APPLICATIONS OF THE LAURENT-STIELTJES CONSTANTS FOR
DIRICHLET L-SERIES

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Abstract. The Laurent Stieltjes constants $\gamma_n(\chi)$ are, up to a trivial coefficient, the coefficients of the Laurent expansion of the usual Dirichlet $L$-series: when $\chi$ is non principal, $(-1)^n \gamma_n(\chi)$ is simply the value of the $n$-th derivative of $L(s, \chi)$ at $s = 1$. In this paper, we give an approximation of the Dirichlet $L$-functions in the neighborhood of $s = 1$ by a short Taylor polynomial. We also prove that the Riemann zeta function $\zeta(s)$ has no zeros in the region $|s - 1| \leq 2.2093$, with $0 \leq \Re(s) \leq 1$. This work is a continuation of [24].

1. Introduction and main results

Let $\gamma_n(\chi)$ denote the $n$-th Laurent-Stieltjes coefficients around $s = 1$ of the associated Dirichlet $L$-series for a given primitive Dirichlet character $\chi$ modulo $q$. These constants are defined by

$$L(s, \chi) = \frac{\delta_{\chi}}{s - 1} + \sum_{n \geq 0} \frac{(-1)^n \gamma_n(\chi)}{n!} (s - 1)^n,$$

where $\delta_{\chi} = 1$ when $\chi$ is principal and $\delta_{\chi} = 0$ otherwise. We may regard $\zeta(s)$ as the Dirichlet $L$-functions to the principal character $\chi_0$ modulo 1. Then, we call the coefficients $\gamma_n(\chi_0) = \gamma_n$ in this series the Laurent-Stieltjes constants for the Riemann zeta function. When $\chi$ is non-principal, $(-1)^n \gamma_n(\chi)$ is simply the value of the $n$-th derivative of $L(s, \chi)$ at $s = 1$. In this case, we call these derivatives by Laurent-Stieltjes constants for the Dirichlet $L$-functions.

The interest in Laurent-Stieltjes constants has a long history, started by Dirichlet in 1837. For a nice survey on these constants see [25] or [23]. When $\chi$ is non-principal, Dirichlet produced a finite expansion for $L(1, \chi)$. Berger [3], Lerch [20], Gut [11] and Deninger [9] gave representations $\gamma_1(\chi)$ by elementary functions. In 1989, Kanemitsu [15] obtained similar results for $\gamma_n(\chi)$ with $n \geq 2$. Toyoizumi [26] and Ishikawa [12] gave explicit upper bounds for these constants.

When $\chi$ is a principal character modulo 1, Stieltjes in 1885 was the first to propose the following definition of $\gamma_n$

$$\gamma_n = \lim_{T \to \infty} \left( \frac{\sum_{m=1}^{T} (\log m)^n}{m} - \frac{(\log T)^{n+1}}{(n+1)} \right).$$

These constants have been studied by many authors, among them, Ramanujan [22], Jensen [14], Verma [27], Ferguson [10], Briggs and Chowla [6], Kluwyer [16], Zhang and Williams [28], and more recently, Adell [2], Adell and Lekuona [1], Coffey [7], [8], Knessl and Coffey [17]. The first explicit upper bound for $|\gamma_n|$ has been given by Briggs [5], that is later improved by Berndt [4] and Israilov [13]. In 1985, the theory made a huge progress via an asymptotic...
expansion produced by Matsuoka [21], for these constants. Matsuoka gave the best upper bound for $|\gamma_n|$ for $n \geq 10$. He proved that

$$|\gamma_n| \leq 10^{-4} e^{n \log \log n}.$$  

Thanks to this result, Matsuoka showed that zeta function $\zeta(s)$ has no zeros in the region $|s - 1| \leq \sqrt{2}$, with $0 \leq \Re(s) \leq 1$.

Many authors have tried to improve on the Matsuoka bound, with few success. Matsuoka’s work relied on a formula that is essentially a consequence of Cauchy’s Theorem and the functional equation. More recently, the author [24], [25] extended this formula to Dirichlet $L$-functions. We gave the following upper bound for $|\gamma_n(\chi)|$ with $1 \leq q < \frac{\pi e^{(n+1)/2}}{2^{n+1}}$.

