LIFESPAN OF SOLUTIONS TO A FOURTH ORDER PARABOLIC PDE INVOLVING THE HESSIAN MODELING EPITAXIAL GROWTH

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Abstract. This paper deals with a fourth order parabolic PDE arising in the theory of epitaxial growth, which was studied in [4]. We estimated the lifespan under the blow-up conditions given in [4]. Moreover, we extend the blow-up conditions of [4] from subcritical initial energy to critical initial energy.

1. Introduction. Epitaxial growth is a technique by means of which the deposition of new material on existing layers of the same material takes place under high vacuum conditions. This technique is used in the semiconductor industry for the growth of thin films [2]. In recent years, much attention has been devoted to models of epitaxial growth (see [9, 15, 11, 10, 14, 12, 19, 20] and references therein). In this paper, we consider the following fourth order parabolic PDE modeling epitaxial growth:

\[ \begin{cases} \frac{\partial u}{\partial t} + \Delta^2 u = \det(D^2 u), & (x, y) \in \Omega, t > 0, \\ u((x, y), 0) = u_0(x, y), & (x, y) \in \Omega, \\ u((x, y), t) = u_\nu((x, y), t) = 0, & (x, y) \in \partial \Omega, t > 0, \end{cases} \]  

(1.1)

where \( \Omega \subset \mathbb{R}^2 = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\} \) is an open, bounded domain with smooth boundary \( \partial \Omega \),

\[ \begin{cases} \Delta^2 u := \Delta (\Delta u) = u_{xxxx} + u_{yyyy} + u_{xxyy} + u_{xyxx}, \\ \det(D^2 u) := \begin{vmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{vmatrix} = u_{xx}u_{yy} - u_{xy}u_{yx}, \end{cases} \]

the initial value \( u_0(x) \in W_0^{2,2}(\Omega) \) and \( \nu \) is the unit out normal vector on \( \partial \Omega \). The model in (1.1) is a special case of

\[ u_t = 2K_1 \det(D^2 u) - K_2 \Delta^2 u + \xi(x, y, t), (x, y) \in \Omega, t > 0, \]  

(1.2)

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which was derived in [6] by means of the variational formulation developed in [16]. Here, \( K_1 \) and \( K_2 \) are two positive constants, \( \xi \) is a function belonging to some Lebesgue space.

The derivation of the model (1.2) is briefly described below. Let the function \( u : \Omega \times [0, +\infty) \to \mathbb{R} \) describe the height of the growing interface at the spatial point \((x, y) \in \Omega \) at time \( t \in [0, +\infty) \). Then the model (1.2) was derived by assuming that \( u \) obeys a gradient flow equation with a forcing term

\[
  u_t = \sqrt{1 + (\nabla u)^2} \left[ -\frac{\delta V(u)}{\delta u} + \xi(x, y, t) \right]
\]

and minimization the functional

\[
  V(u) = \int_{\Omega} \left( K_1 H + \frac{K_2}{2} \right) \sqrt{1 + |\nabla u|^2} \, dx \, dy,
\]

where \( H \) is the mean curvature of the graph of \( u \), in the context of non-equilibrium statistical mechanics of surface growth [16], see [6] for details.

There are two types of boundary conditions were considered [6]: the first one is Dirichlet boundary condition:

\[
  u((x, y), t) = u_\nu((x, y), t) = 0, (x, y) \in \partial \Omega, t > 0,
\]

and the second one is Navier boundary condition:

\[
  u((x, y), t) = \Delta u((x, y), t) = 0, (x, y) \in \partial \Omega, t > 0.
\]

Since the following equality

\[
  \int_{\Omega} \phi \text{det} \left( D^2 \phi \right) \, dx \, dy = 3I(\phi)
\]

with

\[
  I(\phi) := \int_{\Omega} \phi_x \phi_y \phi_{xy} \, dx \, dy
\]

only holds for \( \phi \in W_0^{2,2}(\Omega) \) (see [4] or [5]), as said in [5], problem (1.2) with Dirichlet boundary condition has a variational formulation, then problem (1.2) admits an energy functional and it can be dealt with by potential well method (see [4]).

However, for the Navier boundary condition, as said in [4] (see the open question of [4]: other boundary conditions), (1.2) is no longer of variational type, it is not clear whether an energy functional can be defined and if the same proofs of the paper [4] may be applied.

