SCALAR FIELD COSMOLOGIES WITH VISCOSOUS FLUID

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Abstract

We investigate cosmological models with a free scalar field and a viscous fluid. We find exact solutions for a linear and nonlinear viscosity pressure. Both yield singular and bouncing solutions. In the first regime, a de Sitter stage is asymptotically stable, while in the second case we find power-law evolutions for vanishing cosmological constant.

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1 Introduction

Studies of the dynamics of a scalar field together with other forms of matter usually assume that this behaves as a perfect fluid. Few authors have taken into account dissipative effects [1] [2] [3] [4], even though processes like particle creation might have been important in the early universe. In this paper we study the evolution of a universe filled with a massless minimally coupled scalar field and a viscous fluid. We find exact solutions of the Einstein equations in a Robertson-Walker metric and we analyse their asymptotic stability by means of the Lyapunov method [5].

2 The model

We investigate the evolution of a universe filed with a scalar field and a viscous fluid. The scalar field $\phi$ is free and minimally coupled to gravity, so that it obeys the equation $\Box \phi = 0$. In the case of the homogeneous isotropic Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

(1)

the scalar field equation becomes

$$\ddot{\phi} + 3H \dot{\phi} = 0$$

(2)

Besides, only the bulk viscosity needs to be considered. Thus we replace in the Einstein equations the equilibrium pressure $p$ by an effective pressure

$$3H^2 = \frac{1}{2} \dot{\phi}^2 + \rho - 3 \frac{k}{a^2} + \Lambda$$

(3)

$$2\dot{H} + 3H^2 = -\frac{1}{2} \dot{\phi}^2 - p - \sigma - \frac{k}{a^2} + \Lambda$$

(4)

where $H = \dot{a}/a$, $\dot{} = d/dt$, $\rho$ is the energy density, $\sigma$ is the viscous pressure, $\Lambda$ is the cosmological constant and we use units $c = 8\pi G = 1$. As equation of state we take

$$p = (\gamma - 1)\rho$$

(5)
with a constant adiabatic index $0 \leq \gamma \leq 2$, and we assume that $\sigma$ has the constitutive equation

$$\sigma = -3\zeta H$$

(6)

where $\zeta \geq 0$ is the bulk viscosity coefficient. Thus, we must solve equation (2) together with the Einstein equations (3)(4). Equation (2) has the first integral

$$\dot{\phi} = \frac{C}{a^3}$$

(7)

where $C$ is an arbitrary integration constant, and we are interested in the case that $C \neq 0$. Then, adding (3) and (4), and eliminating $\rho$, we arrive at

$$2\dot{H} + 3\gamma H^2 - 3\zeta H = -\frac{2 - \gamma}{2} \frac{C^2}{a^6} + (2 - 3\gamma) \frac{k}{a^2} + \gamma\Lambda$$

(8)

To find exact solutions of this equation we make the change of variable $a = v^\nu$, and we consider two cases: $\zeta$ constant and $\zeta \sim H$. We assume now that $\zeta$ is a constant. This is a good approximation when the thermodynamical variables do not change too much. This has been studied in several papers [7] [8] [9] [10] [11]. Choosing $\nu = \frac{2}{(3\gamma)}$, the equation (8) becomes

$$\ddot{v} + \frac{3}{2} \zeta \dot{v} + M v^m + N v^n - \frac{\gamma\Lambda}{2\nu} v = 0$$

(9)

where $M = (3/8)\gamma (2 - \gamma) C^2$, $m = 1 - 4/\gamma$, $N = (3/4)\gamma (3\gamma - 2) k$, $n = 1 - 4/(3\gamma)$. It is linear for $\gamma = 2$ and $k = 0$, so that we can find its general solution. In this case, the evolution of the scale factor is the same as in a model without the scalar field and a bulk viscosity coefficient $\zeta/2$ [7] [8]. Thus, we just quote the solutions without further analysis.

