The Global Well-posedness of Strong Solutions to 2D MHD Equations in Lei-Lin Space

Bao-quan YUAN\(^1\), Ya-min XIAO\(^1\)

\(^1\)School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, China
(\(^1\)E-mail: bqyuan@hpu.edu.cn)

Abstract In this paper, we study the Cauchy problem of the 2D incompressible magnetohydrodynamic equations in Lei-Lin space. The global well-posedness of a strong solution in the Lei-Lin space \(\chi^{-1}(\mathbb{R}^2)\) with any initial data in \(\chi^{-1}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)\) is established. Furthermore, the uniqueness of the strong solution in \(\chi^{-1}(\mathbb{R}^2)\) and the Leray-Hopf weak solution in \(L^2(\mathbb{R}^2)\) is proved.

Keywords 2D MHD equations; strong solutions; Lei-Lin space; weak-strong uniqueness

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1 Introduction

In this paper, we consider the following incompressible magnetohydrodynamic equations

\[
\begin{aligned}
\partial_t u - \mu \Delta u + (u \cdot \nabla) u + \nabla p &= (b \cdot \nabla)b, \\
\partial_t b - \nu \Delta u + (u \cdot \nabla)b &= (b \cdot \nabla)u, \\
\nabla \cdot u &= \nabla \cdot b = 0, \\
u(0, x) &= u_0(x), \ b(0, x) = b_0(x),
\end{aligned}
\tag{1.1}
\]

for \(t \geq 0, \ x \in \mathbb{R}^2\). We denote \(u = u(t, x), \ b = b(t, x)\) and \(p = p(t, x)\) the velocity field, magnetic field and scalar pressure of the fluid, respectively. The positive constants \(\mu\) and \(\nu\) are the viscosity and the resistivity coefficients, and \(u_0(x)\) and \(b_0(x)\) are the initial velocity and initial magnetic field satisfying \(\text{div} u_0 = \text{div} b_0 = 0\), respectively.

When \(b = 0\), it reduces to the classical Navier-Stokes equation, which has been investigated extensively with many interesting results. Leray\(^8\) and Hopf\(^7\) established the global existence of weak solutions. Kato\(^6\), Fujita and Kato\(^5\) obtained the local well-posedness for any initial data and the global well-posedness for small initial data in \(L^n(\mathbb{R}^n)\) and \(\dot{H}^s(\mathbb{R}^n)\) with \(s \geq \frac{n}{2} - 1\), respectively. Recently, Lei and Lin\(^9\) constructed the global mild solution with small initial data in the critical space \(\chi^{-1}(\mathbb{R}^3)\), and Zhang and Yin\(^10\) studied the local well-posedness with large initial data and the global well-posedness with small initial data by the semi-group method.

Theorem A ([9] and [15]) Let \(u_0\) be in \(\chi^{-1}(\mathbb{R}^3)\). There exists a positive time \(T\) such that the Navier-Stokes equation has a unique solution \(u\) in \(L^2(0, T; \chi^0(\mathbb{R}^3))\) which also belongs to \(C([0, T]; \chi^{-1}(\mathbb{R}^3)) \cap L^1(0, T; \chi^1(\mathbb{R}^3)) \cap L^\infty(0, T; \chi^{-1}(\mathbb{R}^3))\).

Let \(T_{u_0}\) denote the maximal time of existence of such a solution. Then:

- There exists a constant \(C\) such that, if \(\|u_0\|_{\chi^{-1}} \leq C\), then

\[T_{u_0} = \infty.\]
-If $T_{u_0}$ is finite, then
\[
\int_0^{T_{u_0}} \|u\|_{\chi^0}^2 dt = \infty.
\]

There are several important global well-posedness and decay results for the 3D Navier-Stokes equation and MHD equations (1.1) in Lei-Lin space $\chi^1(\mathbb{R}^3)$ (see, e.g., [1–3, 10–13]). In particular, Ye and Zhao \cite{14} proved the global well-posedness for the $n$ dimensional generalized MHD equations with small initial data. As we know that, in 2D case, the global Leray-Hopf weak solution to the MHD equations (1.1) is a unique one in $L^2(\mathbb{R}^2)$ space. We wonder if there is a global strong solution to the 2D MHD equations (1.1) in the Lei-Lin space $\chi^1(\mathbb{R}^2)$ without the smallness condition, and if it is a unique solution with the Leray-Hopf weak solution with a same initial data in the space $L^2(\mathbb{R}^2) \cap \chi^1(\mathbb{R}^2)$.

