ASYMPTOTIC FORMULA FOR 
SUBORDINATED RANDOM WALKS

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Abstract. We consider a random walk \((S_n : n \in \mathbb{N})\) obtained from the simple random walk \((S_n : n \in \mathbb{N})\) by a discrete time version of Bochner’s subordination. Under certain conditions on the subordinator \((\tau_n : n \in \mathbb{N})\) the increments of \((S_n : n \in \mathbb{N})\) belong to the domain of attraction \(\mathcal{D}(\alpha)\) of the symmetric \(\alpha\)-stable law. We prove an asymptotic formula for the transition function of \((S_n : n \in \mathbb{N})\) similar to that of the symmetric \(\alpha\)-stable process. Whether this asymptotic formula holds for arbitrary random walk having increments in \(\mathcal{D}(\alpha)\) is very much open question at the present writing.

1. Introduction

It is well-known and due to Pólya (see [10]) that the transition density \(p_\alpha(x,t)\), \(\alpha \in (0,2)\), of the one-dimensional symmetric \(\alpha\)-stable process has the following asymptotic expression
\[
p_\alpha(x,t) \sim c_\alpha t|^{-1-\alpha}, \quad \text{as} \quad t|x|^{-\alpha} \to 0,
\]
where
\[
c_\alpha = \alpha 2^{\alpha-1} \pi^{-3/2} \Gamma\left(\frac{\alpha+1}{2}\right) \Gamma\left(\frac{\alpha}{2}\right) \sin\left(\frac{\pi \alpha}{2}\right).
\]
For multidimensional symmetric \(\alpha\)-stable process similar asymptotic formula has been found by Blumenthal and Getoor (see [6]). Both results were obtained by using two main ingredients: the method of characteristic functions and the scaling property of symmetric \(\alpha\)-stable distributions. An elegant proof of the Pólya–Blumenthal–Getoor asymptotic formula was proposed by Bendikov in the paper [2]. There the main idea was to use the fact that the symmetric \(\alpha\)-stable process is obtained from the standard Brownian motion by means of the subordination, the notion introduced in the theory of Markov semigroups by Bochner in 1949 (see [7]).

From the probabilistic point of view, the \(\alpha\)-stable process \(B^\alpha = (B^\alpha_t : t > 0)\) is obtained from the Brownian motion \(B = (B_t : t > 0)\) by setting \(B^\alpha_t = B_{\sigma^\alpha_t}\) where the subordinator \((\sigma^\alpha_t : s > 0)\) is independent of \(B\) non-decreasing Lévy process defined by
\[
E(e^{-\lambda \sigma^\alpha_s}) = e^{-s\lambda^\alpha}, \quad \lambda \geq 0,
\]
see e.g. [8, Section X.7].

From the analytical point of view, the transition density \(p_\alpha(x,t)\) of \(B^\alpha\) is obtained as a time average of the transition density \(p(x,t)\) of \(B\), i.e.
\[
p_\alpha(x,t) = \int_{[0,\infty)} p(x,s) \, d\nu_t^\alpha(s),
\]
where \(\nu_t^\alpha\) is the distribution function of the random variable \(\sigma^\alpha_t\). In particular, the minus infinitesimal generator of the process \(B^\alpha\) is \((-\Delta)^{\alpha/2}\) where \(\Delta\) is the classical Laplace operator.
The purpose of this paper is to study asymptotic behavior of transition functions of random walks driven by measures belonging to the domain of attraction of the $\alpha$-stable law. We conjecture that a discrete time and space counterpart of the formula (1) holds true. We are not able to perform the programme in such a generality, instead we consider a class of random walks obtained from finite range random walks, in particular the simple random walk, by means of discrete time subordination recently developed in [4]. In this more restrictive setting we prove the conjecture.

To illustrate the main result of the paper we consider the simple random walk $S$ in $\mathbb{Z}^r$, a discrete space and time counterpart of the Brownian motion in $\mathbb{R}^r$, and let $S^\alpha$ for $\alpha \in (0,2)$ be the $\alpha$-subordinated random walk defined via relation

$$P_\alpha = I - (I - P)^{\alpha/2},$$

where $P$ and $P_\alpha$ are the transition operator of $S$ and $S^\alpha$, respectively. From the probabilistic point of view $S^\alpha_n = S_{\tau^\alpha_n}$ where the discrete subordinator $\tau^\alpha_n$ is uniquely determined by the formula

$$E(e^{-\lambda \tau^\alpha_n}) = \left(1 - (1 - e^{-\lambda})^{\alpha/2}\right)^n, \quad \lambda \geq 0.$$

Let us denote by $p_\alpha(\cdot, n)$ the $n$th step transition function of $S^\alpha$. In Section 3 we prove the following theorem (or even more general version of it).

**Theorem.** If $\alpha \in (0,2)$ then

$$p_\alpha(x, n) \sim C_{\tau,\alpha} n \|x\|^{r-\alpha}, \quad \text{as} \quad n \|x\|^{\alpha} \to 0,$$

where

$$C_{\tau,\alpha} = 2^{\alpha/2+1} \pi^{r/2-1} \Gamma\left(\frac{\alpha}{2}\right) \frac{r+\alpha}{2} \sin\left(\frac{\pi \alpha}{2}\right).$$

Let us observe that the constant $C_{\tau,\alpha}$ has a factor $r^{r+\alpha/2}$ which is not present in the continuous case and is a consequence of the asymptotic behavior of the simple random walk on $\mathbb{Z}^r$, see [15].

The main ingredients in the proof are certain asymptotic formulas for the discrete $\alpha$-stable subordinator (see Subsection 2.3 for a version concerning $\alpha$-regular subordinators) and for the transition function of the simple random walk (see Subsection 2.1).

