Toward explaining black hole entropy quantization in loop quantum gravity

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Abstract

In a remarkable numerical analysis of the spectrum of states for a spherically symmetric black hole in loop quantum gravity, Corichi, Diaz-Polo and Fernandez-Borja found that the entropy of the black hole horizon increases in what resembles discrete steps as a function of area. In the present article we reformulate the combinatorial problem of counting horizon states in terms of paths through a certain space. This formulation sheds some light on the origins of this step-like behavior of the entropy. In particular, using a few extra assumptions we arrive at a formula that reproduces the observed step-length to a few tenths of a percent accuracy.

However, in our reformulation the periodicity ultimately arises as a property of some complicated process, the properties of which, in turn, depend on the properties of the area spectrum in loop quantum gravity in a rather opaque way. Thus, in some sense, a deep explanation of the observed periodicity is still lacking.

1 Introduction

Recently, a large computerized analysis of the spectrum of states for a spherically symmetric black hole in loop quantum gravity was carried out by Corichi, Diaz-Polo and Fernandez-Borja \[1, 2, 3, 4\]. The analysis focused in particular on the entropy for the black hole. The theory \[6, 7, 8, 9, 10, 11, 12\] predicts the area-dependence of the entropy as

\[
S(A) = \frac{\gamma c}{\gamma} \frac{A}{4l_p^2} - \frac{1}{2} \ln \left( \frac{A}{l_p^2} \right) + O(A^0). \tag{1}
\]
\( \gamma \) is the Barbero-Immirzi-parameter, and \( \gamma_c \) a numerical constant\(^1\). Fixing \( \gamma \) to be equal to \( \gamma_c \) insures consistency with the Bekenstein-Hawking entropy law. The numerical analysis \([1, 2, 3, 4]\) was confined to relatively modest black hole sizes, up to a few hundred Planck-lengths squared in terms of the black hole area. Still, it required the calculation of up to \( 10^{58} \) states. The work beautifully confirmed the leading order behavior linear in \( A \) and it was even powerful enough to confirm the next to leading order contribution. But what is more, it discovered a completely unexpected periodic behavior of the entropy: There is a very distinctive periodic signal superimposed onto the linear growth \([1]\). Its period was found to be proportional to \( \gamma \),

\[ \Delta A = \gamma \chi l_P^2, \tag{2} \]

where \( \chi \) was estimated in \([2]\) to be

\[ \chi \approx \chi_{CDF} = 8.80. \tag{3} \]

This signal leads to a stair-case like behavior for the entropy as a function of the area and was hence called \textit{entropy quantization} in \([1]\).

The observation of this period is remarkable in several ways. For one, \((2)\) can be read as a sort of effective equidistant quantization of area \([1]\). Thus although the area spectrum in loop quantum gravity is \textit{not} equidistant but rather consists of sums of terms of the form

\[ A_j = 8\pi \gamma l_P^2 \sqrt{j(j+1)}, \quad j \in \mathbb{N}/2. \tag{4} \]

\((2)\) does provide a point of contact with the ideas of Bekenstein (see for example \([13, 14]\)) on area quantization. This point gets even more pronounced when one takes into account that, as first observed in \([1]\), the value \( \chi_{CDF} \) is very close to \( 8 \ln(3) \approx 8.788898 \). Based on a heuristic quantization of the black hole and the Bekenstein-Hawking-relation for entropy and area, an equidistant area spectrum had been conjectured \([14]\), with

\[ \Delta A = 4 \ln(k) l_P^2 \tag{5} \]

and \( k \) an integer. Later, \( k = 3 \) was suggested by Hod \([15]\) using a correspondence to the frequencies of quasinormal modes of the black hole in a

