JACOBI INVERSION FORMULAE FOR A CURVE IN WEIERSTRASS NORMAL FORM

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Abstract. We consider a pointed curve \((X, P)\) which is given by the Weierstrass normal form,
\[
y^r + A_1(x)y^{r-1} + A_2(x)y^{r-2} + \cdots + A_{r-1}(x)y + A_r(x)
\]
where \(x\) is an affine coordinate on \(\mathbb{P}^1\), the point \(\infty\) on \(X\) is mapped to \(x = \infty\), and each \(A_j\) is a polynomial in \(x\) of degree \(\leq js/r\) for a certain coprime positive integers \(r\) and \(s\) \((r < s)\) so that its Weierstrass non-gap sequence at \(\infty\) is a numerical semigroup. It is a natural generalization of Weierstrass’ equation in the Weierstrass elliptic function theory. We investigate such a curve and show the Jacobi inversion formulae of the strata of its Jacobian using the result of Jorgenson [Jo].

Dedicated for Emma Previato’s 65th birthday.

1. Introduction

The Weierstrass \(\sigma\) function is defined for an elliptic curve of Weierstrass’ equation \(y^2 = 4x^3 - g_2x - g_3\) in Weierstrass’ elliptic function theory [WW]; \((\wp(u) = -\frac{d^2}{du^2} \ln \sigma(u), \frac{d\wp(u)}{du}\) is identical to a point \((x, y)\) of the curve. The \(\sigma\) function is related to Jacobi’s \(\theta\) function. As the \(\theta\) function was generalized by Riemann for any Abelian variety, its equivalent function \(\mathcal{A}_1\) was defined for any hyperelliptic Jacobian by Weierstrass [W1] and was refined by Klein [K1, K2] as a generalization of the elliptic \(\sigma\) function such that it satisfied a modular invariance under the action of \(\text{Sp}(2g, \mathbb{Z})\) (up to a root of unity).

Recently the studies on the \(\sigma\) functions in the XIXth century have been reevaluated and reconsidered. Grant and Onishi gave their modern perspective and showed precise structures of hyperelliptic Jacobians from a viewpoint of number theory using the \(\sigma\) functions [CJ, O], whereas Buchstaber, Enolski and Leikin [BEL] and Eilbeck, Enolski and Leikin [EEL] investigated the hyperelliptic \(\sigma\) functions for their applications to the integrable system. One of these authors also applied Baker’s results to the dynamics of loops in a plane [P3, MP6] associated with the modified Korteweg-de Vries hierarchy [M1, M3, MP6]. Further Buchstaber, Leikin and Enolski [BLE] and Eilbeck, Enolski and Leikin [EEL] generalized the \(\sigma\) function to more general plane curves. Nakayashiki connected these \(\sigma\) functions with the \(\theta\) functions in Fay’s study [F] and the \(\tau\) functions in the Sato universal Grassmannian theory [N1, N2].

Weierstrass’ elliptic function theory provides the concrete and explicit descriptions of the geometrical, algebraic and analytic properties of elliptic curves and their
related functions \([W,W]\), and thus it has strong effects on various fields in mathematics, physics and technology. We have reconstructed the theory of Abelian functions of curves with higher genus to give the vantage point like Weierstrass’s elliptic function theory \([MK, MP1-5, KMP2-4]\).

In the approach of the \(\sigma\) function, the pointed curve \((X,P)\) is crucial, since the relevant objects are written in terms of \(H^0(X,\mathcal{O}(\ast P))\). The representation of the affine curve \(X \setminus P\) is therefore also relevant, and so is the Weierstrass semigroup (W-semigroup) at \(P\). It is critical to find the proper basis of \(H^0(X,\mathcal{O}(\ast P))\) to connect the (transcendental) \(\theta\) (\(\sigma\)) function with the (algebraic) functions and differentials of the curve; in Mumford’s investigation, the basis corresponds to \(U\) function of the Mumford triplet, \(UVW\) \([Mu]\), which is identical to a square of Weierstrass \(\sigma\) function \([W1]\). The connection means the Abel-Jacobi map and its inverse correspondence, or the Jacobi inversion formula.

Using the connection, we represent the affine curve \(X \setminus P\) in terms of the \(\sigma\) (\(\theta\)) functions and find these explicit descriptions of the geometrical, algebraic and analytic properties of the curve and their related functions. These properties are given as differential equations and related to the integrable system as in \([B2, Mu, Pr1, Pr2, EEMOP2]\). Since the additional structure of the hyperelliptic curves \([EEMOP1]\) is closely related to Toda lattice equations and classical Pon- cedel’s problem, the additional structures were also revealed as dynamical systems \([KMP2]\).

Using the \(\sigma\) functions, we have the Jacobi inversion formulae of the Jacobians, e.g., for hyperelliptic curves \([W1, B1]\), for cyclic \((n,s)\) curves (super-elliptic curves) including the strata of the Jacobians \([MP1, MP4]\), for curves with “telescopic” Weierstrass semigroups \([A]\) and cyclic trigonal curves \([MK, KMP2, KMP3, KMP4]\). In this paper, we consider the Jacobi inversion problem for a general Riemann surface. The main effort in these studies is directed toward explicitness.

Since it is known that every compact Riemann surface is birationally-equivalent to a curve given by the Weierstrass normal form \([W2, B1, Ka]\), in this paper, we investigate the Jacobi inversion formula of such a curve, which is called Weierstrass curve; its definition is given in Proposition \(2.1\). We reevaluate Jorgensen’s results for the \(\theta\) divisor of a Jacobian by restricting ourselves to the Jacobian of a Weierstrass curve. Then we show explicit Jacobi inversion formulae even for the strata of the \(\theta\) divisor in Theorem \(5.2\) as our main theorem of this paper.

This approach enables us to investigate the properties of image of the Abelian maps of a compact Riemann surface more precisely. In the case of the hyperelliptic Jacobian, the sine-Gordon equation gives the relation among the meromorphic functions on the Jacobian, which is proved by the Jacobi inversion formula. Further the Jacobi inversion formula of the stratum of the hyperelliptic Jacobian generalizes the sine-Gordon equation to differential relations in the strata \([MP2]\). Very recently, Ayano and Buchstaber found novel differential equations that characterizes the stratum of the hyperelliptic Jacobian of genus three \([AB]\). Accordingly, it is expected that the results in this paper might also lead a generalization of the integrable system and bring out the geometric and algebraic data of the strata of the Jacobians.

We use the word “curve” for a compact Riemann surface: on occasion, we use a singular representation of the curve; since there is a unique smooth curve with the
same field of meromorphic function, this should not cause confusion. We let the non-negative integer denoted by $\mathbb{N}_0$ and the positive integer by $\mathbb{N}$.

The contents of this paper are organized as follows: in Section 2, we collect definitions, properties, and examples of Weierstrass curves and Weierstrass semigroups. Section 3 provides the Abelian map (integral) and the $\theta$ functions as the transcendental properties of the Abelian map (integral) image. We also mentioned Jorgenson’s result there. In Section 4 we show the properties of holomorphic one forms of Weierstrass curve $(X, P)$ in terms of the proper basis of $H^0(X, \mathcal{O}(P))$ as the algebraic property of the curve $X$. In Section 5 we applied these data to the result of Jorgenson and show our main theorem, the explicit representations of Jacobi inversion formulae of a general Weierstrass curve in Theorem 5.2.

