A simple solvable model of body motion in a one-dimensional resistive medium

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Abstract

We introduce and solve in closed form a simple model of a macroscopic body propagating in a one-dimensional resistive medium at temperature \( T \). The assumption of completely inelastic collisions between the body and the particles composing the medium leads to a resistive force that is opposite and proportional to the square of the body’s velocity.

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The topic of macroscopic bodies moving through resistive media, such as air or viscous fluids, gives rise to one aspect that students of Introductory Physics courses find rather mysterious: The origin of the ‘force law’ that describes the effective force on the moving body as it propagates through the resistive medium. The student is usually told that the effective force on the body is either proportional to the speed of the body or to the square of the body’s speed, according to whether the body has, or does not have, a small cross-sectional area, or whether it is moving at low or high speeds\(^1\). Under further questioning, the instructor might tell the student that these ‘laws’ are based on ‘experimental observations’ which are difficult to obtain analytically. There are basic models, however, that show in a simple manner how the energy and momentum exchange between the moving body and the particles composing the resistive medium lead to some of these ‘force laws’. In this article, we present an extremely simplified model that leads to a very well-known ‘force law’: a resistive force that is opposite and proportional to the square of the body’s velocity: \(F = -\gamma V^2\).

Consider a body, represented by a heavy point ‘particle’ of mass \(M_0\) and initial speed \(V_0\), that propagates inside a one-dimensional medium composed of identical point particles of mass \(m\), with \(m \ll M_0\) which are in thermal equilibrium at temperature \(T\) (Fig.1). We will consider here the ‘short’ time scale where the body does not have enough time to reach thermal equilibrium with the surrounding medium. The ‘brownian motion’ case where the body is in thermal equilibrium with the medium have been nicely discussed by de Groot\(^3\). Let us denote by \(v_j\), the velocity of the \(j\)th medium particle. Since
the medium is one-dimensional the particles can be labelled unambiguously. For instance, the particles to the right of the body could be labelled by odd values of $j$, while the ones to the left, by even $j$ values. Because of thermal equilibrium the $\{v_j\}$ are random quantities whose values are taken from a gaussian distribution of width proportional to the medium temperature $T$. We will assume for simplicity that the body undergoes completely inelastic collisions with the medium particles.

After the first collision we have, because of momentum conservation,

$$M_0 V_0 + m v_1 = (M_0 + m) V_1$$

i.e., the speed of the body after its first collision is

$$V_1 = \left( \frac{M_0}{M_0 + m} \right) V_0 + \left( \frac{m}{M_0 + m} \right) v_1,$$

where $v_1$ denotes the velocity of the medium particle with which the body collides first (this particle could come from the left or right of $M$). Some time afterwards, the body (now with mass $M_0 + m$) will suffer a second inelastic collision from which will emerge with velocity:

$$V_2 = \left( \frac{M_0}{M_0 + 2m} \right) V_0 + \left( \frac{m}{M_0 + 2m} \right) (v_1 + v_2).$$

where $v_2$ is velocity of the medium particle who suffers the second collision with $M_0$, and so on. After $n$ of these collisions, the speed of the body will be

$$V_n = \left( \frac{M_0}{M_0 + nm} \right) V_0 + \left( \frac{m}{M_0 + nm} \right) \sum_{j=1}^{n} v_j,$$

where we remind the reader that the $\{v_j\}$ are random with $\langle v_j \rangle = 0$, $\langle v_j^2 \rangle = kT/m$ and $\langle ... \rangle$ denotes a thermal average. This implies,

$$\langle V_n \rangle = \left( \frac{M_0}{M_0 + nm} \right) V_0.$$  (1)
On the other hand,

\[
\langle V_n^2 \rangle = \frac{M_0^2 V_0^2}{(M_0 + nm)^2} + \frac{2mM_0 V_0}{(M_0 + nm)^2} \left\langle \sum_i v_i \right\rangle + \frac{m^2}{(M_0 + nm)^2} \left\langle \sum_{i,j} v_i v_j \right\rangle
\]

\[
= \langle V_n \rangle^2 + \frac{n m k T}{(M_0 + nm)^2}
\]

\[
= \left(1 - \frac{k T}{M_0 V_0^2}\right) \langle V_n \rangle^2 + \frac{k T}{M_0 V_0} \langle V_n \rangle.
\]

(2)

We note that, as the number of collisions tends to infinity (i.e., after a ‘long’ time), \(\langle V_n^2 \rangle \to (kT/M_0 V_0)\langle V_n \rangle = (kT/M(n))\), where \(M(n) = M_0 + nm\) is the effective body mass after \(n\) collisions. This is nothing else but equipartition: \(M(n)\langle V_n^2 \rangle \to kT = m\langle v^2 \rangle\), where \(v\) is the velocity of a medium particle.

