STATIONARY SOLUTIONS TO THE BOUNDARY VALUE PROBLEM FOR THE RELATIVISTIC BGK MODEL IN A SLAB

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Abstract. In this paper, we are concerned with the boundary value problem in a slab for the stationary relativistic BGK model of Marle type, which is a relaxation model of the relativistic Boltzmann equation. In the case of fixed inflow boundary conditions, we establish the existence of unique stationary solutions.

1. Introduction. In this paper, we address the existence of stationary solutions for a relativistic BGK model defined on a unit interval:

\[ q_1 \frac{\partial f}{\partial x} = w(J_f - f), \quad (x, q) \in [0, 1] \times \mathbb{R}^3, \tag{1} \]

equipped with a fixed inflow data at the boundary:

\[ f(0, q) = f_L(q) \text{ for } q_1 > 0, \quad f(1, q) = f_R(q) \text{ for } q_1 < 0, \]

for some given functions \( f_L \) and \( f_R \). The momentum distribution function \( f(x, q) \) represents the number density of relativistic particles at position \( x \in [0, 1] \) with momentum \( q \in \mathbb{R}^3 \). On the r.h.s of (1), \( w > 0 \) is a collision frequency, and \( J_f \) denotes the local relativistic Maxwellian defined by

\[ J_f = \frac{n}{M(\beta)} e^{-\beta \left( \sqrt{1 + |u|^2} \sqrt{1 + |q|^2} - u \cdot q \right)}, \]

where \( M(\beta) \) is

\[ M(\beta) = \int_{\mathbb{R}^3} e^{-\beta \sqrt{1 + |p|^2}} dp, \]

and the proper particle density \( n \), velocity four-vector \( (\sqrt{1 + |u|^2}, u) \) and the equilibrium temperature \( 1/\beta \) are defined by the following relations: (in the following,

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\( q_0 \) denotes \( \sqrt{1 + |q|^2} \) for \( q \in \mathbb{R}^3 \).

\[
\begin{align*}
    n^2 &= \left( \int_{\mathbb{R}^3} f dq \right)^2 - \sum_{i=1}^{3} \left( \int_{\mathbb{R}^3} \frac{f q_i}{q_0} dq \right)^2, \\
    n \sqrt{1 + |u|^2} &= \int_{\mathbb{R}^3} f dq, \\
    nu &= \int_{\mathbb{R}^3} f \frac{q}{q_0} dq, \\
    \frac{K_1}{K_2}(\beta) &= \frac{1}{n} \int_{\mathbb{R}^3} f \frac{1}{q_0} dq,
\end{align*}
\]

where \( K_i \) denotes a modified Bessel function of the second kind:

\[
K_i(\beta) = \int_0^\infty \cosh(\beta r) e^{-\beta \cosh(r)} dr \quad (i = 1, 2).
\]

Note that \( \beta \) is determined through a nonlinear relation.

The kinetic theory of relativistic particles began with Jüttner [22] in 1911 when he derived a relativistic version of the Maxwellian distribution, which is often called the Jüttner equilibrium. The relativistic generalization of the celebrated Boltzmann equation was made by Lichnerowicz and Marrot in 1941 [24].

The complicated structure of the relativistic collision operator, however, has long been a major obstacle in the application of the relativistic Boltzmann equation to various flow problems. To circumvent this difficulty, two types of relaxation in time approximation were suggested [2, 29, 30] to develop a numerically amenable model equation which still shares essential features of the collision operator such as the conservation laws and H-theorem. The first one was proposed by Marle [29, 30] where the macroscopic fields are represented using the Eckart decomposition, and the other one by Anderson and Witting [2] where the Landau-Lifshitz decomposition was employed for the representation of the macroscopic fields.

The relativistic BGK models then have been widely used for various purposes [1, 2, 8, 9, 10, 14, 18, 19, 25, 26, 27, 28, 30, 31, 37, 38], but rigorous mathematical studies have just started and lots of issues still remain to be addressed. In 2012, Bellouquid et al considered the determination of equilibrium parameters, various formal scaling limits and the analysis of the linearized problem of the Marle model in [4]. Then, the existence of mild solutions and asymptotic stability of the Marle model near global relativistic equilibrium were proved in [5]. To the best knowledge of authors, these two works are the only mathematical literatures treating the relativistic BGK model analytically.

1.1. **Main result.** In this paper, we consider the stationary relativistic BGK model of Marle type posed on a bounded interval with fixed inflow boundary data at both ends, and establish the existence of unique stationary solutions.
Definition 1.1. A non-negative function $f \in L^1 \left([0,1] \times \mathbb{R}^3\right)$ is called a mild solution of (1) if
$$f(x,q) = \left( e^{-\frac{w}{|q_1|}x} f_L + \frac{w}{|q_1|} \int_0^x e^{-\frac{w}{|q_1|}(x-y)} f_y dy \right)_{1,q_1>0}$$
$$+ \left( e^{-\frac{w}{|q_1|}(1-x)} f_R + \frac{w}{|q_1|} \int_x^1 e^{-\frac{w}{|q_1|}(x-y)} f_y dy \right)_{1,q_1<0}.$$  

For brevity, we denote
$$f_{LR} = f_L 1_{q_1>0} + f_R 1_{q_1<0},$$
$$f_{LR}^e = e^{-\frac{w}{|q_1|}x} f_L 1_{q_1>0} + e^{-\frac{w}{|q_1|}(1-x)} f_R 1_{q_1<0}.$$  

We then define quantities $a_l, a_u$ and $\lambda$ by
$$a_l = \int_{\mathbb{R}^3} e^{-\frac{w}{|q_1|}} f_{LR} \frac{1}{q_0} dq, \quad a_u = 2 \int_{\mathbb{R}^3} f_{LR} dq,$$  
and
$$\lambda = \left( \int_{\mathbb{R}^3} f_{LR}^e \frac{1}{q_0} dq \right) \left( \int_{\mathbb{R}^3} f_{LR} dq \right) \int_{\mathbb{R}^3} f_{LR}^e \frac{1}{q_0} dq \right)^{-\frac{1}{2}}. \quad (3)$$

Our main result is as follows:

**Theorem 1.2.** Suppose the inflow boundary data $f_{LR}$ is non-negative, not trivially zero, and belongs to $L^1(\mathbb{R}^3)$. Assume further that $a_l > 0$. Then, we can find $\varepsilon > 0$ such that if $w < \varepsilon$, then there exists a unique mild solution $f$ to (1) such that
$$\int_{\mathbb{R}^3} f \frac{1}{q_0} dq \geq a_l, \quad \int_{\mathbb{R}^3} f dq \leq a_u, \quad \frac{1}{n} \int_{\mathbb{R}^3} f \frac{1}{q_0} dq \leq \sqrt{\lambda}.$$  

**Remark 1.** (1) By a trivially zero function, we mean that its value is zero on almost every points in the domain.

(2) Note that conditions on $f_{LR}$ guarantee that $a_l, a_u$ are well-defined and $a_l > 0$. The latter in turn guarantees that the divisor in the ratio $\lambda$ is strictly positive.

