Excitation modes of vortices in sub-micron magnetic disks

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Classical and quantum theory of spin waves in the vortex state of a mesoscopic sub-micron magnetic disk has been developed with account of the finite mass density of the vortex. Oscillations of the vortex core resemble oscillations of a charged string in a potential well in the presence of the magnetic field. Conventional gyroscopic frequency appears as a gap in the spectrum of spin waves of the vortex. The mass of the vortex has been computed that agrees with experimental findings. Finite vortex mass generates a high-frequency branch of spin waves. Effects of the external magnetic field and dissipation have been addressed.

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I. INTRODUCTION

Recent advances in optical and electron-beam lithography offered possibility to fabricate arrays of micron and submicron-size magnetic structures with controlled magnetic properties. Among such structures are mesoscopic circular disks of soft ferromagnetic materials. Arrays of such disks, as well as individual disks, have been intensively studied\textsuperscript{1–7} due to their unusual magnetic properties and potential for technological and biomedical applications\textsuperscript{8–10}.

Micron-size circular disks exhibit a wide variety of magnetic equilibrium configurations due to geometric constraints on the spin field\textsuperscript{11}. Their applications are based on static and dynamic properties of one of the essentially non-uniform ground states, the vortex state. It is characterized by the curling of the magnetization in the plane of the disk, leaving virtually no magnetic “charges”. The very weak uncompensated magnetic moment of the disk sticks out of a small area confined to the vortex core (VC). The diameter of the core is comparable to the material exchange length\textsuperscript{3,12}. The low frequency dynamics of the vortex state is due to the gyrotropic mode, consisting of the spiral-like precessional motion of the VC as a whole\textsuperscript{13–17}, and it is intrinsically distinct from conventional spin wave excitations.

Because of the strong exchange interaction among the out-of-plane spins in the VC, it behaves as an independent entity. The research on excitation modes of vortices has focused on the low-frequency gyroscopic mode that describes circular motion of the vortex about the center of the disk. It can also be viewed as the uniform precession of the magnetic moment of the disk due to the vortex. The natural question is whether the gyroscopic mode allows spatial dispersion similar to spin waves of finite wavelength in ferromagnets. The aim of this paper is to study spin waves related to the gyroscopic motion of the vortex. Such a wave is shown in Fig. 1. It must exist due to finite elasticity of the vortex provided by the exchange interaction.

![Figure 1: Gyroscopic spin wave in the vortex state of a mesoscopic magnetic disk.](image)

Most of the research on the gyroscopic motion of vortices in circularly polarized disks ignores the inertial mass of the VC. Such a mass has a dynamical origin stemming from the variation of the shape of the VC as it moves inside the disk. Meantime, experimental studies of vortex oscillations in micrometer permalloy rings\textsuperscript{18} hinted towards a non-negligible vortex mass of order $10^{-24}$ kg. On a theoretical side the vortex mass has been previously computed in a two-dimensional Heisenberg model with anisotropic exchange interaction\textsuperscript{19–21}. In disks made of soft magnetic materials that have been experimented with, the exchange interaction is isotropic. We will show that in this case the finite mass density of the vortex originates from the geometrical confinement of the spin field and the magnetic dipole-dipole interactions.

In this paper we will derive the generalized Thiele equation that describes spin waves in the vortex core of finite mass density and will obtain the spectrum of such waves. We will show that the conventional gyroscopic mode, $\omega_G$, appears as a gap in the spectrum of the spin waves in the vortex, $\omega(q) = \omega_G + \alpha q^2$, when the vortex mass is neglected. From the mathematical point of view the above problem resembles the problem of the motion of a charged string in a potential well in the presence of the magnetic field. The latter problem is a generaliza-
tion of the problem of Landau levels of an electron in a two-dimensional potential well in the magnetic field. We will show that this problem has a nice exact solution for quantized oscillations of the string, thus providing the spectrum of magnons in the vortex in the quantum regime as well. Classical and quantum solutions for the spectrum of excitations of the vortex lead to the same dispersion law, $\omega(q)$, in the limit of small $q$.

The paper is structured as follows. In Sec. II a Lagrangian formulation of the problem is presented. Formal derivation of the massive elastic Thiele’s equation that allows deformations of the vortex line is given in Section III. The spectrum of spin waves in the vortex core is obtained in Section IV. We show that a finite mass of the vortex results in the additional excitation mode that is absent in the case of zero mass. Quantum mechanical treatment of magnons in the vortex core is developed in Section V. The vortex mass in a circularly polarized disk is computed in Sec. VI and is shown to be in good agreement with experimental findings. The field dependence of the vortex excitation modes and effects of dissipation are discussed in Sec. VII. Sec. VIII contains final conclusions and suggestions for experiment.

II. LAGRANGIAN MECHANICS OF THE VORTEX CORE

We shall describe the vortex line by the vector field $\vec{X} = (x, y)$, where $x(t, z)$ and $y(t, z)$ are coordinates of the center of the vortex core in the $XY$ plane. Landau-Lifshitz dynamics of the fixed-length magnetization vector $\vec{M}(\Theta, \Phi) = M_s(\cos \Phi \sin \Theta, \sin \Phi \sin \Theta, \cos \Theta)$ follows from the Lagrangian

$$\mathcal{L} \left[ t; \Theta, \Phi, \partial_\Theta, \partial_\Phi \right] =$$

$$\int dz \left[ \frac{M_s}{\gamma} \left( \dot{D}(\Phi) \cos \Theta - \mathcal{E}(\Theta, \Phi, \partial_\Theta, \partial_\Phi) \right) \right]$$

(1)

where $\mathcal{E}(\Theta, \Phi, \partial_\Theta, \partial_\Phi)$ is the energy density. The dependence of the Lagrangian on the partial derivatives $\partial_\Phi$ of the angular coordinates comes from the elastic nature of the vortex core. It is contained in the total energy, $\mathcal{E}$, that takes into account interaction between different layers of the vortex line, see below.

The spatial dependence of angular coordinates $(\Theta, \Phi)$ for the vortex state is given by $\Theta = \Theta(t; \vec{r}, z)$ and $\Phi = \Phi(t; \vec{r}, z) = \Phi(\vec{r} - \vec{X}(t, z), t)$. We only consider long-wave solutions that do not deform the vortex core in any $z$-cross-section of the disk. This means that the angular coordinates depend on $t$ and $z$ via the coordinates of the vortex core $\vec{X}(t, z)$. The covariant derivative with respect to time, $D_t \Phi$, along the vortex core is given by

$$D_t \Phi = \nabla_{\vec{X}(t, z)} \Phi = -\dot{\vec{X}}(t, z) \cdot \nabla_{\vec{r}} \Phi(\vec{r} - \vec{X}(t, z))$$

(2)

where “dot” denotes partial derivative with respect to $t$.

Taking all these considerations into account, the above Lagrangian becomes

$$\mathcal{L} \left[ t; \vec{X}, \dot{\vec{X}}, \partial_\vec{X} \right] = \int dz \mathcal{L} \left[ t, z; \vec{X}, \dot{\vec{X}}, \partial_\vec{X} \right]$$

(3)

with the Lagrangian density being

$$\mathcal{L} \left[ t, z; \vec{X}, \dot{\vec{X}}, \partial_\vec{X} \right] = \int d^2 \vec{r} \left[ \frac{M_s}{\gamma} ( -\dot{\vec{X}}(t, z) \cdot \nabla_{\vec{r}} \Phi) \right] \cos \Theta$$

$$- \mathcal{E}(\Theta, \Phi, \partial_\Theta, \partial_\Phi)$$

(4)

Thus the generalized momentum densities are given by

$$\Pi_t \left[ t, z; \vec{X}, \dot{\vec{X}}, \partial_\vec{X} \right] = \frac{\delta \mathcal{L}}{\delta \dot{X}(t, z)}$$

$$= -\frac{M_s}{\gamma} \int d^2 \vec{r} (\nabla_{\vec{r}} \Phi) \cos \Theta$$

(5)

$$\Pi_z \left[ t, z; \vec{X}, \dot{\vec{X}}, \partial_\vec{X} \right] = \frac{\delta \mathcal{L}}{\delta \partial_z \vec{X}(t, z)}$$

$$= - \frac{\delta \omega(\vec{X}, \partial_\vec{X})}{\delta (\partial_\vec{X}(t, z))}$$

(6)

with $\omega(\vec{X}, \partial_\vec{X}) = \int d^2 \vec{r} \mathcal{E}(\Theta, \Phi, \partial_\Theta, \partial_\Phi)$ being the linear energy density. The dynamics of the vortex core is governed by the Euler-Lagrange equation,

$$D_t \Pi_t + \partial_z \Pi_z - \frac{\delta \mathcal{L}}{\delta X(t, z)} = 0$$

(7)

