IMPROVED STABILITY FOR ODD-DIMENSIONAL ORTHOGONAL GROUP.

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Abstract. We compute the kernel of the stabilization map for $K_1$-functors modeled on split Chevalley groups of types $B_l, C_l, E_l$ one step below the stable range. For the groups of type $B_l$ this implies early injective stability for $K_1(B_l, R)$ over a certain class of rings.

1. Introduction

Let $R$ be a commutative ring with a unit and $\Phi_l$ be a reduced irreducible root system of rank $l > 1$. Denote by $K_1(\Phi_l, R) = G(\Phi_l, R)/E(\Phi_l, R)$ the quotient of the simply connected Chevalley group of type $\Phi_l$ over $R$ by the elementary subgroup. Following [15], we call this group the value of unstable $K_1$-functor modeled on Chevalley group of type $\Phi_l$ on a ring $R$. An embedding of root systems $\Psi \hookrightarrow \Phi$ induces a map between the corresponding groups $\theta^K_{\Psi \hookrightarrow \Phi} : K_1(\Psi, R) \to K_1(\Phi, R)$.

Finding conditions on $R$ sufficient for the injectivity or surjectivity of $\theta$ is a classical problem, which dates back to H. Bass’ paper [3]. In the case $A_{l-1} \hookrightarrow A_l$ such conditions are stated in terms of the stable rank of $R$ (see [2], [18]). More precisely, the map $\theta^K_{A_{l-1} \hookrightarrow A_l}$ is surjective when $sr(R) \leq l$ and is injective when $sr(R) \leq l - 1$. The case of a general $\Phi$ has been exhaustively studied by M. Stein in [15].

It is natural to attempt to find additional assumptions on $R$ which imply bijectivity of $\theta^K_{\Psi \hookrightarrow \Phi}$ below the stable range, i.e. in the situation when it is not possible to apply the stability theorems of Bass, Vaserstein and Stein directly. For example, in [11] it has been shown that the map $\theta^K_{A_{l-1} \hookrightarrow A_l}$ is bijective, when $R$ is a nonsingular affine algebra of dimension $d$ over a perfect $C_1$-field and $l \geq d + 1$.

On the other hand, one may attempt to describe the generators of the kernel of $\theta^K_{\Psi \hookrightarrow \Phi}$ explicitly. Let us state (the absolute case) of the main result of [7], the so-called “prestabilization” theorem.

Denote by $\bar{E}(n, R)$ the normal closure (inside $GL(n, R)$) of the subgroup spanned by the subgroup $E(n, R)$, the mixed commutator subgroup $[E(n, R), GL(n, R)]$ (which is not contained in $E(n, R)$ only for $n = 2$), and the generators of the form $(e_n + xy)(e_n + yx)^{-1}$, where $y = \text{diag}(\xi, 1, \ldots, 1)$, $\xi \in R$, $x \in M(n, R)$ are such that $e_n + xy \in GL(n, R)$.

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Theorem 1.1. Let \( R \) be a commutative ring and \( \max(sr(R), 2) \leq n \). Then one has
\[
\text{GL}(n, R) \cap E(n + 1, R) = \tilde{E}(n, R).
\]
In other words, the kernel of \( \theta^K_{A_{n-1} \to A_n} \) is generated by \( \tilde{E}(n, R) / E(n, R) \).

The main purpose of the present article is to obtain an analogue of this theorem for split simple Chevalley groups of type \( B_l, C_l, E_l \).

Fix a basis of simple roots \( \Pi = \{ \alpha_1, \ldots, \alpha_l \} \) for \( \Phi_l \). The simple roots are numbered as in [5]. Denote by \( \Delta_i \) the subsystem of \( \Phi_l \) spanned by all simple roots except the \( i \)-th one.

Set \( i = l, j = 1 \) for \( \Phi_l = B_l, C_l \) and \( i = 1, j = l \) for \( \Phi_l = E_l \). We use this notation throughout the rest of the article. Clearly, the subsystems \( \Delta_i \) are of type \( A_{l-1} \) in the case \( \Phi_l = B_l, C_l \) and of type \( D_{l-1} \) in the case \( \Phi_l = E_l \). On the other hand, the subsystems \( \Delta_j \) have the same type as \( \Phi_l \) (except for \( \Phi_l = B_2, C_2, E_6 \)).

