What power of two divides a weighted Catalan number?

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Abstract

Given a sequence of integers \( b = (b_0, b_1, b_2, \ldots) \) one gives a Dyck path \( P \) of length \( 2n \) the weight

\[
\text{wt}(P) = b_{h_1} b_{h_2} \cdots b_{h_n},
\]

where \( h_i \) is the height of the \( i \)th ascent of \( P \). The corresponding weighted Catalan number is

\[
C_n^b = \sum_P \text{wt}(P),
\]

where the sum is over all Dyck paths of length \( 2n \). So, in particular, the ordinary Catalan numbers \( C_n \) correspond to \( b_i = 1 \) for all \( i \geq 0 \). Let \( \xi(n) \) stand for the base two exponent of \( n \), i.e., the largest power of 2 dividing \( n \). We give a condition on \( b \) which implies that \( \xi(C_n^b) = \xi(C_n) \). In the special case \( b_i = (2i + 1)^2 \), this settles a conjecture of Postnikov about the number of plane Morse links. Our proof generalizes the recent combinatorial proof of Deutsch and Sagan of the classical formula for \( \xi(C_n) \).
1 Introduction

The Catalan numbers

\[ C_n = \frac{1}{n+1} \binom{2n}{n} \]

have many interesting arithmetic properties. For example, the following result which essentially dates back to Kummer (see Dickson’s book \cite{3} for details) describes their divisibility by powers of 2. Let \( s(n) \) be the sum of digits in the binary expansion of \( n \). Also, let \( \xi(n) \) denote the base two exponent of \( n \), i.e., the largest power of two dividing \( n \).

**Theorem 1.1.** We have

\[ \xi(C_n) = s(n + 1) - 1. \]

A combinatorial proof of this result was recently given by Deutsch and Sagan \cite{2} using group actions. In this paper, we extend this result and their proof to various weighted Catalan numbers. Note that, unlike the Catalan numbers, weighted Catalan numbers need not have simple multiplicative formulas. So determining their divisibility properties is more subtle than for the usual Catalan numbers.

Let \( b = (b_0, b_1, b_2, \ldots) \) be a fixed infinite sequence of integers. Define the **weighted Catalan numbers** \( C^b_n \), as the coefficients of the expansion of the continued fraction:

\[
\frac{1}{1 - \frac{b_0 x}{1 - \frac{b_1 x}{1 - \frac{b_2 x}{1 - \frac{b_3 x}{1 - \ldots}}}}}
= \sum_{n \geq 0} C^b_n x^n. \tag{1}
\]

If \( b = (1, 1, 1, \ldots) \), then \( C^b_n \) is the usual Catalan number \( C_n \).

Combinatorially, the \( C^b_n \) count Dyck paths with certain weights. Recall that a **Dyck path** \( P \) of length \( 2n \) is a sequence of points in the upper half-plane of the integer lattice

\[(x_0, y_0) = (0, 0), (x_1, y_1), \ldots, (x_{2n}, y_{2n}) = (2n, 0),\]

such that each step \( s_i = [x_i - x_{i-1}, y_i - y_{i-1}] \) has the form \([1, 1]\) or \([1, -1]\). Let us say that step \( s_i \) has **height** \( y_{i-1} \). Define the **weight** of a Dyck path \( P \) to be the product

\[ \text{wt}(P) = b_{h_1} b_{h_2} \cdots b_{h_n}, \]

where \( h_1, \ldots, h_n \) are the heights of its steps of the form \([1, 1]\). Then the following proposition is well known and easy to prove, i.e., see the book of Goulden and Jackson \cite{4} Ch. 5.
Proposition 1.2. We have
\[ C_n^b = \sum P \text{wt}(P), \]
where the sum is over all Dyck paths of length \(2n\).

For example, we have
\[ C_3^b = b_0 b_0 b_0 + b_0 b_0 b_0 + b_0 b_0 b_0 + b_0 b_0 b_0 + b_0 b_0 b_1 + b_0 b_1 b_0 + b_0 b_1 b_1 + b_0 b_1 b_2, \]
where the five terms correspond to the five Dyck paths of length 6. As another example, if \(b = (1, q, q^2, q^3, \ldots)\), then the weighted Catalan number \(C_n^b\) is equal to the \(q\)-Catalan number
\[ C_n(q) = \sum P q^{\text{area}(P)}, \]
where the sum is over Dyck paths \(P\) of length \(2n\) and \(\text{area}(P)\) denotes the area between \(P\) and the lowest possible path. The continued fraction (1) with \(b_i = q^i\) is known as the Ramanujan continued fraction.

