Entropy of radical ideal of a tropical curve

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Abstract

The entropy of a tropical ideal is introduced. The radical of a tropical ideal consists of all tropical polynomials vanishing on the tropical prevariety determined by the ideal. We prove that the entropy of the radical of a tropical bivariate polynomial with vanishing coefficients equals zero. An example of a non-radical tropical ideal having a positive entropy is exhibited.

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Introduction

One can find the basic concepts of tropical mathematics in [5].

Let \( f = \min_{1 \leq j \leq m} \{ a_j + \sum_{1 \leq i \leq n} t_{j,i} X_i \} \) be a tropical polynomial. Consider a family of linearizations of \( f \):

\[
\min_{1 \leq j \leq m} \{ a_j + u(t_{j,1} + s_1, \ldots, t_{j,n} + s_n) \}
\]

in \( \mathbb{N}^n \) variables \( u(k_1, \ldots, k_n) \), \( 1 \leq k_1, \ldots, k_n \leq N \) for some \( N \) where \( s_1, \ldots, s_n \in \mathbb{Z} \), provided that \( 1 \leq t_{j,1} + s_1, \ldots, t_{j,n} + s_n \leq N \). Observe that if a point \( (x_1, \ldots, x_n) \in \mathbb{R}^n \) satisfies \( f \) (i.e. the minimum in \( f \) is attained at least twice [5]) then point

\[
u := \{ u(k_1, \ldots, k_n) = k_1 x_1 + \cdots + k_n x_n : 1 \leq k_1, \ldots, k_n \leq N \} \in \mathbb{R}^{N^n}
\]

satisfies the linearizations of \( f \). Denote by \( U_N \subset \mathbb{R}^{N^n} \) a tropical linear prevariety [5] of the points satisfying all the linearizations of \( f \).
We establish existence of the limit

\[ H := H(f) := \lim_{N \to \infty} \frac{\dim(U_N)}{N^n} \]

and call it the (tropical) entropy of \( f \). Evidently, \( 0 \leq H \leq 1 \). In the univariate case \( (n = 1) \) the tropical entropy was introduced and studied in [2]. In a similar way one extends the definition of the entropy \( H(I) \) to tropical ideals \( I \).

Informally speaking, the tropical entropy plays a role similar to the coefficient at \( n \)-th power of Hilbert’s polynomial of an ideal. In the classical commutative algebra this coefficient obviously vanishes for any non-zero ideal (Hilbert’s polynomial has the degree at most \( n - 1 \) being equal the dimension of the variety determined by the ideal). This is not the case in the tropical setting: the tropical entropy can be positive, an example of this phenomenon is provided in section [3].

For a tropical ideal \( I \) its tropical prevariety \( V(I) \subset \mathbb{R}^n \) consists of all tropical solutions of \( I \) (recall that \( V(I) \) is a finite union of convex polyhedra [5]). We define the radical \( \text{rad}(I) \) of a tropical ideal \( I \) as the set of all tropical polynomials vanishing on \( V(I) \). Unlike Hilbert’s strong Nullstellensatz which describes the radical of an ideal in the classical commutative algebra, the structure of the radical of a tropical ideal is more complicated. We mention also that a tropical version of Hilbert’s weak Nullstellensatz was obtained in [3].

The main result of this paper (the Theorem in section [2]) states that for a tropical bivariate polynomial \( f := \min_{1 \leq j \leq m} \{ t_{j,1}X + t_{j,2}Y \} \) with zero coefficients (so, whose prevariety is a tropical curve with a single vertex) the entropy of its radical \( H(\text{rad}(I)) = 0 \) vanishes. In [2] it was proved in the univariate case \( (n = 1) \) that \( H(f) = 0 \) iff \( f = \text{rad}(f) \) and moreover, when \( H(f) > 0 \) it holds \( H(f) \geq 1/6 \).

It would be interesting to clarify, whether one can generalize the Theorem to an arbitrary tropical polynomial \( f \)?

1 Radical of a tropical ideal and entropy

Consider a tropical polynomial

\[ f = \min_{1 \leq j \leq m} \{ a_j + \sum_{1 \leq i \leq n} t_{j,i}X_i \} \]

(1)

where \( a_j + \sum_{1 \leq i \leq n} t_{j,i}X_i \) being linear functions (tropical monomials) with integers \( t_{j,i} \), \( 1 \leq i \leq n \) and \( a_j \in \mathbb{R} \). A point \( (x_1, \ldots, x_n) \in \mathbb{R}^n \) is a tropical solution of \( f \) if the minimum in (1) is attained at least twice [5]. The set of all tropical
solutions of \( f \) is called the tropical prevariety \( V(f) \subset \mathbb{R}^n \) of \( f \). More generally, one defines the tropical prevariety \( V(I) \subset \mathbb{R}^n \) of a tropical ideal \( I \).

