A closed model structure for $n$-categories, internal $Hom$, $n$-stacks and generalized Seifert-Van Kampen

Carlos Simpson
CNRS, UMR 5580, Université Paul Sabatier, 31062 Toulouse CEDEX, France.

1. Introduction

The purpose of this paper is to develop some additional techniques for the weak $n$-categories defined by Tamsamani in [27] (which he calls $n$-nerves). The goal is to be able to define the internal $Hom(A, B)$ for two $n$-nerves $A$ and $B$, which should itself be an $n$-nerve. This in turn is for defining the $n + 1$-nerve $nCAT$ of all $n$-nerves conjectured in [27], which we can do quite easily once we have an internal $Hom$. It is essentially clear a priori that we cannot just take an internal $Hom$ on all of the $n$-nerves of Tamsamani, and in fact some simple examples support this: any strict $n$-category may be considered in an obvious way as an $n$-nerve i.e. a presheaf of sets over $\Delta^n$ satisfying certain properties, but the morphisms of the resulting presheaves are the same as the strict morphisms of the original strict $n$-categories; on the other hand one can see that these strict morphisms are not enough to reflect all of the “right” morphisms. Our strategy to get around this problem will be based on the idea of closed model category [20]. We will construct a closed model category containing the $n$-nerves of Tamsamani. Then we can simply take as the “right” $n$-nerve of morphisms, the internal $Hom(A, B)$ whenever $A$ and $B$ are fibrant objects in the closed model category (all objects will be cofibrant in our case). This strategy is standard practice for topologists.

As usual, in order to define a closed model category we first have to enlarge the class of objects under consideration. Instead of $n$-nerves as defined by Tamsamani we look at $n$-pre-nerves (i.e. presheaves of sets over the cartesian product of $n$ copies of the standard simplicial category) which satisfy the constancy condition—C1 in Tamsamani’s definition of $n$-nerve—and call these $n$-precats (this notion being in between the pre-nerves and nerves of [27], we take a different notation). An $n$-precat may be interpreted as a presheaf on a certain quotient $\Theta^n$ of $\Delta^n$, in particular we obtain a category $PC_n$ of objects closed under all limits, with internal $Hom$ etc. We follow the method of constructing

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1The simplest example which shows that the strict morphisms are not enough is where $G$ is a group and $V$ an abelian group and we set $A$ equal to the category with one object and group of automorphisms $G$, and $B$ equal to the strict $n$-category with only one $i$-morphism for $i < n$ and group $V$ of $n$-automorphisms of the unique $n − 1$-morphism; then for $n = 1$ the equivalence classes of strict morphisms from $A$ to $B$ are the elements of $H^1(G, V)$ so we would expect to get $H^n(G, V)$ in general, but for $n > 1$ there are no nontrivial strict morphisms from $A$ to $B$. 

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a closed model category developed by Jardine-Joyal \cite{13, 15} in the case of simplicial presheaves. The cofibrations are essentially just monomorphisms (however we cannot—and don’t—require injectivity for top-degree morphisms, just as sets or categories with monomorphisms are not closed model categories \cite{24}). The main problem is to define a notion of weak equivalence. Our key construction is the construction of an \( n \)-nerve \( \text{Cat}(A) \) for any \( n \)-precat \( A \), basically by throwing in freely all of the elements which are required by the definition of nerve \cite{27} (although to make things simpler we use here a definition of nerve modified slightly to “easy nerve”). Then we say that a morphism of \( n \)-precats \( A \to B \) is a weak equivalence if \( \text{Cat}(A) \to \text{Cat}(B) \) is an exterior equivalence of \( n \)-nerves in the sense of \cite{27}. The fibrant morphisms are characterized in terms of cofibrations and weak equivalences by a lifting property, in the same way as in \cite{13}.

One new thing that we obtain in the process of doing this is the notion of pushout. The category of \( n \)-precats is closed under direct limits and in particular under pushouts. Applying the operation \( \text{Cat} \) then gives an \( n \)-categorical pushout: if \( A \to B \) and \( A \to C \) are morphisms of \( n \)-nerves then the categorical pushout is \( \text{Cat}(B \cup^A C) \).

The main lemma which we need to prove (Lemma 3.2 below) is—again just as in \cite{13}—the statement that a pushout by a trivial cofibration (i.e. a cofibration which is a weak equivalence) is again a trivial cofibration. After that the rest of the arguments needed to obtain the closed model structure are relatively standard following \cite{13} when necessary.

Once the closed model structure is established, we can go on to define internal \( \text{Hom} \) and construct the \( n+1 \)-nerve \( n\text{CAT} \). Using these we can, in principal, define the notion of \( n \)-stack. Our discussion of \( n \)-stacks is still at a somewhat speculative stage in the present version of the paper, because there are several slightly different notions of a family of \( n \)-categories parametrized by a 1-category \( X \) and ideally we would like to—but don’t yet—know that they are all the same (as happens for 1-stacks).

The notion of categorical pushout which we developed as a technical tool actually has a geometric consequence: we obtain a generalized Seifert-Van Kampen theorem (Theorem 9.1 below) for the Poincaré \( n \)-groupoids \( \Pi_n(X) \) of a space \( X \) which were defined by Tamsamani \cite{27} §2.3 ff. If \( X \) is covered by open sets \( U \) and \( V \) then \( \Pi_n(X) \) is equivalent to the category-theoretic pushout of \( \Pi_n(U) \) and \( \Pi_n(V) \) along \( \Pi_n(U \cap V) \). We define the nonabelian cohomology of \( X \) with coefficients in a fibrant \( n \)-precat \( A \) as \( H(X, A) := \text{Hom}(\Pi_n(X), A) \). The generalized Seifert-Van Kampen theorem implies a Mayer-Vietoris statement for this nonabelian cohomology.

There are many possible approaches to the notion of \( n \)-category and, without pretending to be exhaustive, I would like to point out some of the other possibilities here for comparison.

—One of the pioneering works in the search for an algebraic approach to homotopy of spaces is the notion of \( \text{Cat}^n \)-groups of Brown and Loday. This is what is now known as
the “cubical” approach where the set of objects can itself have a structure for example of
$n − 1$-category, so it isn’t quite the same as the approach we are looking for (commonly
called the “globular” case).
—Gordon, Powers and Street have intensively investigated the cases $n = 3$ and $n = 4$
[11], following the path set out by Benabou for 2-categories [5].
—In [12] A. Grothendieck doesn’t seem to have hit upon any actual definition but gives
a lot of nice intuition about $n$-categories.
—On p. 41 of [12] starts a reproduction of a letter from Grothendieck to Breen dated July
1975, in which Grothendieck acknowledges having received a proposed definition of non-
strict $n$-category from Breen, a definition which according to loc. cit “…has certainly the
merit of existing...”. It is not clear whether this proposed construction was ever worked
out.
—In [26], R. Street proposes a definition of weak $n$-category as a simplicial set satisfying a
certain variant of the Kan condition where one takes into account the directions of arrows.
—Kapranov and Voevodsky in [16] construct, for a topological space $X$, a “Poincaré $\infty$-
groupoid” which is a strictly associative $\infty$-groupoid but where the arrows are invertible
only up to equivalence. This of course raises the question to know if strictly associative
$n$-categories would be a sufficient class to yield the correct $n + 1$-category $n\mathbf{CAT}$. As
pointed out in the footnote above, one wonders in particular whether there is a closed
model structure to go along with these strict $n$-categories.
—In his recent preprint [4] M. Batanin develops some ideas towards a definition of weak
$\infty$-category based on operads. In the introduction he mentions a letter from Baez and
Dolan to Street dating to November 29, 1995 which contains some ideas for a definition
of weak $n$-category; and he states that Makkai, Hermida and Power have worked on the
idea contained in this letter.
—M. Rosellen told me in September 1996 that he was working on a version using the
theory of operads (cf [1] for example). Just as our current effort is based on Segal’s de-
looping machine, there should probably be an $n$-category machine analogous to any of
the other various delooping machines, and in fact the problems are almost identical: the
basic problem of doing $n$-categories comes down to doing delooping while keeping track
of the non-connected case and not requiring things to be invertible up to homotopy (cf
the last section of [27] for some arguments relating $n$-categories and delooping machines).
—J. Baez and J. Dolan have developed their theory originating in the letter refered to
above, a definition of $n$-categories based on operads, in a preprint [3] of February 1997.
In this preprint they discuss operads, give their definition of $n$-category and of certain
morphisms of $n$-categories, and define the homotopy category of $n$-categories which they
conjecture to be equivalent to the homotopy category for other definitions such as the
category $\mathbf{Ho} − n − \mathbf{Cat}$ mentionned in [27].

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The main problem which needs to be accomplished in any of these points of view is to obtain an $n + 1$-category (hopefully within the same point of view) $n\text{CAT}$ parametrizing the $n$-categories of that point of view. This is the main thing we are doing here for Tamsamani's point of view. As far as I know, the present one is the first precise construction of the $n + 1$-category $n\text{CAT}$.

Once several such points of view are up and running, the comparison problem will be posed: to find an appropriate way to compare different points of view on $n$-categories and (one hopes) to say that the various points of view are equivalent and in particular that the various $n+1$-categories $n\text{CAT}$ are equivalent via these comparisons. It is not actually clear to me what type of general setup one should use for such a comparison theory, although the first thing to try would be to explore a theory of “internal closed model category”, a closed model category with internal $\text{Hom}$: any reasonable point of view on $n$-categories should probably yield an internal closed model category $n - - C$ (such as the $PC_n$ we obtain below) and furthermore $n\text{CAT}$ should be an object in $(n + 1) - - C$. Comparison between the theories might then be possible using a version of Quillen's adjoint functor approach [20]. We give an indication of how to start on comparison in §11 by sketching how to obtain a functor from any internal closed model category containing $\text{Cat}$, to the our closed model category of $n$-categories.

Having a good theory of $n$-categories should open up the possibility to pursue any of the several programs such as that outlined by Grothendieck [12], the generalization to $n$-stacks and $n$-gerbs of the work of Breen [7], or the program of Baez and Dolan in topological quantum field theory [2]. Once the theory of $n$-stacks is off the ground this will give an algebraic approach to the “geometric $n$-stacks” considered in [24].

We clarify the pretentions to rigor of the various sections of this paper: §§2–7 are supposed to be a first version of something precise and correct (although at the time of this first version I haven’t checked all of the details in a very thorough way). The same holds for §9 on Seifert-Van Kampen. On the other hand, the discussion of §8 on $n$-stacks is blatantly speculative; and the discussion of §10 on nonabelian cohomology is very incomplete.

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I would also like to thank R. Brown for pointing out the importance of the notion of push-out and Seifert-Van Kampen, and G. Maltsiniotis and A. Bruguières for helpful discussions.

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2. Preliminaries

Let $\Delta$ be the standard category of ordered finite sets. Let $\Theta^n$ be the quotient of the cartesian product $\Delta^n$ obtained by identifying all of the objects $(M,0,M')$ for fixed $M = (m_1,\ldots,m_k)$ and variable $M' = (m'_1,\ldots,m'_{n-k-1})$. The object of $\Theta^n$ corresponding to the class of $(M,0,M')$ with all $m_i > 0$ will be denoted $M$. The object $(1,\ldots,1)$ ($k$ times) will be denoted $1^k$. We permit concatenation in our notation for objects, thus $M,m$ denotes the object $(m_1,\ldots,m_k,m)$ (when this makes sense, that is when $k < n$).

The class of $(0,\ldots,0)$ will be denoted by $0$.

We give the explicit construction of $\Theta^n$. If $M = (m_1,\ldots,m_k)$ and $M' = (m'_1,\ldots,m'_l)$ then set $M,0$ equal to the concatenation of $M$ with $(0,\ldots,0)$ in $\Delta^n$ and similarly for $M',0$. We define an equivalence relation on morphisms $\varphi = (\varphi_1,\ldots,\varphi_n)$ from $M,0$ to $M',0$ by saying $\varphi \sim \varphi'$ whenever there exists $j$ such that $\varphi_i = \varphi'_i$ for $i \leq j$ and $\varphi_j : m_j \to m'_j$ factors through the object $0 \in \Delta$ (which is the one-point set). This equivalence relation is compatible with composition so we obtain a category $\Theta^n$ by taking as morphisms the quotient of the morphisms in $\Delta^n$ by this equivalence relation. There is an obvious projection from $\Delta^n$ to $\Theta^n$.

We assume familiarity with [27]. An $n$-precat is a presheaf of sets on $\Theta^n$. This corresponds to an $n$-prenerve in Tamsamani’s notation (i.e. presheaf of sets on $\Delta^n$) which satisfies his axiom C1 in the definition of $n$-nerve. Let $PC_n$ denote the category of $n$-precats. An $n$-precat is an $n$-category (or $n$-nerve in the notation of Tamsamani [27] which is the sense which we will always assign to the terminology “$n$-category” below) if it satisfies certain additional conditions [27]. We give an easier version of these conditions which we call an easy $n$-category. We start with the notion of easy equivalence between two easy $n$-categories—this is not circular because the notion of easy $n$-category will only use the notion of easy equivalence for morphisms of $n-1$-categories. If $A$ and $B$ are easy $n$-categories then a morphism $f : A \to B$ (of $n$-precats, i.e. of presheaves on $\Theta^n$) is an easy equivalence if for all $v \in B_{1^k}$ (called a $k$-morphism of $B$) and all $a,a' \in A_{1^{k-1}}$ with $s(v) = f(a)$ and $t(v) = f(a')$ and $s(a) = s(a')$ and $t(a) = t(a')$ (here $s$ and $t$ denote the morphisms “source” and “target” from $T_{1^k}$ to $T_{1^{k-1}}$ for any $n$-precat $T$), there exists $u \in A_{1^k}$ with $s(u) = a$ and $t(u) = a'$ and $f(u) = v$. A marked easy equivalence is the data of a morphism $f$ together with choices $u(a,a',v)$ in every situation as above.

The reader is cautioned that we will still need Tamsamani’s notion of equivalence (which he calls “équivalence extérieure” [27] §1.3) for our closed model category structure
below. The notion of easy equivalence is mainly just used when it is an ingredient in the notion of $n$-category.

Before giving the definition of easy $n$-category we introduce the following notation. If $T$ is an $n$-precat then for any $M = (m_1, \ldots, m_k)$ we denote by $T_{M/}$ the $n-k$-precat obtained by restricting $T$ to the subcategory of objects of $\Theta^n$ of the form $(M, M')$ for variable $M'$. This differs from the notation of Tamsamani who called this just $T_M$; our notation with a slash is necessitated by the notation $M$ for objects of $\Theta^n$. (Sorry about these slight notational changes but is is much easier for us to use $\Theta^n$ for what will be done below).

With these notations, an $n$-precat $A$ is an *easy $n$-category* if:

— for each $m$, $A_{m/}$ is an easy $n-1$-category; and

— the morphisms

$$A_{m/} \to A_{1/} \times_{A_0} \cdots \times_{A_0} A_{1/}$$

are easy equivalences. Note here that $A_0$ is set which is the fiber over the object $0 \in \Theta^n$ (which exits slightly from our notational convention; it is the class of objects $0, M'$ of $\Delta^n$ but here there is no “$M$” to put into the notation so we put “0” instead).

A *marked easy $n$-category* is an easy $n$-category provided with the addtional data of markings for the $A_{m/}$ and markings for the easy equivalences going into the definition. These two conditions amount (recursively) to saying that we have markings for all of the morphisms of the form

$$A_{M, m/} \to A_{M, 1/} \times_{A_M} \cdots \times_{A_M} A_{M, 1/}.$$

The notion of marking as we have defined above actually makes sense for any $n$-precat, and an $n$-precat with a marking is automatically an easy $n$-category. For this reason, arbitrary inverse limits of marked easy $n$-categories (indexed by systems of morphisms which preserve the markings) are again marked easy $n$-categories.

Suppose $A$ is an $n$-precat. We define the *marked easy $n$-category generated by $A$* denoted $\text{Cat}(A)$ by

$$\text{Cat}(A) = \lim_{\leftarrow \mathcal{C}} T$$

where the limit is taken over the category $\mathcal{C}$ whose objects are triples $(T, \mu, f)$ with $(T, \mu)$ a marked easy $n$-category ($\mu$ denotes the marking) and $f : A \to T$ is a morphism of $n$-precats. The morphisms of $\mathcal{C}$ are morphisms of $n$-precats (i.e. morphisms of presheaves on $\Theta^n$) required to preserve $f$ and the marking $\mu$. By the principle given in the previous paragraph, this inverse limit is again a marked easy $n$-category.

The construction $\text{Cat}(A)$ is the key to the rest of what we are going to say. The description of $\text{Cat}(A)$ given above is one of cutting it down to size. There is also a creative description. In order to explain this we first discuss certain push-outs of $n$-precats. An object of $\Theta^n$ represents a presheaf (i.e. $n$-precat). If $M$ is an object we
denote the $n$-precat represented by $M$ as $h(M)$. A morphism of $n$-precats $h(M) \to A$ is the same thing as an element of $A_M$. Note that direct limits exist in the category of $n$-precats (as in any category of presheaves). In particular push-outs exist.

We construct the following standard $n$-precats. Let $M = (m_1, \ldots, m_l)$ with $l \leq n - 1$, and let $m \geq 1$ (although by the remark below we could also restrict to $m \geq 2$). Let $-1 \leq k \leq n - l - 1$. We will state the constructions by universal properties (although we give an explicit construction later). Note that these universal properties admit solutions because we work in the category of presheaves over a given category $\Theta^n$ so the necessary limits exist.

Define $\Sigma = \Sigma(M, [m], \langle k, k + 1 \rangle)$ to be the universal $n$-precat with $k$-morphisms $a, b$ in $\Sigma_{M,m/}$ (i.e. $a, b \in \Sigma_{M,m,1^k}$) and a $k + 1$-morphism

$$v = (v_1, \ldots, v_m) \in (\Sigma_{M,1/} \times \Sigma_{M,0} \ldots \times \Sigma_{M,1/})_{1^{k+1}}$$

such that $s(a) = s(b)$, $t(a) = t(b)$, and such that the images of $a$ and $b$ by the usual map to the product of $\Sigma_{M,1/}$ are $s(v)$ and $t(v)$ respectively. Note that $h = h(M, m, 1^{k+1})$ is the universal $n$-precat with a $k + 1$-morphism $u$ in $\Sigma_{M,m/}$ (i.e. $u \in \Sigma_{M,m,1^{k+1}}$). Note that setting $a$ to $s(u)$, $b$ to $t(u)$ and $v$ to the image of $u$ by the usual map to the product, we obtain (by the universal property of $\Sigma$) a morphism

$$\varphi = \varphi(M, [m], \langle k, k + 1 \rangle) : \Sigma(M, [m], \langle k, k + 1 \rangle) \to h(M, m, 1^{k+1})$$

We will show below that $\varphi$ is a cofibration, but the same time giving an explicit construction of $\Sigma$ as a pushout of representable presheaves. Before doing that, we mention the modifications to the above definition necessary for the boundary cases $k = -1$ and $k = n - l - 1$.

For $k = -1$, $\Sigma(M, [m], \langle -1, 0 \rangle)$ is the universal $n$-precat with an object

$$v = (v_1, \ldots, v_m) \in \Sigma_{M,1} \times \Sigma_{M,0} \ldots \times \Sigma_{M,1}$$

As $h(M, m, 0)$ is the universal $n$-precat with an object $u \in h_{M,m}$ we have an object $v$ as above for $h$ (the image of $u$ by the usual map) so we obtain $\Sigma \to h$.

For $k = n - l - 1$, $\Sigma(M, [m], \langle n - l - 1, n - l \rangle)$ is the universal $n$-precat with $a, b \in \Sigma_{M,m,1^n}$ such that $s(a) = s(b)$ and $t(a) = t(b)$ and such that $a$ and $b$ map to the same elements of

$$(\Sigma_{M,1/} \times \Sigma_{M,0} \ldots \times \Sigma_{M,1/})_{1^n}.$$

Note that $h = h(M, m, 1^{n-l})$ is normally speaking not defined because the length of the multiindex $(M, m, 1^{n-l})$ is $n + 1$. Thus we formally define this $h$ to be equal to

$\footnote{The definition, from §3 below, is that a morphism $A \to B$ of $n$-precats is a cofibration if for every $M = (m_1, \ldots, m_k)$ with $k < n$, the morphism $A_M \to B_M$ is injective.}$
\( h(M, m, 1^{n-l-1}) \) and take the elements \( a = b \) equal to the canonical \( n-l-1 \)-morphism in \( h_{M,m/} \). This gives a morphism \( \Sigma \to h \).

We will now give an explicit construction of \( \Sigma \) and use this to show that \( \Sigma \to h \) is a cofibration. (The boundary cases will be left to the reader). In general the universal \( n \)-precat with a collection of elements with certain equalities required, is a quotient of the disjoint union of the representable \( n \)-precats corresponding to the elements we want, by identifying pairs of morphisms from the representable \( n \)-precats corresponding to the elements which need to be equal. We do this in several steps. First, the universal \( n \)-precat \( \Upsilon = \Upsilon(M, [m], 1^k) \) with element

\[ v = (v_1, \ldots, v_m) \in (\Upsilon_{M,1/} \times \Upsilon_{M,0} \times \ldots \times \Upsilon_{M,0} \Upsilon_{M,1/})_1^k \]

is constructed as the quotient of the disjoint union of \( m \) copies of \( h(M, 1, 1^k) \) making \( m-1 \) identifications over pairs of maps

\[ h(M, 1, 1^k) \leftarrow h(M) \rightarrow h(M, 1, 1^k). \]

This is the same as taking the pushout of the diagram

\[ h(M, 1, 1^k) \leftarrow h(M) \rightarrow h(M, 1, 1^k) \leftarrow \ldots \leftarrow h(M) \rightarrow h(M, 1, 1^k). \]

Now \( \Sigma(M, [m], (k, k+1)) \) is the quotient of the disjoint union

\[ h(M, m, 1^k)^a \sqcup h(M, m, 1^k)^b \sqcup \Upsilon(M, [m], 1^{k+1}) \]

by the following identifications (the superscripts \( a \) and \( b \) in the above notation are there to distinguish between the two components, which correspond respectively to choosing \( a \) and \( b \)). There are two maps (dual to \( s \) and \( t \)) \( s^*, t^* : h(M, m, 1^{k-1}) \to h(M, m, 1^k) \), and we identify over the pairs of morphisms

\[ h(M, m, 1^k)^a \xleftarrow{s^*} h(M, m, 1^{k-1}) \xrightarrow{s^*} h(M, m, 1^k)^b \]

and

\[ h(M, m, 1^k)^a \xleftarrow{t^*} h(M, m, 1^{k-1}) \xrightarrow{t^*} h(M, m, 1^k)^b. \]

Then we also identify over the pairs of maps

\[ h(M, m, 1^k)^a \leftarrow \Upsilon(M, [m], 1^k) \xrightarrow{s^*} \Upsilon(M, [m], 1^{k+1}) \]

and

\[ h(M, m, 1^k)^b \leftarrow \Upsilon(M, [m], 1^k) \xrightarrow{t^*} \Upsilon(M, [m], 1^{k+1}) \]
where the left maps are induced by the collection of principal morphisms \(1 \to m\). The result of all these identifications is \(\Sigma(M, [m], (k, k+1))\). This gives an explicit construction for those wary of just defining things by the universal property.

We now show that the morphism \(\varphi : \Sigma \to h\) is a cofibration in the sense of §3 below, i.e. injective on all levels except the top one.

We will say that a diagram of \(n\)-precats of the form

\[ A \xrightarrow{f} B \to C \]

(where the two compositions are the same) is semiexact if the morphism from the coequalizer of the two arrows to \(C\) is a cofibration in the sense of §3 below. Our above construction gives \(\Sigma\) as the coequalizer of

\[ \Upsilon' \sqcup \Upsilon' \sqcup h' \sqcup h' \xrightarrow{\sim} \Upsilon \sqcup h^a \sqcup h^b \]

where

\[ \Upsilon = \Upsilon(M, [m], 1^{k+1}), \quad \Upsilon' = \Upsilon(M, [m], 1^k), \]

\[ h^a (\text{resp. } h^b) := h(M, m, 1^k), \quad h' := h(M, m, 1^{k-1}). \]

We would like to prove that

\[ \Upsilon' \sqcup \Upsilon' \sqcup h' \sqcup h' \xrightarrow{\sim} \Upsilon \sqcup h^a \sqcup h^b \to h(M, m, 1^{k+1}) \]

is semiexact. To prove this it suffices (by a simple set-theoretic consideration) to show that

\[ h' \sqcup h' \xrightarrow{\sim} h^a \sqcup h^b \to h(M, m, 1^{k+1}) \]

\[ \Upsilon' \xrightarrow{\sim} \Upsilon \sqcup h^a \to h(M, m, 1^{k+1}) \]

and

\[ \Upsilon' \xrightarrow{\sim} \Upsilon \sqcup h^b \to h(M, m, 1^{k+1}) \]

are semiexact.