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2. Weierstrass normal form and Weierstrass curve

We set up the notations for Weierstrass semigroups and Weierstrass normal form, and recall the results we use.

2.1. Numerical semigroups and Weierstrass semigroups. A pointed curve is a pair $(X, P)$, with $P$ a point of a curve $X$; the Weierstrass semigroup for $X$ at $P$, which we denote by $H(X, P)$, is the complement of the Weierstrass gap sequence $L$, namely the set of natural numbers $\{\ell_0 < \ell_1 < \cdots < \ell_{g-1}\}$ such that $H^0(K_X - \ell_i P) \neq H^0(K_X - (\ell_i - 1)P)$, for $K_X$ a representative of the canonical divisor (we identify divisors with the corresponding sheaves). By the Riemann-Roch theorem, $H(X, P)$ is a numerical semigroup. In general, a numerical semigroup $H$ has a unique (finite) minimal set of generators, $M = M(H)$ and the finite cardinality $g$ of $L(M) = \mathbb{N}_0 \setminus H$; $g$ is the genus of $H$ or $L$. We let $a_{\min}(H)$ be the smallest positive integer of $M(H)$. For example,

$$L(M) = \{1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18, 23\}, \quad a_{\min} = 5, \text{ for } M = \langle 5, 7 \rangle,$$

$$L(M) = \{1, 2, 3, 4, 6, 8, 9, 13\}, \quad a_{\min} = 5, \text{ for } M = \langle 5, 7, 11 \rangle,$$

$$L(M) = \{1, 2, 4, 5\}, \quad a_{\min} = 3, \text{ for } M = \langle 3, 7, 8 \rangle.$$

The Schubert index of the set $L(M(H))$ is

$$\alpha(H) := \{a_0(H), a_1(H), \ldots, a_{g-1}(H)\},$$

where $a_i(H) := \ell_i - i - 1$. By letting the row lengths be $\Lambda_i = a_{g-i} + 1, i \leq g$, we have the Young diagram of the semigroup, $\Lambda := (\Lambda_1, \ldots, \Lambda_g)$. If for a numerical semigroup $H$, there exists a curve whose Weierstrass non-gap sequence is identical to $H$, we call the semigroup $H$ Weierstrass. It is known that every numerical semigroup is not Weierstrass. A Weierstrass semigroup is called symmetric when $2g - 1$ occurs in the gap sequence. It implies that $H(X, P)$ is symmetric if and only if its Young diagram is symmetric, in the sense of being invariant under reflection across the main diagonal.
2.2. Weierstrass normal form and Weierstrass curve. We now review the “Weierstrass normal form”, which is a generalization of Weierstrass’ equation for elliptic curves [WW]. Baker [B1, Ch. V, §§60-79] gives a complete review, proof and examples of the theory, though he calls it “Weierstrass canonical form”. Here we refer to Kato [Ka], who also produces this representation, with proof.

Let \( m = a_{\min}(H(X, P)) \) and let \( n \) be the least integer in \( H(X, P) \) which is prime to \( m \). By denoting \( P = \infty \), \( X \) can be viewed as an \( m : 1 \) cover of \( \mathbb{P}^1 \) via an equation of the following form: \( f(x, y) = y^m + A_1(x)y^{m-1} + \cdots + A_{m-1}(x)y + A_m(x) = 0 \), where \( x \) is an affine coordinate on \( \mathbb{P}^1 \), the point \( \infty \) on \( X \) is mapped to \( x = \infty \), and each \( A_j \) is a polynomial in \( x \) of degree \( \leq jm/m \), with equality being attained only for \( j = m \). The algebraic curve \( \text{Spec} \mathbb{C}[x, y]/f(x, y) \), is, in general, singular and we denote by \( X \) its unique normalization. Since [Ka] is only available in Japanese, and since the meromorphic functions constructed in his proof will be used in our examples in [2.3] below, we reproduce his proof in sketch: it shows that the affine ring of the curve \( X \setminus \infty \) can be generated by functions that have poles at \( \infty \) corresponding to the \( g \) non-gaps.

**Proposition 2.1.** [Ka] For a pointed curve \((X, \infty)\) with Weierstrass semigroup \( H(X, \infty) \) for which \( a_{\min}(H(X, \infty)) = m \), we let \( m_i := \min\{ h \in H(X, \infty) \setminus \{0\} \mid h \equiv i \mod m \}, i = 0, 1, 2, \ldots, m-1, m_0 = m \) and \( n = \min\{ m_j \mid (m, j) = 1 \} \).

\((X, \infty)\) is defined by an irreducible equation,

\[
(2) \quad f(x, y) = 0,
\]

for a polynomial \( f \in \mathbb{C}[x, y_n] \) of type,

\[
(3) \quad f(x, y) := y^m + A_1(x)y^{m-1} + A_2(x)y^{m-2} + \cdots + A_{m-1}(x)y + A_m(x),
\]

where the \( A_i(x) \)'s are polynomials in \( x \),

\[
A_i = \sum_{j=0}^{\lfloor mn/m \rfloor} \lambda_{i, j}x^j,
\]

and \( \lambda_{i, j} \in \mathbb{C}, \lambda_{m, m} = 1 \).

We call the pointed curve given in Proposition 2.1 Weierstrass curve in this paper.

**Proof.** We let \( I_m := \{m_1, m_2, \ldots, m_{m-1}\} \setminus \{m_{i_0}\} =: \{m_{i_1}, m_{i_2}, \ldots, m_{i_{m-2}}\} \), where \( i_0 \) is such that \( n = m_{i_0} \). Let \( y_{m_i} \) be a meromorphic function on \( X \) whose only pole is \( \infty \) with order \( m_i \), taking \( x = y_m \) and \( y = y_n \). From the definition of \( X \), we have, as \( \mathbb{C} \)-vector spaces,

\[
H^0(X, \mathcal{O}_X(\ast \infty)) = \mathbb{C} \oplus \sum_{i=0}^{m-1} \sum_{j=0}^{\lfloor mn/m \rfloor} \mathbb{C}x^jy_{m_i}.
\]

Thus for every \( i_j \in I_m \) \((j = 1, 2, \ldots, m-2)\), we obtain the following equations

\[
(4) \quad \begin{cases}
 y_{n_1}y_{m_{i_1}} &= A_{1, 0} + A_{1, 1}y_{m_{i_1}} + \cdots + A_{1, m-2}y_{m_{i_{m-2}}} + A_{1, m-1}y_{n_1} \\
y_{n_2}y_{m_{i_2}} &= A_{2, 0} + A_{2, 1}y_{m_{i_1}} + \cdots + A_{2, m-2}y_{m_{i_{m-2}}} + A_{2, m-1}y_{n_2} \\
\vdots \\
y_{n_{m-2}}y_{m_{i_{m-2}}} &= A_{m-2, 0} + A_{m-2, 1}y_{m_{i_1}} + \cdots + A_{m-2, m-2}y_{m_{i_{m-2}}} + A_{m-2, m-1}y_{n_{m-2}}.
\end{cases}
\]
function field of the curve can be generated by these

\[ y_n^2 = A_{m-1,0} + A_{m-1,1}y_{m-1} + \cdots + A_{m-1,m-2}y_{m-2} + A_{m-1,m-1}y_n, \]

where \( A_{i,j} \in \mathbb{C}[x] \).

