Linear bounds for constants in Gromov’s systolic inequality and related results

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Abstract. Gromov’s systolic inequality asserts that the length, \( \text{sys}_1(M^n) \), of the shortest non-contractible curve in a closed essential Riemannian manifold \( M^n \) does not exceed \( c(n) \text{vol}^{\frac{1}{n}}(M^n) \) for some constant \( c(n) \). (Essential manifolds is a class of non-simply connected manifolds that includes all non-simply connected closed surfaces, tori, and projective spaces.)

Here we prove that all closed essential Riemannian manifolds satisfy \( \text{sys}_1(M^n) \leq n \text{vol}^{\frac{1}{n}}(M^n) \). (The best previously known upper bound for \( c(n) \) was exponential in \( n \).)

We similarly improve a number of related inequalities. The paper also contains a qualitative strengthening of Guth’s theorem from [Gu11], [Gu17] asserting that if volumes of all metric balls of radius \( r \) in a closed Riemannian manifold \( M^n \) do not exceed \( \left(\frac{r}{c(n)}\right)^n \), then the \((n-1)\)-dimensional Urysohn width of the manifold does not exceed \( r \). In our version the assumption of Guth’s theorem is relaxed to the assumption that for each \( x \in M^n \) there exists \( \rho(x) \in (0, r] \) such that the volume of the metric ball \( B(x, \rho(x)) \) does not exceed \( \left(\frac{\rho(x)}{c(n)}\right)^n \), where one can take \( c(n) = \frac{n}{2} \).

0. Introduction.

Let \( M^n \) be a closed Riemannian manifold. Larry Guth ([Gu 17]) proved that there exists \( c(n) \) with the following property: if for some \( r > 0 \) the volume of each metric ball of radius \( r \) is less than \( \left(\frac{r}{c(n)}\right)^n \), then there exists a continuous map from \( M^n \) to a \((n-1)\)-dimensional simplicial complex such that the inverse image of each point can be covered by a metric ball of radius \( r \) in \( M^n \). It was previously proven by Gromov that this result implies two by now famous Gromov’s inequalities: \( \text{FillRad}(M^n) \leq c(n) \text{vol}(M^n)^{\frac{1}{n}} \) (Theorem 1.2.A in [Gr]) and, if \( M^n \) is essential, then also \( \text{sys}_1(M^n) \leq 6c(n) \text{vol}(M^n)^{\frac{1}{n}} \) (Theorem 0.1.A in [Gr]) with the same constant \( c(n) \).

Here \( \text{sys}_1(M^n) \) denotes the length of a shortest non-contractible closed curve in \( M^n \).

Here we prove that these results hold with \( c(n) = \left(\frac{n}{2}\right)^\frac{n}{2} \leq \frac{n}{2} \). We demonstrate that for essential Riemannian manifolds \( \text{sys}_1(M^n) \leq n \text{vol}^{\frac{1}{n}}(M^n) \). All previously known upper bounds for \( c(n) \) were exponential in \( n \).
Moreover, we present a qualitative improvement: In Guth’s theorem the assumption that the volume of every metric ball of radius \( r \) is less than \((\frac{r}{c(n)})^n\) can be replaced by a weaker assumption that for every point \( x \in M^n \) there exists a positive \( g(x) \leq r \) such that the volume of the metric ball of radius \( g(x) \) centered at \( x \) is less than \((\frac{g(x)}{c(n)})^n\) (for \( c(n) = (\frac{n!}{2^n})^{\frac{1}{n}} \)).

Also, if \( X \) is a boundedly compact metric space such that for some \( r > 0 \) and an integer \( n \geq 1 \) the \( n \)-dimensional Hausdorff content of each metric ball of radius \( r \) in \( X \) is less than \((\frac{r}{4n})^n\), then there exists a continuous map from \( X \) to a \((n-1)\)-dimensional simplicial complex such that the inverse image of each point can be covered by a metric ball of radius \( r \). This provides a (significant) quantitative improvement of a result from [LLNR] and [P]. (Recall that a metric space is called boundedly compact if all closed metric balls in this space are compact.)

Most other papers in systolic geometry follow Gromov’s approach based on the isoperimetric inequality in Banach spaces proven using “cutting off of thin fingers”. We follow Schoen-Yau style approach , i.e. the inductive dimension reduction. This approach was introduced to systolic geometry by Guth ([Gu10]) and later greatly improved and strengthened by Papasoglu ([P]) who used some ideas from [LLNR]. Our approach in the present paper essentially follows [P], yet we provide a number of modifications, strengthenings and simplifications of proofs there.

Our paper is almost self-contained. Without a proof we use only two (well-known) facts that go beyond material taught in standard graduate courses, namely, the existence of smooth approximations of the distance function (cf. [Ga]) and the coarea formula (cf. [BZ], Theorem 3.2.4).

1. Results.

1.1. Definitions and historical context. Given a bounded metric space \( X \) its Kuratowski embedding into \( L^\infty(X) \) sends each point \( x \) to the distance function to \( x \). Gromov defined the filling radius of a closed Riemannian manifold \( M^n \), \( \text{FillRad}(M^n) \) as the infimum of \( r \) such that the image of \( M^n \) in \( L^\infty(M^n) \) under the Kuratowski embedding bounds in its \( r \)-neighbourhood ([Gr], section 1). In [Gr] Gromov gave a proof of the inequality \( \text{FillRad}(M^n) \leq c(n)\text{vol}(M^n)^{\frac{1}{n}} \) with the constant that behaves as \((Cn)^{\frac{1}{2n}}\). (On the other hand Misha Katz’s paper [K] contains a short proof the inequality \( \text{FillRad}(M^n) \leq \frac{1}{3}\text{diam}(M^n) \) with the optimal constant.) Gromov’s proof was later somewhat simplified by Stefan Wenger ([W]). (More precisely, Wenger simplified the proof of Gromov’s filling volume inequality which is the main ingredient of the proof of inequality \( \text{FillRad}(M^n) \leq c(n)\text{vol}(M^n)^{\frac{1}{n}} \) in [Gr].)

An \( n \)-dimensional simplicial complex \( X^n \) is essential, if there is no map \( f : X^n \rightarrow K(\pi_1(X^n),1) \) such that \( f \) induces the isomorphism of the fundamental groups, and the image of \( f \) is contained in the \((n-1)\)-skeleton of \( K(\pi_1(X^n),1) \). It is easy to
see that $X^n$ is essential if the classifying map $X^n \to K(\pi_1(X^n), 1)$ induces the homomorphism of $n$th homology groups with non-trivial image (for some group of coefficients).

The paper [Gr] contains a short and elegant proof of the inequality $sys_1(M^n) \leq 6\text{FillRad}(M^n)$ for all essential closed Riemannian manifolds (Lemma 1.2.B in [Gr]). Combining this inequality with $\text{FillRad}(M^n) \leq c(n)\text{vol}_{n/2}(M^n)$ Gromov proves that all closed essential Riemannian manifolds satisfy $sys_1(M^n) \leq c(n)\text{vol}_{n/2}(M^n)$. This inequality generalizes earlier results by Loewner, Pu, Accola, Blatter, Hebda, and Yu. Burago and V. Zalgaller for surfaces. In particular, Yu. Burago and V. Zalgaller and, independently, Hebda proved that for all closed Riemannian surfaces $\Sigma$, $sys_1(\Sigma) \leq \sqrt{2\text{Area}(\Sigma)}$ (cf. [BZ]).