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\(^1\gamma_c \) depends on the precise definition of which states are counted as surface states. There are two possibilities, leading to two slightly different values for \( \gamma_c \): Just counting the \( m \)- and \( b \)-labels (ex. \([8, 9]\)) leads to \( \gamma_c \approx 0.237 \), whereas also counting the \( j \)-labels (ex. \([12]\)) leads to \( \gamma_c \approx 0.274 \).
certain limit. Dreyer [16] observed that in loop quantum gravity one could reinterpret (5) as the smallest nonzero area eigenvalue, making a certain choice for the parameter $\gamma$ that seemed natural at that time, but it was later realized that that $\gamma$ would not lead to the Bekenstein-Hawking-relation for entropy and area. Given all this, it is intriguing that an equidistant spectrum with $\Delta A$ involving $\ln(3)$ seems to reappear here. We should not fail to mention however, that there is an additional factor of $2\gamma$ in (2) as compared to (5), so ultimately it is not clear whether there is a deep correspondence with the ideas of Bekenstein and Hod. For more comments on this see [1], for some considerations of physical consequences we refer to [4]. Secondly the phenomenon seems to be robust against the way the split between surface and bulk degrees of freedom is done. As indicated in (2), $\Delta A$ just depends on $\gamma$, and $\chi$ (and not on $\gamma_c$), and is hence independent of the way the states were counted.\footnote{It is only \textit{a posteriori} that one might decide to adjust the freely specifiable parameter $\gamma$ to take the value $\gamma_c$ and hence make (1) consistent with the Bekenstein-Hawking entropy formula.}

Finally, there does not seem to be a straightforward way to explain this phenomenon from the theory. After all, as (4) shows, the area spectrum is quite complicated, non-equidistant, and does not, in any obvious way, determine the constant $\chi$. We should note that the $A_j$ of (4) do become approximately equidistant for large $j$ (for details see [18]), so it might at first seem that this gives an explanation for the observed periodicity. However, the spacing in (4) becomes close to multiples of $4\pi\gamma l_P^2 \approx 12.56 \gamma l_P^2$. Comparing this with (3), it is clear that this can not furnish a simple explanation for the value of $\chi$.

With the present paper we aim to contribute to an explanation of this phenomenon of entropy quantization. We use two main ideas. The first is to reformulate the combinatorial problem of enumerating the physical states for the black hole horizon in terms of paths built from a set of elementary steps. The second idea is to use a statistical description of the set of paths, very similar to a random walk.

These ideas, together with some assumptions on the numerical distribution of steps will enable us to calculate an approximation to the period $\Delta A$ that reproduces the observed step-length to a few tenths of a percent accuracy. $\Delta A$ arises as some sort of ‘resonance’ in the area spectrum (4). While we think that this is a nice result, it is not a complete analysis and expla-
nation of the phenomenon. To start with, since we have to make some assumption and approximations, our result for $\Delta A$ is not exact, and its uncertainty is hard to determine. Thus we are unable to confirm or rule out that $\chi = 8 \ln(3)$. Furthermore our analysis will not determine whether the phenomenon will persist for arbitrarily large areas. We will discuss these points in more detail, below.

After the present work was finished, we became aware of results [17] by Ansari. His is a very nice analysis of the spectrum of the full area operator, breaking it down into an infinite set of equidistant sub-spectra. For the black hole horizon, only a subset of that spectrum is relevant, due to the horizon boundary conditions, and the exclusion of edges that are a submanifold of the horizon, as well edges that pierce the horizon from within the black hole. Ansari’s methods break down for this “reduced” area operator. Still it is not inconceivable that his results are connected to the entropy quantization, and thus may represent another way of looking at the problem tackled in the present work.

Finally we would like to bring to the reader’s attention that there is independent very interesting work on the way [5] which, using somewhat similar methods, will shed more light on the phenomenon of entropy quantization.

The paper is organized as follows: In the next section we review the combinatorial problem of enumerating horizon states of the black hole and formulate it in terms of steps and paths. Section 3 is concerned with statistical considerations and with a computation of $\chi$. We finish with a discussion of the results and future perspectives in Section 4.

2 Counting states by counting paths

In the present article, we will not review any of the physics behind the description of an isolated horizon in loop quantum gravity. We refer the interested reader to [7,8]. Here it will suffice to spell out the combinatorial problem to which finding physical states of the black hole horizon is ultimately reduced. The quantity of greatest interest to us is $N(I)$, the number of horizon states that are eigenstates of horizon area with eigenvalue in the interval $I$. As we have said in the introduction, there are two ways to count states, depending on where one draws the line between bulk and boundary degrees of freedom. Both lead to qualitatively similar results for $N(I)$. 