In this paper, we will prove the global well-posedness of a strong solution to (1.1) in $\chi^{-1}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ for large initial data. Our main results are presented as follows.

**Theorem 1.1.** Let $(u_0, b_0)$ be in $\chi^{-1}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. A unique solution $(u, b)$ then exists in the space $L^2(\mathbb{R}^+; \chi^0(\mathbb{R}^2))$ which also belongs to
\[
C(\mathbb{R}^+; \chi^{-1}(\mathbb{R}^2)) \cap L^1(\mathbb{R}^+; \chi^1(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+; \dot{H}^1(\mathbb{R}^2)),
\]
and satisfies the energy equality
\[
\|(u, b)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla u\|_{L^2}^2 d\tau + 2\nu \int_0^t \|\nabla b\|_{L^2}^2 d\tau = \|(u_0, b_0)\|_{L^2}^2.
\] (1.2)

$(u, b)$ also satisfies the global a priori estimate
\[
\|(u, b)\|_{L^\infty(\chi^{-1})} + \frac{1}{2} \min\{\mu, \nu\} \int_0^t \|(u, b)\|_{\chi^1} d\tau
\leq \|(u_0, b_0)\|_{\chi^{-1}} + \frac{C}{2 \min\{\mu\nu\}} \|(u_0, b_0)\|_{L^2}^4.
\] (1.3)

The weak-strong uniqueness theorem is in order.

**Theorem 1.2.** Let $(u_0, b_0)$ and $(v_0, h_0)$ be the divergence-free vector field in $L^2(\mathbb{R}^2)$ and $(u_0, b_0)$ also be in $\chi^{-1}(\mathbb{R}^2)$. Let $(v, h) \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+; \dot{H}^1(\mathbb{R}^2))$ be a Leray-Hopf weak solution associated with $(v_0, h_0)$, and let $(u, b)$ be a strong solution constructed in Theorem 1.1 associated with $(u_0, b_0)$ and
\[
(u, b) \in L^2(\mathbb{R}^+; \chi^0(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+; \dot{H}^1(\mathbb{R}^2)).
\]

Write $(w, g) \triangleq (u - v, b - h)$, then we have for $0 \leq t < \infty$
\[
\|(w, g)\|_{L^2}^2 + \frac{1}{2} \min\{\mu, \nu\} \left( \int_0^t \|\nabla w\|_{L^2}^2 d\tau + \int_0^t \|\nabla g\|_{L^2}^2 d\tau \right)
\leq \|(u_0, b_0) - (v_0, h_0)\|_{L^2} \times \exp \left( C \int_0^t \|(u, b)\|_{\chi^0}^2 d\tau \right).
\] (1.4)

**Remark 1.3.** If two solutions $(u, b)$ and $(v, h)$ constructed in Theorem 1.1 have a same initial data $(u_0, b_0) = (v_0, h_0) \in L^2(\mathbb{R}^2) \cap \chi^{-1}(\mathbb{R}^2)$, then the estimate inequality (1.4) implies the uniqueness of solutions $(u, b) \equiv (v, h)$. 
**Remark 1.4.** The definitions and notations used in this paper are presented as follows:

1. For $s \in \mathbb{R}$, the functional space $\chi^s(\mathbb{R}^n)$ is defined by
   \[
   \chi^s := \left\{ f \in \mathcal{D}'(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} |\xi|^s \hat{f}(\xi)|d\xi < \infty \right\}.
   \]

2. Let $s \in \mathbb{R}$ and $p \in [1, \infty]$. $f(t, x) \in L^p([0,T];\chi^s(\mathbb{R}^n))$ if and only if
   \[
   \|f\|_{L^p(\chi^s)} \triangleq \left(\int_0^T \|f(t, \cdot)\|_{\chi^s}^p dt\right)^{\frac{1}{p}} < \infty;
   \]
   then $f(t, x) \in \tilde{L}^p([0,T];\chi^s(\mathbb{R}^n))$ if and only if
   \[
   \|f\|_{\tilde{L}^p(\chi^s)} \triangleq \left(\int_0^T \left(\int_{\mathbb{R}^n} |\xi|^s \hat{f}(t, \xi)|d\xi\right)^p dt\right)^{\frac{1}{p}} < \infty.
   \]

3. We will use $\|\cdot\|_X$ to denote $\|\cdot\|_{X(\mathbb{R}^n)}$, $\|\cdot\|_{L^q(\chi^s)}$ to denote $\|\cdot\|_{L^q([0,T];\chi^s(\mathbb{R}^n))}$ and $\|(u, b)\|_{X}^p$ to denote $\|u\|_{X}^p + \|b\|_{X}^p$ for conciseness. And throughout the paper, $C$ stands for a generic positive constant, which may be different from line to line.

