A FACTORIZATION THEOREM FOR THE COINVARIANT ALGEBRA
OF A UNITARY REFLECTION GROUP

G.I. LEHRER

Abstract. We prove the following theorem. Let \( G \) be a finite group generated by unitary reflections in a complex Hermitian space \( V = \mathbb{C}^\ell \) and let \( G' \) be any reflection subgroup of \( G \). Let \( \mathcal{H} = \mathcal{H}(G) \) be the space of \( G \)-harmonic polynomials on \( V \). There is a degree preserving isomorphism \( \mu : \mathcal{H}(G') \otimes \mathcal{H}(G) \rightarrow \mathcal{H}(G) \) of graded \( \mathcal{N} \)-modules, where \( \mathcal{N} := N_{GL(V)}(G) \cap N_{GL(V)}(G') \) and \( \mathcal{H}(G') \) is the space of \( G' \)-fixed points of \( \mathcal{H}(G) \). This generalises a result of Douglass and Dyer for parabolic subgroups of real reflection groups. An application is given to counting rational conjugates of reductive groups over \( \mathbb{F}_q \).

1. Background and notation

Much of the background material in this section may be found in \[8, Ch. 9\]. Let \( G \) be a finite group generated by (pseudo)reflections in a complex vector space \( V \) of dimension \( \ell > 0 \). It is well known that if \( S \) denotes the coordinate ring of \( V \) (identified with the symmetric algebra \( S(V^*) \) on the dual \( V^* \)) then \( G \) acts contragrediently on \( V^* \), and hence on \( S \), and the ring \( S^G \) of polynomial invariants of \( G \) on \( S \) is free; if \( F_1, F_2, \ldots, F_\ell \) is a set of homogeneous free generators of \( S^G \), then the degrees \( d_i = \deg F_i (i = 1, \ldots, \ell) \) are determined by \( G \), and are called the invariant degrees of \( G \). If \( F \) is the ideal of \( S \) generated by the elements of \( S^G \) which vanish at \( 0 \in V \), then \( S/F := S_G \) realises the regular representation of \( G \). The space \( S_G \) is called the coinvariant algebra of \((G, V)\). Since the ideal \( F \) is graded, \( S_G \) is clearly graded, and for each \( i \), the graded component \((S_G)_i \) of degree \( i \) is a \( G \)-module. By a classical result of Chevalley \[8, Cor 3.31, p. 52\], taking products of polynomials defines an isomorphism of \( \mathbb{C}G \)-modules

\[
S \rightarrow S^G \otimes_\mathbb{C} S_G.
\]

For any \( \mathbb{N} \)-graded \( \mathbb{C} \)-vector space \( W = W_0 \oplus W_1 \oplus W_2 \oplus \ldots \) where the \( W_i \) are finite dimensional, we write

\[
Poin_W(t) = \sum_{i=0}^{\infty} \dim_{\mathbb{C}}(W_i)t^i \in \mathbb{C}[[t]]
\]

for its Poincaré series. If \( \dim(W) < \infty \) then \( Poin_W(t) \) is a polynomial. For properties of these series see \[8, Ch. 4\].

1.1. The space of harmonic polynomials. If \( \mathcal{C} \) is any graded \( G \)-stable complement of \( \mathcal{F} \) in \( S \), then evidently \( \mathcal{C} \cong S_G \) as graded \( G \)-module. The space of \( G \)-harmonic polynomials is a canonical such complement, which we now describe. Let \( \mathcal{A} := A_G \) be the set of reflecting

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\]

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hyperplanes of $G$ and for each hyperplane $H \in \mathcal{A}$, let $L_H \in S$ be a linear form such that $H = \ker(L_H)$ and let $\epsilon_H := |G_H|$, where $G_H$ is the (cyclic) group of reflections in $H$. Define
\[(1.3) \quad \Pi = \Pi_G := \prod_{H \in \mathcal{A}_G} L_H^{-1} \in S.\]

Then [8] Lemma 9.9] asserts that $\Pi$ is skew, that is, for $g \in G$, $g \Pi = \det_V(g) \Pi$, and further if $A \in S$ is any skew polynomial, then
\[(1.4) \quad A = B \Pi, \text{ where } B \in S(V^*)^G.\]

