Algebraic duality for partially ordered sets

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Abstract

For an arbitrary partially ordered set $P$ its dual $P^*$ is built as the collection of all monotone mappings $P \to 2$ where $2 = \{0, 1\}$ with $0 < 1$. The set of mappings $P^*$ is proved to be a complete lattice with respect to the pointwise partial order. The second dual $P^{**}$ is built as the collection of all morphisms of complete lattices $P^* \to 2$ preserving universal bounds. Then it is proved that the partially ordered sets $P$ and $P^{**}$ are isomorphic.

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Introduction

The results presented in this paper can be considered as the algebraic counterpart of the duality in the theory of linear spaces. The outline of the construction looks as follows.

Several categories occur in the theory of partially ordered sets. The most general is the category $\mathcal{POSET}$ whose objects are partially ordered sets and the morphisms are the monotone mappings. Another category which will be used is $\mathcal{BCL}$ whose objects are (bounded) complete lattices and the morphisms are the lattice homomorphisms preserving universal bounds. Evidently $\mathcal{BCL}$ is the subcategory of $\mathcal{POSET}$.
To introduce the algebraic duality (I use the term ‘algebraic’ to avoid confusion with the traditional duality based on order reversal) the two element partially ordered set $2$ is used:

$$2 = \{0, 1\} \quad 0 < 1$$

Let $P$ be an object of $\mathcal{POSET}$. Consider its dual $P^*$:

$$P^* = \text{Mor}_{\mathcal{POSET}}(P, 2) \quad (1)$$

The set $P^*$ has the pointwise partial order. Moreover, it is always the complete lattice with respect to this partial order (section 1). Furthermore, starting from $P^* \in \mathcal{BCL}$ (bounded complete lattices) consider the set $P^{**}$ of all morphisms in the appropriate category:

$$P^{**} = \text{Mor}_{\mathcal{BCL}}(P^*, 2) \quad (2)$$

And again, the set of mappings $P^{**}$ is pointwise partially ordered. Finally, it is proved in section 2 that $P^{**}$ is isomorphic to the initial partially ordered set $P$ (the isomorphism lemma 3):

$$P^{**} \simeq P$$

The account of the results is organized as follows. First it is proved that $P^*$ (1) is complete lattice. Then the embeddings $p \to \lambda_p$ and $p \to \upsilon_p$ of the poset $P$ into $P^*$ are built (3). Then it is shown that the principal ideals $[0, \lambda_p]$ in $P^*$ are prime for all $p \in P$ (lemma 3). Moreover, it is shown that there is no more principal prime ideals in $P^*$. Finally, it is observed that the principle prime ideals on $P^*$ are in 1-1 correspondence with the elements of $P^{**}$.

1 The structure of the dual space

First define the pointwise partial order on the elements of $P^*$ (4). For any $x, y \in P^*$

$$x \leq y \Leftrightarrow \forall p \in P \quad x(p) \leq y(p) \quad (3)$$
Evidently the following three statements are equivalent for \( x, y \in P^* \):

\[
\begin{align*}
\forall p \in P & \quad x(p) = 1 \quad \Rightarrow \quad y(p) = 1 \\
\forall p \in P & \quad y(p) = 0 \quad \Rightarrow \quad x(p) = 0
\end{align*}
\] (4)

To prove that \( P^* \) is complete lattice, consider its arbitrary subset \( K \subseteq P^* \) and define the following mappings \( u, v : P \to 2 \):

\[
\begin{align*}
u(p) = \begin{cases} 0, \quad \exists k \in K \quad k(p) = 1 \\
1, \quad \forall k \in K \quad k(p) = 0
\end{cases}
\end{align*}
\]

\[
\begin{align*}
u(p) = \begin{cases} 0, \quad \exists k \in K \quad k(p) = 0 \\
1, \quad \forall k \in K \quad k(p) = 1
\end{cases}
\end{align*}
\] (5)

The direct calculations show that both \( u \) and \( v \) are monotone mappings: \( u, v \in P^* \) and

\[
\begin{align*}
u = \sup_{P^*} K \quad , \quad v = \inf_{P^*} K
\end{align*}
\]

which proves that \( P^* \) is the complete lattice. Denote by \( 0,1 \) the universal bounds of the lattice \( P^* \):

\[
\forall p \in P \quad 0(p) = 0 \quad , \quad 1(p) = 1
\]

Let \( p \) be an element of \( P \). Define the elements \( \lambda_p, \upsilon_p \in P^* \) associated with \( p \): for all \( q \in P \)

\[
\begin{align*}
\lambda_p(q) = \begin{cases} 0, \quad q \leq p \\
1, \quad \text{otherwise}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\upsilon_p(q) = \begin{cases} 1, \quad q \geq p \\
0, \quad \text{otherwise}
\end{cases}
\end{align*}
\] (6)

