ON EXTENSIONS AND SPECTRAL PROBLEMS FOR FOURTH ORDER DIFFERENTIAL OPERATOR EQUATION

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Abstract  The aim of the paper is firstly to study domains of definitions in terms of boundary conditions of minimal and maximal operators, as well as selfadjoint extensions of a minimal operator associated with the fourth-order differential operator equation. Further, we give necessary and sufficient conditions for that operators to have a purely discrete or continuous spectrum, to exist extension with resolvent from \( \sigma_p \), study asymptotics of spectrum in case pure discrete spectrum. Finally, give the new and more general method for evaluations of regularized traces of operators with discrete spectrum associated with one class boundary value problems.

Keywords: Hilbert space, differential operator equation, selfadjoint extensions with exit from space, spectrum, eigenvalues, trace class operators, regularized trace.

Mathematics Subject Classification: 34B05, 34G20, 34L20, 34L05, 47A05, 47A10.

1. Introduction

Our aim is first to study domains of definition of minimal and maximal operators generated by a differential operator expression in a space which is larger than one where the differential expression is considered. Such operators arise upon consideration of boundary value problems for differential equations when boundary conditions also contain an eigenvalue parameter. Secondly, to give boundary conditions for defining selfadjoint extensions, extensions with a discrete or continuous spectrum. Thirdly, to derive an asymptotic formula for a spectrum in the case of purely discrete spectrum, and finally, to give a new method for finding regularized trace of the operator associated with the corresponding boundary value problem in one special case. We show a new treat for the deriving the trace formula, which is more general in comparison with one applied in our previous works and might be applied in future studies.

Consider in \( L_2(H, (0,1)) \), where \( H \) is an abstract separable Hilbert space, the following differential expression with operator coefficients

\[
ly \equiv y^{IV}(t) + Ay(t) + q(t)y(t)
\]

(1.1)

Here \( A \) and \( q(t) \) are operator coefficients. Our assumptions about them are the followings (later, when deriving trace formula we will put some additional requirements on \( A \) and \( q(t) \)):

1. \( A \) is a selfadjoint operator in \( H \), moreover \( A > I \), where \( I \) is an identity operator in \( H \), and \( A^{-1} \in \sigma_{\infty} \).

2. \( q(t) \) is a weakly measurable, selfadjoint, bounded operator -valued function in \( H \) for each \( t \in [0,1] \).

So, \( q(t) \) is bounded in \( H \), while the operator \( A \) is bounded only from below. Under that conditions \( q(t) \) is bounded also in \( L_2(H, (0,1)) \).

Consider the direct sum space \( H_1 = L_2(H, (0,1)) \oplus H_2^Q \) with the elements \( Y = (y(t), y_1, y_2) \), \( Z = (z(t), z_1, z_2) \), where \( y_1, y_2, z_1, z_2 \in H \). A scalar product in \( H_1 \) is defined by

\[
(Y, Z)_1 = (y(t), z(t))_{L_2(H, (0,1))} + \left( Q_1^{-\frac{1}{2}} y_1, Q_1^{-\frac{1}{2}} z_1 \right) + \left( Q_2^{-\frac{1}{2}} y_2, Q_2^{-\frac{1}{2}} z_2 \right),
\]

(1.2)
$(\cdot, \cdot)$ is a scalar product in $H$, $Q_1$ and $Q_2$ are self-adjoint positive-definite operators in $H$.

Define in $H_1$ an operator $L_0'$ in the following way:

$$D\left( L_0' \right) = \left\{ Y \mid Y = \left\{ y(t), Q_1 y(1), Q_2 y'(1), \right\}, y(1) \in D(Q_1), y'(1) \in D(Q_2), L_0' Y = \left\{ ly, -y'''(1), y'(1) \right\}, q(t) \equiv 0 \right\}$$

where $C_0^\infty(H_{\infty}, (0, 1])$ is a class of vector functions with the values from $H_{\infty} \equiv \bigcap_{j=1}^{\infty} D(A^j)$ and finite in the vicinity of zero. By integrating by parts it might be easily verified that $L_0'$ is symmetric in $H_1$. Denote its closure by $L_0$ and call it a minimal operator. Adjoint of $L_0$ is denoted by $L_0^*$ and called a maximal operator.

In [1] the following boundary value problem was considered:

1) The author defines a minimal symmetric operator associated with problem (1.3)-(1.5) in space $L_2(H, (0, b))$ as a closure of the symmetric operator $L_0'$ with domain $D\left( L_0' \right) = C_0^\infty(H_{\infty}, (0, b])$, which is a set of infinitely many times differentiable vector functions with the values from $H_{\infty}$, finite in the vicinity of $b$ and $L_0 y(t) \equiv ly$. It is stated in the work that the domain of closure of $L_0'$ is given by (1.5), which obviously is not true, since the closure in $L_2(H, (0, b))$ are the functions satisfying $y(b) = y'(b) = y''(b) = y'''(b) = 0$ which are just a part of the set of functions satisfying (1.5).

The adjoint operator is denoted by $L_0^*$.

2) In Theorem 1 from [1] it is stated that if $Q_1 = Q_2$ then the closure of the operator $L_B'$ with domain of definition $\{ Y = \{ y(t), y_1, y_2 \}, y_1, y_2 \in H_1, y(t) \in C^\infty(H_{\infty}, (0, b]) \}$, where $y_1 = Q_1 y(0), y_2 = Q_2 y'(0)$ and $L_B' = \{ L_0 y(t), y''(0), -y''(0) \}$ gives operator whose domain of functions satisfying conditions (1.5) and which is selfadjoint. But this is obviously not true, since the indicated closure consists of the vectors $Y = \{ y(t), y_1, y_2 \}, y(t) \in W_2^4(H, (0, b)), t y \in L_2(H, (0, b))$ and $q(t) \in D(A)$.

For that reason, we decide don’t refer to that but give definitions of the minimal and maximal operators and selfadjoint extensions, then treat some spectral questions for operator generated by $l[y]$ in direct sum space. Thus:

1. Define a minimal symmetric operator corresponding to differential expression (1.1) with exit to direct sum space and give boundary conditions defining selfadjoint extensions of that operator.

2. Give conditions for that extension to be discrete or to have spectrum filling some interval from the real axis. Also, define selfadjoint extensions whose resolvents are from $\sigma_p$ which is Schatten von Neumann class of functions. For that, we will follow a succession of steps similar to steps [3], where the domains of minimal and maximal operators are studied. Note here also [4,5], where selfadjoint extensions and eigenvalue asymptotics for Sturm-Liouville operator equation by exiting to a larger space and [2], where selfadjoint
extensions of operators generated by $2n$-th order differential operator expressions (ones having unbounded operator coefficients) without exit to a larger space are studied.

3. Consider the eigenvalue problem

$$ly = \lambda y \quad (1.6)$$

$$y(0) = y''(0) = 0 \quad (1.7)$$

$$-y'''(1) = \lambda Q_1 y(1), \quad y''(1) = \lambda Q_2 y'(1). \quad (1.8)$$

The operator corresponding to this problem is one of selfadjoint extensions of the minimal operator corresponding to (1.1) with an exit to a larger space. We study its eigenvalue distribution.

4. Give a trace formula for an operator associated with (1.6)-(1.8). For traces, a more general method than one used in our previous works will be suggested. The suggested method will let to treat these problems from unique point of view.

Results for such problems are applicable to boundary value problems for some classes of partial differential equations.

Recall that since $q(t)$ is bounded in $L_2(H, (0, 1))$, the existence of $q(t)$ in $l[y]$ is not essential for domain of definitions of minimal, maximal operators and self-adjoint extensions, that is why when studying these questions we will take $q(t) \equiv 0$.

We will use the following notations: $H_j$ (a scale of Hilbert spaces generated by $A$) as always denotes $(j > 0)$ a completion of $D(A^j)$ with respect to the scalar product $(f, g)_j = (A^j f, A^j g)$ (see [3]) for $j > k$, $H_j \subseteq H_k \subseteq H$. $H_{-j}$ is a space with a negative norm constructed with respect to $H$, $H_j$. $H_{-j}$ is the completion of $H$ in the norm $\|A^{-j} f\|$. $H_{-j}$ is usually considered as an adjoint to $H_j$ with respect to the scalar product $(,)$, so that for $g \in H_{-j}$, $f \in H_j$, $g(f)$ will be written as $(f, g)$. The operator $A$ is an isometric operator from $H_1$ to $H$. The adjoint of $A$ denoted by $A$ acts from $H$ to $H_{-1}$ and is the extension of $A$.

2. Domains of Definition of Adjoint Operator, Selfadjoint Extensions, Selfadjoint Extensions with Compact Resolvents

Recall that $q(t) \equiv 0$.

**Theorem 2.1.** The domain $D(L_0^*)$ of $L_0$ consists of those elements $Y = (y(t), Q_1y(1), Q_2y'(1))$ of space $H_1 = L_2(H, (0, 1)) \oplus H_2$, where

$$y(t) = e^{\alpha_1 \sqrt{A} t} f_1 + e^{\alpha_2 \sqrt{A} t} f_2 + e^{-\alpha_1 \sqrt{A} (t-1)} g_1 + e^{-\alpha_2 \sqrt{A} (t-1)} g_2 + \int_0^1 G(t, s) h(s) ds, \quad (2.1)$$

$$G(t, s) = \frac{e^{\alpha_1 \sqrt{A} (t-s)}}{4\alpha_1^3} + \frac{e^{\alpha_2 \sqrt{A} (t-s)}}{4\alpha_2^3} A^{-\frac{s}{2}}, \quad (2.2)$$

$$f_1, f_2 \in H_{-\frac{3}{2}}, g_1, g_2 \in H_{\frac{3}{2}} \quad (or \quad A^\frac{3}{2} g_i \in H, \quad i = 1, 2), \quad (2.3)$$

$Q_1 g_i \in H, \quad A^\frac{3}{2} Q_2 g_i \epsilon H$, for $i = 1, 2$,

$\alpha_1, \alpha_2$ are the roots of the equation $\alpha^4 = -1$ with negative real parts, so, $\alpha_1 = e^{\frac{3i\pi}{4}}, \alpha_2 = e^{\frac{i\pi}{4}}$ and

$$L_0^* Y = \left(l y, -y'''(1), y''(1)\right). \quad (2.4)$$
Since \( g_1, g_2 \in H \) and \( f(A)g = f(A^*)g \) for a bounded function \( f \) on \( H \), then in (2.1) in third and fourth terms we take \( A \) but not \( A^* \).

**Proof.** In [2], \( ly = (-1)^ny^{2n} + Ay \) in \( L_2^0(\mathbb{R}, (0, 1)) \) is considered, and there it was shown that the values of \( y(t) \) at endpoints of the interval are from a larger space than \( H \), namely, \( f_1, f_2, g_1, g_2 \in H_{-\frac{1}{2}} \). But we take \( f_1, f_2, H_{-\frac{1}{2}} \), and \( g_1, g_2 \in H_{-\frac{1}{2}}, Q_1g_i \in H, A^\dagger Q_2g_i \in H \), for \( i = 1, 2 \), because we define an operator in \( H_1 = L_2^0(\mathbb{R}, (0, 1)) \oplus H_2^0 \) and for that reason \( y_1'(1), y_1''(1), Q_1y_1(1), Q_2y_1(1) \) must be from \( H \).

As it follows from Theorem 2.1 (relations (2.1),(2.3)) values of \( y(t) \) at zero are distributions.

Let \( Y_0 \) and \( Y_0' \) be defined in \( H^2 \) by

\[
Y_0 = \left\{ y_0, y_0' \right\}, \quad Y_0' = \left\{ y_0'', y_0''' \right\}
\]

where \( y_0, y_0', y_0'', y_0''' \) are regularized values of \( y(t) \) and its derivatives at zero which are obtained from [2], by taking \( n = 2 \)

\[
y_0 = A^{-\frac{1}{2}}y(0), \quad y_0' = A^{-\frac{1}{2}}y'(0), \quad y_0'' = A^\frac{1}{2} \left(y''(0) - \sqrt{2}A^\frac{1}{2}y'(0) + A^\frac{1}{2}y(0) \right)
\]

\[
y_0''' = A^\frac{1}{2} \left(-y'''(0) + A^\frac{1}{2}y''(0) + \sqrt{2}A^\frac{1}{2}y'(0) \right)
\]  (2.6)

Now let \( \tilde{y} = \left\{ y_0, y_0' \right\}, \quad \tilde{y}' = \left\{ y_0'', y_0''' \right\} \) be arbitrary vectors from \( H^2 \). By the similar way done in [4], [3] the following lemma might be easily proved

**Lemma 2.1** For each \( \{\tilde{y}, \tilde{y}'\} \in H^4 \) there exists \( Y = \left\{ y(t), Q_1y_1(1), Q_2y_1'(1) \right\} \in D(L_0^0) \) so that \( y_0, y_0', y_0'', y_0''' \) are defined by (2.6)

By the methods of the work [6] (where condition for binary relations to be hermitian is given), [3],[2] and [4] (where Sturm-Liouville operator with unbounded operator coefficient and with exit to larger space is defined) the following theorem might be easily verified.

