To hear the shape of a quantum drum is not a mission impossible.

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(Dated: January 7, 2022)

To hear the shape of a quantum drum is in principle not a mission impossible, where the shape means the conformal field theory (CFT) description of certain quantum phases and phase transitions in a $(2+1)d$ manifold and the drum beat means some characteristic (such as spectral) measurements of the quantum system whose structure reveals the CFT information. In realistic quantum many-body lattice models, however, measurements such as the finite size scaling of the entanglement entropy with reliable data to extract the CFT characteristics, are rare due to the numerical difficulties in the precise measurement of entanglement with replicas of the partition functions. To overcome this problem, we design a generic computation protocol to obtain the Rényi entanglement entropy of the aforementioned systems with efficiency and precision. Our method, dubbed “Qiu Ku” algorithm, is based on the massive parallelization of the nonequilibrium increment processes in quantum Monte Carlo simulations. To demonstrate its power, we show the results on a few important yet difficult $(2+1)d$ quantum lattice models, ranging from Heisenberg quantum antiferromagnet with spontaneous symmetry breaking, quantum critical point with $O(3)$ CFT to the toric code $Z_2$ topological ordered state and the Kagome $Z_2$ quantum spin liquid model with frustration and multi-spin interactions. In all these cases, our method reveals either the precise CFT data from the logarithmic correction or quantum dimension of the topological order, with unprecedentedly large system sizes, controlled errorbars and minimal computational costs. Our “Qiu Ku” algorithm therefore helps one to clearly hear the shapes of various difficult yet important quantum drums.

**I. INTRODUCTION**

To hear the shape of a quantum drum is in principle not a mission impossible, as successfully done by its classical predecessor in identifying the logarithmic contribution appearing in the determinant of the Laplacian operator on a 2d manifold (“hearing the shape of a drum”) [4, 5]. And many efforts, in order to understand the generic form of the free energy and the entanglement entropy in $(1+1)d$ and $(2+1)d$ in gapped systems (with and without topological order), systems with spontaneous continuous symmetry breaking and critical systems in the settings of quantum many-body systems have been put forwarded [2, 3, 6-28].

In the case of entanglement entropy (EE) created by partitioning a manifold into regions $A$ and $B$, as shown in Fig. 1, the following form of a universal subleading logarithmic contribution — in addition to the area law contribution proportional to the boundary length $l$ of the entangling region $A$ — has been obtained

$$S = al - s \ln(l) - \gamma,$$

where $a$ is the non-universal area law coefficient. The coefficient of the logarithmic correction term $s$ is a universal number determined by the geometric properties of the partition (i.e., if $A$ is a rectangle surrounded by $\bar{A}$) and intrinsic properties of the underlying physical system (i.e., central charge of the conformal field theory at certain critical points) [3]. The last coefficient $\gamma$ is in general of $O(1)$ and if the system becomes gapped, it represents the topological entanglement entropy (TEE) which is given by the logarithm of the quantum dimension of the underlying topological order [7, 8].

The attempts in obtaining the values of the universal coefficients, especially $s$ and $\gamma$, for free scalar theory and few simple $(2+1)d$ CFTs, for example that of $O(N)$ models [2, 18-20, 23, 25] and systems with $Z_2$ topological order [14, 15, 29], have been carried out with quantum Monte Carlo (QMC) simulations in the form of the 2nd Rényi EE. However, the finite size scaling of EE in QMC is still difficult due to the lack of a stable estimator, especially for the challenging lattice models with multi-spin interactions, frustrations and not to mention the even more complex fermionic models where the computational complexity scales with the system size to a higher power [31, 32]. The difficulties of finite size scaling of EE stem from the fact that in the computation of the Rényi entanglement entropy one has to enlarge the configuration space — to create replicas — and perform partial connection of the partition functions between the replicas during the sampling processes.

These are the difficulties we have overcome in this work. Here, based on the recent nonequilibrium measurement of the Rényi EE [27], we developed a new nonequilibrium increment method, dubbed the "Qiu Ku" algorithm, that makes use of the divide-and-conquer proce-
dure of the nonequilibrium process to improve the speed of the simulation and the data quality of the entanglement measurement. We demonstrate the strength and versatility of the method with few difficult yet representative examples of \((2+1)d\) quantum many-body lattice models in which the EEs are notoriously hard to obtain.

Before diving into the physical systems, let’s first explain the meaning of Qiu Ku in the name of our algorithm. As one might has already guessed, Qiu Ku is a Chinese word, where Qiu means autumn and Ku stands for trousers or a pair of pants. Literally, Qiu Ku is a pair of trousers one wears when autumn arrives. It is basically a long leggings made of cotton with the most popular colors grey, white and beige. If one has ever seen a pair of Qiu Ku, as shown in Fig. 1(b), one would immediately know that it has nothing to do with fashion or style. The only reason for its existence is to keep one warm. When jeans can no longer resist the freezing air in winter, just wear Qiu Ku under it, that solves the problem! Qiu Ku, as a pair of robust and reliable pants that efficiently keep you warm in the tough winter, so does our Qiu Ku algorithm, as a robust and reliable means to obtain the entanglement entropy for the difficult yet unavoidable quantum many-body lattice models.

To be more precise, in the case of the computation of the 2nd Rényi EE, Qiu Ku actually captures the geometrical construction of the partition function \([2]\), as shown in Fig. 1 (a) and (b), the geometrical similarity between the 2nd Rényi EE \(S^{(2)}_\Lambda\) and a pair of Qiu Ku is obvious. Such topological unit provides a stable and efficient way of carrying out the computation both conceptually and technically. Beyond the 2-leg geometry, a general \(n\)-leg Qiu Ku manifold can even give rise to the low-lying entanglement spectrum for quantum many-body systems \([3, 33]\).

The structure of the paper is then organized in the following manner. In Sec. II we explain the methodology of our Qiu Ku algorithm, which is the nonequilibrium increment estimator for the Rényi EE \([27, 28]\). Then in Sec. III we show the results of finite size scaling of EE on three representative non-trivial systems, all in \((2+1)d\), starting from the Néel state of the antiferromagnetic Heisenberg model with spontaneously broken continuous symmetry, to the quantum critical point of \(O(3)\) CFT and eventually arriving at the scaling form on toric code toy model with \(Z_2\) topological order and Kagome \(Z_2\) quantum spin liquid state on a torus geometry, where in the former we also compared the results with density-matrix renormalization group (DMRG) on cylinder geometry. In both cases, the TEE of \(2\gamma = 2\ln(2)\) is for the first time being obtained unambiguously in QMC lattice model simulations. And in all these cases, our algorithm provides reliable data with high efficiency and very large system sizes. We then concluded in Sec. IV with the outline of few immediate directions.

