Fermionic matrix product states and one-dimensional short-range entangled phases with antiunitary symmetries

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We extend the formalism of matrix product states (MPS) to describe one-dimensional gapped systems of fermions with both unitary and antiunitary symmetries. Additionally, systems with orientation-reversing spatial symmetries are considered. The short-ranged entangled phases of such systems are classified by three invariants, which characterize the projective action of the symmetry on edge states. We give interpretations of these invariants as properties of states on the closed chain. The relationship between fermionic MPS systems at a renormalization group fixed point and equivariant algebras is exploited to derive a group law for the stacking of fermionic phases. The result generalizes known classifications to symmetry groups that are nontrivial extensions of fermion parity and time-reversal.

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I. INTRODUCTION AND OVERVIEW

Matrix product states (MPS) have proven useful at describing the ground states of gapped local Hamiltonians in one spatial dimension [1,2]. This representation leads to a classification of interacting short-range-entangled (SRE) bosonic phases with a symmetry \( G \) in terms of the group cohomology of \( G \) [3,4]. One-dimensional systems of fermions are related to these bosonic systems by the Jordan-Wigner transformation, and this fact has been exploited to classify fermionic SRE phases [4,5]. The language of MPS was recently extended to an intrinsically fermionic formalism in what has been dubbed fermionic MPS (fMPS) [6,7]. This approach has the benefits of straightforwardly describing systems on a closed chain with twisted boundary conditions and allowing one to derive a group law for the stacking of fermionic SRE phases. It also makes manifest the relation to spin-TQFT, to derive a group law for the stacking of fermionic SRE phases in the symmetry class BDI (\( T^2 = 1 \)) and the \( \mathbb{Z}/2 \) classification in the class DIII (\( T^2 = P \)).

II. FERMIONIC MPS

A. Unitary symmetries

We begin by briefly recalling the fMPS formalism, leaving many of the details to Ref. [7].

The symmetries of a fermionic system form a finite supergroup \( (\mathcal{G}, \rho) \); that is, a finite group \( \mathcal{G} \) with a distinguished central involution \( \rho \in \mathcal{G} \) called fermion parity. Every supergroup arises as a central extension of a group \( G_b \) of bosonic symmetries by \( \mathbb{Z}/2 = \{1, -1\} \):

\[
\mathbb{Z}_2 \cong \mathcal{G} \rightarrow \mathcal{G}/\mathbb{Z}_2 \cong G_b.
\]  

The structure of the paper and its main results are as follows. In Sec. II, we review the formalism of \( \mathcal{G} \)-equivariant fMPS for a unitary on-site symmetry \( \mathcal{G} \). We recall how fermionic SRE phases are classified by Morita classes of equivariant algebras and how the invariants \( \alpha, \beta, \gamma \) that characterize these algebras appear in the action of \( \mathcal{G} \) on edge degrees of freedom. We then derive interpretations of the invariants on the closed chain that extend the results of Ref. [9], which were discovered in the context of spin-TQFT. Next, time-reversing symmetries and their relation to spatial parity are discussed. The generalizations of the three invariants to phases with such symmetries are derived and interpreted. In Sec. III, a general stacking law (44) is derived for fermionic SRE phases with a symmetry \( \mathcal{G} \) that is a central extension by fermion parity of a bosonic symmetry group that may contain antiunitary symmetries. We contrast this result with the bosonic group structure and emphasize the origin of the difference. In Sec. IV, we demonstrate our result with several examples, recovering the \( \mathbb{Z}/8 \) classification of fermionic SRE phases in the symmetry class BDI (\( T^2 = 1 \)) and the \( \mathbb{Z}/2 \) classification in the class DIII (\( T^2 = P \)).
equivariance condition: for all \( a \in A, g \in G \),
\[
T[R(g)a] = Q(g)T(a)Q(g)^{-1}.
\]
From this data, one may construct a gapped, symmetric, and frustration-free lattice Hamiltonian [11].

For our purposes, we are only interested in MPS at a renormalization group (RG) fixed point, where a system exhibits the physics universal to its gapped phase. Physical sites are blocked together under real-space RG. At a fixed point, this procedure endows the space \( A \) with a product \( A \otimes A \rightarrow A \), making \( A \) into a finite-dimensional associative algebra and the space \( V \) into a faithful module over \( A \) with structure tensor \( T \) [12]; that is,
\[
T(a)T(b) = T[m(a \otimes b)]
\]
for all \( a, b \in A \). It follows that \( T[m[R(g)a \otimes R(g)b]] = T[R(g)m(a \otimes b)] \). Since \( T \) is injective, \( m \) satisfies
\[
R(g)m(a \otimes b) = m[R(g)a \otimes R(g)b].
\]
We say \( A \) is a \( G \)-equivariant algebra. Because \( T \) can be put into a canonical form, \( A \) is semisimple [12]. It can also be viewed as the algebra of linear operators on the space of low energy boundary states [4].

Our approach is to work with ground states of MPS systems, rather than their Hamiltonians. Every ground state of an MPS Hamiltonian has the form of a generalized MPS, which we now describe. Given a \((G, p)\)-equivariant MPS system described by an algebra \( A \) and tensor \( T: A \rightarrow \text{End}(V) \), one obtains a generalized MPS by choosing an observable \( X \in \text{End}(V) \) that supercommutes with \( T(a) \) [13]; that is,
\[
XT(a) = (-1)^{|a||X|} T(a)X
\]
for all \( a \in A \) such that \( R(p)a = (-1)^{|a|}a \), where the parity of \( X \) is defined by \( PXP^{-1} = (-1)^{|X|}X \), for \( P := Q(p) \). A linear map \( X \) satisfying this condition is called an even or odd \( Z/2 \)-graded module endomorphism, depending on its parity.

For \( X \) satisfying the supercommutation rule (5), the generalized MPS has conjugate wave function
\[
|\psi_{T, x}^X\rangle = \sum_{i_1, \ldots, i_N} \text{Tr}[U(Q(g)XT(e_{i_1}) \cdots T(e_{i_N}))][i_1 \cdots i_N]
\]
in the \( g \)-twisted sector. When \( G \) splits as \( G_b \times \{1, p\} \), the \( g \)-twisted spin structure for \( t(g) = 0 \) consists of states on the circle with NS spin structure and a \( G_b \) gauge field of holonomy \( b(g) \), while the \( g \)-twisted sector for \( t(g) = 1 \) consists of states on the circle with R spin structure and a \( G_b \) gauge field of holonomy \( b(g) \). We emphasize that one must choose a splitting \( t \) to make sense of the \( g \)-twisted sector as a \( b(g) \)-twisted NS or R sector. When \( G \) is nonsplit, one cannot speak of independent spin structures and gauge fields [14].

The MPS description of a gapped system is not unique. For this reason, fixed point systems—and therefore gapped phases—are classified not by the algebras themselves but by their Morita classes [12]. After all, it is the module category (the \( T \)'s and \( X \)'s) of \( A \) that determines the system's ground states.

B. Equivariant algebras for fermionic SRE phases
For the remainder of the paper, we will restrict our attention to SRE phases. The algebras that correspond to fixed points in such phases are either of the form \( \text{End}(U) \) for \( U \) a projective representation \( Q \) of \( G \) or of the form \( \text{End}(U_b) \otimes \mathbb{C} \langle \ell(1) \rangle \) for \( U_b \) a projective representation of \( G_b \), with the following actions of \( G \) [7]. We refer to algebras of the first (second) type as “even” (‘odd’). Odd algebras are only present when the extension for \( G \) splits.

The action of \( G \) on an even algebra \( A = \text{End}(U) \) is simply
\[
R(g) \cdot M = Q(g)M Q(g)^{-1}.
\]
Two even algebras are Morita equivalent if their projective representations have the same \([\alpha] \in H^2[G, \text{End}(U)] \) [15]. It is shown in the Appendix that \([\alpha]\) is equivalent to a pair \((\alpha, \beta) \in C^2[G_b, U(1)] \times C^1[G_b, \mathbb{Z}/2] \) that satisfies \( \delta \alpha = \langle \beta, \rho \rangle_\mu \) and \( \delta \beta = 0 \), up to coboundaries [16]. In particular, when \( G \) splits, \( \rho \) is trivial and the equivalence classes defining the phase are \([\alpha], [\beta] \in H^2[G_b, U(1)] \times H^1[G_b, \mathbb{Z}/2] \). The invariant \([\alpha] \) is simply \([\alpha] \) pulled back by \( s \) to \( G_b \), while \( \beta \) measures the parity of \( Q(g) \):
\[
PQ(g)P^{-1} = e^{\pi i \beta_{[\beta|g][\alpha]}} Q(g).
\]
Let \( \Gamma \) denote the generator of \( \mathbb{C} \langle \ell(1) \rangle \) with \( \Gamma^2 = +1 \). The action of \( G \) on an odd algebra is
\[
R(g) \cdot M \otimes \Gamma^m = (-1)^{|\beta|+|\gamma|} Q(g)M Q(g)^{-1} \otimes \Gamma^m.
\]
Morita classes of odd algebras are classified by the class \([\alpha] \in H^2[G_b, U(1)] \) that describes the projective \( G_b \)-action \( Q \) and the class \([\beta] \in H^1[G_b, \mathbb{Z}/2] \) that describes the action of \( G_b \) on \( \Gamma \).

