Mappings between the lattices of saturated submodules with respect to a prime ideal

Morteza Noferesti, Hosein Fazaeli Moghimi*, Mohammad Hossein Hosseini

Department of Mathematics, University of Birjand, P.O.Box 97175-615, Birjand, Iran

Abstract

Let $\mathcal{S}_p(RM)$ be the lattice of all saturated submodules of an $R$-module $M$ with respect to a prime ideal $p$ of a commutative ring $R$. We examine the properties of the mappings $\eta: \mathcal{S}_p(RR) \to \mathcal{S}_p(RM)$ defined by $\eta(I) = \mathcal{S}_p(IM)$ and $\theta: \mathcal{S}_p(RM) \to \mathcal{S}_p(RR)$ defined by $\theta(N) = (N : M)$, in particular considering when these mappings are lattice homomorphisms. It is proved that if $M$ is a semisimple module or a projective module, then $\eta$ is a lattice homomorphism. Also, if $M$ is a faithful multiplication $R$-module, then $\eta$ is a lattice epimorphism. In particular, if $M$ is a finitely generated faithful multiplication $R$-module, then $\eta$ is a lattice isomorphism and its inverse is $\theta$. It is shown that if $M$ is a distributive module over a semisimple ring $R$, then the lattice $\mathcal{S}_p(RM)$ forms a Boolean algebra and $\eta$ is a Boolean algebra homomorphism.

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1. Introduction

We assume throughout this paper that all rings are commutative with nonzero identity and all modules are unitary. Let $R$ be a ring and $M$ be an $R$-module. For any submodule $N$ of $M$, we denote the annihilator of the $R$-module $M/N$ by $(N : M)$, i.e., $(N : M) = \{ r \in R \mid rM \subseteq N \}$.

It is well-known that the collection of all submodules of $M$ forms a lattice with respect to the operations $\lor$ and $\land$ defined by

\[ L \lor N = L + N \quad \text{and} \quad L \land N = L \cap N. \]

Note that this lattice, denoted $\mathcal{L}(RM)$, is bounded with the least element $(0)$ and greatest element $M$. Recently, P.F. Smith has studied several mappings between $\mathcal{L}(RR)$ and $\mathcal{L}(RM)$ [22–24]. For instance, in [22], he examined conditions under which the mappings $\lambda: \mathcal{L}(RR) \to \mathcal{L}(RM)$ defined by $\lambda(I) = IM$ and $\mu: \mathcal{L}(RM) \to \mathcal{L}(RR)$ defined by $\mu(N) = (N : M)$ are injective, surjective or lattice homomorphisms. An $R$-module $M$ is called a $\lambda$-module (respectively $\mu$-module), if $\lambda$ (respectively $\mu$) is a lattice homomorphism. The study of the mappings $\lambda$ and $\mu$ continued in [23], considering when these mappings

*Corresponding Author.

Email addresses: mortezanoferesti@birjand.ac.ir (M. Noferesti), hfazaeli@birjand.ac.ir (H.F. Moghimi), mhhosseini@birjand.ac.ir (M.H. Hosseini)

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are complete lattice homomorphisms.

A proper submodule \( P \) of \( M \) is called a \textit{prime submodule} if for \( r \in R \) and \( x \in M \), \( rx \in P \) implies that \( r \in (P : M) \) or \( x \in P \) (see, for example, \([2, 6, 18, 19]\)). For a proper submodule \( N \) of an \( R \)-module \( M \), the intersection of all prime submodules of \( M \) containing \( N \) is called the \textit{radical} of \( N \) and denoted by \( \text{rad} \) \( N \); if there are no such prime submodules, \( \text{rad} N \) is \( M \) (see, for example, \([11, 14, 17]\)). A submodule \( N \) of \( M \) is called a \textit{radical submodule} if \( \text{rad} N = N \). The collection of all radical submodules of \( M \) which is denoted by \( \mathcal{R}(R)M \) forms a lattice with respect to the following operations:

\[
L \lor N = \text{rad}(L + N) \quad \text{and} \quad L \land N = L \cap N.
\]

Note that \( \mathcal{R}(R)M \) is a bounded lattice with the least element \( \text{rad}(0) \) and the greatest element \( M \).

In \([20]\), H.F. Moghimi and J.B. Harehdashti have studied the properties of the mappings \( \rho : \mathcal{R}(R) \rightarrow \mathcal{R}(R)M \) defined by \( \rho(I) = \text{rad}(IM) \) and \( \sigma : \mathcal{L}(R) \rightarrow \mathcal{L}(R)M \) defined by \( \sigma(N) = (N : M) \), in particular considering when these mappings are lattice monomorphisms or epimorphisms. Later in \([9]\), they investigated conditions under which these mappings are complete homomorphisms. Note that \( \rho \) is always a lattice homomorphism, but not necessarily a complete lattice homomorphism. An \( R \)-module \( M \) is called a \( \sigma \)-module if \( \sigma \) is a lattice homomorphism.

Let \( M \) be an \( R \)-module. For a prime ideal \( p \) of \( R \) and a submodule \( N \) of \( M \), the set \( S_p(N) = \{ m \in M \mid cm \in N \text{ for some } c \in R \setminus \{p\} \} \) is called the \textit{saturation} of \( N \) with respect to \( p \). It is clear that \( N \subseteq S_p(N) \). It is said that \( N \) is \textit{saturated} with respect to \( p \), if \( N = S_p(N) \). It is easily seen that \( S_p(N) \) is a saturated submodule of \( M \) (see \([15, 16]\), for more details about saturation of submodules). The collection of all saturated submodules of an \( R \)-module \( M \) with respect to a fixed prime ideal \( p \) of \( R \) is a lattice with the following operations:

\[
L \lor N = S_p(L + N) \quad \text{and} \quad L \land N = L \cap N.
\]

We shall denote this lattice by \( \mathfrak{S}_p(R)M \), or by \( \mathfrak{S}_p(M) \) if there is no ambiguity about \( R \). Note that \( \mathfrak{S}_p(M) \) is bounded, with the least element \( S_p(0) \) and the greatest element \( M \).