Theorem 1.2. Let \( R \) be a commutative ring. Assume that one of the following holds:

1. \( \Phi_l = B_l, C_l \) and \( \max(sr(R), 2) \leq l - 1 \);
2. \( \Phi_l = E_l, l = 6, 7, 8 \) and \( \text{asr}(R) \leq l - 2 \).

Consider the following diagram of unstable \( K_1 \)-groups induced by the natural embeddings of root systems.

\[
\begin{array}{c}
\text{Ker}(\theta_1) \downarrow \quad \text{K}_1(\Delta_i \cap \Delta_j, R) \downarrow \quad \text{K}_1(\Delta_i, R) \\
\theta_2' \downarrow \\
\text{Ker}(\theta'_1) \downarrow \quad \text{K}_1(\Delta_j, R) \downarrow \quad \text{K}_1(\Phi_l, R)
\end{array}
\]

Then the map \( \theta_2' \) is surjective.

Denote by \( \theta \) the map \( G(A_{l-1}, R) \hookrightarrow G(\Phi_l, R) \) induced by the embedding \( A_{l-1} = \Delta_l \hookrightarrow D_l \) of root systems. From theorems 1.1–1.2 and the main result of [11] one can immediately deduce the following statement.

Corollary 1.3.

1. Assume that \( \max(2, sr(R)) \leq l \) and \( \Phi_l = B_l, C_l \). Then
\[
G(B_l, R) \cap E(B_{l+1}, R) = \theta(\tilde{E}(l, R)) \cdot E(B_l, R).
\]

2. Assume that \( R \) is a nonsingular algebra of dimension \( d \geq 2 \) over a perfect \( C_1 \)-field. Then the map \( K_1(B_{d+1}, R) \to K_1(B_{d+2}, R) \) is an isomorphism.

The definition of \( C_1 \)-field can be found in [13, 3.2 Ch. II]. In fact, the main result of [11] (and hence the above statement) holds under some weaker assumptions on \( R \) (see [11, Prop. 3.1]).

Another consequence of the Theorem 1.2 is the following result which establishes a connection between stability problems for the embeddings \( D_5 \hookrightarrow E_6 \) and \( D_6 \hookrightarrow E_7 \).

Corollary 1.4. Let \( R \) be a commutative noetherian ring such that \( \dim \text{Max} \leq 4 \).
(1) There exists the following commutative diagram with exact rows.

\[
\begin{array}{ccc}
\text{Ker}(\theta_1) & \xrightarrow{\theta_1^2} & \text{K}_1(\text{D}_5, R) \\
\downarrow & & \downarrow \\
\text{Ker}(\theta_1^2) & \xrightarrow{\theta_2} & \text{K}_1(\text{E}_6, R) \\
& & \downarrow \theta_2 \\
& & \text{K}_1(\text{E}_7, R)
\end{array}
\]

(2) The map \(\text{Ker}(\theta_2) \to \text{Ker}(\theta_1')\) induced by \(\theta_1\) is an epimorphism.

(3) The map \(\text{K}_1(\text{E}_6, R)/\text{Im}(\theta_2) \cong \text{K}_1(\text{E}_7, R)/\text{Im}(\theta_1')\) induced by \(\theta_1'\) is an isomorphism of pointed sets.

**Proof.** Under the above assumption on \(R\) the maps \(\theta_1, \theta_1'\) are epimorphisms by [15, Cor. 3.2], [9, Th. 1]. Statements 2–3 follow from the nonabelian snake lemma. \(\square\)

The proof of our main results follows [15] and is essentially based on the calculations with elementary root unipotents and so-called “stable” calculations in the representations of Chevalley groups, i.e. the calculations with the highest weight vector.

**2. Principal notation**

By commutator \([x, y]\) of elements \(x, y\) we always mean the left-normed commutator \(xyx^{-1}y^{-1}\). We denote by \(x^*\) the conjugate element \(y^{-1}xy\).

For any collection of subsets \(H_1, \ldots, H_n\) of a group \(G\) we denote by \(H_1 \cdots H_n\) their Minkowski set-product, i.e. the set consisting of arbitrary products \(h_1 \cdots h_n\) of elements \(h_i \in H_i\). In particular, the equality \(G = H_1 \cdots H_n\) means that every element \(g \in G\) can be presented as a product \(h_1 \cdots h_n\) for \(h_i \in H_i\).

We denote by \(H \ltimes N\) the semidirect product of groups \(H\) and \(N\) such that \(N\) is a normal subgroup in \(H \ltimes N\).

