A short proof of the existence of supercuspidal representations for all reductive $p$-adic groups

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Résumé

Let $G$ be a reductive $p$-adic group. It can be important for certain global arguments on the trace formula to know that $G$ admits supercuspidal complex representations. We prove that it is always the case. This result has already been established by A. Kret in [K]. Our argument is of a different nature and is based on the Harish-Chandra theory of cusp forms. It ultimately relies on the existence of elliptic maximal tori in $G$.

Let $p$ be a prime number and let $F$ be a $p$-adic field (i.e. a finite extension of $\mathbb{Q}_p$). We denote by $\mathcal{O}$ the ring of integers of $F$ and we fix a uniformizer $\varpi \in \mathcal{O}$. We also denote by $\text{val} : F^\times \to \mathbb{Z}$ the normalized valuation. Let $G$ be a connected reductive group defined over $F$. We will denote by $\mathfrak{g}$ the Lie algebra of $G$. A sentence like "Let $P = MN$ be a parabolic subgroup of $G$" will mean as usual that $P$ is a parabolic subgroup of $G$ defined over $F$, that $N$ is its unipotent radical and that $M$ is a Levi component of $P$ also defined over $F$. Also, all subgroups of $G$ that we consider will be implicitly assumed to be defined over $F$.

Recall that a smooth representation of $G(F)$ is a pair $(\pi, V_\pi)$ where $V_\pi$ is a complex vector space (usually infinite dimensional) and $\pi$ is a morphism $G(F) \to GL(V_\pi)$ such that for all vector $v \in V_\pi$ the stabilizer $\text{Stab}_{G(F)}(v)$ of $v$ in $G(F)$ is an open subgroup. Let $(\pi, V_\pi)$ be a smooth representation of $G(F)$ and let $P = MN$ be a parabolic subgroup of $G$. The Jacquet module of $(\pi, V_\pi)$ with respect to $P$ is the space

$$V_{\pi,N} = V_\pi / V_\pi(N)$$

where $V_\pi(N)$ is the subspace of $V_\pi$ generated by the elements $v - \pi(n)v$ for all $v \in V_\pi$ and all $n \in N(F)$. It is also the bigger quotient of $V_\pi$ on which $N(F)$ acts trivially. There is a natural linear action $\pi_N$ of $M(F)$ on $V_{\pi,N}$ and $(\pi_N, V_{\pi,N})$ is a smooth representation of $M(F)$. The functor $V_\pi \mapsto V_{\pi,N}$ is an exact functor from the category of smooth representations of $G(F)$ to the category of smooth representations of $M(F)$. Indeed, this follows from the following fact (cf proposition III.2.9 of [Re])

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(1) Let \((N(F)_k)_{k \geq 0}\) be an increasing sequence of compact-open subgroups of \(N(F)\) such that \(N(F) = \bigcup_{k \geq 0} N(F)_k\) (such sequence always exists). Then a vector \(v \in V_\pi\) belongs to \(V_{\pi,N}\) if and only if there exists \(k \geq 0\) such that
\[
\int_{N(F)_k} \pi(n)v dn = 0
\]
Let \((\pi, V_\pi)\) be an irreducible smooth representation of \(G(F)\) (irreducible means that \(V_\pi\) is nonzero and that it has no non-trivial \(G(F)\)-invariant subspace). We say that \((\pi, V_\pi)\) is supercuspidal if for all proper parabolic subgroup \(P = MN\) of \(G\), the Jacquet module \(V_{\pi,N}\) is zero. An equivalent conditions is that the coefficients of \((\pi, V_\pi)\) are compactly supported modulo the center (cf theorem VI.2.1 of [Re]).