**Lemma 1** For any \( x \in P^* , \quad p \in P \)

\[
\begin{align*}
x(p) = 0 & \quad \Leftrightarrow \quad x \leq \lambda_p \text{ in } P^* \\
x(p) = 1 & \quad \Leftrightarrow \quad x \geq \upsilon_p \text{ in } P^*
\end{align*}
\] (7)

**Proof.** Rewrite the left side of the first equivalency as \( \forall q \quad q \leq p \Rightarrow x(q) = 0 \), hence \( \forall q \quad \lambda_p(q) = 0 \Rightarrow x(q) = 0 \), therefore \( x \leq \lambda_p \) by virtue of (4). The second equivalency is proved likewise. \( \square \)

We shall focus on the 'inner' characterization of the elements \( \lambda_p, \upsilon_p \) in mere terms of the lattice \( P^* \) itself. To do it, recall the necessary definitions.
Let $L$ be a complete lattice. An element $a \in L$ is called join-irreducible (meet-irreducible) if it can not be represented as the join (resp., meet) of a collection of elements of $L$ different from $a$. To make this definition more verifiable introduce for every $a \in L$ the following elements of $L$:

$$
\begin{align*}
\tilde{a} &= \inf_L \{x \in L \mid x > a\} \\
\hat{a} &= \sup_L \{y \in L \mid y < a\}
\end{align*}
$$

(8)

which do exist since $L$ is complete. Clearly, $\tilde{a} \geq a \geq \hat{a}$ and the equivalencies

$$
\begin{align*}
a \neq \tilde{a} &\iff a \text{ is meet-irreducible} \\
\hat{a} &\neq a \iff a \text{ is join-irreducible}
\end{align*}
$$

(9)

follow directly from the above definitions.

**Lemma 2** An element $w \in P^*$ is meet irreducible if and only if it is equal to $\lambda_p$ for some $p \in P$. Dually, $v \in P^*$ is join irreducible iff $v = \nu_p$ for some $p \in P$.

**Proof.** First prove that every $\lambda_p$ is meet irreducible. To do it we shall use the criterion (9). Let $p \in P$. Define $u \in P^*$ as:

$$
u(q) = \begin{cases} 
0 & , q < p \\
1 & , \text{otherwise}
\end{cases}
$$

then the following equivalency holds:

$$
x \leq y \iff (\forall q \ x(q) = 0 \Rightarrow q < p) 
$$

(10)

Now let $x > \lambda_p$, then $x(p) = 1$ (otherwise (7) would enable $x \leq \lambda_p$). Then $x > \lambda_p$ implies $x \geq \lambda_p$, hence $\forall q \ x(q) = 0 \Rightarrow q \leq p$, although $q = p$ is excluded, hence we get exactly the right side of (10). That means that

$$
u = \inf_{P^*} \{x \mid x > \lambda_p\} = \tilde{\lambda}_p
$$

differs from $\lambda_p$, hence $\lambda_p$ is meet irreducible by virtue of (8). The second dual statement is proved quite analogously.

Conversely, suppose we have a meet irreducible $w \in P^*$, hence, according to (8), there exists $p \in P^*$ such that $\tilde{w}(p) \neq 0$ while $w(p) = 0$. The latter
means \( w \leq \lambda_p \) for this \( p \). To disprove \( w < \lambda_p \), rewrite \( \bar{w}(p) \neq 0 \) as \( \neg(\inf \{ x \mid w < x \} = \lambda_p) \) which is equivalent to

\[
\exists y (\forall x \ w < x \Rightarrow y \leq x) \quad \& \quad \neg(y \leq \lambda_p)
\]

In particular, it must hold for \( x = \lambda_p \), thus the assumption \( w < \lambda_p \) implies \( \exists y \ y \leq \lambda_p \& \neg(y \leq \lambda_p) \), and the only remaining possibility for \( w \) is to be equal to \( \lambda_p \).

\[\blacksquare\]

**Dual statement.** The join irreducibles of \( P^* \) are the elements \( \upsilon_p, p \in P \) and only they.

## 2 Second dual and the isomorphism lemma

Introduce the necessary definitions. Let \( L \) be a lattice. An **ideal** in \( L \) is a subset \( K \subseteq L \) such that

- \( k \in K, x \leq k \Rightarrow x \in K \)
- \( a, b \in K \Rightarrow a \lor b \in K \)

Replacing \( \leq \) by \( \geq \) and \( \lor \) by \( \land \) the notion of **filter** is introduced. An ideal (filter) \( K \subseteq L \) is called **prime** if its set complement \( L \setminus K \) is a filter (resp., ideal) in \( L \). Now return to the lattice \( P^* \).