It is apparent from (9) that changing the trivialization \( t \) shifts \( \beta \) by an arbitrary \( \mu = (t' - t) \in Z^1[G_b; \mathbb{Z}/2] \). Therefore, unless \( t \) is fixed, \( \beta \) is not a well-defined invariant of odd algebras. On the other hand, for even algebras, redefining the action of \( g_b \) to \( Q(g_b)P^{\mu(\delta s)} \) leaves \( \beta \) unchanged but shifts \( \alpha \) by \( 1/2 \beta \cup \mu \).

In summary, when \( G \) splits as \( G_b \times \mathbb{Z}_2^x \), the \( G \)-symmetric fermionic SRE phases are classified by \([\alpha], [\beta], [\gamma] \in H^2[G_b, U(1)] \times H^1[G_b, \mathbb{Z}/2] \times \mathbb{Z}/2 \), where \( \gamma \in \mathbb{Z}/2 \) tells us whether the algebra is even (\( \gamma = 0 \)) or odd (\( \gamma = 1 \)). When \( G \) does not split, only the invariants \([\alpha] \) (not a cocycle) and \([\beta] \) are present.

A system in an SRE phase has exactly one state per twisted sector. To see this from the algebra, count independent solutions \( X \). The unique simple module over an even algebra \( \text{End}(U) \) is \( U \), and by Schur’s lemma its only endomorphism is \( X = 1 \). The unique faithful simple module over an odd algebra \( \text{End}(U_b) \otimes \mathbb{C} \langle \ell(1) \rangle \) is \( U_b \otimes \mathbb{C}^{[1]} \). It has two endomorphisms—an even one \( X = 1 \otimes 1 \) and an odd one \( X = 1 \otimes \sigma_b \). The former appears in the wave function of the NS sector MPS state and the latter for the R sector MPS state.

C. Invariants of fermionic SRE phases
The invariants \( \alpha, \beta, \gamma \) can also be extracted from an SRE fermionic MPS system without reference to the algebra \( A \). Below we give a physical interpretation of these invariants as observable quantities.

We begin by studying how the MPS in the \( g \)-twisted sector transforms under the action of a unitary symmetry \( h \in G_b \). Let
\( \omega \) be the cocycle that characterizes the projective action \( Q \) on the module. Then
\[
R(h) \cdot \text{Tr} Q(g) X T^i |i\rangle = \text{Tr} Q(g) X Q(h)^{-1} T^i Q(h) |i\rangle = e^{2\pi i [\omega(h, g) + \omega(hg, h^{-1}) - \omega(h, h^{-1})]} \text{Tr} Q(hg^{-1}) [Q(h) X Q(h^{-1})] T^i |i\rangle.
\]

This surface can also be evaluated as a torus with holonomies \( h \) and \( g^{-1} \), respectively, yielding
\[
\omega(g^{-1}, h) + \omega(g^{-1} h, g) - \omega(g^{-1}, g) = \omega(h, g) + \omega(g^{-1}, h) - \omega(g^{-1}, g)
\]
\[
= \omega(h, g) - \omega(g, h).
\]

These are equal, as is required by consistency of the TQFT. In terms of states, the \( h \)-charge of the \( g \)-twisted sector is the same as the \( g^{-1} \)-charge of the \( h \)-twisted sector, as long as \( g \) and \( h \) commute. There is no analogous statement for systems with \( \gamma = 1 \). Recall that \( \beta(g) \) measures whether or not \( g \) acts as \( \sigma_z \) on the second factor of \( \text{End}(U) \otimes \mathbb{C}[\ell(1)] \). Then \( Q(g) \) anticommutes with \( X = 1 \otimes \sigma_z \), and so the state picks up an extra charge of \( \beta(g) \) which cancels with the sign \( (12) \) for a total \( g \)-charge of \( +1 \) in the R sector.

### 3. Alpha

Consider the MPS state on a circle with two adjacent domain walls, parametrized by bosonic symmetries \( g_b, h_b \in G_b \), as in Fig. 1. Upon fusing them, the state picks up a phase
\[
\text{Tr} Q[s(g_b)] Q[s(h_b)] T^i |i\rangle = e^{2\pi i \alpha(s(g_b), s(h_b))} \text{Tr} Q[s(g_b) s(h_b)] T^i |i\rangle.
\]

These phases define a \( G_b \)-cochain
\[
\alpha(g_b, h_b) = \omega(s(g_b), s(h_b)).
\]

If \( G \) splits, then the fact that \( \omega \) is a cocycle implies that \( \alpha \) is as well. If the extension \( G \) is instead defined by a nontrivial \( \rho \), then \( \alpha \) has coboundary \( 1/2 \beta \cup \rho \). See the Appendix for details. Redefining each \( X = 1 \) by a sector-dependent phase shifts \( \alpha \) by a \( G \)-coboundary with arguments in \( G_b \), as expected.

Note that when \( \beta \) and \( \gamma \) are trivial, there are no fermionic states and the system is insensitive to spin structure. In this sense, \( \alpha \) captures purely bosonic features of the system.

![FIG. 1. Fusion of domain walls.](image)
4. Summary

1. $(-1)^γ$ is the fermion parity of the untwisted R sector.
2. If $γ = 0$, $(-1)^{β(g)}$ is the fermion parity of the $g$-twisted sector for either of the two $g$’s with $b(g) = gb$. Alternatively, $(-1)^{β(g)}$ is the $g$-charge of the untwisted R sector. If $γ = 1$, $(-1)^{β(g)}$ is the fermion parity of the $gb$-twisted NS sector, as determined by the choice of splitting.
3. $e^{2πιω(γ,b)}$ is the phase due to fusing $gb$ and $h_b$ domain walls.

D. Antiunitary and orientation-reversing symmetries

More generally, a fermionic system may be invariant under antiunitary symmetries as well as unitary ones. In this case, the full symmetry group $G$ is a central extension by $Z_2^F$ of a bosonic symmetry group $G_b$, which is itself an extension of $Z_2^I$ by a finite group $G_0$, as in Fig. 2. The symmetry class $(G, p, x)$ is determined by a central $p \in G$ and a map $x : G_b \rightarrow Z/2$ that encodes whether a bosonic symmetry is unitary or antiunitary. Note that the composition $x \circ b$, which we also call “$x$,” satisfies $x(p) = 0$. Let $G_0$ denote its kernel.

A fixed point MPS system of symmetry class $(G, p, x)$ consists of a finite-dimensional semisimple associative algebra $A$ and a faithful module $T : A \rightarrow \text{End}(V)$, satisfying the equivariance conditions (4), (2) as before, only now the group action may be antiunitary. In particular, the projective action on $V$ is given by a unitary operator $Q(g)$ for each $g \in G_0$ and an antiunitary operator $Q(g)$ for each $g \notin G_0$ that satisfy

$$ Q(g)Q(h) = e^{2πιω(g,h)}Q(gh) $$

(18)

for phases $ω(g, h)$. By comparing $[Q(g)Q(h)]Q(k)$ and $Q(g)[Q(h)Q(k)]$, we find the $x$-twisted cocycle condition

$$ ω(g, h) + ω(gh, k) = (-1)^{x(g)}ω(h, k) + ω(g, hk). $$

(19)

Redefining each $Q(g)$ by a $g$-dependent phase corresponds to shifting $ω$ by an $x$-twisted coboundary. Therefore the action of $G$ on $V$ is characterized by a twisted cohomology class $[ω] \in H^2(G, U(1)_γ)$. The group action $R$ on $A$ is defined via (2).

