CONVEX FUNCTIONS ON GRASSMANNIAN MANIFOLDS AND LAWSON-OSSERMAN PROBLEM

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ABSTRACT. We derive estimates of the Hessian of two smooth functions defined on Grassmannian manifold. Based on it, we can derive curvature estimates for minimal submanifolds in Euclidean space via Gauss map as [24]. In this way, the result for Bernstein type theorem done by Jost and the first author could be improved.

1. INTRODUCTION

The celebrated theorem of Bernstein [2] says that the only entire minimal graphs in Euclidean 3-space are planes. Its higher dimensional generalization was finally proved by J. Simons [19], which says that an entire minimal graph has to be planar for dimension ≤ 7, while Bombieri- De Giorgi-Giusti [3] shortly afterwards produced a counterexample to such an assertion in dimension 8 and higher.

Schoen-Simon-Yau [18] gave us a direct proof for Bernstein type theorems for n ≤ 5 dimensional minimal graphs with the aid of curvature estimates for stable minimal hypersurfaces.

There is a weak version of Bernstein type theorem in arbitrary dimension. It was J. Moser [15] who proved that the entire solution f to the minimal surface equation is affine linear, provided |\nabla f| is uniformly bounded. Afterward Ecker-Huisken [9] obtained curvature estimates by a geometric approach, as a corollary Moser’s result had been improved for the controlled growth of |\nabla f|.

Moser’s theorem had been generalized to certain higher codimensional cases by Chern-Osserman [6] for dimension 2 and Babosa, Fischer-Colbrie for dimension 3 [1] [10]. But the counterexample constructed by Lawson-Osserman [13] prevents us going further. They also raised in the same paper a question for finding the ”best” constant possible in the theorem.

In contrast, the first author with J. Jost [12] proved the following Bernstein type theorem without the restriction of dimension and codimension, which is an improvement of the work done by Hildebrandt-Jost-Widman [11].

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Theorem 1.1. Let \( z^\alpha = f^\alpha(x^1, \cdots, x^n) \), \( \alpha = 1, \cdots, m \), be smooth functions defined everywhere in \( \mathbb{R}^n \). Suppose their graph \( M = (x, f(x)) \) is a submanifold with parallel mean curvature in \( \mathbb{R}^{n+m} \). Suppose that there exists a number \( \beta_0 \) with

\[
(1.1) \quad \beta_0 < \begin{cases} 
2 & \text{when } m \geq 2, \\
\infty & \text{when } m = 1;
\end{cases}
\]

such that

\[
(1.2) \quad \Delta f = \left[ \det \left( \delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \right) \right]^{\frac{1}{2}} \leq \beta_0.
\]

Then \( f^1, \cdots, f^m \) has to be affine linear representing an affine \( n \)-plane.

The key point of the proof is to find a geodesic convex set

\( B_{JX}(P_0) = \{ P \in \mathbf{G}_{n,m} : \text{sum of any two Jordan angles between } P \text{ and } P_0 < \frac{\pi}{2} \} \)

in a geodesic polar coordinate of the Grassmannian manifold, where \( P_0 \) denotes a fixed \( n \)--plane. It is larger than the largest geodesic convex ball of radius \( \sqrt{\frac{2}{4}} \pi \) in \( \mathbf{G}_{n,m} \). The geometric meaning of the condition of the above result is that the image under the Gauss map of \( M \) lies in a closed subset \( S \subset B_{JX}(P_0) \).

Recently, the authors [24] studied complete minimal submanifolds whose Gauss image lies in an open geodesic ball of radius \( \sqrt{\frac{2}{4}} \pi \). They carried out the Schoen-Simon-Yau type curvature estimates and the Ecker-Huisken type curvature estimates, and on the basis, the corresponding Bernstein type theorems with dimension limitation or growth assumption could be derived.

It is natural to study the situation when \( \beta_0 \) in the condition (1.1) and (1.2) of the Theorem 1.1 approach to 2. The present paper will devote to this problem. We shall follow the main idea of our previous paper [24]. But, we view now the Grassmannian manifolds as submanifolds in Euclidean space via Plücker imbedding. The auxiliary functions are constructed from this viewpoint. As shown before, \( B_{JX}(P_0) \) is defined in a coordinate neighborhood \( U \) of the Grassmannian \( \mathbf{G}_{n,m} \). We introduce two functions \( v \) and \( u \) in \( U \). Via the Gauss map we can obtain useful functions on our minimal \( n \)--submanifold \( M \) in \( \mathbb{R}^{m+n} \) with \( m \geq 2 \). Then, we can carry out the Schoen-Simon-Yau type curvature estimates and the Ecker-Huisken type curvature estimates, which enable us to get the corresponding Bernstein type theorems and other geometrical conclusions.

In Section 2, we give some facts of a Grassmannian manifold \( \mathbf{G}_{n,m} \), which can be isometric imbedding into a Euclidean space. There is the height function for a submanifold in Euclidean space. Such a height function is called \( w \)--function on \( \mathbf{G}_{n,m} \). Then we have an open domain of \( U \subset \mathbf{G}_{n,m} \), where the \( w \)--function is positive. Every point in \( U \) has a one-to-one correspondence to an \( n \times m \) matrix. We describe canonical metric and the corresponding connection on \( U \) with respect
to the coordinate. On the basis, the Hessian of an arbitrary smooth function could be calculated.

In Section 3 we define \( v = \frac{1}{w} \) on \( U \). We also define another function \( u \) on \( U \). In the section, we shall show \( v \) and \( u \) are convex on \( B_{f\lambda}(P_0) \) and give estimates of the Hessian of them. The estimates are quite delicate. We use the radial compensation technique to accurate the estimates.

In Section 4, we define four auxiliary functions, \( \tilde{h}_1, \tilde{h}_2, \tilde{h}_3 \) and \( \tilde{h}_4 \). They are defined on the minimal submanifolds of \( \mathbb{R}^{n+m} \) whose Gauss image is confined, and they are expressed in term of \( v \) and \( u \). We also estimate the Laplacian of them, which is useful for the next sections.

Later in Section 5, not only we give the Schoen-Simon-Yau type curvature estimates with the aid of \( \tilde{h}_1 \) and \( \tilde{h}_3 \), but also we obtain the Ecker-Huisken type curvature estimates with the aid of \( \tilde{h}_2 \) and \( \tilde{h}_4 \). Our method is completely similar to the previous paper [24], so we only describe the outline of process. From the estimates several geometrical conclusions follow, including the following Bernstein type theorems.

**Theorem 1.2.** Let \( M = (x, f(x)) \) be an \( n \)-dimensional entire minimal graph given by \( m \) functions \( f^\alpha(x^1, \cdots, x^n) \) with \( m \geq 2, n \leq 4 \). If

\[
\Delta_f = \left[ \det \left( \delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \right) \right]^\frac{1}{2} < 2,
\]

then \( f^\alpha \) has to be affine linear functions representing an affine \( n \)-plane.

**Theorem 1.3.** Let \( M = (x, f(x)) \) be an \( n \)-dimensional entire minimal graph given by \( m \) functions \( f^\alpha(x^1, \cdots, x^n) \) with \( m \geq 2 \). If

\[
\Delta_f = \left[ \det \left( \delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \right) \right]^\frac{1}{2} < 2,
\]

and

\[
(2 - \Delta_f)^{-1} = o(R^4),
\]

where \( R^2 = |x|^2 + |f|^2 \). Then \( f^\alpha \) has to be affine linear functions and hence \( M \) has to be an affine linear subspace.

Those are what shall be done in Section 6. It is worthy to note that Theorems 1.2-1.3 still hold true when \( M \) is a submanifold with parallel mean curvature. Dong generalized Chern’s result [5] [4] to higher codimension, which states that a graphic submanifold \( M = (x, f(x)) \) with parallel mean curvature has to be minimal if the slope of \( f \) is uniformly bounded. Hence our results improve Theorem 1.1.

It is natural to ask what is the relations between the results here and that of the previous paper [24]. Since the \( v \)-function varies in \( \left( \sec^p \left( \frac{\pi}{2\sqrt{2p}} \right), \sec \left( \frac{\sqrt{2}}{4} \pi \right) \right) \) on the open geodesic ball of radius \( \frac{\sqrt{2}}{4} \) in \( G_{n,m} \), where \( p = \min(n,m) \), the results
of the present article do not generalize those in the previous one. Both results are complementary.

2. Preliminaries on the Grassmannian manifold $G_{n,m}$

Let $\mathbb{R}^{n+m}$ be an $n + m$-dimensional Euclidean space. All oriented $n$-subspaces constitute the Grassmannian manifolds $G_{n,m}$, which is an irreducible symmetric space of compact type.

Fix $P_0 \in G_{n,m}$ in the sequel, which is spanned by a unit $n$-vector $\varepsilon_1 \wedge \cdots \wedge \varepsilon_n$. For any $P \in G_{n,m}$, spanned by a $n$-vector $e_1 \wedge \cdots \wedge e_n$, we define an important function on $G_{n,m}$

$$w \overset{\text{def.}}{=} \langle P, P_0 \rangle = \langle e_1 \wedge \cdots \wedge e_n, \varepsilon_1 \wedge \cdots \wedge \varepsilon_n \rangle = \det W,$$

where $W = (\langle e_i, \varepsilon_j \rangle)$. It is well known that

$$W^TW = O^T\Lambda O,$$

where $O$ is an orthogonal matrix and

$$\Lambda = \begin{pmatrix} \mu_1^2 & 0 & \cdots & 0 \\ 0 & \mu_2^2 & \cdots & 0 \\ & & \ddots & \cdots \\ 0 & 0 & \cdots & \mu_p^2 \end{pmatrix}, \quad p = \min(m, n),$$

where each $0 \leq \mu_i^2 \leq 1$. The Jordan angles between $P$ and $P_0$ are defined by

$$\theta_i = \arccos(\mu_i).$$

Denote

$$U = \{P \in G_{n,m} : w(P) > 0\},$$

let $\{\varepsilon_{n+\alpha}\}$ be $m$-vectors such that $\{\varepsilon_i, \varepsilon_{n+\alpha}\}$ form an orthonormal basis of $\mathbb{R}^{m+n}$. Then we can span arbitrary $P \in U$ by $n$ vectors $f_i$:

$$f_i = \varepsilon_i + z_{i\alpha} \varepsilon_{n+\alpha},$$

where $Z = (z_{i\alpha})$ are the local coordinate of $P$ in $U$. Here and in the sequel we use the summation convention and agree the range of indices:

$$1 \leq i, j, k, l \leq n; \quad 1 \leq \alpha, \beta, \gamma, \delta \leq m.$$

The canonical metric on $G_{n,m}$ in the local coordinate can be described as (see [22] Ch. VII)

$$g = \text{tr}((I_n + ZZ^T)^{-1}dZ(I_m + Z^TZ)^{-1}dZ^T).$$

Let $P \in U$ determined by an $n \times m$ matrix $Z_0 = (\lambda_\alpha \delta_{i\alpha})$, where $\lambda_\alpha = \tan \theta_\alpha$ and $\theta_1, \cdots, \theta_m$ be the Jordan angles between $P$ and $P_0$. (Here and in the sequel we
assume \( n \geq m \) without loss of generality; for it is similar for \( n < m \).) Let \( X, Y, W \) denote arbitrary \( n \times m \) matrices. Then (2.1) tells us

\[
(X, Y)_P = \text{tr}((I_n + Z_0 Z_0^T)^{-1} X (I_m + Z_0^T Z_0)^{-1} Y^T)
\]

