BOUNDDEDNESS OF FRACTIONAL INTEGRAL OPERATORS ON NON-HOMOGENEOUS METRIC MEASURE SPACES

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ABSTRACT. In this paper, the fractional integral operator on non homogeneous metric measure spaces is introduced, which contains the classical fractional integral operator, fractional integral with non-doubling measures and fractional integral with fractional kernel of order $\alpha$ and regularity $\epsilon$ introduced by García-Cuerva and Gatto as special cases. And the $(L^p(\mu), L^q(\mu))$-boundedness for fractional integral operators on non-homogeneous metric measure spaces is established. From this, the $(L^p(\mu), L^q(\mu))$-boundedness for commutators and multilinear commutators generated by fractional integral operators with $RBMO(\mu)$ function are further obtained. These results in this paper include the corresponding results on both the homogeneous spaces and non-doubling measure spaces.

1. INTRODUCTION

As we know, the theory on spaces of homogeneous type need to assume that measure $\mu$ of metric spaces $(X, d, \mu)$ satisfies the doubling measure condition, i.e. there exists a constant $C > 0$ such that $\mu(B(x, 2r)) \leq C \mu(B(x, r))$ for all $x \in \text{supp} \mu$ and $r > 0$. Recently, many classical theory have been proved still valid without the assumption of doubling measure condition, see [2,4-6,8-10,18,19,21-24] and their references. In case of non-doubling measures, a Radon measure $\mu$ on $\mathbb{R}^n$ only need to satisfy the polynomial growth condition, i.e., for all $x \in \mathbb{R}^n$ and $r > 0$, there exists a constants $C > 0$ and $k \in (0, n]$ such that,

\begin{equation}
\mu(B(x, r)) \leq C_0 r^k,
\end{equation}

where $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$. The analysis on non-doubling measures has important applications in solving the long-standing open Painlevé’s problem (see [21]).
However, as pointed out by Hytönen in [12], the measure satisfying (1.1) do not include the doubling measure as special cases. For this reason, a kind of metric measure spaces $(X, d, \mu)$ satisfying geometrically doubling and the upper doubling measure condition (see Definition 1.1 and 1.2) is introduced by Hytönen in [12], which is called non-homogeneous metric measure spaces. The advantage of this kind of spaces is that it includes both the homogeneous spaces and metric spaces with polynomial growth measures as special cases. From then on, some results paralleled to homogeneous spaces and non-doubling measure spaces are obtained (see [1,3,11-17] and the references therein). For example, Hytönen et al. in [14] and Bui and Duong in [1] independently introduced the atomic Hardy space $H^1(\mu)$ and obtained that the dual space of $H^1(\mu)$ is $RBMO(\mu)$. Hytönen and Martikainen [13] established the $T_b$ theorem in this surroundings. In [1], Bui and Duong also obtained that Calderón-Zygmund operator and commutators of Calderón-Zygmund operators with $RBMO(\mu)$ function are bounded in $L^p(\mu)$ for $1 < p < \infty$. Later, Lin and Yang [16] introduced the space $RBLO(\mu)$ and proved the maximal Calderón-Zygmund operators is bounded from $L^\infty(\mu)$ into $RBLO(\mu)$. Recently, some equivalent characterizations are established by Liu, Yang Da. and Yang Do. in [17] for the boundedness of Calderón-Zygmund operators on $L^p(\mu)$ for $1 < p < \infty$. The boundedness and weak type endpoint estimate of multilinear commutators of Calderón-Zygmund operators on non-homogeneous metric spaces is established by Fu, Yang and Yuan in [3]. And weighted estimate for multilinear Calderón-Zygmund operators on non-homogeneous metric spaces is also obtained by Hu, Meng and Yang in [11].

The purpose of this paper is to establish the theory of fractional integral operators on non-homogeneous metric measure spaces. At first, fractional integral operators on non-homogeneous metric measure spaces is introduced. This kind of fractional integral operators contains the classical fractional integral operator, fractional integral with non-doubling measures and fractional integral with fractional kernel of order $\alpha$ and regularity $\epsilon$ introduced by García-Cuerva and Gatto in [4] as special cases. The $(L^p(\mu), L^q(\mu))$-boundedness for fractional integral operators on non-homogeneous metric measure spaces is obtained. From this result, the boundedness of commutators and multilinear commutators generated by fractional integral operators with $RBMO(\mu)$ function are also established. The results in this paper include the corresponding results on both the homogeneous spaces and non-doubling measure spaces.

For the sake of the reader’s convenience, let us give some references about previous results of fractional integral operators which are closely related to results in this paper. Classical fractional integral theory can be found in [20]. In the circumstance of non-doubling measures, $(L^p(\mu), L^q(\mu))$-boundedness for fractional integral operators is established in [4,5]. The boundedness of commutators and multilinear commutators generated by fractional integral operators with $RBMO(\mu)$ function were established in [2] and [8] respectively.
In addition, it’s worth mentioning that the Besicovitch covering lemma is only applicable to \( R^n \), but is not applicable to non-homogeneous metric measure spaces. Therefore, in the process of proving Lemma 2.2 in this paper, we will adopt a covering lemma introduced in [7].

Before stating the main results, we firstly recall some notations and definitions.

**Definition 1.1.**\(^{[12]}\) A metric spaces \((X, d)\) is called geometrically doubling if there exist some \( N_0 \in \mathbb{N} \) such that, for any ball \( B(x, r) \subset X \), there exists a finite ball covering \( \{B(x_i, r/2)\}_i \) of \( B(x, r) \) such that the cardinality of this covering is at most \( N_0 \).

**Definition 1.2.**\(^{[12]}\) A metric measure space \((X, d, \mu)\) is said to be upper doubling if \( \mu \) is a Borel measure on \( X \) and there exists a dominating function \( \lambda : X \times (0, +\infty) \rightarrow (0, +\infty) \) and a constant \( C_\lambda > 0 \) such that for each \( x \in X, r \mapsto \lambda(x, r) \) is non-decreasing, and for all \( x \in X, r > 0 \),

\[
(1.2) \quad \mu(B(x, r)) \leq \lambda(x, r) \leq C_\lambda \lambda(x, r/2)
\]

**Remark 1.1.** (i) A space of homogeneous type is a special case of upper doubling spaces, where one can take the dominating function \( \lambda(x, r) \equiv \mu(B(x, r)) \).

(ii) Let \((X, d, \mu)\) be an upper doubling space and \( \lambda \) be a dominating function on \( X \times (0, +\infty) \) as in Definition 1.2. In [15], it was showed that there exists another dominating function \( \bar{\lambda} \) such that for all \( x, y \in X \) with \( d(x, y) \leq r \),

\[
(1.3) \quad \bar{\lambda}(x, r) \leq \bar{C} \lambda(y, r).
\]

Thus, in this paper, we always suppose that \( \lambda \) satisfies (1.3).

**Definition 1.3.** Let \( \alpha, \beta \in (1, +\infty) \). A ball \( B \subset X \) is called \((\alpha, \beta)\)-doubling if \( \mu(\alpha B) \leq \beta \mu(B) \).

As stated in Lemma 2.3 of [1], there exist plenty of doubling balls with small radii and with large radii. In the rest of the paper, unless \( \alpha \) and \( \beta \) are specified otherwise, by an \((\alpha, \beta)\) doubling ball we mean a \((6, \beta_0)\)-doubling with a fixed number \( \beta_0 > \max\{C_{\lambda \log^6, 6^n}\} \), where \( n = \log_2 N_0 \) be viewed as a geometric dimension of the spaces.

**Definition 1.4.** Let \( 0 \leq \beta < n \) and \( N_{B,Q} \) be the smallest integer satisfying \( 6^{N_{B,Q}} r_B \geq r_Q \), then we set

\[
K_{B,Q}^{(\beta)} = 1 + \sum_{k=1}^{N_{B,Q}} \left[ \frac{\mu(6^k B)}{\lambda(x_B, 6^k r_B)} \right]^{1-\beta/n}.
\]

When \( \beta = 0 \), then we denote \( K_{B,Q}^{(0)} \) by \( K_{B,Q} \), which is firstly defined in [22].
Definition 1.5. Let \( \rho > 1 \) be some fixed constant. A function \( b \in L^1_{\text{loc}}(\mu) \) is said to belong to \( \text{RBMO}(\mu) \) if there exists a constant \( C > 0 \) such that for any ball \( B \)

\[
\frac{1}{\mu(\rho B)} \int_B |b(x) - m_{\bar{B}} b| d\mu(x) \leq C,
\]

and for any two doubling balls \( B \subset Q \),

\[
|m_B(b) - m_Q(b)| \leq CK_{B,Q}.
\]

\( \bar{B} \) is the smallest \((\alpha, \beta)\)-doubling ball of the form \( 6^k B \) with \( k \in \mathbb{N} \cup \{0\} \), and \( m_{\bar{B}}(b) \) is the mean value of \( b \) on \( \bar{B} \), namely,

\[
m_{\bar{B}}(b) = \frac{1}{\mu(\bar{B})} \int_{\bar{B}} b(x) d\mu(x).
\]

The minimal constant \( C \) appearing in (1.5) and (1.6) is defined to be the \( \text{RBMO}(\mu) \) norm of \( b \) and denoted by \( ||b||_* \). The norm \( ||b||_* \) is independent of \( \rho > 1 \).

