Characteristic energy-dissipating functions of muon penetrating through matters

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Abstract. The characteristic energy-dissipating functions of muon corresponding to positron-electron pair production, bremsstrahlung, and photonuclear interactions are expressed in series-expansions, to investigate transmission properties of muon effectively by analytical methods developed in the cascade shower theory. Accuracies of the new expression are examined by comparing one of the characteristic functions with that derived by the traditional analytical method. We apply the new-obtained characteristic functions to derive the survival probability of muon and compare the result with that of electron, discussing the discrepancies between the both.

1. Introduction

Dense structures of matter are now inspected by measuring atmospheric muons penetrating through the matter [1, 2, 3, 4], where the energy dissipations of muon during the passage should be accurately evaluated. The energy dissipation of muon among its passage is well predicted by the cascade theory improved by Nishimura [5], where the dissipation is described through M-function \( M(s, q, t) \) with the characteristic energy-dissipating functions \( A(s) \) [5, 6].

Unfortunately, the characteristic functions \( A(s) \) cannot be expressed by explicit functions applicable conveniently, so that we describe them for muon in series-expansions with the several coefficients specific to the matter. Contributions of the cross-section elements of positron-electron pair production (P), bremsstrahlung (B), and photonuclear interaction (N) to the characteristic functions are compared and discussed quantitatively and qualitatively. The difference of the characteristic functions between muon’s and electron’s is investigated by applying the functions to evaluate the survival probabilities.

2. Derivation of the characteristic energy-dissipating functions of muon

2.1. The characteristic energy-dissipating functions expressed in series-expansions

The characteristic energy-dissipating function is defined as

\[
A(s) \equiv b^{-1} \int_0^1 \{1 - (1 - v)^s\} \phi(v) dv,
\]

(1)
where \( \phi(v) dv \) denotes the probability of fractional energy loss per g/cm\(^2\) and \( b \approx 4.0 \times 10^{-6} \) g\(^{-1}\)cm\(^2\) denotes the reciprocal of the radiation unit of muon for Standard Rock \((Z = 11, A = 22)\) [7]. As the probability density \( \phi(v) \) for muon is additive as

\[
\phi(v) \equiv \phi_P(v) + \phi_B(v) + \phi_N(v)
\]

among the cross-section elements of \( P, B, \) and \( N \) indicated in figures 1-3, the characteristic function \( A(s) \) is also additive as

\[
A(s) \equiv A_P(s) + A_B(s) + A_N(s),
\]

where each term \( A_s(s) \) of the right-hand side is defined as

\[
A_s(s) \equiv b^{-1} \int_0^1 \{1 - (1 - v)^s\} \phi_s(v) dv.
\]

We express \( A_s(s) \)’s in series-expansions. Expanding \( \{1 - (1 - v)^s\} \) with \( v \) at \( v = 0 \), we have

\[
\{1 - (1 - v)^s\} = sv - \frac{s(s - 1)v^2}{2} + \frac{s(s - 1)(s - 2)v^3}{6} - \cdots = -\sum_{k=1}^{\infty} \frac{(-)^k}{k!} s[k] v^k,
\]

where \( s[0] \equiv 1 \) and \( s[k] \equiv s(s - 1) \cdots (s - k + 1) \),

thus we have

\[
A_s(s) \equiv \sum_{k=1}^{\infty} \frac{(-)^k}{k!} \alpha_{s,k} s[k] \quad \text{with} \quad \alpha_{s,k} = -b^{-1} \int_0^1 v^k \phi_s(v) dv.
\]

Likewise we can express the derivatives of \( A_s(s) \) as

\[
A'_s(s) \equiv \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \alpha'_{s,k} s[k] \quad \text{with} \quad \alpha'_{s,k} = -b^{-1} \int_0^1 v^k \phi_s(v) \ln(1 - v) dv,
\]

\[
A''_s(s) \equiv \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \alpha''_{s,k} s[k] \quad \text{with} \quad \alpha''_{s,k} = -b^{-1} \int_0^1 v^k \phi_s(v) \ln^2(1 - v) dv.
\]