Based on the above reasons, as in the Section 4 of [4], we consider (1.2) with Dirichlet boundary condition, and assume \( K_1 = 1/2, K_2 = 1, \xi = 0 \) for simplicity, i.e., the problem (1.1), and the well-posedness, conditions on global existence and finite time blow-up of solutions to (1.1) were studied in [4].

In this paper, we will continue to study the model on the basis of [4]. Let’s firstly introduce some notations, sets and functionals given in [4]. For \( p \in [1, +\infty) \), we denote by \( \|\cdot\|_p \) the \( L^p \)-norm. The norm of \( W^{m,p}(\Omega) \) is denoted by \( \|\cdot\|_{W^{m,p}(\Omega)} \) except for the \( W_0^{2,2} \)-norm, which is denoted by \( \|\cdot\|_{W_0^{2,2}} \). By the Poincaré’s inequality, one can easily to see \( \|\cdot\|_{W_0^{2,2}} \) is equivalent to \( \|\Delta(\cdot)\|_2 \), i.e., there exists two positive constants \( C_1 \) and \( C_2 \) such that

\[
  C_1 \|\Delta \phi\|_2 \leq \|\phi\|_{W_0^{2,2}} \leq C_2 \|\Delta \phi\|_2, \forall \phi \in W_0^{2,2}(\Omega).
\]
So, throughout the paper, we denote the norm $W^{2,2}_0(\Omega)$ by $\|\Delta(\cdot)\|_2$, and as in [4], we use the notation $\|\phi\| = \|\Delta\phi\|_2$ for simplicity.

The energy functional (i.e., the potential energy of the system in physics) related to the stationary equation of (1.1) is given by

$$J(\phi) := \frac{1}{2}\|\phi\|^2 - I(\phi), \quad \forall \phi \in W^{2,2}_0(\Omega),$$

where $I$ is defined in (1.6).

It is shown in [4, 5] that $J$ has a mountain pass geometry and that the corresponding mountain pass level is given by

$$d = \inf_{\gamma \in \Gamma} \max_{0 \leq s \leq 1} J(\gamma (s)) = \inf_{\phi \in \mathcal{N}} J(\phi),$$

where

$$\Gamma := \left\{ \gamma \in C\left([0,1], W^{2,2}_0(\Omega) \right): \gamma(0) = 0, J(\gamma(1)) < 0 \right\},$$

and $\mathcal{N}$ is the so-called Nehari manifold defined by

$$\mathcal{N} := \left\{ \phi \in W^{2,2}_0(\Omega) \setminus \{0\} : \langle J'(\phi), \phi \rangle = \|\phi\|^2 - 3I(\phi) = 0 \right\},$$

where $J': W^{2,2}_0(\Omega) \to W^{-2,2}_0(\Omega)$ is the Fréchet derivative of $J$, and for any $\phi, \psi \in W^{2,2}_0(\Omega)$,

$$\langle J'(\phi), \psi \rangle = \left. \frac{d}{dt} J(\phi + t\psi) \right|_{t=0} = \int_\Omega \Delta \phi \Delta \psi \, dx \, dy - \int_\Omega \left[ \psi_x \phi_y \phi_{xy} + \phi_x \psi_y \phi_{xy} + \phi_x \phi_y \psi_{xy} \right] \, dx \, dy.$$

By [4, Theorem 2.6], we know that $d$ can be lower bounded in terms of the best constant for embedding $W^{2,2}_0(\Omega) \hookrightarrow W^{1,4}_0(\Omega)$, namely

$$d \geq \frac{8}{27} \min_{\phi \in W^{2,2}_0(\Omega) \setminus \{0\}} \frac{\|\phi\|^4}{\|\nabla \phi\|_4^4} > 0.$$  

Moreover, by [4], $\mathcal{N}$ is an unbounded manifold (of codimension 1), which separates two regions $\mathcal{N}_\pm$ defined by

$$\mathcal{N}_+ := \left\{ \phi \in W^{2,2}_0(\Omega) : \|\phi\|^2 - 3I(\phi) > 0 \right\}$$

and

$$\mathcal{N}_- := \left\{ \phi \in W^{2,2}_0(\Omega) : \|\phi\|^2 - 3I(\phi) < 0 \right\}.$$  