The first statement follows from the claim that for any \(M = (m_1, \ldots, m_l)\),

\[ h(M) \sqcup h(M) \xrightarrow{\sim} h(M, 1) \sqcup h(M, 1) \to h(M, 1, 1) \]

is semi-exact. Let \(P = (p_1, \ldots, p_n)\) be an element of \(\Theta^n\) (some of the \(p_j\) may be zero). A morphism \(P \to (M, 1)\) corresponds to a collection of morphisms \(f_i : p_i \to m_i\) for \(i \leq l\) and \(f_{l+1} : p_{l+1} \to 1\), up to equivalence. The equivalence relation is obtained by saying that if one of the morphisms factors through 0 then the subsequent ones don’t matter. The first thing to note is that the two morphisms \((I, s), (I, t) : h(M, 1) \to h(M, 1, 1)\) are injective, as follows directly from the above description. Now suppose that \(f, g : P \to (M, 1)\) are
two morphisms such that \((I, s) \circ f = (I, t) \circ g\). Since \(s : 0 \to 1\) composed with anything is different from \(t : 0 \to 1\) composed with the same, this means that one of the \(f_i\) must factor through 0 for \(i \leq l + 1\), and that \(f_j = g_j\) for \(j \leq i\). If it is the case for \(i \leq l\) then \(f\) and \(g\) both come from the morphism \((f_1, \ldots, f_i) : P \to M\), which is equivalent to \((g_1, \ldots, g_i)\), via either one of the morphisms \(M \to (M, 1)\). If \(i = l + 1\) then \(f_{l+1} = g_{l+1}\) factors through one of the two morphisms \(0 \to 1\), so \(f\) and \(g\) both come from the morphism \((f_1, \ldots, f_i) = (g_1, \ldots, g_i)\) via the morphism \(M \to (M, 1)\) corresponding to the morphism \(0 \to 1\) occurring above. Thus \(f\) and \(g\) are equivalent in the coequalizer, giving the claim for this paragraph and thus the first of our semiexactness statements.

For the next semiexactness statement, we first note that \(\Upsilon \to h(M, m, 1^{k+1})\) is cofibrant. In fact we can describe \(\Upsilon\) as a subsheaf of \(h(M, m, 1^{k+1})\) as follows. For \(P = (p_1, \ldots, p_n)\) the morphisms from \(P\) to \((M, m, 1^{k+1})\) are the sequences of morphisms \(f = (f_1, \ldots, f_{t+k+2})\) with \(f_i : p_i \to m_i\) (or taking values in \(m\) or 1 as appropriate depending on \(i\)). Such a morphism is contained in \(\Upsilon_P\) if and only if the morphism \(f_{t+1} : p_{t+1} \to m\) factors through one of the principal morphisms \(1 \to m\) (we leave to the reader to verify that \(\Upsilon\) is equal to this subsheaf). Suppose \(f \in \Upsilon_P\) and \(g = (g_1, \ldots, g_{t+k+1}) \in h_P^a\), projecting to the same element of \(h(M, m, 1^{k+1})\). Note that \(g\) projects to the element \((g_1, \ldots, g_{t+k+1}, s)\) where \(s : 0 \to 1\) denotes the source map (or really its dual but for purposes of the present argument we omit the dual notation). In particular \(f\) is equivalent to \((g_1, \ldots, g_{t+k+1}, s)\), which implies that (up to changing \(f\) and \(g\) in their equivalence classes) \(g_{t+1}\) factors through one of the principal maps \(1 \to m\) and \(f_{t+k+2} = s\). This exactly means that \(f\) comes from \(\Upsilon_P' \to \Upsilon_P\) and \(g\) from \(\Upsilon_P' \to h_P^a\). Thus \(f\) and \(g\) are equivalent in the coequalizer, giving the second of our semiexactness statements.

The proof of the third semiexactness statement is the same as that of the second (although \(s\) above would be replaced by \(t\)). This completes the proof that the standard morphisms \(\Sigma \to h\) are cofibrations (modulo the boundary cases which we have left to the reader).

An \(n\)-category \(A\) is an easy \(n\)-category if and only if every morphism \(\Sigma(M, [m], \langle k, k + 1 \rangle) \to A\) extends to a morphism \(h(M, m, 1^{k+1}) \to A\). A marked easy \(n\)-category is an \(n\)-category \(A\) together with choice of extension for every morphism \(\Sigma(M, [m], \langle k, k + 1 \rangle) \to A\). Finally, we say that a partially marked \(n\)-category is an \(n\)-category provided with a distinguished subset \(\mu\) of the set of all morphisms of the form \(f : \Sigma(M, [m], \langle k, k + 1 \rangle) \to A\), and for each such morphism, a chosen extension \(f^\mu\) to \(h(M, m, 1^{k+1})\).

If \((A, \mu)\) is a partially marked \(n\)-category, then we define a new partially marked \(n\)-category \(\text{Raj}(A, \mu)\) by taking the pushout via \(\varphi(M, [m], \langle k, k + 1 \rangle)\) for all morphisms

\[
\Sigma(M, [m], \langle k, k + 1 \rangle) \to A
\]

which are not in the subset \(\mu\) of marked ones.
Remark: In the above notations, if \( m = 1 \) then
\[
\Sigma(M, [1], \{k, k + 1\}) = \Upsilon(M, [1], 1^{k+1}) = h(M, 1, 1^{k+1})
\]
so the pushout by \( \varphi(M, [1], \{k, k + 1\}) \) is trivial and we can ignore these cases if we like in the previous notation (and also in the notion of marking).

Lemma 2.1 If \( A \) is an \( n \)-precat then the marked easy \( n \)-category \( \text{Cat}(A) \) is obtained by iterating infinitely many times (i.e. over the first countable ordinal) the operation \((A', \mu') \mapsto \text{Raj}(A', \mu')\), starting with \((A, \emptyset)\).

Proof: If \((B, \nu)\) is a marked easy \( n \)-category and \((A', \mu') \to (B, \nu)\) is a morphism compatible with the partial marking of \( A' \), then there is a unique extension to a morphism \( \text{Raj}(A', \mu') \to B \) compatible with the partial marking of \( \text{Raj}(A', \mu') \). It follows that if we set \( \text{Cat}'(A) \) equal to the result of the iteration described in the lemma, then there is a unique morphism \( \text{Cat}'(A) \to B \) compatible with the partial marking of \( \text{Cat}(A) \) and extending the given morphism \( A \to B \). But \( \text{Cat}'(A) \) is fully marked. By the universal property of \( \text{Cat}(A) \) this implies that \( \text{Cat}(A) = \text{Cat}'(A) \).

We will often have a need for the following construction. If \( A \) is an \( n \)-precat then iterate (over the first countable ordinal) the operation \((A', \mu') \mapsto \text{Raj}(A', \emptyset)\). Call this \( \text{BigCat}(A) \). Another way to describe this construction is that we throw in an infinite number of times the pushouts of all of the required diagrams (which is in some sense a more obvious way to obtain an \( n \)-category). There is an obvious morphism \( \text{Cat}(A) \to \text{BigCat}(A) \). One of the advantages of the \( \text{BigCat} \) construction is that \( \text{BigCat}(A) \cong \text{BigCat}(\text{BigCat}(A)) \) (although the natural maps are not this isomorphism). More generally, we will use the terminology “reordering” below to indicate that a sequence of pushouts can be done in any order (subject to the obvious condition that the things over which the pushouts are being done exist at the time they are done!), which yields isomorphisms such as \( \text{BigCat}(A) \cong \text{BigCat}(\text{BigCat}(A)) \).

If \( B \leftarrow A \to C \) is a diagram of \( n \)-categories, then we define the \textit{category-theoretic pushout} to be \( \text{Cat}(B \cup^A C) \). It is again an \( n \)-category. We will also often use just the pushout of \( n \)-precats, i.e. the pushout of presheaves over \( \Theta^n \).

3. The closed model category structure

We now come to the first main definition. A morphism \( A \to B \) of \( n \)-precats (that is, a morphism of presheaves on \( \Theta^n \)) is a \textit{weak equivalence} if the induced morphism \( \text{Cat}(A) \to \text{Cat}(B) \) is an exterior equivalence of \( n \)-categories in the sense of Tamsamani ([27] §1.3). Note in particular that we don’t require it to be an easy equivalence—which would be too strong a condition.
The second main definition is relatively easy: we would like to say that a morphism \(A \rightarrow B\) of \(n\)-precats is a cofibration if it is a monomorphism of presheaves on \(\Theta^n\). However, this doesn’t work out well at the top degree (for example, the category of sets with isomorphisms as weak equivalences and injections as cofibrations, is not a closed model category \([20]\)). Thus we leave the top level alone and say that a morphism \(A \rightarrow B\) of \(n\)-precats is a cofibration if for every \(M = (m_1, \ldots, m_k)\) with \(k < n\), the morphism \(A_M \rightarrow B_M\) is injective. A cofibration which is a weak equivalence is called a trivial cofibration.

The third definition which goes along automatically with these two is that a morphism \(A \rightarrow B\) of \(n\)-precats is a fibration if it satisfies the lifting property for trivial cofibrations, that is if every time \(U \hookrightarrow V\) is a trivial cofibration and \(U \rightarrow A\) and \(V \rightarrow B\) are morphisms inducing the same \(U \rightarrow B\) then there exists a lifting to \(V \rightarrow A\) compatible with the first two morphisms.

We recall from \([20]\) the definition of closed model category, as well as from \([21]\) an equivalent set of axioms.

**Theorem 3.1** The category of \(n\)-precats with the weak equivalences, cofibrations and fibrations defined above, is a closed model category.

**Some lemmas**

The proof of Theorem 3.1 is by induction on \(n\). Thus we may assume that the theorem and all of the lemmas contained in the present section and \(\S\) 4-6 are true for \(n'\)-precats for all \(n' < n\). In view of this, we state all (or most) of the lemmas before getting to the proofs.

Our proof will be modelled on the proof of Jardine that simplicial presheaves on a site form a closed model category \([13]\). The main lemma that we need (which corresponds to the main lemma in \([13]\)) is

**Lemma 3.2** Suppose \(A \rightarrow B\) is a trivial cofibration and \(A \rightarrow C\) is any morphism. Let \(D = B \cup^A C\) be the push-out of these two morphisms (the push-out of \(n\)-precats). Then the morphism \(C \rightarrow D\) is a weak equivalence.

This lemma speaks of push-out of \(n\)-precats. Applying the construction \(\text{Cat}\) we obtain a notion of push-out of \(n\)-categories: if \(A \rightarrow B\) and \(A \rightarrow C\) are morphisms of \(n\)-categories (i.e. morphisms of the corresponding \(n\)-precats) then define the push-out \(n\)-category to be \(\text{Cat}(B \cup^A C)\).
Corollary 3.3 If $A \to B$ is an equivalence of $n$-categories then
\[
C \to \text{Cat}(B \cup^A C)
\]
is an equivalence of $n$-categories.

We will come back to push-out below in the section on Siefert-Van Kampen.

Going along with the previous lemma is something that we would like to know:

Lemma 3.4 If an $n$-precat $A$ is an $n$-category in the sense of [27] then the morphism $A \to \text{Cat}(A)$ (resp. the morphism $A \to \text{BigCat}(A)$) is an equivalence of $n$-categories in the sense of [27].

Another lemma which is an important technical point in the proof of everything is the following. An $n$-precat $A$ can be considered as a collection $\{A_m/\}$ of $n-1$-precats (functor of $\Delta$ and the first element is a set). We obtain the collection $\{\text{Cat}(A_m/)\}$ which is a functor from $\Delta$ to the category of $n-1$-precats. Divide by the equivalence relation setting the 0-th element to a constant $n-1$-precat, in this way we obtain a new $n$-precat denoted $\text{Cat}_{\geq 1}(A)$.

Lemma 3.5 Suppose that $A$ and $B$ are $n$-precats and $f: A \to B$ is a morphism which induces an equivalence on the $n-1$-categories $\text{Cat}(A_m/) \to \text{Cat}(B_m/)$. Then $\text{Cat}(A) \to \text{Cat}(B)$ (resp. $\text{BigCat}(A) \to \text{BigCat}(B)$) is an equivalence.

Corollary 3.6 The morphism
\[
\text{Cat}(A) \to \text{Cat}(\text{Cat}_{\geq 1}(A))
\]
is an equivalence of $n$-categories.

Proof: The morphism $A \to \text{Cat}_{\geq 1}(A)$ satisfies the hypotheses of the previous lemma so the corollary follows from the lemma.

Lemma 3.7 For any $n$-precat $A$, the morphism $A \to \text{Cat}(A)$ (resp. $A \to \text{BigCat}(A)$) is a weak equivalence.

The closed model structure that we already have by induction for $n-1$-precats allows us to deduce some things about $n$-categories. Let $HC_{n-1}$ denote the localization of $PC_{n-1}$ by inverting the set weak equivalences, which is also (see 3.9 below) the localization of $n-1$-categories by inverting the set of equivalences. We know from the closed model structure
that this is equivalent to the category of fibrant (and automatically cofibrant) objects
where we take as morphisms, the homotopy classes of morphisms. We also know that a
morphism in $PC_{n-1}$ is a weak equivalence if and only if it projects to an isomorphism in
$HC_{n-1}$ ([20], Proposition 1, p. 5.5). In particular by [3.4] in degree $n - 1$ we know that a
morphism of $n - 1$-categories is an equivalence if and only if it projects to an isomorphism
in $HC_{n-1}$.

Suppose $A$ is an $n$-category. For $x, y \in A_0$ we have an $n - 1$-category $A_1(x, y)$ which
we could denote by $Hom_A(x, y)$. Let $LHom_A(x, y)$ denote the image of this object in the
localization $HC_{n-1}$. On the other hand, the truncation $T^{n-1}A$ is a 1-category. We claim
that for $x$ fixed, the mapping $y \mapsto LHom_A(x, y)$ is a functor from $T^{n-1}A$ to $HC_{n-1}$.
Similarly we claim that for $y$ fixed the mapping $x \mapsto LHom_A(x, y)$ is a contravariant functor
from $T^{n-1}A$ to $HC_{n-1}$. These claims give some meaning at least in a homotopic sense to
the notion of “composition with $f : y \to z$” as a map $LHom_A(x, y) \to LHom_A(x, z)$.

We prove the first of the two claims, the proof for the second one being identical. Note
that these arguments are generalizations of what is mentionned in [27] Proposition 2.2.8 and
the following remark. Suppose $f \in A_1(y, z)$. Then let $A_2(x, y, f)$ be the homotopy
fiber of $A_2(x, y, z) \to A_1(y, z)$ over the object $f$ (this is calculated by replacing the above
map by a fibrant map and taking the fiber). The condition that $A_2$ be equivalent to
$A_1 \times_{A_0} A_1$ implies that this homotopy fiber maps by an equivalence to $A_1(x, y)$. On
the other hand it maps to $A_1(x, z)$ and this diagram gives a morphism $LHom_A(x, y) \to
LHom_A(x, z)$ in the localized category $H_{n-1}$. We just have to check that this morphism
is independent of the choice of $f$ in its equivalence class. For this we use Proposition 3.5
below (there is no circularity because we are discussing $n - 1$-categories here). If $f$ is
equivalent to $g$ as elements of the $n - 1$-category $A_1(y, z)$ then let $K$ denote the $n - 1$
category given by [6.3]; there is a morphism $K \to A_1(y, z)$ sending 0 to $f$ and 1 to $g$, and
since $K$ is a contractible object (weakly equivalent to $\ast$) this proves that the homotopy
fibers over $f$ or $g$ are equivalent to the homotopy fiber product with $K$; we have a single
map from here to $A_1(x, z)$ so our two maps induced by $f$ and $g$ are homotopic.

Associativity is given by a similar argument using $A_3$ which we omit.

Once we have our functors $T^{n-1}A \to HC_{n-1}$ we obtain the following type of state-
ment: suppose $f$ is an equivalence between $u$ and $x$, then composition with $f$ induces an
equivalence $LHom_A(x, y) \cong LHom_A(u, y)$ (and similarly for composition in the second
variable).

**Lemma 3.8** If

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is a pair of morphisms of $n$-categories such that any two of $f$, $g$ or $g \circ f$ are equivalences
in the sense of [27] then the third is also an equivalence.
If \( f : A \to B \) and \( g : B \to A \) are two morphisms of \( n \)-categories such that \( fg \) is an equivalence and \( gf \) is the identity then \( f \) and \( g \) are equivalences.

Proof: The fact that composition of equivalences is an equivalence is \([27]\) Lemme 1.3.5. The statement concluding that \( f \) is an equivalence if \( g \) and \( gf \) are, is a direct consequence of Tamsamani’s interpretation of equivalence in terms of truncation operations (\([27]\) Proposition 1.3.1).

For the conclusion for \( g \), note first of all that on the level of truncations \( T^n A \to T^n B \to T^n C \) the fact that \( f \) and \( gf \) are isomorphisms of sets implies that \( g \) is an isomorphism of sets. This gives essential surjectivity. Now suppose \( x, y \) are objects of \( B \). Choose objects \( u, v \) of \( A \) and equivalences \( f(u) \sim x \) and \( f(v) \sim y \). Then composition with these equivalences induces an isomorphism in the localized category \( HC_{n-1} \) between \( L\text{Hom}_B(x,y) \) and \( L\text{Hom}_B(f(u),f(v)) \) (see the discussion preceding this lemma). The image under \( g \) of this isomorphism is the same as composition with the images of the equivalences, so we have a diagram

\[
\begin{array}{ccc}
\text{LHom}_B(x,y) & \to & \text{LHom}_C(g(x),g(y)) \\
\downarrow & & \downarrow \\
\text{LHom}_B(f(u),f(v)) & \to & \text{LHom}_C(gf(u),gf(v))
\end{array}
\]

in the category \( HC_{n-1} \). The horizontal arrows are the localizations of the arrows \( g : B_1(x,y) \to C_1(g(x),g(y)) \) etc., and the vertical arrows are composition with our chosen equivalences, isomorphisms in \( HC_{n-1} \). On the other hand, the bottom arrow fits into a diagram

\[
\text{LHom}_A(u,v)\text{LHom}_B(f(u),f(v)) \to \text{LHom}_C(gf(u),gf(v))
\]

where the first arrow induced by \( f \) is an isomorphism, and the composed arrow induced by \( gf \) is an isomorphism; thus the bottom arrow of the previous diagram is an isomorphism therefore the top arrow is an isomorphism in \( H_{n-1} \). This implies that the morphism \( B_1(x,y) \to C_1(g(x),g(y)) \) is an equivalence of \( n-1 \)-categories. This is what we needed to prove to complete the proof that \( g \) is an equivalence.

We turn now to the second paragraph of the lemma: suppose \( f \) and \( g \) are morphisms of \( n \)-categories such that \( fg \) is an equivalence and \( gf \) is the identity. The corresponding fact for sets shows that \( T^n f \) and \( T^n g \) are isomorphisms between the sets of equivalence classes of objects \( T^n A \) and \( T^n B \). Suppose \( x, y \in A_0 \). Note that \( gf(x) = x \) and \( gf(y) = y \). We obtain morphisms

\[
A_1(x,y) \xrightarrow{f} B_1(fx, fy) \xrightarrow{g} A_1(gfx, gfy) = A_1(x,y) \xrightarrow{f} B_1(fgf x, fgfy) = B_1(fx, fy).
\]

We have again that \( gf \) is the identity on \( A_1(x,y) \) and \( fg \) is an equivalence on \( B_1(fx, fy) \). Inductively by our statement for \( n-1 \)-categories, the morphism \( f : A_1(x,y) \to B_1(fx, fy) \)
is an equivalence. This implies that \( f : A \to B \) is an equivalence and hence that \( g \) is an equivalence (by the first paragraph of the lemma).

Corollary 3.9 The localized category of \( n \)-precats modulo weak equivalence is equivalent to the category \( \text{Ho} - n - \text{Cat} \) of \( n \)-categories localized by equivalence defined in [27].

Proof: The functor \( \text{Cat} \) sends weak equivalences to equivalences (by Lemma 3.4 together with Lemma 3.8). Thus it induces a functor \( c \) on localizations. Let \( i \) be the functor induced on localizations by the inclusion of \( n \)-categories in \( n \)-precats. The natural transformation \( A \to \text{Cat}(A) \) gives a natural isomorphism \( 1 \cong c \circ i \) of functors on the localization of \( n \)-categories. On the other hand, Lemma 3.7 says that the same natural transformation induces a natural isomorphism \( 1 \cong i \circ c \) of functors on the localization of \( n \)-precats.

We will prove lemmas 3.2, 3.4 and 3.5 all at once in one big induction on \( n \). Thus we may assume that they hold for \( n' < n \). All lemmas from here until the end of the big induction presuppose that we know the inductive statement for \( n' < n \).

Remark on the passage between \( \text{Cat} \) and \( \text{BigCat} \) in 3.4 and 3.5: The statements for \( \text{Cat} \) and \( \text{BigCat} \) are equivalent. Take Lemma 3.4 for example. If \( A \to \text{Cat}(A) \) is an equivalence for any \( n \)-category \( A \) then \( \text{BigCat}(A) \) can be constructed as the iteration over the first countable ordinal of the operation \( A' \to \text{Cat}(A') \) (and starting at \( A \)). The morphisms at each stage in the iteration are equivalences, so it follows that the morphism \( A \to \text{BigCat}(A) \) is an equivalence. On the other hand, suppose we know that \( A \to \text{BigCat}(A) \) is an equivalence for any \( n \)-category \( A \). Then \( \text{Cat}(A) \to \text{BigCat} \) is an equivalence, but by reordering \( \text{BigCat} = \text{BigCat}(A) \). Thus the hypothesis also gives that \( A \to \text{BigCat}(A) \) is an equivalence. Lemma 3.8 then implies that \( A \to \text{Cat}(A) \) is an equivalence. We obtain the required statement concerning 3.5 by using the fact that \( \text{Cat}(A) \to \text{BigCat}(A) \) (resp. \( \text{Cat}(B) \to \text{BigCat}(B) \)) is an equivalence—note that our proof of 3.5 comes after our proof of 3.4 below—and applying 3.8.

A simplified point of view

We started to see, in the proof of Lemma 3.8, a simplified or “derived” point of view on \( n \)-categories. We will expand on that a bit more here. When we use the statements of the above lemmas for \( n - 1 \)-categories, they may be considered as proved in view of our global induction. The homotopy or localized category \( HC_{n-1} \) of \( n - 1 \)-precats modulo weak equivalence, also equal to the localization of Tamsamani’s \( (n - 1) - \text{Cat} \) by equivalences, admits direct products. There is a functor \( T^{n-1} : HC_{n-1} \to \text{Sets} \) related to the inclusion \( i : \text{Sets} \subset HC_{n-1} \) by morphisms \( iT^{n-1}(X) \to X \) and \( T^{n-1} iS \cong S \) (the first is only well defined in the localized category). Thus if \( A \times B \to C \) is a morphism in \( HC_{n-1} \) then we obtain the map \( A \times iT^n B \to C \).
Note that $HC_{n-1}$ admits fibered products over objects of the form $i(S)$ for $S$ a set, since these are essentially just direct products. (However the homotopy category does not admit general fibered products nor, dually, does it admit pushouts.)

We can define the notion of $HC_{n-1}$-category, as simply being a category in the category $HC_{n-1}$ such that the object object is a set. Applying the functor $T^{n-1}$ yields a category, and this category acts on the morphism objects of the previous one, using the above remark.