When \( m = 2 \), (4) equals (5). We assume that \( m > 2 \) and then (4) is reduced to

\[
\begin{pmatrix}
A_{1,1} - y_n & A_{1,2} & \cdots & A_{1,m-2} \\
A_{2,1} & A_{2,2} - y_n & \cdots & A_{2,m-2} \\
\vdots & \vdots & \ddots & \vdots \\
A_{m-2,1} & A_{m-2,2} & \cdots & A_{m-2,m-2} - y_n
\end{pmatrix}
\begin{pmatrix}
y_{m-1} \\
y_{m-2} \\
\vdots \\
y_{m-m-2}
\end{pmatrix}
= -
\begin{pmatrix}
A_{1,0} + A_{1,m-1}y_n \\
A_{2,0} + A_{2,m-1}y_n \\
\vdots \\
A_{m-2,0} + A_{m-2,m-1}y_n
\end{pmatrix}.
\]

One can check that the determinant of the matrix on the left-hand side of (4) is not equal to zero by computing the order of pole at \( \infty \) of the monomials \( B_iy_n^{m-2-i} \) in the expression,

\[
P(x, y_n) := \left| A_{1,1} - y_n & A_{1,2} & \cdots & A_{1,m-2} \\
A_{2,1} & A_{2,2} - y_n & \cdots & A_{2,m-2} \\
\vdots & \vdots & \ddots & \vdots \\
A_{m-2,1} & A_{m-2,2} & \cdots & A_{m-2,m-2} - y_n
\right| = y_n^{m-2} + B_1y_n^{m-3} + \cdots + B_{m-3}y_n + B_{m-4},
\]

which is \( n(m - 2 - i) + m \cdot \deg B_i \). The fact that \( (m, n) = 1 \) shows that \( n(m - 2 - i) + m \cdot \deg B_i \neq n(m - 2 - j) + m \cdot \deg B_j \) for \( i \neq j \).

Hence by solving equation (4) we have

\[
y_{j_i} = \frac{Q_i(x, y_n)}{P(x, y_n)},
\]

where \( j_i \in I_m, Q_i(x, y_n) \in \mathbb{C}[x, y_n] \) and a polynomial of order at most \( m - 2 \) in \( y_n \). Note that the equations (4) are not independent in general but in any cases the function field of the curve can be generated by these \( y_{j_i} \)'s, and its affine ring can be given by \( \mathbb{C}[x, y_n, y_{a_2}, \ldots, y_{a_k}] \) for \( j_i \in M_g \), where \( M_g := \{ a_1, a_2, \ldots, a_k \} \subset \mathbb{N}^l \) with \( (a_{k'}, a_k) = 1 \) for \( k' \neq k \), \( a_1 = a_{\min} = m, a_2 = n \), is a minimal set of generators for \( H(X, \infty) \).

By putting (4) into (5), we have (3) since (3) is irreducible. \( \square \)

Remark 2.2. Since every compact Riemann surface of genus \( g \) has a Weierstrass point whose Weierstrass gap sequence with genus \( g \) [ACGH], it is characterized by the behavior of the meromorphic functions around the point and thus there is a Weierstrass curve which is birationally equivalent to the compact Riemann surface. The Weierstrass curve admits a local \( \mathbb{Z}/m\mathbb{Z} \)-action at \( \infty \), in the following sense. We assume that a curve in this article is a Weierstrass curve which is given by (3). We consider the generator \( M_g \) of the Weierstrass semigroup \( H = H(X, \infty) \) in the proof. For a polynomial ring \( \mathbb{C}[Z] := \mathbb{C}[Z_1, Z_2, \cdots, Z_l] \) and a ring homomorphism,

\[
\varphi : \mathbb{C}[Z] \rightarrow \mathbb{C}[t^{a_1}, t^{a_2}, \cdots, t^{a_k}], \quad (\varphi(Z_i) = t^{a_i}, \ a_i \in M_g),
\]

\[ y_n^2 = A_{m-1,0} + A_{m-1,1}y_{m-1} + \cdots + A_{m-1,m-2}y_{m-2} + A_{m-1,m-1}y_n, \]
we consider a monomial ring $B_H := \mathbb{C}[z]/\ker \varphi$. The action is defined by sending $Z_{i}$ to $z_{m}^{i}$, where $\zeta_{m}$ is a primitive $m$-th root of unity. Sending $Z_{i}$ to $1/x$ and $Z_{i}$ to $1/y_{0}$, the monomial ring $B_H$ determines the structure of gap sequence \cite{He, Pi}.

There are natural projections, 
\[ \varpi_{x} : X \to \mathbb{P}, \quad \varpi_{y} : X \to \mathbb{P}, \]
to obtain $x$-coordinate and $y$-coordinate such that $\varpi_{x}(\infty) = \infty$ and $\varpi_{y}(\infty) = \infty$.

### 2.3. Examples: Pentagonal, Trigonal non-cyclic

We clarify that the words “trigonal” and “pentagonal” are used here only as an indication of the fact that the pointed curve $(X, \infty)$ has Weierstrass semigroup of type three, five, respectively; a different convention requires that $k$-gonal curves not be $j$-gonal for $j < k$ (as in trigonal curves, which by definition are not hyperelliptic); of course this cannot be guaranteed in our examples. Baker \cite[Ch. V, §70]{Baker} also points this out.

I. Nonsingular affine equation \cite{I}: $y^5 + A_1y^4 + A_3y^3 + A_2y^2 + A_1y + A_0 = 0$, $n = 7$ case: \cite{I} corresponds to
\[
\begin{pmatrix}
-y & 0 & 1 \\
1 & -y & 0 \\
-A_3 & -A_2 & -A_4 - y
\end{pmatrix}
\begin{pmatrix}
y^3 \\
y^2 \\
y^4
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
A_1y + A_0
\end{pmatrix},
\]
\[ y^{2+i} = \frac{(A_1y + A_0)y^i}{y^3 + A_1y^2 + A_3y + A_2}. \]

The curve has 5-semigroup at $\infty$ but is not necessarily cyclic.