Let $\lambda_1$ be the least eigenvalue of the following eigenvalue problem:

$$\begin{cases}
\Delta^2 \phi = \lambda \phi, \\
\phi(x, y) = \phi_y(x, y) = 0, \quad (x, y) \in \partial \Omega.
\end{cases}$$

By [8], we know that $\lambda_1$ is positive, simple and it can be characterized as:

$$\lambda_1 = \inf_{\phi \in W^{2,2}_0(\Omega) \setminus \{0\}} \frac{\|\phi\|^2}{\|\phi\|_2^2},$$

which implies

$$\lambda_1 \|\phi\|^2 \leq \|\phi\|^2, \quad \forall \phi \in W^{2,2}_0(\Omega).$$
We also denote by $\phi_1(x)$ the corresponding eigenfunction of $\lambda_1$ such that
\[ I(\phi_1) \geq 0, \]
then
\[ \lambda_1 \| \phi_1 \|^2 = \| \phi_1 \|^2. \]

**Remark 1.** The eigenfunction satisfying (1.16) can be chosen. In fact, if $I(\phi_1) < 0$, since $I(-\phi_1) = -I(\phi_1)$, we can use $-\phi_1$ replacing $\phi_1$.

The well-posedness of solution to problem (1.1) was studied in [4, Theorem 3.2], which can be stated as the following theorem:

**Theorem 1.1.** Assume $u_0 \in W^{2,2}_0(\Omega)$, then there exists a positive constant $\hat{T}$ such that problem (1.1) admits a unique solution in
\[ X_T := C \left( [0, \hat{T}] ; W^{2,2}_0(\Omega) \right) \cap L^2 \left( 0, \hat{T} ; W^{4,2}(\Omega) \right) \cap W^{1,2} \left( 0, \hat{T} ; L^2(\Omega) \right) \]
The solution $u$ can be extended to a maximal interval $[0, T_{\text{max}}]$ such that either
1. $T_{\text{max}} = +\infty$, i.e., the problem admits a global solution; or
2. $T_{\text{max}} < +\infty$, and $\lim_{t \to T_{\text{max}}^-} \| u(t) \| = +\infty$, i.e., the solution blows up at a finite time $T$.

The blow-up results in [4] can be formulated as the following theorem:

**Theorem 1.2.** Assume that $u_0 \in W^{2,2}_0(\Omega)$ satisfies
(i): $u_0 \in N_-$ and $J(u_0) \leq d$; or
(ii): $6J(u_0) < \lambda_1 \| u_0 \|^2$.
Then the solution of (1.1) blows up in finite time.

When blow-up occurs, the blow-up time $T_{\text{max}}$ cannot usually be computed exactly. It is therefore of great importance in practice to determine the lifespan, i.e., the upper bound of $T_{\text{max}}$ (see [1, 3, 7, 17, 18, 22] and references therein for this topic). So it is natural to study the lifespan under the blow-up conditions given in Theorem 1.2, which is done in [21, 23] for the case $J(u_0) < 0$ and Theorem 1.2 (ii) respectively. However, the lifespan was not studied when Theorem 1.2 (i) is satisfied.

The main purpose of this paper is to study the lifespan under the blow-up condition (i) of Theorem 1.2, and the main result of this paper is the following theorem:

**Theorem 1.3.** Assume that $u_0 \in W^{2,2}_0(\Omega)$ satisfies $u_0 \in N_-$ and $J(u_0) < d$. Then the lifespan of the blow-up solution got in Theorem 1.2 can be estimated by
\[ T_{\text{max}} \leq \frac{8 \| u_0 \|^2}{3(d - J(u_0))}. \] (1.18)

Next, we concern the blow-up conditions given in Theorem 1.2. For the blow-up condition (i), the subcritical case ($J(u_0) < d$) and critical case ($J(u_0) = d$) are both studied. However, for the blow-up condition (ii), only the subcritical case ($6J(u_0) < \lambda_1 \| u_0 \|^2$) was studied. Our final theorem shows the solution can blow up in finite time for the critical case $6J(u_0) = \lambda_1 \| u_0 \|^2$.

