An asymptotic formula for models with caustics

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Abstract

We introduce an asymptotic formula for calculating quantum mechanical and quantum theoretical models with caustics, like the Nambu–Jona-Lasinio(NJL) model. This asymptotic formula is given by the form attached the extra term, which suppresses the divergence induced because of caustics, to the leading term of the WKB approximation. This formula guarantees validity of the mean field approximation of models with caustics.

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1 Introduction

There is the WKB approximation method as one of the nonperturbative approximation methods: the action is expanded around the classical solution, and terms with derivatives of greater than third order are treated as perturbations, to obtain a series expansion expressed in terms of a loop expansion parameter, such as $\hbar$ or $1/N$. Then it is very important that positivity for all eigenvalues of a second derivative of the action at the classical solution, that is, the convexity condition of the action must be satisfied. When this approximation method is utilized to the Nambu–Jona-Lasinio (NJL) model, which is one of the most famous field theoretical models exhibiting dynamical chiral symmetry breaking phenomena, many interesting results are made in a level of the mean field approximation. For example, T. Inagaki et al. [2] have investigated the phase structure of the NJL model at finite temperature and chemical potential in an external magnetic field and gained the interesting results such as the magnetic catalysis.

Nevertheless the mean field approximation in the NJL model is open to question: for example, in reference [3], we had calculated the contribution of one loop diagrams of the auxiliary fields for the the effective potential and Gap equation of the NJL model. Then the infrared divergence occurs from the loop diagram of the massless auxiliary field. This fact implies that we always have to introduce a current mass in order to suppress this infrared divergence, and either do the mean field approximation. Otherwise the WKB approximation fails because the convexity condition is not satisfied. This discourages the interesting results obtained in the chiral limit. Therefore it is important to investigate whether these mean field approximations is justified in terms of the asymptotic expansion.

Generally when one and more eigenvalues of the second derivative of action vanish at the classical solution, higher order terms of the WKB approximation diverge. This situation is often called ‘caustics’. L. S. Shulman [4] says that this divergence occurs approximately because the third derivative of action does not vanish at the classical solution. In fact, he has given the ad hoc treatment of caustics for a Green function in quantum mechanics. Unfortunately, This method cannot apply to calculate the effective potential for the NJL model in the chiral limit. We need to consider other method for the treatment of caustics.

It is well-known that similar circumstance like caustics occurs when we calculate an asymptotic expansion of an integral using the saddle point method [5] [6]. For example, let us consider the Bessel function

$$J_N(N\alpha) = \frac{1}{2\pi i} \int_{\infty - \pi i}^{\infty + \pi i} e^{N(\alpha \sinh z - z)} dz \quad (\text{Re} \quad \alpha > 0),$$

(1)

where $\text{Re} \quad \alpha$ stands for the real part of $\alpha$. If $\alpha = 1$, the second derivative of the exponent becomes zero at the saddle point $z = 0$. In such a case, there are two $\alpha$-dependent and nearly coincident saddle points in the neighborhood of $\alpha = 1$. Utilizing the two saddle points and replacing the exponent $\alpha \sinh z - z$ with a cubic function with respect to a new variable, C. Chester et al. have presented the method giving a uniform asymptotic expansion as $\alpha \to 1$ and $N \to \infty$ [7] [8] [9]. The application for an integral with multi-variables had been presented by N. Bleistein [10].

We would like to apply their method to the NJL model. However this method cannot use directly to the NJL model, because the classical equation of the NJL model has only one classical solution or at least we do not know whether the another solution, which coincides with the classical solution in the chiral limit, exists. We need to make the method that we can use for the model having only one classical solution so that we can obtain the same property as the asymptotic expansion presented by C. Chester et al..

In reference [11], an other method obtaining an asymptotic expansion of an integral which the second derivative of the exponent of the integrand becomes zero at the saddle point is introduced. Although this method does not give a uniform asymptotic expansion, it needs only one expansion point and the result coincides with the one based on the method presented by C. Chester et al. in the
neighborhood of caustics. It is the most remarkable thing in this method that this expansion point always exists even if we do not know whether the other saddle point exists. Utilizing this fact, we introduce another approach for obtaining the asymptotic formula of the models with caustics.

