LIVŠIC THEOREM FOR BANACH RINGS

GENADY YA. GRABARNIK AND MISHA GUYSINSKY

ABSTRACT. We prove the Livšic Theorem for Hölder continuous cocycles with values in Banach rings. We consider a transitive homeomorphism \( \sigma : X \to X \) that satisfies the Anosov Closing Lemma, and a Hölder continuous map \( a : X \to B^\times \) from a compact metric space \( X \) to the set of invertible elements of some Banach ring \( B \). We show that it is a coboundary with a Hölder continuous transition function if and only if \( a(\sigma^{-1}p) \ldots a(\sigma)p) = e \) for each periodic point \( p = \sigma^n p \).

1. Introduction

We assume that \( X \) is a compact metric space, \( G \) a complete metric group, and \( \sigma : X \to X \) a homeomorphism.

We say that a map \( a : Z \times X \to G \) is a cocycle over \( \sigma \) if

\[
a(n, x) = a(n - k, \sigma^k x) a(k, x) \quad \text{for any} \quad n, k \in \mathbb{Z}
\]

Every map \( a : X \to G \) generates a cocycle \( a(n, x) \) defined as

\[
a(n, x) = a(\sigma^{n-1} x) a(\sigma^{n-2} x) \ldots a(x) \quad n > 0
\]

\[
a(0, x) = \text{Id}
\]

\[
a(n, x) = a^{-1}(\sigma^{-1} x) \ldots a^{-1}(\sigma^{-2} x) a^{-1}(\sigma^{-1} x) \quad n < 0
\]

We see that \( a(1, x) = a(x) \). In this paper we consider only cocycles generated by Hölder continuous maps \( a : X \to G \).

We say that a Hölder continuous map \( a : X \to G \) is a coboundary (or more precisely generates a cocycle which is a coboundary) if there is a Hölder continuous function \( t : X \to G \) such that

\[
a(x) = t(\sigma x) t^{-1}(x)
\]

The function \( t(x) \) is a called a transition map. If \( a(x) \) is a coboundary then it is clear that

\[
a(n, x) = t(\sigma^n x) t^{-1}(x)
\]

A question whether some cocycle is a coboundary or not appears naturally in many important problems in dynamical systems. There is a simple necessary condition for a cocycle to be a coboundary. If \( a(x) \) is a coboundary and \( p \in X \) is a periodic point \( \sigma^n p = p \) then

\[
a(\sigma^{n-1} p) \ldots a(\sigma)p a(p) = a(n, p) = t(\sigma^n p) t^{-1}(p) = e
\]

where \( e \) is the identity element in the group \( G \).

We say that for a cocycle \( a(n, x) \) periodic obstruction vanish if

\[
a(\sigma^{n-1} p) \ldots a(\sigma)p a(p) = e \quad \forall p \in X \quad \text{with} \quad \sigma^n p = p, n \in \mathbb{N}
\]

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A. Livšic (see [7, 8]) proved that when $\sigma$ is a transitive Anosov map and the group $G$ is $\mathbb{R}$ or $\mathbb{R}^n$ then a cocycle $a(x)$ is a coboundary if and only if the periodic obstruction vanish. This result is called Livšic theorem. The proof of the Livšic theorem for other groups turned out to be harder. Nevertheless, in the last twenty years in the series of papers (see [2], [18], [9], [10], [17], [15]) it was shown that for some groups under an additional assumption on the growth rates of the cocycle $a(n, x)$ the condition (1) is also sufficient. For example, in [2] it was shown that if $G = B \times$ the set of invertible elements of some Banach algebra then if periodic obstruction vanish and $a(x)$ is close to the identity element $e$ then it is a coboundary. The question remained if this additional assumption will follow from the fact that the products along periodic points are equal to $e$. In 2011 B. Kalinin in [5] made a breakthrough by proving the Livšic theorem for functions with values in $GL(n, \mathbb{R})$ and more generally in a connected Lie group assuming only that condition (1) is satisfied.