**Theorem 2.2.** The domain of self-adjoint extensions \( L_0^0 \) of operator \( L_0 \) in \( H_1 \) consists of those \( Y \in D \left( L_0^0 \right) \) which satisfy also

\[
\cos CY_0' - \sin CY_0 = 0
\]  (2.7)

with a selfadjoint operator \( C \) on \( H^2 : C = (C_1, C_2), C_i \) act in \( H \) for \( i = 1, 2 \), and \( Y_0, Y_0' \) are defined by (2.5), (2.6). Without lost of generalization for simplifying notations we will take \( C = (C, C) \).

**Note 2.1.** Since \( q(t) \) is a selfadjoint and bounded operator in \( H_1 \) the statement of the theorem remains true also for \( L = L_0 + Q \), where \( Q = \{ q(t), 0, 0 \} \)

Denote selfadjoint extension of \( L \) by \( L_s \).

**Theorem 2.3.** A spectrum of selfadjoint extensions \( L_0^0 \) of minimal operator \( L_0 \) is discrete if and only if \( \cos C, Q_1A^{-\frac{1}{2}}, Q_2A^{-\frac{1}{2}} \) are compact.

**Proof.** Let \( \lambda \) be non-real, then for selfadjoint extension \( L_0^0 \) and \( \tilde{h} = (h(t), h_1, h_2) \in H_1, Y \in D \left( L_0^0 \right), \) we consider the equation

\[
L_0^0Y - \lambda Y = \tilde{h}
\]  (2.8)

or in the equivalent form

\[
y^{IV} + Ay - \lambda y = h(t)
\]  (2.9)
- \frac{y'''}{1} - \lambda Q_1 y(1) = h_1, \quad (2.10)
\frac{y''}{1} - \lambda Q_2 y'(1) = h_2 \quad (2.11)

moreover, \( Y \) as a vector from the domain of \( L_0^s \) satisfies the condition (2.7). From (2.8)-(2.11) the resolvent \( R_\lambda(L_0^s) \) of \( L_0^s \) is

\[
R_\lambda(L_0^s) \tilde{h} = Y = \begin{pmatrix} y(t, \lambda) \\ Q_1 y(1) \\ Q_2 y'(1) \end{pmatrix} \quad (2.12)
\]

where \( y(t, \lambda) \) is the solution of (2.9) defined by

\[
y(t, \lambda) = e^{\alpha_1 \sqrt{A - \lambda t}} A^\frac{1}{2} f_1 + e^{\alpha_2 \sqrt{A - \lambda t}} A^\frac{1}{2} f_2 + e^{-\alpha_1 \sqrt{A - \lambda (t-1)}} A^{-\frac{3}{4}} g_1 +
+ e^{-\alpha_2 \sqrt{A - \lambda (t-1)}} A^{-\frac{3}{4}} g_2 + \int_0^1 G(t, s, \lambda) h(s) ds, \quad (2.13)
\]

where

\[
f_1, f_2 \in H, g_1, g_2 \in H, Q_1 A^{-\frac{3}{4}} g_1 \in H, \quad Q_2 A^{-\frac{3}{4}} g_i \in H, \quad \text{for } i = 1, 2. \quad (2.14)
\]

Introduce the notations:

\[
\omega_j(t, \lambda) = \begin{cases} e^{\alpha_1 \sqrt{A - \lambda t}} A^\frac{1}{2}, & i = 1, 2, j = 1, 2 \\ e^{-\alpha_1 \sqrt{A - \lambda (t-1)}} A^{-\frac{3}{4}}, & i = 1, 2, j = 3, 4 \end{cases} \quad (2.15)
\]

where, \( \omega_j(t, \lambda) f_i, (j = 1, 2, \ i = 1, 2) \) and \( \omega_j(t, \lambda) g_i (j = 3, 4, \ i = 1, 2) \) from a fundamental system of solutions of the homogeneous equation corresponding to (2.9)

Let also \( Y_0 = \{y_0, y_0'\}, \quad Y_0'' = \{y_0'', y_0''\} \) whose elements are defined by (2.6). With (2.13)-(2.15) in mind we can write:

\[
R_\lambda(L_0^s) \tilde{h} = \begin{pmatrix} y(t) \\ Q_1 y(1) \\ Q_2 y'(1) \end{pmatrix} =
\begin{pmatrix}
\omega_1(t, \lambda) f_1 + \omega_2(t, \lambda) f_2 + \omega_3(t, \lambda) g_1 + \omega_4(t, \lambda) g_2 + \int_0^1 G(t, s, \lambda) h(s) ds \\
Q_1 \omega_1(1, \lambda) f_1 + Q_1 \omega_2(1, \lambda) f_2 + Q_1 A^{-\frac{3}{4}} g_1 + Q_1 A^{-\frac{3}{4}} g_2 + Q_1 \int_0^1 G(1, s, \lambda) h(s) ds \\
Q_2 \omega_1(1, \lambda) f_1 + Q_2 \omega_2(1, \lambda) f_2 - Q_2 k_1 A^{-\frac{3}{4}} g_1 - Q_2 k_2 A^{-\frac{3}{4}} g_2 + Q_2 \int_0^1 G_i(t, s, \lambda) h(s) ds \\
\end{pmatrix}
\]

\[
k_i = \alpha_i \sqrt{A - \lambda}, \quad i = 1, 2. \quad (2.16)
\]

Rewrite the last relation in the following matrix form:

\[
R_\lambda(L_0^s) \tilde{h} = \begin{pmatrix} \omega_1(t, \lambda) & \omega_2(t, \lambda) & \omega_3(t, \lambda) & \omega_4(t, \lambda) \\ Q_1 \omega_1(1, \lambda) & Q_1 \omega_2(1, \lambda) & Q_1 A^{-\frac{3}{4}} & Q_1 A^{-\frac{3}{4}} \\ Q_2 k_1 \omega_1(1, \lambda) & Q_2 k_2 \omega_2(1, \lambda) & -Q_2 k_1 A^{-\frac{3}{4}} & -Q_2 k_2 A^{-\frac{3}{4}} \\ \end{pmatrix}
\begin{pmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{pmatrix}
\]

\[
+ \begin{pmatrix} \int_0^1 G(t, s, \lambda) h(s) ds \\ Q_1 \int_0^1 G(1, s, \lambda) h(s) ds \\ Q_2 \int_0^1 G_i(t, s, \lambda) h(s) ds \end{pmatrix} \quad (2.16)
\]

Define the vector \( \begin{pmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{pmatrix} \) from equalities (2.7), (2.10), (2.11) by substituting \( y(t, \lambda) \) from (2.13) into them. Introduce some notations.
Firstly, let

\[ B_j(t, \lambda) = \begin{cases} \frac{e^{-\alpha_i \sqrt{A - \lambda t}}}{4\alpha_i^2} A^{-\frac{3}{2}}, & i = 1, 2, \ j = 1, 2 \\ \frac{e^{\alpha_i \sqrt{A - \lambda t}}}{4\alpha_i^2} A^{-\frac{3}{2}}, & i = 1, 2, \ j = 3, 4 \end{cases} \]

Note that with this notation

\[ G(t, s, \lambda) = \begin{cases} B_3(t, \lambda)e^{\alpha_1 \sqrt{A - \lambda t} - s} + B_4(t, \lambda)e^{\alpha_2 \sqrt{A - \lambda t} - s}, & s \leq t \\ B_1(t, \lambda)e^{\alpha_1 \sqrt{A - \lambda t} s} + B_2(t, \lambda)e^{\alpha_2 \sqrt{A - \lambda t} s}, & t \leq s \end{cases} \]  

(2.17)

Now we will write boundary conditions (2.7),(2.10),(2.11) in the matrix form with \( y(t) \) defined from (2.13) and with (2.17) in mind. For simplifying \( G^{(n)}(0, s, \lambda), n = 1, 4, \) (since \( t = 0 \leq s \) we have to use the second row expressions from (2.17)). Introduce the following notations, obtained by taking \( B_j(t, \lambda) \) in (2.6) instead of \( y(t) \):

\[ B'_{1,0} = A_{\frac{1}{2}} \left[ -\frac{1}{4\alpha_1^2} (\alpha_1 \sqrt{A - \lambda t}) A^{-\frac{1}{2}} + \frac{1}{4\alpha_1^2} \sqrt{A - \lambda t} A^{-\frac{1}{2}} + \frac{\sqrt{2}}{4\alpha_1^2} \right] \]

\[ B''_{1,0} = A_{\frac{1}{2}} \left[ -\frac{1}{4\alpha_1^2} (\alpha_2 \sqrt{A - \lambda t}) A^{-\frac{1}{2}} + \frac{1}{4\alpha_1^2} \sqrt{A - \lambda t} A^{-\frac{1}{2}} + \frac{\sqrt{2}}{4\alpha_1^2} \right] \]

\[ B'''_{1,0} = A_{\frac{1}{2}} \left[ \frac{1}{4\alpha_1^2} (\alpha_1 \sqrt{A - \lambda t}) A^{-\frac{1}{2}} - \frac{\sqrt{2}}{4\alpha_1^2} \sqrt{A - \lambda t} A^{-\frac{1}{2}} + \frac{\sqrt{2}}{4\alpha_1^2} A^{-\frac{1}{2}} \right] \]

\[ B_{1,0} = \frac{1}{4\alpha_1^2} A^{-\frac{1}{2}}, B_{2,0} = \frac{1}{4\alpha_2^2} A^{-\frac{1}{2}}, B'_{1,0} = -A^{-\frac{1}{2}} \frac{\alpha_1 \sqrt{A - \lambda t}}{4\alpha_1^2} A^{-\frac{1}{2}} \]

\[ B''_{2,0} = -A^{-\frac{1}{2}} \frac{\alpha_2 \sqrt{A - \lambda t}}{4\alpha_2^2} A^{-\frac{1}{2}} \]  

(2.18)

Regularized values of \( \omega_j(t, \lambda) \) and its derivatives at zero defined by (2.6) denote by \( \omega_{j,0}, \omega'_{j,0}, \omega''_{j,0}, \omega'''_{j,0} \) and for a shortcut of notations denote the values of \( \omega_j(t, \lambda) \) and its derivatives at 1 by \( \omega_j(1), \omega'_j(1), \omega''_j(1), \omega'''_j(1) \). Hence,

\[ \omega_j(1, \lambda) \equiv \omega_j(1) = \begin{cases} e^{\alpha_i \sqrt{A - \lambda t}} A_{\frac{1}{2}}, & i = 1, 2, \ j = 1, 2 \\ A_{\frac{1}{2}}, & i = 1, 2, \ j = 3, 4 \end{cases} \]

\[ \omega'_j(1, \lambda) \equiv \omega'_j(1) = \begin{cases} \alpha_i \sqrt{A - \lambda t} e^{\alpha_i \sqrt{A - \lambda t}} A_{\frac{1}{2}}, & i = 1, 2, \ j = 1, 2 \\ -\alpha_i \sqrt{A - \lambda t} A^{-\frac{1}{2}}, & i = 1, 2, \ j = 3, 4 \end{cases} \]

\[ \omega''_j(1, \lambda) \equiv \omega''_j(1) = \begin{cases} (\alpha_i \sqrt{A - \lambda t})^2 e^{\alpha_i \sqrt{A - \lambda t}} A_{\frac{1}{2}}, & i = 1, 2, \ j = 1, 2 \\ (\alpha_i \sqrt{A - \lambda t})^2 A^{-\frac{1}{2}}, & i = 1, 2, \ j = 3, 4 \end{cases} \]

\[ \omega'''_j(1, \lambda) \equiv \omega'''_j(1) = \begin{cases} (\alpha_i \sqrt{A - \lambda t})^3 e^{\alpha_i \sqrt{A - \lambda t}} A_{\frac{1}{2}}, & i = 1, 2, \ j = 1, 2 \\ (\alpha_i \sqrt{A - \lambda t})^3 A^{-\frac{1}{2}}, & i = 1, 2, \ j = 3, 4 \end{cases} \]

