A determinantal formula for the hyper-sums of powers of integers

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Abstract

For non-negative integers \( r \) and \( m \), let \( S_m^{(r)}(n) \) denote the \( r \)-fold summation (or hyper-sum) over the first \( n \) positive integers to the \( m \)th powers, with the initial condition \( S_m^{(0)}(n) = n^m \). In this paper, we derive a new determinantal formula for \( S_m^{(r)}(n) \). Specifically, we show that, for all integers \( r \geq 0 \) and \( m \geq 1 \), \( S_m^{(r)}(n) \) is proportional to \( S_m^{(r)}(1)(n) \) times the determinant of a lower Hessenberg matrix of order \( m - 1 \) involving the Bernoulli numbers and the variable \( N_r = n + \frac{r}{2} \). Furthermore, whenever \( r \geq 1 \), evaluating this determinant gives us \( S_m^{(r)}(n) \) as \( S_m^{(r)}(1)(n) \) times an even or odd polynomial in \( N_r \) of degree \( m - 1 \), depending on whether \( m \) is odd or even.

1 Introduction

For integers \( m, r \geq 0 \) and \( n \geq 1 \), the hyper-sums of powers of integers \( S_m^{(r)}(n) \) are defined recursively as

\[
S_m^{(r)}(n) = \begin{cases} 
n^m & \text{if } r = 0, \\
\sum_{i=1}^{n} S_m^{(r-1)}(i) & \text{if } r \geq 1.
\end{cases}
\]

In particular, \( S_m^{(1)}(n) = \sum_{i=1}^{n} S_m^{(0)}(i) \) is the ordinary sum of powers of integers \( S_m(n) = 1^m + 2^m + \cdots + n^m \). Furthermore, we have [9, p. 281]

\[
S_1^{(r)}(n) = \binom{n + r}{r + 1}, \quad \text{and} \quad S_2^{(r)}(n) = \frac{2n + r}{r + 2} S_1^{(r)}(n). \tag{1}
\]

For its intrinsic interest, next we give an explicit representation of \( S_m^{(r)}(n) \) as a polynomial in \( n \) of degree \( m + r \) without constant term. To this end, we first recall that the hyper-sums \( S_m^{(r+1)}(n) \), \( S_m^{(r)}(n) \), and \( S_m^{(r)}(n) \) satisfy the recurrence relation [2, Theorem 1]

\[
S_m^{(r+1)}(n) = \frac{n + r}{r} S_m^{(r)}(n) - \frac{1}{r} S_m^{(r)}(n+1), \tag{2}
\]

which applies to any integers \( m \geq 0 \) and \( r \geq 1 \). (For the sake of completeness, we provide a proof of the recurrence (2) in the appendix.) Solving this recurrence gives us [2, Theorem 3]

\[
S_m^{(r+1)}(n) = \frac{1}{r!} \sum_{i=0}^{r} (-1)^i q_{r,i}(n) S_m^{(r)}(n+i), \tag{3}
\]

where \( q_{r,i}(n) \) is the following polynomial in \( n \) of degree \( r - i \):

\[
q_{r,i}(n) = \sum_{j=0}^{r-i} \binom{i + j}{i} \binom{r + 1}{i + j + 1} n^j, \tag{4}
\]
and the \( \binom{m}{n} \)'s are the (unsigned) Stirling numbers of the first kind. (Note that (3) holds for \( r = 0 \) if we set \( q_{0,0}(n) = 1 \). Let us also observe that \( q_{r,i}(n) \) is equal to \( \binom{\lfloor r+n+1 \rfloor}{i+n+1} \) \( n+1 \), where \( \binom{m}{n} \) denotes the \( r \)-Stirling numbers of the first kind [1]. The formula \( S_{m+1}(n) = 1 \over \pi^2 \sum_{i=0}^{r} (-1)^i \binom{\lfloor r+n+1 \rfloor}{i+n+1} S_{m+i}(n) \) has been derived recently and independently by Kargın [8, Equation (4.7)].)

On the other hand, as is well-known, the power sum polynomial \( S_{m+i}(n) \) can be expressed by means of the Bernoulli formula

\[
S_{m+i}(n) = \frac{1}{m+i+1} \sum_{t=1}^{m+i+1} (-1)^{m+i+1-t} \binom{m+i+1}{t} B_{m+i+1-t} n^t,
\]

where the \( B_j \)'s are the Bernoulli numbers \( B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, \) etc. Thus, by substituting (4) and (5) into (3), it can be shown that

\[
S^{(r)}_m(n) = \sum_{k=1}^{m+r} c_{m,r}^k n^k,
\]

where the coefficient \( c_{m,r}^k \) is given by

\[
c_{m,r}^k = \frac{(-1)^{m+1-k} r^{-1} \sum_{i=0}^{r-1} \sum_{j=0}^{k-1} (-1)^{j} \binom{m+i+1}{i} \binom{m+i+1}{k-j} \left[ \binom{r}{i+j+1} \right] B_{m+i+j+1-k}}{(r-1)!(r-i)!}
\]

for integers \( m \geq 0, r \geq 1, \) and \( k = 1, 2, \ldots, m + r \), and where \( B_i = 0 \) whenever \( t \) is a negative integer. In particular, we find that

\[
c_{m,r}^1 = \frac{(-1)^{m} \sum_{i=0}^{r-1} \left[ \frac{r}{i+1} \right] B_{m+i}}{(r-1)!(r-i)!}
\]

**Remark 1.** From (2), it is immediate to derive the following recurrence relation between the coefficients of the hyper-sums polynomials \( S^{(r+1)}_m(n), S^{(r)}_m(n) \), and \( S^{(r)}_{m+1}(n) \), namely,

\[
c_{m,r+1}^k = c_{m,r}^k + \frac{1}{r} \left( c_{m,r}^{k-1} - c_{m+1,r}^k \right),
\]

with \( m \geq 0, r \geq 1, \) and \( k = 1, 2, \ldots, m + r + 1, \) and where it is understood that \( c_{m,r}^0 = c_{m+1,r}^0 = 0 \).

