Complementarity in the Multiverse

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ABSTRACT: In the multiverse, as in AdS, light-cones relate bulk points to boundary scales. This holographic UV-IR connection defines a preferred global time cut-off that regulates the divergences of eternal inflation. An entirely different cut-off, the causal patch, arises in the holographic description of black holes. Remarkably, I find evidence that these two regulators define the same probability measure in the multiverse. Initial conditions for the causal patch are controlled by the late-time attractor regime of the global description.
1. Introduction

In global approaches to the measure problem of eternal inflation, one constructs a time variable $t$ that foliates the multiverse (Fig. 1). The relative probability for observations $i$ and $j$ is defined by

$$\frac{p_i}{p_j} = \lim_{t \to \infty} \frac{N_i(t)}{N_j(t)}, \quad (1.1)$$

where $N_i(t)$ is the number of times that $i$ is observed somewhere in the multiverse prior to the time $t$. In the late time limit, both $N_i$ and $N_j$ diverge—this, of course, is the origin of the measure problem. But one expects their ratio to remain finite and to converge to a well-defined relative probability.

This prescription is ambiguous. There are many ways to define a time variable, and the probabilities in Eq. (1.1) are highly sensitive to its choice. In a beautiful recent paper, Garriga and Vilenkin [1] have outlined a novel approach to identifying a preferred
Figure 1: A global cut-off on the multiverse. The curvy lines represent hypersurfaces of constant $t$, with $t \to \infty$ at the top of the diagram. The relative probability of events 1 and 2 is defined by $p_1/p_2 = \lim_{t \to \infty} N_1(t)/N_2(t)$, where $N_i(t)$ is the number of times a given event has occurred by the time $t$. This definition is ambiguous, because it depends on the choice of foliation, i.e., of the time coordinate $t$.

In the spirit of the AdS/CFT correspondence [2], the idea is to specify a short-distance cut-off $\epsilon$ on the future boundary of the multiverse. A conjectured duality between a boundary theory and the bulk physics should relate this ultraviolet cut-off to a late-time cut-off $t$ in the bulk, with the property that $t \to \infty$ as $\epsilon \to 0$.

In order to realize this idea, one needs to construct a precise relation between the boundary scale $\epsilon$ and some finite portion of the bulk. One might be concerned that this task is no less ambiguous than the original problem of picking a time slicing. In this paper I will argue, however, that the UV-IR relation of the multiverse, like that of AdS/CFT, is unambiguously determined by causality (Fig. 2). I will thus construct a preferred foliation $t$ and, via Eq. (1.1), the associated probability measure.

A different response to the ambiguities of Eq. (1.1) has been to abandon the global description of the multiverse. Motivated by black hole complementarity [3], Refs. [4, 5] advocated that no more than one causally connected region (causal diamond, or causal patch) should be considered. This led to the causal patch measure [4] (Fig. 3), which has had considerable phenomenological success [6, 7].

The global and causal patch approaches appear to be radically different. Remarkably, I will find that they make the same predictions: The global probability measure (1.1) arising from the ultraviolet boundary cut-off $\epsilon$ is equivalent to the causal patch measure. Thus, the global and causal patch viewpoints can be reconciled. They are
Figure 2: The future light-cone of any event (black dots) defines a scale on the asymptotic boundary of the multiverse (top edge). (Like in AdS, this scale is physically infinite but can be regulated as described in the text.) Conversely, constant light-cone size on the boundary defines a hypersurface of constant “light-cone time” in the bulk (green horizontal lines). Taking the boundary scale to zero generates a preferred foliation of the multiverse. Remarkably, the resulting global probability measure is equivalent to the causal patch measure (below).

Figure 3: The causal patch measure abandons the global description of the multiverse. Relative probabilities are defined by ratios of the expected numbers of events of different types taking place within the causal past of a single worldline (shaded region), averaging over possible histories and initial conditions. (The darker-shaded narrow triangle is discussed in Sec. 5.)

not contradictory but complementary, or dual.

Outline  Before considering the multiverse, I begin in Sec. 2 by reviewing the UV-IR
connection of AdS/CFT. I show that the geometric relation between bulk points and boundary scales [8] is determined by causality and can be constructed without detailed knowledge of the duality. This construction has a natural analogue in the multiverse, which I present in Sec. 3. The “light-cone time” \( t \) associated to a bulk point \( p \) is given by

\[
t(p) = -\frac{1}{3} \log \epsilon(p) ,
\]

where \( \epsilon(p) \) is the (suitably defined) volume of the chronological future of \( p \) on the boundary.

In Ref. [1], Garriga and Vilenkin propose a different UV-IR connection in the multiverse.\(^1\) This connection involves an extra parameter \( \lambda \) that has no analogue in AdS/CFT, and which is somewhat loosely defined. Here, I will treat \( \lambda \) as a free parameter that can be used to clarify relations between different measures. If \( \lambda \) is constant, then the Garriga-Vilenkin connection reduces to the well-known scale factor measure [9] (see also Refs. [10–14]). In Sec. 4, I generalize this result to the case where \( \lambda \) depends on the bulk point \( p \). Under rather weak assumptions, the resulting class of measures is related to the scale factor measure in a simple and specific way.

