Some stochastic comparison results for frailty and resilience models

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Abstract
Frailty and resilience models provide a way to introduce random effects in hazard and reversed hazard rate modeling by random variables, called frailty and resilience random variables, respectively, to account for unobserved or unexplained heterogeneity among experimental units. This paper investigates the effects of frailty and resilience random variables on the baseline random variables using some shifted stochastic orders based on some ageing properties of the baseline random variables. Relevant examples are provided to illustrate the results. Some results are illustrated with real-world data.

Keywords: Frailty model; Resilience model; Continuous mixture; Stochastic order; Stochastic ageing.
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1 Introduction:
Heterogeneity is a very common issue in many areas including reliability, survival analysis, demography and epidemiology. For instance, in mechanical systems, heterogeneity occurs due to unit-to-unit variability, changes in operating environments, the diversity of tasks and workloads during its lifetime. For example, as mentioned in Özekici and Soyer (2004) that a complex device like an airplane has large number of components where the failure structure of each component depends on a set of environmental conditions (e.g. the levels of vibration, atmospheric pressure, temperature, etc.) that vary during take-off, cruising and landing. So incorporating heterogeneity into hazard (failure) rate modeling is a common practice to achieve accuracy in the estimation. The proportional hazard (PHR) model is the most applied model in the case where factors (covariates) influencing the environment/operating condition are known and can be quantified. In such case, hazard rate of an individual is considered to be constant

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multiplicative to the baseline hazard. However, in many practical situations it may happen that some factors influencing the operating condition are unknown, and heterogeneity occurs in an unpredicted and unexplained manner. A component may subject to different levels of operating environment (e.g. voltage, temperature) which is not fixed but changes over time. Component lifetimes and reliability depend on these random environmental variations. Frailty models (Cha and Finkelstein, 2014, Da et al., 2020, Gupta et al., 2011, Gupta and Peng, 2014, Hougaard, 2000, Li and Li, 2008, Vaupel et al., 1979, Zaki et al., 2022) provide a way to introduce random effects in the model by a random variable (r.v.), called frailty r.v., to account for unobserved (unexplained) heterogeneity among experimental units in their hazard (failure) rates. For instance, Vaupel et al. (1979) discussed that in survival analysis, mortality of individuals differ due to large number of factors beyond age, e.g. the individual’s susceptibility to causes of death, response to treatment and various risk factors. They considered a frailty r.v. to cope with the unobserved individual differences in mortality rates while defining the force of mortality of individuals. Cha et al. (2018) considered a frailty r.v. in the model for mission abort/continuation policy for heterogeneous systems, to justify heterogeneity which may occur due to various reasons such as quality of resources used in the production process, operation and maintenance history, and human errors.

Let $X$ be a r.v. with distribution function $F$ and survival function $\bar{F} = 1 - F$, and $\Lambda$ be a continuous r.v. with distribution function $H$, probability density function (pdf) $h$ and hazard function $r_F$. A r.v. $X^*$ is said to follow multiplicative frailty model with baseline distribution $F$ and frailty r.v. $\Lambda$ if its survival function is given by

$$\bar{F}^*(t) = \int_0^\infty \bar{F}^\Lambda(t)dH(\lambda) \quad (1.1)$$

Here the frailty r.v. $\Lambda$ serves as an unobserved random factor that modifies multiplicatively the underlying hazard function $r_F$ of an individual such that the individual is supposed to have hazard rate $r_{F^\Lambda}(t)$ at age $t$, so that given $\Lambda = \lambda$, the conditional hazard rate function of $X^*$ will be $r_{F^\Lambda}(t|\lambda) = \lambda r_F(t)$, $t \geq 0$.

In analogy to the frailty model, to account for unobserved/unexplained heterogeneity in the reversed hazard rates of the experimental units, the resilience model (reversed frailty models) is introduced. A r.v. $X^*$ is said to follow resilience model with baseline distribution $F$ and resilience r.v. $\Lambda$ if its distribution function is given by

$$F^*(t) = \int_0^\infty F^\Lambda(t)dH(\lambda) \quad (1.2)$$

The model (1.1) is also regarded as mixture (continuous) distribution of the PHR model with baseline distribution function $F$, and mixing r.v. $\Lambda$ (Da et al., 2020). Similarly model (1.2) is regarded as the mixture distribution of the proportional reversed hazard model (Li and Li, 2008).
Ageing properties and stochastic comparisons of frailty models, arising from different choices of frailty/baseline distributions, have been studied by Gupta and Kirmani (2006), Misra and Francis (2020), Kayid et al. (2017a,b), Xie et al. (2016) and Xu and Li (2008). On the other hand ageing properties and stochastic comparison of resilience models have been studied by Gupta et al. (2007) and Li and Li (2008) considering different baseline distributions and/or resilience distributions. He and Xie (2020) derived comparison results for general weighted frailty models with respect to some relative stochastic orders. Another important study in this area is to compare $X^*$ and $X$ which helps us to understand the effect of frailty/resilience r.v. on the underlying original (baseline) distribution. Misra and Francis (2020) studied the ageing of r.v. following frailty (resilience) model ($X^*$) relative to the ageing of corresponding baseline r.v. $X$. Da et al. (2020) compared $X^*$ with certain frailty, i.e. $X^*|\Lambda = \lambda$ and $X$ satisfying some conditions on the mean of the frailty.

In our work we study the effects of frailty and resilience r.v.’s on the baseline r.v. ($X$) using some shifted stochastic orders based on some ageing properties of $X$. In Section 2, we provide definitions of sifted stochastic orders along with their usefulness in stochastic comparisons and superiority over their usual versions. In Section 3, we study the effect of frailty r.v. on the baseline r.v., i.e. compare $X^*$ and $X$ with respect to some shifted stochastic orders, where in Section 4, a similar study is carried out in case of resilience model. In Section 5, we illustrate some of our derived results with real-world data.

2 Notations and definitions

Let $X$ and $Y$ be two absolutely continuous and non-negative r.v.’s with distribution functions $F$ and $G$; survival (/reliability) functions $\bar{F}$ and $\bar{G}$; pdf $f$ and $g$; hazard (failure) rate functions $r_X$ and $r_Y$; reversed hazard rate functions $\bar{r}_X$ and $\bar{r}_Y$, respectively.

Definition 2.1 $X$ is said to be smaller than $Y$ in the
(a) up (down) shifted hazard rate order denoted by $X \leq_{hr} Y$ ($X \leq_{hr_\downarrow} Y$), if $\bar{G}(x)/\bar{F}(x+t)$ ($G(x+t)/\bar{F}(x)$) is increasing in $x$ for all $t > 0$ (Lillo et al., 2000).