This paper is organized as follows: in section 2, we mention the method obtaining the asymptotic formula of an integral with one variable, which had been utilized in the paper [1]. This method utilizes a point at which the second derivative of the exponent becomes zero as an expansion point of the exponent, rather than the saddle point. Then we rewrite the obtained formula with respect to the saddle point and show the formula is the form attached the extra term which suppresses the divergence to the result of the saddle point method. In the end of section 2, we show that the formula coincides with the leading order of the asymptotic expansion given by C. Chester et al. in the neighborhood of caustics. In section 3, we apply this method to obtain an asymptotic formula of an integral with multi-variables. In the same way as the result of section 2, we show that the formula is the form attached the extra term which suppresses the divergence to the result of the WKB approximation. This implies that the mean field approximation of this formula coincides with the one of the WKB approximation and gives the important result that the mean field approximation of the model with caustics, like the NJL model, is justified in the sense of the formula whose result converges. The final section is devoted to conclusions and discussions.

2 An asymptotic formula of an integral with one variable

In this section, we establish an asymptotic formula as \( N \to \infty \) of an integral

\[
I(\alpha, N) = \int_c g(z) e^{NF(z, \alpha)} dz,
\]

(2)

where the real functions \( g(z), f(z, \alpha) \) are analytic functions of their arguments, \( N \) is a large positive parameter and \( c \) is some contour in the complex \( z \) plane. The function \( f(z, \alpha) \) has the saddle point \( z_0 \) which is dependent on \( \alpha \). The second derivative on the steepest descent, \( \partial^2 f(z, \alpha) / \partial z^2 |_{z=z_0(\alpha)} \), becomes zero at some value of \( \alpha \), say \( \hat{\alpha} \), and takes negative value for \( \alpha > \hat{\alpha} \).

We expand \( f(z, \alpha) \), \( g(z) \) around \( \hat{z} \) which satisfy with \( f^{(2)}(\hat{z}, \alpha) = 0 \) (and \( f^{(1)}(\hat{z}, \hat{\alpha}) = 0 \)):

\[
I(\alpha, N) \sim \int_c (g(\hat{z}) + O(z - \hat{z}))
\]

\[
\times \exp \left[ N \left\{ f(\hat{z}, \alpha) + f^{(1)}(\hat{z}, \alpha)(z - \hat{z}) + \frac{1}{3!} f^{(3)}(\hat{z}, \alpha)(z - \hat{z})^3 + O((z - \hat{z})^4) \right\} \right] dz,
\]

(3)

where we utilized the abbreviations \( f^{(n)}(z, \alpha) = \partial^n f(z, \alpha) / \partial z^n \). We assume that \( f^{(3)}(\hat{z}, \hat{\alpha}) \neq 0 \). Considering only in the neighborhood of \( \hat{\alpha} \), we can regard \( f^{(1)}(\hat{z}, \alpha) \) as very small quantity, especially, we set that \( |f^{(1)}(\hat{z}, \alpha)| \sim O(1/N^{\frac{2}{3}}) \) for sufficiently large \( N \). So we can consider both terms \( f^{(1)}(\hat{z}, \alpha)(z - \hat{z}) \) and \( f^{(3)}(\hat{z}, \alpha)(z - \hat{z})^3 \) as same order terms, in the neighborhood of \( \hat{z} \) on the \( z \) space. We retain terms, \( g(\hat{z}), f(\hat{z}, \alpha), f^{(1)}(\hat{z}, \alpha)(z - \hat{z}) \) and \( f^{(3)}(\hat{z}, \alpha)(z - \hat{z})^3 \) as the principle part, and remaining terms with higher derivatives are treated as negligible quantities, to obtain

\[
I(\alpha, N) \sim g(\hat{z}) \left( \frac{2}{f^{(3)}(\hat{z}, \alpha)} \right)^{\frac{1}{3}} e^{Nf(\hat{z}, \alpha)} e^{N^\frac{2}{3}} \int_{c_1} dt e^{\frac{2}{3}t^3 - \alpha t},
\]

