Uniquely dimensional graphs

Behrooz Bagheri Gh., Mohsen Jannesari, Behnaz Omoomi

Department of Mathematical Sciences
Isfahan University of Technology
84156-83111, Isfahan, Iran

Abstract

A set \( W \subseteq V(G) \) is called a resolving set, if for each two distinct vertices \( u, v \in V(G) \) there exists \( w \in W \) such that \( d(u, w) \neq d(v, w) \), where \( d(x, y) \) is the distance between the vertices \( x \) and \( y \). A resolving set for \( G \) with minimum cardinality is called a metric basis. A graph with a unique metric basis is called a uniquely dimensional graph. In this paper, we study some properties of uniquely dimensional graphs.

Keywords: Resolving set; Metric basis; Uniquely dimensional.

1 Introduction

Throughout the paper, \( G = (V, E) \) is a finite, simple, and connected graph of order \( n \). The distance between two vertices \( u \) and \( v \), denoted by \( d(u, v) \), is the length of a shortest path between \( u \) and \( v \) in \( G \). For a vertex \( v \in V(G) \), \( \Gamma_i(v) = \{ u \mid d(u, v) = i \} \). The diameter of \( G \) is \( \text{diam}(G) = \max\{d(u, v) \mid u, v \in V(G)\} \). The girth of \( G \) is the length of a shortest cycle in \( G \). The set of all adjacent vertices to a vertex \( v \) is denoted by \( N(v) \) and \( |N(v)| \) is the degree of a vertex \( v \), \( \text{deg}(v) \). The maximum degree and the minimum degree of a graph \( G \), are denoted by \( \Delta(G) \) and \( \delta(G) \), respectively. The notations \( u \sim v \) and \( u \not\sim v \) denote the adjacency and non-adjacency relations between \( u \) and \( v \), respectively.

For an ordered set \( W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G) \) and a vertex \( v \) of \( G \), the \( k \)-vector

\[
r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))
\]

is called the metric representation of \( v \) with respect to \( W \). The set \( W \) is called a resolving set for \( G \) if distinct vertices have different metric representations. A resolving set for \( G \) with minimum cardinality is called a metric basis, and its cardinality is the metric dimension of \( G \), denoted by \( \beta(G) \). If \( \beta(G) = k \), then \( G \) is said to be \( k \)-dimensional.

In [14], Slater introduced the idea of a resolving set and used a locating set and the location number for what we call a resolving set and the metric dimension, respectively. He described
the usefulness of these concepts when working with U.S. Sonar and Coast Guard Loran stations. Independently, Harary and Melter [7] discovered the concept of the location number as well and called it the metric dimension. For more results related to these concepts see [3, 4, 6, 11]. The concept of a resolving set has various applications in diverse areas including coin weighing problems [13], network discovery and verification [1], robot navigation [11], mastermind game [3], problems of pattern recognition and image processing [12], and combinatorial search and optimization [13].

It is obvious that to see whether a given set $W$ is a resolving set, it is sufficient to consider the vertices in $V(G) \setminus W$, because $w \in W$ is the unique vertex in $G$ for which $d(w, w) = 0$. When $W$ is a resolving set for $G$, we say that $W$ resolves $G$. In general, we say an ordered set $W$ resolves a set $T \subseteq V(G)$, if for each two distinct vertices $u, v \in T$, $r(u|W) \neq r(v|W)$.

The following bound is the known upper bound for the metric dimension.

**Theorem A.** [5] If $G$ is a connected graph of order $n$ and diameter $d$, then $\beta(G) \leq n - d$.

In [9, 10], the properties of $k$-dimensional graphs in which every $k$ subset of vertices is a metric basis are studied. Such graphs are called randomly $k$-dimensional graphs. In the opposite point there are graphs which have a unique metric basis.

**Definition.** A graph $G$ is called uniquely dimensional if $G$ has a unique metric basis. A uniquely dimensional graph $G$ with $\beta(G) = k$ is called a uniquely $k$-dimensional graph.