(2.2)

\[
= \sum_{i, \alpha} (1 + \lambda_1^2)^{-1} (1 + \lambda_\alpha^2)^{-1} X_{i \alpha} Y_{i \alpha}.
\]

(Note that if \( m + 1 \leq i \leq n, \lambda_i = 0 \).) Furthermore, from

\[
(I_n + (Z_0 + tW)(Z_0 + tW)^T)^{-1} X (I_m + (Z_0 + tW)^T(Z_0 + tW))^{-1} Y^T
\]

\[
= (I_n + Z_0 Z_0^T + t(W Z_0^T + Z_0 W^T) + O(t^2))^{-1} X
\]

\[
= (I_m + Z_0^T Z_0 + t(W^T Z_0 + Z_0^T W) + O(t^2))^{-1} (I_n + Z_0 Z_0^T)^{-1} X
\]

\[
= (I_n - t(I_n + Z_0 Z_0^T)^{-1}(W Z_0^T + Z_0 W^T) + O(t^2))(I_n + Z_0 Z_0^T)^{-1} X
\]

\[
= (I_n - t(I_n + Z_0 Z_0^T)^{-1}(W Z_0^T + Z_0 W^T) + O(t^2))(I_n + Z_0 Z_0^T)^{-1} X
\]

\[
= (I_n - t(I_n + Z_0 Z_0^T)^{-1}(W Z_0^T + Z_0 W^T) + O(t^2))(I_m + Z_0^T Z_0)^{-1} Y^T
\]

\[
= (I_n + Z_0 Z_0^T)^{-1} X (I_m + Z_0^T Z_0)^{-1} Y^T
\]

\[
- t(I_n + Z_0 Z_0^T)^{-1} (W Z_0^T + Z_0 W^T)(I_n + Z_0 Z_0^T)^{-1} X (I_m + Z_0^T Z_0)^{-1} Y^T
\]

\[
+ (I_n + Z_0 Z_0^T)^{-1} X (I_m + Z_0^T Z_0)^{-1} (W Z_0^T + Z_0^T W)(I_m + Z_0^T Z_0)^{-1} Y^T + O(t^2),
\]

we have

\[
W \langle X, Y \rangle_P = - \text{tr} \left[ (I_n + Z_0 Z_0^T)^{-1}(W Z_0^T + Z_0 W^T) \right.
\]

\[
\left. (I_n + Z_0 Z_0^T)^{-1} X (I_m + Z_0^T Z_0)^{-1} Y^T \right]
\]

(2.3)

\[
+ (I_n + Z_0 Z_0^T)^{-1} X (I_m + Z_0^T Z_0)^{-1} (W Z_0^T + Z_0^T W)(I_m + Z_0^T Z_0)^{-1} Y^T \right].
\]

We let \( E_{i \alpha} \) be the matrix with 1 in the intersection of row \( i \) and column \( \alpha \) and 0 otherwise. Denote \( g_{i \alpha, j \beta} = \langle E_{i \alpha}, E_{j \beta} \rangle \) and let \( (g^{i \alpha, j \beta}) \) be the inverse matrix of \( (g_{i \alpha, j \beta}) \). Denote by \( \nabla \) the Levi-Civita connection with respect to the canonical matrix on \( G_{n,m} \), and by

\[
\nabla_{E_{i \alpha}} E_{j \beta} = \Gamma_{i \alpha, j \beta}^{k \gamma} E_{k \gamma}.
\]

Then from (2.2),

\[
g_{i \alpha, j \beta}(P) = (1 + \lambda_1^2)^{-1}(1 + \lambda_\alpha^2)^{-1}\delta_{i \alpha} \delta_{j \beta}
\]

(2.4)

and obviously

\[
g^{i \alpha, j \beta}(P) = (1 + \lambda_1^2)(1 + \lambda_\alpha^2)\delta_{i \alpha} \delta_{j \beta}.
\]

(2.5)

Moreover, a direct calculation from (2.3) and (2.5) shows

\[
\Gamma_{i \alpha, j \beta}^{k \gamma} = \frac{1}{2} g^{k \gamma, l \delta} \left( -E_{l \delta} \langle E_{i \alpha}, E_{j \beta} \rangle + E_{l \delta} \langle E_{j \beta}, E_{i \alpha} \rangle + E_{l \delta} \langle E_{j \beta}, E_{i \alpha} \rangle + E_{j \beta} \langle E_{l \delta}, E_{i \alpha} \rangle \right)
\]

(2.6)

\[
= -\lambda_\alpha (1 + \lambda_\alpha^2)^{-1}\delta_{i \alpha} \delta_{j \beta} \delta_{l \delta} - \lambda_\beta (1 + \lambda_\beta^2)^{-1}\delta_{i \alpha} \delta_{j \beta} \delta_{l \delta}.
\]
From (2.5), we see that

\[(2.7) \quad (1 + \lambda^2_i)\frac{1}{2}(1 + \lambda^2_\alpha)\frac{1}{2}E_{i\alpha} \quad (1 \leq i \leq n, 1 \leq \alpha \leq m)\]

form an orthonormal basis of $T_PG_{n,m}$. Denote its dual basis in $T^*_P G_{n,m}$ by

\[(2.8) \quad \omega_{i\alpha} \quad (1 \leq i \leq n, 1 \leq \alpha \leq m),\]

then

\[(2.9) \quad g = \sum_{i,\alpha} \omega_{i\alpha}^2\]

at $P$.

3. Hessian estimates of two smooth functions on $G_{n,m}$

On $U$, $w > 0$, then we can define

\[(3.1) \quad v = w^{-1} \quad \text{on } U.\]

For arbitrary $Q \in U$ determined by an $n \times m$ matrix $Z$, it is easily seen that

\[(3.2) \quad v(Q) = \left[ \det(I_n + ZZ^T) \right]^\frac{1}{2} = \prod_{\alpha=1}^{m} \sec \theta_\alpha.\]

where $\theta_1, \cdots, \theta_m$ denotes the Jordan angles between $Q$ and $P_0$.

Now we calculate the Hessian of $v$ at $P$ whose corresponding matrix is $Z_0$. At first, by noting that for any $n \times n$ orthogonal matrix $U$ and $m \times m$ orthogonal matrix $V$, $Z \mapsto UZV$ induces an isometry of $U$ which keeps $v$ invariant, we can assume $Z_0 = (\lambda_\alpha \delta_{i\alpha})$ without loss of generality, where $\lambda_\alpha = \tan \theta_\alpha$ and $\theta_1, \cdots, \theta_m$ denotes the Jordan angles between $P$ and $P_0$. We also need a Lemma as follows.

**Lemma 3.1.** Let $M$ be a manifold, $A$ be a smooth nonsingular $n \times n$ matrix-valued function on $M$, $X, Y$ be local tangent fields, then

\[(3.3) \quad \nabla_X \log \det A = tr(\nabla_X A \cdot A^{-1})\]

and

\[(3.4) \quad \nabla_Y \nabla_X \log \det A = tr((\nabla_Y \nabla_X A \cdot A^{-1}) - (\nabla_X A \cdot A^{-1} \cdot \nabla_Y A \cdot A^{-1})).\]

**Proof.** Assume that $e_1, \cdots, e_n$ is a standard basis in $\mathbb{R}^n$, then

\[
\det A \ e_1 \wedge \cdots \wedge e_n = Ae_1 \wedge \cdots \wedge Ae_n.
\]
Hence
\[ \nabla_X \det A \, e_1 \wedge \cdots \wedge e_n = \sum_i A e_1 \wedge \cdots \wedge A e_i \wedge \nabla_X A e_i \wedge A e_{i+1} \wedge \cdots \wedge A e_n \]
\[ = \sum_i A e_1 \wedge \cdots \wedge A e_i \wedge (\nabla_X A \cdot A^{-1}) A e_i \wedge A e_{i+1} \wedge \cdots \wedge A e_n \]
\[ = \text{tr}(\nabla_X A \cdot A^{-1}) A e_1 \wedge \cdots \wedge A e_n \]
\[ = \text{tr}(\nabla_X A \cdot A^{-1}) \det A \, e_1 \wedge \cdots \wedge e_n. \]

Thereby (3.3) immediately follows.

(3.4) follows from (3.3) and
\[ A \cdot \nabla_Y A^{-1} + \nabla_Y A \cdot A^{-1} = \nabla_Y (A A^{-1}) = 0. \]

Now we let \( M = U, A(Z) = I_n + ZZ^T, \) then \( \log v = \frac{1}{2} \log \det A. \) A direct calculation shows
\[ \nabla_X A = XZ^T + ZX^T, \quad \nabla_Y \nabla_X A = XY^T + YX^T. \]

Hence we compute from Lemma 3.1 that at \( P \)
\[ \nabla_X \log v = \frac{1}{2} \text{tr} ((XZ_0^T + Z_0X^T)(I_n + Z_0Z_0^T)^{-1}) \]
\[ = \sum_{\alpha} \lambda_\alpha (1 + \lambda_\alpha^2)^{-1} X_{\alpha \alpha}, \]

\[ \nabla_X \nabla_Y \log v = \frac{1}{2} \text{tr} ((XY^T + YX^T)(I_n + Z_0Z_0^T)^{-1}) \]
\[ - \frac{1}{2} \text{tr} ((XZ_0^T + Z_0X^T)(I_n + Z_0Z_0^T)^{-1}(YZ_0^T + Z_0Y^T)(I_n + Z_0Z_0^T)^{-1}) \]
\[ = \sum_{i, \alpha} (1 + \lambda_i^2)^{-1} X_{\alpha \alpha} Y_{i \alpha} \]
\[ - \frac{1}{2} \sum_{i, j} (XZ_0^T + Z_0X^T)_{ij}(1 + \lambda_j^2)^{-1}(YZ_0^T + Z_0Y^T)_{ji}(1 + \lambda_j^2)^{-1} \]
\[ = \sum_{m+1 \leq i \leq n, \alpha} X_{i \alpha} Y_{i \alpha} + \frac{1}{2} \sum_{\alpha, \beta} (1 + \lambda_\alpha^2)^{-1} X_{\alpha \beta} Y_{\alpha \beta} + \frac{1}{2} \sum_{\alpha, \beta} (1 + \lambda_\beta^2)^{-1} X_{\beta \alpha} Y_{\beta \alpha} \]
\[ - \frac{1}{2} \sum_{\alpha, \beta} (\lambda_\beta X_{\alpha \beta} + \lambda_\alpha X_{\beta \alpha})(1 + \lambda_\beta^2)^{-1}(\lambda_\alpha Y_{\beta \alpha} + \lambda_\beta Y_{\alpha \beta})(1 + \lambda_\alpha^2)^{-1} \]
\[ - \sum_{m+1 \leq i \leq n, \alpha} \lambda_\alpha^2 (1 + \lambda_\alpha^2)^{-1} X_{i \alpha} Y_{i \alpha} \]
\[ = \sum_{m+1 \leq i \leq n, \alpha} (1 + \lambda_\alpha^2)^{-1} X_{i \alpha} Y_{i \alpha} + \sum_{\alpha, \beta} (1 + \lambda_\alpha^2)^{-1}(1 + \lambda_\beta^2)^{-1} X_{\alpha \beta} Y_{\alpha \beta} \]
Then, from (2.6), (3.5) and (3.6) we obtain

\[ -\sum_{\alpha, \beta} \lambda_\alpha \lambda_\beta (1 + \lambda_\alpha^2)^{-1}(1 + \lambda_\beta^2)^{-1} X_{\alpha \beta} Y_{\beta \alpha}. \]

