Graph homology of moduli space of pointed real curves of genus zero

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Abstract

The moduli space $\mathcal{M}_S^\sigma(\mathbb{R})$ parameterizes the isomorphism classes of $S$-pointed stable real curves of genus zero which are invariant under relabeling by the involution $\sigma$. This moduli space is stratified according to the degeneration types of $\sigma$-invariant curves. The degeneration types of $\sigma$-invariant curves are encoded by their dual trees with additional decorations. We construct a combinatorial graph complex generated by the fundamental classes of strata of $\mathcal{M}_S^\sigma(\mathbb{R})$. We show that the homology of $\mathcal{M}_S^\sigma(\mathbb{R})$ is isomorphic to the homology of our graph complex. We also give a presentation of the fundamental group of $\mathcal{M}_S^\sigma(\mathbb{R})$.

1 Introduction

Moduli of pointed real curves of genus zero

Let $S = \{s_1, \ldots, s_n\}$ be a finite set, and $\sigma$ be an involution acting on it. The moduli space $\mathcal{M}_S^\sigma(\mathbb{R})$ of $\sigma$-invariant curves is the fixed point set of the real structure $\mathcal{M}_S^\sigma(\mathbb{R})$ of $\sigma$-invariant curves which are invariant under relabeling by the involution $\sigma$.

The moduli space of $\sigma$-invariant curves has recently attracted attention in various contexts such as the representation theory of quantum groups [8, 14, 16], multiple $\zeta$-motives [12], open Gromov-Witten/Welschinger invariants and related problems [3, 9, 21, 24, 25, 26, 27]. All of these applications require descriptions of the (co)homology groups and/or the fundamental group of $\mathcal{M}_S^\sigma(\mathbb{R})$.

Graph complex of $\mathcal{M}_S^\sigma(\mathbb{R})$

The moduli space $\mathcal{M}_S^\sigma(\mathbb{R})$ has a natural combinatorial stratification. Each stratum $C_{\sigma}$ is determined by the degeneration type of $\sigma$-invariant curves. Degen-
peration types of $\sigma$-invariant curves are encoded by trees $\tau$ with corresponding decorations.

In this work, we introduce a combinatorial graph complex $G_\bullet$ where

$$G_d = \bigoplus_{\tau | \dim C_\tau = d} \mathbb{Z} [\overline{C}_\tau] / I_d.$$ 

are Abelian groups generated by the relative fundamental classes $[\overline{C}_\tau]$ of the strata $\overline{C}_\tau$ of $\overline{M}_S^\sigma(\mathbb{R})$. The additive relations between the generators of $G_d$ are spanned by the relations introduced by Keel in [17] (and studied in detail [10, 19]) and by some additional natural relations which are known as the Cardy relations in physics literature. The differential $\partial: G_d \to G_{d-1}$ is given by

$$\partial [\overline{C}_\tau] = \sum_{\gamma / \gamma = \tau} \pm [\overline{C}_\gamma]$$

where the $\gamma$'s are the degeneration types of $\sigma$-invariant curves lying in the codimension one faces of $\overline{C}_\tau$. Then, we prove that;

**Theorem.** $H_\ast(\overline{M}_S^\sigma(\mathbb{R}); \mathbb{Z})$ is isomorphic to $H_\ast(G_\bullet)$.

**Cohomology versus graph homology**

In their recent preprint [8], Etingof and his collaborators calculated the cohomology algebra $H^\ast(\overline{M}_S^\sigma(\mathbb{R}); \mathbb{Q})$ in terms of generators and relations for the $\sigma = \text{id}$ case. Until this work, there was little known about the topology of $\overline{M}_S^\sigma(\mathbb{R})$ (see, [1, 4, 7, 14, 16]).

The graph homology, in a sense, treats the homology of the moduli space $\overline{M}_S^\sigma(\mathbb{R})$ in complementary directions to [8]: The graph complex provides a recipe to calculate the homology of $\overline{M}_S^\sigma(\mathbb{R})$ in $\mathbb{Z}$ coefficients for all possible involutions $\sigma$, and it reduces to the cellular complex of the moduli space $\overline{M}_S^\sigma(\mathbb{R})$ for $\sigma = \text{id}$ (see [7, 16]). In particular, when $\sigma$ has no fixed elements, the moduli space $\overline{M}_S^\sigma(\mathbb{R})$ parameterizes $\sigma$-invariant curves having both non-empty and empty real parts. To the best of our knowledge, this case has never been treated in the literature, since most of the recent literature on the moduli of real curves has been adopted from the moduli space of pseudoholomorphic discs.

Moreover, our presentation is based on a stratification of $\overline{M}_S^\sigma(\mathbb{R})$. All possible homological relations between the strata of $\overline{M}_S^\sigma(\mathbb{R})$ can be deduced from the graph homology. One can easily see that these relations include those introduced in [10, 18, 19] (which are responsible for the higher associativity of $\text{Comm}_\infty$-algebras), $A_\infty$-type relations (arising from the image of the differential of the graph complex) and finally Cardy type relations. The graph homology of $\overline{M}_S^\sigma(\mathbb{R})$ allows us to define the quantum cohomology of real varieties as an open-closed homotopy algebra which is very similar to [15]. Studying quantum cohomology of real varieties is one of the motivations of this work. This study is presented in a subsequent paper [3].
However, the graph homology comes with a lack of product structure. It is expected to give a product formula similar to the Kontsevich-Manin description in [19].

Plan of this paper
Section 2 contains a brief overview of basic facts about the moduli space $\overline{M}_S(\mathbb{C})$ of $S$-pointed stable complex curves. In Section 3 we define $\sigma$-invariant curves and provide a combinatorial description of their degeneration types. In Section 4 we introduce a set of real structures on $\overline{M}_S(\mathbb{C})$, and their real parts $\overline{M}_S(\mathbb{R})$ as moduli spaces of $\sigma$-invariant curves. In Section 5 we give the stratification of $\overline{M}_S(\mathbb{R})$ according to degeneration types of $\sigma$-invariant curves. In Section 6 we calculate the homology of the strata of $\overline{M}_S(\mathbb{R})$ relative to the union of their substrata of codimension one and higher. In Section 7 we define the graph complex and prove that its homology is isomorphic to the homology of $\overline{M}_S(\mathbb{R})$. Finally, we give a presentation of the fundamental group of $\overline{M}_S(\mathbb{R})$ by using the groupoid of paths transversal to codimension one strata.

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Notation/Convention
We denote the finite set $\{s_1, \ldots, s_n\}$ by $S$, and the symmetric group consisting of all permutations of $S$ by $S_n$. For an involution $\sigma \in S_n$, we denote the subset $\{s \in S \mid s = \sigma(s)\}$ by $\text{Fix}(\sigma)$ and its complement $S \setminus \text{Fix}(\sigma)$ by $\text{Perm}(\sigma)$.

In this paper, the genus of all curves is zero except when the contrary is stated. Therefore, we usually omit mentioning the genus of curves.

2 Pointed complex curves and their moduli
This section reviews the basic facts about pointed complex curves of genus zero and their moduli spaces.
2.1 Pointed complex curves

An S-pointed curve \((\Sigma; p)\) is a connected complex algebraic curve \(\Sigma\) with distinct, smooth, labeled points \(p = (p_{s_1}, \cdots , p_{s_n}) \subset \Sigma\) satisfying the following conditions:

- \(\Sigma\) has only nodal singularities.
- The arithmetic genus of \(\Sigma\) is equal to zero.

The nodal points and labeled points are called special points. Each irreducible component, the number of singular points plus the number of labeled points is at least three).

A family of S-pointed curves over a complex manifold \(B(\mathbb{C})\) is a proper, flat holomorphic map \(\pi_B : U_B(\mathbb{C}) \rightarrow B(\mathbb{C})\) with \(n\) sections \(p_{s_i} \cdots , p_{s_n}\) such that each geometric fiber \((\Sigma(b); p(b))\) is an S-pointed curve.

Two S-pointed curves \((\Sigma; p)\) and \((\Sigma'; p')\) are isomorphic if there exists a bi-holomorphic equivalence \(\Phi : \Sigma \rightarrow \Sigma'\) mapping \(p_s\) to \(p'_s\) for all \(s \in S\).

An S-pointed curve is stable if its automorphism group is trivial (i.e., on each irreducible component, the number of singular points plus the number of labeled points is at least three).

2.1.1 Graphs

A graph \(\gamma\) is a pair of finite sets of vertices \(V_\gamma\) and flags (or half edges) \(F_\gamma\) with a boundary map \(\partial_\gamma : F_\gamma \rightarrow V_\gamma\) and an involution \(j_\gamma : F_\gamma \rightarrow F_\gamma\) (\(j_\gamma^2 = \text{id}\)). We call \(E_\gamma = \{(f_1, f_2) \in F_\gamma^2 \mid f_1 = j_\gamma(f_2) \& f_1 \neq f_2\}\) the set of edges, and \(T_\gamma = \{f \in F_\gamma \mid f = j_\gamma(f)\}\) the set of tails. For a vertex \(v \in V_\gamma\), let \(F_\gamma(v) = \partial_\gamma^{-1}(v)\) and \(|v| = |F_\gamma(v)|\) be the valency of \(v\).

We think of a graph \(\gamma\) in terms of its geometric realization \(|\gamma||\) as follows: Consider the disjoint union of closed intervals \(\bigsqcup_{f_i \in F_\gamma} [0, 1] \times f_i\), and identify \((0, f_i)\) with \((0, f_j)\) if \(\partial_\gamma(f_i) = \partial_\gamma(f_j)\), and identify \((t, f_i)\) with \((1-t, j_\gamma(f_i))\) for \(t \in [0, 1]\) and \(f_i \neq j_\gamma(f_i)\). The geometric realization of \(\gamma\) has a piecewise linear structure.

**Definition 2.1.** A tree is a graph whose geometric realization is connected and simply-connected. If \(|v| > 2\) for all vertices, then such a tree is called stable.

There are only finitely many isomorphism classes of stable trees whose set of tails \(T_\gamma\) is equal to \(S\) (see, 22 or 11). We call the isomorphism classes of such trees S-trees.

2.1.2 Dual trees of S-pointed curves

Let \((\hat{\Sigma}; \hat{p})\) be an S-pointed stable curve and \(\eta : \hat{\Sigma} \rightarrow \Sigma\) be its normalization. Let \((\Sigma_v; \hat{p}_v)\) be the following pointed stable curve: \(\Sigma_v\) is a component of \(\Sigma\), and \(\hat{p}_v\) is the set of points consisting of the pre-images of special points on \(\Sigma_v := \eta(\Sigma_v)\). The points \(\hat{p}_v = (p_{f_1}, \cdots , p_{f_{|v|}})\) on \(\Sigma_v\) are ordered by the elements \(f_s\) in the set \(\{f_1, \cdots , f_{|v|}\}\).
2.2 Moduli space of S-pointed curves

The moduli space \( \overline{M}_S(\mathbb{C}) \) is the space of isomorphism classes of S-pointed stable curves. This space is stratified according to the degeneration types of S-pointed stable curves. The degeneration types of S-pointed stable curves are combinatorially encoded by S-trees. In particular, the principal stratum \( M_S(\mathbb{C}) \) parameterizes S-pointed irreducible curves; i.e., it is associated to the one-vertex S-tree. Therefore, \( M_S(\mathbb{C}) \) is the quotient of the product \( (\mathbb{P}^1(\mathbb{C}))^n \) minus the diagonals \( \Delta = \bigcup_{k<l}(\{p_s,\ldots,p_s\})p_k = p_l \) by \( \text{Aut}(\mathbb{P}^1(\mathbb{C})) = \text{PSL}_2(\mathbb{C}) \).

Theorem 2.3 (Knudsen & Keel, 20 \cite{KnudsenKeel1998}). (a) For any \( |S| \geq 3 \), \( \overline{M}_S(\mathbb{C}) \) is a smooth projective algebraic variety of (real) dimension \( 2|S| - 6 \).

(b) Any family of S-pointed stable curves over \( B(\mathbb{C}) \) is induced by a unique morphism \( \kappa : B(\mathbb{C}) \to \overline{M}_S(\mathbb{C}) \). The universal family of curves \( \overline{U}_S(\mathbb{C}) \) of \( \overline{M}_S(\mathbb{C}) \) is isomorphic to \( \overline{M}_{S,\{s_{n+1}\}}(\mathbb{C}) \).

(c) For any S-tree \( \gamma \), there exists a quasi-projective subvariety \( D_\gamma(\mathbb{C}) \subset \overline{M}_S(\mathbb{C}) \) parameterizing the S-pointed curves whose dual tree is given by \( \gamma \). The subvariety \( D_\gamma(\mathbb{C}) \) is isomorphic to \( \prod_{v \in \gamma} \overline{M}_{f(v),\gamma}(\mathbb{C}) \). Its (real) codimension (in \( \overline{M}_S(\mathbb{C}) \)) is \( 2|E_\gamma| \).

(d) \( \overline{M}_S(\mathbb{C}) \) is stratified by pairwise disjoint subvarieties \( D_\gamma(\mathbb{C}) \). The closure \( \overline{D}_\gamma(\mathbb{C}) \) of any stratum \( D_\gamma(\mathbb{C}) \) is stratified by \( \{D_{\gamma'}(\mathbb{C}) \mid \gamma' \leq \gamma \} \).
2.3 Forgetful morphisms

We say that \((\Sigma; p_1, \cdots, p_{s-1})\) is obtained by forgetting the labeled point \(p_s\) of an \(S\)-pointed curve \((\Sigma; p_1, \cdots, p_n)\). However, the resulting pointed curve may well be unstable. This happens when the component \(\Sigma_v\) of \(\Sigma\) supporting \(p_s\) has only two additional special points. In this case, we contract this component to its intersection point(s) with the components adjacent to \(\Sigma_v\).

With this stabilization, we extend this map to the whole space and obtain \(\pi_{\{s_n\}} : \overline{M}_S(\mathbb{C}) \to \overline{M}_{S'}(\mathbb{C})\) where \(S' = S \setminus \{s_n\}\). There exists a canonical isomorphism \(\overline{M}_S(\mathbb{C}) \to \overline{U}_{S'}(\mathbb{C})\) commuting with the projections to \(\overline{M}_{S'}(\mathbb{C})\). In other words, \(\pi_{\{s_n\}} : \overline{M}_S(\mathbb{C}) \to \overline{M}_{S'}(\mathbb{C})\) can be identified with the universal family of curves.

A very detailed study on the moduli space \(\overline{M}_S(\mathbb{C})\) can be found in Chapter 3.2 and 3.3 in [22], and also in [17, 20].

3 Pointed real curves of genus zero

This section reviews some basic facts on \(S\)-pointed real curves and their degeneration types.

3.0.1

A real structure on a complex variety \(X := X(\mathbb{C})\) is an anti-holomorphic involution \(c_X : X \to X\). The fixed point set \(X(\mathbb{R}) = \text{Fix}(c_X)\) of the involution \(c_X\) is called the real part of the variety \(X(\mathbb{C})\) (or of the real structure \(c_X\)).

3.1 \(\sigma\)-invariant curves and their families

An \(S\)-pointed stable curve \((\Sigma; p)\) is called \(\sigma\)-invariant if it admits a real structure \(c_\Sigma : \Sigma \to \Sigma\) such that \(c_\Sigma(p_s) = p_{\sigma(s)}\) for all \(s \in S\).

Let \(\pi_B : U_B(\mathbb{C}) \to B(\mathbb{C})\) be a family of \(S\)-pointed stable curves with a pair of real structures

\[
\begin{align*}
U_B(\mathbb{C}) \xrightarrow{c_U} & U_B(\mathbb{C}) \\
\pi_B \downarrow & \downarrow \pi_B \\
B(\mathbb{C}) \xrightarrow{c_B} & B(\mathbb{C}).
\end{align*}
\]

Such a family is called \(\sigma\)-equivariant if the following conditions are met:

- if \(\pi^{-1}(b) = \Sigma\), then \(\pi^{-1}(c_B(b)) = \Sigma\) for every \(b \in B\);
- \(c_U : z \in \Sigma = \pi^{-1}(b) \mapsto z \in \Sigma = \pi^{-1}(c_B(b))\).