(b) up (down) shifted reversed hazard rate order denoted by $X \leq_{rh} Y$ ($X \leq_{rh_\downarrow} Y$), if $G(x)/F(x+t)$ ($G(x+t)/F(x)$) is increasing in $x$ for all $t > 0$ (Di Crescenzo and Longobardi, 2001).

(c) up (down) shifted likelihood ratio order denoted by $X \leq_{lr} Y$ ($X \leq_{lr_\downarrow} Y$), if $g(x+t)/f(x)$ is increasing in $x$ for all $t > 0$ (Lillo et al., 2006, Shaked and Shanthikumar, 2007).

(d) up (down) shifted mean residual life order denoted by $X \leq_{mrl} Y$ ($X \leq_{mrl_\downarrow} Y$), if $\int_{x+t}^{\infty} G(u)du/\int_{x}^{\infty} F(u)du$ ($\int_{x+t}^{\infty} G(u)du/\int_{x}^{\infty} F(u)du$) is increasing in $x$ for all $t > 0$ (Nanda et al., 2010).
(c) up (down) shifted mean inactivity time order (also known as reversed mean residual life order) denoted by $X \leq_{mitr} Y$ ($X \leq_{mitl} Y$), if $\int_0^{t+\kappa} F(u) \frac{f(u)}{G(u)} \frac{G(u)du}{F(u)}$ is decreasing in $x$ for all $t > 0$ \cite{Nanda2000, Kayid2017b}.

It is worth to mention that $X \leq_{lr} Y \Rightarrow X \leq_{hr} Y$, $X \leq_{hr} Y \Rightarrow X \leq_{hr} Y$, and $X \leq_{mrl} Y \Rightarrow X \leq_{mrl} Y$. For positive support, these results also hold true for respective down shifted orders. That means these shifted stochastic orders are stronger than their respective usual versions of stochastic orders. Also, these shifted orders can be considered as generalization of their usual counterparts in some aspects. For instance, unlike likelihood ratio ordering, shifted likelihood ratio ordering preserves the order under convolution \cite{Lillo2000}. If $X \leq_{lr} Y$, then $\kappa_X(t_1) \leq \kappa_Y(t_2)$ for $t_1 \geq t_2 \geq 0$, where $\kappa_X \equiv f'/f$ and $\kappa_Y \equiv g'/g$ \cite{DiCrescenzo2001, Lillo2000}. Note that if $X \leq_{lr} Y$, then $\kappa_X(t) \leq \kappa_Y(t)$ for all $t \geq 0$. It is shown by \cite{DiCrescenzo2001, Lillo2000} that $X \leq_{hr} Y \iff r_X(t_1) \geq r_Y(t_2)$ for $t_1 \geq t_2 \geq 0$. Note that $X \leq_{hr} Y$ implies $r_X(t) \geq r_Y(t)$ for all $t \geq 0$. Similarly, $X \leq_{hr} Y \iff \tilde{r}_X(t_1) \leq \tilde{r}_Y(t_2)$ for $t_1 \geq t_2 \geq 0$ \cite{DiCrescenzo2001}. Note that $X \leq_{hr} Y$ implies $\tilde{r}_X(t) \leq \tilde{r}_Y(t)$ for all $t \geq 0$. Also if $X \leq_{hr} Y$, then $F(t_1) \leq \tilde{G}(t_2)$ for $t_1 \geq t_2 \geq 0$. If $X \leq_{mrl} Y$, then $m_X(t_1) \leq m_Y(t_2)$ for $t_1 \geq t_2 \geq 0$, where $m_X(t) = \int_{t}^{\infty} \frac{F(u)du}{F(t)}$ is the mean residual life (mrl) of $X$ \cite{Nanda2006}. If $X \leq_{mitr} Y$, then $mit_X(t_1) \leq mit_Y(t_2)$ for $t_1 \geq t_2 \geq 0$, where $mit_X(t) = \int_{0}^{t} F(u)du/\int_{0}^{\infty} F(t)du$ is known as mean inactivity time (or reversed mean residual life) of $X$. Similar results are also shown for down shifted orders, e.g., if $X \leq_{hr} Y$, then $r_X(t_1) \geq r_Y(t_2)$ for $t_2 \geq t_1 \geq 0$ \cite{Lillo2000}. Thus these shifted stochastic orders give us the flexibility that even at different points of time for the two variables, we can compare their hazard rates, reversed hazard rates, survival functions, mean residual life, etc.

One such specific instance is that we can compare the reliability of an used device and a new device using the shifted stochastic orders. For more discussion on those shifted orders including their applications and preservations properties, we refer to \cite{boukalam2007, Naqvi2021, Kayid2017b} and references therein.

Next we give the definitions of some ageing classes \cite{Lai2006}.

**Definition 2.2** $X$ is said to have

(a) increasing (decreasing) likelihood ratio (ILR (DLR)) if $f$ is log-concave (log-convex) or equivalently for any $t > 0$, $f(x+t)/f(x)$ is decreasing (increasing) in $x$.

(b) increasing (decreasing) failure rate (IFR (DFR)) if $\tilde{F}$ is log-concave (log-convex) or equivalently for any $t > 0$, $\tilde{F}(x+t)/\tilde{F}(x)$ is decreasing (increasing) in $x$.

(c) decreasing (increasing) reversed failure rate (DRFR (IRFR)) if $F$ is log-concave (log-convex) or equivalently for any $t > 0$, $F(x+t)/F(x)$ is decreasing (increasing) in $x$.

(d) decreasing (increasing) mean residual life (IMRL(DMRL)) if $\int_{x+t}^{\infty} \tilde{F}(u)du$ is log-convex (log-concave) or equivalently for any $t > 0$, $\int_{x}^{\infty} \tilde{F}(u)du/\int_{x}^{\infty} \tilde{F}(u)du$ increasing (decreasing) in $x$.  

4
(e) increasing mean inactivity time (IMIT) if \( \int_0^x F(u)du \) is log-concave or equivalently for any \( t > 0 \), \( \int_0^{x+t} F(u)du / \int_0^x F(u)du \) decreasing in \( x \).

3 Results for frailty model

Here we study effect of frailty r.v. on the baseline r.v. with respect to some shifted stochastic ordering based on some ageing properties of concerned baseline r.v.’s. Throughout this section, we consider \( X \) and \( X^* \) be two r.v.’s as defined in Section 1 for which the survival function of \( X^* \) is given by the equation (1.1). Also consider that \( X \) be an absolutely continuous non-negative r.v.