This result will be useful in Sec. 5, where I investigate the probability measure defined by the light-cone time \( t \). I show that the light-cone construction of Sec. 3 can be regarded as selecting a particular choice of \( \lambda(p) \), in the language of Sec. 4. This makes it possible to quantify precisely how the light-cone measure differs from the scale factor measure. I show that in a wide class of multiverse regions, this difference is precisely the difference between the causal patch measure and the scale factor measure. Thus, the light-cone cut-off is equivalent to the causal patch measure. This result is discussed in Sec. 6.

\[\text{2. The UV-IR relation of AdS/CFT from light-cones}\]

In this section, I will show that the geometric aspects of the bulk-boundary relation of AdS/CFT are determined by causality. The AdS/CFT correspondence is the statement that quantum gravity in asymptotically Anti-de Sitter spacetimes is nonperturbatively defined by a conformal field theory that can be thought of as living on the spatial boundary of Anti-de Sitter space. If the AdS curvature radius is large, the spacetime will be accurately described by general relativity as well. In this regime, the AdS/CFT

\(^1\)The general idea of a boundary cut-off [1] forms the basis of the present paper and is left untouched. I depart from Garriga and Vilenkin [1] only in that I argue that a careful analogy with AdS/CFT leads to a different implementation of the novel approach they have proposed.
correspondence implies a highly nontrivial equivalence between the boundary quantum field theory and the classical dynamics in the bulk.

The metric of AdS in $D$ space time dimensions is

$$ds^2 = \frac{L_{\text{AdS}}^2}{\cos^2 \rho} \left(-d\tau^2 + d\rho^2 + \sin^2 \rho \, d\Omega_{D-2}^2 \right), \quad 0 \leq \rho < \frac{\pi}{2}, \quad (2.1)$$

(I will not include the extra product space factors that appear in supergravity solutions, such as $\text{AdS}_5 \times S^5$, because they play no role in this analysis.) The proper area-radius of spheres in the bulk is related to the coordinate $\rho$ by

$$r = L_{\text{AdS}} \tan \rho. \quad (2.2)$$

By dropping the conformal factor $L_{\text{AdS}}^2/\cos^2 \rho$, one sees that the conformal boundary of AdS has the structure $\mathbb{R} \times S^{D-2}$. It can be thought of as a unit $D-2$ sphere sitting at $\rho = \pi/2$ at all times $\tau$ (Fig. 4, right).

Both AdS and the CFT contain an infinite number of degrees of freedom. Now, let us introduce an infrared cut-off in the bulk, keeping a spatially finite portion of AdS space, $r < r_c$, and removing points closer to the boundary. I will assume that $r_c \gg L_{\text{AdS}}$. The surviving region has finite information content, limited by the entropy of the largest black hole that can fit: $S \sim r^{D-2}$ in $D$ spacetime dimensions.

A finite portion of the Hilbert space of the conformal field theory should suffice to describe this region. Indeed, there is overwhelming evidence that the truncation of the bulk corresponds to an ultraviolet cut-off in the CFT, at a distance scale

$$\delta_c \sim \frac{L_{\text{AdS}}}{r_c} \quad (2.3)$$

on the unit sphere. (Note that $\delta_c \ll 1$.) For example, correlators begin to differ from the conformal scaling at this distance. Moreover, the IR bulk cut-off regulates the divergent energy of a string stretched across the bulk in precisely as the UV boundary cut-off regulates its boundary dual (a point charge). Crucially, the maximum information content of the UV-regulated boundary theory is of order $L_{\text{AdS}}^{D-2}/\delta^{D-2}$, which matches the entropy $r^{D-2}$ of the largest black hole allowed by the IR bulk cut-off [8].

In the above examples, the consistency of the UV-IR relation, Eq. (2.3), depends on detailed properties of the AdS/CFT correspondence in string theory. In hindsight, however, the relation itself could have been obtained from a geometric analysis alone, under two rather weak assumptions: (1) Bulk and boundary theories are both causal; and (2) As a localized bulk excitation approaches a boundary point $p$, its dual boundary
Figure 4: Every point $p$ in Anti-de Sitter space is associated to a boundary scale $\delta$. This relation is completely determined by causality (left; only a small portion of the boundary is shown). It implies the well-known UV-IR connection of the AdS/CFT correspondence (right; here the global geometry is shown).

degrees of freedom also become localized at $p$. Both of these assumptions hold true in AdS/CFT, but they are of course far weaker than the full duality.\(^2\)

Let us define a coordinate $\delta \equiv L_{\text{AdS}}/r$. By Eq. (2.2), $\delta \approx \frac{\pi}{2} - \rho$ near the boundary, where we can approximate the AdS metric as

$$ds^2 = \frac{1}{\delta^2} \left( -d\tau^2 + d\delta^2 + dx^2 \right), \quad 0 \leq \delta \ll 1, \quad |x| \ll 1; \quad (2.4)$$

see Fig. 4 (left). In otherwise empty AdS space, consider an excitation localized at $\tau = 0, x = 0$ on the boundary, $\delta = 0$. As this excitation propagates into the bulk, causality requires that it remain localized within the light-cone, $\delta^2 + |x|^2 \leq \tau^2$. As it propagates on the boundary, causality of the CFT requires that it remain within $|x|^2 \leq \tau^2$. It will suffice to consider values of $\tau \ll 1$.