The purpose of this short article is to show the following

**Theorem 1** \(G(F)\) admits irreducible supercuspidal representations.

This theorem has already been proved by A.Kret ([K]). We propose here a different proof. Namely, we will deduce theorem 1 from the existence of nonzero compactly supported cusp forms, in the sense of Harish-Chandra, for the group \(G(F)\). Before stating this existence result, we need to introduce some more definitions and notations. We will denote by \(C^\infty_c(G(F))\) the space of complex-valued functions on \(G(F)\) that are smooth, i.e. locally constant, and compactly supported. We say that a function \(f \in C^\infty_c(G(F))\) is a cusp form if for all proper parabolic subgroup \(P = MN\) of \(G\) we have
\[
\int_{N(F)} f(xn) dn = 0, \quad \forall x \in G(F)
\]
(thes functions are called supercusp forms in [H-C]). We denote by \(C^\infty_{c,cusp}(G(F)) \subseteq C^\infty_c(G(F))\) the subspace of cusp forms. As we said, theorem 1 follows from the following proposition.

**Proposition 1** We have \(C^\infty_{c,cusp}(G(F)) \neq 0\).

Proof that proposition 1 implies theorem 1: Let us denote by \(\rho\) the action of \(G(F)\) on \(C^\infty_c(G(F))\) given by right translation. Then, \((\rho, C^\infty_c(G(F)))\) is a smooth representation of \(G(F)\). Moreover, it is easy to see that the subspace \(C^\infty_{c,cusp}(G(F)) \subseteq C^\infty_c(G(F))\) is \(G(F)\)-invariant. We claim the following

(2) For all proper parabolic subgroup \(P = MN\) of \(G\), we have
\[
C^\infty_{c,cusp}(G(F))_N = 0
\]
Let $P = MN$ be a proper parabolic subgroup of $G$ and let us fix an increasing sequence $(N(F)_k)_{k \geq 0}$ of compact-open subgroups of $N(F)$ such that $N(F) = \bigcup_{k \geq 0} N(F)_k$. Let $f \in C_{c, \text{cusp}}(G(F))$. By (1), it suffices to show the existence of an integer $k \geq 0$ such that

$$\int_{N(F)_k} \rho(n) f dn = 0$$

or what amounts to the same

$$\int_{N(F)_k} f(xn) dn = 0, \quad \forall x \in G(F)$$

Since $\text{Supp}(f)$ (the support of the function $f$) is compact, there exists $k \geq 0$ such that

$$\text{Supp}(f) \cap N(F)_k = \emptyset$$

We now show that (3) is satisfied for such a $k$. Let $x \in G(F)$. If $x \notin N(F)_k$, the term inside the integral (3) is always zero and there is nothing to prove. Assume now that $x \in N(F)_k$. Up to translating $x$ by an element of $N(F)_k$, we may as well assume that $x \in \text{Supp}(f)$. Then, by (4) we have $xn \notin \text{Supp}(f)$ for all $n \in N(F) \setminus N(F)_k$. It follows that

$$\int_{N(F)_k} f(xn) dn = \int_{N(F)} f(xn) dn$$

But by definition of $C_{c, \text{cusp}}(G(F))$, this last integral is equal to zero. This proves (3) and ends the proof of (2).

We now show how to deduce from (2) that proposition \[\square\] implies theorem \[\square\]. Assume that proposition \[\square\] is satisfied. Then, we can find $f \in C_{c, \text{cusp}}(G(F))$ which is nonzero. Denote by $V_f$ the $G(F)$-invariant subspace of $C_{c, \text{cusp}}(G(F))$ generated by $f$ and let $V \subseteq V_f$ be a maximal $G(F)$-invariant subspace among those not containing $f$ (Zorn’s lemma). Then, $V_f/V$ is a smooth irreducible representation of $G(F)$. We claim that it is supercuspidal. Indeed, let $P = MN$ be a proper parabolic subgroup of $G$. By (2) and since the Jacquet module’s functor is left exact, we have $V_{f,N} = 0$. Hence, since the Jacquet module’s functor is also right exact, we have $(V_f/V)_N = 0$. Thus, $V_f/V$ is indeed a supercuspidal representation and this proves theorem \[\square\].