Notice that

$$D_t \Lambda(t, z; \vec{r}, \vec{v}) = \xi(t, z)(\Lambda) = \frac{\partial}{\partial t} \Lambda(\vec{r} - \vec{X}(t, z), \vec{X}(t, z))$$

$$= -\dot{\vec{X}}(t, z) \cdot \nabla_{\vec{r}} \Lambda(\vec{r} - \vec{X}(t, z), \vec{X}(t, z)) +$$

$$\dot{\vec{X}}(t, z) \cdot \nabla_{\vec{r}} \Lambda(\vec{r} - \vec{X}(t, z), \vec{X}(t, z))$$

(8)

means covariant derivative along the curve that is tangent to the vortex core, $\xi(t, z)$. All terms involving $\dot{\vec{X}}(t, z)$ and $\vec{X}(t, z)$ in the Euler-Lagrange equation come from $D_t \Pi_t$, which is given by

$$D_t \Pi_t = -\frac{M_s}{\gamma} \int d^2 \vec{r} D_t (\cos \Theta \nabla_{\vec{r}} \Phi)$$

$$= -\frac{M_s}{\gamma} \int d^2 \vec{r} \left( \nabla_{\vec{r}} [ -\dot{X}_j \partial_j \Phi + \dot{\vec{X}} \partial_j \Phi] \cos \Theta + \right.$$

$$+ \nabla_{\vec{r}} \Phi \left[ -\dot{X}_j \partial_j \cos \Theta + \dot{\vec{X}} \partial_j \cos \Theta \right)$$

$$= \epsilon_i M_{ij} \dot{X}_j - \epsilon_i K_{ij} \dot{X}_j,$$

(9)
where
\[ M_{ij} = -\frac{M_s}{\gamma} \int d^2 \vec{r} \left[ (\partial_i \hat{\partial}_j \Phi) \cos \Theta + (\partial_j \Phi) \partial_i \cos \Theta \right], \]
\[ K_{ij} = -\frac{M_s}{\gamma} \int d^2 \vec{r} \left[ (\partial_i \partial_j \Phi) \cos \Theta + (\partial_j \Phi) \partial_i \cos \Theta \right] \]
\[ \tag{10} \]
and \( \partial_j \equiv \nabla_{\tau_j}, \hat{\partial}_j = \nabla_{\nu_j} \).

We want \( \hat{e}_j K_{ij} \hat{X}_j \) to be of the form \( \vec{\rho}_G \times \vec{X} \) which results in the identity \( \epsilon_{ikj} \vec{\rho}_{G,j} = K_{ik} \). From this we obtain \( \rho_{G,j} = -\frac{1}{2} \epsilon_{ikj} K_{ik} \), which translates into the vector form as
\[ \vec{\rho}_G = \frac{M_s}{2\gamma} \int d^2 \vec{r} \epsilon_{ikj} \left[ (\partial_i \partial_k \Phi) \cos \Theta + (\partial_i \Phi) (\partial_k \cos \Theta) \right] \hat{e}_j \]
\[ = \frac{M_s}{2\gamma} \int d^2 \vec{r} \left[ (\nabla_{\tau} \times \nabla_{\tau} \Phi) \cos \Theta + \right. \]
\[ \nabla_{\tau} \Phi \times \nabla_{\tau} \cos \Theta \] \[ \tag{11} \]
To compute the mass tensor and the gyrovector we have to find the solutions \( (\Theta, \Phi) \) of the Landau-Lifshitz equation in the low dynamics regime that is characterized by the condition \( |\vec{X}| \ll 1 \). In this regime solutions can be expanded as a perturbative series on the differential speed, \( |\vec{X}| \), of the vortex core
\[ \Theta(t, z; \vec{r}) = \Theta^{(0)}(z; \vec{r}) + \Theta^{(1)}(t, z; \vec{r}) + \ldots \]
\[ \Phi(t, z; \vec{r}) = \Phi^{(0)}(z; \vec{r}) + \Phi^{(1)}(t, z; \vec{r}) + \ldots \] \[ \tag{12} \]
Notice that the zero-th order is time independent, otherwise the gyrovortex would depend on time.

The approach that neglects deformation of the vortex core in any z-cross-section of the disk is correct only for weak deviations of the centerline of the vortex core from the straight line along the Z-axis. We now proceed to the study of the Landau-Lifshitz equation for the set of variables \( (\Theta, \Phi) \) in such weak bending regime. It can be obtained by applying the variational principle to the Lagrangian density
\[ \mathcal{L} = \int d^2 \vec{r} \left[ \frac{M_s}{\gamma} (D_t \Phi) \cos \Theta \right. \]
\[ - \mathcal{E}(\Theta, \Phi, \nabla_{\tau} \Theta, \nabla_{\tau} \Phi, \partial_2 \Theta, \partial_2 \Phi) \] \[ \tag{13} \]
\[ \text{being the elastic energy in which } A_{e,ff} \text{ is a constant. It describes contribution of the exchange and dipolar forces to the elasticity of the vortex line, with the exchange playing a dominant role. Consequently, with good accuracy, } A_{e,ff} \text{ can be identified with the exchange constant } A. \]

The set of dynamical equations for \( (\Theta, \Phi) \) is
\[ D_t \left( \frac{\delta \mathcal{L}}{\delta (D_t \Phi)} \right) + \partial_z \left( \frac{\delta \mathcal{L}}{\delta (\partial_z \Phi)} \right) - \frac{\delta \mathcal{L}}{\delta \Phi} = 0 \]
\[ -2A_{e,ff} \left[ \sin 2\Theta \partial_z \partial_2 \Phi + \sin^2 \Theta \partial_2^2 \Phi \right] + \frac{M_s}{\gamma} \frac{d \cos \Theta}{dt} + \frac{\delta \mathcal{E}_{XY}}{\delta \Phi} = 0 \] \[ \tag{16} \]
and
\[ D_t \left( \frac{\delta \mathcal{L}}{\delta (D_t \Theta)} \right) + \partial_z \left( \frac{\delta \mathcal{L}}{\delta (\partial_z \Theta)} \right) - \frac{\delta \mathcal{L}}{\delta \Theta} = 0 \]
\[ -2A_{e,ff} \left[ \partial_2^2 \Theta - \frac{\sin 2\Theta}{2} (\partial_2 \Phi)^2 \right] + \frac{M_s}{\gamma} \frac{d \sin \Theta}{dt} + \frac{\delta \mathcal{E}_{XY}}{\delta \Phi} = 0 \] \[ \tag{17} \]
Performing a Fourier transform
\[ \Phi(t, z; \vec{r}) = \frac{1}{\sqrt{2\pi}} \int dq \hat{\Phi}(t, q; \vec{r}) e^{iqz} \]
\[ \Theta(t, z; \vec{r}) = \frac{1}{\sqrt{2\pi}} \int dq \hat{\Theta}(t, q; \vec{r}) e^{iqz} \] \[ \tag{18} \]
we obtain the following set of equations for the pair \( (\hat{\Theta}, \hat{\Phi}) \):
\[ \frac{A_{e,ff} q^2}{\pi} \left[ \sin 2\Theta \star \hat{\Theta} \star \hat{\Phi} + \sin \Theta \star \sin \Theta \star \hat{\Phi} \right] + \frac{M_s}{\gamma} \frac{d \cos \Theta}{dt} + \frac{\delta \mathcal{E}_{XY}}{\delta \Phi} = 0 \] \[ \tag{19} \]
\[ \frac{1}{\sqrt{2\pi}} \frac{M_s}{\gamma} \frac{d \sin \Theta}{dt} + \frac{\delta \mathcal{E}_{XY}}{\delta \Phi} + 2A_{e,ff} q^2 \left[ \hat{\Theta} - \frac{\sin 2\Theta}{4\pi} \star \hat{\Phi} \star \hat{\Phi} \right] = 0 \] \[ \tag{20} \]
where \( \star \) means Fourier convolution.

