LIMITING DYNAMICS FOR NON-AUTONOMOUS STOCHASTIC RETARDED REACTION-DIFFUSION EQUATIONS ON THIN DOMAINS

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Abstract. A system of stochastic retarded reaction-diffusion equations with multiplicative noise and deterministic non-autonomous forcing on thin domains is considered. Relations between the asymptotic behavior for the stochastic retarded equations defined on thin domains in \(\mathbb{R}^{n+1}\) and an equation on a domain in \(\mathbb{R}^n\) are investigated. We first show the existence and uniqueness of tempered random attractors for these equations. Then, we analyze convergence properties of the solutions as well as the attractors.

1. Introduction. Let \(Q \subset \mathbb{R}^n\) be a bounded \(C^2\)-domain and \(O_\varepsilon \subset \mathbb{R}^{n+1}\) be the domain
\[
O_\varepsilon = \{ x = (x^*, x_{n+1}) | x^* = (x_1, \ldots, x_n) \in Q \text{ and } 0 < x_{n+1} < \varepsilon g(x^*) \},
\]
where \(g \in C^2(\overline{Q}, (0, +\infty))\) and \(0 < \varepsilon \leq 1\). Since \(g \in C^2(\overline{Q}, (0, +\infty))\), there exist two positive constants \(\gamma_1\) and \(\gamma_2\) such that
\[
\gamma_1 \leq g(x^*) \leq \gamma_2, \quad \forall x^* \in \overline{Q}. \tag{1}
\]

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Denote \( O = Q \times (0,1) \) and \( \overline{O} = Q \times (0,\gamma_2) \) which contains \( O_\varepsilon \) for \( 0 < \varepsilon \leq 1 \). Given \( \tau \in \mathbb{R} \), in this paper, we study the asymptotical behavior of the following stochastic reaction-diffusion equation with multiplicative noise defined on the thin domain \( O_\varepsilon \):

\[
\begin{aligned}
d\hat{u}_\varepsilon - \Delta \hat{u}_\varepsilon dt &= (H(t,x,\hat{u}_\varepsilon(t)) + f(t,x,\hat{u}_\varepsilon(t) - \rho_0(t))) + G(t,x) dt \\
+ \sum_{j=1}^{m} c_j \hat{u}_\varepsilon \circ dw_j, & \quad x \in O_\varepsilon, \ t > \tau,
\end{aligned}
\]

(2)

with the initial condition

\[
\hat{u}_\varepsilon(\tau + s, x) = \hat{\phi}(s,x), \quad s \in [-\rho,0], \ x \in O_\varepsilon,
\]

(3)

where \( \nu_\varepsilon \) is the unit outward normal vector to \( \partial O_\varepsilon \), \( H \) is a superlinear source term, \( f : \mathbb{R} \times \overline{O} \times \mathbb{R} \to \mathbb{R} \) is a nonlinearity capturing the time delay, \( \rho_0 : \mathbb{R} \to [0,\rho] \) is an adequate given delay function, where \( \rho \) is a positive constant, \( G \) is a function defined on \( \mathbb{R} \times \overline{O} \), \( c_j \in \mathbb{R} \) for \( j = 1,\ldots,m \), \( w_j \), \( j = 1,\ldots,m \), are independent two-sided real-valued Wiener processes on a probability space, and the symbol \( \circ \) indicates that the equation is understood in the sense of Stratonovich integration.

The domain \( O_\varepsilon \) is the so-called thin domain when \( \varepsilon \) is small. We will investigate the limiting behavior of (2) as \( \varepsilon \to 0 \). For the deterministic reaction-diffusion equations without delay, in [26, 27], Hale and Raugel first studied this problem. Some extensions of their results can be also found in [1, 3, 4, 5, 19, 36]. However, systems are always subject to environmental noise. The environmental noise is an intrinsic effect in a variety of settings and spatial scales. It is worth mentioning that the ergodicity of stochastic 3D Navier-Stokes equations in a thin domain was recently investigated in [17, 18], the synchronization of semilinear parabolic stochastic equations in thin bounded tubular domains was studied in [11], and the upper semicontinuity of random attractors for reaction-diffusion equations in thin domains was established in [32, 34]. However, as far as the author is aware, the limiting dynamics for stochastic random attractors on thin domains are not well studied, even for deterministic retarded equations on thin domains. In this paper, we will investigate this problem.

As \( \varepsilon \to 0 \), we will show in certain sense that the limiting behavior of (2) is governed by the following equation:

\[
\begin{aligned}
du_0 - \frac{1}{\varepsilon} \sum_{i=1}^{n} (gdu_{y_i})_{y_i} dt &= (H(t,(y^*),0),u_0(t)) + f(t,(y^*),0),u_0(t) - \rho_0(t)) \\
+ G(t,(y^*,0)) dt + \sum_{j=1}^{m} c_j u_0 \circ dw_j, & \quad y^* = (y_1,\ldots,y_n) \in Q, \ t > \tau,
\end{aligned}
\]

(4)

with the initial condition

\[
u_0(\tau + s,y^*) = \phi_0(s,y^*), \quad s \in [-\rho,0], \ y^* \in Q,
\]

(5)

where \( \nu_0 \) is the unit outward normal to \( \partial Q \). Note that \( u_{y_i}^0 \) means \( \frac{\partial u_0}{\partial y_i} \) in (4) and similar notation will be used throughout this paper.

In this paper, we first prove the existence and uniqueness of tempered random attractors for systems (2)-(3) and (4)-(5), and establish the periodicity of these attractors when \( H, f, G \) and \( \rho_0 \) are periodic functions with respect to \( t \). Then, we compare the continuous cocycle \( \Phi_\varepsilon \) generated by (2)-(3) with the continuous cocycle \( \Phi_0 \) generated by (4)-(5). Finally, we establish upper semicontinuity result for the
corresponding family of random attractors. Random attractors have been investigated in [6, 7, 10, 20, 21, 25, 28, 29, 30, 31, 37, 38, 39, 45, 50] in the autonomous stochastic case, and in [8, 9, 14, 24, 33, 40, 41, 42, 49] in the non-autonomous stochastic case. Recently, attractors of delay equations have been considered in [15, 43, 44] in the deterministic case, and in [12, 13, 16, 22, 46, 48, 51] in the stochastic case, where delay equations are driven by additive noise. In particular, the existence of a random attractor for non-autonomous stochastic delay lattice systems driven by a multiplicative white noise was proved recently in [47].

Let $X$ be a Banach space. The norm of $X$ is written as $\| \cdot \|_X$. For each fixed $\rho > 0$, we use $C([-\rho, 0], X)$ to denote the set of all continuous functions from $[-\rho, 0]$ to $X$ with norm $\| \varphi \|_{C([-\rho, 0], X)} = \sup_{-\rho \leq s \leq 0} \| \varphi \|_X$ for $\varphi \in C([0, \rho], X)$. Let $M = C \left( [-\rho, 0], L^2(\mathcal{O}) \right)$ and $\mathcal{N} = C \left( [-\rho, 0], L^2(\mathcal{O}) \right)$. Given $\tau \in \mathbb{R}$, $T > \tau$ and a function $u : [\tau - \rho, T] \to X$, for each $t \in [\tau, T)$, let $u_t : [-\rho, 0] \to X$ denote the function defined by $u_t(s) = u(t + s)$ for $s \in [-\rho, 0]$. We denote by $\langle \cdot, \cdot \rangle_Y$ the inner product in a Hilbert space $Y$. The letter $c$ is a generic positive constant which may change its values from line to line.

We organize the paper as follows. In the next section, we establish the existence of a continuous cocycle in $\mathcal{N}$ for the stochastic retarded equation defined on the fixed domain $\mathcal{O}$, which is converted from (2)-(3). We also describe the existence of a continuous cocycle in $M$ for the stochastic retarded equation (4)-(5). Section 3 contains all necessary uniform estimates of the solutions. We then prove the existence and uniqueness of tempered random attractors for the stochastic retarded equations in section 4, and analyze convergence properties of the solutions as well as the random attractors in section 5.

2. Cocycles associated with stochastic retarded equations. Here we show that there is a continuous cocycle generated by the delay reaction-diffusion equation defined on $\mathcal{O}_x$ with multiplicative noise and deterministic non-autonomous forcing:

$$
\begin{aligned}
\left\{ \begin{array}{l}
\dot{u}_x^\varepsilon - \Delta u^\varepsilon dt = (H(t,x,u_\varepsilon^x(t)) + f(t,x,u_\varepsilon^x(t - \rho_0(t))) + G(t,x)) dt \\
+ \sum_{j=1}^{m} c_j \dot{u}_x^\varepsilon \circ dw_j,
\end{array} \right.
\end{aligned}$$

with the initial condition

$$u_x^\varepsilon(s,x) = \hat{\varphi}_x(s,x), \quad s \in [-\rho, 0], x \in \mathcal{O}_x,$$

where $\nu_x$ is the unit outward normal to $\partial \mathcal{O}_x$, $G : \mathbb{R} \times \tilde{\mathcal{O}} \to \mathbb{R}$ belongs to $L^2_{\text{loc}}(\mathbb{R}, L^\infty(\tilde{\mathcal{O}}))$, $c_j \in \mathbb{R}$, $w_j$ ($j = 1, 2, \ldots, m$) are independent two-sided real-valued Wiener processes on a probability space which will be specified later. $f : \mathbb{R} \times \tilde{\mathcal{O}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and has the property that there exist a positive constant $K$ and a function $\psi_1 \in L^2_{\text{loc}}(\mathbb{R}, L^\infty(\tilde{\mathcal{O}}))$ such that

$$|f(t,x,s)|^2 \leq K^2 |s|^2 + |\psi_1(t,x)|^2, \quad \text{for all} \ t \in \mathbb{R}, x \in \tilde{\mathcal{O}} \text{ and } s \in \mathbb{R},$$

and there exists $L_f > 0$ such that

$$|f(t,x,s_1) - f(t,x,s_2)| \leq L_f |s_1 - s_2|, \quad \text{for all} \ t \in \mathbb{R}, x \in \tilde{\mathcal{O}} \text{ and } s_1, s_2 \in \mathbb{R}. \quad \rho_0 : \mathbb{R} \to [0, \rho] \text{ is a continuously differentiable function with } |\rho_0'(t)| \leq \rho^* < 1 \text{ for all } t \in \mathbb{R}. \quad H \text{ is a nonlinear function satisfying}$$
the following conditions: for all \( x \in \tilde{\Omega} \) and \( t, s \in \mathbb{R} \),
\[
H (t, x, s) s \leq -\lambda_1 |s|^p + \varphi_2(t, x), \\
|H (t, x, s)| \leq \lambda_2 |s|^{p-1} + \varphi_3(t, x), \\
\frac{\partial H (t, x, s)}{\partial s} \leq \lambda_3, \\
\frac{\partial H (t, x, s)}{\partial x} \leq \psi_4(t, x),
\]
where \( p \geq 2, \; \lambda_1, \; \lambda_2 \) and \( \lambda_3 \) are positive constants, \( \varphi_2 \in L_{loc}^\infty (\mathbb{R}, L_{loc}^\infty (\tilde{\Omega})) \) and \( \varphi_3, \psi_4 \in L_{loc}^\infty (\mathbb{R}, L_{loc}^\infty (\tilde{\Omega})) \).

Throughout this paper, we fix a positive number \( \lambda \in (0, \lambda_1) \) and write
\[
h(t, x, s) = H(t, x, s) + \lambda s \tag{14}
\]
for all \( x \in \tilde{\Omega} \) and \( t, s \in \mathbb{R} \). Then it follows from (10)-(13) that there exist positive numbers \( \alpha_1, \alpha_2, \beta, b_1 \) and \( b_2 \) such that
\[
h (t, x, s) s \leq -\alpha_1 |s|^p + \psi_2(t, x), \\
|h (t, x, s)| \leq \alpha_2 |s|^{p-1} + \psi_3(t, x), \\
\frac{\partial h (t, x, s)}{\partial s} \leq \beta, \\
\frac{\partial h (t, x, s)}{\partial x} \leq \psi_4(t, x),
\]
where \( \psi_2(t, x) = \varphi_2(t, x) + b_1 \) and \( \psi_3(t, x) = \varphi_3(t, x) + b_2 \) for \( x \in \tilde{\Omega} \) and \( t, s \in \mathbb{R} \).

Substituting (14) into (6) we get for \( t > \tau \),
\[
\begin{cases}
d\hat{u}^x - (\Delta \hat{u}^x - \lambda \hat{u}^x) \ dt = (h(t, x, \hat{u}^x(t)) + f (t, x, \hat{u}^x(t) - \rho_0(t))) + G(t, x) \ dt \\
\quad + \sum_{j=1}^m c_j \hat{u}^x \circ d\omega_j, & x = (x^*, x_{n+1}) \in \Omega, \\
\frac{\partial \hat{u}^x}{\partial \nu_x} = 0, & x \in \partial \Omega_x,
\end{cases}
\tag{19}
\]
with the initial condition
\[
\hat{u}^x (s, x) = \hat{\phi}^x(s, x), \quad s \in [-\rho, 0], \; x \in \Omega_x.
\tag{20}
\]

We now transfer problem (19)-(20) into an initial boundary value problem on the fixed domain \( \Omega \). To that end, we introduce a transformation \( T_\varepsilon : \Omega_e \to \Omega \) by \( T_\varepsilon (x^*, x_{n+1}) = \left( x^*, \frac{x_{n+1}}{\varepsilon y^*} \right) \) for \( x = (x^*, x_{n+1}) \in \Omega_e \). Let \( y = (y^*, y_{n+1}) = T_\varepsilon (x^*, x_{n+1}) \). Then we have
\[
x^* = y^*, \quad x_{n+1} = \varepsilon g (y^*) y_{n+1}.
\]

After some calculations, we find that the Jacobian matrix of \( T_\varepsilon \) is given by
\[
J = \frac{\partial (y_1, \ldots, y_{n+1})}{\partial (x_1, \ldots, x_{n+1})} = \\
\begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots \\
0 & 0 & \cdots & 1 & 0 \\
-\frac{y_{n+1}}{g(y^*)} y_1 & -\frac{y_{n+1}}{g(y^*)} y_2 & \cdots & -\frac{y_{n+1}}{g(y^*)} y_n & \frac{1}{\varepsilon g(y^*)}
\end{pmatrix}.
\]
The determinant of \( J \) is \( |J| = \frac{1}{\varepsilon g(y^*)} \). Let \( J^* \) be the transport of \( J \). Then we have

\[
JJ^* = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{\varepsilon^{n+1}}{g(y^*)}g_{y_1} & -\frac{\varepsilon^{n+1}}{g(y^*)}g_{y_2} & \cdots & -\frac{\varepsilon^{n+1}}{g(y^*)}g_{y_n} \\
\frac{\varepsilon^{n+1}}{g(y^*)}g_{y_2} & \frac{\varepsilon^{n+1}}{g(y^*)}g_{y_1} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\varepsilon^{n+1}}{g(y^*)}g_{y_n} & \frac{\varepsilon^{n+1}}{g(y^*)}g_{y_{n-1}} & \cdots & \frac{\varepsilon^{n+1}}{g(y^*)}g_{y_1} \\
\end{pmatrix}.
\]

It follows from [35] (see also [26]) that the gradient operator and the Laplace operator in the new variable \( y \) and in the original variable \( x \) are related by

\[
\nabla_x \hat{u}(x) = J^* \nabla_y u(y) \quad \text{and} \quad \Delta_x \hat{u}(x) = |J| \operatorname{div}_y(|J|^{-1} J^* \nabla_y u(y)) = \frac{1}{g} \operatorname{div}_y(P_{\varepsilon} u(y)),
\]

where we denote by \( u(y) = \hat{u}(x) \), \( \nabla_x \) and \( \Delta_x \) are the gradient and the Laplace operator in \( x \in \mathcal{O}_\varepsilon \) respectively, \( \operatorname{div}_y \) and \( \nabla_y \) are the divergence and the gradient operator in \( y \in \mathcal{O} \) respectively, and \( P_{\varepsilon} \) is the operator given by

\[
P_{\varepsilon} u(y) = \begin{pmatrix}
g u_{y_1} - g y_1 y_{n+1} u_{y_{n+1}} \\
g u_{y_2} - g y_2 y_{n+1} u_{y_{n+1}} \\
\vdots \\
- \sum_{i=1}^{n} y_{n+1} g y_i u_{y_i} + \frac{1}{\varepsilon g} (1 + \sum_{i=1}^{n} (\varepsilon y_{n+1} g y_i)^2) u_{y_{n+1}} \end{pmatrix}.
\]

In the sequel, we abuse the notation a little bit by writing \( h(t, x, s,f(t, x, s) \) and \( G(t, x) \) as \( h(t, x^*, x_{n+1}, s), f(t, x^*, x_{n+1}, s) \) and \( G(t, x^*, x_{n+1}) \) for \( x = (x^*, x_{n+1}) \), respectively. With this agreement, for any function \( F(t, y, s) \), we introduce

\[
F_{\varepsilon} (t, y^*, y_{n+1}, s) = F(t, y^*, \varepsilon g(y^*) y_{n+1}, s), \quad F_0 (t, y^*, s) = F(t, y^*, 0, s),
\]

where \( y = (y^*, y_{n+1}) \in \mathcal{O} \) and \( t, s \in \mathbb{R} \). Then problem (19)-(20) is equivalent to the following system for \( t > \tau \),

\[
\begin{cases}
d u^\varepsilon - \frac{1}{g} \operatorname{div}_y(P_{\varepsilon} u^\varepsilon - \lambda u^\varepsilon) dt = (h_{\varepsilon} (t, y, u^\varepsilon(t)) + f_{\varepsilon} (t, y, u^\varepsilon(t) - \rho_0(t))) \\
+ G_{\varepsilon} (t, y)) dt + \sum_{j=1}^{m} c_j u^\varepsilon \circ dw_j, \quad y = (y^*, y_{n+1}) \in \mathcal{O}, \\
P_{\varepsilon} u^\varepsilon \cdot \nu = 0, \quad y \in \partial \mathcal{O},
\end{cases}
\]

with the initial condition

\[
u^\varepsilon (s, y) = \hat{\phi}^\varepsilon (s, y) = \hat{\phi}^\varepsilon (s) \circ T_{-1}^{-1} (y), \quad s \in [-\rho, 0], \ y \in \mathcal{O},
\]

where \( \nu \) is the unit outward normal to \( \partial \mathcal{O} \). Note that the boundary condition in (21) follows from the original boundary condition in (19) (see [34] for the details).