One can define the $(n - 1)$-dimensional Urysohn width, $UW_{n-1}(X)$, of a metric space $X$ as infimum of $t$ such that there exists a continuous map $f : X \to K^{n-1}$ to a $(n - 1)$-dimensional polyhedron $K^{n-1}$ such that for each $k \in K^{n-1}$ $f^{-1}(k)$ has diameter $\leq t$. We will need also a closely related notion of $(n - 1)$-dimensional Alexandrov width, $UR_{n-1}(X)$, that is defined almost as the Urysohn width, but with condition $\text{diam}(f^{-1}(k)) \leq t$ replaced by the condition that $f^{-1}(x)$ is contained in a metric ball of radius $t$. It is obvious that $UR_{n-1} \leq UW_{n-1}(X) \leq 2UR_{n-1}(X)$. The $(n - 1)$-dimensional Alexandrov width of a compact metric space $X$ can also be defined as the infimum of $t$ such that there exists a covering of $X$ by connected open sets $U_\alpha$ of radius $\leq t$ such that no $n + 1$ sets $U_\alpha$ have a non-empty intersection. (The equivalence of these two definitions is well-known. For the sake of completeness, we include a sketch of a proof of the equivalence in section 1.3 below.)

Gromov also provided a proof of the inequalities $\text{FillRad}(M^n) \leq \frac{1}{2}UW_{n-1}(M^n)$ (the combination of Proposition (D) in Appendix 1 in [Gr] with the inequality in the example at the end of section (B) in Appendix 1 of [Gr]). Therefore, any upper bound for $UW_{n-1}$ automatically leads to upper bounds to $\text{FillRad}$ and, in the essential case, for $sys_1$. Now a natural question (posed by Gromov in [Gr]) is whether or not $UW_{n-1}(M^n) \leq c(n)\text{vol}(M^n)^{1/2}$. This question was solved in the affirmative by Larry Guth in [Gu11], [Gu17]. In fact, Guth proved more. He demonstrated that there exists $\delta(n)$ such that if for some $r > 0$ all metric balls of radius $r$ have volume less than $\delta(n)r^n$, then $UW_{n-1}(M^n) \leq r$. To recover the previous inequality one can take here $r = \frac{\text{vol}(M^n)^{1/2}}{\delta(n)^{1/n}}$. The assumption will automatically hold, and one sees that $UW_{n-1}(M^n) \leq \delta(n)^{-1/n} \text{vol}(M^n)^{1/n}$.

Recall that the $m$-dimensional Hausdorff content of a compact metric space $X$ is the infimum over all coverings of $X$ by metric balls with radii $r_i$ of the sum $\sum_i r_i^m$. It is denoted by $HC_m(X)$. (If one requires here that all $r_i$ do not exceed $\delta$, and then takes the limit as $\delta \to 0$, one obtains the $m$-dimensional Hausdorff
measure of \( X \).) Guth asked if one can replace the volume in these inequalities by the \( n \)-dimensional Hausdorff content, and if such estimates will be true for all (not necessarily \( n \)-dimensional) compact metric spaces (Questions 5.1 and 5.2 in [Gu 17]).

In [LLNR] we proved that this is, indeed, so. For example, we proved that for each compact metric space \( X \) and each integer \( m > 1 \), \( UW_{m-1}(X) \leq C(m)HC_{m}^{\frac{1}{m}}(Y) \). As a corollary, we immediately see that if \( X \) is a compact \( m \)-essential smooth polyhedron endowed with the structure of the length space, \( sys_{1}(X) \leq 3C(m)HC_{m}^{\frac{1}{m}}(X) \).

Recently, Panos Papasoglu wrote a paper [P] with a much shorter proof of these results than the proof in [LLNR]. His proof did not contain an estimate for \( C(m) \), but our analysis of his proof yields \( C(m) \sim const^{m} \) leading to exponential in \( m \) estimates for constants in the previous inequalities. He learned about [LLNR] from my talk and our conversations at the conference at Barcelona. While his proof draws on several ideas of [LLNR], it also contains a central observation that is quite different from the ideas of [LLNR]. Roughly speaking, the amazing in its strength and simplicity Papasoglu’s insight was to consider an (almost) minimal “hypersurface” dividing a compact metric space into subsets of a small diameter and to observe that the “area” (or, more precisely, the appropriate Hausdorff content) of the intersection of this hypersurface with any metric ball cannot exceed the Hausdorff content of the corresponding metric sphere. Indeed, otherwise, one could just replace the part of the minimal hypersurface inside the metric ball by the metric sphere preserving the same upper bound for diameter for each component of the complement. Thus, the (almost) minimal hypersurface inherits the main property of the metric space, namely, that its intersections with metric balls of a certain size are “small”. This observation enables one to run an induction argument, where the result for the metric space and the \( n \)-dimensional Hausdorff content would follow from the same result for the minimal hypersurface and its \((n-1)\)-dimensional Hausdorff content.

Of course, this approach is strongly reminiscent of the Schoen - Yau approach to scalar curvature that was introduced to systolic geometry by Guth, who in [Gu10] proved that \( sys_{1}(M^{n}) \leq 8\text{vol}(M^{n})^{\frac{1}{n}} \) for Riemannian tori. The proof of our main theorem below is heavily based on Papasoglu’s idea. Yet it contains a number of modifications and simplifications:

First, we observed that the dependence of the constant in the inequalities could be improved from exponential to linear by (a) carefully choosing the radius of the metric ball (in the argument above) and (b) improving an argument of the end of the proof of Lemma 2.4 in [P] so as not to decrease the constant by a constant factor on each step - compare our Lemma 2.5, where the upper bound in the assumption and the conclusion is the same. (We observed that it is more convenient to use \( UR \) instead of the previously used \( UW \) here.) In order to accomplish (a), we could have
used the trick used by Larry Guth at the end of [Gu10] (as it was done in the first version of the present paper).

However, we noticed that there is a better (point dependent) way to choose the radii as in Lemma 2.4 below. Not only this observation leads to an improvement of the estimate by a constant factor, but it also yields a quantitative improvement of all the previous results that was mentioned in the abstract: the radius of a small ball centered at a point is allowed to depend on the center as long as it does not exceed a fixed $r$. It is interesting to note that we do not see how to achieve this quantitative improvement, if one follows the approach of [P] via Hausdorff contents, as in this approach one needs to restrict the radii of the considered balls by a quantity that depends on the radii of small balls $r$ and becomes wildly variable, if these radii are allowed to depend on the centers. (See the remark at the end of section 3 for more details.)

Third, the approach of Papasoglu to the classical systolic geometry was through results about Hausdorff contents (the same as in [LLNR]). He mentioned that instead one can directly use the Hausdorff measure and Eilenberg’s inequality. We adopted this approach and discovered that not only it leads to a much simpler proof, but also one can save an extra $\sqrt{n}$ factor in comparison with first establishing the inequality for Hausdorff content with linear constant (Theorem 1.4), and then using the obvious inequality relating Hausdorff measure and Hausdorff content.

Fourth, we were careful about the values of the numerical constants in our proof. This is, probably, not that important in the long run, as one expects that the optimal dimensional constants in the above inequalities should behave as $\text{const} \sqrt{n}$ and not as $\text{const} n$. Still, as the result, we derive aesthetically pleasing and convenient to use inequalities $\text{sys}_1(M^n) \leq n \text{vol}^{\frac{1}{n}}(M^n)$ for all closed essential manifolds, and $\text{FillRad}(M^n) \leq \frac{n}{2} \text{vol}^{\frac{1}{n}}(M^n)$ for all closed manifolds. In fact, I am not aware of any previously published specific value of the constant at $\text{vol}(M^n)^{\frac{1}{2}}$ in the general case of Gromov’s systolic inequality for $n = 3$ other than Gromov’s $1296\sqrt{6}$ or Wenger’s 118098. For $n = 3$ our value $2 \times 3^{\frac{1}{2}} = 2.88\ldots$ of the systolic constant is within the factor of 2 of the (unknown) optimal value.