**2.1. Chevalley groups and Steinberg groups.** Our treatment of Steinberg groups, Chevalley groups and their representations follows [10], [20], [21].

Let \(\Phi\) be a reduced irreducible system of rank \(l\) and let \(\Pi = \{\alpha_1, \ldots, \alpha_i\}\) be some fixed basis of simple roots of \(\Phi\). Denote by \(\Phi^+\) and \(\Phi^-\) the subsets of positive and negative roots of \(\Phi\) with respect to \(\Pi\). Denote by \(m_r(\alpha)\) the \(r\)-th coefficient of the expansion of \(\alpha\) in \(\Pi\), i.e. \(\alpha = \sum r m_r(\alpha)\alpha_r\). Obviously, the condition \(m_r(\alpha) = 0\) is equivalent to \(\alpha \in \Delta_r\).

Denote by \(G(\Phi, -)\) the simply connected Chevalley—Demazure group scheme with the root system \(\Phi\), and by \(E(\Phi, -)\) its elementary subfunctor. For \(l \geq 2\) Taddei’s normality theorem (see [17, Th. 0.3]) asserts that \(E(\Phi, R) \leq G(\Phi, R)\).

The elementary subgroup is generated by the **elementary root unipotents** \(t_\alpha(\xi)\) for all \(\alpha \in \Phi, \xi \in R\). These elements satisfy the Steinberg relations:

\[
(2.1) \quad t_\alpha(s)t_\alpha(t) = t_\alpha(s + t),
\]

\[
(2.2) \quad [t_\alpha(s), t_\beta(t)] = \prod t_{p\alpha + q\beta}(N_{\alpha, \beta, p, q}s^{p}t^{q}), \quad \alpha \neq -\beta.
\]

In the above formula \(\alpha, \beta \in \Phi, s, t \in R\), and \(p, q\) run over all positive integers such that \(p\alpha + q\beta \in \Phi\). The constants \(N_{\alpha, \beta, p, q}\) are small integers which can be explicitly computed and only depend on \(\Phi\).
The Steinberg group $\text{St}(\Phi, R)$ is defined by the generators $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$ and relations 2.1, 2.2 with the $t_\alpha(\xi)$’s replaced by $x_\alpha(\xi)$’s. Basic properties of these groups are discussed in [14].

We denote by $X_\alpha$ the root subgroup corresponding to the root $\alpha$, i.e. the subgroup consisting of the root unipotents $x_\alpha(\xi)$, $\xi \in R$.

An embedding of root systems $\Psi \subseteq \Phi$ induces the natural transformations

$$\theta_{\Psi \hookrightarrow \Phi} : G(\Psi, -) \to G(\Phi, -), \quad \theta_{\Psi \hookrightarrow \Phi}^{\text{St}} : \text{St}(\Psi, -) \to \text{St}(\Phi, -).$$

Notice that the maps $\theta_{\Psi \hookrightarrow \Phi}^{\text{St}}(R)$ are not injective in general.

For $1 \leq r \leq l$ and $s \neq r$ consider the following subgroups of $\text{St}(\Phi, R)$:

$$\hat{P}_r(\Phi_l, R) = \{x_\alpha(\xi), m_r(\alpha) \geq 0\}, \quad \hat{L}_r(\Phi_l, R) = \{x_\alpha(\xi), m_r(\alpha) = 0\},$$

$$\hat{U}_r(\Phi_l, R) = \{x_\alpha(\xi), m_r(\alpha) > 0\}, \quad \hat{U}_r^-(\Phi_l, R) = \{x_\alpha(\xi), m_r(\alpha) < 0\},$$

$$\hat{U}_r(\Phi_l, R) = \hat{U}_r \cap \hat{U}_s^-.\]$$

Clearly $\hat{L}_r(\Phi_l, R)$ normalizes both $\hat{U}_r(\Phi_l, R)$ and $\hat{U}_r^-(\Phi_l, R)$, hence $\hat{P}_r(\Phi_l, R)$ admits the Levi decomposition:

$$\hat{P}_r(\Phi_l, R) = \hat{L}_r(\Phi_l, R) \ltimes \hat{U}_r(\Phi_l, R).$$

Denote by $\hat{U}(\Phi, R)$ (respectively, by $\hat{U}^-(\Phi, R)$) the subgroup spanned by $x_\alpha(\xi)$ for $\xi \in R$ and $\alpha \in \Phi^+$ (respectively, $\alpha \in \Phi^-$). If the choice of $\Phi_l$ is clear from the context, we shorten the notation for $\hat{Z}(\Phi_l, R)$ to $Z$, where

$$Z = G, E, \hat{U}, \hat{U}^-, \hat{P}_r, \hat{L}_r, \hat{U}_r, \hat{U}^-, \hat{U}_r^- \hat{U}_s^-.$$