Let us consider first that $\Lambda = 0$. We find

$$a(t) = \left[ A \exp \left( \frac{3}{2} \frac{\zeta t}{\tilde{t}} \right) + B \right]^{1/3}$$

(10)

$$\Delta \phi(t) = \frac{2C}{3\zeta B} \ln \left[ \frac{\exp \left( \frac{3}{2} \zeta \tilde{t} \right)}{A \exp \left( \frac{3}{2} \zeta \tilde{t} \right) + B} \right]$$

(11)
\[ \rho(t) = \frac{\frac{3}{2}\zeta^2 A^2 \exp(3\zeta t) - C^2}{2 \left[ A \exp\left(\frac{3}{2}\zeta t\right) + B \right]^2} \] (12)

where \( A \) and \( B \) are arbitrary integration constants. The requirement \( \rho \geq 0 \) implies that there is a minimum time for the validity of this solution. Thus it describes a geodesically incomplete manifold, which in this sense is singular [12]. Clearly, in its asymptotically de Sitter regime it is physically justified the approximation of constant \( \zeta \). However, for shorter times, this assumption may not be justified.

The asymptotically de Sitter stage occurs for any other value of \( \gamma \) and \( k \). In effect, assuming \( a \sim \exp(H_0 t) \) for \( t \to \infty \), with \( H_0 \) a constant, we find that \( H_0 = \zeta/\gamma \). To verify the asymptotic stability of this solution we use \( a \) as the independent variable and we take the Liapunov function \( L = (H - H_0)^2 \). Then, we find that

\[ L' = -3\gamma \frac{L}{a} + O\left(\frac{1}{a^3}\right) \] (13)

is negative for large times.

Let us consider now that \( \Lambda \neq 0 \). There is a critical value for the cosmological constant \( \Lambda_0 = -3\zeta^2/16 \). Thus, we must distinguish several cases: Two kinds of solutions appear for \( \Lambda > \Lambda_0 \). Singular evolutions

\[ a(t) = a_0 e^{3\Delta t/4} \left[ \sinh (\omega \Delta t) \right]^{1/3} \] (14)

or nonsingular evolutions

\[ a(t) = a_0 e^{3\Delta t/4} \left[ \cosh (\omega \Delta t) \right]^{1/3} \] (15)

where \( \omega = \left[ 3(\Lambda - \Lambda_0) \right]^{1/2} \), \( \Delta t = t - t_0 \) and \( a_0, t_0 \) are integration constants.

For the scalar field we find

\[ \Delta \phi(t) = -\frac{C}{B^2} e^{(\omega - 3\zeta/4)\Delta t} F_1 \left( 1, \frac{1}{2}, \frac{3\zeta}{8\omega}, \frac{3\zeta}{8\omega}, -\frac{A}{B} e^{2\omega \Delta t} \right) \] (16)

While \( \rho \to (3/8)\zeta^2 + (3/2)\zeta \omega > 0 \) for \( t \to \infty \), in the nonsingular solutions \( \rho \) becomes negative before a minimum time. On the other hand, for singular solutions, we find
\[ \rho \sim \left( \frac{1}{3} - \frac{C^2}{2a_0^3 \omega^2} \right) \frac{1}{\Delta t^2} \quad \Delta t \to 0 \]  

so that the energy density may remain positive along all the evolution.

In the case \( \Lambda = \Lambda_0 \), the evolution is always singular

\[ a(t) = a_0 (\Delta t)^{1/3} e^{3\Delta t/4} \]  

In the case \( \Lambda \leq \Lambda_0 \), the scale factor recollapses to a second singularity

\[ a(t) = a_0 e^{3\Delta t/4} [\sin (|\omega| \Delta t)]^{1/3} \]  

All singular evolutions have particle horizons because \( a \sim \Delta t^{1/3} \) as \( \Delta t \to 0 \).

For other values of \( \gamma \) and \( k \), there is also a critical value of the cosmological constant \( \Lambda_0 = -3\zeta^2/(4\gamma^2) \), so that for \( \Lambda > \Lambda_0 \) an asymptotically de Sitter evolution occurs for two values of \( H \)

\[ H_\pm = \frac{1}{2\gamma} \left[ \zeta \pm \left( \zeta^2 + \frac{4}{3} \gamma^2 \Lambda \right)^{1/2} \right] \]  

We verify that the behavior \( a \sim \exp(H_+ t) \) is asymptotically stable by means of the Liapunov function \( L_+ = (H - H_+)^2 \), which satisfies \( L'_+ < 0 \) for large times when \( H > 0 \).

Nonlinear viscous effects has been shown to arise as a phenomenological description of the effect of particle creation in the early universe [13], and cosmological models with this kind of fluids has been considered in [14] and [15].