Given \( t \in \mathbb{R} \), define a translation \( \theta_{1, t} \) on \( \mathbb{R} \) by

\[
\theta_{1, t} (\tau) = \tau + t, \quad \text{for all} \ \tau \in \mathbb{R}.
\]

Then \( \{\theta_{1, t}\}_{t \in \mathbb{R}} \) is a group acting on \( \mathbb{R} \). We now specify the probability space. Denote by

\[
\Omega = \{ \omega \in C \left( \mathbb{R}, \mathbb{R} \right) : \omega(0) = 0 \}.
\]

Let \( \mathcal{F} \) be the Borel \( \sigma \)-algebra induced by the compact-open topology of \( \Omega \), and \( P \) be the corresponding Wiener measure on \( (\Omega, \mathcal{F}) \). There is a classical group \( \{\theta_{t}\}_{t \in \mathbb{R}} \) acting on \( (\Omega, \mathcal{F}, P) \), which is defined by

\[
\theta_{t} \omega (\cdot) = \omega (\cdot + t) - \omega (t), \quad \omega \in \Omega, \ t \in \mathbb{R}.
\]
Then \((\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})\) is a metric dynamical system (see [2]). On the other hand, let us consider the one-dimensional stochastic differential equation

\[
dz + \alpha z \, dt = dw(t),
\]

for \(\alpha > 0\). This equation has a random fixed point in the sense of random dynamical systems generating a stationary solution known as the stationary Ornstein-Uhlenbeck process (see [12, 23] for more details). In fact, we have

**Lemma 2.1.** There exists a \(\{\theta_t\}_{t \in \mathbb{R}}\)-invariant subset \(\Omega' \in \mathcal{F}\) of full measure such that

\[
\lim_{t \to \pm \infty} \frac{|\omega(t)|}{t} = 0 \quad \text{for all} \quad \omega \in \Omega',
\]

and, for such \(\omega\), the random variable given by

\[
z^*(\omega) = -\alpha \int_{-\infty}^{0} e^{\alpha s} \omega(s) \, ds
\]

is well defined. Moreover, for \(\omega \in \Omega'\), the mapping

\[
(t, \omega) \to z^*(\theta_t \omega) = -\alpha \int_{-\infty}^{0} e^{\alpha s} \omega(s) \, ds = -\alpha \int_{-\infty}^{0} e^{\alpha s} \omega(t + s) \, ds + \omega(t)
\]

is a stationary solution of (25) with continuous trajectories. In addition, for \(\omega \in \Omega'\)

\[
\lim_{t \to \pm \infty} \frac{1}{t} \int_{0}^{t} |z^*(\theta_s \omega)| \, ds = 0,
\]

(26)

\[
\lim_{t \to \pm \infty} \frac{1}{t} \int_{0}^{t} |z^*(\theta_s \omega)| \, ds = \mathbb{E}[|z^*|] < \infty.
\]

(27)

Denote by \(z^*_j\) the associated Ornstein-Uhlenbeck process corresponding to (25) with \(\alpha = 1\) and \(w\) replaced by \(w_j\) for \(j = 1, \ldots, m\). Then for any \(j = 1, \ldots, m\), we have a stationary Ornstein-Uhlenbeck process generated by a random variable \(z_j^*\) on \(\Omega'_j\), with properties formulated in Lemma 2.1 defined on a metric dynamical system \((\Omega'_j, \mathcal{F}_j, P, \{\theta_t\}_{t \in \mathbb{R}})\). We set

\[
\hat{\Omega} = \Omega'_1 \times \cdots \times \Omega'_m, \quad P = P_1 \times \cdots \times P_m \quad \text{and} \quad \mathcal{F} = \bigotimes_{j=1}^m \mathcal{F}_j.
\]

Then \((\hat{\Omega}, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})\) is a metric dynamical system.

Denote by

\[
S_{C_j}(t)u = e^{c_j t} u, \quad \text{for} \quad u \in L^2(\mathcal{O}),
\]

and

\[
\mathcal{T}(\omega) := S_{C_1}(z_1^*(\omega)) \circ \cdots \circ S_{C_m}(z_m^*(\omega)) = e^{\sum_{j=1}^m c_j z_j^*(\omega)} \text{Id}_{L^2(\mathcal{O})}, \quad \omega \in \hat{\Omega}.
\]

Then for every \(\omega \in \hat{\Omega}, \mathcal{T}(\omega)\) is a homeomorphism on \(L^2(\mathcal{O})\), and its inverse operator is given by

\[
\mathcal{T}^{-1}(\omega) := S_{C_m}(-z_m^*(\omega)) \circ \cdots \circ S_{C_1}(-z_1^*(\omega)) = e^{-\sum_{j=1}^m c_j z_j^*(\omega)} \text{Id}_{L^2(\mathcal{O})}.
\]

These operators can be easily extended to linear homeomorphisms \(\mathcal{T}(\omega)\) and \(\mathcal{T}^{-1}(\omega)\) on \(\mathcal{N}\). Indeed, for any \(\xi \in \mathcal{N}\), let us define

\[
(\mathcal{T}(\omega)\xi)(s) = \mathcal{T}(\theta_s \omega)\xi(s), (\mathcal{T}^{-1}(\omega)\xi)(s) = \mathcal{T}^{-1}(\theta_s \omega)\xi(s), \quad \text{for} \ s \in [-\rho, 0].
\]
It follows that \( \|T^{-1}(\theta_0)\| \) has sub-exponential growth as \( t \to \pm \infty \) for any \( \omega \in \Omega \). Hence \( \|T^{-1}\| \) is tempered. Analogously, \( \|T\| \) is also tempered. Obviously, \( \sup_{s \in [s_0-a,s_0+a]} \|T(\theta_s\omega)\| \) is still tempered for every \( s_0 \in \mathbb{R} \) and \( a \in \mathbb{R}^+ \).

On the other hand, since \( z^*_j, j = 1, \ldots, m, \) are independent Gaussian random variables, by the ergodic theorem we still have a \( \{\theta_t\}_{t \in \mathbb{R}} \)-invariant set \( \tilde{\Omega} \in \mathcal{F} \) of full measure such that

\[
\lim_{t \to \pm \infty} \frac{1}{t} \int_0^t \|T(\theta_s\omega)\|^2 \, ds = \mathbb{E} \|T\|^2 = \prod_{j=1}^m \mathbb{E}(e^{2c_jz^*_j}) < \infty,
\]

and

\[
\lim_{t \to \pm \infty} \frac{1}{t} \int_0^t \|T^{-1}(\theta_s\omega)\|^2 \, ds = \mathbb{E} \|T^{-1}\|^2 = \prod_{j=1}^m \mathbb{E}(e^{-2c_jz^*_j}) < \infty.
\]

**Remark 1.** We now consider \( \theta \) defined in (24) on \( \tilde{\Omega} \cap \tilde{\Omega} \) instead of \( \Omega \). This mapping possesses the same properties as the original one if we choose \( \mathcal{F} \) as the trace \( \sigma \)-algebra with respect to \( \Omega \cap \tilde{\Omega} \). The corresponding metric dynamical system is still denoted by \( (\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}}) \) throughout this paper.

Next, we define a continuous cocycle for system (21)-(22) in \( \mathcal{N} \). This can be achieved by transferring the stochastic system into a deterministic one with random parameters in a standard manner. Let \( u^\varepsilon \) be a solution to (21)-(22) and denote by

\[
v^\varepsilon(t) = T^{-1}(\theta_\varepsilon\omega) u^\varepsilon(t) \quad \text{and} \quad \delta(\omega) = \sum_{j=1}^m c_j z^*_j(\omega).
\]

Then \( v^\varepsilon \) satisfies

\[
\begin{cases}
\frac{dv^\varepsilon}{d\tau} - \frac{1}{2} \text{div}_y(P_x v^\varepsilon) = (-\lambda + \delta(\theta_\varepsilon\omega)) v^\varepsilon + T^{-1}(\theta_\varepsilon\omega) G_{\varepsilon}(t, y) \\
\quad + T^{-1}(\theta_\varepsilon\omega) f_x(t, y, T(\theta_{t-\rho_0(y)}\omega) v^\varepsilon(t - \rho_0(t))), \quad y \in \mathcal{O}, \ t > \tau,
\end{cases}
\]

with the initial conditions

\[
v^\varepsilon(s, y) = \psi^\varepsilon(s, y), \quad s \in [-\rho, 0], \ y \in \mathcal{O},
\]

where \( \psi^\varepsilon = (T^{-1}(\theta_\varepsilon\omega)) \phi^\varepsilon \).

Since (28) is a deterministic equation, by the Galerkin method, one can show that if \( f \) satisfies (8)-(9) and \( H \) satisfies (10)-(13), then for every \( \varepsilon \in \Omega \), \( \tau \in \mathbb{R} \) and \( \psi^\varepsilon \in \mathcal{N} \), (28)-(29) has a unique solution \( v^\varepsilon(\cdot, \tau, \omega, \psi^\varepsilon) \in C([\tau - \rho, \tau + T], L^2(\mathcal{O})) \cap L^2(\mathcal{O}) \) with \( v^\varepsilon(s, \tau, \omega, \psi^\varepsilon) = \psi^\varepsilon(s) \) for every \( T > 0 \) and \( s \in [-\rho, 0] \). Furthermore, one may show that \( v^\varepsilon(\cdot, \tau, \omega, \psi^\varepsilon) \) is \( \mathcal{F} \mathcal{B}(\mathcal{N}) \)-measurable in \( \omega \in \Omega \) and continuous with respect to \( \psi^\varepsilon \in \mathcal{N} \) for all \( t \geq \tau \). Since \( u^\varepsilon(\cdot, \tau, \omega, \phi^\varepsilon) = T(\theta_{t+\tau}\omega) v^\varepsilon(\cdot, \tau, \omega, \psi^\varepsilon) \) with \( \phi^\varepsilon = (T(\theta_\varepsilon\omega)) \psi^\varepsilon \), then we find that \( u^\varepsilon \) is continuous in both \( t \geq \tau \) and \( \phi^\varepsilon \in \mathcal{N} \) and is \( \mathcal{F} \mathcal{B}(\mathcal{N}) \)-measurable in \( \omega \in \Omega \). In addition, it follows from (28) that \( u^\varepsilon \) is a solution of problem (21)-(22). We now define \( \Phi^\varepsilon : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathcal{N} \to \mathcal{N} \) by

\[
\Phi^\varepsilon(t, \tau, \omega, \phi^\varepsilon)(\cdot) = u^\varepsilon(t, \tau, \theta_{t+\tau}\omega, \phi^\varepsilon) = T(\theta_{t+\tau}\omega) v^\varepsilon(t, \tau, \theta_{-\tau}\omega, \psi^\varepsilon),
\]

for all \( (t, \tau, \omega, \phi^\varepsilon) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathcal{N} \).

By the properties of \( u^\varepsilon \), we find that \( \Phi^\varepsilon \) is a continuous cocycle on \( \mathcal{N} \) over \( (\mathbb{R}, \{\theta_t\}_{t \in \mathbb{R}}) \) and \( (\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}}) \), where \( \{\theta_t\}_{t \in \mathbb{R}} \) and \( \{\theta_t\}_{t \in \mathbb{R}} \) are given by (23) and (24), respectively.
Let \( R_\varepsilon : C \left([-\rho, 0], L^2(\Omega)\right) \rightarrow C \left([-\rho, 0], L^2(\Omega)\right) \) be an affine mapping of the form
\[
(R_\varepsilon \hat{\phi}_\varepsilon)(y) = \hat{\phi}_\varepsilon(T_\varepsilon^{-1}(y)), \quad \forall \hat{\phi}_\varepsilon \in C([-\rho, 0], L^2(\Omega_\varepsilon)).
\]
Given \( t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega \) and \( \hat{\phi}_\varepsilon \in C([-\rho, 0], L^2(\Omega_\varepsilon)) \), we can define a continuous cocycle \( \hat{\Phi}_\varepsilon \) for problem (6)-(7) by the formula
\[
\hat{\Phi}_\varepsilon(t, \tau, \omega, \hat{\phi}_\varepsilon) = R_\varepsilon^{-1} \hat{\Phi}_\varepsilon(t, \tau, \omega, R_\varepsilon \hat{\phi}_\varepsilon),
\]
where \( \hat{\Phi}_\varepsilon \) is the continuous cocycle for problem (21)-(22) on \( \mathcal{N} \).

The same argument as above allows us to prove that problem (4) and (5) generates a continuous cocycle \( \Phi_0(t, \tau, \omega, \Phi^0) \) in the space \( \mathcal{M} \).

Now we want to write equation (28)-(29) as an abstract evolutionary equation. We introduce the inner product \((\cdot, \cdot)_{H_0(\Omega)}\) on \( L^2(\Omega) \) defined by
\[
(u, v)_{H_0(\Omega)} = \int_{\Omega} uv dy, \quad \text{for all} \ u, v \in L^2(\Omega)
\]
and denote by \( H_0(\Omega) \) the space equipped with this inner product. Since \( g \) is a continuous function on \( \overline{\Omega} \) and satisfies (1), one easily shows that \( H_0(\Omega) \) is a Hilbert space with norm equivalent to the natural norm of \( L^2(\Omega) \).