### 2.2. Representations of Chevalley groups

In the present article we work with the irreducible fundamental representations of $G(\Phi_l, R)$ corresponding to the highest weight $\varpi_j$, i.e. $\varpi_1$ in the case of a classical $\Phi_l$ and $\varpi_l$ in the case $\Phi_l = E_l$. The former are the natural vector representations of the classical groups acting on the free modules $V = R^n$ of dimension $n = 2l, 2l + 1, 2l, 2l$ for $\Phi = A_l, B_l, C_l, D_l$ respectively. The latter act on the free modules $V = R^n$ of dimension $n = 27, 56, 248$ for $l = 6, 7, 8$ respectively. All these representations are basic in the sense of [10]. Moreover, all the representations except for $(B_l, \varpi_1)$ and $(E_8, \varpi_8)$ are microweight, i.e. they do not have zero weights.

For classical groups we use the standard numbering of the weights of representations (cp. [15, §1B]). In particular, the coordinates of the elements of $V$ are indexed as follows:

- $1, 2, \ldots, l + 1$ in the case $\Phi = A_l$,
- $1, 2, \ldots, l, 0, -l, \ldots, -2, -1$ in the case $\Phi = B_l$,
- $1, 2, \ldots, l, -l, \ldots, -2, -1$ in the cases $\Phi = C_l, D_l$.

As for exceptional groups, for our purposes it suffices to refer explicitly the weights of the representations $(E_l, \varpi_l)$, arising from subrepresentation $(D_{l-1}, \varpi_1)$ corresponding to the same highest weight $\varpi_l$. For such weights we keep the natural numbering introduced above. Our partial numbering of the weights in the case $(E_6, \varpi_6)$ is illustrated in Figure 1.

We denote by $v^+$ the highest weight vector of a representation. Clearly, in the above notation the coordinate $(v^+)_1$ is equal to 1 and all other coordinates are zero.

The following easy observation directly follows from the description of the action of elementary root unipotents on $V$ (see [8, Lemma 2.3], [20, §4.3]).
Remark 2.3. Let $\Phi_l = E_l$, $l = 6, 7, 8$ and assume that $v \in V$ is such that all coordinates, except maybe $v_1, \ldots, v_{l-1}$ are zero. Then $x_{-a_1}(\xi)$ fixes $v$ for any $\xi \in R$.

2.3. Stability conditions. Recall that a column $(a_1, \ldots, a_n)^t \in R^n$ is called unimodular if $Ra_1 + \ldots + Ra_n = R$. A unimodular column $(a_1, \ldots, a_{n+1})^t \in R^{n+1}$ is called stable if there exist $b_1, \ldots, b_n \in R$ such that the column $(a_1 + b_1 a_{n+1}, a_2 + b_2 a_{n+1}, \ldots, a_n + b_n a_{n+1})^t$ is unimodular.

By definition, the stable rank of $R$ is the smallest natural number $k$ for which any unimodular column of height $> k$ is stable. If such $k$ does not exist we assume the stable rank of $R$ to be equal to $\infty$.

For a column $a = (a_1, \ldots, a_n)^t \in R^n$ denote by $\mathcal{L}(a)$ the intersection of the left maximal ideals of $R$ containing $a_1, \ldots, a_n$. Clearly, $u \in R^n$ is unimodular if and only if $\mathcal{L}(u) = R$.

By definition, the absolute stable rank is the smallest natural $k$ such that for any column $a = (a_1, \ldots, a_{n+1})^t$ of height $n + 1 > k$ there exist $b_1, \ldots, b_n$ such that

$$\mathcal{L}(a_1 + b_1 a_{n+1}, \ldots, a_n + b_n a_{n+1})^t = \mathcal{L}a.$$ 

The stable rank and the absolute stable rank of $R$ are denoted by $sr(R)$ and $asr(R)$ respectively. Clearly, $sr(R) \leq asr(R)$. If $R$ is commutative and its maximal spectrum is Noetherian of dimension $d$ (i.e. $\dim Max(R) = d$) then by [6, Th. 2.3] one has $asr(R) \leq d + 1$.