Here a complex curve \(\Sigma\) is regarded as a pair \(\Sigma = (C, J)\), where \(C\) is the underlying two-dimensional manifold, \(J\) is a complex structure on \(C\), and \(\Sigma = (C, -J)\) is its complex conjugated pair.
Remark 3.1. If $\pi_B : U_B(\mathbb{C}) \to B(\mathbb{C})$ is a $\sigma$-equivariant family, then each $(\Sigma(b), p(b))$ for $b \in B(\mathbb{R})$ is $\sigma$-invariant. It follows from the fact that the group of automorphisms of $S$-pointed stable curves is trivial.

3.2 Combinatorial types of $\sigma$-invariant curves

Degeneration types of $\sigma$-invariant curves are combinatorially encoded by $S$-trees with additional decorations. This section contains definitions of relevant combinatorial structures.

3.2.1 Topological types of $\sigma$-invariant curves.

Let $(\Sigma; p)$ be a $\sigma$-invariant curve. Each irreducible real component $\Sigma_v$ of $\Sigma$ is isomorphic to $\mathbb{P}^1(\mathbb{C})$ with a real structure that is either $z \mapsto \bar{z}$ or $z \mapsto -1/\bar{z}$. Note that the real structure $z \mapsto -1/\bar{z}$ has empty real part. Therefore, $(\Sigma; p)$ is either one of the following topological types:

- **Type 1**: $\Sigma(\mathbb{R})$ is a tree of real projective spaces having only nodal singularities,
- **Type 2**: $\Sigma(\mathbb{R})$ is the empty set,
- **Type 3**: $\Sigma(\mathbb{R})$ is a solitary nodal point.

This fact directly follows from the classification of real structures on $\mathbb{P}^1(\mathbb{C})$ (see, for instance [23]) and Lefschetz’s fixed point theorem.

Remark 3.2. The terminology ‘type 1’ and ‘type 2’ is commonly used in real algebraic geometry. However, ‘type 3’ has not yet been used in the literature. In this paper it is used due to its convenience.

Remark 3.3. If $\text{Fix}(\sigma) \neq \emptyset$, then all $\sigma$-invariant curves are of type 1. This follows from the fact that real parts of $\sigma$-invariant curves cannot be the empty set (i.e., they can’t be type 2) and all special points must be distinct (i.e., they are not type 3 either). By contrast, $\sigma$-invariant curves can be of type 1, type 2 or type 3 when $\text{Fix}(\sigma) = \emptyset$.

3.2.2 Oriented combinatorial types

$\sigma$-invariant curves inherit additional structures on their sets of special points. In this subsection, we introduce ‘oriented’ versions of these structures for different topological types of $\sigma$-invariant curves separately.

Let $(\Sigma; p)$ be a $\sigma$-invariant curve, and let $\gamma$ be its dual tree. We denote the set of real components $\{v | c_\Sigma(\Sigma_v) = \Sigma_v\}$ of $(\Sigma; p)$ by $V_\gamma^R$.

$\sigma$-invariant curves of type 1.

Let $(\hat{\Sigma}; \hat{p})$ be the normalization of a $\sigma$-invariant curve $(\Sigma; p)$ of type 1. By identifying $\Sigma_v$ with $\Sigma_v \subset \Sigma$, we obtain a real structure on $\hat{\Sigma}_v$ for a real component $\Sigma_v$. The real part $\Sigma_v(\mathbb{R})$ of this real structure divides $\Sigma_v$ into two halves: two
2-dimensional open discs, $\Sigma_v^+$ and $\Sigma_v^-$, having $\hat{\Sigma}_v(\mathbb{R})$ as their common boundary in $\hat{\Sigma}_v$. The real structure $c_{\Sigma_v} : \Sigma_v \to \Sigma_v$ interchanges $\Sigma_v^\pm$, and the complex orientations of $\Sigma_v^\pm$ induce two opposite orientations on $\hat{\Sigma}_v(\mathbb{R})$, called its complex orientations.

If we fix a labeling of halves of $\hat{\Sigma}_v$ by $\Sigma_v^\pm$ (or equivalently, if we orient $\hat{\Sigma}_v(\mathbb{R})$ with one of the complex orientations), then the set of pre-images of special points $\hat{p}_v \in \hat{\Sigma}_v$ admits the following structures:

- An oriented cyclic ordering on the set of special points lying in $\Sigma_v(\mathbb{R})$: For any point $p_f \in (\hat{p}_v \cap \hat{\Sigma}_v(\mathbb{R}))$, there is a unique $p_{f_{i+1}} \in (\hat{p}_v \cap \hat{\Sigma}_v(\mathbb{R}))$ which follows the point $p_f$ in the positive direction of $\hat{\Sigma}_v(\mathbb{R})$ (the direction which is determined by the complex orientation induced by the orientation of $\Sigma_v^+$).

- An ordered two-partition of the set of special points lying in $\Sigma_v \setminus \Sigma_v(\mathbb{R})$. The subsets $\hat{p}_v \cap \Sigma_v^\pm$ of $\hat{p}_v$ give an ordered partition of $\hat{p}_v \cap (\Sigma_v \setminus \Sigma_v(\mathbb{R}))$ into two disjoint subsets.

The relative positions of the special points lying in $\hat{\Sigma}_v(\mathbb{R})$ and the complex orientation of $\hat{\Sigma}_v(\mathbb{R})$ give an oriented cyclic ordering on the corresponding labeling set $\mathbf{F}_\gamma^R(v) := (\hat{p}_v \cap \hat{\Sigma}_v(\mathbb{R}))$. We denote this oriented cyclic ordering by $\{f_{r_1} < \cdots < f_{r_{l-1}} < f_{r_l}\}$. Moreover, the partition $\{p_f \in \Sigma_v^\pm\}$ gives an ordered two-partition $\mathbf{F}_\gamma^\pm(v) := \{f \mid p_f \in \Sigma_v^\pm\}$ of $\mathbf{F}_\gamma(v) \setminus \mathbf{F}_\gamma^R(v)$.

The oriented combinatorial type of a real component $\Sigma_v$ with a fixed complex orientation is the following set of data:

$$o_v := \{\text{type 1; two partition } \mathbf{F}_\gamma^R(v); \text{oriented cyclic ordering on } \mathbf{F}_\gamma^R(v)\}.$$ 

If we consider a $\sigma$-invariant curve $(\Sigma; p)$ with a fixed complex orientation at each real component, then the set of oriented combinatorial types of real components

$$o := \{o_v \mid v \in \mathbf{V}_\gamma^R\}$$

is called an oriented combinatorial type of $(\Sigma; p)$.

**$\sigma$-invariant curves of type 2.**

For a $\sigma$-invariant curve $(\Sigma; p)$ of type 2, $\Sigma$ has a unique real component, $\Sigma_v$, since the intersection of a pair of real components of a $\sigma$-invariant curve must be a real point, and $\Sigma$ has none. Moreover, $\mathbf{Fix}(\sigma)$ must be the empty set since $\Sigma(\mathbb{R}) = \emptyset$.

In this case, the oriented combinatorial type of $(\Sigma; p)$ is the following set of data:

$$o := \{\text{type 2; } \mathbf{V}_\gamma^R = \{v\}\}.$$
\(\sigma\)-invariant curves of type 3.

For a \(\sigma\)-invariant curve \((\Sigma; p)\) of type 3, the real part \(\Sigma(\mathbb{R})\) of \((\Sigma; p)\) divides \(\Sigma\) into two connected pointed complex curves \((\Sigma^{\pm}; p^{\pm})\) having \(\Sigma(\mathbb{R})\) as their intersection point. We denote the sets of components of \((\Sigma^{\pm}; p^{\pm})\) by \(V_{\gamma}^{\pm}\), and the sets of their flags \(\bigcup_{v \in V_{\gamma}^{\pm}} \partial_{\gamma}^{-1}(v)\) by \(F_{\gamma}^{\pm}\).

An oriented combinatorial type of \((\Sigma; p)\) is the following set of data:

\[
o := \{\text{type 3; two partitions } V_{\gamma}^{\pm} \text{ and } F_{\gamma}^{\pm}\}.
\]

3.2.3 Unoriented combinatorial types

The definition of oriented combinatorial types requires additional choices (such as complex orientations) which are not determined by real structures of \(\sigma\)-invariant curves. By identifying the oriented combinatorial types for such different choices, we obtain unoriented combinatorial types of \(\sigma\)-invariant curves.

\(\sigma\)-invariant curves of type 1.

For each real component \(\Sigma_v\) of a \(\sigma\)-invariant curve \((\Sigma; p)\) of type 1, there are two possible ways of choosing \(\Sigma_v^+\) in \(\Sigma_v\). These two different choices give the opposite oriented combinatorial types \(o_v\) and \(\bar{o}_v\). Namely, the oriented combinatorial type \(\bar{o}_v\) is obtained from \(o_v\) by reversing the cyclic ordering of \(F_{\gamma}^R(v)\) and swapping \(F_{\gamma}^{+}(v)\) and \(F_{\gamma}^{-}(v)\).

The unoriented combinatorial type of a real component \(\Sigma_v\) of \((\Sigma; p)\) is the pair of opposite oriented combinatorial types \(u_v := \{o_v, \bar{o}_v\}\). The set of unoriented combinatorial types of real components

\[
u := \{u_v \mid v \in V_{\gamma}^R\}
\]

is called the unoriented combinatorial type of \((\Sigma; p)\).

\(\sigma\)-invariant curves of type 2.

For a \(\sigma\)-invariant curve \((\Sigma; p)\) of type 2, the unoriented combinatorial type is the same set of data as the oriented combinatorial type i.e., \(u := o = \{\text{type 2; } V_{\gamma}^R = \{v\}\}\).

\(\sigma\)-invariant curves of type 3.

For a \(\sigma\)-invariant curve \((\Sigma; p)\) of type 3, there are two possible ways of choosing \((\Sigma_v^+; p_v^+)\) in \((\Sigma; p)\). These choices give two opposite oriented combinatorial types, \(o\) and \(\bar{o}\). Namely, the oriented combinatorial type \(\bar{o}\) is obtained from \(o\) by swapping \(V_{\gamma}^{+}\) and \(V_{\gamma}^{-}\), and swapping \(F_{\gamma}^{+}\) and \(F_{\gamma}^{-}\).

The unoriented combinatorial type of \((\Sigma; p)\) is the pair of opposite oriented combinatorial types \(u := \{o, \bar{o}\}\).
3.2.4 Dual trees of \(\sigma\)-invariant curves

The combinatorial types of \(\sigma\)-invariant curves can be encoded on their dual trees.

O-planar trees.

Let \((\Sigma; p)\) be an \(\sigma\)-invariant curve and let \(\gamma\) be its dual tree.

An oriented locally planar (o-planar) structure on \(\gamma\) is a set of data which encodes an oriented combinatorial type of \((\Sigma; p)\). O-planar structures for different topological types are explicitly given as follows:

- For a \(\sigma\)-invariant curve of type 1,
  - \((\Sigma; p)\) is of type 1 (i.e., \(\Sigma(\mathbb{R})\) is a tree of real projective spaces).
  - \(V_\gamma^R \subset V_\gamma\) is the set of real components of \(\Sigma\) (i.e., the set of real vertices).
  - \(F^R_\gamma(v) \subset F_\gamma(v)\) is the set of the pre-images of special points in \(\Sigma_v(\mathbb{R})\) (i.e., the set of real flags adjacent to the real vertex \(v \in V_\gamma^R\)).
  - \(F^R_\gamma(v)\) carries an oriented cyclic ordering for every \(v \in V_\gamma^R\).
  - \(F_\gamma(v) \setminus F^R_\gamma(v)\) carries an ordered two-partition for every \(v \in V_\gamma^R\).

- For a \(\sigma\)-invariant curve of type 2,
  - \((\Sigma; p)\) is of type 2 (i.e., \(\Sigma(\mathbb{R}) = \emptyset\)).
  - \(V_\gamma^R = \{v\} \subset V_\gamma\) is the set of real components of \(\Sigma\) (i.e., the set of real vertices contains only one element).

- For a \(\sigma\)-invariant curve of type 3,
  - \((\Sigma; p)\) is of type 3 (i.e., \(\Sigma(\mathbb{R})\) is a point).
  - \(e = (f_e, f^e)\) is the edge corresponding to the solitary nodal point of \(\Sigma\).
  - \(F_\gamma\) and \(V_\gamma\) carry two partitions \(F^\pm_\gamma\) and \(V^\pm_\gamma\) respectively.

We denote S-trees \(\gamma, \tau, \mu\) with o-planar structures by \((\gamma, o), (\tau, o), (\mu, o)\) or by bold Greek characters with tilde \(\tilde{\gamma}, \tilde{\tau}, \tilde{\mu}\). When it is necessary to indicate different o-planar structures on the same S-tree, we use indices in parentheses (e.g., \(\tilde{\tau}(i)\)).
Notations.

For each vertex $v \in V_{\tilde{\gamma}}^\pm$ (resp. $v \in V_\gamma \setminus V_{\tilde{\gamma}}^\pm$) of an o-planar tree $\tilde{\gamma}$, we associate a subtree $\tilde{\gamma}_v$ (resp. $\gamma_v$) which is given by $V_{\gamma_v} = \{v\}$, $F_{\gamma_v} = F_{\gamma}(v)$, $\partial_{\gamma_v} = \partial_\gamma$, and by the o-planar structure $\alpha_v$ of $\tilde{\gamma}$ at the vertex $v \in V_{\tilde{\gamma}}^\pm$.

A pair of vertices $v, \tilde{v} \in V_\gamma \setminus V_{\tilde{\gamma}}^\pm$ is said to be conjugate if $c_{\gamma}(\Sigma_v) = \Sigma_{\tilde{v}}$. Similarly, we call a pair of flags $f, \tilde{f} \in F_\gamma \setminus F_{\tilde{\gamma}}$ conjugate if $c_{\gamma}$ swaps the corresponding special points.

To each o-planar tree $\gamma$ of type 1, we associate the subsets of vertices $V_\gamma^\pm$ and flags $F_\gamma^\pm$ as follows: Let $v_1 \in V_\gamma \setminus V_{\tilde{\gamma}}^\pm$, and let $v_2 \in V_{\tilde{\gamma}}^\pm$ be the closest vertex to $v_1$ in $||\gamma||$. Let $f(v_1) \in F_{\tilde{\gamma}}(v_2)$ be in the shortest path connecting the vertices $v_1$ and $v_2$. The sets $V_\gamma^\pm$ are the subsets of vertices $v_1 \in V_\gamma \setminus V_{\tilde{\gamma}}^\pm$ such that the corresponding flags $f(v_1)$ are respectively in $F_{\tilde{\gamma}}^\pm(v_2)$. The subsets of flags $F_\gamma^\pm$ are defined as $\partial_{\gamma}^{-1}(V_\gamma^\pm)$.

U-planar trees.

A u-planar structure on the dual tree $\gamma$ of $(\Sigma; p)$ is the set of data encoding the unoriented combinatorial type of $(\Sigma; p)$. It is given by

$$
\{ (\gamma_v, o_v), (\gamma_v, \tilde{o}_v) \mid v \in V_{\tilde{\gamma}}^\pm \} \quad \text{if } (\Sigma; p) \text{ is of type 1},
$$

$$
\{ \text{type 2; } V_{\gamma}^\gamma = \{v\} \} \quad \text{if } (\Sigma; p) \text{ is of type 2},
$$

$$
\{ \text{special real edge } e = (f_\gamma, f^\gamma) \} \quad \text{if } (\Sigma; p) \text{ is of type 3}.
$$

We denote S-trees $\gamma, \tau, \mu$ with u-planar structures by $(\gamma, u), (\tau, u), (\mu, u)$ or simply by bold Greek characters $\gamma, \tau, \mu$. O-planar planar trees $\tilde{\gamma}, \tilde{\tau}, \tilde{\mu}$ give representatives of u-planar trees $\gamma, \tau, \mu$ respectively.

