TENSOR AND CONVOLUTION DIRECT IMAGE OF ℓ-ADIC SHEAVES

ANTONIO ROJAS-LEÓN

Abstract. Given a Galois étale map of varieties \( \pi : Y \to X \) and an ℓ-adic sheaf or derived category object \( P \in D^b_c(Y, \bar{Q}_\ell) \), we study two cohomological operations: the tensor direct image and the convolution direct image, which give objects of \( D^b_c(X, \bar{Q}_\ell) \), and can be used to improve the estimates on some partial exponential sums and the number of rational points on certain varieties.

1. Introduction

Let \( k \) be a finite field of cardinality \( q = p^n \), and \( X \) a commutative group variety defined over \( k \). Fix a prime \( \ell \neq p \), and let \( k_r \) be the degree \( r \) extension of \( k \) in a fixed algebraic closure \( \bar{k} \). Given an ℓ-adic constructible sheaf \( F \) on \( X \) (or, more generally, an object \( P \) in the derived category \( D^b_c(X, \bar{Q}_\ell) \)), one can consider “trace and norm Frobenius sums” of the following form, for \( a \in X(k) \):

\[
S(P, a) = \sum_{y \in X(k_r) : N_{X/k}(y) = x} N(F_y | P_{\bar{y}})
\]

where \( F_y \) is a geometric Frobenius element of \( X \) at \( y \) and \( P_{\bar{y}} \) is the stalk of \( P \) at a geometric point over \( y \). These sums were studied in [RL12], where we developed a cohomological construction (the convolution Adams operation) that, given an object \( P \), produces another object whose (regular) Frobenius traces are precisely the sums \( S(P, a) \). This was used, for example, to give sharp estimates for the number of rational points on certain Artin-Schreier and superelliptic curves.

Now let \( Y = X \times_{\text{Spec } k} \text{Spec } k_r \) be the extension of scalars, and suppose that \( P \) is only defined on \( Y \) (but not on \( X \)). Then the sums \( S(P, a) \) still make sense, so one could ask whether they are given by the Frobenius traces of some sheaf or derived category object on \( X \). The main goal of this article is to construct such an object, which will be called the convolution direct image of \( P \).

Given a finite index subgroup \( H \) of a group \( G \) and a representation \( \rho \) of \( H \) on some finitely generated free module, the tensor induction of \( \rho \) is a representation of \( G \) derived from it. Roughly speaking, it is constructed in a similar way to the induced representation, but replacing the direct sum with the tensor product. See [CR81, Section 13] for details. Using the fact that the category of ℓ-adic lisse sheaves on \( X \) (respectively on \( Y \)) is equivalent to the category of continuous ℓ-adic representations of its fundamental group \( \pi_1(X, \bar{x}) \) with base point a given geometric point \( \bar{x} \) of \( X \) (resp. representations of \( \pi_1(Y, \bar{y}) \)) and, for a finite étale map \( \pi : X \to Y \) such that \( \pi(\bar{y}) = \bar{x} \) the group \( \pi_1(Y, \bar{y}) \) is a finite index subgroup of \( \pi_1(X, \bar{x}) \), this was used by N. Katz in [Kat90, 10.3-6] to define the tensor direct image of a lisse sheaf on \( Y \). In [FW14, Proposition 2.1], L. Fu and D. Wan showed that, if \( Y = X \times \text{Spec } k_r \) and \( a \in X(k) \), the trace of a geometric Frobenius element \( F \) at \( a \) acting on the stalk at \( a \)

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of the tensor direct image of a lisse sheaf $\mathcal{F}$ is precisely the trace of $F^r$ acting on the stalk of $\mathcal{F}$ at $a$. This was used to give estimates for some partial exponential sums. In this article we start by generalizing this construction to arbitrary constructible sheaves and derived category objects, so the number of Frobenius trace sums to which this result can be applied is greatly increased.

In section 2 we start by recalling the definition and main properties of the tensor induction of representations, and we give an alternative characterization when $H$ is a normal subgroup of $G$. This characterization can be generalized to the category of constructible sheaves, and this is what we do in section 3. Given a finite Galois étale cover $\pi : Y \to X$, we define the tensor direct image $\pi_{\otimes*}\mathcal{F}$ of a constructible étale sheaf $\mathcal{F}$ on $Y$ with respect to $\pi$, and show that it generalizes the construction given in [Kat90] for locally constant objects. We also show that, in the case $Y = X \times \text{Spec} k_r$, its Frobenius traces are given by the same formula that was proved in [FW14] for lisse sheaves (Proposition 7):

$$\text{Tr}(F_x|\pi_{\otimes*}\mathcal{F})_x = \text{Tr}(F^r_x|\mathcal{F}_x).$$

In section 4 we extend this construction to the derived category of étale sheaves on $Y$, and show that essentially the same properties hold for these objects. In section 5 we work on commutative group schemes, and work out the same construction but replacing the tensor product operation with the convolution in the derived category of étale sheaves. This gives rise to a new operation: the convolution direct image of an object $P \in D^b_c(Y, \mathbb{Q}_l)$, which is an object $\pi_{cv*}P \in D^b_c(X, \mathbb{Q}_l)$. The main result is Proposition 12 in this case $Y = X \times \text{Spec} k_r$, the Frobenius traces of $\pi_{cv*}P$ on a point $x \in X(k)$ are given by

$$\text{Tr}(F_x|\pi_{cv*}P_x) = \sum_{y \in X(k_r)|\text{Tr}(y) = x} \text{Tr}(F^r_y|P_y).$$

Finally, in the last section we briefly describe how this result can be applied to estimate exponential sums defined by trace constraints and the number of rational points on some types of curves, and we will compare these estimates with the ones obtained in [RL12] using convolution Adams power.

2. Tensor Induction of Representations

Let us start by recalling the tensor induction operation for group representations [CRS1 Section 13]. Let $G$ be a group, $H \subseteq G$ a subgroup of finite index, $A$ a ring and $\rho : H \to \text{GL}(M)$ a representation of $H$ on the automorphism group of a finitely generated free $A$-module $M$. Fix a set $Hg_1, \ldots, Hg_d$ of representative for the right cosets of $H$ in $G$. Given an element $g \in G$, let $g = h \cdot g_\tau(i)$ with $h \in H$. Then $\tau$ is a permutation of the set $\{1, \ldots, d\}$. The tensor induction of $\rho$ maps $g$ to the element of $\text{GL}(\otimes^d A)$ given by

$$m_1 \otimes \cdots \otimes m_d \mapsto \rho(h_1)m_{\tau(1)} \otimes \cdots \otimes \rho(h_d)m_{\tau(d)}.$$  

It is a representation $\otimes \to \text{Ind}(\rho)$ of $G$ whose isomorphism class is independent of the choice of $g_1, \ldots, g_d$. If $G$ is profinite and $\rho$ is continuous, so is $\otimes \to \text{Ind}(\rho)$.

Let $X$ be a connected scheme and $\pi : Y \to X$ a finite étale map of degree $d$. In [Kat90 10.5], the tensor induction of representations is used to construct the tensor direct image of a lisse $\ell$-adic sheaf $\mathcal{F}$ on $Y$: choosing geometric points $y$ of $Y$ and $\bar{x} = \pi(y)$ of $X$, and viewing $\mathcal{F}$ as a finite dimensional $\ell$-adic representation of $\pi_1(Y, \bar{y})$, which is an open subgroup of $\pi_1(X, \bar{x})$ of index $d$, the tensor direct image of $\mathcal{F}$ is the lisse sheaf $\pi_{\otimes*}\mathcal{F}$ on $X$ corresponding to the representation $\otimes \to \text{Ind}(\mathcal{F})$ of $\pi_1(X, \bar{x})$. It is well defined up to isomorphism.

A particularly important case is when $X$ is a variety over a finite field $k$ of characteristic $p \neq \ell$, $k_d$ is a finite extension of $k$ of degree $d$, and $Y = X \times \text{Spec}(k_r)$
is the extension of scalars of $X$ to $k_n$. In that case, if $\mathcal{F}$ is a lisse sheaf on $Y$, $t \in X(k)$ is a $k$-valued point of $X$ and $F_t \in \pi_1(X)$ denotes a geometric Frobenius element of $X$ at $t$, we have $\text{FW13}$ Proposition 2.1

$$\text{Tr}(F_t(\pi_{\otimes}(\mathcal{F}_t)) = \text{Tr}(F_{t'}(\mathcal{F}_t)).$$

This is exploited in $\text{FW13}$ to give estimates for some partial character sums.

Suppose that $H$ is normal in $G$. We will give a characterization of the tensor induction of a representation $\rho : H \to \text{GL}(M)$ of $H$ that will be useful later to motivate the definition of the tensor direct image of a constructible sheaf. For every $g \in G$, let $g^*\rho : H \to \text{GL}(M)$ be the representation given by

$$(g^*\rho)(h)(m) = \rho(ghg^{-1})(m).$$

It is clear that $(gg')^*\rho = g^*g^*\rho$.

If $\phi : M \to M'$ is a homomorphism between the $H$-representations $\rho : H \to \text{GL}(M)$ and $\rho' : H \to \text{GL}(M')$, then it is also a homomorphism between $g^*\rho$ and $g^*\rho'$, that we will denote by $g^*\phi$. So $\rho \mapsto g^*\rho$ defines an auto-equivalence of the category of representations of $H$ (over finitely generated free $A$-modules). If $g \in H$, then $\rho(g)$ is a morphism between $\rho$ and $g^*\rho$. More generally, if $g^g g^{-1} \in H$, then $\rho(g^g g^{-1})$ is an isomorphism between $g^*\rho$ and $g^*\rho$.

For every $\sigma \in G/H$, pick a representative $\tilde{\sigma} \in \sigma$, and let $h_{\sigma,\tau} = \tilde{\sigma}\tilde{\tau}\tilde{\sigma}^{-1} \in H$ for every $\sigma, \tau \in G/H$. We define the following category $\mathcal{C}_{G,H}$: the objects of $\mathcal{C}_{G,H}$ are representations $\rho : H \to \text{GL}(M)$ together with isomorphisms $\Psi_\sigma : \rho \to \tilde{\sigma}^*\rho$ for each $\sigma \in G/H$ satisfying the following cocycle condition for every $\sigma, \tau \in G/H$:

$$(1) \quad (\tilde{\tau}^*\Psi_\sigma) \circ \Psi_\tau = \rho(h_{\sigma,\tau}) \circ \Psi_{\sigma\tau} : \rho \xrightarrow{\sim} \tilde{\tau}^*\tilde{\sigma}^*\rho$$

A morphism in $\mathcal{C}_{G,H}$ is a homomorphism of representations $\Phi : \rho \to \rho'$ such that

$$\Psi_{\rho'} \circ \Phi = (\tilde{\sigma}^*\Phi) \circ \Psi_\sigma : \rho \to \tilde{\sigma}^*\rho'$$

for every $\sigma \in G/H$.

