Transition Amplitudes within the Stochastic Quantization Scheme

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Abstract

Quantum mechanical transition amplitudes are calculated within the stochastic quantization scheme for the free nonrelativistic particle, the harmonic oscillator and the nonrelativistic particle in a constant magnetic field; we close with free Grassmann quantum mechanics.
1 Introduction

Stochastic quantization [1] was introduced more than a decade ago as a novel method for quantizing field theories and provides an intriguing connection between statistical mechanics and quantum field theory [2–3].

It is the aim of this paper to explore the somewhat more elementary issue of how to calculate within the stochastic scheme the transition amplitudes of ordinary quantum mechanics. Amusingly this problem has passed practically unnoticed over the years (see however [4–5]) so we would like to present it in a rather elementary way: We study the nonrelativistic free particle, the harmonic oscillator and the nonrelativistic particle in a constant magnetic field, presenting a phase space as well as a configuration space formulation; we close with the free Grassmann quantum mechanics [6].

At the beginning we recall basic facts on stochastic quantization: Given some arbitrary functional $f$ of coordinates $x(t)$, its vacuum expectation value can be obtained as the equilibrium limit $s \to \infty$ of the stochastic correlation function

$$\langle f[x]; s \rangle = \lim_{s \to \infty} \langle f[x]; s \rangle_0$$

which can be defined as

$$\langle f[x]; s \rangle = \int Dxf[x]P[x; s].$$

Here $P[x; s]$ is a normalized (generalized) probability distribution, obeying the Fokker–Planck equation

$$\frac{\partial}{\partial s}P[x; s] = \int_{-\infty}^{\infty} dt \frac{\delta}{\delta x(t)} \left( \frac{\delta}{\delta x(t)} - i \frac{\delta S}{\delta x(t)} \right) P[x; s].$$

We work in real–time quantum mechanics and implicitly assume a convenient $i\epsilon$–insertion in the action $S$ to guarantee the convergence of the stochastic processes [7–10]. Alternatively the stochastic correlations (1.2) can be obtained by assigning to $x(t)$ an additional stochastic time dependence $x(t) \to x(t, s)$ via the Langevin equation

$$\frac{\partial x(t, s)}{\partial s} = i \frac{\delta S}{\delta x(t, s)} + \eta(t, s).$$

Its solution has to be inserted into the given functional $f$ and the average over the white noise $\eta$, characterized by

$$\langle \eta(t, s)\eta(t', s') \rangle = 2\delta(t - t')\delta(s - s')$$

finally has to be worked out.

In quantum mechanics in contrast to the calculation of vacuum expectation values one frequently is interested in the normalized transition elements of time ordered products of operators between Heisenberg state vectors $|x, t\rangle$ which are eigenstates of Heisenberg operators $x(t)$ belonging to the eigenvalues $x$. We therefore introduce new stochastic correlation functions, taking care of the boundary values $x_i$ at $t_i$ and $x_f$ at $t_f$. We consider for example

$$\frac{\langle x_f, t_f | T(x(t_1)x(t_2)) | x_i, t_i \rangle}{\langle x_f, t_f | x_i, t_i \rangle} = \lim_{s \to \infty} \langle x(t_1, s)x(t_2, s) \rangle_{bc}$$

\footnote{Actually we would like to avoid a general discussion of complex stochastic processes but we note that this prescription is sufficient for the case of the quadratic systems under consideration.}
and define in analogy to (1.2)
\[ \langle x(t_1, s)x(t_2, s) \rangle_{bc} = \int_{x(t_e) = x_i, x(t_f) = x_f} D x \ x(t_1)x(t_2) P[x; s]. \] (1.7)

\( P \) now satisfies a Fokker–Planck equation (1.3) with the time integration restricted to the interval \( t \in [t_i, t_f] \); the normalization condition reads \( \langle 1 \rangle_{bc} = 1 \). It follows that the stationary solution is modulo given by \( e^{iS} \) so that the standard path integral representation emerges at the equilibrium limit of (1.7).

Throughout this paper we prefer to use a Langevin equation approach which seems favourable given the rich technical experience already acquired in the case of field theories \([2–3]\); we will give the details of how to implement the boundary conditions in the Langevin scheme in the next section. Furthermore our ultimate goal (not yet addressed in this paper) will be the discussion of constrained quantum mechanical systems and we expect intriguing advantages of the Langevin approach over the rather complicated Fokker–Planck formulation \([4]\).

