Self-Similar Collapse of Conformally Coupled Scalar Fields

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Abstract

A massless scalar field minimally coupled to the gravitational field in a simplified spherical symmetry is discussed. It is shown that, in this case, the solution found by Roberts [1], describing a scalar field collapse, is in fact the most general one. Taking that solution as departure point, a study of the gravitational collapse for the self-similar conformal case is presented.
Introduction

Gravitational collapse has been exhaustively studied in the last thirty years. Basically, the complexity of the field equations is responsible for the lack of exact solutions which could provide us with some insight to understand the collapse phenomena. For this reason, one of the most studied simplified models is that of a massless scalar field coupled to gravity in a spherically symmetric context.

In several papers [2], Christodoulou studied in detail the solutions of the spherically symmetric Einstein-scalar field equations from an analytical point of view. On the other hand, an intriguing result came from a numerical study.

More precisely, Choptuik [5] exhibited the occurrence of critical phenomena in the formation of black holes by numerically integrating the corresponding set of partial differential equations for several initial-space-distributions of the scalar field, parameterized say by $p$. He showed that there is a generally critical value, $p^*$, that separates solutions containing black holes from those which do not, and that the mass of the black hole formed near this critical point satisfies a power law $M_{BH} \propto |p - p^*|^\gamma$, where $\gamma \sim 0.37$ seems to be a universal exponent. Remarkably, the same kind of critical behavior was found for the collapse of gravitational waves in axisymmetric spacetimes[6], and all this may be seen as an indication that the observed critical phenomena are independent of the collapsing matter as well as of the symmetries considered.

Currently, it is a fascinating task to look for analytical solutions that exhibit exactly the above-mentioned critical behavior. Husain et al[4] obtained a result where there is, for some cases, formation of an apparent horizon, but the black hole mass evaluated on it grows without bound. The same situation was found by Roberts[7] (self-similar case), and in some other models related to the self-similar one [8].

In this paper we analyze both the solution of the field equations in the case treated by Roberts and the collapse phenomena for conformally coupled massless scalar fields.

Besides several attractive features of models with conformally coupled fields[2], Choptuik states that these models have a critical behavior such as that described above[5], and our aim is to study these phenomena, but from an analytical point of view. Our analysis is based on the technique of generation of solutions for conformally coupled scalar fields, starting from those associated to the ordinary scalar fields, first developed by Bekenstein[10].

The paper is organized as follows: in section 1 we present the basic equations for the Einstein-scalar field system and the general solution for the case of a simplified spherically symmetric background. It is shown that among two possible solutions, that leading to the scalar field collapse is the same as the one already found by Roberts. Section 2 is devoted to the mapping of Roberts’ solution into a solution for the self-similar collapse of a conformally coupled scalar field, followed by an analysis of the critical phenomena related to the formation of black holes. Finally, the Conclusions contain a brief discussion about this work and some of its possible extensions.

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1Indeed, these phenomena occur for general non-minimally coupled scalar fields. The exponent $\gamma$ depends weakly on the coupling constant $\xi$ of the $\frac{1}{2} \xi \psi^2 R$ term. The conformal case is characterized by $\xi = 1/6$. 

2
1 Field Equations

Let us consider the line element for spherically symmetric spacetimes written as:

\[ ds^2 = h(u, v) \, du \, dv - r^2(u, v) \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \]  

where \( u \) and \( v \) are null coordinates. The field equations in which the ordinary massless scalar field is taken as the source of curvature are:

\[ R_{\mu\nu} = -\phi_{,\mu} \phi_{,\nu} \]  

where \( \phi \) is the scalar field. The Einstein-scalar field equations are:

\[ \frac{2}{r} r_{,uu} - \frac{2}{r^2} h_{,u} h_{,u} = - (\phi_{,u})^2 \]  
\[ \frac{2}{r} r_{,vv} - \frac{2}{r^2} h_{,v} h_{,v} = - (\phi_{,v})^2 \]  
\[ \frac{h_{,uv}}{h} - \frac{h_{,u} h_{,v}}{h^2} + \frac{2}{r^2} r_{,uv} = - \phi_{,u} \phi_{,v} \]  
\[ (r^2)_{,uv} = - \frac{h}{2} \]  

