SEELEY’S THEORY OF PSEUDODIFFERENTIAL OPERATORS ON ORBIFOLDS

BOGDAN BUCICOVSCHI

Abstract. In this paper we extend the results of Robert Seeley concerning the complex powers of elliptic pseudodifferential operators and the residues of their zeta function to operators acting on vector orbibundles over orbifolds.

We present the theory of pseudodifferential operators acting on a vector orbibundle over an orbifold, construct the zeta function of an elliptic pseudodifferential operator and show the existence of a meromorphic extension to \( \mathbb{C} \) with at most simple poles. We give formulas for generalized densities on the orbifold whose integrals compute the residues of the zeta function.

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0. Introduction

In this paper we extend the results of Robert Seeley concerning the complex powers of elliptic pseudodifferential operators and the residues of their zeta function to operators acting on vector orbibundles over orbifolds. We present the theory of pseudodifferential operators acting on a vector orbibundle over an orbifold, construct the zeta function of an elliptic pseudodifferential operator and show the existence of a meromorphic extension to $\mathbb{C}$ with at most simple poles. We give formulas for generalized densities on the orbifold whose integrals compute the residues of the zeta function.

Recall that a pseudodifferential operator $A$ of order $d$ acting on a hermitian vector bundle $E \overset{p}{\to} M$ over a Riemannian manifold $M$ of dimension $m$ becomes an unbounded operator on the space of $L^2$ sections $L^2(M; E)$. If $A$ is an elliptic pseudodifferential operator with spectrum in $(\epsilon, \infty)$ for a sufficiently small $\epsilon > 0$, by using the functional calculus one can define the complex powers $A^s$, $s \in \mathbb{C}$, as
\begin{equation}
A^s = \frac{1}{2\pi i} \int_{\gamma} \lambda^s(\lambda - A)^{-1} d\lambda \quad \text{when } \mathrm{Re}(s) < 0
\end{equation}
(1)
where $\gamma$ is a contour in the complex plane obtained by joining two parallel half-lines to the negative real axis by a circle around the origin) and
\begin{equation}
A^s = A^{s-k}A^k \quad \text{for } \mathrm{Re}(s) \geq 0
\end{equation}
(2)
for a large enough $k \in \mathbb{Z}$ that makes $s - k < 0$.

The condition on the spectrum of $A$ can be relaxed; in order to apply the above construction, it is sufficient for the spectrum of $A$ to be inside an angle with the vertex the origin and away from the negative real axis.

In particular one can define the complex powers of a selfadjoint positive pseudodifferential operator $A$. Seeley showed in his paper [Se] that $A^s$ are pseudodifferential operators of complex order $sd$ and gave a local description of their symbols. For $s$ in the half-plane $\mathrm{Re}(s) < -\frac{m}{d}$, the operators $A^s$ are of trace class. Seeley also showed that the zeta function of $A$ defined as
\begin{equation}
\zeta_A(s) = \mathrm{Tr}(A^s) \quad \text{for } \mathrm{Re}(s) < -\frac{m}{d}
\end{equation}
(3)
has a meromorphic extension to the complex plane with at most simple poles at $-\frac{m}{d} + k$, with $k \in \{0, 1, \ldots\}$ and residues computable as integrals on $M$ of quantities that depend only on the total symbol of the operator $A$. These results will be referred to as Seeley Theory.

An $m$-dimensional orbifold $M$ is locally the quotient of $\mathbb{R}^m$ by a finite group of diffeomorphisms $\Gamma$ and globally it is the quotient of a manifold $M^{m+q}$ by a compact Lie group $G$ of dimension $q$ with finite isotropy groups. The dimension of the Lie group can be a priori quite large, like for example $\frac{m(m+1)}{2}$. Similarly, a rank $k$ vector orbibundle $E \overset{p}{\to} M$ can be represented locally as the quotient of a rank $k$ vector bundle by a finite group of bundle diffeomorphisms $\Gamma$, and globally as the quotient of a $G$ vector bundle of rank $k$, $\tilde{E} \overset{\tilde{p}}{\to} \tilde{M}$, by a compact Lie group $G$ (cf. Proposition 2.4 and Theorem 2.5). We will refer to the local description of the orbifolds and orbibundles using the quotient by a finite group $\Gamma$ as Perspective 1 and to the global description using the quotient by a Lie group $G$ as Perspective 2.

From these perspectives the space of smooth sections in a vector orbibundle can be thought of as the space of invariant smooth sections in a genuine vector bundle.
over a manifold. One defines a pseudodifferential operator \( A \) in a vector orbibundle as an operator acting on the space of invariant sections in \( \tilde{E} \overset{\tilde{p}}{\to} \tilde{M} \) which is the restriction of a \( \Gamma \) (respectively \( G \)) equivariant pseudodifferential operator \( \tilde{A} \) acting on \( \tilde{E} \overset{\tilde{p}}{\to} \tilde{M} \). In the case of a finite group \( \Gamma \), the operator \( \tilde{A} \) is unique up to a smoothing operator, cf. Proposition \[.3\.3\.\] If \( \dim(G) \geq 1 \) then the operator \( \tilde{A} \) which induces \( A \) is far from being unique.

For the definition and the discussion of the basic properties of the pseudodifferential operators we will use perspective 1, while for some global properties and spectral theory of the elliptic pseudodifferential operators we will use perspective 2. Seeley theory is about elliptic operators which is a global theory. However, to formulate and prove the results concerning the pseudodifferentiability of the complex powers and the residues of the zeta function we use the local theory. Of particular importance are the explicit formulae in coordinate charts for the residues of the zeta function. This makes clear the need of perspective 1. Nevertheless, the use only of perspective 1 to extend Seeley theory in the context of orbibundles and orbifolds will require the repetition of the entire theory of elliptic pseudodifferential operators acting in orbibundles. The results are probably implicit in the existing literature, but not explicit enough for our needs in connection with the operators acting in orbibundles.

In the first section we consider a pseudodifferential operator of classical type \( A \) on the space of smooth sections of a vector bundle \( E \overset{p}{\to} M \) endowed with the a smooth action of a finite group \( \Gamma \) such that \( A \) commutes with this action and extend Seeley’s results to this case. This extension, which is of independent interest, is also crucial for the study of the pseudodifferential operators acting on vector orbibundles. The results are probably implicit in the existing literature, but not explicit enough for our needs in connection with the operators acting in orbibundles.

The \( \Gamma \) action on \( E \overset{p}{\to} M \) induces the decomposition:

\[
C^\infty(M; E) = \bigoplus_{i=1}^{l} C^\infty(M; E)_i
\]

with \( C^\infty(M; E)_i = V_i \otimes \text{Hom}_\Gamma(V_i; C^\infty(M; E)) \), where \( (V_i, \rho_i)_{i=1}^{l} \) is a complete set of irreducible representations of \( \Gamma \), with \( \rho_1 \) being the trivial 1-dimensional representation. The \( \Gamma \) equivariance of \( A \) induces the decomposition

\[
A = \bigoplus_{i=1}^{l} A_i
\]

with \( A_i : C^\infty(M; E)_i \to C^\infty(M; E)_i \). This decompositions will be referred to as the decomposition along the irreducible representations of \( \Gamma \). We restrict our attention to an elliptic positive pseudodifferential \( A \) of order \( d > 0 \). We are interested in the traces of the complex powers \( (A^*)_\gamma = (A_i)^* \) which can be recovered from \( Tr(A^* \cdot T_\gamma) \) (as prescribed by formula \( [\Pi] \)), where \( T_\gamma \) is the operator by which \( \gamma \in \Gamma \) acts on \( C^\infty(M; E) \). Although \( T_\gamma \) are not pseudodifferential operators, we showed that the
trace functional \( \zeta_{A,\gamma}(s) = Tr(A^s \cdot \mathcal{T}_\gamma) \), and therefore \( \zeta_{A,i}(s) = Tr(A_i^s) \), defined for \( \Re(s) < -\frac{\dim}{2} \) can be extended to a meromorphic function on \( \mathbb{C} \) with at most simple poles. We construct a sequence of smooth densities \( \eta_k^\gamma \) on the fixed point sets \( M^\gamma = \{ x \in M \mid \gamma x = x \} \) such that the residue of the meromorphic extension of \( \zeta_{A,\gamma} \) at \( s = -\frac{m+k}{d} \), \( k \in \{ 0, 1, \ldots \} \), is computed as an integral on \( M^\gamma \) of the density \( \eta_k^\gamma \) (see Theorem 1.3 and 1.4). Here \( m = \dim(M) \) and \( d = \text{ord}(A) \). These densities can be interpreted as Dirac-type generalized densities on \( M \) (as in [GS], Chap. VI). For \( k < (\dim(M) - \dim(M^\gamma)) \) we have \( \eta_k^\gamma = 0 \). As a direct consequence we will show that the trace functionals \( \zeta_{A,i}(s) = Tr(A_i^s) \) can be extended to meromorphic functions with at most simple poles at \( s = -\frac{m+k}{d} \), \( k \in \{ 0, 1, \ldots \} \) and compute the residues in Theorem 3.7. Of particular interest is \( Tr(A_1^s) \) – the component corresponding to the trivial one dimensional representation \( \rho_1 \), because the \( \Gamma \) equivariant operator \( A \) induces an elliptic pseudodifferential operator acting on sections of the orbibundle \( E/\Gamma \overset{p}{\to} M/\Gamma \) which can be identified with \( A_1 \). The trace functional \( Tr(A_1^s) \) is the zeta function of this operator.

In the second section we present the basic definitions and properties of the orbifolds and orbibundles. It is a known fact that any orbifold \( M^m \) is the quotient of a smooth manifold \( M^{m+q} \) by a compact Lie group \( G \) of dimension \( q \) acting on \( M \) with finite isotropy groups. We complete this result by proving that a vector orbibundle \( E \overset{p}{\to} M \) is the quotient by \( G \) of a \( G \) vector bundle \( \tilde{E} \overset{\tilde{p}}{\to} \tilde{M} \) in Theorem 2.3. This will allow us to give a global characterization for the spaces of smooth sections in orbibundles in Proposition 2.4. We also describe the space of generalized Dirac-type densities on orbifolds and the canonical stratification associated with an orbifold structure.

In the third section we show how one can extend the theory of pseudodifferential operators acting on smooth sections in a vector bundle over a manifold to pseudodifferential operators acting on smooth sections in a vector orbibundle over an orbifold. The basic definitions and elementary properties were stated, though with some gaps, in [GN1] and [GN2]. Next we discuss the complex powers and the zeta function of such an operator which is selfadjoint and elliptic and extend Seeley theory to this context. We show that the zeta function is well defined and has a meromorphic extension to the complex plane with at most simple poles located at \( s = -\frac{m+k}{d} \), \( k \in \{ 0, 1, \ldots \} \). In Theorem 3.8, using the tools presented in the previous sections, we construct a sequence of generalized Dirac-type densities \( \eta_k \), \( k \in \{ 0, 1, \ldots \} \), on the base orbifold \( M \) and compute the residues of the zeta function as integrals of these densities. In Theorem 3.8 we reinterpret the computations we make as integrals of genuine smooth densities on the strata of the canonical stratification of \( M \).

The results of this paper provide the analytic foundation for the study of indices, signatures and torsion for orbifolds equipped with a Riemannian metric (Riemannian orbifolds), which will provide the topic of a subsequent paper. We recall the reader that many of the moduli spaces have a canonical structure of an orbifold (rather than manifold) and their topological and geometric invariants are of legitimate interests.

Finally, I want to thank D. Burghelea for the help and advice in preparing this work. In many respects he is the coauthor of this paper.

1. \( \Gamma \) Equivariant Pseudodifferential Operators
1.1. \( \Gamma \) Vector Bundles.

**Definition 1.1.** Let \((\Gamma, \cdot)\) be a finite group of order \(|\Gamma|\). A real or complex smooth vector bundle \(E \to M\) endowed with a left smooth \(\Gamma\) action by bundle isomorphisms is called a \(\Gamma\) vector bundle.

For each \(\gamma \in \Gamma\) we have a diffeomorphism \(t_\gamma : M \to M\) of the base space and a linear isomorphism \((T_\gamma)_x : E_x \to E_{t_\gamma(x)}\) between the fiber above \(x\) and the fiber above \(t_\gamma(x)\). As usual, we will denote \(t_\gamma(x)\) by \(\gamma x\). \(\Gamma\) acts on the space of sections \(C^\infty(M; E)\) by \((\gamma \cdot f)(x) = (T_\gamma)_x f(\gamma^{-1}x)\). We will denote the action of \(\gamma\) on the space of sections by \(T_\gamma\).

**Example 1.** A particular case of a \(\Gamma_0\) vector bundle is the trivial vector bundle \(U \times V \cong U\) endowed with a product \(\Gamma_0\) action \(\mu \times \rho\) on \(U \times V\) and with the \(\Gamma_0\) action \(\mu\) on \(U\) where \(\mu : \Gamma_0 \times U \to U\) is the restriction of a linear action \(\mu : \Gamma_0 \times \mathbb{R}^m \to \mathbb{R}^m\) to an open invariant neighborhood of the origin \(U \subset \mathbb{R}^m\) and \(\rho : \Gamma_0 \times V \to V\) is a linear representation on the vector space \(V\).

**Example 2.** If \(\Gamma_0 \subset \Gamma\) is an inclusion of groups, then the bundle \(\Gamma \times_{\Gamma_0} (U \times V) \to \Gamma \times U\) is a \(\Gamma\) vector bundle, with \(\Gamma\) acting on the total space and base space by left translations.

The following proposition states that any \(\Gamma\) vector bundle is locally diffeomorphic to a \(\Gamma\) vector bundle as above.

**Proposition 1.1.** Let \(E \to M\) be a \(\Gamma\) vector bundle and \(x \in M\). Let \(\Gamma_x\) be the isotropy group of \(x\). Then there exists \(U\) a \(\Gamma_x\) invariant neighborhood of \(x\) in \(M\) and \(O\) a neighborhood of the origin in \(\mathbb{R}^m\) such that the restriction bundle \(E|_U \to \Gamma U\) is isomorphic to the \(\Gamma\) vector bundle \(\Gamma \times_{\Gamma_x} (O \times V) \to \Gamma \times O\) by a \(\Gamma\) equivariant isomorphism.

The proof of this proposition is elementary and rather standard. For the sake of completeness we included it in the Appendix A.

Let \(E \to M\) be a \(\Gamma\) vector bundle. Let \((V_1, \rho_1), (V_2, \rho_2), \ldots, (V_l, \rho_l)\) be a complete set of irreducible non-isomorphic complex representations of \(\Gamma\), with \(\rho_1\) being the trivial one dimensional representation. Then \(C^\infty(M; E)\) decomposes as the direct sum of multiples of \(V_i\)

\[
C^\infty(M; E) = \bigoplus_{i=1}^l C^\infty(M; E)_i
\]

\(C^\infty(M; E)_i\) is spanned by submodules of the form \(h(V_i)\) where \(h : V_i \to C^\infty(M; E)\) are \(\Gamma\) equivariant maps. We have the isomorphism

\[
V_i \otimes \text{Hom}_\Gamma(V_i; C^\infty(M; E)) \cong C^\infty(M; E)_i
\]

where \(e\) is the evaluation map, \(e(v, \phi) = \phi(v)\). If \(M\) is a closed Riemannian manifold endowed with a \(\Gamma\) invariant metric, and we have a \(\Gamma\) invariant Hermitian structure on the bundle \(E \to M\), then \(\Gamma\) acts on the space of \(L^2\) integrable sections \(L^2(M; E)\), and we get an analogous decomposition

\[
L^2(M; E) = \bigoplus_{i=1}^l L^2(M; E)_i
\]
such that the spectrum of \( L \) and \( \text{ers} \) respectively

First observe that the operators \( \text{ers} \) and \( \text{ers} \) respectively the action of \( \gamma \) and \( \gamma \) respectively the fixed point set of the diffeomorphism \( \gamma \circ T \) for any \( \gamma \in \Gamma \).

**Proposition 1.2.** If \( \chi_i, i = 1, \ldots, l \) is the complete set of irreducible characters of the group \( \Gamma \) corresponding to the representations \( (V_i, \rho_i) \) and \( k_i = \text{dim}_\mathbb{C}V_i \) are the corresponding dimensions, then the projection on the \( i^{th} \) factor is given by

\[
pr_i = \frac{k_i}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_i(\gamma^{-1}) \cdot T_\gamma
\]

(10)

The proof can be found in [Sr], Part 1, Theorem 8.

### 1.2. \( \Gamma \) Equivariant Pseudodifferential Operators.

**Definition 1.2.** A pseudodifferential operator \( A \) acting on the space of sections \( C^\infty(M; E) \) of a \( \Gamma \) vector bundle is called \( \Gamma \) equivariant if \( A \) commutes with the action of \( \Gamma \) on \( C^\infty(M; E) \) i.e. \( T_\gamma \cdot A = A \cdot T_\gamma \) for any \( \gamma \in \Gamma \).

Throughout this chapter we suppose that \( A \) is elliptic \( \Gamma \) equivariant pseudodifferential operator of classical type (as described in [Sh], Section 3.7), of positive order \( d \). We suppose that \( \pi \) is an Agmon angle for \( A \), i.e. there exists \( \varepsilon > 0 \) such that the spectrum of \( A \) is disjoint from the region in the complex plane \( \{z \mid \text{arg}(z) \in (\pi - \varepsilon, \pi + \varepsilon)\} \cup \{z \mid |z| < \varepsilon\} \). Then \( A \) and all its complex powers \( A^s, s \in \mathbb{C} \), will preserve the decompositions \( C^\infty(M; E) = \bigoplus_{i=1}^l C^\infty(M; E)_i \) and \( L^2(M; E) = \bigoplus_{i=1}^l L^2(M; E)_i \). Consequently, we can consider the restrictions \( A^s_i : C^\infty(M; E)_i \to C^\infty(M; E)_i \) and \( A^s_i : L^2(M; E)_i \to L^2(M; E)_i \) for \( s \in \mathbb{C} \). The goal of this section is the study of the trace of these operators \( \zeta_{A, i}(s) = Tr(A^s_i) \).

First observe that \( A^s_i = A^s \circ pr_i \). Using Proposition 1.2, we get

\[
A^s_i = \frac{k_i}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_i(\gamma^{-1}) A^s \circ T_\gamma
\]

(11)

so, in order to study the trace of \( A^s_i \), it is convenient to study the trace of \( A^s \circ T_\gamma \).

We will show that these operators are of trace class for all complex numbers \( s \) in a half-plane \( \text{Re}(s) < -K \), and that the associated trace function, \( Tr(A^s \circ T_\gamma) \), as a function of \( s \), has a meromorphic extension to \( \mathbb{C} \) with at most simple poles.

Let us fix an element \( \gamma \in \Gamma \) and denote by \( T = T_\gamma \), \( t = t_\gamma \) and \( \mathcal{T} = \mathcal{T}_\gamma \) respectively the action of \( \gamma \) on the total space, on the base space and on the space of sections of the bundle. Let \( M^\gamma \) be the fixed point set of the diffeomorphism \( t \) and \( M^\gamma = \bigcup_{i \in I} M^\gamma_i \) be the decomposition in connected components of dimensions respectively \( n_i \). Let \( m = \text{dim}(M) \).

**Theorem 1.3.** The operators \( A^s \circ T \) are of trace class for \( s \) in the half-plane \( \text{Re}(s) < -\frac{m}{d} \). The associated zeta function \( \zeta_{A, \gamma}(s) = Tr(A^s \circ T_\gamma) \), defined for \( \text{Re}(s) < -\frac{m}{d} \), has a meromorphic continuation to the whole complex plane \( \mathbb{C} \) with at most simple poles at \( -\frac{m+n+k}{d} \) for \( k \in \{0, 1, 2, \ldots\} \). One can construct positive numbers \( d_i^\gamma \) and densities \( \{\eta_{i,k}^\gamma\} \) for \( k \in \{0, 1, 2, \ldots\} \) on the submanifolds \( M^\gamma_i \) such
that $\eta_{i,k}^\gamma = 0$ for $k < m - n_i \gamma$ and the residue of the function $\zeta_{A,\gamma}$ at $s = -\frac{m+k}{d}$ is equal to

$$\text{res}_{s = -\frac{m+k}{d}} \zeta_{A,\gamma} = \sum_i d_i \int_{\mathcal{M}_i} \eta_{i,k}^\gamma$$

The details of the proof of this Theorem are contained in Appendix B. We will give the description of the coefficients $d_i$ and the densities $\eta_{i,k}^\gamma$.

For $\gamma \in \Gamma$ and $\mathcal{M}_i^\gamma$ a connected component of the fixed point set $M^\gamma$, let $x \in \mathcal{M}_i^\gamma$. Using a $\Gamma$ invariant metric in the tangent space $T(M)$ one decomposes $T_x(M) = T_x(M^\gamma) \oplus T_x(\mathcal{M}_i^\gamma)^\perp$ and the action of $\gamma$ on $T_x(M)$ as $\text{Id} \oplus \bar{T}$. Then we define:

$$d_i = |\text{det}(\bar{T} - \text{Id})|^{-1}.$$ 

The quantity above does not depend on the particular choice of $x \in \mathcal{M}_i^\gamma$ and decomposition $T_x(M) = T_x(M^\gamma) \oplus T_x(\mathcal{M}_i^\gamma)^\perp$.

In order to define the densities $\eta_{i,k}^\gamma$ near $x \in \mathcal{M}_i^\gamma$ we choose a coordinate chart $\phi: (0,0) \to (\phi(O),x)$ in an open neighborhood $\phi(O)$ of $x \in M$ such that the induced action of $\Gamma_x$ on $O \subset \mathbb{R}^m$ is given by a linear orthogonal maps. This can be realized with the help of a $\Gamma$ invariant metric on $M$ and the associated exponential map at $x$. Denote the action of $\gamma$ on $O$ by $t$. Let $(x_1,x_2)$ be coordinates given by this chart, $x_1 \in O^t$ the fixed point set of $t$, and $x_2 \in O^t$. Let $(\xi_1,\xi_2)$ be the corresponding coordinates in the cotangent bundle space. Then $t = \text{Id} \oplus \bar{T}$. Observe that $d_i = |\text{det}(\bar{T} - \text{Id})|^{-1}$. Let $a_s(x_1,x_2,\xi_1,\xi_2)$ be the total symbol of $A^s$ in $O$ and

$$a_s(x_1,x_2,\xi_1,\xi_2) \sim \sum_{k \geq 0} a_{s,k}(x_1,x_2,\xi_1,\xi_2)$$

be its asymptotic expansion (as defined in [35], Section 3.7). The component $a_{s,k}$ is homogeneous in $(\xi_1,\xi_2)$ of degree of homogeneity $sd - k$.

Let $\pi_{s,k}(x_1,w,\xi_1,\xi_2) = a_{s,k}(x_1, (\bar{T} - \text{Id})^{-1}w, \xi_1, \xi_2)$.

Consider the homogeneous symbol $\hat{b}_{s,j}(x_1,\xi_1)$ of degree of homogeneity $sd - j$ given by

$$\hat{b}_{s,j}(x_1,\xi_1) = \sum_{|\alpha| + k = j} \frac{1}{\alpha!} (D_x^\alpha \partial_{\xi_1} \pi_{s,k})(x_1,0,\xi_1,0)$$

Then the density on $O^t$ whose integral computes the residue of $\zeta_{A,\gamma}$ at $s = -\frac{m+k}{d}$ is given by

$$\eta_{i,k}^\gamma(x_1) = -\frac{1}{d} \text{Tr}(\int_{S^{n-1}} \hat{b}_{s,n_i,-m+k}(x_1,\xi) d\vec{\xi} \circ T) \, dx_1$$

if $k \geq m - n_i$

$$\eta_{i,k}^\gamma = 0$$

if $k < m - n_i$

Here $n_i = \text{dim}(O^t) = \text{dim}(\mathcal{M}_i^\gamma)$ and $T = T_{\gamma,x}$ is the map by which $\gamma$ acts in the fiber $E_x$ above $x \in M$. The $(n - 1)$ form $d\vec{\xi}$ is the canonical volume form on $S^{n-1}$ induced from $\mathbb{R}^n$, rescaled by a factor of $(2\pi)^{-n}$. $dx_1$ is the canonical volume form on $O^t$.

Using Theorem 1.3 and the formula (11) linking the operators $A^s: C^\infty(M;E) \to C^\infty(M;E)$, and $A^s \circ T$, for $\gamma \in \Gamma$, we can formulate the following theorem

**Theorem 1.4.** The operator $A^s$ is of trace class and the trace functional

$$\zeta_{A,s}(s) = \text{Tr}(A^s)$$
is a holomorphic function in $s$ on the half-plane $\text{Re}(s) < -\frac{m}{d}$. The function $\zeta_{A,i}$ has a meromorphic continuation to the whole complex plane with at most simple poles situated at $s = -\frac{m+k}{d}$ for $k \in \{0, 1, 2, \ldots \}$. For each $k \in \{0, 1, 2, \ldots \}$ there exist smooth densities $\eta^k_\gamma$ on the fixed point set $M^\gamma$ such that the residue of $\zeta_{A,i}$ at $s = -\frac{m+k}{d}$ is equal to

$$\text{res}_{s = -\frac{m+k}{d}} \zeta_{A,i} = k_i \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_i(\gamma^{-1}) \int_{M^\gamma} \eta^k_\gamma$$

The smooth density $\eta^k_\gamma$ depends only on a finite number of terms in the asymptotic expansion of the total symbol of the operator $A^g$ and the action of $\Gamma$ on a neighborhood of $M^\gamma$.

Proof. Formula (11) implies:

$$\zeta_{A,i}(s) = \frac{k_i}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_i(\gamma^{-1}) \zeta_{A,\gamma}(s)$$

The first part of the statement in the theorem follows directly from Theorem 1.3. If we define $\eta^k_\gamma = \sum_i n_d \xi^i \eta^i_{\gamma,k}$ then (19) is a direct consequence of the equality (12).

1.3. Dirac-Type Densities on $\Gamma$ Manifolds. Let us consider $C^\infty(M)$ the space of smooth functions on a compact manifold $M$ and endow it with the topology of uniform convergence together with a finite number of derivatives on $M$.

**Definition 1.3.** Let $N$ be a smooth submanifold which is a closed subset of $M$. A smooth density $\eta$ on $N$ defines a continuous functional on the space $C^\infty(M)$ by

$$< \eta, f > = \int_N f \eta \quad \text{for} \quad f \in C^\infty(M)$$

We call such a functional a Dirac-type distribution on $M$.

The singular support of this distribution is equal to $N$ if $\text{dim}(N) < \text{dim}(M)$ and it is empty if $\text{dim}(N) = \text{dim}(M)$.