**Remark 1.5.** The space $L^2(\mathbb{R}^2) \cap \chi^{-1}(\mathbb{R}^2)$ is not empty. Indeed, let $f = \mathcal{F}^{-1}\left(\frac{1}{|\xi|}\mathbf{1}_{\{|\xi|>2\}}\right)$, then we have
   \[
   \|f\|_{\chi^{-1}} = 2\pi \int_2^{+\infty} \frac{1}{r^2} dr < \infty
   \]
   and
   \[
   \|f\|_{L^2} = 2\pi \int_2^{+\infty} \frac{1}{r^3} dr < \infty.
   \]
   Therefore, $f \in L^2(\mathbb{R}^2) \cap \chi^{-1}(\mathbb{R}^2)$.

**2 Preliminaries**

In this Preliminary section, we present some elementary lemmas which will be used in our proofs.

**Lemma 2.1.** Let $s_1 < s_0 < s_2$. If $f \in \chi^{s_1}(\mathbb{R}^n) \cap \chi^{s_2}(\mathbb{R}^n)$, then $f \in \chi^{s_0}(\mathbb{R}^n)$, and
   \[
   \|f\|_{\chi^{s_0}} \leq \|f\|_{\chi^{s_1}}^{\frac{s_2-s_0}{s_1-s_2}} \|f\|_{\chi^{s_2}}^{\frac{s_0-s_1}{s_2-s_1}}.
   \]

**Proof.**
   \[
   \|f\|_{\chi^{s_0}} = \int_{\mathbb{R}^n} |\xi|^{s_0} \hat{f}(\xi)|d\xi
   = \int_{|\xi| \leq \lambda} |\xi|^{s_0-s_1} |\xi|^{s_1} \hat{f}(\xi)|d\xi + \int_{|\xi| > \lambda} |\xi|^{s_0-s_2} |\xi|^{s_2} \hat{f}(\xi)|d\xi
   \leq \lambda^{s_0-s_1} \|f\|_{\chi^{s_1}} + \lambda^{s_0-s_2} \|f\|_{\chi^{s_2}}.
   \]

Let $\lambda = \left(\frac{\|f\|_{\chi^{s_2}}}{\|f\|_{\chi^{s_1}}}\right)^{\frac{1}{s_2-s_1}}$, then the Lemma 2.1 follows from (2.1).

**Lemma 2.2.** Let $f \in L^2(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2)$, then $f \in \chi^{-\frac{1}{2}}(\mathbb{R}^2)$, and $\|f\|_{\chi^{-\frac{1}{2}}} \leq C \|f\|_{L^2} \|f\|_{\dot{H}^1}$.\]
Proof. We write
\[ \|f\|_{\mathcal{X}^{\alpha}} = \int_{\mathbb{R}^2} |\xi|^{-\frac{1}{2}} |\hat{f}(\xi)|d\xi := I_1(\lambda) + I_2(\lambda), \] (2.2)
where
\[ I_1(\lambda) = \int_{|\xi| \leq \lambda} |\xi|^{-\frac{1}{2}} |\hat{f}(\xi)|d\xi \leq C\lambda^{\frac{1}{2}} \|f\|_{L^2} \]
and
\[ I_2(\lambda) = \int_{|\xi| > \lambda} |\xi|^{-\frac{1}{2}-1} |\hat{f}(\xi)|d\xi \leq C\lambda^{-\frac{1}{2}} \|\hat{\mu}^{1}\| \]
by the Hölder’s inequality. Choosing \( \lambda = \frac{\|f\|_{L^2}}{\|f\|_{L^2}} \) completes the proof of Lemma 2.2. \( \square \)

**Lemma 2.3.** Let \( f, g \in L^2([0, T]; \chi^0(\mathbb{R}^n)) \), then \( fg \in L^1([0, T]; \chi^0(\mathbb{R}^n)) \) and
\[ \|fg\|_{L^1(\chi^0)} \leq \|f\|_{L^2(\chi^0)} \|g\|_{L^2(\chi^0)}. \]
In particular, \( \|f^2\|_{L^1(\chi^0)} \leq \|f\|^2_{L^2(\chi^0)}. \)