We shall restrict our study to the case in which \( h = 1 \). It is not difficult to see that a general solution for eq. (3) can be written as:

\[ r^2(u, v) = - \frac{u v}{2} + g_1(u) + g_2(v) \]  

where \( g_1(u) \) and \( g_2(v) \) are arbitrary functions. Now, it is possible to explicitly determine \( g_1(u) \) and \( g_2(v) \) by building an equation involving only \( r \) from Eqs. (3,4,5),

\[ r_{,uv}^2 - r_{,uu} r_{,vv} = 0, \]  

introducing Eq. (7) into Eq. (8) and expanding \( g_1(u) \) and \( g_2(v) \) in power series. We find, by induction, that the latter equation restricts the form of \( g_1 \) and \( g_2 \) to second order polynomials,

\[ g_1(u) = a_0 + a_1 u + a_2 u^2, \quad g_2(v) = b_0 + b_1 v + b_2 v^2 \]  

where the \( a \) and \( b \) coefficients must satisfy:

\[ 4 b_2 a_1^2 - 16 b_2 b_0 a_2 - 16 b_2 a_0 a_2 + 4 b_1^2 a_2 + 2 b_1 a_1 + b_0 + a_0 = 0 \]  

The relation above has two different solutions for \( \{a_0, a_1, a_2, b_1, b_2\} \). The first one is given by

\[ a_0 = 2 \frac{2 a_2 b_1^2 + 2 a_1^2 b_2 + a_1 b_1}{16 a_2 b_2 - 1} \]  

in which case we are assuming \( 1 - 16 a_2 b_2 \neq 0 \). Note that, although Eq. (11) apparently leads to four independent constants, \( \{a_1, a_2, b_1, b_2\} \), since \( 1 - 16 a_2 b_2 \neq 0 \) it is always possible to\(^2\) we take \( b_0 = 0 \) without loss of generality.
remove the linear (in \( u \) and \( v \)) and independent terms of Eqs. (7,9) by introducing \( u \to u - u_0 \) and \( v \to v - v_0 \), with suitable constant values for \( u_0 \) and \( v_0 \). After making that shift, we arrive at a solution for \( r^2(u,v) \) with only two independent constants, which coincides with the one already found by Roberts\( \textsuperscript{1} \) (self-similar case).

Another solution to Eq. (10) is given by:

\[

b_2 = \frac{1}{16 a_2}, \quad b_1 = -\frac{a_1}{4 a_2}
\]  

that is,

\[

1 - 16 a_2 b_2 = 0
\]  

and there are only three independent constants: \( \{a_0, a_1, a_2\} \)\( \textsuperscript{1} \).

This solution has a killing vector \( k_\mu = (1/4 b_2, 1, 0, 0) \) besides those associated to the spherical symmetry. In turn, this killing vector can be timelike or spacelike, depending on the sign of \( b_2 \), leading to a static or time-dependent (cosmological model) solution that cannot describe the scalar field collapse.

Let us now solve the field equations Eqs. (3,4,5) for the scalar field \( \phi \). Taking Eq. (3), for instance, along with the solution found for \( r(u,v) \) in which \( 1 - 16 a_2 b_2 \neq 0 \), we arrive at:

\[

\phi(u,v) = \pm \frac{1}{\sqrt{2}} ln \left| \frac{4 a_2 u + 2 a_1 - (1 + \sqrt{1 - 16 a_2 b_2}) v + 2 \frac{a_1 + 4 a_2 b_1}{\sqrt{1 - 16 a_2 b_2}}}{4 a_2 u + 2 a_1 - (1 - \sqrt{1 - 16 a_2 b_2}) v + 2 \frac{a_1 + 4 a_2 b_1}{\sqrt{1 - 16 a_2 b_2}}} \right|, \quad (14)
\]

For the case in which \( 1 - 16 a_2 b_2 = 0 \), we have:

\[

\phi(u,v) = \pm \frac{1}{\sqrt{2}} ln \left| \frac{u - 2 b_1 - 4 b_2 v - 2 \sqrt{b_1^2 - 4 a_0 b_2}}{u - 2 b_1 - 4 b_2 v + 2 \sqrt{b_1^2 - 4 a_0 b_2}} \right|, \quad (15)
\]

The first case has already been analyzed\( \textsuperscript{1} \) in order to reproduce analytically the critical behavior obtained by Choptuik\( \textsuperscript{1} \). By adjusting a characteristic parameter of the solution, say \( \alpha \)\( \textsuperscript{1} \) (\( b_2 \) in our notation), three classes are obtained: subcritical \((0 < \alpha < 1/4)\), where the scalar field collapses and disperses, leaving behind a flat spacetime; critical \((\alpha = 0)\), where the final result is an asymptotically flat spacetime with a null singularity; and supercritical \((\alpha < 0)\), corresponding to the formation of black holes. Among these classes, the last is the most interesting, despite the undesirable fact that the mass of the black hole grows without bound as \( v \to \infty \).