Because of the above, we are now left with proving proposition \[\square\]. The strategy is to prove first an analog result on the Lie algebra and then lift it to the group by mean of the exponential map. Let $C_c^\infty(g(F))$ be the space of complex-valued smooth and compactly supported functions on $g(F)$. We say that a function $\varphi \in C_c^\infty(g(F))$ is a cusp form if for all proper parabolic subgroup $P = MN$ of $G$ we have

$$\int_{n(F)} \varphi(X + N) dN = 0, \quad \forall X \in g(F)$$
We denote by $C^\infty_{c,\text{cusp}}(\mathfrak{g}(F)) \subseteq C^\infty_c(\mathfrak{g}(F))$ the subspace of cusp forms. The analog of proposition [1] for the Lie algebra is the following lemma

**Lemma 1** We have $C^\infty_{c,\text{cusp}}(\mathfrak{g}(F)) \neq 0$.

Before proving this lemma, we first show how it implies proposition [1].

**Proof that Lemma 1 implies Proposition 1:** Assume that Lemma 1 holds. Then, we can find a nonzero function $\varphi \in C^\infty_{c,\text{cusp}}(\mathfrak{g}(F))$. The idea is to lift $\varphi$ to a function on $G(F)$ using the exponential map. Of course, the exponential map is not necessarily defined on the support of $\varphi$. Hence, we need first to scale the function $\varphi$ so that its support becomes small. Let us fix an element $\lambda \in F^\times$ in all what follows. We define the function $\varphi_\lambda$ by

$$\varphi_\lambda(X) = \varphi(\lambda^{-1}X), \quad X \in \mathfrak{g}(F)$$

We easily check that $\varphi_\lambda$ is still a cusp form. Recall that there exists an open neighborhood $\omega \subseteq \mathfrak{g}(F)$ of 0 on which the exponential map $\exp$ is defined and such that it realizes an $F$-analytic isomorphism

$$\exp : \omega \cong \Omega$$

where $\Omega = \exp(\omega)$. Since $\text{Supp}(\varphi_\lambda) = \lambda \text{Supp}(\varphi)$, for $\lambda$ sufficiently small, we have

$$\text{Supp}(\varphi_\lambda) \subseteq \omega$$

We henceforth assume that $\lambda$ is that sufficiently small. This allows us to define a function $f_\lambda$ on $G(F)$ by setting

$$f_\lambda(g) = \begin{cases} \varphi_\lambda(X) & \text{if } g = \exp(X) \text{ for some } X \in \omega \\ 0 & \text{otherwise} \end{cases}$$

for all $g \in G(F)$. Note that we have $f_\lambda \in C^\infty_c(G(F))$ and obviously the function $f_\lambda$ is nonzero. Hence, we will be done if we can prove the following

(5) If $\lambda$ is sufficiently small, the function $f_\lambda$ is a cusp form.

Let us denote by $\log : \Omega \to \omega$ the inverse of the exponential map. Then, by the Campbell-Hausdorff formula, it is easy to see that we can find a lattice $L \subseteq \omega$ that satisfies the following condition

(6) $\log(e^X e^Y) \in X + Y + \omega^{\text{val}_L(X) + \text{val}_L(Y)} L$

for all $X, Y \in L$ and where we have set $\text{val}_L(X) = \sup\{k \in \mathbb{Z} : X \in \omega^k L\}$ for all $X \in \mathfrak{g}(F)$. For all integer $n \geq 0$, we set $K_n = \exp(\omega^n L)$. It is easy to infer from (6) that $K_n$ is an open-compact subgroup of $G(F)$ for all $n \geq 0$. Since $\varphi$ is smooth and compactly supported, there exists $n_0 \geq 0$ such that translation by $\omega^{n_0} L$ leaves $\varphi$ invariant. Also, since $\varphi$ is compactly
supported, there exists $n_1 \geq 0$ such that $\text{Supp}(\varphi) \subseteq \varpi^{-n_1}L$. We will show that (5) is true provided $\text{val}(\lambda) \geq 2n_1 + n_0$. Assume this is so and set $n = \text{val}(\lambda) - n_1$. Then, we have

\[(7) \quad \text{Supp}(\varphi_\lambda) = \lambda \text{Supp}(\varphi) \subseteq \lambda \varpi^{-n}L = \varpi^n L\]

Hence, it follows that

\[(8) \quad \text{Supp}(f_\lambda) \subseteq K_n\]