Now let us impose an infrared cut-off $r \lesssim L_{\text{AdS}}/\tilde{\delta}_c$ on the bulk, ignoring all points with $\delta \lesssim \tilde{\delta}_c$. The above excitation first enters the surviving portion of the bulk at the time $\tau_1 \sim \tilde{\delta}_c$, at the point $x = 0$. Let us also impose an ultraviolet cut-off $|\Delta x| \gtrsim \delta_c$ on the boundary, ignoring modes smaller than $\delta_c$. Then the above excitation is first

\(^2\)A closely related argument was presented in Ref. [15] in a different context; see in particular Sec. 5 therein.
resolved by the boundary theory at the time $\tau_2 \sim \delta_c$, when the boundary light-cone becomes larger than $\delta_c$. Unless $\tau_1 = \tau_2$, there will be times when the excitation is described by the boundary theory but absent from the bulk, or present in the bulk but not in the boundary theory. For the two descriptions (bulk with IR cut-off, boundary with UV cut-off) to be approximately equivalent, it follows that one must choose $\delta_c \sim \bar{\delta}$. This implies the UV-IR relation of Eq. (2.3).³

Let us summarize this result. Each point (event) $p$ in Anti-de Sitter space can be associated with a length scale $\delta(p)$ on its spatial boundary, such that $\delta \to 0$ as $p$ approaches the boundary. Given a point $p$, $\delta(p)$ can be constructed by considering the causal future $I^+(y)$ of a boundary point $y$ chosen so that $p$ is barely contained in $I^+(y)$ (i.e., $\partial I^+(y)$ passes through $p$). Then $\delta(p)$ is the spatial size of the causal future of $y$ on the boundary, at the time when the light-cone reaches $p$ (Fig. 4). This relation defines a UV-IR connection: The boundary theory with a UV cut-off $\delta_c$ describes the portion of the bulk whose points satisfy $\delta(p) > \delta_c$. We thus see that the UV-IR relation of AdS/CFT is determined by causality alone and requires no detailed knowledge of the rich structures on both sides of the duality.

### 3. The UV-IR relation of the multiverse

I have shown that null hypersurfaces uniquely relate the bulk and boundary of AdS, and that they constrain the UV-IR connection quite independently of the details of the correspondence. Though neither fact, perhaps, is widely appreciated, neither is surprising: In general spacetimes, holography can only be defined in terms of null hypersurfaces [16], since this is the only setting in which entropy is always bounded by area [17, 18]. In the multiverse, the details of any holographic correspondence remain obscure at best, so it is encouraging that the bulk-boundary connection of AdS/CFT can be established without such knowledge. We will now see that a robust causal

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³It may appear that this conclusion depends on the way the boundary time coordinate $\tau$ is extended into the bulk. (Suppose we used a coordinate $\tilde{\tau}$ that agrees with $\tau$ on the boundary but not away from it. Then we might obtain a different relation between $\delta$ and $\tilde{\delta}$.) In fact, there is no ambiguity. A short-distance cut-off breaks the symmetries of the boundary and picks out a preferred time coordinate $\tau$. This coordinate must be extended to the bulk by assigning to any point $p$ the value $\tau = \frac{1}{2}[\tau_+(p)+\tau_-(p)]$, where $\tau_+$ (\tau_-) is the earliest (latest) value of $\tau$ at which the future (past) light-cone of $p$ reaches the boundary. (Any other choice would break time-reversal invariance.) Alternatively, one could make the construction of $\delta(p)$ explicitly independent of the choice of bulk time, by introducing a second excitation localized on the boundary at the time $\tau_c > 0$ and considering its past-directed causal propagation into the bulk. Then $\delta_c = \tau_c/2$ is unambiguously the innermost bulk point at which both signals are present, and also the largest boundary distance on which both signals simultaneously have support.
Figure 5: The boundary scale $\epsilon$ associated with the multiverse event $p$ is defined as the volume (thick bars), on $\Sigma_0$, of those geodesics (thin vertical lines) that enter the future of $p$ (shaded). Other features in this diagram are discussed in Sec. 3.2.

construction naturally relates bulk points to the future boundary of the multiverse, establishing a UV-IR connection.

3.1 Light-cone time

The goal, by analogy, is to associate to each bulk event $p$ a scale $\epsilon(p)$ on the future boundary of the multiverse, such that $\epsilon \to 0$ as $p$ approaches the boundary. If the relation $\epsilon(p)$ is known, then $\epsilon$ can be used to generate a foliation of the bulk, where points of equal $\epsilon$ form spacelike hypersurfaces of equal time $t(\epsilon)$, with $t \to \infty$ as $\epsilon \to 0$.

The map $p \to \epsilon$ cannot be exactly the same as in AdS, because the causal structure of the boundary is different. However, it seems appropriate to require that it be obtained, like the map $p \to \delta$ in AdS/CFT, from causality alone. In fact, there is a very simple construction that achieves this goal: Given a point $p$ in the multiverse, let $\epsilon(p)$ be the volume, on the boundary, of its chronological future $I^+(p)$ (Fig. 5).

$I^+(p)$ is the set of points that can be reached from $p$ by a future-directed timelike curve. It can be visualized as the interior of a future light-cone whose tip is at $p$. To define the volume of $I^+(p)$ on the boundary, consider a timelike geodesic congruence $\gamma$ orthogonal to a finite spacelike hypersurface $\Sigma_0$ in the bulk.\footnote{Here the analogy with AdS/CFT is less clear. It would be nice to quantify UV scales directly on the} I will define $\epsilon$ as the
volume of the starting points, on $\Sigma_0$, of all geodesics in $\gamma$ that eventually enter $I^+(p)$. Note that it is not necessary to assume that the congruence $\gamma$ be everywhere expanding.

The choice of $\Sigma_0$ is arbitrary except for the requirement that at least one geodesic in $\gamma$ must be eternally inflating, i.e., must never enter a vacuum with $\Lambda \leq 0$. The remaining freedom in $\Sigma_0$ will not affect the UV-IR relation in the small $\epsilon$ (i.e., late time) limit. Therefore, it will not affect the resulting probability measure.

