Perturbation theory and linear partial differential equations with delay

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Abstract

Functional evolution equations are used in the modeling of numerous physical processes. In this work, our main tool is perturbation theory of strongly continuous semigroups. The advantage of this technique is that one can provide functional evolution equations with the explicit representation formulas of the solution. First, we introduce a closed form of the fundamental solution of the evolution equation with a discrete delay using the delayed Dyson-Phillips series. Then we set up the analytical representation formulas of the classical solutions of linear homogeneous/non-homogeneous evolution equations with a constant delay in a Banach space. In the special case, when a strongly continuous group \( \{T(t)\}_{t \in \mathbb{R}} \) commutes with a bounded linear operator \( A_1 \), we obtain an elegant formula for the fundamental solution using the powers of the resolvent operator of \( A_0 \). Furthermore, we consider delay evolution equations with permutable/non-permutable linear bounded operators and derive crucial results in terms of non-commutative analysis. Finally, we present an example, in the context of a one-dimensional heat equation with a discrete delay to demonstrate the applicability of our theoretical results and give some comparisons with existing results.

Keywords: Delay evolution equation; perturbation theory; a discrete delay; a delayed Dyson-Phillips series; heat equation with delay
1 Introduction

Differential equations with time delays are called functional differential equations or time-delay systems. A distinguishing feature of the functional differential equations under consideration is that the evolution rate of the processes described by such equations depends on the pre-history. Time-delay systems are central to all areas of science, particularly the physical and biological sciences, such as immunology, medicine, nuclear power generation, heat transfer, track signal processing, regulation systems, etc. (see, [1]-[3]).

One of the main problems of qualitative theory for functional differential equations is the explicit representation formulas for the solutions of linear delay differential equations in finite and infinite dimensional spaces. Let us briefly summarise what has been done to solve the problem of representing solutions of linear time-delay systems in $\mathbb{R}^n$ by delayed matrix-valued functions. We mention the pioneering work [4, 5] in which the first results for linear time-delay systems were found. In [4], Khusainov and Shuklin studied the problem of relative controllability for a linear control system with a single constant delay using the pure delayed exponential matrix function. In [5], Khusainov et al. proposed an explicit representation formula of the solution for a linear delay differential system with permutable matrix coefficients. Furthermore, in [6, 7], the classical results are extended to time-delay systems of fractional order with permutable [6] and non-permutable [7] matrices using delayed perturbation of Mittag-Leffler matrix functions.

In recent years, there has been increasing interest in the study of partial differential equations with delay [8-11]. Partial differential equations of the parabolic type with a delayed argument are widely used to model and study various problems that arise in the study of population dynamics in ecological systems (see, [12]).
Khusainov et al. studied the existence and uniqueness of a classical solution to the initial-boundary value problem for the one-dimensional heat equation with a constant delay. For the construction of a solution, the authors in [8, 10] used the method of separation of variables or the Fourier method. In [11], Samoilenko and Serheeva have described an algorithm for the construction of global solutions of the delayed one-dimensional heat equation with time-varying coefficients.

Perturbation theory for strongly continuous operator families is a useful tool for evolution equations, in particular, for partial differential equations in modeling many physical phenomena [13, 14, 15]. The perturbation of linear operators in a Banach space has been studied to a considerable extent, most notably by Phillips [13], Travis & Webb [16], and Lutz [17]. In [13], Phillips first studied the implication for a linear abstract Cauchy problem with the infinitesimal generator $A_0 : \mathcal{D}(A_0) \subseteq X \rightarrow X$ of a strongly continuous semigroup which is conserved under bounded perturbation $A_1 \in \mathcal{L}(X)$ in a Banach space $X$:

$$\begin{align*}
\frac{d}{dt} u(t) &= (A_0 + A_1) u(t), \quad t \geq 0, \\
u(0) &= x \in \mathcal{D}(A_0).
\end{align*}$$

In [16], Travis and Webb have established sufficient conditions for perturbed strongly continuous cosine operator families. In [17], Lutz has intended to perturb the infinitesimal generator $A_0$ by adding to it a linear time-varying bounded operator $A_1(\cdot) : \mathbb{R} \rightarrow \mathcal{L}(X)$ and investigating some perturbation properties for infinitesimal generators of strongly continuous cosine and sine operator families. In terms of fractional sense, some perturbation results for fractional abstract initial value problems of order $1 < \alpha < 2$ in [15, 18] have been studied in several aspects and the obtained results agree with the classical ones when $\alpha = 2$.

In the general case, partial differential equations with delay can be written in an abstract way as follows:

$$\begin{align*}
\frac{d}{dt} u(t) &= A_0 u(t) + A_1 u_t, \quad t \geq 0, \\
u(0) &= x, \\
u_0 &= f,
\end{align*}$$

where

- $x \in X$, $X$ is a Banach space;
- $A_0 : \mathcal{D}(A_0) \subseteq X \rightarrow X$ is a closed and densely-defined linear operator;
• $A_1 : \mathbb{L}^p((-\tau,0],X) \to X$ is the delay operator which is a bounded linear operator;

• $u(\cdot) : [-\tau,\infty) \to X$ and $u_t(\cdot) : [-\tau,0] \to X$ is the history function defined by $u_t(s) = u(t+s)$ for $s \in [-\tau,0]$;

• $f(\cdot) : \mathbb{L}^p([-\tau,0],X)$ for $1 \leq p < \infty$.

To analyse such equations, the first step is to choose a suitable state space. There are two different directions to solve this kind of equations. One way is to apply the semigroup theoretical methods in the state space $X = C([-\tau,0],X)$. In this case, the relation between the solutions of the delay equations (2) and a corresponding translation semigroup has been widely studied (see, [19], Section VI.6).

In the second direction, Bátkai and Piazzera in [20, 21] have studied the equivalency between the partial differential equation with delay (2) and the following abstract Cauchy problem, associated with the operator $(A, D(A))$:

$$
\begin{align*}
\frac{d}{dt} U(t) &= AU(t), \quad t \geq 0, \\
U(0) &= (f^i) \in D(A),
\end{align*}
$$

on the product state space $\mathcal{X} := X \times \mathbb{L}^p((-\tau,0],X)$ with $U(t) = (u(t))^i \in \mathcal{X}$ for $t \geq 0$, and a linear operator $A = \begin{pmatrix} A_0 & A_1 \\ 0 & \frac{d}{ds} \end{pmatrix}$. Then, in [20] using the Miyadera-Voigt’s perturbation theorem, the authors give sufficient conditions for $A = A_0 + A_1 = \begin{pmatrix} A_0 & 0 \\ 0 & \frac{d}{ds} \end{pmatrix} + \begin{pmatrix} 0 & A_1 \\ 0 & 0 \end{pmatrix}$ to be the infinitesimal generator of a $C_0$-semigroup on a Banach space $\mathcal{X}$. Note that this reformulation of the delay evolution equation (2) has the advantage that the question of its well-posedness is reduced to the question of whether or not the operator matrix $A$ generates a strongly continuous semigroup on the Banach space $\mathcal{X}$.

Motivated by Phillips [13] and Bátkai & Piazzera [20], we consider the following abstract Cauchy problem for an evolution equation with a discrete delay in a Banach space $X$:

$$
\begin{align*}
\frac{d}{dt} u(t) &= A_0 u(t) + A_1 u(t-\tau) + g(t), \quad t \geq 0, \\
u(t) &= \varphi(t), \quad -\tau \leq t \leq 0,
\end{align*}
$$

where $A_0 : D(A_0) \subseteq X \to X$ is a unbounded linear operator, $A_1 \in \mathcal{L}(X)$ and $\tau > 0$ is a fixed time delay. Moreover, an initial function $\varphi(\cdot) : [-\tau,0] \to X$ is describing
the prehistory of the system and a forcing term $g(\cdot) : [0, \infty) \to X$ is describing the external forces of the system.

Unlike the authors in [20], we consider an abstract differential equation with a discrete delay (4) on the state space $X$ rather than the product state space. In this work, our main tool is the perturbation theory for the strongly continuous semigroups and the advantage of this technique is that one can provide the functional evolution equation (4) with the explicit representation formulas of the solution. Exploiting the perturbation theorem of Phillips [13], we derive the closed-form of a fundamental solution $S(t; \tau)$, $t \geq -\tau$ via a delayed Dyson-Phillips series. Moreover, this result agrees with the delayed perturbation of an operator-valued exponential function for a delay evolution equation with bounded linear operator coefficients. As an application of our theoretical results, we derive an explicit representation formula of the classical (mild) solution of the one-dimensional heat equation with a discrete delay and give some comparisons between the closed-form solutions and existing works such as [8, 22].

The paper includes significant updates in the theory of abstract delay differential equations and is structured as follows. Section 2 is a preparatory section in which we recall the main definitions and results from functional analysis, mainly, operator theory and evolution equations. Section 3 is devoted to the closed-form of a fundamental solution of the abstract Cauchy problem for a linear homogeneous delay evolution equation (10) in terms of a delayed Dyson-Phillips series. In the special case, when a strongly continuous group $\{T(t)\}_{t \in \mathbb{R}}$ commutes with a bounded linear operator $A_1$, we obtain an elegant formula for the fundamental solution using powers of the resolvent operator of $A_0$. Moreover, we provide the analytical representation formulas of classical solutions to linear homogeneous and non-homogeneous evolution equations with a discrete delay. In Section 4 we consider functional evolution equations with linear bounded operators and some special cases of them. Thus, we propose closed-form solutions for delay evolution equations with permutable and non-permutable linear bounded operators. Finally, Section 5 is devoted to the presentation of an illustrative example on one-dimensional heat equation with delay to show the efficiency and validity of the theoretical results, and we give some important comparisons with the existing works.

2 Mathematical description

We embark on this section by briefly presenting some notations and fundamental preliminaries from functional analysis, especially, operator theory and abstract differential equations [19 23 24 25] which are used in this paper.
Some notations. Let us fix some notations. Let $X$ be a complex Banach space with norm $\|\cdot\|$ and $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators on $X$ to itself. The identity and zero (null) operators on $X$ are denoted by $I \in \mathcal{L}(X)$ and $\Theta \in \mathcal{L}(X)$, respectively.

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of real and complex numbers, respectively, $[-\tau, T] = [-\tau, 0] \cup \bigcup_{i=0}^{n} (i\tau, (i+1)\tau] \subset \mathbb{R}$ and $[0, T] = \{0\} \cup \bigcup_{i=0}^{n} (i\tau, (i+1)\tau] \subset \mathbb{R}$ where $T = (n + 1)\tau$ is a pre-fixed positive number for a fixed $n \in \mathbb{N}_0$ and $\tau > 0$.

In the representation of delayed operator-valued functions, we will use the following indicator or characteristic function $1_{t \geq n\tau} : \mathbb{R} \to \{0, 1\}$ defined by:

$$1_{t \geq n\tau} = \begin{cases} 1, & t \geq n\tau, \\ 0, & t < n\tau, \end{cases}$$

where $n \in \mathbb{N}_0$ and $\tau > 0$.

Alternatively, for the delayed operator-valued functions, we will also use the ramp or positive part function $(t)_+ : \mathbb{R} \to [0, \infty)$ defined by the formula:

$$(t)_+ = \max (t, 0) = \begin{cases} t, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Semigroup theory. $C_0$-semigroups serve to describe the time evolution of autonomous linear systems. We provide some fundamental facts concerning $C_0$-semigroups, their generators and their applications to the abstract differential equations in a Banach space $X$ which are used throughout this paper.

We shall be concerned with a family of bounded linear operators $\{\mathcal{T}(t)\}_{t \geq 0}$ on a half-line $[0, \infty)$ to $\mathcal{L}(X)$ is called a $C_0$-semigroup or a strongly continuous semigroup satisfying the following hypotheses:

- $\mathcal{T}(0) = I$;
- $\mathcal{T}(t + s) = \mathcal{T}(t)\mathcal{T}(s)$ for all $t, s \geq 0$;
- $\lim_{t \to 0^+} \|\mathcal{T}(t)x - x\| = 0$ for every $x \in X$.

Note that the condition $(iii)$ is a condition of strong continuity of the function $t \mapsto \mathcal{T}(t)$ at point $t = 0$. Alternatively, the last condition can be given as follows:

- $\mathcal{T}(t)x$ is continuous in $t$ on $[0, \infty)$ for each $x \in X$. 

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If $\mathcal{T}(t)$ is defined for $t \in \mathbb{R}$, the second condition holds for all $t, s \in \mathbb{R}$ and the function $t \to \mathcal{T}(t)x$ is continuous with respect to $t$ on $\mathbb{R}$ for every $x \in X$, then $\{\mathcal{T}(t)\}_{t \in \mathbb{R}}$ is called a $C_0$-group.

A linear operator $A_0 : X \to X$ defined by

$$A_0x = \lim_{t \to 0^+} \frac{\mathcal{T}(t)x - x}{t}, \text{ for every } x \in \mathcal{D}(A_0),$$

where

$$\mathcal{D}(A_0) = \left\{ x \in X : \lim_{t \to 0^+} \frac{\mathcal{T}(t)x - x}{t} \text{ exists} \right\},$$

is the infinitesimal generator of the $C_0$-semigroup $\{\mathcal{T}(t)\}_{t \geq 0}$, defined on its domain $\mathcal{D}(A_0)$. It is known that the infinitesimal generator $A_0 : \mathcal{D}(A_0) \subseteq X \to X$ of a strongly continuous semigroup $\{\mathcal{T}(t)\}_{t \geq 0}$ is a closed and densely-defined linear operator on $X$, i.e., $\overline{\mathcal{D}(A_0)} = X$.

For a strongly continuous semigroup $\{\mathcal{T}(t)\}_{t \geq 0}$, the following assertions are equivalent:

• If $x \in \mathcal{D}(A_0)$, then $\mathcal{T}(t)x \in \mathcal{D}(A_0)$ and

$$\frac{d}{dt} \mathcal{T}(t)x = A_0 \mathcal{T}(t)x = \mathcal{T}(t)A_0x, \quad t \geq 0;$$

• For every $x \in X$, one has

$$\int_0^t \mathcal{T}(s)xds \in \mathcal{D}(A_0), \quad t \geq 0;$$

• For every $t \in [0, \infty)$, the following identities hold true:

$$\mathcal{T}(t)x - x = A_0 \int_0^t \mathcal{T}(s)xds, \quad x \in X,$$

$$= \int_0^t \mathcal{T}(s)A_0xds, \quad x \in \mathcal{D}(A_0).$$

It is known that for every strongly continuous semigroup $\{\mathcal{T}(t)\}_{t \geq 0}$, there exists constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$\|\mathcal{T}(t)\| \leq M \exp(\omega t), \quad t \geq 0. \quad (5)$$
The resolvent \( R(\lambda_0; A_0) = (\lambda_0 I - A_0)^{-1} \) of \( A_0 \) is defined on the resolvent set \( \rho(A_0) \) for \( A_0 \):
\[
\rho(A_0) = \{ \lambda_0 \in \mathbb{C} : \lambda_0 I - A_0 : \mathcal{D}(A_0) \to X \text{ is bijective} \}
\]
and belongs to \( \mathcal{L}(X) \). The resolvent \( R(\lambda_0; A_0) \) of \( A_0 \) satisfies the following identities:
\[
(\lambda_0 I - A_0) R(\lambda_0; A_0) = I, \\
R(\lambda_0; A_0) (\lambda_0 I - A_0) x = x, \quad x \in \mathcal{D}(A_0).
\]
Moreover, if \( \text{Re}(\lambda_0) > \omega \), then \( \lambda_0 \in \rho(A_0) \), and
\[
R(\lambda_0; A_0)x = \int_0^\infty \exp(-\lambda_0 t) T(t)x\ dt, \quad x \in X. \tag{6}
\]
By the formulae (5) and (6), we derive the following estimation:
\[
\|R(\lambda_0; A_0)\| \leq \frac{M}{\text{Re}(\lambda_0) - \omega}.
\]
In the particular case, if the infinitesimal generator \( A_0 : X \to X \) is bounded, i.e., there exists \( M > 0 \) such that \( \|A_0x\| \leq M \|x\| \) for all \( x \in \mathcal{D}(A_0) \), then the following relations hold true:

- The domain \( \mathcal{D}(A_0) \) is all of \( X \), i.e., \( \mathcal{D}(A_0) = X \);
- The \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) is uniformly continuous at the point \( t = 0 \):
  \[
  \lim_{t \to 0^+} \|T(t) - I\| = 0.
  \]
- The \( C_0 \)-semigroup is defined by the formula
  \[
  T(t) = \exp(A_0 t) = \sum_{k=0}^\infty A_0^k \frac{t^k}{k!}, \quad t \geq 0.
  \]
Consider the following abstract Cauchy problem for a linear homogeneous evolution equation in a Banach space \( X \):
\[
\begin{cases}
  u'(t) = A_0 u(t), \quad t \geq 0, \\
  u(0) = x,
\end{cases} \tag{7}
\]
where \( A_0 : \mathcal{D}(A_0) \subseteq X \to X \) is an infinitesimal generator of \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \). Then,
for every $x \in \mathcal{D}(A_0)$, the function
$$u(\cdot) : [0, \infty) \ni t \mapsto u(t) = \mathcal{T}(t)x \in \mathcal{D}(A_0)$$
is the unique classical or strong solution of (7) with initial value $x$.

for every $x \in X$, the function
$$u(\cdot) : [0, \infty) \ni t \mapsto u(t) = \mathcal{T}(t)x \in X$$
is the unique mild solution of (7) with initial value $x$.

Perturbation theory. In many concrete situations, the evolution equation is given as a sum of several terms having different physical meaning and different mathematical properties. In such situations perturbation theory plays a very useful tool in the hands of both analyst and physicist. In this part, we provide some fundamental properties on bounded time-independent perturbations of $C_0$-semigroups (i.e., generator $A_0$ perturbed by the bounded linear operator $A_1$).

