Macroscopic Simulation of Violation of Bells Inequality *

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Abstract

A macroscopic quantum model of a two-level system (the analogue of a half-spin particle) is described. The model is employed for simulating not only the system under study, but the measurement process as well. Single- and two-particle state models of a quantum system are constructed. The Einstein-Podolsky-Rosen paradox and Bells inequality are discussed within the framework of the model.

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1 INTRODUCTION

Bohr's collaborator A. Petersen cites a statement [1] he had once heard from Bohr while discussing problems in quantum mechanics: "There is no quantum world. There only is an abstract physical quantum description."

Although this assertion is in no way obvious, we take it to serve us in describing a physical model that possesses, on one hand, typical quantum properties and consists of macroscopic components, on the other.

Such a model can be quite useful in studying problems that still provoke considerable controversy in the physics community. These include, e.g., the problem of locality in quantum measurements, the causality problem, the existence of a physical reality that may affect results of measurement, etc.

According to widespread belief, the main feature of quantum systems is their specific quantum dynamics. In the latter, a significant role is played by processes in which the components of a physical system exchange portions of action comparable to the atomic unit of action $\hbar$.

In contrast to this point of view, we try to show in this work that quantum dynamics, which determines the interactions among different parts of a system studied, is in many cases inessential for the onset of typical quantum features, whereas the peculiarities of the interaction between a quantum system explored and an external system, which serves the purpose of either the formation of the state of a system under study or proper measurement, are important.

Since the result of such an interaction depends not only on the characteristics of a system under investigation, but also on the properties of an external system, a likelihood exists

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that results will be obtained that are typical of quantum phenomena even in the case of macroscopic dimensions of a system examined only by employing appropriate parameters of the external system. We shall try to implement this program for a two-level physical system that is analogous to a half-spin particle.

Further consideration will be carried out within the framework of a special version of the algebraic approach to the quantum theory, which is described in [2, 3]. In the latter references, a phenomenological basis for the postulates of this approach is given, along with a description of how to develop the standard mathematical apparatus of quantum mechanics starting from these postulates.

This approach has a number of distinctive features that will be very useful in what follows. First, quantum and classical systems are considered from a unified point of view and, moreover, the quantum system can be regarded as a set of open classical subsystems. Second, the dynamics is not fixed in this approach. Therefore, it is equally applicable to both quantum and classical systems. Third, the mathematical apparatus of the approach is constructed as a mathematical description of the physical measurement process. Recall that, within the traditional approach, one tries (not very successfully) to develop a theory of quantum measurements that is consistent with the a priori mathematical apparatus of quantum mechanics.

2 BASICS OF THE APPROACH

In this section the fundamental axioms of the approach will be listed and briefly commented on. More detailed accounts and justifications can be found in [2, 3].

Within the framework of the algebraic approach, the primary building blocks of the quantum theory are observables (see, e.g., [4, 5, 6]). The latter are characteristics of a physical system that can be given numerical values by using certain measurement processes. Upon choosing a specific system of units, all the observables can be considered dimensionless. The main property of the observables is that they can be multiplied by real numbers and form products between and add up to each other. Therefore, the following postulate is adopted in the algebraic approach.

**Postulate 1.** Observables \( \hat{A} \) of a physical system are Hermitian elements of some \( C^* \)-algebra \( \mathfrak{A} \), \( \hat{A} \in \mathfrak{A}_+ \subset \mathfrak{A}, \hat{A}^* = \hat{A} \).

Here \( \mathfrak{A}_+ \) is the set of observables.

Recall that algebra is a set that is a linear space with a defined operation of multiplication of the elements. The algebra \( \mathfrak{A} \) is called \( C^* \)-algebra (see, e.g., [7]), if a conjugation (an involution) operation \( \hat{U} \rightarrow \hat{U}^* \) \( (\hat{U} \in \mathfrak{A}, \hat{U}^* \in \mathfrak{A}) \) is defined, and the norm of an arbitrary element \( \hat{U} \) satisfies the following condition \( \|\hat{U}\hat{U}^*\| = \|\hat{U}\|^2 \).

The fundamental difference between quantum and classical systems lies in the fact that any pair of observables in a classical system can be measured in a compatible manner, which means that multiple measurements of two observables \( \hat{A} \) and \( \hat{B} \) will give rise to identical results for each of them regardless of the sequence of measurements undertaken. For quantum systems, only certain groups of observables can be measured in such a way. The observables are called compatible if they belong to the same group. Mark each such group with the
index $\xi$ that takes its values in the set $\Xi$ and distinguishes one such group from another. The same observable $\hat{A}$ can belong to many groups $\Omega_\xi$. Each group $\Omega_\xi$ can be treated as a set of observables of some classical subsystem of a quantum system studied. As a consequence, the following two postulates are adopted.

**Postulate 2.** A set of compatible (simultaneously measurable) observables is a maximal real associative commutative subalgebra $\Omega_\xi$ of the algebra $\mathfrak{A}$ ($\Omega_\xi \subset \mathfrak{A}^+$).

**Postulate 3.** The state of a classical subsystem, with its observables being the elements of subalgebra $\Omega_\xi$, is described by the character $\varphi_\xi(\cdot)$ of this subalgebra.

The latter means that $\hat{A} \xrightarrow{\varphi_\xi} \varphi_\xi(\hat{A})$ ($\hat{A} \in \Omega_\xi$) is a homomorphism of the algebra $\Omega_\xi$ into the set of real numbers.

Let us now consider a quantum system to be a set of open classical subsystems with their corresponding observables being the elements of the corresponding subalgebra $\Omega_\xi$. Bearing in mind that each observable that belongs to the algebra $\mathfrak{A}$ is also associated with some subalgebra $\Omega_\xi$, we adopt the following postulate.

**Postulate 4.** The result of each individual measurement of the observable of a physical system is defined by the elementary state $\varphi = [\varphi_\xi]$ of this system.

Here $\varphi = [\varphi_\xi]$ is the set of characters of all subalgebras $\Omega_\xi$, with each of them represented by just one character $\varphi_\xi(\cdot)$ in $\varphi = [\varphi_\xi]$.

Instrument readings are due to interaction between an instrument and a physical system. This result may depend on the characteristics of both the system studied and the instrument. The latter is highly undesirable when examining the system under scrutiny. The problem of unification of the readings obtained from different instruments is usually solved with the help of a calibration procedure.

This procedure is schematically as follows. The instrument is taken to be a template that provides reproducible measurement of some observable $\hat{A}$. This instrument measures the observable $\hat{A}$. Hereinafter, the result of measurement is denoted by the same symbol as an observable itself, but without a hat. If the first measurement was carried out in a reproducible manner, then the duplicate measurement of the same observable by the instrument subject to calibration should yield the same value. Unification of readings of both the template and the instrument to be calibrated works in this way.

In a quantum case, however, such a procedure is feasible only within each group of compatible observables. Let us say that the instruments allowing for such measurements within a group $\Omega_\xi$, are related to the $\xi$-type. In the general case, reading unification for different-type instruments is impossible to realize. For a quantum system, even in the ideal case, dependence of the result of measurement on the type of instrument cannot be excluded.

If the observable $\hat{A}$ belongs simultaneously to two different subalgebras ($\hat{A} \in \Omega_\eta \cap \Omega_{\eta'}$, $\eta, \eta' \in \Xi$), then the different type ($\eta$ and $\eta'$) instruments can be used to measure it and the equality

$$\varphi_\eta(\hat{A}) = \varphi_{\eta'}(\hat{A})$$

of $\varphi_\eta, \varphi_{\eta'} \in \varphi = [\varphi_\xi], \eta \neq \eta'$

may be violated (see [2]).
If for some $\varphi = [\varphi_\xi]$ equality (1) holds for all $Q_\xi$ containing the observable $\hat{A}$, then we say that the elementary state $\varphi = [\varphi_\xi]$ is stable with respect to the observable $\hat{A}$.