For \( 0 < \varepsilon \leq 1 \), we introduce a bilinear form \( a_\varepsilon (\cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R} \), given by
\[
a_\varepsilon (u, v) = (J^* \nabla_y u, J^* \nabla_y v)_{H_0(\Omega)},
\]
where
\[
J^* \nabla_y u = (u_{y_1} - \frac{g_{y_1}}{g} y_{n+1} u_{y_{n+1}}, \ldots, u_{y_n} - \frac{g_{y_n}}{g} y_{n+1} u_{y_{n+1}}, 1 \frac{g_{y_n}}{\varepsilon^2} u_{y_{n+1}}).
\]

By introducing on \( H^1(\Omega) \) the equivalent norm, for every \( 0 < \varepsilon \leq 1 \),
\[
\|u\|_{H^1_\varepsilon(\Omega)} = \left( \int_{\Omega} (|\nabla_y u|^2 + |u|^2 + \frac{1}{\varepsilon^2} u_{y_{n+1}}^2) dy \right)^{1/2},
\]
we see that there exist positive constants \( \varepsilon_0, \eta_1 \) and \( \eta_2 \) such that for all \( 0 < \varepsilon \leq \varepsilon_0 \) and \( u \in H^1(\Omega) \),
\[
\eta_1 \|u\|_{H^1_\varepsilon(\Omega)}^2 \leq a_\varepsilon (u, u) + \|u\|_{L^2(\Omega)}^2 \leq \eta_2 \|u\|_{H^1(\Omega)}^2.
\]
Denote by \( A_\varepsilon \) an unbounded operator on \( H_0(\Omega) \) with domain \( D(A_\varepsilon) = \{v \in H^2(\Omega), P_\varepsilon v \cdot \nu = 0 \text{ on } \partial \Omega\} \) as defined by
\[
A_\varepsilon v = -\frac{1}{g} \text{div} P_\varepsilon v, \quad v \in D(A_\varepsilon).
\]
Then we have

\[ a_\varepsilon(u, v) = (A_\varepsilon u, v)_{H_\varepsilon(O)}, \forall u \in D(A_\varepsilon), \forall v \in H^1(O). \]  

(36)

Using \( A_\varepsilon \), (28)-(29) can be written as

\[
\begin{aligned}
\frac{dv^\varepsilon}{dt} + A_\varepsilon v^\varepsilon &= (-\lambda + \delta(\theta_t \omega))v^\varepsilon + \mathcal{T}^{-1}(\theta_t \omega) h_\varepsilon(t, y, \mathcal{T}(\theta_t \omega) v^\varepsilon(t)) + \mathcal{T}^{-1}(\theta_t \omega) f_\varepsilon(t, y, \mathcal{T}(\theta_t \rho_0(\tau) \omega) v^\varepsilon(t - \rho_0(t))) \\
v_\varepsilon^\varepsilon(s) &= \psi^\varepsilon(s), \quad s \in [-\rho, 0].
\end{aligned}
\]

(37)

To reformulate system (31)-(32), we introduce the inner product \( \langle \cdot, \cdot \rangle_{H_\varepsilon(Q)} \) on \( L^2(Q) \) defined by

\[ \langle u, v \rangle_{H_\varepsilon(Q)} = \int_Q guvdy^*, \quad \text{for all } u, v \in L^2(Q), \]

and denote by \( H_g(Q) \) the space equipped with this inner product. Let \( a_0(\cdot, \cdot): H^1(Q) \times H^1(Q) \to \mathbb{R} \) be a bilinear form given by

\[ a_0(u, v) = \int_Q g(\nabla y^* u, \nabla y^*)dy^*. \]

Denote by \( A_0 \) an unbounded operator on \( H_g(Q) \) with domain \( D(A_0) = \{ v \in H^2(Q), \frac{\partial v}{\partial n_0} = 0 \text{ on } \partial Q \} \) as defined by

\[ A_0v = -\frac{1}{g} \sum_{i=1}^{n} (gv_{y_i})_{y_i}, \quad v \in D(A_0). \]

Then we have

\[ a_0(u, v) = (A_0u, v)_{H_\varepsilon(Q)}, \quad \forall u \in D(A_0), \forall v \in H^1(Q). \]

Using \( A_0 \), (31)-(32) can be written as

\[
\begin{aligned}
\frac{dv^0}{dt} + A_0 v^0 &= (-\lambda + \delta(\theta_t \omega))v^0 + \mathcal{T}^{-1}(\theta_t \omega) h_0(t, y^*, \mathcal{T}(\theta_t \omega) v^0(t)) + \mathcal{T}^{-1}(\theta_t \omega) f_0(t, y^*, \mathcal{T}(\theta_t \rho_0(\tau) \omega) v^0(t - \rho_0(t))) \\
v^0(s) &= \psi^0(s), \quad s \in [-\rho, 0].
\end{aligned}
\]

(38)

Hereafter, we set \( X_0 = \mathcal{M}, X_\varepsilon = C([-\rho, 0], L^2(O_\varepsilon)) \) and \( X_1 = \mathcal{N} \). For every \( i = \varepsilon, 0 \) or 1, a family \( B_i = \{ B_i(t, \omega) : t \in \mathbb{R}, \omega \in \Omega \} \) of nonempty subsets of \( X_i \) is called tempered if for every \( \varepsilon > 0 \), we have:

\[ \lim_{t \to -\infty} \varepsilon^\tau \| B_i(t + \varepsilon, \theta_t \omega) \|_{X_i} = 0, \]

where \( \| B_i \|_{X_i} = \sup_{x \in B_i} \| x \|_{X_i} \). The collection of all families of tempered nonempty subsets of \( X_i \) is denoted by \( \mathcal{D}_i \), i.e.,

\[ \mathcal{D}_i = \{ B_i = \{ B_i(t, \omega) : t \in \mathbb{R}, \omega \in \Omega \} : B_i \text{ is tempered in } X_i \}. \]

Our main purpose of the paper is to prove that the continuous cocycles \( \hat{\Phi}_\varepsilon \) and \( \Phi_0 \) possess a unique \( \mathcal{D}_\varepsilon \)-pullback attractor \( \mathcal{A}_\varepsilon \) in \( X_\varepsilon \) and a unique \( \mathcal{D}_0 \)-pullback attractor \( \mathcal{A}_0 \) in the space \( \mathcal{M} \), respectively. Furthermore \( \mathcal{A}_\varepsilon \) is upper-semicontinuous at \( \varepsilon = 0 \), that is, for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[ \lim_{\varepsilon \to 0} \sup_{u_\varepsilon \in \mathcal{A}_\varepsilon} \inf_{u_0 \in \mathcal{A}_0} \varepsilon^{-1} \| u_\varepsilon - u_0 \|^2_{X_\varepsilon} = 0. \]

(39)
To prove (39), we only need to show that the cocycle $\Phi_\varepsilon$ has a unique $D_1$-pullback attractor $\mathcal{A}_\varepsilon$ in $\mathcal{N}$ and it is upper-semicontinuous at $\varepsilon = 0$ in the sense that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{\varepsilon \to 0} \text{dist}_{\mathcal{N}}(\mathcal{A}_\varepsilon(\tau, \omega), \mathcal{A}_0(\tau, \omega)) = 0,$$

which will be established in the last section of the paper.

Furthermore, we suppose that there exists $\lambda_0 > 0$ such that

$$\gamma \triangleq \lambda_0 - 2\mathbb{E}(|\delta(\omega)|) - \frac{1}{\sqrt{1-\rho^*}}Ke^{\frac{1}{2}\lambda o}(\int \frac{m}{j=1} \mathbb{E}(e^{2\gamma_j \tau_j}) + \int \frac{m}{j=1} \mathbb{E}(e^{-2\gamma_j \tau_j})) > 0. \quad (40)$$

Let us consider the mapping

$$\gamma(\omega) = \lambda_0 - 2|\delta(\omega)| - \frac{1}{\sqrt{1-\rho^*}}Ke^{\frac{1}{2}\lambda o}(\int \|T(\omega)\|^2 + \|T^{-1}(\omega)\|^2). \quad (41)$$

By the ergodic theory and (40) we have

$$\lim_{t \to \pm \infty} \frac{1}{t} \int_0^t \gamma(\theta\omega) dl = \mathbb{E}\gamma = \gamma > 0. \quad (42)$$

The following condition will be needed when deriving uniform estimates of solutions:

$$\int_{-\infty}^T e^{\frac{1}{2}\gamma \tau}(\|G(s, \cdot)\|_{L^\infty(\tilde{\omega})}^2 + \|\psi_1(s, \cdot)\|_{L^\infty(\tilde{\omega})}^2 + \|\varphi_2(s, \cdot)\|_{L^\infty(\tilde{\omega})}^2 + \|\psi_4(s, \cdot)\|_{L^\infty(\tilde{\omega})}^2) ds < \infty, \quad \forall \tau \in \mathbb{R}. \quad (43)$$

When constructing tempered pullback attractors, we will assume

$$\lim_{r \to \infty} e^{\sigma r} \int_0^r e^{\frac{1}{2}\gamma \tau} \left(\|G(s + r, \cdot)\|_{L^\infty(\tilde{\omega})}^2 + \|\psi_1(s + r, \cdot)\|_{L^\infty(\tilde{\omega})}^2 + \|\varphi_2(s + r, \cdot)\|_{L^\infty(\tilde{\omega})}^2 + \|\psi_4(s + r, \cdot)\|_{L^\infty(\tilde{\omega})}^2\right) ds = 0, \quad \forall \sigma > 0. \quad (44)$$

Notice that condition (43) does not require that $G(s, \cdot)$ be bounded in $L^\infty(\tilde{\omega})$, when $s \to \pm \infty$. Since $\psi_2 = \varphi_2 + b_1$ for some positive constant $b_1$, it is evident that (43) and (44) imply

$$\int_{-\infty}^T e^{\frac{1}{2}\gamma \tau}(\|G(s, \cdot)\|_{L^\infty(\tilde{\omega})}^2 + \|\psi_1(s, \cdot)\|_{L^\infty(\tilde{\omega})}^2 + \|\psi_2(s, \cdot)\|_{L^\infty(\tilde{\omega})}^2 + \|\psi_4(s, \cdot)\|_{L^\infty(\tilde{\omega})}^2) ds < \infty, \quad \forall \tau \in \mathbb{R} \quad (45)$$

and

$$\lim_{r \to \infty} e^{\sigma r} \int_0^r e^{\frac{1}{2}\gamma \tau} \left(\|G(s + r, \cdot)\|_{L^\infty(\tilde{\omega})}^2 + \|\psi_1(s + r, \cdot)\|_{L^\infty(\tilde{\omega})}^2 + \|\psi_2(s + r, \cdot)\|_{L^\infty(\tilde{\omega})}^2 + \|\psi_4(s + r, \cdot)\|_{L^\infty(\tilde{\omega})}^2\right) ds = 0, \quad (46)$$

for any $\sigma > 0$. 
3. Uniform estimates of solutions. In this section, we derive uniform estimates of solutions of problem (37) which are needed for proving the existence of $D_1$-pullback absorbing sets and the $D_1$-pullback asymptotic compactness of the continuous cocycle $\Phi$. The estimates of solutions of problem (37) in $C([-\rho, 0], H_g(O))$ are provided below.

**Lemma 3.1.** Assume that \((8)-(13), (40)\) and \((43)\) hold. Then for every $0 < \varepsilon \leq \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D_1 = \{D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \subset D_1$, there exists $T = T(\tau, \omega, D_1) > \rho$, independent of $\varepsilon$, such that for all $t \geq T$, $\lambda_1 > \lambda_0$ and $\psi^\varepsilon \in D_1(\tau - t, \theta - \varepsilon \omega)$, the solution $v^\varepsilon$ of (37) with $\omega$ replaced by $\theta - \varepsilon \omega$ satisfies

$$
\|v^\varepsilon(\cdot, \tau - t, \theta - \varepsilon \omega, \psi^\varepsilon)\|^2_{C([-\rho, 0], H_g(O))} \leq R_1(\tau, \omega),
$$

where $R_1(\tau, \omega)$ is determined by

$$
R_1(\tau, \omega) = c \int_{-\infty}^{\infty} e^{\int_0^r \gamma(\theta_i(t))dt} \|\psi_1(r + \tau, \cdot)\|^2_{L^\infty(\Omega)} dr
$$

$$
+c \int_{-\infty}^{\infty} e^{\int_0^r \gamma(\theta_i(t))dt} \|T^{-1}(\theta_i(t), \omega)\|^2 \|\psi_2(r + \tau, \cdot)\|^2_{L^\infty(\Omega)} dr
$$

$$
+c \int_{-\infty}^{\infty} e^{\int_0^r \gamma(\theta_i(t))dt} \|T^{-1}(\theta_i(t), \omega)\|^2 \|v^\varepsilon(r + \tau, \cdot)\|^2_{L^\infty(\Omega)} dr,
$$

where $c$ is independent of $\varepsilon$.

**Proof.** Since $\lambda_1 > \lambda_0$, we can take the $\lambda$ in (14) satisfying $\lambda \geq \lambda_0$. Taking the inner product of (37) with $v^\varepsilon$ in $H_g(O)$, we find that

$$
\frac{1}{2} \frac{d}{dt} \|v^\varepsilon\|^2_{H_g(O)} \leq -\alpha_\varepsilon \langle v^\varepsilon, v^\varepsilon \rangle + (-\lambda_0 + \delta(\theta_i(t))) \|v^\varepsilon\|^2_{H_g(O)}
$$

$$
+ (T^{-1}(\theta_i(t), \omega)h_c(t, y, T(\theta_i(t))v^\varepsilon(t)), v^\varepsilon)_{H_g(O)}
$$

$$
+ (T^{-1}(\theta_i(t), \omega)G_c(t, y, v^\varepsilon), v^\varepsilon)_{H_g(O)}
$$

For the third term on the right-hand side of (49), by (15), we have

$$
(T^{-1}(\theta_i(t), \omega)G_c(t, y, v^\varepsilon), v^\varepsilon)_{H_g(O)} = \int_O T^{-2}(\theta_i(t))gh(t, y^*, \varepsilon g(y^*)y_{n+1}, u^\varepsilon) \, dy
$$

$$
\leq -\alpha_1 \gamma_1 \|T^{-1}(\theta_i(t))\|^2 \int_O |u^\varepsilon|^p \, dy + \gamma_2 |O| \|T^{-1}(\theta_i(t))\|^2 \|\psi_2(t, \cdot)\|^2_{L^\infty(\Omega)},
$$

where $|O|$ stands for the Lebesgue measure of $O$. For the last term on the right-hand side of (49), we have

$$
(T^{-1}(\theta_i(t), \omega)G_c(t, y^*, v^\varepsilon), \psi^\varepsilon)_{H_g(O)} \leq \frac{\lambda_0}{4} \|v^\varepsilon\|^2_{H_g(O)} + \frac{1}{\lambda_0} \gamma_2 |O| \|T^{-1}(\theta_i(t))\|^2 \|G(t, \cdot)\|^2_{L^\infty(\Omega)}.
$$

Consequently, it follows from (49)-(51) that

$$
\frac{d}{dt} \|v^\varepsilon\|^2_{H_g(O)} + 2\alpha_\varepsilon \langle v^\varepsilon, v^\varepsilon \rangle + \frac{\lambda_0}{2} \|v^\varepsilon\|^2_{H_g(O)} + 2\alpha_1 \gamma_1 \|T^{-1}(\theta_i(t))\|^2 \|u^\varepsilon\|^p_{L^p(O)}
$$

$$
\leq (-\lambda_0 + \delta(\theta_i(t))) \|v^\varepsilon\|^2_{H_g(O)}
$$

$$
+ 2 (T^{-1}(\theta_i(t), \omega)G_c(t, y, T(\theta_i(t)), v^\varepsilon(t - \rho_0(t))), v^\varepsilon)_{H_g(O)}
$$
Now we deal with the third term on the right-hand side of (54) by means of (8).

\[ \epsilon > \]

\[ \sigma \leq \| \gamma \|_{L^\infty (\tilde{\omega})} + 2 \gamma_2 \| T^{-1} (\theta_1 \omega) \|^2 \| \psi_2 (t, \cdot) \|_{L^\infty (\tilde{\omega})}. \]  \tag{52} 

It follows from (52) that

\[ \frac{d}{dt} \left( e^{\int_\tau^\sigma \gamma (\theta_1 \omega) dt} \| v^\varepsilon (\sigma) \|^2_{H_2 (\Omega)} \right) + 2 \epsilon e^{\int_\tau^\sigma \gamma (\theta_1 \omega) dt} a_\varepsilon (v^\varepsilon, v^\varepsilon) \]
\[ + \frac{\lambda_0}{2} e^{\int_\tau^\sigma \gamma (\theta_1 \omega) dt} \| v^\varepsilon \|_{L^2 (\Omega)}^2 + 2 \alpha_1 \gamma_1 \| T^{-1} (\theta_1 \omega) \|^2 e^{\int_\tau^\sigma \gamma (\theta_1 \omega) dt} \| u^\varepsilon \|^2_{L^p (\Omega)} \]
\[ \leq e^{\int_\tau^\sigma \gamma (\theta_1 \omega) dt} \left( - \lambda_0 + 2 \delta (\theta_1 \omega) + \gamma (\theta_1 \omega) \right) \| v^\varepsilon \|_{H^1 (\Omega)} + 2 \epsilon e^{\int_\tau^\sigma \gamma (\theta_1 \omega) dt} \| T^{-1} (\theta_1 \omega) \|^2 \| G (t, \cdot) \|^2_{L^\infty (\tilde{\omega})} \]
\[ + 2 \gamma_2 \| e^{\int_\tau^\sigma \gamma (\theta_1 \omega) dt} \| T^{-1} (\theta_1 \omega) \|^2 \| \psi_2 (t, \cdot) \|_{L^\infty (\tilde{\omega})}. \]  \tag{53} 