The central object on which we would like to concentrate, however, is the transition amplitude \( \langle x_f, t_f|x_i, t_i \rangle \) itself, which is one of the fundamental quantities in quantum theory. Due to the normalization condition for the probability density \( P \) we cannot directly express the transition amplitude in terms of it. Stochastic expectation values calculated either with Fokker–Planck or Langevin equation techniques are always normalized automatically and it seemed difficult to find a way to reproduce such quantities as transition amplitudes within the framework of the stochastic quantization. It is, however, indeed possible to relate \( \langle x_f, t_f|x_i, t_i \rangle \) to the normalized stochastic expectation value of the Hamiltonian \( H \). This follows, as in the case of a regular quantum mechanical system the Heisenberg state vectors fulfill
\[ |x_i, t_i \rangle = T \exp[i \int_{t_0}^{t_e} H(t) dt] |x_i, t_0 \rangle \] (1.8)
and
\[ \frac{\partial}{\partial t_i} |x_i, t_i \rangle = iH(t_i) |x_i, t_i \rangle. \] (1.9)

Therefore we get
\[ \frac{\partial}{\partial t_i} \ln \langle x_f, t_f|x_i, t_i \rangle = i \frac{\langle x_f, t_f|H(t_i)|x_i, t_i \rangle}{\langle x_f, t_f|x_i, t_i \rangle}. \] (1.10)

The right hand side is a normalized expectation value of the Hamiltonian at \( t_i \), which implies that this quantity is obtainable as a limit of the stochastic average \( \langle H(t_i, s) \rangle_{bc} \). Here note that for conservative systems the stochastic average of the Hamiltonian is \( t \)-independent at equilibrium \( s = \infty \); for calculational simplicity we evaluate it at the initial time \( t = t_i \). That is, we have as a first step
\[ \langle x_f, t_f|x_i, t_i \rangle = \tilde{c} \exp[i \int_{t_i}^{t_f} \lim_{s \to \infty} \langle H(t_i, s) \rangle_{bc} dt_i] \] (1.11)
with \( \tilde{c} \) a constant independent of \( t_i \). For a transparent presentation we restrict ourselves to Hamiltonians quadratic in the canonical variables. In this case we can separate off very simply contributions from solutions to the classical equations of motion and arrive at
\[ \langle x_f, t_f|x_i, t_i \rangle = \tilde{c} \exp \left[ i \int_{t_i}^{t_f} \tilde{H}_cl(t_i) dt_i \right] \exp \left[ i \int_{t_i}^{t_f} \lim_{s \to \infty} \langle H_Q(t_i, s) \rangle_{bc} dt_i \right] \]
\[ = \tilde{c} \exp[iS_c] \exp \left[ i \int_{t_i}^{t_f} \lim_{s \to \infty} \langle H_Q(t_i, s) \rangle_{bc} dt_i \right]. \] (1.12)
Here the classical action $S_{\text{cl}}$ has appeared as a result of the famous relation $\partial S_{\text{cl}}/\partial t_i = H_{\text{cl}}(t_i)$ \[11\] and is composed of the classical path $x_{\text{cl}}(t)$ as usual. We remark that all the quantum contributions are contained in the second exponential factor.

The constant $c$ in principle could depend on $x_f$ and $x_i$ which, however, is not the case. This follows from a similar variational principle as above for $\langle x_f, t_f | x_i, t_i \rangle$ w.r.t. $x_f$ and $x_i$ and the relations $\partial S_{\text{cl}}/\partial x_f = p_{\text{cl}}(t_f)$ and $\partial S_{\text{cl}}/\partial x_i = -p_{\text{cl}}(t_i)$. So the constant $c$ is indeed $t_f, t_i, x_f, x_i$ independent and can be fixed by requiring the transition amplitude to approach a Dirac $\delta$–function $\delta(x_f - x_i)$ in the limit of $t_i = t_f$ (compare also with Ref. \[11\]).

2 Phase Space Formulation

2.1 General Procedure

In the last section we related the transition amplitude to an expectation value of the Hamiltonian, so it appears appropriate to rely on a phase space formulation of stochastic quantization \[12\]. We put

\[
x(t) = x_{\text{cl}}(t) + x_Q(t)
\]

and implement the boundary conditions for the quantum fields $x_Q(t_i) = x_Q(t_f) = 0$ by the Fourier decompositions

\[
x_Q(t) = \sum_{n=1}^{\infty} x_n \sin \frac{n\pi}{T}(t - t_i),
\]

\[
p_Q(t) = \sum_{n=1}^{\infty} p_n \cos \frac{n\pi}{T}(t - t_i) + \frac{p_0}{2}, \quad T = t_f - t_i.
\]