(4)

\[\text{In the case of the integral with one variable, it is expected that the another saddle point, say } z_0^{\star}, \text{ which coincides with the saddle point } z_0 \text{ at } \alpha = \hat{\alpha} \text{ always exists. In this paper, as we would like to derive the method utilized only one saddle point, we assume that we cannot deform the integration path so that it traces the steepest descent of the point } z_0^{\star} \text{ when } \alpha > \hat{\alpha} \text{ and we do not utilize the point } z_0^{\star}. \text{ Also, we do not consider about } \alpha < \hat{\alpha} \text{ because the saddle points } z_0 \text{ and } z_0^{\star} \text{ become the complex conjugate each other and it is out of our aim.} \]
where we have set
\[
\left( \frac{f^{(3)}(\tilde{z}, \alpha)}{2} \right)^{\frac{1}{3}} (z - \tilde{z}) \equiv \frac{t}{N^{\frac{1}{3}}} - \left( \frac{2}{f^{(3)}(\tilde{z}, \alpha)} \right)^{\frac{1}{3}} f^{(1)}(\tilde{z}, \alpha) \equiv \zeta(\alpha)
\] (5)

and \( \zeta' = N^{\frac{2}{3}} \zeta \). The integral part of Eq. (4) just corresponds to the Airy integral
\[
\text{Ai}(\zeta') = \frac{1}{2\pi i} \int_{cA} dt e^{\frac{4}{3}t^3 - \zeta' t} .
\] (6)

Therefore we can obtain the asymptotic formula
\[
I(\alpha, N) \sim (2\pi i)g(\tilde{z}) \left( \frac{2}{f^{(3)}(\tilde{z}, \alpha)} \right)^{\frac{1}{3}} e^{Nf(\tilde{z}, \alpha)} N^{\frac{1}{3}} \text{Ai}(\zeta')
\] (7)

We can rewrite the asymptotic formula with the saddle point \( z_0 \), not the point \( \tilde{z} \). When \( \alpha \) is in a neighborhood of \( \hat{\alpha} \), a distance between \( \tilde{z} \) and \( z_0, |\tilde{z} - z_0| \), becomes very small. So we can expand \( g(\tilde{z}), f(\tilde{z}, \alpha) \) and its derivatives at the point \( \tilde{z} \) around the saddle point \( z_0 \):
\[
g(\tilde{z}) = g(z_0) + \cdots,
\] (8)
\[
f(\tilde{z}, \alpha) = f(z_0, \alpha) + \frac{1}{2} f^{(2)}(z_0, \alpha)(\tilde{z} - z_0)^2 + \frac{1}{3!} f^{(3)}(z_0, \alpha)(\tilde{z} - z_0)^3 + \cdots,
\] (9)
\[
f^{(1)}(\tilde{z}, \alpha) = f^{(2)}(z_0, \alpha)(\tilde{z} - z_0) + \frac{1}{2} f^{(3)}(z_0, \alpha)(\tilde{z} - z_0)^2 + \cdots,
\] (10)
\[
0 = f^{(2)}(\tilde{z}, \alpha) = f^{(2)}(z_0, \alpha) + f^{(3)}(z_0, \alpha)(\tilde{z} - z_0) + \cdots,
\] (11)
\[
f^{(3)}(\tilde{z}, \alpha) = f^{(3)}(z_0, \alpha) + \cdots.
\] (12)

If we assume that \( f^{(2)}(z_0, \alpha) \sim O(1/N^{\frac{1}{4}}) \) and that \( N \) is sufficiently large, Eq. (11) gives
\[
\tilde{z} - z_0 \approx -\frac{f^{(2)}(z_0, \alpha)}{f^{(3)}(z_0, \alpha)} \sim O\left( \frac{1}{N^{\frac{1}{2}}} \right)
\] (13)

and Eqs. (8), (9), (10) and (12) are approximately given by
\[
g(\tilde{z}) \approx g(z_0),
\] (14)
\[
f(\tilde{z}, \alpha) \approx f(z_0, \alpha) - \frac{1}{3} \left( -\frac{f^{(2)}(z_0, \alpha)}{f^{(3)}(z_0, \alpha)} \right)^2,
\] (15)
\[
f^{(1)}(\tilde{z}, \alpha) \approx -\frac{1}{2} \frac{(\tilde{z} - z_0)^2}{f^{(3)}(z_0, \alpha)} \sim O\left( \frac{1}{N^{\frac{1}{2}}} \right),
\] (16)
\[
f^{(3)}(\tilde{z}, \alpha) \approx f^{(3)}(z_0, \alpha),
\] (17)