For a small bending of the vortex core, the boundary conditions on the angle \( \Theta \) are the same as in the rigid VC case, i.e.: \( \Theta \approx 0 \) or \( \pi \) in the limit \( \vec{r} \ll \Delta_0 \) and \( \Theta \approx \pi/2 \) in the limit \( \vec{r} \gg \Delta_0 \), with \( \Delta_0 = \sqrt{A/M_s^2} \) being the exchange length of the material and where \( \vec{r} = ||\vec{r} - \vec{X}(t, z)||_2 \) is the radial distance from the VC center at any height \( z \). Considering this two limits Eq. \[ 19 \] becomes:

- Limit \( \vec{r} \ll \Delta_0 \). In this case, \( \sin \Theta \approx 0 \) and thus \( \sin 2\Theta \partial_2 \partial_2 \Phi + \sin^2 \Theta \partial_2^2 \Phi \approx 0 \). So we have the following equation in the Fourier space
\[ \frac{M_s}{\gamma} \frac{d \cos \Theta}{dt} + \frac{\delta \mathcal{E}_{XY}}{\delta \Phi} = 0 \] \[ \tag{21} \]
III. ELASTIC THIELE EQUATION

We now proceed to the computation of the gyrovector and the mass density tensor. Using (23) we obtain that the first term of Eq. (11) equals $(\pi n_{v}p M_{s}/\gamma)\hat{e}_{z}$, where $p = \cos \Theta(0) = \pm 1$ defines the direction of the polarization of the vortex core (\Theta(0) = 0 or \pi). The second term is evaluated at the zero-th order of the perturbative expansion of the angular coordinates in the low dynamics regime, Eq. (24), taking into account that in the weak bending regime the deformation of the vortex core is small and one can consider $\tilde{r} \simeq r$ because $||\tilde{X}(t, z)||z \ll 1$. In doing so we obtain $(\pi n_{v}p M_{s}/\gamma)\hat{e}_{z}$ again. The gyrovector becomes $\tilde{\rho}_{G} = \rho_{G}p n_{v}\hat{e}_{z}$ with

$$\rho_{G} = 2\pi M_{s}/\gamma$$

Notice that $\tilde{\rho}_{G}$ is the gyrovector linear density as compared to the gyrovector in the Thiele equation for a rigid vortex (24).

Computation of the mass density tensor will be performed in Section VI. For circular polarized disks we show that this tensor reduces to a scalar, $M_{ij} = \rho_{M}\delta_{ij}$, with the vortex core mass density given by $\rho_{M} = \frac{1}{4\gamma^{2}}\ln(R/\Delta_{0})$, where $R$ is the radius of the disk. Only $\omega(\tilde{X}, \partial_{z}\tilde{X})$ contributes to the partial derivative $\partial\tilde{L}/\partial\tilde{X}$ in the slow dynamics regime, because $\tilde{\Pi}_{t} = \rho_{M}\tilde{X}$ and so the term $\tilde{X} \cdot \tilde{\Pi}_{t}$ equals $\rho_{M}\tilde{X}^{2}$. Consequently, the generalized Thiele equation becomes

$$\rho_{M}\tilde{X}(t, z) + \tilde{X}(t, z) \times \tilde{\rho}_{G} + \partial_{t}\tilde{\Pi}_{z} + \nabla_{\tilde{X}}\omega = 0$$

The linear energy density $\omega(\tilde{X}, \partial_{z}\tilde{X})$ is the sum of the magnetostatic and exchange contributions in the $z$-cross-section, $\omega_{XY}(\tilde{X})$, and an elastic contribution due to the deformation of the vortex core line, $\omega_{el}(\partial_{z}\tilde{X})$. Zeeman contribution will be considered later. The dependence on the vortex core coordinates on the $\omega_{XY}(\tilde{X})$ term for small displacements (23, 24)

$$\omega_{XY}(\tilde{X}) = \frac{1}{2}\rho_{M}\omega_{M}e_{0}\tilde{X}^{2},$$

where $\omega_{M} = \rho_{G}/\rho_{M}$ is the characteristic frequency of the system and $e_{0} = \omega_{G}/\omega_{M}$ is a dimensionless parameter. Recall that the conventional gyrofrequency $\omega_{G}$ is defined as

$$\omega_{G} = \frac{\omega_{XY}(\tilde{X} = 0)}{\rho_{G}} \simeq \frac{20}{9}\gamma M_{s}\beta$$

where $\beta = L/R$ is the ratio of the thickness and the radius of the disk. This last expression is valid in the limit $\beta \ll 1$.

From the continuous spin-field model we know that

$$\omega_{el}(\partial_{z}\tilde{X}) = A_{eff} \int d^{2}r\left[(\partial_{z}\Theta)^{2} + \sin^{2}(\Theta(\partial_{z}\Phi))^{2}\right]$$

Noticing that $\partial_{z}\Theta = -\nabla_{r}\Theta \cdot \partial_{z}\tilde{X}$ and $\partial_{z}\Phi = -\nabla_{r}\Phi \cdot \partial_{z}\tilde{X}$, and taking into account the vector identity, $(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$, with either $\vec{A} = \vec{D} = \nabla_{r}\Theta$, $\vec{B} = \vec{C} = -\partial_{z}\tilde{X}$ or $\vec{A} = \vec{D} = \nabla_{r}\Phi$, $\vec{B} = \vec{C} = -\partial_{z}\tilde{X}$, we obtain the following relations

$$\left(\nabla_{r}\Theta \cdot \partial_{z}\tilde{X}\right)^{2} = \left(\nabla_{r}\Theta\right)^{2} \left(\partial_{z}\tilde{X}\right)^{2} - \left(\nabla_{r}\Theta \times \partial_{z}\tilde{X}\right)^{2}$$

$$\left(\nabla_{r}\Phi \cdot \partial_{z}\tilde{X}\right)^{2} = \left(\nabla_{r}\Phi\right)^{2} \left(\partial_{z}\tilde{X}\right)^{2} - \left(\nabla_{r}\Phi \times \partial_{z}\tilde{X}\right)^{2}$$

The main contribution to the integral comes from the zero-th order in the perturbative expansion (24). Notice
Recalling that the angular dependence of \( \Theta \) and the elastic energy density as discussed before, in the weak bending regime we can use approximation \( \hat{r} \approx r \), so that \( \nabla_r \theta_0 \times \partial_x \hat{X} = -\frac{n \cdot \partial_x \hat{X}}{r} \) and \( \nabla_r \Theta_0 \times \partial_x \hat{X} = \frac{\partial \Theta_0}{\partial r} \partial_x \hat{X} + \lambda \), where \( \hat{X} = e_r \cdot \hat{X} = x \cos \theta + y \sin \theta \) and \( \Theta_0 = e_\phi \cdot \hat{X} = -x \sin \theta + y \cos \theta \). The elastic energy density finally becomes

\[
\omega_{el}(\partial_x \hat{X}) = A_{eff} \int d^2 r \left[ \left( \nabla_r \Theta_0 \right)^2 + \frac{\sin^2 \Theta_0}{r^2} \right] \left( \frac{\partial \hat{X}}{\partial z} \right)^2
- \frac{A_{eff}}{2} \int d^2 r \left[ \frac{\partial \Theta_0}{\partial r} \right] \left( \frac{\partial X_\phi}{\partial z} \right) \left( \frac{\partial \hat{X}}{\partial z} \right)
- \frac{A_{eff}}{2} \int d^2 r \left[ \frac{\sin \Theta_0}{r^2} \right] \left( \frac{\partial X_\phi}{\partial z} \right)^2
= \pi A_{eff} \int \left[ \frac{\partial \Theta_0}{\partial r} \right] \left( \frac{\partial \hat{X}}{\partial z} \right) \left( \frac{\partial \hat{X}}{\partial z} \right)
\]

where the angular dependence of \( \left( \partial_x X_r \right)^2 \) and \( \left( \partial_x X_\phi \right)^2 \) has been integrated on \( \theta \). We can recast this energy density as \( \omega_{el}(\partial_x \hat{X}) = \frac{1}{2} \lambda \left( \frac{\partial \hat{X}}{\partial z} \right)^2 \), where \( \lambda \) is the elastic constant given by

\[
\lambda = 2\pi A_{eff} \int r d\theta \left[ \frac{1}{2} \left( \frac{\partial \hat{X}}{\partial \theta} \right)^2 + \frac{\sin^2 \Theta_0}{r^2} \right]
\]

Making use of the variable \( m_0(r) = \cos \Theta_0(r) \), we can rewrite the above equation as

\[
\lambda = 2\pi A_{eff} \int r d\theta \left[ \frac{1}{2} \left( \frac{d m_0}{dr} \right)^2 \frac{1 - m_0^2}{r^2} \right]
\]

and using the spatial dependence \( \frac{1}{r} \), we get

\[
\frac{1}{2} \left( \frac{d m_0}{dr} \right)^2 = \frac{1}{2} m_0^2 = \frac{1}{2} \lambda \left( \frac{\partial \hat{X}}{\partial \theta} \right)^2 + \frac{1}{2} m_0^2
\]

Recalling that \( C_1 = \frac{3}{2} \) and \( C_2 = \frac{1}{2} \rho G r e \) and computing the integral \( \frac{\Delta^2}{2C_1 r^2} \), by splitting into two regions, \( [0, \Delta_0] \) and \( [\Delta_0, R] \), we obtain

\[
\lambda = 2\pi A_{eff} \left[ \frac{50}{49} \ln(R/\Delta_0) \right]
\]