If $A_0 : \mathcal{D}(A_0) \subseteq X \to X$ is an infinitesimal generator of $C_0$-semigroup $\{\mathcal{T}(t)\}_{t \geq 0}$ and $A_1 \in \mathcal{L}(X)$, then the semigroup of bounded linear operators $\{\mathcal{S}(t)\}_{t \geq 0}$ generated by $A_0 + A_1$ defined on $\mathcal{D}(A_0)$ can be represented by an absolutely and uniformly (in every compact subset of $[0, \infty)$) convergent series as follows:

$$\mathcal{S}(t) = \sum_{n=0}^{\infty} S_n(t), \quad t \geq 0, \quad (8)$$

where

$$S_0(t) = \mathcal{T}(t), \quad t \geq 0,$$

$$S_n(t) = \int_0^t \mathcal{T}(t-s)A_1S_{n-1}(s)ds,$$

$$= \int_0^t S_{n-1}(t-s)A_1S_0(s)ds, \quad n \in \mathbb{N}, \quad t \geq 0.$$  

Moreover, the strongly continuous semigroup $\{\mathcal{S}(t)\}_{t \geq 0}$ for some $\omega \in \mathbb{R}$ and $M \geq 1$ is satisfying

$$\|\mathcal{S}(t)\| \leq M \exp(\omega_1 t), \quad \omega_1 := \omega + M \|A_1\|, \quad t \geq 0.$$  

For $\text{Re}(\lambda_0) > \omega_1$, we have

$$\mathcal{R}(\lambda_0; A_0 + A_1)x = \int_0^{\infty} \exp(-\lambda_0 t) \mathcal{S}(t)x dt, \quad x \in X.$$
If $A_0$ is an infinitesimal generator of $C_0$-semigroup $\{\mathcal{T}(t)\}_{t \geq 0}$ and $A_1$ is a bounded linear operator, then the resolvent $\mathcal{R}(\lambda_0; A_0 + A_1)$ of $A_0 + A_1$ satisfies the following identities:

$$\mathcal{R}(\lambda_0; A_0 + A_1)x = \sum_{n=0}^{\infty} \mathcal{R}(\lambda_0; A_0) \left[A_1 \mathcal{R}(\lambda_0; A_0)\right]^n x$$

$$= \sum_{n=0}^{\infty} \left[A_1 \mathcal{R}(\lambda_0; A_0)\right]^n \mathcal{R}(\lambda_0; A_0)x, \quad x \in X.$$ 

Consider the following abstract Cauchy problem for a linear non-homogeneous evolution equation in a Banach space $X$:

$$\begin{aligned}
\frac{d}{dt} u(t) &= A_0 u(t) + A_1 u(t) + g(t), \quad t \geq 0, \\
u(0) &= x.
\end{aligned} \tag{9}$$

Let $A_0 : \mathcal{D}(A_0) \subseteq X \to X$ be the infinitesimal generator of a $C_0$-semigroup $\{\mathcal{T}(t)\}_{t \geq 0}$, $A_1 \in \mathcal{L}(X)$ and $g(\cdot) \in C^1 ([0, \infty), X)$. Then, for each $x \in \mathcal{D}(A_0)$, there exists a unique continuously differentiable solution $u(\cdot) : [0, \infty) \to X$ of (9) which is satisfying $u(t) \in \mathcal{D}(A_0)$ for $t \in [0, \infty)$ with $u(0) = x$. This solution has a closed form:

$$u(t) = S(t)x + \sum_{n=0}^{\infty} w_n(t), \quad t \geq 0,$$

where $S(t)$ is given by (8), and

$$w_0(t) = \int_{0}^{t} \mathcal{T}(t-s) g(s) ds, \quad t \geq 0,$$

$$w_n(t) = \int_{0}^{t} \mathcal{T}(t-s) A_1 w_{n-1}(s) ds,$$

$$= \int_{0}^{t} w_{n-1}(t-s) A_1 \mathcal{T}(s) ds, \quad n \in \mathbb{N}, \quad t \geq 0.$$ 

### 3 Main results: a delayed Dyson-Phillips series

In this section, first, we consider the following abstract Cauchy problem for a linear homogeneous evolution equation with a discrete delay $\tau > 0$ in a Banach space $X$:

$$\begin{aligned}
\frac{d}{dt} u(t) &= A_0 u(t) + A_1 u(t-\tau), \quad t \geq 0, \\
u(t) &= \varphi(t) \in X, \quad -\tau \leq t \leq 0.
\end{aligned} \tag{10}$$
where $A_0 : D(A_0) \subseteq X \to X$ is the infinitesimal generator of a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ of bounded linear operators, $A_1 \in \mathcal{L}(X)$ and the initial function $\varphi(\cdot) \in \mathbb{C}([-\tau, 0], X)$.

Our main aim is to determine a closed-form of the classical (strong) solution of an abstract Cauchy problem (10).

A classical (strong) solution of the abstract Cauchy problem (10) is understood as an operator-valued function $u(\cdot) : [-\tau, \infty) \to X$ -continuously defined for all $t \geq -\tau$, continuously differentiable for all $t \geq 0$, $u(t) \in D(A_0)$ for all $t \geq 0$ with $\varphi(0) \in D(A_0)$ and satisfying (10) for all $t \geq 0$.

Meanwhile, this definition can be generalized to mild sense as follows:

A mild solution of the abstract Cauchy problem (10) is understood as an operator-valued function $u(\cdot) : [-\tau, \infty) \to X$ -continuously defined for all $t \geq -\tau$, continuously differentiable for all $t \geq 0$, $u(t) \in X$ for all $t \geq 0$ and satisfying (10) for all $t \geq 0$.

The fundamental solution of the abstract initial value problem (10) which is the main part of the solution of delay evolution equation can be defined as follows.

**Definition 1.** If $S(\cdot; \tau) : [-\tau, \infty) \to \mathcal{L}(X)$ satisfies the following linear homogeneous abstract differential equation with linear operator coefficients:

$$\frac{d}{dt}S(t; \tau) = A_0 S(t; \tau) + A_1 S(t - \tau; \tau), \quad t \geq 0,$$

under initial conditions

$$S(t; \tau) = \begin{cases} \Theta, & -\tau \leq t < 0, \\ I, & t = 0, \end{cases}$$

then $S(t; \tau), \ t \geq -\tau$ is called the corresponding fundamental solution of an abstract differential equation (10) with a constant delay.

**Remark 2.** The fundamental solution $S(t; \tau), \ t \geq -\tau$ also satisfies the following differential equation with operator coefficients and initial conditions

$$\begin{cases} \frac{d}{dt}S(t; \tau) = S(t; \tau)A_0 + S(t - \tau; \tau)A_1, & t \geq 0, \\ S(t; \tau) = \begin{cases} \Theta, & -\tau \leq t < 0, \\ I, & t = 0. \end{cases} \end{cases}$$

This does not mean that $S(t; \tau)$ commutes individually with the coefficient operators $A_i, \ i = 0, 1$ for any $t \in [0, \infty)$. 

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Proof. To verify this remark, it is sufficient to compare the Laplace image of the fundamental solution as a solution of differential equation with operator coefficients (11) with the abstract delay differential equation (13).

A fundamental solution is an operator-valued function with values in $L(X)$ which can be found with the help of Laplace transform technique. Let $\lambda_0 \in \rho(A_0)$. Then, applying the Laplace integral transform each side of (11) with initial conditions (12) and using integration by substitution, we obtain

$$\lambda_0 \int_0^\infty \exp(-\lambda_0 t) S(t; \tau) x dt - x = A_0 \int_0^\infty \exp(-\lambda_0 t) S(t; \tau) x dt$$
$$+ A_1 \int_0^\infty \exp(-\lambda_0 t) S(t - \tau; \tau) x dt$$
$$= A_0 \int_0^\infty \exp(-\lambda_0 t) S(t; \tau) x dt$$
$$+ A_1 \int_0^\infty \exp(-\lambda_0 (t + \tau)) S(t; \tau) x dt$$
$$= A_0 \int_0^\infty \exp(-\lambda_0 t) S(t; \tau) x dt$$
$$+ A_1 \exp(-\lambda_0 \tau) \int_0^\infty \exp(-\lambda_0 t) S(t; \tau) x dt, \quad x \in X.$$

Therefore, for sufficiently large $Re(\lambda_0)$, the Laplace transform of a fundamental solution is defined by

$$\int_0^\infty \exp(-\lambda_0 t) S(t; \tau) x dt = \left(\lambda_0 I - A_0 - A_1 \exp(-\lambda_0 \tau)\right)^{-1} x, \quad x \in X. \quad (14)$$

Moreover, the right-hand side of (14) is the perturbation of the infinitesimal generator $A_0$ with a bounded linear operator $A_1 := A_1 \exp(-\lambda_0 \tau)$:

$$\left(\lambda_0 I - A_0 - A_1 \exp(-\lambda_0 \tau)\right)^{-1} x = \mathcal{R} \left(\lambda_0; A_0 + \hat{A}_1\right) x, \quad x \in X. \quad (15)$$

Therefore, for any $x \in X$, the Laplace transform of a fundamental solution of (10) can be determined by the resolvent of $A_0 + \hat{A}_1$ as follows:

$$\int_0^\infty \exp(-\lambda_0 t) S(t; \tau) x dt = \mathcal{R} \left(\lambda_0; A_0 + \hat{A}_1\right) x, \quad \hat{A}_1 := A_1 \exp(-\lambda_0 \tau) \in L(X). \quad (16)$$

The following lemma is given in more general case for closed linear operators and plays a significant role in the proof of Theorem 5.

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Lemma 3. Let $A_0 : \mathcal{D}(A_0) \subseteq X \to X$ be closed linear operator on $X$ and assume $A_1 \in \mathcal{L}(X)$ is such that $\|A_1 R(\lambda_0; A_0) \exp(-\lambda_0 \tau)\| < 1$ for some $\lambda_0 \in \rho(A_0)$. Then, $A_0 + A_1$ where $\hat{A}_1 := A_1 \exp(-\lambda_0 \tau) \in \mathcal{L}(X)$ is a closed linear operator with domain $\mathcal{D}(A_0)$ and $R \left( \lambda_0; A_0 + \hat{A}_1 \right)$ exists, and the following identity holds true:

$$R \left( \lambda_0; A_0 + \hat{A}_1 \right) = \sum_{n=0}^{\infty} \mathcal{R}(\lambda_0; A_0) \left[ A_1 \mathcal{R}(\lambda_0; A_0) \right]^n \exp(-n\lambda_0 \tau).$$

(17)

Proof. It is obvious that $A_0 + \hat{A}_1$ with $\hat{A}_1 = A_1 \exp(-\lambda_0 \tau) \in \mathcal{L}(X)$ is a closed linear operator with domain $\mathcal{D}(A_0)$. Since the Neumann series $\sum_{n=0}^{\infty} \left[ A_1 \mathcal{R}(\lambda_0; A_0) \exp(-\lambda_0 \tau) \right]^n$ converges under the hypotheses $\|A_1 \mathcal{R}(\lambda_0; A_0) \exp(-\lambda_0 \tau)\| < 1$, we note that

$$\mathcal{R} \equiv \sum_{n=0}^{\infty} \mathcal{R}(\lambda_0; A_0) \left[ A_1 \mathcal{R}(\lambda_0; A_0) \right]^n \exp(-n\lambda_0 \tau)$$

$$= \mathcal{R}(\lambda_0; A_0) \sum_{n=0}^{\infty} \left[ A_1 \mathcal{R}(\lambda_0; A_0) \exp(-\lambda_0 \tau) \right]^n$$

$$= \mathcal{R}(\lambda_0; A_0) \left( I - A_1 \mathcal{R}(\lambda_0; A_0) \exp(-\lambda_0 \tau) \right)^{-1},$$

and hence that

$$\left( \lambda_0 I - A_0 - A_1 \exp(-\lambda_0 \tau) \right) \mathcal{R}$$

$$= \left( I - A_1 \mathcal{R}(\lambda_0; A_0) \exp(-\lambda_0 \tau) \right) \mathcal{R}(\lambda_0; A_0)^{-1} \mathcal{R}$$

$$= \left( I - A_1 \mathcal{R}(\lambda_0; A_0) \exp(-\lambda_0 \tau) \right) \left( I - A_1 \mathcal{R}(\lambda_0; A_0) \exp(-\lambda_0 \tau) \right)^{-1}$$

$$= I. \quad (18)$$

Furthermore, the range of $\mathcal{R}$ is precisely $\mathcal{D}(A_0)$ since the range of $\left( I - A_1 \mathcal{R}(\lambda_0; A_0) \exp(-\lambda_0 \tau) \right)^{-1}$ is $X$. Thus, for a given $x \in \mathcal{D}(A_0)$ there exists a $y \in X$ such that $x = \mathcal{R}y$. Therefore, by (18), we attain that

$$\mathcal{R} \left( \lambda_0 I - \left( A_0 + A_1 \exp(-\lambda_0 \tau) \right) \right) x$$

$$= \mathcal{R} \left( \lambda_0 I - \left( A_0 + A_1 \exp(-\lambda_0 \tau) \right) \right) \mathcal{R} y$$

$$= \mathcal{R} y$$

$$= x,$$

so that $\mathcal{R}$ is both a left and a right inverse. The proof is complete. \qed
Remark 4. Note that under the condition $\|R(\lambda_0; A_0)A_1 \exp(-\lambda_0 \tau)\| < 1$ for some $\lambda_0 \in \rho(A_0)$, the following identity also holds true:

$$R(\lambda_0; A_0 + \hat{A}_1) = \sum_{n=0}^{\infty} \left[ R(\lambda_0; A_0)A_1 \right]^n R(\lambda_0; A_0) \exp(-n\lambda_0 \tau).$$

The closed-form of a fundamental solution to delay evolution equation (10) can be expressed with the help of a delayed Dyson-Phillips series.

Theorem 5. Let $A_0 : \mathcal{D}(A_0) \subseteq X \to X$ be the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t \geq 0}$ and $A_1 \in \mathcal{L}(X)$. Then there is a unique one-parameter family of bounded linear operators $S(t; \tau)$ strongly continuous on $[-\tau, \infty)$ such that $S(t; \tau) = \Theta, -\tau \leq t < 0$ and $S(0; \tau) = I$; strongly continuously differentiable on $[0, \infty)$ to $\mathcal{L}(X)$ and satisfying

$$\frac{d}{dt} S(t; \tau) = A_0 S(t; \tau) + A_1 S(t - \tau; \tau), \quad t \geq 0. \tag{19}$$

This solution has an explicit representation formula:

$$S(t; \tau) = \sum_{n=0}^{\infty} S_n(t, n\tau) 1_{t \geq n\tau}, \quad t \geq 0, \tag{20}$$

where

$$S_0(t, 0) = T(t), \quad t \geq 0,$$

$$S_n(t, n\tau) = \int_{n\tau}^{t} T(t - s) A_1 S_{n-1}(s - \tau, (n - 1)\tau) ds, \quad t \geq n\tau, \quad n \in \mathbb{N}. \tag{21}$$

Proof. We divide by the proof some parts:

1. Since a fundamental solution is satisfying initial conditions (12), i.e., $S(t; \tau) = \Theta, -\tau \leq t < 0$ and $S(0; \tau) = I$, it is strongly continuous on the initial interval $[-\tau, 0]$.

It is obvious that $S_0(t, 0) = T(t)$ is strongly continuous on $[0, \infty)$ that $\|S_0(t, 0)\| \leq M \exp(\omega t)$ by (15). Suppose $S_n(t, n\tau) 1_{t \geq n\tau}$ is like-wise strongly continuous on $[0, \infty)$ and that

$$\|S_n(t, n\tau)\| \leq M \left( M \|A_1\| \exp(-\omega\tau) \right)^n \frac{(t - n\tau)^n}{n!} \exp(\omega t), \quad t \geq n\tau. \tag{22}$$
Then $T(t-s)A_1S_n(s-\tau, n\tau)$ will be strongly continuous on $[(n+1)\tau, t]$ so that the integral defining $S_{n+1}(t, (n+1)\tau)1_{t\geq(n+1)\tau}$ exists in the strong topology for $t \in [0, \infty)$, by induction principle. Furthermore, by (21) and (22), we have:

$$\|S_{n+1}(t, (n+1)\tau)\| \leq M \|A_1\| \int_{(n+1)\tau}^{t} \exp(\omega(t-s)) \|S_n(s-\tau, n\tau)\| \, ds$$

$$\leq M \left( M \|A_1\| \exp(-\omega\tau) \right)^{n+1} \exp(\omega t) \int_{(n+1)\tau}^{t} \frac{(s-(n+1)\tau)^n}{n!} \, ds$$

$$= M \left( M \|A_1\| \exp(-\omega\tau) \right)^{n+1} \frac{(t-(n+1)\tau)^{n+1}}{(n+1)!} \exp(\omega t), \; t \geq (n+1)\tau.$$

Finally, for $t_1 < t_2$, we have

$$\|S_{n+1}(t_2, (n+1)\tau)x - S_{n+1}(t_1, (n+1)\tau)x\|$$

$$\leq \int_{(n+1)\tau}^{t_2} \|\left( T(t_2-s) - T(t_1-s) \right)A_1S_n(s-\tau, n\tau)x\| \, ds$$

$$+ \int_{t_1}^{t_2} \|T(t_2-s)\| \|A_1\| \|S_n(s-\tau, n\tau)x\| \, ds, \; x \in X. \quad (23)$$

As $t_1 \to t_2$, the integrand in the first term on the right of (23) converges to zero boundedly, the integrand of the second term is bounded and the second integral converges to zero boundedly, too. It follows that $S_{n+1}(t, (n+1)\tau)1_{t\geq(n+1)\tau}$ is strongly continuous on $[0, \infty)$. Therefore, with the help of mathematical induction principle, we have showed that $S_n(t, n\tau)1_{t\geq n\tau}$ is well-defined, strongly continuous, and satisfies (22) for all $n \in \mathbb{N}_0$. Hence, the series (20) is a strongly continuous function on $[0, \infty)$ with values in $L(X)$.