If the elementary state $\varphi = [\varphi_\xi]$ of the system studied was known to us, then we would be able to predict unambiguously the result of measurement of any observable. However, this is impossible. Only compatible observables can simultaneously be measured, for example, those associated with the subalgebra $Q_\eta$. Therefore, out of all the characters $\varphi_\xi$ inherent to the elementary state $\varphi$, only the functional $\varphi_\eta(\cdot)$ can experimentally be determined. In this connection, it is convenient to introduce the notion of $\varphi_\eta$-equivalence.

Two elementary states $\varphi$ and $\varphi'$ are called $\varphi_\eta$-equivalent if for all $\hat{A} \in Q_\eta$ the following equality holds:

$$\varphi_\eta(\hat{A}) = \varphi'_\eta(\hat{A}),$$

where $\varphi_\eta \in \varphi$, $\varphi'_\eta \in \varphi'$.

Accordingly, what we can at most learn of the elementary state in experiment is that it belongs to the certain equivalence class $\{\varphi\}_{\varphi_\eta}$. If the observable $\hat{A}$ belongs to the subalgebra $Q_\eta$ and elementary $\varphi \in \{\varphi\}_{\varphi_\eta}$ is stable with respect to the observable $\hat{A}$, then as a result of measurement we definitely obtain $A = \varphi_\eta(\hat{A})$. If $\hat{A} \notin Q_\eta$, then it is impossible to predict definitely the result of measurement, since these may differ for different elementary states $\varphi \in \{\varphi\}_{\varphi_\eta}$.

To put it differently, the equivalence class has the physical properties that are ascribed to the quantum state in the traditional convenient approach. Therefore, the set of equivalent elementary states stable with respect to all the observables $\hat{A} \in Q_\eta$ is identified with the quantum state $\Psi_\varphi \equiv \{\varphi\}_{\varphi_\eta}$.

The elementary state meets the requirements for the elementary events in Kolmogorov probability theory [8] ((see also [9]). Namely, only one elementary event is realized in each test. Different elementary events are mutually exclusive. Therefore, there is no need to devise some artificial quantum probability theory; instead, the well-developed classical one can be used. As the probabilistic properties of a quantum system are fixed by its quantum states, it is natural to adopt the following postulate.

**Postulate 5.** The equivalence class $\{\varphi\}_{\varphi_\eta}$ can be equipped with the structure $(\Omega, \mathcal{F}, P)$ of the probability space.

Here $\Omega$ is the space of elementary events for which it is in general impossible to define the probabilistic measure $P(F)$ in Kolmogorov probability theory. In order to define the latter, it is necessary to render the space $\Omega$ measurable. To this end, besides elementary events, the so-called (probabilistic) events $F_\cdot$ should be introduced. These are subsets of the $\Omega$ set for which the probabilistic measure $P(F)$ can be defined. It is assumed that the event $F$ has occurred once an elementary event belonging to the subset $F$ is realized. The subsets $F$ must be the elements of so-called Boolean $\sigma$-algebra $\mathcal{F}$. This means that the set $\mathcal{F}$ is equipped with three algebraic operations: the union of subsets $F$, their intersection, and the complement of each subset to the full set $\Omega$. The set $\mathcal{F}$ must necessarily include the set $\Omega$ itself along with the empty set $\emptyset$. In addition, this set $\mathcal{F}$ must be closed with respect to the complement operation and a countable number of unions and intersections. The latter property means that, under any of those operations, we obtain an element of the same set. The space $\Omega$ equipped with the described $\sigma$-algebra $\mathcal{F}$ is called measurable space.
The choice of certain Boolean algebra $F$ corresponds in terms of physics to the choice of instrument type. The probabilistic measures $P(F)$ are defined only for the events $F \in F$.

A distinctive feature of the quantum measurements is that it is possible to use simultaneously only those instruments which allow for measurements of compatible observables, i.e., those which belong to a certain $\xi$-type. There is certain Boolean algebra $F_\xi$ that corresponds to each type of such instruments. It is essential that, in a quantum case, there is no Boolean algebra $F_0$ with the following properties: for all $F \in F_0$ there are probabilistic measures $P(F)$ and the algebras $F_\xi$ are subalgebras of the algebra $F_0$.

Postulate 5 allows defining the average value of the observable $\hat{A}$ in a quantum state $\Psi_\eta$ using the probabilistic average over a space $\{\varphi\}_{\varphi_\eta}$ of elementary (states) events:

$$\Psi_{\varphi_\eta}(\hat{A}) = \int_{\varphi \in \{\varphi\}_{\varphi_\eta}} P_A(d\varphi) \varphi_\xi(\hat{A}).$$

(2)

Here $P_A(d\varphi) = P(\varphi : \varphi_\xi(\hat{A}) \leq A + dA) - P(\varphi : \varphi_\xi(\hat{A}) \leq A)$, $\hat{A} \in Q_\xi$, $\varphi_\xi \in \varphi \in \{\varphi\}_{\varphi_\eta}$.

In order for formula (2) to define the quantum average, the probabilistic measure $P_A(\varphi)$ must satisfy the following postulates.

**POSTULATE 6.** The functional $\Psi_{\varphi_\eta}(\hat{A})$ is linear over the algebra $A$.

**POSTULATE 7.** The functional $\Psi_{\varphi_\eta}(\hat{A})$ does not depend on the particular choice $\xi$.

The latter statement is to be understood as follows. The observable $\hat{A}$ can simultaneously belong to several subalgebras $Q_{\xi_1}, Q_{\xi_2}, \ldots$. For all $\xi_1, \xi_2, \ldots$ formula (2) must define the same functional. Since $\varphi_\xi(\cdot)$ is the character of the subalgebra $Q_\xi$, the functional $\Psi_{\varphi_\eta}$ will automatically be positive and normalized to unity, i.e., $\Psi_{\varphi_\eta}(\hat{I}) = 1$, where $\hat{I}$ is the identity element of the algebra $A$.

With the $C^*$-algebra $A$ and the linear positive normalized functional $\Psi_{\varphi_\eta}(\cdot)$ defined over this algebra, we can construct a representation of the algebra $A$ by using the canonical Gelfand-Naimark-Segal construction (see, e.g., [10, 4]). In other words, we can construct Gilbert space $\mathcal{H}$, in which there is an operator $\Pi(\hat{A})$ acting over a space $\mathcal{H}$ that corresponds to each element $\hat{A} \in A$, while the expectation value $\langle \Phi | \Pi(\hat{A}) | \Phi \rangle$, where $| \Phi \rangle \in \mathcal{H}$ is the corresponding vector in Gilbert space — to the quantum average $\Psi_{\varphi_\eta}(\hat{A})$. This is the way the standard mathematical apparatus of quantum mechanics is reproduced.

From the point of view of quantum calculations, the mathematical apparatus based on Gilbert space usually turns out to be more convenient. However, it bears no clear physical interpretation and, therefore, is poorly adapted for macroscopic simulation of quantum processes. In contrast, the formalism based on the elementary state is applicable to both quantum and classical systems. That is why it is convenient for such a simulation to be performed.

**3 A TWO-LEVEL SYSTEM**

In this section we discuss an application of basics given in the previous section to a certain quantum model. For the latter, let us take a two-level system analogous to a half-spin particle. In what follows the spin terminology will be used for the model considered. At the
same time, we emphasize that this does not imply that the dynamics of the system under study is identical to that of a half-spin particle.