It will be convenient to define linear maps $M(g)$ by

$$ M(g) = \begin{cases} Q(g), & g \in G_0, \\ Q(g)K, & g \notin G_0, \end{cases} $$

(20)

where $K$ denotes complex conjugation.

Unitary symmetries that reverse the orientation of one-dimensional space can also be described in this language. Let $x$ measure whether a symmetry reverses orientation. The natural generalization of (2) is

$$ T[R(g)a] = M(g)T(a)M(g)^{-1} \quad \text{for } g \in G_0, $$

$$ T[R(g)a] = M(g)T(a)^T M(g)^{-1} \quad \text{for } g \notin G_0. $$

(21)

Let us introduce the following shorthand. For a matrix $O \in \text{End}(V)$, define

$$ O^{T_0} = O, \quad O^{T_1} = OT, $$

$$ \{O\}_0 = O, \quad \{O\}_1 = [O^{-1}]^T. $$

(22)

Since $R$ is a group homomorphism,

$$ M(g)[M(h)]^{x(g)}T(a)^{x(gh)}M(h)^{x(g)}M(g)^{-1} = T[R(g)R(h)a] $$

$$ = T[R(gh)a] $$

$$ = M(gh)T(a)^{x(gh)}M(gh)^{-1}. $$

(23)

This implies there exists a number $ω(g, h) \in \mathbb{R}/\mathbb{Z}$ such that

$$ M(g)[M(h)]^{x(g)} = e^{2πiω(g,h)}M(gh). $$

(24)

By comparing the two equal expressions $M(g)[M(h)]^{x(g)} \{k\}^{x(gh)}$ and $M(g)[M(h)M(k)]^{x(g)}\{k\}^{x(gh)}$, one recovers the $x$-twisted cocycle condition (19) for $ω$.

From the perspective of two-dimensional spacetime, it is not surprising that time-reversal [19] and space-reversal should be treated similarly. To make the connection more explicit, note that the physical Hilbert space carries the action of an antilinear involution $*$, which we regard as CPT (see Ref. [12]). Using the equivariance of the multiplication and (anti)unitarity of $R(g)$ with respect to the inner product on the Hilbert space, it may be shown that $*$ commutes with $R(g)$ for all $g \in G$. With respect to the product on $A$, this map is an antiautomorphism. If $R(g)$ denotes the action of a time-reversing symmetry, $R(g)\cong$ is a unitary symmetry that reverses the orientation of space. Then

$$ T[R(g)\ast a] = M(g)T(\ast a)M(g)^{-1} = M(g)T(a)^TM(g)^{-1}. $$

(25)

Moreover, since $\ast$ commutes with $R(g)$, the equivariance condition (2) implies that $M(g)$ is unitary (up to a phase), so (18) and (24) are equivalent (up to a coboundary). For the remainder of the paper, we suppress $\ast$ and simply write $R$ to denote a time-reversing or space-reversing symmetry.

E. Invariants of fermionic SRE phases with antiunitary symmetries

As in the case of unitary symmetries, fermionic SRE systems at fixed points correspond to even algebras of the form $\text{End}(U)$ and odd algebras of the form $\text{End}(U_b) \otimes \mathbb{C}ℓ(1)$. However, when the symmetries may act antiunitarily, the cohomology class characterizing the Morita class (and hence the SRE phase) is twisted.

We now discuss the meaning of the invariants $α, β, \gamma$ in the antiunitary context, following the previous analysis. The form of the MPS conjugate wave function is (6) as before. Consider the action of an antiunitary symmetry $h \notin G_0$ on an
MPS in the $g$-twisted ($g \in \mathcal{G}_0$) sector:

$$R(h) \cdot \text{Tr}[Q(g) XT^i]/[i| = \text{Tr}[M(g)X M(h^{-1})(T^i)^T M(h^{-1})^{-1}]/[i| = \text{Tr}[M(h^{-1})^T X T M(h^{-1})^{-1} T^i]/[i| = e^{2\pi i\omega(h^{-1}, h^{-1})} \text{Tr}[(M(h^{-1})^T M(h^{-1})^{-1})^T]/[i| = e^{2\pi i\omega(h^{-1}, h^{-1})} \text{Tr}[(M(h^{-1})^T M(h^{-1})^{-1})^T]/[i| = e^{2\pi i\omega(h^{-1}, h^{-1})} \text{Tr}[(M(h^{-1})^T M(h^{-1})^{-1})^T]/[i| = e^{2\pi i\omega(h^{-1}, h^{-1})} \text{Tr}[(M(h^{-1})^T M(h^{-1})^{-1})^T]/[i|.$$

where in the last line we use the fact that

$$\omega(h^{-1}, h^{-1}, h^{-1}) = -\omega(h, g^{-1} - \omega(h, g^{-1}) - \omega(h, h^{-1}) - \omega(g^{-1}, h^{-1}) - 2\omega(h^{-1}, h).$$

which can be verified by repeated application of the twisted cocycle condition. We see that under the action of an antunitary symmetry $h$ the following occurs:

1. The $g$-twisted sector maps to the $h^{-1}g^{-1}$-twisted sector.
2. The operator $X$ is transposed, then conjugated by $M(h^{-1})$.
3. States also pick up a phase of $e^{2\pi i\omega(h^{-1}, h^{-1})}$.

The phase matches Eq. (4.12) of Ref. [18]. In particular, when $g$ acts on the R sector, it is

$$\beta(g) := \omega(g, p) - \omega(p, g) + \omega(p, p), \quad g \notin \mathcal{G}_0.$$  

This phase satisfies $\beta(pg) = \beta(g)$, takes values in $\mathbb{Z}/2$, and, together with (13), is a $G_b$-cocycle. Refer to the Appendix for a proof. When $\gamma = 0$, this is the $g$-charge of the R sector. However, when $\gamma = 1$, the charge receives an additional contribution from the transformation of $X$. Similar to in the unitary case detailed above, the total charge is the $\beta$-independent quantity $(-1)^{\chi(g)}$, so this interpretation of $\beta$ fails.

The invariant $\beta$ also has an interpretation in terms of edge states, like (8) [21]. A time-reversing symmetry $g \notin \mathcal{G}_0$ maps $V$ to its dual space $V^*$, on which $p$ acts as $P^{-1}$, so the parity of $Q(g)$ is read off of

$$P^{-1} Q(g) P^{-1} = e^{i\beta(g)} Q(g), \quad g \notin \mathcal{G}_0.$$  

A similar interpretation holds if $g$ reverses the orientation of space. Let $V^* \otimes V$ represent the tensor product of left and right edge state spaces. On this space, $g$ acts as

$$\psi L \otimes \psi R \mapsto Q(g)^{-1} (\psi L \otimes \psi R)^T Q(g) = Q(g)^{-1} \psi R \otimes Q(g) \psi L.$$  

Then $\beta$ appears as the result of acting by $P \otimes P^{-1}$, $g$, then $P^{-1} \otimes P$: 

$$\psi \otimes 1 \mapsto 1 \otimes P Q(g) P \psi = e^{\pi \beta(\hat{g})}(1 \otimes \psi).$$  

The meaning of $\alpha$ (17) is more difficult to describe in Hamiltonian language [22]. The lack of twisted sectors for antiunitary symmetries means that $\alpha(g_b, h_b)$ has an interpretation as the phase due to fusing domain walls only when $g_b$ and $h_b$ are unitary. The rest of $\alpha$ appears in other places. It is convenient to first describe the invariant $\omega$. For two unitary symmetries $g, h \in \mathcal{G}_0$, the phase $\omega(g, h)$ is due to fusing domain walls. It was shown in Ref. [18] that two extra families of phases—which we now describe—together with $\omega$ restricted to $\mathcal{G}_0$, determine the full $\omega$ on $\mathcal{G}$. The first family is the phases (28) due to acting on the $g$-twisted sector by an antunitary symmetry $h$. The second family consists of the relative phases due to comparing, for each antunitary symmetry $g \notin \mathcal{G}_0$, the crosscap state (see Refs. [8,18]) $[Q(g) T^i]/[i|$ to the MPS state in the $g^2$-twisted sector. These phases have the simple form $\omega(g, g)$. Note that these data are not gauge invariant, and the equivalence classes of them under shifting $\omega$ by a twisted coboundary do not take a simple form. Now that we described the full $\omega$, the full $\alpha$ can be recovered by restricting to $G_b$. As we demonstrate in the Appendix, the result is a $G_b$ cochain whose $x$-twisted coboundary is $\beta \cup \rho$.