**Theorem 1.4.** The solution of problem (1.1) with $u_0 \in B$ will blow up in finite time, where
\[ B := \begin{cases} N^*, & \text{if } I(\phi_1) = 0; \\ N^* \setminus \{ \lambda \phi_1 \}, & \text{if } I(\phi_1) > 0. \end{cases} \] (1.19)
Here,
\[ \mathcal{N}^* := \left\{ \phi \in W_0^{2,2}(\Omega) \setminus \{0\} : 6J(\phi) = \lambda_1 \| \phi \|_2^2 \right\}, \quad (1.20) \]
\[ \lambda_* := \| \phi_1 \|^2 \frac{3I(\phi_1)}{\mathcal{N}(\phi_1)} \quad (1.21) \]
and \( \phi_1 \) is the eigenfunction corresponding to \( \lambda_1 \) which satisfies (1.16).

The rest of the paper is organized as follows. In Section 2, we will give some preliminaries. In Section 3, we prove Theorems 1.3 and 1.4.

2. Preliminaries. In this section, we will give some lemmas, which will be used in the proof.

Lemma 2.1. [4, Theorem 2.5] Let \( \phi \in W_0^{2,2}(\Omega) \), then the following implication hold:
(i): \( 0 < \| \phi \|^2 < 6d \Rightarrow \phi \in \mathcal{N}_+ \);
(ii): \( \phi \in \mathcal{N}_+ \), \( J(\phi) < d \Rightarrow 0 < \| \phi \|^2 < 6d \);
(iii): \( \phi \in \mathcal{N}_- \Rightarrow \| \phi \|^2 > 6d \).

Lemma 2.2. [4, Lemmas 4.1 and 4.4] If \( u = u(t) \) solves problem (1.1) then
\[ \int_0^t \| u_+(\tau) \|^2 d\tau + J(u(t)) = J(u_0) \quad (2.1) \]
and
\[ \frac{1}{2} \frac{d}{dt} \| u(t) \|^2 + \| u(t) \|^2 - 3I(u(t)) = 0. \quad (2.2) \]

Lemma 2.3. [4, Lemma 4.2] Let \( u_0 \in W_0^{2,2}(\Omega) \) be such that \( J(u_0) < d \). Then:
(i): if \( u_0 \in \mathcal{N}_- \) the solution \( u = u(t) \) to problem (1.1) satisfies \( J(u(t)) < d \) and \( u(t) \in \mathcal{N}_- \) for all \( t \in (0, T_{\max}) \);
(ii): if \( u_0 \in \mathcal{N}_+ \) the solution \( u = u(t) \) to problem (1.1) satisfies \( J(u(t)) < d \) and \( u(t) \in \mathcal{N}_+ \) for all \( t \in (0, T_{\max}) \).

Lemma 2.4. [4, Theorem 4.10] If \( u_0 \in W_0^{2,2}(\Omega) \) and let \( u(t) \) be the local solution to problem (1.1). If \( u \) is global, then the \( \omega \)-limit set \( \omega(u_0) \) consists of a solution to the following problem
\[ \begin{cases} 
\Delta^2 u = \det(D^2 u), & (x, y) \in \Omega, \\
u(x, y) = u_\nu(x, y) = 0, & (x, y) \in \partial \Omega.
\end{cases} \quad (2.3) \]
This means that there exists a solution \( \overline{u} \) to problem (2.3) and an increasing sequence \( \{ t_n \}_{n=1}^{+\infty} \) such that \( u(t_n) \to \overline{u} \) in \( W_0^{2,2}(\Omega) \) as \( n \to +\infty \). This convergence is, in fact, also in \( W^{4,2}(\Omega) \). Here
\[ \omega(u_0) = \bigcap_{t \geq 0} \left\{ u(s) : s \geq t \right\} \]
the \( \omega \)-limit set of \( u_0 \in W_0^{2,2}(\Omega) \), and the closure is taken in \( W_0^{2,2}(\Omega) \).

The next lemma is about the set \( \mathcal{N}_\ast \) defined in (1.20).

Lemma 2.5. Let \( \phi_1 \) be the eigenfunction corresponding to \( \lambda_1 \) satisfying (1.16). Then
(i): if \( I(\phi_1) = 0 \), then \( \mathcal{N}_\ast \subset \mathcal{N}_- \);
(ii): if \( I(\phi_1) > 0 \), then \( N^* \cap N = \{ \lambda_* \phi_1 \} \) and \( N^* \setminus \{ \lambda_* \phi_1 \} \subset N_- \), where \( \lambda_* \) is given in (1.21),

where \( N, N_- \) and \( N^* \) are the sets defined in (1.9), (1.12) and (1.20) respectively.