The map $p \to \epsilon(p)$ defines a UV-IR relation exactly as in AdS/CFT: Given a cut-off $\epsilon_c$ on the boundary, remove from the bulk all points $p$ with $\epsilon(p) < \epsilon_c$. Like $\delta$ in the AdS/CFT case, $\epsilon$ can be considered both a bulk and a boundary coordinate. Unlike $\delta$, which is a spatial coordinate in the bulk, $\epsilon$ is a time coordinate in the bulk. That is, the vector field $\nabla^a \epsilon$ is everywhere timelike (except for degeneracies discussed in Sec. 3.2).

To prove this, consider two points connected by a timelike curve, with $q$ lying in the future of $p$. Intuitively, the future light-cone of $q$ should be nested within the light-cone of $p$ and so should be smaller on the future boundary. This can be made precise. Since any timelike curve from $q$ to a point in $I^+(q)$ can be extended by the timelike curve connecting $p$ to $q$, $I^+(q)$ is a subset of $I^+(p)$. Therefore, every timelike curve that intersects with $I^+(q)$ must also enter $I^+(p)$. It follows that

$$\epsilon(p) \geq \epsilon(q).$$

This result applies, in particular, as $q$ approaches $p$ along any timelike curve, so the gradient of $\epsilon$ is everywhere timelike and past-directed (or zero; see Sec. 3.2).

Since $\epsilon$ vanishes at the future boundary and increases towards the past, it will be convenient to define a new time coordinate $t$ that increases towards the boundary. $t$ should depend only on $\epsilon$, so that it will define the same spacelike hypersurfaces. The detailed relation is irrelevant, because it affects only the rate at which the boundary is approached. I will choose

$$t(p) = -\frac{1}{3} \log \epsilon(p)$$

so that $t \to \infty$ as $\epsilon \to 0$. $t$ will be called light-cone time. Via Eq. (1.1), $t$ defines the “light-cone measure” on the multiverse (Fig. 2).

### 3.2 Hat domains and singular domains

Though I have shown that $t$ increases monotonically along any timelike curve, Eq. (3.1) does not guarantee that it is strictly monotonic, nor that it is continuous. In fact, in future boundary of the multiverse, without reference to the congruence $\gamma$ and/or initial hypersurface $\Sigma_0$. Because the future boundary is a fractal, this task is not straightforward and will be left to future work.
Figure 6: Examples of singular domains and hat domains. The light-cone time increases monotonically towards the future, but not strictly so. Because all geodesics in the hat domain (blue spacetime region) end at $i_+$, the light-cone time is constant there, and has fractal-like bends and discontinuities in the bulk. The dashed diagonals are included to guide the eye.

the presence of “hats”, it will be neither. Hats are the portions of the future boundary corresponding to supersymmetric regions with $\Lambda = 0$. Each hat is a portion of the future conformal boundary of Minkowski space. All geodesics in $\gamma$ that enter the past domain of dependence of a hat, $D^-_{\text{hat}}$, end up at the tip of the hat, $i_+$ (Fig. 5). I will refer to $D^-_{\text{hat}}$ as the “hat domain”; an example is shown in Fig. 6. For all points $p$ in a hat domain, $I^+(p)$ contains $i_+$ and does not contain any other endpoints of geodesics. Therefore, $\epsilon$ (and thus, $t$) is constant in the entire hat domain. It is set by the volume, on $\Sigma_0$, of the geodesics that enter the hat domain.

This has two implications. First, a hypersurface of constant $t$, $\Sigma_t$, will have a jagged, fractal structure, exemplified by the solid brown line in Fig. 6. It will have null portions, becoming disconnected along portions of the boundary of the past of a hat. This does not matter as far as Eq. (1.1) is concerned. The smoothness of slices is irrelevant since $N_i$ counts events taking place in a spacetime region, defined by the set of points $p$ occuring before the cut-off, $\{p : t(p) < t_c\}$.

Second, and more importantly, it can lead to a divergence at finite cut-off. As the IR cut-off $t_c \equiv t(\epsilon_c)$ is increased above the value of $t$ in a particular hat, this means that the entire spacetime four-volume of a $\Lambda = 0$ bubble suddenly becomes included below the cut-off. If the bubble contains observations of type $i$, then by the $SO(3,1)$ symmetry of the bubble interior, it will contain infinitely many. Thus, $N_i(t)$ will diverge at finite $t$, and the cut-off will have failed to regulate some of the fractions appearing
in Eq. (1.1).

It is interesting that exactly the same divergence arises in the causal patch measure. This is a first hint that the equivalence shown in Sec. 5 below is more general than I will prove here. This is encouraging, since the global viewpoint may admit a natural modification that eliminates this divergence, as I will now discuss.

Among potential resolutions, perhaps the least satisfying is that the divergence is not realized in practice, because the number of observations is strictly zero in all hat domains. For example, unbroken supersymmetry may be incompatible with sufficiently complex structures. An infinite number of observations would still appear in bubbles of anthropic vacua that collide with the hat domain. However, the domain walls between a $\Lambda = 0$ vacuum, and one with sufficiently small $\Lambda$ to fit an observer, would likely have to be of the VIS type, accelerating away from both vacua. This would imply that only microphysical slivers of superexponentially redshifted observers are contained in the hat domain, rendering their contribution to $N_i$ unclear.

A more interesting possibility is to modify the definition of the volume of $I^+(p)$. For example, one could explore definitions in terms of the area of the boundary of $I^+(p)$ on the future boundary of the multiverse, or in terms of the volumes of $I^-(I^+(p))$ and $I^-(p)$ on $\Sigma_0$. (This would have the additional advantage of dispensing with the congruence $\gamma$.) Hats are uniquely distinguished on the future boundary in that they constitute its only light-like portions. Thus, it is quite plausible that there exists a definition of $\epsilon$ which reduces to the one I have given, except in hat domains.

