Skyrmions in Higher Landau Levels

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Abstract

We calculate the energies of quasiparticles with large numbers of reversed spins ("skyrmions") for odd integer filling factors $\nu = 2k + 1, k \geq 1$. We find, in contrast with the known result for $\nu = 1 (k = 0)$, that these quasiparticles always have higher energy than the fully polarized ones and hence are not the low energy charged excitations, even at small Zeeman energies. It follows that skyrmions are the relevant quasiparticles only at $\nu = 1, 1/3$ and $1/5$. 
Two advances in device fabrication have, in recent years, focussed the attention of theorists and experimentalists on the role of the spin degree of freedom in the physics of systems that exhibit the quantum Hall effect (QHE). These are the advent of higher mobility samples that exhibit the QHE at relatively “low” magnetic fields and correspondingly small values of the Zeeman energy, and that of double layer systems where the layer index defines an approximately $SU(2)$ symmetric “pseudospin” that does not couple to a Zeeman term at all. The description of the physics of the spin was given a new twist in recent work [1] on ferromagnetic filling factors in the QHE [4] in which it was argued that the low energy, long wavelength physics was distinguished by a novel linkage between the density fluctuations and those of the topological density of the spins; this linkage has been extended to double layer systems and exploited to great effect by Yang et al. [3] in their theory of a novel phase transition in the latter [4].

Our focus in this communication is on the specific use of the density–topological density relation by Sondhi et al., namely their demonstration that at $\nu = 1, 1/3$ and $1/5$, and in the small Zeeman energy ($g \to 0$) limit, the quasiparticles involve a large number of reversed spins and are skyrmions. While their specific work was restricted to these filling factors and is corroborated both by their finite size studies and by subsequent Hartree-Fock calculations for $\nu = 1$ [3], it was suggested in [1] that the conclusions would generalize to all ferromagnetic fillings. It was subsequently pointed out by Jain and Wu [6] that finite size studies at $\nu = 3$ yield fully polarized quasiparticles and that this would continue to be the case at all higher odd integer filling factors as well. Our purpose here is to argue that 1) there are skyrmions in the quasiparticle spectrum at odd integer $\nu \geq 3$, but that 2) they have, in agreement with [6], higher energies than the fully polarized quasiparticles and hence are not the relevant low energy excitations in any regime. Combining this information with general consequences of the composite fermion description of the QHE, we conclude that the only fillings at which skyrmions are the lowest energy excitations at small Zeeman energies are $\nu = 1, 1/3$ and $1/5$.

**Skyrmions Energies from Effective Lagrangian:** We briefly recapitulate the procedure
used to calculate the energies of the skyrmions in [1]. We begin by noting that the long
wavelength physics, at ferromagnetic fillings and at small Zeeman energies, can be described
by a purely magnetic effective lagrangian of the form,

\[ \mathcal{L}_{\text{eff}} = \alpha \mathbf{A}(\mathbf{n}(\mathbf{r})) \cdot \partial_t \mathbf{n}(\mathbf{r}) + \alpha' (\nabla \mathbf{n}(\mathbf{r}))^2 + g\mu_B \mathbf{n}(\mathbf{r}) \cdot \mathbf{B} \]

\[-\frac{1}{2} \int d^2r' V(\mathbf{r} - \mathbf{r}') q(\mathbf{r}) q(\mathbf{r}'). \]  

(1)

Here \( \mathbf{n}(\mathbf{r}) \) is a field of unit magnitude and describes the orientation of the local spins, \( \mathbf{A} \)
is the vector potential of a unit monopole, i.e. \( \epsilon^{ijk} \partial_j \mathbf{A}^k = n^i \), and \( q(\mathbf{r}) = \epsilon^{ij} \epsilon^{abc} \partial_i n^a \partial_j n^b \partial_c n^c / 8\pi \) is the topological density of the \( \mathbf{n} \) field. The first three terms are present in any ferromagnet; however, the last term is specific to the quantum Hall problem and arises from the relation,

\[ \delta \rho(\mathbf{r}) = \nu q(\mathbf{r}), \]  

(2)

between the fluctuations of the density and those of the topological density of the spins. The value of \( \nu \) here is the filling factor of the spins that constitute the magnetic degrees of freedom and is therefore equal to 1 for all odd integer \( \nu \).

Skyrmions (antiskyrmions) are topologically nontrivial excitations of the spin field that carry topological charge \( Q \equiv \int d^2 r q(\mathbf{r}) = \pm 1 \). To compute their energies in the small Zeeman limit we need to fix the values of \( \alpha \) and \( \alpha' \) which can be done from a knowledge of Larmor’s theorem and the spin-wave dispersion. More precisely, if \( \hbar \omega(q) \sim g\mu_B B + \kappa(e^2/\epsilon l)(ql)^2 \) at small \( q \), then

\[ \alpha = \frac{1}{4} \hbar \rho \quad \text{and} \quad \alpha' = \frac{\kappa e^2}{8\pi \epsilon l}, \]  

(3)

where \( l \) is the magnetic length and \( \rho \) is the density of electrons with free spins; at odd integer \( \nu \) this equals \( 1/(2\pi l^2) \). Finally, the limiting value of the energy of the skyrmions at small Zeeman energies is \( E_s = 8\pi \alpha' = \kappa(e^2/\epsilon l) \).