With all that notations substituting \( y(t, \lambda) \) into (2.7),(2.10),(2.11) and then writing them in the matrix form we have:
\[
\begin{pmatrix}
\cos C & 0 & 0 & 0 \\
0 & \cos C & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
\omega''_{1,0} & \omega''_{2,0} & \omega''_{3,0} & \omega''_{4,0} \\
\omega'_{1,0} & \omega'_{2,0} & \omega'_{3,0} & \omega'_{4,0} \\
\omega''_{1}(1) & \omega''_{2}(1) & \omega''_{3}(1) & \omega''_{4}(1) \\
\omega'_{1}(1) & \omega'_{2}(1) & \omega'_{3}(1) & \omega'_{4}(1)
\end{pmatrix}
\begin{pmatrix}
\sin C & 0 & 0 & 0 \\
0 & \sin C & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{pmatrix}
\times
\begin{pmatrix}
\omega'_{1,0} & \omega'_{2,0} & \omega'_{3,0} & \omega'_{4,0} \\
\lambda Q_{1}\omega_{1}(1) & \lambda Q_{1}\omega_{2}(1) & \lambda Q_{1}\omega_{3}(1) & \lambda Q_{1}\omega_{4}(1) \\
\lambda Q_{2}\omega'_{1}(1) & \lambda Q_{2}\omega'_{2}(1) & \lambda Q_{2}\omega'_{3}(1) & \lambda Q_{2}\omega'_{4}(1)
\end{pmatrix}
\begin{pmatrix}
f_1 \\
f_2 \\
g_1 \\
g_2
\end{pmatrix}
\begin{pmatrix}
\cos CA^{-\frac{1}{2}}B''_{1,0} & \cos CA^{-\frac{1}{2}}B''_{2,0} \\
\cos CA^{-\frac{1}{2}}B'_{1,0} & \cos CA^{-\frac{1}{2}}B'_{2,0} \\
O & O \\
O & O
\end{pmatrix}
\begin{pmatrix}
A^{-\frac{1}{2}}\int_{0}^{1} e^{\alpha_{1}\sqrt{A-\lambda} t} h(s) ds \\
A^{-\frac{1}{2}}\int_{0}^{1} e^{\alpha_{2} \sqrt{A-\lambda} t} h(s) ds \\
A^{-\frac{1}{2}}\int_{0}^{1} e^{\alpha_{1} \sqrt{A-\lambda} (1-s) h(s) ds} \\
A^{-\frac{1}{2}}\int_{0}^{1} e^{\alpha_{2} \sqrt{A-\lambda} (1-s) h(s) ds}
\end{pmatrix}
\begin{pmatrix}
A^{-\frac{1}{2}} \int_{0}^{1} e^{\alpha_{1} \sqrt{A-\lambda} t} h(s) ds \\
A^{-\frac{1}{2}} \int_{0}^{1} e^{\alpha_{2} \sqrt{A-\lambda} t} h(s) ds \\
A^{-\frac{1}{2}} \int_{0}^{1} e^{\alpha_{1} \sqrt{A-\lambda} (1-s) h(s) ds} \\
A^{-\frac{1}{2}} \int_{0}^{1} e^{\alpha_{2} \sqrt{A-\lambda} (1-s) h(s) ds}
\end{pmatrix}
\begin{pmatrix}
h_1 \\
h_2
\end{pmatrix}
\]
as the adjoint to the operator $f \rightarrow e^{o\sqrt{A-M}sf}$ which is continuously acts from $H^{-\frac{1}{2}}$ to $L_2(H,(0,1))$ (see [3]). Hence,

$$
\begin{pmatrix}
  f_1 \\
  f_2 \\
  g_1 \\
  g_2
\end{pmatrix} = D^{-1} \left( \tilde{B}_1 - \tilde{B}_2 \right) \tilde{H},
$$

(2.20)

Denote in (2.16), the matrix in front of the vector

$$
\begin{pmatrix}
  f_1 \\
  f_2 \\
  g_1 \\
  g_2
\end{pmatrix}
$$

by $J$ and the second term by $G$

$$
R_\lambda \left( L_0^s \right) \tilde{h} = JD^{-1}(B_1 - \tilde{B}_2)\tilde{H} + G,
$$

(2.21)

In a similar way as in [7] it might be easily verified that $D^{-1}$ is bounded in $H$. In the matrix operator $\tilde{B}_2$ the term $\sin CA^{-\frac{1}{2}}B_{i,0}$ is $\sin CA^{-\frac{1}{2}}B_{i,0} = \frac{\sin CA^{-\frac{1}{2}}}{40}i$ and in $\tilde{B}_1$ the term $\cos CA^{-\frac{1}{2}}B_{i,0}$ (in other terms too) is representable as $\cos CA^{-\frac{1}{2}}B_{i,0} = \cos \beta(I + F)$ where $\beta$ is a number defined by the coefficients of the terms $\tilde{B}_1$, $F$ is a bounded operator in $H$. It follows that $R_\lambda \left( L_0^s \right)$ is compact if and only if $\cos C, Q_1A^{-\frac{1}{2}}, Q_2A^{-\frac{1}{2}}$ are compact. □

**Note 2.2.** Since $q(t)$ is bounded the in $L_2(H,(0,1))$, then in virtue of relation

$$
R_\lambda \left( L_0^s \right) = R_\lambda \left( L_s \right) - R_\lambda \left( L_0 \right) QR_\lambda \left( L_0^s \right),
$$

(2.22)

where $L_s = L_0^s + Q$,

$$
QY = \{ q(t), y(t), 0, 0 \},
$$

(2.23)

statement of Theorem 2.3 holds also for selfadjoint extensions $L_s$ of the minimal operator $L = L_0 + Q$.

3. **ASYMPTOTICS OF EIGENVALUE DISTRIBUTION OF ONE CLASS OF SELFADJOINT EXTENSIONS AND DEFINITION OF THE DOMAIN OF SELFADJOINT EXTENSIONS WHOSE RESOLVENTS ARE FROM $\sigma_p$ AND EXTENSIONS WITH CONTINUOUS SPECTRUM**

Take in boundary conditions (2.7) $C = (C_1,C_2)$, where $C_1$ and $C_2$ are operators on $H$, moreover $C_1 = \frac{\pi}{2}I$ ($I$ is an identity operator in $H$), $C_2 = \arctg(-\sqrt{2}A)$, then the corresponding selfadjoint extension will be given by the boundary conditions $y(0) = y''(0) = 0$. The eigenvalue problem corresponding to that operator is:

$$
l y = \lambda y
$$

(3.1)

$$
y(0) = y''(0) = 0.
$$

(3.2)

$$
- y'''(1) = \lambda Q_1 y(1), \quad y''(1) = \lambda Q_2 y'(1)
$$

(3.3)

Note here that boundary conditions (3.2),(3.3) are obtained from (1.4),(1.5) with indicated above choice of $C$ and by the setting $b = 1$ and making a change of variable $1 - t = x$.

The operator corresponding to that problem for $q(t) \equiv 0$ denote by $L_1^0$ which in virtue of Theorems 2.2 and 2.3 is self-adjoint and discrete.

Now we study the asymptotics of eigenvalues of that operator.
The solution of (3.1) is

\[ y(t, \lambda) = e^{\alpha_1 \sqrt{A-\lambda} t} A^{\frac{1}{2}} f_1 + e^{\alpha_2 \sqrt{A-\lambda} t} A^{\frac{1}{2}} f_2 + e^{-\alpha_1 \sqrt{A-\lambda} (t-1)} A^{-\frac{1}{2}} g_1 + e^{-\alpha_2 \sqrt{A-\lambda} (t-1)} A^{-\frac{1}{2}} g_2 \]  

(3.4)

with \( f_1, f_2, g_1, g_2 \in H \) defined as in (2.14).

Substituting the function (3.4) in the boundary conditions (3.2),

\[ A^{\frac{1}{2}} f_1 + A^{\frac{1}{2}} f_2 + e^{\alpha_1 \sqrt{A-\lambda} t} A^{-\frac{1}{2}} g_1 + e^{\alpha_2 \sqrt{A-\lambda} t} A^{-\frac{1}{2}} g_2 = 0 \]

(3.5)

\[ (\alpha_1 \sqrt{A-\lambda t})^2 A^{\frac{1}{2}} f_1 + \left( \alpha_2 \sqrt{A-\lambda t} \right)^2 A^{\frac{1}{2}} f_2 + \left( \alpha_1 \sqrt{A-\lambda t} \right)^2 e^{\alpha_1 \sqrt{A-\lambda} t} \]

\[ \times A^{-\frac{3}{2}} g_1 + \left( \alpha_2 \sqrt{A-\lambda t} \right)^2 e^{\alpha_2 \sqrt{A-\lambda} t} A^{-\frac{3}{2}} g_2 = 0 \]

(3.6)

rewrite (3.5), (3.6) as

\[ \left( A^{\frac{1}{2}} f_1 + e^{\alpha_1 \sqrt{A-\lambda} t} A^{-\frac{1}{2}} g_1 \right) + \left( A^{\frac{1}{2}} f_2 + e^{\alpha_2 \sqrt{A-\lambda} t} A^{-\frac{1}{2}} g_2 \right) = 0 \]

\[ \left( \alpha_1 \sqrt{A-\lambda t} \right)^2 \left( A^{\frac{1}{2}} f_1 + e^{\alpha_1 \sqrt{A-\lambda} t} A^{-\frac{1}{2}} g_1 \right) + \left( \alpha_2 \sqrt{A-\lambda t} \right)^2 \left( A^{\frac{1}{2}} f_2 + e^{\alpha_2 \sqrt{A-\lambda} t} A^{-\frac{1}{2}} g_2 \right) = 0 \]

hence

\[ f_1 = -A^{-\frac{3}{2}} e^{\alpha_1 \sqrt{A-\lambda} t} A^{-\frac{1}{2}} g_1 = -A^{-\frac{3}{2}} e^{\alpha_1 \sqrt{A-\lambda} t} g_1 = -A^{-\frac{3}{2}} e^{\alpha_1 \sqrt{A-\lambda} t} \]

(3.7)

\[ f_2 = -A^{-\frac{3}{2}} e^{\alpha_2 \sqrt{A-\lambda} t} A^{-\frac{1}{2}} g_2 = -A^{-\frac{3}{2}} e^{\alpha_2 \sqrt{A-\lambda} t} g_2 = \]

\[ = -A^{-\frac{3}{2}} e^{\alpha_1 \sqrt{A-\lambda} t} g_2 = -A^{-\frac{3}{2}} e^{\alpha_2 \sqrt{A-\lambda} t} \]

(3.8)

Taking in (3.4) \( f_1 \) and \( f_2 \) as in (3.7), (3.8) yields

\[ y(t, \lambda) = -e^{\alpha_1 \sqrt{A-\lambda} t} A^{\frac{1}{2}} e^{\frac{\sqrt{A-\lambda} t}{2}} g_1 - e^{\alpha_2 \sqrt{A-\lambda} t} A^{\frac{1}{2}} e^{\frac{\sqrt{A-\lambda} t}{2}} g_2 + \]

\[ + e^{-\sqrt{A-\lambda} (t-1)} A^{-\frac{3}{2}} g_1 + e^{-\sqrt{A-\lambda} (t-1)} A^{-\frac{3}{2}} g_2 = -e^{\sqrt{A-\lambda} t} A^{-\frac{3}{2}} e^{\frac{\sqrt{A-\lambda} t}{2}} g_1 + \]

\[ + e^{-\sqrt{A-\lambda} (t-1)} A^{-\frac{3}{2}} g_1 - e^{\sqrt{A-\lambda} t} A^{-\frac{3}{2}} e^{\frac{\sqrt{A-\lambda} t}{2}} g_2. \]

Denoting \( F_1 = -2e^{\sqrt{A-\lambda} t} A^{-\frac{3}{2}} g_1, \quad F_2 = 2i e^{\sqrt{A-\lambda} t} A^{-\frac{3}{2}} g_2 \), \( F_1, F_2, e^{i A t} \) we have

\[ y(t) = sh \sqrt{A} - At F_1 + sin \sqrt{A} - At F_2 \]

(3.9)

Writing that solution in boundary conditions (3.3), from expansion of a selfadjoint operator with discrete spectrum \( A = \sum_{k=1}^{\infty} \gamma_k \psi_k \), where \( \gamma_k \) are eigenvalues and \( \varphi_k \) are orthonormal basis formed by eigenvectors of \( A \), we have

\[ -\sqrt{\lambda - \gamma_k} e^{\sqrt{\lambda - \gamma_k}} (F_1, \varphi_k) + \sqrt{\lambda - \gamma_k} cos \sqrt{\lambda - \gamma_k} (F_2, \varphi_k) = \lambda sh \sqrt{\lambda - \gamma_k} (F_1, Q_1 \varphi_k) + \lambda sin \sqrt{\lambda - \gamma_k} (F_2, Q_1 \varphi_k) \]

(3.10)

and

\[ \sqrt{\lambda - \gamma_k} sh \sqrt{\lambda - \gamma_k} (F_1, \varphi_k) - \sqrt{\lambda - \gamma_k} sin \sqrt{\lambda - \gamma_k} (F_2, \varphi_k) = \lambda \sqrt{\lambda - \gamma_k} e^{\sqrt{\lambda - \gamma_k}} (F_1, Q_2 \varphi_k) + \lambda \sqrt{\lambda - \gamma_k} cos \sqrt{\lambda - \gamma_k} (F_2, Q_2 \varphi_k). \]
For simplicity of calculations take here $Q_1 = Q_2 = A^\alpha$, and $0 < \alpha < \frac{1}{2}$, which is important for holding the hypothesis of theorem 2.3.