II. METHODOLOGY

A. Replica trick

In a quantum many-body system, the entanglement of a subsystem \(A\) with the rest of the system \(\bar{A}\) is most commonly identified by the von-Neumann entropy \(S_A^{(1)} = -\text{Tr}\rho_A \ln \rho_A\). Here \(\rho_A = \text{Tr}_\bar{A}\) is a reduced density matrix defined as the partial trace of the total density matrix \(\rho\) over a complete basis of subsystem \(\bar{A}\). The calculation of the von-Neumann entropy directly from the reduced density matrix in QMC simulations is difficult as usually one does not have the wave-function of a generic quantum many-body system, say, in \((2+1)d\).

\[ S_A^{(n)} = 1 - \frac{1}{n} \ln \left( \frac{Z_A^{(n)}(\Lambda)}{Z^{(n)}} \right), \]

which can be regarded as a generalization of the von-Neumann entropy, approaching the latter when \(n \to 1\), can be calculated by the QMC method. As \(Z^{(n)} = [\text{Tr}(e^{-\beta H})]^{n}\) is the ordinary partition function of \(n\) replicas of the system while \(Z_A^{(n)}(\Lambda)\) is a modified partition function with the boundary condition of area \(A\) of the \(n\) replicas changed according to the value of \(n\). In \(n = 2\), \(Z_A^{(2)}\) is two replicas with sites in \(A\) glued together as depicted in Fig. 1 (a). However for sites in \(\bar{A}\) for each replica, the usual periodical boundary condition is maintained. It is from here one sees \(Z_A^{(2)}\) share the same geometry as the “Qiu Ku”, as shown in Fig. 1 (b).

Based on Eq. (2), various estimators have been introduced to calculate the Rényi entanglement entropy \([12, 18, 23]\). However, the data quality of these estimators has severely limited their usage in higher dimensions and larger sizes. The recent proposal of the nonequilibrium method \([27]\), which borrows Jarzynski’s equality \([35]\) of a nonequilibrium process, has made substantial progress.
in this regard, as it can obtain the 2nd Rényi entanglement with improved data quality on larger system sizes. However as we will show below, this method can still be further improved by the increment trick which gives rise to a nonequilibrium increment method – the Qiu Ku algorithm – for the more complicated quantum many-body systems with frustration, multi-spin interactions, etc.

\[ \frac{\partial \ln Z_A^{(n)}(\lambda)}{\partial \lambda} \] over \( \lambda \in [0,1] \):

\[ S_A^{(n)} = \frac{1}{1-n} \int_0^1 d\lambda \frac{\partial \ln Z_A^{(n)}(\lambda)}{\partial \lambda}. \]

Actually, \( \frac{\partial \ln Z_A^{(n)}(\lambda)}{\partial \lambda} \) can be seen as an Monte Carlo expectation value of \( \langle \frac{\partial \ln Z_A^{(n)}(\lambda)}{\partial \lambda} \rangle \). Ref. [27] puts forward a nonequilibrium process where the system evolves from a configuration of \( Z_A^{(n)} \) to a configuration of \( Z_A^{(n)} \). The total work done in this process is

\[ W_A^{(n)} = -\frac{1}{\beta} \int_{t_i}^{t_f} dt \frac{d\lambda}{dt} \ln g_A(\lambda(t), N_B(t)) \]

\[ = -\frac{1}{\beta} \int_{t_i}^{t_f} dt \frac{d\lambda}{dt} \langle N_B - N_B / (1 - \lambda) \rangle \lambda, \]

where \( \lambda(t_i) = 0 \) and \( \lambda(t_f) = 1 \).

According to the Jarzynski’s equality [35], the free energy difference \( \Delta F \) accumulated in such a path can be extracted by the ensemble average of nonequilibrium measurement by the following relation:

\[ \Delta F = -\beta^{-1} \ln e^{-\beta W}, \]

the overbar refers to the average of an ensemble of nonequilibrium paths. By regarding the Rényi entropy as the free energy difference \( \Delta F \) which can also be defined as the logarithmic of the ratio of two partition functions, the Rényi entanglement entropy can be estimated by

\[ S_A^{(n)} = \frac{1}{1-n} \ln \left( \left( e^{-\beta W_A^{(n)}} \right) \right) \]

when the total quench time \( t_f - t_i \) is finite.

B. Nonequilibrium measurement

In this section the nonequilibrium method [27] will be reviewed as the foundation of our further development. The first step of the nonequilibrium method is to introduce a partition function \( Z_A^{(n)}(\lambda) \) parameterized by \( \lambda \). \( Z_A^{(n)}(\lambda) \) is defined as the sum of a collection of partition functions \( Z_B^{(n)} \) weighted by a binomial distribution \( g_B(\lambda, N_B) = \lambda^N(1 - \lambda)^{N_A - N_B} \). According to this definition, \( Z_A^{(n)}(\lambda) \) can be written as

\[ Z_A^{(n)}(\lambda) = \sum_{B \subseteq A} \lambda^{N_B} (1 - \lambda)^{N_A - N_B} Z_B^{(n)} \]

\[ = \sum_{B \subseteq A} g_A(\lambda, N_B) Z_B^{(n)} \]

where \( Z_A^{(n)}(1) = Z_A^{(n)} \) and \( Z_A^{(n)}(0) = Z_B^{(n)} \). Then the Rényi entanglement entropy can be reexpressed as an integral of \( \frac{\partial \ln Z_A^{(n)}(\lambda)}{\partial \lambda} \) over \( \lambda \in [0,1] \):

\[ S_A^{(n)} = \frac{1}{1-n} \int_0^1 d\lambda \frac{\partial \ln Z_A^{(n)}(\lambda)}{\partial \lambda}. \]

C. Nonequilibrium increment method

Above is the basic concept of the nonequilibrium measurement, our optimization of this method – the Qiu Ku algorithm – takes advantage of the fact that the partition function \( Z_A^{(n)}(\lambda) \) is well defined at every \( \lambda \in [0,1] \), so that we can insert a set of identity elements \( Z_A^{(n)}(kh) \) \( (k = 1, 2, ..., N - 1) \) to the ratio of the two partition functions and rewrite \( \frac{Z_A^{(n)}(1)}{Z_A^{(n)}(0)} \) as

\[ \frac{Z_A^{(n)}(1)}{Z_A^{(n)}(0)} = \frac{Z_A^{(n)}(N h)}{Z_A^{(n)}((N - 1) h)} \frac{Z_A^{(n)}((N - 1) h)}{Z_A^{(n)}((N - 2) h)} \cdots \frac{Z_A^{(n)}(h)}{Z_A^{(n)}(0)} \]

FIG. 2. The schematic plot of the Qiu Ku (nonequilibrium increment) algorithm. It splits a con- currence of \( \lambda \)-parameterized nonequilibrium process into \( N \) independent smaller pieces. For each piece we start from a thermalized state of the partition function \( Z_A^{(n)}(\lambda = kh) \) and carry out the nonequilibrium measurement from \( \lambda = kh \) to \( \lambda = (k + 1)h \). For the \( k = 0 \) piece (the left column), the starting configuration is two independent replicas with ordinary periodical conditions and as the system evolves to \( \lambda = h \) the configuration becomes two modified replicas with some sites in region \( A \) are glued to- gether. The \( k = h \) (the middle column) and \( k = (N - 1)h \) (the right column) pieces are carried in parallel. The final en- tergence entropy is obtained from the summation of these independent pieces, as shown in Eq. [10].
where $N$ is an integer and $\lambda = 1/N$. The increment trick is discussed in Ref. [18] as a way to overcome the limitation of equilibrium measurement of the Rényi entanglement entropy with large entangling regions. Our Qiu Ku algorithm is the nonequilibrium version of the increment trick. According to Eq. (8) and Eq. (4) the Rényi entanglement entropy can be reexpressed as

$$S^{(n)}_A = \frac{1}{1-n} \sum_{k=0,1,\ldots,N-1} \int_{kh}^{(k+1)h} \partial \ln Z^{(n)}_A(\lambda) \frac{d\lambda}{d\lambda}.$$  

Jarzynski’s equality [35] can be carried out on each piece $\frac{Z^{(n)}_A((k+1)h)}{Z^{(n)}_A(kh)}$ ($k = 1, 2, \ldots, N-1$) and the corresponding small piece of integral $\int_{kh}^{(k+1)h} \partial \ln Z^{(n)}_A(\lambda) \frac{d\lambda}{d\lambda}$ regardless of the lower and upper limits of the integral. As a result, we obtain

$$S^{(n)}_A = \frac{1}{1-n} \sum_{k=0,1,\ldots,N-1} \ln \left( \frac{e^{-\beta W^{(n)}_{k,A}}}{\ln Z^{(n)}_A(\lambda)} \right)$$

where $W^{(n)}_{k,A}$’s are defined as Eq. [5] but $\lambda(t_i) = kh$ and $\lambda(t_f) = (k+1)h$ for each small piece in the nonequilibrium process, as shown in Fig. 2. In this way, our Qiu Ku (nonequilibrium increment) measurement of the 2nd Rényi entropy follows the protocol:

1. We perform quantum Monte Carlo simulation on the partition function $Z$ and store the thermalized QMC configuration (one replica) for later use.