Let \( R \) be a ring, \( p \) a fixed prime ideal of \( R \) and \( M \) an \( R \)-module. Now consider the mappings \( \eta : \mathfrak{S}_p(R) \rightarrow \mathfrak{S}_p(M) \) defined by

\[
\eta(I) = S_p(IM),
\]

for every saturated ideal \( I \) of \( R \), and \( \theta : \mathfrak{S}_p(M) \rightarrow \mathfrak{S}_p(R) \) defined by

\[
\theta(N) = (N : M),
\]

for every saturated submodule \( N \) of \( M \). It will be convenient for us to call the module \( M \) an \( \eta \)-module (resp. a \( \theta \)-module) in case the above mapping \( \eta \) (resp. \( \theta \)) is a lattice homomorphism.

In this paper, we investigate conditions under which \( \eta \) and \( \theta \) are lattice homomorphisms, in particular considering when \( \eta \) and \( \theta \) are Boolean algebra homomorphisms. It is shown that modules over Prüfer domains (Corollary 2.4), projective modules (Corollary 2.6) and semisimple \( R \)-modules (Corollary 2.7) are three classes of \( \eta \)-modules. It is proved that if \( M \) is a faithful multiplication \( R \)-module, then \( \eta \) is a lattice epimorphism, and in particular \( \mathfrak{S}_p(M) \) is isomorphic to a quotient of \( \mathfrak{S}_p(R) \) (Theorem 2.8) for all prime ideals \( p \) of \( R \). It is shown that a finitely generated module \( M \) is a \( \theta \)-module if and only if it is a multiplication module (Corollary 2.11). In particular, every cyclic \( R \)-module is a \( \theta \)-module (Corollary 2.10). Moreover, if \( M \) is a finitely generated faithful multiplication \( R \)-module then \( \eta \) and \( \theta \) are lattice isomorphisms (Corollary 2.17).

An \( R \)-module \( M \) is called \textit{distributive} if \( \mathcal{L}(R)M \) is a distributive lattice (see, for example, \([8]\)). A ring \( R \) is called \textit{arithmetical} if it is a distributive \( R \)-module. We say that an \( R \)-module \( M \) is \( \mathfrak{S} \)-\textit{distributive} with respect to a prime ideal \( p \) of \( R \) if \( \mathfrak{S}_p(M) \) is a distributive
lattice. It is proved that an $R$-module $M$ is distributive if and only if it is $\mathcal{S}$-distributive with respect to any prime ideal of $R$ (Corollary 3.4). In particular, every multiplication module over an arithmetical ring $R$ is $\mathcal{S}$-distributive with respect to any prime ideal of $R$ (Corollary 3.5). It is shown that if $M$ is a distributive module over a semisimple ring $R$, then $\mathcal{S}_p(M)$ forms a Boolean algebra (Theorem 3.7) and $\eta$ is a Boolean algebra homomorphism (Theorem 3.13). In particular, if $M$ is a multiplication module over a semisimple ring $R$, then $\eta$ is a Boolean algebra epimorphism (Corollary 3.14).

2. $\eta$-modules and $\theta$-modules

We start with a lemma which collects some facts about saturation of submodules.

Lemma 2.1. Let $R$ be a ring, $p$ a prime ideal of $R$ and $M$ an $R$-module. Then

1. $S_p(L \cap N) = S_p(L) \cap S_p(N)$ for all submodules $L$ and $N$ of $M$;
2. $S_p(S_p(IM) + S_p(JM)) = S_p(S_p(I + J)M) = S_p(IM + JM)$ for all ideals $I$ and $J$ of $R$.

Proof. (1) Clear.

Proof. (2) Since $IM \subseteq (I + J)M \subseteq S_p(I + J)M$, we conclude that $S_p(IM) \subseteq S_p(S_p(I + J)M)$. Similarly, $S_p(IM) \subseteq S_p(S_p(I + J)M)$. Therefore, we have $S_p(IM) + S_p(JM) \subseteq S_p(IM) + S_p(JM)$. Hence we have $S_p(S_p(IM) + S_p(JM)) \subseteq S_p(S_p(I + J)M)$. Now, let $x \in S_p(S_p(I + J)M)$. Then there exists $c \in R \setminus p$ such that $cx \in S_p(I + J)M$. Therefore $cx = \sum_{i=1}^{k} r_i x_i$ for some $r_i \in S_p(I + J)$ and $x_i \in M$ ($1 \leq i \leq k$). Thus there are $c_i \in R \setminus p$ ($1 \leq i \leq k$) such that $c_i r_i \in I + J$, and so $c_1 \ldots c_k x \in (I + J)M$. It follows that $x \in S_p((I + J)M)$. Hence we have $S_p(S_p(I + J)M) \subseteq S_p(IM + JM)$. It is also clear that $S_p(IM + JM) \subseteq S_p(S_p(IM) + S_pJM))$.