A third possibility is that the direct analogy with AdS/CFT breaks down in hat domains. Then the measure of Eq. (1.1), with the cut-off defined by Eq. (3.2), should simply not be applied to events in hat regions. In fact, it is plausible that it may break down also in “singular domains”, defined as the past domains of dependence of singular portions of the future boundary. This would exclude from consideration all bubbles with negative cosmological constant, and the interior of black holes. The remaining set $M - D^-(all \ hats) - D^-(all \ future \ singularities)$, where the cut-off does apply, is the “eternal domain”, consisting of the set of points whose future includes at least one eternally inflating geodesic. This includes most events taking place in metastable de Sitter vacua (such as, presumably, ours).

For the remainder of this paper, I will follow Garriga and Vilenkin in adopting this last, most conservative viewpoint. Unlike GV, I will not attempt to fill the resulting gaps and formulate a separate cut-off for hat domains and singular domains. Eqs. (1.1) and (3.2) suffice to compute the relative probabilities of events that occur in the eternal domain, which is all I will need in Sec. 5 below. However, in order to develop the light-

\footnote{The last two sentences were pointed out to me by S. Shenker, and by B. Freivogel, respectively.}
cone measure further it will clearly be important to explore alternative formulations in hat domains, and possibly also in singular domains. I leave this to future work.

4. The Garriga-Vilenkin parameter $\lambda$ and the scale factor cut-off

The light-cone construction of the previous section is a concrete realization of the general idea of a boundary cut-off for the multiverse [1]. However, it differs from the bulk-boundary connection that Garriga and Vilenkin themselves have outlined. The GV proposal does not involve future light-cones; in fact, the future boundary plays no role at all in assigning a scale to each bulk point. GV introduce a physical length scale $\lambda$, which depends on the physical process whose probability one would like to compute, and which is defined loosely as the “resolution” required to “describe” or identify this process.

The parameter $\lambda$ has no analogue in the AdS/CFT correspondence. Moreover, its definition remains too vague for a meaningful comparison of the GV proposal to the light-cone proposal. However, it is convenient to treat $\lambda$ as a free parameter that may or may not dependent on the spacetime point $p$. In this section, I will establish some properties of this “generalized” GV construction. These results will be useful in Sec. 5, where I will show that the light-cone construction defines a particular choice of $\lambda$, and that this choice reproduces the causal patch measure.

In Sec. 4.1, I will review the GV construction. For fixed $\lambda$, it reproduces the scale factor time in the bulk [1] (Sec. 4.2). In Sec. 4.3, I will generalize this result. I will consider the GV relation in the case where $\lambda$ does depend on the type of bulk event in question. I find the resulting class of measures to be simply related to the scale factor measure: The relative probability $p_i/p_j$ differs by $\lambda_i^3/\lambda_j^3$ from the prediction of the scale factor measure.

6Of course, the fact that it is more sharply defined favors the light-cone proposal. So do its economy, its greater generality (it does not require $\gamma$ to be everywhere expanding), and its AdS/CFT pedigree. Moreover, under at least one plausible interpretation of “resolution”, the proposed definition of $\lambda$ contradicts observation. For example, consider the probability distribution of the frequency of CMB photons. To “resolve” a given photon, $\lambda$ should be chosen of order its wavelength. By Eq. (4.10), the detection of high energy photons would then be suppressed relative to the black body curve, in proportion to the third power of the wavelength. (GV consider the possibility that on subhorizon scales, $\lambda$ may not be related to the scales one would naively associate with various phenomena. But this further obscures its definition precisely for those phenomena we can actually measure. And it may not help: In our universe, the entire CMB spectrum will be stretched to superhorizon scales.)
Garriga and Vilenkin consider a physical length scale \( \lambda \) at the multiverse event \( p \) (purple intervals). This length is transported along the congruence \( \gamma \) (thin vertical lines) back to the initial surface \( \Sigma_0 \), thereby contracting to physical size \( \xi \) (thick green bars). This construction of \( \xi(p) \) does not actually involve the future boundary or asymptotic regime (shaded), though one can choose to picture \( \xi \) as a boundary scale by transporting the interval to the future boundary along \( \gamma \). In the example shown, \( \lambda \) is chosen the same at all points, so \( \xi \) will define slices of constant scale factor.

**4.1 The Garriga-Vilenkin bulk-to-boundary relation**

Again, one begins with the (arbitrary) choice of a finite initial hypersurface, \( \Sigma_0 \) and its orthogonal congruence of timelike geodesics, \( \gamma \), at least one member of which must be eternal. Instead of the future light-cone at \( p \), GV consider a physical length scale \( \lambda(p) \), which is supposed to be chosen somewhat smaller than the characteristic scale of the physical process at \( p \) whose probability one would like to compute.

To associate a boundary length scale \( \xi \) to the bulk event \( p \), one transports a spatial interval of length \( \lambda \) from \( p \) to the future boundary along the congruence \( \gamma \), allowing it to expand with the congruence along the way. GV define \( \xi \) as the projection, onto \( \Sigma_0 \), of the resulting boundary interval, again along the congruence \( \gamma \). Viewed as a bulk coordinate, \( \xi \) is past-directed, so it is convenient to define the time coordinate

\[
\hat{\eta} = -\log \xi ,
\]

which satisfies \( \hat{\eta} \to \infty \) as \( \xi \to 0 \). A given UV cut-off \( \xi_c \) on the boundary thus defines a late time cut-off \( \hat{\eta}_c \) in the bulk. Via Eq. (1.1), the limit \( \xi_c \to 0 \) defines a measure in the multiverse.
In order for $\dot{\eta}$ to increase monotonically as $\xi$ is decreased, it is necessary that the congruence $\gamma$ be everywhere expanding. This condition is restrictive; it is violated, for example, in gravitationally collapsed regions such as our galaxy. Note that it was not needed for defining the light-cone time $t$ in the previous section.