Proof. For convenience, in the following, we let

\[
J^*(\phi) := J(\phi) - \frac{\lambda_1}{6} ||\phi||^2_2, \quad \forall \phi \in W^{2,2}_0(\Omega). \tag{2.4}
\]

Firstly, we consider the case \( I(\phi_1) = 0 \). For any \( u \in N^* \), we claim that \( u \) cannot be represented by \( \lambda \phi_1 \) for some \( \lambda \in \mathbb{R} \). In fact if there exits \( \lambda \in \mathbb{R} \) such that \( u = \lambda \phi_1 \). Since \( u \in N^* \), we get \( \lambda \neq 0 \). Then it follows from \( u \in N^* \) and \( I(\phi_1) = 0 \) that

\[
0 = J^*(\lambda \phi_1) = \left( \frac{1}{2} ||\phi_1||^2 - I(\phi_1)\lambda - \frac{1}{6} ||\phi_1||^2_2 \lambda_1 \right) \lambda^2 = \frac{1}{3} ||\phi_1||^2 \lambda^2 > 0,
\]

a contradiction.

So,

\[
\lambda_1 ||u||^2_2 < ||u||^2 \quad \text{and} \quad J(u) = \frac{\lambda_1}{6} ||u||^2_2.
\]

Therefore, it follows from (1.7) and the above two relations that

\[
3I(u) = 3 \left( \frac{1}{2} ||u||^2 - J(u) \right) = 3 \left( \frac{1}{2} ||u||^2 - \frac{\lambda_1}{6} ||u||^2_2 \right)
\]

\[
> 3 \left( \frac{1}{2} ||u||^2 - \frac{1}{6} ||u||^2 \right) = ||u||^2. \tag{2.5}
\]

Then by the definition of \( N_- \) in (1.12) we get \( u \in N_- \). So \( N^* \subset N_- \).

Secondly, we consider the case \( I(\phi_1) > 0 \).

At first, we consider the set \( N^* \cap N \). By the definition of \( \lambda_* \) in (1.21), it is obvious that \( \lambda_* \phi_1 \in N \). Furthermore, it follows from (1.17), (2.4) and (1.21) that

\[
J^*(\lambda_* \phi_1) = \frac{1}{2} ||\phi_1||^2 \lambda_*^2 - I(\phi_1)\lambda_*^2 - \frac{1}{6} ||\phi_1||^2_2 \lambda_*^2
\]

\[
= \frac{1}{2} ||\phi_1||^2 \lambda_*^2 - \frac{1}{3} ||\phi_1||^2 \lambda_*^2 - \frac{1}{6} ||\phi_1||^2_2 \lambda_*^2
\]

\[
= \frac{1}{6} \lambda_*^2 (||\phi_1||^2 - \lambda_1 ||\phi_1||^2_2) = 0.
\]

So, \( \lambda_* \phi_1 \in N^* \), and then \( \lambda_* \phi_1 \in N^* \cap N \).

On the other hand, for any \( u \in N^* \cap N \), we have \( u \neq 0 \),

\[
J(u) = \frac{1}{2} ||u||^2 - I(u) = \frac{1}{6} ||u||^2 \tag{2.6}
\]

and

\[
0 = J^*(u) = J(u) - \frac{\lambda_1}{6} ||u||^2_2 = \frac{1}{6} \left( ||u||^2 - \lambda_* ||u||^2_2 \right). \tag{2.7}
\]

Then \( u \) is an eigenfunction corresponding to \( \lambda_1 \), and then there exists \( \lambda_0 \in \mathbb{R} \) such that \( u = \lambda_0 \phi_1 \). Moveover since \( u \in N \), we get \( \lambda_0 = \lambda_* \), i.e., \( u = \lambda_* \phi_1 \). So, \( N^* \cap N = \{ \lambda_* \phi_1 \} \).

