On Integral Upper Limits Assuming Power-law Spectra and the Sensitivity in High-energy Astronomy

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Abstract

The high-energy non-thermal universe is dominated by power-law-like spectra. Therefore, results in high-energy astronomy are often reported as parameters of power-law fits, or, in the case of a non-detection, as an upper limit assuming the underlying unseen spectrum behaves as a power law. In this paper, I demonstrate a simple and powerful one-to-one relation of the integral upper limit in the two-dimensional power-law parameter space into the spectrum parameter space and use this method to unravel the so-far convoluted question of the sensitivity of astroparticle telescopes.

Key words: gamma rays: general – methods: statistical

1. Introduction

Power-law spectra are a universal feature in the non-thermal high-energy universe. Often results in high-energy astronomy are reported as parameters of power-law fits (e.g., Aharonian et al. 2006b; Ackermann et al. 2016), or, if no source was detected, as a flux limit (e.g., Aliu et al. 2012, 2014a; Abramowski et al. 2014; Aleksić et al. 2015; Aliu et al. 2015; Ahnen et al. 2016b, 2016a; Archambault et al. 2016) assuming the unseen spectrum $dN/dE$ is a power law

$$
\frac{dN}{dE} = f_0 \left( \frac{E}{E_0} \right)^\Gamma
$$

(1)

In Equation (1) $E_0$ denotes a freely chosen reference energy, $f_0$ represents the flux normalization, and $\Gamma$ is known as the spectral index.

The ability of gamma-ray, neutrino, and cosmic-ray telescopes to detect a source depends on their intrinsic parameters and on the external source spectrum (Bertou et al. 2002; Aharonian et al. 2006d; Murase & Takami 2008; Aartsen et al. 2015; Aleksić et al. 2016). However, even when observing and not detecting any significant signal counts, one would like to put limits on the observed signal flux in order to infer physical results because observing nothing is fundamentally different from not observing at all. However, what does an upper limit on the signal count rate mean for the spectrum? This question is tightly related to the detection limit of a given instrument. After all, every observer is interested to know, “How long does it take to detect my source with an assumed average flux?”

Currently, there are only convoluted or partial answers to these two fundamental questions in high-energy astronomy. However, the universal observation of power laws in gamma-ray sources allows one to frame the questions in the $\{f_0, \Gamma\}$ parameter space of the power law.

In the following sections, I first summarize current methods for upper limits in the context of common astroparticle measurements. Second, I introduce a simple and powerful one-to-one relation of the integral upper limit in the two-dimensional power-law parameter space into the spectrum parameter space. Third, I apply this method to the question of the sensitivity of astroparticle telescopes.

2. Upper Limits in the Context of On/Off Measurements

In the low-count regime of high-energy astronomy one experimental method is well established: the On/Off measurement (Li & Ma 1983; Rolke et al. 2005; Abraham et al. 2007; Berge et al. 2007; Abbasi et al. 2011; Knoetig 2014). In it, the experimentalist would like to measure an imprecisely known background count rate as well as a potential signal count rate. The measurement consists of two observations. The first counts $N_{\text{on}}$ events in a region with potential signal, the second counts $N_{\text{off}}$ events in a usually adjacent region, which is assumed to be signal free. Both regions are related to one another through the ratio of exposures $\alpha$. If the data show no evidence for a source, an upper limit on the signal counts $\lambda_{\text{lim}}$ can be calculated (Rolke et al. 2005; Knoetig 2014) using $N_{\text{on}}, N_{\text{off}}$, and $\alpha$. This limit can be used to infer constraints about the potential source spectrum.

The most constraining upper limits can be extracted from methods known as integral upper limits. A widely used integral upper limit is calculated with the average signal counts $\lambda_s$ integrated over all energies

$$
\lambda_s = t \int \frac{dN}{dE} A_{\text{eff}}(E) \, dE,
$$

(2)

$$
\lambda_s = t f_0 \int \frac{E}{E_0} \Gamma \, A_{\text{eff}}(E) \, dE,
$$

(3)

assuming a certain power law with a common spectral index $\Gamma$. (Abramowski et al. 2014; Aliu et al. 2014a, 2015; Aleksić et al. 2015; Ahnen et al. 2016b). The integrated average signal counts $\lambda_s$ depend, assuming a steady source, on the instrument effective area $A_{\text{eff}}(E)$ as a function of the energy $E$, the observation time $t$, and the assumed source spectrum parameter $\Gamma$. The limit flux normalization $f_0$ is then calculated so that the resulting power-law spectrum would yield the limit

$$
\lambda_s = \lambda_{\text{lim}}.
$$

(4)
The result is often reported as integrated source flux \( F_{E_{\text{thr}}} \) above a certain threshold energy \( E_{\text{thr}} \)

\[
F_{E_{\text{thr}}} = \int_{E_{\text{thr}}}^{\infty} \frac{dN}{dE} \, dE.
\]