If $A$ is an $n$-category then taking $A_0$ as set of objects and using the object $L\text{Hom}_A(x, y)$ as morphism object in $HC_{n-1}$ we obtain an $HC_{n-1}$-category which we denote $HC_{n-1}(A)$.

We can write

$$\text{Hom}_{HC_{n-1}(A)}(x, y) := L\text{Hom}_A(x, y)$$

which in turn is, we recall, the image of $A_1(x, y)$ in the localization of the category of $n-1$-precats. This is what we used in the proof of Lemma 3.8 above. The truncation operation $T^{n-1}$ applied to $HC_{n-1}(A)$ gives the 1-category $T^{n-1}A$. We obtain again the action of this category on the morphism objects in $HC_{n-1}(A)$.

In the next section we will be interested in the notion of $HC_{n-1}$-precategory, a functor $F : \Delta \to HC_{n-1}$ sending 0 to a set. An $HC_{n-1}$-precategory $F$ is an $HC_{n-1}$-category if and only if the usual morphisms

$$F_p \to F_1 \times_{F_0} \ldots \times_{F_0} F_1$$

are isomorphisms. If $A$ is an $n$-precat then let $HC_{n-1}(A)$ denote the $HC_{n-1}$-precategory which to $p \in \Delta$ associates the image of $A_p/\sim$ in the localized category $HC_{n-1}$.

Here is a small remark which is sometimes useful.

**Lemma 3.10** Suppose $f : A \to B$ is a morphism of $n$-categories and suppose that $HC_{n-1}(A) \to HC_{n-1}(B)$ is an equivalence in $HC_{n-1}\text{Cat}$. Then $f$ is an equivalence.

///

We can also make a similar statement for $HC_{n-1}$-precats under the condition of requiring an isomorphism on the set of objects.

**Lemma 3.11** Suppose $f : A \to B$ is a morphism of $n$-precats and suppose $HC_{n-1}(A) \to HC_{n-1}(B)$ is an isomorphism of functors $\Delta \to HC_{n-1}\text{Cat}$. Then $f$ is a weak equivalence.

**Proof:** This is just a restatement of Lemma 3.5 (in particular it is not available for use in degree $n$ until we have proved 3.3 below). ///

It would have been nice to be able to have an operation on $HC_{n-1}$-precategories which, when applied to $HC_{n-1}(A)$ yields $HC_{n-1}(\text{Cat}(A))$. This doesn’t seem to be possible.
(although I don’t have a counterexample) because the construction we discuss in the next section relies heavily on pushouts but these don’t exist in \( HC_{n-1} \). If this had been possible we would have been able to formulate a notion of weak equivalence for \( HC_{n-1} \)-precats and in particular we would have been able to give a stronger formulation in the previous lemma.

We end this discussion by pointing out that information is lost in passing from \( A \) to \( HC_{n-1}(A) \). (See the next paragraph for some counterexamples but I don’t have counterexamples for all of the nonexistence statements which are made.) Let \( HC_{n-1}Cat \) (resp. \( HC_{n-1}PreCat \)) denote the categories of \( HC_{n-1} \)-categories (resp. \( HC_{n-1} \)-precategories). The functors \( n-Cat \to HC_{n-1}Cat \) and \( n-Cat \to HC_n \) do not enter into a commutative triangle with a morphism between \( HC_n \) and \( HC_{n-1}Cat \) in either direction. The only thing we can say is that there is an obvious notion of equivalence between two \( HC_{n-1} \)-categories, and if we let \( Ho - HC_{n-1} - Cat \) denote the category of \( HC_{n-1} \)-categories localized by inverting these equivalences, then there is a factorization

\[
 n - CAT \to HC_n \to Ho - HC_{n-1} - Cat
\]

but the second arrow in the factorization is not an isomorphism. In particular, when we pass from \( A \) to \( HC_{n-1}(A) \) we lose information. Nonetheless, it may be helpful especially from an intuitive point of view to think of an \( n \)-category in terms of its associated object \( HC_{n-1}(A) \) which is a category in the homotopy category of \( n - 1 \)-categories.

The topological analogy of the above situation (which can be made precise using the Poincaré groupoid and realization constructions \[27\]—thus providing some counterexamples to support some of the the nonexistence statements made in the previous paragraph) is the following: if \( X \) is a space then for each \( x, y \in X \) we can take as \( h(x, y) \) the space of paths from \( x \) to \( y \) viewed as an object in the homotopy category \( Ho(Top) \). We obtain a category in \( Ho(Top) \). If \( X \) is connected it is a groupoid with one isomorphism class, thus essentially a group in \( Ho(Top) \). This group is just the loop space based at any choice of point, viewed as a group in \( Ho(Top) \). It is well known (\[1\] \[28\]) that this object does not suffice to reconstitute the homotopy type of \( X \), thus our functor from \( Top \) to the category of groupoids in \( Ho(Top) \) does not yield a factorization of the localization functor \( Top \to Ho(Top) \). On the other hand, since there is no way to canonically choose a collection of basepoints for an object in \( Ho(Top) \), there probably is not a factorization in the other direction either.

\textit{Another simplified point of view}

We now give another set of remarks relating the present approach to \( n \)-categories with the usual standard ideas. This is based on the following observation. The proof of the lemma is based on some ideas from the next section so the reader should look there before trying to follow the proof. We have put the lemma here for expository reasons.
Lemma 3.12 If $A$ is a fibrant $n$-precat then the $A_p/\cdot$ are fibrant $n-1$-precats.

Proof: Fix objects $x_0, \ldots, x_p \in A_0$. We show that $A_p/(x_0, \ldots, x_p)$ is fibrant. Suppose $U \to V$ is a trivial cofibration of $n-1$-precats. Let $B$ (resp. $C$) be the $n$-precat with objects $0, \ldots, p$ and such that $B_q/(i_0, \ldots, i_q)$ (resp. $C_q/(i_0, \ldots, i_q)$) is the disjoint union of $U$ (resp. $V$) over all morphisms $f : q \to p$ such that $f(q) = i_q$. Then (as can be seen by the discussion of the next section) $B \to C$ is a trivial cofibration. A morphism $B \to A$ (resp. $C \to A$) is the same thing as a morphism $U \to A_p/(x_0, \ldots, x_p)$ (resp. $V \to A_p/(x_0, \ldots, x_p)$). It follows immediately that if $A$ is fibrant then $A_p/(x_0, \ldots, x_p)$ has the required lifting property to be fibrant.

Now we can use the closed model category structure on $PC_{n-1}$ to analyze the collection of $A_p/\cdot$ when $A$ is fibrant. Recall that morphisms in the localized category between fibrant and cofibrant objects are represented by actual morphisms [20]. Thus the morphism

$$A_2/ \to A_1/ \times_{A_0} A_1/$$

which is an equivalence, can be inverted and then followed by the projection to the third edge of the triangle to give

$$A_1/ \times_{A_0} A_1/ \to A_2/ \to A_1/.$$

We get a morphism “composition”

$$m : A_1(x,y) \times A_1(y,z) \to A_1(x,z)$$

which represents the composition

$$LHom_A(x,y) \times LHom_A(y,z) \to LHom_A(x,z)$$

of the previous “simplified point of view”. Of course our composition morphism $m$ is not uniquely determined but depends on the choice of inversion of the original equivalence. In particular $m$ will not in general be associative. However $A_3/\cdot$ gives a homotopy in the sense of Quillen between $m(m(f,g),h)$ and $m(f,m(g,h))$. This can be turned into a homotopy in the sense of the $n-1$-categories of morphisms (an exercise left to the reader).

4. Calculus of “generators and relations”

For the proofs of 3.4 and 3.5 we need a close analysis of an operation which when iterated yields $BigCat$. This analysis will lead us to a point of view which generalizes the idea of generators and relations for an associative monoid. At the end we draw as a consequence one of the main special lemmas needed to treat the special case 6.2 in the proof of 3.2.
The overall goal of this section is to investigate the operation \( A \mapsto \text{Cat}(A) \) in the spirit of looking at the simplicial collection of \( n-1 \)-precats \( A_p/ \) as a functor from \( \Delta \) to our closed model category in degree \( n-1 \). We would like to understand the transformation which this functor undergoes when we apply the operation \( \text{Cat} \) to \( A \).

We first describe a general type of operation which we often encounter. Suppose \( A \) is an \( n \)-precat and suppose \( A_m/ \rightarrow B \) is a cofibration of \( n-1 \)-precats provided with a morphism \( \pi : B \rightarrow A_0 \times \ldots \times A_0 \) making the composition

\[
A_m/ \rightarrow B \rightarrow A_0 \times \ldots \times A_0
\]
equal to the usual morphism (there are \( m+1 \) factors \( A_0 \) in the product). We can alternatively think of this as a collection of cofibrations

\[
A_m/(x_0, \ldots, x_m) \rightarrow B(x_0, \ldots, x_m)
\]
for all sequences of objects \( x_i \in A_0 \). Then we define the cofibration of \( n \)-precats

\[
A \rightarrow \mathcal{I}(A; A_m/ \rightarrow B)
\]
as follows (the projection \( \pi \) is part of the data even though it is not contained in the notation). For any \( p \), \( \mathcal{I}(A; A_m/ \rightarrow B)_p/ \) is the multiple pushout of \( A_p/ \) and \( A_m/ \rightarrow B \) over all morphisms \( A_m/ \rightarrow A_p/ \) coming from morphisms \( p \rightarrow m \) which do not factor through 0. Functoriality is defined as follows: if \( q \rightarrow p \) is a morphism then for any \( f : p \rightarrow m \) such that the composition \( q \rightarrow m \) doesn’t factor through 0, we define the morphism of functoriality on the part of the pushout corresponding to \( f \) as the identity in the obvious way; on the other hand, if \( f : p \rightarrow m \) is a morphism such that the composition \( q \rightarrow m \) factors through 0 \( \rightarrow m \) then we obtain (from the projection \( \pi \)) a morphism \( B \rightarrow A_0 \) extending the morphism \( A_m/ \rightarrow A_0 \) and so that part of the pushout is sent into the image of \( A_0 \) in \( A_q/ \).

We call \( A \rightarrow \mathcal{I}(A; A_m/ \rightarrow B) \) the *pushout of \( A \) induced by \( A_m/ \rightarrow B \).* Using Lemma 3.2 in degree \( n-1 \) we find that if \( A_m/ \rightarrow B \) is a trivial cofibration then the morphisms

\[
A_p/ \rightarrow \mathcal{I}(A; A_m/ \rightarrow B)_p/
\]
are trivial cofibrations.

This operation occurs notably in the process of doing \( \text{Cat} \) or \( \text{BigCat} \) to \( A \). Fix \( m \geq 1, M = (m_1, \ldots, m_l), m' \) and \( k \). Let

\[
\Sigma := \Sigma(m, M, [m'], \langle k, k+1 \rangle),
\]
and

\[
\varphi := \varphi(m, M, [m'], \langle k, k+1 \rangle) : \Sigma \rightarrow h(m, M, m', 1^{k+1}).
\]
Suppose again \( a : \Sigma \to A \) is a morphism and let \( C \) be the pushout \( n \)-precat of \( A \) and \( \varphi \) over \( a \). In this case note that \( \Sigma \) and \( h(m, M, m', 1^{k+1}) \) are pushouts of diagrams of objects entirely within the category \((m, \Theta^{n-1})\) of objects of the form \((m, M')\). The restriction of \( A \) to this category is just \( A_{m/} \). Let \( \psi : A_{m/} \to F \) be the pushout \( n - 1 \)-precat of \( \varphi \) over \( a \) considered in this way. Then \( C = \mathcal{I}(A; A_{m/} \to F) \) is the pushout of \( A \) induced by \( \psi \) (note that \( \psi \) admits a projection \( \pi \) in an obvious way). The proof is that \( h(m, M, m', 1^{k+1}) \) has exactly the same description as a pushout of \( \Sigma \).

Suppose \( A \) is an \( n \)-precat. Define a new \( n \)-precat \( \text{Fix}(A) \) by iterating the above operation of pushout by all standard cofibrations \( \varphi(m, M, [m'], (k, k+1)) \), over all possible values of \( m, M, m' \) and \( k \), and repeating this operation a countable number of times. By reordering, \( \text{Fix}(A) \) may be seen as obtained from \( A \) by a sequence of standard pushouts of the form

\[
A' \to \mathcal{I}(A', A'_{m/} \to \text{BigCat}(A'_{m/})).
\]

In particular it is clear that each \( A_{p/} \to \text{Fix}(A)_{p/} \) is a trivial cofibration. On the other hand it is also clear that the \( \text{Fix}(A)_{p/} \) are \( n - 1 \)-categories (they are obtained by iterating operations of the form, taking \( \text{BigCat} \) then taking a bunch of pushouts then taking \( \text{BigCat} \) and so on an infinite number of times—and such an iteration is automatically an easy \( n - 1 \)-category).

In order to get to \( \text{Cat}(A) \) or \( \text{BigCat}(A) \) we need another type of operation which relates the different \( A_{m/} \). Suppose \( A \) is an \( n \)-precat, fix \( m \geq 2 \) and suppose that we have a diagram

\[
A_{m/} \xrightarrow{f} Q \xrightarrow{g} A_{1/} \times_{A_0} \cdots \times_{A_0} A_{1/}
\]

with the first arrow cofibrant. Then we define the pushout \( A \to \mathcal{J}(A; f, g) \) as follows. \( \mathcal{J}(A; f, g)_{p/} \) is the multiple pushout of \( A_{m/} \to A_{p/} \) and \( A_{m/} \to Q \) over all maps \( p \to m \) not factoring through any of the principal maps \( 1 \to m \). The morphisms of functoriality are defined in the same way as for the construction \( \mathcal{I} \) using the map \( g \) here.

Remark: This pushout changes the object over \( 1 \in \Delta \) because there are morphisms \( 1 \to m \) (the faces other than the principal ones) which don’t factor through the principal face maps.

The remaining of our standard pushouts which are not covered by the operation \( \mathcal{I} \) are covered by this operation \( \mathcal{J} \). Fix some \( m \geq 2 \) and \( k \). Write

\[
\Sigma \quad \text{for} \quad \Sigma([m], (k, k + 1)),
\]

and

\[
\varphi := \varphi([m], (k, k + 1)) : \Sigma \to h(m, 1^{k+1}).
\]

We have a diagram of \( n - 1 \)-precats

\[
\Sigma_{m/} \xrightarrow{f(m,k)} h(1^{k+1}) \xrightarrow{g(m,k)} \Sigma_{1/} \times_{\Sigma_0} \cdots \times_{\Sigma_0} \Sigma_{1/},
\]

21
and via this diagram

\[ h(m, 1^{k+1}) = J(\Sigma; f(m, k), g(m, k)). \]

It follows that if \( A \) is an \( n \)-precat and \( \Sigma \to A \) is a morphism then the standard pushout \( B \) of \( A \) along \( \varphi \) is of the form \( B = J(A; f, g) \) for appropriate maps \( f \) and \( g \) induced by the above ones.

We need to have some information about decomposing and commuting the operations \( I \) and \( J \). Suppose

\[ A \xrightarrow{\varphi} P \xrightarrow{\pi} A_0 \times \ldots \times A_0 \]

is a morphism. Let

\[ \eta : A_1/ \to B \]

denote the multiple pushout of \( A_1/ \) by \( \varphi \) over all of the principal morphisms \( 1 \to m \) (with projection \( \nu : B \to A_0 \times A_0 \)). We obtain a factorization

\[ A \xrightarrow{\varphi} P \xrightarrow{\psi} B \times_{A_0} \ldots \times_{A_0} B \]

and we have

\[ I(A; \varphi, \pi) = J(I(A; \eta, \nu); \varphi, \psi). \]

In this way we turn an operation of the form \( I \) for \( m \) into an operation of the form \( I \) for \( 1 \) followed by an operation of the form \( J \) for \( m \).

We define a type of operation combining operations of the form \( J \) for \( m \) with operations of the form \( I \) for \( 1 \). However, we would like to keep track of certain sub-\( n-1 \)-precats of \( A \). So we say that an \((m, 1)\)-painted \( n \)-precat (or just painted \( n \)-precat in the current context where \( m \) is fixed) is an \( n \)-precat \( A \) together with cofibrations of \( n-1 \)-precats \( A_1^* / \to A_m/ \) and \( A_1^* / \to A_1/ \). We require a lifting of the standard morphism to

\[ A_1^* / \to A_1^* / \times_{A_0} \ldots \times_{A_0} A_1^* / \]

Suppose \((A, A_1^*, A_1^*)\) is a painted \( n \)-precat, and suppose that we have morphisms

\[ A_1^* / \xrightarrow{\eta'} P \xrightarrow{\nu'} A_0 \times A_0 \]

and

\[ A_m/ \xrightarrow{\varphi} P \xrightarrow{\psi'} B \times_{A_0} \ldots \times_{A_0} B \]

compatible with the previous lifting of the standard morphism to the painted parts. Let \( \eta'', \nu'', \varphi' \) and \( \psi' \) be obtained by taking the pushouts of the above with \( A_1/ \) or \( A_m/ \). Then we define a new painted \( n \)-precat

\[ J'(A; \eta, \nu; \varphi, \psi) := J(I(A; \eta', \nu'); \varphi', \psi'). \]
with painted parts \((P, B)\) replacing \((A_{m/}^*, A_{1/}^*)\). This operation now behaves well under iteration: the composition of two such operations is again an operation of the same form. Furthermore, our operations \(I\) and \(J\) coming from standard trivial cofibrations can be interpreted as operations of the above type if the original \(\Sigma \to A\) sends the arrows \((a, b, v_i)\) into the painted parts \(A_{m/}^*, A_{1/}^*\). These operations are exactly designed to do two things: replacing the painted parts by their associated \(n\)-categories; and getting the standard map to being an equivalence. In particular, starting with \(A_{1/}^* = A_{1/}\) and \(A_{m/}^* = A_{m/}\), there is a sequence of operations coming from standard trivial cofibrations (concerning only \(m\) and \(1\)) such that, when interpreted as operations on painted \(n\)-precats, combine into one big operation of the form \(J'\) where \(\eta : A_{1/}^* \to B\) is a trivial cofibration to an \(n - 1\)-category, and where the morphism

\[ P \xrightarrow{\psi} B \times A_0 \times \cdots \times A_0 \ B \]

is an equivalence of \(n\)-categories.

Going back to the original definition of the operation \(J'\) in terms of \(J\) and \(I\) we find that an appropriate sequence of trivial cofibrations can be reordered into an operation of the form \(A \mapsto A' = I(A, \eta, \nu)\) followed by \(J(A'; f, g)\) for

\[ A_{m/}^* \xleftarrow{f} G[m](A) \xrightarrow{g} A_{1/}^* \times A_0 \times \cdots \times A_0 A_{1/}^* \]

where \(g\) is a weak equivalence. Recall that the morphism \(A \to A'\) coming before the operation \(J\) has the property that the \(A_{p/} \to A_{p/}^*\) are weak equivalences. Note also that we can assume that \(A_{1/}^*\) is an \(n - 1\)-category, because it is equal to \(B\)—there are no morphisms \(1 \to 1\) other than the identity and those which factor through \(0\)—and \(B\) can be chosen to be an \(n - 1\)-category. (This paragraph is the conclusion we want; the discussion of painted \(n\)-precats was just a means to arrive here and will not be used any further below.)

Let \(Gen[m](A)\) denote the result of the previous operation, which we can thus write as

\[ Gen[m](A) = J(A'; A_{m/}^* \xleftarrow{f} G[m](A) \xrightarrow{g} A_{1/}^* \times A_0 \times \cdots \times A_0 A_{1/}^*) \]

The pushouts chosen as above may be assumed to contain, in particular, all of the standard pushouts of the second type for \(m\).

Put \(Gen_1(A) = Fix(A)\) and \(Gen_i(A) := Fix(Gen[i](Gen_{i-1}(A)))\) for \(i \geq 2\). Let \(Gen(A)\) be the inductive limit of the \(Gen_i(A)\). Finally, iterate the operation \(A' \mapsto Gen(A')\) a countable number of times. It is clear that, by reordering, this yields \(BigCat(A)\), since on the one hand all of the necessary pushouts occur, whereas on the other hand only the standard pushouts are used.

\textit{Proofs of 3.4 and 3.5}
The above description yields immediately the proofs of these two lemmas.

**Proof of 3.4:**
Suppose $A$ is an $n$-precat such that $A_{1/}$ is an $n - 1$-category and such that $(\ast)$ for all $m$ the morphisms

$$A_{m/} \to A_{1/} \times_{A_0} \cdots \times_{A_0} A_{1/}$$

are weak equivalences of $n - 1$-precats. Fix $m$ and apply the operation $Gen[m](A)$. Let $A'$ be the intermediate result of doing the preliminary operations $\mathcal{I}$. The morphism

$$f : A'_{m/} \to \mathcal{G}[m](A)$$

is a trivial cofibration of $n$-precats, using:

1. the fact that $\mathcal{G}[m](A) \to A'_{1/} \times_{A_0} \cdots \times_{A_0} A'_{1/}$ is a weak equivalence;
2. the fact that $A'_{1/}$ are $n$-categories equivalent to $A_{1/}$, and noting that direct products of $n$-categories (or fibered products over sets) preserve equivalences; and
3. the hypothesis that $A_{m/}$ is weakly equivalent to the product of the $A_{1/}$, again coupled with the fact that $A'_{m/}$ is weakly equivalent to $A_{m/}$ because the operations $\mathcal{I}$ preserve the weak equivalence type of the $A_{p/}$.

It now follows from the definition of the operation $\mathcal{J}(A'; f, g)$ that the morphisms

$$A_{p/} \to A'_{p/} \to \mathcal{J}(A'; f, g)_{p/}$$

are weak equivalences. Thus (under the hypothesis $(\ast)$ above) the morphism $A \to Gen[m](A)$ induces weak equivalences $A_{p/} \to Gen[m](A)_{p/}$. The same holds always for the operation $Fix$, and iterating these we obtain the conclusion (under hypothesis $(\ast)$) that

$$A_{p/} \to BigCat(A)_{p/}$$

are weak equivalences.

In the hypotheses of 3.4, $A$ is an $n$-category, so $A_{1/}$ is an $n - 1$-category and hypothesis $(\ast)$ is satisfied. It follows from above that the morphisms

$$A_{p/} \to BigCat(A)_{p/}$$

are weak equivalences of $n - 1$-precats, but since both sides are $n - 1$-categories this implies that they are equivalences of $n - 1$-categories (using Lemma 3.4 in degree $n - 1$). Therefore $A \to BigCat(A)$ is an equivalence of $n$-categories, completing the proof of 3.4.

///

**Proof of 3.5:**
Suppose $A \to B$ is a morphism of $n$-precats which induces weak equivalences of $n - 1$-precats $A_{m/} \to B_{m/}$ for all $m$. Replacing $A$ by $Fix(A)$ and $B$ by $Fix(B)$ conserves the
hypothesis. We show that replacement of $A$ by $\text{Gen}[m](A)$ and $B$ by $\text{Gen}[m](B)$ conserves the hypothesis. Let $A'$ (resp. $B'$) denote the intermediate $n$-precats used in the definition of $\text{Gen}[m](A)$ (resp. $\text{Gen}[m](B)$). These are the results of applying operations $\mathcal{I}$ to $A$ and $B$, and we may assume as in the previous proof that $A'_{1/}$ and $B'_{1/}$ are $n-1$-categories. We do these operations in a canonical way so as to preserve morphisms $A' \to B'$ and $\text{Gen}[m](A) \to \text{Gen}[m](B)$. Note that $A_{1/} \to A'_{1/}$ is a weak equivalence and the same for $B$. Our hypothesis now implies that the morphism $A'_{1/} \to B'_{1/}$ is an equivalence of $n-1$-categories. Therefore

$$A'_{1/} \times A'_{0} \times \cdots \times A'_{0} A'_{1/} \to B'_{1/} \times B'_{0} \times \cdots \times B'_{0} B'_{1/}$$

is a weak equivalence.