We define the radical \( \text{rad}(I) \) of \( I \) as the set (a tropical ideal) of all tropical polynomials vanishing on \( V(I) \). Unlike Hilbert’s strong Nullstellensatz the radical of a tropical ideal is not exhausted by extracting roots of elements of the ideal (we’ll see some examples below). We mention that a tropical version (in a dual form) of Hilbert’s weak Nullstellensatz was established in [3].

In [2] the entropy of a tropical polynomial \( f \) was introduced as follows. For an integer \( N \) consider a tropical prevariety \( U_N \subset \mathbb{R}^N \) consisting of points \( \{u(k_1, \ldots, k_n) \in \mathbb{R} : 1 \leq k_1, \ldots, k_n \leq N\} \) satisfying tropical linear equations

\[
\min_{1 \leq j \leq m} \{a_j + u(t_{j,1} + s_1, \ldots, t_{j,n} + s_n)\}
\]

over the variables \( \{u(k_1, \ldots, k_n) : 1 \leq k_1, \ldots, k_n \leq N \} \) for any vector \( (s_1, \ldots, s_n) \in \mathbb{Z}^n \), provided that \( 1 \leq t_{j,1} + s_1, \ldots, t_{j,n} + s_n \leq N, 1 \leq j \leq m \).

We call (2) the linearization of the tropical polynomial

\[
\min_{1 \leq j \leq m} \{a_j + \sum_{1 \leq i \leq n} (t_{j,i} + s_i)x_i\}. \tag{3}
\]

Observe that if a point \( x := (x_1, \ldots, x_n) \in \mathbb{R}^n \) is a solution of \( f \) then \( x \) satisfies also (3) and the point

\[
\{u(k_1, \ldots, k_n) = k_1x_1 + \cdots + k_nx_n : 1 \leq k_1, \ldots, k_n \leq N\} \in U_N
\]
due to (1), (2), (3). On the other hand, \( U_N \) can contain points not arising from tropical solution of \( f \) (we’ll see examples below).

Consider a partition of \( n \)-dimensional grid \( T_N := \{(k_1, \ldots, k_n) : 1 \leq k_1, \ldots, k_n \leq N\} \subset \mathbb{Z}^n \) with the side \( N \) into subgrids with sides \( q_1, \ldots, q_R \), respectively. Then the number of points in \( T_N \) equals \( N^n = q_1^n + \cdots + q_R^n \).

Denote by \( p_r : \mathbb{R}^N \rightarrow \mathbb{R}^{q_r}, 1 \leq r \leq R \) the projection of the coordinates from \( T_N \) onto the coordinates from the \( r \)-th subgrid. Then \( p_r(U_N) \subset U_q \), and \( U_N \subset U_{q_1} \times \cdots \times U_{q_R} \). Hence

\[
\dim(U_N) \leq \dim(U_{q_1}) + \cdots + \dim(U_{q_R}). \tag{4}
\]

Therefore, similar to the proof of Fekete’s subadditivity lemma [6] one can verify that there exists a limit

\[
H := H(f) := \lim_{N \rightarrow \infty} \dim(U_N)/N^n = \inf \dim(U_N)/N^n. \tag{5}
\]

Indeed, for any fixed \( q \) partition grid \( T_N \) into \( \lfloor N/q \rfloor^n \) subgrids equal \( T_q \) which fill grid \( T_q \lfloor N/q \rfloor \subset T_N \) and \( N^n - (q \cdot \lfloor N/q \rfloor)^n = O(N^{n-1}) \) subgrids each equal \( T_1 \). Due to (4)

\[
\dim(U_N) \leq \lfloor N/q \rfloor^n \cdot \dim(U_q) + N^n - (q \cdot \lfloor N/q \rfloor)^n.
\]
With $N$ tending to the infinity, we conclude that

$$\limsup_{N \to \infty} \frac{\dim(U_N)}{N^n} \leq \frac{\dim(U_q)}{q^n}$$

which implies (5).