Implicit in this method is the choice of the power-law index \( \Gamma \), which can be extrapolated (Ahnen et al. 2016b), physically motivated (Aliu et al. 2014a, 2015), or hedged to have little impact on the result (Abramowski et al. 2014; Aleksic et al. 2015). However, extrapolating is not always possible and the power-law index is often varying strongly from source to source (Ackermann et al. 2016), which propagates to a potentially large systematic error in the flux limits.

A method to reduce the systematic error of the choice of the power-law index and to infer spectral information in the absence of signal is to calculate differential upper limits. Although they are called differential, no observation can measure differentials directly. Therefore, differential upper limits are approximated by upper limits integrated in small bins, assuming a certain power-law index \( \Gamma \) and using Equations (3) and (4), just as before. The limit curve obtained in this way still depends on the choice of \( \Gamma \) (Ahnen et al. 2016b), admittedly less than the full integral upper limit. The main issue with differential upper limits is their lower sensitivity due to the binning, which falls short of the real detection capabilities of the instrument.

A more robust approach to the influence of the power-law index \( \Gamma \) is the so-called decorrelation energy method (Aliu et al. 2012; Archambault et al. 2016), which is also an integral upper limit method. Here, three different limit power laws are calculated assuming three different power-law indices. The energy at which their flux and therefore their upper limit estimate depends the least on the power-law index is then designated as decorrelation energy. In the final step, the least constraining of the three upper limit power-law fluxes at the decorrelation energy (which turns out to be the upper limit of the central power-law index) is reported. While this method takes variations in \( \Gamma \) better into account, it is still not universal as three common power-law indices have to be chosen, none of which are physically motivated.

In this paper, I demonstrate a more useful approach to integral upper limits than the previously discussed methods. The key insight comes from investigating integral upper limits in the two-dimensional \( \{f_{0}, \Gamma\} \) parameter space of the power law.

### 3. Generalized Integral Upper Limits

Assuming Power-law Spectra

In the following, I derive my novel integral upper limit method on, and compare it to, a data set from the VERITAS Cherenkov telescope (McCutcheon 2012), where upper limits from an observation of the globular cluster M13 are reported. M13 is measured to host millisecond pulsars (Hessels et al. 2007) and is a popular, yet undetected, target for Cherenkov telescopes (Anderhub et al. 2009; McCutcheon 2012).

When investigating spectra of sources, it is important to take the difference between estimated energy \( E_{\text{est}} \) and true (or simulated true) energy \( E \) into account. This relationship is the migration matrix. This means that the effective area measured in estimated energy \( A_{\text{eff}}(E_{\text{est}}) \) is a convolution of the effective area in true energy \( A_{\text{eff}}(E) \), the migration matrix, and the source spectrum. However, as the proposed method is an integral limit method, it needs the effective area as a function of the true (simulated) energy \( A_{\text{eff}}(E) \). In this case, because no suitable effective areas are published together with the necessary data components, I use the commonly reported \( A_{\text{eff}}(E_{\text{est}}) \) (McCutcheon 2012) as an approximation to \( A_{\text{eff}}(E) \).

To summarize, the following components must be specified to calculate any integral upper limit for a specific On/Off measurement.

1. \( A_{\text{eff}}(E) \): Effective area as a function of true energy after all cuts.
2. \( t \): Effective observation time.
3. \( \lambda_{\text{lim}} \): Effective observation time.
4. \( \lambda_{\text{lim}} \): Choice of criterion on the signal counts limit, e.g., Rolke et al. (2005), Knoetig (2014), Helene (1983), calculation relies on
   - (a) C.I.: Choice of credible interval or confidence interval e.g., 95%, 99%.
   - (b) \( \alpha \): On/Off ratio.
   - (c) \( N_{\text{on}} \): Number of events in the On region.
   - (d) \( N_{\text{off}} \): Number of events in the Off region.