He used Lyapunov exponents for different invariant measures to estimate the rate of the cocycle growth and then approximated Lyapunov exponents for all invariant measures by Lyapunov exponents at periodic points. To do the latter the Oseledets Theorem was used. In this paper, we are proving that a cocycle with values in invertible elements of Banach ring is a coboundary if and only if periodic obstructions vanish. There is no analogs of the Oseledets Theorem for Banach rings ( or even Banach algebras). Still we can define analogs of the highest and lowest Lyapunov exponents and using a different argument show that they could be approximated by the values of the cocycle at periodic points. Various examples of Banach rings include, Banach algebras, and Banach algebras with $F$ as a field of scalars, where $F$ is a local field. For them it is a new result. Also several already known results follow: Livšic Theorem for cocycles with values in $GL(n, \mathbb{R})$ (see [5]) and $GL(n, F)$ (see [13]).

As in [5] we require that the map $\sigma$ was transitive and had the following property

**Definition 1.** We say that a homeomorphism $\sigma : X \to X$ has a closing property if there exist positive numbers $\delta_0, \lambda, C$ such that for any $x \in X$ and $n > 0$ with $\text{dist}(x, \sigma^n x) \leq \delta_0$ we can find points $p, z \in X$ where

$$\sigma^n p = p$$

and for every $i = 0, 1, \ldots, n$

$$\text{dist}(\sigma^i p, \sigma^i z) \leq e^{-i\lambda} C \text{dist}(x, \sigma^n x) \quad \text{dist}(\sigma^i x, \sigma^i z) \leq e^{-n-i\lambda} C \text{dist}(x, \sigma^n x)$$

We will call $\lambda$ the expansion constant for the map $\sigma$.

Anosov maps and shifts of finite types are main examples of maps with closing property.

**Definition 2.** An associative (non–commutative) ring $B$ with the unity element $e$ is called Banach ring if there is a function $\| \cdot \| : B \to \mathbb{R}$ such that

1. $\|a\| \geq 0$ and $\|a\| = 0$ if and only if $a = 0$.
2. $\|a + b\| \leq \|a\| + \|b\|$.
3. $\|ab\| \leq \|a\| \cdot \|b\|$.
4. The ring $B$ is a complete metric space with respect to the distance defined as $\text{dist}(a, b) = \|a - b\|$.
We denote as $B^\times$ the set of invertible elements of a Banach ring $B$. The main result of this paper is:

**Main Theorem.** Let $X$ be a compact metric space, $\sigma : X \to X$ a transitive homeomorphism with closing property. If $a : X \to B^\times$ is an $\alpha$-Hölder continuous function such that

$$a(\sigma^{n-1}p) \ldots a(\sigma p) a(p) = e \quad \forall p \in X, n \in \mathbb{N} \text{ with } \sigma^n p = p$$

then there exists an $\alpha$-Hölder function $t : X \to B^\times$ such that

$$a(x) = t(\sigma x) t^{-1}(x)$$

2. **Subadditive Cocycles**

Let $\sigma : X \to X$ be a continuous function. We will call a continuous function $s(n,x) : \mathbb{Z} \times X \to \mathbb{R}$ a subadditive cocycle over the function $\sigma$ if

$$s(n + m, x) \leq s(n, \sigma^m x) + s(m, \sigma x)$$

From the Kingman’s Theorem about subadditive cocycles [6, 1] follows that for every $\sigma$-invariant measure $\mu$ and for almost all $x$ there exists a number

$$r(x) = \lim_{n \to \infty} \frac{s(n,x)}{n}$$

If $\mu$ is ergodic then this number is the same for a.a $x$ and equals $\inf \int_X s(n,x) d\mu$. For an ergodic $\mu$ we will call this number $r_\mu$. The set of all $\sigma$-invariant ergodic measures we denote as $\mathcal{M}$. The set of points $x \in X$ for which limit (2) exists we call regular and denote as $\mathcal{R}$.