Our main result, Theorem 2.1 below, gives a necessary condition on the sequence \(b\) so that
\[ \xi(C_n^b) = \xi(C_n) = s(n + 1) - 1. \]

As a special case, we obtain a conjecture of Postnikov [5] about plane Morse links. A plane Morse curve is a simple curve \(f : S^1 \to \mathbb{R}^2\) (i.e., a smooth injective map) such that, for the height function \(h : (x, y) \to y\), the map \(h \circ f\) has a finite number of isolated nondegenerate critical points with distinct values. All curves are oriented clockwise. See, for example, Figure 1. As one goes around a Morse curve, the sequence formed by the critical values is alternating and returns to where it started. So the number of critical values must be an even integer \(2n\), and \(n\) is called the order of the curve. The combinatorial type of a Morse curve is its connected component in the space of all Morse curves. So Figure 1(a) depicts the four plane Morse curves of order 2 up to combinatorial type.

A plane Morse link is a disjoint union of plane Morse curves. All our definitions for Morse curves carry over in the natural way to links. In particular, the order of a link is the sum of the orders of its components. Figure 1(b) shows the six disconnected Morse links of order 2, again up to combinatorial type. Let \(L_n\) be the number of combinatorial types of plane Morse links of order \(n\). Then we have just seen that \(L_2 = 4 + 6 = 10\). The connection with weighted Catalan numbers is made by the following theorem.
Theorem 1.3 ([5]). The numbers \( L_n \) satisfy

\[
\sum_{n \geq 0} L_n x^n = \frac{1}{1 - \frac{1^2 x}{1 - \frac{3^2 x}{1 - \frac{5^2 x}{1 - \frac{7^2 x}{\ldots}}}}}
\]

and so

\[ L_n = C_n^{(1^2, 3^2, 5^2, 7^2, \ldots)}. \]

An easy corollary of our main theorem will be a proof of Conjecture 3.1 from [5]. It can also be found listed as Problem 6.C5(c) in the Catalan Addendum to the second volume of Stanley’s *Enumerative Combinatorics* [7].

Conjecture 1.4 ([5, 7]). We have

\[ \xi(L_n) = \xi(C_n) = s(n + 1) - 1. \]

2 The main theorem

Let \( \mathbb{Z}_{\geq 0} \) denote the nonnegative integers. The *difference operator*, \( \Delta \), acts on functions \( f : \mathbb{Z}_{\geq 0} \to \mathbb{Z} \) by

\[ (\Delta f)(x) = f(x + 1) - f(x). \]

Note that we can regard our sequence \( b \) as such a function where \( b(x) = b_x \). We can now state our main result, using \( c \mid d \) as usual to mean that \( c \) divides evenly into \( d \).

**Theorem 2.1.** Assume that the sequence \( b \) satisfies
1. \(b(0)\) is odd, and
2. \(2^{n+1} \mid (\Delta^n b)(x)\) for all \(n \geq 1\) and \(x \in \mathbb{Z}_{\geq 0}\).

Then

\[
\xi(C_n^b) = \xi(C_n) = s(n + 1) - 1.
\]

To prove this, we will also have to consider the shift operator, \(S\), acting on functions \(f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}\) by

\[(Sf)(x) = f(x + 1).\]

It is well known and easy to verify that we have the product rule

\[
\Delta(f \cdot g) = \Delta(f) \cdot g + S(f) \cdot \Delta(g).
\]

which generalizes to

\[
\Delta^n(f \cdot g) = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k}(S^k(f)) \cdot \Delta^k(g).
\]

(2)

Let \(\mathcal{F}\) be the set of functions \(f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}\) such that

(a) \(f(x)\) is odd for all \(x \in \mathbb{Z}_{\geq 0}\), and

(b) \(2^{n+1} \mid (\Delta^n f)(x)\) for all \(n \geq 1\) and \(x \in \mathbb{Z}_{\geq 0}\).

Note that because of (b), we can replace (a) by the seemingly weaker condition that \(f(0)\) is odd. We will need the following lemma.

**Lemma 2.2.** The set \(\mathcal{F}\) is closed under the three operations

\[f \mapsto Sf, \quad (f, g) \mapsto f \cdot g, \quad \text{and} \quad (f, g) \mapsto \langle f, g \rangle := \frac{f(x + 1)g(x) + f(x)g(x + 1)}{2}.\]

**Proof.** Closure under \(S\) is obvious. Now assume that \(f(x), g(x) \in \mathcal{F}\). Then the value \((f \cdot g)(x)\) is clearly odd. And (2) shows that the divisibility criterion is satisfied. Thus \(f \cdot g \in \mathcal{F}\).

