathbb{R} \times S^1 \to M$. In this chapter we study the linear parabolic PDE

$$\nabla_s \xi - \nabla_t \xi - R(\xi , \partial_t u) \partial_t u - \mathcal{H}_V(u) \xi = 0$$

for vector fields $\xi$ along $u$. Throughout $R$ denotes the Riemannian curvature tensor associated to the closed Riemannian manifold $M$ and the covariant Hessian $\mathcal{H}_V$ of $\mathcal{V}$ at a loop $u(s, \cdot)$ is defined by (4).

In section 2.1 we show that strong solutions, that is solutions of class $W^{1,p}_u$, are automatically smooth. More generally, for $\xi \in L^p_u$ we define the notion of weak solution and show that even weak solutions are smooth. In section 2.2 we derive pointwise estimates of $\xi$ and certain partial derivatives in terms of the $L^2$ norm of $\xi$ over small backward cylinders. In section 2.3 we establish asymptotic exponential decay of the slicewise $L^2$ norm $\| \xi_s \|_{L^2(S^1)}$ of a solution $\xi$ whenever the covariant Hessian $A_{\xi,s}$ given by (10) is asymptotically injective. Still assuming asymptotic injectivity we prove in section 2.4 that the linear operator

$$\mathcal{D}_u : W^{1,p}_u \to L^p_u$$

given by the left hand side of (13) is Fredholm.

Observe that if $u$ solves the (nonlinear) heat equation (6) then $\xi := \partial_s u$ solves the linear equation (13). Hence the results of this chapter will be useful in chapter 3 on solutions of the nonlinear heat equation.

2.1 Regularity

Define the operator $\mathcal{D}_u^*$ by the left hand side of (13) with $\nabla_u$ replaced by $-\nabla_u$.

**Theorem 2.1 (Local regularity of weak solutions).** Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V3) and constants $q > 1$ and $a < b$. Let $u : (a, b] \times S^1 \to M$ be a smooth map with bounded derivatives of all orders. Then the following is true. If $\eta$ is a vector field along $u$ of class $L^q_{\text{loc}}$ such that

$$\langle \eta, \mathcal{D}_u^* \xi \rangle = 0$$

for every smooth vector field $\xi$ along $u$ of compact support in $(a, b) \times S^1$, then $\eta$ is smooth. Here $\langle \cdot , \cdot \rangle$ denotes integration over the pointwise inner products.

**Remark 2.2.** Theorem 2.1 remains true if we replace $\mathcal{D}_u^*$ by $\mathcal{D}_u$ and define $u$ on $[a, b) \times S^1$. This follows by the variable substitution $s \mapsto -s$.

**Proof.** It suffices to prove the conclusion in a neighborhood of any point $z \in (a, b) \times S^1$. Shifting the $s$ and $t$ variables, if necessary, we may assume that $z \in \Omega_r = (-r^2, 0] \times (-r, r)$ for some sufficiently small $r > 0$. Now choose local coordinates on the manifold $M$ around the point $u(z)$ and fix $r > 0$ sufficiently small such that $u(\Omega_r)$ is contained in the local coordinate patch. In these local
coordinates the vector field \( \eta \) is represented by the map \( (\eta^1, \ldots, \eta^n) : \Omega_r \to \mathbb{R}^n \) of class \( \mathcal{L}^q_{\text{loc}} \) and the Riemannian metric \( g \) by the matrix with components \( g_{ij} \). Throughout we use Einstein’s sum convention. By induction we will prove that

\[
v_\mu := g_{\mu \eta} \eta^j \in \bigcap_{m=1}^{\infty} \mathcal{W}^{m,q}_{\text{loc}}(\Omega_r), \quad \mu = 1, \ldots, n.
\]

Note that the intersection of spaces equals \( \mathcal{C}^\infty(\Omega_r) \); see e.g. [MS04, app. B.1]. Now apply the inverse metric matrix to obtain that \( \eta^j = g^{\mu j} v_\mu \in \mathcal{C}^\infty(\Omega_r) \) and this proves the theorem.

**Step \( m = 1 \)**. Fix \( \mu \in \{1, \ldots, n\} \) and consider vector fields of the form

\[
\xi^{(\mu, \phi)} = (0, \ldots, 0, \phi, 0, \ldots, 0) : \Omega_r \to \mathbb{R}^n
\]

where a function \( \phi \in \mathcal{C}^\infty_0(\text{int } \Omega_r) \) occupies slot \( \mu \). Via extension by zero we view \( \xi^{(\mu, \phi)} \) as a compactly supported smooth vector field along \( u \). Now our assumption implies that \( \langle \eta, D_\mu^* \xi^{(\mu, \phi)} \rangle = 0 \) for every \( \phi \in \mathcal{C}^\infty_0(\text{int } \Omega_r) \). By straightforward calculation this is equivalent to

\[
\int_{\Omega_r} v_\mu (-\partial_u \phi - \partial_t \partial_t \phi) = \int_{\Omega_r} f_\mu \phi - \int_{\Omega_r} h_\mu \partial_t \phi
\]

for every \( \phi \in \mathcal{C}^\infty_0(\text{int } \Omega_r) \), where \( h_\mu = -2v_k \Gamma^k_{ij} \partial_i u^j \) and

\[
\begin{align*}
f_\mu = v_k \left( \Gamma^k_{ij} \partial_i u^j + \frac{\partial^2}{\partial u^r} \partial_i u^r \partial_i u^j + \Gamma^k_{ij} \partial_i \partial_j u^j \right. \\
+ \Gamma^k_{ij} \partial_i u^j \Gamma^j_{\mu r} \partial_r u^r + R^k_{ij \mu r} \partial_i u^j \partial_r u^r \left. + H^k_{ij} \right) +
\end{align*}
\]

Here \( R^k_{ij} \) represents the Riemann curvature operator and \( H^k_{ij} \) the Hessian \( H_{ij}(u) \) in local coordinates. The Christoffel symbols associated to the Levi Civita connection \( \nabla \) are denoted by \( \Gamma^k_{ij} \).

From now on the domain of all spaces will be \( \Omega_r \), unless specified differently. Observe that \( v_\mu \in \mathcal{L}^q_{\text{loc}} \subset \mathcal{L}^1_{\text{loc}} \) by smoothness of the metric, compactness of \( M \), and the fact that \( \eta^k \in \mathcal{L}^q_{\text{loc}} \) by assumption. It follows that \( h_\mu \) and \( f_\mu \) are in \( \mathcal{L}^q_{\text{loc}} \). Here we used in addition boundedness of the derivatives of \( u \) and axiom (V1). Hence \( \partial_t v_\mu \in \mathcal{L}^1_{\text{loc}} \) by theorem A.2 b) and this implies that \( \partial_t h_\mu \in \mathcal{L}^q_{\text{loc}} \). Now integration by parts shows that

\[
\int_{\Omega_r} v_\mu (-\partial_u \phi - \partial_t \partial_t \phi) = \int_{\Omega_r} (f_\mu + \partial_t h_\mu) \phi
\]

for every \( \phi \in \mathcal{C}^\infty_0(\text{int } \Omega_r) \) and therefore \( v_\mu \in \mathcal{W}^{1,q}_{\text{loc}} \) by theorem A.2 a).

**Induction step \( m \Rightarrow m + 1 \)**. Assume that \( v_\mu \in \mathcal{W}^{m,q}_{\text{loc}} \). Then \( f_\mu, h_\mu \in \mathcal{W}^{m,q}_{\text{loc}} \) by compactness of \( M \), boundedness of the derivatives of \( u \), and axiom (V3). Hence \( \partial_t v_\mu \in \mathcal{W}^{m,q}_{\text{loc}} \) by theorem A.2 b). But this implies that \( \partial_t h_\mu \) is in \( \mathcal{W}^{m,q}_{\text{loc}} \) and so is \( f_\mu + \partial_t h_\mu \). Therefore \( v_\mu \in \mathcal{W}^{m+1,q}_{\text{loc}} \) by theorem A.2 a). \( \square \)
2.2 Apriori estimates

Theorem 2.3. Fix a perturbation $V : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V2) and a constant $C_0 > 0$. Then there is a constant $C = C(C_0, V) > 0$ such that the following is true. Assume $u : \mathbb{R} \times S^1 \to M$ is a smooth map with $\|\partial_s u\|_\infty \leq C_0$ and $\xi$ is a smooth vector field along $u$ satisfying the linear heat equation (13). Then

$$|\xi(s, t)| \leq C \|\xi\|_{L^2([s-\frac{1}{2}, s] \times S^1)}$$

for every $(s, t) \in \mathbb{R} \times S^1$. If in addition $\|\partial_t u\|_\infty + \|\nabla \partial_t u\|_\infty \leq C_0$, then

$$|\nabla_t \xi(s, t)| \leq C \|\xi\|_{L^2([s-1, s] \times S^1)}$$

for every $(s, t) \in \mathbb{R} \times S^1$.

Theorem 2.4. Fix a perturbation $V : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V2) and a constant $C_0 > 0$. Then there is a constant $C = C(C_0, V) > 0$ such that the following is true. Assume $u : \mathbb{R} \times S^1 \to M$ is a smooth map with

$$\|\partial_t u\|_\infty + \|\partial_t \partial_s u\|_\infty + \|\nabla \partial_t u\|_\infty + \|\nabla \partial_t \partial_s u\|_\infty \leq C_0$$

and $\xi$ is a smooth vector field along $u$ satisfying the linear heat equation (13). Then

$$|\nabla_t \nabla_t \xi(s, t) + \nabla_s \xi(s, t)| \leq C \|\xi\|_{L^2([s-2, s] \times S^1)}$$

for every $(s, t) \in \mathbb{R} \times S^1$.

Remark 2.5. If in theorem 2.3 or theorem 2.4 the vector field $\xi$ solves $D_u^\ast \xi = 0$, then $\eta(s, t) := \xi(-s, t)$ solves (13). The apriori estimates for $\eta$ then translate into apriori estimates for $\xi$. For example, it follows that

$$|\xi(s, t)| \leq C \|\xi\|_{L^2([s, s+\frac{1}{2}] \times S^1)}$$

for every $(s, t) \in \mathbb{R} \times S^1$ and similarly for the higher order derivatives.

The proof of theorem 2.3 and theorem 2.4 is based on the following mean value inequalities. Consider the parabolic domain defined for $r > 0$ by

$$P_r := (-r^2, 0) \times (-r, r).$$

Lemma 2.6 ([SW03, lemma B.1]). There is a constant $c_1 > 0$ such that the following holds for all $r \in (0, 1]$ and $a \geq 0$. If $w : P_r \to \mathbb{R}$, $(s, t) \mapsto w(s, t)$, is $C^1$ in the $s$-variable and $C^2$ in the $t$-variable such that

$$(\partial_t \partial_t - \partial_s)w \geq -aw, \quad w \geq 0,$$

then

$$w(0) \leq C_1 e^{ar^2} \int_{P_r} w.$$
Corollary 2.7. Let $c_1$ be the constant of lemma 2.6 and fix two constants $r \in (0, 1]$ and $\mu \geq 0$. Then the following is true. If $F : [-r^2, 0] \to \mathbb{R}$ is a $C^1$ function such that 
\[-F' + \mu F \geq 0, \quad F \geq 0,\]
then
\[F(0) \leq \frac{2c_1e^{\mu r^2}}{r^2} \int_{-r^2}^{0} F(s) \, ds.\]

Proof. Lemma 2.6 with $w(s, t) := F(s)$. \qed

Lemma 2.8 ([SW03, lemma B.4]). Let $R, r > 0$ and $u : P_{R+r} \to \mathbb{R}, (s, t) \mapsto u(s, t)$, be $C^1$ in the $s$-variable and $C^2$ in the $t$-variable and $f, g : P_{R+r} \to \mathbb{R}$ be continuous functions such that
\[(\partial_t \partial_t - \partial_s) u \geq g - f, \quad u \geq 0, \quad f \geq 0, \quad g \geq 0.\]
Then
\[\int_{P_R} g \leq \int_{P_{R+r}} f + \left(\frac{4}{r^2} + \frac{1}{Rr}\right) \int_{P_{R+r} \setminus P_R} u.\]

Corollary 2.9. Fix two positive constants $r, R$ and three functions $U, F, G : [-\sqrt{(R + r)^2}, 0] \to \mathbb{R}$ such that $U$ is $C^1$ and $F, G$ are continuous. If
\[-U' \geq G - F, \quad U \geq 0, \quad F \geq 0, \quad G \geq 0,\]
then
\[\int_{-R^2}^{0} G(s) \, ds \leq \frac{R + r}{R} \left(\int_{-\sqrt{(R + r)^2}}^{0} F(s) \, ds + \left(\frac{4}{r^2} + \frac{1}{Rr}\right) \int_{-\sqrt{(R + r)^2}}^{0} U(s) \, ds\right).\]

Proof. Lemma 2.8 with $u(s, t) = U(s), f(s, t) = F(s)$, and $g(s, t) = G(s)$. \qed

Proof of theorem 2.3. We prove the theorem in three steps. The idea is to prove in step 1 the desired pointwise estimate in its integrated form (slicewise estimate). In steps 2 and 3 this is then used to prove the pointwise estimates. Note that in step 3 we provide an estimate which is not used in the current proof, but later on in the proof of theorem 2.4. Occasionally we denote $\xi(s, t)$ by $\xi_s(t)$ and in this case $\|\xi_s\|$ abbreviates $\|\xi_s\|_{L^2(S^1)}$.

Step 1. There is a constant $C_1 = C_1(C_0, \mathcal{V}) > 0$ such that
\[\int_0^1 |\xi(s, t)|^2 \, dt + \int_{s - \frac{1}{4}}^{s + \frac{1}{4}} \int_0^1 |\nabla_t \xi(s, t)|^2 \, dt \, ds \leq C_1 \|\xi\|_{L^2([s - \frac{1}{4}, s] \times S^1)}^2\]
for every $s \in \mathbb{R}$.

Define the functions $f, g : \mathbb{R} \times S^1 \to \mathbb{R}$ and $F, G : \mathbb{R} \to \mathbb{R}$ by
\[2f := |\xi|^2, \quad 2g := |\nabla_t \xi|^2, \quad F(s) := \int_0^1 f(s, t) \, dt, \quad G(s) := \int_0^1 g(s, t) \, dt,\]
and abbreviate
\[ L := \partial_t \partial_t - \partial_s, \quad \mathcal{L} := \nabla_t \nabla_t - \nabla_s. \]

Then
\[ Lf = 2g + U, \quad U := \langle \xi, \mathcal{L} \xi \rangle. \] (14)

Assume that \( U \) satisfies the pointwise inequality
\[ |U| \leq \mu f + \frac{1}{2} \|\xi_s\|^2 \] (15)

for a suitable constant \( \mu = \mu(C_0, V) > 0 \). Hence \( Lf + \mu f + F \geq 2g \) by (14) and integration over the interval \( 0 \leq t \leq 1 \) shows that
\[ -F' + (\mu + 1)F \geq 2G. \]

Step 1 follows by Corollary 2.7 with \( r = \frac{1}{2} \) and corollary 2.9 with \( R = r = \frac{1}{4} \).

It remains to prove (15). Since \( \xi \) solves the linear heat equation (13), it follows that
\[ |U| = |\langle \xi, \nabla_t \nabla_t \xi - \nabla_s \xi \rangle| 
  = |\langle \xi, R(\xi, \partial_t u) \partial_t u + H_V(u)\xi \rangle| 
  \leq \|R\|_{L^\infty} \|\partial_t u\|^2 \|\xi\|^2 + c_1 |\xi| (\|\xi\|_{L^1(S^1)} + \|\xi_s\|_{L^1(S^1)}) 
  \leq (2C_0^2 \|R\|_{L^\infty} + 2c_1 + c_1^2) \frac{1}{2} |\xi|^2 + \frac{1}{2} \|\xi_s\|^2. \]

Here we used the assumption on \( \partial_t u \), axiom (V1) with constant \( c_1 \), and the fact that \( \|\cdot\|_{L^1(S^1)} \leq \|\cdot\|_{L^2(S^1)} \) by Hölder’s inequality. This proves (15).

Step 2. We prove the estimate for \( |\xi| \) in theorem 2.3.

Note that \( Lf \geq -|U| \) by (15). Hence the estimate (15) for \( |U| \) and the slicewise estimate for \( \xi_s \) provided by step 1 prove the pointwise inequality
\[ Lf \geq -\mu f - 2C_1 \|\xi\|^2_{L^2([s_0 - \frac{1}{2}, s_0] \times S^1)} \]
for all \( s \) and \( t \). Fix \( (s_0, t_0) \) and set \( a = a(s_0) := \frac{2C_1}{\mu} \|\xi\|^2_{L^2([s_0 - \frac{1}{2}, s_0] \times S^1)} \). Then
\[ L(f + a) \geq -\mu (f + a) \]
for all \( t \) and \( s \in [s_0 - \frac{1}{2}, s_0] \). Hence lemma 2.6 with \( r = \frac{1}{2} \) applies to the function \( w(s, t) := f(s_0 + s, t_0 + t) + a \) and we obtain that
\[ f(s_0, t_0) \leq 8c_1 e^{\mu/4} \int_{-\frac{1}{2}}^{0} \int_{0}^{1} (f(s_0 + s, t_0 + t) + a) \, dt \, ds \]
\[ \leq 8c_1 e^{\mu/4} \left( \frac{1}{2} + \frac{C_1}{2\mu} \right) \|\xi\|^2_{L^2([s_0 - \frac{1}{2}, s_0] \times S^1)}. \]

Since \( s_0 \in \mathbb{R} \) and \( t_0 \in S^1 \) were chosen arbitrarily, this proves step 2.
Step 3. There is a constant $C_3 = C_3(C_0, \mathcal{V}) > 0$ such that

$$
\int_{s-\frac{r}{2}}^{s} \int_{0}^{1} |\nabla_t \nabla_t \xi(s, t)|^2 \, dt \, ds \leq C_3 \|\xi\|_{L^2([s-\frac{r}{2}, s] \times S^1)^2}^2
$$

for every $s \in \mathbb{R}$. Moreover, the estimate for $|\nabla_t \xi|$ in theorem 2.3 holds true.

Define the functions $f_1, g_1 : \mathbb{R} \times S^1 \to \mathbb{R}$ by

$$2f_1 := |\nabla_t \xi|^2, \quad 2g_1 := |\nabla_t \nabla_t \xi|^2$$

and the functions $F_1, G_1 : \mathbb{R} \to \mathbb{R}$ by

$$F_1(s) := \int_{0}^{1} f_1(s, t) \, dt, \quad G_1(s) := \int_{0}^{1} g_1(s, t) \, dt.$$  

Then

$$L f_1 = 2g_1 + U_t, \quad U_t := \langle \nabla_t \xi, \mathcal{L} \nabla_t \xi \rangle. \quad (16)$$

Since $\xi$ solves the linear heat equation (13), it follows that

$$\mathcal{L} \nabla_t \xi = \nabla_t (\nabla_t \nabla_t \xi - \nabla_t \xi) - [\nabla_t, \nabla_t] \xi$$

$$= \nabla_t (-R(\xi, \partial_t u) \partial_t u - \mathcal{H}_t(u) \xi) - R(\partial_t u, \partial_t u) \xi$$

$$= -\langle \nabla_t R(\xi, \partial_t u) \partial_t u - R(\nabla_t \xi, \partial_t u) \partial_t u - R(\xi, \nabla_t \partial_t u) \partial_t u$$

$$- R(\xi, \partial_t u) \nabla_t \partial_t u - \nabla_t \mathcal{H}_t(u) \xi - R(\partial_t u, \partial_t u) \xi \rangle.$$  

Now take the pointwise inner product of this identity and $\nabla_t \xi$ and estimate the resulting six terms separately using the $L^\infty$ boundedness assumption of the various derivatives of $u$. For instance, term five satisfies the estimate

$$|\langle \nabla_t \xi, \nabla_t \mathcal{H}_t(u) \xi \rangle| \leq c_2 |\nabla_t \xi| \left( |\nabla_t \xi| + (1 + |\partial_t u|) \left( |\xi| + \|\xi_s\|_{L^2(S^1)}^2 \right) \right)$$

by the second inequality of axiom (V2) with constant $c_2$. It follows that $U_t$ satisfies the pointwise inequality

$$|U_t| \leq \mu f_1 + \mu |\xi|^2 + \mu \|\xi_s\|_{L^2(S^1)}^2$$

for a suitable constant $\mu = \mu(C_0, \mathcal{V}) > 0$. Hence

$$L f_1 \geq 2g_1 - \mu f_1 - \mu |\xi|^2 - \mu \|\xi_s\|_{L^2(S^1)}^2 \quad (17)$$

pointwise for all $s$ and $t$. Integrate this inequality over $t \in [0, 1]$ to obtain that

$$-F'_1 \geq 2G_1 - \mu F_1 - 2\mu F$$

pointwise for every $s \in \mathbb{R}$. Then corollary 2.9 with $R = r = \frac{1}{2}$ shows that

$$\int_{s_0}^{s_1} \|\nabla_t \nabla_t \xi_s\|^2 \, ds \leq (\mu + 20) \int_{s_0}^{s_1} \|\nabla_t \xi_s\|^2 \, ds + 2\mu \int_{s_0}^{s_1} \|\xi_s\|^2 \, ds$$

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for every \( s_0 \in \mathbb{R} \). Now

\[
\int_{s_0-1}^{s_0} \|\nabla_t \xi_s\|^2 \, ds \leq 16C_1 \int_{s_0-\frac{1}{4}}^{s_0} \|\xi_s\|^2 \, ds
\]

by step 1 and this proves the first assertion of step 3. (We need this result only in the proof of theorem 2.4 below.)

To prove the second assertion of step 3, that is the estimate for \( |\nabla_t \xi| \), note that estimate (17), step 1, and step 2 imply the pointwise estimate

\[
L f_1 \geq -\mu f_1 - \mu \|\xi\|^2_{L^2([s_0-\frac{1}{4}, s_0] \times S^1)}
\]

for all \( s \) and \( t \). Here we have chosen a larger value for the constant \( \mu \). Fix \((s_0, t_0) \in \mathbb{R} \times S^1\) and set \( a = a(s_0) := \|\xi\|^2_{L^2([s_0-1, s_0] \times S^1)} \). Then

\[
L (f_1 + a) \geq -\mu (f_1 + a)
\]

for all \( t \) and \( s \in [s_0 - \frac{1}{4}, s_0] \). Hence lemma 2.6 with \( r = \frac{1}{2} \) applies to the function \( w(s, t) := f_1(s_0 + s, t_0 + t) + a \) and proves the desired estimate, namely

\[
f_1(s_0, t_0) \leq 8C_1e^{\mu/4} \int_{s_0-\frac{1}{4}}^{s_0} \int_{t_0}^{t_0+1} (f_1(s_0 + s, t_0 + t) + a) \, dtds
\]

\[
= 8C_1e^{\mu/4} \left( \frac{1}{2} \int_{s_0-\frac{1}{4}}^{s_0} \int_{t_0}^{t_0+1} |\nabla_t \xi(s, t)|^2 \, dtds + a \right)
\]

\[
\leq 8C_1e^{\mu/4} \left( 2 \|\xi\|^2_{L^2([s_0-\frac{1}{4}, s_0] \times S^1)} + \frac{1}{4} \|\xi\|^2_{L^2([s_0-1, s_0] \times S^1)} \right)
\]

for all \( s_0 \in \mathbb{R} \) and \( t_0 \in S^1 \). The final inequality uses the estimate of step 1. This concludes the proof of step 3 and theorem 2.3. \( \square \)

**Proof of theorem 2.4.** Occasionally we denote \( \xi(s, t) \) by \( \xi_s(t) \). Define the functions \( f_2, g_2 : \mathbb{R} \times S^1 \to \mathbb{R} \) by

\[
f_2 := \frac{1}{2} |\nabla \nabla_t \xi|^2, \quad g_2 := \frac{1}{2} |\nabla_t \nabla \nabla_t \xi|^2
\]

and abbreviate \( L := \partial_t \partial_t - \partial_s \) and \( \mathcal{L} := \nabla_t \nabla_t - \nabla_s \). Then

\[
L f_2 = 2g_2 + U_{tt}, \quad U_{tt} := \langle \nabla_t \nabla_t \xi, \mathcal{L} \nabla_t \nabla_t \xi \rangle.
\]

We estimate \( |U_{tt}| \). Since \( \xi \) solves the linear heat equation (13), it follows that

\[
\mathcal{L} \nabla_t \xi = \nabla_t \nabla_t (\nabla_t \nabla_t \xi - \nabla_s \xi) + \nabla_t \nabla_t \nabla_t \xi
\]

\[
= \nabla_t \nabla_t (\xi - \nabla_t \nabla_s \xi) - \nabla_t \nabla_t \nabla_s \xi + \nabla_t \nabla_s \nabla_t \xi
\]

\[
= \nabla_t \left( -\langle \nabla_t R \rangle (\xi, \partial_t u) \partial_t u - \nabla_t \nabla_s \nabla_t \xi - \nabla_t \nabla_t \nabla_s \xi \right)
\]

\[
= \nabla_t \left( -\langle \nabla_t R \rangle (\xi, \partial_t u) \partial_t u - \nabla_t \nabla_t \nabla_s \xi - \nabla_t \nabla_t \nabla_s \xi \right)
\]

\[
+ R(\partial_t u, \partial_s u) \nabla_t \xi + R(\partial_s u, \partial_t u) \nabla_t \xi + 2R(\partial_t u, \partial_s u) \nabla_t \xi.
\]
Now take the pointwise inner product of this identity and \( \nabla \nabla \xi \). Estimate the resulting sum term by term and use the assumption that various derivatives of \( u \) are bounded in \( L^\infty \). It follows that

\[
|U_t| \leq \mu_1 |\nabla \nabla \xi| (|\xi| + |\nabla \xi| + |\nabla \nabla \xi|) + |\nabla \xi| \cdot |\nabla \nabla \mathcal{H} \nu(u) | \xi|
\]

for some positive constant \( \mu_1 \) which depends only on the \( L^\infty \) bound \( C_0 \). Note that by axiom (V3) there is a positive constant \( c_3 = c_3(\nu) \) such that

\[
|\nabla \nabla \mathcal{H} \nu(u) | \xi| \leq c_3 |\nabla \xi| + c_3 (1 + |\partial_t u|) |\nabla \xi|
\]

\[
+ c_3 \left( 1 + |\partial_t u|^2 + |\nabla \partial_t u| \right) \left( |\xi| + \|\xi\|_{L^2(S^1)} \right).
\]

Hence there is a positive constant \( \mu_2 = \mu_2(C_0, \nu) \) such that

\[
|U_t| \leq \mu_2 \left( f_2 + |\nabla \xi|^2 + |\xi|^2 + \|\xi\|_{L^2(S^1)}^2 \right).
\]

Theorem 2.3 applied to the last three terms of this sum implies that

\[
|U_t| \leq \mu_2 f_2 + \mu \|\xi\|_{L^2([s_0-1, s] \times S^1)}^2
\]

pointwise for all \( s \) and \( t \) with a suitable constant \( \mu = \mu(C_0, \nu) > 0 \). Now \( L f_2 \geq -|U_t| \) by (18) and therefore

\[
L f_2 \geq -\mu f_2 - \mu \|\xi\|_{L^2([s_0-1, s] \times S^1)}^2
\]

pointwise for all \( s \) and \( t \). Fix \( s_0 \in \mathbb{R} \) and set \( a := \|\xi\|_{L^2([s_0-2, s_0] \times S^1)}, \) then

\[
L (f_2 + a) \geq -\mu (f_2 + a)
\]

for all \( t \in S^1 \) and \( s \in [s_0 - 1, s_0] \). Fix \( t_0 \in S^1 \) and apply lemma 2.6 with \( r = 1 \) to the function \( w(s, t) := f_2(s_0 + s, t_0 + t) + a \) to obtain that

\[
f_2(s_0, t_0) \leq c_1 e^\mu \int_{-1}^{0} \int_{-1}^{+1} (f_2(s_0 + s, t_0 + t) + a) \, dtds
\]

\[
= c_1 e^\mu \left( \int_{-1}^{0} \int_{-1}^{1} |\nabla \nabla \xi(s, t)|^2 \, dtds + 2a \right)
\]

\[
\leq c_1 e^\mu (4C_3 + 2) \|\xi\|_{L^2([s_0-2, s_0] \times S^1)}^2.
\]

Here the last inequality follows by the estimate of step 3 in the proof of theorem 2.3 with constant \( C_3 = C_3(C_0, \nu) > 0 \). Since \( s_0 \in \mathbb{R} \) and \( t_0 \in S^1 \) were chosen arbitrarily, the proof of the first estimate of theorem 2.4 is complete.

The second estimate, that is the one for \( |\nabla \xi| \), follows easily from the fact that \( \xi \) solves the linear heat equation (13), the estimate for \( |\nabla \nabla \xi| \) which we just proved, the estimate for \( |\xi| \) of theorem 2.3, and the estimate for \( |\mathcal{H} \nu(u) | \xi| \) provided by axiom (V1). This concludes the proof of theorem 2.4. \( \square \)
2.3 Exponential decay

Given a smooth loop \( x : S^1 \to M \) consider the linear operator defined by
\[
A_x \xi = -\nabla \nabla \xi - R(\xi, \partial_x \partial_x) \partial_x - H_{\mathcal{V}}(x) \xi
\]  
(19)
on \( L^2(S^1, x^*TM) \) with dense domain \( W^{2,2}(S^1, x^*TM) \). With respect to the \( L^2 \) inner product \( \langle \cdot, \cdot \rangle \) this operator is self-adjoint; see e.g. [W02] for the case of geometric perturbations \( V_t \) and use lemma 2.14 in the general case.

**Theorem 2.10** (Backward exponential decay). Fix a perturbation \( \mathcal{V} : \mathcal{L}M \to \mathbb{R} \) that satisfies (V0)–(V2) and a constant \( c_0 > 0 \). Then there exist positive constants \( \delta, \rho, C \) such that the following holds. Let \( x : S^1 \to M \) be a smooth loop such that \( A_x \) given by (19) is injective and \( \| \partial_x x \|_2 + \| \nabla_x \partial_x x \|_2 \leq c_0 \). Assume \( u : (-\infty, 0] \times S^1 \to M \) is a smooth map and \( T_0 > 0 \) is a constant such that
\[
\| \eta_s \|_{W^{2,2}} \leq \delta, \quad \| \partial_x u_s \|_2 + \| \nabla_x \partial_x u_s \|_2 \leq \delta,
\]
whenever \( s \leq -T_0 \). Assume further that \( \xi \) is a smooth vector field along \( u \) such that the function \( s \mapsto \| \xi_s \|_2 \) is bounded by a constant \( c = c(\xi) \) and \( \xi \) solves one of two equations
\[
\pm \nabla_x \xi - \nabla \nabla \xi - R(\xi, \partial_x \partial_x) \partial_x - H_{\mathcal{V}}(u) \xi = 0.
\]  
(20)
Then
\[
\| \xi_s \|_2^2 \leq e^{\rho(s+T_0)} \| \xi_{-T_0} \|_2^2 \leq C^2 e^{\rho(s+T_0)}
\]
and
\[
\| \xi \|_{L^2([\infty, s] \times S^1)}^2 \leq \frac{C^2}{\rho} e^{\rho(s+T_0)} \| \xi \|_{L^2([-T_0-1,-T_0] \times S^1)}^2
\]
for every \( s \leq -T_0 \).

Note the weak assumption \( \langle L^2 \text{ versus } L^\infty \rangle \) on the \( s \)-derivatives of \( \partial_x u_s \) and its base component \( u_s \). To prove theorem 2.10 we need two lemmas.

**Remark 2.11** (Forward exponential decay). If the domain of \( u \) is the forward half cylinder \([0, \infty) \times S^1 \) and the vector field \( \xi \) along \( u \) solves \( \pm(20) \), then theorem 2.10 applies to \( v(\sigma, t) := u(-\sigma, t) \) and \( \eta(\sigma, t) := \xi(-\sigma, t) \), since \( \eta \) solves \( \mp(20) \). The estimates obtained for \( \eta \) provide estimates for \( \xi \), for instance
\[
\| \xi \|_{L^2([\sigma, \infty] \times S^1)}^2 \leq \frac{C^2}{\rho} e^{\rho(-\sigma+T_0)} \| \xi \|_{L^2([-T_0, T_0+1] \times S^1)}^2
\]
for every \( \sigma \geq T_0 \).

**Lemma 2.12** (Stability of injectivity). Fix a perturbation \( \mathcal{V} : \mathcal{L}M \to \mathbb{R} \) that satisfies (V0)–(V2) and a constant \( c_0 > 1 \). Then there are constants \( \mu, \delta_0 > 0 \) such that the following holds. If \( x \) and \( \gamma \) are smooth loops in \( M \) such that the operator \( A_x \) is injective and
\[
\gamma = \exp_x(\eta), \quad \| \eta \|_{W^{2,2}} \leq \delta_0, \quad \| \partial_x x \|_2 + \| \nabla_x \partial_x x \|_2 \leq c_0,
\]
then
\[
\| \xi \|_2 + \| \nabla \xi \|_2 + \| \nabla \nabla \xi \|_2 \leq \mu \| A_{\gamma} \xi \|_2
\]
for every \( \xi \in \Omega^0(S^1, \gamma^*TM) \).
Lemma 2.13. Let $A_x$ be a $C^2$ function on the interval $(-\infty, -T_0]$. If $f$ is bounded by a constant $c$ and satisfies the differential inequality $f'' \geq \rho^2 f$ for some constant $\rho \geq 0$, then

$$f(s) \leq e^{\rho(s+T_0)} f(-T_0)$$

for every $s \leq -T_0$.

Proof. Although the argument is standard, see e.g. [DS94], we provide the details for the sake of completeness. The main point is to observe that $f'(s) - \rho f(s) \geq 0$ for every $s \leq -T_0$. To see this assume by contradiction that $f'(s_0) - \rho f(s_0) < 0$ for some time $s_0 \leq -T_0$. Note that the function $g(s) = e^{\rho s} (f'(s) - \rho f(s))$ satisfies $g' \geq 0$ on $(-\infty, -T_0]$. Hence $g(s) \leq g(s_0)$, or equivalently

$$f'(s) \leq e^{\rho(s-s_0)} (f'(s_0) - \rho f(s_0)) + \rho c$$

for every $s \leq s_0$. It follows that $f'(s) \to -\infty$ as $s \to -\infty$ and therefore

$$\int_s^{s_0} f'(\sigma) \, d\sigma \to -\infty, \quad as \ s \to -\infty.$$  

But this contradicts the fact that by boundedness of $f$

$$\int_s^{s_0} f'(\sigma) \, d\sigma = f(s_0) - f(s) \geq -c$$

for every $s \leq s_0$. To conclude the proof consider the function $h(s) = e^{-\rho s} f(s)$ on the interval $(-\infty, -T_0]$. It follows from the observation above that $h' \geq 0$. Hence $h(s) \leq h(-T_0)$ for every $s \leq -T_0$ and this proves the lemma. \qed
To prove theorem 2.10 it is useful to denote \( \exp_u(\xi) \) by \( E(u, \xi) \) and define linear maps

\[
E_i(u, \xi) : T_u M \to T_{\exp u \xi} M, \quad E_{ij}(u, \xi) : T_u M \times T_u M \to T_{\exp u \xi} M
\]

for \( \xi \in T_x M \) and \( i, j \in \{1, 2\} \). If \( u : \mathbb{R} \to M \) is a smooth curve and \( \xi, \eta \) are smooth vector fields along \( u \), then the maps \( E_i \) and \( E_{ij} \) are characterized by the identities

\[
\frac{d}{ds} \exp_u(\xi) = E_1(u, \xi) \partial_s u + E_2(u, \xi) \nabla u, \xi \\
\nabla_s (E_1(u, \xi) \eta) = E_{11}(u, \xi) (\eta, \partial_s u) + E_{12}(u, \xi) (\eta, \nabla u, \xi) + E_1(u, \xi) \nabla \eta, \xi \\
\nabla_s (E_2(u, \xi) \eta) = E_{21}(u, \xi) (\eta, \partial_s u) + E_{22}(u, \xi) (\eta, \nabla u, \xi) + E_2(u, \xi) \nabla \eta, \xi.
\]

These maps satisfy the symmetry properties

\[
E_{12}(u, \xi) (\eta, \eta') = E_{21}(u, \xi) (\eta', \eta), \quad E_{22}(u, \xi) (\eta, \eta') = E_{22}(u, \xi) (\eta', \eta),
\]

and the identities

\[
E_{11}(u, 0) = E_{12}(u, 0) = E_{22}(u, 0) = 0, \quad E_1(u, 0) = E_2(u, 0) = 1.
\]

Alternatively \( E_2 \) can be defined by

\[
E_2(u, \xi) \eta := \left. \frac{d}{d\tau} \right|_{\tau=0} \exp_u(\xi + \tau \eta)
\]

for \( \xi, \eta \in T_u M \) and \( \tau \in \mathbb{R} \). An explicit definition of \( E_1 \) and the maps \( E_{ij} \) can be given in local coordinates.