Since the work of the German mathematician Johann Faulhaber (1580-1635), it is known that \( S^{(r)}_{2m-1}(n) \) [respectively, \( S^{(r)}_{2m}(n) \)] can alternatively be expressed as \( S^{(r)}_1(n) \) \( S^{(r)}_2(n) \)\( S^{(r)}(n) \) times a polynomial in \( n(n + r) \). This result was rigorously proved by Knuth (see [9, Theorem, p. 280]), and can be established as the following theorem.

**Theorem 1** (Faulhaber-Knuth). For all positive integers \( r \) and \( m \), there exist polynomials \( F^{(r)}_{2m-1} \) and \( F^{(r)}_{2m} \) in \( n(n + r) \) of degree \( m - 1 \) such that

\[
S^{(r)}_{2m-1}(n) = S^{(r)}_1(n) F^{(r)}_{2m-1}(n(n + r)), \quad \text{and} \quad S^{(r)}_{2m}(n) = S^{(r)}_2(n) F^{(r)}_{2m}(n(n + r)).
\]

It is important to notice that, because of the relation \( n(n + r) = \left( n + \frac{r}{2} \right)^2 - \frac{1}{4} r^2 \), the above theorem can equivalently be stated as follows.
**Theorem 2** (Faulhaber-Knuth). For all positive integers \( r \) and \( m \), there exist even polynomials \( K_{2m-1}^{(r)} \) and \( K_{2m}^{(r)} \) in \( N_r = n + \frac{r}{2} \) of degree \( 2m - 2 \) such that

\[
S_{2m-1}^{(r)}(n) = S_1^{(r)}(n) K_{2m-1}^{(r)}(N_r), \quad \text{and} \quad S_{2m}^{(r)}(n) = S_2^{(r)}(n) K_{2m}^{(r)}(N_r).
\]

**Remark 2.** Since \( S_2^{(r)}(n) \) is proportional to \( N_r S_1^{(r)}(n) \), from Theorem 2 it follows that \( S_{2m}^{(r)}(n) \) can equally be expressed as \( S_1^{(r)}(n) \) times an odd polynomial in \( N_r \) of degree \( 2m - 1 \).

In view of Theorem 2 and Remark 2, the Faulhaber-Knuth result can be formulated in another equivalent way, as stated in the next theorem.

**Theorem 3** (Faulhaber-Knuth). For all positive integers \( r \) and \( m \), there exist polynomials \( G_m^{(r)} \) in \( N_r = n + \frac{r}{2} \) of degree \( m - 1 \) such that

\[
S_m^{(r)}(n) = S_1^{(r)}(n) G_m^{(r)}(N_r),
\]

where the \( G_m^{(r)} \)'s are even or odd, according as \( m \) is odd or even.

In this paper, we give a determinantal version of Theorem 3. More precisely, in section 2, we show that, for all integers \( r \geq 0 \) and \( m \geq 1 \), \( S_m^{(r)}(n) \) can be expressed (leaving aside a factor involving \( m \) and \( r \)) as \( S_1^{(r)}(n) \) times a determinant of order \( m - 1 \) depending on \( N_r \) (see Theorem 4 below). In other words, we provide an explicit determinantal formula for the polynomials \( G_m^{(r)}(N_r) \). Furthermore, in section 3, we offer a new proof of the fact that, whenever \( r \geq 1 \), the polynomials \( G_m^{(r)}(N_r) \) are even or odd depending on the parity of \( m \) (see Theorem 5 below).

### 2 A determinantal formula for \( S_m^{(r)}(n) \)

Next, we establish the following theorem which constitutes the main result of this paper.

**Theorem 4.** Let \( N_r = n + \frac{r}{2} \). Then, for all integers \( r \geq 0 \) and \( m \geq 1 \), \( S_m^{(r)}(n) \) can be expressed as

\[
S_m^{(r)}(n) = S_1^{(r)}(n) \frac{(-1)^{m-1}}{(r + 2)^{m-1}} \det H_m^{(r)}(N_r),
\]

where \( r^m = r(r+1) \ldots (r+m-1) \) and \( r^m = 1 \), and where \( H_m^{(r)}(N_r) \) is the following lower Hessenberg matrix of order \( m - 1 \):

\[
H_m^{(r)}(N_r) = \begin{pmatrix}
-2N_r & r + 2 & 0 & \ldots & \ldots & 0 \\
(-1)^1 B_2 & -3N_r & r + 3 & 0 & \ldots & 0 \\
(-1)^2 B_3 & (-1)^2 B_2 & -4N_r & r + 4 & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & 0 \\
(-1)^{m-1} B_{m-2} & (-1)^{m-1} B_{m-3} & (-1)^{m-1} B_{m-4} & \ldots & -(m-1)N_r & r + m - 1 \\
(-1)^{m-1} B_{m-1} & (-1)^{m-1} B_{m-2} & (-1)^{m-1} B_{m-3} & \ldots & (-1)^{m-1} B_{m-2} & -mN_r
\end{pmatrix},
\]

with the \( B_j \)'s denoting the Bernoulli numbers.
Remark 3. It is understood that $\det H_1^{(r)}(N_r) = 1$, so that (8) holds trivially for $m = 1$. On the other hand, it is easily seen that $\det H_m^{(0)}(N_0) = (-1)^{m-1}2^{m-1}n^{m-1}$, and then (8) gives us $S_m^{(0)}(n) = n^m$, as it should. Furthermore, from (8) it follows that $S_2^{(r)}(n) = \frac{2n+r}{r+2}S_1^{(r)}(n)$, in accordance with (1).