The observables of the two-level quantum system can be represented by the Hermitian matrices $2 \times 2$. In this case, the set of all the matrices of the type

$$\hat{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

can be considered to be the algebra $\mathfrak{A}$, in which algebraic operations coincide with the corresponding matrix ones.

For such a system, it is not difficult to construct all the elementary states. Let $\hat{A}$ be the Hermitian matrix; then, $a^* = a$, $d^* = d$, and $c = b^*$. Any matrix of such a type can be represented in the form

$$\hat{A} = g_0\hat{I} + g\hat{\tau}(r).$$

Here $r$ is the three-dimensional unit vector, $\tau_i$ are the Pauli matrices, and $\hat{\tau}(r) = (\tau \cdot r)$. In order for formula (3) to hold, it is required to set

$$g = ((a - d)^2/4 + b b^*)^{1/2}, \quad g_0 = (a + d)/2, \quad r_1 = (b + b^*)/(2g), \quad r_2 = (b - b^*)/(2i g), \quad r_3 = (a - d)/(2g).$$

It is obvious that $\hat{\tau}(-r) = -\hat{\tau}(r)$. For $r' \neq \pm r$, the commutator of matrices $\hat{\tau}(r)$ and $\hat{\tau}(r')$ is nonzero. Therefore, each (up to the sign) matrix $\hat{\tau}(r)$ is the generatrix for the real maximal commutative subalgebra $\Omega_r$. Since $\hat{\tau}(r)\hat{\tau}(r) = \hat{I}$, then the element spectrum $\hat{\tau}(r)$ consists of two points: $\pm 1$.

Let $\varphi^\alpha = [\varphi^\alpha_r]$ be an elementary state. Here $\varphi^\alpha_r$ is the character of the subalgebra $\Omega_r$ and the index $\alpha$ distinguishes one elementary state from the other. Consider the function $f^\alpha(r)$ such that $f^\alpha(-r) = -f^\alpha(r)$ and for each $r$ the function takes either $+1$ or $-1$, with the index $\alpha$ once again distinguishing the functions from each other. Then we can apparently assume that $\varphi^\alpha_r(\hat{\tau}(r)) = f^\alpha(r)$. Taking into account that $\hat{\tau}(r)$ is the generatrix of the subalgebra $\Omega_r$, we obtain

$$\varphi^\alpha_r(\hat{A}) = g_0(\hat{A}) + g(\hat{A})f^\alpha(r)$$

we obtain $\hat{A} \in \Omega_r$.

Note that for the quantum system in question each observable $\hat{A}$ (aliquant to $\hat{I}$) belongs to the same maximal subalgebra $\Omega_r$. Of course, this in general is not correct. By taking advantage of the mentioned feature inherent to the system at hand, we can represent the elementary state in the form of a universal functional defined over the entire set $\mathfrak{A}_+$:

$$\varphi^\alpha(\hat{A}) = g_0(\hat{A}) + g(\hat{A})f^\alpha(\hat{r}(\hat{A}))$$

for any observable $\hat{A}$. Here, not only $g$ and $g_0$, but also $r$ should be considered as functions of $\hat{A}$ (see formula (4)). This functional is explicitly extended over the whole algebra $\mathfrak{A}$. This example is evidence that the proof by von Neumann [11] of nonexistence of hidden parameters is inapplicable to elementary events. Let us draw attention to the fact that the functional $\varphi^\alpha(\hat{A})$ defined by formula (5) is nonlinear. In a general case, the elementary state can also be formally represented in the form of a nonlinear functional defined over the whole algebra $\mathfrak{A}$. However, this functional will in general be ambiguous, because the same observable $\hat{A}$ may belong to several maximal commutative subalgebras $\Omega_\xi$. 
From formula (5) it follows that, in order to define the functional $\varphi^\alpha(\hat{A})$, it is sufficient to define the function $f^\alpha(r)$. This means that each function $f^\alpha(r)$ is in one-to-one correspondence with a certain elementary state. Let us designate the value of the function $f^\alpha(r)$ equal to $+1$ with a black dot and $-1$ with a white one. Then, each of such elementary states can be visualized as a sphere of unit radius the surface of which is spotted by black and white dots, with those of different colors occupying the ends of each diameter. By using these spheres, we will be able to model elementary states of a half-spin particle.

Let the system be in a definite elementary state to which the sphere (the elementary state sphere (ESS)) described above corresponds. To define the outcome of measurement of spin projection onto a definite direction, it is required to draw a unit vector $r$ from the center of the ESS along that particular direction. If this vector encounters a black (white) dot on a sphere, then the result of measurement will be $+1/2 (-1/2)$.

Thus, consider a half-spin quasiparticle the elementary states of which can be described by pointwise painted ESSs. Very close points may have different colors. Therefore, an infinitesimal rotation of the instrument may give rise to a drastic change in the result registered. At the same time, experience shows that, if we study some quantum state, i.e., some set of ESSs, then on average a small rotation of the instrument causes a mild change in the result. This means that, from the point of view of finding average values, we can substitute a set of pointwise painted spheres in the quantum state for a set of continuously painted ones. Of course, both the coloring and ensemble of such spheres should be selected properly.

Let us assume that all the spheres have identical coloring and differ solely in their orientations. The sphere of the north magnetic pole ($+1$) is painted black, and that of the south magnetic pole ($-1$) white. Intermediate areas are gray, their darkness varying according to the law

$$\rho = (rR). \quad (6)$$

Here $r$ is the radius-vector drawn toward a current gray dot, while $R$ denotes the radius-vector drawn toward the north magnetic pole.

Transition to gray-painted spheres can resolve the problem of a smooth change of registered average values caused by the instrument rotation, but, at the same time, gives rise to the new problem. Each measurement must yield either $+1/2$ or $-1/2$ for the spin projection, not some intermediate gray value.

To overcome this obstacle, assume that the elementary state of a particle is characterized by a multilayer gray-painted sphere (MGS), with each layer painted according to the law given in (6) and with different layers having different orientations. The orientation of the $k$ layer is denoted by the vector $R(k)$. In addition, assume that the elementary state is characterized by the values of auxiliary variable $\hat{\varepsilon}$. A (failed) attempt to use such a variable was undertaken in [12]. Let $r$ be the radius-vector drawn toward a current dot on a sphere. Then, each value of $r$ corresponds to the value $\varepsilon(r)$ of the variable $\hat{\varepsilon}$. Of the variable $\varepsilon(r)$ take, independently of each other, random values from within the interval

$$-1/2 < \varepsilon^{(k)}(r) < +1/2$$

and

$$\varepsilon^{(k)}(-r) = -\varepsilon^{(k)}(r).$$
Now, out of all such MGSs, select a subset $\Upsilon$ that consists of those $\varepsilon(r)$ of which related to each layer satisfy one of the following conditions:

$$|R^{(k)}r + \varepsilon^{(k)}(r)| > 1/2,$$  \hspace{1cm} (7) 

or

$$|R^{(k)}r + \varepsilon^{(k)}(r)| < 1/2.$$  \hspace{1cm} (8)

It is easy to verify that such functions (discontinuous in general) exist almost for each $r$ at an arbitrary orientation of the layer. It is also clear that the subset $\Upsilon$ is invariant with respect to the spherical transformations. Below, we shall deal with only those MGSs that belong to a subset $\Upsilon$.

