Intransitive geometries and fused amalgams
Extended arXiv version

Ralf Gramlich  Max Horn  Antonio Pasini  Hendrik Van Maldeghem

February 1, 2008

Abstract
We study geometries that arise from the natural $G_2(K)$ action on the geometry of one-dimensional subspaces, of nonsingular two-dimensional subspaces, and of nonsingular three-dimensional subspaces of the building geometry of type $C_3(K)$ where $K$ is a perfect field of characteristic 2. One of these geometries is intransitive in such a way that the non-standard geometric covering theory from [GVM06] is not applicable. In this paper we introduce the concept of fused amalgams in order to extend the geometric covering theory so that it applies to that geometry. This yields an interesting new amalgamation result for the group $G_2(K)$.

1 Introduction

Tits’ lemma [Tit86] (see also [Pas85, Lemma 5] or [Pas94, Theorem 12.28]) provides a geometric way to prove that certain groups can be identified as universal enveloping groups of certain amalgams. More precisely, the universal enveloping group of the amalgam of parabolic subgroups of a group $G$ acting flag-transitively on a geometry $\Gamma$ equals $G$ if and only if $\Gamma$ is simply connected. Obviously, this technique to compute amalgams of groups is limited. In order to include more amalgams with this geometric technique, one can generalise Tits’ lemma to intransitive geometries. Roughly speaking, two difficulties have to be overcome in this generalisation process: (1) the reconstruction of the geometry from the various parabolic subgroups, (2) finding the right amalgam and getting control over its universal enveloping group by using the simple connectivity of the geometry.

In [GVM06] a theory was established for geometries with an automorphism group admitting possibly more than one vertex orbit per type. For the reconstruction of the geometry from the stabiliser data the authors used a result of Stroppel [Str93]. Their work was motivated by the geometries arising from non-isotropic elements with respect to an orthogonal polarity in projective space. Another sporadic amalgam, considered by Hoffman and Shpectorov [HS05], related to the group $G_2(3)$ could be handled very elegantly with that theory. Yet another application of the new covering theory is a local characterisation of the group $SL_{n+1}(\mathbb{F}_q)$ via centralisers of root subgroups using the local characterisation of the graph on incident point-hyperplane pairs obtained in [GPP].

In general, it seems that non-standard amalgams related to exceptional groups of Lie type cannot be treated with Tits’ original lemma [Tit86]. But also the intransitive theory developed in [GVM06] often falls short, as it requires that

(‡) for every flag $F$ of rank two or three of $\Gamma$, the action of $G$ on the orbit $G.F$ is flag-transitive.

In the present paper, we consider some rather natural amalgams related to Dickson’s groups of type $G_2$. For some of them, the existing theory suffices to get control over the universal enveloping group. For others, we need to modify the theory. This will lead us to fused amalgams. Roughly speaking, fused amalgams occur when the corresponding group acts intransitively on the set of maximal flags of the corresponding geometry as in [GVM06] and Stroppel’s reconstruction of the geometry fails because Property (‡) above is not satisfied. In our example, the group acts transitively on each type of vertex (and there are three types), but there are two orbits on the
set of chambers. As a result, there seems to be no purely group-theoretic way to reconstruct the geometry. Instead, we use the properties of the diagram to settle incidences that cannot be recovered by the group. In particular, we exploit the fact that the diagram is a string of length three, and that hence the residues of the elements belonging to the middle node are generalised digons. We only consider geometries belonging to tree diagrams, because circuits introduce ambiguity on how to define incidence in residues that are generalised digons. The main covering theoretic results of this article are the Reconstruction Theorem 3.4 and the Covering Theorem 3.11. The main applications of this covering theory contained in this paper are the simple connectivity result Theorem 2.4 and the amalgamation result Theorem 3.13.

The paper is organised as follows. In Section 2 we define the geometries that are relevant for the rest of the paper and in particular for the amalgams that we will consider in Section 2.3. In Section 3, we develop a theory of intransitive geometries and amalgams, which we call fused amalgams, that allows us to tackle the amalgam described in Section 2.3. We immediately apply the theory to our situation. In the final Section 4, we establish the simple connectivity for the investigated geometries.

2 Some geometries related to $G_2$

2.1 The split Cayley hexagon and its properties

We consider the Chevalley group $G_2(K)$, with $K$ any (commutative) field. Naturally associated with each Chevalley group is a building, in the sense of Tits [Tit74]. In the case of $G_2(K)$, this building is a bipartite graph, which is the incidence graph of a pair of generalised hexagons (and we may freely consider one of those by choosing one of the bipartition classes as set of points, and the other class as set of lines). We use the standard notions from incidence geometry and the theory of building geometries, like distances between elements and opposition of elements, cf. [Pas94] and [Tit74]. We recall that a generalised hexagon is a point-line incidence geometry with the properties that

(GH1) every two elements (points or lines) are contained in an ordinary hexagon (i.e., a cycle of 12 distinct consecutively incident elements), and

(GH2) there are no ordinary $n$-gons for $n < 6$.

The generalised hexagon related to $G_2(K)$ is called the split Cayley hexagon and can be represented on the parabolic quadric $Q(6, K)$, which is a nondegenerate quadric in $PG(6, K)$ of (maximal) Witt index 3. The points of the hexagon are all points of $Q(6, K)$, while the lines (the hexagon lines) are only some well-chosen lines on $Q(6, K)$. If $Q(6, K)$ has the standard equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$, then a line on $Q(6, K)$ with Grassmannian coordinates $(p_0, p_1, p_2, \ldots, p_6, p_{12}, p_{13}, \ldots, p_{56})$ is a hexagon line if and only if $p_{12} = p_{34}$, $p_{56} = p_{33}$, $p_{45} = p_{23}$, $p_{01} = p_{56}$, $p_{02} = -p_{35}$ and $p_{46} = -p_{13}$. This line set has the following properties (see e.g. [VMal98]).

(i) The set of lines of $H(K)$ through a fixed point fills up a projective plane on $Q(6, K)$. We call such a plane a hexagonal plane.

(ii) Any plane of $Q(6, K)$ that contains at least one hexagon line is a hexagonal one. Any other plane of $Q(6, K)$ will be called an ideal plane.

(iii) Every line of $Q(6, K)$ that does not belong to the hexagon is contained in a unique hexagonal plane. We call such a line an ideal line. The point of the corresponding hexagonal plane that is the intersection of all hexagon lines in that plane is called the ideal centre of the ideal line.
The ideal centres of all ideal lines of an ideal plane \( \pi \) form again an ideal plane \( \pi' \), which, together with \( \pi \), generates a hyperplane \( H \) of \( \text{PG}(6, \Bbbk) \) that intersects \( Q(6, \Bbbk) \) is a non-degenerate (hyperbolic) quadric of Witt index 3. The point set of \( \pi \cup \pi' \) is the point set of a non-thick ideal subhexagon of \( \text{H}(\Bbbk) \). The hyperplane \( H \) is called a hyperbolic hyperplane. Every hyperplane that intersects \( Q(6, \Bbbk) \) in a hyperbolic quadric arises in this way. In particular, every hyperplane \( H \) that contains a plane of \( Q(6, \Bbbk) \) is either a tangent hyperplane or a hyperbolic hyperplane. The former does not contain disjoint planes of \( Q(6, \Bbbk) \).

If \( \Bbbk \) has characteristic two, and \( \Bbbk \) is perfect (which means that the mapping \( x \mapsto x^2 \) is surjective), then the projection of the point set of \( Q(6, \Bbbk) \) from the point \((0,0,0,1,0,0,0)\) onto the hyperplane \( \text{PG}(5, \Bbbk) \) with equation \( X_3 = 0 \) embeds \( Q(6, \Bbbk) \) bijectively onto a symplectic space \( \text{W}(5, \Bbbk) \), so that we also obtain an embedding of \( \text{H}(\Bbbk) \) into \( \text{W}(5, \Bbbk) \). The lines of \( Q(6, \Bbbk) \) are projected onto totally isotropic lines with respect to the corresponding symplectic polarity (we will call such lines symplectic lines), cf. 2.4.14 of [VMa98]. The lines of \( \text{PG}(5, \Bbbk) \) that are not symplectic will be called non-symplectic lines. The projection of hexagonal planes and ideal planes will be called hexagonal and ideal, respectively. Likewise, the projection of hexagon and ideal lines will be called hexagon and ideal, respectively; both are symplectic lines. A nonsingular plane is a nonsingular plane containing a hexagonal line.