Here, we assume that \( \zeta = \alpha H \), with a constant \( \alpha \) such that \( \gamma > \alpha > 0 \), and we consider only expanding evolutions. Following the same procedure as before, we arrive at the equation

\[ \ddot{v} + Mv^m + Nv^n - \frac{\gamma \Lambda}{2\nu} v = 0 \]  

where now \( \nu = 2/[3(\gamma - \alpha)] \), \( M = (3/8)(\gamma - \alpha)(2 - \gamma) \), \( m = 1 - 4/(\gamma - \alpha) \), \( N = (3/4)(\gamma - \alpha)(3\gamma - 2)k \) and \( n = 1 - 4/[3(\gamma - \alpha)] \). As this case behaves as a conservative mechanical system, we may obtain (at least formally) its general solution:
\[ \Delta t = \int \frac{dv}{\sqrt{2 \left[ E - V(v) \right]}} \]  

(22)

where \( E \) is an integration constant and

\[ V(v) = \frac{M}{m+1} v^{m+1} + \frac{N}{n+1} v^{n+1} - \frac{\gamma \Lambda}{4 \nu} v^2 \quad \gamma - \alpha \neq \frac{2}{3} \]  

(23)

A qualitative analysis of (22) shows that expanding evolutions may begin either at a singularity or a bounce. Both bounded and unbounded singular solutions occur, but only for \( \Lambda < 0 \) are there bouncing solutions that reach a maximum.

We quote two cases for which the equation (21) becomes linear, so that we may obtain its general solution in closed form (for simplicity we take \( \Lambda = 0 \)). The first one is \( \gamma = 2 \) and \( k = 0 \)

\[ a(t) = (A \Delta t)^\nu \]  

(25)

\[ \Delta \phi(t) = \frac{C \Delta t^{1-3\nu}}{A^{3\nu} (1-3\nu)} \]  

(26)

\[ \rho(t) = \frac{3\nu^2}{\Delta t^2} - \frac{C^2}{2(A \Delta t)^{6\nu}} \]  

(27)

The other solution is for \( \gamma = 2, \alpha = 2/3 \) and \( k \neq 0 \). We find

\[ a(t) = \left| A \Delta t - 2k \Delta t^2 \right|^{1/2} \]  

(28)

\[ \Delta \phi(t) = - \frac{2C}{A^2} \frac{A - 4k \Delta t}{\left| A \Delta t - 2k \Delta t^2 \right|^{1/2}} \]  

(29)

\[ \rho(t) = \frac{3}{4} \frac{(A - 4k \Delta t)^2}{\left| A \Delta t - 2k \Delta t^2 \right|^2} - \frac{1}{2} \frac{C^2}{\left| A \Delta t - 2k \Delta t^2 \right|^3} + \frac{3k}{\left| A \Delta t - 2k \Delta t^2 \right|} \]  

(30)

We find also solutions of the form \( a = a_0 \Delta t^\sigma \) for special values of \( \alpha \): a. \( \gamma = 2, \sigma = 1, \alpha = (4/3)(1+k/a_0^2) \); b. \( k = 0, \sigma = 1/3, \alpha = \gamma - 3(2-\alpha)C^2/(2a_0^3) - 2 \).
3 Conclusions

We have found exact solutions of the Einstein equations with a free scalar field and a viscous fluid source, in a homogeneous isotropic metric. We have considered cosmological models with two forms of the bulk viscosity pressure: linear, with a constant coefficient, and quadratic in the Hubble variable. For the first case the picture is similar to the case without scalar field and we show that the de Sitter evolution is asymptotically stable. Also, we find that the physical requirement of positivity of the energy density of the fluid makes some of these solutions geodesically incomplete. For the case of a nonlinear viscous pressure we reduce the equations of this model to that of a conservative mechanical system. Thus, we are able to give its general solution in implicit form, and besides we show several cases for which an explicit solution arise. Both, singular and bouncing solutions arise, and we find cases in which the scale factor reach a maximum or grow without bound. In the latter case, when $\Lambda = 0$, the scale factor has a power law behavior for large times.

Dissipative effects like particle production may have been important in the early universe, for instance in the reheating period at the end of the inflationary era. It is generally assumed that this stage of accelerated evolution was driven by a self-interacting scalar field. Thus, we consider of interest to investigate further the models of this paper, taking into account also an interaction potential.
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