**Proof of theorem 2.10.** Fix \( c_0 \) and \( V \) and let \( \mu \) and \( \delta_0 \) be the constants of lemma 2.12 and \( C \) be the constant of theorem 2.3 with this choice. Set \( \delta := \delta_0 \) and suppose \( u, x, T_0, \xi \) satisfy the assumptions of the theorem. Then lemma 2.12 for \( \gamma = u_s \) and vector fields \( \eta = \eta_s \) and \( \xi = \xi_s \) asserts that

\[
\|\xi_s\|_2^2 + \|\nabla \xi_s\|_2^2 + \|\nabla \nabla \xi_s\|_2^2 \leq \mu^2 \|A_{us} \xi_s\|_2^2 = \mu^2 \|\nabla \xi_s\|_2^2
\]

whenever \( s \leq -T_0 \). The last step uses the consequence \( \nabla \xi_s = \mp A_{us} \xi_s \) of (19) and (20). From now on we assume that \( s \leq -T_0 \). Observe that

\[
\partial_s u_s = E_1(x, \eta_s) \partial_t x + E_2(x, \eta_s) \nabla \eta_s \\
\nabla_t \partial_s u_s = E_{11}(x, \eta_s) (\partial_t x, \partial_s x) + 2E_{12}(x, \eta_s) (\partial_t x, \nabla \eta_s) + E_1(x, \eta_s) \nabla_t \partial_t x \\
\quad + E_{22}(x, \eta_s) (\nabla \eta_s, \nabla \eta_s) + E_2(x, \eta_s) \nabla_t \nabla \eta_s.
\]

By the identities (23) we can choose \( \delta > 0 \) smaller, if necessary, such that

\[
\|\partial_t u_s\|_2 \leq \|E_1(x, \eta_s)\|_\infty \|\partial_t x\|_2 + \|E_2(x, \eta_s)\|_\infty \|\nabla \eta_s\|_2 \leq 2c_0.
\]

and, similarly, that \( \|\nabla \partial_t u_s\|_2 \leq 2c_0 \).
Claim. Consider the function
\[ F(s) := \frac{1}{2} \| \xi_s \|_2^2 = \frac{1}{2} \int_0^1 |\xi(s,t)|^2 \, dt. \]

Then there is a sufficiently small constant \( \delta > 0 \) such that
\[ F''(s) \geq \frac{1}{\mu^2} F(s) \]
whenever \( s \leq -T_0 \).

Before proving the claim we show how it implies the conclusions of theorem 2.10. Set \( \rho = \rho(c_0, \mathcal{V}) := 1/\mu \), then \( F'' \geq \rho^2 F \) on \(( -\infty, T_0 )\). Hence lemma 2.13 proves the first conclusion of theorem 2.10. Use this conclusion, the fact that \( \| \cdot \|_2 \leq \| \cdot \|_\infty \) on the domain \( S^1 \), and theorem 2.3 with constant \( C = C(c_0, \mathcal{V}) \) to obtain that
\[ \| \xi_s \|_2^2 \leq e^{\rho(s+T_0)} \| \xi_{-T_0} \|_\infty^2 \leq C^2 e^{\rho(s+T_0)} \| \xi \|_{L^2([-T_0-1,-T_0] \times S^1)}^2 \]
whenever \( s \leq -T_0 \). Fix \( \sigma \leq -T_0 \) and integrate this estimate over \( s \in ( -\infty, \sigma ] \). This proves the final conclusion of theorem 2.10.

It remains to prove the claim. In the following calculation we drop the subindex \( s \) for simplicity and denote the \( L^2(S^1) \) inner product by \( \langle \cdot, \cdot \rangle \). By straightforward computation it follows that
\[ F''(s) = \| \nabla_s \xi_s \|_2^2 + \langle \xi, \nabla_s \nabla_s \xi \rangle \]
and
\[
\langle \xi, \nabla_s \nabla_s \xi \rangle = \pm \langle \xi, \nabla_s ( \nabla_s \nabla_t \xi + R(\xi, \partial_t u) \partial_t u + \mathcal{H}(u) \xi) \rangle \\
= \pm \langle \xi, \nabla_s ( \nabla_s \nabla_t \xi + \nabla_s \nabla_t \nabla_s \xi + \nabla_s (R(\xi, \partial_t u) \partial_t u + \mathcal{H}(u) \xi) \rangle \\
= \pm \langle \nabla_s \nabla_s \xi, \nabla_s \xi \rangle \\
= \pm \langle \nabla_s \xi - R(\xi, \partial_t u) \partial_t u - \mathcal{H}(u) \xi, \nabla_s \xi \rangle \\
= \pm \langle \xi, (\nabla_s R)(\partial_s u, \partial_t u) \xi + R(\nabla_s \partial_s u, \partial_t u) \xi + R(\partial_s u, \nabla_s \partial_t u) \xi + 2R(\partial_s u, \partial_t u) \nabla_s \xi + \nabla_s (\nabla_s R)(\xi, \partial_t u) \partial_t u + R(\xi, \nabla_s \partial_t u) \partial_t u + R(\xi, \nabla_s \partial_t u) \nabla_s \partial_t u + \nabla_s \mathcal{H}(u) \xi) \rangle \\
= \| \nabla_s \xi \|_2^2 \pm \langle \xi, \nabla_s \mathcal{H}(u) \xi - \mathcal{H}(u) \nabla_s \xi \rangle \\
= \pm \langle \xi, (\nabla_s R)(\partial_s u, \partial_t u) \xi + 2R(\xi, \partial_t u) \nabla_s \partial_s u + R(\partial_s u, \nabla_s \partial_t u) \xi + 2R(\partial_s u, \partial_t u) \nabla_s \xi + (\nabla_s R)(\xi, \partial_t u) \partial_t u \rangle.
\]

To obtain the first and the fourth step we replaced \( \xi \) according to (20). The third step is by integration by parts. In the final step we used twice the first Bianchi identity and lemma 2.14 on symmetry of the Hessian. Note that the term \( \nabla_s \partial_s u \) forces us to assume \( W^{1,2} \) and not only \( L^\infty \) smallness of \( \partial_s u \).
Abbreviate \( \| \cdot \|_{1,2} := \| \cdot \|_{W^{1,2}(S^1)} \) and assume from now on that \( s \leq -T_0 \). Recall that \( \| \partial_s u_s \|_\infty \leq c_1 \| \partial_\zeta u_s \|_{1,2} \leq 4c_0 c_1 \) where \( c_1 \) is the Sobolev constant of the embedding \( W^{1,2}(S^1) \to C^0(S^1) \). Then the former two identities imply that
\[
F''(s) \geq 2 \| \nabla_s \zeta_s \|_2^2 - C_1 (\| \partial_s u_s \|_\infty + \| \nabla_\partial \partial_s u_s \|_2) \left( \| \zeta_s \|_\infty^2 + \| \zeta_s \|_\infty \| \nabla \zeta \|_2 \right)
\]
\[
\geq 2 \| \nabla_s \zeta_s \|_2^2 - C_2 \| \partial_s u_s \|_{1,2} \| \zeta_s \|_{1,2}^2
\]
for positive constants \( C_1 = C_1(c_0, c_1, V, \| R \|_{C^2}) \) and \( C_2 = C_2(c_1, C_1) \). Choose \( \delta > 0 \) again smaller, if necessary, namely such that \( \delta < 1/(2\mu^2 C_2) \). Hence
\[
\| \partial_s u_s \|_{1,2} \leq \delta < \frac{1}{2\mu^2 C_2}
\]
where the first inequality is by assumption. Therefore
\[
F''(s) \geq 2 \| \nabla_s \zeta_s \|_2^2 - \frac{1}{2\mu^2} \| \zeta_s \|_{1,2}^2 \geq \| \nabla_s \zeta_s \|_2^2
\]
where the second inequality is by (24). But
\[
\| \nabla_s \zeta_s \|_2^2 \geq \frac{1}{\mu^2} \| \zeta_s \|_2^2 = \frac{2}{\mu^2} F(s)
\]
again by (24) and definition of \( F \). This proves the claim and theorem 2.10. \( \square \)

**Lemma 2.14 (Symmetry of the Hessian).** Fix a smooth map \( \mathcal{V} : \mathcal{L} \to \mathbb{R} \) and let \( x : S^1 \to M \) be a smooth loop. Then
\[
\langle \mathcal{H}_x(x) \xi, \eta \rangle = \langle \xi, \mathcal{H}_x(x) \eta \rangle
\]
for all smooth vector fields \( \xi \) and \( \eta \) along \( x \).

**Proof.** Let \( h : \mathbb{R}^2 \to \mathcal{L} \), \( (\sigma, \tau) \mapsto h(\sigma, \tau) \) be a smooth map such that
\[
h(0, 0) = x, \quad \frac{\partial}{\partial \sigma} \bigg|_{0} h(\sigma, 0) = \xi, \quad \frac{\partial}{\partial \tau} \bigg|_{0} h(0, \tau) = \eta.
\]
Observe that
\[
\frac{\partial^2}{\partial \tau \partial \sigma} \bigg|_{(0,0)} \mathcal{V}(h(\sigma, \tau))
\]
\[
= \frac{d}{d\tau} \bigg|_{0} \mathcal{V} \bigg|_{h(0, \tau)} \left( \frac{\partial}{\partial \sigma} \bigg|_{0} h(\sigma, \tau) \right)
\]
\[
= \frac{d}{d\tau} \bigg|_{0} \left( \text{grad} \mathcal{V} \bigg|_{h(0, \tau)}, \frac{\partial}{\partial \sigma} \bigg|_{0} h(\sigma, \tau) \right)
\]
\[
= \left( \frac{D}{d\tau} \bigg|_{0} \text{grad} \mathcal{V} \bigg|_{h(0, \tau)}, \frac{\partial}{\partial \sigma} \bigg|_{0} h(\sigma, 0) \right) + \left( \text{grad} \mathcal{V} \bigg|_{x}, \frac{D}{d\tau} \bigg|_{0} \frac{\partial}{\partial \sigma} \bigg|_{0} h(\sigma, \tau) \right)
\]
\[
= \langle \mathcal{H}_x(x) \eta, \xi \rangle + \left( \text{grad} \mathcal{V} \bigg|_{x}, \frac{D}{d\tau} \bigg|_{0} \frac{\partial}{\partial \sigma} \bigg|_{0} h(\sigma, \tau) \right).
\]
Now interchange the order of partial differentiation and use the fact that this is still valid for two-parameter maps. \( \square \)
2.4 The Fredholm operator

**Hypothesis 2.15.** Throughout this section we fix a perturbation $V$ that satisfies (V0)–(V3) and two nondegenerate critical points $x^\pm$ of $S_V$. Fix a smooth map $u : \mathbb{R} \times S^1 \to M$ such that $u_s$ converges to $x^\pm$ in $W^{2,2}(S^1)$ and $\partial_s u_s$ converges to zero in $W^{1,2}(S^1)$, as $s \to \pm\infty$. Moreover, assume that $\|\nabla^2 u_s\|_2$ is bounded, uniformly in $s \in \mathbb{R}$; see footnote below. Set $x = x^-$ and $y = x^+$.

Note that by theorem 1.9, proved in section 3.4 below, these assumptions are satisfied if $S_V$ is Morse and $u$ is a finite energy solution of the heat equation (6). On the other hand, the hypothesis guarantees that the assumptions of the exponential decay theorem 2.10 and the local regularity theorem 2.1 – only here (V3) is needed – are satisfied. More precisely, set $a = \max\{S_V(x), S_V(y)\}$.

Then (5) and (7) imply that

$$
\|\partial_s u_s\|_2^2 = 2a + 2V(x) \leq 2(a + C_0), \quad \|\nabla \partial_s u_s\|_2 = \|\nabla V(x)\|_2 \leq C_0.
$$

Here $C_0 > 0$ is the constant in axiom (V0). Similar estimates hold true for $y$. Precisely as in the proof of theorem 2.10 it follows that $T = T(u) > 0$ can be chosen sufficiently large such that

$$
\|\partial_t u_s\|_2^2 \leq 2c_0, \quad \|\nabla \partial_s u_s\|_2 = \|\nabla \nabla V(x)\|_2 \leq 2c_0
$$

whenever $|s| \geq T_0$ and where $c_0 = 2(|a| + C_0)$. Hence by smoothness of $u$ and compactness of the remaining domain $[-T, T] \times S^1$ we conclude that

$$
\|\partial_t u_s\|_\infty \leq c_1 \|\partial_t u_s\|_{W^{1,2}} \leq c_2
$$

(25)

for every $s \in \mathbb{R}$ and where $c_2 = c_2(x, y, u, V)$. Similarly it follows that

$$
\|\partial_s u_s\|_\infty \leq c_1 \|\partial_s u_s\|_{W^{1,2}} \leq c_3
$$

(26)

for every $s \in \mathbb{R}$ and some constant $c_3 = c_3(x, y, u, V)$.

Now consider the linear operator $D_u$ given by

$$
D_u \xi = \nabla \xi - \nabla \nabla \xi - R(\xi, \partial_t u)\partial_t u - H_V(u)\xi
$$

(27)

for smooth vector fields $\xi$ along $u$. Recall that $R$ denotes the Riemannian curvature tensor on $M$. The operator $D_u$ arises, for instance, by linearizing the heat equation (6) at a solution $u$; see [W99, app. A.2]. Recall the definition of the Banach spaces $L^p_u$ and $W^{1,p}_u$ and their norms in (12). The goal of this section is to prove that $D_u : W^{1,p}_u \to L^p_u$ is a **Fredholm operator** whenever $p > 1$ and $u$ satisfies nondegenerate asymptotic boundary conditions as in hypothesis 2.15. By definition this means that $D_u$ is a bounded linear operator with closed range and finite dimensional kernel and cokernel. The difference of these dimensions is called the **Fredholm index** of $D_u$ and denoted by index $\text{ind} D_u$. The **formal adjoint operator** $D^*_u : W^{1,p}_u \to L^p_u$ with respect to the $L^2$-inner product has the form

$$
D^*_u \xi = -\nabla \xi - \nabla \nabla \xi - R(\xi, \partial_t u)\partial_t u - H_V(u)\xi.
$$

(28)
We proceed as follows. In the case $p = 2$ we show that our situation suits the assumptions of [RS95] where the Fredholm property is proved. Then we reduce the case $p > 1$ to the case $p = 2$ by proving that the kernel and the cokernel do actually not depend on $p$. The argument is based on exponential decay and local regularity, theorem 2.10 and theorem 2.1, respectively.

**Fredholm property and index for $p = 2$**

To prove that $D_u$ is Fredholm it is useful to choose a representation with respect to an orthonormal frame along $u$. However, since $M$ is not necessarily orientable, a frame which is periodic in the $t$-variable might not exist. Hence, given a smooth map $u : R \times S^1 \to M$, we define

$$
\sigma = \sigma(u) := \begin{cases} 
+1, & \text{if } u^*TM \to R \times S^1 \text{ is trivial} \\
-1, & \text{else}
\end{cases}
$$

and $E_\sigma := \text{diag}(\sigma, 1, \ldots, 1) \in R^{n \times n}$. The orthogonal group $O(n)$ has two connected components, one contains $E_1 = I$ and the other one $E_{-1}$. Hence there exists an orthonormal frame $\phi = \phi_\sigma : R \times [0, 1] \to u^*TM$ such that $\phi(s, 1) = \phi(s, 0)E_\sigma$ for all $s \in R$. The vector space of smooth sections of $u^*TM$ is isomorphic to the space $C^\infty_\sigma$ of all maps $X \in C^\infty(R \times [0, 1], R^n)$ such that $X(s, 1) = E_\sigma X(s, 0)$, for every $s \in R$, and such that this condition also holds for all derivatives of $X$ with respect to the $t$-variable.

Denote by $W$ the closure of $C^\infty_\sigma$ with respect to the Sobolev $W^{2,2}$ norm and by $H$ its closure with respect to the $L^2$ norm. Then $D_u : W^{1,2} \to L^2$ given by (27) is represented by the Atiyah-Patodi-Singer type operator

$$
D_{A+C} := \phi^{-1}D_u\phi = \frac{d}{ds} + A(s) + C(s)
$$

from $W^{1,2} := L^2(R, W) \cap W^{1,2}(R, H)$ to $L^2(R, H)$. Here $A(s)$ is the family of symmetric second order operators on $H$ with dense domain $W$ given by

$$
A(s) = -\frac{d^2}{dt^2} - B(s, t) - Q(s, t)
$$

where

$$
Q = \phi^{-1}R(\phi, \partial_t u)\partial_t u + \phi^{-1}\mathcal{V}(u)\phi
$$

and

$$
B = (\partial_t P) + 2P\partial_t + P^2.
$$

The families of skew-symmetric matrices $P(s, t)$ and $C(s, t)$ are determined by the identities

$$
\phi^{-1}\nabla_t \phi = \partial_t + P, \quad \phi^{-1}\nabla_s \phi = \partial_s + C.
$$

Hypothesis 2.15 implies that $\partial_s u_s$ converges to zero in $C^0(S^1)$, as $s \to \pm \infty$, and therefore $\lim_{s \to \pm \infty} C(s, t) = 0$, uniformly in $t$. It follows that the family $C(s)$ of bounded operators on $H$ - defined pointwise by matrix multiplication with
The inclusion of Hilbert spaces \( W \hookrightarrow H \) is compact with dense image.

(ii) The operator \( A(s) : H \to H \) with dense domain \( W \) is unbounded and self-adjoint for every \( s \).

(iii) The norm of \( W \) is equivalent to the graph norm of \( A(s) \) for every \( s \).

(iv) The map \( \mathbb{R} \to \mathcal{L}(W, H) : s \mapsto A(s) \) is continuously differentiable with respect to the weak operator topology.

(v) There exist invertible operators \( A^\pm \in \mathcal{L}(W, H) \) which are the limits of \( A(s) \) in the norm topology, as \( s \) tends to \( \pm \infty \).

Statements (i) and (ii) follow by the Sobolev embedding theorem, the well known fact that the 1-dimensional Laplacian \( -d^2/dt^2 \) on \([0, 1]\) with periodic boundary conditions is self-adjoint, and by the Kato-Rellich Theorem since the perturbation \( B + Q \) is of relative bound zero; see [ReS75]. To prove (iii) one has to establish that the \( W \) norm is bounded above by a constant times the graph norm and vice versa. The first inequality uses the elliptic estimate for the operator \( A(s) \) and the second one follows since \( \|\partial_t u_s\|_\infty \) and \( \|\nabla \partial_t u_s\|^2 \) are bounded by (25) and the Hessian \( \mathcal{H}_V(u_s) \) is a bounded linear operator on \( L^2(S^1, u_s^* TM) \) by axiom (V1). To prove (iv) we need to show that, given any \( \xi \in W \) and \( \eta \in H \), the map \( s \mapsto \langle \eta, A(s)\xi \rangle \) is in \( C^1(\mathbb{R}, \mathbb{R}) \). This follows by the bounds in (25) and (26), by the final estimate in axiom (V2), and the apparently unnatural\(^2\) assumption in hypothesis 2.15 that \( \nabla \nabla \partial_t u_s \) be uniformly \( L^2 \) bounded. Statement (v) is true, since the critical points \( x^\pm \) are nondegenerate and \( u_s \) and \( \partial_t u_s \) converge in \( C^0 \) to \( x^\pm \) and \( \partial_t x^\pm \), respectively, and \( \nabla \partial_t u_s \) converges in \( L^2 \) to \( \nabla \partial_t x^\pm \), all as \( s \to \pm \infty \).

The properties (i–v) are precisely the assumptions of theorem A in [RS95] which asserts that the operator \( D_A : W^{1,2} \to L^2 \) is Fredholm and its index is given by the spectral flow of the operator family \( A(s) \). The spectral flow represents the net change in the number of negative eigenvalues of \( A(s) \) as \( s \) runs from \(- \infty \) to \( \infty \). It is equal to \( \text{ind}(A^+) - \text{ind}(A^-) \) where \( \text{ind}(A^\pm) \) denotes the Morse index, i.e. the number of negative eigenvalues of the self-adjoint operator \( A^\pm \). To see this observe that \( \text{ind}(A^+) \) equals \( \text{ind}(A^-) \) plus the number of eigenvalues changing from positive to negative minus the number of those changing sign in the opposite direction. Finally, the Fredholm indices of \( D_A \) and \( D_{A+C} \) are equal, since \( \{D_{A+C}\}_{\tau \in [0,1]} \) is an interpolating family of Fredholm operators. This proves theorem 1.10 in the case \( p = 2 \).

\(^2\)If in [RS95, thm.A], hence in (iv), \textit{continuously differentiable} could be replaced by \textit{continuous}, then the assumption on \( \|\nabla \nabla \partial_t u_s\|_2 \) can be dropped in hypothesis 2.15 and theorem 1.10.
Remark 2.16 (The formal adjoint). If $\mathcal{D}_u : \mathcal{W}^1_2 \to \mathcal{L}^2_u$ is represented with respect to an orthonormal frame by the operator $D_{A+C}$ in (29), then $\mathcal{D}_u^* \equiv -D_{-A-C}$. Above we proved that $A$ satisfies (i-v), hence so does $-A$. Thus $D_{-A}$ is a Fredholm operator again by [RS95, thm. A] and its index is given by minus the spectral flow of the operator family $A = A(s)$. But if $D_{-A}$ is Fredholm, so is its negative $-D_{-A}$ and both Fredholm indices are equal, since both kernels and both cokernels coincide. Now $-D_{-A}$ and $-D_{-A-C}$ are homotopic through the family $\{ -D_{-A-\tau C} \}_{\tau \in [0,1]}$ of Fredholm operators. This proves that the formal adjoint operator $\mathcal{D}_u^* : \mathcal{W}^1_2 \to \mathcal{L}^2_u$ is Fredholm and index $\text{ind}(\mathcal{D}_u^*) = -\text{ind}(\mathcal{D}_u)$.

Fredholm property and index for $p > 1$

Still assuming hypothesis 2.15 consider the vector space given by

$$X_0 := \left\{ \xi \in C^\infty(\mathbb{R} \times S^1, u^* TM) \mid \mathcal{D}_u \xi = 0, \exists c, \delta > 0 \forall s \in \mathbb{R} : \|\xi\|_\infty + \|\nabla_s \xi\|_\infty + \|\nabla_\tau \xi\|_\infty + \|\nabla_\tau \xi\|_\infty \leq ce^{\delta |s|} \right\}.$$ 

Define $X_0^*$ by using $\mathcal{D}_u^*$ in the definition. Note that $p$ does not enter.

Proposition 2.17. Let $p > 1$, then

$$\ker \left[ \mathcal{D}_u : \mathcal{W}^{1,p}_u \to \mathcal{L}^p_u \right] = X_0,$$

and

$$\ker \left[ \mathcal{D}_u^* : \mathcal{W}^{1,p}_u \to \mathcal{L}^p_u \right] = X_0^*.$$ 

Proof. The inclusion $\supset$ is trivial. To prove the inclusion $\subset$ assume that $\xi \in \mathcal{W}^{1,p}_u$ solves $\mathcal{D}_u \xi = 0$ almost everywhere. Being a local property smoothness of $\xi$ follows from theorem 2.1 using integration by parts. Exponential $L^\infty$ decay follows by combining the apriori estimates theorem 2.3 and theorem 2.4 with the $L^2$ exponential decay results theorem 2.10 and remark 2.11. The last two results require nondegeneracy of the critical points $x^\pm$ and boundedness of the map $s \mapsto \|\xi_s\|_2$. To see the latter note that $\|\xi_s\|_p$ and $\|\nabla_s \xi_s\|_p$ converge to zero as $s \to \pm \infty$, because $\xi$ and $\nabla_s \xi$ are $L^p$ integrable on $\mathbb{R} \times S^1$. Hence $\|\xi_s\|_p + \|\nabla_s \xi_s\|_p \leq C$ for some constant $C = C(p, \xi)$. Now observe that

$$\|\xi_s\|_2 \leq \|\xi_s\|_\infty \leq c_d \left( \|\xi_s\|_p + \|\nabla_s \xi_s\|_p \right)$$

by the Sobolev embedding $W^{1,p}(S^1) \hookrightarrow C^0(S^1)$ with constant $c_d$. This proves that $X_0$ is the kernel of $\mathcal{D}_u$. The result for $\mathcal{D}_u^*$ follows by reflection $s \mapsto -s$. □

Proposition 2.18. The range of $\mathcal{D}_u, \mathcal{D}_u^* : \mathcal{W}^{1,p}_u \to \mathcal{L}^p_u$ is closed whenever $p > 1$.

Proof. The structure of proof is standard; see e.g. [S99, sec. 2]. We sketch the two key steps for $\mathcal{D}_u$. Step one is the linear estimate

$$\|\xi\|_{W^{1,p}} \leq c_p \left( \|\mathcal{D}_u \xi\|_p + \|\xi\|_p \right)$$

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for compactly supported vector fields $\xi$ along $u$. This follows immediately from proposition A.4, lemma A.3, the $L^\infty$ bound for $\partial_t u$ in (25) and axiom (V1). Step two is to prove bijectivity of $D_u$ in the case of the constant cylinder $u(s, t) = x(t)$, whenever $x$ is a nondegenerate critical point of $S_V$. We give a proof for $p \geq 2$ in the related situation of half cylinders in theorem 7.5 below. The case $1 < p \leq 2$ follows by duality; see [S99, exc. 2.5]. Both steps are then combined by a cutoff function argument; see [S99, thm 2.2].

Proposition 2.18 enables us to define the cokernels of $D_u : \mathcal{W}^{1,p}_u \rightarrow \mathcal{L}^p_u$ and $D^*_u : \mathcal{W}^{1,p}_u \rightarrow \mathcal{L}^p_u$ as Banach space quotients, namely for $p > 1$ set

$$\text{coker } D_u := \frac{\mathcal{L}^p_u}{\text{im } D_u},$$

and

$$\text{coker } D^*_u := \frac{\mathcal{L}^p_u}{\text{im } D^*_u}.$$ 

The next result shows that these spaces are again independent of $p$.

**Proposition 2.19.** Let $p > 1$, then

$$\text{coker } [D_u : \mathcal{W}^{1,p}_u \rightarrow \mathcal{L}^p_u] = X^*_0,$$

and

$$\text{coker } [D^*_u : \mathcal{W}^{1,p}_u \rightarrow \mathcal{L}^p_u] = X_0.$$ 

**Proof.** We prove the second identity. The other one follows by reflection $s \mapsto -s$. Note that there is a natural complement of the image of $D^*_u$ in $\mathcal{L}^p_u$, namely its orthogonal complement with respect to the $L^2$ inner product. Hence we identify

$$\text{coker } D^*_u \simeq (\text{im } D^*_u)^\perp.$$ 

The inclusion $\supset$ is trivial. To prove the inclusion $\subseteq$ assume that $\xi \in (\text{im } D^*_u)^\perp$. This means that $\xi \in \mathcal{L}^p_u$ and that $\langle \xi, D^*_u \eta \rangle = 0$ for all $\eta \in C_0^\infty(\mathbb{R} \times S^1)$. Hence $\xi$ is smooth by theorem 2.1. Integration by parts then shows that $D_u \xi = 0$. Exponential decay follows by combining theorem 2.3 and theorem 2.4 with theorem 2.10 and remark 2.11 as explained in the proof of proposition 2.17.

**Remark 2.20.** It is an easy but important consequence of proposition 2.19 that if $D_u : \mathcal{W}^{1,p}_u \rightarrow \mathcal{L}^p_u$ is surjective for some $p > 1$, then it is surjective for all $p > 1$. This justifies the phrase “$D_u$ is surjective” encountered occasionally.

**Proof of theorem 1.10.** The range of $D_u : \mathcal{W}^{1,p}_u \rightarrow \mathcal{L}^p_u$ is closed by proposition 2.18. Moreover, by proposition 2.17 and proposition 2.19 the kernel and the cokernel of $D_u : \mathcal{W}^{1,p}_u \rightarrow \mathcal{L}^p_u$ are given by $X_0$ and $X^*_0$, respectively. Now these vector spaces do not depend on $p > 1$. But for $p = 2$ we proved in the previous subsection that they are finite dimensional and the difference of their dimensions equals $\text{ind}_V(x^-) - \text{ind}_V(x^+)$. The claim for $D^*_u$ follows similarly.
3 Solutions of the nonlinear heat equation

3.1 Regularity and compactness

Throughout this subsection we embed the compact Riemannian manifold $M$ isometrically into some Euclidean space $\mathbb{R}^N$ and view any continuous map $u : Z = (-T,0] \times S^1 \to M$ as a map into $\mathbb{R}^N$ taking values in the embedded manifold. We indicate this by the notation $u : Z = (-T,0] \times S^1 \to M$ as a map into $\mathbb{R}^N$ taking values in the embedded manifold. Then the heat equation (6) is of the form

$$\partial_s u - \partial_t \partial_t u = \Gamma(u) (\partial_t u, \partial_t u) + F.$$  \hspace{1cm} (30)

Here and throughout this section $\Gamma$ denotes the second fundamental form associated to the embedding $M \hookrightarrow \mathbb{R}^N$ and the map $F : Z \to \mathbb{R}^N$ is given by

$$F(s,t) := (\text{grad} V(u_t))(t).$$  \hspace{1cm} (31)

Recall the definition of the $W^{k,p}$ and the $C^k$ norm in (98) and (99), respectively.

**Proposition 3.1.** Fix a perturbation $V : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V3), constants $p > 2$ and $\mu_0 > 0$, and cylinders

$$Z = (-T,0] \times S^1, \quad Z' = (-T',0] \times S^1, \quad T > T' > 0.$$

Then for every integer $k \geq 1$ there is a constant $c_k = c_k(p,\mu_0,T,T',V)$ such that the following is true. If $u : Z \to M \hookrightarrow \mathbb{R}^N$ is a $W^{1,p}$ map such that

$$\|u\|_p + \|\partial_s u\|_p + \|\partial_t u\|_p + \|\partial_t \partial_t u\|_p \leq \mu_0$$

and which satisfies the heat equation (30) almost everywhere, then

$$\|u\|_{W^{k,p}(Z',\mathbb{R}^N)} \leq c_k.$$

Proposition 3.1 follows by induction from the bootstrap proposition A.7 using all axioms (V0)–(V3) and a product estimate, lemma 3.4 below. By standard arguments proposition 3.1 immediately implies theorem 3.2 on regularity and theorem 3.3 on compactness.

**Theorem 3.2 (Regularity).** Fix a perturbation $V : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V3) and constants $p > 2$ and $a < b$. Let $u$ be a map $(a,b] \times S^1 \to M \hookrightarrow \mathbb{R}^N$ which is of Sobolev class $W^{1,p}$ and solves the heat equation (30) almost everywhere. Then $u$ is smooth.

**Theorem 3.3 (Compactness).** Fix a perturbation $V : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V3) and constants $p > 2$ and $a < b$. Let $u^\nu : (a,b] \times S^1 \to M \hookrightarrow \mathbb{R}^N$ be a sequence of smooth solutions of the heat equation (30) such that

$$\sup \|\partial_t u^\nu\|_\infty + \sup \|\partial_s u^\nu\|_p < \infty.$$

Then there is a smooth solution $u : (a,b] \times S^1 \to M$ of (90) and a subsequence, still denoted by $u^\nu$, such that $u^\nu$ converges to $u$, uniformly with all derivatives on every compact subset of $(a,b] \times S^1$.  

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Lemma 3.4. Fix a constant $p > 2$ and a bounded open subset $\Omega \subset \mathbb{R}^2$ with area $|\Omega|$. Then for every integer $k \geq 1$ there is a constant $c = c(k, |\Omega|)$ such that

$$\|\partial_s u \cdot v\|_{W^{k,p}} \leq c \|\partial_s u\|_{W^{k,p}} \|v\|_{\infty} + c (\|u\|_{C^k} + \|\partial_t u\|_{C^k}) \|v\|_{W^{k,p}}$$

for all functions $u, v \in C^\infty(\Omega)$.

Proof. The proof is by induction on $k$. By definition of the $W^\ell,p$ norm

$$\|\partial_s u \cdot v\|_{W^{\ell+1,p}} \leq \|\partial_s u \cdot v\|_{W^{\ell,p}} + \|\partial_t \partial_s u \cdot v + \partial_s u \cdot \partial_s v\|_{W^{\ell,p}}$$

$$+ \|\partial_t \partial_s u \cdot v + 2\partial_t \partial_s u \cdot \partial_t v + \partial_s u \cdot \partial_t v\|_{W^{\ell,p}}$$

$$+ \|\partial_t \partial_s u \cdot v + \partial_s u \cdot \partial_s v\|_{W^{k,p}}.$$  \hspace{1cm} (33)

Step $k = 1$. Estimate (33) for $\ell = 0$ shows that

$$\|\partial_s u \cdot v\|_{W^{1,p}} \leq \left(\|\partial_s u\|_p + \|\partial_t \partial_s u\|_p + \|\partial_t \partial_t \partial_s u\|_p + \|\partial_t \partial_s u\|_p\right) \|v\|_{\infty}$$

$$+ 2 \|\partial_t \partial_s u\|_{\infty} \|\partial_t v\|_p$$

$$+ \|\partial_s u\|_{\infty} \left(\|\partial_t v\|_p + \|\partial_t \partial_t v\|_p + \|\partial_t v\|_p\right).$$

Since $\partial_t \partial_s u = \partial_t \partial_t u$ this proves the lemma for $k = 1$.

Induction step $k \Rightarrow k + 1$. Consider estimate (33) for $\ell = k$, then inspect the right hand side term by term using the induction hypothesis for the appropriate functions to conclude the proof. To illustrate this we give full details for the last term in (33), namely

$$\|\partial_s u \cdot \partial_s v\|_{W^{k+2,p}} \leq c \|\partial_s u\|_{W^{k+1,p}} \|\partial_s v\|_{\infty} + c (\|u\|_{C^k} + \|\partial_t u\|_{C^k}) \|\partial_s v\|_{W^{k+1,p}}$$

$$\leq c_1 |\Omega| \|\partial_s u\|_{C^k} \|\partial_s v\|_{W^{1,p}} + c (\|u\|_{C^k} + \|\partial_t u\|_{C^k}) \|\partial_s v\|_{W^{k+1,p}}$$

$$\leq c_1 |\Omega| \|u\|_{C^{k+1}} \|\partial_s v\|_{W^{2,p}} + c (\|u\|_{C^k} + \|\partial_t u\|_{C^k}) \|\partial_s v\|_{W^{k+1,p}}.$$  \hspace{1cm}

The first step is by the induction hypothesis for the function $\partial_s v$. In the second step we pulled out the $L^\infty$ norms of all derivatives of $\partial_s u$ and for the term $\partial_s v$ we applied the Sobolev embedding $W^{1,p} \subset W^{1,p} \hookrightarrow C^0$ with constant $c_1$. Here our assumptions $p > 2$ and $\Omega$ bounded enter. Step three is obvious. Note that $k \geq 1$ implies that $W^{k+1,p} \hookrightarrow W^{2,p}$.

Proof of proposition 3.1. Consider the family

$$T_r := T' + \frac{T - T'}{r}, \quad r \in [1, \infty),$$

and the corresponding nested sequence of cylinders $Z_r := (-T_r, 0) \times S^1$ with

$$Z = Z_1 \supset Z_2 \supset Z_3 \supset \ldots \supset Z'.$$  \hspace{1cm}

Denote by $C_0$ the constant in (V0). More generally, for $\ell \geq 1$ choose $C_\ell$ larger than $C_{\ell-1}$ and larger than all constants $C(k', \ell', \mathcal{V})$ in (V3) for which $2k' + \ell' \leq \ell$.  \hspace{1cm}
Claim. The map $F$ given by (31) is in $W^{k,p}(Z_{t+1})$ for every integer $t \geq 1$.