Note that in [2] the entropy was defined by means of considering parallelepipeds (rather than cubes as in the present paper). One can verify that these two definitions of the entropy coincide.

We call $H(f)$ the \textit{(tropical) entropy} of $f$. More generally, one defines in a similar way the entropy $H(I)$ of a tropical ideal $I$. Clearly, if $I \subseteq I_1$ then $H(I) \geq H(I_1)$. Obviously, $0 \leq H \leq 1$. In [2] it was shown that when the support of a tropical polynomial $f$ is located in $T_R$ then $H(f) \leq 1 - 1/R^n$.

For any $N$ and a tropical polynomial $g = \min_{1 \leq j \leq l}\{h_j + \sum_{1 \leq i \leq n} b_{j,i}X_i\} \in \text{rad}(I)$ from the radical of $I$ consider its linearization $\min_{1 \leq j \leq l}\{h_j + u(b_{j,1}, \ldots, b_{j,n})\}$, provided that $1 \leq b_{j,1}, \ldots, b_{j,n} \leq N$, $1 \leq j \leq l$. Denote by $W_N \subset \mathbb{R}^{N^n}$ the set of points satisfying these linearizations for all $g \in \text{rad}(I)$.

In [4] an example of intersection of an infinite number of tropical linear prevarieties was produced being not a tropical prevariety. Therefore, it is not clear whether $W_N$ is a tropical linear prevariety. Nevertheless, we define $\dim(W_N)$ as the minimum of dimensions of tropical linear prevarieties containing $W_N$.

One can prove the existence of the limit

$$H(\text{rad}(f)) := \lim_{N \to \infty} \frac{\dim(W_N)}{N^n} = \inf \dim(W_N)/N^n$$

slightly modifying the above argument which was used to prove the existence of the limit in (5). Indeed, consider a partition of $n$-dimensional grid $T_N$ into subgrids with sides $q_1, \ldots, q_R$, respectively. Let $W_{q_r} \subset D_r$, $1 \leq r \leq R$ where for suitable tropical linear prevarieties $D_r$ hold $\dim(W_{q_r}) = \dim(D_r)$. Then $W_N \subset D_{q_1} \times \cdots \times D_{q_R}$, hence $\dim(W_N) \leq \dim(W_{q_1}) + \cdots + \dim(W_{q_R})$ which entails as above (cf. (5)) the existence of the limit in (6).

Similar to $U_\infty$ one can consider the projective limit $W_\infty \subset \mathbb{R}^{Z^n}$ of $\{W_N\}_{N<\infty}$.

\textbf{Remark 1.1} i) Is $W_N$ a tropical linear prevariety?

ii) How to describe the radical of a tropical ideal and $W_N$, $W_\infty$ explicitly?
2 Vanishing of entropy of radical of ideal of a tropical curve

In this section we prove that in case of two variables \( n = 2 \) if all the coefficients of \( f \) vanish (i.e. \( a_j = 0 \), \( 1 \leq j \leq m \), see (1)) then \( H(\text{rad}(f)) = 0 \). In section 3 an example is exhibited of a non-radical tropical ideal with a positive entropy.

**Theorem 2.1** For a tropical bivariate polynomial

\[
 f = \min_{1 \leq j \leq m} \{t_{j,1}X + t_{j,2}Y\} \tag{7}
\]

the entropy of its radical \( H(\text{rad}(f)) = 0 \) equals zero.

**Proof.** Consider Newton polygon of \( f \) being the convex hull \( P \subset \mathbb{R}^2 \) of the points \((t_{1,1}, t_{1,2}), \ldots, (t_{m,1}, t_{m,2})\) (see (7)). Let \( P \) be \( e \)-gon. Then Newton polyhedron \( \mathcal{P} \) of \( f \) is an infinite cylinder in \( \mathbb{R}^3 \) (with the coordinates \( X, Y, Z \)) such that \( \mathcal{P} = \{ (x, y, z) : (x, y) \in P, z \geq 0 \} \). The tropical prevariety \( V(f) \subset \mathbb{R}^2 \) being moreover, a tropical variety [1] (a tropical curve) consists of supporting planes to \( P \) in \( \mathbb{R}^3 \) (not containing lines parallel to the axis \( Z \)) which intersect \( \mathcal{P} \) in at least two points, and thereby, contain an edge of \( \mathcal{P} \). In fact, \( V(f) \) consists of a vertex which corresponds to the face \( P \) of \( \mathcal{P} \) together with \( e \) rays emanating from the vertex which correspond to the edges of \( \mathcal{P} \) (being simultaneously the edges of \( P \)).