Now let $S(V)$ be the symmetric algebra on $V$. For $v \in V$, we have the derivation $D_v$ of $S$ defined by
\[D_v(A)(x) = \lim_{t \to 0} \frac{A(x + tv) - A(x)}{t}.\]

It is explained in [8, §9.5] how the map $v \mapsto D_v$ extends to an algebra homomorphism from $S(V)$ to the algebra of differential operators on $S$. Thus for $a \in S(V)$ we have $D_a : S \to S$; for homogeneous $a$, this operator has degree $-\deg(a)$. The properties of this algebra of operators are summarised in the next statement, whose proof may be found in [8, Ch. 9].

**Theorem 1.5.** Let $G$, $V$, $S = S(V^*)$, $S(V)$ etc be as above.

1. Let $a \in S(V)$, $P \in S(V^*)$ and $g \in GL(V)$. Then $g(D_a(P)) = D_{ga}(gP)$.
2. Define a bilinear pairing $[-,-] : S(V) \times S(V^*) \to \mathbb{C}$ by $[a,P] := D_a(P)(0)$ for $a \in S(V)$, $P \in S$. This pairing is non-degenerate in both variables.
3. The pairing in (ii) is respected by the action of $GL(V)$; i.e., for $a \in S(V)$, $P \in S(V^*)$ and $g \in GL(V)$, we have $[ga,gP] = [a,P]$.

Theorem [13] shows that $[-,-]$ puts the spaces $S = S(V^*)$ and $S(V)$ in $GL(V)$-equivariant duality. Now let $S(V)^G$ be the algebra of invariants of $G$ on $S(V)$. Denote by $\mathfrak{f} \subset S(V)$ the analogue of $\mathcal{F}$ for $S(V)$. That is, $\mathfrak{f}$ is the ideal of $S(V)$ generated by the elements of $S(V)^G$ with no constant term.

**Definition 1.6.** Define the space $\mathcal{H} = \mathcal{H}(G)$ of $G$-harmonic polynomials in $S$ by $\mathcal{H}(G) := \mathfrak{f}^\perp$. That is, $P \in \mathcal{H}$ if and only if $[a,P] = 0$ for all $a \in \mathfrak{f}$.

The main facts concerning the space $\mathcal{H}$ which we shall require are summarised in the next statement.

**Theorem 1.7.** Maintain the above notation.

1. We have $P \in \mathcal{H}$ if and only if $D_a(P) = 0$ for all $a \in S(V)^G$. That is, $\mathcal{H}$ is the space of functions which are annihilated by the invariant differential operators.
2. The space $\mathcal{H}$ coincides with $\{D_a(\Pi) \mid a \in S(V)\}$, where $\Pi$ is as defined in (1.3).
3. The space $\mathcal{H}$ is a $G$-stable complement of $\mathcal{F}$ in $S(V^*)$. In particular, we have the $N_{GL(V)}(G)$-equivariant decomposition
\[(1.8) \quad S(V^*) = \mathcal{H} \oplus \mathcal{F}.\]

**2. Reflection subgroups—the main theorem.**

A reflection subgroup of the reflection group $G$ in $V$ is a subgroup of $G$ which is generated by some of the reflections in $G$. For background concerning such groups see [4, 12]. They include the parabolic subgroups of $G$ [10, 7], but many other subgroups as well. Let $G'$ be such a subgroup and write $\mathcal{H}' = \mathcal{H}(G')$ for its space of harmonic polynomials $\mathcal{F}' = \mathcal{F}(G')$, and so on. For any $G$-module $M$, write $M^{G'}$ for its subspace of $G'$-fixed elements. Our objective is to prove the following theorem.
Theorem 2.1. Let $G$ be a finite group generated by unitary reflections in a complex Hermitian space $V = \mathbb{C}^l$ and let $G'$ be any reflection subgroup of $G$. Let $\mathcal{H} = \mathcal{H}(G)$ be the space of $G$-harmonic polynomials on $V$. There is a degree preserving isomorphism
\[ \xi : \mathcal{H}' \otimes \mathcal{H}' \to \mathcal{H} \]
of graded $\mathcal{N}$-modules, where $\mathcal{N} := N_{GL(V)}(G) \cap N_{GL(V)}(G')$.