Then, we have for any \( \sigma \geq \tau, \)

\[ e^{\int_\tau^\sigma \gamma (\theta_1 \omega) dt} \| v^\varepsilon (\sigma) \|^2_{H_2 (\Omega)} + 2 \int_\tau^\sigma e^{\int_\tau^r \gamma (\theta_1 \omega) dt} a_\varepsilon (v^\varepsilon (r), v^\varepsilon (r)) dr \]
\[ + \frac{\lambda_0}{2} \int_\tau^\sigma e^{\int_\tau^r \gamma (\theta_1 \omega) dt} \| v^\varepsilon (r) \|^2_{H_2 (\Omega)} dr \]
\[ + 2 \alpha_1 \gamma_1 \int_\tau^\sigma \| T^{-1} (\theta_1 \omega) \|^2 e^{\int_\tau^r \gamma (\theta_1 \omega) dt} \| u^\varepsilon (r) \|^2_{L^p (\Omega)} dr \]
\[ \leq \| v^\varepsilon (\tau) \|^2_{H_2 (\Omega)} + \int_\tau^\sigma e^{\int_\tau^r \gamma (\theta_1 \omega) dt} \left( - \lambda_0 + 2 \delta (\theta_1 \omega) + \gamma (\theta_1 \omega) \right) \| v^\varepsilon (r) \|^2_{H^1 (\Omega)} dr \]
\[ + 2 \epsilon \int_\tau^\sigma e^{\int_\tau^r \gamma (\theta_1 \omega) dt} \| T^{-1} (\theta_1 \omega) \|^2 \| G (r, \cdot) \|^2_{L^\infty (\tilde{\omega})} dr \]
\[ + 2 \gamma_2 \| e^{\int_\tau^\sigma \gamma (\theta_1 \omega) dt} \| T^{-1} (\theta_1 \omega) \|^2 \| \psi_2 (r, \cdot) \|^2_{L^\infty (\tilde{\omega})}. \]  \tag{54} 

Now we deal with the third term on the right-hand side of (54) by means of (8). Given \( \epsilon > 0 \) we have

\[
2 \int_\tau^\sigma e^{\int_\tau^r \gamma (\theta_1 \omega) dt} \left( T^{-1} (\theta_1 \omega) f_\omega (r, y, T (\theta_1 - \rho_0 (r)) \omega v^\varepsilon (r - \rho_0 (r))) \right) a_\varepsilon (v^\varepsilon (r), v^\varepsilon (r)) \| T^{-1} (\theta_1 \omega) \|^2 \| v^\varepsilon (r) \|^2_{H^1 (\Omega)} dr 
\]
\[ \leq \epsilon \int_\tau^\sigma e^{\int_\tau^r \gamma (\theta_1 \omega) dt} \| T^{-1} (\theta_1 \omega) \|^2 \| v^\varepsilon (r) \|^2_{H_2 (\Omega)} dr \]
\[ + \epsilon^{-1} \frac{1}{1 - \rho^2} K^2 e^{\lambda_0 \rho} \int_\tau^\sigma e^{\int_\tau^r \gamma (\theta_1 \omega) dt} \| T (\theta_1 \omega) \|^2 \| v^\varepsilon (r) \|^2_{H_2 (\Omega)} dr \]
\[ + \epsilon^{-1} \gamma_2 \| e^{\int_\tau^\sigma \gamma (\theta_1 \omega) dt} \| T^{-1} (\theta_1 \omega) \|^2 \| \psi_1 (r, \cdot) \|^2_{L^\infty (\tilde{\omega})} dr, \]  \tag{55} 

where we use the fact that \( \gamma (\omega) \leq \lambda_0. \) Let \( \epsilon = \frac{1}{\sqrt{1 - \rho^2}} K e^{\frac{\lambda_0 \rho}{2}}, \) then we obtain from (41), (54) and (55) that

\[ \| v^\varepsilon (\sigma) \|^2_{H_2 (\Omega)} + 2 \int_\tau^\sigma e^{\int_\tau^r \gamma (\theta_1 \omega) dt} a_\varepsilon (v^\varepsilon (r), v^\varepsilon (r)) dr \]
\[ + \frac{\lambda_0}{2} \int_{\tau}^{\sigma} e^{\int_{\tau}^{\sigma}} \gamma(\theta(t))dt \| v^\varepsilon (r) \|_{H^1(\Omega)}^2 dr \\
aleq \| v^\varepsilon (\tau) \|_{H^1(\Omega)}^2 e^{-\int_{\tau}^{\sigma} \gamma(\theta(t))dt} \\
+ 2 \alpha_1 \gamma_1 \int_{\tau}^{\sigma} \| T^{-1} (\theta(t)) \|_r^2 e^{\int_{\tau}^{\sigma}} \gamma(\theta(t))dt \| u^\varepsilon (r) \|_{L^p(\Omega)}^p dr \\
\]

Considering time \( \tau - t \) instead of \( \tau \) with \( t \geq 0 \), and then replacing \( \omega \) by \( \theta - \omega \), we have

\[ \| v^\varepsilon (\omega, \theta - \tau, \omega, \psi^\varepsilon) \|_{H^1(\Omega)}^2 \\
+ 2 \int_{\tau - t}^{\sigma} e^{\int_{\tau - t}^{\sigma}} \gamma(\theta(t - \omega))dt a_\varepsilon (v^\varepsilon (r, \tau - t, \theta - \omega, \psi^\varepsilon), v^\varepsilon (r, \tau - t, \theta - \omega, \psi^\varepsilon)) dr \\
+ \frac{\lambda_0}{2} \int_{\tau - t}^{\sigma} e^{\int_{\tau - t}^{\sigma}} \gamma(\theta(t - \omega))dt \| v^\varepsilon (r, \tau - t, \theta - \omega, \psi^\varepsilon) \|_{H^1(\Omega)}^2 dr \\
+ 2 \alpha_1 \gamma_1 \int_{\tau - t}^{\sigma} \| T^{-1} (\theta - \omega) \|_r^2 e^{\int_{\tau - t}^{\sigma}} \gamma(\theta(t - \omega))dt \| u^\varepsilon (r, \tau - t, \theta - \omega, \phi^\varepsilon) \|_{L^p(\Omega)}^p dr \\
\]

Replacing \( \tau \) by \( \tau + s \) in (57) and taking supremum when \( s \in [-\rho, 0] \), then we obtain

\[ \| v^\varepsilon (\omega, \tau - t, \theta - \omega, \psi^\varepsilon) \|_{C([-\rho, 0], H^1(\Omega))} \leq \sup_{-\rho \leq s \leq 0} \left\{ e^{\int_{\sigma}^{0}} \gamma(\theta(t))dt \left[ e^{-\int_{\tau}^{\sigma} \gamma(\theta(t))dt} \right] \left[ e^{\int_{\tau}^{\sigma}} \gamma(\theta(t))dt \right] \right\} \\
+ \frac{1}{\sqrt{1 - \rho^2}} K e^{\frac{1}{2} \lambda_0 \rho^2} \int_{\tau - t}^{\tau} e^{\int_{\tau - t}^{\tau}} \gamma(\theta(t - \omega))dt \| T (\theta(t)) \|_r^2 dr \left\| \psi^\varepsilon \right\|_{C([-\rho, 0], H^1(\Omega))}^2 \\
+ \frac{\sqrt{1 - \rho^2}}{K} e^{-\frac{1}{2} \lambda_0 \rho^2 \gamma_2(\Omega)} \int_{\tau - t}^{\tau} e^{\int_{\tau - t}^{\tau}} \gamma(\theta(t - \omega))dt \| \psi_1 (r, \cdot) \|_{L^\infty(\bar{\Omega})}^2 dr \\
+ \frac{2}{\lambda_0 \gamma_2(\Omega)} \int_{\tau - t}^{\tau} e^{\int_{\tau - t}^{\tau}} \gamma(\theta(t - \omega))dt \| T^{-1} (\theta - \omega) \|_r^2 \| G (r, \cdot) \|_{L^\infty(\bar{\Omega})}^2 dr \\
+ \frac{2}{\lambda_0 \gamma_2(\Omega)} \int_{\tau - t}^{\tau} e^{\int_{\tau - t}^{\tau}} \gamma(\theta(t - \omega))dt \| T^{-1} (\theta - \omega) \|_r^2 \| \psi_2 (r, \cdot) \|_{L^\infty(\bar{\Omega})}^2 dr. 
\]
\[ + 2\gamma_2 |O| \int_{-\infty}^{0} e^{\int_{0}^{\tau} \gamma(\theta_1)} d\theta_1 \left\| \mathcal{T}^{-1}(\theta_1) \right\|^2 \left\| \psi_2 (r + \tau, \cdot) \right\|_{L^\infty(\partial)}^2 \, dr \right\}. \] (58)

Notice that \( \gamma(\omega) \leq \lambda_0 \), and we get
\[ \sup_{-\rho \leq \tau \leq 0} e^{\int_{0}^{\tau} \gamma(\theta_1)} d\theta_1 \leq e^{\lambda_0 \rho}. \] (59)

By (42), for any \( \varepsilon_1 > 0 \) and \( \omega \in \Omega \), there is a \( T_1 = T_1(\varepsilon_1, \omega) > 0 \) such that for all \( |t| \geq T_1 \), we have
\[ \left| \int_{0}^{t} (\gamma(\theta, -\tau) - \tau) d\tau \right| \leq \varepsilon_1 |t|. \] (60)

Since \( \| \mathcal{T}(\theta, \omega) \|^2 \) and \( \| \mathcal{T}^{-1}(\theta, \omega) \|^2 \) are tempered, for any \( \varepsilon_2 > 0 \) and \( \omega \in \Omega \) there exists a \( T_2 = T_2(\varepsilon_2, \omega) \geq T_1 \) such that for \( |t| \geq T_2 \)
\[ \| \mathcal{T}(\theta, \omega) \|^2 \leq e^{\varepsilon_2 |t|} \] and \( \| \mathcal{T}^{-1}(\theta, \omega) \|^2 \leq e^{\varepsilon_2 |t|}. \] (61)

Taking \( \varepsilon_1 = \varepsilon_2 = \frac{1}{4} \gamma_t \), we get from (60) and (61) that for all \( t \geq T_2 \)
\[ e^{-\int_{0}^{\theta} \gamma(\theta_1)} d\theta_1 \leq e^{-\int_{0}^{\theta} (\gamma(\theta_1) - \gamma_1) d\theta_1} \leq e^{\frac{1}{4} \gamma_t}. \] (62)

and
\[ \int_{-t-\rho}^{-t} e^{\int_{0}^{\theta} \gamma(\theta_1)} d\theta_1 \| \mathcal{T} (\theta, \omega) \|^2 \, d\theta \leq \int_{-t-\rho}^{-t} e^{\frac{1}{4} \gamma_1} \, d\theta \leq \frac{2}{\gamma_t} e^{-\frac{1}{4} \gamma_t}. \] (63)

Since \( \psi^\varepsilon \in D_1(\tau - t, \theta, \omega) \) and \( D_1 \in D_1 \), we find from (62) and (63) that
\[ \limsup_{t \to +\infty} \left| e^{-\int_{0}^{\theta} \gamma(\theta_1)} d\theta_1 + \frac{K e^{\frac{1}{4} \lambda_0 \rho}}{\sqrt{1 - \rho^2}} \int_{-t-\rho}^{-t} e^{\int_{0}^{\theta} \gamma(\theta_1)} d\theta_1 \| \mathcal{T} (\theta, \omega) \|^2 \, d\theta \right| ||\psi^\varepsilon||_{L^2([-\rho, 0], H^1(\Omega))}^2 \]
\[ = 0. \] (64)

Furthermore, by (43), (61) and (62), we find the last three terms on the right-hand side of (58) are convergent. This completes the proof of Lemma 3.1.

Notice that \( \gamma(\omega) \leq \lambda_0 \), we have \( \int_{0}^{\tau} \gamma(\theta) d\theta \geq -\lambda_0(2\rho+1) \) for \( r \in [\tau - t, \tau] \) and \( t \geq 2\rho + 1 \). As a consequence of Lemma 3.1, we have the following inequality which is useful for deriving the uniform estimates of solutions in \( C \left([-\rho, 0], H^1_{x} (\Omega) \right) \).

**Lemma 3.2.** Assume that (8)-(13), (40) and (43) hold. Then for every \( 0 < \varepsilon \leq \varepsilon_0, \tau \in \mathbb{R}, \omega \in \Omega, \) and \( D_1 = \{D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D_1 \), there exists \( T = T(\tau, \omega, D_1) \geq 2\rho + 1 \), independent of \( \varepsilon \), such that for all \( t \geq T, \lambda_1 \geq \lambda_0 \) and \( \psi^\varepsilon \in D_1(\tau - t, \theta, \omega), \) the solution \( \psi^\varepsilon \) of (37) with \( \omega \) replaced by \( \theta, \omega \) satisfies
\[ \int_{\tau - 2\rho - 1}^{\tau} a_\varepsilon (r, \tau - t, \theta, \omega, \psi^\varepsilon, \psi^\varepsilon) + v^\varepsilon (r, \tau - t, \theta, \omega, \psi^\varepsilon) \, dr \]
\[ + \int_{\tau - 2\rho - 1}^{\tau} \| v^\varepsilon (r, \tau - t, \theta, \omega, \psi^\varepsilon) \|^2_{H^1(\Omega)} \, dr \]
\[ + \int_{\tau - 2\rho - 1}^{\tau} \| \mathcal{T}^{-1}(\theta, \omega) \|^2 \| u^\varepsilon (r, \tau - t, \theta, \omega, \phi^\varepsilon) \|^2_{L^\infty(\Omega)} \, dr \]
\[ \leq c R_1(\tau, \omega), \] (65)
where \( R_1(\tau, \omega) \) is given by (48) and \( c \) is independent of \( \varepsilon \).

We need the following inequality to derive the uniform estimates of solutions in \( C \left([-\rho, 0], H^1_{x} (\Omega) \right) \).
Lemma 3.3. [34] Assume that (17)-(18) hold. Then we have for all $u \in D(A_{\varepsilon})$,
\[
(h_{\varepsilon}(t, y, u), A_{\varepsilon}u)_{H_{\varepsilon}(\Omega)} \leq c \left( a_{\varepsilon}(u, u) + \|\psi_{4}(t, \cdot)\|_{L_{\infty}(\tilde{\Omega})}^{2} \right),
\]
where $c$ is independent of $\varepsilon$.

Now, we estimate the solution of (37) in $C \left([-\rho, 0], H_{\varepsilon}^{1}(\Omega)\right)$.

Lemma 3.4. Assume that (8)-(13), (40) and (43) hold. Then for every $0 < \varepsilon \leq \varepsilon_{0}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D_{1} = \{D_{1}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \subset D_{1}$, there exists $T = T(\tau, \omega, D_{1}) \geq 2\rho + 1$, independent of $\varepsilon$, such that for all $t \geq T$, $\lambda_{1} \geq \lambda_{0}$ and $\psi_{\varepsilon} \in D_{1}(\tau - t, \theta_{-\omega})$, the solution $v_{\varepsilon}$ of (37) with $\omega$ replaced by $\theta_{-\omega}$ satisfies
\[
\|v_{\varepsilon}(\cdot, t, \theta_{-\omega}, \psi_{\varepsilon})\|_{C([-\rho, 0], H_{\varepsilon}^{1}(\Omega))}^{2} \leq R_{2}(\tau, \omega),
\]
where $R_{2}(\tau, \omega)$ is determined by
\[
R_{2}(\tau, \omega) = r_{1}(\omega)R_{1}(\tau, \omega) + ce^{{\tau}(\varepsilon+1)} \int_{-\infty}^{0} e^{\tau r} \|T^{-1}(\theta_{r}, \omega)\|^{2} \|\psi_{4}(\rho + r, \cdot)\|_{L_{\infty}(\tilde{\Omega})}^{2} dr,
\]
where $r_{1}(\omega)$ is a tempered function, $R_{1}(\tau, \omega)$ is given by (48), and $c$ is independent of $\varepsilon$.