3.2.5 Contraction morphism of o/u-planar trees

Contraction morphism of o-planar trees

Consider a family of $\sigma$-invariant curves which is a deformation of a real node of the central fiber $(\Sigma(b_0), p(b_0))$ with a given oriented combinatorial type. Let $\tau, \tilde{\gamma}$ be the o-planar trees associated respectively to a generic fiber $(\Sigma(b), p(b))$ and the central fiber $(\Sigma(b_0), p(b_0))$ of this family. Let $e$ be the edge corresponding to the nodal point that is deformed. We say that $\tilde{\tau}$ is obtained by contracting the edge $e$ of $\tilde{\gamma}$, and to indicate this we use the notation $\tilde{\gamma} < \tilde{\tau}$.

Contraction morphism of u-planar trees

The definition of contraction morphisms of u-planar trees is the same as that of contraction morphisms of o-planar trees. By contrast, the contraction of an edge of an u-planar tree is not a well-defined operation: We can think of a deformation of a real node as the family $\{ x \cdot y = t \mid t \in \mathbb{R} \}$. According to the sign of the deformation parameter $t$, we obtain two different unoriented combinatorial types of $\sigma$-invariant curves, see Figure 1. Different u-planar trees $\gamma_{(i)}$ that are
obtained by contraction of the same edge of \( \tau \) correspond to different signs of deformation parameters.

Further details of contraction morphisms for o/u-planar trees can be found in [1].

4 Moduli of \(\sigma\)-invariant curves

The moduli space \(\overline{M}_S(\mathbb{C})\) comes equipped with a natural real structure. The involution

\[
c : (\Sigma; p) \mapsto (\Sigma; p)
\]

(4.1)
gives the principal real structure of \(\overline{M}_S(\mathbb{C})\).

On the other hand, the permutation group \(S_n\) acts on \(\overline{M}_S(\mathbb{C})\) via relabeling: For each \(\varrho \in S_n\), there is an holomorphic map \(\psi_\varrho\) defined by \((\Sigma; p) \mapsto (\Sigma; \varrho(p)) := (\Sigma; p_{\varrho(s_1)}, \ldots, p_{\varrho(s_n)})\). For each involution \(\sigma \in S_n\), we have an additional real structure

\[
c_\sigma := c \circ \psi_\sigma : (\Sigma; p) \mapsto (\Sigma; \sigma(p))
\]

(4.2)
of \(\overline{M}_S(\mathbb{C})\). The real part \(\overline{M}_S(\mathbb{R})\) of the real structure \(c_\sigma : \overline{M}_S(\mathbb{C}) \to \overline{M}_S(\mathbb{C})\) gives the moduli space of \(\sigma\)-invariant curves:

Theorem 4.1 (Ceyhan, [1]). (a) For any \(|S| \geq 3\), \(\overline{M}_S(\mathbb{R})\) is a smooth projective real manifold of dimension \(|S| - 3\).

(b) Any \(\sigma\)-equivariant family \(\pi_B : U_B(\mathbb{C}) \to B(\mathbb{C})\) of \(S\)-pointed stable curves is induced by a unique pair of real morphisms

\[
\begin{array}{ccc}
U_B(\mathbb{C}) & \xrightarrow{\hat{\kappa}} & \overline{U}_S(\mathbb{C}) \\
\pi_S & & \pi \\
B(\mathbb{C}) & \xrightarrow{\kappa} & \overline{M}_S(\mathbb{C}).
\end{array}
\]

(c) Let \(\mathcal{M}_\sigma(\mathbb{C})\) be the contravariant functor that sends each real variety \((B(\mathbb{C}), c_B)\) to the set of \(\sigma\)-equivariant families of curves over \(B\). The moduli functor \(\mathcal{M}_\sigma(\mathbb{C})\) is represented by the real variety \((\overline{M}_S(\mathbb{C}), c_\sigma)\).

(d) Let \(\mathcal{M}_\sigma(\mathbb{R})\) be the contravariant functor that sends each real analytic manifold \(R\) to the set of families of \(\sigma\)-invariant curves over \(R\). The moduli functor \(\mathcal{M}_\sigma(\mathbb{R})\) is represented by the real part \(\overline{M}_S(\mathbb{R})\) of \((\overline{M}_S(\mathbb{C}), c_\sigma)\).
Remark 4.2. The group of holomorphic automorphisms of $\overline{M}_g(\mathbb{C})$ that respect its stratification is isomorphic to $S_n$. Therefore, the real structures preserving the stratification of $\overline{M}_g(\mathbb{C})$ are of the form (4.2) (see [2]).

However, we don’t know whether there exist real structures other than (4.2), since the whole group of holomorphic automorphisms $\text{Aut}(\overline{M}_g(\mathbb{C}))$ is not necessarily isomorphic to $S_n$. For example, the automorphism group of $\overline{M}_g(\mathbb{C})$ is $\text{PSL}_2(\mathbb{C})$ when $|\mathbb{S}| = 4$.

It is believed that $\text{Aut}(\overline{M}_g(\mathbb{C})) \cong S_n$ for $|\mathbb{S}| \geq 5$. In fact, it is true for $|\mathbb{S}| = 5$ and a proof can be found in [6]. To the best of our knowledge, there is no systematic exposition of $\text{Aut}(\overline{M}_g(\mathbb{C}))$ for $|\mathbb{S}| > 5$.

5 Stratification of moduli of $\sigma$-invariant curves

The moduli space $\overline{M}_g(\mathbb{R})$ can be stratified according to degeneration types of $\sigma$-invariant curves. This stratification is given in terms of spaces of $\mathbb{Z}_2$-equivariant point configurations on $\mathbb{P}^1(\mathbb{C})$ and moduli spaces of complex curves $\overline{M}_g(\mathbb{C})$.

5.1 Spaces of $\mathbb{Z}_2$-equivariant point configurations on $\mathbb{P}^1(\mathbb{C})$

Let $z := [z : 1]$ be an affine coordinate on $\mathbb{P}^1(\mathbb{C})$. Consider the upper half-plane $\mathbb{H}^+ = \{z \in \mathbb{P}^1(\mathbb{C}) | \Im(z) > 0\}$ (resp. lower half plane $\mathbb{H}^- = \{z \in \mathbb{P}^1(\mathbb{C}) | \Im(z) < 0\}$) as a half of the $\mathbb{P}^1(\mathbb{C})$ with respect to $z \mapsto \bar{z}$, and the real part $\mathbb{P}^1(\mathbb{R})$ as its boundary. We denote the compactified disc $\mathbb{H}^+ \cup \mathbb{P}^1(\mathbb{R})$ by $\mathbb{H}$.

Let $F$ be a finite set and $\varrho$ be an involution acting on $F$. Denote the fixed point set of $\varrho$ by $F^\varrho$.

5.1.1 $\mathbb{Z}_2$-equivariant configurations on $(\mathbb{P}^1(\mathbb{C}), z \mapsto \bar{z})$

The configuration space of $k = |F \setminus F^\varrho|/2$ distinct pairs of conjugate points in $\mathbb{H}^+ \bigsqcup \mathbb{H}^-$ and $l = |F^\varrho|$ distinct points in $\mathbb{P}^1(\mathbb{R})$ is

$$\overline{\text{Conf}}_{(F, \varrho)} := \{(z_{f_1}, \ldots, z_{f_k} ; x_{g_1}, \ldots, x_{g_l}) \mid z_f \in \mathbb{H}^+ \bigsqcup \mathbb{H}^- \text{ for } f \in F \setminus F^\varrho, \text{ where } f \neq \varrho(f), \quad \begin{array}{c} z_f = z_{f'} \Rightarrow f = f', \quad z_f = \bar{z}_{f'} \Leftrightarrow f = \varrho(f) \Leftrightarrow \varrho(g) = g' \Leftrightarrow g \in F^\varrho, \quad x_g \in \mathbb{P}^1(\mathbb{R}) \end{array} \}.$$ 

The number of connected components of $\overline{\text{Conf}}_{(F, \varrho)}$ is $2^k(l - 1)!$ when $l \geq 2$, and $2^k$ when $l = 0, 1$. They are all pairwise diffeomorphic; the natural diffeomorphisms are given by $\varrho$-invariant relabelings.

The action of $SL_2(\mathbb{R})$ on $\mathbb{H}$ is given by

$$SL_2(\mathbb{R}) \times \mathbb{H} \rightarrow \mathbb{H}, \quad (\Lambda, z) \mapsto \Lambda(z) = \frac{az + b}{cz + d}, \quad \Lambda = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{R}).$$

It induces an isomorphism $SL_2(\mathbb{R})/\pm I \rightarrow Aut(\mathbb{H})$. The automorphism group $Aut(\mathbb{H})$ acts on $\overline{\text{Conf}}_{(F, \varrho)}$ by

$$\Lambda : (z_{f_1}, \ldots, z_{f_k} ; x_{g_1}, \ldots, x_{g_l}) \mapsto (\Lambda(z_{f_1}), \ldots, \Lambda(z_{f_k});\Lambda(x_{g_1}), \ldots, \Lambda(x_{g_l})).$$
This action preserves each connected component of $\widehat{Conf}_1(F, \varrho)$. It is free when $2k + l \geq 3$, and it commutes with diffeomorphisms given by $\varrho$-invariant relabellings. Therefore, the quotient space $\overline{C}_1(F, \varrho) := \widehat{Conf}_1(F, \varrho)/\text{Aut}(\mathbb{H})$ is a manifold of dimension $2k + l - 3$ whose connected components are pairwise diffeomorphic.

In addition to the automorphisms considered above, there is a diffeomorphism $\overline{\Xi} : \bigcup_{|V_\gamma| = 1} C_\gamma \to M_1^2(\mathbb{R})$ (5.1) which maps $\mathbb{Z}_2$-equivariant point configurations to the corresponding isomorphisms classes of irreducible $\varrho$-invariant curves.
Lemma 5.1. (a) The map $Ξ$ is a diffeomorphism.
(b) Let $|\text{Perm}(σ)| = 2k$ and $\text{Fix}(σ) = l$. The configuration space $C_γ$ is diffeomorphic to

- $((\mathbb{C}^+)^k \setminus Δ) \times \mathbb{R}^{l-3}$ when $l > 2$,
- $((\mathbb{C}^+ \setminus \{\sqrt{l-1}\})^k \setminus Δ) \times \mathbb{R}^{l-1}$ when $l = 1, 2$,
- $((\mathbb{C}^+ \setminus \{-l, l-1, 1\})^k \setminus Δ) \times \mathbb{R}$ when $l = 0$ and $γ$ of type 1,
- $((\mathbb{P}^1(\mathbb{C}) \setminus \{-l, 1, 0\})^k \setminus (Δ ∪ Δ')) \times \mathbb{R}$ when $l = 0$ and $γ$ is of type 2.

Here, $Δ$ is the union of all diagonals where $z_s = z_{s'}$ ($s \neq s'$), and $Δ'$ is the union of all cross-diagonals where $z_s = -\frac{1}{z_{s'}}$ ($s \neq s'$).

The proof of Lemma 5.1 can be found in [1].

5.2 Configuration spaces of o/u-planar trees and stratification of $\overline{\mathcal{M}}_S^1(\mathbb{R})$

We associate a product of configuration spaces $C_γ_v$ and moduli spaces of pointed complex curves $\overline{\mathcal{M}}_{F_v, (v)}(\mathbb{C})$ to each o-planar tree $γ_v$:

$$C_γ = \begin{cases} 
\prod_{v \in V_γ} C_γ_v \times \prod_{\{v, v\}' \in V_γ} M_{F_v, (v)}(\mathbb{C}) & \text{if } γ \text{ is of type 1}, \\
\prod_{\{v, v\} \in V_γ \setminus V_γ^}\ M_{F_v, (v)}(\mathbb{C}) & \text{if } γ \text{ is of type 2}, \\
\prod_{\{v, v\}' \in V_γ} M_{F_v, (v)}(\mathbb{C}) & \text{if } γ \text{ is of type 3}.
\end{cases}$$

Here, the products $\prod M_{F_v, (v)}(\mathbb{C})$ run over unordered conjugate pairs of vertices.

For each u-planar $γ$, we first choose an o-planar representative $γ_v$, and then we set $C_γ := C_γ_v$. Note that $C_γ$ does not depend on the o-planar representative.

Theorem 5.2 (Ceyhan, [1]). (a) $\overline{\mathcal{M}}_S^1(\mathbb{R})$ is stratified by $C_γ$.
(b) A stratum $C_γ_v$ is contained in the boundary of $\overline{C}_γ$ if and only if $γ$ is obtained by contracting an invariant set of edges of $γ$. The codimension of $C_γ$ in $\overline{C}_γ$ is $|E_γ| - |E_γ'|$.

Example. The first nontrivial example is $\overline{\mathcal{M}}_S^1(\mathbb{C})$ where $S = \{s_1, s_2, s_3, s_4\}$. There are three real structures: \(c_{σ_1}, c_{σ_2}, c_{σ_3}\), where

$$σ_1 = \text{id}, \quad σ_2 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_2 & s_1 & s_3 & s_4 \end{pmatrix} \quad \text{and} \quad σ_3 = \begin{pmatrix} s_4 & s_3 & s_2 & s_1 \\ s_3 & s_4 & s_1 & s_2 \end{pmatrix}.$$  

Note that, all other real structures which preserve the stratification of $\overline{\mathcal{M}}_S^1(\mathbb{C}) = \mathbb{P}^1(\mathbb{C})$ are conjugate to $c_{σ_i}$, $i = 1, 2, 3$.

The main stratum $M_{S_0}^{31}(\mathbb{R})$ of $\overline{\mathcal{M}}_S^{31}(\mathbb{R})$ is the configuration space of four distinct points in $\mathbb{P}^1(\mathbb{R})$ up to the action of $\text{PSL}_2(\mathbb{R})$. Due to Lemma 5.1, $σ_1$-invariant curves $(Σ; p) \in M_{S_0}^{31}(\mathbb{R})$ are identified with tuples $(0, x_{s2}, 1, \infty)$ where
\( x_{s_2} \in \mathbb{P}^1(\mathbb{R}) \setminus \{0, 1, \infty\}. \) Hence, the main stratum \( M^2_S(\mathbb{R}) \) of \( \overline{M}^2_S(\mathbb{R}) \) is the space of distinct configurations of two points in \( \mathbb{P}^1(\mathbb{R}) \) and a pair of complex conjugate points in \( \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}) \) up to the action of \( PSL_2(\mathbb{R}) \). Due to Lemma 5.1, \( \sigma_2 \)-invariant curves \( (\Sigma; p) \in M^2_S(\mathbb{R}) \) are identified with tuples \((\sqrt{-1}, -\sqrt{-1}, x_{s_3}, \infty, -\infty < x_3 < \infty)\). Hence, \( M^2_S(\mathbb{R}) = \mathbb{P}^1(\mathbb{R}) \setminus \{\infty\} \) and its compactification \( \overline{M}^2_S(\mathbb{R}) \) is \( \mathbb{P}^1(\mathbb{R}) \).