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Here we will consider only one of these ways, the one that was laid out in [8] in detail. In [9] this problem was revisited and given a very simple formulation. It was shown that $N(I)$ is the number of ordered sequences $(m_i)_i$, $m_i \in \mathbb{Z}/2$ such that

$$\sum_i m_i = 0 \quad \text{and} \quad 8\pi\gamma l^2 \sum_i \sqrt{|m_i||m_i| + 1} \in I. \quad (6)$$

The connection to the quantum geometry of the horizon is that sequences $(m_i)_i$ with these properties are labels of physical states of the horizon. Let us simplify even further and get rid of all the units, by defining $n(a) = N(8\pi\gamma l^2 a)$. It will also be useful to introduce the shorthand $a(m) = \sqrt{|m||m| + 1}$, $m \in \mathbb{Z}/2$.

Finally we take from [9, 10] the idea that the counting problem can be simplified by implementing the two conditions of (6) in separate steps. We define

$$n(a, j) = \left| \left\{ (m_1, m_2, \ldots), m_i \in \mathbb{Z}/2 : \sum_i m_i = j, \sum_i a(m_i) = a \right\} \right|.$$ 

It will be instructive to reinterpret the state labels as paths. To this end, introduce the space $S = \mathbb{R}_+ \times \mathbb{Z}/2$. Let us call a sequence of points $(p_i)_i$ in this space a path, and the differences $p_{i+1} - p_i$ the steps of the path. Now let us call a path $(p_i)_i$ allowed if

- it starts at 0, i.e. $p_1 = (0, 0)$, and
- the steps $p_{i+1} - p_i$ are of the form $v(m_i) = (a(m_i), m_i)$ for some $m_i \in \mathbb{Z}/2$.

Obviously, then, we can associate to a state labeled by $(m_1, \ldots, m_n)$ the allowed path $(0, v(m_1), v(m_1) + v(m_2), \ldots)$.

What we are really interested in is the number $n(I)$ of states with the area in an interval $I$. In the language of paths this is given as

$$n(I) = \text{the number of allowed path with endpoint in } I \times \{0\}.$$ 

Given any number $R \geq 0$ there is a finite number of allowed paths that end within $[0, R] \times \mathbb{Z}/2 \subset S$. It is easy to enumerate them, and we have written a little Mathematica routine that does this for us. In Figure 1 we show the
results, for \( R = 4 \) and \( R = 8 \), by plotting, for each path, its endpoint \((a, j)\) as a dot in the diagram. In effect these diagrams contain all information about the functions \( n(I) \) and \( n(a, j) \). The most obvious feature of the results is the striking regularity that they exhibit. We should mention that many of the points plotted in Figure 1 lie on top of each other. For \( R = 8 \) there is for example a total of 76619 points to plot. So the regularity in the concentration of points becomes even more striking if one plots the density of endpoints of paths. We have done so in Figure 2 for \( R = 8 \).

Now an important question is the following: Is the period we see in these figures the one that was observed in [1, 2, 3, 4]? The answer is yes: The \( a = \text{const.} \) lines drawn in the Figures 1 and 2 are regularly spaced at intervals \( \Delta a = 0.35 \).

Their correspondence to the period exhibited in the data is clear. Moreover, physical states are associated to the points on the line \( \mathbb{R}_+ \times \{0\} \) in the diagrams. Thus the periodicity on that line corresponds directly to the periodicity seen in [1, 2, 3, 4]. We have also plotted density on this line and logarithm of density in a sliding interval \([a − \Delta a/2, a + \Delta a/2]\) (Figure 3). The latter starts to show the staircase of [1, 2, 3, 4] near \( a = 8 \).

It must be said that the amount of data that we have assembled is much smaller than that handled in the much more sophisticated analysis [1, 2, 3, 4]. Based on our data alone it would be premature to conclude that a
Figure 2: Density (left) and logarithm of density (right) of paths that end below $R = 8$ (in fiducial units).

Figure 3: Logarithm of density of physical states, in a small interval around $a$ (left) and in an interval adapted to the periodicity (right).
periodicity in the entropy is present. We are however confident that our plots show the onset of the pattern that [1, 2, 3, 4] has demonstrated much more clearly and to much higher values of area.