Of course, there could be points for which the limit in (2) does not exist.

We can also consider numbers $s_n = \max s(n,x)$. It is a subadditive sequence of numbers $s_{n+m} \leq s_n + s_m$ and we denote as $r$ the following number:

$$r = \lim_{n \to \infty} \frac{s_n}{n} = \inf_n \frac{s_n}{n}$$

It is known (see [11]) that if $\sigma$ is continuous and $X$ is compact then

$$r = \sup_{x \in \mathcal{R}} r(x) = \sup_{\mu \in \mathcal{M}} r_\mu$$

For a periodic point $p = \sigma^k p$ we denote as $r_p$ the following quantity

$$r_p = \frac{s(k,p)}{k}$$

It is easy to see that $r(p)$ exists (but can be $-\infty$) and $r(p) \leq r_p$.

We will show that if $\sigma$ has a closing property we can prove that:

**Theorem 3.** Let $X$ be a compact metric space, $\sigma : X \to X$ a homeomorphism with closing property. We denote as $\mathcal{P}$ the set of all periodic points. If $a : X \to B^\times$ is an $\alpha$-Hölder continuous function, $a(n,x)$ is a cocycle generated by it and $s(n,x) = \ln \|a(n,x)\|$ then

$$r = \sup_{x \in \mathcal{R}} r(x) = \sup_{\mu \in \mathcal{M}} r_\mu \leq \sup_{p \in \mathcal{P}} r_p$$

An easy corollary of this theorem is the following important for us result.
Corollary 4. Let $X$ be a compact metric space, $\sigma : X \to X$ a homeomorphism with closing property. If $a(n, p) = e$ for every periodic point $p$ with period $n$ then for any $\varepsilon > 0$ there exists $C$ such that for all integer positive $n$ and all $x \in X$
\begin{align*}
\|a(n, x)\| &\leq Ce^{\varepsilon n} \\
\|a(-n, x)\| &\leq Ce^{\varepsilon n} \\
\|a(n, x)^{-1}\| &\leq Ce^{\varepsilon n}
\end{align*}

Proof. The first inequality follows from the fact that if $s(n, x) = \ln \|a(n, x)\|$ then for this subadditive cocycle $r_p = 0$ for every periodic point $p$ and from (5) follows that $r = 0$. For the second inequality we can consider a cocycle $b(n, x)$ over $\sigma^{-1}$ generated by $a^{-1}(x)$. Below, we will prove that if $a(x)$ is Hölder continuous then $a^{-1}(x)$ is also Hölder continuous, and therefore we can apply Theorem 3 to the cocycle $b(n, x)$ also. But $a(-n, x) = b(n, x)$ and if $a(n, p) = e$ for every periodic point then

$$b(n, p) = a(-n, p) = a(-n, \sigma^n p) = \left[a(n, p)^{-1}\right] = e$$

So the rate of growth $r$ for $b(n, x)$ is also 0.

The last inequality follows from the fact that

$$\left[a(n, x)^{-1}\right] = b(n, \sigma^n x)$$

The only thing left to show is that if $a(x)$ is $\alpha$-Hölder continuous then $a^{-1}(x)$ is also $\alpha$-Hölder continuous. For normed rings the operation of taking the inverse element is continuous (see [10]). Therefore the function $a^{-1}(x)$ is bounded. But

$$\|a^{-1} - b^{-1}\| = \|b^{-1}(b - a)a^{-1}\| \leq \|b^{-1}\| \cdot \|(b - a)\| \cdot \|a^{-1}\|$$

Therefore, if the function $a(x)$ is $\alpha$-Hölder continuous, then $a^{-1}(x)$ is also $\alpha$-Hölder continuous. $\Box$

3. PROOF OF THE THEOREM 3

The following result proven in [12, Proposition 4.2] will be used.