Next, let us introduce fractional integral operator on nonhomogeneous metric measure spaces.

Definition 1.6. Let \( 0 < \alpha < n \) and \( 0 < \epsilon \leq 1 \). A function \( K_{\alpha}(\cdot, \cdot) \in L^1_{\text{loc}}(X \times X \setminus \{(x, y) : x = y\}) \) is said to be a fractional kernel of order \( \alpha \) and regularity \( \epsilon \) if it satisfies the following two conditions:

(i)

\[
|K_{\alpha}(x, y)| \leq \frac{C}{\lambda(x, d(x, y))^{1-\alpha/n}}
\]

for all \( x \neq y \).

(ii) There exists \( 0 < \epsilon \leq 1 \) such that

\[
|K_{\alpha}(x, y) - K_{\alpha}(x', y)| + |K_{\alpha}(y, x) - K_{\alpha}(y, x')| \leq \frac{Cd(x, x')^\epsilon}{d(x, y)^\epsilon \lambda(x, d(x, y))^{1-\alpha/n}},
\]

proved that \( Cd(x, x') \leq d(x, y) \).

A operator \( I_{\alpha} \) is called a fractional integral operator on non-homogeneous metric measure spaces with the above fractional kernel \( K_{\alpha} \) satisfying (1.7) and (1.8) if, for \( f \in L^\infty \) functions with compact support and \( x \notin \text{supp} f \),

\[
I_{\alpha} f(x) = \int_X K_{\alpha}(x, y) f(y) d\mu(y).
\]

Remark 1.2. By taking \( \lambda(x, d(x, y)) = Cd(x, y)^\alpha \), it is easy to see that Definition 1.6 in this paper contains Definition 3.1 and Definition 4.1 introduced.
by García-Cuerva and Gatto in [4]. Obviously, it also contains the classical fractional integral operator
\[
I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|y|^{n-\alpha}} d\mu(y)
\]
as special case.

**Definition 1.7.** The commutators \([b, I_\alpha]\) generated by fractional integral operator \(I_\alpha\) with \(RBM\) function \(b\) is defined by
\[
[b, I_\alpha](f)(x) = b(x)I_\alpha(f)(x) - I_\alpha(bf)(x).
\]

**Definition 1.8.** The multilinear commutators \(I_{\alpha,\vec{b}}\) is formally defined by
\[
I_{\alpha,\vec{b}} f(x) = [b_k, [b_{k-1}, \ldots, [b_1, I_\alpha]]] f(x)
\]
where \(\vec{b} = (b_1, b_2, \ldots, b_k)\), and
\[
[b_1, I_\alpha] f(x) = b_1(x)I_\alpha f(x) - I_\alpha(b_1f)(x).
\]

For \(1 \leq i \leq k\), we denote by \(C^i_k\) the family of all finite subsets \(\sigma = \{\sigma(1), \sigma(2), \ldots, \sigma(i)\}\) of \(\{1, 2, \ldots, k\}\) with \(i\) different elements. For any \(\sigma \in C^i_k\), the complementary sequences \(\sigma'\) is given by \(\sigma' = \{1, 2, \ldots, k\} \setminus \sigma\). Let \(\vec{b} = (b_1, b_2, \ldots, b_k)\) be a finite family of locally integrable functions. For all \(1 \leq i \leq k\) and \(\sigma = \{\sigma(1), \ldots, \sigma(i)\} \in C^i_k\), we set \(b_\sigma = (b_{\sigma(1)}, \ldots, b_{\sigma(i)})\) and the product \(b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(i)}\). With this notation, we write
\[
||\vec{b}_\sigma||_* = ||b_{\sigma(1)}||_* \cdots ||b_{\sigma(i)}||_*.
\]
In particular, for \(i \in \{1, \ldots, k\}\) and \(\sigma = \{\sigma(1), \ldots, \sigma(i)\} \in C^i_k\),
\[
[b(y) - b(z)]_\sigma = [b_{\sigma(1)}(y) - b_{\sigma(1)}(z)] \cdots [b_{\sigma(i)}(y) - b_{\sigma(i)}(z)],
\]
and
\[
[b(y) - m_B(b)]_\sigma = [b_{\sigma(1)}(y) - m_B(b_{\sigma(1)})] \cdots [b_{\sigma(i)}(y) - m_B(b_{\sigma(i)})],
\]
where \(B\) is any ball in \(X\) and \(y, z \in X\). For the product of all the functions, we simply write
\[
||\vec{b}||_* = ||b_1||_* \cdots ||b_k||_*.
\]
With any \(\sigma \in C^i_k\), we set
\[
I_{\alpha, b_\sigma} f(x) = [b_{\sigma(i)}, [b_{\sigma(i-1)}, \ldots, [b_{\sigma(1)}, I_\alpha]]] f(x).
\]
In particular, when \(\sigma = \{1, \ldots, k\}\), we denote \(I_{\alpha, \vec{b}}\) simply by \(I_{\alpha, \vec{b}}\).

The main results of this paper are stated as follows.

**Theorem 1.1.** Let \(0 < \alpha < n\), \(1 < p < n/\alpha\) and \(1/q = 1/p - \alpha/n\). \(I_\alpha\) is defined by (1.9). Then there exists a constant \(C > 0\) such that for all \(f \in L^p(\mu)\),
\[
||I_\alpha f||_{L^q(\mu)} \leq C||f||_{L^p(\mu)}.
\]
Theorem 1.2. Let $0 < \alpha < n$, $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Suppose that $b \in RBMO(\mu)$ and $[b, I_\alpha]$ is defined by (1.10). Then there exists a constant $C > 0$ such that for all $f \in L^p(\mu)$,

$$||[b, I_\alpha](f)||_{L^q(\mu)} \leq C||b||_{*}||f||_{L^p(\mu)}.$$  

Theorem 1.3. Let $0 < \alpha < n$, $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Suppose that $b_i \in RBMO(\mu), i = 1, 2, \cdots, k$ and $I_{\alpha, \vec{b}}$ is defined by (1.11). Then there exists a constant $C > 0$ such that for all $f \in L^p(\mu)$,

$$||I_{\alpha, \vec{b}}(f)||_{L^q(\mu)} \leq C||\vec{b}||_{*}||f||_{L^p(\mu)}.$$

Throughout this paper, $\chi_E$ denotes the characteristic function of set $E$. $C$ always denotes a positive constant independent of the main parameters involved, but it may be different from line to line. And $p'$ is the conjugate index of $p$, namely, $\frac{1}{p} + \frac{1}{p'} = 1$.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we need to give the following notations and lemmas. Let $0 \leq \beta < n$ and $f \in L^1_{loc}(\mu)$, the sharp maximal operator is defined by

$$M^{\ast, (\beta)}f(x) = \sup_{B \ni x} \frac{1}{\mu(6B)} \int_B |f(y) - m_B(f)| d\mu(y)$$

$$+ \sup_{(B, Q) \in \Delta_x} \frac{|m_B(f) - m_Q(f)|}{K_{B,Q}^{(\beta)}},$$

where $\Delta_x := \{(B, Q) : x \in B \subset Q$ and $B, Q$ are two doubling balls} and the non centered doubling maximal operator is denoted by

$$N_{\ast}f(x) = \sup_{B \ni x, B \text{ doubling}} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y).$$

By the Lebesgue differentiation theorem, it is easy to see that for any $f \in L^1_{loc}(\mu)$ and almost every $x \in X$,

$$|f(x)| \leq N_{\ast}f(x).$$

Moreover, $N$ is of weak type $(1,1)$ and bounded on $L^p(\mu), 1 < p \leq +\infty$.