II. Singular affine equation \cite{II}: $y^5 = k_{2}(x)^{2}k_{3}(x)$, where $k_{2}(x) = (x - b_1)(x - b_2)$, $k_{3}(x) = (x - b_3)(x - b_4)(x - b_5)$ for pairwise distinct $b_i \in \mathbb{C}$; \cite{II} corresponds to
\[
\begin{pmatrix}
-y & 0 & 0 \\
0 & -y & 0 \\
0 & 0 & -y
\end{pmatrix}
\begin{pmatrix}
w \\
w y \\
y^2
\end{pmatrix}
= \begin{pmatrix}
0 \\
-k_{2}k_{3} \\
-k_{2}w
\end{pmatrix}.
\]

The affine ring is $H^0(\mathcal{O}_X(\ast \infty)) = \mathbb{C}[x, y, w]/(y^3 - k_2w, w^2 - k_3y, y^2w - k_2k_3)$. Here \cite{II} is reduced to
\[
w = \frac{k_2k_3}{y^2}, \quad yw = \frac{k_2k_3}{y}, \quad y^2 = \frac{k_2w}{y}.
\]
This is a pentagonal cyclic curve $(X, \infty)$ with $H(X, \infty) = \langle 5, 7, 11 \rangle$.

III. Singular affine equation \cite{III}: $y^3 + a_1k_2(x)y^2 + a_2k_2(x)k_3(x)y + k_2(x)^2k_3(x) = 0$, where $k_{2}(x) = (x - b_1)(x - b_2)$, $k_{3}(x) = (x - b_3)(x - b_4)(x - b_5)$, $k_{2}(x) = (x - b_6)(x - b_7)$, for pairwise distinct $b_i \in \mathbb{C}$ and $a_j$ generic constants. Here \cite{III} and \cite{III} correspond to
\[
y^2 + a_1k_2y + k_2a_2k_3 + k_2w = 0, \quad yw = k_2k_3.
\]

Multiplying the first equation by $y$ and using the second equation gives the curve’s equation. Multiplying that by $w^2$ gives
\[
w^3 + a_2k_2w^2 + a_1k_2k_3w + k_2k_3^2 = 0,
\]
which is reduced to the less order relation with respect to $w$,
\[
w^2 + a_2k_2w + a_1k_2k_3 + k_3y = 0.
\]
This curve is trigonal with $H(X, \infty) = \langle 3, 7, 8 \rangle$ but not necessarily cyclic.
2.4. Weierstrass non-gap sequence. Let the commutative ring \( R \) be that of the affine part of \((X, \infty)\), \( R := H^0(X, \mathcal{O}(\infty)) \), which is also obtained by several normalizations of \( \mathbb{C}[x, y]/(f(x, y)) \) of \([3]\) as a normal ring.

**Proposition 2.3.** The commutative ring \( R \) has the basis \( S_R := \{ \phi_i \} \subset R \) as a set of monomials of \( R \) such that

\[
R = \bigoplus_{i=0}^{\infty} \mathbb{C}\phi_i
\]

and \( \text{wt}_\infty \phi_i < \text{wt}_\infty \phi_j \) for \( i < j \), where \( \text{wt}_\infty : R \to \mathbb{Z} \) is the order of the singularity at \( \infty \).

The examples are following.

**Table 1.** The \( \phi \)'s of Examples \([2,3]\)

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| I |   | - | - | - | x | - | y | - | x^2 | - | xy | - | y^2 |   |   |
| II|   | - | - | - | x | - | y | - | x^2 | w | xy | - | y^2 |   |   |
| III| 1 | - | - | - | x^2 | y | w | x^3 | xy | xw | x^3 | y | w^2 | x | w^2 |

\[15 \quad 16 \quad 17 \quad 18 \quad 19 \quad 20 \quad 21 \quad 22 \quad 23 \quad 24\]

3. Abelian integrals and \( \theta \) functions

3.1. Abelian integrals and \( \theta \) functions. Let us consider a compact Riemann surface \( X \) of genus \( g \) and its Jacobian \( J^\circ(X) := \mathbb{C}^g/\Gamma^0 \) where \( \Gamma^0 := \mathbb{Z}^g + \tau \mathbb{Z}^g \); \( \kappa : \mathbb{C}^g \to J^\circ(X) \). Let \( \tilde{X} \) be an Abelian covering of \( X \) \((\varpi : \tilde{X} \to X)\). Since the covering space \( \tilde{X} \) is constructed by a quotient space of path space (contour of integral) fixing a point \( P \in X \), for \( \gamma_{P, P'} \in \tilde{X} \) whose ending points are \( P' \in X \) and \( P \), \( \varpi \gamma_{P, P'} \) is equal to \( P' \), and we consider an embedding \( X \) into \( \tilde{X} \) by a map \( \iota : X \to \tilde{X} \) such that \( \varpi \circ \iota = \text{id} \). We fix \( \iota \). For \( \gamma_{P, P'} \in \tilde{X} \), we define the Abelian integral \( \tilde{w}^\circ \) and the Abel-Jacobi map \( w^\circ \),

\[
\tilde{w}^\circ : S^k \tilde{X} \to \mathbb{C}^g, \quad w^\circ(\gamma_{P_1, P}, \cdots, \gamma_{P_k, P}) = \sum_{i=1}^{k} \tilde{w}^\circ(\gamma_{P_i, P}) = \sum_{i=1}^{k} \int_{\gamma_{P_i, P}} \nu^\circ,
\]
where $\mathcal{S}^k \bar{X}$ and $\mathcal{S}^k X$ are $k$-symmetric products of $\bar{X}$ and $X$ respectively, and $\nu^o$ is the column vector of the normalized holomorphic one forms $\nu^o_i$ ($i = 1, 2, \ldots, g$) of $X$; for the standard basis $(\alpha_i, \beta_i)_{i=1,\ldots,g}$ of Homology of $X$ such that $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0$, $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$, we have
\[
\int_{\alpha_i} \nu^o_j = \delta_{ij}, \quad \int_{\beta_i} \nu^o_j = \tau_{ij}.
\]

The Abel theorem shows $\kappa \circ \tilde{w}^o = w^o \circ \omega$. We fix the point $P \in X$ as a marked point $\infty$ of $(X, \infty)$.

The map $w^o$ embeds $X$ into $\mathcal{J}^o(X)$ and generalizes to a map from the space of divisors of $X$ into $\mathcal{J}^o(X)$ as $(w^o)(\sum_i n_i P_i) := \sum_i n_i w^o(P_i)$, $P_i \in X$, $n_i \in \mathbb{Z}$.

Similarly we define $\tilde{w}^o(\iota D)$ for a divisor $D$ of $X$.

The Riemann $\theta$ function, analytic in both variables $z$ and $\tau$, is defined by
\[
\theta(z, \tau) = \sum_{n \in \mathbb{Z}^g} \exp \left( 2\pi i (nz + \frac{1}{2} i \tau n) \right).
\]

The zero-divisor of $\theta$ modulo $\Gamma^o$ is denoted by $\Theta := \kappa \text{ div}(\theta) \subset \mathcal{J}^o(X)$.

The $\theta$ function with characteristic $\delta', \delta'' \in \mathbb{R}^g$ is defined as:
\[
\theta \left[ \left[ \begin{array}{c} \delta' \\ \delta'' \end{array} \right] \right](z, \tau) = \sum_{n \in \mathbb{Z}^g} \exp \left[ \pi i \left( 'n + \delta' \right) \tau' (n + \delta') + 2 \left( 'n + \delta' \right) (z + \delta'') \right].
\]

If $\delta = (\delta', \delta'') \in \{0, \frac{1}{2}\}^{2g}$, then $\theta[\delta](z, \tau) := \theta \left[ \left[ \begin{array}{c} \delta' \\ \delta'' \end{array} \right] \right](z, \tau)$ has definite parity in $z$, $\theta[\delta](-z, \tau) = e(\delta) \theta[\delta](z, \tau)$, where $e(\delta) := e^{2\pi i \delta \cdot \delta''}$. There are $2^{2g}$ different characteristics of definite parity.