The components used in this section are shown in Table 1. The resulting signal limit \( \lambda_{\text{lim}} \) is calculated using a 95% credible interval on the signal posterior using the method from Knoetig (2014).

When looking at the family of excluded power laws in the \( \{f_{0}, \Gamma\} \) parameter space for the VERITAS M13 observation, the result is Figure 1(a). It shows the average signal counts as a function of the power-law parameters \( \lambda_{s}(f_{0}, \Gamma) \) (Equation (3)) together with the implicitly defined curve where the average signal counts \( \lambda_{s} \) are equal to the signal limit \( \lambda_{\text{lim}} \) (Equation (4)). The power laws on this curve represent the border of the excluded parameter space—the region with higher fluxes to the right is excluded. This plot can already be used to compare models with limits because sources populate the power-law parameter space (Ackermann et al. 2016). However, most physicists compare source spectra and models in the spectrum parameter space.

The following translation of the implicitly defined power-law exclusion curve (Figure 1(a)) into the spectrum parameter space (Figure 1(b)) is key. When looking at a set of power laws from the family of curves on the exclusion line it becomes apparent that they produce an envelope curve when regarding flux upper limits in the spectrum parameter space. This means that for every assumed power-law index \( \Gamma \) there is an energy \( E_{\text{max}}(\Gamma) \) in the spectrum parameter space. At the sensitive energy \( E_{\text{max}}(\Gamma) \), the limit power law with \( \Gamma \) has, locally, the maximum flux \( \frac{dN}{dE} \) (Equation (1)), compared to the other possible limit power laws on the power-law exclusion curve. Therefore, one specific integral flux upper

| \( A_{\text{eff}} \) | \( t \) (hr) | C.I. | \( \alpha \) | \( N_{\text{on}} \) | \( N_{\text{off}} \) | \( \lambda_{\text{lim}} \) |
|---|---|---|---|---|---|---|
| (1) | 7 | 0.95 | 1/10 | 55 | 670 | 11.3 |

Note. The On/Off measurement parameters from Table 5.2 and Figure 5.10 of McCutcheon (2012). The author of McCutcheon (2012) uses the method from Helene (1983) in order to calculate the count upper limit. However, the \( \lambda_{\text{lim}} \) criterion chosen in this paper is from Knoetig (2014). The resulting \( \lambda_{\text{lim}} \) is shown in bold.

Reference. (1) McCutcheon (2012).
limit is distinguished at $E_{\text{sens}}(\Gamma)$, which can be used to construct a one-to-one relation from points on the power-law exclusion curve to points in the spectrum parameter space by

$$\left( \frac{dN}{dE} \right)_{\text{lim}}(E) = \max_{f_i(t)} \frac{dN}{dE}(E),$$

subject to $\lambda_s = \lambda_{\text{lim}}$.  (6)

The curve $(dN/dE)_{\text{lim}}(E)$ in the spectrum parameter space represents the border of the region I call the integral spectral exclusion zone. The necessary conditions for the existence of the integral spectral exclusion zone can be deduced from the method of Lagrange multipliers, which is done in the Appendix. A fundamental analytical result of this is

$$E_{\text{sens}}(\Gamma) = \exp(\mu(\Gamma)), \quad (7)$$

where $\mu(\Gamma)$ is defined as the average of the natural logarithm of the true energy $E$ over the power-law-weighted sensitive area

$$\mu(\Gamma) = \frac{\int E^\Gamma A_{\text{eff}}(E) \ln(E) dE}{\int E^\Gamma A_{\text{eff}}(E) dE}. \quad (8)$$

The example VERITAS $E_{\text{sens}}(\Gamma)$ is shown in Figure 2(a)). In order to calculate the integral spectral exclusion zone $(dN/dE)_{\text{lim}}(E)$, one has to determine the power law $(f_i(t), \Gamma)$, which is locally tangent to the border of the zone at the energy $E$. This can be done in two ways.
1. Use the Lagrange multiplier formalism results.
   (a) Calculate $\Gamma'$ numerically from inverting $E = E_{sens}(\Gamma')$ (using Equation (7)).
   (b) Calculate $f'_0$ from solving the power-law exclusion line constraint Equation (4)
   \[
   f'_0 = \frac{\lambda_{\text{lim}}}{c(\Gamma')},
   \]
   where $c(\Gamma')$ is defined as the spectrum weighted acceptance
   \[
   c(\Gamma') = \int \left( \frac{E}{E_0} \right)^{\Gamma'} A(E) dE.
   \]
   (c) Insert parameters
   \[
   \left( \frac{dN}{dE} \right)_{\text{lim}}(E) = \frac{dN}{dE}(E)|_{f'_0, \Gamma'}.\]