Let $P = MN$ be a proper parabolic subgroup of $G$ and let $x \in G(F)$. Consider the integral

\[(9) \quad \int_{N(F)} f_\lambda(xn)dn\]

If $xN(F) \cap K_n = \emptyset$, then by (8) the term inside the integral above is identically zero and it follows that the integral is itself equal to zero. Assume now that $xK_n \cap N(F) \neq \emptyset$. Up to translating $x$ by an element of $N(F)$, we may assume that $x \in K_n$. Then, we may write $x = e^X$ for some $X \in \varpi^n L$. Using again (8), and since $K_n$ is a subgroup of $G(F)$, we see that the integral (9) is supported on $K_n \cap N(F)$. Thus, we have

\[(10) \quad \int_{N(F)} f_\lambda(xn)dn = \int_{K_n \cap N(F)} f_\lambda(e^X n)dn\]

Set $L_N = L \cap n(F)$. Then, if we normalize measures correctly, the exponential map induces a measure preserving isomorphism $\varpi^n L_N \simeq K_n \cap N(F)$ so that we have

\[(11) \quad \int_{K_n \cap N(F)} f_\lambda(e^X n)dn = \int_{\varpi^n L_N} f_\lambda(e^X e^N)dn = \int_{\varpi^n L} \varphi_\lambda(\log(e^X e^N))dN\]

By (6), for all $N \in \varpi^n L_N$ we have

\[(12) \quad \log(e^X e^N) \in X + N + \varpi^{2n} L\]

Moreover, since $\varphi$ is invariant by translation by $\varpi^n L$, the function $\varphi_\lambda$ is invariant by translation by $\lambda \varpi^n L = \varpi^{n+n_1+n_0} L$ (recall that $n = \text{val}(\lambda) - n_1$). As $\text{val}(\lambda) \geq 2n_1 + n_0$, we also have $n \geq n_1 + n_0$. So finally, the function $\varphi_\lambda$ is invariant by translation by $\varpi^{2n} L$. Thus, by (12), we have

\[\varphi_\lambda(\log(e^X e^N)) = \varphi_\lambda(X + N)\]

for all $N \in \varpi^n L_N$. By (10) and (11), it follows that

\[(13) \quad \int_{N(F)} f_\lambda(xn)dn = \int_{\varpi^n L} \varphi_\lambda(X + N)dN\]

By (7) and since $X \in \varpi^n L$, the function $N \in n(F) \mapsto \varphi_\lambda(X + N)$ is supported on $\varpi^n L_N$. Hence, we have
Since $\varphi_\lambda$ is a cusp form, this last integral is zero. Hence, by (13) the integral (9) is also zero. Since it is true for all $x \in G(F)$ and all proper parabolic subgroup $P = MN$ of $G$, this shows that $f_\lambda$ is a cusp form. Hence, (5) is indeed satisfied as soon as $val(\lambda) \geq 2n_1 + n_0$ and this ends the proof that lemma 1 implies proposition 1. ■

It now only remains to prove lemma 1. Recall that a maximal torus $T$ in $G$ is said to be elliptic if $A_T = A_G$, where $A_T$ and $A_G$ denotes the maximal split subtorus in $T$ and the center of $G$ respectively. The proof of lemma 1 will ultimately rely on the following existence result (cf Theorem 6.21 of [PR]) :

**Theorem 2** $G$ admits an elliptic maximal torus.

**Proof of lemma 1** : Let us fix a symmetric non-degenerate bilinear form $B$ on $\mathfrak{g}(F)$ which is $G(F)$-invariant. Such a bilinear form is easy to construct. On $\mathfrak{g}_{\text{der}}(F)$, the derived subalgebra of $\mathfrak{g}(F)$, we have the Killing form $B_{\text{Kil}}$ which is symmetric $G(F)$-invariant and non-degenerate. Hence, we may take $B = B_\text{j} \oplus B_{\text{Kil}}$ where $B_\text{j}$ is any non-degenerate symmetric bilinear form on $\mathfrak{z}_G(F)$, the center of $\mathfrak{g}(F)$. Let us also fix a non-trivial continuous additive character $\psi : F \to \mathbb{C}^\times$. Using those, we can define the Fourier transform on $C_c^\infty(\mathfrak{g}(F))$ by \[
\hat{\varphi}(Y) = \int_{\mathfrak{g}(F)} \varphi(X)\psi(B(X,Y))dX, \quad \varphi \in C_c^\infty(\mathfrak{g}(F)), \ Y \in \mathfrak{g}(F)\]