If $N = M$ then a Dirac-type distribution on $M$ is given by smooth density on $M$. If $N$ is a proper subset of $M$ then a Dirac-type distribution on $N$ can be described locally by a density that is zero on open sets disjoint from $N$ and by a smooth density on $N$. This will not be a smooth density on $M$ anymore, the singular set being exactly $N$. We call this density a Dirac-type density and denote it with the same letter as the smooth density on $N$.

If $\phi : M \to M$ is a diffeomorphism and $\eta$ is a Dirac-type density with singular support $N$, then the push-forward $\phi_*(\eta)$ is a Dirac-type density associated with the smooth density on $\phi(N)$ which is the push-forward of the smooth density $\eta$ on $N$ by the diffeomorphism $\phi|_N : N \to \phi(N)$. We have the following identity

$$< \phi_*(\eta), \phi_*(f) > = < \eta, f > .$$

**Definition 1.4.** If $\Gamma \times M \to M$ is a $\Gamma$ smooth manifold, a $\Gamma$ Dirac-type density on $M$ is a collection of Dirac-type densities $\eta = \{\eta^\gamma\}_{\gamma \in \Gamma}$ indexed by the elements of the group $\Gamma$ with singular support respectively the fixed point sets $M^\gamma$ and such that $\gamma_{\phi}(\eta^\gamma) = \eta^\gamma \gamma^{-1}$ for any $\gamma, \gamma' \in \Gamma$ (observe that, in general, $\gamma'(M^\gamma) = (M^\gamma \gamma^{-1})$).
Definition 1.5. If \( \eta = \{ \eta_\gamma \}_{\gamma \in \Gamma} \) is a Dirac-type density on a \( \Gamma \) manifold \( M \) and \( \chi = \chi_\gamma \) is the character of an irreducible representation of \( \Gamma \) we define the associated distribution \( \eta^\chi \) by

\[
< \eta^\chi, f > = \frac{k_i}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_i(\gamma^{-1}) \cdot \int_{M^\gamma} f_{|M^\gamma} \eta^\gamma
\]

Using the above definitions, Theorem 1.4 can be reformulated as follows:

Theorem 1.5. Let \( A \) be a \( \Gamma \) equivariant elliptic pseudodifferential operator acting on the smooth sections of a \( \Gamma \) bundles manifold and \( A_i \) be the restriction of \( A \) to the component of the space of sections \( C^\infty(M; E) \) corresponding to the irreducible representation \((V_i, \rho_i)\) with the character \( \chi_i \). The trace functional \( \zeta_{A,i}(s) = Tr(A_i^s) \) defines a holomorphic function on the half-plane \( \Re(s) < -m \frac{d}{d} \) which has a meromorphic extension to the whole complex plane with at most simple poles situated at \( s = -m + k \frac{d}{d} \), for \( k \in \{0, 1, 2, \ldots\} \). There exists a family \( \{ \eta_k \}_{k=0}^\infty \) of Dirac-type densities on \( M \) with \( \eta_k = \{ \eta_k^\gamma \}_{\gamma \in \Gamma} \) so that the residue of \( \zeta_{A,i} \) at \( s = -m + k \frac{d}{d} \) is equal to \( < \eta^\chi_k, 1 > \).

Proof. We only have to prove the existence of the representation of the residues of \( \zeta_{A,i} \) as stated in the theorem.

For \( k \in \{0, 1, 2, \ldots\} \) and \( \gamma \in \Gamma \) let \( \eta_k^\gamma \) be the Dirac-type distribution given by the sum of the smooth densities \( \sum_i d_i^\gamma \eta^\gamma_{i,k} \) each defined on the connected component \( M_i^\gamma \) of the fixed point set \( M^\gamma \) as described in Theorem 1.3 and by the formulas (13), (16) and (17). We have \( \gamma^{-1}(M_i^{\gamma \gamma \gamma^{-1}}) = M_i^{\gamma} \) for any \( \gamma, \gamma' \in \Gamma \), and after a convenient reindexing of the connected components of each fixed point set, we can suppose that \( \gamma^{-1}(M_i^{\gamma \gamma \gamma^{-1}}) = M_i^{\gamma} \) as well. Then \( d_i^{\gamma \gamma} = d_i^{\gamma} \) because, as defined by formula (13), they are determinants of two linear maps conjugated by a diffeomorphism defined at the tangent space level by \( \gamma' \). Because the operator \( A \) is \( \Gamma \) equivariant, a straightforward computation shows that the smooth densities \( \eta^\gamma_{i,k} \) and \( \eta^{\gamma \gamma \gamma^{-1}}_{i,k} \) are conjugated by the the map induced at the cotangent space level by \( \gamma' \). Then the collection \( \{ \eta_k^\gamma \}_{\gamma \in \Gamma} \) is a \( \Gamma \) Dirac-type distribution, as described in the definition 1.4. Theorem 1.4 states that the residue of \( \zeta_{A,i} \) at \( s = -m + k \frac{d}{d} \) is equal to

\[
\frac{k_i}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_i(\gamma^{-1}) \cdot \sum_{i} d_i^\gamma \int_{M_i^{\gamma}} \eta^\gamma_{i,k}
\]

which can be rewritten as \( < \eta^\chi_k, 1 > \). \( \square \)
2. Orbifolds and Orbibundles

2.1. Orbifolds. Let \( M \) be a Hausdorff topological space.

**Definition 2.1.** An orbifold chart on \( M \) is given by \( \mathcal{R} = (\tilde{U}, \Gamma, \mu, U, \pi) \) where \( U \) is an open set in \( M \), \( \tilde{U} \) is an open set in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), \( \Gamma \) is a finite group, \( \mu : \Gamma \times \tilde{U} \to \tilde{U} \) is a faithful smooth action of \( \Gamma \) on \( \tilde{U} \) and \( \pi : \tilde{U} \to U \) is a continuous map that factors through a homeomorphism \( \pi \) from the orbit space \( \tilde{U}/\Gamma \) to \( U \) and makes the following diagram commutative:

\[
\begin{array}{ccc}
\tilde{U} & \xrightarrow{\pi} & U \\
\downarrow & & \downarrow \\
\tilde{U}/\Gamma & \xrightarrow{\sim} & U
\end{array}
\]

The open set \( U \) is called a coordinate neighborhood.

If the group action \( \mu \) is the restriction of a linear representation of \( \Gamma \) on \( \mathbb{R}^n \) to a neighborhood of the origin \( \tilde{U} \) and \( x = \pi(0) \), we call \( \mathcal{R} \) a linear chart at \( x \).

**Definition 2.2.** Two orbifold charts \( \mathcal{R}_i = (\tilde{U}_i, \Gamma_i, \mu_i, U_i, \pi_i) \), \( i = 1, 2 \) are called compatible if for any two points \( \tilde{x}_i \in \tilde{U}_i \) such that \( \pi_1(\tilde{x}_1) = \pi_2(\tilde{x}_2) = x \in U_1 \cap U_2 \) there exists a diffeomorphism \( h \) from a neighborhood of \( x_1 \) in \( \tilde{U}_1 \) to a neighborhood of \( x_2 \) in \( \tilde{U}_2 \) such that \( \pi_2 \circ h = \pi_1 \).

**Remark 2.1.** If \( \mathcal{R} = (\tilde{U}, \Gamma, \mu, U, \pi) \) is an orbifold chart and \( x \in U \), there exist a linear orbifold chart \( \mathcal{R}_x \) at \( x \) which is compatible with \( \mathcal{R} \). The proof follows the same idea as the one in the proof of Proposition 1.1 contained in Appendix A. Let \( \tilde{x} \in \tilde{U} \) such that \( \pi(\tilde{x}) = x \) and \( \Gamma_{\tilde{x}} \subset \Gamma \) be the isotropy group of \( \tilde{x} \). Let \( \tilde{V} \) be a \( \Gamma_{\tilde{x}} \) invariant neighborhood of \( \tilde{x} \) such that \( \tilde{V} \cap \gamma \cdot \tilde{V} = \emptyset \) for any \( \gamma \in \Gamma \setminus \Gamma_{\tilde{x}} \). Then \( (\tilde{V}, \Gamma_{\tilde{x}}, \mu, V = \pi(\tilde{V}), \pi|_{\tilde{V}}) \) is an orbifold chart which is compatible with \( \mathcal{R} \). The linearization of the action of \( \Gamma_{\tilde{x}} \) on \( \tilde{V} \) gives us an action \( \mu' \) of \( \Gamma_{\tilde{x}} \) on \( T_{\tilde{x}}(\tilde{V}) \), the tangent space of \( \tilde{V} \) at \( \tilde{x} \). The exponential map \( exp : T_{\tilde{x}}(\tilde{V}) \to \tilde{V} \) associated with a \( \Gamma_{\tilde{x}} \) invariant metric on \( \tilde{V} \) is \( \Gamma_{\tilde{x}} \) equivariant. One can choose a smaller \( \tilde{V} \) so that \( exp \) is an equivariant diffeomorphism between a neighborhood \( \tilde{W} \) of 0 in \( T_{\tilde{x}}(\tilde{V}) \) and \( \tilde{V} \). Then \( (\tilde{W}, \Gamma_x, \mu', V, \pi|_{\tilde{V}} \circ exp) \) is an linear orbifold chart at \( x \) which is compatible with \( \mathcal{R} \).

**Definition 2.3.** An orbifold atlas \( \mathcal{A} \) on \( M \) is a collection of compatible orbifold charts on \( M \) such that the corresponding coordinate neighborhoods form an open cover of \( M \). Two orbifold atlases \( \mathcal{A}_i \), \( i = 1, 2 \) on \( M \) are compatible if their reunion is an orbifold atlas of \( M \).

**Definition 2.4.** An orbifold structure on \( M \) is given by a maximal orbifold atlas \( \mathcal{A} \) on \( M \).

Though an orbifold is given by a topological space and a maximal atlas, in the future we will drop the atlas from the notation of an orbifold and use only the letter designated for the underlying topological space. In order to keep our notations simple we will also drop the representation \( \mu \) from the notation of an orbifold chart.
Example 3. If \( W \) is an open subset of \( M \), then \( W \) inherits an orbifold structure whose charts are all orbifold charts \((\tilde{U}, \Gamma, U, \pi)\) of \( M \) for which \( U \subset W \).

Example 4. Let \( M \) be a differentiable manifold and \( \Gamma \) a finite group of diffeomorphisms of \( M \). Let \( \pi \) be the projection map onto the orbit space \( M/\Gamma \). Then \( M/\Gamma \) has a canonical orbifold structure (cf. [Hf]). An atlas of \( M/\Gamma \) can be obtained as follows: let \( x \in M \) and \( \overline{x} \in M/\Gamma \) its orbit. Let \( \Gamma_x \subset \Gamma \) be the isotropy group of \( x \). Because \( \Gamma \) is finite, there exists an open neighborhood \( \tilde{U} \) of \( x \) in \( M \) which is \( \Gamma_x \) invariant and \( \tilde{U} \cap \gamma \cdot \tilde{U} = \emptyset \) for \( \gamma \in \Gamma \backslash \Gamma_x \). The set \( U = \tilde{U}/\Gamma_x \subset M/\Gamma \) is an open neighborhood of \( \overline{x} \). We call a neighborhood like \( \tilde{U} \) a slice at \( x \). Choose \( \tilde{U} \) small enough such that \( \tilde{U} \xrightarrow{\phi} O \subset \mathbb{R}^n \) is a smooth chart for \( M \). The action of the group \( \Gamma \) can be transported to a smooth action on \( O \) via \( \phi \) and the collection \((O, \Gamma_x, U, \pi \circ \phi^{-1})\) is an orbifold chart of \( M/\Gamma \). All the charts obtained as above are compatible (cf. [Hf]) and the maximal atlas containing them defines the canonical orbifold structure on \( M/\Gamma \).

We have a generalization of the previous example:

**Proposition 2.1.** Let \( M \) be a differentiable manifold and \( \mu : G \times M \to M \) a smooth action of a compact Lie group \( G \) with finite isotropy groups \( G_x \subset G \) for any \( x \in M \). Then the quotient topological space \( M/G \) has a canonical structure of an orbifold.

**Proof.** Let us fix a \( G \) invariant metric on \( M \). This can be done by averaging any metric over the compact group \( G \).

Let \( \overline{x} \in M/G \) be a point in the quotient space and \( x \in M \) such \( \overline{x} = Gx \). Let \( G_x \) be the isotropy group at \( x \). Then \( G_x \) acts on the tangent space \( T_x(M) \) and keeps invariant the tangent space \( T_x(Gx) \) to the \( G \)-orbit of \( x \). Let \( V \subset T_x(M) \) be the orthogonal complement of \( T_x(Gx) \) in \( T_x(M) \), \( V \oplus T_x(Gx) = T_x(M) \). Because the metric on \( M \) is \( G \) invariant, \( G_x \) will act on \( V \) by restriction. Also, all its translations \( gV \subset T_{gx}(M) \) will be \( G_{gx} \) invariant and \( gV \oplus T_{gx}(Gx) = T_{gx}(M) \).

Let \( T(M)_{Gx} \to Gx \) be the restriction of the tangent vector bundle to the orbit \( Gx \), and \( V \to Gx \) be the subbundle whose fiber above \( gx \) is \( gV \). This subbundle has a natural \( G \) vector bundle structure coming from the action of \( G \) on \( M \).

Let us consider the principal bundle \( G_x \to G \to G/G_x \), where \( G_x \) acts by right translations on \( G \), and the associated vector bundle \( V \to G \times G_x \) \( V \to G/G_x \). This vector bundle has a natural \( G \) vector bundle structure, \( G \) acting by left translations on \( G \) and \( G/G_x \).

The maps \( \Phi : G \times G_x, V \to V, \Phi(g, v) = gv \) and \( \phi : G/G_x \to Gx, \phi(gG_x) = gx \) give a \( G \)-equivariant isomorphism \((\Phi, \phi)\) between the two \( G \) vector bundles considered above.

If \( \text{exp} : T(M) \to M \) is the exponential map associated with the \( G \) invariant metric on \( M \), then \( \text{exp} \) realizes a \( G \) equivariant diffeomorphism between a neighborhood \( U \) of the zero section in \( V \) and an open tubular neighborhood \( N \) of the orbit \( Gx \) in \( M \). Because \( G \) is compact one can find an open \( G_x \) invariant neighborhood \( U \) of the origin in \( V \) such that \( \text{exp} \circ \Phi : G \times G_x, U \to N \) is a \( G \) equivariant diffeomorphism. We will then take \( U = \Phi(G \times G_x, U) \). Passing to the \( G \)-orbit spaces, we get a homeomorphism \( \text{exp} \circ \Phi : (G \times G_x, U)/G \to N/G, \text{where } N/G \text{ is an open neighborhood of } \overline{x} = Gx \text{ in } M/G \). We will construct an orbifold chart over \( N/G \) around \( \overline{x} \).
The map $\iota : U \to G \times G_x U$, $\iota(u) = (e, u)$ gives a homeomorphism when passing to the orbit spaces $\tau : U/G_x \xrightarrow{\sim} (G \times G_x U)/G$. Denote by $\pi$ the composition $U \to G \times G_x U \xrightarrow{\exp_\Phi} N \xrightarrow{\text{proj}} N/G$. Then $(U, G_x, N/G, \pi)$ is a linear orbifold chart at $\overline{\tau}$, where the action of $G_x$ on $U \subset V$ is the restriction of the linear representation of $G_x$ on $V$. As shown above, the induced map to the orbit spaces $\overline{\tau} : U/G_x \to N/G$ is a homeomorphism.

We have to show that any two different charts defined as above are compatible.

Let $\mathcal{R}_i = (U_i, G_{x_i}, N_i/G, \pi_i)$ be two orbifold charts around $\overline{\tau}_i$, $i = 1, 2$. Let $u_i \in U_i$ such that $\pi_i(u_1) = \pi_2(u_2) = \overline{\tau} \in M/G$. Then, using the definition of $\pi$ in the construction of the charts $\mathcal{R}_i$, one can find $g_i \in G$, $i = 1, 2$ and $x \in M$, $Gx = \overline{\tau}$, so that we have $\exp_{g_1x_1}(g_1u_1) = \exp_{g_2x_2}(g_2u_2) = x \in M$.

By replacing $x$ with $g_1^{-1}x$ we can assume that $g_1 = e \in G$. Moreover, the map $G \times G_{x_i} U_i \ni (g, u) \mapsto \exp_{g_1x_1}(gu) \in M$ is a $G$ equivariant local diffeomorphism, so one gets a local $G$ equivariant diffeomorphism $\Psi$ between a neighborhood of $(e, u_1)$ in $G \times G_{x_1} U_1$ and a neighborhood of $(g_2, u_2)$ in $G \times G_{x_2} U_2$. The tangent space to $G \times G_{x_1} U_1$ at $(g_1, u_1)$ is equal to $T_{g_1}(G) \oplus T_{u_1}(U_1)$ and the derivative of $\Psi$ will have a block decomposition corresponding to this direct sum as $T_{(g_1, u_1)} \Psi = (A \, B \, C \, D)$. Because $G$ acts by left translations on the first component of $G \times G_{x_i} U_i$ and $\Psi$ is $G$ equivariant, a straightforward computation shows that $A = Id$ and $C = 0$, so $D$ is a diffeomorphism between $T_{U_1}(U_1)$ and $T_{U_2}(U_2)$. Then there exist neighborhoods $U'_i \subset U_i$ of $u_i$ such that the map $U'_i \ni u \mapsto \text{proj} \circ \Psi(e, u) \in U'_2$ is a diffeomorphism, whose derivative at $1_i$ is $D$. Denote this map by $h$. The horizontal dotted lines in the following diagram represent locally defined maps that are local diffeomorphisms:

\[
\begin{array}{ccc}
G \times G_{x_1} U_1 & \xrightarrow{\exp_{\Phi_1}} & G \times G_{x_2} U_2 \\
\downarrow{\psi} & \cong & \downarrow{\exp_{\Phi_2}} \\
M & \xrightarrow{\text{proj}} & M/G \\
\pi_1 & \xrightarrow{h} & \pi_2 \\
\downarrow & & \downarrow \pi_2 \\
U_1 & \xrightarrow{\text{pr}_2} & U_2
\end{array}
\]

The map $\text{pr}_2$ is locally defined in a neighborhood of $(g_2, u_2) \in G \times G_{x_2} U_2$ as $\text{pr}_2(g, u) = u$. Using the fact that the upper triangle, the left and right quadrilaterals and the large trapezoid are commutative, we get that $\pi_1 = \pi_2 \circ h$ on a neighborhood near $u_1$. Because $u_1$ was chosen arbitrarily, we conclude that the orbifold charts $\mathcal{R}_1$ and $\mathcal{R}_2$ are compatible.

Later we will show that any connected orbifold can be obtained as the orbit space of a smooth manifold endowed with a smooth action of a compact Lie group.

Remark 2.2. If $M$ is an orbifold and $x \in M$ consider an orbifold chart $(\tilde{U}, \Gamma, U, \pi)$ in a neighborhood of $x$ and $\tilde{x} \in \tilde{U}$ such that $\pi(\tilde{x}) = x$. The isomorphism class of the isotropy group $\Gamma_{\tilde{x}}$ depends only on $x$ and not on a particular choice of $\tilde{x}$ or chart around $x$. We will denote this isomorphism class by $G_x$. 


Definition 2.5. A point $x \in M$ is called smooth if $G_x$ is the isomorphism class of the trivial group (1) and it is called singular otherwise.

We will denote by $M_{reg}$ the set of regular points of $M$ and by $M_{sing}$ the set of singular points. $M_{reg}$ is an open and dense subset of $M$ whose induced orbifold structure is a genuine manifold structure.

Definition 2.6. Let $M$ and $N$ be two orbifolds. A map $\phi : M \to N$ is an orbifold diffeomorphism if $\phi$ is a homeomorphism and for any orbifold chart $R = (\tilde{U}, \Gamma, U, \pi)$ of $M$, the collection $\phi(R) = (\tilde{U}, \Gamma, \phi(U), \phi \circ \pi)$ is an orbifold chart of $N$ and for any orbifold chart $R' = (\tilde{U}', \Gamma', U', \pi')$ of $N$, $\phi^{-1}(R') = (\tilde{U}', \Gamma', \phi^{-1}(U'), \phi^{-1}\pi')$ is an orbifold chart of $M$.

To see whether a homeomorphism $\phi$ is a diffeomorphisms one must check that $\phi$ and $\phi^{-1}$ take orbifold charts of an atlas (not necessarily the maximal atlas) into orbifold charts.

Example 5. If $M$ is an orbifold and $(\tilde{U}, \Gamma, U, \pi)$ is an orbifold chart, then $\tilde{U}/\Gamma$ has a canonical orbifold structure as described in Example 2 and $U$ has an induced orbifold structure from the orbifold structure of $M$. Then the induced map $\overline{\pi} : \tilde{U}/\Gamma \to U$ is an orbifold diffeomorphism.

Definition 2.7. Let $M$ and $N$ be two orbifolds. A continuous map $f : M \to N$ is smooth if for any $x \in M$ one can find orbifold charts $R = (\tilde{U}, \Gamma, U, \pi)$ around $x$ and $R' = (\tilde{U}', \Gamma', U', \pi')$ around $f(x)$ and a smooth map $\tilde{f} : \tilde{U} \to \tilde{U}'$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
\tilde{U} & \xrightarrow{f} & \tilde{U}' \\
\pi \downarrow & & \downarrow \pi' \\
U & \xrightarrow{f} & U'
\end{array}
\]

Remark 2.3. The spaces $\mathbb{F}^k$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ have a canonical structure of an orbifold, given by the atlas with a unique chart $R = (\mathbb{F}^k, \Gamma = (e), \mathbb{F}^k, \pi = Id)$. Then a continuous map $f : M \to \mathbb{F}^k$ is called smooth if it is smooth as a map between orbifolds.

Remark 2.4. If $M_i$, $i = 1, 2$ are two orbifolds, $U_i \subset M_i$ are open subsets and $\phi : U_1 \cong U_2$ is an orbifold diffeomorphism, then the topological space $M_1 \amalg M_2 / \sim$ where we identify $x \in U_1$ with $\phi(x) \in U_2$ has a canonical structure of an orbifold, and the inclusion maps $\iota_i : M_i \to M_1 \amalg M_2 / \sim$ are smooth. An atlas for $M_1 \amalg M_2 / \sim$ is given by the reunion of two atlases for respectively $M_1$ and $M_2$.

This procedure allows us to create an orbifold by gluing orbifold charts along orbifold diffeomorphisms.

We will need the following statement, whose proof can be found in [GN1], Theorem 2.1.

Proposition 2.2. Let $M$ be a smooth orbifold and $\{U_\alpha\}$ an open cover of $M$. Then there exists a countable partition of unity $\{h_\alpha, i \in \mathbb{N}\}$ subordinated to $\{U_\alpha\}$ such that each $h_\alpha$ is a smooth function with compact support.
2.2. Vector Orbibundles. Let \( p : E \to M \) be a continuous map between two orbifolds. Let \( V \) be a fixed vector space over \( \mathbb{R} \) or \( \mathbb{C} \).

**Definition 2.8.** A vector orbibundle chart is a collection \( \mathcal{R} = (\tilde{U}, V, \Gamma, U, \Pi, \pi) \) such that \((\tilde{U}, \Gamma, U, \pi)\) is an orbifold chart for \( M \), \((\tilde{U} \times V, \Gamma, p^{-1}(U), \Pi)\) is an orbifold chart for \( E \) such that the induced action of \( \Gamma \) on the trivial vector bundle \((\tilde{U} \times V \overset{pr_2}{\to} \tilde{U})\) is by vector bundle isomorphisms, and the following diagram is commutative:

\[
\begin{array}{ccc}
\tilde{U} \times V & \overset{\Pi}{\longrightarrow} & p^{-1}(U) \\
pr_1 \downarrow & & \downarrow p \\
\tilde{U} & \overset{\pi}{\longrightarrow} & U
\end{array}
\]

Though every vector orbibundle chart comes with the actions of a group \( \Gamma \) on \( \tilde{U} \) and \( \tilde{U} \times V \), for the sake of simplicity we will not show them explicitly in the notation of the chart.

**Definition 2.9.** A vector orbibundle chart \( \mathcal{R} = (\tilde{U}, V, \Gamma, U, \Pi, \pi) \) is called a linear chart at \( x \in M \) if \((\tilde{U}, \Gamma, U, \pi)\) is a linear orbifold chart at \( x \in M \) with \( \mu : \Gamma \times \tilde{U} \to \tilde{U} \) the restriction of a linear representation of \( \Gamma \) and there exists a linear representation \( \rho : \Gamma \times V \to V \) such that the action of \( \Gamma \) on the trivial bundle \( \tilde{U} \times V \overset{pr_2}{\to} \tilde{U} \) is equal to the diagonal action \( \mu \otimes \rho \).

**Definition 2.10.** Two vector orbibundle charts \( \mathcal{R}_i = (\tilde{U}_i, V, \Gamma_i, U_i, \Pi_i, \pi_i), \ i = 1, 2, \) are compatible if for any two points \( \tilde{x}_1 \in \tilde{U}_1 \) such that \( \pi(\tilde{x}_1) = (\tilde{x}_2) = x \in U_1 \cap U_2 \) there exists a neighborhood \( \tilde{W}_1 \) of \( \tilde{x}_1 \) in \( \tilde{U}_1 \), a neighborhood \( \tilde{W}_2 \) of \( \tilde{x}_2 \) in \( \tilde{U}_2 \) and a vector bundle diffeomorphism \((H, h)\) between \((\tilde{W}_1 \times V \overset{pr_2}{\to} \tilde{W}_1)\) and \((\tilde{W}_2 \times V \overset{pr_2}{\to} \tilde{W}_2)\) such that \( \pi_2 \circ h = \pi_1 \) and \( \Pi_2 \circ H = \Pi_1 \).

**Definition 2.11.** A vector orbibundle atlas on \( E \overset{p}{\to} M \) is a collection of compatible vector orbibundle charts such that the corresponding coordinate neighborhoods of \( M \) form an open cover of \( M \). A structure of vector orbibundle on \( E \overset{p}{\to} M \) is given by a maximal vector orbibundle atlas.

Using the ideas in the proof of Proposition 1.1 and Remark 2.1, one can prove that for any vector orbibundle chart \( \mathcal{R} = (\tilde{U}, V, \Gamma, U, \Pi, \pi) \) and \( x \in U \) there exist a linear vector orbibundle chart at \( x \), \( \mathcal{R}_x \), which is compatible with \( \mathcal{R} \). As a consequence, any vector orbibundle has an atlas consisting of linear vector orbibundle charts.