Fifth, in the last section of the paper we also similarly improve the main result of [LLNR] and [P]. We provide a four-page long self-contained proof of the inequality $UW_{m-1}(X) \leq 8mHC_{m}^{\frac{1}{m}}(X)$ for a compact metric space $X$ as well as a local version of this result for boundedly compact $X$. This section heavily relies on [P] but contains several improvements and simplifications.

1.2. Results. Our first main theorem is:
Theorem 1.1. Let $M^n$ be a closed Riemannian manifold, and $r > 0$ a real number. Assume that for every $x \in M^n$ there exists $t = t(x) \in (0, r]$ such that the volume of the metric ball of radius $t$ centered at $x$ is less than $\frac{2n}{n!}$. Then $UR_{n-1}(M^n) < r$.

The relationships between $UR_{n-1}$, $UW_{n-1}$, $FillRad$ and $sys_1$ stated above immediately imply that:

Theorem 1.2. Assume that $M^n$ is a compact $n$-dimensional smooth Riemannian manifold. Then

\[ UR_{n-1}(M^n) \leq \left(\frac{n!}{2}\right)^{\frac{1}{n}} \text{vol}(M^n)^{\frac{1}{n}} \leq \frac{n}{2} \text{vol}^{\frac{1}{n}}(M^n), \quad (1) \]

\[ UW_{n-1}(M^n) \leq 2\left(\frac{n!}{2}\right)^{\frac{1}{n}} \text{vol}(M^n)^{\frac{1}{n}} \leq n \text{ vol}^{\frac{1}{n}}(M^n), \quad (2) \]

\[ FillRad(M^n) \leq \left(\frac{n!}{2}\right)^{\frac{1}{n}} \text{vol}(M^n)^{\frac{1}{n}} \leq \frac{n}{2} \text{vol}^{\frac{1}{n}}(M^n), \quad (3) \]

Inequality (1) and the inequality $sys_1(X^n) \leq 6UR_{n-1}(X^n)$ for essential Riemannian polyhedra which is a combination of two inequalities: $sys_1(X^n) \leq 6FillRad(X^n)$ and $FillRad(X^n) \leq UR_{n-1}(X^n)$ that were proven in [Gr] immediately imply the following version of Gromov’s systolic inequality with linear in $n$ dimensional constant: For each essential Riemannian manifold $M^n$ $sys_1(M^n) \leq 3n\text{vol}^{\frac{1}{n}}(M^n)$ that appeared in the first version of the present paper. However, Roman Karasev e-mailed to me a very short proof of a stronger inequality:

\[ sys_1(X^n) \leq 2UR_{n-1}(X^n). \quad (4) \]

for essential polyhedral length spaces that does not involve the Kuratowski embedding or the filling radius. Karasev’s proof is based on work of Albert Schwartz [S], and in a nutshell goes as follows: The second definition of Alexandrov width implies that there is an open covering of $M^n$ by connected open sets of radius $\leq UR_{n-1}(M^n)$ with multiplicity of intersections $\leq n$. If $sys_1(M^n) > 2UR_{n-1}(M^n)$, then each loop in $U_\alpha$ is contractible in $M^n$. Therefore, each $U_\alpha$ lifts to $\tilde{M^n}$ as the collection of sets $\{(\tilde{U}_\alpha)_\beta\}_{\beta \in \pi_1(M^n)}$ homeomorphic to $U_\alpha$. The collection of all these sets forms a covering of $\tilde{M^n}$ of multiplicity $\leq n$. Theorem 14 in [S] asserts that the existence of such a covering of $\tilde{M^n}$ implies that $M^n$ is not essential. With Karasev’s permission I will present a self-contained proof of inequality (4) at the end of section 2.

As an immediate corollary:

Theorem 1.3. If $M^n$ is a closed essential Riemannian manifold, then $sys_1(M^n) \leq 2(n!)^{\frac{1}{n}} \text{vol}^{\frac{1}{n}}(M^n) \leq n \text{vol}^{\frac{1}{n}}(M^n)$.

Remark. As $(n!)^{\frac{1}{n}} = \frac{1}{e} (1 + o(1))n$, for all sufficiently large $n$ $sys_1(M^n) < 0.74 n \text{vol}(M^n)^{\frac{1}{n}}$. If $n = 2$, the inequality in the theorem is well-known, and a
better estimate can be found in section 1.4.3 of [BZ]. If \( n = 3 \), then the constant at 
\( \text{vol}^{\frac{1}{3}}(M^3) \) in Theorem 1.3 is equal to \( 2 \times 3^{\frac{1}{2}} = 2.88 \ldots \). On the other hand, we see 
that the optimal value of this constant for \( n = 3 \) cannot be less than \( \pi^{\frac{1}{3}} = 1.46 \ldots \),
as this is the value that one gets in the case of \( RP^3 \) with the canonical metric. So, for 
\( n = 3 \), our constant is within the factor of 1.97 from the optimal systolic constant.

We will prove Theorems 1.1, 1.2 in a somewhat greater generality, namely for compact Riemannian polyhedra (i.e. finite polyhedra endowed with a smooth Riemannian metric on each maximal simplex, so that Riemannian metrics on two simplices that have a common face match on this face). Note that all previous definitions and quoted results by Gromov can be directly extended to Riemannian polyhedra, which was observed by Gromov in [Gr].

Below a subpolyhedron will always mean a compact subpolyhedron with smoothly embedded faces endowed with the Riemannian metric of the ambient Riemannian polyhedron (and the corresponding intrinsic distance). Also, below \( |X| \) will denote the volume of \( X \). Sometimes we write it as \( |X|_n \), when we want to emphasize the dimension.

In the last section, we give a self-contained proof of the following quantitative improvement of a result that first appeared in [LLNR] and then was reproven in [P]:

**Theorem 1.4.** 1. Let \( X \) be a compact metric space, \( r > 0 \), \( n \) a positive integer. Assume that for each metric ball \( B \) of radius \( r \) in \( X \), \( HC_n(B) < \left( \frac{r}{4n} \right)^n \). Then \( UR_{n-1}(X) < r \).

2. Let \( X \) be a compact metric space. Then \( UR_{n-1}(X) \leq 4n HC_n(X)^{\frac{1}{n}} \).

3. Let \( X \) be boundedly compact. Assume that for some positive \( \mu \) and each metric ball \( B \) of radius \( r \), \( HC_n(B) \leq \left( \frac{r}{6n} \right)^n - \mu \). Then \( UR_{n-1}(X) < r \).

**Remark.** As \( UW_{n-1}(X) \leq 2UR_{n-1}(X) \), we also immediately obtain the corresponding upper bounds for the Urysohn width of \( X \), \( UW_{n-1}(X) \), that differ from the upper bounds for \( UR_{n-1}(X) \) by a factor of 2. For example, if \( X \) is compact, then \( UW_{n-1}(X) \leq 8n HC_n^{\frac{1}{n}}(X) \).

1.3. Equivalence of the two definitions of the \((n - 1)\)-dimensional Alexandrov widths. To see that the two definitions of Alexandrov width given in section 1.1 are equivalent for all compact \( X \), denote \( UR_{n-1}(X) \) in the sense of the first definition by \( g \), and the second by \( r \). We will first demonstrate that \( r \leq g + \varepsilon \) for an arbitrarily small positive \( \varepsilon \), and then demonstrate that \( g \leq r \). To prove the first inequality choose \( K^{n-1} \) and \( f \) such that \( \max_{k \in K^{n-1}} \text{rad}(f^{-1}(k)) \) is very close to \( g \). Now consider a very fine covering of \( K^{n-1} \) by open sets \( V_\beta \) such that each \((n + 1)\)-tuple of these sets has the empty intersection. Finally, define the collection \( U_\alpha \) as the collection of all connected components of open sets \( f^{-1}(V_\beta) \). Note that the radii
of $U_\alpha$ do not exceed $\max_{k \in K^{n-1}} \text{rad}(f^{-1}(k)) + \varepsilon$, where $\varepsilon > 0$ can be made arbitrarily small by choosing the covering $V_\beta$ sufficiently fine. To see that $\varrho \leq r$, consider the nerve $K^{n-1}$ of the covering $U_\alpha$, and the standard map $f : X \to K^{n-1}$ defined using a partition of unity subordinate to the covering $\{U_\alpha\}$. Now note that the inverse images of each point of $K^{n-1}$ under $f$ will be contained in one of the sets $U_\alpha$.