One could have obtained $\xi(p)$ and $\dot{\eta}(p)$ more directly by transporting the physical interval $\lambda$ from $p$ back in time to the initial hypersurface $\Sigma_0$, without first making an excursion forward in time to the boundary (Fig. 7). Thus, in the end, no properties of the boundary play any role in the construction. This contrasts with the light-cone time, which cannot be defined without knowledge of the asymptotic future. I will work with this simpler, more direct definition, because it makes it easy to see that $\dot{\eta}$ is closely related to the scale factor time.

### 4.2 Constant $\lambda$ and the scale factor time

The scale factor $a$ at a point $p$ in the congruence $\gamma$ is defined by

$$a(p)^3 = \frac{dV_p}{dV_{\Sigma_0}}.$$  \hspace{1cm} (4.2)

Here, $dV$ is the physical volume element spanned by the geodesic that includes $p$ and its infinitesimally neighboring geodesics in the congruence. Thus, the local scale factor $a$ measures how much this physical volume has expanded along the geodesic connecting $p$ to the initial surface $\Sigma_0$. The scale factor time is defined by

$$\eta \equiv \log a.$$  \hspace{1cm} (4.3)

Let us assume isotropic expansion and relax the requirement that $dV$ be infinitesimal. Then it follows from the above definitions of $\xi$ and $a$ that

$$a(p) = \frac{\lambda}{\xi},$$  \hspace{1cm} (4.4)

so $\dot{\eta}$ is related to the scale factor time by a $\lambda$-dependent shift:

$$\dot{\eta}(p) = \eta(p) - \log \lambda(p).$$  \hspace{1cm} (4.5)

If $\lambda$ is constant, then $\eta$ and $\dot{\eta}$ define the same hypersurfaces and therefore the same probability measure, the scale factor measure.

### 4.3 General $\lambda$

Suppose now that $\lambda$ is not constant, but that every type of event $i$ whose probability we would like to compute is associated to a unique value of $\lambda$, $\lambda_i$. (For example, this
would be the case if, as Garriga and Vilenkin suggest, $\lambda$ is related to the bulk resolution required to describe the physics of $i$. Moreover, I will show below that the light-cone cut-off is recovered, $\hat{\eta} = t$, for some choice of $\lambda_i$.) This means that in effect, one is using different scale factor times, shifted by $\log \lambda_i$, to define the number of such events below the cut-off, $\hat{N}_i^{GV}(\hat{\eta})$:

$$N_i^{GV}(\hat{\eta}) = N_i^{SF}(\eta_i),$$

(4.6)

where

$$\eta_i \equiv \hat{\eta} + \log \lambda(p).$$

(4.7)

The scale factor cut-off exhibits attractor behavior. At late times, the number of events $i$ grows exponentially,

$$\lim_{\eta \to \infty} \frac{N_i^{SF}(\eta + \Delta \eta)}{N_i^{SF}(\eta)} = e^{(3 - q)\Delta \eta},$$

(4.8)

where $q \ll 1$ in a realistic landscape of vacua. This universal behavior was found in Ref. [19] for the special case where $i$ is the observation of a particular metastable vacuum in the landscape. It holds regardless of how the event $i$ is defined. For example, $i$ could refer to the observation of a particular CMB temperature. The only exception are terminal states, whose volume fraction grows as $1 - e^{-q\eta}$ at late times (this defines $q$), but which are not observed in any case.\(^7\)

The previous three equations imply that

$$\lim_{\hat{\eta} \to \infty} \frac{N_i^{GV}(\hat{\eta})}{N_i^{SF}(\hat{\eta})} = \lambda_i^{3 - q}.$$  

(4.9)

By Eq. (1.1), it follows that the relative probabilities defined by the GV cut-off, and those computed from the SF cut-off, are simply related by powers of $\lambda_i$:

$$\left( \frac{p_i}{p_j} \right)_{GV} = \left( \frac{p_i}{p_j} \right)_{SF} \left( \frac{\lambda_i}{\lambda_j} \right)^{3 - q}. $$

(4.10)

For $q$ sufficiently small, $1 - (\lambda_i/\lambda_j)^q \ll 1$ for all pairs $i, j$, and one can set $q \to 0$ in the above formula.

\(^7\)Terminal vacua can contain observations. I am assuming only that they will cease to do so after some finite time. This is clearly the case for regions with negative cosmological constant (which, however, cannot be described by an everywhere-expanding congruence), and it may also be the case for regions with zero cosmological constant. A terminal state is one in whose future no observations of any type occur.
5. Equivalence between light-cone and causal patch measures

In this section, I will consider the probability measure defined by the light-cone cut-off of Sec. 3. I will show that in a broad regime of bulk regions, the resulting measure is equivalent to the causal patch measure, with initial conditions dictated by the attractor behavior of the light-cone slicing. This relation is remarkable, since the causal patch measure was formulated as a “local” measure without reference to a global geometry.

The argument will proceed by formulating the light-cone cut-off as a special case of the GV cut-off corresponding to a particular spacetime-dependent choice of $\lambda(p)$. (This is purely for convenience; in particular, this choice of $\lambda$ does not appear to meet the criteria Garriga and Vilenkin have described: As we shall see, the light-cone does not relate $\lambda$ to the scale of experiments that may be taking place at $p$, nor does it necessarily relate $\lambda$ to the Hubble radius at $p$.) The advantage of this viewpoint is that the dependence of $\lambda$ on $p$ quantifies how the light-cone cut-off differs from the scale factor cut-off, which corresponds to constant $\lambda$. I will show that this difference is the same as the (known) difference between the causal patch measure and the scale factor measure. Therefore, the light-cone cut-off must be equivalent to the causal patch measure.