Finally, $\gamma$ is the fermion parity of the untwisted Ramond sector, as in the unitary case.

### III. FERMIONIC STACKING LAW

Gapped fermionic phases form a commutative monoid under the operation of stacking. The result of stacking fixed point systems corresponding to algebras $A_1$ and $A_2$ is the system corresponding to the supertensor product $A_1 \otimes A_2$, defined by the multiplication law $(a_1 \otimes a_2)(b_1 \otimes b_2) = (-1)^{\chi(a_1,b_2)}a_1b_1 \otimes a_2b_2$ [6,7]. SRE phases are precisely those that are invertible under stacking, and so they form a group. The goal of this section is to derive this group structure on the set of SRE phases in terms of the invariants $\alpha$, $\beta$, and $\gamma$. Our plan is to follow the argument presented in Appendix D of Ref. [7], while taking into account that $Q(g)$ is antilinear when $x(g) = 1$. We summarize the results at the end of the section.

The following discussion relies on a result proven in the Appendix: that one can choose a gauge such that the twisted cocycle $\omega$ is related to $\alpha$ and $\beta$ by, for all $g, h \in \mathcal{G}$, where $\bar{g}$ is short for $b(g)$,

$$\omega(g, h) = \alpha(\bar{g}, h) - i/2 \beta(\bar{g}) \chi(h).$$  

There are three cases to consider: The stacking of (1) two even algebras, (2) an even and an odd algebra, and (3) two odd algebras. When $\mathcal{G}$ does not split, there are no odd algebras so we need only consider the first case.
A. Even-even stacking

Consider the even algebras \(\text{End}(U_1)\) and \(\text{End}(U_2)\). Their tensor product is \(\text{End}(U_1 \otimes U_2)\), where \(U_1 \otimes U_2\) carries a projective representation \(Q = Q_1 \otimes Q_2\). Then

\[
Q(g)Q(h) = [Q_1(g) \otimes Q_2(g)][Q_1(h) \otimes Q_2(h)]
\]

\[
= (-1)^\beta_1(h, h) Q_1(g)Q_1(h) \otimes Q_2(g)Q_2(h)
\]

\[
= (-1)\beta_1(h, h) ^{\epsilon_{2\pi i(a_1(h, h) + 1/2\beta_1(h, h))}} Q_1(gh) \otimes Q_2(gh).
\]

Thus the invariants of the stacked phase are \(\alpha = \alpha_1 + \alpha_2 + 1/2(\beta_1 + \beta_2)\) [23] and \(\beta = \beta_1 + \beta_2\). Since the stacked algebra is again even, \(\gamma = 0\). The presence of antiunitary symmetries does not affect even-even stacking.

B. Even-odd stacking

Now consider the even algebra \(A_1 = \text{End}(U_1)\), where \(U_1\) carries a projective representation \(Q_1\) of \(G_1\), and the odd algebra \(A_2 = \text{End}(U_2) \otimes \mathbb{C} \ell(1)\), where \(U_2\) carries a projective representation \(Q_2\) of \(G_2\). Their tensor product \(\text{End}(U_1 \otimes U_2) \otimes \mathbb{C} \ell(1)\) is isomorphic as an algebra to the odd algebra \(\text{End}(U_1 \otimes U_2) \otimes \mathbb{C} \ell(1)\) by the map

\[
JW : M_1 \otimes (M_2 \otimes \Gamma^m) \mapsto M_1 P^m \otimes M_2 \otimes \Gamma^{m+|M_1|},
\]

which has inverse

\[
JW^{-1} : M_1 \otimes M_2 \otimes \Gamma^m \mapsto M_1 P^{m+|M_1|} \otimes (M_2 \otimes \Gamma^{m+|M_1|}),
\]

where the parity of \(M_1\) is defined by \(Q_1\): \(P_1 M_1 P_1 = (-1)^{|M_1|} M_1\). This isomorphism respects the \(\mathbb{Z}/2\)-grading defined by the standard action of fermion parity on even and odd algebras.

It remains to determine the \(G_b\) action on the odd algebra. For \(g \in G\) with \(t(g) = 0\),

\[
JW \circ g \circ JW^{-1} : (M_1 \otimes M_2 \otimes \Gamma^m) = JW \circ g \cdot (M_1 P^{m+|M_1|} \otimes (M_2 \otimes \Gamma^{m+|M_1|}))
\]

\[
= JW \cdot [Q_1(g)M_1P^{m+|M_1|}Q_1(g)^{-1} \otimes [Q_2(\bar{g})M_2Q_2(\bar{g})^{-1} \otimes (-1)^{|\epsilon_{2\pi i(\bar{g}h)}M_1|}]]
\]

\[
= (-1)^{|\epsilon_{2\pi i(\bar{g}h)}M_1|}|M_1| |\beta_1(\bar{g})| \Gamma^{m+|M_1|}.
\]

To read off the invariants from this group action, we must rewrite it in the standard form by defining \(\tilde{Q}_1(g) = Q_1(g)P^{\beta_1(\bar{g})+\beta_1(\bar{g})} [24]\) and \(Q(g) = \tilde{Q}_1(g)Q_2(\bar{g}).\) Then, continuing from (37),

\[
g \cdot (M_1 \otimes M_2 \otimes \Gamma^m) = (-1)^{m|\beta_1(\bar{g})+\beta_1(\bar{g})|}[\tilde{Q}_1(g) \otimes Q_2(\bar{g})]M_1 \otimes M_2[\tilde{Q}_1(g)^{-1} \otimes Q_2(\bar{g})^{-1}] \otimes \Gamma^m,
\]

from which we read off the stacked invariant \(\beta = \beta_1 + \beta_2\). And

\[
Q(g)Q(h) = [\tilde{Q}_1(g) \otimes Q_2(\bar{g})][\tilde{Q}_1(h) \otimes Q_2(\bar{h})]
\]

\[
= \tilde{Q}_1(g)P^{\beta_1(\bar{g})+\beta_1(\bar{g})} \tilde{Q}_1(h)P^{\beta_1(h)+\beta_1(h)} \otimes Q_2(\bar{g})Q_2(\bar{h})
\]

\[
= (-1)^{\beta_1(h, h)+1/2(\beta_1(h, h))} \tilde{Q}_1(gh)P^{\beta_1(h)+\beta_1(h)} \otimes Q_2(\bar{g})Q_2(\bar{h})
\]

\[
= e^{2\pi i(a_1(h, h) + 1/2\beta_1(h, h))} \tilde{Q}_1(gh)P^{\beta_1(h)+\beta_1(h)} \otimes Q_2(\bar{g})Q_2(\bar{h}).
\]

where we see \(\alpha = \alpha_1 + \alpha_2 + 1/2(\beta_1 + \beta_2 + 1/2(\beta_1 + \beta_2)).\) There is no asymmetry: the \(1/2\beta_1 + \beta_1\) term always comes from the \(\beta\) of the even algebra [25]. Finally, \(\gamma = 1\) since the stacked algebra is odd.

C. Odd-odd stacking

Consider the odd algebras \(A_1 = \text{End}(U_1) \otimes \mathbb{C} \ell(1)\), where \(U_1\) carries a projective representation \(Q_1\) of \(G_b\), and \(A_2 = \text{End}(U_2) \otimes \mathbb{C} \ell(1)\), where \(U_2\) carries a projective representation \(Q_2\) of \(G_b\). Their tensor product is given by \(A_1 \otimes A_2 \simeq \text{End}(U_1 \otimes U_2 \otimes \mathbb{C}^{11})\), since \(\mathbb{C} \ell(1) \otimes \mathbb{C} \ell(1) \simeq \mathbb{C} \ell(2) \simeq \text{End}(\mathbb{C}^{11})\), via an isomorphism

\[
(M_1 \otimes \Gamma^m) \otimes (M_2 \otimes \Gamma^2) \mapsto M_1 \otimes M_2 \otimes \sigma_1^m \sigma_2^n,
\]

where \(\sigma_1\) and \(\sigma_2\) are any two distinct Pauli matrices. With respect to the action of fermion parity on the \(\text{End}(\mathbb{C}^{11})\) factor as conjugation by \(\sigma_3 = -i\sigma_1\sigma_2\), this map is an isomorphism of \(\mathbb{Z}/2\)-graded algebras.