In the limit \( R \gg \Delta_0 \) the logarithmic term dominates and \( \lambda \) becomes

\[
\lambda = 2\pi A_{eff} \ln(R/\Delta_0)
\]

Finally, for the total energy density we obtain

\[
\omega(\hat{X}, \partial_x \hat{X}) = \frac{1}{2} \rho_M \omega_M^2 \epsilon_0 \hat{X}^2 + \frac{1}{2} \lambda \left( \frac{\partial \hat{X}}{\partial z} \right)^2
\]

and thus the generalized Thiele equation for an elastic vortex core line becomes

\[
\rho_M \ddot{\hat{X}}(t, z) - \lambda \partial_z^2 \hat{X}(t, z) + \ddot{\hat{X}}(t, z) \times \hat{\mu} + \rho_M \omega_M^2 \epsilon_0 \hat{X}(t, z) = 0
\]

**IV. SPIN WAVES IN THE VORTEX CORE**

Introducing the complex variable \( \chi = x - iy \) we can recast Eq. (40) as the following complex partial differential equation:

\[
\rho_M \ddot{\chi} - \lambda \partial_z^2 \chi + \rho_G \dot{\chi} + \rho_M \omega_M^2 \epsilon_0 \chi = 0
\]

Let \( \chi_0(z) \) be the equilibrium complex center of the straight vortex core line. In the presence of the wave it gets perturbed and becomes \( \chi(t, z) = \chi_0(z) + \chi_w(t, z) \), with \( ||\chi_w||z \ll ||\chi_0||z \). Switching to the Fourier transform,

\[
\chi_w(t, z) = \frac{1}{2\pi} \int d\omega dq \chi_w(\omega, q) e^{i(\omega t - qz)}
\]

we obtain the following equation for \( \chi_w(\omega, q) \):

\[
[-\rho_M \omega^2 + \lambda q^2 - \rho_G \omega + \rho_M \omega_M^2 \epsilon_0] \chi_w(\omega, q) = 0
\]

For non-zero amplitude of the wave the expression in the square parenthesis must vanish. This determines the spectrum of the waves:

\[
\rho_M \omega^2 - \lambda q^2 + \rho_G \omega - \rho_M \omega_M^2 \epsilon_0 = 0
\]

At \( \rho_M \neq 0 \) one can normalize Eq. (44) to get

\[
\omega^2 + \omega_M \omega - \omega_M^2 \left( \epsilon_0 + \frac{\lambda}{\rho_M \omega_M^2} q^2 \right) = 0
\]

Solving this equation we obtain the spectrum of vortex core excitations:

\[
\omega_{\pm}(q) = \frac{\omega_M}{2} \left[ \sqrt{(1 + 4\epsilon_0) + \frac{4\lambda q^2}{\rho_M \omega_M^2}} \pm 1 \right]
\]

In the weak bending regime we have \( 4\lambda q^2/\rho_M \omega_M^2 \ll 1 \) and so we can expand the square root and obtain the following expression for the frequencies

\[
\omega_{\pm}(q) = \frac{\omega_M}{2} \left( \sqrt{1 + 4\epsilon_0} \pm 1 \right) + \frac{1}{\sqrt{1 + 4\epsilon_0} \rho_G q^2}
\]

where we have used the relation \( \rho_M \omega_M = \rho_G \).
As will be shown in Section IV the parameter $\epsilon_0 = \omega_G/\omega_M$ is normally small due to the smallness of $\beta = L/R$. Consequently

$$\omega_-(q) = \omega_G + \frac{\lambda}{\rho_G} q^2 \quad (48)$$

$$\omega_+(q) = \omega_M + \frac{\lambda}{\rho_G} q^2 \quad (49)$$

With the help of Eqs. (26) and (38) with $A_{eff} \approx A = M^2 \Delta_0^2$ the above equations can be written in a transparent form:

$$\omega_-(q) = \omega_G + \gamma M_\rho (q \Delta_0)^2 \ln(R/\Delta_0) \quad (50)$$

$$\omega_+(q) = \omega_M + \gamma M_\rho (q \Delta_0)^2 \ln(R/\Delta_0) \quad (51)$$

Note that the weak bending regime corresponds to $q \Delta_0 \ll 1$.

V. QUANTUM MECHANICS OF THE EXCITATIONS IN THE VORTEX CORE

In this section we will show that excitations of the vortex core can be also obtained in a rather non-trivial way from the quantum theory as well. This problem is interesting on its own as it turns out to be equivalent to the problem of quantum excitations of a charged string confined in a parabolic potential and subjected to the magnetic field.

It is straightforward to prove that the generalized Thiele equation (40) is the Euler-Lagrange equation associated with the following effective Lagrangian density that can be derived from Eq. (4)

$$\mathcal{L}(t, \{ \tilde{X}_n \}, \{ \tilde{\Pi}_n \}) = \frac{1}{2} \rho_M \tilde{X}^2 + \tilde{\Pi} \cdot \tilde{A}_{\rho G} - \omega(\tilde{X}, \partial_z \tilde{X}) \quad (52)$$

where $\tilde{A}_{\rho G}$ is the gyrovortex potential satisfying $\nabla \times \tilde{A}_{\rho G} = -\tilde{G}$. Thus the total Lagrangian becomes

$$\mathcal{L} = \int dz \, \mathcal{L} = \int dz \left[ \frac{1}{2} \rho_M \tilde{X}^2 + \tilde{\Pi} \cdot \tilde{A}_{\rho G} - \omega(\tilde{X}, \partial_z \tilde{X}) \right] \quad (53)$$

Noticing that $\{ \varphi_n(z) \}_{n \in \mathbb{N}} = \left\{ \frac{\sqrt{2}}{L} \sin \left( \frac{2 \pi n z}{L} \right) \right\}_{n \in \mathbb{N}}$, with $q_n = \frac{2 \pi}{L}$, is a Hilbert basis of the function subspace $W = \{ \varphi \in L^2(0, L), \varphi(0) = \varphi(L) = 0 \}$, we can expand $\tilde{X}$ as

$$\tilde{X}(t, z) = \tilde{X}_0(t) + \sum_n \tilde{X}_n(t) \varphi_n(z), \quad (54)$$

where $\tilde{X}_0(t)$ is the center of the undisturbed vortex and $\tilde{X}_n(t) = \langle \tilde{X}(t, z), \varphi_n(z) \rangle \in L^2(0, L)$. Introducing this expansion in Eq. (53) and taking into account the orthonormality of the Hilbert basis (and its spatial derivatives) we obtain the following identity:

$$\mathcal{L} \left( t, \{ \tilde{X}_n \}_{n \in \mathbb{Z}^+}, \{ \tilde{\Pi}_n \}_{n \in \mathbb{Z}^+} \right) = \left[ \frac{1}{2} \rho_M \tilde{X}_0^2 + \tilde{\Pi}_0 \cdot \tilde{A}_0 - \frac{1}{2} M \omega_M^2 \epsilon_0 \tilde{X}_0^2 \right] + \sum_{n>0} \left[ \frac{1}{2} \rho_M \tilde{X}_n^2 + \tilde{\Pi}_n \cdot \tilde{A}_n - \frac{1}{2} M \omega_M^2 \epsilon_0 \tilde{X}_n^2 - \frac{1}{2} \eta \rho^2 \tilde{X}_n^2 \right] \quad (55)$$

where $\mathbb{Z}^+ = \{ 0 \} \cup \mathbb{N}$, $M = \rho_M L$ is the total mass of the rigid vortex line and $\tilde{A}_n$ is the gyrovortex potential associated to the $n$-th coordinate $\tilde{X}_n$, which satisfies $\nabla \tilde{X}_n \times \tilde{A}_n = -\tilde{G}$ and $\nabla \tilde{X}_n \times \tilde{A}_n = -\tilde{G}$, $n > 0$, with $\tilde{G} = \tilde{G}_C L$ being the gyrovortex of the rigid vortex.

