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Numerical approach to MHD flow of power-law fluid on a stretching sheet with non-uniform heat source

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Abstract: In the present study the flow of power-law fluid, due to a stretching sheet embedded in a saturated porous medium, is considered. This study also accounts for the variable thermal conductivity in the process of heat transfer along with dissipation due to Joule heating. The thermal conductivity is assumed to vary as a linear function of temperature. The similarity transformation is used to convert nonlinear partial differential equations to nonlinear ordinary equations. The numerical method, Runge-Kutta method with shooting technique has been applied to solve the resulting equations. The power-law fluid exhibits a dual property in the presence of magnetic field. The limiting cases \( n \to 0 \) and \( n \to \infty \) have been discussed. For large \( n \) the solution becomes unstable which leads to flow instability (Shown in the graph).

Keywords: Power-law fluid; heat generation/absorption; Runge-Kutta method

1 Introduction

The boundary-layer flow and heat transfer over a stretching surface have many industrial applications. A variety of constitutive equations have been suggested to predict the behaviour of non-Newtonian fluids in industry and engineering. Among such non-Newtonian fluids, some fluids such as commercial carboxymethyl cellulose in water, cement rock in water, napalm in kerosene, lime in water and Illinois yellow clay in water are power-law fluid. Schowalter [1] has introduced the concept of the boundary-layer in the theory of non-Newtonian power-law fluids. Acrovos et al. [2] have investigated the steady laminar flow of non-Newtonian fluids over a plate. Howell et al. [3] and Rao et al. [4] have studied the momentum and heat transfer on a continuous moving surface in a power-law fluid. Kumar and Nath [5] discussed over a continuously moving surface with a parallel free stream. Shah et al. [6] have studied exact solutions of a power-law fluid model in post treatment analysis of wire coating with linearly varying boundary temperature. Mahmoud and Mahmoud [7] have given the analytical solutions of hydromagnetic boundary-layer flow of a non-Newtonian power-law fluid past a continuously moving surface. Ishak et al. [8] investigated the steady boundary-layer flow of a power-law fluid over a flat plate in a moving fluid.

The subject of boundary-layer flow on a continuously moving surface traveling through a quiet ambient fluid is currently important in view of its relevance to a number of engineering processes. Flows due to a continuously moving surface is encountered in several processes for thermal and moisture treatment of materials, particularly in processes involving continuous pulling of a sheet through a reaction zone, as in metallurgy, in textile and paper industry, in the manufacture of polymeric sheets, sheet glass and crystalline materials. An example for a continuously moving surface is a polymer sheet or filament extruded continuously from die, or a long thread traveling between a feed roll and wind-up roll. Sakiadis [9] was the first to investigate the flow due to sheet issuing with constant speed from a slit into a fluid at rest; he has considered the problem of forced convection along an isothermal moving plate. Tsou et al. [10] studied flow and heat transfer in the boundary-layer on a continuously moving surface whereas, Soundalgekar and Murty [11] studied the heat transfer problem by assuming the plate temperature to be variable. Mahapatra et al. [12] discussed an
MHD stagnation-point flow of a power-law fluid towards a stretching surface. Hassanien et al. [13] investigated the flow and heat transfer in a power-law fluid over a non-isothermal stretching sheet with suction/injection.

The study of magnetohydrodynamic (MHD) flow of an electrically conducting fluid is of considerable interest in modern metallurgical and metal-working processes such as drawing of continuous filaments through quiescent fluids, and annealing and tinning of copper wires, the properties of the end product depend greatly on the rate of cooling involved in these processes. This type of flow has also attracted many investigators due to its application in various engineering problems such as MHD generators, nuclear reactors, geothermal energy extraction. Numerous attempts have been made to analyze the effect of transverse magnetic field on boundary-layer flow characteristics. Vajravelu and Rollins [14] studied heat transfer in an electrically conducting fluid over a stretching surface by taking into account the magnetic field. Baag et al. [15] studied the MHD flow of a viscoelastic fluid through a porous medium between infinite parallel plates with time dependent suction. Reddy et al. [16] investigated on Oldroyd-B type non-Newtonian fluid in a wedge incorporating non-uniform heat source/sink. Gireesha and his co-workers [17, 18] have studied the influence of various flow phenomena by considering the effects of buoyancy, thermal radiation, chemical reaction etc. Further, Mahanthesh and his co-workers [19–21] have developed their work on heat transfer phenomena where the flow past a stretching surface.