2. Meanwhile, by making use of the estimation (22) and closed-form of a pure delayed exponential function, in accordance with the comparison test for functional series, the series representing in (20) converges in the uniform operator topology uniformly for $t$ in every compact subset of $[0, \infty)$:

$$\left\| \sum_{n=0}^{\infty} S_n(t, n\tau)1_{t\geq n\tau} \right\| \leq \sum_{n=0}^{\infty} \left\| S_n(t, n\tau)1_{t\geq n\tau} \right\|$$

$$\leq M \exp(\omega t) \sum_{n=0}^{\infty} \left( M \|A_1\| \exp(-\omega\tau) \right)^n \frac{(t-n\tau)^n}{n!} 1_{t\geq n\tau}$$

$$= M \exp(\omega t) \exp(\hat{\omega} t), \; t \geq 0, \; \hat{\omega} := M \|A_1\| \exp(-\omega\tau),$$

15
where \( \exp_{\tau}(\cdot) : \mathbb{R} \to \mathbb{R} \) is a \textit{pure delayed} real-valued exponential function defined by
\[
\exp_{\tau}(t) = \sum_{n=0}^{\infty} \frac{(t-n\tau)^n}{n!} 1_{t \geq n\tau}, \quad t \in \mathbb{R}.
\]

On the other hand, the series (20) is majorized by the series expansion of \( M \exp(\omega_1 t) \) where \( \omega_1 := \omega + \hat{\omega} \) with \( \hat{\omega} := M \|A_1\| \exp(-\omega \tau) \)
\[
\left\| \sum_{n=0}^{\infty} S_n(t, n\tau) 1_{t \geq n\tau} \right\| \leq M \exp(\omega t) \sum_{n=0}^{\infty} \hat{\omega}^n \frac{(t-n\tau)^n}{n!} 1_{t \geq n\tau}
\]

\[
\leq M \exp(\omega t) \sum_{n=0}^{\infty} \hat{\omega}^n \frac{t^n}{n!}
\]

\[
= M \exp \left( (\omega + \hat{\omega}) t \right)
\]

\[
= M \exp(\omega_1 t), \quad t \geq 0.
\]

Then, for \( \text{Re}(\lambda_0) > \omega_1 \), we can attain that
\[
\int_{0}^{\infty} \exp(-\lambda_0 t) \sum_{n=0}^{\infty} \left[ S_n(t, n\tau) 1_{t \geq n\tau} x \right] dt = \sum_{n=0}^{\infty} \int_{0}^{\infty} \exp(-\lambda_0 t) S_n(t, n\tau) 1_{t \geq n\tau} x dt
\]

\[
= \sum_{n=0}^{\infty} \int_{n\tau}^{\infty} \exp(-\lambda_0 t) S_n(t, n\tau) x dt, \quad t \geq n\tau, \quad x \in X,
\]

where the interchanging of the summation and integration is justified by the \textit{uniform convergence} of the series in the \textit{uniform operator topology}.

Now, if \( x^* \in X^* \), it is a consequence of the uniform convergence of the integral and of the Fubini’s theorem that
\[
x^* \left[ \int_{n\tau}^{\infty} \exp(-\lambda_0 t) S_n(t, n\tau) x dt \right]
\]

\[
= \int_{n\tau}^{\infty} \exp(-\lambda_0 t) x^* \left[ S_n(t, n\tau) x \right] dt
\]

\[
= \int_{n\tau}^{\infty} \exp(-\lambda_0 t) \int_{n\tau}^{t} x^* \left[ T(t-s)A_1 S_{n-1}(s-\tau, (n-1)\tau) x \right] ds dt
\]

\[
= \int_{n\tau}^{\infty} \exp(-\lambda_0 s) \int_{n\tau}^{\infty} \exp(-\lambda_0 (t-s)) x^* \left[ T(t-s)A_1 S_{n-1}(s-\tau, (n-1)\tau) x \right] dt ds
\]

\[
= \int_{n\tau}^{\infty} \exp(-\lambda_0 s) \int_{0}^{\infty} \exp(-\lambda_0 t) x^* \left[ T(t) A_1 S_{n-1}(s-\tau, (n-1)\tau) x \right] dt ds
\]
\[
\int_{n\tau}^{\infty} \exp(-\lambda_0 s)x^* \left[ \mathcal{R} (\lambda_0; A_0) A_1 S_{n-1} (s - \tau, (n-1)\tau) x \right] ds
\]
\[
x^* \left[ \mathcal{R} (\lambda_0; A_0) A_1 \left\{ \int_{n\tau}^{\infty} \exp(-\lambda_0 s) S_{n-1} (s - \tau, (n-1)\tau) x ds \right\} \right], \quad t \geq n\tau, \quad x \in X.
\]

Hence, by induction, we derive that
\[
\int_{n\tau}^{\infty} \exp(-\lambda_0 t) S_n (t, n\tau) x dt = \mathcal{R} (\lambda_0; A_0) \left[ A_1 \mathcal{R} (\lambda_0; A_0) \right]^n \exp(-n\lambda_0 \tau) x, \quad t \geq n\tau, \quad x \in X.
\]

Therefore, the Laplace transform of the series (20) for any \( t \in [0, \infty) \) is defined by
\[
\int_0^{\infty} \exp(-\lambda_0 t) \left[ \sum_{n=0}^{\infty} S_n(t, n\tau) 1_{t \geq n\tau} x \right] dt = \sum_{n=0}^{\infty} \mathcal{R} (\lambda_0; A_0) \left[ A_1 \mathcal{R} (\lambda_0; A_0) \right]^n \exp(-n\lambda_0 \tau) x, \quad x \in X.
\]

On the other hand, for \( Re(\lambda_0) > \omega_1 = \omega + M \| A_1 \| \exp (-\omega \tau) \), we have
\[
\| A_1 \mathcal{R} (\lambda_0; A_0) \exp(-\lambda_0 \tau) \| = \| A_1 \| \left\| \int_0^{\infty} \exp(-\lambda_0 (t + \tau)) T(t) dt \right\|
\]
\[
= \| A_1 \| \left\| \int_\tau^{\infty} \exp(-\lambda_0 t) T(t - \tau) dt \right\|
\]
\[
\leq \| A_1 \| \int_0^{\infty} \exp \left( - Re(\lambda_0) t \right) \| T(t - \tau) \| dt
\]
\[
\leq M \| A_1 \| \exp(-\omega \tau) \int_0^{\infty} \exp \left( -(Re(\lambda_0) - \omega) t \right) dt
\]
\[
= \frac{M \| A_1 \| \exp(-\omega \tau)}{Re(\lambda_0) - \omega} < 1. \quad (24)
\]

Therefore, for the infinitesimal generator \( A_0 \) of a strongly continuous semigroup of bounded linear operators \( \{ T(t) \}_{t \geq 0} \) and the bounded linear operator \( A_1 \), by Lemma 3 we have:
\[
\mathcal{R} \left( \lambda_0; A_0 + \hat{A}_1 \right) x = \int_0^{\infty} \exp(-\lambda_0 t) S(t; \tau) x dt
\]
\[
= \sum_{n=0}^{\infty} \mathcal{R} (\lambda_0; A_0) \left[ A_1 \mathcal{R} (\lambda_0; A_0) \right]^n \exp(-n\lambda_0 \tau) x, \quad x \in X.
\]

Therefore, for \( Re(\lambda_0) > \omega_1 \), the Laplace integral transforms of both \( S(t; \tau) \) and \( \sum_{n=0}^{\infty} S_n(t, n\tau) 1_{t \geq n\tau} \) are equal on \([0, \infty)\) and hence by the uniqueness theorem of
Laplace transform ([19], pp. 530), these two functions are equal for any \( t \in [0, \infty) \). Therefore, \( \sum_{n=0}^{\infty} S_n(t, n\tau) 1_{t \geq n\tau} \) converges to a strongly continuous function \( S(t; \tau) \) uniformly with respect to \( t \) in the uniform operator topology on every compact subsets of \([0, \infty)\) and \( S(t; \tau) \) is a fundamental solution of (10) for \( t \geq 0 \), under the initial conditions (12).

3. Alternatively, we can prove that \( t \mapsto S(t; \tau) \) satisfies (10) for all \( t \geq 0 \) with initial conditions \( S(t; \tau) = \Theta, -\tau \leq t < 0 \) and \( S(0; \tau) = I \). Since \( T(0) = I \), \( S_n(0, n\tau) = \Theta \) for \( n \in \mathbb{N} \), we have \( S(t; \tau) = \Theta, -\tau \leq t < 0 \) and \( S(0; \tau) = I \), i.e., the initial conditions (12) are satisfied. Applying (20), (21) and interchanging of the summation and integration which is justified by the uniform convergence of the series in the uniform operator topology, it follows that

\[
S(t; \tau) = \sum_{n=0}^{\infty} S_n(t, n\tau) 1_{t \geq n\tau} = S_0(t, 0) + \sum_{n=1}^{\infty} S_n(t, n\tau) 1_{t \geq n\tau}
\]

\[
= T(t) + \sum_{n=1}^{\infty} \int_{0}^{t} T(t-s)A_1 S_{n-1}(s-\tau, (n-1)\tau) 1_{s \geq n\tau} ds
\]

\[
= T(t) + \int_{0}^{t} T(t-s)A_1 \sum_{n=1}^{\infty} S_{n-1}(s-\tau, (n-1)\tau) 1_{s \geq n\tau} ds
\]

\[
= T(t) + \int_{0}^{t} T(t-s)A_1 S_n(s-\tau, n\tau) 1_{s \geq (n+1)\tau} ds
\]

\[
= T(t) + \int_{0}^{t} T(t-s)A_1 S(s-\tau; \tau) ds, \quad t \geq 0.
\]

First, we need to show that the function \([0, \infty) \ni t \mapsto S(t, \tau) \in \mathcal{L}(X)\) is strongly continuously differentiable. Since \( t \mapsto T(t) \) is strongly continuously differentiable for all \( t \geq 0 \), this implies that \( t \mapsto S_0(t, 0) \) is strongly continuously differentiable for all \( t \geq 0 \). Assuming that this is true for \( t \mapsto S_n(t, n\tau) \) for all \( t \geq 0 \), then by induction, it is easily shown to be true for \( t \mapsto S_{n+1}(t, (n+1)\tau) \) for all \( t \geq 0 \). Furthermore, by Leibniz integral rule, we derive that

\[
S_0'(t, 0) = A_0 T(t) = T(t)A_0, \quad t \geq 0
\]

\[
S_n'(t, n\tau) = \int_{n\tau}^{t} T(t-s)A_1 S_{n-1}'(s-\tau, (n-1)\tau) ds, \quad t \geq n\tau, \quad n \in \mathbb{N}.
\]

By using (13), for \( x \in \mathcal{D}(A_0) \), we obtain by induction from these equations the following bound:

\[
\|S_n'(t, n\tau)x\| \leq M \left( M \|A_1\| \exp(-\omega \tau) \right)^n \exp(\omega t) \frac{(t-n\tau)^n}{n!} \|A_0x\|, \quad t \geq n\tau, \quad n \in \mathbb{N}_0.
\]
From this bound it follows that the series $\sum_{n=0}^{\infty} S_n(t, n\tau) \mathbb{1}_{t \geq n\tau}$ converges uniformly in every compact interval of $[0, \infty)$ to a continuous function which, by the usual argument, is $S(t; \tau)$.

To show $S(t; \tau)$ is a fundamental solution of (10) for $t \geq 0$ under initial conditions (12), we differentiate the last expression with the help of Leibniz integral rule (differentiation under integral sign), as follows:

$$\frac{d}{dt} S(t; \tau) = \frac{d}{dt} \left[ T(t) + \int_0^t T(t-s) A_1 S(s-\tau) ds \right]$$

$$= A_0 \left[ T(t) + \int_0^t T(t-s) A_1 S(s-\tau) ds \right] + A_1 S(t-\tau; \tau)$$

$$= A_0 S(t; \tau) + A_1 S(t-\tau; \tau), \quad t \geq 0.$$

4. In the uniqueness proof, it will be sufficient to show that if $U(t)$ solves the abstract functional differential equation (10) for $t \geq 0$ with zero initial conditions $U(t) = 0$ for $t \in [-\tau, 0]$, then $U(t) \equiv 0$ for all $t \in [0, \infty)$, since $A_0$ is densely-defined in $X$.

In other words, it is sufficient to consider a strongly continuously differentiable function $U(t)$ on $[0, \infty)$ to $L(X)$ such that $U(t) = 0$ for $-\tau \leq t \leq 0$ and $\frac{d}{dt} U(t) = A_0 U(t) + A_1 U(t-\tau)$ for $t \geq 0$. Operating on both sides of this equation by $T(t-s)$ and integrating on $[0, t]$ gives

$$\int_0^t T(t-s) \frac{\partial}{\partial s} U(s) ds = \int_0^t T(t-s) A_0 U(s) ds$$

$$+ \int_0^t T(t-s) A_1 U(s-\tau) ds, \quad t \geq 0. \quad (25)$$

It can be easily shown for $s \in [0, t]$ that

$$\frac{\partial}{\partial s} \left[ T(t-s) U(s) \right] = -T(t-s) A_0 U(s) + T(t-s) \frac{\partial}{\partial s} U(s). \quad (26)$$

Therefore, by virtue of (25), (26) and the second fundamental theorem of calculus, we attain that

$$U(t) = \int_0^t T(t-s) A_1 U(s-\tau) ds, \quad t \geq 0. \quad (27)$$

Let $V(t) := \sup \{ U(t+h) : h \in [-\tau, 0] \}$. For a fixed time-delay $\tau > 0$, we are setting $m_t := \sup \{ \| V(s) \| : s \in [0, t] \}$ and we see that

$$m_t \leq M \| A_1 \| \frac{\omega t}{\omega} - 1 m_t,$$
and for chosen sufficiently small $t$ such that $M \|A_1\| \frac{|t-1|}{\omega} < 1$. This implies that $m_t = 0$. Thus, $U(t) = 0$ on $[0, t_0]$ with $t_0 > 0$. Iteration of this argument leads to $U(t) \equiv 0$ on $[0, \infty)$.

Assume that a strongly continuously differentiable function $C(t; \tau)$ on $[0, \infty)$ to $\mathcal{L}(X)$ such that $C(t; \tau) = \Theta$ for $-\tau \leq t < 0$, $C(0; \tau) = I$ and

$$\frac{d}{dt} C(t; \tau) = A_0 C(t; \tau) + A_1 C(t - \tau; \tau), \quad t \geq 0. \tag{28}$$

For operating on both sides of (28) by $T(t - s)$ and integrating on $[0, t]$ gives for any $t \geq$:

$$\int_0^t T(t - s) \frac{\partial}{\partial s} C(s; \tau) ds = \int_0^t T(t - s) A_0 C(s; \tau) ds$$

$$+ \int_0^t T(t - s) A_1 C(s - \tau; \tau) ds. \tag{29}$$

It can be easily shown for $s \in [0, t]$ that

$$\frac{\partial}{\partial s} [T(t - s) C(s; \tau)] = -T(t - s) A_0 C(s; \tau) + T(t - s) \frac{\partial}{\partial s} C(s; \tau). \tag{30}$$

Therefore, from (29) and (30) and by the second theorem of fundamental calculus, we obtain that

$$C(t; \tau) = T(t) + \int_0^t T(t - s) A_1 C(s - \tau; \tau) ds, \quad t \geq 0. \tag{31}$$

On the other hand, the method of variation of constant formula yields $S(t; \tau)$ on $[0, \infty)$ to $\mathcal{L}(X)$ is also satisfying

$$S(t; \tau) = T(t) + \int_0^t T(t - s) A_1 S(s - \tau; \tau) ds, \quad t \geq 0. \tag{32}$$

The difference $U(t) = S(t; \tau) - C(t; \tau), \ t \geq 0$ satisfies (27) and vanishes at the points $t \in [-\tau, 0]$. Thus, the uniqueness argument shows that this difference is identically zero for any $t \in [0, \infty)$. The proof is complete.

\begin{remark}
Note that the solution has can also represented by

$$S(t; \tau) = \sum_{n=0}^{\infty} S_n(t, n\tau) 1_{t \geq n\tau}, \quad t \geq 0,$$
\end{remark}
via the following successive iterations:

\[ S_0(t, 0) = \mathcal{T}(t), \quad t \geq 0, \]
\[ S_n(t, n\tau) = \int_0^{t-n\tau} S_{n-1}(t-s-\tau, (n-1)\tau)A_1\mathcal{T}(s)ds, \quad t \geq n\tau, \quad n \in \mathbb{N}. \]

If we consider abstract differential equation with a discrete delay (10) on \([-\tau, T]\) where \(T = (n+1)\tau\) for a fixed \(n \in \mathbb{N}_0\), then we can introduce a piece-wise construction for a fundamental solution \(\mathcal{S}(\cdot; \tau) : [-\tau, T] \to \mathcal{L}(X)\) of (10) under the initial conditions (12).

**Corollary 7.** Let \(A_0 : \mathcal{D}(A_0) \subseteq X \to X\) be the infinitesimal generator of a strongly continuous semigroup of bounded linear operators \(\mathcal{T}(t), 0 \leq t \leq T\) and \(A_1 \in \mathcal{L}(X)\). Then, the abstract Cauchy problem for functional evolution equation (10) admits an uniquely determined strongly continuous fundamental solution \(\mathcal{S}(\cdot; \tau) : [-\tau, T] \to \mathcal{L}(X)\) on \([-\tau, T]\) which is satisfying initial conditions \(\mathcal{S}(t; \tau) = \Theta, -\tau \leq t < 0\) and \(\mathcal{S}(0; \tau) = I\) and strongly continuously differentiable on \([0, T]\) and

\[ \mathcal{S}(t; \tau) = \sum_{k=0}^{n} \mathcal{S}_{k}(t, k\tau), \quad n\tau < t \leq (n+1)\tau, \quad n \in \mathbb{N}_0, \quad (33) \]

where

\[ \mathcal{S}_0(t, 0) = \mathcal{T}(t), \quad t \geq 0, \]
\[ \mathcal{S}_k(t, k\tau) = \int_{k\tau}^{t} \mathcal{T}(t-s)A_1\mathcal{S}_{k-1}(s-\tau, (k-1)\tau)ds \]
\[ = \int_{0}^{t-k\tau} \mathcal{S}_{k-1}(t-s-\tau, (k-1)\tau)A_1\mathcal{T}(s)ds, \quad t \geq k\tau, \quad k = 1, 2, \ldots, n. \]

In the particular case, by using the powers of \(\mathcal{R}(\lambda_0; A_0)\), we can derive the following elegant representation formula for a fundamental solution \(\mathcal{S}(t; \tau), t \geq -\tau\) of (10) under the initial conditions (12).