The uppermost layer that satisfies condition (7) is called active. Now the associate ESS with each MGS equipped with the functions $\varepsilon(r)$ according to the following rule. If the active layer of the MGS considered has the number $k$, then the radius-vector $r$ of the corresponding ESS is set against either a black dot $(j = +1)$ if $R^{(k)}r > 0$ or a white dot $(j = -1)$ or a white dot $R^{(k)}r < 0$. Recall that the dot color on the ESS describes the result of measurement of the spin projected onto the corresponding direction when a particle is in the elementary state under consideration. Accordingly, assume that the instrument that measures the spin projection onto the direction of $r$ operates as follows. First, it checks if the uppermost layer of the MGS corresponding to the elementary event considered is active. If it is, then the instrument registers the event according to the rule outlined above. If it is passive, then the instrument fails to register any event and proceeds to study the next layer in the same fashion. This process continues until there is a definite result registered. We shall see later that the probability of the active layer having a number greater than $k$ rapidly decreases with increasing $k$. In this way, a definite result will be registered for nearly every elementary event.

Having in mind the correspondence between the MGS and ESS, we shall construct the quantum states we are interested in by using sets of MGSs rather than ESSs.

The quantum state with the doubled spin projection (equal to $+1$) onto the direction of $n$ corresponds to the set of ESSs, with a black dot found at the end of the radius of each member sphere directed along $n$. The set of MGSs ($\Upsilon_n$) that satisfies the following conditions is associated with this quantum state. The radius-vector $R^{(k)}$ in each layer $k'$ pointing toward the north magnetic pole belongs to the upper hemisphere $R^+_n$ with a central direction $n$. The orientations of the vectors $R^{(k')}_{n}$ for different layers are random and distributed over $R^+_n$ independently of each other. Each MGS included into $\Upsilon_n$ is equipped with the functions $\varepsilon^{(k')}(r)$. The latter functions are random as well and satisfy either condition (7) or (8).

Any individual MGS with given functions $\varepsilon^{(k')}(r)$ define some elementary state. In probability theory, the latter corresponds to some elementary event. According to probability theory [8], it is not at all necessary that an elementary event be associated with some probabilistic measure. The probabilistic measures must correspond to (probabilistic) events that are some subsets of the set of elementary events.

Therefore, the notion of probability will only be associated with certain subsets of the set of elementary states. For every value of $r$ we construct a proper system of such subsets, a proper $\sigma$-algebra $\mathcal{F}_r$. This is consistent with the quantum case, in which there is proper $\sigma$-algebra (see [2]) for each group of compatible observables. In the case at hand, such a
The \( \sigma \)-algebras \( \mathcal{F}_r \) generated are defined by the following conditions.

1. The vector \( r \) fixes \( \sigma \)-algebra \( \mathcal{F}_r \). The remaining conditions fix the elements of this algebra.

2. The interval \((\varepsilon, \varepsilon + d\varepsilon)\) is fixed, where the parameter \( \varepsilon \) satisfies inequalities \(-1/2 < \varepsilon < +1/2\). Then, the interval itself and its length will be denoted by a single symbol \( d\varepsilon \).

3. Either \( \varepsilon(k')(r) \in d\varepsilon \) or \(-\varepsilon(k')(r) \in d\varepsilon \). Both alternatives correspond to the same subset (one element of \( \sigma \)-algebra).

4. The number \( k \) of the active layer is fixed.

5. Each subset consists of all the MGSs the layers of which satisfy one of the following conditions:

\[
|\mathbf{R}(k')r + \varepsilon(k')(r)| > 1/2 \quad k' = k, \tag{9}
\]
\[
|\mathbf{R}(k')r + \varepsilon(k')(r)| \leq 1/2 \quad k' \leq k \quad (k' \text{ fixed}). \tag{10}
\]

The conditions given in (9) and (10) correspond to different subsets.

6. If the condition in (9) is satisfied, the subsets are distinguished by another feature: either \( \mathbf{R}(k)r > 0 \) or \( \mathbf{R}(k)r < 0 \).

It is easy to verify that, irrespective of \( r \), any elementary event from \( \Upsilon_n \) belongs to any listed subset at some \( \varepsilon \) and \( k \).

Now construct the probabilistic measures for each such subset, assuming that the random values of \( \mathbf{R}(k')r \) are evenly distributed on a hemisphere \( \mathcal{H}_n^+ \).

Let \( r \) and \( d\varepsilon \) fixed and \( k' = 1 \). The probability for inequality (9) to hold with the additional condition of either \( \mathbf{R}(k)r > 0 \) (\( j = +1 \)) or \( \mathbf{R}(k)r < 0 \) (\( j = -1 \)) is described by the following expression:

\[
P_n^{(1)}(r, \varepsilon, j)d\varepsilon = d\varepsilon \frac{N}{2} \int d\mathbf{R} \Theta(\mathbf{R}n) \left[ \Theta[j(\mathbf{R}r + \varepsilon) - 1/2] + \Theta[j(\mathbf{R}r - \varepsilon) - 1/2] \right]. \tag{11}
\]

Here \( N \) is the normalization factor, \( d\mathbf{R} = d\phi d\vartheta \sin \vartheta \), and \( \Theta(x) \) is the Heaviside step function (\( \Theta(x) = 0 \) for \( x < 0 \), \( \Theta(x) = 1 \) for \( x > 0 \)). From (11) it follows that

\[
P_n^{(1)}(r, \varepsilon) \equiv \sum_{j=\pm1} P_n^{(1)}(r, \varepsilon, j) = N\pi. \tag{12}
\]

The probability for inequality (10) to hold is given by

\[
P_n^{(1)}(r, \varepsilon)d\varepsilon = d\varepsilon \frac{N}{2} \int d\mathbf{R} \Theta(\mathbf{R}n) \left[ \Theta[1/2 + \mathbf{R}r + \varepsilon] \Theta[1/2 - \mathbf{R}r - \varepsilon] + \Theta[1/2 + \mathbf{R}r - \varepsilon] \Theta[1/2 - \mathbf{R}r + \varepsilon] \right]. \tag{13}
\]

\[
= d\varepsilon \frac{N}{2} \int d\mathbf{R} \left[ \Theta[1/2 + \mathbf{R}r + \varepsilon] \Theta[1/2 - \mathbf{R}r - \varepsilon] \right] = N\pi d\varepsilon.
\]
From (12) and (13), we obtain
\[ N = (2\pi)^{-1} \]
and
\[ \tilde{P}_{n}^{(1)}(r, \varepsilon) = 1/2. \] (14)

Now consider the case \( k' = 2 \). The second layer has the same properties as the first one; at the same time, its orientation and the corresponding function \( \varepsilon^{(k')}(r) \) are independent of the first layer. As a result, we can repeat previous arguments for the second layer. The only thing to take into account is that, because of formula (14), the instrument will deal with the second layer with a probability of 1/2. This means that for a given \( \varepsilon \) the probability of obtaining the number of the active layer equal to 2 and the fixed value of \( j \) as a result is described by the following formula:
\[ P_{n}^{(2)}(r, \varepsilon, j) = \frac{1}{2 \pi 2^k} \int dR \Theta(jRn) \left[ \Theta[Rr + \varepsilon - 1/2] + \Theta[Rr - \varepsilon - 1/2] \right], \]
while the probability for the active layer to have a number greater than 2 is as follows:
\[ \tilde{P}_{n}^{(2)}(r, \varepsilon) = (1/2)^2. \]

Continuing this process we obtain that, for fixed \( r \), \( d\varepsilon \) and \( j \), the probability for the active layer to have the number \( k \) is equal to
\[ P_{n}^{(k)}(r, \varepsilon, j) = \frac{1}{2 \pi 2^k} \int dR \Theta(jRn) \left[ \Theta[Rr + \varepsilon - 1/2] + \Theta[Rr - \varepsilon - 1/2] \right], \] (15)
whereas the probability of finding an active layer with a number greater than \( k \) is represented by
\[ \tilde{P}_{n}^{(k)}(r, \varepsilon) = (1/2)^k. \] (16)

It is noteworthy that the latter probability depends on neither \( r \) nor \( \varepsilon \) and rapidly decreases with increasing \( k \). Formulas (15) and (16) describe a system of probabilistic measures corresponding to the \( \sigma \)-algebra chosen.