**Proof.**
\[ \|fg\|_{L^1(\chi^0)} = \int_0^T \int_{\mathbb{R}^n} |\hat{f}(\tau, \xi)| * |\hat{g}(\tau, \xi)|d\xi d\tau \leq \int_0^T \|f\|_{\mathcal{X}^0} \|g\|_{\mathcal{X}^0} d\tau \leq \|f\|_{L^2(\chi^0)} \|g\|_{L^2(\chi^0)}. \] (2.3)

\( \square \)

**Lemma 2.4.** There exists a constant \( C \) such that
\[ \|B(u, b)\|_{L^2(\chi^0)} \leq \frac{C}{\min\{\mu, \nu\}^{\frac{1}{2}}} \|(u, b)\|^2_{L^2(\chi^0)}, \]
where \( B(u, b) \triangleq \int_0^T [e^{\mu(t-\tau)\Delta} \nabla \cdot (u \otimes u + b \otimes b) + e^{\nu(t-\tau)\Delta} \nabla \cdot (u \otimes b + b \otimes u)] d\tau. \)

**Proof.**
\[ \|B(u, b)\|_{L^2(\chi^0)} \leq \left\| \int_{\mathbb{R}^2} \int_0^T e^{-\min\{\mu, \nu\}(t-\tau)}|\xi|^2 |\hat{u} * \hat{b}| + |\hat{b} * \hat{u}| + |\hat{u}| * |\hat{b}| + |\hat{b}| * |\hat{u}| d\tau d\xi \right\|_{L^2} \]
\[ \leq \int_{\mathbb{R}^2} \left( \int_0^T e^{-2\min\{\mu, \nu\}t|\xi|^2} d\tau \right)^{\frac{1}{2}} \int_0^T \left( |\hat{u}| * |\hat{b}| + |\hat{b}| * |\hat{u}| + |\hat{u}| * |\hat{b}| + |\hat{b}| * |\hat{u}| \right) d\tau d\xi \]
\[ \leq \frac{C}{\min\{\mu, \nu\}^{\frac{1}{2}}} \int_0^T \left( \|u\|^2_{\mathcal{X}^0} + \|b\|^2_{\mathcal{X}^0} + 2\|u\|_{\mathcal{X}^0} \|b\|_{\mathcal{X}^0} \right) d\tau \]
\[ \leq \frac{C}{\min\{\mu, \nu\}^{\frac{1}{2}}} \|(u, b)\|^2_{L^2(\chi^0)}. \] (2.4)

\( \square \)

We also need the following Banach contraction mapping principle (see [4, Lemma 5.5]).
Lemma 2.5. Let $E$ be a Banach space, $B$ a continuous bilinear map from $E \times E$ to $E$, and $\alpha$ a positive real number such that

$$\alpha < \frac{1}{4\|B\|} \text{ with } \|B\| = \sup_{\|f\|, \|g\| \leq 1} \|B(f, g)\|.$$ 

For any $a$ in the ball $B(0, \alpha)$ (i.e., with center 0 and radius $\alpha$) in $E$, a unique $x$ then exists in $B(0, 2\alpha)$ such that

$$x = a + B(x, x).$$

Finally, we show some estimates of the heat equations in the space $C([-T]; \chi^s(\mathbb{R}^n)) \cap L^1((0,T]; \chi^{s+2}(\mathbb{R}^n))$.

Lemma 2.6. For $s \in \mathbb{R}$, let $v$ be a solution of the Cauchy problem

$$\begin{cases}
\partial_t v - \kappa \Delta v = f,
\v(0,0) = v_0(x),
\end{cases}$$

with $f \in L^1([0,T]; \chi^s(\mathbb{R}^n))$ and $v_0 \in \chi^s(\mathbb{R}^n)$. Then $v$ belongs to $L^\infty([0,T]; \chi^s(\mathbb{R}^n)) \cap L^1((0,T]; \chi^{s+2}(\mathbb{R}^n)) \cap C([0,T]; \chi^s(\mathbb{R}^n))$, and satisfies the following estimate

$$\|v\|_{L^\infty(\chi^s)} + \kappa \|v\|_{L^1(\chi^{s+2})} \leq C(\|v_0\|_{\chi^s} + \|f\|_{L^1(\chi^s)}).$$