Theorem 2.2. Let $R$ be a ring, $p$ a prime ideal of $R$ and $M$ an $R$-module. Then the following statements are equivalent:

1. $M$ is an $\eta$-module over $R$;
2. $S_p((I \cap J)M) = S_p(IM) \cap S_p(JM)$ for all ideals $I$ and $J$ of $R$;
3. $(I_p \cap J_p)M_p = I_pM_p \cap J_pM_p$ for all ideals $I$ and $J$ of $R$;
4. $M_p$ is a $\lambda$-module over $R_p$.

Proof. (1) $\Rightarrow$ (2) By definition.

(2) $\Rightarrow$ (1) Let $I, J \in \mathcal{S}_p(R)$. By the assumption, $\eta(I \wedge J) = \eta(I) \wedge \eta(J)$.

By using Lemma 2.1, we have

$$\eta(I \wedge J) = S_p((I \wedge J)M) = S_p(S_p(I + J)M) = S_p(S_p(IM) + S_p(JM)) = S_p(IM) \vee S_p(JM) = \eta(I) \wedge \eta(J).$$

(2) $\Rightarrow$ (3) Let $z \in I_pM_p \cap J_pM_p$. Then $z = \sum_{i=1}^{k} a_i x_i/s_i = \sum_{i=1}^{k} b_i y_i/t_i$ for some $a_i \in I$, $b_i \in J$, $x_i, y_i \in M$, $s_i, t_i \in R \setminus p$. Hence we have $s_1 \ldots s_k t_1 \ldots t_k z \in IM \cap JM$ which follows that $z \in S_p(IM) \cap S_p(JM)$. Therefore by (2), $z \in S_p((I \cap J)M)$. Thus $cz \in (I \cap J)M$ for some $c \in R \setminus p$, and so $z \in (I_p \cap J_p)M_p$ as desired. The reverse inclusion is clear.

(3) $\Rightarrow$ (2) Let $x \in S_p(IM) \cap S_p(JM)$. Then $cx \in IM$ and $dx \in JM$ for some $c, d \in R \setminus p$. Therefore $cx = \sum_{i=1}^{k} c_i x_i$ and $dx = \sum_{j=1}^{k} d_j x_j'$ for some $c_i \in I$, $d_j \in J$ and $x_i, x_j' \in M$ ($1 \leq i, j \leq k$). Thus $c_1 dx = \sum_{j=1}^{k} c_1 d_j x_j'$ and hence $c_1 dx \in (I \cap J)M$ such that $c_1 d \in R \setminus p$. Thus $x \in S_p((I \cap J)M)$. The reverse inclusion is clear.

(3) $\Leftrightarrow$ (4) Follows from [22, Lemma 2.1 (ii)].

\[\Box\]
Let $R$ be a domain with the field of fractions $K$. A non-zero ideal $I$ of $R$ is called invertible provided $I^{-1} = R$ where $I^{-1} = \{ k \in K : kI \subseteq R \}$. A domain $R$ is called Prüfer if every non-zero finitely generated ideal of $R$ is invertible (see, for more details, [13]).

**Corollary 2.3.** Let $R$ be a domain, $p$ a prime ideal of $R$ and $M$ an $R$-module. Then the following statements are equivalent:

1. $R_p$ is Prüfer;
2. Every $R_p$-module is a $\lambda$-module;
3. Every $R$-module is an $\eta$-module.

**Proof.** (1) $\iff$ (2) By [22, Theorem 2.3].
(2) $\iff$ (3) By Theorem 2.2. □

**Corollary 2.4.** Let $R$ be any Prüfer domain. Then every $R$-module is an $\eta$-module.

**Proof.** Let $R$ be a Prüfer domain and $p$ be a prime ideal of $R$. Then by [13, Theorem 6.6], $R_p$ is a valuation ring. Thus by [22, Proposition 2.4], every $R_p$-module is a $\lambda$-module and hence by Corollary 2.3, every $R$-module is an $\eta$-module. □

**Theorem 2.5.** Let $R$ be any ring. Then

1. Every direct summand of an $\eta$-module is an $\eta$-module.
2. Every direct sum of $\lambda$-modules is an $\eta$-module.

**Proof.** (1) Let $K$ be a direct summand of an $\eta$-module $M$. Let $I$ and $J$ be any ideals of $R$ and $p$ be a prime ideal of $R$. Then by Lemma 2.1 (1) and Theorem 2.2, we have

\[
S_p(IK) \cap S_p(JK) = S_p(K \cap IM) \cap S_p(K \cap JM) \\
= S_p(K) \cap S_p(IM) \cap S_p(JM) \\
= S_p(K) \cap S_p((I \cap J)M) \\
= S_p(K \cap (I \cap J)M) \\
= S_p((I \cap J)K).
\]

Thus by Theorem 2.2, $K$ is an $\eta$-module.

(2) Let $M_i (i \in I)$ be any collection of $\lambda$-modules and let $M = \oplus_{i \in I} M_i$. Given any ideals $I$ and $J$ of $R$, by [22, Lemma 2.1], we have

\[
S_p(IM) \cap S_p(JM) = S_p(\oplus_{i \in I} IM_i) \cap S_p(\oplus_{i \in I} JM_i) \\
= S_p(\oplus_{i \in I} IM_i \cap \oplus_{i \in I} JM_i) \\
= S_p(\oplus_{i \in I} (IM_i \cap JM_i)) \\
= S_p(\oplus_{i \in I} (I \cap J)M_i) \\
= S_p((I \cap J)M).
\]

Thus by Theorem 2.2, $M$ is an $\eta$-module. □

**Corollary 2.6.** For any ring $R$, every projective $R$-module is an $\eta$-module.

**Proof.** By [22, Lemma 2.1], every ring $R$ is a $\lambda$-module. Thus by [10, Theorem IV.2.1] and Theorem 2.5(2), every free $R$-module is an $\eta$-module, and therefore by [10, Theorem IV.3.4] and Theorem 2.5(1), every projective $R$-module is an $\eta$-module. □

**Corollary 2.7.** For any ring $R$, every semisimple $R$-module is an $\eta$-module.

**Proof.** Clearly every simple module is a $\lambda$-module. Since any semisimple module is a direct sum of a family of simple submodules, the result follows from Theorem 2.5(2). □
An $R$-module $M$ is called a multiplication module if the mapping $\lambda$ is surjective, i.e., 
for each submodule $N$ of $M$ there exist an ideal $I$ of $R$ such that $N = IM$. In this case, 
we can take $I = (N : M)$ (see, for example, [4,7]).