**Remark 2.5.** A vector orbibundle \( E \overset{p}{\to} M \) is usually not a vector bundle. If \( x \in M_{\text{sing}} \) is a singular point, then \( p^{-1}(x) \) might not have a vector space structure. If \((\tilde{U}, V, \Gamma, U, \Pi, \pi)\) is a chart around \( x \) then \( p^{-1}(x) \) is the quotient of \( V \) by the action of the isotropy group \( \Gamma_x \) with \( \pi(\tilde{x}) = x \). The isomorphism class of the representation of \( \Gamma_x \in G_x \) on \( V \) depends only on \( x \) and not on a particular choice of vector orbibundle chart and \( \tilde{x} \). Denote this isomorphism class by \( V_x \). Then the restriction of \( E \overset{p}{\to} M \) to \( \{x \in M \mid V_x \text{ is trivial}\} \) is a genuine vector bundle.

**Example 6.** Let \( E \overset{p}{\to} M \) be a smooth vector bundle and suppose the finite group \( \Gamma \) acts on the vector bundle by bundle diffeomorphisms. Denote by \( \Pi \), resp. \( \pi \),
the canonical projections onto the orbit spaces $E \xrightarrow{\Pi} E/\Gamma$ and $M \xrightarrow{\pi} M/\Gamma$. For $x \in M$, let $\mathfrak{r} = \Gamma x \in M/\Gamma$ be its orbit and $\Gamma_x$ the isotropy group of $x$. Choose $\tilde{U}$ a $\Gamma_x$ invariant neighborhood of $x$ in $M$, as we did in the Example 4, such that $\tilde{U} \cap \gamma \tilde{U} = \emptyset$ for $\gamma \in \Gamma \backslash \Gamma_x$. The group $\Gamma_x$ acts by diffeomorphisms on the restriction of the initial bundle $E|_{\tilde{U}} = p^{-1}(\tilde{U}) \xrightarrow{\rho} \tilde{U}$ and Proposition 4 provides us with a linear vector orbibundle chart at $x$, $R_x = (O, E_x, \Gamma_x, \tilde{U}/\Gamma_x, \Pi, \pi)$.

We also have a generalization of the above example, analogous to Proposition 2.3.

**Proposition 2.3.** Let $E \xrightarrow{\rho} M$ be a smooth vector bundle endowed with a smooth action of a compact Lie group $G$ such that any $x \in M$ has a finite isotropy group $G_x \subset G$. Then $E/G \xrightarrow{\overline{\rho}} M/G$ has a canonical structure of a vector orbibundle.

**Proof.** Let us fix a $G$ invariant Riemannian metric on the base space and a $G$ invariant linear connection $\nabla$ in the vector bundle.

For a fixed $x = Gx \in M/G$ with $x \in M$, consider, as described in the proof of Proposition 2.1, the isotropy group $G_x$ and the direct sum decomposition of the tangent space at $x$ as $G_x$ modules $V \oplus T_x(Gx) = T_x(M)$. Denote by $\mu_x : G_x \times V \rightarrow V$ the action of $G_x$ on the vector space $V$. Let $V \rightarrow Gx$ be the restriction to $Gx$ of the subbundle of the tangent bundle to $M$ whose fiber above $gx$ is $gV$. The map $\Phi : G \times G_x V \rightarrow V$, $\Phi(g, v) = gv$ is $G$ equivariant and realizes an isomorphism between the vector bundles $G \times G_x V \rightarrow G/Gx$ and $V \rightarrow Gx$. Let $U \subset V$ be an open $G_x$ invariant neighborhood of the origin such that the map $G \times G_x U \ni (g, u) \mapsto \exp \circ \Phi(g, u) = \exp_{Gx}(gu) \in M$ is a $G$ equivariant map and realizes a diffeomorphism onto its image $N$. We showed in the proof of Proposition 2.3 that $(U, G_x, N/G, \pi)$ is a linear orbifold chart at $\mathfrak{r}$ for $M/G$. We will construct a vector orbibundle chart for $E/G \xrightarrow{\overline{\rho}} M/G$ at $\mathfrak{r}$.

Let $\tilde{\nabla} \xrightarrow{\overline{\rho}} \nabla$ be the pull-back of the vector bundle $E|_{Gx} \xrightarrow{\rho} Gx$ via the natural projection map $T(M)|_{Gx} \xrightarrow{\text{proj}} Gx$ restricted to $\nabla$. The vector bundle $\tilde{\nabla} \xrightarrow{\overline{\rho}} \nabla$ has a $G$ vector bundle structure coming from the action of $G$ on $E$ and $M$; indeed $\tilde{\nabla} = \{(v, Y); v \in \nabla, Y \in E|_{Gx} \text{ with } \text{proj}(v) = p(Y) \in Gx\}$ and $g \in G$ acts by the diagonal action $g \cdot (v, Y) = (gv, gY)$.

The group $G_x$ fixes the point $x$, so it acts on the fiber above $x$ by a linear representation $\rho_x : G_x \times E_x \rightarrow E_x$. Let $G_x$ act on $V \times E_x$ by the product action $\mu_x \otimes \rho_x$. Consider the vector bundle $G \times G_x (V \times E_x) \xrightarrow{\text{proj}} G \times G_x V$ with fiber $E_x$. The group $G$ acts on this bundle by left translations. We already showed that $\Phi : G \times G_x V \rightarrow \nabla$ given by $\Phi(g, v) = gv \in \nabla|_{Gx}$ is a $G$ equivariant diffeomorphism.

Let $\overline{\Phi} : G \times G_x (V \times E_x) \rightarrow \tilde{\nabla}$ defined as $\overline{\Phi}(g, v, Y) = (gv, gY)$. Then we have the following commutative diagram in which the horizontal maps are $G$ equivariant diffeomorphisms

$$
\begin{array}{ccc}
G \times G_x (V \times E_x) & \xrightarrow{\Phi} & \tilde{\nabla} \\
\text{proj} \downarrow & & \downarrow \overline{\rho} \\
G \times G_x V & \xrightarrow{\text{proj}} & \nabla
\end{array}
$$

Indeed $\overline{\rho} \circ \overline{\Phi}(g, v, Y) = \overline{\rho}(gv, gY) = gv = \Phi \circ \text{proj}(g, v, Y) = \Phi(g, v)$. $\overline{\Phi}$ is surjective because any $(v', Y') \in \tilde{\nabla}$ with $\text{proj}(v') = p(Y) = gx \in Gx$ is of the form
\( \tilde{\Phi}(g, g^{-1}v', g^{-1}Y') \). Also, if \( \tilde{\Phi}(g, v, Y) = \tilde{\Phi}(g', v', Y') \) then \( \text{proj}(gv) = \text{proj}(g'v') \in Gx \), so \( g^{-1}g' = g'' \in Gx \) and \( g' = gg'' \). But \( gv = g'v' = gg''v' \) and \( gY' = g'Y' = gg''Y' \) so \( v = g'v' = Y = g''Y' \) with \( gg'' = g' \). We conclude that \( (g, v, Y) = (g', v', Y') \in G \times Gx (V \times E_x) \), so \( \tilde{\Phi} \) is injective as well. The pair \((\tilde{\Phi}, \Phi)\) defines a \( G \) equivariant isomorphism of \( G \) vector bundles.

In the proof of Proposition 2.1 we considered the exponential map \( \exp : V \to M \) with respect to the \( G \) invariant metric on \( M \), which realizes a \( G \) equivariant diffeomorphism between a neighborhood \( U \) of the zero section in \( V \to Gx \) and \( N \)–a tubular neighborhood of \( Gx \) in \( M \). For \( v \in gV \subset T_{gx}(M) \) and \( X \in E_{gx} \), let \( s(t), t \in [0, 1] \) be the path that realizes the parallel transport in \( E \overset{g}{\to} M \) with respect to the \( G \) invariant connection \( \nabla \) above the path \( \exp_{gx}(tv) \in M \), with \( s(0) = X \). Then we define \( \tilde{\exp}(v, X) = s(1) \), \( \tilde{\exp} : V \to E \). The map \( \tilde{\exp} \) is \( G \) equivariant and \( \tilde{\exp}(v, \cdot) \) is a linear isomorphism. Then the restriction of \( \tilde{\exp} \) to \( \tilde{\pi}^{-1}(U) \subset V \) together with the restriction of \( \exp \) to \( U \) give us a \( G \) equivariant isomorphism between the restriction of the bundle \( \tilde{\nu} \overset{\tilde{\nu}}{\to} V \) to \( U \) and the \( G \) vector bundle \( E_{\tilde{\pi}} \overset{\tilde{\nu}}{\to} N \). If we choose a smooth \( G \) equivariant neighborhood \( U \) of the origin in \( V \) as in Proposition 2.1, the pair of maps \( (\tilde{\exp}, \exp) \circ (\tilde{\Phi}, \Phi) \) gives us a \( G \) equivariant isomorphism between the \( G \) vector bundles \( G \times Gx(1) \overset{\tilde{\pi}}{\to} G \times GxU \) and \( E_{\tilde{\pi}} \overset{\tilde{\pi}}{\to} N \). Passing to the \( G \) orbits we get a homeomorphism between \( G \times Gx(U \times E_x) \overset{\tilde{\pi}}{\to} G \times GxU/G = U/Gx \) and \( E_{\tilde{\pi}} \overset{\tilde{\pi}}{\to} N/G \).

We will describe a linear vector orbibundle chart around \( \bar{\tau} = Gx \subset M/G \). Let \( \iota : U \to G \times Gx \), \( \iota(u) = (e, u) \) and \( \pi \) be the composition \( U \overset{\gamma}{\to} G \times Gx (1) \overset{\exp_{\bar{\tau}}}{\to} G \times GxU \overset{\tilde{\pi}}{\to} U/Gx \). Also let \( \iota : U \times E_x \to G \times Gx (U \times E_x) \), \( \iota(u, X) = (e, u, X) \) and \( \Pi \) be the composition \( U \times E_x \overset{\iota}{\to} G \times Gx(U \times E_x) \overset{\exp_{\bar{\tau}}}{\to} E_{\tilde{\pi}} \overset{\tilde{\pi}}{\to} E_{\tilde{\pi}} \overset{\tilde{\pi}}{\to} N/G \). Then \( (U, E_x, Gx, N/G, \Pi, \pi) \) is a linear vector orbibundle chart around \( Gx \).

As in Proposition 2.1, we have to prove that any two vector orbibundle charts defined above are compatible.

Let \( \mathcal{R}_i = (U_i, E_{x_i}, Gx_i, N_i/G, \Pi_i, \pi_i), i = 1, 2 \) be two vector orbibundle charts around \( \bar{\tau} \). Let \( u_i \in U_i \) such that \( \pi_i(u_i) = \pi_2(u_2) = \bar{\tau} \in M/G \). As described in Proposition 2.1, we can find \( g_2 \in G \) and \( x \in M \) so that \( \exp_{x_1}(u_1) = \exp_{gx_2}(g_2u_2) = x \) and \( Gx = \bar{\tau} \). The maps \( G \times Gx_i(U_i \times E_{x_i}) \ni (g, u, X) \mapsto \tilde{\exp}(g, u, X) \in E \) for \( i = 1, 2 \) define \( G \) equivariant diffeomorphisms. Then one can choose \( G \) invariant neighborhoods \( W_i \) of \((g_i, u_i) \) in \( G \times Gx_i.U_i \) so that the composition of the previous diffeomorphisms defines a \( G \) equivariant diffeomorphism \( \Psi \) between \( \tilde{\pi}^{-1}(W_1) \subset G \times Gx_1(U_1 \times E_{x_1}) \) and \( \tilde{\pi}^{-1}(W_2) \subset G \times Gx_2(U_2 \times E_{x_2}) \). \( \Psi \) induces a \( G \) equivariant diffeomorphism \( \Psi \) between \( W_1 \) and \( W_2 \). Using the fact that \( \Psi \) is \( G \) equivariant, one can show, as in the proof of Proposition 2.1, that the composition

\[
(28) \quad H = (U_1 \times E_{x_1}, \iota \circ G \times Gx_i(U_1 \times E_{x_1}) \overset{\Psi}{\to} G \times Gx_2(U_2 \times E_{x_2}) \overset{\tilde{\pi}}{\to} U_2 \times E_{x_2})
\]

is a local diffeomorphism which together with the induced local diffeomorphism \( h : U_1 \to U_2 \) between neighborhoods of \( u_1 \) in \( U_1 \) and \( u_2 \) in \( U_2 \) define a vector bundle diffeomorphism \((H, h)\) which realizes the compatibility of the charts \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) in a neighborhood of \( u_1 \).

**Definition 2.12.** Let \( E_i \overset{\tilde{\pi}}{\to} M_i, i = 1, 2 \), be two vector orbibundles. The pair \((\Phi, \phi)\) defines a vector bundle diffeomorphism if \( \Phi : E_1 \to E_2 \) and \( \phi : M_1 \to M_2 \)
are smooth diffeomorphisms, \( p_2\Phi = \phi p_1 \), for any vector orbibundle chart
\( \mathcal{R} = (\bar{U}, V, \Gamma, \Pi, \pi) \) of \( E \xrightarrow{p_1} M_1 \) the collection \((\Phi, \phi)\mathcal{R} = (\bar{U}, V, \Gamma, \phi(U), \Phi\Pi, \phi\pi)\) is a vector orbibundle chart of \( E_2 \xrightarrow{p_2} M_2 \) and for any vector orbibundle chart
\( \mathcal{R}' = (\bar{U}', V', \Gamma', \Pi', \pi') \) of \( E_2 \xrightarrow{p_2} M_2 \) the collection
\((\Phi^{-1}, \phi^{-1})\mathcal{R}' = (\bar{U}', V', \Gamma', \phi^{-1}(U), \Phi^{-1}\Pi, \phi^{-1}\pi)\)
is a vector orbibundle chart of \( E_1 \xrightarrow{p_1} M_1 \).

**Example 7.** If \( E \xrightarrow{p_2} M \) is a vector orbibundle and \((\bar{U}, V, \Gamma, U, \Pi, \pi)\) is a chart, then
\(((\bar{U} \times V) / \Gamma \xrightarrow{\mu} \bar{U} / \Gamma)\) is a vector orbibundle which is isomorphic to the restriction to \( U \) of the initial orbibundle \( E |_U \xrightarrow{p_2} U \) by an isomorphism induced by \((\Pi, \pi)\) to the \( \Gamma \) orbit spaces.

**Definition 2.13.** Let \( E_i \xrightarrow{p_i} M_i \), \( i = 1, 2 \) be two vector orbibundles. The pair
\((\Phi, \phi)\) defines a vector orbibundle morphism if \( \Phi : E_1 \rightarrow E_2 \) and \( \phi : M_1 \rightarrow M_2 \) are smooth, \( p_2\Phi = \phi p_1 \), and for any \( x \in M_1 \) one can find vector orbibundle charts
\( \mathcal{R} = (\bar{U}, V, \Gamma, U, \Pi, \pi) \) around \( x \) and \( \mathcal{R}' = (\bar{U}', V', \Gamma', U', \Pi', \pi') \) around \( \phi(x) \) and a smooth vector bundle map \( (\Phi, \phi) \) from \( \bar{U} \times V \xrightarrow{\mu} \bar{U} \) to \( \bar{U}' \times V' \xrightarrow{\mu} \bar{U}' \) such that
\((\Pi', \pi') \circ (\Phi, \phi) = (\Phi, \phi) \circ (\Pi, \pi)\).

**Remark 2.6.** If \( E_i \xrightarrow{p_i} M_i \), \( i = 1, 2 \) are two vector orbibundles and \( U_i \subset M_i \) are two open sets such that there exists a vector orbibundle diffeomorphism \( (\Phi, \phi) \) between
\( E_1|_{U_1} \xrightarrow{p_{i,1}} U_1 \) and \( E_2|_{U_2} \xrightarrow{p_{i,2}} U_2 \) then \( E_1 \Pi \xrightarrow{\mu} M_1 \Pi M_2 \) has a canonical structure of a vector orbibundle, an atlas being given by the reunion of two atlases \( \mathcal{A}_i \) for \( E_i \xrightarrow{p_i} M_i \), \( i = 1, 2 \).

In view of the above observation, one can apply to vector orbibundles the usual operations that were done on ordinary vector bundles: duality, direct sum, tensor product, symmetric and exterior powers. If \( E_i \xrightarrow{p_i} M_i \), \( i \in I \), is a collection of vector orbibundles and if \( \mathcal{R}_{i,\alpha} \) are vector orbibundle charts, then we can apply the operations on the vector bundles \( U_{i,\alpha} \times V_i \xrightarrow{\mu} U_{i,\alpha} \) and extend the action of the group \( \Gamma_{i,\alpha} \) to these vector bundles. We will get new vector orbibundle charts and the vector orbibundles of orbit spaces associated to these new charts can be glued together by diffeomorphisms prescribed by the way the initial vector orbibundle charts were glued to yield the initial vector orbibundles. Because the initial spaces were Hausdorff, the underlying topological spaces for the resulting orbibundle will be Hausdorff as well.

**Example 8.** Let \( M \) be an orbifold. The tangent vector orbibundle \( T(M) \) can be constructed starting from an atlas \( \mathcal{A} = (\mathcal{R}_i) \). If \( \mathcal{R}_i = (\bar{U}_i, \Gamma_i, U_i, \pi_i) \) is an orbifold chart, with \( \bar{U} \subset \mathbb{R}^m \), then \((\bar{U}_i, \mathbb{R}^m, \Gamma_i, U_i, \pi_i, \pi_i) \) will be a vector orbibundle chart for \( T(M) \), where the action of \( \Gamma_i \) on \( \bar{U} \times \mathbb{R}^m \cong T(\bar{U}_i) \) is the induced action to the tangent space level. The vector orbibundles \( \bar{U}_i \times \mathbb{R}^m / \Gamma_i \xrightarrow{\mu_\Gamma} \bar{U}_i / \Gamma_i \) are glued together along diffeomorphisms prescribed by the identifications of \( \bar{U}_i / \Gamma \cong U_i \) inside \( M \) to form the tangent vector orbibundle \( T(M) \).

**Definition 2.14.** A metric on the orbifold \( M \) is a smooth map \( \mu : T(M) \otimes T(M) \to \mathbb{R} \) such that for each orbifold chart \((\bar{U}, \Gamma, U, \pi)\) the map \( \mu \) is induced by a \( \Gamma \) invariant metric \( \bar{\mu} : T(\bar{U}) \otimes T(\bar{U}) \to \mathbb{R} \).

An orbifold endowed with a metric is called a Riemannian orbifold.
On any paracompact orbifold one can construct a metric by first defining metrics on orbifold charts and then gluing them together using a smooth partition of unity.

**Definition 2.15.** A hermitian structure on a complex vector orbibundle $E \overset{p}{\to} M$ is given by a smooth map $<,> : E \otimes E \to \mathbb{C}$ such that for each vector orbibundle chart $(\tilde{U}, V, \Gamma, \Pi, \pi)$ there exists a $\Gamma$ invariant hermitian structure on $(\tilde{U} \times V \overset{p_r}{\to} \tilde{U})$ which induces $<,>$ above $\tilde{U}/\Gamma \cong U$.

**Remark 2.7.** We can think of a vector orbibundle as of a smooth family of linear representations of groups $\Gamma_x \in G_x$ on $E_x$ parameterized by $x \in M$. Then $E_x \equiv \tilde{E}_x$. A hermitian structure is a smooth family of hermitian products in $\tilde{E}_x$ which are $\Gamma_x$ invariant and induce $<,>$ above $\tilde{U}/\Gamma \cong U$.

**Definition 2.16.** Let $E \overset{p}{\to} M$ be a vector orbibundle. A map $f : M \to E$ is a smooth section if $p \circ f = Id_M$ and if $f$ is smooth as a map between the orbifolds $M$ and $E$. We will denote the space of smooth sections by $\text{C}^\infty(M; E)$.

**Remark 2.8.** If $A = (\mathcal{R}_i)$ is a vector orbibundle atlas with $\mathcal{R}_i = (\tilde{U}_i, V, \Gamma_i, U_i, \Pi_i, \pi_i)$ then the vector orbibundles $\tilde{U}_i \times V/\Gamma_i \overset{p_i}{\to} \tilde{U}/\Gamma_i$ and $E|_{U_i} \overset{p}{\to} U_i$ are isomorphic via the map induced by $(\Pi_i, \pi_i)$ to the orbit spaces. A map $f : M \to E$ is a smooth section if the restriction to each $U_i$ can be identified via the above vector bundle diffeomorphisms to sections induced to the $\Gamma_i$ orbit spaces by $\Gamma_i$ invariant smooth sections $f_i$ of the $\Gamma_i$ vector bundles $\tilde{U}_i \times V/\Gamma_i \overset{p_i}{\to} \tilde{U}_i$.

In Proposition 2.3 we showed that starting with a $G$ vector bundle $E \overset{p}{\to} M$ where the Lie group $G$ acts on $M$ with finite isotropy groups and passing to the $G$ orbit spaces we obtain a vector orbibundle $E/G \overset{\mathcal{P}}{\to} M/G$. In this case we have a global characterization of the smooth sections of the vector orbibundle.

**Proposition 2.4.** The canonical map

$$\text{Sect}(M; E)^G \to \text{Sect}(M/G; E/G)$$

identifies the $G$ invariant smooth sections of the $G$ vector bundle $E \overset{p}{\to} M$ with the smooth sections of the vector orbibundle of $G$ orbits $E/G \overset{\mathcal{P}}{\to} M/G$.

**Proof.** It is obvious that any $G$ invariant section $f \in \text{Sect}(M; E)^G$ defines a section $\overline{f} \in \text{Sect}(M/G; E/G)$ by $\overline{f}(Gx) = Gf(x)$.

Let $f_1$ and $f_2$ be two $G$ invariant sections in $\text{Sect}(M; E)^G$ which equal induce sections $\overline{f}_1 = \overline{f}_2 \in \text{Sect}(M/G; E/G)$. If $f_1 \neq f_2$ then there exists $x \in M$ such that $f_1(x) \neq f_2(x)$. Nevertheless, $\overline{f}_1(\overline{x}) = \overline{f}_2(\overline{x}) \in E/G$. Then there exists $g \in G$ such that $f_1(x) = gf_2(x) \in E_x$. But $g f_2(x) = f_2(gx) \in E_x$ so $gx = x$. So $f_1(x) = f_1(g^{-1}x) = g^{-1} f_1(x) = g^{-1} f_2(x) = f_2(x)$. We reached a contradiction so the map $\text{Sect}(M; E)^G \to \text{Sect}(M/G; E/G)$ is injective.

Let $h \in \text{C}^\infty(M/G; E/G)$ be a smooth section. We need to construct a $G$ invariant smooth section $f \in \text{C}^\infty(M; E)^G$ such that $\overline{f} = h$. We will construct $f$ from local data. As we showed in Proposition 2.3, the local model of passing from a $G$ vector bundle to a vector orbibundle is given by passing from the $G$ vector bundle $G \times G\backslash (U \times V) \overset{p}{\to} G \times G\backslash U$ to the vector orbibundle $(U \times V)/G' \overset{\mathcal{P}}{\to} U/G'$. $G' \subset G$ is a finite group which acts on the left on the open set $U \subset \mathbb{R}^n$ and on
a vector space $V$ and by right translations on $G$. The action of $G'$ on $G \times U$ and $G \times (U \times V)$ is given by $g' \cdot (g, u) = (gg'^{-1}, g' u)$ and $g' \cdot (g, u, v) = (gg'^{-1}, gu, gv)$. In fact the vector bundle $G \times (U \times V) \xrightarrow{p} G \times U$ is a $G'$ vector bundle whose associated vector orbibundle of $G'$ orbits is the local model $G \times_{G'} (U \times V) \xrightarrow{p} G \times_{G'} U$, which happens to be a genuine vector bundle because the action of $G'$ is free. A section $h \in C^\infty(U/G'; (U \times V)/G')$ is induced by a smooth section $h \in C^\infty(U; U \times V)$. Let $\tilde{f} \in C^\infty(G \times U; G \times (U \times V))$ be defined by $\tilde{f}(g, u) = (g, h(u))$. This is a $G'$ invariant smooth section which is also $G'$ invariant and it induces a smooth section $f \in C^\infty(G \times_{G'} U; G \times_{G'} (U \times V))$. It is obvious that the map $f$ induces $h$ when passing to the $G'$ orbit spaces. Because of the first part of the proof such an $f$ is unique, so we can glue different sections $f$ to get a global $G$ invariant smooth section in $C^\infty(M; E)$ which will induce $h \in C^\infty(M/G; E/G)$.

The problem of proving that if $f \in \text{Sect}(M; E)^G$ is smooth then its image $\tilde{f} \in \text{Sect}(M/G; E/G)$ is smooth is a local one. A smooth section $f \in C^\infty(G \times_{G'} U; G \times_{G'} (U \times V))$ is induced by a $G'$ invariant smooth section $\tilde{f} \in C^\infty(G \times U; G \times (U \times V))$. If $f$ is $G$ invariant then $\tilde{f}$ is $G'$ invariant so there exists a $G'$ smooth section $h$ of the $G'$ vector bundle $U \times V \xrightarrow{\pi_2} U$ such that $\tilde{f}(g, u) = (g, h(u))$. The induced section $\tilde{f}$ of the vector orbibundle $(U \times V)/G' \xrightarrow{p} U/G'$ is the same as the one induced by $h$. So $\tilde{f}$ is smooth.

Remark 2.9. Let $\tilde{E} \xrightarrow{\tilde{p}} \tilde{M}$ be a $\Gamma$ vector bundle and $E \xrightarrow{p} M$ be the associated vector orbibundle of $\Gamma$ orbits. If $U \subset \tilde{M}$ is an open subset and $\Gamma' \subset \Gamma$ a subgroup such that $\tilde{U}$ is $\Gamma'$ invariant and $\gamma \tilde{U} \cap \tilde{U} = \emptyset$ for $\gamma \in \Gamma \setminus \Gamma'$, then $\tilde{E}_{|\tilde{U}} \xrightarrow{\tilde{p}} \tilde{U}$ is a $\Gamma'$ vector bundle and the associated vector orbibundle of $\Gamma'$ orbits is a subbundle of $E \xrightarrow{p} M$. Let $U = \tilde{U}/\Gamma'$. Denote by $\iota$ the inclusion map between the two vector orbibundles. $\iota$ induces a map between the spaces of compactly supported sections $\iota_* : C_0^\infty(U; E_{|U}) \rightarrow C_0^\infty(M; E)$ given by extension by zero on $M \setminus U$ and a restriction map $\iota^* : C^\infty(M; E) \rightarrow C^\infty(U; E_{|U})$. We have $C^\infty(M; E) \cong C^\infty(M; \tilde{E})'$ and $C^\infty(U; E_{|U}) \cong C^\infty(U; \tilde{E}_{|U})'$. Then the map $\iota^*$ is induced by the restriction map $C^\infty(M; \tilde{E}) \rightarrow C^\infty(U; \tilde{E}_{|U})$. The map $\iota_*$ between $C^\infty(\tilde{U}; \tilde{E}_{|\tilde{U}})$ and $C^\infty(\tilde{M}; \tilde{E})$ is given by

$$\iota_*(\tilde{f}) = \sum_{i=1}^{l} \gamma_i \tilde{f}$$

where $\{\gamma_1 = e, \gamma_2, \ldots, \gamma_l\}$ with $l = [\Gamma : \Gamma']$ is a complete system of left coset representatives for $\Gamma/\Gamma'$, or

$$\iota_*(\tilde{f}) = \frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma} \gamma \tilde{f}$$

for $\tilde{f} \in C^\infty(\tilde{U}; \tilde{E}_{|\tilde{U}})'$.  