As \( \varepsilon^{(k')} \) depend on \( r \), the subsets of elementary states corresponding to the same interval \( d\varepsilon \) will be different for different \( r \), which means that different instruments correspond to different \( \sigma \)-algebras. In other words, different instruments split the set of elementary events into subsets in different ways. Only within a single such partition is it meaningful to speak about the probability of a specific elementary event falling into one such subset. On the other hand, it is meaningless to compare the probabilities of the elementary events falling into subsets that belong to different partitions, since such subsets are elements of different \( \sigma \)-algebras. In their turn, these \( \sigma \)-algebras are not subalgebras of some universal \( \sigma \)-algebra to which some system of probabilistic measures corresponds. This is a distinctive feature of the quantum measurements (see [2]).

From (15) it follows that the probability of obtaining a fixed value for \( j \) at fixed \( r \) and an arbitrary number of the active layer and \( \varepsilon \) is given by the formula
\[ P_{n}(r, j) = \sum_{k=1}^{\infty} \int_{-1/2}^{1/2} d\varepsilon P_{n}^{(k)}(r, \varepsilon, j) \]
\[ = \frac{1}{2 \pi} \int_{-1/2}^{1/2} d\varepsilon \int dR \Theta(jRn) \left[ \Theta[Rr + \varepsilon - 1/2] + \Theta[Rr - \varepsilon - 1/2] \right] \]
\[ = \frac{1}{\pi} \int dR \Theta[jRn](Rr)\Theta[Rr]. \]
It follows then that
\[ \sum_j P_n(r, j) = \frac{1}{\pi} \int dR \Theta[Rr] = 1. \tag{17} \]
and
\[ \sum_j j P_n(r, j) = \frac{1}{\pi} \sum_j \int dR \Theta[jRn] \Theta[Rr] = \int dR \Theta[Rn](Rr) = (rn). \tag{18} \]
From (17) and (18), we derive
\[ P_n(r, j) = \frac{1}{2} \left( 1 + j(rn) \right). \tag{19} \]
This distribution coincides with the quantum one.

4 EPR PARADOX, BELLS INEQUALITY

In their famous work [13], Einstein, Podolsky, and Rosen formulated the principles to be satisfied by a complete physical theory: (a) "each element of physical reality must have a counterpart in a complete physical theory" and (b) "if we are able with certainty (i.e., with a unit probability) to predict the value of a physical quantity without perturbing a system, then it means that there is an element in the real world that corresponds to this quantity."

The standard mathematical apparatus of quantum mechanics does not meet this requirement. An individual experiment happens to have no adequate counterpart in this approach. This leads to paradoxical conclusions if one works within the framework of the standard formalism of quantum mechanics. One such conclusion is the renowned Einstein-Podolsky-Rosen (EPR) paradox. In the original work [13] this paradox was formulated using an example with coordinate and momentum measurements, whereas Bohm proposed a simpler physical model [14] in which the same old problem is discussed using the example of measurement of spin projections onto various directions. Here we shall be concerned with the Bohm model.

It appears as follows. A zero-spin particle decays into two half-spin particles \( A \) and \( B \), which are scattered far apart. The spin state of this system is described by the following formula of standard quantum mechanics:
\[ |\Psi\rangle = \frac{1}{\sqrt{2}} \left[ |A_z^{(+)}\rangle |B_z^{(-)}\rangle - |A_z^{(-)}\rangle |B_z^{(+)}\rangle \right], \tag{20} \]
where \( |A_z^{(\pm)}\rangle \), \( |B_z^{(\pm)}\rangle \) are the eigenvectors of the operators of spin projections onto the \( z \) axis with eigenvalues of \( +1/2 \) and \( -1/2 \), respectively. This is the so-called entangled state, in which neither particle \( A \), nor \( B \) has a certain value for the spin projection onto \( z \) axis.

Once the \( A \) and \( B \) particles move far apart, the \( B \) particles spin projection onto the \( z \) axis is measured. Let the result of this measurement be \( +1/2 \). Then, according to the postulate of a quantum state collapse (projection principle), the state \( |\Psi\rangle \) is instantly replaced by the state \( |\tilde{\Psi}\rangle = -|A_z^{(-)}\rangle |B_z^{(+)}\rangle \). This means that the subsequent measurement of the \( A \) particles spin projection onto \( z \) axis the value \( -1/2 \) will be obtained with unit probability, thus describing the experiment very well. However, from the physics point of view, this result seems to be paradoxical.
Indeed, the following reasoning appears to be most natural: at the moment of decay, the particles $A$ and $B$ acquire certain spin projections onto the $z$ axis (opposite in sign), but until $B$'s spin projection is measured we do not know their definite values. Once $B$'s spin projection is measured, that of the $A$ particle is automatically known. However, such an explanation contradicts the general concept of the standard quantum mechanics.

The point is that the same quantum state $|\Psi\rangle$ can be represented in the form

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left[ |A_x^{(+)}\rangle |B_x^{(-)}\rangle - |A_x^{(-)}\rangle |B_x^{(+)}\rangle \right],$$

where the notation is the same as in formula (20), except for the projection onto the $x$ rather than the $z$ axis. Now we can repeat all the arguments given after formula (20), with the $z$ axis replaced by the $x$ one. As a result, we obtain that, at the moment of decay, the particles must acquire definite values for the spin projections onto the $x$ axis. However, the observables corresponding to the spin projections onto the $z$ and $x$ axes are mutually incompatible and, therefore, cannot simultaneously have definite values, according to standard quantum mechanics.

An alternative might look as follows. After decay the particles $A$ and $B$ did not acquire definite values for the spin projections onto any axis. Once those projections onto certain axis were measured, they acquired definite values for the former. It is not difficult to imagine that this mechanism is feasible for the particle $B$ that interacted with the instrument. On the other hand, the particle $A$ is in the spacelike region with respect to the instrument and, as such, cannot be influenced by the measurement process without violating the principles of the special theory of relativity. Thus, both ways of explaining the physical mechanism appear to be inconsistent, and this is exactly what is called the paradox.

In arguing with the authors of [13], Bohr wrote in article [15] that they incorrectly understood the notion of "physical reality." According to Bohr, it is impossible for a system with correlations to be considered as consisting of two separate physical realities. Those correlations are physical realities as well. Therefore, any measurement involving part of a system should be considered as a measurement involving the whole system. However, Bohr failed to back up his argument by providing any clear physical interpretation. Of course, one can comfort oneself with the fact that quantum mechanics is intrinsically not interpretable in a commonsense way; nonetheless, dissatisfaction with the situation still remains.

To circumvent this difficulty, Fock suggested considering that in the quantum case the concept of a "state" should not be ascribed objective meaning [16]. Rather, it should be understood as "information about the state." Here the question arises "is there something objective we receive the information from?" In the approach described in the second section of this article, it is suggested to consider an elementary state to be that "something objective."