If we now assume that \(\rho\), the density of medium particles per unit length is constant, then we can express \(n\) as \(n = \rho x\) where \(x\) is the distance travelled by the body between its first and \(n\)-th collision. We are assuming here, as in hydrodynamics, that an element of length \(\Delta x\) while ‘small’ will contain a great number of medium particles. By re-expressing \(n\) in terms of \(x\) in (1), we can express the average velocity of the body after it has travelled a distance \(x\) as

\[
\langle V(x) \rangle = \left(\frac{M_0}{M_0 + \rho m x}\right) V_0,
\]

(3)

and the average of the velocity squared as

\[
\langle V(x)^2 \rangle = \left(1 - \frac{k T}{M_0 V_0^2}\right) \langle V(x) \rangle^2 + \frac{k T}{M_0 V_0} \langle V(x) \rangle.
\]

(4)

The average velocity decreases monotonically with distance. Its explicit time dependence can be found from (3): \(dx/dt = M_0 V_0/(M_0 + \rho m x)\), which can be integrated to give

\[
\frac{X(t)}{X_0} = -1 + \sqrt{1 + 2(t/t_0)}
\]

(5)
where $X_0 \equiv M_0/(\rho m)$ and $t_0 \equiv X_0/V_0$ constitute natural length and time scales. Finally, after replacing (5) into (3), or by direct differentiation of (5), one obtains

$$\langle V(t) \rangle = \frac{1}{\sqrt{1 + 2(t/t_0)}}$$

and

$$\frac{M(t)}{M_0} = 1 + \left(\frac{\rho m}{M_0}\right) X(t) = \sqrt{1 + (2t/t_0)}$$

is the effective body mass as a function of time. In Fig.2 we show $M(t)$, $X(t)$ and $V(t)$, all of which diverge at long times.

**Average resistive force.** As the body propagates, it is being hit from front and back by medium particles which stick completely to it after colliding. This accretion process is rather akin to the opposite process that occurs in the propulsion of a rocket engine: instead of expelling matter our body absorbs it. One process is the time-reversal of the other. The average effective force on the body can be directly computed from $\langle F \rangle = M(t)\langle V(t) \rangle/dt$. From Eqs.(6) and (7) one obtains:

$$\langle F \rangle = -\frac{M_0V_0}{t_0} \frac{1}{1 + 2(t/t_0)}$$

which can be recast as

$$\langle F \rangle = -\gamma V(t)^2$$

with $\gamma \equiv \rho m$.

Another way to compute $\langle F \rangle$ is to start from conservation of momentum during an infinitesimal collision, $M(x)V(x)+dM(x)v = (M(x)+dM(x))(V(x)+dV(x))$. This implies that the instantaneous force on the body is

$$M \frac{dV}{dt} = -\left(\frac{dM}{dt}\right)(V - v)$$
where \( v \) is random. The average force on the moving body will then be
\[
\langle F \rangle = - \left( \frac{dM}{dt} \right) \langle V \rangle = - \left( \frac{dM}{dx} \right) \langle V \rangle^2.
\] (11)

Since \( M(x) = M_0 + \rho m x \), we now have \( \langle F \rangle = -\gamma \langle V \rangle^2 \), with \( \gamma \equiv \rho m \) as before.

**Stopping power.** The stopping power \( \langle S \rangle \) of a medium is defined by the average energy per unit length, lost by a projectile while traversing a resistive medium:
\[
\langle S \rangle = \langle \frac{dE}{dx} \rangle = \frac{d}{dx} \left\{ \frac{1}{2} M(x) \langle V(x) \rangle^2 \right\}.
\] (12)

From Eqs. (4), (3) and the relations \((d/dx)\langle V(x) \rangle = -(\rho m/M_0 V_0)\langle V(x) \rangle^2\) and \( M(x) = M_0 V_0 / \langle V(x) \rangle \), we have
\[
\langle S \rangle = -\frac{\rho m}{2} \left( 1 - \frac{kT}{M_0 V_0^2} \right) \langle V(x) \rangle^2
\]
\[
= -\frac{\rho m}{2} \left( 1 - \frac{kT}{M_0 V_0^2} \right) \left( \frac{M_0}{M_0 + \rho m x} \right)^2 V_0^2
\]
\[
= -\frac{\rho m}{2M_0} \frac{(2E - kT)^2}{M_0 V_0^2 - kT}.
\] (13)

Figure 2 shows \( \langle S \rangle \) as a function of distance traversed inside the medium for several temperatures. Note that, since we are assuming the body’s initial kinetic energy to be higher than the average thermal energy, the body will always lose energy to the medium, on average. This energy loss becomes smaller and smaller as the body traverses the medium. Only after an infinite amount of time, or distance travelled, will the body’s average energy loss reach zero, where a thermalization process will occur.

In summary, we have introduced and solved in closed form the dynamics of a simple model of a body moving through a resistive medium. We find that
the effective resistive force is opposite and proportional to the square of the body’s speed\(^2\).
References

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2 A related model, where the collisions between the body and the medium particles is assumed to be completely elastic, leads to the same force law (but a different $\gamma$) and is reported in: M. I. Molina, “Body Motion in a One-Dimensional Resistive Medium”, M.I. Molina, Am. J. of Phys. 66, 973–975 (1998).

3 Bart G. de Grooth, “A Simple model for Brownian motion leading to the Langevin equation”, Am. J. Phys. 67, pp. 1248–1252.
Figure Captions

FIG 1: Macroscopic body of mass $M_0$ propagating inside a one-dimensional resistive medium composed by identical particles of mass $m << M_0$ in thermal equilibrium at temperature $T$.

FIG 2: Effective body mass, average velocity and distance travelled as a function of time, for body moving through our resistive medium ($S_0 \equiv \rho m V_0^2 / 2$).

FIG 3: Stopping power of the one-dimensional resistive medium as a function of the distance traversed by the body, for several medium temperatures.
Fig. 1
A graph showing the functions $M(t)/M$, $X(t)/X_0$, and $V(t)/V_0$ as a function of $t/t_0$. The graph illustrates the behavior of these quantities over the range of $t/t_0$ from 0.0 to 2.0.
FIG. 3