2. Preliminaries

2.1. Finite range random walks. Let $S = (S_n : n \in \mathbb{N})$ be a random walk driven by the probability $p$. Let $p(\cdot, n)$ be the $n$-fold convolution of $p$. We say that $S \in \mathcal{F}$ if the following properties hold

(i) $p$ is supported by a finite set $V \subset \mathbb{Z}^r$;
(ii) for each $x \in \mathbb{Z}^r$ there is $n \in \mathbb{N}$ such that $p(x, n) > 0$;
(iii) $\sum_{v \in V} p(v) v = 0$;
(iv) a set $U = \{\xi \in [-\pi, \pi]^r : |\kappa(i\xi)| = 1\}$ has finite number of elements where $\kappa$ is defined by

$$\kappa(z) = \sum_{v \in V} p(v) e^{i\langle z, v \rangle}, \quad z \in \mathbb{C}^r.$$

However, studying the transition function $p^{(n)}(x) = p(x, n)$ of $S \in \mathcal{F}$ one applies Laplace method to the characteristic function of $p$ (see for instance [13, Chapter II, Section 7], [9, Section 2.3]), all known results provide asymptotics for $p(x, n)$ valid in the range $\|x\| \leq n^{2/3}$. We emphasize that it is not sufficient for our application, as may be seen in the proof of Theorem 4. For our purpose this range is too small to prove asymptotic for subordinated random walk. A satisfactory asymptotic is provided by the recent paper by Trojan, see [14, 15].

Before we formulate the asymptotic of $p(x, n)$, we need to introduce some notation. We fix $S \in \mathcal{F}$. Let $\mathcal{M}$ be the interior of the convex hull of $V$. For each $\delta \in \mathcal{M}$, we defined a function on $\mathbb{R}^r$ by the
function if for all \( \phi \) attains its maximum. Set \( \eta = f(\delta, \phi) = f(s, \delta, \langle B_0u, u \rangle) = D_s^2 \log \kappa(s) \) and \( \|x\|^2 = \langle B_0^{-1}x, x \rangle \). Then the following properties hold true:\footnote{Theorem 1 for every \( \epsilon > 0 \) we obtain provided \( \text{dist}(\kappa) \geq \epsilon \).}

\begin{itemize}
    \item[(i)] \( \phi(\delta) = \frac{1}{2} \|\delta\|^2 (1 + \mathcal{O}(\|\delta\|)) \) at zero;
    \item[(ii)] there is \( A > 0 \) such that for all \( \delta \in \mathcal{M} \), \( A^{-1}\|\delta\|^2 \leq \phi(\delta) \leq A\|\delta\|^2 \);
    \item[(iii)] \( \nabla \phi(\delta) = \sigma \); \[ (iii) \log \kappa(s) = \frac{1}{2} \|\delta\|^2 (1 + \mathcal{O}(\|\delta\|)) \) at zero.
\end{itemize}

Now, we may state the asymptotic for \( p(x, n) \).

**Theorem 1.** There is \( \eta \geq 1 \) such that for all \( x \in \mathbb{Z}^r \) and \( n \in \mathbb{N} \), if \( p(x, n) > 0 \) then

\[ p(x, n) = (2\pi)^{-r/2} (\det B_0)^{-1/2} n^{-r/2} \exp(-n\phi(x/n)) \left( \#U + \mathcal{O}(n^{-1} \text{dist}(x/n, \partial M)^{-2\eta}) \right). \]

In particular, by Theorem 1 for every \( \epsilon > 0 \) we obtain

\[ p(x, n) = (2\pi)^{-r/2} (\det B_0)^{-1/2} n^{-r/2} \exp(-n\phi(x/n)) \left( \#U + \mathcal{O}(n^{-1}) \right) \]

provided \( \text{dist}(x/n, \partial M) \geq \epsilon \).

### 2.2. Discrete time subordination

For continuous time Markov processes, subordination is a useful procedure of obtaining new Markov processes. The later may differ very much from the original one, but its properties can be understood in terms of the old process. For example, the symmetric stable process can be obtained from the Brownian motion (see e.g. [2]).

From a probabilistic point of view, a new process \( (Y_t : t > 0) \) is obtained from the process \( (X_t : t > 0) \) by setting \( Y_t = X_{\sigma_t} \), where the subordinator \( \sigma_t \) is a non-decreasing Lévy process taking values in \((0, +\infty)\) and independent of \( (X_t : t > 0) \) (see e.g. [8, Section X.7]). From an analytical point of view, the transition function \( h_\sigma(x, B, t) \) of the new process is obtained as a time average of the transition function \( h(x, B, t) \) of the original process, i.e.

\[ h_\sigma(x, B, t) = \int_0^t h(x, B, s) \, d\nu_t(s) \]

where \( \nu_t \) is the distribution function of the random variable \( \sigma_t \). Subordination was introduced by Bochner in the context of semigroup theory (see [8, footnote, p. 347]). Formally, the minus infinitesimal generator \( B \) of the process \( (Y_t : t > 0) \) is a function of the minus infinitesimal generator \( A \) of the process \( (X_t : t > 0) \), that is, \( B = \psi(A) \) for some function \( \psi \).

A discrete time version of subordination, where the functional calculus equation \( B = \psi(A) \) serves as the defining starting point, has been considered by Bendikov and Saloff-Coste in [4]. To describe the procedure, let \( R \) be the operator of convolution by \( p \), a probability density on \( \mathbb{Z}^r \). Then \( L = I - R \) may be considered as minus the Markov generator of the associated random walk. For a proper function \( \psi \) we may define a subordinated random walk as the Markov process with the generator \( -\psi(L) \). As it has been shown in [4], the appropriate class consists of Bernstein functions.