In the following theorem we derived that, for a baseline r.v. \( X \) with ILR (resp. DLR) property, effect of a frailty r.v. \( \Lambda \) with \( P(0 < \Lambda \leq 1) = 1 \) (resp. \( P(\Lambda \geq 1) = 1 \)) on \( X \) is that, \( X^* \) will be greater than (resp. less than) \( X \) in the sense of the up shifted likelihood ratio order.

Theorem 3.1

(i) \( X^* \geq_{lr\uparrow} X \) if \( X \) is ILR, provided \( 0 < \Lambda \leq 1 \) with probability 1;
(ii) \( X^* \leq_{lr\uparrow} X \) if \( X \) is DLR, provided \( \Lambda \geq 1 \) with probability 1.

Proof:

(i) We have

\[
\frac{f^*(x)}{f(x+t)} = \frac{f(x)}{f(x+t)} \times \int_0^\infty \lambda \bar{F}^{\Lambda-1}(x)dH(\lambda)
= E \left[ \frac{f(x)\lambda \bar{F}^{\Lambda-1}(x)}{f(x+t)} \right]
\]

(3.1)

Now \( X \) is ILR implies \( \frac{f(x)}{f(x+t)} \) is increasing in \( x \) for any \( t > 0 \). Again \( \lambda \bar{F}^{\Lambda-1}(x) \) will be increasing in \( x \) for any \( 0 < \lambda \leq 1 \). Now if we consider \( \Lambda \) such that \( P(0 < \Lambda \leq 1) = 1 \) the result follows immediately.

(ii) Similarly \( X \) is DLR implies \( \frac{f(x)}{f(x+t)} \) is decreasing in \( x \). Again \( \lambda \bar{F}^{\Lambda-1}(x) \) will be decreasing in \( x \) for any \( \lambda \geq 1 \). Now if we consider \( \Lambda \) such that \( P(\Lambda \geq 1) = 1 \), the result follows immediately.

Examples 3.1 and 3.2 illustrate Theorem 3.1(i) and 3.1(ii) respectively.

Example 3.1 Let \( X \) be a gamma r.v. with pdf \( f(x) = xe^{-x} \), \( x \geq 0 \). Then clearly \( X \) is ILR. Consider the frailty r.v. \( \Lambda \) to be uniformly distributed on \([0,1]\). Then it is easy to check that \( f^*(x)/f(x+t) \) is increasing in \( x \) for all \( t > 0 \), giving \( X \leq_{lr\uparrow} X^* \).

Example 3.2 Let \( X \) be a Weibull r.v. with pdf \( f(x) = 3x^2e^{-x^3} \), \( x \geq 0 \). Then clearly \( X \) is ILR. Consider the frailty r.v. \( \Lambda \) to be uniformly distributed on \([1,3]\). Then it is easy to check that \( f^*(x)/f(x+t) \) is decreasing in \( x \) for all \( t > 0 \), giving \( X \geq_{lr\uparrow} X^* \).
The following corollary follows immediately in case \( \Lambda \) is a degenerate r.v.

**Corollary 3.1**

(i) \( X^* \geq \text{lr} \uparrow X \) if \( X \) is ILR, provided \( 0 < \lambda \leq 1 \);

(ii) \( X^* \leq \text{lr} \uparrow X \) if \( X \) is DLR, provided \( \lambda \geq 1 \).

**Theorem 3.2**

(i) \( X^* \geq \text{lr} \downarrow X \) if \( X \) is DLR, provided \( 0 < \Lambda \leq 1 \) with probability 1;

(ii) \( X^* \leq \text{lr} \downarrow X \) if \( X \) is ILR, provided \( \Lambda \geq 1 \) with probability 1.

**Proof:**

(i) We have

\[
\frac{f^*(x + t)}{f(x)} = \frac{f(x + t)}{f(x)} \times \int_0^\infty \lambda \bar{F}^{-1}(x + t) dH(\lambda) = E \left[ \frac{f(x + t) \Lambda \bar{F}^{-1}(x + t)}{f(x)} \right] (3.2)
\]

Now \( X \) is DLR implies \( \frac{f(x + t)}{f(x)} \) is increasing in \( x \) for any \( t > 0 \). Again \( \lambda \bar{F}^{-1}(x + t) \) will be increasing in \( x \) for any \( 0 < \lambda \leq 1 \). Now if we consider \( \Lambda \) such that \( P(0 < \Lambda \leq 1) = 1 \) the result follows immediately.

(ii) Similarly \( X \) is ILR implies \( \frac{f(x + t)}{f(x)} \) is decreasing in \( x \). Again \( \lambda \bar{F}^{-1}(x + t) \) will be decreasing in \( x \) for any \( \lambda \geq 1 \). Now if we consider \( \Lambda \) such that \( P(\Lambda \geq 1) = 1 \) the result follows immediately.

**Remark 3.1** Theorem 3.2(i) implies that under the stated assumptions on \( X \) and \( \Lambda \), \( \kappa_{X^*}(t) \geq \kappa_X(t') \) for \( t \geq t' \geq 0 \). Similarly, Theorem 3.2(ii) implies that \( \kappa_{X^*}(t) \leq \kappa_X(t') \) for \( t' \geq t \geq 0 \).

The following theorem shows that, for a baseline r.v. \( X \) with IFR (resp. DFR) property, effect of a frailty r.v. \( \Lambda \) with \( P(0 < \Lambda \leq 1) = 1 \) (resp. \( P(\Lambda \geq 1) = 1 \)) on \( X \) is that, \( X^* \) will be greater than (resp. less than) \( X \) in the sense of the up shifted hazard rate order.

**Theorem 3.3**

(i) \( X^* \geq \text{hr} \uparrow X \) if \( X \) is IFR, provided \( 0 < \Lambda \leq 1 \) with probability 1;

(ii) \( X^* \leq \text{hr} \uparrow X \) if \( X \) is DFR, provided \( \Lambda \geq 1 \) with probability 1.

**Proof:**

(i) We have

\[
\frac{\bar{F}^*(x)}{\bar{F}(x + t)} = \frac{\bar{F}(x)}{\bar{F}(x + t)} \times \int_0^\infty \bar{F}^{-1}(x) dH(\lambda) = E \left[ \frac{\bar{F}(x) \Lambda \bar{F}^{-1}(x)}{\bar{F}(x + t)} \right] (3.3)
\]
Now $X$ is IFR implies $\frac{\bar{F}(x)}{\bar{F}(x+t)}$ is increasing in $x$ for any $t > 0$. Again $\bar{F}^{\lambda-1}(x)$ will be increasing in $x$ for any $0 < \lambda \leq 1$. Now if we consider $\Lambda$ such that $P(0 < \Lambda \leq 1) = 1$ the result follows immediately.