Furthermore,

\[
\nabla_X v = v \nabla_X \log v = \left( \sum_{\alpha} \lambda_\alpha (1 + \lambda_\alpha^2)^{-1} X_{\alpha \alpha} \right) v,
\]

\[
\nabla_X \nabla_Y v = v(\nabla_X \nabla_Y \log v + \nabla_X \log v \cdot \nabla_Y \log v)
= \left( \sum_{m+1 \leq i \leq n, \alpha} (1 + \lambda_\alpha^2)^{-1} X_{i \alpha} Y_{i \alpha} + \sum_{\alpha, \beta} \lambda_\alpha \lambda_\beta (1 + \lambda_\alpha^2)^{-1}(1 + \lambda_\beta^2)^{-1} X_{\alpha \beta} Y_{\beta \alpha}
+ \sum_{\alpha, \beta} \lambda_\alpha \lambda_\beta (1 + \lambda_\alpha^2)^{-1}(1 + \lambda_\beta^2)^{-1} (X_{\alpha \alpha} Y_{\beta \beta} - X_{\alpha \beta} Y_{\beta \alpha}) \right) v
= \left( \sum_{i, \beta} (1 + \lambda_i^2)^{-1}(1 + \lambda_{\beta}^2)^{-1} X_{i \beta} Y_{i \beta}
+ \sum_{\alpha, \beta} \lambda_\alpha \lambda_\beta (1 + \lambda_\alpha^2)^{-1}(1 + \lambda_\beta^2)^{-1} (X_{\alpha \alpha} Y_{\beta \beta} - X_{\alpha \beta} Y_{\beta \alpha}) \right) v.
\]

In particular,

(3.5) \quad \nabla_{E_{i \alpha}} v(P) = \lambda_\alpha (1 + \lambda_\alpha^2)^{-1} v \delta_{i \alpha}

and

(3.6) \quad \nabla_{E_{i \alpha}} \nabla_{E_{j \beta}} v(P) = \begin{cases} 
(1 + \lambda_i^2)^{-1}(1 + \lambda_{\alpha}^2)^{-1} v & i = j, \alpha = \beta; \\
-\lambda_\alpha \lambda_\beta (1 + \lambda_\alpha^2)^{-1}(1 + \lambda_\beta^2)^{-1} v & i = \beta, j = \alpha, \alpha \neq \beta; \\
\lambda_\alpha \lambda_\beta (1 + \lambda_\alpha^2)^{-1}(1 + \lambda_\beta^2)^{-1} v & i = \alpha, j = \beta, \alpha \neq \beta; \\
0 & \text{otherwise.}
\end{cases}

Then, from (2.6), (3.5) and (3.6) we obtain

\[
\text{Hess}(v)(E_{i \alpha}, E_{j \beta})(P) = \nabla_{E_{i \alpha}} \nabla_{E_{j \beta}} v - (\nabla_{E_{i \alpha}} E_{j \beta}) v
= \nabla_{E_{i \alpha}} \nabla_{E_{j \beta}} v - \Gamma_{i \alpha, j \beta}^{k \gamma} \nabla_{E_{k \gamma}} v
\]

(3.7) \quad \begin{cases} 
(1 + \lambda_i^2)^{-1}(1 + \lambda_{\alpha}^2)^{-1} v & i = j, \alpha = \beta, i \neq \alpha; \\
(1 + 2\lambda_i^2)(1 + \lambda_{\alpha}^2)^{-2} v & i = j = \alpha = \beta; \\
\lambda_\alpha \lambda_\beta (1 + \lambda_\alpha^2)^{-1}(1 + \lambda_\beta^2)^{-1} v & i = \beta, j = \alpha, \alpha \neq \beta; \\
\lambda_\alpha \lambda_\beta (1 + \lambda_\alpha^2)^{-1}(1 + \lambda_\beta^2)^{-1} v & i = \alpha, j = \beta, \alpha \neq \beta; \\
0 & \text{otherwise.}
\end{cases}
In other words
\[ \text{Hess}(v)_P = \sum_{i \neq \alpha} v \omega_{\alpha i}^2 + \sum_\alpha (1 + 2\lambda_\alpha^2) v \omega_{\alpha \alpha}^2 + \sum_{\alpha \neq \beta} \lambda_\alpha \lambda_\beta v (\omega_{\alpha \alpha} \otimes \omega_{\beta \beta} + \omega_{\alpha \beta} \otimes \omega_{\beta \alpha}) \]
\[ = \sum_{m+1 \leq i \leq n, \alpha} v \omega_{\alpha i}^2 + \sum_\alpha (1 + 2\lambda_\alpha^2) v \omega_{\alpha \alpha}^2 + \sum_{\alpha \neq \beta} \lambda_\alpha \lambda_\beta v \omega_{\alpha \alpha} \otimes \omega_{\beta \beta} \]
\begin{equation}
(3.8)
\end{equation}
\[ + \sum_{\alpha < \beta} [(1 + \lambda_\alpha \lambda_\beta) v \left(\frac{\sqrt{2}}{2}(\omega_{\alpha \beta} + \omega_{\beta \alpha})\right)^2 \]
\[ + (1 - \lambda_\alpha \lambda_\beta) v \left(\frac{\sqrt{2}}{2}(\omega_{\alpha \beta} - \omega_{\beta \alpha})\right)^2].\]

(3.8) could be simplified further. Please note (3.5), which also tells us
\begin{equation}
dv = \sum_\alpha \lambda_\alpha v \omega_{\alpha \alpha};\]
\end{equation}
then
\begin{equation}
dv \otimes dv = \sum_\alpha \lambda_\alpha^2 v^2 \omega_{\alpha \alpha}^2 + \sum_{\alpha \neq \beta} \lambda_\alpha \lambda_\beta v^2 \omega_{\alpha \alpha} \otimes \omega_{\beta \beta}.\]
\end{equation}
Substituting (3.10) into (3.8) yields
\begin{equation}
\text{Hess}(v)_P = \sum_{m+1 \leq i \leq n, \alpha} v \omega_{\alpha i}^2 + \sum_\alpha (1 + \lambda_\alpha^2) v \omega_{\alpha \alpha}^2 + v^{-1} \ dv \otimes dv \]
\begin{equation}
(3.11)
\end{equation}
\[ + \sum_{\alpha < \beta} [(1 + \lambda_\alpha \lambda_\beta) v \left(\frac{\sqrt{2}}{2}(\omega_{\alpha \beta} + \omega_{\beta \alpha})\right)^2 \]
\[ + (1 - \lambda_\alpha \lambda_\beta) v \left(\frac{\sqrt{2}}{2}(\omega_{\alpha \beta} - \omega_{\beta \alpha})\right)^2].\]

Note that \( \lambda_\alpha \geq 0 \) and \( 1 - \lambda_\alpha \lambda_\beta = 1 - \tan \theta_\alpha \tan \theta_\beta = \frac{\cos(\theta_\alpha + \theta_\beta)}{\cos \theta_\alpha \cos \theta_\beta}, \) which implies that \( \text{Hess}(v)_P \) is positive definite if and only if \( \theta_\alpha + \theta_\beta < \frac{\pi}{2} \) for arbitrary \( \alpha \neq \beta, \) i.e., \( P \in B_{JX}(P_0). \)

By (3.2), \( v = \prod_\alpha (1 + \lambda_\alpha^2)^{\frac{1}{2}}, \) then
\[ \lambda_\alpha \lambda_\beta \leq \left[(1 + \lambda_\alpha^2)(1 + \lambda_\beta^2)\right]^{\frac{1}{2}} - 1 \leq v - 1, \]
the equality holds if and only if \( \lambda_\alpha = \lambda_\beta \) and \( \lambda_\gamma = 0 \) for each \( \gamma \neq \alpha, \beta. \) Hence, we have \( 1 - \lambda_\alpha \lambda_\beta \geq 2 - v. \) Finally we arrive at an estimate
\begin{equation}
(3.12)
\end{equation}
\[ \text{Hess}(v) \geq v(2 - v)g + v^{-1} dv \otimes dv. \]

Now we introduce another smooth function on \( U. \) For any \( Q \in U, \)
\begin{equation}
(3.13)
\end{equation}
\[ u \overset{def.}{=} \sum_\alpha \tan \theta_\alpha^2. \]
where $\theta_1, \ldots, \theta_m$ denotes the Jordan angles between $Q$ and $P_0$. Denote by $Z$ the coordinate of $Q$, then it is easily seen that

$$u(Q) = \text{tr}(ZZ^T).$$

We can calculate the Hessian of $u$ at $P \in \mathbb{U}$ whose corresponding matrix is $Z_0$ in the same way. Similar to above, we can assume $Z_0 = (\lambda_\alpha \delta_\alpha)$, where $\lambda_\alpha = \tan \theta_\alpha$ and $\theta_1, \ldots, \theta_m$ are the Jordan angles between $P$ and $P_0$.