As before

$$\text{Gen}[m](A) = \mathcal{J}(A'; A'_{m/} \xrightarrow{f_A} \mathcal{G}[m](A) \xrightarrow{g_A} A'_{1/} \times A'_{0} \times \cdots \times A'_{0} A'_{1/}),$$

with $g_A$ being a weak equivalence. Similarly

$$\text{Gen}[m](B) = \mathcal{J}(B'; B'_{m/} \xrightarrow{f_B} \mathcal{G}[m](B) \xrightarrow{g_B} B'_{1/} \times B'_{0} \times \cdots \times B'_{0} B'_{1/}),$$

with $g_B$ a weak equivalence. It follows immediately that

$$\mathcal{G}[m](A) \to \mathcal{G}[m](B)$$

is a weak equivalence and hence (using the description of $\mathcal{J}$ as well as the fact that weak equivalences on the components induce weak equivalences of pushouts, which we know by induction for $n-1$-precats) the morphism $\text{Gen}[m](A) \to \text{Gen}[m](B)$ induces weak equivalences

$$\text{Gen}[m](A)_{p/} \to \text{Gen}[m](B)_{p/}.$$ 

This shows that the operation $\text{Gen}[m]$ preserves the hypothesis of 3.5. Taking limits we get that $\text{Gen}$ and finally that $\text{BigCat}$ preserve the hypothesis: we get that for all $p$,

$$\text{BigCat}(A)_{p/} \to \text{BigCat}(B)_{p/}$$

is a weak equivalence. This implies that $\text{BigCat}(A) \to \text{BigCat}(B)$ is an equivalence of $n$-categories, finishing the proof of 3.5. 

1-free ordered precats
Suppose $A$ is an $n$-precat. We say that $A$ is 1-free ordered if there is a total order on the set $A_0$ of objects (which we suppose for simplicity to be finite) such that the following properties are satisfied:

(FO1)—for any sequence $x_0,\ldots,x_n$ which is out of order (i.e. some $x_i$ is strictly bigger than $x_{i+1}$), $A_m/(x_0,\ldots,x_m) = \emptyset$;

(FO2)—for any sequence $x_0,\ldots,x_n$ with $x_{i-1} \leq x_i$ the morphism

$$A_m/(x_0,\ldots,x_m) \to A_1/(x_0,x_m)$$

is a weak equivalence; and

(FO3)—for any stationary sequence $A_m/(x,\ldots,x)$ is weakly equivalent to $\ast$.

Properties (FO1) and (FO2) properties are preserved under the operation $\text{BigCat}$. Indeed, the standard cofibrations $\Sigma \to h$ go between $n$-precats which satisfy these conditions, and these conditions are preserved by pushouts over diagrams of morphisms which are order-respecting (i.e. morphisms respecting $\leq$) between 1-free ordered $n$-precats. Note that if $A$ satisfies (FO1) then any morphism $\Sigma \to A$ must respect the order.

The condition (FO3) is preserved by pushouts of morphisms which are strictly order-preserving, and also by the operation $\text{Fix}$ (which is essentially the same as $\text{Cat}_{\leq 1}$). If $A$ is 1-free ordered, using (FO1) and (FO3) we get that in order to obtain $\text{Cat}(A)$ it suffices to use the operation $\text{Fix}$ and pushouts for trivial cofibrations $\Sigma \to h(m,1^{k+1})$ via morphisms $\Sigma \to A$ which are strictly order preserving. Thus $\text{BigCat}(A)$ again satisfies (FO3). Furthermore, note that these trivial cofibrations do not change the homotopy type for adjacent objects, so if $x,y$ are adjacent in the ordering then

$$A_1/(x,y) \to \text{BigCat}(A)_{1/}(x,y)$$

is a weak equivalence. We obtain the following conclusion:

**Lemma 4.1** Suppose $A$ is a 1-free ordered $n$-precat with finite object set. For two objects $x,y \in A_0$ let $x = x_0,\ldots,x_m = y$ be the maximal strictly increasing ordered sequence going from $x$ to $y$. Let $A' := \text{Bicat}(A)$. Then the morphisms

$$A'_1/(x,y) \leftarrow A'_m/(x_0,\ldots,x_m) \to A_1/(x_0,x_1) \times_{A_0} \cdots \times_{A_0} A_1/(x_{m-1},x_m)$$

are weak equivalences. In particular if $A \to B$ is a morphism of 1-free ordered $n$-precats (with finite object sets) preserving the ordering, inducing an isomorphism on object sets and inducing equivalences $A_1/(x,y) \to B_1/(x,y)$ for all pairs of adjacent objects $(x,y)$ then $A \to B$ is a weak equivalence.

**Proof:** In the diagram

$$A'_1/(x,y) \leftarrow A'_m/(x_0,\ldots,x_m) \to A'_1/(x_0,x_1) \times_{A_0} \cdots \times_{A_0} A'_1/(x_{m-1},x_m)$$
the left arrow is a weak equivalence by property (FO2) for $A'$ (we have shown that that property is preserved under passage from $A$ to $A'$); and the right arrow is a weak equivalence because $A'$ is an $n$-category. The product on the right may be replaced by that appearing in the statement of the lemma since $A_1/(x_{i-1}, x_i) \to A_1'(x_{i-1}, x_i)$ is an equivalence because $x_{i-1}$ and $x_i$ are adjacent. This gives the first statement of the lemma. For the second statement, note that (using the same notation for objects of $A$ and $B$, and using the notations $A'$ and $B'$ for associated $n$-categories) the first statement implies that $A_1'(x, y) \to B_1'(x, y)$ is a weak equivalence for any pair of objects $x, y$. This implies that $A' \to B'$ is an equivalence of $n$-categories.

Characterization of weak equivalence

We close this section by mentioning a proposition which gives a sort of uniqueness for the notion of weak equivalence.

**Proposition 4.2** Suppose $F : PC_n \to PC_n$ is a functor with natural transformation $i_A : A \to F(A)$ such that:

(a)—for all $A$, $F(A)$ is an $n$-category;

(b)—if $A$ is an $n$-category then $i_A$ is an iso-equivalence of $n$-categories (recall that this means an equivalence inducing an isomorphism on sets of objects); and

(c)—for any $n$-precat $A$ the morphism $F(i_A) : F(A) \to F(F(A))$ is an equivalence of $n$-categories.

Then for any $n$-precat $A$ the morphism $A \to F(A)$ is a weak equivalence.

**Proof:** Put $F'(A) := \text{Cat}(F(A))$. It is a marked easy $n$-category. We have a morphism $k_A := \text{Cat}(i_A) : \text{Cat}(A) \to F'(A)$. Letting $j_A : A \to \text{Cat}(A)$ denote the inclusion and $i_A'$ the map $A \to F'(A)$, note that $k_A j_A = i_A'$.

The functor $F'$ again satisfies the properties (a), (b) and (c) above. For property (c) note that the map $F'(i_A')$ is obtained by applying $\text{Cat}$ to the composed map

$$F(A) \xrightarrow{F(i_A)} F(F(A)) \xrightarrow{F(j_F(A))} F(\text{Cat}(F(A)))$$

but the first map is an equivalence by hypothesis and the second is an equivalence because of the diagram

$$
\begin{array}{ccc}
F(A) & \rightarrow & F(F(A)) \\
\downarrow & & \downarrow \\
\text{Cat}(F(A)) & \rightarrow & F(\text{Cat}(F(A)))
\end{array}
$$

where the top arrow is $i_{F(A)}$ which is an equivalence by (b) (but it is different from $F(i_A)$!), the left vertical arrow is the equivalence $i_{F(A)}$ (by (a)) and the bottom arrow is the equivalence $i_{\text{Cat}(F(A))}$ again an equivalence by (b).
We have the following diagram:

\[
\begin{array}{ccc}
Cat(A) & \rightarrow & Cat(Cat(A)) \\
\downarrow & & \downarrow \\
F'(A) & \rightarrow & F'(Cat(A)) \rightarrow F'(F'(A))
\end{array}
\]

The morphism on the top is \(Cat(j_A)\), and the vertical morphisms are \(k_A\) and \(k_{Cat(A)}\) respectively. The morphisms on the bottom are \(F'(j_A)\) and \(F'(k_A)\) respectively. The diagram comes from naturality of \(k\). The top arrow \(Cat(j_A)\) and the middle vertical arrow \(k_{Cat(A)}\) are equivalences, as is the composition along the bottom \(F'(i_A')\). On the other hand, all of the arrows are identities on the sets of objects. Thus, when morphisms are equivalences they are in fact iso-equivalences, and in particular equivalences on the level of the \(n-1\)-categories \((\cdot)_{\mathcal{P}/}\). The closed model structure for \(n-1\) implies that a morphism of \(n-1\)-categories is a weak equivalence (hence an equivalence) if and only if it projects to an isomorphism in the localized category. Look at the images of the above diagram in the localized category of \(n-1\)-precats after applying the operation \((\cdot)_{\mathcal{P}/}\). The equivalences that we know show that the bottom left arrow goes to an arrow which has a left and right inverse. It follows that it goes to an invertible arrow, i.e. an isomorphism, in the localized category. Its left inverse, the left vertical arrow, must also go to an isomorphism. This implies that for each \(\mathcal{P}\), the map \(Cat(A)_{\mathcal{P}/} \rightarrow F'(A)_{\mathcal{P}/}\) is an equivalence. By definition then \(A \rightarrow F(A)\) is a weak equivalence.

\[\]

5. Compatibility with products

The goal of the present section is to prove the following theorem.

**Theorem 5.1** Suppose \(A\) and \(B\) are \(n\)-precats. Then the morphism

\[A \times B \rightarrow Cat(A) \times Cat(B)\]

is a weak equivalence.

Before getting to the proof, we give some corollaries.

**Corollary 5.2** Suppose \(B\) is an \(n\)-precat. Let \(\mathcal{T}\) be the 1-category with two isomorphic objects denoted 0 and 1, considered as an \(n\)-precat. Then the morphisms

\[Cat(B) \xrightarrow{i_0,i_1} Cat(B \times \mathcal{T}) \xrightarrow{p} Cat(B)\]

are equivalences of \(n\)-categories, where \(i_0\) and \(i_1\) come from the inclusions \(0 \rightarrow \mathcal{T}\) and \(1 \rightarrow \mathcal{T}\) and \(p\) comes from the projection on the first factor.
Proof: Note that the morphism $\mathcal{T} \to \text{Cat}(\mathcal{T})$ is a weak equivalence by Lemma 3.4. Thus Theorem 5.1 says that $B \times \mathcal{T} \to \text{Cat}(B) \times \mathcal{T}$ is a weak equivalence. On the other hand, the morphism $\text{Cat}(B) \times \mathcal{T} \to \text{Cat}(B)$ is a weak equivalence, so $B \times \mathcal{T} \to \text{Cat}(B)$ is a weak equivalence. The morphism $B \to \text{Cat}(B)$ is of course a weak equivalence, so Lemma 3.8 implies that the two morphisms

$$B \xrightarrow{\iota_0, \iota_1} B \times \mathcal{T} \xrightarrow{p} B$$

are weak equivalences, which is the same statement as the corollary. 

Corollary 5.3 Suppose $A \to A'$ is a weak equivalence. Then for any $B$, $A \times B \to A' \times B$ is a weak equivalence.

Proof: By Theorem 5.1 we have that

$$A \times B \to \text{Cat}(A) \times \text{Cat}(B), \quad A' \times B \to \text{Cat}(A') \times \text{Cat}(B)$$

are weak equivalences. By hypothesis, the map $\text{Cat}(A) \to \text{Cat}(A')$ is an equivalence of $n$-categories. It follows (from any of several characterizations of equivalences of $n$-categories, see for example [27] Proposition 1.3.1) that $\text{Cat}(A) \times \text{Cat}(B) \to \text{Cat}(A') \times \text{Cat}(B)$ is an equivalence, which gives the corollary.

Proof of Theorem 5.1

We start by making some preliminary reductions. First we claim that it suffices to prove that if $B$ is any $n$-precat and $\Sigma = \Sigma(M, [m], \langle k, k + 1 \rangle)$ and $h = h(M, m, 1^{k+1})$ then the morphism $\Sigma \times B \to h \times B$ is a weak equivalence. Suppose that we know this statement. Then, noting that the proof of 3.2 below doesn’t use Theorem 5.1 in degree $n$ in the case for a pushout by a trivial cofibration which is an isomorphism on objects (which is the case for $\Sigma \times B \to h \times B$) we obtain that for any $A$ and any morphism $\Sigma \to A$ the morphism

$$A \times B \to A \times B \cup^{\Sigma \times B} h \times B$$

is a weak equivalence. The morphism $A \times B \to \text{Cat}(A) \times \text{Cat}(B)$ is obtained by iterating operations of this form (either on the variable $A$ or on the variable $B$ which works the same way). Therefore it would follow from the hypothesis of our claim that the morphism of 5.1 is a weak equivalence.

We are now reduced to proving that $\Sigma \times B \to h \times B$ is a weak equivalence. In the previous notations if $M$ has length strictly greater than 0 then the $\Sigma_{p/} \to h_{p/}$ are weak equivalences. Thus

$$(\Sigma \times B)_{p/} = \Sigma_{p/} \times B_{p/} \to h_{p/} \times B_{p/} = (h \times B)_{p/}$$

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is a weak equivalence, and by Lemma 3.5 it follows that $\Sigma \times B \to h \times B$ is a weak equivalence. Thus we are reduced to treating the case where $M$ has length zero, that is

$$\Sigma = \Sigma([m], \langle k, k+1 \rangle) \quad \text{and} \quad h = h(m, 1^{k+1}).$$

Let $\Sigma^u = \Sigma^u([m], 1^{k+1})$ denote the pushout of $m$ copies of $h(1, 1^{k+1})$ over the standard $h(0)$. We claim that it suffices to prove that

$$\Sigma^u \times B \to h \times B$$

is a weak equivalence. This claim is proved by induction on $k$. Note that $\Sigma$ is obtained from $\Sigma^u$ by a sequence of standard cofibrations over $\Sigma' \to h'$ which are for smaller values of $k$. Assuming that we have treated all of the cases $\Sigma^u \times B \to h \times B$ and assuming our present claim for smaller values of $k$, we obtain that $\Sigma^u \times B \to \Sigma \times B$ is a trivial cofibration. It follows from Lemma 3.8 that $\Sigma \times B \to h \times B$ will be a trivial cofibration.

We are now reduced to proving that the morphism

$$\Sigma^u([m], 1^k) \to h(m, 1^k)$$

induces a weak equivalence $\Sigma^u \times B \to h \times B$ (we have changed the indexing $k$ here from the previous paragraphs). Our next reduction is based on the following observation. Suppose we know this statement for $n$-precats $A$, $B$ and $C$ with cofibrations $A \to B$ and $A \to C$. Let $P := B \cup^A C$. The morphisms $\Sigma^u \times A \to h \times A$ (resp. $\Sigma^u \times B \to h \times B$, $\Sigma^u \times C \to h \times C$) are weak equivalences inducing isomorphisms on objects. As remarked above, the proof of 3.2 below doesn’t use Theorem 5.1 in degree $n$ when concerning pushouts by weak equivalences which are isomorphisms on objects. On the other hand, the morphism $\Sigma^u \times P \to h \times P$ may be obtained as a successive coproduct by these previous morphisms; thus it is a weak equivalence.

We may apply this observation to infinite iterations of cofibrant pushouts. But note that any $n$-precat $B$ may be expressed as an iterated pushout of representable objects $h(M)$ over the boundaries $\partial h(M) \to h(M)$. The boundaries are in turn iterated pushouts over representable objects. From the remark of the previous paragraph, it follows that we are reduced to proving that $\Sigma^u \times B \to h \times B$ is a weak equivalence when $B$ is a representable object. We will write $B = h(u, M)$, distinguishing the first variable from the rest.

We next define the following operation. Suppose $C$ is an $n-1$-precat and suppose $D$ is a 1-precat; then we define $D \oplus C$ to be the $n$-precat with $(D \oplus C)_0 := D_0$ and for any $p$, $(D \oplus C)_p$ is the union for $f \in D_p$ of $C$ when $f$ is not totally degenerate and $*$ when $f$ is totally degenerate (we say that $f \in D_p$ is totally degenerate if it is in the image of the morphism $D_0 \to D_p$). The morphisms of functoriality for $p \to q$ are obtained by
projecting $C$ to the final $n - 1$-precat $*$ for elements $f \in D_\ell$ going to totally degenerate elements in $D_p$.

This notation is useful because $h(u, M) = h(u) \oplus h(M)$. Thus $h(m, 1^k) = h(m) \oplus h(1^k)$; and finally

$$\Sigma^{nu}([m], 1^k) = \Sigma^{nu}([m]) \oplus h(1^k).$$

The operation $\oplus$ is compatible with pushouts: if $B \leftarrow A \rightarrow C$ is a diagram of $n - 1$-precats then for any 1-precat $X$,

$$X \oplus (B \cup^A C) = (X \oplus B) \cup^{X \oplus A} (X \oplus C).$$

On the other hand, if $C \rightarrow C'$ is a weak equivalence then $X \oplus C \rightarrow X \oplus C'$ is a weak equivalence (by applying 3.3). The object $h(M)$ is weak equivalent to a pushout of objects of the form $h(1^k)$, with the pushouts being over boundaries which are themselves weak equivalent to pushouts of objects of the same form. Since, as we have seen above, changing things by pushouts or by weak equivalences (which are isomorphisms on objects) in the second variable, preserves the statement in question. Thus it suffices to treat, in the previous notations, the case $M = 1^j$. We have now boiled down to the basic case which needs to be treated: we must show that

$$\Sigma^{nu}([m], 1^k) \times h(u, 1^j) \rightarrow h(m, 1^k) \times h(u, 1^j)$$

is a weak equivalence.

To do this, use the standard subdivision of the product of simplices $h(m) \times h(u)$ into a coproduct of simplices identified over their boundaries. In this last part of the proof there was an error in version 1: on p. 31 the line claiming that “$B^{(a,b)} = B^{(a,b)} \cup^{B^a} B^x$” was not true. Furthermore the notation of this part of the proof was relatively difficult to follow. Thus we rewrite things in the present version 2.

This error and its correction were found during discussions with R. Pellissier, so I would like to thank him.

The basic idea remains the same as what was said in version 1. The objects of $h(m, 1^k) \times h(u, 1^j)$ may be denoted by $(i, j)$ with $i = 0, \ldots, m$ and $j = 0, \ldots, u$. These should be arranged into a rectangle. The problem is to understand the composition as we go from $(0, 0)$ to $(m, u)$. There are many different paths (i.e. sequences of points which are adjacent on the grid and where $i$ and $j$ are nondecreasing) and the goal is to say

---

3 This is basically the only place in the paper where we really use the fact that we have taken the category $\Delta$ and not some other category such as the semisimplicial category or a truncation of $\Delta$ using only objects $m$ for $m \leq m_0$. One can see for example that the statement 5.1 for 1-precats is no longer true if we try to replace $\Delta$ by the semisimplicial category throwing out the degeneracy maps—this example comes down to saying that the product of two free monoids on two sets of generators is not the free monoid on the product of the sets of generators.
that the composition along the various different paths is the same. The reason is that
when one changes the path by the smallest amount possible at a single square, that is
to say changing “up then over” to “over then up”, the composition doesn’t change. This
elementary step was done correctly in the original proof (see in v1 the statements that the
morphisms $A^x \to B^x$ and $A^x \to B^x$ are weak equivalences). Then one has to put these
elementary steps together to conclude the desired statement for the big rectangular grid.
This requires an inductive argument along the lines of what was done in v1 but in v1 the
induction wasn’t organized correctly and was based on a mistaken claim as pointed out
above.

So let’s rewrite things and hope for the best! Put

$$A := \Sigma^n m([m], 1^k) \times h(u, 1^j)$$

and

$$B := h(m, 1^k) \times h(u, 1^j).$$

Note that $A$ is a coproduct of things of the form $h(1, 1^k) \times h(u, 1^j)$. We want to show
that the morphism $A \to B$ is a weak equivalence. The precats $A$ and $B$ share the same
set of objects which we denote by $Ob$, equal to the set of pairs $(i, j)$ with $0 \leq i \leq m$
and $0 \leq j \leq u$. Suppose $S \subset Ob$ is a subset of objects. We denote by $A\{S\}$ (resp. $B\{S\}$) the
full sub-precat of $A$ (resp. $B$) whose object-set is $S$. By “full sub-precat” we mean that
for any sequence $x_0, \ldots, x_k$ in $S$,

$$A\{S\} \times_{(x_0, \ldots, x_k)} (x_0, \ldots, x_k) := A_{k/}(x_0, \ldots, x_k)$$

and the same for $B\{S\}$. We will use this for subsets $S$ of the form “notched sub-rectangle
plus a tail”. By a “sub-recatangle” we mean a subset of the form

$$S' = \{ (i, j) : 0 \leq i \leq m', 0 \leq j \leq u' \}$$

and by a “tail” we mean a subset of the form

$$S'' = \{ (i_k, j_k) : 0 \leq k \leq r \}$$

where $i_k \leq i_{k+1} \leq i_{k+1}$ and $j_k \leq j_{k+1} \leq j_k + 1$. A tail $S''$ that goes with a rectangle $S'$
as above, is assumed to have $i_0 = m', j_0 = u'$, $i_r = m$, $j_r = u$. In other words, the tail is
a path going from the upper corner of $S'$ to the upper corner of $Ob$, and the path goes by
steps of at most one in both the $i$ and the $j$ directions.

Finally, a “notched sub-rectangle” is a subset of the form

$$S' = \{ (i, j) : (0 \leq i \leq m' \text{ and } 0 \leq j \leq u' - 1) \text{ or } (v' \leq i \leq m' \text{ and } j = u') \}.$$
We call \((m', u', v')\) the parameters of \(S'\), and if necessary we denote \(S'\) by \(S'(m', u', v')\). Note that \(u' \geq 1\) here, and \(0 \leq v' \leq m'\). A rectangle with \(u' = 0\) may be considered as part of a tail; thus, modulo the initial case where all of \(S\) is a tail, which will be treated below, it is safe to assume \(u' \geq 1\). If \(v' = 0\) then \(S'\) is just the rectangle of size \(m' \times u'\). If \(v' = m'\) then \(S'\) is a rectangle of size \(m' \times (u' - 1)\), plus the first segment of a tail going from \((m', u' - 1)\) to \((m', u')\). Thus if \(S''\) is a tail from \((m', u')\) to \((m, u)\), we get that

\[
S'(m', u', 0) \cup S''
\]

is a rectangle of size \(m' \times u'\) plus a tail, whereas

\[
S'(m', u', m') \cup S''
\]

is a rectangle of size \(m' \times (u' - 1)\) plus a tail.

We prove by induction on \((m', u')\) that if \(R\) is of the form \(R = S' \cup S''\) for \(S'\) a rectangle of size \(m' \times u'\) and \(S''\) a tail from \((m', u')\) to \((m, u)\), then \(A\{R\} \to B\{R\}\) is a weak equivalence. We treat the case of \(m', u'\) and suppose that it is known for all strictly smaller rectangles (i.e. for \((m'', u'')\) \(\neq (m', u')\) with \(m'' \leq m'\) and \(u'' \leq u'\)) and all tails. In the current part of the induction we assume that \(u' \geq 1\). The case \(u' = 0\) (which is really the case of a tail \(S''\) going all the way from \((0, 0)\) to \((m, u)\)) will be treated below.

In particular, we know that

\[
A\{S'(m', u', m') \cup S''\} \to B\{S'(m', u', m') \cup S''\}
\]

is a weak equivalence (cf the above description of \(S'(m', u', m')\)). Now we treat the case where \(S\) is a notched rectangle plus tail of the form

\[
S = S'(m', u', v') \cup S''
\]

for \(0 \leq v' \leq m'\). For an \(S\) of this notched form, we claim again that \(A\{S\} \to B\{S\}\) is a weak equivalence. We prove this by descending induction on \(v'\), the initial case \(v' = m'\) being obtained above from the case of rectangles of size \(m' \times (u' - 1)\). Thus we may fix \(v' < m'\) and assume that it is known for

\[
\overline{S} = S'(m', u', v' + 1) \cup S'',
\]

in other words we may assume that \(A\{\overline{S}\} \to B\{\overline{S}\}\) is a weak equivalence.