In this paper, we first obtain some upper bounds for the metric dimension of uniquely dimensional graphs. Then, we give some construction for uniquely $k$-dimensional graphs of the given order. Finally, we obtain a lower bound and an upper bound for the minimum order of uniquely $k$-dimensional graphs in terms of $k$.

**2 Some upper bounds**

In this section we obtain some upper bounds for the metric dimension of uniquely dimensional graphs.

Two vertices $u, v \in V(G)$ are called twin vertices if $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. It is known that, if $u$ and $v$ are twin vertices, then every resolving set $W$ for $G$ contains at least one of the vertices $u$ and $v$. Moreover, if $u \notin W$ then $(W \setminus v) \cup \{u\}$ is also a resolving set for $G$. [8]

For a uniquely dimensional graph we have the following fact.

**Lemma 1.** If $G$ is a uniquely dimensional graph, then $G$ contains no twin vertices.

**Proof.** Let $B$ be the unique metric basis of $G$. If $u, v \in V(G)$ are twin vertices, then $u, v \in B$; otherwise we can replace the one in $B$ with the other one. Now, since $B \setminus \{u\}$ is not a basis of $G$, there is exactly one vertex $w \in V(G) \setminus B$ such that $r(u|B \setminus \{u\}) = r(w|B \setminus \{u\})$. Consequently, $(B \setminus \{u\}) \cup \{w\}$ is a metric basis of $G$ different from $B$, which is a contradiction.  


Theorem 1. If $G$ is a uniquely dimensional graph of order $n$ and diameter $d$, then $\beta(G) \leq n - d - 1$.

Proof. Let $(v_0, v_1, \ldots, v_d)$ be a path of length $d$ in $G$. Two sets $V(G) \setminus \{v_1, v_2, \ldots, v_d\}$ and $V(G) \setminus \{v_0, v_1, \ldots, v_{d-1}\}$ are two resolving set of $G$ of size $n - d$. Hence, if $G$ is uniquely dimensional, then $\beta(G) \leq n - d - 1$. To complete the proof we show that $\beta(G) \neq n - d - 1$.

Let $\beta(G) = n - d - 1$ and for each $i$, $1 \leq i \leq d$, $\Gamma_i = \Gamma_i(v_0)$. We claim that for each $i$, $1 \leq i \leq d$, $\Gamma_i$ is an independent set or a clique; otherwise there exists an $i$ for which $\Gamma_i$ contains vertices $x, y, z$ such that $x \sim y$ and $x \sim z$. Therefore, $V(G) \setminus \{y, z, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\}$ is a metric basis of $G$. Now, if $y \sim z$, then $V(G) \setminus \{x, z, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\}$ and if $y \sim z$, then $V(G) \setminus \{x, y, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\}$ is another metric basis of $G$, respectively, which both are contradictions. Consequently, for each $i$, $1 \leq i \leq d$, $\Gamma_i$ is an independent set or a clique.

Now let for some $i$, $1 \leq i \leq d$, $|\Gamma_i| \geq 2$. Then, all vertices in $\Gamma_i$ are adjacent to all vertices in $\Gamma_{i-1}$; otherwise there exist $a \in \Gamma_{i-1}$ and $x \in \Gamma_i$ such that $a \sim x$. Therefore, $x$ has a neighbor in $\Gamma_{i-1}$, say $b$. Assume that $y \in \Gamma_i$ and $y \neq x$. Clearly $i \geq 2$. Thus, $V(G) \setminus \{a, b, y, v_1, v_2, \ldots, v_{i-2}, v_{i+1}, \ldots, v_d\}$ is a metric basis of $G$. Now, if $y \sim a$, then $V(G) \setminus \{b, x, y, v_1, v_2, \ldots, v_{i-2}, v_{i+1}, \ldots, v_d\}$, and if $y \sim b$, then $V(G) \setminus \{a, x, y, v_1, v_2, \ldots, v_{i-2}, v_{i+1}, \ldots, v_d\}$ is another metric basis of $G$. These contradictions imply that $y \sim a$ and $y \sim b$. Hence, $V(G) \setminus \{a, b, x, v_1, v_2, \ldots, v_{i-2}, v_{i+1}, \ldots, v_d\}$ is a metric basis of $G$, which is also a contradiction. Consequently, all vertices in $\Gamma_i$ are adjacent to all vertices in $\Gamma_{i-1}$.