Obviously

$$\nabla_X u = \text{tr}(XZ^T) + \text{tr}(ZX^T),$$

$$\nabla_X \nabla_Y u = \text{tr}(XY^T) + \text{tr}(YX^T).$$

Then, at $P$

$$\text{Hess}(u)(E_{i\alpha}, E_{j\beta}) = \nabla_{E_{i\alpha}} \nabla_{E_{j\beta}} u - (\nabla_{E_{i\alpha}} E_{j\beta}) u$$

$$= \nabla_{E_{i\alpha}} \nabla_{E_{j\beta}} u - \Gamma_{i\alpha,j\beta}^{k\gamma} \nabla_{E_{k\gamma}} u$$

$$= 2\delta_{ij}\delta_{\alpha\beta} + (\lambda_\alpha(1 + \lambda_\alpha^2)^{-1}\delta_{\alpha j}\delta_\beta \delta_{ik} + \lambda_\beta(1 + \lambda_\beta^2)^{-1}\delta_{\beta i}\delta_\alpha \delta_{jk})$$

$$\cdot 2\lambda_\gamma \delta_{k\gamma}$$

$$= 2\delta_{ij}\delta_{\alpha\beta} + 2\lambda_\alpha \lambda_\beta [(1 + \lambda_\alpha^2)^{-1} + (1 + \lambda_\beta^2)^{-1}] \delta_{\alpha j}\delta_\beta i$$

$$= \begin{cases} 
2 & i = j, \alpha = \beta, i \neq \alpha; \\
2 + 4\lambda_\alpha^2(1 + \lambda_\beta^2)^{-1} & i = j = \alpha = \beta; \\
2\lambda_\alpha \lambda_\beta [(1 + \lambda_\alpha^2)^{-1} + (1 + \lambda_\beta^2)^{-1}] & i = \beta, j = \alpha, \alpha \neq \beta.
\end{cases}$$

In other words

$$\text{Hess}(u)_P = \sum_{i \neq \alpha} 2(1 + \lambda_\alpha^2)(1 + \lambda_\alpha^2)\omega_{i\alpha}^2 + \sum_{\alpha} (2 + 6\lambda_\alpha^2)(1 + \lambda_\alpha^2)\omega_{\alpha\alpha}^2$$

$$+ \sum_{\alpha \neq \beta} 2\lambda_\alpha \lambda_\beta(2 + \lambda_\alpha^2 + \lambda_\beta^2)\omega_{\alpha\beta} \otimes \omega_{\beta\alpha}$$

$$= \sum_{m+1 \leq i \leq n, \alpha} 2(1 + \lambda_\alpha^2)(1 + \lambda_\alpha^2)\omega_{i\alpha}^2 + \sum_{\alpha} (2 + 6\lambda_\alpha^2)(1 + \lambda_\alpha^2)\omega_{\alpha\alpha}^2$$

$$+ 2[(1 + \lambda_\alpha^2)(1 + \lambda_\beta^2) + \lambda_\alpha \lambda_\beta(2 + \lambda_\alpha^2 + \lambda_\beta^2)] \left[\frac{\sqrt{2}}{2}(\omega_{\alpha\beta} + \omega_{\beta\alpha})\right]^2$$

$$+ 2[(1 + \lambda_\alpha^2)(1 + \lambda_\beta^2) - \lambda_\alpha \lambda_\beta(2 + \lambda_\alpha^2 + \lambda_\beta^2)] \left[\frac{\sqrt{2}}{2}(\omega_{\alpha\beta} - \omega_{\beta\alpha})\right]^2$$

By computing,

$$2[(1 + \lambda_\alpha^2)(1 + \lambda_\beta^2) - \lambda_\alpha \lambda_\beta(2 + \lambda_\alpha^2 + \lambda_\beta^2)] = 2(1 - \lambda_\alpha \lambda_\beta)(\lambda_\alpha^2 + \lambda_\beta^2 - \lambda_\alpha \lambda_\beta + 1).$$

It is positive if and only if $1 - \lambda_\alpha \lambda_\beta = 1 - \tan \theta_\alpha \tan \theta_\beta = \frac{\cos(\theta_\alpha + \theta_\beta)}{\cos \theta_\alpha \cos \theta_\beta} \geq 0$, i.e., $\theta_\alpha + \theta_\beta < \frac{\pi}{2}$. Hence $\text{Hess}(u)_P$ is positive definite if and only $P \in B_{JX}(P_0)$. 

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Moreover, the right side of (3.18) can be estimated by
\[ 2(1 - \lambda_\alpha \lambda_\beta)(\lambda_\alpha^2 + \lambda_\beta^2 - \lambda_\alpha \lambda_\beta + 1) \geq 2\left(1 - \frac{\lambda_\alpha^2 + \lambda_\beta^2}{2}\right)\left(\frac{\lambda_\alpha^2 + \lambda_\beta^2}{2} + 1\right) \]
\[ = 2\left(1 - \frac{(\lambda_\alpha^2 + \lambda_\beta^2)^2}{4}\right) \]
\[ \geq 2\left(1 - \frac{1}{4}u^2\right) = 2 - \frac{1}{2}u^2. \]
(Here we used the fact \( u = \sum \alpha \tan^2 \theta_\alpha = \sum \alpha \lambda_\alpha^2 \).) By combining it with (3.17) and (2.9), we arrive that
\[ \text{Hess}(u) \geq \left(2 - \frac{1}{2}u^2\right) g. \]

For later applications the estimates (3.12) and (3.19) are not accurate enough. Using the radial compensation technique we could refine those estimates which are based on the following lemmas.

**Lemma 3.2.** Let \( V \) be a real linear space, \( h \) be a nonnegative definite quadratic form on \( V \) and \( \omega \in V^* \). \( V = V_1 \oplus V_2 \), \( h \) is positive definite on \( V_1 \), \( h(V_1, V_2) = 0 \) and \( \omega(V_2) = 0 \). Denote by \( \omega^* \) the unique vector in \( V_1 \) such that for any \( z \in V_1 \),
\[ \omega(z) = h(\omega^*, z). \]
Then we have
\[ h \geq \omega(\omega^*)^{-1}\omega \otimes \omega. \]

**Proof.** For arbitrary \( y \in V \), there exist \( \lambda \in \mathbb{R} \), \( z_1 \in V_1 \) and \( z_2 \in V_2 \), such that \( y = \lambda \omega^* + z_1 + z_2 \) and \( h(\omega^*, z_1) = 0 \). Then
\[ h(y, y) = \lambda^2 h(\omega^*, \omega^*) + h(z_1, z_1) + h(z_2, z_2) \geq \lambda^2 h(\omega^*, \omega^*) = \lambda^2 \omega(\omega^*) \]
and
\[ \omega(\omega^*)^{-1}\omega \otimes \omega(y, y) = \omega(\omega^*)^{-1}\omega(y)^2 = \lambda^2 \omega(\omega^*). \]
Hence (3.20) holds.

**Lemma 3.3.** Let \( \Omega \) be a compact and convex subset of \( \mathbb{R}^k \), such that for every \( \sigma \in \mathcal{S}(k) \) and \( x = (x^1, \cdots, x^k) \in \Omega \),
\[ T_\sigma(x) = (x^{\sigma(1)}, \cdots, x^{\sigma(k)}) \in \Omega; \]
where \( \mathcal{S}(k) \) denotes the permutation group of \( \{1, \cdots, k\} \). If \( f : \Omega \to \mathbb{R} \) is a symmetric \( C^2 \) function, and \( (D^2 f) \) is nonpositive definite everywhere in \( \Omega \), then there exists \( x_0 = (x_0^1, \cdots, x_0^k) \in \Omega \), such that \( x_0^1 = x_0^2 = \cdots = x_0^k \) and
\[ f(x_0) = \sup_{\Omega} f. \]
Proof. By the compactness of $\Omega$, there exists $x = (x^1, \cdots, x^k) \in \Omega$, such that $f(x) = \sup_{\Omega} f$. Furthermore we have
\[
f(T_\sigma(x)) = f(x) = \sup_{\Omega} f \quad \sigma \in \mathcal{I}(k)
\]
from the fact that $f$ is symmetric. Denote by $C_\sigma(x)$ the convex closure of $\{T_\sigma(x) : \sigma \in \mathcal{I}(k)\}$, then $C_\sigma(x) \subseteq \Omega$ and $f(y) \geq \sup_{\sigma \in \mathcal{I}(k)} f(T_\sigma(x)) = \sup_{\Omega} f$ for arbitrary $y \in C_\sigma(x)$, since $(D^2 f) \leq 0$; which implies
\[
f|_{C_\sigma(x)} \equiv \sup_{\Omega} f.
\]
Denote $x_0^1 = \cdots = x_0^k = \frac{1}{k} \sum_{i=1}^k x^i$, then
\[
x_0 = (x_0^1, \cdots, x_0^k) = \frac{1}{k} \sum_{s=1}^k (x^s, x^{s+1}, \cdots, x^k, x^1, x^2, \cdots, x^{s-1}) \in C_\sigma(x);
\]
From which (3.22) follows.

By (3.12),
\[
h \overset{\text{def}}{=} \text{Hess}(v) - v(2 - v)g - v^{-1} dv \otimes dv
\]
is nonnegative definite on $T_P G_{n,m}$. Denote
\[
V_1 = \bigoplus \alpha E_{\alpha\alpha}, \quad V_2 = \bigoplus_{i \neq \alpha} E_{i\alpha};
\]
then $T_P G_{n,m} = V_1 \oplus V_2$, and (3.11), (2.9), (3.9) tell us
\[
h(V_1, V_2) = 0, \quad dv(V_2) = 0
\]
and
\[
h|_{V_1} = \sum \alpha (v - 1 + \lambda^2_{\alpha})v \omega^2_{\alpha\alpha}.
\]
is positive definite. Denote by $\tilde{\nabla} v$ the unique element in $V_1$ such that for any $X \in V_1$,
\[
h(\tilde{\nabla} v, X) = dv(X).
\]
From (3.25) and (3.9), it is not difficult to obtain
\[
\tilde{\nabla} v = \sum \alpha \frac{\lambda_{\alpha}(1 + \lambda^2_{\alpha})}{v - 1 + \lambda^2_{\alpha}} E_{\alpha\alpha}
\]
and
\[
dv(\tilde{\nabla} v) = \sum \alpha \frac{\lambda^2_{\alpha}}{v - 1 + \lambda^2_{\alpha}} v.
\]
Then Lemma 3.2 and (3.23) tell us

\[(3.27) \quad \text{Hess}(v) \geq v(2 - v)g + \left[1 + \left(\sum_{\alpha} \frac{\lambda_{\alpha}^2}{v - 1 + \lambda_{\alpha}^2}\right)^{-1}\right]v^{-1}dv \otimes dv.\]

It is necessary to estimate the upper bound of \(\sum_{\alpha} \frac{\lambda_{\alpha}^2}{v - 1 + \lambda_{\alpha}^2}\). Denote

\[(3.28) \quad \nu_{\alpha} = \log(1 + \lambda_{\alpha}^2),\]

then \(\lambda_{\alpha}^2 = -1 + e^{\nu_{\alpha}}\); since \(v = \prod_{\alpha} (1 + \lambda_{\alpha}^2)^{\frac{1}{2}}\),

\[
\log v = \frac{1}{2} \sum_{\alpha} \log(1 + \lambda_{\alpha}^2) = \frac{1}{2} \sum_{\alpha} \nu_{\alpha}
\]

and

\[
\sum_{\alpha} \frac{\lambda_{\alpha}^2}{v - 1 + \lambda_{\alpha}^2} = \sum_{\alpha} \frac{-1 + e^{\nu_{\alpha}}}{v - 2 + e^{\nu_{\alpha}}}.\]

Now we define

\[(3.29) \quad \Omega = \{(\nu_1, \ldots, \nu_m) \in \mathbb{R}^m : \nu_{\alpha} \geq 0, \sum_{\alpha} \nu_{\alpha} = 2 \log v\},\]

and \(f : \Omega \to \mathbb{R}\) by

\[
(\nu_1, \ldots, \nu_m) \mapsto \sum_{\alpha} \frac{-1 + e^{\nu_{\alpha}}}{v - 2 + e^{\nu_{\alpha}}}.\]

Then obviously \(\Omega\) is compact and convex, \(T_{\sigma}(\Omega) = \Omega\) for every \(\sigma \in \mathcal{S}(m)\) (cf. Lemma 3.3), \(f\) is a symmetric function and a direct calculation shows

\[
\frac{\partial^2 f}{\partial \nu_{\alpha} \partial \nu_{\beta}} = \frac{(v - 1)e^{\nu_{\alpha}}(v - 2 - e^{\nu_{\alpha}})}{(v - 2 + e^{\nu_{\alpha}})^3} \delta_{\alpha\beta};
\]

i.e.,

\[
(D^2 f) \leq 0 \quad \text{when } v \in (1, 2].
\]

Then from Lemma 3.3,

\[
\sup_{\Omega} f = f\left(\frac{2 \log v}{m}, \ldots, \frac{2 \log v}{m}\right) = m\left(-1 + \frac{2}{m}\right)
\]

which is an upper bound of \(\sum_{\alpha} \frac{\lambda_{\alpha}^2}{v - 1 + \lambda_{\alpha}^2}\). Substituting it into (3.27) gives

\[
\text{Hess}(v) \geq v(2 - v)g + \left(\frac{v - 1}{mv(v^\frac{2}{m} - 1)} + \frac{m + 1}{mv}\right)dv \otimes dv.
\]

In summary, we have the following Proposition.