2.2. The characteristic functions for positron-electron pair production

The probability density \( \phi_P(E, v) \) of muon with energy \( E \) to dissipate fractional energy of \( v \) by positron-electron pair production is described by Bugaev et al. [8] as

\[
\phi_P(E, v) = \frac{16}{\pi} Z(Z + 1)(\alpha v)^2 N \frac{1.7 \times 10^{-4}(v + 1.05 \times 10^{-4})}{v^2(v + 0.006)^2} \frac{1 + 2v}{1 + k/(vE)}
\]

\[
\times \left\{ 1 - \frac{\exp[-0.025 \ln^2(E/\mu)]}{1 + 0.323 \ln(10^5v)} \right\}, \quad k = 0.02\text{GeV},
\]

for \( 10^{-3} < v < 0.2 \) as indicated in figure 1. This density is approximated at \( E \rightarrow \infty \) by

\[
\phi_P(v) \approx 2.20 \times 10^{-9} \frac{Z(Z + 1)}{v(v + 0.006)^2} \text{ g}^{-1}\text{cm}^2,
\]

regarding \( 1.05 \times 10^{-4} \ll v \), where we take 1 instead of \( (1 + 2v)/(1 + k/(vE)) \) at \( 0.2 < v \). Note that Kobayakawa took the same \( v \)-dependence in the probability density [9], \( \alpha_{P,k} \), \( \alpha'_{P,k} \), and \( \alpha''_{P,k} \) derived by Eqs. (7)-(9) for Standard Rock from \( \phi_P(v) \) are indicated in Table 1-3.
2.3. The characteristic functions for bremsstrahlung

The probability density $\phi_B(E,v)$ of muon with energy $E$ to dissipate fractional energy of $v$ by bremsstrahlung is described in GEANT [10] based on Kelber et al. [11] as

$$\phi_B(E, v) = \frac{16}{3} N \left( \frac{m}{\mu} \right)^2 \frac{1}{v A} Z \left( Z \Phi_n + \Phi_e \right) \left\{ 1 - v + \frac{3}{4} v^2 \right\}$$

$$= 3.27 \times 10^{-8} \frac{Z \left( Z \Phi_n + \Phi_e \right)}{A} \frac{(4/3)(1 - v) + v^2}{v} \text{ g}^{-1} \text{cm}^2,$$

with

$$\Phi_n = \ln \left( \frac{B Z^{-1/3} \left( \frac{\mu + \delta (D'_n \sqrt{e} - 2)}{D_n (m + \sqrt{e} B Z^{-1/3})} \right)}{D'_n (m + \sqrt{e} B Z^{-1/3})} \right),$$

$$\Phi_e = \ln \left( \frac{B' Z^{-2/3} \mu}{\left( 1 + \frac{\mu^2}{m^2 \sqrt{e}} \right) (m + \sqrt{e} B' Z^{-2/3})} \right),$$

$$\delta = \frac{\mu^2 v}{2E(1 - v)}, \quad D'_n = D_n^{(1-1/2)}, \quad D_n = 1.54A^{0.27},$$

$$B = 183, \quad B' = 1429, \quad \sqrt{e} = 1.648 \cdots,$$

Table 1. Coefficients $\alpha_{s,k}$ and $\alpha_k$ applied to the series expansion of the characteristic function $A(s)$.

| $k$ | $\alpha_{P,k}$ | $\alpha_{B,k}$ | $\alpha_{N,k}$ | $\alpha_k$ |
|-----|----------------|----------------|----------------|-----------|
| 1   | -0.54672       | -0.43          | -0.0817365     | -1.05846  |
| 2   | -0.0136222     | -0.203056      | -0.0237638     | -0.240442 |
| 3   | -0.00311685    | -0.133778      | -0.0119711     | -0.14866  |
| 4   | -0.00161211    | -0.100333      | -0.00749212    | -0.109437 |
| 5   | -0.00108054    | -0.0805397     | -0.0052516     | -0.0868718|
| 6   | -0.000811975   | -0.0674008     | -0.00394304    | -0.0721558|
| 7   | -0.000650217   | -0.0580159     | -0.00309962    | -0.0617657|
| 8   | -0.000542168   | -0.050963      | -0.00251783    | -0.054023 |
| 9   | -0.000468999   | -0.0454613     | -0.00209615    | -0.0480223|
| 10  | -0.000406902   | -0.0410455     | -0.00177882    | -0.0432312|
Table 2. Coefficients $\alpha'_{s,k}$ and $\alpha'_{k}$ applied to the series expansion of the characteristic function $A'(s)$.