Proposition 3.1 immediately follows: Given any integer $k \geq 1$, then $F \in W^{k,p}(Z_{k+1})$ by the claim. Furthermore, by inclusion $Z_{k+1} \subset Z$ and (32)

$$\|u\|_{W^{0,p}(Z_{k+1})} \leq \|u\|_{W^{0,p}(Z)} \leq \mu_0.$$  

Hence by corollary A.8 for the pair $Z_{k+2} \subset Z_{k+1}$ there is a constant $c_{k+1}$ depending on $p, \mu_0, Z_{k+2}, Z_{k+1}, ||\Gamma||_{C^{2k+2}}$, and $||F||_{W^{k,p}(Z_{k+1})}$ such that

$$\|u\|_{W^{k+1,p}(Z_{k+2})} \leq \|u\|_{W^{k+1,p}(Z_{k+1})} \leq c_{k+1}.$$  

It remains to prove the claim. The proof is by induction.

**Step $t = 1$.** We need to prove that $F, \partial_t F, \partial_x F$, and $\partial_t \partial_x F$ are in $L^p(Z_2)$. The domain of all norms of $\Gamma$ and its derivatives is the compact manifold $M$. The domain of all other norms is the cylinder $Z$ unless indicated differently. By axiom (V0) with constant $C_0$ it follows (even on the larger domain $Z$) that

$$\|F\|_\infty = \sup_{s \in (-T,0]} \|\text{grad} \mathcal{V}(u_s)\|_{L^\infty(S^1)} \leq C_0$$  

and therefore

$$\|F\|_p \leq \|F\|_\infty (\text{Vol } Z)^{1/p} \leq C_0 T^{1/p}.$$  

Next we use axiom (V1) with constant $C_1 \geq C_0$ to obtain that

$$\|\partial_t F\|_p \leq \|\nabla \text{grad} \mathcal{V}(u)\|_p + ||\Gamma(u)(\partial_t u, \text{grad} \mathcal{V}(u))\|_p$$

$$\leq C_1 \left(1 + \|\partial_t u\|_p\right) + \|\Gamma\|_\infty \|\partial_t u\|_p \|F\|_\infty$$

$$\leq C_1 (1 + \mu_0) + ||\Gamma||_\infty \mu_0 C_0.$$  

Here we used the assumption (32) in the last step. Now by the bootstrap proposition A.7 (i) for $k = 1$ and the pair $Z_{4/3} \subset Z$ there is a constant $a_1$ depending on $p, \mu_0, Z_{4/3}, Z, ||\Gamma||_{C^4}$, and the $L^p(Z)$ norms of $F$ and $\partial_t F$ such that $||\partial_t u||_{W^{1,p}(Z_{4/3})} \leq a_1$. Then by the Sobolev embedding $W^{1,p} \hookrightarrow C^0$ with constant $c' = c'(p, Z_{5/3})$ it follows that $\partial_t u$ is continuous on $Z_{4/3}$ and

$$\|\partial_t u\|_{C^0(Z_{5/3})} \leq c' \|\partial_t u\|_{W^{1,p}(Z_{5/3})} \leq a_1 c'.$$

Again using axiom (V1) we obtain similarly that

$$\|\partial_x F\|_p \leq \|\nabla \text{grad} \mathcal{V}(u)\|_p + ||\Gamma(u)(\partial_x u, \text{grad} \mathcal{V}(u))\|_p$$

$$\leq 2C_1 \|\partial_x u\|_p + ||\Gamma||_\infty \|\partial_x u\|_p \|F\|_\infty$$

$$\leq \mu_0 (2C_1 + ||\Gamma||_\infty C_0).$$  

In order to estimate $\partial_t \partial_x F$ observe first that

$$\|\nabla \partial_t u\|_{L^p(Z_{5/3})} \leq \|\partial_t \partial_t u\|_{L^p(Z_{5/3})} + ||\Gamma||_\infty \|\partial_t u\| \|\partial_t u\|_{L^p(Z_{5/3})}$$

$$\leq \mu_0 + \|\Gamma\|_\infty \|\partial_t u\|_{C^0(Z_{5/3})} \|\partial_t u\|_{L^p(Z_{5/3})}$$

$$\leq \mu_0 + \|\Gamma\|_\infty a_1 c' \mu_0.$$  

32
Here the last step uses assumption (32) and the $C^0$ estimate (35) for $\partial_1 u$ which requires shrinking of the domain. Now by axiom (V3) for $k = 0$ and $\ell = 2$ there is a constant still denoted by $C_1 = C_1(V)$ such that
\[
|\nabla_\ell F| \leq C_1 \left( 1 + |\partial_1 u| + |\nabla \partial_1 u| \right) \quad (36)
\]
pointwise for every $(s, t)$. Integrating this inequality to the power $p$ implies that
\[
\|\nabla_\ell F\|_{L^p(Z_{5/3})} \leq C_1 \left( 1 + \|\partial_1 u\|_{L^p(Z_{5/3})} + \|\nabla \partial_1 u\|_{L^p(Z_{5/3})} \right) 
\leq C_1 (1 + 2\mu_0 + \|\Gamma\|_c a_1^\ell \mu_0) .
\]

Straightforward calculation shows that
\[
\|\partial_1 F\|_{L^p(Z_{5/3})} \leq \|\nabla_\ell F\|_{L^p} + \|\partial \|_{\infty} \|\partial_1 u\|_{C^0} \|\partial_1 u\|_{L^p} \|F\|_{C^0} 
+ \|\Gamma\|_c \|\partial_1 u\|_{L^p} \|F\|_{C^0} + 2 \|\Gamma\|_c \|\partial_1 u\|_{C^0} \|\partial_1 F\|_{L^p} 
+ \|\Gamma\|_c \|\partial_1 u\|_{C^0} \|\partial_1 u\|_{L^p} \|F\|_{C^0}
\]
is bounded by a constant $c = c(p, \mu_0, c', C_1, \|\Gamma\|_{C^1})$. Here all $C^0$ and $L^p$ norms are on the domain $Z_{5/3}$. We used again assumption (32), the estimates for $F$ and its derivatives obtained earlier, and (35).

**Induction step** $\ell \Rightarrow \ell + 1$. Let $\ell \geq 1$ and assume that the claim is true for $\ell$. This means that $F$ is in $W^{\ell, p}(Z_{\ell+1})$, hence
\[
\alpha_\ell := \|F\|_{W^{\ell, p}(Z_{\ell+1})} < \infty.
\]
Therefore by corollary A.8 for the integer $\ell$ and the pair of sets $Z_{\ell+1} \supset Z_{\ell+3/2}$ there is a constant $c_\ell = c_\ell(p, \mu_0, T_{\ell+1}, T_{\ell+3/2}, \|\Gamma\|_{C^{2\ell+2}}, \alpha_\ell)$ such that
\[
\|u\|_{W^{\ell+1, p}(Z_{\ell+3/2})} \leq c_\ell, \quad \|u\|_{C^r(Z_{\ell+3/2})} \leq c_\ell . \quad (37)
\]
The second inequality follows from the first by the Sobolev embedding $W^{1, p} \hookrightarrow C^0$ applied to each term in the $C^0$ norm. Then choose $c_\ell$ larger, if necessary. It remains to prove that the $W^{\ell, p}(Z_{\ell+1})$ norms of $\partial_1 F$, $\partial_1 F$, and $\partial_1 \partial_1 F$ are finite. Similarly as in step $\ell = 1$ we obtain that
\[
\|\partial_1 F\|_{W^{\ell, p}(Z_{\ell+3/2})} \leq \|\nabla_\ell F\|_{W^{\ell, p}} + \|\Gamma(u)(\partial_1 u, F)\|_{W^{\ell, p}} 
\leq C_1 \left( \|1\|_{W^{\ell, p}} + \|\partial_1 u\|_{W^{\ell, p}} \right) + \tilde{c} \|\partial_1 u\|_{W^{\ell, p}} \|F\|_{\infty} + \|u\|_{C^\ell} \|F\|_{W^{\ell, p}} 
\leq C_1 (T^{1/p} + c\ell) + \tilde{c} \|\Gamma\|_{C^\ell} \left( c\ell C_0 + c\ell \alpha_\ell \right) .
\]
Here the domain of all norms, except the one of $\Gamma$, is $Z_{\ell+3/2}$. The first step is by definition of the covariant derivative and the triangle inequality. Step two uses axiom (V1) and lemma A.9 with constant $\tilde{c}$. The last step uses the estimates (34), (37), and the definition of $\alpha_\ell$ in the induction hypothesis. Now by the refined bootstrap proposition A.7 there is a constant $a_{\ell+1}$ such that
\[
\|\partial_1 u\|_{W^{\ell+1, p}(Z_{\ell+2})} \leq a_{\ell+1}, \quad \|\partial_1 u\|_{C^r(Z_{\ell+2})} \leq a_{\ell+1} . \quad (38)
\]
Next observe that
\[
\|\partial_s F\|_{W^{\ell,p}(Z_{t+2})}
\leq \|\nabla F\|_{W^{\ell,p}} + \|\Gamma(u) (\partial_s u, F)\|_{W^{\ell,p}}
\leq 2C_1 \|\partial_s u\|_{W^{\ell,p}} + C' \|\Gamma\|_{C^t} (\|\partial_s u\|_{W^{\ell,p}} \|F\|_{\infty} + (\|u\|_{C^t} + \|\partial_t u\|_{C^t}) \|F\|_{W^{\ell,p}})
\leq 2C_1 c_\ell + C' \|\Gamma\|_{C^t} (c_\ell c_0 + (c_\ell + a_{\ell+1})\alpha_\ell).
\]

Here the domain of all norms, except the one of \(\Gamma\), is \(Z_{t+2}\). Again the first step is by definition of the covariant derivative and the triangle inequality. Step two uses axiom (V1) and lemma 3.4 with constant \(C'\). The last step uses the estimates (34), (37), (38), and the definition of \(\alpha_\ell\) in the induction hypothesis. Similarly as in step \(\ell = 1\) we obtain that
\[
\|\partial_t \partial_t F\|_{W^{\ell,p}(Z_{t+2})}
\leq \|\nabla \nabla_t F\|_{W^{\ell,p}} + \|d\Gamma(u) (\partial_t u, \partial_t u, F)\|_{W^{\ell,p}}
+ \|\Gamma(u) (\partial_t \partial_t u, F)\|_{W^{\ell,p}} + 2 \|\Gamma(u) (\partial_t u, \partial_t u)\|_{W^{\ell,p}}
+ \|\Gamma(u) (\partial_t u, \Gamma(u) (\partial_u F))\|_{W^{\ell,p}}
\leq C_1 \left(T^{1/p} + \|\partial_t u\|_{W^{\ell,p}} + \|\partial_t \partial_t u\|_{W^{\ell,p}} + \|\Gamma\|_{C^t} \|\partial_t u\|_{C^t} \|\partial_t u\|_{W^{\ell,p}}\right)
+ \|d\Gamma\|_{C^t} \|\partial_t u\|_{C^t} \|F\|_{W^{\ell,p}}
+ \tilde{c}\|\Gamma\|_{C^t} (\|\partial_t \partial_t u\|_{W^{\ell,p}} \|F\|_{\infty} + \|\partial_t u\|_{C^t} \|F\|_{W^{\ell,p}})
+ 2 \|\Gamma\|_{C^t} \|\partial_t u\|_{C^t} \|\partial_t F\|_{W^{\ell,p}}
+ \|\Gamma\|_{C^t} \|\partial_t u\|_{C^t} \|F\|_{W^{\ell,p}}.
\]

Here the domain of all norms, except the one of \(\Gamma\), is \(Z_{t+2}\). In the second step we used axiom (V2) with constant \(C_1\) to estimate the term \(\nabla \nabla_t F\) and we spelled out the covariant derivative arising in \(\nabla_t \partial_t u\). Moreover, crudely pulling out \(C^t\) norms worked for all terms but the third one, the one involving \(\partial_t \partial_t u\), here we used lemma 3.4 with constant \(\tilde{c}\) for the functions \(\partial_t \partial_t u\) and \(F\). Now all terms appearing on the right hand side have been estimated earlier. This proves the induction step and therefore the claim and proposition 3.1.

\textit{Proof of theorem 3.2.} Fix any point \(z \in Z = (a, b) \times S^1\) and a subcylinder \(Z' = (a', b) \times S^1\) that contains \(z\) and where \(a' \in (a, b)\). Set \(\mu_0 = \|u\|_{W^{\ell,p}(Z, \mathbb{R}^N)}\), then proposition 3.1 for the function \(\tilde{u}(s, t) := u(s + b, t)\) and the constants \(T = b - a\) and \(T' = b - a'\) implies that
\[
u \in \bigcap_{k \geq 0} W^{k,p}(Z', \mathbb{R}^N) = \bigcap_{k \geq 0} W^{k,p}(Z', \mathbb{R}^N) = C^\infty(\overline{Z'}, \mathbb{R}^N).
\]

See [MS04, app. B.1] for the last step. Hence \(u\) is locally smooth.

\textit{Proof of theorem 1.6.} Theorem 3.2.
Proof of theorem 3.3. Shifting the $s$ variable by $b$ and setting $T = b - a$, if necessary, we may assume without loss of generality that the maps $u^\nu$ are defined on $(-T,0]$ and, furthermore, by composition with the isometric embedding $M \hookrightarrow \mathbb{R}^N$ that they take values in $\mathbb{R}^N$. All norms are taken on the domain $(-T,0] \times S^1$, unless indicated otherwise. To apply proposition 3.1 we need to verify that the maps $u^\nu : (-T,0] \times S^1 \to \mathbb{R}^N$ satisfy the four apriori estimates in (32) for some constant $\mu_0$ independent of $\nu$. To see this observe that
\[
\|u^\nu\|_p \leq \|u^\nu\|_{\infty} \text{Vol}((-T,0] \times S^1) \leq c_1 T^{1/p}
\]
for some constant $c_1$ depending only on the isometric embedding $M \hookrightarrow \mathbb{R}^N$ and the diameter of the compact manifold $M$. By assumption there is a constant $c_2$ independent of $\nu$ such that
\[
\|\partial_t u^\nu\|_p \leq \|\partial_t u^\nu\|_{\infty} T^{1/p} \leq c_2 T^{1/p}
\]
and
\[
\|\partial_s u^\nu\|_p \leq c_2.
\]
Then it follows by the heat equation (30) that
\[
\|\nabla_t \partial_t u^\nu\|_p \leq \|\partial_t u^\nu\|_p + \|\text{grad} \mathcal{V}(u^\nu)\|_p \leq c_2 + C_0 T^{1/p}.
\]
In the second step we used (V0) to estimate grad$\mathcal{V}(u^\nu)$ in $L^\infty$ from above by a constant $C_0 = C_0(\mathcal{V})$. By definition of the covariant derivative
\[
\|\partial_t \partial_t u^\nu\|_p \leq \|\nabla_t \partial_t u^\nu\|_p + \|\partial_t u^\nu\|_{\infty} \|\partial_t u^\nu\|_p \\
\leq c_2 + C_0 T^{1/p} + c_2 T^{1/p} \|\partial_t u^\nu\|_{\infty} \|\partial_t u^\nu\|_p
\]
Now set $\mu_0 := c_2 + C_0 T^{1/p} + c_2 T^{1/p} \|\Gamma\|_{C^0(M)} + (c_1 + c_2) T^{1/p}$. Then proposition 3.1 asserts that for every constant $T' \in (0,T)$ and every integer $k \geq 2$ there is a constant $c_k = c_k(p,\mu_0,T,T',\mathcal{V})$ such that
\[
\|u^\nu\|_{W^{k,p}(Q,\mathbb{R}^N)} \leq c_k
\]
where $Q = [-T',0] \times S^1$. Recall that the inclusion $W^{k,p}(Q) \hookrightarrow C^{k-1}(Q)$ is compact; see e.g. [MS04, B.1.11]. Hence there is a subsequence which converges on $Q$ in the $C^k$ topology. We denote the limit by $u \in C^k(Q)$. Since this is true for every $k \geq 2$ there is a subsequence, still denoted by $u^\nu$, converging on $Q$ to $u$ uniformly with all derivatives. Since this is true for every compact subcylinder $Q$ of $(-T,0] \times S^1$, the theorem follows by choosing a diagonal subsequence associated to an exhausting sequence by such $Q$’s. Because, in particular, the convergence is in $C^0$ and the $u^\nu$ take values in $M$, so does the limit $u$. By $C^k$ convergence with $k \geq 2$ the limit $u$ satisfies the heat equation (30).
3.2 An apriori estimate

**Theorem 3.5.** Fix a perturbation \( V : \mathcal{L}M \to \mathbb{R} \) that satisfies (V0)–(V1) and a constant \( c_0 > 0 \). Then there is a constant \( C = C(c_0, V) > 0 \) such that the following holds. If \( u : \mathbb{R} \times S^1 \to M \) is a smooth solution of (6) such that

\[
\sup_{s \in \mathbb{R}} S_V(u(s, \cdot)) \leq c_0 \tag{39}
\]

then \( \|\partial_t u\|_\infty \leq C \).

The proof of theorem 3.5 is based on the following mean value inequality. For \( r > 0 \) define the open parabolic rectangle \( P_r \subset \mathbb{R}^2 \) by

\[
P_r := (-r^2, r) \times (-r, r).
\]

**Lemma 3.6 ([SW03, lemma B.1]).** There is a constant \( c_1 > 0 \) such that the following holds for all \( r \in (0, 1] \) and \( a \geq 0 \). If \( w : P_r \to \mathbb{R}, (s, t) \mapsto w(s, t) \), is \( C^1 \) in the \( s \)-variable and \( C^2 \) in the \( t \)-variable such that

\[
(\partial_t \partial_t - \partial_s) w \geq -aw, \quad w \geq 0,
\]

then

\[
w(0) \leq c_1 e^{ar^2} \int_{P_r} w.
\]

**Corollary 3.7.** Fix two constants \( r \in (0, 1] \) and \( \mu \geq 0 \). Let \( c_1 \) be the constant of lemma 3.6. If \( F : [-r^2, 0] \to \mathbb{R} \) is a \( C^2 \) function satisfying

\[
-F'' + \mu F \geq 0, \quad F \geq 0,
\]

then

\[
F(0) \leq 2c_1 e^{\mu r^2} \int_{-r^2}^0 F(s) \, ds.
\]

**Proof.** This follows immediately from lemma 3.6 with \( w(s, t) := f(s) \).

**Proof of theorem 3.5.** The idea is to first derive slicewise \( L^2 \) bounds, then verify the differential inequality in lemma 3.6 and apply the lemma using the slicewise bounds on the right hand side. The slicewise bound for \( \partial_t u \) follows easily from the assumption

\[
c_0 \geq S_V(u_s) = \frac{1}{2} \|\partial_t u_s\|^2_{L^2(S^1)} - V(u_s)
\]

where \( u_s(t) := u(s, t) \). Let \( C_0 \) denote the constant in (V0), then this implies

\[
\|\partial_t u_s\|^2_{L^2(S^1)} \leq 2c_0 + 2V(u_s) \leq 2c_0 + 2C_0 \tag{40}
\]

for every \( s \in \mathbb{R} \). Consider the pointwise differential inequality given by

\[
(\partial_t \partial_t - \partial_s) |\partial_t u|^2 = 2 |\nabla_t \partial_t u|^2 + 2 (\nabla_t \nabla_t - \nabla_s) \partial_t u, \partial_t u) \\
= 2 |\nabla_t \partial_t u|^2 - 2 \langle \text{grad} V(u), \partial_t u \rangle \\
\geq -2C_1 (1 + |\partial_t u|) |\partial_t u| \\
\geq -C_1 - 3C_1 |\partial_t u|^2.
\]
To obtain the second step we replaced $\nabla_t \partial_t u$ according to the heat equation (6) and used the fact that $\nabla_t \partial_s u = \nabla_s \partial_t u$. The third step is by condition (V1) with constant $C_1$. Choose $(s_0, t_0) \in \mathbb{R} \times S^1$ and apply lemma 3.6 in the case $r = 1$ and with
\[
w(s, t) := \frac{1}{3} + |\partial_t u(s_0 + s, t_0 + t)|^2
\]and $a = 3C_1$ to obtain
\[
w(0) \leq c_1 e^{a} \int_{-1}^{0} \int_{0}^{+1} \left( \frac{1}{3} + |\partial_t u(s_0 + s, t_0 + t)|^2 \right) \, dt \, ds
\]
\[= c_1 e^{3C_1} \left( \frac{2}{3} + 2 \int_{-1}^{0} \|\partial_t u_{s_0 + s}\|_{L^2(S^1)}^2 \, ds \right).
\]
Theorem 3.5 then follows from the slicewise estimate (40).

**Lemma 3.8.** Fix a constant $c > 0$ and a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0) with constant $C > 0$. If $u : \mathbb{R} \times S^1 \to M$ is a solution of (6) then
\[
sup_{s \in \mathbb{R}} \mathcal{S}_\mathcal{V}(u(s, \cdot)) \leq c \quad \Rightarrow \quad E(u) \leq c + C.
\]

**Proof.** Let $u_s(t) := u(s, t)$ and choose $T > 0$, then
\[
E_{[-T, T]}(u) = \int_{-T}^{T} \int_{0}^{1} |\partial_s u(s, t)|^2 \, dt \, ds
\]
\[= -\int_{-T}^{T} \langle \nabla \mathcal{S}_\mathcal{V}(u_s), \partial_s u_s \rangle_{L^2} \, ds
\]
\[= -\int_{-T}^{T} \frac{d}{ds} \mathcal{S}_\mathcal{V}(u_s) \, ds
\]
\[= \mathcal{S}_\mathcal{V}(u_{-T}) - \mathcal{S}_\mathcal{V}(u_T).
\]
Here we used the fact that the heat equation (6) is the negative $L^2$ gradient flow equation for the action functional. Now the crucial property of the action functional is its boundedness from below, namely $\mathcal{S}_\mathcal{V}(x) \geq -C$ for every $x \in \mathcal{L}M$ by (V0). Hence $\mathcal{S}_\mathcal{V}(u_{-T}) - \mathcal{S}_\mathcal{V}(u_T) \leq c + C$ and this proves the lemma.

### 3.3 Gradient bounds

**Theorem 3.9.** Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V2) and a constant $c_0 > 0$. Then there is a constant $C = C(c_0, \mathcal{V}) > 0$ such that the following holds. If $u : \mathbb{R} \times S^1 \to M$ is a smooth solution of (6) that satisfies (39), i.e. $\sup_{s \in \mathbb{R}} \mathcal{S}_\mathcal{V}(u(s, \cdot)) \leq c_0$, then
\[
|\partial_s u(s, t)|^2 + |\nabla_t \partial_s u(s, t)|^2 \leq CE_{[s-1, s]}(u)
\]
\[|\nabla_s \partial_t u(s, t)|^2 + |\nabla_t \nabla_s \partial_t u(s, t)|^2 \leq CE_{[s-2, s]}(u)
\]for every $(s, t) \in \mathbb{R} \times S^1$. Here $E_I(u)$ denotes the energy of $u$ over the set $I \times S^1$. 37
Proof. By theorem 3.5 there is a constant $C_0 = C_0(c_0, V) > 0$ such that

$$\|\partial_t u\|_{\infty} \leq C_0.$$  

Let $C = C(C_0, V)$ be the constant of theorem 2.3 with this choice of $C_0$. Observe that $\xi := \partial_s u$ solves the linearized heat equation. Hence theorem 2.3 shows that

$$|\partial_s u(s, t)|^2 \leq C^2 E_{|s-1, s|}(u) \leq C^2 (c_0 + c')$$

for every $(s, t) \in \mathbb{R} \times S^1$. Here the last step is by lemma 3.8 and axiom (V0) with constant $c'$. Use that $u$ solves (6) and satisfies axiom (V0) to obtain that

$$\|\nabla_t \partial_t u\|_{\infty} \leq \|\partial_s u\|_{\infty} + \|\text{grad} V(u)\|_{\infty} \leq C\sqrt{c_0 + c'} + c'.$$

Now choose $C_0$ larger than $2C\sqrt{c_0 + c'} + c'$ and let $C = C(C_0, V)$ be the constant of theorem 2.3 with this new choice of $C_0$. Theorem 2.3 then proves the desired estimate for $|\nabla_t \partial_t u|$. It follows that $\|\nabla_t \partial_t u\|_{\infty}$ is bounded. Therefore $|\nabla_t \nabla_s \partial_s u|_{\infty}$ is bounded by (6) and axiom (V1). Hence theorem 2.4 applies with a new choice of $C_0$ and proves the remaining two estimates of theorem 3.9.

Proof of theorem 1.8. Theorem 3.5, theorem 3.9 and lemma 3.8. Only (V0)–(V1) are used. Use (6) and (V0) to obtain the estimate for $\nabla_t \partial_t u$.

3.4 Exponential decay

Theorem 3.10. Fix a perturbation $V : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V2). Suppose $S_V$ is Morse and let $a \in \mathbb{R}$ be a regular value of $S_V$. Then there exist constants $\delta, c, \rho > 0$ such that the following holds. If $u : \mathbb{R} \times S^1 \to M$ is a smooth solution of (6) that satisfies (39), i.e. $\sup_{s \in \mathbb{R}} S_V(u(s, \cdot)) \leq a$, and

$$E_R([-T_0, T_0])(u) < \delta$$

for some $T_0 > 0$, then

$$E_R([-T, T])(u) \leq ce^{-\rho(T-T_0)}E_R([-T_0, T_0])(u)$$

for every $T \geq T_0 + 1$.

Corollary 3.11. Fix a perturbation $V : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V2). Suppose $S_V$ is Morse and let $x^\pm \in \mathcal{P}(V)$. Then there exist constants $\delta, c, \rho > 0$ such that the following holds. Suppose that $u : \mathbb{R} \times S^1 \to M$ is a smooth solution of (6) that satisfies (39) for some $T_0 > 0$. Then

$$|\partial_s u(s, t)|^2 + |\nabla_t \partial_s u(s, t)|^2 \leq ce^{-\rho(s-T_0)}E_R([-T_0, T_0])(u)$$

for every $s \geq T_0 + 2$.

Proof. Theorem 3.9 and theorem 3.10.

The proof of theorem 3.10 is based on the following lemma which asserts existence of a true critical point nearby an approximate one.
Lemma 3.12 (Critical point nearby approximate one). Fix a perturbation $V : L \to \mathbb{R}$ that satisfies (V0) and let $a \in \mathbb{R}$ be a regular value of $S_V$. Then, for every $\delta_0 > 0$, there is a constant $\delta_1 > 0$ such that the following is true. Suppose $x : S^1 \to M$ is a smooth loop such that

$$S_V(x) \leq a, \quad \| \nabla_t \partial_t x + \text{grad} V(x) \|_\infty < \delta_1.$$  

Then there is a critical point $x_0 \in \mathcal{P}^a(V)$ and a vector field $\xi_0$ along $x_0$ such that $x = \exp_{x_0}(\xi_0)$ and

$$\| \xi_0 \|_\infty + \| \nabla \xi_0 \|_\infty + \| \nabla_t \nabla \xi_0 \|_\infty \leq \delta_0.$$

Proof. First note that

$$\| \partial_t x \|_2^2 = \int_0^1 |\partial_t x(t)|^2 \, dt = 2S_V(x) + 2V(x) \leq 2(a + C)$$

where $C$ is the constant in (V0). Now, assuming $\delta_1 \leq 1$, we have

$$\left| \frac{d}{dt} |\partial_t x|^2 \right| = 2 \left| \langle \partial_t x, \nabla_t \partial_t x + \text{grad} V(x) \rangle - \langle \partial_t x, \text{grad} V(x) \rangle \right|$$

$$\leq 2(\delta_1 + C) |\partial_t x| \leq (1 + C)^2 + |\partial_t x|^2.$$  

Integrate this inequality to obtain that

$$|\partial_t x(t_1)|^2 - |\partial_t x(t_0)|^2 \leq (1 + C)^2 + |\partial_t x|_2^2$$

for $t_0, t_1 \in [0, 1]$. Integrating again over the interval $0 \leq t_0 \leq 1$ gives

$$|\partial_t x|_\infty \leq \sqrt{(1 + C)^2 + 2 |\partial_t x|_2^2} \leq c \tag{42}$$

where $c^2 := (1 + C)^2 + 4(a + C)$.

Now suppose that the assertion is wrong. Then there is a constant $\delta_0 > 0$ and a sequence of smooth loops $x_\nu : S^1 \to M$ satisfying

$$S_V(x_\nu) \leq a, \quad \lim_{\nu \to \infty} (\| \nabla_t \partial_t x_\nu + \text{grad} V(x_\nu) \|_\infty) = 0,$$

but not the conclusion of the lemma for the given constant $\delta_0$. By (V0) we have $\sup \| \nabla_t \partial_t x_\nu \|_\infty < \infty$ and (42) implies $\sup \| \partial_t x_\nu \|_\infty < \infty$. Hence, by the Arzela–Ascoli theorem, there exists a subsequence, still denoted by $x_\nu$, that converges in the $C^1$-topology. Let $x_0 \in C^1(S^1, M)$ be the limit. We claim that this subsequence actually converges in the $C^2$-topology. Then $\nabla_t \partial_t x_0 + \text{grad} V(x_0) = 0$. Hence $x_0 \in \mathcal{P}^a(V)$ and $x_\nu$ converges to $x_0$ in the $C^2$-topology. This contradicts our assumption on the sequence $x_\nu$ and proves the lemma.

It remains to prove the claim. For simplicity let us assume that $M$ is isometrically embedded in Euclidean space $\mathbb{R}^N$ for some sufficiently large integer $N$. Since $\sup \| \nabla_t \partial_t x_\nu \|_2 < \infty$, the Banach-Alaoglu Theorem asserts existence
of a subsequence, still denoted by \( x_\nu \), and an element \( v \in L^2 \) such that \( \nabla_t \partial_t x_\nu \) converges to \( v \) weakly in \( L^2 \). In fact \( v \) equals the weak \( t \)-derivative of \( \partial_t x \). Now \( \text{grad} V(x_\nu) \) converges to \( \text{grad} V(x_0) \) in \( L^2 \) and to \( -v \) weakly in \( L^2 \). But the weak limit equals the strong limit, hence \( v = -\text{grad} V(x_0) \in C^1 \). Therefore \( \partial_t x_0 \in C^1 \) and \( \nabla_t \partial_t x_0 \) equals the weak \( t \)-derivative \( v \) of \( \partial_t x_0 \). Now \( x_0 \in C^2 \) satisfies

\[
\nabla_t \partial_t x_0 + \text{grad} V(x_0) = 0, \tag{43}
\]

because \( \nabla_t \partial_t x_0 \) converges to \( v = \nabla_t \partial_t x_0 \) weakly in \( L^2 \) and to \( -\text{grad} V(x_0) \) strongly in \( L^2 \). By induction \( (43) \) implies that \( x_0 \in C^\infty \). Moreover, it follows using \( (43) \) that \( \nabla_t \partial_t x_0 \) converges to \( \nabla_t \partial_t x_0 \) in \( C^0 \) and this proves the claim.

**Proof of theorem 3.10.** Given \( a \) and \( V \), let \( C = C(a, V) \) be the constant of theorem 1.8 and theorem 3.9 with this choice. Let \( C_0 > 1 \) be the constant in \( (V0) \). Then \( E(u) \leq a + C_0 \) by lemma 3.8 and \( \| \partial_s u \|_\infty \leq CE(u) \leq C(a + C_0) \) by theorem 3.9. Hence

\[
\| \partial_t u \|_\infty + \| \nabla_t \partial_t u \|_\infty \leq c_0 \]

by theorem 1.8 and by replacing \( \nabla_t \partial_t u \) according to the heat equation (6). Here \( c_0 = C(a + 2C_0) + C_0 \). Let \( \delta_0 \) and \( \rho_0 \) be the positive constants of theorem 2.10 with this choice of \( c_0 \). Choose \( \delta_0 \) smaller than one quarter the minimal \( C^0 \) distance \( \kappa = \kappa(a) \) of any two elements of \( P^a(V) \). Let \( \delta_1 > 0 \) be the constant of lemma 3.12 associated to \( a \) and \( \delta_0 \) and set

\[
\delta := \min \left\{ \frac{\delta_0^2}{4C}, \frac{\delta_1^2}{4C} \right\}.
\]

Note that \( \delta_0 \), \( \rho_0 \), \( \delta_1 \), and \( \delta \) depend only on \( a \), \( V \), and the constant \( C_0 \) of axiom \( (V0) \). Note furthermore that \( \xi := \partial_t u \) solves the linear heat equation (13) and that the continuous function \( s \mapsto \| \partial_s u_s \|_{L^2(S^1)} \) is bounded, because its integral over \( \mathbb{R} \) is the energy \( E(u) \) which is finite.

If \( |s| \geq T_0 + 1 \), then \( E_{[s-1,s]}(u) \leq E_{[-T_0,T_0]}(u) \) and it follows that

\[
\| \partial_s u_s \|_\infty + \| \nabla_t \partial_s u_s \|_\infty \leq \sqrt{CE_{[s-1,s]}(u)} \leq \sqrt{C\delta} < \min \{ \delta_0, \delta_1 \}. \tag{44}
\]

Here we used theorem 3.9 in step one, assumption (41) in step two, and the definition of \( \delta \) in the last step. Hence, by lemma 3.12, there are critical points \( x^\pm \in P^a(V) \) such that

\[
u_s = \exp_{x^\pm}(\eta_s^\pm), \quad \| \eta_s \|_{C^{2}(S^1)} \leq \delta_0
\]

whenever \( \pm s \geq T_0 + 1 \). Although the critical points \( x^\pm \) apriori depend on \( s \) they are in fact independent, because \( \delta_0 < \kappa/4 \) and \( P^a(V) \) is a finite set by the Morse condition. Moreover, injectivity of the operators \( A_{x^\pm} \) is equivalent to nondegeneracy of the critical points \( x^\pm \) and this is true again by the Morse condition. Now theorem 2.10 and remark 2.11 conclude the proof of theorem 3.10.
Proof of theorem 1.9. We prove exponential decay in three steps.

I) Firstly, the energy of $u$ is finite. In the case (B) this is part of the
assumptions. In the case (F) it follows as in the proof of lemma 3.8 for $u : [0, \infty) \times S^1 \to \mathbb{R}$. Namely, let $C_0 > 0$ be the constant in (V0) and set $u_0(t) := u(0, t)$, then $E(u) \leq S_{\mathcal{V}}(u_0) + C_0$.

II) Secondly, we establish the existence of asymptotic limits. Consider the
forward case (F). We claim that $\partial_s u(s, t) \to 0$ as $s \to \infty$, uniformly in $t$. Let $C > 0$ be the constant in theorem 3.9 and let $s \geq 1$, then

$$|\partial_s u(s, t)| \leq CE_{[s-1, s]}(u) = C \int_{s-1}^s \|\partial_s u_\sigma\|_{L^2(S^1)}^2 d\sigma \to 0.$$

Here the last step follows by finite energy of $u$ and this proves the claim. Because $\partial_s u_s$ converges to zero in $L^\infty(S^1)$ so does $\nabla_t \partial_s u_s + \text{grad} \mathcal{V}(u_s)$ by (6). Hence it follows from lemma 3.12 that there is a critical point $x^+ \in \mathcal{P}(\mathcal{V})$ and, for every sufficiently large $s$, there is a smooth vector field $\xi_s$ along $x^+$ such that

$$u_s = \exp_{x^+}(\xi_s), \quad \|\xi_s\|_\infty + \|\nabla \xi_s\|_\infty + \|\nabla_t \nabla \xi_s\|_\infty \to 0.$$

(Here we used the fact that – since $\mathcal{S}_\mathcal{V}$ is Morse – there are only finitely many elements in $\mathcal{P}(\mathcal{V})$ below any fixed action level.) This and the identities for the
maps $E_{ij}$ in (21) imply that

$$\|\partial_s u\|_\infty + \|\partial_t u\|_\infty + \|\nabla \partial_t u\|_\infty < \infty. \quad (45)$$

The same arguments apply in case (B) with corresponding asymptotic limit $x^-$. 