In the paper [KMP3], we showed that the characteristics are defined for a Weierstrass curve by shifting the Abelian integrals and the Riemann constant as in Proposition 4.14. We recall the basic fact by Lewittes [Le]:

**Proposition 3.1.** (Lewittes [Le])

1. Using the Riemann constant $\xi \in \mathbb{C}^g$, we have the relation between the $\theta$ divisor $\Theta := \text{div}(\theta)$ and the standard $\theta$ divisor $w^o(\mathcal{S}^{g-1} X)$,
\[
\Theta = w^o(\mathcal{S}^{g-1} X) + \xi \mod \Gamma^o,
\]

i.e., for $P_i \in X$, $\theta(w^o \circ \iota(P_1, \ldots, P_{g-1}) + \xi) = 0$.

2. The canonical divisor $\mathcal{K}_X$ of $X$ and the Riemann constant $\xi$ have the relation,
\[
w^o(\mathcal{K}_X) + 2\xi = 0 \mod \Gamma^o.
\]

Further in [Le], Jorgenson found a crucial relation by considering the analytic torsion on the Jacobian:
Proposition 3.2. (Jorgenson [Jo] Theorem 1) For \( P_1, P_2, \ldots, P_\ell \in X, \ell = g - 1, \) and general complex numbers \( a_i \) and \( b_i \) \((i = 1, 2, \ldots, g)\), the following holds:

\[
\begin{array}{c|c|c|c}
\nu^\circ_1(P_1) & \nu^\circ_2(P_1) & \cdots & \nu^\circ_g(P_1) \\
\nu^\circ_1(P_2) & \nu^\circ_2(P_2) & \cdots & \nu^\circ_g(P_2) \\
\vdots & \vdots & \ddots & \vdots \\
\nu^\circ_1(P_\ell) & \nu^\circ_2(P_\ell) & \cdots & \nu^\circ_g(P_\ell) \\
\hline
a_1 & a_2 & \cdots & a_g \\
\end{array}
\begin{array}{c}
= \sum_{i=1}^g b_i \frac{\partial}{\partial z_i} \vartheta(\nu^\circ \circ (P_1, P_2, \ldots, P_\ell) + \xi) \\
\sum_{i=1}^g a_i \frac{\partial}{\partial z_i} \vartheta(\nu^\circ \circ (P_1, P_2, \ldots, P_\ell) + \xi)
\end{array}
\]

In this paper, we investigate this formula for a Weierstrass curve.

4. Holomorphic one forms of a Weierstrass curve

In this section we show several basic properties of a Weierstrass curve \((X, \infty)\) with Weierstrass semigroup \(H(X, \infty)\), including non-symmetrics ones. We consider the sheaf of the holomorphic one-form \(A_X\) and its sections \(H^0(X, A_X(*\infty))\).

The normalized holomorphic one-forms \(\{\nu^\circ_i \mid i = 1, \ldots, g\}\) of the Weierstrass curve \((X, P)\) hold the following relation:

Lemma 4.1. There are \( h \in R, a \) finite positive integer \( N \) and a subset \( \hat{S}_h := \{\phi_\ell, \in S_R \mid i = 1, \ldots, N\} \) satisfying

\[
\nu^\circ_i = \sum_{j=1}^N \alpha_{ij} \phi_\ell \frac{dx}{h}, \quad i = 1, 2, \ldots, g
\]

where \( \alpha_{ij} \) is a certain complex number for \( i = 1, \ldots, g \) and \( j = 1, \ldots, N \).

Proof. The Weierstrass curve has the natural projection \( \varpi_x : X \rightarrow \mathbb{P} \) and Nagata’s Jacobian criterion shows that \( dx \) is the natural one-form. Thus each \( \nu^\circ_i \) has an expression \( \nu^\circ_i = \frac{g_i dx}{h_i} \) for certain \( g_i, h_i \in R \). The \( h \) is the least common multiple of \( h_i \)’s. Due to Riemann-Roch theorem, the numerator must have the finite order of the singularity at \( \infty \) and thus there exists \( N < \infty \). \( \square \)

For the set \( \{(h, \hat{S}_h)\} \) whose element satisfies the condition in Lemma 4.1 we employ the pair \((h, \hat{S}_h)\) whose \( h \) has the least weight and fix it from here. The Riemann-Roch theorem enables us to find the basis of \( H^0(X, A_X(*\infty)) \).

Definition 4.2. Let us define the subset \( \hat{S}_R \) of \( R \) whose element is given by a finite sum of \( \phi_j \in S_R \) with coefficients \( a_{ij} \in \mathbb{C}, \phi_i := \sum_j a_{ij} \phi_j \), satisfying the following relations,

\[
(1) \text{ for the maximum } j \text{ in non-vanishing } a_{ij}, a_{ij} = 1,
(2) \text{ wt}_\infty \phi_i < \text{ wt}_\infty \phi_j \text{ for } i < j, \text{ and}
(3) H^0(X, A_X(*\infty)) = \bigoplus_{i=0}^N \mathbb{C} \phi_i \frac{dx}{h}.
\]
The Riemann-Roch theorem means that \( \nu_i := \frac{\hat{\phi}_i}{h} dx \) is another basis of \( H^0(X, A_X) \). Let \( \hat{S}_R^{(g)} := \{ \hat{\phi}_i \}_{i=0,1,\ldots,g-1} \) which gives the canonical embedding of the curve and we call \( \nu_i \) \( (i = 1, \ldots, g) \) unnormalized holomorphic one forms.

**Definition 4.3.**

1. The matrix \( \omega' \) is defined by

\[

\nu = 2 \omega' \nu^o, \quad \text{for} \quad \nu := \begin{pmatrix}
\nu^o_1 \\
\nu^o_2 \\
\vdots \\
\nu^o_g
\end{pmatrix}, \quad \nu^o := \begin{pmatrix}
\nu^o_1 \\
\nu^o_2 \\
\vdots \\
\nu^o_g
\end{pmatrix}.

2. The unnormalized Abelian integral and the unnormalized Abel-Jacobi map are defined by

\[\tilde{w}(P_1, \ldots, P_k) = 2 \omega' \tilde{w}^o(P_1, \ldots, P_k), \quad w(P_1, \ldots, P_k) = 2 \omega' w^o(P_1, \ldots, P_k).\]

From the definition, the following lemma is obvious:

**Lemma 4.4.** By letting \( \omega'' = \omega' \tau \), we have

\[
(\int_{\alpha_i} \nu_j) = 2 \omega', \quad (\int_{\beta_i} \nu_j) = 2 \omega''.
\]

We introduce the unnormalized lattice \( \Gamma := \langle \omega', \omega'' \rangle \mathbb{Z} \) and Jacobian \( J(X) := \mathbb{C}^g/\Gamma \).