Next, we consider the set \( N^* \setminus \{ \lambda_* \phi_1 \} \). Consider the function \( g(\lambda) := J^*(\lambda \phi_1) \), then by (1.17) and (2.4), we have

\[
g(\lambda) = \left( \frac{1}{2} ||\phi_1||^2 - I(\phi_1)\lambda - \frac{1}{6} ||\phi_1||^2_2 \lambda_1 \right) \lambda^2 = \left( \frac{1}{3} ||\phi_1||^2 - I(\phi_1)\lambda \right) \lambda^2.
\]
So 0 and \( \lambda_\ast \) are all roots of \( g(\lambda) = 0 \). For any \( u \in N^* \setminus \{ \lambda_\ast \phi_1 \} \), it follows from the above discussions and \( 0 \not\in N^* \) that for any \( \lambda \in \mathbb{R} \), \( u \neq \lambda \phi_1 \). So,

\[
\lambda_1 \|u\|_2^2 < \|u\|^2 \quad \text{and} \quad J(u) = \frac{\lambda_1}{6} \|u\|_2^2.
\]

Then by the same proof as (2.5), we get \( u \in N_- \). So, \( N^* \setminus \{ \lambda_\ast \phi_1 \} \subset N_- \). \( \square 

**Lemma 2.6.** [13] Suppose that \( 0 < T \leq +\infty \) and suppose a nonnegative function \( \mathcal{F}(t) \in C^2[0,T] \) satisfies

\[
\mathcal{F}''(t) \mathcal{F}(t) - (1 + \gamma)(\mathcal{F}'(t))^2 \geq 0
\]

for some constant \( \gamma > 0 \) and \( t \in [0,T] \). If \( \mathcal{F}(0) > 0 \), \( \mathcal{F}'(0) > 0 \), then

\[
T \leq \frac{\mathcal{F}(0)}{\mathcal{F}'(0)} < +\infty
\]

and \( \mathcal{F}(t) \to +\infty \) as \( t \to T^- \).

3. Proofs of the theorems.

**Proof of Theorem 1.2.** Let \( u = u(t) \) be the solution of problem (1.1) with initial value \( u_0 \in W^{2,2}_0(\Omega) \) satisfying \( u_0 \in N_- \) and \( J(u_0) < d \). By Theorem 1.2, \( u(t) \) blows up at a finite time \( T_{\max} \).

Next, we will prove (1.18) by contradiction argument. Assume (1.18) is not true, i.e.,

\[
+\infty > T_{\max} > \frac{8\|u_0\|_2^2}{3(d - J(u_0))}.
\]

Let

\[
T := \frac{1}{2} \left( T_{\max} + \frac{8\|u_0\|_2^2}{3(d - J(u_0))} \right),
\]

then we have

\[
\frac{8\|u_0\|_2^2}{3(d - J(u_0))} < T < T_{\max}, \quad (3.1)
\]

and (by Theorem 1.1)

\[
u \in X_T := C \left( [0,T]; W^{2,2}_0(\Omega) \right) \cap L^2(0,T; W^{4,2}(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega)). \quad (3.2)
\]

By Lemma 2.3 (a), \( u(t) \in N_- \) for all \( t \in [0,T] \). Then it follows from (2.2) and the definition of \( N_- \) in (1.12) that \( \|u(t)\|_2^2 \) is strictly increasing.

Consider the following functional

\[
\mathcal{F}(t) := \int_0^t \|u(\tau)\|_2^2 d\tau + (T - t)\|u_0\|_2^2 + \beta(t + \alpha)^2, \quad \forall t \in [0,T], \quad (3.3)
\]

where

\[
\alpha = \frac{4\|u_0\|_2^2}{3(d - J(u_0))}, \quad \beta = \frac{3}{2} \left( d - J(u_0) \right). \quad (3.4)
\]

Then by \( J(u_0) < d \) and (3.1), we get \( \beta > 0 \) and

\[
0 < \alpha < \frac{T}{2}. \quad (3.5)
\]

Moreover, it follows from (3.2) that

\[
\mathcal{F}(\bar{T}) = \int_0^{\bar{T}} \|u(\tau)\|_2^2 d\tau + (T - \bar{T})\|u_0\|_2^2 + \beta(\bar{T} + \alpha)^2 < +\infty, \quad \forall \bar{T} \in [0,T]. \quad (3.6)
\]
Since \( \|u(t)\|^2 \) is strictly increasing, we get
\[
\mathcal{F}'(t) = \|u(t)\|^2 - \|u_0\|^2 + 2\beta(t + \alpha) \geq 2\beta(t + \alpha) > 0, \forall t \in [0, T].
\] (3.7)

Then
\[
\mathcal{F}(t) \geq \mathcal{F}(0) = T\|u_0\|^2 + \beta \alpha^2 > 0, \forall t \in [0, T].
\] (3.8)