The main stratum \( M^3_S(\mathbb{R}) \) of \( \overline{M}^3_S(\mathbb{R}) \) has different pieces parameterizing \( \sigma_3 \)-invariant curves with non-empty and empty real parts: The elements of the subspace of \( M^3_S(\mathbb{R}) \) parameterizing \( \sigma_3 \)-invariant curves with \( \Sigma(\mathbb{R}) \neq \emptyset \) are identified with tuples \((\lambda \sqrt{-1}, \sqrt{-1}, -\lambda \sqrt{-1}, -\sqrt{-1})\) where \( \lambda \in [1, 1]\setminus\{0\} \). Similarly, the elements of the subspace of \( M^3_S(\mathbb{R}) \) parameterizing \( \sigma_3 \)-invariant curves with \( \Sigma(\mathbb{R}) = \emptyset \) are identified with tuples \((\lambda \sqrt{-1}, \sqrt{-1}, -\sqrt{-1})/\lambda, -\sqrt{-1})\), where \( \lambda \in [1, 1]\). Note that, the strata parameterizing \( \Sigma(\mathbb{R}) \neq \emptyset \) and \( \Sigma(\mathbb{R}) = \emptyset \) are joined through the boundary points corresponding to \( \sigma_3 \)-invariant curves of type 3. The compactification of \( M^3_S(\mathbb{R}) \) is again \( \mathbb{P}^1(\mathbb{R}) \).

**Example.** Let \( S = \{s_1, \ldots, s_5\} \). The moduli space \( \overline{M}_S(\mathbb{C}) \) has three different types of real structures that are given by

\[
\sigma_1 = \text{id}, \quad \sigma_2 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \\ s_2 & s_1 & s_3 & s_4 & s_5 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_1 & s_2 & s_5 \end{pmatrix}.
\]

All other real structures of \( \overline{M}_S(\mathbb{C}) \) are conjugate to \( c_{\sigma_i} \), since the automorphism group of \( \overline{M}_S(\mathbb{C}) \) is \( S_5 \) in this case (see [5]).

The main stratum \( M^3_S(\mathbb{R}) \) is identified with the configuration space of five distinct points in \( \mathbb{P}^1(\mathbb{R}) \) modulo \( PSL_2(\mathbb{R}) \), following Lemma 5.1. It is (\( \mathbb{P}^1(\mathbb{R}) \setminus \{0, 1, \infty\}\))^2 \( \setminus \Delta \), where \( \Delta \) is union of all diagonals. Each connected component of \( M^3_S(\mathbb{R}) \) is isomorphic to a two-dimensional open simplex. The closure of each cell can be obtained by adding the boundaries described in Theorem 5.2 (for an example see Fig. [3]). The cells corresponding to u-planar trees \( \tau_1 \) and \( \tau_2 \) are glued along their common stratum corresponding to \( \gamma \), which gives \( \tau_i, i = 1, 2 \) by contracting an edge of \( \gamma \). The moduli space \( \overline{M}^2_S(\mathbb{R}) \) is a torus with 3 points blown up (see Fig. [3]).

The main stratum \( M^2_S(\mathbb{R}) \) is diffeomorphic to the configuration space of a conjugate pair of points on \( \mathbb{P}^1(\mathbb{C}) \). Following Lemma 5.1, \( \sigma_2 \)-invariant curves in
Figure 3: (a) Stratification of $C_\tau$. (b) The stratification of $\overline{M}_S(\mathbb{R})$ for $|S| = 5$ and $\sigma = \text{id}$.

$M^\sigma_S(\mathbb{R})$ are identified with tuples $(z, \bar{z}, 0, 1, \infty)$ where $z \in \mathbb{C} \setminus \mathbb{R}$; i.e., the main stratum is $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$. The moduli space $\overline{M}^\sigma_S(\mathbb{R})$ is a sphere with three points blown up according to the stratification given in Theorem 5.2.

Finally, $\sigma_3$-invariant curves in $M^\sigma_S(\mathbb{R})$ are identified with the configurations of a conjugate pair of points in $\mathbb{H}^+ \setminus \{\sqrt{-1}\} \cup \mathbb{H}^- \setminus \{-\sqrt{-1}\}$ following 5.1. These configurations are given by $(z, \sqrt{-1}, \bar{z}, -\sqrt{-1}, \infty)$, and the main space $M^\sigma_S(\mathbb{R})$ is identified with $\mathbb{P}^1(\mathbb{C}) \setminus (\mathbb{P}^1(\mathbb{R}) \cup \{\sqrt{-1}, -\sqrt{-1}\})$. The moduli space $\overline{M}^\sigma_S(\mathbb{R})$ is a sphere with a point blown up.

6 Homology of the strata of $\overline{M}^\sigma_S(\mathbb{R})$

In this section, we calculate the homology of the strata of $\overline{M}^\sigma_S(\mathbb{R})$ relative to the union of their substrata of codimension one and higher.

6.1 Forgetful morphism revisited

This section discusses the restriction of the forgetful morphism to $\overline{M}^\sigma_S(\mathbb{R})$.

6.1.1 The forgetful morphism and u-planar trees

Let $S' \subset S$ such that $\sigma(S') = S'$. Denote the restriction $\sigma$ on $S'$ by $\sigma'$. In this case, the morphism $\pi_{S \setminus S'} : \overline{M}_S(\mathbb{C}) \to \overline{M}_{S'}(\mathbb{C})$ forgetting the points labelled by $S \setminus S'$ is a real morphism; i.e., $\pi_{S \setminus S'} \circ c_\sigma = c_{\sigma'} \circ \pi_{S \setminus S'}$. Therefore, $\pi_{S \setminus S'}$ maps the real part of $(\overline{M}_S(\mathbb{C}), c_\sigma)$ onto the real part of $(\overline{M}_{S'}(\mathbb{C}), c_{\sigma'})$.

Let $\gamma^*$ be the u-planar tree of $(\Sigma^*: p^*) \in \overline{M}_{S'}(\mathbb{R})$, and let $\gamma$ be the u-planar tree of $\pi_{S \setminus S'}((\Sigma^*: p^*))$. Then, we say that $\gamma$ is obtained by forgetting the tails $S \setminus S'$ of $\gamma^*$.
6.1.2 Forgetting a conjugate pair of labeled points

Let $S = \{s_1, \ldots, s_n\}$ and $\sigma \neq \text{id}$. Let $S' = S \setminus \{s, \bar{s}\}$ for a pair $s, \bar{s} := \sigma(s) \in \text{Perm}(\sigma)$. Let $\sigma'$ be the restriction of $\sigma$ on $S'$. From now on, we denote the morphism $\pi_{(s, \bar{s})} : \overline{\mathcal{M}}_S^\text{M}(\mathbb{R}) \to \overline{\mathcal{M}}_{S'}^\text{M}(\mathbb{R})$, forgetting the labeled points $p_s, p_{\bar{s}}$ by $\pi$.

Let $C_{\gamma^*}$ be a stratum of $\overline{\mathcal{M}}_S^\text{M}(\mathbb{R})$, and identify it with $C_{\gamma^*}$ where $\gamma^*$ is an o-planar representative of $\gamma$. Let $\pi : C_{\gamma^*} \to C_{\gamma^*}$ be the restriction of the forgetful map $\pi$ to $C_{\gamma^*}$. Let $v_s$ be the vertex adjacent to the tail $s$, i.e., $v_s := \partial_s\gamma^*(s)$. Whenever $v_s \in V_{\gamma^*}^R$ and $|v_s| = 3$, there is a unique vertex in $V_{\gamma^*}^R$ next to $v_s$ since $v_s$ supports both $s$ and $\bar{s}$ and a real edge connecting $v_s$ to the rest of $\gamma^*$. We denote this vertex closest to $v_s$ by $v_c$.

We will denote the fibers of the forgetful map $\pi : C_{\gamma^*} \to C_{\gamma^*}$ by $A_{\gamma^*}$.

**Lemma 6.1.** (a) Let $(\Sigma; p) \in C_{\gamma^*}$ be a $\sigma'$-invariant curve of type 1, and $s \in F_{\gamma^*}^\Sigma$. (resp. $s \in F_{\gamma^*}^{-\Sigma}$). Then, the fiber $A_{\gamma^*}$ over $(\Sigma; p)$ is

1. a two-dimensional open disc $\Sigma_{v_s}$ (resp. $\Sigma_{v_{\bar{s}}}$) minus the special points $p_f$ where $f \in F_{\gamma^*}^R(v_s) \setminus \{s\}$ (resp. $f \in F_{\gamma^*}^{-R}(v_s) \setminus \{s\}$) if $v_s \in V_{\gamma^*}^R$ and $|F_{\gamma^*}^R(v_s)| \geq 5$;
2. a two-dimensional sphere $\Sigma_{v_s}$ minus the special points $p_f$ where $f \in F_{\gamma^*}(v_s) \setminus \{s\}$ if $v_s \notin V_{\gamma^*}^{-R}$ and $|F_{\gamma^*}(v_s)| \geq 4$;
3. an open interval if $v_s \in V_{\gamma^*}^R$ and $|F_{\gamma^*}(v_s)| = 4$;
4. an open interval if $v_s \in V_{\gamma^*}^{-R}$, $|F_{\gamma^*}(v_s)| = 3$, $|F_{\gamma^*}(v_c)| \geq 4$ and $|F_{\gamma^*}^R(v_c)| > 1$;
5. a point if $v_s \in V_{\gamma^*}^R$, $|F_{\gamma^*}(v_s)| = 3$, $|F_{\gamma^*}(v_c)| \geq 4$ and $|F_{\gamma^*}^R(v_c)| = 1$;
6. a point if $v_s \in V_{\gamma^*}^{-R}$, $|F_{\gamma^*}(v_s)| = 3$;
7. a point if $v_s \notin V_{\gamma^*}^R$, and $|F_{\gamma^*}(v_s)| = 3$.

(b) Let $(\Sigma; p) \in C_{\gamma^*}$ be a $\sigma'$-invariant curve of type 2. Then the fiber $A_{\gamma^*}$ over $(\Sigma; p)$ is

1. a two-dimensional sphere $\Sigma_{v_s}$ minus the special points $p_f$ where $f \in F_{\gamma^*}(v_s) \setminus \{s, \bar{s}\}$ if $v_s \in V_{\gamma^*}^R$ and $|F_{\gamma^*}(v_s)| \geq 6$;
2. a two-dimensional sphere $\Sigma_{v_s}$ minus the special points $p_f$ where $f \in F_{\gamma^*}(v_s) \setminus \{s\}$ if $v_s \notin V_{\gamma^*}^R$, and $|F_{\gamma^*}(v_s)| \geq 4$;
3. a point if $|F_{\gamma^*}(v_s)| = 3$.

(c) Let $(\Sigma; p) \in C_{\gamma^*}$ be a $\sigma'$-invariant curve of type 3. Then the fiber $A_{\gamma^*}$ over $(\Sigma; p)$ is

1. a two-dimensional sphere $\Sigma_{v_s}$ minus the special points $p_f$ where $f \in F_{\gamma^*}(v_s) \setminus \{s\}$ if $|F_{\gamma^*}(v_s)| \geq 4$;

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2. a point if $|F_{\gamma'}(v_s)| = 3$.

Proof. Here, we prove only (a). The proofs of all other cases are essentially the same.

Let $\tilde{\gamma}$ and $\tilde{\gamma}'$ be of type 1. Pick a $\sigma'$-invariant curve $(\Sigma; p) \in C_{\tilde{\gamma}}$. Let $(\Sigma^*, p^*)$ denote the points of the fiber $A_{\tilde{\gamma}}$ over $(\Sigma; p)$. If $(\Sigma^*, p^*) \in A_{\tilde{\gamma}}$ does not require a contraction of its component $\Sigma^*_s$, after removing the labeled points $p_s, p_{\bar{s}}$, then there are two possible subcases due to the stability condition:

1. $\Sigma^*_s$ is a real component and supports five or more special points, or
2. $\Sigma^*_s$ is not a real component and supports four or more special points.

If $(\Sigma^*, p^*) \in A_{\tilde{\gamma}'}$ requires a contraction after removing the labeled points $p_s, p_{\bar{s}}$, then the component $\Sigma^*_s$ supports only three or four special points. Here, there are three possible subcases:

3. $\Sigma^*_s$ is a real component and supports four special points,
4-5-6. $\Sigma^*_s$ is a real component and supports three special points, or
7. $\Sigma^*_s$ is not a real component and supports three special points.

In these three cases which require stabilization, we implicitly assumed that $|V_{\gamma'}| > 1$ so as not to violate the stability condition for $\gamma$.

We consider these subcases separately.

We first note that the positions of the special points lying in the components other than $\Sigma_v$ and $\Sigma_{v_c}$ are fixed, since we consider the fiber over a fixed $\sigma'$-invariant curve $(\Sigma; p)$.

1. If $v_s \in V_{\gamma}^\mathbb{R}$ and $|v_s| \geq 5$, then $(\Sigma^*, p^*) \in A_{\tilde{\gamma}'}$ do not require the contraction of $\Sigma^*_s$ after removing $p_s, p_{\bar{s}}$. For all such $(\Sigma^*, p^*)$, $\Sigma^* = \Sigma$, and so $\Sigma^*_s = \Sigma_s$. Assume that $s \in F_{\gamma'}^+(v_s)$. For $f \in F_{\gamma'}^+(v_s) \setminus \{s\}$, the special points $p_f$ are fixed and distinct in $\Sigma^*_s$. Therefore, the elements $(\Sigma^*, p^*)$ of $A_{\tilde{\gamma}'}$ are determined by the position of the point $p_s$ in $\Sigma^*_s$. Since all special points are distinct, $p_s$ must be in $\Sigma^*_s \setminus \bigcup \{p_f\}$; i.e., the fiber is $\Sigma^*_s \setminus \bigcup \{p_f\}$ where $f \in F_{\gamma'}^+(v_s) \setminus \{s\}$.

2. If $v_s \not\in V_{\gamma}^\mathbb{R}$ and $|v_s| \geq 4$, then $(\Sigma^*, p^*) \in A_{\tilde{\gamma}'}$ do not require the contraction of $\Sigma^*_s$ after forgetting $p_s, p_{\bar{s}}$. Similarly to the above case, $\Sigma^*_s = \Sigma_{v_c}$, and $(\Sigma^*, p^*) \in A_{\tilde{\gamma}'}$ are determined by the position of the point $p_s$ in $\Sigma_{v_c}$. Hence, the fiber is $\Sigma_{v_c} \setminus \bigcup \{p_f\}$ where $f \in F_{\gamma'}^+(v_s) \setminus \{s\}$.

3. Since all special points other than $p_s, p_{\bar{s}}$ are fixed, a fiber of $\mathcal{F}$ is a family of $\sigma$-invariant curves which, in this case, is the deformation of the irreducible component $(\Sigma^*_{v_c}, p_{v_c}^*)$ with two real special points and the conjugate pair $p_s, p_{\bar{s}}$. It clearly gives an open interval (see examples in Section 5).

4-5-6. In these cases, $(\Sigma^*_{v_c}, p_{v_c}^*)$ cannot be deformed since $|v_s| = 3$. Here, the family of $\sigma$-invariant curves along the fiber $A_{\tilde{\gamma}}$ is the deformation of $(\Sigma^*_{v_c}, p_{v_c}^*)$ (instead of $(\Sigma_{v_c}, p_{v_c}^*)$). The fiber parameterizes the nodal points $\Sigma^*_{v_c} \cap \Sigma_{v_c}$ which disappear after forgetting $p_s, p_{\bar{s}}$ and contracting $\Sigma^*_{v_c}$. Here, there are
three subcases: (6) The fiber \( A_\gamma^* \) is a point when \(|v_c| = 3\) since \((\Sigma^*_v, p^*_v)\) cannot deformed in this case. (5) The fiber \( A_\gamma^* \) is a circle when \( F_{R\gamma}(v_c) = 1 \). Each \((\Sigma^*_v, p^*_v)\) is given by a different position of the nodal point \( \Sigma^*_v \cap \Sigma^*_v \) in \( \Sigma_v(\mathbb{R}) \). (4) The fiber \( A_\gamma^* \) is an open interval when \( F_{R\gamma}^>(v_c) > 1 \). Each \((\Sigma^*_v, p^*_v)\) is given by a different position of the nodal point \( (\Sigma^*_v \cap \Sigma^*_v) \in \Sigma_v(\mathbb{R}) \) which can vary between two other special points lying in the real part of \( \Sigma^*_v \).