Remark 4. From (7) and (8), it follows immediately that, for all integers $r \geq 0$ and $m \geq 1$, the polynomials $G_m^{(r)}(N_r)$ are given by the determinantal formula

$$G_m^{(r)}(N_r) = \frac{(-1)^{m-1}}{(r+2)^{m-1}} \det H_m^{(r)}(N_r),$$

where $H_m^{(r)}(N_r)$ is the matrix (9), and $G_1^{(r)}(N_r) = 1$.

The proof of Theorem 4 is based on the following lemma.

Lemma 1. Let $N_r = n + \frac{r}{2}$. Then, for all integers $r \geq 0$ and $m \geq 2$, the following recurrence relation holds true

$$(m + r)S_m^{(r)}(n) = mN_rS_{m-1}^{(r)}(n) - r \sum_{k=1}^{m-2} \binom{m}{k} B_{m-k}S_k^{(r)}(n),$$

where the summation on the right-hand side is zero if $m = 2$.

Proof. We start with the Bernoulli formula for the power sum polynomial $S_m(i) = \sum_{j=1}^{i} j^{m-1}$, namely,

$$S_m(i) = \frac{1}{m}i^m + \frac{1}{2}i^{m-1} + \frac{1}{m} \sum_{k=1}^{m-2} \binom{m}{k} B_{m-k}i^k,$$

which applies to $m \geq 2$. Therefore, it follows that

$$mS_m(i) = i^m + \frac{1}{2}i^{m-1} + \sum_{k=1}^{m-2} \binom{m}{k} B_{m-k}i^k.$$

Summing from $i = 1$ to $n$ on both sides of this equation yields

$$mS_m^{(2)}(n) = S_m(n) + \frac{1}{2}mS_{m-1}(n) + \sum_{k=1}^{m-2} \binom{m}{k} B_{m-k}S_k(n).$$

Clearly, by iterating the procedure $r$ times, we get

$$mS_m^{(r+1)}(n) = S_m^{(r)}(n) + \frac{1}{2}mS_m^{(r)}(n) + \sum_{k=1}^{m-2} \binom{m}{k} B_{m-k}S_k^{(r)}(n).$$

Now, from (2) we have

$$S_m^{(r+1)}(n) = \frac{n+r}{r}S_m^{(r)}(n) - \frac{1}{r}S_m^{(r)}(n),$$

and hence, using (13) in (12), we quickly obtain (11).
For the proof of Theorem 4, rewrite (11) in the form

$$(r + j)S_j^{(r)}(n) = jN_r S_{j-1}^{(r)}(n) - r \sum_{k=1}^{j-2} \binom{j}{k} B_{j-k} S_k^{(r)}(n), \quad (14)$$

for integers $r \geq 0$ and $j \geq 2$. Thus, letting successively $j = 2, 3, 4, \ldots, m$ in (14) gives rise to the following system of equations in the unknowns $S_1^{(r)}(n), S_2^{(r)}(n), \ldots, S_m^{(r)}(n)$:

$$-2N_r S_1^{(r)}(n) + (r + 2) S_2^{(r)}(n) = 0,$$

$$r \binom{3}{1} B_2 S_1^{(r)}(n) - 3N_r S_2^{(r)}(n) + (r + 3) S_3^{(r)}(n) = 0,$$

$$r \binom{4}{1} B_3 S_1^{(r)}(n) + r \binom{4}{2} B_2 S_2^{(r)}(n) - 4N_r S_3^{(r)}(n) + (r + 4) S_4^{(r)}(n) = 0,$$

$$\vdots$$

$$r \binom{m}{1} B_{m-1} S_1^{(r)}(n) + r \binom{m}{2} B_{m-2} S_2^{(r)}(n) + \cdots + r \binom{m}{m-2} B_2 S_{m-2}^{(r)}(n) \quad \vdots$$

$$- mN_r S_{m-1}^{(r)}(n) + (r + m) S_m^{(r)}(n) = 0.$$

This system of equations augmented with the trivial relationship $(r + 1) S_1^{(r)}(n) = (r + 1) S_1^{(r)}(n)$ can be written in matrix form as

$$
\begin{pmatrix}
  r + 1 & 0 & 0 & \cdots & 0 & 0 \\
  -2N_r & r + 2 & 0 & \cdots & 0 & 0 \\
  r \binom{3}{1} B_2 & -3N_r & r + 3 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  r \binom{m-1}{1} B_{m-2} & r \binom{m-1}{2} B_{m-3} & r \binom{m-1}{3} B_{m-4} & \cdots & r + m - 1 & 0 \\
  r \binom{m}{1} B_{m-1} & r \binom{m}{2} B_{m-2} & r \binom{m}{3} B_{m-3} & \cdots & - mN_r & r + m \\
\end{pmatrix}
\begin{pmatrix}
  S_1^{(r)}(n) \\
  S_2^{(r)}(n) \\
  S_3^{(r)}(n) \\
  \vdots \\
  S_{m-1}^{(r)}(n) \\
  S_m^{(r)}(n)
\end{pmatrix}
= 
\begin{pmatrix}
  (r + 1) S_1^{(r)}(n) \\
  0 \\
  0 \\
  \vdots \\
  0 \\
  0
\end{pmatrix}.
$$