An analysis is performed by Dash et al. [22] to study the numerical approach to boundary layer stagnation-point flow past a stretching/shrinking sheet. Chen [23] gave an analytic solution to MHD flow and heat transfer with energy dissipation, internal heat source for viscoelastic fluid. Recently, Abel et al. [24] studied the flow and heat transfer in a power-law fluid over a stretching sheet with variable thermal conductivity and non-uniform heat source. They have not considered the flow through porous medium in spite of the importance in many fields of practical interest, such as petroleum engineering, ground water hydrology and agricultural engineering. Moreover, they have taken care of the electromagnetic force due to interaction of magnetic field and electrically conducting fluid. In MHD, the current produces Joule heating in the field, which depends on the magnetic field strength. But they have not considered the energy loss due to Joules dissipation which is also an important phenomenon affecting the flow and heat transfer processes. Further, Nandeepanavar and his co-workers [25–28] have investigated the heat transfer effects when a flow past a stretching sheet considering in various geometries.

The objective of the present study is not only to study the effect of permeability of the medium but also to account for the energy loss due to Joules dissipation. The Darcy model for flow through porous media, a linear model, which assumes the Reynolds number is small for inertia effects to be unimportant. Another aspect of the present model, as regards to flow through porous media, is to account for flow with large gradient and the curvature effect giving rise to inertial acceleration which Darcy model does not account for. Therefore, here the non-Darcy model due to Forchheimer [33] and Ergun [30] have been considered.

From mathematical model point of view, the present study embraces the nonlinear terms both in momentum equation \((-\frac{\partial p}{\partial x} + \frac{1}{2} \rho u^2)\) and energy equation \((\frac{\partial H}{\partial x} \frac{u^2}{\rho c^2})\) which contribute substantially to both flow and heat transfer phenomena. However, the energy loss due to viscous dissipation has not been taken care of assuming negligible viscous heating due to linear temperature profiles but in magneto fluid dynamics (MFD), the current produces Joule heating in the fluid, which depends on the magnetic field strength, Cramer and Pai [31]. As regard to the areas of application, molten polymers exhibit the behaviour of power-law fluid which is used as wire coating material. The incompressibility of the flow in the present study is justified due to high viscosity of the polymer.

In view of the above discussions, the present study investigates the effect of variable thermal conductivity on power-law fluid over a stretching sheet in the presence of Joule dissipation and non-uniform source/sink of heat. The flow is subjected to a transverse magnetic field normal to the plate. The Forchheimer’s extension is used to describe the fluid flow in the porous medium. Highly nonlinear momentum and heat transfer equations are solved numerically using fourth order Runge–Kutta method with shooting technique. The effects of various parameters on the velocity and temperature profiles are presented graphically. It is hoped that the results obtained from the present investigation will provide useful information for application and also serve as a complement to the previous studies.

2 Mathematical analysis

A steady, two-dimensional flow of an incompressible, electrically conducting, power-law fluid is considered over a flat stretching sheet embedded in a non-Darcian porous medium with the plane \(y = 0\) of a coordinate system (Fig-
(a) Prescribed power-law surface temperature (PST) and (b) Prescribed power-law heat flux (PHF) is adopted (Abel et al. [24]).

In case of PST the boundary conditions are

\[
\begin{align*}
&u = u_w = cx, \quad v = 0, \quad T = T_w = T_\infty + A(x/L)^{n} \quad \text{at} \quad y = 0 \\
&u \to 0, \quad T \to T_\infty, \quad \text{as} \quad y \to \infty
\end{align*}
\]

(5)

and in PHF case the boundary conditions are

\[
\begin{align*}
&u = u_w = cx, \quad v = 0, \quad -k \frac{\partial T}{\partial y} = q_w = D(x/L)^{n} \quad \text{at} \quad y = 0 \\
&u \to 0, \quad T \to T_\infty, \quad \text{as} \quad y \to \infty
\end{align*}
\]

(6)

where \( T_w \) is the stretching sheet temperature and \( T_\infty \) is the temperature far away from the stretching sheet. The thermal conductivity is assumed to vary linearly with temperature and it is of the form, \( \kappa = \kappa_\infty (1 + \varepsilon \left( \frac{T - T_\infty}{T_\infty} \right) ) \).