2.3. The Representation Theorem. In this section we will state and prove the reciprocal of Proposition 2.3.

Theorem 2.5. Any vector orbibundle $E \xrightarrow{p} M$ is isomorphic to the vector orbibundle of $G$ orbits associated with a $G$ vector bundle $\tilde{E} \xrightarrow{\tilde{p}} \tilde{M}$, with $G$ a compact
Lie group acting on $M$ with finite isotropy groups. In particular any orbifold $M$ is diffeomorphic to the orbifold of $G$ orbits associated with a $G$ manifold $M$ where the action of the compact Lie group $G$ has finite isotropy groups.

Proof. Let us fix a metric $\mu$ on the base space $M$. Let $G = O(m)$ be the orthogonal group, $m = \dim(M)$. We will describe the orthonormal frame bundle $F(M)$ and $M$. We will show that $F(M)$ is a manifold and that the group $G$ acts on it with finite isotropy groups. The $G$ vector bundle we need to construct will be essentially the pull-back of $E \xrightarrow{\pi} M$ to $F(M)$ via the canonical projection.

The description of the space of orthonormal frames uses an atlas $A = (R_i)$ of the vector orbibundle $E \xrightarrow{\pi} M$. Let $\mu_i$ be the $\Gamma_i$ invariant metric on $\tilde{U}_i \subset \mathbb{R}^m$ which induces the metric $\mu$ via the map $\tilde{U}_i \xrightarrow{z_i} U_i$. Let $F(\tilde{U}_i)$ be the space of orthonormal frames of $\tilde{U}_i$. Then the orthogonal group $O(m)$ acts by right translations on $F(\tilde{U}_i)$. Because the metric $\mu_i$ is $\Gamma_i$ invariant, the action of $\Gamma_i$ on $\tilde{U}_i$ extends to $F(\tilde{U}_i)$. Observe that this action is free. Indeed, if $\gamma \in \Gamma_i$ and $(x, \lambda) \in F(\tilde{U}_i)$ such that $\gamma(x, \lambda) = (x, \lambda')$ then $\gamma \in \Gamma_x$ and $\gamma$ acts on the tangent space $T_{x}(\tilde{U}_i)$ as the identity map. But the order of $\gamma$ is finite so $\gamma = e \in \Gamma_i$. Then the space of orbits $F(\tilde{U}_i)/\Gamma_i$ is a smooth manifold. The local diffeomorphisms between open sets $\pi_i^{-1}(U_i \cap U_j) \subset \tilde{U}_i$ and $\pi_j^{-1}(U_i \cap U_j) \subset \tilde{U}_j$ preserve the metrics so they will induce local diffeomorphisms between $F\pi_i^{-1}(U_i \cap U_j)$ and $F\pi_j^{-1}(U_i \cap U_j)$ and a diffeomorphism between their images in $(F(\tilde{U}_i))/\Gamma_i$ respectively $F(\tilde{U}_j)/\Gamma_j$. The space of orthonormal frames $F(M)$ is obtained by gluing the manifolds $(F(\tilde{U}_i))/\Gamma_i$ along these diffeomorphisms and it has a natural structure of smooth manifold. The right action of $O(m)$ on $F(\tilde{U}_i)$ commutes with the action of $\Gamma_i$ and with the local diffeomorphisms induced between different $F(\tilde{U}_i)$ and $F(\tilde{U}_j)$ so we will get an induced right action of $O(m)$ on $F(M)$.

Let us consider the $\Gamma_i$ vector bundle $\tilde{U}_i \times V \xrightarrow{pr_2} \tilde{U}_i$ corresponding to the chart $R_i$ and the pull-back to the space of orthonormal frames $F(\tilde{U}_i) \times V \xrightarrow{pr_2} F(\tilde{U}_i)$ together with the induced left action of the group $\Gamma_i$ on it and the right action of group $O(m)$. The action of $\Gamma_i$ is free and commutes with the action of $O(m)$ so if we pass to the spaces of $\Gamma_i$ orbits we get a genuine vector bundle $F(\tilde{U}_i) \times V \xrightarrow{pr_2} F(\tilde{U}_i)/\Gamma_i$ endowed with the action of $O(m)$. By gluing these vector bundles together using diffeomorphisms induced by the identifications made on the initial vector orbibundle charts we get an $O(m)$ vector bundle $E \xrightarrow{\pi} F(M)$.

We will show that the action of $O(m)$ on $F(M)$ has finite isotropy groups and if we pass to the space of $O(m)$ orbits we get a vector orbibundle isomorphic with the initial vector orbibundle $E \xrightarrow{\pi} M$.

Any point in $F(M)$ is the $\Gamma_i$ orbit of a point $(x, \lambda) \in F(\tilde{U}_i)$ for some index $i$. The group $O(m)$ acts freely on $F(\tilde{U}_i)$. If $g \in O(m)$ fixes the $\Gamma_i$ orbit $(x, \lambda)$ then there exists $\gamma \in \Gamma_i$ such that $\gamma(x, \lambda) = \gamma(x, \lambda')$. This is equivalent to $x = \gamma x$ and $\lambda'g = \gamma \lambda'$. The group $O(m)$ acts freely and transitively on the frames at $x \in \tilde{U}_i$. The group $(\Gamma_i)_x$ acts freely on the frames at $x \in \tilde{U}_i$ as well. Then the above equalities imply that the cardinality of the isotropy group of $(x, \lambda) \in F(M)$ with respect to the action of $O(m)$ is equal to the cardinality of the group $(\Gamma_i)_x$, which is finite.
Passing to the vector orbibundle of $O(m)$ orbit spaces $\tilde{E}/O(m) \overset{pr}{\to} F(M)/O(m)$ can be realized by first passing to the $\Gamma_i$ vector bundles $F(U) \times V/O(m) \overset{pr}{\to} F(U)/O(m)$ and then gluing together the resulting vector orbibundles of $\Gamma_i$ orbits. But the $\Gamma_i$ vector bundles $\tilde{U} \times V \overset{pr}{\to} \tilde{U}$ and then $F(U)/O(m) \overset{pr}{\to} F(U)/O(m)$ are canonically isomorphic and the resulting vector orbibundles obtained by gluing the vector orbibundles of $\Gamma_i$ orbits will be isomorphic. So $\tilde{E} \overset{p}{\to} M$ and $\tilde{E}/O(m) \overset{p}{\to} F(M)/O(m)$ are isomorphic.

\[\square\]

2.4. Sobolev Spaces. Consider the Euclidean space $\mathbb{R}^m$ with the canonical scalar product $(x, y) = \sum_{i=1}^{m} x_i y_i$ and let the finite group $\Gamma$ act on $\mathbb{R}^m$ by linear isometries. We have an induced action of $\Gamma$ on the space of function on $\mathbb{R}^m$ defined by $\gamma f(x) = f(\gamma^{-1}x)$.

For compactly supported smooth functions $f \in C_0^\infty(\mathbb{R}^m)$ the Fourier transform $\hat{f} : \mathbb{R}^m \to \mathbb{R}$ is defined by $\hat{f}(\xi) = \int e^{-i<x,\xi>} f(x) \, dx$. Then $(\gamma f)(\xi) = \int e^{-i<y,\gamma^{-1}\xi>} f(\gamma^{-1}x) \, dx$ and if we replace $x$ by $\gamma y$ we get

\begin{equation}
(\hat{\gamma f})(\xi) = \int e^{-i<y,\gamma^{-1}\xi>} f(y) \, dy = \hat{f}(\gamma^{-1}\xi) = \gamma \hat{f}(\xi)
\end{equation}

(we used the fact that $\gamma$ was an isometry and $\det(\gamma) = 1$).

This shows that the Fourier transform commutes with the action of $\Gamma$ on the space of functions.

For $s \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R}^m)$ the Sobolev norm $\| \cdot \|_s$ is defined by $\| f \|^2_s = \int (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi$. The Sobolev space $H_s(\mathbb{R}^m)$ is the completion of $C_0^\infty(\mathbb{R}^m)$ with respect to the norm $\| \cdot \|_s$. Because the group $\Gamma$ acts by isometries on $\mathbb{R}^m$ and the Fourier transform is $\Gamma$ equivariant we conclude that the action of $\Gamma$ on the space of smooth functions preserves the Sobolev norms and extends to an action by isometries on the Sobolev spaces $H_s(\mathbb{R}^m)$. The closure of the space of smooth $\Gamma$ invariant functions $C_0^\infty(\mathbb{R}^m)^\Gamma$ coincides with the subspace of $\Gamma$ invariant elements $H_s(\mathbb{R}^m)^\Gamma = \{ f \in H_s(\mathbb{R}^m) \mid \gamma f = f \text{ for all } \gamma \in \Gamma \}$.

In order to define the Sobolev norm of the smooth functions on the orbifold $\mathbb{R}/\Gamma$ it is convenient to replace the Sobolev norm of the $\Gamma$ invariant smooth functions with an equivalent norm.

**Definition 2.17.** For $f \in C_0^\infty(\mathbb{R}^m)^\Gamma \equiv C_0^\infty(\mathbb{R}^m/\Gamma)$ let $\| f \|^\Gamma_s = \frac{1}{|\Gamma|} \| f \|_s$.

The Sobolev space $H_s(\mathbb{R}^m/\Gamma)$ is the completion of $C^\infty(\mathbb{R}^m/\Gamma)$ with respect to the norm $\| \cdot \|_s^\Gamma$.

If $\tilde{U} \subset \mathbb{R}^m$ is a $\Gamma$ invariant neighborhood of the origin with compact closure let $H_s(\tilde{U}/\Gamma)$ be the closure of $C_0^\infty(\tilde{U})^\Gamma$ in $H_s(\mathbb{R}^m/\Gamma)$.

We observe that $H_s(\mathbb{R}^m/\Gamma) \cong H_s(\mathbb{R}^m)^\Gamma$.

Our goal is to define the Sobolev spaces $H_s(M)$ for $M$ a compact orbifold. The above definitions will describe the elements of the Sobolev space $H_s(M)$ whose support is included in the image of a linear chart $(\tilde{U}, \Gamma, \tilde{U}/\Gamma \cong \pi(\tilde{U}), \pi)$. In order for this description to be independent of the chosen orbifold chart we need to show that the definition of the spaces $H_s(\tilde{U}/\Gamma)$ is compatible with the following two operations occurring in the compatibility of the linear orbifold charts:
1. If \( \phi: \tilde{U}_1 \to \tilde{U}_2 \) is a diffeomorphism intertwining the two linear actions of the group \( \Gamma \) on two open neighborhoods of the origin \( \tilde{U}_i, i = 1, 2 \) in \( \mathbb{R}^m \) then the two orbifold charts \((\tilde{U}_i, \Gamma, \tilde{U}_i/\Gamma, \pi_i), i = 1, 2 \) are compatible with the map \( \phi \) inducing a diffeomorphism between the orbifolds \( \tilde{U}_1/\Gamma \) and \( \tilde{U}_2/\Gamma \).

2. If \( \Gamma_x \subset \Gamma \) is the isotropy group of \( x \in \mathbb{R}^m \) and \( \tilde{V} \subset \tilde{U} \) is a \( \Gamma_x \) invariant neighborhood of \( x \) such that \( \gamma \tilde{V} \cap \tilde{V} = \emptyset \) for \( \gamma \in \Gamma \setminus \Gamma_x \) then the restricted chart \((\tilde{V}, \Gamma_x, \pi(\tilde{V}), \pi|_{\tilde{V}}) \) and the initial chart \((\tilde{U}, \Gamma, \tilde{U}/\Gamma, \pi) \) are compatible via the canonical inclusion \( i: \tilde{V} \hookrightarrow \tilde{U} \) inducing a smooth map between the associated orbifolds \( \tilde{V}/\Gamma_x \) and \( \tilde{U}/\Gamma \).

In the first case the diffeomorphism \( \phi \) induces a continuous bijection with a continuous inverse between the spaces of smooth functions \( C^\infty_0(\tilde{U}_i), i = 1, 2 \), endowed with the Sobolev norms, (cf. [11] Lemma [1.3.3]) and permutes with the action of the group \( \Gamma \). As a consequence \( \phi \) will induce a continuous bijection with a continuous inverse between the Sobolev spaces \( H_s(U_i/\Gamma) \cong H_s(\tilde{U}_i)^{\Gamma} \), \( i = 1, 2 \).

In the second case the inclusion \( i: \tilde{V} \to \tilde{U} \) induces the map \( i_*: C^\infty_0(\tilde{V})^{\Gamma_v} \to C^\infty_0(\tilde{U})^{\Gamma} \) with \( i_*(f) = \sum_{i=1}^k \gamma_i f \) where \( \{\gamma_i\}_{i=1}^k \) is a complete system of left coset representatives for \( \Gamma/\Gamma_x \). We extend the compactly supported function \( f \) by zero outside \( \tilde{V} \). Then \( ||i_*(f)||_s^\Gamma = \frac{1}{|\gamma|} ||i_*(f)||_s = \frac{1}{|\gamma|} \sum_{i=1}^k ||\gamma_i f||_s = \frac{1}{|\gamma|} ||f||_s^\Gamma \). So \( i_* \) induces an isometric embedding of \( H_s(\tilde{V}/\Gamma_x) \) into \( H_s(\tilde{U}/\Gamma) \).

We will need the following statement as well:

**Lemma 2.6.** If \( f \in C^\infty_0(\tilde{U})^{\Gamma} \cong C^\infty_0(\tilde{U}/\Gamma) \) is a compactly supported smooth function then the multiplication operator by \( f \) is continuous operator on \( C^\infty_0(\tilde{U}/\Gamma) \) endowed with the Sobolev norm \( || \cdot ||_s \).

**Proof.** The multiplication operator by \( f \) is a continuous map on \( C^\infty_0(\tilde{U})^{\Gamma} \) endowed with the Sobolev norm \( || \cdot ||_s \), being a differential operator of order zero. The restriction to the \( \Gamma \) invariant smooth functions remains a continuous operator.

We will define the Sobolev spaces \( H_s(M) \) for a compact orbifold \( M \).

Let \( \mathcal{A} = (\tilde{U}_i, \Gamma_i, U_i, \pi_i) \) be a finite atlas of the orbifold \( M \) consisting of linear charts and \( \{\phi_i\}_i \) be a smooth partition of unity associated with the open cover \( (U_i)_i \).

For \( f \in C^\infty(M) \) define the Sobolev norm
\[
||f||^A_{\mathcal{A}(\phi)} = \sum_i ||\pi_i^*(\phi_i f)||^\Gamma_i,
\]
where \( \pi_i^*(\phi_i f) \in C^\infty_0(\tilde{U}_i)^{\Gamma_i} \) is the unique \( \Gamma_i \) invariant map inducing \( \phi_i f \in C^\infty(U_i) \). As in the case of manifolds, this norm depends on the choice of the atlas and the partition of unity, but two different choices lead to two equivalent norms. This is a direct consequence of the equivalence of Sobolev norms for equivalent orbifold charts as described above and the continuity of the multiplication operators by smooth functions. The Sobolev space \( H_s(M) \) is the completion of the space of smooth functions with respect to any of the norms \( || \cdot ||^A_{\mathcal{A}(\phi)} \).

In the case of vector orbibundles, we define the Sobolev norms \( || \cdot ||^F_s \) for the local model \( (\tilde{U} \times \mathbb{R}^k)/\Gamma \xrightarrow{\varphi} \tilde{U}/\Gamma \), with \( \tilde{U} \subset \mathbb{R}^m \), by \( ||f||^F = \frac{1}{|\pi|} ||f||_s \) where \( f \in C^\infty_0(\tilde{U}/\Gamma; (\tilde{U} \times \mathbb{R}^k)/\Gamma)) \cong C^\infty_0(\tilde{U}; \tilde{U} \times \mathbb{R}^k)^{\Gamma} \) and \( || \cdot ||_s \) is the usual Sobolev norm restricted to the \( \Gamma \) invariant sections. If \( E \overset{f}{\to} M \) is a vector orbibundle, we use a vector orbifold atlas and a partition of unity to define the Sobolev norm of smooth sections as in the case of smooth functions. All Sobolev norms defined using different
atlases and partitions of unity will be equivalent and the Sobolev space $H_s(M; E)$ is the completion of $C^\infty(M; E)$ with respect to any of these equivalent norms.

**Remark 2.10.** We showed in Proposition 2.9 that any vector orbibundle $E \xrightarrow{p} M$ is isomorphic to the associated vector orbibundle of a $G$ vector bundle $\tilde{E} \xrightarrow{\tilde{p}} \tilde{M}$ with the compact Lie group $G$ acting on $\tilde{M}$ with finite isotropy groups. Then $C^\infty(M; E) \cong C^\infty(\tilde{M}; \tilde{E})^G$. Also $H_s(M; E) = H_s(\tilde{M}; \tilde{E})^G$. To prove this, we use a $G$ invariant Sobolev norm $\| \cdot \|_s$ on $C^\infty(M; E)$. The fact that the restriction of the Sobolev norm $\| \cdot \|_s$ to the $G$ invariant smooth sections is equivalent to the Sobolev norm on $C^\infty(M; E)$ can be seen directly when looking at the local model $G \times_G (U \times V) \xrightarrow{\tilde{p}} G \times_G U$ and passing to the vector orbibundle $(U \times V)/G' \xrightarrow{\tilde{p}} U/G'$. Then the completion of $C^\infty(\tilde{M}; \tilde{E})^G \cong C^\infty(M; E)$ with respect to the induced Sobolev norm $\| \cdot \|_s$ is equal to $H_s(M; E)$.

2.5. **Dirac-Type Densities on Orbifolds.** In this chapter we will construct a certain family of Dirac-type densities and the associated Dirac-type distributions.

**Definition 2.18.** Let $\Gamma \times \tilde{M} \rightarrow \tilde{M}$ be a faithful action of a finite group on a smooth manifold and $M = \tilde{M}/\Gamma$ the associated orbifold. A Dirac-type density $\eta$ on the orbifold $M$ is given by a $\Gamma$ Dirac-type density $\tilde{\eta}$ in $C^\infty(\tilde{M}; E)$, where $\tilde{\eta}$ is invariant under the $\gamma \in \Gamma$ action on $\tilde{M}$.

The Dirac-type density $\eta$ defines a continuous functional on $C^\infty_0(M)$ by

$$< \eta, f >= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \int_{\tilde{M}/\gamma} \tilde{f}|_{\tilde{M}/\gamma} \eta_{\gamma}$$

where $\tilde{f} \in C^\infty_0(\tilde{M})^\Gamma$ induces $f \in C^\infty_0(M)$. We call the above functional the Dirac-type distribution associated with $\eta$.

We want to extend the above notions to a general orbifold $M$, which is not the quotient of a smooth manifold by a finite group.

**Remark 2.11.** Let $\Gamma_i$, $i = 1, 2$ be two isomorphic groups acting respectively on two smooth manifolds $\tilde{M}_1$ and $\tilde{M}_2$ as above. Suppose that the actions are conjugated by a diffeomorphism $\phi : \tilde{M}_1 \rightarrow \tilde{M}_2$. Then the orbifolds $M_1 = \tilde{M}_1/\Gamma_1$ and $M_2 = \tilde{M}_2/\Gamma_2$ are diffeomorphic by the diffeomorphism $\phi$ and for any Dirac-type density $\eta_1 = \{ \eta_{\gamma}\}_{\gamma \in \Gamma_1}$ on $M_1$ the collection $\phi_* (\eta_1) = \{ \phi_* (\eta_{\gamma})\}_{\gamma \in \Gamma_2}$ is a Dirac-type density on $M_2$. The corresponding Dirac-type distributions on $C^\infty_0(\tilde{M}_i)$, $i = 1, 2$ are conjugated by the isomorphism induced by $\phi$ between the spaces of smooth functions.

**Remark 2.12.** Let $\eta$ be a Dirac-type density on the orbifold $M = \tilde{M}/\Gamma$. Let $\tilde{U} \subset \tilde{M}$ be an open subset and $\Gamma' \subset \Gamma$ a subgroup such that $\tilde{U}$ is $\Gamma'$ invariant and $\gamma \tilde{U} \cap \tilde{U} = \emptyset$ for $\gamma \in \Gamma \setminus \Gamma'$. Then we have a $\Gamma$ equivariant diffeomorphism $\Gamma \times \Gamma' \tilde{U} \cong \Gamma' \tilde{U}$. The orbifold $U = \tilde{U}/\Gamma'$ is an open suborbifold of $M$. Denote by $\iota : U \rightarrow M$ the inclusion map. The restriction of the Dirac-type densities $\eta_{\gamma'}$ on $U$ for $\gamma' \in \Gamma'$ define a $\Gamma'$ Dirac-type density on $U$ and so a Dirac-type density on the orbifold $U$ which we denote by $\iota^*(\eta)$ or simply $\eta_U$. Let $\iota_* : C^\infty_0(U) \rightarrow C^\infty_0(M)$ be the map given by the extension by zero on $M \setminus U$. Then

$$< \eta_U, f > = < \eta, \iota_*(f) > \quad \text{for any } f \in C^\infty_0(U)$$

Indeed, if $\tilde{f} \in C^\infty_0(\tilde{U})$ induces $f$, then $\iota_*(\tilde{f}) = \sum_{i=1}^l \gamma_i \tilde{f}_i$, with $\{ \gamma_1, \gamma_2, \ldots, \gamma_l \}$ a complete system of left coset representatives for $\Gamma/\Gamma'$, as proved in Remark 2.9.
The left-hand side of (35) is equal to
\[ <\eta, U, f > = \frac{1}{|\Gamma|} \sum_{\gamma' \in \Gamma} \int_{\tilde{U} \gamma'} \tilde{f}(\tilde{\gamma}', \eta') \]
and the right-hand side is equal to
\[ <\eta, \iota \ast (\tilde{f}) > = 1_{|\Gamma|} \sum_{\gamma \in \Gamma} \int_{\tilde{M} \gamma} \tilde{M} \gamma f \gamma \gamma \]

Let \(\phi : \tilde{M}_1 / \Gamma_1 \to \tilde{M}_2 / \Gamma_2\) be a diffeomorphism between two orbifolds. We will define the action of \(\phi\) on Dirac-type densities. Because \(\phi\) is a diffeomorphism, there exist open covers \(\{U_{\alpha}\}_\alpha\) of the two orbifolds, finite groups \(\Gamma_{\alpha}\) acting on open subsets \(\tilde{U}_{\alpha}\) of the Euclidean space \(\mathbb{R}^m\) such that \(\tilde{U}_{\alpha} / \Gamma_{\alpha} \cong U_{\alpha}\). The diffeomorphism \(\phi\) between \(U_{1,\alpha}\) and \(U_{2,\alpha}\) is induced by a diffeomorphism \(\tilde{\phi}_{\alpha} : \tilde{U}_{1,\alpha} \to \tilde{U}_{2,\alpha}\) which is \(\Gamma_{\alpha}\) equivariant. If \(\eta\) is a Dirac-type density on \(M_1 = \tilde{M}_1 / \Gamma_1\) then the Dirac-type density \(\phi_\ast(\eta)\) on \(M_2 = \tilde{M}_2 / \Gamma_2\) is given locally on each \(U_{2,\alpha}\) by the Dirac-type density \(\phi_\ast(\eta_{U_{1,\alpha}})\). Because of the two remarks we made above, the Dirac-type distributions associated with \(\eta\) and \(\phi_\ast(\eta)\) are conjugated by the isomorphism between the spaces of functions induced by \(\phi\).

**Definition 2.19.** Let \(M\) a smooth orbifold. A Dirac-type density on \(M\) is given by a collection of Dirac-type densities \(\eta_{\alpha}\) on the orbifolds \(U_{\alpha} = \tilde{U}_{\alpha} / \Gamma_{\alpha}\) such that \(\phi_{\alpha\beta\ast}(\eta_{\alpha}|_{U_{\alpha} \cap U_{\beta}}) = \eta_{\beta}|_{U_{\alpha} \cap U_{\beta}}\), where \(\phi_{\alpha\beta}\) are the transition maps associated with an orbifold atlas \(A = (\tilde{U}_{\alpha}, \Gamma_{\alpha}, U_{\alpha}, \pi_{\alpha})_\alpha\).

The distribution associated with the Dirac-type density \(\eta\) on \(M\) is defined using a partition of unity associated with the orbifold atlas; if \(f \in C_0^\infty(M)\) then
\[ <\eta, f > = \sum_{\alpha} <\eta_{\alpha}, \psi_\alpha f > \]
where \(\{\psi_\alpha\}_\alpha\) is a partition of unity associated with the above atlas. We will denote \(<\eta, f >\) by \(\int_M f \eta\).

**Definition 2.20.** The integral of the Dirac-type density \(\eta\) on \(M\) is equal to \(<\eta, 1 >\).
2.6. The Canonical Stratification of an Orbifold. The goal of this section to describe the canonical stratification $\mathcal{S}(M)$ of an orbifold $M$. Let $\Gamma$ be a finite group. Let $\mathcal{O}(\Gamma)$ be the set of equivalence classes of subgroups of $\Gamma$ with respect to the conjugation relation: $\Gamma_1 \sim \Gamma_2$ if there exists $\gamma \in \Gamma$ such that $\gamma \Gamma_1 \gamma^{-1} = \Gamma_2$. Denote by $[\Gamma']$ the class of $\Gamma' \subseteq \Gamma$. On $\mathcal{O}(\Gamma)$ we consider the order relation $[\Gamma_1] \preceq [\Gamma_2]$ if there exists $\gamma \in \Gamma$ such that $\gamma \Gamma_1 \gamma^{-1} \subseteq \Gamma_2$. Then $(\mathcal{O}(\Gamma), \preceq)$ is a partially ordered set.