In such a case, the physical phenomenon leading to the EPR paradox can be given clear physical interpretation. Once an initial particle has decayed, the physical system is characterized by stable (zero) values of the observables $\hat{S}_n$ (a total spin projection onto $n$). As a result, the values of observables $\hat{A}_n$ and $\hat{B}_n$ (spin projections onto $n$ direction for $A$ and $B$ particles, respectively) satisfy the following relation:

$$A_n + B_n = S_n = 0.$$  \hspace{1cm} (21)

In contrast to the quantum state, incompatible observables in the elementary state may simultaneously have definite values, though the latter cannot simultaneously be measured.
by means of a classical instrument. In the experiment we can measure the observable \( \hat{B}_n \) for an arbitrary (although only one) \( n \) direction, because, for different directions \( n \) and \( n' \) the observables \( \hat{B}_n \) and \( \hat{B}_{n'} \) are incompatible. By virtue of equality (21), we automatically measure the value of the observable \( A_n \) in such a measurement. This is so-called indirect measurement. Thus, in such an approach, the EPR paradox is trivially resolved.

One of the most frequently cited arguments in favor of physical reality in the EPR sense being nonexistent is violation of Bells inequality \[17, 18\]. Bell derived his inequality inspired by the EPR proposition. After Bell numerous versions of analogous inequalities were proposed. We shall focus on the version proposed in \[19\].

In this work the same physical system as in the present article dealing with the EPR paradox is studied. A zero-spin particle is considered that decays into two half-spin particles \( A \) and \( B \). These particles fly apart to a large distance and are detected by the instruments \( D_a \) and \( D_b \), respectively. The instrument \( D_a \) measures the spin projection of particle \( A \) onto the \( a \) direction, while \( D_b \) does the same over the \( B \) particle. Corresponding observables will be denoted by \( \hat{A} \) and \( \hat{B} \), while the results of measurement are denoted by by \( A_a \) and \( B_b \).

Suppose that the state of the initial particle is characterized by some physical reality that can be denoted by the parameter \( \lambda \). This same parameter will be used to describe physical realities that characterize decay products. Respectively, the results of measurement of the observables \( \hat{A} \) and \( \hat{B} \) can be considered to be the functions \( A_a(\lambda) \) and \( B_b(\lambda) \) of the parameter \( \lambda \).

Let the event distribution as a function of the parameter \( \lambda \) be characterized by the probabilistic measure \( P(\lambda) \):

\[
\int P(\lambda) = 1, \quad 0 \leq P(\lambda) \leq 1.
\]

Introduce the correlation function \( E(a, b) \):

\[
E(a, b) = \int P(\lambda) A_a(\lambda) B_b(\lambda)
\]

and consider the following combination:

\[
M = |E(a, b) - E(a', b')| + |E(a', b) + E(a', b')| = \quad (22)
\]

\[
= \left| \int P(\lambda) A_a(\lambda) [B_b(\lambda) - B_b'(\lambda)] \right| + \left| \int P(\lambda) A_a'(\lambda) [B_b(\lambda) + B_b'(\lambda)] \right|.
\]

For any directions \( a \) and \( b \)

\[
A_a(\lambda) = \pm 1/2, \quad B_b(\lambda) = \pm 1/2.
\]

Therefore,

\[
M \leq \int P(\lambda) [A_a(\lambda) |B_b(\lambda) - B_b'(\lambda)| + A_a'(\lambda) |B_b(\lambda) + B_b'(\lambda)| = \quad (23)
\]

\[
= 1/2 \int P(\lambda) [B_b(\lambda) - B_b'(\lambda)| + |B_b(\lambda) + B_b'(\lambda)|].
\]

Due to equalities (24), one of the expressions

\[
|B_b(\lambda) - B_b'(\lambda)|, \quad |B_b(\lambda) + B_b'(\lambda)|
\]
for arbitrary $\lambda$ is equal to zero, while the other is equal to unity. Note that both expressions contain the same value of $\lambda$.

Taking into account the property of expressions (26), Bells inequality is deduced from (25),

$$M \leq 1/2 \int P(d\lambda) = 1/2. \quad (27)$$

In standard quantum mechanics, the correlation function is easily calculated to yield

$$E(a, b) = -1/4 \cos \theta_{ab},$$

where $\theta_{ab}$ is the angle between the directions $a$ and $b$. For the directions $a = 0$, $b = \pi/8$, $a' = \pi/4$, and $b' = 3\pi/8$ we have

$$M = 1/\sqrt{2},$$

which contradicts inequality (27).

The results of the experiment [20] are consistent with quantum mechanical calculations and do not confirm Bells inequality. As a rule, these results are viewed as evidence that a quantum mechanical system does not correspond to any physical reality that would pre-determine the results of the measurement.

Now we shall see that, in fact, application of probability theory to quantum systems does not allow for such a derivation of Bells inequality.

Since, in the quantum case, $\sigma$-algebra and, respectively, the probabilistic measure depend on the instrument used, the replacement $P(d\lambda) \rightarrow P_{A}\hat{B}(d\varphi)$ should be done in definition (22). And if we consider the correlation function $E(a', b')$, then in (22) the replacement $P(d\lambda) \rightarrow P_{\hat{A}'\hat{B}'}(d\varphi)$ is in order. Although in both cases the same symbol $d\varphi$ is used to denote an elementary volume in the space of elementary states, it should be borne in mind that the set of elementary states corresponding to $d\varphi$ will be different. The point is that these sets must be elements of $\sigma$. If the observables $\hat{A}$ and $\hat{B}$ are incompatible with the observables $\hat{A}'$ and $\hat{B}'$, $\sigma$-algebra will be different. Moreover, there is no physically justified $\sigma$-algebra the subalgebras of which could be identified with these algebras.

In addition, in the process of determining the value of the correlation function $E(a, b)$ in the experiment we deal with a random countable sample $\{\varphi\}_{\hat{A}\hat{B}}$ selected out of the total probability space $\Omega(\varphi_{\xi})$, rather than with that complete space itself. Finally, formula (22) must be replaced by

$$E(a, b) = \int_{\{\varphi\}_{\hat{A}\hat{B}}} P_{\hat{A}B}(d\varphi) \varphi(\hat{A}\hat{B}).$$

Accordingly, formula (23) takes the following form:

$$M = \left| \int_{\{\varphi\}_{\hat{A}\hat{B}}} P_{\hat{A}\hat{B}} d\varphi(\hat{A}\hat{B}) - \int_{\{\varphi\}_{\hat{A}\hat{B}'}} P_{\hat{A}\hat{B'}} d\varphi(\hat{A}\hat{B'}) \right| +$$

$$+ \left| \int_{\{\varphi\}_{\hat{A}'\hat{B}}} P_{\hat{A}'\hat{B}} d\varphi(\hat{A}'\hat{B}) - \int_{\{\varphi\}_{\hat{A}'\hat{B}'}} P_{\hat{A}'\hat{B}'} d\varphi(\hat{A}'\hat{B}') \right|.$$

If the directions $a$ and $a'$ ($b$ and $b'$) are not parallel to each other, then the observables $\hat{A}\hat{B}$, $\hat{A}'\hat{B}'$, $\hat{A}'\hat{B}$, and $\hat{A}'\hat{B}'$ are mutually incompatible. Hence, there is no universal physically justified $\sigma$-algebra, which would correspond to the measurement of all these observables.
Moreover, since the sets \( \{ \varphi \}_{\xi}^{AB}, \{ \varphi \}^{AB'}_{\xi}, \{ \varphi \}^{A'B}_{\xi} \) are different random samples from the continuum space \( \Omega(\varphi_{\xi}) \), the probability of their pairwise intersection is equal to zero. Consequently, the probability of combinations of type (26) taking place is equal to zero as well. As a result, the arguments that led to inequality (27) turn out to be incorrect for the elementary states.

Thus, in a quantum case, the hypothesis of the existence of an elementary state does not lead to Bell's inequality. Therefore, numerous experimental verifications of this latter inequality conducted so far become largely devoid of a theoretical basis.