**Proposition 3.1.** \(v\) is a convex function on \(B_{JX}(P_0) \subset U \subset G_{n,m}\), and

\[(3.30) \quad \text{Hess}(v) \geq v(2 - v)g + \left(\frac{v - 1}{pv(v^\frac{2}{p} - 1)} + \frac{p + 1}{pv}\right)dv \otimes dv\]

on \(\{P \in U : v(P) \leq 2\}\), where \(p = \min(n, m)\).
Similarly, we consider
\begin{equation}
\tilde{h} \overset{\text{def}}{=} \text{Hess}(u) - (2 - \frac{1}{2}u^2)g;
\end{equation}
which is nonnegative definite on $T_PG_{n,m}$. The definition of $V_1$ and $V_2$ is similar to above. It is easily seen from (3.17) and (2.9) that
\begin{equation}
\tilde{h}(V_1, V_2) = 0
\end{equation}
and
\begin{equation}
\tilde{h}|_{V_1} = \sum_\alpha (8\lambda_\alpha^2 + 6\lambda_\alpha^4 + \frac{1}{2}u^2)\omega_{\alpha\alpha}
\end{equation}
is positive definite. By (3.15),
\begin{equation}
du = \sum_\alpha 2\lambda_\alpha (1 + \lambda_\alpha^2)\omega_{\alpha\alpha},
\end{equation}
then
\begin{equation}
du(V_2) = 0.
\end{equation}
Hence Lemma 3.2 can be applied for us to obtain
\begin{equation}
\text{Hess}(u) \geq (2 - \frac{1}{2}u^2)g + (du(\tilde{\nabla}u))^{-1}du \otimes du.
\end{equation}
where $\tilde{\nabla}u$ denotes the unique element in $V_1$ such that for arbitrary $X \in V_1$,
\begin{equation}
\tilde{h}(\tilde{\nabla}u, X) = du(X).
\end{equation}
From (3.32) and (3.33), we can derive
\begin{equation}
\tilde{\nabla}u = \sum_\alpha \frac{2\lambda_\alpha (1 + \lambda_\alpha^2)^2}{8\lambda_\alpha^2 + 6\lambda_\alpha^4 + \frac{1}{2}u^2}E_{\alpha\alpha},
\end{equation}
and hence
\begin{equation}
du(\tilde{\nabla}u) = \sum_\alpha \frac{2\lambda_\alpha^2 (1 + \lambda_\alpha^2)^2}{3\lambda_\alpha^4 + 4\lambda_\alpha^2 + \frac{1}{4}u^2}.
\end{equation}
(3.34) tells us it is necessary for us to estimate the upper bound of the right side of (3.35).

Define $\Omega = \{(\nu_1, \cdots, \nu_m) \in \mathbb{R}^m : \sum_\alpha \nu_\alpha = u\}$ and $f : \Omega \rightarrow \mathbb{R}$
\begin{equation}
(\nu_1, \cdots, \nu_m) \mapsto \sum_\alpha \frac{2\nu_\alpha (1 + \nu_\alpha)^2}{3\nu_\alpha^2 + 4\nu_\alpha + C} \quad \text{where} \ C = \frac{1}{4}u^2.
\end{equation}
Then it is easy to see that sup $f$ is an upper bound of $du(\tilde{\nabla}u)$, since $u = \sum_\alpha \tan^2 \theta_\alpha = \sum_\alpha \lambda_\alpha^2$.

Obviously $\Omega$ is compact and convex, $T_\sigma(\Omega) = \Omega$ for every $\sigma \in \mathcal{S}(m)$, $f$ is a symmetric function and a direct calculation shows
\begin{equation}
\frac{\partial^2 f}{\partial \nu_\alpha \partial \nu_\beta} = -4\left(\frac{3C - 1)(\nu_\alpha^3 + 6C\nu_\alpha^2 + (9C - 3C^2)\nu_\alpha + 4C - 2C^2)}{(3\nu_\alpha^2 + 4\nu_\alpha + C)^3}\delta_{\alpha\beta}\right).
\end{equation}
To show $(D^2 f) \leq 0$ when $u \in (0, 2]$, it is sufficient to prove $F : [0, u] \to \mathbb{R}$

$$t \mapsto (3C - 1)t^3 + 6Ct^2 + (9C - 3C^2)t + 4C - 2C^2$$

is a nonnegative function, where $C = \frac{u^2}{4} \in (0, 1]$. If $F$ attains its minimum at $t_0 \in (0, u)$, then

$$0 = F'(t_0) = 3(3C - 1)t_0^2 + 12Ct_0 + 9C - 3C^2;$$

$$0 \leq F''(t_0) = 6(3C - 1)t_0 + 12C. \quad (3.36)$$

On the other hand, when $3C - 1 \geq 0$, we have $F'(t_0) \geq 9C - 3C^2 > 0$, which causes a contradiction; when $3C - 1 < 0$, from (3.37), $t_0 \leq \frac{2C}{1-3C}$, then $F'(t_0) \geq F'(0) = 9C - 3C^2 > 0$, which also causes a contradiction. Therefore

$$\min_{[0,u]} F = \min \{F(0), F(u)\}. \quad (3.37)$$

In conjunction with

$$F(0) = 4C - 2C^2 > 0$$

$$F(u) = (3C - 1)u^3 + 6Cu^2 + (9C - 3C^2)u + 4C - 2C^2$$

$$= \frac{9}{16} u^5 + \frac{11}{8} u^4 + \frac{5}{4} u^3 + u^2 > 0,$$

$F$ is a nonnegative function. Thereby applying Lemma 3.3 we have

$$du(\nabla u) \leq \sup f = f\left(\frac{u}{m}, \cdots, \frac{u}{m}\right) = \frac{2(u + m)^2}{(3 + \frac{1}{2}m^2)u + 4m}. \quad (3.38)$$

Substituting (3.38) into (3.34) gives

$$\text{Hess}(u) \geq \left(2 - \frac{1}{2}u^2\right)g + \frac{(3 + \frac{1}{2}m^2)u + 4m}{2(u + m)^2}du \otimes du.$$

We rewrite the conclusion as the following Proposition.

**Proposition 3.2.** $u$ is a convex function on $B_{JX}(P_0) \subset \mathbb{U} \subset \mathbf{G}_{n,m}$ and

$$\text{Hess}(u) \geq \left(2 - \frac{1}{2}u^2\right)g + \frac{(3 + \frac{1}{2}p^2)u + 4p}{2(u + p)^2}du \otimes du \quad (3.39)$$

on $\{P \in \mathbb{U} : u(P) \leq 2\}$, where $p = \min(n, m)$.

4. **The construction of auxiliary functions**

Let

$$h_1 = v^{-k}(2 - v)^k, \quad (4.1)$$
where \( k > 0 \) to be chosen, then
\[
\begin{align*}
    h'_1 &= -kv^{-k-1}(2-v)^k - kv^{-k}(2-v)^{k-1} \\
    &= -2kv^{-k-1}(2-v)^{k-1}, \\
    h''_1 &= 2k(k+1)v^{-k-2}(2-v)^{k-1} + 2k(k-1)v^{-k-1}(2-v)^{k-2} \\
    &= 4kv^{-k-2}(2-v)^{k-2}(k+1-v).
\end{align*}
\]

Here \( ' \) denotes derivative with respect to \( v \). Hence, from (3.30)
\[
\text{Hess}(h_1) = -2kv^{-k-1}(2-v)^k g - 2kv^{-k-2}(2-v)^{k-2}(k+1-v) dv \otimes dv \leq -2kv^{-k-1}(2-v)^k g
\]
\[
\leq -2kv^{-k-1}(2-v)^k g - 2kv^{-k-2}(2-v)^{k-2} \left[ \frac{(v-1)(2-v)}{p(v^\frac{2}{p} - 1)} + \frac{p+1}{p} (2-v) - 2(k+1-v) \right] dv \otimes dv.
\]

Please note that \( \frac{v-1}{v^\frac{2}{p} - 1} \) is an increasing function on \([1, 2] \): it is easily seen when \( p \) is even, since
\[
\frac{v-1}{v^\frac{2}{p} - 1} = 1 + v^\frac{2}{p} + v^\frac{4}{p} + \cdots + v^{1-\frac{2}{p}};
\]
otherwise, when \( p \) is odd,
\[
\frac{v-1}{v^\frac{2}{p} - 1} = \frac{v^{1-\frac{2}{p}} - 1}{v^\frac{2}{p} - 1} + \frac{v - v^{1-\frac{1}{p}}}{v^\frac{2}{p} - 1} = 1 + v^\frac{2}{p} + v^\frac{4}{p} + \cdots + v^{1-\frac{2}{p}} + \frac{v^{1-\frac{1}{p}}}{v^\frac{2}{p} + 1}
\]

it follows from
\[
\left( \frac{v^{1-\frac{1}{p}}}{v^\frac{2}{p} + 1} \right)' = \frac{1 - \frac{2}{p} + \left(1 - \frac{1}{p}\right)v^{-\frac{1}{p}}}{(v^\frac{2}{p} + 1)^2} \geq 0.
\]
Hence
\[
\frac{v-1}{v^\frac{2}{p} - 1} \geq \frac{p}{2},
\]
and moreover
\[
\frac{(v-1)(2-v)}{p(v^\frac{2}{p} - 1)} + \frac{p+1}{p} (2-v) - 2(k+1-v)
\[
\geq \left(1 + \frac{p+1}{p}\right)(2-v) - 2(k+1-v)
\[
= \left(1 - \frac{1}{p}\right)v + \left(3 + \frac{2}{p}\right) - 2(k+1)
\[
\geq \frac{3}{2} + \frac{1}{p} - 2k.
\]