Let $\Gamma \times \tilde{M} \to \tilde{M}$ be a faithful action on a manifold $\tilde{M}$. Let $M = \tilde{M}/\Gamma$ be the associated orbifold of $\Gamma$ orbits. Let $\mathcal{O}(M) \subseteq \mathcal{O}(\Gamma)$ given by

$$\mathcal{O}(M) = \{[\Gamma'] \mid \text{there exists } \tilde{x} \in \tilde{M} \text{ such that } \Gamma \tilde{x} \sim \Gamma'\}$$

**Definition 2.21.** $\mathcal{O}(M)$ is called the set of orbit types of $M$.

We have the map $\mathcal{O} : M \to \mathcal{O}(M)$ given by $\mathcal{O}(x) = [\Gamma \tilde{x}]$ where $\tilde{x} \in \tilde{M}$ is a representative for the class $x \in \tilde{M}/\Gamma$. The image of $x$ in $\mathcal{O}(M)$ is called the orbit type of $x$.

We will define the set of orbit types of a general orbifold $M$. Let $\mathcal{A} = (\mathcal{R}_\alpha)$ an atlas with $\mathcal{R}_\alpha = (\tilde{U}_\alpha, \Gamma_\alpha, U_\alpha, \pi_\alpha)$. On the disjoint reunion $\bigsqcup_{\alpha} \mathcal{O}(U_\alpha)$ we consider the equivalence relation generated by $[\Gamma_\alpha'] \sim [\Gamma_\beta']$ if $\Gamma_\alpha' = (\Gamma_\alpha \tilde{x}_\alpha) \subseteq \Gamma_\alpha$ and $\Gamma_\beta' = (\Gamma_\beta \tilde{x}_\beta) \subseteq \Gamma_\beta$ where $\tilde{x}_\alpha \in \tilde{U}_\alpha$ and $\tilde{x}_\beta \in \tilde{U}_\beta$ are such that $\pi_\alpha(\tilde{x}_\alpha) = \pi_\beta(\tilde{x}_\beta) = x \in U_\alpha \cap U_\beta$. Let $\mathcal{O}(M) = \bigsqcup_{\alpha} \mathcal{O}(U_\alpha)/\sim$ be the set of orbit types of $M$. This definition does not depend on the choice of the atlas because any addition of an extra chart to $\mathcal{A}$ will not change $\mathcal{O}(M)$. There exists a map $\mathcal{O} : M \to \mathcal{O}(M)$ which associates to each $x \in M$ its orbit type, given by $\mathcal{O}(x) = [(\Gamma_\alpha \tilde{x})]$ where $\tilde{x} \in U_\alpha$ with $\pi_\alpha(\tilde{x}) = x$. We have a partial order $\preceq$ on $\mathcal{O}(M)$ induced by the partial order relations on $\mathcal{O}(U_\alpha) \subseteq \mathcal{O}(\Gamma_\alpha)$.

For each $v \in \mathcal{O}(M)$ let

$$M_v = \{x \in M \mid \mathcal{O}(x) = v\}$$

Observe that $M_{[(\varepsilon)]}$ is equal to the set of regular points $M_{reg}$.

**Lemma 2.7.** For any $v \in \mathcal{O}(M)$ the set $M_v$ is a smooth submanifold of $M$.

**Proof.** It is sufficient to prove that any point $x \in M_v$ has a neighborhood $U$ such that $M_v \cap U$ is a submanifold in $U$. Let $(\tilde{U}, \Gamma, U, \pi)$ be an linear orbifold chart at $x$, $\pi(0) = x$. Observe that $[\Gamma] = v$. Then for any $\tilde{y} \in \tilde{U}$ we have $\Gamma \tilde{y} \subseteq \Gamma = \Gamma_0$. So $M_{[\Gamma]} \cap U = \pi(\tilde{U})$. The fixed point set $\tilde{U}^\Gamma \subset \tilde{U}$ is a submanifold given by linear equations on which $\Gamma$ acts trivially. Then $M_{[\Gamma]} \cap U \cong \tilde{U}^\Gamma$ is a smooth submanifold of $U$. \hfill $\Box$

**Definition 2.22.** The decomposition

$$M = \bigsqcup_{v \in \mathcal{O}(M)} M_v$$

is called the canonical stratification on $M$ and will be denoted by $\mathcal{S}(M)$.

A strata $M_v$ is not necessarily connected. Let $M_v = \cup_a M^a_v$ be the decomposition in connected components of $M_v$. 

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*Pseudodifferential Operators on Orbifolds*
Proposition 2.8. Let \( \eta \) be a Dirac-type density on \( M \). There exist densities \( \eta_\alpha \) on the strata \( M_\alpha \) such that for any \( f \in C_0^\infty(M) \) we have:

\[
< \eta, f > = \sum_{\alpha \in \Omega(M)} \int_{M_\alpha} f_{M_\alpha} \eta_\alpha
\]

Proof. We will give the construction of the density \( \eta_{[\Gamma']} \) on a linear orbifold chart \( R = (\tilde{U}, \Gamma, U, \pi) \). Let \( \{\eta^\gamma\}_{\gamma \in \Gamma} \) be the \( \Gamma \) Dirac-type density on \( \tilde{U} \) that represents \( \eta_U \).

For \( \Gamma' \subseteq \Gamma \) let \( \tilde{U}_{\Gamma'} = \{\tilde{x} \mid \tilde{x} \in \tilde{U} \text{ and } \Gamma \tilde{x} = \Gamma'\} \). Then \( \tilde{U}_{\Gamma'} \) is a submanifold of \( \tilde{U} \).

We have \( \pi(\tilde{U}_{\Gamma'}) = U_{[\Gamma']} \) and \( \pi^{-1}(U_{[\Gamma']}) = \Gamma \tilde{U}_{\Gamma'} = \bigcup_{\gamma \in \Gamma} \tilde{U}_{\gamma \Gamma' \gamma^{-1}} \).

We will need the following lemma:

Lemma 2.9. The restriction of the projection map

\[
\pi_{[\Gamma \tilde{U}_{\Gamma'}]} : \Gamma \tilde{U}_{\Gamma'} \to U_{[\Gamma']}
\]

is a covering map with \( [\Gamma : \Gamma'] \) sheets.

Proof. \( \Gamma \tilde{x} \) is locally constant on \( \Gamma \tilde{U}_{\Gamma'} \), being either \( \Gamma' \) or a conjugate of it by an element in \( \Gamma \). For each \( \tilde{x} \in \tilde{U}_{\Gamma'} \), one can choose a small neighborhood \( \tilde{V}_x \) of \( \tilde{x} \) in \( \tilde{U}_{\Gamma'} \) such that \( \Gamma' \) acts trivially on \( \tilde{V}_x \) and \( \gamma \tilde{V}_x \cap \tilde{V}_x = \emptyset \) for \( \gamma \in \Gamma \setminus \Gamma' \). Let \( x = \pi(\tilde{x}) \) and \( V_x = \pi(\tilde{V}_x) \). The map \( \pi_{|\tilde{V}_x} : \tilde{V}_x \to V_x \) is a homeomorphism. For any other \( \tilde{y} = \gamma \tilde{x} \in \pi^{-1}(x) \) the map \( \pi_{|\gamma \tilde{V}_x} : \gamma \tilde{V}_x \to V_x \) is a homeomorphism between a neighborhood of \( \tilde{y} \) and \( V_x \). Observe also that \( \pi^{-1} = \Gamma \tilde{x} \) has exactly \( [\Gamma : \Gamma'] \) points.

Let us fix \( \gamma \in \Gamma \). Then \( \tilde{U}^\gamma = \{\tilde{x} \mid \gamma \tilde{x} = \tilde{x}\} \) is a smooth submanifold of \( \tilde{U} \) and \( \tilde{U}^\gamma = \bigcup_{\gamma \in \Gamma'} \tilde{U}_{\Gamma'}^\gamma \) is a stratification of \( \tilde{U}_{\Gamma'} \). Then

\[
\bigcup_{\gamma \in \Gamma'} \tilde{U}_{\Gamma'}^\gamma \subset \tilde{U}^\gamma
\]

is an open and dense set in \( \tilde{U}^\gamma \) (\( \{\tilde{U}^\gamma_\alpha\}_\alpha \) is the decomposition in connected components of \( \tilde{U}_{\Gamma'} \)). Let \( \eta_{\gamma' \alpha} \) be the restriction of the density \( \eta^\gamma \) to \( \tilde{U}^\gamma_\alpha \), and

\[
\eta_{\gamma' \alpha} = \frac{1}{[\Gamma']^\gamma} \sum_{\gamma \in \Gamma'} \eta_{\gamma \alpha} \quad \text{dim}(\tilde{U}^\gamma) = \text{dim}(\tilde{U}^\gamma_\alpha)
\]

Let \( U_{[\Gamma']}^\gamma = \pi(\tilde{U}^\gamma_\alpha) \). We showed in the previous lemma that \( \pi \) realizes a local diffeomorphism between \( \tilde{U}_{\Gamma'}^\gamma \) and \( U_{[\Gamma']}^\gamma \). Let \( \eta^\gamma_{\gamma' \alpha}(x) = \pi_*(\eta_{\gamma \alpha})(\tilde{x}) \) for some \( \tilde{x} \in \tilde{U}_{\Gamma'}^\gamma \), such that \( \pi(\tilde{x}) = x \). This definition is independent of the choice of \( \tilde{x} \) and of \( \Gamma' \subset \Gamma \). Indeed, let \( \gamma \in \Gamma \) be such that \( \gamma \Gamma' \gamma^{-1} \) is another representative for \( [\Gamma'] \) and \( \pi(\gamma \tilde{x}) = x \). After a reordering of the indices \( \alpha \) we can suppose that \( \tilde{U}^\gamma_{\Gamma' \gamma^{-1}} = \gamma \tilde{U}_{\Gamma'}^\gamma \). Because \( \{\eta^\gamma\}_{\gamma' \alpha} \) is a \( \Gamma \) Dirac-type density on \( \tilde{U} \) we have \( \eta_{\gamma' \gamma^{-1}, \alpha}(\gamma \tilde{x}) = \gamma_*(\eta_{\gamma \alpha}(\tilde{x})) \) and because \( \pi \) commutes with the action of \( \Gamma \) on \( \tilde{U} \) we get the independence of \( \eta_{\gamma' \gamma^{-1}, \alpha}(x) \) of the choices we made.

If \( \eta^\gamma_{\gamma' \alpha} \) was not defined on \( U_{[\Gamma']}^\gamma \) because there were no \( \gamma \in \Gamma \) such that \( \tilde{U}^\gamma_{\Gamma' \gamma^{-1}} \) is an open submanifold in \( \tilde{U}^\gamma \), we take \( \eta^\gamma_{[\Gamma']} = 0 \).
We define $\eta_{[\Gamma']}$ on $M_{[\Gamma']}$ to be the density whose restriction to $U_{[\Gamma']}$ is equal to $\eta_{[\Gamma']}$. We need to show that

$$<\eta, f> = \sum_{[\Gamma'] \in \mathcal{O}(\Gamma)} \int_{M_{[\Gamma']}} f_{M_{[\Gamma']}} \eta_{[\Gamma']} \quad \text{for any } f \in C_{0}^{\infty}(M). \quad (51)$$

It is sufficient to prove this for $f \in C_{0}^{\infty}(U)$. If $\tilde{f} \in C_{0}^{\infty}(\tilde{U})$ is such that $\tilde{f} = \pi f$ then

$$<\eta, f> = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \int_{U_{\gamma}} \tilde{f} \eta_{\gamma} =$$

$$= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \left( \sum_{\gamma' \supset \gamma} \int_{\tilde{U}_{\gamma'}} \tilde{f} \eta_{\gamma', \alpha} \right) =$$

$$= \frac{1}{|\Gamma|} \sum_{\alpha, \Gamma' \subset \Gamma} \left( \sum_{\gamma' \supset \gamma} \int_{\tilde{U}_{\gamma'}} \tilde{f} \eta_{\gamma', \alpha} \right) =$$

$$= \frac{1}{|\Gamma|} \sum_{v \in \mathcal{O}(\Gamma)} \left( \sum_{\alpha, \Gamma' \subset \Gamma} \int_{\tilde{U}_{\gamma'}} \tilde{f} \eta_{\gamma', \alpha} \right) = \left( \text{by } (50) \right)$$

$$= \frac{1}{|\Gamma|} \sum_{\Gamma'} \int_{\pi^{-1}(U_{[\Gamma']})} \tilde{f} \eta_{\Gamma', \alpha} = \sum_{[\Gamma'] \in \mathcal{O}(U)} \int_{U_{[\Gamma']}} f \eta_{[\Gamma']} \quad (56)$$

which is equal to the right side of (51). In the last step we used the fact that $\pi : \pi^{-1}(U_{[\Gamma']}) \to U_{[\Gamma']}$ is a covering map with $[\Gamma : \Gamma']$ leaves. □
3. Pseudodifferential Operators in Orbibundles

3.1. Pseudodifferential Operators – The Local Model. We remind the reader that locally any vector orbibundle over an orbifold is obtained as Γ orbits of a Γ vector bundle, with Γ finite. We begin by analyzing the case where Γ acts freely on a vector bundle $E \xrightarrow{\rho} M$, in which case the associated vector orbibundle of Γ orbits $E/\Gamma \xrightarrow{\rho/\Gamma} B/\Gamma$ is a genuine vector bundle over a manifold. Let $R : C^\infty(M; E) \to C^\infty(M/\Gamma; E/\Gamma)$ be defined as

$$(57) \quad R(f)(\tau) = \frac{1}{|\Gamma|} \sum_{x \in \tau} f(x) \quad \text{for } f \in C^\infty(M; E) \text{ and } \tau \in M/\Gamma$$

and $I : C^\infty(M/\Gamma; E/\Gamma) \to C^\infty(M; E)$ by

$$(58) \quad I(f)(x) = v \in E_x \quad \text{for } f \in C^\infty(M/\Gamma; E/\Gamma) \text{ and } x \in M$$

where $v \in E_x$ is such that $f(\Gamma x) = \Gamma v$. $v$ is unique in $p^{-1}(x) = E_x$ because the action of Γ is free. $I$ identifies $C^\infty(M/\Gamma; E/\Gamma)$ with the space of Γ invariant sections $C^\infty(M; E)^\Gamma$ and then $R$ is the averaging operator over the group Γ. Obviously, $R \cdot I = Id$ and $I \cdot R$ is the averaging operator over the group Γ. If we endow the quotient vector bundle with the trivial Γ action then both $R$ and $I$ are Γ equivariant.

Let $\Psi(M; E)$ and $\Psi(M/\Gamma; E/\Gamma)$ be the spaces of pseudodifferential operators on $C^\infty(M; E)$ respectively $C^\infty(M/\Gamma; E/\Gamma)$. Let $R : \Psi(M; E) \to \Psi(M/\Gamma; E/\Gamma)$ given by

$$(59) \quad R(A) = RAI \quad \text{for } A \in \Psi(M; E)$$

$R(A)$ is a pseudodifferential operator acting on $C^\infty(M/\Gamma; E/\Gamma)$. To see this, observe that if the distributional kernel of the operator $A$ is equal to $K_A(x, y)$ then the distributional kernel of $R(A)$ is given by the formula

$$(60) \quad K_{R(A)}(\tau, \varphi) = \frac{1}{|\Gamma|} \sum_{x \in \tau} K_A(x, y)$$

In a coordinate chart $(U, \phi)$ the total symbol of the operator $R(A)$ is given by

$$(61) \quad \tau(\tau, \xi) = \frac{1}{|\Gamma|} \sum_{x \in \tau} a(x, \xi)$$

where $a(x, \xi)$ is the total symbol of $A$ in the chart $(p^{-1}U, \phi p)$. If $B$ is the average of $A$ with respect to the action of Γ by conjugation on $\Psi(M; E)$, $B = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma A \gamma^{-1}$, then $R(A) = R(B)$. Moreover, $R(B)$ can be seen as the restriction of the Γ invariant operator $B$ to $C^\infty(M; E)^\Gamma \cong C^\infty(M/\Gamma; E/\Gamma)$.

Lemma 3.1. There exists a $\Gamma$ equivariant operator $I : \Psi(M/\Gamma; E/\Gamma) \to \Psi(M; E)$ so that $RI = Id$ on $\Psi(M/\Gamma; E/\Gamma)$.

Proof. If we define $I$ by $I(A) = IAR$ then $RI = Id$, but $IAR$ will not be, in general, a pseudodifferential operator, unless $A$ is smoothing and then the smooth kernel $K_{I(A)}$ of $I(A)$ is defined uniquely by $K_{I(A)}(x, y) = K_A(\tau, \varphi)$.

We will define $I$ using a partition of unity of $M/\Gamma$ with smooth functions $\{\phi_\lambda\}_{\lambda \in A}$ subordinated to a cover of $M/\Gamma$ with coordinate chart neighborhoods $(U_\lambda)_{\lambda \in I}$ which
are slices with respect to the \( \Gamma \) action on \( M \). We will choose \( \phi_{\lambda} \) so that for \( \lambda, \lambda' \in \Lambda \) at least one of the following conditions holds:

(i) \( \text{supp}(\phi_{\lambda}) \cap \text{supp}(\phi_{\lambda'}) = \emptyset \)

(ii) \( \text{supp}(\phi_{\lambda}) \cup \text{supp}(\phi_{\lambda'}) \) is included in the same coordinate neighborhood.

If \( A \in \Phi(M;\Gamma; E/\Gamma) \) then \( A = \sum_{\lambda,\lambda'} \phi_{\lambda} A_{\lambda\lambda'} \). Denote \( A_{\lambda\lambda'} = \phi_{\lambda} A_{\lambda\lambda'} \). We will define \( I_{\lambda\lambda'}(A) = I(A_{\lambda\lambda'}) \) and then \( I = \sum_{\lambda,\lambda'} I_{\lambda\lambda'} \).

If \( \text{supp}(\phi_{\lambda}) \cap \text{supp}(\phi_{\lambda'}) = \emptyset \), then \( A_{\lambda\lambda'} \) is smoothing and we define \( I_{\lambda\lambda'}(A) = IA_{\lambda\lambda'} R \).

Otherwise, choose \( i \in I \) such that \( \text{supp}(\phi_{\lambda}) \cup \text{supp}(\phi_{\lambda'}) \subset U_i \). Because \( U_i \) is a slice, its preimage \( \tilde{U}_i \subset M \) can be identified with \( \Gamma \times U_i \) endowed with the \( \Gamma \) action by left translations. We have the induced \( \Gamma \) equivariant isomorphism

\[
C^\infty(\tilde{U}_i; E_{|\tilde{U}_i}) \cong \Gamma \times C^\infty(U_i; E/\Gamma_{|U_i})
\]

The operator \( A_{\lambda\lambda'} \) is localized above \( U_i \) and we define \( I(A_{\lambda\lambda'}) \in \Psi(\tilde{U}_i; E_{|\tilde{U}_i}) \) to be the block-diagonal operator with diagonal cells equal to \( A_{\lambda\lambda'} \), where the block representation is with respect to the decomposition given by the isomorphism (62).

Then \( I(A_{\lambda\lambda'}) \) is a \( \Gamma \) invariant pseudodifferential operator on \( C^\infty(M; E) \) localized above \( \tilde{U}_i \), and \( RI(A_{\lambda\lambda'}) = A_{\lambda\lambda'} \).

Define \( I \) to be the sum \( \sum_{\lambda,\lambda'} I_{\lambda\lambda'} \). Then \( RI = Id \) and \( Im(I) \) is a subset of the \( \Gamma \) equivariant pseudodifferential operators in \( C^\infty(M; E) \). So \( I \) is \( \Gamma \) equivariant. \( \square \)

Thus every pseudodifferential operator \( A \in C^\infty(M; \Gamma; E/\Gamma) \) can be represented by the restriction to the \( \Gamma \) invariant sections of a \( \Gamma \) equivariant pseudodifferential operator in \( C^\infty(M; E) \) given by \( I(A) \).

**Remark 3.1.** In the presence of the action of another finite group \( \Gamma' \) on the vector bundle \( E \xrightarrow{\gamma'} M \) which is free and commutes with the action of \( \Gamma \), we have an induced natural action of \( \Gamma' \) on the quotient vector bundle \( E/\Gamma' \xrightarrow{\gamma' \Gamma} M/\Gamma \). \( \gamma' \in \Gamma' \) acts on \( \Gamma x \) by \( \gamma' \Gamma x = \Gamma \gamma' x \). This induced action might not be free. The maps \( R \) and \( I \) between the spaces of smooth sections in the two vector bundles are \( \Gamma' \) equivariant. Indeed, the \( \Gamma' \) action on the \( \Gamma \) invariant sections is the restriction of the \( \Gamma' \) action on all sections, and because \( R \) and \( I \) are essentially the projection onto the \( \Gamma \) invariant sections and the inclusion of the \( \Gamma \) invariant sections into the space of all sections, the two maps are \( \Gamma' \) equivariant. As a consequence, \( R \) is \( \Gamma' \) invariant. The components \( I_{\lambda,\lambda'} \) of \( I \) for \( \text{supp}(\phi_{\lambda}) \cap \text{supp}(\phi_{\lambda'}) = \emptyset \) are \( \Gamma' \) equivariant as well. To show that \( I_{\lambda,\lambda'} \) are \( \Gamma' \) equivariant for \( \text{supp}(\phi_{\lambda}) \cup \text{supp}(\phi_{\lambda'}) \subset U_i \) for a slice \( U_i \subset M/\Gamma \), observe that the identification of the preimage \( \tilde{U}_i \subset M \) of \( U_i \) with \( \Gamma \times U_i \) is \( \Gamma' \) equivariant as well. The map \( I_{\lambda,\lambda'} \) from \( \Psi(U_i, E_{|U_i}) \) to \( \Psi(\tilde{U}_i; E_{|\tilde{U}_i}) \) given by the block-diagonal construction as above is \( \Gamma' \) equivariant. So \( I \) is \( \Gamma' \) equivariant.

We will use the above facts to define the pseudodifferential operators acting on the space of smooth sections of a vector orbibundle. We will start with the local construction.

Let \( \tilde{U} \times V \xrightarrow{\pi_2} \tilde{U} \) be a \( \Gamma \) vector bundle, with \( \Gamma \) a finite group. Suppose that \( \tilde{U} \subset \mathbb{R}^m \) is an open subset with compact closure. The group \( \Gamma \) acts on the space of sections \( C^\infty(\tilde{U}, \tilde{U} \times V) \). Let \( \Gamma \) act on the space of operators on \( C^\infty(\tilde{U}, \tilde{U} \times V) \) by conjugation: if \( A \) is an operator then \( (\gamma A)(f) = \gamma A(\gamma^{-1} f) \) for any \( f \in C^\infty(\tilde{U}, \tilde{U} \times V) \) and \( \gamma \in \Gamma \).

Let \( (\tilde{U} \times V)/\Gamma \xrightarrow{\pi} \tilde{U}/\Gamma \) be the associated vector orbibundle.
Definition 3.1. An operator $A$ acting on $C_0^\infty(\tilde{U}/\Gamma; (\tilde{U} \times V)/\Gamma)$ is called a pseudodifferential operator if $A$ is the restriction to the $\Gamma$ invariant sections of a $\Gamma$ invariant pseudodifferential operator $\tilde{A}$ acting on $C_0^\infty(\tilde{U}; \tilde{U} \times V)$.

We will show later in Proposition 3.3 in a greater generality, that the operator $\tilde{A}$ whose restriction to the invariant sections is equal to $A$ is unique up to smoothing operators.

We define the order of $A$ to be equal to the order of $\tilde{A}$. We denote the space of pseudodifferential operators of order $d$ by $\Psi^d(\hat{U}/\Gamma; (\hat{U} \times V)/\Gamma)$ or simply by $\Psi^d$ and the space of all pseudodifferential operators by $\Psi$. The space $\Psi$ is a filtered algebra with respect to the composition of operators, where the filtration is given by the degree of the operators. An operator $A$ is smoothing if the operator $\tilde{A}$ is smoothing. The space of smoothing operators will be denoted by $\Psi^{-\infty}$.

Definition 3.2. $A$ is called a classical pseudodifferential operator if $\tilde{A}$ is classical.

Let $\tilde{a}(x, \xi)$ be the total symbol of $\tilde{A}$. $\tilde{a}$ is a section of the pull-back of the endomorphism vector bundle to the tangent space, $\tilde{a} \in C^\infty(T^*(\tilde{U}), T^*(\tilde{U}) \times \text{End}(V))$. The principal symbol $\tilde{a}_{pr}$ is a section of the same vector bundle. The group $\Gamma$ acts naturally on the vector bundle $T^*(\tilde{U}) \times \text{End}(V) \xrightarrow{\rho} T^*(\tilde{U})$. Then $\Gamma$ acts on the sections of this vector bundle and $\gamma \tilde{a}_{pr}$ is the principal symbol of the operator $\gamma \tilde{A}$ (cf. Lemma 1.1.3). This is the direct consequence of the invariance of the principal symbol with respect to changes of coordinates. Because $\tilde{A}$ is $\Gamma$ invariant we conclude that $\tilde{a}_{pr}$ is a $\Gamma$ invariant section so it defines a smooth section in the vector orbibundle $(\tilde{U} \times \text{End}V)/\Gamma \xrightarrow{\rho} \hat{U}/\Gamma$. Because $\tilde{A}$ is unique up to smoothing operators, $\tilde{a}_{pr}$ depends only on $A$ and will be called the principal symbol of the operator $A$.

In the case when the $\Gamma$ action on the vector bundle $\tilde{U} \times V \xrightarrow{\rho} \tilde{U}$ is given by the restriction to $\tilde{U} \subset \mathbb{R}^m$ of a linear action of $\Gamma$ on $\mathbb{R}^m$ and by a $\Gamma$ representation $\rho : \Gamma \times V \rightarrow V$, we will be able to define the total symbol of the pseudodifferential operator $A$. This is the consequence of the invariance of the total symbol of $\tilde{A}$ with respect to linear changes of coordinates. The total symbol of $A$ is the given by $\tilde{a} \in C_0^\infty((T^*(\tilde{U}) \times V)/\Gamma; T^*(\tilde{U}))$ and is unique up to a smoothing symbol.

If $\tilde{A}$ is classical and

$$\tilde{a}(x, \xi) \sim \sum_{i \geq 0} \tilde{a}_{d-i}(x, \xi)$$

is an asymptotic expansion of the total symbol with $\tilde{a}(x, \xi)_{d-i}$ being homogeneous symbols of degree of homogeneity $d - i$, then each homogeneous component is $\Gamma$ invariant and will define the homogeneous component $a_{d-i}$ of the asymptotic expansion of the total symbol of the operator $A$. The formal sum in the right-hand side of the equality (63) is called the asymptotic symbol of the operator $A$.