The following theorem is given in more general case, for \(C_0\)-groups. If \(C_0\)-group \(\{\mathcal{T}(t)\}_{t \in \mathbb{R}}\) commutes with \(A_1 \in \mathcal{L}(X)\), one can obtain more elegant formula for a fundamental solution as below.

**Theorem 8.** Let \(A_0 : \mathcal{D}(A_0) \subseteq X \to X\) be the infinitesimal generator of a strongly continuous group \(\{\mathcal{T}(t)\}_{t \in \mathbb{R}}\) of bounded linear operators. If a strongly continuous
group \( \{T(t)\}_{t \in \mathbb{R}} \) commutes with \( A_1 \in \mathcal{L}(X) \), then, the strongly continuous one-parameter family of bounded linear operators \( S(t; \tau) \) which is satisfying \( S(t; \tau) = \Theta \) for \(-\tau \leq t < 0\) and \( S(0; \tau) = I \), has a closed-form on \([0, \infty)\) as follows:

\[
S(t; \tau) = \exp^{[A_1 T(\tau)^{-1}]t} T(t) = \sum_{n=0}^{\infty} \left[ A_1 T(\tau)^{-1} \right]^n \frac{(t-n\tau)^n}{n!} 1_{t \geq n\tau} T(t)
\]

\[
= T(t) \exp^{[A_1 T(\tau)^{-1}]t} = T(t) \sum_{n=0}^{\infty} \left[ A_1 T(\tau)^{-1} \right]^n \frac{(t-n\tau)^n}{n!} 1_{t \geq n\tau}, \quad t \geq 0. \quad (34)
\]

**Proof.** Since a \( C_0 \)-group \( T(\cdot) : \mathbb{R} \to \mathcal{L}(X) \) commutes with \( A_1 \in \mathcal{L}(X) \), then by the virtue of Lemma 3 and relation (24), for \( \Re(\lambda_0) > \omega_1 = \omega + M \|A_1\| \exp(-\omega \tau) \) the following identity holds true:

\[
\mathcal{R}(\lambda_0; A_0 + \hat{A}_1) = \int_0^{\infty} \exp(-\lambda_0 t) S(t; \tau) dt
\]

\[
= \sum_{n=0}^{\infty} \mathcal{R}(\lambda_0; A_0) \left[ A_1 \mathcal{R}(\lambda_0; A_0) \exp(-\lambda_0 \tau) \right]^n
\]

\[
= \sum_{n=0}^{\infty} A_1^n \left[ \mathcal{R}(\lambda_0; A_0) \right]^{n+1} \exp(-n\lambda_0 \tau), \quad \hat{A}_1 = A_1 \exp(-\lambda_0 \tau) \in \mathcal{L}(X).
\]

It is known that \([12], \text{Corollary 1.11, pp. 56}\), for \( \Re(\lambda_0) > \omega \) and \( n \in \mathbb{N}_0 \), the following identity is true:

\[
\left[ \mathcal{R}(\lambda_0; A_0) \right]^{n+1} x = \int_0^{\infty} \exp(-n\lambda_0 t) \frac{t^n}{n!} T(t) x dt, \quad x \in X. \quad (36)
\]

By the identity (36) and using integration by substitution, we get

\[
\left[ \mathcal{R}(\lambda_0; A_0) \right]^{n+1} \exp(-n\lambda_0 \tau) x = \int_0^{\infty} \exp(-\lambda_0 t) \frac{(t-n\tau)^n}{n!} T(t-n\tau) 1_{t \geq n\tau} x dt, \quad x \in X. \quad (37)
\]

Therefore, from (35) and (37), taking inverse Laplace transform, we derive a desired result:

\[
S(t; \tau) = \sum_{n=0}^{\infty} A_1^n \frac{(t-n\tau)^n}{n!} T(t-n\tau) 1_{t \geq n\tau}
\]

\[
= \sum_{n=0}^{\infty} \left[ A_1 T(\tau)^{-1} \right]^n \frac{(t-n\tau)^n}{n!} 1_{t \geq n\tau} T(t)
\]

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\[ T(t) = \sum_{n=0}^{\infty} \left( A_1 T(\tau)^{-1} \right)^n \frac{t-n\tau^n}{n!} \mathbb{1}_{t \geq n\tau} \]
\[ = \exp_t^{[A_1 T(\tau)^{-1}]t} T(t) \]
\[ = T(t) \exp_t^{[A_1 T(\tau)^{-1}]t}, \quad t \geq 0, \]

where by the properties of \( C_0 \)-group \( \{T(t)\}_{t \in \mathbb{R}} \) we have used that

\[ T(t-n\tau) = T(t) T(-n\tau) = T(t) \left[ T(-\tau) \right]^n = T(t) \left[ T(\tau)^{-1} \right]^n. \]

The proof is complete. \( \square \)

Furthermore, the following corollary deals with the special case where the \( C_0 \)-semigroup and \( A_1 \in \mathcal{L}(X) \) are permutable.

**Corollary 9.** Let \( A_0 : \mathcal{D}(A_0) \subseteq X \to X \) be the infinitesimal generator of a strongly continuous semigroup \( \{T(t)\}_{t \geq 0} \) of bounded linear operators. If a strongly continuous group \( \{T(t)\}_{t \geq 0} \) commutes with \( A_1 \in \mathcal{L}(X) \), then, the strongly continuous one-parameter family of bounded linear operators \( S(t;\tau) \) which is satisfying \( S(t;\tau) = \Theta \) for \( -\tau \leq t < 0 \) and \( S(0;\tau) = I \), has a closed-form on \( [0, \infty) \) as follows:

\[ S(t;\tau) = \sum_{n=0}^{\infty} A_1^n \frac{(t-n\tau)^n}{n!} T(t-n\tau) \mathbb{1}_{t \geq n\tau} \]
\[ = \sum_{n=0}^{\infty} T(t-n\tau) \frac{(t-n\tau)^n}{n!} \mathbb{1}_{t \geq n\tau} A_1^n, \quad t \geq 0. \quad (38) \]

If we consider this particular case on \([-\tau,T]\) where \( T = (n+1)\tau \) for a fixed \( n \in \mathbb{N}_0 \), then we can use the following piece-wise construction for a fundamental solution of (10) with the initial conditions (12).

**Corollary 10.** Let \( A_0 : \mathcal{D}(A_0) \subseteq X \to X \) be the infinitesimal generator of a strongly continuous group \( \{T(t)\}_{t \in \mathbb{R}} \) of bounded linear operators. If a strongly continuous group \( \{T(t)\}_{t \in \mathbb{R}} \) commutes with \( A_1 \in \mathcal{L}(X) \), then, the fundamental solution of (10) has a representation on \([0,T]\) as follows:

\[ S(t;\tau) = \exp_t^{[A_1 T(\tau)^{-1}]t} T(t) = \sum_{k=0}^{n} \left[ A_1 T(\tau)^{-1} \right]^k \frac{(t-k\tau)^k}{k!} T(t) \]
\[ = T(t) \exp_t^{[A_1 T(\tau)^{-1}]t} = T(t) \sum_{k=0}^{n} \left[ A_1 T(\tau)^{-1} \right]^k \frac{(t-k\tau)^k}{k!}, \quad n\tau < t \leq (n+1)\tau. \]
Next, we derive an explicit representation formula is known as the Cauchy formula for the classical (strong) solution of the abstract initial value problem to linear homogeneous functional evolution equation \((10)\) via the method of variation of constants formula.

**Theorem 11.** Let \(A_0 : \mathcal{D}(A_0) \subseteq X \rightarrow X\) be infinitesimal generator of a strongly continuous semigroup \(\{T(t)\}_{t \geq 0}\) of bounded linear operators, \(A_1 \in \mathcal{L}(X)\) and the initial function \(\varphi(\cdot) \in \mathcal{C}^1([-\tau, 0], X)\). Then, the classical (strong) solution \(u_0(\cdot) \in \mathcal{C}^1([-\tau, \infty), X)\) of the abstract Cauchy problem \((10)\) for a linear homogeneous functional evolution equation is satisfying \(u_0(t) \in \mathcal{D}(A_0)\) for \(t \geq 0\) with \(\varphi(t) \in \mathcal{D}(A_0)\) for all \(t \in [-\tau, 0]\) and it can be represented in the integral form

\[
\begin{align*}
\varphi(t) &= S(t + \tau; \tau) \varphi(-\tau) + \int_{-\tau}^{0} S(t - s; \tau) \left[ \varphi'(s) - A_0 \varphi(s) \right] ds, \quad t \geq -\tau. \quad (39)
\end{align*}
\]

**Proof.** By using the variation of constants formula, any solution \(u_0(\cdot) : [-\tau, \infty) \rightarrow X\) of linear homogeneous delay evolution equation \((10)\) should be satisfied in the form:

\[
\begin{align*}
u_0(t) &= S(t + \tau; \tau) c + \int_{-\tau}^{0} S(t - s; \tau) g(s) ds, \quad t \geq -\tau,
\end{align*}
\]

where \(c\) is an unknown vector, \(g(\cdot) : [-\tau, 0] \rightarrow X\) is an unknown continuously differentiable operator-valued function and that it satisfies the initial conditions \(u_0(t) = \varphi(t)\) for \(-\tau \leq t \leq 0\):

\[
\begin{align*}
\varphi(t) &= S(t + \tau; \tau) c + \int_{-\tau}^{0} S(t - s; \tau) g(s) ds, \quad t \in [-\tau, 0]. \quad (40)
\end{align*}
\]

If \(t = -\tau\), then in accordance with the following result

\[
S(-\tau - s; \tau) = \begin{cases} 
\Theta, & -\tau < s \leq 0, \\
I, & s = -\tau,
\end{cases}
\]

we acquire \(c = \varphi(-\tau) \in \mathcal{D}(A_0)\).

On the interval \(-\tau \leq t \leq 0\), we split \((10)\) into two integrals:

\[
\begin{align*}
\varphi(t) &= S(t + \tau; \tau) \varphi(-\tau) + \int_{-\tau}^{t} S(t - s; \tau) g(s) ds \\
&\quad + \int_{t}^{0} S(t - s; \tau) g(s) ds, \quad t \in [-\tau, 0].
\end{align*}
\]
Furthermore, by making use of the following relations:

\[ S(t-s;\tau) = T(t-s), \quad -\tau \leq s \leq t \quad \text{and} \quad S(t-s;\tau) = \begin{cases} \Theta, & t < s \leq 0, \\ I, & s = t, \end{cases} \]

one can easily derive that

\[ \varphi(t) = T(t+\tau)\varphi(\tau) + \int_{-\tau}^{\tau} T(t-s)g(s)ds, \quad t \in [-\tau, 0]. \quad (41) \]

After differentiating the equation (41) by Leibniz integral rule, we derive

\[ \varphi'(t) = A_0 \left[ T(t+\tau)\varphi(\tau) + \int_{-\tau}^{t} T(t-s)g(s)ds \right] + g(t) \]

\[ = A_0\varphi(t) + g(t), \quad t \in [-\tau, 0]. \]

Hence, it follows that \( g(t) = \varphi'(t) - A_0\varphi(t), \) \( t \in [-\tau, 0]. \) The proof is complete. \( \square \)

The integrand occurring in the expression (39) is the derivative of the initial function \( \varphi(t), \) \( -\tau \leq t \leq 0, \) i.e., the initial conditions are continuously differentiable on the initial interval \([-\tau, 0]. \) This condition can be "weakened" by requiring \( \varphi(t) \) only to be continuous for \( t \in [-\tau, 0]. \)

**Theorem 12.** Let \( A_0 : \mathcal{D}(A_0) \subseteq X \to X \) be infinitesimal generator of a strongly continuous semigroup of bounded linear operators \( \{T(t)\}_{t \geq 0}, A_1 \in \mathcal{L}(X) \) and the initial function \( \varphi(\cdot) \in \mathbb{C}([[-\tau, 0], X). \) Then, the classical (strong) solution \( u_0(\cdot) \in \mathbb{C}([-\tau, \infty), X) \cap C^1([0, \infty), X) \) of the abstract Cauchy problem (10) for a linear homogeneous functional evolution equation is satisfying \( u_0(t) \in \mathcal{D}(A_0) \) for all \( t \geq 0 \) with \( \varphi(0) \in \mathcal{D}(A_0) \) and it can be represented in the integral form:

\[ u_0(t) = S(t;\tau)\varphi(0) + \int_{-\tau}^{0} S(t-\tau-s;\tau)A_1\varphi(s)ds, \quad t \geq 0. \quad (42) \]

**Proof.** Indeed, by performing integration by parts, we derive that

\[ \int_{-\tau}^{0} S(t-s;\tau)\left[ \varphi'(s) - A_0\varphi(s) \right]ds = S(t;\tau)\varphi(0) - S(t+\tau;\tau)\varphi(\tau) \]

\[ - \int_{-\tau}^{0} \left[ \frac{\partial}{\partial s}S(t-s;\tau) - S(t-s;\tau)A_0 \right] \varphi(s)ds \]

\[ = S(t;\tau)\varphi(0) - S(t+\tau;\tau)\varphi(\tau) \]
∫_0^{−τ} S(t − τ − s; τ)A_1 ϕ(s)ds.

By putting (43) in (39), one can reduce the expression (39) to the form (42).

Remark 13. It is interesting to note that these results are the natural extension of the results attained in [26] for first-order impulsive time-delay linear systems. Secondly, we consider the following abstract Cauchy problem for a linear non-homogeneous functional evolution equation in a Banach space $X$:

$$
\begin{align*}
\frac{du(t)}{dt} &= A_0 u(t) + A_1 u(t − τ) + g(t), \quad t \geq 0, \\
u(t) &= ϕ(t), \quad −τ ≤ t ≤ 0,
\end{align*}
$$

(44)

where $A_0 : D(A_0) ⊆ X → X$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t≥0}$, $A_1 ∈ \mathcal{L}(X)$, $τ$ is a positive time-delay, $ϕ(·) : [−τ, 0] → X$ is a continuous operator-valued function on $[−τ, 0]$ and $g(·) : [0, ∞) → X$ is a continuously differentiable operator-valued function on $[0, ∞)$.

We will apply a superposition principle for a construction of solution to the linear non-homogeneous abstract differential equation (44) with a constant delay. This principle is clear from linearity of the problem, but it is a very useful principle in solving linear non-homogeneous differential equations.

Superposition principle: It is known that if $u_0(t)$ is a solution of linear homogeneous abstract Cauchy problem (10) with non-homogeneous initial conditions $u_0(t) = ϕ(t)$, $−τ ≤ t ≤ 0$ and $u_1(t)$ is a solution of linear non-homogeneous abstract Cauchy problem (44) with homogeneous initial conditions $u_1(t) = 0$, $−τ ≤ t ≤ 0$, then $u(t) := u_0(t) + u_1(t)$ is a solution of linear non-homogeneous abstract Cauchy problem (44) with non-homogeneous initial conditions $u(t) = ϕ(t)$, $−τ ≤ t ≤ 0$.

Before finding a closed-form of solution to (44), we prove the following auxiliary lemma plays a crucial role in the proof of Theorem 15.

Lemma 14. Let $\{T(t)\}_{t≥0}$ be a strongly continuous semigroup of bounded linear operators. For continuous $f(t)$ on $[0, ∞)$ to $X$, $g(t) = ∫_0^t T(t−s)f(s)ds = ∫_0^t T(s)f(t−s)ds$ exists and is itself continuous on $[0, ∞)$ to $X$. If $f(t)$ is continuously differentiable on $[0, ∞)$, then $g(t)$ is also continuously differentiable on $[0, ∞)$ and the following relations hold true on $[0, ∞)$:

$$
\frac{dg(t)}{dt} = T(t)f(0) + ∫_0^t T(t−s)f'(s)ds,
$$

(45)

$$
= f(t) + A_0 ∫_0^t T(t−s)f(s)ds.
$$

(46)
Proof. Since \( \mathcal{T}(t)x \) is continuous with respect to \( t \) on \([0, \infty)\) for a fixed \( x \in X \), it is obvious that \( \mathcal{T}(t-s)f(s) \) is also continuous in \( s \in [0, t] \) whenever the same is true of \( f(s) \). In this case, \( g(t) = \int_0^t \mathcal{T}(t-s)f(s)ds \) will exists in the strong topology and be equal to \( \int_0^t \mathcal{T}(s)f(t-s)ds \).