Given a representation $\pi : G \to \text{GL}(M)$, the restriction $\rho = \pi_{|H}$ of $\pi$ to $H$ together with the isomorphisms $\Psi_\sigma : \rho \to \tilde{\sigma}^*\rho$ given by $\Psi_\sigma(m) = \pi(\tilde{\sigma})(m)$ is an object of $\mathcal{C}_{G,H}$, and every homomorphism of representations of $G \to \pi' \in \mathcal{C}_{G,H}$ induces a morphism in $\mathcal{C}_{G,H}$ between their restrictions. This defines a (covariant) functor $F$ from the category $\text{Rep}_A(G)$ of representations of $G$ to $\mathcal{C}_{G,H}$.

**Proposition 1.** The functor $F$ is an equivalence of categories.

**Proof.** $F$ is clearly flat: a morphism between two representations is zero if and only if it is zero when restricted to a given subgroup. Let us check that it is faithful: let $\Phi : \pi_{|H} \to \pi'_{|H}$ be a homomorphism in $\mathcal{C}_{G,H}$, we need to show that it is also a homomorphism of representations between $\pi$ and $\pi'$.

Let $g \in G$, and write $g = h\tilde{h}$, where $\sigma$ is the class of $g$ in $G/H$ and $h \in H$. Then

$$\Phi \circ \pi(g) = (\tilde{\sigma}^*\Phi) \circ \pi(g) = (\tilde{\sigma}^*\Phi) \circ \pi(\tilde{h}) = (\tilde{\sigma}^*\Phi) \circ \pi(h) =$$

$$= \Psi_{\rho'} \circ \Phi \circ \pi(h) = \Psi_{\rho'} \circ \pi'(h) \circ \Phi = \pi'(\tilde{h}^*\phi) \circ \Phi.$$

Finally, let us check that $F$ is essentially surjective. Given an object $(\rho, \{\Psi_\sigma\})$ of $\mathcal{C}_{G,H}$ and $g \in G$, write $g = h\tilde{h}$ as above, and let $\pi(g) : M \to M$ be the automorphism given by $\pi(g) = \rho(h) \circ \Psi_\sigma$. This defines a representation of $G$: if $g' = h'\tilde{h}'$ with $h' \in H$, then $gg' = h\tilde{h}'\tilde{h}^{-1}(\tilde{h}^*\tilde{h})^{-1} = h\tilde{h}'\tilde{h}^{-1}h_{\sigma,\sigma'}\sigma\sigma'$, so

$$\pi(gg') = \rho(h\tilde{h}'\tilde{h}^{-1}h_{\sigma,\sigma'}) \circ \Psi_{\sigma\sigma'} = \rho(h) \circ \rho(h\tilde{h}'\tilde{h}^{-1}) \circ \rho(h_{\sigma,\sigma'}) \circ \Psi_{\sigma\sigma'} =$$

$$= \rho(h) \circ \rho(h\tilde{h}'\tilde{h}^{-1}) \circ (\tilde{\sigma}^*\Psi_\sigma) \circ \Psi_{\sigma'} = \rho(h) \circ \Psi_\sigma \circ \rho(h') \circ \Psi_{\sigma'} = \pi(g) \circ \pi(g')$$
for every $\sigma, \tau$ with the isomorphisms $\Psi_{\sigma \tau}$ the natural transformation $(\Phi_{\sigma \tau})^* F \to \tau^* \sigma^* F$, given that $\Psi_{\sigma}$ is an isomorphism between $\rho$ and $\tilde{\sigma}^* \rho$.

This gives an object of $\text{Rep}_{\mathbb{A}}(G)$ whose associated object of $\mathcal{C}_{G,H}$ is $(\rho, \{\Psi_{\sigma}\})$. □

Using this equivalence of categories, we can characterize the tensor induction of $\rho : H \to \text{GL}(M)$ as the representation of $G$ corresponding to the object of $\mathcal{C}_{G,H}$ given by the representation

$$\mathcal{M} := \bigotimes_{\tau \in G/H} \tilde{\tau}^* M$$

of $H$ and the isomorphisms

$$\Psi_{\sigma} : \mathcal{M} \to \tilde{\sigma}^* \mathcal{M} = \bigotimes_{\tau \in G/H} (\tilde{\tau}^* \sigma)^* M = \bigotimes_{\tau \in G/H} (h_{\tau,\sigma} \tilde{\sigma})^* M = \bigotimes_{\tau \in G/H} \tilde{\sigma}^* h_{\tau,\sigma}^* M$$

given by

$$\bigotimes_{\tau \in G/H} m_{\tau} \mapsto \bigotimes_{\tau \in G/H} \rho(h_{\tau,\sigma})(m_{\tau}).$$

An easy but tedious computation shows that this is indeed an object of $\mathcal{C}_{G,H}$.

3. Tensor direct image of sheaves

In this section we will give a geometric definition of the tensor direct image of a $\ell$-adic sheaf which extends the one given in [Kat90, 10.5] for lisse sheaves to arbitrary constructible sheaves and even to the derived category.

Fix a prime $\ell$, let $E_{\ell}$ be a finite extension of $\mathbb{Q}_{\ell}$, $\mathfrak{o}_{\ell}$ its ring of integers, with maximal ideal $\mathfrak{m}_{\ell}$, and let $A = \mathfrak{o}_{\ell}/\mathfrak{m}_{\ell}^n$ for some $n \geq 1$. Let $\pi : Y \to X$ be a Galois finite étale morphism of schemes on which $\ell$ is invertible, with Galois group $G = \text{Aut}(Y/X)$. By [Mil80] 1.5.4], this means that the morphism

$$\prod_{\sigma \in G} Y_{\sigma} \to Y \times_X Y$$

given by $y \in Y_{\sigma} \cong Y \mapsto (y, \sigma y)$ is an isomorphism.

Following the construction given in the previous section for group representations, let $\mathcal{C}_{X,Y}$ be the following category: the objects in $\mathcal{C}_{X,Y}$ consist of a constructible étale sheaf of free $A$-modules $F$ on $Y$, together with isomorphisms $\Psi_{\sigma} : F \to \sigma^* F$ for every $\sigma \in G$ satisfying the cocycle condition

$$(\tau^* \Psi_{\sigma}) \circ \Phi_{\sigma \tau} = \Phi_{\sigma \tau \sigma} \circ \Psi_{\sigma \tau} : F \to \tau^* \sigma^* F$$

every $\sigma, \tau \in G$, where $\Phi_{\sigma \tau} : (\sigma \tau)^* F \to \tau^* \sigma^* F$ is the isomorphism induced by the natural transformation $(\sigma \tau)^* \to \tau^* \sigma^*$ (note that this implies that $\Psi_{1_{G}} = \text{Id}_F$).

A morphism between two objects $(F, \{\Psi_{\sigma}\}_{\sigma \in G})$ and $(F', \{\Psi'_{\sigma}\}_{\sigma \in G})$ is a morphism $\rho : F \to F'$ such that $\Psi'_{\sigma} \circ \rho = (\sigma \rho)^* \circ \Psi_{\sigma}$ for every $\sigma \in G$.

For every étale sheaf of free $A$-modules $G$ on $X$, the sheaf $F := \pi^* G$ together with the isomorphisms $\Psi_{\sigma} : F \to \sigma^* F$ induced by the natural transformations $\pi^* = (\pi \sigma)^* \to \sigma^* \pi^*$ is an object of $\mathcal{C}_{X,Y}$, and for every morphism $\rho : G \to G'$ of sheaves on $X$ the restriction $\pi^* \rho : \pi^* G \to \pi^* G'$ is clearly a morphism in $\mathcal{C}$. This defines a functor $F$ from the category of étale sheaves of free $A$-modules on $X$ to $\mathcal{C}_{X,Y}$. The analogue to Proposition 1 is the following

**Proposition 2.** $F$ is an equivalence of categories.

**Proof.** Consider the diagram

$$Y \times_X Y \times_X Y \xrightarrow{p_{12} \circ \overline{p_{13}}} Y \times_X Y \xrightarrow{p_{12}} Y \xrightarrow{\pi} X$$
By descent theory (see e.g. [AD+11] Lemma 7.25.5), the pull-back $F \to \pi^* F$ is an equivalence of categories between the category of étale sheaves of free $A$-modules on $X$ and the category of descent data of étale sheaves of free $A$-modules for the étale cover $\pi : Y \to X$ (that is, an étale sheaf $\mathcal{F}$ of free $A$-modules on $Y$ endowed with an isomorphism $\Psi : p_1^* \mathcal{F} \to p_2^* \mathcal{F}$ such that $p_1^* \Psi = (p_2^* \Psi) \circ (p_1^* \Psi)$ modulo the natural transformations $p_{13}^* p_1^* \cong p_{12}^* p_1^*$ and $p_{13}^* p_2^* \cong p_{23}^* p_2^*$).

Remember that there is a natural isomorphism

$$\prod_{\sigma \in G} Y_\sigma \xrightarrow{\cong} Y \times X$$

where $Y_\sigma \cong Y$, given by $y \in Y_\sigma \mapsto (y, \sigma y)$. Under this isomorphism, $p_1^* \mathcal{F}$ and $p_2^* \mathcal{F}$ correspond to $\mathcal{F}$ and $\pi^* \mathcal{F}$ on $Y_\sigma$, so giving an isomorphism $p_1^* \mathcal{F} \to p_2^* \mathcal{F}$ is equivalent to giving an isomorphism $\mathcal{F} \to \pi^* \mathcal{F}$ for every $\sigma \in G$. So every object $(\mathcal{F}, \{\Psi_\sigma\})$ of $\mathcal{C}_{X,Y}$ determines an isomorphism $\Psi : p_1^* \mathcal{F} \to p_2^* \mathcal{F}$.