Thus, solving for $S_m^{(r)}(n)$ and applying Cramer’s rule to the above $m \times m$ lower triangular system of linear equations yields

$$S_m^{(r)}(n) = \frac{1}{(r + 1)^m} \begin{vmatrix}
  r + 1 & 0 & 0 & \cdots & 0 & (r + 1) S_1^{(r)}(n) \\
  -2N_r & r + 2 & 0 & \cdots & 0 & 0 \\
  r \binom{3}{1} B_2 & -3N_r & r + 3 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  r \binom{m-1}{1} B_{m-2} & r \binom{m-1}{2} B_{m-3} & r \binom{m-1}{3} B_{m-4} & \cdots & r + m - 1 & 0 \\
  r \binom{m}{1} B_{m-1} & r \binom{m}{2} B_{m-2} & r \binom{m}{3} B_{m-3} & \cdots & - mN_r & 0 \\
\end{vmatrix}.$$ 

Finally, by expanding the above determinant with respect to the last column, we get (8).

As a concrete example, next we quote the results obtained from (10) for $G_5^{(r)}(N_r)$ and $G_6^{(r)}(N_r)$,
namely,
\[
G_5^{(7)}(N_7) = \frac{1}{9^5} \begin{vmatrix} 
-2N_7 & 9 & 0 & 0 \\
\frac{7}{2} & -3N_7 & 10 & 0 \\
0 & 7 & -4N_7 & 11 \\
-\frac{7}{6} & 0 & \frac{35}{3} & -5N_7 \\
\end{vmatrix} = \frac{N_7^4}{99} - \frac{35N_7^2}{198} + \frac{7}{16},
\]
and
\[
G_6^{(7)}(N_7) = -\frac{1}{9^6} \begin{vmatrix} 
-2N_7 & 9 & 0 & 0 & 0 \\
\frac{7}{2} & -3N_7 & 10 & 0 & 0 \\
0 & 7 & -4N_7 & 11 & 0 \\
-\frac{7}{6} & 0 & \frac{35}{3} & -5N_7 & 12 \\
0 & -\frac{7}{2} & 0 & \frac{35}{2} & -6N_7 \\
\end{vmatrix} = \frac{2N_7^5}{429} - \frac{49N_7^3}{429} + \frac{6419N_7}{10296},
\]
from which it follows that
\[
S_5^{(7)}(n) = \frac{1}{1584} \left( n + \frac{7}{8} \right) \left[ 16 \left( n + \frac{7}{2} \right)^4 - 280 \left( n + \frac{7}{2} \right)^2 + 693 \right],
\]
and
\[
S_6^{(7)}(n) = \frac{1}{10296} \left( n + \frac{7}{8} \right) \left[ 48 \left( n + \frac{7}{2} \right)^5 - 1176 \left( n + \frac{7}{2} \right)^3 + 6419 \left( n + \frac{7}{2} \right) \right],
\]
respectively.
As another simple example, let us observe that, for \(m = 3\), equation (8) gives us
\[
S_3^{(r)}(n) = \frac{S_1^{(r)}(n)}{(r+2)(r+3)} \begin{vmatrix} 
-2N_r & r + 2 \\
\frac{1}{2}r & -3N_r \\
\end{vmatrix} = \frac{(n + r)}{r + 1} \frac{6n^2 + 6rn + r(r - 1)}{(r + 2)(r + 3)}.
\]
Of course, when \(r = 1\), the last formula reduces to the well-known identity
\[
S_3(n) = 1^3 + 2^3 + \cdots + n^3 = \left( \frac{n + 1}{2} \right)^2.
\]

**Remark 5.** Clearly, the term of maximum degree in \(N_r\) of the polynomial \(G_m^{(r)}(N_r)\) corresponds to the product of the entries on the main diagonal of (9). Thus, using (8), and recalling that \(N_r = n + \frac{r}{2}\) and \(S_1^{(r)}(n) = \binom{n+r}{r+1}\), it is easily seen that the leading coefficient of the hyper-sum polynomial (6) is given by \(c_{m+1}^m = \frac{m!}{(m+r)!}\), which holds for any integers \(m \geq 0\) and \(r \geq 1\). In particular, for \(r = 1\), we retrieve the well-known result \(c_{m+1}^m = \frac{1}{m+1}\).

### 3 A further characterization of the polynomials \(G_m^{(r)}(N_r)\)

For \(r = 0\), the polynomial \(G_m^{(r)}(N_r)\) is equal to \(G_m^{(0)}(N_0) = N_r^{m-1}\) for any integer \(m \geq 1\). However, whenever \(r \geq 1\), \(G_m^{(r)}(N_r)\) involves powers of \(N_r\) other than \(m - 1\). In this section, we show that, for all integers \(r \geq 1\) and \(m \geq 1\), the polynomials \(G_m^{(r)}(N_r)\) are necessarily of the form given by the following theorem.
Theorem 5. Let \( N_r = n + \frac{r}{2} \) and \( S_m^{(r)}(n) = S_1^{(r)}(n) G_m^{(r)}(N_r) \). Then, for all integers \( r \geq 1 \) and \( m \geq 1 \), we have

\[
G_{2m-1}^{(r)}(N_r) = \sum_{j=0}^{m-1} g_{2m-1,j}^{(r)} N_r^{2j},
\]

and

\[
G_{2m}^{(r)}(N_r) = \sum_{j=0}^{m-1} g_{2m,j}^{(r)} N_r^{2j+1},
\]

for certain non-zero (rational) coefficients \( g_{2m-1,j}^{(r)} \) and \( g_{2m,j}^{(r)} \), \( j = 0, 1, \ldots, m - 1 \). Furthermore, the coefficients \( \{g_{2m-1,m-1}^{(r)}, g_{2m-1,m-2}^{(r)}, \ldots, g_{2m-1,0}^{(r)}\} \) have alternating signs, with the sign of the leading coefficient \( g_{2m-1,m-1}^{(r)} \) being positive. The same happens with the coefficients in the set \( \{g_{2m,m-1}^{(r)}, g_{2m,m-2}^{(r)}, \ldots, g_{2m,0}^{(r)}\} \).

