ON AN ALGEBRAIC ESSENTIAL OF SUBMANIFOLD QUANTUM MECHANICS

SHIGEKI MATSUTANI

ABSTRACT. The submanifold quantum mechanics was opened by Jensen and Koppe (Ann. Phys. 63 (1971) 586-591) and has been studied for these three decades. This article gives its more algebraic definition and show what is the essential of the submanifold quantum mechanics from an algebraic viewpoint.

MCS Codes: 34L40, 35Q40, 81T20, 32C25

Key Words: Laplacian, Schrödinger operator, Submanifold

The submanifold quantum mechanics we have called was given by Jensen and Koppe [18] and de Casta [8] and has been developed by Duclos, Exner, Krejčiřík, Šeba and Štovíček [11, 10, 13, 19], Ikegami, Nagao, Takagi and Tanzawa [16, 17, 33], Clark and Bracken [6, 7], Goldstone and Jaffe [14], Bergress and Jensen [5], Encinosa and Etemadi [12], Mladenov [29], Suzuki, Tsuru and this author [20, 21, 22, 23, 24, 25, 26, 28, 30, 31, 32] and so on. In these theories, we obtain differential operators over a submanifold $S$ in an euclidean space using a confinement potential and taking squeezing limit of the potential under some approximating theories.

However the obtained differential operators do not strongly depend upon the shape of confinement potentials or ways to take squeezing limits. Further they exhibit geometrical nature of the submanifold. In fact, the Dirac operators obtained in the scheme are related to the Frenet-Serret and the generalized Weierstrass relations [21, 22, 23, 24, 25, 26, 27, 28]: they recover all geometrical data of submanifold. Thus we believe that it should be obtained beyond an approximation and defined more algebraically.

In this article, we will give a more algebraic definition of the submanifold quantum mechanics, which is free from any approximation theories, and show what is the essential of the submanifold quantum mechanics from an algebraic point of view.
As shown in later, in submanifold quantum mechanics, a non-unitary transformation plays a key role, which makes a not self-adjoint operator self-adjoint. Thus as a preparation, we should investigate the self-adjoint operator precisely. However the concept of “adjointness”of an operator is subtle even in study of algebra of differential operators as in [Remark 1.2.16 in [3]]. Let us consider a differential operator \( \partial/\partial z^\alpha \) defined over a \( n \)-dimensional differential manifold \( M \) with local coordinate \( z \). Let \( \partial/\partial z^\alpha \) act a function from left hand side conventionally. Provided that it is equipped with a metric \( (g) \) and a volume form \( g^{1/2}d^n z \), for smooth wavefunctions \( f_1 \) and \( f_2 \) whose support is compact, we have a natural pairing, a map to the complex number,

\[
<f_1|f_2>_g = \int_M g^{1/2}d^n z \ f_1^*(z)f_2(z).
\]

Using the pairing, an expectation value of \( \partial/\partial z^\alpha \) is also naturally defined by,

\[
<f_1|\frac{\partial}{\partial z^\alpha}f_2>_g = \int_M g^{1/2}d^n z \ f_1^*(z)\frac{\partial}{\partial z^\alpha}f_2(z).
\]

The adjoint operator of \( \partial/\partial z^\alpha \) is given as \( (\partial/\partial z^\alpha)^* = -\partial/\partial z^\alpha - 1/2(\partial \log g/\partial z^\alpha) \), which depends upon the measure. For another measure \( g^{1/2}d^n z \) such that \( g^{1/2}d^n z \ f_1^*(z)f_2(z) = g^{1/2}d^n z (\alpha f_1)^*(z)(\alpha f_2)(z) \) where \( \alpha := (g/g')^{1/4} \), we might have another expectation value,

\[
<f_1|\frac{\partial}{\partial z^\alpha}f_2>_g' = \int_M g'^{1/2}d^n z \ f_1^*(z)\frac{\partial}{\partial z^\alpha}f_2(z).
\]

we have a different adjoint operator \( (\partial/\partial z^\alpha)^* \). The adjointness has such an ambiguity.

Using the ambiguity, we can introduce the half-density and then any \( \partial/\partial z^\alpha \) can be self-adjoint by setting \( g' = 1 \) (Theorem 18.1.34 in [15]). However in general the measure \( g^{1/2}d^n z \), e.g., Haar measure, exhibits a geometrical nature of the space. Thus the measureless expression such as the half-density does sometimes have less effective [N1].