(ii) Similarly $X$ is DFR implies $\frac{\bar{F}(x)}{\bar{F}(x+t)}$ is decreasing in $x$. Again $\bar{F}^{\lambda-1}(x)$ will be decreasing in $x$ for any $\lambda \geq 1$. Now if we consider $\Lambda$ such that $P(\Lambda \geq 1) = 1$ the result follows immediately.

**Remark 3.2** Theorem 3.3(i) implies that under the stated assumptions on $X$ and $\Lambda$, $r_{X^*}(t) \leq r_X(t')$ for $t \geq t' \geq 0$. Similarly, Theorem 3.3(ii) implies that $r_{X^*}(t) \geq r_X(t')$ for $t \geq t' \geq 0$.

Examples 3.3 and 3.4 illustrate Theorem 3.3(i) and 3.3(ii) respectively.

**Example 3.3** Let $X$ follows Weibull distribution with sf $\bar{F}(x) = e^{-x^2}$, $x \geq 0$. Clearly, $X$ is IFR. Let the frailty r.v. $\Lambda$ to be uniformly distributed on $[0, 1]$. Then it is easy to check that $\bar{F}^*(x)/\bar{F}(x+t)$ is increasing in $x$ for all $t > 0$.

**Example 3.4** Let $X$ follows Weibull distribution with sf $\bar{F}(x) = e^{-x^{0.5}}$, $x \geq 0$. Clearly, $X$ is DFR. Let the frailty r.v. $\Lambda$ to be uniformly distributed on $[2, 5]$. Then it is easy to check that $\bar{F}^*(x)/\bar{F}(x+t)$ is decreasing in $x$ for all $t > 0$.

**Theorem 3.4**

(i) $X^* \geq_{hr\downarrow} X$ if $X$ is DFR, provided $0 < \Lambda \leq 1$ with probability 1;

(ii) $X^* \leq_{hr\downarrow} X$ if $X$ is IFR, provided $\Lambda \geq 1$ with probability 1.

**Proof:**

(i) We have

\[
\frac{\bar{F}^*(x+t)}{\bar{F}(x)} = \frac{\bar{F}(x+t)}{\bar{F}(x)} \times \int_{0}^{\infty} \bar{F}^{\lambda-1}(x+t)dH(\lambda) = E\left[\frac{\bar{F}(x+t)\bar{F}^{\lambda-1}(x+t)}{\bar{F}(x)}\right]
\]

Now $X$ is DFR implies $\frac{\bar{F}(x+t)}{\bar{F}(x)}$ is increasing in $x$ for any $t > 0$. Again $\bar{F}^{\lambda-1}(x+t)$ will be increasing in $x$ for any $0 < \lambda \leq 1$. Now if we consider $\Lambda$ such that $P(0 < \Lambda \leq 1) = 1$ the result follows immediately.

(ii) Similarly $X$ is IFR implies $\frac{\bar{F}(x+t)}{\bar{F}(x)}$ is decreasing in $x$. Again $\bar{F}^{\lambda-1}(x+t)$ will be decreasing in $x$ for any $\lambda \geq 1$. Now if we consider $\Lambda$ such that $P(\Lambda \geq 1) = 1$ the result follows immediately.

**Remark 3.3** Theorem 3.4(i) implies that under the stated assumptions on $X$ and $\Lambda$, $r_{X^*}(t) \leq r_X(t')$ for $t \geq t' \geq 0$. Similarly, Theorem 3.4(ii) implies that $r_{X^*}(t) \geq r_X(t')$ for $t' \geq t \geq 0$. 
Theorem 3.5

(i) $X^* \geq_{mrl} X$ if $X$ is IMRL, provided $0 < \Lambda \leq 1$ with probability 1;
(ii) $X^* \leq_{mrl} X$ if $X$ is DMRL, provided $\Lambda \geq 1$ with probability 1.

Proof:

(i) We have

$$
\int_x^\infty \bar{F}^*(u)du / \int_x^\infty \bar{F}(u)du = \frac{\int_0^\infty \frac{\bar{F}(u)dH(\lambda)}{\int_x^\infty \bar{F}(u)du}}{\int_x^\infty \bar{F}(u)du} = E \left[ \int_{x+t}^\infty \bar{F}(u)du / \int_x^\infty \bar{F}(u)du \right]
$$

(3.5)

Now if $X$ is IMRL then $\int_{x+t}^\infty \bar{F}(u)du / \int_x^\infty \bar{F}(u)du$ increasing in $x$ for any $t > 0$. That is we have

$$
\bar{F}(x+t) \int_{x+t}^\infty \bar{F}(u)du \leq \bar{F}(x) \int_x^\infty \bar{F}(u)du
$$

(3.6)

Let us define a function $\alpha(\lambda) = \frac{\bar{F}(x+t)}{\int_{x+t}^\infty \bar{F}(u)du}$. $\lambda > 0$.

$$
\alpha'(\lambda) \equiv \int_{x+t}^\infty \bar{F}(u)[\log(\bar{F}(x+t)) - \log(\bar{F}(u))]du
$$

(3.7)

Therefore from (3.6) and (3.7) we have for any $0 < \lambda \leq 1$ we have

$$
\bar{F}(x+t) \int_{x+t}^\infty \bar{F}(u)du \leq \bar{F}(x) \int_x^\infty \bar{F}(u)du
$$

(3.8)

Hence from (3.8) we can easily conclude that (3.5) is increasing in $x$ if $P(0 < \Lambda \leq 1) = 1$.

(ii) Since $X$ is DMRL hence $\int_{x+t}^\infty \bar{F}(u)du / \int_x^\infty \bar{F}(u)du$ decreasing in $x$. That is we have

$$
\bar{F}(x+t) \int_{x+t}^\infty \bar{F}(u)du \geq \bar{F}(x) \int_x^\infty \bar{F}(u)du
$$

(3.9)

Therefore from (3.9) and (3.7) we have for any $\lambda \geq 1$ we have

$$
\bar{F}(x+t) \int_{x+t}^\infty \bar{F}(u)du \geq \bar{F}(x) \int_x^\infty \bar{F}(u)du
$$

(3.10)

Hence from (3.9) we can easily conclude that (3.5) is decreasing in $x$ if $P(\Lambda \geq 1) = 1$.