By replacing

$$\sqrt{\lambda - \gamma_k} = z_i(F_1, \varphi_k) = c_{1k}, \quad (F_2, \varphi_k) = c_{2k} \quad (3.11)$$

in virtue of $Q_i\varphi_k = \gamma_k^a\varphi_k$, $i = 1, 2$ we have

$$-z^3chzc_{1k} + z^3 \cos zc_{2k} = (z^4 + \gamma_k) shz\gamma_k^a c_{1k} + (z^4 + \gamma_k) \sin z\gamma_k^a c_{2k} \quad (3.12)$$

$$zshz_{1k} - z \sin zc_{2k} = (z^4 + \gamma_k) chz\gamma_k^a c_{1k} + (z^4 + \gamma_k) \cos z\gamma_k^a c_{2k} \quad (3.13)$$

which is a system of equations in $c_{1k}, c_{2k}$ and has nonzero roots if and only if the characteristic determinant $\Delta(z)$ of boundary value problem (3.1)-(3.3) (determinant formed by coefficients of equations (3.12), (3.13)) is zero:

$$\Delta(z) = \left| \begin{array}{ccc} -z^3chz - (z^4 + \gamma_k) shz \gamma_k^a & z^3 \cos z - (z^4 + \gamma_k) \sin z \gamma_k^a \\ zshz - (z^4 + \gamma_k) chz \gamma_k^a & -z \sin z - (z^4 + \gamma_k) \cos z \gamma_k^a \end{array} \right| = 0 \quad (3.14)$$

After simplifications in (3.14)

$$tgz = \frac{-2z^3(z^4 + \gamma_k) \gamma_k^a + z^4\sin z - (z^4 + \gamma_k)^2 \gamma_k^{2a}\gamma_k}{z^4 + 2z(z^4 + \gamma_k) \gamma_k^{2a}\gamma_k - (z^4 + \gamma_k)^2 \gamma_k^{2a}} \quad (3.15)$$

Since $\lambda$ as an eigenvalue of a selfadjoint operator must be real, feasible values for $z$ are $y, -y, iy, -iy(y > 0)$ or $\pm y \pm iy$.

Thus, (3.15) might have only real, imaginary roots and roots of the form $y \pm iy, y$ is real. It can’t have other complex roots with exception of these roots, because complex roots will give complex eigenvalues for a selfadjoint operator, which is impossible.

Hence, setting $\sqrt{\lambda_k, j - \gamma_k} = z_k, j$ from (3.15) for real roots as $|z| \to \infty$ we have $z = z_{k, j} \sim \frac{\pi}{4} + \pi j + O\left(\frac{1}{k}\right), k$ is an entire number large in modulus.

Writing in (3.15) $iz$ in place of $z$ shows that if $z$ is a real root, then $iz$ is also a root of that equation, thus for imaginary roots

$$z = iz_{k, j} \sim \left(\frac{\pi}{4} + \pi j + O\left(\frac{1}{k}\right)\right) i,$$

But in virtue of (3.11)

$$\lambda = z^4 + \gamma_k$$

which shows that real and imaginary roots of (3.15) result in the same eigenvalues but in linearly dependent eigenvectors of the operator and that is why algebraic multiplicity of each eigenvalue corresponding those roots is 2.

Writing in (3.15) $y \pm iy$ in place of $z$ after simplifications we get

$$-4y^4 \left[ \frac{\sin 2y}{2} - \frac{sh 2y}{2i} \right] + 4iy^2 (y + iy) (\gamma_k - 4y^4) \gamma_k^a \left[ \frac{\cos 2y}{2i} + \frac{ch 2y}{2} \right] + 2y (y + iy) (\gamma_k - 4y^4) \gamma_k^a \left[ \frac{\cos 2y}{2i} - \frac{ch 2y}{2i} \right] +$$

$$+ 2y^4 (\gamma_k - 4y^4)^2 \gamma_k^{2a} \left[ \frac{sh 2y}{2} + \frac{isin 2y}{2} \right] - (\gamma_k - 4y^4)^2 \gamma_k^{2a} \left[ \frac{sh 2y}{2} + \frac{isin 2y}{2} \right] = 0 \quad (3.16)$$

It is easy to see that the right side of (3.16) has the form $K = iK$, where $K$ is real and

$$K = -4y^4 \frac{\sin 2y}{2} - 4y^3 (\gamma_k - 4y^4) \gamma_k^a \left[ \frac{\cos 2y}{2} + \frac{ch 2y}{2} \right] + 2 (\gamma_k - 4y^4) \gamma_k^a y \left[ \frac{\cos 2y}{2} - \frac{ch 2y}{2} \right] +$$
\[ +4y^4 \sin 2y - \frac{1}{2} (\gamma_k - 4y^4)^2 \frac{1}{2} (\gamma_k - 4y^4)^2 \frac{1}{2} \]

Equation (3.16) has roots if and only if

\[ K = 0. \tag{3.17} \]

But for large values of \( y \) (3.17) is equivalent to

\[ 4y^7 \gamma_k^2 e^{2y} + y^2 \gamma_k e^{2y} + 8y^8 \gamma_k e^{2y} + 4y^7 \gamma_k e^{2y} + y^5 \gamma_k e^{2y} \]

which has no positive roots. By Descartes’s rule of signs (3.16) has small in modulus roots, but they don’t change asymptotics of eigenvalues.

Hence, we get the next theorem

**Theorem 3.1.** The algebraic multiplicity of eigenvalues \( \lambda_{k,j} \) of the operator \( L_1^0 \) is two and the following asymptotic formula is true:

\[ \lambda_{k,j} = \gamma_k + z_{k,j}, \quad z_{k,j} \sim \left\{ \begin{array}{l} \pi j + \frac{\pi}{4} + O\left( \frac{1}{n} \right), \\ i (\pi j + \frac{\pi}{4} + O\left( \frac{1}{n} \right)) \end{array} \right. \tag{3.18} \]

and \( \lambda_{k,j} = \gamma_k + \eta_{k,j} \), where \( \eta_{k,j} \) are small in modulus roots of (3.15).

Analogous to [5,8,9] the following statements might be justified:

**Lemma 3.1.** For distribution function \( N(\lambda) = \sum_{n<\lambda} \lambda \) of the eigenvalues of the operator \( L_1^0 \) the following relation is valid

\[ N(\lambda) \sim C_1 \lambda^{\frac{4n}{4+n}} \tag{3.19} \]

for sufficiently large \( \lambda \).

**Lemma 3.2.** For large values of \( n \) the following asymptotic formula is true

\[ \lambda_n \sim C_2 n^{\frac{4n}{4+n}}, \quad n \to \infty. \tag{3.20} \]

Setting \( L_1 = L_1^0 + Q \), one can easily see that hypotheses of Theorem 2.2 and Theorem 2.3 hold also for \( L_1 \). Denote the eigenvalues of \( L_1 \) by \( \{ \mu_n \} : \mu_1 < \mu_2 < \ldots \).

**Note 3.1** From (3.19) it follows that inverse of \( L_1^0 \) is from Neumann von Shcetten class \( \sigma_p \), if and only if \( p \cdot \frac{4n}{4+n} > 1 \) or \( \alpha > \frac{4}{4p-1} \), which means that \( A^{-\frac{1}{p}} \in \sigma_{4p-1} \).

**Lemma 3.3.** The operator \( (L_1^0)^{-1} \) is from \( \sigma_p \) if and only if \( A^{-\frac{1}{p}} \in \sigma_{4p-1} \).

Because of relation (2.22) that statement holds also for the operator \( L_1 \).

Now we can prove the next theorem.

**Theorem 3.2** Let \( A^{-\frac{1}{p}} \in \sigma_{4p-1} \). Then \( R_\lambda(L_0^0), \quad R_\lambda(L_0) \in \sigma_p(H_1) \) if and only if

\[ \cos C, \quad Q_1 A^{-\frac{1}{p}}, \quad Q_2 A^{-\frac{1}{p}} \]

are from \( \sigma_p \).

**Proof.** Setting in formula (2.21) \( C = (C_1, C_2), \quad C_1 = \frac{\pi}{4} I, \quad C_2 = \arctg(-\sqrt{2} A) \) and subtracting obtained by that way formula from (2.21) we get

\[ R_\lambda(L_0^0) \hat{h} = R_\lambda(L_0) \hat{h} + JD^{-1} (\hat{B}_1 - \hat{B}_2) \hat{H} - JD_0^{-1} (\hat{B}_{01} - \hat{B}_{02}) \hat{H} \tag{3.21} \]

where \( D_0 \) and \( \hat{B}_{01} - \hat{B}_{02} \) are obtained from \( D \) and \( \hat{B}_1 - \hat{B}_2 \) by taking there \( C = (C_1, C_2), \)

\[ C_1 = \frac{\pi}{4} I, \quad C_2 = \arctg(-\sqrt{2} A) \]. Writing (3.21) in the open form and with lemma 3.3 in mind one can easily see that \( R_\lambda(L_0^0) \) is from \( \sigma_p \) if and only if \( \cos C, \quad Q_1 A^{-\frac{1}{p}}, \quad Q_2 A^{-\frac{1}{p}} \) are from \( \sigma_p \). Since \( Q \) is bounded in \( H_1 \), statement of theorem is true also for \( R_\lambda(L_0) \). \( \Box \)
**Theorem 3.3.** If the inverse of the operator $A$ is compact, then for any closed set $F$ of the real axis, there exists a selfadjoint extension of the minimal operator, whose spectrum coincides with $F$.

*Proof.* Let $C = \left( \begin{array}{cc} \frac{\pi}{2}I & O \\ O & f(A) \end{array} \right)$, where $f(\mu)$ is any function Borel measurable on $(1, \infty)$. Then boundary conditions (2.7) take on the form

$$y(0) = 0,$$

$$\cos f(A) A^\frac{3}{4} \left(y''(0) - \sqrt{2} A^\frac{1}{4}y'(0) + A^\frac{1}{4}y(0) \right) - \sin f(A) A^{-\frac{1}{4}}y'(0) = 0$$

(3.22) (3.23)

Let us corresponding to $f$ selfadjoint extension denote by $L_f$. Obviously, $\lambda$ is an eigenvalue of $L_f$, if in addition to (3.22) and (3.23) holds (3.3). After substituting $y(t, \lambda)$ from (3.4) into those relations and denoting by $K^t_\lambda(A)$ the determinant of the matrix formed by the coefficients of $f_1$, $f_2$, $g_1$, $g_2$ in relations (3.22), (3.23), (3.3), the further justifications are lead as in Theorem 5 from [3]. \qed

4. **Orthonormal eigenvectors of operator** $L^0_1$

Reminding that by \{ \varphi_k \} we denote orthonormal eigenvectors of the operator $A$, then orthogonal eigenvectors of $L^0_1$ will be as

$$Y_{k,j} = \left\{ c_{1k,j} \sinh \sqrt{\lambda_{k,j} - \gamma_k \varphi_k} + c_{2k,j} \sin \sqrt{\lambda_{k,j} - \gamma_k \varphi_k}, c_{1k,j} \gamma_k \alpha \sinh \sqrt{\lambda_{k,j} - \gamma_k \varphi_k} + \\
+ c_{2k,j} \gamma_k \alpha \sin \sqrt{\lambda_{k,j} - \gamma_k \varphi_k}, c_{1k,j} \gamma_k \alpha \cosh \sqrt{\lambda_{k,j} - \gamma_k \varphi_k} + \\
+ c_{2k,j} \sqrt{\lambda_{k,j} - \gamma_k \varphi_k} \right\}, \quad k = 1, \infty, \; j = 1, \infty. \quad (4.1)$$

Coefficients $c_{1k,j}$ and $c_{2k,j}$ are the values of $c_{1k}$ and $c_{2k}$ in relations (3.12), (3.13) obtained by taking in (3.11) $\lambda = \lambda_{k,j}$. From (3.13)

$$c_{1k} = \frac{z \sin z + (z^4 + \gamma_k) \cos z \gamma_k^\frac{1}{2}}{z \sinh z - (z^4 + \gamma_k) \cosh z \gamma_k^\frac{1}{2}} \quad (4.2)$$

The multiplier at $c_{2k}$ we denote by $H(z)$:

$$c_{1k} = H(z)c_{2k} \quad (4.3)$$

Thus,

$$c_{1k,j} = H(z_{k,j})c_{2k,j} \quad (4.4)$$

Assuming for shortcut of notations $c_{2k,j} = c_{k,j}$ with all above in mind we have the following expressions for $Y_{k,j}$:

$$Y_{k,j} = c_{k,j} \left\{ \sin z_{k,j} \sqrt{\lambda_{k,j} - \gamma_k \varphi_k} + H(z_{k,j}) \sin \gamma_k \sqrt{\lambda_{k,j} - \gamma_k \varphi_k} + \\
+ H(z_{k,j}) \gamma_k \alpha \sin \sqrt{\lambda_{k,j} - \gamma_k \varphi_k}, z_{k,j} \gamma_k \alpha \cos \sqrt{\lambda_{k,j} - \gamma_k \varphi_k} + H(z_{k,j}) \gamma_k \alpha \cosh \sqrt{\lambda_{k,j} - \gamma_k \varphi_k} \right\} = \\
= c_{k,j} \varphi_k \left\{ \sin z_{k,j} \sqrt{\lambda_{k,j} - \gamma_k \varphi_k}, \gamma_k \alpha \sin z_{k,j} + H(x_{k,j}) \gamma_k \alpha \cos z_{k,j} + \\
+ H(z_{k,j}) \gamma_k \alpha \cosh z_{k,j} \right\}, \quad k = 1, \infty, \; j = 1, \infty \quad (4.5)$$

Introduce the direct sum space $\Lambda = \mathbb{R}((0, 1)) \oplus C^2$, ($C$ is a complex space) with a scalar product of elements $u = (u(t), u_1, u_2), v = (v(t), v_1, v_2)$ defined as

$$(u, v)_\Lambda = \int_0^1 u \overline{v} dt + \gamma_k^\alpha u(1) \overline{v}(1) + \gamma_k^\alpha u'(1) \overline{v}'(1)$$
Thus,

\[ Y_{k,j} = c_{k,j} \Psi_{k,j} \varphi_k \]  

(4.6)

where by \( \Psi_{k,j} \in \Lambda \) we denote the vector in parenthesis in the right side of (4.5).