My analysis will not be completely general.8 Because I will use the scale factor cut-off as an intermediary, I will need to assume that the congruence $\gamma$ is everywhere expanding. (Neither the causal patch cut-off nor the light-cone cut-off require this assumption.) Moreover, for simplicity, I will only consider observations that take place in approximately homogeneous, isotropic regions with positive cosmological constant and negligible spatial curvature (such as ours). In such regions, the congruence $\gamma$ will be comoving with the homogeneous matter or radiation. Moreover, over distances less than the curvature scale, hypersurfaces of constant scale factor time will have constant density [20], and will thus agree with the usual Friedmann-Robertson-Walker slices.

The boundary scale $\epsilon(p)$ is the volume, on $\Sigma_0$, of the geodesics entering the chronological future $I^+(p)$. By homogeneity and isotropy, a geodesic on the constant density hypersurface containing $p$ will enter $I^+(p)$ if and only if it is inside the event horizon of the geodesic passing through $p$ (see Fig. 8). Because curvature is negligible, this constant density hypersurface agrees with the hypersurface defined by the scale factor at $p$, $a = a_p$, at least on the scale of the event horizon. Thus, $\epsilon(p)$ can be obtained by scaling the physical volume of the event horizon at $p$ back to $\Sigma_0$:

$$\epsilon(p) = a(p)^{-3} V_{EH}(p)$$  \hspace{1cm} (5.1)

---

8See, however, the Note added below.
Figure 8: Light-cone time as a special case of the generalized Garriga-Vilenkin construction. In homogeneous, isotropic, spatially flat regions, the future light-cone of $p$ (shaded) and the event horizon at $p$ have the same comoving volume. At $p$, the corresponding physical volume is $V_{\text{EH}}$ (thick dashed). Thus, by choosing the GV parameter to be $\lambda = V_{\text{EH}}^{1/3}$, one recovers the light-cone time slicing. Combined with earlier results, this implies that the light-cone measure is equivalent to the causal diamond measure.

In order to obtain the same result from the GV prescription (i.e., in order for $\epsilon(p) = \xi(p)^3$), one would need to choose

$$\lambda(p)^3 = V_{\text{EH}}(p),$$

by Eq. (4.4).\(^9\) Whereas with fixed $\lambda$, the GV prescription reproduces the scale factor cut-off, we now see that with $\lambda$ as in Eq. (5.2), it reproduces the light-cone cut-off.

Now consider the relative probability of events of type $i$ and $j$. Without loss of generality I will assume that $i$ has been sufficiently narrowly specified that the physical volume of the event horizon is the same, $V_{i_{\text{EH}}}$, at all points $p$ at which $i$ takes place;

\(^9\)The light-cone defines a volume on the boundary, whereas $\lambda$ defines a length scale. In fact, however, both prescriptions use a congruence of timelike geodesics to quantify scales on the boundary, and therefore both fundamentally define a volume. The expansion along a congruence is generically anisotropic, so the length $\xi$ of the projection of the bulk scale $\lambda$ onto the initial surface $\Sigma_0$ depends on its spatial orientation. To avoid this ambiguity, $\lambda$ and $\xi$ should be understood as third roots of volumes, say of cubes of linear size $\lambda$ at $p$. It is interesting that the holographic construction of a measure does not seem to require a metric structure (definition of lengths) on the boundary, but only a notion of volumes, which is weaker.
similarly for $j$. (For example, since $V^\text{EH}_i$ is observable, this can be accomplished by including it in the specification of $i$.) By Eqs. (4.10) and (5.2), the relative probability of $i$ and $j$ computed from the light-cone measure is related to that computed from the scalefactor measure by the ratio of event horizon volumes:

$$\left(\frac{p_i}{p_j}\right)_\text{LC} = \left(\frac{p_i}{p_j}\right)_\text{SF} \frac{V^\text{EH}_i}{V^\text{EH}_j},$$

where I have used $q \ll 1$.

I will now show that the right hand side is the relative probability defined by the causal patch measure. In Ref. [20], the scale factor measure was reformulated in terms of the evolution of a single geodesic of infinitesimal thickness $dV$ (the dark shaded region in Fig. 3):

$$\left(\frac{p_i}{p_j}\right)_\text{SF} = \langle \frac{dN_i}{dV} \rangle \langle \frac{dN_j}{dV} \rangle$$

where $dN_i$ are the number of events of type $i$ occurring in the worldtube of thickness $dV$. The $\langle \rangle$ signs indicate that one must average over possible histories, to account for the fact that a geodesic starting with given initial conditions has finite branching ratios for transitions into different vacua. For this “local” formulation to be equivalent to the scale factor measure as defined in Eqs. (1.1) and (4.3), the initial conditions for the geodesic must reflect the attractor behavior of the scale factor slicing. That is, another average is taken in which geodesics starting in the state $i$ receive weight $V_i$, where $V_i$ is the volume fraction occupied by $i$ on constant scale factor hypersurfaces at late times. As shown in Ref. [21], the nonterminal volume distribution in the attractor regime is dominated by the metastable vacuum $\ast$ with the slowest decay rate,

$$\tilde{V}_\ast \gg \tilde{V}_i \quad \text{for all } i \neq \ast.$$  

Thus, to excellent approximation, the geodesic can be taken to start out in the dominant vacuum, $\ast$.