\section{2 Self-Similar Solutions with Conformal Scalar Field}

In this section, we apply the technique of generating solutions for conformally coupled scalar fields first developed by Bekenstein\( \textsuperscript{1} \), departing from Roberts’ solution. The actions for a massless conformally coupled scalar field \( \psi \) and for the ordinary one, \( \phi \), are given respectively by

\footnote{Note that, in this case, since \( 1 - 16 a_2 b_2 = 0 \), it is not possible to recover Roberts’ solution by a constant shift on the \( u \) and \( v \) variables of Eqs. (11,12).}
\[ S_\psi = \int d^4 x \sqrt{-g} \left( \frac{R}{2} - \frac{1}{2} g^{\alpha \beta} \partial_\alpha \psi \partial_\beta \psi - \frac{\xi}{2} \psi^2 R \right) \] (16)

\[ S_\phi = \int d^4 x \sqrt{-\tilde{g}} \left( \frac{\tilde{R}}{2} - \frac{1}{2} \tilde{g}^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi \right) \] (17)

where \( \xi = 1/6 \), and \( \tilde{g}_{\mu \nu} \) and \( g_{\mu \nu} \) are the metric tensors associated to ordinary and conformally coupled scalar fields, respectively. These actions are connected by the conformal transformation:

\[
g_{\mu \nu} = \Omega^{-2} \tilde{g}_{\mu \nu} = \begin{cases} 
\cosh^2 (\sqrt{\xi} \phi) \tilde{g}_{\mu \nu} \\
\sinh^2 (\sqrt{\xi} \phi) \tilde{g}_{\mu \nu}
\end{cases} \tag{18}
\]

where \( \Omega^2 = |1 - \xi \psi^2| \) and \( \xi = 1/6 \), and the relation between \( \psi \) and \( \phi \) is given by:

\[
\psi = \begin{cases} 
\pm \frac{1}{\sqrt{\xi}} \tanh (\sqrt{\xi} \phi) \\
\pm \frac{1}{\sqrt{\xi}} \coth (\sqrt{\xi} \phi)
\end{cases} \tag{19}
\]

As we can see from the above, in this manner we generate two distinct types of conformally coupled scalar field solutions. Henceforth, we will denote the solutions for the first case type \( A \) and type \( B \) for the second. Before going on with the determination and analysis of the conformal solutions, it is useful to define the mass function \( m(u, v) \) as:

\[
m = \frac{\Sigma}{2} \left( 1 + g^{\alpha \beta} \Sigma_{\alpha} \Sigma_{\beta} \right) \tag{20}
\]

where \( \Sigma(u, v) \) denotes the radius of the two-sphere. From the equation above, we can write a relation between the mass function \( m \) associated to the conformally coupled scalar field solutions and the corresponding one associated to the ordinary scalar field, \( \tilde{m}(u, v) \). After direct calculation, the following expression arises:

\[
\frac{m}{\Sigma} = \frac{\tilde{m}}{r} + \frac{\Omega_\phi^2}{2 \Omega^2} \tilde{g}^{\mu \nu} \phi_{,\mu} \phi_{,\nu} r^2 + \frac{\Omega_\phi}{\Omega} r \tilde{g}^{\mu \nu} \phi_{,\mu} r_{,\nu} \tag{21}
\]

where \( \Omega_\phi = \frac{d\Omega}{d\phi} \) and \( r(u, v) \) is the radius of the two-sphere given by the ordinary scalar field solution.

Now, applying the transformations Eqs. (18,19) to Roberts’ solution, the metric tensor components and the conformally coupled scalar field appear, respectively, as:

\[
h = \frac{1}{4} (M \sqrt{\Sigma} \pm M^{-1} \sqrt{\Sigma})^2 \tag{22}
\]

\[
\Sigma^2 = \frac{1}{4} (M \sqrt{\psi} \pm M^{-1} \sqrt{\psi})^2 \tag{23}
\]

\[
\psi = \pm \sqrt{6} \frac{1 \mp M^{-1}}{1 \pm M^{-1}} \tag{24}
\]

where \( r^2 = -\frac{1}{2} u v + a_2 \ u^2 + b_2 \ v^2 \) and \( M \) is given by:
\[ M = \frac{4 a_2 u - (1 + \sqrt{1 - 16 a_2 b_2}) v}{4 a_2 u - (1 - \sqrt{1 - 16 a_2 b_2}) v}. \]  
(25)

The solution above is self-similar since it brings about a homothetic killing vector

\[ k_{\mu} = \frac{\Omega^{-2}}{2} (v, u, 0, 0) \]