Set

\[ \Phi_{k,j} = \Psi_{k,j} \varphi_k, \]  

(4.7)

obviously \( \Phi_{k,j} \in H_1 \). \( Y_{k,j} \) will form an orthonormal system of eigenvectors by putting in place of \( c_{k,j} \) the norming constants:

\[ c_{k,j}^2 = \frac{1}{\| \Phi_{k,j} \|^2_1} = \frac{1}{\| \Psi_{k,j} \|^2_\Lambda}, \]  

(4.8)

where:

\[
\| \Phi_{k,j} \|^2_1 = \left[ \int_0^1 \sin^2 z_{k,j} t dt + H(z_{k,j})^2 \int_0^1 s h^2 z_{k,j} t dt + \right. \\
\left. + 2H(z_{k,j}) \int_0^1 s h z_{k,j} t s i n z_{k,j} t dt + H(z_{k,j})^2 s h^2 z_{k,j} \right] + \\
+ 2H(z_{k,j}) \gamma_{k,j}^2 s h z_{k,j} s i n z_{k,j} + \gamma_{k,j}^2 \sin^2 z_{k,j} + H(z_{k,j})^2 \gamma_{k,j}^2 z_{k,j} c h^2 z_{k,j} + \\
+ 2z_{k,j}^2 \gamma_{k,j}^2 H(z_{k,j}) c h z_{k,j} c o s z_{k,j} + \gamma_{k,j}^2 z_{k,j}^2 c o s^2 z_{k,j} \]  

(4.9)

Here \( (\varphi_k, \varphi_k)_H = 1 \) was used.

5. SOME IMPORTANT RELATIONS

Recalling that \( \gamma_1 \leq \gamma_2 \leq \ldots \) are eigenvalues and \( \{ \varphi_k \}, \; k = 1, \infty \) are orthonormal eigenvectors of the operator \( A \), in virtue of expansion theorem any \( y(t) \) from \( L_2 (H, (0, 1)) \) is expanded as

\[ y(t) = \sum_{k=1}^{\infty} (y(t) \varphi_k, \varphi_k) \varphi_k \]

Denoting \( (y(t) \varphi_k, \varphi_k) = y_k(t) \), from (3.1)-(3.2) we have the following spectral problem for scalar functions \( y_k(t) \):

\[ l_k y_k(t) \equiv y_k^{IV}(t) + \gamma_k y_k(t) = \lambda y_k(t) \]  

(5.1)

\[ y_k(0) = y_k'(0) = 0 \]  

(5.2)

\[ -y_k''(1) = \lambda \gamma_k^a y_k(1) \]  

(5.3)

\[ y_k''(1) = \lambda \gamma_k^a y_k'(1) \]  

(5.4)

For each fixed \( k \) \((k = 1, \infty)\) denote the eigenvalues of that problem by \( \lambda_{k,j} \), and solutions of (5.1) by \( y_k(t, \lambda - \gamma_k) \). Obviously, \( \lambda_{k,j}, \; k = 1, \infty, \; j = 1, \infty \) are the eigenvalues of problem (3.1)-(3.3)

It is easy to see that vectors \( \{y_k, y_k'(t, \lambda - \gamma_k), \gamma_k^a y_k, (1, \lambda - \gamma_k), \gamma_k^a y_k'(1, \lambda - \gamma_k)\} \), form a set of orthogonal eigenvectors of the operator \( L_{0k} \) associated with the scalar problem (5.1)-(5.4) in space \( \Lambda \) and acting as \( L_{0k} \) \( (y_k(t, \lambda - \gamma_k), \gamma_k^a y_k(1, \lambda - \gamma_k), \gamma_k^a y_k'(1, \lambda - \gamma_k)) \) =

\[ = (l_k y_k(t, \lambda - \gamma_k), -y_k''(1, \lambda - \gamma_k), y_k''(1, \lambda - \gamma_k)) \]

Obviously, it coincide, with \( \Psi_{k,j} \):

\[ \Psi_{k,j} = c_{k,j} \Psi_{k,j} \] and \( Y_{k,j} = c_{k,j} \Phi_{k,j} \) are orthonormal eigenvectors of problems and (5.1)-(5.4) and (3.1)-(3.3), respectively.
Introduce the notations

\[
\omega_1(z) \equiv y_k''(1,\lambda - \gamma_k) + \lambda \gamma_k \alpha y_k(1,\lambda - \gamma_k) = \\
y_k''(1,z^4) + (z^4 + \gamma_k) \gamma_k \alpha y_k(1,z^4) \quad (5.5)
\]

\[
\omega_2(z) \equiv y_k''(1,\lambda - \gamma_k) - \lambda \gamma_k \alpha y_k'(1,\lambda - \gamma_k) = \\
y_k''(1,z^4) - (z^4 + \gamma_k) \gamma_k \alpha y_k'(1,z^4) \quad (5.6)
\]

The eigenvalues of (5.1)-(5.4) are defined from the system

\[
\omega_1(z) = 0 \quad \omega_2(z) = 0 \tag{5.7-5.8}
\]

—Introduce the following function:

\[
f_k(z) \equiv y_k^2(1,z^4) \left[ \frac{\omega_1(z)}{y_k(1,z^4)} \right] - y_k^2(1,z^4) \left[ \frac{\omega_2(z)}{y_k'(1,z^4)} \right] \tag{5.9}
\]

Prove the next theorem which has an important role in deriving the trace formula.

**Theorem 5.1.**

\[
f_k'(z_{k,j}) = 4z_{k,j}^3 \| \gamma_{k,j} \|^2 = 4z_{k,j}^3 \frac{1}{c_{k,j}^2}
\]

where \( c_{k,j}^2 \) are norming constants.

**Proof.** Let \( y_k(t,\lambda - \gamma_k) \) and \( y_k(t,\lambda_{k,j} - \gamma_k) \) be solutions of equation (5.1) with \( \lambda \) and \( \lambda_{k,j} \), respectively:

\[
y_k^{IV}(t,\lambda - \gamma_k) + \gamma_k y_k(t,\lambda - \gamma_k) = \lambda y_k(t,\lambda - \gamma_k) \quad (5.10)
\]

\[
y_k^{IV}(t,\lambda_{k,j} - \gamma_k) + \gamma_k y_k(t,\lambda_{k,j} - \gamma_k) = \lambda_{k,j} y_k(t,\lambda_{k,j} - \gamma_k) \quad (5.11)
\]

Multiply (5.10) by \( y_k(t,\lambda_{k,j} - \gamma_k) \), (5.11) \( y_k(t,\lambda - \gamma_k) \), then subtract the second one from the first, integrate both sides of obtained relation along (0, 1), and to the obtained results add the term

\[
(\lambda - \gamma_k - (\lambda_{k,j} - \gamma_k)) \gamma_k \alpha y_k(1,\lambda - \gamma_k) y_k(1,\lambda_{k,j} - \gamma_k) + \\
(\lambda - \gamma_k - (\lambda_{k,j} - \gamma_k)) y_k'(1,\lambda - \gamma_k) y_k'(1,\lambda_{k,j} - \gamma_k) \gamma_k \alpha \tag{5.12}
\]

Recall here the notations \( \sqrt{\lambda - \gamma_k} = z \), \( \sqrt{\lambda_{k,j} - \gamma_k} = z_{k,j} \) from section 3. Note that addition of the last term is needed for finding the norm of the eigenvector of the operator \( L_{0k} \) in the direct sum space \( \Lambda_1 = L_2((0,1)) \oplus \mathbb{C}^2 \) as it will become clear in next derivations.

Thus,

\[
N \equiv \int_0^1 y_k^{IV}(t,z^4) y_k(t,z_{k,j}^4) dt - \int_0^1 y_k^{IV}(t,z_{k,j}^4) y_k(t,z^4) dt + \\
+ (z^4 - z_{k,j}^4) \gamma_k^2 y_k(1,z^4) y_k(1,z_{k,j}^4) + (z^4 - z_{k,j}^4) y_k'(1,z^4) y_k'(1,z_{k,j}^4) \gamma_k^2 \gamma_k^\alpha = \\
(z^4 - z_{k,j}^4) \int_0^1 y_k(t,z^4) y_k(t,z_{k,j}^4) dt + (z^4 - z_{k,j}^4) \gamma_k^2 y_k(1,z^4) y_k(1,z_{k,j}^4) + \\
+ (z^4 - z_{k,j}^4) y_k'(1,z^4) y_k'(1,z_{k,j}^4) \gamma_k^\alpha \tag{5.13}
\]

Integration by parts on the left side of that relation yields

\[
N = y_k'''(1,z^4) y_k(1,z_{k,j}^4) - \\
y_k'''(1,z_{k,j}^4) y_k(1,z^4) - y_k''(1,z^4) y_k'(1,z_{k,j}^4) + y_k''(1,z_{k,j}^4) y_k'(1,z^4) + \\
+ (z^4 - z_{k,j}^4) \gamma_k^\alpha y_k(1,z^4) y_k(1,z_{k,j}^4) + (z^4 - z_{k,j}^4) y_k'(1,z^4) y_k'(1,z_{k,j}^4) \gamma_k^\alpha = \\
\]
\[ \begin{align*}
&= y_k(1, z_{k,j}^i) y_k(1, z^4) \left[ \frac{y_k'''(1, z^4)}{y_k(1, z^4)} - \frac{y_k''(1, z_{k,j}^i)}{y_k(1, z_{k,j}^i)} \right] + (z^4 - z_{k,j}^i) \gamma_k^\alpha \
&- y_k'(1, z^4) y_k(1, z_{k,j}^i) \left[ \frac{y_k''(1, z^4)}{y_k'(1, z^4)} - \frac{y_k'(1, z_{k,j}^i)}{y_k'(1, z_{k,j}^i)} \right] + (z^4 - z_{k,j}^i) \gamma_k^\alpha \\
&= 4z_{k,j}^3 \int_0^1 y_k(t, z_{k,j}^4) dt + \gamma_k^\alpha y_k^2(1, z_{k,j}^i) + \gamma_k^2 y_k^2(1, z_{k,j}^i) + \gamma_k^2 y_k^2(1, z_{k,j}^i) \\
&\lim_{z \to z_{k,j}} \left( \frac{y_k(1, z_{k,j}^4) y_k(1, z^4)}{z - z_{k,j}} \left[ \frac{y_k'''(1, z^4)}{y_k(1, z^4)} - \frac{y_k''(1, z_{k,j}^i)}{y_k(1, z_{k,j}^i)} \right] + z^4 \gamma_k^\alpha \right) \\
&- \frac{y_k'(1, z_{k,j}^i) y_k(1, z^4)}{z - z_{k,j}} \left[ \frac{y_k''(1, z^4)}{y_k'(1, z^4)} - \frac{y_k'(1, z_{k,j}^i)}{y_k'(1, z_{k,j}^i)} \right] + z^4 \gamma_k^\alpha = 0 \\
&= y_k^2(1, z_{k,j}^i) \left[ \frac{y_k''(1, z^4)}{y_k(1, z^4)} \right] \bigg|_{z = z_{k,j}} + 4z_{k,j}^3 \gamma_k^\alpha y_k^2(1, z_{k,j}^i) \\
&- y_k^2(1, z_{k,j}^i) \left[ \frac{y_k''(1, z^4)}{y_k(1, z^4)} \right] \bigg|_{z = z_{k,j}} - 4z_{k,j}^3 \gamma_k^\alpha y_k^2(1, z_{k,j}^i) \\
&= y_k^2(1, z_{k,j}^i) \left[ \frac{y_k''(1, z^4)}{y_k(1, z^4)} + z^4 \gamma_k^\alpha \right] \bigg|_{z = z_{k,j}} - y_k^2(1, z_{k,j}^i) \left[ \frac{y_k''(1, z^4)}{y_k(1, z^4)} + z^4 \gamma_k^\alpha \right] \bigg|_{z = z_{k,j}} \quad (5.15)
\end{align*} \]