Notice that no boundary conditions are imposed on the momentum variable $p_Q(t)$ and that any function defined in $[t_i, t_i + T]$ can continuously be extended to $[t_i - T, t_i + T]$ as an even function like the above $p_Q$. Stochastic quantization proceeds by introducing the $s$-dependence for the Fourier modes $x_n \rightarrow x_n(s), p_n \rightarrow p_n(s)$ according to the phase-space Langevin equations

\[
\frac{d}{ds}x_n = i\frac{\delta S_Q}{\delta x_n} + \xi_n, \quad \frac{d}{ds}p_n = i\frac{\delta S_Q}{\delta p_n} + \eta_n
\]

where the noises fulfill

\[
\langle \xi_n(s)\xi_m(s') \rangle = \langle \eta_n(s)\eta_m(s') \rangle = 2\delta_{nm}\delta(s - s'),
\]

\[
\langle \eta_n(s)\xi_m(s') \rangle = \langle \xi_n(s)\eta_m(s') \rangle = 0.
\]

For the quadratic case (2.4) is explicitly solved by

\[
\begin{pmatrix}
  x_n \\
  p_n
\end{pmatrix}(s) = \int_0^s e^{iA_n(s-\sigma)}
\begin{pmatrix}
  \xi_n \\
  \eta_n
\end{pmatrix}(\sigma)d\sigma
\]

where $A_n$ is a model-dependent matrix. Working in Stratonovich calculus we easily derive

\[
\langle x_n(s)\xi_m(s) \rangle = \langle p_n(s)\eta_m(s) \rangle = \delta_{nm},
\]

\[
\langle x_n(s)\eta_m(s) \rangle = \langle p_n(s)\xi_m(s) \rangle = 0
\]
and find that correlations containing stochastic time derivatives vanish in the equilibrium limit 
\( s \to \infty \)
\[
\langle x_n(s) \frac{dx_m(s)}{ds} \rangle = \langle x_n(s) \frac{dp_m(s)}{ds} \rangle = \langle p_n(s) \frac{dx_m(s)}{ds} \rangle = \langle p_n(s) \frac{dp_m(s)}{ds} \rangle \to 0. \tag{2.8}
\]

In the following examples, we avoid solving the Langevin equations explicitly and instead extract
the two-point correlations indirectly by using (2.8).

2.2 Free Particle

The easiest example to discuss is the nonrelativistic free particle with the Hamiltonian
\[
H = \frac{p^2}{2M}. \tag{2.9}
\]

We have now
\[
S_{cl} = \frac{M}{2T} (x_f - x_i)^2, \quad S_Q = \sum_{n=1}^{\infty} \left( \frac{n\pi}{T} x_n p_n - \frac{p_n^2}{2M} \right) \frac{T}{2} - p_0^2 \frac{T}{8M}\tag{2.10}
\]
and get
\[
\begin{align*}
\frac{dx_n}{ds} &= \frac{i n\pi}{2} p_n + \xi_n, \\
\frac{dp_n}{ds} &= i \left( \frac{n\pi}{T} x_n - \frac{p_n}{M} \right) \frac{T}{2} + \eta_n, \\
\frac{dp_0}{ds} &= -i \frac{T}{4M} p_0 + \eta_0.
\end{align*}
\tag{2.11}
\]

From \( \langle p_m(s) \frac{dx_n(s)}{ds} \rangle \), \( \langle p_m(s) \frac{dp_0(s)}{ds} \rangle \) and \( \langle p_0(s) \frac{dp_0(s)}{ds} \rangle \) we immediately find in the equilibrium limit
\[
\langle p_m p_n \rangle = \lim_{s \to \infty} \langle p_m(s) p_n(s) \rangle = 0, \quad \langle p_m p_0 \rangle = 0, \quad \langle p_0^2 \rangle = -\frac{4iM}{T}, \tag{2.12}
\]
respectively. Furthermore,
\[
\lim_{s \to \infty} \langle H_Q(t_i, s) \rangle_{bc} = \lim_{s \to \infty} \frac{\langle p_Q^2(t_i, s) \rangle_{bc}}{2M} = \frac{\langle p_0^2 \rangle}{8M} = -\frac{i}{2T}, \tag{2.13}
\]
so that
\[
i \int_{s=\infty}^{t_f} \langle H_Q(t_i, s) \rangle_{bc} dt_i = -i \int_{0}^{T} (-\frac{i}{2T}) dT = -\frac{1}{2} \ln T \tag{2.14}
\]
and finally
\[
\langle x_f, t_f | x_i, t_i \rangle = c \frac{1}{\sqrt{T}} e^{is_{cl}}, \quad c = \sqrt{\frac{M}{2\pi i}}. \tag{2.15}
\]
2.3 Harmonic Oscillator