In order to facilitate the analysis following dimensionless variables and parameters are introduced.

\[
\begin{align*}
&X = x/L, \quad Y = \left( \frac{\rho u_w^2}{K} \right)^{\frac{1}{n+1}} \frac{y}{T}, \quad U = \frac{u}{u_w}, \\
&\theta(\eta) = \frac{T-T_\infty}{T_w-T_\infty}, \quad \alpha^* = \frac{k_\infty A^*}{K}, \quad M = \frac{\sigma B_p^2}{\rho C_p}, \quad \alpha = \frac{x}{\kappa L}, \quad k_p = \frac{y}{\kappa x}, \\
&V = \left( \frac{\rho u_w^2}{K} \right)^{\frac{1}{n+1}} \frac{y}{U}, \quad \beta^* = \frac{k_\infty B'_p}{k_\infty}, \\
&\tau_{xy} = \frac{\partial U}{\partial Y} \frac{\partial U}{\partial Y}, \quad \nabla = \frac{\partial^2 T}{\partial Y^2}, \quad \rho U_{\infty} = \frac{\rho u_w^{n+1}}{K}, \\
&\rho_{\infty} = \frac{\rho u_w^{n+1}}{K}, \quad \kappa_{\infty} = \kappa_{\infty} \nu_{\infty} \frac{\kappa_{\infty}}{\kappa_{\infty} \nu_{\infty}} \mathfrak{m}_{\infty}
\end{align*}
\]

(7)

The reference velocity is of the form \( U_0 = cL \) and

\[
T_w = T_\infty = \begin{cases} AX^\lambda & \text{in PST case} \\ \frac{d}{dL} R_{eL}^{-1/(n+1)}X^\lambda & \text{in PHF case} \end{cases}
\]

(8)

In view of (7), the equations (1), (2) and (3) transform into

\[
\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0
\]

(9)

\[
U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = \frac{\partial}{\partial Y} \left[ \frac{\partial U}{\partial Y} \right] - MU - k_p U - FU^2
\]

(10)

subject to boundary conditions

\[
\begin{align*}
&U = U_w = X, \quad V = 0, \quad \theta = 1 \quad \text{at} \quad Y = 0 \\
&U \to 0, \quad \theta \to 0, \quad \text{as} \quad Y \to \infty
\end{align*}
\]

(12)
\[ U = U_w = X, \quad V = 0, \quad \frac{\partial \psi}{\partial \psi} = \frac{\psi'(1/(1-n))}{1+n(2(1-n))} \quad \text{at } Y = 0 \quad \text{as } Y \rightarrow \infty \]

in PHF case (13)

In order to convert the partial differential equations into ordinary differential equations the following similarity transformation is adopted [24]:

\[
\psi(X, Y) = X^{2n} f(\eta), \quad \theta(X, Y) = \begin{cases} \theta(\eta) & \text{in PST} \\ \phi(\eta) & \text{in PHF} \end{cases}
\]

\[ \eta = X^{1-n} Y \] (14)

\[ \psi(x, y) \] is the stream function satisfying the continuity equation (9) such that

\[ U = \frac{\partial \psi}{\partial \psi'}, \quad V = -\frac{\partial \psi}{\partial X} \] (15)

Using the above relations a system of nonlinear ordinary differential equations with appropriate boundary conditions are obtained.