7. The element \((\Sigma^*_v, p^*_v)\) is unique when \( v_s \notin V^R_{\gamma} \) and \(|v_s| = 3\), since the contracted component supports only three special points.

The o-planar trees associated to \((\Sigma^*_v, p^*_v)\) are simply obtained by considering the cases above.

Consider the forgetful map for closed strata \( \pi: C_{\gamma^*} \to C_{\gamma^*} \). We denote the fiber \( \pi^{-1}(\Sigma; p) \) over \( (\Sigma; p) \in C_{\gamma^*} \) by \( A_{\gamma^*} \) since it is the closure of the fiber \( A_{\gamma^*} \) of \( \pi: C_{\gamma} \to C_{\gamma} \) over the same point \( (\Sigma; p) \). By using the stratification of \( C_{\gamma^*} \), we obtain a stratification of the fiber \( A_{\gamma^*} \).

**Lemma 6.2.** Let \( \tilde{A}_{\gamma^*} \) be the fibers of \( \pi: C_{\gamma^*} \to C_{\gamma} \) over the same point \( (\Sigma; p) \in C_{\gamma^*} \). Then, \( \tilde{A}_{\gamma^*} \subset \tilde{A}_{\gamma^*} \) if and only if \( \tilde{\gamma}_1^* \) produces \( \tilde{\gamma}_2^* \) by contracting one of its invariant edges or a pair of conjugate edges.

**Proof.** This statement is a direct corollary of Theorem 5.2.

### 6.2 Homology of the fibers of the forgetful morphism

Let \( \gamma^* \) be a one-vertex o-planar tree, and let \( \pi: C_{\gamma^*} \to C_{\gamma} \) be the morphism forgetting the labeled points \( p_s, p_s^* \), which is discussed in Section 6.1. Assume that the fibers are two-dimensional; i.e., a punctured disc or a punctured sphere (see corresponding cases of Lemma 6.1).

#### 6.2.1 Case of type 1

Let \( \gamma^* \) be a one-vertex o-planar tree of type 1. Assume that \( s \in F^+_{\gamma^*} \) (resp. \( s \in F^-_{\gamma^*} \)). Then, each fiber \( A_{\gamma^*} \) of \( \pi \) is homotopy equivalent to a bouquet of \(|F^+_{\gamma^*}| - 1\) circles \( S^1 \vee \cdots \vee S^1 \). The cohomology of \( A_{\gamma^*} \) is generated by the logarithmic differentials:

\[
H^0(A_{\gamma^*}) = \mathbb{Z},
\]

\[
H^1(A_{\gamma^*}) = \bigoplus_f \mathbb{Z} \omega_{sf}
\]

where

\[
\omega_{sf} = \frac{1}{2\pi \sqrt{-1}} d\log(z_s - z_f)
\]

for \( f \in F^+_{\gamma^*} \setminus \{s\} \) (resp. \( f \in F^-_{\gamma^*} \setminus \{s\} \)).

The homology with closed support \( H^1_c(A_{\gamma^*}) \) is isomorphic to the cohomology group \( H^1(A_{\gamma^*}) \) and generated by the duals of the logarithmic forms; i.e., by the
arcs connecting the punctures \( z_f \) to a point in the boundary of the closure \( \overline{A}_{\gamma^*} \) of the fiber \( A_{\gamma^*} \).

We denote the dual of the generator \( \omega_{sf} \) by \( \mathcal{E}_{sf} \). The homology group \( H_2^*(A_{\gamma^*}) \) is isomorphic to \( H^0(\overline{A}_{\gamma^*}) \) and generated by the relative fundamental class of \( \overline{A}_{\gamma^*} \). Hence,

\[
H_2^*(A_{\gamma^*}) = \mathbb{Z}[\overline{A}_{\gamma^*}],
H_1^*(A_{\gamma^*}) = \bigoplus_f \mathbb{Z} \mathcal{E}_{sf}
\]

where \( f \in F_{\gamma^*}^+ \setminus \{s\} \) (resp. \( f \in F_{\gamma^*}^- \setminus \{s\} \)).

### 6.2.2 Case of type 2

Let \( \gamma^* \) be a one-vertex o-planar tree of type 2. Then, each fiber \( A_{\gamma^*} \) of \( \pi \) is homotopy equivalent to a bouquet of \( |F_{\gamma^*}|-3 \) circles \( S^1 \vee \cdots \vee S^1 \). The cohomology of \( A_{\gamma^*} \) is generated by the logarithmic differentials:

\[
H^0(A_{\gamma^*}) = \mathbb{Z},
H^1(A_{\gamma^*}) = \bigoplus_f \mathbb{Z} \omega_{sf}
\]

where

\[
\omega_{sf} = \frac{1}{2\pi \sqrt{-1}} d \log(z_s - z_f)
\]

for \( f \in F_{\gamma^*} \setminus \{s, \bar{s}, s_n\} \).

The homology with closed support \( H_1^c(A_{\gamma^*}) \) is isomorphic to the cohomology group \( H^1(A_{\gamma^*}) \) and generated by the arcs connecting the pairs of punctures \( z_{f_1}, z_{f_2} \). These arcs are the duals of the cohomology classes \( \omega_{sf_1} - \omega_{sf_2} \). We denote them by \( \mathcal{F}_{s,f_1,f_2} \). Hence,

\[
H_2^c(A_{\gamma^*}) = \mathbb{Z}[\overline{A}_{\gamma^*}],
H_1^c(A_{\gamma^*}) = \left( \bigoplus_{f_1 \neq f_2} \mathbb{Z} \mathcal{F}_{s,f_1,f_2} \right) / \mathcal{J}_s
\]

where \( f_i \in F_{\gamma^*} \setminus \{s, \bar{s}\} \), and the subgroup \( \mathcal{J}_s \) is generated by

\[
\mathcal{F}_{s,f_1,f_2} + \mathcal{F}_{s,f_2,f_3} + \mathcal{F}_{s,f_3,f_1}.
\]

### 6.2.3 Homology of the fibers of \( \pi_{\{s\}} : M_S(\mathbb{C}) \to M_{S'}(\mathbb{C}) \)

Let \( |S| = n \geq 4 \) and \( s \in S \) be different than \( s_n \). Let \( S' = S \setminus \{s\} \). Then, each fiber \( A_s \) of \( \pi_{\{s\}} \) is homotopy equivalent to a bouquet of \( |S|-2 \) circles \( S^1 \vee \cdots \vee S^1 \).
The cohomology of $A_s$ is generated by the logarithmic differentials:

$$H^0(A_s) = \mathbb{Z},$$
$$H^1(A_s) = \bigoplus_f \mathbb{Z} \omega_{sf}$$

where

$$\omega_{sf} = \frac{1}{2\pi i} d \log(z_s - z_f)$$

for $f \in \mathcal{S} \setminus \{s, s_n\}$.

The homology with closed support $H^c_1(A_s)$ is isomorphic to the cohomology group $H^1(A_s)$ and generated by the arcs connecting the pairs of punctures $z_{f_1}, z_{f_2}$. These arcs are the duals of the cohomology classes $\omega_{sf_1} - \omega_{sf_2}$. We denote them by $G_{s,f_1f_2}$. Hence,

$$H^c_2(A_s) = \mathbb{Z} [\mathcal{T}_s],$$
$$H^c_1(A_s) = \left( \bigoplus_{f_1 \neq f_2} \mathbb{Z} G_{s,f_1f_2} \right) / \mathcal{J}_s$$

where $f_i \in \mathcal{S} \setminus \{s\}$, and the subgroup $\mathcal{J}_s$ is generated by

$$G_{s,f_1f_2} + G_{s,f_2f_3} + G_{s,f_3f_1}.$$

6.3 Homology of the strata

**Lemma 6.3.** Let $\pi : C_{\gamma^*} \to C_{\gamma}$ be the fibration discussed in Section 6.1.2. Then,

$$H^c_0(C_{\gamma^*}; \mathbb{Z}) = \bigoplus_{p+q=d} H^p(C_{\gamma}; \mathbb{Z}) \otimes H^q(A_{\gamma^*}; \mathbb{Z}).$$

**Proof.** We first consider the subcases where $\dim A_{\gamma^*} = 2$. Assume that $\gamma^*$ is of type 1. The strata $C_{\gamma^*}$ and $C_{\gamma}$ are given by the products

$$\prod_{v \in \mathcal{V}_{\gamma^*}^+} C_{\gamma^* v} \times \prod_{v \in \mathcal{V}_{\gamma^*}^-} M_{F_{\gamma^*}(v)}(\mathbb{C}), \quad \prod_{v \in \mathcal{V}_{\gamma}^+} C_{\gamma v} \times \prod_{v \in \mathcal{V}_{\gamma}^-} M_{F_{\gamma}(v)}(\mathbb{C})$$

respectively (see, Section 5.2). The forgetful map $\pi$ preserves the component $(\Sigma_v^*, \mathbf{p}_v^*)$ of $(\Sigma^*, \mathbf{p}^*)$ for $v \neq v_s$. Hence, it is the identity map on the factors

$$C_{\gamma^* v} \to C_{\gamma v}, \quad M_{F_{\gamma^*}(v)}(\mathbb{C}) \to M_{F_{\gamma}(v)}(\mathbb{C})$$

for $v \neq v_s$. On the other hand, it gives a fibration

$$\pi_{res} : C_{\gamma^*_v} \to C_{\gamma v}, \quad M_{F_{\gamma^*}(v)}(\mathbb{C}) \to M_{F_{\gamma}(v)}(\mathbb{C}),$$

when $v_s \in \mathcal{V}_{\gamma^*}^R$, and

$$\pi_{res} : C_{\gamma^*_v} \to C_{\gamma v}, \quad M_{F_{\gamma^*}(v)}(\mathbb{C}) \to M_{F_{\gamma}(v)}(\mathbb{C}),$$

when $v_s \not\in \mathcal{V}_{\gamma^*}^R$, (6.1)
with the same fibers $A_{\gamma^*}$ of $\pi : C_{\gamma^*} \to C_\gamma$. Therefore, we only need to consider the fibrations in (6.1) to calculate the homology.

The strata $C_{\gamma^*}$ and $C_{\gamma}$ are diffeomorphic to the products of simplices with the products of the upper half plane minus the diagonals (see, Lemma 5.1). The map $\pi_{res}$ forgets the coordinate subspace $\mathbb{H}^+$ corresponding to the labeled points $p_s$. For instance, when $|F_{\gamma^*}(v_s)| \geq 3$, the map $\pi_{res} : C_{\gamma^*_{v_s}} \to C_{\gamma_{v_s}}$ is

$$((\mathbb{H}^+)_{F_{\gamma^*}(v_s)} \setminus \Delta^*) \times \mathbb{R}|F_{\gamma^*}(v_s)|^{-3} \to ((\mathbb{H}^+)_{F_{\gamma^*}(v_s)} \setminus \Delta) \times \mathbb{R}|F_{\gamma^*}(v_s)|^{-3}$$

forgetting the coordinate subspace $\mathbb{H}^+$ of the labeled point $p_s$. Similarly, $\pi_{res} : M_{F_{\gamma^*}(v_s)}(\mathbb{C}) \to M_{F_{\gamma^*}(v_s)}(\mathbb{C})$ is

$$((\mathbb{C} \setminus \{0, 1\})_{F_{\gamma^*}(v_s)}^{-3} \setminus \Delta^*) \to ((\mathbb{C} \setminus \{0, 1\})_{F_{\gamma^*}(v_s)}^{-3} \setminus \Delta$$

forgetting the coordinate subspace $\mathbb{C}$ of the labeled point $p_s$.

The logarithmic forms $d \log(z_s - z_f)$ give a set of cohomology classes of the total space $C_{\gamma_{v_s}}$ (resp. $M_{F_{\gamma^*}(v_s)}(\mathbb{C})$). On the other hand, the restrictions of these logarithmic forms to each fiber generate the cohomology of that fiber (see, Section 6.2). By using the Leray-Hirsch theorem, we obtain

$$H^d(C_{\gamma^*}) = \bigoplus_{p+q=d} H^p(C_{\gamma}) \otimes H^q(A_{\gamma^*}).$$

If $\dim A_{\gamma^*} = 1$ and the fiber is an open interval, then we directly have

$$H^p(C_{\gamma^*}) = H^p(C_{\gamma}) \otimes H^0(A_{\gamma^*}) = H^p(C_{\gamma}) \otimes H^0(A_{\gamma^*}).$$

If $\dim A_{\gamma^*} = 1$ and the fiber is a circle, then $C_{\gamma^*}$ is $C_{\gamma} \times S^1$, and the claim follows from the Kunneth formula.

If $\dim A_{\gamma^*} = 0$, then each fiber is a single point and the statement is obvious.

Finally, the duality between cohomology and homology with closed support gives us the isomorphisms which we need to complete the proof.

The same arguments apply to o-planar trees of type 2 and type 3.

Let $Q_{\gamma}$ be the union of the substrata of $\overline{C}_{\gamma}$ of codimension one and higher.

**Proposition 6.4.** (a) Let $\gamma$ be an u-planar tree of type 1. The relative homology group $H_{\dim(C_{\gamma})-d}(\overline{C}_{\gamma}, Q_{\gamma})$ is generated by

$$\mathcal{E}_{s_1 s_2} \otimes \cdots \otimes \mathcal{E}_{s_4 s_5}$$

where $j_s < i_s$ and $i_1 < \cdots < i_d \leq |F^+_\gamma|$. In particular,

$$H_{\dim(C_{\gamma})}(\overline{C}_{\gamma}, Q_{\gamma}; \mathbb{Z}) = \mathbb{Z}[\overline{C}_{\gamma}]$$

where $[\overline{C}_{\gamma}]$ is the relative fundamental class of $\overline{C}_{\gamma}$. 

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(b) Let $\gamma$ be an u-planar tree of type 2. Then, $H_{\dim(C_\gamma)-d}(\overline{C_\gamma}, Q_\gamma)$ is generated by

$$\mathcal{G}_{s_{i_1}s_{i_2}s_{i_k}} \otimes \cdots \otimes \mathcal{G}_{s_{i_d} s_{j_d} s_{k_d}}$$

where $\sigma(s_{j_+}), \sigma(s_{k_+}) \neq s_{i_+}$, $j_+ < i_+$ and $2 < i_1 < \cdots < i_d \leq |F_\gamma|$. In particular,

$$H_{\dim(C_\gamma)}(\overline{C_\gamma}, Q_\gamma; \mathbb{Z}) = \mathbb{Z}[\overline{C_\gamma}]$$

where $[\overline{C_\gamma}]$ is the relative fundamental class of $\overline{C_\gamma}$.

(c) Let $W_S$ be the union of the substrata of $\overline{M}_S(\mathbb{C})$ of codimension one and higher. The relative homology group $H_{\dim(M_S(\mathbb{C}))-d}(\overline{M}_S(\mathbb{C}), W_S)$ is generated by

$$\mathcal{G}_{s_{i_1}s_{i_2}s_{i_k}} \otimes \cdots \otimes \mathcal{G}_{s_{i_d} s_{j_d} s_{k_d}}$$

where $j_+ < i_+$ and $i_1 < \cdots < i_d \leq |S|$. In particular,

$$H_{\dim(M_S(\mathbb{C}))}(\overline{M}_S(\mathbb{C}), W_S; \mathbb{Z}) = \mathbb{Z}[\overline{M}_S(\mathbb{C})]$$

where $[\overline{M}_S(\mathbb{C})]$ is the fundamental class of $\overline{M}_S(\mathbb{C})$.

Proof. We obtain the result by applying the forgetful morphism successively and using Lemma 6.3 and the generators of the homologies of closed support of the fibers given in Section 6.2.