We calculate the spin-wave stiffness \( \kappa \) using the results of Kallin and Halperin [1]. Their Eq. (4.11) yields, for \( \nu = 2k + 1 \), the dispersion relation
$$\hbar \omega(q) = g \mu_B B - \Sigma_k - \tilde{V}_{kkkk}^{(1)}(q)$$

(4)

where \( \Sigma_k \) is given by the expressions [1],

$$\Sigma_k = -\frac{e^2}{\ell} V(k, k)$$

$$V(l, m) = \frac{1}{\sqrt{2} m!} \sum_{r=0}^{l} \binom{l}{r} (-1)^r \frac{\Gamma(r + 1/2) \Gamma(m - r + 1/2)}{r! \Gamma(1/2 - r)}$$

(5)

and the matrix element is defined as the integral

$$\tilde{V}_{kkkk}^{(1)}(q) = \frac{e^2}{\ell} \int \frac{d^2x}{2\pi} \frac{1}{|x - ql|} [L_k(x^2/2)]^2 e^{-x^2/2}.$$

(6)

Two features of \( \tilde{V}_{kkkk}^{(1)}(q) \) follow from these expressions at once: first that \( \tilde{V}_{kkkk}^{(1)}(0) = \Sigma_k \) in order that the Larmor theorem holds, and second that \( \tilde{V}_{kkkk}^{(1)}(q) \) vanishes as \( q \to \infty \) whence the gap to creating a polarized quasiparticle-quasihole pair is exactly \(-\Sigma_k\).

We have not carried out the integral in (6) as it stands. Instead we expand the Laguerre polynomials as

$$L_k(x) = \sum_{m=0}^{k} (-1)^m \binom{k}{m} \frac{x^m}{m!}$$

$$= \sum_{m=0}^{k} (-1)^m \frac{k}{m} x^m$$

(7)

and find that

$$\tilde{V}_{kkkk}^{(1)}(q) = \sqrt{\frac{\pi}{2}} \sum_{l} \sum_{m} c_l^k c_m^k \frac{d^{l+m}}{d\tau^{l+m}} \frac{1}{\sqrt{\tau}} e^{-\tau q^2 l^2/4} I_\nu(\alpha q^2 l^2/4)|_{\tau=1}.$$ 

(8)

This is evidently an expansion in powers of \((ql)^2\) and using Mathematica it is straightforward to obtain the coefficient of the quadratic term for a given filling factor. For the first few odd integer fillings we find \( \kappa = \frac{1}{4} \sqrt{\frac{2}{\nu}} \) (\( \nu = 1 \)), \( \frac{7}{16} \sqrt{\frac{2}{\nu}} \) (\( \nu = 3 \)), \( \frac{145}{256} \sqrt{\frac{2}{\nu}} \) (\( \nu = 5 \)). These are also, in units of \( e^2/\ell \), the energies of the skyrmions/antiskyrmions and therefore half the gap to making an (infinite) skyrmion-antiskyrmion pair.

For the corresponding fillings we obtain the gap to a fully polarized quasiparticle-quasihole pair from Eq. 5; these are, in units of \( e^2/\ell \), \( \sqrt{\frac{2}{\nu}} \) (\( \nu = 1 \)), \( \frac{3}{4} \sqrt{\frac{2}{\nu}} \) (\( \nu = 3 \)),
Finally, we find that the ratios of the interaction energy of an (infinite) skyrmion-antiskyrmion pair to that of a fully polarized quasiparticle-quasihole pair are:

\[ \frac{\Delta_{\text{sk--ask}}}{\Delta_{\text{pqh--pqe}}} = \frac{1}{2} \ (\nu = 1), \ \frac{7}{6} \ (\nu = 3) \ \text{and} \ \frac{145}{82} \ (\nu = 5). \] (9)

It follows then, that of the odd integer filling factors only \( \nu = 1 \) has large skyrmions as its lowest energy quasiparticles at small Zeeman energies.

