Parametric Dynamics of Level Spacings in Quantum Chaos

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Abstract

We identify parametric (radial) Bessel-Ornstein-Uhlenbeck stochastic processes as primitive dynamical models of energy level repulsion in irregular quantum systems. Familiar GOE, GUE, GSE and non-Hermitian Ginibre universality classes of spacing distributions arise as special cases in that formalism.

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In case of complex quantum systems, affinities with the statistical theory of random-matrix spectral problems support the viewpoint [1, 2], that certain features of a fully developed classical chaos can be elevated to the quantum level as universality classes of spectral fluctuations. However, in those cases, an ensemble of large (size is to grow to infinity) random matrices is usually set in correspondence with one quantum system.

Then, an immediate question arises: how may one justify a comparison of a statistical ensemble of randomly disordered spectral series with one only specific (and generally being or "looking" random) energy level sequence of an a priori chosen quantum system. If specialized to the quantum chaos context, the basic hypothesis behind previous arguments is that quantum Hamiltonian may be represented by just one random-matrix representative drawn from an ensemble of suitable random ones, provided things happen in the large matrix size regime (needed to conform with the semiclassical regime in quantum theory). That particular issue of an individual versus ensemble spectral information is the major objective of our investigation.

To our knowledge this immediate conceptual obstacle, except for preliminary investigations of Ref. [3], has not been seriously addressed in the quantum chaos literature. A partial answer to that question, [3, 4], points towards certain ergodicity properties appropriate for models of the parametric level dynamics (Coulomb gas, plasma or else), evolving in "fictitious time" and ultimately approaching suitable equilibria, characterized by invariant probability measures.

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Let us mention that the possibly troublesome "fictitious" time parameter may be consistently interpreted as the running coupling constant measuring the strength of the chaotizing perturbation, or more generally as a "complexity parameter" whose growth to infinity gives account of a complexity increase in the quantum system, [5].

That amounts to a reinterpretation of random-matrix theory in terms of an equilibrium statistical mechanics for a fictitious \( n \)-particle system, where \( n \) needs to grow indefinitely to meet (or satisfactorily approximate) spectral predictions for semiclassically analyzed irregular quantum systems.

A common mathematical structure behind this argument is related to the so-called Calogero-Moser Hamiltonian system, [2, 5], for \( n \) particles on a real line (actually energy levels, here \( n \) also corresponds to the matrix size in the corresponding random-matrix theory reasoning and we need \( n \) to be large) which interact via pairwise inverse square interaction and a harmonic attraction. In particular, the probability distribution of \( n \)-particle (energy level) coordinates in the ground state of the Calogero-Moser Hamiltonian is known to coincide with the Wigner-Dyson distribution of energy levels, regarded as a statistical state of equilibrium in the large "fictitious time" asymptotic of the corresponding random-matrix dynamical model, cf. [2, 5].

It is the level repulsion which is routinely interpreted as a quantum manifestation of classical chaos. Normally that is quantified by means of polynomial modifications of the Gaussian probability law (in association with the Wigner-Dyson statistics of adjacent level spacings for e.g. unitary, orthogonal and symplectic random matrix ensembles).

For completeness, let us list the standard level spacing formulas: \( P_1(s) = s^\pi \frac{2}{\pi} e^{(-s^2/4)} \), \( P_2(s) = s^2 \frac{5}{\pi} e^{(-s^2/4)} \) and \( P_4(s) = s^4 \frac{218}{85\pi^2} e^{(-2^6s^4/9\pi^4)} \), corresponding respectively to the GOE, GUE and GSE random-matrix theory predictions. The cubic repulsion case \( P_3(s) = s^3 \frac{4032}{125\pi^2} e^{(-s^2/4)} \) is related to the non-Hermitian Ginibre ensemble, [2] and was not covered by the alternative technique for complex spectra analysis, as developed in Ref. [5].

Once we have encountered probability densities on the positive half-line in \( \mathbb{R}^1 \), it is rather natural to investigate a general issue of parametric stochastic processes which would provide a dynamical model of level repulsion in an irregular quantum system and generate at the same time spacing densities as those of asymptotic invariant (equilibrium) probability measures. Such random processes clearly must run with respect to the previously mentioned "fictitious" time-parameter and take values in the set of all level spacings which are appropriate for a complex quantum system or the corresponding random-matrix ensemble.

Effectively, we wish to introduce a Markovian diffusion-type process which might stand for a reliable approximation of a random walk over level spacing sizes.