One choice [26] of \(G_b\)-action \(Q\) on \(U_1 \otimes U_2 \otimes \mathbb{C}^2\), with respect to which (40) is equivariant, is

\[
g : U_1 \otimes U_2 \otimes v \mapsto Q_1(\bar{g})u_1 \otimes Q_2(\bar{g})u_2 \otimes \sigma_1^{\beta_1(\bar{g})} \sigma_2^{\beta_1(\bar{g})} K^{x(\bar{g})} v,
\]

where \(x(h)\) is a function with \(x(h) = 0\) for \(h \in \mathbb{Z}/2\).
for \( g \in \mathbb{G} \) with \( t(g) = 0 \), where \( K \) denotes complex conjugation in a basis in which \( \sigma_1 \) and \( \sigma_2 \) are real. Then

\[
Q(\tilde{g})Q(\tilde{h}) = \left[ Q_1(\tilde{g}) \otimes Q_2(\tilde{g}) \otimes \sigma_1^{\tilde{g}}(\tilde{g}) \sigma_2^{\tilde{g}}(\tilde{g}) K^{\tilde{g}(\tilde{g})} \right] \left[ Q_1(\tilde{h}) \otimes Q_2(\tilde{h}) \otimes \sigma_1^{\tilde{h}}(\tilde{h}) \sigma_2^{\tilde{h}}(\tilde{h}) K^{\tilde{h}(\tilde{h})} \right]
\]

\[
= e^{2\pi i \alpha(\tilde{g},\tilde{h})} Q_1(\tilde{g}\tilde{h}) \otimes e^{2\pi i \alpha(\tilde{g},\tilde{h})} Q_2(\tilde{g}\tilde{h}) (-1)^{\beta(\tilde{g}\tilde{h})} \sigma_1^{\tilde{g}\tilde{h}} \sigma_2^{\tilde{g}\tilde{h}} K^{\tilde{g}(\tilde{h})}
\]

(42)

from which we see that \( \alpha = \alpha_1 + \alpha_2 + \frac{1}{2} \beta_1 + \beta_2 \). Since \( U_1 \otimes U_2 \) is purely even, the parity of \( Q \) comes from

\[
P\sigma_1^{\tilde{g}}(\tilde{g}) \sigma_2^{\tilde{g}}(\tilde{g}) K^{\tilde{g}(\tilde{g})} P = (-i \sigma_1 \sigma_2) \sigma_1^{\tilde{g}}(\tilde{g}) \sigma_2^{\tilde{g}}(\tilde{g}) K^{\tilde{g}(\tilde{g})} (-i \sigma_1 \sigma_2)
\]

\[
= (-1)^{\beta(\tilde{g})+\beta(\tilde{h})} \sigma_1^{\tilde{g}}(\tilde{g}) \sigma_2^{\tilde{g}}(\tilde{g}) K^{\tilde{g}(\tilde{g})} (-1)^{\sigma(\tilde{g})}
\]

(43)

We read off \( \beta = \beta_1 + \beta_2 + x \). Finally, the stacked algebra is even, so \( \gamma = 0 \).

D. Summary

The stacking law for the invariants \((\alpha, \beta, \gamma)\) is given by

\[
(\alpha_1, \beta_1, 0) \cdot (\alpha_2, \beta_2, 0) = (\alpha_1 + \alpha_2 + \frac{1}{2} \beta_1 + \beta_2, \beta_1 + \beta_2, 0),
\]

\[
(\alpha_1, \beta_1, 0) \cdot (\alpha_2, \beta_2, 1) = (\alpha_1 + \alpha_2 + \frac{1}{2} \beta_1 + \beta_2 + \frac{1}{2} \beta_1 \cup \beta_1 \cup \beta_1, \beta_1 + \beta_2, 1),
\]

\[
(\alpha_1, \beta_1, 1) \cdot (\alpha_2, \beta_2, 1) = (\alpha_1 + \alpha_2 + \frac{1}{2} \beta_1 + \beta_2, \beta_1 + \beta_2 + x, 0).
\]

This group law inherits the properties of commutativity and associativity from the tensor product of algebras. When \( G \) does not split, \( \gamma \) is not present, and the stacking law is simply

\[
(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2 + \frac{1}{2} \beta_1 + \beta_2, \beta_1 \cup \beta_2).
\]

IV. EXAMPLES

A. Class BDI fermions: \( G = \mathbb{Z}_2^F \times \mathbb{Z}_2^T \)

Let us consider SRE phases with symmetry \( G = \mathbb{Z}_2^F \times \mathbb{Z}_2^T \). The two classes \( \alpha \in H^2[\mathbb{Z}_2^T, U(1)_T] = \mathbb{Z}/2 \), two classes \( \beta \in H^1[\mathbb{Z}_2^T, \mathbb{Z}/2] = \mathbb{Z}/2 \), and two classes \( \gamma \in \mathbb{Z}/2 \) make for a total of eight phases. A straightforward application of the general stacking law (44) reveals that these phases stack like the cyclic group \( \mathbb{Z}/8 \). In this section, we will reproduce this group law by exploiting the relationship between \( G \)-equivariant algebras, real superdivision algebras, and Clifford algebras, which have Bott periodicity \( \mathbb{Z}/8 \).

We begin by describing simple \( G \)-equivariant algebras. The matrix algebra \( M_2^G \) represents the sole Morita class of simple complex algebras. This algebra has a unitary structure \(*\) given by conjugate transposition. Its action fixes a basis \( \{1, X, Y, Z = -iXY\} \). On \( Cl_{2} \simeq M_2^G \), \(*\) acts by Cliffor transposition and complex conjugation of coefficients with respect to a pair of generators that square to \(+1\).

There are two distinct real structures on \( M_2^G \) given by complex conjugation \( T \) on the second component of \( M_2^G \simeq M_2^\mathbb{R} \otimes \mathbb{R}^2 \) and \( M_2^\mathbb{C} \simeq \mathbb{H} \otimes \mathbb{C} \). The unitary structure \(*\) of \( M_2^\mathbb{C} \) acts by transposition on \( M_2^\mathbb{R} \), complex conjugation on \( \mathbb{C} \), and inversion of the generators \( \hat{i} \) and \( \hat{j} \) of \( \mathbb{H} \); that is, its fixed bases are

\[
\{1 \otimes 1, X \otimes 1, iY \otimes i, Z \otimes 1\} \in M_2^\mathbb{R} \otimes \mathbb{R}^2 \mathbb{C},
\]

\[
\{1 \otimes 1, \hat{i} \otimes i, \hat{j} \otimes i, \hat{k} \otimes i\} \in \mathbb{H} \otimes \mathbb{C}.
\]

These bases have the same \( T \)-eigenvalues as they do \(*\)-eigenvalues, where \(*\) acts as transposition on \( M_2^\mathbb{R} \) and inversion of generators on \( \mathbb{H} \). Under the algebra isomorphisms \( M_2^\mathbb{R} \simeq Cl_{1,1} \mathbb{R} \simeq Cl_{2,0} \mathbb{R} \) and \( \mathbb{H} \simeq Cl_{0,2} \mathbb{R} \), \(*T\) acts by inverting the generators and products of generators that square to \(-1\).

Let us derive the invariants \( \alpha \) of these real structures. Pulled back from \( M_2^\mathbb{R} \otimes \mathbb{R} \mathbb{C} \) to \( M_2^\mathbb{C} \), \( T \) acts like complex conjugation and \(*T\) like transposition; that is \( M(t) = I \). Then

\[
M(t)M(t)^{-1T} = 1,
\]

(47)

which means \( \alpha(t, t) = 0 \). Pulled back from \( \mathbb{H} \otimes \mathbb{R} \mathbb{C} \), \( T \) acts like complex conjugation and conjugation by \( Y \), while \(*T\) acts like transposition and conjugation by \( Y \); that is \( M(t) = Y \). Then \( \alpha(t, t) = 1/2 \) since

\[
M(t)M(t)^{-1T} = e^{i\pi}1.
\]

(48)

By the Skolem-Noether theorem, a superalgebra structure on \( M_2^\mathbb{C} \) is given by conjugation by an element that squares to \(+1\). If this element is \( I \), the \( \mathbb{Z}/2 \)-grading is purely even; otherwise, it has two even dimensions and two odd. All structures of the last type are isomorphic in the absence of a real structure.