**Theorem 2.8.** Let $M$ be a faithful multiplication $R$-module. Then $\eta$ is a lattice epimorphism. 
In particular, $\mathcal{E}_p(M)$ is isomorphic to a quotient of $\mathcal{E}_p(R)$ for all prime ideals $p$ of $R$.

**Proof.** Since $M$ is a faithful multiplication $R$-module, $M$ is a $\lambda$-module by [22, Theorem 2.12]. Thus by [22, Lemma 2.1], $(I \cap J)M = IM \cap JM$ for all ideals $I$ and $J$ of $R$. It follows that, by Lemma 2.1 (1),

$$S_p((I \cap J)M) = S_p(IM \cap JM) = S_p(IM) \cap S_p(JM)$$

for all ideals $I$ and $J$ and prime ideals $p$ of $R$. Hence by Theorem 2.2, $\eta$ is a lattice homomorphism. Now, let $p$ be a prime ideal of $R$ and $N \in \mathcal{E}_p(M)$. Since $M$ is a multiplication module, we have

$$\eta((N : M)) = S_p((N : M)M) = S_p(N) = N$$

and therefore $\eta$ is an epimorphism. Now, we define the relation $\sim$ on $\mathcal{E}_p(R)$ by

$$I \sim J \iff S_p(IM) = S_p(JM).$$

It is evident that $\sim$ is an equivalence relation on $\mathcal{E}_p(R)$. We show that $\sim$ is a congruence relation. Assume that $I_1 \sim I_2$ and $I_2 \sim J_2$. Thus we have $S_p(I_1M) = S_p(J_1M)$ and $S_p(I_2M) = S_p(J_2M)$. Since $M$ is a faithful multiplication module,

$$S_p((I_1 \cap J_1)M) = S_p(I_1M) \cap S_p(J_1M)$$

$$= S_p(I_2M) \cap S_p(J_2M)$$

$$= S_p((I_2 \cap J_2)M),$$

and therefore $I_1 \cap J_1 \sim I_2 \cap J_2$. Also, by Lemma 2.1 (2),

$$S_p(S_p(I_1 + J_1)M) = S_p(S_p(I_1M) + S_p(J_1M))$$

$$= S_p(S_p(I_2M) + S_p(J_2M))$$

$$= S_p(S_p(I_2 + J_2)M)$$

which follows that $I_1 \lor J_1 \sim I_2 \lor J_2$. Thus $\mathcal{E}_p(R)/\sim$, the set of equivalence classes with respect to $\sim$, is a lattice with the following operations:

$$I \lor J/\sim = I \lor J/\sim \quad \text{and} \quad I \land J/\sim = I \land J/\sim.$$

Now, the mapping $\bar{\eta} : \mathcal{E}_p(R)/\sim \to \mathcal{E}_p(M)$ given by $\bar{\eta}(I/\sim) = \eta(I) = S_p(IM)$ is a lattice isomorphism. \hfill \Box

Recall that $\theta : \mathcal{E}_p(M) \to \mathcal{E}_p(R)$ defined by $\theta(N) = (N : M)$ is the restriction of the mapping $\mu : \mathcal{L}(R)M \to \mathcal{L}(R)$ to $\mathcal{E}_p(M)$ given in [22]. Thus every $\mu$-module is a $\theta$-module.

**Theorem 2.9.** Let $R$ be a ring and $M$ an $R$-module. Consider the following statements:

1. $M$ is a $\theta$-module over $R$;
2. $(L + N : M) = (L : M) + (N : M)$ for all saturated submodules $L$ and $N$ of $M$;
3. $(L_p + N_p : M_p) = (L_p : M_p) + (N_p : M_p)$ for all submodules $L$ and $N$ of $M$ and for all prime ideals $p$ of $R$;
4. $(L + N : M) = (L : M) + (N : M)$ for all submodules $L$ and $N$ of $M$;
5. $M$ is a $\mu$-module over $R$.

Then (1) $\iff$ (2) and (4) $\iff$ (5).

