A NOTE ON MUMFORD-TATE GROUPS AND DOMAINS I

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Abstract. We use some classical results on nonabelian Galois cohomology to explain Mumford-Tate groups and Domains in Hodge theory, as automorphism groups and moduli of polarized Hodge structures.

1. Introduction

In this short note we remark some connections between non-abelian Galois cohomology $H^1$ and basic symmetry groups of Hodge tensors. The observation is comparison of classical results on Galois cohomology in 'Local fields’ of J. Serre and Mumford-Tate groups in 'Mumford-Tate Domains' by M. Green-P Griffiths-M. Kerr. We first briefly introduce non-abelian cohomology of groups as the cohomology of groups with coefficients in non-commutative $G$-module, as a set with a distinguished element, namely trivial element. Then we explain a Galois descent procedure on Hodge tensors which connects the above theory to Hodge theory. Mumford-Tate groups can be understood as basic symmetry groups of Hodge structures. Mumford-Tate domains parametrize the set of Hodge structures whose generic points have a fixed Mumford-Tate group.

2. Non-abelian cohomology

Let $G$ be a group and $A$ (not necessarily abelian) another group on which $G$ acts on the left. Write $A$ multiplicatively. $H^0(G, A)$ is by definition the group $A^G$ of elements of $A$ fixed by $G$. A 1-cocycle would be a map $s \mapsto a_s$ from $G \rightarrow A$ such that $a_{st} = a_s.s(a_t)$. Two cocycles $a_s$ and $b_s$ are equivalent if there exists $a \in A$ such that $b_s = a^{-1}.a_s.s(a)$ for all $s \in G$. This defines an equivalence relation and the quotient set is denoted by $H^1(G, A)$. It is a pointed set with a distinguished element of the unit cocycle $a_s = 1$. Here $s(-)$ means the action of $s \in G$, and $a_s$ is the value of the cocycle $a$ at $s \in G$. These two definitions agree with the usual definitions of cohomology of $G$ when $A$ is abelian. These constructions are also functorial in $A$.

1991 Mathematics Subject Classification. 14Lxx.

Key words and phrases. Mumford-Tate group, Mumford-Tate domain, non-abelian Galois cohomology, Hodge tensors, Galois descent.
and $G$. If $0 \to A \to B \to C \to 0$ is an exact sequence of non-abelian $G$-modules, then we have the following exact sequence of pointed sets

$$H^0(G, A) \to H^0(G, B) \to H^0(G, C) \to H^1(G, A) \to H^1(G, B) \to H^1(G, C)$$

when the group $G$ is Galois group of a (not necessarily finite) field extension and $A$ a topological $G$-module such that

$$A = \bigcup A^H$$

when $H$ runs through the open normal subgroups of $G$, we define

$$H^1(G, A) := \lim_{\to} H^1(G/H, A^H)$$

There is a simple geometric interpretation for $H^1(G, A_K)$: It is the set of classes of principal homogeneous spaces for $A$, defined over $k$ which have a rational point over $K$, page 123.

As an example, take $G = \mathbb{Z}/2 = Gal(\mathbb{C}/\mathbb{R})$ acting naturally on a set of tensors $A_{\mathbb{R}}$. Then any element in $H^1(\mathbb{Z}/2, A_{\mathbb{C}})$ is determined by an involution map $-1 : a \mapsto * \in A_{\mathbb{R}}$. This shows

$$H^1(\mathbb{Z}/2, A_{\mathbb{C}}) = A_{\mathbb{R}}$$

### 3. Galois descent for Hodge tensors

If $V$ be a vector space over $k$, provided with a fixed tensor of $x$ of type $(p, q)$, i.e. $x \in \bigotimes^p V \otimes \bigotimes^q V^*$ where $V^*$ is the dual of $V$. two pairs $(V, x), (V, x')$ are called $k$-isomorphic if there is a $k$-linear isomorphism $f : V \to V'$ such that $f(x) = x'$. Denote by $A_k$ the group of these automorphisms. Let $K/k$ be a Galois extension with Galois group $G$. Write $E_{V,x}(K, k)$ for the set of $k$-isomorphism classes that are $K$-isomorphic to $(V, x)$. The group $G$ acts on $V_K$ by $s.(x \otimes \lambda) = x \otimes s.\lambda$. It also acts on $f : V \to V'$ by $s.f = s \circ f \circ s^{-1}$. If we put

$$p_s = f^{-1} \circ s \circ f \circ s^{-1}, \quad s \in G$$

the map $s \mapsto p_s(f)$ is a 1-cocycle in $H^1(G, A_K)$. 
Theorem 3.1. ([S] page 153) The map \( \theta : E_{V,x}(K/k) \to H^1(G, A_K) \) defined by

\[
f \mapsto p_s(f)
\]

is a bijection.