respectively. Substituting Eqs. (14)-(17) into Eq. (21), We obtain the asymptotic formula of the integral in terms of the saddle point \( z_0 \):
\[
I(\alpha, N) \sim (2\pi i)g(z_0) \left( \frac{2}{f^{(3)}(z_0, \alpha)} \right)^{\frac{1}{3}} e^{Nf(z_0, \alpha) - \frac{2}{3} N^{\frac{1}{3}} \zeta(\alpha)} \text{Ai}(\zeta')
\] (18)

and \( \zeta(\alpha) \) is deformed as
\[
\zeta(\alpha) \equiv \left[ \frac{(-f^{(2)}(z_0, \alpha))^3}{2(f^{(3)}(z_0, \alpha))^2} \right]^{\frac{1}{3}}.
\] (19)
The form of the asymptotic formula (13) is the form attached the extra term to the leading order of the asymptotic expansion based on the saddle point method, so that the divergence due to the term \( f^{(2)}(z_0, \alpha) \) at \( \alpha = \hat{\alpha} \) is suppressed. Expanding the Airy function (6) by use of the saddle point method as follows

\[
\text{Ai}(\zeta') \sim \frac{N^\frac{1}{2}}{2\pi i} e^{\frac{2}{3}i\zeta'^3} \sqrt{-\frac{\pi}{N^\frac{1}{2}}} \zeta' = \frac{e^{\frac{2}{3}i\zeta'^3}}{(2\sqrt{\pi})^\frac{1}{4}} \zeta',
\]

(20)

and substituting this expansion in the formula (13) takes off the extra term and leaves the leading term of the asymptotic expansion of the integral (2) by the saddle point method. We find that, as in Schulman’s textbook [4], the term \( f^{(3)}(z_0, \alpha) \) plays an important role in order to converge the result of the WKB approximation.

Finally, we show that the asymptotic formula (13) coincides with the uniform asymptotic expansion of the integral (2) based on the method presented by C. Chester et al. in the neighborhood of \( \hat{\alpha} \). The leading order of the asymptotic expansion due to their method is given as

\[
I(\alpha, N) \sim (2\pi i)a_0(\alpha)\frac{e^{NA(\alpha)}}{N^{\frac{3}{2}}} \text{Ai}(\zeta'),
\]

(21)

where

\[
A(\alpha) = \frac{1}{2}(f(z_0, \alpha) + f(z^*_0, \alpha)),
\]

(22)

\[
\zeta(\alpha) = \left[\frac{3}{4}(f(z_0, \alpha) - f(z^*_0, \alpha))\right]^{\frac{2}{3}},
\]

(23)

\[
a_0(\alpha) = \frac{1}{2} \left(g(z_0)\sqrt{\frac{2\zeta^\frac{1}{2}}{-f^{(2)}(z_0, \alpha)}} + g(z^*_0)\sqrt{\frac{2\zeta^\frac{1}{2}}{f^{(2)}(z^*_0, \alpha)}}\right).
\]

(24)

A point \( z^*_0 \) represents the another saddle point that coincides with the saddle point \( z_0 \) when \( \alpha \) approaches to \( \hat{\alpha} \). When \( \alpha \) is in a neighborhood of \( \hat{\alpha} \), a distance between \( z^*_0 \) and \( z_0 \), \( |z^*_0 - z_0| \), becomes very small. Therefore we can expand \( g(z^*_0), f(z^*_0, \alpha) \) and its derivatives at the another saddle point \( z^*_0 \) around the saddle point \( z_0 \):

\[
g(z^*_0) = g(z_0) + \cdots,
\]

(25)

\[
f(z^*_0, \alpha) = f(z_0, \alpha) + \frac{1}{2}f^{(2)}(z_0, \alpha)(z^*_0 - z_0)^2 + \frac{1}{3!}f^{(3)}(z_0, \alpha)(z^*_0 - z_0)^3 + \cdots,
\]

(26)

\[
0 = f^{(1)}(z^*_0, \alpha) = f^{(2)}(z_0, \alpha)(z^*_0 - z_0) + \frac{1}{2}f^{(3)}(z_0, \alpha)(z^*_0 - z_0)^2 + \cdots,
\]

(27)

\[
f^{(2)}(z^*_0, \alpha) = f^{(2)}(z_0, \alpha) + f^{(3)}(z_0, \alpha)(z^*_0 - z_0) + \cdots,
\]

(28)

\[
f^{(3)}(z^*_0, \alpha) = f^{(3)}(z_0, \alpha) + \cdots.
\]