Another standard example is given by the harmonic oscillator

\[ H = \frac{\dot{p}^2}{2M} + \frac{1}{2}M\omega^2x^2 \]  

in which case

\[ S_{\text{cl}} = \frac{\omega M}{2\sin \omega T}[(x_i^2 + x_f^2) \cos \omega T - 2x_ix_f], \]  

\[ S_Q = \sum_{n=1}^{\infty} \left( \frac{n\pi}{T}x_n p_n - \frac{p_n^2}{2M} - \frac{1}{2}M\omega^2x_n^2 \right) \frac{T}{2} - \frac{p_0^2}{8M} \]  

so that

\[ \frac{dx_n}{ds} = i \left( \frac{n\pi}{T}p_n - M\omega^2x_n \right) \frac{T}{2} + \xi_n, \]
\[ \frac{dp_n}{ds} = i \left( \frac{n\pi}{T}x_n - \frac{p_n}{M} \right) \frac{T}{2} + \eta_n, \]
\[ \frac{dp_0}{ds} = -i \frac{T}{4M}p_0 + \eta_0. \]  

(2.19)

From \( \langle p_n \frac{dp_0}{ds} \rangle, \langle p_0 \frac{dp_n}{ds} \rangle \) we find in the equilibrium limit

\[ \langle p_n p_0 \rangle = 0, \quad \langle p_0^2 \rangle = -i\frac{4M}{T} \]  

(2.20)

and from \( \langle p_m \frac{dx_n}{ds} \rangle \) combined with \( \langle p_m \frac{dp_n}{ds} \rangle \)

\[ \langle p_n p_m \rangle = \frac{2im}{T} \frac{\delta_{nm}}{\left( \frac{n\pi}{T\omega} \right)^2 - 1}. \]  

(2.21)

Therefore (recalling the boundary condition \( x_Q(t_i) = 0 \))

\[ \lim_{s \to \infty} \langle H_Q(t_i, s) \rangle_{\text{bc}} = \frac{1}{2M} \left( \sum_{n,m=1}^{\infty} \langle p_n p_m \rangle + \frac{\langle p_0^2 \rangle}{4} \right) = -\frac{i\omega}{2} \cot T\omega \]  

(2.22)

and the transition amplitude is obtained as

\[ \langle x_f, t_f | x_i, t_i \rangle = c \frac{1}{\sqrt{\sin T\omega}} e^{iS_{\text{cl}}}, \quad c = \sqrt{\frac{M\omega}{2\pi i}} \]  

(2.23)

2.4 Constant Magnetic Field

A less trivial example of our approach is the nonrelativistic charged particle in a constant magnetic field:

\[ H = \frac{1}{2}(\vec{p} - \vec{A})^2, \quad \vec{p} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \quad \vec{A} = \begin{pmatrix} 0 \\ Bx \\ 0 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \]  

(2.24)
We split again
\[ \vec{x}(t) = \vec{x}_{cl}(t) + \vec{x}_Q(t), \quad \vec{p}(t) = \vec{p}_{cl}(t) + \vec{p}_Q(t) \]  
(2.25)
and obtain
\[ S_{cl} = \frac{1}{2} \left\{ \frac{(z_f - z_i)^2}{T} + \frac{B}{2} \cot{\frac{BT}{2}} [(x_f - x_i)^2 + (y_f - y_i)^2] + B(x_i + x_f)(y_f - y_i) \right\} \]  
(2.26)
as well as
\[ S_Q = -\frac{T}{8} \vec{p}_0^2 + \sum_{n=1}^{\infty} \left\{ \frac{n\pi}{T} \vec{x}_n \cdot \vec{p}_n - \frac{1}{2} \vec{p}_n^2 - \frac{B^2}{2} \vec{x}_n^2 \right. \]
\[ - \frac{BT}{\pi} \left[ x_n q_0 \left( -\frac{1}{n} - 1 \right) - \sum_{m=1}^{\infty} x_n q_m \frac{( - 1 )^{n+m} - 1}{n^2 - m^2} \right] \}. \]  
(2.27)
The Langevin equations read as
\[ \frac{d\vec{x}_n}{ds} = i \left( \frac{n\pi}{2} \vec{x}_n + \vec{a}_n \right) + \vec{\xi}_n \]
\[ \frac{d\vec{p}_n}{ds} = i \left( \frac{n\pi}{2} - \vec{p}_n + \vec{b}_n \right) + \vec{\eta}_n \]
\[ \frac{\vec{p}_0}{ds} = i \left( -\frac{T}{4} \vec{p}_0 + \vec{c} \right) + \vec{\eta}_0 \]  
(2.28)
where we have introduced
\[ \vec{a}_n = \frac{BT}{2} \begin{pmatrix} \alpha_n \\ 0 \\ 0 \end{pmatrix}, \quad \vec{b}_n = \begin{pmatrix} 0 \\ \beta_n \\ 0 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \]  
(2.29)
with
\[ \alpha_n = -Bx_n - q_0 \frac{(-1)^n - 1}{n\pi} + \frac{2}{\pi} \sum_{m=1}^{\infty} q_m n \frac{(-1)^{n+m} - 1}{n^2 - m^2} \]
\[ \beta_n = -\frac{BT}{\pi} \sum_{m=1}^{\infty} n \frac{(-1)^{n+m} - 1}{n^2 - m^2} \]
\[ \gamma = -\frac{BT}{2\pi} \sum_{m=1}^{\infty} x_m \frac{(-1)^m - 1}{m} \].