The above properties also translate to the situation in \( \text{PG}(5, \Bbbk) \), when the characteristic of \( \Bbbk \) is equal to two. For instance, every ideal line is contained in a unique hexagonal plane and the ideal centre is not contained in that ideal line.

Moreover, we have the following:

(v) Let \( l \) be a non-symplectic line of \( \text{PG}(5, \Bbbk) \). Then the set of hexagon lines at hexagon-distance three from all points of \( l \) form a distinguished regulus \( R \) of a hyperbolic quadric \( Q(3, \Bbbk) \) in the orthogonal space \( l^\perp \) of \( l \) with respect to the symplectic polarity. This follows immediately from the regulus property (see [Ron80]) and the fact that opposition of points in the hexagon corresponds to non-perpendicularity in the symplectic polar space. Moreover, every pair of opposite lines (in the hexagon) is contained in such a regulus.

### 2.2 Some geometries

We now consider four different infinite classes of geometries \( \Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3 \), all of rank 3 and with type set \( \{1,2,3\} \). We call elements of type 1 points, of type 2 lines, and of type 3 planes.

To define \( \Gamma_0, \Gamma_1, \Gamma_2 \) let \( \Bbbk \) be perfect and of characteristic two. The geometries \( \Gamma_i, i \in \{0,1,2\} \), have as set of elements of type 1 the set of points of \( \text{PG}(5, \Bbbk) \), and as set of elements of type 2 the set of non-symplectic lines of \( \text{PG}(5, \Bbbk) \), with natural incidence. The elements of type 3 of the geometries \( \Gamma_0, \Gamma_1, \Gamma_2 \) are all nonsingular planes, all nonspecial nonsingular planes, and all special nonsingular planes, respectively. Incidence between elements of type 2 and 3 is natural, and incidence between elements of type 1 and 3 is given by the following rule: \( p \) is incident with \( \pi \) if and only if there is a type 2 element \( l \) incident with both. The geometry \( \Gamma_0 \) is flag-transitive for the symplectic group \( S_6(\Bbbk) \) and has been considered by Cuypers [Cuy94] and Hall [Hal88] and, more recently, by Blok and Hoffman [BH]; see also [Gra04].

The geometry \( \Gamma_0 \) is in a certain sense a join of the geometries \( \Gamma_1 \) and \( \Gamma_2 \). Indeed, the point-line truncations of \( \Gamma_0, \Gamma_1, \) and \( \Gamma_2 \) coincide, while the plane set of \( \Gamma_0 \) consists of the disjoint union of the plane sets of \( \Gamma_1 \) and \( \Gamma_2 \).

The following gives a relation between covers of two connected rank 3 geometries \( \Delta_1, \Delta_2 \) having identical point-line truncations and the join \( \Delta \) of \( \Delta_1 \) and \( \Delta_2 \). For \( i = 1,2 \) let \( \Delta_i \) be the universal cover of \( \Delta_i \) and let \( t_i \) be number of layers of the covering projection from \( \Delta_i \) to \( \Delta_i \). Furthermore, let \( \Delta \) be the universal cover of \( \Delta \) and let \( t \) be number of layers of the covering projection from \( \Delta \) to \( \Delta \). (We refer the reader to [ST34] for a thorough introduction to the covering theory of simplicial complexes, in particular for the definition of a universal cover of a connected pure simplicial complex.)
Proposition 2.1
Assume that the planes of $\Delta_1$ and $\Delta_2$ are connected. If, for some $i \in \{1, 2\}$, we have $t_i < \infty$, then $t | t_i$.

Proof. Let $\pi : \hat{\Delta} \to \Delta$ be a universal covering of $\Delta$ and, for $i = 1, 2$, let $\Sigma_i$ be the pre-image of $\Delta_i$ under $\pi$. Since the planes of $\Delta_i$ are connected and since $\Delta_i$ and $\Delta$ have identical connected point-line truncations, the preimage $\Sigma_i$ is connected, so $\pi$ induces a covering from $\Sigma_i$ to $\Delta_i$. Hence $t | t_i$, if $t_i$ is finite. 

Corollary 2.2
Assume that the planes of $\Delta_1$ and $\Delta_2$ are connected.

(i) If one of $\Delta_1$, $\Delta_2$ is simply connected, then $\Delta$ is simply connected.

(ii) Suppose $t_1, t_2 < \infty$. Then $t | \gcd(t_1, t_2)$. In particular, $\Delta$ is simply connected, if $t_1$ and $t_2$ are coprime.

Remark 2.3 The join of two geometries $\Delta_1$ and $\Delta_2$ can be simply connected, even if $\Delta_1$ and $\Delta_2$ are isomorphic and admit infinite universal covers. A nice example for this behaviour is the geometry studied in [HS05], which in fact also occurs in [AS83], [DWHVM05], [Kan85] in different guise. In [HS05] Hoffman and Shpectorov study an amalgam of maximal subgroups of $\hat{G} = \text{Aut}(G_2(3))$ given by a certain choice of subgroups $\hat{L} = 2^4 \cdot L_3(2) : 2$, $\hat{N} = 2_+^{1+4}(S_3 < S_4)$, $M = G_2(2) = U_3(3) : 2$ which corresponds to an amalgam of subgroups of $G = G_2(3)$ given by $L = \hat{L} \cap G = 2^3 \cdot L_3(2)$, $N = \hat{N} \cap G = 2_+^{1+4}(S_3 < S_4)$, $M = G_2(2) = U_3(3) : 2$, $K = \epsilon M e^{-1}$ for $e \in O_2(L) \setminus O_2(L)$.

The groups $\hat{G}_1 = \hat{L}$, $\hat{G}_2 = \hat{N}$, $\hat{G}_3 = M$ define a flag-transitive coset geometry $\Gamma$ of rank three for $\hat{G} = \text{Aut}(G_2(3))$, which is simply connected by [HS05]. The subgroup $G = G_2(3)$ of $\hat{G}$ does not act flag-transitively on $\Gamma$. Nevertheless, the groups $G_\mu = L$, $G_\lambda = N$, $G_{\pi_1} = M$, $G_{\pi_2} = K$ define an intransitive coset geometry of rank three for $\hat{G} = G_2(3)$ satisfying Property (1) from the introduction, which is isomorphic to $\Gamma$ by [HS05] and, hence, simply connected, so that non-standard covering theory as in [GVM06] is applicable.

The coset geometries $(G_p, G_l, G_{\pi_1}, *)$ and $(G_p, G_l, G_{\pi_2}, *)$ are isomorphic to the GAB — Geometry that is Almost a Building — studied in [AS83] Table 1, Example 4], in [DWHVM05] Section 6.1, and in [Kan85] with diagram

```
6
```

This GAB is very far from being simply connected. In fact, by [Kan85] the amalgam of $L$, $N$, $M$ admits the group $G_2(\mathbb{Q}_2)$ as a universal enveloping group — while by [HS05] the amalgam of $\hat{L}$, $\hat{N}$, $\hat{M}$ admits the group Aut($G_2(3)$) as its universal enveloping group. We conjecture that Kantor’s description [Kan85] of the universal cover of $(G_p, G_l, G_{\pi_1}, *) \cong (G_p, G_l, G_{\pi_2}, *)$ can be used to give an alternative proof of the simple connectivity of the join of $(G_p, G_l, G_{\pi_1}, *)$ and $(G_p, G_l, G_{\pi_2}, *)$ by studying those quotients of the group $G_2(\mathbb{Q}_2)$ that admit an involutory outer automorphism. However, the combinatorial simple connectivity proof given by Hoffman and Shpectorov [HS05] is short and clear and likely to be shorter than any group-theoretic proof of simple connectivity.