Now we take
\[
(4.3) \quad k = \frac{3}{4} + \frac{1}{2p},
\]
then \( \frac{(v-1)(2-v)}{p(v^p-1)} + \frac{p+1}{p}(2-v) - 2(k + 1 - v) \geq 0 \) and then (4.2) becomes

\[
(4.4) \quad \text{Hess}(h_1) \leq -2kh_1 \ g = - \left( \frac{3}{2} + \frac{1}{p} \right) h_1 \ g.
\]

Denote

\[
(4.5) \quad h_2 = h_1^{-\frac{6p}{3p+2}} = v^\frac{4}{p}(2-v)^{-\frac{2}{p}},
\]

then

\[
(4.6) \quad \text{Hess}(h_2) = -\frac{6p}{3p+2} h_1^{-\frac{6p}{3p+2} - 1} \text{Hess}(h_1) + \frac{6p}{3p+2} \left( \frac{6p}{3p+2} + 1 \right) h_1^{-\frac{6p}{3p+2} - 2} dh_1 \otimes dh_1
\]

\[
\geq 3h_1^{-\frac{6p}{3p+2}} g + \left( \frac{3}{2} + \frac{1}{3p} \right) h_1^{-\frac{6p}{3p+2}} dh_2 \otimes dh_2
\]

\[
= 3h_2 \ g + \left( \frac{3}{2} + \frac{1}{3p} \right) h_2^{-1} dh_2 \otimes dh_2.
\]

Let

\[
(4.7) \quad h_3 = (u + \alpha)^{-1}(2 - u),
\]

where \( \alpha > 0 \) to be chosen. A direct calculation shows

\[
h_3' = -(u + \alpha)^{-2}(2 - u) - (u + \alpha)^{-1}
\]

\[
= -(2 + \alpha)(u + \alpha)^{-2},
\]

\[
h_3'' = 2(2 + \alpha)(u + \alpha)^{-3}.
\]

Here ’ denotes derivative with respect to \( u \). Combining with (3.39), we have

\[
\text{Hess}(h_3) = -(2 + \alpha)(u + \alpha)^{-2}\text{Hess}(u) + 2(2 + \alpha)(u + \alpha)^{-3} du \otimes du
\]

\[
\leq - \frac{(2 + \alpha)(u + 2)}{2(u + \alpha)} h_3 \ g
\]

\[
- (2 + \alpha)(u + \alpha)^{-3}\left[ \frac{(u + \alpha)((3 + \frac{1}{4}p^2)u + 4p)}{2(u + p)^2} - 2 \right] du \otimes du.
\]

Choose

\[
(4.9) \quad \alpha = p,
\]

then

\[
\frac{(u + \alpha)((3 + \frac{1}{4}p^2)u + 4p)}{2(u + p)^2} - 2 = \frac{(3 + \frac{1}{4}p^2)u + 4p}{2(u + p)} - 2 \geq 2 - 2 \geq 0,
\]

and

\[
\frac{(2 + \alpha)(u + 2)}{2(u + \alpha)} \geq \frac{2 + p}{p} = 1 + \frac{2}{p}.
\]

Thereby (4.8) becomes

\[
(4.10) \quad \text{Hess}(h_3) \leq -(1 + \frac{2}{p})h_3 \ g.
\]
Denote
\[ h_4 = h_3^{\frac{3p}{p+2}} = (u + p)^{\frac{3p}{p+2}}(2 - u)^{-\frac{3p}{p+2}}, \]
then
\[
\text{Hess}(h_4) = -\frac{3p}{p+2} h_3^{\frac{3p}{p+2}-1} \text{Hess}(h_3) + \frac{3p}{p+2} \left( \frac{3p}{p+2} + 1 \right) h_3^{\frac{3p}{p+2}-2} dh_3 \otimes dh_3 \]
\[
\geq 3 h_3^{\frac{3p}{p+2}} g + \left( \frac{4}{3} + \frac{2}{3p} \right) h_3^{\frac{3p}{p+2}} dh_4 \otimes dh_4 \]
\[ = 3h_4^4 g + \left( \frac{4}{3} + \frac{2}{3p} \right) h_4^{-1} dh_4 \otimes dh_4. \]

Let \( M \) be an \( n \)-dimensional submanifold in \( \mathbb{R}^{n+m} \) with \( m \geq 2 \). The Gauss map \( \gamma : M \to G_{n,m} \) is defined by
\[
\gamma(x) = T_x M \in G_{n,m}
\]
via the parallel translation in \( \mathbb{R}^{n+m} \) for arbitrary \( x \in M \). The energy density of the Gauss map (see [21] Chap.3, §3.1) is
\[
e(\gamma) = \frac{1}{2} \langle \gamma_* e_i, \gamma_* e_i \rangle = \frac{1}{2} |B|^2.
\]
Ruh-Vilms proved that the mean curvature vector of \( M \) is parallel if and only if its Gauss map is a harmonic map [17].

If the Gauss image of \( M \) is contained in \( \{ P \in U \subset G_{n,m} : v(P) < 2 \} \), then the composition function \( \tilde{h}_1 = h_1 \circ \gamma \) of \( h_1 \) with the Gauss map \( \gamma \) defines a function on \( M \). Using composition formula, we have
\[
\Delta \tilde{h}_1 = \text{Hess}(h_1)(\gamma_* e_i, \gamma_* e_i) + dh_1(\tau(\gamma)) \]
\[
\leq -\left( \frac{3}{2} + \frac{1}{p} \right) |B|^2 \tilde{h}_1,
\]
where \( \tau(\gamma) \) is the tension field of the Gauss map, which is zero, provided \( M \) has parallel mean curvature by the Ruh-Vilms theorem mentioned above. Similarly, for \( \tilde{h}_2 = h_2 \circ \gamma \) defined on \( M \), we have
\[
\Delta \tilde{h}_2 = \text{Hess}(h_2)(\gamma_* e_i, \gamma_* e_i) + dh_2(\tau(\gamma)) \]
\[
\geq 3 \tilde{h}_2 |B|^2 + \left( \frac{3}{2} + \frac{1}{3p} \right) \tilde{h}_2^{-1} |\nabla \tilde{h}_2|^2.
\]

If the Gauss image of \( M \) is contained in \( \{ P \in U \subset G_{n,m} : u(P) < 2 \} \), we can defined composition function \( \tilde{h}_3 = h_3 \circ \gamma \) and \( \tilde{h}_4 = h_4 \circ \gamma \) on \( M \). Again using composition formula, we obtain
\[
\Delta \tilde{h}_3 \leq -\left( 1 + \frac{2}{p} \right) |B|^2 \tilde{h}_3
\]
and

\begin{equation}
\Delta \tilde{h}_4 \geq 3 \tilde{h}_4 |B|^2 + \left(\frac{4}{3} + \frac{2}{3p}\right) \tilde{h}_4^{-1} |\nabla \tilde{h}_4|^2.
\end{equation}

With the aid of \( \tilde{h}_1 \) and \( \tilde{h}_3 \), we immediately have the following lemma.

**Lemma 4.1.** Let \( M \) be an \( n \)-dimensional minimal submanifold of \( \mathbb{R}^{n+m} \) (\( M \) needs not be complete), if the Gauss image of \( M \) is contained in \( \{ P \in U \subset G_{n,m} : v(P) < 2 \} \) (or respectively, \( \{ P \in U \subset G_{n,m} : u(P) < 2 \} \)), then we have

\begin{equation}
\int_M |\nabla \phi|^2 * 1 \geq \left(\frac{3}{2} + \frac{1}{p}\right) \int_M |B|^2 \phi^2 * 1
\end{equation}

(or respectively, \( \int_M |\nabla \phi|^2 * 1 \geq (1 + \frac{2}{p}) \int_M |B|^2 \phi^2 * 1 \))

for any function \( \phi \) with compact support \( D \subset M \).

**Remark 4.1.** For a stable minimal hypersurface there is the stability inequality, which is one of main ingredient for Schoen-Simon-Yau’s curvature estimates for stable minimal hypersurfaces. For minimal submanifolds with the Gauss image restriction we have stronger inequality as shown in (4.17). Our proof is similar to [23] and [24], so we omit the detail of it.

5. Curvature estimates

We are now in a position to carry out the curvature estimates of Schoen-Simon-Yau type.

Let \( M \) be an \( n \)-dimensional minimal submanifold in \( \mathbb{R}^{n+m} \). Assume that the estimate

\begin{equation}
\int_M |\nabla \phi|^2 * 1 \geq \lambda \int_M |B|^2 \phi^2 * 1
\end{equation}

holds for arbitrary function \( \phi \) with compact support \( D \subset M \), where \( \lambda \) is a positive constant.

Replacing \( \phi \) by \( |B|^{1+q} \phi \) in (5.1) gives

\begin{equation}
\int_M |B|^{4+2q} \phi^2 * 1 \leq \lambda^{-1} \int_M |\nabla (|B|^{1+q} \phi)|^2 * 1
\end{equation}

\begin{equation}
= \lambda^{-1}(1+q)^2 \int_M |B|^{2q} |\nabla |B||^2 \phi^2 * 1 + \lambda^{-1} \int_M |B|^{2+2q} |\nabla \phi|^2 * 1
+ 2\lambda^{-1}(1+q) \int_M |B|^{1+2q} \nabla |B| \cdot \phi \nabla \phi * 1.
\end{equation}
Using Bochner technique, the estimate done in [14][7], and the Kato-type inequality derived in [24], we obtain

\begin{equation}
\Delta |B|^2 \geq 2(1 + \frac{2}{mn})|\nabla |B||^2 - 3|B|^4.
\end{equation}

(For the detail, see [24] Section 2.) It is equivalent to

\begin{equation}
\frac{2}{mn} |\nabla |B||^2 \leq |B| \Delta |B| + \frac{3}{2}|B|^4.
\end{equation}

Multiplying $|B|^{2q} \phi^2$ with both sides of (5.4) and integrating by parts, we have

\begin{equation}
\frac{2}{mn} \int_M |B|^{2q} |\nabla |B||^2 \phi^2 \ast 1
\end{equation}

\begin{equation}
\leq -(1 + 2q) \int_M |B|^{2q} |\nabla |B||^2 \phi^2 \ast 1
\end{equation}

\begin{equation}
- 2 \int_M |B|^{1+2q} \nabla |B| \cdot \phi \nabla \phi \ast 1 + \frac{3}{2} \int_M |B|^{1+2q} \phi^2 \ast 1.
\end{equation}

By multiplying $\frac{3}{2}$ with both sides of (5.2) and then adding up both sides of it and (5.5), we have

\begin{equation}
\left(\frac{2}{mn} + 1 + 2q - \frac{3}{2} \lambda^{-1}(1 + q)^2\right) \int_M |B|^{2q} |\nabla |B||^2 \phi^2 \ast 1
\end{equation}

\begin{equation}
\leq \frac{3}{2} \lambda^{-1} \int_M |B|^{2+2q} |\nabla \phi|^2 \ast 1 + (3 \lambda^{-1}(1 + q) - 2) \int_M |B|^{1+2q} \nabla |B| \cdot \phi \nabla \phi \ast 1.
\end{equation}