Proof. Taking the inner product of (37) with $A_{\varepsilon}v_{\varepsilon}$ in $H_{\varepsilon}(\Omega)$, we find that
\[
\frac{1}{2} \frac{d}{dt} a_{\varepsilon}(v_{\varepsilon}, v_{\varepsilon}) + \|A_{\varepsilon}v_{\varepsilon}\|_{H_{\varepsilon}(\Omega)}^{2} \leq (-\lambda_{0} + \delta(\theta_{\omega}))a_{\varepsilon}(v_{\varepsilon}, v_{\varepsilon}) + (T^{-1}(\theta_{\omega})h_{\varepsilon}(t, y, T(\theta_{\omega})v_{\varepsilon}(t)), A_{\varepsilon}v_{\varepsilon})_{H_{\varepsilon}(\Omega)} + (T^{-1}(\theta_{\omega})f_{\varepsilon}(t, y, T(\theta_{-\rho_{0}(\omega)}v_{\varepsilon}(t - \rho_{0}(t))), A_{\varepsilon}v_{\varepsilon})_{H_{\varepsilon}(\Omega)} + (T^{-1}(\theta_{\omega})G_{\varepsilon}(t, y), A_{\varepsilon}v_{\varepsilon})_{H_{\varepsilon}(\Omega)}.
\]
We first estimate the second term on the right-hand side of (68), by Lemma 3.3, we have
\[
(T^{-1}(\theta_{\omega})h_{\varepsilon}(t, y, T(\theta_{\omega})v_{\varepsilon}), A_{\varepsilon}v_{\varepsilon})_{H_{\varepsilon}(\Omega)} \leq c\|T^{-1}(\theta_{\omega})\|^{2} \left( a_{\varepsilon}(v_{\varepsilon}, v_{\varepsilon}) + \|\psi_{4}(t, \cdot)\|_{L_{\infty}(\tilde{\Omega})}^{2} \right) = ca_{\varepsilon}(v_{\varepsilon}, v_{\varepsilon}) + c\|T^{-1}(\theta_{\omega})\|^{2} \|\psi_{4}(t, \cdot)\|_{L_{\infty}(\tilde{\Omega})}^{2}.
\]
We now estimate the nonlinear delay term in (68) for which, by (8) we have
\[
(T^{-1}(\theta_{\omega})f_{\varepsilon}(t, y, T(\theta_{-\rho_{0}(\omega)}v_{\varepsilon}(t - \rho_{0}(t))), A_{\varepsilon}v_{\varepsilon})_{H_{\varepsilon}(\Omega)} \leq \frac{1}{4} \|A_{\varepsilon}v_{\varepsilon}\|_{H_{\varepsilon}(\Omega)}^{2} + K^{2}\|T^{-1}(\theta_{\omega})\|^{2} \|T(\theta_{-\rho_{0}(\omega)})\|^{2} \|v_{\varepsilon}(t - \rho_{0}(t))\|_{H_{\varepsilon}(\Omega)}^{2} + c\|T^{-1}(\theta_{\omega})\|^{2} \|\psi_{1}(t, \cdot)\|_{L_{\infty}(\tilde{\Omega})}^{2}.
\]
On the other hand, the last term on the right-hand side of (68) is bounded by
\[
(T^{-1}(\theta_{\omega})G_{\varepsilon}(t, y), A_{\varepsilon}v_{\varepsilon})_{H_{\varepsilon}(\Omega)} \leq \frac{1}{4} \|A_{\varepsilon}v_{\varepsilon}\|_{H_{\varepsilon}(\Omega)}^{2} + c\|T^{-1}(\theta_{\omega})\|^{2} \|G(t, \cdot)\|_{L_{\infty}(\tilde{\Omega})}^{2}.
\]
By (68)-(71) we get that
\[
\frac{d}{dt} \alpha_e (v^\varepsilon, v^\varepsilon) + \| A_e v^\varepsilon \|_{\mathcal{H}_s (\mathcal{O})} \leq (c + 2 \delta (\theta_\omega)) \alpha_e (v^\varepsilon, v^\varepsilon) + 2 K^2 \| \mathcal{T}^{-1} (\theta_\omega) \|_2 \| \mathcal{T} (\theta_{t-\rho_0 (t) \omega}) \|_2 \| v^\varepsilon (t - \rho_0 (t)) \|^2_{\mathcal{H}_s (\mathcal{O})} + c \| \mathcal{T}^{-1} (\theta_\omega) \|_2 (\| \psi_1 (t, \cdot) \|^2_{L^\infty (\mathcal{O})} + \| G (t, \cdot) \|^2_{L^\infty (\mathcal{O})} + \| \psi_4 (t, \cdot) \|^2_{L^\infty (\mathcal{O})}),
\]
which implies that
\[
\frac{d}{dt} \alpha_e (v^\varepsilon, v^\varepsilon) \leq (c + 2 \delta (\theta_\omega)) \alpha_e (v^\varepsilon, v^\varepsilon) + 2 K^2 \| \mathcal{T}^{-1} (\theta_\omega) \|_2 \| \mathcal{T} (\theta_{t-\rho_0 (t) \omega}) \|_2 \| v^\varepsilon (t - \rho_0 (t)) \|^2_{\mathcal{H}_s (\mathcal{O})} + c \| \mathcal{T}^{-1} (\theta_\omega) \|_2 (\| \psi_1 (t, \cdot) \|^2_{L^\infty (\mathcal{O})} + \| G (t, \cdot) \|^2_{L^\infty (\mathcal{O})} + \| \psi_4 (t, \cdot) \|^2_{L^\infty (\mathcal{O})}).
\]
Given \( \tau \in \mathbb{R}, \omega \in \Omega, \) and \( \varpi \in (\tau + s - 1, \tau + s), \) where \( s \in [-\rho, 0], \) integrating (73) on \( (\varpi, \tau + s), \) we get
\[
a_e (v^\varepsilon (\tau + s, \tau - t, \omega, \psi^\varepsilon), v^\varepsilon (\tau + s, \tau - t, \omega, \psi^\varepsilon)) \\
\leq a_e (v^\varepsilon (\varpi, \tau - t, \omega, \psi^\varepsilon), v^\varepsilon (\varpi, \tau - t, \omega, \psi^\varepsilon)) + \int_{\varpi}^{\tau + s} \left( c + 2 \delta (\theta_\omega (r)) \right) a_e (v^\varepsilon (r, \tau - t, \omega, \psi^\varepsilon), v^\varepsilon (r, \tau - t, \omega, \psi^\varepsilon)) dr + 2 K^2 \int_{\varpi}^{\tau + s} \| \mathcal{T}^{-1} (\theta_r \omega) \|_2 \| \mathcal{T} (\theta_{r-\rho_0 (r) \omega}) \|_2 \| v^\varepsilon (r - \rho_0 (r), \tau - t, \omega, \psi^\varepsilon) \|^2_{\mathcal{H}_s (\mathcal{O})} dr + c \int_{\varpi}^{\tau + s} \| \mathcal{T}^{-1} (\theta_r \omega) \|_2 (\| \psi_1 (r, \cdot) \|^2_{L^\infty (\mathcal{O})} + \| G (r, \cdot) \|^2_{L^\infty (\mathcal{O})} + \| \psi_4 (r, \cdot) \|^2_{L^\infty (\mathcal{O})}).
\]
Now integrating the above with respect to \( \varpi \) over \( (\tau + s - 1, \tau + s) \) and replacing \( \omega \) by \( \theta_{-\tau} \omega, \) we find that
\[
\alpha_e (v^\varepsilon (\tau + s, \tau - t, \theta_{-\tau} \omega, \psi^\varepsilon), v^\varepsilon (\tau + s, \tau - t, \theta_{-\tau} \omega, \psi^\varepsilon)) \\
\leq \int_{\tau-\rho}^{\tau + s} \left( 1 + c + 2 \delta (\theta_{-\tau} \omega) \right) a_e (v^\varepsilon (r, \tau - t, \theta_{-\tau} \omega, \psi^\varepsilon), v^\varepsilon (r, \tau - t, \theta_{-\tau} \omega, \psi^\varepsilon)) dr + 2 K^2 \int_{\tau-\rho}^{\tau + s} \| \mathcal{T}^{-1} (\theta_{-\tau} \omega) \|_2 \| \mathcal{T} (\theta_{r-\rho_0 (r) - \tau} \omega) \|_2 \times \| v^\varepsilon (r - \rho_0 (r), \tau - t, \theta_{-\tau} \omega, \psi^\varepsilon) \|^2_{\mathcal{H}_s (\mathcal{O})} dr + c \int_{\tau-\rho}^{\tau + s} \| \mathcal{T}^{-1} (\theta_{-\tau} \omega) \|_2 (\| \psi_1 (r, \cdot) \|^2_{L^\infty (\mathcal{O})} + \| G (r, \cdot) \|^2_{L^\infty (\mathcal{O})} + \| \psi_4 (r, \cdot) \|^2_{L^\infty (\mathcal{O})}) dr \\
\leq \left( 1 + c + 2 \max_{\nu \in [-\rho_0 - 1] \delta (\theta_\nu \omega) \right) \times \int_{\tau-\rho}^{\tau} a_e (v^\varepsilon (r, \tau - t, \theta_{-\tau} \omega, \psi^\varepsilon), v^\varepsilon (r, \tau - t, \theta_{-\tau} \omega, \psi^\varepsilon)) dr + 2 - \frac{1}{\rho^*} K^2 \max_{\nu \in [-\rho_0 - 1] \| \mathcal{T}^{-1} (\theta_\nu \omega) \|^2} \times \int_{\tau-\rho}^{\tau} \| v^\varepsilon (r, \tau - t, \theta_{-\tau} \omega, \psi^\varepsilon) \|^2_{\mathcal{H}_s (\mathcal{O})} dr.
\]
Lemma 3.5. Assume that (8)-(13), (40) and (43) hold. Then for every 
\[ \varepsilon \]
where 
\[ \omega \]
Let \( T = T(\tau, \omega, D_1) \geq 2\rho + 1 \) be the positive number found in Lemma 3.2. Then it follows from the above inequality and Lemma 3.1 that, for all \( t \geq T \) and for all \( \omega \in \Omega \),
\[ a_\varepsilon (v^\varepsilon _{\tau}(\cdot, \tau - t, \theta_{-\tau} \omega, \psi^\varepsilon ), v^\varepsilon _{\tau}(\cdot, \tau - t, \theta_{-\tau} \omega, \psi^\varepsilon )) \]
\[ \leq c \left( 1 + \max \limits_{\nu \in [-\rho - 1,0]} \| \delta (\theta_{\nu} \omega) \| \right) R_1 (\tau, \omega) \]
\[ + \frac{cK^2}{1 - \rho^2} \max \limits_{\nu \in [-\rho - 1,0]} \| T^{-1} (\theta_{\nu} \omega) \| ^2 \max \limits_{\nu \in [-2\rho - 1,0]} \| T (\theta_{\nu} \omega) \| ^2 R_1 (\tau, \omega) \]
\[ + c \int_{-\rho - 1}^{0} \| T^{-1} (\theta_{\nu} \omega) \| ^2 \| \psi_1 (r + \tau, \cdot) \| _{L^\infty (\partial)} + \| G (r + \tau, \cdot) \| _{L^\infty (\partial)} \]
\[ + \| \psi_4 (r + \tau, \cdot) \| _{L^\infty (\partial)} \] 
\[ \] 
(74)
The last term on the right-hand side of (74) satisfies
\[ c \int_{-\rho - 1}^{0} \| T^{-1} (\theta_{\nu} \omega) \| ^2 \| \psi_1 (r + \tau, \cdot) \| _{L^\infty (\partial)} + \| G (r + \tau, \cdot) \| _{L^\infty (\partial)} \]
\[ + \| \psi_4 (r + \tau, \cdot) \| _{L^\infty (\partial)} \] 
\[ \leq ce^{-\tau(\rho+1)} \int_{-\infty}^{0} e^{-\tau r} \| T^{-1} (\theta_{\nu} \omega) \| ^2 \| \psi_1 (r + \tau, \cdot) \| _{L^\infty (\partial)} \]
\[ + \| G (r + \tau, \cdot) \| _{L^\infty (\partial)} + \| \psi_4 (r + \tau, \cdot) \| _{L^\infty (\partial)} \] 
which is bounded as in Lemma 3.1. Thus, Lemma 3.4 follows from Lemma 3.1. □

The following estimate is needed to obtain the equicontinuity of the solution of (37).

Lemma 3.5. Assume that (8)-(13), (40) and (43) hold. Then for every 0 < \( \varepsilon \leq \varepsilon_0 \), \( \tau \in \mathbb{R} \), \( \omega \in \Omega \), and \( D_1 = \{ D_1 (\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \subset D_1 \), there exists \( T = T(\tau, \omega, D_1) \geq 2\rho + 1 \) independent of \( \varepsilon \), such that for all \( t \geq T \), \( \lambda_1 \geq \lambda_0 \), \( s_1, s_2 \in [-\rho,0] \) and \( \psi^\varepsilon \in D_1 (\tau - t, \theta_{-\tau} \omega) \), the solution \( v^\varepsilon \) of (37) with \( \omega \) replaced by \( \theta_{-\tau} \omega \) satisfies
\[ \int_{\tau + s_1}^{\tau + s_2} \| A_{\varepsilon} v^\varepsilon (r, \tau - t, \theta_{-\tau} \omega, \psi^\varepsilon ) \| _{H^s_\varepsilon (\Omega)} dr \leq R_3 (\tau, \omega), \] 
(75)
where \( R_3 (\tau, \omega) \) is determined by
\[ R_3 (\tau, \omega) = r_2 (\omega) (R_1 (\tau, \omega) + R_2 (\tau, \omega)) \]
\[ + ce^{-\tau\rho} \int_{-\infty}^{0} e^{-\tau r} \| T^{-1} (\theta_{\nu} \omega) \| ^2 \| \psi_1 (r + \tau, \cdot) \| _{L^\infty (\partial)} \]
\[ + \| G (r + \tau, \cdot) \| _{L^\infty (\partial)} + \| \psi_4 (r + \tau, \cdot) \| _{L^\infty (\partial)} \] 
\[ ] 
(76)
where \( r_2 (\omega) \) is a tempered function, \( R_1 (\tau, \omega) \) and \( R_2 (\tau, \omega) \) is given by (48) and (67), respectively, and \( c \) is independent of \( \varepsilon \).
Proof. Without loss of generality, we assume $s_1 < s_2$. Integrating (72) over the interval $[\tau + s_1, \tau + s_2]$ leads to

$$\int_{\tau + s_1}^{\tau + s_2} \|A_\varepsilon v^\varepsilon (r, \tau - t, \omega, \psi^\varepsilon)\|_{H_\delta(O)}^2 \, dr$$

$$\leq a_\varepsilon (v^\varepsilon (\tau + s_1, \tau - t, \omega, \psi^\varepsilon), v^\varepsilon (\tau + s_1, \tau - t, \omega, \psi^\varepsilon))$$

$$+ \int_{\tau + s_1}^{\tau + s_2} (c + 2 \delta (\theta, \omega)) a_\varepsilon (v^\varepsilon (r, \tau - t, \omega, \psi^\varepsilon), v^\varepsilon (r, \tau - t, \omega, \psi^\varepsilon)) \, dr$$

$$+ 2K^2 \int_{\tau + s_1}^{\tau + s_2} \|T^{-1} (\theta, \omega)\|\|T (\theta - \rho_0 (r) \omega)\|^2 \|v^\varepsilon (r - \rho_0 (r), \tau - t, \omega, \psi^\varepsilon)\|_{H_\delta(O)}^2 \, dr$$

$$+ c \int_{\tau + s_1}^{\tau + s_2} \|T^{-1} (\theta, \omega)\|\|\psi_1 (r, \cdot, \omega)\|_{L^\infty (\hat{\partial})}^2 + \|G (r, \cdot, \omega)\|_{L^\infty (\hat{\partial})}^2 + \|\psi_4 (r, \cdot)\|_{L^\infty (\hat{\partial})}^2 \, dr.$$  

Replacing $\omega$ with $\theta - \omega$, we find that

$$\int_{\tau + s_1}^{\tau + s_2} \|A_\varepsilon v^\varepsilon (r, \tau - t, \theta - \omega, \psi^\varepsilon)\|_{H_\delta(O)}^2 \, dr$$

$$\leq a_\varepsilon (v^\varepsilon (\tau + s_1, \tau - t, \theta - \omega, \psi^\varepsilon), v^\varepsilon (\tau + s_1, \tau - t, \theta - \omega, \psi^\varepsilon))$$

$$+ \left( c + 2 \max_{\nu \in [-\rho, 0]} \|\delta (\theta, \nu)\| \right)$$

$$\times \int_{\tau + s_1}^{\tau + s_2} a_\varepsilon (v^\varepsilon (r, \tau - t, \theta - \omega, \psi^\varepsilon), v^\varepsilon (r, \tau - t, \theta - \omega, \psi^\varepsilon)) \, dr$$

$$+ \frac{1}{1 - \rho^2} K^2 \max_{\nu \in [-\rho, 0]} \|T^{-1} (\theta, \nu)\|^2 \max_{\nu \in [-2\rho, 0]} \|T (\theta, \nu)\|^2$$

$$\times \int_{\tau - 2\rho}^{\tau} \|v^\varepsilon (r, \tau - t, \theta - \omega, \psi^\varepsilon)\|_{H_\delta(O)}^2 \, dr$$

$$+ c \int_{\tau - 2\rho}^{\tau} \|T^{-1} (\theta, \nu)\|\|\psi_1 (r, \tau, \cdot)\|_{L^\infty (\hat{\partial})}^2 + \|G (r, \tau, \cdot)\|_{L^\infty (\hat{\partial})}^2 \, dr.$$  

(77)

For the last term on the right hand side of (77), we have

$$\int_{-\rho}^{\rho} \|T^{-1} (\theta, \nu)\|^2 \|\psi_1 (r + \tau, \cdot)\|_{L^\infty (\hat{\partial})}^2 + \|G (r + \tau, \cdot)\|_{L^\infty (\hat{\partial})}^2 \, dr$$

$$\leq e^\tau \int_{\infty}^{0} e^{\tau r} \|T^{-1} (\theta, \nu)\|^2 \|\psi_1 (r + \tau, \cdot)\|_{L^\infty (\hat{\partial})}^2 + \|G (r + \tau, \cdot)\|_{L^\infty (\hat{\partial})}^2 \, dr,$$

which together (67), (77), Lemma 3.2 and Lemma 3.4 completes the proof.