Since \( f(s) \) is continuously differentiable for any \( s \in [0, t] \), by Leibniz integral rule, we obtain that
\[
\frac{dg(t)}{dt} = \frac{d}{dt} \int_0^t \mathcal{T}(s)f(t-s)ds = \mathcal{T}(t)f(0) + \int_0^t \mathcal{T}(s)\frac{\partial}{\partial t}f(t-s)ds
\]
\[
= \mathcal{T}(t)f(0) + \int_0^t \mathcal{T}(t-s)f'(s)ds, \quad t \geq 0. \quad (47)
\]

Next, we prove the equivalence of the equations (45) and (46). Since \( \int_0^t \mathcal{T}(t-s)f(s)ds \in \mathcal{D}(A_0) \), by the integration by parts formula, it follows that
\[
\int_0^t \mathcal{T}(t-s)f'(s)ds = f(t) - \mathcal{T}(t)f(0) + A_0 \int_0^t \mathcal{T}(t-s)f(s)ds, \quad t \geq 0. \quad (48)
\]
As a consequence, from (47) and (48), we attain the desired result:
\[
\frac{dg(t)}{dt} = f(t) + A_0 \int_0^t \mathcal{T}(t-s)f(s)ds, \quad t \geq 0.
\]

Further, the continuity of \( \frac{dg(t)}{dt} \) follows from the first part of the lemma using the first representation of \( \frac{dg(t)}{dt} \) given by (47). The proof is complete. \( \square \)

**Theorem 15.** Let \( A_0 \) be the infinitesimal generator of a strongly continuous semigroup of bounded linear operators \( \{\mathcal{T}(t)\}_{t \geq 0} \). Let \( A_1 \in \mathcal{L}(X) \) and \( g(t) \) be continuously differentiable function on \([0, \infty)\) to \( X \). Then there exists a unique continuously differentiable solution \( u(\cdot) : [-\tau, \infty) \to X \) of the abstract Cauchy problem (43) for a linear non-homogeneous delay evolution equation which is satisfying \( u(t) \in \mathcal{D}(A_0) \) for all \( t \geq 0 \) with \( \varphi(0) \in \mathcal{D}(A_0) \). This solution has a closed form:
\[
u(t) = u_0(t) + u_1(t), \quad t \geq 0, \quad (49)
\]
where \( u_0(t) \) is given by (12), and
\[
u_1(t) = \sum_{n=0}^{\infty} w_n(t, n\tau) \mathbb{1}_{t \geq n\tau}, \quad t \geq 0,
\]
\[
u_0(t, 0) = \int_0^t \mathcal{T}(t-s)g(s)ds, \quad t \geq 0,
\]
\[
u_n(t, n\tau) = \int_{n\tau}^t \mathcal{T}(t-s)A_1w_{n-1}(s-\tau, (n-1)\tau)ds, \quad t \geq n\tau, \quad n \in \mathbb{N}.
\]

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Proof. It follows from Lemma 14 that \( w_0(t, 0) \) is continuously differentiable on \([0, \infty)\) and hence by induction that \( w_n(t, n\tau) \mathbb{1}_{t \geq n\tau} \) is like-wise for any \( n \in \mathbb{N} \) on \([0, \infty)\). In fact, by Lemma 14, we have

\[
\begin{align*}
w'_0(t, 0) &= T(t)g(0) + \int_0^t T(t-s)g'(s)ds, \quad t \geq 0, \\
w'_n(t, n\tau) &= \int_{n\tau}^t T(t-s)A_1w_{n-1}(s-\tau, (n-1)\tau)ds, \quad t \geq n\tau, \quad n \in \mathbb{N}.
\end{align*}
\]

By using (5) it is now easy to acquire the following estimations for \( t \geq n\tau \), \( n \in \mathbb{N}_0 \):

\[
\begin{align*}
\|w_n(t, n\tau)\| &\leq M \left( M \|A_1\| \exp(-\omega\tau) \right)^n N \exp(\omega t) \frac{(t-n\tau)^{n+1}}{(n+1)!}, \\
\|w'_n(t, n\tau)\| &\leq M \left( M \|A_1\| \exp(-\omega\tau) \right)^n N \exp(\omega t) \left[ \frac{(t-n\tau)^n}{n!} + \frac{(t-n\tau)^{n+1}}{(n+1)!} \right],
\end{align*}
\]

where \( N := \sup_{t \geq 0} \{ \|g(t)\|, \|g'(t)\| \} \).

From these bounds, as in Theorem 5, this implies that the functional series \( \sum_{n=0}^{\infty} w_n(t, n\tau) \mathbb{1}_{t \geq n\tau} \) and \( \sum_{n=0}^{\infty} w'_n(t, n\tau) \mathbb{1}_{t \geq n\tau} \) converge uniformly in each compact subset of \([0, \infty)\) to continuous functions which are \( u_1(t) \) and \( u'_1(t) \), respectively. Furthermore, \( u_1(t) = 0 \), for \( -\tau \leq t \leq 0 \). From the definition of \( w_n(t, n\tau) \), \( n \in \mathbb{N}_0 \) and uniform convergence of the series \( \sum_{n=0}^{\infty} w_n(t, n\tau) \mathbb{1}_{t \geq n\tau} \) in every finite interval, it follows that

\[
\begin{align*}
\sum_{n=0}^{\infty} w_n(t, n\tau) \mathbb{1}_{t \geq n\tau} &= w_0(t, 0) + \sum_{n=1}^{\infty} w_n(t, n\tau) \mathbb{1}_{t \geq n\tau} \\
&= w_0(t, 0) + \sum_{n=1}^{\infty} \int_0^t T(t-s)A_1w_{n-1}(s-\tau, (n-1)\tau) \mathbb{1}_{s \geq n\tau} ds \\
&= w_0(t, 0) + \int_0^t T(t-s)A_1 \sum_{n=1}^{\infty} w_{n-1}(s-\tau, (n-1)\tau) \mathbb{1}_{s \geq n\tau} ds \\
&= w_0(t, 0) + \int_0^t T(t-s)A_1 \sum_{n=0}^{\infty} w_n(s-\tau, n\tau) \mathbb{1}_{s \geq (n+1)\tau} ds \\
&= w_0(t, 0) + \int_0^t T(t-s)A_1 u_1(s-\tau) ds, \quad t \geq 0.
\end{align*}
\]

Since \( u_1(t) \) is continuously differentiable on \([0, \infty)\), by Lemma 14 we can differentiate
term-wise as follows:
\[
\frac{du_1(t)}{dt} = \frac{d}{dt}\left(w_0(t, 0) + \int_0^t \mathcal{T}(t - s)A_1u_1(s - \tau)ds\right)
\]
\[
= \frac{d}{dt}\left(\mathcal{T}(t) \ast g(t)\right) + \frac{d}{dt}\left(\mathcal{T}(t) \ast (A_1u_1(t - \tau))\right)
\]
\[
= A_0\mathcal{T}(t) \ast g(t) + g(t) + A_0\mathcal{T}(t) \ast (A_1u_1(t - \tau)) + A_1u_1(t - \tau)
\]
\[
= A_0\left[w_0(t, 0) + \int_0^t \mathcal{T}(t - s)A_1u_1(s - \tau)ds\right] + A_1u_1(t - \tau) + g(t)
\]
\[
= A_0u_1(t) + A_1u_1(t - \tau) + g(t), \quad t \geq 0.
\]

This shows immediately that \(u_1(t)\) is a particular solution of linear non-homogeneous abstract Cauchy problem (44). In other words, \(u_1(t)\) is a solution of (44) with zero initial conditions, i.e., \(u_1(t) = 0, -\tau \leq t \leq 0\). Therefore, by Theorem 5, \(u(t) = u_0(t) + u_1(t)\) is a solution for linear non-homogeneous abstract Cauchy problem (44). The uniqueness of a particular solution follows precisely as in the uniqueness proof of Theorem 5. The proof is complete.

Remark 16. Note that the particular solution of (44) has can also expressed by
\[
\begin{align*}
\frac{d}{dt}u_1(t) &= \frac{d}{dt}\left(w_0(t, 0) + \int_0^t \mathcal{T}(t - s)A_1u_1(s - \tau)ds\right) \\
&= \frac{d}{dt}\left(\mathcal{T}(t) \ast g(t)\right) + \frac{d}{dt}\left(\mathcal{T}(t) \ast (A_1u_1(t - \tau))\right) \\
&= A_0\mathcal{T}(t) \ast g(t) + g(t) + A_0\mathcal{T}(t) \ast (A_1u_1(t - \tau)) + A_1u_1(t - \tau) \\
&= A_0\left[w_0(t, 0) + \int_0^t \mathcal{T}(t - s)A_1u_1(s - \tau)ds\right] + A_1u_1(t - \tau) + g(t) \\
&= A_0u_1(t) + A_1u_1(t - \tau) + g(t), \quad t \geq 0.
\end{align*}
\]

(51)

where
\[
\begin{align*}
w_0(t, 0) &= \int_0^t \mathcal{T}(t - s)g(s)ds, \quad t \geq 0, \\
w_n(t, n\tau) &= \int_0^{t-n\tau} w_{n-1}(t - s - \tau, (n - 1)\tau)A_1\mathcal{T}(s)ds, \quad t \geq n\tau, \quad n \in \mathbb{N}.
\end{align*}
\]

If we consider linear non-homogeneous abstract differential equation with a discrete delay (44) on \([-\tau, T]\), then we can introduce a piece-wise construction for a particular solution of (44) as follows.

Corollary 17. Let \(\mathcal{T}(t)\) be a strongly continuous semigroup of bounded linear operators with infinitesimal generator \(A_0\) on \([0, T]\). Let \(A_1 \in \mathcal{L}(X)\) and \(g(t)\) be continuously differentiable operator-valued function on \([0, T]\) to \(X\). Then, there exists a unique continuously differentiable function \(u_1(\cdot) : [0, T] \rightarrow X\) is a particular solution of (44) with zero initial conditions \(u_1(t) = 0, -\tau \leq t \leq 0\) and satisfying
$u(t) \in \mathcal{D}(A_0)$ for all $0 \leq t \leq T$. This solution has a closed form:

$$u_1(t) = \sum_{k=0}^{n} w_k(t, k\tau), \quad n\tau < t \leq (n+1)\tau, \quad n \in \mathbb{N}_0, \quad (52)$$

where

$$w_0(t, 0) = \int_0^t \mathcal{T}(t - s)g(s)ds, \quad t \geq 0,$$

$$w_k(t, k\tau) = \int_{k\tau}^t \mathcal{T}(t - s)A_1 w_{k-1}(s - \tau, (k-1)\tau)ds$$

$$= \int_0^{t-k\tau} w_{k-1}(t - s - \tau, (k-1)\tau)A_1 \mathcal{T}(s)ds, \quad t \geq k\tau, \quad k = 1, 2, \ldots, n.$$

The particular solution $u_1(t)$, $t \geq 0$ of (44) can also be put in a more suggestive form. By making use of the series representation of fundamental solution to the abstract Cauchy problem (10) and Fubini’s theorem, we attain:

$$\int_0^t S_n(t - s, n\tau)1_{t-s \geq n\tau}g(s)ds$$

$$= \int_0^t \int_0^{t-s} \mathcal{T}(t - s - \sigma)A_1 S_{n-1}(\sigma - \tau, (n-1)\tau)1_{\sigma \geq n\tau}g(s)d\sigma ds$$

$$= \int_0^t \int_s^t \mathcal{T}(t - \sigma)A_1 S_{n-1}(\sigma - s - \tau, (n-1)\tau)1_{\sigma-s \geq n\tau}g(s)d\sigma ds$$

$$= \int_0^t \mathcal{T}(t - \sigma)A_1 \left[ \int_0^\sigma S_{n-1}(\sigma - s, n\tau)1_{\sigma-s \geq n\tau}g(s)ds \right]d\sigma, \quad n \in \mathbb{N}, \quad t \geq 0.$$
\[
= \sum_{n=0}^{\infty} \int_{0}^{t} S_n(t-s,n\tau) \mathbb{1}_{t-s \geq n\tau} g(s) ds \\
= \int_{0}^{t} \sum_{n=0}^{\infty} S_n(t-s,n\tau) \mathbb{1}_{t-s \geq n\tau} g(s) ds \\
= \int_{0}^{t} S(t-s;\tau) g(s) ds, \quad t \geq 0. \tag{53}
\]

4 Delay evolution equations with bounded linear operators

In this section, we consider different important cases of delay evolution equation (10) with bounded linear operators \(A_0, A_1 \in \mathcal{L}(X)\) in a Banach space \(X\). We will take a domain of equation as \([0, T]\) where \(T = (n + 1)\tau\) for a fixed \(n \in \mathbb{N}_0\) and use a piece-wise construction for the delayed operator-valued functions. It is known that, in this case, the domain \(D(A_0)\) coincides with the state space \(X\), i.e., \(D(A_0) = X\), and a one-parameter \(C_0\)-semigroup family of bounded linear operators \(\mathcal{T}(t), 0 \leq t \leq T\) which is continuous with respect to uniform operator topology defined by

\[
\mathcal{T}(t) = \exp(A_0 t) = \sum_{k=0}^{\infty} A_0^k \frac{t^k}{k!}.
\tag{54}
\]

For a fixed \(n \in \mathbb{N}\) and time-delay \(\tau > 0\), we define the following sequence of operator-valued functions via a recursive way:

\[
S_0(t,0) := \exp(A_0 t), \quad t \geq 0, \\
S_1(t,\tau) := \begin{cases} 
\int_{\tau}^{t} \exp(A_0(t-s)) A_1 S_0(s - \tau, 0) ds, & t \geq \tau, \\
\Theta, & t < \tau,
\end{cases} \\
\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
S_n(t,n\tau) := \begin{cases} 
\int_{n\tau}^{t} \exp(A_0(t-s)) A_1 S_{n-1}(s - \tau, (n-1)\tau) ds, & t \geq n\tau, \\
\Theta, & t < n\tau.
\end{cases} \tag{55}
\]

Therefore, in this case, the fundamental solution of abstract Cauchy problem (10) with bounded linear operators which is represented by delayed Dyson-Phillips series becomes the delayed perturbation of an operator-valued exponential function.
Definition 18. The delayed perturbation of an operator-valued exponential function $S(\cdot; \tau) : \mathbb{R} \to \mathcal{L}(X)$ generated by linear operators $A_0, A_1 \in \mathcal{L}(X)$ is defined by

$$
S(t; \tau) := \begin{cases} 
\Theta, & -\infty < t < 0, \\
I, & t = 0, \\
\exp(A_0 t) + S_1(t, \tau) + \cdots + S_n(t, n\tau), & n\tau < t \leq (n+1)\tau, \quad n \in \mathbb{N}_0.
\end{cases}
$$

(56)

Depending on the relation between bounded linear operators $A_0$ and $A_1$, one can obtain various representation formulae for the delayed perturbation of an operator-valued exponential function.

The following theorem deals with the fundamental solution of delay evolution equation (10) with non-permutable bounded linear operators.

Theorem 19. Let $A_0, A_1 \in \mathcal{L}(X)$ with non-zero commutator $[A_0, A_1] := A_0 A_1 - A_1 A_0 \neq 0$. Then a fundamental solution $S(\cdot; \tau) : [−\tau, T] \to \mathcal{L}(X)$ of linear homogeneous evolution equation with a constant delay (10) which is satisfying the following initial conditions

$$
S(t; \tau) = \begin{cases} 
\Theta, & -\tau \leq t < 0, \\
I, & t = 0,
\end{cases}
$$

(57)

can be represented by

$$
S(t; \tau) = \sum_{k=0}^{\infty} \sum_{l=0}^{n} Q_{k+1}(l\tau) \frac{(t-l\tau)^k}{k!}, \quad n\tau < t \leq (n+1)\tau, \quad n \in \mathbb{N}_0,
$$

(58)

where a linear operator family $\{Q_{k+1}(l\tau) : k, l \in \mathbb{N}_0\} \subset \mathcal{L}(X)$ is given by

$$
Q_{k+1}(0) := A_0^k, \quad k \in \mathbb{N}_0, \quad Q_{k+1}(l\tau) := \sum_{m=l}^{k} A_0^{k-m} A_1 Q_m((l-1)\tau), \quad k, l \in \mathbb{N}.
$$

(59)

Proof. By making use of the recursive sequence of operator-valued functions (55), we derive the basis case for $n = 0$:

$$
S_0(t, 0) = \exp(A_0 t) = \sum_{k=0}^{\infty} A_0^k \frac{t^k}{k!} = \sum_{k=0}^{\infty} Q_{k+1}(0) \frac{t^k}{k!}, \quad t \geq 0,
$$

$$
Q_{k+1}(0) := A_0^k, \quad k = 0, 1, \ldots
$$
For \( n = 1 \), by making use of the well-known Cauchy product formula for double infinite series and interchanging the order of summation and integration which is permissible in accordance with the uniform convergence of the series \([54]\), we get:

\[
S_1(t, \tau) = \int_{\tau}^{t} \exp (A_0(t - s)) A_1 \exp (A_0(s - \tau)) \, ds
\]

\[
= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} A_0^k A_1 A_0^m \int_{\tau}^{t} \frac{(t-s)^k(s-\tau)^m}{k!m!} \, ds
\]

\[
= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} A_0^k A_1 A_0^m \frac{(t-\tau)^{k+m+1}}{(k+m+1)!}
\]

\[
= \sum_{k=0}^{\infty} \sum_{m=0}^{k-1} A_0^{k-m} A_1 A_0^m \frac{(t-\tau)^k}{k!}
\]

\[
= \sum_{k=0}^{\infty} \sum_{m=1}^{k} A_0^{k-m} A_1 A_0^{m-1} \frac{(t-\tau)^k}{k!}
\]

\[
= \sum_{k=0}^{\infty} \sum_{m=1}^{k} A_0^{k-m} A_1 Q_m(0) \frac{(t-\tau)^k}{k!}
\]

\[
= \sum_{k=1}^{\infty} Q_{k+1}(\tau) \frac{(t-\tau)^k}{k!}, \quad t \geq \tau,
\]

\[
Q_{k+1}(\tau) := \sum_{m=1}^{k} A_0^{k-m} A_1 Q_m(0), \quad k = 1, 2, \ldots
\]

In a recursive way, for \( n = 2 \), by using \( Q_1(\tau) = \Theta \), we assure that

\[
S_2(t, 2\tau) = \int_{2\tau}^{t} \exp (A_0(t - s)) A_1 S_1(s - \tau, \tau) \, ds
\]

\[
= \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} A_0^k A_1 Q_{m+1}(\tau) \int_{2\tau}^{t} \frac{(t-s)^k(s-2\tau)^m}{k!m!} \, ds
\]

\[
= \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} A_0^k A_1 Q_{m+1}(\tau) \frac{(t-2\tau)^{k+m+1}}{(k+m+1)!}
\]
\[
\sum_{k=1}^{\infty} \sum_{m=1}^{k} A_0^{k-m} A_1 \mathcal{Q}_{m+1}(\tau) \frac{(t - 2\tau)^{k+1}}{(k+1)!}
\]

\[
\sum_{k=2}^{\infty} \sum_{m=1}^{k-1} A_0^{k-m-1} A_1 \mathcal{Q}_{m+1}(\tau) \frac{(t - 2\tau)^k}{k!}
\]

\[
\sum_{k=2}^{\infty} \sum_{m=2}^{k} A_0^{k-m} A_1 \mathcal{Q}_m(\tau) \frac{(t - 2\tau)^k}{k!}
\]

\[
\sum_{k=2}^{\infty} \mathcal{Q}_{k+1}(2\tau) \frac{(t - 2\tau)^k}{k!}, \quad t \geq 2\tau,
\]

\[
\mathcal{Q}_{k+1}(2\tau) := \sum_{m=2}^{k} A_0^{k-m} A_1 \mathcal{Q}_m(\tau), \quad k = 2, 3, \ldots
\]