We analyze how to go from \(\overline{S}\) to \(S\). Note that \(S\) has exactly one object more than \(\overline{S}\), the object

\[
x := (v', u').
\]
Let $S^x$ denote the subset of objects $(i, j) \in S$ such that either $(i, j) \leq x$ or $(i, j) \geq x$. Here we define the order relation by

$$(i, j) \leq (k, l) \iff i \leq k \text{ and } j \leq l.$$ 

With respect to this order relation, note that for a sequence of objects $(x_0, \ldots, x_p)$, we have that $B_p/(x_0, \ldots, x_p)$ is nonempty, only if $x_0 \leq x_1 \leq \ldots \leq x_p$. (The same remark holds a fortiori for $A$ because of the map $A \to B$.) In particular, if $(x_0, \ldots, x_p)$ is a sequence of objects of $S$ such that $B_p/(x_0, \ldots, x_p)$ is nonempty and such that some $x_a = x$ then all of the $x_b$ are in $S^x$.

Let $S^x = S \cup S^x$. We claim (*) that

$$A\{S\} = A\{S\} \cup^{A(S^x)} A\{S^x\},$$

and similarly that

$$B\{S\} = B\{S\} \cup^{B(S^x)} B\{S^x\}.$$ 

These are the statements that replace the faulty lines of the proof in version 1. To prove the claim, suppose $(x_0, \ldots, x_p)$ is a sequence of objects of $S$. It suffices to show that

$$B\{S\}_p/(x_0, \ldots, x_p)$$

is the pushout of

$$B\{S\}_p/(x_0, \ldots, x_p)$$

and

$$B\{S^x\}_p/(x_0, \ldots, x_p)$$

over

$$B\{S^x\}_p/(x_0, \ldots, x_p)$$

(the proof is the same for $A$, we give it for $B$ here).

If none of the objects $x_a$ is equal to $x$, then either the sequence stays inside $S^x$, in which case:

$$B\{S\}_p/(x_0, \ldots, x_p) = B\{S\}_p/(x_0, \ldots, x_p) = B\{S^x\}_p/(x_0, \ldots, x_p) = B\{S^x\}_p/(x_0, \ldots, x_p);$$

or else the sequence doesn’t stay inside $S^x$, in which case

$$B\{S\}_p/(x_0, \ldots, x_p) = B\{S\}_p/(x_0, \ldots, x_p)$$

but

$$B\{S^x\}_p/(x_0, \ldots, x_p) = B\{S^x\}_p/(x_0, \ldots, x_p) = \emptyset.$$ 

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In both of these cases one obtains the required pushout formula. On the other hand, if some \( x_a \) is equal to \( x \), then either the sequence doesn’t stay inside \( S^x \) in which case all terms are empty (cf the above remark), or else it stays inside \( S^x \) in which case

\[
B\{S\}_{p/}(x_0, \ldots, x_p) = B\{S^x\}_{p/}(x_0, \ldots, x_p)
\]

but

\[
B\{\overline{S}\}_{p/}(x_0, \ldots, x_p) = B\{\overline{S}^x\}_{p/}(x_0, \ldots, x_p) = \emptyset.
\]

Again one obtain the required pushout formula. This proves the claim (*).

We now note that both \( S^x \) and \( \overline{S}^x \) are of the form, a rectangle of size \( v' \times (u' - 1) \), plus a tail. The first step of the tail for \( S^x \) goes from \( (v', u' - 1) \) to \( x = (v', u') \). The next step goes to \( (v' + 1, u') \). The first step of the tail for \( \overline{S}^x \) goes from \( (v', u' - 1) \) directly to \( (v' + 1, u') \). Both tails continue horizontally from \( (v' + 1, u') \) to \( (m', u') \) and then continue as the tail \( S'' \) from there to \( (m, u) \). By our induction hypothesis, we know that

\[
A\{S^x\} \to B\{S^x\}
\]

and

\[
A\{\overline{S}^x\} \to B\{\overline{S}^x\}
\]

are weak equivalences. Recall from above that we also know that \( A\{\overline{S}\} \to B\{\overline{S}\} \) is a weak equivalence. These morphisms from full sub-precats of \( A \) to full sub-precats of \( B \) are all isomorphisms on sets of objects, so by the remark at the start of the proof of Theorem 5.1, we can use Lemma 3.2 for pushouts along these morphisms. A standard argument shows that the morphism

\[
A\{\overline{S}\} \cup^{A\{\overline{S}^x\}} A\{S^x\} \to B\{\overline{S}\} \cup^{B\{\overline{S}^x\}} B\{S^x\}
\]

is a weak equivalence. For clarity we now give this standard argument. First,

\[
A\{\overline{S}\} \to A\{\overline{S}\} \cup^{A\{\overline{S}^x\}} B\{\overline{S}^x\}
\]

is a trivial cofibration (which again induces an isomorphism on sets of objects). One can verify that the morphism

\[
A\{\overline{S}\} \cup^{A\{\overline{S}^x\}} B\{\overline{S}^x\} \to B\{\overline{S}\}
\]

is a cofibration. It is a weak equivalence by Lemma 3.8. Thus it is a trivial cofibration inducing an isomorphism on sets of objects. Similarly, the morphism

\[
B\{\overline{S}^x\} \cup^{A\{\overline{S}^x\}} A\{S^x\} \to B\{S^x\}
\]
is a trivial cofibration inducing an isomorphism on sets of objects. Thus the pushout morphism (pushing out by these two morphisms at once)

\[
(A \{S\} \cup^{A(S^x)} B \{S^x\}) \cup^{B(S^x)} (B \{S^x\} \cup^{A(S^x)} A \{S^x\})
\]

\[
\rightarrow B \{S\} \cup^{B(S^x)} B \{S^x\}
\]

is a weak equivalence. Finally, note that

\[
(A \{S\} \cup^{A(S^x)} B \{S^x\}) \cup^{B(S^x)} (B \{S^x\} \cup^{A(S^x)} A \{S^x\})
\]

\[
= (A \{S\} \cup^{A(S^x)} A \{S^x\}) \cup^{A(S^x)} B \{S^x\}
\]

so the morphism from \(A \{S\} \cup^{A(S^x)} A \{S^x\}\) to this latter, is also a weak equivalence. Putting these all together we have shown that the morphism

\[
A \{S\} = A \{S\} \cup^{A(S^x)} A \{S^x\}
\]

\[
\rightarrow B \{S\} \cup^{B(S^x)} B \{S^x\} = B \{S\}
\]

is a weak equivalence. This completes the proof of the inductive step for the descending induction on \(v\), so we obtain the result for \(v' = 0\), in which case \(S\) is a rectangle of size \(m' \times u'\) plus a tail; in turn, this gives the inductive step for the induction on \((m', u')\).

After all of this induction we are left having to treat the initial case \(u' = 0\), where all of \(S\) is a tail going from \((0, 0)\) to \((m, u)\). This part of the proof is exactly the same as in version 1: what we call the “tail” here corresponds to the sequence which was denoted \(x\) in version 1. If \(S\) is a tail, then \(A \{S\}\) and \(B \{S\}\) are 1-free ordered \(n\)-precats, so by Lemma 4.1 it suffices to check that for two adjacent objects \(x, y\) in the sequence corresponding to \(S\), the morphisms

\[
A_1/(x, y) \rightarrow B_1/(x, y)
\]

are weak equivalences. In fact these morphisms are isomorphisms. If \(x = (i, j)\) and \(y = (i, j + 1)\) then

\[
A_1/(x, y) = B_1/(x, y) = h(1^j);
\]

if \(x = (i, j)\) and \(y = (i + 1, j)\) then

\[
A_1/(x, y) = B_1/(x, y) = h(1^k);
\]

and if \(x = (i, j)\) and \(y = (i + 1, j + 1)\) then

\[
A_1/(x, y) = B_1/(x, y) = h(1^k) \times h(1^j).
\]
Thus the criterion of §4.1 implies that $A\{S\} \to B\{S\}$ is a weak equivalence. This completes the initial case of the induction.

Combining this initial step with the inductive step that was carried out above, we obtain the result in the case where $S = Ob$ is the whole rectangle of size $m \times u$; in this case $A\{S\} = A$ and $B\{S\} = B$, so we have completed the proof that

$$A = \Sigma^{n\mu}([m], 1^k) \times h(u, 1^j) \to h(m, 1^k) \times h(u, 1^j) = B$$

is a weak equivalence. This completes the proof of Theorem 5.1.

The above proof went basically along the same lines as the proof of version 1, but here we met all possible tails going from $(0, 0)$ to $(m, u)$ along the way in our induction, whereas in version 1 only some of the tails were met. This should have been taken as an indication that the proof in version 1 was incorrect. I would like again to thank R. Pellissier for an ongoing careful reading which turned up this problem. His reading has also turned up numerous other problems in the exposition or organisation of the argument (the reader has no doubt noticed!); however, I have chosen in the present version 2 to make only a minimalist correction of the above problem.

6. Proofs of the remaining lemmas and Theorem 3.1

We can assume 3.4, 3.5 and the corollary 3.6 for degree $n$ also.

Proof of 3.7: We have to prove that the morphism $f : \text{Cat}(A) \to \text{Cat}(\text{Cat}(A))$ is an equivalence. Note that this is not the same morphism as the standard inclusion, rather it is the morphism induced by $A \to \text{Cat}(A)$. In particular, 3.7 is not just an immediate corollary of 3.4. To obtain the proof, note that $\text{Cat}(A)$ is marked, so we have a morphism

$$r : \text{Cat}(\text{Cat}(A)) \to \text{Cat}(A)$$

of marked easy $n$-categories, inducing the identity on the standard map $i : \text{Cat}(A) \to \text{Cat}(\text{Cat}(A))$. The morphism $r$ is an equivalence because the standard map $i$ is an equivalence by 3.4. On the other hand, the morphism of $n$-precats $A \to \text{Cat}(A)$ induces the morphism $f$ of marked easy $n$-categories. We obtain a morphism $r \circ f$ of marked easy $n$-categories $\text{Cat}(A) \to \text{Cat}(A)$ extending the standard map $A \to \text{Cat}(A)$. By the universal property of $\text{Cat}(A)$, $r \circ f$ is the identity. Thus (applying our usual Lemma 3.8) the morphism $f$ is an equivalence.

Proof of 3.2: We first treat the following special case of 3.2. Say that a morphism of $n$-categories $A \to B$ is an iso-equivalence if it is an equivalence and an isomorphism on objects. This is equivalent to the condition that for all $m$, the morphism $A_m \to B_m$ is an equivalence of $n - 1$-categories.
Lemma 6.1 Suppose \( A \rightarrow B \) is an iso-equivalence of \( n \)-categories and \( A \rightarrow C \) is a morphism of \( n \)-categories. Then the morphism

\[
C \rightarrow \text{Cat}(B \cup^A C)
\]

is an equivalence (in fact, even an iso-equivalence).

Proof: By our inductive hypotheses, the morphisms

\[
C_m/ \rightarrow \text{Cat}(B_m/ \cup^{A_m/} C_m/)
\]

are equivalences of \( n-1 \)-categories. Setting \( D = B \cup^A C \) we have

\[
D_m/ = B_m/ \cup^{A_m/} C_m/
\]

and note that \( D_0 = C_0 \). Hence the morphism of \( n \)-precats

\[
C \rightarrow \text{Cat}_{\geq 1}(D)
\]

is an equivalence on the level of each \( C_m/ \). But the condition of being an \( n \)-category depends only on the equivalence type of the \( C_m/ \), in particular \( \text{Cat}_{\geq 1}(D) \) is an \( n \)-category (in this special case only—this is not a general principle!). Note in passing that the morphism

\[
C \rightarrow \text{Cat}_{\geq 1}(D)
\]

is an equivalence of \( n \)-categories. By Lemma 3.4 in degree \( n \), the morphism

\[
\text{Cat}_{\geq 1}(D) \rightarrow \text{Cat}(\text{Cat}_{\geq 1}(D))
\]

is an equivalence. On the other hand, by Lemma 3.6 in degree \( n \) the morphism

\[
\text{Cat}(D) \rightarrow \text{Cat}(\text{Cat}_{\geq 1}(D))
\]

is an equivalence. By Lemma 3.8 at the start of the proof of all of the lemmas, this shows that \( C \rightarrow \text{Cat}(D) \) is an equivalence. This proves Lemma 6.1. ///

The next lemma is the main special case which has to be treated by hand. This proof uses Corollary 5.2, which in turn is where we use the full simplicial structure of \( \Delta \). It seems likely (from some considerations in topological examples) that the following lemma would not be true if we looked only at functors from \( \Delta_{\leq k} \) to \( n \)-precats.

Lemma 6.2 Suppose \( \mathcal{T} \) is the category with two objects and exactly one isomorphism between them. Let \( 0 \) denote one of the objects. Then for any \( n \)-category \( A \) and object \( c \in A_0 \), if \( 0 \rightarrow A \) denotes the corresponding morphism, the push-out morphism

\[
A \rightarrow \text{Cat}(\mathcal{T} \cup^0 A)
\]

is an equivalence.
Proof: Let $T^2$ denote $T \times T$. There is a morphism $h : T^2 \rightarrow T \times T$ equal to the identity on $T \times \{0\}$ and on $\{0\} \times T$, and sending $T \times \{1\}$ to $(0, 1)$. Let $B := T \cup^0 A$. Then

$$B \times T = T^2 \cup \{0\} \times T \times T.$$

Using $h$ on the first part of this pushout we obtain a map

$$f : B \times T \rightarrow B \times T$$

such that $f|_{B \times \{0\}}$ is the identity and $f|_{B \times \{1\}}$ is the projection $B \rightarrow A$ obtained from the projection $T \rightarrow \{0\}$. By Corollary 5.2 the morphisms

$$i_0 : \text{Cat}(B) \times \{0\} \rightarrow \text{Cat}(B \times T)$$

and

$$i_1 : \text{Cat}(B) \times \{1\} \rightarrow \text{Cat}(B \times T)$$

are equivalences of $n$-categories. Next, the morphism $f$ induces a morphism

$$g : \text{Cat}(B \times T) \rightarrow \text{Cat}(B)$$

such that the composition with $i_0$ is the identity $\text{Cat}(B) \rightarrow \text{Cat}(B)$ and the composition with $i_1$ is the factorization $\text{Cat}(B) \rightarrow \text{Cat}(A) \rightarrow \text{Cat}(B)$. Looking at $g \circ i_0$ we conclude that $g$ is an equivalence of $n$-categories. Therefore (since $i_1$ is an equivalence) the composition $g \circ i_1$ is an equivalence of $n$-categories. Now we have morphisms $\text{Cat}(A) \rightarrow \text{Cat}(B)$ and $\text{Cat}(B) \rightarrow \text{Cat}(A)$ such that the composition in one direction is the identity, and the composition in the other direction is an equivalence of $n$-categories. This implies that the two morphisms are equivalences of $n$-categories by Lemma 3.8. ///

In preparation for the next corollary, we discuss a sort of “versal semi-interval” $J$. Ideally we would like to have an $n$-precat which is weakly equivalent to $\ast$, containing two objects 0 and 1 such that for any $n$-category (easy, perhaps) $A$ with two equivalent objects $a$ and $b$, there exists a morphism from our “interval” to $A$ taking 0 to $a$ and 1 to $b$. I didn’t find an easy way to make this construction. The problem is somewhat analogous to the problem of finding a canonical inverse for a homotopy equivalence, solved in a certain topological context in [23] but which seems quite complicated to put into action here in view of the fact that our $n$-category $A$ might not be fibrant (we don’t yet have the closed model structure!). Thus we will be happy with a cruder version. Let $J$ be the universal easy $n$-category with two objects 0 and 1 and a “marked inner equivalence” $u : 0 \rightarrow 1$. The quasi-inverse of $u$ will be denoted by $v$. The marking means a structure of choice of morphism whenever necessary for the definition of inner equivalence, as well as a choice of diagram (i.e. a partial marking in the sense defined at the start of the paper) whenever
necessary for things to make sense. In practice this means that we start with objects 0 and 1, add the morphisms \( u \) and \( v \), add the diagrams over \( 2 \in \Delta \) mapping to \((u, v)\) and \((v, u)\), and (letting \( w \) and \( y \) denote the compositions resulting from these diagrams) add (inductively by the same construction for \( n - 1 \)-categories) equivalences between \( w \) and \( e \) (resp. \( y \) and \( e \)) where \( e \) denote the identities. Let \( \mathcal{L} \subset \mathcal{J} \) be the full-sub-\( n \)-category whose object set is \( \{0\} \). The morphism \( \mathcal{L} \to \mathcal{J} \) is automatically an equivalence since it is an isomorphism on morphism \( n - 1 \)-categories and is essentially surjective since by construction \( 1 \in \mathcal{J} \) is equivalent to 0. By the universal property of \( \mathcal{J} \) we obtain a morphism \( \mathcal{J} \to \mathcal{L} \) sending \( u \) to \( e \) and \( v \) to \( w \), sending our 2-diagrams to degenerate diagrams in \( \mathcal{L} \) and sending our homotopies to the corresponding homotopies in \( \mathcal{L} \). The composition

\[
\mathcal{L} \to \mathcal{J} \to \mathcal{L}
\]

is the identity. On the other hand we have an obvious map \( \mathcal{J} \to \mathcal{I} \), so we obtain a map

\[
\mathcal{J} \to \mathcal{L} \times \mathcal{I}.
\]

This map is compatible with the inclusions of \( \mathcal{L} \) and hence is an equivalence of \( n \)-categories. It is also an isomorphism on objects so it is an iso-equivalence.

**Corollary 6.3** Let \( \mathcal{L} \subset \mathcal{J} \) be as above. Then for any \( n \)-category \( A \) and morphism \( \mathcal{L} \to A \), the push-out morphism

\[
A \to \text{Cat}(\mathcal{J} \cup^\mathcal{L} A)
\]

is an equivalence.

**Proof**: Let \( B = (\mathcal{L} \times \mathcal{I}) \cup^\mathcal{L} A \) and \( C = \mathcal{J} \cup^\mathcal{L} A \) The morphism \( \mathcal{J} \to (\mathcal{L} \times \mathcal{I}) \) is an iso-equivalence so it satisfies the hypothesis of \( \#3 \). By \( \#2 \) for \( n - 1 \)-precats, the morphism \( C \to B \) also satisfies the hypothesis of \( \#3 \). Therefore the morphism \( \text{Cat}(C) \to \text{Cat}(B) \) is an equivalence. It suffices to show that \( A \to \text{Cat}(B) \) is an equivalence. For this, note that

\[
\mathcal{L} \cup^0 \mathcal{I} \to \mathcal{L} \times \mathcal{I}
\]

is a weak equivalence by Lemma \( \#2 \). Similarly

\[
A \to A \cup^0 \mathcal{L}
\]

is a weak equivalence, and

\[
B = (A \cup^0 \mathcal{L}) \cup^{(\mathcal{L}, \mathcal{I})} \mathcal{L} \times \mathcal{I}
\]

so composing these two gives that \( A \to B \) is a weak equivalence. ///
Corollary 6.4 Suppose $A \to B$ is a cofibrant equivalence of $n$-categories such that the objects of $A$ form a subset of the objects of $B$ whose complement has one object. Then the push-out morphism

$$C \to \text{Cat}(B \cup^A C)$$

is an equivalence.

Proof: We may replace $B$ by $\text{Cat}(B)$ since the morphism $B \to \text{Cat}(B)$ is a sequence of standard pushouts, so the corresponding morphism on pushouts of $C$ is also a sequence of standard pushouts so the conclusion for $\text{Cat}(B)$ implies the conclusion for $B$ (by Lemma 3.8). Thus we may assume that $B$ is an easy $n$-category.

Let $A'$ be the full sub-$n$-category of $B$ consisting of the objects of $A$. The pushout of $C$ from $A$ to $A'$ is a weak equivalence by Lemma 6.1. Thus we may assume that $A = A'$.

Let $b$ denote the single new object of $B$. It is equivalent to an object $a \in A$. By the universal property of $J$ there is a morphism $\overline{J} \to B$ sending 0 to $a$ and 1 to $b$. Since $A$ is now a full sub-$n$-category of $B$, this morphism sends $\overline{L}$ to $A$. Let $E$ denote the push-out

$$E := \text{Cat}(A \cup^{\overline{L}} \overline{J}).$$

By the previous corollary, $A \to E$ is an equivalence. Our morphism $\overline{J} \to B$ gives a morphism

$$E \to B$$

(use the marking of $B$ to go from $\text{Cat}(B)$ back to $B$) and this is an equivalence since

$$A \to B \to \text{Cat}(B) \quad \text{and} \quad A \to E$$

are equivalences. But the morphism $E \to B$ induces an isomorphism on objects. Now we have

$$C \cup^A (A \cup^{\overline{L}} \overline{J}) = C \cup^{\overline{L}} \overline{J}$$

so

$$C \to \text{Cat}(C \cup^A (A \cup^{\overline{L}} \overline{J}))$$

is an equivalence by the previous corollary. It is obvious from the construction of $\text{Cat}$ (resp. $\text{BigCat}$) via pushouts, together with the reordering of these pushouts, that

$$\text{BigCat}(C \cup^A (A \cup^{\overline{L}} \overline{J})) = \text{BigCat}(C \cup^A \text{Cat}(A \cup^{\overline{L}} \overline{J})) = \text{BigCat}(C \cup^A E).$$

Thus (since taking $\text{BigCat}$ is equivalent to taking $\text{Cat}$ by Lemma 3.4—which we now know—and the reordering principle)

$$C \to \text{Cat}(C \cup^A E)$$
is an equivalence. Now
\[ \text{Cat}(C \cup^A E) \rightarrow \text{Cat}(C \cup^A B) \]
is an equivalence because \( E_{p/} \rightarrow B_{p/} \) is an equivalence so by 3.2 in degree \( n - 1 \),
\[ (C \cup^A E)_{p/} \rightarrow (C \cup^A B)_{p/} \]
is an equivalence and by Lemma 3.3 we get the desired statement. Combining, we get that
\[ C \rightarrow \text{Cat}(C \cup^A B) \]
is an equivalence.

\[ \text{Proof of Lemma 3.2:} \] Suppose \( A \rightarrow B \) is a cofibration of \( n \)-categories which is an equivalence. By applying the previous corollary inductively (adding one object at a time) we conclude that the push-out is an equivalence.

Finally we treat the case where \( A, B \) and \( C \) are only \( n \)-precats rather than \( n \)-categories. If \( \Sigma \rightarrow h \) is one of our standard pushout diagrams and if \( \Sigma \rightarrow A \) is a morphism then
\[ (B \cup^\Sigma h) \cup^{A \cup^\Sigma h} (C \cup^\Sigma h) = (B \cup^A C) \cup^\Sigma h. \]
This implies that
\[ \text{BigCat}(B) \cup^{\text{BigCat}(A)} \text{BigCat}(C) \]
is obtained by a collection of standard pushouts from \( B \cup^A C \), so in particular (by re-ordering)
\[ \text{BigCat}(\text{BigCat}(B) \cup^{\text{BigCat}(A)} \text{BigCat}(C)) = \text{BigCat}(B \cup^A C). \]
Now our hypothesis is that \( \text{BigCat}(A) \rightarrow \text{BigCat}(B) \) is an equivalence (note also that it is a cofibration since \( A \rightarrow B \) is a cofibration). By our proof of 3.2 for the case of \( n \)-categories (and the equivalence between \( \text{Cat} \) and \( \text{BigCat} \) which we now know by 3.4) we conclude that
\[ \text{BigCat}(C) \rightarrow \text{BigCat}(\text{BigCat}(B) \cup^{\text{BigCat}(A)} \text{BigCat}(C)) \]
is an equivalence, which is to say that
\[ \text{BigCat}(C) \rightarrow \text{BigCat}(B \cup^A C) \]
is an equivalence. Thus \( C \rightarrow B \cup^A C \) is a weak equivalence.

\[ \text{Remark:} \] The semi-interval \( \overline{J} \) we have constructed above is not contractible (i.e. equivalent to \( * \)). However for some purposes we would like to have such an object. We have the following fact (which is not used in the proof of Theorem 3.1 in degree \( n \)—but which we put here for expository reasons):
Proposition 6.5 There is an $n$-category $K$ such that $K \to \ast$ is an equivalence, together with objects $0, 1 \in K$ such that if $A$ is an $n$-category and if $a, b$ are two equivalent objects of $A$ then there is a morphism $K \to A$ sending $0$ to $a$ and $1$ to $b$.