2. Prepare two replica configurations as the thermalized configuration of $Z^{(2)}_A$ and then send them to $N$ parallel processes.

3. As shown in Fig. 2 for process $k$ we set the initial value of $\lambda$ to be $\lambda(t_i) = kh$. The value of $\lambda$ controls the probability of sites in $A$ (the entanglement region) joining or leaving the glued geometry of the replicas, with the following probabilities:

$$P_{\text{join}} = \min\{\frac{\lambda}{1 - \lambda}, 1\} \quad P_{\text{leave}} = \min\{\frac{1 - \lambda}{\lambda}, 1\}.$$  

Each MC sweep is then consisted of the following steps:

- Each site in $A$ can choose whether to stay or leave the region according to Eq. (11). After the decision is made, changing the topology, i.e. update the connectivity of the entangling region $A$.
- After the trace structure is determined, carry out the MC updates on the replicas.

4. At this step, we fix $\lambda = kh$ for process $k$ and conduct several MC sweeps to thermalize the configuration at the beginning of the nonequilibrium measurement.

5. Start the nonequilibrium measurement. Increase the value of $\lambda$ by $\Delta \lambda$ and record the value of $\frac{\partial A(\lambda(t_{m+1}), N_{\beta}(t_{m}))}{\partial A(\lambda(t_{m}), N_{\beta}(t_{m}))}$ for each $t_m$ where $\lambda(t_m) = kh + m \Delta \lambda$ and $\lambda(t_0) = \lambda(t_i = kh)$. Then carry out a MC sweep. Repeat this process until $\lambda(t_m)$ reaches the value of $(k+1)h$. For each process, $\left\langle e^{-\beta W^{(n)}_{k,A}} \right\rangle = \left\langle \prod_{m=0}^{\Delta \lambda - 1} \frac{g_A(\lambda(t_{m+1}), N_{\beta}(t_{m}))}{g_A(\lambda(t_{m}), N_{\beta}(t_{m}))} \right\rangle$.

6. In the end, we collect the observable from all the $N$ parallel processes and sum then to get the Rényi entanglement entropy according to Eq. (10).

![FIG. 3. The 2nd Rényi entanglement entropy \( S^{(2)}_A \) of a 2D antiferromagnetic Heisenberg model on a \( 16 \times 8 \) torus versus the quench time \( t_f - t_i \) for the nonequilibrium increment method with the total number of nonequilibrium pieces taken to be \( N = 1, 24, \) and \( N = 48 \). The entangling region \( A \) is a \( 8 \times 8 \) lattice which is chosen as depicted in the inset of Fig. 2.](image)
controlled computation of Rényi EE of larger and more complicated systems such as the Kagome spin model discussed below.

We propose the Qiu Ku algorithm in order to solve this problem. By splitting the \( \lambda \in [0, 1] \) process into hundreds of short processes \( \lambda \in [kh, (k+1)h] \) with \( h = 1/N \), we manage to greatly decrease the simulation time and obtain better convergence and data quality. Fig. 3 shows the comparison of the results of \( S_A^{(2)} \) versus the total quench time for \( N = 1, N = 24 \), and \( N = 48 \). We find that the results of \( N = 24 \) and \( N = 48 \) converge faster and have smaller errorbars than the results of \( N = 1 \). In practice, we will test the convergence of many different values of \( N \) to get the best convergence speed for every model we encounter.

We would like to end this section by mentioning, that, the nonequilibrium increment measurement of entanglement, implemented here and in Ref. [27], could also be applied on other physics systems beyond quantum many-body lattice model and even condensed matter physics [36, 37].

III. RESULTS

In this section, we demonstrate the strength of the Qiu Ku algorithm with three representative directions all computed in \((2 + 1)d\) lattice models. i) Entanglement entropy at the quantum phases with spontaneous continuous symmetry breaking; ii) Entanglement entropy at the quantum critical point with conformal field theory description; iii) Entanglement entropy at the topological order phase with fractional excitations.

A. Entanglement of spontaneous continuous symmetry breaking state

For the system with spontaneously broken continuous symmetry, it is known that the entanglement entropy follows the scaling form of Eq. (1) with the coefficient of the logarithmic correction \( s = N_G(d-1)/2 \), where \( N_G \) is the number of Goldstone modes and \( d \) is the spatial dimension [13]. The origin of this logarithmic correction lies in the interplay between gapless Goldstone modes and restoration of symmetry in a finite system which we are simulating.

We employ the Qiu Ku measurement of \( S_A^{(2)}(L) \) for \( d = 2 \) square lattice antiferromagnetic Heisenberg model whose ground state at the thermodynamic limit breaks the SU(2) spin rotational symmetry, with \( N_G = 2 \). Therefore, we expect the \( s = 1 \) in the present case. The simulation setup is shown in the inset of Fig. 4 that we chose the \( L \times L \) periodic boundary conditions and the entangling region \( A \) of \( L/2 \times L/2 \), in this way, there is no corner between \( A \) and \( \bar{A} \) and the boundary of \( A \) is of length (volume) \( L \). We then obtain the \( S_A^{(2)}(L) \) with Qiu Ku measurement in the finite temperature stochastic series expansion QMC [38, 39] for different \( L \) and fit its scaling form with Eq. (1). From the fitting with \( L \in [40, 160] \), we see \( s = 1.00(9) \), very well consistent with the theoretical prediction and previous similar numerical measurement with the sequential nonequilibrium method [27]. However, we emphasize that here we do not need to use the projection version of the QMC simulation into the valence bond basis to suppress the thermal fluctuations but only use the most basic finite temperature simulation \( 1/T = L \) in the \( s^2 \) basis, and have successfully achieved controlled data quality in shorter simulation time.

B. Entanglement at \((2 + 1)d\) quantum critical point

The second example of the strength of our method is that of the O(3) quantum critical point of \((2 + 1)d\) CFT. For this purpose, following previous literature [28, 40, 41], we design the square lattice \( J_1-J_2 \) Heisenberg model (the columnar dimer lattice model), shown in the inset of Fig. 5. The Hamiltonian reads

\[
H_{J_1-J_2} = J_1 \sum_{\langle ij \rangle} S_i \cdot S_j + J_2 \sum_{\langle ij \rangle'} S_i \cdot S_j,
\]

where \( \langle ij \rangle \) denotes the thin \( J_1 \) bond and \( \langle ij \rangle' \) denotes the thick \( J_2 \) bond and the QCP at \( (J_2/J_1)_c = 1.9095(1) \) is known to fall within the \((2 + 1)d\) O(3) universality class. In the model, the presence of strong \( J_2 \) and weak \( J_1 \) bonds breaks the lattice translation symmetry, i.e. the columnar dimers. Because of the translation symmetry breaking, we choose the entangling region \( A \) so that its boundary avoids strong dimer bonds to better extract
the scaling behavior of the entanglement entropy, similar approaches have also been shown in recent studies of the disorder operator (related with the entanglement entropy) in the same model [41].