Since \( u(t) \in \mathcal{N} \) for all \( t \in [0, T] \), we get from Lemma 2.1 (iii) that
\[
\|u(t)\|^2 > 6d, \forall t \in [0, T].
\] (3.9)

Then, it follows from (3.7), (2.2), (1.7), (3.9) and (2.1) that
\[
\mathcal{F}''(t) = \frac{d}{dt} \left( \|u(t)\|^2 \right) + 2\beta = 2 \left( 3I(u(t)) - \|u(t)\|^2 \right) + 2\beta
= \|u(t)\|^2 - 6J(u(t)) + 2\beta \geq 6d - 6J(u_0)
+ 6 \int_0^t \|u_\tau(\tau)\|^2 d\tau + 2\beta, \forall t \in [0, T].
\] (3.10)

For any constant \( \mu > 0 \), it follows from (3.3) and (3.7) that for any \( t \in [0, T] \),
\[
\mathcal{F}(t) \mathcal{F}''(t) - \mu (\mathcal{F}'(t))^2
= \mathcal{F}(t) \mathcal{F}''(t) - 4\mu \left( \frac{1}{2} \int_0^t \frac{d}{d\tau} \|u(\tau)\|^2 d\tau + \beta(t + \alpha) \right)^2
= \mathcal{F}(t) \mathcal{F}''(t) + 4\mu \left[ \phi(t) - (\mathcal{F}(t) - (T - t)\|u_0\|^2) \left( \int_0^t \|u_\tau(\tau)\|^2 d\tau + \beta \right) \right]
\geq \mathcal{F}(t) \mathcal{F}''(t) + 4\mu \left[ \phi(t) - \mathcal{F}(t) \left( \int_0^t \|u_\tau(\tau)\|^2 d\tau + \beta \right) \right],
\] (3.11)

where
\[
\phi(t) := \left( \int_0^t \|u(\tau)\|^2 d\tau + \beta(t + \alpha)^2 \right) \left( \int_0^t \|u_\tau(\tau)\|^2 d\tau + \beta \right)
- \left( \frac{1}{2} \int_0^t \frac{d}{d\tau} \|u(\tau)\|^2 d\tau + \beta(t + \alpha) \right)^2.
\]

Let
\[
\xi := \left( \int_0^t \|u(\tau)\|^2 \right)^{1/2}, \eta := \left( \int_0^t \|u_\tau(\tau)\|^2 \right)^{1/2},
\]
then by Hölder’s inequality, we have
\[
\frac{1}{2} \int_0^t \frac{d}{d\tau} \|u(\tau)\|^2 d\tau = \int_0^t \int_\Omega u(\tau) u_\tau(\tau) dx dy d\tau \leq \int_0^t \|u(\tau)\|_2 \|u_\tau(\tau)\|_2 d\tau \leq \xi \eta.
\]

So,
\[
\phi(t) \geq (\xi^2 + \beta(t + \alpha)^2) (\eta^2 + \beta) - (\xi \eta + \beta(t + \alpha))^2
= \beta \xi^2 - 2\beta(t + \alpha) \xi \eta + \beta(t + \alpha)^2 \eta^2 = \left( \sqrt{\beta} \xi - \sqrt{\beta} \eta \right)^2 \geq 0.
\]

Then it follows from (3.11) that for any \( \mu > 0 \),
\[
\mathcal{F}(t) \mathcal{F}''(t) - \mu (\mathcal{F}'(t))^2
\geq \mathcal{F}(t) \mathcal{F}''(t) - 4\mu \mathcal{F}(t) \left( \int_0^t \|u_\tau(\tau)\|^2 d\tau + \beta \right), \forall t \in [0, T],
\] (3.12)
which, together with (3.10), implies
\[
F(t,F'(t)) - \mu(F'(t))^2 \\
\geq F(t) \left[ 6d - 6J(u_0) + (6-4\mu) \int_0^t \|u_\tau(\tau)\|_2^2 2d\tau - (4\mu - 2)\beta \right], \quad \forall t \in [0,T]. \tag{3.13}
\]

Taking \(\mu = \frac{3}{2}\), we get from above inequality and the value of \(\beta\) (see (3.4)) that
\[
F(t,F''(t) - \frac{3}{2}F'(t)^2 \geq (6d - 6J(u_0) - 4\beta) F(t) = 0, \quad \forall t \in [0,T]. \tag{3.14}
\]

Since, by (3.4) and (3.7),
\[
\hat{T} := \frac{2F(0)}{F'(0)} \leq \frac{\|u_0\|^2_2}{\beta \alpha} T + \alpha = \frac{T}{2} + \alpha < T,
\]
it follows from Lemma 2.6 that \(\lim_{t \to \hat{T}} F(t) = F(\hat{T}) = +\infty\), which contradicts (3.6).