It follows from \( \langle q_m \frac{dy_n}{ds} \rangle, \langle q_0 \frac{dy_0}{ds} \rangle, \langle p_n \frac{dp_0}{ds} \rangle, \langle r_m \frac{dz_n}{ds} \rangle \) and \( \langle r_n \frac{dr_m}{ds} \rangle \) that
\[ \langle q_n q_m \rangle = \langle q_n q_0 \rangle = \langle p_n p_0 \rangle = \langle r_n r_m \rangle = \langle r_n r_0 \rangle = 0. \]  
(2.30)
Therefore
\[ \lim_{s \to \infty} \langle H_Q(t_i, s) \rangle_{bc} = \frac{1}{8} \left( \langle \vec{p}_0^2 \rangle + \frac{1}{2} \sum_{n,m=1}^{\infty} \langle p_n p_m \rangle \right). \]  
(2.31)
Using $\left< p_0 \frac{dp_0}{ds} \right>$, $\left< r_0 \frac{dr_0}{ds} \right>$ we have as previously

$$\langle p_0^2 \rangle = -\frac{4i}{T}. \tag{2.32}$$

From $\left< q_0 \frac{dp_n}{ds} \right>$ we relate $\langle q_0 p_n \rangle$ to $\langle x_n q_0 \rangle$; then $\left< q_0 \frac{dx_n}{ds} \right>$ connects $\langle x_n q_0 \rangle$ to $\langle q_0^2 \rangle$ and from $\left< q_0 \frac{dq_0}{ds} \right>$ we find

$$\langle q_0^2 \rangle = -i2B \cot \frac{BT}{2}. \tag{2.33}$$

From $\left< p_m \frac{dp_n}{ds} \right>$ we relate $\langle x_n p_m \rangle$ to $\langle p_n p_m \rangle$; from $\left< p_n \frac{dx_n}{ds} \right>$ relating $\langle x_n p_m \rangle$, $\langle q_0 p_n \rangle$ and $\langle q_0^2 \rangle$ we obtain

$$\langle p_n p_{n'} \rangle = 2i \frac{B^2 T}{\pi^2} \left[ \frac{\delta_{nn'}}{n^2 - (\frac{BT}{\pi})^2} - \frac{BT}{\pi^2} \frac{(-1)^n - 1}{n^2} \frac{(-1)^{n'} - 1}{(\frac{BT}{\pi})^2} \cot \frac{BT}{2} \right]. \tag{2.34}$$

Straightforwardly we finally arrive at

$$\lim_{s \to \infty} \langle H_{Q}(t_i, s) \rangle_{bc} = -i B \cot \frac{BT}{2} + i \frac{B^2}{2T} \tag{2.35}$$

and

$$\langle x_f, t_f | x_i, t_i \rangle = c \frac{1}{\sqrt{T \sin \frac{BT}{2}}} e^{i S_{cl}}, \quad c = \frac{B}{2} \left( \frac{B}{2\pi i} \right)^{3/2}. \tag{2.36}$$

### 3 Configuration Space Formulation

#### 3.1 General Procedure

Although as outlined in the last section the phase space formulation is the natural one, it is for technical reasons preferable to rely on the simpler configuration space formulation. If

$$H = \frac{p^2}{2M} + V(x), \quad \mathcal{L} = \frac{M}{2} \dot{x}^2 - V(x) \tag{3.1}$$

we have, of course,

$$\langle H_{Q}(t_i, s) \rangle_{bc} = \left< p_Q(t_i, s) \frac{\partial x_Q(t_i, s)}{\partial t} \right>_{bc} - \langle \mathcal{L}_{Q}(t_i, s) \rangle_{bc}. \tag{3.2}$$