We employ the torus geometry of $L \times L$ and create the entangling region with four corners and boundary length $L$. The corners contribute the universal log-correction with the coefficient $\beta$ related with the central charge of the underlying CFT of the $O(3)$ transition. The fitting result is $S_A^{(2)}(L) = 0.167(2)L - 0.081(4)\ln(L) + 0.124(7)$.

We present our results for TEE in two lattice models exhibiting $\mathbb{Z}_2$ topological order. As will be shown below, the Qiu Ku algorithm is able to go beyond limitations in previous Monte-Carlo studies and for the first time the expected value of TEE is obtained in the Balents-Fisher-Girvin kagome spin-1/2 model to unambiguously prove that the ground state is a $\mathbb{Z}_2$ spin liquid. For clarify, in this section we fix $\gamma = \ln 2$.

1. Toy model

To test the efficiency of the Qiu Ku algorithm on systems with topological order, we start with a toy model of $\mathbb{Z}_2$ topological phase on 2d checkerboard lattice [12][44], with the following Hamiltonian,

$$H_{\text{toy}} = -J_p \sum B_i - h \sum \sigma_i^z,$$  (13)

with $B_i \equiv \prod_{x \in A_i} \sigma_i^x$ and $\square$ being the red-colored plaquettes in Fig. 6(a). Note that this Hamiltonian has an extensive number of conserved quantities:

$$A_{\square} = \prod_{x \in \square} \sigma_i^x,$$  (14)

where $\square$ are the white plaquettes in Fig. 6(a). In fact, $-\sum_{x \in \square} B_x - \sum_{\square} A_{\square}$ is the celebrated toric code model.

Below, we will combine finite-size DMRG [45] and QMC methods, to show that the low-energy physics of this model [c.f. Eq.(13)] is equivalent to the well-known toric code model under transverse field [15].

$$H_{\text{TCM}} = -J_s \sum_{\square} A_{\square} - J_p \sum B_i - h \sum \sigma_i^z.$$  

Our DMRG simulations reveal $\langle A_{\square} \rangle = 1$ for the range of $h$ we considered in this section, in the low-temperature regime of $H_{\text{toy}}$. Thus the ground state wavefunctions of $H_{\text{toy}}$ and $H_{\text{TCM}}$ are identical for $J_s > 0$. In the toric code model, the transverse field induces dynamics of $\square$-plaquette excitations. As $h$ increases, such excitations condense and drive a transition to a trivial confined phase. Thus, the toy model $H_{\text{toy}}$ should also experience a continuous phase transition at $h = h_c \approx 0.333$ [46], from a $\mathbb{Z}_2$ deconfined phase to a confined phase.

Within DMRG, the ground state properties of $H_{\text{toy}}$, to be precise, the $L_x \times L_y$ cylinder geometry with open/periodic boundary condition along the horizontal($x$)/vertical($y$) direction, are computed. We can directly calculate the von Neumann entropy

$$S_{\text{NN}} = -\text{tr}(\rho_A \ln \rho_A),$$

where the reduced density matrix $\rho_A = \text{tr}_B |\psi\rangle \langle \psi|$ and the subsystem $A$ and $B$ are both cylinders of size $\frac{L_x}{2} \times L_y$. Fig. 6(b) shows, the plaquette energy $\langle B_{\square}\rangle$ is monotonically decrease as the transverse field $h$ grows, and a
tributions as well as possible corrections from corners. The entanglement area law, in order to directly extract the correction to TEE. Since TEE is the correction to von Neumann entropy for cylinder geometry with \( L_y = 72 \) and \( L_y = 10 \) and 12. The deconfine-confine transition happens at \( h_c = 0.33 \), indicating by the vertical dashed line, which is determined by the dip of the slope \( d(\mathcal{B}_p)/dh \) for both circumferences. (c) The EE at \( h = 0.3 < h_c \) of von Neumann and 2nd Rényi obtained from DMRG with \( 72 \times 12 \) cylinders and the Qiu Ku algorithm of QMC measurements on torus of size \( L \times L \) with \( L = 4, 6, 8, 10 \). (d) The entangling region \( A \) for QMC is \( L \times L/2 \) (similar with Fig. 7 (b) and the boundary length \( l = 2L \). The extrapolation for both DMRG and QMC gives rise to the TEE of values \( \gamma = 0.69(1) \) for cylinder geometry and \( 2\gamma = 1.35(8) \) for torus geometry, respectively.

The change of curve can be seen around the dashed vertical line which highlights the \( h = h_c \approx 0.333 \) (the inset shows \( d(B_p) \) and a dip is found around \( h \approx 0.338 \) for both \( L_y = 10 \) and 12 systems). In Fig. 6 (c), for \( 72 \times 10 \) and \( 72 \times 12 \) cylinders, \( S^{\text{vN}} \) are shown versus \( h \) and exhibit peaks around \( h_c \). The above DMRG data strongly suggest the expected deconfinement-confinement transition at \( h = h_c \approx 0.333 \), consistent with the literature [15, 46].

To further extract the nature of the lower-\( h \) phase, we perform the finite-size scaling and extrapolated the topological entanglement entropy (TEE), i.e. \( \gamma \) in Eq. (1) and note here there is no logarithmic correction, in cylinder geometry for DMRG and torus geometry with Qiu Ku algorithm in QMC. As shown in Fig. 6 (d), we measure both von Neumann entropy \( S^{\text{vN}} \) and Rényi entropy \( S^{(2)} \) in cylinder geometry via DMRG simulations. With circumferences \( L_y = 6, 8, 10, 12 \) and fixed length \( L_x = 72 \), the entanglement entropies are measured and extrapolated with boundary length \( l = 2L_y \), where the extrapolated value of \( \gamma = 0.69(1) \) is obtained. For torus geometry of size \( L \times L \), the Rényi entropy versus \( l = 2L \) is also obtained in QMC simulations, and the extrapolation gives \( 2\gamma = 1.35(8) \). Both of two results are indistinguishable from \( \gamma = \ln(2) \approx 0.693 \).

To this end, the kind of entanglement cut adopted in our simulations, i.e. bipartition of a torus into two cylinders, has the advantage of maximally enlarging the entangling region and also having no corners on the boundary. Naively, the TEE should be \( 2\gamma \) since the boundary has two disconnected components. However, because the boundary curve is topologically nontrivial (a non-contractible cycle on torus), the value of the TEE now depends on which ground state is used for the calculation of the entanglement entropy. More specifically, for \( Z_2 \) toric code and a given entanglement cut (e.g. along the \( y \) direction), only certain choices of ground states on torus yields the expected value \( 2\gamma = 2\ln 2 \). Any other ground state gives a smaller \( \gamma \) and thus a larger \( S \) for the same size. For this reason, these special ground states which saturate \( 2\gamma = 2\ln 2 \) are called minimum entropy states (MES) [48, 49]. Physically, the MES are characterized by a definite anyon flux through the non-contractible entanglement cut, so are in one-to-one correspondence.