2. **Proof of Theorems 1.1 and 1.3.**

The well-known coarea inequality for Lipschitz functions of Riemannian manifolds immediately generalizes to Riemannian polyhedra ([BZ]) and implies that given a Riemannian polyhedron $X^n$, a real $r$, and a metric ball $B$ of radius $r$ centered at a point $x \in X^n$, $\int_0^r |S_s|_{n-1} ds \leq |B|_n$, where $S_s$ denotes the metric sphere of radius $s$ centered at $x$.

We prefer to work in the situation when for almost all $s$, $S_s$ is a subpolyhedron. One well-known way to achieve this is to approximate the distance function by $(1+\tau)$-Lipschitz function that is smooth on each open simplex (cf. section 3 of [Ga]), and to replace the distance function by this approximation. (Here $\tau$ can be chosen to be arbitrarily small.) Starting from Lemma 2.2 “metric spheres” will really mean the level sets of a sufficiently close smooth approximation of the distance function. Now Sard’s theorem implies that almost all geodesic spheres are subpolyhedra. (Sard’s theorem will separately apply to the restriction of the smooth approximation of the distance function to each open simplex.) However, the coarea inequality above and all inequalities below will hold only up to a factor of $1 + f(\tau)$, where $f$ will be some specific function such that $\lim_{\tau \to 0} f(\tau) = 0$. Eventually, one will pass to the limit as $\tau \to 0$. For the sake of readability we will not be mentioning terms of the form $1 + f(\tau)$ in the inequalities, and will just pretend that the distance function is smooth on each open simplex.

**Lemma 2.1.** Let $X$ be a compact Riemannian polyhedron of dimension $\leq 1$. Assume that there exists $r > 0$ such that for each $x \in X$ there exists a metric ball $B$ centered at $x$ of radius $t(x) \in (0, r]$ such that $|B| < 2t(x)$. Then $UR_\alpha(X) < r$. In other words each connected component of $X$ can be covered by a a metric ball of radius $< r$.

**Proof.** First, note that without any loss of generality we can assume that $X$ is connected.

Second, observe that the lemma can be reduced to its particular case, when $X$ is tree (endowed with Riemannian metric on each edge). Indeed, by disconnecting some of the edges of $X$ from one of their endpoints to destroy cycles in $X$ we can transform $X$ into a Riemannian tree $Y$. We also have a (quotient) map $f : Y \to X$, obtained by identifying new vertices of $Y$ with the corresponding old ones. The distances in $Y$ are not less than distances between the images of the same points in $X$. Therefore, each metric ball with center $y$ in $Y$ is a subset of the metric ball in $X$.
with the center \( f(y) \) and the same radius. Therefore, the assumption of the lemma holds for \( Y \). On the other, if \( Y \) can be covered by a metric ball with center \( y \), then the metric ball in \( X \) with the center \( f(x) \) and the same radius will cover \( X \).

Therefore, we can assume that \( X \) is a Riemannian tree. Let \( p \) be a center of \( X \), that is, a point in \( Y \) realizing the minimum, \( R \), of \( \max_{q \in X} \text{dist}_X(p, q) \). Let \( x \in Y \) denote one of the most distant points of \( Y \) from \( p \) (in the metric of \( Y \)). (So, \( R \) is the radius of \( X \).) As \( X \) is contained in the metric ball of radius \( R \) centered at \( p \), we need to prove that \( R < r \).

Observe that the definition of \( p \) implies that there exists another point \( x' \in Y \) such that \( \text{dist}_Y(p, x') = \text{dist}_Y(p, x) \), and the (unique) shortest path from \( x \) to \( x' \) passes through \( p \) (as \( Y \) is a tree). Therefore, \( \text{dist}(x, x') = 2\text{dist}(p, x) = 2R \). This implies that for each \( t \leq R \) the length of \( X \cap B(x, t) \geq 2t \), and therefore \( R < r \).

\[ \square \]

**Lemma 2.2.** Assume that \( B \) is a metric ball of radius \( r \) centered at \( x \in X^{n+1} \), \( \varepsilon \in (0, 1) \). Assume that \( |B| \leq cr^{n+1}(1 - \varepsilon) \) for some \( c \). Let \( \lambda = r(\frac{\varepsilon}{3})^{n+1} \). There exists a subset \( A \) of the open interval \((\lambda, r)\) of positive measure such that for each metric sphere \( S \) centered at \( x \) with radius \( t \in A \), \( |S| < (n+1)c\varepsilon t^n(1 - \frac{1}{3} \varepsilon) \), and \( S \) is a subpolyhedron of \( X^{n+1} \).

**Proof.** Assume that the set of radii \( \lambda > \lambda \) such that \( |S| < (n+1)c\varepsilon t^n(1 - \frac{1}{3} \varepsilon) \) has measure zero. The coarea inequality implies \( |B| \geq \int_0^\varepsilon c(n+1)t^n(1 - \varepsilon/3)dt > cr^{n+1}(1 - \varepsilon/3) - c\lambda^{n+1} = cr^{n+1}(1 - \frac{2}{3} \varepsilon) \), yielding a contradiction with our assumption. To ensure that \( S \) is a subpolyhedron, we apply Sard’s theorem on each open simplex of \( X^{n+1} \). (Recall, that by distance function we actually mean a smooth approximation to the distance function.) \[ \square \]

**Definition 2.3.** A compact subpolyhedron \( Y^{m-1} \) of \( X^m \) is called \( d \)-separating if each connected component of its complement \( X^m \setminus Y^{m-1} \) can be covered by a metric ball of radius \( \leq d \). Denote the infimum of \( |Y|_{m-1} \) over all \( d \)-separating sets \( Y \) in \( X^m \) by \( I_X(d, m-1) \). If \( \delta > 0 \) is a positive real number, a \( d \)-separating set \( Y^{m-1} \) is called \( \delta \)-minimal if \( |Y^{m-1}| \leq I_X(d, m-1) + \delta \).

**Lemma 2.4.** Assume that \( X^{n+1} \) is a Riemannian polyhedron of dimension \( \leq n + 1 \) such that for some positive \( r \), \( \varepsilon \) and \( \tau \) each each \( x \in X^{n+1} \) there exists \( t(x) \in (\tau, r] \) such that the metric ball \( B \) of radius \( t(x) \) centered at \( x \) satisfies the inequality \( |B|_{n+1} < \frac{2n(x)^{n+1}}{(n+1)!}(1 - \varepsilon) \). Then there exists \( \delta = \delta(n, \varepsilon, \tau) \) such that for every \( \delta \)-minimal \( r \)-separating set \( Z \) and each \( x \in X^{n+1} \) there exists \( \rho \in (t(x)(\frac{\varepsilon}{3})^{n+1}, t(x)) \) such that:

(a) The metric ball \( \beta \) in \( X^{n+1} \) of radius \( \rho \) centered at \( x \) satisfies

\[ |Z \cap \beta|_n < \frac{2\rho^n}{n!}(1 - \frac{\varepsilon}{6}); \]
(b) If \( x \in Z \), and \( \alpha \) denotes the metric ball in \( Z \) of radius \( \rho \) centered at \( x \), where \( Z \) is endowed with the intrinsic metric, then the volume of \( \alpha \) is less than \( \frac{2\rho^n}{n!}(1 - \frac{\rho}{\delta}) \).