Proof. By the Duhamel’s formula in Fourier space, the equations (2.5) can be written as

$$|\hat{\v}(t, \xi)| \leq e^{-\kappa t|\xi|^2}|\hat{v}_0(\xi)| + \int_0^t e^{-\kappa(t-\tau)|\xi|^2}|\hat{f}(\tau, \xi)|d\tau.$$ 

(2.7)

Multiplying (2.7) by $|\xi|^s$ and taking the $L^\infty$ norm in time, one has

$$\sup_{0 \leq \tau \leq T} |\xi|^s|\hat{\v}(t, \xi)| \leq |\xi|^s|\hat{v}_0(\xi)| + \int_0^t |\xi|^s|\hat{f}(\tau, \xi)|d\tau.$$ 

(2.8)

Taking the $L^1$ norm in $\xi$ to (2.8), we get

$$\|v\|_{L^\infty(\chi^s)} \leq \int_{\mathbb{R}^n} |\xi|^s|\hat{v}_0(\xi)|d\xi + \int_{\mathbb{R}^n} \int_0^t |\xi|^s|\hat{f}(\tau, \xi)|d\tau d\xi 
\leq \|v_0\|_{\chi^s} + \|f\|_{L^1(\chi^s)}.$$ 

(2.9)

Because, for almost all fixed $\xi \in \mathbb{R}^n$, the map $t \mapsto \hat{\v}(t, \xi)$ is continuous over $[0,T]$, thus by the Lebesgue dominated convergence theorem, it implies that $v \in C([0,T]; \chi^s(\mathbb{R}^n))$. Next we estimate $\|v\|_{L^1(\chi^{s+2})}$. Multiplying (2.7) by $|\xi|^{s+2}$ and taking the $L^1$ norm in time, applying the Young’s inequality in time and noting $\int_0^T |\xi|^2 e^{-\kappa t|\xi|^2} dt = \frac{1}{\kappa}$, we deduce that

$$\int_0^T |\xi|^{s+2}|\hat{\v}(t, \xi)|dt \leq \int_0^T e^{-\kappa t|\xi|^2}|\xi|^{s+2}|\hat{v}_0(\xi)|dt + \int_0^T \int_0^t e^{-\kappa(t-\tau)|\xi|^2}|\xi|^{s+2}|\hat{f}(\tau, \xi)|d\tau dt 
\leq \frac{1}{\kappa} (|\xi|^s|\hat{v}_0(\xi)| + \int_0^T |\xi|^s|\hat{f}(t, \xi)|dt),$$

(2.10)
then taking $L^1$ norm in $\xi$ to (2.10), we obtain
\[
\|v\|_{L^1(\chi^+)} \leq \frac{1}{K} \left( \int_{\mathbb{R}^n} |\xi|^s |v_0(\xi)| d\xi + \int_0^T \int_{\mathbb{R}^n} |\xi|^s |\hat{f}(t, \xi)| dt d\xi \right)
\]
\[
\leq \frac{1}{K} \left( \|v_0\|_{\chi^+} + \|f\|_{L^1(\chi^+)} \right).
\]  
(2.11)

Combining (2.9) and (2.11), we get (2.6).

3 Proof of Theorems 1.1 and 1.2

In this section, we focus on the global well-posedness of a mild solution to (1.1) in $L^2(\mathbb{R}^+; \chi^0(\mathbb{R}^2))$. First, we give the following lemma on the well-posedness and blow-up criterion, which plays a key role in proving our theorem.

**Lemma 3.1.** Assume $(u_0, b_0)$ be in $\chi^{-1}(\mathbb{R}^2)$. Then there exists a positive time $T$ such that the equations (1.1) have a unique local solution $(u, b)$ in the space $L^2([0, T]; \chi^0(\mathbb{R}^2))$, which also belongs to $L^\infty([0, T]; \chi^{-1}(\mathbb{R}^2)) \cap L^1([0, T]; \chi^1(\mathbb{R}^2)) \cap C([0, T]; \chi^{-1}(\mathbb{R}^2))$.

