Coulomb drag Between One Dimensional Wigner Crystal Rings

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We consider the Coulomb drag between two metal rings in which the long range Coulomb interaction leads to the formation of a Wigner crystal. The first ring is threaded by an Aharonov Bohm flux creating a persistent current $J_0$. The second ring is brought in close proximity to the second and due to the Coulomb interaction between the two rings a drag current $J_D$ is produced in the second. We investigate this system at zero temperature for perfect rings as well as the effects of impurities. We show that the Wigner crystal state can in principle lead to a higher ratio of drag current to drive current $J_D/J_0$ than in weakly interacting electron systems.

I. INTRODUCTION

Coulomb Drag between current carrying systems has been studied in numerous papers both experimentally and theoretically. The earliest experiments were done in two dimensional systems. In this configurations the electrons are combined to two-dimensional layers. In one such layer a current is driven and another conducting layer is brought to within close proximity of the first. The Coulomb interaction between the electrons in these two layers causes a transfer of momentum and the second layer acquires a current. This current is referred to as the drag current.

Recently, the phenomenon of Coulomb drag in one dimensional systems has attracted much attention, particularly with regards to nanowires and nanotechnology. The basic description of the one dimensional case is identical to that in two dimensions. In all of these systems the time averaged charge density of the systems are translationally invariant. The coupling of the two systems is due to charge fluctuations in the two systems.

It is known however that the long range character of the Coulomb interaction can lead to the formation of a Wigner crystal in which the electrons become localized and form a periodic lattice. Numerical simulations on one dimensional systems indicate that an arbitrarily weak long range interaction will lead to the formation of a Wigner crystal. This long range nature of the Coulomb force should be apparent for low electron densities.

In this paper we consider the drag between two such Wigner crystals. It is evident that this system has a non-uniform charge distributions and we will show that this in principle can lead to a much higher value of the drag current.

II. CLASSICAL DRAG

It is the aim of this section to analyze from a classical perspective some aspects of current drag that we may expect in a quantum mechanical treatment of the Wigner crystal. The Wigner crystal is as a system where the electrons are effectively localized, so it is reasonable to expect that some aspects of the current drag between Wigner crystals may be exhibited in a classical system.

Here, we discuss the current drag between two one dimensional wires consisting of classical particles interacting via an unscreened Coulomb potential. The two systems are close enough that they are coupled by the Coulomb interaction between the wires. This system is illustrated in Figure 1. To get an idea of

![Figure 1](https://example.com/figure1.png)

**FIG. 1:** Illustration of the current drag setup for considered in this classical analysis. A constant force $f$ drives system one. $U(x-y)$ is the interwire interaction and $d$ is the distance of separation.
addition, the first chain is driven by a constant force \( f \). In steady state the equations of motion of the two chains are given by

\[
\begin{align*}
\alpha \dot{x}_1 &= -A \sin\left(2\pi/a(x_1 - x_2)\right) + f \\
\alpha \dot{x}_2 &= A \sin\left(2\pi/a(x_1 - x_2)\right)
\end{align*}
\]

Setting \( \phi = (x_1 - x_2) \), results in the following equation for the difference between the velocities of the two chains.

\[
\dot{\phi} = f/\alpha - \frac{2A}{\alpha} \sin(2\pi\phi/a)
\]

with solution

\[
T_0 = \int_{\phi_0}^{\phi + \Delta \phi} \frac{d\phi}{\sqrt{\frac{2}{\alpha} \sin(2\pi\phi/a)}}
\]

where \( T_0 \) is the time for the two chains to slip one complete cycle (distance \( a/2 \)).

Carrying out the integral results in

\[
T_0 = \left\{ \begin{array}{ll}
\frac{\infty}{2(f/\alpha)^2 - (2A/\alpha)^2} & \text{for } f < 2A/\alpha \text{ (locking)} \\
\frac{1}{2(f/\alpha)^2 - (2A/\alpha)^2} & \text{for } f \geq 2A/\alpha \text{ (slipping)}
\end{array} \right.
\]

This simple analysis shows the interesting result that for suitable conditions of the parameters \( f \) and \( A \), the two chains will be locked together with the drag current equal to the drive current.