The following lemma is a direct consequence of the definition of stable rank.

Lemma 2.4. Let $sr(R) \leq l - 1$, then for any unimodular column $v \in R^l = V$ there exist matrices $x = \left(\begin{smallmatrix} e_{i-1} & * \\ 0 & 1 \end{smallmatrix}\right)$, $y = \left(\begin{smallmatrix} e_{i} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ such that the $l$-th coordinate of the vector $yx \cdot v$ is zero.

Denote by $p$ the matrix which has 1’s along its secondary diagonal and zeroes elsewhere. Call a matrix $a \in M(l, R)$ antipersymmetric if $pa^tp = a$.

Lemma 2.5. Assume that $asr(R) \leq l - 1$, then for any columns $u^+, u^- \in R^l$ such that $(u^+, u^-)^t \in R^{2l}$ is unimodular there exists an antipersymmetric matrix $a \in M(l, R)$ such that $u^+ + au^- \in R^l$ is unimodular.
Proof. The statement of the lemma is a special case of [1, Th. 1.2] applied to $R$ viewed as a form ring with trivial involution and zero form parameter. □

Denote by $H$ the hyperbolic embedding $H : \text{GL}(l, R) \to \text{O}(2l, R)$ of a general linear group into the even-dimensional orthogonal group, i.e. the map which sends $g$ to \((g \begin{pmatrix} 0 & 1 \\ 0 & p(g)^{-1}p \end{pmatrix})\).

**Lemma 2.6.** Assume that \(\text{asr}(R) \leq l - 1\). Then for any unimodular column \(v \in R^{2l} = V\) there exist orthogonal matrices \(x = \begin{pmatrix} e_l \\ e_l \end{pmatrix}, y = \begin{pmatrix} e_l \\ e_l \end{pmatrix}\) such that \((yx \cdot v)_k = 0\) for \(k = -l, \ldots, -(l - 1)\).

Proof. Applying Lemma 2.5 we obtain \(x = \begin{pmatrix} e_l \\ a \\ e_l \end{pmatrix}\) such that the first \(l\) coordinates of \(x \cdot v\) form a unimodular column of height \(l\).

The group \(E(l, R)\) acts transitively on the set of unimodular columns of height \(l\), therefore we can choose an appropriate \(g \in E(l, R)\) such that \(H(g)x \cdot v = (1, 0, \ldots, 0, *, \ldots, *)^t\), where the number of stars equals \(l\). Choose \(y' = \begin{pmatrix} e_l \\ 0 \\ e_l \end{pmatrix}\) such that \(y' H(g)x \cdot v = e_1 = (1, 0, \ldots, 0)^t\). Clearly \(y = y'^{-1}H(g)\) is an orthogonal matrix of the required form and \(yx \cdot v = y'^{-1}H(g)x \cdot v = H(g)^{-1}e_1 = (*, \ldots, *, 0, \ldots, 0)^t\). □

3. Proof of the main results.

The main ingredient needed in the proof of the Theorem 1.2 is the following factorization of the Steinberg group.

**Proposition 3.1.** Let \(\Phi_l, R, i, j\) be as in the statement of 1.2. Then one has the following factorization

\[(3.2) \quad \text{St}(\Phi_l, R) = \hat{U} \cdot \hat{U}^- \cdot \hat{L}_i \cdot \hat{P}_j.\]

From the Levi decomposition it follows that the decomposition 3.2 can be rewritten in the form \(\text{St}(\Phi_l, R) = \hat{P}_i \cdot \hat{U}_j \cdot \hat{P}_j\), i.e. the Proposition 3.1 can be thought of as an analogue of the "Dennis—Vaserstein decomposition" in the sense of [16] (cf. Lemma 2.1).

**Remark 3.3.** The cases \(\Phi_l = B_l, C_l\) of the factorization 3.1 directly follow from [15, Th. 2.5]. Nevertheless, we reprove these cases below for the sake of self-containedness.

The group \(\hat{L}_i\) coincides with the image of the stabilization map \(\theta_{\Delta_i \to \Phi_l}\). Denote by \(\hat{S}\) the subset of \(\hat{L}_i\) consisting of elements \(g \in \hat{L}_i\) such that

- the coordinate \((\varphi(g) \cdot v^+_i)_i\) is zero in the cases \(\Phi_l = B_l, C_l\);
- the coordinates \((\varphi(g) \cdot v^+_i)_k\) are zero for \(k = -(l - 1), \ldots, -(l - 1)\) in the case \(\Phi_l = E_l\).