Observe that if Newton polygon of a tropical polynomial with zero coefficients

\[
 g = \min_{1 \leq j \leq r} \{b_{j,1}X + b_{j,2}Y\} \tag{8}
\]

whose Newton polygon contains \( e \) edges parallel to the edges of \( P \) (and perhaps, in addition, other edges) then \( g \in \text{rad}(f) \). This differs the tropical situation from the classical commutative algebra with respect to Hilbert’s strong Nullstellensatz (providing the structure of the radical of an ideal).

To prove the Theorem it suffices to verify that \( \dim(W_N) = o(N^2) \). Moreover, we’ll prove that \( \dim(W_N) = O(N) \).

Denote by \( W_N^{(d)} \subset W_N \) the subset of points \( \{w(k_1, k_2) : 1 \leq k_1, k_2 \leq N\} \subset W_N \) such that among the values of \( w(k_1, k_2) \), \( 1 \leq k_1, k_2 \leq N \) there are at most \( d \) different. We’ll prove that \( W_N^{(d)} = \emptyset \) for \( d > c_0 \cdot N \) for an appropriate constant \( c_0 > 0 \), in other words, \( W_N = W_N^{(d)} \) for \( d = \lfloor c_0 \cdot N \rfloor \). Then \( W_N^{(d)} \) is contained in a tropical linear prevariety (i.e. in a finite union of polyhedra)

\[
 \{w(k_1, k_2) : 1 \leq k_1, k_2 \leq N, \text{ among the values of } w(k_1, k_2) \text{ are at most } d \text{ different}\}.
\]
Since the dimension of the latter set does not exceed \( d \), one can talk about upper bounds on \( \dim(W_N) \) (see section [II], and thereby we’ll prove an upper bound \( \dim(W_N) = O(N) \) (which suffices for the proof of the Theorem).

Fix a point \( w_0 = \{w(k_1, k_2) : 1 \leq k_1, k_2 \leq N\} \in W_N^{(d_0)} \) for some \( d_0 \). We describe a recursive process in the course of which it modifies (more precisely, shrinks) a polygon \( Q \). As a base of recursion we take as \( Q \) the square \( \{(x, y) : 1 \leq x, y \leq N\} \). At every step of the recursion take an edge \( E \) of (Newton) polygon \( P \). Denote by \( L := L(E) \) the line containing edge \( E \). First we move \( L \) parallel to itself outwards \( P \) until we reach a line \( L_0 := L_0(E) \) such that \( P \) and the (current) \( Q \) lie on the same side of \( L_0 \). Then we move \( L_0 \) parallel to itself in the direction towards \( P \) (let us call it for definiteness, the inwards direction) until we reach a line \( L_1 := L_1(E) \) which for the first time contains an integer point from \( Q \). Therefore, all the integer points from \( Q \) are situated in the half-plane bounded by \( L_1 \). If \( w_0(k_1, k_2) \) takes at all the integer points \( (k_1, k_2) \in L_1 \cap Q \) at most two different values, then we move further \( L_1 \) parallel to itself in the inwards direction until the resulting line \( L_2 \) reaches an integer point of \( Q \) (unless there are no other integer points in \( Q \), and the process terminates). Thus, all the integer points from \( Q \) lie either on \( L_1 \) or in the half-plane \( S \) bounded by \( L_2 \). For the next step of the recursion we shrink \( Q \) intersecting it with \( S \).

Now alternatively, we suppose that for each edge \( E \) of \( P \) the constructed above line \( L_1(E) \cap Q \) intersected with \( Q \) contains at least three integer points having different values of \( w_0 \). Choose a triple of such points for each edge (the triples may intersect). Denote by \( A \) the union over all the edges of \( P \) of these triples of points. Among the points from \( A \) choose a point \( (k_1, k_2) \) with the minimal value of \( w_0(k_1, k_2) \). Then for each edge \( E \) of \( P \) choose two integer points from \( L_1(E) \cap A \) having values of \( w_0 \) greater than \( w_0(k_1, k_2) \). The set of chosen points (including point \( (k_1, k_2) \)) denote by \( B \subset A \).