(3) Considering the conditions under which the identity holds in the Hölder inequality, we see that $a_l > 0$ implies $0 < \lambda < 1$. Therefore, $\sqrt{\lambda}$ is strictly less than 1 (see the following paragraph below to see why this condition is important).

To prove our main result, we adopt and make a relativistic extension of the Banach fixed point framework developed in [3] where one of the authors considered the stationary problem of the classical ellipsoidal BGK model for classical particles. The relativistic nature of the equation complicates the problem at virtually every point, and makes the adaptation nontrivial. One of the key difficulties arises in the way the relativistic counterpart of the local temperature $1/\beta$ is defined, which is implicitly defined through a nonlinear functional relation:
$$\frac{K_1}{K_2}(\beta) = \frac{1}{n} \int_{\mathbb{R}^3} f \frac{1}{q_0} dq.$$  

This implies that we need to control $(K_1/K_2)(\beta)$ to get a proper bound on $\beta$. In view of this, we first note that we already have some control on it: $0 < (K_1/K_2)(\beta) < 1$, which holds trivially by the definitions of $K_1$ and $K_2$. This trivial bound, however, gives no information on the size of $\beta$ since $(K_1/K_2)^{-1}(1) = \infty$. And without the information on the size of $\beta$, we cannot guarantee that our solution space is invariant under our solution map, which is essential to close the fixed point
argument. Therefore, we need to bound $K_1/K_2$ by a constant that is strictly less than 1 (see the remark 1.3 (2)). This is accomplished in Lemma 3.1 using the following estimate controlling the relativistic Maxwellian by the collision frequency and the boundary data:

$$
\int_{q_1 > 0} \frac{w}{|q_1|} \int_0^x e^{-\frac{w}{|v_1|} (x-y)} f d\nu d\lambda + \int_{q_1 < 0} \frac{w}{|q_1|} \int_x^1 e^{-\frac{w}{|v_1|} (y-x)} f d\nu d\lambda
\leq \frac{16C_1}{C_2^2} \left(2w \ln \frac{1}{w} + (1 + e)w + \frac{\sqrt{2}u^2}{C_2} e^{-\frac{C_2}{\sqrt{2}u^2}}\right),
$$

which is established in Lemma 2.3.

The paper is organized as follows: In Section 2, we define the solution space and present several preliminary technical estimates. In Section 3, we define the solution operator and show that it maps the solution space into itself. In Section 4, we show that the solution operator is a contraction mapping on the solution space under our assumptions.

2. Estimates in solution space. We define our solution space $\Omega$ by

$$\Omega = \{f(x, q) \in L^1([0, 1] \times \mathbb{R}^3) \mid f \text{ satisfies } (A)\}$$

where the property $(A)$ denotes

$$f \geq 0, \quad a_t \leq \int_{\mathbb{R}^3} f \frac{1}{q_0^2} dq, \quad \int_{\mathbb{R}^3} f dq \leq a_0, \quad \frac{1}{n} \int_{\mathbb{R}^3} f \frac{1}{q_0^2} dq \leq \sqrt{\lambda}.$$ 

We will show that the solution to the boundary value problem (1) is given as a unique fixed point in $\Omega$ of a solution operator, which will be defined later. First, we need to establish several preliminary estimates.

**Lemma 2.1.** Let $f \in \Omega$, then we have

$$a_t \leq n \leq a_u, \quad |u| \leq \frac{a_u}{a_t}, \quad \frac{a_t}{a_u} \leq \frac{1}{n} \int_{\mathbb{R}^3} f \frac{1}{q_0^2} dq \leq \sqrt{\lambda}.$$ 

**Proof.** First, we see from the definition of $n$ that

$$n^2 = \left(\int_{\mathbb{R}^3} f dq\right)^2 - \sum_{i=1}^3 \left(\int_{\mathbb{R}^3} f \frac{q_i}{q_0} dq\right)^2 \leq \left(\int_{\mathbb{R}^3} f dq\right)^2 \leq a_u^2.$$ 

For the lower bound of $n$, we employ the Hölder inequality as follows:

$$\sum_{i=1}^3 \left(\int_{\mathbb{R}^3} f \frac{q_i}{q_0} dq\right)^2 \leq \sum_{i=1}^3 \int_{\mathbb{R}^3} f dq \int_{\mathbb{R}^3} f \frac{q_i^2}{q_0^2} dq$$

$$\quad = \int_{\mathbb{R}^3} f dq \sum_{i=1}^3 \int_{\mathbb{R}^3} f \frac{q_i^2}{q_0^2} dq$$

$$\quad = \int_{\mathbb{R}^3} f dq \int_{\mathbb{R}^3} f \frac{|q|^2}{q_0^2} dq.$$ 

to get

$$n^2 = \left(\int_{\mathbb{R}^3} f dq\right)^2 - \sum_{i=1}^3 \left(\int_{\mathbb{R}^3} f \frac{q_i}{q_0} dq\right)^2.$$
\[ \geq \left( \int_{\mathbb{R}^3} f dq \right)^2 - \int_{\mathbb{R}^3} f dq \int_{\mathbb{R}^3} \frac{|q|^2}{q_0^2} dq \]

\[ = \left( \int_{\mathbb{R}^3} f dq \right) \left( \int_{\mathbb{R}^3} f dq - \int_{\mathbb{R}^3} \frac{|q|^2}{q_0^2} dq \right) \]

\[ = \int_{\mathbb{R}^3} f dq \int_{\mathbb{R}^3} f \frac{1}{q_0^2} dq \]

\[ \geq \left( \int_{\mathbb{R}^3} f \frac{1}{q_0^2} dq \right)^2 \]

\[ = a_l^2. \] (4)