From Theorem 4.2 in [1] or Lemma 4 in [2], it is easy to obtain that

Lemma 2.1. Let $0 \leq \beta < n$ and $f \in L^1_{loc}(\mu)$ with $\int_X f(x) d\mu(x) = 0$ if $||\mu|| < \infty$. For $1 < p < \infty$, if $\inf(1, Nf) \in L^p(\mu)$, then there exists a constant $C > 0$ such that

$$||N(f)||_{L^p(\mu)} \leq C||M^{\ast, (\beta)}(f)||_{L^p(\mu)}.$$
Lemma 2.2. Let \( \eta \geq 5 \), \( 0 \leq \beta < n/p \), \( r < p < n/\beta \) and \( 1/q = 1/p - \beta/n \). Then

\[
\| M^{(\beta)}_{r,(\eta)} f \|_{L^q(\mu)} \leq C \| f \|_{L^p(\mu)},
\]

where the non-centered maximal operator \( M^{(\beta)}_{r,(\eta)} \) is defined by

\[
M^{(\beta)}_{r,(\eta)} f(x) = \sup_{B \ni x} \left\{ \frac{1}{\mu(\eta B)^{1-\beta r/n}} \int_B |f(y)|^r d\mu(y) \right\}^{1/r}.
\]

Remark 2.1. If \( \beta = 0 \), we denote \( M^{(\beta)}_{r,(\eta)} \) simply by \( M_{r,(\eta)} \). From [1], we obtain that if \( \eta \geq 5 \) and \( q > r \), then

\[
\| M_{r,(\eta)} f \|_{L^q(\mu)} \leq C \| f \|_{L^p(\mu)}.
\]

Proof. (of Lemma 2.2) Let us firstly prove the following result.

\[
\mu\{ x : M^{(\beta)}_{r,(\eta)} f(x) > \lambda \} \leq \left( \frac{C}{\lambda} \| f \|_{L^p(\mu)} \right)^{nr/(n-\beta r)}.
\]

Let \( E_\lambda = \{ x : M^{(\beta)}_{r,(\eta)} f(x) > \lambda \} \). By the definition of \( M^{(\beta)}_{r,(\eta)} \), for any \( x \in E_\lambda \), there exists a ball \( B_x \) containing \( x \) such that

\[
\frac{1}{\mu(\eta B_x)^{1-\beta r/n}} \int_{B_x} |f(x)|^r d\mu(x) \geq \lambda^r.
\]

Note that for \( \eta \geq 5 \). By Theorem 1.2 in [7], we can pick a disjoint collection \( \{B_{x_i}\} \) with \( x_i \in E_\lambda \) and \( E_\lambda \subset \bigcup_{x_i \in E_\lambda} B_x \subset \bigcup_{i} 5B_{x_i} \subset \bigcup_{i} \eta B_{x_i} \). Let \( q = \frac{n r}{n - \beta r} \) and \( r/q < 1 \), then

\[
\mu(E_\lambda)^{r/q} \leq \mu(\bigcup_i 5B_{x_i})^{r/q} \leq \mu(\bigcup_i \eta B_{x_i})^{r/q} \leq \sum_i \mu(\eta B_{x_i})^{r/q}.
\]

Since \( r/q = 1 - \beta r/n \), then

\[
\sum_i \mu(\eta B_{x_i})^{r/q} \leq \frac{1}{\lambda^r} \int_X |f|^r (\sum_i \chi_{B_{x_i}}) d\mu.
\]

Therefore,

\[
\mu(E_\lambda) \leq \frac{C}{\lambda^r} \| f \|_{L^p(\mu)}^{q/n}.
\]

If \( r < s < n/\beta \), using the Hölder’s inequality, we deduce

\[
M^{(\beta)}_{s,(\eta)} f(x) \leq M^{(\beta)}_{r,(\eta)} f(x).
\]

By the preceding arguments, then we obtain

\[
\mu(E_\lambda) \leq \mu(\{ x : M^{(\beta)}_{s,(\eta)} f(x) > \lambda \}) \leq \left( \frac{C}{\lambda} \| f \|_{L^p(\mu)} \right)^{ns/(n-\beta s)}.
\]

By the Marcinkiewicz interpolation theorem, the proof of Lemma 2.2 is completed. \( \square \)
With the similar method to proof of Lemma 3 in [2] or Lemma 2.1 in [22], it is easy to obtain the following Lemma 2.3. Here we omit the details.

**Lemma 2.3.** For $0 \leq \beta < n$, we have the following properties:

1. If $B \subset Q \subset R$ are balls in $X$, then $K_{B,Q}^{(\beta)} \leq K_{B,R}^{(\beta)}$, $K_{Q,R}^{(\beta)} \leq K_{B,R}^{(\beta)}$ and $K_{B,R}^{(\beta)} \leq C(K_{B,Q}^{(\beta)} + K_{Q,R}^{(\beta)})$.

2. If $B \subset Q$ have comparable sizes, then $K_{B,Q}^{(\beta)} \leq C$.

3. If $N$ is a positive integer and the balls $6B, 6^2B, \ldots, 6^{N-1}B$ are non doubling balls, then $K_{B,6^NB}^{(\beta)} \leq C$.

Lemma 2.4 and Lemma 2.5 can be obtained by analogue to Lemma 3.11 and Lemma 3.12 in [3] or Lemma 5 and Lemma 6 in [2] respectively. Here we omit the details of proof.

**Lemma 2.4.** For $0 \leq \beta < n$, there exists a positive constant $P_\beta$ (big enough), depending on $\beta$, $n$ and $C_\lambda$, such that if $B_1 \subset B_2 \subset \cdots \subset B_m$ are concentric balls with $K_{B_i,B_{i+1}}^{(\beta)} > P_\beta$ for $i = 1, 2, \ldots, m - 1$, then there exists a positive constant $C$, depending on $\beta$, $n$ and $C_\lambda$, such that $\sum_{i=1}^{m-1} K_{B_i,B_{i+1}}^{(\beta)} \leq CK_{B_1,B_m}^{(\beta)}$.

**Lemma 2.5.** For $0 \leq \beta < n$, there exists a positive constant $P_{\beta}'$ (big enough), depending on $\beta$, $n$ and $C_\lambda$, such that if $x \in X$ is some fixed point and $\{f_B\}_{B \ni x}$ is a set of numbers such that $|f_B - f_Q| \leq C_\lambda$ for all doubling balls $B \subset Q$ with $x \in B$ such that $K_{B,Q}^{(\beta)} \leq P_{\beta}'$, then there exists a positive constant $C$, depending on $\beta$, $n$ and $C_\lambda$, such that for all doubling balls $B \subset Q$ with $x \in B$, $|f_B - f_Q| \leq CK_{B,Q}^{(\beta)}C_\lambda$.

**Proof.** (of Theorem 1.1) For all $1 < p < n/\alpha$, we firstly establish the following sharp maximal function estimate

$$M^{2,(\alpha)}(I_\alpha f)(x) \leq CM^{(\alpha)}_{r,(5)} f(x).$$

Suppose (2.14) is valid for a moment. Choosing $r$ such that $1 < r < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. By Lemma 2.1 and Lemma 2.2, we have

$$||I_\alpha f||_{L^q(\mu)} \leq ||N(I_\alpha f)||_{L^q(\mu)} \leq C||M^{2,(\alpha)}(I_\alpha f)||_{L^q(\mu)}$$

$$\leq C||M^{(\alpha)}_{1,5,B} f||_{L^q(\mu)} \leq C||f||_{L^p(\mu)}.$$ 

As in the proof of Theorem 9.1 in [22], to obtain (2.14), by Lemma 2.4 and Lemma 2.5, it suffices to show that

$$\frac{1}{\mu(6B)} \int_B |I_\alpha f(y) - m_B(I_\alpha(f_X \chi_{\frac{y}{6}B}))| \, d\mu(y) \leq CM^{(\alpha)}_{r,(5)} f(x).$$
holds for any \( x \) and ball \( B \) with \( x \in B \), and

\[
|m_B(I_{\alpha}(f x x \chi_B)) - m_Q(I_{\alpha}(f x x \chi_Q))| \leq CK_{B,Q}^{r(\alpha)}M^{(\alpha)}_{r(\alpha)}f(x)
\]

for all balls \( B \subset Q \) with \( x \in B \), where \( B \) is an arbitrary ball and \( Q \) is a doubling ball.

For any ball \( B \), it is easy to see that

\[
\frac{1}{\mu(6B)} \int_B |I_{\alpha}f(y) - m_B(I_{\alpha}(f x x \chi_B))|d\mu(y) \\
\leq \frac{1}{\mu(6B)} \int_B |I_{\alpha}(f x x \chi_B)(y)|d\mu(y) \\
+ \frac{1}{\mu(6B)} \int_B |I_{\alpha}(f x x \chi_B)(y) - m_B(I_{\alpha}(f x x \chi_B))|d\mu(y) \\
=: I_1 + I_2.
\]