2. Minimal Entropy State

We now discuss an important subtlety in our prescription for extracting TEE. Since TEE is the correction to the entanglement area law, in order to directly extract TEE complex prescriptions on planar partitions have been designed [7, 8] to cancel the leading area-law contributions as well as possible corrections from corners. However, when numerically implemented on lattice such prescriptions typically suffer from severe finite-size effects because they require several non-overlapping regions. As an example, the Levin-Wen prescription has been employed to study the TEE of the \( Z_2 \) quantum spin liquid state, which in principle should yield a universal correction of \( 2\gamma = 2\ln 2 \). The best results by now only find \( 2\ln 2 \) instead of \( 2\ln 2 \) [14, 30]. This is because within the allowed system sizes for large-scale numerical simulations, the size of each of the four partitions is at most \( l \approx 10 \), which is too small compared with the characteristic length-scale of the problem (inverse vison gap [47], as discussed in the next section), and the cancellation of the entanglement entropies between different partitions are too noisy to give the TEE in a controlled manner.

FIG. 6. (a) The checkerboard lattice \( Z_2 \) gauge field model with plaquette term \( B_p \) and transverse field \( h \) term as in Eq. (13). (b) and (c), the DMRG results on the plaquette energy term and the von Neumann entropy for cylinder geometry with \( L_z = 72 \) and \( L_y = 10 \) and 12. The deconfine-confine transition happens at \( h_c = 0.33 \), indicating by the vertical dashed line, which is determined by the dip of the slope \( d(B_p)/dh \) for both circumferences. (c) The EE at \( h = 0.3 < h_c \) of von Neumann and 2nd Rényi obtained from DMRG with \( 72 \times 12 \) cylinders and the Qiu Ku algorithm of QMC measurements on torus of size \( L \times L \) with \( L = 4, 6, 8, 10 \). (d) The entangling region \( A \) for QMC is \( L \times L/2 \) (similar with Fig. 7 (b) and the boundary length \( l = 2L \). The extrapolation for both DMRG and QMC gives rise to the TEE of values \( \gamma = 0.69(1) \) for cylinder geometry and \( 2\gamma = 1.35(8) \) for torus geometry, respectively.
with anyon excitations of the topological order.

Numerical simulations in DMRG for the toric code toy model in the previous section and quantum spin liquid system \[15\], have shown that actually due to the energy minimization process in the DMRG for quasi-one-dimensional systems with finite accuracy, that the algorithm actually finds the desired MES with \( \gamma = \ln 2 \) for the cylinder geometry (also shown in Ref. \[15\]) as shown in Fig. 6(d).

Our numerical results in the previous section for \( Z_2 \) toric code toy model suggest that the Monte Carlo sampling processes in the Qiu Ku algorithm serve the same purpose in finding the expected TEE of \( 2\gamma = 2\ln 2 \), as shown in Fig. 6(d). In the next section we further strengthen this point using an even more nontrivial test of the kagome \( Z_2 \) spin liquid lattice model.

3. \( Z_2 \) quantum spin liquid on kagome lattice

The last and yet the most challenging case for the entanglement entropy measurement is that of the Balents-Fisher-Girvin (BFG) model with \( Z_2 \) quantum spin liquid (QSL) ground state \[14\] \[50\] \[57\]. As far as we are aware of, the expected value of the TEE for such kagome \( Z_2 \) QSL model has never been observed in previous QMC computations of the 2nd R"enyi EE.

As illustrated in Fig. 7(a) and (b), the model is defined on a kagome lattice with the following Hamiltonian

\[
H = -J_x \sum_{\langle i,j \rangle} (S_i^+ S_j^- + \text{H.c.}) + J_z \sum_x \left( \sum_{i \in O} S_i^z \right)^2, \tag{15}
\]

where \( J_x \) is the ferromagnetic transverse nearest neighbor interaction and \( J_z \) is the antiferromagnetic longitudinal interactions between any two spins in the hexagon of the kagome plane. The model is known from previous intensive QMC simulations to host a \( Z_2 \) QSL and the transition from ferromagnetic phase to the \( Z_2 \) QSL is at \((J_x/J_z)_c = 0.07076(2)\) \[51\] \[58\]. The identification of a \( Z_2 \) QSL ground state for \((J_x/J_z) < (J_x/J_z)_c\) has been supported by several different types of measurements, such as the unconventional quantum phase transition between ferromagnetic phase and the \( Z_2 \) QSL belonging to the \((2+1)d\) XY* (instead of XY) universality class \[51\] \[52\] \[65\], signifying the existence of the fractional anyon excitations in the \( Z_2 \) QSL phase, and the v-spin-pair spectra with translation symmetry fractionalization \[54\], the vestigial anyon condensation transition towards other topological ordered state \[57\] and the fractional conductivity at the \((2+1)d\) XY* critical point \[59\].

This system is also relevant for on-going experimental efforts in synthesizing and identifying QSL materials in the kagome based quantum magnets such as Zn-paratacamite \( \text{Zn}_x \text{Cu}_{4-x} \text{(OH)}_6 \text{Cl}_2 \) \((0 \leq x \leq 1)\) with Herbertsmithite its full Zn end \[58\] \[62\] and Zn-doped Barlowite \( \text{Zn}_x \text{Cu}_{4-x} \text{(OH)}_6 \text{FBr} \) and Zn-doped Claringbullite \( \text{Zn}_x \text{Cu}_{4-x} \text{(OH)}_6 \text{FCl} \) \((0 \leq x \leq 1)\) \[63\] \[67\] with NMR \[63\], neutron scattering \[68\] \[69\], \( \mu \text{SR} \) \[69\] and thermodynamic measurements \[70\] \[71\]. Better theoretical characterization of the \( Z_2 \) QSL states with fundamental probe such as the EE, and its future connection with the experimental reality such as the existences of magnetic impurities \[71\], will certainly help to eventually reveal the existence of fractionalized excitations and anyonic statistics in kagome based quantum magnets \[72\] \[73\].

However, previous attempts of measuring the second R"enyi EE in this model \[11\] and other similar frustrated kagome spin model \[80\] using the Levin-Wen prescription, were not successful in completely revealing the TEE correction \( \gamma = \ln 2 \) in Eq. (1). As discussed in the previous section, although the Levin-Wen construction \[7\] \[8\] can in principle remove the area law contribution and expose the universal constant correction of \( 2\gamma \), what has been observed at best is a plateau of approximately \( \gamma \) for finite size systems \((e.g. 3 \times 8 \times 8 \) with 3 sites per unit cell of the K"agome lattice), at an intermediate temperature below the vison energy scale \( \sim J_z \), but still comparable or higher than the vison energy scale of \( \sim J_z^2/J_x^2 \). It is understood that the data quality and the computational complexity in simulating even larger sizes and lower temperatures prohibit more precise determination of the 2nd R"enyi EE of the system \[14\].

We found that our Qiu Ku algorithm successfully overcomes these difficulties. We consider two kinds of geometry as shown in Fig. 7(a) and (b), with periodic boundary conditions in both lattice directions and aspect ratios \( L_x/L_y = 2 \) and \( L_x/L_y = 1 \). We choose a cylindrical entangling region \( A \) without corners but with two disconnected boundaries. In this way, the second R"enyi entropy (for a MES) should scale as

\[
S_A^{(2)} = 2\alpha L_y - 2\gamma, \tag{16}
\]

where \( 2L_y \) is now the total length of the boundary and the TEE is \( 2\gamma = 2\ln 2 \approx 1.386 \). We carried out the non-equilibrium increment measurement with \( N = 240 \) parallel pieces and gradually increase the system sizes \( L_x = L \) (so the two torus are \( L \times L/2 \) and \( L \times L \), respectively), with \( J_x/J_z = 0.0625 \) (inside the QSL phase) and \( T = 1/480 \) (below the anyon gaps) \[53\] \[56\].