## 5 SINGLET STATE SIMULATION

In the previous section, it was shown that the usually cited proofs of Bell's inequality are substantially flawed, for they do not account for a very important requirement of probability theory: that the space of elementary events must be countable. Otherwise, the very concept of probability itself loses a strict mathematical sense. We shall see in this section that it is possible to construct a macroscopic model in which Bell's inequality will be violated.

To this end, let us construct a singlet state of a two-particle system. A characteristic feature of a singlet state is that equality (21) always holds during measurement of the spin projection onto an arbitrary \( n \) direction.

This equality assumes a hard correlation between the results of measurements for the first and second particles regardless of how far they are located from each other at the moment of measurement. In the approach to follow in the present article, this assumes a rigid correlation between the elementary states of the first and second particles. In terms of the ESS, such a correlation is easily noticed: the ESS for the second particle must be a negative copy of the ESS for the first one. However, it is not clear in the case of a singlet state what kind of the distribution in a set of ESSs for the first particle should look like. It is simpler to pin down such a distribution by using the MGS framework. Keeping in mind the relation between the MGS and ESS mentioned earlier, it can be expected that, in a singlet state, the MGS for the second particle will be a mirror image of the MGS for the first one. By this we mean that the layers in the MGS for the second particle mimic those for the first one up to a change of orientation to the opposite, i.e.,

\[
R_{1}^{(k)} + R_{2}^{(k)} = 0,
\]

(28)

Here \( R_{1}^{(k)} \) and \( R_{2}^{(k)} \) are the orientation vectors of the \( k \)-th layer for the first and second particles, respectively. In addition, the functions \( \varepsilon^{(k)}(r) \) for these particles satisfy the condition \( \varepsilon_{1}^{(k)}(r) = \varepsilon_{2}^{(k)}(-r) \). The latter condition, along with equality (28), means that the numbers of active layers in MGS for the first and second particles coincide.

From this it immediately follows that the equality \( R_{1}^{(k)} r = -R_{2}^{(k)} r \) holds for arbitrary \( r \), which, in turn, means that \( j_{1} = -j_{2} \). That is, regardless of the distance between particles, the spin projections onto the arbitrary direction of \( r \) registered at the moment of measurement satisfy relation (21). This is the way the EPR paradox is realized. The correlation in (21) does not appear at the moment of measurement; instead, it is a consequence of the MGS ensemble structure that corresponds to a singlet state. This means that the correlation appears at the moment the ensemble is formed.

Note that this state of affairs does not contradict Einstein's remark on the incompleteness of quantum mechanics. In the model proposed, the traditional mathematical apparatus of
quantum mechanics is supplemented by a novel concept — that of an elementary state.

Due to strong correlation between MGSs, it is sufficient to define a set of elementary states for just one of two particles, say, for the first one, so as to fix a set of elementary states corresponding to a singlet quantum state.

We assume that the elementary states of the first particle are described by MGSs that are elements of the set $\Upsilon$ and satisfy the following conditions. The radius-vectors $R^{(k)}$ in each MGS, which set a layer orientation, are randomly distributed on the entire sphere $\mathfrak{R}$. The layer orientations in the individual MGSs are distributed independently of each other.

Just like in a single-particle system, an elementary state (elementary event) of a two-particle system cannot be ascribed any probabilistic measure. Probability can only be attributed to events that are elements of some $\sigma$-algebra. This latter algebra will be constructed in analogy to the single-particle case.

Demand that the generatrices of $\sigma$-algebra satisfy the following conditions.

1. The generatrices form a subset of the set of two-particle elementary states.

2. Every two-particle elementary state is described by two MGSs.

3. In each of such MGSs, the conditions $R_1^{(k') r} = -R_2^{(k') r}$ and $\varepsilon_1^{(k') r} = \varepsilon_2^{(k') r}$ are satisfied for the layers with the same number $k'$.

4. $r_1$ and $r_2$ are fixed; $r_1(r_2)$ is the direction in which the spin projection of the first (second) particle is measured. These vectors $r_1$ and $r_2$ fix $\sigma$-$. The remaining conditions define elements of this algebra.

5. The interval $d\varepsilon, -1/2 < \varepsilon < +1/2$ is fixed for the first particle. Either $\varepsilon^{(k') r_1} \in d\varepsilon, -\varepsilon^{(k') r_1} \in d\varepsilon$. Both options correspond to the same subset (the same element of $\sigma$-algebra).

6. The number $k$ of active layer is fixed.

7. Each subset of elementary states for the first particle consists of all MGSs that belong to $\Upsilon$ the layers of which satisfy one of the following conditions:

$$|R_1^{(k') r_1} + \varepsilon^{(k') r_1}| > 1/2 \quad k' = k, \quad (29)$$

$$|R_1^{(k') r_1} + \varepsilon^{(k') r_1}| \leq 1/2 \quad k' \leq k \quad (k' \text{ fixed}). \quad (30)$$

These two conditions (29) and (30) correspond to two different subsets.

8. In the case of (29), the subsets differ in two more parameters, $j_1$ and $j_2$: $j_1 = +1$, if $R_1^{(k) r_1} > 0$; $j_1 = -1$, if $R_1^{(k) r_1} < 0$; $j_2 = +1$, if $R_2^{(k) r_2} > 0$; $j_2 = -1$, if $R_2^{(k) r_2} < 0$.

Similarly to the single-particle case, it is easy to verify that any elementary event from the set of singlet elementary states belongs to any subset listed above irrespective of $r_1$ and $r_2$.

Now, construct the probabilistic measure for each such subset assuming that the random values of $R_1$ are evenly distributed on a sphere $\mathfrak{R}$, while the random values of $\varepsilon$ are within the interval $(-1/2, +1/2)$.
Let \( r_1, r_2, d\varepsilon, j_1 \) and \( j_2 \) be fixed and \( k' = 1 \). Then, the probability for inequality (29) to be satisfied is described by the following expression:

\[
P^{(1)}(r_1, r_2, \varepsilon, j_1, j_2) d\varepsilon = d\varepsilon N/2 \int dR \left[ \Theta[j_1(Rr_1 + \varepsilon) - 1/2] + \Theta[j_1(Rr_1 - \varepsilon) - 1/2] \right]
\]

\[
= d\varepsilon N/2 \int dR \left[ \Theta[\Theta[j_1(Rr_1 + \varepsilon) - 1/2] + \Theta[\Theta[j_1(Rr_1 - \varepsilon) - 1/2] \right].
\]

From the latter we obtain

\[
P^{(1)}(r_1, r_2, \varepsilon) = \sum_{j_1, j_2} P^{(1)}(r_1, r_2, \varepsilon, j_1, j_2) = N2\pi.
\]

The probability for inequality (30) to be realized is given as follows:

\[
\tilde{P}^{(1)}(r_1, r_2, \varepsilon) d\varepsilon = d\varepsilon N/2 \int dR \left[ \Theta[1/2 + Rr_1 + \varepsilon] \Theta[1/2 - Rr_1 - \varepsilon] + \Theta[1/2 + Rr_1 - \varepsilon] \Theta[1/2 - Rr_1 + \varepsilon] \right] = N2\pi d\varepsilon.
\]

From (31) and (32), we deduce

\[
N = (4\pi)^{-1}, \quad \tilde{P}^{(1)}(r_1, r_2, \varepsilon) = 1/2.
\]

Furthermore, repeating the calculations carried out for a single-particle system, we obtain that, for fixed \( r_1, r_2, d\varepsilon, j_1, \) and \( j_2 \) the probability for the active layer to have the number \( k \) is described by the formula

\[
P^{(k)}(r_1, r_2, \varepsilon, j_1, j_2) = \frac{1}{4\pi^2} \int dR \Theta[-j_1, j_2 Rr_2] \left[ \Theta[Rr_1 + \varepsilon - 1/2] + \Theta[Rr_1 - \varepsilon - 1/2] \right],
\]

while the probability of having a number greater than \( k \) is given by

\[
\tilde{P}^{(k)}(r_1, r_2, \varepsilon) = 2^{-k}.
\]

Formulas (33) and (34) define the probabilistic measures for the generatrices of \( \sigma \)-algebra Proceeding with calculations along the lines of (17) and (18), we find the expression

\[
P(r_1, r_2, j_1, j_2) = \frac{1}{4} \left( 1 - j_1 j_2(r_1 r_2) \right),
\]

which describes the probability of finding the spin projection of the first particle onto the \( r_1 \) direction equal to \( j_1/2 \), while for the second particle it is found to be \( j_2/2 \).