Let us check that this isomorphism satisfies the cocycle condition. Notice that

$$(Y \times X Y) \times X Y \cong \left( \prod_{\sigma, \tau \in G} Y_\sigma \times X Y \right) \cong \prod_{\sigma, \tau \in G} (Y_\sigma \times X Y) \cong \prod_{\sigma, \tau \in G} Y_{(\sigma, \tau)}$$

where, under this isomorphism, $y \in Y_{(\sigma, \tau)}$ corresponds to $(y, \sigma y, \tau y) \in Y \times X Y \times X Y$. In particular, $p_{12}, p_{13}$, and $p_{23}$, viewed as maps $\prod_{\sigma, \tau \in G} Y_{(\sigma, \tau)} \to \prod_{\sigma \in G} Y_\sigma$, take $y \in Y_{(\sigma, \tau)}$ to $y \in Y_\sigma, y \in Y_\tau$, and $\sigma y \in Y_{\sigma^{-1}}$, respectively, and the natural transformations $p_{13}^* p_1^* \cong p_{12}^* p_1^*$ and $p_{13}^* p_2^* \cong p_{23}^* p_2^*$ correspond to the identity and the natural transformation $\tau^* \cong \pi^* (\tau \sigma^{-1})^*$ respectively. Then (after a “change of variables” $(\sigma, \tau) \mapsto (\tau, \sigma\tau)$):

$$(p_{12}^* \Psi)_{Y_{(\tau, \sigma\tau)}} = \Psi_\tau : \mathcal{F} \to \tau^* \mathcal{F},$$

$$(p_{13}^* \Psi)_{Y_{(\tau, \sigma\tau)}} = \Psi_{\sigma\tau} : \mathcal{F} \to (\tau \sigma)^* \mathcal{F}$$

and

$$(p_{23}^* \Psi)_{Y_{(\tau, \sigma\tau)}} = \tau^* \Psi_\sigma : \mathcal{F} \to \tau^* \sigma^* \mathcal{F}$$

so

$$(p_{23}^* \phi) \circ (p_{12}^* \phi),$$

in the connected component $Y_{(\tau, \sigma\tau)}$, translates to $\Phi_{\sigma\tau} \circ \Psi_{\sigma\tau} = (\tau^* \Psi_\sigma) \circ \Psi_\tau$, which holds for all objects of $\mathcal{C}_{X,Y}$ by definition.

This defines an equivalence of categories between $\mathcal{C}_{X,Y}$ and the category of descent data for $\pi : Y \to X$, which is equivalent via $\pi^*$ to the category of sheaves on $X$.

It is clear that, under this correspondence, locally constant objects of $\mathcal{C}_{X,Y}$ (i.e. objects whose sheaf is locally constant) correspond to locally constant sheaves on $X$. By passing to the limit, this equivalence of categories also applies to sheaves with coefficients on $\mathcal{O}_X$ and on $E_\lambda$.

The equivalence of categories is compatible with pull-backs in the following sense: Let $g : Z \to X$ be a morphism of schemes and $\tilde{\pi} : Z \times_X Y \to Z$ the base change of $\pi$ with respect to $g$. Then $\tilde{\pi}$ is also a Galois étale morphism with Galois group $G$, with the elements of $G$ acting on $Z \times_X Y$ via the second factor. Let $(\mathcal{F}, \{\Psi_\sigma\})_{\sigma \in G}$ be an object of $\mathcal{C}_{X,Y}$. Taking pull-backs with respect the projection map $\tilde{g} : Z \times_X Y \to Y$ we get a sheaf $\tilde{g}^* \mathcal{F}$ on $Z \times_X Y$ and isomorphisms $\tilde{g}^* \Psi_\sigma : \tilde{g}^* \mathcal{F} \to \tilde{g}^* \pi^* \mathcal{F}$ for every $\sigma \in G$. By composition with the natural transformations $\tilde{g}^* \sigma^* \cong (\sigma \circ \tilde{g})^* = (\tilde{g} \circ (1 \times \sigma))^* \cong (1 \times \sigma)^* (\tilde{g}^*)$ we get isomorphisms $\tilde{g}^* \Psi_\sigma : \tilde{g}^* \mathcal{F} \to (1 \times \sigma)^* (\tilde{g}^*) \mathcal{F}$ for every $\sigma \in G$.

**Proposition 3.** $(\tilde{g}^* \mathcal{F}, \{\tilde{\Psi}_\sigma\})_{\sigma \in G}$ is an object of $\mathcal{C}_{Z,Z \times_X Y}$. If $(\mathcal{F}, \{\Psi_\sigma\})$ corresponds to the sheaf $\mathcal{G}$ on $X$, $(\tilde{g}^* \mathcal{F}, \{\tilde{\Psi}_\sigma\})$ corresponds to $g^* \mathcal{G}$ on $Z$.

**Proof.** Given the equivalence of categories above, we can assume that $(\mathcal{F}, \{\Psi_\sigma\})$ is associated to a sheaf $\mathcal{G}$ on $X$, that is, $\mathcal{F} = \pi^* \mathcal{G}$ and $\Psi_\sigma : \mathcal{F} \to \pi^* \mathcal{F}$ is given by the natural transformation $\pi^* = (\pi \sigma)^* \cong \sigma^* \pi^*$ for every $\sigma \in G$. Then its pull-back
Proposition 4. If \( \	ilde{\tau} \) to the representation \( \tau_\sigma \) isomorphisms induced by the natural transformations \( \sigma, \pi, \rho \) be the natural permutation isomorphism transformation \( G \) constant

Proposition 5. Definition 1. The corresponding to this object of \( C \) that \( (\sigma \tau)^* \rightarrow \tau^* \sigma^* \) in the category of representations of \( \pi_1(Y,\bar{y}) \) given, for every \( \rho \), by the isomorphism of representations \( \tilde{\tau}^* \rho \rightarrow \tilde{\tau}^* \sigma^* \sigma \rho \) defined by \( m \mapsto \rho(h_{\sigma,\tau}(m)). \)

Suppose now that \( Y \) (and therefore \( X \)) is connected. Pick a geometric point \( \tilde{y} \mapsto Y \), and let \( \tilde{x} = f(\tilde{y}) \mapsto X \). Let \( \pi_1(Y,\tilde{y}) \) and \( \pi_1(X,\tilde{x}) \) be the fundamental groups of \( Y \) and \( X \) with base points \( \tilde{y} \) and \( \tilde{x} \) respectively. We have an exact sequence

\[ 1 \rightarrow \pi_1(Y,\tilde{y}) \rightarrow \pi_1(X,\tilde{x}) \rightarrow G \rightarrow 1. \]

There is an equivalence between the categories of locally constant sheaves of free A-modules on \( X \) (respectively \( Y \)) and of continuous representations of \( \pi_1(X,\tilde{x}) \) (resp. \( \pi_1(Y,\tilde{y}) \)) defined on A-modules \( C(X,Y) \) [Mii80, V.1.2].

For every \( \sigma \in G \), pick an element \( \tilde{\sigma} \in \pi_1(X,\tilde{x}) \) that lifts \( \sigma \) and, for \( \sigma, \tau \in G \), denote by \( h_{\sigma,\tau} \) the element \( \bar{\tau} \cdot \tilde{\tau} \cdot \bar{\tau}^{-1} \in \pi_1(Y,\tilde{y}) \). If \( \tilde{\sigma} \cdot \tilde{\tau} \cdot \bar{\tau}^{-1} \in \pi_1(Y,\tilde{y}) \), then \( \sigma \tau \) corresponds to the permutation \( \rho \) of \( Y \), then \( \sigma \tau \) corresponds to \( \tilde{\sigma} \tilde{\tau} \) by \( \tilde{\sigma} \tilde{\tau} (h) = \rho(h \tilde{\sigma}^{-1}) \). The natural transformation \( (\sigma \tau)^* \rightarrow \tau^* \sigma^* \) corresponds to the natural transformation \( \tilde{\sigma} \tilde{\tau}^* \rightarrow \tilde{\tau}^* \tilde{\sigma}^* \) in the category of representations of \( \pi_1(Y,\tilde{y}) \) given, for every \( \rho \), by the isomorphism of representations \( \sigma \tilde{\tau}^* \rho \rightarrow \tilde{\tau}^* \sigma \tilde{\tau} \rho \) defined by \( m \mapsto \rho(h_{\sigma,\tau}(m)) \).

Let \( \{ F, \Psi_\sigma \}_{\sigma \in G} \) be an object of \( C(X,Y) \) with \( F \) locally constant, \( \rho \) the representation of \( \pi_1(Y,\tilde{y}) \) corresponding to \( F \), and \( \tilde{\Psi}_\sigma : \rho \rightarrow \tilde{\sigma} \rho \) the isomorphism corresponding to \( \Psi_\sigma \). Then the paragraph above and the cocycle equation \([\ref{eqn:3}]\) imply that \( \{ \rho, \tilde{\Psi}_\sigma \}_{\sigma \in G} \) is an object of \( C(X,Y) \), \( \pi_1(X,\tilde{x}) \). If \( F = \pi^* G \) for a locally constant \( G \) on \( X \), then \( \rho \) is the restriction to \( \pi_1(Y,\tilde{y}) \) of the representation \( \tilde{\rho} \) of \( \pi_1(X,\tilde{x}) \) associated to \( G \), and the natural isomorphism \( F \rightarrow \sigma^* F \) translates to the isomorphism of representations \( \tilde{\rho} \rightarrow \tilde{\tau}^* \tilde{\rho} \) given by \( m \mapsto \tilde{\rho}(h_{\sigma,\tau}(m)) \).

Proposition 4. If \( \{ F, \Psi_\sigma \}_{\sigma \in G} \) is an object of \( C(X,Y) \) corresponding to the locally constant sheaf \( G \) on \( Y \), then the object \( \{ \rho, \tilde{\Psi}_\sigma \}_{\sigma \in G} \) of \( C(X,Y) \) corresponds to the representation \( \tilde{\rho} \) of \( \pi_1(X,\tilde{x}) \) defined by \( G \).

Now let \( F \) be a constructible sheaf of free A-modules on \( Y \), where \( A \) is either \( E_\lambda, o_\lambda \) or \( o_\lambda/m_\lambda^n \) for some \( n \geq 1 \). Take

\[ \mathcal{G} = \bigotimes_{\tau \in G} \tau^* F \]

and, for every \( \sigma \in G \), let

\[ \Psi_\sigma : \mathcal{G} = \bigotimes_{\tau \in G} \tau^* F \rightarrow \bigotimes_{\tau \in G} (\tau \sigma)^* F \cong \bigotimes_{\tau \in G} \sigma^* \tau^* F \cong \sigma^* \mathcal{G} \]

be the natural permutation isomorphism \( \otimes \tau m_{\tau} \mapsto \otimes \tau m_{\tau \sigma} \), composed with the isomorphisms induced by the natural transformations \( (\tau \sigma)^* \rightarrow \tau^* \sigma^* \). These isomorphisms satisfy the cocycle condition, so they define an object of \( C(X,Y) \), which corresponds to a constructible sheaf of A-modules on \( X \).