Let us recall that a smooth and non-negative function \( \psi : [0, +\infty) \to [0, +\infty) \) is called a Bernstein function if for all \( n \in \mathbb{N} \) and \( x \geq 0 \)

\[ (-1)^n \psi^{(n)}(x) \geq 0. \]
For a Bernstein function \( \psi \) there are constants \( a, b \geq 0 \) and a measure \( \nu \) on \([0, +\infty)\) satisfying
\[
\int_0^\infty \min\{1, s\} \, d\nu(s) < \infty,
\]
and such that
\[
\psi(x) = a + bx + \int_0^\infty (1 - e^{-xs}) \, d\nu(s).
\]
Given a Bernstein function \( \psi \) with \( \psi(0) = 0 \) and \( \psi(1) = 1 \) we set
\[
c(\psi, k) = \begin{cases} 
\frac{1}{k!} \int_0^\infty t^k e^{-t} \, d\nu(t) & \text{if } k = 1, \\
0 & \text{otherwise}.
\end{cases}
\]
Then the random walk with the generator \(-\psi(L)\) has the probability density (see [4, Proposition 2.3])
\[
p_\psi(x) = \sum_{k \geq 1} p(x, k) c(\psi, k).
\]
Formula (4) has the following probabilistic interpretation. Let \((R_k : k \in \mathbb{N})\) be a sequence of independent and identically distributed integer-valued random variables also independent of \(S\) and such that \(P(R_i = k) = c(\psi, k)\). Then for \(\tau_n = R_1 + \ldots + R_n\) we have
\[
P(\tau_n = k) = \sum_{k_1 + \ldots + k_n = k} c(\psi, k_1) \cdots c(\psi, k_n).
\]
In particular, the law of \(S_{\tau_n}\) is given by
\[
p_\psi(x, n) = \sum_{k \geq n} p(x, k) P(\tau_n = k).
\]

2.3. Asymptotics related to subordinators. For a given \(S \in \mathcal{S}\) and a Bernstein function \(\psi\), the relations (5) and (6) defining the transition function \(p_\psi(x, n)\) of the subordinated random walk are rather complicated. Therefore, in the course of study, we are going to restrict ourselves by considering Bernstein functions \(\psi\) regularly varying at zero of index \(\alpha/2\), \(\alpha \in (0, 2)\).

Let us recall that a function \(f\) defined in \((0, a)\), \(a > 0\), is regularly varying of index \(\beta\) at zero if
\[
f(x) = x^\beta \ell(1/x)
\]
where \(\ell\) satisfies
\[
\lim_{x \to +\infty} \frac{\ell(\lambda x)}{\ell(x)} = 1,
\]
for each \(\lambda > 0\). The function \(\ell\) is called slowly varying at infinity. For the detailed exposition of the theory of regular variation we refer the reader to [5]. Here, we only mention Potter bounds (see [11], see also [5, Theorem 1.5.6]). If \(\ell\) varies slowly at infinity then for every \(\epsilon > 0\) and \(C > 1\) there exists \(x_0 > 0\) such that for all \(x, y \geq x_0\)
\[
\ell(y) \leq C \ell(x) \max\{y/x, x/y\}^\epsilon.
\]
The main result of this subsection is the following theorem.

**Theorem 2.** Let \(F_n(t) = P(\tau_n \leq t)\). Whenever \(n\psi(t^{-1})\) tends to zero,
\[
1 - F_n(t) \sim \frac{n\psi(t^{-1})}{\Gamma(1 - \alpha/2)}.
\]

**Proof.** The statement of the theorem may be expressed as follows: for any sequence \((t_n : n \in \mathbb{N})\) satisfying
\[
\lim_{n \to \infty} n\psi(t_n^{-1}) = 0,
\]
one has
\[
\lim_{n \to \infty} \frac{1 - F_n(t_n)}{n\psi(t_n^{-1})} = \frac{1}{\Gamma(1 - \alpha/2)}.
\]
Let us observe that the Laplace transform of the function $F_1$ is equal to 
\[ \mathcal{L}\{F_1\}(\lambda) = \lambda^{-1} (1 - \psi(1 - e^{-\lambda})). \]

Hence, for any $n \in \mathbb{N}$
\[ (10) \quad \mathcal{L}\{1 - F_n\}(\lambda) = \lambda^{-1} (1 - (1 - \psi(1 - e^{-\lambda}))^n). \]

In particular,
\[ (11) \quad \mathcal{L}\{1 - F_n\}(\lambda) = \lambda^{-1} n \psi(\lambda) (1 + O(n \psi(\lambda))]. \]

Next, we define
\[ \mathcal{F}_n(x) = \int_0^x 1 - F_n(y) \, dy. \]

for $x > 0$. Then, by (10)
\[ \mathcal{F}_n(x) = \int_0^x \mathcal{F}_n(y) \leq e \int_0^x e^{-y/x} \, d\mathcal{F}_n(y) \]
\[ \leq C \mathcal{L}(d\mathcal{F}_n)(1/x) \]
\[ \leq C x n \psi(1 - e^{-1/x}). \]

(12)

In what follows $C$ denotes an absolute positive constant which may vary from line to line.

For a given sequence $(t_n : n \in \mathbb{N})$ satisfying (8) we define a sequence of tempered distribution on $[0, \infty)$ by setting
\[ \Lambda_n(f) = \frac{1}{n t_n \psi(t_n^{-1})} \int_0^\infty f(x) \mathcal{F}_n(t_n x) \, dx \]

for any $f \in \mathcal{S}([0, \infty))$. Let us recall that the space $\mathcal{S}([0, \infty))$ consists of Schwartz functions on $\mathbb{R}$ restricted to $[0, \infty)$ and $\mathcal{S}'([0, \infty))$ consists of tempered distributions supported by $[0, \infty)$, see [16] for details.