Concerning the fourth class of geometries, let $K$ be any field. Then the rank 3 geometry $\Gamma_3$ consists of the points of the split Cayley hexagon $H(K)$, the ideal lines, and the ideal planes, with natural incidence. The amalgam and corresponding geometry $\Gamma_3$ considered here has also been treated by Baumeister, Shpectorov and Stroth in an unpublished manuscript [BSS01]. We have found an independent proof which we include here so that a proof of this fact is made available in the literature. Moreover there exists a result [Sip06] by Shpectorov dealing with the simple connectivity of hyperplane complements in arbitrary dual polar spaces with line size at least five, thus independently implying simple connectivity of $\Gamma_3$, but only for $|K| \geq 4$.

In this article we prove the following results:
Theorem 2.4

(i) The geometry $\Gamma_0$ is simply connected.

(ii) The geometry $\Gamma_1$ is flag-transitive. Moreover, it is simply connected, whenever $|K| > 2$.

(iii) The geometry $\Gamma_2$ is simply connected.

(iv) The geometry $\Gamma_3$ is simply connected.

Proof.

(i) This is proved in Proposition 4.1 also [BH] or Proposition 2.1 plus Proposition 4.3.

(ii) See Propositions 2.5 and 4.2.

(iii) Cf. Proposition 4.3.

(iv) This follows from Proposition 4.4, also [BSS01], or [Shp06] for $|K| \geq 4$.

2.3 Amalgams for $\Gamma_2$

The geometries $\Gamma_0$ and $\Gamma_3$ have been extensively studied. Moreover, the geometry $\Gamma_1$ is flag-transitive, cf. Proposition 2.5, so that classical covering theory applies. Hence we concentrate on the amalgam of parabolics given by the $G_2(K)$ action on $\Gamma_2$.

Let $p, l, \pi$ be a chamber of $\Gamma_2$ and denote by $G_p, G_l, G_\pi, G_{p,l}$, etc., the respective stabilisers.

We now collect information about $\Gamma_2$ and these stabilisers, most of which are based on the following proof of the flag-transitivity of $\Gamma_1$.

Proposition 2.5

The action of $G_2(K)$ on the geometry $\Gamma_1$ is flag-transitive.

Proof. Set $G := G_2(K)$. All non-symplectic lines are determined by two opposite points of $H(K)$. The fact that $G$ acts transitively on pairs of opposite points of $H(K)$ (see e.g. Chapter 4 of [VMa98]) implies that $G$ acts transitively on the point-line pairs of $\Gamma_1$. So we may fix such a point-line pair $(x, l)$ and it suffices to prove that $H := G_{x,l}$ acts transitively on the planes of $\Gamma_1$ containing $l$. The group $H$ stabilises the pole $\Sigma$ of $l$ with respect to the symplectic polarity related to the $C_3$ building $W(5, K)$. The pole $\Sigma$ is a projective 3-space, so that every plane $\pi$ of $PG(5, K)$ containing $l$ meets $\Sigma$ in a unique point $x_\pi$. Viewed in $H(K)$, the space $\Sigma$ is determined by the lines at distance three from all the points of $l$. These lines form a distinguished regulus $R$ of a hyperbolic quadric $Q(3, K)$ in $\Sigma$, cf. Section 2.1 item (v). Clearly, $\pi$ is nonsingular. Also, it is easy to see that $\pi$ contains a hexagon line if and only if $x_\pi$ is contained in the quadric $Q(3, K)$. Let $R'$ be the complementary regulus of $R$ on $Q(3, K)$. Then every point $y$ on $l$ uniquely determines a line $m$ of $R'$ by the fact that all hexagon lines through $y$ meet $m$. Now, the stabiliser in $G$ of $Q(3, K)$ contains the group $L_2(K) \times L_2(K)$. Hence the assertion is equivalent with saying that in $\Sigma$, the group $L_2(K) \times L_2(K)$ stabilising the hyperbolic quadric $Q(3, K)$ acts transitively on the pairs $(p, k)$, where $p$ is a point off the quadric, and $k$ is a line of a fixed regulus of the quadric, which is a true statement as one can easily verify.

Proposition 2.6

The action of $G_2(K)$ on the geometry $\Gamma_2$ is transitive on the incident point-line pairs and transitive on the incident line-plane pairs, but it is intransitive on the incident point-plane pairs and has two incident point-plane orbits instead. Moreover, the stabiliser of an incident line-plane pair $(l, \pi)$ has two orbits on the points incident to $l$.

Proof.
(i) Point-line-transitivity: The point-line truncations of $\Gamma_1$ and $\Gamma_2$ coincide (see the discussion before Proposition 2.1) and $G_2(\mathbb{K})$ is flag-transitive on $\Gamma_1$ by Proposition 2.3 so that $G_2(\mathbb{K})$ acts transitively on the incident point-line pairs of $\Gamma_2$.

(ii) Point-plane-intransitivity: The planes of $\Gamma_2$ are those rank 2 planes of the $C_5$ building geometry $W(5, \mathbb{K})$ which contain a (unique) hexagon line. Therefore the plane stabiliser $G_\pi$ has to fix this hexagon line and consequently cannot map a point on that line onto a point in the plane not on the line. Hence $G_\pi$ is not transitive on the set of points incident with $\pi$, whence $\Gamma_2$ is not point-plane-transitive.

(iii) Line-plane-transitivity: Let $l$ be a line of $\Gamma_2$ as in the proof of Proposition 2.3. A plane $\pi$ of $PG(5, \mathbb{K})$ containing $l$ has rank two with respect to the symplectic form, and intersects the pole $\Sigma$ of $l$ with respect to the symplectic form in a point $x_\pi$. As in the proof of Proposition 2.3 the space $\Sigma$ carries the structure of a $Q(3, \mathbb{K})$, and $\pi$ contains a hexagon line if and only if $x_\pi$ is contained in $\Sigma$. Line-plane-transitivity now is a consequence of line-transitivity and transitivity of $G_l$ on the points of $Q(3, \mathbb{K})$.

(iv) Two orbits: Denote the hexagon line contained in $\pi$ by $h$. The pole $l^t$ of $l$ with respect to the symplectic polarity contains a regulus of hexagon lines, cf. Section 2.1, item (v). The map sending a point of $\pi$ onto the set of points of $\mathcal{R}$ at distance three in $H(\mathbb{K})$ is a bijection of the points of $l$ onto the lines of the complementary regulus of $\mathcal{R}$. Since the pointwise stabiliser $H$ in $G_2(\mathbb{K})$ of $\mathcal{R}$ acts two-transitively on the complementary regulus, the Moufang property, and the regulus condition, cf. [Ron80, VMal98 Proposition 4.5.11]) and since $H$ stabilises the line $l$, we see that $H_x$ acts transitively on $\mathcal{R}\setminus\{x\}$ where $x = l \cap h$. Hence $G_2(\mathbb{K})$ has two orbits on the incident point-plane pairs as well.

\[\square\]

3 Fused amalgams and intransitive geometries

The nonstandard notions that we will need below were introduced in [GVM06], to which we refer for more details and results.