Definition 1. The tensor direct image of \( F \) by \( \pi \) is the sheaf \( \pi_\sigma^* F \) on \( X \) corresponding to this object of \( C(X,Y) \).

We now enumerate some of the basic properties of this construction.

Proposition 5. Let \( F \) and \( F' \) be constructible sheaves on \( Y \).
(1) \(\pi^!(\pi_{0*}F) \cong \bigotimes_{\tau \in G} \tau^*F\)
(2) \(\pi_{0*}(F \otimes F') \cong (\pi_{0*}F) \otimes (\pi_{0*}F')\)
(3) (base change) Let \(\bar{g} : \bar{Z} \to \bar{X}\) be a morphism of schemes, and

\[
\begin{array}{ccc}
Z \times_X Y & \xrightarrow{\bar{g}} & Y \\
\downarrow \pi & & \downarrow \pi \\
Z & \xrightarrow{g} & X
\end{array}
\]

the pull-back. Then \(g^!(\pi_{0*}F) \cong \pi_{0*}\bar{g}^*F\).

**Proof.** The first property is clear by construction. For the second one, note that there is a natural isomorphism

\[
\bigotimes_{\tau \in G} \tau^*(F \otimes F') \cong \bigotimes_{\tau \in G} (\tau^*F) \otimes (\tau^*F') \cong (\bigotimes_{\tau \in G} \tau^*F) \otimes (\bigotimes_{\tau \in G} \tau^*F')
\]

which commutes with the isomorphisms \(\Psi_{\sigma}\).

The third property is a particular case of Proposition 6 via the natural isomorphism

\[
\bar{g}^* \left( \bigotimes_{\tau \in G} \tau^*F \right) \cong \bigotimes_{\tau \in G} \bar{g}^*\tau^*F \cong \bigotimes_{\tau \in G} (1 \times \tau)^*\bar{g}^*F
\]

The next result shows that this operation does in fact generalize the tensor direct image of locally constant sheaves defined in [Kat90, 10.5] to arbitrary (constructible) sheaves.

**Proposition 6.** Suppose that \(Y\) is connected, and let \(F\) be a locally constant sheaf on \(Y\), corresponding to the representation \(\rho\) of \(\pi_1(Y, y)\). Then \(\pi_{0*}F\) is a locally constant sheaf on \(Y\) corresponding to the representation \(\otimes = \text{Ind}(\rho)\) of \(\pi_1(X, \bar{x})\).

**Proof.** By Proposition 6 and the construction of \(\pi_{0*}F\), the representation associated to \(\pi_{0*}F\) is the representation corresponding to the object of \(\mathcal{C}_{\pi_1(X, \bar{x}), \pi_1(Y, y)}\) given by \(\bigotimes_{\tau \in G} \tilde{\tau}^*\rho\), with the isomorphisms

\[
\tilde{\Psi}_{\sigma} : \bigotimes_{\tau \in G} \tilde{\tau}^*\rho \to \bigotimes_{\tau \in G} \bar{g}^*\tilde{\tau}^*\rho
\]

being the composition of the permutation map \(\bigotimes_{\tau \in G} \tilde{\tau}^*\rho \to \bigotimes_{\tau \in G} \tilde{\sigma}^*\rho\) and the tensor product of the natural isomorphisms \(\tau\tilde{\sigma}^*\rho \to \tilde{\sigma}^*\tilde{\tau}^*\rho\) given by \(m \mapsto \rho(h_{\tau, \sigma})(m)\). But this is precisely the tensor induction of \(\rho\), according to the characterization given in the previous section.

We will be mainly interested in the following situation: \(X\) is an algebraic variety over the finite field \(k\), \(k_r\) is a finite extension of \(k\) of degree \(r\) and \(Y = X \times \text{Spec} k_r\) is the extension of scalars of \(X\). Then the projection \(\pi : Y \to X\) induced by the inclusion \(k \hookrightarrow k_r\) is a Galois étale map, whose Galois group is cyclic and generated by the Frobenius automorphism \(\sigma\) of \(\text{Spec} k_r/\text{Spec} k\). Here

\[
\mathcal{G} = F \otimes \sigma^*F \otimes \cdots \otimes \sigma^{(r-1)*}F
\]

and

\[
\Psi_{\sigma} : \mathcal{G} \to \sigma^*\mathcal{G} = \sigma^*F \otimes \sigma^{2*}F \otimes \cdots \otimes \sigma^{r*}F = \sigma^*F \otimes \sigma^{2*}F \otimes \cdots \otimes F
\]

is given by \(m_0 \otimes m_1 \otimes \cdots \otimes m_{r-1} \mapsto m_1 \otimes m_2 \otimes \cdots \otimes m_0\).

Let \(x : \text{Spec} k \to X\) be a \(k\)-valued point of \(X\), \(x_r : \text{Spec} k_r \to X\) the corresponding \(k_r\)-valued point and \(\bar{x} : \text{Spec} k \to X\) a geometric point over \(x\). Let \(F_{\bar{x}} \in \pi_1(X, \bar{x})\) be a geometric Frobenius element at \(x\), then \(F_{\bar{x}}^{(r)}\) is in \(\pi_1(Y, \bar{x})\).
(since the quotient is cyclic of degree \( r \)) and is a geometric Frobenius element of \( Y \) at \( x_r \). The following result generalizes [FW14, Proposition 2.1]

**Proposition 7.** *With the previous notation, we have*

\[
\text{Tr}(F_x|_{\pi_{G,ctf}}) = \text{Tr}(F_x^r|_{F_x,ctf})
\]

**Proof.** We have a cartesian diagram

\[
\begin{array}{ccc}
\text{Spec } k_r & \xrightarrow{x_r} & Y \\
\downarrow \bar{\pi} & & \downarrow \pi \\
\text{Spec } k & \xrightarrow{x} & X
\end{array}
\]

where \( \bar{\pi} : \text{Spec } k_r \to \text{Spec } k \) is the canonical projection.

By Proposition [3], we get an isomorphism of sheaves on Spec \( k \) (i.e. of \( \text{Gal}(\bar{k}/k) \)-modules) \( (\pi_{G,ctf})_x \cong \bar{\pi}_{G,F_x} \). Since every sheaf on Spec \( k \) is locally constant (and therefore corresponds to a representation of \( \pi_1(\text{Spec } k, \text{Spec } k) = \text{Gal}(\bar{k}/k) \)), as a representation of \( \text{Gal}(\bar{k}/k) \), the tensor induction of the representation \( F_{x_r} \) of \( \text{Gal}(\bar{k}/k_r) \). By [FW14, Proposition 2.1] we conclude that \( \text{Tr}(F_x|_{\pi_{G,ctf}}) = \text{Tr}(F_x^r|_{F_x,ctf}) \). \( \square \)

4. **Tensor direct image in the derived category**

In this section we keep the same geometric setup as in the previous one: \( \pi : Y \to X \) is a Galois finite étale morphism with Galois group \( G \), defined over a base ring good enough for the derived category or étale sheaves to be well behaved (e.g. a field).

Let \( \ell \) be a prime invertible in \( X, E, \phi, m \), and \( A = \phi_n, m_n^A \) for some \( n \geq 1 \). Let \( D_X = D^\alpha(X, A) \) (respectively \( D_Y = D^\alpha(Y, A) \)) be the derived category of \( ctf \)-complexes of sheaves of \( A \)-modules on \( X \) (resp. on \( Y \)), that is, the full subcategory of the derived category consisting of objects which are quasi-isomorphic to a bounded complex of \( A \)-flat sheaves [KW01, II.5]. Let \( \mathcal{D}_{X,Y} \) be the following category: the objects of \( \mathcal{D}_{X,Y} \) consist of an object \( P \in \mathcal{D}_Y \) together with isomorphisms \( \Psi_\sigma : P \to \sigma^*P \) for every \( \sigma \in G \) satisfying the cocycle condition

\[
\tau^*\psi_\sigma \circ \psi_\tau = \psi_{\sigma\tau} : P \to \tau^*\sigma^*P
\]

for every \( \sigma, \tau \in G \), where \( \psi_{\sigma,\tau} : (\sigma\tau)^*P \to \tau^*\sigma^*P \) is the isomorphism defined by the natural transformation \((\sigma\tau)^* \to \tau^*\sigma^* \). A morphism between two objects \((P, \{\Psi_\sigma\}_{\sigma \in G}) \) and \((P', \{\Psi'_\sigma\}_{\sigma \in G}) \) is a morphism \( \rho : P \to P' \) such that \( \psi_\rho = \rho = (\sigma^*\rho) \circ \psi_\sigma \) for every \( \sigma \in G \).

**Proposition 8.** *The category \( \mathcal{D}_{X,Y} \) is equivalent to the full subcategory of \( ctf \)-complexes in the derived category of \( \mathcal{C}_{X,Y} \).*

**Proof.** It is clear that giving a \( ctf \) complex of objects in \( \mathcal{C}_{X,Y} \) is equivalent to giving a \( ctf \) complex \( P \) of sheaves on \( X \) together with isomorphisms \( \Psi_\sigma : P \to \sigma^*P \) satisfying the cocycle condition. So the category of \( ctf \) objects in the derived category of \( \mathcal{C}_{X,Y} \) can be identified with the latter category quotiented by the quasi-isomorphisms. Let us call it \( \mathcal{D}'_{X,Y} \).