| $k$ | $\alpha'_{P,k}$ | $\alpha'_{B,k}$ | $\alpha'_{N,k}$ | $\alpha'_{k}$ |
|-----|-----------------|-----------------|-----------------|---------------|
| 0   | 0.555921        | 0.692262        | 0.103377        | 1.35156       |
| 1   | 0.0168017       | 0.406111        | 0.0367687       | 0.459682      |
| 2   | 0.00520665      | 0.303588        | 0.0211748       | 0.329969      |
| 3   | 0.00323692      | 0.248126        | 0.0145632       | 0.265926      |
| 4   | 0.00243597      | 0.212372        | 0.0109579       | 0.225766      |
| 5   | 0.00198732      | 0.186987        | 0.00870106      | 0.197675      |
| 6   | 0.00169481      | 0.167829        | 0.00716098      | 0.176684      |
| 7   | 0.00148659      | 0.152749        | 0.00604624      | 0.160282      |
| 8   | 0.0013296       | 0.14051         | 0.0052042       | 0.147043      |
| 9   | 0.00120634      | 0.130338        | 0.00454725      | 0.136092      |
| 10  | 0.00110659      | 0.121726        | 0.00402153      | 0.126854      |

as indicated in figure 2. As they approach

$$\Phi_n \to \ln \frac{BZ^{-1/3} \mu}{D'_n m} \to 8.59,$$

$$\Phi_e \to \ln (B'Z^{-2/3} \mu/m) \to 11.0$$

at $E \to \infty$, the density is approximated by

$$\phi_B(v) = 1.72 \times 10^{-6} \frac{(4/3)(1-v) + v^2}{v} g^{-1}cm^2.$$

$\alpha_{B,k}, \alpha'_{B,k}, \text{ and } \alpha''_{B,k}$ derived by Eqs. (7)-(9) for Standard Rock from $\phi_B(v)$ are indicated in Table 1-3.

2.4. The characteristic functions for photonuclear interaction

The probability density $\phi_N(E,v)$ of muon with energy $E$ to dissipate fractional energy of $v$ by photonuclear interaction is described by Borog and Petrukhin [12] and applied to GEANT [10],

Table 3. Coefficients $\alpha''_{s,k}$ and $\alpha''_{k}$ applied to the series expansion of the characteristic function $A''(s)$.

| $k$ | $\alpha''_{P,k}$ | $\alpha''_{B,k}$ | $\alpha''_{N,k}$ | $\alpha''_{k}$ |
|-----|------------------|------------------|------------------|----------------|
| 0   | -0.0237038       | -0.984192        | -0.0631867       | -1.07108       |
| 1   | -0.0105516       | -0.820185        | -0.0415348       | -0.872272      |
| 2   | -0.0078061       | -0.720482        | -0.0312109       | -0.759499      |
| 3   | -0.00650595      | -0.650553        | -0.0250611       | -0.68212       |
| 4   | -0.0059665       | -0.597565        | -0.0209362       | -0.624198      |
| 5   | -0.00512585      | -0.555418        | -0.0179593       | -0.578503      |
| 6   | -0.00469349      | -0.520758        | -0.0157016       | -0.541153      |
| 7   | -0.00435046      | -0.491548        | -0.0139269       | -0.509825      |
| 8   | -0.00406926      | -0.466466        | -0.0124937       | -0.483029      |
| 9   | -0.00383309      | -0.444607        | -0.0113112       | -0.459751      |
| 10  | -0.00363095      | -0.425324        | -0.0103188       | -0.439274      |
as
\[
\phi_N(E, v) = \frac{\alpha A_{\text{eff}} N}{\pi A} \sigma_{\gamma N} \frac{1}{v} \left\{ v - 1 + \left[ 1 - v + \frac{v^2}{2} \left( 1 + \frac{2\mu^2}{A^2} \right) \right] \ln \frac{E^2(1-v)}{v^2} \left( 1 + \frac{\mu^2 v^2}{\Lambda^2(1-v)} \right) \right\},
\]
(19)
where \( \Lambda^2 = 0.4 \text{ GeV}^2 \), \( \sigma_{\gamma N} = 100 \mu\text{b/nuc} \), and \( A_{\text{eff}} = 0.22A + 0.78A^{0.89} \),
(20)
as indicated in figure 3. This density is approximated at \( E \to \infty \) by
\[
\phi_N(v) \simeq 1.40 \times 10^{-7} A_{\text{eff}} \frac{1}{v} \left\{ v - 1 + [1 - v + 0.528 v^2] \ln \left[ 1 + \frac{1 - v}{0.0279 v^2} \right] \right\} \text{ g}^{-1}\text{cm}^2.
\]
(21)
\( \alpha_{N,k} \), \( \alpha'_{N,k} \), and \( \alpha''_{N,k} \) derived by Eqs. (7)-(9) for Standard Rock from \( \phi_N(v) \) are indicated in Table 1-3.