Applying the Laguerre transformation to the above Lagrangian we obtain the following expression for the Hamiltonian

$$\mathcal{H} \left( t, \{ \tilde{X}_n \}_{n \in \mathbb{Z}^+}, \{ \tilde{\Pi}_n \}_{n \in \mathbb{Z}^+} \right) = \left[ \frac{1}{2M} (\tilde{\Pi}_0 - \tilde{A}_0)^2 + \frac{1}{2} M \omega_M^2 \epsilon_0 \tilde{X}_0^2 \right] + \sum_{n>0} \left[ \frac{1}{2\rho_M} (\tilde{\Pi}_n - \tilde{A}_n)^2 + \frac{1}{2} \rho_M \omega_M^2 \left( \epsilon_0 + \frac{\lambda}{\rho_M \omega_M^2 \rho^2} \right) \tilde{X}_n^2 \right] \quad (56)$$

where $\tilde{\Pi}_0 = M \dot{\tilde{X}}_0 + \tilde{A}_0$ and $\tilde{\Pi}_n = \rho_M \dot{\tilde{X}}_n + \tilde{A}_n$, $n > 0$ are the corresponding canonical momenta. Notice that Eq. (56) shows that $\mathcal{H}$ splits into the direct sum $\oplus_{n \in \mathbb{Z}^+} \mathcal{H}_n$, with $\mathcal{H}_n$ being the Hamiltonian defined over the phase space ($\tilde{X}_n, \tilde{\Pi}_n$). It has a structure of the form

$$\mathcal{H} = \frac{1}{2\eta} (\tilde{\Pi} - \tilde{A})^2 + \frac{1}{2 \eta} \omega_M^2 \xi \tilde{X}^2, \quad (57)$$

where $(\tilde{X}, \tilde{\Pi})$ are the canonically conjugate variables, $\eta, \xi$ are constants and $\tilde{A}$ is the gyrovortex satisfying $\nabla \tilde{X} \times \tilde{A} = -\chi \dot{z}$, with $\chi$ being a constant. It is important to point out that $\chi/\eta = \omega_M$ in all cases.

From now on we consider the case of the vortex core of a non-zero mass, ($\eta \neq 0$). It is convenient to choose a "symmetric gauge" given by

$$\tilde{A} = \frac{1}{2} (-\chi \hat{z}) \times \tilde{X} = \frac{\chi y}{2} \hat{x} - \frac{\chi x}{2} \hat{y} \quad (58)$$

Firstly, we define the kinetic momentum operators as $\hat{\tilde{p}} = \eta \hat{\tilde{X}}$, so that $\hat{\tilde{P}} = \hat{\tilde{P}} + \hat{\tilde{A}}$. Notice the following non vanishing commutators

$$[p_j, p_k] = -i h \epsilon_{jk} \quad j, k \in \{ x, y \}, \quad (59)$$

where $\epsilon_{jk}$ is the antisymmetric tensor $\epsilon_{xy} = -\epsilon_{yx} = 1.$
Secondly, we introduce the operators
\[ a = \frac{1}{2\hbar\chi} (p_y + ip_x) \]
\[ a^\dagger = \frac{1}{2\hbar\chi} (p_y - ip_x) \]
which satisfy standard commutation relations for Bose operators, \([a, a^\dagger] = 1\). The number operator \(N_a = a^\dagger a\) satisfies commutation relations \([N_a, a] = -a, \ [N_a, a^\dagger] = a^\dagger\) and we have the identity
\[ \frac{1}{2\hbar\chi} (\bar{T} - \bar{A})^2 = \hbar\omega_M \left( N_a + \frac{1}{2} \right), \]
(61)
In analogy with the case of a charged particle in the electromagnetic field\(^{20}\), we obtain that the gyrotrropic translational group is generated by \(\bar{T} = \bar{\Pi} + \bar{A}\),
\[ T_x = p_x + \chi y, \quad T_y = p_y - \chi x \]
(62)
which satisfies the following commutation relations,
\[ [T_j, p_k] = 0, \quad [T_j, T_k] = i\hbar\chi\epsilon_{jk}, \quad j, k \in \{x, y\} \]
(63)
Now we introduce another set of Bose operators
\[ b = \sqrt{\frac{1}{2\hbar\chi}} (T_y - iT_x) \]
\[ b^\dagger = \sqrt{\frac{1}{2\hbar\chi}} (T_y + iT_x), \]
(64)
which satisfy commutation relations
\[ [b, b^\dagger] = 1, \quad [M_b, b] = -b, \quad [M_b, b^\dagger] = b^\dagger, \]
where \(M_b = b^\dagger b\) is the corresponding number operator. Notice that the commutation relations \([a, b] = [a, b^\dagger] = 0\) also hold.

Coordinates \(x\) and \(y\) can be expressed in terms of the above Bose operators:
\[ x = \frac{1}{\chi}(p_y - T_y) \]
\[ y = -\frac{1}{\chi}(p_x - T_x), \]
(65)
so that
\[ \frac{1}{2} \hbar\omega_M^2 (x^2 + y^2) = \hbar\omega_M (N_a + M_b - ab - a^\dagger b^\dagger + 1). \]
(66)
Consequently, the Hamiltonian \(^{[57]}\) becomes
\[ H' = \hbar\omega_M \left[ (1 + \xi)N_a + \xi M_b - (ab + a^\dagger b^\dagger) + \xi + \frac{1}{2} \right] \]
(67)
It can be diagonalized with the help of Bogoliubov transformations
\[ \tilde{\alpha} = ua - vb^\dagger, \quad \tilde{\beta} = ub - va^\dagger \]
(68)
with \(u, v\) being real numbers. These new operators satisfy Bose commutation relations if \(u^2 - v^2 = 1\). Substituting the above equations into Eq. (67) we obtain
\[ H' = \hbar\omega_M \left[ \tilde{\alpha}^\dagger \tilde{\alpha} (u^2 (1 + \xi) + \xi v^2 - 2\xi uv) + \tilde{\beta}^\dagger \tilde{\beta} (v^2 (1 + \xi) + \xi u^2 - 2\xi uv) + \right. \]
\[ \left. (\tilde{\alpha}^\dagger \tilde{\beta} + \tilde{\beta}^\dagger \tilde{\alpha}) (uv(1 + 2\xi) - \xi (u^2 + v^2)) \right] \]
(69)
To get a Hamiltonian in the oscillator form, the coefficient related to \((\tilde{\alpha}^\dagger \tilde{\beta} + \tilde{\beta}^\dagger \tilde{\alpha})\) should be zero, which requires
\[ uv(1 + 2\xi) - \xi (u^2 + v^2) = 0 \]
(70)
The solution is \(u = \cosh(\theta), \ v = \sinh(\theta), \)
\[ \tanh(2\theta) = \frac{2\xi}{1 + 2\xi} \]
(71)
Finally, the coefficients of the terms \(\tilde{\alpha}^\dagger \tilde{\alpha}\) and \(\tilde{\beta}^\dagger \tilde{\beta}\) become
\[ u^2 (1 + \xi) + \xi v^2 - 2\xi uv = \frac{1}{2} \left[ 1 + \frac{2\xi}{\cosh(2\theta)} + 1 \right] \]
\[ = \frac{1}{2} \sqrt{1 + 4\xi} + 1 \]
\[ v^2 (1 + \xi) + \xi u^2 - 2\xi uv = \frac{1}{2} \left[ 1 + \frac{2\xi}{\cosh(2\theta)} - 1 \right] \]
\[ = \frac{1}{2} \sqrt{1 + 4\xi} - 1 \]
(72)
and, consequently, the Hamiltonian in the second quantization formalism becomes
\[ H' = \hbar\omega_+ \left( \tilde{\alpha}^\dagger \tilde{\alpha} + \frac{1}{2} \right) + \hbar\omega_- \left( \tilde{\beta}^\dagger \tilde{\beta} + \frac{1}{2} \right) \]
(73)
where
\[ \omega_{\pm} = \frac{1}{2} \sqrt{1 + 4\xi} \pm 1 \]
(74)
Noticing that for any \(n \in \mathbb{Z}^+\) we have \(\xi = \epsilon_0 + \frac{\lambda}{\rho_M \omega_M^2} q_n^2\), the second quantization procedure yields the following form of Hamiltonian \(^{[56]}\)
\[ H = \sum_{n \geq 0} \hbar\omega_{n^+} \left( \tilde{\alpha}_n^\dagger \tilde{\alpha}_n + \frac{1}{2} \right) + \sum_{n \geq 0} \hbar\omega_{n^-} \left( \tilde{\beta}_n^\dagger \tilde{\beta}_n + \frac{1}{2} \right) \]
(75)
where \(\omega_{n^\pm}\) are the eigenfrequencies of the vortex state given by
\[ \omega_{n^\pm} = \frac{1}{2} \left[ (1 + 4\epsilon_0) + \frac{4\lambda}{\rho_M \omega_M^2} q_n^2 \pm 1 \right] \]
(76)
which coincides with \(^{[40]}\).
VI. COMPUTATION OF THE VORTEX MASS

As it has been discussed in Section II, to calculate the vortex mass density tensor (see Eq. 10) we need to find solution \((\Phi, t), (\Theta, t)\) of the Landau-Lifshitz equation in the slow dynamics regime, i.e., in the first order on \(\dot{X}\). A more convenient set of variables for this problem is the pair \((\Phi, m)\), where \(m \equiv m_z = \frac{M_z}{M_s} = \cos \Theta\) is the projection of the magnetic moment onto the \(z\) axis. Notice that Landau-Lifshitz equation can be recast as the set of equations

\[
\frac{d\Phi}{dt} = \frac{\gamma}{M_s} \frac{\delta E}{\delta m} \quad \frac{dm}{dt} = -\frac{\gamma}{M_s} \frac{\delta E}{\delta \Phi} \tag{77}
\]