We can write the second operation as

\[
\langle f, g \rangle = f \cdot g + \frac{\Delta(f) \cdot g + f \cdot \Delta(g)}{2}.
\]

Since \(\Delta(f)\) and \(\Delta(g)\) are divisible by 4 and \((f \cdot g)(x)\) is odd, we deduce that \(\langle f, g \rangle(x)\) is odd. By (2), \(\Delta^n(\Delta(f) \cdot g)\) and \(\Delta^n(f \cdot \Delta(g))\) are divisible by \(2^{n+2}\). Thus \(\langle f, g \rangle \in \mathcal{F}\).
Deutsch and Sagan used the interpretation of $C_n$ in terms of binary trees to prove Theorem 1.1. So we will need to review this method and translate our weight function into this setting. A binary tree, $T$, is a rooted tree where every vertex has a right child, a left child, both children, or no children. We also consider the empty tree to be a binary tree. Let $\mathcal{T}_n$ be the set of binary trees with $n$ vertices. Then one of the standard interpretations of the Catalan numbers is that

$$C_n = \# \mathcal{T}_n. \quad (3)$$

Let $G_n$ be the group of symmetries of the binary tree which is complete to depth $n$ (having all of its leaves at distance $n$ from the root). The group $G_n$ is generated by reflections which exchange the left and right subtrees associated with a vertex. Then $G_n$ acts on $\mathcal{T}_n$ with two trees being in the same orbit if they are isomorphic as rooted trees if we forget about the information concerning left and right children. Deutsch and Sagan show in [2, Section 2] that $\#G_n$ is a power of 2, so the cardinality of any $G_n$-orbit is as well. They also show that the minimal size of a $G_n$-orbit is $2^s$ where $s = s(n+1) - 1$. Moreover, orbits of the minimal size can be identified with binary total partitions on the set $\{1, 2, \ldots, s\}$, whose number is $(2s - 1)!! = 1 \cdot 3 \cdot 5 \cdots (2s - 1)$ as shown by Schröder [6], see also [8, Example 5.2.6]. For ease of reference, we summarize these facts in the following lemma.

**Lemma 2.3 ([2]).** Let $O$ be an orbit of $G_n$ acting on $\mathcal{T}_n$ and let $s = s(n+1) - 1$. Then

1. $\#O = 2^t$ for some $t \geq s$, and
2. $\#O = 2^s$ for exactly $(2s - 1)!!$ orbits. $\square$

It is now easy to prove Theorem 1.1 using this lemma and equation (3). To generalize the proof, consider any fixed function $b \in \mathcal{F}$ and define the corresponding weight of a binary tree $T$ to be the function

$$w_b(T) = w_b(T; x) = \prod_{v \in T} b(x + l_v),$$

where the product is over all vertices $v$ of $T$, and $l_v$ is the number of left edges on the unique path from the root of $T$ to $v$. If binary tree $T$ corresponds to Dyck path $P$ under the usual depth-first search bijection, then it is easy to see that

$$\text{wt}(P) = \prod_{v \in T} b(l_v) = w_b(T; 0). \quad (4)$$

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We need one last lemma for the proof of Theorem 2.1. If \( \mathcal{O} \) is an orbit of \( G_n \) acting on \( \mathcal{T}_n \) then we define it’s weight to be
\[
w_b(\mathcal{O}) = w_b(\mathcal{O}; x) = \sum_{T \in \mathcal{O}} w_b(T).
\] (5)

**Lemma 2.4.** For any \( b \in \mathcal{F} \) and any orbit \( \mathcal{O} \) we have
\[
w_b(\mathcal{O}; x) = \#\mathcal{O} \cdot r_b(\mathcal{O}; x),
\]
where \( r_b(\mathcal{O}; x) \in \mathcal{F} \).

**Proof.** Write
\[
r_b(\mathcal{O}; x) = \frac{w_b(\mathcal{O}; x)}{\#\mathcal{O}}.
\] (6)
We induct on \( n \), the number of vertices in a tree of \( \mathcal{O} \). If \( n = 0 \) then \( r_b(\mathcal{O}; x) = 1 \) for all \( x \) which is clearly in \( \mathcal{F} \).