In the submanifold quantum mechanics [8, 18], we partially use the half-density in the framework of a theory with the Haar measure as we will show later.

From quantum mechanical point of view, the first problem on quantum mechanics over a curved system is to search a proper metric and a proper measure. This problem is easily solved in quantum mechanics over a curved object in our euclidean space \( \mathbb{E}^3 \). In our euclidean space \( \mathbb{E}^3 \), the ordinary Lebesgue measure is natural because it is the Haar measure for the translation. As the quantum mechanics is established
in $\mathbb{E}^3$ and the concept of the adjoint operator plays essential roles [9], we will use an induced metric on the curved object from that in $\mathbb{E}^3$.

Next we will review properties of a self-adjoint operator precisely. We deal with function spaces $\Omega$ and its dual $\Omega^*$ with $L^2$-type paring $\langle , \rangle : \Omega^* \times \Omega \to \mathbb{C}$ using the Lebesgue measure in $\mathbb{E}^3$. We consider an operator $Q$ whose domain is $\Omega$. Then if exists, we could define a right-adjoint operator, $\text{Ad}(Q)$, with the domain $\Omega^*$ by

$$\langle f, Qg \rangle = \langle f, \text{Ad}(Q)g \rangle, \quad \text{for } (f, g) \in \Omega^* \times \Omega.$$ (3)

Assume that there is an isomorphism $\varphi$ between domains $\Omega$ and $\Omega^*$ as a vector space [N2]. Then triplet $(\Omega^* \times \Omega, \langle , \rangle, \varphi)$ becomes a Hilbert space $\mathcal{H}$ by introducing the inner product $(,): \Omega \times \Omega \to \mathbb{C}$ be $(f, g) := \langle \varphi(f), g \rangle$ after completion in $(,)$.

Then the ordinary adjoint operator $Q^*$, $(Q^*f, g) = (f, Qg)$, is given by $Q^*f := \varphi^{-1}(\varphi(f)\text{Ad}(Q))$.

Suppose that the operator $Q$ is self-adjoint, i.e., the domains of $Q^*$ and $Q$ coincide and $Q^* = Q$ over there. Then we have the following properties:

1. The kernel of $Q$, $\ker(Q)$, is isomorphic to the $\ker(\text{Ad}(Q))$ i.e.,

$$\ker(Q)^* := \varphi(\ker(Q)) = \ker(\text{Ad}(Q)).$$ (4)

2. The projection $\pi_Q$ from $\Omega^* \times \Omega$ to $(\ker(Q))^* \times \ker(Q)$ is commutative with $\varphi$, i.e.,

$$\varphi\pi_Q|_\Omega = \pi_Q|_\Omega \varphi, \quad (\varphi(\pi_Q|_\Omega f) = \pi_Q|_\Omega \varphi(f) \equiv \varphi(f)\text{Ad}(\pi_Q|_\Omega)).$$ (5)

3. $(\ker(Q))^* \times \ker(Q), \langle , \rangle, \varphi)$ becomes a Hilbert space.

For $\pi_Q$ satisfying (5), we will say that $\pi_Q$ is consistent with the inner product. In fact (5) means that $\varpi_Q := \pi_Q|_\Omega$ is a projection operator as $\ast$-algebra [1]: $\varpi_Q^2 = \varpi_Q$ and $\varpi_Q^* = \varpi_Q$ due to the relation $\varpi_Q f \equiv \varphi^{-1}(\varphi(f)\text{Ad}(\varpi_Q)) = \varpi_Q f$.

As we finished the review of the properties of self-adjoint operators, let us give a physical setting on the submanifold quantum mechanics. Though we can do more general, we will investigate only a case of a surface in three dimensional euclidean space $\mathbb{E}^3$. For a smooth surface $S$ embedded in the euclidean space $\mathbb{E}^3$ with the induced metric of $S$ from that in $\mathbb{E}^3$, we consider a Schrödinger equation over a tubular neighborhood $T_S$ of $S$, $\pi_{T_S} : T_S \to S$, with the $L^2$-type Hilbert space $\mathcal{H} = (\Omega^* \times \Omega, \langle , \rangle_{\text{gr}_S}, \varphi)$,

$$-\Delta \psi = E\psi \quad \text{over } T_S.$$ (6)
Here $\Omega^*$ and $\Omega$ consist of smooth compact support functions over $T_S$ and $\Delta$ is the Laplacian in $\mathbb{E}^3$. In the ordinary methods $\cite{8, 18}$, we add a potential to the left hand side in (6), which confines a particle in the tubular neighborhood $T_S$. The potential makes the support of the wavefunctions in $T_S$. By taking a squeezing limit of the potential, we decompose the system to the normal and the tangential modes, suppress the normal mode, and obtain the submanifold Schrödinger equation along the surface $S$. Instead of the scheme, we will give another definition of the Schrödinger operator by (16).