If a pseudodifferential operator $A$ of order $d$ acts on the sections of the vector orbibundle $(\tilde{U} \times V)/\Gamma \xrightarrow{\rho} \hat{U}/\Gamma$ then, as in the case of genuine vector bundles, it extends to a continuous operator between Sobolev spaces $A : H_s(\hat{U}/\Gamma; (\hat{U} \times V)/\Gamma) \rightarrow H_{s-d}(\hat{U}/\Gamma; (\hat{U} \times V)/\Gamma)$ (cf. Lemma 1.2.1).

Remark 3.2. If $A \in \Psi^{-\infty}$ then there exists a $\Gamma$ invariant smoothing operator $\tilde{A}$ whose restriction to the $\Gamma$ invariant sections is equal to $A$. Because $\gamma \tilde{A} = \tilde{A}\gamma$ for all
\( \gamma \in \Gamma \), the kernel \( \tilde{K}(\tilde{x}, \tilde{y}) \) of \( \tilde{A} \) satisfies the equality \( \gamma \tilde{K}(\gamma^{-1} \tilde{x}, \gamma^{-1} \tilde{y}) = \tilde{K}(\tilde{x}, \tilde{y}) \) so it defines a \( \Gamma \) invariant section in the vector bundle \((\tilde{U} \times \tilde{U}) \times \text{End}(V) \overset{\rho_2}{\to} (\tilde{U} \times \tilde{U}) \). Here \( \Gamma \) acts by the diagonal action on \( \tilde{U} \times \tilde{U} \). If we define

\[
\tilde{K}'(\tilde{x}, \tilde{y}) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \tilde{K}(\tilde{x}, \gamma^{-1} \tilde{y})
\]

then \( \tilde{K}' \) is the kernel of the \( \Gamma \) invariant smoothing operator \( \tilde{A}' = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \tilde{A}\gamma \) whose restriction to the \( \Gamma \) invariant sections induces \( \tilde{A} \). \( \tilde{K}' \) defines a \( \Gamma \times \Gamma \) invariant smooth section of the vector bundle \((\tilde{U} \times \tilde{U}) \times \text{End}(V) \overset{\rho_2}{\to} (\tilde{U} \times \tilde{U}) \) where \( \Gamma \times \Gamma \) acts by the product action on \( \tilde{U} \times \tilde{U} \). Then \( \tilde{U} \times \tilde{U} / (\Gamma \times \Gamma) \cong U \times U \) and \( \tilde{K}' \) defines a smooth section of the endomorphism vector orbibundle \((\tilde{U} \times \tilde{U}) \times \text{End}(V) / \Gamma \times \Gamma \overset{\rho_2}{\to} U \times U \) which will be the smooth kernel of the operator \( A \).

**Definition 3.3.** The pseudodifferential operator \( A \) of order \( d \) is called elliptic above an open set \( U_1 \subset \tilde{U} / \Gamma \) if the operator \( \tilde{A} \) is elliptic above the preimage of \( U_1 \) in \( \tilde{U} \).

Let \( \tilde{U}_1 \subset \tilde{U} \) be the preimage of \( U_1 \). The above definition implies the existence of a parametrix \( \tilde{B}' \) for \( \tilde{A} \) above \( \tilde{U}_1 \). Then \( \tilde{B} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \tilde{B}' \gamma^{-1} \) is a \( \Gamma \) invariant parametrix for \( \tilde{A} \) and defines a parametrix \( B \in \Psi^{-d} \) for \( A \), \( BA - 1d \in \Psi^{-\infty} \), \( AB - 1d \in \Psi^{-\infty} \).

### 3.2. Pseudodifferential Operators in Orbibundles

Let \( E \overset{p}{\to} M \) be a vector orbibundle and \( A \) be a linear operator acting on the space of smooth sections \( C^\infty(M; E) \). If \( \mathcal{R} = (\tilde{U}, V, \Gamma, U, \Pi, \pi) \) is an orbibundle chart for \( E \overset{p}{\to} M \) above \( U \subset M \) then \( \tilde{U} \times V \overset{\rho_2}{\to} \tilde{U} \) is a \( \Gamma \) vector bundle and passing to the orbit space yields the orbibundle \( \tilde{U} \times V / \Gamma \overset{\rho_2}{\to} \tilde{U} / \Gamma \). This orbibundle is isomorphic to the restriction of \( E \overset{p}{\to} M \) to \( U \) via the orbibundle map \((\Pi, \pi)\):

\[
\begin{array}{ccc}
\tilde{U} \times V / \Gamma & \overset{\Pi}{\rightarrow} & E|_U \\
\pi_1 & \downarrow & p \\
\tilde{U} / \Gamma & \overset{\pi}{\rightarrow} & U
\end{array}
\]

The orbibundle \( E \overset{p}{\to} M \) can be covered by such orbibundle maps.

Let \( \phi \) and \( \psi \) two smooth functions on \( M \) such that their support is included in \( U \). We will denote by the same letter the multiplication operator by the respective functions. Then the operator \( \phi \circ A \circ \psi \) has support in \( \tilde{U} \) and takes sections with support in \( U \) to sections with support in \( U \). Using the orbibundle isomorphism \( \pi \), one can define the operator \( A_{\mathcal{R}} \) acting on the sections of \( \tilde{U} \times V / \Gamma \to \tilde{U} / \Gamma \) as

\[
A_{\mathcal{R}} \ f = (\pi(\phi \circ A \circ \psi))^{-1} \ f.
\]

**Definition 3.4.** \( A \) is a pseudodifferential operator if \( A_{\mathcal{R}} \) is a pseudodifferential operator for any orbibundle chart \( \mathcal{R} \).

In order for the previous definition to be consistent, we need to show that this property is independent of the choice of a chart. More precisely:
Proposition 3.2. Let $\tilde{E}_i \xrightarrow{p_i} \tilde{M}_i$ be two $\Gamma_i$ vector bundles, $i = 1, 2$, such that the corresponding orbibundles are isomorphic via an isomorphism $(T, t)$. Let $A_1$ be an operator acting on sections of $\tilde{E}_1/\Gamma_1 \xrightarrow{p_1} \tilde{M}_1/\Gamma_1$ and $A_2 = (T, t)_* \circ A_1 \circ (T, t)^*$ the corresponding operator acting on $\tilde{E}_2/\Gamma_2 \xrightarrow{p_2} \tilde{M}_2/\Gamma_2$.

Then there exists a $\Gamma_1$ equivariant pseudodifferential operator $\tilde{A}_1$ acting on sections of $\tilde{E}_1 \xrightarrow{p_1} \tilde{M}_1$ whose restriction to $\Gamma_1$ invariant sections is equal to $A_1$ if and only if there exists a $\Gamma_2$ equivariant pseudodifferential operator $\tilde{A}_2$ acting on sections of $\tilde{E}_2 \xrightarrow{p_2} \tilde{M}_2$ whose restriction to $\Gamma_2$ invariant sections is equal to $A_2$.

Proof. For each $x_i \in \tilde{M}_i/\Gamma_i$ such that $t(x_1) = x_2$ choose $\tilde{x}_i \in \tilde{M}_i$ with $\pi_i(\tilde{x}_i) = x_i$. Because the orbibundles are isomorphic, there exists a group isomorphism $\alpha$ between $(\Gamma_i)_{\tilde{x}_i}$, the corresponding isotropy groups of $\tilde{x}_i$ in $\Gamma_i$, and one can find neighborhoods $\tilde{W}_i$ of $\tilde{x}_i$ in $\tilde{M}_i$ which are $(\Gamma_i)_{\tilde{x}_i}$ invariant and an $\alpha$-equivariant bundle diffeomorphism $(L, l)$ that makes the following diagram commutative:

Let $\{\phi_\lambda\}_{\lambda \in \Lambda}$ be a partition of unity of $\tilde{M}_i/\Gamma_i$ subordinated to the open cover $\tilde{W}_i/(\Gamma_i)_{\tilde{x}_i}$ (because $\tilde{M}_i/\Gamma_i$ are diffeomorphic and the diffeomorphism $t$ permutes the two open covers, we will denote the two partition of unity with the same letters, thought we will refer to functions on two different, but diffeomorphic orbifolds). We can choose this partition so that for any $\lambda, \lambda' \in \Lambda$ at least one of the following conditions holds:

(i) $\text{supp}(\phi_\lambda) \subset \tilde{W}_i/(\Gamma_i)_{\tilde{x}_i}$, $\text{supp}(\phi_{\lambda'}) \subset \tilde{W}_i'/(\Gamma_i)_{\tilde{x}_i'}$ and
\[ \tilde{W}_i/(\Gamma_i)_{\tilde{x}_i} \cap \tilde{W}_i'/(\Gamma_i)_{\tilde{x}_i'} = \emptyset \]

(ii) $\text{supp}(\phi_\lambda) \cup \text{supp}(\phi_{\lambda'}) \subset \tilde{W}_i/(\Gamma_i)_{\tilde{x}_i}$.

We will show only one implication of the proposition, the other implication can be proved similarly. Let $\tilde{A}_1$ be a $\Gamma_1$ equivariant pseudodifferential operator acting on the smooth sections of $\tilde{E}_1 \xrightarrow{p_1} \tilde{M}_1$ whose restriction to the invariant sections is equal to $A_1$. We will construct a lift $\tilde{A}_2$ as required in the proposition.
Moreover

\[ A_1 = \sum_{\lambda, \lambda'} \phi_{\lambda} A_{\lambda} \phi_{\lambda'} \]

and because a lift for the operator \( \phi_{\lambda} A_{\lambda} \phi_{\lambda'} \) is provided by \( \hat{\phi}_{\lambda} \hat{A}_{\lambda} \hat{\phi}_{\lambda'} \), we reduced our problem to the operators of the form \( \phi_{\lambda} A_{\lambda} \phi_{\lambda'} \) which we will denote by \( (A_1)_{\lambda, \lambda'} \) for simplicity (we denoted by \( \hat{\phi} \) the lift of the smooth function \( \phi \)). Denote the lifts \( \hat{\phi}_{\lambda} \hat{A}_{\lambda} \hat{\phi}_{\lambda'} \) by \( (A_1)_{\lambda, \lambda'} \). In the first case the operator \( (A_1)_{\lambda, \lambda'} \) is smoothing. In the second case the operator \( (A_1)_{\lambda, \lambda'} \) is localized to the open set \( \Gamma_1 \cdot \hat{W}_1 \cong \Gamma_1 \times (\Gamma_1 \cdot \hat{W}_1) \). We will treat the two cases separately.

In the first case define:

\[ \overline{A}_1 = \sum_{\gamma \in \Gamma_1} (\hat{A}_1)_{\lambda, \lambda'} \gamma \]

Because \( (\hat{A}_1)_{\lambda, \lambda'} \) was smoothing and \( \Gamma_1 \) equivariant, \( \overline{A}_1 \) is smoothing and \( \Gamma_1 \) bi-invariant:

\[ \mu \overline{A}_1 = \overline{A}_1 \nu = \overline{A}_1 \quad \text{for any } \mu, \nu \in \Gamma_1 \]

Moreover \( \overline{A}_1 \) induces the same operator \( (A_1)_{\lambda, \lambda'} \) at the orbibundle level because its action on the \( \Gamma_1 \) invariant sections is not changed. If we choose a \( \Gamma_1 \) invariant metric on \( M_1 \) then the kernel \( \overline{K}_1(x, y) \) of the operator \( \overline{A}_1 \) is a smooth section in \( C^{\infty}(M_1 \times M_1; \text{End}(\hat{E}_1)) \) and because of the equalities \([\ref{5}]\), it satisfies the condition

\[ \mu K_1(\mu^{-1} x, y) = \nu K(x, \nu^{-1} y) \quad \text{for all } \mu, \nu \in \Gamma_1. \]

This implies that \( K_1 \) is a \( \Gamma_1 \times \Gamma_1 \) invariant smooth section in the \( \Gamma_1 \times \Gamma_1 \) vector bundle \( \text{End}(\hat{E}_1) \to M_1 \times M_1 \) and so it defines a smooth section in the vector orbibundle \( \text{End}(\hat{E}_1) / \Gamma_1 \to M_1 / \Gamma_1 \times M_1 / \Gamma_1 \).

Because the vector orbibundles \( \hat{E}_i / \Gamma_i \cong \hat{M}_i / \Gamma_i, \ i = 1, 2, \) are isomorphic, there exists a unique smooth \( \Gamma_2 \times \Gamma_2 \) invariant section \( \overline{K}_2 \in \text{End}(\hat{E}_2) \to \hat{M}_2 \times \hat{M}_2 \) which induces the same section via the isomorphism \((T, t) \mapsto \Gamma_1 \cdot (\hat{W}_1 \times V) \to \Gamma_1 \times \hat{W}_1 \), \( \Gamma \times \hat{W}_1 / \Gamma \) which is isomorphic to \( \hat{W}_1 \times V \to \hat{W}_1 \).

Because \( (\hat{A}_1)_{\lambda, \lambda'} \) was \( \Gamma_1 \equiv \Gamma' \) equivariant, \( (\hat{A}_1)_{\lambda, \lambda'} \) is \( \Gamma' \) equivariant. We have the natural isomorphisms between the spaces of invariant sections

\[ C^{\infty}(\hat{W}_1; \hat{W}_1 \times V)^{\Gamma_1} \cong C^{\infty}(\Gamma_1 \times \hat{W}_1; \Gamma_1 \times (\hat{W}_1 \times V))^{\Gamma_1 \times \Gamma_1} \cong C^{\infty}(\Gamma_1 \times \Gamma_1 \times \hat{W}_1 \times V)^{\Gamma_1 \times \Gamma_1} \]

In the case (ii) \( (\hat{A}_1)_{\lambda, \lambda'} \) is a \( \Gamma_1 \) equivariant operator in \( \Psi(\Gamma_1 \cdot \hat{W}_1; \hat{E}_1 | \Gamma_1 \cdot \hat{W}_1) \). Let us denote \( \Gamma'_1 = (\Gamma_1 \cdot \hat{W}_1 \times V) \). The \( \Gamma_1 \) vector bundle \( \hat{E}_1 \) is isomorphic with \( \Gamma_1 \times \Gamma'_1 \) which is \( \Gamma' \) equivariant as well. An element \( \gamma' \in \Gamma'_1 \) acts on \( \Gamma_1 \times \hat{W}_1 \) by right multiplication on \( \Gamma_1 \) by \( \gamma'^{-1} \) and left multiplication on \( \hat{W}_1 \) by \( \gamma' \). Let \( RI((\hat{A}_1)_{\lambda, \lambda'}) \) be the induced operator to the vector bundle \( \Gamma \times \hat{W}_1 / \Gamma \) which is isomorphic to \( \hat{W}_1 \times V \to \hat{W}_1 \). Because \( I((\hat{A}_1)_{\lambda, \lambda'}) \) was \( \Gamma_1 \) equivariant, cf. remark \([\ref{5}]\), the operator \( RI((\hat{A}_1)_{\lambda, \lambda'}) \) is \( \Gamma' \) equivariant. We have the natural isomorphisms between the spaces of invariant sections

\[ C^{\infty}(\hat{W}_1; \hat{W}_1 \times V)^{\Gamma_1} \cong C^{\infty}(\Gamma_1 \times \hat{W}_1; \Gamma_1 \times (\hat{W}_1 \times V))^{\Gamma_1 \times \Gamma_1} \cong C^{\infty}(\Gamma_1 \times \Gamma_1 \times \hat{W}_1 \times V)^{\Gamma_1 \times \Gamma_1} \]
which can be identified with the space of smooth sections in the vector orbibundle \( \tilde{E}_1/\Gamma_1 \overset{\rho_1}{\to} \tilde{M}_1/\Gamma_1 \) above the open set \( \tilde{W}_1/\Gamma_1 \). The operators \( RI((A_1)_{\lambda \lambda'}) \), \( I((A_1)_{\lambda \lambda'}) \) and \( (A_1)_{\lambda \lambda'} \) restricted to the above spaces of invariant sections will be equal via the natural isomorphisms. In particular they induce \( (A_1)_{\lambda \lambda'} \) at the level of sections in the associated vector orbibundle \( \tilde{E}_1/\Gamma_1 \overset{\rho_1}{\to} \tilde{M}_1/\Gamma_1 \).

We define the \( \Gamma'_2 \) equivariant operator \( \overline{A}_2 = (L,l)_* \circ RI((A_1)_{\lambda \lambda'}) \circ (L,l)^* \) in \( \Psi(\tilde{W}_2; \tilde{W}_2 \times V) \) using the isomorphisms \( (L,l) \) provided in the diagram (66). (here, as before, \( \Gamma_2 = (\Gamma_2)^{\tilde{\varepsilon}} \)). Let \( I(\overline{A}_2) \in \Psi(\Gamma_2 \times \tilde{W}_2; \Gamma_2 \times (\tilde{W}_2 \times V)) \) be the \( \Gamma_2 \) equivariant operator which induces \( \overline{A}_2 \) at the \( \Gamma_2 \) orbit level, and let \( RI(\overline{A}_2) \in \Psi(\Gamma_2 \times \Gamma'_2; \Gamma_2 \times \Gamma'_2 (\tilde{W}_2 \times V)) \) be the operator induced at the \( \Gamma'_2 \) orbit level. Cf. remark [3], this operator is \( \Gamma_2 \) equivariant. Using the sequence of isomorphisms:

\[
C^\infty(\tilde{W}_2; \tilde{W}_2 \times V)^{\Gamma_2} \cong C^\infty(\Gamma_2 \times \tilde{W}_2; \Gamma_1 \times (\tilde{W}_2 \times V))^{\Gamma_2 \times \Gamma_2} \cong C^\infty(\Gamma_2 \times \Gamma'_2 \tilde{W}_2; \Gamma_2 \times \Gamma'_2 (\tilde{W}_2 \times V))^{\Gamma_2}
\]

observe that the operators \( RI(\overline{A}_2) \) and \( \overline{A}_2 \) act the same way on the corresponding spaces of invariant sections and induce \( (A_2)_{\lambda \lambda'} = (T,t)_* \circ (A_1)_{\lambda \lambda'} \circ (T,t)^* \) at the vector orbibundle level. If we extend \( RI(\overline{A}_2) \) by 0 outside the neighborhood \( \Gamma_2 \tilde{W}_2 \) we obtain a lift of \( (A_2)_{\lambda \lambda'} \). We proved the proposition in the case (ii) as well. \( \Box \)

We also have the uniqueness up to smoothing operators of the lift of a pseudodifferential operator.

**Proposition 3.3.** Let \( \tilde{E} \overset{\rho}{\to} \tilde{M} \) a \( \Gamma \) vector bundle, with \( \Gamma \) a finite group and \( E \overset{\pi}{\to} M \) the associated orbibundle. If \( A_i \), \( i = 1,2 \) are two classical pseudodifferential operators on \( C^\infty(M; E) \) which are \( \Gamma \) equivariant and induce the same pseudodifferential operator on \( C^\infty(M; E) \), then \( A_1 - A_2 \) is a smoothing operator.

**Proof.** It is sufficient to prove that if \( \tilde{A} \) is a classical pseudodifferential operator that induces the zero operator on \( C^\infty(M; E) \), then \( \tilde{A} \) is smoothing.

Let \( \tilde{x} \in \tilde{M} \) be a point such that its projection \( \pi(\tilde{x}) \) onto \( M = \tilde{M}/\Gamma \) is a smooth point. We will show that the total symbol of the operator \( \tilde{A} \) on a neighborhood of \( \tilde{x} \) is smoothing (though the total symbol of the operator is not well-defined, the property of being a smoothing symbol is independent of the chart around \( \tilde{x} \)). Let \( \tilde{W} \) be a slice at \( \tilde{x} \) with respect to the action of the group \( \Gamma \). Because the point \( x = \pi(\tilde{x}) \) is smooth, the isotropy group \( (\Gamma)_x \) is trivial. Let \( \phi \) be a smooth function on \( M \) with support inside \( \tilde{W} \) which is equal to 1 on a neighborhood of a smaller slice \( \tilde{V} \) at \( \tilde{x} \). Let \( \psi \) be a smooth function on \( \tilde{M} \) which is equal to 1 on a neighborhood of \( supp(\phi) \) and zero outside \( \tilde{W} \).

\( \phi \tilde{A} \phi \) is a pseudodifferential operator which is localized on \( \tilde{W} \). We will prove that this operator is smoothing. Then \( \tilde{A} \) will be smoothing above \( \tilde{V} \) where \( \phi = 1 \).

Let \( \tilde{\phi} = \sum_{\gamma \in \Gamma} \gamma \phi \) be the \( \Gamma \) invariant extension of \( \phi \) to \( \tilde{M} \). Then \( \tilde{\phi} \tilde{A} \tilde{\phi} \) is a \( \Gamma \) equivariant extension to \( C^\infty(\tilde{M}; \tilde{E}) \) of the zero operator on \( C^\infty(M; E) \). Let \( f \in C^\infty(\tilde{M}; \tilde{E}) \) be an arbitrary section with support in a neighborhood of \( supp(\psi) \) that vanishes outside the set \( \{ \tilde{x} | \psi(\tilde{x}) = 1 \} \). Then \( \tilde{f} = \sum_{\gamma \in \Gamma} \gamma f \) is a \( \Gamma \) invariant section that extends \( f \). Because \( \tilde{\phi} \tilde{A} \tilde{\phi} \) restricted to the \( \Gamma \) invariant section is equal
to zero, we have:

\[(72) \quad 0 = \hat{\phi} \hat{A} \hat{\phi}(\hat{f}) = \psi \hat{\phi} \hat{A} \hat{\phi}(\sum_{\gamma \in \Gamma} \gamma f) = \]

\[(73) \quad = \psi \hat{\phi} \hat{A} \hat{\phi}(\sum_{\gamma \in \Gamma} \gamma f) + \psi \hat{\phi} \hat{A} \hat{\phi}(1 - \psi)(\sum_{\gamma \in \Gamma} \gamma f) = \]

\[(74) \quad = \hat{\phi} \hat{A} \hat{\phi}(f) + \left( \sum_{\gamma \in \Gamma} \hat{\phi} \hat{A} \hat{\phi}(1 - \psi)\gamma \right)(f) = 0 \]

But $\hat{\phi}(1 - \psi)$ has support inside $\cup_{\gamma \neq e} \hat{W}$ which is disjoint from $\hat{W}$ so $\hat{\phi} \hat{A} \hat{\phi}(1 - \psi)$ and so $\sum_{\gamma \in \Gamma} \hat{\phi} \hat{A} \hat{\phi}(1 - \psi)\gamma$ are smoothing. Because $f$ was chosen arbitrary, we conclude that $\hat{\phi} \hat{A} \hat{\phi}$ is smoothing so $\hat{A}$ is smoothing above $\hat{V}$.

Then $\hat{A}$ is smoothing on the regular set $\hat{M}_{\text{reg}} = \{ \hat{x} | (\Gamma)\hat{x} = (e) \}$. If $\hat{x}_1 \in \hat{M}_{\text{sing}}$ is a singular point then, on an Euclidean chart around $\hat{x}_1$ the total symbol of $\hat{A}$ will have the asymptotic expansion

\[(75) \quad a(\hat{x}, \xi) \sim \sum_{k \geq 0} a_{d-k}(\hat{x}, \xi) \]

But $\hat{M}_{\text{reg}}$ is dense in $\hat{M}$ so $a_{d-k}(\hat{x}, \xi) = 0$ on a dense set because $a(\hat{x}, \xi)$ is smoothing on $\hat{M}_{\text{reg}}$. We conclude that $a_{d-k}(\hat{x}, \xi) = 0$ on a neighborhood of $\hat{x}_1$ so $\hat{A}$ is smoothing on that neighborhood and thus on the whole manifold $\hat{M}$.

We denote by $\Psi(M; E)$ or simply by $\Psi$ the space of pseudodifferential operators acting on the smooth sections of the vector orbibundle $E \overset{p}{\rightarrow} M$. Most of the definitions and constructions for the pseudodifferential operators acting on genuine vector bundles can be extended to the operators in $\Psi(M; E)$.

The symbol of a pseudodifferential operator $A$ in an linear vector orbibundle chart $\mathcal{R}$ is a section in the vector orbibundle $\text{End}(E) \overset{p}{\rightarrow} T^*(U)$ and is unique up to a smoothing symbol.

The order of a pseudodifferential operator is the maximum of the orders of the restrictions to vector orbibundle charts and denote by $\Psi^d$ the subspace of operators of order less or equal to $d$. $\Psi$ becomes a filtered algebra with respect to the compositions of operators.

**Definition 3.5.** A pseudodifferential operator $A$ of order $d$ is called elliptic if it is elliptic when restricted to all vector orbifold charts.

An elliptic pseudodifferential operator $A \in \Psi^d$ has a parametrix $B \in \Psi^{-d}$ such that $AB - Id$ and $BA - Id$ are both smoothing. The construction of $B$ can be done locally, in each vector orbibundle chart, as in the case of pseudodifferential operators on manifolds.

In Proposition 3.3 we showed that any vector orbibundle $E \overset{p}{\rightarrow} M$ can be realized as a map between the orbit spaces of a $G$ vector bundle $\tilde{E} \overset{\tilde{p}}{\rightarrow} \tilde{M}$, with $G$ a compact Lie group. If $G$ was finite, Proposition 3.2 shows that any pseudodifferential operator acting on $C^\infty(M; E)$ can be realized as a $G$ equivariant pseudodifferential operator acting on $C^\infty(\tilde{M}; \tilde{E})$. The following proposition extend this result to the case of Lie groups.

**Proposition 3.4.** Let $\tilde{E} \overset{\tilde{p}}{\rightarrow} \tilde{M}$ be a $G$ vector bundle over a compact smooth manifold $M$ with $G$ a compact Lie group such that any $x \in M$ has a finite isotropy group
section induces \( A \) invariant sections induces the smoothing operator \( K \). We will transport it using the isomorphism between the previous \( \tilde{G} \) and \( \tilde{M} \) to \( N_\alpha \) is diffeomorphic with the \( G \) vector bundle \( G \times G_\alpha (U \times V) \rightarrow G \times G_\alpha U \) by a \( G \) equivariant vector bundle diffeomorphism. The cover \(( U_\alpha )_\alpha \) can be chosen to be finite because \( M \) is compact.

Using a partition of unity, any pseudodifferential operator \( A \) acting on sections of the vector orbibundle \( E \rightarrow M \) can be written as \( A = \sum_A A_\alpha + K \) where \( K \) is a smoothing operator with smooth kernel which is zero on a neighborhood of the diagonal and \( A_\alpha \) are pseudodifferential operators which vanish on sections that are supported outside the open set \( U_\alpha \) and take sections with support inside \( U_\alpha \) into sections with support inside \( U_\alpha \). For each \( \alpha \) we will construct a \( G \) equivariant pseudodifferential operator \( A_\alpha \) acting on sections of \( E \rightarrow M \) whose restriction to the \( G \) invariant sections induces \( A_\alpha \).