(derivatives of expressions within square bracket in the last relation are taken with respect to \( z \))

Note that \( \int_0^1 y_k(t, z_{k,j}^4) dt + \gamma_k^\alpha y_k^2(1, z_{k,j}^i) + \gamma_k^2 y_k^2(1, z_{k,j}^i) \) standing on the left side of (5.15) is square of the norm of eigenvectors of an operator associated with problem (5.1)-(5.4) in \( \Lambda \):

\[ \int_0^1 y_k(t, z_{k,j}^4) dt + \gamma_k^\alpha y_k^2(1, z_{k,j}^i) + \gamma_k^2 y_k^2(1, z_{k,j}^i) = ||\Psi_{k,j}||_\Lambda^2 \quad (5.16) \]

Using (5.16) on the left side of (5.15) and notations (5.5), (5.6), we arrive at

\[ 4z_{k,j}^3 ||\Psi_{k,j}||_\Lambda^2 = y_k^2(1, z_{k,j}^i) [\omega_1(z)] \bigg|_{z = z_{k,j}} - y_k^2(1, z_{k,j}^i) [\omega_2(z)] \bigg|_{z = z_{k,j}} \quad (5.17) \]

Obviously, the solution of problem (5.1) satisfying (5.2) is

\[ y_k(t, z^4) = c_{1k} sh \sqrt{\lambda - \gamma_k t} + c_{2k} sin \sqrt{\lambda - \gamma_k t} \]

or

\[ y_k(t, z^4) = c_{1k} sh z t + c_{2k} sin z t \quad (5.18) \]
For this function to be the first component of eigenvector \(\Psi_{k,j}\) of problem (5.1)-(5.4), the function \(y_k(t, z^4)\) must satisfy also (5.3) and (5.4) or equivalently (5.7), (5.8). Substituting it into (5.15), we get again (3.11), (3.12).

Obviously, writing \(y_k(t, z^4)\) from (5.8) with \(c_{1k}\) defined from (4.2) into (5.4) yields (3.11) from which \(\lambda_{k,j}\) are found as \(\lambda_{k,j} = z_{k,j}^4 + \gamma_k\)

Evaluate the derivative of \(f_k(z)\) at \(z_{k,j}\)

\[
 f_k'(z_{k,j}) = \left( y_k^2(1, z^4) \right)' \bigg|_{z=z_{k,j}} = \left( y_k^2(1, z^4) \right)' \bigg|_{z=z_{k,j}} - \left( y_k^2(1, z^4)' \right) \bigg|_{z=z_{k,j}}
\]

which in virtue of \(\omega_1(z_{k,j}) = 0, \omega_2(z_{k,j}) = 0\) yields

\[
 f_k'(z_{k,j}) = y_k^2(1, z_{k,j}^4) \left[ \omega_1(z) \right]' \bigg|_{z=z_{k,j}} - y_k^2(1, z_{k,j}^4) \left[ \omega_2(z) \right]' \bigg|_{z=z_{k,j}}
\]

From which with (5.16) in mind we get

\[
 f_k'(z_{k,j}) = 4 z_{k,j}^3 \left\| \Psi_{k,j} \right\|^2_\Lambda = 4 z_{k,j}^3 \frac{1}{c_{k,j}^2}
\]

where

\[
 c_{k,j}^2 = \frac{1}{\left\| \Psi_{k,j} \right\|^2_\Lambda} = \frac{1}{\left\| \Phi_{k,j} \right\|^2_1}
\]

which completes the proof.

Thus, (5.18), (5.20) relates the characteristic determinant or \(\omega_1(z)\) and \(\omega_2(z)\) whose common zeros define the eigenvalues and norms of orthogonal eigenvectors. The function \(f_k(z)\) and its analogous for the problem (5.1), (5.2) but with boundary conditions different than (5.3), (5.4) have essential role in our derivation of trace formula described in the next section.

If \(c_{1k}\) is defined from (5.8), then \(f_k(z)\) from (5.18) is simplified to the form

\[
 f_k(z) \equiv y_k(1, z^4) \omega_1(z)
\]

\[\Upsilon_{k,j} = c_{k,j} \Psi_{k,j}\] are orthonormal eigenvectors of the operator \(L_{0k}\) associated with problem (5.1)-(5.4) in the space \(\Lambda\), and

\[ Y_{k,j} = c_{k,j} \Phi_{k,j} = c_{k,j} \Psi_{k,j} \varphi_k \]

are orthonormal eigenvectors of the operator \(L_0\) associated with problem (3.1)-(3.3) in \(H_1\).

6. Evaluation of regularized trace

Before passing to derivations, put on \(q(t)\) the next condition:

\[
 \sum_{k=1}^\infty (q(t) \varphi_k, \varphi_k)_H < \infty
\]

From Theorem 2.3 and Note 2.1 the operator \(L_1 = L_0^1 + Q\) is discrete. Recall that eigenvalues of \(L_0\) and \(L_1\) are denoted by \(\lambda_1 \leq \lambda_2 \leq \ldots\) and \(\mu_1 \leq \mu_2 \ldots\), respectively. In
virtue of Theorem 3.1 from section 3 and Theorem 1 from [10](application to our case is justified as in our work [9])

$$\lim_{m \to \infty} \sum_{n=1}^{n_m} (\mu_n - \lambda_n - (QY_{k_n,j_n}, Y_{k_n,j_n})_1) = 0$$  (6.2)

for some subsequence of natural numbers \(\{n_m\}\) satisfying conditions of Lemma 3 from [10].

In virtue of the asymptotic formula for \(z_{k,j}\) the next lemma is proved (proof is similar to one, for example in [9], Lemma 4.2).

**Lemma 6.1.** The series

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (QY_{k,j}, Y_{k,j})_1$$

is absolutely convergent.

From (6.2) and Lemma 5.1

$$\lim_{m \to \infty} \sum_{n=1}^{n_m} (\mu_n - \lambda_n) = \lim_{m \to \infty} \sum_{n=1}^{n_m} (QY_{k_n,j_n}, Y_{k_n,j_n})_1 = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (QY_{k,j}, Y_{k,j})_1$$  (6.3)

Denote by \(\sum' \sum_{n=1}^{\infty} (\mu_n - \lambda_n)\) the limit on the left side of (6.3) and call it regularized trace of \(L_1\).

$$\sum' \sum_{n=1}^{\infty} (\mu_n - \lambda_n) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (QY_{k,j}, Y_{k,j})_1$$  (6.4)

From lemma 6.1 and from (6.4), with (4.5) in mind, we get:

$$\sum_{n=1}^{\infty} (\mu_n - \lambda_n) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{k,j}^2 \int_0^1 q_k(t) \ y_k^2(t, z_{k,j}^4) dt = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{k,j}^2 \int_0^1 q_k(t) \left[ \sin^2 z_{k,j} t + 2H(z_{j,k}) \ sh z_{k,j} t \ sin z_{k,j} t + H(z_{j,k})^2 sh^2 z_{k,j} t \right] dt,$$

where

$$q_k(t) = (q(t) \varphi_k, \varphi_k)$$  (6.5)

Without loss of generality, putting \(\int_0^1 q_k(t) dt = 0\) with Theorem 5.1 in mind we come to

$$\sum_{n=1}^{\infty} (\mu_n - \lambda_n) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{k,j}^2 \int_0^1 q_k(t) \left[ -\cos 2z_{k,j} t + 4H(z_{j,k}) \ sh z_{k,j} t \ sin z_{k,j} t + H(z_{j,k})^2 sh^2 z_{k,j} t \right] dt \over 2$$  (6.6)

Consider, the \(N\)-th partial sum of the inner series:

$$\sum_{k=1}^{N} c_{k,j}^2 \int_0^1 q_k(t) \left[ -\cos 2z_{k,j} t + 4H(z_{j,k}) \ sh z_{k,j} t \ sin z_{k,j} t + H(z_{j,k})^2 sh^2 z_{k,j} t \right] dt \over 2$$  (6.7)
Our aim in that section is to find the sum of the series in (6.6). For that sake in our previous works, for example, [8,9], for evaluating the value of the right side of (6.4) we use Cauchy’s residue theorem, further tending contour of integration to infinity and using asymptotic formulas for the integrand. Namely, each time we have selected a function of a complex variable with poles at \( z_{j,k} \) (zeros of characteristic determinant): they are the functions, whose denominators are defined by the characteristic determinants \( \Delta(z) \) corresponding to the problem (whose equivalent in the present work is equation (3.14)) and numerators are suggested by numerators of series the sum of which to be evaluated (here numerator of (6.7)). Usually, residues at poles of that function give terms of sum analog of which here is (6.6) which indicates on some relation between characteristic determinant of the associated operator and norming constants. Further, using asymptotic formulas found for \( z_{k,j} \) on the integration contour, we arrived at the desired formulas.

In this work, since the norming constants defined by (4.8) and (4.9) and \( \Delta(z) \) from (3.14) have too long expressions, and that is why to manipulate with them by using the above indicated methods is impossible.

For that reason, by (5.19) and (5.20) we establish the indicated above relation between \( \omega_1(z) \), \( \omega_2(z) \) and norming constants existence of which was intuitively clear for us in all previous works. Remind that by determining, for example, \( c_{1k} \) from \( \omega_1(z) = 0 \) and substituting in \( \omega_2(z) = 0 \) yields an equation equivalent to \( \Delta(z) = 0 \) (see (3.14)). Note that this a is more general method and might be used in studies of regularized traces in the future.

Interchange the integral and the sum in (6.7) and denote by \( S_N(t) \) the following expression

\[
S_N(t) = \sum_{j=1}^{N} c_{k,j}^2 \int_0^1 q_k(t) y_k^2(t, z_k,j^4)dt
\]

Now using (5.20),(5.21), we see that the following functions of a complex variable \( z \)

\[
F_k(z, t) = \frac{4z^3y_k^2(t, z^4)}{f_k(z)} = \frac{4z^3y_k^2(t, z^4)}{y_k(1, z^4) \omega_1(z)}
\]

have poles at common roots \( z = z_{k,j} \) of system (5.7),(5.8) and the residues of \( F_k(z, t) \) at these poles are the terms of the sum (6.7):

\[
res_{z=z_{k,j}}F_k(z, t) = c_{k,j}^2 y_k^2(t, z^4)
\]

In virtue of (6.6), (6.10) we arrive at the next lemma

**Lemma 6.2.**

\[
\sum_{n=1}^{\infty} \left( \mu_n - \lambda_n \right) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_0^1 res_{z=z_{k,j}}F_k(z, t) q_k(t)dt
\]

where \( F_k(z, t) \) is defined by (5.8)

(4.8) and (4.9) show that expressions for norming constants \( c_{k,j} \) are too long. In our previous works, we write \( c_{k,j} \) in an open form and write concrete form for expressions analogous to \( y_k(1, \lambda) \omega_1(z) \) in choice of \( F_k(z, t) \). But since here those functions have quite long expressions, it is impossible to arrive at results or manipulate formulas writing them in an open form. That is why there arise a need in derivations (5.10)-(5.17) which let us relate the norms of eigenvectors with \( \omega_1(z) \), \( \omega_2(z) \) or characteristic determinant \( \Delta(z) \) whose zeros are related to eigenvalues of \( L_0 \).
But the function $F_k(z,t)$ together with $z_{k,j}$ has poles also at zeros of $y_k(1, z^4)$. Denoting the zeros of $y_k(1, z^4)$ by $z = \beta_{k,j}$ we have

$$res_{z=\beta_{k,j}} F_k(z,t) = \frac{4 \beta_{k,j}^3 y_k^2(t, \beta_{k,j}^4)}{y_k(1, \beta_{k,j}^4) \omega_1(\beta_{k,j})}$$

where dot indicate derivative with respect to $z$.