2. Calculate Equation (6) by maximizing the local flux under constraints directly.

When repeating one of those procedures for many values of energy one can draw the integral spectral exclusion zone.

What is the physical meaning of $E_{sens}(\Gamma)$ and the integral spectral exclusion zone? In case a power law with index $\Gamma_{\text{cross}}$ crosses into the integral spectral exclusion zone, its related implicitly defined (Equation (4)) upper limit flux normalization $f'_0(\Gamma_{\text{cross}})$ at the sensitive energy $E_{sens}(\Gamma_{\text{cross}})$ is lower. Therefore, the border of the integral spectral exclusion zone is the physical boundary, where every source with a power-law spectrum crossing it would have been detected, independent of $\Gamma_{\text{cross}}$. From this point of view, $E_{sens}(\Gamma_{\text{cross}})$ is the energy at which a source with a given power-law index $\Gamma_{\text{cross}}$ is best constrained. As the sensitive energy is a monotonically increasing function of $\Gamma$ (see the Appendix), this yields the intuitive result that harder spectra are better constrained at higher energies and that softer spectra are better constrained at lower energies.

Applying this method to the VERITAS M13 data set results in Figure 2(b)). Here, the integral spectral exclusion zone is calculated maximizing the logarithm of the flux under constraints directly (see Equation (6)) using the scipy.optimize Python library (Jones et al. 2001). The integral spectral exclusion zone is compatible with the reported integral limit at the decorrelation energy and is about five times more sensitive than the reported differential limit. The VERITAS method of reporting an integral upper limit at the decorrelation energy (see Section 2) is approximating the integral spectral exclusion zone at one point, which is also apparent from the method. In this sense, the integral spectral exclusion zone may be seen as a generalization of the decorrelation energy method.

4. Application: The Sensitivity of Astroparticle Telescopes

One important application of the integral spectral exclusion zone is the fundamental question: “What potential sources can be detected with a certain instrument?” After all, upper limits are a statement of sensitivity. Therefore, the current methods tackling this question are equivalent to the current methods of calculating upper limits described in Section 2 and can be classified into two categories.

First, there are integral methods assuming a certain power-law index $\Gamma$. Among the possible power laws, one calculates the power law with the minimum flux normalization $f'_0$ that is required for a detection in a certain observation time. This is often reported as a function of observation time (Aharonian et al. 2006d). A similar approach is to give the integral sensitivity as a function of the minimum flux required for detection above a given threshold energy (Aleksić et al. 2016). However, this method does not constitute a full analysis because the last cut (in energy) is unspecified. Therefore, it cannot be compared to other sensitivities easily. Furthermore, the power-law index problem persists: “what if the source has a different $\Gamma$ than expected?”

Second, authors report sensitivities as differential upper limits (Bernlörð et al. 2013; Aleksić et al. 2016) solving the power-law index issue. It is also used to compare instrument sensitivities (Funk et al. 2013) but as discussed in Section 2, being a differential limit, it falls short of the real detection capabilities of the instruments.

I demonstrate my novel method using the latest published performance data for dark, moonless nights from the MAGIC Cherenkov telescope, as validated on a northern winter 2013/2014 data set from the Crab Nebula (Aleksić et al. 2016). For a given On/Off measurement analysis, the following components define the sensitivity of an astroparticle telescope to detect sources with power-law spectra.

1. $A_{\text{eff}}(E)$: Effective area as a function of true energy after all cuts.
2. $\lambda_{\text{lim}}$: The sensitivity consensus criterion is, unlike the upper limit criterion, five standard deviations calculated according to Li & Ma (1983), subsequently required are
   (a) $\alpha$: On/Off normalization factor.
   (b) $\sigma_{\text{bg}}$: Background event rate in the On region, estimated from simulations or independent Off region data.