First let us fix a \( G \) bi-invariant pseudodifferential operator \( Q \) acting on \( C^\infty (G) \) of order \( d = \text{ord}(A) \) such that \( Qf = 0 \) on constant functions \( f \). In all the situations \( Q \) can be taken to be the \( d/2 \) power of the Laplacian \( \Delta \) on \( G \) with respect to a bi-invariant metric on \( G \). Using Proposition 3.2 we can find a \( G_\alpha \) equivariant pseudodifferential operator \( B_\alpha \) acting on the space of sections \( C^\infty (\tilde{U}_\alpha ; V) \) of the \( G_\alpha \) bundle \( \tilde{U}_\alpha \times V \rightarrow \tilde{U}_\alpha \) which induces \( A_\alpha \) when restricted to \( G_\alpha \) invariant sections. The space of smooth sections of the \( G \) vector bundle \( G \times (\tilde{U}_\alpha \times V) \rightarrow G \times \tilde{U}_\alpha \) is equal to \( C^\infty (G \times \tilde{U}_\alpha ; V) \cong C^\infty (G) \otimes C^\infty (\tilde{U}_\alpha ; V) \) so the pseudodifferential operator \( Q \otimes 1 + 1 \otimes B_\alpha \) acts on this space and it has order \( d \). It is \( G \) equivariant. The finite group \( G_\alpha \) acts by a diagonal action on \( G \times (\tilde{U} \times V) \) and \( G \times \tilde{U} \) so we can replace the above pseudodifferential operator with the \( G_\alpha \) equivariant pseudodifferential operator \( (\sum_{g \in G_\alpha} gQ \otimes 1 + 1 \otimes B_\alpha) \). This will induce a pseudodifferential operator on the sections of the \( G \) bundle of \( G_\alpha \) orbits \( G \times G_\alpha (\tilde{U}_\alpha \times V) \rightarrow G \times G_\alpha \tilde{U}_\alpha \).

We will transport it using the isomorphism between the previous \( G \) vector bundle and \( \tilde{E}|_{N_\alpha} \rightarrow N_\alpha \) to a pseudodifferential operator \( \tilde{A}_\alpha \) acting on \( \tilde{E} \rightarrow M \), extending it by \( 0 \) outside \( N_\alpha \).

We have to compare the actions of \( A_\alpha \) and \( \tilde{A}_\alpha \). Let \( f \) be a \( G \) invariant smooth section in \( C^\infty (M; \tilde{E}) \) which induces \( \tilde{f} \in C^\infty (M; \tilde{E}) \). It is enough to take \( f \) with support in \( N_\alpha \). In this case \( f \) is induced by a \( G_\alpha \) invariant section \( \tilde{f} \in C^\infty (\tilde{U}_\alpha ; G \times (\tilde{U}_\alpha \times V)) \cong C^\infty (G) \otimes C^\infty (\tilde{U}; V) \) which is \( G \) invariant as well. Then \( \tilde{f}(g, u) = gh(u) \) with \( h \in C^\infty (\tilde{U}; V) \), \( h \) induces \( \tilde{f} \in C^\infty (M; \tilde{E}) \) and \( \tilde{A}_\alpha \tilde{f} \) corresponds via the \( G \) equivariant vector bundle diffeomorphism to \( (Q \otimes 1 + 1 \otimes B_\alpha)\tilde{f} = B_\alpha(h) \) and this section induces \( A_\alpha \tilde{f} \in C^\infty (M; \tilde{E}) \).

We will construct a smoothing operator \( \hat{K} \) in \( C^\infty (\hat{M}; \tilde{E}) \) whose action on \( G \) invariant sections induces the smoothing operator \( K \). The kernel of \( K \) is given by
a smooth section \( S(x, y) \) of the endomorphism bundle \( \text{End}(E) \to M \times M \) which corresponds to a smooth \( G \) invariant section \( \tilde{S}(\tilde{x}, \tilde{y}) \) of the endomorphism bundle \( \text{End}(\tilde{E}) \to \tilde{M} \times \tilde{M} \). \( \tilde{S} \) is the kernel of a smoothing operator \( \tilde{K} \), which induces \( K \) when restricted to the \( G \) invariant sections of \( \tilde{E} \stackrel{p}{\to} \tilde{M} \).

The operator \( \tilde{A} \) will be the sum of the operators \( \tilde{A}_\alpha \) and \( \tilde{K} \).

If the operator \( A \) is classical, we can choose \( Q \) to be classical, then \( B_\alpha, Q \otimes 1 + 1 \otimes B_\alpha \) and \( \tilde{A}_\alpha \) are classical so \( A \) is classical.

### 3.3. Zeta Function of an Elliptic Pseudodifferential Operator

Let \( A \) be a pseudodifferential operator acting on the space of smooth sections in a vector orbibundle \( E \stackrel{p}{\to} M \). Suppose that \( M \) is compact and it is endowed with a Riemannian metric \( g \) and that we have a hermitian structure \( \langle , \rangle \) in the vector orbibundle.

As in the case of hermitian vector bundles over a closed Riemannian manifold, we define the scalar product on \( C^\infty(M; E) \) by the formula

\[
\langle f, g \rangle = \int_M \langle f_1(x), f_2(x) \rangle \, d\text{vol} \quad \text{for } f_1, f_2 \in C^\infty(M; E).
\]

The integration with respect to the volume form \( d\text{vol} \) can be defined locally on vector orbibundle charts. If \( (\tilde{U}, \Gamma, U, \pi) \) is such a chart then the metric \( g \) and the hermitian structure \( \langle , \rangle \) above \( U \) are induced by \( \Gamma \) invariant metric \( \tilde{g} \) and respectively a \( \Gamma \) invariant hermitian structure \( \langle , \rangle \) in the \( \Gamma \) vector bundle \( \tilde{U} \times V \overset{\text{pr}}{\to} \tilde{U} \). Let \( d\text{vol} \) be the volume form on \( \tilde{U} \) associated with \( \tilde{g} \). Then define

\[
\int_U \alpha(x) \, d\text{vol} = \frac{1}{|\Gamma|} \int_{\tilde{U}} \tilde{\alpha}(\tilde{x}) \, d\text{vol}
\]

for \( \tilde{\alpha} \in C^\infty(\tilde{U})^\Gamma \) that induces \( \alpha \in C^\infty(U) \) (we chose \( \alpha = \langle f_1, f_2 \rangle \)). It is easy to see that the above formula is consistent with respect to the equivalence of charts.

The \( L^2 \) completion of \( C^\infty(M; E) \) with respect to the above scalar product will be denoted by \( L^2(M; E) \). A pseudodifferential operator \( A \) defines an unbounded operator on \( L^2(M; E) \).

**Definition 3.6.** The pseudodifferential operator \( A \) is called symmetric if

\[
\langle Af_1, f_2 \rangle = \langle f_1, Af_2 \rangle \quad \text{for all } f_1, f_2 \in C^\infty(M; E).
\]

\( A \) is positive if

\[
\langle Af, f \rangle \geq 0 \quad \text{for any } f \in C^\infty(M; E).
\]

**Remark 3.3.** If \( A \) is symmetric, then the restriction \( A_R \) to any vector orbibundle chart \( R \) as defined in [65] is symmetric. This implies that \( A_R \) is induced by a \( \Gamma \) equivariant pseudodifferential operator \( A_R \) which can be chosen to be symmetric with respect to the \( \Gamma \) invariant metric \( \tilde{g} \) and \( \Gamma \) invariant hermitian structure inducing \( g \) and \( \langle , \rangle \) above \( U \). Indeed, the formal adjoint \( \tilde{A}_R^* \) of the operator \( A_R \) induces \( A_R \) as well, so the symmetric operator \( \frac{1}{2}(\tilde{A}_R + \tilde{A}_R^*) \) induces \( A_R \).

Then the principal symbol of \( A \), which is induced by the \( \Gamma \) invariant principal symbol of \( \tilde{A} \), is selfadjoint.

**Theorem 3.5.** Let \( A \) be a positive symmetric elliptic pseudodifferential operator acting on the space of sections of a vector orbibundle \( E \stackrel{p}{\to} M \) with compact base
space. Then the operator $A$ acting on $L^2(M;E)$ is essentially selfadjoint and its spectrum is discrete.

**Proof.** Let $m = \dim(M)$ and denote by $G$ the orthogonal group $O(m)$. Let $\tilde{E} \xrightarrow{p} \tilde{M}$ be the $G$ vector bundle, with $\tilde{M} = \mathcal{F}(M)$, such that the vector orbibundle of orbits is canonically isomorphic to $E \xrightarrow{p} M$, as in Proposition 2.3. Let us fix a $G$ bi-invariant metric $g_m$ on $G$ so that $\int_G dg_m = 1$ and a $G$ bi-invariant positive selfadjoint pseudodifferential operator $Q$ of order $d$ acting on $C^\infty(G)$. As a metric $g_m$ we can choose the left translations of the opposite of the Killing form on the Lie algebra $o(m)$ and $Q = \Delta^{d/2}$ where $\Delta$ is the Laplace operator associated with the metric $g_m$.

We will construct a $G$ invariant metric on $\tilde{M}$ which induces the given metric on $M$, a $G$ invariant hermitian structure on $\tilde{E} \xrightarrow{p} \tilde{M}$ which induce the given hermitian metric on $E \xrightarrow{p} M$ and a $G$ equivariant elliptic positive selfadjoint pseudodifferential operator $\tilde{A}$ acting on $C^\infty(\tilde{M};\tilde{E})$ whose restriction to the invariant sections is equal to $A$.

Let us fix a finite atlas $\mathcal{A} = (\mathcal{R}_\alpha)_\alpha$ of the vector orbibundle $E \xrightarrow{p} M$, with $\mathcal{R}_\alpha = (\tilde{U}_\alpha,V,\Gamma_\alpha,U,\Pi_\alpha,\pi_\alpha)$. Then $\tilde{\mathcal{R}}_\alpha = (G \times \tilde{U}_\alpha,V,\Gamma_\alpha,G \times \Gamma_\alpha,\tilde{U}_\alpha,Id \times \Pi_\alpha,Id \times \pi_\alpha)$ forms an atlas for $\tilde{E} \xrightarrow{p} \tilde{M}$. Let $\tilde{g}_\alpha$ be the $\Gamma_\alpha$ invariant metric on $\tilde{U}_\alpha$ which induces $g$ on $U$. Then the collection of $\Gamma_\alpha$ invariant metrics $g_m \otimes \tilde{g}_\alpha$ on $G \times \tilde{U}_\alpha = \mathcal{F}(U)$ induces the $G$ invariant metric $\tilde{g}$ on $\tilde{M}$. Because $\tilde{E} \xrightarrow{p} \tilde{M}$ is the pull-back of $E \xrightarrow{p} M$ with respect to the canonical map $\tilde{M} = \mathcal{F}(M) \xrightarrow{p} M$, we define the hermitian structure on $\tilde{E} \xrightarrow{p} \tilde{M}$ to be the pull-back hermitian structure. It will be $G$ invariant by construction and induces the initial hermitian structure when passing to the $G$ orbit spaces. Consider the $G$ invariant scalar product on $C^\infty(\tilde{M};\tilde{E})$ associated with hermitian structure and the metric $\tilde{g}$ constructed above. When restricted to the $G$ invariant sections, this scalar product is equal with the scalar product on $C^\infty(M;E)$ constructed with the help of the metric $g$ and the hermitian structure on $E \xrightarrow{p} M$. Also $L^2(\tilde{M};\tilde{E})^G \cong L^2(M;E)$.

The pseudodifferential operator $\tilde{A}$ acting on $C^\infty(\tilde{M};\tilde{E})$ constructed as in Proposition 2.3 will be symmetric with respect to the scalar product on $C^\infty(\tilde{M};\tilde{E})$. By construction, the principal symbol of $\tilde{A}$ is equal to $q_{pr} \otimes Id + a_{pr}$ where $q_{pr}$ is the principal symbol of $Q$ and $a_{pr}$ is the principal symbol of $A$. Because $Q$ and $A$ are positive symmetric elliptic pseudodifferential operators, the operator $\tilde{A}$ is elliptic as well. Because $\tilde{M}$ is compact, the operator $\tilde{A}$ acting on $L^2(\tilde{M};\tilde{E})$ is essentially selfadjoint. The restriction of $\tilde{A}$ to the closed subspace of $G$ invariant $L^2$ sections $L^2(\tilde{M};\tilde{E})^G \cong L^2(M;E)$ which is equal to $A$ is essentially selfadjoint. The spectrum of $\tilde{A}$ is real and discrete so the spectrum of $A$ is a discrete subset of $\mathbb{R}$.

To show that the complex powers of the elliptic positive selfadjoint operator $A$ are pseudodifferential operators, one can apply the same local approach as in [Sc], and use approximations of the resolvent of $(A - \lambda)$ in order to show that $A^s$ are, up to a smoothing operator, pseudodifferential operators of complex order $s$. Because all the proofs follow in the same way with minimal complications, we will not repeat them in our paper. For $Re(s) < -\frac{m}{d}$ the distributional kernel of $A^s$ is continuous and $A^s$ is of trace class.
We can relax the conditions imposed on the operator $A$, and prove an analogous statement to Theorem 3.5. Suppose that $\pi$ is an Agmon angle for $A$. Then one can define the complex powers of $A$ as

$$A^s = \frac{1}{2\pi i} \int_\varphi \lambda^s (\lambda - A)^{-1} d\lambda \quad \text{when } \Re(s) < 0$$

(80)

(where $\varphi$ is a contour in the complex plane obtained by joining two parallel half-lines to the negative real axis by a circle around the origin) and

$$A^s = A^{s-k} A^k \quad \text{for } \Re(s) \geq 0$$

(81)

for large enough $k \in \mathbb{Z}$ that makes $s - k < 0$. Obviously, if $A$ is positive selfadjoint then $\pi$ is an Agmon angle for $A$.

**Proposition 3.6.** Let $A$ be an elliptic pseudodifferential operator of positive order $d$ acting on the space of sections of a vector orbibundle $E^p \to M$ with compact base space. Suppose that $\pi$ is an Agmon angle for $A$. Then the operator $A$ acting on $L^2(M; E)$ has a discrete spectrum and its complex powers $A^s$ are pseudodifferential operators of complex order $sd$.

**Proof.** The proof is analogous to the proof of Theorem 3.5. We construct the operator $\tilde{A}$ acting on sections of the $G$ vector bundle $\tilde{E} \to \tilde{M}$ with the help of a $G$ bi-invariant elliptic selfadjoint pseudodifferential operator $Q$ acting on $C^\infty(G)$. The principal symbol of $\tilde{A}$ is equal to $\tilde{a}_{pr} = q_{pr} \otimes \text{Id} + a_{pr}$ with $q_{pr}$ the principal symbol of $Q$ and $a_{pr}$ the principal symbol of $A$. Because $\pi$ is an Agmon angle for $A$, the spectrum of $a_{pr}(x, \xi)$ is disjoint from the region in the complex plane $C = \{z \mid \arg(z) \in (\pi - \varepsilon, \pi + \varepsilon)\} \cup \{z \mid |z| < \varepsilon\}$ for a small enough $\varepsilon > 0$. Because $q_{pr}(x, \xi)$ is selfadjoint the spectrum of $\tilde{a}(x, \xi)$ is disjoint from $C$. We conclude that $\pi$ is an Agmon angle for $\tilde{A}$. Then $\tilde{A}$ is elliptic and has a discrete spectrum. $A$ being the restriction to the $G$ invariant sections will have a discrete spectrum as well. The complex powers $A^s_n$ are well defined and are pseudodifferential operators of order $sd$ so the restriction to the $G$ invariant sections, which are equal to $A_n^s$, are pseudodifferential operators of order $sd$.

Throughout the rest of the chapter we suppose that $\pi$ is an Agmon angle for the elliptic pseudodifferential operator $A$. The complex powers $A^s = A^s_n$ will be defined as in (80) and (81).

**Definition 3.7.** The zeta function of the operator $A$ is equal to

$$\zeta_A(s) = \text{Trace}(A^s) \quad \text{for } \Re(s) < \frac{-m}{d}.$$  

(82)

We will prove the following theorem:

**Theorem 3.7.** The zeta function of an elliptic pseudodifferential operator $A$ acting on the sections of a vector orbibundle $E^p \to M$ can be extended to a meromorphic function on $\mathbb{C}$ with at most simple poles at $s = -\frac{m+k}{d}$, $k = 0, 1, 2, \ldots$. The residues are integrals on $M$ of Dirac-type densities which can be explicitly computed in terms of the asymptotic expansion of the total symbol of the operator $A$.

**Proof.** Let $A = (U_\alpha, V, \Gamma_\alpha, U_\alpha, \Pi_\alpha, \pi_\alpha)_\alpha$ be a fixed finite atlas of linear orbibundle charts. Using a partition of unity we can decompose the complex powers of the
operator $A$ as
\begin{equation}
A^s = \sum_\alpha A^s_\alpha + K_s
\end{equation}
where $A^s_\alpha$ are pseudodifferential operators of order $sd$ with support inside $U_\alpha$ and $K_s$ is a smoothing operator. We can arrange for $A^s_\alpha$ to take smooth sections with support inside $U_\alpha$ into smooth sections with support inside $U_\alpha$. The trace $\text{Tr}(K_s)$ is defined for $s \in \mathbb{C}$ and it is a holomorphic function on the complex plane. In order to prove the theorem, we will show that the statement in the theorem holds for the trace of the operators $A^s_\alpha$. We will show that $\text{Tr}(A^s_\alpha)$ is a meromorphic function on $\mathbb{C}$ with at most simple poles at $s = -\frac{\text{m}+k}{d}$, $k = 0, 1, 2, \ldots$, and that the residues are computed as integrals on $M$ of Dirac-type densities.

$A^s_\alpha$ can be seen as a holomorphic family of pseudodifferential operators of order $sd$ acting on the smooth sections of the vector orbibundle $E_{\mathcal{U}_\alpha} \overset{\mathcal{P}}{\to} U_\alpha$. For the sake of simplicity, we will drop the index $\alpha$ and denote $A^s_\alpha$ by $A^s$. There exists a holomorphic family of $\Gamma$ equivariant pseudodifferential operators $\tilde{A}^s$ of order $sd$ acting on the sections of the $\Gamma$ vector bundle $\tilde{U} \times V \overset{\mathcal{P}}{\to} \tilde{U}$ which induce $A^s$ on the $\Gamma$ invariant sections. This family is unique up to smoothing operators, cf. Proposition 3.3. Then the component corresponding to the trivial representation of $\Gamma$ Dirac-type densities on $\tilde{U}$ constructed from the total symbol of $\tilde{A}^s$ which on the chosen linear vector orbibundle chart coincides with the total symbol of the pseudodifferential operator $A^s$ on $U$.

Though the statements of Theorems 1.4 and 1.5 refer to the complex powers of a pseudodifferential operator, their proofs can be slightly changed and adapted to include the case of holomorphic families of pseudodifferential operators and so the conclusions listed above are still true.

The $\Gamma$ Dirac-type density $\{\eta^\gamma_k\}_{\gamma \in \Gamma}$ defines a Dirac-type density on the orbifold $U$ which we will denote by $\eta_k$. Observe that the formula (84) is equivalent to
\begin{equation}
\text{res}_{s=-\frac{\text{m}+k}{d}} \text{Tr}(\tilde{A}^s_{\chi_0}) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \int_{\tilde{U}^\gamma} \eta^\gamma_k
\end{equation}
where $\{\eta^\gamma_k\}_{\gamma \in \Gamma}$ are $\Gamma$ Dirac-type densities on $\tilde{U}$ constructed from the total symbol of $\tilde{A}^s$ which on the chosen linear vector orbibundle chart coincides with the total symbol of the pseudodifferential operator $A^s$ on $U$.

The Dirac-type density whose integral on $M$ is equal to the residues of the zeta function $\zeta_A$ at $s = -\frac{\text{m}+k}{d}$ will be equal to the sum of the Dirac-type densities $\eta_k$ constructed as above on the charts of the vector orbibundle.

Using Proposition 2.3, we can reformulate Theorem 3.7 and express the residues of the zeta function of $A$ in terms of integrals of smooth densities on submanifolds of $S(M)$ associated with the canonical stratification.

\textbf{Theorem 3.8.} The zeta function of an elliptic pseudodifferential operator $A$ acting on the sections of a vector orbibundle $E \overset{\mathcal{P}}{\to} M$ can be extended to a meromorphic
function on $\mathbb{C}$ with at most simple poles at $s = -\frac{m+k}{d}$, $k = 0, 1, 2, \ldots$. For each $k \geq 0$ there exist densities $\eta_{\upsilon,k}$ on the strata $M_{\upsilon}$ of the canonical stratification of $M$ such that the residue of the zeta function at $s = -\frac{m+k}{d}$ is the sum of integrals of these densities. On each strata $M_{\upsilon}$ the density $\eta_{\upsilon,k}$ can be explicitly computed in terms of the asymptotic expansion of total symbol of the operator $A$ in a neighborhood of $M_{\upsilon}$.

Proof. For each $\upsilon \in \mathcal{O}(M)$ and $k \geq 0$, the density $\eta_{\upsilon,k}$ can be obtained as follows: let $x \in M_{\upsilon}$ and $(\tilde{U}, V, \Gamma, U, \Pi, \pi)$ be a linear vector orbibundle chart centered at $x$. Observe that $\Gamma \in \upsilon$ and the map $\pi : \tilde{U}^\Gamma \to U \cap M_{\upsilon}$ is a local diffeomorphism. Theorem 3.7 gives us Dirac-type densities $\eta_k$ on $M$ whose integrals are equal to the residues of the zeta function and can be computed explicitly in terms of the total symbol of $A$. These densities can be described above $U$ as families of densities $\{\eta^\gamma_k\}$ such that $\eta^\gamma_k$ is a smooth density on $\tilde{U}^\gamma$. Consider the density on $\tilde{U}^\Gamma$ given by

\begin{equation}
\eta^\Gamma_{\upsilon,k} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma_{\upsilon}^{\Gamma}} \eta^\gamma_k \mid_{\tilde{U}^\gamma}
\end{equation}

Define $\eta_{\upsilon,k}$ on $M_{\upsilon}$ to be the density whose restriction to $U$ is equal to $\pi_\ast(\eta^\Gamma_{\upsilon,k})$.

As proved in Proposition 2.8, the integral on $M$ of the Dirac-type density $\eta_k$ and the sum of the integrals of the densities $\eta_{\upsilon,k}$ on the strata $M_{\upsilon}$ for $\upsilon \in \mathcal{O}(M)$ are equal, so they are equal to the residue of $\zeta_A$ at $s = -\frac{m+k}{d}$.

$\square$
APPENDIX A. PROOF OF THEOREM 1.1

Proof. Let $U$ be a $\Gamma_x$-invariant open neighborhood of $x$ in $M$ such that $\gamma U \cap U = \emptyset$ for $\gamma \in \Gamma \setminus \Gamma_x$. Let $\mu : \Gamma_x \times T_x(M) \to T_x(M)$ be the linear representation of $\Gamma_x$ on the tangent space at $x$ generated by the action of $\Gamma_x$ on $U$. Choose a $\Gamma_x$ invariant metric on $U$ (this can always be done by averaging a metric over the finite group $\Gamma_x$). Then the exponential map $exp : T_x(M) \to U$ is $\Gamma_x$ equivariant. Indeed, if $s(t)$ is a geodesic curve with $s(0) = x$ and $s'(0) = X \in T_x(M)$ then, for $\gamma \in \Gamma_x$, $\gamma \cdot s(t)$ is again a geodesic curve and $(\gamma \cdot s)'(0) = \mu(\gamma)X$. So $exp(\mu'(\gamma)X) = \gamma \cdot s(1) = \gamma \cdot \text{exp}(X)$. One can choose a smaller $U$ such that the exponential map realizes a diffeomorphism between a $\Gamma_x$ invariant neighborhood of $0$ in $T_x(M)$ and $U$. If we fix a linear isomorphism $T_x(M) \cong \mathbb{R}^m$ and transport the action of $\Gamma_x$ on $T_x(M)$ via this isomorphism, we obtain a linear representation $\mu : \Gamma_x \times \mathbb{R}^m \to \mathbb{R}^m$ and a $\Gamma_x$ equivariant diffeomorphism between a neighborhood $O$ of the origin and $U$.

Using the pull-back via this diffeomorphism we can replace the bundle $E_{\mid U} \xrightarrow{\mu} U$ by a new $\Gamma_x$ vector bundle such that the action of $\Gamma_x$ on the base will be the restriction of a linear representation to an open neighborhood of the origin. The point $x$ corresponds to the origin $0$. For the sake of simplicity we will use the same notation for the new bundle.

Consider now a linear connection $\nabla$ in the $\Gamma$ vector bundle $E_{\mid U} \xrightarrow{\mu} U$. The group $\Gamma_x$ acts on the space of connections, and we can make $\nabla$ to be invariant with respect to $\Gamma_x$ by replacing it with the connection $\nabla + \frac{1}{\mu}(\gamma_x, \nabla - \nabla)$. We will define the bundle map $\Psi$ between the trivial $\Gamma$ bundle $U \times E_x \rightarrow U$ and $E_{\mid U} \xrightarrow{\mu} U$ as follows:

for a point $y \in U$ and a vector $X \in E_x$, let $\Psi(y, X) = Y$ be the vector in the fiber of $E$ above $y$ obtained by parallel transport of $X$ along the curve $\phi(t) = ty$, $t \in [0, 1]$. The vector space $V$ as in the statement of the proposition will be equal to $E_x$ with the action of $\Gamma_x$ induced from the action of $\Gamma$ on the total space of the bundle.

$\Psi$ is a vector bundle isomorphism and we have to show that $\Psi$ is $\Gamma_x$ equivariant. If $\phi(t)$ is the path in $E$ that realizes the parallel transport from $X$ to $Y = \Psi(y, X)$ above $\phi(t)$, and $\gamma \in \Gamma_x$, then, because the connection is $\Gamma_x$ invariant, the path $\gamma \cdot \phi(t)$ realizes the parallel transport between $\gamma X$ and $\gamma Y$ above the curve $p(\gamma \cdot \phi(t)) = \gamma \cdot \phi(t)$, which is linear. Then $\gamma Y = \Psi(\gamma y, \gamma X)$, so $\Psi$ is $\Gamma_x$ equivariant.