It is clear that the top dimensional relative homologies are generated by the (relative) fundamental classes. \qed

7 Graph homology of $\overline{M}_S^\sigma(\mathbb{R})$

In this section, we give a combinatorial complex whose homology is the homology of $\overline{M}_S^\sigma(\mathbb{R})$.

7.1 A graph complex of $\overline{M}_S^\sigma(\mathbb{R})$: Fix($\sigma$) $\neq \emptyset$ case

Let $\sigma \in S_n$ be an involution such that Fix($\sigma$) $\neq \emptyset$. We define a graded group

$$G_d = \left( \bigoplus_{\gamma: |E_\gamma| = |S| - d - 3} H_{\dim(C_\gamma)}(\overline{C_\gamma}, Q_\gamma; \mathbb{Z}) \right) / I_d, \quad (7.1)$$

$$= \left( \bigoplus_{\gamma: |E_\gamma| = |S| - d - 3} \mathbb{Z}[\overline{C_\gamma}] \right) / I_d \quad (7.2)$$

where $[\overline{C_\gamma}]$ are the (relative) fundamental classes of the strata $\overline{C_\gamma}$ of $\overline{M}_S^\sigma(\mathbb{R})$.

For $|\text{Perm}(\sigma)| < 4$, the subgroup $I_d$ (for degree $d$) is the trivial subgroup. In all other cases (i.e., for $|\text{Perm}(\sigma)| \geq 4$), the subgroup $I_d$ is generated by the following elements.
The generators of the ideal of the graph complex.

The following subsections \( \text{R-1} \) and \( \text{R-2} \) describe the generators of the ideal of the graph complex.

\( \text{R-1. Degeneration of a real vertex.} \)

Consider an o-planar representative \( \tilde{\gamma} \) of a u-planar tree \( \gamma \) of type 1 such that \( |E_{\gamma}| = |S| - d - 5 \), and consider one of its vertices \( v \in V^R_\gamma \) with \( |v| \geq 5 \) and \( |F^+_\gamma(v)| \geq 2 \). Let \( f_i, f_j \in F^\gamma \) be conjugate pairs of flags for \( i = 1, 2 \) such that \( f_1, f_2 \in F^+_\gamma(v) \) of \( \tilde{\gamma} \), and let \( f_3 \in F^-_\gamma \). Put \( F = F^\gamma \) \( \cup \{f_1, f_2, f_3\} \).

We define two u-planar trees \( \gamma_1 \) and \( \gamma_2 \) as follows.

The o-planar representative \( \tilde{\gamma}_1 \) of \( \gamma_1 \) is obtained by inserting a pair of conjugate edges \( e = (f_1, f_2) \), \( \tilde{\gamma} \) at \( v \) in such a way that \( \tilde{\gamma}_1 \) gives \( \tilde{\gamma} \) when we contract the edges \( e, \tilde{e} \). Let \( \partial_{\gamma_1}(e) = \{\tilde{v}, v_1\} \), \( \partial_{\gamma_1}(\tilde{e}) = \{\tilde{v}, v_1\} \).

Then, the distribution of flags \( \gamma_2 \) is given by \( F_{\gamma_2}(\tilde{v}) = F_1 \cup \{f_3, f_1, f_2\} \), \( F_{\gamma_2}(v_1) = F_2 \cup \{f_1, f_2, f_3\} \) and \( F_{\gamma_2}(v_2) = F_3 \cup \{f_1, f_2, f_3\} \) where \( (F_1, F_2, F_3) \) is an equivariant partition of \( F \).

The u-planar trees \( \gamma_1, \gamma_2 \) are the equivalence classes represented by \( \tilde{\gamma}_1, \tilde{\gamma}_2 \) given above.

Then, we define

\[
\mathcal{R}(\gamma; v, f_1, f_2, f_3) := \sum_{\gamma_1} [C_{\gamma_1}] - \sum_{\gamma_2} [C_{\gamma_2}],
\]

where the summation is taken over all possible \( \gamma_i, i = 1, 2 \) for a fixed set of flags \( \{f_1, f_2, f_3\} \).

\( \text{R-2. Degeneration of a conjugate pair of vertices.} \)

Consider an o-planar representative \( \tilde{\gamma} \) of a u-planar tree \( \gamma \) of type 1 such that \( |E_{\gamma}| = |S| - d - 5 \), and a pair of its conjugate vertices \( v, \tilde{v} \in V^R_\gamma \) with \( |v| = |\tilde{v}| \geq 4 \). Let \( f_i \in F_{\gamma}(v), i = 1, \ldots, 4 \) and \( \tilde{f}_i \in F_{\gamma}(\tilde{v}) \) be the flags conjugate to \( f_i, i = 1, \ldots, 4 \). Put \( F = F_{\gamma}(v) \) \( \{f_1, \ldots, f_4\} \). Let \( (F_1, F_2) \) be a two-partition of \( F \), and \( F_1, F_2 \) be the sets of flags that are conjugate to the flags in \( F_1, F_2 \) respectively.

We define two u-planar trees \( \gamma_1 \) and \( \gamma_2 \) as follows.

The o-planar representative \( \tilde{\gamma}_1 \) of \( \gamma_1 \) is obtained by inserting a pair of conjugate edges \( e = (f_1, f_2), \tilde{e} = (\tilde{f}_1, \tilde{f}_2) \) to \( \tilde{\gamma} \) at \( v, \tilde{v} \) in such a way that \( \tilde{\gamma}_1 \) gives \( \tilde{\gamma} \) when we contract the edges \( e, \tilde{e} \). Let \( \partial_{\gamma_1}(e) = \{v_1, v_2\} \), \( \partial_{\gamma_1}(\tilde{e}) = \{v_3, v_4\} \).
The sets of flags of \( \tilde{\gamma}_1 \) are \( F_{\gamma_1}(v_\epsilon) = F_1 \cup \{ f_1, f_2, f_3 \} \), \( F_{\gamma_1}(v^e) = F_2 \cup \{ f_3, f_4, f^e \} \) and \( F_{\gamma_1}(v^e_\bar{\epsilon}) = F_1 \cup \{ f_1, f_2, f_3 \} \), \( F_{\gamma_1}(v^{e_\bar{\epsilon}}) = F_2 \cup \{ f_3, f_4, f^e \} \).

The o-planar representative \( \tilde{\gamma}_2 \) of \( \gamma_2 \) is also obtained by inserting a pair of conjugate edges into \( \tilde{\gamma} \) at the same vertices \( v, \bar{v} \), but the flags are distributed differently on vertices. Let \( \partial_\gamma(\epsilon) = \{ v_\epsilon, v^e_\epsilon \} \), \( \partial_\gamma(\bar{\epsilon}) = \{ v^{\bar{\epsilon}}, v^{e_\bar{\epsilon}} \} \). Then, the distribution of the flags of \( \tilde{\gamma}_2 \) is given by \( F_{\gamma_2}(v_\epsilon) = F_1 \cup \{ f_1, f_3, f_\epsilon \} \), \( F_{\gamma_2}(v^e_\epsilon) = F_2 \cup \{ f_2, f_4, f^\epsilon \} \), \( F_{\gamma_2}(v^{\bar{\epsilon}}) = F_1 \cup \{ f_1, f_3, f_{\bar{\epsilon}} \} \), and \( F_{\gamma_2}(v^{e_{\bar{\epsilon}}}) = F_2 \cup \{ f_2, f_4, f^{e_{\bar{\epsilon}}} \} \).

The u-planar trees \( \gamma_1, \gamma_2 \) are the equivalence classes represented by \( \tilde{\gamma}_1, \tilde{\gamma}_2 \) given above.

We define
\[
\mathcal{R}(\gamma; f_1, f_2, f_3, f_4) := \sum_{\gamma_1} |C_{\gamma_1}| - \sum_{\gamma_2} |C_{\gamma_2}|,
\]
where the summation is taken over all \( \gamma_1, \gamma_2 \) for a fixed set of flags \( \{ f_1, \ldots, f_4 \} \).

The ideal \( I_d \) is generated by \( \mathcal{R}(\gamma; f_1, f_2, f_3) \) and \( \mathcal{R}(\gamma; f_1, f_2, f_3, f_4) \) for all \( \gamma \) and \( v \) satisfying the required conditions above.

**The boundary homomorphism of the graph complex**

We define the *graph complex* \( G_* \) of the moduli space \( \overline{M}_d(S)(\mathbb{R}) \) by introducing a boundary map \( \partial : G_d \to G_{d-1} \)
\[
\partial \left[ C_\gamma \right] = \sum_{\gamma} \pm \left[ C_\gamma \right],
\]
where the summation is taken over all u-planar trees \( \gamma \) which give \( \tau \) after contracting one of their real edges.

**Theorem 7.1.** The homology of the graph complex \( G_* \) is isomorphic to the singular homology of \( \overline{M}_d(S)(\mathbb{R}) \) for \( \text{Fix}(\sigma) \neq \emptyset \).

**Proof.** First, we note that the statement directly follows when \( |\text{Perm}(\sigma)| = 0 \). In this case, the stratification of \( \overline{M}_d(S)(\mathbb{R}) \) is a cell decomposition. Details of this case can be found in [7] and [10]. Similarly, the stratifications of \( \overline{M}_d(S)(\mathbb{R}) \) are cell decompositions for the cases \( |S| = 4 \) and \( |\text{Perm}(\sigma)| = 2 \), and \( |S| = 3 \) and \( |\text{Perm}(\sigma)| = 2 \) (see examples in Section 6.2).

We prove the statement for the \( \sigma \neq \text{id} \) cases by induction on the cardinality of \( \text{Perm}(\sigma) \).

Let \( \pi : \overline{M}'_d(S)(\mathbb{R}) \to \overline{M}_d'(S)(\mathbb{R}) \) be the morphism forgetting the points labeled by \( s, \bar{s} \in \text{Perm}(\sigma) \). Here, we use the notations introduced in Section 6.1.2.

Let \( B_d \) denote the union of \( d \)-dimensional strata of \( \overline{M}_d'(S)(\mathbb{R}) \) (i.e., \( \bigcup_\gamma C_\gamma \) where \( |E_\gamma| = |S'| - d - 3 \)). The moduli space \( \overline{M}_d(S)(\mathbb{R}) \) is filtered as follows
\[
\emptyset = B_{-1} \subset B_0 \subset \cdots \subset B_{|S'|-4} \subset B_{|S'|-3} = \overline{M}_d'(S)(\mathbb{R}).
\]
The forgetful morphism $\pi$ induces a filtration of $\overline{M}_S^r(\mathbb{R})$:
\[
\emptyset = E_{-1} \subset E_0 \subset \cdots \subset E_{(|S^r|-4)} \subset E_{(|S^r|-3)} = \overline{M}_S^r(\mathbb{R})
\]
where $E_d = \pi^{-1}(B_d)$. Then, the spectral sequence of this filtration gives us
\[
E^1_{p,q} = H_{p+q}(E_p, E_{p-1}) \Rightarrow H_{p+q}(\overline{M}_S^r(\mathbb{R}); \mathbb{Z}). \quad (7.6)
\]
We prove this theorem by writing down the spectral sequence (7.6) explicitly. As a first step, we calculate the homology groups $H_{p+q}(E_{p-1})$.

**Step 1.**

We can write the homology of a pair $(E_p, E_{p-1})$ as a direct sum of the homology of its pieces:
\[
H_{p+q}(E_p, E_{p-1}) = \bigoplus_{\gamma \in E_r = |S^r| - d - 3} H_{p+q}(\pi^{-1}(\overline{C}_\gamma), \pi^{-1}(Q_\gamma)).
\]

Consider the following filtration of $\pi^{-1}(\overline{C}_\gamma)$:
\[
\emptyset \subset Y_0 \subset Y_1 \subset Y_2 = \pi^{-1}(\overline{C}_\gamma)
\]
where $Y_j$'s are the unions of strata
\[
Y_0 = \bigcup_{\gamma \in \gamma_m} \overline{C}_\gamma, \quad Y_1 = \bigcup_{\gamma \in \zeta_\gamma} \overline{C}_\gamma, \quad Y_2 = \bigcup_{\gamma \in \tau_\gamma} \overline{C}_\gamma,
\]
such that $\pi$ maps each of these stratum onto $\overline{C}_\gamma$, and the dimension of the fibers of $\pi : Y_j \to \overline{C}_\gamma$ is $j$ for $j = 0, 1, 2$.

By using this filtration, we obtain the following spectral sequence
\[
Y^1_{i,j} = H_{i+j}(Y_i, Y_{i-1} \bigcup_{\gamma} (Y_i \cap \pi^{-1}(Q_\gamma))) \Rightarrow H_{i+j}(E_p, E_{p-1}).
\]

Clearly, $Y_i$ contains strata of dimension $p + i$, and $Y_i \cap \pi^{-1}(Q_\gamma)$ contains the substrata that map to $B_{p-1}$ (i.e., substrata of codimension one or higher in $\overline{C}_\gamma$). Hence, we have
\[
\begin{align*}
Y^1_{0,j} &= \bigoplus_{\gamma \in \gamma_m} H_j(\overline{C}_\gamma, Q_{\gamma_m}), \\
Y^1_{1,j} &= \bigoplus_{\gamma \in \zeta_\gamma} H_{j+1}(\overline{C}_\gamma, Q_{\zeta_\gamma}), \\
Y^1_{2,j} &= \bigoplus_{\gamma \in \tau_\gamma} H_{j+2}(\overline{C}_\gamma, Q_{\tau_\gamma}).
\end{align*}
\]
By using Lemma 6.3 (and the isomorphism between relative homology and homology with closed support), we can write the groups $\mathcal{Y}_{1,j}$ as products of homology groups of the base and fiber:

The dimension of the fibers $\mathcal{A}_{m^*}$ of $\pi : \mathcal{C}_{m^*} \to \mathcal{C}$ is zero, hence

$$\mathcal{Y}_{0,j} = \bigoplus_{\gamma_m} H_j(\mathcal{C}_{\gamma}, Q_{\gamma}) \otimes H_0^s(A_{m^*}).$$

The dimension of the fibers $\mathcal{A}_{i^*}$ of $\pi : \mathcal{C}_{i^*} \to \mathcal{C}$ is one, hence

$$\mathcal{Y}_{1,j} = \bigoplus_{\xi_i} H_j(\mathcal{C}_{\gamma}, Q_{\gamma}) \otimes H_1^s(A_{i^*}).$$

Finally, the dimension of the fibers $\mathcal{A}_{k^*}$ of $\pi : \mathcal{C}_{k^*} \to \mathcal{C}$ is two, hence

$$\mathcal{Y}_{2,j} = \bigoplus_{\tau_k} H_j(\mathcal{C}_{\gamma}, Q_{\gamma}) \otimes H_2^s(A_{k^*})$$

$$\mathcal{Y}_{2,j-1} = \bigoplus_{\tau_k} H_j(\mathcal{C}_{\gamma}, Q_{\gamma}) \otimes H_1^s(A_{k^*}).$$

Then, the differentials $d_1^\gamma : \mathcal{Y}_{1,j} \to \mathcal{Y}_{0,j}$ and $d_2^\gamma : \mathcal{Y}_{1,j} \to \mathcal{Y}_{0,j}$ are given respectively by the differentials

$$\partial_* : H_0^s(A_{m^*}) \to \bigoplus_{\gamma_m} H_1^s(A_{m^*}),$$

$$\partial_* : H_1^s(A_{i^*}) \to \bigoplus_{\gamma_m < \zeta_i} H_0^s(A_{m^*}).$$

(7.7)

Hence, the differential $d_1^\gamma$ maps the fundamental class $[\mathcal{C}_{\tau_k}]$ to $\sum \pm [\mathcal{C}_{\xi_i}]$, and similarly $[\mathcal{C}_{\tau_k}]$ to $\sum \pm [\mathcal{C}_{\gamma_m}]$.