**Table 2.** The \( \hat{\phi} \)'s in \( \hat{S}_R^{(g)} \) of Examples 2.3

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|----|----|
| H | - | - | - | - | y | - | - | - | - | - | - |
| H | - | - | - | - | y | - | - | - | - | - | - |
| H | - | - | - | - | y | - | - | - | - | - | - |

Then we have the following lemma:

**Lemma 4.5.**

1. The matrix \( \omega' \) is regular.

2. The divisor of \( \frac{dx}{h} \) can be expressed by \( (2g - 2 + d_1)\infty - \mathfrak{B} \) for a certain effective divisor \( \mathfrak{B} \) whose degree is \( d_1 \geq 0 \), i.e., \( \left( \frac{dx}{h} \right) = (2g - 2 + d_1)\infty - \mathfrak{B} \).

3. \( \text{wt}_\infty \hat{\phi}_g = (2g - 2) + d_1 \).

4. \( (\hat{\phi}_i) \geq (\mathfrak{B} - (2g - 2 + d_1)\infty) \) for every \( \hat{\phi}_i \in \hat{S}_R^{(g)} \), \( (i = 0, 1, 2, \ldots, g - 1) \).

5. \( (\hat{\phi}_i) \geq (\mathfrak{B} - (g - 1 + d_1 + i)\infty) \) for every \( \hat{\phi}_i \in \hat{S}_R \), \( (i \geq g) \).

**Proof.** (1) is trivial. (2) is obvious because degree of meromorphic one-form is \( 2g - 2 \) and \( h \) is the element of \( R \). From the Riemann-Roch theorem, \( \text{wt}_\infty \nu_g = 0 \) and thus, we obtain (3), (4) and (5). \( \square \)

**Remark 4.6.** The degree of \( \mathfrak{B} \) vanishes, if and only if the Weierstrass semigroup is symmetric and \( 2g - 1 \) is a gap.
Now we introduce meromorphic functions over $S^n X$, which can be regarded as a natural generalization of $U$-function of Mumford triplet $UVW$ [Mu].

**Definition 4.7.** We define the Frobenius-Stickelberger (FS) matrix with entries in $S_R$: let $n$ be a positive integer and $P_1, \ldots, P_n$ ($1 \leq n$) be in $X \setminus \infty$,

$$\hat{\Psi}_n(P_1, P_2, \ldots, P_n) := \begin{pmatrix} \hat{\phi}_0(P_1) & \hat{\phi}_1(P_1) & \hat{\phi}_2(P_1) & \cdots & \hat{\phi}_{n-1}(P_1) \\ \hat{\phi}_0(P_2) & \hat{\phi}_1(P_2) & \hat{\phi}_2(P_2) & \cdots & \hat{\phi}_{n-1}(P_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\phi}_0(P_n) & \hat{\phi}_1(P_n) & \hat{\phi}_2(P_n) & \cdots & \hat{\phi}_{n-1}(P_n) \end{pmatrix}.$$ 

The Frobenius-Stickelberger (FS) determinant is

$$\hat{\psi}_n(P_1, \ldots, P_n) := \det(\hat{\Psi}_n(P_1, \ldots, P_n)).$$

We define the meromorphic function,

$$\hat{\mu}_n(P) := \hat{\mu}_n(P; P_1, \ldots, P_n) := \lim_{P_i' \to P_i} \frac{1}{\hat{\psi}_n(P_1', \ldots, P_n')} \hat{\psi}_{n+1}(P_1', \ldots, P_n', P),$$

where the $P_i'$ are generic, the limit is taken (irrespective of the order) for each $i$, and the meromorphic functions $\hat{\mu}_{n,k}$'s by

$$\hat{\mu}_n(P) = \hat{\phi}_n(P) + \sum_{k=0}^{n-1} (-1)^{n-k} \hat{\mu}_{n,k}(P_1, \ldots, P_n) \hat{\phi}_k(P),$$

with the convention $\mu_{n,n}(P_1, \ldots, P_n) = \mu_{n,n}(P_1, \ldots, P_n) = 1$.

**Remark 4.8.** When $X$ is a hyperelliptic curve, by letting $P \in X$ and $P_t \in X$ expressed by $(x, y)$ and $(x_t, y_t)$, $\hat{\mu}_k$ is identical to $U$ in [Mu]:

$$\hat{\mu}_k(P; P_1, \ldots, P_k) = (x - x_1)(x - x_2) \cdots (x - x_k)$$

and each $\hat{\mu}_{k,i}$ is an elementary symmetric polynomial of $x_i$'s.

We mention the behavior of the meromorphic function $\mu_{n,k}(P_1, \ldots, P_n)$, which is obvious:

**Lemma 4.9.** For $k < n$, the order of singularity $\mu_{n,k}(P_1, \ldots, P_n)$ as a function of $P_n$ at $\infty$ is $\text{wt} \phi_{n} - \text{wt} \phi_{n-1}$.

Using $\mu_n(P; P_1, \ldots, P_n)$, we have an addition structure in Jacobin $J(X)$ as the linear system; “$\sim$” means the linear equivalence.

**Proposition 4.10.** The divisor of $\hat{\mu}_{g-1}(P; P_1, \ldots, P_{g-1})$ with respect to $P$ is given by

$$[\hat{\mu}_{g-1}] = \sum_{i=1}^{g-1} P_i + \sum_{i=1}^{g-1} Q_i + \mathfrak{B} - (2g - 2 + d_1) \infty,$$

where $Q_i$'s are certain points of $X$.

**Proof.** It is obvious that there is a non-negative number $N$ satisfying the relation,

$$[\hat{\mu}_{g-1}] = \sum_{i=1}^{g-1} P_i + \sum_{i=1}^{N} Q_i + \mathfrak{B} - (g - 1 + N + d_1) \infty,$$

and Lemma 4.5 (3) means $N = g - 1$. \qed
**Lemma 4.11.** There is a certain divisor \(\mathfrak{B}_0\) such that \(\mathfrak{B}_1 - d_1\infty = 2(\mathfrak{B}_0 - d_0\infty)\) and thus we have the following linear equivalence,

\[
\sum_{i=1}^{g-1} P_i + \mathfrak{B}_0 - 2(g-1 + d_0)\infty \sim - \left( \sum_{i=1}^{g-1} Q_i + \mathfrak{B}_0 - 2(g-1 + d_0)\infty \right).
\]

Proof. The Abel-Jacobi theorem implies that since the Abel-Jacobi map is surjective, the existence \(\mathfrak{B}_0\) is obvious. As the linear equivalence, we have

\[
\sum_{i=1}^{g-1} P_i + \sum_{i=1}^{g-1} Q_i + 2\mathfrak{B}_0 - 2(g-1 + d_0)\infty \sim 0,
\]

where \(Q_i\)'s are certain divisors of \(X\).