\(\square\)

**Proof of Theorem 1.4.** According to Lemma 2.5, we know that \(u_0 \in \mathcal{N}_- \cap \mathcal{N}^*\) if \(u_0 \in \mathcal{B}\). Let \(u(t)\) be the solution whose maximum existence time is denoted by \(T_{\text{max}}\). Next we claim \(u(t) \in \mathcal{N}_-\) for \(t \in [0,T_{\text{max}}]\). In fact, if the claim is not true, then a \(t_0 \in (0,T)\) exists such that
\[
\|u(t_0)\|^2 = 3I(u(t_0)) < \|u(t)\|^2 < 3I(u(t)) \quad \text{for } t \in [0,t_0).
\]

Then by (2.2), \(\|u(t)\|_2^2\) is strictly increasing on \([0,t_0]\). Thus it follows from \(u_0 \in \mathcal{N}^*\) and (2.4) that
\[
J(u_0) = J^*(u_0) + \frac{\lambda_1}{6} \|u_0\|^2_2 = \frac{\lambda_1}{6} \|u_0\|^2_2 < \frac{\lambda_1}{6} \|u(t_0)\|^2_2. \tag{3.15}
\]

On the other hand, since \(\|u(t_0)\|^2_2 > \|u_0\|^2_2 > 0\), it follows from the definition of \(\mathcal{N}\) in (1.9) that \(u(t_0) \in \mathcal{N}\). Then by (1.7), (1.15) and (2.1), we obtain
\[
J(u_0) = \int_0^{t_0} \|u_\tau(\tau)\|_2^2 d\tau + J(u(t_0)) \geq J(u(t_0)) = \left( \frac{1}{2} - \frac{1}{3} \right) \|u(t_0)\|^2_2 \geq \frac{\lambda_1}{6} \|u(t_0)\|^2_2,
\]
which contradicts (3.15). So the claim is true, and then by (2.2), we know that \(\|u(t)\|^2_2\) is strictly increasing on \([0,T_{\text{max}}]\).

If there exists \(t_1 \in [0,T_{\text{max}}]\) such that \(J(u(t_1)) \leq d\), then it follows from Theorem 1.2 and the above claim that \(T_{\text{max}} < +\infty\). So in the following we assume \(J(u(t)) > d\) for all \(t \in [0,T_{\text{max}}]\), and prove \(T_{\text{max}} < +\infty\).

Suppose on the contrary that \(T_{\text{max}} = +\infty\), i.e., \(u(t)\) is a global solution. Then by Lemma 2.4, there exists \(\pi \in W^{2,2}_0(\Omega)\), which is the solution of problem (2.3), and an increasing sequence \(\{t_n\}_{n=1}^\infty\) such that \(u(t_n) \to \pi\) in \(W^{2,2}_0(\Omega)\) as \(n \to +\infty\). Since \(\|u(t)\|^2_2\) is strictly increasing on \([0, +\infty)\), similar to the proof of (3.15), we get
\[
J(u_0) < \frac{\lambda_1}{6} \lim_{n \to +\infty} \|u(t_n)\|^2_2 = \frac{\lambda_1}{6} \|\pi\|^2_2. \tag{3.16}
\]

In view of \(J(u_0) > d\), we get from (3.16) that \(\pi \neq 0\). Furthermore, since \(\pi\) is a solution of (2.3), by using of (1.5), we have \(\|\pi\|^2 = 3I(\pi)\). So it following the definition of \(\mathcal{N}\) in (1.9) that \(\pi \in \mathcal{N}\). Then it follows from Lemma 2.1 and (1.15) that
\[
J(u_0) \geq \lim_{n \to +\infty} J(u_n) = J(\pi) = \frac{1}{6} \|\pi\|^2_2 \geq \frac{\lambda_1}{6} \|\pi\|^2_2,
\]
which contradicts (3.16).

\(\square\)
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