Finally, the differential $d_2^\gamma : \mathcal{Y}_{1,j} \to \mathcal{Y}_{0,j+1}$ is given by the differentials

$$\partial_* : H_1^s(A_{\tau_k}) \to \bigoplus_{\gamma_m < \tau_k} H_0^s(A_{m^*}).$$

(7.8)

For each pair of zero dimensional fibers $\mathcal{A}_{m^*}$ and $\mathcal{A}_{n^*}$ lying in the same two dimensional fiber $\mathcal{A}_{\tau_k}$, there is a generator in $H_1^s(A_{\tau_k})$ whose image under $\partial_*$ gives the difference of these points (see Section 6.2). Therefore, each pair of strata $\mathcal{C}_{m^*}$, $\mathcal{C}_{n^*}$ which are zero dimensional fibrations over $\mathcal{C}$ are homologous relative to $\pi^{-1}(Q_{\gamma})$ i.e.,

$$[\mathcal{C}_{\gamma_m}] - [\mathcal{C}_{\gamma_n}] = 0$$

(7.9)

in $H_*(E_p, E_{p-1})$.

It is important to note that the kernel of the differential $d_2^\gamma$ is trivial. This follows from the fact that the kernel of $\partial_*$ given in (7.8) is trivial, as a consequence of the homology of the fibers given in Section 6.2. Therefore, the homology $H_*(E_p, E_{p-1})$ is given by the total homology of the spectral sequence $(\mathcal{Y}_{i,j}/I_0, d_1)$ where $I_0$ is the ideal of the relations given by (7.3).
Step 2.

The calculations in Step 1 imply that the term $E_{1\ast}$, is generated by the relative fundamental classes of the strata. Moreover, it admits the relations that are imposed in the definition of $G_{\ast}$:

The forgetful morphism maps the chains defined in $R_{-1}$ and $R_{-2}$ for $f_i \neq s$ onto the chains of the same type in $\overline{M}_{S'}(R)$. Hence, if $f_i \neq s$ for all $i$, then $R(\gamma; v, f_1, f_2, f_3)$ and $R(\gamma; v, f_1, \cdots, f_4)$ are homologous to zero since we have assumed that the statement is true for $M_{S'}(R)$.

On the other hand, the same statement for $f_1 = s$ comes as a consequence of the calculation of Step 1. For each relation (7.9) in relative homology $H_{\ast}(E_{p}, E_{p-1})$, there is a relation in $H_{\ast}(\overline{M}_{S}(R))$. In fact, the sums (7.3) (and also (7.4)) are mapped onto a difference given in (7.9) by the relativization map $rel : E_{p} \to E_{p} / E_{p-1}$.

We need to confirm that the sums defined in $R_{-1}$ and $R_{-2}$ are indeed homologous to zero. We can show this by using certain forgetful maps.

The chains of type $R_{-1}$. Consider the composition of projection $\overline{C}_{\tau^\ast} \to \overline{C}_{\mu^\ast}$ onto the factor corresponding to vertex $v \in V_{\tau^\ast}$, and forgetful map $\overline{C}_{\tau^\ast} \to \overline{C}_{\mu^\ast}$ where $\mu^\ast$ is a one-vertex o-planar tree with $F_{\mu^\ast} = \{ f_1, f_2 \}$ and $F_{\tau^\ast} = \{ f_3 \}$ which is obtained by forgetting all tails of $\tau^\ast$ but $f_1, f_1, f_2, f_2, f_3$.

The configuration space $\overline{C}_{\mu^\ast}$ is a two-dimensional disc with a puncture, and it is stratified as in Figure 4.

![Figure 4: The strata of $\overline{C}_{\mu^\ast}$.](image)

If a codimension two stratum $\overline{C}_{\gamma^\ast}$ of $\overline{C}_{\tau^\ast}$ is in the fiber over the codimension two stratum of $\overline{C}_{\mu^\ast}$, which is lying in its boundary (see, Figure 3), then $\gamma^\ast$ is obtained from $\tau^\ast$ by inserting a pair of real edges at vertex $v$. Similarly, if a codimension two stratum $\overline{C}_{\gamma^\ast}$ of $\overline{C}_{\tau^\ast}$ is in the fiber over the codimension two stratum of $\overline{C}_{\mu^\ast}$ which is lying inside of $\overline{C}_{\mu^\ast}$ (see, Figure 3), then $\gamma^\ast$ is obtained from $\tau^\ast$ by inserting a pair of conjugate edges at vertex $v$ as above. Moreover, since $\overline{C}_{\mu^\ast}$ is a punctured disc, the fibers of the forgetful map over any two points of $\overline{C}_{\mu^\ast}$ are homologous; i.e., $R(\gamma; v, f_1, f_2, f_3)$ is homologous to zero.

The chains of type $R_{-2}$. We show that the sum (7.4) is homologous to zero by
using the same method as in the 9t-1 case. Here, we use a projection map \( \overline{\mathcal{M}}_{\tau^*} \to \overline{\mathcal{M}}_{\tau^*}(\mathbb{C}) \). The relations in the complex moduli space \( \overline{\mathcal{M}}_{\tau^*}(\mathbb{C}) \) which are introduced by Kontsevich and Manin induce the relations \( \mathcal{R}(\gamma; v, f_1, \cdots, f_4) = 0 \) (see, \cite{19} or \cite{22}).

**Step 3.**

We have a complete description of generators and relations in \( \mathbb{E}_{+,s}^1 \). We need to calculate the differentials.

The first differential \( d_1^R : \mathbb{E}_{p,q}^1 \to \mathbb{E}_{p-1,q}^1 \) is given by the sum boundary homomorphisms

\[
\partial_* : H_{\dim \tau}^*(\overline{\mathcal{M}}_{\tau^*}) \to \oplus_{\gamma^*} H_{\dim \tau}^* (\overline{\mathcal{M}}_{\gamma^*})
\]

where \( \overline{\mathcal{M}}_{\tau^*} \subset \mathbb{E}_p \) and \( \overline{\mathcal{M}}_{\gamma^*} \subset \mathbb{E}_{p-1} \), and, by contracting a real edge \( e \) of each \( \gamma^* \), we obtain \( \tau^* \).

In order to complete the proof, we only need to show that the higher differentials \( d_2^R \) and \( d_3^R \) of \( \mathbb{E}_{+,s} \) vanish.

For dimensional reasons, the differential \( d_2^R \) is zero except \( d_2^R : \mathbb{E}_{p,1}^1 \to \mathbb{E}_{p-2,2}^1 \) and \( d_2^R : \mathbb{E}_{p,0}^1 \to \mathbb{E}_{p-2,1}^1 \).

On the other hand,

- **I** if \( \overline{\mathcal{M}}_{\tau^*} \) is in \( \mathbb{E}_{p,2} \), then either \( v_s \in V_{r_s}^E \) and \( |v_s| \geq 5 \), or \( v_s \notin V_{r_s}^E \) and \( |v_s| \geq 4 \);

- **II** if \( \overline{\mathcal{M}}_{\tau^*} \) is in \( \mathbb{E}_{p,1} \), then \( v_s \in V_{r_s}^E \) and either \( |v_s| = 4 \), or \( |v_s| = 3 \) and \( |v_c| \geq 4 \);

- **III** if \( \overline{\mathcal{M}}_{\tau^*} \) is in \( \mathbb{E}_{p,0} \), then either \( v_s \in V_{r_s}^E \) and \( |v_s| = |v_c| = 3 \), or \( v_s \notin V_{r_s}^E \) and \( |v_s| = 3 \)

(see Lemma \(6.1\)).

Now, assume that

\[
d_2^R(\overline{\mathcal{M}}_{\tau^*}) = \sum \pm |\overline{\mathcal{M}}_{\tau^*}| \neq 0
\]

for \( \overline{\mathcal{M}}_{\tau^*} \) in \( \mathbb{E}_{p,1} \). Each \( \tau^* \) must produce \( \zeta^*_i \) by contracting one of its real edges, due to the stratification given in Theorem \(5.2\) Note that, the contraction of a real edge of \( \tau^*_i \) increases or preserves the valency of the vertex \( v_s \). However, according to condition **II**, the valency \( |v_s| \) must decrease after a contraction i.e., this gives a contradiction. Hence, \( d_2^R : \mathbb{E}_{p,1} \to \mathbb{E}_{p-2,2}^1 \) must be zero.

By using similar arguments (i.e., comparing valencies of vertices \( v_s \) and \( v_c \)), we show that the other higher differentials \( d_2^R : \mathbb{E}_{p,0}^1 \to \mathbb{E}_{p-2,1}^1 \) and \( d_3^R : \mathbb{E}_{p,0}^1 \to \mathbb{E}_{p-3,2}^1 \) also vanish.

The proof is completed by observing that the differential of the graph complex is the sum of the differentials \( d_1^R \) and \( d_2^R \) that are defined in \(7.6\), \(7.10\) respectively.
Remark 7.2. If $|S| > 4$ and $|\text{Fix}(\sigma)| \neq 0$, then the moduli space $\overline{M}_S'(\mathbb{R})$ is not orientable. A combinatorial construction of the orientation double covering of $\overline{M}_S'(\mathbb{R})$ is given in [1]. A stratification of the orientation cover is given in terms of certain equivalence classes of o-planar trees. By following the same ideas above, it is possible to construct a graph complex generated by fundamental classes of the strata that calculates the homology of the orientation double cover of $\overline{M}_S'(\mathbb{R})$.

7.2 A graph complex of $\overline{M}_S'(\mathbb{R})$: $\text{Fix}(\sigma) = \emptyset$ case

Let $\sigma \in S_n$ be an involution such that $\text{Fix}(\sigma) = \emptyset$. We define a graded group

$$G_d = \left( \bigoplus_{\gamma; |E_\gamma| = |S| - d - 3} H_{\dim(C_\gamma)}(\overline{C}_\gamma, Q_\gamma; \mathbb{Z}) \right) / I_d, \quad (7.11)$$

$$\quad = \left( \bigoplus_{\gamma; |E_\gamma| = |S| - d - 3} \mathbb{Z}[\overline{C}_\gamma] \right) / I_d \quad (7.12)$$

where $[\overline{C}_\gamma]$ are the (relative) fundamental class of the strata $\overline{C}_\gamma$ of $\overline{M}_S'(\mathbb{R})$.

The subgroup $I_d$ (for degree $d$) is generated by the following elements.

The generators of the ideal of the graph complex.

G-1. Degeneration of a real vertex: Case of type 1.

G-1.1. $|F_\gamma| = 0$ case Consider an o-planar representative $\tilde{\gamma}$ of an u-planar tree $\gamma$ of type 1 such that $|E_\gamma| = |S| - d - 5$ and $|F_\gamma| = 0$. Let $v$ be its real vertex, and assume that $|v| \geq 6$. Let $f_1 \in F_{\gamma_1}^+, i = 1, 2, 3$, and let $\bar{f}_i \in F_{\gamma_2}^-$ be their conjugate flags. Put $F = F_{\gamma_1}(v) \setminus \{f_1, f_2, f_3, \bar{f}_1, \bar{f}_2, \bar{f}_3\}$.

We define two u-planar trees $\gamma_1, \gamma_2$ as follows.

The o-planar representative $\gamma_1$ of $\gamma_1$ is obtained by inserting a pair of conjugate edges $e = (f_\epsilon, f^\epsilon), \bar{e} = (f_\bar{\epsilon}, f^{\bar{\epsilon}})$ into $\tilde{\gamma}$ at $v$ in such a way that $\gamma_1$ produces $\gamma$ when we contract the edges $e, \bar{e}$. Let $\partial_{\gamma_1}(e) = \{\bar{v}, v^\epsilon\}$, $\partial_{\gamma_1}(\bar{e}) = \{\bar{v}, v^{\bar{\epsilon}}\}$. The sets of flags are $F_{\gamma_1}(v) = F_1 \cup \{f_1, f_2, f_3, f_\bar{1}, f_\bar{2}, f_\bar{3}\}$, $F_{\gamma_1}(v^\epsilon) = F_2 \cup \{f_2, f_3, f^{\bar{\epsilon}}\}$ and $F_{\gamma_1}(v^{\bar{\epsilon}}) = F_3 \cup \{f_3, f_\bar{2}, f^{\epsilon}\}$, where $F$ is the disjoint union of $F_1, F_2$ and $F_3$. The set $F_2$ contains the flags that are conjugate to the flags in $F_2$ and vice versa.

The o-planar representative $\gamma_2$ of $\gamma_2$ is obtained in a similar way. First, we swap $f_1$ and $f_\bar{1}$ (i.e., put $f_1$ in $F_2^\epsilon$ and $f_{\bar{1}}$ in $F_2^{\bar{\epsilon}}$). Then, we obtain $\gamma_2$ by inserting a pair of conjugate edges at the vertex $v$ in the same way, but the flags are distributed differently $F_{\gamma_2}(v) = F_1 \cup \{f_3, f_\bar{2}, f_\bar{3}, f_2, f_3, f^{\epsilon} \}$. The u-planar trees $\gamma_1, \gamma_2$ are the equivalence classes represented by $\gamma_1, \gamma_2$ given above.

Then, we define

$$R(\gamma; v, f_1, f_2, f_3) := \sum_{\gamma_1}[\overline{C}_{\gamma_1}] - \sum_{\gamma_2}[\overline{C}_{\gamma_2}], \quad (7.13)$$
where the summation is taken over all possible $\gamma_i$, $i = 1, 2$ for a fixed set of flags $\{f_i, \bar{f}_i \mid i = 1, 2, 3\}$.

\textbf{$\mathcal{G}$-1.2. $|F^n_\mathcal{R}| \neq 0$ case} The definition of generators of ideal for this case is the same as $\mathcal{R}$-1. Instead of repeating this definition here, we will refer to [7,3] when it is needed.

\textbf{$\mathcal{G}$-2. Degeneration of a real vertex: Case of type 2.}

Consider an u-planar tree $\gamma$ of type 2 such that $|E_\gamma| = |S| - d - 5$. Let $v$ be its real vertex, and assume that $|v| \geq 6$. Let $f_i, \bar{f}_i \in F_\gamma(v)$ be conjugate pairs of flags for $i = 1, 2, 3$. Put $F = F_\gamma(v) \setminus \{f_1, f_2, f_3, \bar{f}_1, \bar{f}_2, \bar{f}_3\}$.

We define two u-planar trees $\gamma_1, \gamma_2$ as follows.

The first tree, $\gamma_1$, is obtained by inserting a pair of conjugate edges $e = (f, f')$, $\bar{e} = (\bar{f}, f')$ to $\gamma$ at $v$ with boundaries $\partial_{\gamma_1}(e) = \{\bar{v}, v', \bar{e}, e\}$, $\partial_{\gamma_1}(\bar{e}) = \{\bar{v}, v', e, \bar{e}\}$. The sets of flags are given by $F_{\gamma_1}(v) = F_1 \cup \{f_1, f_2, f_3, \bar{f}_1, \bar{f}_2, \bar{f}_3\}$, $F_{\gamma_1}(e) = F_2 \cup \{f_1, f_2, f_3, \bar{f}_1, \bar{f}_2, \bar{f}_3\}$ and $F_{\gamma_1}(\bar{e}) = F_2 \cup \{f_1, f_2, f_3, \bar{f}_1, \bar{f}_2, \bar{f}_3\}$, where $F$ is a disjoint union of $F_1$, $F_2$ and $F_3$. $F_2$ contains the flags that are conjugate to the flags in $F_1$ and $F_3$.

Then, we define

$$\mathcal{R}(\gamma; v, f_1, f_2, f_3) := \sum_{\gamma_1} [C_{\gamma_1}] - \sum_{\gamma_2} [C_{\gamma_2}]. \quad (7.14)$$

Here, the summation is taken over all possible $\gamma_i$, $i = 1, 2$ for a fixed set of flags $\{f_i, \bar{f}_i \mid i = 1, 2, 3\}$.