Using this, we can estimate

\[ |u| = \frac{|nu|}{n} \leq \frac{1}{a_l} \int_{\mathbb{R}^3} f \frac{q}{q_0} dq \leq \frac{1}{a_l} \int_{\mathbb{R}^3} f dq \leq \frac{a_u}{a_l}, \]

and

\[ \sqrt{\lambda} \geq \frac{1}{n} \int_{\mathbb{R}^3} f \frac{1}{q_0} dq \geq \frac{1}{a_u} \int_{\mathbb{R}^3} f \frac{1}{q_0} dq \geq \frac{a_l}{a_u}. \]

**Lemma 2.2.** For \( f \in \Omega \), there exist constants \( C_1 \) and \( C_2 \) depending on \( a_l, a_u \) such that

\[ J_f \leq C_1 e^{-C_2 \sqrt{1 + |q|^2}}. \]

**Proof.** It is shown in [4] that \((K_1/K_2)(\beta)\) is strictly increasing, and the range is \([0,1]\) for \( \beta \in (0, \infty) \). Therefore, from the last estimate of Lemma 2.1, and the definition of \( \beta \) given in (2), we can guarantee the existence of \( \beta \), and can specify the range of \( \beta \) as follows:

\[ \beta_l \equiv \left( \frac{K_1}{K_2} \right)^{-1} \left( \frac{a_l}{a_u} \right) \leq \beta \leq \left( \frac{K_1}{K_2} \right)^{-1} \left( \sqrt{\lambda} \right) \equiv \beta_u. \] (5)

This, together with the upper bound of \( n \) in Lemma 2.1 and the fact that \( M(\beta) \) is a decreasing function, gives

\[ \frac{n}{M(\beta)} \leq \frac{a_u}{M(\beta_u)}. \] (6)

On the other hand, we recall \( 0 \leq |u| \leq a_u/a_l \) from Lemma 2.1, and use the fact that \( h(x) \equiv \sqrt{1 + x^2} - x \) is a non-negative decreasing function to conclude that

\[ \sqrt{1 + |u|^2 \sqrt{1 + |q|^2} - u \cdot q} \geq \sqrt{1 + |u|^2 \sqrt{1 + |q|^2} - |u||q|} \]

\[ \geq (\sqrt{1 + |u|^2} - |u|) \sqrt{1 + |q|^2} \]

\[ \geq C_0 \sqrt{1 + |q|^2}, \] (7)

for

\[ C_0 = h \left( \frac{a_u}{a_l} \right) = \sqrt{1 + \left( \frac{a_u}{a_l} \right)^2} - \frac{a_u}{a_l} > 0. \]

Combining (6) and (7), we obtain the desired result:

\[ J_f = \frac{n}{M(\beta)} e^{-\beta \sqrt{1 + |u|^2 \sqrt{1 + |q|^2} - u \cdot q}} \leq \frac{a_u}{M(\beta_u)} e^{-\beta \sqrt{1 + |q|^2}} \equiv C_1 e^{-C_2 \sqrt{1 + |q|^2}}. \]
Lemma 2.3. Let \( f \in \Omega \). Assume \( 0 < w < 1 \). Then we have

\[
\int_{q_1 > 0} \frac{w}{|q_1|} \int_{0}^{x} e^{-\frac{w}{|q_1|}(x-y)} J_f dy dq + \int_{q_1 < 0} \frac{w}{|q_1|} \int_{0}^{1} e^{-\frac{w}{|q_1|}(y-x)} J_f dy dq \\
\leq 16C_1 e^C \left( 2w \ln \frac{1}{w} + (1 + e)w + \sqrt{2w^2 e^{-\frac{C_1}{2w}}} \right).
\]

Proof. We only consider \( \int_{q_1 > 0} \) to avoid the repetition. From Lemma 2.2,

\[
\int_{q_1 > 0} \frac{w}{|q_1|} \int_{0}^{x} e^{-\frac{w}{|q_1|}(x-y)} J_f dy dq \leq C_1 \int_{q_1 > 0} \frac{w}{|q_1|} \int_{0}^{x} e^{-\frac{w}{|q_1|}(x-y)} e^{-C_2 \sqrt{1+|q_1|^2}} dy dq \\
\leq \frac{8C_1}{C_2^2} \int_{q_1 > 0} \frac{w}{|q_1|} \int_{0}^{x} e^{-\frac{w}{|q_1|}(x-y)} e^{-C_2 \sqrt{1+|q_1|^2}} dy dq_1.
\]

Here we used

\[
\int e^{-C_2 \sqrt{1+|q_1|^2}} dq_2 dq_3 \leq \int e^{-C_2 \sqrt{|q_1|+|q_2|+|q_3|}} dq_2 dq_3 \\
= e^{-C_2 |q_1|} \int e^{-C_2 \sqrt{|q_2|+|q_3|}} dq_2 dq_3 \\
= \frac{8}{C_2^2} e^{-C_2 |q_1|}.
\]

Now we split integral on the r.h.s of (8) into the following two parts:

\[
\int_{q_1 > 0} = \int_{0 < q_1 \leq \frac{1}{w}}^I + \int_{q_1 > \frac{1}{w}}^I.
\]

• (Estimate for I): We split I further as

\[
I = \int_{0 < q_1 \leq \frac{1}{w}} e^{-\frac{C_2}{2} |q_1|} (1 - e^{-\frac{w}{|q_1|} x}) dq_1 \\
= \left( \int_{0 < q_1 \leq \frac{1}{w}}^I + \int_{\frac{1}{w} < q_1 \leq \frac{1}{w}}^I \right) e^{-\frac{C_2}{2} |q_1|} (1 - e^{-\frac{w}{|q_1|} x}) dq_1.
\]

For \( I_1 \) we have

\[
I_1 \leq \int_{0 < q_1 \leq \frac{1}{w}} 1 - e^{-\frac{w}{|q_1|} x} dq_1 \leq \int_{0 < q_1 \leq \frac{1}{w}} dq_1 \leq w.
\]