Every object of \( \mathcal{D}'_{X,Y} \) represents an object in \( \mathcal{D}_{X,Y} \) in the obvious way, and a morphism of complexes that commutes with the isomorphisms \( \Psi_\sigma \) induces a morphism in the derived category \( \mathcal{D}_{X,Y} \), which is an isomorphism if the original morphism was a quasi-isomorphism. Therefore, the obvious functor from the category of \( ctf \) complexes of sheaves on \( X \) together with isomorphisms satisfying the cocycle condition to \( \mathcal{D}_{X,Y} \) factors through the quotient category \( \mathcal{D}'_{X,Y} \) and induces a functor \( F : \mathcal{D}'_{X,Y} \to \mathcal{D}_{X,Y} \).
To check that $F$ is an equivalence of categories, let us first see that every object of $\mathcal{D}'_{X,Y}$ and of $\mathcal{D}_{X,Y}$ is isomorphic to an object whose complex is injective (i.e. which consists of injective sheaves). Let $(P^\bullet, \{\Psi_\sigma\}_{\sigma \in G})$ be a $\mathcal{F}$ object in $\mathcal{D}_{X,Y}$, and let $\theta : I^\bullet \to P^\bullet$ be an injective resolution of $P^\bullet$ (which is an isomorphism in the derived category). Then $\sigma^\ast \theta : \sigma^\ast P^\bullet \to \sigma^\ast I^\bullet$ is an injective resolution of $\sigma^\ast I^\bullet$ for every $\sigma \in G$. Let $\tilde{\Psi}_\sigma = (\sigma^\ast \theta) \circ \Psi_\sigma \circ \theta^{-1} : I^\bullet \to \sigma^\ast I^\bullet$. Then, for every $\sigma, \tau \in G$,

$$(\tau^\ast \tilde{\Psi}_\sigma) \circ \tilde{\Psi}_\tau = \tau^\ast ((\sigma^\ast \theta) \circ \Psi_\sigma \circ \theta^{-1}) \circ ((\tau^\ast \theta) \circ \Psi_\tau \circ \theta^{-1}) =$$

$$= \tau^\ast \sigma^\ast \theta \circ \tau^\ast \Psi_\sigma \circ \tau^\ast \theta^{-1} \circ \tau^\ast \theta \circ \Psi_\tau \circ \theta^{-1} = \tau^\ast \sigma^\ast \theta \circ \tau^\ast \Psi_\sigma \circ \Psi_\tau \circ \theta^{-1} =$$

$$= \tau^\ast \sigma^\ast \theta \circ \Psi_{\sigma, \tau} \circ \Psi_\sigma \circ \theta^{-1} = \Phi_{\sigma, \tau} \circ \sigma^\ast \theta \circ \Psi_{\sigma, \tau} \circ \theta^{-1} = \tilde{\Phi}_{\sigma, \tau} \circ \tilde{\Psi}_{\sigma, \tau}$$

where $\tilde{\Phi}_{\sigma, \tau} : (\sigma \tau)^\ast I^\bullet \to \tau^\ast \sigma^\ast I^\bullet$ is the isomorphism defined by the natural transformation $(\sigma \tau)^\ast \to \tau^\ast \sigma^\ast$, so $\tau^\ast \sigma^\ast \theta \circ \Phi_{\sigma, \tau} = \tilde{\Phi}_{\sigma, \tau} \circ (\sigma \tau)^\ast \theta$ by naturality.

Therefore $(I^\bullet, \{\tilde{\Psi}_\sigma\}_{\sigma \in G})$ is an object of $\mathcal{D}_{X,Y}$ isomorphic to $(P^\bullet, \{\Psi_\sigma\}_{\sigma \in G})$ (via $\theta$). The proof for $\mathcal{D}'_{X,Y}$ is similar.

In particular, to show that $F$ is faithfully flat, it suffices to prove the equality $\text{Hom}_{\mathcal{D}_{X,Y}}(P^\bullet, Q^\bullet) = \text{Hom}_{\mathcal{D}'_{X,Y}}((P^\bullet, Q^\bullet)$ for injective objects $P^\bullet$ and $Q^\bullet$ in $\mathcal{D}_{X,Y}$. But, in that case, any morphism between them in $\mathcal{D}_{X,Y}$ and in $\mathcal{D}'_{X,Y}$ is an actual map of complexes [KW01, II.1.8] which commutes with the isomorphisms $\Psi_\sigma$ for every $\sigma \in G$, so the statement is clear.

Finally, $F$ is essentially surjective: as we have just seen, every object $(P^\bullet, \{\Psi_\sigma\}_{\sigma \in G})$ in $\mathcal{D}_{X,Y}$ is isomorphic to an injective object $(I^\bullet, \{\tilde{\Psi}_\sigma\}_{\sigma \in G})$, and, since $\sigma^\ast I^\bullet$ is injective for every $\sigma \in G$, the morphisms $\Psi_\sigma$ are actual maps of complexes [KW01, II.1.8]. So $(I^\bullet, \{\tilde{\Psi}_\sigma\}_{\sigma \in G})$ is an object in $\mathcal{D}'_{X,Y}$ whose image by $F$ is isomorphic to $(P^\bullet, \{\Psi_\sigma\}_{\sigma \in G})$.

Since, by Proposition 1, the category $\mathcal{C}_{X,Y}$ is equivalent to the category of constructible sheaves of free $A$-modules on $X$, we conclude

**Corollary 1.** The pull-back $\pi^\ast$ induces an equivalence of categories between the derived category $D_b(X, A)$ and the category $\mathcal{D}_{X,Y}$.

By taking projective limits, the same statement is true if we replace $A$ with the full ring of integers $\mathfrak{a}_\lambda$ of $E_\lambda$ and, by tensoring with $\mathbb{Q}$, with $E_\lambda$ itself. This allows us to define the tensor direct image of derived category objects in the same way we did for sheaves

**Definition 2.** Let $P \in D_b(Y, A)$ (where $A = E_\lambda$, $\mathfrak{a}_\lambda$ or $\mathfrak{a}_{\lambda,n}$ for some $n \geq 1$), the tensor direct image of $P$ is the object $\pi^\ast P \in D_b(X, A)$ corresponding, under the previous equivalence of categories, to the object

$$\bigotimes_{\tau \in G} \tau^\ast P \in D_b(X, A)$$

and the natural commutation isomorphisms

$$\Psi_\sigma : \bigotimes_{\tau \in G} \tau^\ast P \to \bigotimes_{\tau \in G} (\tau^\ast \sigma^\ast) P \cong \bigotimes_{\tau \in G} \sigma^\ast \tau^\ast P \cong \sigma^\ast \bigotimes_{\tau \in G} \tau^\ast P.$$

If $P = \mathcal{F}[0]$ for a single sheaf $\mathcal{F}$, then $\pi^\ast \mathcal{F} = (\pi^\ast \mathcal{F})[0]$ by construction. The basic properties stated in Proposition 3 are still valid for derived category objects, with a similar proof.

Let us now focus on the setup where $X$ is an algebraic variety over the finite field $k$, $k_0$ is a finite extension of $k$ of degree $r$ and $\pi : Y = X \times \text{Spec } k_0 \to X$ is the projection. Let $F_\pi \in \pi_1(X, \bar{x})$ be a geometric Frobenius element at $x$, then $F^\pi_\pi \in \pi_1(Y, \bar{x})$ is a geometric Frobenius element of $Y$ at $\bar{x}$. Given an element
\( P^* \in D^b_{ctf}(X, A) \), there is a well-defined value of \( \text{Tr}(F_x|P^*) \), given as the alternating sum \( \sum_i (-1)^i \text{Tr}(F_x|P^*_i) \) if \( P^* \) is a \( ct_f \) complex (which is well defined since it is invariant under quasi-isomorphism). Then we have the following result similar to Proposition\(^7\)

**Proposition 9.** For any \( P^* \in D^b(Y, A) \) we have

\[
\text{Tr}(F_x|\pi_{\otimes x}P^*) = \text{Tr}(F_x^*|P^*)
\]

**Proof.** By the usual passage to the limit and torsion killing process, it suffices to prove it when \( A = \mathbb{A}_{mn} \). By base change, we may assume that \( X = \text{Spec} \ k \) and \( Y = \text{Spec} \ k_r \). In that case we can view \( P^* \) as a bounded complex of representations of \( \text{Gal}(\bar{k}/k) \) on flat (or, equivalently, free, see [KW01, 5.1]) \( A \)-modules. Since \( \{F_x^1\}_{i=0}^{r-1} \) is a set of representatives of the cosets of \( \text{Gal}(k/k_r) \) in \( \text{Gal}(k/k) \), \( \pi_{\otimes x}P^* \), as a representation of \( \text{Gal}(k/k_r) \), is given by

\[
Q^* = P^* \otimes F_x^*P^* \otimes \cdots \otimes F_x^{(r-1)*}P^*
\]

with the isomorphism

\[
\Psi_{F_x} : Q^* = P^* \otimes F_x^*P^* \otimes \cdots \otimes F_x^{(r-1)*}P^* \rightarrow F_x^*Q^* = F_x^*P^* \otimes F_x^{(r-2)*}P^* \otimes \cdots \otimes F_x^{r*}P^*
\]

given by the commutation map \( P^* \otimes F_x^*P^* \otimes \cdots \otimes F_x^{(r-1)*}P^* \rightarrow F_x^*P^* \otimes F_x^{(r-2)*}P^* \otimes \cdots \otimes F_x^{r*}P^* \) followed by the isomorphism induced by the action of \( F_x^* : P^* \rightarrow F_x^{r*}P^* \).

Let \( m \in \mathbb{Z} \); then, as \( A \)-modules,

\[
Q^m = \bigoplus_{i_0 + \cdots + i_{r-1} = m} P^{i_0} \otimes P^{i_1} \otimes \cdots \otimes P^{i_{r-1}}
\]

with \( F_x \) acting on it by

\[
m_0 \otimes m_1 \otimes \cdots \otimes m_{r-1} \in P^{i_0} \otimes P^{i_1} \otimes \cdots \otimes P^{i_{r-1}} \mapsto (-1)^{i_0 \sum (i_1 + \cdots + i_{r-1})} m_1 \otimes m_2 \otimes \cdots \otimes m_{r-1} \otimes F_x^r(m_0) \in P^{i_0} \otimes P^{i_2} \otimes \cdots \otimes P^{i_{r-1}}.
\]

In particular, if we fix bases \( B_j \) of \( P^{i_j} \) and \( m_j \in B_j \) for every \( j = 0, \ldots, r-1 \), then in the expression of \( F_x \cdot (m_0 \otimes m_1 \otimes \cdots \otimes m_{r-1}) \) with respect to the tensor product of the \( B_j \)’s, the element \( m_0 \otimes m_1 \otimes \cdots \otimes m_{r-1} \) appears with non-zero coefficient if and only if \( m_j = m_{j+1} \) for all \( j = 0, \ldots, r-2 \) (so, in particular, \( i_j = i_{j+1} \) and \( m = i_0r \)) and \( m_0 \) appears with non-zero coefficient in the expansion of \( F_x^r \cdot m_0 \) with respect to \( B_0 \). Additionally, in that case, these two non-zero coefficients concide up to multiplication by \( (-1)^{i_0}(-1)^{i_1} = (-1)^{i_0+i_1} \). Putting all this together, we get:

\[
\text{Tr}(F_x|Q^m) = \begin{cases} 0 & \text{if } r \nmid m \\ (-1)^{i_0+i_1} \text{Tr}(F_x^r|P^i) & \text{if } m = i_0r \\ \end{cases}
\]

so

\[
\text{Tr}(F_x|\pi_{\otimes x}P^*) = \sum_m (-1)^m \text{Tr}(F_x|Q^m) = \sum_i (-1)^i (-1)^{i(r-1)} \text{Tr}(F_x^r|P^i) = \sum_i (-1)^i \text{Tr}(F_x^r|P^i) = \text{Tr}(F_x^r|P^*)
\]

\( \square \)

5. Convolution direct image

In this section we will assume that \( Y \) and \( X \) are commutative group schemes over a base field \( k \), and \( \pi : Y \rightarrow X \) a finite Galois étale map as before. Given a coefficient ring \( A \) as in the previous section, we have an exact convolution bi-functor

\[-* - \text{ from } D^b_c(Y, A) \times D^b_c(Y, A) \text{ to } D^b_c(Y, A) \]

given by

\[
K \ast L = R\mu_0(K \boxtimes L)
\]
where $\mu : Y \times_k Y \to Y$ is the multiplication map. See [Kat90, 8.1] for an exhaustive account of its basic properties. For our purposes, the most important ones are the existence of natural commutativity

$$K \ast_l L \to L \ast_l K$$

and associativity

$$(K \ast_l L) \ast_l M \to K \ast_l (L \ast_l M)$$

isomorphisms from which one can construct natural multi-commutativity isomorphisms

$$K_1 \ast_l K_2 \ast_l \cdots \ast_l K_r \to K_{\sigma(1)} \ast_l K_{\sigma(2)} \ast_l \cdots \ast_l K_{\sigma(r)}$$

for every $\sigma \in \mathfrak{S}_r$.