In the presence of the real structure \( M_2^\mathbb{R} \otimes \mathbb{R} \mathbb{C} \), there are three distinct gradings. First, there is the purely even grading, given by \( P = \Pi \). This structure has \( \frac{1}{2} \beta(t) = \omega(t, P) = 0 \). Second, there is conjugation by \( Z \) (or \( X \)), which gives \( M_2^\mathbb{R} \) the superalgebra structure of \( Cl_{1,1} \mathbb{R} \). Again, \( \beta(t) = 0 \) since \( P = Z \) means

\[
PM(t)P^T = Z1Z^T = 1.
\]

(49)
The matching of the invariants alludes to the fact that the real superalgebra structures $M_2\mathbb{R}$ and $C\ell_{1,1}\mathbb{R}$ are graded Morita equivalent. Third, there is conjugation by $Y$; that is, $P = Y$. Then $\beta(t) = 1$ since

$$PM(t)P^T = Y1Y^T = e^{\pi}1. \quad (50)$$

The corresponding real Clifford algebra is $C\ell_{2,0}\mathbb{R}$ and represents a distinct Morita class.

On the real structure $H \otimes \mathbb{R} C$, there are two distinct gradings. First, there is the purely even grading $P = 1$, which has $\beta(t) = 0$. The second grading is given by conjugation by $Z$ (or $X$ or $Y$) on $M_2\mathbb{C}$ and gives $H$ the superalgebra structure of $C\ell_{0,2}\mathbb{R}$. Then $\beta(t) = 1$ since

$$PM(t)P^T = ZYZ^T = e^{\pi}Y. \quad (51)$$

Now consider algebras of the form $M_2\mathbb{C} \otimes C\ell_1\mathbb{C}$. The second component $C\ell_1\mathbb{C}$ has a unitary structure given by complex conjugation of coefficients of the generator $\Gamma$ that squares to $+1$. There are two distinct real structures on $C\ell_1\mathbb{C}$ given by complex conjugation $T$ on the second component of $C\ell_{1,0}\mathbb{R} \otimes \mathbb{C}$ and $C\ell_{0,1}\mathbb{R} \otimes \mathbb{C}$. The unitary structure $*$ of $C\ell_1\mathbb{C}$ acts by complex conjugation on $C$ and inversion of generators that square to $-1$; that is, the fixed bases are $\{1 \otimes 1, \gamma \otimes 1\}$ for $C\ell_{1,0}\mathbb{R} \otimes \mathbb{C}$ and $\{1 \otimes 1, \gamma \otimes t\}$ for $C\ell_{0,1}\mathbb{R} \otimes \mathbb{C}$. The map $*T$ is trivial on $C\ell_{1,0}\mathbb{R}$ and inversion of the generator on $C\ell_{0,1}\mathbb{R}$. Therefore, pulled back from $C\ell_{1,0}\mathbb{R} \otimes \mathbb{C}$ to $C\ell_1\mathbb{C}$, $T$ is complex conjugation and $*T$ is inversion of $\Gamma$.

As discussed in Sec. II and in Ref. [7], we need only to consider a single $Z/2$-grading on $M_2\mathbb{C} \otimes C\ell_1\mathbb{C}$—the one where $M_2\mathbb{C}$ is purely even and the generator of $C\ell_1\mathbb{C}$ is odd. The algebra $M_2\mathbb{C} \otimes C\ell_1\mathbb{C}$ has four real structures: a choice of $M_2\mathbb{R}$ or $H$ for the first component and $C\ell_{1,0}\mathbb{R}$ or $C\ell_{0,1}\mathbb{R}$ for the second. As was true for even algebras, the first choice determines whether $M(t)$ is $1$ or $Y$; that is, whether $\alpha(t, t) = 0$ or $1/2$. The second choice determines whether $*T$ inverts the odd generator; this is $\beta(t)$.

Due to the Morita equivalence $M_2\mathbb{R} \sim R$, several of the eight Morita classes are represented by algebras of lower dimension; for example, $C\ell_{1,0}\mathbb{R}$ instead of $M_2\mathbb{R} \otimes C\ell_{1,0}\mathbb{R}$. Up to this substitution, the eight real-structured superalgebras we found are complexifications of the eight central real superdivision algebras—real superalgebras with center $\mathbb{R}$ that are invertible under a supertensor product up to graded Morita equivalence [27,28]. They constitute a set of representatives of the eight graded Morita classes of real superalgebras. These algebras appear in the second column of Table I, next to their invariants in the third column.

Another set of Morita class representatives is the Clifford algebras $C\ell_{n,0}\mathbb{R}$. In terms of these algebras, stacking is simple, as

$$C\ell_{n,0}\mathbb{R} \otimes C\ell_{m,0}\mathbb{R} \simeq C\ell_{n+m,0}\mathbb{R} \quad (52)$$

and

$$C\ell_{n,0}\mathbb{R} \sim C\ell_{m,0}\mathbb{R} \quad \text{for } n \equiv m \text{ mod } 8. \quad (53)$$

Each central superdivision algebra can be matched with the Clifford algebra $C\ell_{n,0}\mathbb{R}$, $n < 8$ in its Morita class [28], as in the first column of Table I. This determines a $Z/8$ stacking law on central superdivision algebras and their invariants that agrees with the more general law (44).

Physically speaking, the $Z/8$ classification is generated by the time-reversal-invariant Majorana chain [4,29]. While the symmetry protects pairs of dangling Majorana zero modes from being gapped out, turning on interactions can gap out these modes in groups of eight. Fidkowski and Kitaev formulated their stacking law in terms of three invariants that are equivalent to $\alpha, \beta$, and $\gamma$. Their results match ours.

For contrast, we list the invariants of the corresponding bosonic phases in the rightmost column of Table I. There, $H$ denotes the subgroup of unbroken symmetries and $\omega$ denotes 2-cocycle characterizing the SPT order. These invariants can be obtained from the fermionic invariants [7]. We observe that invertibility is not preserved by bosonization; in particular, only the fermionic SREs with $\gamma = 0$ become bosonic SREs. The four bosonic SRE phases have a $Z/2 \times Z/2$ stacking law. We also include the two noncentral superdivision algebras $C\ell_3$ and $C\ell_1$ at the bottom of the table. These correspond to symmetry-breaking (SB) phases.

### B. Class DIII fermions: $G = Z^4_T$

In the following, $G = Z^4_T$ denotes the notriivial extension of $G_p = Z^4_T$ by fermion parity. Let us consider fermionic SRE phases with this symmetry. There are two distinct classes $\beta \in H^1[Z^2_T, Z/2]$, determined by $\beta(t) = 0$ and $\beta(t) = 1$. The trivial $\beta$ has a single $\alpha$, the trivial one, that satisfies $\delta T = 1/\beta \cup \rho$, up to the proper equivalence [30]. The nontrivial $\beta$ has also a single compatible $\alpha$, up to equivalence $\alpha(t, t) = 1/4$.

The trivial phase is represented by the algebra $C\ell_3$ with trivial actions of $p$ and $t$, as always. For the nontrivial phase, consider $A = \text{End}(U)$, where $P$ and $T$ act on $U$ as

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad M(t) = \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}. \quad (54)$$

Then the invariants can be recovered as follows:

$$M(t)M(t)^{-1T} = e^{2\pi i/4}P \quad \Rightarrow \quad \alpha(t, t) = 1/4, \quad (55)$$

$$P M(t)P^T = e^{i\pi} M(t) \quad \Rightarrow \quad \beta(t) = 1, \quad (56)$$

| $C\ell_{n,0}$ | $A_{\text{div}}$ | $\alpha, \beta, \gamma$ | Fermionic | Bosonic | $(H, \omega)$ |
|----------------|------------------|--------------------------|------------|---------|-------------|
| $\mathbb{R}$   | 0,0,0            | trivial                  | trivial    | trivial | $(G, 0)$    |
| $C\ell_{1,0}$  | 0,0,1            | SRE                      | SB         | $(Z^2_T, 0)$ | |
| $C\ell_{2,0}$  | 0,1,0            | SRE                      | SPT        | $(G, o_1)$ | |
| $H \otimes C\ell_{0,1}$ | 1,1,1 | SRE                      | mixed      | $(Z^2_{d=1}, \alpha)$ | |
| $H$            | 1,0,0            | SRE                      | SPT        | $(G, o_2)$ | |
| $H \otimes C\ell_{1,0}$ | 1,0,1 | SRE                      | mixed      | $(Z^2_T, \alpha)$ | |
| $C\ell_{0,2}$  | 1,1,0            | SRE                      | SPT        | $(G, o_1 + o_2)$ | |
| $C\ell_{0,1}$  | 0,1,1            | SRE                      | SB         | $(Z^2_{d=1}, 0)$ | |
| $C\ell_1$      | -                | DB                       | SB         | $(Z^2_T, 0)$ | |
| $C\ell_1$      | -                | SB                       | $(Z^2_T, 0)$ | |