We now wish to investigate the behavior of this classical system when internal dynamics are included. We take therefore as a model of the dynamics of these classical particles the Fokker-Planck equation

\[
\frac{\partial}{\partial t} P(\vec{x}, \vec{y}, t) = \sum_i D \left( \frac{\partial^2}{\partial x_i^2} + \frac{1}{k_B T} \frac{\partial}{\partial x_i} F_i \right) P(\vec{x}, \vec{y}, t) + \sum_j D \left( \frac{\partial^2}{\partial y_j^2} + \frac{1}{k_B T} \frac{\partial}{\partial y_j} F_j \right) P(\vec{x}, \vec{y}, t) \tag{1}
\]

Here, \( \vec{x} = (x_1, x_2, \ldots, x_N) \) labels the positions of \( N \) particles in wire 1, \( \vec{y} = (y_1, y_2, \ldots, y_N) \) labels the positions of particles in wire 2, \( D \) is the diffusion constant and \( T \) is the temperature. \( P(\vec{x}, \vec{y}, t) \) is then the probability of having the \( N \) particles in layer 1 at positions \( \vec{x} \) and the \( N \) particles in layer 2 at position \( \vec{y} \) at time \( t \). The forces present are

\[
F_i = f + \frac{\partial}{\partial x_i} V(\vec{x}) + \frac{\partial}{\partial x_i} U(\vec{x}, \vec{y})
\]

in wire 1 and

\[
F_j = \frac{\partial}{\partial y_j} V(\vec{y}) + \frac{\partial}{\partial y_j} U(\vec{x}, \vec{y})
\]

in wire 2. The interaction \( U \) is taken to be the unscreened Coulomb interaction between the wires, \( V \) is the unscreened Coulomb interaction within each wire and \( f \) is a constant force driving the particles in wire 1. The basic problem is to determine the drag current produced in wire 2 due to the moving charges in wire 1.

In the numerical analysis of this problem we performed Monte Carlo simulations on discrete systems. The conducting systems are partitioned into a lattice with the particles occupying positions at the lattice points. The dynamics of the simulation are determined by equation \( (\text{1}) \) which is approximated numerically as follows. For each particle, one of two possible random directions is chosen. If the particle lowers its energy by moving one lattice site in that direction, then the move is accepted. Otherwise the move is accepted with probability \( e^{-\beta \Delta E} \), where \( \beta = 1/k_B T \) and \( \Delta E \) is the change in the particles energy upon making this move. This process is completed iteratively for each particle until a steady state condition is reached. Simulations were performed to determine the dependence of the drag current on separation between the two systems. The results are shown in Figure \( (\text{3}) \).

The plot of Drag Current vs. Separation shows the fact that the sum of the drive and drag currents is a constant, \( \dot{J}_1 + \dot{J}_2 = \left(\frac{\partial}{\partial x_i} V(\vec{x}) + \frac{\partial}{\partial x_i} U(\vec{x}, \vec{y})\right) \), as can be seen by averaging the right hand side of equation \( (\text{1}) \).
III. WIGNER CRYSTAL RING

A. Disorder Free Case

In this section we calculate the non-dissipative drag between two perfect Wigner crystal rings at zero temperature. The basics of non-dissipative drag effects has been considered in previous papers. The basic system of interest here consists of two metallic rings one of which is threaded by an Ahronov-Bohm flux, $\Phi$. This flux shifts the ground state, $E_g$, of the ring to a current carrying state which is calculated according to

$$J = \frac{\partial E_g}{\partial \Phi}.$$

If another ring is brought close to the first one, then the Coulomb interaction will cause a drag current to be produced in the second ring. Since this is the ground state of the two ring system this current is non-dissipative and persists as long as the magnetic flux is present in the first ring.