Let $P \in D^b_c(Y,A)$. Given the commutative and associative properties of the convolution, there is a well defined (up to natural isomorphism) object

$$Q = \bigoplus_{\tau \in \mathcal{G}} \tau^* P \in D^b_c(Y,A)$$

with natural isomorphisms $Q \to \sigma^* Q$ for every $\sigma \in \mathcal{G}$

$$\Psi_\sigma : Q = \bigoplus_{\tau \in \mathcal{G}} \tau^* P \to \bigoplus_{\tau \in \mathcal{G}} (\tau \sigma)^* P \cong \bigoplus_{\tau \in \mathcal{G}} \sigma^* \tau^* P \cong \sigma^* Q \in D^b_c(Y,A)$$

given by the natural commutation isomorphism and the natural transformation $(\tau \sigma)^* \to \sigma^* \tau^*$, which satisfy the cocycle condition.

**Definition 3.** Let $P \in D^b_c(Y,A)$ (where $A = E_\lambda$, $o_\lambda$ or $o_\lambda,n$ for some $n \geq 1$), the convolution direct image of $P$ is the object $\pi_{cv} P \in D^b_c(X,A)$ corresponding, under the equivalence of categories given in Proposition 8, to the object

$$\bigoplus_{\tau \in \mathcal{G}} \tau^* P \in D^b_c(Y,A)$$

and the isomorphisms $\Psi_\sigma$.

If $k \hookrightarrow L$ is a finite Galois extension of fields and $X$ is a commutative group scheme over Spec $k$, then $Y := X \otimes_k L$ is a finite Galois étale cover of $X$ via the natural map $\pi : Y \to X$ induced by the projection $\pi : \text{Spec } L \to \text{Spec } k$. Given an object $P \in D^b_c(Y,A)$, its cohomology with compact supports $R\Gamma_c(\bar{Y},P)$ with its $\text{Gal}(k^{sep}/L)$-action (where $\bar{Y} = Y \otimes_k k^{sep}$) can be viewed as an object on $D^b_c(\text{Spec } L,A)$, and similarly for objects in $D^b_c(X,A)$.

**Proposition 10.** With the previous notation, for every $P \in D^b_c(Y,A)$ we have an isomorphism in $D^b_c(\text{Spec } k,A)$

$$R\Gamma_c(\bar{X},\pi_{cv} P) \cong \varpi_{\otimes_L} R\Gamma_c(\bar{Y},P)$$

*Proof.* We will check that the corresponding objects on $D_{\text{Spec } k,\text{Spec } L}$ are isomorphic. The object corresponding to $\varpi_{\otimes_L} R\Gamma_c(\bar{Y},P)$ is, by definition,

$$Q := \bigotimes_{\tau \in \text{Gal}(L/k)} \tau^* R\Gamma_c(\bar{Y},P)$$

with the isomorphisms $\Psi_\sigma : Q \to \sigma^* Q$ induced by the commutation isomorphisms of the tensor product. On the left hand side, the object is

$$\varpi^* R\Gamma_c(\bar{X},\pi_{cv} P)$$

with the natural isomorphisms

$$\tilde{\Psi}_\sigma : \varpi^* R\Gamma_c(\bar{X},\pi_{cv} P) \to \sigma^* \varpi^* R\Gamma_c(\bar{X},\pi_{cv} P) \cong \varpi^* R\Gamma_c(\bar{X},\pi_{cv} P).$$

By proper base change, there are natural isomorphisms

$$\varpi^* R\Gamma_c(\bar{X},\pi_{cv} P) \cong R\Gamma_c(\bar{Y},\pi^* \pi_{cv} P) = R\Gamma_c(\bar{Y}, \bigotimes_{\tau \in \mathcal{G}} \tau^* P)$$
and, by the Künneth formula \[\text{[Kat90] 8.1.8}],
\[
\Gamma_c(\bar{Y}, \oplus_{\tau \in G} \tau^* P) \cong \bigotimes_{\tau \in G} \Gamma_c(\bar{Y}, \tau^* P) \cong \bigotimes_{\tau \in G} \tau^* \Gamma_c(\bar{Y}, P)
\]
where this isomorphism is compatible with the action of \(G\) on both the convolution and the tensor product via permutation of the factors. Similarly, for every \(\sigma \in G\) we have
\[
\sigma^* \tau^* \Gamma_c(\bar{X}, \pi_{\text{cv}} P) \cong \bigotimes_{\tau \in G} \sigma^* \tau^* \Gamma_c(\bar{Y}, P)
\]
and, by naturality, the diagrams
\[
\begin{align*}
\xymatrix{
\tau^* \Gamma_c(\bar{X}, \pi_{\text{cv}} P) \ar[r]_{\Psi_{\sigma}} & \bigotimes_{\tau \in G} \tau^* \Gamma_c(\bar{Y}, P) \\
\sigma^* \tau^* \Gamma_c(\bar{X}, \pi_{\text{cv}} P) \ar[r]_{\Psi_{\sigma}} ^{\cong} & \bigotimes_{\tau \in G} \sigma^* \tau^* \Gamma_c(\bar{Y}, P)
}
\end{align*}
\]
commute. In other words, the objects \((Q, \{\Psi_{\sigma}\})\) and \((\tau^* \Gamma_c(\bar{X}, \pi_{\text{cv}} P), \{\Psi_{\sigma}\})\) in \(\mathcal{DC}_{\text{Spec } k, \text{Spec } l}\) are isomorphic.

Suppose now that \(Y\) and \(X\) are commutative group varieties over a finite field \(k\), let \(\chi : X(k) \to A^\times\) be a character and \(\chi_{\pi} : Y(k) \to A^\times\) its pull-back character on \(Y\). Let \(\mathcal{L}_X\) and \(\mathcal{L}_{\chi_{\pi}}\) the associated rank 1 locally constant sheaves on \(X\) and \(Y\) respectively (cf. \[\text{[Del77] 1.4-1.8}]).

**Proposition 11.** For every \(P \in D^b(Y, A)\), there is an isomorphism of objects on \(D^b(X, A)\):
\[
\pi_{\text{cv}}(P \otimes \mathcal{L}_X) \cong (\pi_{\text{cv}} P) \otimes \mathcal{L}_X
\]

**Proof.** We will prove the isomorphism by showing that the corresponding objects on \(\mathcal{DC}(X, Y)\) are isomorphic. The object corresponding to \(\pi_{\text{cv}}(P \otimes \mathcal{L}_{\chi_{\pi}})\) is
\[
\bigoplus_{\tau \in G} \tau^* (P \otimes \mathcal{L}_{\chi_{\pi}})
\]
with the natural isomorphisms \(\Psi_{\sigma}\). On the other hand, to object corresponding to \((\pi_{\text{cv}} P) \otimes \mathcal{L}_X\) is
\[
\pi^*(\pi_{\text{cv}} P) \otimes \mathcal{L}_X = (\pi^* \pi_{\text{cv}} P) \otimes \mathcal{L}_{\chi_{\pi}} = (\bigoplus_{\tau \in G} \tau^* P) \otimes \mathcal{L}_{\chi_{\pi}}
\]
with the isomorphisms
\[
\tilde{\Psi}_{\sigma} : (\bigoplus_{\tau \in G} \tau^* P) \otimes \mathcal{L}_{\chi_{\pi}} \to \sigma^*(\bigoplus_{\tau \in G} \tau^* P) \otimes \sigma^* \mathcal{L}_{\chi_{\pi}}
\]
induced by the commutation isomorphism \(\bigoplus_{\tau \in G} \tau^* P \to \sigma^*(\bigoplus_{\tau \in G} \tau^* P)\) and the natural isomorphism \(\mathcal{L}_{\chi_{\pi}} \to \sigma^* \mathcal{L}_{\chi_{\pi}}\). Since tensoring with \(\mathcal{L}_{\chi_{\pi}}\) is an auto-equivalence of the tensor category \(D^b(Y, A)\) (with the convolution operation) by \[\text{[Kat90] 8.1.10}\] and the commutation isomorphisms are natural, we conclude that the diagrams
\[
\begin{align*}
\xymatrix{
\pi^*(\pi_{\text{cv}} P) \otimes \mathcal{L}_X \ar[r]_{\tilde{\Psi}_{\sigma}} ^{\cong} & (\pi_{\text{cv}} P) \otimes \mathcal{L}_X \\
\pi^*(\pi_{\text{cv}} P) \otimes \mathcal{L}_X \ar[r]_{\tilde{\Psi}_{\sigma}} ^{\cong} & (\pi_{\text{cv}} P) \otimes \mathcal{L}_X
}
\end{align*}
\]
are commutative and, in particular, both objects are isomorphic in \(\mathcal{DC}(X, Y)\). \(\square\)

Now let \(Y = X \times \text{Spec } k_{\ell}\) and let \(A\) be a finite extension of \(\mathbb{Q}_{\ell}\). For every \(y \in X(k_{\ell}) = Y(k_{\ell})\), let \(N(y) \in X(k)\) be its norm, defined as \(\prod_{\sigma \in G} \sigma(y)\) (which is a point in \(X(k_{\ell})\) invariant under \(G\), and therefore in \(X(k)\))
Proposition 12. Under the previous hypothesis, let $P \in D^b_{\mathbb{C}}(Y, A)$ and let $x \in X(k)$. Then

$$
\text{Tr}(F_x | \pi_{\text{cv}*} P_x) = \sum_{y \in X(k_v)} \text{Tr}(F_y | P_y)
$$