By using Young’s inequality, (5.6) becomes

\begin{equation}
\left(\frac{2}{mn} + 1 + 2q - \frac{3}{2} \lambda^{-1}(1 + q)^2 - \varepsilon\right) \int_M |B|^{2q} |\nabla |B||^2 \phi^2 \ast 1
\end{equation}

\begin{equation}
\leq C_1(\varepsilon, \lambda, q) \int_M |B|^{2+2q} |\nabla \phi|^2 \ast 1.
\end{equation}

If

\begin{equation}
\lambda > \frac{3}{2} \left(1 - \frac{2}{mn}\right),
\end{equation}

then

\begin{equation}
\frac{2}{mn} + 1 + 2q - \frac{3}{2} \lambda^{-1}(1 + q)^2 > 0
\end{equation}

whenever

\begin{equation}
q \in \left[0, -1 + \frac{2}{3} \lambda + \frac{1}{3} \sqrt{4 \lambda^2 - 6(1 - \frac{2}{mn}) \lambda}\right).
\end{equation}

Thus we can choose $\varepsilon$ sufficiently small, such that

\begin{equation}
\int_M |B|^{2q} |\nabla |B||^2 \phi^2 \ast 1 \leq C_2 \int_M |B|^{2+2q} |\nabla \phi|^2 \ast 1
\end{equation}

where $C_2$ only depends on $n$, $m$, $\lambda$ and $q$. 

Combining with (5.2) and (5.10), we can derive
\begin{equation}
\int_M |B|^{4+2q} \phi^2 \ast 1 \leq C_3(n, m, \lambda, q) \int_M |B|^{2+2q} |\nabla \phi|^2 \ast 1
\end{equation}
by again using Young’s inequality.

By replacing \( \phi \) by \( \phi^{2+q} \) in (5.11) and then using Hölder inequality, we have
\begin{equation}
\int_M |B|^{4+2q} \phi^{4+2q} \ast 1 \leq C \int_M |\nabla \phi|^{4+2q} \ast 1.
\end{equation}
where \( C \) is a constant only depending on \( n, m, \lambda \) and \( q \).

Similarly, replacing \( \phi \) by \( \phi^{1+q} \) in (5.11) and then again using Hölder inequality yields
\begin{equation}
\int_M |B|^{4+2q} \phi^{2+2q} \ast 1 \leq C' \int_M |\nabla \phi|^{2+2q} \ast 1.
\end{equation}
where \( C' \) is a constant only depending on \( n, m, \lambda \) and \( q \).

Let \( r \) be a function on \( M \) with \( |\nabla r| \leq 1 \). For any \( R \in [0, R_0] \), where \( R_0 = \sup_M r \), suppose
\[ M_R = \{ x \in M, \ r \leq R \} \]
is compact.

(5.12) and Lemma 4.1 enable us to prove the following results by taking \( \phi \in C^\infty_c(M_R) \) to be the standard cut-off function such that \( \phi \equiv 1 \) in \( M_{\theta R} \) and \( |\nabla \phi| \leq C(1-\theta)^{-1}R^{-1} \).

**Theorem 5.1.** Let \( M \) be an \( n \)-dimensional minimal submanifolds of \( \mathbb{R}^{n+m} \). If the Gauss image of \( M_R \) is contained in \( \{ P \in U \subset G_{n,m} : u(P) < 2 \} \), then we have the estimate
\begin{equation}
\| |B| \|_{L^s(M_{\theta R})} \leq C(n, m, s)(1-\theta)^{-1}R^{-1} \text{Vol}(M_R)^{\frac{1}{s}}
\end{equation}
for arbitrary \( \theta \in (0, 1) \) and
\[ s \in \left[ 4, 4 + \frac{4}{3p} + \frac{2}{3} \sqrt{(3 + \frac{2}{p})(\frac{6}{mn} + \frac{2}{p})} \right] .
\]

If \( p \leq 4 \), and the Gauss image of \( M_R \) is contained in \( \{ P \in U \subset G_{n,m} : u(P) < 2 \} \), then (5.14) still holds for arbitrary \( \theta \in (0, 1) \) and
\[ s \in \left[ 4, 2 + \frac{4}{3} + \frac{8}{3p} + \frac{2}{3} \sqrt{(1 + \frac{2}{p})(\frac{12}{mn} + \frac{8}{p} - 2)} \right] .
\]

We can also fulfill the curvature estimates of Ecker-Huisken type.

Assume that \( h \) is a positive function on \( M \) satisfying the following estimate
\begin{equation}
\Delta h \geq 3h^g + c_0 h^{-1} dh \otimes dh,
\end{equation}
where
\[ c_0 > \frac{3}{2} - \frac{1}{mn} \]
is a positive constant.

We compute from (5.15) and (5.3):
\[ \Delta (|B|^{2s} h^q) \]
\[ \geq 3(q - s)|B|^{2s+2} h^q + 2s(2s - 1 + \frac{2}{mn})|B|^{2s-2} \nabla |B|^{2} h^q + q(q + c_0 - 1)|B|^{2s} h^{q-2} \nabla |B|^{2} h^q \]
\[ + 4sq|B|^{2s-1} \nabla |B| \cdot h^{q-1} \nabla h. \]

By Young’s inequality, when \( 2s(2s - 1 + \frac{2}{mn}) \cdot q(q + c_0 - 1) \geq (2sq)^2 \), i.e.,
\[ (5.16) \]
\[ q \geq s \geq \frac{1}{2} - \frac{1}{mn} + \frac{1}{c_0 - 1}\left(\frac{1}{2} - \frac{1}{mn}\right)q, \]
the inequality
\[ (5.17) \]
\[ \Delta (|B|^{2s} h^q) \geq 3(q - s)|B|^{2s+2} h^q \]
holds. Especially,
\[ (5.18) \]
\[ \Delta (|B|^{s-1} h^{\frac{s}{2}}) \geq \frac{3}{2}|B|^{s+1} h^{\frac{s}{2}} \]
whenever
\[ (5.19) \]
\[ s \geq \frac{2 - \frac{2}{mn}}{1 - \frac{1}{c_0 - 1}\left(\frac{1}{2} - \frac{1}{mn}\right)}. \]

Let \( \eta \) be a smooth function with compact support. Integrating by parts in conjunction with Young’s inequality lead to
\[ (5.20) \]
\[ \frac{2}{3} \int_M |B|^{2s} h^s \eta^{2s} * 1 \leq \frac{2}{3} \int_M |B|^{s-1} h^{\frac{s}{2}} \eta^{2s} \nabla (|B|^{s-1} h^{\frac{s}{2}}) * 1 \]
\[ = -\frac{2}{3} \int_M \left| \nabla (|B|^{s-1} h^{\frac{s}{2}}) \right|^2 \eta^{2s} * 1 \]
\[ - \frac{2}{3} \int_M |B|^{s-1} h^{\frac{s}{2}} \cdot 2s \eta^{2s-1} \nabla \eta \cdot \nabla (|B|^{s-1} h^{\frac{s}{2}}) * 1 \]
\[ \leq \frac{s^2}{3} \int_M |B|^{2s-2} h^s \eta^{2s-2} |\nabla \eta|^2 * 1. \]

By Hölder inequality,
\[ (5.21) \]
\[ \int_M |B|^{2s-2} h^s \eta^{2s-2} |\nabla \eta|^2 * 1 \leq \left( \int_M |B|^{2s} h^s \eta^{2s} * 1 \right)^{\frac{s-1}{s}} \left( \int_M h^s |\nabla \eta|^{2s} * 1 \right)^{\frac{1}{s}}. \]

Substituting (5.21) into (5.20), we finally arrive at
\[ (5.22) \]
\[ \left( \int_M |B|^{2s} h^s \eta^{2s} * 1 \right)^{\frac{1}{s}} \leq \frac{2}{3} s^2 \left( \int_M h^s |\nabla \eta|^{2s} * 1 \right)^{\frac{1}{s}}. \]
Take \( \eta \in C^\infty_c(M_{\theta R}) \) to be the standard cut-off function such that \( \eta \equiv 1 \) in \( M_{\theta R} \) and \( |\nabla \eta| \leq C(1-\theta)^{-1}R^{-1} \), then from (5.22) we have the following estimate.

**Theorem 5.2.** Let \( M \) be an \( n \)-dimensional minimal submanifolds of \( \mathbb{R}^{n+m} \). If there exists a positive function \( h \) on \( M \) satisfying (5.15), then there exists \( C_1 = C_1(n, m, c_0) \), such that

\[
(B^2h)_{L^s(M_{\theta R})} \leq C_2(s)(1-\theta)^{-2}R^{-2}h_{L^s(M_R)}
\]

whenever \( s \geq C_1 \) and \( \theta \in (0, 1) \).

By (4.14) and (4.16), if the Gauss image of \( M \) is contained in \( \{ P \in U \subset G_{n,m} : v(P) < 2 \} \), or \( p \leq 4 \) and the Gauss image of \( M \) is contained in \( \{ P \in U \subset G_{n,m} : u(P) < 2 \} \), there exists a positive function on \( M \), which is \( \tilde{h}_2 \) or respectively \( \tilde{h}_4 \), satisfying (5.15). Hence the estimate (5.23) holds for both cases.

Furthermore, the mean value inequality for any subharmonic function on minimal submanifolds in \( \mathbb{R}^{m+n} \) (ref. [8], [16]) can be applied to yield an estimate of the upper bound of \( |B^2| \). We write the results as the following theorem without detail of proof, for it is similar to [24]. Please note that \( B_R(x) \subset \mathbb{R}^{m+n} \) denotes a ball of radius \( R \) centered at \( x \in M \) and its restriction on \( M \) is denoted by

\[
D_R(x) = B_R(x) \cap M.
\]

**Theorem 5.3.** Let \( x \in M, R > 0 \) such that the image of \( D_R(x) \) under the Gauss map lies in \( \{ P \in U \subset G_{n,m} : v(P) < 2 \} \). Then, there exists \( C_1 = C_1(n, m) \), such that

\[
|B|^{2s}(x) \leq C(n, s)(n+2s)(\sup_{D_R(x)} \tilde{h}_2)^s \text{Vol}(D_R(x)),
\]

for arbitrary \( s \geq C_1 \).

If \( p \leq 4 \), the image of \( D_R(x) \) under the Gauss map lies in \( \{ P \in U \subset G_{n,m} : u(P) < 2 \} \), then there exists \( C_2 = C_2(n, m) \) such that

\[
|B|^{2s}(x) \leq C(n, s)(n+2s)(\sup_{D_R(x)} \tilde{h}_4)^s \text{Vol}(D_R(x)),
\]

holds for any \( s \geq C_2 \).