3.1 Diagram coset pregeometries

Definition 3.1 (Diagram Coset Pregeometry) Let $I$ be a finite set, let $\Delta = (I, \sim)$ be a tree, and let $(T_i)_{i \in I}$ be a family of pairwise disjoint sets. Also, let $G$ be a group and let $(G^{i,t})_{t \in T_i, i \in I}$ be a family of subgroups of $G$. Then the diagram coset pregeometry of $G$ with respect to $(G^{i,t})_{t \in T_i, i \in I}$ equals the pregeometry

\[
\left\{ (C, t) : t \in T_i \text{ for some } i \in I, C \in G/G^{i,t} \right\}, *, \text{typ}
\]

over $I$ with $\text{typ}(C, t) = i$ if $t \in T_i$, and

(DCos) $g G^{i,t} * h G^{s,j}$ if

- $i = j$ and $t = s$ and $g G^{i,t} \cap h G^{s,j} \neq \emptyset$,
- $i, j$ adjacent in $\Delta$ and $g G^{i,t} \cap h G^{s,j} \neq \emptyset$, or
- $i, j$ not adjacent in $\Delta$ and there exists a geodesic $i = x_0, \ldots, x_k = j$ in $\Delta$ and cosets $g_{x_1, x_2} G^{x_1, x_2}$ with $g G^{i,t} = g_{x_0, x_0} G^{x_0, x_0}, h G^{s,j} = g_{x_k, x_k} G^{x_k, x_k}$ and $g_{x_1, x_2, x_3} G^{x_1, x_2, x_3}$
Since the type function is completely determined by the indices, we also denote the coset pregeometry of $G$ with respect to $(G^{t,i})_{t \in T_i, i \in I}$ by
\[(G/G^{t,i} \times \{t\})_{t \in T_i, i \in I, *}.)\]
If the diagram coset pregeometry happens to be a geometry, then $\Delta$ is its basic diagram if and only if at least one of the residues corresponding to adjacent $i, j$ is not a generalised digon.

**Theorem 3.2 (inspired by Buekenhout & Cohen [BC])**
Let $|I| > 1$. The diagram coset geometry $((G/G^{t,i} \times \{t\})_{t \in T_i, i \in I, *})$ is connected if and only if
\[G = \langle G^{t,i} \mid i \in I, t \in T_i \rangle.\]

**Proof.** Suppose that $\Gamma$ is connected. Take $i \in I$ and $t \in T_i$. If $a \in G$, then there is a path
\[1G^{t,i}, a_0G^{t_0,i_0}, a_1G^{t_1,i_1}, a_2G^{t_2,i_2}, \ldots, a_mG^{t_m,i_m}, aG^{t,i}\]
in the geometry connecting the elements $1G^{t,i}$ and $aG^{t,i}$ of $\Gamma$. Extending that path, if necessary, we can assume that the types $i_j$ and $i_{j+1}$ are adjacent in $\Delta$ for all $j$. Therefore
\[a_kG^{t_k,i_k} \cap a_{k+1}G^{t_{k+1},i_{k+1}} \neq \emptyset,\]
so
\[a_k^{-1}a_{k+1} \in G^{t_k,i_k}G^{t_{k+1},i_{k+1}}\]
for $k = 0, \ldots, m - 1$. Hence
\[a = (1^{-1}a_0)(a_0^{-1}a_1) \cdots (a_{m-1}^{-1}a_m)(a_m^{-1}a) \in G^{t,i}G^{t_0,i_0} \cdots G^{t_{m-1},i_{m-1}}G^{t_m,i_m}G^{t,i},\]

and so $a \in (G^{t,i} \mid i \in I, j \in T_i)$. The converse is obtained by reversing the above argument. The only difficulties that can occur are the occasions in which $g_1G^{t_1,i_1} \cap g_2G^{t_2,i_2} \neq \emptyset$, where $i_1 = i_2$ or $i_1, i_2$ not neighbors in $\Delta$. However, this can be remedied by including some suitable chain of cosets between $g_1G^{t_1,i_1}$ and $g_2G^{t_2,i_2}$ into the chain of incidences. \(\square\)

**Definition 3.3 (Sketch)** Let $\Gamma = (X, *, \text{typ})$ be a geometry over a finite set $I$ whose basic diagram $\Delta$ is a tree, let $G$ be a group of automorphisms of $\Gamma$, and let $W \subset X$ be a set of $G$-orbit representatives of $X$. We write
\[W = \bigcup_{i \in I} W_i\]
with $W_i \subseteq \text{typ}^{-1}(i)$. The **sketch of $\Gamma$ with respect to $(G, W, \Delta)$** is the diagram coset geometry
\[((G/G_w \times \{w\})_{w \in W, i \in I, *})\]

Let $\phi : G \rightarrow \text{Sym} X$ be a group action. Then we denote by $G \times X$ the corresponding permutation group, called a $G$-set. Two $G$-sets $G \times X$ and $G \times X'$ are said to be equivalent if there is a bijection $\psi : X \rightarrow X'$ such that $\psi \circ \phi(g) = \psi \circ \phi'(g)$ for each $g \in G$ or, equivalently, $\psi \circ \phi(g) = \phi'(g) \circ \psi$ for all $g \in G$. In this case, we shall also say that $G \times X$ and $G \times X'$ are isomorphic $G$-sets.

Recall also from [GVM06, Definition 2.2] that a **lounge of a geometry $\Gamma = (X, *, \text{typ})$ over $I$ is a set $W \subset X$ of elements such that each subset $V \subset W$ for which $\text{typ}_{|V} : V \rightarrow I$ is an injection, is a flag. A **hall** is a lounge $W$ with $\text{typ}(W) = I$.

Finally, recall that, for geometries $\Gamma_1 = (X_1, *, \text{typ}_1)$ over $I$ and $\Gamma_2 = (X_2, *, \text{typ}_2)$ over $I'$, the direct sum $\Gamma_1 \oplus \Gamma_2$ is the geometry $(X_1 \sqcup X_2, *, \text{typ}_1 \oplus \text{typ}_2)$ over $I \sqcup I'$ with $\text{typ}_1 \oplus \text{typ}_2|X_1 \times X_1 = \text{typ}_1$ and $\text{typ}_1 \oplus \text{typ}_2|X_2 \times X_2 = \text{typ}_2$ and $\text{typ}_1 \oplus \text{typ}_2|X_1 \times X_2 = X_1 \times X_2$ and $\text{typ}_1\oplus|X_1 = \text{typ}_1$ and $\text{typ}_1\oplus|X_2 = \text{typ}_2$. A geometry $\Gamma$ is said to have the direct sum property, if for each flag $F$ of $\Gamma$, the residue of $\Gamma$ in $F$ is isomorphic to the direct sum of its truncations to the connected components of its diagram, where the direct sum of more than two geometries is defined iteratively. Note that residual connectivity is a sufficient condition for the direct sum property, see [Pas94, Theorem 4.2].
Theorem 3.4 (Reconstruction theorem)
Let $\Gamma = (X, *, \text{typ})$ be a geometry over a finite set $I$ with the direct sum property whose basic diagram $\Delta$ is a tree. Let $G$ be a group of automorphisms of $\Gamma$. For each $i \in I$ let

$$w_i^1, \ldots, w_i^i$$

be $G$-orbit representatives of the elements of type $i$ of $\Gamma$ such that

(i) $W := \bigcup_{i \in I} \{w_i^1, \ldots, w_i^i\}$ is a hall and,

(ii) if $V \subseteq W$ is a flag, the action of $G$ on the pregeometry over $\text{typ}(V)$ consisting of all elements of the $G$-orbits $G.x$, $x \in V$, is transitive on the flags of type $\{i, j\}$ for all $i, j \in \text{typ}(V)$ corresponding to adjacent nodes of the diagram $\Delta$.

Then the bijection $\Phi$ between the sketch of $\Gamma$ with respect to $(G, W, \Delta)$ and the pregeometry $\Gamma$ given by

$$gG_{w_i^1} \mapsto gw_i^1$$

is an isomorphism between geometries and an isomorphism between $G$-sets.