**Power-law Surface Temperature**

\[ n(-f')^{n-1}f''' - f'^2 + \left(\frac{2n}{n+1}\right) f'' - Mf' - k_p f'' - Ff'^2 = 0 \] (16)

\[ (1 + \varepsilon \theta)\theta' + P_r \left( \left(\frac{2n}{n+1}\right) f\theta - \lambda f\theta' \right) + E_c P_r Mf'^2 + P_r (1 + \varepsilon \theta)(\alpha f' + \beta \theta) + \varepsilon \theta'^2 = 0 \] (17)

\[ f = 0, f' = 1, \theta = 1 \quad \text{at } \eta = 0 \]
\[ f' \rightarrow 0, \theta \rightarrow 0 \quad \text{as } \eta \rightarrow \infty \] (18)

**Power-law Heat Flux**

\[ n(-f'^{-n})f''' - f'^2 + \left(\frac{2n}{n+1}\right) f'' - Mf' - k_p f'' - Ff'^2 = 0 \] (19)

\[ (1 + \varepsilon \phi)\phi' + P_r \left( \left(\frac{2n}{n+1}\right) f\phi - \lambda f\phi' \right) + E_c P_r Mf'^2 + P_r (1 + \varepsilon \phi)(\alpha f' + \beta \phi) + \varepsilon \phi'^2 = 0 \] (20)

\[ f = 0, f' = 1, \phi' = -1/(1 + \varepsilon \phi) \quad \text{at } \eta = 0 \]
\[ f' \rightarrow 0, \phi \rightarrow 0 \quad \text{as } \eta \rightarrow \infty \] (21)

where the primes denote differentiation with respect to \( \eta \).

\[ Pr = \frac{\rho C_p u_w X}{\kappa_x (Re_x)^2/(n + 1)}, \quad E_c = \frac{c_\varepsilon^2 L^2}{C_p (T_w - T_\infty)} \quad \text{and} \quad F = \frac{c_h X}{\sqrt{K_p}} \]

The local skin friction coefficient \( C_f \) and the local Nusselt number \( Nu_x \) at the wall are given by:

\[ C_f = -2Re_x^{-n} + 1 \left[ -f''(0)^n \right] \]

\[ Nu_x = \begin{cases} \frac{1}{Re_x^{-n} + 1 \theta'(0)} & \text{in PST} \\ \frac{1}{Re_x^{-n} + 1 \phi'(0)} & \text{in PHF} \end{cases} \] (23)

\[ \text{where } Re_x = \frac{\rho u_w^2 n x^n}{\mu}. \]

### 3 Solution of the problem

The set of coupled nonlinear governing boundary-layer equations (16) and (17) together with the boundary conditions, equation (18) are solved numerically using Runge-Kutta method along with shooting technique. First of all, higher order nonlinear differential equations (16) - (17) are converted into simultaneous nonlinear differential equations of first order and they are further transformed into initial value problem by applying the shooting technique. The resultant initial value problem is solved by employing Runge-Kutta fourth order method. The step size \( \Delta \eta = 0.001 \) is used to obtain the numerical solution with five decimal place accuracy as the criterion of convergence. From the process of numerical computation, the skin-friction coefficient, the Nusselt number which are respectively proportional to \(-f''(0)\), \(-\theta'(0)\) and \(\phi(0)\) are also sorted out and their numerical values are presented in the tables 1-4 respectively. The numerical procedure as follows:

Let, \( f = y_1, f' = y_2, f'' = y_3, \theta = y_4, \theta' = y_5 \)

Hence,

\[ f'' = y_5 = \frac{1}{n(y_1)^n+1} \left( y_2 - \frac{2n}{n+1} y_1 y_3 + (M + k_p) y_2 + F y_2^2 \right) \]

\[ \theta' = y_5' = -P_r \left( \frac{2n}{n+1} y_1 y_3 - \lambda y_2 y_5 + E_c M y_5^2 \right) + (1 + \varepsilon y_4)(\alpha y_2 + \beta y_5 + \varepsilon y_5^2) \]

with the boundary conditions for **Power-law Surface Temperature**

\[ \begin{cases} y_a(1) = 0, & y_a(2) = 1, \quad y_a(4) = 1 \\ y_b(2) = 0, & y_b(4) = 0, \end{cases} \]

Also the boundary conditions for **Power-law Heat Flux**

\[ \begin{cases} y_a(1) = 0, & y_a(2) = 1, \quad y_a(4) = -1/(1 + \varepsilon y_4) \\ y_b(2) = 0, & y_b(4) = 0, \end{cases} \]

where \( a \) and \( b \) are used for initial and boundary conditions respectively.
4 Results and discussion