To estimate \( I_1 \), by (i) in Definition 1.6, the properties of function \( \lambda \) and the Hölder’s inequality, we get

\[
I_1 \leq \frac{C}{\mu(6B)} \int_B \int_{\frac{y}{6B}} \frac{|f(z)|}{\lambda(y, d(y, z))^{1-\alpha/n}}d\mu(z)d\mu(y) \\
\leq \frac{C}{\mu(6B)} \int_B \int_{\frac{y}{6B}} \frac{|f(z)|}{\lambda(y, r_{6B})^{1-\alpha/n}}d\mu(z)d\mu(y) \\
\leq \frac{C}{\mu(6B)} \int_B \int_{\frac{y}{6B}} \frac{|f(z)|}{\mu(6B)^{1-\alpha/n}}d\mu(z)d\mu(y) \\
\leq \frac{C}{\mu(6B)} \int_B \left[ \frac{1}{\mu(6B)^{1-\alpha/n}} \int_{\frac{y}{6B}} |f(z)|^r d\mu(z) \right]^{\frac{1}{r}} \left[ \frac{\mu(6B)}{\mu(6B)} \right]^{1-\frac{1}{r}} d\mu(y) \\
\leq CM^{(\alpha)}_{r(\alpha)}f(x).
\]
For \( y, y_0 \in B \), by (ii) in Definition 1.6 and the properties of function \( \lambda \) and the Hölder’s inequality, we obtain

\[
|I_\alpha(f \chi_{X \setminus \frac{B}{2}})(y) - I_\alpha(f \chi_{X \setminus \frac{B}{2}})(y_0)| \\
\leq C \int_{X \setminus \frac{B}{2}} \frac{d(y, y_0)^\varepsilon}{d(y, z)^\varepsilon \lambda(y, d(y, z))^{1-\alpha/n}} |f(z)| d\mu(z) \\
\leq C \sum_{k=1}^{\infty} \int_{6^k \frac{B}{2} \setminus 6^{k-1} \frac{B}{2}} 6^{-k\varepsilon} \frac{|f(z)| d\mu(z)}{\lambda(y, r_{5 \times 6^k \frac{B}{2}})^{1-\alpha/n}} \\
\leq C \sum_{k=1}^{\infty} 6^{-k\varepsilon} \left[ \frac{1}{\mu(5 \times 6^k \frac{B}{2})^{1-\frac{\alpha}{n}}} \int_{6^k \frac{B}{2}} |f(z)| d\mu(z) \right]^{\frac{1}{\varepsilon}} \left[ \frac{\mu(6^k \frac{B}{2})}{\mu(5 \times 6^k \frac{B}{2})} \right]^{1-\frac{\alpha}{n}} \\
\leq CM_{r,(5)}^{(\alpha)} f(x).
\]

Taking the mean over \( y_0 \in B \), then we have

\[
I_2 \leq CM_{r,(5)}^{(\alpha)} f(x).
\]

So (2.16) holds from (2.18) to (2.21).

Now we prove (2.17). Consider two balls \( B \subset Q \) with \( x \in B \), where \( B \) is an arbitrary ball and \( Q \) is a doubling ball. Let \( N = N_B, Q + 1 \), then we have

\[
m_B \left[ I_\alpha(f \chi_{X \setminus \frac{B}{2}}) - m_Q \left[ I_\alpha(f \chi_{X \setminus \frac{Q}{2}}) \right] \right] \\
\leq m_B \left[ I_\alpha(f \chi_{X \setminus 6^N B}) - m_Q \left[ I_\alpha(f \chi_{X \setminus 6^N B}) \right] \right] + m_B \left[ I_\alpha(f \chi_{6^N B \setminus \frac{B}{2}}) \right] + m_Q \left[ I_\alpha(f \chi_{6^N B \setminus \frac{Q}{2}}) \right] \\
= : J_1 + J_2 + J_3.
\]

With the same method to estimate \( I_2 \), we immediately get

\[
J_1 \leq CM_{r,(5)}^{(\alpha)} f(x).
\]

To estimate \( J_2 \), for \( z \in B \), it is easy to see that

\[
|I_\alpha(f \chi_{6^N B \setminus \frac{B}{2}})(z)| \leq |I_\alpha(f \chi_{6^N B \setminus 6^N B})(z)| + |I_\alpha(f \chi_{6^N B \setminus \frac{Q}{2}})(z)|.
\]
By (i) in Definition 1.6 and the Hölder’s inequality, we deduce

\begin{equation}
|I_{\alpha}(f \chi_{6N_B \setminus 6B})(z)| \leq C \int_{6N_B \setminus 6B} \frac{|f(y)|}{\lambda(z, d(z, y))^{1-\alpha/n}} d\mu(y)
\end{equation}

\begin{equation}
\leq C \sum_{k=1}^{N-1} \int_{6^{k+1}B \setminus 6^kB} \frac{|f(y)|}{\lambda(z, r_{6^{k+1}B})^{1-\alpha/n}} d\mu(y)
\end{equation}

\begin{equation}
\leq C \sum_{k=1}^{N-1} \left[ \frac{\lambda(5 \times 6^{k+1}B)}{\lambda(z, 5 \times 6^{k+1}B)} \right]^{1-\alpha/n} \int_{6^{k+1}B} \frac{|f(y)|}{\lambda(z, r_{6^{k+1}B})^{1-\alpha/n}} d\mu(y)
\end{equation}

\begin{equation}
\leq C \left[ \frac{5 \times 6^{k+1}B}{\lambda(5 \times 6^{k+1}B)} \right]^{1-\alpha/n} \int_{6^{k+1}B} \frac{|f(y)|}{\lambda(z, r_{6^{k+1}B})^{1-\alpha/n}} d\mu(y)
\end{equation}

\begin{equation}
\leq C \int_{6^{k+1}B} \frac{|f(y)|}{\lambda(z, r_{6^{k+1}B})^{1-\alpha/n}} d\mu(y)
\end{equation}

\begin{equation}
\leq C \int_{6^{k+1}B} \frac{|f(y)|}{\lambda(z, r_{6^{k+1}B})^{1-\alpha/n}} d\mu(y)
\end{equation}

\begin{equation}
\leq C \left[ \frac{\lambda(6B)}{\lambda(z, 6B)} \right]^{1-\alpha/n} \int_{6B} |f(y)|^{1-\alpha/n} d\mu(y)
\end{equation}

\begin{equation}
\leq CM_{r,(5)}^{\alpha} f(x).
\end{equation}

Also, we obtain that

\begin{equation}
|I_{\alpha}(f \chi_{6N_B \setminus 6B})(z)| \leq C \int_{6N_B \setminus 6B} \frac{|f(y)|}{\lambda(z, r_B)^{1-\alpha/n}} d\mu(y)
\end{equation}

\begin{equation}
\leq C \left[ \frac{\lambda(6B)}{\lambda(z, 6B)} \right]^{1-\alpha/n} \int_{6B} |f(y)|^{1-\alpha/n} d\mu(y)
\end{equation}

\begin{equation}
\leq C \frac{1}{\lambda(6B)} \int_{6B} |f(y)| d\mu(y)
\end{equation}

\begin{equation}
\leq C \frac{1}{\lambda(6B)} \int_{6B} |f(y)| d\mu(y)
\end{equation}

\begin{equation}
\leq CM_{r,(5)}^{\alpha} f(x).
\end{equation}

Then

\begin{equation}
|I_{\alpha}(f \chi_{6N_B \setminus 6B})(z)| \leq CK_{B,Q}^{\alpha} M_{r,(5)}^{\alpha} f(x).
\end{equation}

Taking mean for z over B, we have

\begin{equation}
J_2 \leq CK_{B,Q}^{\alpha} M_{r,(5)}^{\alpha} f(x).
\end{equation}

Similarly, we also obtain that

\begin{equation}
J_3 \leq CK_{B,Q}^{\alpha} M_{r,(5)}^{\alpha} f(x).
\end{equation}

From (2.22) to (2.29), we yield (2.17) holds. Hence the proof of Theorem 1.1 is completed. \qed
3. Proof of Theorem 1.2

Let us firstly give the equivalent definition of RBMO(μ), which is useful in proving Theorem 1.2.

**Definition 3.1.**[8] Let ρ > 1 be some fixed constant. A function b ∈ L^1_{loc}(μ) is said to belong to RBMO(μ) if there exists a constant C > 0 such that for any ball B, a number b_B such that
\[
\frac{1}{\mu(ρB)} \int_B |b(x) - b_B| dμ(x) \leq C,
\]
and for any two balls B ⊂ Q,
\[
|b_B - b_Q| \leq CK_{B,Q}.
\]
The minimal constant C appearing in (3.1) and (3.2) is defined to be the RBMO(μ) norm of f and denoted by ||b||_*. The norm ||b||_* is independent of ρ > 1.