Proof. The theorem can be easily proved by means of the recurrence (11) and the use of complete mathematical induction. To this end, we need the following couple of well-known properties of the Bernoulli numbers:

Property 1: \( B_{2j+1} = 0 \), for all \( j = 1, 2, \ldots \),

Property 2: \( \text{sign} \ B_{2j} = (-1)^{j+1} \), for all \( j = 1, 2, \ldots \).

First we show the validity of (15). Putting \( S_{2m-1}^{(r)}(n) = S_1^{(r)}(n) G_{2m-1}^{(r)}(N_r) \) in (11), and factoring out the term \( S_1^{(r)}(n) \), we obtain

\[
(2m - 1 + r)G_{2m-1}^{(r)}(N_r) = (2m - 1)N_rG_{2m-2}^{(r)}(N_r) - r \sum_{k=1}^{2m-3} \binom{2m-1}{k} B_{2m-k-1} G_k^{(r)}(N_r).
\]

By Property 1, this can be expressed equivalently as

\[
(2m - 1 + r)G_{2m-1}^{(r)}(N_r) = (2m - 1)N_rG_{2m-2}^{(r)}(N_r) - r \sum_{k=1}^{m-1} \left( \frac{2m - 1}{2k - 1} \right) B_{2m-2k} G_{2k-1}^{(r)}(N_r),
\]

which applies to any integers \( r \geq 0 \) and \( m \geq 2 \). When \( r = 0 \), (17) reduces to \( G_{2m-1}^{(0)}(N_0) = N_0 G_{2m-2}^{(0)}(N_0) \), and then, assuming that \( G_{2m-2}^{(0)}(N_0) = N_0^{2m-3} \), we get \( G_{2m-1}^{(0)}(N_0) = N_0^{2m-2} \). Next, we consider the non-trivial case where \( r \) is any arbitrary integer \( r \geq 1 \). Thus, the recurrence (17) gives us \( G_{2m-1}^{(r)}(N_r) \) in terms of the lower-degree polynomials \( G_1^{(r)}(N_r), G_3^{(r)}(N_r), \ldots, G_{2m-3}^{(r)}(N_r) \), and \( G_{2m-2}^{(r)}(N_r) \). As the complete inductive hypothesis, we assume that \( G_{2k-1}^{(r)}(N_r) \) is of the form (15) for all \( k = 1, 2, \ldots, m - 1 \). Also, we assume that \( G_{2k}^{(r)}(N_r) \) is of the form (16) for \( k = m - 1 \). (Please note that, when we say that a polynomial “is of the form” (15) or (16), it is understood that the corresponding coefficients are such that they satisfy the associated properties stipulated...
after equation (16).) Therefore, we have

\[
(2m - 1 + r)G_{2m-1}^{(r)}(N_r) = (2m - 1)N_r \sum_{j=0}^{m-2} g_{2m-2,j}^{(r)} N_r^{2j+1}
- r \sum_{k=1}^{m-1} \left( \frac{2m - 1}{2k - 1} \right) B_{2m-2k} \sum_{j=0}^{k-1} g_{2k-1,j}^{(r)} N_r^{2j}
= (2m - 1)g_{2m-2,m-2}^{(r)} N_r^{2m-2} + (2m - 1) \sum_{j=1}^{m-2} g_{2m-2,j-1}^{(r)} N_r^{2j}
- r \sum_{j=0}^{m-2} \left[ \sum_{k=j+1}^{m-1} \left( \frac{2m - 1}{2k - 1} \right) B_{2m-2k} g_{2k-1,j}^{(r)} \right] N_r^{2j},
\]

from which we deduce that

\[
G_{2m-1}^{(r)}(N_r) = \sum_{j=0}^{m-1} g_{2m-1,j}^{(r)} N_r^{2j},
\]

where

\[
g_{2m-1,j}^{(r)} = \begin{cases} 
\frac{2m - 1}{2m - 1 + r} g_{2m-2,m-2}^{(r)}, & \text{for } j = m - 1; \\
\frac{1}{2m - 1 + r} \left( (2m - 1)g_{2m-2,j-1}^{(r)} - r \sum_{k=j+1}^{m-1} \left( \frac{2m - 1}{2k - 1} \right) B_{2m-2k} g_{2k-1,j}^{(r)} \right), & \text{for } j = 1, 2, \ldots, m - 2; \\
- \frac{r}{2m - 1 + r} \sum_{k=1}^{m-1} \left( \frac{2m - 1}{2k - 1} \right) B_{2m-2k} g_{2k-1,0}^{(r)}, & \text{for } j = 0.
\end{cases}
\] 