What we have done so far certainly does not amount to an explanation of the periodicity. We merely reformulated the problem of enumerating surface states of the black hole into one concerning paths in the space $\mathcal{S}$. We then observed that the periodicity found in [1, 2, 3, 4] does apparently not just govern paths corresponding to physical states (the line $j = 0$ in the figures) but a larger class of paths. In the next section we will show that using the image of steps and paths can be very helpful in the analysis of the pattern.

### 3 $\Delta A$ and the statistics of the steps

Now that we have exhibited the pattern in our reformulation through paths and steps, let us return to its explanation. The reformulation can shed new light on the issue as follows: Imagine for a moment that all the steps that would be allowed in allowed paths were just integer multiples of one basic step. Then regularity in a diagram like Figure 1 would obviously result. This is not the case for the system at hand: The allowed steps $\nu(m) = (a(m), m)$ are not integer multiples of one another. Moreover, even if the steps were approximately multiples of one basic step (as could be argued is the case for the area spectrum [18]), stringing together many steps would in general lead arbitrarily far away from points in a regular pattern. So at first sight the consideration of a situation with just integer multiples of a basic step does not seem to lead anywhere in terms of explaining the pattern observed here and in [1, 2, 3, 4]. We should however keep in mind that what we want to explain is not a completely rigid phenomenon. It is not so that there are no states that fall outside the pattern observed in Figure 1, it is rather that the majority clusters around some evenly spaced points. Moreover the observed pattern is really the result of thousands (or in the case of [1, 2, 3, 4], more like of $10^{40}$) points, which in our language are each obtained by taking many steps. So what we should be looking for is not a formula that describes all the details of the spectrum, but something that explains why it is statistically likely for a point to lie in one of the clusters.

Coming back to the steps and paths, our idea is as follows: If we can demonstrate that the steps are multiples of a single step on average in a suit-
able sense and moreover that the variance of the steps around this average is very small, then we can at least explain that a pattern formed at low areas will reproduce itself for some time. Let us try to make this argument more precise.

Let us assume that there is a well defined probability distribution $p(m)$ for the occurrence of a step $v(m)$ in a path corresponding to a physical state. Let us furthermore assume that we can treat the individual steps in a given path corresponding to a physical state as independently distributed and with the distribution $p(m)$, to a good approximation. Then let us write

$$a(m) = I(m)\Delta a + \epsilon(m)$$  \hspace{1cm} (7)$$

where $I(m)$ shall be an integer and $\epsilon(m) < \Delta a$. Now we consider a path with $n$ steps $v(m_i)$ that starts somewhere on the lattice $\Delta a\mathbb{Z} \times \mathbb{Z}/2$. The distance of the endpoint of this path to the corresponding lattice point is

$$\delta(n, \{v(m_i)\}) = \sum_{i=1}^{n} \epsilon(m_i).$$

Now we look at this quantity under the probability distribution. Because of our assumptions we can use the central limit theorem to approximate

$$\langle \delta(n) \rangle \approx n\langle \epsilon(m) \rangle, \quad \langle (\delta(n))^2 - (\langle \delta(n) \rangle)^2 \rangle \approx n\langle \epsilon(m)^2 - (\langle \epsilon(m) \rangle)^2 \rangle, \quad (8)$$

where the averages on the left of these approximate equalities are expectation values in the ensemble of physical paths with $n$ steps, whereas the averages on the right are in the ensemble of steps,

$$\langle f \rangle \doteq \sum_{m \in \mathbb{Z}/2} f(m)p(m)$$

for $f$ a function on $\mathbb{Z}/2$. \textcolor{red}{(5)} shows that if we can choose $\Delta a$ in (7) such that $\langle \epsilon(m) \rangle = 0$ then we can expect the path to remain near the lattice as long as

$$\sqrt{n\langle \epsilon(m)^2 \rangle} < \Delta a, \quad \text{or} \quad n < \frac{(\Delta a)^2}{\langle \epsilon(m)^2 \rangle}.$$

Let us apply this reasoning to the problem at hand. What we need is (a) information about the statistics of the steps involved, and (b) we will have to make an ansatz for the function $I(m)$ of (7).