Proof. For every $x \in X(k)$, let $\alpha(x) = \text{Tr}(F_x | \pi_{\text{cv}*} P)$ and $\beta(x) = \sum_{y \in X(k_v)} N(y) \text{Tr}(F_y | P)$. View $\alpha$ and $\beta$ as $A$-valued functions on the finite abelian group $X(k)$. Then $\alpha$ and $\beta$ are equal if and only if their discrete Fourier transforms $\hat{\alpha}$ and $\hat{\beta}$ are. For every character $\chi : X(k) \to \mathbb{C}^\times$ we have

$$
\hat{\alpha}(\chi) = \sum_{x \in X(k)} \chi(x) \text{Tr}(F_x | \pi_{\text{cv}*} P_x) = \sum_{x \in X(k)} \text{Tr}(F_x | (\pi_{\text{cv}*} P) \otimes \mathcal{L}_\chi)
$$

and

$$
\hat{\beta}(\chi) = \sum_{x \in X(k)} \chi(x) \text{Tr}(F_y | P_y) = \sum_{y \in X(k_v)} \chi(N_{k_v}/k(y)) \text{Tr}(F_y, P) = \sum_{y \in X(k_v)} \text{Tr}(F_y, P \otimes \mathcal{L}_\chi)
$$

If $F$ denotes the geometric Frobenius element of $\text{Gal}(\bar{k}/k)$ then, by the trace formula, we have

$$
\hat{\alpha}(\chi) = \text{Tr}(F | R\Gamma_c(X, (\pi_{\text{cv}*} P) \otimes \mathcal{L}_\chi))
$$

and

$$
\hat{\beta}(\chi) = \text{Tr}(F | R\Gamma_c(Y, P \otimes \mathcal{L}_\chi))
$$

and, by propositions 9, 10 and 11

$$
\hat{\alpha}(\chi) = \text{Tr}(F | R\Gamma_c(X, (\pi_{\text{cv}*} P) \otimes \mathcal{L}_\chi) = \text{Tr}(F | R\Gamma_c(X, \pi_{\text{cv}*} (P \otimes \mathcal{L}_\chi))) = \text{Tr}(F | \pi_{\otimes} R\Gamma_c(Y, P \otimes \mathcal{L}_\chi)))) = \text{Tr}(F | R\Gamma_c(Y, P \otimes \mathcal{L}_\chi))) = \hat{\beta}(\chi).
$$

\hfill \Box

6. Applications and examples

In this section we will describe how the convolution direct image construction can be applied to obtain improved estimates for the number of rational points on some varieties over finite fields or for certain exponential sums. We will give two important examples to illustrate this: Artin-Schreier curves and superelliptic curves. These examples were worked out directly in [RL10], here we will show how they can be obtained as a simple application of our convolution direct image construction.

In order to use this construction for obtaining estimates for the number of rational points on varieties or for partial exponential sums, we need a way to control the weight of the convolution direct image of an object. Suppose again that $X$ is a commutative group variety over a finite field $k$ and $Y = X \times \text{Spec} \ k_v$. We will fix an embedding $i : \mathbb{Q}_l \to \mathbb{C}$ so we can freely speak about weights of $\ell$-adic objects (by which we will mean $\ell$-weights for the given embedding $i$). Let $q = p^n$ denote the cardinality of $q$.

Proposition 13. Let $P \in D^b_{\mathbb{C}}(Y, \mathbb{Q}_l)$ be an object mixed of weights $\leq w$. Then $\pi_{\text{cv}*} P \in D^b_{\mathbb{C}}(X, \mathbb{Q}_l)$ is mixed of weights $\leq rw$.

Proof. Since being mixed is invariant under finite extension of the base field, it suffices to prove it for $\pi^* \pi_{\text{cv}*} P$. But

$$
\pi^* \pi_{\text{cv}*} P \cong P \ast_1 (F^* P) \ast_1 \cdots (F^{(r-1)*} P)
$$

so this is a consequence of [BBD82] 5.1.14. \hfill \Box
Corollary 2. Suppose that $P$ is mixed of weights $\leq w$. Let $x \in X(k)$, and suppose that $\mathcal{H}^i(P \ast_1 (F^*P) \ast_2 \cdots \ast_{i-1} (F^{(r-1)}*P))x = 0$ for $i > n$. Then

$$\sum_{y \in X(k) \setminus X(y) = x} \text{Tr}(F_y | P_y) \leq C \cdot q^{(rw+n)/2}$$

where $C = \sum_i \dim \mathcal{H}^i(P \ast_1 (F^*P) \ast_2 \cdots \ast_{i-1} (F^{(r-1)}*P))x$.

Proof. By Proposition 12 we have

$$\sum_{y \in X(k) \setminus X(y) = x} \text{Tr}(F_y | P_y) = |\text{Tr}(F_2 | (\pi_{cv*}P)x)\rangle = \sum_i (-1)^i \text{Tr}(F_2 | (\mathcal{H}^i(\pi_{cv*}P))x) \leq$$

$$\leq \sum_i \dim \mathcal{H}^i(\pi_{cv*}P)x \cdot q^{(rw+n)/2} \leq \left( \sum_i \dim \mathcal{H}^i(\pi_{cv*}P)x \right) q^{(rw+n)/2} = C \cdot q^{(rw+n)/2}$$

since $\pi_{cv*}P$ is mixed of weights $\leq rw$, so $\mathcal{H}^i(\pi_{cv*}P)$ is mixed of weights $\leq rw + i$ for every $i$.

\[ \square \]

Let us now specialize to the case $X = A^1_k$.

Proposition 14. Let $f \in k[x]$ be a polynomial of degree $d$ prime to $p$, and let $F$ be the $\breve{\mathbb{Q}}_L$-sheaf on $A^1_k$, given by the kernel of the trace morphism $f_! \breve{\mathbb{Q}}_L \to \breve{\mathbb{Q}}_L$. Let $S \subseteq k$ be the set of critical values of $f$, and suppose that $a \notin S + F(S) + \cdots + F^{r-1}(S)$ (where $F$ is the Frobenius automorphism $x \mapsto x^d$ of $k$). Then

$$\sum_{y \in k \setminus X(y) = a} \text{Tr}(F_y | F_y) \leq (d - 1)^r \cdot q^{(r-1)/2}.$$  

Proof. Let $P = F^{|x|} \in D^b(Y, \breve{\mathbb{Q}}_L)$. Then $P$ is a perverse sheaf [Kat98 2.3.1], which is lisse on $A^1_k \setminus S$ and tamely ramified at infinity (and, in fact, its monodromy at infinity is the direct sum of all non-trivial characters with trivial $d$-th power). By Laumon’s local Fourier transform theory [Lau97 2.4], its Fourier transform (with respect to a fixed additive character of $k$) is of the form $\mathcal{G}[1]$, where $\mathcal{G}$ is lisse on $\mathbb{G}_m$ of rank $d - 1$, tamely ramified at zero and whose monodromy at infinity is a direct sum of representations of the form $\mathcal{L}^\psi_{\psi(\tau)} \otimes K_t$ for $t \in S$, where $\mathcal{L}^\psi_{\psi(\tau)}$ is an Artin-Schreier character and $K_t$ have slopes $< 1$.

Similarly, for every integer $i$ the Fourier transform $G_i$ of $F^rP$ is lisse on $\mathbb{G}_m$ of rank $d - 1$, tamely ramified at zero, and its monodromy at infinity is a direct sum of representations of the form $\mathcal{L}^\psi_{\psi((t_0 + t_1 + \cdots + t_{r-1})x)} \otimes K_{t_0 \cdots t_{r-1}-1}$. Since Fourier transform interchanges convolution and tensor product (up to a shift), the Fourier transform of $P \ast_1 (F^rP) \ast_2 \cdots \ast_{i-1} (F^{(r-1)}*P)$ is $G_0 \otimes G_1 \otimes \cdots \otimes G_{r-1}^1[1]$.

This tensor product is lisse on $\mathbb{G}_m$ of rank $(d - 1)^r$, tamely ramified at zero, and its monodromy at infinity is a direct sum of representations of the form $\mathcal{K} \otimes \mathcal{L}^\psi_{\psi((t_0 + t_1 + \cdots + t_{r-1})x)}$, where $\mathcal{K}$ has slopes $< 1$. In particular, its tensor product with $\mathcal{L}^\psi_{\psi(-a x)}$ is totally wild at infinity, since $a \notin S + F(S) + \cdots + F^{r-1}(S)$. Again by local Fourier transform theory, this implies that $P \ast_1 (F^rP) \ast_2 \cdots \ast_{i-1} (F^{(r-1)}*P)$ is concentrated in degree $-1$ and unramified at $a$, and its rank there is the dimension of $H^1_1(A^1_k, G_0 \otimes G_1 \otimes \cdots \otimes G_{r-1} \otimes \mathcal{L}^\psi_{\psi(ax)})$, which by the Ogg-Shafarevic formula is $(d - 1)^r$ (its Swan conductor at infinity).
We conclude that $\mathcal{H}(P \star (F^* P) \star \cdots \star (F^{(r-1)*P})) = 0$ for $i > -1$. Since $F$ is pure of weight 0, $P$ is mixed of weights $\leq 1$. Then $\pi_{e_*} P$ is mixed of weights $\leq r$ by Proposition 13. Applying corollary 2, we get

$$\sum_{y \in k, \Tr(y) = a} \Tr(F_y | P_y) = \sum_{y \in k, \Tr(y) = a} \Tr(F_y | F_y) \leq (d-1)^r \cdot q^{(r-1)/2}$$

□

Applying this to $a = 0$ and using the fact that the number of solutions in $k_\ell$ to the equation $y^q - y = t$ is $q$ if $\Tr_{k_\ell/k}(t) = 0$ and 0 otherwise, we get

**Corollary 3.** Under the previous hypotheses, the number $N$ of $k_\ell$-valued points on the Artin-Schreier curve $y^q - y = f(x)$ is bounded by

$$|N - q^r| \leq (d-1)^r \cdot q^{(r+1)/2}.$$  

**Proof.** The number $N$ can be expressed as

$$N = \sum_{x \in k_\ell} \# \{y \in k_\ell; y^q - y = f(x)\} = q \cdot \# \{x \in k_\ell; \Tr_{k_\ell/k}(f(x)) = 0\} =$$

$$= q \cdot \sum_{y \in k_\ell, \Tr(y) = 0} \# \{x \in k_\ell; f(x) = y\} = q \cdot \sum_{y \in k_\ell, \Tr(y) = 0} \Tr(F_y | (F_\ell Q_\ell)_y) =$$

$$= q \left( q^{r-1} + \sum_{y \in k_\ell, \Tr(y) = 0} \Tr(F_y | F_y) \right) = q^r + q \sum_{y \in k_\ell, \Tr(y) = 0} \Tr(F_y | F_y)$$

and we just need to apply the previous proposition. □

We now study the same example on the multiplicative group $G_{m,k}$.