We claim that there is $C > 0$ such that for any $f \in \mathcal{S}([0, \infty))$ and $n \in \mathbb{N}$
\[ (13) \quad \left| \int_0^\infty f(x) \Lambda_n(x) \, dx \right| \leq C \sup_{x \geq 0} \{(1 + x^2) |f(x)|\}. \]

Indeed, by (12) we have
\[ |\Lambda_n(f)| \leq \frac{C}{\psi(t_n^{-1})} \int_0^\infty x |f(x)| \psi(1 - e^{-1/(t_n x)}) \, dx. \]

To bound the right-hand side, first, we estimate
\[ \frac{1}{\psi(t_n^{-1})} \int_0^{t_n^{-1/2}} x |f(x)| \psi(1 - e^{-1/(t_n x)}) \, dx \leq C \sup_{x \geq 0} |f(x)|. \]

Let $x_n = -1/(t_n \cdot \log (1 - 1/t_n))$ and
\[ A_n(x) = t_n (1 - e^{-1/(t_n x)}). \]

For $t_n^{-1/2} \leq x \leq x_n$ we have
\[ 1 \leq A_n(x) \leq t_n^{1/2}. \]

Recall that $\psi(\lambda) = \lambda^{\alpha/2} \ell(1/\lambda)$ for some $\alpha \in (0, 2)$. Therefore, by (7), there is $N \geq 1$ such that for $n \geq N$
\[ \ell\left(A_n(x)^{-1} t_n\right) \leq \ell(t_n) A_n(x)^{(1 - \alpha/2)/2}, \]

thus
\[ \frac{\psi(A_n(x) \cdot t_n^{-1})}{A_n(x) \psi(t_n^{-1})} \leq A_n(x)^{(\alpha/2 - 1)/2} \leq 1. \]
Since \( xA_n(x) \leq 1 \) whenever \( x > 0 \), we obtain

\[
\frac{1}{\psi(t_n^{-1})} \int_{t_n^{-1}/2}^{t_n} x|f(x)|\psi(A_n(x) \cdot t_n^{-1}) \, dx \leq \int_{t_n^{-1}/2}^{t_n} x|f(x)|A_n(x) \, dx \\
\leq C \sup_{x \geq 0} \{(1 + x)^2|f(x)|\}.
\]

Finally, if \( x \geq x_n \) then \( A_n(x) \leq 1 \). Therefore, by monotonicity we get

\[
\psi(A_n(x) \cdot t_n^{-1}) \leq \psi(t_n^{-1}).
\]

Hence,

\[
\frac{1}{\psi(t_n^{-1})} \int_{x_n}^{\infty} x|f(x)|\psi(A_n(x) \cdot t_n^{-1}) \, dx \leq \int_{x_n}^{\infty} x|f(x)| \, dx \leq C \sup_{x \geq 0} \{(1 + x)^2|f(x)|\},
\]

what finishes the proof of the claim \((13)\).

Let us observe, that \((13)\) implies that the family of distributions \((\Lambda_n : n \in \mathbb{N})\) is equicontinuous. The integration by parts for \( f_r(x) = e^{-\tau x} \) yields

\[
\Lambda_n(f_r) = \frac{1}{nt_n\psi(t_n^{-1})} \int_0^{\infty} e^{-\tau x} \mathcal{F}_n(t_n, x) \, dx = \frac{1}{n\tau t_n\psi(t_n^{-1})} \mathcal{L}(\mathcal{D}F_n)(\tau/t_n),
\]

whence, by \((11)\),

\[
\lim_{n \to \infty} \Lambda_n(f_r) = r^{\alpha/2-2} = \frac{1}{\Gamma(2 - \alpha/2)} \int_0^{\infty} e^{-\tau x} x^{1-\alpha/2} \, dx.
\]

Finally, density of \( \mathcal{B} \) and equicontinuity of \((\Lambda_n : n \in \mathbb{N})\) allows us to extend the limit in \((14)\) over all of \( f \in \mathcal{S}([0, \infty)) \) giving

\[
\lim_{n \to \infty} \Lambda_n(f) = \frac{1}{\Gamma(2 - \alpha/2)} \int_0^{\infty} f(x)x^{1-\alpha/2} \, dx.
\]

For the completeness of our presentation we give a sketch of the proof that the linear span \( \mathcal{B} \) of the set \( \{f_r : \tau > 0\} \) is dense in \( \mathcal{S}([0, \infty)) \) (see e.g. [16, 17]). By the Hahn–Banach theorem it is enough to show that if \( \Lambda \in \mathcal{S}'([0, \infty)) \) and \( \Lambda(\phi) = 0 \) for all \( \phi \in \mathcal{B} \), then \( \Lambda \) is the zero functional. Assume that \( \Lambda \) vanishes on \( \mathcal{B} \) and let \( \mathcal{L}\Lambda(z) = \Lambda(e^{-z}) \), \( \text{Re} \, z > 0 \), be the Laplace transform of \( \Lambda \). Since \( \Lambda = 0 \) on \( \mathcal{B} \) we get \( \mathcal{L}\Lambda(\lambda) = 0 \) for \( \lambda > 0 \). But the Laplace transform is analytic in the half-plane \( \text{Re} \, z > 0 \) whence \( \mathcal{L}\Lambda(z) = 0 \), for \( \text{Re} \, z > 0 \). Using the connection between Laplace and Fourier transforms,

\[
\hat{\Lambda}(\xi) = \lim_{\lambda \to 0^+} \mathcal{L}\Lambda(\lambda + i\xi), \quad \text{in } \mathcal{S}'([0, \infty)),
\]

we obtain that \( \hat{\Lambda} = 0 \), i.e. \( \Lambda \) is the zero functional as desired.