But from $y(1, \beta_{k,j}^4) = 0$

$$res_{z=\beta_{k,j}} F_k(z,t) = \frac{4 \beta_{k,j}^3 y_k^2(t, \beta_{k,j}^4)}{y_k(1, \beta_{k,j}^4) y''(1, \beta_{k,j}^4)}.$$ 

(6.12)

Note that $\beta_{k,j}^4 + \gamma_k$ are the eigenvalues of problem (5.1),(5.2) and (6.13),(6.14) for each fixed $k$

$$y_k(1) = 0$$

(6.13)

$$y_k^n(1) - \lambda \gamma_k^0 y_k(1) = 0$$

(6.14)

and the collection \( \{ \beta_{k,j}^4 + \gamma_k \}_{j=1}^\infty \) are the eigenvalues of problem (3.1), (3.2), and (6.15), (6.16)

$$y(1) = 0$$

(6.15)

$$y^n(1) - \lambda A^n y(1) = 0$$

(6.16)

Selecting rectangular contour $l_N$ including inside it $z_{k,j}$ and $\beta_{k,j}$ for each fixed $k$ and $j = \frac{1}{N}$ (we can choose such a contour because of asymptotics of $z_{k,j}$, $\beta_{k,j}$), see, for example, [8,9]) and applying the Cauchy theorem about residues we have

$$\sum_{j=1}^N \int_0^1 res_{z=z_{k,j}} F_k(z,t) q_k(t) dt = -\sum_{j=1}^N \int_0^1 res_{z=\beta_{k,j}} F_k(z,t) + \int_{l_N} F_k(z,t) \, dz$$

(6.17)

Multiplying by $q_k(t)$, integrating along $[0,1]$ and passing to limit in (6.17) as $N \to \infty$ yields

$$\sum_{j=1}^\infty \int_0^1 res_{z=z_{k,j}} F_k(z,t) q_k(t) dt =$$

$$-\sum_{j=1}^{\infty} \int_0^1 res_{z=\beta_{k,j}} F_k(z,t) q_k(t) dt + \lim_{N \to \infty} \int_{l_N} \int_0^1 F_k(z,t) \, dz \, q_k(t) \, dt \, dz.$$ 

(6.18)

By using the asymptotics of $F_k(z,t)$ for large $|z|$ values it can be shown that as $N$ tends to infinity, the integral along the extended contours approaches zero. So, with (6.11) and (6.12) in mind

$$\sum_{n=1}^{\infty} (\mu_n - \lambda_n) = \sum_{n=1}^{\infty} \int_0^1 res_{z=z_{k,j}} F_k(z,t) q_k(t) dt =$$

$$= -\sum_{n=1}^{\infty} \int_0^1 res_{z=\beta_{k,j}} F_k(z,t) q_k(t) dt - \sum_{j=1}^{\infty} \frac{4 \beta_{k,j}^3 \int_0^1 y_k^2(t, \beta_{k,j}^4) q_k(t) dt}{2 y_k(1, \beta_{k,j}^4) y''(1, \beta_{k,j}^4)}.$$ 

(6.19)

Let $L_{11} = L_{01} + Q$, where $L_{01}$ is an operator corresponding to (3.1) (3.2),(6.15),(6.16) which is defined in space $H_2 = L_2(H,(0,1) \oplus H$ of vectors $Y = (y(t), y_1), Z = (z(t), z_1)$ where $y_1, z_1 \in H$, with scalar product defined as, $(Y, Z)_2 = (y(t), z(t))_{L^2(H,(0,1))} + (A^* \Phi y_1, A^* \Phi z_1), L_{01} = \{ Y \in D_{L_0^*}, y(1) = 0, y_1 = A^* y(1) \}$, $L_0 Y = \{ q(t), y''(1) \}$ and $Q$ this time is defined as $QY = \{ q(t)y(t), 0 \}$. Moreover, denote by $L_{01k}$ the operator defined by $L_{01k}(y_k(t), \gamma_k^0 y_k'(1)) = \{ l_k y_k(t), y_k''(1) \}$ in space $\Lambda_2 = L_2(0,1) \oplus C$.

**Theorem 6.1**

$$res_{z=\beta_{k,j}} F_k(z,t) = -c_{k,j}^2 y_k^2(t, \beta_{k,j}^4).$$
with \( c_{k,j}^2 = \frac{1}{\|\Phi_{k,j}\|_{H_2}^2} = \frac{1}{\|\Psi_{k,j}\|_{H_2}^2} \), where \( \{\Phi_{k,j}\} \) this time are orthogonal eigenvectors of the operator \( L_{01} \) in \( H_2 \) associated with problem (3.1) (3.2),(6.15),(6.16), and \( \Psi_{k,j} \) are orthogonal eigenvectors of problem (5.1) (5.2),(6.13),(6.14).

To prove it, from the right side of (6.19) we see that, it must be shown that

\[
c_{k,j}^2 = -\frac{4\beta_{k,j}^4}{y_k^m(1,\beta_{k,j}^4)}
\]

**Proof.** Again multiplying (5.10) by \( y_k(t,\lambda_{k,j} - \gamma_k) \), (5.11) \( y_k(t,\lambda - \gamma_k) \), then subtracting the second relation from the first, integrating the both sides of obtained the relation along (0,1), adding to the obtained results the term

\[
(y - \lambda_{k,j})y_k(1,\lambda - \gamma_k) y_k'(1,\lambda_{k,j} - \gamma_k) \gamma_k^\alpha
\]

keeping in mind

\[
y_k(1,\lambda_{k,j} - \gamma_k) = y_k(1,\beta_{k,j}^4) = 0
\]

we get

\[
y_k(1,\lambda_{k,j} - \gamma_k) = y_k(1,\beta_{k,j}^4) = 0
\]

(we again keep notations \( \sqrt{\lambda_{k,j} - \gamma_k} = z \), \( \sqrt{\lambda_{k,j} - \gamma_k} = \beta_{k,j} \), where this time \( \lambda_{k,j} = \beta_{k,j}^4 + \gamma_k \), \( j = 1, \infty \) are eigenvalues of the problem (5.1),(5.2),(6.13), (6.14) we get:

\[
E \equiv \int_0^1 y_k^IV(t,z^4) y_k(t,\beta_{k,j}^4) dt - \int_0^1 y_k^IV(t,\beta_{k,j}^4) y_k(t,z^4) dt + (y - \lambda_{k,j}) y_k'(1,z^4) y_k'(1,\beta_{k,j}^4) \gamma_k^\alpha = (z^4 - \beta_{k,j}^4) \int_0^1 y_k(t,z^4) y_k(t,\beta_{k,j}^4) dt + (z^4 - \beta_{k,j}^4) y_k'(1,z^4) y_k'(1,\beta_{k,j}^4) \gamma_k^\alpha
\]

Integration by parts gives

\[
E = y_k''(1,z^4) y_k(1,\beta_{k,j}^4) - y_k''(1,\beta_{k,j}^4) y_k(1,z^4) - y_k''(1,z^4) y_k'(1,\beta_{k,j}^4)+ y_k''(1,\beta_{k,j}^4) y_k(1,z^4) + (z^4 - \beta_{k,j}^4) y_k'(1,\beta_{k,j}^4) \gamma_k^\alpha = -y_k''(1,\beta_{k,j}^4) y_k(1,z^4) - y_k''(1,\beta_{k,j}^4) y_k'(1,\beta_{k,j}^4) \gamma_k^\alpha + (z^4 - \beta_{k,j}^4) y_k'(1,\beta_{k,j}^4) \gamma_k^\alpha - y_k''(1,\beta_{k,j}^4) \left[ y_k(1,z^4) - y_k(1,\beta_{k,j}^4) \right] - y_k'(1,z^4) y_k'(1,\beta_{k,j}^4) \left[ y_k''(1,z^4) - y_k''(1,\beta_{k,j}^4) \right] + (z^4 - \beta_{k,j}^4) y_k'(1,\beta_{k,j}^4) \gamma_k^\alpha
\]

So,

\[
(z^4 - \beta_{k,j}^4) \int_0^1 y_k(t,z^4) y_k(t,\beta_{k,j}^4) dt + (z^4 - \beta_{k,j}^4) y_k'(1,z^4) y_k'(1,\beta_{k,j}^4) \gamma_k^\alpha = -y_k''(1,\beta_{k,j}^4) \left[ y_k(1,z^4) - y_k(1,\beta_{k,j}^4) \right] - y_k'(1,z^4) y_k'(1,\beta_{k,j}^4) \left[ y_k''(1,z^4) - y_k''(1,\beta_{k,j}^4) \right] + (z^4 - \beta_{k,j}^4) y_k'(1,\beta_{k,j}^4) \gamma_k^\alpha
\]

Recall that the term \( y_k(1,\beta_{k,j}^4) \) can appear on the left side of (6.23) because of (6.21)

Note here \( \lambda_{k,j} = \beta_{k,j}^4 + \gamma_k \).

Dividing the both sides of (6.23) by \( z - \beta_{k,j} \), passing to the limit as \( z \to \beta_{k,j} \) and denoting orthogonal eigenvectors of the problem (5.1),(5.2),(6.13),(6.14) again by \( \Psi_{k,j} \), we get

\[
\frac{4\beta_{k,j}^2}{\|\Psi_{k,j}\|_{L_2}^2} = -y_k''(1,\beta_{k,j}^4) y_k(1,\beta_{k,j}^4) - \frac{\gamma_k^\alpha}{\beta_{k,j}^4}
\]
\[-y_k' \left( 1, \beta_{k,j}^4 \right)^2 \lim_{z \to \beta_{k,j}} \left[ \frac{y_k''(1,z^4)}{y_k'(1,z^4)} \frac{y_k''(1,\beta_{k,j}^4)}{y_k'(1,\beta_{k,j}^4)} \right] = \]

\[= -y_k'' \left( 1, \beta_{k,j}^4 \right) y_k(1,z^4) |_{z = \beta_{k,j}} - y_k' \left( 1, \beta_{k,j}^4 \right)^2 \frac{y_k''(1,z^4)}{y_k'(1,z^4)} - z^4 \gamma_k^\alpha \] \(|_{z = \beta_{k,j}} \) \quad (6.24)

If \(c_{2k}\) is defined from \(\omega_2(z) = 0\), then in right side of (6.24) \(\frac{y_k''(1,z^4)}{y_k'(1,z^4)} - z^4 \gamma_k^\alpha \equiv 0\) and (6.24) simplifies to

\[4\beta_{k,j}^3 \|\Psi_{k,j}\|_2^2 = -y_k'' \left( 1, \beta_{k,j}^4 \right) y_k(1,z^4) |_{z = \beta_{k,j}} \] \(\text{(6.25)}\)

or

\[\|\Phi_{k,j}\|_2^2 = \frac{\|\Psi_{k,j}\|_2^2}{4\beta_{k,j}^3} \] \(\text{(6.26)}\)

We have (see (5.12), (6.21))

\[\omega_1(\beta_{k,j}) = -y_k''(1,\lambda_{k,j}) \] \(\text{(6.27)}\)

So, for norming constants of the problem (3.1),(3.2),(6.15),(6.16)

\[\frac{1}{c_{k,j}^2} = \frac{\|\Phi_{k,j}\|_2^2}{\|\Psi_{k,j}\|_2^2} = \omega_1(\beta_{k,j}) y_k(1,z^4) |_{z = \beta_{k,j}} = \] \(\text{(6.28)}\)

or

\[\frac{-y_k''(1,4\beta_{k,j}^4) y_k(1,z^4) |_{z = \beta_{k,j}}}{4\beta_{k,j}^3} \] \(\text{(6.29)}\)

From (6.12) and (6.28)

\[\text{res}_{z = \beta_{k,j}} F_k(z,t) = \frac{4\beta_{k,j}^3 y_k^2(t,\beta_{k,j}^4)}{y_k(1,\beta_{k,j}^4) \omega_1(\beta_{k,j})} = \frac{4\beta_{k,j}^3 y_k^2(t,\beta_{k,j}^4)}{2y_k(1,\beta_{k,j}^4) y_k''(1,\beta_{k,j}^4)} = \] \(\text{(6.30)}\)

\[\frac{-2c_{k,j}^2 y_k^2(t,\beta_{k,j}^4)}{\|\Psi_{k,j}\|_2^2} = -c_{k,j}^2 \gamma_k^\alpha \] \(\text{(6.30)}\)

\(c_{k,j}\) (if \(k\) is fixed) are now norming constants of the problem (5.11), (5.12), (6.13), (6.14) or for varying \(k\) of the problem (3.1),(3.2), (6.15), (6.16).

Denoting the eigenvalues of \(L_{01}\) and \(L_{11}\) by \(\lambda_{n1}\), \(\mu_{n1}\) respectively, we have for the regularized trace of \(L_{11}\) as in (6.4)

\[\sum_{n=1}^{\infty} (\mu_{n1} - \lambda_{n1}) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (Q Y_{k,j}, Y_{k,j})_2, \] \(\text{(6.31)}\)

where \(\{Y_{k,j}\}\) are now orthonormal eigenvectors of the operator \(L_{01}\).