Before giving a novel definition, we will give a geometrical preliminairy. Let $S$ be locally expressed by a coordinate system $(s^1, s^2)$ and $q$ be a normal coordinate of $T_S$ whose absolute value is the distance from the surface $S$; $dq$ is an infinitesimal length in $E^3$ and belongs to kernel of $\pi_{T_S*}$ and $dq(\partial_\alpha) = 0$ ($\alpha = 1, 2, \partial_\alpha := \partial/\partial s^\alpha$). A point in $T_S$ expressed by the affine coordinate $x := (x^1, x^2, x^3)$ in $E^3$ can be uniquely represented by

$$x = \pi_{T_S} x + q e_3,$$

where $e_3$ is the normal unit vector at $S$. The moving frame $E^i_\alpha = \partial_\mu x^i$, ($\mu = 1, 2, 3 : i = 1, 2, 3$) is written by,

$$E^i_\alpha = e^i_\alpha + q^3 \gamma^\beta_{3\alpha} e^i_\beta, \quad E^i_3 = e^i_3,$$

where $\alpha, \beta = 1, 2$, $e^i_\alpha := \partial_\alpha (\pi_{T_S} x^i)$, and

$$\partial_\alpha e_3 = \gamma^\beta_{3\alpha} e_\beta.$$

Thus the induced metric, $g_{T_S \mu \nu} := \delta_{ij} E^i_\mu E^j_\nu$, ($\mu, \nu = 1, 2, 3$), from that in the euclidean space $E^3$ is given as

$$g_{T_S} = \begin{pmatrix} g_{S_{\alpha \beta}} & 0 \\ 0 & 1 \end{pmatrix},$$

$$g_{S_{\alpha \beta}} = g s_{\alpha \beta} + [\gamma^\gamma_{3\alpha} g s_{\gamma \beta} + g s_{\alpha \gamma} \gamma^\gamma_{3\beta}] q + [\gamma^\delta_{3\alpha} g s_{\delta \gamma} \gamma^\gamma_{3\beta}] q^2,$$

where $g s_{\alpha \beta} := \delta_{ij} e^i_\alpha e^j_\beta$. The determinant of the metric is expressed as,

$$\det g_{T_S} = \rho \det g_S, \quad \rho = (1 + \tr (\gamma^\alpha_{3\beta}) q + \det (\gamma^\alpha_{3\beta}) q^2),$$

and thus the pairing $\langle , \rangle_{g_{T_S}}$ is expressed by

$$\langle \psi_1, \psi_2 \rangle_{g_{T_S}} = \int (\det g_S)^{1/2} \rho^{1/2} d^2 q d S \psi_1^* \psi_2.$$

As the titles in $\cite{8, 18}$, we wish to establish the quantum mechanics of a particle restricted at $S \subset E^3$. The restriction of the particle into the surface requires that momentum and position of the particle
for the normal direction vanish. In order to realize the vanishing momentum, we will consider the momentum of the normal direction. We have the canonical commutation relation for the normal direction \([N3]\),

\[
\sqrt{-1} \partial_q, q \equiv \sqrt{-1} \partial_q q - q \sqrt{-1} \partial_q = \sqrt{-1}
\]

and wish to consider kernel of \(p_q := \sqrt{-1} \partial_q\). However \(p_q\) is not self-adjoint in general due to the existence of \(\rho\) in \((12)\) and thus a projection to kernel of \(\partial_q\) must not be consistent with the inner product in the above sense.

Thus we deform the Hilbert space so that \(p_q\) becomes a self-adjoint operator by using the half-density theory (Theorem 18.1.34 in \([15]\)). As we will show \((13)\) and \((14)\), there exist a Hilbert space \(\mathcal{H}' = (\tilde{\Omega}^* \times \tilde{\Omega}, <,>_{g_S}, \tilde{\varphi})\) and self-adjointization: \(\eta_{sa} : \mathcal{H} \to \mathcal{H}'\) satisfying the following properties.