In order to have a clear insight into the physics of the flow of power-law fluid, a visco-inelastic fluid, a parametric study is performed. The power-law fluid model is capable of subdividing the fluid model into three classes of fluids depending upon the power index $n < 1$, $n = 1$, and $n > 1$. Moreover, the deviation of $n$ from unity indicates the degree of deviation from Newtonian behaviour Andersson and Irgen [23]. Therefore, non-Newtonian property of the power-law fluid commensurate with decrease/increase in $n$ from unity. The main focus of the discussion lies with the contributions made by permeability parameter due to presence of porous medium and Joulian dissipation parameter which accounts for the dissipative electromagnetic energy in the form of heat.

For limiting cases the equations (16) and (17) reduce to following form.

Case I: $n \to 0$ (Shear thinning)

\[
-f'^2 - Mf' - k_p f' - Ff' = 0
\]

\[
(1+\varepsilon\theta)\theta'' - P_r Mf' + E_c P_r M f'^2 + P_r (1+\varepsilon\theta)(af' + \beta \theta) + \varepsilon \theta'^2 = 0
\]

For large $n$ (shear-thickening) the momentum equation (16) leads to inconsistency. This implies that the present non-Newtonian model is unsuitable for fluids with large deviation from Newtonian behaviour. On substituting $n=1$ in equation (16), (case of Newtonian fluid), no reduction of order of equation occurs. In other words, the present non-Newtonian visco-inelastic model does not give rise to higher order equation unlike viscoelastic model (Walters $B'$) which creates difficulty for solving the governing equations due to insufficient boundary conditions. For determining the surface criterion, the skin friction and Nusselt number are calculated. The comparison of the present results with that of Hassanien [13], Abel et al. [18], Anderson [24] and Chaim [35] are made with the help of Tables 1–4 as particular cases.

Figs. 2(a, b) and 3 exhibit the effects of $n$, $k_p$ and $M$ with two layer profiles. Fig. 2(a) shows that in the absence of magnetic field an increase in power index, exhibiting the pseudoplastic, Newtonian and dilatant properties, decreases the velocity at all points. This result agrees well with Abel et al. [18]. On careful observation it is found that in the presence of magnetic field (Fig. 2(b)), opposite behaviour is marked in the neighborhood of the plate which was not shown in [18]. This shows that due to interplay of electromagnetic force and the shear drag, the velocity increases with the increase in the value of $n$ in a few layers near the leading edge of the plate, thereafter velocity decreases. Another important observation is that for large $n$, flow instability is marked which has been discussed above analyzing the limiting cases of the governing equation. Further, another contribution is well marked due to the presence of porous medium (Fig. 3). Contribution of porous medium has two parts:

(i) $k_p$, linear Darcy model 
(ii) $F$, nonlinear Forchheimer’s contribution.

This shows that $k_p$ clearly reduces velocity for three classes of fluids ($n < 1$, $n = 1$, $n > 1$) but the effect of $F$ is not very much significant. However, on close observation it is found that $F$ also has a decreasing effect on the velocity field. To sum up, power-law fluid under study exhibits a dual property in the presence of magnetic field near and far off the leading edge. This model fails to represent the highly non linear (with large $n$) i.e. shear thick-
Table 1: Values of skin friction $f''(0)$ for various values of power-law index $n$ with $M = 0$

| $n$  | Hassanien et al. [13] | Andersson [2] | Abel et al. [18] | Present study |
|------|-----------------------|----------------|------------------|--------------|
| 0.2  | 1.9287                | 1.943695       | 1.944442         |              |
| 0.5  | 1.16524               | 1.1605         | 1.16774          | 1.167125     |
| 1    | 1.0                   | 1.0            | 1.0              | 1.000000     |
| 1.2  | 0.98737               | 0.9874         | 0.987372         | 0.987372     |
| 1.5  | 0.98090               | 0.9806         | 0.980653         | 0.98057      |

Figure 3: Horizontal velocity profiles

Figure 4: Horizontal velocity profiles

Figure 5: Horizontal velocity profiles

Figure 6: Temperature profiles (PST case)