The total energy \(E(\Phi, m)\) splits into the sum

\[
E(\Phi, m) = E_{xx}(\Phi, m) + E_{an}(\Phi, m) + E_{demag}(\Phi, m)
\]

\[
= A \left[ (\nabla \Theta)^2 + \sin^2 \Theta (\nabla \Phi)^2 \right] - K_n \frac{M_s^2}{M_s^2} + K_L \frac{M_s^2}{M_s^2} - \frac{1}{2} \tilde{M} \cdot \tilde{H}_d
\]

\[
= A \left[ \frac{1}{1 - m^2} (\nabla m)^2 + (1 - m^2)(\nabla \Phi)^2 \right] - K_n \cos^2 \Phi (1 - m^2) + K_L m^2 - \frac{1}{2} \tilde{M} \cdot \tilde{H}_d \tag{80}
\]

This gives for the total energy

\[
E(\Phi, m) = A \left[ \frac{1}{1 - m^2} (\nabla m)^2 + (1 - m^2)(\nabla \Phi)^2 \right] - K_n \cos^2 \Phi (1 - m^2) + (K_L + 2\pi M_s^2) m^2 \tag{82}
\]

and the equations of motion (77) become:

\[
\frac{M_s}{\gamma} \frac{d\Phi}{dt} = -\frac{2A}{(1 - m^2)^2} \frac{\nabla m}{\delta m} - 2A \frac{A}{1 - m^2} \tilde{H}_d
\]

\[
- 2A (\nabla \Phi)^2 + 2K_n \cos^2 \Phi m + 2(\tilde{K_L} + 2\pi M_s^2) m
\]

\[
\frac{M_s}{\gamma} \frac{dm}{dt} = -K_n \sin(2\Phi) (1 - m^2) - 4Am \nabla m \cdot \nabla \Phi + 2A (1 - m^2) \Phi \tag{83}
\]

In the slow dynamics regime (\(|\tilde{X}| \ll 1\) solutions \((\Phi, m)\) can be split into \(\Phi = \Phi_0 + \Phi_1\) and \(m = m_0 + m_1\), where \(\Phi_0\) and \(m_0\) are the static solutions of the Landau-Lifshitz equation (we consider the anisotropy interaction to be weak enough so that the static solutions of the Hamiltonian \(E_{xx} + E_{demag}\) are valid for our problem) and where \(\Phi_1\) and \(m_1\) are linear on \(|\tilde{X}|\). Static solutions are given by Eqs. (24). As discussed in Section III, in the weak bending regime we can approximate \(\tilde{r} \approx r\) so that

\[
\nabla \Phi_0 = \frac{\delta \phi}{\delta r} \quad \text{and} \quad \nabla m_0 = \frac{\delta m_0}{\delta r} \frac{d \Phi}{dr}
\]

Linearizing Eqs. (83) and taking into account that

\[
\frac{d \Phi}{dr} = -\tilde{\Phi} \cdot \nabla \Phi \quad \text{and} \quad \frac{dm}{dr} = -\tilde{\Phi} \cdot \nabla m,
\]

we obtain the equations of motion

\[
\frac{M_s}{\gamma} \frac{\tilde{X} \cdot \tilde{e}_r}{r} = -\frac{2A}{1 - m_0^2} \Delta m_1 - \left[ \frac{2A}{1 - m_0^2} \frac{\delta m_0}{\delta r} \frac{d \Phi}{dr} \right]^2
\]

\[
+ \frac{2A}{r^2} - 2K_n \cos^2 \Phi_0 - 2(\tilde{K_L} + 2\pi M_s^2) m_1
\]

\[
- \frac{A m_0}{(1 - m_0^2)^2} \frac{dm_0}{dr} \frac{d \Phi}{dr} \cdot \nabla m_1 - 4A m_0 \frac{\tilde{e}_\phi}{r} \cdot \nabla \Phi_1
\]

\[
- 2K_n m_0 \sin(2\Phi_0) \Phi_1
\]

\[
\frac{M_s}{\gamma} \frac{\tilde{X} \cdot \tilde{e}_r}{r} \frac{dm_0}{dr} = 2A (1 - m_0^2) \Delta \Phi_1
\]

\[
- 4A m_0 \frac{\tilde{e}_\phi}{r} \cdot \nabla m_1 - 4A m_0 \frac{dm_0}{dr} \frac{d \Phi}{dr} \cdot \nabla \Phi_1
\]

\[
+ 2K_n \sin(2\Phi_0) m_0 m_1 - 2K_n \cos(2\Phi_0)(1 - m_0^2) \Phi_1 \tag{84}
\]

Asymptotic expressions for the \(O(|\tilde{X}|)\) corrections to the out-of-plane vortex shape can be determined by substituting Eqs. (24) into Eqs. (84). In doing so we obtain

\[
m_1 = \frac{M_s}{2\gamma} \frac{\tilde{X} \cdot \tilde{e}_r}{r} \frac{d \Phi}{dr} \frac{d \Phi}{dr}
\]

\[
\Phi_1 = \frac{C_2 M_s}{2\gamma A} \delta_\phi (\tilde{X} \cdot \tilde{e}_r) \exp \left( -\frac{r}{\Delta_0} \right) \tag{85}
\]

We are dealing with a two-dimensional micrometric object, so the surface energy term dominates over the volume energy term. We can approximate this surface term by an effective easy plane anisotropy contribution given by

\[
E_{demag,S} = \int_V d^3r 2\pi M_s^2 (\tilde{r}) \tag{81}
\]
for $r \gg \Delta_0$, and

$$m_1 = \frac{M_sC_1n_v}{3\gamma A\Delta_0^2} (\hat{X} \cdot \hat{e}_\phi) r^3$$

$$\Phi_1 = \frac{M_s}{\gamma} \frac{p}{18A} \hat{X} \cdot \mathbf{r}$$

(86)

for $r \ll \Delta_0$. Computation of the mass of the vortex core can be made via $\hat{\Pi}_t$, which should be proportional to $\hat{X}$ in this limit:

$$\hat{\Pi}_t = -\frac{M_s}{\gamma} \int d^2r (\nabla \Phi m)$$

$$= -\frac{M_s}{\gamma} \int d^2r (\nabla \Phi_0) m_0 - \frac{M_s}{\gamma} \int d^2r (\nabla \Phi_0) m_1$$

$$- \frac{M_s}{\gamma} \int d^2r (\nabla \Phi_1) m_0 - \frac{M_s}{\gamma} \int d^2r (\nabla \Phi_1) m_1$$

(87)

Notice that $\frac{M_s}{\gamma} \int d^2r (\nabla \Phi_0) m_0 = 0$ because it corresponds to the momentum of the static solution. The last term of Eq. (87) can be neglected because it is quadratic in $|\hat{X}|$. Therefore it remains to calculate the second and third terms, which are given by

$$-\frac{M_s}{\gamma} \int d^2r (\nabla \Phi_0) m_1 = -\frac{M_s}{\gamma} \int_{r \leq \Delta_0} d^2r (\nabla \Phi_0) m_1$$

$$- \frac{M_s}{\gamma} \int_{r \geq \Delta_0} d^2r (\nabla \Phi_0) m_1 =$$

$$2\pi \frac{M_s^2}{K_\perp + 2\pi M_s^2} \ln(R/\Delta_0) - \frac{1}{56} \hat{X}$$

and

$$-\frac{M_s}{\gamma} \int d^2r (\nabla \Phi_1) m_0 = -\frac{M_s}{\gamma} \int_{r \leq \Delta_0} d^2r (\nabla \Phi_1) m_0$$

$$- \frac{M_s}{\gamma} \int_{r \geq \Delta_0} d^2r (\nabla \Phi_1) m_0$$

$$= \frac{2\pi}{\gamma^2} \left( -\frac{11}{504} + \frac{2}{49} (1 - \Xi \cdot \epsilon^2) \right) \hat{X},$$

(88)

respectively. Notice that $\Xi = \int_{r \geq \Delta_0} d^2r \exp(-2x) / x \simeq \int_{1}^{\infty} dx \exp(-2x) = 0.049$ because we are interested in the limit $R \gg \Delta_0$. Collecting all terms for the momentum, we get for the total mass density

$$\rho_M = \frac{2\pi}{\gamma^2} \left[ \frac{M_s^2}{K_\perp + 2\pi M_s^2} \ln(R/\Delta_0) - \frac{1}{56} \right]$$

(90)

Notice that we are interested in the limit $R \gg \Delta_0$, so that the term involving $\ln(R/\Delta_0)$ is the dominant one. Furthermore, redefining the exchange length by a factor close to unity we can always absorb the small numerical constant in Eq. (90) into the logarithmic term. Magneto-crystalline anisotropies, if they are sufficiently large, destroy the circularly polarized state. Consequently, materials like permalloy, used in the studies of the vortex state, have negligible magneto-crystalline anisotropy energy as compared to the demagnetizing energy. This means that the above expression for the vortex mass density can be reduced to

$$\rho_M \simeq \frac{1}{4\gamma^2} \ln(R/\Delta_0)$$

(91)

With account of this formula one obtains the following expressions for the parameters $\omega_M$ and $\epsilon_0$ that determine eigenfrequencies in the equation (70):

$$\omega_M = \frac{8\pi\gamma M_s}{\ln(R/\Delta_0)}$$

$$\epsilon_0 = \frac{5L}{18\pi R} \ln(R/\Delta_0)$$

(92)

VII. EFFECTS OF THE MAGNETIC FIELD AND DISSIPATION

In this section we study the effects of a magnetic field on the excitation modes of the vortex state. Arbitrary directed magnetic field can be split into two components, one being in the plane of the disk and the other one being perpendicular to it. The effects of these two components can be investigated separately.