For future reference let us mention that in the regime of equilibrium (when an invariant measure appears in the large "time" asymptotic), a sample path of such random walk would take the form of an
ordered sequence of spacings which are sampled (drawn) according to the prescribed invariant probability distribution. That is precisely one explicit example of the ladder of energy levels, understood as a random sample drawn from a suitable ensemble.

An analysis of statistical features of this spectral sequence involves an ergodicity notion to stay in conformity with the ensemble evaluation of various averages (carried out with respect to the invariant density), [6, 7].

We shall consider the previously listed GOE, GUE and GSE probability densities on $R^+$ (up to suitable rescalings!) as, distorted in view of the spacing size normalization, asymptotic invariant densities of certain parametric Markovian stochastic processes whose uniqueness status can be unambiguously settled.

Let is begin from the observation that probability densities on $R^+$, of the characteristic form $f(x) \sim x \exp(-x^2/4)$, $g(x) \sim x^2 \exp(-x^2/4)$ and $h(x) \sim x^4 \exp(-x^2)$ appear notoriously in various quantum mechanical contexts (harmonic oscillator or centrifugal-harmonic eigenvalue problems), cf. [8, 9, 10]. Notwithstanding, as notoriously they can be identified in connection with special classes of stationary Markovian diffusion processes on $R^+$, [11].

Anticipating further discussion, let us consider a Fokker-Planck equation on the positive half-line in the form:

$$\partial_t \rho = \frac{1}{2} \Delta \rho - \nabla \left[ \frac{\beta}{2x} - x \right] \rho$$

which may be set in correspondence with the stochastic differential equation

$$dX_t = \left( \frac{\beta}{2X_t} - X_t \right) dt + dW_t$$

formally valid for a random variable $X_t$ with values contained in $(0, \infty)$. Here $\beta \geq 0$ and $W_t$ represents the Wiener process.

Accordingly, if $\rho_0(x)$ with $x \in R^+$ is regarded as the density of distribution of $X_0$ then for each $t > 0$ the function $\rho(x, t)$, solving Eq. (1), is the density of $X_t$. In view of a singularity of the forward drift at the origin, we refrain from looking for strong solutions of the stochastic differential equation (2) and confine attention to weak solutions only and the associated tractable parabolic problem (1) with suitable boundary data, cf. [11].

In all those cases a mechanism of repulsion is modeled by the $\frac{1}{x}$ term in the forward drift expression. The compensating harmonic attraction which is modeled by the $-x$ term, saturates the long distance effects of repulsion-induced scattering and ultimately yields asymptotic steady (stationary) probability densities.

To interpret a density $\rho(x)$ as an asymptotic (invariant) density of a well defined Markovian diffusion process we shall utilize the rudiments of so-called Schrödinger boundary and stochastic interpolation problem, [6, 10, 12].
Let us notice that both in case of the standard Ornstein-Uhlenbeck process and its Bessel (radial) variant, we have emphasized the role of a stochastic process with an asymptotic invariant density. To deduce such processes, in principle we can start from an invariant density and address an easier issue of the associated measure preserving stochastic dynamics and next consider whether the obtained process would drive a given initial density towards a prescribed invariant measure. That feature involves the notion of exactness of the related stochastic process, whose straightforward consequence are the properties of mixing and ergodicity of the involved random dynamics, [6].

There is a general formula [9, 12] relating the forward drift of the sought for stationary process with an explicit functional form of an invariant probability density. We confine our attention to Markov diffusion processes with a constant diffusion coefficient, denoted $D > 0$. Then, the pertinent formula reads:

$$b(x) = 2D \frac{\nabla \rho^{1/2}}{\rho^{1/2}}. \quad (3)$$

In particular, for the familiar Ornstein-Uhlenbeck process we have $\rho^{1/2}(x) = \left(\frac{1}{\pi}\right)^{1/4} \exp(-\frac{x^2}{2})$ and $D = \frac{1}{2}$, so we clearly arrive at $b(x) = -x$ as should be. Quite analogously, in case of the GUE-type spacing density, we have $\rho^{1/2}(x) = \frac{2}{\pi^{1/4}} x \exp(-\frac{x^2}{2})$. Thus, accordingly $b(x) = \frac{1}{2}x - x$.

The very same strategy allows us to identify a forward drift of the Markovian diffusion process supported by the GOE-type spacing density. By employing $\rho^{1/2}(x) = \sqrt{2x} \exp(-\frac{x^2}{2})$ and setting $D = \frac{1}{2}$ we arrive at the formula: $b(x, t) = \frac{1}{2x} - x$.