Proof. We begin with the observation that using (1.1) applied to the reflection group $G'$ acting on $V$, we have the linear isomorphism
\[ \mathcal{H}' \otimes S^{G'} \to S, \]
given by multiplying the polynomials in the tensor factors.

Next, applying (1.8), we have the $G$-equivariant decomposition
\[ S = \mathcal{H} \oplus \mathcal{F}. \]
Since both summands on the right side of (2.3) are stable under $G$ and hence a fortiori under $G'$, it follows that we have a graded linear isomorphism
\[ S^{G'} \cong \mathcal{H}^{G'} \oplus \mathcal{F}^{G'}. \]
Substituting (2.4) into (2.2) we obtain a linear isomorphism
\[ \mathcal{H}' \otimes \left( \mathcal{H}^{G'} \oplus \mathcal{F}^{G'} \right) \cong \left( \mathcal{H}' \otimes \mathcal{H}^{G'} \right) \oplus \left( \mathcal{H}' \otimes \mathcal{F}^{G'} \right) \to \mathcal{S}. \]
Now evidently the summand $\mathcal{H}' \otimes \mathcal{F}^{G'}$ is mapped in the multiplication isomorphism (2.5) to a subspace of $\mathcal{F}$, since $\mathcal{F}$ is an ideal of $\mathcal{S}$. It follows by restricting the map in (2.6) to the summand $\mathcal{H}' \otimes \mathcal{H}^{G'}$ that we have a surjective degree preserving linear map
\[ \xi : \mathcal{H}' \otimes \mathcal{H}^{G'} \to \mathcal{S} / \mathcal{F} \cong S_G. \]
But since $\mathcal{H}$ realises the regular representation of $G$, we have $\dim(\mathcal{H}^{G'}) = |G| / |G'|$ while evidently $\dim(\mathcal{H}') = |G'|$, so that the dimension of the left side of (2.6) is $|G| = \dim(S_G)(= \dim(\mathcal{H}))$. It follows that $\xi$ is a graded isomorphism, and since $S / \mathcal{F} \cong S_G \cong \mathcal{H}$ as graded $G$-modules, it follows that $\xi$ is a graded linear isomorphism.
It remains only to observe that all homomorphisms above evidently respect the action of $\mathcal{N}$, and the proof is complete. \hfill \Box

Remark 2.7. One of the key ingredients of the proof is (1.1), which asserts that $S$ is free as module over $S^G$. Notice that taking $G'$-invariants in (1.1) yields further that $S^{G'}$ is free over $S^G$. This was first noticed by Dyer [3].

Remark 2.8. The special case of Theorem 2.1 where $G$ is a finite Coxeter group and $G'$ is a parabolic subgroup was treated in [2, Thm. 2.1], where the author acknowledges input from M. Dyer, who pointed out that the result is connected to the discussion of the cohomology of the flag variety in [1]. We believe that our proof is significantly simpler and our result is more general than that in op. cit.

3. Complements

3.1. Poincaré polynomials. Since Poincaré polynomials are multiplicative on tensor products, the following statement is clear. In the statement we use the convention that for any unitary reflection group $G$, $\text{Poin}_G(t) := \text{Poin}_{S / \mathcal{F}}(t) = \text{Poin}_{\mathcal{H}}(t)$ (cf. (1.2)).
Corollary 3.1. With notation as in Theorem 2.7 we have
\[ \text{Poin}_G(t) = \text{Poin}_{G'}(t) \text{Poin}_{H^{G'}}(t). \]

We can be a little more explicit about the second factor in the right side above. Recall that for any finite dimensional \( CG \)-module \( M \), the fake degree
\[ f_M^{(G)}(t) = f_M(t) := \sum_{i=0}^{N} (\mathcal{H}_i, M)_G t^i, \]
where \( (\mathcal{H}_i, M)_G \) is the intertwining number of \( M \) with the degree \( i \) graded component \( \mathcal{H}_i \) of \( \mathcal{H} = \mathcal{H}(G) \).