Consider first the case of a spatially uniform in-plane magnetic field, $\vec{H}_{in} = h_x \hat{e}_x + h_y \hat{e}_y$. For small displacements along the disk, the magnetic vortex develops an in-plane magnetization density given by

$$\vec{M}(\hat{X}) = -\mu \left[ \hat{z} \times \hat{X} \right], \quad \mu = (2\pi/3)M_s n_v R.$$

(93)

The Zeeman energy density term is

$$\omega_Z(\hat{X}) = -\vec{M}(\hat{X}) \cdot \vec{H}_{in} = -\mu \left[ \hat{z} \times \vec{H}_{in} \right] \cdot \hat{X}$$

$$= \mu h_y x - \mu h_x y$$

(94)

and thus the total in-plane potential energy becomes

$$\omega_{XY}(\hat{X}) = \frac{1}{2} \rho_M \omega_M^2 \epsilon_0 \left( x^2 + y^2 \right) + \mu h_y x - \mu h_x y$$

$$= \frac{1}{2} \rho_M \omega_M^2 \epsilon_0 \left[ x + \frac{\mu h_y}{\rho_M \omega_M^2 \epsilon_0} \right]^2$$

$$+ \left( y - \frac{\mu h_x}{\rho_M \omega_M^2 \epsilon_0} \right)^2 - \frac{1}{2} \rho_M \omega_M^2 \epsilon_0 \vec{H}_{in}^2$$

(95)
Notice that by shifting the origin of the coordinate system we retrieve the original in-plane term of the total energy density \( \frac{1}{2} \rho_M \omega_M^2 M_0 \), except for the constant term \( -\frac{1}{2} \frac{\mu^2}{\rho_M \omega_M^2 M_0} \vec{H}_n^2 \), which is field dependent. Consequently, the application of an in-plane magnetic field does not modify the excitation modes given by (76).

Consider now the effect of the magnetic field perpendicular to the plane of the disk. \( \vec{H}_\perp = \vec{H}_n \). Application of such a field results in the precession of the magnetic moment about the direction of the field, described by the Landau-Lifshitz equation\(^{[22]}\),

\[
\frac{\partial \vec{M}(t, \vec{X})}{\partial t} = -\gamma \left[ \vec{M}(t, \vec{X}) \times \vec{H}_\perp \right],
\]

where \( \gamma \) is the electron gyromagnetic ratio. Formally, this effect can be accounted for by adding an extra term to the gyrovector. Indeed, integration of Eq. \(^{[27]}\) (with no potential energy) on time gives \( \vec{X} = \alpha \left[ \vec{X} \times \vec{\rho}_G \right] \), where \( \alpha = -1/\rho_M \). With account of Eq. (93), we have

\[
\begin{align*}
\dot{\vec{X}} & = -\gamma \left[ \vec{X} \times \vec{\rho}_G \right] + \frac{\rho_G}{\rho_M} \vec{M} \times \vec{H}_\perp \quad (97) \\
\alpha \left[ \dot{\vec{X}} \times \vec{\rho}_G \right] & = -\gamma \left[ \vec{X} \times \vec{\rho}_G \right] + \frac{\rho_G}{\rho_M} \vec{M} \times \vec{H}_\perp \quad (98)
\end{align*}
\]

The vector identity \( \vec{a} \times \vec{b} \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \) leads to \( \alpha \vec{M} \vec{H}_\perp = -\gamma \vec{H}_\perp \). This adds the Larmor frequency to \( \omega_M \):

\[
\omega_M(H) = \frac{\rho_G \text{tot}}{\rho_M} = \omega_M + \frac{\rho_G \vec{M} \cdot \vec{H}_\perp}{\rho_M} = \omega_M + \gamma H
\]

so that the eigenfrequencies \(^{[76]}\) become

\[
\omega^\pm_n(H) = \frac{1}{2} \left[ \sqrt{1 + 4\epsilon(H)} + \frac{4\lambda}{\rho_M \omega_M^2(H)q^2} \pm 1 \right] \omega_M(H)
\]

with \( \epsilon(H) \) given by

\[
\epsilon(H) = \frac{\omega_G(H)}{\omega_M(H)} = \frac{\omega_{XY}(\vec{X} = \vec{0})}{\rho_M \omega_M^2(H)} = \frac{\epsilon_0}{(1 + \gamma H/\omega_M)^2}
\]

Introducing dimensionless variables \( h = \gamma H/\omega_M \) and \( \tilde{\omega}^\pm_n(h) = \omega^\pm_n(H)/\omega_M \) we can rewrite Eqs. (100) as

\[
\begin{align*}
\tilde{\omega}^+_n(h) & = \frac{1}{2} \left[ \sqrt{1 + \frac{4\epsilon_0}{(1 + h)^2}} + \frac{4\lambda}{\rho_M \omega_M^2(1 + h)^2}q^2 \pm 1 \right] \times \\
& \approx \frac{1}{2} \left[ \sqrt{1 + \frac{4\epsilon_0}{(1 + h)^2}} \pm 1 \right](1 + h) + \frac{\lambda q^2}{\sqrt{(1 + h)^2 + 4\epsilon_0 \rho_M \omega_M^2}}
\end{align*}
\]

The distance between \( \omega^+_n \) and \( \omega^-_n \) equals \( \Delta \omega = \omega_M + \gamma H \).

To conclude this Section, we investigate the effects of the dissipation on the excitation modes of magnetic vortices. We consider only the zero field case. Derivation of the corresponding expressions when a magnetic field is applied is straightforward. The way to introduce dissipation into our equations is by adding a damping term of the form \( -D \dot{\vec{X}} \) (\( D \) being the damping constant) to Eq. \(^{[27]}\). Therefore, the elastic Thiele’s equation becomes

\[
\rho_M \dot{\vec{X}} - \lambda \rho_M^2 \vec{X} \times \vec{\rho}_G - D \dot{\vec{X}} + \rho_M \omega_M^2 \epsilon(x) \vec{X} = 0
\]

Repeating the procedure of Sec. IV with the above equation in the massive vortex case \( (\rho_M \neq 0) \) we obtain the following equation for the frequency modes

\[
\omega^2 + (\omega_M + i d) \omega - \omega_M^2 \epsilon(q) = 0
\]

with \( d = D/\rho_M \) and \( \epsilon(q) = \epsilon_0 + \frac{\lambda}{\rho_M \omega_M^2 q^2} \). The (complex) roots of this equation, \( \omega_\pm = \text{Re}(\omega_\pm) + i \text{Im}(\omega_\pm) \), are given by

\[
\begin{align*}
\text{Re}(\omega_\pm) & = \mp \frac{1}{2} \cos(\theta/2) - \frac{\omega_M}{2} \\
\text{Im}(\omega_\pm) & = \mp \frac{1}{2} \sin(\theta/2) - \frac{d}{2}
\end{align*}
\]

with

\[
\theta = \arg \left( \frac{1 + 4\epsilon(q)\omega_M^2 - d^2}{1 + 4\epsilon(q)\omega_M^2 - d^2} + i \left[ 2d\omega_M \right] \right)
\]

\[
= \arctan \frac{2d\omega_M}{\left(1 + 4\epsilon(q)\omega_M^2 - d^2\right)}
\]