We immediately identify the above forward drifts with the ones appropriate for the time homogeneous radial Ornstein-Uhlenbeck processes, with a corresponding family of ($N > 1$ and otherwise arbitrary integer) transition probability densities, [11]:

$$p_t(y, x) = p(y, 0, x, t) = 2x^{N-1} \exp(-x^2). \quad (4)$$

$$\frac{1}{1 - \exp(-2t)} \exp\left[-\frac{(x^2 + y^2) \exp(-2t)}{1 - \exp(-2t)}\right] \cdot \left[xy \exp(-t)\right]^{-\alpha} I_{\alpha}\left(\frac{2xy \exp(-t)}{1 - \exp(-2t)}\right)$$

where $\alpha = \frac{N-2}{2}$ and $I_{\alpha}(z)$ is a modified Bessel function of order $\alpha$:

$$I_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+\alpha}}{(k!)^2 \Gamma(k + \alpha + 1)} \quad (5)$$

while the Euler gamma function has a standard form $\Gamma(x) = \int_0^\infty \exp(-t)t^{x-1}dt$. We remember that $\Gamma(n + 1) = n!$ and $\Gamma(1/2) = \sqrt{\pi}$.

The resultant forward drift has the general form:

$$b(x) = \frac{N - 1}{2x} - x. \quad (6)$$
and corresponds to $\beta = N - 1$ in the notation of Eqs. (1), (2).

By setting $N = 2$ in Eq. (4), and then employing the series representation of $I_0(z)$, we easily recover the asymptotic invariant density for the process: $\lim_{t \to \infty} p(y, 0, x, t) = 2x \exp(-x^2)$.

We can also analyze the large time asymptotic of $p(y, 0, x, t)$, Eq. (4) in case of $N = 3$ which gives rise to an invariant density in the form: $\frac{4}{\sqrt{\pi}} x^2 \exp(-x^2)$. That obviously corresponds to the GUE-type case with $b(x) = \frac{1}{2}x - x$.

When passing to the GSE case, we are interested in the Markovian diffusion process which is supported by an invariant probability density $\rho(x) = \frac{2}{\Gamma(3/2)} x^4 \exp(-x^2)$. Let us evaluate the forward drift of the sought for process in accordance with the recipe (3) (we set $\delta = \frac{1}{2}$): $b(x, t) = \frac{2}{x} - x$.

A comparison with the definition (6) shows that we deal with a radial Ornstein-Uhlenbeck process corresponding to $N = 5$. Accordingly, the transition probability density of the process displays an expected asymptotic: $\lim_{t \to \infty} p(y, 0, x, t) = \frac{4}{\sqrt{\pi}} x^4 \exp(-x^2)$. Here we have exploited $\Gamma(1/2) = \sqrt{\pi}$ to evaluate $\Gamma(3/2) = \frac{3}{2}\sqrt{\pi}$.

The formulas (4) and (6) allow us to formulate a hypothesis that novel universality classes may possibly be appropriate for quantifying quantum chaos. Straightforwardly, one can verify that transition probability densities (4) refer to asymptotic invariant densities of the form:

$$\rho(x) = \frac{2}{\Gamma(N/2)} x^{N-1} \exp(-x^2).$$  \hspace{1cm} (7)

In particular we get a direct evidence in favor of $N = 4$, i.e. $b(x) = \frac{3}{2}x - x$, universality class which in fact corresponds to the Ginibre ensemble of of non-Hermitian random matrices, \[3\], where a cubic level repulsion appears: $\rho(x) = 2x^3 \exp(-x^2)$.

The formulas (6), (7) allow us to expect that processes corresponding to any $N > 5$ may be realizable as well, and thus the related higher-power level repulsion might have relevance in the realm of quantum chaos.

In all considered cases, an asymptotic invariance of probability measures (densities) is sufficient to yield ergodic behaviour. For each value of $N > 1$ we deal with an independent repulsion mechanism, albeit all of them belong to the radial Ornstein-Uhlenbeck family.

We have thus identified a universal stochastic law (in fact, a family of the like) behind the functional form of basic, random-matrix theory inspired and named generic, spacing probability densities appropriate for quantum chaos.

A common feature of those parametric processes is an asymptotic balance between the radial (Bessel-type) repulsion and the harmonic confinement (attraction), as manifested in the general form of forward drifts $b(x) = \frac{N-1}{2x} - x$ with $N \geq 1$. Here $N = 2, 3, 5$ correspond respectively to the familiar GOE, GUE and GSE cases while $N = 4$ to the cubic level repulsion associated with the non-Hermitian Ginibre ensemble.
Let us emphasize at this point that one should keep in mind a number of possible reservations coming from the fact that neither of "universal" or "generic" laws can be regarded as a faithful representation of a real state of affairs. Usually exact laws are derived for two by two (hence of the small size!) random matrices, and are known to reappear again as approximate spacing formulas in the large random-matrix size regime. That in turn allows to achieve a correspondence with semiclassical quantum spectra of complex systems.