Theorem 3.2. ([S] page 153) The set \( H^1(G, \text{Aut}(Q, K)) \) is in bijective correspondence with the classes of quadratic \( k \)-forms that are \( K \)-isomorphic to \( Q \).

The above definition can be generalized in this way that instead of considering a single tensor \( T^{p,q} = \bigotimes^p V \otimes \bigotimes^q V^* \) one may consider a sum of such tensors that is a subset \( T \) as

\[
T \subset T^{\bullet \bullet} = \bigoplus_{p,q} T^{p,q}
\]

The proofs will proceed exactly the same and we obtain

Theorem 3.3. Assume \( T \) is a set of tensors for a vector space \( V \). The map \( \theta : E_{V,T}(K/k) \to H^1(G, A_{T,K}) \) defined by

\[
f_T \mapsto p_s(f_T)
\]

is a bijection, where \( A_{T,K} \) is the group of \( K \)-automorphisms of all the tensors in \( T \).

4. Mumford-Tate groups and Domains

Mumford-Tate groups are basic symmetry groups of Hodge structures. To begin with let \( V \) be finite dimensional \( \mathbb{Q} \)-vector space, and \( Q \) a non-degenerate bilinear map \( Q : V \otimes V \to \mathbb{Q} \) which is \((-1)^n\)-symmetric for some fixed \( n \). A Hodge structure is given by a representation \( \phi : \mathbb{U}(\mathbb{R}) \to \text{Aut}(V,Q)_{\mathbb{R}} \),

\[
\mathbb{U}(\mathbb{R}) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad a^2 + b^2 = 1
\]

It decomposes over \( \mathbb{C} \) into eigenspaces \( V^{p,q} \) such that \( \phi(t).u = t^{p,q}.u \) for \( u \in V^{p,q} \), and \( \overline{V^{p,q}} = V^{q,p} \). The Hodge tensors \( Hg_{a,b}^\phi \) are given by the subspace of \( T^{a,b} \) such that \( \mathbb{U}(\mathbb{R}) \) acts trivially. Set
The period domain $D$, associated to the above data is the set of all polarized Hodge structures $\phi$ with a given Hodge numbers. The real Lie group $G(\mathbb{R})$ acts transitively on $D$. The compact dual $\hat{D}$ of $D$ is the set of flags $F^• = \{ F^n \subset ... \subset F^0 = V_C \}$ with $\dim F^p = \sum_{r \geq p} h_{r,s}$ and where the first Hodge-Riemann bilinear relation, $Q(F^p, F^{n-p+1}) = 0$ holds.

**Definition 4.1.** [MGK] The Mumford-Tate group of the Hodge structure $\phi$ denoted $M_{\phi}(\mathbb{R})$ is the smallest $\mathbb{Q}$-algebraic subgroup of $G = \text{Aut}(V, Q)$ with the property

$$\phi(\mathbb{U}(\mathbb{R})) \subset M_{\phi}(\mathbb{R})$$

$M_{\phi}$ is a simple, connected, reductive $\mathbb{Q}$-algebraic group.

**Theorem 4.2.** [MGK] If $F^• \in \hat{D}$ the Mumford-Tate group $M_{F^•}$ is the subgroup of $G_{\mathbb{R}}$ that fixes the Hodge tensors in $Hg_{F^•}^•$. 

**Remark 4.3.** It is probable that the Mumford-Tate group be all of $G_{\mathbb{R}} = \text{Aut}(Q, \mathbb{R})$, and of course in this case the Hodge structure is not polarized. The Mumford-Tate group of product of Hodge structures is not in general product of their Mumford-Tate groups.

5. Relation with non-abelian cohomology

We are going to investigate the relation between nonabelian cohomology discussed in sec. 2, and classifying spaces for Hodge structures. By definition $H^0(G_{\mathbb{R}}, Hg_{F^•}^•) = M_{F^•}$. The relation $H^1(\mathbb{Z}/2, G_C) = G_{\mathbb{R}}$ is trivial. By applying Theorem 3.3 to the above construction we get the following results.