The above two facts imply that, if $|\Gamma_i| \geq 2$ and $|\Gamma_{i+1}| \geq 2$, then all vertices in $\Gamma_i$ have the same neighbors in $\Gamma_{i-1} \cup \Gamma_i \cup \Gamma_{i+1}$. Therefore, all vertices $u, v \in \Gamma_i$ are twin vertices, which by Lemma 1 this is impossible. Thus, $|\Gamma_i| \geq 2$ implies that $|\Gamma_{i+1}| = 1$ and $|\Gamma_{i-1}| = 1$. Hence, if $|\Gamma_i| > 2$, then since $\Gamma_{i+1} = \{v_{i+1}\}$, by the Pigenhole principle there are two vertices $u, v \in \Gamma_i$ with the same adjacency relation with $v_{i+1}$. Therefore, $u$ and $v$ are twin vertices, which is impossible. That is, for each $i$, $1 \leq i \leq d$, $|\Gamma_i| \leq 2$. Now let $j$ be the largest integer in $\{1, 2, \ldots, d\}$ with $|\Gamma_j| = 2$ and $\Gamma_j = \{v_j, y_j\}$, where $y_j$ is the vertex with no neighbor in $\Gamma_{j+1}$. Therefore, the sets $\{v_0, v_d\}$ and $\{v_0, y_j\}$ are two metric bases of $G$. This contradiction implies that $\beta(G) \neq n - d - 1$.

Theorem 2. If $G$ is a uniquely dimensional graph of order $n$ and girth $g$, then $\beta(G) \leq n - g + 1$.

Proof. Let $C_g = (v_1, v_2, \ldots, v_g, v_1)$ be a shortest cycle in $G$. Then $V(G) \setminus \{v_3, v_4, \ldots, v_g\}$ and $V(G) \setminus \{v_2, v_3, \ldots, v_{g-1}\}$ are two resolving set of $G$ of size $n - g + 2$. Since $G$ has a unique basis, none of these two sets is a metric basis of $G$. Therefore, $\beta(G) \leq n - g + 1$.

Theorem 3. If $G$ is a uniquely dimensional graph of order $n$, then $\beta(G) < \frac{n}{2}$.

Proof. By the contrary assume that $G$ has a unique metric basis $B = \{v_1, v_2, \ldots, v_k\}$ and $n \leq 2k$. Since $k \leq n - 1$, $W = (V(G) \setminus B) \cup \{v_1, v_2, \ldots, v_{2k-n}\} \neq B$ with $|W| = k$. Therefore, $W$ is not a basis of $G$ and there exist vertices $x, y \in V(G) \setminus W \subseteq B$ such that $r(x|W) = r(y|W)$. Say $x = v_i$ and $y = v_j$. Hence, for each $v \in V(G) \setminus B$, $d(v, v_i) = d(v, v_j)$. By this reason, $B \setminus \{v_i\}$ resolves $V(G) \setminus B$. Therefore, there is exactly one vertex $u \in V(G) \setminus B$ such that $r(u|B \setminus \{v_i\}) = r(v_i|B \setminus \{v_i\})$. Consequently, $(B \setminus \{v_i\}) \cup \{u\}$ is a metric basis of $G$, which is a contradiction. Thus, $2\beta(G) < n$. 