**Proof.** The distances between two points of \( Z \) endowed with the inner metric cannot be less than the distance between these points in the metric of \( X^{n+1} \). Therefore, assertion (b) of the lemma follows from assertion (a) simply because each metric ball in \( Z \) endowed with the inner metric is contained in the metric ball in \( X^{n+1} \) with the same center and the same radius. Thus, it is sufficient to prove (a).

We apply the previous lemma to \( B \). There exists a metric sphere \( S \) of radius \( s \in (t(x)(\frac{\delta}{3})^{\frac{1}{n-1}}, t(x)) \) that is a subpolyhedron and satisfies \( |S| < (1 - \frac{1}{3})\epsilon \frac{2n^3}{n!} \). Let \( \lambda = \tau(\frac{\delta}{3})^{\frac{1}{n-1}}, \delta = \frac{\epsilon \lambda^n}{3n!} \). Observe that

\[
\frac{2s^n}{n!}(1 - \frac{1}{6}\epsilon) - |S| > \frac{2s^n \epsilon}{n!} \frac{1}{6} > \delta. \quad (\ast)\]

The proof is by contradiction. Let \( Z \) be a \( \delta \)-minimal \( r \)-separating set. We assume that for some \( x \) and each \( \rho \in (t(x)(\frac{\delta}{3})^{\frac{1}{n-1}}, t(x)) \), the metric ball \( \beta \) of radius \( \rho \) centered at \( x \) does not satisfy the inequality in part (a) of the lemma. In particular, this is true for the metric ball \( \beta \) of radius \( s \) bounded by \( S \): \( |Z \cap \beta| \geq \frac{\frac{2n^3}{n!}}{2}(1 - \frac{\epsilon}{\delta}) \). We are going to modify \( Z \) to obtain another \( r \)-separating set \( Z' \) with volume less than \( |Z| - \delta \) arriving to a contradiction.

To construct \( Z' \) we remove from \( Z \) all points inside the metric ball bounded by \( S \), and take the union of the resulting set \( Z_1 \) with \( S \). It is obvious that \( Z' \) is \( r \)-separating. Indeed, all components of \( X^{n+1} \) \( \setminus \) \( Z \) outside of \( S \) became smaller or unchanged when we replace \( Z \) by \( Z' \), and the “new” component or components of \( X^{n+1} \) \( \setminus \) \( Z' \) inside of \( S \) can clearly be covered by the metric ball of radius \( r \) centered at \( x \).

It follows from formula (\ast) that \( |Z'| < |Z| - \delta \).

\( \square \)

**Lemma 2.5.** Assume that \( Y^n \) is a \( d \)-separating subpolyhedron in \( X^{n+1} \) such that for some \( d \) \( U \cap_{n-1}(Y^n) \leq d \). Then \( U \cap_{n}(X^{n+1}) \leq d \).

**Proof.** First proof. Let \( \{U_\alpha\}_{\alpha \in A} \) be the set of all connected components of \( X \setminus Y \). Observe that for each \( \alpha \), \( \beta \cap_{n-1}(Y^n) \leq d \). Consider the map \( f \) of \( Y \) to a \( (n-1) \)-dimensional simplicial complex \( K \) such that for each \( x \in K \), \( f^{-1}(x) \) can be covered by a metric ball \( b(x) \) of radius \( d + \delta \) in the intrinsic metric of \( Y^n \), where \( \delta \) is arbitrarily small. Observe that the metric ball in \( X^{n+1} \) with the same center and radius will contain \( b(x) \), and, therefore, \( f^{-1}(x) \). For each \( \alpha \) take a copy \( CK_\alpha \) of the cone \( CK \) over \( K \). Using the version of Tietze extension theorem for maps into contractible simplicial complexes, we can extend the restriction of \( f \) to \( \beta \cap_{n-1}(Y^n) \) to a continuous map \( g \) of the closure of \( U_\alpha \) to \( CK_\alpha \) (compare with a similar argument in section 6.1 of [LLNR]).

We would like to change this map so that the images of all points of \( \beta \) would remain
unchanged, and all points in \( \bar{U}_\alpha \setminus \partial U_\alpha \) will be mapped to \( CK_\alpha \setminus K_\alpha \) (that is, to the interior of the cone). For this purpose endow each top-dimensional simplex in \( CK_\alpha \) by the metric of the Euclidean regular simplex with side length \( d \). For each \( x \in \bar{U}_\alpha \) \( g(x) \) will be either the tip of the cone, or a point on the unique generator of the cone passing through \( x \) (which is a straight line segment with one end at \( K_\alpha \) and another end at the tip of the cone). If \( g(x) \) is the tip of the cone, the new map \( h(x) = g(x) \). Otherwise, let \( \phi(x) \) denote the distance from \( g(x) \) to the tip of the cone. Now move \( g(x) \) towards the tip of the cone by \( \min\{\phi(x), \text{dist}(x, \partial U)\} \). Now glue all copies of \( CK_\alpha \) into one \( n \)-dimensional simplicial complex \( L \) by identifying all copies of \( K_\alpha \) at the boundaries into one copy of \( K \).

The resulting map \( h: X \rightarrow L \) is a continuous map. By construction, its restriction to \( Y \) coincides with \( f \), and for each point \( x \in Y \), \( f^{-1}(f(x)) \) is in \( Y \). For each \( x \in U_\alpha \), \( f(x) \in CK_\alpha \setminus K_\alpha \), and \( f^{-1}(f(x)) \in U_\alpha \). In both cases \( f^{-1}(f(x)) \) can be covered by a ball of radius \( d + \delta \).

**Second proof.** After reading the first version of this paper ([N]) Roman Karasev suggested the following very simple proof of Lemma 2.5. His proof uses the definition of \( UR_n(X) \) as the infimum of \( r \) such that there exists a cover of \( X \) by open sets with radii \( \leq r \) with multiplicity of the covering \( \leq n + 1 \). Start with an open covering of \( Y^n \) of multiplicity \( \leq n \) with radii of the sets in the intrinsic metric \( \leq d \). Then we convert this covering of \( Y^n \) into an open covering of a very small open neighbourhood of \( Y^n \) in \( X^{n+1} \) without increasing the multiplicity and increasing the radii by not more than an arbitrarily small amount. Finally, we add all open sets \( U_\alpha \) to the covering increasing its multiplicity by 1.

\[ \square \]

**Proposition 2.6.** Let \( r, \varepsilon, \tau < r \) be positive real numbers, and \( X^n \) is a compact \( n \)-dimensional Riemannian polyhedron. Assume that for each \( x \in X^n \) there exists \( t(x) \in (\tau, r) \) such that the metric ball \( B \) in \( X^n \) of radius \( t(x) \) centered at \( x \) satisfies \( |B| < \frac{2t(x)^n}{n!} \) \( (1 - \varepsilon) \). Then \( UR_{n-1}(X) < r \).

**Proof.** We are going to prove this proposition using the induction with respect to the dimension \( n \). Lemma 2.1 is the base of induction. To prove the induction step assume that the theorem is true for \( n \). To prove it for \( n + 1 \) choose a sufficiently small positive \( \delta \) (as in Lemma 2.4) and consider a \( \delta \)-minimal \( r \)-separating \( n \)-dimensional subpolyhedron \( Z \) of \( X^{n+1} \). Lemma 2.4 immediately implies that Riemannian subpolyhedron \( Z \) satisfies the assumptions of the proposition the following values of the parameters \( n, r, \varepsilon, \tau, t(x) \): We take \( n \) and \( r \) as in the text of the proposition, \( \frac{\varepsilon}{6} \) as a new value of \( \varepsilon \), and \( \tau(\frac{\varepsilon}{3})^{n+1} \) as a new value of \( \tau \). Finally, we take \( g \in (\tau(\frac{\varepsilon}{3})^{n+1}, r) \) provided by Lemma 2.4 as the value of \( t(x) \).