Lemma (A. Karlsson, G. A. Margulis). Let $\sigma : X \to X$ be a measurable map, $\mu$ an ergodic measure, $s(n, x)$ a subadditive cocycle. For any $\varepsilon > 0$, let $E_\varepsilon$ be the set of $x$ in $X$ for which there exist an integer $K(x)$ and infinitely many $n$ such that

$$s(n, x) - s(n - k, \sigma^k x) \geq (r_\mu - \varepsilon)k$$

for all $k$, $K(x) \leq k \leq n$. Let $E = \cap_{\varepsilon > 0} E_\varepsilon$ then $\mu(E) = 1$.

If $s(n, x) = \ln \|a(n, x)\|$ then the inequality in the lemma could be rewritten as

(6) $$\|a(n - k, \sigma^k x)\| \leq \|a(n, x)\|e^{-(r_\mu - \varepsilon)k}$$

Definition 5. Let $\gamma, \delta$ be some positive numbers and $n$ is a natural number. We say that a point $y$ is $(\gamma, \delta, n)$—close to $x$ if

$$\text{dist}(\sigma^k x, \sigma^k y) \leq \delta e^{-\gamma k} \quad \text{for all} \quad 0 \leq k \leq n$$

Proposition 6. Let $\sigma : X \to X$ be a homeomorphism and $a : X \to B^\infty$ be an $\alpha$-Hölder continuous function, and $s(n, x) = \ln \|a(n, x)\|$. For any $\gamma > 0$ let $S_\gamma$ be the set of points $x$ in $X$ for which there exist a number $\delta(x) > 0$ and infinitely many $n$ such that for any point $y$ which is $(\gamma, \delta, n)$—close to $x$

(7) $$\|a(n, y)\| \geq \frac{1}{2}\|a(n, x)\|$$
Then $\mu(S_\gamma) = 1$ for any ergodic invariant measure $\mu$ with $r - r_\mu < \alpha \gamma$.

**Proof.** Let $\mu$ be an ergodic invariant measure with $r - r_\mu < \alpha \gamma$. We choose a number $0 < \varepsilon < \frac{1}{3}(\gamma \alpha - (r - r_\mu))$. Almost all points with respect to this measure and $\varepsilon$ satisfy Karlsson-Margulis Lemma and for almost all points the number $r(x) = r_\mu$.

Take a point $x$ from the intersection of those two sets. Using the identity

$$b_n a_{n-1} \ldots b_1 - a_n a_{n-1} \ldots a_1 = \sum_{k=1}^n b_n \ldots b_{k+1}(b_k - a_k)a_{k-1} \ldots a_1$$

we can see that

$$\|a(n, x) - a(n, y)\| = \left\| \sum_{k=0}^{n-1} a(n-k-1, \sigma^{k+1}x)[a(\sigma^k x) - a(\sigma^k y)]a(k, y) \right\| \leq$$

$$(8) \quad \sum_{k=0}^{n-1} \|a(n-k-1, \sigma^{k+1}x)\| \cdot \|a(\sigma^k x) - a(\sigma^k y)\| \cdot \|a(k, y)\|$$

Our goal is to show that if we choose a sufficiently small $\delta$ then for infinitely many numbers $n$ and for every $(\gamma, \delta, n)$ close to $x$ the sum $B$ is smaller than $\frac{1}{2}\|a(n, x)\|$. Let $K(x, \varepsilon)$ and $n$ be as in the Karlsson-Margulis Lemma and a point $y$ is $(\gamma, \delta, n)$-close to $x$ for some $\delta$ that we specify later. By definition $r = \lim_{k \rightarrow \infty} s_k/k$, so we can find $K \geq K(x, \varepsilon)$ such that $s_k < k(r + \varepsilon)$ for all $k \geq K$, or $\|a(k, x)\| < e^{k(r+\varepsilon)}$. For every $k > K$ factors in the product