The results are shown in Fig. 7(c). One can see that as \( L \) increases, \( S_A^{(2)}(L) \) clearly develops a linear behavior, with a converged slope \((\alpha \text{ in Eq. (16)})\) and more importantly, converged intercept 1.4(2), for both systems with different aspect ratios. It is worth noting that the system size where the converged behavior starts to emerge is at \( L \sim 10 \), which is precisely where the previous works in the entanglement entropy measurements have been carried out in this system \[14\]. This observation clearly shows that one really needs to go beyond such finite size limitations to be able to observe the full TEE. Such unprecedented large-scale computation of entanglement entropy only becomes possible because of the Qiu Ku algorithm, which can be easily and robustly implemented for such complicated models resulting in significantly improved
FIG. 7. The second Rényi entanglement entropy of the 2D Kagome $\mathbb{Z}_2$ QSL model in Eq. (16) for different aspect ratios, with the torus geometry of size $L \times \frac{L}{2}$ in (a) and $L \times L$ in (b), $J_x/J_z = 0.0625$ (inside the QSL phase) and $1/T = 480$ (below the anyon gaps). The entangling region $A$ is chosen to be a $\frac{L}{2} \times \frac{L}{2}$ in (a) and $L \times \frac{L}{2}$ in (b), which are half of the system without corner. Therefore the entanglement entropy is expected to only have area law contribution plus a constant $\gamma$ — the topological entanglement entropy. (c). For the $\mathbb{Z}_2$ QSL state on Kagome lattice with torus geometry, the $\gamma = 2 \ln(2)$ signifying the fractionalized statistics of the topological ordered QSL state. Our fitting results with $L = 10, 12, 14, 16, 18$ in type-I geometry (a) and $L = 6, 8, 10$ in type-II geometry (b), give rise to $\gamma = 1.4(2)$.

IV. CONCLUSIONS

To hear the shape of the quantum drum [2, 3] is to employ advanced physical measurement such as the EE on different manifolds to directly reveal the fundamental nature of the quantum states of interests, as in its classical counterpart [1, 4, 5]. These measurements, appear to be different from the traditional probes such as order parameters, structure factors and various correlation functions, actually reveal the deeper information of the wave function, CFT or statistics of the quantum many-body systems. And it is perhaps for the same reason that they are both conceptually and technically hard to be measured and understood, their wider applications in generic investigations in the phases with spontaneous symmetry breaking, at quantum critical point and inside the topological state of quantum matter, are prohibited.

The situation is changing in recent years, new measurements, such as the symmetry domain walls or field lines of emergent gauge field and disorder operators to probe and characterize phases and phase transitions and the associated condensation of extended objects, spontaneously breaking of the so-called higher-form symmetry [29, 41, 74–82] have been successfully designed and implemented in many exotic quantum many-body systems. For example, the recent measurement of disorder operator at the deconfined quantum critical point (DQCP) has unambiguously exhibited the difference between the DQC and other QCPs with unitary CFT description [28, 41].

So does the Qiu Ku algorithm for the entanglement entropy. Here we show that previously difficult EE measurements can be greatly optimized and improved via the nonequilibrium increment method, and in this way, one can investigate the scaling behavior of EE in many $(2 + 1)d$ quantum many-body systems with unprecedentedly large system sizes, controlled errorbars and minimal computational costs. Starting from the three representative cases shown here, one can foresee the implementation and measurement of EE via the Qiu Ku algorithm for other topological ordered phases and phase transitions, interacting fermionic systems such as the Gross-Neveu QCPs with critical Dirac fermions [83], the deconfined QED$_3$ problems of gauge fields coupled to fermion matter fields [84, 85] and the more complicated situations of non-Fermi-liquid and quantum critical metals [87, 95] and hopefully make further suggestions to the on-going experimental search for these strongly entangled quantum matter.
ACKNOWLEDGEMENT

J.R.Z., B.B.C., Z.Y. and Z.Y.M. would like to thank Jonathan D’Emidio for encouraging and fruitful discussion on the details of the quench time of the Jarzynski estimator, and its applications in other subjects. They also thank Chenjie Wang for valuable discussions on the theoretical meaning of MES and related issues. They acknowledge support from the RGC of Hong Kong SAR of China (Grant Nos. 17303019, 17301420, 17301721 and AoE/P-701/20), the Strategic Priority Research Program of the Chinese Academy of Sciences (Grant No. XDB33000000), the K. C. Wong Education Foundation (Grant No. GJTD-2020-01) and the Seed Funding "Quantum-Inspired explainable-AI" at the HKU-TCL Joint Research Centre for Artificial Intelligence. Y.C.W. acknowledges the supports from the NSFC under Grant Nos. 11804383 and 11975024, and the Fundamental Research Funds for the Central Universities under Grant No. 2018QNA39. M.C. acknowledges support from NSF under award number DMR-1846109. We thank the Computational Initiative at the Faculty of Science and the Information Technology Services at the University of Hong Kong and the Tianhe platforms at the National Supercomputer Centers in Tianjin and Guangzhou for their technical support and generous allocation of CPU time. The authors also acknowledge Beijing PARATERA Tech CO.,Ltd. (https://www.paratera.com/) for providing HPC resources that have contributed to the research results reported within this paper.

[1] John L. Cardy and Ingo Peschel, “Finite-size dependence of the free energy in two-dimensional critical systems,” Nuclear Physics B 300, 377–392 (1988).

[2] Pasquale Calabrese and John Cardy, “Entanglement entropy and quantum field theory,” Journal of Statistical Mechanics: Theory and Experiment 2004, P06002 (2004).

[3] Eduardo Fradkin and Joel E. Moore, “Entanglement entropy of 2d conformal quantum critical points: Hearing the shape of a quantum drum,” Phys. Rev. Lett. 97, 050404 (2006).

[4] Mark Kac, “Can one hear the shape of a drum?” The American Mathematical Monthly 73, 1–23 (1966). https://doi.org/10.1080/00029890.1966.11970915.

[5] Jr. H. P. McKean and I. M. Singer, “Curvature and the eigenvalues of the Laplacian,” Journal of Differential Geometry 1, 43 – 69 (1967).

[6] H. Casini and M. Huerta, “Universal terms for the entanglement entropy in 2+1 dimensions,” Nucl. Phys. B 764, 183–201 (2007), arXiv:hep-th/0606256.

[7] Alexei Kitaev and John Preskill, “Topological entanglement entropy,” Phys. Rev. Lett. 96, 110404 (2006).

[8] Michael Levin and Xiao-Gang Wen, “Detecting topological order in a ground state wave function,” Phys. Rev. Lett. 96, 110405 (2006).

[9] Michael M. Wolf, “Violation of the entropic area law for fermions,” Phys. Rev. Lett. 96, 010404 (2006).

[10] Yu-Cheng Lin, Fernec Iglói, and Heiko Rieger, “Entanglement entropy at infinite-randomness fixed points in higher dimensions,” Phys. Rev. Lett. 99, 147202 (2007).

[11] Rong Yu, Hubert Saléur, and Stephan Humeniuk, “Entanglement entropy in the two-dimensional random transverse field ising model,” Phys. Rev. B 77, 140402 (2008).