**Theorem 5.1.** $H^1(\mathbb{Z}/2, M_{F^•}) = M_{\phi}$.

This follows from the example at the end of sec 2, and the fact that $M_{\phi}$ is the subgroup of $G$ that fixes pointwise the algebra of Hodge tensors, cf. [MGK].
Theorem 5.2. \( H^1(\text{Gal}(\mathbb{C}/\mathbb{Q}), \text{Aut}(\mathbb{Q}, \mathbb{C})) = \tilde{D} \)

Follows from Theorem 3.3, noting that \( \tilde{D} = G_\mathbb{C}/\text{Stab}_{G_\mathbb{C}}(F_0) \).

Theorem 5.3. \( H^1(\text{Gal}(\mathbb{R}/\mathbb{Q}), \text{Aut}(\mathbb{Q}, \mathbb{R})) = D \)

This also follows from Theorem 3.3, and similar identity \( D = G_\mathbb{R}/\text{Stab}_{G_\mathbb{R}}(F_0) \). We have the short exact sequence

\[
0 \to \mathbb{Z}/2 \to \text{Gal}(\mathbb{C}/\mathbb{Q}) \to \text{Gal}(\mathbb{R}/\mathbb{Q}) \to 0
\]

where the second non-zero map is the restriction

\[
0 \to H^1(\mathbb{Z}/2, \text{Aut}(\mathbb{Q}, \mathbb{C})^{\text{Gal}(\mathbb{R}/\mathbb{Q})}) \to H^1(\text{Gal}(\mathbb{C}/\mathbb{Q}), \text{Aut}(\mathbb{Q}, \mathbb{C})) \to H^1(\text{Gal}(\mathbb{R}/\mathbb{Q}), \text{Aut}(\mathbb{Q}, \mathbb{C}))^{\mathbb{Z}/2}
\]

where the second map is the restriction. The first map is called inflation map. The first item is \( \text{Aut}(\mathbb{Q}, \mathbb{R}) \) by the discussion in section 1, and the second item is \( \tilde{D} \) by Theorem 5.2. Thus we have

\[
0 \to \text{Aut}(\mathbb{Q}, \mathbb{R}) \to \tilde{D} \to H^1(\text{Gal}(\mathbb{R}/\mathbb{Q}), \text{Aut}(\mathbb{Q}, \mathbb{C}))^{\mathbb{Z}/2}
\]

as exact sequence of sets with distinguished unit elements.

Theorem 5.4. \( H^1(\mathbb{U}_\mathbb{R}, G_\mathbb{R}) = D \), where \( G_\mathbb{R} \) acts on \( G_\mathbb{R} = \text{Aut}(\mathbb{Q}, \mathbb{R}) \) by \( g : T \mapsto g^Tg \).

Proof. The cocycle condition is equivalent to \( \mathbb{U}_\mathbb{R} \to G_\mathbb{R} \) being a homomorphism and the boundary condition is when two such homomorphism are conjugate by an element of \( G_\mathbb{R} \). Then, the theorem is consequence of that, \( D \) is isomorphic to the set of conjugacy classes of the isotropy group of a fixed Hodge structure. \( \square \)

For each point \( \phi \in D \), the adjoint representation

\[
\text{Ad}\phi : \mathbb{U}(\mathbb{R}) \to \text{Aut}(\mathfrak{g}_\mathbb{R}, B)
\]

induces a Hodge structure of weight 0 on \( \mathfrak{g} \). This Hodge structure is polarized by the killing form \( B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \). and it is a sub-Hodge structure of \( \tilde{V} \times V \).
**Theorem 5.5.** $H^1(\mathbb{U}(\mathbb{R}), \text{Aut}(\mathfrak{g}_\mathbb{R}, B)) = D$, where $\mathbb{U}_\mathbb{R}$ acts by $g.X = g^t X g$, $X \in \mathfrak{g}_\mathbb{R}$, and $B$ is the killing form.

**Proof.** Again the cocycle condition is the cocycles are $\text{Hom}(\mathbb{U}_\mathbb{R}, \text{Aut}(B))$, and the coboundray condition becomes when two such homomorphisms are conjugate by an automorphism of $B$. Regarding $B$ as a tensor then the theorem follows from the known fact that $M_\phi$ is the subgroup of $G$ with the property that $M_\phi$-stable subspaces $W \subset T^{a,b}_\phi$ are exactly the sub-Hodge structures of these tensor space. \qed

**References**

[S] J. P. Serre, Local fields, Graduate texts in mathematics 67, Springer Verlag, 1979

[MGK] M. Green, P. Griffiths, M. Kerr, Mumford-tate domains, Bollettino dell’ UMI (9) III (2010), 281-307.

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