At the moment we cannot propose a definitive explanation of a physical meaning of the integer parameter $N$ in the radial stochastic process scenario. One hypothesis comes from the random-matrix theory, where $\beta = N - 1 = 1, 2, 4$ would correspond to a number of components of a typical matrix entry which is decided by the underlying symmetry of the problem (GOE, GUE, GSE). That can be extended to the case of $N = 4$, but there is no obvious explanation of that sort for $N > 5$. This issue needs further investigation.

Previously we have indicated that a common mathematical basis for various level repulsion mechanisms appropriate to quantum chaos is set by the Calogero-Moser Hamiltonian. At the first glance, our stochastic arguments may leave an impression that something completely divorced from that setting has been obtained in the present paper. However things look otherwise and our theoretical framework proves to be compatible with standard techniques for spectral analysis of complex quantum systems.

It is peculiar to the general arguments of Refs. that invariant probability densities give rise to measure preserving stochastic processes in a fully controlled way. One of basic ingredients of the formalism is a proper choice of Feynman-Kac kernel functions, which are the building block for the construction of transition probability densities of the pertinent Markov processes. Feynman-Kac semigroup operators (and their kernels) explicitly involve one particle Hamiltonian operators as generators (in less technical terms one may think at this point about rather standard transformation from the Fokker-Planck operator to the associated self-adjoint one).

For stationary processes, a general formula relating forward drifts $b(x)$ of the stochastic process with potentials of the conservative Hamiltonian system reads (we choose a diffusion coefficient to be equal $\frac{1}{2}$): $V(x) = \frac{1}{2} (b^2 + \nabla \cdot b)$. Upon substituting $b(x)$ according to Eq. (6) we arrive at:

$$V(x) = \frac{1}{2} \left[ \beta (\beta - 2) - (\beta + 1) + x^2 \right]$$

where $\beta = N - 1$. This potential function enters a standard definition of the one particle Hamiltonian operator (physical parameters have been scaled away):

$$H = -\frac{1}{2} \Delta + V(x)$$
where $\Delta = \frac{d^2}{dx^2}$. The operator (10) with $V(x)$ defined by (9) is an equivalent form of a two-particle (actually two-level) version of the Calogero-Moser Hamiltonian, cf. [8] and compare e.g. the formula (1) in Ref. [5].

Indeed, the classic Calogero-type problem is defined by

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 + \frac{\beta(\beta - 2)}{8x^2}$$

with the well known spectral solution. The eigenvalues read $E_n(\beta) = 2n + 1 + \frac{1}{2}[1 + \beta(\beta - 2)]^{1/2}$, where $n \geq 0$ and $\beta > -1$.

By inspection we can check that all previously considered $N = 2, 3, 4, 5$ radial processes correspond to the Calogero operator of the form $H - E_0$ where $E_0$ is the ground state ($n=0$) eigenvalue. Its explicit form relies on the choice of $\beta$ and by substituting $\beta = 1, 2, 3, 4$ we easily check that $E_0(\beta) = 1 + \frac{1}{2}[1 + \beta(\beta - 2)]^{1/2} = \frac{1}{2}(\beta + 1)$ as should be to conform with Eq. (9).

Accordingly, all considered radial processes arise as the so-called ground state processes associated with the Calogero Hamiltonians. Let us recall that the classic Ornstein-Uhlenbeck process can be regarded as the ground state process of the harmonic oscillator Hamiltonian operator. That by the way corresponds to choosing $N = 1$ i.e. $\beta = 0$ in the above.

It is useful to mention that our discussion can be readily extended beyond the quantum chaos context and then to random processes running in a real time. Incidentally, Bessel processes with the parameter $\beta$ being not necessarily integer were exploited in the nonequilibrium statistical formulation for grain growth, [14, 15]. The apparently unlimited growth can be tamed. By introducing the harmonic confinement (a substitute for various surface tension effects) we can produce a family of grain growth models with calibrated finite mean grain size approached in the asymptotic limit. In the present problem, we have involved a random walk over grain sizes which can randomly increase or decrease (that remains in close parallel to the previously discussed random walk over erratically varying level spacings), with obvious ergodicity connotations. Indeed, we may literally think about a history and statistics of sizes for a single grain evolution in the course of the growth process (along a single sample trajectory in the space of sizes), or equivalently about an ensemble statistics, appropriate in the large time asymptotic.

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