Remark 3.4 Theorem 3.5(i) implies that under the stated assumptions on $X$ and $\Lambda$, $m_X(t) \geq m_X(t')$ for $t' \geq t \geq 0$. Similarly, Theorem 3.5(ii) implies that $m_X(t) \leq m_X(t')$ for $t \geq t' \geq 0$. 
Theorem 3.6

(i) \( X^* \geq_{mrl} X \) if \( X \) is DMRL, provided \( 0 < \Lambda \leq 1 \) with probability 1.

(ii) \( X^* \leq_{mrl} X \) if \( X \) is IMRL, provided \( \Lambda \geq 1 \) with probability 1;

Proof:

(i) We have

\[
\int_{x+t}^{\infty} \frac{\bar{F}^*(u)du}{\int_{x+t}^{\infty} \bar{F}(u)du} = \frac{\int_{0}^{\infty} \frac{\bar{F}^*(u)dH(\lambda)}{\int_{x+t}^{\infty} \bar{F}(u)du}}{\int_{x}^{\infty} \bar{F}(u)du} = E \left[ \int_{x}^{\infty} \frac{\bar{F}^*(u)du}{\int_{x+t}^{\infty} \bar{F}(u)du} \right] \tag{3.11}
\]

Now, if \( X \) is DMRL then \( \int_{x+t}^{\infty} \frac{\bar{F}^*(u)du}{\int_{x+t}^{\infty} \bar{F}(u)du} \) decreasing in \( x \). That is we have

\[
\frac{\bar{F}(x+t)}{\int_{x+t}^{\infty} \bar{F}(u)du} \leq \frac{\bar{F}(x)}{\int_{x}^{\infty} \bar{F}(u)du} \tag{3.12}
\]

Therefore from (3.12) and (3.15) we have for any \( 0 < \lambda \leq 1 \) we have

\[
\frac{\bar{F}(x+t)}{\int_{x+t}^{\infty} \bar{F}(u)du} \geq \frac{\bar{F}(x)}{\int_{x}^{\infty} \bar{F}(u)du} \geq \frac{\bar{F}^\lambda(x)}{\int_{x}^{\infty} \bar{F}^\lambda(u)du}. \tag{3.13}
\]

Hence from (3.12) we can easily conclude that (3.11) is increasing in \( x \) if \( P(0 < \Lambda \leq 1) \).

(ii) If \( X \) is IMRL then \( \int_{x+t}^{\infty} \frac{\bar{F}(u)du}{\int_{x}^{\infty} \bar{F}(u)du} \) decreasing in \( x \). That is we have

\[
\frac{\bar{F}(x+t)}{\int_{x+t}^{\infty} \bar{F}(u)du} \leq \frac{\bar{F}(x)}{\int_{x}^{\infty} \bar{F}(u)du} \tag{3.14}
\]

Let us define a function \( \beta(\lambda) = \frac{\bar{F}^\lambda(x)}{\int_{x}^{\infty} \bar{F}^\lambda(u)du} \), \( \lambda > 0 \).

\[
\beta'(\lambda) \overset{sgn}{=} \int_{x}^{\infty} \bar{F}^\lambda(u)[\log(\bar{F}(x)) - \log(\bar{F}(u))]du \overset{sgn}{=} \geq 0. \tag{3.15}
\]

Therefore from (3.14) and (3.15) we have for any \( \lambda \geq 1 \) we have

\[
\frac{\bar{F}(x+t)}{\int_{x+t}^{\infty} \bar{F}(u)du} \leq \frac{\bar{F}(x)}{\int_{x}^{\infty} \bar{F}(u)du} \leq \frac{\bar{F}^\lambda(x)}{\int_{x}^{\infty} \bar{F}^\lambda(u)du}. \tag{3.16}
\]

Hence from (3.16) we can easily conclude that (3.11) is decreasing in \( x \) if \( P(\Lambda \geq 1) = 1 \).

Remark 3.5 Theorem 3.6(i) implies that under the stated assumptions on \( X \) and \( \Lambda \), \( m_{X^*}(t) \geq m_X(t') \) for \( t \geq t' \geq 0 \). Similarly, Theorem 3.6(ii) implies that \( m_{X^*}(t) \leq m_X(t') \) for \( t' \geq t \geq 0 \).
4 Results for resilience model:

Here we study some shifted stochastic ordering of resilience models based on some ageing properties of concerned baseline r.v.’s. Let \( X^* \) follow resilience model with baseline distribution \( G \), and resilience r.v. \( \Omega \) having distribution function \( K \) so that the distribution function of \( X^* \) is given by

\[
G^*(x) = \int_0^\infty G(\omega) dK(\omega)
\]

(4.1)

Throughout this section, we consider \( X \) be a r.v. with distribution function \( G \) and \( X^* \) be the r.v.’s as defined above for which the distribution function is given by equation (4.1). Also consider that \( X \) be an absolutely continuous non-negative r.v.

Following theorem shows that, for a baseline r.v. \( X \) with ILR (resp. DLR) property, effect of a resilience r.v. \( \Omega \) with \( P(\Omega \geq 1) = 1 \) (resp. \( P(0 < \Omega \leq 1) = 1 \)) on \( X \) is that, \( X^* \) will be greater than (resp. less than) \( X \) in the sense of the up shifted likelihood ratio order.

**Theorem 4.1**

(i) \( X^* \geq_{lr}^\uparrow X \) if \( X \) is ILR, provided \( \Omega \geq 1 \) with probability 1;  
(ii) \( X^* \leq_{lr}^\uparrow X \) if \( X \) is DLR, provided \( 0 < \Omega \leq 1 \) with probability 1.

**Proof:**

(i) We have

\[
\frac{g^*(x)}{g(x+t)} = \frac{g(x)}{g(x+t)} \times \int_0^\infty \omega G^{\omega-1}(x) dK(\omega)
\]

(4.2)

Now \( X \) is ILR implies \( \frac{g(x)}{g(x+t)} \) is increasing in \( x \) for any \( t > 0 \). Again \( \omega G^{\omega-1}(x) \) will be increasing in \( x \) for any \( \omega \geq 1 \). Now if we consider \( \Omega \) such that \( P(\Omega \geq 1) = 1 \) the result follows immediately.

(ii) Similarly \( X \) is DLR implies \( \frac{g(x)}{g(x+t)} \) is decreasing in \( x \) for any \( t > 0 \). Again \( \omega G^{\omega-1}(x) \) will be decreasing in \( x \) for any \( 0 < \omega \leq 1 \). Now if we consider \( \Omega \) such that \( P(0 < \Omega \leq 1) = 1 \) the result follows immediately.

The following corollary follows immediately in case \( \Omega \) is a degenerate r.v.