In these works the two rings have uniform charge densities and the drag effect is due to coupling of charge fluctuations in each ring. In a system with zero disorder, the relative magnitude of the drag current produced is typically a small fraction of the driving current. For two such systems separated by a distance $\sim 200\text{Å}$ the drag current is on the order of $\sim 10^{-4}$ times the current in the driving system. Based on the analysis of the last section, we expect that the drag current of two Wigner crystals could in principle be as large as the driving current.

The model of the one dimensional Wigner crystal that we use here is that proposed in references and . In this model the Wigner crystal is viewed as an elastic chain of spinless electrons. In the continuum limit, the Wigner crystal of length $L$, in the presence of an Ahronov Bohm flux, $\Phi$, is described by the Lagrangian

$$\mathcal{L} = \int dx \left[ \frac{m}{2a} (u'^2 - s^2 u'^2) + \frac{\hbar}{L \Phi_0} \frac{\Phi}{\Phi_0} \dot{u} \right].$$

(2)

where $a$ is the crystal period, $m$ is the electron mass, $s$ is the velocity of sound in the crystal and $\Phi_0$ is the flux quantum. The field variable $u(x, t)$ describes the local displacement of the chain at a point $x$ at time $t$. If the field variable is expanded in a Fourier series
FIG. 3: Schematic illustration of the two ring Wigner crystal setup. The crystal period is \( a \) and \( u_1(x, t) \) denotes the local displacement of ring 1 and \( u_2(x, t) \) denotes the local displacement of ring 2. Ring 1 has an Aharonov-Bohm flux \( \Phi_1 \), present which creates a persistent current \( J_0 \) in ring 1. The interaction between the two crystals, \( V(u_1 - u_2) \), is a function of the relative displacement of the two crystals and creates a drag current \( J_D \) in ring 2.

ignored and only the rotation of the ring as a whole is relevant. We wish to study the interaction between two such Wigner crystal rings as shown in Figure (3). We take as a model for two disorder free interacting Wigner crystal rings the following Hamiltonian.

\[
H = \frac{1}{2M} \left( -i\hbar \frac{\partial}{\partial u_1} - \frac{e}{c} \Phi_1 \right)^2 + \frac{1}{2M} \left( -i\hbar \frac{\partial}{\partial u_2} - \frac{e}{c} \Phi_2 \right)^2 + V_0 \cos \frac{2\pi}{a}(u_1 - u_2) \tag{3}
\]

where \( M = Nm \) is the total mass of each ring, \( V_0 \) represents the strength of the inter-ring interaction, \( \Phi_1 \) is the flux through ring 1, \( \Phi_2 \) is the flux through ring 2 and \( u_{1,2} \) represents the displacement of ring 1, 2 respectively. The first two terms in Equation (3) represent the individual dynamics of rings 1 and 2 respectively. The third term represents the interaction between the two rings. It is evident that the interaction between the chains will be periodic in the relative displacement between the two crystals. For simplicity we model this interaction as a cosine term, however the basic features of this analysis should not depend on the particular form of the periodic potential used.

Transforming to center of mass and relative angular coordinates \( \phi = \frac{2\pi}{L}(u_1 - u_2) \) and \( \theta = \frac{\pi}{L}(u_1 + u_2) \), we have in dimensionless form

\[
\left[ \left( \frac{i}{\partial \phi} + \frac{\alpha}{2} \right)^2 + \frac{1}{4} \left( \frac{i}{\partial \phi} + \beta \right)^2 + q \cos(N\phi) \right] \Psi = \epsilon \Psi \tag{4}
\]

where \( \alpha = \Phi_1 - \Phi_2, \beta = \Phi_1 + \Phi_2, q = V_0(MR^2/\hbar^2) \) and \( \epsilon = E(MR^2/\hbar^2) \). Our task is now to solve equation (4) for the eigenenergy \( \epsilon \).