Of course, this Fourier transform also depends on the choice of a Haar measure on $\mathfrak{g}(F)$. More generally, if $V$ is a subspace of $\mathfrak{g}(F)$ and $V^\perp$ denotes the orthogonal of $V$ with respect to $B$, we can also define a Fourier transform $C_c^\infty(V) \to C_c^\infty(\mathfrak{g}(F)/V^\perp)$, $\varphi \mapsto \hat{\varphi}$, by setting \[
\hat{\varphi}(Y) = \int_V \varphi(X)\psi(B(X,Y))dY, \quad X \in \mathfrak{g}(F)/V^\perp
\]

where again we need to choose a Haar measure on $V$. It is easy to check that for compatible choices of Haar measures, the following diagram commutes

\[
\begin{array}{ccc}
C_c^\infty(\mathfrak{g}(F)) & \xrightarrow{\text{FT}} & C_c^\infty(\mathfrak{g}(F)) \\
\downarrow \text{res}_V & & \downarrow \text{f}_{V^\perp} \\
C_c^\infty(V) & \xrightarrow{\text{FT}} & C_c^\infty(\mathfrak{g}(F)/V^\perp)
\end{array}
\]

where the horizontal arrows are Fourier transforms, the left vertical arrow is given by restriction to $V$ and the right vertical arrow is given by integration over the cosets of $V^\perp$. For $P = MN$ a parabolic subgroup of $G$, we have $p(F)^\perp = n(F)$. The commutation of the above diagram in this particular case gives us (for some compatible choices of Haar measures) the following formula
\[
\int_{n(F)} \hat{\varphi}(X + N)dN = \int_{p(F)} \varphi(Y)\psi(B(X, Y))dY
\]
for all $\varphi \in C_c^\infty(\mathfrak{g}(F))$ and all $X \in \mathfrak{g}(F)$.

Let $T_{\text{ell}}$ be an elliptic maximal torus of $G$ whose existence is insured by theorem 2. Let $t_{\text{ell}}$ be its Lie algebra and $t_{\text{ell,reg}} = t_{\text{ell}} \cap \mathfrak{g}_{\text{reg}}$ be the subset of $G$-regular elements in $t_{\text{ell}}$. Denote by $t_{\text{ell,reg}}(F)^G$ the subset of elements in $\mathfrak{g}_{\text{reg}}(F)$ that are $G(F)$-conjugated to an element of $t_{\text{ell,reg}}(F)$. Then, $t_{\text{ell,reg}}(F)^G$ is an open subset of $\mathfrak{g}(F)$ (since the map $T_{\text{ell}}(F) \setminus G(F) \times t_{\text{ell,reg}}(F) \to \mathfrak{g}(F)$, $(g, X) \mapsto g^{-1}Xg$, is everywhere submersive). In particular, we can certainly find a non-zero function $\varphi \in C_c^\infty(\mathfrak{g}(F))$ whose support is contained in $t_{\text{ell,reg}}(F)^G$. Let us fix such a function $\varphi$. We claim the following

(15) The function $\hat{\varphi}$ is a cusp form.

Indeed, let $P = MN$ be a proper parabolic subgroup of $G$ and let $X \in \mathfrak{g}(F)$. Then, we need to see that the following integral

\[
\int_{n(F)} \hat{\varphi}(X + N)dN
\]
is zero. By (14), this integral is equal to

\[
\int_{p(F)} \varphi(Y)\psi(B(X, Y))dY
\]
Hence, we only need to show that $\text{Supp}(\varphi) \cap p(F) = \emptyset$. By definition of $\varphi$, it even suffices to see that $t_{\text{ell,reg}}(F)^G \cap p(F) = \emptyset$. But this follows immediately from the fact that $P$ being proper, it doesn’t contain any elliptic maximal torus of $G$. ■

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