3 Construction of uniquely \( k \)-dimensional graphs

In this section, we provide some construction for uniquely \( k \)-dimensional graphs of given order. Then we end with giving a lower bound and an upper bound for the minimum number of vertices in such graphs in terms of \( k \).

Remark 1. Note that, if \( G \) is a graph of diameter \( d \), then every \( W \subseteq V(G) \) can resolve at most \( d|W| \) vertices of \( V(G) \setminus W \). Hence, every \( k \)-dimensional graph of diameter \( d \) has at most \( k + d^k \) vertices.

In [2], Buczkowski et al. constructed a uniquely \( k \)-dimensional graph with diameter 2 and order \( k + 2^k \).

Theorem B. [2] For \( k \geq 2 \), there exists a uniquely \( k \)-dimensional graph of order \( n = k + 2^k \), diameter 2, and maximum degree \( n - 1 \).

In the following theorem regarding to constructing uniquely \( k \)-dimensional graphs with diameter \( d \), we obtain two necessary conditions for the existence of \( k \)-dimensional graphs with diameter \( d \) and order \( k + d^k \).

Theorem 4. If \( G \) is a \( k \)-dimensional graph with diameter \( d \) and order \( k + d^k \), then

(i) \( d \leq 3 \).

(ii) For a basis \( B \) and every \( v \in B \), \( |\Gamma_d(v)| \geq d^{k-1} \).

Proof. (i) Let \( G \) be a \( k \)-dimensional graph of diameter \( d \geq 4 \) and order \( k + d^k \). Thus, \( V(G) = U \cup B \), where \( U = \{u_1, u_2, \ldots, u_{d^k}\} \) and the ordered set \( B = \{v_1, v_2, \ldots, v_k\} \) is a basis of \( G \). Clearly, \( \{r(u_i|B) \mid 1 \leq i \leq d^k\} = [d]^k \), where \([d]^k \) denotes the set of all \( k \)-tuples with entries in \( \{1, 2, \ldots, d\} \). Without loss of generality, suppose that \( r(u_1|B) = (1, 1, \ldots, 1) \) and \( r(u_2|B) = (4, 1, \ldots, 1) \). Therefore, \( d(v_1, v_2) \leq 2 \) and \( d(u_2, v_1) \leq d(u_2, v_2) + d(v_2, v_1) \leq 3 \), a contradiction. Thus, \( d \leq 3 \).

(ii) Let \( B = \{v_1, v_2, \ldots, v_k\} \). By the order and diameter of \( G \), each \( k \)-vector with coordinates in \( \{1, 2, \ldots, d\} \) is the metric representation of a vertex \( u \in V(G) \setminus B \) with respect to \( B \). Therefore, for each \( v \in B \), there are \( d^{k-1} \) vertices of \( G \) that the \( i \)-th coordinate of their metric representations is \( d \). Thus, \( |\Gamma_d(v)| \geq d^{k-1} \). \( \blacksquare \)

In the following, we give a construction for uniquely \( k \)-dimensional graphs of diameter 3 and order \( k + 3^k \).

Theorem 5. For every integer \( k \geq 2 \), there exists a uniquely \( k \)-dimensional graph of diameter 3 and order \( k + 3^k \).
Proof. Let $G$ be a graph with vertex set $U \cup W$, where $U = \{u_1, u_2, \ldots, u_k\}$ is an independent set and $W$ is the set of all $k$-tuples with entries in $\{1, 2, 3\}$ and two vertices $x, y \in W$ are adjacent if they are different in exactly one coordinate and this difference is one. Moreover, the vertex $(2, 2, \ldots, 2)$ is adjacent to all vertices in $W$. Also, $w \in W$ is adjacent to $u_i \in U$ if the $i$-th coordinate of $w$ is $1$.