1. There exists an isomorphism \(\Omega^* \times \Omega \to \tilde{\Omega}^* \times \tilde{\Omega}\) as a vector space. We denote it by the same \(\eta_{sa}\).
2. By defining a pairing \(<\circ, \times>_{gs} := <\eta_{sa} \circ, \eta_{sa} \times>_{g_{TS}}\), we set \(\tilde{\varphi} := \eta_{sa} \varphi \eta_{sa}^{-1}\).
3. An operator \(Q\) of \(\mathcal{H}\) is transformed into \(\eta_{sa}(Q) = \eta_{sa} Q \eta_{sa}^{-1}\).
4. \(p_q\) itself (not \(\eta_{sa}(p_q)\)) becomes a self-adjoint operator in \(\mathcal{H}'\).

Of course, the self-adjointization is not a unitary operation and due to the operation, the inner product changes from \(<,>_{g_{TS}}\) to \(<,>_{g_S}\).

Using the dependence of the adjoint operator upon the measure as mentioned above, the self-adjointization \(\eta_{sa}\) is realized as follows:

\[
\eta_{sa}(\psi_1^*) = \rho^{-1/4} \psi_1^*, \quad \eta_{sa}(\psi_2) = \rho^{-1/4} \psi_2, \quad \eta_{sa}(\Delta) = \rho^{-1/4} \Delta \rho^{1/4}, \quad (13)
\]

\[
<\psi_1, \psi_2>_{gs} = \int (\text{det } g_S)^{1/2} d^2 s d^2 q \psi_1^* \psi_2. \quad (14)
\]

In \(T_S\), \(\rho\) does not vanishes and \((13)\) give isomorphisms. Since in the measure of \((14)\), \(q\)-dependence disappears, \(p_q := \sqrt{-1} \partial_q\) itself becomes self-adjoint in \(\mathcal{H}'\) \([N4]\). Hence the projection,

\[
\pi_{p_q} : \tilde{\Omega}^* \times \tilde{\Omega} \to (\text{Ker}(Ad(p_q)))^* \times \text{Ker}(p_q), \quad (15)
\]

is consistent with the inner product \((, )\). Due to self-adjointness, \(\tilde{\varphi} : \text{Ker}(p_q) \to \text{Ker}(Ad(p_q))\) is isomorphic each other \([N5]\). In other words, \(\mathcal{H}_{p_q} := ((\text{Ker}(Ad(p_q))) \times \text{Ker}(p_q), (, ), \tilde{\varphi})\) becomes a small quantum mechanical system.

We should note that as a Goldstone mode is sometimes given as a zero mode of differential operator exhibiting a symmetry \([4]\), \(\psi \in \text{Ker}(p_q)\) behaves like a Goldstone mode for the normal translation mode. In fact, \(p_q\) mode does not contribute in \(\eta_{sa}(\Delta)\) over the small Hilbert space \(\mathcal{H}_{p_q}\). The projection to \(\mathcal{H}_{p_q}\) means vanishing momentum of the normal direction and kills a normal translation freedom. Hence
we can choose a position $q$ as an ordinary symmetry breaking [N6]. After choosing it as $q = 0$, the Laplacian in (6) becomes

$$\Delta_{S \to \mathbb{E}^n} := \eta_{sa}(\Delta)|_{\text{Ker} (p_q)|_{q=0}},$$

(16)
as an operator in $\mathcal{H}_{p_q}|_{q=0}$ [N7, N8]. By letting $K$ and $H$ denote the Gauss and mean curvatures of $S \subset \mathbb{E}^3$, we obtain the well-known operator [8, 18],

$$\Delta_{S \to \mathbb{E}^3} = \Delta_S + H^2 - K,$$

(17)
and the submanifold Schrödinger equation,

$$-\Delta_{S \to \mathbb{E}^3} \psi = E \psi \text{ over } S,$$

(18)
over the Hilbert space $\mathcal{H}_{p_q}|_{q=0}$. Here $\Delta_S$ is the Beltrami-Laplace operator on $S$ which exhibits the intrinsic properties of the surface $S$, whereas the second and the third terms in (17) represent the extrinsic properties of $S \subset \mathbb{E}^3$.

Here we emphasize that the definition (16) is very algebraic. Particularly in this construction, we did not use any approximation theories nor limit-theorems. Physically speaking, the above requirement of vanishing momentum and position might be contradict with the uncertainly principle. However the vanishing normal momentum naturally leads a symmetry breaking. Thus if we regard the normal direction as an inner space, we believe that the above requirement is natural. In fact our construction is consistent with that in [17], in which (18) was obtained by means of the Dirac constraint quantization scheme under the constraint condition of vanishing momentum.