Now

$$\langle \mathcal{L}_{Q}(t_i, s) \rangle_{bc} = \frac{M}{2} \left< \frac{\partial x_Q^2}{\partial t}(t_i, s) \right>_{bc} = \frac{M}{2} \lim_{t_i, t_2 \to t_i} \partial_{t_1} \partial_{t_2} \langle x_Q(t_1, s) x_Q(t_2, s) \rangle_{bc} \tag{3.3}$$

where we have used Feynman’s time splitting procedure to properly define the kinetic energy contributions [11]. The transition to a pure configuration space formulation is achieved by observing that generally

$$\frac{\partial p_Q}{\partial s}(t, s) = i \left( \frac{\partial x_Q(t, s)}{\partial t} - \frac{\partial H_Q}{\partial p_Q(t, s)} \right) + \xi(t, s). \tag{3.4}$$
From $\langle \frac{\partial x_Q(t', s)}{\partial t} \frac{\partial p_Q(t, s)}{\partial s} \rangle_{bc}$ we find

$$\lim_{s \to \infty} \left[ \langle \frac{\partial x_Q(t', s)}{\partial t} p_Q(t, s) \rangle_{bc} - M \langle \frac{\partial x_Q(t', s)}{\partial t} \frac{\partial x_Q(t, s)}{\partial t} \rangle_{bc} \right] = 0 \quad (3.5)$$

and finally obtain

$$\lim_{s \to \infty} \langle H_Q(t_i, s) \rangle_{bc} = \frac{M}{2} \lim_{t_1, t_2 \to t_i} \partial_{t_1} \partial_{t_2} \lim_{s \to \infty} \langle x_Q(t_1, s)x_Q(t_2, s) \rangle_{bc}, \quad (3.6)$$

which coincides with the very definition of the Hamiltonian in the path-integral formulation [11] and can be calculated entirely in configuration space. We start from the regular Lagrangian $\mathcal{L}$ and separate the coordinate $x(t)$ into a classical and quantum part, the latter being Fourier expanded as in (2.2). The configuration-space Langevin equations for the Fourier modes are associated as in the original Parisi-Wu approach and the transition amplitudes can be calculated using (1.12) and (3.6). We remark that the transition to configuration space in the case of the particle in a constant magnetic field arrives at the same expression (3.6), even though it corresponds to a case of velocity-dependent quantum mechanical potential.

### 3.2 Free Particle

To demonstrate the configuration space approach we start from the free particle action

$$S = \int_{t_i}^{t_f} dt \frac{M}{2} x^2, \quad S_Q = \sum_{n=1}^{\infty} \frac{n^2 \pi^2 M}{4T} x_n^2, \quad (3.7)$$

and get

$$\frac{dx_n}{ds} = i \frac{n^2 \pi^2 M}{2T} x_n + \eta_n \quad (3.8)$$

so that

$$\langle x_n x_m \rangle = \frac{2i}{n^2 \pi^2 M} \delta_{nm}. \quad (3.9)$$

From this we immediately find

$$\lim_{s \to \infty} \langle x_Q(t_1, s)x_Q(t_2, s) \rangle_{bc} = \frac{i}{MT} [t_f - \max(t_1, t_2)][\min(t_1, t_2) - t_i] \quad (3.10)$$

and have from (3.6)

$$\lim_{s \to \infty} \langle H_Q(t_i, s) \rangle_{bc} = -\frac{i}{2T}. \quad (3.11)$$

Here we have to keep in mind that we should first sum over Fourier modes $n$ to obtain $\lim_{s \to \infty} \langle x_Q(t_1, s)x_Q(t_2, s) \rangle_{bc} = \langle x_Q(t_1)x_Q(t_2) \rangle_{bc}$, differentiate w.r.t. $t_1$ and $t_2$ and finally take the limit $t_1, t_2 \to t_i$. This ordering corresponds to Feynman’s time splitting procedure [11], which avoids infinite contributions arising from the velocity correlation at the same point. If the order were reversed or changed, for example, if we differentiated $x_Q(t)$ before the summation over $n$, we would have an infinite contribution $\sum_{n=1}^{\infty} = \infty$, as is easily seen from (3.9).
3.3 Harmonic Oscillator

With
\[ S = \int_{t_i}^{t_f} dt \left( \frac{M}{2} \dot{x}^2 - \frac{1}{2} M \omega^2 x^2 \right), \quad S_Q = \sum_{n=1}^{\infty} \frac{M \pi^2}{4T} \left[ n^2 - \left( \frac{T \omega}{\pi} \right)^2 \right] x_n^2, \]
we obtain now
\[ \frac{dx_n}{ds} = i \frac{M \pi^2}{2T} \left[ n^2 - \left( \frac{T \omega}{\pi} \right)^2 \right] x_n + \eta_n \]
and
\[ \langle x_n x_m \rangle = i \frac{2T}{M \pi^2} \frac{\delta_{nm}}{n^2 - \left( \frac{T \omega}{\pi} \right)^2}. \]