Moreover, there exists a constant $C(\mu, \nu)$ such that if $\|(u_0, b_0)\|_{\chi^{-1}} \leq C(\mu, \nu)$, then the solution is a global one; if $T^* < \infty$ is the maximal time of existence, then
\[
\lim_{T \to T^*} \int_0^T \|(u, b)\|_{\chi^0}^2 dt = \infty.
\]  
(3.1)

**Proof.** It is easy to see that the equations (1.1) can be rewritten as the integral form
\[
\begin{cases}
  u(t, x) = e^{\mu t} u_0 - \int_0^t e^{\mu(t-\tau)} \Delta [\mathbb{P} \nabla \cdot (u \otimes u) - \mathbb{P} \nabla \cdot (b \otimes b)] d\tau, \\
  b(t, x) = e^{\nu t} b_0 - \int_0^t e^{\nu(t-\tau)} \nabla \cdot (u \otimes b) - \nabla \cdot (b \otimes u)] d\tau.
\end{cases}
\]  
(3.2)

By applying the Banach contraction mapping principle Lemma 2.5, we will carry out the proof of global or local well-posedness of the Cauchy problem (3.2) in the Lei-Lin space $\chi^{-1}(\mathbb{R}^2)$. First, one has
\[
\|(e^{\mu t} u_0, e^{\nu t} b_0)\|_{L^2(\chi^0)}^2
\]
\[
= \left\| \int_{\mathbb{R}^2} e^{-\mu t|\xi|^2} |\hat{u}_0(\xi)|^2 d\xi \right\|_{L^2}^2 + \left\| \int_{\mathbb{R}^2} e^{-\nu t|\xi|^2} |\hat{b}_0(\xi)|^2 d\xi \right\|_{L^2}^2
\]
\[
\leq \left( \int_{\mathbb{R}^2} \left( \int_0^t e^{-2\mu t|\xi|^2} |\hat{u}_0(\xi)|^2 d\tau \right)^{\frac{1}{2}} d\xi \right)^2 + \left( \int_{\mathbb{R}^2} \left( \int_0^t e^{-2\nu t|\xi|^2} |\hat{b}_0(\xi)|^2 d\tau \right)^{\frac{1}{2}} d\xi \right)^2
\]
\[
\leq \frac{1}{2 \min\{\mu, \nu\}} \|(u_0, b_0)\|_{\chi^{-1}}^2.
\]  
(3.3)

Thus, Combining the estimate (3.3) with Lemma 2.4, we can obtain that if $\|(u_0, b_0)\|_{\chi^{-1}} \leq \frac{\min\{\mu, \nu\}}{2^2 C_0}$, then
\[
\|(e^{\mu t} u_0, e^{\nu t} b_0)\|_{L^2(\chi^0)} \leq \frac{\min\{\mu, \nu\}^\frac{1}{2}}{4C_0} < \frac{\min\{\mu, \nu\}^\frac{1}{2}}{4C}.
\]  
(3.4)
with $C_0 > C$, and we obtain the global solution.

Now, we consider the case of a large initial data $(u_0, b_0)$ in $\chi^{-1}(\mathbb{R}^2)$. Setting

$$(u_0, b_0) = (u^L_0, b^L_0) + (u^R_0, b^R_0),$$

where

$$(u^L_0, b^L_0) \triangleq F^{-1}(1_{\{\xi \leq \rho_{u_0, b_0}\}}(\tilde{u}_0, \tilde{b}_0)) \quad \text{and} \quad (u^R_0, b^R_0) \triangleq F^{-1}(1_{\{\xi > \rho_{u_0, b_0}\}}(\tilde{u}_0, \tilde{b}_0)).$$

One fixes some positive real number $\rho_{u_0, b_0}$ such that

$$\|(u^R_0, b^R_0)\|_{\chi^{-1}} = \int_{\xi > \rho_{u_0, b_0}} |\xi|^{-1}(|\tilde{u}_0| + |\tilde{b}_0|) d\xi \leq \frac{\min\{\mu, \nu\} \frac{1}{2}}{2C_0}. \quad (3.5)$$

By (3.3) we derived that

$$\|(e^{\mu t \Delta} u_0, e^{\nu t \Delta} b_0)\|_{L^2(\chi^0)} \leq \frac{\min\{\mu, \nu\} \frac{1}{2}}{8C_0} + \|(e^{\mu t \Delta} u^L_0, e^{\nu t \Delta} b^L_0)\|_{L^2(\chi^0)}, \quad (3.6)$$

and

$$\|(e^{\mu t \Delta} u^L_0, e^{\nu t \Delta} b^L_0)\|_{L^2(\chi^0)} \leq \rho_{u_0, b_0} T \frac{1}{2} \|(u_0, b_0)\|_{\chi^{-1}}. \quad (3.7)$$

Hence, if we choose

$$T \leq \left(\frac{\min\{\mu, \nu\} \frac{1}{2}}{8\rho_{u_0, b_0} C_0 \|(u_0, b_0)\|_{\chi^{-1}}}\right)^2, \quad (3.8)$$

then we have a unique solution $(u, b)$ in the ball $B(0, \frac{\min\{\mu, \nu\} \frac{1}{2}}{8C_0})$ of the space $L^2([0, T]; \chi^0(\mathbb{R}^2))$.