**Corollary 4.1**

(i) \( X^* \geq_{lr}^\uparrow X \) if \( X \) is ILR, provided \( \omega \geq 1 \);  
(ii) \( X^* \leq_{lr}^\uparrow X \) if \( X \) is DLR, provided \( 0 < \omega \leq 1 \).

**Theorem 4.2**
(i) $X^* \geq_{lr\uparrow} X$ if $X$ is DLR, provided $\Omega \geq 1$ with probability 1;
(ii) $X^* \leq_{lr\uparrow} X$ if $X$ is ILR, provided $0 < \Omega \leq 1$ with probability 1.

Proof:

(i) We have

$$\frac{g^*(x+t)}{g(x)} = \frac{g(x+t)}{g(x)} \times \int_0^\infty \omega G^{\omega-1}(x+t)dK(\omega)$$

$$= E \left[ \frac{g(x+t)\Omega G^{\Omega-1}(x+t)}{g(x)} \right] \quad (4.3)$$

Now $X$ is DLR implies $\frac{g(x+t)}{g(x)}$ is increasing in $x$ for any $t > 0$. Again $\omega G^{\omega-1}(x)$ will be increasing in $x$ for any $\omega \geq 1$. Now if we consider $\Omega$ such that $P(\Omega \geq 1) = 1$ the result follows immediately.

(ii) Similarly $X$ is ILR implies $\frac{g(x+t)}{g(x)}$ is decreasing in $x$. Again $\omega G^{\omega-1}(x)$ will be decreasing in $x$ for any $0 < \omega \leq 1$. Now if we consider $\Omega$ such that $P(0 < \Omega \leq 1) = 1$ the result follows immediately.

Following theorem shows that, for a baseline r.v. $X$ with DRFR (resp. IRFR) property, effect of a resilience r.v. $\Omega$ with $P(\Omega \geq 1) = 1$ (resp. $P(0 < \Omega \leq 1) = 1$) on $X$ is that, $X^*$ will be greater than (resp. less than) $X$ in the sense of the up shifted reversed hazard rate order.

Theorem 4.3

(i) $X^* \geq_{rh\uparrow} X$ if $X$ is DRFR, provided $\Omega \geq 1$ with probability 1;
(ii) $X^* \leq_{rh\uparrow} X$ if $X$ is IRFR, provided $0 < \Omega \leq 1$ with probability 1.

Proof:

(i) We have

$$\frac{G^*(x)}{G(x+t)} = \frac{G(x)}{G(x+t)} \times \int_0^\infty G^{\omega-1}(x)dK(\omega)$$

$$= E \left[ \frac{G(x)G^{\Omega-1}(x)}{G(x+t)} \right] \quad (4.4)$$

Now $X$ is DRFR implies $\frac{G(x)}{G(x+t)}$ is increasing in $x$ for any $t > 0$. Again $G^{\omega-1}(x)$ will be increasing in $x$ for any $\omega \geq 1$. Now if we consider $\Omega$ such that $P(\Omega \geq 1) = 1$ the result follows immediately.

(ii) Similarly $X$ is IRFR implies $\frac{G(x)}{G(x+t)}$ is decreasing in $x$. Again $G^{\omega-1}(x)$ will be decreasing in $x$ for any $0 < \omega \leq 1$. Now if we consider $\Omega$ such that $P(0 < \Omega \leq 1) = 1$ the result follows immediately.
Remark 4.1 Theorem 4.3(i) implies that under the stated assumptions on $X$ and $\Omega$, $\tilde{r}_{X^*}(t) \geq \tilde{r}_{X}(t')$ for $t' \geq t \geq 0$. Similarly, Theorem 4.3(ii) implies that $\tilde{r}_{X^*}(t) \leq \tilde{r}_{X}(t')$ for $t' \geq t \geq 0$.

Example 4.1 Let $X$ follows Weibull r.v. with cdf $G(x) = 1 - e^{-2x^2}$, $x \geq 0$. Clearly, $X$ is DRFR. Let $\Omega$ to be uniformly distributed on $[2,5]$. Then it is easy to check that $G^*(x)/G(x+t)$ is increasing in $x$ for all $t > 0$.

Theorem 4.4

(i) $X^* \geq_{r\downarrow} X$ if $X$ is IRFR, provided $\Omega \geq 1$ with probability 1;
(ii) $X^* \leq_{r\downarrow} X$ if $X$ is DRFR, provided $0 < \Omega \leq 1$ with probability 1.

Proof:

(i) We have

$$
\frac{G^*(x+t)}{G(x)} = \frac{G(x+t)}{G(x)} \times \int_0^\infty G^{-1}(x+t) dK(\omega)
\quad = E \left[ \frac{G(x+t)G^{-1}(x+t)}{G(x)} \right]
\quad (4.5)
$$

Now $X$ is IRFR implies $G(x+t)/G(x)$ is increasing in $x$ for any $t > 0$. Again $G^{-1}(x)$ will be increasing in $x$ for any $\omega \geq 1$. Now if we consider $\Omega$ such that $P(\Omega \geq 1) = 1$ the result follows immediately.

(ii) Similarly $X$ is DRFR implies $G(x+t)/G(x)$ is decreasing in $x$. Again $\omega G^{-1}(x)$ will be decreasing in $x$ for any $0 < \omega \leq 1$. Now if we consider $\Omega$ such that $P(0 < \Omega \leq 1) = 1$ the result follows immediately.

Remark 4.2 Theorem 4.4(i) implies that under the stated assumptions on $X$ and $\Omega$, $\tilde{r}_{X^*}(t) \geq \tilde{r}_{X}(t')$ for $t' \geq t \geq 0$. Similarly, Theorem 4.4(ii) implies that $\tilde{r}_{X^*}(t) \leq \tilde{r}_{X}(t')$ for $t' \geq t \geq 0$.

Example 4.2 Let $X$ follows Weibull distribution with cdf $G(x) = 1 - e^{-x^3}$, $x \geq 0$ so that $X$ is DRFR. Let $\Omega$ to be uniformly distributed on $[0,1]$. Then it is easy to check that $G^*(x+t)/G(x)$ is decreasing in $x$ for all $t > 0$.

Theorem 4.5

(i) $X^* \leq_{mit\uparrow} X$ if $X$ is IMIT, provided $\Omega \geq 1$ with probability 1;
(ii) $X^* \geq_{mit\downarrow} X$ if $X$ is IMIT, provided $0 < \Omega \geq 1$ with probability 1.