III) The third step is to prove exponential decay of the $C^k$ norm of $\partial_s u$.
Consider the forward case (F). We prove by induction that for every $k \in \mathbb{N}$ there is a constant $c_k' > 0$ such that

$$\|\partial_s u\|_{W^{k, 2}([s, \infty) \times S^1)} \leq c_k' \|\partial_s u\|_{L^2([s-k, \infty) \times S^1)}$$

for every $s \geq k$. This estimate, the definition of the energy in (9), and theorem 3.10 with constants $\delta, c, \rho, T_0 > 0$, where $T_0$ is chosen sufficiently large such that (41) holds true, then show that

$$\|\partial_s u\|_{W^{k, 2}([s, \infty) \times S^1)} \leq c_k' \sqrt{E_{[s-k, \infty]}(u)} \leq c_k' \rho e^{-\rho(s-k-T_0)/2}$$

whenever $s \geq k + T_0 + 1$. The Sobolev embedding $W^{k, 2} \hookrightarrow C^{k-2}$, e.g. on the compact set $[s, s+1] \times S^1$, concludes the proof of forward exponential decay (F).

It remains to carry out the induction argument. It is based on the identity

$$(\nabla_s - \nabla_t \nabla_t) \partial_s u = R(\partial_s u, \partial_t u) \partial_t u + \mathcal{H}_{\mathcal{V}}(u) \partial_s u \quad (46)$$

– which follows by linearizing the heat equation (6) in the $s$-direction to obtain that $\partial_s u \in \ker D_u$ in the notation of section 2.4 – and the subsequent estimate.
Proposition A.4 with $p = 2$ applies\(^3\) by (45) and shows that there is a constant $c' > 0$ with the following significance. If $s_0 \geq 1$ then
\[
\|\nabla \xi\|_{L^2([s_0, \infty) \times S^1)} + \|\nabla \nabla \xi\|_{L^2([s_0, \infty) \times S^1)} + \|\nabla t \nabla \xi\|_{L^2([s_0, \infty) \times S^1)}
\leq c' \left( \left\| \nabla \xi - \nabla t \nabla \xi \right\|_{L^2([s_0-1, \infty) \times S^1)} + \|\xi\|_{L^2([s_0-1, \infty) \times S^1)} \right)
\]  
(47)
for every $\xi \in \Omega^0([0, \infty) \times S^1)$ of compact support. To see this fix a smooth nondecreasing cutoff function $\beta : \mathbb{R} \to [0, 1]$ which equals zero for $s \leq s_0 - 1$ and one for $s \geq s_0$ and whose slope is at most two. Via extension by zero we interpret $\beta \xi$ as a smooth compactly supported vector field along the extended cylinder $u : \mathbb{R} \times S^1 \to M$. Now proposition A.4 applies to $\beta \xi$ and proves (47). Note that $c'$ depends on the $L^\infty$ norms of $\partial_s \beta$, $\partial_t \beta$, and $\partial_s \partial_t \beta$. We also used lemma A.3 to deal with the term $\nabla \xi$ which appears on the right hand side.

We prove the induction hypothesis in the case $k = 1$. Let $s \geq 1$ and denote by $C_1 > 0$ the constant in (V1). By (47) with $\xi = \partial_s u$ and (46) it follows that
\[
\|\nabla_s \partial_s u\|_{L^2([s, \infty) \times S^1)} + \|\nabla_t \partial_s u\|_{L^2([s, \infty) \times S^1)} + \|\nabla \nabla \partial_s u\|_{L^2([s, \infty) \times S^1)}
\leq c' \left( \left\| \nabla_s - \nabla \nabla \right\|_{L^2([s-1, \infty) \times S^1)} + \|\partial_s u\|_{L^2([s-1, \infty) \times S^1)} \right)
\leq c' \left( \|R\partial_s u, \partial_t u\| + H_V(u)\partial_s u\|_{L^2([s-1, \infty) \times S^1)} + \|\partial_s u\|_{L^2([s-1, \infty) \times S^1)} \right)
\leq c' \left( \left\| \nabla_s (R\partial_s u, \partial_t u)\| + H_V(u)\partial_s u\|_{L^2([s-1, \infty) \times S^1)} \right) + \|\partial_s u\|_{L^2([s-1, \infty) \times S^1)} \right) + \|\partial_s u\|_{L^2([s-1, \infty) \times S^1)} \right) .
\]

We prove the induction hypothesis for $k = 2$. Assume $s \geq 2$. Then by (47) with $\xi = \nabla_s \partial_s u$ and (46) it follows that
\[
\|\nabla_s \nabla_s \partial_s u\|_{L^2([s, \infty) \times S^1)} + \|\nabla_s \nabla_t \partial_s u\|_{L^2([s, \infty) \times S^1)} + \|\nabla \nabla \nabla_s \partial_s u\|_{L^2([s, \infty) \times S^1)}
\leq c' \left( \left\| \nabla_s (R\partial_s u, \partial_t u)\| + H_V(u)\partial_s u\|_{L^2([s-1, \infty) \times S^1)} \right)
\leq c' \left( \left\| \nabla_s (R\partial_s u, \partial_t u)\| + H_V(u)\partial_s u\|_{L^2([s-1, \infty) \times S^1)} \right) .
\]

Now use $s \geq 2$, the apriori estimates (45), axiom (V2), and the case $k = 1$ to bound the right hand side by a constant times $\|\partial_s u\|_{L^2([s-2, \infty) \times S^1)}$. An $L^2$ bound for $\nabla \nabla \partial_s u$ was obtained earlier in the case $k = 1$ and the identity
\[
\nabla_s \nabla \partial_s u = \nabla_t \nabla \partial_s u - R(\partial_t u, \partial_s u)\partial_s u
\]
implies one for $\nabla_s \nabla \partial_s u$.

Proving the induction hypothesis in the case $k = 3$ requires additional information: Theorem 3.5 and theorem 3.9 only assume an upper action bound for the heat flow solution. In the case at hand this is provided by $S_V(u(0, \cdot))$. This reproves (45) and in addition shows that $\|\nabla t \nabla \partial_s u\|_{\infty} < \infty$. This estimate is crucial, since (47) with $\xi = \nabla_s \nabla_s \partial_s u$ and (46) lead to terms of the form
\[
\|R(\nabla_s \partial_s u, \nabla_s \partial_s u)\|_{L^2([s, \infty) \times S^1)},
\]
\(^3\)Formally add to $u$ any smooth half cylinder imposing a uniform limit as $s \to -\infty$. 

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but our induction hypothesis in the case $k = 2$ only provides a $C^0$ bound for $\partial_s u$. The remaining part of proof follows the same pattern as in the case $k = 2$. Here we use axiom (V3).

Fix an integer $k \geq 3$ and assume the induction hypothesis is true for every $\ell \in \{1, \ldots, k\}$. In particular, we have $W^{k,2}$ and $C^{k-2}$ bounds for $\partial_s u$ on the appropriate domains. Apply (47) with $\xi = \nabla_s^k \partial_s u$ and (46) to obtain $L^2$ bounds for $\nabla_s^{k+1} \partial_s u$ and $\nabla \nabla_s^k \partial_s u$. Here we use axiom (V3) and the induction hypothesis for $\ell \in \{1, \ldots, k\}$. A problem of the type encountered in the case $k = 3$ does not arise, since we have $C^{k-2}$ bounds for $\partial_s u$ with $k \geq 3$. To obtain $L^2$ estimates for the remaining terms of the form $\nabla_s^{j} \partial_s u$ with $j \geq 2$ use (46) to treat any $\nabla \nabla_s^j$ for one $\nabla_s$. This reduces the order of the term, hence the induction hypothesis can be applied. This completes the induction step and proves (F).

The backward case (B) follows similarly. This proves theorem 1.9.

**Lemma 3.13.** Fix a perturbation $V : LM \to R$ that satisfies (V0)–(V3), a constant $p > 1$, and nondegenerate critical points $x^\pm$ of $S_V$. If $u \in M(x^-; x^+; V)$, then the operators $D_u, D_u^* : W_1^1,p \to L_p$ are Fredholm and

$$\text{index } D_u = \text{ind}_V(x^-) - \text{ind}_V(x^+) = -\text{index } D_u^*.$$

**Proof.** By theorem 1.9 on exponential decay $u$ satisfies the assumptions of the Fredholm theorem 1.10.

**3.5 Compactness up to broken trajectories**

**Proposition 3.14 (Convergence on compact sets).** Assume that the perturbation $V : LM \to R$ satisfies (V0)–(V3) and that $S_V$ is Morse. Fix critical points $x^\pm \in P(V)$ and a sequence of connecting trajectories $u^\nu \in M(x^-, x^+; V)$. Then there is a pair $x_0, x_1 \in P(V)$, a connecting trajectory $u \in M(x_0, x_1; V)$, and a subsequence, still denoted by $u^\nu$, such that the following hold:

(i) The subsequence $u^\nu$ converges to $u$, uniformly with all derivatives on every compact subset of $\mathbb{R} \times S^1$.

(ii) For all $s \in \mathbb{R}$ and $T > 0$

$$S_V(u(s, \cdot)) = \lim_{\nu \to \infty} S_V(u^\nu(s, \cdot))$$

$$E_{[-T, T]}(u) = \lim_{\nu \to \infty} E_{[-T, T]}(u^\nu).$$

**Proof.** Since the flow lines $u^\nu$ connect $x^-$ to $x^+$ and the action $S_V$ decreases along flow lines, it follows that

$$\sup_{s \in \mathbb{R}} S_V(u^\nu(s, \cdot)) = S_V(x^-) =: c_0.$$

Hence by the apriori estimates theorem 3.5 and theorem 3.9 there is a constant $C = C(c_0, V)$ such that

$$|\partial_t u^\nu(s, t)| \leq C,$$

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|∂suν(s, t)| ≤ C√SV(x−) − SV(x+),

for every (s, t) ∈ ℝ × S1. To obtain the second estimate we used the energy identity (9) for connecting orbits. Now fix a constant p > 2 and pick an integer ℓ ≥ 2. Then the assumptions of theorem 3.3 are satisfied for the sequence uν restricted to the cylinder Zℓ = (−ℓ, ℓ] × S1. Hence there is a smooth solution u : Zℓ → M of the heat equation (6) and a subsequence, still denoted by uν, such that uν converges to u, uniformly with all derivatives on the compact subset [−ℓ + 1, ℓ] × S1 of Zℓ. Now (i) follows by choosing a diagonal subsequence associated to the exhausting sequence Z2 ⊂ Z3 ⊂ ... of ℝ × S1.

To prove (ii) note that

\[ E_{[-T,T]}(u) = \lim_{\nu \to \infty} \int_{-T}^{T} \int_{0}^{1} |∂su|^2 \, dt \, ds \]

for every T > 0. Here the first step uses that, by (i), the sequence ∂suν converges to ∂su, uniformly on compact sets. The second step is by definition of the energy and the last step is again by the energy identity (9). Hence the limit u : ℝ × S1 → M has finite energy and so, by theorem 1.9, belongs to the moduli space M(x0, x1; V) for some x0, x1 ∈ P(V).

Now (i) follows by choosing a diagonal subsequence associated to the exhausting sequence Z2 ⊂ Z3 ⊂ ... of ℝ × S1.

Lemma 3.15 (Compactness up to broken trajectories). Assume that the perturbation V : LM → ℝ satisfies (V0)–(V3) and that SV is Morse. Fix distinct critical points x± ∈ P(V) and let uν ∈ M(x−, x+; V) be a sequence of connecting trajectories. Then there exist a subsequence, still denoted by uν, finitely many critical points x0, ..., x_m with x0 = x+ and x_m = x−, finitely many solutions

\[ u_k \in M(x_k, x_{k-1}; V), \quad ∂su_k \neq 0, \quad k = 1, \ldots, m, \]

and finitely many sequences s_k, such that the shifted sequence uν(s_k + s, t) converges to u_k(s, t), uniformly with all derivatives on every compact subset of ℝ × S1. Moreover, these limit solutions satisfy \[ \sum_{k=1}^{m} E(u_k) = SV(x−) − SV(x+) \].

Proof. In [SW03, Proof of lemma 10.3] replace lemma 10.2 by prop. 3.14. □
4 The implicit function theorem

Throughout this section we fix a smooth perturbation $V : LM \to \mathbb{R}$ that satisfies (V0)--(V3) and two nondegenerate critical points $x^\pm$ of $S_V$. The idea to prove the manifold property and the dimension formula in theorem 1.11 is to construct a smooth Banach manifold which contains the moduli space $M(x^-, x^+; V)$ and then prove these statements locally near each element of the moduli space.

Fix a real number $p > 2$ and denote by

$$B^{1,p} = B^{1,p}(x^-, x^+)$$

the space of continuous maps $u : \mathbb{R} \times S^1 \to M$, which satisfy the limit conditions (8), are locally of class $W^{1,p}$, and satisfy the asymptotic conditions $\xi^- \in W^{1,p}([-\infty, -T] \times S^1, u^*TM)$ and $\xi^+ \in W^{1,p}([T, \infty) \times S^1, u^*TM)$ for some sufficiently large $T > 0$. Here $\xi^\pm$ are defined pointwise by the identity $\exp_x \xi(s, t) = u(s, t)$. For $p > 2$ the space $B^{1,p}$ carries the structure of a smooth infinite dimensional Banach manifold. The tangent space $T_u B^{1,p}$ is given by the Banach space $W^{1,p}$ whose norm is defined in (12). Around any smooth map $u$ local coordinates are provided by the inverse of the map $\varphi_u^{-1} : V_u \to B^{1,p}$ given by $\xi \mapsto [(s, t) \mapsto \exp_u(s, t) \xi(s, t)]$ where $V_u \subset W^{1,p}$ is a sufficiently small neighborhood of zero. By abuse of notation we shall denote this map again by $\xi \mapsto \exp_u \xi$. Moreover, note that if some $u \in B^{1,p}$ satisfies the heat equation (6) almost everywhere, then $u$ is smooth by theorem 1.6, hence $u \in M(x^-, x^+; V)$.

For $x \in M$ and $\xi \in T_x M$ denote parallel transport with respect to the Levi-Civita connection along the geodesic $\tau \mapsto \exp_x (\tau \xi)$ by

$$\Phi(x, \xi) : T_x M \to T_{\exp_x(\xi)} M.$$

For $u \in B^{1,p}$ the map $F_u : W^{1,p} \to L^p_u$ defined by

$$F_u(\xi) := \Phi(u, \xi)^{-1} \left( \partial_s (\exp_u \xi) - \nabla \partial_t (\exp_u \xi) - \text{grad} V(\exp_u \xi) \right)$$

(49)
is induced by pointwise evaluation at $(s, t)$. Its significance lies in the following three facts. Firstly, it is a smooth map between Banach spaces, hence the implicit function theorem for Banach spaces applies. Secondly, the differential $dF_u(0) : W^{1,p} \to L^p_u$ is given by the linear operator $D_u$; see [W99, app. A.3]. Thirdly, the map $\xi \mapsto \exp_u \xi$ identifies a neighborhood $V$ of zero in $F_u^{-1}(0)$ with a neighborhood of $u$ in $M(x^-, x^+; V)$. Now theorem 1.11 follows immediately.

Proof of theorem 1.11. Fix $p > 2$. Then the operator $dF_u(0) = D_u : W^{1,p} \to L^p_u$ is Fredholm by theorem 1.10 and surjective by assumption. Since every surjective Fredholm operator admits a right inverse, the implicit function theorem for Banach spaces, see e.g. [MS04, thm A.3.3], applies to $F_u$ restricted to a small neighborhood $V$ of zero. It asserts that $F_u^{-1}(0) \cap V$ is a smooth manifold whose tangent space at zero is given by the kernel of $D_u$. Since $D_u$ is onto, it follows that $\dim \ker D_u = \text{ind}_V(x^-) - \text{ind}_V(x^+)$ by definition of the Fredholm index. On the other hand, the Fredholm index equals $\text{ind}_V(x^-) - \text{ind}_V(x^+)$ by theorem 1.10. \qed

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Proof of proposition 1.12. Set $c_s = \frac{1}{2}(S_V(x^-) - S_V(x^+))$ and identify

$$\tilde{M}(x^-, x^+; V) \simeq M^* := \{ u \in M(x^-, x^+; V) \mid S_V(u(0, \cdot)) = c_s \}.$$ 

Here we use that the action $S_V$ is strictly decreasing along nonconstant (in the $s$-variable) heat flow trajectories. This is standard and follows from the first variation formula for the functional $S_V$; see e.g. [M69, sec. 12]. Now choose a sequence $u^\nu$ in $M^*$. By lemma 3.15 there is a subsequence, still denoted by $u^\nu$, finitely many critical points $x_0 = x^+, x_1, \ldots, x_m = x^-$, finitely many connecting trajectories $u_k \in M(x_k, x_{k-1}; V)$ and sequences $s_k^\nu$ where $k = 1, \ldots, m$, such that each shifted sequence $u^\nu(s_k^\nu + s, t)$ converges to $u_k(s, t)$ in $C^\infty_{loc}$. Note that $m \geq 1$. By the Morse–Smale assumption theorem 1.11 applies to all moduli spaces. Since $\partial_s u_k \not= 0$ and the heat equation (6) is $s$-shift invariant this implies

$$\text{ind}_V(x_k) - \text{ind}_V(x_{k-1}) = \dim \mathcal{M}(x_k, x_{k-1}; V) \geq 1, \quad \forall k \in \{1, \ldots, m\}.$$ 

Use these inequalities to obtain that $\text{ind}_V(x^-) - \text{ind}_V(x^+) \geq m \geq 1$. But by assumption the index difference is one and therefore $m = 1$. Now this means that the subsequence $u^\nu$ converges in $C^\infty_{loc}$ to $u := u_1 \in M(x^-, x^+; V)$. In fact, convergence of the action functional for fixed time $s = 0$, see proposition 3.14 (ii), shows that $u \in M^*$. Hence $M^*$ is compact in the $C^\infty_{loc}$ topology. On the other hand, the moduli space $\mathcal{M}(x^-, x^+; V)$ is a manifold of dimension one by theorem 1.11. Now the $\mathbb{R}$ action is free and therefore the quotient, hence $M^*$, is a manifold of dimension zero. But a zero dimensional compact manifold consists of finitely many points. \hfill $\Box$

The refined implicit function theorem

**Proposition 4.1** (The estimate for the right inverse). Fix a perturbation $V : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V3) and nondegenerate critical points $x^\pm$ of $S_V$. Assume $u \in \mathcal{M}(x^-, x^+; V)$ and $D_u$ is onto. Then, for every $p > 1$, there is a positive constant $c = c(p, u)$ invariant under $s$-shifts of $u$ such that

$$\|\xi^*\|_{W^1_p} \leq c \|D_u\xi^*\|_p$$

for every $\xi^* \in \text{im}(D_u : W^2_{p} \to W^1_{p})$. Here $W^2_{p} := \{ \xi \in W^1_{p} \mid D_u\xi \in W^1_{p} \}$.

**Proof of proposition 4.1.** The proof of [DS94, lemma 4.5] carries over. We include it for the sake of completeness. Fix $p > 1$ and let $1/q + 1/p = 1$. By lemma 3.13 the operators $D_u$ and $D^*_u$ are Fredholm. Since $D_u$ is onto, the operator $D^*_u$ is injective by proposition 2.17 and proposition 2.19 (hypothesis 2.15 is satisfied by theorem 1.9 on exponential decay). Hence by the open mapping theorem $D^*_u$ satisfies the injectivity estimate

$$\|\eta\|_q + \|
abla_s\eta\|_q + \|\nabla_s\nabla\eta\|_q \leq c_1 \|D^*_u\eta\|_q$$

for every $\eta \in W^1_{q}$ and with shift invariant constant $c_1 = c_1(q, u) > 0$. Next observe that

$$\frac{\langle D^*_u\xi, D^*_u\eta \rangle_{q}}{\|D^*_u\eta\|_q} = \frac{\langle D_uD^*_u\xi, \eta \rangle}{\|D^*_u\eta\|_q} \leq \frac{\|D_uD^*_u\xi\|_p}{\|D^*_u\eta\|_q} \leq c_1 \|D_uD^*_u\xi\|_p$$

for every $\xi \in W^1_{p}$ and with shift invariant constant $c_1 = c_1(q, u) > 0$. Next observe that
for all $\xi \in \mathcal{W}^{2,p}_u$ and $\eta \in \mathcal{W}^{1,q}_u$. Here the first step is by definition of the formal adjoint and the second one by Hölder’s inequality. The third step is by (51).

Now there is a shift invariant constant $c_2 = c_2(p, u) > 0$ such that

$$\|D^*_u \xi\|_p \leq c_2 \sup_{\eta \in \mathcal{W}^{1,q}_u} \langle D^*_u \xi, D^*_u \eta \rangle \|D^*_u \eta\|_q$$

(53)

for every $\xi \in \mathcal{W}^{2,p}_u$. The argument uses that $D_u$ is onto and $\dim \ker D_u < \infty$. The constant $c_2$ depends also on the choice of an $L^2$ orthonormal basis of $\ker D_u$.

Full details are given in step 2 of the proof of lemma 4.5 in [DS94]. Now the linear estimate proposition A.4 for $\xi^* := D^*_u \xi$ shows that

$$\|\xi^*\|_{\mathcal{W}^{1,p}_u} \leq c_3\left(\|D_u \xi^*\|_p + \|\xi^*\|_p\right)$$

where the constant $c_3(p, u)$ is again shift invariant. To estimate the second term in the sum apply (53) and (52) to obtain that $\|\xi^*\|_p \leq c_1 c_2 \|D_u \xi^*\|_p$.

Proposition 4.2 (Quadratic estimate). Fix a perturbation $V : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V1). Let $\iota > 0$ be the injectivity radius of $M$ and fix constants $1 < p < \infty$ and $c_0 > 0$. Then there is a constant $C = C(p, c_0) > 0$ such that the following is true. If $u : \mathbb{R} \times S^1 \to M$ is a smooth map and $\xi$ is a compactly supported smooth vector field along $u$ such that

$$\|\partial_s u\|_{\infty} + \|\partial_t u\|_{\infty} + \|\nabla \partial_t u\|_{\infty} \leq c_0, \quad \|\xi\|_{\infty} \leq \iota,$$

then

$$\|F_u(\xi) - F_u(0) - dF_u(0)\xi\|_p \leq C \|\xi\|_{\infty} \|\xi\|_{\mathcal{W}^{1,p}_u} \left(1 + \|\xi\|_{\mathcal{W}^{1,p}_u}\right).$$

Proof. Recall the definition (21) of the maps $E_i$ and $E_{ij}$ and write

$$F_u(\xi) - F_u(0) - \frac{d}{dt}\bigg|_{t=0} F_u(\tau \xi) = f(\xi) - g(\xi) - h(\xi)$$

where

$$f(\xi) := \Phi(u, \xi)^{-1} \partial_s E(u, \xi) - \partial_s u - \frac{d}{dt}\bigg|_{\tau=0} \Phi(u, \tau \xi)^{-1} \partial_s u$$

$$- \frac{d}{dt}\bigg|_{\tau=0} \partial_s E(u, \tau \xi)$$

$$g(\xi) := \Phi(u, \xi)^{-1} \nabla \partial_t E(u, \xi) - \nabla \partial_t u + (\nabla_2 \Phi(u, 0) \xi) \nabla \partial_t u$$

$$- \frac{d}{dt}\bigg|_{\tau=0} \nabla_\xi \partial_t E(u, \tau \xi)$$

$$h(\xi) := \Phi(u, \xi)^{-1} \text{grad} V(E(u, \xi)) - \text{grad} V(u) + (\nabla_2 \Phi(u, 0) \xi) \text{grad} V(u)$$

$$- \frac{d}{dt}\bigg|_{\tau=0} \text{grad} V(E(u, \tau \xi)).$$

47
Here we used that $\Phi(u, 0) = \mathbb{I}$. Straightforward calculation using the identities (23) shows that $f(\xi) = f_1(\xi)\nabla_s \xi + f_2(\xi)$ where

$$f_1(\xi)\nabla_s \xi = (\Phi(u, \xi)^{-1}E_2(u, \xi) - \mathbb{I})\nabla_s \xi$$

$$f_2(\xi)\partial_s u = (\Phi(u, \xi)^{-1}E_1(u, \xi) - \mathbb{I} + \nabla_2 \Phi(u, 0)\xi)\partial_s u,$$

that

$$g = g_1 \nabla_t \partial_t u + g_2 \nabla_t \partial_t u + g_3 \nabla_t \nabla_t \xi + g_4 \nabla_t \partial_t \xi + g_5 \nabla_t \nabla_t \xi$$

where

$$g_1(\xi) = \Phi(u, \xi)^{-1}E_1(u, \xi) - \mathbb{I} + \nabla_2 \Phi(u, 0)\xi$$

$$g_2(\xi) = \Phi(u, \xi)^{-1}E_1(u, \xi) - \mathbb{I} + \frac{d\Phi}{dt}(u, \xi)$$

$$g_3(\xi) = \Phi(u, \xi)^{-1}E_2(u, \xi) - \mathbb{I}$$

$$g_4(\xi) = 2\Phi(u, \xi)^{-1}E_1(u, \xi)$$

$$g_5(\xi) = \Phi(u, \xi)^{-1}E_2(u, \xi),$$

and that

$$h(\xi) = \Phi(u, \xi)^{-1}\nabla(E(u, \xi)) - (\mathbb{I} - (\nabla_2 \Phi(u, 0)\xi))\nabla u - H_V(u)\xi.$$ 

Here $H_V$ denotes the covariant Hessian of $V$ given by (4). It follows by inspection using the identities (23) that the maps $f_2, g_1, g_2, h$ together with their first derivative are zero at $\xi = 0$. Therefore there exists a constant $c > 0$ which depends continuously on $|\xi|$ and the constant in (V1) such that

$$|(f_2 + g_1 + g_2 + h)(\xi)| \leq c|\xi|^2 \left( |\partial_s u| + |\nabla_t \partial_t u| + |\partial_t u| + 1 \right)$$

pointwise at every $(s, t)$. Similarly, it follows that the remaining functions are zero at $\xi = 0$ and therefore

$$|(f_1 + g_3 + g_4 + g_5)(\xi)| \leq c|\xi| \left( |\nabla_s \xi| + |\nabla_t \nabla_t \xi| + |\nabla_t \xi| |\partial_t u| + |\nabla_t \xi|^2 \right).$$

Take these pointwise estimates to the power $p$, integrate them over $\mathbb{R} \times S^1$ and pull out $L^\infty$ norms of $\partial_s u, \partial_t u, \nabla_t \partial_t u$ to obtain the conclusion of proposition 4.2. The term $|\xi| \cdot |\nabla_t \xi|^2$ involving a product of first order terms is taken care of by the product estimate lemma A.5 and remark A.6. Here we use the fact that the (compact) support of $\xi$ is contained in some set $(a, b) \times S^1$. \qed

**Proof of the refined implicit function theorem 1.13**

Assume the result is false. Then there exist constants $p > 2$ and $c_0 > 0$ and a sequence of smooth maps $u_\nu : \mathbb{R} \times S^1 \to M$ such that $\lim_{s \to \pm \infty} u_\nu(s, \cdot) = x^{\pm}(\cdot)$ exists, uniformly in $t$, and

$$|\partial_s u_\nu(s, t)| \leq \frac{c_0}{1 + s^2}, \quad ||\partial_t u_\nu||_\infty \leq c_0, \quad ||\nabla_t \partial_t u_\nu||_\infty \leq c_0, \quad \text{(54)}$$
for all \((s, t) \in \mathbb{R} \times S^1\) and

\[
\|\partial_s u_\nu - \nabla_t \partial_t u_\nu - \text{grad} \mathcal{V}(u_\nu)\|_p \leq \frac{1}{\nu},
\]

(55)

but which does not satisfy the conclusion of theorem 1.13 for \(c = \nu\). This means that for every \(u_\nu \in \mathcal{M}(x^-, x^+; \mathcal{V})\) and every \(\xi_\nu \in \text{im} \mathcal{D}^*_u \cap \mathcal{W}_u\), the following holds. If \(u_\nu = \exp_{u_\nu}(\xi_\nu)\) then

\[
\|\partial_s u_\nu - \nabla_t \partial_t u_\nu - \text{grad} \mathcal{V}(u_\nu)\|_p < \frac{1}{\nu} \|\xi_\nu\|_W.
\]

(56)

The **time shift** of a smooth map \(u : \mathbb{R} \times S^1\) by \(\sigma \in \mathbb{R}\) is defined pointwise by

\[
u\sigma(s, t) := u(s + \sigma, t).
\]

Set \(a_0 := 2c_0^2\) and observe that

\[
\mathcal{S}_\nu(x^-) = \lim_{x \to -\infty} \mathcal{S}_\nu(u_\nu(s, \cdot)) = \frac{1}{2} \|\partial_t u_\nu(s, \cdot)\|^2_2 - \mathcal{V}(u_\nu(s, \cdot)) \leq \frac{1}{2} c_0^2 + C_0 \leq a_0.
\]

Here we used the assumption on asymptotic \(W^{1,2}\) convergence, estimate (54), and our choice of the constant \(c_0 > 1\) larger than the constant \(C_0\) in (V0). Now fix a regular value \(c_*\) of \(\mathcal{S}_\nu\) between \(\mathcal{S}_\nu(x^+)\) and \(\mathcal{S}_\nu(x^-)\). Here we use that the set \(\mathcal{P}^{a_0}(\mathcal{V})\) is finite, because \(\mathcal{S}_\nu\) is Morse–Smale below level \(a_0\). Applying time shifts, if necessary, we may assume without loss of generality that

\[
\mathcal{S}_\nu(u_\nu(0, \cdot)) = c_*.
\]

(57)

Furthermore we set \(\tilde{c}_0 = a\) and let \(C_0 = C_0(a, \mathcal{V}) > 0\) be the constant in theorem 1.8 with that choice. Then we have the apriori estimates

\[
\|\partial_s u\|_\infty + \|\partial_t u\|_\infty + \|\nabla \partial_t u\|_\infty \leq C_0
\]

(58)

for all \(u \in \mathcal{M}(x, y; \mathcal{V})\) and \(x, y \in \mathcal{P}^{a}(\mathcal{V})\).

**Claim.** There is a subsequence, still denoted by \(u_\nu\), a constant \(C > 0\), a trajectory \(u \in \mathcal{M}(x^-, x^+; \mathcal{V})\), and a sequence of times \(\sigma_\nu\) such that the sequence \(\eta_\nu\) determined by the identity

\[
u\eta_\nu = \exp_{u_{\sigma_\nu}}(\eta_\nu)
\]

satisfies \(\eta_\nu \in \text{im} \mathcal{D}^*_u \cap \mathcal{W}_u\) and

\[
\lim_{\nu \to \infty} \left(\|\eta_\nu\|_\infty + \|\eta_\nu\|_p\right) = 0, \quad \|\eta_\nu\|_W \leq C.
\]

(59)

Before we prove the claim we show how it leads to a contradiction. Consider the trajectories \(u_{\sigma_\nu} \in \mathcal{M}(x^-, x^+; \mathcal{V})\) and vector fields \(\eta_\nu\) provided by the claim. They satisfy the assumptions of the quadratic estimate, proposition 4.2, by (58).
and by choosing a further subsequence, if necessary, to achieve that \( \| \eta_\nu \|_\infty < \iota \).

Set \( c'_0 = C_0(a, \nu) \) and let \( C_2 = C_2(p, c'_0) \) be the constant in proposition 4.2 with that choice. Furthermore, since \( \mathcal{M}(x^-, x^+; \nu)/\mathbb{R} \) is a finite set by proposition 1.12 (and \( \mathcal{P}^a(\nu) \) is a finite set as well) the estimate for the right inverse, proposition 4.1, applies with constant \( C_1 \) depending only on \( p, \nu \), and \( \nu \). Now by the definition (49) of the map \( F_\hat{\nu} \) and the fact that parallel transport is an isometry we obtain the first step in the following estimate, namely

\[
\| \partial_s u_\nu - \nabla_t \partial_t u_\nu - \nabla \nu (u_\nu) \|_p = \| F_\hat{\nu}(\eta_\nu) \|_p \\
\geq \| D_\nu \eta_\nu \|_p - \| F_\hat{\nu}(\eta_\nu) - F_\hat{\nu}(0) \eta_\nu \|_p \\
\geq \| \eta_\nu \|_\nu \left( \frac{1}{C_1} - C_2 \| \eta_\nu \|_\infty (1 + \| \eta_\nu \|_\nu) \right) \\
\geq \frac{1}{2C_1} \| \eta_\nu \|_\nu.
\]

Step two uses that \( F_\hat{\nu}(0) = \partial_s \hat{u} - \nabla_\nu \partial_t \hat{u} - \nabla \nu (\hat{u}) = 0 \) and \( dF_\hat{\nu}(0) = D_\nu \). Step three is by proposition 4.1 and proposition 4.2. By (59) the last step holds for sufficiently large \( \nu \). For \( \nu > 2C_1 \) the estimate contradicts (56) and this proves theorem 1.13. It only remains to prove the claim. This takes four steps.

**Step 1.** There is a subsequence of \( u_\nu \), still denoted by \( u_\nu \), and a trajectory \( u \in \mathcal{M}(x^-, x^+; \nu) \) such that

\[
u \rightarrow \infty \left( \| \xi_\nu \|_\infty + \| \xi_\nu \|_p \right) = 0. \quad (60)
\]

**Proof.** We embed the compact Riemannian manifold \( M \) isometrically into some Euclidean space \( \mathbb{R}^N \) and view any continuous map to \( M \) as a map into \( \mathbb{R}^N \) taking values in the embedded manifold. By translation we may assume that the embedded \( M \) contains the origin. Now \( L^p \) and \( L^\infty \) norms of \( u_\nu \) are provided by the ambient Euclidean space. By compactness of \( M \) and, in particular, the \( L^\infty \) bounds in (54) we obtain on every compact cylindrical domain \( Z_T := [-T, T] \times \nu \) the estimates

\[
\| u_\nu \|_{L^p(Z_T)} \leq (2T)^\frac{p}{2} \text{diam } M, \quad \| \partial_t u_\nu \|_{L^p(Z_T)} + \| \nabla_\nu \partial_t u_\nu \|_{L^p(Z_T)} \leq 2c_0(2T)^\frac{p}{2},
\]

and

\[
\| \partial_s u_\nu \|_r \leq 4c_0 \quad \forall r \in (1, \infty]. \quad (61)
\]

The latter follows by the estimate

\[
\int_{-\infty}^{\infty} \left( \frac{1}{1 + s^2} \right)^r ds \leq 2 + 2 \int_{1}^{\infty} \frac{1}{s^{2r}} ds = \frac{4}{2 - 1/r} < 4
\]

whenever \( r > 1 \). Hence the sequence \( u_\nu \) is uniformly bounded in \( \mathcal{W}^{1, p}(Z_T) \). Thus by the Arzela-Ascoli and the Banach-Alaoglu theorem a suitable subsequence, still denoted by \( u_\nu \), converges strongly in \( C^0 \) and weakly in \( \mathcal{W}^{1, p} \) on
every compact cylindrical domain \( Z_T \) to some continuous map \( u : \mathbb{R} \times S^1 \to M \) which is locally of class \( \mathcal{W}^{1,p} \). Hence \( \partial_u u_\nu - \nabla t u_\nu \to \partial_T u - \nabla t u - \text{grad} \mathcal{V}(u) \) converges weakly in \( L^p \) to \( \partial_T u - \nabla t u - \text{grad} \mathcal{V}(u) \). On the other hand, by (55) it converges to zero in \( L^p \). By uniqueness of limits \( u \) satisfies the heat equation (6) almost everywhere. Thus \( u \) is smooth by theorem 1.6.