(29)

If we assume that \( f^{(2)}(z_0, \alpha) \sim O(1/N^{\frac{3}{2}}) \) and that \( N \) is sufficiently large, Eq. (27) gives

\[
z^*_0 - z_0 \approx \frac{2f^{(2)}(z_0, \alpha)}{f^{(3)}(z_0, \alpha)} \sim O\left(\frac{1}{N^{\frac{3}{2}}}\right),
\]

(30)

and Eqs. (25), (26), (28) and (29) are approximately given by

\[
f(z^*_0, \alpha) \approx f(z_0, \alpha) - \frac{2}{3}\frac{(-f^{(2)}(z_0, \alpha))^3}{f^{(3)}(z_0, \alpha)^2},
\]

(31)

\[
f^{(2)}(z^*_0, \alpha) \approx -f^{(2)}(z_0, \alpha),
\]

(32)

\[
f^{(3)}(z^*_0, \alpha) \approx f^{(3)}(z_0, \alpha),
\]

(33)

\footnote{In making change of variables, the saddle points \( z_0 \) and \( z^*_0 \) are corresponded to \(-\zeta^\frac{1}{2}\) and \( \zeta^\frac{1}{2}\), respectively.}
If we can find out such a point $\tilde{x}$, without loss of generality, an expansion of

$$A(\alpha) \approx f(z_0, \alpha) - \frac{1}{3} \left( \frac{-f^{(2)}(z_0, \alpha)}{f^{(3)}(z_0, \alpha)} \right)^3, \quad \zeta(\alpha) \approx \left[ \frac{1}{2} \left( \frac{-f^{(2)}(z_0, \alpha)}{f^{(3)}(z_0, \alpha)} \right)^3 \right]^\frac{3}{2},$$

(34)

and substitution of these expressions in the asymptotic expansion (21) gives the asymptotic formula (18). This fact shows that, even if we do not know whether the another saddle point exists, we can derive the asymptotic formula (18) consistent with the result of C. Chester et al. at least in the neighborhood of $\hat{\alpha}$.

3 An asymptotic formula of an integral with multi-variables

Next we investigate an asymptotic formula of an integral with $n$ component variables, which is formed as

$$I_n(\alpha, N) = \int_D e^{NF(x, \alpha)} d^n x,$$

(36)

where $x = (x_1, x_2, \cdots, x_n)$, $d^n x = dx_1 dx_2 \cdots dx_n$ and $D$ is a region in $n$ dimensional space. A saddle point of $F(x, \alpha)$ is expressed as

$$x_0(\alpha) = (x_{01}(\alpha), x_{02}(\alpha), \cdots, x_{0n}(\alpha)).$$

(37)

Without loss of generality an expansion of $F(x, \alpha)$ around the saddle point $x_0$ can be given by

$$F(x, \alpha) = F(x_0, \alpha) + \frac{1}{2} \sum_{i=1}^n \lambda_i(\alpha)(x_i - x_{0i})^2$$

$$+ \frac{1}{3!} \sum_{i,j,k=1}^n a_{ijk}(\alpha)(x_i - x_{0i})(x_j - x_{0j})(x_k - x_{0k}) + \cdots,$$

(38)

where we assume that $\lambda_1(\alpha)$ becomes zero at $\alpha = \hat{\alpha}$ and takes negative value for $\alpha > \hat{\alpha}$, $\lambda_i(\alpha) < 0$ ($i = 2, \cdots, n$) for $\alpha \geq \hat{\alpha}$ and $a_{111}(\hat{\alpha}) \neq 0$. We would like to give an asymptotic formula of the integral (36) by extension of the method discussed in Section 2. The most direct method is to find a point $\tilde{x}$ so that $F(x, \alpha)$ is expanded as

$$F(x, \alpha) = F(\tilde{x}, \alpha) + a_1(\alpha)(x_1 - \tilde{x}_1) + \frac{1}{2} \sum_{i=2}^n \lambda_i(\alpha)(x_i - \tilde{x}_i)^2$$

$$+ \frac{1}{3!} \sum_{i,j,k=1}^n \tilde{a}_{ijk}(\alpha)(x_i - \tilde{x}_i)(x_j - \tilde{x}_j)(x_k - \tilde{x}_k) + \cdots.$$

(39)