**Lemma 3.1.**[22] For any ball B, we have
\[
|b_B - b_{\frac{B}{2}}| \leq Ck||b||_*.
\]

**Proof.** (of Theorem 1.2) For all 1 < p < n/α, we firstly establish the following sharp maximal function estimate
\[
M^{\sharp, (\alpha)}(|b, I_\alpha f|)(x) \leq C||b||_* [M^{(\alpha)}_{r, (5)} f(x) + M_{r, (6)}(I_\alpha f)(x) + I_\alpha(|f|)(x)].
\]
Suppose (3.4) holds for a moment. By Lemma 3.3 in [22], we can assume that b ∈ L^∞(μ). Choosing r such that 1 < r < p < n/α and \(\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}\). By Lemma 2.1, Lemma 2.2 and Theorem 1.1 in this paper, we obtain
\[
||[b, I_\alpha f]|L^q(\mu)|| \leq ||N([b, I_\alpha f])|L^q(\mu)|| \leq CM^{\sharp, (\alpha)}(b, I_\alpha f)|L^q(\mu)|
\]
\[
\leq C\{||M^{(\alpha)}_{r, (5)} f||L^q(\mu) + ||M_{r, (6)}(I_\alpha f)||L^q(\mu) + ||I_\alpha(|f|)||L^q(\mu)\}
\]
\[
\leq C\{||f||L^p(\mu) + ||I_\alpha f||L^q(\mu)\} \leq C||f||L^p(\mu).
\]

Let \{b_B\} be a collection of numbers satisfying for ball B,
\[
\int_B |b - b_B| dμ \leq 2\mu(6B)||b||_*,
\]
and for two balls B ⊂ Q,
\[
|b_B - b_Q| \leq 2K_{B,Q}||b||_*.
\]

As in the proof of Theorem 1 in [2], to obtain (3.4), by Lemma 2.4 and Lemma 2.5 in Section 2 of this paper, it suffices to deduce that
\[
\frac{1}{\mu(6B)} \int_B |b, I_\alpha f(y) - h_B| dμ(y) \leq C||b||_* (M^{(\alpha)}_{r, (5)} f(x) + M_{r, (6)}(I_\alpha f)(x)).
\]
holds for any $x$ and ball $B$ with $x \in B$, and

\[(3.9) \quad |h_B - h_Q| \leq C ||b||_* K_{B,Q} K_{B,Q}^{(\alpha)}(M_{r,(5)}^{(\alpha)} f(x) + I_\alpha(|f|)(x))\]

for all balls $B \subset Q$ with $x \in B$, where $B$ is an arbitrary ball, $Q$ is a doubling ball and for any ball $B$, we denote

$$h_B := m_B(\{ b - b_B \} f \chi_{X \setminus \frac{B}{5}}).$$

To obtain (3.8), we write $[b, I_\alpha]f$ as follows.

\[(3.10) \quad [b, I_\alpha]f(y) = (b(y) - b_B)I_\alpha f(y) - (b - b_B)I_\alpha((b - b_B)f_1)(y) - I_\alpha((b - b_B)f_2)(y),\]

where $f_1 = f \chi_{\frac{B}{5}}$ and $f_2 = f - f_1$. Now, by the Hölder's inequality, we have

\[(3.11) \quad \leq \frac{1}{\mu(6B)} \int_B |(b(y) - b_B)I_\alpha f(y)| d\mu(y) \leq \frac{1}{\mu(6B)} \int_B |I_\alpha f(y)|^{1/r} d\mu(y) \leq C ||b||_* M_{r,(6)}^{(\alpha)} I_\alpha f(x).\]

We take $s = \sqrt{r}$ and let $1/t = 1/s - \alpha/n$. Using the Hölder’s inequality, the result of Theorem 1.1 and Definition 3.1, we obtain

\[
\leq \frac{1}{\mu(6B)} \int_B |I_\alpha((b - b_B)f_1)(y)| d\mu(y) \\
\leq \frac{\mu(B)^{1-1/t}}{\mu(6B)} ||I_\alpha((b - b_B)f_1)||_{L^r(\mu)} \\
\leq \frac{\mu(B)^{1-1/t}}{\mu(6B)} ||(b - b_B)f_1||_{L^r(\mu)} \\
\leq \left( \frac{1}{\mu(6B)} \int_{\frac{B}{5}} |b - b_B|^{ss'} d\mu(y) \right)^{\frac{1}{ss'}} \\
x \left( \frac{1}{\mu(6B)^{1-\alpha r/n}} \int_{\frac{B}{5}} |f(y)|^{r} d\mu(y) \right)^{\frac{1}{r}} \\
\leq C ||b||_* M_{r,(5)}^{(\alpha)} f(x).
\]
Next, to prove (3.8), we only need to compute \(|I_\alpha((b - b_B)f_2)(y) - h_B|\).

For \(y, y_0 \in B\), by Lemma 3.1, we have

\[
|I_\alpha((b - b_B)f_2)(y) - I_\alpha((b - b_B)f_2)(y_0)|
\leq C \int_{\mathbb{R}^n} \frac{d(y, y_0)^\varepsilon}{d(y, z)^\varepsilon \lambda(y, d(y, z))^{1 - \frac{\varepsilon}{\alpha}}} |b(z) - b_B| |f(z)| d\mu(z)
\]

\[
\leq C \sum_{k=1}^{\infty} \int_{6^k B \setminus 6^{k-1} B} \lambda(y, 6^k B)^{1 - \frac{\varepsilon}{\alpha}} 
\times (|b(z) - b_D| + |b_B - b_D|) |f(z)| d\mu(z)
\]

\[
\leq C \sum_{k=1}^{\infty} \frac{6^{-k\varepsilon}}{\mu(5 \times 6^k B)^{1 - \frac{\varepsilon}{\alpha}}} \int_{6^k B} |b(z) - b_D| |f(z)| d\mu(z)
\]

\[
+ C \sum_{k=1}^{\infty} \frac{6^{-k\varepsilon}}{\mu(5 \times 6^k B)^{1 - \frac{\varepsilon}{\alpha}}} \int_{6^k B} |f(z)| d\mu(z)
\]

(3.13)

\[
\leq C \sum_{k=1}^{\infty} \frac{6^{-k\varepsilon}}{\mu(5 \times 6^k B)^{1 - \frac{\varepsilon}{\alpha}}} \left[ \frac{1}{\mu(5 \times 6^k B)^{1 - \frac{\varepsilon}{\alpha}}} \int_{6^k B} |f(z)|^{r/\alpha} d\mu(z) \right]^{1/r}
\]

\[
\times \left[ \frac{1}{\mu(5 \times 6^k B)^{1 - \frac{\varepsilon}{\alpha}}} \int_{6^k B} |b(z) - b_D|^{r/\alpha} d\mu(z) \right]^{1/r'}
\]

\[
+ C \sum_{k=1}^{\infty} \frac{6^{-k\varepsilon}}{\mu(5 \times 6^k B)^{1 - \frac{\varepsilon}{\alpha}}} \left[ \frac{1}{\mu(5 \times 6^k B)^{1 - \frac{\varepsilon}{\alpha}}} \int_{6^k B} |f(z)|^{r/\alpha} d\mu(z) \right]^{1/r}
\]

\[
\times \left[ \frac{\mu(5 \times 6^k B)^{1 - \frac{\varepsilon}{\alpha}}}{\mu(5 \times 6^k B)^{1 - \frac{\varepsilon}{\alpha}}} \int_{6^k B} |b(z) - b_D|^{r/\alpha} d\mu(z) \right]^{1/r'}
\]

\[
\leq C |b|_r M_{r,(5)}^{(\alpha)} f(x).
\]

Taking the mean over \(y_0 \in B\), we obtain

(3.14) \quad |I_\alpha((b - b_B)f_2)(y) - h_B| \leq C |b|_r M_{r,(5)}^{(\alpha)} f(x).

So (3.8) is proved.
To prove (3.9), we consider two balls $B \subset Q$ with $x \in B$, where $B$ is an arbitrary ball, $Q$ is a doubling ball. Denote $N = N_{B,Q} + 1$, we write

$$
|m_B(I_\alpha((b - b_B)f_{X \setminus 6B})) - m_Q(I_\alpha((b - b_Q)f_{X \setminus 6Q}))| \\
\leq m_B(I_\alpha((b - b_B)f_{6B \setminus \frac{5}{5}B})) \\
+ |m_B(I_\alpha((b_b - b_Q)f_{X \setminus 6B}))| \\
+ |m_B(I_\alpha((b - b_Q)f_{6B \setminus \frac{5}{5}B}))| \\
+ |m_Q(I_\alpha((b - b_Q)f_{6B \setminus \frac{5}{5}B}))| \\
=: L_1 + L_2 + L_3 + L_4 + L_5.
$$

(3.15)

For $y \in B$, by the Hölder’s inequality, we get

$$
|I_\alpha((b - b_B)f_{6B \setminus \frac{5}{5}B})(y)| \\
\leq C \int_{6B} \frac{|b(z) - b_B||f(z)|}{X(y, d(y, z))^{1 - \alpha/n}} d\mu(z) \\
\leq C \left[ \frac{1}{\mu(5 \times 6B)^{1 - \alpha/n}} \int_{6B} |f(z)|^{\frac{n}{\alpha}} d\mu(z) \right]^{\frac{1}{\frac{n}{\alpha}}} \\
\times \left[ \frac{1}{\mu(5 \times 6B)} \int_{6B} |b(z) - b_B|^{r} d\mu(z) \right]^{\frac{1}{r}} \\
\leq C ||b||_* M_{r,(5)}^{(\alpha)} f(x).
$$

(3.16)

Then we get $L_1 \leq C ||b||_* M_{r,(5)}^{(\alpha)} f(x)$.