The vertex $(2, 2, \ldots, 2)$ is adjacent to all vertices in $W$ and $(1, 1, \ldots, 1)$ is adjacent to all vertices in $U$, thus $\text{diam}(G) \leq 3$. On the other hand, $d((3, 3, \ldots, 3), u_1) = 3$. Therefore, $\text{diam}(G) = 3$. Since $\text{diam}(G) = 3$ and the order of $G$ is $k + 3^k$, by Remark 1, $\beta(G) \geq k$. For each $w \in W$, $r(w \mid U) = w$, thus, $U$ is a resolving set for $G$ of size $k$. Hence, $U$ is a metric basis of $G$.

Now since $\text{diam}((W)) = 2$, for each $w \in W$, $|\Gamma_1(w) \cup \Gamma_2(w)| \geq 3^k - 1$ and hence $|\Gamma_3(w)| \leq k < 3^{k-1}$. Therefore, by Theorem 4(ii), no vertex of $W$ is in a metric basis of $G$. Consequently, $U$ is the unique metric basis of $G$.

By Theorems 1 and 3 if $G$ is a uniquely $k$-dimensional graph of order $n$, then $n \geq k + d + 2$ and $n \geq 2k + 1$. Let

$$n_0(k) = \min\{n \mid \text{there exists a uniquely } k\text{-dimensional graph of order } n\}.$$ 

Hence, we have $\max\{2k + 1, k + d + 2\} \leq n_0(k)$.

The following theorem shows that if a uniquely $k$-dimensional graph of order $n_0$ exists, then for every $n \geq n_0$, a uniquely $k$-dimensional graph of order $n$ exists.

**Theorem 6.** If $G$ is a uniquely $k$-dimensional graph of order $n_0$, then for every $n \geq n_0$, there exists a uniquely $k$-dimensional graph of order $n$.

**Proof.** Let $G$ be a given uniquely $k$-dimensional graph of order $n_0$ and $u$ be a vertex in the basis $B$. Assume that $v_0 \in V(G) \setminus B$ is a vertex that $d(v_0, u) = \max\{d(v, u) \mid v \in V(G) \setminus B\}$. We construct a graph $G'$ by identifying an end vertex of a path $P$ of length $n - n_0$ by $v_0$. By the property of $v_0$, $B$ is also a resolving set for $G'$. Thus, $\beta(G') \leq k$. On the other hand, since every basis of $G'$ contains at most one vertex of the path $P$, by replacing that vertex by $v_0$, we obtain a basis for $G$. Thus, $G'$ is also a uniquely $k$-dimensional graph.

In the following theorem we give a recursive construction for uniquely dimensional graphs to obtain an upper bound for $n_0(G)$.

**Theorem 7.** If $G_i$, $i = 1, 2$, is a uniquely $k_i$-dimensional graph of order $n_i$ with $\Delta(G_i) = n_i - 1$, then there exists a uniquely $(k_1 + k_2)$-dimensional graph $G$ of order $n_1 + n_2 - 1$ with $\Delta(G) = n_1 + n_2 - 2$.

**Proof.** Let $G_i$ be a uniquely $k_i$-dimensional graph of order $n_i$ with the basis $B_i$ and $v_i \in V(G_i)$ such that $\deg(v_i) = n_i - 1$, for $i = 1, 2$. Let $G$ be a graph that obtained from joining $G_1$ and $G_2$, and then identifying $v_1$ and $v_2$, say $v_0$. Thus, $\deg(v_0) = n_1 + n_2 - 2$. Since for every $u \in V(G_1) \setminus \{v_1\}$ and $v \in V(G_2) \setminus \{v_2\}$, $d(u, v) = 1$, if $B$ is a basis of $G$, then $B \cap V(G_i)$ is a basis of $G_i$, for $i = 1, 2$. Therefore, $B$ is the unique basis of $G$. 


Proposition 1. There exists a uniquely 3-dimensional graph of order 9 and maximum degree 8.