The induction assumption implies that \( UR_{n-1}(Z) \leq g < r \) in the intrinsic metric on \( Z \). The same inequality will automatically be true for the “shorter” extrinsic...
Now the induction step follows from Lemma 2.5 applied for $Y^n = Z$, $d = r$. \hfill \Box

Now we are going to establish Theorem 1.1 for the class of all compact Riemannian polyhedra (and not only Riemannian manifolds):

**Theorem 2.7.** Let $M^n$ be a compact Riemannian polyhedron (for example, a closed Riemannian manifold), and $r > 0$ a real number. Assume that for every $x \in M^n$ there exists $t = t(x) \in (0, r]$ such that the volume of the metric ball of radius $t$ centered at $x$ is less than $\frac{2^n}{n!}$. Then $UR_{n-1}(M^n) < r$.

**Proof.** We are going to deduce this theorem from Proposition 2.6 by proving that its assumption can be relaxed in two ways.

First, we would like to demonstrate that the assumption of existence of $\varepsilon > 0$ such that for each $x$ there exists $t \in [\tau, r]$ such that the ball $B = B(x, t)$ satisfies $|B| < \frac{2^n}{n!}(1 - \varepsilon)$ is equivalent to the assumption that for each $x$ there exists $t \in [\tau, r]$ such that $|B| < \frac{2^n}{n!}$. Observe that $\frac{|B(x, R)|}{R^n}$ is a continuous function of the center $x \in M^n$ of the ball $B(x, R)$ and its radius $R$. Note that $g(x) = \min_{R \in [\tau, r]} \frac{|B(x, R)|}{R^n}$ is a continuous function of $x$ such that its value at every point is strictly less than $\frac{2^n}{n!}$. Hence, the maximum of $g(x)$ over all $x \in M^n$ will be attained at some point, and will be strictly less than $\frac{2^n}{n!}$. Now one can choose $\varepsilon$ as $0.5 (\frac{2^n}{n!} - \max_{x \in M^n} g(x))$.

Second, we are going to demonstrate that one does not need the condition that for some $x$ and some $t$ as in Proposition 2.6 $t \geq \tau$, as this condition automatically holds. For all $x \in M^n$ formally define $t(x)$ as $\sup_{t \in (0, r]} \{t \mid \frac{|B(x, t)|}{t^n} < \frac{2^n}{n!}\}$. The assumption of the theorem is that the set of $t$ is non-empty, and, therefore, $t(x)$ is defined and positive for all $x$. Observe that $t(x)$ is lower-semicontinuous, i.e. $t(x) \leq \liminf_{y \to x} t(y)$. Indeed, as $\frac{|B(x, t)|}{t^n}$ is continuous, if $\frac{|B(x, t)|}{t^n} < \frac{2^n}{n!}$ for some $t$, then the same inequality will be true for all $y$ sufficiently close to $x$. Therefore, for each positive $\delta$ the inequality $t(y) \geq t(x) - \delta$ holds for all $y$ sufficiently close to $x$. This observation immediately implies the lower-semicontinuity of $t$. Hence, $t(x)$ attains its positive minimum on $M^n$ which can be chosen as $\tau$. Now the theorem follows from Proposition 2.6. \hfill \Box

**Proof of the inequality (4) in section 1.2:** Finally, I am going to present an elementary proof of the inequality $sys_1(X^n) \leq 2UR_{n-1}(X^n)$ for essential polyhedral length spaces $X^n$. I learned the idea of this proof from Roman Karasev. This proof is based on ideas from [S].

Recall that $UR_{n-1}(X^n)$ can be defined as the lower bound of $r$ such that there exists a covering of $X^n$ by connected open sets with radii $\leq r$ with multiplicity $\leq n$. 


Assume that $\text{sys}_1(X^n) > 2r$, where $r = UR_{m-1}(X^n)$. Choose a covering of $X^n$ of multiplicity $\leq n$ by connected open sets $U_\alpha$ with radii $\leq r + \delta$, where $\delta < 1/3(\text{sys}_1(M^n) - 2r)$. Consider a closed curve $\gamma$ in some $U_\alpha$. Let $p$ be the center of a ball of radius $r + \delta$ covering $U_\alpha$. We can homotope $\gamma$ into a concatenation of many thin triangles $p\gamma(t_i)\gamma(t_{i+1})$, where the length of the arc of $\gamma$ between $\gamma(t_i)$ and $\gamma(t_{i+1})$ does not exceed $\delta$, and two other sides are minimizing geodesics. The length of each of these triangles is less than $\text{sys}_1(X^n)$. Therefore, these triangles are contractible, and so is $\gamma$. Thus, the inclusion homomorphisms $\pi_1(U_\alpha) \rightarrow \pi_1(X^n)$ are trivial, and each $U_\alpha$ lifts to a collection of disjoint open sets $(\tilde{U}_\alpha)_{g} \subset \tilde{X}^n$, where $g$ runs over $\pi_1(X^n)$, and $\tilde{X}$ denotes the universal covering of $X^n$ endowed with the pullback metric.

Consider the nerves $N$ of the covering $\{U_\alpha\}$ of $X^n$, and $\tilde{N}$ of the covering $\{\{\tilde{U}_\alpha\}_g\}$ of $\tilde{X}^n$. It is easy to see that there exists a commutative square with horizontal sides $\tilde{X}^n \rightarrow \tilde{N}$ and $X^n \rightarrow N$, where vertical sides $\tilde{X}^n \rightarrow X^n$, and $\tilde{N} \rightarrow N$ are the universal covering maps. This easily implies that the map $X^n \rightarrow N$ induces an injective homomorphism $\pi_1(X^n) \rightarrow \pi_1(N)$. As this homomorphism is obviously surjective, it is an isomorphism. Thus, the classifying map $X^n \rightarrow K(\pi_1(M^n), 1)$ factors through the nerve $N$ that has dimension $\leq n-1$. (Recall that all maps $X^n \rightarrow K(\pi_1(X^n), 1)$ that induce the isomorphism of fundamental groups are homotopic.) Therefore, $X^n$ is not essential. Equivalently, if $X^n$ is essential, then $\text{sys}_1(X^n) \leq 2r$.

3. Hausdorff Content

Similar ideas can be applied to majorize $UR_{m-1}$ of a compact or even a boundedly compact metric space $X$ in terms of the Hausdorff content, $HC_m$, of metric balls in $X$. The key is the coarea inequality proven in [LLNR]. (Note that in this section we are no longer assuming that $X$ is a Riemannian polyhedron or even a length space.)

Recall that a metric space is called boundedly compact if all its closed and bounded subsets are compact. Given a boundedly compact metric space $X$, its bounded subset $A$, and a positive real $m$, one defines the $m$-dimensional Hausdorff content, $HC_m(A)$, of $A$ as $\inf \sum_i r^m_i$, where the infimum is taken over all coverings of $A$ by closed metric balls $\beta_i$ of radii $r_i$ in $X$. If $A$ is empty, then it can be covered by the empty set of metric balls, and $HC_m(A) = 0$.