$$(9) \quad \|a(n-k-1, \sigma^{k+1}x)\| \cdot \|a(\sigma^k x) - a(\sigma^k y)\| \cdot \|a(k, y)\|$$

could be bounded from above as

$$\|a(n-k-1, \sigma^k x)\| \leq \|a(n, x)\|e^{-(r_\mu-r)(k+1)} \quad \text{It follows from the Karlsson-Margulis Lemma.}$$

$$\|a(\sigma^k x) - a(\sigma^k y)\| \leq H^\delta \alpha e^{-k\gamma \alpha} \text{ where } H \text{ is some positive constant. It follows from the fact that } a(x) \text{ is } \text{Hölder continuous and } y \text{ is } (\gamma, \delta, n)-\text{close to } x.$$  

$$\|a(k, y)\| \leq e^{\varepsilon k} \leq e^{k(r+\varepsilon)} \text{ It follows from the definition of } K.$$

If we combine those inequalities we can see that that the number in the product $B$ is smaller than

$$\|a(n, x)\|e^{-(r_\mu-r)(k+1)} \cdot H^\delta \alpha e^{-k\gamma \alpha} \cdot e^{k(r+\varepsilon)} \leq \|a(n, x)\|H^\delta \alpha e^{-k(\gamma \alpha - (r-r_\mu) - 2\varepsilon)}$$

After simplification we can write that

$$\|a(n-k-1, \sigma^{k+1}x)\| \cdot \|a(\sigma^k x) - a(\sigma^k y)\| \cdot \|a(k, y)\| \leq \|a(n, x)\|H^\delta \alpha e^{-k\varepsilon}$$

If we add those inequalities for $k \geq K$ we can see that

$$\| \sum_{k=K}^n a(n-k-1, \sigma^{k+1}x)[a(\sigma^k x) - a(\sigma^k y)]a(k, y) \| \leq$$

$$\|a(n, x)\|H^\delta \alpha \sum_{k=0}^{\infty} e^{-k\varepsilon} = \|a(n, x)\| \frac{H^\delta \alpha \varepsilon}{1-e^{-2\varepsilon}}$$

To estimate the number in the formula $f$ for $k < K$ we denote as

$$M = 1 + \max_x \|a(x)\|$$

$$m = 1 + \max_x \|a^{-1}(x)\|$$
Then as before
\[ \|a(\sigma^k x) - a(\sigma^k y)\| \leq H\delta^\alpha e^{-k\gamma_\alpha} \]
but
\[ \|a(n - k + 1, \sigma^{k+1} x)\| = \|a(-k + 1, \sigma^n x) a(n, x)\| \leq \|a(n, x)\| \cdot m^k \]
and
\[ \|a(k, y)\| \leq M^k \]
So for \( k < K \) the expression (9) is bounded by
\[ \|a(n, x)\| m^k \cdot H\delta^\alpha e^{-k\gamma_\alpha} \cdot M^k \leq \|a(n, x)\| H\delta^\alpha (mM)^K \]
Finally,
\[ \|a(n, x) - a(n, y)\| \leq \|a(n, x)\| \delta^\alpha \left( \frac{H}{1 - e^{-\varepsilon}} + K(mM)^K \right) = \|a(n, x)\| \delta' \]
By choosing \( \delta \) sufficiently small we can make \( \delta' < 1/2 \). Then
\[ \|a(n, y)\| = \|a(n, x) - (a(n, x) - a(n, y))\| \geq \|a(n, x)\| - \|a(n, x) - a(n, y)\| \geq \frac{1}{2} \|a(n, x)\| \]

To finish the proof of the Theorem we will need the following features of the maps with closing property.