The hamiltonian, equation (3), separates into two parts \( H = H_\phi + H_\theta \) with energies \( \epsilon = \epsilon_\phi + \epsilon_\theta \). The ground state wavefunction is a product \( \Psi(\phi, \theta) = S(\theta)T(\phi) \). The equation for the center of mass coordinates is readily solved giving

\[
S(\theta) = \frac{1}{\sqrt{2\pi}} e^{in'\theta}
\]

\[
\epsilon_\theta = \frac{1}{4}(n' + \beta)^2; n' = 0, \pm 1, \pm 2, \ldots \tag{5}
\]

The equation for the relative coordinate wavefunction is

\[
\left[ - \left( \frac{i}{\partial \phi} + \frac{\alpha}{2} \right)^2 - q \cos(N\phi) + \epsilon_\phi \right] T(\phi) = 0 \tag{6}
\]
FIG. 4: Plots of drive and drag current against the interaction strength. $V_0$ is shown in units of $\tilde{V} = \frac{\hbar^2}{mR^2}$ and the current $\tilde{J} = J/J_1(0)$ where $J_1(0)$ denotes the current in ring 1 at $V_0 = 0$.

The energy eigenvalues may be found numerically to any desired accuracy by truncating the set of equations for some $c_{N}$ and searching for the value of $\epsilon$ which gives a zero value for the appropriate determinant of coefficients.

The energy eigenvalues determined in this way, combined with the results of equation (5) allow the currents in each ring to be determined from $J_1 = \frac{\partial \epsilon}{\partial \Phi_1}$ and $J_2 = \frac{\partial \epsilon}{\partial \Phi_2} |_{\Phi_2 = 0}$. The results are shown in Figure (4).

Figure (4) shows that the drag current does approach the drive current in magnitude as the interaction $V_0$ is increased which is qualitatively similar to the classical result. For large values of the interaction strength, $q/N \gtrsim 3$ or

$$V_0 \gtrsim 3\hbar^2 N/mR^2,$$

the two Wigner crystals have essentially the same current with each having half the current that one ring would have if isolated.

In order to relate the interaction in equation (3) to the true Coulomb interaction we approximate the amplitude $V_0$ as follows. We want to calculate the potential felt by an electron a distance $d$ from a finite periodic system of length $L$ and period $a$. We model our system by taking a length $L$ of an infinite line of electrons centered at $x$. This system is shown schematically in Figure (5).

FIG. 5: Schematic model of a periodic system of electrons. The potential on an electron centered at $x$ and a distance $d$ from an infinite line of electrons is obtained by taking a finite segment of length $L$ of the infinite chain centered at $x$.

This potential energy is given by

$$V = \frac{1}{\pi} \sum_{n=\pm \infty}^{\frac{1}{2} \left( \frac{L}{2} + x \right)} \frac{e^2}{\sqrt{(x - na)^2 + d^2}},$$

which can be rewritten as

$$\int_{\frac{1}{2} \left( \frac{L}{2} + x \right)}^{\infty} \frac{e^2}{\sqrt{(x - na)^2 + d^2}}.$$
\[ V(\phi_1 - \phi_2) \] (a) 

\[ V_p \]

\[ \phi_2 \]

\[ \phi_1 \]

\[ \text{FIG. 6: Schematic illustration of the two ring Wigner crystal setup. The phase } \phi_{1,2} = \frac{2\pi}{a} u_{1,2}(x, t) \text{ where } u_{1,2} \text{ is the local displacement of the Wigner crystal. The interaction between the two rings is } V(\phi_1 - \phi_2) = W[1 - \cos(\phi_1 - \phi_2)]. \]

Ring 2 has an impurity of strength \( V_p \) that couples to the phase \( \phi_2 \). Figure (6b) shows a phase shift in the relative coordinate \( \phi_1 - \phi_2 \) of \( \pi \) relative to Figure (6a).

\[ \int_{-L/2}^{L/2} \frac{dz'}{a} \frac{e^{2}}{\sqrt{z'^{2} + d^{2}}} + 2 \sum_{l=1}^{\infty} \int_{-L/2}^{L/2} \frac{dz'}{a} \frac{e^{2}}{\sqrt{z'^{2} + d^{2}}} \cos(\frac{2\pi l}{a} z') \cos(\frac{2\pi l}{a} x) \]