For convenience of the reader we present a version of the coarea inequality for Hausdorff content proven in [LLNR]:

**Lemma 3.1.** ([LLNR]) Let $m, r_1, r_2$ be real numbers such that $0 \leq r_1 < r_2, m > 0$, $X$ a metric space, $Y$ a subset of $X$, $x$ a point in $X$. For $j \in \{1, 2\}$ let $B_j$ be closed metric balls of radius $r_j$ centered at $x$, $A$ the annulus $B_2 \setminus B_1$, and $S_2$, $s \in (r_1, r_2)$, metric spheres of radius $s$ centered at $x$. Then there exists $s \in (r_1, r_2)$ such that
\[ \text{HC}_{m-1}(S \cap Y) \leq \frac{2}{r_2-r_1} \text{HC}_m(A \cap Y). \] In particular, when \( r_1 = 0 \), we see that for each \( r \) and metric ball \( B \) of radius \( r \) centered at \( x \) there exists a positive \( s < r \) such that \( \text{HC}_{m-1}(S \cap Y) \leq \frac{2}{r} \text{HC}_m(B \cap Y) \).

**Proof.** For an arbitrarily small \( \varepsilon > 0 \) choose a covering of \( A \cap Y \) by a countable collection of closed metric balls \( \beta_i \subset X \) with radii \( r_i \) such that \( \sum_i r_i^m \leq \text{HC}_m(A \cap Y) + \varepsilon \). Observe that \( \text{HC}_{m-1}(S \cap Y) \leq \text{HC}^*_m(S \cap Y) \) defined as \( \sum_{i \in I_s} r_i^{m-1} \), where \( I_s \) is the set of indices \( i \) such that \( \beta_i \cap S \cap Y \neq \emptyset \). We are going to prove that for some \( s \) \( \text{HC}^*_m(S \cap Y) \leq \frac{2}{r_2-r_1} \sum_i r_i^m \), which implies the lemma. For this purpose it is sufficient to prove that

\[ \int_{r_1}^{r_2} \text{HC}^*_m(S \cap Y) ds \leq 2 \sum_i r_i^m. \]

For each \( i \) and \( s \) define \( \chi_i(s) \) as 1, if \( B_i \cap S \cap Y \neq \emptyset \), and 0 otherwise. Using this notation

\[ \int_{r_1}^{r_2} \text{HC}^*_m(S \cap Y) ds = \int_{r_1}^{r_2} \sum_i r_i^{m-1} \chi_i(s) ds = \sum_i \int_{r_1}^{r_2} r_i^{m-1} \chi_i(s) ds \leq \sum_i r_i^{m-1}(2r_i) = 2 \sum_i r_i^m. \]

**Remark.** The constant 2 in the right hand side of the coarea inequality is optimal. Indeed, let \( m = 1 \), \( r_1 = 0 \), \( r_2 = 1 \), \( X = Y = [0,1] \), \( x = 0 \), and \( A = (0,1] \). \( \text{HC}_1(A \cap Y) = \frac{1}{2} \), as \( A \) can be covered by a ball of radius 0.5 centered at 0.5, and \( \text{HC}_0 \) of the intersection of \( Y \) with any geodesic sphere of radius in \((0,1)\) centered at \( x = 0 \) is 1.

**Lemma 3.2.** Let \( X \) be a metric space, \( Y \) a subset of \( X \). Assume that for every metric ball \( B \) of radius \( r \) in \( X \) \( \text{HC}_1(B \cap Y) < \frac{r}{2} \). Then \( UR_0(Y) < r \).

**Proof.** We are going to prove that each connected component \( C \) of \( Y \) is contained in the interior of a closed metric ball \( B(C) = B(x, \rho) \) of radius \( \rho < r \) with a center \( x = x(C) \in C \). (In fact, we will see that we can choose any point \( x \in C \) as \( x(C) \).) If so, we can map \( Y \) into the set of centers of balls \( B(C) \) by sending all points of \( C \) to the center \( x(C) \) of \( B(C) \).

Given \( x \in C \) apply the coarea inequality (Lemma 3.1) to the closed ball \( B(x, r) \subset X \) of radius \( r \) centered at \( x \) (minus \( x \)) regarded as the annulus with radii \( r_1 = 0 \) and \( r_2 = r \). We are going to obtain a geodesic sphere \( S \) centered at \( x \) of a positive radius \( \rho < r \) such that \( \text{HC}_0(S \cap Y) \leq \frac{2}{r} \text{HC}_1(B(x, r) \cap Y) < 1 \). Note that \( \text{HC}_0(S \cap Y) \) is just the minimal number of metric balls in \( X \) required to cover \( S \cap Y \), and so it is either equal to 1, if \( S \cap Y \) is non-empty, or to 0, if \( S \cap Y \) is empty. Therefore, we conclude that \( \text{HC}_0(S \cap Y) = 0 \), and \( S \cap Y \) is empty. Therefore, \( B(x, \rho) \cap Y \) coincides with the intersection of \( Y \) with the open metric ball of radius \( \rho \) centered at \( x \), and is the union of some collection of connected components of \( Y \), one of which coincides with \( C \). Now we can define \( B(C) \) as \( B(x, \rho) \). \qed

**Definition 3.3.** Given a subset \( W \) of \( X \), positive real \( n \), and \( \delta > 0 \), a collection of metric balls \( B_i \) in \( X \) with radii \( r_i \) is called a \((n, \delta)\)-optimal covering of \( W \), if they cover \( W \), and

\[ \sum_i r_i^n \leq \text{HC}_n(W) + \delta. \]
**Definition 3.4.** Let $X$ be a metric space, $Y$ its compact subset. A compact subset $Z$ of $Y$ is called $d$-separating for $Y$ if each connected component of its complement $Y \setminus Z$ can be covered by a metric ball in $X$ of radius $\leq d$. Let $HC_n^{(b)}(Z)$ denote the infimum of $\sum_i r_i^n$ over all coverings of $Z$ by closed metric balls with radii $r_i \leq b$ in $X$. Denote the infimum of $HC_n^{(b)}(Z)$ over all $d$-separating sets $Z$ by $I_Y(d, b, n)$. If $\delta > 0$ is a positive real number, a $d$-separating set $Z$ is called $(b, n, \delta)$-minimal if $HC_n^{(b)}(Z) \leq I_Y(d, b, n) + \delta$.

Using $HC_n^{(b)}(Y)$ instead of $HC_n(Y)$ here is another simple but beautiful idea of Papasoglu from [P] designed to overcome non-additivity of Hausdorff content (and strongly reminiscent of ideas earlier used for the same purpose in [LLNR]).

**Lemma 3.5.** Let $X$ be a metric space, $Y$ its compact subset, $r, \mu$ positive real numbers, $n \geq 1$ an integer. Assume that for each closed metric ball $B$ of radius $r$ in $X$, $HC_{n+1}(B \cap Y) < (\frac{r}{4(n+1)})^{n+1}-\mu$. Let $\mu_1 = (\frac{r}{4(n+1)})^n - (\frac{r}{4(n+1)})^{n+1}-\mu$. Assume that $Z$ is a $(\frac{r}{4(n+1)}-\mu_1, n, 4(n+1)^\mu)$-minimal $r$-separating set for $Y$. Then for each ball $\beta$ of radius $\rho = r(1-\frac{1}{n+1})$ in $X$

$$HC_n(Z \cap \beta) < (\frac{\rho}{4n})^n.$$ 

**Proof.** The proof is by contradiction. We assume that there exists a ball $\beta$ or radius $\rho$ centered at a point $x$ that does not satisfy the above inequality. We are going to modify $Z$ to obtain another $r$-separating set $Z'$ with a significantly lower $HC_n^{(\frac{r}{4(n+1)}-\mu_1)}$ than $Z$, obtaining a contradiction that proves the lemma.

Using Lemma 3.1 one can choose a sphere $S$ centered at $x$ of radius in the interval $(r(1-\frac{1}{2(n+1)}), r)$ such that $HC_n(S \cap Y) < (\frac{r}{4(n+1)})^n - 4(n+1)^\mu = (\frac{r}{4(n+1)}-\mu_1)^n$. Therefore, $HC_n^{(\frac{r}{4(n+1)}-\mu_1)}(S \cap Y) = HC_n(S \cap Y)$.