Denote by \( \text{Irr}(G) \) the set of equivalence classes of irreducible \( CG \)-modules. For \( M \in \text{Irr}(G) \) define integers \( m(M) \geq 0 \) by
\[ \text{Ind}^{G'}_G(1) = \sum_{M \in \text{Irr}(G)} m(M) M. \]

Proposition 3.2. We have
\[ \text{Poin}_{H^{G'}}(t) = \sum_{M \in \text{Irr}(G)} m(M) f_M(t), \]
where \( f_M(t) \) is the fake degree of \( M \) defined above.

Proof. For any \( M \in \text{Irr}(G) \), \( \dim(M^{G'}) = \dim(M) \). For \( \dim(M^{G'}) = (M, 1)_G = (M, \text{Ind}^{G'}_G(1))_G \) by Frobenius reciprocity. It follows that \( \dim(\mathcal{H}_i)^{G'} = \sum_{M \in \text{Irr}(G)} m(M)(\mathcal{H}_i, M)_G \).

Multiplying this last relation by \( t^i \) and summing over \( i \) and \( M \) gives the stated formula. \( \square \)

We give two examples where \( G' \) is a non-parabolic reflection subgroup.

Example 3.4. Let \( G = \mu_e \) be the group of \( e \)th roots of unity acting on \( V = \mathbb{C} \) in the obvious way. Let \( G' = \mu_d \subseteq G \) be the subgroup of \( d \)th roots of unity, where \( d \) divides \( e \). Then \( \Pi = X^{e-1}, \Pi' = X^{d-1}, \mathcal{H} = \langle 1, X, X^2, \ldots, X^{e-1} \rangle, \mathcal{H}' = \langle 1, X, X^2, \ldots, X^{d-1} \rangle \) and \( \mathcal{H}^{G'} = \langle 1, X^d, X^{2d}, \ldots, X^{e-d} \rangle \), so that \( \xi : \mathcal{H}' \otimes \mathcal{H}(G)^{G'} \to \mathcal{H} \) is given here by simple multiplication of the basis elements.

Example 3.5. Let \( G \) be the Weyl group of type \( B_2 \). Let \( V \) have orthonormal basis \( \{x, y\} \), with \( G' = \langle r_x, r_y \rangle \), where \( r_x \) is the reflection in \( x^2 \) and similarly for \( r_y \). Let \( V^* \) have dual basis \( X, Y \), so that in the above notation \( \varphi(x) = X \) and \( \varphi(y) = Y \).

Further, we have
\[ \Pi = XY(X^2 - Y^2), \quad \Pi' = XY, \text{ and} \]
\[ \text{Moreover } S(V)^{G'} = \langle x^2, y^2 \rangle \text{ and } S(V)^G = \langle x^2 + y^2, x^2y^2 \rangle. \]

Using this data, it is straightforward to compute that \( \mathcal{H} \) has a basis
\[ 1, X, Y, XY, X^2 - Y^2, 3X^2Y - Y^3, X^3 - 3XY^2, X^3Y - XY^3, \]
and that \( \mathcal{H}' \) has basis \( 1, X, Y, XY \). Evidently \( \mathcal{H}^{G'} \) has basis \( 1, X^2 - Y^2 \).

Now consider \( \xi : \mathcal{H}' \otimes \mathcal{H}^{G'} \to \mathcal{H} \), given by multiplication of polynomials, followed by projection to \( \mathcal{H} \). Clearly \( 1 \otimes 1, X \otimes 1, Y \otimes 1, XY \otimes 1 \mapsto 1, X, Y, XY \). Now \( 1 \otimes (X^2 - Y^2) \mapsto \text{proj}_\mathcal{H}(X^2 - Y^2) = X^2 - Y^2, \) where \( \text{proj}_\mathcal{H} : S/F \to \mathcal{H} \) is the projection with respect to the decomposition \( S = \mathcal{H} \otimes F \). But \( X \otimes (X^2 - Y^2) \mapsto \text{proj}_\mathcal{H}(X^3 - XY^2) \). Since \( X^3 - XY^2 = \frac{1}{2}(X^3 - 3XY^2) + \frac{3}{2}X^2(Y^2 + 2Y^2), \) with the second summand being in \( F; \) \( \xi(X \otimes (X^2 - Y^2)) = \frac{1}{2}(X^3 - 3XY^2). \) Similarly, \( \xi(Y \otimes (X^2 - Y^2)) = \frac{1}{2}(3X^2Y - Y^3) \) and \( \xi(XY \otimes (X^2 - Y^2)) = XY(X^2 - Y^2). \) The group \( \mathcal{N} \) is generated by the interchange of \( X \) and \( Y \), and the \( \mathcal{N} \)-equivariance is easily checked.
Remark 3.6. Our main result implies some constraints on which groups could occur as reflection subgroups. Here is one superficial one.