In particular, if $M$ is a finitely generated $R$-module, then all of the above statements are equivalent.
Proof. (1) $\Leftrightarrow$ (2) Follows from definition.
(4) $\Leftrightarrow$ (5) Follows from [22, Lemma 3.1].
(4) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (3) Suppose that $M$ is finitely generated. Then $M = Rm_1 + \ldots + Rm_k$ for some $m_i \in M$ ($1 \leq i \leq k$). Let $L$ and $N$ be two submodules of $M$. First we show that $(S_p(L) + S_p(N) : M)_p = ((L + N)_p : M_p)$ for all prime ideals $p$ of $R$. Let $p$ be a prime ideal of $R$ and assume that $r/1 \in (S_p(L) + S_p(N) : M)_p$. It follows that $rM \subseteq S_p(L) + S_p(N)$. Thus $rm_i = x_i + y_i$ for some $x_i \in S_p(L)$, $y_i \in S_p(N)$ ($1 \leq i \leq k$). Therefore $c_ix_i \in L$ and $d_iy_i \in N$ for some $c_i, d_i \in R \setminus \{ 0 \}$ ($1 \leq i \leq k$). Now, since $c_1 \ldots c_k d_1 \ldots d_k rM \subseteq L + N$, we have $r/1 \in ((L + N)_p : M_p)$, as requested. Hence, by using [15, Theorem 2.1], we have
\[
(L_p : M_p) + (N_p : M_p) = (S_p(L) : M)_p + (S_p(N) : M)_p = ((S_p(L) : M) + (S_p(N) : M))_p = (S_p(L) + S_p(N) : M)_p = ((L + N)_p : M_p) = (L_p + N_p : M_p).
\]
(3) $\Rightarrow$ (4) Follows from [3, Proposition 3.8 and Corollaries 3.4 and 3.15].
(4) $\Rightarrow$ (3) Follows from [3, Corollary 3.4 and Corollary 3.15].

Corollary 2.10. For any ring $R$, every cyclic $R$-module is a $\theta$-module.

Proof. Follows from [22, Corollary 3.7] and Theorem 2.9.

Corollary 2.11. Let $M$ be a finitely generated $R$-module. Then the following statements are equivalent:

1. $M$ is a $\theta$-module over $R$;
2. $M_p$ is a $\theta$-module over $R_p$ for every prime ideal $p$ of $R$;
3. $M_m$ is a $\theta$-module over $R_m$ for every maximal ideal $m$ of $R$;
4. $M$ is a $\mu$-module over $R$;
5. $M$ is a $\sigma$-module over $R$;
6. $M$ is a multiplication module over $R$.

Proof. (1) $\Leftrightarrow$ (4) By Theorem 2.9.
(4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) By [20, Theorem 2.11 and Theorem 2.19].
(6) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) By [4, Lemma 2 (ii)], [20, Theorem 2.11] and Theorem 2.9.

Corollary 2.12. Let $R$ be a ring. If $M$ is a finitely generated $\theta$-module over $R$ and $((0) : M) = Re$ for some idempotent $e$ of $R$, then $M$ is an $\eta$-module over $R$. In particular, every finitely generated faithful $\theta$-module is an $\eta$-module.

Proof. By Corollary 2.11 $M$ is a multiplication $R$-module, and then by [21, Theorem 11] $M$ is a projective $R$-module. Thus by Corollary 2.6, $M$ is an $\eta$-module over $R$.

Now, we investigate conditions under which $\eta$ and $\theta$ are injective or surjective.

Theorem 2.13. Let $\eta$ and $\theta$ be as before. Then

1. $\eta\theta\eta = \eta$;
2. $\theta\eta\theta = \theta$.

Proof. (1) Let $p$ be a prime ideal of $R$ and $I \in \mathfrak{S}_p(R)$. Since $\eta\theta\eta(I) = S_p((S_p(IM) : M)M)$, we must show that $S_p((S_p(IM) : M)M) = S_p(IM)$. First note that, since $I \subseteq (S_p(IM) : M)$, we have $IM \subseteq (S_p(IM) : M)M$ and thus $S_p(IM) \subseteq S_p((S_p(IM) : M)M)$. The reverse inclusion follows from $S_p((S_p(IM) : M)M) \subseteq S_p(S_p(IM)) = S_p(IM)$.
(2) Let $p$ be a prime ideal of $R$ and $N \in \mathfrak{S}_p(M)$. Now, since $\theta \eta(N) = (S_p((N : M)M) : M)$, we must show that $(S_p((N : M)M) : M) = (N : M)$. Since $(N : M)M \subseteq S_p((N : M)M)$, we have $(N : M) \subseteq (S_p((N : M)M) : M)$. The reverse inclusion follows from

$$(S_p((N : M)M) : M) \subseteq (S_p(N) : M) = (N : M).$$

□

Corollary 2.14. Let $\eta$ and $\theta$ be as before, and $p$ be a prime ideal of $R$. Then the following statements are equivalent:

1. $\eta : \mathfrak{S}_p(R) \to \mathfrak{S}_p(M)$ is a surjection;
2. $\eta \theta = 1$;
3. $S_p((N : M)M) = N$ for all $N \in \mathfrak{S}_p(M)$;
4. $\theta : \mathfrak{S}_p(M) \to \mathfrak{S}_p(R)$ is an injection.

Proof. $(1) \Rightarrow (2)$ and $(4) \Rightarrow (2)$ follows from Theorem 2.13.

$(2) \iff (3)$, $(2) \Rightarrow (1)$ and $(2) \Rightarrow (4)$ are clear. □

Corollary 2.15. Let $\eta$ and $\theta$ be as before, and $p$ be a prime ideal of $R$. Then the following statements are equivalent:

1. $\eta : \mathfrak{S}_p(R) \to \mathfrak{S}_p(M)$ is an injection;
2. $\theta \eta = 1$;
3. $(S_p(IM) : M) = I$ for all $I \in \mathfrak{S}_p(R)$;
4. $\theta : \mathfrak{S}_p(M) \to \mathfrak{S}_p(R)$ is a surjection.

Proof. $(1) \Rightarrow (2)$ and $(4) \Rightarrow (2)$ follows from Theorem 2.13.

$(2) \iff (3)$, $(2) \Rightarrow (1)$ and $(2) \Rightarrow (4)$ are clear. □

Corollary 2.16. Let $\eta$ and $\theta$ be as before. Then $\eta$ is a bijection if and only if $\theta$ is a bijection. In this case $\eta$ and $\theta$ are inverse of each other.