Proof: Since this proposition is not used in degree $n$ in the proof of Theorem 3.1, we can apply Theorem 3.1. We say that $K$ is contractible if the morphism $K \to \ast$ is an equivalence. In view of the versal property of $J$, it suffices to construct a contractible $K$ with objects $0, 1$ and a morphism $K \to J$ sending $0$ to $0$ and $1$ to $1$. From the original discussion of $J$ we have an equivalence $J \to \mathcal{L} \times \mathcal{I}$. Using the closed model structure, factor the constant morphism as

$$\mathcal{I} \to M \to \mathcal{L} \times \mathcal{I}$$

into a composition of a trivial cofibration followed by a fibration. Note that $M$ is contractible. Set

$$K := J \times_{\mathcal{L} \times \mathcal{I}} M.$$  

The morphism $J \to \mathcal{L} \times \mathcal{I}$ is an isomorphism on objects, so for each $p$, $J_p/ \to (\mathcal{L} \times \mathcal{I})_{p/}$ is an equivalence of $n-1$-categories. Note also that $M_{p/} \to (\mathcal{L} \times \mathcal{I})_{p/}$ are fibrations. By the fact that weak equivalences are stable under fibrant pullbacks for $n-1$-categories (Theorem 6.7), we have that

$$K_{p/} = J_{p/} \times_{(\mathcal{L} \times \mathcal{I})_{p/}} M_{p/} \to M_{p/}$$

are weak equivalences, which in turn implies that $K \to M$ is a weak equivalence. In particular, $K$ is contractible. Since the morphism

$$M \to \mathcal{L} \times \mathcal{I}$$

is surjective on objects, there are objects $0, 1 \in K$ mapping to $0, 1 \in J$. This completes the construction. ///

Corollary 6.6 If $A$ is a fibrant $n$-precat then $A$ is automatically an easy $n$-category.

Proof: To show this it suffices to show that the morphisms

$$\varphi : \Sigma(M, [m], \langle k, k+1 \rangle) \to h(M, [m], 1^{k+1})$$

are trivial cofibrations. But $\varphi$ is a cofibration which is the first step in an addition of arbitrary pushouts of our standard morphisms $\varphi$, so by reordering of these pushouts the above inclusion extends to an isomorphism

$$\text{BigCat}(\Sigma(M, [m], \langle k, k+1 \rangle)) \cong \text{BigCat}(h(M, [m], 1^{k+1})).$$
Since $\text{Cat}(B) \to \text{BigCat}(B)$ is an equivalence by Lemma 3.4 applied to $\text{Cat}(B)$ plus reordering, this implies that the morphism $\varphi$ above is a trivial cofibration. By the definition of fibrant, $A$ must then satisfy the extension property to be an easy $n$-category. 

\[\begin{array}{c}
\end{array}\]

The proof of Theorem 3.1

We follow the proof of Jardine-Joyal that simplicial presheaves form a closed model category, as described in [13]. The proof is based on the axioms CM1–CM5 of [21].

Proof of CM1: The category of $n$-precats is a category of presheaves so it is closed under finite (and even arbitrary) direct and inverse limits.

Proof of CM2: Given composable morphisms

\[\begin{array}{c}
X \xrightarrow{f} Y \xrightarrow{g} Z,
\end{array}\]

if any two of $f$ or $g$ or $g \circ f$ are weak equivalences then the same two of $\text{Cat}(f)$, $\text{Cat}(g)$ or $\text{Cat}(g \circ f)$ are equivalences of $n$-categories in the sense of [27] and by Lemma 3.8 the third is also an equivalence; thus the third of our original morphisms is a weak equivalence.

Proof of CM3: This axiom says that “the classes of cofibrations, fibrations and weak equivalences are closed under retracts”. Jardine [13] doesn’t actually discuss the retract condition other than to say that it is obvious in his case, and a look at Quillen yields only the conclusion that the diagram on p. 5.5 of [20] for the definition of retract is wrong (that diagram has no content related to the word “retract”, it just says that one arrow is the composition of three others). Thus—since I am not sufficiently well acquainted with other possible references for this—we are reduced to speculation about what Quillen means by “retract”. Luckily enough, this speculation comes out to be non-speculative in the end. We say that $f : A \to B$ is a weak retract of $g : X \to Y$ if there is a diagram

\[\begin{array}{c}
A \xrightarrow{i} X \xleftarrow{j} A \\
\downarrow \downarrow \downarrow \\
B \xrightarrow{r} Y \xleftarrow{s} B
\end{array}\]

such that $r \circ i = 1_A$ and $s \circ j = 1_B$. There is also another notion which we call strong retract obtained by using the same diagram but with the arrows going in the opposite direction on the bottom. It turns out that if $f$ is a strong retract of $g$ then $f$ is also a weak retract of $g$: the strong retract condition can be stated as the condition $j \circ f \circ r = g$ (along with the retract conditions $ri = 1$ and $sj = 1$). Applying $s$ on the left we obtain $fr = sg$ and applying $i$ on the right we obtain $jf = gi$, these two conditions giving the weak retract condition. Thus for our purposes, if we can show that the classes of maps in question are closed under weak retract, this implies that they are also closed under strong retract, and we don’t actually care which of the two definitions was intended in [20]!
We start out, then, with the condition that \( f \) is a weak retract of \( g \) using the notations of the diagram given above. If \( g \) satisfies any lifting property then \( f \) satisfies the same lifting property, using the retractions. This shows that if \( g \) is a fibration then \( f \) is a fibration. Furthermore, if \( g \) is a cofibration then it is injective over any object \( M = (m_1, \ldots, m_k) \) with \( k < n \). It follows from the retractions that \( f \) satisfies the same injectivity conditions (one has the same diagram of retractions on the values of all of the presheaves over the object \( M \)). Thus \( f \) is a cofibration.

Suppose \( g \) is a weak equivalence, we would like to show that \( f \) is a weak equivalence. Replacing the whole diagram by \( \text{Cat} \) there is an object \( g \) of \( \text{Cat} \). Using the retractions. This shows that if \( g \) is a weak equivalence then it is contained in an \( n \)-precat which is weakly equivalent to \( A \), using the notations of \( \text{Cat} \) (i.e. a pair of elements \( v \in X(g,u), i(w) \)) such that their compositions, which are well defined in the truncation \( T_{n-1}X(g(u), g(u)) \) and \( T_{n-1}i(w), i(w) \) are the identities in these truncations). Applying the retractions \( r \) and \( s \) we obtain an element \( r(u) \in A_0 \) and an equivalence \( s(e) \) between \( fr(u) \) and \( si(w) = w \). This proves essential surjectivity of \( f \), completing the verification of CM3.

**Proof of CM4:** The first part of CM4 is exactly Lemma 3.2. The second part follows from the first by the same trick as used by Jardine (13) pp 64-65) and ascribed by him to Joyal [15].

**Proof of CM5(1):** For our situation, the cardinal \( \alpha \) refered to in Jardine is the countable infinite one \( \omega \). Suppose \( A \to C \) is a trivial cofibration. We claim that if \( B \) is an \( \omega \)-bounded subobject of \( C \) (by this we mean a sub-presheaf over \( \Theta^n \)) then there is an \( \omega \)-bounded subobject \( B_\omega \subset C \) as well as an \( \omega \)-bounded subobject \( A_\omega \subset A \times_B B_\omega \) such that \( B \subset B_\omega \subset C \) and such that \( A_\omega \to B_\omega \) is a trivial cofibration. (Note that in our situation cofibrations are not necessarily injective morphisms of presheaves, so \( A_\omega \) is not necessarily equal to \( A \times_B B_\omega \) the latter of which could be uncountable).

To prove the claim, note that for a given element in \( B_M \) for some \( M \), the statement that it is contained in an \( n \)-precat which is weakly equivalent to \( A \) can in principal be written out explicitly involving only a countable number of elements of various \( A_M' \) and \( B_M' \). Iterate this operation starting with all of the elements of \( B \) and repeatedly applying it to all of the new elements that are added. The iteration takes place a countable number of times, and each time we add on a countable union of countable objects. At the end we
arrive at $A_\omega \subset B_\omega$ which is an $\omega$-bounded trivial cofibration.

Using this claim, the rest of Jardine’s arguments of ([13], Lemmas 2.4 and 2.5) work and we obtain the statement that every morphism $f : X \rightarrow Y$ of $n$-precats can be factored as $f = p \circ i$ where $i$ is a trivial cofibration and $p$ is fibrant—[13] Lemma 2.5, which is CM5(1). Note that the only sentence in Jardine’s argument which needs further verification is the fact that filtered colimits of trivial cofibrations are again trivial cofibrations; and this holds in our case too.

Proof of CM5(2): We have to prove that any morphism $f$ may be factored as $f = q \circ j$ where $q$ is a fibrant weak equivalence and $j$ a cofibration.

It suffices to construct a factorization $f = q \circ j$ with $j$ a cofibration and $q$ a weak equivalence, for then we can apply CM5(1) to factor $q$ as a product of a trivial cofibration and a fibration, the latter of which is automatically also a weak equivalence by CM2. Thus we now search for $f = q \circ j$ with $q$ a weak equivalence and $j$ a cofibration.

The reader may wish to think about this in the case of 1-categories to get an idea of what is happening and to see why this part is actually easy modulo some small details: we multiply the number of objects in each isomorphism class in the target category to have the morphism injective on the sets of objects.

If $f : A \rightarrow B$ is a morphism of $n$-precats then we define a canonical factorization $A \rightarrow N(A, B) \rightarrow B$ in the following way. Let $L(A)$ denote the 1-category (considered as an $n$-category) whose set of objects is equal to $A_0$ and which has exactly one morphism between any pair of objects. Note that $L(A) \rightarrow *$ is a weak equivalence. The tautological map $A_0 \rightarrow L(A)_0$ lifts to a unique map of $n$-precats $t : A \rightarrow L(A)$. Set $N(A, B) := L(A) \times B$ with the diagonal map $(t, f) : A \rightarrow N(A, B)$ and the second projection $p : N(A, B) \rightarrow B$. Note that $p$ is a weak equivalence (by an appropriate generalization of Corollary 5.2) and $(t, f)$ is injective on objects.

Now suppose by induction that we have constructed for every morphism $f' : A' \rightarrow B'$ of $n-1$-precats a factorization $A' \rightarrow M(A', B') \rightarrow B'$ as a composition of a weak equivalence and a cofibration, functorial in $f'$.

(To start the induction for $n = 0$ we set $M(A', B') := B'$ recalling that all morphisms are cofibrations in this case.)

Suppose $f : A \rightarrow C$ is a morphism of $n$-precats such that $A_0 \hookrightarrow C_0$ is injective. Define a presheaf on $\Delta \times \Theta^{n-1}$ denoted $P(A, C)$, with factorization $A \rightarrow P(A, C) \rightarrow C$ as follows. Put

$$P(A, C)_{p/} := M(A_{p/}, C_{p/}).$$

By functoriality this is a functor from $\Delta^0$ to $n-1$-precats, and we have a factorization

$$A_{p/} \rightarrow P(A, C)_{p/} \rightarrow C_{p/}.$$

The second morphisms in the factorization are equivalences, and the first morphisms are cofibrations. The only problem is that $P(A, C)_{0/}$ is not a set: it is an $n$-category which
is equivalent to the set $C_0$.

For any $p$ there is a morphism $\psi_p : P(A, C)_0/ \to P(A, C)_p/ \to P(A, C)_{p/}$ which, because it is a section of any one of the morphisms back to 0, is a cofibration and in fact even injective in the top degree. If $p \to q$ is a morphism in $\Delta$ then $P(A, C)_q/ \to P(A, C)_p/$ composed with $\psi_q$ is equal to $\psi_p$. Hence if we set

$$Q(A, C)_p/ := P(A, C)_p/ \cup P(A, C)_0/ \cap C_0$$

then $Q(A, C)_p/$ is functorial in $p \in \Delta$. Now $Q(A, C)_0/ = C_0$ is a set rather than an $n-1$-precat so $Q(A, C)$ descends to a presheaf on $\Theta^n$. We have a morphism $A \to Q(A, C)$ projected from the above morphism into $P(A, C)$. We also have a morphism $Q(A, C) \to C$ because the composed morphism $P(A, C)_0/ \to C_0$ is a weak equivalence and $P(A, C)_0/ \to P(A, C)_{p/}$ a cofibration imply (inductively using the closed model structure for $n-1$-precats) that $P(A, C)_{p/} \to Q(A, C)_p/$ is a weak equivalence. Now $P(A, C)_{p/} \to C_{p/}$ being a weak equivalence implies that $Q(A, C)_{p/} \to C_{p/}$ is a weak equivalence. This proves the claim.

Finally we claim that $A \to Q(A, C)$ is a cofibration. It suffices to prove that the $A_{p/} \to Q(A, C)_{p/}$ are cofibrations. We know by the inductive hypothesis that $A_{p/} \to P(A, C)_{p/}$ are cofibrations. By the pushout definition of $Q(A, C)_{p/}$ and using the fact that $P(A, C)_0/$ is a sub-presheaf of $P(A, C)_{p/}$, it suffices to prove that the map

$$A_{p/} \times P(A, C)_{p/} P(A, C)_0/ \to C_0$$

is cofibrant. In fact we show below that for any $M$ of length $< n-1$,

$$A_{p,M} \times P(A, C)_{p,M} P(A, C)_{0,M} = A_0 \subset A_{p,M}$$

which implies what we want, since we have assumed that $A_0 \to C_0$ is injective. Note that the notation $P(A, C)_{0,M}$ means $(P(A, C)_0/)_{M}$, and we don’t have in this case that this is a constant $n-1$-category so the usual rule saying that $P(A, C)_{0,M}$ should be equal to $P(A, C)_{0,0}$ doesn’t apply.

Fix any one of the maps $e : p \to 0 \to p$. This gives a map $A_{p/} \to A_{p/}$ whose image is automatically $A_0$. This implies that the fixed subsheaf of the endomorphism $e$ is equal to $A_0$. The endomorphism acts compatibly on $P(A, C)_{p/}$ and the fixed point subsheaf there
is $P(A, C)_{0/}$. For any $M$ of length $< n - 1$ we have an inclusion $A_{p, M} \hookrightarrow P(A, C)_{p, M}$. This is compatible with the endomorphisms $e$ on both sides, so the intersection of $A_{p, M}$ with the fixed point set $P(A, C)_{0, M} \subset P(A, C)_{p, M}$ is the fixed point set $A_0 \subset A_{p, M}$. This shows the statement of the previous paragraph.

This completes the proof that $A \to Q(A, C) \to C$ is a factorization of the desired type, when $A_0 \to C_0$ is injective on objects. Note also that $Q(A, C)$ is functorial in the morphism $A \to C$. Suppose now that $A \to B$ is any morphism. We put

$$M(A, B) := Q(A, N(A, B)).$$

We have the factorization $A \to N(A, B) \to B$ with the first arrow injective on objects and the second arrow a weak equivalence, discussed at the start of the proof. The first arrow is then factored as $A \to M(A, B) \to N(A, B)$ with the first arrow a cofibration and the second arrow a weak equivalence. The factorization $A \to M(A, B) \to B$ therefore has the desired properties, and furthermore it is functorial in $A \to B$ (this is needed in order to continue with the induction on $n$). This completes the proof of CM5(2).

We refer to [20] for all of the consequences of Theorem 3.1. Recall also that a closed model category is said to be proper if it satisfies the following two axioms:

**Pr(1)** If $A \to B$ is a weak equivalence and $A \to C$ a cofibration then $C \to B \cup^A C$ is a weak equivalence;

**Pr(2)** If $B \to A$ is a weak equivalence and $C \to A$ a fibration then $B \times_A C \to C$ is a weak equivalence.

**Theorem 6.7** The closed model category $PC_n$ satisfies axiom **Pr(1)**; and it satisfies axiom **Pr(2)** for equivalences $B \to A$ between $n$-categories; however it doesn’t satisfy axiom **Pr(2)** in general.

**Proof:** We will prove stability of weak equivalences under coproducts. Suppose $A \to B$ is a cofibration, and suppose $A \to C$ is a weak equivalence. We would like to show that $B \to B \cup^A C$ is a weak equivalence. For this we use a version of the “mapping cone”. Recall that $\overline{T}$ is the category with two isomorphic objects 0, 1 and no other morphisms. The morphism $B \times \{1\} \to B \times \overline{T}$ is a trivial cofibration, so

$$B \cup^A C \to D := (B \cup^A C) \cup^{B \times \{1\}} B \times \overline{T}$$

is a trivial cofibration. It follows that the projection $D \to B \cup^A C$ deduced from $B \times \overline{T} \to B$ is a weak equivalence. Let

$$E := (B \times \{0\}) \cup^{A \times \{0\}} A \times \overline{T}$$

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and note that $B \times \{0\} \to E$ is a weak equivalence (since it is pushout by the trivial cofibration $A \times \{0\} \to A \times T$) hence $E \to B \times T$ is a trivial cofibration. Thus the morphism
\[ E \cup^{A \times \{1\}} C \to B \times T \cup^{A \times \{1\}} C \]
is a trivial cofibration. But note that $B \times T \cup^{A \times \{1\}} C = D$ so
\[ E \cup^{A \times \{1\}} C \to D \]
is a weak equivalence. Finally,
\[ E \cup^{A \times \{1\}} C = B \times \{0\} \cup^{A \times \{0\}} (A \times T \cup^{A \times \{1\}} C) \]
and the morphism
\[ A \times \{0\} \to A \times T \cup^{A \times \{1\}} C \]
is a weak equivalence because it projects to $A \to C$ which is by hypothesis a weak equivalence. Therefore the map
\[ B \times \{0\} \to E \cup^{A \times \{1\}} C \]
is a weak equivalence, and from above $B \times \{0\} \to D$ is a weak equivalence. Following by the projection $D \to B \cup^A C$ which we have seen to be a weak equivalence, gives the standard map $B \to B \cup^A C$ which is therefore a weak equivalence. This proves the first half of properness.

We now prove the second statement, proceeding as usual by induction on $n$. Factoring $B \to A$ into a cofibration followed by a fibration and treating the fibration, we can assume that the morphism is a cofibration (note that a fibration which is a weak equivalence, over an $n$-category, is again an $n$-category so the hypotheses are preserved). Let $A' \subset A$ be the full sub-$n$-category consisting of the objects which are in the image of $B_0$. Let $C' := A' \times_A C$. The morphism $B \to A'$ is an iso-equivalence so the $B_p/ \to A'_p/$ are equivalences of $n-1$-categories. The morphisms $C'_{p/} \to A'_{p/}$ are fibrant, so by the inductive hypothesis
\[ (C' \times_{A'} B)_{p/} = C'_{p/} \times_{A'_{p/}} B_{p/} \to C'_{p/} \]
are equivalences. This implies that
\[ C \times_A B = C' \times_{A'} B \to C' \]
is an equivalence. Now $C'$ is a full sub-$n$-category of $C$ (meaning that for any objects $x_0, \ldots, x_m$ of $C'$ the morphism $C'_m/(x_0, \ldots, x_m) \to C_m/(x_0, \ldots, x_m)$ is an isomorphism), so to prove that $C' \to C$ is an equivalence it suffices to prove essential surjectivity. Suppose $x$ is an object of $C$. It projects to an object $y$ in $A$ which is equivalent to an object $y'$
coming from $A'$. By 5.3 there is a morphism $K \to A$ sending 0 to $y$ and 1 to $y'$. The object $x$ provides a lifting to $C$ over $\{0\}$, so by the condition that $C \to A$ is fibrant there is a lifting to $K \to C$ sending 0 to $x$ and 1 to an object lying over $y'$. In particular 1 goes to an object of $C'$. This shows that $x$ is equivalent to an object of $C'$, the essential surjectivity we needed.

We now sketch an example showing why axiom $\Pr(2)$ can’t be true in general. Let $A$ be the category $I^{(2)}$ with objects 0, 1, 2 and one morphism $i \to j$ for $i \leq j$ ($i, j = 0, 1, 2$). Let $B$ be the sub-1-precat obtained by removing the morphism $0 \to 2$ (it is the pushout of two copies of $I$ over the object 1). The morphism $B \to A$ is a weak equivalence. Let $C$ be a 1-category with three objects $x_0, x_1, x_2$ and morphisms from $x_i$ to $x_j$ only when $i \leq j$. There is automatically a unique morphism $C \to A$ sending $x_i$ to $i$. One can see that this morphism is fibrant. We can choose $C$ so that the composition morphism $C_1(x_0, x_1) \times C_1(x_1, x_2) \cong C_2(x_0, x_1, x_2) \to C_1(x_0, x_2)$ is not an isomorphism. Let $D := C\text{at}(B \times_A C)$. There is a unique morphism $D \to C$ extending the second projection morphism, and this morphism takes $D_1(x_0, x_1)$ (resp. $D_1(x_1, x_2)$) isomorphically to $C_1(x_0, x_1)$ (resp. $C_1(x_1, x_2)$). However, the composition morphism for $D$ is an isomorphism

$$D_1(x_0, x_1) \times D_1(x_1, x_2) \xrightarrow{\text{comp}} D_1(x_0, x_2).$$

Thus the morphism $D_1(x_0, x_2) \to C_1(x_0, x_2)$ is not an isomorphism; thus $D \to C$ is not an equivalence and $B \times_A C \to C$ is not a weak equivalence.

As one last comment in this section we note the following potentially useful fact.

**Lemma 6.8** If $f : A \to A'$ is a fibrant morphism of $m$-precats then it is again fibrant when considered as a morphism of $n$-precats for $n \geq m$.

**Proof:** Suppose $m < n$. Define the brutal truncation denoted $\beta\tau_{\leq m}$ from $n$-precats to $m$-precats as follows. If $B$ is an $n$-precat then put

$$\beta\tau_{\leq n-1}(B)_M := B_M$$

for $M = (m_1, \ldots, m_k)$ with $k < m$ whereas for $M$ of length $m$ put

$$\beta\tau_{\leq n-1}(B)_M := B_M / \langle B_{M,1} \rangle$$

where $\langle B_{M,1} \rangle$ denotes the equivalence relation on $B_M$ generated by the image of $B_{M,1} \to B_M \times B_M$. This should not be confused with the “good” truncation operation $T^{n-m}$ of [27], as in general they will not be the same (however they are equal in the case of

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n-groupoids). We claim that brutal truncation is compatible with the operation \(\text{BigCat} \), that is

\[
\text{BigCat}(\beta \tau \leq_m B) = \beta \tau \leq_m (\text{BigCat}(B)).
\]

To prove this claim, we note two things:

1. that if \(\Sigma \to h\) is one of our standard cofibrations for \(n\)-precats then \(\beta \tau \leq_m \Sigma \to \beta \tau \leq_m h\) is a standard cofibration for \(m\)-precats; and
2. that any map \(\beta \tau \leq_m \Sigma \to \beta \tau \leq_m B\) comes from a map \(\Sigma \to \text{BigCat}(B)\) or—in the top degree case—at least from a map \(\Sigma \to \text{BigCat}(B)\).

By reordering we find that the two sides of the above equation are the same, which gives the claim. Next we claim that brutal truncation preserves weak equivalences. From the previous claim it suffices to note that it preserves equivalences of \(n\)-categories, and this follows from the fact that brutal truncation of 1-categories takes equivalences to isomorphisms of sets.

Finally, it is immediate from the definitions that brutal truncation takes cofibrations of \(n\)-precats to cofibrations of \(m\)-precats (using of course the fact that there is no injectivity on the top degree morphisms for cofibrations of \(m\)-precats).

Suppose \(A\) is an \(m\)-precat, and let \(\text{Ind}^n_m(A)\) denote \(A\) considered as an \(n\)-precat (for this we simply set \(\text{Ind}^n_m(A)_{M,M'} := A_M\) for \(M\) of degree \(m\) and any \(M'\) or for \(M\) of degree \(< m\) and empty \(M'\)). Then (speaking of absolute \(\text{Hom}\) here rather than internal \(\text{Hom}\) as in the next section)

\[
\text{Hom}(\beta \tau \leq_m B, A) = \text{Hom}(B, \text{Ind}^n_m(A)),
\]

in other words \(\beta \tau \leq_m\) and \(\text{Ind}^n_m\) are adjoint functors.