Examining the three-case equation above, we first note that the coefficient \( g_{2m-2,m-2}^{(r)} \) corresponds to the leading coefficient of \( G_{2m-2}^{(r)}(N_r) \) which, by the induction hypothesis, is a positive (rational) number. Hence, so is the leading coefficient \( g_{2m-1,m-1}^{(r)} \) of \( G_{2m-1}^{(r)}(N_r) \). Likewise, by the induction hypothesis, the coefficient \( g_{2m-2,j-1}^{(r)} \) appearing in \((18b)\) is a non-zero (rational) number with sign \((-1)^{m-1-j}\). Furthermore, by invoking Property 2, it can be seen that the sign of the (non-zero) term \( B_{2m-2k} g_{2k-1,j}^{(r)} \) is equal to \((-1)^{m-k+1}(-1)^{k-1-j} = (-1)^{m-j}\), which does not depend on \( k \). Therefore, the overall expression in \((18b)\) turns out to be a non-zero (rational) number with sign \((-1)^{m-1-j}\). Similarly, the sign of the (non-zero) term \( B_{2m-2k} g_{2k-1,0}^{(r)} \) in \((18c)\) is equal to \((-1)^m\), and thus the sign of \( g_{2m-1,0}^{(r)} \) is \((-1)^{m-1}\). Putting all these things together, it follows that, for each \( j = 0, 1, \ldots, m - 1 \), the coefficient \( g_{2m-1,j}^{(r)} \) is a non-zero (rational) number with sign \((-1)^{m-1-j}\).

On the other hand, it is obvious that, for \( m = 1 \), \( G_{2m-1}^{(r)}(N_r) \) is of the form \((15)\) since \( G_1^{(r)}(N_r) = 1 \). Hence, we conclude that, for all integers \( r \geq 1 \) and \( m \geq 1 \), the polynomial \( G_{2m-1}^{(r)}(N_r) \) is of the form \((15)\).

The validity of \((16)\) can be proved analogously. Now, the counterpart of \((17)\) reads

\[
(2m + r)G_{2m}^{(r)}(N_r) = 2mN_r G_{2m-1}^{(r)}(N_r) - r \sum_{k=1}^{m-1} \left( \frac{2m}{2k} \right) B_{2m-2k} G_{2k}^{(r)}(N_r),
\]

which applies to any integers \( r \geq 0 \) and \( m \geq 1 \). As before, when \( r = 0 \), from \((19)\) we have \( G_0^{(0)}(N_0) = N_0^{2m-1} \), provided that \( G_0^{(0)}(N_0) = N_0^{2m-2} \). Furthermore, when \( m = 1 \), from \((19)\)
we obtain $G_2^{(r)}(N_r) = \frac{2}{r^2} N_r$, which is of the form (16). Considering the non-trivial case where $r$ and $m$ are arbitrary integers $r \geq 1$ and $m \geq 2$, the recurrence (19) gives us $G_{2m}^{(r)}(N_r)$ in terms of the lower-degree polynomials $G_2^{(r)}(N_r), G_4^{(r)}(N_r), \ldots, G_{2m-2}^{(r)}(N_r)$, and $G_{2m-1}^{(r)}(N_r)$. Thus, assuming that $G_{2k}^{(r)}(N_r)$ is of the form (16) for all $k = 1, 2, \ldots, m - 1$, and that $G_{2k-1}^{(r)}(N_r)$ is of the form (15) for $k = m$, implies that $G_{2m}^{(r)}(N_r) = \sum_{j=0}^{m-1} g_{2m,j} N_r^{2j+1}$, where

$$g_{2m,j}^{(r)} = \begin{cases} \frac{2m}{2m + r} g_{2m-1,m-1}^{(r)}, & \text{for } j = m - 1; \\ \frac{1}{2m + r} \left[ 2mg_{2m-1,j}^{(r)} - r \sum_{k=j+1}^{m-1} \binom{2m}{2k} B_{2m-2k} g_{2k,j}^{(r)} \right], & \text{for } j = 0, 1, \ldots, m - 2. \end{cases} \quad (20a)$$

Likewise, by the induction hypothesis, the coefficient $g_{2m-1,m-1}^{(r)}$ in (20a) is a positive (rational) number. Furthermore, by invoking the induction hypothesis and Property 2, one can easily deduce that the overall expression in (20b) is a non-zero (rational) number with sign $(-1)^{m-1-j}$. This means that, for each $j = 0, 1, \ldots, m - 1$, the coefficients $g_{2m,j}^{(r)}$ conform to the required properties, and, consequently, the polynomial $G_{2m}^{(r)}(N_r)$ is of the form (16) for all integers $r \geq 1$ and $m \geq 1$. \qed

Equations (18a)-(18c) [respectively, (20a)-(20b)] provide us with a concrete procedure to obtain $G_{2m-1}^{(r)}(N_r) [G_{2m}^{(r)}(N_r)]$ from the lower-degree polynomials $G_1^{(r)}(N_r), G_3^{(r)}(N_r), \ldots, G_{2m-3}^{(r)}(N_r)$, and $G_{2m-2}^{(r)}(N_r) [G_2^{(r)}(N_r), G_4^{(r)}(N_r), \ldots, G_{2m-2}^{(r)}(N_r)$, and $G_{2m-1}^{(r)}(N_r)]$. For example, for $m = 5$, equations (18a)-(18c) give

$$g_{9,0}^{(r)} = \frac{3}{19} g_{1,0}^{(r)} - \frac{20}{19} g_{3,0}^{(r)} + \frac{42}{19} g_{5,0}^{(r)} - \frac{60}{19} g_{7,0}^{(r)},$$
$$g_{9,1}^{(r)} = \frac{9}{19} g_{9,0}^{(r)} - \frac{20}{19} g_{3,1}^{(r)} + \frac{42}{19} g_{5,1}^{(r)} - \frac{60}{19} g_{7,1}^{(r)},$$
$$g_{9,2}^{(r)} = \frac{9}{19} g_{9,1}^{(r)} + \frac{42}{19} g_{5,2}^{(r)} - \frac{60}{19} g_{7,2}^{(r)},$$
$$g_{9,3}^{(r)} = \frac{9}{19} g_{9,2}^{(r)} - \frac{60}{19} g_{7,3}^{(r)},$$
$$g_{9,4}^{(r)} = \frac{9}{19} g_{9,3}^{(r)},$$

from which we can get $G_9^{(r)}(N_r)$ if we know the coefficients of the polynomials $G_1^{(r)}(N_r), G_3^{(r)}(N_r), G_5^{(r)}(N_r), G_7^{(r)}(N_r)$, and $G_8^{(r)}(N_r)$.