In Figure 1 these coordinates are marked with black circles.

**Lemma 3.4.** For every \(a \in \hat{L}_i\) there exist \(x \in \hat{L}_i \cap \hat{U}, y \in \hat{L}_i \cap \hat{U}^-\) such that \(yx a \in \hat{S}\).
**Proof.** The restriction of the projection $\varphi$ to $\hat{U}(\Phi_t, R)$ or $\hat{U}^-(\Phi_t, R)$ is an isomorphism, hence $x$ and $y$ are determined by the matrices $\varphi(x), \varphi(y)$. It remains to apply Lemma 2.4 in the cases $\Phi_t = B_t, C_t$ and Lemma 2.6 in the case $\Phi_t = E_t$ taking $v = \varphi(a) \cdot v^+$. □

**Lemma 3.5.**

(a) The following inclusions hold

\[ X_{-\alpha_i} \cdot \hat{U} \subseteq \hat{U} \cdot X_{-\alpha_i} \cdot X_{\alpha_i}, \]
\[ X_{\alpha_i} \cdot \hat{U}^- \subseteq \hat{U}^- \cdot X_{\alpha_i} \cdot X_{-\alpha_i}, \]
\[ X_{-\alpha_i} \cdot \hat{U} \cdot \hat{U}^- \subseteq \hat{U} \cdot \hat{U}^- \cdot X_{\alpha_i} \cdot X_{-\alpha_i}. \]

(b) For any element $a$ of $\hat{S}$ one has $(X_{-\alpha})^a \subseteq \hat{L}_j$.

**Proof.** Denote by $\hat{U}'$ the subgroup generated by the elements $x_\alpha(\xi)$ for $\alpha \in \Phi^+ \setminus \{\alpha_i\}$, $\xi \in R$. Let $u$ be an arbitrary element of $\hat{U}$. Express $u$ as a product $vx_\alpha(\eta)$ for some $v \in \hat{U}', \eta \in R$. From the Chevalley commutator formula 2.2 it follows that $v^{x_{-\alpha}(\eta)} \in \hat{U}$. Thus

\[ x_{-\alpha}(\eta)u = v^{x_{-\alpha}(\eta)} \cdot x_{-\alpha}(\eta) \cdot x_\alpha(\xi) \in \hat{U} \cdot X_{-\alpha_i} \cdot X_{\alpha_i}. \]

The second inclusion can be demonstrated in a similar way. The third inclusion follows from the first two.

From the Levi decomposition it follows that $z = x_{-\alpha_i}(\xi)^a$ lies in $\hat{U}_i^-$. On the other hand, the element $t_{-\alpha_i}(\xi)$ fixes $a \cdot v^+$ (cf. Remark 2.3), hence $v^+$ is fixed by $\varphi(z)$. From the description of the action of elementary root unipotents on $V$ (see [8, Lemma 2.3], [20, § 4.3]) we conclude that $z \in \hat{U}_i^- \cap \hat{L}_j$ as claimed. □

**Proof of the Proposition 3.1.** Call a decomposition $g = u \cdot v \cdot a \cdot p$ of the form 3.2 reduced if $a \in \hat{S}$. Let us check that our stability assumptions imply that every such $g$ can be rewritten as a reduced decomposition. Indeed, take $g = u \cdot v \cdot a \cdot p \in \hat{U} \cdot \hat{U}^- \cdot \hat{L}_i \cdot \hat{P}_j$. Without loss of generality we may assume that $u \in \hat{U}_i$, $v \in \hat{U}_i^-$. It remains to choose $x, y$ from the statement of Lemma 3.4 and rewrite

\[ u \cdot v \cdot a \cdot p = ux^{-1} \cdot (xvx^{-1})y^{-1} \cdot (yxa) \cdot p \in \hat{U} \cdot \hat{U}^- \cdot \hat{S} \cdot \hat{P}_j. \]

Since the group $St(\Phi_t, R)$ is spanned by the root subgroups $X_{\pm \alpha_k}$, $1 \leq k \leq l$ it suffices to show that the decomposition 3.2 is stable under left multiplication by $X_{-\alpha_k}$. For $k \neq i$ this easily follows from the Levi decomposition.