**Lemma 7.** Let \( \sigma : X \to X \) be a homeomorphism with closing property and the expansion constant \( \lambda \), then for any positive numbers \( \varepsilon \) and \( \delta \) there is a number \( \delta' \) such that if \( \text{dist}(x, \sigma^k x) \leq \delta' \) and \( k \geq n(1 + \varepsilon) \) then there is a point \( p \) such that \( \sigma^k p = p \) and \( p \) is \((\gamma, \delta, n)\)-close to \( x \), where \( \gamma = \varepsilon \lambda \).

**Proof.** It follows from the definition of the closing property that for \( 0 \leq i \leq k \)
\[ \text{dist}(\sigma^i x, \sigma^i p) \leq \text{dist}(\sigma^i x, \sigma^i z) + \text{dist}(\sigma^i z, \sigma^i p) \leq 2C\delta' e^{-\lambda \min(i, k - i)} \]

The function \( -\lambda \min(x, k - x) \) is convex downward so the segment connecting points \((0, 0)\) and \((n, -\lambda \min(n, k - n))\) on the graph of this function stays above the graph. The linear function that corresponds to this segment is \(-\gamma x\) where
\[ \gamma = \frac{k - n}{n} \lambda > \varepsilon \lambda \]
Therefore the point \( p \) satisfies the following inequalities:
\[ \text{dist}(\sigma^i x, \sigma^i p) \leq 2C\delta' e^{-\gamma i} \quad 0 \leq i \leq n \]
If we take \( \delta' = \frac{\delta}{2C} \) we can see that \( p \) is \((\gamma, \delta, n)\)-close to \( x \). \( \Box \)

**Lemma 8.** Let \( \sigma : X \to X \) be a homeomorphism. For any \( \varepsilon, \delta > 0 \) let \( P_{\varepsilon, \delta} \) be the set of points \( x \) in \( X \) for which there is an integer number \( N = N(x, \varepsilon, \delta) \) such that if \( n > N \) then there is an integer \( n(1 + \varepsilon) < k < n(1 + 2\varepsilon) \) for which
\[ \text{dist}(x, \sigma^k x) < \delta \]
If \( P = \cap_{\varepsilon > 0, \delta > 0} P_{\varepsilon, \delta} \) then \( \mu(P) = 1 \) for any invariant measure \( \mu \).
Proof. It is enough to prove it only for ergodic invariant measures. Let \( \mu \) be an invariant ergodic measure. The support of a measure is the the set of all points in \( X \) such that the measure of any open ball centered at \( x \) is not 0. The support of a measure on a compact metric space has always full measure (see [3]). \( X \) is compact so there is a sequence of balls \( B_i \) which is a base of the topology. If we define as \( f_i(n, x) \) the number of such \( k \) that \( \sigma^k x \in B_i \) and \( 1 \leq k \leq n \) then by Birkhoff’s Ergodic Theorem \( \lim_{n \to \infty} \frac{f_i(n, x)}{n} \) exists and equals \( \mu(B_i) \) for almost all \( x \). It is easy to see that any \( x \) that belongs to the support of the measure and satisfies Birkhoff’s Ergodic Theorem for all \( i \) will belong to the set \( P \). Indeed, if we choose \( \delta > 0 \) then we know that the ball \( B_\delta \) centered at \( x \) should have measure greater than 0. This ball is a countable union of some of the balls \( B_i \), therefore there exists at least one ball \( B_{i_0} \) such that \( \mu(B_{i_0}) > 0 \) and \( B_{i_0} \subset B_\delta \). Now, using the numbers \( \varepsilon \) and \( \mu(B_{i_0}) \) we choose a very small \( \varepsilon \). How small we specify later. For this \( \varepsilon > 0 \) we can find \( N \) such that if \( n > N \) then \( |f_{i_0}(n, x) - \mu(B_{i_0})| < \varepsilon n \). If \( n > N \) and there is no \( k \) such that \( n(1 + \varepsilon) < k < n(a + 2\varepsilon) \) and \( \sigma^k x \in B_{i_0} \) then \( f_{i_0}(n(1 + \varepsilon), x) = f_{i_0}(n(1 + 2\varepsilon), x) \). It is impossible if we choose \( \varepsilon \) very small because in this case