**Remark 6.** For any integer $r \geq 1$, the binomial coefficient $\binom{n+r}{r+1}$ admits the following representation involving the (unsigned) Stirling numbers of the first kind $\left[ \begin{array}{c} r \\ j \end{array} \right]$ and the power sum polynomials $S_j(n)$ for each $j = 1, 2, \ldots, r$, namely (see, e.g., [6, Equation (8)] and [4, Equation (10)]):

$$\binom{n+r}{r+1} = \frac{1}{r!} \sum_{j=1}^{r} \left[ \begin{array}{c} r \\ j \end{array} \right] S_j(n).$$
Therefore, from (7) and Theorem 5, it follows that, for all integers \( r \geq 1 \) and \( m \geq 1 \), the hyper-sum polynomial \( S_m^{(r)}(n) \) can be expressed in the form

\[
r!S_{2m-1}^{(r)}(n) = \left( \sum_{j=1}^{r} \binom{r}{j} S_j(n) \right) \times \left( \sum_{j=0}^{m-1} g_{2m-1,j}^{(r)} \left( n + \frac{r}{2} \right)^{2j} \right),
\]

for odd powers \( 2m - 1 \), and

\[
r!S_{2m}^{(r)}(n) = \left( \sum_{j=1}^{r} \binom{r}{j} S_j(n) \right) \times \left( \sum_{j=0}^{m-1} g_{2m,j}^{(r)} \left( n + \frac{r}{2} \right)^{2j+1} \right),
\]

for even powers \( 2m \). It is left as an open problem to determine an explicit formula for the coefficients \( g_{2m-1,j}^{(r)} \) and \( g_{2m,j}^{(r)} \).

4 Conclusion

Summarizing, in this paper we have derived a new determinantal formula for the hyper-sums of powers of integers \( S_m^{(r)}(n) \), embodied in equation (8) of Theorem 4. Central to the derivation of this determinantal formula is the discovery of the recurrence relation (11). Moreover, by virtue of Theorem 5, it turns out that, whenever \( r \geq 1 \) and \( m \geq 1 \), the determinant of the lower Hessenberg matrix (9) is invariably an even or odd polynomial in \( N_r \) of degree \( m - 1 \), according as \( m \) is odd or even. Consequently, whenever \( r \geq 1 \) and \( m \geq 1 \), the said determinantal formula (8) gives us \( S_m^{(r)}(n) \) as \( S_1^{(r)}(n) \) times an even or odd polynomial in \( n + \frac{r}{2} \) of degree \( m - 1 \), depending on whether \( m \) is odd or even.

We conclude the main text with the following additional remarks.

For the special case in which \( r = 1 \), the recurrence (11) becomes

\[
(m + 1)S_m(n) = m \left( n + \frac{1}{2} \right) S_{m-1}(n) - \sum_{k=1}^{m-2} \binom{m}{k} B_{m-k} S_k(n),
\]

from which one can deduce the following determinantal formula for the power sum polynomial \( S_m(n) = 1^m + 2^m + \cdots + n^m \), namely,

\[
S_m(n) = \frac{(-1)^{m+1}}{(m+1)!} \left( N^2 - \frac{1}{4} \right) \times \begin{vmatrix}
-2N & 3 & 0 & \cdots & \cdots & 0 \\
\binom{3}{1}B_2 & -3N & 4 & 0 & \cdots & 0 \\
\binom{4}{1}B_3 & \binom{4}{2}B_2 & -4N & 5 & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\binom{m-1}{1}B_{m-2} & \binom{m-1}{2}B_{m-3} & \binom{m-1}{3}B_{m-4} & \cdots & -(m-1)N & m \\
\binom{m}{1}B_{m-1} & \binom{m}{2}B_{m-2} & \binom{m}{3}B_{m-3} & \cdots & \binom{m}{m-2}B_2 & -mN
\end{vmatrix},
\]

where \( N \) is a shorthand for \( n + \frac{1}{2} \). The above determinantal formula applies to any integer \( m \geq 1 \), with the convention that the displayed determinant of order \( m - 1 \) is equal to 1 when \( m = 1 \).
Incidentally, both the recurrence (21) and the determinantal formula (22) were obtained by the author in [3]. As an example, for \( m = 7 \) and \( m = 8 \), formula (22) yields

\[
S_7(n) = \frac{N^8}{8} - \frac{7N^6}{24} + \frac{49N^4}{192} - \frac{31N^2}{384} + \frac{17}{2048},
\]

and

\[
S_8(n) = \frac{N^9}{9} - \frac{7N^7}{3} + \frac{49N^5}{120} - \frac{31N^3}{144} + \frac{127N}{3840},
\]

respectively. In general, \( S_m(n) \) can be expressed as an even or odd polynomial in \( n + \frac{1}{2} \) depending on whether \( m \) is odd or even [7]. Specifically, for any integer \( m \geq 1 \), we have

\[
S_{2m-1}(n) = \sum_{j=0}^{m} f_{2m-1,j} \left( n + \frac{1}{2} \right)^{2j},
\]

and

\[
S_{2m}(n) = \sum_{j=0}^{m} f_{2m,j} \left( n + \frac{1}{2} \right)^{2j+1}.
\]

Theorem 5 ensures that the coefficients \( f_{2m-1,j} \) and \( f_{2m,j} \), \( j = 0, 1, \ldots, m \), are non-zero (rational) numbers with alternating signs. Explicit formulas for \( f_{2m-1,j} \) and \( f_{2m,j} \) in terms of the Bernoulli numbers can be found elsewhere.