If we can find out such a point $\tilde{x}$, we obtain the asymptotic formula

$$I_n(\alpha) \sim (2\pi i)^{\frac{3}{2}} \left( \frac{2\pi}{N} \right)^{-\frac{n}{4}} \left( \frac{2}{N a_{111}(\alpha)} \right)^\frac{1}{3} \left( \prod_{i=2}^n (\lambda_i) \right)^{-\frac{1}{2}} \text{Ai}(\zeta'),$$

(40)

where

$$\zeta(\alpha) \equiv -\left( \frac{2}{a_{111}(\alpha)} \right)^{\frac{1}{3}} \lambda_1(\alpha).$$

(41)
and we have only to rewrite this formula by $F(x_0, \alpha)$ and its derivatives at the saddle point $x_0$.

Although such a point $\hat{x}$ always exists, it is difficult to exactly find out the point. Fortunately, we do not need to derive explicitly the point. As the same way in section 2, when $\alpha$ is in a neighborhood of $\tilde{\alpha}$, a distance between $\hat{x}$ and $x_0$, $|\hat{x} - x_0|$, becomes very small. Therefore we can expand $F(\hat{x}, \alpha)$ and its derivatives at the point $\hat{x}$ around the saddle point $x_0$:

$$F(\hat{x}, \alpha) = F(x_0, \alpha) + \frac{1}{2} \lambda_1(\hat{x} - x_0)^2 + \frac{1}{3!} a_{111}(\hat{x} - x_0)^3 + \cdots,$$

$$F_1^{(1)}(\hat{x}, \alpha) = \lambda_1(\hat{x} - x_0) + \frac{1}{2} a_{111}(\hat{x} - x_0)^2 + \cdots,$$

$$0 = F_i^{(1)}(\hat{x}, \alpha) = \lambda_i(\hat{x} - x_0) + \frac{1}{2} a_{i11}(\hat{x} - x_0)^2 + \cdots, \quad (i = 2, \cdots, n),$$

$$0 = F_i^{(2)}(\hat{x}, \alpha) = a_{1i1}(\hat{x} - x_0) + \cdots = F_i^{(1)}(\hat{x}, \alpha), \quad (i = 2, \cdots, n),$$

$$F_{ij}^{(2)}(\hat{x}, \alpha) = \lambda_i \delta_{ij} + a_{ij1}(\hat{x} - x_0) + \cdots, \quad (i, j = 2, \cdots, n),$$

$$F_{111}^{(3)}(\hat{x}, \alpha) = a_{111} + \cdots,$$

where we have used the abbreviations

$$F_{i1j \cdots k}^{(k)}(\hat{x}, \alpha) = \frac{\partial^k F(x, \alpha)}{\partial x_i \partial x_j \cdots \partial x_k} \bigg|_{x = \hat{x}}.$$

We assume that $\lambda_1(\alpha) \sim O(1/N^{3})$ and that $N$ is sufficiently large. As in an integral with one variable, Eqs. (44, 45) give

$$\hat{x}_1 - x_0 \approx -\frac{\lambda_1}{a_{111}}, \quad \hat{x}_i - x_0 \approx -\frac{a_{i11}}{2\lambda_i a_{111}} \lambda_i^2 \quad (i = 2, \cdots, n),$$

and Eqs. (42), (43), (47) and (48) are approximately given by

$$F(\hat{x}, \alpha) \approx F(x_0, \alpha) - \frac{1}{3} \frac{(-\lambda_1)^3}{a_{111}^3},$$

$$F_1^{(1)}(\hat{x}, \alpha) \approx -\frac{1}{2} \frac{(-\lambda_1)^2}{a_{111}},$$

$$F_{ij}^{(2)}(\hat{x}, \alpha) \approx \lambda_i(\alpha) \delta_{ij}, \quad (i, j = 2, \cdots, n),$$

$$F_{111}^{(3)}(\hat{x}, \alpha) \approx a_{111}(\alpha),$$

respectively. Although the terms $F_i^{(2)}(\hat{x}, \alpha)$ ($i = 2, \cdots, n$) remain finite, these turns out to be suppressed as higher order terms. The use of Eqs. (46), (47), (48) gives

$$I_n(\alpha, N) \sim \int d^n x \exp \left[ N \left\{ F(x_0, \alpha) - \frac{1}{3} \frac{(-\lambda_1)^3}{a_{111}^3} 
- \frac{1}{2} \frac{(-\lambda_1)^2}{a_{111}} (x_1 - \hat{x}_1) + \frac{a_{111}}{3!} (x_1 - \hat{x}_1)^3 + \frac{1}{2} \lambda_i(x_i - \hat{x}_i)^2 + \cdots \right\} \right].$$