**Theorem 1.** Let $\chi$ be a primitive Dirichlet character to modulus $q$. Then, for every $1 \leq q < \frac{\pi e^{(n+1)/2}}{2^{n+1}}$ and $n \geq 2$, we have

$$\frac{|\gamma_n(\chi)|}{n!} \leq q^{-1/2} C(n, q) \min \left(1 + D(n, q), \frac{\pi^2}{6}\right),$$

with

$$C(n, q) = 2\sqrt{2} \exp \left\{-(n+1) \log \theta(n, q) + \theta(n, q) \log \left(\frac{2q\theta(n, q)}{\pi e}\right)\right\},$$

and

$$\theta(n, q) = \frac{n+1}{\log \left(\frac{2q(n+1)}{\pi}\right)} - 1,$$

$$D(n, q) = 2^{-\theta(n, q)-1} \frac{\theta(n, q) + 1}{\theta(n, q) - 1}.$$

In the case when $\chi = \chi_0$ and $q = 1$, this leads to a sizable improvement of the Matsuoka bound and of previous results. The aim of this paper is to use this result to give applications of the Laurent-Stieltjes constants. This work is a continuation of [24]. We shall show that this result enables us to approximate $L(s, \chi)$ in the neighborhood of $s = 1$ by a short Taylor polynomial. We have

**Application A.** Let $\chi$ be a primitive Dirichlet character to modulus $q$. For $N = 4 \log q$ and $q \geq 150$, we have

$$\left|L(s, \chi) - \sum_{n \leq N} \frac{(-1)^n \gamma_n(\chi)}{n!}(s-1)^n\right| \leq \frac{32.3}{q^{2.5}},$$

where $|s - 1| \leq e^{-1}$.

We also prove that

**Application B.** $\zeta(s)$ has no zeros in the region $|s - 1| \leq 2.2093$ with $0 \leq \Re(s) \leq 1$.

This result is an improvement on the Matsuoka result. In order to do this we apply the same technique used in [19] and [21] by giving the best possible choice of the radius of $|s - 1|$ in which $\zeta(s)$ has no zeros in.
2. Proofs

2.1. **Proof of Application A.** From Theorem [1] for \( n + 1 \geq 4 \log q \), we note that the function \( \theta(n, q) \) is non-decreasing function of \( n \), it follows that the function \( D(n, q) \) is decreasing function of \( \theta \). For \( n + 1 \geq 4 \log q \) and \( q \geq 150 \) we find that

\[
\theta(n, q) \geq \frac{4 \log q}{\log \left( \frac{8q \log q}{n} \right)} - 1 \geq 1.65,
\]

and

\[
D(n, q) \leq 0.65.
\]

On the other hand, we have

\[
\log \theta(n, q) + \log \frac{2q}{\pi e} \leq \log \left( \frac{2q(n+1)}{\pi e} \right).
\]

Putting \( H = 2q(n+1)/\pi \), we obtain that

\[
\theta(n, q) \left( \log \theta(n, q) + \log \frac{2q}{\pi e} \right) \leq \frac{n+1}{\log H} \log \left( \frac{H}{e} \right).
\]

For \( H \geq 1.45 \), we infer that

\[
\theta(n, q) \left( \log \theta(n, q) + \log \frac{2q}{\pi e} \right) \leq n + 1.
\]

Hence

\[
C(n, q) \leq 2\sqrt{2} \exp \left\{ -(n+1) \log \theta(n, q) + (n+1) \right\}.
\]

That is

\[
C(n, q) \leq 2\sqrt{2} \left( \frac{e}{\theta(n, q)} \right)^{n+1}.
\]

For \( n + 1 \geq N \), we have \( \theta(n, q) \geq \theta(N, q) \) and then

\[
\frac{|\gamma_n(\chi)|}{n!} \leq 3.3 \sqrt{2} \left( \frac{e}{\theta(N, q)} \right)^{n+1}.
\]

Now, we recall that

\[
L(s, \chi) = \sum_{n \geq 1} \frac{(-1)^n \gamma_n(\chi)}{n!} (s - 1)^n.
\]

Put

\[
\left| L(s, \chi) - \sum_{n \leq N-2} \frac{(-1)^n \gamma_n(\chi)}{n!} (s - 1)^{n+1} \right| = I_1,
\]

and let \( \varepsilon > 0 \) such that \( |s - 1| \leq \varepsilon \). Then, for \( n + 1 \geq N = 4 \log q \), we get

\[
I_1 \leq \sum_{n \geq N-1} \frac{|\gamma_n(\chi)|}{n!} |s - 1|^n
\]

\[
\leq 3.3 \sqrt{2} \varepsilon \sqrt{q} \sum_{n \geq N-1} \left( \frac{e \varepsilon}{\theta(N, q)} \right)^{n+1}
\]

\[
\leq 3.3 \sqrt{2} \varepsilon \sqrt{q} \left( \frac{e \varepsilon}{\theta(N, q)} \right)^N \left( 1 - \frac{1}{e \varepsilon} \right).
\]
Taking \( \varepsilon = e^{-1} \), we get
\[
I_1 \leq 3.3 \frac{e^{\sqrt{2}}}{\sqrt{q}} \left( \frac{1}{q} \cdot \frac{1}{4 \log \left( \frac{4 \log q}{\log(8q \log q / \varepsilon)} \right)} \right) \left( \frac{1}{1 - \frac{1}{1.65}} \right).
\]

For \( q \geq 150 \), we conclude that
\[
I_1 \leq \frac{32.3}{q^{2.5}}.
\]

This completes the proof.