\[
(\mu(B_{i_0}) + \varepsilon)n(1 + \varepsilon) \geq f_{i_0}(n(1 + \varepsilon), x) = f_{i_0}(n(1 + 2\varepsilon), x) \geq (\mu(B_{i_0}) - \varepsilon)n(1 + 2\varepsilon)
\]

or

\[
\frac{\mu(B_{i_0}) + \varepsilon}{\mu(B_{i_0}) - \varepsilon} \geq \frac{1 + 2\varepsilon}{1 + \varepsilon}
\]

When \( \varepsilon \) is small the left side is as close to 1 as we want, so we get a contradiction. It means, if \( N \) is sufficiently big and \( n > N \) then there is \( k \) such that \( \sigma^k \in B_{i_0} \subset B_\delta \) and \( n(1 + \varepsilon) \leq k \leq n(1 + 2\varepsilon) \). Therefore the set \( P \) includes the intersection of two sets of full measure and has the full measure. \( \square \)

Proof of the Theorem [3] Choose any \( \varepsilon > 0 \). We can find an ergodic invariant measure \( \mu \) such that \( r - r_\mu < \min(\varepsilon, \varepsilon \alpha \lambda) \). Choose a point \( x \) such that \( r(x) = r_\mu \) and \( x \) belongs to the set \( S_{\varepsilon \lambda} \cap P \) where \( S_{\varepsilon \lambda} \) as in the Proposition [3] and \( P \) as in the Lemma [8] All those sets have full support, so their intersection is not empty. For the point \( x \) we can find \( \delta \) such that for infinitely many \( n_i \) if a point \( p \) is \( (\varepsilon \lambda, \delta, n_i) \)-close to \( x \) then

\[
(10) \quad ||a(n_i, p)|| \geq \frac{1}{2} ||a(n_i, x)||
\]

For this \( \delta \) we can find \( \delta' \) from Lemma [7]. Using this \( \delta' \) and \( \varepsilon \) we can find \( N = N(\varepsilon, \delta') \) from the Lemma [8] such that if \( n_i > N \) then there is \( k \) such that \( n_i(1 + \varepsilon) \leq k \leq n_i(1 + 2\varepsilon) \) and \( \text{dist}(\sigma^k x, x) < \delta' \), then from Lemma [7] follows that there is a periodic point \( p \) with the period \( k \) such that it is \( (\varepsilon \lambda, \delta, n_i) \)-close to \( x \) and therefore satisfies the inequality (10).

Now, we estimate \( ||a(k, p)|| \). Let \( N' \) be a number such that if \( n > N' \) then

\[
||a(n, x)|| \geq e^{n(r_\mu - \varepsilon)} \geq e^{n(r - 2\varepsilon)}
\]

We always can choose \( n_i \) bigger not only than \( N \) but also and \( N' \). Denote as

\[ m = \ln \max_y ||a^{-1}(y)||. \]

Then

\[
||a(n_i, p)|| = ||a(-(k - n_i), p)|| \leq ||a(-(k - n_i), p)|| \cdot ||a(k, p)||
\]
so
\[ \|a(k, p)\| \geq \frac{\|a(n_i, p)\|}{e^{m(k-n_i)}} \geq \frac{1}{2} \frac{\|a(n_i, x)\|}{e^{2m\varepsilon n_i}} \geq \frac{1}{2} e^{(r-2\varepsilon-2m\varepsilon)n_i}. \]
We see that
\[ r_p = \frac{\ln \|a(k, p)\|}{k} \geq \frac{(r - 2\varepsilon - 2m\varepsilon)n_i - \ln 2}{(1 + 2\varepsilon)n_i}. \]