Lemma 4.11 implies that the image of the Abel-Jacobi map has the symmetry of the Jacobian as in the following lemma:

**Lemma 4.12.** By defining the shifted Abel-Jacobi map,

\[
w_s(P_1, \ldots, P_k) := w(P_1, \ldots, P_k) + w(\mathfrak{B}_0),
\]

the following relation holds:

\[-w_s(S^{g-1}X) = w_s(S^{g-1}X).\]

Proof. Lemma 4.11 means \(w_s(P_1, \ldots, P_{g-1}) = -w_s(Q_1, \ldots, Q_{g-1})\) and thus \(-w_s(S^{g-1}X) \subset w_s(S^{g-1}X)\). It leads the relation.

**Proposition 4.13.** The canonical divisor \(K_X \sim (2g - 2 + 2d_0)\infty - 2\mathfrak{B}_0\).

For \((\gamma_1, \gamma_2, \ldots, \gamma_k) \in S^k \overline{X}\), we also define the shifted Abelian integral \(w_s\) by

\[
\tilde{w}_s(\gamma_1, \gamma_2, \ldots, \gamma_k) := \tilde{w}(\gamma_1, \gamma_2, \ldots, \gamma_k) + \tilde{w} \circ \iota(\mathfrak{B}_0).
\]

We recall our previous results on the Riemann constant [KMP3]:

**Proposition 4.14.**

1. If \(d_1 > 0\), the Riemann constant \(\xi\) is not a half period of \(\Gamma^0\).
2. The shifted Riemann constant \(\xi_s := \xi - \frac{1}{2}\omega^{0-1}\tilde{w} \circ \iota(\mathfrak{B}_0)\) is the half period of \(\Gamma^0\).
3. By using the shifted Abel-Jacobi map, we have

\[
\Theta = \frac{1}{2} \omega^{0-1}w_s(S^{g-1}X) + \xi_s \quad \text{modulo} \quad \Gamma^0,
\]

i.e., for \(P_i \in X\), \(\theta \left( \frac{1}{2}\omega^{0-1}\tilde{w} \circ \iota(P_1, \ldots, P_{g-1}) + \xi_s \right) = 0\).
4. There is a \(\theta\)-characteristic \(\delta\) of a half period which represents the shifted Riemann constant \(\xi_s\), i.e., \(\theta[\delta] \left( \frac{1}{2}\omega^{0-1}\tilde{w} \circ \iota(P_1, \ldots, P_{g-1}) \right) = 0\).

In order to describe the Jacobi inversion formulae for the strata of the Jacobian, we define the Wirtinger variety \(W_k := w_s(S^kX)\) and

\[
\Theta_k := W_k \bigcup [-1]W_k,
\]

where \([-1]\) is the minus operation on the Jacobian \(J(X) := \Theta_g\). We also define the strata: \(W^k_{g-1} := w_s(S^k_\infty X)\) \((W^k_i := (S^k_\infty X))\), where \(S^k_\infty(X) := \{D \in S^n(X) \mid \dim|D| \geq m\}\).

The Abel-Jacobi theorem and Lemma 4.12 mean that \(W_g = J(X)\) and \([-1]W_i = W_i\) for \(i = g - 1, g\) but in general, \([-1]W_i\) does not equal to \(W_i\) for \(i < g - 1\).
5. Jacobi inversion formulae for any Weierstrass curve

The \( \mu \) function enables us to rewrite Jorgenson’s relation in Proposition 3.2. Since any element \( u = (u_1, u_2, \ldots, u_g) \) of \( \mathbb{C}^g \) is given as \( u = \tilde{\omega}(\gamma_1, \ldots, \gamma_g) \) for a certain element \( (\gamma_1, \ldots, \gamma_g) \in S^g \tilde{X} \), we consider the differential \( \partial_i := \frac{\partial}{\partial u_i} \) of a function on \( \mathbb{C}^g \) and have the following proposition:

**Proposition 5.1.** For \( P_1, P_2, \ldots, P_\ell \in X \) and \( \ell = g - 1 \), we have

\[
\tilde{\mu}_g(P_1, P_2, \ldots, P_\ell) = \phi_\ell(P) + \sum_{i=1}^{g} \partial_i \theta \left( \frac{1}{2} \tilde{\omega}^{-1} \tilde{\omega} \circ \iota(P_1, P_2, \ldots, P_\ell) + \xi \right) \phi_{\ell-1}(P).
\]

The right hand side does not depend on the choice of the embedding \( \iota \).

**Proof.** First we rewrite Proposition 3.2 in terms of the unnormalized holomorphic one forms. It is obvious that for general complex numbers \( a_i \) and \( b_i \) (\( i = 1, 2, \ldots, g \)), we have

\[
\begin{vmatrix}
\nu_1(P_1) & \nu_2(P_1) & \cdots & \nu_g(P_1) \\
\nu_1(P_2) & \nu_2(P_2) & \cdots & \nu_g(P_2) \\
\vdots & \vdots & \ddots & \vdots \\
\nu_1(P_{\ell}) & \nu_2(P_{\ell}) & \cdots & \nu_g(P_{\ell}) \\
\end{vmatrix}
= \sum_{i=1}^{g} a_i \partial_i \theta \left( \frac{1}{2} \tilde{\omega}^{-1} \tilde{\omega} \circ \iota(P_1, P_2, \ldots, P_\ell) + \xi \right)
\]

By letting \( (a_1, \ldots, a_g) = (\nu_1(P), \ldots, \nu_g(P)) \) and \( (b_1, \ldots, b_g) = (0, \ldots, 0, \nu_g(P)) \) for a generic point \( P \in X \), the relation is reduced to the proposition. \( \square \)

Using Jorgenson’s result with the ordering of \( \phi_1 \), we show our main theorem:

**Theorem 5.2.** For \( (P_1, \ldots, P_k) \in S^kX \setminus S^kX \) \( (k < g) \) and a positive integer \( \ell \leq k \), we have the relation,

\[
\tilde{\mu}_{k,\ell-1}(P_1, \ldots, P_k) = \frac{\partial_\ell \theta \left( \frac{1}{2} \tilde{\omega}^{-1} \tilde{\omega} \circ \iota(P_1, P_2, \ldots, P_k) + \xi \right)}{\partial_{k+1} \theta \left( \frac{1}{2} \tilde{\omega}^{-1} \tilde{\omega} \circ \iota(P_1, P_2, \ldots, P_k) + \xi \right)},
\]

and for a certain \( \theta \)-characteristic \( \delta \) for \( \xi \),

\[
\tilde{\mu}_{k,\ell-1}(P_1, \ldots, P_k) = \frac{\partial_\ell \theta[\delta] \left( \frac{1}{2} \tilde{\omega}^{-1} \tilde{\omega} \circ \iota(P_1, P_2, \ldots, P_k) \right)}{\partial_{k+1} \theta[\delta] \left( \frac{1}{2} \tilde{\omega}^{-1} \tilde{\omega} \circ \iota(P_1, P_2, \ldots, P_k) \right)},
\]
especially $k = 1$ case,
\[
\frac{\hat{\phi}_1(P_1)}{\hat{\phi}_0(P_1)} = \frac{\partial_1 \theta[\delta] \left( \frac{1}{2} \omega'^{-1} \bar{w}_s \circ \iota(P_1) \right)}{\partial_2 \theta[\delta] \left( \frac{1}{2} \omega'^{-1} \bar{w}_s \circ \iota(P_1) \right)}.
\]