In virtue of (6.31) and application Theorem 6.1 to \(L_{01}\) yields

\[\sum_{n=1}^{\infty} (\mu_{n1} - \lambda_{n1}) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} c_{k,j}^2 \int_0^1 q_k(t) y_k^2(t,\beta_{k,j}^4) dt = -\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_0^1 \text{res}_{z = \beta_{k,j}} F_k(z,t) q_k(t) dt \] \(\text{(6.32)}\)

By comparing (6.32) and (6.19) we get \(\square\)

**Corollary 6.1.**

\[\sum_{n=1}^{\infty} (\mu_n - \lambda_n) = \sum_{n=1}^{\infty} (\mu_{n1} - \lambda_{n1}) \]

Thus, the problem is reduced to evaluating a regularized trace of corresponding operator \(L_{11}\).
To find the sum on the right side of (6.32), again apply the technique used above: select function of a complex variable with the poles at \( \beta_{k,j} \) and residues equal to the terms of the series on the left of side (6.32). Really, setting

\[
K(z) = -y_k(1, \lambda) y_k''(1, \lambda) - y_k'(1, \lambda) [y_k''(1, \lambda) - \lambda \gamma_k \gamma_k' y_k(1, \lambda)]
\]

and in the solution of (5.1), (5.2) defining \( c_{2k} \) from condition (5.4) \( (y_k'1, \lambda) - \lambda \gamma_k y_k'(1, \lambda) = 0 \) we have

\[
K'(\beta_{k,j}) = -y_k'(1, \beta_{k,j}^4) y_k'''(1, \beta_{k,j}^4)
\]

(6.33)

So, if

\[
F_{1k}(z, t) = \frac{4z^3 y_k^2(t, z^4)}{K(z)}
\]

then in virtue of (6.33) and Theorem 6.1 (or relation 6.29)

\[
res_{z=\beta_{k,j}} F_{1k}(z, t) = \frac{4 \beta_{k,j}^2 y_k^2(t, \beta_{k,j}^4)}{K'(\beta_{k,j})} = \frac{4 \beta_{k,j}^2 y_k^2(t, \beta_{k,j}^4)}{y_k'(1, \beta_{k,j}^4) y_k'''(1, \beta_{k,j}^4)}
\]

(6.34)

Now, if we define in the last relation \( c_{2k} \) from \( y_k(1, \lambda) = 0 \), then \( F_{1k}(z, t) \) will be simplified to the form

\[
F_{1k}(z, t) = \frac{4z^3 y_k^2(t, z^4)}{-2y_k'(1, z^4) [y_k''(1, z^4) - \lambda \gamma_k y_k'(1, z^4)]}
\]

Thus,

\[
\sum_{n=1}^{\infty} (\mu_n - \lambda_{n1}) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left[ \int_0^1 res_{z=\beta_{k,j}} F_{1k}(z, t) q_k(t) dt \right]
\]

(6.35)

But \( F_{1k}(z) \) together with \( \beta_{k,j} \) has poles also at the zeros of the function \( y_k'(1, z^4) \). Denote them by \( \delta_{k,j} \). Thus,

\[
res_{z=\delta_{k,j}} F_{1k}(z, t) = \frac{4 \delta_{k,j}^2 y_k^2(t, z^4)}{-[y_k''(1, z^4)]^3 |_{z=\delta_{k,j}} y_k''(1, \delta_{k,j}^4)}
\]

Again taking the contour \( l_N \ (j = 1, N) \) including \( \beta_{k,j} \) and \( \delta_{k,j} \) and extending it to infinity, we will have

\[
\sum_{j=1}^{\infty} \int_0^1 res_{z=\beta_{k,j}} F_{1k}(z, t) q_k(t) dt = -\sum_{j=1}^{\infty} \int_0^1 res_{z=\delta_{k,j}} F_{1k}(z, t) q_k(t) dt =
\]

\[
\sum_{j=1}^{\infty} \int_0^1 \frac{4 \delta_{k,j}^2 y_k^2(t, z^4) q_k(t) dt}{[y_k''(1, \lambda)]^3 |_{z=\delta_{k,j}} y_k''(1, \delta_{k,j}^4)}
\]

(6.36)

where \(-[y_k''(1, z^4)]^3 |_{z=\delta_{k,j}} y_k''(1, \delta_{k,j}^4)\) is the norm of orthogonal eigenvectors of the operator corresponding to problem (5.1), (5.2) with the additional conditions

\[
y_k(1) = 0,
\]

(6.37)

\[
y_k'(1) = 0
\]

(6.38)

or the norm of orthogonal eigenvectors of the operator \( L_{02} \). Corresponding perturbed operator \( L_{12} = L_{02} + q(t) \) corresponds to problem (3.1), (3.2) and

\[
y(1) = 0,
\]

(6.39)

\[
y'(1) = 0
\]

(6.40)
Dividing both sides of this relation by $Y_k = y_k(1, z^4)$ where $y_k$ is a solution from (5.1), (5.2), (6.37), (6.38) in $H_3 = L_2(H, (0, 1))$ again use the above technique, but this time we will not add any additional terms like the term (6.20) in (6.22) or term (5.12) in (5.13) (there it was done for defining the norm in direct sum space, because of $\lambda$ in the boundary conditions. The last boundary conditions don’t depend on $\lambda$ and that condition define a selfadjoint operator in original space).

For not complicating notations denoting the eigenvectors again by $\Phi_{k,j}$ we have by above illustrated in (5.13)-(5.17) or (6.20)-(6.26) technique and defining $c_{2k}$ from (6.37):

\[
(z^4 - \delta_{k,j}^4) \int_0^1 y_k^2(t, z^4) \, dt = y''(1, z^4) y_k(1, \delta_{k,j}^4) - y''(1, \delta_{k,j}^4) y_k(1, z^4) - y''(1, z^4) y_k'(1, \delta_{k,j}^4) + y''(1, \delta_{k,j}^4) y_k'(1, z^4)
\]

(6.41)

Dividing the both sides of this relation by $z - \delta_{k,j}$, letting $z \to \delta_{k,j}$, defining $c_{2k}$ from (5.34) and taking into consideration $y_k'(1, \delta_{k,j}^4)$ yields

\[
4\delta_{k,j}^4 \|\Phi_{k,j}\|_3^2 = 4\delta_{k,j}^4 \|\Psi_{k,j}\|_{\Lambda_3}^2 = [y_k'(1, z^4)]_{z=\delta_{k,j}} y''(1, \delta_{k,j}^4)
\]

(6.42)

Thus,

\[
\sum_{n=1}^{\infty} (\mu_n - \lambda_n) = -\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_0^1 \text{res}_{z=\delta_{k,j}} F_{1k}(z, t) \, q_k(t) \, dt = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (QY_{k,j}, Y_{k,j})_3
\]

where $Y_{k,j}$ are orthonormal eigenvectors of the operator $L_{02}$ in $H_3$.

Denoting eigenvalues of $L_{12}, L_{02}$ by $\mu_{n2}, \lambda_{n2}$, respectively,

\[
\sum_{n=1}^{\infty} (\mu_n - \lambda_n) = \sum_{n=1}^{\infty} (\mu_{n1} - \lambda_{n1}) = \sum_{n=1}^{\infty} (\mu_{n2} - \lambda_{n2})
\]

Now we come to the evaluation of the sum of series

\[
\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_0^1 4\delta_{k,j}^4 y_k^2(t, z^4) \, q_k(t) \, dt = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (QY_{k,j}, Y_{k,j})_3
\]

(6.43)

For that sake select the following function of complex variable

\[
F_{2k}(z, t) = \frac{4z^2 y_k^2(t, z^4)}{y_k''(1, z^4) y_k'(1, z^4) + y_k'(1, z^4) y_k''(1, z^4)}
\]

(6.44)

whose residues at $\delta_{k,j}$ give terms of series (6.43). Selecting $c_{2k}$ the solution of boundary value problem from $y_k(1) = 0$, $F_{2k}(z, t)$ takes the form

\[
F_{2k}(z, t) = \frac{4z^2 y_k^2(t, z^4)}{y_k'(1, \lambda) y_k''(1, \lambda)}
\]

(6.45)

and

\[
\text{res}_{z=\delta_{k,j}} F_{2k}(z, t) = \frac{4\delta_{k,j}^3 y_k^2(t, \delta_{k,j}^4)}{[y_k(1, \delta_{k,j}^4)]_{z=\delta_{k,j}}}
\]

(6.46)
 Obviously $F_{2k}(z,t)$ will have poles also at roots of the equation $y''_k(1,\lambda) = 0$. Denote these roots by $\rho_{k,j}$. Thus, $\rho_{k,j}$ are common roots of the equations

\[ y_k(0,\lambda) = 0, \quad y'_k(0,\lambda) = 0, \quad y''_k(1,\lambda) = 0 \]  

(6.47)

(6.48)

moreover,

\[ \text{res}_{z=\rho_{k,j}} F_{2k}(z,t) = \frac{4\rho^3_{k,j}y^2_k(t,\rho^4_{k,j})}{[y'_k(1,\rho^4_{k,j})]'' y''_k(1,\rho^4_{k,j})]} \]  

(6.49)

But

\[ \frac{\rho''(1,\rho_{k,j})}{4\rho^2_{k,j}} y_k''(1,\rho_{k,j}) \mid_{z=\rho_{k,j}} = \|\Phi_{k,j}\|^2_3 \]

where $\Phi_{k,j}$ are the eigenvectors of problem (3.1), (3.2) with additional boundary conditions

\[ y(1) = 0, y''(1) = 0 \]  

(6.50)

Really,

\[ (z^4 - \rho^4_{k,j}) \int_0^1 y_k(t, z^4)^2 dt = y''_k(1, z^4) y_k(1, \rho^4_{k,j}) - y'''_k(1, \rho^4_{k,j}) y_k(1, z^4) - \]

\[-y''_k(1, z^4) y'_k(1, \rho^4_{k,j}) + y''_k(1, \rho^4_{k,j}) y'_k(1, z^4) = -y'''_k(1, \rho^4_{k,j}) [y_k(1, z^4) - y_k(1, \rho^4_{k,j})] - \]

\[ -y''_k(1, \rho^4_{k,j}) \left[ y''_k(1, z^4) - y''_k(1, \rho^4_{k,j}) \right] \]  

(6.51)

If $c_{2k}$ is defined from (6.47), then from (6.51) as $z \to \rho_{k,j},$

\[ 4\rho^3_{k,j}||\Psi_{k,j}||^2_\Lambda = -y''_k(1, \rho^4_{k,j}) y_k(1, z^4) \mid_{z=\rho_{k,j}} - y''_k(1, \rho^4_{k,j}) y'_k(1, z^4) \mid_{z=\rho_{k,j}} \]  

(6.52)

Denoting the eigenvalues of $L_{03}$ and $L_{03} + q(t)$ in $L_2(H,(0,1))$ by $\lambda_{n3}$, $\mu_{n3}$, we come to the next theorem

**Theorem 6.2.** \( \sum_{n=1}^{n_1'} (\mu_n - \lambda_n) = \sum_{n=1}^{n_1'} (\mu_{n1} - \lambda_{n1}) = \sum_{n=1}^{n_2'} (\mu_{n2} - \lambda_{n2}) \)

\[ = \sum_{n=1}^{n_3'} (\mu_{n3} - \lambda_{n3}) \]

Hence,

\[ \sum_{n=1}^{n_1'} (\mu_{n2} - \lambda_{n2}) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_0^1 \text{res}_{z=\delta_{k,j}} F_{2k}(z,t) q_k(t) dt = \]

\[ = -\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_0^1 \text{res}_{z=\rho_{k,j}} F_{2k}(z,t) q_k(t) dt = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (QY_{k,j}, Y_{k,j})_4 \]

where $Y_{k,j}$ are now the set of orthonormal eigenvectors of the operator $L_{03}$.

But on the other hand, since the solution satisfying conditions (5.2) is given by (5.18), then from (6.47), (6.48) we have

\[ c_{1k} \sin z + c_{2k} \sinh z = 0 \]

\[ -z^2 c_{1k} \sin z + c_{2k} z^2 \sinh z = 0 \]

from which $c_{2k} = 0$ and orthogonal eigenvectors are $c_{1k} \sin z t$
From boundary conditions (6.47), (6.48) follows $\sin z = 0$ or $z = \pi j$, and eigenvalues are $\lambda_{k,j} = (\pi j)^4 + \gamma_k$ and orthonormal eigenvectors of $L_{03}$ are $Y_{k,j} = \sqrt{2} \sin \pi j t \varphi_k$, $k, j = 1, \infty$

Thus, taking into consideration also the requirement (6.1)

$$\sum_{n=1}^{\infty} (\mu_{n3} - \lambda_{n3}) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (QY_{k,j}, Y_{k,j})_3 = -\sum_{k=1}^{\infty} q_k(1) + q_k(0)$$

**Theorem 6.3.**

$$\sum_{n=1}^{\infty} (\mu_n - \lambda_n) = \sum_{n=1}^{\infty} (\mu_{n1} - \lambda_{n1}) = \sum_{n=1}^{\infty} (\mu_{n2} - \lambda_{n2}) =$$

$$= \sum_{n=1}^{\infty} (\mu_{n3} - \lambda_{n3}) = -\sum_{k=1}^{\infty} q_k(1) + q_k(0)$$

(6.53)

If we put on $q(t)$ a stronger condition than (6.1), namely would $q(t)$ belong to the trace class $\sigma_1$, then from (6.53) we get

**Corollary 6.2.**

$$\sum_{n=1}^{\infty} (\mu_n - \lambda_n) = \sum_{n=1}^{\infty} (\mu_{n1} - \lambda_{n1}) = \sum_{n=1}^{\infty} (\mu_{n2} - \lambda_{n2}) =$$

$$= \sum_{n=1}^{\infty} (\mu_{n3} - \lambda_{n3}) = -\text{tr} q(1) + \text{tr} q(0)$$

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