For $x, y \in B$, it is easy to see that

$$
|I_\alpha(f_{X \setminus 6B})(y)| \leq I_\alpha(|f|)(x) + CM_{r,(5)}^{(\alpha)} f(x).
$$

(3.17)

Then, by Definition 3.1, we get

$$
L_2 \leq CK_{B,Q} ||b||_* (I_\alpha(|f|)(x) + M_{r,(5)}^{(\alpha)} f(x)).
$$

(3.18)
Lemma 4.1. Let us estimate $L_3$. For $y \in B$, we have
\[ |D_y((b-b_Q)f \chi_{6^N B \setminus GB})(y)| \]
\[ \leq C \sum_{k=1}^{N-1} \left\{ \frac{|b(z)-b_Q||f(z)|}{\lambda(y, d(y,z))^{1-\alpha/n}} \right\} \]
\[ \leq C \sum_{k=1}^{N-1} \left\{ \frac{\mu(5 \times 6^{k+1} B)}{\lambda(y, r_{5 \times 6^{k+1} B})^{1-\alpha/n}} \right\} \]
\[ \leq C \sum_{k=1}^{N-1} \left\{ \frac{\mu(5 \times 6^{k+1} B)}{\lambda(y, r_{(5 \times 6^{k+1} B)})^{1-\alpha/n}} \right\} \]
\[ \times \left\{ \int_{6^{k+1} B} |b(z) - b_Q||f(z)| \, d\mu(z) \right\}^{1/2} \]
\[ \leq CK_{B,Q}^{(\alpha)} \left[ \frac{1}{\mu(5 \times 6^{k+1} B)} \int_{6^{k+1} B} |f(z)| \, d\mu(z) \right]^{1/2} \]
\[ \times \left[ \int_{6^{k+1} B} |b(z) - b_Q|^r \, d\mu(z) \right]^{1/2} \]
\[ \leq CK_{B,Q}^{(\alpha)} M_{r,(5)}^{(\alpha)} f(x) \]
\[ \leq CK_{B,Q}^{(\alpha)} |b|_\ast M_{r,(5)}^{(\alpha)} f(x). \]
Here we have used the fact that $|b_{6^{k+1} B} - b_Q| \leq CK_{B,Q} ||b||_\ast$.

Taking the mean over $B$, we have
\[ L_3 \leq CK_{B,Q}^{(\alpha)} |b|_\ast M_{r,(5)}^{(\alpha)} f(x). \]

For $L_4$. Operating as in (3.13), for any $y \in B$ and $z \in Q$, we obtain
\[ |T((b-b_Q)f \chi_{6^N B})(y) - T((b-b_Q)f \chi_{6^N B})(z)| \leq CK_{B,Q}^{(\alpha)} M_{r,(5)}^{(\alpha)} f(x). \]

Taking the mean over $B$ for $y$ and over $Q$ for $z$, we have
\[ L_4 \leq C ||b||_\ast M_{r,(5)}^{(\alpha)} f(x). \]

For $L_5$, similar to $L_1$, we can deduce $L_5 \leq C ||b||_\ast M_{r,(5)}^{(\alpha)} f(x).$ Thus (3.9) is valid and the proof of Theorem 1.2 is finished. \qed

4. PROOF OF THEOREM 1.3

To prove Theorem 1.3, we need the following some lemmas.

Lemma 4.1. \cite{5,22} Let $1 \leq p < \infty$ and $1 < \rho < \infty$. If $b \in RBMO(\mu)$, then for any ball $B \in X$,
\[ \left\{ \frac{1}{\mu(\rho B)} \int_B |b(x) - m_B(b)|^p \, d\mu(x) \right\}^{1/p} \leq C ||b||_\ast. \]
Lemma 4.2. [9] For any ball \( B \), we have

\[
|m_{\frac{1}{6} B}(b) - m_B(b)| \leq C_j \|b\|.
\]  

\[ (4.2) \]

Proof. (of Theorem 1.3) We prove the theorem by induction on \( k \). If \( k = 1 \), the result of Theorem 1.2 asserts that \( [b, I_\alpha] \) is bounded from \( L^p(\mu) \) to \( L^q(\mu) \) for any \( 1 < p < n/\alpha \), \( 0 < \alpha < n \) and \( 1/q = 1/p - \alpha/n \). Now we assume that \( k \geq 2 \) is an integer and that for any \( 1 \leq i \leq k-1 \) and any subset \( \sigma = \{\sigma(1), \cdots, \sigma(i)\} \) of \( \{1, \cdots, k\} \), \( I_{\alpha,\sigma} \) is bounded from \( L^p(\mu) \) to \( L^q(\mu) \) for any \( 1 < p < n/\alpha \), \( 0 < \alpha < n \) and \( 1/q = 1/p - \alpha/n \). We next claim that for any \( 1 < r < \infty \), \( I_{\alpha,\tau} \) satisfies the following sharp maximal function estimate

\[
M_{\ell,\tau}(I_{\alpha,\tau}f)(x) \leq C\|\tilde{b}\|_* \{M_{r,6}(I_\alpha f)(x) + M_{r,5}^{(\alpha)} f(x)\}
\]

\[ (4.3) \]

Suppose (4.3) holds for a moment. Now we prove \( T_b \) satisfies (1.14). By Lemma 3.3 in [22], we can assume that \( b_i \in L^\infty(\mu) \) for \( 1 \leq i \leq k \). Choosing \( r \) such that \( 1 < r < n/\alpha \) and \( 1/q = 1/p - \alpha/n \). By Lemma 2.1, Lemma 2.2, Theorem 1.1 and Theorem 1.2 in this paper, we deduce

\[
\|I_{\alpha,\sigma}f\|_{L^p(\mu)} \leq C\|N(I_{\alpha,\sigma}f)\|_{L^p(\mu)} \leq C\|M_{\ell,\tau}(I_{\alpha,\sigma}f)\|_{L^p(\mu)}
\]

\[
\leq C\|\tilde{b}\|_* \{\|M_{r,6}(I_\alpha f)(x)\|_{L^p(\mu)} + \|M_{r,5}^{(\alpha)} f(x)\|_{L^p(\mu)}\}
\]

\[ \leq C\|\tilde{b}\|_* \{\|I_\alpha f\|_{L^p(\mu)} + \|f\|_{L^p(\mu)}\} + C\sum_{i=1}^{k-1} \|\tilde{b}_{\sigma,i}\|_* \|I_{\alpha,\sigma,i}f\|_{L^p(\mu)}
\]

\[ (4.4) \]

As in the proof of Theorem 2 in [9], to obtain (4.3), by Lemma 2.4 and Lemma 2.5 in this paper, it only need to show that

\[
\frac{1}{\mu(B)} \int_B |I_{\alpha,\sigma}f(y) - h_B| d\mu(y)
\]

\[ \leq C\|\tilde{b}\|_* \{M_{r,6}(I_\alpha f)(x) + M_{r,5}^{(\alpha)} f(x)\} + C\sum_{i=1}^{k-1} \|\tilde{b}_{\sigma,i}\|_* M_{r,6}(I_{\alpha,\sigma,i}f)(x)
\]

\[ (4.5) \]
hold for all \( x \) and \( B \) with \( x \in B \), and
\[
|h_B - h_Q| \leq C(K_{B,Q})^k K_{B,Q}^{(a)} |\bar{b}|_\sigma \{ M_{r,(6)}(I_\alpha f)(x) + M'_{r,(5)} f(x) \}
\]
(4.6) 
\[+ C(K_{B,Q})^k \sum_{i=1}^{k-1} \sum_{\sigma \in C_1^k} ||b_\sigma||_\sigma M_{r,(6)}(I_{\alpha,b_\sigma}) f(x) \]
holds for any cube \( B \subset Q \) with \( x \in B \), where \( B \) is an arbitrary cube and \( Q \) is a doubling cube. We denote
\[
h_B = m_B \left( I_\alpha \left[ (b_1 - m_{\overline{B}}(b_1)) \cdots (b_k - m_{\overline{B}}(b_k)) f_{X \setminus \frac{3}{4}B} \right] \right),
\]
and
\[
h_R = m_R \left( I_\alpha \left[ (b_1 - m_R(b_1)) \cdots (b_k - m_R(b_k)) f_{X \setminus \frac{3}{4}R} \right] \right).
\]
Let us firstly estimate (4.5). It is easy to see that
\[
\prod_{i=1}^{k} [b_i(z) - m_{\overline{B}}(b_i)] = \sum_{\sigma = 0}^{k} \sum_{\sigma \in C_1^k} [b(z) - b(y)]_{\sigma'} [b(y) - m_{\overline{B}}(b)]_\sigma
\]
for \( y, z \in X \), where if \( i = 0 \), then \( \sigma' = \{1, 2, \ldots, k\} \), \( \sigma = \emptyset \) and \( [b(y) - m_{\overline{B}}(b)]_\emptyset = 1 \).