From these expressions and Eq. (21), the mass function is determined directly after some calculation as:

\[
m = \frac{1}{2r} \left[ -\frac{1}{4} (1 - 16 a_2 b_2) u v - \frac{1}{12} (1 - 16 a_2 b_2) u v \left( \frac{M^{\frac{1}{2}} \sqrt{3} \mp M^{-\frac{1}{2}} \sqrt{3}}{M^{\frac{1}{2}} \sqrt{3} \pm M^{-\frac{1}{2}} \sqrt{3}} \right)^2 \\
+ \frac{1}{\sqrt{3}} \left( \frac{M^{\frac{1}{2}} \sqrt{3} \mp M^{-\frac{1}{2}} \sqrt{3}}{M^{\frac{1}{2}} \sqrt{3} \pm M^{-\frac{1}{2}} \sqrt{3}} \right) \sqrt{1 - 16 a_2 b_2} (b_2 v^2 - a_2 u^2) \right] \]  
(26)

Type A solutions are the ones with the upper sign in the above expressions, whereas the lower sign case stands for what we have called type B solutions. We now choose \( a_2 = 1/4 \), without loss of generality, in order to restrict ourselves only to the collapse case, and analyze the following cases: (a) \( 0 < b_2 < 1/4 \), (b) \( b_2 = 0 \) and (c) \( b_2 < 0 \).

For type A solutions, the area of the two-sphere, \( 4 \pi \Sigma^2 \), vanishes in the same regions in which \( r^2 \) does, since the conformal factor \( \cosh^2(\sqrt{\xi} \phi) \) is always different from zero. The invariants \( R^{\alpha\beta} R_{\alpha\beta} \) and \( \psi_{,\alpha} \psi_{,\alpha} \) diverge in these regions, indicating the existence of singularities. Due to the fact that an everywhere conformal transformation does not alter the spacetime structure, these solutions exhibit the same phases as Roberts’ solution.

Concerning the apparent horizon dynamics, we start from the definition of the locus of such structures:

\[ g^{\alpha\beta} \Sigma_{,\alpha} \Sigma_{,\beta} = 0. \]  
(27)

Two relations are obtained: \( \Sigma_{,u} = 0 \) and \( \Sigma_{,v} = 0 \). In neither case is it possible to express \( u_{AH} \) as an explicit function of \( v \). A numerical plot is thus necessary, indicating that the relevant expression is \( \Sigma_{,v} = 0 \), given by:

\[
\sqrt{1 - 4 b_2 (1 - M^{\frac{1}{2}} \sqrt{3}) v} - \sqrt{3} (1 + M^{\frac{1}{2}} \sqrt{3}) (-u + 4 b_2 v) = 0
\]  
(28)

For \( 0 < b_2 < 1/4 \) and \( b_2 = 0 \), there is no apparent horizon. Only for \( b_2 < 0 \) is the apparent horizon present (see fig. 2). It is also possible to show that \( u_{AH} \cong 2.64 b_2 v \), whereas for Roberts’ solution \( u_{AH} = 4 b_2 v \). Therefore, the only modification is the slope of the curve. The mass function evaluated on the apparent horizon, i.e. the mass of the black hole, is:

\[
m_{AH} = \frac{1}{8} \left( M_{AH}^{\frac{1}{2}} \sqrt{3} \mp M_{AH}^{-\frac{1}{2}} \sqrt{3} \right) \sqrt{(1.7424 b_2 - 0.32) b_2 v}
\]  
(29)

where \( M_{AH} = \frac{2.64 b_2 - 1 - \sqrt{1 - 16 b_2}}{2.64 b_2 - 1 + \sqrt{1 - 16 b_2}} \). Again, the mass of the black hole grows without bound for \( v \to \infty \). Whether or not we consider the self-similar collapse for \( 0 < v < v_0 \), as done in [7],
the important thing is to exhibit the power law of \( m_{AH} \) for near-critical evolution \((b_2 = 0)\). We obtain:

\[
m_{AH} \simeq \text{Const.} b_2^{0.21} v.
\]

The exponent differs from 0.37, claimed by Choptuik and others to be universal. In fact, we could argue that the value of the coupling constant \( \xi \) is responsible for this difference. However, as pointed out in the numerical work, the exponent depends only weakly on \( \xi \). Thus, as in the ordinary scalar field collapse (the exponent is 0.5), the origin of this difference is unclear within our model.