Fix \( s \in \mathbb{R} \) and observe that by (54) there are uniform \( C^1(S^1) \) bounds for the sequence \( \partial_t u_\nu(s, \cdot) \). Hence by Arzela-Ascoli a suitable subsequence, still denoted by \( \partial_t u_\nu(s, \cdot) \), converges in \( C^0(S^1) \) to \( \partial_t u(s, \cdot) \). Thus

\[
\lim_{\nu \to \infty} \mathcal{S}_\nu(u_\nu(s, \cdot)) = \mathcal{S}_\nu(u(s, \cdot))
\]

and therefore \( \mathcal{S}_\nu(u(0, \cdot)) = c_* \) by (57). Recall that \( \partial_t u = \nabla t u + \text{grad} \mathcal{V}(u) \). When restricted to \( s = 0 \) this means that the vector field \( \partial_t u(0, \cdot) \) is equal to the \( L^2 \) gradient of \( \mathcal{S}_\nu \) at the loop \( u(0, \cdot) \). But \( \mathcal{S}_\nu(u(0, \cdot)) = c_* \) and \( c_* \) is a regular value. Hence \( \partial_t u(0, \cdot) \) cannot vanish identically.

On the other hand, by (54) and axiom (V0) with constant \( C_0 \) it follows exactly as above that

\[
\sup_{\nu} \mathcal{S}_\nu(u_\nu(s, \cdot)) = \sup_{\nu} \frac{1}{2} \| \partial_t u_\nu(s, \cdot) \|^2 - \mathcal{V}(u_\nu) \leq a_0.
\]

This shows that all relevant trajectories including relevant limits over \( s \) or \( \nu \) lie in the sublevel set \( \mathcal{L}^{u,M} \) on which \( \mathcal{S}_\nu \) is Morse–Smale by assumption. In particular, we have that \( \sup_{s \in \mathbb{R}} \mathcal{S}_\nu(u(s, \cdot)) \leq a_0 \) and therefore the energy of \( u \) is finite by lemma 3.8. Hence by the exponential decay theorem 1.9 there are critical points \( y^\pm \in \mathcal{P}^{au}(\mathcal{V}) \) such that \( u(s, \cdot) \) converges to \( y^\pm \) in \( C^2(S^1) \), as \( s \to \pm \infty \). Moreover, the limits \( y^- \) and \( y^+ \) are distinct, because the action along a nonconstant trajectory is strictly decreasing and the trajectory is nonconstant because \( \partial_t u \) is not identically zero as observed above.

More generally, a standard argument shows the following, see e.g. [SW03, lemma 10.3]. There exist critical points \( x^- = x^0, x^1, \ldots, x^\ell = x^+ \in \mathcal{P}^{au}(\mathcal{V}) \) and trajectories \( u^k \in \mathcal{M}(x^{k-1}, x^k; \mathcal{V}) \), \( \partial_t u^k \neq 0 \), for \( k \in \{1, \ldots, \ell\} \), a subsequence, still denoted by \( u_\nu \), and sequences \( s^k_\nu \in \mathbb{R} \), \( k \in \{1, \ldots, \ell\} \), such that the shifted sequence \( u_\nu(s^k_\nu + s, t) \) converges to \( u^k(s, t) \) in an appropriate topology. The point here is that \( \partial_t u^k \neq 0 \) and therefore the Morse index strictly decreases along the sequence \( x^- = x^0, x^1, \ldots, x^\ell = x^+ \). Namely, by the Morse–Smale condition each Fredholm operator \( \mathcal{D}_u^k \) is onto, hence its Fredholm index is equal to the dimension of its kernel. But this is strictly positive because the kernel contains the nonzero element \( \partial_t u^k \). On the other hand, by lemma 3.13 the Fredholm index is given by the difference of Morse indices \( \text{ind}_\mathcal{V}(x^{k-1}) - \text{ind}_\mathcal{V}(x^k) \). Our assumption that the pair \( x^\pm \) has Morse index difference one then implies that \( \ell = 1 \) and this proves that \( u \in \mathcal{M}(x^-, x^+; \mathcal{V}) \). The first assertion of step 1.

It remains to prove the second assertion, that is (60). The key fact to prove (60) is that \( u_\nu(s, \cdot) \) not only converges in \( \mathcal{W}^{1,2}(S^1) \) to \( x^\pm \), as \( s \to \pm \infty \), but that the rate of convergence is independent of \( \nu \). More precisely, we prove that for every \( \varepsilon > 0 \) there is a time \( T = T(\varepsilon) > 1 \) such that

\[
s > T \implies d \left( u_\nu(s, t), x^+(t) \right) < \varepsilon
\]
for all $t \in S^1$ and $\nu \in \mathbb{N}$. Recall that $M$ is embedded isometrically in $\mathbb{R}^N$. By the fundamental theorem of calculus and uniform decay (54) we have that

$$
|x^+(t) - u_\nu(\sigma, t)|_{\mathbb{R}^N} = \left| \int_{\sigma}^{\infty} \partial_s u_\nu(s, t) \, ds \right|_{\mathbb{R}^N} \leq \int_{\sigma}^{\infty} \frac{c_0}{s^2} \, ds = \frac{c_0}{\sigma} \tag{63}
$$

for all $t \in S^1$, $\nu \in \mathbb{N}$, and $\sigma > 1$ sufficiently large. The Riemannian distance $d$ in $M$ and the restriction of the Euclidean distance in $\mathbb{R}^N$ to the compact manifold $M$ are locally equivalent. Hence (63) implies (62). Let $Z_T^\pm := [T, \infty) \times S^1$ denote the positive end of the cylinder $\mathbb{R} \times S^1$ and $Z_T^-$ the negative end. Let $t > 0$ be the injectivity radius of $M$. Now fix $\varepsilon \in (0, t/2)$ and choose $T = T(\varepsilon) > 0$ such that the ends $u(Z_T^\pm)$ and $u_\nu(Z_T^\pm)$ for all $\nu$ are contained in the $(\varepsilon/6)$-neighborhood of $x^+(S^1)$. Such $T$ exists by (62). Since $u_\nu$ converges to $u$ uniformly on $Z_T$, there exists $\nu_0 = \nu_0(T(\varepsilon)) \in \mathbb{N}$ such that $\|\xi_\nu\|_{L^\infty(Z_T)} < \varepsilon/3$ for every $\nu \geq \nu_0$. Hence

$$
\|\xi_\nu\|_{L^\infty} = \|\xi_\nu\|_{L^\infty(Z_T^\pm)} + \|\xi_\nu\|_{L^\infty(Z_T^-)} + \|\xi_\nu\|_{L^\infty(Z_T^+)} \leq \sup_{Z_T^\pm} \left( d(u_\nu, x^-) + d(x^-, u) \right) + \sup_{Z_T^-} \left( d(u_\nu, x^+) + d(x^+, u) \right) \leq \varepsilon
$$

for every $\nu \geq \nu_0$. This proves that the $L^\infty$ limit in (60) is zero. To prove that the $L^p$ limit is zero one uses again the decomposition of $\mathbb{R} \times S^1$ into the compact part $Z_T$ and the two ends $Z_T^\pm$. The left hand side of (63) is $p$-integrable over the ends $Z_T^\pm$. The key fact is that the value of this integral does not depend on $\nu$ and converges to zero as $|T| \to \infty$. A similar integral is needed in the case of $u$. Here the exponential decay theorem 1.9 shows that the integral exists and converges to zero as $|T| \to \infty$. This concludes the proof of step 1. \qed

**Step 2.** Set $\varepsilon_\nu := \|\xi_\nu\|_{L^\infty} + \|\xi_\nu\|_p$ and let $C_0$ be the constant in (58). Then there is a constant $\sigma_0 > 0$ and integer $\nu_0 \geq 1$ such that $\eta = \eta(\sigma, \nu)$ is determined by the identity $u_\nu = \exp_{u^\sigma} \eta$ and satisfies $\|\eta\|_{L^\infty} < \varepsilon/2$ for all $\sigma \in [-\sigma_0, \sigma_0]$ and $\nu \geq \nu_0$. Furthermore, there is a constant $c_2 = c_2(\sigma_0, \sigma_0) > 0$ such that

$$
\|\eta\|_{L^\infty} \leq \varepsilon_\nu + C_0 |\sigma|, \quad \|\eta\|_p \leq 2\varepsilon_\nu + c_2 |\sigma|
$$

and

$$
\|\nabla \eta\|_p \leq c_2, \quad \|\nabla \eta\|_{L^\infty} \leq c_2, \quad \|\nabla^2 \nabla \eta\|_p \leq c_2
$$

for all $\sigma \in [-\sigma_0, \sigma_0]$ and $\nu \geq \nu_0$.

**Proof.** Existence of $\sigma_0$ and $\nu_0$ follows from the fact that $\eta(\nu, 0) = \xi_\nu$, continuity of time shift, and the $L^\infty$ limit in (60). Now denote by $L$ the length functional. Then for every $\sigma \in \mathbb{R}$ and $\gamma(r) := u(s + r\sigma, t)$ for $r \in [0, 1]$ we have that

$$
d(u(s, t), u(s + \sigma, t)) \leq L(\gamma) = |\sigma| \int_0^1 |\partial_s u(s + r\sigma, t)| \, dr \leq |\sigma| \|\partial_s u\|_{L^\infty}. \tag{65}
$$
Similarly, since $M$ covariantly with respect to $t$, each of the two linear terms to which $E$ is uniformly bounded. To prove the last estimate of step 2 differentiate (65) and the first estimate in (66) with $\hat{s} = s + r\sigma$

$$d(u(s,t), u(s+\sigma,t)) \leq |\sigma| \int_0^1 |\partial_s u(s+r\sigma, t)| \, dr \leq |\sigma| c_3 e^{\rho \sigma} e^{-\rho|s|}.$$ 

Hence the left hand side is $L^p$ integrable. This concludes the proof of the second estimate of step 2. To prove the next two estimates we differentiate the identity $\exp u^\sigma, \eta = u^\nu$ with respect to $s$ and $t$ to obtain that

$$E_1(u^\sigma, \eta)\partial_s u^\sigma + E_2(u^\sigma, \eta)\nabla_s \eta = \partial_s u^\nu,$$

$$E_1(u^\sigma, \eta)\partial_t u^\sigma + E_2(u^\sigma, \eta)\nabla_t \eta = \partial_t u^\nu.$$ 

Here the maps $E_i$ are defined by (21). Since $\|\partial_s u^\sigma\|_p \leq c_3$ by (66) and $\|\partial_t u^\sigma\|_p \leq 4c_0$ by (61), the $L^p$ norm of $\nabla_s \eta$ is uniformly bounded as well. Similarly, since $\|\partial_t u^\sigma\|_\infty \leq C_0$ by (58) and $\|\partial_t u^\nu\|_\infty \leq c_0$ by (54), the $L^\infty$ norm of $\nabla_t \eta$ is uniformly bounded. To prove the last estimate of step 2 differentiate (66) covariantly with respect to $t$ and abbreviate $E_{ij} = E_{ij}(u^\sigma, \eta)$ to obtain

$$E_{11}(u^\sigma, \eta)(\partial_t u^\sigma, \partial_t u^\sigma) + E_{12}(u^\sigma, \eta)(\partial_t u^\sigma, \nabla_t \eta) + E_1(u^\sigma, \eta)\nabla_t \partial_t u^\sigma$$
$$+ E_{21}(u^\sigma, \eta)(\nabla_s \eta, \partial_t u^\sigma) + E_{22}(u^\sigma, \eta)(\nabla_t \eta, \nabla_t \eta) + E_2(u^\sigma, \eta)\nabla_t \nabla_t \eta$$
$$\begin{equation}
\begin{aligned}
&= \nabla_t \partial_t u^\sigma + \text{grad } V(u^\sigma) - \partial_t u^\nu.
\end{aligned}
\end{equation}$$

This identity implies a uniform $L^p$ bound for $\nabla_t \nabla_t \eta$ as follows. The right hand side is bounded in $L^p$ by $1/\nu$ and the last term of the left hand side by $4c_0$ according to (61). Since $E_{ij}(u^\sigma, 0) = 0$ and we have uniform $L^\infty$ bounds for each of the two linear terms to which $E_{ij}(u^\sigma, \eta)$ is applied, we can estimate the $L^p$ norm by a constant times $\|\eta\|_p$. The only terms left are term three and term seven of the left hand side. By the heat equation (6) their sum equals

$$E_1(u^\sigma, \eta)\partial_t u^\sigma - E_1(u^\sigma, \eta)\text{grad } V(u^\sigma) + \text{grad } V(u^\nu).$$

Since $\|\partial_t u^\sigma\|_p \leq c_3$ by (66), the $L^p$ norm of the first term is uniformly bounded. Consider the remaining two terms as a function $f$ of $\eta$. Then $f(0) = 0$, because $E_1(u^\sigma, 0) = 1$ and $\eta = 0$ means $u^\nu = u^\sigma$. Hence $\|f\|_p$ is uniformly bounded by a constant times $\|\eta\|_p$. Here we used axiom (V0). This proves step 2. \qed
Step 3. For $\sigma \in [-\sigma_0, \sigma_0]$ consider the function $\theta_\nu(\sigma) := -\langle \partial_s u^\sigma, \eta \rangle$ where $\eta = \eta(\sigma, \nu)$ has been defined in step 2 by the identity $u_\nu = \exp_{\nu^*} \eta$ and where $\langle \cdot, \cdot \rangle$ denotes the $L^2(\mathbb{R} \times S^1)$ inner product. This function has the property that

$$\theta_\nu(\sigma) = 0 \iff \eta \in \text{im } D^*_u.$$ 

Moreover, there exist new constants $\sigma_0 > 0$ and $\nu_0 \in \mathbb{N}$ such that

$$|\theta_\nu(0)| \leq c_3 \varepsilon_\nu, \quad \frac{d}{d\sigma} \theta_\nu(\sigma) \geq \frac{\mu}{2},$$

for all $\sigma \in [-\sigma_0, \sigma_0]$ and $\nu \geq \nu_0$ where $\mu := \mathcal{S}_V(x^-) - \mathcal{S}_V(x^+) > 0$.

Proof. $'\Leftarrow'$ follows by definition of the formal adjoint operator using that $\partial_s u^\sigma \in \ker D_{u^\sigma}$. We prove $'\Rightarrow'$. The kernel of the linear operator $D_{u^\sigma}$ is 1-dimensional: It is Fredholm of index one by theorem 1.10 and it is onto by the Morse–Smale condition. This kernel is spanned by the (nonzero) element $\partial_s u^\sigma$. Now consider $D^*_{u^\sigma}$ on the domain $\mathcal{W}^{2,p}$ and apply proposition 2.19 to obtain that $\mathcal{W}^{1,p} = \ker D_{u^\sigma} \oplus \text{im } D^*_{u^\sigma}$. The implication $'\Rightarrow'$ now follows immediately by contradiction.

Set $1/q + 1/p = 1$. By (66) and the definition of the sequence $\varepsilon_\nu \to 0$ in step 2 it follows that

$$|\theta_\nu(0)| = |\langle \partial_s u, \xi_\nu \rangle|_{L^2} \leq ||\partial_s u||_q ||\xi_\nu||_p \leq c_3 \varepsilon_\nu.$$ 

Abbreviate $E_i = E_i(u^\sigma, \eta)$. Then straightforward calculation using the identity (67) for $\nabla \eta$ shows that

$$\frac{d}{d\sigma} \theta_\nu(\sigma) = -\langle \nabla_s \partial_s u^\sigma, \eta \rangle_{L^2} - \langle \partial_s u^\sigma - \partial_s u^\sigma + \partial_s u^\sigma - E^{-1}_1 \partial_s u^\sigma \rangle_{L^2}$$

$$\geq - \|\nabla \partial_s u^\sigma\|_q \|\eta\|_p + \|\partial_s u^\sigma\|_q^2 - \|\partial_s u^\sigma\|_q \|\partial_s u^\sigma\|_\infty c_4 \|\eta\|_p$$

$$= \|\partial_s u\|_2^2 - \|\eta\|_p \left( \|\nabla \partial_s u\|_q + c_4 \|\partial_s u\|_q \right)$$

$$\geq \|\partial_s u\|_2^2 - (2\varepsilon_\nu + c_2)\sigma((c_5 + c_3 c_4))$$

for some constant $c_4 = c_4(a_0, \sigma_0) > 0$. The last step is by (66) with constant $c_3$. We also used that $\|\nabla \partial_s u\|_q \leq c_5$ for some positive constant $c_5 = c_5(a_0)$, which follows from exponential decay of $\nabla \partial_s u$ according to theorem 1.9. The energy identity (9) shows that $\|\partial_s u\|_2^2 = \mu > 0$. Now choose $\sigma_0 > 0$ sufficiently small and $\nu_0$ sufficiently large to conclude the proof of step 3.

Step 4. We prove the claim.

Proof. By step 3 there exists, for every sufficiently large $\nu$, an element $\sigma_\nu \in [-\sigma_0, \sigma_0]$ such that $\theta_\nu(\sigma_\nu) = 0$ and $|\sigma_\nu| \leq \varepsilon_\nu(2c_3/\mu)$. Set $\eta_\nu := \eta(\sigma_\nu, \nu)$. Then $\eta_\nu \in \text{im } D^*_{u^\sigma}$ again by step 3 and

$$||\eta_\nu||_\infty + ||\eta_\nu||_p \leq \varepsilon_\nu(3 + (c_2 + C_0)2c_3/\mu), \quad ||\eta_\nu||_V \leq C,$$

by step 2. This proves (59), hence the claim, and therefore theorem 1.13.
5 Unique Continuation

To prove unique continuation for the nonlinear heat equation we slightly extend a result of Agmon and Nirenberg [AN67] (to the case \(C_1 \neq 0\)). This generalization is needed to deal with the nonlinear heat equation (6), since here nonzero order terms appear on the right hand side of (69). For the linear heat equation the original result for \(C_1 = 0\) is sufficient.

**Theorem 5.1.** Let \(H\) be a real Hilbert space and let \(A(s) : \text{dom } A(s) \to H\) be a family of symmetric linear operators. Assume that \(\zeta : [0, T] \to H\) is continuously differentiable in the weak topology such that \(\zeta(s) \in \text{dom } A(s)\) and

\[
\|\zeta'(s) - A(s)\zeta(s)\| \leq c_1 \|\zeta(s)\| + C_1 \|\langle A(s)\zeta(s), \zeta(s)\rangle\|^{1/2} \tag{69}
\]

for every \(s \in [0, T]\) and two constants \(c_1, C_1 \geq 0\). Here \(\zeta'(s) \in H\) denotes the derivative of \(\zeta\) with respect to \(s\). Assume further that the function \(s \mapsto \langle \zeta(s), A(s)\zeta(s) \rangle\) is also continuously differentiable and satisfies

\[
\frac{d}{ds} \langle \zeta, A\zeta \rangle - 2\langle \zeta', A\zeta \rangle \geq -c_2 \|A\zeta\| \|\zeta\| - c_3 \|\zeta\|^2 \tag{70}
\]

pointwise for every \(s \in [0, T]\) and constants \(c_2, c_3 > 0\). Then the following holds.

1. If \(\zeta(0) = 0\) then \(\zeta(s) = 0\) for all \(s \in [0, T]\).
2. If \(\zeta(0) \neq 0\) then \(\zeta(s) \neq 0\) for all \(s \in [0, T]\) and, moreover,

\[
\log \|\zeta(s)\|^2 \geq \log \|\zeta(0)\|^2 - \left(2 \frac{\langle \zeta(0), A(0)\zeta(0) \rangle}{\|\zeta(0)\|^2} + \frac{b}{a} \right) c_3^a s - \frac{1}{a} - 2c_1 s
\]

where \(a = 2C_1^2 + c_2\) and \(b = 4c_1^2 + c_2^2/2 + 2c_3\).

**Proof.** A beautiful exposition in the case \(C_1 = 0\) was given by Salamon in [S97, appendix E] in the case \(C_1 = 0\). It generalizes easily. A key step is to prove that the function

\[
\varphi(s) := \log \|\zeta(s)\|^2 - \int_0^s 2\frac{2\langle \zeta(\sigma), \zeta'(\sigma) - A(\sigma)\zeta(\sigma) \rangle}{\|\zeta(\sigma)\|^2} d\sigma
\]

satisfies the differential inequality

\[
\varphi'' + a |\varphi'| + b \geq 0 \tag{71}
\]

for two constants \(a, b > 0\).

In [S97] it is shown that assumption (70) implies the inequality

\[
\varphi'' \geq 2 \|\eta - \langle \eta, \xi \rangle \xi\|^2 - 2 \frac{2\|\zeta' - A\zeta\|^2}{\|\zeta\|^2} - 2c_2 \|\eta\| - 2c_3
\]

where

\[
\xi := \frac{\zeta}{\|\zeta\|}, \quad \eta := \frac{A\zeta}{\|\zeta\|}.
\]
Now it follows by assumption (69) that

\[
\frac{2 \| \zeta' - A\zeta \|^2}{\|\zeta\|^2} \leq 4c_1^2 + 4C_1^2 \frac{| \langle A\zeta, \zeta \rangle |}{\|\zeta\|^2} = 4c_1^2 + 4C_1^2 |\langle \eta, \xi \rangle |
\]

and therefore

\[
\varphi'' \geq 2 \| \eta - \langle \eta, \xi \rangle \xi \|^2 - 4c_1^2 - 4C_1^2 |\langle \eta, \xi \rangle| - 4c_2 \| \eta \| - 2c_3.
\]

To obtain the inequality (71) it remains to prove that

\[
2 \| \eta - \langle \eta, \xi \rangle \xi \|^2 - 4c_1^2 - 4C_1^2 |\langle \eta, \xi \rangle| - 2c_2 \| \eta \| - 2c_3 \geq -a |\varphi'| - b.
\]

Since \( \varphi' = 2 \langle \xi, \eta \rangle \) this is equivalent to

\[
c_2 \| \eta \| \leq \| \eta - \langle \eta, \xi \rangle \xi \|^2 + (a - 2C_1^2) |\langle \eta, \xi \rangle| + (b/2 - 2c_1^2 - c_3).
\]

Abbreviate

\[ u := \| \eta - \langle \eta, \xi \rangle \xi \|^2, \quad v := |\langle \eta, \xi \rangle|, \]

then \( \| \eta \|^2 = u^2 + v^2 \) and the desired inequality has the form

\[
c_2 \sqrt{u^2 + v^2} \leq u^2 + (a - 2C_1^2)v + (b/2 - 2c_1^2 - c_3).
\]

Since \( c_2 \sqrt{u^2 + v^2} \leq c_2 u + c_2 v \leq u^2 + c_2 v + c_2^2/4 \) this is satisfies with

\[
a = 2C_1^2 + c_2, \quad b = 4c_1^2 + c_2^2/2 + 2c_3.
\]

This proves the inequality (71). The remaining part of the proof of theorem 5.1 carries over from \([S97]\) unchanged. \( \square \)

## 5.1 Linear equation

Unique continuation for the linearized heat equation is used to prove proposition 6.7 on transversality of the universal section and the unstable manifold theorem 7.1.

**Proposition 5.2.** Fix a perturbation \( \mathcal{V} : \mathcal{L}M \to \mathbb{R} \) that satisfies (V0)–(V2) and two constants \( a < b \). Let \( u : [a, b] \times S^1 \to M \) be a smooth map and let \( \xi = \xi(s, t) \) be a smooth vector field along \( u \) such that \( \mathcal{D}_u \xi = 0 \) or \( \mathcal{D}^*_u \xi = 0 \), where the operators are defined by (27) and (28), respectively. Abbreviate \( \xi(s, \cdot) \) by \( \xi(s) \). Then the following is true.

(a) If \( \xi(s) = 0 \) for some \( s_* \), then \( \xi(s) = 0 \) for all \( s \in [a, b] \).

(b) If \( \xi(s) \neq 0 \) for some \( s_* \), then \( \xi(s) \neq 0 \) for all \( s \in [a, b] \).

**Proof.** We represent \( \mathcal{D}_u \) by the operator \( D_{A+C} = \frac{d}{ds} + A(s) + C(s) \) given by (29). Here the family \( A(s) \) consists of self-adjoint operators on the Hilbert space \( H := L^2(S^1, \mathbb{R}^n) \) with dense domain \( W \); see (ii) and (iv) in section 2.4. The space
W has been defined prior to (29). Recall that if the vector bundle $u^*TM \to [a, b] \times S^1$ is trivial then $W = W^{2,2}(S^1, \mathbb{R}^n)$ and otherwise some boundary condition enters. In either case $W =: \text{dom} A(s)$ is independent of $s$.

(b) Let $\xi \in \ker D_{A+C}$ satisfy $\xi(s_*) \neq 0$. Assume by contradiction that $\xi(s_0) = 0$ for some $s_0 \in [a, b]$. Now if $s_0 > s_*$, then replace $\xi(s)$ by $\xi(s + s_*)$ and set $T = b - s_*$ and $s_1 = s_0 - s_*$, otherwise replace $\xi(s)$ by $\xi(s + s_*)$ and set $T = a + s_*$ and $s_1 = -s_0 + s_*$. Hence we may assume without loss of generality that $\xi \in \ker D_{A+C}$ maps $[0, T]$ to $H$ and satisfies $\xi(0) \neq 0$ and $\xi(s_1) = 0$ for some $s_1 \in (0, T]$.

Next we check that the conditions in theorem 5.1 are satisfied: Firstly, the vector field $\xi$ is smooth by assumption. Secondly, the family $A(s)$ consists of self-adjoint operators by (ii) in section 2.4. Thirdly, the function $s \mapsto \langle \xi(s), A(s)\xi(s) \rangle$ is continuously differentiable. Here we use the first condition in axiom (V2), which tells that the Hessian $H$ is a zeroth order operator, and the fact that by compactness of the domain the vector fields $\partial_t u, \partial_s u, \nabla_i \partial_s u,$ and $\nabla_j \nabla_i \partial_s u$ are bounded in $L^\infty([0, T] \times S^1)$ by a constant $c_T > 0$. Next assumption (69) is satisfied with $C_1 = 0$, because

$$\|\xi'(s) - A(s)\xi(s)\| = \|C(s)\xi(s)\| \leq c'_T \|\xi(s)\|$$

where the constant $c'_T = \sup_{[0, T] \times S^1} \|C(s, t)\|_{L^\infty(\mathbb{R}^n)}$ is finite by compactness of the domain. To verify the inequality (70) note that its left hand side is given by $\langle \xi(s), A'(s)\xi(s) \rangle$; see [AN67, Rmk. in sec. 1] and [S97, Rmk. F.3]. Now

$$\langle \xi(s), A'(s)\xi(s) \rangle \geq -\|\xi(s)\| \|A'(s)\xi(s)\| \geq -c'_T \|\xi(s)\| (\|\xi(s)\| + \|\partial_t \xi(s)\|).$$

where the second step is by straightforward calculation of $A'(s)$. Replacing $\|\partial_t \xi(s)\|$ according to the elliptic estimate for $A(s)$ yields (70).

Now the Agmon-Nirenberg theorem 5.1 applies and part (2) tells that $\xi(s) \neq 0$ for all $s \in [0, T]$. This contradiction proves (b) for elements in the kernel of $D_u$. The same argument covers the case of the operator $D_{u_*}$, since it is represented by $-D_{-A-C}$ according to remark 2.16.

(a) This follows either by a time reversing argument (see proof of the Agmon-Nirenberg Theorem in [S97]) and application of (b) or by a line of argument analogous to the proof of (b) given above, where in the final step part (2) of theorem 5.1 is replaced by part (1).

\[\square\]

5.2 Nonlinear equation

Unique continuation for the nonlinear heat equation is used to prove the unstable manifold theorem 7.1.

Theorem 5.3 (Unique Continuation for compact cylindrical domains). Fix two constants $a < b$ and a perturbation $V : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0) and (V1). If two smooth solutions $u, v : [a, b] \times S^1 \to M$ of the heat equation (6) coincide along one loop, then $u = v$. 

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Proof. Abbreviate \( u_s = u(s, \cdot) \) and assume \( u_s = v_\sigma : S^1 \to M \) for some \( \sigma \in [a, b] \). Moreover, we may assume without loss of generality that \( \partial_s u \) is nonzero at some point \((s, t)\). Otherwise \( u \) coincides with a critical point \( x \) of the action functional \( S_V \) and, since \( v_\sigma = u_\sigma = x \), so does \( v \) and we are done. It follows similarly that \( \partial_s v \) is nonzero somewhere. Hence

\[
\delta := \frac{\ell}{2 + \|\partial_s u\|_\infty + \|\partial_s v\|_\infty} \in (0, \ell/2).
\]

Here \( \ell > 0 \) denotes the injectivity radius of our compact Riemannian manifold.

The first step is to prove that the restrictions of \( u \) and \( v \) to \([\sigma - \delta, \sigma + \delta] \times S^1\) are equal. (In fact we should take the intersection with \([a, b] \times S^1\), but suppress this throughout for simplicity of notation.) The key idea is to express the difference of \( u \) and \( v \) near \( \sigma \) with respect to geodesic normal coordinates based at \( u_\sigma \) and show that this difference \( \zeta \) and a suitable operator \( A \) satisfy the requirements of theorem 5.1 (with nonzero constant \( C_1 \)). Then, since \( \zeta(\sigma) = 0 \), part (1) of the theorem shows that \( \zeta = 0 \) and therefore \( u = v \) on \([\sigma - \delta, \sigma + \delta] \times S^1\).

Once the above has been achieved we successively restrict \( u \) and \( v \) to cylinders of the form \([\sigma + (2k - 1)\delta, \sigma + (2k + 1)\delta] \times S^1\), where \( k \in \mathbb{Z} \), and use that \( u \) and \( v \) coincide along one of the two boundary components to conclude by the same argument as above that \( u = v \) on each of these cylinders. Due to compactness of \( Z \) the same constants \( c_1 \) and \( C_1 \) can be chosen in (69) for all cylinders. After finitely many steps the union of these cylinders covers \([a, b] \times S^1\) and this proves the theorem.

It remains to carry out the first step. Consider the interval \( I = [\sigma - \delta, \sigma + \delta] \) and the cylinder

\[
Z = I \times S^1 = [\sigma - \delta, \sigma + \delta] \times S^1.
\]

From now on \( u \) and \( v \) are restricted to the domain \( Z \). Note that the Riemannian distance between \( u(\sigma, t) \) and \( u(s, t) \) is less than half the injectivity radius \( \ell \) for every \((s, t) \in Z\). Hence the identities

\[
u(s, t) = \exp_{u(\sigma, t)} \xi(s, t), \quad v(s, t) = \exp_{u(\sigma, t)} \eta(s, t)
\]

for \((s, t) \in Z\) uniquely determine smooth families of vector fields \( \xi \) and \( \eta \) along the loop \( u_\sigma \). The domain of \( \xi \) and \( \eta \) is \( Z \), they satisfy the estimates

\[
\|\xi\|_\infty < \frac{\ell}{2}, \quad \|\eta\|_\infty < \frac{\ell}{2},
\]

and \( \xi(\sigma, t) = 0 = \eta(\sigma, t) \) for every \( t \in S^1 \). Moreover, since \( \xi(s, t) \) and \( \eta(s, t) \) live in the same tangent space \( T_{u(\sigma, t)} M \) their difference \( \zeta = \xi - \eta \) is well defined.

Now consider the Hilbert space \( H = L^2(S^1, u_\sigma^* TM) \) and the symmetric differential operator \( A = \nabla_s \nabla_t \) with domain \( W = W^{2,2}(S^1, u_\sigma^* TM) \). Here \( \nabla_t \) denotes the covariant derivative along the loop \( u_\sigma \). Hence the operator \( A \) is independent of \( s \) and condition (70) in the Agmon-Nirenberg theorem 5.1 is vacuous. If we can verify condition (69) as well, then \( \zeta(\sigma) = 0 \) implies that \( \zeta(s) = 0 \) for every \( s \in I \) by theorem 5.1 (1). Since \( \zeta \) is smooth, this means that on \( Z \) we have

\[
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\]
\( \xi = \eta \) pointwise and therefore \( u = v \).

It remains to verify (69). Use (21) to obtain the identities

\[
\partial_s u = E_2(u_\sigma, \xi) \partial_s \xi \\
\nabla \partial_t u = E_{11}(u_\sigma, \xi) \overbrace{(\partial_t u_\sigma, \partial_t u_\sigma)} + 2E_{12}(u_\sigma, \xi) \overbrace{(\partial_t u_\sigma, \nabla \xi)} + E_1(u_\sigma, \xi) \nabla \partial_t u_\sigma + E_{22}(u_\sigma, \xi) \overbrace{(\nabla \xi, \nabla \xi)} + E_2(u_\sigma, \xi) \nabla \nabla \xi
\]

(73)

pointwise for \((s, t) \in Z\) and similarly for \(v\) and \(\eta\). To obtain the second identity we used the symmetry property (22) of \(E_{12}\). Now consider the heat equation (6) and replace \(\partial_s u\) and \(\nabla \partial_t u\) according to (73), then solve for \(\partial_s \xi - \nabla \xi \nabla \xi\). Do the same for \(v\) and \(\eta\) to obtain a similar expression for \(-\partial_s \eta + \nabla \nabla \eta\). Add both expressions to get the pointwise identity

\[
(\partial_s - \nabla \nabla) (\xi - \eta) = (E_2(u_\sigma, \xi)^{-1} E_{11}(u_\sigma, \xi) - E_2(u_\sigma, \eta)^{-1} E_{11}(u_\sigma, \eta)) (\partial_t u_\sigma, \partial_t u_\sigma) \\
+ (E_2(u_\sigma, \xi)^{-1} E_1(u_\sigma, \xi) - E_2(u_\sigma, \eta)^{-1} E_1(u_\sigma, \eta)) \nabla \partial_t u_\sigma \\
+ 2 (E_2(u_\sigma, \xi)^{-1} E_{21}(u_\sigma, \xi) \nabla \xi - E_2(u_\sigma, \eta)^{-1} E_{21}(u_\sigma, \eta) \nabla \eta) \partial_t u_\sigma \\
+ E_2(u_\sigma, \xi)^{-1} \nabla \nabla (\exp \sigma, \xi) - E_2(u_\sigma, \eta)^{-1} \nabla \nabla (\exp \sigma, \eta) \\
+ E_2(u_\sigma, \xi)^{-1} E_{22}(u_\sigma, \xi) (\nabla \xi, \nabla \xi) - E_2(u_\sigma, \eta)^{-1} E_{22}(u_\sigma, \eta) (\nabla \eta, \nabla \eta).
\]

Now by compactness of the domain \(Z\) there is a constant \(C > 0\) such that

\[
\| \partial_t u_\sigma \|_{L^\infty(S^1)} \leq \| \partial_t u \|_{L^\infty(Z)} < C, \quad \| \nabla \partial_t u_\sigma \|_{L^\infty(S^1)} < C.
\]

Moreover, since the maps \(E_i\) and \(E_{ij}\) are uniformly continuous on the radius \(\ell/2\) disk tangent bundle \(\mathcal{O} \subset TM\) in which \(\xi\) and \(\eta\) take their values, there exists a constant \(c_1 > 0\) such that

\[
\| \partial_s (\xi - \eta) - \nabla \nabla (\xi - \eta) \| \leq (c_1 C^2 + c_1 C) \| \xi - \eta \| \\
+ 2C \| E_2(u_\sigma, \xi)^{-1} E_{21}(u_\sigma, \xi) \nabla \xi - E_2(u_\sigma, \eta)^{-1} E_{21}(u_\sigma, \eta) \nabla \eta \| \\
+ \| E_2(u_\sigma, \xi)^{-1} \nabla \nabla (\exp \sigma, \xi) - E_2(u_\sigma, \eta)^{-1} \nabla \nabla (\exp \sigma, \eta) \| \\
+ \| E_2(u_\sigma, \xi)^{-1} E_{22}(u_\sigma, \xi) (\nabla \xi, \nabla \xi) - E_2(u_\sigma, \eta)^{-1} E_{22}(u_\sigma, \eta) (\nabla \eta, \nabla \eta) \| 
\]

pointwise for \((s, t) \in Z\). It remains to estimate the last three terms in the sum. First we estimate term three. Use linearity and the symmetry property (22) of \(E_{22}\) to obtain the first identity in the pointwise estimate

\[
\| E_2(u_\sigma, \xi)^{-1} E_{22}(u_\sigma, \xi) (\nabla \xi, \nabla \xi) - E_2(u_\sigma, \eta)^{-1} E_{22}(u_\sigma, \eta) (\nabla \eta, \nabla \eta) \| \\
= \| E_2(u_\sigma, \xi)^{-1} E_{22}(u_\sigma, \xi) (\nabla \xi - \nabla \eta, \nabla \xi) \\
+ E_2(u_\sigma, \eta)^{-1} E_{22}(u_\sigma, \eta) (\nabla \xi - \nabla \eta, \nabla \eta) \\
+ (E_2(u_\sigma, \xi)^{-1} E_{22}(u_\sigma, \xi) - E_2(u_\sigma, \eta)^{-1} E_{22}(u_\sigma, \eta)) (\nabla \xi, \nabla \eta) \| \\
\leq \| E_2^{-1} E_{22} \|_{L^\infty(C)} \||\nabla \xi\|_\infty + \|\nabla \eta\|_\infty \| (\nabla \xi - \nabla \eta) \| \\
+ c_1 \|\nabla \xi\|_\infty \|\nabla \eta\|_\infty \| \xi - \eta \| \\
\leq \mu_1 \|\nabla \xi - \eta\| + \mu_2 \| \xi - \eta \|
\]
where \( \mu_1 = 2c_2^2 C(1 + c_2) \), \( \mu_2 = c_1 c_2^2 C^2 (1 + c_2)^2 \), and the constant \( c_2 > 0 \) is chosen sufficiently large such that for \( j = 0, 1 \) we have
\[
\| E_j \|_{L^\infty(O)} + \| E_j^{-1} \|_{L^\infty(O)} + \| E_2^{-1} E_{22} \|_{L^\infty(O)} + \| E_2^{-1} E_{21} \|_{L^\infty(O)} \leq c_2.
\]
Moreover, we used that by the first identity in (21)
\[
\nabla t \xi = E_2 (u_\sigma, \xi)^{-1} (\partial_t u - E_1 (u_\sigma, \xi) \partial_t u_\sigma).
\]
Hence \( \| \nabla t \xi \|_\infty \leq c_2 C (1 + c_2) \) and similarly for \( \nabla t \eta \). Next we estimate term one. Replace \( \nabla t \xi \) by \( \nabla t \xi - \nabla t \eta + \nabla t \eta \), then similarly as above we obtain that
\[
2C | E_2 (u_\sigma, \xi)^{-1} E_{21} (u_\sigma, \xi) \nabla t \xi - E_2 (u_\sigma, \eta)^{-1} E_{21} (u_\sigma, \eta) \nabla t \eta | \\
\leq 2c_2 C | \nabla t (\xi - \eta) | + 2c_1 c_2 C^2 (1 + c_2) | \xi - \eta |\
\]
pointwise for \( (s, t) \in Z \). Next rewrite term two setting \( X := \eta - \xi \) and replacing \( \eta \) accordingly to obtain pointwise at \( (s, t) \in Z \) the identity
\[
E_2 (u_\sigma, \xi)^{-1} \nabla \nabla \mathcal{V} (\exp_{u_\sigma} \xi) - E_2 (u_\sigma, \xi + X)^{-1} \nabla \nabla \mathcal{V} (\exp_{u_\sigma} \xi + X)
\]
\[
=: f(X)
\]
\[
= f(0) + \frac{d}{d\tau} f(\tau X)
\]
\[
= \frac{d}{d\tau} (E_2 (u_\sigma, \xi + \tau X)^{-1} \nabla \nabla \mathcal{V} (\exp_{u_\sigma} \xi + \tau X))
\]
for some \( \tau \in [0, 1] \). Since \( f(0) = 0 \), this implies that
\[
| f(X) | \leq \| E_2^{-1} E_{22} \|_{L^\infty(O)} | X | \cdot \| E_2^{-1} \|_{L^\infty(O)} \| \nabla \nabla \mathcal{V} (\exp_{u_\sigma} (\xi + \tau X)) | \\
+ \| E_2^{-1} \|_{L^\infty(O)} \| \nabla \tau \nabla \mathcal{V} (\exp_{u_\sigma} (\xi + \tau X)) | \\
\leq c_2 C_0 | X | + c_2 C_1 \left( | X | + \| X_s \|_{L^1(S^1)} \right)
\]
pointwise at \( (s, t) \in Z \). Here \( C_0 \) and \( C_1 \) denote the constants in axiom (V0) and (V1), respectively. To obtain the final step we applied the first estimate in axiom (V1) to the curve \( \tau \mapsto \exp_{u_\sigma} (\xi_s + \tau X_s) \) in the loop space \( LM \). Now replace \( X \) by \( \eta - \xi \).