We can now prove the lemma. If \(f\) is a fibrant morphism of \(m\)-precats and \(B \to C\) is a trivial cofibration of \(n\)-precats then \(\beta \tau \leq_m B \to \beta \tau \leq_m C\) is a trivial cofibration of \(m\)-precats, so \(f\) has the lifting property for this latter. By adjointness \(\text{Ind}^n_m(f)\) has the lifting property for \(B \to C\). Therefore \(\text{Ind}^n_m(f)\) is fibrant.

### 7. Internal \(\text{Hom}\) and \(n\text{CAT}\)

Recall the result of Corollary 5.3: that direct product with any \(n\)-precat preserves weak equivalences. Direct product also preserves cofibrations, so it preserves trivial cofibrations. This property is not a standard property of any closed model category, it is one of the nice things about our present situation which allows us to obtain the right thing by looking at internal \(\text{Hom}\) of \(n\)-precats.

**Theorem 7.1** Suppose \(A\) is an \(n\)-precat and \(B\) is a fibrant \(n\)-precat. Then the internal \(\text{Hom}(A,B)\) of presheaves over \(\Theta^n\) is a fibrant easy \(n\)-category. Furthermore if \(B' \to B\) is
a fibrant morphism then $\text{Hom}(A, B') \to \text{Hom}(A, B)$ is fibrant. Similarly if $A \to A'$ is a cofibration and if $B$ is fibrant then $\text{Hom}(A', B) \to \text{Hom}(A, B)$ is fibrant.

Proof: Note that it suffices to prove that $\text{Hom}(A, B)$ is fibrant, for 6.6 then shows that it is an easy $n$-category. A morphism $S \to \text{Hom}(A, B)$ is the same thing as a morphism $S \times A \to B$. Suppose $S \to T$ is a trivial cofibration. Then $S \times A \to T \times A$ is a trivial cofibration. It follows immediately from the definition of $B$ being fibrant that any map $S \times A \to B$ extends to a map $T \times A \to B$. Thus $\text{Hom}(A, B)$ is fibrant. Similarly if $B' \to B$ is fibrant then any map $T \times A \to B$ with lifting $S \times A \to B'$ admits a compatible lifting $T \times A \to B$. Thus $\text{Hom}(A, B') \to \text{Hom}(A, B)$ is fibrant.

Suppose $A \to A'$ is cofibrant, and $B$ fibrant. We show that $\text{Hom}(A', B) \to \text{Hom}(A, B)$ satisfies the lifting property to be fibrant. Say $S \to S'$ is a trivial cofibration, and suppose we have maps $S' \to \text{Hom}(A, B)$ and lifting $S \to \text{Hom}(A', B)$. These are by definition maps $S' \times A \to B$ and $S \times A' \to B$ which agree over $S \times A$. These give a morphism $f : S \times A' \cup^{S \times A} S' \times A \to B$.

The morphism $g : S \times A' \cup^{S \times A} S' \times A \to S' \times A'$ is a cofibration. Lemma 3.2 applied to the trivial cofibration $S \times A \to S' \times A$ implies that the morphism $S \times A' \to S \times A' \cup^{S \times A} S' \times A$ is a weak equivalence. On the other hand the morphism $S \times A' \to S' \times A'$ is a weak equivalence by 3.3, so by Lemma 3.3 the morphism $g$ is a weak equivalence. Thus the fact that $B$ is fibrant means that our morphism $f$ extends to a morphism $S' \times A' \to B$, and this gives exactly the desired lifting property for the last statement of the theorem. //

Lemma 7.2 Suppose $A \to A'$ is a weak equivalence, and $B$ fibrant. Then $\text{Hom}(A', B) \to \text{Hom}(A, B)$ is an equivalence of $n$-categories.

If $B \to B'$ is an equivalence of fibrant $n$-precats then $\text{Hom}(A, B) \to \text{Hom}(A, B')$ is an equivalence.

Suppose $A \to B$ and $A \to C$ are cofibrations. Then

$$\text{Hom}(B \cup^A C, D) = \text{Hom}(B, D) \times_{\text{Hom}(A, D)} \text{Hom}(C, D).$$

Proof: The last statement is of course immediate, because for any $S$ we have $(B \cup^A C) \times S = (B \times S) \cup^{(A \times S)} (C \times S)$. We treat the other statements.

Suppose first that $A \to A'$ is a trivial cofibration. Suppose that $S \to T$ is any cofibration. Suppose we have maps $T \to \text{Hom}(A, B)$ lifting over $S$ to $S \to \text{Hom}(A', B)$. 52
We claim that the lifting extends to $T$; then the characterization of weak equivalences in ([20] §5, Definition 1, Property M6, part (c)) will imply that $\text{Hom}(A', B) \to \text{Hom}(A, B)$ is a weak equivalence. To prove the claim, note that our data correspond to a morphism

$$T \times A \cup^{S \times A} S \times A' \to B.$$  

The morphism $S \times A \to S \times A'$ is a trivial cofibration, so the morphism

$$T \times A \to T \times A \cup^{S \times A} S \times A'$$

is a trivial cofibration, and since $T \times A \to T \times A'$ is a weak equivalence we get that the morphism

$$T \times A \cup^{S \times A} S \times A' \to T \times A'$$

is a trivial cofibration. The fibrant property of $B$ implies that our map extends to a map $T \times A' \to B$, so we get the required lifting to $T \to \text{Hom}(A', B)$. This implies that $\text{Hom}(A', B) \to \text{Hom}(A, B)$ is a fibrant weak equivalence.

Next we treat the case of any weak equivalence $A \to A'$. Let $C$ be the $n$-precat pushout of $A \times 0 \to A \times T$ and $A \times 0 \to A' \times 0$. Since $0 \to T$ is a trivial cofibration, the various morphisms

$$A \hookrightarrow C, \quad A' \hookrightarrow C, \quad C \to A'$$

(the first sending $A$ to $A \times 1$, the second sending $A'$ to $A' \times 0$ and the third coming from the projection $A \times T \to A'$) are all weak equivalences. We have a composable pair of morphisms

$$\text{Hom}(A', B) \to \text{Hom}(C, B) \to \text{Hom}(A', B)$$

composing to the identity, and where the second arrow is an equivalence by the previous paragraph since $A' \to C$ is a trivial cofibration. Therefore the first arrow is an equivalence. Next, the morphism $\text{Hom}(C, B) \to \text{Hom}(A, B)$ obtained from the trivial cofibration $A \to C$ (going to $A \times 1$) is an equivalence, so the composed map $\text{Hom}(A', B) \to \text{Hom}(A, B)$ is an equivalence. This is the map induced by our original $A \to A'$. This completes the proof of the first part of the lemma.

We now turn to the second part and treat first a fibrant weak equivalence $B \to B'$. Note first that such a morphism satisfies the lifting property for any cofibrations (this is the other half of CM4 which comes from Joyal’s trick). We prove that $\text{Hom}(A, B) \to \text{Hom}(A, B')$ satisfies lifting for any cofibration (which as above implies that it is a fibrant weak equivalence). Suppose $S \to T$ is a cofibration and $T \to \text{Hom}(A, B')$ is a map with lifting over $S$ to a map $S \to \text{Hom}(A, B)$. These correspond to maps $T \times A \to B'$ and lifting to $S \times A \to B$. The morphism $S \times A \to T \times A$ is a cofibration so by the lifting property of $B \to B'$ for any cofibration, there is a lifting to $T \times A \to B$ compatible
with the given map on $S$. This establishes the necessary lifting property to conclude that $\text{Hom}(A, B) \to \text{Hom}(A, B')$ is a fibrant equivalence.

Next suppose that $i : B \to B'$ is a trivial cofibration of fibrant $n$-precats. The lifting property for $B$ lets us choose a retraction $r : B' \to B$ such that $ri = 1_B$. Let $p : B' \cup^B B' \to B' \to B'$ be the projection which induces the identity on both of the components $B'$. Note that $B' \to B' \cup^B B'$ is a trivial cofibration by 3.2 so the projection $p$ is a weak equivalence (using 3.8). Choose a factorization

$$B' \cup^B B' \to P \to B'$$

with the first morphism cofibrant and the second morphism fibrant (whence $P$ fibrant itself); and both morphisms weak equivalences. Let $q : B' \cup^B B' \to B'$ be the morphism inducing the retraction $r$ on the first copy of $B'$ and the identity on the second copy. Since $B'$ is fibrant this extends to a morphism we again denote $q : P \to B'$. The result of the previous paragraph implies that the morphism

$$p : \text{Hom}(A, P) \to \text{Hom}(A, B')$$

is an equivalence, which implies that either of the two morphisms

$$j_0, j_1 : \text{Hom}(A, B') \to \text{Hom}(A, P)$$

(referring to the two inclusions $j_0, j_1 : B' \to P$) are equivalences. Now we have that the composition

$$\text{Hom}(A, B') \xrightarrow{j_1} \text{Hom}(A, P) \xrightarrow{q} \text{Hom}(A, B')$$

is the morphism induced by $qj_1 = 1_{B'}$ thus it is the identity. The fact that the morphism induced by $j_1$ (the first of the above pair) is an equivalence implies that the morphism induced by $q$ (the second in the above sequence) is an equivalence. But since the morphism induced by $j_0$ is an equivalence, we get that the morphism induced by $qj_0 = ir$ is an auto-equivalence of $\text{Hom}(A, B')$. The morphism induced by $ri = 1_B$ is of course the identity. The last part of Lemma 3.8 now implies that the morphism

$$i : \text{Hom}(A, B) \to \text{Hom}(A, B')$$

is an equivalence. This proves the statement in case of a trivial cofibration.

Finally note that any equivalence of $n$-categories $B \to B'$ decomposes as a composition $B \to C \to B'$ where the first arrow is a trivial cofibration and the second a fibration and weak equivalence. Note that $C$ is fibrant since by hypothesis $B'$ is fibrant. Thus our two previous discussions apply to give that the two morphisms

$$\text{Hom}(A, B) \to \text{Hom}(A, C) \to \text{Hom}(A, B')$$

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are equivalences, their composition is therefore an equivalence. This completes the proof of the first paragraph of Lemma 7.2.

For any fibrant $n$-categories $A$, $B$ and $C$ we have composition morphisms

$$\text{Hom}(A, B) \times \text{Hom}(B, C) \to \text{Hom}(A, C),$$

which are associative.

Define a simplicial $n$-category $n\text{CAT}$ by setting $n\text{CAT}_0$ equal to a set of representatives for isomorphism classes of fibrant $n$-categories, and by setting

$$n\text{CAT}_m(A_0, \ldots, A_m) := \text{Hom}(A_0, A_1) \times \ldots \times \text{Hom}(A_{m-1}, A_m),$$

with simplicial structure given by the above compositions. Since $n\text{CAT}_0$ is a set considered as $n$-precat, this simplicial $n$-precat (presheaf on $\Delta \times \Theta^n$) descends to a presheaf on $\Theta^{n+1}$, in other words it is an $n+1$-precat. The composition gives the necessary conditions in the first degree, and in higher degrees the fact that $\text{Hom}(A, B)$ are $n$-categories completes what we need to know to conclude that $n\text{CAT}$ is an $n+1$-category.

If $A$ and $B$ are $n$-categories but not necessarily fibrant then let $\text{Fib}(A)$ and $\text{Fib}(B)$ be their fibrant replacements (given by the above construction for Theorem 3.1 for example). We call these the fibrant envelopes. The “right” $n$-category of morphisms from $A$ to $B$ is $\text{Hom}(\text{Fib}(A), \text{Fib}(B))$. We will sometimes use the notation

$$HOM(A, B) := \text{Hom}(\text{Fib}(A), \text{Fib}(B)).$$

We obtain an $n+1$-category equivalent to $n\text{CAT}$ by taking all $n$-categories as objects and taking the $HOM(A, B)$ as morphism $n$-categories.

**Question:** Describe the fibrant envelope of the $n+1$-category $n\text{CAT}$. This would be important if one wants to consider weak morphisms $A \to n\text{CAT}$ as families of $n$-categories indexed by $A$ in a meaningful way.

We have almost proved Conjecture ([27] between 1.3.6 and 1.3.7) on the existence of $n\text{CAT}$. We just have to check that the truncation of $n\text{CAT}$ down to a 1-category is equivalent to the localization of the category of $n$-categories by the subcategory of morphisms which are equivalences.

In Corollary 3.9 above, we have seen that the localization in question is equal to the localization of $PC_n$ by the weak equivalences. We now know that $PC_n$ is a closed model category, and Quillen shows in this case that the $Hom$ in the localized category is equal to the set of homotopy classes of morphisms between fibrant and cofibrant objects. In our (second) definition above, we took $n\text{CAT}$ to be the category of fibrant (and automatically cofibrant) $n$-precats. The $Hom$ in the localized category is thus the set of homotopy classes of maps. On the other hand, the truncation of $n\text{CAT}$ down to a 1-category is obtained
by replacing the $\text{Hom}(A,B)$ $n$-categories by their sets of equivalence classes of objects. Thus, to prove the conjecture we simply need to show that for $A$ and $B$ fibrant, the set of equivalence classes of objects in the $n-1$-category $\text{Hom}(A,B)$ is equal to the set of homotopy classes of maps from $A$ to $B$. Note that the objects of $\text{Hom}(A,B)$ are again just the maps from $A$ to $B$ so we are reduced to showing the following lemma.

**Lemma 7.3** If $A$ and $B$ are fibrant $n$-precats then two morphisms $f,g : A \to B$ are homotopic in the sense of [20] if and only if the corresponding elements of the $n$-category $\text{Hom}(A,B)$ are equivalent.

**Proof:** Suppose $f$ and $g$ are homotopic ([20] p. 0.2). Then there is an object $A'$ with morphisms $i, j : A \to A'$ each inducing a weak equivalence, with a projection $p : A' \to A$ such that the compositions $pi$ and $pj$ are the identity, and a morphism $h : A' \to B$ such that $hi = f$ and $hj = g$. We may assume that $A'$ is fibrant. Then we obtain pullback morphisms on the $\text{Hom}$ $n$-categories and in particular, two morphisms $i^*, j^* : \text{Hom}(A',B) \to \text{Hom}(A,B)$ and a morphism $p^* : \text{Hom}(A,B) \to \text{Hom}(A',B)$ which are weak equivalences by Lemma 7.2. These induce isomorphisms on the sets of equivalence classes which we denote $T^n\text{Hom}(A,B)$ etc., so we have

$$T^n i^*, T^n j^* : T^n \text{Hom}(A',B) \cong T^n \text{Hom}(A,B)$$

and

$$T^n p^* : T^n \text{Hom}(A,B) \cong T^n \text{Hom}(A',B).$$

Here, as before, we have that $T^n i^* \circ T^n p^*$ and $T^n j^* \circ T^n p^*$ are equal to the identity. This implies that $T^n i^* = T^n j^*$ and hence that their applications to the class of $h$ give the same equivalence class. The results are respectively the classes of $f$ and of $g$, hence $f$ is equivalent to $g$.

Conversely suppose $f$ and $g$ are equivalent as objects in $\text{Hom}(A,B)$. Then by Proposition 6.5 there is a contractible $K$ with $0,1 \in K$ and a morphism $K \to \text{Hom}(A,B)$ taking $0$ to $f$ and $1$ to $g$. This yields (by the universal property of the internal $\text{Hom}$) a morphism $h : A \times K \to B$. This morphism together with the various others gives a homotopy from $f$ to $g$. ///

**Corollary 7.4** ([27] Conjecture 1.3.6-7) The $n+1$-category $\text{nCAT}$ yields when truncated down to a 1-category the localization $Ho \text{Cat}$ of [27].
8. *n*-stacks

We can give a preliminary discussion of the notion of *n*-stack, following the lines that are already well known for simplicial presheaves and even *n*-stacks of *n*-groupoids (approached via topological spaces in [23], discussed for *n* = 2, 3 in [7]). Our present discussion will be incomplete, basically for the following reason: if *X* is a 1-category, there are several natural types of objects which represent the idea of a family of *n*-categories indexed (contravariantly) by *X*, and we would like to know that all of these notions are equivalent. The main possible versions are:

1. A functor *X* → *nCAT*, which if we take the second point of view on *nCAT* presented above, is the same thing as a presheaf of fibrant *n*-categories over *X*;
2. A weak functor from *X* to *nCAT*, in other words a functor from *X* to Fib(*nCAT*) or (what is basically the same thing) an element of HOM(*X*, *nCAT*) i.e. a morphism in (*n* + 1)CAT;
3. A “fibered *n*-category over *X”*, which would be a morphism of *n*-categories *F* → *X* (note that a 1-category considered as an *n*-category is automatically fibrant by 3.8) satisfying some condition analogous to the definition of fibered 1-category—I haven’t written down this condition (note however that it is distinct from the condition that the morphism be fibrant in the sense we use in this paper).

Here is what I currently know about the relationship between these points of view. From (1) one automatically gets (2) just by composing with the morphism *nCAT* → Fib(*nCAT*) to the fibrant envelope. From (2) one should be able to get (3) by pulling back a universal fibered *n* + 1-category over Fib(*nCAT*). To construct this universal object, first construct a universal *n* + 1-category *U* → *nCAT* (with fibers the *n*-categories being parametrized—in particular this morphism is relatively *n*-truncated) then replace the composed morphism *U* → Fib(*nCAT*) by a fibrant morphism. Finally, from (3) one should be able to get (1) by applying the “sections functor”: if *F* → *X* is a fibered *n*-category then define Γ(*X*, *F*) to be the *n*-category fiber (calculated in the correct homotopic sense) over 1*X* ∈ HOM(*X*, *X*) of

\[ \text{HOM}(\mathcal{X}, \mathcal{F}) \rightarrow \text{HOM}(\mathcal{X}, \mathcal{X}). \]

Require now that *F* → *X* be a fibrant morphism (if this doesn’t come into the condition of being fibered already). Then

\[ X \in \mathcal{X} \mapsto \Gamma(\mathcal{X}/X, \mathcal{F} \times_{\mathcal{X}} (\mathcal{X}/X)) \]
should be strictly functorial in the variable \(X\) yielding a presheaf \(X^n \to nCAT\) which is notion (1). The condition of being a fibered category should imply (as it does in the case \(n = 1\)) that the morphism

\[
\Gamma(\mathcal{X}/X, \mathcal{F} \times_X (\mathcal{X}/X)) \to \mathcal{F}_X := \mathcal{F} \times \{X\}
\]

be an equivalence of \(n\)-categories (one might even try to take this condition as the definition of being fibered but I’m not sure if that would work). Finally we would like to show that doing these three constructions in a circle results in an essentially equivalent object.

The previous paragraph is for the moment speculative, the main questions left open being the definition of “fibered \(n\)-category” and the construction of the universal family. However, for the rest of this section we will discuss the theory of \(n\)-stacks supposing that the above equivalences are known. Denote by \(\int\) the operation going from (1) to (3).

Suppose \(\mathcal{X}\) is a site. There are a couple of different ways of approaching the notion of \(n\)-stack over \(\mathcal{X}\). Our first definition will be modelled on what was done in [23]. A fibered \(n\)-category \(\mathcal{F} \to \mathcal{X}\) is an \(n\)-stack if for any \(X \in \mathcal{X}/X\) and any sieve \(B \subset \mathcal{X}/X\) the morphism

\[
\Gamma(\mathcal{X}/X, \mathcal{F} \times_X (\mathcal{X}/X)) \to \Gamma(B, \mathcal{F} \times_X B)
\]

is an equivalence of \(n\)-categories. If \(\mathcal{F} \to \mathcal{X}\) is a fibered \(n\)-category then we (should be able to) construct the associated \(n\)-stack by iterating \(n + 2\) times the operation

\[
L(\mathcal{F}) := \int \left( X \mapsto \lim_{\to, B \subset \mathcal{X}/X} \Gamma(B, \mathcal{F} \times_X B) \right)
\]

This conjecture is based on the corresponding result for flexible sheaves in [23].

The second main type of approach is to combine the theory of simplicial presheaves of Jardine-Joyal-Brown-Gersten (cf [13]) with the discussion in the present paper to obtain a closed model structure for the category of presheaves of \(n\)-precats over \(\mathcal{X}\). In this case the fibrant condition would imply the condition of being an \(n\)-stack. To give the definitions (without proving that we get a closed model category) it suffices to define weak equivalence—the cofibrations being just the maps which over each object of \(\mathcal{X}\) are cofibrations of \(n\)-precats, and the fibrations then being defined by the lifting property for trivial cofibrations. (As usual the main problem would then be to prove that pushout by a trivial cofibration is again a trivial cofibration—and for this we could probably just combine the proofs of Jardine/Joyal [13] and the present paper.) If \(A \to B\) is a morphism of presheaves of \(n\)-precats over \(\mathcal{X}\) then we obtain a morphism \(\text{Cat}(A) \to \text{Cat}(B)\) of presheaves of easy \(n\)-categories (where \(\text{Cat}(A)(X) := \text{Cat}(A(X))\)). We will say that \(A \to B\) is a weak equivalence if \(\text{Cat}(A) \to \text{Cat}(B)\) is a weak equivalence of presheaves of \(n\)-categories, notion which we now define. Let \(T\) denote Tamsamani’s truncation operation [27] which is functorial so it extends to presheaves of \(n\)-categories. A morphism

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of presheaves of $n$-categories $A \to B$ is a \textit{weak equivalence in top degree} if for every $n$-
-morphism of $B$ and lifting of its source and target to $n-1$-morphisms in $A$, there exists a
unique lifting to an $n$-morphism in $A$. Now we say that a morphism $A \to B$ of presheaves
of $n$-categories is a \textit{weak equivalence} if for every $k$ the morphism

$$Sh(T^k A) \to Sh(T^k B)$$

is a weak equivalence in top degree, where $Sh$ denotes the stupid sheafification operation
(i.e. sheafify each of the presheaves $A_M$).

In this point of view, if $A$ is a presheaf of $n$-categories over $X$ then we define the
\textit{associated $n$-stack} to be the fibrant object equivalent to $A$ in the previous presumed
closed model category.

$n$-categories as $n-1$-stacks

Heuristically we can define a structure of \textit{site} on $\Delta$ where the coverings of an object $m$ are
the collections of morphisms $\lambda_i : m$ where $\lambda_i = \{a_i, a_i + 1, \ldots, b_i\}$ such that $a_{i+1} = b_i$.
If $A$ is an $n$-precat then the collection $\{A_p\}$ may be thought of as a presheaf of $n-1$-
precats over $\Delta$. The condition to be an $n$-category is that this should be a presheaf of
$n-1$-categories which should satisfy descent for coverings, i.e. it should be an $n-1$-stack
of $n-1$-categories over this site. The construction $Cat$ is essentially just finding the
$n-1$-stack associated to an $n-1$-prestack by an operation similar to that described in
$[23]$. The main problems above are caused by the fact that this site doesn’t admit fiber
products. It might be a good idea to replace this site by its associated topos, the category
of categories, which would lead to the yoga: \textit{that an $n$-category is an $n-1$-stack over the
topos of categories.}

It might be possible, by treating $n$-stacks at the same time as $n$-categories, to simplify
the arguments of the present paper by recursively defining $n$-categories as $n-1$-stacks. I
haven’t thought about this any further.

9. The generalized Seifert-Van Kampen theorem

Our closed model category structure allows us (with a tiny bit of extra work) to
obtain the analogue of the Siefert-Van Kampen theorem for the Poincaré $n$-groupoid of a
topological space $\Pi_n(X)$ defined by Tamsamani ($[27]$, §2.3).

\textbf{Theorem 9.1} If $X$ is a space covered by open subsets $X = U \cup V$ then (setting $W := U \cap V$) $\Pi_n(X)$ is equivalent to the category-theoretic pushout of the diagram

$$\Pi_n(U) \leftarrow \Pi_n(W) \to \Pi_n(V).$$
In order to prove this theorem we recall Tamsamani’s realization functor from \( n \)-precats to topological spaces (\cite{27} \S 2.5). There is a covariant functor \( R : \Delta^n \to \text{Top} \) which associates to \( M = (m_1, \ldots, m_n) \) the product \( R^{m_1} \times \cdots \times R^{m_n} \) where \( R^m \) denotes the usual topological \( m \)-simplex. If \( A : (\Delta^n)^0 \to \text{Sets} \) is a presheaf of sets then Tamsamani defines the realization of \( A \) in the standard way combining \( R \) and \( A \). We denote this \( \langle R, A \rangle \) because it is a sort of pairing of functors. If \( A \) is an \( n \)-precat in our notations then pull it back to a presheaf on \( \Delta^n \) and apply the realization (we still denote this as \( \langle R, A \rangle \)). The functor \( \langle R, \cdot \rangle \) obviously preserves pushouts.