For \( I_2 \), we use Taylor expansion to estimate

\[
I_2 = \int_{\frac{1}{w} < q_1 \leq \frac{1}{w}} e^{-\frac{C_2}{2} |q_1|} \left( \frac{w}{q_1} x - \frac{1}{2!} \left( \frac{w}{q_1} x \right)^2 + \frac{1}{3!} \left( \frac{w}{q_1} x \right)^3 - \frac{1}{4!} \left( \frac{w}{q_1} x \right)^4 + \cdots \right) dq_1 \\
\leq \int_{\frac{1}{w} < q_1 \leq \frac{1}{w}} \left( \frac{w}{q_1} x + \frac{1}{2!} \left( \frac{w}{q_1} x \right)^2 + \frac{1}{3!} \left( \frac{w}{q_1} x \right)^3 + \frac{1}{4!} \left( \frac{w}{q_1} x \right)^4 + \cdots \right) dq_1 \\
= 2w \ln \frac{1}{w} + \frac{w}{2!} (1 - w^2) + \frac{w}{2 \cdot 3!} (1 - w^4) + \frac{w}{3 \cdot 4!} (1 - w^6) + \cdots \ldots
\]
\[
\leq 2w \ln \frac{1}{w} + w \left(1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \right)
\]
\[
= 2w \ln \frac{1}{w} + \text{we}.
\]

Therefore, we have
\[
I \leq 2w \ln \frac{1}{w} + (1 + e)w.
\]  

(9)

• (Estimate for II): Since \(0 \leq x \leq 1\), we can bound II as
\[
II \leq w^2 \int_{q_1 > \frac{1}{2}} e^{-\frac{C_2}{2} |q_1|} dq_1 = \frac{\sqrt{2}w^2}{C_2} e^{-\frac{C_2}{2} x}.
\]  

(10)

Combining (9) and (10), we get the desired result:
\[
\int_{q_1 > 0} \int_0^x e^{-\frac{w}{|q_1|} (x-y)} J_f dy dq \leq \frac{8C_1}{C_2} \left(2w \ln \frac{1}{w} + (1 + e)w + \sqrt{2}w^2 e^{-\frac{C_2}{2} x} \right). \quad \square
\]

3. \(\Phi\) maps \(\Omega\) into itself. For \(f \in \Omega\), we define our solution operator \(\Phi(f)\) as follows:

\[
\Phi(f)(x, q) \equiv \left( e^{-\frac{w}{|q_1|} x} f_L + \frac{w}{|q_1|} \int_0^x e^{-\frac{w}{|q_1|} (x-y)} J_f dy \right) 1_{q_1 > 0}
\]
\[
+ \left( e^{-\frac{w}{|q_1|} (1-x)} f_R + \frac{w}{|q_1|} \int_x^1 e^{-\frac{w}{|q_1|} (y-x)} J_f dy \right) 1_{q_1 < 0}
\]
\[
= \Phi^+(f) 1_{q_1 > 0} + \Phi^-(f) 1_{q_1 < 0}.
\]

The main goal of this section is to prove that if \(f\) belongs to \(\Omega\), \(\Phi(f)\) also belongs to \(\Omega\):

**Proposition 1.** The solution operator maps the solution space \(\Omega\) into itself for sufficiently small \(w\).

The above proposition follows immediately once we prove the following lemma

**Lemma 3.1.** Let \(f \in \Omega\). Then, for sufficiently small \(w\), \(\Phi(f)\) satisfies
\[
\Phi(f) \geq 0, \quad \int_{\mathbb{R}^3} \Phi(f) \frac{q_0}{q_0} dq \geq a_l, \quad \int_{\mathbb{R}^3} \Phi(f) dq \leq a_u, \quad \frac{1}{n_\Phi} \int_{\mathbb{R}^3} \Phi(f) \frac{q_0}{q_0} dq \leq \sqrt{\lambda}
\]

where \(n_\Phi\) denotes the proper particle density with respect to \(\Phi\):
\[
n_\Phi^2 = \left( \int_{\mathbb{R}^3} \Phi(f) dq \right)^2 - \sum_{i=1}^3 \left( \int_{\mathbb{R}^3} \Phi(f) \frac{q_i}{q_0} dq \right)^2.
\]

**Proof.** (1) The non-negativity of \(\Phi(f) \geq 0\) follows directly from Lemma 2.1-(1) and the definition of \(\Phi(f)\).

(2) We note that
\[
\Phi(f) \frac{q_0}{q_0} \geq e^{-\frac{w}{|q_1|} x} f_L \frac{q_0}{q_0} 1_{q_1 > 0} + e^{-\frac{w}{|q_1|} (1-x)} f_R \frac{q_0}{q_0} 1_{q_1 < 0}
\]
\[
\geq e^{-\frac{w}{|q_1|} L} \frac{q_0}{q_0} 1_{q_1 > 0} + e^{-\frac{w}{|q_1|} R} \frac{q_0}{q_0} 1_{q_1 < 0}
\]
\[
= e^{-\frac{w}{|q_1|} L} R \frac{q_0}{q_0},
\]
yielding
\[ \int_{\mathbb{R}^3} \Phi(f) \frac{1}{q_0^2} dq \geq \int_{\mathbb{R}^3} e^{\frac{-w}{M}} f_{LR} \frac{1}{q_0^2} dq = a_t. \]

(3) We recall Lemma 2.3 to see
\[ \int_{\mathbb{R}^3} \Phi(f) dq = \int_{\mathbb{R}^3} e^{\frac{-w}{M}} f_{L1} 1_{q_1>0} + e^{\frac{-w}{M}} (1-x) f_{R1} 1_{q_1<0} dq \]
\[ + \int_{\mathbb{R}^3} \frac{w}{|q_1|} \int_0^x e^{\frac{-w}{M}} (x-y) J_f dy 1_{q_1>0} + \frac{w}{|q_1|} \int_x^1 e^{\frac{-w}{M}} (y-x) J_f dy 1_{q_1<0} dq \]
\[ \leq \int_{\mathbb{R}^3} f_{LR} dq + \frac{16C_1}{C_2^2} \left( 2w \ln \frac{1}{w} + (1+e)w + \frac{\sqrt{2}w^2}{C_2} e^{-\frac{C_2}{2w}} \right). \]

It remains to show that, for sufficiently small \( w \), we have
\[ \frac{16C_1}{C_2^2} \left( 2w \ln \frac{1}{w} + (1+e)w + \frac{\sqrt{2}w^2}{C_2} e^{-\frac{C_2}{2w}} \right) \leq \int_{\mathbb{R}^3} f_{LR} dq \]  \[ (11) \]

to get the desired result:
\[ \int_{\mathbb{R}^3} \Phi(f) dq \leq 2 \int_{\mathbb{R}^3} f_{LR} dq = a_u. \]