[12] Matthew B. Hastings, Iván González, Ann B. Kallin, and Roger G. Melko, “Measuring rényi entanglement entropy in quantum monte carlo simulations,” Phys. Rev. Lett. 104, 157201 (2010).

[13] Max A. Metlitski and Tarun Grover, “Entanglement Entropy of Systems with Spontaneously Broken Continuous Symmetry,” arXiv e-prints, arXiv:1112.5166 (2011), arXiv:1112.5166 [cond-mat.str-el].

[14] Sergei V. Isakov, Matthew B. Hastings, and Roger G. Melko, “Topological entanglement entropy of a bose-hubbard spin liquid,” Nature Physics 7, 772 – 775 (2011).

[15] Hong-Chen Jiang, Zhenghan Wang, and Leon Balents, “Identifying topological order by entanglement entropy,” Nature Physics 8, 902 – 905 (2012).

[16] H. Casini and M. Huerta, “Positivity, entanglement entropy, and minimal surfaces,” Journal of High Energy Physics 2012, 87 (2012).

[17] Brian Swingle and T. Senthil, “Structure of entanglement at deconfined quantum critical points,” Phys. Rev. B 86, 155131 (2012).

[18] Stephan Humeniuk and Tommaso Roscilde, “Quantum monte carlo calculation of entanglement rényi entropies for generic quantum systems,” Phys. Rev. B 86, 235116 (2012).

[19] I. A. Kovács and F. Iglói, “Universal logarithmic terms in the entanglement entropy of 2d, 3d and 4d random transverse-field ising models,” EPL 97, 67009 (2012).

[20] Stephen Inglis and Roger G. Melko, “Wang-landau method for calculating rényi entropies in finite-temperature quantum monte carlo simulations,” Phys. Rev. E 87, 013306 (2013).

[21] Stephen Inglis and Roger G. Melko, “Entanglement at a two-dimensional quantum critical point: a T = 0 projector quantum Monte Carlo study,” New J. Phys. 15, 073048 (2013), arXiv:1305.1069.

[22] Ann B. Kallin, Katharine Hyatt, Rajiv R. P. Singh, and Roger G. Melko, “Entanglement at a two-dimensional quantum critical point: A numerical linked-cluster expansion study,” Phys. Rev. Lett. 110, 135702 (2013).

[23] David J. Luitz, Xavier Plat, Nicolas Lallouere, and Fabien Alet, “Improving entanglement and thermodynamic rényi entropy measurements in quantum monte carlo,” Phys. Rev. B 90, 125105 (2014).

[24] A. B. Kallin, E. M. Stoudenmire, P. Fendley, R. R. P. Singh, and R. G. Melko, “Corner contribution to the entanglement entropy of an O(3) quantum critical point in 2 + 1 dimensions,” J. Stat. Mech. 2014, 06009 (2014), arXiv:1401.3504.

[25] Johannes Helmes and Stefan Wessel, “Entanglement entropy scaling in the bilayer heisenberg spin system,” Phys. Rev. B 89, 245120 (2014).

[26] Nicolas Lallouere, “Quantum entanglement in condensed matter systems,” Physics Reports 646, 1–59.
quantum entanglement in condensed matter systems.

[27] Jonathan D’Emidio, “Entanglement entropy from nonequilibrium work,” Phys. Rev. Lett. 124, 110602 (2020).

[28] Jiarui Zhao, Yan-Cheng Wang, Zheng Yan, Meng Cheng, and Zi Yang Meng, “Scaling of entanglement entropy at deconfined quantum criticality,” Phys. Rev. Lett. 128, 010601 (2022).

[29] Jiarui Zhao, Zheng Yan, Meng Cheng, and Zi Yang Meng, “Higher-form symmetry breaking at ising transitions,” Phys. Rev. Research 3, 033024 (2021).

[30] Matthew S. Block, Jonathan D’Emidio, and Ribhu K. Kaul, “Kagome model for a $\pm J$ quantum spin liquid,” Phys. Rev. B 101, 020402 (2020).

[31] Tarun Grover, “Entanglement of interacting fermions in quantum monte carlo calculations,” Phys. Rev. Lett. 111, 130402 (2013).

[32] Fakher F. Assaad, “Stable quantum monte carlo simulations for entanglement spectra of interacting fermions,” Phys. Rev. B 91, 125146 (2015).

[33] Zheng Yan and Zi Yang Meng, “Extract low-lying entanglement spectrum from quantum monte carlo simulation,” (2021), arXiv:2112.05886 [cond-mat.str-el].

[34] L. Tagliacozzo, G. Evenbly, and G. Vidal, “Simulation of two-dimensional quantum systems using a tree tensor network that exploits the entropic area law,” Phys. Rev. B 80, 235127 (2009).

[35] C. Jarzynski, “Nonequilibrium equality for free energy differences,” Phys. Rev. Lett. 78, 2690–2693 (1997).

[36] Michael R. Shirts, Eric Bair, Giles Hooker, and Vineet B. Shenoy, “Dynamical signa- lization in an easy-axis kagome antiferromagnet,” Phys. Rev. Lett. 113, 5347 (2021).

[37] Matteo Palassini and Felix Ritort, “Improving free-energy estimates from unidirectional work measurements: Theory and experiment,” Phys. Rev. Lett. 107, 060601 (2011).

[38] Anders W. Sandvik, “Stochastic series expansion method with operator-loop update,” Phys. Rev. B 59, R14157–R14160 (1999).

[39] Olav F. Syljuåsen and Anders W. Sandvik, “Quantum monte carlo with directed loops,” Phys. Rev. E 66, 046701 (2002).

[40] Nvsen Ma, Phillip Weingber, Hui Shao, Wenan Guo, Dao-Xin Yao, and Anders W. Sandvik, “Anomalous quantum-critical scaling corrections in two-dimensional antiferromagnets,” Phys. Rev. Lett. 121, 117202 (2018).

[41] Yan-Cheng Wang, Nvsen Ma, Meng Cheng, and Zi Yang Meng, “Scaling of disorder operator at deconfined quantum criticality,” arXiv e-prints, arXiv:2106.01380 (2021), arXiv:2106.01380 [cond-mat.str-el].

[42] F. F. Assaad and Tarun Grover, “Simple fermionic model of deconfined phases and phase transitions,” Phys. Rev. X 6, 041049 (2016).

[43] Chuang Chen, Xiao Yan Xu, Yang Qi, and Zi Yang Meng, “Metal to orthogonal metal transition,” Chinese Physics Letters 37, 047103 (2020).

[44] Chuang Chen, Tian Yuan, Yang Qi, and Zi Yang Meng, “Fermi arcs and pseudogap in a lattice model of a doped orthogonal metal,” Phys. Rev. B 103, 165131 (2021).

[45] Steven R. White, “Density matrix formulation for quantum renormalization groups,” Phys. Rev. Lett. 69, 2863–2866 (1992).

[46] Fengcheng Wu, Youjin Deng, and Nikolay Prokof’ev, “Phase diagram of the toric code model in a parallel magnetic field,” Phys. Rev. B 85, 195104 (2012).

[47] Zheng Yan, Yan-Cheng Wang, Nvsen Ma, Yang Qi, and Zi Yang Meng, “Topological phase transition and single/multi anyon dynamics of $Z_2$ spin liquid,” npj Quantum Mater. 39 (2021).

[48] Shiying Dong, Eduardo Fradkin, Robert G Leigh, and Sean Nowling, “Topological entanglement entropy in chern-simons theories and quantum hall fluids,” Journal of High Energy Physics 2008, 016–016 (2008).