Table 2: Values of $-\theta'(0)$ for various values of $\varepsilon$

| $\varepsilon$ | Chiam [35] | Abel et al. [18] | Present study |
|---------------|------------|------------------|--------------|
| 0             | 0.5819767  | 0.5819767        | 0.582011     |
| 0.01          | 0.5775650  | 0.5768627        | 0.577599     |
| 0.1           | 0.5411268  | 0.5406564        | 0.541162     |
| 0.2           | 0.5064329  | 0.5061888        | 0.506469     |
| 0.5           | 0.4274450  | 0.4277759        | 0.427485     |

The resistive forces such as permeability of the medium and magnetic parameter reduce the velocity individually whereas their combined effect is not that significant. Fig. 3 exhibits the velocity variation for dilatant class of fluid $n > 1$. It is observed that flow instability is marked far off the plate in the presence of porous medium.

Fig. 4 shows the effect of magnetic parameter in the presence /absence of porous medium for shear thinning and Newtonian fluid. The effect of Lorentz force due to in-
Table 3: Values of skin friction $-f''(0)$ for various values of $nM$, $F$ and $k_p$

| Parameters   | $n=0.5$ ($-f''(0)$) | $n=1.0$ ($-f''(0)$) | $n=1.5$ ($-f''(0)$) |
|--------------|----------------------|----------------------|----------------------|
| $M$          | $F = 0$, $k_p = 0$   | $F = 0$, $k_p = 0$   | $F = 0$, $k_p = 0$   |
| 0            | 1.167124 (Abel et al. [18]) | 1.0000 | 0.9807 |
| 1            | 1.97449               | 1.1412               | 1.25708              |
| 2            | 2.63558               | 1.7320               | 1.4357               |
| $F = 0$, $k_p = 10$ |                      |                      |                      |
| 0            | 6.4247                | 3.3166               | 2.4359               |
| 1            | 6.81346               | 3.4641               | 2.5185               |
| 2            | 7.19135               | 3.6055               | 2.6636               |
| $F = 1$, $k_p = 0$ |                      |                      |                      |
| 0            | 1.71381               | 1.2818               | 1.17103              |
| 1            | 2.4194                | 1.62918              | 1.3992               |
| 2            | 3.0286                | 3.6965               | 1.96004              |
| $F = 1$, $k_p = 10$ |                      |                      |                      |
| 0            | 6.68453               | 3.4153               | 2.4963               |
| 1            | 7.06609               | 3.5587               | 2.5794               |
| 2            | 7.43761               | 3.6965               | 2.9152               |

The interaction of magnetic field with conducting fluid opposes the fluid motion in both absence and presence of porous medium and it is further decreased in case of Newtonian fluid. The resistive force offered by the Lorentz force opposes the fluid motion or has a retarding effect on the velocity distribution resulted in the velocity boundary layer thickness decreases.

The speciality of Fig. 5 is to show that reducing effect of $M$ overrides the effect of other parameter $k_p$ which is clear for the profile when $M = 3.0$. From the above discussion it may be inferred that the additional body force due to porous medium characterized by permeability parameter and non-Darcian term $F$ has a decelerating effect on the velocity with a significant contribution due to permeability parameter $k_p$. The fact is very much straight forward as similar to magnetic field. The additional body force, porosity is also a resistive force and as described earlier $k_p$ augmented with high value of magnetic parameter retards the velocity profiles significantly.

Figs. 6 and 7 exhibit the variation of temperature for various values of Eckert number $E_c$ when $Pr = 3$, $M = 1$, $\varepsilon = 0.1$, $\alpha^* = -0.05 = \beta^*$, $\lambda = 1$. It is evident that $E_c$ contributes to enhance the thermal boundary-layer thickness. This is quite consistent with the present model which includes the energy loss due to Joule heating. The loss of energy resulted in increase in temperature and hence increased the thermal boundary-layer thickness. This observation agrees well with Chen [36]. Moreover, the increase in temperature occurs when fluid behaviour changes from pseudo plastic to dilatant. Another striking feature is that, when $E_c=0$, $k_p = 0$ and $F=0$, the curves IX and X of Fig. 6
represent Abel et al. [18] in the absence of porous medium and ohmic dissipation. The same remark holds good for dilatant fluid, Fig. 7.