Furthermore we obtain
\[ \lim_{s \to \infty} \langle x_Q(t_1, s)x_Q(t_2, s) \rangle_{bc} = -i \frac{\omega}{2} \cot T \omega. \]

3.4 Constant Magnetic Field

Despite the simple looking form
\[ L = \frac{\dot{x}^2}{2} + \dot{x} \cdot \vec{A}, \]
\[ S_Q \]
takes an involved expression
\[ S_Q = \frac{\pi^2}{4T} \sum_{n,n'=1}^{\infty} (x_n, y_n, z_n) \begin{pmatrix} D & M & 0 \\ -M & D & 0 \\ 0 & 0 & D \end{pmatrix}_{nm'} \begin{pmatrix} x_{n'} \\ y_{n'} \\ z_{n'} \end{pmatrix}, \]
where
\[ D_{nn'} = n^2 \delta_{nn'}, \quad M_{nn'} = -\frac{(1)^{n+n'}-1}{n^2-n'^2} \frac{2BT}{\pi^2}. \]

As the z-components decouple, we first investigate the \((x_n, y_n)\) contributions only. We start from the Langevin equations
\[ \frac{\partial}{\partial s} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = i \frac{\pi^2}{2T} \begin{pmatrix} D & M \\ -M & D \end{pmatrix}_{nn'} \begin{pmatrix} x_{n'} \\ y_{n'} \end{pmatrix} + \begin{pmatrix} \eta_n \\ \xi_n \end{pmatrix} \]
and multiply with \((x_{n''}, y_{n''})\) to extract in the equilibrium limit
\[ -i \frac{\pi^2}{2T} \begin{pmatrix} D & M \\ -M & D \end{pmatrix}_{nn'} \begin{pmatrix} x_{n'} \\ y_{n'} \end{pmatrix} \begin{pmatrix} x_{n''} \\ y_{n''} \end{pmatrix} = \begin{pmatrix} 1_{nn''} & 0 \\ 0 & 1_{nn''} \end{pmatrix}. \]
We eliminate the off-diagonal expectation values and find for \( x_n \) (we remark that an identical expression holds for \( y_n \))

\[
- \frac{i\pi^2}{2T} (D + MD^{-1})_{nn'} \langle x_n x_{n'} \rangle = \delta_{nn'}
\]

which explicitly evaluates to

\[
- \frac{i\pi^2}{2T} \left\{ n^2 - \left( \frac{BT}{\pi} \right)^2 \right\} \delta_{nn'} + 2 \left( \frac{BT}{\pi^2} \right)^2 \frac{[(-1)^n - 1][(-1)^{n'} - 1]}{nn'} \langle x_{n'} x_{n''} \rangle = \delta_{nn''}.
\]

We find

\[
\langle x_n x_{n'} \rangle = \frac{2iT}{\pi^2} \left\{ \frac{\delta_{nn'}}{n^2 - \left( \frac{BT}{\pi} \right)^2} - \frac{(BT)^3}{\pi^4} \frac{[(-1)^n - 1][(-1)^{n'} - 1]}{nn'} \frac{\cot BT}{2} \right\}.
\]

Therefore after the summations over \( n \) and \( n' \), we get the correlation function

\[
\langle x_Q(t_1) x_Q(t_2) \rangle_{bc} = \frac{i}{B \sin BT} \left\{ \sin B(t_f - \text{max}(t_1, t_2)) \sin B(\text{min}(t_1, t_2) - t_i) \right.
\]

\[
- \left[ \cos \frac{B}{2}(t_f + t_i - 2t_1) - \cos \frac{BT}{2} \right]\left[ \cos \frac{B}{2}(t_f + t_i - 2t_2) - \cos \frac{BT}{2} \right],
\]

from which follows

\[
\lim_{t_1, t_2 \to t_i} \partial_{t_1} \partial_{t_2} \lim_{s \to \infty} \langle x_Q(t_1, s) x_Q(t_2, s) \rangle_{bc} = - \frac{iB}{2} \cot \frac{BT}{2}.
\]

Adding an identical \( y_n \)-contribution and the trivial \( z_n \)-contribution we get in total as before

\[
\lim_{s \to \infty} \langle H(t_i, s) \rangle_{bc} = - \frac{i}{2B} \cot \frac{BT}{2} - i \frac{1}{2T}.
\]