The causal patch measure [4] is similar to the “local” formulation of the scale factor measure I have just reviewed, with two differences: First, the “thickness” of the geodesic is not fixed, nor is it infinitesimal. Instead, one includes events in the entire causal past of its future endpoint, i.e., in the interior of its event horizon (the entire shaded region in Fig. 3).\footnote{One can restrict the causal patch further by including only points which are also contained in the future of the geodesic’s starting point. In a causal diagram this gives the surviving four-volume a diamond shape. However, this additional restriction tends to affect only the earliest portions of the four-volume and is irrelevant here.}

$$\left(\frac{p_i}{p_j}\right)_\text{CD} = \langle \frac{N^\text{EH}_i}{N^\text{EH}_j} \rangle$$

10
Secondly, initial conditions were not specified in Ref. [4], but were left to a future theory of initial conditions.

Combining Eqs. (5.3) and (5.4), it follows that

\[
\left( \frac{p_i}{p_j} \right)_{LC} = \frac{\langle dN_i/dV \rangle V_{EH}^i}{\langle dN_j/dV \rangle V_{EH}^j}.
\]

(5.7)

Assuming homogeneity, \( \frac{dN_i}{dt} V_{EH}^i \) is just the number \( N_{iEH} \) of events of type \( i \) that occur within the event horizon of the geodesic. Therefore, by Eq. (5.6), the light-cone cut-off is equivalent to the causal patch measure:

\[
\left( \frac{p_i}{p_j} \right)_{LC} = \left( \frac{p_i}{p_j} \right)_{CD}.
\]

(5.8)

At the level of approximation I have used, the initial conditions for the geodesic spanning the causal patch are determined by the attractor regime of the scale factor measure. Strictly speaking, they are determined by the attractor regime defined by hypersurfaces of equal light-cone time \( t \), which will be slightly different for small but finite \( q \). Either way, in a realistic landscape, it is an excellent approximation to start the geodesic in the longest lived metastable vacuum, \(*\), and to average over possible decoherent histories.11

6. Discussion

That two measures are equivalent does not mean that they are right. Ref. [20] showed that the scale factor measure admits both a global and a local formulation. Eq. (4.10) generalizes this global-local duality to an infinite family of pairs of equivalent measures, parameterized by different choices of \( \lambda \). Yet, the pair we have singled out, the causal patch measure and the light-cone time cut-off, occupy a special place in this family. Both arose from specific (but different) approaches related to holography. Each of these two approaches was developed independently of the other, and independently of the duality that has now been shown to relate them.

In fact, three different roads have led us to the same place: (1) The causal patch measure [4] was motivated by black hole complementarity, well before (2) its numerous phenomenological successes were understood, such as the absence of Boltzmann brains [22], Boltzmann babies [23], a “Q-catastrophe” [4], and of certain runaway problems [7, 24]; and its excellent agreement with the observed value of the cosmological constant [7]. We can now add to this a new piece of evidence: (3) The global “holographic” cut-off advocated by Garriga and Vilenkin [1], and motivated by the UV-IR

11This would modify the analysis of Ref. [6], where Planck scale initial conditions were assumed.
connection of AdS/CFT, reproduces the relative probabilities obtained from the causal patch measure.

The causal patch and global approaches complement each other. The restriction to a causal patch avoids the apparent cloning of quantum information by Hawking radiation, which plagues the global description of any spacetime with evaporating black holes, such as the multiverse. But this is a very subtle quantum effect, and it need not preclude the usefulness of the global picture for other purposes. For example, the causal patch can be constructed with any initial conditions. Although it is perfectly conceivable that an independent theory determines initial conditions, the causal patch measure has been criticized for being less predictive, on this count, than global measures whose predictions are largely independent of initial conditions. Indeed, it now appears that the initial conditions for the causal patch are completely determined by the duality with the global picture. For the duality to hold, one must weight all possible initial states in the causal patch in proportion to their volume fraction in the attractor regime of the light-cone time slicing.

The multiverse complementarity uncovered here is not like black hole complementarity, if the latter is interpreted as the statement that complete information about the physics behind an observer’s horizon is available in his causal patch in scrambled form. (This property, in any case, seems peculiar to black holes arising from scattering experiments in otherwise empty space, and would not appear to generalize to cosmology. For example, the interior of a small black hole cannot contain information about a large distant galaxy, scrambled or not.) In the multiverse, neither the global nor the causal patch picture contains the whole truth. Perhaps a holographic theory on the future boundary will unify the global and causal patch descriptions of the bulk, allowing each to emerge in a suitable limit.

Many questions remain open. How should hat domains be treated? It may be possible to eliminate the divergences discussed in Sec. 3.2 by a modification that affects the measure only in hats. This may reveal a connection to the approach of Freivogel et al. [25], who have focussed on the two-dimensional rim of the hat; or it might suggest that the null portions of the future boundary play a nontrivial role. How should singular domains be treated? Here the situation is, in a sense, reversed: Light-cone time is well-defined, but it is totally unclear how to approach the formulation of a quantum gravity theory on the boundary. It seems unlikely to me that this problem is any simpler for big crunch singularities than it is for the description of the interior of a Schwarzschild black hole. Perhaps light-cone time will yield a fresh perspective on the problem of spacelike singularities.

Note added The equivalence between the causal patch and light-cone cut-offs holds
independently of the assumptions made in Sec. 5 [26].

Acknowledgments

I thank B. Freivogel and I. Yang for very helpful discussions. This work was supported by the Berkeley Center for Theoretical Physics, by a CAREER grant (award number 0349351) of the National Science Foundation, and by the US Department of Energy under Contract DE-AC02-05CH11231.

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