A new combinatorial characterization of the minimal cardinality of a subset of $\mathbb{R}$ which is not of first category

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Abstract

Let $\mathcal{M}$ denote the ideal of first category subsets of $\mathbb{R}$. We prove that $\min\{\text{card } X : X \subseteq \mathbb{R}, X \not\in \mathcal{M}\}$ is the smallest cardinality of a family $S \subseteq \{0, 1\}^\omega$ with the property that for each $f : \omega \rightarrow \bigcup_{n \in \omega} \{0, 1\}^n$ there exists a sequence $\{a_n\}_{n \in \omega}$ belonging to $S$ such that for infinitely many $i \in \omega$ the infinite sequence $\{a_{i+n}\}_{n \in \omega}$ extends the finite sequence $f(i)$.

We inform that $S \subseteq \{0, 1\}^\omega$ is not of first category if and only if for each $f : \omega \rightarrow \bigcup_{n \in \omega} \{0, 1\}^n$ there exists a sequence $\{a_n\}_{n \in \omega}$ belonging to $S$ such that for infinitely many $i \in \omega$ the infinite sequence $\{a_{i+n}\}_{n \in \omega}$ extends the finite sequence $f(i)$.

Let $\mathcal{M}$ denote the ideal of first category subsets of $\mathbb{R}$. Let $\mathcal{M}(\{0, 1\}^\omega)$ denote the ideal of first category subsets of the Cantor space $\{0, 1\}^\omega$. Obviously:

\begin{equation} (*) \nonumber \end{equation}

\begin{equation} \text{non}(\mathcal{M}) = \min\{\text{card } X : X \subseteq \mathbb{R}, X \not\in \mathcal{M}\} = \min\{\text{card } X : X \subseteq \{0, 1\}^\omega, X \not\in \mathcal{M}(\{0, 1\}^\omega)\} \nonumber \end{equation}

Let $\forall^\infty$ abbreviate ”for all except finitely many”. It is known (see [1], [2] and also [3]) that:

\begin{equation} \text{non}(\mathcal{M}) = \min\{\text{card } F : F \subseteq \omega^\omega \text{ and } \neg \exists g \in \omega^\omega \forall f \in F \forall^\infty k g(k) \neq f(k)\} \nonumber \end{equation}

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For $S \subseteq \{0, 1\}^\omega$ we define the following property (**):

for each $f : \omega \to \bigcup_{n \in \omega} \{0, 1\}^n$ there exists a sequence $\{a_n\}_{n \in \omega}$ belonging to $S$ such that for infinitely many $i \in \omega$ the infinite sequence $\{a_{i+n}\}_{n \in \omega}$ extends the finite sequence $f(i)$.

**Theorem 1** If $S \subseteq \{0, 1\}^\omega$ is not of first category then $S$ has the property (**).

**Proof.** Let us fix $f : \omega \longrightarrow \bigcup_{n \in \omega} \{0, 1\}^n$. Let $S_k(f)$ ($k \in \omega$) denote the set of all sequences $\{a_n\}_{n \in \omega}$ belonging to $\{0, 1\}^\omega$ with the property that there exists $i \geq k$ such that the infinite sequence $\{a_{i+n}\}_{n \in \omega}$ extends the finite sequence $f(i)$.

Sets $S_k(f)$ ($k \in \omega$) are open and dense. In virtue of the Baire category theorem $\bigcap_{k \in \omega} S_k(f) \cap S$ is non-empty i.e. there exists a sequence $\{a_n\}_{n \in \omega}$ belonging to $S$ such that for infinitely many $i \in \omega$ the infinite sequence $\{a_{i+n}\}_{n \in \omega}$ extends the finite sequence $f(i)$. This completes the proof.

**Note.** The author recently proved that if $S \subseteq \{0, 1\}^\omega$ has the property (**)) then $S$ is not of first category; the proof will appear in a separate preprint. From this result and Theorem 1 we obtain the following characterization:

$S \subseteq \{0, 1\}^\omega$ is not of first category if and only if $S$ has the property (**)).