We need to be cautious in the derivation of (11) since \( C_1 \) and \( C_2 \) depend on \( w \).
We, therefore, need to check that \( C_1 \) and \( C_2 \) do not vanish nor get singular as \( w \) tends to zero, to justify (11). For this, we assume without loss of generality that \( 0 < w < 1 \). Then, we have from the definition of \( a_u \) and \( a_t \) that
\[ 2 \leq \frac{a_u}{a_t} \leq \left( 2 \int_{\mathbb{R}^3} f_{LR} dq \right) \left( \int_{\mathbb{R}^3} e^{\frac{-w}{M}} f_{LR} \frac{1}{q_0^2} dq \right)^{-1} \equiv C^* \]  \[ (12) \]

Therefore, we can treat \( a_u/a_t \) as a fixed constant in the limit \( w \to 0 \). Similarly, \( \lambda \) can be considered to be a fixed constant in the limit \( w \to 0 \) with no harm, and so is \( C_1 \) and \( C_2 \) since they are defined by
\[ C_1 = \frac{a_u}{M(\beta_u)} = \frac{a_u}{M\left( \frac{K_1}{K_2} \right)^{-\frac{1}{2}} \left( \sqrt{\lambda} \right)}, \]
and
\[ C_2 = C_0 \beta_l = \left\{ \sqrt{1 + (a_u/a_t)^2} - a_u/a_t \right\} \left( \frac{K_1}{K_2} \right)^{-1} \left( \frac{a_t}{a_u} \right). \]

This completes the proof.

(4) Estimating similarly as in (4), we get
\[ n_\Phi \geq \left( \int_{\mathbb{R}^3} \Phi(f) dq \int_{\mathbb{R}^3} \Phi(f) \frac{1}{q_0^2} dq \right)^{\frac{1}{2}}. \]

But we have from the positivity of \( J_f \)
\[ f_{LR}(x,q) = e^{\frac{-w}{M}} f_{L1} 1_{q_1>0} + e^{\frac{-w}{M}} (1-x) f_{R1} 1_{q_1<0} \]
\[ \leq \left( e^{\frac{-w}{M}} f_{L} + \frac{w}{|q_1|} \int_0^x e^{\frac{-w}{M}} (x-y) J_f dy \right) 1_{q_1>0} \]
$\left( e^{-\frac{w}{|q_1|}(1-x)} f_R + \frac{w}{|q_1|} \int_x^1 e^{-\frac{w}{|q_1|}(y-x)} J_f dy \right) 1_{q_1 < 0} = \Phi(f)(x, q),$

so that we can bound $n_\Phi$ further from below as

$$n_\Phi \geq \left( \int_{\mathbb{R}^3} f_{LR} d\mathbf{q} \int_{\mathbb{R}^3} f_{LR} \frac{1}{q_0^2} d\mathbf{q} \right)^{\frac{1}{2}}.$$  

Therefore, we get

$$\frac{1}{n_\Phi} \int_{\mathbb{R}^3} \Phi(f) \frac{1}{q_0} d\mathbf{q} \leq \left( \int_{\mathbb{R}^3} \Phi(f) d\mathbf{q} \int_{\mathbb{R}^3} \Phi(f) \frac{1}{q_0^2} d\mathbf{q} \right)^{-\frac{1}{2}} \int_{\mathbb{R}^3} \Phi(f) \frac{1}{q_0} d\mathbf{q}$$

$$\leq \left( \int_{\mathbb{R}^3} f_{LR} d\mathbf{q} \int_{\mathbb{R}^3} f_{LR} \frac{1}{q_0^2} d\mathbf{q} \right)^{-\frac{1}{2}} \int_{\mathbb{R}^3} \Phi(f) \frac{1}{q_0} d\mathbf{q},$$

which, in view of the definition of $\lambda$ in (3), leads to

$$\frac{1}{n_\Phi} \int_{\mathbb{R}^3} \Phi(f) \frac{1}{q_0} d\mathbf{q} \leq \lambda \left( \int_{\mathbb{R}^3} f_{LR} \frac{1}{q_0^2} d\mathbf{q} \right)^{-1} \int_{\mathbb{R}^3} \Phi(f) \frac{1}{q_0} d\mathbf{q}. \quad (13)$$

Now, since

$$\int_{\mathbb{R}^3} \Phi(f) \frac{1}{q_0} d\mathbf{q} = \int_{\mathbb{R}^3} f_{LR} \frac{1}{q_0^2} d\mathbf{q} + \int_{q_1 > 0} \frac{w}{|q_1|} \int_x^1 e^{-\frac{w}{|q_1|}(x-y)} J_f dy \frac{1}{q_0} d\mathbf{q}$$

$$+ \int_{q_1 < 0} \frac{w}{|q_1|} \int_x^1 e^{-\frac{w}{|q_1|}(y-x)} J_f dy \frac{1}{q_0} d\mathbf{q},$$

Lemma 2.3 implies

$$\int_{\mathbb{R}^3} \Phi(f) \frac{1}{q_0} d\mathbf{q} \leq \int_{\mathbb{R}^3} f_{LR} \frac{1}{q_0^2} d\mathbf{q} + \frac{16C_1}{C_2^2} \left( 2w \ln \frac{1}{w} + (1 + e)w + \frac{\sqrt{2w^2} e^{-\frac{C_2}{2w}}}{C_2} \right).$$

We then note that, as $w$ decreases,

$$f_{LR} = e^{-\frac{w}{|q_1|}} f_{LR1_{q_1 > 0}} + e^{-\frac{w}{|q_1|}(1-x)} f_{LR1_{q_1 < 0}}$$

increases, which enables one to find $w$ sufficiently small such that (Recall that it was shown in the proof of (3) that $C_1$, $C_2$ and $\lambda$ can be treated as a fixed constant in the limit $w \to 0$)

$$\frac{16C_1}{C_2^2} \left( 2w \ln \frac{1}{w} + (1 + e)w + \frac{\sqrt{2w^2} e^{-\frac{C_2}{2w}}}{C_2} \right) \leq \left( \frac{1}{\sqrt{\lambda}} - 1 \right) \int_{\mathbb{R}^3} f_{LR} \frac{1}{q_0} d\mathbf{q},$$

yielding

$$\int_{\mathbb{R}^3} \Phi(f) \frac{1}{q_0} d\mathbf{q} \leq \int_{\mathbb{R}^3} f_{LR} \frac{1}{q_0^2} d\mathbf{q} + \frac{16C_1}{C_2^2} \left( 2w \ln \frac{1}{w} + (1 + e)w + \frac{\sqrt{2w^2} e^{-\frac{C_2}{2w}}}{C_2} \right)$$

$$\leq \frac{1}{\sqrt{\lambda}} \int_{\mathbb{R}^3} f_{LR} \frac{1}{q_0} d\mathbf{q}.$$  

Inserting this into (13) gives the desired result. \[\square\]
4. **Contraction mapping.** In this section, we establish the Lipschitz continuity of our solution operator. We first need to set up preliminary computations. The following lemma can be found in [4], but we provide a detailed proof for reader’s convenience.