Consider a tropical polynomial \( g := \min_{(b_1, b_2) \in B} \{b_1X + b_2Y\} \). Then \( g \in rad(f) \) because for each edge \( E \) of \( P \) Newton polygon of \( g \) contains an edge parallel to \( E \). We claim that point \( w_0 \in W_N^{(d_0)} \subset W_N \) does not satisfy the linearization \( \min_{(b_1, b_2) \in B} \{w(b_1, b_2)\} \) of \( g \) (cf. (2)). Assume the contrary. We have \( w_0(k_1, k_2) = \min_{(b_1, b_2) \in B} \{w(b_1, b_2)\} \). There is a point \( B \ni (a_1, a_2) \neq (k_1, k_2) \) such that \( w_0(a_1, a_2) = w_0(k_1, k_2) \). This contradicts to the choice of \( B \).

Thus, the described above recursive process terminates only with exhausting all the integer points of grid \( T_N \). Observe that there are at most \( O(N) \) steps of the process because consecutive parallel lines \( L_1, L_2 \) (see above) contain integer points, and taking into account that there are at most \( O(N) \) lines parallel to \( E \) and intersecting \( T_N \) (a constant hidden in big "\( O \)" is determined by the denominators of the slopes of the edges of \( P \), so depends on \( t_{j,1}, t_{j,2}, 1 \leq j \leq m \)). At each step of the process at most two different values of \( w_0 \) occur. Hence \( w_0 \) has at most \( O(N) \) different values, i. e. \( d_0 = O(N) \)
which completes the proof of the Theorem. □

Remark 2.2 We have shown in the proof that \( \dim(W_N) = O(N) \). Does there exist the limit \( \lim_{N \to \infty} \dim(W_N)/N \)? If it were the case this limit would play a role of a tropical version of the leading coefficient of Hilbert’s polynomial of the radical ideal \( \text{rad}(f) \).

3 Bounds on entropy of tropical polynomial

\[
\min\{0, X, Y, X + Y\}
\]

In this section we provide more precise bounds on dimensions \( \dim(W_N) \) for tropical polynomial \( f := \min\{0, X, Y, X + Y\} \) and show that \( H(f) > 0 \) (unlike \( H(\text{rad}(f)) = 0 \) according to the Theorem). Note that Newton polygon \( P \) of \( f \) is \( 1 \times 1 \) square \( \{(x, y) : 0 \leq x, y \leq 1\} \).

Following the recursive process described in the proof of the Theorem one can observe that in case of \( f \) all the intermediate polygons in the course of the process are rectangles with horizontal and vertical sides. Moreover, horizontal sides of rectangles can be moved at most \( N \) times (as well as vertical sides). Thus, the recursive process runs in at most \( 2N \) steps. At every step at most two values of \( w_0 \) occur. Hence \( \dim(W_N) \leq 4N \).

On the other hand, to establish a lower bound on \( \dim(W_N) \), consider the following point \( \{w_\infty(x, y) = c(x) : x, y \in \mathbb{Z}\} \in W_\infty \subset \mathbb{R}^{\mathbb{Z}^2} \) from the infinite-dimensional space on \( \mathbb{Z}^2 \) for an arbitrary concave function \( c \). One can verify that \( w_\infty \) satisfies all the linearizations of \( \text{rad}(f) \). Thus, restricting \( w_\infty \) on \( N \times N \) grid \( T_N \subset \mathbb{Z}^2 \) we conclude that \( W_N \) contains a tropical linear prevariety of dimension \( N \), therefore \( \dim(W_N) \geq N \).

Another example is \( \{w_\infty(x, y) = 0, w_\infty(x, x) \geq 0 : x, y \in \mathbb{Z}, x \neq y\} \subset W_\infty \subset \mathbb{R}^{\mathbb{Z}^2} \), providing a lower bound \( \dim(W_N) \geq N + 1 \).

Finally, we show that \( H(f) > 0 \), in other words, the condition of a tropical ideal to be radical in the Theorem, is essential. Consider the following set

\[
\{u_\infty(2x, y) = 0, u_\infty(2x + 1, y) \geq 0 : x, y \in \mathbb{Z}\} \subset U_\infty \subset \mathbb{R}^{\mathbb{Z}^2}.
\]

Restricting \( u_\infty \) on grid \( T_N \) (obtaining a subset of \( U_N \subset \mathbb{R}^{N^2} \)) we get that \( \dim(U_N) \geq N \cdot \lceil N/2 \rceil \), therefore \( H(f) \geq 1/2 \).

More exotic examples of points from \( U_\infty \) one can find in [3].

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