Hence
\[
I_{\alpha,b_\sigma} f(y) = \left( \prod_{i=1}^{k} [b_i - m_{\overline{B}}(b_i)] f \right) (y) - \sum_{i=1}^{k} \sum_{\sigma \in C_1^k} [b(y) - m_{\overline{B}}(b)]_\sigma I_{\alpha,b_\sigma} f(y),
\]
and if \( i = k \), we denote \( I_\alpha f(y) \) by \( I_{\alpha,b_k} f(y) \). Thus,
\[
\frac{1}{\mu(6B)} \int_B |I_{\alpha,b_\sigma} f(y) - h_B| d\mu(y)
\]
\[\leq \frac{1}{\mu(6B)} \int_B \left| I_\alpha \left[ \prod_{i=1}^{k} [b_i - m_{\overline{B}}(b_i)] f_{X \setminus \frac{3}{4}B} \right] (y) \right| d\mu(y)
\]
(4.8) 
\[+ \sum_{i=1}^{k} \sum_{\sigma \in C_1^k} \frac{1}{\mu(6B)} \int_B \left| [b(y) - m_{\overline{B}}(b)]_\sigma \right| \left| I_{\alpha,b_\sigma} f(y) \right| d\mu(y)
\]
\[+ \frac{1}{\mu(6B)} \int_B \left| I_\alpha \left[ \prod_{i=1}^{k} [b_i - m_{\overline{B}}(b_i)] f_{X \setminus \frac{3}{4}B} \right] (y) - h_B \right| d\mu(y)
\]
\[=: II_1 + II_2 + II_3.\]
Write
\[
b_i(y) - m_{\frac{3}{4}Q}(b_i) = b_i(y) - m_{\frac{3}{4}Q}(b_i) + m_{\frac{3}{4}Q}(b_i) - m_{\frac{3}{4}Q}(b_i)
\]
for $i = 1, \ldots, k$. By Lemma 4.1, we obtain

\begin{equation}
(4.9) \quad \int_{B} \prod_{i=1}^{k} |b_i(y) - m_{B}(b_i)|^{s_i'} d\mu(y) \leq C \|\vec{b}\|_{s'}^{s'} \mu(6B).
\end{equation}

Take $s = \sqrt{r}$ and let $1/t = 1/s - \alpha/n$. By Theorem 1.1, the Hölder’s inequality and (4.9), we deduce

\begin{align}
II_1 & \leq \frac{\mu(B)^{1-1/t}}{\mu(6B)} \left\| I_{\alpha} \left( \prod_{i=1}^{k} |b_i - m_{B}(b_i)| f \chi_{\tilde{B}} \right) \right\|_{L^{r}(\mu)} \\
& \leq C \frac{\mu(B)^{1-1/t}}{\mu(6B)} \left\| \prod_{i=1}^{k} |b_i - m_{B}(b_i)| f \chi_{\tilde{B}} \right\|_{L^{r}(\mu)} \\
& \leq C \left( \frac{1}{\mu(6B)} \int_{\tilde{B}} |b_i(y) - m_{B}(b_i)|^{s_i'} d\mu(y) \right)^{s_i'} \\
& \times \left( \frac{1}{\mu(6B)^{1-s}} \int_{\tilde{B}} |f(y)|^{s} d\mu(y) \right)^{1/s}
\end{align}

\begin{equation}
(4.10)
\leq C \|\vec{b}\|_{s} M_{r,(5)}(f)(x).
\end{equation}

From the Hölder’s inequality and Lemma 4.1, it follows that

\begin{align}
II_2 & \leq \sum_{i=1}^{k} \sum_{\sigma \in C_i^{0}} \left( \frac{1}{\mu(6B)} \int_{B} |[b_i(y) - m_{B}(b_i)] \mu(y) \right)^{s_i'} \\
& \times \left( \frac{1}{\mu(6B)} \int_{B} |I_{\alpha,\tilde{B},\sigma}(f(y)) \mu(y) \right)^{s_i'} \\
& \leq C \sum_{i=1}^{k} \sum_{\sigma \in C_i^{0}} \|\vec{b}\|_{s} M_{r,(6)}(I_{\alpha,\tilde{B},\sigma}(f))(x)
\end{align}

\begin{equation}
(4.11)
= C \sum_{i=1}^{k-1} \sum_{\sigma \in C_i^{k}} \|\vec{b}\|_{s} M_{r,(6)}(I_{\alpha,\tilde{B},\sigma}(f))(x) + C \|b\|_{s} M_{r,(6)}(I_{\alpha}(f))(x).
\end{equation}
Let us estimate $I_{III}$. For $y, y_0 \in B$, by the condition (ii) in Definition 1.6 and the Hölder’s inequality, we have

$$\left| I_{\alpha} \left( \prod_{i=1}^{k} [b_i - m_B(b_i)] f \chi_{X \setminus \oplus B} \right)(y) \right|$$

$$- I_{\alpha} \left( \prod_{i=1}^{k} [b_i - m_B(b_i)] f \chi_{X \setminus \oplus B} \right)(y_0) \leq C \int_{X \setminus \oplus B} \frac{d(y, y_0)^r}{d(y, z)^r} \lambda(y, d(y, z)) |f(z)| d\mu(z) \prod_{i=1}^{k} |b_i(z) - m_B(b_i)| \left| f(z) \right| d\mu(z)$$

$$(4.12)$$

$$\leq C \sum_{j=0}^{\infty} \sum_{\sigma \in C_1^*} \sum_{i=0}^{\infty} 6^{-jr} j^{k-i} \|\vec{b}_\sigma\|_* \times \frac{1}{\mu(5 \times 6^i \oplus B)^{1-\alpha/n}} \int_{6^i \oplus B} b(z) - m_{6^i \oplus B}(b_i) \left| f(z) \right| d\mu(z)$$

$$\leq C \sum_{i=0}^{\infty} \sum_{\sigma \in C_1^*} \sum_{j=1}^{\infty} 6^{-jr} j^{k-i} \|\vec{b}_\sigma\|_* \|\vec{b}_\sigma\|_* M_{r,(5)}^{(\alpha)} f(x)$$

$$\leq C \|\vec{b}\|_* M_{r,(5)}^{(\alpha)} f(x).$$

From the above estimate and the definition of $h_B$, we have

$$\left| I_{\alpha} \left( \prod_{i=1}^{k} [b_i - m_B(b_i)] f \chi_{X \setminus \oplus B} \right)(y) \right|$$

$$- I_{\alpha} \left( \prod_{i=1}^{k} [b_i - m_Q(b_i)] f \chi_{X \setminus \oplus B} \right)(y) \leq C \|\vec{b}\|_* M_{r,(5)}^{(\alpha)} f(x).$$

$$(4.13)$$

Then

$$II_3 \leq C \|\vec{b}\|_* M_{r,(5)}^{(\alpha)} f(x).$$

The estimate for $II_1, II_2$ and $II_3$ yields (4.5).
Now we turn to the estimate for (4.6). For any balls $B \subset Q$ with $x \in B$ and $Q$ is a doubling ball, we denote $N_{B,Q} + 1$ simply by $N$.

\begin{equation}
\frac{|m_B \left[ I_\alpha \left( \prod_{i=1}^{k} [b_i - m_{B}(b_i)] f_{\chi_{X \setminus \frac{1}{2}B}} \right) \right]|}{|m_Q \left[ I_\alpha \left( \prod_{i=1}^{k} [b_i - m_{Q}(b_i)] f_{\chi_{X \setminus \frac{1}{2}Q}} \right) \right]|} \leq \frac{|m_Q \left[ I_\alpha \left( \prod_{i=1}^{k} [b_i - m_{B}(b_i)] f_{\chi_{X \setminus \frac{1}{2}B}} \right) \right]|}{|m_Q \left[ I_\alpha \left( \prod_{i=1}^{k} [b_i - m_{B}(b_i)] f_{\chi_{X \setminus \frac{1}{2}B}} \right) \right]|} + m_B \left[ I_\alpha \left( \prod_{i=1}^{k} [b_i - m_{B}(b_i)] f_{\chi_{X \setminus \frac{1}{2}B}} \right) \right] + m_Q \left[ I_\alpha \left( \prod_{i=1}^{k} [b_i - m_{B}(b_i)] f_{\chi_{X \setminus \frac{1}{2}B}} \right) \right] + m_B \left[ I_\alpha \left( \prod_{i=1}^{k} [b_i - m_{B}(b_i)] f_{\chi_{X \setminus \frac{1}{2}B}} \right) \right] + m_Q \left[ I_\alpha \left( \prod_{i=1}^{k} [b_i - m_{Q}(b_i)] f_{\chi_{X \setminus \frac{1}{2}B}} \right) \right] + m_B \left[ I_\alpha \left( \prod_{i=1}^{k} [b_i - m_{B}(b_i)] f_{\chi_{X \setminus \frac{1}{2}B}} \right) \right] + m_Q \left[ I_\alpha \left( \prod_{i=1}^{k} [b_i - m_{Q}(b_i)] f_{\chi_{X \setminus \frac{1}{2}B}} \right) \right] =: JJ_1 + JJ_2 + JJ_3 + JJ_4.
\end{equation}