Next, we claim that

\[
\lim_{n \to \infty} \frac{\mathcal{F}_n(t_n)}{nt_n\psi(t_n^{-1})} = \frac{1}{\Gamma(2 - \alpha/2)}.
\]

For a fixed \( \epsilon > 0 \), we choose \( \phi_+ \in \mathcal{S}([0, \infty)) \) such that \( 0 \leq \phi_+ \leq 1 \) and

\[
\phi_+(x) = \begin{cases} 
1 & \text{for } 0 \leq x \leq 1, \\
0 & \text{for } 1 + \epsilon \leq x.
\end{cases}
\]

Then we have

\[
\mathcal{F}_n(t_n) = \int_0^{t_n} d\mathcal{F}_n(s) \leq \int_0^{t_n} \phi_+(s/t_n) \, d\mathcal{F}_n(s) \leq \int_0^{\infty} \phi_+(s/t_n) \, d\mathcal{F}_n(s).
\]
Hence, by the integration by parts we get
\[ \frac{F_n(t_n)}{nt_n \psi(t_n^{-1})} \leq -\Lambda_n (\phi'_+). \]

Therefore, we may estimate
\[
\limsup_{n \to \infty} \frac{F_n(t_n)}{nt_n \psi(t_n^{-1})} \leq \frac{-1}{\Gamma(2 - \alpha/2)} \int_0^\infty s^{1-\alpha/2} \phi'_+(s) \, ds
\]
\[
= \frac{1}{\Gamma(1 - \alpha/2)} \int_0^\infty s^{-\alpha/2} \phi'_+(s) \, ds \leq \frac{1}{\Gamma(2 - \alpha/2)} (1 + \epsilon)^{1-\alpha/2}.
\]

Similarly, by taking \( \phi_- \in \mathcal{S}([0, \infty)), 0 \leq \phi_- \leq 1 \) satisfying
\[
\phi_-(x) = \begin{cases} 
1 & \text{for } 0 \leq x \leq 1 - \epsilon, \\
0 & \text{for } 1 \leq x,
\end{cases}
\]
one can show that
\[
\liminf_{n \to \infty} \frac{F_n(t_n)}{nt_n \psi(t_n^{-1})} \geq \frac{1}{\Gamma(2 - \alpha/2)} (1 - \epsilon)^{1-\alpha/2}.
\]

Since \( \epsilon \) was arbitrary, we conclude (15).

To prove (9) we follow the line of reasoning from [5, Theorem 1.7.2]. Let \( \epsilon > 0 \). The function \( 1 - F_n(s) \) is non-increasing therefore
\[
F_n(t) - F_n((1 - \epsilon)t) = \int_{(1-\epsilon)t}^t (1 - F_n(s)) \, ds \geq \epsilon t (1 - F_n(t))
\]
and
\[
F_n((1 + \epsilon)t) - F_n(t) = \int_t^{(1+\epsilon)t} (1 - F_n(s)) \, ds \leq \epsilon t (1 - F_n(t)).
\]

By (15),
\[
\lim_{n \to \infty} \frac{F_n((1 - \epsilon)t_n)}{nt_n \psi(t_n^{-1})} = \frac{(1 - \epsilon)^{1-\alpha/2}}{\Gamma(2 - \alpha/2)}.
\]

Hence, (16) implies
\[
\limsup_{n \to \infty} \frac{1 - F_n(t_n)}{n \psi(t_n^{-1})} \leq \frac{1}{\Gamma(2 - \alpha/2)} \frac{1 - (1 - \epsilon)^{1-\alpha/2}}{\epsilon}.
\]

Similarly, by (17) we get
\[
\liminf_{n \to \infty} \frac{1 - F_n(t_n)}{n \psi(t_n^{-1})} \geq \frac{1}{\Gamma(2 - \alpha/2)} \frac{(1 + \epsilon)^{1-\alpha/2} - 1}{\epsilon}.
\]

Finally, by taking \( \epsilon \) tending to zero we obtain (9). \( \square \)

2.4. The domain of attraction. Let us recall that a distribution \( \mu \) belongs to the domain of attraction \( \mathcal{D}(\alpha) \) of the \( \alpha \)-stable law \( G_\alpha \) if for every \( n \in \mathbb{N} \) there are a vector \( a_n \in \mathbb{R}^r \) and a constant \( b_n > 0 \) such that
\[
\left( \sum_{j=1}^n X_j - a_n \right)/b_n
\]
converges in distribution to \( G_\alpha \) where \( (X_n : n \in \mathbb{N}) \) is a sequence of independent random variables with distribution \( \mu \). The main result of this subsection is the following theorem.

**Theorem 3.** The law of \( S_1^\psi \) belongs to the domain of attraction \( \mathcal{D}(\alpha) \) of the symmetric \( \alpha \)-stable law \( G_\alpha \).
We start with two elementary lemmas. Let $\Phi$ and $\Phi_\psi$ denote the characteristic functions of $p$ and $p_\psi$, respectively.

**Lemma 1.** For all $\xi \in \mathbb{R}^r$

$$\Phi_\psi(\xi) = 1 - \psi(1 - \Phi(\xi)).$$

*Proof.* Notice that $\Phi(\xi)$ may take complex values, thus we need to use the holomorphic extension of the Bernstein function $\psi$. For any $z \in \mathbb{C}$ such that $\text{Re} \, z \geq 0$ we may write (see [12, Proposition 3.5])

$$\psi(z) = a + bz + \int_0^\infty (1 - e^{-zt}) \, d\nu(t).$$

The function $\psi(z)$ is continuous for $\text{Re} \, z \geq 0$ and holomorphic for $\text{Re} \, z > 0$. Thus, for any $z \in \mathbb{C}$ such that $\text{Re} \, z \leq 1$, we have