Proof. The isomorphism as $G$-sets is clear from the fundamental theorem of permutation representations as $W$ is a transversal with respect to the action of $G$. Therefore let us turn to the isomorphism as geometries. For adjacent $i, j$ we have $gG_{w_i^1} \cap hG_{w_j^1} \neq \emptyset$ if and only if $gw_i^1 \ast hw_j^1$ by the isomorphism theorem for incidence-transitive geometries. If $i$ and $j$ are non-adjacent, then each pair of incident $gw_i^k \ast hw_j^j$ is contained in a chamber of $\Gamma$, hence the basic diagram $\Delta$ implies incidence of $gG_{w_i^j}, hG_{w_j^j}$ in the sketch. If $gw_i^j, hw_j^j$ are not incident, then $gG_{w_i^j}, hG_{w_j^j}$ cannot be incident by the direct sum property. \hfill $\Box$

The direct sum property in the hypothesis of Theorem 3.4 is necessary. Indeed, let $\Gamma_1$ and $\Gamma_2$ be isomorphic geometries of rank 4 with a string basic diagram over the type set $\{0, 1, 2, 3\}$. Let $f : \Gamma_1 \rightarrow \Gamma_2$ be an isomorphism and glue $\Gamma_1$ to $\Gamma_2$ via $f$ restricted to the set of elements of type 1. Denote the resulting geometry by $\Gamma$. Two elements of $\Gamma$ of type distinct from 1 are incident if and only if they are contained by the same $\Gamma_1$ and are incident in $\Gamma_1$. For $x \in \Gamma_1$ of type 1 and $y \in \Gamma_1$, we have $x \ast y$ in $\Gamma$ if and only if $x \ast y$ in $\Gamma_1$ or $f(x) \ast y$ in $\Gamma_2$. The basic diagram of $\Gamma$ also is a string, but the residue of an element of type 1 does not split into the direct sum of two geometries, so $\Gamma$ does not satisfy the direct sum property. Moreover, if the $\Gamma_1$ are flag-transitive, then $\Gamma$ is flag-transitive. Altogether, all hypotheses of Theorem 3.4 are satisfied. Nevertheless, $\Gamma$ cannot be recovered from its sketch (considered as a diagram coset geometry), because, given an element $x$ of type 1, Definition 3.4 forces all elements of type 0 incident with $x$ to be incident to all elements of type 2 incident with $x$, which is not the case in $\Gamma$. Of course, $\Gamma$ can be reconstructed in the classical way from its sketch as a flag-transitive geometry, emphasising that our reconstruction approach in the present paper is not a generalisation of the classical reconstruction or the reconstruction by Stroppel [Str93], cf. also [GVM06].

3.2 Fused amalgams
In the present paper we will work with the following definition of an amalgam.

**Definition 3.5 (Amalgam)** Let $\mathcal{J} = (J, \leq)$ be a finite graded poset with grading function $\tau : J \rightarrow I = \{1, 2, \ldots, n\}$ such that every maximal chain has length $n - 1$ (namely, it contains an element of every grade). Then an **amalgam of shape** $\mathcal{J}$ is a pair $\mathcal{A} = ((G_j)_{j \in J}, (\phi_{i,j})_{i < j})$ such that $G_j$ is a group for every $j \in J$ and, for any $i, j \in J$ with $i < j$, the map $\phi_{i,j} : G_i \rightarrow G_j$ is a monomorphism satisfying $\phi_{i,j} \circ \phi_{k,i} = \phi_{i,k}$ for any choice of $i, j, k \in J$ with $i < j < k$.

The fibers $\tau^{-1}(i)$ are denoted by $J_i$, for $i \in I$.
Example 3.6 In the following diagram we depict an amalgam with \( I = \{0, 1, 2\} \), \( J_0 = \{1, 2\} \), \( J_1 = \{1, 2, 3, 4, 5\} \), \( J_2 = \{1, 2, 3, 4\} \). The maps \( \phi_{i,j} \) are given by arrows and compositions of arrows.

\[
\begin{array}{c}
G_{1,1} \rightarrow G_{1,2} \\
G_{2,1} \uparrow \quad G_{2,2} \downarrow \\
G_{1,0} \quad G_{3,1} \quad G_{3,2} \\
G_{2,0} \quad G_{4,1} \quad G_{4,2} \\
G_{5,1} \\
\end{array}
\]

In terms of stabilisers of a group \( G \) acting on a geometry \( \Gamma \) with orbit representatives \( p, l, \pi_1, \pi_2 \) this example might concretely arise as

\[
\begin{array}{c}
G_{p,l} \rightarrow G_p \\
G_{p,\pi_1} \uparrow \quad G_l \downarrow \\
G_{p,\pi_2} \quad G_{\pi_1} \\
G_{p,l,\pi_2} \quad G_{l,\pi_1} \quad G_{\pi_2} \\
G_{l,\pi_2} \\
\end{array}
\]

If in the above example \( \pi_1 \) and \( \pi_2 \) happen to be contained in the same \( G \)-orbit, then of course we have \( G_{\pi_2} = gG_{\pi_1}g^{-1} \) for some \( g \in G \). But it may happen that this element \( g \) cannot be described in terms of the amalgam, as \( G_{\pi_1} \) and \( G_{\pi_2} \) might not be conjugate in the universal enveloping group of this amalgam, so that in this case it is very difficult to establish a nice correspondence between amalgams and coverings of geometries, as done in geometric covering theory. If, however, \( G_{\pi_2} = gG_{\pi_1}g^{-1} \) for some \( g \in G_l \), then such a correspondence exists. In this case automatically \( G_{l,\pi_2} = G_l \cap G_{\pi_2} = G_l \cap gG_{\pi_1}g^{-1} = gG_{l,\pi_1}g^{-1} \). Furthermore, \( g \in G_l \) is an element of the amalgam, and we can fuse \( G_{\pi_1} \) and \( G_{\pi_2} \) via conjugation with \( g \). We call such an amalgam \( \mathcal{A} \) a \textit{fused amalgam}. 
of parabolics and depict it by

\[
\begin{array}{c}
G_{p,1} \\
\downarrow \\
G_{p,\pi_1} \\
\downarrow \\
G_{p,\pi_2} \\
\downarrow \\
G_{\pi_1} \\
\downarrow \\
G_{\pi_2} \\
\downarrow \\
G_{1,\pi_1} \\
\downarrow \\
G_{1,\pi_2}
\end{array}
\]

The next definition formalises the concept of a fused amalgam. In this paper we only define fused amalgams sufficiently general for our purposes, although a number of possible generalisations come to mind immediately.

**Definition 3.7 (Fused Amalgam)** Let \( A = ((G_j)_{j \in J}, (\phi_{ij})_{i < j}) \) be an amalgam, with underlying graded poset \( J = (J, \leq) \) with grading function \( \tau : J \rightarrow I = \{1, 2, \ldots, n\} \). A fusion of \( A \), turning \( A \) into a fused amalgam, consists of three indices \( j_0, j_1, j_2 \in J \) such that a lower neighbor \( i_1 \) of \( j_0 \) and \( j_1 \), a lower neighbor \( i_2 \) of \( j_0 \) and \( j_2 \), and an isomorphism \( \gamma : G_{j_1} \rightarrow G_{j_2} \) such that the following properties hold:

1. \( \phi_{i_2,j_2}(gxg^{-1}) = (\gamma \circ \phi_{i_1,j_1})(x) \);
2. \( \phi_{i,j_2}(x) = (\gamma \circ \phi_{i,j_1})(x) \) for each \( i < j_1, j_2 \).

**Definition 3.8 (Enveloping Group)** Let \( A \) be a fused amalgam. A pair \((G, \pi)\) consisting of a group \( G \) and a map \( \pi : \sqcup A \rightarrow G \) is called an enveloping group of \( A \), if

1. for all \( j \in J \) the restriction of \( \pi \) to \( G_j \) is a homomorphism of \( G_j \) to \( G \);
2. \( \pi|_{G_j} \circ \phi_{i,j} = \pi|_{G_i} \) for all \( i < j \);
3. \( \pi \) preserves fusion, i.e., \( \pi(\gamma(x)) = \pi(g)\pi(x)\pi(g)^{-1} \) for every \( x \in G_{j_1} \) and \( g, \gamma, j_1 \) as in Definition 3.7 and
4. \( \pi(\sqcup A) \) generates \( G \).