Further our study reveals why we need deform the Hilbert space in the construction of the submanifold quantum mechanics [8, 18] [N9]. Hence we show the algebraic essential of the submanifold quantum mechanics.

By means of the construction, we can give a more algebraic representation of generalized Weierstrass relation in terms of the submanifold Dirac operators [24, 25, 26, 28], which is closely related to the extrinsic Polyakov string [27].

**Notes**

[N1] For example, in the theory of the ordinary second order differential equation related to orthonormal polynomial functions, the concept of the half-density implicitly appears (p.424 in [2]) but gives only very rough estimations, such as the asymptotic expressions of the functions. On the other hand, their expressions with the proper measure as the orthonormal polynomials
give more precise information. In other words, measureless expression is not sometimes effective to concrete problems well.

[N2] In general, the dual space of \( \Omega \) is greater than \( \Omega \) itself, thus we regard \( \Omega^* \) as the image of \( \varphi \). In this construction, the Dirac \( \delta \) functions are elements in the complement of the image of \( \varphi \) in the dual space.

[N3] We note that the canonical commutation relation has an ambiguity. For a function \( h(q) \) of \( q \), \( [h(q), q] = 0 \) and thus we have \( [\sqrt{-1} \partial_q + h(q), q] = \sqrt{-1} \). Using the ambiguity, we can optimize \( h \) so that \( \sqrt{-1} \partial_q + h(q) \) is self-adjoint. However in this article, we give another equivalent scheme called self-adjointization.

[N4] Here we should notice that the proper treatment of self-adjoint operator and a prototype of self-adjointization are implicitly written in the study of hydrogen atom in the text book of Dirac [9]. In the book, instead of \( pq \), the formally self-adjoint operator \( \eta^{-1}(pq) \) is treated. \( \eta^{-1}(pq) \) determines an addition term \( h(q) \) to \( \sqrt{-1} \partial_q \) [N3] so that it is self-adjoint. Corresponding to the property (4) of \( \eta_{sa} \), the transformation from \( \sqrt{-1} \partial_q \) to \( \eta^{-1}_{sa}(pq) \) is not unitary.

[N5] Precisely we should write the map \( \tilde{\varphi} \big|_{Ker (pq)} \) instead of \( \tilde{\varphi} \).

[N6] Precisely speaking, since \( \eta_{sa} (\Delta) \big|_{Ker (pq)} \) has \( q \)-dependence and the energy weakly depends upon \( q \), it slightly differs from the ordinary symmetry breaking. However as the dependence is not strong, it can be justified. If we choose \( q = q_0 \) for all points in \( S \), we have submanifold Schrödinger equation over a surface \( S_{q_0} \) given by \( q = q_0 \) instead of the surface \( S \) for \( q = 0 \).

[N7] The first restriction \( |_{Ker p_q} \) means the restriction of domain as an operator and the second one \( |_{q=0} \) should be regarded as an restriction in the meaning of presheaf theory.

[N8] Since (16) can be written by

\[
\Delta_{S\hookrightarrow \mathbb{R}^n} := (\rho^{1/4} \Delta \rho^{-1/4})|_{\partial_q = 0, q = 0},
\]

it might be expected that the definition should be expressed in differential ring theory [3].

[N9] If we did not deform the Hilbert space using \( \eta_{sa} \), \( \varpi_{pq} \) cannot becomes the projection operator in the sense of \(*\)-algebra [1].

References

[1] Araki H., Mathematical Theory of Quantum Fields, (International Series of Monographs on Physics, No. 101), Oxford Univ. Press, Oxford, 1999.

[2] Arfken G., mathematical methods for physicists, Academic Press, New York, 1970.
[3] Björk J-E., *Analytic D-Modules and Applications*, Kluwer, Dordrecht, 1992.

[4] Burgess C. P., *Goldstone and Psued-Goldstone Bosons in Nuclear, Particle and Condensed-Matter Physics*, Phys. Rep., **330** (2000) 193-261.

[5] Burgess M. and Jensen B., *Fermions near two-dimensional sufraces*, Phys. Rev. A, **48** (1993) 1861-1866.

[6] Clark I. J., *More on effective potentials of quantum strip waveguides*, J. Phys. A: Math. Gen., **31** (1998) 2103-2107.