6. Bernstein type theorems and related results

If \( M \) is a submanifold in \( \mathbb{R}^{n+m} \), then the function \( w \) defined on \( G_{n,m} \) (see Section 2) and the Gauss map \( \gamma \) could be composed, yielding a smooth function on \( M \), which is also denoted by \( w \). By studying the properties of \( w \)-function, we can obtain:
Proposition 6.1. [24] Let $M$ be a complete submanifold in $\mathbb{R}^{n+m}$. If the $w$-function is bounded below by a positive constant $w_0$. Then $M$ is an entire graph with Euclidean volume growth. Precisely,

\begin{equation}
\text{Vol}(D_R(x)) \leq \frac{1}{w_0} C(n) R^n.
\end{equation}

Now we let $M$ be a complete minimal submanifold in $\mathbb{R}^{n+m}$ whose Gauss image lies in $\{ P \in U \subset G_{n,m} : v(P) < 2 \}$. Then $w = v^{-1} > \frac{1}{2}$ on $M$ and Proposition 6.1 tells us $M$ is an entire graph. Precisely, the immersion $F : M \to \mathbb{R}^{m+n}$ is realized by a graph $(x, f(x))$ with

$$f : \mathbb{R}^n \to \mathbb{R}^m.$$ 

At each point in $M$ its image $n$-plane $P$ under the Gauss map is spanned by

$$f_i = \varepsilon_i + \frac{\partial f^\alpha}{\partial x^i} \varepsilon_\alpha.$$ 

Hence the local coordinate of $P$ in $U$ is

$$Z = \left( \frac{\partial f^\alpha}{\partial x^i} \right).$$ 

By (3.2),

$$v(P) = \left[ \det \left( \delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \right) \right]^{\frac{1}{2}}.$$ 

Hence

\begin{equation}
\left[ \det \left( \delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \right) \right]^{\frac{1}{2}} < 2
\end{equation}

at each $x \in \mathbb{R}^n$. Conversely, if $M = (x, f(x))$ is a minimal graph given by $f : \mathbb{R}^n \to \mathbb{R}^m$ which satisfy (6.2), then the Gauss image of $M$ lies in $\{ P \in U \subset G_{n,m} : v(P) < 2 \}$.

Let $P \in U$ such that $u(P) = \sum_\alpha \tan^2 \theta_\alpha < 2$, then $\cos^2 \theta_\alpha = (1 + \tan^2 \theta_\alpha)^{-1} > \frac{1}{3}$ and

$$w(P) = \prod_\alpha \cos \theta_\alpha > 3^{-\frac{w}{4}}.$$ 

Hence Proposition 6.1 could be applied when $M$ is a complete minimal submanifold in $\mathbb{R}^{n+m}$ whose Gauss image lies in $\{ P \in U \subset G_{n,m} : v(P) < 2 \}$; which is hence a minimal graph given by $f : \mathbb{R}^n \to \mathbb{R}^m$. Thereby (3.14) shows

\begin{equation}
\sum_{i,\alpha} \left( \frac{\partial f^\alpha}{\partial x^i} \right)^2 < 2.
\end{equation}

And vice versa.

Theorem 5.1 and Proposition 6.1 give us the following Bernstein-type theorem.
Theorem 6.1. Let $M = (x, f(x))$ be an $n$-dimensional entire minimal graph given by $m$ functions $f^\alpha(x^1, \cdots, x^n)$ with $m \geq 2, n \leq 4$. If
\[ \Delta_f = \left[ \det \left( \delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \right) \right]^{\frac{1}{2}} < 2 \]
or
\[ \Lambda_f = \sum_{i, \alpha} \left( \frac{\partial f^\alpha}{\partial x^i} \right)^2 < 2, \]
then $f^\alpha$ has to be affine linear functions representing an affine $n$-plane.

Proof. If $\Delta_f < 2$, then the Gauss image of $M$ is contained in $\{ P \in U \subset G_{n,m} : v(P) < 2 \}$. We choose
\[ s = 4 + \frac{4}{3p} > 4. \]
Fix $x \in M$ and let $r$ be the Euclidean distance function from $x$ and $M_R = D_R(x)$. Hence, letting $R \to +\infty$ in (5.14) yields
\[ ||B||_{L^s(M)} = 0. \]
i.e., $|B| = 0$. $M$ has to be an affine linear subspace.

For the case $\Lambda_f < 2$, the proof is similar. \qed

Theorem 5.3 and Proposition 6.1 yield Bernstein type results as follows.

Theorem 6.2. Let $M = (x, f(x))$ be an $n$-dimensional entire minimal graph given by $m$ functions $f^\alpha(x^1, \cdots, x^n)$ with $m \geq 2$. If
\[ \Delta_f = \left[ \det \left( \delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \right) \right]^{\frac{1}{2}} < 2, \]
and
\[ (2 - \Delta_f)^{-1} = o(R^\frac{4}{p}), \]
where $R^2 = |x|^2 + |f|^2$. Then $f^\alpha$ has to be affine linear functions and hence $M$ has to be an affine linear subspace.

Theorem 6.3. Let $M = (x, f(x))$ be an $n$-dimensional entire minimal graph given by $m$ functions $f^\alpha(x^1, \cdots, x^n)$ with $p = \min\{n, m\} \leq 4$. If
\[ \Lambda_f = \sum_{i, \alpha} \left( \frac{\partial f^\alpha}{\partial x^i} \right)^2 < 2, \]
and
\[ (2 - \Lambda_f)^{-1} = o(R^{2(p+2)/sp}), \]
where $R^2 = |x|^2 + |f|^2$. Then $f^\alpha$ has to be affine linear functions and hence $M$ has to be an affine linear subspace.
Proof. Here we only give the proof of Theorem 6.2, for the proof of Theorem 6.3 is similar.

From (4.5), it is easily seen that
\[ h_2 \leq C(2 - v)^{-\frac{3}{2}}, \]
where \( C \) is a positive constant. Thus, for any point \( q \in M \), by Theorem 5.3 and Proposition 6.1, we have
\[ |B|^{2s}(q) \leq C(n, s) R^{-2s}(2 - v \circ \gamma)^{-\frac{3}{2}s} \]
Letting \( R \to +\infty \) in the above inequality forces \( |B(q)| = 0 \).

\[ \Box \]

Remark 6.1. If \( n = 2 \) or 3, the conclusion of Theorem 6.1-6.3 could be inferred from the work done by Chern-Osserman [6], Babosa [1] and Fischer-Colbrie [10].

From (5.13) it is easy to obtain the following result.

Theorem 6.4. Let \( M = (x, f(x)) \) be an \( n \)-dimensional entire minimal graph given by \( m \) functions \( f^\alpha(x^1, \cdots, x^n) \). Assume \( M \) has finite total curvature. If \( \Delta f < 2 \), or \( p \leq 4 \) and \( \Lambda f < 2 \), then \( M \) has to be an affine linear subspace.

There are other applications of the strong stability inequalities (4.17), besides its key role in S-S-Y’s estimates. We state following results, whose detail proof can be found in the previous paper of the first author [23].

Theorem 6.5. Let \( M = (x, f(x)) \) be an \( n \)-dimensional entire minimal graph given by \( m \) functions \( f^\alpha(x^1, \cdots, x^n) \). If \( \Delta f < 2 \) or \( \Lambda f < 2 \), then any \( L^2 \)-harmonic 1-form vanishes.

Theorem 6.6. Let \( M \) be one as in Theorem 6.5, \( N \) be a manifold with non-positive sectional curvature. Then any harmonic map \( f : M \to N \) with finite energy has to be constant.

References

1. J.L.M.Babosa: An extrinsic rigidity theorem for minimal immersion from \( S^2 \) into \( S^n \). J. differential Geometry 14(3) (1980), 355-368.
2. S. Bernstein: Sur un théorème de géométrie et ses application aux équations aux dérivés partielles du type elliptique. Comm. de la Soc Math. de Kharkov (2é sér.) 15 (1915-1917), 38-45.
3. E. Bombieri, E. De Giorgi and E. Guusti: Minimal cones and Bernstein problem. Invent. Math. 7(1969), 243-268.
4. Yuxin Dong: On graphic submanifolds with parallel mean curvature in Euclidean space. Preprint.
5. S. S. Chern: On the curvature of a piece of hypersurfaces in Euclidean space, Abh. Math. Sem. Univ. Hamberg 29 (1965), 77-91.
6. S. S. Chern and R. Osserman: Complete minimal surfaces in Euclidean \( n \)-space. J. d’Anal. Math. 19(1967), 15-34.
7. Qing Chen and Senlin Xu: Rigidity of compact minimal submanifolds in a unit sphere. Geom. Dedicata 45 (1)(1993), 83-88.
8. S. Y. Cheng, P. Li and S. T. Yau: Heat equations on minimal submanifolds and their applications. Amer. J. Math. 103(1981), 1021-1063.
9. K. Ecker and G. Huisken: A Bernstein result for minimal graphs of controlled growth. J. Diff. Geom. 31(1990), 397-400.
10. D. Fischer-Colbrie: Some rigidity theorems for minimal submanifolds of the sphere. Acta math. 145(1980), 29-46.
11. S. Hildebrandt, J. Jost, J and K. O. Widman: Harmonic mappings and minimal submanifolds. Invent. math. 62 (1980), 269-298.
12. J. Jost and Y. L. Xin: Bernstein type theorems for higher codimension. Calculus. Var. PDE 9 (1999), 277-296.
13. H. B. Lawson and R. Osserman: Non-existence, non-uniqueness and irregularity of solutions to the minimal surface system. Acta math. 139(1977), 1-17.
14. An-Min Li and Jimin Li: An intrinsic rigidity theorem for minimal submanifolds in a sphere. Arch. Math.58 (1992), 582-594.
15. J. Moser: On Harnack's theorem for elliptic differential equations. Comm. Pure Appl. Math. 14 (1961), 577-591.
16. Lei Ni: Gap theorems for minimal submanifolds in $\mathbb{R}^{n+1}$. Comm. Analy. Geom. 9 (3)(2001), 641-656.
17. E. A. Ruh and J. Vilms: The tension field of Gauss map. Trans. Amer. Math. 149(1970), 569-573.
18. R. Schoen, L. Simon and S. T. Yau: Curvature estimates for minimal hypersurfaces. Acta Math. 134 (1975), 275-288.
19. J. Simons: Minimal varieties in Riemannian manifolds. Ann. Math. 88 (1968), 62-105.
20. K. Smoczyk, Guofang Wang and Y. L. Xin: Bernstein type theorems with flat normal bundle. Calc. Var. and PDE. 26(1)(2006), 57-67.
21. Y. L. Xin: Geometry of harmonic maps. Birkhäuser PNLDE 23, (1996).
22. Yuanlong Xin: Minimal submanifolds and related topics. World Scientific Publ., (2003).
23. Y. L. Xin: Bernstein type theorems without graphic condition. Asia J. Math. 9(1)(2005), 31-44.
24. Y. L. Xin and Ling Yang: Curvature estimates for minimal submanifolds of higher codimension. arXiv:0709.3686.

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