Unfortunately, it is not common for Cherenkov telescope collaborations to publish $A_{\text{eff}}(E)$, $\alpha$, and $\sigma_{\text{bg}}$ coherently for selected and specific analyses. This means it is, outside of a telescope collaboration, not possible to calculate the time until an analysis detects a power-law source without further assumptions. I therefore make the following assumptions in order to use the published MAGIC results (Aleksić et al. 2016).

1. $\alpha = 1/5$: possible values for $\alpha$ stated in that publication are one-third and one-fifth. Given that similar other Cherenkov telescopes use even more Off regions (see Table 1), I assume the more sensitive value of one-fifth is reasonable.
2. $A_{\text{eff}}(E)$: the reported effective area is not given after all cuts. The authors explain using an additional energy cut at $\sim 300 \text{ GeV}$ to achieve a more sensitive instrument. Therefore, to approximate $A_{\text{eff}}(E)$ I use the reported $A_{\text{eff}}'(E_{\text{calc}})$ with a threshold at $300 \text{ GeV}$ convoluted with the stated energy resolution.

These assumptions are summarized in Table 2. The table further contains the MAGIC analyses specific zenith distance domain names (low Zd $\in [0^\circ, 30^\circ]$ and medium Zd $\in [30^\circ, 45^\circ]$) and background rates $\sigma_{\text{bg}}$. The authors (Aleksić et al. 2016) explain that their reported effective area after cuts was optimized for differential sensitivity. Therefore, the total on region background rates $\sigma_{\text{bg}}$ are calculated by summing over the bin-wise background rates above the additional energy threshold in the tables of differential limit results.
In this way, the integral spectral exclusion zone method can be applied to the MAGIC sensitivity question. When investigating the sensitivity, most authors (Aharonian et al. 2006a; Aleksić et al. 2016; Park 2016) use five standard deviations calculated according to Equation (17) in Li & Ma (1983) as sensitivity limit criterion.

The connection between this criterion and the signal counts limit \( \lambda_{\text{lim}} \) can be constructed from assuming a constant background flux \( \sigma_{\text{bg}} \) in an area with On region exposure

\[
N_{\text{on}} = (\sigma_{\text{lim}} + \sigma_{\text{bg}}) t, \tag{12}
\]

\[
N_{\text{off}} = \sigma_{\text{bg}} t/\alpha, \tag{13}
\]

where the (to be inferred) signal rate limit \( \sigma_{\text{lim}} \) is related to the signal counts limit via the observation time

\[
\lambda_{\text{lim}} = \sigma_{\text{lim}} t. \tag{14}
\]

When inserting Equations (12) and (13) into the limit condition

\[
\frac{1}{2} \lambda_{\text{LiMa}} (N_{\text{on}}, N_{\text{off}}, \alpha) = 5 \quad \text{one can numerically solve for} \quad \sigma_{\text{lim}}.
\]

Then, Equation (14) can be used to calculate \( \lambda_{\text{lim}} \). The Li&Ma criterion is only valid for count numbers that are “not too few” (Li & Ma 1983). Therefore, it is common (Aharonian et al. 2006a; Aleksić et al. 2016; Park 2016) to use an additional limit, which is imposing to measure at least 10 gamma-ray signal counts in the On region \( \lambda_{\text{lim}} \geq 10 \).

In this way, the instrument and analysis specific sensitivity can be calculated for any observation time \( t \) using the integral spectral exclusion zone method. The result, applied to the MAGIC telescope analyses from Table 2, is shown in Figure 3. It illustrates when and at what energy a certain source spectrum would be detected: at the sensitive energy when the spectrum touches the integral spectral exclusion zone. The results agree well with the published (Aleksić et al. 2016) result of MAGIC being able to detect a Crab-Nebula-like source with 1% Crab Nebula flux within \( \sim 25 \) hr.

In the case of specific sources, it is useful to return to the power-law parameter space \( (f_0, \Gamma) \). Table 3 shows eight selected TeV emitting gamma-ray sources. The selected sources are of galactic origin in order to simplify the analysis by excluding flux variability. They are observable from the MAGIC site (Latitude 28°76'N) at a low or medium zenith distance. This implies source declinations \(|-16°24', 73°76'|\).