**Lemma 4.1.** For the modified Bessel function of the second kind \( K_i(\beta) \), the following holds

\[
\left( \frac{K_1}{K_2} \right)'(\beta) = \frac{3}{\beta} \frac{K_1}{K_2}(\beta) + \left( \frac{K_1}{K_2} \right)^2(\beta) - 1.
\]

**Proof.** We recall that

\[
K_i(\beta) = \int_0^\infty \cosh(\beta r)e^{-\beta \cosh(r)}dr,
\]

and use change of variable \( x = \sinh r \) to get

\[
K_0(\beta) = \int_0^\infty \exp\left\{-\beta \cosh(r)\right\}dr = \int_0^\infty \frac{1}{\sqrt{1+x^2}} \exp\left\{-\beta \sqrt{1+x^2}\right\}dx,
\]

\[
K_1(\beta) = \int_0^\infty \cosh(r) \exp\left\{-\beta \cosh(r)\right\}dr = \int_0^\infty \exp\left\{-\beta \sqrt{1+x^2}\right\}dx,
\]

\[
K_2(\beta) = \int_0^\infty \cosh(2r) \exp\left\{-\beta \cosh(r)\right\}dr = \int_0^\infty \frac{2x^2+1}{\sqrt{1+x^2}} \exp\left\{-\beta \sqrt{1+x^2}\right\}dx.
\]

We then observe that

\[
(K_1)'(\beta) = -\int_0^\infty \sqrt{1+x^2} \exp\left\{-\beta \sqrt{1+x^2}\right\}dx
\]

\[
= -\frac{1}{2} \left( K_2(\beta) + K_0(\beta) \right)
\]

and

\[
(K_2)'(\beta) = -\int_0^\infty (2x^2+1) \exp\left\{-\beta \sqrt{1+x^2}\right\}dx
\]

\[
= \frac{2}{\beta} \int_0^\infty x \sqrt{1+x^2} \frac{d}{dx} \left( \exp\left\{-\beta \sqrt{1+x^2}\right\} \right)dx - K_1(\beta)
\]

\[
= \frac{2}{\beta} \int_0^\infty \frac{2x^2+1}{\sqrt{1+x^2}} \exp\left\{-\beta \sqrt{1+x^2}\right\}dx - K_1(\beta)
\]

\[
= -\frac{4}{\beta^2} - 1)K_1(\beta) - \frac{2}{\beta} K_0(\beta).
\]

Here we used

\[
K_2(\beta) = \int_0^\infty \frac{2x^2+1}{\sqrt{1+x^2}} \exp\left\{-\beta \sqrt{1+x^2}\right\}dx
\]

\[
= \int_0^\infty \frac{2x^2}{\sqrt{1+x^2}} \exp\left\{-\beta \sqrt{1+x^2}\right\}dx + K_0(\beta)
\]

\[
= \int_0^\infty -\frac{2x}{\beta} \frac{d}{dx} \left( \exp\left\{-\beta \sqrt{1+x^2}\right\} \right)dx + K_0(\beta)
\]
These identities then give

\[
(K_1)'(β)K_2(β) - K_1(β)(K_2)'(β) = \frac{3}{β}K_1(β)K_2(β) + (K_1)^2(β) - (K_2)^2(β),
\]

which, upon dividing both sides by \((K_2)^2(β)\), gives the desired result.

The following lemma shows that the r.h.s of the identity in Lemma 4.1 is strictly positive. The proof can be found in [4].

**Lemma 4.2.** [4] For the modified Bessel function of the second kind, the following inequality holds

\[
\frac{3K_1}{βK_2}(β) + \left(\frac{K_1}{K_2}\right)^2(β) - 1 \geq ℓ(β),
\]

where \(ℓ(β)\) is defined by

\[
ℓ(β) = \begin{cases} 
\frac{2-β}{(β+2)^2} & 0 < β < 2 \\
\frac{3(6565β^2 + 2419β^2 + 726)}{(128β^2 + 240β^2 + 106β - 66)} & β \geq 2.
\end{cases}
\]

Note that \(ℓ(β)\) is strictly positive.

**Lemma 4.3.** Let \(f, g \in Ω\), then there exist positive constants \(C_8\) and \(C_9\) such that

\[
|J_f - J_g| \leq C_0e^{C_8\sqrt{1+|q|^2}}∥f - g∥_{L_1^q},
\]

where \(∥·∥_{L_1^q}\) denotes the usual \(L_1^q\) norm:

\[
∥f∥_{L_1^q} = \int_{R^3} |f(q)|dq.
\]

**Proof.** For the convenience of computation, we introduce a new variable \(α\) (see [4]) defined by

\[
α = \frac{1}{n} \int_{R^3} \frac{f}{q_0}dq.
\]

Since \((K_1/K_2)(β)\) is strictly increasing with the range is \([0, 1]\) for \(β \in (0, ∞)\), and since \(α\) satisfies from Lemma 2.1 that

\[
0 < \frac{a_f}{a_u} \leq α \leq \sqrt{α} < 1,
\]

there is a one-to-one correspondence between \(α\) and \(β\):

\[
β = X(α) = \left(\frac{K_1}{K_2}\right)^{-1}(α).
\]

In view of this, we consider \(J(n, u, β)\) as a functional of \((n, u, α)\), and apply the mean value theorem to get

\[
J(n_f, u_f, α_f) - J(n_g, u_g, α_g) = \nabla_{n, u, α} J(θ) \cdot (n_f - n_g, u_f - u_g, α_f - α_g),
\]

for some \(0 ≤ θ ≤ 1\), where the abbreviate notation \(J(θ)\) denotes

\[
J(θ) = J((1-θ)n_f + θn_g, (1-θ)u_f + θu_g, (1-θ)α_f + θα_g).
\]
We need to estimate \( \nabla_{n,u,\alpha} J \) and \((n_f - n_g, u_f - u_g, \alpha_f - \alpha_g)\).