Corollary 3.7. Let the degrees of $G$ be $d_1, d_2, \ldots, d_\ell$ and those of $G'$ be $d'_1, d'_2, \ldots, d'_\ell$. Then

(i) $\prod_{i=1}^{\ell}(1 + t + \ldots + t^{d'_i-1})$ divides $\prod_{i=1}^{\ell}(1 + t + \ldots + t^{d_i-1})$.

(ii) For each $n \in \mathbb{N}$ we have $|\{i \text{ such that } n|d'_i\}| \leq |\{i \text{ such that } n|d_i\}|$.

The relation (i) is evident because $\text{Poin}_G(t) = \prod_{i=1}^{\ell}(1 + t + \ldots + t^{d'_i-1})$ and similarly for $\text{Poin}_{G'}(t)$, and (ii) follows using the fact that $t^k - 1 = \prod_{n|k} \Phi_n(t)$, where $\Phi_n(t)$ is the $n$th cyclotomic polynomial.

4. Duality.

The main result may be formulated without recourse to the projection $S \rightarrow H$ by using the dual reflection structure. We briefly indicate how this is done.

The key point is the fact that $G$ also acts as a reflection group on $V^*$. We denote the $(G, V^*)$-analogues of $H$, $F$ and $\Pi$ for $G$ on $V$ by $h$, $f$ and $\varpi$ and write $h'$ etc for their $G'$-analogues. All the statements in §1.1 remain true with $H$, $F$ and $\Pi$ replaced by $h$, $f$ and $\varpi$ respectively. The following Lemma is an easy consequence of the basic facts outlined in §1.

Lemma 4.1. (i) There is a linear isomorphism $d : h \rightarrow H$, defined for $h$ by $d(h) = D_h(\Pi)$. If $N := \deg(\Pi)$, then for homogeneous $h \in h$, $\deg(d(h)) = N - \deg(h)$.

(ii) There is an isomorphism $e : h(G)^G \rightarrow H(G)^G$ defined, for $a \in h^G$, by $e(a) = D_\varpi^a(\Pi)$.

The maps $d$ and $e$ of Lemma 4.1 are evidently $N$-equivariant, and our main result may now be formulated as follows.

Proposition 4.2. The isomorphism $\xi : H' \otimes \mathcal{H}^G \rightarrow \mathcal{H}$ of Theorem 2.7 may be explicitly realised as follows. Let $H = D_h(\Pi') \in H'$ and $K = D_{\varpi^a}(\Pi) \in \mathcal{H}^G$ where $h \in h'$ and $a \in h^G$ are uniquely defined as in Lemma 4.1. Then $\xi(H \otimes K) = D_{ah}(\Pi) \in \mathcal{H}$.

Equivalently, the projection of $HK$ onto $H(G)$ is $D_{ah}(\Pi)$.