$$1 - \psi(1 - z) = 1 - b(1 - z) - \int_0^\infty (1 - e^{-t(1-z)}) \, d\nu(t)$$

$$= bz + \int_0^\infty e^{-t} \sum_{n \geq 1} \frac{t^n z^n}{n!} \, d\nu(t)$$

$$= bz + \sum_{n \geq 1} \frac{z^n}{n!} \int_0^\infty e^{-t} t^n \, d\nu(t)$$

$$= \sum_{n \geq 1} c(\psi, n) z^n$$

where the coefficients $c(\psi, n)$ were defined by (3). On the other hand

$$\Phi_\psi(\xi) = \sum_{x \in \mathbb{Z}^r} p_\psi(x) e^{i\xi x} = \sum_{x \in \mathbb{Z}^r} e^{i\xi x} \sum_{k \geq 1} p(x, k) c(\psi, k)$$

$$= \sum_{k \geq 1} c(\psi, k) (\Phi(\xi))^k,$$

what finishes the proof. \hfill \Box

**Lemma 2.** For $r > 0$ and $\text{Re} \, z \geq 0$,

$$|\psi(r(1 + z)) - \psi(r)| \leq |z| \psi(r).$$

In particular, we have

$$\psi(r(1 + i\epsilon)) = \psi(r)(1 + O(\epsilon)).$$

*Proof.* From the representation of the Bernstein function we get

$$\psi(r(1 + z)) - \psi(r) = brz + \int_0^\infty e^{-rt} (1 - e^{-rzt}) \, d\nu(t).$$

Since $|1 - e^{-z}| \leq |z|$ for $\text{Re} \, z \geq 0$, we obtain

$$|\psi(r(1 + z)) - \psi(r)| \leq br|z| + |z| \int_0^\infty rte^{-rt} \, d\nu(t)$$

$$\leq br|z| + |z| \int_0^\infty (1 - e^{-rt}) \, d\nu(t) = |z| \psi(r),$$

where we used the inequality $e^x > 1 + x$. \hfill \Box
Proof of Theorem 3. Let \( a_n \equiv 0 \). For \( (b_n : n \in \mathbb{N}) \) we choose a sequence of real numbers such that

\[
\lim_{n \to +\infty} n \psi(b_n^{-2}) = 1.
\]

We are going to show that

\[
\lim_{n \to +\infty} E(e^{\xi(S_n^\psi-a_n)/b_n}) = \exp\{-2^{-\alpha/2} B_0(\xi, \xi)^{\alpha/2}\},
\]

where the quadratic form \( B_0 \) is defined in Proposition 2.1. We have

\[
E(e^{\xi S_n^\psi/b_n}) = (\Phi_\psi(\xi/b_n))^n.
\]

Since \( E(S_1) = 0 \), we obtain

\[
\Phi(\xi) = 1 - \frac{B_0(\xi, \xi)}{2} + O(\|\xi\|^2),
\]

thus, for any \( \xi \in \mathbb{R}^r \)

\[
1 - \Phi(\xi/b_n) = \frac{B_0(\xi, \xi)}{2b_n^2} \left(1 + O(b_n^{-1})\right).
\]

By Lemma 2, we have

\[
\psi(1 - \Phi(\xi/b_n)) = \psi(B_0(\xi, \xi)/(2b_n^2)) \left(1 + O(b_n^{-1})\right).
\]

Hence, by Lemma 1

\[
\log \Phi_\psi(\xi/b_n) = \log \left(1 - \psi(1 - \Phi(\xi/b_n))\right)
\]

\[
= -\psi(1 - \Phi(\xi/b_n)) \left(1 + O(\psi(1 - \Phi(\xi/b_n)))\right)
\]

\[
= -\psi(B_0(\xi, \xi)/(2b_n^2)) \left(1 + O(b_n^{-1}) + O(\psi(B_0(\xi, \xi)/(2b_n^2)))\right).
\]

Since

\[
\lim_{n \to +\infty} \frac{\psi(B_0(\xi, \xi)/(2b_n^2))}{\psi(b_n^{-2})} = 2^{-\alpha/2} B_0(\xi, \xi)^{\alpha/2},
\]

by (18), we conclude

\[
\lim_{n \to +\infty} n \log \Phi_\psi(\xi/b_n) = -2^{-\alpha/2} B_0(\xi, \xi)^{\alpha/2}.
\]

3. Transition function asymptotics

Let \( S_n^\psi \) be the \( \psi \)-subordinated random walk defined in Section 2. In this section we find the asymptotic of the transition density of \( S_n^\psi \).

Theorem 4. If \( n \psi(\|x\|^{-2}) \) tends to zero then

\[
p_\psi(x, n) \sim C_{r, \alpha, n} \|x\|^{-\alpha} \psi(\|x\|^{-2}),
\]

where \( \|x\| = \langle B_0^{-1} x, x \rangle \), and

\[
C_{r, \alpha} = \alpha 2^{\alpha/2} \pi^{-r/2-1} (\det B_0)^{-1/2} \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{r + \alpha}{2}\right) \sin \left(\frac{\pi \alpha}{2}\right) \#U.
\]

Proof. Let us recall that \( \psi \) varies regularly of index \( \alpha/2 \) at zero, i.e. \( \psi(\lambda) = \lambda^{\alpha/2} \ell(1/\lambda) \) for \( \alpha \in (0, 2) \). Given \( k_0 > 0 \), we consider a sum

\[
S_1(x, n) = \sum_{k=1}^{k_0} p(x, k) \mathbb{P}(\tau_n = k).
\]