As the MAGIC background rate parameters \( \sigma_{\text{bg}} \) from Table 2 are valid for point-like sources, only MAGIC quasi-point-like sources (up to a mean extension of \(<0.1'\)) are considered. In the following, I am going to use the LS 5039 low-state flux as an example for how to estimate the necessary observation time \( t_{\text{est}} \) until detection.

The average time to detect a certain power law in the power-law parameter space must be calculated by numerically inverting the signal rate limit \( \sigma_{\text{lim}}(t) \). Figure 4(a) shows \( \sigma_{\text{lim}}^{-1} \) as a function of the signal rate for the considered low Zd and medium Zd MAGIC analyses.

Then, the power-law limit curve Equation (4) can be reformulated using the average signal counts function \( \lambda_c \) (Equation (3)), the power-law spectrum \( dN/dE \) (Equation (1)), and \( \sigma_{\text{lim}} \) as

\[
t = \sigma_{\text{lim}}^{-1}(c(\Gamma)f_0). \tag{15}
\]

Finally, the average estimated observation time \( t_{\text{est}} \) until detection (Equation (15)) can be calculated as a function of the power-law parameters \( (f_0, \Gamma) \), which is demonstrated in Figure 4(b) together with the measured LS 5039 parameter confidence intervals (Aharonian et al. 2006a) assuming uncorrelated normal distributed errors in \( f_0 \) and \( \Gamma \). The data indicates that MAGIC, using the analysis parameters from Table 2, would need \( t_{\text{est}} \sim 4–7 \) hr in order to detect this source.

In order to quantify \( t_{\text{est}} \), one can sample from the parameter space according to the LS 5039 parameter errors and calculate the histogram of times to detection. I report the time to detection as the median time with asymmetric errors calculated from the 16th percentile and the 84th percentile. In the case of LS 5039, this results in an estimated MAGIC time to detection of \( t_{\text{est}} = 5.69^{+0.72}_{-0.70} \) hr. The other seven \( t_{\text{est}} \) are reported in Table 3. Four out of eight \( t_{\text{est}} \) calculated are predictions because MAGIC has not yet\(^1\) reported detections on HESS J1837-069, LS 5039, W 49B, and CTB 87.

Finally, it is illustrative to show the estimated time to detection using the integral spectral exclusion zone. Figure 5 shows spectra drawn from the parameter space according to the LS 5039 parameter errors. The integral spectral exclusion zone for \( t = 5.69 \) hr and MAGIC medium Zd parameters is drawn.

\(^1\) As of late 2016.

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\( \text{Example Table 2: Inputs for Sensitivity Calculations of the MAGIC Cherenkov Telescope} \)

| Mode    | \( A_{\text{eff}} \) | \( \alpha \) | \( \sigma_{\text{bg}} \) (1/hr) | Extra Cut |
|---------|---------------------|-------------|-------------------------------|-----------|
| low Zd  | (1)                 | 1/5        | 7.37                          | \( E_{\text{est}} > 300 \text{ GeV} \) |
| med. Zd | (1)                 | 1/5        | 28.83                         | \( E_{\text{est}} > 300 \text{ GeV} \) |

Note. MAGIC published low zenith distance parameters (Zd \( \in [0°, 30°] \)) and medium zenith distance parameters (Zd \( \in [30°, 45°] \)). The On/Off normalization factor in this work is assumed to be \( \alpha = 0.2 \), Aleksić et al. (2016) report doing another cut in energy to achieve a more sensitive instrument. Therefore, I also perform an additional cut in energy, which translates into a cutoff in the effective area and a reduced background rate.

**Reference.** (1) Aleksić et al. (2016).
Table 3
Selected Northern Hemisphere Galactic TeV Sources with Estimated Time to Detections for MAGIC