Contrary to the previous case, type \( B \) solutions exhibit distinct spacetime structure. The singular regions, in general characterized by \( \Sigma^2 = 0 \), are described by \( r^2 = 0 \), or by \( \sinh^2(\sqrt{\xi} \phi) = 0 \), i.e., the conformal factor vanishes. In this way, an additional singular region given by \( v = 0 \) is present. Such a region separates two different spacetimes, which we denote by \( B_+ \) and \( B_- \), that are characterized by \( v < 0 \) and \( v > 0 \), respectively. Another new feature is that the region \( J^- \) is singular \((R^\mu\nu R_{\mu\nu} \text{ as well as } \psi^\alpha \psi_{,\alpha} \text{ diverge on } J^-)\), even though \( \Sigma^2 \) is finite in it. Such a region can be interpreted as a cosmological null singularity at the null infinity past, where the scalar field diverges. The structure of the spacetime \( B_+ \) depends on \( b_2 \), as we can see from fig. 2. The spacetime \( B_- \) is not altered by changing the parameter \( b_2 \), and, as shown in fig. 3, this spacetime is limited by three singular regions. As before, the dynamics of the apparent horizon is described by \( \Sigma, v = 0, \) or:

\[
\sqrt{1 - 4 b_2 (1 + M^{-\sqrt{3}})} u - \sqrt{3 (1 - M^{-\sqrt{3}})} (4 b_2 v - u) = 0
\]

For \( B_+ \), we find the same behavior as in the type \( A \) solution (or Roberts’ solution): there is an apparent horizon, and consequently black hole formation, only for \( b_2 < 0 \). The mass of the formed black hole tends to infinity as \( v \to \infty \), and for near-critical behavior the power law displays the same value as before, i.e. 0.21. The structure of \( B_- \) is not altered by changing the parameter \( b_2 \) (fig. 3); therefore, there is no critical behavior related to the black hole formation, and the timelike singularity is surrounded by an apparent horizon. The evolution of the mass of the “black hole”, \( m_{AH} \), is distinct from the previous case: in the very beginning \((u v = -\infty)\), \( m_{AH} \) is infinity; then, it diminishes gradually to become equal to zero at \( v = 0 \).

3 Conclusions

We have analyzed, for the first time, self-similar collapse of conformally coupled scalar fields. The main motivation was to reproduce analytically the critical behavior discovered numerically by Choptuik. For this task, we have used the technique which permits the generation of solutions for conformally coupled scalar fields departing from those for ordinary scalar fields. Thus, from Roberts’ solution (self-similar), we obtained two types of solutions which we denoted type \( A \) and \( B \).

Type \( A \) solutions have no new qualitative features if compared with Roberts’ solution. The parameter \( b_2 \) plays the central role: the value \( b_2 = 0 \) separates the solutions which do not form black holes \((0 < b_2 < 1/4)\) from those that do \((b_2 < 0)\). However, as a characteristic of the continuous self-similar regime, the mass of the black hole tends to infinity for \( v \to \infty \). Despite this undesirable and unphysical behavior (asymptotically all spacetime becomes
trapped), we found a power law for the mass of the black hole for near critical evolution. The exponent is 0.21. This value is not close to 0.37, obtained in the numerical work, and it is not clear we can expect such a strong influence of the coupling parameter $\xi = 1/6$ on the exponent. According to Choptuik, the exponent depends only weakly on $\xi$. It would be interesting to check this conjecture in the more general case of a non-minimally coupled scalar field.

Type $B$ solutions, on the other hand, reveal some new characteristics. Due to an additional singular region described by $v = 0$, two distinct spacetimes have to be taken into account. We called them $B_+$ and $B_-$, and they are characterized by $v > 0$ and $v < 0$, respectively (see figs. (2) and (3)). The critical behavior was found to take place only for solutions $B_+$, and the exponent for the power law associated with $m_{AH}$ is the same as that for type $A$ solutions. As a final remark, the dynamics of type $B_-$ solutions is independent of $b_2$: the spacetime is limited by three singular regions (fig. (3)), and an apparent horizon encloses the timelike singularity. Thus, this situation is not relevant with respect to the critical behavior in the gravitational collapse.

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• fig. 1 Apparent horizon for Roberts’ solution, conformally coupled scalar field collapse $A$ and $B$ for the case $b_2 < 0$.

• fig. 2 Causal diagrams for type $B_+$ solution in the cases (a) $0 < b_2 < 1/4$, (b) $b_2 = 0$ and (c) $b_2 < 0$. Although the region $J^-(u \to -\infty)$ has finite area $\Sigma^2(-\infty, v) = \frac{1}{12} (1 - 4 b_2) v^2$, it is singular.

• fig. 3 Causal diagram for $B_-$ solution. The timelike singularity is enclosed by an apparent horizon. According with some numerical work with respect to eq. (27) there is a region beyond the apparent horizon where the mass function becomes negative. Then, this model must be considered unphysical.
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