On the other hand, for integers \( r \geq 0 \) and \( m \geq 1 \), one can readily deduce from (12) the relations

\[
S_{2m-1}(n) = \frac{1}{2^n} S_{2m-1}^{(r+1)}(n) + \frac{1}{2m} \sum_{k=1}^{m} \binom{2m}{2k} B_{2m-2k} S_{2k}^{(r)}(n), \quad (23)
\]

and

\[
S_{2m}(n) = \frac{1}{2^n} S_{2m}^{(r+1)}(n) + \frac{1}{2m+1} \sum_{k=1}^{m+1} \binom{2m+1}{2k-1} B_{2m+2-2k} S_{2k-1}^{(r)}(n), \quad (24)
\]

which gives us the \((r+1)\)-fold summation \( S_{2m-1}^{(r+1)}(n) \) [respectively, \( S_{2m}^{(r+1)}(n) \)] in terms of the \( r \)-fold summations \( S_2^{(r)}(n), S_4^{(r)}(n), \ldots, S_{2m}^{(r)}(n), \) and \( S_{2m-1}^{(r)}(n) \) [\( S_1^{(r)}(n), S_3^{(r)}(n), \ldots, S_{2m+1}^{(r)}(n), \) and \( S_{2m}^{(r)}(n) \)]. It is to be mentioned that relations (23) and (24) have been previously derived (in a slightly different form) by Coffey and Lettington using a different method (see Equations (1.11) and (1.10) in [5]). For example, considering the specific case when \( r = m = 3 \), and employing (23) in combination with (8) leads to

\[
S_5^{(4)}(n) - \frac{1}{2} S_5^{(3)}(n) = \frac{1}{240} n(n+1)(n+2)(n+3)(2n+3)
\]

\[
\times \left[ \frac{5}{126} \left( n + \frac{3}{2} \right)^4 - \frac{5}{252} \left( n + \frac{3}{2} \right)^2 - \frac{859}{2016} \right],
\]

which should be compared with the (equivalent) result obtained in [5], namely,

\[
S_5^{(4)}(n) - \frac{1}{2} S_5^{(3)}(n) = \frac{1}{240} n(n+1)(n+2)(n+3)(2n+3)
\]

\[
\times \left[ \frac{5}{126} n^2(n+3)^2 + \frac{10}{63} n(n+3) - \frac{17}{63} \right].
\]
Appendix

Here, we give a proof of the recurrence (2) by mathematical induction. For this, let us consider in the first place the series \( s_n = \sum_{j=1}^{n} a_j \) \((n \geq 1)\), with the general term \( a_j \) being quite arbitrary. Then, the following relation can be easily derived (see, e.g., [10, Lemma]):

\[
\sum_{j=1}^{n} ja_j = (n + 1)s_n - \sum_{j=1}^{n} s_j. \tag{A1}
\]

For the purpose of our argument, we choose \( a_j = S_m^{(r-1)}(j) \) where \( m \) and \( r \) are taken to be arbitrary but fixed integers with \( m \geq 0 \) and \( r \geq 1 \). Hence, in the language of hyper-sums of powers of integers, (A1) can be written as

\[
\sum_{j=1}^{n} jS_m^{(r-1)}(j) = (n + 1)S_m^{(r)}(n) - S_m^{(r+1)}(n). \tag{A2}
\]

Note that, for \( r = 1 \), (A2) becomes

\[
S_{m+1}(n) = (n + 1)S_m(n) - S_m^{(2)}(n),
\]

or,

\[
S_m^{(2)}(n) = (n + 1)S_m(n) - S_{m+1}(n),
\]

which is just the recurrence (2) for \( r = 1 \). In general, for any arbitrary integer \( j \geq 1 \), we have

\[
S_m^{(2)}(j) = (j + 1)S_m(j) - S_{m+1}(j).
\]

Thus, summing each side of this equation from \( j = 1 \) to \( n \) yields

\[
S_m^{(3)}(n) = \sum_{j=1}^{n} jS_m(j) + S_m^{(2)}(n) - S_m^{(2)}(m+1)(n).
\]

When \( r = 2 \), from (A2) we obtain

\[
\sum_{j=1}^{n} jS_m(j) = (n + 1)S_m^{(2)}(n) - S_m^{(3)}(n),
\]

and then

\[
S_m^{(3)}(n) = \frac{n + 2}{2}S_m^{(2)}(n) - \frac{1}{2}S_m^{(2)}(m+1)(n),
\]

which is just the recurrence (2) for \( r = 2 \).

Now, as the induction hypothesis, let us assume that, for any given integer \( r \geq 2 \), the following relationship holds true

\[
S_m^{(r)}(j) = \frac{j + r - 1}{r - 1}S_m^{(r-1)}(j) - \frac{1}{r - 1}S_m^{(r-1)}(m+1)(j),
\]

where \( j \) stands for any arbitrary integer \( j \geq 1 \). Hence, summing over the first \( n \) integers on both sides of the last equation, we find

\[
S_m^{(r+1)}(n) = \frac{1}{r - 1} \sum_{j=1}^{n} jS_m^{(r-1)}(j) + S_m^{(r)}(n) - \frac{1}{r - 1}S_m^{(r)}(m+1)(n).
\]
Finally, by replacing $\sum_{j=1}^{n} jS_{m}^{(r-1)}(j)$ with (A2), and after a trivial rearrangement, we obtain (2).

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