Putting things together we have proved that due to compactness of the domain \( Z \) there exists a positive constant \( \mu = \mu (Z, g) \) such that for every \( s \in I \)
\[
\| \zeta'(s) - A \zeta(s) \| \leq \mu (| \zeta(s) | + \| \nabla \zeta(s) \|).
\]
Here the norm is in \( L^2(S^1, u_\sigma^* TM) \). Now by integration by parts
\[
\| \nabla \zeta \|^2 = \langle \nabla \zeta, \nabla \zeta \rangle = - \langle A \zeta, \zeta \rangle \leq | \langle A \zeta, \zeta \rangle |.
\]
Hence (69) is satisfied and this concludes the proof of theorem 5.3. \( \square \)

In the proof of the unstable manifold theorem 7.1 we use backward unique continuation for the nonlinear heat equation.

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Theorem 5.4 (Forward and backward unique continuation). Fix a perturbation $\mathcal{V} : L^\infty M \to \mathbb{R}$ that satisfies $(V0)\sim(V1)$.

(F) Let $u$ and $v$ be smooth solutions of the heat equation (6) defined on the forward halfcylinder $[0, \infty) \times S^1$. If $u$ and $v$ agree along the loop at $s = 0$, then $u = v$.

(B) Let $u$ and $v$ be smooth solutions of the heat equation (6) defined on the backward halfcylinder $(-\infty, 0] \times S^1$. Assume further that
\[
\sup_{s \in (-\infty, 0]} \mathcal{S}_\mathcal{V}(u(s, \cdot)) \leq c_0, \quad \sup_{s \in (-\infty, 0]} \mathcal{S}_\mathcal{V}(v(s, \cdot)) \leq c_0,
\]
for some constant $c_0 > 0$. Then the following is true. If $u$ and $v$ agree along the loop at $s = 0$, then $u = v$.

Proof. The idea is the same as in the proof of theorem 5.3, namely to decompose the halfcylinder into small cylinders of width $\delta$ and then show $u = v$ on each piece (by the method developed in the first step of the proof of theorem 5.3). The only additional problem is noncompactness of the domain. One way to deal with this is to choose the same width for each piece (in order to arrive at any given time $s$ in finitely many steps). Here we need uniform bounds for $|\partial_s u|$ and $|\partial_v v|$. Once we have these we can define $\delta$ again by (72). Check the proof of theorem 5.3 to see that the only further ingredients in proving $u = v$ on each small cylinder are uniform bounds for the first two $t$-derivatives of $u$ and of $v$.

Hence to complete the proof it remains to show that
\[
\|\partial_s u\|_\infty + \|\partial_t u\|_\infty + \|\nabla_\ell \partial_s u\|_\infty + \|\partial_s v\|_\infty + \|\partial_t v\|_\infty + \|\nabla_\ell \partial_t v\|_\infty \leq C
\]
for some constant $C > 0$.

ad (F) Let $C_0$ be the constant in axiom $(V0)$ and observe that $\mathcal{S}_\mathcal{V} \geq -C_0$. Now by theorem 3.9 with constant $C_1$, more precisely, by checking its proof
\[
|\partial_s u(s, t)|^2 \leq C_1 E_{[s-1, s]}(u) = C_1 (\mathcal{S}_\mathcal{V}(u_{s-1}) - \mathcal{S}_\mathcal{V}(u_s)) \leq C_1 (\mathcal{S}_\mathcal{V}(u_0) + C_0)
\]
for $(s, t) \in [1, \infty) \times S^1$. In the second and the last step we used that $u$ is a negative gradient flow line and the action decreases along $u$. Note that the proof of theorem 3.9 shows that the estimate at a point depends on its past. This is why we get the above estimate only on $[1, \infty) \times S^1$. However, the missing part $[0, 1] \times S^1$ is compact and $u$ is smooth. Hence $\|\partial_s u\|_\infty \leq C$ and
\[
\|\nabla_\ell \partial_t u\|_\infty \leq \|\partial_s u\|_\infty + \|\nabla_\ell \mathcal{V}(u)\|_\infty \leq C + C_0.
\]
Here we used the heat equation (6) and axiom $(V0)$ with constant $C_0$. It follows similarly by (checking the proof of) theorem 3.5 that $|\partial_t u(s, t)|$ is uniformly bounded on $[1, \infty) \times S^1$. The corresponding estimates for $v$ are analogous.

ad (B) The proof of the $L^\infty$ estimates follows the same steps as in (F). We even get all estimates right away on the whole backward halfcylinder, because this halfcylinder contains the past of each of its points. \qed
6 Transversality

In section 6.1 we construct a separable Banach space $Y$ of abstract perturbations satisfying axioms (V0)–(V3). In section 6.2 we fix a perturbation $V$ such that (V0)–(V3) hold and $S_V$ is Morse. We choose a closed $L^2$ neighborhood $U$ of the critical points of the function $S_V$ and define the subspace $Y(V, U) \subset Y$ of those perturbations supported away from $U$. Then, given a regular value $a$ of $S_V$, we define a separable Banach manifold $O^a = O^a(V, U)$ of admissible perturbations. In fact $O^a$ is the open ball about zero in the Banach space $Y(V, U)$ for some sufficiently small radius $r(a)$. For any admissible perturbation $v$ it holds that $P_a(V) = P_a(V + v)$ – in particular $a$ is also a regular value of $S_{V+v}$ – and the sublevel sets $\{S_V \leq a\}$ and $\{S_{V+v} \leq a\}$ are homologically equivalent. For such a triple $(V, U, a)$ we prove in section 6.3 that there is a residual subset $O^a_{reg} \subset O^a$ of regular perturbations $v$. These, in addition, have the property that the perturbed functional $S_{V+v}$ is Morse–Smale below level $a$. The crucial step is to prove proposition 6.7 on surjectivity of the universal section $F$. Here unique continuation for the linear heat equation enters. A further key ingredient in the ‘no return’ part of the proof is the (negative) gradient flow property which implies that the functional is strictly decreasing along nonconstant heat flow solutions.

6.1 The universal Banach space of perturbations

We fix, once and for all, the following data.

a) A dense sequence $(x_i)_{i \in \mathbb{N}}$ in $\mathcal{L}M = C^\infty(S^1, M)$.

b) For every $x_i$ a dense sequence $(\eta_{ij})_{j \in \mathbb{N}}$ in $C^\infty(S^1, x_i^*TM)$.

c) A smooth cutoff function $\rho : \mathbb{R} \to [0, 1]$ such that $\rho = 1$ on $[-1, 1]$ and $\rho = 0$ outside $[-4, 4]$ and such that $\|\rho'\|_\infty < 1$. Then set $\rho_{1/k}(r) = \rho(rk^2)$ for $k \in \mathbb{N}$ (figure 1).

Moreover, let $\iota > 0$ denote the injectivity radius of the closed Riemannian manifold $M$ and fix a smooth cutoff function $\beta$ such that $\beta = 1$ on $[-(\iota/2)^2, (\iota/2)^2]$ and $\beta = 1$ outside $[-\iota^2, \iota^2]$ (figure 2).

Figure 1: The cutoff function $\rho_{1/k}$

Figure 2: The cutoff function $\beta$
Then for any choice of \(i, j, k \in \mathbb{N}\) there is a smooth function on the loop space given by

\[
V^i_j(x) = V^{ijk}(x) = \rho_1/k \left( \|x - x_i\|_{L^2} \right) \int_0^1 V^{ij}(t, x(t)) \, dt, \tag{74}
\]

where \(V^{ij}\) is the smooth function on \(S^1 \times M\) defined by

\[
V^{ij}(t, q) := \begin{cases} 
\beta(\|\xi^i(t)\|^2) \langle \xi^i(t), \eta^{ij(t)} \rangle & \text{if } \|\xi^i(t)\| < \iota, \\
0 & \text{else.}
\end{cases}
\]

Here the vector \(\xi^i(t)\) is determined by the identity

\[
q = \exp_{x_i(t)} \xi^i(t)
\]

whenever the Riemannian distance between \(q\) and \(x_i(t)\) is less than \(\iota\). To simplify notation we fixed a bijection \(\ell \colon \mathbb{N}^3 \to \mathbb{N}_0\). Observe that the support of \(V^{ijk}\) is contained in the \(L^2\) ball of radius \(2/k\) about \(x_i\). Each function \(V^{i\ell} : L^2 M \to \mathbb{R}\) is uniformly continuous with respect to the \(C^0\) topology and satisfies (V0)–(V3). This follows by compactness of \(M\), smoothness of the potential \(V\), and by the identity

\[
\langle \nabla V(u), \partial_s u \rangle_{L^2} = \frac{d}{ds} V(u)
\]

which determines \(\nabla V\). Here \(\mathbb{R} \to L^2 M : s \mapsto u(s, \cdot)\) is any smooth map.

Given \(V_i\), we fix a constant \(C^0_i \geq 1\) which is greater than its constant of uniform continuity and for which (V0) holds true. Then we fix a constant \(C^1_i \geq C^0_i\) for which both estimates in (V1) hold true and a constant \(C^2_i \geq C^1_i\) to cover the three estimates of (V2). Furthermore, for every integer \(i \geq 3\), we choose a constant \(C^i_i \geq C^i_{i-1}\) that covers all estimates in (V3) with \(k' + \ell' = i\) (here \(k'\) and \(\ell'\) denote the integers \(k\) and \(\ell\) that appear in (V3)). To summarize, for each integer \(\ell \geq 0\) we have fixed a sequence of constants

\[
1 \leq C^0_\ell \leq C^1_\ell \leq ... \leq C^i_\ell \leq ... \quad \forall \ell \in \mathbb{N}_0. \tag{75}
\]

The \textit{universal space of perturbations} is the normed linear space

\[
Y = \left\{ v_\lambda := \sum_{\ell=0}^{\infty} \lambda_\ell V_\ell \mid \lambda = (\lambda_\ell) \subset \mathbb{R} \text{ and } ||v_\lambda|| := \sum_{\ell=0}^{\infty} |\lambda_\ell| C^i_\ell < \infty \right\}. \tag{76}
\]

**Proposition 6.1.** The universal space \(Y\) of perturbations is a separable Banach space and every \(v_\lambda \in Y\) satisfies the axioms (V0)–(V3).
Proof. The map $v_\lambda \mapsto (\lambda_t C_t^i)_{t \in \mathbb{N}_0}$ provides an isomorphism from $Y$ to the separable Banach space $\ell^1$ of absolutely summable real sequences. This proves that $Y$ is a separable Banach space. That every element $v_\lambda = \sum \lambda_t V_t$ of $Y$ satisfies (V0)–(V3) follows readily from the corresponding property of the generators $V_t$. To explain the idea we give the proof of the second estimate in (V2), namely

$$|\nabla L \cdot \text{grad} v_\lambda(u)| \leq \sum_{\ell=0}^{\infty} |\lambda_\ell| \cdot |\nabla \nabla L \cdot \text{grad} V_t(u)|$$

$$\leq \left( |\lambda_0| C_0^2 + |\lambda_1| C_1^2 + \sum_{\ell=2}^{\infty} |\lambda_\ell| C_\ell^2 \right) f(u)$$

$$\leq \left( |\lambda_0| C_0^2 + |\lambda_1| C_1^2 + \|v_\lambda\| \right) f(u)$$

for every smooth map $\mathbb{R} \to \mathcal{L}M : s \mapsto u(s, \cdot)$ and every $(s, t) \in \mathbb{R} \times S^1$. We abbreviated $f(u) = (|\nabla \partial_t u| + (1 + |\partial_s u|)|\partial_s u| + |\partial_s u|_{\mathcal{L}^1})$. Step two uses the second estimate in (V2) for each $V_t$ with constant $C_t^2$. Step three follows from $C_t^k \leq C_t^2$ whenever $\ell \geq k$, see (75). The remaining estimates in (V0)–(V3) follow by the same argument. Continuity of $v_\lambda$ with respect to the $C^0$ topology follows similarly using uniform continuity of the functions $V_t$. \hfill \Box

6.2 Admissible perturbations

Throughout we fix a perturbation $Y$ that satisfies (V0)–(V3) and such that $S_Y : \mathcal{L}M \to \mathbb{R}$ is Morse. Denote the critical values $c_0$ of $S_Y$ by

$$c_0 < c_1 < c_2 < \ldots < c_k < a < c_{k+1} < \ldots$$

and recall that there is no accumulation point, because $S_Y$ admits only finitely many critical points on each sublevel set. Now fix a regular value $a > c_0$ (otherwise $\{S_Y \leq a\} = \emptyset$ and we are done) and let $c_k$ be the largest critical value smaller than $a$. If there are critical values larger than $a$ let $c_{k+1}$ be the smallest such, otherwise set $c_{k+1}$ at the same distance above $a$ as $c_k$ sits below $a$, that is $c_{k+1} := a + (a - c_k)$. The idea to prove the transversality theorem 1.14 is to perturb $S_Y$ outside some $L^2$ neighborhood $U$ of its critical points in such a way that no new critical points arise on the sublevel set $\{S_Y < c_{k+1}\}$. To achieve this we fix for every critical point $x$ a closed $L^2$ neighborhood $U_x$ such that $U_x \cap U_y = \emptyset$ whenever $x \neq y$. This is possible, because on any sublevel set there are only finitely many critical points ($S_Y$ is Morse and satisfies the Palais-Smale condition; see e.g. [W02, app. A]). Set

$$U = U(V) := \bigcup_{x \in \mathcal{P}(V)} U_x$$

and consider the Banach space of perturbations $Y$ given by (76). We are interested in the subset of those perturbations supported away from $U$, namely

$$Y(V, U) := \left\{ v_\lambda = \sum_{\ell=0}^{\infty} \lambda_\ell V_\ell \in Y \mid \text{supp} V_\ell \cap U \neq \emptyset \Rightarrow \lambda_\ell = 0 \right\}.$$
Lemma 6.2. $Y(V, U)$ is a closed subspace of the separable Banach space $Y$.

Proof. Let $\alpha, \beta \in \mathbb{R}$ and let $v_\lambda$ and $v_\mu$ be elements of $Y(V, U)$. By definition of $Y(V, U)$ the following is true for every $\ell \in \mathbb{N}_0$. If $\text{supp} V_\ell \cap U \neq \emptyset$, then $\lambda_\ell = 0$ and $\mu_\ell = 0$. Hence $\alpha \lambda_\ell + \beta \mu_\ell = 0$ and therefore $\alpha v_\lambda + \beta v_\mu \in Y(V, U)$. To see that the subspace $Y(V, U)$ is closed let $v_\lambda^i = \sum \lambda_\ell V_\ell$ be a sequence in $Y(V, U)$ which converges to some element $v_\lambda = \sum \lambda_\ell V_\ell$ of $Y$. This means that $\lambda_\ell^i \to \lambda_\ell$, as $i \to \infty$, for every $\ell$. Now assume $\text{supp} V_\ell \cap U \neq \emptyset$. It follows that $\lambda_\ell^i = 0$, because $v_\lambda^i \in Y(V, U)$, and this is true for all $i$. Hence the limit $\lambda_\ell$ is zero and therefore $v_\lambda \in Y(V, U)$.

For $c_k < a < c_{k+1}$ as above set

$$\delta^a = \delta^a(V) := \frac{1}{2} \min \{a - c_k, c_{k+1} - a\} > 0, \quad a_\pm := a \pm \delta^a. \quad (78)$$

Hence the distance between any two of the five reals

$$c_k < a_- < a < a_+ < c_{k+1}$$

is at least $\delta^a$.

Lemma 6.3. Fix a perturbation $V$ that satisfies (V0–V3) and assume $S_V$ is Morse. Let $U$ be given by (77). Fix a regular value $a$ of $S_V$ and consider the reals $c_k$, $c_{k+1}$, $a_-$, $a_+$, and $\delta^a$, defined above. Then the following is true. If $v_\lambda \in Y(V, U)$ and $\|v_\lambda\| < \delta^a$, then there are inclusions

$$\{S_V \leq c_k\} \subset \{S_{V+v_\lambda} \leq a_-\} \subset \{S_V \leq a\} \subset \{S_{V+v_\lambda} \leq a_+\} \subset \{S_V < c_{k+1}\} \subset \{S_{V+v_\lambda} \leq a\} \subset \{S_V \leq a_+\}. $$

Proof. Fix $v_\lambda \in Y(V, U)$ with $\|v_\lambda\| < \delta^a$. Observe that for each $\gamma \in LM$

$$|v_\lambda(\gamma)| \leq \sum_{\ell=0}^{\infty} |\lambda_\ell V_\ell(\gamma)| \leq \sum_{\ell=0}^{\infty} |\lambda_\ell| C_\ell^0 \leq \sum_{\ell=0}^{\infty} |\lambda_\ell| C_\ell^0 = \|v_\lambda\| < \delta^a.$$ 

Here we used that $v_\lambda$ is of the form $\sum \lambda_\ell V_\ell$, axiom (V0) with constant $C_\ell^0$ for $V_\ell$, the fact that $C_\ell^0 \leq C_\ell^1$ by (75), and definition (76) of the norm on $Y$. Observe further that

$$S_{V+v_\lambda} = S_V - v_\lambda.$$ 

The proofs of the asserted inclusions all follow the same pattern. We only provide details for the last two inclusions in the first line of the assertion of the lemma. Assume $S_V(\gamma) \leq a$, then $S_{V+v_\lambda}(\gamma) = S_V(\gamma) - v_\lambda(\gamma) < a + \delta^a = a_+$ where the last step is by definition of $a_+$. Now assume $S_{V+v_\lambda}(\gamma) \leq a_+$, then $S_V(\gamma) \leq a_+ + v_\lambda(\gamma) < a + 2\delta^a \leq c_{k+1}$ again by definition of $a_+$. The last step is by definition of $\delta^a$. \qed
Consider the positive constants given by
\[ \kappa^a = \kappa^a(\mathcal{V}, U) := \inf_{\gamma \in \{S_{\mathcal{V}} < c_{k+1}\} \setminus U} \|\text{grad} S_{\mathcal{V}}(\gamma)\|_2 > 0 \]
and
\[ r^a = r^a(\mathcal{V}, U) := \frac{1}{2} \min\{\delta^a, \kappa^a\} > 0. \tag{79} \]

To prove the strict inequality \( \kappa^a > 0 \) assume by contradiction that \( \kappa^a = 0 \). Then by Palais-Smale there exists a sequence \( \{\gamma_k\} \subset \{S_{\mathcal{V}} < c_{k+1}\} \setminus U \) converging in the \( W^{1,2} \) topology to a critical point \( x \). It follows that \( x \in U \), because \( U \) contains all critical points. Since \( W^{1,2} \) convergence implies \( L^2 \) convergence and \( U \) is a \( L^2 \) neighborhood of the critical points, we arrive at a contradiction to \( \gamma_k \not\in U \) for every \( k \in \mathbb{N} \).

**Proposition 6.4.** Fix a perturbation \( \mathcal{V} \) that satisfies (V0–V3) and assume \( S_{\mathcal{V}} \) is Morse and \( a \) is a regular value. Then the following is true. If \( v_\lambda \in Y(\mathcal{V}, U) \) and \( \|v_\lambda\| \leq r^a \), then
\[
\mathcal{P}^a(\mathcal{V}) = \mathcal{P}^a(\mathcal{V} + v_\lambda), \quad \mathcal{H}_* \left( \{S_{\mathcal{V}} \leq a\} \right) \cong \mathcal{H}_* \left( \{S_{\mathcal{V}+v_\lambda} \leq a\} \right).
\]

**Proof.** Fix \( v_\lambda \in Y(\mathcal{V}, U) \) with \( \|v_\lambda\| \leq \frac{1}{2} \min\{\delta^a, \kappa^a\} \). Define \( a_+ \) by (78).

1) We prove that \( \mathcal{P}^a(\mathcal{V}) = \mathcal{P}^{a+}(\mathcal{V} + v_\lambda) \) and this immediately implies the first assertion of the proposition. On \( U \) both functionals \( S_{\mathcal{V}} \) and \( S_{\mathcal{V}+v_\lambda} \) coincide, because \( S_{\mathcal{V}+v_\lambda} = S_{\mathcal{V}} - v_\lambda \) and \( v_\lambda \) is not supported on \( U \). Now \( S_{\mathcal{V}} \) does not admit any critical point on \( \{S_{\mathcal{V}+v_\lambda} < c_{k+1}\} \setminus U \) by definition of \( U \). Assume the same holds true for \( S_{\mathcal{V}+v_\lambda} \). Then, since \( \{S_{\mathcal{V}+v_\lambda} \leq a_+\} \subset \{S_{\mathcal{V}} < c_{k+1}\} \) by lemma 6.3, it follows that all critical points of \( S_{\mathcal{V}+v_\lambda} \) below level \( a_+ \) are contained in \( U \). But there it coincides with \( S_{\mathcal{V}} \). Hence \( \mathcal{P}^{a+}(\mathcal{V} + v_\lambda) = \mathcal{P}^{a+}(\mathcal{V}) \).

It remains to prove the assumption. Suppose by contradiction that there is a critical point \( x \) of \( S_{\mathcal{V}+v_\lambda} \) on \( \{S_{\mathcal{V}+v_\lambda} < c_{k+1}\} \setminus U \). Hence
\[ 0 = \text{grad} S_{\mathcal{V}+v_\lambda}(x) = \text{grad} S_{\mathcal{V}}(x) - \text{grad} v_\lambda(x) \]
and therefore \( \|\text{grad} v_\lambda(x)\|_2 = \|\text{grad} S_{\mathcal{V}}(x)\|_2 \geq \kappa^a \) by definition of \( \kappa^a \). On the other hand, since \( v_\lambda \) is of the form \( \sum_{\ell=0}^{\infty} |\lambda_\ell| V_{\ell} \) it follows that
\[
\|\text{grad} v_\lambda(x)\|_2 \leq \sum_{\ell=0}^{\infty} |\lambda_\ell| \cdot \|\text{grad} V_{\ell}(x)\|_\infty \leq \sum_{\ell=0}^{\infty} |\lambda_\ell| C_{\ell}^0 \leq \|v_\lambda\| \leq \frac{1}{2} \kappa^a.
\]
Here we used the inequality \( \|\cdot\|_2 \leq \|\cdot\|_\infty \), axiom (V0) with constant \( C_{\ell}^0 \) for \( V_{\ell} \) and the fact that \( C_{\ell}^0 \leq C_{\ell}^0 \) by (75). The last two lines are by definition (76) of the norm on \( Y \) and the assumption on \( \|v_\lambda\| \).
2) We prove that $H_*\left(\{S_{V+v_\lambda} \leq a\}\right) \cong H_*\left(\{S_{V+v_\lambda} \leq a\}\right)$. Observe that all elements of the intervall $[a_-, a_+]$ are regular values of $S_{V+v_\lambda}$ by step 1). Hence classical Morse theory for the negative $W^{1,2}$ gradient flow on the loop space shows that

$$H_*\left(\{S_{V+v_\lambda} \leq a_-\}\right) \cong H_*\left(\{S_{V+v_\lambda} \leq a_+\}\right).$$

On the other hand, using the inclusions provided by lemma 6.3 this isomorphism factors through the inclusion induced homomorphisms

$$H_*\left(\{S_{V+v_\lambda} \leq a_-\}\right) \to H_*\left(\{S_{V} \leq a\}\right) \to H_*\left(\{S_{V+v_\lambda} \leq a_+\}\right).$$

Therefore the first homomorphism is injective and the second one surjective. Since $a$ lies in the intervall of regular values of $S_{V+v_\lambda}$, the first one leads to an injective homomorphism $H_*\left(\{S_{V+v_\lambda} \leq a\}\right) \to H_*\left(\{S_{V} \leq a\}\right)$. By construction the intervall $[a_-, a_+]$ consists of regular values of $S_{V}$. Hence the same argument using again lemma 6.3 to obtain the inclusion induced homomorphisms

$$H_*\left(\{S_{V} \leq a_-\}\right) \to H_*\left(\{S_{V+v_\lambda} \leq a\}\right) \to H_*\left(\{S_{V} \leq a_+\}\right)$$

provides a surjection $H_*\left(\{S_{V+v_\lambda} \leq a\}\right) \to H_*\left(\{S_{V} \leq a\}\right)$.

By definition the set of admissible perturbations is given by the open ball in the Banach space $Y(V, U)$ of radius $r^a$ defined in (79). We denote this set by

$$O^a = O^a(V, U) := \{v_\lambda \in Y(V, U) : \|v_\lambda\| \leq r^a\}.\quad (80)$$

Since $Y(V, U)$ is a separable Banach space by lemma 6.2, the closed subset $O^a$ inherits the structure of a complete metric space. Proposition 6.4 then concludes the proof of the first part of theorem 1.14. Namely, if $v_\lambda \in O^a$, then $S_{V}$ and $S_{V+v_\lambda}$ have homologically equivalent sublevel sets with respect to $a$ and the same critical points when restricted to these sublevel sets.

**Remark 6.5.** If $a < b$ are regular values of $S_{V}$ and $v \in O^b$ satisfies $\|v\| \leq \delta^b/2$, then $v \in O^a$. To see this note that $\kappa^b \leq \kappa^a$ and therefore $\|v\| \leq r^b \leq \kappa^b/2 \leq \kappa^a/2$. Hence $\|v\| \leq \frac{1}{2}\min\{\delta^a, \kappa^a\} := r^a$.

**Remark 6.6.** Since we chose to cut off our abstract perturbations in section 1.1 with respect to the $L^2$ norm, we cannot naturally control the support of $v \in O^a$ in terms of sublevel sets of $S_{V}$. This would be possible if we had cut off with respect to the $W^{1,2}$ norm, because the action functional $S_{V}$ is continuous with respect to the $W^{1,2}$ topology.

### 6.3 Surjectivity

**Proof of theorem 1.14.** Assume that the perturbation $V$ satisfies (V0)–(V3) and the function $S_{V} : M \to \mathbb{R}$ is Morse. Consider the neighborhood $U$ of the critical points of $S_{V}$ defined by (77) in the previous section and fix a regular value $a$ of $S_{V}$. For $O^a = O^a(V, U)$ given by (80) the first part of theorem 1.14 is true by proposition 6.4. To prove the second part fix in addition a constant
p > 2 and two critical points \( x, y \in \mathcal{P}^a(V) \). We denote by \( \mathcal{B}_{x,y}^{1,p} \) the smooth Banach manifold of cylinders between \( x \) and \( y \) defined by (48) in section 4. This manifold is separable and admits a countable atlas. Now consider the smooth Banach space bundle

\[
\mathcal{E}^p \to \mathcal{B}_{x,y}^{1,p} \times \mathcal{O}^a
\]

whose fibre over \((u, v_\lambda)\) are the \( L^p \) vector fields along \( u \). The formula

\[
F(u, v_\lambda) = \partial_s u - \nabla_t \partial_t u - \text{grad}(V + v_\lambda)(u)
\]  

defines a smooth section of this bundle. Its zero set

\[
\mathcal{Z} = \mathcal{Z}(x, y; V, U, a) = F^{-1}(0)
\]

is called the **universal moduli space**. It does not depend on \( p > 2 \), since all solutions of the heat equation (6) are smooth by theorem 1.6. Now zero is a regular value of \( F \). By definition this means that either there is no zero of \( F \) at all or \( dF(u, v_\lambda) \) admits a topological complement, whenever \( F(u, v_\lambda) = 0 \). In the first case it is natural to set \( O_{\text{reg}}^a(x, y) = O^a \). The second case naturally decomposes into two classes.

The first class consists of constant solutions \( u \) and transversality holds true automatically, since \( S_V \) is Morse. More precisely, if \( x = y \), then \( u(s, \cdot) := x(\cdot) \) is a zero of \( F \). In fact it solves (6) since each \( v_\lambda \in O^a \) is supported away from the elements of \( \mathcal{P}(V) \). Now the linearization \( D_u \) of (6) reduces to the covariant Hessian \( A_x \) of \( S_V \) given by (10). This Hessian is injective by the Morse assumption on \( S_V \). It is also surjective, because the cokernel of \( A_x \) coincides with the kernel of its formal adjoint operator with respect to the \( L^2 \) inner product. But by symmetry of \( A_x \) this kernel is equal to \( \ker A_x = \{0\} \). Hence \( D_u \), and therefore \( dF(u, v_\lambda) \), is automatically surjective at constant solutions. Hence \( O_{\text{reg}}^a(x, x) = O^a \).

The second class consists of zeroes \((u, v_\lambda)\) of (81) where \( u \) depends on \( s \). In this case surjectivity of \( dF(u, v_\lambda) \) is the content of proposition 6.7 below and existence of a topological complement follows (see e.g. [W02, prop. 3.3]) from surjectivity and the fact that by theorem 1.10 and theorem 1.9 the operator

\[
D_u \xi = \nabla_s \xi - \nabla_t \nabla_t \xi - R(\xi, \partial_t u) \partial_t u - H_{V + v_\lambda}(u) \xi
\]

is Fredholm. (Note that \( S_{V + v_\lambda} \) is Morse below level \( a \) by proposition 6.4 and the fact that \( v_\lambda \) is not supported near the critical points.) Hence \( Z \) is a smooth Banach manifold by the implicit function theorem. Consider the projection onto the second factor

\[
\pi : Z \to O^a.
\]

By standard Thom-Smale transversality theory (see e.g. [MS04, lemma A.3.6]) \( \pi \) is a smooth Fredholm map whose index is given by the Fredholm index of \( D_u \). This index is equal to the difference of the Morse indices of \( x \) and \( y \), again by theorem 1.10. Since \( Z \) is separable and admits a countable atlas, we can apply the Sard-Smale theorem [Sm73] to countably many coordinate representatives
of $\pi$. It follows that the set of regular values of $\pi$ is residual in $O^a$. We denote this set by $O^a_{reg}(x, y)$ and observe that

$$O^a_{reg}(x, y) = \{ v_\lambda \in O^a \mid D_u \text{ onto } \forall u \in M(x; y; V + v_\lambda) \}$$

again by standard transversality theory; see e.g. [W02, prop. 3.4].

We define the set of regular perturbations by

$$O^a_{reg} := \bigcap_{x, y \in P^a(V)} O^a_{reg}(x, y).$$

It is a residual subset of $O^a$, since it consists of a finite intersection of residual subsets. This proves theorem 1.14 up to proposition 6.7.

**Proposition 6.7 (Surjectivity).** Fix a perturbation $V$ that satisfies (V0)–(V3) and assume $S_V$ is Morse. Fix a regular value $a$, critical points $x, y \in P^a(V)$, and a constant $p > 2$. Let $U$ be defined by (77) and consider the section $F$ given by (81). Then the following is true. The linearization $dF(u, v_\lambda) : W^1_p u \times Y(V, U) \to L^p u$ is onto at every zero $(u, v_\lambda) \in B^1_p(x, y) \times O^a(V, U)$ of the section $F$.