**Finite Size Study at \( \nu = 3 \):** For the particular case of \( \nu = 3 \), we have expanded on the \( g = 0 \) finite size work of Jain and Wu on the sphere by studying systems with sizes ranging from 4 to 10 particles for the quasihole and 6 to 12 particles for the quasielectron. Here we assume that the lowest Landau level states of both spins are filled and inert and hence we restrict ourselves to the states in the second Landau level alone. We confirm that the ground states in the quasihole and quasielectron sectors, i.e. ±1 flux quantum away from commensuration (\( \nu = 3 \)), are maximally spin polarized consistent with the restricted Hilbert space and Fermi statistics. We find that in both sectors there is a low lying state with quantum numbers \( L = 0 \) and \( S = 0 \) in the spectrum. In fact these states are related, they are particle-hole conjugates of each other; consequently, we only describe here the results for the quasiholes (antiskyrmion) [8]. The candidate antiskyrmion state does in fact display spin correlations characteristic of the infinite antiskyrmion (Fig 1). While the creation energy of this state as well as that of the fully polarized quasihole have considerable finite size dependence (Fig 2), the difference of their energies is quite linear in \( 1/N \) except at the smallest system size (Fig 3). For the quasihole the extrapolated values for \( N = \infty \) are \( 0.163e^2/\epsilon l \) for a linear fit and \( 0.153e^2/\epsilon l \) for a quadratic one which compare favorably with the analytic value of \( \frac{1}{8} \sqrt{\frac{2}{\pi}} (\approx 0.157)e^2/\epsilon l \). Hence even though the antiskyrmion is not the ground state in the quasihole sector, it is well described by the long wavelength action. (The same is evidently true of the skyrmion.)

**Discussion:** We have shown that at odd integer filling factors greater than one, there are skyrmions in the spectrum of the quasiparticle sectors as a consequence of the ferromagnetic ground states. We have also shown that these have higher energy than the fully polarized
quasiparticles. Consequently, we do not expect them to show up in activation energies for transport, and more generally, in the asymptotically low temperature behavior of the system. However, the spin polarization at finite temperatures will be quite sensitive even to a small density of large skyrmions. For example at \( \nu = 3 \) the energy difference between the fully polarized quasiparticles and the skyrmions is approximately \( 4(B[T])^{1/2} \) K and so one should expect to see an anomalous decrease of the spin polarization at temperatures of about 10 K in currently used GaAs samples.

The reader might be somewhat puzzled that a Lagrangian that is claimed to capture the low energy, long wavelength fluctuations does not appear to yield the correct quasiparticles. The problem here is not that the true quasiparticles are absent from the Landau-Ginzburg approach but that their energies cannot be calculated reliably by means of the effective Lagrangian. More precisely, the polarized quasiparticles are contained in the Landau-Ginzburg description as microscopic skyrmions \[9\]. However, the long wavelength physics that (1) captures correctly, is the physics of slowly varying spin textures which only carry a small amount of local charge. To compute accurately the energy of small skyrmions, whose density profile varies rapidly on the scale of the magnetic length, we would have to keep higher derivative terms and we have no practical way of doing that. Consequently, there is no way to tell, from within the effective Lagrangian approach, that the lowest energy quasiparticles are not slowly varying textures.

Finally, it was already noted in \[9\] that numerical studies and long wavelength calculations at \( \nu = 1/3 \) and \( 1/5 \) indicate that skyrmions are the relevant quasiparticles in the small Zeeman energy limit. This conclusion also follows from the composite fermion interpretation of the spectrum for fractional fillings \[10,11\]. The latter however also implies that skyrmions are not the lowest energy quasiparticles at all other polarized fractional fillings \[12\] for they correspond to odd integer filled Landau levels of composite fermions; a direct numerical examination of the simplest such fraction, \( \nu = 3/5 \), suggests that this is indeed the case.

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[8] Our choice is motivated by the better scaling of the data as a function of the number of particles \( N \). On account of the the symmetry one cannot fit both sets independently, i.e. the quasielectron data scales equally well as a function of \( N + 2 \).

[9] There is a subtlety here. Strictly speaking, the polarized quasihole is not a microscopic antiskyrmion but is a vortex in the parent bosonic Chern-Simons Lagrangian \([1]\). This is not important for our purposes.

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Strictly speaking this conclusion applies only at the lowest level of the Jain hierarchy but that does cover the cases of practical interest at this point.
FIGURES

FIG. 1. Spin (solid curve) and density (dashed curve) correlation functions in the second Landau level antiskyrmion (quasihole). The one-quasihole sector has, in this case, 10 electrons and 8 flux quanta (11 available states). \( \sigma_z \) is twice the spin density and \( \rho \) the density of the electrons, in units in which the filled second Landau level has density \( \rho_0 = 1 \). \( \theta \) is the polar angle.

FIG. 2. Polarized quasihole gaps (filled circles) and antiskyrmion gaps (open circles) for systems with \( N = 4, 6, 8 \) and 10 particles. These gaps are the differences between the energies in the one-quasihole sector states and the ground state with one fewer flux quantum and the same number of particles.

FIG. 3. Difference in energy between the polarized quasihole and the antiskyrmion for systems with \( N = 4, 6, 8 \) and 10 particles. The solid line is a linear fit to the three largest system sizes and extrapolates to \( 0.163 e^2 / \ell \) at \( N = \infty \). (A quadratic fit yields \( 0.153 e^2 / \ell \) and gives a rough estimate of the error in the extrapolation.) The Landau-Ginzburg analysis gives \( 0.157 e^2 / \ell \).