As for (a), luckily we have at least some information on the statistics of the horizon states. In \textcolor{red}{[9]} it was shown that upon picking a path at random out
of all physical paths ending below some $a_0$, the probability to find $v(m)$ as
the first step is

$$p(m) \approx \exp \left( -2\pi \gamma_M \sqrt{|m|(|m|+1)} \right)$$

with $\gamma_M \approx 0.2375$ the numerical constant that makes the sum of the $p(m)$
over $m$ equal 1, and the approximation good as long as $a(m)$ is small compared
with $a_0$. What we would rather like to know is a different probability,
namely that of finding $v(m)$ as first step in a path among all the paths of $n$
steps and ending below $a_0$. Now, as long as $n$ is large, (but not as large as
$a_0$), the dependence of this probability on $n$ should be rather weak, and
thus we assume that it is proportional to the $p(m)$ above. Thus we will
work with an ensemble of steps with the above probability distribution,
i.e. we will define the average for a function $f$ on $\mathbb{Z}_* / 2$ by

$$\langle f \rangle \equiv \sum_{m \in \mathbb{Z}_* / 2} f(m) \exp \left( -2\pi \gamma_M \sqrt{|m|(|m|+1)} \right).$$

Now we turn to part (b): We have to make an Ansatz for the function I of
(7). To that end, we inspect Figure 4 which shows the basic steps, and how
they fit into the pattern of the allowed paths. One sees very clearly that
the regularity in the pattern of the paths and of the steps are related. More
precisely \( a(m + 1/2) - a(m) \approx 3\Delta a/2 \). Moreover there is a shift of one unit, independent of \( m \). Altogether, we will write

\[
a(m) = \left( \frac{3}{2} \cdot 2m + 1 \right) \Delta a + \epsilon(m).
\]

which in turn defines the quantities \( \epsilon(m) \), once \( \Delta a \) is fixed.

Now we can proceed as outlined above. We want to determine \( \Delta a \), and we want to do it in such a way that \( \langle \epsilon(m) \rangle \) is zero, otherwise any pattern would be washed out. Taking averages of \( \eqref{eq:9} \) and using the condition \( \langle \epsilon(m) \rangle = 0 \) indeed determines \( \Delta a \):

\[
\Delta a = \frac{\langle a(m) \rangle}{3\langle m \rangle + 1}
\]

This in turn fixes

\[
\epsilon(m) = a(m) - \left( 3m + 1 \right) \frac{\langle a(m) \rangle}{3\langle m \rangle + 1}.
\]

Numerical evaluation of these formula can be done very easily on a computer. We find

\[
\Delta a \approx 0.34952, \quad \langle \epsilon(m)^2 \rangle \approx 0.00019156
\]

Let us put these numbers in perspective and into context. First of all, we find that the standard deviation for the \( \epsilon(m) \) is very small:

\[
\frac{\Delta a}{\sqrt{\langle \epsilon(m)^2 \rangle}} \approx 25
\]

That means that only after a number \( n \) of steps of the order of 625 (= 25\(^2\)) do we expect to deviate from the pattern substantially, as we have argued before. This means that at the very least our results are significant for the black holes of small area as considered here and in \([1, 2, 3, 4]\). Secondly, our value for \( \Delta a \) compares nicely with the one obtained in \([1, 2, 3, 4]\). Let us illustrate this in terms of the parameter \( \chi \). It is related to our \( \Delta a \) by \( \chi = 8\pi\Delta a \), which compares to the result \( \chi_{\text{CDF}} \) of \([1, 2, 3, 4]\) as follows:

\[
\chi \approx 8.7843, \quad \chi_{\text{CDF}} \approx 8.80 \quad \frac{\chi_{\text{CDF}} - \chi}{\chi_{\text{CDF}}} \approx 0.00129,
\]

so we agree with the quoted reference to within a fraction of a percent. What is more, we seem to be even closer to the conjectured value \( 8\ln(3) \):

\[
8\ln(3) \approx 8.7889, \quad \frac{8\ln(3) - \chi}{\chi} \approx 0.00053.
\]

We will discuss these findings further in the next section.
Discussion and outlook