**Proof.** Assume that $(u, v_\lambda)$ is a zero of $F$. The case of constant $u$ has been treated in the proof of theorem 1.14. Hence we assume that $u$ depends on $s \in \mathbb{R}$. Since the action $S_{V + v_\lambda}$ decreases strictly along nonconstant zeroes of (81), it follows that

$$c_k \geq S_V(x) = S_{V + v_\lambda}(x) > S_{V + v_\lambda}(u_s) > S_{V + v_\lambda}(y) = S_V(y). $$

Here the two identities are due to the fact that $v_\lambda$ is not supported near $x$ and $y$. In particular, this shows that $x \neq y$. Now define $1 < q < 2$ by $1/p + 1/q = 1$. By the regularity theorem 1.6 the map $u$ is smooth and by theorem 1.9 on exponential decay all derivatives of $\partial_s u$ are bounded. The linearization of $F$ at the zero $(u, v_\lambda)$ is given by

$$dF(u, v_\lambda) (\xi, \dot{V}) = dF_{v_\lambda}(u) \xi + dF_u(v_\lambda) \dot{V}$$

$$= D_u \xi - \text{grad} \dot{V}(u)$$

where $F_{v_\lambda}(u) := F(u, v_\lambda) =: F_u(v_\lambda)$ and $D_u$ is given by (82). Recall that $S_{V + v_\lambda}$ is Morse below level $a$ by proposition 6.4 and the fact that $v_\lambda$ is not supported near the critical points. Hence by theorem 1.9 the Fredholm theorem 1.10 shows that the operator $D_u$ is Fredholm. Moreover, the second operator

$$Y(V, U) \to L^p_u : \dot{V} \mapsto -\text{grad} \dot{V}(u)$$

is bounded. To see this observe that, since the support of $\dot{V}$ is disjoint to the neighborhood $U$ of $x$ and $y$, there is a constant $T = T(u) > 0$ such that
\[ \text{grad} \hat{\mathcal{V}}(u_s) = 0 \text{ whenever } |s| > T. \text{ Now } \hat{\mathcal{V}} \text{ is of the form } \sum_{\ell=0}^{\infty} \mu_{\ell} \mathcal{V}_{\ell}. \text{ Hence} \]

\[ \left\| \text{grad} \hat{\mathcal{V}}(u) \right\|_{L^p(\mathbb{R} \times S^1)} = \left( \int_{-T}^{T} \left\| \text{grad} \hat{\mathcal{V}}(u_s) \right\|^p ds \right)^{1/p} \]

\[ \leq (2T)^{1/p} \sum_{\ell=0}^{\infty} |\mu_{\ell}| \cdot \| \text{grad} \mathcal{V}_{\ell}(u_s) \|_\infty \]

\[ \leq (2T)^{1/p} \sum_{\ell=0}^{\infty} |\mu_{\ell}| C_{\ell}^0 \]

\[ \leq (2T)^{1/p} \| \hat{\mathcal{V}} \| \]

where for each \( \mathcal{V}_{\ell} \) we used the last condition in (V0) with constant \( C_{\ell}^0 \leq C_{\ell}^f \).

The last step uses the definition (76) of the norm in \( Y \).

Hence the range of \( d\mathcal{F}(u, \lambda) \) is closed by standard arguments; see e.g. [W02, proposition 3.3]. Therefore it suffices to prove that it is dense. We use that density of the range is equivalent to trivality of its annihilator: By definition this means that, given \( \eta \in \mathcal{L}^1_u \), then

\[ \langle \eta, \mathcal{D}_u \xi \rangle = 0, \quad \forall \xi \in \mathcal{W}^{1,p}_u, \quad (84) \]

and

\[ \langle \eta, \text{grad} \hat{\mathcal{V}}(u) \rangle = 0, \quad \forall \hat{\mathcal{V}} \in Y(\mathcal{V}, U), \quad (85) \]

imply that \( \eta = 0 \).

Assume by contradiction that \( \eta \in \mathcal{L}^1_u \) satisfies (84) and \( \eta \neq 0 \). In five steps we derive a contradiction to (85). Steps 1–3 are preparatory, in step 4 we construct a model perturbation \( \mathcal{V}_{\varepsilon} \) violating (85) and in step 5 we approximate \( \mathcal{V}_{\varepsilon} \) by the fundamental perturbations \( \mathcal{V}_{ijk} \) of the form (74). To start with observe that \( \eta \) is smooth by (84) and theorem 2.1. Furthermore, integrating (84) by parts for \( \xi \in C^\infty_0(\mathbb{R} \times S^1, u^*T\mathcal{M}) \) shows that \( \mathcal{D}^*_u \eta = 0 \) pointwise, where the operator \( \mathcal{D}^*_u \) arises by replacing \( \nabla_s \) by \( -\nabla_s \) in (82). Throughout we use the notation \( \eta_s(t) = \eta(s, t) \). Hence \( \eta_s \) is a smooth vector field along the loop \( u_s \).

**Step 1.** (Unique Continuation) \( \eta_s \neq 0 \) and \( \partial_s u_s \neq 0 \) for every \( s \in \mathbb{R} \). Because \( \eta \) is smooth, nonzero, and \( \mathcal{D}^*_u \eta = 0 \), proposition 5.2 on unique continuation shows that \( \eta_s \neq 0 \) for every \( s \in \mathbb{R} \). Next observe that \( \partial_s u \) is smooth, because \( u \) is smooth, and that \( 0 = \frac{d}{ds} \mathcal{F}_{\lambda}(u) = \mathcal{D}_u \partial_s u \). Since \( u \) connects different critical points, the derivative \( \partial_s u \) cannot vanish identically on \( \mathbb{R} \times S^1 \). Hence \( \partial_s u_s \neq 0 \) for every \( s \in \mathbb{R} \) by proposition 5.2 for \( \xi(s) := \partial_s u_s \).

**Step 2.** (Slicewise Orthogonal) \( \langle \eta_s, \partial_s u_s \rangle = 0 \) for every \( s \in \mathbb{R} \).

Note that here \( \langle \eta_s, \partial_s u_s \rangle \) denotes the \( L^2(S^1) \) inner product. Now

\[ \frac{d}{ds} \langle \eta_s, \partial_s u_s \rangle = \langle \nabla_s \eta_s, \partial_s u_s \rangle + \langle \eta_s, \nabla_s \partial_s u_s \rangle \]

\[ = \langle -\nabla_t \eta_s - R(\eta_s, \partial_t u_s) \partial_t u_s - \mathcal{H}_{\nu_v}(u_s) \eta_s, \partial_s u_s \rangle \]

\[ + \langle \eta_s, \nabla_t \partial_s u_s - R(\partial_s u_s, \partial_t u_s) \partial_t u_s - \mathcal{H}_{\nu_v}(u_s) \partial_t u_s \rangle \]

\[ = 0 \]
by straightforward calculation. In the second equality we replaced \( \nabla_s \eta_s \) according to the identity \( D^*_u \eta = 0 \) and \( \nabla_s \partial_s u_s \) according to \( D_u \partial_s u = 0 \); see (82).

The last step is by integration by parts, symmetry of the Hessian \( H \), and the first Bianchi identity for the curvature operator \( R \). It follows that \( \langle \eta_s, \partial_s u_s \rangle \) is constant in \( s \). Now this constant, say \( c \), must be zero, because

\[
\int_{-\infty}^{\infty} c \, ds = \int_{-\infty}^{\infty} \langle \eta_s, \partial_s u_s \rangle \, ds = \langle \eta, \partial_s u \rangle
\]

and the right hand side is finite, because \( \eta \in L^p \) and \( \partial_s u \in L^q \) with \( \frac{1}{p} + \frac{1}{q} = 1 \).

This proves step 2. Note that \( \eta_s \) and \( \partial_s u_s \) are linearly independent for every \( s \in \mathbb{R} \) as a consequence of step 1 and step 2.

**Step 3.** (No Return) Assume the loop \( u_{s_0} \) is different from the asymptotic limits \( x \) and \( y \) and let \( \delta > 0 \). Then there exists \( \varepsilon > 0 \) such that for every \( s \in \mathbb{R} \)

\[
\| u_s - u_{s_0} \|_2 < 3\varepsilon \quad \implies \quad s \in (s_0 - \delta, s_0 + \delta).
\]

In words, once \( s \) leaves a given \( \delta \)-interval about \( s_0 \) the loops \( u_s \) cannot return to some \( L^2 \) \( \varepsilon \)-neighborhood of \( u_{s_0} \).

Key ingredients in the proof are smoothness of \( u \), existence of asymptotic limits, and the gradient flow property. Recall the footnote in remark 1.4 concerning the difference of loops \( u_s - u_{s_0} \). Now assume by contradiction that there is a sequence of positive reals \( \varepsilon_i \to 0 \) and a sequence of reals \( s_i \) which satisfy \( \| u_{s_i} - u_{s_0} \|_2 < 3\varepsilon_i \) and \( s_i \notin (s_0 - \delta, s_0 + \delta) \). In particular, it follows that

\[
u_{s_i} \xrightarrow{L^2} u_{s_0} \quad \text{as} \quad i \to \infty.
\]

Assume first that the sequence \( s_i \) is unbounded. Hence we can choose a subsequence, without changing notation, such that \( s_i \) converges to \( +\infty \) or \( -\infty \). In either case \( u_{s_i} \) converges to one of the critical points \( x \) or \( y \) and the convergence is in \( C^0(S^1) \) by theorem 1.9. By (86) and uniqueness of limits it follows that \( u_{s_0} \) is equal to one of the critical points \( x \) or \( y \) contradicting our assumption.

Assume now that the sequence \( s_i \) is bounded. Then we can choose a subsequence, without changing notation, such that \( s_i \) converges to some element \( s_1 \notin (s_0 - \delta, s_0 + \delta) \). Since \( u \) is smooth, it follows that \( u_{s_i} \) converges to \( u_{s_1} \) in \( C^0(S^1) \). Again by uniqueness of limits \( u_{s_1} = u_{s_0} \). On the other hand, the action functional is strictly decreasing along nonconstant negative gradient flow lines. Therefore \( s_1 = s_0 \) and this contradiction concludes the proof of step 3.

**Step 4.** There is a time \( s_0 \in \mathbb{R} \) such that the loop \( u_{s_0} \) is not element of \( U \). Moreover, there is a constant \( \varepsilon > 0 \) and a smooth function \( V_0 : \mathbb{L} \mathbb{M} \to \mathbb{R} \) supported in the \( L^2 \) ball of radius \( 2\varepsilon \) about \( u_{s_0} \) such that

\[
V_0(u_{s_0}) = 0, \quad dV_0(u_{s_0}) \eta_{s_0} = \| \eta_{s_0} \|_2^2, \quad \langle \text{grad} V_0(u), \eta \rangle \neq 0.
\]

The first assertion follows from \( x \neq y \) and the fact that the closed sets \( U_z \), where \( z \in \mathcal{P}(\mathcal{V}) \), are pairwise disjoint. Now observe that the graph \( t \mapsto (t, u_{s_0}(t)) \) of
the loop $u_{s_0}$ is embedded in $S^1 \times M$. We define a smooth function $V$ on $S^1 \times M$ supported near this graph as follows. Denote by $\epsilon > 0$ the injectivity radius of the closed Riemannian manifold $M$. Pick a smooth cutoff function $\beta : \mathbb{R} \to [0, 1]$ such that $\beta = 1$ on $[-(\epsilon/2)^2, (\epsilon/2)^2]$ and $\beta = 0$ outside $[-\epsilon^2, \epsilon^2]$; see figure 2. Then define

$$V_t(q) := V(t, q) := \begin{cases} \beta(|\xi_q(t)|^2) \langle \xi_q(t), \eta_{s_0}(t) \rangle, & |\xi_q(t)| < \epsilon, \\ 0, & \text{else}, \end{cases}$$

(87)

where the vector $\xi_q(t)$ is determined by the identity $q = \exp_{u_{s_0}(t)} \xi_q(t)$ whenever the Riemannian distance between $q$ and $u_{s_0}(t)$ is less than $\epsilon$. Note that the function $V$ vanishes on the graph of the loop $u_{s_0}$.

Since all maps involved are smooth, we can choose a constant $\delta > 0$ sufficiently small such that for every $s \in (s_0 - \delta, s_0 + \delta)$ the following is true

i) $d\mathcal{C}^0(u_s, u_{s_0}) = ||\xi_s||_\infty < \frac{1}{2} \mu_1$, where the vector field $\xi_s$ along the loop $u_{s_0}$ is uniquely determined by the pointwise identity $u_s = \exp_{u_{s_0}} \xi_s$,

ii) $\langle E_2(u_{s_0}, \xi_s)^{-1} \eta_s, \eta_{s_0} \rangle \geq \frac{1}{2} \mu_0$, where $\mu_0 := ||\eta_{s_0}||_2 > 0$,

iii) $\frac{1}{2} \mu_1 \leq \frac{||u_s - u_{s_0}||}{||\xi_s||} \leq \frac{3}{2} \mu_1$, where $\mu_1 := ||\partial_s u_{s_0}||_2 > 0$.

Recall the definition (21) of $E_2$ and the identities (23). For $s \in (s_0 - \delta, s_0 + \delta)$, we obtain that

$$dV_t(u_s) \eta_s = \frac{d}{dr} \bigg|_{r=0} V_t(\exp_{u_s} r \eta_s)$$

$$= 2 \beta'(||\xi_s||^2) \langle \xi_s, E_2(u_{s_0}, \xi_s)^{-1} \eta_s \rangle \cdot \langle \xi_s, \eta_{s_0} \rangle$$

$$+ \beta(||\xi_s||^2) \langle E_2(u_{s_0}, \xi_s)^{-1} \eta_s, \eta_{s_0} \rangle$$

$$= \langle E_2(u_{s_0}, \xi_s)^{-1} \eta_s, \eta_{s_0} \rangle$$

(88)

pointwise for every $t \in S^1$. The final step uses i) and the definition of $\beta$. Note that $dV_t(u_{s_0}) \eta_{s_0} = ||\eta_{s_0}||^2$ pointwise.

Integrating $V$ along a loop defines a smooth function on the loop space which vanishes on $u_{s_0}$. To cut this function off with respect to the $L^2$ distance fix a smooth cutoff function $\rho : \mathbb{R} \to [0, 1]$ such that $\rho = 1$ on $[-1, 1]$, $\rho = 0$ outside $[-4, 4]$, and $||\rho'||_\infty < 1$. Then, for the constant $\delta$ fixed above, choose $\varepsilon > 0$ according to step 3 (No Return) and set $\rho_\varepsilon(r) = \rho(r/\varepsilon^2)$; see figure 1 for $\varepsilon = \frac{1}{4}$. Note that $||\rho_\varepsilon'||_\infty < \varepsilon^{-2}$. Observe that we can choose $\varepsilon > 0$ smaller and the assertion of step 3 remains true. Now define a smooth function on $\mathcal{L}M$ by

$$V_0(x) := \rho_\varepsilon \left( ||x - u_{s_0}||_2^2 \right) \int_0^1 V(t, x(t)) \, dt$$
where \( V \) is given by (87). The function \( \mathcal{V}_0 \) vanishes on the loop \( u_{s_0} \) and satisfies

\[
d\mathcal{V}_0(u_s) \eta_s = \frac{d}{dr} \mathcal{V}_0(\exp_{u_s} r \eta_s)
\]

\[
= 2 \rho'(\|u_s - u_{s_0}\|^2_2) \langle u_s - u_{s_0}, \eta_s \rangle \int_0^1 V_i(u_s(t)) \, dt
\]

\[
+ \rho \varepsilon(\|u_s - u_{s_0}\|^2_2) \int_0^1 dV_i(u_s(t)) \eta_s(t) \, dt.
\]

Hence \( d\mathcal{V}_0(u_{s_0}) \eta_{s_0} = \|\eta_{s_0}\|_2^2 \) and this proves another assertion of step 4.

To prove the final assertion of step 4 observe that \( s \notin (s_0 - \delta, s_0 + \delta) \) implies that \( \|u_s - u_{s_0}\|_2 \geq 3\varepsilon \), by step 3, and therefore \( u_s \notin \text{supp} \mathcal{V}_0 \). It follows that

\[
\langle \text{grad} \mathcal{V}_0(u), \eta \rangle = \int_{s_0 - \delta}^{s_0 + \delta} d\mathcal{V}_0(u_s) \eta_s \, ds
\]

\[
= \int_{s_0 - \delta}^{s_0 + \delta} 2 \rho'(\|u_s - u_{s_0}\|^2_2) \langle u_s - u_{s_0}, \eta_s \rangle \langle \xi_s, \eta_{s_0} \rangle \, ds
\]

\[
+ \int_{s_0 - \delta}^{s_0 + \delta} \rho \varepsilon(\|u_s - u_{s_0}\|^2_2) \langle E_2(u_{s_0}, \xi_s)^{-1} \eta_s, \eta_{s_0} \rangle \, ds.
\]  

(89)

We shall estimate the two terms in the sum separately. Let \( s_2 > s_0 \) be such that \( \|u_{s_2} - u_{s_0}\|_2 = \varepsilon \) and \( \|u_s - u_{s_0}\|_2 < \varepsilon \) whenever \( s \in (s_0, s_2) \). This means that \( s_2 \) is the forward exit time of \( u_s \) with respect to the \( L^2 \) ball of radius \( \varepsilon \) about \( u_{s_0} \). Let \( s_1 < s_0 \) be the corresponding backward exit time; see figure 3. Then, by ii) and \( \rho \varepsilon \geq 0 \), it holds that

\[
\int_{s_0 - \delta}^{s_0 + \delta} \rho \varepsilon(\|u_s - u_{s_0}\|^2_2) \langle E_2(u_{s_0}, \xi_s)^{-1} \eta_s, \eta_{s_0} \rangle \, ds
\]

\[
\geq \int_{s_1}^{s_2} 1 \cdot \frac{\mu_0}{2} \, ds = \frac{\mu_0}{2} (s_2 - s_0 + s_0 - s_1)
\]

\[
\geq \frac{\mu_0}{3 \mu_1} (\|u_{s_2} - u_{s_0}\|_2 + \|u_{s_0} - u_{s_1}\|_2) = \frac{2 \mu_0}{3 \mu_1} \varepsilon.
\]

Here the second inequality uses iii). To estimate the other term in (89) let \( \sigma_1 \) be the time of first entry into the \( L^2 \) ball of radius \( 2\varepsilon \) starting from \( s_0 - \delta \) and let \( \sigma_2 \) be the corresponding time when time runs backwards and we start from \( s_0 + \delta \); see figure 3. Then it follows that

\[
\int_{s_0 - \delta}^{s_0 + \delta} 2 \rho'(\|u_s - u_{s_0}\|^2_2) \langle u_s - u_{s_0}, \eta_s \rangle \langle \xi_s, \eta_{s_0} \rangle \, ds
\]

\[
\geq -2 \int_{\sigma_1}^{\sigma_2} \|\rho'\|_{\infty} \langle (u_s - u_{s_0}, \eta_s) \cdot |(\xi_s, \eta_{s_0}) \rangle \, ds
\]

\[
\geq -2 c_1 c_2 \varepsilon^{-2} \int_{\sigma_1}^{\sigma_2} (s - s_0)^4 \, ds
\]

\[
= -\frac{2 c_1 c_2}{5 \varepsilon^2} (\sigma_2 - s_0 + s_0 - \sigma_1)^5 \geq \frac{2 c_1 c_2 \varepsilon^5}{5 \mu_1^2}.\]

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It remains to explain the second and the final inequality. In the final one we use that by iii) there is the estimate \( \sigma_2 - s_0 \leq 2\|u_{s_2} - u_{s_0}\|_2/\mu_1 = 4\varepsilon/\mu_1 \) and similarly for \( s_0 - \sigma_1 \). The second inequality is based on the geometric fact that \( \partial_x u \) and \( \eta \) are slicewise orthogonal by step 2. Namely, let \( f(s) = \langle u_s - u_{s_0}, \eta_s \rangle \) and \( h(s) = \langle \xi_s, \eta_{s_0} \rangle \), then \( f(s_0) = h(s_0) = 0 \) and

\[
\begin{align*}
  f'(s) &= \langle \partial_x u_s, \eta_s \rangle + \langle u_s - u_{s_0}, \nabla_s \eta_s \rangle = \langle u_s - u_{s_0}, \nabla_s \eta_s \rangle \\
  h'(s) &= \langle E_2(u_{s_0}, \xi_s)^{-1} \partial_x u_s, \eta_{s_0} \rangle.
\end{align*}
\]

Hence \( f'(s_0) = h'(s_0) = 0 \) and so there exist constants \( c_1 = c_1(f) > 0 \) and \( c_2 = c_2(h) > 0 \) depending continuously on \( \delta \) such that for every \( s \in (s_0 - \delta, s_0 + \delta) \)

\[
|f(s)| \leq c_1(s - s_0)^2, \quad |h(s)| \leq c_2(s - s_0)^2.
\]

This proves the second inequality. Now choose \( \varepsilon > 0 \) sufficiently small such that \( \varepsilon^2 < \mu_0 \mu_1^2/c_1 c_2 \). This implies that \( (\text{grad} V_0(u), \eta) > 0 \) and proves step 4.

Recall that \( u_{s_0} \notin U \). Now we choose \( \varepsilon > 0 \) again smaller such that the \( L^2 \) ball of radius \( 3\varepsilon \) about \( u_{s_0} \) is disjoint from the \( L^2 \) closed set \( U \), that \( 3\varepsilon \) is smaller than the injectivity radius \( \nu \) of \( M \), and that \( \varepsilon = 1/k \) for some integer \( k \).

**Step 5.** Given \( k = 1/\varepsilon \) as in the paragraph above, there exist integers \( i, j > 0 \) such that the function \( \tilde{V} := V_{ijk} \) given by (74) lies in \( Y(V, U) \) and satisfies

\[
(\text{grad} V_{ijk}(u), \eta) > 0.
\]

This contradicts (85) and thereby proves proposition 6.7.

Let \( u_{s_0} \) be as in step 4. In section 6.1 we fixed a dense sequence \( (x_i) \) in \( C^\infty(S^1, M) \) and for each \( i \) a dense sequence \( (\eta^{ij}) \) in \( C^\infty(S^1, x_i^* TM) \). Choose a subsequence, still denoted by \( (x_i) \), such that

\[
x_i \to u_{s_0} \quad \text{as} \quad i \to \infty.
\]

Now we may assume without loss of generality that every \( x_i \) lies in \( B_\varepsilon(u_{s_0}) \) the \( L^2 \) ball of radius \( \varepsilon \) about \( u_{s_0} \). Hence \( B_{2\varepsilon}(x_i) \subset B_{3\varepsilon}(u_{s_0}) \). Let \( \xi_{s_0} \) be defined
by the identity \( u_{s_0} = \exp_x \xi_{s_0} \) pointwise for every \( t \in S^1 \). Choose a diagonal subsequence, denoted for simplicity by \((\eta^{(i)})_i\), such that
\[
\Phi_{x_i}(\xi_{s_0}^{(i)})\eta^{(i)} \to \eta_{s_0} \quad \text{as } i \to \infty.
\]
Here \( \Phi_x(\xi) \) is parallel transport from \( x \) to \( \exp_x \xi \) along \( t \mapsto \exp_x t\xi \) pointwise for every \( t \in S^1 \). Let \( (\mathcal{V}_{iik})_{i \in \mathbb{N}} \) be the corresponding sequence of functions where each \( \mathcal{V}_{iik} \) is given by (74). Now observe that
\[
\operatorname{supp} \mathcal{V}_{iik} \subset B_{2/k}(x_i) = B_{2\varepsilon}(u_{s_0}) \subset B_{3\varepsilon}(u_{s_0}).
\]
But \( B_{3\varepsilon}(u_{s_0}) \cap U = \emptyset \) by the choice of \( \varepsilon \) in the paragraph prior to step 4, and therefore \( \mathcal{V}_{iik} \in Y(V, U) \). Next recall that the constant \( \delta > 0 \) has been chosen in the proof of step 4 in order to exclude any return of the trajectory \( s \mapsto u_s \) to the ball \( B_{3\varepsilon}(u_{s_0}) \) once \( s \) has left the interval \((s_0 - \delta, s_0 + \delta)\). Since \( \operatorname{supp} \mathcal{V}_{iik} \subset B_{3\varepsilon}(u_{s_0}) \), this shows that \( \mathcal{V}_{iik}(u_s) = 0 \) whenever \( s \not\in (s_0 - \delta, s_0 + \delta) \).

Hence
\[
\langle \text{grad} \mathcal{V}_{iik}(u), \eta \rangle = \int_{s_0 - \delta}^{s_0 + \delta} 2\rho_{i/k}(||u_s - x_i||^2)\langle u_s - x_i, \eta_s \rangle \langle \xi_s^{(i)}, \eta^{(i)} \rangle \, ds
+ \int_{s_0 - \delta}^{s_0 + \delta} \rho_{i/k}(||u_s - x_i||^2)\langle E_2(x_i, \xi_s^{(i)})^{-1} \eta_s, \eta^{(i)} \rangle \, ds
\]
where \( \xi_s^{(i)} \) is determined by \( u_s = \exp_x \xi_s^{(i)} \). Now the right hand side converges as \( i \to \infty \) to the right hand side of (89), which equals \( \langle \text{grad} \mathcal{V}_0(u), \eta \rangle > 0 \). This proves step 5 and proposition 6.7.

7 Heat flow homology

In section 7.1 we define the unstable manifold of a critical point \( x \) of the action functional \( S_V : LM \to \mathbb{R} \) as the set of endpoints at time zero of all backward halfcylinders solving the heat equation (6) and emanating from \( x \) at \(-\infty\). The main result is theorem 7.1 saying that if \( x \) is nondegenerate, then this is a submanifold of the loop space and its dimension equals the Morse index of \( x \).

Section 7.2 puts together the results proved so far to construct the Morse complex for the negative \( L^2 \) gradient of the action functional on the loop space.

7.1 The unstable manifold theorem

Fix a perturbation \( \mathcal{V} : LM \to \mathbb{R} \) that satisfies (V0)-(V3) and let \( Z^- \) be the backward halfcylinder \((-\infty, 0] \times S^1\). Given a critical point \( x \) of the action functional \( S_V \) the moduli space
\[
\mathcal{M}^-(x; \mathcal{V})
\]
is, by definition, the set of all smooth solutions \( u^- : Z^- \to M \) of the heat equation (6) such that \( u^-(s, t) \to x(t) \) as \( s \to -\infty \), uniformly in \( t \in S^1 \). Note
that the moduli space is not empty, since it contains the stationary solution $u^{-}(s,t) = x(t)$. The **unstable manifold of $x$** is defined by

$$W^{u}(x;V) = \{ u^{-}(0,\cdot) \mid u^{-} \in \mathcal{M}^{-}(x;V) \}.$$  

**Theorem 7.1.** Let $V : \mathcal{L}M \to \mathbb{R}$ be a perturbation that satisfies $(V0)$–$(V3)$. If $x$ is a nondegenerate critical point of the action functional $S_{V}$, then the unstable manifold $W^{u}(x;V)$ is a smooth contractible embedded submanifold of the loop space and its dimension is equal to the Morse index of $x$.

The idea to prove theorem 7.1 is to first show in proposition 7.2 that non-degeneracy of $x$ implies that the moduli space $\mathcal{M}^{-}(x;V)$ is a smooth manifold of the desired dimension. A crucial ingredient is proposition 7.3 on surjectivity of the operator $D_{u^{-}} : \mathcal{W}^{1,p} \to \mathcal{L}^{p}$ whenever $u^{-} \in \mathcal{M}^{-}(x;V)$ and $p \geq 2$. Here the operator $D_{u^{-}}$ given by (27) arises by linearizing the heat equation at the backward trajectory $u^{-}$. A further key result to prove theorem 7.1 is unique continuation for the linear and the nonlinear heat equation, proposition 5.2 and theorem 5.4. Namely, unique continuation implies that the evaluation map

$$ev_{0} : \mathcal{M}^{-}(x;V) \to \mathcal{L}M, \quad u^{-} \mapsto u^{-}(0,\cdot)$$

is an injective immersion. It is even an embedding by the gradient flow property.

**Proposition 7.2** (Moduli space). Let $V : \mathcal{L}M \to \mathbb{R}$ be a perturbation satisfying $(V0)$–$(V3)$ and suppose that $x$ is a nondegenerate critical point of $S_{V}$. Then the moduli space $\mathcal{M}^{-}(x;V)$ is a smooth contractible manifold of dimension $\text{ind}_{V}(x)$. Its tangent space at $u^{-}$ is equal to the vector space $X^{-}$ given by (91).

**Proposition 7.3** (Surjectivity). Fix a constant $p > 2$, a perturbation $V$ that satisfies $(V0)$–$(V3)$, and a nondegenerate critical point $x$ of $S_{V}$. If $u^{-} \in \mathcal{M}^{-}(x;V)$, then the operator $D_{u^{-}} : \mathcal{W}^{1,p} \to \mathcal{L}^{p}$ is onto and its kernel is given by

$$X^{-} := \left\{ \xi \in C^{\infty}(Z^{-}, u^{-}TM) \mid D_{u^{-}}\xi = 0, \exists c, \delta > 0 \forall s \leq 0 : \right.$$  

$$\left. \|\xi_{s}\|_{\infty} + \|\nabla_{\xi}\xi_{s}\|_{\infty} + \|\nabla_{s}\nabla_{\xi}\xi_{s}\|_{\infty} + \|\nabla_{s}\xi_{s}\|_{\infty} \leq ce^{\delta s} \right\}. \tag{91}$$

Moreover, the dimension of $X^{-}$ is equal to the Morse index of $x$.

Proposition 7.3 is in fact a corollary of theorem 7.5 below which asserts surjectivity in the special case of a stationary solution $u^{-}(s,t) = x(t)$, where $x$ is a nondegenerate critical point of $S_{V}$. The idea is that if a solution $u^{-}$ is nearby the stationary solution $x$ in the $\mathcal{W}^{1,p}$ topology, then the corresponding linearizations $D_{u^{-}}$ and $D_{x}$ are close in the operator norm topology. But surjectivity is an open condition with respect to the norm topology. The case of a general solution reduces to the nearby case by shifting the $s$-variable.

**Remark 7.4.** Abbreviate $H = L^{2}(S^{1}, \mathbb{R}^{n})$ and $W = W^{2,2}(S^{1}, \mathbb{R}^{n})$ and consider the operator

$$A_{S} = -\frac{d^{2}}{dt^{2}} - S : H \to H$$
with dense domain $W$. Here we assume that $S : W \to H$ is a symmetric and compact linear operator. Under these assumptions it is well known (see (ii) in section 2.4) that $A_S$ is self-adjoint and that its Morse index $\text{ind}(A_S)$, that is the dimension of the negative eigenspace $E^-$ of $A_S$, is finite.

**Theorem 7.5.** Let $S$ and $A_S$ be as in remark 7.4. Fix $p \geq 2$ and assume that the linear operator $S : W^{1,p}(S^1, \mathbb{R}^n) \to L^p(S^1, \mathbb{R}^n)$ is bounded with bound $c_S$. Then the following is true. If $A_S$ is injective, then the operator

$$D = \partial_s - \partial_t \partial_t - S : W^{1,p}(Z^-, \mathbb{R}^n) \to L^p(Z^-, \mathbb{R}^n)$$

is onto. In the case $p = 2$ the map $E^- \to \ker D, v \mapsto e^{-sA_S}v$ is an isomorphism.

**Proof of theorem 7.5.** The proof takes four steps. Step 1 proves the theorem for $p = 2$. The proof by Salamon [S99, lemma 2.4 step 1] of the corresponding result in Floer theory carries over with minor but important modifications. These are due to the fact that our domain $Z^-$ does have a boundary. The proof uses the theory of semigroups. We recall the details for convenience of the reader. The generalization of surjectivity in step 4 to $p > 2$ follows an argument due to Donaldson [Do02]. It uses the case $p = 2$ and the estimates provided by step 2 and step 3. Again we follow the presentation in [S99, lemma 2.4 steps 2–4] up to minor but subtle modifications. One subtlety is related to the parabolic estimate of step 2. Here in contrast to the elliptic case the domain needs to be increased only towards the past. Hence the estimates of step 3 work precisely for the backward halfcylinder. Throughout the proof, unless indicated differently, the domain of all spaces is the backward halfcylinder $Z^-$ and the target is $\mathbb{R}^n$.

**Step 1.** The theorem is true for $p = 2$.

The operator $A_S$ is unbounded and self-adjoint on the Hilbert space $H$ with dense domain $W$. Denote the negative and positive eigenspaces of $A_S$ by $E^-$ and $E^+$, respectively. Note that $\dim E^- < \infty$ by remark 7.4. By assumption $A_S$ is injective, hence zero is not an eigenvalue and there is a splitting $H = E^- \oplus E^+$. Denote by $P^\pm : H \to E^\pm$ the orthogonal projections and set $A^\pm = A_S|_{E^\pm}$. The self-adjoint negative semidefinite operators $A^-$ and $-A^+$ generate contraction semigroups on $E^-$ and $E^+$, respectively, by the Hille-Yosida theorem; see e.g. [ReS75, sec. X.8 ex. 1]. We denote them by $s \mapsto e^{sA^-}$ and $s \mapsto e^{-sA^+}$, respectively. Both are defined for $s \geq 0$. Define the map $K : \mathbb{R} \to \mathcal{L}(H)$ by

$$K(s) = \begin{cases} -e^{-sA^-}P^-, & \text{for } s \leq 0, \\ e^{-sA^+}P^+, & \text{for } s > 0. \end{cases}$$

This function is strongly continuous for $s \neq 0$ and satisfies

$$\|K(s)\|_{\mathcal{L}(H)} \leq e^{-\delta|s|}$$

(92)

where $\delta = \min\{-\lambda^-, \lambda^+\} > 0$. Here $\lambda^-$ denotes the largest eigenvalue of $A^-$ and $\lambda^+$ the smallest eigenvalue of $A^+$. Abbreviate $\mathbb{R}^- = (-\infty, 0]$. For
\( \eta \in L^2(\mathbb{R}^-, H) \) consider the operator

\[
(Q\eta)(s) := \int_{-\infty}^{0} K(s - \sigma)\eta(\sigma) \, d\sigma.
\]

Now the operator \( Q \) maps \( L^2(\mathbb{R}^-, H) \) to the intersection of Banach spaces \( \mathcal{W}^{1,2}(\mathbb{R}^-, H) \cap L^2(\mathbb{R}^-, W) \) and it is a right inverse of \( D \). To prove the latter set \( \xi := Q\eta \). Then \( \xi = \xi^- + \xi^+ \), where

\[
\xi^+(s) = \int_{-\infty}^{s} e^{-(s-\sigma)A^+}P^+\eta(\sigma) \, d\sigma, \quad \xi^-(s) = -\int_{s}^{0} e^{-(s-\sigma)A^-}P^-\eta(\sigma) \, d\sigma.
\]

Calculation shows that \( D\xi^\pm = P^\pm \eta \) pointwise for every \( s \in \mathbb{R}^- \). It follows that

\[
DQ\eta = D\xi = D\xi^- + D\xi^+ = P^-\eta + P^+\eta = \eta.
\]

Since the space \( \mathcal{W}^{1,2}(\mathbb{R}^-, H) \cap L^2(\mathbb{R}^-, W) \) agrees with \( \mathcal{W}^{1,2} \), this proves that \( Q \) is a right inverse of \( D \). Hence \( Q \) is injective and \( D \) is onto. To calculate the kernel of \( D \) fix \( \xi \in \mathcal{W}^{1,2} \) and set \( \eta := D\xi \). Then by straightforward calculation

\[
(QD\xi)(s) = (Q\eta)(s) = \xi^+(s) + \xi^-(s) = \int_{-\infty}^{s} \frac{d}{d\sigma} \left( e^{-(s-\sigma)A^+}P^+\xi(\sigma) \right) \, d\sigma - \int_{s}^{0} \frac{d}{d\sigma} \left( e^{-(s-\sigma)A^-}P^-\xi(\sigma) \right) \, d\sigma = P^+\xi(s) - e^{-sA^+}P^-\xi(0) + P^-\xi(s) = \xi(s) - e^{-sA^-}P^-\xi(0).
\]

To obtain the third identity replace \( \eta(\sigma) \) in \( \xi^\pm(s) \) by \( \eta'(\sigma) + A_S\xi(\sigma) \) and use the fact that \( A^\pm P^\pm = P^\pm A_S \). Now observe that \( \xi \in \ker D \) is equivalent to \( D\xi \in \ker Q \), because \( Q \) is injective. But \( QD\xi = 0 \) means that \( \xi(s) = e^{-sA^-}P^-\xi(0) \) for every \( s \in \mathbb{R}^- \). This shows that the map

\[
E^- \to \ker \left[ D : \mathcal{W}^{1,2} \to L^2 \right] : v_k \mapsto e^{-sA^+}v_k
\]

induces an isomorphism. Here \( v_1, \ldots, v_N \) is an orthonormal basis of \( E^- \) consisting of eigenvectors of \( A_S \) with eigenvalues \( \lambda_1, \ldots, \lambda_N \) and where \( N = \text{ind}(A_S) \).

**Step 2.** Fix a constant \( p \geq 2 \). Then there is a constant \( c_1 = c_1(p, c_S) \) such that

\[
\|\xi\|_{\mathcal{W}^{1,p}([-1,0] \times S^1)} \leq c_1 \left( \|D\xi\|_{L^p([-3,0] \times S^1)} + \|\xi\|_{L^2([-3,0] \times S^1)} \right)
\]

for \( \xi \in C^\infty([-3,0] \times S^1) \). Moreover, if \( \xi \in \mathcal{W}^{1,2} \) and \( D\xi \in L^p_{\text{loc}}, \) then \( \xi \in \mathcal{W}^{1,p}_{\text{loc}} \).