5. An application to reductive groups.

Let $G$ be a connected reductive algebraic group defined over the finite field $\mathbb{F}_q$ of $q$ elements, and let $F : G \rightarrow G$ be the corresponding Frobenius endomorphism, as in [6], whose notation we adopt here. For any $F$-stable subset $H \subseteq G$ we write $H^F$ for the (finite) set of $F$-fixed points of $H$. Let $T_0$ be an $F$-stable maximally $F$-split maximal torus of $G$ and $B \supseteq T_0$ be a Borel subgroup. This data determines the Weyl group $W := N_G(T_0)/T_0$, together with its reflection representation in $V := Y_0 \otimes \mathbb{C}$, where $Y_0$ is the cocharacter group of $T_0$ and its root system $\Phi \subseteq Y_0$ as well as a positive subsystem $\Phi^+ \subseteq \Phi$ and its corresponding simple system $\Pi \subseteq \Phi^+$. Now take $L$ to be any $F$-stable connected reductive subgroup of $G$ which has maximal rank. It is well known that such $L$ are characterised as the connected centralisers of semisimple elements of $G^F$. They include Levi components of parabolic subgroups. We shall be concerned with the set $L$ of $G$-conjugates of $L$ and in particular the set $L^F$ of $F$-stable conjugates of $L$. We may therefore assume, without loss of generality, that $L \supseteq T_0$. It is always the case that such $L$ is the centraliser of an element of $T_0$, but we shall not use this fact. Let $\Phi' \subseteq \Phi$ be the root system of $L$ with respect to $T_0$. Then $\Phi'_+ := \Phi' \cap \Phi^+$ is a positive system in $\Phi'_+$ and there is a unique corresponding simple system $\Pi' \subseteq \Phi'_+$. Note that unlike in the case treated in [2], it is not generally the case that $\Pi' \subseteq \Pi$. Let $W'$ be the Weyl group of $\Phi'$; this is a reflection
Theorem 2.1. We note also that the conjugacy class in product of class functions on \( N \) denoted the connected component of the group \( N \),
\begin{equation}
    \frac{N_G(L)}{N_G(L)^o} \simeq C,
\end{equation}
where \( C \) is as above. Now the conjugate \( gL := gLg^{-1} \) is \( F \)-stable precisely when \( g^{-1}F(g) \in N_G(L) \). We therefore have a map \( \mathcal{L}^F \rightarrow C \) obtained by taking the image in \( C \) of \( g^{-1}F(g) \) in \( C \) (cf. (5.1)). It is easily checked that this image is uniquely determined up to \( F \)-conjugacy, where the \( F \)-conjugate of \( L \) is the product \( xcF(x)^{-1} \). We denote this image by \( \omega(gL) \).

The following facts are standard and may be found, e.g. in [9], [11], [6] or [2].

Lemma 5.2. Maintain the above notation.

(i) The map \( \omega : \mathcal{L}^F \rightarrow \{ F \text{-conjugacy classes of } C \} \) described above induces a bijection from the set of \( G \)-orbits on \( \mathcal{L}^F \) to the set of \( F \)-conjugacy classes of \( C \).

(ii) Let \( gL \in \mathcal{L}^F \) and let \( Z \subseteq C \) be the \( F \)-centraliser of \( \omega(gL) \). Then \( N_{G^F}(gL) \) is the semidirect product of \( (\bar{F}^N)^{\bar{F}} \) with \( Z \).

Next, recall that \( F \) acts on \( V = qF_0 \), where \( F_0 \in GL(V) \) fixes both \( \Pi \) and \( \Pi' \). The methods of [5] show that \( C \) is the connected component of the group \( |\Phi_+| \) and \( N' = |\Phi'_+| \).

Corollary 5.4. Maintaining the above notation, the number of \( F \)-stable conjugates of \( L \) is equal to
\begin{equation}
    q^{2(N-N')} |F_0| \sum_{d=0}^{N-N'} \langle H(W)^d, \gamma_c \rangle \bar{q}^{-d},
\end{equation}
where \( H(W)^d \) is the \( d \)-th graded component of \( H(W)^{W'} \), \( \langle -,- \rangle_{\overline{C}} \) denotes the usual inner product of class functions on \( \overline{C} \), \( N = |\Phi_+| \) and \( N' = |\Phi'_+| \).

The proof of this result is exactly as in [6], where the result is proved for \( L \) equal to a torus, and [2] where the result is proved for \( L \) a Levi factor. The crucial difference is that our Theorem 2.1 was not available for arbitrary reflection subgroups of \( W \) in [2]. We remark finally that the usual corollaries concerning the number of \( F \)-stable conjugates of \( L \) are now available in the wider generality of our result. As an example, we have the following result.

Theorem 5.3. Let \( L \) be any \( F \)-stable connected reductive subgroup of maximal rank of \( G \) and let \( W, W', C \) etc. be as above.