Now take $k = i$ and let $a$ be an arbitrary element of $\hat{S}$. In view of Lemma 3.5 we have

\[ X_{-\alpha_i} \cdot \hat{U} \cdot \hat{U}^- \cdot a \cdot \hat{P}_j \subseteq \hat{U} \cdot \hat{U}^- \cdot X_{\alpha_i} \cdot a \cdot (X_{-\alpha_i})^a \cdot \hat{P}_j \subseteq \hat{U} \cdot \hat{U}^- \cdot a \cdot \hat{P}_j. \]

Set $G' = G(\Delta_j, R)$. We identify $G'$ with the corresponding subgroup of $G(\Phi_t, R)$. The proof of the following technical statement is similar to [15, Th. 3.1].

**Lemma 3.6.** Assume that $St(\Phi_t, R)$ admits decomposition 3.2. Then one has

\[ (3.7) \quad St(\Phi_t, R) \cap \varphi^{-1}(G') \subseteq (\hat{L}_i(\Phi_t, R) \cap \varphi^{-1}(G')) \cdot \hat{L}_j(\Phi, R). \]
Proof. Denote by $X$ the left hand side of the inclusion 3.7. Applying Proposition 3.1 we get that

$$X = \left( \hat{U}_i \cdot \hat{U}_i^{-} \cdot \hat{L}_i \cdot \hat{P}_j \right) \cap \varphi^{-1}(G') = \left( \hat{U}_i \cdot \hat{U}_i^{-} \cdot \hat{U}_i \cdot \hat{L}_j \right) \cap \varphi^{-1}(G').$$

Let $g$ be an arbitrary element of $X$, write its decomposition $g = u_1 \cdot l_1 \cdot v_2 \cdot u_2 \cdot l_2$ for some $u_1, l_1, v_2, u_2, l_2$ belonging to the corresponding subgroups.

Since the elements $\varphi(g), \varphi(u_1), \varphi(u_2), \varphi(l_2)$ fix $v^+$, we get that $\varphi(v_2) \cdot v^+ = \varphi(l_1^{-1}) \cdot v^+$. Comparing the coordinates of these vectors we conclude that $\varphi(v_2) \cdot v^+ = v^+$, therefore $v_2 \in \hat{U}_i^{-} \cap \hat{L}_j$ and from the Levi decomposition it follows that $X \subseteq (\hat{L}_i \cdot \hat{U}_i \cdot \hat{L}_j) \cap \varphi^{-1}(G')$.

Denote by $\sigma$ the automorphism of $St(\Phi_t, R)$ induced by the automorphism $\alpha \mapsto -\alpha$ of $\Phi_t$. Clearly $\sigma(X) \subseteq (\hat{L}_i \cdot \hat{U}_i^{-} \cdot \hat{L}_j) \cap \varphi^{-1}(G(\Phi_t, R))$.

Consider an element $g \in \sigma(X)$. As in the previous case, decompose $g = l'_1 \cdot v'_1 \cdot l'_2$ and notice that $\varphi(v'_1) \cdot v^+ = \varphi(l'_1^{-1}) \cdot v^+$, hence $v'_1 \in \hat{L}_i$ and $X \subseteq (\hat{L}_i \cdot \hat{L}_j) \cap \varphi^{-1}(G') \subseteq (\hat{L}_i \cap \varphi^{-1}(G')) \cdot \hat{L}_j$ as claimed. \hfill $\square$

Proof of the Theorem 1.2. Applying $\varphi$ to both sides of 3.7 we get the inclusion

$$E(\Phi_t, R) \cap G(\Delta_j, R) \subseteq \left( E(\Delta_i, R) \cap G(\Delta_j, R) \right) \cdot E(\Delta_j, R) = \theta_{\Delta_i \to \Phi_t}(E(\Delta_i, R) \cap G(\Delta_i \cap \Delta_j, R)) \cdot E(\Delta_j, R)$$

from which the theorem follows. \hfill $\square$

4. Concluding remarks

We note that no explicit description of the stabilization kernel similar to Theorem 1.1 is known for $\Phi = D_t$. Moreover, no result similar to Corollary 1.3 may hold for $\Phi = D_t$ in general as the following negative result has been demonstrated in [12].

Theorem 4.1. Let $R$ be a nonsingular affine algebra of dimension $d \geq 2$ over a perfect $C_1$-field. Assume that $2 \in R^*$.