Choose a smooth compactly supported cutoff function \( \rho : (-2, 0) \to [0, 1] \) such that \( \rho = 1 \) on \([-1, 0]\) and \( \|\partial_s \rho\|_\infty \leq 2 \). Now apply proposition A.4 for the backward halfcylinder \( Z^- \), Euclidean space \( \mathbb{R}^n \), covariant derivatives replaced by partial derivatives, and with constant \( c \) to the function \( \rho \xi \) to obtain that

\[
\|\xi\|_{\mathcal{W}^{1,p}([-1,0] \times S^1)} \leq c \left( 2 \|\partial_s - \partial_t \partial_t\xi\|_{L^p([-2,0] \times S^1)} + \|\xi\|_{L^p([-2,0] \times S^1)} \right)
\]

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for every $\xi \in C^\infty([-2,0] \times S^1)$. To obtain the first estimate in step 3 for the backward half cylinder it will be crucial that the domain on the right hand side does not extend to the future. Now write $\partial_\tau - \partial_t \partial_k = D + S$ and use that the operator $S : W^{1,p}(S^1) \to L^p(S^1)$ is bounded to obtain that

$$\|\xi\|_{W^{1,p}([-1,0] \times S^1)} \leq c \left( \|D\xi\|_{L^p([-2,0] \times S^1)} + (1 + c_S) \|\xi\|_{L^p([-2,0] \times S^1)} \right)$$

for every $\xi \in C^\infty([-2,0] \times S^1)$ and some constant $\tilde{c} = \tilde{c}(p, c_S)$. Now integrate the estimate in lemma A.3 over $s \in [-2,0]$ and chose $\delta > 0$ sufficiently small in order to throw the arising term $\partial_\tau \partial_k \xi$ to the left hand side. It follows that

$$\|\xi\|_{W^{1,p}([-1,0] \times S^1)} \leq \tilde{c} \left( \|D\xi\|_{L^p([-2,0] \times S^1)} + \|\xi\|_{L^p([-2,0] \times S^1)} \right) \quad (94)$$

for every $\xi \in C^\infty([-2,0] \times S^1)$ and some constant $\tilde{c} = \tilde{c}(p, c_S)$. It remains to replace the $L^p$ norm of $\xi$ by the $L^2$ norm. Since $p \geq 2$, there is the Sobolev inequality $\|\xi\|_{L^p} \leq c_p \|\xi\|_{W^{1,2}}$ for $\xi \in W^{1,2}$; see e.g. [LL97, theorem 8.5 (ii)] for the domain $\mathbb{R}^2$. The first step is to replace the last term in (94) according to the Sobolev inequality. Then use (94) with $p = 2$ and on increased domains to complete the proof of the estimate in step 2 (use Hölder’s inequality to estimate the $L^2$ norm of $D\xi$ by the $L^p$ norm).

To conclude the proof of step 2 assume $\xi \in W^{1,2}$, then of course $\xi \in L^2$ and $D\xi \in L^2$. If in addition $D\xi$ is locally $L^p$ integrable, then the estimate of step 2 which we just proved shows that $\xi \in W^{1,p}_\text{loc}$.

**Step 3.** Fix a constant $p \geq 2$ and consider the norm

$$\|\xi\|_{2,p} = \left( \int_0^1 \|\xi(s,\cdot)\|_{L^2(S^1)}^p \, ds \right)^{1/p}.$$

Then there exist constants $c_2$ and $c_3$ both depending on $p$ and $c_S$ such that the following is true. If $\xi \in W^{1,2}$ and $D\xi \in L^p$, then $\xi \in W^{1,p}$ and

$$\|\xi\|_{W^{1,p}} \leq c_2 \left( \|D\xi\|_{L^p} + \|\xi\|_{2,p} \right), \quad \|QD\xi\|_{2,p} \leq c_3 \|D\xi\|_{L^p}.$$

Fix $\xi \in W^{1,2}$ such that $D\xi \in L^p$. Then $\xi \in W^{1,p}_\text{loc}$ by step 2. Moreover, the estimate of step 2 implies that

$$\|\xi\|_{W^{1,p}([k,k+1] \times S^1)} \leq 3^{p/2 - 1} 2^p c_1 \int_{k-2}^{k+1} \left( \|D\xi\|_{L^p(S^1)} + \|\xi\|_{L^2(S^1)} \right) \, ds$$

for every integer $k < 0$; see [S99, lemma 2.4 step 3] for details. Now take the sum over all such $k$ to obtain the first estimate of step 3.

Next observe that $\eta := D\xi$ lies in $L^2(\mathbb{R}^-, H)$ and in $L^p(\mathbb{R}^-, H)$. Here $H = L^2(S^1)$ and we used that by Hölder’s inequality

$$\|\cdot\|_{L^2(S^1)} \leq \|\cdot\|_{L^p(S^1)}.$$ 

(95)
Since $\eta$ is in the domain $L^2(\mathbb{R}^-, H)$ of the operator $Q$ from step 1, we obtain

$$Q D\xi = Q\eta = K \ast \eta.$$  

Now Young’s inequality applies to $K \ast \eta$, because $\eta \in L^p(\mathbb{R}^-, H)$. Hence

$$\|K \ast \eta\|_{L^p} \leq \|K\|_{L^1(\mathbb{R}^-, L^1(H))} \|\eta\|_{L^p(\mathbb{R}^-, H)} \leq C \|D\xi\|_{L^p}$$  \hspace{1cm} (96) 

where $C$ depends on the constant $\delta$ in estimate (92) for the norm of $K$; see [S99]. The last step uses (95) again. This proves the second estimate of step 3.

It remains to prove that $\xi \in W^{1, p}$. The two estimates of step 3 imply that

$$\|\xi\|_{W^{1, p}} \leq c_2 \left( (1 + c_3) \|D\xi\|_{L^p} + \|\xi - Q D\xi\|_{L^p} \right).$$

To see that the right hand side is finite recall that $D\xi \in L^p$ by assumption and $\xi - Q D\xi$ lies in the kernel of $D : W^{1, 2} \rightarrow L^2$ by (the proof of) step 1. Moreover, by (93) every element of this kernel is a finite sum of functions of the form $\xi_k = e^{-s\lambda_k} v_k$ and $\|\xi_k\|_{L^p} \leq 1/\lambda_k$ by calculation.

**Step 4. The theorem is true for $p > 2$.**

Fix $p > 2$ and set $X^- := \ker[D : W^{1, 2} \rightarrow L^2]$. Then the linear operator

$$\pi : W^{1, p} \rightarrow (X^-, \|\cdot\|_{L^2}), \quad \xi \mapsto \xi - Q D\xi,$$

is well defined, bounded and of finite rank, hence compact. To prove this observe that $\pi$ is well defined on the dense subset $C_0^\infty(Z^-)$ of $W^{1, p}$. Since $C_0^\infty(Z^-)$ is also dense in $W^{1, 2}$, step 1 shows that $\xi - Q D\xi \in X^-$. To see that $\pi$ is bounded on $C_0^\infty(Z^-)$ let $\xi \in C_0^\infty(Z^-)$. Then

$$\|\pi\xi\|_{L^p} = \|\xi - Q D\xi\|_{L^p} \leq \|\xi\|_{L^p} + c_3 \|D\xi\|_{L^p} \leq (1 + c_3 c_4) \|\xi\|_{W^{1, p}}$$

by definition of $\pi$, the triangle inequality, the estimate (95), and the second estimate of step 3. The last inequality follows from the estimate

$$\|D\xi\|_{L^p} \leq \|\partial_1 \xi\|_{L^p} + \|\partial_2 \xi\|_{L^p} + \|S\xi\|_{L^p} \leq c_4 \|\xi\|_{W^{1, p}}$$

with suitable constant $c_4 = c_4(p, c_S)$. Here we used that $\|S\|_{L^p} \leq c_S(\|\xi\|_{L^p} + \|\partial_2 \xi\|_{L^p})$ by boundedness of $S$. Now being bounded on a dense subset the operator $\pi$ extends to a bounded linear operator on $W^{1, p}$.

To prove that $D : W^{1, p} \rightarrow L^p$ is onto we show first that the range is closed and then that it is dense. By the two estimates of step 3 we have that

$$\|\xi\|_{W^{1, p}} \leq c_2 \left( (1 + c_3) \|D\xi\|_{L^p} + \|\pi\xi\|_{L^p} \right)$$

for every $\xi \in C_0^\infty$, hence for every $\xi \in W^{1, p}$ by density. Since $\pi$ is compact, the range of $D$ is closed by the abstract closed range lemma. To prove density of the range fix $\eta \in L^p \cap L^2$ and note that the subset $L^p \cap L^2$ is dense in $L^p$, because it contains the dense subset $C_0^\infty$ of $L^p$. Now by surjectivity of $D$ in the case $p = 2$ (step 1) and since $\eta \in L^2$, there exists an element $\xi \in W^{1, 2}$ such that $D\xi = \eta$. But then $\xi \in W^{1, p}$ by step 3, because $D\xi = \eta \in L^p$ by the choice of $\eta$. Hence $\eta$ is in the range of $D : W^{1, p} \rightarrow L^p$. This proves theorem 7.5.  \hspace{1cm} $\square$
The arguments in the proof of proposition 2.17 show that the kernel of \( D_{u^-} : W^{1,p} \to L^p \) is equal to \( X^- \) and \( X^- \) does not depend on \( p \). On the other hand, for \( p = 2 \) the dimension of the kernel is equal to the Morse index of \( x \) by theorem 7.5. Surjectivity of \( D_{u^-} \) follows in three stages.

**The stationary case.** Consider the stationary solution \( u^-(s,t) = x(t) \), then \( D_{x} \) is onto by theorem 7.5. To see this represent \( D_{x} \) with respect to an orthonormal frame along \( x \); see section 2.4.

**The nearby case.** Surjectivity is preserved under small perturbations with respect to the operator norm. Moreover, the operator family \( D_{u^-} \) depends continuously on \( u^- \) with respect to the \( W^{1,p} \) topology (here we use \( p > 2 \)). Hence, if \( u^- \in \mathcal{M}^-(x; \mathcal{V}) \) satisfies \( u^- = \exp_x(\eta) \) and \( ||\eta||_{W^{1,p}} \) is sufficiently small, it follows that \( D_{u^-} \) is onto.

**The general case.** Given \( u \in \mathcal{M}^-(x; \mathcal{V}) \) and \( \sigma < 0 \), consider the shifted solution \( u^\sigma(s,t) := u(s + \sigma, t) \). Then \( (D_u \xi)^\sigma = D_{u^\sigma} \xi^\sigma \) by shift invariance of the linear heat equation. This means that surjectivity of \( D_u \) is equivalent to surjectivity of \( D_{u^\sigma} \). But the latter is true by the nearby case above, because \( u^\sigma \) converges to \( x \) in the \( W^{1,p} \) topology as \( \sigma \to -\infty \). To see this apply theorem 3.10 (B) on exponential decay to \( u \) and note that \( u^\sigma(0, t) = u(\sigma, t) \).

**Proof of proposition 7.2.** The proof follows the same (standard) pattern as the proof of theorem 1.11; see also the introduction to section 4. The first key step is the definition of a Banach manifold \( \mathcal{B} = \mathcal{B}_\mathcal{V}^1 \) of backward halfcylinders emanating from \( x \) such that \( \mathcal{B} \) contains the moduli space \( \mathcal{M}^-(x; \mathcal{V}) \) whenever \( p > 2 \). The second key step is to define a smooth map \( \mathcal{F}_{u^-} \) between Banach spaces as in (49). Its significance lies in the fact that its zeroes correspond precisely to the elements of the moduli space near \( u^- \) and that \( d\mathcal{F}_{u^-}(0) = D_{u^-} \). By proposition 7.3 this operator is surjective and the dimension of its kernel is equal to the Morse index of \( x \). Hence \( \mathcal{M}^{-}(x; \mathcal{V}) \) is locally near \( u^- \) modeled on \( \ker D_{u^-} \) by the implicit function theorem for Banach spaces. To see that the moduli space is a contractible manifold observe that backward time shift provides a contraction

\[
\begin{align*}
    h : \mathcal{M}^-(x; \mathcal{V}) \times [0, 1] &\to \mathcal{M}^-(x; \mathcal{V}) \\
    (u, r) &\mapsto u(\cdot - \sqrt{r/(1-r)}, \cdot)
\end{align*}
\]

onto the stationary solution \( x \), that is \( h \) is continuous and satisfies \( h(u, 0) = u \) and \( h(u, 1) = x \) for every \( u \in \mathcal{M}^-(x; \mathcal{V}) \).

**Proof of theorem 7.1.** We abbreviate \( \mathcal{M}^- = \mathcal{M}^-(x; \mathcal{V}) \) and \( W^u = W^u(x; \mathcal{V}) \). Recall that the moduli space \( \mathcal{M}^- \) is a smooth manifold of dimension equal to \( \text{ind}_{\mathcal{V}}(x) \) by proposition 7.2 and, furthermore, by definition the unstable manifold \( W^u \) is equal to the image of the evaluation map \( ev_0 : \mathcal{M}^- \to \mathcal{L} \mathcal{M} \). We use the notation \( ev_0(u) =: u_0 \), hence \( u_0(t) = u(0, t) \). It remains to prove that \( ev_0 \) and its linearization are injective and that \( ev_0 \) is a homeomorphism onto \( W^u \).

To prove that \( ev_0 \) is injective let \( u, v \in \mathcal{M}^- \) and assume that \( ev_0(u) = ev_0(v) \), that is \( u_0 = v_0 \). Hence \( u = v \) by theorem 5.4 on backward unique continuation.
We prove that the linearization \( d(ev_0)_u \) of \( ev_0 \) at \( u \in M^- \) is injective. Let \( \xi, \eta \in T_uM^- \). Hence \( D_u \xi = 0 = D_u \eta \) by proposition 7.2. Now assume that \( d(ev_0)_u \xi = d(ev_0)_u \eta \). This means that \( \xi_0 = \eta_0 \). Therefore \( \xi = \eta \) by application of proposition 5.2 (a) on linear unique continuation to the vector field \( \xi - \eta \).

We prove that \( ev_0 : M^- \to L^M \) is a homeomorphism onto its image. Fix \( u \in M^- \) and recall that every immersion is locally an embedding. Hence there is an open disk \( D \) in \( M^- \) containing \( u \) such that \( ev_0|_D : D \to L^M \) is an embedding. It remains to prove that there is an open neighborhood \( U \) of \( u_0 = ev_0(u) \) in \( L^M \) such that

\[
U \cap W^u = U \cap ev_0(D). \tag{97}
\]

Now there are two cases. In case one \( u \) is constant in \( s \) and therefore \( u \equiv x \). Here we exploit the (negative) gradient flow property that the restricted function \( S_V|W^u \) takes on its maximum precisely at the critical point \( x \). Case two is the complementary case in which \( u \) depends on \( s \). Here we use a convergence argument based on the compactness theorem 3.3.

**Case 1.** \( (u \equiv x) \) Set \( c = S_V(x) \), then a set \( U \) having the desired property (97) is given by

\[
U := \{ c - \varepsilon < S_V < c + \varepsilon \},
\]

where

\[
2\varepsilon := \min_{u \in clD \setminus D} (S_V(x) - S_V(u_0)).
\]

Here the compact set \( clD \setminus D \) is the topological boundary of the open disc \( D \). Note that the elements of \( W^u \setminus ev_0(D) \) have action at most \( c - 2\varepsilon \).

**Case 2.** \( (u \not\equiv x) \) Assume by contradiction that there is no \( U \) which satisfies (97). Then there is a sequence \( \gamma^\nu \in W^u \setminus ev_0(D) \) that converges to \( u_0 \) in \( L^M \) as \( \nu \to \infty \). Note that \( \gamma^\nu = ev_0(u^\nu) \) where \( u^\nu \in M^- \setminus D \). In particular, each heat trajectory \( u^\nu \) converges in backward time asymptotically to \( x \). Thus we obtain that

\[
\sup_{s \in (-\infty,0]} S_V(u^\nu_s) \leq S_V(x) =: c
\]

for every \( \nu \). Together with the energy identity this implies that

\[
E(u^\nu) = S_V(x) - S_V(u^\nu_0)
= c - \frac{1}{t} \| \partial_t u^\nu \|^2_{L^2(S^1)} + V(u^\nu_0)
\leq c + C_0
\]

where \( C_0 > 1 \) is the constant in axiom (V0). Adapting the proofs of the apriori theorem 3.5 and the gradient bound theorem 3.9 to cover the case of backward halfcylinders it follows that there is a constant \( C = C(c, \mathcal{V}) > 0 \) such that

\[
\| \partial_t u^\nu \| \leq C,
\]

and

\[
\| \partial_s u^\nu \| \leq C \sqrt{E(u^\nu)} \leq C(c + C_0)
\]
for every $\nu$. Here the norms are taken on the domain $(-\infty, 0] \times S^1$. Adapting also the proof of the compactness theorem 3.3 we obtain – in view of the uniform a priori $L^\infty$ bounds for $\partial_t u^\nu$ and $\partial_s u^\nu$ just derived – the existence of a smooth heat flow solution $v : (-\infty, 0] \times S^1 \to M$ and a subsequence, still denoted by $u^\nu$, such that $u^\nu$ converges to $v$ in $C^\infty_{loc}$. In particular, this implies that $u_0 = v_0$ and that $\partial_t u^\nu_s$ converges to $\partial_t v_s$, as $\nu \to \infty$, uniformly with all derivatives on $S^1$ and for each $s$. This and our earlier uniform action bound for $u^\nu_s$ show that $S_V(v_s) = \lim_{\nu \to \infty} S_V(u^\nu_s) \leq c$

for every $s$. To summarize, we have two backward flow lines $u$ and $v$ defined on $(-\infty, 0] \times S^1$ along which the action is bounded from above by $c$ and which coincide along the loop $u_0 = v_0$. Hence theorem 5.4 (B) on backward unique continuation asserts that $u = v$. Because $u^\nu$ converges to $v$ in $C^\infty_{loc}$, this means that $u^\nu \in D$ whenever $\nu$ is sufficiently large. For such $\nu$ we arrive at the contradiction $\gamma^\nu = ev_0(u^\nu) \in ev_0(D)$ and this proves theorem 7.1.

7.2 The Morse complex

Assume that the action $S_V$ is a Morse function on the loop space. This is true for a generic potential $V \in C^\infty(S^1 \times M)$ by [W02]. Fix a regular value $a$ of $S_V$ and, furthermore, for each critical point $x \in P_a(V)$ fix an orientation $\langle x \rangle$ of the tangent space at $x$ to the (finite dimensional) unstable manifold $W_u(x; V)$. By $\nu = \nu(V, a)$ we denote a choice of orientations for all $x \in P_a(V)$. The Morse chain groups are the $\mathbb{Z}$-modules

$$CM_a^k = CM_a^k(V, \nu) := \bigoplus_{x \in P_a(V) \text{ ind}_V(x) = k} \mathbb{Z} \langle x \rangle, \quad k \in \mathbb{Z}.$$  

These modules are finitely generated and graded by the Morse index. We set $C_a^0 = \{0\}$ whenever the direct sum is taken over the empty set. We define

$$CM_a^k := \bigoplus_{k=0}^N CM_a^k$$

where $N$ is the largest Morse index of an element of the finite set $P_a(V)$.

Set $V(x) = \int_0^1 V_t(x(t)) \, dt$ and note that $V$ satisfies (V0)–(V3). Now consider the associated set of admissible perturbations $\mathcal{O}^a$ of $V$ defined by (80) and the dense subset $\mathcal{O}^a_{reg}$ of regular perturbations provided by theorem 1.14. (The ambient Banach space $\mathcal{Y}$ given by (76) provides the metric on $\mathcal{O}^a$.) Now for any $v \in \mathcal{O}^a_{reg}$ we have the following key facts: The functionals $S_V$ and $S_{V+v}$ coincide near their critical points and have the same sublevel set with respect to $a$. Moreover, the perturbed functional $S_{V+v}$ is Morse-Smale below level $a$. Here and throughout we sometimes denote $V + v$ in abuse of notation by $V + v$ to emphasize that we are actually perturbing a geometric potential.
To define the Morse boundary operator \( \partial \) on \( \text{CM}^* \) it suffices to define it on the set of generators \( \mathcal{P}^* (V) \) and then extend linearly. Fix a regular perturbation \( v \in \mathcal{O}_{\text{reg}} \). Note that each chosen orientation \( \langle x \rangle \) orients the perturbed unstable manifold \( W^u(x; V + v) \). This is because the tangent spaces at \( x \) to \( W^u(x; V) \) and \( W^u(x; V + v) \) coincide (\( v \) is not supported near \( x \)) and unstable manifolds are finite dimensional and contractible, hence orientable, by theorem 7.1. Now given two critical points \( x^\pm \) of action less than \( a \), consider the heat moduli space \( \mathcal{M}(x^-, x^+; V + v) \) of solutions \( u \) of the heat equation (6) with \( V \) replaced by \( V + v \) and subject to the boundary condition (8). Jointly with D. Salamon we proved in [SW03, ch. 11] that a choice of orientations for all unstable manifolds determines a system of coherent orientations on the heat moduli spaces in the sense of Floer–Hofer [FH93].

From now on we assume that \( x^\pm \) are of Morse index difference one. In this case \( \mathcal{M}(x^-, x^+; V + v) \) is a smooth 1-dimensional manifold by theorem 1.11 and its quotient \( \mathcal{M}(x^-, x^+; V + v)/\mathbb{R} \) by the (free) time shift action consists of finitely many points by proposition 1.12. For \( [u] \in \mathcal{M}(x^-, x^+; V + v)/\mathbb{R} \) time shift naturally induces an orientation of the corresponding component of \( \mathcal{M}(x^-, x^+; V + v) \); compare [SW03] and note that \( \partial_u u \in \ker D_u = \det(D_u) \). We set \( n_u = 1 \), if the time shift orientation coincides with the coherent orientation, and we set \( n_u = -1 \) otherwise. One calls \( n_u \) the characteristic sign of the heat trajectory \( u \). It depends on the orientations \( \langle x^- \rangle \) and \( \langle x^+ \rangle \). Consider the (finite) sum of characteristic signs corresponding to all heat trajectories from \( x^- \) to \( x^+ \), namely

\[
n(x^-, x^+) := \sum_{[u] \in \mathcal{M}(x^-, x^+; V + v)/\mathbb{R}} n_u.
\]

If the sum runs over the empty set, we set \( n(x^-, x^+) = 0 \). For \( x \in \mathcal{P}^*(V) \) define the Morse boundary operator \( \partial = \partial(V, a, \nu, v) \) by the (finite) sum

\[
\partial x := \sum_{y \in \mathcal{P}^*(V)} n(x, y) y.
\]

Set \( \partial x = 0 \), if the sum runs over the empty set.

Proof of theorem 1.1. The main result of [SW03] is that for each heat flow line \( u \) between critical points of Morse index difference one there is precisely one Floer trajectory in the loop space of the cotangent bundle between corresponding critical points of the symplectic action functional; see [SW03, cor. 10.4 (ii)]. Moreover, we proved that the characteristic sign of \( u \) coincides with the characteristic sign of the corresponding Floer trajectory. In other words, both chain complexes are equal (up to natural identification). Hence \( \partial \circ \partial = 0 \) follows immediately from the well known analogue for the Floer boundary operator; see e.g. [F89b, S99]. (The required, but in case of our nongeometric potentials \( V \) slightly nonstandard apriori \( C^0 \) estimate is provided by [SW03, thm. 5.1] with \( \varepsilon = 1 \).)
The fact that heat flow homology is independent of the choice of orientations \( \nu \) and the regular perturbation \( v \) follows from the homotopy argument which is standard in Floer theory; see again e.g. [F89b, S99]. Here it is crucial to observe that our admissible perturbations \( v \in \mathcal{O}^a \) are supported away from the level set \( \{ S_V = a \} \) on which the \( L^2 \) gradient of \( S_V \) (hence of \( S_V + v \)) is nonvanishing and inward pointing with respect to \( \mathcal{L}^0 M \). Likewise independence follows by theorem 1.15.

A Parabolic regularity

Proofs of all results collected in this appendix are given in [W09], unless specified differently. By \( \mathbb{H}^- \) we denote the closed lower half plane, that is the set of pairs of reals \((s, t)\) with \( s \leq 0 \). In this section, unless specified differently, all maps are real-valued and the domains of the various Banach spaces which appear are understood to be either open subsets \( \Omega \) of \( \mathbb{R}^2 \) or \( \mathbb{H}^- \) or (cylindrical subsets of) the cylinder \( Z = \mathbb{R} \times S^1 \). To deal with the heat equation it is useful to consider the anisotropic Sobolev spaces \( W^{k,p}_p \). We call them parabolic Sobolev spaces and denote them by \( \mathcal{W}^{k,p} \). For constants \( p \geq 1 \) and integers \( k \geq 0 \) these spaces are defined as follows. Set \( W^{0,p} = L^p \) and denote by \( W^{1,p} \) the set of all \( u \in L^p \) which admit weak derivatives \( \partial_s u, \partial_t u, \partial_t \partial_t u \) in \( L^p \). For \( k \geq 2 \) define

\[
\mathcal{W}^{k,p} = \{ u \in W^{1,p} \mid \partial_s u, \partial_t u, \partial_t \partial_t u \in \mathcal{W}^{k-1,p} \}
\]

where the derivatives are again meant in the weak sense. The norm

\[
\| u \|_{\mathcal{W}^{k,p}} := \left( \int \int \sum_{2\nu + \mu \leq 2k} |\partial_\nu s \partial_\mu t u(s, t)|^p \, dt \, ds \right)^{1/p}
\]

(98)

gives \( \mathcal{W}^{k,p} \) the structure of a Banach space. Here \( \nu \) and \( \mu \) are nonnegative integers. For \( k = 1 \) we obtain that

\[
\| u \|_{\mathcal{W}^{1,p}} = \| u \|_p + \| \partial_s u \|_p + \| \partial_t u \|_p + \| \partial_t \partial_t u \|_p
\]

and occasionally we abbreviate \( \mathcal{W} = \mathcal{W}^{1,p} \). Note the difference to (standard) Sobolev space \( W^{k,p} \) where the norm is given by

\[
\| u \|_{W^{k,p}} = \sum_{\nu + \mu \leq k} \| \partial_\nu s \partial_\mu t u \|_p
\]

A rectangular domain is a set of the form \( I \times J \) where \( I \) and \( J \) are bounded intervals. For rectangular (or more generally Lipschitz) domains \( \Omega \) the parabolic Sobolev spaces \( \mathcal{W}^{k,p} \) can be identified with the closure of \( C^\infty(\overline{\Omega}) \) with respect to the \( W^{k,p} \) norm; see e.g. [MS04, appendix B.1]. Similarly, we define the \( C^k \) (or \( \mathcal{W}^{k,\infty} \)) norm by

\[
\| u \|_{C^k} := \sum_{2\nu + \mu \leq 2k} \| \partial_\nu s \partial_\mu t u \|_\infty
\]

(99)
The following parabolic analogue of the Calderon-Zygmund inequality is used to prove theorem A.2 on local regularity.

**Theorem A.1** (Fundamental $L^p$ estimate, [SW03]). For every $p > 1$, there is a constant $c = c(p)$ such that
\[ \|\partial_s v\|_p + \|\partial_t \partial_t v\|_p \leq c\|\partial_s v - \partial_t \partial_t v\|_p \]
for every $v \in C_0^\infty(\mathbb{R}^2)$. The same statement is true for the domain $\mathbb{H}^-$.

**Theorem A.2** (Local regularity). Fix a constant $1 < q < \infty$, an integer $k \geq 0$, and an open subset $\Omega \subset \mathbb{H}^-$. Then the following is true.

a) If $u \in L^1_{\text{loc}}(\Omega)$ and $f \in W^{k,q}_{\text{loc}}(\Omega)$ satisfy
\[ \int_{\Omega} u (-\partial_s \phi - \partial_t \partial_t \phi) = \int_{\Omega} f \phi \] (100)
for every $\phi \in C_0^\infty(\text{int } \Omega)$, then $u \in W^{k+1,q}_{\text{loc}}(\Omega)$.

b) If $u \in L^1_{\text{loc}}(\Omega)$ and $f, h \in W^{k,q}_{\text{loc}}(\Omega)$ satisfy
\[ \int_{\Omega} u (-\partial_s \phi - \partial_t \partial_t \phi) = \int_{\Omega} f \phi - \int_{\Omega} h \partial_t \phi \] (101)
for every $\phi \in C_0^\infty(\text{int } \Omega)$, then $u$ and $\partial_t u$ are in $W^{k,q}_{\text{loc}}(\Omega)$.

**Lemma A.3** ([SW03, lemma D.4]). Let $x \in C^\infty(S^1, M)$ and $p > 1$. Then
\[ \|\nabla_t \xi\|_p \leq \kappa_p \left( \delta^{-1} \|\xi\|_p + \delta \|\nabla \nabla \xi\|_p \right) \]
for $\delta > 0$ and smooth vector fields $\xi$ along $x$. Here $\kappa_p$ equals $p/(p-1)$ for $p \leq 2$ and it equals $p$ for $p \geq 2$.

**Proposition A.4.** Assume $u : \mathbb{R} \times S^1 \to M$ is a smooth map such that $\|\partial_s v\|_\infty$, $\|\partial_t u\|_\infty$, and $\|\nabla_t \partial_t u\|_\infty$ are finite and $\lim_{s \to \pm \infty} u(s, t)$ exists, uniformly in $t$. Then, for every $p > 1$, there is a constant $c = c(p, u, M)$ such that
\[ \|\nabla_s \xi\|_p + \|\nabla_t \xi\|_p + \|\nabla \nabla \xi\|_p \leq c \left( \|\nabla_s \xi - \nabla_t \partial_t \xi\|_p + \|\xi\|_p \right) \] (102)
for every smooth compactly supported vector field $\xi$ along $u$. Estimate (102) remains valid for $-\nabla_s$ replacing $\nabla_s$. Estimate (102) also remains valid if $u$ is defined on the backward halfcylinder $(-\infty, 0] \times S^1$.

**Proof.** The proof of (102) for $\mathbb{R} \times S^1$ and $(-\infty, 0] \times S^1$ is based on theorem A.1 for $\mathbb{R}^2$ and $\mathbb{H}^-$, respectively, using a covering argument. Full details in the case $\mathbb{R} \times S^1$ are provided by [SW03, prop. D.2]. Lemma A.3 allows to add the term $\nabla_s \xi$ to the left hand side of (102). The underlying reason is periodicity in the $t$ variable. The statement for $-\nabla_s$ follows by reflection $s \mapsto -s$. \qed
Applications of proposition A.4 include closedness of the range of the linearized operator, proposition 2.18, estimate (47) in the proof of the exponential decay theorem 1.9, and step 2 in the proof of theorem 7.5.

**Lemma A.5** (Product estimate). Let \( N \) be a Riemannian manifold with Levi-Civita connection \( \nabla \) and Riemannian curvature tensor \( R \). Fix constants \( 2 \leq p < \infty \) and \( c_0 > 0 \). Then there is a constant \( C = C(p, c_0, \|R\|_{\infty}) \) such that the following holds. If \( u : (a, b] \times S^1 \to N \) is a smooth map such that

\[
\|\partial_s u\|_{\infty} + \|\partial_t u\|_{\infty} \leq c_0,
\]

then

\[
\left( \int_a^b \int_0^1 \left( |\nabla \xi| \cdot |\nabla X| \right)^p \, dt \, ds \right)^{1/p} \leq C \|\xi\|_{W^{1,p}} \left( \|\nabla X\|_p + \|\nabla \nabla X\|_p \right)
\]

for all smooth compactly supported vector fields \( \xi \) and \( X \) along \( u \).

**Remark A.6.** Lemma A.5 continues to hold for smooth maps \( u \) that are defined on the whole cylinder \( \mathbb{R} \times S^1 \). In this case the (compact) supports of \( \xi \) and \( X \) are contained in an interval of the form \( (a, b] \).

Now we fix a closed smooth submanifold \( M \hookrightarrow \mathbb{R}^N \) and a smooth family of vector-valued symmetric bilinear forms \( \Gamma : M \to \mathbb{R}^{N \times N \times N} \). Abbreviate \( W^{k,p}(Z) = W^{k,p}(Z, \mathbb{R}^N) \). Moreover, for \( T > T' > 0 \) we abbreviate

\[
Z = Z_T = (-T,0] \times S^1, \quad Z' = Z_{T'} = (-T',0] \times S^1.
\]

**Proposition A.7** (Parabolic regularity). Fix constants \( p > 2, \mu_0 > 1, \) and \( T > 0 \). Fix a map \( F : Z \to \mathbb{R}^N \) such that \( F \) and \( \partial F \) are of class \( L^p \). Assume that \( u : Z \to \mathbb{R}^N \) is a \( W^{1,p} \) map taking values in \( M \) with \( \|u\|_{W^{1,p}} \leq \mu_0 \) and such that the perturbed heat equation

\[
\partial_s u - \partial_t \partial_t u = \Gamma(u) (\partial_s u, \partial_t u) + F
\]

is satisfied almost everywhere. Then the following is true for every integer \( k \geq 1 \) such that \( F, \partial_t F \in W^{k-1,p}(Z) \) and every \( T' \in (0, T) \).

(i) There is a constant \( a_k \) depending on \( p, \mu_0, T, T', \|\Gamma\|_{C^{2k+2}}, \) and the \( W^{k-1,p}(Z) \) norms of \( F \) and \( \partial_t F \) such that

\[
\|\partial_s u\|_{W^{k,p}(Z')} \leq a_k.
\]

(ii) If \( \partial_s F \in W^{k-1,p}(Z) \) then there is a constant \( b_k \) depending on \( p, \mu_0, T, T', \|\Gamma\|_{C^{2k+2}}, \) and the \( W^{k-1,p}(Z) \) norms of \( F, \partial_t F, \) and \( \partial_s F \) such that

\[
\|\partial_s u\|_{W^{k,p}(Z')} \leq b_k.
\]
(iii) If $\partial \partial_t u \partial_t F \in W^{k-1,p}(Z)$ then there is a constant $c_k$ depending on $p$, $\mu_0$, $T$, $T'$, $\|\Gamma\|_{C^{k+2}}$, and the $W^{k-1,p}(Z)$ norms of $F$, $\partial_t F$, and $\partial_t \partial_t F$ such that

$$\|\partial_t \partial_t u\|_{W^{k,p}(Z')} \leq c_k.$$ 

Note that by the Sobolev embedding theorem the assumption $p > 2$ guarantees that every $W^{1,p}$ map $u$ is continuous. Hence it makes sense to specify that $u$ takes values in the submanifold $M$ of $\mathbb{R}^N$.

**Corollary A.8.** Under the assumptions of proposition A.7 the following is true. For every integer $k \geq 1$ such that $F \in W^{k,p}(Z_T)$ and every $T' \in (0,T)$ there is a constant $c_k = c_k(k, p, \mu_0, T - T', \|\Gamma\|_{C^{k+2}(M)}, \|F\|_{W^{k,p}(Z_T)})$ such that

$$\|\partial_t u\|_{W^{k+1,p}(Z_T')} \leq c_k.$$ 

**Proof.** The $W^{k+1,p}$ norm of $u$ is equivalent to the sum of the $W^{k,p}$ norms of $u$, $\partial_t u$, $\partial_s u$, and $\partial_t \partial_t u$. Apply proposition A.7 (i–iii).

**Lemma A.9 ([W09]).** Fix a constant $p > 2$ and a bounded open subset $\Omega \subset \mathbb{R}^2$ with area $|\Omega|$. Then for every integer $k \geq 1$ there is a constant $c = c(k, |\Omega|)$ such that

$$\|\partial_t u \cdot v\|_{W^{k,p}} \leq c (\|\partial_t u\|_{W^{k,p}} \|v\|_\infty + \|u\|_{C^k} \|v\|_{W^{k,p}})$$

for all functions $u, v \in C^\infty(\overline{\Omega})$.

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