Next, we prove the persistence that if $(u, b)$ is a solution to (1.1) in $L^2([0, T]; \chi^0(\mathbb{R}^2))$ with initial data $(u_0, b_0) \in \chi^{-1}(\mathbb{R}^2)$, then $(u, b)$ also belongs to

$$\mathcal{C}([0, T]; \chi^{-1}(\mathbb{R}^2)) \cap L^1([0, T]; \chi^1(\mathbb{R}^2)).$$

In fact, let $s = -1$ and $n = 2$ in Lemma 2.6 and by Lemma 2.3 the result is followed.

Finally, we prove the blow-up criterion (3.1). Assume that we have a solution to the equations (1.1) on a time interval $[0, T)$ such that

$$\int_0^T \|(u, b)\|^2_{\chi^0} dt < \infty.$$

We claim that the lifespan $T^*$ of $(u, b)$ is larger than $T$. Indeed, due to the estimate (2.9) in Lemma 2.6 and by Lemma 2.3, we have

$$\int_{\mathbb{R}^n} \sup_{0 \leq t \leq T} |\xi|^{-1}(|\tilde{u}| + |\tilde{b}|)(t, \xi) d\xi \leq \|(u_0, b_0)\|_{\chi^{-1}} + C\|(u, b)\|^2_{L^2(\chi^0)} < \infty. \quad (3.9)$$

Thus, a positive number $\rho$ exists such that

$$\forall t \in [0, T), \quad \int_{|\xi| > \rho} |\xi|^{-1}(|\tilde{u}| + |\tilde{b}|)(t, \xi) d\xi \leq \frac{\min\{\mu, \nu\}}{2C_0}. \quad (3.10)$$

The condition (3.8) now implies that for any $t \in [0, T)$, the lifespan for a solution to (1.1) with initial data $(u(t), b(t))$ is bounded from below by a positive real number $C$ which is independent of $t$. Thus the lifespan $T^* > T$, and the whole proof of Lemma 3.1 is finished.
3.1 Proof of Theorem 1.1

Taking the $L^2$ inner products of the equations (1.1)$_{1,2}$ with $u$ and $b$, respectively, adding the results and integrating by parts, we obtain, for any $t \in (0, +\infty)$, the energy equality

$$
\|(u, b)\|^2_{L^2} + 2\mu \int_0^t \|\nabla u\|^2_{L^2} \, dt + 2\nu \int_0^t \|\nabla b\|^2_{L^2} \, dt = \|(u_0, b_0)\|^2_{L^2},
$$

(3.11)

for details refer to [4, Theorem 5.14]. And by Lemma 2.2, it yields that, for any $t \in (0, +\infty)$

$$
\int_0^t \|(u, b)\|^4_{\chi^{-\frac{1}{2}}} \, dt \leq \|(u, b)\|^2_{L^2} \int_0^t \|(u, b)\|^2_{H^1} \, dt
$$

$$\leq \frac{1}{2\min\{\mu\nu\}} \|(u_0, b_0)\|^4_{L^2}.
$$

(3.12)

According to Lemmas 2.6, 2.3 and 2.1, by the Young’s inequality, we deduce that

$$
\|(u, b)\|_{L^\infty(\chi^{-1})} + \mu \int_0^t \|u\|_{\chi^1} \, dt + \nu \int_0^t \|b\|_{\chi^1} \, dt
$$

$$\leq \|(u_0, b_0)\|_{\chi^{-1}} + C \int_0^t \|(u, b)\|^2_{\chi^0} \, dt
$$

$$\leq \|(u_0, b_0)\|_{\chi^{-1}} + C \int_0^t \|(u, b)\|^4_{\chi^{-\frac{1}{2}}} \, dt + \frac{1}{2} \min\{\mu, \nu\} \int_0^t \|(u, b)\|_{\chi^1} \, dt.
$$