Lemma 3.6. Assume that (8)-(13), (40) and (43) hold. Then for every $0 < \varepsilon \leq \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D_1 = \{D_1 (\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D_1$, there exists $T = T (\tau, \omega, D_1) \geq 2\rho + 1$, independent of $\varepsilon$, such that for all $t \geq T$, $\lambda_1 \geq \lambda_0$ and $\psi^\varepsilon \in D_1 (\tau - t, \theta - \omega)$, the solution $v^\varepsilon$ of (37) with $\omega$ replaced by $\theta - \omega$ satisfies for
all $s \in [-\rho, 0]$,
\[
\|v^s(\tau + s, \tau - t, \theta_{-r}(\tau, \omega), \psi^\varepsilon)\|_{L^p(O)}^p + \int_{-\rho}^{\tau} \|v^s(\tau - r, \theta_\tau(\tau, \omega), \psi^\varepsilon)\|_{L^{2p-2}(O)}^{2p-2} \, dr \\
\leq R_4(\tau, \omega),
\] (78)
where $R_4(\tau, \omega) < \infty$ for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

Proof. Taking the inner product of (37) with $|v^s|^{p-2}v^\varepsilon$ in $H_\varepsilon(O)$, we get
\[
\frac{1}{p} \frac{d}{dt} \int_O g |v^\varepsilon|^p \, dy = -\left( A_\varepsilon v^\varepsilon, |v^s|^{p-2} v^\varepsilon \right)_{H_\varepsilon(O)} \, dy + \left( T^{-1}(\theta_\omega) \, h_x(t, y, T(\theta_\omega) \, v^\varepsilon(t)), |v^s|^{p-2} v^\varepsilon \right)_{H_\varepsilon(O)} \\
+ \left( T^{-1}(\theta_\omega) \, f_x(t, y, T(\theta_{-\rho_0}(t) \, \omega) \, v^\varepsilon(t - \rho_0(t))), |v^s|^{p-2} v^\varepsilon \right)_{H_\varepsilon(O)} \\
+ \left( T^{-1}(\theta_\omega) \, G_x(t, y), |v^s|^{p-2} v^\varepsilon \right)_{H_\varepsilon(O)}. \tag{79}
\]
It follows from (33) and (36) that
\[
(A_\varepsilon v^\varepsilon, |v^s|^{p-2} v^\varepsilon)_{H_\varepsilon(O)} = a_\varepsilon(v^\varepsilon, |v^s|^{p-2} v^\varepsilon) = (J^* \nabla_y v^\varepsilon, J^* \nabla_y |v^s|^{p-2} v^\varepsilon)_{H_\varepsilon(O)} \\
= (p - 1) \int_O g |v^\varepsilon|^{p-2} \left[ \sum_{i=1}^n (v_{y_i}^\varepsilon - \frac{y_{y_i}}{|y|} y_{n+1} v_{y_{n+1}}^\varepsilon)^2 + \frac{1}{\varepsilon^2 g^2} |v_{y_{n+1}}^\varepsilon|^2 \right] \, dy \geq 0. \tag{80}
\]
By (15) and Young’s inequality, we have
\[
\left( T^{-1}(\theta_\omega) \, h_x(t, y, T(\theta_\omega) \, v^\varepsilon(t)), |v^s|^{p-2} v^\varepsilon \right)_{H_\varepsilon(O)} \\
\leq \gamma_2 \|T^{-1}(\theta_\omega)\|_2 \int_O \left( -\alpha_1 |u^s|^p + \psi_2(t, y, g(y^*) y_{n+1}) |v^s|^{p-2} \right) \, dy \\
\leq -\frac{1}{2} \gamma_2 \alpha_1 \|T(\theta_\omega)\|^{p-2} \|v^\varepsilon\|^{2p-2}_{L^{2p-2}(O)} \\
+ c \|T(\theta_\omega)\| \|v^\varepsilon\|_{L^\infty(O)} \|v^\varepsilon\|_{L^{2p-2}(O)} \tag{81}
\]
By (8), we get
\[
\left( T^{-1}(\theta_\omega) \, f_x(t, y, T(\theta_{-\rho_0}(t) \, \omega) \, v^\varepsilon(t - \rho_0(t))), |v^s|^{p-2} v^\varepsilon \right)_{H_\varepsilon(O)} \\
\leq \frac{1}{8} \gamma_2 \alpha_1 \|T(\theta_\omega)\|^{p-2} \|v^\varepsilon\|^{2p-2}_{L^{2p-2}(O)} \\
+ c \|T^{-1}(\theta_\omega)\| \int_O \left| f_x(t, y, T(\theta_{-\rho_0}(t) \, \omega) \, v^\varepsilon(t - \rho_0(t))) \right|^2 \, dy \\
\leq \frac{1}{8} \gamma_2 \alpha_1 \|T(\theta_\omega)\|^{p-2} \|v^\varepsilon\|^{2p-2}_{L^{2p-2}(O)} + c \|T^{-1}(\theta_\omega)\| \|u^\varepsilon(t - \rho(t))\|_{L^2(O)}^2 \\
+ c \|T^{-1}(\theta_\omega)\| \|\psi_1(t, \cdot)\|_{L^\infty(O)}^2. \tag{82}
\]
Similarly, the last term in right hand side of (79) is bounded by
\[
\frac{1}{8} \gamma_2 \alpha_1 \|T(\theta_\omega)\|^{p-2} \|v^\varepsilon\|^{2p-2}_{L^{2p-2}(O)} + c \|T^{-1}(\theta_\omega)\| \|G(t, \cdot)\|_{L^\infty(O)}^2. 
\]
Therefore, we get that
\[
\frac{d}{dt} \| v^\varepsilon \|_{L^p(\Omega)}^p + c \| T (\theta \omega) \|^{p-2} \| v^\varepsilon \|_{L^{2p-2}(\Omega)}^{2p-2} \leq c \| \delta (\theta \omega) \| \| v^\varepsilon \|_{L^p(\Omega)}^p \\
+ c \| T^{-1} (\theta \omega) \|^{p} \| u^\varepsilon (t - \rho_0 (t)) \|_{L^2(\Omega)}^2 + c \| T (\theta \omega) \| \frac{s^2 + 2p - 4}{p} \| \psi_2 (t, \cdot) \|_{L^\infty(\Omega)}^{2p-2} \\
+ c \| T^{-1} (\theta \omega) \|^{p} \| \psi_1 (t, \cdot) \|_{L^\infty(\Omega)}^2 + c \| T^{-1} (\theta \omega) \|^{p} \| G (t, \cdot) \|_{L^\infty(\Omega)}^2 .
\] (83)

Let \( \tau \in \mathbb{R}, t \geq 2\rho + 1, \omega \in \Omega, \) and \( \sigma \in (\tau + s - 1, \tau + s) \) for \( s \in [-\rho, 0]. \) Integrating (83) over \( (\sigma, \tau + s) \) we find that
\[
\| v^\varepsilon (\tau + s, \tau - t, \omega, \psi^\varepsilon) \|_{L^p(\Omega)}^p \\
\leq \| v^\varepsilon (\sigma, \tau - t, \omega, \psi^\varepsilon) \|_{L^p(\Omega)}^p + c \int_{\tau - \rho - 1}^{\tau} \| \delta (\theta \omega) \| \| v^\varepsilon (r, \tau - t, \omega, \psi^\varepsilon) \|_{L^p(\Omega)}^p dr \\
+ c \int_{\tau - \rho - 1}^{\tau} \| T^{-1} (\theta \omega) \|^{p} \| u^\varepsilon (r - \rho_0 (r), \tau - t, \omega, \phi^\varepsilon) \|_{L^2(\Omega)}^2 dr \\
+ c \int_{\tau - \rho - 1}^{\tau} \| T (\theta \omega) \| \frac{s^2 + 2p - 4}{p} \| \psi_2 (r, \cdot) \|_{L^\infty(\Omega)}^{2p-2} dr \\
+ c \int_{\tau - \rho - 1}^{\tau} \| T^{-1} (\theta \omega) \|^{p} \| \psi_1 (r, \cdot) \|_{L^\infty(\Omega)}^2 dr \\
+ c \int_{\tau - \rho - 1}^{\tau} \| T^{-1} (\theta \omega) \|^{p} \| G (r, \cdot) \|_{L^\infty(\Omega)}^2 dr .
\] (84)

Integrating (84) with respect to \( \sigma \) over \( (\tau + s - 1, \tau + s) \) and replacing \( \omega \) by \( \theta_{-\tau} \omega, \) we get for all \( s \in [-\rho, 0], \)
\[
\| v^\varepsilon (\tau + s, \tau - t, \theta_{-\tau} \omega, \psi^\varepsilon) \|_{L^p(\Omega)}^p \\
\leq c \left( 1 + \max_{\rho \in [-\rho, 0]} \| \delta (\theta \omega) \| \right) \max_{\rho \in [-\rho, 0]} \| T^{-1} (\theta \omega) \|^{p-2} \\
\times \int_{\tau - \rho - 1}^{\tau} \| T^{-1} (\theta \omega) \|^{2} \| u^\varepsilon (r, \tau - t, \theta_{-\tau} \omega, \phi^\varepsilon) \|_{L^p(\Omega)}^p dr \\
+ c \left( 1 - \frac{1}{\rho} \right) \max_{\rho \in [-\rho, 0]} \| T^{-1} (\theta \omega) \|^{p} \max_{\rho \in [-\rho, 0]} \| T (\theta \omega) \|^{2} \\
\times \int_{\tau - 2\rho - 1}^{\tau} \| v^\varepsilon (r, \tau - t, \theta_{-\tau} \omega, \psi^\varepsilon) \|_{L^2(\Omega)}^2 dr \\
+ c \max_{\rho \in [-\rho, 0]} \| T (\rho \omega) \| \frac{s^2 + 2p - 4}{p} e^{\frac{1}{2} \tau (\rho + 1)} \int_{-\infty}^{0} e^{\frac{1}{2} \tau r} \| \psi_2 (r + \tau, \cdot) \|^{2p-2}_{L^\infty(\Omega)} dr \\
+ c \max_{\rho \in [-\rho, 0]} \| T^{-1} (\theta \omega) \|^{p} e^{\frac{1}{2} \tau (\rho + 1)} \int_{-\infty}^{0} e^{\frac{1}{2} \tau r} \| \psi_1 (r + \tau, \cdot) \|_{L^\infty(\Omega)}^2 dr \\
+ \max_{\rho \in [-\rho, 0]} \| T^{-1} (\theta \omega) \|^{p} e^{\frac{1}{2} \tau (\rho + 1)} \int_{-\infty}^{0} e^{\frac{1}{2} \tau r} \| G (r + \tau, \cdot) \|_{L^\infty(\Omega)}^2 dr .
\] (85)

Next, integrating (83) over \( (\tau - \rho, \tau) \) and replacing \( \omega \) by \( \theta_{-\tau} \omega, \) we get that for all \( t \geq T = T(\tau, \omega, D_1) \geq 2\rho + 1, \)
\[
\min_{\rho \in [-\rho, 0]} \| T (\rho \omega) \|^{p-2} \int_{\tau - \rho}^{\tau} \| v^\varepsilon (r, \tau - t, \theta_{-\tau} \omega, \psi^\varepsilon) \|_{L^{2p-2}(\Omega)}^{2p-2} dr \\
\leq \| v^\varepsilon (\tau - \rho, \tau - t, \theta_{-\tau} \omega, \psi^\varepsilon) \|_{L^p(\Omega)}^p .
\]
of the stochastic equation (21) and (22) by using those estimates for the solution \( u^\varepsilon \). Assume that (8)-(9), (40) and (43) hold. Then for every \( \varepsilon \), which along with Lemma 3.2 and (85) yields (78).

Given \( \varepsilon \), we see that

\[
\| u^\varepsilon (\cdot, \tau - t, \theta_{-\tau} \omega, \phi^\varepsilon) \|_{L^p(\Omega)} \leq \eta(\varepsilon) R_2(\tau, \omega),
\]

where \( r_3(\omega) \) is a tempered function and \( R_2(\tau, \omega) \) is given by (67), and for \( s_1, s_2 \in [-\rho, 0] \),

\[
\| u^\varepsilon (s_2, \tau - t, \theta_{-\tau} \omega, \phi^\varepsilon) - u^\varepsilon (s_1, \tau - t, \theta_{-\tau} \omega, \phi^\varepsilon) \|_{L^2(\Omega)} \leq \eta_2(s_1, s_2),
\]

where the function \( \eta_2(s_1, s_2) \) has the property that \( \eta_2(s_1, s_2) \to 0 \) as \( |s_2 - s_1| \to 0 \), i.e., \( u^\varepsilon (\cdot, \tau - t, \theta_{-\tau} \omega, \phi^\varepsilon) : [-\rho, 0] \to L^2(\Omega) \) is equicontinuous.

**Proof.** Given \( D_1 = \{ D_1(\tau, \omega) : \tau \in R, \omega \in \Omega \} \in \mathcal{D}_1 \), define a new family \( \hat{D}_1 \) for \( D_1 \) as

\[
\hat{D}_1 = \{ \hat{D}_1(\tau, \omega) : v \in N : \| v \|_N \leq \| \hat{T}^{-1}(\omega) \| \| D_1(\tau, \omega) \|_N : \tau \in R, \omega \in \Omega \}. \tag{88}
\]

Since \( D_1 \in \mathcal{D}_1 \) and \( \| \hat{T}^{-1} \| \) is tempered, one can check that \( \hat{D}_1 \) also belongs to \( \mathcal{D}_1 \), i.e., \( \hat{D}_1 \) is tempered. Since \( \phi^\varepsilon \in D_1(\tau - t, \theta_{-\tau} \omega) \), we find that \( \psi^\varepsilon = \hat{T}^{-1}(\theta_{-\tau} \omega)\phi^\varepsilon \) satisfies

\[
\| \psi^\varepsilon \|_N \leq \| \hat{T}^{-1}(\theta_{-\tau} \omega) \| \| D_1(t - \tau, \theta_{-\tau} \omega) \|_N. \tag{89}
\]

By (88) and (89) we see that \( \psi^\varepsilon \in \hat{D}_1(\tau - t, \theta_{-\tau} \omega) \). Since \( \hat{D}_1 \in \mathcal{D}_1 \), by Lemmas 3.4, there exists \( T = T(\tau, \omega, D_1) \geq 2\rho + 1 \) such that for all \( t \geq T \),

\[
\| \tilde{v}^\varepsilon (\cdot, \tau - t, \theta_{-\tau} \omega, \psi^\varepsilon) \|_{L^2(\Omega)} \leq R_2(\tau, \omega),
\]

Notice that

\[
u^\varepsilon (\sigma, \tau - t, \theta_{-\tau} \omega, \phi^\varepsilon) = T(\theta_{\sigma - \tau} \omega) \varepsilon (\sigma, \tau - t, \theta_{-\tau} \omega, \psi^\varepsilon). \tag{90}\]
So we find for all \( t \geq T \),
\[
\| u^\varepsilon_r (\cdot, \tau - t, \theta_{-t\omega}, \phi^\varepsilon) \|_{C([-\rho, 0], H^1(\Omega))}^2 \\
\leq \max_{\nu \in [-\rho, 0]} \| T (\theta_\nu \omega) \|_{2}^2 \| v^\varepsilon_r (\cdot, \tau - t, \theta_{-t\omega}, \psi^\varepsilon) \|_{C([-\rho, 0], H^1(\Omega))}^2 \\
\leq \max_{\nu \in [-\rho, 0]} \| T (\theta_\nu \omega) \|_{2}^2 R_2 (\tau, \omega).
\]
Consequently, we get (86).