Proof:
(i) We have
\[
\frac{\int_0^{x+t} G^*(u)du}{\int_0^x G(u)du} = \frac{\int_0^{x+t} \int_0^{\infty} G^\omega(u)dK(\omega)du}{\int_0^x G(u)du} = E \left[ \frac{\int_0^{x+t} G^\Omega(u)du}{\int_0^x G(u)du} \right]
\] (4.6)

Now \( X \) is IMIT implies \( \frac{\int_0^{x+t} G(u)du}{\int_0^x G(u)du} \) decreasing in \( x \) for any \( t > 0 \). That is we have for any \( t > 0 \)
\[
\frac{G(x+t)}{\int_0^{x+t} G(u)du} \leq \frac{G(x)}{\int_0^x G(u)du}
\] (4.7)

Also it is easy to verify that for any \( \omega > 0 \), \( \frac{G^\omega(x)}{\int_0^x G^\omega(u)du} \) is increasing function of \( \omega \). Hence we have from (4.7) for any \( 0 < \omega \leq 1 \)
\[
\frac{G^\omega(x+t)}{\int_0^{x+t} G^\omega(u)du} \leq \frac{G(x+t)}{\int_0^x G(u)du} \leq \frac{G(x)}{\int_0^x G(u)du}
\] (4.8)

Hence from (4.8) we can conclude that (4.6) is decreasing in \( x \).

(ii). Again we have
\[
\frac{\int_0^x G(u)du}{\int_0^{x+t} G^*(u)du} = \frac{\int_0^x G(u)du}{\int_0^{x+t} \int_0^{\infty} G^\omega(u)dH(\omega)du} = E \left[ \frac{\int_0^x G(u)du}{\int_0^{x+t} G^\Omega(u)du} \right]
\] (4.9)

As \( \frac{G^\omega(x)}{\int_0^x G^\omega(u)du} \) is increasing function of \( \omega \) we have for any \( \omega \geq 1 \)
\[
\frac{G(x)}{\int_0^x G(u)du} \leq \frac{G(x+t)}{\int_0^{x+t} G^\omega(u)du} \leq \frac{G^\omega(x+t)}{\int_0^x G^\omega(u)du}
\] (4.10)

Consequently from (4.10) we can conclude that (4.9) is decreasing in \( x \).

**Remark 4.3** Theorem 4.5(i) implies that under the stated assumptions on \( X \) and \( \Omega \), \( \text{mit}_X(t) \geq \text{mit}_X(t') \) for \( t \geq t' \geq 0 \). Similarly, Theorem 4.5(ii) implies that \( \text{mit}_X(t) \leq \text{mit}_X(t') \) for \( t \geq t' \geq 0 \).

### 5 Illustration with real-world data

Here we illustrate some of our results in two real scenarios considering two data sets, namely “Survival times in leukaemia” and “Fatigue-life failures” data (Hand et al., 1993). In scenario
I, we establish our results (Theorems 3.2 and 3.4) for frailty model. In scenario II, we establish our results for resilience model (Theorems 4.1 and 4.3).

**Scenario I:** We consider the data set “Survival times in leukaemia” (Hand et al., 1993) which contains the survival times of 43 patients suffering from chronic granulocytic leukaemia, measured in days from the time of diagnosis. From the quantile-quantile (Q-Q) plot (Figure 1) and results of Anderson-Darling test (Table 1) for the observed samples, it is observed that Weibull distribution fits well. Estimated values of parameters of the fitted baseline Weibull ($X$) with the distribution function

$$F(x) = 1 - e^{(-x/\beta)^k}, \quad x \geq 0, \beta > 0, k > 0$$

are presented in Table 2.

| AD-value | p-value | Critical value(cv) |
|----------|---------|--------------------|
| 0.3616   | 0.8852  | 2.4978             |

**Table 2: Estimated parameters of Weibull distribution**

| Parameters | Estimated value | 95% confidence interval |
|------------|-----------------|-------------------------|
| Scale ($\beta$) | 986.672 | [766.52, 1270.06] |
| Shape ($k$) | 1.24044 | [0.973535, 1.58052] |

Figure 1: QQ plot of sample data vs Weibull distribution

With shape parameter $k > 1$, this baseline Weibull distribution is ILR and so is IFR. Next we consider well known Gamma-frailty i.e. $\Lambda \sim \Gamma(1/a^2, 1/a^2)$ where $\Lambda \geq 1$ with probability 1. According to Theorem 3.2(ii), the effect of considered gamma frailty on $X$ is that, $X^* \leq_{hr^*} X$, which implies that $\kappa_{X^*}(t) \leq \kappa_X(t')$ for $t' \geq t \geq 0$. Similarly, according to Theorem 3.4(ii), $X^* \leq_{hr^*} X$, which implies that $r_{X^*}(t) \geq r_X(t')$ for $t' \geq t \geq 0$. Also, we have $X^* \geq_{disp} X$, where ‘disp’ stands for dispersive order (Shaked and Shanthikumar, 2007). It follows from the fact
that for two non-negative r.v.’s $X$ and $Y$, $X \leq_{hr} Y \Rightarrow X \leq_{disp} Y$ (Lillo et al. 2000).

To demonstrate the above mentioned stochastic orders, we proceed as follows. The survival function and probability density function of the above frailty model (Gamma-frailty Weibull-baseline) are, respectively

$$F^*(t) = \frac{(1/a^2)^{-1/2} \left(a^2 + \frac{t_k}{\beta^k}\right) \zeta_1(\frac{1}{a^2}, a^2 + \frac{t_k}{\beta^k})}{\Gamma(\frac{1}{a^2}) \left(1 - \zeta_2(\frac{1}{a^2}, 0, a^2)\right)}$$ (5.1)

and

$$f^*(t) = \frac{k \frac{t_k}{\beta^k} \left(1/a^2\right)^{-1/2} \left(a^2 + \frac{t_k}{\beta^k}\right)^{-1/2} \zeta_1(\frac{1}{a^2}, a^2 + \frac{t_k}{\beta^k})}{\Gamma(\frac{1}{a^2}) \left(1 - \zeta_2(\frac{1}{a^2}, 0, a^2)\right)}$$ (5.2)