Let \( g \) be a function on \( C \) which is constant on \( F \)-conjugacy classes. Then
\begin{equation}
    \sum_{L' \in \mathcal{L}^G} g(L') = q^{2(N-N')} \sum_{d=0}^{N-N'} \langle H(W)^d, \gamma_c \rangle \bar{q}^{-d},
\end{equation}
where \( H(W)^d \) is the \( d \)-th graded component of \( H(W)^{W'} \), \( \langle -,- \rangle_{\overline{C}} \) denotes the usual inner product of class functions on \( \overline{C} \), \( N = |\Phi_+| \) and \( N' = |\Phi'_+| \).
where \( \chi_{C:F_0^{-1}} \) is the characteristic function of the coset \( CF_0^{-1} \) in \( C \).

In particular, if \( G \) is split, \( C = C \) and this number is just the Poincaré polynomial

\[
q^{2(N-N')} \sum_{d=0}^{N-N'} \dim \mathcal{H}(W)^d, 1_C \circ q^{-d} = q^{2(N-N')} \sum_{d=0}^{N-N'} \dim \mathcal{H}(W)^d q^{-d}.
\]

We close with two examples where \( L \) is not a Levi factor.

**Example 5.6.** Take \( G = Sp_4(\mathbb{F}_q) \) and let \( L \) be the reductive subgroup of maximal rank corresponding to the unique subsystem of the root system \( \Phi \) of type \( A_1 \times A_1 \). Thus \( L \) has semisimple part of type \( SL_2 \times SL_2 \). One sees easily that in this case \( W' \) is normal in \( W \), so that \( W'C = W \). Further, \( |N| = 4 \) and \( |N'| = 2 \). We may therefore apply (5.5) to conclude that the number of \( F \)-stable conjugates of \( L \) is \( q^4 \).

**Example 5.7.** Take \( G = Sp_6(\mathbb{F}_q) \) and let \( L \) be a reductive subgroup of maximal rank with semisimple part isomorphic to \( SL_2 \times Sp_4 \). Thus the corresponding root subsystem is of type \( A_1 \times C_2 \). In this case we have \( |\Phi_+| = N = 9, |\Phi'_+| = N' = 5 \) and \( |C| = 2 \), so that \( N_W(W') = W'C \) is isomorphic to \( \text{Sym}_3 \times (\mathbb{Z}_q^2)^3 \). To compute \( \dim \mathcal{H}(W)^d \), observe that for any \( W \)-module \( M \) and subgroup \( W_1 \subseteq W \), we have \( \dim M_{W_1} = \langle \text{Res}^W_{W_1}(M), 1 \rangle_{W_1} = \langle M, \text{Ind}_{W_1}^W(1) \rangle_W \) by Frobenius reciprocity.

In our case, it is easily verified that \( \text{Ind}_{W'C}^W(1) = 1 + \rho \), where \( \rho \) is the two dimensional representation of \( W \) obtained by pulling back the two dimensional irreducible representation of \( \text{Sym}_3 \) via the map \( W = \text{Sym}_3 \times (\mathbb{Z}_q^2)^3 \rightarrow \text{Sym}_3 \). Further it is well known and easily verified that

\[
\langle \mathcal{H}(W)^d, 1 \rangle_W = \begin{cases} 
0 & \text{if } d \neq 0 \\
1 & \text{if } d = 0,
\end{cases}
\]

\[
\langle \mathcal{H}(W)^d, \rho \rangle_W = \begin{cases} 
0 & \text{if } d \neq 2 \text{ or } 4 \\
1 & \text{if } d = 2 \text{ or } 4.
\end{cases}
\]

We may therefore apply (5.5) to deduce that the number of \( F \)-stable conjugates of \( L \) is equal to

\[
q^4 \sum_{d=0} \left( \langle \mathcal{H}(W)^d, 1 \rangle_W + \sum_{d=0}^4 \langle \mathcal{H}(W)^d, \rho \rangle_W \right) q^{-d} = q^4 (1 + q^4 + q^2 + q) = 1 + q^2 + q^4.
\]

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School of Mathematics and Statistics, University of Sydney, N.S.W. 2006, Australia, Fax: +61 2 9351 4534

E-mail address: gustav.lehrer@sydney.edu.au