By (8), (11), and Lemmas 3.4-3.6, it follows from (37) that there exists \( T = T(\tau, \omega, D_1) \geq 2\rho + 1 \) such that for all \( t \geq T \),
\[
\int_{\tau - \rho}^{\tau} \frac{d}{dr} v^\varepsilon_r (r, \tau - t, \theta_{-t\omega}, \psi^\varepsilon) \|_{L^2(\Omega)}^2 dr \leq c,
\]
where \( c = c(\tau, \omega) \) is a positive number. Therefore, for any \( s_1, s_2 \in [-\rho, 0] \),
\[
\| v^\varepsilon_r (s_2, \tau - t, \theta_{-t\omega}, \psi^\varepsilon) - v^\varepsilon_r (s_1, \tau - t, \theta_{-t\omega}, \psi^\varepsilon) \|_{L^2(\Omega)} \\
= \| \int_{\tau + s_1}^{\tau + s_2} \frac{d}{dr} v^\varepsilon_r (r, \tau - t, \theta_{-t\omega}, \psi^\varepsilon) dr \|_{L^2(\Omega)} \\
\leq |s_2 - s_1|^\frac{1}{2} \left( \int_{\tau + s_1}^{\tau + s_2} \frac{d}{dr} v^\varepsilon_r (r, \tau - t, \theta_{-t\omega}, \psi^\varepsilon) \|_{L^2(\Omega)}^2 dr \right)^{\frac{1}{2}} \\
\leq |s_2 - s_1|^\frac{1}{2} \left( \int_{\tau - \rho}^{\tau} \frac{d}{dr} v^\varepsilon_r (r, \tau - t, \theta_{-t\omega}, \psi^\varepsilon) \|_{L^2(\Omega)}^2 dr \right)^{\frac{1}{2}} \leq c |s_2 - s_1|^\frac{1}{2}.
\]
By (90) and (91) we get, for all \( t \geq T \) and \( s_1, s_2 \in [-\rho, 0] \),
\[
\| u^\varepsilon_r (s_2, \tau - t, \theta_{-t\omega}, \phi^\varepsilon) - u^\varepsilon_r (s_1, \tau - t, \theta_{-t\omega}, \phi^\varepsilon) \|_{L^2(\Omega)} \\
\leq \| T (\theta_{s_2} \omega) \| \| v^\varepsilon_r (s_2, \tau - t, \theta_{-t\omega}, \psi^\varepsilon) - v^\varepsilon_r (s_1, \tau - t, \theta_{-t\omega}, \psi^\varepsilon) \|_{L^2(\Omega)} \\
+ \| T (\theta_{s_2} \omega) - T (\theta_{s_1} \omega) \| \| v^\varepsilon_r (\cdot, \tau - t, \theta_{-t\omega}, \psi^\varepsilon) \|_{N} \\
\leq c |s_2 - s_1|^\frac{1}{2} + c R_1^2 (\tau, \omega) \| T (\theta_{s_2} \omega) - T (\theta_{s_1} \omega) \|,
\]
which together with the continuity of \( T (\theta_\nu \omega) \) completes the proof. \( \square \)

4. Existence of pullback attractors. In this subsection, we establish the existence of \( D_1 \)-pullback attractor for the cocycle \( \Phi_\varepsilon \) associated with the stochastic problem (21)-(22). We first show that problem (21)-(22) has a tempered pullback absorbing set as stated below.

**Lemma 4.1.** Suppose (8)-(13), (40) and (44) hold. Then the continuous cocycle \( \Phi_\varepsilon \) associated with problem (21)-(22) has a closed measurable \( D_1 \)-pullback absorbing set \( K = \{ K (\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D_1 \).

**Proof.** We first notice that, by Lemma 3.7, \( \Phi_\varepsilon \) has a closed \( D_1 \)-pullback absorbing set \( K \in \mathcal{N} \). More precisely, given \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), let
\[
K (\tau, \omega) = \left\{ u \in \mathcal{N} : \| u \|_{\mathcal{N}}^2 \leq L (\tau, \omega) \right\},
\]
where \( L (\tau, \omega) \) is the constant given by the right-hand side of (86). It is evident that, for each \( \tau \in \mathbb{R} \), \( L (\tau, \cdot) : \Omega \to \mathbb{R} \) is \( (\mathcal{F}, \mathcal{B}(\mathbb{R})) \)-measurable. In addition, for every \( \tau \in \mathbb{R} \), \( \omega \in \Omega \), and \( D_1 \in D_1 \), there exists \( T = T(\tau, \omega, D_1) \geq 2\rho + 1 \), independent of \( \varepsilon \), such that for all \( t \geq T \),
\[
\Phi_\varepsilon (t, \tau - t, \theta_{-t\omega}, D_1 (\tau - t, \theta_{-t\omega})) \subseteq K (\tau, \omega).
\]
Thus we find that \( K = \{ K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) is a closed measurable set and pullback-attracts all elements in \( D_1 \). We now verify that \( K \) is tempered. For an arbitrary positive number \( \sigma \) we have

\[
e^{\sigma t} R_2(\tau + t, \theta_t \omega) = c e^{\sigma t} \left| r_1(\theta_t \omega) \right| \int_{-\infty}^{0} e^{\int_{t}^{\tau} \gamma(\theta_t \omega) dt} \| \psi_1 (r + \tau + t, \cdot) \|_{L^\infty(\bar{\Omega})}^2 dr + c e^{\sigma t} \left| r_1(\theta_t \omega) \right| \int_{-\infty}^{0} e^{\int_{t}^{\tau} \gamma(\theta_t \omega) dt} \| T^{-1} (\theta_{r+t} \omega) \|^2 \\
\times \left( \| G (r + \tau + t, \cdot) \|_{L^\infty(\bar{\Omega})}^2 + \| \psi_2 (r + \tau + t, \cdot) \|_{L^\infty(\bar{\Omega})}^2 \right) dr \\
+ c e^{\sigma t} \left| r_1(\theta_t \omega) \right| \int_{-\infty}^{0} e^{\int_{t}^{\tau} \gamma(\theta_t \omega) dt} \| T^{-1} (\theta_{r+t} \omega) \|^2 \left( \| \psi_1 (r + \tau + t, \cdot) \|_{L^\infty(\bar{\Omega})}^2 \right) dr \\
+ \| G (r + \tau + t, \cdot) \|_{L^\infty(\bar{\Omega})}^2 + \| \psi_4 (r + \tau + t, \cdot) \|_{L^\infty(\bar{\Omega})}^2 dr. \tag{93}\]

Take \( \varepsilon_1 = \varepsilon_2 = \epsilon \) in (60) and (61) with \( 0 < \epsilon < \min\{ \frac{1}{4} \sigma, \frac{1}{4} \tau \} \). Consequently, by (60), (61) and the temperedness of \( |r_1(\theta_t \omega)| \) we have

\[
\limsup_{t \to -\infty} e^{\sigma t} R_2(\tau + t, \theta_t \omega) \leq \limsup_{t \to -\infty} e^{(\sigma - \epsilon) t} \int_{-\infty}^{0} e^{\int_{t}^{\tau} \gamma(\theta_t \omega) dt} \| \psi_1 (r + \tau + t, \cdot) \|_{L^\infty(\bar{\Omega})}^2 dr \\
\times \left( \| G (r + \tau + t, \cdot) \|_{L^\infty(\bar{\Omega})}^2 + \| \psi_2 (r + \tau + t, \cdot) \|_{L^\infty(\bar{\Omega})}^2 \right) dr \\
+ \limsup_{t \to -\infty} c e^{\sigma t} \int_{-\infty}^{0} e^{\int_{t}^{\tau} \gamma(\theta_t \omega) dt} \| \psi_1 (r + \tau + t, \cdot) \|_{L^\infty(\bar{\Omega})}^2 dr \\
+ \| G (r + \tau + t, \cdot) \|_{L^\infty(\bar{\Omega})}^2 + \| \psi_4 (r + \tau + t, \cdot) \|_{L^\infty(\bar{\Omega})}^2 dr. \tag{94}\]

This together with (44) yields that

\[
\lim_{t \to -\infty} e^{\sigma t} R_2(\tau + t, \theta_t \omega) = 0.
\]

Hence \( r_3(\omega) R_2(\tau, \omega) \) is tempered, i.e., \( K = \{ K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) is tempered. \( \square \)

**Lemma 4.2.** Assume that (8)-(13), (40) and (44) hold. Then, the cocycle \( \Phi_\varepsilon \) is \( D_1 \)-pullback asymptotically compact in \( N \); that is, for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), \( \{ \Phi_\varepsilon(t_n, \tau - t_n, \theta_{-t_n} \omega, x_n) \}_{n=1}^{\infty} \) has a convergent subsequence in \( N \) whenever \( t_n \to \infty \), and \( x_n \in D_1(\tau - t_n, \theta_{-t_n} \omega) \) with \( \{ D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D_1 \).

**Proof.** Since \( t_n \to \infty \), by Lemma 3.7, there exists \( n_0 = n_0(D_1, \tau, \omega) \in N \) such that for all \( n \geq n_0 \) and \( s \in [-\rho, 0] \),

\[
\left\| u^\varepsilon(s, \tau - t_n, \theta_{-t_n} \omega, x_n) \right\|_{H^1(\Omega)}^2 \leq r_3(\omega) R_2(\tau, \omega). \tag{95}\]
By the compactness of imbedding $H^1(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$, it follows from (95) that for each $s \in [-\rho, 0]$ the sequence $\{u^\varepsilon_t (s - t_n, \theta^{-\tau} \omega, x_n)\}_{n=1}^{\infty}$ is relatively compact in $L^2(\mathcal{O})$. On the other hand, by Lemma 3.7, we also find that the sequence $\{u^\varepsilon_t (\cdot - t_n, \theta^{-\tau} \omega, x_n)\}_{n=1}^{\infty}$ is equicontinuous. By Ascoli-Arzelà theorem, the sequence $\{u^\varepsilon_t (\cdot - t_n, \theta^{-\tau} \omega, x_n)\}_{n=1}^{\infty}$ is relatively compact in $\mathcal{N}$. This completes the proof.

**Theorem 4.3.** Assume that (8)-(13), (40) and (44) hold. Then, the cocycle $\Phi$ has a unique $\mathcal{D}_1$-pullback attractor $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$ in $\mathcal{N}$. If, in addition, $H, G, f, \rho_0, \psi_1, \psi_2$ and $\psi_4$ are $T$-periodic with respect to $t$, where $T > 0$, then the attractor $\mathcal{A}_\varepsilon$ is also $T$-periodic.

**Proof.** First, we know from Lemma 4.1 that $\Phi_\varepsilon$ has a a closed measurable $\mathcal{D}_1$-pullback absorbing set $K(\tau, \omega)$. Second, it follows from Lemma 4.2 that $\Phi_\varepsilon$ is $\mathcal{D}_1$-pullback asymptotically compact in $\mathcal{N}$. Hence, the existence of a unique $\mathcal{D}_1$-pullback attractor for the cocycle $\Phi_\varepsilon$ follows from [40] immediately. If $H, G, f, \rho_0, \psi_1, \psi_2$ and $\psi_4$ are $T$-periodic with respect to $t$, then the continuous cocycle $\Phi_\varepsilon$ and the absorbing set $K$ are also $T$-periodic, which implies the $T$-periodicity of the attractor.

Analogous results also hold for the solution of (4) and (5). In particular, we have:

**Theorem 4.4.** Assume that (8)-(13), (40) and (44) hold. Then, the cocycle $\Phi_0$ has a unique $\mathcal{D}_{\tau}$-pullback attractor $\mathcal{A}_0 = \{\mathcal{A}_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_0$ in $\mathcal{M}$. If, in addition, $H, G, f, \rho_0, \psi_1, \psi_2$ and $\psi_4$ are $T$-periodic with respect to $t$, where $T > 0$, then the attractor $\mathcal{A}_0$ is also $T$-periodic.

5. **Upper semicontinuity of attractors.** The following estimates are needed when we derive the convergence of pullback attractors.

**Lemma 5.1.** Assume that (8)-(13) and (40) hold. Then for every $0 < \varepsilon \leq \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$, and $\lambda_1 \geq \lambda_0$, the solution $v^\varepsilon$ of (37) satisfies, for all $t \in [\tau, \tau + T]$,

$$\int_\tau^t \|v^\varepsilon(r, \tau, \omega, \psi^\varepsilon)\|_{H^1(\mathcal{O})}^2 dr \leq c \|\psi^\varepsilon\|_{\mathcal{N}}^2 \quad + \quad c \int_{\tau}^{\tau + T} \left( \|G(r, \cdot)\|_{L^\infty(\Omega)}^2 + \|\psi_1(r, \cdot)\|_{L^\infty(\Omega)}^2 + \|\psi_2(r, \cdot)\|_{L^\infty(\Omega)}^2 \right) dr,$$

where $c$ is a positive constant depending on $\tau, \omega, \lambda_0$ and $T$, but independent of $\varepsilon$.

**Proof.** It follows from (56) that for $t \in [\tau, \tau + T]$,

$$\int_\tau^t e^{\int_r^\tau \gamma(\theta(t)) dt} \|H_x^\varepsilon(\tau)(r)\|_{H^1(\mathcal{O})}^2 \leq \frac{1}{\lambda_1} \int_\tau^t e^{\int_r^\tau \gamma(\theta(t)) dt} \|\psi^\varepsilon\|_{\mathcal{N}}^2 \quad + \quad \frac{1}{\lambda_1} \int_\tau^t e^{\int_r^\tau \gamma(\theta(t)) dt} \|G(r, \cdot)\|_{L^\infty(\Omega)}^2 \|T(\theta(t))\|_{H^1(\mathcal{O})}^2 dr \quad + \quad \frac{1}{\lambda_1} \int_\tau^t e^{\int_r^\tau \gamma(\theta(t)) dt} \|\psi_1(r, \cdot)\|_{L^\infty(\Omega)}^2 \|T^{-1}(\theta(t))\|_{H^1(\mathcal{O})}^2 dr \quad + \quad \frac{1}{\lambda_1} \int_\tau^t e^{\int_r^\tau \gamma(\theta(t)) dt} \|\psi_2(r, \cdot)\|_{L^\infty(\Omega)}^2 \|T^{-1}(\theta(t))\|_{H^1(\mathcal{O})}^2 dr$$

$$\leq \quad \frac{1}{\lambda_1} \int_\tau^t e^{\int_r^\tau \gamma(\theta(t)) dt} \|\psi^\varepsilon\|_{\mathcal{N}}^2 \quad + \quad \frac{1}{\lambda_1} \int_\tau^t e^{\int_r^\tau \gamma(\theta(t)) dt} \|G(r, \cdot)\|_{L^\infty(\Omega)}^2 \|T(\theta(t))\|_{H^1(\mathcal{O})}^2 dr \quad + \quad \frac{1}{\lambda_1} \int_\tau^t e^{\int_r^\tau \gamma(\theta(t)) dt} \|\psi_1(r, \cdot)\|_{L^\infty(\Omega)}^2 \|T^{-1}(\theta(t))\|_{H^1(\mathcal{O})}^2 dr \quad + \quad \frac{1}{\lambda_1} \int_\tau^t e^{\int_r^\tau \gamma(\theta(t)) dt} \|\psi_2(r, \cdot)\|_{L^\infty(\Omega)}^2 \|T^{-1}(\theta(t))\|_{H^1(\mathcal{O})}^2 dr$$
Lemma 5.2. Assume that (8)-(13) and (40) hold. Then for every $\gamma \in t,s$ where $c$ is a positive constant depending on $\epsilon$.

Moreover, notice that $\gamma(\omega) \leq \lambda_0$, and we have for $t \in [\tau, \tau + T]$:

$$2e^{-\lambda_0 T} \int_{\tau}^{t} a_\varepsilon (v^\varepsilon (r), v^\varepsilon (r))dr + \lambda_0 \int_{\tau}^{t} \|v^\varepsilon (r)\|^2_{H_{s}(\Omega)} dr \leq 2 \int_{\tau}^{t} e^{\int_{\tau}^{r} \gamma(\theta,\omega) dt} a_\varepsilon (v^\varepsilon (r), v^\varepsilon (r))dr + \lambda_0 \int_{\tau}^{t} \|v^\varepsilon (r)\|^2_{H_{s}(\Omega)} dr,$$

which together with (96) and the continuity of $T^{-1}(\theta,\omega)$ and $\gamma(\theta,\omega)$ completes the proof.

Similarly, one can prove

Lemma 5.2. Assume that (8)-(13) and (40) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$, and $\lambda_1 \geq \lambda_0$, the solution $v^0$ of (31) satisfies, for all $t \in [\tau, \tau + T]$:

$$\int_{\tau}^{t} \|v^0 (r, \tau, \omega, \phi^0)\|^2 \leq c \|\phi^0\|^2 M$$

$$+ c \int_{\tau}^{T} \left(\|G (r, \cdot)\|^2_{L^\infty(\Omega)} + \|v_1 (r, \cdot)\|^2_{L^\infty(\Omega)} + \|v_2 (r, \cdot)\|^2_{L^\infty(\Omega)}\right) dr,$$

where $c$ is a positive constant depending on $\tau$, $\omega$, $\lambda_0$ and $T$, but independent of $\varepsilon$.