Using Gaussian bounds of Alexopoulos [1, Theorem 1.8],

\[
p(x, k) \leq C k^{-r/2} \exp\{-2C_0^\prime \|x\|^2/k\},
\]

where \( C = \frac{\alpha}{2} \).
we estimate
\[ S_1(x, n) \leq C_0 ||x||^{-r} \exp \left\{ - C_0' ||x||^2 / k_0 \right\} \sum_{k=1}^{k_0} ||x||^r \exp \left\{ - C_0' ||x||^2 / k \right\} P(\tau_n = k) \]
\[ \leq C_1 ||x||^{-r} \exp \left\{ - C_0' ||x||^2 / k_0 \right\}. \]
For \( x \in \mathbb{Z}^r \) and \( n \in \mathbb{N} \), we define
\[ k_0 = \frac{\min \{ C_0', A^{-1} \}}{4} \frac{||x||^2}{\log \{ n\psi(||x||^{-2}) \}} \]
where \( A \) is the constant from Proposition 2.1(ii). Then we have
\[ S_1(x, n) = o \left( n ||x||^{-r} \psi \left( ||x||^{-2} \right) \right). \]
**Claim 1.** There are \( C > 0 \) and \( R > 0 \) such that for all \( n \in \mathbb{N} \) and \( x \in \mathbb{Z}^r \), if \( ||x|| \geq R \) then
\[ k_0 \geq C \|x\|^2 / \log ||x||, \]
\[ n\psi(k_0^{-1}) \leq C (n\psi(||x||^{-2}))^{(1-\alpha)/2}. \]
Assuming for a moment the validity of Claim 1, we proceed with the proof of the theorem. We need to find the asymptotic of a sum
\[ S_2(x, n) = \sum_{k > k_0} p(x, k) P(\tau_n = k). \]
To do so we apply Theorem 1 (see Trojan [15]). Since (21) implies that for \( x \in \mathbb{Z}^r \) such that \( ||x|| \) is sufficiently large,
\[ \frac{||x||}{k_0} \leq \frac{1}{2 \sqrt{r}}, \]
by (2), there is \( C > 0 \) such that
\[ \left| p(x, k) - (2\pi)^{-r/2} k^{-r/2} (\det B_0)^{-1/2} \# U \cdot k^{-r/2} \exp \{ -k\phi(x/k) \} \right| \leq C k^{-r/2-1} \exp \{ -k\phi(x/k) \} \]
for all \( k > k_0 \). Then, by setting
\[ I(x, n) = (2\pi)^{-r/2} (\det B_0)^{-1/2} \# U \sum_{k \geq k_0} k^{-r/2} \exp \{ -k\phi(x/k) \} P(\tau_n = k), \]
we obtain
\[ |S_2(x, n) - I(x, n)| \leq C \sum_{k \geq k_0} k^{-r/2-1} \exp \{ -k\phi(x/k) \} P(\tau_n = k) \]
\[ \leq C_1 k_0^{-1} I(x, n). \]
Next, for \( G_n(t) = 1 - P(\tau_n \geq t) \) we can write
\[ I(x, n) = (2\pi)^{-r/2} (\det B_0)^{-1/2} \# U \int_{[k_0, \infty)} t^{-r/2} \exp \{ -t\phi(x/t) \} \ dG_n(t). \]
Let
\[ I(x, n) = (2\pi)^{-r/2} (\det B_0)^{-1/2} \# U \int_{k_0}^{\infty} \frac{d}{dt} \left( t^{-r/2} \exp \{ -t\phi(x/t) \} \right) G_n(t) \ dt. \]
The integration by parts together with estimates (ii) from Proposition 2.1 yields
\[ |I(x, n) + I(x, n)| \leq C_1 k_0^{-r/2} \exp \{ -k_0\phi(x/k_0) \} \]
\[ \leq C_2 ||x||^{-r} \exp \{ -A^{-1} ||x||^2 / k_0 \} = o \left( ||x||^{-n} \psi \left( ||x||^{-2} \right) \right), \]
where in the last step we have used the definition of \(k_0\). To find the asymptotic behavior of \(I(n, x)\) we compute

\[
\frac{d}{dt}(t^{-r/2}\exp\{-t\phi(x/t)\}) = -\frac{r}{2}t^{-r/2-1}\exp\{-t\phi(x/t)\} - t^{-r/2}\exp\{-t\phi(x/t)\}\frac{d}{dt}(t\phi(x/t)).
\]

In view of (23), we have \(x/t \in \mathcal{M}\), for \(t \geq k_0\). Therefore, using the notation from Proposition 2.1 we write \(s_t = s(x/t)\). Thus

\[
\frac{d}{dt}(t\phi(x/t)) = \phi(x/t) - \langle \nabla \phi(x/t), x/t \rangle = -\log \kappa(s_t).
\]

Next, let us define two integrals

\[
J_1 = \frac{r}{2}\int_{k_0}^{\infty} t^{-r/2-1}\exp\{-t\phi(x/t)\}G_u(t)\, dt,
\]

\[
J_2 = \int_{k_0}^{\infty} t^{-r/2}\exp\{-t\phi(x/t)\}\log \kappa(s_t)G_u(t)\, dt.
\]

Then, by (26), we have

\[
I(x, u) = (2\pi)^{-r/2}(\det B_0)^{-1/2}#U(J_2 - J_1).
\]

To show the asymptotic of \(J_1\) and \(J_2\) we need the following lemma.

**Lemma 3.** For any \(\beta > 1\)

\[
\int_{k_0}^{\infty} t^{-\beta} \ell(t) \exp\{-t\phi(x/t)\} \, dt \sim 2^{\beta-1}\Gamma(\beta - 1)\|x\|^{-2(\beta-1)}\ell(\|x\|^2),
\]

as \(n\psi(\|x\|^{-2})\) tends to zero.