[7] Clark I. J. and Bracken A. J., *Bound states in tubular quantum waveguides with torsion*, J. Phys. A: Math. Gen., **29** (1996) 4527-4535.

[8] da Costa R. C. T., *Quantum mechanics of a constrained particle*, Phys. Rev. A, **23** (1981) 1982-7.

[9] Dirac P. A. M., *The principles of Quantum Mechanics*, fourth edition, Oxford Univ. Press, Oxford, 1958.

[10] Duclos P., Exner P. and Štovíček P., *Curvature-induced resonances in a two-dimensional Dirichlet tube*, Ann. Inst. Henri Poincaré, **62** (1995) 81-101.

[11] Duclos P., Exner P. and Krejčířík D., *Bound states in curved quantum layers*, Commun. Math. Phys., **223** (1995) 13-28.

[12] Encinosa M. and Etemadi B., *Surface distortion effects on quantum dot helium*, Physica B, **266** (1999) 361-367.

[13] Exner P. and Šeba P., *Bound states in curved quantum waveguides*, J. Math. Phys., **30** (1989) 2574-2580.

[14] Goldstone J. and Jaffe R. L., *Bound states in twisting tubes*, Phys. Rev. B, **45** (1992) 14100-1407.

[15] Hörmander L., *The analysis of linear partial differential operators III*, Springer-Verlag, Berlin, 1985.

[16] Ikegami M. and Nagaoka Y., *Quantum Mechanics of Electron on a Curved Interface*, Prog. Theor. Phys. Suppl., **106** (1991) 235-248.

[17] Ikegami M., Nagaoka Y., Takagi S. and Tanzawa T., *Quantum Mechanics of a Particle on a Curved Surface - Comparision of Three Different Approaches*, Prog. Theor. Phys., **88** (1992) 229-249.

[18] Jensen H. and Koppe H., *Quantum Mechanics with Constraints*, Ann. Phys., **63** (1971) 586-591.

[19] Krejčířík D., *Quantum strips on surfaces*, J. Geom. Phys., **45** (2003) 203-217.

[20] Matsutani S., *The Physical meaning of the embedded effect in the quantum submanifold system*, J. Phys. A: Math. Gen., **26** (1993) 5133-5143.

[21] , *The Relation between the Modified Korteweg-de Vries Equation and Anomaly of Dirac Field on a Thin Elastic Rod*, Prog. Theor. Phys., **105** (1994) 1005-1037.

[22] , *On the physical relation between the Dirac equation and the generalized mKdV equation on a thin elastic rod*, Phys. Lett. A, **189** (1994) 27-31.

[23] , *Anomaly on a Submanifold System: New Index Theorem related to a Submanifold System*, J. Phys. A: Math. Gen., **28** (1995) 1399-1412.

[24] , *Constant Mean Curvature Surface and Dirac Operator*, J. Phys. A: Math. Gen., **30** (1997) 4019-4029.
[25] _______, Immersion Anomaly of Dirac Operator on Surface in $\mathbb{R}^3$, Rev. Math. Phys., 11 (1999) 171-186.
[26] _______, Dirac Operator of a Conformal Surface Immersed in $\mathbb{R}^4$: Further Generalized Weierstrass Relation, Rev. Math. Phys., 12 (2000) 431-444.
[27] _______, On Density of State of Quantized Willmore Surface: A Way to a Quantized Extrinsic String in $\mathbb{R}^3$, J. Phys. A: Math. Gen., 31 (1998) 3595-3606.
[28] _______, Generalized Weierstrass Relation for a Submanifold $S^k$ in $\mathbb{R}^n$ Coming from Submanifold Dirac Operator, to appear in Adv. Stud. Pure Math., (2003).
[29] Mladenov I. M., Quantization on Curved Surfaces, Int. J. Quan. Chem., 89 (2002) 248-254.
[30] Matsutani S. and Tsuru H., Reflectionless Quantum Wire, J. Phys. Soc. Jpn., 60 (1991) 3640-3644.
[31] _______ Physical relation between quantum mechanics and soliton on a thin elastic rod, Phys. Rev. A, 46 (1992) 1144-1147.
[32] Suzuki A. and Matsutani S., Confinement of a Particle in a Ring with Finite Potential, Nuovo Cimento, 111 B (1996) 593-606.
[33] Takagi S. and Tanzawa T., Quantum mechanics of a particle confined to a twisted ring, Prog. Theor. Phys., 87 (1992) 561-568.