The first term is a constant in \( x \) therefore we ignore it. For small separation distances \( d/a \lesssim 1 \), higher \( l \) terms may be ignored and we keep only the \( l = 1 \) term. This yields finally

\[ V \approx \frac{4}{a} \int_{0}^{L/2} \frac{dz'}{\sqrt{z'^{2} + d^{2}}} \cos(\frac{2\pi}{a} z') \cos(\frac{2\pi}{a} x) \]

We can now identify the amplitude \( V_0 \) in Equation (3) for a crystal with \( N \) electrons as

\[ V_0 = N \frac{4}{a} \int_{0}^{L/2} \frac{dz'}{\sqrt{z'^{2} + d^{2}}} \cos(\frac{2\pi}{a} z') \]

\[ = N \frac{4}{a} \int_{0}^{L/2a} \frac{dz}{\sqrt{z^{2} + d^{2}}} \cos(2\pi z) \]

As mentioned in the introduction, for a one dimensional system, a long range interaction will lead to the formation of a Wigner crystal\(^{27,28}\). The long range nature of the Coulomb force should be apparent for electron densities, \( n \lesssim \frac{1}{a_B} \) where \( a_B \) is the Bohr radius for the particular ring material\(^{29}\). For a typical mesoscopic ring radius of 100\( \mu \)m, crystal lattice constant \( a = 10a_B \) and \( a_B \approx 2\text{Å} \) we have \( L/a > 10^5 \) so the upper limit in the integral in Equation (8) may be taken to infinity which yields

\[ V_0 = \frac{4Ne^2}{a} K_0(2\pi d/a) \]

where \( K_0 \) is the zeroth order Bessel function. This implies an exponential decrease in the amplitude of the interaction with increasing separation between the two Wigner crystals. Comparison with Equation (3) shows that the two crystals will be "locked" together at a distance \( d/a \approx 4 \). So two very low density perfect Wigner crystals will be essentially locked together when separated by a distance of four times the lattice constant or less.

**B. Impurity Effects**

We now consider the effects of impurities on the dynamics of the Wigner crystal. In the presence of an impurity, the phase of the Wigner crystal will be pinned. In this case the crystal ring cannot rotate as a whole so the current as presented in the last subsection is not possible. It was shown, however, by Rice et al.\(^{18}\), that a new type of current is possible, namely the tunneling of solitons through the impurity barrier. It is the tunneling of solitons that we will investigate in this section.
Where

\[ L_1 = \frac{ma}{8\pi^2} \left\{ \frac{\partial \phi_1}{\partial t}^2 - s^2 \frac{\partial \phi_1}{\partial x}^2 \right\}, \tag{10} \]

\[ L_2 = \frac{ma}{8\pi^2} \left\{ \left( \frac{\partial \phi_2}{\partial t} \right)^2 - s^2 \left( \frac{\partial \phi_2}{\partial x} \right)^2 \right\} - V_p \delta(x) \left[ 1 - \cos(\phi_2) \right], \tag{11} \]

and the phase \( \phi_{1,2} = u_{1,2}(x,t)(2\pi/a) \).

The first two terms in Equation (9) are the Lagrangians describing the dynamics of rings 1 and 2 respectively. The last term describes the interaction between the two rings. We follow the work of Krive et al. and introduce the dimensionless parameter

\[ \alpha = \frac{\pi \hbar}{msa} \]

which characterizes the magnitude of the quantum fluctuations of the Wigner crystal, and the parameter

\[ T_s = \hbar s \]

which characterizes the energy scale in a Wigner crystal of length \( L \). We will be concerned with stiff crystals that are weakly fluctuating characterized by \( \alpha \ll 1 \).