**Proof of Lemma 3.** First, we change variables by setting \(u = \|x\|^{-2}t\)

\[
\int_{k_0}^{\infty} t^{-\beta} \ell(t) \exp\{-t\phi(x/t)\} \, dt = \|x\|^{-2(\beta-1)}\ell(\|x\|^2) \int_{\|x\|^{-2}k_0}^{\infty} u^{-\beta} \frac{\ell(\|x\|^2u)}{\ell(\|x\|^2)} \exp\left\{ -u\|x\|^2\phi(x\|x\|^{-2}u^{-1}) \right\} \, du.
\]

Proposition 2.1(ii) together with (7) implies

\[
u^{-\beta} \frac{\ell(\|x\|^2u)}{\ell(\|x\|^2)} \exp\left\{ -u\|x\|^2\phi(x\|x\|^{-2}u^{-1}) \right\} \leq C' u^{-\beta} \max\left\{ u^{-\frac{2r+1}{2}}, u^{-\frac{1}{2}} \right\} \exp\left\{ -C' u^{-1} \right\}.
\]

We observe that, by Proposition 2.1(i), for any \(u > 0\)

\[
\lim_{\|x\| \to +\infty} u\|x\|^2\phi(x\|x\|^{-2}u^{-1}) = 2^{-1} u^{-1},
\]

thus applying the dominated convergence we obtain

\[
\int_{\|x\|^{-2}k_0}^{\infty} u^{-\beta} \frac{\ell(\|x\|^2u)}{\ell(\|x\|^2)} \exp\left\{ -u\|x\|^2\phi(x\|x\|^{-2}u^{-1}) \right\} \, du \sim \int_{0}^{\infty} u^{-\beta} \exp\left\{ -2^{-1} u^{-1} \right\} \, du = 2^{\beta-1}\Gamma(\beta - 1),
\]

as \(n\psi(\|x\|^{-2})\) → 0, and the proof of the lemma is finished. □
Now, we return to showing the asymptotic of $J_1$ and $J_2$. For $J_1$, by inequality (22), when $t \geq k_0$ we have
\[ n\psi(t^{-1}) \leq C(n\psi(\|x\|^{-2}))^{(1-\alpha/2)/2}, \]
whence, applying Theorem 2, we obtain
\[ J_1 \sim \frac{nr}{2^{\Gamma(1-\alpha/2)}} \int_{k_0}^{\infty} t^{-r/2-\alpha/2-1} \ell(t) \exp\{-t\phi(x/t)\} \, dt, \]
as $n\psi(\|x\|^{-2}) \to 0$. Therefore, by Lemma 3 we get
\[ J_1 \sim \frac{r \cdot 2^{r+2}}{2^{\Gamma(1-\alpha/2)}} \cdot \frac{\Gamma((r+\alpha)/2)}{\Gamma(1-\alpha/2)} \cdot n\|x\|^{-r} \psi(\|x\|^{-2}). \]
In the case of $J_2$, by Proposition 2.1(iv), we get
\[ \log \kappa(s) = \frac{\|x\|^2}{2t^2} \left(1 + O(\|x\|/t)\right), \]
whence
\[ J_2 = \frac{\|x\|^2}{2} \int_{k_0}^{\infty} t^{-r/2-2} \exp\{-t\phi(x/t)\} G_n(t) \, dt \leq C\|x\|^3 \int_{k_0}^{\infty} t^{-r/2-3} \exp\{-t\phi(x/t)\} G_n(t) \, dt, \]
for some absolute constant $C > 0$. Therefore, by Lemma 3, we obtain
\[ J_2 \sim \frac{r \cdot 2^{r+2}}{2^{\Gamma(1-\alpha/2)}} \cdot \frac{\Gamma((r+\alpha)/2)}{\Gamma(1-\alpha/2)} \cdot n\|x\|^{-r} \psi(\|x\|^{-2}), \]
as $n\psi(\|x\|^{-2})$ approaches zero. Putting (27) and (28) together we conclude
\[ I(x, n) \sim \alpha 2^{\alpha/2} \pi^{-r/2-1} (\det B_0)^{-1/2} \#U \sin \left( \frac{\pi \alpha}{2} \right) \Gamma \left( \frac{\alpha}{2} \right) \Gamma \left( \frac{r+\alpha}{2} \right). \]
Finally, estimates (20), (24), (25) and (29) yield the desired asymptotic for $p_\psi(x, n)$.

To finish the proof of the theorem we need to justify Claim 1. First, by (7), there is $R > 0$ such that for $\|x\| \geq R$ and $n \in \mathbb{N}$
\[ n\psi(\|x\|^{-2}) \geq \psi(\|x\|^{-2}) \geq \|x\|^{-3\alpha}. \]
Hence
\[ -\log \left(n\psi(\|x\|^{-2})\right) \leq 3\alpha \log \|x\|, \]
what easily implies (21). For the proof of (ii), again using (7), there are $R > 0$ and $C > 1$ such that for all $n \in \mathbb{N}$ and $\|x\| \geq R$
\[ \ell(k_0) \leq C\ell(\|x\|^2) \left( -\log \left(n\psi(\|x\|^{-2})\right) \right)^{(1-\alpha/2)/2}. \]
Therefore we obtain
\[ n\psi(k_0^{-1}) = nk_0^{-\alpha/2} \ell(k_0) \leq Cn\psi(\|x\|^{-2}) \left( -\log \left(n\psi(\|x\|^{-2})\right) \right)^{(1+\alpha/2)/2} \]
\[ \leq C \left(n\psi(\|x\|^{-2})\right)^{(1-\alpha/2)/2}, \]
and the proof of Claim 1 is finished. \[\square\]

Remark 1. The next step of the study would be to apply Theorem 4 in order to find an asymptotic formula for the Green function
\[ G_\psi(x) = \sum_{n \geq 0} p_\psi(x, n). \]
Clearly, formula (19) does not provide enough information to solve the task. Under certain special assumptions on $\psi$ this problem was solved in the recent paper of Bendikov and Cygan [3]. In this connection we also mention here the closely related paper of Williamson [18].
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