In the regime of weak dissipation, \( d \ll \omega_M \), we have \( \theta \simeq \arctan \frac{2d}{(1 + 4\epsilon(q)\omega_M^2)} \) and \( r \simeq (1 + 4\epsilon(q)\omega_M^2) \). As \( \cos[\arctan(x)/2] \approx 1 - x^2/8 + o(x^4) \) and \( \sin[\arctan(x)/2] \approx x^2/2 + o(x^3) \) if \( |x| \ll 1 \), we finally obtain

\[
\begin{align*}
\text{Re}(\omega_\pm) & \approx \mp \frac{1}{2} \left( \sqrt{1 + 4\epsilon(q)\pm 1} \right) - \frac{1}{4} \frac{(d/\omega_M)^2}{(1 + 4\epsilon(q)/3/2)} \omega_M \\
& \approx \frac{\omega_M}{2} \left( \sqrt{1 + 4\epsilon_0} \pm 1 \right) - \frac{\omega_M}{2} \frac{(d/\omega_M)^2}{4 (1 + 4\epsilon_0)/3/2} \\
& + \frac{\lambda}{\sqrt{1 + 4\epsilon_0}} \left( \frac{3}{2} \frac{(d/\omega_M)^2}{(1 + 4\epsilon_0)/3/2} \right) q^2
\end{align*}
\]

and

\[
\begin{align*}
\text{Im}(\omega_+) & = \frac{1}{2} \sqrt{1 + 4\epsilon(q) - 1} \frac{d}{2} \\
\text{Im}(\omega_-) & = \frac{1 + \sqrt{1 + 4\epsilon(q)}}{1 - \sqrt{1 + 4\epsilon(q)}} = \frac{(1 + \sqrt{1 + 4\epsilon(q)^2)}^2}{4 \epsilon(q)}
\end{align*}
\]
VIII. CONCLUSIONS

We have studied excitation modes of vortices in circularly polarized mesoscopic magnetic disks that correspond to the string-like gyroscopic waves in the vortex core. This problem was studied by classical treatment based on Landau-Lifshitz equation and by quantum treatment based upon Hamiltonian approach. The quantum problem is interesting on its own as it is equivalent to the problem of quantum oscillations of a charged string confined in a parabolic potential and subjected to the magnetic field, which in its turn, is a generalization of the problem of the field-induced orbital motion of the electron in a potential well. Both treatments rendered identical results. Our solution generalizes the expression for the frequency of the gyroscopic motion of the vortex for the case of the finite wave number $q$, as

$$\omega_-(q) = \omega_G + \gamma M_s(q\Delta_0)^2 \ln(R/\Delta_0),$$

where $\omega_G$ is the conventional gyrofrequency, $\gamma$ is the gyromagnetic ratio, $M_s$ is the saturation magnetization, $\Delta_0$ is the exchange length, and $R$ is the radius of the disk. This expression is valid in the long-wave limit $q\Delta_0 \ll 1$. The wave number is quantized, $q_n = 2\pi n/L$, where $L$ is the thickness of the disk and $n$ is an integer. For a disk of radius $R \sim 1 \mu m$, thickness $L \sim 100 nm$, exchange length $\Delta_0 \sim 5 nm$, and saturation magnetization $M_s \sim 10^4 emu$, the $n = 1$ mode is separated from $\omega_G$ by a few GHz. It could be excited by, e.g., a tip of a force microscope or a micro-SQUID placed at the center of the disk. Such measurement, while challenging, is definitely within experimental reach.

Throughout this paper we considered of a non-zero mass of the vortex. In addition to the gyroscopic mode the finite provides a new excitation mode, $\omega_+(q) = \omega_M + \gamma M_s(q\Delta_0)^2 \ln(R/\Delta_0)$. The gap, $\omega_M$, is higher than the gyroscopic frequency $\omega_G$. It depends explicitly on the vortex mass. The vortex mass density has been computed by us as a coefficient of proportionality, $\rho_M$, in the kinetic energy of the moving vortex $\rho_M v^2/2$. It is given by $\rho_M \approx 1/(4\gamma^2) \ln(R/\Delta_0)$, where $R$ is the radius of the disk, $\Delta_0$ is the exchange length, and $\gamma$ is the gyromagnetic ratio. For a 25nm thick, micron size permalloy disk this gives the vortex mass in the ball park of $10^{-23} kg$, which is close to the experimental value estimated for a comparable size permalloy ring [15]. Our result for the mass gives $\omega_M = 8\pi \gamma M_s/\ln(R/\Delta_0)$. This is in the ballpark of, or below, the uniform ferromagnetic resonance of the disk. It would be interesting to investigate this frequency range experimentally alongside with the low-frequency gyroscopic mode. One can also test in experiment the explicit field dependence of the vortex modes, computed in this paper. So far we have done it for the low field that only slightly disturbs the vortex state formed in a zero field. However, the statement concerning the existence of the additional mode due to the finite vortex mass should apply to higher fields as well. This case, however, defies analytical study and must employ full-scale numerical micromagnetic calculations. When the field is sufficient to fully polarize the disk in the perpendicular direction, we expect the high frequency mode to evolve into the uniform ferromagnetic resonance.

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1. R. P. Cowburn, D. K. Koltsov, A. O. Adeyeye, M. E. Welland, and D. M. Tricker. Phys. Rev. Lett. 83, 1042 (1999).
2. T. Shinjo, T. Okuno, R. Hassdorf, K. Shigeto, and T. Ono. Science 289, 930 (2000).
3. V. Novosad, K. Yu. Guslenko, H. Shima, Y. Otani, S. G. Kim, K. Fukamichi, N. Kikuchi, O. Kitakami and Y. Shimada. Phys. Rev. B 65, 060402(R) (2002).
4. V. Novosad, F. Y. Fradin, P. E. Roy, K. S. Buchanan, K. Yu. Guslenko and S. D. Bader. Phys. Rev. B 72, 024455 (2005).
5. G. de Loubens, A. Riegler, B. Pigeau, F. Lochner, F. Boust, K. Y. Guslenko, H. Hurdequint, L. W. Molenkamp, G. Schmidt, A. N. Slavin, V. S. Tiberkevich, N. Vukadinovic and O. Klein. Phys. Rev. Lett. 102, 177602 (2009).
6. V. Castel, J. Ben Youssif, F. Boust, R. Weil, B. Pigeau, G. de Loubens, V. V. Naletov, O. Klein and N. Vukadinovic. Phys. Rev. B 85, 184419 (2012).
7. R. Zarzuela, S. Vélez, J. M. Hernandez, J. Tejada and V. Novosad. Phys. Rev. B 85, 180401(R) (2012).
8. S. S. P. Parkin, M. Hayashi, and L. Thomas. Science 320, 190 (2008).
9. E. A. Rozhikova, V. Novosad, D.-H. Kim, J. Pearson, R. Divan, T. Rajh and S. D. Bader. J. Appl. Phys. 105, 07B306 (2009).
10. D.-H. Kim, E. A. Rozhikova, I. V. Ulasov, S. D. Bader, T. Rajh, M. S. Lesniak and V. Novosad. Nature Materials 9, 165-171 (2010).
11. J. Kin Ha, R. Hertel and J. Kirschner. Phys. Rev. B 67, 224432 (2003).
12. K. L. Metlov and K. Yu. Guslenko. J. Mag. Mag. Mat. 242-245, 1015 (2002).
13. S.-B. Choe, Y. Acremann, A. Bauer, A. Doran, J. Stöhr and H. A. Padmore. Science 304, 420-422 (2004).
14. K. Yu. Guslenko. Appl. Phys. Lett. 89, 022510 (2006).
15. K. Yu. Guslenko, B. A. Ivanov, V. Novosad, Y. Otani, H. Shima and K. Fukamichi. J. Appl. Phys. 91, 8037 (2002).
16. K. Yu. Guslenko, X. F. Han, D. J. Keavney, R. Divan and S. D. Bader. Phys. Rev. Lett. 96, 067205 (2006).
17. K.-S. Lee and S.-K. Kim. Appl. Phys. Lett. 91, 132511
18 D. Bedau, M. Kläui, S. Krzyk, U. Rüdiger, G. Faini and L. Vila. Phys. Rev. Lett. 99, 146601 (2007).
19 M. E. Gouvêa, G. M. Wysin, A. R. Bishop and F. G. Mertens. Phys. Rev. B 39, 11840 (1989).
20 G. M. Wysin. Phys. Rev. B 54, 15156 (1996).
21 B. A. Ivanov and G. M. Wysin. Phys. Rev. B 65, 134434 (2002).
22 E.M. Chudnovsky and J. Tejada. Lectures on Magnetism (Rinton Press, Princeton, NJ, 2006).
23 A. A. Thiele. Phys. Rev. Lett. 30, 230 (1973).
24 D. L. Huber, Phys. Rev. B 26, 3758 (1982).
25 R. B. Laughlin. Phys. Rev. B 27, 3383 (1983).
26 J. Zak. Phys. Rev. 134, 6A (1964).