Given $u \in L^2(\Omega)$, let $Gu$ be the average function of $u$ in $y_{n+1}$ as defined by

$$Gu = \int_{0}^{1} u (y^*, y_{n+1}) dy_{n+1}.$$

The following result on the average function can be found in [26].

Lemma 5.3. If $u \in H^1(\Omega)$, then $Gu \in H^1(\Omega)$ and

$$\|u - Gu\|_{H^1(\Omega)} \leq c \|u\|_{H^1(\Omega)},$$

where $c$ is a constant, independent of $\varepsilon$.

In the sequel, we further assume the functions $G$, $f$ and $H$ satisfy that for all $t, s \in \mathbb{R}$,

$$\|G_{\varepsilon} (t, \cdot) - G_0 (t, \cdot)\|_{L^2(\Omega)} \leq \kappa_1 (t) \varepsilon, \quad (97)$$

$$\|f_{\varepsilon} (t, \cdot, s) - f_0 (t, \cdot, s)\|_{L^2(\Omega)} \leq \kappa_2 (t) \varepsilon, \quad (98)$$

and

$$\|H_{\varepsilon} (t, \cdot, s) - H_0 (t, \cdot, s)\|_{L^2(\Omega)} \leq \kappa_3 (t) \varepsilon, \quad (99)$$

where $\kappa_1 (t), \kappa_2 (t), \kappa_3 (t) \in L^2_{\text{loc}} (\mathbb{R})$.

By (14) and (99) we have, for all $x \in \tilde{\Omega}$ and $t, s \in \mathbb{R}$,

$$\|h_{\varepsilon} (t, \cdot, s) - h_0 (t, \cdot, s)\|_{L^2(\Omega)} \leq \kappa_3 (t) \varepsilon. \quad (100)$$

Since $M$ can be embedded naturally into $N$ as the subspace of functions independent of $y_{n+1}$, we can consider the cocycle $\Phi_0$ as a mapping from $M$ into $N$. Therefore we can compare $\Phi_0$ with $\Phi_\varepsilon$.

Theorem 5.4. Suppose (8)-(13), (40), and (97)-(99) hold. Given $\tau \in \mathbb{R}$, $\omega \in \Omega$ and a positive number $L(\tau, \omega)$, if $\phi^\varepsilon \in C([-\rho, 0], H^1_s (\Omega))$ is such that $\|\phi^\varepsilon\|_{C([-\rho, 0], H^1_s (\Omega))} \leq L(\tau, \omega)$, then we have, for any $t \geq \tau$,

$$\lim_{\varepsilon \to 0} \|\Phi_\varepsilon (t, \tau, \omega, \phi^\varepsilon) - \Phi_0 (t, \tau, \omega, G\phi^\varepsilon)\|_{N} = 0.$$
Proof. Taking inner product of (31) with $g_l$, where $l \in H^1(\mathcal{Q})$, we find that
\[
\int_{\mathcal{Q}} g \frac{dv^0}{dt} ldy^* + \sum_{i=1}^{n} \int_{\mathcal{Q}} g_{y_i}^0 l_{y_i} dy^* + (\lambda - \delta(\theta(\omega))) \int_{\mathcal{Q}} g^0 ldy^* \\
= \int_{\mathcal{Q}} g T^{-1}(\theta(\omega)) h_0(t, y^*, T(\theta(\omega)) v^0) ldy^* \\
+ \int_{\mathcal{Q}} g T^{-1}(\theta(\omega)) f_0(t, y^*, T(\theta_{t-\rho_0(t)} \omega) v^0(t-\rho_0(t))) ldy^* \\
+ \int_{\mathcal{Q}} g T^{-1}(\theta(\omega)) G_0(t, y^*) ldy^*.
\]
As $\int_0^1 \zeta(y^*, y_{n+1}) dy_{n+1}$ belongs to $H^1(\mathcal{Q})$ if $\zeta$ is in $H^1(\mathcal{O})$, the above equality becomes, for any $\zeta \in H^1(\mathcal{O})$,
\[
\left( \frac{dv^0}{dt}, \zeta \right)_{H_{s}(\mathcal{O})} + \sum_{i=1}^{n} \left( v_{y_i}^0, \zeta_{y_i} \right)_{H_{s}(\mathcal{O})} + (\lambda - \delta(\theta(\omega))) \left( v^0, \zeta \right)_{H_{s}(\mathcal{O})} \\
= \left( T^{-1}(\theta(\omega)) h_0(t, y^*, T(\theta(\omega)) v^0), \zeta \right)_{H_{s}(\mathcal{O})} \\
+ \left( T^{-1}(\theta(\omega)) f_0(t, y^*, T(\theta_{t-\rho_0(t)} \omega) v^0(t-\rho_0(t))), \zeta \right)_{H_{s}(\mathcal{O})} \\
+ \left( T^{-1}(\theta(\omega)) G_0(t, y^*), \zeta \right)_{H_{s}(\mathcal{O})} - \sum_{i=1}^{n} \left( g_{y_i} \frac{v^0}{g_{y_i}}, y_{n+1} \zeta_{y_{n+1}} \right)_{H_{s}(\mathcal{O})}.
\]
Since $v^0$ is independent of $y_{n+1}$, the above equality gives, for any $\zeta \in H^1(\mathcal{O})$,
\[
\left( \frac{dv^0}{dt}, \zeta \right)_{H_{s}(\mathcal{O})} + a_\varepsilon \left( v^0, \zeta \right)_{H_{s}(\mathcal{O})} + (\lambda - \delta(\theta(\omega))) \left( v^0, \zeta \right)_{H_{s}(\mathcal{O})} \\
= \left( T^{-1}(\theta(\omega)) h_0(t, y^*, T(\theta(\omega)) v^0), \zeta \right)_{H_{s}(\mathcal{O})} \\
+ \left( T^{-1}(\theta(\omega)) f_0(t, y^*, T(\theta_{t-\rho_0(t)} \omega) v^0(t-\rho_0(t))), \zeta \right)_{H_{s}(\mathcal{O})} \\
+ \left( T^{-1}(\theta(\omega)) G_0(t, y^*), \zeta \right)_{H_{s}(\mathcal{O})} - \sum_{i=1}^{n} \left( g_{y_i} \frac{v^0}{g_{y_i}}, y_{n+1} \zeta_{y_{n+1}} \right)_{H_{s}(\mathcal{O})}.
\]
Due to (37) and (101), the function $v^\varepsilon - v^0$ satisfies the equation, for any $\zeta \in H^1(\mathcal{O})$,
\[
\left( \frac{dv^\varepsilon}{dt} - \frac{dv^0}{dt}, \zeta \right)_{H_{s}(\mathcal{O})} + a_\varepsilon \left( v^\varepsilon - v^0, \zeta \right)_{H_{s}(\mathcal{O})} + (\lambda - \delta(\theta(\omega))) \left( v^\varepsilon - v^0, \zeta \right)_{H_{s}(\mathcal{O})} \\
= \left( T^{-1}(\theta(\omega)) (h_\varepsilon(t, y^*, y_{n+1}, T(\theta(\omega)) v^\varepsilon)) - h_0(t, y^*, T(\theta(\omega)) v^0), \zeta \right)_{H_{s}(\mathcal{O})} \\
+ \left( T^{-1}(\theta(\omega)) (f_\varepsilon(t, y^*, y_{n+1}, T(\theta_{t-\rho_0(t)} \omega) v^\varepsilon(t-\rho_0(t)))) - f_0(t, y^*, T(\theta_{t-\rho_0(t)} \omega) v^0(t-\rho_0(t))), \zeta \right)_{H_{s}(\mathcal{O})} \\
+ \left( T^{-1}(\theta(\omega)) (G_\varepsilon(t, y^*, y_{n+1}) - G_0(t, y^*)), \zeta \right)_{H_{s}(\mathcal{O})} \\
+ \sum_{i=1}^{n} \left( g_{y_i} \frac{v^0}{g_{y_i}}, y_{n+1} \zeta_{y_{n+1}} \right)_{H_{s}(\mathcal{O})}.
\]
We replace $\zeta$ by $v^\varepsilon - v^0$ in this expressing and estimate all the terms in the right-hand side of this equality. By (17) and (99) we obtain
\[
T^{-1}(\theta(\omega)) (h_\varepsilon(t, y^*, y_{n+1}, T(\theta(\omega)) v^\varepsilon)) - h_0(t, y^*, T(\theta(\omega)) v^0), v^\varepsilon - v^0)_{H_{s}(\mathcal{O})} \\
= T^{-1}(\theta(\omega)) (h (t, y^* \varepsilon g (y^*) y_{n+1}, T (\theta(\omega)) v^\varepsilon))
\]
We have from (9) and (98) that
\[ t \]
Taking into account the estimates (103)-(106), as well as (102), we obtain, for
\[ \int \]
\[ \leq \beta \| \nu - \nu^0 \|^2_{H^1(\mathcal{O})} + \kappa_3(\epsilon) \| \theta \| \| \nu - \nu^0 \|^2_{H^1(\mathcal{O})}. \]
(103)

Using estimate (97) of \( G_\epsilon - G_0 \), we easily obtain
\[ \mathcal{T}^{-1}(\theta_\omega) \left( (G_\epsilon(t, t^*, y^*, y^{n+1}) - G_0(t, t^*)) , \nu - \nu^0 \right)_{H^1(\mathcal{O})} \]
\[ \leq \| \mathcal{T}^{-1}(\theta_\omega) \| \| G_\epsilon(t, t^*, y^*, y^{n+1}) - G_0(t, t^*) \|_{H^1(\mathcal{O})} \| \nu - \nu^0 \|_{H^1(\mathcal{O})} \]
\[ \leq \kappa_1(\epsilon) \| \mathcal{T}^{-1}(\theta_\omega) \| \| \nu - \nu^0 \|_{H^1(\mathcal{O})}. \]
(105)

Finally, by (34), we have
\[ \sum_{i=1}^{n} \left( \frac{g_{y_i}}{g} v_{g_i}^0(y^{n+1} + v_{g_{n+1}}^0 - v_{g_{n+1}}^0) \right)_{H^1(\mathcal{O})} \]
\[ = \sum_{i=1}^{n} \left( g_{y_i} v_{g_i}^0, y^{n+1}(v_{g_{n+1}}^0 - v_{g_{n+1}}^0) \right)_{L^2(\mathcal{O})} \]
\[ \leq \epsilon \| v_0 \|^2_{H^1(\mathcal{Q})} \| \nu - \nu^0 \|^2_{H^1(\mathcal{Q})} \leq \epsilon \left( \| \nu - \nu^0 \|^2_{H^1(\mathcal{O})} + \| \nu - \nu^0 \|^2_{H^1(\mathcal{Q})} \right). \]
(106)

Taking into account the estimates (103)-(106), as well as (102), we obtain, for
\[ t \in [\tau, \tau + T], \]
\[ \frac{d}{dt} \| \nu - \nu^0 \|^2_{H^1(\mathcal{O})} \leq 2(\beta + \delta(\theta_\omega)) \| \nu - \nu^0 \|^2_{H^1(\mathcal{O})} \]
\[ + 2L^2_\epsilon \| \mathcal{T}^{-1}(\theta_\omega) \| \| \mathcal{T} \left( \theta_{t-p_0(t)} \omega \right) \| \| \nu - \nu^0 \|^2_{H^1(\mathcal{O})} \]
\[ + 2 \| \nu - \nu^0 \|^2_{H^1(\mathcal{O})} \]
\[ + c(\kappa_1(t) + \kappa_2(t) + \kappa_3(t)) \epsilon \| \mathcal{T}^{-1}(\theta_\omega) \| \| \nu - \nu^0 \|_{H^1(\mathcal{O})} \]
\[ + \epsilon \left( \| \nu - \nu^0 \|^2_{H^1(\mathcal{Q})} + \| \nu - \nu^0 \|^2_{H^1(\mathcal{O})} \right). \]
(107)

Integrating (107) on \( (\tau, t) \), we obtain
\[ \| \nu - \nu^0 \|^2_{H^1(\mathcal{O})} \leq \| \nu^0 \|_{H^1(\mathcal{O})}^2 \]
\[ + 2 \int_{\tau}^{t} (\beta + \delta(\theta_\omega)) + L^2_\epsilon \| \mathcal{T}^{-1}(\theta_\omega) \| \| \mathcal{T} \left( \theta_{t-p_0(s)} \omega \right) \| \| \nu - \nu^0 \|_{H^1(\mathcal{O})}^2 ds \]
Theorem 5.5. Assume that (8)-(13), (40), (44), and (97)-(99) hold. The attractors $A_\varepsilon$ are upper-semicontinuous at $\varepsilon = 0$, that is, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, 
\[
\lim_{\varepsilon \to 0} \text{dist}_N(A_\varepsilon(\tau, \omega), A_0(\tau, \omega)) = 0.
\]

Proof. Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, by the invariance of $A_\varepsilon$ and (86) we find that there exists $\varepsilon_0 > 0$ such that
\[
u||u||_{C([-\rho, 0], H^1(\Omega))} \leq L(\tau, \omega) \quad \text{for all } 0 < \varepsilon < \varepsilon_0 \text{ and } u \in A_\varepsilon(\tau, \omega),
\]
where $L(\tau, \omega)$ is the positive constant given by the right-hand side of (86) which is independent of $\varepsilon$. Let $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be the $D_1$-pullback absorbing set of $\Phi$, obtained in Lemma 4.1 and denote by $K_0 = \{K_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ with $K_0(\tau, \omega) = \{Gu : u \in K(\tau, \omega)\}$. Then $K_0$ is tempered in $\mathcal{M}$ and hence $K_0 \in \mathcal{D}_0$. Since $A_0$ is the $D_0$-pullback attractor of $\Phi_0$ in $\mathcal{M}$, given $\eta > 0$, we infer that there exists $T = T(\eta, \tau, \omega) \geq 2\rho + 1$ such that

$$\text{dist}_{\mathcal{M}} (\Phi_0(T, \tau - T, \theta_{-T}\omega, K_0(\tau - T, \theta_{-T}\omega)), A_0(\tau, \omega)) < \frac{1}{2}\eta. \quad (112)$$

By the invariance of $A_\varepsilon(\tau, \omega)$, we see that for any $x_\varepsilon \in A_\varepsilon(\tau, \omega)$, there exists $y_\varepsilon \in A_\varepsilon(\tau - T, \theta_{-T}\omega)$ such that

$$x_\varepsilon = \Phi_\varepsilon(T, \tau - T, \theta_{-T}\omega, y_\varepsilon). \quad (113)$$

By (111) and Theorem 5.4 we get

$$\lim_{\varepsilon \to 0} \|\Phi_\varepsilon(T, \tau - T, \theta_{-T}\omega, y_\varepsilon) - \Phi_0(T, \tau - T, \theta_{-T}\omega, G\varepsilon y_\varepsilon)\|_\mathcal{N} = 0,$$

and hence there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that for all $\varepsilon < \varepsilon_1$,

$$\|\Phi_\varepsilon(T, \tau - T, \theta_{-T}\omega, y_\varepsilon) - \Phi_0(T, \tau - T, \theta_{-T}\omega, G\varepsilon y_\varepsilon)\|_\mathcal{N} < \frac{1}{2}\eta. \quad (114)$$

Since $y_\varepsilon \in A_\varepsilon(\tau - T, \theta_{-T}\omega)$ and $A_\varepsilon(\tau - T, \theta_{-T}\omega) \subseteq K(T - T, \theta_{-T}\omega)$, which along with (112) implies

$$\text{dist}_{\mathcal{M}} (\Phi_0(T, \tau - T, \theta_{-T}\omega, G\varepsilon y_\varepsilon), A_0(\tau, \omega)) < \frac{1}{2}\eta. \quad (115)$$

By (114) and (115) we have, for all $\varepsilon < \varepsilon_1$,

$$\text{dist}_{\mathcal{N}} (\Phi_\varepsilon(T, \tau - T, \theta_{-T}\omega, y_\varepsilon), A_0(\tau, \omega)) < \eta. \quad (116)$$

By (113) and (116) we obtain, for all $\varepsilon < \varepsilon_1$,

$$\text{dist}_{\mathcal{N}} (x_\varepsilon, A_0(\tau, \omega)) < \eta \text{ for all } x_\varepsilon \in A_\varepsilon(\tau, \omega).$$

This indicates that for all $\varepsilon < \varepsilon_1$,

$$\text{dist}_{\mathcal{N}} (A_\varepsilon(\tau, \omega), A_0(\tau, \omega)) \leq \eta,$$

as desired. \hfill \Box

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