Recursively, for the \(n\)-th case, it yields that

\[
S_n(t, n\tau) = \int_{n\tau}^{t} \exp \left( A_0(t - s) \right) A_1 S_{n-1}(s - \tau, (n - 1)\tau) ds
\]

\[
= \sum_{k=n}^{\infty} \mathcal{Q}_{k+1}(n\tau) \frac{(t - n\tau)^k}{k!}, \quad t \geq n\tau,
\]

\[
\mathcal{Q}_{k+1}(n\tau) := \sum_{m=n}^{k} A_0^{k-m} A_1 \mathcal{Q}_m((n - 1)\tau), \quad k = n, n + 1, \ldots
\]

Therefore, the fundamental solution \(S(\cdot; \tau) : [-\tau, T] \to \mathcal{L}(X)\) of (10) which is satisfying the initial conditions (57) is a summation of the finite number of above terms and we attain the following desired result:

\[
S(t; \tau) = \sum_{k=0}^{\infty} \mathcal{Q}_{k+1}(0) \frac{t^k}{k!} + \sum_{k=1}^{\infty} \mathcal{Q}_{k+1}(\tau) \frac{(t - \tau)^k}{k!} + \ldots + \sum_{k=n}^{\infty} \mathcal{Q}_{k+1}(n\tau) \frac{(t - n\tau)^k}{k!}
\]

\[
= \sum_{k=1}^{n} \sum_{l=0}^{n} \mathcal{Q}_{k+1}(l\tau) \frac{(t - l\tau)^k}{k!}, \quad n\tau < t \leq (n + 1)\tau, \quad n \in \mathbb{N}_0.
\]

The proof is complete. \(\square\)

**Remark 20.** It should be noted that this particular case is a natural extension of the results obtained by Mahmudov in [7] concerning non-commutative matrix coefficients for time-delay systems of fractional order.
A linear operator family \( \{ Q_{k+1}(l) \tau : k, l \in \mathbb{N}_0 \} \subset \mathcal{L}(X) \) plays a role as a kernel for the delayed perturbation of operator-valued exponential functions. With the help of formulae (59), simple calculations show that

| \( Q_{k+1}(l) \tau \) | \( l = 0 \) | \( l = 1 \) | \( l = 2 \) | \( l = 3 \) | ... | \( l = n \) |
|---|---|---|---|---|---|---|
| \( k = 0 \) | \( I \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | ... | \( \Theta \) |
| \( k = 1 \) | \( A_0 \) | \( A_1 \) | \( \Theta \) | \( \Theta \) | ... | \( \Theta \) |
| \( k = 2 \) | \( A_0^2 \) | \( A_0 A_1 + A_1 A_0 \) | \( A_1^2 \) | \( \Theta \) | ... | \( \Theta \) |
| \( k = 3 \) | \( A_0^3 \) | \( A_0 (A_0 A_1 + A_1 A_0) + A_1 A_0^2 \) | \( A_1 A_0^2 + A_1 (A_0 A_1 + A_1 A_0) \) | \( A_1^3 \) | ... | \( \Theta \) |
| ... | ... | ... | ... | ... | ... | ... |
| \( k = n \) | \( A_0^n \) | ... | ... | ... | ... | \( A_1^n \) |
| ... | ... | ... | ... | ... | ... | ... |

From this table, we can derive the following important result:

- If \( l \geq k + 1 \) for \( k \geq 0 \), then \( Q_{k+1}(l) \tau = \Theta \). Therefore, a linear operator family \( \{ Q_{k+1}(l) \tau : k, l \in \mathbb{N}_0 \} \subset \mathcal{L}(X) \) is a lower triangular operator-matrix;

- If \( A_1 = \Theta \), then for \( k \in \mathbb{N}_0 \), we have:

\[
Q_{k+1}(l) \tau = \begin{cases} 
  A_0^k, & l = 0, \\
  \Theta, & l \in \mathbb{N},
\end{cases}
\]

and a fundamental solution becomes an operator-valued exponential function:

\[
S(t; \tau) = \exp (A_0 t), \quad t \geq 0.
\]

- If \( A_0 = \Theta \), then for \( k, l \in \mathbb{N}_0 \), we have:

\[
Q_{k+1}(l) \tau = \begin{cases} 
  A_1^k, & k = l, \\
  \Theta, & k \neq l,
\end{cases}
\]

and a fundamental solution becomes the pure delayed operator-valued exponential function as below:

\[
S(t; \tau) = \exp_x (A_1 t) = Q_1(0) + Q_2(\tau)(t - \tau) + \ldots + Q_{n+1}(n\tau) \frac{(t - n\tau)^n}{n!}
\]

\[
= I + A_1(t - \tau) + \ldots + A_1^n \frac{(t - n\tau)^n}{n!}
\]

\[
= \sum_{l=0}^{n} A_1^l \frac{(t - l\tau)^l}{l!}, \quad n\tau < t \leq (n+1)\tau, \quad n \in \mathbb{N}_0.
\]
Note that a piece-wise construction of the pure delayed operator-valued exponential function \( \exp_{\tau}(A_1 \cdot) : [-\tau,T] \rightarrow \mathcal{L}(X) \) generated by a linear operator \( A_1 \in \mathcal{L}(X) \) can be written via the following explicit formula:

\[
\exp_{\tau}(A_1 t) = \begin{cases} 
\Theta, & -\tau \leq t < 0, \\
I, & t = 0, \\
I + A_1 (t - \tau) + A_1^2 \frac{(t - 2\tau)^2}{2!} + \ldots + A_1^n \frac{(t - n\tau)^n}{n!}, & n\tau < t \leq (n+1)\tau, \quad n \in \mathbb{N}_0.
\end{cases}
\] (60)

A linear operator family \( \{Q_{k+1}(l\tau) : k, l \in \mathbb{N}_0\} \subset \mathcal{L}(X) \) has the following properties depending on the commutativity condition of \( A_0, A_1 \in \mathcal{L}(X) \).

**Theorem 21.** A linear operator family \( \{Q_{k+1}(l\tau) : k, l \in \mathbb{N}_0\} \subset \mathcal{L}(X) \) has the following properties:

(i) For any (permutable and non-permutable) linear operators \( A_0, A_1 \in \mathcal{L}(X) \), we have

\[
Q_{k+1}(l\tau) = A_0 Q_k(l\tau) + A_1 Q_k((l - 1)\tau), 
\] (61)

\[
Q_0(l\tau) = Q_k(-\tau) = \Theta, \quad k, l \in \mathbb{N}_0.
\]

(ii) If \( A_0 A_1 = A_1 A_0 \), then we have

\[
Q_{k+1}(l\tau) = \binom{k}{l} A_0^{k-l} A_1^l, \quad k, l \in \mathbb{N}_0.
\] (62)

**Proof.** (i) By making use of the mathematical induction principle, we can prove (61) is true for all \( l \in \mathbb{N}_0 \) for a fixed \( k \in \mathbb{N}_0 \). It is obvious that the formula (61) holds true for \( l = 0 \). Since \( Q_{k+1}(0) = A_0^k \) and \( Q_k(-\tau) = \Theta \) for any \( k \in \mathbb{N}_0 \), for \( l = 0 \), we obtain:

\[
Q_{k+1}(0) = A_0 Q_k(0) + A_1 Q_k(-\tau).
\]

Suppose that the formula (61) is true for \( l \in \mathbb{N}_0 \). Then by applying the formula (59) for \( l \)-th case, we prove the statement is true for \( (l + 1) \in \mathbb{N}_0 \) as follows:

\[
Q_{k+1}((l + 1)\tau) = \sum_{m=l+1}^{k} A_0^{k-m} A_1 Q_m(l\tau)
= A_0 \sum_{m=l+1}^{k-1} A_0^{k-1-m} A_1 Q_m(l\tau) + A_1 Q_k(l\tau)
= A_0 Q_k((l + 1)\tau) + A_1 Q_k(l\tau).
\]
This implies that the formula (61) is true for arbitrary \( l \in \mathbb{N}_0 \).

To show (ii), we will use proof by induction with regard to \( l \in \mathbb{N}_0 \) for a fixed \( k \in \mathbb{N}_0 \) via the formula (59). Since \( A_0 A_1 = A_1 A_0 \), for \( l = 0 \), we have

\[
Q_k(0) = A_0^k = \binom{k}{0} A_0^k A_1^0.
\]

Assume that the formula (62) is true for \( l = n \in \mathbb{N}_0 \):

\[
Q_{k+1}(n\tau) = \binom{k}{n} A_0^{k-n} A_1^n, \quad k \in \mathbb{N}_0.
\]

Let us prove it for \( l = n + 1 \) as below:

\[
Q_{k+1}((n + 1)\tau) = \sum_{m=n+1}^{k} A_0^{k-m} A_1 A_m(n\tau)
\]

\[
= \sum_{m=n+1}^{k} A_0^{k-m} A_1 \binom{m-1}{n} A_0^{m-1-n} A_1^n
\]

\[
= \sum_{m=n+1}^{k} \binom{m-1}{n} A_0^{k-1-n} A_1^{n+1}
\]

\[
= \binom{k}{n+1} A_0^{k-(n+1)} A_1^{n+1},
\]

where we have used for \( k \geq n + 1 \) the following identity:

\[
\sum_{m=n+1}^{k} \binom{m-1}{n} = \binom{n}{n} + \binom{n+1}{n} + \ldots + \binom{k-1}{n} = \binom{k}{n+1}.
\]

It follows that the formula (62) is true for any \( l \in \mathbb{N}_0 \). The proof is complete. \( \square \)

**Remark 22.** Using the formulae (61) and (62) together, we obtain the following relation for permut able bounded linear operators \( A_0 \) and \( A_1 \):

\[
\binom{k}{l} A_0^{k-l} A_1^l = A_0 \binom{k-1}{l} A_0^{k-l-1} A_1^l + A_1 \binom{k-1}{l-1} A_0^{k-l-1} A_1^{l-1}
\]

\[
= \binom{k-1}{l} A_0^{k-l} A_1 + \binom{k-1}{l-1} A_0^{k-l-1} A_1^l, \quad k, l \in \mathbb{N}_0.
\]
It is easy to see that (63) is a Pascal’s rule of binomial coefficients for bounded linear operators (especially matrices). Analogously, the decomposition formula (61) is a generalisation of Pascal’s rule for non-permutable bounded linear operators (especially, matrices).

Remark 23. It should be noted that a similar operator family \( \{Q_{k,l} : k, l \in \mathbb{N}_0\} \subset \mathcal{L}(X) \) is proposed to investigate the properties of solutions to fractional-order multi-term evolution equations and functional evolution equations in [27, 28], respectively. Furthermore, a linear operator family \( \{Q_{k,l} : k, l \in \mathbb{N}_0\} \subset \mathcal{L}(X) \) is satisfying the following corresponding properties:

(i) For any (permutable and non-permutable) linear operators \( A_0, A_1 \in \mathcal{L}(X) \), we have

\[
Q_{k,l} = A_0 Q_{k-1,l} + A_1 Q_{k,l-1}, \quad k, l \in \mathbb{N}_0.
\]

(ii) If \( A_0 A_1 = A_1 A_0 \), then we have

\[
Q_{k,l} = \binom{k+l}{l} A_0^k A_1^l, \quad k, l \in \mathbb{N}_0.
\]

(iii) For non-permutable linear operators \( A_0, A_1 \in \mathcal{L}(X) \), we have:

\[
\sum_{l=0}^{k} Q_{k-l,l} = \left( A_0 + A_1 \right)^k, \quad k \in \mathbb{N}_0.
\]

(iv) For permutable linear operators \( A_0, A_1 \in \mathcal{L}(X) \), we have:

\[
\sum_{l=0}^{k} \binom{k}{l} A_0^{k-l} A_1^l = \left( A_0 + A_1 \right)^k, \quad k \in \mathbb{N}_0.
\]

Therefore, by Theorem 19, we can use another suggestive explicit formula for the delayed perturbation of an operator-valued exponential function as follows.

Definition 24. The delayed perturbation of an operator-valued exponential function \( S(\cdot; \tau) : \mathbb{R} \to \mathcal{L}(X) \) generated by linear operators \( A_0, A_1 \in \mathcal{L}(X) \) is defined by

\[
S(t; \tau) := \begin{cases} 
\Theta, & -\infty \leq t < 0, \\
I, & t = 0, \\
\sum_{k=0}^{\infty} Q_{k+1}(0) \frac{t^k}{k!} + \sum_{k=1}^{\infty} Q_{k+1}(\tau) \frac{(t-\tau)^k}{k!} \\
+ \ldots + \sum_{k=n}^{\infty} Q_{k+1}(n\tau) \frac{(t-n\tau)^k}{k!}, & n\tau < t \leq (n+1)\tau, \quad n \in \mathbb{N}_0.
\end{cases}
\]
It is obvious that operator coefficients of the first series of (64) are the elements of the first column \((l = 0)\) of the above table, operator coefficients of the second series of (64) are elements of the second column \((l = 1)\) of the table and so on.

Rearranging the terms we can write the delayed perturbation of an operator-valued exponential function as follows:

\[
S(t; \tau) = Q_1(0) + \left( Q_2(0)t + Q_2(\tau)(t-\tau) \right) \\
+ \left( Q_3(0) \frac{t^2}{2!} + Q_3(\tau) \frac{(t-\tau)^2}{2!} + Q_3(2\tau) \frac{(t-2\tau)^2}{2!} \right) \\
+ \ldots + \left( Q_{n+1}(0) \frac{t^n}{n!} + Q_{n+1}(\tau) \frac{(t-\tau)^n}{n!} + \ldots + Q_{n+1}(n\tau) \frac{(t-n\tau)^n}{n!} \right) \\
+ \left( Q_{n+2}(0) \frac{t^{n+1}}{(n+1)!} + Q_{n+2}(\tau) \frac{(t-\tau)^{n+1}}{(n+1)!} + \ldots + Q_{n+2}(n\tau) \frac{(t-n\tau)^{n+1}}{(n+1)!} \right) + \ldots + \\
= \sum_{k=0}^{\infty} \sum_{l=0}^{k} Q_{k+1}(l\tau) \frac{(t-l\tau)^k}{k!}, \quad n\tau < t \leq (n+1)\tau, \quad n \in \mathbb{N}_0. \quad (65)
\]

We can propose an elegant representation formula for the delayed perturbation of an operator-valued exponential function that is equivalent to the formula (65) and this behaviour helps us to understand the delayed construction of operator-valued functions.

To do this, we introduce a shift operator also known as translation operator from operational calculus is an operator \(T_\tau\) for \(\tau \in \mathbb{R}\) that takes a function \(t \mapsto f(t)\) on \(\mathbb{R}\) to its translation \(t \mapsto f(t + \tau)\):

\[
T_\tau f(t) := f(t + \tau).
\]

For the notation of our results, we will use the following Lagrange translation formula for shift operators:

\[
T_\tau f(t) = f(t + \tau) := \exp \left( \tau \frac{d}{dt} \right) f(t).
\]

Using the shift operator we rewrite the explicit formula (65) as below:

\[
S(t; \tau) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} Q_{k+1}(l\tau) \frac{(t-l\tau)^k}{k!} \\
= \sum_{k=0}^{\infty} \sum_{l=0}^{k} Q_{k+1}(l\tau) \exp \left( -l\tau \frac{d}{dt} \right) \frac{(t)^k}{k!}, \quad n\tau < t \leq (n+1)\tau, \quad n \in \mathbb{N}_0. \quad (66)
\]
Based on the following lemma, we introduce a new representation formula for the delayed perturbation of an operator-valued exponential function via a translation operator. This identity is the delayed analogue of the generalisation of binomial theorem for non-permutable linear operators $A_0, A_1 \in \mathcal{L}(X)$.

**Lemma 25.** A linear operator family $\{Q_{k+1}(l\tau) : k, l \in \mathbb{N}_0\} \subset \mathcal{L}(X)$ satisfies the following identity:

$$\sum_{l=0}^{k} Q_{k+1}(l\tau) \exp\left(-l\tau \frac{d}{dt}\right) = \left(A_0 + A_1 \exp\left(-\tau \frac{d}{dt}\right)\right)^k, \ k \in \mathbb{N}_0. \quad (67)$$

**Proof.** To prove the identity (67), we use the mathematical induction principle with respect to $k \in \mathbb{N}_0$. For $k = 0$, we have $Q_1(0) = I$. For $k = 1$, using the first property of $Q_{k+1}(l\tau)$ and $Q_1(\tau) = \Theta$, we proceed as follows:

$$\sum_{l=0}^{1} Q_{2}(l\tau) \exp\left(-l\tau \frac{d}{dt}\right) = \sum_{l=0}^{1} A_0 Q_1(l\tau) \exp\left(-l\tau \frac{d}{dt}\right)$$

$$+ \sum_{l=1}^{1} A_1 Q_1((l-1)\tau) \exp\left(-l\tau \frac{d}{dt}\right)$$

$$= A_0 Q_1(0) + A_1 Q_1(\tau) \exp\left(-\tau \frac{d}{dt}\right) + A_1 Q_1(0) \exp\left(-\tau \frac{d}{dt}\right)$$

$$= A_0 + A_1 \exp\left(-\tau \frac{d}{dt}\right).$$

Then, assuming that the identity (67) is true for the case of $k = n$:

$$\sum_{l=0}^{n} Q_{n+1}(l\tau) \exp\left(-l\tau \frac{d}{dt}\right) = \left(A_0 + A_1 \exp\left(-\tau \frac{d}{dt}\right)\right)^n, \ n \in \mathbb{N}_0. \quad (68)$$

and we prove it for $k = n + 1$ by using (68):

$$\left(A_0 + A_1 \exp\left(-\tau \frac{d}{dt}\right)\right)^{n+1} = \left(A_0 + A_1 \exp\left(-\tau \frac{d}{dt}\right)\right)\left(A_0 + A_1 \exp\left(-\tau \frac{d}{dt}\right)\right)^n$$

$$= \left(A_0 + A_1 \exp\left(-\tau \frac{d}{dt}\right)\right) \sum_{l=0}^{n} Q_{n+1}(l\tau) \exp\left(-l\tau \frac{d}{dt}\right)$$

$$= \sum_{l=0}^{n} A_0 Q_{n+1}(l\tau) \exp\left(-l\tau \frac{d}{dt}\right)$$
\[ + \sum_{l=0}^{n} A_1 Q_{n+1}(l \tau) \exp \left( -(l+1) \tau \frac{d}{dt} \right) \]
\[ = \sum_{l=0}^{n} A_0 Q_{n+1}(l \tau) \exp \left( -l \tau \frac{d}{dt} \right) \]
\[ + \sum_{l=1}^{n+1} A_1 Q_{n+1}((l-1) \tau) \exp \left( -l \tau \frac{d}{dt} \right) \]
\[ = A_0^{n+1} + A_1^{n+1} \exp \left( -(n+1) \tau \frac{d}{dt} \right) \]
\[ + \sum_{l=1}^{n} \left( A_0 Q_{n+1}(l \tau) + A_1 Q_{n+1}((l-1) \tau) \right) \exp \left( -l \tau \frac{d}{dt} \right) \]
\[ = \sum_{l=0}^{n+1} \left( A_0 Q_{n+1}(l \tau) + A_1 Q_{n+1}((l-1) \tau) \right) \exp \left( -l \tau \frac{d}{dt} \right) \]
\[ = \sum_{l=0}^{n+1} Q_{n+2}(l \tau) \exp \left( -l \tau \frac{d}{dt} \right). \]

Therefore, the identity (67) holds true for any \( k \in \mathbb{N}_0 \). The proof is complete. \( \square \)

**Corollary 26.** The delayed analogue of the binomial theorem for permutable linear operators \( A_0, A_1 \in \mathcal{L}(X) \) is defined by

\[ \sum_{l=0}^{k} \binom{k}{l} A_0^{k-l} A_1^l \exp \left( -l \tau \frac{d}{dt} \right) = \left( A_0 + A_1 \exp \left( -\tau \frac{d}{dt} \right) \right)^k, \quad k \in \mathbb{N}_0. \]

**Remark 27.** The multivariate analogue of this identity is proved for the representation formula of a multi-delayed perturbation of Mittag-Leffler type matrix function generated by \( A_0, A_1, \ldots, A_n \in \mathbb{R}^{n \times n} \) which is a fundamental solution of the fractional-order multi-delay system in [29].