**Table I.** The 10-fold way of $Z^2_T \times Z^2_T$-symmetric fermionic phases.
TABLE II. Fermionic phases with unitary and antiunitary symmetries.

| Symmetry class | Fermionic | Bosonic |
|----------------|-----------|---------|
| $Z_2 \times Z_2^T \times Z_2^I$ | $Z_4 \times Z_4^t$ | $(Z_4)^4$ |
| $Z_2 \times Z_2^T$ | $Z_2 \times Z_2$ | $Z_2 \times Z_2$ |
| $Z_2^I \times Z_2^I$ | $Z_4$ | $Z_2 \times Z_2$ |
| $Z_2^I$ | $Z_2 \times Z_2^I$ | $Z_2 \times Z_2$ |

According to the rule (44), stacking two copies of this phase results in the trivial phase

$$(1/4, 1) \cdot (1/4, 1) = (1/4 + 1/4 + 1/2 \cdot 1, 1 + 1) = (0, 0).$$

We find that fermionic SRE phases with symmetry $Z_2^T \times Z_2$ have a $Z/2$ classification, in agreement with the condensed matter literature [31,32]. The nontrivial phase appears as a Majorana chain with two dangling modes protected by the symmetry.

C. Unitary $Z/2$ symmetry

As a last set of examples, let us consider systems with a unitary bosonic symmetry group $G_0 = Z/2$, in addition to time-reversal and fermion parity. There are many ways to organize these symmetries into a full symmetry class $(\mathcal{G}, p, x)$. Here we consider the five abelian possibilities, which are listed with their fermionic and bosonic phase classifications in Table II. The first three have $G_0 = Z \times Z_2^T$, the last two $G_0 = Z_2^I$. In the two cases where the central extension of $G_0$ by $Z_2^I$ splits, we use a superscript $\gamma$ to denote the subgroup of the fermionic classification that contains the odd phases.

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where

\[ \omega(g, p) = \omega(g, h), \quad (A6) \]

\[ \omega(g, ph) = \omega(g, h) + \omega(p, g). \quad (A7) \]

**Proof.** For an arbitrary 2-cocycle \( W \in \mathbb{Z}^2[\mathcal{G}, U(1)_{\mathcal{T}}] \), define

\[ \omega = W - \delta_T L, \quad (A8) \]

where \( L \in C^1[\mathcal{G}, U(1)_{\mathcal{T}}] \) satisfies

\[ L(1) = 0, \]

\[ L(p) = \frac{1}{2} W(p, p) \] or \( \frac{1}{2} W(p, p) + \frac{1}{2}, \quad (A9) \]

\[ L(p\tilde{g}) = L(\tilde{g}) - W(p, \tilde{g}) + L(p). \]

Here we abuse notation by letting \( \tilde{g} \) denote a \( g \in \mathcal{G} \) with \( t(g) = 0 \). This implies \( L(p) = \frac{1}{2} W(p, p) \). We fixed \( L(p\tilde{g}) \) in terms of \( L(\tilde{g}) \) and \( L(p) \), but left \( L(\tilde{g}) \) undetermined, while \( L(p) \) is fixed up to a \( \frac{1}{2} \).

We see that

\[ \omega(p, p) = W(p, p) - (-1)^{t(p)} L(p) - L(1) \]

\[ = W(p, p) - 2 \cdot \frac{1}{2} W(p, p) = 0 \quad (A10) \]

and

\[ \omega(p, \tilde{g}) = W(p, \tilde{g}) - [(-1)^{t(p)} L(\tilde{g}) + L(p) - L(p\tilde{g})] \]

\[ = W(p, \tilde{g}) - \frac{1}{2} W(p, p) + \frac{1}{2} W(p, \tilde{g}) - \frac{1}{2} W(p, p) \]

\[ = 0. \quad (A11) \]

Next we show that any \( \omega \) satisfying \((A10)\) and \((A11)\) must also satisfy the gauge conditions \((A6)\) and \((A7)\). First,

\[ \omega(p, p\tilde{g}) = -\delta_T \omega(p, p, \tilde{g}) - (-1)^{t(p)} \omega(p, \tilde{g}) + \omega(p, g). \quad (A12) \]

Similarly, computing \( 0 = \delta_T \omega(p, \tilde{g}, \tilde{h}) \) shows that \( \omega(p\tilde{g}, \tilde{h}) = \omega(p, \tilde{h}) \) and computing \( 0 = \delta_T \omega(p, \tilde{g}, p\tilde{h}) \) shows that \( \omega(p\tilde{g}, p\tilde{h}) = \omega(p, \tilde{g}) \). Putting these together, we get that \((A6)\) is satisfied.

Now we compute \( 0 = \delta_T \omega(p, \tilde{g}, \tilde{h}) \) which shows that \( \omega(p\tilde{g}, \tilde{h}) = \omega(p, \tilde{h}) \), \( \omega(p\tilde{h}, p\tilde{g}) = \omega(p, \tilde{g}) \), and \( 0 = \delta_T \omega(p, \tilde{g}, p\tilde{h}) \) which shows that \( \omega(p\tilde{g}, \tilde{h}) = \omega(p, \tilde{h}) \). Putting these together, we see that \((A7)\) is satisfied.

**Lemma A.3.** Given a trivialization \( t \), the map

\[ \omega(g, h) = \alpha(\tilde{g}, \tilde{h}) + \frac{1}{2} \beta(\tilde{g}) t(h) \quad (A13) \]

defines a bijection from pairs \((\alpha, \beta) \in C^2[G_b, U(1)_{\mathcal{T}}] \times C^1[G_b, \mathbb{Z}/2] \) that satisfy \( \delta_T \alpha = \frac{1}{2} \beta \cup \alpha \) and \( \delta T \beta = 0 \) where \( \frac{1}{2} \beta \) is regarded as a \( U(1)_{\mathcal{T}} \)-valued cocycle] twisted cocycles \( \omega \in \mathbb{Z}^2[\mathcal{G}, U(1)_{\mathcal{T}}] \) that satisfy \((A6)\) and \((A7)\) for all

\[ g, h \in \mathcal{G}. \]

In particular, for all \( g_b, h_b \in G_b \), this map has an inverse

\[ \alpha(g_b, h_b) = \omega(s(g_b), s(h_b)), \]

\[ \frac{1}{2} \beta(g_b) = \omega(s(g_b), p). \quad (A14) \]

**Proof.** First we show that \( \omega \) is a twisted cocycle:

\[ (\delta_T \omega)(g, h, k) \]

\[ = (-1)^{t(g)} \omega(h, k) + \omega(g, h k) - \omega(g, h) - \omega(g h, k) \]

\[ = (-1)^{t(g)} \alpha(\tilde{h}, \tilde{k}) + \alpha(\tilde{g}, \tilde{h} k) - \alpha(\tilde{g}, \tilde{h}) - \alpha(\tilde{g} h, \tilde{k}) \]

\[ + \frac{1}{2}(\delta(1)^{t(g)} \beta(\tilde{h}) t(k) + \frac{1}{2} \beta(\tilde{g}) t(h k) \]

\[ = \frac{1}{2} \beta(\tilde{g}) t(h) \]

\[ - \frac{1}{2} \beta(\tilde{g}) t(h k) \]

\[ = \frac{1}{2} \beta(\tilde{g}) (\delta t(h)) \]

\[ = 0. \quad (A15) \]

Next we verify that \( \omega \) satisfies the gauge conditions

\[ \omega(p, g, h) = \alpha(\tilde{p} g, \tilde{h} + \tilde{g} p) t(h) \]

\[ = \alpha(\tilde{g}, \tilde{h} + \frac{1}{2} \beta(\tilde{g}) t(h) \]

\[ = \omega(g, h), \quad (A16) \]

\[ \omega(p, g, ph) = \alpha(\tilde{g}, p h) + \frac{1}{2} \beta(\tilde{g}) t(\tilde{p} h) \]

\[ = \alpha(\tilde{g}, \tilde{h} + \frac{1}{2} \beta(\tilde{g}) t(h) \]

\[ = \omega(g, h) + \omega(g, p). \quad (A17) \]

Then we check the conditions for \( \alpha \) and \( \beta \). For these two calculations, let \( \tilde{g} \) denote a \( g \in \mathcal{G} \) with \( t(g) = 0 \). This implies \( L(p) = \frac{1}{2} W(p, p) \). We fixed \( L(p\tilde{g}) \) in terms of \( L(\tilde{g}) \) and \( L(p) \), but left \( L(\tilde{g}) \) undetermined, while \( L(p) \) is fixed up to a \( \frac{1}{2} \).