| name         | assoc. tevcat | R.A. (°)  | Decl. (°) | Zd range | $f_0$ (cm$^2$ s TeV$^{-1}$) | $\Gamma$ | References | $t_{est}$ (hr) |
|--------------|---------------|-----------|-----------|----------|----------------------------|---------|------------|--------------|
| Crab         | TeV J0534+220 | 83.629    | 22.022    | low      | (2.83 ± 0.64) · 10$^{-11}$ | −2.62 ± 0.07 | (1)        | 0.036 ± 0.019 |
| HESS J1837-069 | TeV J1837-069 | 279.410   | −6.950    | medium   | (5.0 ± 0.3) · 10$^{-12}$  | −2.27 ± 0.06 | (2)        | 0.356 ± 0.031 |
| W 41         | TeV J1834-087 | 278.690   | −8.780    | medium   | (3.7 ± 0.6) · 10$^{-12}$  | −2.5 ± 0.2  | (3)        | 0.489 ± 0.124 |
| Cas A        | TeV J2323+588 | 350.806   | 58.808    | medium   | (1.45 ± 0.11) · 10$^{-12}$ | −2.75 ± 0.3 | (4)        | 2.04 ± 0.51 |
| LS 5396      | TeV J1826-148 | 276.563   | −14.842   | medium   | (9.1 ± 0.7) · 10$^{-13}$  | −2.53 ± 0.07 | (5)        | 5.69 ± 0.72 |
| W 49B        | TeV J1911+090 | 287.780   | 9.087     | low      | (3.2 ± 1.1) · 10$^{-13}$  | −3.14 ± 0.34 | (6)        | 16.39 ± 0.6 |
| CTB 87       | TeV J2016+372 | 304.008   | 37.220    | low      | (3.1 ± 1.5) · 10$^{-13}$  | −2.3 ± 0.5  | (7)        | 28.5 ± 0.54 |
| 3C 58        | TeV J0209+648 | 31.379    | 64.841    | medium   | (2.0 ± 1.0) · 10$^{-13}$  | −2.4 ± 0.4  | (8)        | 100 ± 2.30 |

Note. Table showing selected galactic TeV gamma-ray sources, used in combination with the method to estimate the observation time until detection $t_{est}$ for MAGIC. R.A. and decl. are given for the epoch J2000. Sources are identified by common names and by their TeVCat designation (Wakely & Horan 2008). The individual flux normalizations $f_0$ and the power-law indices $\Gamma$ are taken from the stated references. Calculated values of $t_{est}$ are given in bold. Stars indicate predictions as these sources have not yet been reported by MAGIC.

References. (1) Aharonian et al. (2004), (2) Aharonian et al. (2006c), (3) Albert et al. (2006), (4) Kumar (2016), (5) Aharonian et al. (2006a) low state, (6) Abdalla et al. (2016), (7) Aliu et al. (2014b), (8) Aleksic et al. (2014).

as a solid line. The dotted lines indicate how the integral exclusion zone evolves over time, given half or double the observation time.

### 5. Discussion

All calculations in this paper are done using the true energy $E$ because I only consider integral results (see Section 3). This means that limited energy resolution and migration do not affect the integral spectral exclusion zone because a power-law source does indeed emit with its true individual spectrum and an instrument measures the events according to $A_{eff}(E)$. This is the case even when one is unable to reconstruct the individual event energies at all. Therefore, it is possible to know the sensitivity of an instrument to detect power-law sources without reconstructing the energy of individual events. It further shows that the fundamental effective area of an astroparticle telescope analysis is its effective area over true energy after all cuts $A_{eff}(E)$.

Assuming power-law spectra is systematically limited. There are several cases when sources reveal additional curvature, besides the general power-law behavior, after measuring them with high significance and low statistical uncertainty (Acero et al. 2015). However, the upper limit question and the sensitivity question for an individual telescope analysis only consider what happens at the detection threshold and only what happens in the limited energy domain of $A_{eff}(E)$ (see Equation (3)). There, basically every source in astroparticle physics appears as power law initially. Therefore, I argue that the integral spectral exclusion zone is applicable, even when source spectra deviate slightly from power laws.

In this manuscript, I only considered On/Off measurements so far, which infer results in view of Poissonian signal-and-noise experiments (see Section 2), but there may be other cases when, for instance, no background is expected and $\lambda_{lim}$ can be calculated from a single Poisson process producing $N_{on}$. In general, there are plenty of statistical approaches to choose from, including maximum likelihood methods (Li & Ma 1983; Feldman & Cousins 1998) and Bayesian approaches with various assumed priors (Helene 1983; Narsky 2000; Kashyap et al. 2010; Knoetig 2014).

However, the details of calculating $\lambda_{lim}$ are up to each individual experimentalist and what method they believe in. The signal count limit $\lambda_{lim}$, formulated as a constraint in Equation (6), is only the interface of my integral spectral exclusion zone method with the world of statistics: it stays the same while the statistical methods may be exchanged.