(1) The bijectivity of the map

$$SO(2(d + 1), R)/EO(2(d + 1), R) \to SO(2(d + 2), R)/EO(2(d + 2), R)$$

implies the transitivity of the action of $E(d + 1, R)$ on the unimodular columns of height $d + 1$.

(2) There exists an algebra from the considered class of rings for which the latter condition fails.

On the other hand, in fact, a much stronger version of the 2nd statement of Corollary 1.3 holds for $\Phi_t = C_t$ (see [4, Th. 2]).

We conclude the text with several problems related to the main result of the paper.

Problem 4.2. Generalize Theorem 1.1 to the case of the quadratic group from [1] using the subgroup $\hat{E}U$ defined in [19] as an analogue of $\hat{E}(n, R)$.

Problem 4.3. Obtain a relative version of Theorem 1.2.

Problem 4.4. In the assumptions of Theorem 1.2 describe the kernel of $\theta_2''$. 

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References

[1] A. Bak, V. A. Petrov and G. Tang Stability for quadratic $K_1$. K-theory. 30, 1 (2003), 1–11.
[2] H. Bass, J. Milnor and J.-P. Serre Solution of the congruence subgroup problem for the groups of type $SL_n$ ($n \geq 3$) and $Sp_{2n}$, ($n \geq 2$). Publ. Inst. Hautes Et. Sci., 33 (1967), 59–137.
[3] H. Bass K-theory and stable algebra. Publ. Inst. Hautes Et. Sci., 22, (1964), 5–60.
[4] R. Basu and R. Rao Injective Stability for $K_1$ of classical modules. J. Algebra 323, 4, (2010), 867–877.
[5] N. Bourbaki Lie groups and Lie algebras. Chapters 4–6. Springer-Verlag, Berlin (2002).
[6] D. R. Estes, J. Ohm Stable range in commutative rings. J. Algebra 7, 343–362 (1967).
[7] W. van der Kallen Vaserstein prestability theorem for commutative rings. Comm. Algebra 15, 3 (1987), 657–663.
[8] H. Matsumoto Sur les sous-groupes arithmetiques des groupes semi-simples deployes. Ann. Sci. Ecole Norm. Sup. 4, 2 (1969), 1–62.
[9] E. Plotkin On the stability of the $K_1$-functor for Chevalley groups of type $E_7$. J. Algebra, 210, (1998), 67–85.
[10] E. Plotkin, A. Semenov, N. A. Vavilov Visual basic representations: an atlas. Internat. J. Algebra Computat., 8, 1 (1998), 61–95.
[11] R. Rao and W. van der Kallen Improved stability for $SK_1$ and $WMS_d$ of a non-singular affine algebra. Astérisque 226 (1994), 411–420.
[12] R. Rao, R. Basu and S. Jose Injective stability for $K_1$ of orthogonal group. J. Algebra 323, 2, (2010), 393–396.
[13] J.-P. Serre Galois cohomology. Springer, (2002), 210p.
[14] M. R. Stein Generators, relations and coverings of Chevalley groups over commutative rings. Amer. J. Math., 93, 4 (1971), 965–1004.
[15] M. R. Stein Stability theorems for $K_1$, $K_2$ and related functors modeled on Chevalley groups. Japan J. Math., 4, 1 (1978), 77–108.
[16] A. A. Suslin, M. S. Tulenbaev Stabilization theorem for the Milnor $K_2$-functor. J. Soviet Math., 17, 2 (1981), 1804–1819.
[17] G. Taddei Normalité des groupes élémentaires dans les groupes de Chevalley sur un anneau. Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983.), Providence, RI: Amer. Math. Soc., 1986., Vol. 55 of Contemp. Math., 693–710.
[18] L. N. Vaserstein On the stabilization of the general linear group over a ring. Math. USSR Sbornik, 8, 3 (1969), 383–400.
[19] L. N. Vaserstein Stabilization for unitary and orthogonal groups over a ring with involution, Math. USSR Sbornik, 10, 307–326.
[20] N. A. Vavilov Structure of Chevalley groups over commutative rings. Nonassociative algebras and related topics (Hiroshima, 1990), World Sci., Publ., River Edge, NJ, (1991), 219–335.
[21] N. A. Vavilov, E. Plotkin Chevalley groups over commutative rings: I. Elementary Calculations. Acta Applicandae Math., 45, (1996), 73–113.

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