1. Estimates for \( \nabla_{n,u,\alpha} J \): A direct computation gives

\[
\frac{\partial J}{\partial n} = \frac{1}{n} J, \quad \nabla_u J = \beta (q - \sqrt{1 + |q|^2} u) J, \quad \frac{\partial J}{\partial \alpha} = -\frac{\partial \beta}{\partial \alpha} \left( \frac{M'(\beta)}{M(\beta)} + \sqrt{1 + |q|^2} \sqrt{1 + |u|^2} \right) J, \tag{15}
\]

Using Lemma 2.1, Lemma 2.2 and (5), we can show that \( \frac{\partial J}{\partial n} \) and \( \nabla_u J \) are bounded as

\[
|\frac{\partial J}{\partial n}| \leq C_1 e^{-C_2 \sqrt{1 + |q|^2}},
\]

\[
|\nabla_u J| \leq \beta_u \left( |q| + \sqrt{1 + |q|^2} \frac{|u|}{\sqrt{1 + |u|^2}} \right) |J|
\]

\[
\leq 2C_1 \beta_u \sqrt{1 + |q|^2} e^{-C_2 \sqrt{1 + |q|^2}}
\]

\[
\leq 2C_1 \beta_u C_3 e^{-C_4 \sqrt{1 + |q|^2}}
\]

where \( C_3 \) and \( C_4 \) are given in the following manner:

\[
\sqrt{1 + |q|^2} e^{-C_2 \sqrt{1 + |q|^2}} \leq C_3 e^{-C_4 \sqrt{1 + |q|^2}} \tag{16}
\]

The estimate for \( \frac{\partial J}{\partial \alpha} \) is more involved. First, we use differentiation rule for inverse functions and Lemma 4.1 to get

\[
\frac{\partial \beta}{\partial \alpha} = \frac{\partial}{\partial \alpha} \left( \frac{K_1}{K_2} \right)^{-1} (\alpha) = \frac{1}{\frac{K_1}{K_2} (\beta)} = \frac{1}{3 \frac{K_1}{K_2} (\beta) + \left( \frac{K_1}{K_2} \right)^2 (\beta) - 1}.
\]

We then recall Lemma 4.2 that

\[
\frac{3}{\beta} \frac{K_1}{K_2} (\beta) + \left( \frac{K_1}{K_2} \right)^2 (\beta) - 1 \geq \ell(\beta).
\]

Note that \( \ell(\beta) \) is a strictly positive function. Therefore, we can conclude that the continuous function \( \left( \frac{K_1}{K_2} \right)'(\beta) \) (since the modified Bessel function of the second kind is continuous) is strictly positive on a closed and bounded interval \([\beta_l, \beta_u]\). This implies that \( \left( \frac{K_1}{K_2} \right)'(\beta) \) possesses a strictly positive minimum on \([\beta_l, \beta_u]\). If we denote it by \( \frac{1}{C_5} > 0 \), we have

\[
\frac{3}{\beta} \frac{K_1}{K_2} (\beta) + \left( \frac{K_1}{K_2} \right)^2 (\beta) - 1 \geq \frac{1}{C_5}.
\]

In conclusion, we obtain

\[
|\frac{\partial \beta}{\partial \alpha}| = \left| \frac{3}{\beta} \frac{K_1}{K_2} (\beta) + \left( \frac{K_1}{K_2} \right)^2 (\beta) - 1 \right|^{-1}
\]

\[
= \left( \frac{3}{\beta} \frac{K_1}{K_2} (\beta) + \left( \frac{K_1}{K_2} \right)^2 (\beta) - 1 \right)^{-1}
\]

\[
\leq C_5.
\]
On the other hand, it is clear that there exists a constant $C_6 > 0$ such that

$$|M'/(\beta)| = \frac{\int_{\mathbb{R}^3} \sqrt{1 + |q|^2} \exp \{-\beta \sqrt{1 + |q|^2}\} dq}{\int_{\mathbb{R}^3} \exp \{-\beta \sqrt{1 + |q|^2}\} dq} < C_6,$$

when $\beta$ lies in a closed and bounded range: $\beta \in [\beta_l, \beta_u]$. We return back to (15) with these estimates to get

$$\left| \frac{\partial J}{\partial \alpha} \right| \leq C_5 \left( C_6 + 2\sqrt{1 + |q|^2} \sqrt{1 + a_u^2/a_l^2} \right) C_1 e^{-C_4 \sqrt{1 + |q|^2}}$$

$$\leq C_5 C_6 C_1 e^{-C_4 \sqrt{1 + |q|^2}} + 2C_5 C_3 \sqrt{1 + a_u^2/a_l^2} C_1 e^{-C_4 \sqrt{1 + |q|^2}}$$

$$\leq \max \left\{ C_1 C_5 C_6, 2C_1 C_5 C_5 \sqrt{1 + a_u^2/a_l^2} \right\} e^{-C_4 \sqrt{1 + |q|^2}}$$

where we used (16) and the fact that $C_4$ is less than equal to $C_2$.