What we have done in the present paper is to give an explanation of the phenomenon of entropy quantization in loop quantum gravity by means of a formulation using paths, steps, and their statistics. The periodicity in the spectrum of horizon states arises as some sort of resonance \[^{(9)}\] in the area spectrum. That \[^{(9)}\] works so well has to do with the fact that the area spectrum is nearly equidistant, \(\sqrt{j(j+1)} \approx j + 1/2\). The precise relation \[^{(9)}\] that gives small \(\langle \varepsilon(m)^2 \rangle\) (and hence the value of \(\Delta A\)) does however depend on the details of the area spectrum as well as on the quantum boundary conditions (i.e. the \(j=0\) constraint). Thus it is intimately related to properties of the area quantization of loop quantum gravity, however in a rather opaque way. A nice way to confirm this is to redo the calculation of allowed path, however using a slightly distorted area spectrum \(a'(m) = a(m) + 1/(10m)\). One can see that the details of the result (Figure 5) change quite drastically as compared to the undistorted spectrum. One does however still recognize a lot of regularity in the result. Our interpretation is that because the area spectrum is still approximately equidistant, one again sees regularities.

We should stress that although we talk about a resonance, and our model for the black hole involves some sort of random walk, these are not physical processes. They have nothing to do with the dynamics of the black hole.
emerge, whereas the precise pattern has changed because the resonance condition to achieve small $\langle \epsilon(m)^2 \rangle$ is now different.

Our explanation seems to be quite successful quantitatively, as we recover the results of [1, 2, 3, 4] for $\chi$. However here already one problem of our approach becomes apparent: Several approximations go into the determination of our value for $\chi$, and we have little idea how accurate the result actually is.

As for other aspects of the phenomenon, some can be explained by our approach, while others remain mysterious. In particular, we want to remark the following:

(1) The phenomenon occurs for both ways to count [1, 2, 3, 4] but here we have considered only one. We note however that the other way of counting states (i.e. the inclusion of $j$-labels) can be seen as a refinement of the counting we have done here. To be more precise, each path corresponding to a physical state that we have counted here corresponds to one or more physical states as counted in the other scheme. Thus the pattern that we have observed is bound to appear also in the other scheme, possibly modulated further by some other effects that come from the details of the counting of the $j$-labels. Thus it seems to us that the explanation of the phenomenon given here also applies to the other counting scheme.

(2) It was observed [1, 2, 3, 4] that the phenomenon goes away when not implementing the condition that $\sum_i m_i = 0$. From our Figures [1 and 4] as well as our expression for the area spectrum (9) it appears that the pattern is shifted by $\Delta a/2$ between lines with $2j$ even and lines with $2j$ odd. This explains at least why the pattern gets washed out considerably when summed over all $j$. It could be, however, that a (substantially weaker) pattern with a spacing of $\Delta a/2$ remains, and it would be interesting to look for it in numerical data.

(3) It was observed [1, 2, 3, 4] that $\chi$ is very close to $8 \ln(3)$. The situation here is very tantalizing, in that on the one hand our result for $\chi$ moves even closer to the conjectured value. On the other hand, since our treatment is only approximate, we can not draw any conclusion from this.

(4) It has been conjectured [1, 2, 3, 4] that the phenomenon continues to be present even for macroscopic black holes. From our treatment it does seem that the pattern should start to get washed out once the black hole is so large that the dominant paths are longer than about 600 steps. We do not understand the mechanism at work in the generation of the paths.
well enough to present this as a result, however. Rather, we must leave this question open for future research.

Altogether, we think that the approach taken affords interesting insights, but it does not give answers to some crucial questions. Moreover it may even be questioned wether the present approach can answer these questions even when worked out in more detail, since it uses statistics and some heuristics. Therefor it would be interesting to pursue alternative approaches. One possibility that comes to our mind is to analyze in detail the properties of the Laplace-Fourier-transform of \( n(a, j) \) as determined to a good approximation in [10] (and which can be determined \textit{exactly}, as far as we can see, with similar methods). Its structure, and in particular its poles, should contain information on periodic phenomena of \( n(a, j) \) along lines \( j = \text{const} \). We will pursue this approach elsewhere.

**Acknowledgements**

I would like to thank A. Corichi, J. Diaz-Polo and E. Fernandez-Borja for explaining details of their work to me, for their comments on a draft of this article and their encouragement, and for interesting discussions on black hole entropy in loop quantum gravity. I am grateful to P. Mitra for pointing out an error in the bibliography of an earlier version of this paper, and to C. Fleischhack for comments on the genericity of the phenomenon.

I gratefully acknowledge funding for this work through a Marie Curie Fellowship of the European Union.

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