Proof. Let $U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2, \ldots, w_6\}$. Also let $G$ be graph with $V(G) = U \cup W$ and $E(G) = \{w_iw_j \mid 1 \leq i \neq j \leq 6\} \cup \{u_1w_1 \mid 1 \leq i \leq 3, j = i, i + 1, 6\}$. We show that $U$ is the unique basis of $G$.

Clearly, $\text{diam}(G) = 2$. Since $|V(G)| = 9$, by Remark 1, $\beta(G) \geq 3$. It is easy to see that $U$ is resolving set and consequently is a basis of $G$. Now let $B$ be another basis of $G$. Since $\langle W \rangle$ is a complete graph, $B \not\subseteq W$. Therefore, $|B \cap W| = 1$ or 2. If $|B \cap W| = 1$, then five vertices of $W$ have the same representation with respect to $B \cap W$ while since $\text{diam}(G) = 2$, $B \setminus W$ can not resolve five vertices. If $|B \cap W| = 2$, then four vertices of $W$ have the same representation with respect to $B \cap W$ while $B \setminus W$ can not resolve 4 vertices. These contradictions imply that $U$ is the unique basis of $G$.

In the following theorem, based on the recursive construction in Theorem 7, we obtain an upper bound for $n_0(k)$.

Theorem 8. For every $k$, $k \geq 2$, there exists a uniquely $k$-dimensional graph of order $\lceil \frac{5k}{2} + 1 \rceil$.

Proof. Let $k$ be a positive integer. If $k = 2k'$, then the graph $G$ obtained by the recursive construction given in Theorem 7 from $k'$ copies of the uniquely 2-dimensional graph of order 6, constructed in Theorem 11 is a uniquely $k$-dimensional graph of order $6k' - (k' - 1) = 5k' + 1 = \lceil \frac{5k}{2} + 1 \rceil$.

If $k = 2k' + 1$, then the graph $G$ obtained by the recursive construction given in Theorem 7 from $k' - 1$ copies of the uniquely 2-dimensional graph of order 6, constructed in Theorem 11 and one copy of the uniquely 3-dimensional graph of order 9 given in Proposition 1 is a uniquely $k$-dimensional graph of order $6(k' - 1) - (k' - 2) + 8 = 5k' + 4 = \lceil \frac{5k}{2} + 1 \rceil$.

Although the above theorem provides the recursive construction for uniquely $k$-dimensional graphs of order $\lceil \frac{5k}{2} + 1 \rceil$, to get the more explicit construction, we construct uniquely $k$-dimensional graphs of order $3k$, in the following theorem.

Theorem 9. For each $k \geq 2$, there exists a uniquely $k$-dimensional graph of order $3k$.

Proof. Let $U = \{u_1, u_2, \ldots, u_k\}$ and $W = \{w_1, w_2, \ldots, w_{2k}\}$. Also, let $G$ be a graph with vertex set $V(G) = U \cup W$ such that the induced subgraph $\langle W \rangle$ of $G$ be a complete graph, $U$ be an independent set, $u_k$ be adjacent to $w_{2i}$, $1 \leq i \leq k$, and for each $i$, $1 \leq i \leq k - 1$, $u_i$ be adjacent to $w_{2i - 1}$ and $w_{2i}$. We prove that $G$ is the desired graph.

Let $w_i$ and $w_j$ be two arbitrary vertices of $V(G) \setminus U = W$. If $i$ and $j$ have different parity, then $d(w_i, u_k) \neq d(w_j, u_k)$. If $i$ and $j$ have the same parity, then $\lfloor \frac{i}{2} \rfloor \neq \lfloor \frac{j}{2} \rfloor$ and hence $d(w_i, u_i) \neq d(w_j, u_i)$. Therefore, $U$ is a resolving set for $G$ of size $k$ and $\beta(G) \leq k$.