Proof. By Corollaries 2.14 and 2.15. □

Corollary 2.17. Let $R$ be a ring and $M$ be a finitely generated faithful multiplication $R$-module. Then the mappings $\eta$ and $\theta$ are lattice isomorphisms. In particular, $\eta$ and $\theta$ are inverse of each other, and therefore $\mathfrak{S}_p(R)$ and $\mathfrak{S}_p(M)$ are isomorphic lattices for all prime ideals $p$.

Proof. Since $M$ is a faithful multiplication $R$-module, $\eta$ is an epimorphism by Theorem 2.8, and hence $\theta$ is a monomorphism by Corollary 2.14 and [22, Theorem 3.8]. On the other hand, by [15, Proposition 3.2], we have

$$(S_p(IM) : M) = S_p(IM : M) = S_p(I) = I,$$

for all prime ideals $p$ of $R$ and $I \in \mathfrak{S}_p(R)$. Hence, by Corollary 2.15, $\eta$ is an injection and $\theta$ is a surjection. Hence $\eta$ is an isomorphism and its inverse is $\theta$. □

3. $\mathfrak{S}_p(M)$ as a Boolean algebra

We start this section by recalling the following basic definition.

Definition 3.1. Let $R$ be a ring and $p$ be a prime ideal of $R$. An $R$-module $M$ is called a $\mathfrak{S}$-distributive module with respect to $p$, if $\mathfrak{S}_p(M)$ is a distributive lattice.

First note the following simple fact.

Lemma 3.2. Let $R$ be a ring, $p$ a prime ideal of $R$ and $M$ be an $R$-module. Then the following statements are equivalent:

1. $M$ is $\mathfrak{S}$-distributive with respect to $p$;
2. $K \cap S_p(L + N) = S_p((K \cap L) + (K \cap N))$ for all $K, L, N \in \mathfrak{S}_p(M)$;
(3) \( S_p(K + (L \cap N)) = S_p(K + L) \cap S_p(K + N) \) for all \( K, L, N \in \mathcal{S}_p(M) \).

**Proof.** By [5, Theorem I.3.2]. □

The following example shows that a ring \( R \) may be \( \mathcal{S} \)-distributive with respect to a prime ideal and not with respect to another one.

**Example 3.3.** Let \( R = K[X, Y] \) be the ring of polynomials with independent indeterminates \( X \) and \( Y \) over a field \( K \). It is evident that \( R \) is \( \mathcal{S} \)-distributive with respect to \((0)\), since \( \mathcal{S}_{(0)}(R) = \{ (0), R \} \). However, \( R \) is not \( \mathcal{S} \)-distributive with respect to \( m = RX + RY \).

Let \( p_1 = RX, p_2 = RY, p_3 = R(X + Y) \). Since \( p_1, p_2 \) and \( p_3 \) are prime ideals of \( R \), these ideals are saturated with respect to \( m \) and hence \( p_3 \cap p_1 \) and \( p_3 \cap p_2 \) are saturated with respect to \( m \) by Lemma 2.1 (1). Now, since \( p_3 \cap (p_1 + p_2) \not\subseteq (p_3 \cap p_1) + (p_3 \cap p_2) \), \( R \) is not \( \mathcal{S} \)-distributive with respect to \( m \) by Lemma 3.2.

It is remarked that some classes of \( R \)-modules are characterized by using the localization with respect to all prime ideal of \( R \) (see for example [1]). In the next result, it is seen that the class of distributive modules has this property.

**Corollary 3.4.** Let \( R \) be a ring and \( M \) be an \( R \)-module. Then the following conditions are equivalent:

1. \( M \) is a distributive \( R \)-module;
2. \( M \) is \( \mathcal{S} \)-distributive with respect to any prime ideal \( p \) of \( R \);
3. \( M_p \) is a distributive \( R_p \)-module for all prime ideals \( p \) of \( R \).

**Proof.** (1) ⇒ (2) Let \( p \) be a prime ideal of \( R \) and \( K, L, N \in \mathcal{S}_p(M) \). By Lemma 2.1 (1) and the assumption, we have

\[
S_p(K + L) \cap S_p(K + N) = S_p((K + L) \cap (K + N)) = S_p(K + (L \cap N)).
\]

Thus, the result follows from Lemma 3.2 (3).

(2) ⇒ (3) Let \( p \) be a prime ideal of \( R \) and \( K, L \) and \( N \) be submodules of \( M \). It suffices to show that \((K_p + L_p) \cap (K_p + N_p) \subseteq (K_p + (L_p \cap N_p)) \) or equivalently, by [3, Corollary 3.4], \((K + L) \cap (K + N))_p \subseteq (K + (L \cap N))_p \). For this, let \( x/s \in ((K + L) \cap (K + N))_p \) so that \( x/s = (k_1 + l)/s_1 = (k_2 + n)/s_2 \). It follows that \( us_1s_2x = (k_1 + l) = (k_2 + n) \) for some \( u \in R \setminus p \) so that \( x \in S_p(K + L) \cap S_p(K + N) \). Hence by (2), \( x \in S_p(K + (L \cap N)) \). Therefore \( cx \in K + (L \cap N) \) for some \( c \in R \setminus p \) which implies that \( x/s = cx/cs \in (K + (L \cap N))_p \), as required.

(3) ⇒ (1) Follows from [3, Corollary 3.4 and Proposition 3.8]. □

**Corollary 3.5.** Let \( R \) be an arithmetical ring, and \( M \) be a multiplication \( R \)-module. Then \( M \) is a \( \mathcal{S} \)-distributive \( R \)-module with respect to any prime ideal of \( R \).