(3.13)

Inserting the estimate (3.12) into (3.13) leads to the result

$$
\|(u, b)\|_{L^\infty(\chi^{-1})} + \frac{1}{2} \min\{\mu, \nu\} \int_0^t \|(u, b)\|_{\chi^1} \, dt
$$

$$\leq \|(u_0, b_0)\|_{\chi^{-1}} + \frac{C}{2\min\{\mu\nu\}} \|(u_0, b_0)\|^4_{L^2}.
$$

(3.14)

By Lemma 2.1, we arrive at

$$
\int_0^t \|(u, b)\|^2_{\chi^0} \, dt \leq \int_0^t \|(u, b)\|_{\chi^{-1}} \|(u, b)\|_{\chi^1} \, dt \leq C (\mu, \nu, \|(u_0, b_0)\|_{\chi^{-1}}, \|(u_0, b_0)\|_{L^2}),
$$

(3.15)

for any $0 \leq t \leq T$. Thus, the blow-up criterion (3.1) implies that the strong solution $(u, b)$ is a global one in $L^2([0, T]; \chi^0(\mathbb{R}^2))$ for any $T < \infty$. And by Lemma 2.6 the solution $(u, b) \in C(\mathbb{R}^+; \chi^{-1}(\mathbb{R}^2)) \cap L^1(\mathbb{R}^+; \chi^1(\mathbb{R}^2))$, which completes the proof of Theorem 1.1.

3.2 Proof of Theorem 1.2

Subtracting the two equations satisfied by $(u, b)$ and $(v, h)$, respectively, we get

$$
\begin{align*}
\partial_t w - \mu \Delta w + (w \cdot \nabla)u + (v \cdot \nabla)w + \nabla(p_1 - p_2) &= (g \cdot \nabla)b + (h \cdot \nabla)g, \\
\partial_t g - \nu \Delta g + (w \cdot \nabla)h + (v \cdot \nabla)g &= (g \cdot \nabla)u + (h \cdot \nabla)w,
\end{align*}
$$

(3.16)

where $(w, g) \triangleq (u - v, b - h)$. Taking the $L^2$ inner products of the equations (3.16)$_{1,2}$ with $w$ and $g$, respectively, adding the results and by Hölder’s and Young’s inequalities and the fact $\|f\|_{L^\infty} \leq \|f\|_{L^1} = \|f\|_{L^2}$, and

$$
\int_{\mathbb{R}^2} (v \cdot \nabla)w \cdot w \, dx = 0, \quad \int_{\mathbb{R}^2} (v \cdot \nabla)g \cdot g \, dx = 0
$$
and
\[ \int_{\mathbb{R}^2} (h \cdot \nabla) g \cdot wdx + \int_{\mathbb{R}^2} (h \cdot \nabla) w \cdot gdx = 0, \]
it follows that
\[ \frac{1}{2} \frac{d}{dt} \| (w, g) \|_{L^2}^2 + \mu \| \nabla w \|_{L^2}^2 + \nu \| \nabla g \|_{L^2}^2 \\
= -\int_{\mathbb{R}^2} (w \cdot \nabla) u \cdot wdx + \int_{\mathbb{R}^2} (g \cdot \nabla) v \cdot wdx - \int_{\mathbb{R}^2} (w \cdot \nabla) b \cdot gdx + \int_{\mathbb{R}^2} (g \cdot \nabla) b \cdot wdx \\
\leq C (\| \nabla w \|_{L^2} \| w \|_{L^2} \| u \|_{L^\infty} + \| \nabla w \|_{L^2} \| g \|_{L^2} \| b \|_{L^\infty} + \| \nabla g \|_{L^2} \| w \|_{L^2} \| b \|_{L^\infty} \\
+ \| \nabla g \|_{L^2} \| g \|_{L^2} \| u \|_{L^\infty}) \\
\leq \frac{1}{2} \min\{ \mu, \nu \} (\| \nabla w \|_{L^2} \| g \|_{L^2} \| b \|_{L^\infty})^2 + C \| (w, g) \|_{L^2}^2 \| (u, b) \|_{\chi^0}^2. \tag{3.17} \]

Thanks to the Grönwall’s inequality, we conclude the proof of (1.4). And we thus complete the proof of Theorem 1.2. \qed

Conflict of Interest

The authors declare no conflict of interest.

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