Let $t_1, t_2, ..., t_n$ be the observations under consideration. We now obtain maximum likelihood estimation of the parameter $a$ under the Gamma-frailty Weibull-baseline. The likelihood function is given by

$$L(a|t_1, t_2, ..., t_n) = \left(\frac{(1/a^2)^{-1/2} \left(a^2 + \frac{t_k}{\beta^k}\right) \zeta_1(\frac{1}{a^2}, a^2 + \frac{t_k}{\beta^k})}{\Gamma(\frac{1}{a^2}) \left(1 - \zeta_2(\frac{1}{a^2}, 0, a^2)\right)}\right)^n \prod_{i=1}^{n} \left(t_i \frac{(a^2)^{-1/2} \left(a^2 + \frac{t_k}{\beta^k}\right)^{-1/2} \zeta_1(\frac{1}{a^2}, a^2 + \frac{t_k}{\beta^k})}{\Gamma(\frac{1}{a^2}) \left(1 - \zeta_2(\frac{1}{a^2}, 0, a^2)\right)}\right)$$ (5.3)

$$\times \prod_{i=1}^{n} \zeta_1(\frac{1}{a^2}, a^2 + \frac{t_k}{\beta^k}),$$ (5.4)

where $\zeta_1(a, x) = \int_{x}^{\infty} t^{a-1} e^{-t} dt$ and $\zeta_2(a, x) = \int_{x}^{\infty} t^{a-1} e^{-t} dt / \Gamma(a)$ are upper incomplete gamma functions and regularized lower incomplete gamma functions respectively. Estimated value of $a$ is obtained as 0.784 with $P(\Gamma \geq 1) = 1$.

We then plotted $f^*(x + t) / f(x)$ taking some finite range of $x$ and $t$ as shown in Figure 2 which is clearly showing that the ratio is decreasing in $x$, giving $X^* \leq_{hr} X$. To demonstrate that $X^* \leq_{hr} X$, we plotted $\tilde{F}^*(x + t) / \tilde{F}(x)$ in Figure 3 showing that it is decreasing in $x$.

**Scenario II:** Here we consider the data set “Fatigue-life failures” (Hand et al. 1993) on the fatigue-life failures of ball-bearings. The data give the number of cycles to failure. From the quantile-quantile (Q-Q) plot (Figure 4) and the results of Anderson-Darling test (Table 3) for the observed samples, it is observed that the samples can taken to be from Weibull distribution. Estimated values of parameters of baseline Weibull with the distribution function $G(x) = 1 - e^{(-x/\beta)^k}$, $x \geq 0$, $\beta > 0$, $k > 0$ are given in Table 4.

| AD-value | p-value | Critical value(cv) |
|----------|---------|--------------------|
| 0.1496   | 0.99    | 2.503              |

With shape parameter $k > 1$, this baseline Weibull distribution is ILR and also is DRFR. Next we consider Gamma resilience i.e. $\Omega \sim \Gamma(1/a^2, 1/a^2)$ where $\Omega \geq 1$ with probability 1. According to Theorem 4.1(i), the effect of considered gamma resilience on $X$ is that, $X^* \leq_{hr} X$. Similarly, according to Theorem 4.3(i), $X^* \leq_{hr} X$, which indicates that $\tilde{r}_X(t) \leq \tilde{r}_X(t')$ for
Figure 2: Plot of $f^*(x+t)/f(x)$

| Parameters | Estimated value | 95% confidence interval |
|------------|-----------------|-------------------------|
| Scale      | 232.9           | [198.758, 272.906]      |
| Shape      | 3.0721          | [2.13732, 4.41572]      |

$t \geq t' \geq 0$.

To demonstrate the above mentioned stochastic orders, we proceed as follows. The distribution function and probability density function of the above resilience model (Gamma-resilience Weibull-baseline) are, respectively

$$G^*(t) = \frac{(1/a^2)^{-1/a^2} \left( a^2 - \ln(1 - e^{(\frac{1}{a^2})^{k}}) \right)^{-1/a^2}}{\Gamma \left( \frac{1}{a^2} \right) \left( 1 - \zeta_2 \left( \frac{1}{a^2}, 0, a^2 \right) \right)} \frac{1}{a^2} \zeta_1 \left( 1, \frac{1}{a^2}, a^2 - \ln(1 - e^{(\frac{1}{a^2})^{k}}) \right)$$

and

$$g^*(t) = \frac{k \frac{1}{a^2} (1/a^2)^{-1/a^2} \left( a^2 - \ln(1 - e^{(\frac{1}{a^2})^{k}}) \right)^{-1/a^2} \zeta_1 \left( 1, \frac{1}{a^2}, a^2 - \ln(1 - e^{(\frac{1}{a^2})^{k}}) \right)}{\Gamma \left( \frac{1}{a^2} \right) \left( 1 - \zeta_2 \left( \frac{1}{a^2}, 0, a^2 \right) \right)}$$

Let $t_1, t_2, ..., t_n$ be the observations under consideration. We now obtain maximum likelihood estimate of the parameter $a$ under the Gamma-resilience Weibull-baseline. The likelihood function is given by

$$L(a | t_1, t_2, ..., t_n) = \left( \frac{(1/a^2)^{-1/a^2}}{\Gamma \left( \frac{1}{a^2} \right) \left( 1 - \zeta_2 \left( \frac{1}{a^2}, 0, a^2 \right) \right)} \right)^n \prod_{i=1}^{n} (t_i)^{k-1} \prod_{i=1}^{n} (a^2 - \ln(1 - e^{(\frac{1}{a^2})^{k}}))^{-1-1/a^2} \prod_{i=1}^{n} \zeta_1 \left( 1, \frac{1}{a^2}, a^2 - \ln(1 - e^{(\frac{1}{a^2})^{k}}) \right),$$
where \( \zeta_1(a, x) \) and \( \zeta_2(a, x) \) are defined in previous case. Estimated value of the parameter \( a \) is obtained as 4.0558 with \( P(\Omega \geq 1) = 1 \).

Then we plotted \( g^*(x)/g(x + t) \) taking some finite range of \( x \) and \( t \) as shown in Figure 5, which is clearly showing that the ratio is increasing in \( x \), giving \( X^* \leq_{h\uparrow} X \). To demonstrate that \( X^* \leq_{rh\uparrow} X \), we plotted \( G^*(x)/G(x + t) \) in Figure 6 showing that it is increasing in \( x \).
6 Conclusion

In this study, we have derived results on stochastic comparisons for frailty as well as resilience models to study the effects of frailty and resilience r.v.’s on the baseline r.v.’s based on some ageing properties of concerned baseline r.v.’s. To derive the results we have used some shifted stochastic orders which are stronger than their respective usual counterparts, and also provide more flexibility in stochastic comparisons. As a future study, comparisons for considered frailty or resilience models could be explored using other generalized stochastic orders like proportional
stochastic and shifted proportional stochastic orders.

**Conflict of interest**

On behalf of all authors, the corresponding author states that there is no conflict of interest.

**Data availability statement**

The datasets analysed during the current study are available in [Hand et al. (1993)](#).

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