**Proposition 3.9**
Let \( A \) as above be a fused amalgam of groups, let \( F(A) = \langle (u_g)_{g \in A} \rangle \) be the free group on the elements of \( A \) and let

\[
S_1 = \{u_xu_y = u_z, \text{ whenever } xy = z \text{ in some } G_j\}
\]

and

\[
S_2 = \{u_x = u_y, \text{ whenever } \phi(x) = y \text{ for some identification } \phi\}
\]

and

\[
S_3 = \{u_x = gu_yg^{-1}, \text{ whenever } x \in G_j \text{ and } y \in G_{j'} \text{ are fused by } g\}
\]

be relations for \( F \). Then for each enveloping group \((G, \pi)\) of \( A \) there exists a unique group epimorphism

\[
\tilde{\pi} : U(A) \rightarrow G
\]
with \( \pi = \hat{\pi} \circ \psi \) where
\[
\mathcal{U}(\mathcal{A}) = ((u_g)_{g \in \mathcal{A}} | S_1, S_2, S_3) \quad \text{and} \quad \psi : \sqcup \mathcal{A} \to \mathcal{U}(\mathcal{A}) : g \mapsto u_g.
\]

Proof. As in [GVM06]. \(\square\)

**Definition 3.10 (Universal Enveloping Group)** Let \( \mathcal{A} \) be a fused amalgam of groups. Then
\[
\psi : \sqcup \mathcal{A} \to \mathcal{U}(\mathcal{A}) : g \mapsto u_g
\]
for \( \mathcal{U}(\mathcal{A}) \) as in Proposition 3.9 is called the universal enveloping group of \( \mathcal{A} \).

### 3.3 Some additional theory of intransitive geometries

Note that the covering theory from [GVM06] does not apply to the geometry \( \Gamma_2 \) from Subsection 2.2 by Proposition 2.6. In this section we present a covering theorem making use of fused amalgams in order to tackle that geometry \( \Gamma_2 \). For simplicity, we will state the theorem in such a way that it exactly fits the properties of \( \Gamma_2 \). Generalisations are of course possible.

**Theorem 3.11**
Let \( \Gamma = (X, \ast, \text{typ}) \) be a connected geometry over \( I = \{1, 2, 3\} \) having the direct sum property whose basic diagram \( \Delta \) is \( 1 \rightarrow 2 \rightarrow 3 \). Let \( G \) be a vertex-transitive group of automorphisms of \( G \) that acts transitively on the flags of type \( \{i, j\} \) for all \( i, j \in I \) corresponding to adjacent nodes of the diagram \( \Delta \). Furthermore, let \( F = \{w^1, w^2, w^3\} \) be a flag, let \( w^3 \) and \( gw^3, g \in G_{w^3} \), be orbit representatives of the action of \( G_{w^3} \) on the elements of type 3. Finally, let \( \mathcal{A} = \mathcal{A}(\Gamma, G, F) \) be the fused amalgam of parabolics. Then the diagram coset pregeometry
\[
\hat{\Gamma} = ((\mathcal{U}(\mathcal{A})/G_{w^i} \times \{w^i\})_{i \in I}, \ast)
\]
is a simply connected geometry that admits a universal covering \( \pi : \hat{\Gamma} \to \Gamma \) induced by the natural epimorphism \( \mathcal{U}(\mathcal{A}) \to G \). Moreover, \( \mathcal{U}(\mathcal{A}) \) is of the form \( \pi_1(\Gamma).G \).

Proof. First notice that, since \( \Gamma \) is connected, \( G \) is generated by all its parabolics (different from \( G \)) by Theorem 3.2. As the embedding of \( \mathcal{A} \) in \( G \) preserves fusion, by Definition 3.8 the group \( G \) is an enveloping group of \( \mathcal{A} \) and Proposition 3.9 shows that the natural morphism \( \mathcal{U}(\mathcal{A}) \to G \) is surjective.

The map
\[
\phi : \sqcup \mathcal{A} \to G
\]
and, thus, the map
\[
\hat{\phi} : \sqcup \mathcal{A} \to \mathcal{U}(\mathcal{A})
\]
is injective. Therefore the natural epimorphism
\[
\psi : \mathcal{U}(\mathcal{A}) \to G
\]
induces an isomorphism between the amalgam \( \hat{\phi}(\sqcup \mathcal{A}) \) inside \( \mathcal{U}(\mathcal{A}) \) and the amalgam \( \phi(\sqcup \mathcal{A}) \) inside \( G \). Hence the epimorphism \( \psi : \mathcal{U}(\mathcal{A}) \to G \) induces a quotient map between pregeometries
\[
\pi : \hat{\Gamma} = ((\mathcal{U}(\mathcal{A})/G_{w^i} \times \{w^i\})_{i \in I}, \ast) \to ((G/G_{w^i} \times \{w^i\})_{i \in I}, \ast).
\]
The latter diagram coset pregeometry is isomorphic to $\Gamma$ by the Reconstruction Theorem \textbf{[3.4]}.
Notice that $U(A)$ acts on $\Gamma \cong (\langle G/G_{w^1} \rangle \times \{w^1\})_{i \in I, *}$ via

\[(gG_{w^1}, w^i) \mapsto (\psi(u)gG_{w^1}, w^i) \quad \text{for} \quad u \in U(A),\]

We want to prove that this quotient map actually is a covering map. The pregeometry $\hat{\Gamma}$ is connected by Theorem \textbf{[3.2]} because $U(A)$ is generated by $\partial(\overline{\Lambda})$. Let us start with proving the isomorphism between the residues of elements of type 2. Since the diagram is a string, a coset $xG_{w^1}$ is incident with a coset $yY$ with $Y = G_{w^3}$ or $Y = G_{gG_{w^3}}$, and if and only if there exists a coset $zG_{w^2}$ such that $xG_{w^1} \cap zG_{w^2} \neq \emptyset \neq zG_{w^2} \cap yY$. So, all thats needs to be checked is that there is a bijection between the cosets of $G_{w^1}$ in $U(A)$ that meet $G_{w^2}$ and the corresponding cosets in $G$, and similarly for cosets of $Y$. However, the existence of such bijections is obvious. Indeed, the cosets of $G_{w^1}$ meeting $G_{w^2}$ are precisely those that can be written as $hG_{w^1}$ for $h \in G_{w^2}$, and similarly for cosets of $Y$.