Similarly, computing \( 0 = \delta_T \omega(p, \tilde{g}, \tilde{h}) \) shows that \( \omega(p\tilde{g}, \tilde{h}) = \omega(p, \tilde{h}) \) and computing \( 0 = \delta_T \omega(p, \tilde{g}, p\tilde{h}) \) shows that \( \omega(p\tilde{g}, p\tilde{h}) = \omega(p, \tilde{g}) \). Putting these together, we get that \((A6)\) is satisfied.

Now we compute \( 0 = \delta_T \omega(p, \tilde{g}, \tilde{h}) \) which shows that \( \omega(p\tilde{g}, \tilde{h}) = \omega(p, \tilde{h}) \), \( \omega(p\tilde{h}, p\tilde{g}) = \omega(p, \tilde{g}) \), and \( 0 = \delta_T \omega(p, \tilde{g}, p\tilde{h}) \) which shows that \( \omega(p\tilde{g}, \tilde{h}) = \omega(p, \tilde{h}) \). Putting these together, we see that \((A7)\) is satisfied.

The object \( \frac{1}{2} \beta \) defined in \((A14)\) is the gauge-fixed form of \((A1)\). Then, by \( A.1 \), it defines a \( \beta \in Z^1[\mathcal{G}_b, \mathbb{Z}/2] \).

It remains to show that these maps are indeed inverses. Since \( \frac{1}{2} \beta \) is the image of a \( \mathbb{Z}/2 \)-valued cocycle \( \beta \), \( \omega \) can be written with a minus sign like \( \omega = \alpha - \frac{1}{2} \beta \cup t \). Note also
that \( s(\tilde{g}) = p^{t(s)}g \). Then

\[
\omega(\tilde{g}, h) = \omega(\tilde{g}, \tilde{h}) - \frac{1}{2}\beta(\tilde{g}) T(h)
\]

\[
= \omega(p^{t(s)}g, p^{t(h)}h) - \omega(p^{t(s)}g, p) T(h)
\]

\[
= \omega(g, h).
\]

This proves (A19).

**Theorem A.4.** \( H^2[\mathcal{G}, U(1)_T] \) equals, as a set, the set of pairs \((\alpha, \beta)\) (see A.3) modulo the equivalence \((\alpha', \beta) \sim (\alpha, \beta)\) if \(\alpha' = \alpha + \delta_T \lambda\) with \(\lambda\) a cochain in \(C^1[\mathcal{G}, U(1)_T]\) satisfying \(\lambda[s(\tilde{g})p] = \lambda[s(\tilde{g})] + \lambda(p)\) [34].

**Proof.** The preceding lemmas show that the set of twisted cocycles \(\omega\) satisfying the gauge conditions (A6) and (A7) is equivalent to the set of pairs \((\alpha, \beta)\). After transforming \(\omega\) into this gauge, there remains freedom to choose \(L(g)\) for each \(g \in \mathcal{G}\) such that \(T(g) = 0\), and to shift \(L(p)\) by \(1/2\). We have already seen that \(\beta\) is invariant under an arbitrary gauge transformation. However, there is some residual gauge freedom for \(\alpha\).

Let \(\omega' = \omega + \delta_T \lambda\) be another 2-cocycle satisfying the gauge conditions. It takes the form \(W - \delta_T L', \) with \(L'\) possibly differing from \(L\) in its values on \(s(g_b)\) and \(p\). We see from \(\delta_T \lambda = \omega' - \omega = \delta_T L' - L\) that \(\lambda = L' - L + \kappa\) where \(\kappa\) is a twisted 1-cocycle. Then, by (A9), \(L[s(g_b)p] = \lambda[s(g_b)] + \lambda(p)\). The quantities \(L(p), L'(p), \kappa(p)\), and therefore \(\lambda(p)\), can each be chosen to be 0 or \(1/2\). Finally, by (A14), this freedom in gauge-fixed \(\omega\) translates into the desired freedom in \(\alpha\).

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[10] On the contrary, the trivialization is unphysical from the perspective of TQFT. This subtlety is explored in Ref. [7].

[11] N. Schuch, D. Perez-García, and I. Cirac, Classifying quantum phases using MPS and PEPS, Phys. Rev. B 84, 165139 (2011).

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[13] Herein lies the difference between the bosonic and fermionic MPS formalisms. In the bosonic case, \(X\) commutes with \(T(\alpha)\) regardless of its charge under \(P\) (which is not a distinguished symmetry). For this reason, the twisted sector state spaces and how they are acted on by symmetries typically differ between a fermionic system and its Jordan-Wigner transform.

[14] The appropriate geometric structure—the \(G\)-Spin structure—is discussed in Ref. [7].

[15] V. Ostrik, Module categories, weak Hopf algebras and modular invariants, Transformation Groups 8, 177 (2003).

[16] In our notation, \(\alpha\) takes values in \(\mathbb{R}/\mathbb{Z} = [0, 1)\) and \(\alpha + 1/2 \in [0, 1/2)\) in \(G\)-coboundary with arguments in \(G\)-only when \(G\) splits this is a \(G\)-coboundary. This subtlety is relevant when \(\mathcal{G} = \mathbb{Z}_2^{T}\), see Sec. IV.

[17] We always work in a gauge \(Q(1) = 1\).

[18] A. Kapustin and A. Turzillo, Equivariant topological quantum field theory and symmetry protected topological phases, J. High Energ. Phys. 03 (2017) 006.

[19] By a well-known result of Wigner, an antiunitary symmetry reverses the direction of time.

[20] If \(X\) is Hermitian, this is the same as \(X\) being conjugated by the antilinear operator \(Q(hg)\).

[21] If \(g\) is antilinear, the expression (8) is not invariant under the change of gauge \(\omega \leftrightarrow \omega + \delta\Lambda\).

[22] In the Lagrangian picture, we expect \(\alpha\) to be related to trivalent junctions of possibly orientation-reversing domain walls.

[23] We used the fact that \(\beta_1 \cup \beta_1\) is cohomologous to \(\beta_1 \cup \beta_2\) in \(\mathbb{Z}/2\).

[24] Adding a phase factor to \(\tilde{Q}\) would have shifted the resulting 2-cocycle \(\alpha\) by an irrelevant coboundary. For example, if we had chosen a factor \(i^{h(s)}\) as in Ref. [7], we would have gotten \(\beta_1 \cup x\) instead of \(\beta_1 \cup \beta_1\) in the final answer.

[25] Note that while \(\beta_1 \cup \beta_1\) is an ordinary coboundary and hence could be ignored for phases without time-reversal, it is not a twisted coboundary and so cannot be ignored when time-reversing symmetries are present. By adding a twisted coboundary, we can put it in the form \(\beta_1 \cup x\), which makes the dependence on time-reversal symmetry manifest.
[26] Again, had we chosen a different $G_b$-action on $\mathbb{C}^{|1|}$ compatible with the action $g_b : \Gamma_i \mapsto (-1)^{\ell(i)b} \Gamma_i$ on the $C\ell(1)$ factors, the 2-cocycle $\alpha$ would be shifted by a twisted coboundary.

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[29] L. Fidkowski and A. Kitaev, The effects of interactions on the topological classification of free fermion systems, Phys. Rev. B 81, 134509 (2010).

[30] The cocycle $\alpha(t, t) = \frac{1}{2}$ is nontrivial in $H^2[G_b, U(1)_T]$ but is trivialized by adding a 2-coboundary on $\mathcal{G}$ satisfying the proper conditions. See the Appendix for details.

[31] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, Topological insulators and superconductors: Tenfold way and dimensional hierarchy, New J. Phys. 12, 065010 (2010).

[32] A. Kitaev, Periodic table for topological insulators and superconductors, AIP Conf. Proc. 22, 1134 (2009).

[33] This condition on $L$ ensures that $Q(1) = \mathbb{1}$ is preserved.

[34] Had we chosen a different representative $\rho' = \rho + b\mu$ of $[\rho]$ to describe the extension of $G_b$ by $\mathbb{Z}_2^F$, we would have considered a different set of cochains $\alpha$ (modulo coboundaries), shifted by $\frac{1}{2}b \beta \cup \mu$, but their counting would be the same.