The basic approach that we will use here is a semiclassical approximation used by Larkin and Lee in the context of charge density waves and elaborated by Kleinert for a single particle tunneling through a barrier. In this approach the probability amplitude of a soliton tunneling through an impurity is given by \( e^{A/\hbar} \) where \( A \) is the action associated with the tunneling trajectory obtained by minimizing the function \( \int dt \int dx L(\tau) \), where \( L(\tau) \) is the Lagrangian, Equation (9), with \( t \) replaced by the imaginary time \( \tau = it \). This approach amounts to approximating the path integral of the propagator by its value at the classical trajectory. We now want to answer the following question: given a current in ring 1 that causes a rotation of the phase by \( 2\pi \), what is the probability that the phase of ring 2 will also rotate by \( 2\pi \)? It is apparent that there are three possibilities:

i) \( \phi_1 \rightarrow 2\pi, \phi_2 \rightarrow 0 \): the phase in ring 2 will remain pinned and ring 1 will rotate on top of the relative potential between the two ending with a relative phase difference \( (\phi_1 - \phi_2) = 2\pi \).

ii) \( \phi_1 \rightarrow 2\pi, \phi_2 \rightarrow -2\pi \): the phase of ring 2 will counter-rotate with that in ring 1 ending with a relative phase difference, \( (\phi_1 - \phi_2) = 4\pi \).

Here we have limited our consideration to single tunneling trajectories. In other words, each shift of the phase of each ring occurs due to a single tunneling event and ignore all multistep tunneling trajectories. Thus we only consider strong coupling between rings, \( WL/(\frac{Nh^2}{mR}) \gg 1 \), and strong pinning \( \alpha V_p \gg T_s \). We now calculate the tunneling amplitudes for each case above.

In case (i), the Lagrangian in Equation (9) reduces to

\[ L = -\frac{ma}{8\pi^2} \left\{ \left( \frac{\partial \phi_1}{\partial \tau} \right)^2 + s^2 \left( \frac{\partial \phi_1}{\partial x} \right)^2 \right\} - W \left[ 1 - \cos(\phi_1) \right] \tag{12} \]

This is simply the problem of a single Wigner crystal ring in the presence of a periodic pinning potential. The classical wave equation resulting from this Lagrangian in real time, is

\[ \frac{ma}{8\pi^2} \left\{ \dddot{\phi}_1 - s^2 \ddot{\phi}_1 \right\} + W \sin(\phi_1) = 0 \]

This is the sine-Gordon equation and it admits the exact soliton solution

\[ \phi_1 = 4 \tan^{-1} \left[ \exp \pm \left( \frac{\tilde{x} - u \tilde{t}}{\sqrt{1 - u^2}} \right) \right] \]

where \( \tilde{x} = \frac{ma}{e} x, \tilde{t} = \omega_0 t \) and
This solution has the following interpretation: at $t = 0$ the phase of the crystal is 0, as $t$ increases, the phase at $x = 0$ locally increases to $2\pi$ creating a region around $x = 0$ with phase $2\pi$. As $t$ increases this region with $2\pi$ phase propagates symmetrically from $x = 0$ at speed $u$ towards $x = -L/2$ and $x = -L/2$ until the entire ring is at $\phi = 2\pi$. The Lagrangian, Equation (12) has been considered in the context of charge density wave tunneling by Bogachek et al.\cite{1}, and there it is noted that the homogeneous (space independent) soliton solution is advantageous in the action against any spatially dependent solution for a finite system. The homogenous solution corresponds to a uniform rotation of the Wigner crystal as a whole and has the form:\cite{21}

$$\phi_1(t) = 4\tan^{-1}\exp(\omega_0 t).$$

(13)

This solution describes the uniform increase of the phase of the entire ring from 0 to $2\pi$ in a time $\tau_0 \sim 1/\omega_0$. The action associated with this solution is by Equation (12)

$$A_1 = -\frac{8WL}{\omega_0} = -8\left(\frac{WL}{mR^2}\right)^{1/2}$$

(14)

In case $(ii)$, the relative difference between the two phases does not increase, while the phase of the center of mass of the two ring system increases by $2\pi$. We assume that during the tunneling process the two rings remain in phase to minimize the potential energy between them. If we rewrite Equation (1) using $\phi = \phi_1 - \phi_2$ and $\theta = \phi_1 + \phi_2$, we get the relevant Lagrangian for case $(ii)$ to be

$$L = \frac{ma}{8\pi^2}\left\{\left(\frac{\partial \theta}{\partial \tau}\right)^2 + s^2 \left(\frac{\partial \theta}{\partial x}\right)^2\right\} - V_p\delta(x) \left[1 - \cos\left(\frac{\theta}{2}\right)\right]$$

(15)

This is the problem of a single Wigner crystal ring tunneling through an impurity. We consider the case of strong crystal pinning, $\alpha V_p \gg T_s$.