Making changes of variables

$$x_1 - \hat{x}_1 = \frac{1}{N^\frac{1}{3}} (\frac{2}{a_{111}})^{\frac{1}{3}} t_1, \quad x_i - \hat{x}_i = \frac{1}{N^\frac{1}{3}} t_i, \quad (i = 2, \cdots, n),$$

$$\zeta(\alpha) = \left( \frac{(-\lambda_1(\alpha))^3}{2a_{111}^2(\alpha)} \right)^{\frac{2}{3}},$$

(57)
we obtain an asymptotic formula of the integral (36),

\[ I_n(\alpha, N) \sim e^{NF(x_0, \alpha)} \left( \frac{2\pi}{N} \right)^{\frac{n}{2}} \frac{1}{\det \left\{ -F^{(2)}(x_0, \alpha) \right\}^{\frac{1}{2}}} (2\sqrt{\pi t})^\frac{\zeta'}{4} e^{-\frac{\zeta'}{4}} \text{Ai}(\zeta') , \]  

(58)

where we have used \( \zeta' = N^\frac{2}{3} \zeta \). The divergence of the term \( \det \left\{ -F^{(2)}(x_0, \alpha) \right\} \) at \( \alpha = \hat{\alpha} \) is suppressed by the term \( \zeta'^{\frac{1}{4}} \). Thus we can derive the asymptotic formula without another saddle point which coincides with the saddle point \( x_0 \) at \( \alpha = \hat{\alpha} \). Also, in the same way as section 2, it is not difficult to show that the asymptotic formula (58) coincides with the uniform asymptotic expansion derived by N. Bleistein [10] in the neighborhood of \( \hat{\alpha} \).

Finally, we can simply comment about the mean field approximation. As same way as the asymptotic formula (18), the form of the asymptotic formula (58) is the form attached the extra term to the leading order of the asymptotic expansion based on the WKB approximation. Therefore the mean field approximation based on the asymptotic formula (58) coincides with the one based on the WKB approximation. This fact gives the important result that the mean field approximation of the model with caustics is justified in the sense of the formula (58).

## 4 Conclusion and Discussion

In this paper, we derive the asymptotic formula for the integral with caustics, which we can utilize when the integral has only one saddle point \( x_0 \) or at least when we do not know whether another saddle point, which coincides with \( z_0 \) at \( \alpha = \hat{\alpha} \), exists. This asymptotic formula coincides with the uniform asymptotic expansion in the neighborhood of \( \hat{\alpha} \). Of course, if we can find two saddle points, we should utilize the method presented by C. Chester et al. [7, 8, 9] and N. Bleistein [10] so that we can obtain the uniform asymptotic expansion. Also, we show the important result that the mean field approximation of the model with caustics is justified in the sense of the asymptotic formula whose result converges. This result guarantees the validity for the mean field approximation of the NJL model.

Our next aim is to apply this result to quantum field theoretical models with caustics. We think that carrying out this is very important: in reference [3], we calculated the effective potential and Gap equation of the NJL model. Then in order to suppress the infrared divergence which occurs from the loop diagrams of the massless Nambu-Goldstone boson, we introduced a fermion mass in advance. According to reference [11], the result of the WKB approximation becomes bad not only at \( \alpha = \hat{\alpha} \) but also in the neighborhood of \( \hat{\alpha} \), whereas the result of the approach mentioned in section 2 gives very accurate approximation in this region. Therefore if the introduced mass is sufficiently small in comparison with the typical energy scale, there is possibility that the asymptotic formula derived in this paper gives better result than the WKB approximation. In this sense, re-estimate for quantum (= loop) effects of auxiliary fields in the effective potential and gap equation of the NJL model is very interesting. Also, in reference [12], A. Osipov et al. first calculated the quantum correction of the gap equation and effective potential of the auxiliary fields for the NJL model with the ‘t Hooft interactions and dealt with the change of the vacuum state due to it. Then they suggested that including the quantum correction causes caustics at some values of the coupling constants. The use of the asymptotic formula to these caustics is interesting issue.

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