Now suppose \( n \geq 1 \) so that \( \mathcal{O} \) contains a nonempty tree \( T \). Let \( T_1 \) be the subtree of \( T \) consisting of the left child of the root and all its descendents. (So \( T_1 \) may be empty.) Similarly, define \( T_2 \) for the right child. Suppose \( T_1 \) and \( T_2 \) are in orbits \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \), respectively. If \( \mathcal{O}_1 = \mathcal{O}_2 \) then \( \#\mathcal{O} = \#\mathcal{O}_1 \cdot \#\mathcal{O}_2 \) and
\[
w_b(T) = b(x) \cdot w_b(T_1) \cdot w_b(T_2).
\]
If \( \mathcal{O}_1 \neq \mathcal{O}_2 \), then \( \#\mathcal{O} = 2 \cdot \#\mathcal{O}_1 \cdot \#\mathcal{O}_2 \) and
\[
w_b(T) = b(x) \left[ w_{S(b)}(T_1) \cdot w_b(T_2) + w_b(T_1) \cdot w_{S(b)}(T_2) \right].
\]
In the both cases, it follows from equations (5) and (6) that
\[
r_b(\mathcal{O}; x) = b(x) \cdot \frac{r_b(\mathcal{O}_1; x + 1) \cdot r_b(\mathcal{O}_2; x) + r_b(\mathcal{O}_1; x) \cdot r_b(\mathcal{O}_2; x + 1)}{2}.
\]
So by Lemma 2.2 and induction we have \( r_b(\mathcal{O}; x) \in \mathcal{F} \) as desired. \( \square \)

**Proof of Theorem 2.1** Combining Proposition 1.2, the previous lemma, and equations (4) and (5) gives
\[
C_n^b = \sum_P \text{wt}(P) = \sum_{T \in \mathcal{T}_n} w_b(T; 0) = \sum_\mathcal{O} \#\mathcal{O} \cdot r_b(\mathcal{O}; 0)
\] (7)
where the integers \( r_b(\mathcal{O}; 0) \) are all odd since \( r_b(\mathcal{O}) \in \mathcal{F} \). So \( \xi(\#\mathcal{O} \cdot r_b(\mathcal{O}; 0)) = \xi(\#\mathcal{O}) \) for all orbits \( \mathcal{O} \). It now follows from Lemma 2.3 that \( \xi = s \) for an odd number (namely \((2s - 1)!!\)) of summands in the last summation in (7) and that \( \xi > s \) for the rest. We conclude that \( \xi(C_n^b) = s = \xi(C_n) \). \( \square \)

As corollaries, we can prove Conjecture 1.4 and give information about divisibility of the \( q \)-Catalan numbers.
Corollary 2.5. 1. The number of combinatorial types of plane Morse links of order $n$ satisfies
\[ \xi(L_n) = \xi(C_n) = s(n + 1) - 1. \]
2. If $q \equiv 1 \pmod{4}$ then the $q$-Catalan numbers satisfy
\[ \xi(C_n(q)) = \xi(C_n) = s(n + 1) - 1. \]

Proof. For the first assertion, it suffices to show that the function $b(x) = (2x + 1)^2$ is in $F$. Clearly $b(x)$ is odd for all $x \in \mathbb{Z}_{\geq 0}$. Furthermore $\Delta b = 8(x + 1)$, $\Delta^2 b = 8$, and $\Delta^n b = 0$ for $n \geq 3$. So the divisibility condition also holds.

For the second statement, we need the function $b(x) = q^x$ to be in $F$. Since $q$ is odd, so is $b(x)$. Also $\Delta^n b = (q - 1)^n q^x$ for $n \geq 1$. So the hypothesis on $q$ implies that $\Delta^n b$ is divisible by $4^n = 2^{2n}$ which is more than needed.

3 An open problem

Consider the Catalan sequence
\[ C = (C_0, C_1, C_2, \ldots). \]

Theorem 1.1 implies immediately that $C_n$ is odd if and only if $n = 2^k - 1$ for some $k \geq 0$. It follows that the $k$th block of zeros in the sequence $C$ taken modulo 2 has length $2^k - 1$ (where we start numbering with the first block). Alter and Kubota [1] have generalized this result to arbitrary primes and prime powers. One of their main theorems is as follows.

Theorem 3.1 ([1]). Let $p \geq 3$ be a prime and let $q = (p + 1)/2$. The length of the $k$th block of zeros in $C$ modulo $p$ is
\[ \frac{p^{\xi_q(k) + \delta_{3,p} + 1} - 3}{2} \]
where $\xi_q(k)$ is the largest power of $q$ dividing $k$ and $\delta_{3,p}$ is the Kronecker delta.

Deutsch and Sagan [2] have improved on this theorem when $p = 3$ by giving a complete characterization of the residues in $C$ modulo 3. However, the demonstrations of all these results rely heavily on the expression for $C_n$ as a product. It would be interesting to find analogous theorems for $C_n^b$, but new proof techniques would have to be found.
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