**Caution:** The realization functor does not preserve cofibrations. It takes injective morphisms of presheaves over \( \Theta^n \) to cofibrations of spaces, but the cofibrations which are not injective in the top degree are taken to non-injective morphisms.

Recall Tamsamani’s Proposition 3.4.2(ii):

**Proposition 9.2** If \( A \to B \) is an equivalence of \( n \)-categories then

\[
\langle R, A \rangle \to \langle R, B \rangle
\]

is an equivalence of spaces.

**Proof:** The proof of \cite{27} for \( n \)-groupoids using induction on \( n \) also works for \( n \)-categories. ///

Say that a morphism \( X \to Y \) of topological spaces is an \( n \)-weak equivalence if it is an isomorphism on \( \pi_0 \) and for any choice of basepoint in \( X \) it is an isomorphism on \( \pi_i \) for \( i \leq n \). This is equivalent to saying that it induces a weak equivalence on the Postnikov tower up to stage \( n \).

**Corollary 9.3** The realization functor \( \langle R, \cdot \rangle \) from \( n \)-precats to \( \text{Top} \) takes weak equivalences to \( n \)-weak equivalences of topological spaces.

**Proof:** Realization takes our standard trivial cofibrations \( \Sigma \to h \) to homotopy equivalences of topological spaces. This is essentially the content of the constructions of retractions in the proof of Theorem 2.3.5 (that \( \Pi_n(X) \) is an \( n \)-nerve) of \cite{27}.

For all except the upper boundary cases, the standard trivial cofibrations are taken to cofibrations of topological spaces. Pushout by the injective standard trivial cofibrations becomes pushout by a trivial cofibration of spaces, whence a homotopy equivalence. In order to deal with the upper boundary cases we introduce the following notation:

\[
\langle R, A \rangle_q
\]

denotes the \( q \)-skeleton of the realization of \( A \), that is the realization taken over all \( M \) with \( \sum m_i \leq q \). Then, if \( q \leq n - 1 \) the functor \( \langle R, \cdot \rangle_q \) takes cofibrations to cofibrations of spaces.
Suppose \( \varphi : \Sigma \to h \) is a standard trivial cofibration in one of the boundary cases. Using the notation of \( \S 2 \) we can write \( \Sigma \) as the coequalizer of

\[
h' \sqcup h' \sqcup \Upsilon' \to h^a \sqcup h^a
\]

(the component \( \Upsilon \) which appears on the right in the general case disappears in the upper boundary case). The map \( \Sigma \to h \) is given by the map \( h^a \sqcup h^b \to h \) which in this case is two times the identity (because \( h(M, m, 1^{k+1}) \) which doesn’t exist is replaced by \( h := h(M, m, 1^k) = h^a = h^b \)). The cells in \( \langle R, \Sigma \rangle \) of dimension \( n \) are automatically of the form \( h(1^l, 1, 1^k) \) for maps \( 1^l \to M \) and \( 1 \to m \). There are two such which are identified whenever \( 1 \to m \) is not one of the principal morphisms. The cells coming from principal \( 1 \to m \) occur only once in the realization of \( \Sigma \) already. It follows (since any non-principal \( 1 \to m \) is a path which is homotopic to a concatenation of principal \( 1 \to m \)) that the \( n \)-cells which are identified are homotopic.

Note that on the level of cells of dimension \( < n \) the morphism \( \Sigma \to h \) is an isomorphism. In particular, pushout via \( \varphi \) over any \( \Sigma \to A \) preserves the \( n-1 \)-skeleton of the realization, and the \( n \)-cells which are identified are homotopic. In this boundary case the pushout by \( \varphi \) is surjective, in particular it is surjective on \( n+1 \)-cells. A surjective morphism of cell complexes which is an isomorphism on \( n-1 \)-skeleta and which only identifies \( n \)-cells which are homotopic (relative the \( n-1 \)-skeleton) is an \( n \)-weak equivalence. This completes the proof that pushout by any of our standard trivial cofibrations \( \varphi \) induces an \( n \)-weak equivalence.

It follows by construction of the operation \( \text{Cat} \) that for any \( n \)-precat \( A \) the morphism

\[
\langle R, A \rangle \to \langle R, \text{Cat}(A) \rangle
\]

is an \( n \)-weak equivalence of spaces. Now we can complete the proof: if \( A \to B \) is a weak equivalence then by definition \( \text{Cat}(A) \to \text{Cat}(B) \) is an equivalence of \( n \)-categories so in the diagram

\[
\begin{array}{ccc}
\langle R, A \rangle & \to & \langle R, B \rangle \\
\downarrow & & \downarrow \\
\langle R, \text{Cat}(A) \rangle & \to & \langle R, \text{Cat}(B) \rangle
\end{array}
\]

the vertical arrows are \( n \)-weak equivalences from the previous argument and the bottom arrow is a weak equivalence by the proposition, so the top arrow is an \( n \)-weak equivalence of spaces. //

**Lemma 9.4** If \( A \to B \) and \( A \to C \) are morphisms of \( n \)-groupoids with one being a cofibration, then the category-theoretic pushout \( \text{Cat}(B \cup^A C) \) is an \( n \)-groupoid.
Proof: We say that an \( n \)-precat is a pre-groupoid if its associated \( n \)-category is a groupoid. We prove that the pushout of pre-groupoids is again a pre-groupoid, and we proceed by induction on \( n \) so we may assume this is known for \( n - 1 \)-pre-groupoids.

Suppose now that \( A, B \) and \( C \) are \( n \)-groupoids with morphisms as in the statement of the lemma. Then the \( A_p/; B_p/ \) and \( C_p/ \) are \( n - 1 \)-groupoids, and

\[
(B \cup^A C)_p/ = B_p/ \cup^{A_p/} C_p/.
\]

In particular, \((B \cup^A C)_p/\) are \( n - 1 \)-pre-groupoids. The process of going from this collection of \( n - 1 \)-precats to the collection corresponding to \( \text{Cat}(B \cup^A C) \) as described in §4, uses only iterated pushouts by the various \((B \cup^A C)_p/\) in various combinations. Since we know by induction that pushouts of \( n - 1 \)-pre-groupoids are again \( n - 1 \)-pre-groupoids, it follows that \( \text{Cat}(B \cup^A C)_p/ \) are \( n - 1 \)-groupoids. It suffices now to show that the truncation of \( \text{Cat}(B \cup^A C) \) down to a 1-category is a groupoid. But this truncation is the same as the brutal truncation since we know that the \( \text{Cat}(B \cup^A C)_p/ \) are \( n - 1 \)-groupoids. On the other hand, brutal truncation commutes with the operations \( \text{Cat} \) and pushout, therefore the truncation of \( \text{Cat}(B \cup^A C) \) is the pushout of the truncations of \( A, B \) and \( C \) which are groupoids. Finally, the 1-category pushout of groupoids is again a groupoid, so \( \text{Cat}(B \cup^A C) \) is a groupoid.

To complete the proof it remains to be seen that the pushout of \( n \)-pregroupoids is a pregroupoid. Suppose \( A, B \) and \( C \) are \( n \)-pre-groupoids. Then by reordering

\[
\text{BigCat}(B \cup^A C) = \text{BigCat}(\text{Cat}(B) \cup^{\text{Cat}(A)} \text{Cat}(C)).
\]

Thus \( \text{BigCat}(B \cup^A C) \) is the category-theoretic pushout of \( n \)-groupoids so by the previous argument it is an \( n \)-groupoid. This shows that \( B \cup^A C \) is an \( n \)-pre-groupoid, completing the proof of the induction step. 

Proof of Theorem 9.4: Note first of all that Tamsamani’s proof that \( \Pi_n(X) \) is an \( n \)-category ([27] Theorem 2.3.6) actually shows that it is an easy \( n \)-category. With the notations of the theorem, we have a diagram of easy \( n \)-categories

\[
\begin{array}{ccc}
\Pi_n(W) & \rightarrow & \Pi_n(U) \\
\downarrow & & \downarrow \\
\Pi_n(V) & \rightarrow & \Pi_n(X).
\end{array}
\]

Let \( A \) be the pushout \( n \)-precat of the upper and left arrows. We have a morphism \( A \rightarrow \Pi_n(X) \), and hence (non-uniquely) \( \text{Cat}(A) \rightarrow \Pi_n(X) \) since the latter is an easy \( n \)-category. The realization of \( A \) is the pushout of the realizations of \( \Pi_n(U) \) and \( \Pi_n(V) \) over \( \Pi_n(W) \). These last realizations are \( n \)-weak equivalent to \( U, V \) and \( W \) respectively ([27] 3.3.4), so the realization of \( A \) is \( n \)-weak equivalent to the pushout of \( U \) and \( V \).
over $W$, in other words to $X$. Thus the morphism $A \to \Pi_n(X)$ induces an $n$-weak equivalence on realizations. On the other hand we have seen above that $A \to \text{Cat}(A)$ induces an $n$-weak equivalence on realizations. Thus the morphism $\text{Cat}(A) \to \Pi_n(X)$ induces an $n$-weak equivalence on realizations. Lemma 9.4 implies that $\text{Cat}(A)$ is an $n$-groupoid. Applying the functor $\Pi_n$ again and using Proposition 3.4.4 of [27] we conclude that $\text{Cat}(A) \to \Pi_n(X)$ is an equivalence of $n$-groupoids. This proves the theorem. ///

10. Nonabelian cohomology

If $A$ is a fibrant $n$-category and $X$ a topological space then define the nonabelian cohomology of $X$ with coefficients in $A$ to be $H(X, A) := \text{Hom}(\Pi_n(X), A)$. It is an $n$-category. This satisfies Mayer-Vietoris: if $U, V \subset X$ and $W = U \cap V$ then

$$m : H(X, A) \to H(U, A) \times_{H(W, A)} H(V, A)$$

is an equivalence of $n$-categories (where the fiber product is understood to be the homotopic fiber product obtained by replacing one of the morphisms with a fibrant morphism). To see this, note that if $A$ is fibrant then for any cofibration $B \to C$ the morphism $\text{Hom}(C, A) \to \text{Hom}(B, A)$ is fibrant. To prove this claim it suffices to remark that if $S \to T$ is a trivial cofibration then $S \times C \cup S \times B \to T \times C$ is a trivial cofibration, now apply the universal property of the internal $\text{Hom}$ to obtain the lifting property in question.

In particular, note that $H(U, A) \to H(W, A)$ and $H(V, A) \to H(W, A)$ are fibrations (since open inclusions induce cofibrations of $\Pi_n$). Thus the above fiber product is the homotopic fiber product.

We now prove that the Mayer-Vietoris map $m$ is an equivalence of $n$-categories. It is the same as the map

$$\text{Hom}(\Pi_n(X), A) \to \text{Hom}(\Pi_n(U) \cup^{\Pi_n(W)} \Pi_n(V), A).$$

But we have seen in [7.2] that the morphism

$$\Pi_n(U) \cup^{\Pi_n(W)} \Pi_n(V) \to \Pi_n(X)$$

is a trivial cofibration. Thus, to complete the proof it suffices to note (from [7.2]) that for any trivial cofibration $B \to C$ the morphism $\text{Hom}(C, A) \to \text{Hom}(B, A)$ is an equivalence of $n$-categories.
If we take cohomology of a CW complex $X$ with coefficients in a fibrant groupoid $A$ then $H(X, A)$ is equivalent to $\Pi_n(\underline{\text{Hom}}_{\text{Top}}(X, \langle R, A \rangle))$. To prove this note that $\Pi_n$ is adjoint to the realization $\langle R, A \rangle$, which implies on the level of internal $\text{Hom}$ that for any $n$-precat $B$ and space $U$,

$$\underline{\text{Hom}}(B, \Pi_n(U)) = \Pi_n(\underline{\text{Hom}}_{\text{Top}}(\langle R, B \rangle, U))$$

where $\underline{\text{Hom}}_{\text{Top}}$ denotes the compact-open mapping space. Corollary 9.3 and the adjointness imply that for any space $U$, $\Pi_n(U)$ is fibrant. On the other hand, Tamsamani proves in [27] §3 that for any $n$-groupoid $A$ the morphism

$$A \to \Pi_n(\langle R, A \rangle)$$

is an equivalence of $n$-groupoids. Thus if $A$ is a fibrant $n$-groupoid we have an equivalence

$$\underline{\text{Hom}}(\Pi_n(X), A) \to \underline{\text{Hom}}(\Pi_n(X), \Pi_n(\langle R, A \rangle)) = \Pi_n(\underline{\text{Hom}}_{\text{Top}}(\langle R, \Pi_n(X) \rangle, \langle R, A \rangle)).$$

On the other hand, again from [27] §3 we know that $\langle R, A \rangle$ is $n$-truncated, and there is a space $W$ and diagram

$$X \leftarrow W \rightarrow \langle R, \Pi_n(X) \rangle$$

where the left morphism is a weak homotopy equivalence and the right morphism induces an isomorphism on homotopy groups in degrees $\leq n$. Thus, under the assumption that $X$ is a CW complex (which allows us to obtain weak equivalences when we apply $\underline{\text{Hom}}_{\text{Top}}$) we obtain a diagram of weak homotopy equivalences

$$\underline{\text{Hom}}_{\text{Top}}(X, \langle R, A \rangle) \to \underline{\text{Hom}}_{\text{Top}}(W, \langle R, A \rangle) \leftarrow \underline{\text{Hom}}_{\text{Top}}(\langle R, \Pi_n(X) \rangle, \langle R, A \rangle)).$$

Combining with the above we get a diagram of equivalences

$$\underline{\text{Hom}}(\Pi_n(X), A) \to \Pi_n(\underline{\text{Hom}}_{\text{Top}}(W, \langle R, A \rangle)) \leftarrow \Pi_n(\underline{\text{Hom}}_{\text{Top}}(X, \langle R, A \rangle)).$$

Thus the nonabelian cohomology with coefficients in an $n$-groupoid coincides with the approach using topological spaces.

Of course even the nonabelian cohomology with coefficients in an $n$-category $A$ which isn’t a groupoid doesn’t really give a new homotopy invariant since all of the information is contained in the Poincaré $n$-groupoid $\Pi_n(X)$. However, it might give some interesting special cases to study.

Once the theory of $n$-stacks gets off the ground, we should be able to interpret $H(X, A)$ as the $n$-category of global sections of the $n$-stack associated to the constant presheaf $U \mapsto A$ over the site $\text{Site}(X)$ of disjoint unions of open subsets of $X$.  

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More generally if $\mathcal{X}$ is any site and $\mathbb{A}$ is a presheaf of $n$-categories over $\mathcal{X}$ (or a fibered $n$-category over $\mathcal{X}$) then the $n$-category of global sections of the $n$-stack associated to $\mathbb{A}$ is the nonabelian cohomology $H(\mathcal{X}, \mathbb{A})$.

We now treat the example mentioned in the footnote of the introduction. Suppose $G$ is a group and $U$ an abelian group, and let $\mathbb{A}$ (resp. $\mathbb{B}$) be the strict 1-category with one object and group of automorphisms $G$ (resp. the strict $n$-category with one arrow in each degree $< n$ and group $U$ of automorphisms in degree $n$). Let $\mathbb{A}'$ (resp. $\mathbb{B}'$) be fibrant replacements for $\mathbb{A}$ and $\mathbb{B}$. We would like to show that the set $T^n\text{Hom}(\mathbb{A}', \mathbb{B}')$ is equal to the group cohomology $H^n(G, U)$. For the moment, the only way I see to do this is to pass by topology using the Seifert-Van Kampen theorem. Let $X = K(G, 1)$ and $Y = K(U, n)$. Then $\Pi_n(X)$ is equivalent to $A$ and $\Pi_n(Y)$ is equivalent to $B$. Similarly in the other direction $\langle R, B' \rangle$ is equivalent to $\langle R, B \rangle$ which in turn is equivalent to $Y$. By the above discussion $H(X, B')$ is equivalent to $\Pi_n(\text{Hom}(X, Y))$. The truncation $T^nH(X, B')$ is thus equal to $\tau_0(\text{Hom}(X, Y))$ which (as is well-known) is $H^n(X, U) = H^n(G, U)$. But by definition $H(X, B') = \text{Hom}(\Pi_n(X), B')$ which is equivalent to $\text{Hom}(\mathbb{A}', \mathbb{B}')$. This gives the desired statement.

The above argument is clearly not ideal, since we are looking for a purely algebraic approach to these types of problems. It seems likely that the algebraic techniques of [20] with appropriate small additional lemmas would permit us to give a purely algebraic proof of the result of the previous paragraph.

11. Comparison

As pointed out in the introduction, there are many different theories of weak $n$-categories in the process of becoming reality, and this will pose the problem of comparison. As an initial step we give a construction of functors modeled on Tamsamani’s Poincaré $n$-groupoid construction. We denote our “Poincaré $n$-category” functors by $\Upsilon_n$ to avoid confusion with the $\Pi_n$ (specially on the fact that the $\Upsilon_n$ will not take images in $n$-groupoids).

This section is only a sketch, with many details of proofs missing. In particular the following proposed set of axioms for internal model categories is a preliminary attempt only.

Suppose $\mathcal{C}$ is a closed model category with the following additional properties:

(IM1) $\mathcal{C}$ admits an internal $\text{Hom}$;

(IM2) If $A$ and $B$ are fibrant and cofibrant objects then $\text{Hom}(A, B)$ is fibrant;

(IM3) If $A \to A'$ is a cofibration (resp. trivial cofibration) of fibrant and cofibrant objects, and if $B' \to B$ is a fibration (resp. trivial fibration) of fibrant and cofibrant objects, then $\text{Hom}(A', B') \to \text{Hom}(A, B)$ is a fibration (resp. a trivial fibration);

(IM4) Internal $\text{Hom}$ takes cofibrant pushout in the first variable (resp. fibrant fiber product in the second variable) to fiber product.
We call $\mathcal{C}$ an internal closed model category.

Suppose now that $\mathcal{C}$ is an internal closed model category with an inclusion $i : \text{Cat} \subset \mathcal{C}$ having the following properties:

(a)—$i(\emptyset)$ is the initial object and $i(*)$ is the final object of $\mathcal{C}$; (b)—$i$ is compatible with disjoint union;
(c)—$i$ takes values in the fibrant and cofibrant objects of $\mathcal{C}$;
(d)—$i$ takes the internal $\text{Hom}$ in $\text{Cat}$ to the internal $\text{Hom}$ of $\mathcal{C}$.

(e)—Let $I$ denote the category with two objects 0, 1 and one non-identity morphism from 0 to 1. Let $I^{(m)}$ denote the symmetric product of $m$ copies of $I$, which is the category with objects 0, ..., $m$ and one morphism from $i$ to $j$ when $i \leq j$. Then we require that the morphism from the $\mathcal{C}$-pushout of the diagram

$$i(I) \leftarrow * \rightarrow i(I) \leftarrow * \ldots \rightarrow i(I)$$

to $i(I^{(m)})$ be a cofibrant weak equivalence in $\mathcal{C}$.

Remark: Our closed model categories $PC_n$ are internal, and (for $n \geq 1$) have functors $i : \text{Cat} \hookrightarrow PC_n$ satisfying properties (a)–(e) above.

These seem to be reasonable properties to ask of any closed model category representing a theory of $n$-categories (or $\infty$-categories). However, some of the properties are of a rather technical nature so it is possible that some technically slightly different approach to comparison would be needed—the present section is just a first attempt.

Suppose $(\mathcal{C}, i)$ is an internal closed model category with inclusion $i$ having the above properties. Let $\mathcal{C}_f$ denote the subcategory of fibrant objects. Then for any $n$ we define a functor $\Upsilon_n : \mathcal{C} \to n-Cat \subset PC_n$, which we call the “Poincaré $n$-category” functor. These functors will have the property that they take weak equivalences in $\mathcal{C}_f$ to equivalences of $n$-categories, and will be compatible with direct products (hence with fiber products over sets).

The definition is by induction. First of all, $\Upsilon_0(X)$ is defined to be equal to the set of homotopy classes of maps $* \to X$. Then, supposing that we have defined $\Upsilon_{n-1}$ we define for any $X \in \mathcal{C}_f$ the simplicial object $U$ of $\mathcal{C}$ by: $U_0$ is the set (note that sets are categories so $i$ gives an inclusion of sets into $\mathcal{C}$) of morphisms $* \to X$; and for $x_0, \ldots, x_m \in U_0$, $U_m(x_0, \ldots, x_m)$ is the fiber of

$$\text{Hom}(I^{(m)}, X) \to \text{Hom}(\{0, \ldots, m\}, X)$$

over the point $(x_0, \ldots, x_m)$. Then $U_m$ is the disjoint union of the $U_m(x_0, \ldots, x_m)$ over all sequences of $x_i \in U_0$.

Axiom IM4 and condition (e) imply that the usual morphism

$$U_m \to U_1 \times_{U_0} \ldots \times_{U_0} U_1$$

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is a weak equivalence.

With the above notation set

\[ \Upsilon_n(X)_{m/} := \Upsilon_{n-1}(U_m) . \]

This simplicial \( n-1 \)-category is an \( n \)-category since \( \Upsilon_{n-1} \) is compatible with direct products and preserves weak equivalences. Note that \( \Upsilon_n \) is obviously compatible with direct products.

One has to prove that \( \Upsilon_n \) preserves weak equivalences (we leave this out for now).

**Examples**

Tamsamani’s functor \( \Pi_n \) is essentially an example of the functor functor \( \Upsilon_n \) for \( \mathcal{C} = \text{Top} \) and \( i : \text{Cat} \to \text{Top} \) the realization functor. Our definition of \( \Upsilon_n \) is a generalization of the definition of (\cite{27} §3).

For \( \mathcal{C} = PC_{n'} \) we obtain the functor \( \Upsilon_n \). If \( n = n' \) it is essentially the identity; for \( n < n' \) is is the truncation \( T^{n' - n} \); and for \( n > n' \) it is the induction \( \text{Ind}_{n'}^{n} \). Note however that the induction doesn’t preserve pushouts, so \( \Upsilon_n \) will not necessarily preserve pushouts in general (where by pushouts here we mean the replacement of pushouts by weak-equivalent objects of \( \mathcal{C}_f \)).

For any given theory \( \mathcal{C} \) of \( n \)-categories satisfying the above properties, one would like to check that the functor \( \Upsilon_n \) is an equivalence of homotopy theories in the sense of (\cite{20} (or at least that it induces an isomorphism of localized categories). If \( n \) is correctly chosen to correspond to the level of \( \mathcal{C} \) then one would try to show that \( \Upsilon_n \) preserves pushouts.

There are examples of \( \mathcal{C} \) which are not equivalent to \( PC_n \), such as \( \text{Top} \) or, for example, the category of “Segal categories”, i.e. simplicial spaces whose first object is a set and which satisfy Segal’s condition (cf \cite{24} §3). Even if we look at Segal categories whose elements are \( n-1 \)-truncated, the functor \( \Upsilon_n \) will go into the \( n \)-precats \( \mathcal{A} \) whose \( \mathcal{A}_{m/} \) are \( n-1 \)-groupoids, in particular \( \Upsilon_n \) will not be essentially surjective.

Similarly, one can imagine looking at a theory \( \mathcal{C} \) of \( n \)-categories with extra structure. For example \( \text{Top} \) is basically the theory of \( n \)-categories where the \( i \)-morphisms have essential inverses. Baez and Dolan propose another type of extra structure of “adjoints” rather than inverses, in relation to topological quantum field theory \cite{2}. It is possible in this case that \( \mathcal{C} \) would again be an internal closed model category and that we would have a functor \( i \). The resulting functor \( \Upsilon_n \) would then be essentially the functor of “forgetting the extra structure” and taking the underlying \( n \)-category.

A more fundamental example of the above phenomenon will be the closed model category of \( n \)-stacks. This retracts onto that of \( n \)-categories: the inclusion being the constant stack functor and the morphism \( \Upsilon_n \) being the global section functor. Of course in this situation we don’t expect \( \Upsilon_n \) to be an equivalence of theories. This example shows that more is needed than just the above axioms for \( \mathcal{C} \) in order to prove that the composition in the other direction is the identity.
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