The case of soliton tunneling through an impurity in a strongly pinned Charge Density wave was first considered by Larkin and Lee\cite{19}. In their analysis, the tunneling process is broken into two stages. The first stage consists of a rapid tunneling of a length $l_0$ of the crystal through the impurity in a short time $t_0$ and leaves the crystal in a distorted state of high potential energy. The second stage consists of a slow relaxation of the crystal from this distorted state back into a state that minimizes its potential energy.

To describe the first stage of the process Larkin and Lee proposed the following trial solution to Equation (15)

$$\theta = \frac{2\pi t}{t_0} \left(1 - \frac{|x|}{l_0}\right)$$

for $0 < t < t_0$ and $|x| < l_0$. It is evident that this solution interpolates between the initial state, $\theta = 0$ for all $x$ at $t = 0$, and the final state $\theta = 2\pi$ for $x = 0$ at $t = 2\pi$. The second stage of the process takes place away from the impurity $|x| > l_0$ and obeys the equation of motion

$$\ddot{\theta} + s^2 \theta'' = 0.$$ 

The solution to this stage of the tunneling should obey the periodic boundary conditions of the ring and match the solution for the first stage at $x = -L/2, L/2$. This problem has been considered in the work of Krive et al\cite{18}, where the proposed trajectory for the relaxation stage has the form

$$\phi = \pi \pm 2 \arctan\left(\frac{(\tau - \tau_i)s}{|x|}\right).$$

The action associated with the tunneling stage of the process is

$$A_{2t} = -\frac{\hbar}{\alpha} \left(C_1 + C_2 \frac{\alpha V_p}{T_h}\right).$$

where $C_1, C_2$ are constants. The action associated with the relaxation stage is

$$\frac{\hbar}{\alpha} \left(L\right).$$
where \( C \) is a constant and \( C_3 \) is a constant assumed to be of the order unity. Equation (16) shows the interesting result that the action is only logarithmically dependent on the impurity for strong pinning.

In case (iii) above, the center of mass of the two ring system does not move while the phase of the relative coordinates increases by \( 4\pi \). The relevant Lagrangian for this system is given by

\[
\mathcal{L} = -\frac{ma}{8\pi^2} \left\{ \left( \frac{\partial \phi}{\partial \tau} \right)^2 + s^2 \left( \frac{\partial \phi}{\partial x} \right)^2 \right\} - V_p \delta(x) \left[ 1 - \cos \left( \frac{\phi}{2} \right) \right] - W \left[ 1 - \cos \left( \frac{\phi}{2} \right) \right].
\]

This is identical to the lagrangian in Equation (13) with the addition of the term proportional to \( W \). Since the probability for this process goes as \( e^A \) it is expected that the probability for this case will be reduced relative to that in case (ii) by \( \sim e^{-W} \) and can therefore be neglected.

We are now in a position to analyze the relative magnitude of the drag current \( J_D \) to the drive current \( J_0 \). This ratio is given by

\[
\frac{J_D}{J_0} = \frac{e^{\frac{A_{ST}}{4\pi}}}{e^{\frac{A_{ST}}{4\pi}} + e^{\frac{A_{ST}}{4\pi}}}
\]

It is clear that if \( |A_1| \gg |A_{ST}| \) the two crystal will essentially be locked together. For purposes of illustration we consider a stiff crystal with \( \alpha = .1 \) and consider values of \( V_p \) for stiff pinning \( \alpha V_p \gg T_s \).