Number \( m \) does not depend on the choice of \( x, n_i \) and \( p \), so by choosing \( \varepsilon \) very small and \( n_i \) very big we can make \( r_p \) as close to \( r \) as we want.

\[ \square \]

4. Proof of the Main Theorem

After Theorem 3 is established we can use Corollary 4 to show that the growth of \( \|a(n, x)\| \) is sub-exponential. It allows to use the idea of the original Livšic proof for cocycles with values in Banach rings. H.Bercovici and V.Nitica in [2] (Theorem 3.2) showed that if \( \sigma \) is a transitive Anosov map, periodic obstructions vanish and
\[ \|a(x)\| \leq 1 + \delta \]
\[ \|a^{-1}(x)\| \leq 1 + \delta \]
for some \( \delta \) that depends on \( \sigma \), then \( a(x) \) is a coboundary. From Corollary 4 we can get a little bit less. If periodic obstructions vanish then for any \( \delta > 0 \) there exists \( C > 0 \) such that for any positive integer \( n \)
\[ \|a(n, x)\| \leq C(1 + \delta)^n \]
\[ \|a(n, x)^{-1}\| \leq C(1 + \delta)^n \]
Those inequalities are actually enough to repeat the arguments from [2] with some small changes, but we also refer to more general theorem proven in [4] that considers cocycles over maps that satisfy closing property and with values in abstract groups satisfying some conditions. But we will need couple of more definitions.

**Definition 9.** If \( G \) is a group with metric denoted as \( \text{dist} \) and \( g \in G \) we define the distortion of the element \( g \) as
\[ |g| = \sup_{f \neq g} \left[ \frac{\text{dist}(gf, gh)}{\text{dist}(f, h)}, \frac{\text{dist}(fg, hg)}{\text{dist}(f, h)}, \frac{\text{dist}(g^{-1}f, g^{-1}h)}{\text{dist}(f, h)}, \frac{\text{dist}(fg^{-1}, hg^{-1})}{\text{dist}(f, h)} \right] \]
We say that a group is Lipschitz if \( |g| < \infty \) for all \( g \in G \).

It is easy to see that for Banach rings if we define
\[ \text{dist}(f, h) = \max(\|f - h\|, \|f^{-1} - g^{-1}\|) \]
then \( |g| \leq \max(\|g\|, \|g^{-1}\|) \) and \( B^\times \) is Lipschitz.

**Definition 10.** We call the rate of distortion of a cocycle \( a(x) : X \to G \) the following number
\[ r(a) = \lim_{n \to \infty} \sup_{x \in X} \frac{\ln \|a(n, x)\|}{n} \]

**Theorem 11.** Let \( G \) be a Lipschitz group with the property that there are numbers \( \epsilon \) and \( D \) such that \( \text{dist}(g, e) \leq \epsilon \) implies \( |g| \leq D \), \( \sigma \) be a transitive homeomorphism with \( \lambda \)-closing property. If the rate of the distortion of an \( \alpha \)-Hölder continuous cocycle \( a : X \to G \) is smaller than \( \alpha \lambda/2 \) and the periodic obstructions vanish then \( a(x) \) is a coboundary with \( \alpha \)-Hölder continuous transition function \( t(x) \).
Proof. See [4].

Proof of the Main Theorem. If \(a(n, p) = e\) for every periodic point then it follows from the Corollary 4 that the distortion rate of the \(a(n, x)\) is less or equal than 0. In the group \(B^\times\) if \(\text{dist}(e, g) < \frac{1}{2}\) then \(\|e - g\| < \frac{1}{2}\) and \(\|e - g^{-1}\| < \frac{1}{2}\) so \(\|g\|, \|g^{-1}\| \leq \frac{3}{2}\). We see that \(|g| < \frac{3}{2}\), therefore by Theorem 11 the cocycle \(a(n, x)\) is a coboundary with \(\alpha\)-Hölder continuous transition function.

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