For $U \subset M$ as above the set $\Gamma \cdot U = U \cup \gamma \cup U \subset M$ is the reunion of disjoint open sets $\gamma \cup U$ with $\{\gamma_1, \gamma_2, \ldots\}$ a complete system of left coset representatives for $\Gamma/\Gamma_x$. Then $\Gamma \times_{\Gamma_x} U \rightarrow \Gamma \cdot U$ given by $u(\gamma, y) = \gamma y$ is a diffeomorphism, where $\Gamma \times_{\Gamma_x} U$ is the cross-product $\Gamma \times U/ \sim$, with $(\gamma \gamma', y) \sim (\gamma, \gamma' y)$, for any $\gamma \in \Gamma$, $\gamma' \in \Gamma_x$, $y \in U$. Moreover, $u$ is $\Gamma$ equivariant. The action of $\Gamma$ on the cross-product $\Gamma \times_{\Gamma_x} U$ is by left translations. The restriction of the vector bundle $E_{\mid U} \xrightarrow{\mu} \Gamma \cdot U$ is trivial and a trivialization is given by

$$\Gamma \times_{\Gamma_x} (O \times E_x) \xrightarrow{1 \times \Psi} E_{\mid \Gamma \cdot U}$$

The bundle isomorphism is also $\Gamma$ equivariant. □
Appendix B. Proof of Theorem 1.3

Proof. Let \( \{U_\alpha\} \) be a finite cover of \( M \) with open sets as in Proposition 1.1 so that the restrictions of the \( \Gamma \) bundle \( E \overset{\pi}{\rightarrow} M \) to the subsets \( U_\alpha \) are isomorphic to \( \Gamma_x \) trivial vector bundles for some \( x \in U_\alpha \). Then \( \{\Gamma \cdot U_\alpha\} \) is a finite open cover with \( \Gamma \) invariant open sets. Consider a partition of unity \( \{ \phi_\beta \} \) subordinated to the open cover \( \{\Gamma \cdot U_\alpha\} \) such that the functions \( \phi_\beta \) are \( \Gamma \) equivariant. One can choose the open cover and the partition of unity such that for any \( \beta \) and \( \beta' \) either \( \text{supp}(\phi_\beta) \cap \text{supp}(\phi_{\beta'}) = \emptyset \) or \( \text{supp}(\phi_\beta) \cap \text{supp}(\phi_{\beta'}) \subset \Gamma \cdot U_\alpha \) for some \( \alpha \). Then the operators \( \phi_\beta \cdot A^* \phi_{\beta'} \) are either smoothing (in the first case) or have the support and range included in the space of sections that vanish outside the open set \( \Gamma \cdot U_\alpha \) (in the second case). Consider the trivialization by the \( \Gamma \) vector bundle isomorphism \( (\Gamma \times_{\Gamma_x} (O_\alpha \times E_x), \Gamma \times_{\Gamma_x} O_\alpha) \overset{\sim}{\rightarrow} (E_{\Gamma \cdot U_\alpha}, \Gamma \cdot U_\alpha) \). Let \( a_s(x, \xi) \) the complete symbol of the operator \( \phi_\beta \cdot A^* \phi_{\beta'} \) in this trivialization. Then for a section \( f \) with compact support in \( \Gamma \cdot O_\alpha \) we have:

\[
(\phi_\beta \cdot A^* \phi_{\beta'})(f)(x) = \iint e^{i<x-y,\xi>} a_s(x, \xi) f(y) dy d\xi
\]

where \( a_s(\xi) = (2\pi)^{-m} d\xi \) and the double integral is computed over \( \mathbb{R}^m \times \mathbb{R}^m \). To keep our notation simple, we will drop the indexes \( \alpha \) and \( \beta \) and denote \( \phi_\beta \cdot A^* \phi_{\beta'} \) by \( A_s \). The action of \( T \) on \( f \) is given by \( T f(x) = T f(t^{-1}x) \). Then

\[
(A_s \circ T) f(x) = \iint e^{i<x-y,\xi>} a_s(x, \xi) \circ T f(t^{-1}y) dy d\xi
\]

and after the change of variable \( y \mapsto ty \)

\[
= \iint e^{i<x-ty,\xi>} a_s(x, \xi) \circ T f(y) |\text{det}(t)| dy d\xi
\]

We can choose the trivialization in such a way that \( |\text{det}(t)| = 1 \). The distributional kernel of \( A_s \circ T \) will be equal to \( K_s(x, y) = \iint e^{i<x-y,\xi>} a_s(x, \xi) \circ T \xi \). For \( \text{Re}(s) < -\frac{m}{2} \) the trace of this operator is equal to the integral

\[
\text{Tr}(A_s \circ T) = \iint e^{i<x-tx,\xi>} \text{Tr}(a_s(x, \xi) \circ T) \xi dx
\]

We will choose the open cover \( O_\alpha \) so that either \( t \) has fixed points inside \( O_\alpha \) or \( \|x-tx\| > \varepsilon \) for all \( x \in O_\alpha \) for some fixed \( \varepsilon > 0 \). In the second case we will show that the trace function \( \text{Tr}(A_s \circ T) \) can be extended to the whole complex plane. Indeed, if we denote by \( \|D_\xi\| \) the differential operator \(-\left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_m^2} \right)\) we have \( \|D_\xi\|^{2\nu} e^{i<x-tx,\xi>} = \|x-tx\|^{2\nu} e^{i<x-tx,\xi>} \). Then

\[
\text{Tr}(A_s \circ T) = \iint e^{i<x-tx,\xi>} \text{Tr}(a_s(x, \xi) \circ T) \xi dx = \iint \frac{\|D_\xi\|^{2\nu} e^{i<x-tx,\xi>}}{\|x-tx\|^{2\nu}} \text{Tr}(a_s(x, \xi) \circ T) \xi dx
\]

and after integration by parts

\[
\iint \frac{e^{i<x-tx,\xi>}}{\|x-tx\|^{2\nu}} \|D_\xi\|^{2\nu} \text{Tr}(a_s(x, \xi) \circ T) \xi dx
\]

For a fixed half-plane \( \text{Re}(s) < K \) if we choose a large enough \( \nu \in \mathbb{N} \) the expression \( \|D_\xi\|^{2\nu} (\text{Tr}(a_s(x, \xi) \circ T) \xi \) is a symbol in \( S^{-2m}(O) \) so it is absolutely integrable and
gives after integration a holomorphic function in $s$ in the half-plane $Re(s) < K$. It follows that on open sets where $\|x - tx\| > \varepsilon$ the trace function $Tr(A_s \circ \mathcal{T})$ has a holomorphic extension to the complex plane $\mathbb{C}$.

Consider now a set $O = O_0$ so that $t|_O$ has a nonempty fixed point set. The diffeomorphism $t$ is given by $\gamma$. Denote $N = M^\gamma$. Let $x \in O$ such that $\gamma \in \Gamma_x$. As shown in Proposition 1.1, we can choose $x$ to be the origin in $O \subset \mathbb{R}^m$ and $\Gamma_x$ act on $O$ by linear isometries. Then $t$ is the restriction of a linear isometry and its fixed set $N \cap O$ is the intersection between a linear subspace $F_t$ of $\mathbb{R}^m$ and $O$. The dimension of $F_t$ is one of the dimensions $n_1$ of the connected components of the fixed point set $N$. For the sake of simplicity we will denote it by $n$. Let $(x_1, x_2)$ be linear coordinates on $\mathbb{R}^m$ so that $x_1$ are coordinates along the fixed-point set $F_t$ and $x_2$ are normal coordinates to $F_t$. In these new coordinates $t$ has the form $(\text{Id} \circ \tilde{T})$ with $\tilde{T}$ an $(m - n)$ square matrix whose eigenvalues are different from 1. Let $\xi = (\xi_1, \xi_2)$ be the coordinates in the cotangent space corresponding to the coordinates $x = (x_1, x_2)$. Using the new coordinates in the equation (90) $Tr(A_s \circ \mathcal{T})$ will be equal to

$$
\int e^{i<\xi_1, x_2 > - (x_1, x_2, (\xi_1, \xi_2), x_1, x_2, (\xi_1, \xi_2))} Tr(a_s(x_1, x_2, \xi_1, \xi_2) \circ \mathcal{T})\frac{dx_1 dx_2}{\Delta_2} = \int e^{i<\xi_1, x_2 >} Tr(a_s(x_1, x_2, \xi_1, \xi_2) \circ \mathcal{T})\frac{dx_1 dx_2}{\Delta_2}
$$

Because $\tilde{T}$ is a matrix whose eigenvalues are different from 1, the matrix $(\tilde{T} - \text{Id})$ is non-degenerated and if we use the change of coordinates $w = (\tilde{T} - \text{Id})x_2$ the above integral becomes

$$
\int e^{-i < w, \xi_2 >} Tr(a_s(x_1, (\tilde{T} - \text{Id})^{-1}w, \xi_1, \xi_2) \circ \mathcal{T})\frac{d\xi_1 d\xi_2}{det(\tilde{T} - \text{Id})^{-1}} = \int e^{-i < w, \xi_2 >} Tr(\tilde{a}_s(x_1, w, \xi_1, \xi_2) \circ \mathcal{T})\frac{d\xi_1 d\xi_2}{det(\tilde{T} - \text{Id})^{-1}}
$$

where $\tilde{a}_s(x_1, w, \xi_1, \xi_2) = a_s(x_1, (\tilde{T} - \text{Id})^{-1}w, \xi_1, \xi_2)$ is a classical symbol of order $sd$ in $(\xi_1, \xi_2)$ and with compact support in $(x_1, w)$. For $Re(s) < -\frac{m-n}{d}$ the integrand is absolutely integrable so we can apply Fubini’s theorem and get:

$$
Tr \left[ \left( \int e^{-i < w, \xi_2 >} \tilde{a}_s(x_1, w, \xi_1, \xi_2) \frac{d\xi_1 d\xi_2}{\Delta_2} \right) \circ \mathcal{T} \right] \frac{d\xi_1}{\Delta_2} = \int e^{-i < w, \xi_2 >} Tr(\tilde{a}_s(x_1, w, \xi_1, \xi_2) \circ \mathcal{T})\frac{d\xi_1 d\xi_2}{det(\tilde{T} - \text{Id})^{-1}}
$$

We will need the following result:

**Lemma B.1.** If $a_s(x, \xi) = a_s(x_1, x_2, \xi_1, \xi_2)$ is a classical matrix valued symbol of complex order $sd$ then, for $Re(s) < -\frac{m-n}{d}$, the expression

$$
b_s(x_1, \xi_1) = \int e^{-i < x_2, \xi_2 >} a_s(x_1, x_2, \xi_1, \xi_2)\frac{d\xi_2}{\Delta_2}
$$
is a classical symbol of order $sd$ with the asymptotic expansion
\begin{equation}
    b_s(x_1, \xi_1) \sim \sum_{\alpha} \frac{1}{\alpha!} (D_{x_2}^{\alpha} \partial_{\xi_1}^{\alpha} a_s)(x_1, 0, \xi_1, 0)
\end{equation}
If $a_s$ is a holomorphic family of symbols then there exists a holomorphic family of symbols $\tilde{b}_s$ defined for $s \in \mathbb{C}$ such that $\tilde{b}_s - b_s$ is a holomorphic family of smoothing symbols. The asymptotic expansion of $\tilde{b}_s(x_1, \xi_1)$ is given by (101).

We will postpone the proof until the end of the proof of the main theorem.

Using the above lemma for $\text{Re}(s) < \frac{(m-n)}{d}$, the trace of $A_s \circ T$ becomes:
\begin{equation}
    \text{Tr}(A_s \circ T) = \text{Tr} \left( \int \int b_s(x_1, \xi_1) \partial_{\xi_1} dx_1 \circ T \right) \left| \text{det}(\mathcal{T} - \text{Id}) \right|^{-1}
\end{equation}
where the integration is taken over the cotangent space of the fixed point set $\mathcal{N} \cap \mathcal{O}$. Because $\tilde{b}_s - b_s$ is a smoothing symbol, the difference between the integral above and
\begin{equation}
    \text{Tr} \left( \int \int \tilde{b}_s(x_1, \xi_1) \partial_{\xi_1} dx_1 \circ T \right) \left| \text{det}(\mathcal{T} - \text{Id}) \right|^{-1}
\end{equation}
is a whole function. We will show that, as a function in $s \in \mathbb{C}$, the above expression has a meromorphic extension to the complex plane. Let us consider the asymptotic expansion as a classical symbol
\begin{equation}
    \tilde{b}_s(x_1, \xi_1) \sim \sum_{i \geq 0} \tilde{b}_{s,i}(x_1, \xi_1)
\end{equation}
with $\tilde{b}_{s,i}(x_1, \xi_1)$ homogeneous of degree $sd - i$ in $\xi_1$. Let $\psi(\xi_1)$ be a positive, real valued smooth function which vanishes on a neighborhood of the origin and is equal to 1 for $\|\xi_1\| \geq 1$. Let us fix a half-plane $\text{Re}(s) < K$. For a large enough $\nu$ the difference $\tilde{b}_s(x_1, \xi_1) - \sum_{i=0}^{\nu} \psi(\xi_1) \tilde{b}_{s,i}(x_1, \xi_1)$ is a symbol in $S^{-n}(\mathcal{N} \cap \mathcal{O})$ for any $s$ with $\text{Re}(s) < K$. Then on this half-plane the difference
\begin{equation}
    \text{Tr} \left( \int \int \tilde{b}_s(x_1, \xi_1) \partial_{\xi_1} dx_1 \circ T \right) \left| \text{det}(\mathcal{T} - \text{Id}) \right|^{-1} -
\end{equation}
\begin{equation}
    - \sum_{i=0}^{\nu} \text{Tr} \left( \int \int \psi(\xi_1) \tilde{b}_{s,i}(x_1, \xi_1) \partial_{\xi_1} dx_1 \circ T \right) \left| \text{det}(\mathcal{T} - \text{Id}) \right|^{-1}
\end{equation}
is holomorphic. We will have to prove the existence of a meromorphic extension and find the poles and residues for each of the functions
\begin{equation}
    s \mapsto \text{Tr} \left( \int \int \psi(\xi_1) \tilde{b}_{s,i}(x_1, \xi_1) \partial_{\xi_1} dx_1 \circ T \right) \left| \text{det}(\mathcal{T} - \text{Id}) \right|^{-1}
\end{equation}
Because $\psi(\xi_1) = 1$ for $\|\xi_1\| \geq 1$ and the integral on the compact set $|\xi_1| \leq 1$ yields a holomorphic function in $s$ we need to study the meromorphic extension of
\begin{equation}
    s \mapsto \text{Tr} \left( \int_{\mathcal{N} \cap \mathcal{O}} \int_{|\xi_1| \geq 1} \tilde{b}_{s,i}(x_1, \xi_1) \partial_{\xi_1} dx_1 \circ T \right) \left| \text{det}(\mathcal{T} - \text{Id}) \right|^{-1}
\end{equation}
Let $\xi_1 = \lambda \overline{\xi}$ with $\lambda = ||\xi_1||$ and $\overline{\xi} = \frac{\xi_1}{||\xi_1||} \in S^{n-1}$ be the decomposition in polar coordinates. The degree of homogeneity of $\tilde{b}_{s,i}$ in $\xi_1$ is equal to $sd - i$. Then
\( \tilde{b}_{s,i}(x_1, \lambda \xi) = \lambda^{sd-i} \tilde{b}_{s,i}(x_1, \xi) \) and after passing to polar coordinates, the expression (107) becomes:

\[
\begin{align*}
\text{(108)} & \quad Tr \left( \int_{N \cap O} \int_{S^{n-1}} \int_1^\infty \lambda^{sd-i} \tilde{b}_{s,i}(x_1, \xi) \lambda^{n-1} d\lambda \xi dx_1 \circ T \right) \left| \det(\bar{T} - Id) \right|^{-1} = \\
\text{(109)} & \quad = Tr \left( \int_{N \cap O} \int_{S^{n-1}} \tilde{b}_{s,i}(x_1, \xi) \int_1^\infty \lambda^{sd+n-1-i} d\lambda \xi dx_1 \circ T \right) \left| \det(\bar{T} - Id) \right|^{-1} = \\
\text{(110)} & \quad = -\frac{1}{sd+n-i} Tr \left( \int_{N \cap O} \int_{S^{n-1}} \tilde{b}_{s,i}(x_1, \xi) d\xi dx_1 \circ T \right) \left| \det(\bar{T} - Id) \right|^{-1}.
\end{align*}
\]

The double integral defines a holomorphic function on \( \mathbb{C} \), so the above expression has a meromorphic extension to \( \mathbb{C} \) with a simple pole at \( s = \frac{-n+i}{d} \) and residue:

\[
\frac{1}{d} Tr \left( \int_{N \cap O} \int_{S^{n-1}} \tilde{b}_{s,i}(x_1, \xi) d\xi dx_1 \circ T \right) \left| \det(\bar{T} - Id) \right|^{-1}.
\]

As a consequence, the trace functions \( Tr(A_s \circ T) \) and \( Tr(A^* \circ T) \) have meromorphic extensions on any half plane \( Re(s) < K \) and so on \( \mathbb{C} \).

We will proceed with the computation of the poles and residues of \( Tr(A_s \circ T) \). We observed that on the half space \( Re(s) < K \) the difference

\[
\text{(112)} \quad Tr(A_s \circ T) - \sum_{i=0}^\nu Tr \left( \int_{N \cap O} \int_{|\xi| \geq 1} \tilde{b}_{s,i}(x_1, \xi_1) d\xi \circ T_1 \right) \left| \det(\bar{T} - Id) \right|^{-1}
\]

is holomorphic. Then \( Tr(A_s \circ T) \) has simple poles at \( s = \frac{-n+i}{d} \) for \( i = 0, 1, \ldots \) and the residue at \( s = \frac{-n+i}{d} \) is equal to the expression in (111). Lemma B.1 gives an asymptotic expansion of \( \tilde{b}_s(x_1, \xi_1) \) of the form:

\[
\text{(113)} \quad \tilde{b}_s(x_1, \xi_1) \sim \sum_\alpha \frac{1}{\alpha!} (D^\alpha \xi \bar{\bar{\pi}}_s)(x_1, 0, \xi_1, 0)
\]

with \( \bar{\bar{\pi}}_s(x_1, w, \xi_1, \xi_2) = a_s(x_1, (\bar{T} - Id)^{-1} w, \xi_1, \xi_2) \) If \( a_s \sim \sum a_{s,i} \) is the asymptotic expansion in homogeneous terms, with \( a_{s,i} \) homogeneous of degree \( sd-i \), then the homogeneous component of degree \( sd-i \) of \( \tilde{b}_s \) will be equal to

\[
\text{(114)} \quad \tilde{b}_{s,i}(x_1, \xi_1) = \sum_{|\alpha|+k=i} \frac{1}{\alpha!} (D^\alpha \xi \bar{\bar{\pi}}_{s,k})(x_1, 0, \xi_1, 0)
\]

with \( \bar{\bar{\pi}}_{s,k}(x_1, w, \xi_1, \xi_2) = a_{s,k}(x_1, (\bar{T} - Id)^{-1} w, \xi_1, \xi_2) \) of degree of homogeneity \( sd-k \).

To conclude the proof of the theorem, consider the fixed point set \( N = M^\gamma \) and a coordinate chart \( O \) where \( N \cap O = N_i \cap O \). Let \( n_i = dim(N_i \cap O) \). We define the densities \( \eta_{i,k} \) that compute the residue of the zeta function at \( s = \frac{-m+k}{d} \) as:

\[
\text{(115)} \quad \eta_{i,k}(x_1) = -\frac{1}{d} Tr \left( \int_{S^{n-1}} \tilde{b}_{s,n_i-m+k}(x_1, \xi) d\xi \circ T \right) dx_1 \quad \text{if } k \geq m - n_i
\]

\[
\text{(116)} \quad n_{i,k} = 0 \quad \text{if } k < m - n_i
\]

and

\[
\text{(117)} \quad d^\gamma_i = |\det(\bar{T} - Id)|^{-1}
\]
We will now continue with the proof of Lemma 3.1. We will follow closely the ideas contained in the proof of Theorem 3.1 in [31].

Proof of Lemma B.1. We have \( \|a_s(x, \xi)\| \leq C(1 + \|\xi\|)^{Re(s)}d \) so for \( Re(s) < -\frac{m-n}{d} \), the integral defining \( b_s \) is absolutely convergent so one can change the order of integration in \( [100] \). Consider the Taylor expansion of \( a_s(x_1, x_2, \xi_1, \xi_2) \) near \( \xi_2 = 0 \)

\[
(118) \quad a_s(x_1, x_2, \xi_1, \xi_2) = \sum_{|\alpha| \leq Q-1} \frac{1}{\alpha!} (\partial_{\xi_2}^\alpha a_s)(x_1, x_2, \xi_1, 0)\xi_2^\alpha + R_Q
\]

with the reminder in the integral form

\[
(119) \quad r_Q = \sum_{|\alpha| = Q} \frac{Q\xi_2^\alpha}{\alpha!} \int_0^1 (1-t)^{Q-1} (\partial_{\xi_2}^\alpha a_s)(x_1, x_2, \xi_1, t\xi_2) dt
\]

We have

\[
(120) \quad \int e^{-i<x_2, \xi_2>} \frac{1}{\alpha!} (\partial_{\xi_2}^\alpha a_s)(x_1, x_2, \xi_1, 0)\xi_2^\alpha dx_2 d\xi_2 = \frac{1}{\alpha!} (D_{x_2}^\alpha \partial_{\xi_2}^\alpha a_s)(x_1, 0, \xi_1, 0)
\]

the double integral being the composition of the Fourier transform in \( x_2 \) and the inverse Fourier transform in \( \xi_2 \) evaluated at 0. Because \( a_s \) is a classical symbol of order \( s \), the right-hand side term of the above equality is a classical symbol of order \( s - |\alpha| \).

In order to prove that \( b_s(x_1, \xi_1) \) is a symbol with the asymptotic expansion as in \( [100] \) we need to show that the integral of the remainder \( \iint e^{-i<x_2, \xi_2>} r_Q dx_2 d\xi_2 \) is a symbol of arbitrary negative order if \( Q \) is chosen large enough. After changing the order of integration of \( dt \) and \( dx_2 d\xi_2 \) we see that it will be sufficient to provide a uniform estimate in \( t \in (0, 1] \) for the integrals

\[
(121) \quad R_{\alpha, t}(x_1, \xi_1) = \iint e^{-i<x_2, \xi_2>} \xi_2^\alpha (\partial_{\xi_2}^\alpha a_s)(x_1, x_2, \xi_1, t\xi_2) dx_2 d\xi_2
\]

with \( |\alpha| = Q \). Integrating by parts we get:

\[
(122) \quad R_{\alpha, t}(x_1, \xi_1) = \iint e^{-i<x_2, \xi_2>} (D_{x_2}^\alpha \partial_{\xi_2}^\alpha a_s)(x_1, x_2, \xi_1, t\xi_2) dx_2 d\xi_2
\]

Let \( R_{\alpha, t} = R_{\alpha, t}^1 + R_{\alpha, t}^2 \) where \( R_{\alpha, t}^1 \) is the integral over the set \( D = \{(x_2, \xi_2) \mid \|\xi_2\| \leq \|\xi_1\|\} \) and \( R_{\alpha, t}^2 \) is the integral over the complement. The volume of \( D \) is bounded by \( C\|\xi_1\|^n \) where \( C \) doesn’t depend on \( t, x_1 \) and \( \xi_1 \). On \( D \) we have \( \|\xi_2, t\xi_2\| \leq 2\|\xi_1\| \) so the integrand in \( R_{\alpha, t}^1 \) is bounded by \( C(1 + \|\xi_1\|)^{Re(s) - Q} \). Thus

\[
(123) \quad \|R_{\alpha, t}^1(x_1, \xi_1)\| \leq C(1 + \|\xi_1\|)^{Re(s) - Q + n}
\]

with \( C \) independent of \( t, x_1 \) and \( \xi_1 \).

By using the identity \( (1 + \|\xi_2\|^2)^{-\nu}(1 + \|D_{x_2}\|^2)^{\nu} e^{-i<x_2, \xi_2>} = e^{-i<x_2, \xi_2>} \) and integrating by parts, we can rewrite \( R_{\alpha, t}^2 \) as a sum of terms of the form:

\[
(124) \quad \iint e^{-i<x_2, \xi_2>} (1 + \|\xi_2\|^2)^{-\nu} (\partial_{\xi_2}^\beta D_{x_2}^\beta a_s)(x_1, x_2, \xi_1, t\xi_2) dx_2 d\xi_2
\]

with \( |\beta| \leq 2\nu \). Because \( \|\xi_1\| < \|\xi_2\| \), the expression \( (\partial_{\xi_2}^\beta D_{x_2}^\beta a_s)(x_1, x_2, \xi_1, t\xi_2) \) is bounded from above by \( C\|\xi_2\|^{Re(s) - Q} \) for \( Re(s) \geq Q \) and by \( C \) otherwise. The
constant $C$ doesn’t depend on $\beta$, $t$, $\xi_1$ and $\xi_2$. For $\nu$ large enough the previous integral is bounded by

$$C \int_{\|\xi_2\| > \|\xi_1\|} (1 + \|\xi_2\|)^{-\nu'} \xi_2 \leq C \|\xi_1\|^{-\nu' + n + 1} \int (1 + \|\xi_2\|)^{-n-1} \xi_2 \leq C \|\xi_1\|^{-\nu' + n + 1}$$

with $\nu'$ arbitrary large. Thus

$$\|R_{\alpha,t}(x_1, \xi_1)\| \leq C \|\xi_1\|^{\Re(s) - Q + n}$$

for an arbitrary $Q \in \mathbb{N}$ with the constant $C$ independent of $(x_1, \xi_1)$.

Using Proposition 3.6 and Theorem 3.1 in [Sh] we conclude that $b_s(x_1, \xi_1)$ is a symbol of order $s$ and has the asymptotic expansion given in (101). If $a_s$ is a classical symbol then $b_s$ is classical as well.

For the second part of the proof, let us consider the positive, real valued smooth functions $\psi_i(\xi_1)$, for $i \in \mathbb{N}$, such that $\psi_i \equiv 0$ on the ball of radius $i$ and $\psi_i \equiv 1$ outside the ball of radius $i + 1$. Let

$$\tilde{b}_s(x_1, \xi_1) = \sum_{i=0}^{\infty} \psi_i(\xi_1) (\sum_{|\alpha|=i} \frac{1}{\alpha!} (D_{x_2}^\alpha \partial_{\xi_1}^\alpha a_s)(x_1, 0, \xi_1, 0))$$

It is obvious that $\tilde{b}_s - b_s$ is a smoothing operator. If $a_s$ is a holomorphic family of symbols for $s \in \mathbb{C}$ then $\tilde{b}_s$ defined as above is a holomorphic family as well.
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Bogdan Bucicovschi, Mathematics Department, Ohio State University, 231 W 18th Avenue, Columbus, OH 43210

E-mail address: bogdanb@math.ohio-state.edu