[49] Yi Zhang, Tarun Grover, Ari Turner, Masaki Oshikawa, and Ashvin Vishwanath, “Quasiparticle statistics and braiding from ground-state entanglement,” Phys. Rev. B 85, 235151 (2012).

[50] L. Balents, M. P. A. Fisher, and S. M. Girvin, “Fractionalization in an easy-axis kagome antiferromagnet,” Phys. Rev. B 65, 224412 (2002).

[51] S. V. Isakov, Yong Baek Kim, and A. Paramekanti, “Spin-liquid phase in a spin-1/2 quantum magnet on the kagome lattice,” Phys. Rev. Lett. 97, 207204 (2006).

[52] Sergei V. Isakov, Roger G. Melko, and Matthew B. Hastings, “Universal signatures of fractionalized quantum critical points,” Science 335, 193–195 (2012).

[53] Yan-Cheng Wang, Chen Fang, Meng Cheng, Yang Qi, and Zi Yang Meng, “Topological Spin Liquid with Symmetry- Protected Edge States,” arXiv e-prints, arXiv:1701.01552 (2017), arXiv:1701.01552 [cond-mat.str-el].

[54] Guang-Yu Sun, Yan-Cheng Wang, Chen Fang, Yang Qi, Meng Cheng, and Zi Yang Meng, “Dynamical signature of symmetry fractionalization in frustrated magnets,” Phys. Rev. Lett. 121, 077201 (2018).

[55] Yan-Cheng Wang, Xue-Feng Zhang, Frank Pollmann, Meng Cheng, and Zi Yang Meng, “Quantum spin liquid with even ising gauge field structure on kagome lattice,” Phys. Rev. Lett. 121, 057202 (2018).

[56] Yan-Cheng Wang, Meng Cheng, William Witzczak-Krempa, and Zi Yang Meng, “Fractionalized conductivity and emergent self-duality near topological phase transitions,” Nature Communications 12, 5347 (2021).

[57] Yan-Cheng Wang, Zheng Yan, Chenjie Wang, Yang Qi, and Zi Yang Meng, “Vestigial anyon condensation in kagome quantum spin liquids,” Phys. Rev. B 103, 014408 (2021).

[58] Matthew P. Shores, Emily A. Nytko, Bart M. Bartlett, and Daniel G. Nocera, “A structurally perfect s = 1/2 kagome antiferromagnet,” J. Am. Chem. Soc. 127, 13462–13463 (2005).

[59] J. S. Helton, K. Matan, M. P. Shores, E. A. Nytko, Bart M. Bartlett, Y. Yoshida, Y. Takano, A. Suslov, Y. Qiu, J.-H. Chung, D. G. Nocera, and Y. S. Lee, “Spin dynamics of the spin-1/2 kagome lattice antiferromagnet $\text{ZnCu}_2\text{Cl}_4\text{O}_4\text{Cl}_2$,” Phys. Rev. Lett. 98, 107204 (2007).

[60] T. H. Han, J. S. Helton, S. Chiu, D. G. Nocera, J. A. Rodriguez-Rivera, C. Broholm, and Y. S. Lee, “Fractionalized excitations in the spin-liquid state of a kagome-lattice antiferromagnet,” Nature 492, 406–410 (2012).

[61] Mingxuan Fu, Takashi Imai, Tian-Heng Han, and Young S. Lee, “Evidence for a gapped spin-liquid ground state in a kagome heisenberg antiferromagnet,” Science 350, 655–658 (2015).

[62] M. R. Norman, “Colloquium: Herbstsmithite and the search for the quantum spin liquid,” Rev. Mod. Phys.
Zili Feng, Yuan Wei, Xin Meng, Wei Yi, Shanshan Miao, Jie Ma, Jianlin Luo, Shiliang Li, Zi Yang Meng, Youguo Shi, “From claringbullite to a new spin liquid state of quantum excitations in the kagome lattice antiferromagnet,” ArXiv e-prints (2017), arXiv:2102.06223 [cond-mat.str-el].

Xiao-Chuan Wu, Chao-Ming Jian, and Cenke Xu, “Universal Features of Higher-Form Symmetries at Phase Transitions,” SciPost Phys. 11, 35 (2021).

Xiaoyan Ma, Zili Feng, Devashibhai Adroja, Adrian Hillier, Pabitra Biswas, Anatoliy Senyshyn, Andreas Hoser, Jia-Wei Mei, Zi Yang Meng, Huiqian Luo, Youguo Shi, and Shiliang Li, “Magnetic phase diagram of Cu_{6−x}Zn_{x}(OH)_{6}FBr studied by neutron-diffraction and μSR techniques,” Chinese Physics Letters 37, 107503 (2020).

Yuan Wei, Xiao Yan Xu, Kai Sun, Yoni Schattner, Erez Berg, and Zi Yang Meng, “Non-fermi liquid at (2+1)|Fermi| symmetry – a holographic and entanglement view of topological phase transitions,” (2019), arXiv:1912.13492.

Xiao Yan Xu, Zi Hong Liu, Guo Qing Zhang, and Zi Yang Meng, “Revealing fermionic quantum criticality from new monte carlo techniques,” Journal of Physics: Condensed Matter 33 (2021).

Z. H. Liu, G. Pan, X. Y. Xu, K. Sun, and Z. Y. Meng, “Itinerant quantum critical point with fermion pockets and hotspots,” Proc Natl Acad Sci U S A 116, 16760–16767 (2019).

Xiao Yan Xu, Avraham Klein, Kai Sun, Andrey V. Chubukov, and Zi Yang Meng, “Identification of non-fermi liquid fermionic self-energy from quantum monte carlo data,” npj Quantum Materials 5, 65 (2020).

Avraham Klein, Andrey V. Chubukov, Yoni Schattner, and Erez Berg, “Normal state properties of quantum critical metals at finite temperature,” Phys. Rev. X 10, 031053 (2020).

Bin Shen, Yongjun Zhang, Yashar Komijani, Michael Nicklas, Robert Borth, An Wang, Ye Chen, Zhiyong Nie, Rui Li, Xin Lu, Hanoh Lee, Michael Smidman, Frank Steglich, Piers Coleman, and Huiqiu Yuan, “Strange-metal behaviour in a pure ferromagnetic kondo lattice,” Nature 579, 51 – 55 (2020).
[93] Weilun Jiang, Yuzhi Liu, Avraham Klein, Yuxuan Wang, Kai Sun, Andrey V. Chubukov, and Zi Yang Meng, “Pseudogap and superconductivity emerging from quantum magnetic fluctuations: a Monte Carlo study,” arXiv e-prints, arXiv:2105.03639 (2021), arXiv:2105.03639 [cond-mat.str-el].

[94] Yi Wu, Yongjun Zhang, Feng Du, Bin Shen, Hao Zheng, Yuan Fang, Michael Smidman, Chao Cao, Frank Steglich, Huiqiu Yuan, Jonathan D. Denlinger, and Yang Liu, “Anisotropic $c-f$ hybridization in the ferromagnetic quantum critical metal $\text{CeRh}_6\text{Ge}_4$,” Phys. Rev. Lett. 126, 216406 (2021).

[95] Yuzhi Liu, Weilun Jiang, Avraham Klein, Yuxuan Wang, Kai Sun, Andrey V. Chubukov, and Zi Yang Meng, “The dynamical exponent of a quantum critical itinerant ferromagnet: a Monte Carlo study,” arXiv e-prints, arXiv:2106.12601 (2021), arXiv:2106.12601 [cond-mat.str-el].