From formula (35) we obtain for the correlation function

\[
E(r_1, r_2) = \frac{1}{4} \sum_{j_1, j_2} j_1 j_2 P(r_1, r_2, j_1, j_2) = -\frac{1}{4}(r_1 r_2).
\]

The distribution given in (35) and the correlation function in (36) coincide with those obtained in the standard quantum mechanics and violate Bells inequality. For \( r_1 = r_2 \) and \( j_1 = j_2 \), the right hand side of (35) vanishes. This corresponds to the EPR paradox.
A few words on the locality issue in the model proposed are in order. The reason behind correlations between results for the first and second particles appears at the time of shaping out a singlet state rather than at the moment of measurement. Specifically, this correlation is due to the fact that the numbers of active layers in MGSs for both particles in each elementary event coincide. Of course, it is assumed that each of the two instruments can independently detect this active layer for its own particle.

In this article it is described in some detail how the first instrument can find an active layer by using the functions \( \varepsilon^{(k')}(r) \), while for the second particle it is only stated that its active layer has the same number. In order to render the measurement processes for the first and second particles completely independent, we can take the following approach. Starting from the MGS (and the functions \( \varepsilon^{(k')}(r) \)) studied for the first particle, we construct a corresponding ESS according to the technique given above. For the second particle, the ESS is constructed as a negative copy of the ESS for the first one. All these operations are related to the process of shaping of the state studied, rather than that of a measurement. By using the ESSs of individual particles shaped in this manner, each of the instruments can independently measure the spin projection onto a desired direction.

Such a scheme of measurement is not convenient for computer simulation. More suitable is a scheme that simulates the experiment with a so-called delayed choice. It can be outlined as follows. In the elementary event considered, the number of active layer is fixed for the first particle along, with measurement of the spin projection onto the \( r_1 \) direction. For the second particle in the same elementary event, the inequalities \( R_2r_2 > 0 \) and \( R_2r_2 < 0 \) are verified for a sufficiently large number of layers and the corresponding values of \( j_2 \) with the indicated number of layer are fixed. The accuracy of the final result depends on the number of layers checked. Since the contribution to this result rapidly decreases with increasing number of active layers, this accuracy promptly increases with an increase in the number of layers checked.

Once the observational part of the experiment is over, the measurement data processing stage is set to commence. If the goal of the experiment is to reveal certain correlations between results of measurements for the first and second particles, then the latter must be stored in one place. In this sense this stage will be necessarily nonlocal. However, this is not a distinctive feature of the quantum experiment; the same is true of the classical one.

In the case at hand, only those values of \( j_2 \) obtained for different layers should be chosen while processing the data, which correspond to the layer whose number coincides with that of an active layer found by the observation with the first particle. This procedure can be viewed as a simulation of the coincidence process that is necessary in such a type of experiment. Thus, the entire nonlocality is exclusively associated with processing of the measurement data.

In the process of developing the computer simulation procedure, it became clear that a finer partition of the set of elementary events into subsets of probabilistic events is possible. Namely, events corresponding to \( \varepsilon(r) \in d\varepsilon \) and \( -\varepsilon(r) \in d\varepsilon \) can be regarded as belonging to different elements of \( \sigma \)-algebra. Such a \( \sigma \)-algebra is convenient for computer simulation, while for analytical evaluations it is less favorable.
6 CONCLUSIONS

The procedure described can be considered as a working model of quantum measurement, rather than a mere imitation of it. Consequently, this model is quite suitable for conducting experiments with the aim of verifying any statements about quantum systems.

Like any particular model, it is not very well suited for substantiating positive statements, i.e., for justifying the fact that the statement expressed is unconditionally correct. On the other hand, it is perfectly suited for negative statements, i.e., for disproving that a certain statement is correct.

In particular, this model refutes the frequently adduced statement that experimentally observed violation of Bell's inequality proves the absence of a local physical reality that is a source for quantum phenomena.

The model proposed might appear to be very useful for conducting experiments with so-called quantum teleportation (see, for example, [21]). Hopefully, it will help demystify this physical phenomenon. For more detail, refer to [22].

On the other hand, for the purposes of quantum cryptography, the model proposed is only partially suitable. It can be used for experimental verification of any statements made in this field. However, this model is not suitable for practical use in quantum cryptography. The issue here is that the latter is based on the existence in quantum systems of observables that can in no way be simultaneously measured, whereas in the model proposed there are observables that are incompatible only with respect to a particular measurement process.

References

[1] A. Petersen, Bull. Atomic Sci. 19, 8 (1963).
[2] D. A. Slavnov, Phys. Part. Nucl. 38, 147 (2007).
[3] D. A. Slavnov, Theor. Math. Phys. 149, 1690 (2006).
[4] G. Emch, Algebraic Methods in Statistical Mechanics and Quantum Field Theory (Wiley, New York, 1972).
[5] S. S. Khoruzhii, Introduction to Algebraic Quantum Field Theory (Kluwer Academic, Dordrecht, Boston, 1990).
[6] N. N. Bogolyubov et al., General Principles of Quantum Field Theory (Kluwer, Dordrecht, 1990).
[7] J. Dixmier, C*-algebra et leurs Representations (Gauthier-Villars, 1969).
[8] A. N. Kolmogorov, Foundations of the Theory of Probability (Chelsea, New York, 1956).
[9] J. Neveu, Mathematical Foundations of the Calculus of Probability (Holden-Day, San Francisco, CA, 1965).
[10] M. A. Neimark, Normed Algebras (Wolters-Noordhoff, Groningen, 1972).
[11] J. Von Neumann, Mathematical Foundations of Quantum Mechanics (Princeton Univ. Press, 1955).
[12] Matzkin A. Local hidden-variables can account for EPR quantum correlatios, arXiv: quant-ph/0703271.
[13] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev., Ser. 2, 47, 777 (1935).
[14] D. Bohm, *Quantum Theory* (Dover, New York, 1990).

[15] N. Bohr, *Discussion with Einstein on Epistemological Problems in Atomic Physics* (in Albert Einstein: Philosopher-Scientist, Ed. by PA. Schlipp (La Salle, Open Court, 1949), P. 201).

[16] V. A. Fock, Usp. Fiz. Nauk 16, 436 (1936).

[17] J. S. Bell, Physics 1, 195 (1965).

[18] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics: Collected Paper on Quantum Philosophy* (Cambridge Univ., Cambridge, 1993).

[19] J. F. Clauser, M. A. Horne, A. Shimony, et al., Phys. Rev. Lett. 23, 880 (1969).

[20] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. 49, 1804 (1982).

[21] D. Bouwmeester, J.-W. Pan, H. Weinfurter, and A. Zeilinger, Experimental Quantum Teleportation, in *The Physics of Quantum Information*, (Ed. by D. Bouwmeester, A. Ekert, and A. Zeilinger (Springer, Heidelberg 2000).

[22] D. A. Slavnov, Theor. Math. Phys. 157, 1433 (2008).