### 3.5 Nonrelativistic Grassmann Quantum Mechanics

In this section we want to extend our method to deal with Grassmann quantum systems. We study the simplest possible model, which is a free nonrelativistic Grassmann particle described by [6]

\[
\mathcal{L} = \frac{M}{2} \left( \frac{d\Theta}{dt} \right)^2.
\]

Here we have introduced a Grassmann vector

\[
\Theta = \{ \Theta^\alpha \}, \quad \alpha = 1, 2
\]

and defined an inner product

\[
\Theta^2 = \Theta \cdot \Theta = \Theta^\alpha \Gamma_{\alpha \beta} \Theta^\beta
\]

with the help of a metric tensor

\[
\Gamma_{\alpha \beta} = -\Gamma^{\alpha \beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
Notice that (3.28) differs from Grassmann quantum mechanical models which are inspired from particle physics and have kinetic terms linear in the time derivative [13–15]. These models generally lead to second class constraints, however. Having decided to restrict ourselves in this paper to regular systems we do not include these models in our discussion.

We choose to work in the configuration space formulation and separate the Grassmann variable into a classical and quantum part as before

$$\Theta(t) = \Theta_{cl}(t) + \Theta_Q(t), \quad \Theta_Q(t) = \sum_{n=1}^{\infty} \Theta_n \sin \frac{n\pi}{T} (t - t_i)$$  \hspace{1cm} (3.32)

so that

$$S_{cl} = \frac{M}{2} \frac{(\Theta_f - \Theta_i)^2}{T}, \quad S_Q = \sum_{n=1}^{\infty} \frac{n^2\pi^2 M^2}{4T} \Theta_n^2.$$  \hspace{1cm} (3.33)

The definition of the Langevin equation for the Grassmann mode $\Theta_n$ requires (similarly to the fermionic field theory case [16]) the introduction of a kernel, which we choose to be $\Gamma^{\alpha\beta}$

$$\frac{d\Theta_n^\alpha}{ds} = i\Gamma^{\alpha\beta} \frac{\delta S_Q}{\delta \Theta_n^\beta} + \eta_n^\alpha$$
$$= \frac{i n^2 \pi^2 M}{2T} \Theta_n^\alpha + \eta_n^\alpha.$$  \hspace{1cm} (3.34)

Here the noise fulfills

$$\langle \eta_n^\alpha(s) \eta_m^\beta(s') \rangle = 2\delta_{nm} \Gamma^{\alpha\beta} \delta(s - s')$$  \hspace{1cm} (3.35)

which guarantees the Grassmann anticommutativity. From $\langle \Theta_m(s) \cdot \frac{d\Theta_n}{ds}(s) \rangle$ we find

$$\lim_{s \to \infty} \langle \Theta_n(s) \cdot \Theta_m(s) \rangle = -\frac{4iT}{n^2 \pi^2 M} \delta_{nm}$$  \hspace{1cm} (3.36)

where a crucial minus sign has appeared as a consequence of the definition of the inner product with $\Gamma^{\alpha\beta}$. We easily deduce

$$\lim_{s \to \infty} \langle \Theta_Q(t_1, s) \cdot \Theta_Q(t_2, s) \rangle_{bc} = -\frac{2iT}{MT} [t_f - \max(t_1, t_2)][\min(t_1, t_2) - t_i]$$  \hspace{1cm} (3.37)

so that after differentiation

$$\lim_{s \to \infty} \langle H_Q(t_i, s) \rangle_{bc} = \frac{i}{T}$$  \hspace{1cm} (3.38)

and the desired transition amplitude becomes

$$\langle \Theta_{f_f} | \Theta_{i_i} \rangle = cTe^{iS_{cl}}, \quad c = \frac{i}{M}.$$  \hspace{1cm} (3.39)

Note that the constant $c$ was fixed in this example by requiring the transition amplitude to approach a Berezin delta function in the equal time limit $t_i = t_f$ [6].
4 Conclusions

In this paper we have filled a somewhat historical gap in the applications of stochastic quantization and calculated for the nonrelativistic free particle, the harmonic oscillator, the non-relativistic particle in a constant magnetic field and the free Grassmann particle the quantum mechanical transition amplitudes. Our main observation is the possibility to relate the non-normalized transition amplitudes with the normalized excitation values within the stochastic quantization scheme taking care of the boundary conditions.

Our procedure can equally well be applied to Euclidean quantum mechanics, in which case the corresponding partition functions become straightforwardly calculable.

It remains a challenge to apply our scheme to constrained systems.

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