Let us note that from the above characterization we may deduce all next results; therefore all next proofs are unnecessary.
Theorem 2 If $S \subseteq \{0, 1\}^\omega$ has the property (**) then $\text{card } S \geq \text{non}(\mathcal{M})$.

Proof. For a sequence $\{a_n\}_{n \in \omega}$ belonging to $S$ we define:

$$\tilde{a}_n := \sum_{i=n}^{\infty} \frac{a_i}{2^{i-n+1}} \in [0, 1].$$

The following Observation is easy.

Observation. Assume that $S \subseteq \{0, 1\}^\omega$ has the property (**).

We claim that for each sequence $\{U_k\}_{k \in \omega}$ of non-empty open sets satisfying $U_k \subseteq (0, 1)$ ($k \in \omega$) there exists a sequence $\{a_k\}_{k \in \omega}$ belonging to $S$ such that for infinitely many $k \in \omega$ $\tilde{a}_k \in U_k$.

There exists a sequence $\{(c_i, d_i)\}_{i \in \omega}$ of non-empty pairwise disjoint intervals satisfying

$$\bigcup_{i \in \omega} (c_i, d_i) \subseteq (0, 1).$$

We assign to each $\{a_n\} \in S$ the function $s_{\{a_n\}} : \omega \rightarrow \omega$ according to the following rules (cf.[4]):

1) if $\tilde{a}_k \notin \bigcup_{i \in \omega} (c_i, d_i)$ then $s_{\{a_n\}}(k) = 0$,

2) if $\tilde{a}_k \in \bigcup_{i \in \omega} (c_i, d_i)$ then $s_{\{a_n\}}(k)$ is the unique $i \in \omega$ such that $\tilde{a}_k \in (c_i, d_i)$.

Suppose, contrary to our claim, that $\text{card } S < \text{non}(\mathcal{M})$. It implies that the cardinality of the family $\{s_{\{a_n\}} : \{a_n\} \in S\} \subseteq \omega^\omega$ is also less than

$$\text{non}(\mathcal{M}) = \min\{\text{card } F : F \subseteq \omega^\omega \text{ and } \exists g \in \omega^\omega \forall f \in F \forall k g(k) \neq f(k)\}$$

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Therefore, there exists a function $g: \omega \rightarrow \omega$ such that for each sequence $\{a_n\} \in S \forall k \in \omega \ g(k) \neq s_{\{a_n\}}(k)$. We define $U_k := (c_{g(k)}, d_{g(k)}) \subseteq (0, 1) \ (k \in \omega)$. If $\{a_n\} \in S$ then the set $A_{\{a_n\}} := \{k \in \omega : g(k) = s_{\{a_n\}}(k)\}$ is finite and for each $k \in \omega \setminus A_{\{a_n\}} \ \tilde{a}_k \not\in U_k$. It contradicts the thesis of the Observation which ensures that there exists a sequence $\{a_k\}_{k \in \omega}$ belonging to $S$ such that for infinitely many $k \in \omega \ \tilde{a}_k \in U_k$. This completes the proof of Theorem 2.

**Corollary.** From (*), Theorem 1 and Theorem 2 follows that $\text{non}(M)$ is the smallest cardinality of a family $S \subseteq \{0, 1\}^\omega$ with the property that for each $f: \omega \rightarrow \bigcup_{n \in \omega}\{0, 1\}^n$ there exists a sequence $\{a_n\}_{n \in \omega}$ belonging to $S$ such that for infinitely many $i \in \omega$ the infinite sequence $\{a_{i+n}\}_{n \in \omega}$ extends the finite sequence $f(i)$.

**Remark.** Another (not purely combinatorial) characterizations of $\text{non}(M)$ can be found in [4].

**References**

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