Figure (7) shows that the impurity effects are not that drastic in determining the value of \( W \) where the locking of the two crystals occur. This is due to the fact that the impurity potential enters logarithmically into the tunneling action. Comparison with Figure (b) shows that the distance at which the two Wigner crystals become "locked" together is essentially the same as in the impurity free case. The main effect of the impurity is then in the magnitude of the persistent current.

We can estimate the magnitude of these currents by noting that if the two currents are locked together, then the magnitude of the persistent current for a given flux \( \Phi \) should be the same as a single Wigner crystal ring with twice the mass. This current has been calculated at zero temperature to be

\[
J_0 \sim T_s \left( \frac{T_s}{\alpha V_p} \right)^{1/\alpha} e^{-C_3/\alpha} \cos \left( 2\pi \frac{\Phi}{\Phi_0} \right).
\]

We see that the current in Equation (19) is much smaller than the disorder free persistent current due to the \((1/V_p)^{1/\alpha}\) factor.

In summary we have analyzed the Coulomb drag between two one-dimensional Wigner crystal rings. For sufficiently large interaction between rings the drag current is essentially equal in magnitude to the drive current. For an impurity free ring this "locking" of currents occurs at a separation distance of \( d \approx 4a \) with \( a \) the crystal period. We analyzed the effect of a single impurity in the drag ring and found that the two currents are "locked" at essentially the same distance and the major effect of the impurity is to significantly decrease the magnitude of the persistent current.
10 V. L. Gurevich and M. I. Muradov JETP Lett. 71, 111 (2000).
11 P. Debray, P. Vasilopoulos O. Raichev, R. Perrin, M. Rahman, and W. C. Mitchel Physica E 6, 694 (2000).
12 P. E. Cladis, R. D. Parks and J. M. Daniels Phys. Rev. Lett. 21, 1521 (1968)
13 As pointed out in reference 12, the exact form of the interaction is not important for the results so long as the force is periodic in the vortex lattice.
14 A.G.Rojo and G.D.Mahan Phys. Rev. Lett. 68, 2074 (1992).
15 J. Baker and A.G.Rojo Phys. Rev. B 60 8804 (1999).
16 L.I.Glazman, I. M. Ruzin and B. I. ShklovskiiPhys. Rev. Lett 45 8454 (1992).
17 I.V. Krive, P.Sandström, R. I. Shechter, S.M. Girvin and M. JonsonPhys. Rev. B 52 16451 (1995).
18 M. J. Rice, A.R. Bishop, J. A. Krumhansl, and S. E. Trullinger Phys. Rev. Lett 36 432 (1976).
19 A. I. Larkin and P. A. Lee Phys. Rev. B 17 1596 (1978).
20 H. Kleinert Path Integrals in Quantum Mechanics, Statistics and polymer Physics (World Scientific, Singapore 1995).
21 E. N. Bogacheck, I. V. Krive, I. O. Kulik, and A. S. Rozhavsky Phys. Rev. B 42 7614 (1990).
22 D. W. Jordan and P. Smith Nonlinear ordinary differential equations: An introduction to dynamical systems (Oxford, New York 1977).
23 I. Giaver Phys. Rev. Lett. 15, 825 (1965)
24 J. R. Clem Phys. Rev. B 9, 898 (1974); J. W. Ekin, B. Serin and J. R. Clem Phys. Rev. B 9, 912 (1974)
25 J. R. Clem Phys. Rev. B 12, 1742 (1975); J. W. Ekin, and J. R. Clem Phys. Rev. B 12, 1753 (1975)
26 K. K. Uptry and D. Dominguez Phys. Rev. B 51, 5955 (1995);
27 H.J. Schulz Phys. Rev. Lett 71, 1864 (1993);
28 G. Fano, F. Ortolani, A. Parola ans L. Ziosi Phys. Rev. B 60, 15654 (1999);
29 A.L. Ivanov and H. Haug Phys. Rev. Lett 71, 3182 (1993);
30 E. A. Jagla and C. A. Balseiro Phys. Rev. B 52, 4494 (1995)