3. Accuracies of the characteristic energy-dissipating function expressed by the series-expansion

We confirm accuracies of the reconstructed \( A_\gamma(s) \) of (7) by comparing that for bremsstrahlung with the exact \( A_B(s) \) derived analytically by applying Eq. (4) on the probability density (18):
\[
A_B(s) = \frac{1.72 \times 10^{-6} \text{ g}^{-1}\text{cm}^2}{b} \left\{ \frac{4}{3} \left( \psi(s+2) - \psi(2) \right) - \frac{1}{(s+1)(s+2)} + \frac{1}{2} \right\},
\]
(22)
indicated in figure 4 (thick line).

The reconstructed \( A_B(s) \) by Eq. (7) with \( k \) up to 8, 9, 10 are also indicated in the figure (broken, dot, and thin lines), which are approaching the exact one at \( s > 0 \) with the increase of \( k \).

4. Comparison of the characteristic energy-dissipating function and the survival probability between muon and electron

P, B and N elements of \( A_\gamma(s) \) for muon are indicated in figure 5 (thin solid, broken and dot lines). P and B elements are comparable at \( 0 < s < \sim 1 \); though P element exceed B much at \( \sim 1 < s \), increasing almost linearly. Almost linear increase of \( A_P(s) \) corresponds to almost continuous (less fluctuated) energy-dissipation of P process, as Kobayakawa obtained \( A(s) \simeq s b_{\text{eff}} \) in case without fluctuations [13]. This property is confirmed in narrower distribution of \( \phi_P(v) \) than \( \phi_N(v) \) and \( \phi_N(v) \) indicated in figures 1-3.

The characteristic energy-dissipating function \( A(s) \) of muon and that of electron are indicated in the figure (thick solid and broken lines). \( A(s) \) of muon is comparable with that of electron at \( 0 < s < \sim 1.5 \), though the former exceeds the latter much reflecting the dominating linear increase of \( A_P(s) \) of muon.

We can evaluate the survival probability of muon at \( t \) of radiation length with its energy greater than \( E \) [6] by applying the characteristic energy-dissipating function \( A(s) \) derived above,
\[
\mu(E_0, > E, t) = \frac{1}{2\pi t} \int \frac{ds}{s} \frac{E_0}{E} \left. s^{-A(s)t} \right| \frac{E_0}{E} \right. \left. e^{-A(s)t} \right| \sqrt{2\pi(1 - \bar{s}^2 A''(\bar{s})t)}
\]
with \( t = \ln(E_0/E) - 1/\bar{s} \) \( /A'(\bar{s}) \) and \( 0 < \bar{s} \),
(23)
where \( \bar{s} \) denotes the saddle point. We compare in figure 6 the so obtained survival probability of muon (\( \mu(E_0, > E, t) \), thick line) with that of electron (\( \pi(E_0, > E, t) \), thin line) indicated in Eqs. (3.4) and (3.5) of [6]. We have to take \( 0 < \bar{s} < \sim 10 \) for muon instead of \( 0 < \bar{s} < \sim 2 \) for electron. The survival probability of muon decreases slower than that of electron at the early stage of penetration though the former drops abruptly at the late stage of penetration, which reflects a more continuous property of energy-dissipation for muon than for electron.
5. Conclusion
The characteristic energy-dissipating functions of muon are described with continuous functions with $s$ expressed in series-expansions with the several coefficients indicated in Tables for Standard Rock.

Accuracies of the new expressed function are confirmed in case of bremsstrahlung element by comparing the result with the exact one derived analytically from the definition integral.

Properties of P, B and N elements of energy dissipation are investigated by comparing the characteristic functions among them. Differences of energy dissipation between muon and electron are investigated and discussed by comparing the survival probabilities of them evaluated with the characteristic functions expressed in the series.

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