Proof: Since these $\hat{\phi}_i$’s have the natural ordering for $P_k \to \infty$, we could apply the proof of Theorem 5.1 (3) in [MP1] to this case. It means that we prove it inductively. For $k < g$ and $i < k$, using the property of Lemma 4.9 we consider $\hat{\mu}_{k,i-1}(P_1,P_2,\ldots,P_k)/\hat{\mu}_{k,k-1}(P_1,P_2,\ldots,P_k)$ and its limit $P_k \to \infty$. Then we have the relation for $\hat{\mu}_{k-1,i-1}$. □

Theorem 5.2 means the Jacobi inversion formula of $\Theta_k$ for a general Weierstrass curve. Using the natural basis of the Weierstrass non-gap sequence of the Weierstrass curve, we can consider the precise structure of strata $\Theta_k$ of the Jacobian. The relation contains the cyclic $(r,s)$ curve cases and trigonal curve cases reported in [MP1, MP2, KMP2].

Since for every compact Riemann surface $Y$, there exists a Weierstrass curve $(X,\infty)$ which is birationally-equivalent to $Y$, Theorem 5.2 gives the structure of the strata in the Jacobian of $Y$.

Remark 5.3. Nakayashiki investigated precise structure of the $\theta$ function for a pointed compact Riemann surface whose Weierstrass gap is associated with a Young diagram [N2]. He refined the Riemann-Kempf theory [ACGH] in terms of the unnormalized holomorphic one-forms. Using the results, it is easy to rewrite the diagram [N2]. He refined the Riemann-Kempf theory in terms of the pointed compact Riemann surface whose Weierstrass gap is associated with a Young diagram [N2].

Remark 5.4. Though it is well-known that the sine-Gordon equation is an algebraic relations of the differentials of $\mu_g$-function in the hyperelliptic Jacobian $J(X)$ [Mu, Pt2], the equation can be generalized to one in strata in the Jacobian $J(X)$ using the $\mu_k$ function $k < g$ [M2]. Ayano and Buchstaber investigated similar structure of stratum of the hyperelliptic Jacobian of genus three to find a novel differential equation which characterizes the stratum [AB]. In other words, it is expected that these relations in our theorem might show some algebraic relations in the strata in the Jacobian of a general compact Riemann surface.

For example, for a Weierstrass curve $X$ whose $\hat{\phi}_1/\hat{\phi}_0$ equals to $x$, we have the Burgers equation,
\[
\frac{\partial}{\partial u_i} \frac{\partial_1 \theta[\delta] \left( \frac{1}{2} \omega'^{-1} w_s(P) \right)}{\partial_2 \theta[\delta] \left( \frac{1}{2} \omega'^{-1} w_s(P) \right)} - \frac{\hat{\phi}_1(P)}{\hat{\phi}_0(P)} \frac{\partial}{\partial u_j} \frac{\partial_1 \theta[\delta] \left( \frac{1}{2} \omega'^{-1} w_s(P) \right)}{\partial_2 \theta[\delta] \left( \frac{1}{2} \omega'^{-1} w_s(P) \right)} = 0.
\]

Remark 5.5. Korotkin and Shramchenko defined the $\sigma$ function for a general compact Riemann surface based on the Klein’s investigation [KS]; they assume that the $\theta$ characteristic $\delta$ as a half period of Jacobian $\Gamma^o$ is given by the Riemann constant $\xi$ with $w(D_s)$ of the divisor $D_s$ of the spin structure of the curve $X$. The spin structure corresponds to $\mathfrak{B}_0$ in Propositions 4.14 and 4.13 [At].

We have constructed the $\sigma$ functions for trigonal curves [MP1, MP2, KMP2] using the EEL-construction proposed by Eilbeck, Enolskiĭ and Leĭkin [EEL] whose origin is Baker [BI]; the EEL-construction differs from Klein’s approach. Due to the arguments on this paper, Emma Previato posed a problem whether we could reproduce the $\sigma$ function of Korotkin and Shramchenko by means of the
EEL-construction for the Weierstrass curve. It means an extension of the EEL-construction of the \( \sigma \) function.

Further using the extension and Nakayashiki’s investigation in [N2], we could connect our results to Sato Grassmannian in the Sato universal Grassmannian theory and investigate the \( \sigma \) function of a Weierstrass curve more precisely.

Since it is known that there are infinitely many numerical semigroups which are not Weierstrass, but the distribution of Weierstrass semigroups in the set of the numerical semigroups is not clarified. Thus it is a natural question how the Weierstrass curves are characterized in Sato Grassmannian manifolds.

### 6. An Example

**6.1. Curve X of Example II in [2,3]** Let us consider the curve \( X \) of Example II in [2,3]. For a point \( P_i = (x_i, y_i, w_i) \) (\( i = 1, 2, \ldots, 7 \)) of \( X \), the \( \hat{\mu}_7 \) is given by

\[
\hat{\mu}_7(P_1, P_2, \ldots, P_7) = \begin{vmatrix}
1 & w_1 & x_1 y_1 & y_1^2 & x_1 w_1 & x_1^2 y_1 & w_1 y_1 \\
y_2 & w_2 & x_2 y_2 & y_2^2 & x_2 w_2 & x_2^2 y_2 & w_2 y_2 \\
y_3 & w_3 & x_3 y_3 & y_3^2 & x_3 w_3 & x_3^2 y_3 & w_3 y_3 \\
y_4 & w_4 & x_4 y_4 & y_4^2 & x_4 w_4 & x_4^2 y_4 & w_4 y_4 \\
y_5 & w_5 & x_5 y_5 & y_5^2 & x_5 w_5 & x_5^2 y_5 & w_5 y_5 \\
y_6 & w_6 & x_6 y_6 & y_6^2 & x_6 w_6 & x_6^2 y_6 & w_6 y_6 \\
y_7 & w_7 & x_7 y_7 & y_7^2 & x_7 w_7 & x_7^2 y_7 & w_7 y_7 \\
\end{vmatrix}
\]

Thus we have the following Proposition:

**Proposition 6.1.** 1) For \( (P_1, P_2, \ldots, P_k) \in X \times (S^k(X) \setminus S_1^k(X)) \) for \( k < 8 \), we have

\[
\hat{\mu}_{k,i-1}(P_1, \ldots, P_k) = \frac{\partial_i \theta[\delta] \left( \frac{1}{2} \omega^{i-1} w_i(P_1, P_2, \ldots, P_k) \right)}{\partial_k \theta[\delta] \left( \frac{1}{2} \omega^{i-1} w_i(P_1, P_2, \ldots, P_k) \right)}
\]

especially

\[
\frac{\partial_1 \theta[\delta] \left( \frac{1}{2} \omega^{i-1} w_i(P_1) \right)}{\partial_2 \theta[\delta] \left( \frac{1}{2} \omega^{i-1} w_i(P_1) \right)} = \frac{w_1}{y_1}.
\]

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