**Proposition 15.** Let $f \in k[x]$ be a polynomial of degree $d$ prime to $p$ without multiple roots and such that $f'$ has no roots with multiplicity $\geq p$, and let $F$ be the $Q_\ell$-sheaf on $G_{m,k}$ given by the kernel of the trace morphism $f_* Q_\ell \to Q_\ell$. Let $S \subseteq \bar{k}$ be the set of critical values of $f$, and suppose that $a \notin S \cdot F(S) \cdots F^{r-1}(S)$ (where $F$ is the Frobenius automorphism $x \mapsto x^q$ of $\bar{k}$). Then

$$\left| \sum_{y \in k, \Tr(y) = a} \Tr(F_y | F_y) \right| \leq r(d-1)^r \cdot q^{(r-1)/2}.$$  

**Proof.** Since $f$ does not have multiple roots, 0 is not a critical value of $f$. This implies that $F$ is unramified at 0. Then $F$ has trivial monodromy at 0 and its monodromy at infinity is the direct sum of all non-trivial characters of the inertia group of $G_{m,k}$ at $\infty$ with trivial $d$-th power. In particular, $F$ does not have any subquotient isomorphic to a Kummer sheaf $\mathcal{L}_\chi$ (since such a sheaf would have non-trivial monodromy at zero if $\chi$ is not trivial, and trivial monodromy at infinity otherwise). Then the perverse sheaf $P := F[1]$ has “property $P$” in the terminology of [Kat96, 2.6], and the same happens to $F^{i*} P$ for every integer $i$. Therefore

$$P \star (F^* P) \star \cdots \star (F^{(r-1)*P})$$

is a perverse sheaf on $G_{m,k}$ by [Kat96, 2.6]. By [Kat12, Lemma 19.5], this sheaf is smooth on $G_{m,k} \backslash T$, where $T = S \cdot F(S) \cdots F^{r-1}(S)$. So, if $a \notin T$, we have $\mathcal{H}(P \star (F^* P) \star \cdots \star (F^{(r-1)*P})) = 0$ for
where $C$ is the rank of $\mathcal{H}^{-1}(P \ast_1 (F^s P) \ast_1 \cdots \ast_1 (F^{(r-1)s} P))$ at $a$.

We claim that this rank is $r(d-1)^r$. We will prove this by induction on $r$: for $r = 1$ it is obvious. Suppose that we have shown that $Q := P \ast_1 (F^s P) \ast_1 \cdots \ast_1 (F^{(r-2)s} P)$ is a perverse sheaf whose $\mathcal{H}^{-1}$ has generic rank $(r-1)(d-1)^{r-1}$. The hypotheses on $f$ imply that $F$ (and so $F^s F$ for every $i$) is everywhere tamely ramified. Therefore so is $\mathcal{H}^{-1}(Q)$ by [11] Corollary 24. The rank of $\mathcal{H}^{-1}(Q \ast_1 (F^{(r-1)s} P))$ at $a$ is then

$$\dim H^1_c(G_{m,k}, \mathcal{H}^{-1}(Q) \otimes \tau^*_a F^{(r-1)s} F)$$

where $\tau_a : G_{m,k} \to G_{m,k}$ is the map $t \mapsto a/t$. By the Ogg-Shafarevic formula, since $\mathcal{H}^{-1}(Q) \otimes \tau^*_a F^{(r-1)s} F$ is everywhere tamely ramified and $H^1_c$ is its only non-zero cohomology group, this dimension is given by the sum of the rank drops at the ramification points.

Now, since $a \notin S \cdot F(S) \cdots F^{r-1}(S)$, the sets of ramification points of $\mathcal{H}^{-1}(Q)$ (which is contained in $S \cdot F(S) \cdots F^{r-2}(S)$) and $\tau_a^* F^{(r-1)s} F$ are disjoint, so the set of ramification points of $\mathcal{H}^{-1}(Q) \otimes \tau^*_a F^{(r-1)s} F$ is the union of these two, and the drop at each ramification point coming from one of the factors is the drop of the point in said factor multiplied by the generic rank of the other factor. That is,

$$\dim H^1_c(G_{m,k}, \mathcal{H}^{-1}(Q) \otimes \tau^*_a F^{(r-1)s} F) =$$

$$= \text{rank}(\mathcal{H}^{-1}(Q)) \dim H^1_c(G_{m,k}, \tau^*_a F^{(r-1)s} F) +$$

$$+ \text{rank}(\tau^*_a F^{(r-1)s} F) \dim H^1_c(G_{m,k}, \mathcal{H}^{-1}(Q))$$

$$(r-1)(d-1)^r - (d-1)(d-1)^{r-1} = r(d-1)^r.$$}

since

$$\dim H^1_c(G_{m,k}, \mathcal{H}^{-1}(Q)) = \chi_c(G_{m,k}, Q) = \chi_c(G_{m,k}, P \ast_1 (F^s P) \ast_1 \cdots \ast_1 (F^{(r-2)s} P)) =$$

$$= \chi_c(G_{m,k}, P) \chi_c(G_{m,k}, F^s P) \cdots \chi_c(G_{m,k}, F^{(r-2)s} P) = \chi_c(G_{m,k}, P)^{r-1}$$

and

$$\chi_c(G_{m,k}, P) = -\chi_c(G_{m,k}, F) = (d-1) - \chi_c(A^1_k, F)$$

since $f$ has no multiple roots (so $F$ has rank $d - 1$ at 0), and

$$\chi_c(A^1_k, F) = \chi_c(A^1_k, F, \mathbb{Q}_\ell) - \chi_c(A^1_k, \mathbb{Q}_\ell) = 1 - 1 = 0.$$

\hfill $\square$

Applying this to $a = 1$ and using the fact that the number of solutions in $k_r$ to the equation $y^r = t$ is $q - 1$ if $N_{k_r/k}(t) = 1$ and 0 otherwise, we get

**Corollary 4.** Under the previous hypotheses, the number $N$ of $k_r$-valued points on the superelliptic curve $y^{r-1} = f(x)$ is bounded by

$$|N - q^r - \delta + 1| \leq r(d-1)^r (q-1) \cdot q^{(r-1)/2}$$

where $\delta$ is the number of roots of $f$ in $k_r$. 

Proof. The number $N$ can be expressed as

$$N = \sum_{x \in k_r} \# \{ y \in k_r; y^{q-1} = f(x) \} = \delta + \sum_{x \in k_r} \# \{ y \in k_r; y^{q-1} = f(x) \} =$$

$$= \delta + (q - 1) \cdot \# \{ x \in k_r; N_{k_r/k}(f(x)) = 1 \} =$$

$$= \delta + (q - 1) \cdot \sum_{y \in k_r; N(y) = 1} \# \{ x \in k_r; f(x) = y \} =$$

$$= \delta + (q - 1) \cdot \sum_{y \in k_r; N(y) = 1} \text{Tr}(F_y)(f_\bar{y}(x)y) =$$

$$= \delta + (q - 1) \cdot \left( \frac{q^r - 1}{q - 1} + \sum_{y \in k_r; N(y) = 1} \text{Tr}(F_y|F_y) \right) =$$

$$= \delta + q^r - 1 + (q - 1) \sum_{y \in k_r; N(y) = 1} \text{Tr}(F_y|F_y)$$

and we just need to apply the previous proposition. \qed

To conclude, let us compare the estimates obtained using the convolution direct image with those obtained in [RL12] using the convolution Adams operation. For objects which are defined on $X$, the Adams operation generally gives better estimates (this is to be expected, since the Adams operation is a linear combination of subobjects of the convolution power, so it has lower rank). For instance, if $g \in k[x]$ and $a \in k$ satisfy the hypotheses of Proposition 14, by [RL12, Example 6.5] we have

$$\left| \# \{ x \in k_r; \text{Tr}(f(x)) = a \} - q^r - 1 \right| \leq C_{d,r} \cdot q^{(r-1)/2},$$

where $C_{d,r} = \sum_{i=0}^{r-1} \binom{d+r-2}{i} \binom{r-1}{i}$, which is better than the estimate in Proposition 14 as this table shows (for $d = 5$):

| $r$ | $C_{d,r}$ | $(d-1)^r$ |
|-----|-----------|-----------|
| 2   | 16        | 16        |
| 3   | 44        | 64        |
| 4   | 96        | 256       |
| 5   | 180       | 1024      |
| 10  | 1360      | 1048576   |
| 20  | 10720     | 1099511627776 |

On the other hand, for objects which are not defined on $X$ (only on $Y$), the method in [RL12, Section 8] implies taking the direct sum of all its Galois conjugates (which descends to $X$) and then taking convolution Adams power, which greatly increases the rank. In this case, the estimates here are much better: applying [RL12, Corollary 8.2], for $f \in k_r[x]/k[x]$ we would get

$$\left| \# \{ x \in k_r; \text{Tr}(f(x)) = a \} - q^r - 1 \right| \leq \frac{1}{r} C_{r^d, r+1, r} \cdot q^{(r-1)/2},$$

which is worse than the estimate in [14] as this table shows (again for $d = 5$):

| $r$ | $\left| \frac{1}{r} C_{r^d, r+1, r} \right|$ | $(d-1)^r$ |
|-----|---------------------------------|-----------|
| 2   | 32                              | 16        |
| 3   | 386                             | 64        |
| 4   | 5504                            | 256       |
| 5   | 86401                           | 1024      |
| 10  | 153547568007                    | 1048576   |
| 20  | 1356608411506872363943501       | 1099511627776 |
So the estimates provided by the convolution direct image greatly improve those of [RL12] for objects not defined on $X$. On the other hand, note that the convolution direct image operation does not commute with finite extensions of the base field, so this operation cannot be used to construct a “Trace $L$-function” as it was done in [RL12] for objects defined on $X$ (and, in fact, such a function is not rational in general, as it was shown in [RL12] Section 8).

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Dpto. de Álgebra, Fac. de Matemáticas, Universidad de Sevilla, c/Tarfia, s/n, 41013 Sevilla, Spain, arojas@us.es