Now let $B$ be a metric basis of $G$. If $u_k \notin B$, then to resolve the set $\{u_1, w_1, w_2, w_{2k-1}, w_{2k}\}$, $B$ should contain at least three vertices from this set, since $\langle W \rangle$ is a complete graph, while
replacing these three vertices by \( u_1 \) and \( u_k \) provides a resolving set with smaller size. This contradiction implies that \( u_k \in B \). If for some \( i, \ 1 \leq i \leq k - 1, \ u_i \notin B \), then to resolve the set \( \{u_i, w_{2i-1}, w_{2i}, w_{2k-1}, w_{2k}\} \), \( B \) should contain at least two vertices from \( \{w_{2i-1}, w_{2i}, w_{2k-1}, w_{2k}\} \), because \( \langle W \rangle \) is a complete graph. But replacing these two vertices by \( u_i \) provides a resolving set with smaller size. This contradiction implies that \( U \subseteq B \). Since \( U \) is a resolving set, \( U = B \) is the unique metric basis of \( G \).

By Theorems \( \square \) and \( \square \) we have the following corollary.

**Corollary 1.** Let \( k \geq 2 \) be an integer. Then \( 2k + 1 \leq n_0(k) \leq \left\lceil \frac{5k}{2} + 1 \right\rceil \).

For \( k = 2 \), \( n \geq 4 + d \) implies \( n \geq 6 \). Hence, \( n_0(2) = 6 \). It can be seen, there is no uniquely 3-dimensional graph of order 7. Thus, \( 8 \leq n_0(3) \leq 9 \). The determination of \( n_0(k) \), for every integer \( k \) could be an nontrivial interesting problem.

**References**

[1] Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihalak and L.S. Ram, Network discovery and verification, *IEEE Journal On Selected Areas in Communications* **24**(12) (2006) 2168-2181.

[2] P. Buczkowski, G. Chartrand, C. Poisson and P. Zhang, On \( k \)-dimensional graphs and their bases, *Period. Math. Hungar.* **46**(1) (2003) 9-15.

[3] J. Caceres, C. Hernando, M. Mora, I.M. Pelayo, M.L. Puertas, C. Seara and D.R. Wood, *On the metric dimension of cartesian products of graphs*, SIAM Journal on Discrete Mathematics **21**(2) (2007) 423-441.

[4] G.G. Chappell, J. Gimbel and C. Hartman, *Bounds on the metric and partition dimensions of a graph*, Ars Combinatoria **88** (2008) 349-366.

[5] G. Chartrand, L. Eroh, M.A. Johnson and O.R. Ollermann, *Resolvability in graphs and the metric dimension of a graph*, Discrete Applied Mathematics **105** (2000) 99-113.

[6] G. Chartrand and P. Zhang, *The theory and applications of resolvability in graphs*. A survey. In Proc. 34th Southeastern International Conf. on Combinatorics, Graph Theory and Computing **160** (2003) 47-68.

[7] F. Harary and R.A. Melter, *On the metric dimension of a graph*, Ars Combinatoria **2** (1976) 191-195.

[8] C. Hernando, M. Mora, I.M. Pelayo, C. Seara and D.R. Wood, *Extremal graph theory for metric dimension and diameter*, The Electronic Journal of Combinatorics (2010) #R30.

[9] M. Jannesari and B. Omoomi, *On randomly \( k \)-dimensional graphs*, Applied Mathematics Letters **24** (2011) 1625-1629.
[10] M. Jannesari and B. Omoomi, *Characterization of Randomly k-dimensional graphs*, arXiv:1103.357v1.

[11] S. Khuller, B. Raghavachari and A. Rosenfeld, *Landmarks in graphs*, Discrete Applied Mathematics 70(3) (1996) 217-229.

[12] R.A. Melter and I. Tomescu, *Metric bases in digital geometry*, Computer Vision Graphics and Image Processing 25 (1984) 113-121.

[13] A. Sebo and E. Tannier, *On metric generators of graphs*, Mathematics of Operations Research 29(2) (2004) 383-393.

[14] P.J. Slater, *Leaves of trees*, Congressus Numerantium 14 (1975) 549-559.