### 6. Conclusion

Many high-energy astronomers struggle with the implications of non-detections and instrument limits. The universal observation of power laws in high-energy sources allows one to frame these problems in the parameter space of the power-law model. A simple and powerful one-to-one relation of the integral spectral limit in the power-law parameter space into spectra makes it possible to construct the integral spectral exclusion zone. Given sources with power-law spectra, this integral spectral exclusion zone demonstrates what astroparticle telescopes are actually capable of detecting—and how long it will take. In this way, one can easily compare instrument performances, optimize analysis algorithms, and test model predictions of high-energy astrophysics.

A python package implementing the methods from this manuscript can be found at https://github.com/mahnen/gamma_limits_sensitivity.

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### Appendix

#### Application of the Lagrange Multiplier Method for Calculating the Integral Spectral Exclusion Zone

Motivated by the results shown in Figure 1, I assume the existence of a maximum flux (Equation (1)) given the external constraint (Equation (4)). Then, the necessary conditions for the existence of the integral spectral exclusion zone can be deduced from the method of Lagrange multipliers. The first step is the construction of the problem specific Lagrange function $L(f_0, \Gamma)$ by joining the function to be maximized $dN/dE$ to the external constraint using a Lagrange multiplier $\delta$:

$$L(f_0, \Gamma) = f_0 \left( \frac{E}{E_0} \right)^\Gamma + \delta (f_0 c(\Gamma) - \lambda_{lim}),$$

(16)
where $c(\Gamma)$ is an abbreviation for the spectrum weighted acceptance

$$c(\Gamma) = \int \left( \frac{E}{E_0} \right)^\Gamma A(E) dE.$$  \hspace{1cm} (17)

By taking the gradient with respect to the parameters $\{f_0, \Gamma\}$ and equating it to zero, one gets a set of equations

$$f_0 \left( \frac{E}{E_0} \right)^\Gamma \ln \left( \frac{E}{E_0} \right) + \delta t f_0 \frac{\partial c(\Gamma)}{\partial \Gamma} = 0,$$ \hspace{1cm} (18)

$$\left( \frac{E}{E_0} \right)^\Gamma + \delta t c(\Gamma) = 0.$$ \hspace{1cm} (19)

which, together with the external constraint Equation (4), constitute the necessary conditions. Equation (19) solved for $\delta$ gives

$$\delta = -\left( \frac{E}{E_0} \right)^\Gamma \frac{1}{t c(\Gamma)}.$$ \hspace{1cm} (20)

This equation inserted back into Equation (18) results in

$$\ln \left( \frac{E}{E_0} \right) = \frac{1}{c(\Gamma)} \frac{\partial c(\Gamma)}{\partial \Gamma}.$$ \hspace{1cm} (21)

When interchanging the integral and differentiation in the explicit formula of $\frac{\partial c(\Gamma)}{\partial \Gamma}$ and solving Equation (21) for the energy $E$ the result is

$$E = \exp(\mu(\Gamma)),$$ \hspace{1cm} (22)

where $\mu(\Gamma)$ is an abbreviation for the average of the natural logarithm of the energy $E$ over the spectrum weighted acceptance

$$\mu(\Gamma) := \frac{\int E^\Gamma A_{\text{eff}}(E) \ln(E) dE}{\int E^\Gamma A_{\text{eff}}(E) dE}.$$ \hspace{1cm} (23)

$\mu(\Gamma)$ is independent of $E_0$, which is a reasonable result for a scale factor. I identify the energy at which the Lagrange function has a maximum with the sensitive energy $E_{\text{sens}}$ in Section 3.

Equation (23) shows that the existence of the integral spectral exclusion zone is a direct consequence of the physical argument that there cannot be infinite counts that leads to the normalizability of the spectrum weighted acceptance (finite numerator and denominator in Equation (23)). The average natural logarithm of the energy $\mu(\Gamma)$ is an increasing function of $\Gamma$ as harder spectra produce higher energy events more frequently. Therefore, also $E_{\text{sens}}(\Gamma)$ is an increasing function, which means it is invertible. Finally, one can take Equation (22), solve it numerically for $\Gamma$, and find the last remaining undetermined...
parameter $f_0$ by solving the so-far unused constraint function Equation (4), which is demonstrated in Equation (9).

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