To construct $Z'$ we remove from $Z$ all points inside the ball bounded by $S$, and take the union of the resulting set $Z_1$ with $S \cap Y$. It is obvious that $Z'$ is $r$-separating.

Now we are going to estimate $HC_n^{(\frac{r}{4(n+1)}-\mu_1)}(Z_1)$. First, note that none of the balls of radius $\leq \frac{r}{4(n+1)}-\mu_1$ in $X$ used to cover $Z_1$ in a nearly optimal way can intersect the closed ball $B'$ of radius $r(1-\frac{1}{2(n+1)})$ centered at $x$. On the other hand, every metric ball of radius $\leq \frac{r}{4(n+1)}-\mu_1$ that has a non-empty intersection with $\beta$ is contained in $B'$. Therefore, $HC_n^{(\frac{r}{4(n+1)}-\mu_1)}(Z) \geq HC_n^{(\frac{r}{4(n+1)}-\mu_1)}(Z_1) + HC_n^{(\frac{r}{4(n+1)}-\mu_1)}(Z \cap \beta) \geq HC_n^{(\frac{r}{4(n+1)}-\mu_1)}(Z_1) + HC_n(Z \cap \beta)$. Hence, $HC_n^{(\frac{r}{4(n+1)}-\mu_1)}(Z_1) \leq HC_n^{(\frac{r}{4(n+1)}-\mu_1)}(Z) - HC_n(Z \cap \beta)$, and $HC_n^{(\frac{r}{4(n+1)}-\mu_1)}(Z') \leq HC_n^{(\frac{r}{4(n+1)}-\mu_1)}(Z) - HC_n(Z \cap \beta)$.
Lemma 3.6. Let $X$ be a metric space, $Y$ a closed subset of $X$, $r$ a positive real number, $Z$ a $r$-separating set for $Y$. Assume that $Z$ is compact, and $UR_{n-1}(Z) \leq r$. Then $UR_n(Y) \leq r$.

Proof. Either of the two proofs of Lemma 2.5 can be used with only minor modifications to prove this lemma. For example, if $UR_{n-1}(Z) \leq r$, then for an arbitrary positive $\delta$, there exists a finite collection of open sets of radius $< r + \delta$ in $Z$ that covers $Z$ and has multiplicity \leq n. It is easy to demonstrate that one can modify this collection so that the open sets in $Z$ become restrictions of open subsets of $X$ with radius $< r + 2\delta$, and the multiplicity of the resulting collection of open sets in $X$ does not exceed $n$. Taking the intersections of these open sets with $Y$, we obtain a collection of open subsets of $Y$ of multiplicity \leq n and radius $< r + 2\delta$. After adding all connected components of $Y \setminus Z$ to this collection we obtain a covering of $Y$ by open in $Y$ sets of radius $< r + 2\delta$ such that the multiplicity of the resulting collection of open sets does not exceed $n + 1$. □

Theorem 3.7. Let $r$ be a positive number, $n$ a positive integer, $X$ a metric space, $Y$ a compact subset of $X$ such that for each metric ball $B$ of radius $r$, $HC_n(B \cap Y) < \left(\frac{r}{4n}\right)^n$. Then $UR_{n-1}(Y) < r$.

Proof. We are going to use the induction with respect $n$. Lemma 3.2 provides the base of induction. To prove the induction step assume that the theorem is true for $n$. To prove it for $n + 1$ observe that the assumptions of the theorem and the compactness of $Y$ imply the existence of a positive $\mu$ such that for each metric ball $B$ of radius $r$, $HC_{n+1}(B \cap Y) < \left(\frac{r}{4(n+1)}\right)^{n+1} - \mu$. Define $\mu_1$ as in Lemma 3.5, and consider a $(\frac{r}{4(n+1)} - \mu_1, n, 4(n + 1)\frac{\mu_1}{r})$-separating set $Z$. Lemma 3.5 implies that $Z$ satisfies the conditions of the present theorem for $Y$ with parameters $n$ and $\varrho = r(1 - \frac{1}{n+1})$ (instead of $r$). The induction assumption implies that $UR_{n-1}(Z) < \varrho < r$. Now the induction step follows from Lemma 3.6. □

Proof of Theorem 1.4. The first part of Theorem 1.4 immediately follows from Theorem 3.7 applied for $Y = X$. To prove the second part it is sufficient to take $r = \left(\frac{HC_n(X)}{\varepsilon_n}\right)^\frac{1}{n} + \delta$ for an arbitrarily small $\delta > 0$, apply the first part, and pass to the limit, when $\delta \rightarrow 0$.

To prove the third part observe that the main difficulty in the non-compact case is that every $r$-separating set in $X$ might have infinite Hausdorff content. In this case there will be no (almost) minimal $r$-separating sets, and (key) Lemma 3.5 cannot be
applied. Instead, one can use another trick from [P]: Let $\rho > 0$ be a real number. One chooses a point $x$ of a boundedly compact $X$ and covers $X$ by two overlapping sets of closed annuli centered at $x$. One family of annuli involves radii in the intervals $[8(i - 1)\rho, 8i\rho]$ for all positive integer $i$, another $[(8(i - 1) + 4)\rho, (8i + 4)\rho]$. The idea is that the union of almost minimal $\rho$-separating subsets for all these annuli will be a $\rho$-separating family for $X$. One can even remove the parts of almost minimal separating sets in all annuli that bound a domain only together with a non-empty subset of the boundary of the annulus. Indeed, all points in such domains with “destroyed” boundary will be $2\rho$-close to the boundary of the annulus, and, therefore, $2\rho$-close to the central sphere of an overlapping annulus $A$ in the second family. Therefore, they will be $2\rho$-far from the boundary of $A$ and will be in domains in $A$ of radius $\leq \rho$ such that their closures do not intersect $\partial A$.

Now we can take each annulus (in one of the two collections) as $Y$ in Lemma 3.5, and apply Lemma 3.5. Remove the intersection of the separating set $Z$ (as in Lemma 3.5) with the boundary of the annulus. This remaining part will also satisfy the conclusion of Lemma 3.5. Now take the union of the remaining parts of $Z$ over all annuli. As we observed, we will obtain an $r$-separating set in the whole metric space $X$. Each ball of radius $\rho$ (as in Lemma 3.5) will intersect almost minimal $r$-separating sets, $Z$, coming from at most two annuli. Therefore, the conclusion of Lemma 3.5 will be almost true for the union $U$ of all these separating sets: One will only need the extra factor of $2$ in the right hand side of the inequality in the conclusion of Lemma 3.5. Thus, we obtain an analogue of Lemma 3.5 for non-compact boundedly compact metric spaces for $U$ instead of $Z$ with the extra factor of $2$ in the right-hand side of the inequality.

Going through the rest of the proof, we see that this leads to the appearance of the extra factor of $2$ in the denominator of the right-hand side of the inequality in Theorem 1.4, part 3.

**Remark.** We do not see how to adapt this proof of Theorem 1.4(1) to prove its version, where the assumption that each metric ball $B$ of radius $r$ satisfies the inequality $HC_n(B) < \left(\frac{r}{4n}\right)^n$ is replaced by a weaker assumption that for each $r$ there exists $\rho \in (0, r)$ such that $HC_n(B) \leq c(n)\rho^n$, where one is allowed to choose any positive constant $c(n)$. The reason is that one uses $HC_n(\frac{r}{4n+1})$ in the proof of Lemma 3.5, and it is not clear what is the correct replacement of this quantity if $r$ is allowed to be variable. So, we do not know how to prove a Hausdorff content analog of Theorem 1.1 where the radii of “small” balls can be variable.

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