2.2. **Proof of Application B.** For \( \chi \) is a principal Dirichlet character modulo 1, Eq (1) is rewritten as
\[
\zeta(s) = \frac{1}{s - 1} + \sum_{n \geq 0} \frac{(-1)^n}{n!} \gamma_n (s - 1)^n
\]

Multiplying both sides of this equation by \( s - 1 \), we get
\[
| (s - 1) \zeta(s) | \geq | 1 + \gamma_0 (s - 1) | - \sum_{n \geq 1} \frac{|\gamma_n|}{n!} | s - 1 |^{n+1}
\]

Put
\[
| 1 + \gamma_0 (s - 1) | - \sum_{1 \leq n \leq 11} \frac{|\gamma_n|}{n!} | s - 1 |^{n+1} = I_2.
\]

Here, the above summation is taken over \( 1 \leq n \leq 11 \), that the bound in Theorem 1 is numerically better than Matsuoka’s bound as soon as \( n \geq 11 \).

Now, let \( | s - 1 | \leq T_0 \), where \( T_0 \) is a positive real number to be chosen later such that \( | (s - 1) \zeta(s) | > 0 \). Using the fact that \( 0 \leq \Re(s) \leq 1 \), then \( I_2 \) is estimated by
\[
I_2 \geq 1 - \gamma_0 - \sum_{1 \leq n \leq 11} \frac{|\gamma_n|}{n!} T_0^{n+1}.
\]

Since the function \( \theta(n, q) \) in Theorem 1 is non-decreasing function of \( n \), it follows that the function \( D(n, 1) \) is decreasing function of \( \theta \). For \( n \geq 12 \) we find that
\[
\theta(n, 1) \geq \frac{13}{\log(26/\pi)} - 1 \geq 5.1513,
\]

and
\[
D(n, 1) \leq 0.0209.
\]

Thus, we have
\[
\log \theta(n, 1) + \log \frac{2}{\pi e} \leq \log \left( \frac{\frac{2(n+1)}{\pi e}}{\log \left( \frac{2(n+1)}{\pi} \right)} \right).
\]

Putting \( M = 2(n + 1)/\pi \), we obtain that
\[
\theta(n, 1) \log \left( \frac{2\theta(n, 1)}{\pi e} \right) \leq \frac{n + 1}{\log M} \log \left( \frac{M/e}{\log M} \right).
\]

For \( M \geq 8.2760 \), we infer that
\[
\theta(n, 1) \log \left( \frac{2\theta(n, 1)}{\pi e} \right) \leq 0.1728(n + 1).
\]
Hence, we get
\[ C(n, 1) \leq 2\sqrt{2} \left( \frac{e^{0.1728}}{\theta(n, 1)} \right)^{n+1}, \]
and then
\[ \frac{|\gamma_n|}{n!} \leq 2.8876 \left( \frac{e^{0.1728}}{\theta(n, 1)} \right)^{n+1} \leq 2.8876 \left( \frac{e^{0.1728}}{5.1513} \right)^{n+1}. \]

It follows that
\[ \sum_{n \geq 12} \frac{|\gamma_n|}{n!} |s-1|^{n+1} \leq 2.8876 \sum_{n \geq 12} \left( \frac{T_0 e^{0.1728}}{5.1513} \right)^{n+1}. \]

From Eq (4) and (5), we write
\[ |(s-1)\zeta(s)| \geq 1 - \gamma_0 - \sum_{1 \leq n \leq 11} \frac{|\gamma_n|}{n!} T_0^{n+1} - 2.8876 \sum_{n \geq 12} \left( \frac{T_0 e^{0.1728}}{5.1513} \right)^{n+1}. \]

Using numerical values of \( \gamma_n \) for \( 1 \leq n \leq 11 \) of [18], we find that the best possible choice of \( T_0 \) is 2.2093 in which
\[ |(s-1)\zeta(s)| > 0.000941198 - 0.000924993 > 0. \]

This completes the proof.

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