Therefore, by using the formulae (66) and (67) together, we get a new explicit formula for the delayed perturbation of an operator-valued exponential function as below.

**Definition 28.** The delayed perturbation of an operator-valued exponential function \( S(\cdot; \tau) : \mathbb{R} \to \mathcal{L}(X) \) generated by linear operators \( A_0, A_1 \in \mathcal{L}(X) \) is defined by

\[ S(t; \tau) = \sum_{k=0}^{\infty} \left( A_0 + A_1 \exp \left( -\tau \frac{d}{dt} \right) \right)^k \frac{t^k}{k!}, \quad t \in \mathbb{R}. \]  

(69)
Remark 29. What is the advantage of this alternative definition? This definition retains a delayed analogue of binomial formula. Using this representation formula (69) we can easily determine the binomial expansion of commutative and non-commutative linear bounded operators corresponding to the domain of piece-wise defined delayed operator-valued functions.

Remark 30. Note that, in the case of \( \tau = 0 \), the formula (69) turns into the following perturbed operator-valued exponential function:

\[
S(t) = \sum_{k=0}^{\infty} \left( A_0 + A_1 \right)^k \frac{t^k}{k!} = \exp (A_0 + A_1) t, \quad t \in \mathbb{R}.
\]

The following theorem refers to a fundamental solution of the delay evolution equation (10) with permutable bounded linear operators. Since \( A_0 A_1 = A_1 A_0 \), the fundamental solution is the product of exponential and delayed exponential operator-valued functions.

Theorem 31. Let \( A_0, A_1 \in \mathcal{L}(X) \) with zero commutator \([A_0, A_1] := A_0 A_1 - A_1 A_0 = 0\). Then, the fundamental solution \( S(\cdot; \tau) : [-\tau,T] \to \mathcal{L}(X) \) of linear homogeneous evolution equation (10) with a discrete delay which is satisfying initial conditions (57) for \( t \in [-\tau,0] \) can be expressed by

\[
S(t; \tau) = \exp (A_0 t) \exp_\tau (A_2 t) = \exp_\tau (A_2 t) \exp (A_0 t),
\]

\[
A_2 := A_1 \exp (-A_0 \tau) \in \mathcal{L}(X), \quad n\tau < t \leq (n+1)\tau, \quad n \in \mathbb{N}_0.
\]

Proof. Since bounded linear operators \( A_0, A_1 \in \mathcal{L}(X) \) are permutable, i.e. \( A_0 A_1 = A_1 A_0 \), we have \( \exp (A_0 (t - s)) A_1 = A_1 \exp (A_0 (t - s)) \). By making use of this relation, we derive the following cases, recursively. Firstly, for \( n = 0 \), we attain that

\[
S_0(t,0) = \exp (A_0 t), \quad t \geq 0.
\]

For the case of \( n = 1 \), we have

\[
S_1(t, \tau) = \int_{\tau}^{t} \exp (A_0 (t - s)) A_1 \exp (A_0 (s - \tau)) \, ds
\]

\[
= \exp (A_0 (t - \tau)) A_1 (t - \tau)
\]

\[
= \exp (A_0 t) A_1 \exp (-A_0 \tau) (t - \tau), \quad t \geq \tau.
\]

Similarly, for \( n = 2 \), it follows that

\[
S_2(t, 2\tau) = \int_{2\tau}^{t} \exp (A_0 (t - s)) A_1 S_1(s, \tau) \, ds
\]

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\[
= \int_{2\tau}^{t} \exp \left( A_0(t-s) \right) A_1 \exp \left( A_0(s-2\tau) \right) A_1(s-2\tau) ds
\]
\[
= \exp \left( A_0(t-2\tau) \right) A_1^2 \int_{2\tau}^{t} (s-2\tau) ds
\]
\[
= \exp \left( A_0 t \right) A_1^2 \exp \left( -2A_0 \tau \right) \frac{(t-2\tau)^2}{2!}, \quad t \geq 2\tau.
\]

In a recursive way, for the general \(n\)-th case, we acquire
\[
S_n(t, n\tau) = \exp \left( A_0 t \right) A_1^n \exp \left( -nA_0 \tau \right) \frac{(t-n\tau)^n}{n!}, \quad t \geq n\tau, \quad n \in \mathbb{N}_0.
\]

Therefore, the fundamental solution \(S(\cdot, \tau) : [-\tau, T] \rightarrow \mathcal{L}(X)\) of (10) with initial conditions (57) is a summation of a finite number above terms and we get the following result:
\[
S(t, \tau) = \sum_{l=0}^{n} S_l(t, l\tau) = \exp \left( A_0 t \right) \sum_{l=0}^{n} A_1^l \exp \left( -lA_0 \tau \right) \frac{(t-l\tau)^l}{l!}
\]
\[
= \sum_{l=0}^{n} A_1^l \exp \left( -lA_0 \tau \right) \frac{(t-l\tau)^l}{l!} \exp \left( A_0 t \right)
\]
\[
= \exp \left( A_0 t \right) \exp \left( A_2 t \right) = \exp \left( A_0 t \right) \exp \left( A_2 t \right),
\]
\(A_2 = A_1 \exp \left( -A_0 \tau \right) \in \mathcal{L}(X), \quad n\tau < t \leq (n+1)\tau, \quad n \in \mathbb{N}_0.
\]

The proof is complete. \(\square\)

**Remark 32.** Alternatively, we can prove the formula (70) directly with the result of Corollary 10. By using the formula (54), we derive the following result:
\[
S(t, \tau) = \sum_{l=0}^{n} \left[ A_1 \left( \exp \left( A_0 \tau \right) \right)^{-1} \right]^{l} \frac{(t-l\tau)^l}{l!} \exp \left( A_0 t \right)
\]
\[
= \sum_{l=0}^{n} \left[ A_1 \exp \left( -A_0 \tau \right) \right]^{l} \frac{(t-l\tau)^l}{l!} \exp \left( A_0 t \right)
\]
\[
= \exp \left( A_0 t \right) \sum_{l=0}^{n} \left[ A_1 \exp \left( -A_0 \tau \right) \right]^{l} \frac{(t-l\tau)^l}{l!}
\]
\[
= \exp \left( A_0 t \right) \exp \left( A_2 t \right) = \exp \left( A_0 t \right) \exp \left( A_2 t \right),
\]
\(A_2 = A_1 \exp \left( -A_0 \tau \right) \in \mathcal{L}(X), \quad n\tau < t \leq (n+1)\tau, \quad n \in \mathbb{N}_0.
\]
Remark 33. In addition, we can also prove the formula (70) by using the formulae (58) and (62) together, as follows:

\[
S(t; \tau) = \sum_{k=0}^{\infty} \sum_{l=0}^{n} Q_{k+1}(l\tau) \frac{(t-l\tau)^k}{k!}
\]

\[
= \sum_{k=0}^{\infty} Q_{k+1}(0) \frac{t^k}{k!} + \sum_{k=1}^{\infty} Q_{k+1}(\tau) \frac{(t-\tau)^k}{k!} + \ldots + \sum_{k=n}^{\infty} Q_{k+1}(n\tau) \frac{(t-n\tau)^k}{k!}
\]

\[
= \sum_{k=0}^{\infty} A_0^k \frac{t^k}{k!} + \sum_{k=1}^{\infty} \left( \frac{1}{k} \right) A_0^{k-1} A_1 \frac{(t-\tau)^k}{k!} + \ldots + \sum_{k=n}^{\infty} \left( \frac{k}{n} \right) A_0^{k-n} A_1^n \frac{(t-n\tau)^k}{k!}
\]

\[
= \sum_{k=0}^{\infty} A_0^k \frac{t^k}{k!} + \sum_{k=0}^{\infty} A_0 \frac{(t-\tau)^k}{k!} A_1 (t-\tau) + \ldots + \sum_{k=0}^{\infty} A_0 \frac{(t-n\tau)^k}{k!} A_1^n (t-n\tau)^n
\]

\[
= \exp (A_0 t) \exp \left( A_0 (t-\tau) \right) \exp \left( A_1 (t-\tau) \right) + \ldots + \exp (A_0 (t-n\tau)) \exp \left( A_1^n (t-n\tau)^n \right)
\]

\[
= \exp (A_0 t) \left[ I + A_1 \exp (-A_0 \tau) (t-\tau) + \ldots + \left( A_1 \exp (-A_0 \tau) \right)^n (t-n\tau)^n \right]
\]

\[
= \exp (A_0 t) \sum_{l=0}^{n} \left[ A_1 \exp (-A_0 \tau) \right]^{l} \frac{(t-l\tau)^l}{l!}
\]

\[
= \sum_{l=0}^{n} \left[ A_1 \exp (-A_0 \tau) \right]^{l} \frac{(t-l\tau)^l}{l!} \exp (A_0 t)
\]

Remark 34. It should be noted that this particular case is a natural extension of the results are attained in [37] in terms of commutative matrix coefficients. Moreover, corresponding results are derived for fractional-order time-delay systems in [6, 30].

Directly by means of the formula (70), we can obtain the following special cases, which are the fundamental solutions of the evolution equation with a pure delay and delay-free systems without using the above table.

Corollary 35. Let \( S(t; \tau), t \in [-\tau, T] \) be the fundamental solution of delay evolution equation (10) with initial conditions (57). Then the following assertions hold true:
(i) If $A_0 = \Theta$, then the delayed perturbation of an operator-valued exponential function turns into the pure delayed operator-valued exponential function and it can be represented by

$$S(t; \tau) = \exp(A_1 t) = \sum_{l=0}^{n} A_1^l \frac{(t - l\tau)^l}{l!}, \quad n\tau < t \leq (n+1)\tau, \quad n \in \mathbb{N}_0. \quad (71)$$

(ii) If $A_1 = \Theta$, then the delayed perturbation of an operator-valued exponential function becomes classical operator-valued exponential function and it can be expressed by

$$S(t; \tau) = \exp(A_0 t) = \sum_{k=0}^{\infty} A_0^k \frac{t^k}{k!}, \quad t \geq 0. \quad (72)$$

**Proof.** The proof of this Corollary follows from directly Theorem 31. So, we pass over it here. □

**Remark 36.** Note that the first part is a natural generalisation of pure delayed exponential matrix function which is studied by Khusainov and Shuklin in [4] for linear differential equation with matrix coefficient and pure delay, and the second part is well-known uniformly continuous semigroup in a Banach space $X$ [19].

**Uniform continuity property:** Note that in Chapter 3, we have proved that if $A_0$ is an infinitesimal generator of the $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ and $A_1 \in \mathcal{L}(X)$, then a one-parameter family of bounded linear operators $S(t; \tau)$ is continuous with respect to strong operator topology for each $t \in [0, \infty)$. Analogously, one can show that $S(t; \tau)$ generated by $A_0, A_1 \in \mathcal{L}(X)$ is uniformly continuous with respect to $t$ on $[0, \infty)$.

$$\|S(t; \tau) - I\| = \left\| \sum_{k=1}^{\infty} \sum_{l=0}^{k} Q_{k+1}(l\tau) \frac{(t - l\tau)^{k}}{k!} \right\|$$

$$\leq \sum_{k=1}^{\infty} \sum_{l=0}^{k} \|Q_{k+1}(l\tau)\| \frac{(t - l\tau)^{k}}{k!}$$

$$\leq \sum_{k=1}^{\infty} \left( \sum_{l=0}^{k} \binom{k}{l} \|A_0\|^{k-l} \|A_1\|^l \right) \frac{t^k}{k!}$$

$$= \sum_{k=1}^{\infty} \left( \|A_0\| + \|A_1\| \right)^k \frac{t^k}{k!}$$

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\[
= \exp \left( \left( \|A_0\| + \|A_1\| \right) t \right) - I, \quad t \geq 0,
\]
where we have used that
\[
\|Q_{k+1}(l\tau)\| \leq \binom{k}{l} \|A_0\|^{k-l} \|A_1\|^l,
\]
and hence, it tends to zero as \( t \to 0_+ \). Therefore, \( \lim_{t \to 0_+} \|S(t; \tau) - I\| = 0 \) and the one-parameter operator family \( \{S(t; \tau)\}_{t \geq 0} \) is uniformly continuous with respect to operator norm \( \|\cdot\| \) associated with \( X \).

For delay-free time evolution linear autonomous systems, uniformly continuous one-parameter family of bounded linear operators \( \{S(t)\}_{t \geq 0} \) equals to the perturbed operator-valued exponential function \( \{\exp((A_0 + A_1)t)\}_{t \geq 0} \) which is satisfying the semigroup property. To show this, we use Cauchy product formula for double series as follows:

\[
S(t)S(s) = \sum_{k=0}^{\infty} \left( A_0 + A_1 \right)^k \frac{t^k}{k!} \sum_{n=0}^{\infty} \left( A_0 + A_1 \right)^n \frac{s^n}{n!}
\]
\[
= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} \left( A_0 + A_1 \right)^{k-n} \left( A_0 + A_1 \right)^n \frac{t^k}{(k-n)!} \frac{s^n}{n!}
\]
\[
= \sum_{k=0}^{\infty} \left( \sum_{n=0}^{k} \binom{k}{n} t^k \frac{s^n}{n!} \right) \frac{\left( A_0 + A_1 \right)^k}{k!}
\]
\[
= \sum_{k=0}^{\infty} \left( A_0 + A_1 \right)^k \frac{(t+s)^k}{k!} = S(t+s), \quad t, s \geq 0.
\]

However, in our case, the same property is not working for the operator family of bounded linear operators \( \{S(t; \tau)\}_{t \geq 0} \) which is continuous with respect to uniform operator topology. To show this, we will use the following counterexample:

**Counterexample.** Let \( X = \mathbb{R}^2 \), \( T = 2 \), \( \tau = 1 \), \( A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \). It is obvious that \( A_0 A_1 = A_1 A_0 \). Then, by elementary calculations, we obtain \( \exp(A_0 t) = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} \) for any \( t \in [0, 2] \) and \( A_2 = A_1 \exp(-A_0) = \begin{pmatrix} -1/e & 0 \\ 0 & -1/e \end{pmatrix} \). In addition, by (60), the pure delayed exponential matrix function
exp_1 \left( A_2 t \right) : [-1, 2] \to \mathbb{R}^{n \times n} is defined by

\begin{align*}
\exp_1 \left( A_2 t \right) &= \begin{cases}
\Theta, & -1 \leq t < 0, \\
I, & 0 \leq t \leq 1, \\
I + A_2 (t - 1), & 1 < t \leq 2.
\end{cases}
\end{align*} (74)

By making use of the formula (74), for a left-hand side, we derive that

\begin{align*}
S(t; \tau) S(s; \tau) &= \exp \left( A_0 t \right) \exp_1 \left( A_2 t \right) \exp \left( A_0 s \right) \exp_1 \left( A_2 s \right) \\
&= \exp \left( A_0 (t + s) \right) \exp_1 \left( A_2 (t + s) \right) \\
&= \exp \left( A_0 (t + s) \right) \left[ I + A_2 (t - 1) \right] \left[ I + A_2 (s - 1) \right] \\
&= \exp \left( A_0 (t + s) \right) \left[ I + A_2 (t + s - 2) + A_2^2 (ts - t - s - 1) \right] \\
&= \begin{pmatrix} e^{t+s} & 0 \\
0 & e^{t+s} \end{pmatrix} \begin{pmatrix} -ts + (t + s)(1 - e) + 2e + 2 & 0 \\
0 & e^{t+s} \end{pmatrix} \begin{pmatrix} 0 \\
-(-ts + (t + s)(1 - e) + 2e + 2) \end{pmatrix}, \quad \forall t, s \in [0, 2].
\end{align*}

Similarly, for a right-hand side, we have:

\begin{align*}
S(t + s; \tau) &= \exp \left( A_0 (t + s) \right) \exp_1 \left( A_2 (t + s) \right) \\
&= \exp \left( A_0 (t + s) \right) \left[ I + A_2 (t + s - 1) \right] \\
&= \begin{pmatrix} e^{t+s} & 0 \\
0 & e^{t+s} \end{pmatrix} \begin{pmatrix} - (t + s) + e + 1 & 0 \\
0 & e^{t+s} - 1 \end{pmatrix} \begin{pmatrix} 0 \\
- (t + s) + e + 1 \end{pmatrix}, \quad \forall t, s \in [0, 2].
\end{align*}

It follows that, \( S(t; \tau) S(s; \tau) \neq S(t + s; \tau) \) for any \( t, s \in [0, 2] \).