\textbf{$\mathcal{G}$-3. Degeneration of a conjugate pair of vertices.}

Consider an u-planar tree $\gamma$ (of type 1, type 2 or type 3) such that $|E_\gamma| = |S| - d - 5$, and a pair of its conjugate vertices $v, \bar{v} \in V_\gamma \setminus V^n_\mathcal{R}$ such that $|v| = |\bar{v}| \geq 4$. Let $f_i \in F_\gamma(v), i = 1, \cdots, 4$, and let $\bar{f}_i \in F_\gamma(\bar{v})$ be their conjugate flags. Put $F = F_\gamma(v) \setminus \{f_1, \cdots, f_4\}$. Let $(F_1, F_2)$ be a partition of $F$, and $(\bar{F}_1, \bar{F}_2)$ be the sets of conjugate flags.

We define two u-planar trees $\gamma_1$ and $\gamma_2$ as follows.

The first one, $\gamma_1$, is obtained by inserting a pair of conjugate edges $e = (f, f'), \bar{e} = (f, f')$ into $\gamma$ at $v, \bar{v}$ such that $\partial_{\gamma_1}(e) = \{v, \bar{e}, v', \bar{v}\}$, $\partial_{\gamma_1}(\bar{e}) = \{v, \bar{e}, v', \bar{v}\}$. The sets of flags are given by $F_{\gamma_1}(v) = F_1 \cup \{f_1, f_2, f_3, \bar{f}_1, \bar{f}_2, \bar{f}_3\}$, $F_{\gamma_1}(e) = F_2 \cup \{f_1, f_2, f_3, \bar{f}_1, \bar{f}_2, \bar{f}_3\}$ and $F_{\gamma_1}(\bar{e}) = F_2 \cup \{f_1, f_2, f_3, \bar{f}_1, \bar{f}_2, \bar{f}_3\}$.

The second one, $\gamma_2$, is also obtained by inserting a pair of conjugate edges into $\gamma$ at the same vertices $v, \bar{v}$, but the flags are distributed differently on vertices. Let $\partial_{\gamma_2}(e) = \{v, v', \bar{v}, \bar{v}\}$, $\partial_{\gamma_2}(\bar{e}) = \{v, v', \bar{v}, \bar{v}\}$. Then, the sets of flags are
given by \( F_{\gamma_2}(v) = F_1 \cup \{ f_1, f_3, f_c \} \), \( F_{\gamma_2}(v) = F_2 \cup \{ f_4, f_c \} \), \( F_{\gamma_2}(v) = F_1 \cup \{ f_1, f_3, f_c \} \) and \( F_{\gamma_2}(v) = F_2 \cup \{ f_4, f_c \} \).

We define
\[
\mathcal{R}(\gamma; v, f_1, f_2, f_3, f_4) := \sum_{\gamma_1} [C_{\gamma_1}] - \sum_{\gamma_2} [C_{\gamma_2}],
\] (7.15)

where the summation is taken over all such \( \gamma_i, i = 1, 2 \) for a fixed set of flags \( \{ f_1, \cdots, f_4 \} \).

The ideal \( I_d \) is generated by the chains \( \mathcal{R}(\gamma; v, f_1, f_2, f_3) \) defined in (7.3), (7.13) and (7.14), and \( \mathcal{R}(\gamma; v, f_1, f_2, f_3, f_4) \) defined in (7.15) for all \( \gamma \) and \( v \) satisfying the required conditions above.

The boundary homomorphism of the graph complex.

We define the graph complex \( \mathcal{G}_d \) of the moduli space \( \overline{M}_S^d(R) \) by introducing a boundary map \( \partial : \mathcal{G}_d \rightarrow \mathcal{G}_{d-1} \)
\[
\partial [C_{\tau}] = \sum_{\gamma} [C_{\gamma}] \pm [C_{\gamma}],
\]

where the summation is taken over all u-planar trees \( \gamma \) which give \( \tau \) after contracting one of their real edges.

**Theorem 7.3.** The homology of the graph complex \( \mathcal{G}_d \) is isomorphic to the singular homology of \( \overline{M}_S^d(R) \) for \( \text{Fix}(\sigma) = \emptyset \).

The proof of this theorem is essentially the same as the proof of Theorem 7.1. We will not repeat it here.

8 Fundamental group of \( \overline{M}_S^d(R) \)

In this section, we give a presentation of the fundamental group of \( \overline{M}_S^d(R) \) by using the groupoid of paths which are transversal to the codimension one strata of \( \overline{M}_S^d(R) \). This idea has been used by Kamnitzer and Henriques in [14] to calculate the fundamental group of \( \overline{M}_S^d(R) \) for \( \sigma = \text{id} \). This section extends their description to \( \pi_1(\overline{M}_S(R)) \) for an arbitrary involution \( \sigma \).

8.1 Fundamental groups of open parts of strata

In this section, we consider a particular subset of the set of u-planar trees. Let \( \mathcal{O}_\text{Tree}(\sigma) \) be the set of u-planar trees having no conjugate pairs of edges. If \( \gamma \in \mathcal{O}_\text{Tree}(\sigma) \) is of type 1, then \( V_\gamma = V_\gamma^R \). If \( \gamma \in \mathcal{O}_\text{Tree}(\sigma) \) is of type 2, then \( |V_\gamma| = 1 \). If \( \gamma \in \mathcal{O}_\text{Tree}(\sigma) \) is of type 3, then \( |V_\gamma| = 2 \).

For a u-planar tree \( \gamma \in \mathcal{O}_\text{Tree}(\sigma) \), the open part \( O_\gamma \) of the stratum is the closed stratum \( C_\gamma \) minus the union of the closure of its codimension one strata.
**Proposition 8.1.** For $\gamma \in \mathcal{O}Tree(\sigma)$, the open part of the stratum $\mathcal{C}_\gamma$ is simply connected.

**Proof.** For $\gamma^* \in \mathcal{O}Tree(\sigma)$, the open part of a stratum is

$$O_{\gamma^*} = \begin{cases} \prod_{v \in \gamma^*} O_{\gamma^*} & \text{if } \gamma^* \text{ is of type 1}, \\ O_{\gamma^*} & \text{if } \gamma^* \text{ is of type 2}, \\ \overline{M_{\mathcal{F}_+(v)}(\mathbb{C})} & \text{if } \gamma^* \text{ is of type 3}. \end{cases}$$

This follows from the fact that the open part $O_{\gamma^*}$ of $\mathcal{C}_\gamma$ is the product of the open parts of its factors.

Hence, we only need to consider the factors that correspond to the one-vertex trees.

We prove the statement by induction on the cardinality of $\text{Perm}(\sigma)$. First, we note that the open parts of the strata of $\overline{M_S(\mathbb{R})}$ are contractible for $S = \text{Fix}(\sigma)$, $|S| = 4$ and $|\text{Fix}(\sigma)| = 2$, $|S| = 4$ and $|\text{Fix}(\sigma)| = 0$, and $|S| = 3$ and $|\text{Fix}(\sigma)| = 1$. In these cases, the stratifications are cell decompositions, and the open parts of the strata are open discs (see [14] and examples in Section 5.2).

Let $\gamma^*$ be a one-vertex $u$-planar tree of type 1. Let $|\mathcal{F}^+_{\gamma^*}| > 0$, and $\pi : \overline{\mathcal{C}}_{\gamma^*} \to \overline{\mathcal{C}}_\gamma$ be the morphism forgetting the conjugate pairs of points $p_s, p_{\bar{s}}$. Let $O$ be the subset of the fiber $\pi^{-1}(\Sigma; p)$ such that $(\Sigma^*, p^*) \in O$ does not require any stabilization after forgetting $p_s, p_{\bar{s}}$. For $(\Sigma^*, p^*) \in O$, $\Sigma^* = \Sigma$. Since all special points are fixed in $\Sigma^*$, the different elements of $O$ are given by positions of the labelled point $p_s$. The labelled point $p_s$ is in $(\Sigma \setminus \{(\text{special points}) \cup \Sigma(\mathbb{R})\})/c_\Sigma$. This follows from the fact that all special points must be distinct (hence, we need to remove special points and $\Sigma(\mathbb{R})$ where $p_s$ and $p_{\bar{s}}$ collide and give a real node) and $s$ in either $\mathcal{F}^+_\gamma$ or $\mathcal{F}^-_{\gamma}$ (so that, we need to take the quotient with respect to the real structure $c_\Sigma : \Sigma \to \Sigma$).

A degeneration of $(\Sigma^*, p^*) \in O$, which is obtained as the limit as $p_s$ goes to a special point in $\Sigma \setminus \Sigma(\mathbb{R})$, gives us an element in $O_{\gamma^*}$, since the limit element has an additional conjugate pair of edges. On the other hand, a degeneration of $(\Sigma^*, p^*)$, which is obtained as the limit as $p_s$ goes to a point in $\Sigma(\mathbb{R})$, gives a curve with a real node; i.e., this limit does not lie in $O_{\gamma^*}$. Therefore, the restriction of the forgetful map $\pi : O_{\gamma^*} \to O_\gamma$ has a fiber $(\Sigma \setminus \Sigma(\mathbb{R}))/c_\Sigma$ over $(\Sigma; p) \in O_{\gamma^*}$. It is clear that the fiber is simply connected.

If we assume simply connectedness of $O_\gamma$, then $O_{\gamma^*}$ is clearly simply connected. We prove the statement by induction on the cardinality of the labeling set $\text{Perm}(\sigma)$.

The proofs for $u$-planar trees $\gamma \in \mathcal{O}Tree(\sigma)$ of type 2 and type 3 are the same as for the type 1 case. The fiber of the forgetful map $\pi : O_{\gamma^*} \to O_\gamma$ over $(\Sigma; p)$ is $\Sigma$ when $\gamma^*$ is of type 2, and $\Sigma/c_\Sigma$ when $\gamma^*$ is of type 3. In both cases, the fibers are simply connected. \qed

**Proposition 8.2.** The moduli space $\overline{M^\sigma_S(\mathbb{R})}$ is stratified by simply connected subspaces $O_\gamma$.

**Proof.** This directly follows from the fact that the open parts $O_\gamma$ of the strata $\mathcal{C}_\gamma$ are pairwise disjoint. \qed
8.2 A groupoid of paths in $M_\Sigma(R)$

We consider the following groupoid $P$ of paths in $M_\Sigma(R)$. Choose an element $(\Sigma(i), p(i))$ in every connected component $C_{\tau(i)}$ of the main stratum $M_\Sigma(R)$.

The objects $\text{Ob}(P)$ are these elements $(\Sigma(i), p(i))$.

The morphisms of $P$ are given as follows: Let $C_{\tau(i)}$ and $C_{\tau(j)}$ be a pair of top-dimensional adjacent strata in $M_\Sigma(R)$, and let $C_{\gamma_{ij}}$ be their common codimension one stratum. The morphism $\langle \gamma_{ij} \rangle$ is the homotopy class of oriented paths in $M_\Sigma(R)$ that start from the point $(\Sigma(i), p(i))$, pass through the common codimension one (open) stratum $C_{\gamma_{ij}}$, and end at $(\Sigma(j), p(j))$. Notice that such paths connecting a pair of points $(\Sigma(i), p(i))$ and $(\Sigma(j), p(j))$ are homotopic, since the open parts of $C_{\tau(i)}$, $C_{\tau(j)}$ and $C_{\gamma_{ij}}$ are simply connected (see Proposition 8.1). The homotopy classes of paths that intersect only codimension 1 strata are given by concatenations of $\langle \gamma_{ij} \rangle$'s.

**Theorem 8.3.** The fundamental group $\pi_1(M_\Sigma(R))$ is generated by the loops

$$\langle \gamma_{i_1 i_2} \gamma_{i_2 i_3} \cdots \gamma_{i_{n-1} i_n} \gamma_{i_n i_1} \rangle := \langle \gamma_{i_2 i_1} \gamma_{i_3 i_2} \cdots \gamma_{i_n i_{n-1}} \gamma_{i_1 i_n} \rangle \quad (8.1)$$

which are subject to the following relations:

1. Let $\gamma_{i_1 i_2}$ be a u-planar tree having only one edge, then

$$\langle \gamma_{i_1 i_2} \gamma_{i_2 i_1} \rangle = 1. \quad (8.2)$$

2. Let $\gamma'$ with $|E_{\gamma'}| = 2$, and let $\gamma_{ij}, \tau_{ji}$ be the only u-planar trees that are obtained by contracting one of the edges of $\gamma'$, then

$$\langle \gamma_{ij} \tau_{ji} \rangle = 1. \quad (8.3)$$

3. Let $\gamma'$ with $|E_{\gamma'}| = 2$, and let $\gamma_{i_1 i_2}, \cdots, \gamma_{i_{n-1} i_n}$ be the u-planar trees that are obtained by contracting an edge of $\gamma'$, then

$$\langle \gamma_{i_1 i_2} \gamma_{i_2 i_3} \cdots \gamma_{i_{n-1} i_n} \gamma_{i_n i_1} \rangle = 1. \quad (8.4)$$

**Proof.** Every loop in $M_\Sigma(R)$ is homotopic to a loop which is transversal to codimension one strata. Such transversal loops can be obtained by perturbing the original loops. Hence, we can choose the loops given in (8.1) as representatives of the homotopy classes of loops.

These loops are subject to the following relations:

1. The concatenation of a path $\langle \gamma_{i_1 i_2} \rangle$ with the reverse path $\langle \gamma_{i_2 i_1} \rangle$ is obviously homotopic to a point and gives the relation (8.2).

On the other hand, if two paths in $M_\Sigma(R)$ are homotopic, then they are homotopic by a homotopy of paths that are transversal to the codimension one strata. Therefore, the homotopy relations arise from the passing of the paths...
through codimension two strata; Let $\gamma'$ be a u-planar tree corresponding to a codimension two stratum of $\mathcal{M}_S^\sigma(\mathbb{R})$. The stratum $\mathcal{C}_{\gamma'}$ is contained either in two or in four codimension one strata, since we can obtain two or four u-planar trees by contracting one of the two edges of $\gamma'$.

2. If only two codimension one strata, $\mathcal{C}_{\gamma_{ij}}$ and $\mathcal{C}_{\tau_{ji}}$, intersect along the codimension two stratum $\mathcal{C}_{\gamma'}$, then the concatenation of $\langle \gamma_{ij} \rangle$ and $\langle \tau_{ji} \rangle$ gives the loop $\langle \gamma_{ij} \tau_{ji} \rangle$ around $\mathcal{C}_{\gamma'}$ which is contractible; i.e., gives relations in \( (8.3) \).

3. If there are four codimension one strata $\mathcal{C}_{\gamma_{i1_{i2}}, \cdots, \gamma_{i4_{i1}}}$ intersecting along the codimension two stratum $\mathcal{C}_{\gamma'}$, then the loop $\langle \gamma_{i1_{i2}} \gamma_{i2_{i3}} \gamma_{i3_{i4}} \gamma_{i4_{i1}} \rangle$ around $\mathcal{C}_{\gamma'}$ is contractible and gives the relations \( (8.4) \).

\[ \square \]

**Remark 8.4.** The fundamental group of $\mathcal{M}_S^\sigma(\mathbb{R})$ is particularly interesting for $\sigma = \text{id}$, since $\mathcal{M}_S^\sigma(\mathbb{R})$ is $K(\pi, 1)$ in this case. However, this is not true when $\sigma \neq \text{id}$. For instance, if $|\text{Perm}(\sigma)| = 6$, then the open part of each top dimensional stratum is topologically a 2-dimensional sphere i.e., higher homotopy groups of $\mathcal{M}_S^\sigma(\mathbb{R})$ cannot be trivial in this case.

**Remark 8.5.** A similar presentation of the fundamental group of the orientation double cover of $\mathcal{M}_S^\sigma(\mathbb{R})$ is given in [2].

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