(2) Estimates on $(n_f - n_g, u_f - u_g, \alpha_f - \alpha_g)$

• $n_f - n_g$: From Lemma $2.1$, we get

$$|n_f - n_g| = \left| \frac{n_f^2 - n_g^2}{n_f + n_g} \right|$$

$$\leq \frac{1}{n_f + n_g} \left\{ \left( \int_{\mathbb{R}^3} \left| f - g \right| dq \right) \left( \int_{\mathbb{R}^3} \left| f + g \right| dq \right) \right.$$

$$- \sum_{i=1}^{3} \left( \int_{\mathbb{R}^3} \left| f - g \right| \frac{q_i}{q_0} dq \right) \left( \int_{\mathbb{R}^3} \left| f + g \right| \frac{q_i}{q_0} dq \right) \right\}$$

$$\leq \frac{1}{2a_l} \left( 2a_u \int_{\mathbb{R}^3} \left| f - g \right| dq + \sum_{i=1}^{3} 2a_u \int_{\mathbb{R}^3} \left| f - g \right| dq \right)$$

$$\leq \frac{4a_u}{a_l} \| f - g \|_{L^1}.$$

• $u_f - u_g$: Using the above estimate, Lemma $2.1$ and

$$|n_f u_f - n_g u_g| = \left| \int_{\mathbb{R}^3} (f - g) \frac{q}{q_0} dq \right| \leq \int_{\mathbb{R}^3} \left| f - g \right| dq = \| f - g \|_{L^1},$$

we compute

$$|u_f - u_g| = \frac{n_g(n_f u_f - n_f(n_g u_g))}{n_f n_g}$$

$$= \frac{n_g(n_f u_f - n_g u_g) + n_g u_g(n_f - n_g)}{n_f n_g}$$

$$\leq \frac{1}{a_l^2} \left( a_u \| f - g \|_{L^1} + a_u \frac{4a_u}{a_l} \| f - g \|_{L^1} \right)$$

$$\leq \left( \frac{a_u}{a_l^2} + \frac{4a_u^2}{a_l^2} \right) \| f - g \|_{L^1}. $$
Thus (14) can be estimated as in the previous cases:

\[
|\alpha_f - \alpha_g| = \frac{1}{n_f n_g} \left| n_g \int_{\mathbb{R}^3} f \frac{1}{q_0} dq - n_f \int_{\mathbb{R}^3} g \frac{1}{q_0} dq \right|
= \frac{1}{n_f n_g} \left| n_g \int_{\mathbb{R}^3} (f - g) \frac{1}{q_0} dq - (n_f - n_g) \int_{\mathbb{R}^3} g \frac{1}{q_0} dq \right|
\leq \frac{1}{a_i^2} (a_u \|f - g\|_{L^1} + a_u \frac{4a_u}{a_i} \|f - g\|_{L^1})
\leq \left( \frac{a_u}{a_i^2} + \frac{4a_u^2}{a_i^3} \right) \|f - g\|_{L^1}.
\]

Combining all the estimates we obtained so far, we get

\[
\left| \frac{\partial J}{\partial \alpha}(n_f - n_g) \right| \leq 4a_u C_1 \left( \frac{a_u}{a_i^2} + \frac{4a_u^2}{a_i^3} \right) e^{-C_3 \sqrt{1 + |q|^2} \|f - g\|_{L^1}},
\]

\[
|\nabla u_f \cdot (u_f - u_g)| \leq 2C_1 C_3 \beta u \left( \frac{a_u}{a_i^2} + \frac{4a_u^2}{a_i^3} \right) e^{-C_3 \sqrt{1 + |q|^2} \|f - g\|_{L^1}},
\]

\[
\left| \frac{\partial J}{\partial \alpha}(\alpha_f - \alpha_g) \right| \leq \max \left\{ C_1 C_3 C_5, 2C_1 C_3 C_5 \sqrt{1 + a_u^2 / a_i^2} \right\}
\times \left( \frac{a_u}{a_i^2} + \frac{4a_u^2}{a_i^3} \right) e^{-C_3 \sqrt{1 + |q|^2} \|f - g\|_{L^1}}.
\]

Thus (14) can be estimated as

\[
|J(n_f, u_f, \alpha_f) - J(n_g, u_g, \alpha_g)| \leq C_9 e^{-C_3 \sqrt{1 + |q|^2} \|f - g\|_{L^1}},
\]

where the constant $C_9$ is given by

\[
C_9 = \frac{a_u}{a_i^2} \left( 4C_1 + 2C_1 \beta u C_3 \left( \sqrt{2} + \frac{8a_u}{a_i} \right) + C_7 \left( 1 + \frac{4a_u}{a_i} \right) \right)
\]

This gives the desired result. $\square$

The following proposition, together with Proposition 1 completes the proof of Theorem 1.2.

**Proposition 2.** $\Phi(f)$ is a contraction mapping on $\Omega$ for sufficiently small $w$. That is, we can take $w$ sufficiently small such that there exists $0 < \alpha < 1$ satisfying

\[
\sup_x \|\Phi(f) - \Phi(g)\|_{L^1} \leq \alpha \sup_x \|f - g\|_{L^1}.
\]

**Proof.** We have for $f, g \in \Omega$

\[
\int_{\mathbb{R}^3} |\Phi(f) - \Phi(g)| dq
= \int_{q_1 > 0} |\Phi(f) - \Phi(g)| dq + \int_{q_1 < 0} |\Phi(f) - \Phi(g)| dq
\leq \int_{q_1 > 0} w \int_0^x e^{-\frac{w}{q_1}(x-y)} |J_f - J_g| dy dq + \int_{q_1 < 0} w \int_0^x e^{-\frac{w}{q_1}(y-x)} |J_f - J_g| dy dq.
\]
Then, using Lemma 2.3 and Lemma 4.3, we can control the last expression as follows:

$$\int_{q_1>0} \frac{w}{q_1} \int_0^e e^{-\frac{w}{w_0}(x-y)}|J_f - J_g|dy dq + \int_{q_1<0} \frac{w}{q_1} \int_x^1 e^{-\frac{w}{w_0}(y-x)}|J_f - J_g|dy dq$$

$$\leq C_9 \left\{ \int_{q_1>0} \frac{w}{q_1} e^{-C_S \sqrt{1+|q|^2}} \int_0^e e^{-\frac{w}{w_0}(x-y)} dy dq \right\} \| f - g \|_{L^1_q}$$

$$+ \int_{q_1<0} \frac{w}{q_1} e^{-C_S \sqrt{1+|q|^2}} \int_x^1 e^{-\frac{w}{w_0}(y-x)} dy dq \right\} \| f - g \|_{L^1_q}$$

$$\leq \frac{16C_9}{C_S^2} \left( 2w \ln \frac{1}{w} + (1 + e)w + \frac{\sqrt{2}w^2}{C_S} e^{-\frac{C_S}{w_0}} \right) \| f - g \|_{L^1_q}$$

for sufficiently small $w$. Here we used the fact that $C_S$ and $C_9$ can be considered to be a fixed positive constant in the limit $w \to 0$, that can be verified by a similar argument as in the proof of Lemma 3.1 (3). This gives the desired results.

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