Now, in PHF case displayed in Figs. 8 and 9 the above observation holds good in case of $k_p$, $F$ and $E_c$. To sum up, the effects of permeability of the medium and Joule heating increase the temperature in both cases i.e. PST and PHF. It is to note that contribution of non-Darcy term $F$ is not that significant compared with Darcian term $k_p$. The consistency and validity of the results are verified qualitatively and quantitatively with the results of Chen [36] and Abel et al. [18].

Figs. 10 and 11 exhibit the temperature variation for various values of the magnetic parameter, porous matrix and power index, $n$ in PST case. It is clear to note that both magnetic and porous matrix accelerates the temperature
Figure 9: Temperature profiles (PHF case)

Figure 10: Temperature profiles (PST case)

Figure 11: Temperature profiles (PST case)

Figure 12: Temperature profiles (PST case)

Figure 13: Temperature profiles (PST case)

Figure 14: Temperature profiles (PST case)
profiles whereas reverse effect is encountered for the increasing values of power index. All profiles exhibit fall of temperature asymptotic along the main direction of flow in both PST and PHF cases.

Figs. 12 to 14 exhibit the temperature variation when the plate surface is subjected to power-law variation (PST). It is observed that an increase in permeability parameter and magnetic parameter increases the temperature but the reverse effect is observed in case of power index $n$. Thus, it may be inferred that due to resistance offered by the porous matrix and electromagnetic force, temperature increases in all three classes of fluids, but an increase in index $n$, leading to increase in shear drag, contributes to fall of temperature in all the layers. Fig. 12 shows the influence of Prandtl number on the temperature distribution. It is observed that higher Prandtl number fluid with increasing power index $n$ causes a fall in temperature. It is due to the fact that low thermal diffusivity associated with transition from pseudoplastic to dilatant. From Figs. 13 and 14...
it is interesting to note that the temperature increases with space dependent heat source, and temperature dependent heat source for \(k_p = 0 \) and \(k_p = 5\). Moreover, permeability parameter \(k_p\) increases the temperature irrespective of source/sink. It is also pointed out that temperature dependent heat source/sink affecting significantly by increasing the temperature than space dependent heat source/sink and it is further hiked in case of shear thinning inelastic fluid (\(n < 1\)).

Figs. 15 and 16 show that shear thinning contribute significantly to the rise of temperature irrespective of the presence/absence of porous medium whereas shear thickening fluid (Dilatant fluid) contributes relatively less. Further, it is to note that the increase in temperature is accelerated due to the presence of magnetic field. To sum up, the response of shear thinning fluid to magnetic field and porous medium is considerably higher than its counterpart Newtonian and dilatant fluids. The characteristics of visco-inelastic fluid is that after removing the stress, no recovery of strain energy is marked but in case of viscoelastic fluid some amount of strain energy is released so that some recovery in deformation occurs. In case of shear thinning liquid, increase in thermal boundary-layer occurs which comparatively higher temperature is marked than that of Newtonian and dilatant fluid.

One striking feature of Figs. 17, 18 and 19 is that higher Prandtl number fluid i.e. fluid with low thermal diffusivity, contributes to rise in temperature in thermal boundary-layer irrespective of the permeability of the medium and pseudoplastic and dilatant property of the fluid.

Besides the flow configuration in momentum and thermal boundary-layer, conditions on solid surface past which the flow and heat transfer take place play an important role and hence skin friction and Nusselt number are calculated and discussed as follows. Table 1 and 2 present a comparative study in case of skin friction and Nusselt number as particular case of the present analysis.

Table. 3 shows the following observations depending on the variation of skin friction:

1. Skin friction decreases with an increase in the value of power index \(n\) i.e. valid for \(n < 1\), \(n = 1\), \(n > 1\).
2. Skin friction increases with an increase in magnetic parameter for all \(n\) for \(k_p = 0\), \(k_p = 10\).
3. Contribution of non-Darcy term is to increase the skin friction for all classes of fluid.