Turning to the residue $w^1$, let $hG_{w^3}$ represent a 3-element incident to $G_{w^1}$. Then there exists a coset $h_1G_{w^2}$ with $h_1 \in G_{w^1}$ and $h_1G_{w^2} \cap hG_{w^3} \neq \emptyset$. As $h_1 \in G_{w^1}$, we have $G_{w^1} \cap h_1G_{w^2} = h_1G_{w^1 \cap w^2}$. Turning to $hG_{w^3}$, the condition $h_1G_{w^2} \cap hG_{w^3} \neq \emptyset$ is equivalent to $G_{w^2} \cap h_1^{-1}hG_{w^3} \neq \emptyset$. This shows that we can choose a coset $h_1^{-1}h = h_2 \in G_{w^2}$, namely $h = h_1h_2 \in G_{w^1} \cap G_{w^3}$. Moreover, in view of the hypotheses we have assumed on $G_{w^1} \cap G_{w^3}$, there exists an element $f \in G_{w^1} \cap G_{w^2}$ such that either $fh_2G_{w^3} = G_{w^3}$ or $fh_2G_{w^3} = gG_{w^3}$. In the former case we can choose $h_2 = f^{-1}G_{w^1} \cap G_{w^2}$ and $h = h_1h_2 \in G_{w^1}$. Thus, $G_{w^1} \cap G_{w^2} \cap fh_2G_{w^3} = fh_2G_{w^1 \cap w^3}$. Also, $G_{w^1} \cap G_{w^2} \cap hG_{w^3} \subset G_{w^1} \cap G_{w^2}$ contains $h_2G_{w^3} \neq \emptyset$ in $G_{w^1 \cap w^2}$. Hence $G_{w^1} \cap G_{w^2} \cap hG_{w^3} = f^{-1}G_{w^1 \cap w^2 \cup w^3}$. Accordingly, $G_{w^1} \cap G_{w^2} \cap hG_{w^3} = G_{w^1 \cap w^2 \cup w^3}$. So, these particular $\{2,3\}$-flags of $res(w^1)$ correspond to cosets of $G_{w^1} \cap G_{w^2} \cap G_{w^3}$ in $G_{w^1}$. However, the above holds in $\hat{\Gamma}$ as well as in $G$. Consequently, that the part of $\hat{\Gamma}_{w^1}$ formed by 2-elements and 3-elements in the same orbit of $G_{w^3}$ is isomorphic to the analogous part of $\Gamma_{w^1}$. Suppose now that the latter case occurs, namely $fh_2G_{w^3} = gG_{w^3}$. So, $hG_{w^3} = h_1h_2G_{w^3} = h_1f^{-1}gG_{w^3}$, whence $hG_{w^3}g^{-1} = h_1f^{-1}gG_{w^3}g^{-1}$. As $h_1G_{w^1} = h_1G_{w^1 \cup w^2}$ (because $G_{w^1} \cap G_{w^2} \cap hG_{w^3} \neq \emptyset$ if and only if $h_1G_{w^1 \cap w^2} \cap hG_{w^3} \neq \emptyset$). So, we can repeat the above argument with $hG_{w^3}g^{-1}$ replaced by $hG_{w^3}g^{-1} = h_1f^{-1}gG_{w^3}g^{-1}$, thus obtaining that the flag $\{h_1G_{w^1}, hG_{w^3}\}$ of $res(w^1)$ corresponds to a coset of $G_{w^1} \cap G_{w^2} \cap G_{w^3}$ in $G_{w^1}$. In other words, the part of $res(w^1)$ formed by the 2-elements and the 3-elements of the orbit containing $G_{w^3}$ is isomorphic to the geometry of cosets of $G_{w^1 \cup w^2}$ and $G_{w^1 \cap w^3}$ inside $G_{w^1}$. Again, this is true in $\hat{\Gamma}$ as well as in $\Gamma$. So, in either of these two geometries, that part of the residue of $w^3$ is canonically isomorphic to the same geometry of cosets inside $G_{w^1}$. So, that part of the residue of $w^1$ in $\hat{\Gamma}$ is isomorphic to the corresponding part of the residue of $w^1$ in $\Gamma$. So far, we have proved that each of the two parts of $\hat{\Gamma}_{w^1}$ is isomorphic to the corresponding part in $\Gamma_{w^1}$. Moreover, it is clear from the above that the two ‘partial’ isomorphisms constructed in this way from $\hat{\Gamma}_{w^1}$ to $\Gamma_{w^1}$ agree on the set of 2-elements. Therefore they can be pasted together so that to construct an isomorphism from the whole of $\hat{\Gamma}_{w^1}$ to the whole of $\Gamma_{w^1}$.

A similar argument applies to residues of $w^3$. Hence $\pi : \hat{\Gamma} \to \Gamma$ induces isomorphisms between the residues of flags of rank one, so $\pi$ indeed is a covering of pregeometries. Since $\Gamma$ actually is a geometry the pregeometry $\hat{\Gamma}$ is also a geometry. The universality of the covering $\pi : \hat{\Gamma} \to \Gamma$ induced by the canonical map $U(A) \to G$ is proved as in \cite[Theorem 3.1]{GVM06}. The structure of $\hat{G} \cong U(A)$ is evident by combinatorial topology, cf. Chapter 8 of \cite{ST33}, restated in \cite{GVM06} Section 2.2. \hfill \Box

**Corollary 3.12 (Tits’ lemma)**
Let $\Gamma = (X, * \text{, typ})$ be a connected geometry over $I = \{1, 2, 3\}$ having the direct sum property whose basic diagram $\Delta$ is $1 \rightarrow 2 \rightarrow 3$. Let $G$ be a vertex-transitive group of automorphisms of $\Gamma$ that acts transitively on the flags of type $\{i, j\}$ for all $i, j \in I$ corresponding to
adjacent nodes of the diagram $\Delta$. Furthermore, let $F = \{w^1, w^2, w^3\}$ be a flag, let $w^3$ and $gw^3$, $g \in G_{w^2}$, be orbit representatives of the action of $G_{w^1,w^2}$ on the elements of type $3$. Finally, let $\mathcal{A} = \mathcal{A}(\Gamma, G, F)$ be the fused amalgam of parabolics. The geometry $\Gamma$ is simply connected if and only if the canonical epimorphism

$$U(\mathcal{A}(\Gamma, G, F)) \to G$$

is an isomorphism. $\square$

### 3.4 Amalgamation

By Proposition 2.6 the $G_2(\mathbb{K})$ action on $\Gamma_2$ satisfies the hypotheses of Theorem 3.11 so that by Proposition 3.13 we can apply Corollary 3.12 in order to obtain the following amalgamation result.

**Theorem 3.13**

Let $p, l, \pi_1, \pi_2 = g\pi_1, g \in G_l$ be a set of orbit representatives of the $G_2(\mathbb{K})$ action on $\Gamma_2$ such that $p, l, \pi_1$ is a chamber of $\Gamma_2$. Then $G_2(\mathbb{K})$ is the universal enveloping group of the following fused amalgam of parabolics:

4 Simple connectivity

In this section, we show the simply connectivity of most of the geometries $\Gamma_i$, $i = 0, 1, 2, 3$. To be precise, we prove that all these geometries are simply connected, except for $\Gamma_1$ with $|\mathbb{K}| = 2$.

Collinearity on $Q(6, \mathbb{K})$ and in $W(5, \mathbb{K})$ will be denoted by $\perp$. Also, in $H(\mathbb{K})$, there is a natural distance function on the set of points and lines, with values in $\{0, 1, \ldots, 6\}$ (this function is the graph theoretic distance in the incidence graph). Pairs of elements at distance 6 will be called opposite. In $PG(5, \mathbb{K})$, there are two types of singular planes: the ideal planes, and the hexagonal singular planes. If $\alpha$ is a hexagonal singular plane, then the ideal centres of all ideal lines in $\alpha$ coincide and will be called the hexagonal pole of $\alpha$ (it is the intersection of all hexagon lines in $\alpha$).

The pencil of hexagon lines will be denoted by $P_\alpha$. Also, a nonsingular plane $\beta$ contains a pencil of symplectic lines; this pencil will be denoted by $P_\beta$ and the intersection of the lines of $P_\beta$ shall be called the pole of $\beta$.

First we will start by proving simple connectivity of $\Gamma_0$, $\Gamma_1$, $\Gamma_2$ as defined in Subsection 2.2.

In what follows, we will use over and over the simple observation that, if two symplectic lines $L_1, L_2$ meet in a point $p$, then every line in the plane $\langle L_1, L_2 \rangle$ through $p$ is symplectic. Moreover, if the plane $\langle L_1, L_2 \rangle$ is nondegenerate, then at most one of these lines can be hexagonal, because a pair of intersecting hexagon lines spans a totally isotropic plane, see Section 2.1 item (i). In the following, a geometric triangle is a triangle consisting of points and lines in a plane of the geometry.

The next proposition has also been proved by Blok and Hoffman [BH], but we provide a short proof for sake of completeness.
Proposition 4.1
The geometry $\Gamma_0$ is simply connected.

Proof. Clearly the diameter at type 1 of the truncation of $\Gamma_0$ at the types 1 and 2 is two, so it suffices to show that we can subdivide any triangle, any quadrangle and any pentagon in geometric triangles. Every triangle is geometric, so for triangles there is nothing to prove.