**Proof.** By [8, Proposition 1.2] and Corollary 3.4. □

Our next example shows that \( M \) being a multiplication module is needed in Corollary 3.5.

**Example 3.6.** Let \( K \) be a field and \( V = K \oplus K \) be the usual two-dimensional vector space over \( K \). It is easy to see that every subspace of \( V \) is saturated with respect to \((0)\).

Now if \( W_1 = K(1, 0), W_2 = K(0, 1) \) and \( W_3 = K(1, 1) \). Then \( W_3 \cap (W_1 + W_2) = W_3 \) while \( (W_3 \cap W_1) + (W_3 \cap W_2) = K(0, 0) \). Thus \( V \) is not \( \mathcal{S} \)-distributive.

We recall that a distributive lattice \((L, \lor, \land)\) is a Boolean algebra if there is a unary operation \( ' \) on \( L \) and two constants 0 and 1 such that \( x \land x' = 0 \) and \( x \lor x' = 1 \).

Let \( M \) be a semisimple \( R \)-module and \( N \) a submodule of \( M \). Then, by definition, there is a submodule \( L \) of \( M \) such that \( M = N \oplus L \). We define the unary operation \( ' \) on \( \mathcal{S}_p(M) \) by \( N' = S_p(L) \).
Theorem 3.7. Let $R$ be a semisimple ring, $p$ a prime ideal of $R$ and $M$ a distributive $R$-module. Then the lattice $\mathcal{S}_p(M)$ is a Boolean algebra with the unary operation $'$ defined above, $0 = S_p(0)$ and $1 = M$.

**Proof.** By Corollary 3.4, $M$ is a $\mathcal{S}$-distributive $R$-module. By using Lemma 2.1 (1),
$$N \cap N' = N \cap S_p(N) \cap S_p(L) = S_p(N \cap L) = S_p(0) = 0.$$  
Moreover, $M = N + L \subseteq S_p(N) + S_p(L) \subseteq S_p(S_p(N) + S_p(L))$, which implies
$$N \lor N' = S_p(N + N') = S_p(S_p(N) + S_p(L)) = M.$$  
Hence $\mathcal{S}_p(M)$ is a Boolean algebra. □

From now on, $\mathcal{S}_p(M)$ is assumed to be a Boolean algebra with the above assumptions.

Corollary 3.8. For any semisimple ring $R$, $\mathcal{S}_p(R)$ is a Boolean algebra with respect to any prime ideal $p$ of $R$.

**Proof.** Let $R$ be a semisimple ring and $p$ a prime ideal of $R$. By [12, Exercise 1.2.5] $R$ is an arithmetical ring. Thus by Theorem 3.7, $\mathcal{S}_p(R)$ is a Boolean algebra. □

Corollary 3.9. Let $R$ be a semisimple ring and $M$ be a distributive $R$-module. Then $\mathcal{S}_p(M)$ is a Boolean ring with the following operations:

$$L + N = S_p(L \cap S_p(\bar{N}) + S_p(\bar{L}) \cap N) \quad \text{and} \quad L \cdot N = L \cap N,$$

where $M = L \oplus \bar{L} = N \oplus \bar{N}$.

**Proof.** Follows from Theorem 3.7 and [5, Theorem IV.2.3]. □

Corollary 3.10. Let $R$ be a semisimple ring, $p$ a prime ideal of $R$ and $M$ a multiplication $R$-module. Then $M$ is cyclic and the lattice $\mathcal{S}_p(M)$ is a Boolean algebra.

**Proof.** Since $R$ is a semisimple ring, by [12, Corollary 2.6], $R$ is an Artinian ring. Hence $M$ is cyclic by [7, Corollary 2.9]. Also, by [12, Exercise 1.2.5], $R$ is an arithmetical ring. Thus by [8, Proposition 1.2], $M$ is a distributive $R$-module. Hence by Theorem 3.7, $\mathcal{S}_p(M)$ is a Boolean algebra with respect to any prime ideal $p$ of $R$. □

Theorem 3.11. Let $R$ be a ring, $p$ a prime ideal of $R$, $M$ an $R$-module and $N$ a submodule of $M$. Then the followings hold:

1. For any submodule $L$ containing $N$, $S_p(L/N) = S_p(L)/N$. In particular, the assignment $L \mapsto L/N$ is a one to one corresponding between the set $\{L \mid L \leq M \in \mathcal{S}_p(M), L \supseteq N\}$ and $\mathcal{S}_p(M/N)$;
2. If $M$ is a $\mathcal{S}$-distributive lattice over $R$ with respect to $p$, then $M/N$ is $\mathcal{S}$-distributive over $R$ with respect to $p$;
3. If $R$ is a semisimple ring and $M$ a distributive $R$-module, then $\mathcal{S}_p(M/N)$ is a Boolean algebra.

**Proof.**
1. Clear.
2. Let $\mathcal{S}_p(M)$ be a distributive lattice with the operations $\lor$ and $\land$ and $\mathcal{S}_p(M/N)$ be a lattice with the operations $\lor$ and $\land$. It is seen that $\lor$ and $\land$ are expressed by $\lor$ and $\land$ respectively as follows:

$$L/N \lor K/N = S_p(L/N + K/N) = S_p((L + K)/N) = (L \lor K)/N,$$

and

$$L/N \land K/N = L/N \cap K/N = (L \cap K)/N = (L \land K)/N.$$
By these statements, the distributivity of \( \mathfrak{S}_p(M/N) \) follows immediately from the distributivity of \( \mathfrak{S}_p(M) \).

(3) Follows from Theorem 3.7 and (2).