**Lemma 3** For any \( p \in P \) both the principal ideal \( [0, \lambda_p] \) and the principal filter \( [\upsilon_p, 1] \) are prime in \( P^* \). Moreover,

\[
[v_p, 1] = P^* \setminus [0, \lambda_p]
\]

**Proof.** Fix up \( p \in P \), then for any \( x \in P^* \) the value \( x(p) \) is either 0 (hence \( x \leq \lambda_p \)) or 1 (and then \( x \geq \upsilon_p \)) according to (6). Since \( \lambda_p \) never equals \( \upsilon_p \) (because their values at \( p \) are different), the sets \( [\upsilon_p, 1] \) and \( [0, \lambda_p] \) are disjoint, which completes the proof.

The converse statement is formulated in the following lemma.

**Lemma 4** For any pair \( u, v \in P^* \) such that

\[
[0, u] = P^* \setminus [v, 1]
\]

there exists an element \( p \in P \) such that \( u = \lambda_p \) and \( v = \upsilon_p \).
Proof. It follows from (1) that \( u \) and \( v \) are not comparable, therefore \( u \land v < v \). Thus there exists \( p \in P \) such that \( (u \land v)(p) = 0 \) while \( v(p) = 1 \). Then (7) implies \( u \land v \leq \lambda_p \) and \( v \geq \nu_p \). Suppose \( v \neq \nu_p \), then (11) implies \( \nu_p \leq u \), which together with \( \nu_p \leq v \) implies \( \nu_p \leq u \land v \leq \lambda_p \) which never holds since \( \nu_p \) and \( \lambda_p \) are not comparable. So, we have to conclude that \( v = \nu_p \), thus \( u = \lambda_p \).

Now introduce the second dual \( P^{**} \) as the set of all homomorphisms of complete lattices \( P^* \to 2 \) preserving universal bounds, that is, for any \( p \in P^{**}, K \subseteq P^* \)

\[ p(\sup K) = \sup_{k \in K} p(k) \]
\[ p(\inf K) = \inf_{k \in K} p(k) \]
\[ p(0) = 0; p(1) = 1 \]

with the pointwise partial order as in (3).

Now we are ready to prove the following isomorphism lemma.

**Lemma 5** The partially ordered sets \( P \) and \( P^{**} \) are isomorphic.

**Proof.** Define the mapping \( F : P \to P^{**} \) by putting

\[ F(p) = p : p(x) = x(p) \quad \forall x \in P^{**} \]

Evidently \( F \) is the order preserving injection. To build the inverse mapping \( G : P^{**} \to P \), for any \( p \in P^{**} \) consider the ideal \( p^{-1}(0) \) and the filter \( p^{-1}(1) \) in \( P^* \) both being prime (see [1], II.4). Let \( u = \sup p^{-1}(0) \) and \( v = \inf p^{-1}(1) \). Since \( p \) is the homomorphism of complete lattices, \( u \in p^{-1}(0) \) and \( v \in p^{-1}(1) \), hence \( p^{-1}(0) = [0, u] \) and \( p^{-1}(1) = [v, 1] \). Applying lemma 4 we see that there exists \( p \in P \) such that \( u = \lambda_p \) and \( v = \nu_p \). Put \( G(p) = p. \) The mapping \( G \) is order preserving and injective (since the different principal ideals have different suprema). It remains to prove that \( F, G \) are mutually inverse.

Let \( p \in P \), consider \( G(F(p)) \). Denote \( p = F(p) \), then \( p^{-1}(1) = \{ x \in P^* \mid x(p) = 0 \} = \{ x \mid x \leq \lambda_p \} \). Thus \( \sup p^{-1}(0) = \lambda_p \), then \( G \circ F = \text{id}_P \) which completes the proof.

\[ \square \]
Concluding remarks

The results presented in this paper show that besides the well known duality in partially ordered sets based on order reversal, we can establish quite another kind of duality à la linear algebra. As in the theory of linear topological spaces, we see that the ‘reflexivity’ expressed as $P = P^{**}$ can be achieved by appropriate definition of dual space.

We see that a general partially ordered set have the dual space being a complete lattice. We also see that not every complete lattice can play the rôle of dual for a poset. These complete lattices can be characterized in terms of spaces with two closure operations [4]. For the category of orthoposets this construction was introduced in [2]. Another approach to dual spaces when they are treated as sets of two-valued measures (in terms of this paper, as sub-posets of $P^{**}$) is in [3]. The main feature of the techniques suggested in the present paper is that all the constructions are formulated in mere terms of partially ordered sets and lattices.

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