Furthermore, we can say that the similar scenario holds true for the strongly continuous family of bounded linear operators \( \{S(t; \tau)\}_{t \geq 0} \) (see Section 3), which does not satisfy the semigroup property, i.e., \( S(t; \tau) S(s; \tau) \neq S(t + s; \tau) \) for any \( t, s \geq 0 \).
5 Application: a delayed heat equation

Let us take \( X = L^2([0, \pi], \mathbb{R}) \). We consider the following initial-boundary value problem with homogeneous Dirichlet boundary conditions for a one-dimensional heat equation with a constant delay:

\[
\begin{cases}
\frac{\partial}{\partial t} u(x, t) = a^2 \frac{\partial^2}{\partial x^2} u(x, t) + bu(x, t - \tau) + \psi(x, t), & x \in [0, \pi], \ t \geq 0, \\
u(x, t) = \varphi(x, t), & x \in [0, \pi], \ t \in [-\tau, 0], \\
u(0, t) = u(\pi, t) = 0, & t \geq -\tau,
\end{cases}
\]  

(75)

where \( a, b \in \mathbb{R} \) and \( \tau > 0 \).

We define the following unbounded linear operator \( A_0 : \mathcal{D}(A_0) \subseteq X \rightarrow X \) as follows:

\[
A_0 u = a^2 \frac{\partial^2}{\partial x^2} u, \quad a \in \mathbb{R}, \quad u \in \mathcal{D}(A_0),
\]

with the domain is given by

\[
\mathcal{D}(A_0) = \left\{ u \in X : u, \frac{\partial}{\partial x} u \text{ are absolutely continuous}, \frac{\partial^2}{\partial x^2} u \in X, \ u(0) = u(\pi) = 0 \right\}.
\]

Moreover, a linear bounded operator \( A_1 : X \rightarrow X \) is defined by \( A_1 u = bu \) for all \( b \in \mathbb{R} \) and \( x \in X \).

It is known that \( A_0 \) has a discrete spectrum with eigenvalues of the form \( \lambda_n = -a^2 n^2 \), \( n \in \mathbb{N} \) and the corresponding normalized eigenvectors are given by \( u_n(x) := \sqrt{\frac{2}{\pi}} \sin(nx), x \in [0, \pi] \) for any \( n \in \mathbb{N} \). Moreover, \( \{u_n : n \in \mathbb{N}\} \) is an orthonormal basis for \( X \) and thus, \( A_0 \) and \( A_1 \) has the following spectral representations:

\[
A_0 u = \sum_{n=1}^{\infty} -a^2 n^2 \langle u, u_n \rangle u_n, \quad a \in \mathbb{R}, \quad u \in \mathcal{D}(A_0),
\]

\[
A_1 u = \sum_{n=1}^{\infty} b \langle u, u_n \rangle u_n, \quad b \in \mathbb{R}, \quad u \in X.
\]

Furthermore, \( A_0 \) is a closed, densely-defined linear operator and the infinitesimal generator of \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) on \( X \) which is represented by

\[
T(t)u = \sum_{n=1}^{\infty} \exp(-a^2 n^2 t) \langle u, u_n \rangle u_n, \quad a \in \mathbb{R}, \quad u \in \mathcal{D}(A_0), \quad t \geq 0.
\]
Therefore, the initial-boundary value problem for a one-dimensional heat equation (75) with a constant delay \( \tau > 0 \) can be formulated in abstract sense as follows:

\[
\begin{aligned}
\begin{cases}
\frac{\partial}{\partial t} u(t) = A_0 u(t) + A_1 u(t - \tau) + \psi(t), & t \geq 0, \\
u(t) = \varphi(t), & -\tau \leq t \leq 0,
\end{cases}
\end{aligned}
\]  

(76)

where \( u(t) = u(\cdot, t), \varphi(t) = \varphi(\cdot, t) \) and \( \psi(t) = \psi(\cdot, t) \).

Since a strongly continuous semigroup \( \{T(t)\}_{t \geq 0} \) generated by \( A_0 = \frac{\partial^2}{\partial x^2} \) commutes with \( A_1 \in L(X) \), by making use the formula (38), for any \( u \in X \), a fundamental solution \( S(\cdot; \tau) : [-\tau, \infty) \rightarrow L(X) \) of (76) which is satisfying initial conditions \( S(t; \tau) = \Theta, -\tau \leq t < 0 \) and \( S(0; \tau) = I \) can be represented as follows:

\[
S(t; \tau)u = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} b_k^k T(t - k\tau) \frac{(t - k\tau)^k}{k!} \langle u, u_n \rangle u_n
\]

\[
= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} b_k^k \exp \left( - a^2 n^2 (t - k\tau) \right) \frac{(t - k\tau)^k}{k!} \langle u, u_n \rangle u_n
\]

\[
= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \exp \left( - a^2 n^2 t \right) \left[ b \exp(a^2 n^2 \tau) \right] \frac{k (t - k\tau)^k}{k!} \langle u, u_n \rangle u_n
\]

\[
= \sum_{n=1}^{\infty} \exp \left( - a^2 n^2 t \right) \exp_{\tau} \left( b_n t \right) \langle u, u_n \rangle u_n, & t \geq 0,
\]

(77)

where \( \exp_{\tau}(\cdot) : [0, \infty) \rightarrow \mathbb{R} \) is a pure delayed operator-valued exponential function defined by

\[
\exp_{\tau} \left( b_n t \right) = \sum_{k=0}^{\infty} b_n^k \frac{(t - k\tau)^k}{k!}, \quad b_n = b \exp \left( a^2 n^2 \tau \right), \quad n \in \mathbb{N}, \quad t \geq 0.
\]

Therefore, by using the formulas (42) and (77), the classical or strong solution \( u(\cdot) \in C([-\tau, \infty), X) \cap C^1([0, \infty), X) \) of abstract Cauchy problem (76) which is satisfying \( u(t) \in D(A_0) \) for all \( t \geq 0 \) with \( \varphi(0) \in D(A_0) \) can be expressed with the help of a special case of delayed Dyson-Phillips series as below:

\[
u(t) = \sum_{n=1}^{\infty} \left\{ \exp \left( -a^2 n^2 t \right) \exp_{\tau} \left( b_n t \right) \langle u, u_n \rangle \right\} + b \int_{-\tau}^{0} \exp \left( -a^2 n^2 (t - \tau - s) \right) \exp_{\tau} \left( b_n (t - \tau - s) \right) \langle u, u_n \rangle ds
\]

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\[ + \int_0^t \exp \left( -a^2 n^2 (t-s) \right) \exp_r (b_n(t-s)) < \psi(s), u_n > ds \right\} u_n, \quad t \geq 0. \quad (78) \]

Similarly, with the help of formulae (33) and (77), the classical (strong) solution \( u(\cdot) \in C^1([-\tau, \infty), X) \) of abstract Cauchy problem (76) which is satisfying \( u(t) \in D(A_0) \) for all \( t \geq 0 \) with \( \varphi(t) \in D(A_0) \) for any \( t \in [-\tau, 0] \) can also be expressed by

\[
 u(t) = \sum_{n=1}^{\infty} \left\{ \exp \left( -a^2 n^2 (t+\tau) \right) \exp_r (b_n(t+\tau)) < \varphi(-\tau), u_n > 
  + \int_{-\tau}^0 \exp \left( -a^2 n^2 (t-s) \right) \exp_r (b_n(t-s)) < \varphi'(s) + a^2 n^2 \varphi(s), u_n > ds 
  + \int_0^t \exp \left( -a^2 n^2 (t-s) \right) \exp_r (b_n(t-s)) < \psi(s), u_n > ds \right\} u_n, \quad t \geq -\tau. \quad (79) \]

**Remark 37.** Currently, in [22], Pinto et al. have studied an approximation of a mild solution \( u(\cdot) \in X = L^2 ([0, \pi], \mathbb{R}) \) of first-order linear homogeneous abstract differential problem (76) with a constant delay (where \( \psi(t) = 0, a = 1, \tau = 1, A_1 u = Lu \) with \( L \geq 0 \)), which depends on an initial history condition \( \varphi(t) \in X \) for \( -1 \leq t \leq 0 \) and unbounded closed linear operator \( A_0 = \frac{\partial^2}{\partial x^2} \) generating a \( C_0 \)-semigroup on a Banach space \( X \). The authors have introduced in [22], the mild solution \( u(t) \) of the abstract Cauchy problem (76) associated with \( \varphi(t), -1 \leq t \leq 0 \), for any \( t \in [0, 1] \) is given by

\[
 u(t) = \sum_{n=1}^{\infty} \exp \left( -n^2 t \right) < \varphi(0), u_n > u_n
 + L \sum_{n=1}^{\infty} \frac{\exp \left( -n^2 t \right) + n^2 t - 1}{n^4} < \varphi(0), u_n > u_n, \quad t \geq 0. \]

In contrast to [22], we can easily deduce a mild solution \( u(\cdot) \in C([-\tau, \infty), X) \cap C^1 ([0, \infty), X) \) of (76) by taking \( a = \tau = 1 \) in (78) as follows:

\[
 u(t) = \sum_{n=1}^{\infty} \left\{ \exp \left( -n^2 t \right) \exp_1 (b_n t) < \varphi(0), u_n > 
  + L \int_{-1}^0 \exp \left( -n^2 (t-1-s) \right) \exp_1 (b_n (t-1-s)) < \varphi(s), u_n > ds 
  + \int_0^t \exp \left( -n^2 (t-s) \right) \exp_1 (b_n (t-s)) < \psi(s), u_n > ds \right\} u_n, \quad t \geq 0.
\]
Furthermore, an exact explicit formula of a mild solution \( u(\cdot) \in C^1([-\tau, \infty), X) \) of (76) can be represented by

\[
\begin{align*}
    u(t) &= \sum_{n=1}^{\infty} \left\{ \exp\left(-n^2(t + 1)\right) \exp_1(b_n(t + 1)) < \varphi(-1), u_n > \\
    &+ \int_{-1}^{0} \exp\left(-n^2(t - s)\right) \exp_1(b_n(t - s)) < \varphi'(s) + n^2\varphi(s), u_n > ds \\
    &+ \int_{0}^{t} \exp\left(-n^2(t - s)\right) \exp_\tau(b_n(t - s)) < \psi(s), u_n > ds \right\} u_n, \\
    &\quad t \geq -1.
\end{align*}
\]

where

\[
\exp_1(b_n t) = \sum_{k=0}^{\infty} b_n^k \frac{(t-k)^k}{k!}, \quad b_n = L \exp\left(n^2\right), \quad n \in \mathbb{N}, \quad t \geq 0.
\]

**Remark 38.** Furthermore, one can use Fourier method for the construction of the first boundary value problem (75). Note that the solution of one dimensional heat equation (73) with a discrete delay \( \tau > 0 \) can be expressed with Fourier coefficients which is studied in [8]-[10] for \( x \in [0, \pi] \):

\[
\begin{align*}
    u(x,t) &= \sum_{n=1}^{\infty} \left\{ \exp\left(-a^2n^2(t + \tau)\right) \exp_\tau(b_n(t + \tau)) \Phi_n(-\tau) \\
    &+ \int_{-\tau}^{0} \exp\left(-a^2n^2(t - s)\right) \exp_\tau(b_n(t - s)) \left[ \Phi_n'(s) + a^2n^2\Phi_n(s) \right] ds \\
    &+ \int_{0}^{t} \exp\left(-a^2n^2(t - s)\right) \exp_\tau(b_n(t - s)) \Psi_n(s) ds \right\} \sin(nx), \\
    &\quad t \geq -\tau,
\end{align*}
\]

or

\[
\begin{align*}
    u(x,t) &= \sum_{n=1}^{\infty} \left\{ \exp\left(-a^2n^2t\right) \exp_\tau(b_n t) \Phi_n(0) \\
    &+ b \int_{-\tau}^{0} \exp\left(-a^2n^2(t - \tau - s)\right) \exp_\tau(b_n(t - \tau - s)) \Phi_n(s) ds \\
    &+ \int_{0}^{t} \exp\left(-a^2n^2(t - s)\right) \exp_\tau(b_n(t - s)) \Psi_n(s) ds \right\} \sin(nx), \\
    &\quad t \geq 0,
\end{align*}
\]
where \( b_n = b \exp (a^2 n^2 \tau) \), \( n \in \mathbb{N} \); \( \Phi_n(\cdot) : [-\tau, 0] \to \mathbb{R} \) and \( \Psi_n(\cdot) : [0, \infty) \to \mathbb{R} \) for \( n \in \mathbb{N} \) are Fourier coefficients of \( \varphi(x, t) \) and \( \psi(x, t) \), respectively, that is

\[
\Phi_n(t) = \frac{2}{\pi} \int_0^\pi \varphi(\xi, t) \sin(n\xi) d\xi, \quad t \in [-\tau, 0],
\]

\[
\Psi_n(t) = \frac{2}{\pi} \int_0^\pi \psi(\xi, t) \sin(n\xi) d\xi, \quad t \in [0, \infty).
\]

References

[1] V. Kolmanovskii, A. Myshkis, Applied theory of functional differential equations, Dordrecht, The Netherlands: Kluwer, 1992.

[2] J. K. Hale, Theory of functional differential equations, Applied Mathematical Sciences Series, Vol. 3, 1977.

[3] R. Bellman, K. L. Cooke, Differential-Difference Equations, Academic, New York, 1963.

[4] D. Ya. Khusainov, G. V. Shuklin, On relative controllability in systems with pure delay, Prikladnaya Mekhanika, Int. Appl. Mech., 41(2) (2005) 210–221.

[5] D. Ya. Khusainov, G. V. Shuklin, Linear autonomous time-delay system with permutation matrices solving, Stud. Univ. Zilina, 17(1) (2003) 101–108.

[6] I. T. Huseynov, N. I. Mahmudov, Delayed analogue of three-parameter Mittag–Leffler functions and their applications to Caputo type fractional time delay differential equations, Math. Methods Appl. Sci., (2021), https://doi.org/10.1002/mma.6761.

[7] N. I. Mahmudov, Delayed perturbation of Mittag-Leffler functions their applications to fractional linear delay differential equations, Math. Methods Appl. Sci., (2018) 1–9, https://doi.org/10.1002/mma.5446.

[8] D. Ya. Khusainov, A. F. Ivanov, I. V. Kovarzh, Solution of one heat equation with delay, Nonlinear Oscil., 12(2) (2009) 1–20.

[9] D. Ya. Khusainov, M. Pokojovy, R. Racke, Strong and mild extrapolated \( L^2 \) -solutions to the heat equation with constant delay, SIAM J. Math. Anal., 47(1) (2015) 427–454.
[10] D. Ya. Khusainov, M. Pokojovy, Solving the Linear 1D Thermoelasticity Equations with Pure Delay, Int. J. Math. Math. Sci., 2015, (2015) 1–11.

[11] A. M. Samoilenko, L. M. Serheeva, Construction of global solutions of partial differential equations with deviating arguments in the time variable, J. Math. Sci., 212(4) (2016) 426-441.

[12] A. Okubo, S. A. Levin, Diffusion and ecological problems: Modern Perspectives, Springer Verlag, New York, Berlin, Heidelberg, 2001.

[13] R. S. Phillips, Perturbation theory for semigroups of linear operators, Trans. Am. Math. Soc., 74 (1954) 199-221.

[14] A. Ahmadova, N. I. Mahmudov, J. J. Nieto, Exponential stability and stabilization of fractional stochastic degenerate evolution equations in a Hilbert space: Subordination principle, Evol. Equ. Control Theory, https://doi.org/10.3934/eect.2022008.

[15] I. T. Huseynov, A. Ahmadova, N. I. Mahmudov, Perturbation properties of fractional strongly continuous cosine and sine family operators, ERA, 30(8) (2022) 2911-2940.

[16] C. C. Travis, G. F. Webb, Perturbation of strongly continuous cosine family generators, Colloquium Mathematicae, 45(2) (1981) 277-285.

[17] D. Lutz, On bounded time-dependent perturbations of operator cosine functions, Aequationes Mathematicae, 23 (1981) 197-203.

[18] E. Bazhlekov, Perturbation properties for abstract evolution equations of fractional order, Fract. Cal. Appl. Anal., 2(4) (1999) 359-366.

[19] K.-J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Vol. 194, Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.

[20] A. Bátkai, S. Piazzera, Semigroups and linear partial differential equations with delay, J. Math. Anal. Appl., 264 (2001) 1–20.

[21] A. Bátkai, S. Piazzera, Semigroups for delay equations, Research Notes in Mathematics, 10 A.K. Peters: Wellesley MA, 2005.

[22] M. Pinto, F. Poblete, D. Sepúlveda, Approximation of mild solutions of delay differential equations on Banach spaces, J. Differ. Equ., 303(5) (2021) 156-182.
[23] W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander, Vector-Valued Laplace Transforms and Cauchy Problems, Second Edition, Vol. 96, Monographs in Mathematics, Birkhäuser, Basel, 2010.

[24] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, Berlin, 1983.

[25] R. F. Curtain, H. J. Zwart, An Introduction to Infinite Dimensional Linear Systems Theory, Berlin: Springer-Verlag, 1995.

[26] N. I. Mahmudov, A. M. Almatarneh, Stability of Ulam–Hyers and existence of solutions for impulsive time-delay semi-linear systems with non-permutable matrices, Mathematics, 8 (2020) 1-17.

[27] N. I. Mahmudov, A. Ahmadova, I. T. Huseynov, A new technique for solving Sobolev type fractional multi-order evolution equations, Comp. Appl. Math., 41(71) (2022) 1-35.

[28] I. T. Huseynov, A. Ahmadova, N. I. Mahmudov, On a study of Sobolev type fractional functional evolution equations, Math. Methods Appl. Sci., 45(9) (2022) 5002-5042.

[29] N. I. Mahmudov, Multi-delayed perturbation of Mittag-Leffler type matrix functions, J. Math. Appl. Anal., 505(1) (2021) 125589.

[30] A. Ahmadova, I. T. Huseynov, N. I. Mahmudov, Controllability of fractional stochastic delay dynamical systems, Proceed. Inst. Math. Mech. ANAS, 46(2) (2020) 294–320.