**Theorem 3.12.** Let \( R \) be a ring, \( T \) a multiplicatively closed subset of \( R \), \( M \) an \( R \)-module and \( N \) a submodule of \( M \). Then the followings hold:

\( \begin{align*}
(1) \quad S_{T^{-1}}(T^{-1}N) &= T^{-1}(S_p(N)) \quad \text{for all prime ideals} \ p \ \text{disjoint from} \ T. \quad \text{In particular,} \\
& N \in \mathfrak{S}_p(M) \quad \text{if and only if} \ T^{-1}N \in \mathfrak{S}_{T^{-1}p}(T^{-1}M) \quad \text{for all prime ideals} \ p \ \text{disjoint from} \ T; \\
(2) \quad \text{If} \ M \quad \text{is a} \ \mathfrak{S} \text{-distributive lattice over} \ R \quad \text{with respect to a prime ideal} \ p \quad \text{such that} \ p \cap T = \emptyset, \quad \text{then} \ T^{-1}M \quad \text{is} \ \mathfrak{S} \text{-distributive over} \ T^{-1}R \quad \text{with respect to} \ T^{-1}p; \\
(3) \quad \text{If} \ R \quad \text{is a semisimple ring,} \ p \quad \text{a prime ideal of} \ R \quad \text{with} \ p \cap T = \emptyset \quad \text{and} \ M \quad \text{a distributive} \ \mathfrak{S} \text{-module, then} \ \mathfrak{S}_{T^{-1}p}(T^{-1}M) \quad \text{is a Boolean algebra.}
\end{align*} \)

**Proof.** (1) Clear.

(2) Let \( p \) be a prime ideal of \( R \) such that \( p \cap T = \emptyset \). Let \( \mathfrak{S}_p(M) \) be a distributive lattice with the operations \( \lor \) and \( \land \) and \( \mathfrak{S}_{T^{-1}p}(T^{-1}M) \) be a lattice with the operations \( \tilde{\lor} \) and \( \tilde{\land} \). It is seen that \( \tilde{\lor} \) and \( \tilde{\land} \) are expressed by \( \lor \) and \( \land \) respectively as follows:

\[
T^{-1}L \tilde{\lor} T^{-1}N = S_{T^{-1}p}(T^{-1}L + T^{-1}N) = S_{T^{-1}p}(T^{-1}(L + N)) = T^{-1}(S_p(L + N)) = T^{-1}(L \lor N),
\]

and

\[
T^{-1}L \tilde{\land} T^{-1}N = T^{-1}L \land T^{-1}N = T^{-1}(L \land N).
\]

By these statements, the distributivity of \( \mathfrak{S}_{T^{-1}p}(T^{-1}M) \) follows immediately from the distributivity of \( \mathfrak{S}_p(M) \).

(3) Since \( R \) is a semisimple ring, then so is \( T^{-1}R \). Thus the result follows from Theorem 3.7 and (2). \( \Box \)

Let \( A \) and \( B \) be Boolean algebras. A function \( f : A \to B \) is called a **Boolean algebra homomorphism**, if \( f \) is a lattice homomorphism, \( f(0) = 0, f(1) = 1 \) and \( f(a') = f(a)' \) for all \( a \in A \). It is easily proved that a lattice homomorphism \( f \) preserves \( 0 \) and \( 1 \) if and only if it preserves \( ' \). Thus, in order to show that a function \( f \) between two Boolean algebras is a Boolean algebra homomorphism, it suffices to check that \( f \) preserves lattice operations \( \lor \) and \( \land \) and constants \( 0, 1 \).

**Theorem 3.13.** Let \( R \) be a semisimple ring, \( p \) a prime ideal of \( R \) and \( M \) a distributive \( R \)-module. Then \( \eta : \mathfrak{S}_p(R) \to \mathfrak{S}_p(M) \) is a Boolean algebra homomorphism.

**Proof.** First note that \( \mathfrak{S}_p(M) \) and \( \mathfrak{S}_p(R) \) are Boolean algebras, by Theorem 3.7 and Corollary 3.8 respectively. By Corollary 2.7, \( \eta \) is a lattice homomorphism. Also, \( \eta(0) = \eta(S_p(0)) = S_p(S_p(0)M) = S_p(0) = 0, \)

and

\( \eta(1) = \eta(R) = S_p(RM) = S_p(M) = M = 1. \)

Hence, as noted above, \( \eta \) is a Boolean algebra homomorphism. \( \Box \)

**Corollary 3.14.** Let \( R \) be a semisimple ring, \( p \) a prime ideal of \( R \) and \( M \) a multiplication \( R \)-module. Then \( \eta : \mathfrak{S}_p(R) \to \mathfrak{S}_p(M) \) is a Boolean algebra epimorphism.
Proof. By Corollaries 3.8 and 3.10, $\mathcal{S}_p(R)$ and $\mathcal{S}_p(M)$ are Boolean algebras respectively. Also, by the proof of Corollary 3.10, $M$ is distributive. Thus by Theorem 3.13, $\eta$ is a Boolean algebra homomorphism. Moreover, if $N \in \mathcal{S}_p(M)$, then $(N : M) \in \mathcal{S}_p(R)$ and
$$\eta(N : M) = S_p((N : M)M) = S_p(N) = N.$$  
Thus, $\eta$ is an epimorphism. \hfill $\square$

Finally, we remark that if $M$ is a faithful multiplication module over a semisimple ring $R$, then since $M$ is cyclic by Corollary 3.10, we conclude that $M$ is isomorphic to $R$. So it clearly follows that $\eta$ and $\theta$ are Boolean algebra isomorphisms.

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