On careful observation Table 3 reveals that the skin friction coefficient decreases with an increase in power index \(n\) representing all three classes of fluids whereas it increases with an increase in the value of magnetic parameter and permeability parameter (both Darcy and non-Darcy terms). This clearly supports our earlier observation in respect of velocity distribution i.e. the parameters \(M\), \(k_p\) and \(F\), the resistive forces reduce the velocity and hence increases the skin friction at the bounding surface. In order to reduce the skin friction, which is desirable the dilatant fluid is more suitable in reducing the skin friction.

Table-4 shows the rate of heat transfer for PST and PHF cases. One striking feature is that the Nusselt number bears a common feature in response to the variation of the index \(n\), representing shear drag and characterizing the three classes of fluids under study. The constitutive properties of the pseudoplastic, Newtonian and dilatant fluids are not affected qualitatively but quantitatively by the physical parameters governing heat transfer at the solid surface. Now, coming to the effects of individual parameters the following observations are recorded in favour of Nusselt number. The Nusselt number decreases with an increase in \(M\), \(\alpha^*\), \(\beta^*\), \(\epsilon\), \(F\) and \(E_c\) whereas it increases for \(P_r\) and \(\lambda\) in the case of PST. On the other hand the opposite effect is observed for all the parameters in the case of PHF besides \(\epsilon\) for which the effect remains same. An increase in magnetic parameter \(M\) and Joule’s heating \(E_c\) leads to decrease the Nusselt number. The decrease of Nusselt number implies the rate of heat transfer decreases at the bounding surface. Consequently, the temperature in the thermal boundary-layer increases with an increase in \(E_c\). This is in confirmation with our earlier discussion relating to Figs. 10 and 7. On the other hand, an increase in \(E_c\) increases the Nusselt number which leads to higher surface heating consequently lowering down the boundary-layer temperature distribution. This is well supported by Fig. 12.

5 Conclusions

1. The power-law fluid under study exhibits a dual property in the presence of magnetic field near and far off the leading edge.
2. The porous medium characterized by permeability parameter \(k_p\) and non Darcian term \(F\) has a decelerating effect on the velocity.
3. The dissipative term \(E_c\) contributes to enhance the thermal boundary-layer thickness.
4. The effect of Joule’s heating is to increase the temperature at all points.
5. The higher Prandtl fluid contributes to rise in temperature in thermal boundary-layer irrespective of
the permeability of the medium and pseudoplastic and dilatant property of the fluid.

6. The dilatant fluid favours significant reduction in heat source/sink

7. An increase in $P_r$ increases the Nusselt number which leads to greater surface heating consequently giving rise to thinning of thermal boundary-layer.

c a small parameter

Greek symbols

* $A^*$ space dependent heat source/sink parameter

* $B^*$ temperature dependent heat source/sink parameter

$\theta(\eta)$, $\phi(\eta)$ non-dimensional temperature parameter both
PS and PHF case

$\lambda$ temperature parameter

$\eta$ similarity variable

Nomenclature

$A^*$, $B^*$ coefficients of space and temperature dependent heat source/sink

$A$ and $D$ are constants

$c$ stretching parameter

$B_o$ magnetic field of constant strength

$c_b$ drag coefficient

$C_f$ local skin friction coefficient

$C_p$ Specific heat [kJkg$^{-1}$ K$^{-1}$]

$E_c$ Eckert number

$F$ inertia-coefficient

$g$ acceleration due to gravity [ms$^{-2}$]

$K$ consistency coefficient

$k_p$ permeability of the porous medium [m$^2$]

$k_p$ porous parameter

$\kappa$ thermal conductivity [Wm$^{-1}$ K$^{-1}$]

$L$ characteristic length

$M$ magnetic parameter

$Nu$ local Nusselt number

$n$ power law index

$P_{rL}$ uniform Prandtl number

$P_r$ generalized Prandtl number

$q''''$ non-uniform heat source/sink

$Re_s$ local Reynolds number

$T$ temperature of the fluid [K]

$T_w$ stretching sheet temperature

$T_{\infty}$ ambient temperature

$U_0$ reference velocity [ms$^{-1}$]

$u$ velocity along $x$-direction [ms$^{-1}$]

$v$ velocity along $y$-direction [ms$^{-1}$]

$x$, $y$ coordinates

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