With the similar estimate for $II_3$, we easily get that

\[ JJ_1 \leq C||\tilde{b}||_1 M_{r,(3)}^{(\alpha)} f(x). \]
To estimate $JJ_2$, with the help of (4.7), we deduce that

$$
|I_\alpha \left( \prod_{i=1}^{k} [b_i - m_Q(b_i)] f_{X^{N\backslash \mathcal{B}^{N^{*} \cap B}}(y) \right) - I_\alpha \left( \prod_{i=1}^{k} [b_i - m_{\mathcal{B}}(b_i)] f_{X^{N\backslash \mathcal{B}^{N^{*} \cap B}}(y) \right)
$$

$$
= I_\alpha \left( \prod_{i=1}^{k} [b_i - m_Q(b_i)] f_{X^{N\backslash \mathcal{B}^{N^{*} \cap B}}(y) \right)
$$

$$
- \sum_{i=0}^{k-1} \sum_{\sigma \in C_i} (K_{B,Q})^{k-i-1} |\vec{b}_{\sigma'}|_{*} \left| I_\alpha \left( \prod_{i=1}^{i} [b_i - m_Q(b_i)] f_{X^{N\backslash \mathcal{B}^{N^{*} \cap B}}(y) \right) \right|
$$

$$
\leq C \sum_{i=0}^{k-1} \sum_{\sigma \in C_i} (K_{B,Q})^{k-i-1} |\vec{b}_{\sigma'}|_{*} \left\{ I_\alpha \left( \sum_{\eta \in C_j} \sum_{j=0}^{i} |[b(y) - m_Q(b)]_{\eta'} I_{\alpha,\vec{b}_{\eta}} f(y) \right) \right\}
$$

$$
\leq C \sum_{i=0}^{k-1} \sum_{\sigma \in C_i} (K_{B,Q})^{k-i-1} |\vec{b}_{\sigma'}|_{*} \left\{ \sum_{\eta \in C_j} \sum_{j=0}^{i} |[b(y) - m_Q(b)]_{\eta'} I_{\alpha,\vec{b}_{\eta}} f(y) \right\}
$$

Using the Hölder’s inequality and the fact that $Q$ is a doubling ball, it follows that

$$
\frac{1}{\mu(Q)} \int_{Q} |[b(y) - m_Q(b)]_{\eta'} I_{\alpha,\vec{b}_{\eta}} f(y)| \, d\mu(y)
$$

$$
\leq C \mu(6Q) \mu(Q) \left[ \frac{1}{\mu(6Q)} \int_{Q} |b(y) - m_Q(b)| \, d\mu(y) \right]^{1/r'}
$$

$$
\times \left[ \frac{1}{\mu(6Q)} \int_{Q} |I_{\alpha,\vec{b}_{\eta}} f(y)| \, d\mu(y) \right]^{1/r}
$$

$$
\leq C |\vec{b}_{\eta'}|_{*} M_{r,(6)}(I_{\alpha,\vec{b}_{\eta}} f)(x).
$$
By the Hölder’s inequality, Lemma 4.1 and the condition (i) in Definition 1.6, it is easy to see that for \( y \in Q \),

\[
\left| I_{\alpha} \left( [b - m_Q(b)]_\sigma f \chi_{6N_B \setminus \frac{5}{6}Q} \right)(y) \right|
\leq C \int_{6N_B \setminus \frac{5}{6}Q} \frac{1}{\lambda(y, \delta(y, z))^{1-\alpha/n}} |b(z)| |f(z)| |z - m_Q(b)| |z - m_Q(b)|^\sigma d\mu(z)
\leq C \left( \frac{1}{\mu(5 \times 6N_B)^{1-\alpha/n}} \int_{6N_B} |[b(z) - m_Q(b)]_\sigma |f(z)| |z - m_Q(b)|^\sigma d\mu(z) \right)^{1/2}
\times \left( \frac{1}{\mu(5 \times 6N_B)^{1-\alpha/n}} \int_{6N_B} |f(z)| |z - m_Q(b)|^\sigma d\mu(z) \right)^{1/2}
\leq C \|b_\sigma\|_s M_{r, (5)}(f(x)).
\]

Taking the mean over \( y \in Q \), it easily obtains that

\[
m_Q \left[ I_{\alpha} \left( [b - m_Q(b)]_\sigma f \chi_{6N_B \setminus \frac{5}{6}Q} \right) \right] \leq C \|b_\sigma\|_s M_{r, (5)}(f(x)).
\]

With the similar method to estimate \( II_1 \), we deduce

\[
m_Q \left[ I_{\alpha} \left( [b - m_Q(b)]_\sigma f \chi_{\frac{5}{6}Q} \right) \right] \leq C \|b_\sigma\|_s M_{r, (5)}(f(x)).
\]

Therefore,

\[
JJ_2 \leq C(K_{B,Q})^k \sum_{i=0}^{k-1} \sum_{\sigma \in C_i^k} \|b_\sigma\|_s \left( M_{r, (6)}(I_{\alpha, b_\sigma} f)(x) + \|b_\sigma\|_s M_{r, (5)}(f(x)) \right)
\]

\[
= C(K_{B,Q})^k \sum_{i=1}^{k-1} \sum_{\sigma \in C_i^k} \|b_\sigma\|_s \left( M_{r, (6)}(I_{\alpha, b_\sigma} f)(x) + M_{r, (5)}(f(x)) \right).
\]
Now we turn to estimate \( JJ_3 \). By the Hölder’s inequality, Lemma 4.1 and Lemma 4.2, for \( y \in B \), we get

\[
(4.21) \quad \left| I_\alpha \left( \prod_{i=1}^k [b_i - m_B(b_i)] f \chi_{6^\alpha B(\frac{y}{B})} \right)(y) \right| \\
\leq C \sum_{j=1}^{N-1} \frac{1}{\lambda(y, r_{6^j B})} \int_{6^{j+1}B} |b_i(z) - m_B(b_i)||f(z)|d\mu(z) \\
+ \frac{1}{\lambda(y, r_B)} \int_{6^jB \cap \frac{y}{B}} \prod_{i=1}^k |b_i(z) - m_B(b_i)||f(z)|d\mu(z) \\
\leq C \sum_{j=1}^{N-1} \left[ \frac{\mu(5 \times 6^{j+1}B)}{\lambda(y, r_{5 \times 6^{j+1}B})} \right]^{1 - \frac{1}{r}} \left[ \int_{6^{j+1}B} 1 \int_{6^{j+1}B} |f(z)|r d\mu(z) \right]^{1/r} \\
\times \left[ \frac{1}{\mu(5 \times 6B)} \int_{6B} \prod_{i=1}^k |b_i(z) - m_{6^{j+1}B}(b_i)|r d\mu(z) \right]^{1/r'} \\
+ C \left[ \frac{1}{\mu(5 \times 6B)} \int_{6B} \prod_{i=1}^k |b_i(z) - m_B(b_i)|r d\mu(z) \right]^{1/r'} \\
\times \left[ \int_{6B} |f(z)|r d\mu(z) \right]^{1/r} \\
\leq CK_{B,Q}^{(\alpha)} (K_{B,Q})^k ||\vec{b}||_* M_{r,(5)}^{(\alpha)} f(x) + C ||\vec{b}||_* M_{r,(5)}^{(\alpha)} f(x) \\
\leq CK_{B,Q}^{(\alpha)} (K_{B,Q})^k ||\vec{b}||_* M_{r,(5)}^{(\alpha)} f(x).
\]

Taking the mean over \( y \in Q \), we obtain

\[
JJ_3 \leq CK_{B,Q}^{(\alpha)} (K_{B,Q})^k ||\vec{b}||_* M_{r,(5)}^{(\alpha)} f(x).
\]
Finally, we estimate $JJ_4$. For $y \in Q$,

\[
\left| I_\alpha \left( \prod_{i=1}^{k} [b_i - m_Q(b_i)] \chi_{6^N B \setminus \xi Q} \right)(y) \right| \\
\leq C \int_{6^N B \setminus \xi Q} \frac{1}{\lambda(y, d(y, z))^{1 - \alpha/n}} \prod_{i=1}^{k} |b_i(z) - m_Q(b_i)||f(z)| d\mu(z) \\
\leq C \left[ \frac{1}{\mu(5 \times 6^N B)} \int_{6^N B} \prod_{i=1}^{k} |b_i(y) - m_B(b_i)|^r d\mu(y) \right]^{1/r'} \\
\times \left[ \frac{1}{\mu(5 \times 6^N B)^{1 - \alpha/n}} \int_{6^N B} |f(y)|^r d\mu(y) \right]^{1/r} \\
\leq C \|\vec{b}\|_r M_{r,(5)}^{(\alpha)} f(x).
\]

(4.22)

Taking the mean over $y \in Q$, it follows that

\[ JJ_4 \leq C \|\vec{b}\|_r M_{r,(5)}^{(\alpha)} f(x). \]

The estimate for $JJ_1, JJ_2, JJ_3$ and $JJ_4$ yields (4.6). Thus the proof of Theorem 1.3 is completed.

\[ \square \]

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