Let $a, b, c, d$ be a quadrangle of type 1 elements (points of $\text{PG}(5,q)$). We may assume that $ac$ and $bd$ are symplectic lines, since otherwise the quadrangle automatically decomposes into triangles. Choose any point $e$ on $ab$. If $ce$ were symplectic, then so would $cb$. Hence $ce$, and similarly also $de$, are non-symplectic. We have subdivided our quadrangle in the triangles $a, d, e$ and $c, d, e$ and $b, c, e$.

Now let $a, b, c, d, e$ be a pentagon of type 1 elements. Again we may assume that all of $ac$, $bd$, $ce$, $da$ and $eb$ are symplectic, since otherwise the pentagon decomposes automatically. Since $a, c, d$ is a triangle in $\text{PG}(5,q)$, the corresponding symplectic hyperplanes $a^\perp$, $c^\perp$, $d^\perp$ do not meet in a 3-space, whence their union cannot cover the whole space. Therefore there is a point $f$ with $cf$, $df$ and $af$ non-symplectic lines and we have subdivided our pentagon into the null-homotopic circuits $a, b, c, f$ and $c, d, f$ and $d, e, a, f$. $\square$

Proposition 4.2
The geometry $\Gamma_1$ is simply connected, whenever $|K| > 2$.

Proof. Since the point-line truncations of $\Gamma_0$ and $\Gamma_1$ coincide, by the proof of Proposition 4.1 it suffices to show that every triangle is null-homotopic.

Let $a, b, c$ be a triangle, and suppose it is not geometric. Hence $a, b, c$ are three pairwise opposite points in $H(K)$ and the plane $\langle a, b, c \rangle$ contains a (unique) hexagon line $l$. We may assume that $a$ is not incident with $l$. Then there exists a hexagon line $m$ through $a$ not concurrent with $l$ and not concurrent with a hexagon line that is concurrent with $l$. It follows that the 3-space $\Xi := \langle l, m \rangle$ is nondegenerate and contains a regulus, consisting of hexagon lines, of a ruled nondegenerate quadric $Q$, cf. Section 2.1 item (v). Denote the unique line of $Q$ in $\langle a, b, c \rangle$ different from $l$ by $l'$ (and note that $l'$ is incident with $a$ and meets $l$), and for each point $z \in l'$, denote by $l_z$ the unique hexagon line on $Q$. To prove the claim, it suffices to find a point $x$ such that the planes $\langle a, b, x \rangle$, $\langle a, c, x \rangle$ and $\langle b, c, x \rangle$ do not contain any hexagon line and such that the lines $ax, bx, cx$ are not symplectic. The latter is satisfied whenever $x \in \Xi$ is not contained in the union $U_0$ of the planes $\pi_a, \pi_b, \pi_c$, where $\pi_a$ is generated by the symplectic lines through $a$ inside $\Xi$, and likewise for $\pi_b$ and $\pi_c$. Note that $\langle a, b, c \rangle = \langle l, l' \rangle \subseteq \Xi$. The former is satisfied whenever $x$ does not lie in the union $U_1$ of the planes $\langle a, b, c \rangle$, $\langle a, c, l_a \rangle$, $\langle b, c, l_b \rangle$, $\langle a, c, l_c \rangle$ and $\langle b, c, l_c \rangle$. Since $l'$ contains $a$, we see that $l_a \cap l_c = l_b \cap l_c = m$. Since $\Xi$ cannot be the union of seven planes if $|K| \geq 8$, we may suppose $|K| = 4$. In that case $U_2$, which is the union of four planes no three of which meet in a line, i.e., a tetrahedron, covers exactly 58 points, leaving a set $S$ of 85 − 58 = 27 possibilities for $x$. If a plane contained in $U_1$ does not contain an intersection line of two planes in $U_2$, then it meets $S$ in 7 or 6 points. If, on the other hand, such a plane does contain such an intersection line, it contains 9 points of $S$. Consequently we may assume that the planes in $U_1$ partition $S$ and each plane contains some intersection line of planes in $U_2$. For all three planes, this line must be the same, as otherwise the three intersections would have to be pairwise distinct. Hence in that case each of the three planes would have to contain two lines of the tetrahedron, which would imply that they actually belong to $U_2$, a contradiction. But then the planes $\pi_a, \pi_b, \pi_c$ contain a common line, which implies that $a, b, c$ are collinear in $\text{PG}(5, K)$ (since $\Xi$ is nondegenerate), another contradiction. $\square$

The above proof fails for $|K| = 2$. In fact, a computation using GAP reveals that $\Gamma_1$ admits in this case a 3-fold universal cover. The source code of the program we used for verification of this fact can be found in Appendix A and also on the website [WWW].

Proposition 4.3
The geometry $\Gamma_2$ is simply connected.
Proof. As in the case of $\Gamma_1$ we only need to prove that every triangle is null-homotopic. Suppose $|\mathbb{K}| > 2$ (hence $|\mathbb{K}| \geq 4$, as the characteristic of $\mathbb{K}$ is two). Given a triangle $p_0, p_1, p_2$ of the collinearity graph of $\Gamma_2$, not contained in one line of $\Gamma_2$, set $\pi := \langle p_0, p_1, p_2 \rangle$. Clearly, the plane $\pi$ is nondegenerate. Let $p$ be its pole. We may assume that $\pi$ is a plane of $\Gamma_1$, as otherwise our triangle is geometric. So all the symplectic lines of $\pi$ are ideal. Given a line $l \in \mathcal{P}_\pi$, let $\pi_l$ be the hexagonal plane on $l$ and let $p_l = p_{\pi_l}$ be the hexagonal pole of $\pi_l$. Suppose, by way of contradiction, that $\pi \subset p_l$. So, the singular planes on $pp_l$ are precisely those spanned by $p_l$ and a line $m \in \mathcal{P}_\pi$. Since $pp_l$ is a hexagon line, all of these planes are hexagonal and the map sending a line $m \in \mathcal{P}_\pi$ to the hexagonal pole $p_m$ of $\langle m, p_l \rangle$ is a bijection from $\mathcal{P}_\pi$ to the set of points of $pp_l$. Therefore, $p = p_m$ for some line $m \in \mathcal{P}_\pi$. Hence $m$ is a hexagon line, contrary to our assumptions. As a consequence, $p_l \cap \pi = l$ for every line $l \in \mathcal{P}_\pi$. We can now choose the line $l$ such that $l \in \mathcal{P}_\pi \setminus \{pp_0, pp_1, pp_2\}$ (noting that we have assumed $|\mathbb{K}| > 2$). For $1 \leq i < j \leq 3$, none of the planes $\pi_{ij} = \langle p_i, p_j, p_l \rangle$ is singular (since they contain the non-symplectic line $pp_j$), but each of them contains a hexagon line, namely the line $a_{ij}p_i$, where $a_{ij} = l \cap p_ip_j$. So, we have decomposed $p_0, p_1, p_2$ into triangles $p_i, p_j, a$, each of which is contained in a plane of $\Gamma_2$.

For $|\mathbb{K}| = 2$, a computer based argument proves the claim. Again, the source code for the GAP program we used can be found in Appendix A and also on the website [WWW].

Proposition 4.4
The geometry $\Gamma_3$ is simply connected.

Proof. We prove this in a series of lemmas. The strategy is to show that every cycle in the collinearity graph of $\Gamma_3$ is null homotopic. We begin by noting that the diameter of that graph is equal to two. Indeed, if two points $a, b$ are incident with the same hexagon line $l$, then we can choose a hexagonal plane $\pi$ through $l$ such that neither $a$ nor $b$ is incident with at least two hexagonal lines of $\pi$. Consequently, for any point $c$ in $\pi$ not on $l$ the lines $ac$ and $bc$ are ideal. If two points $a, b$ are at distance two in the collinearity graph of $Q(6, \mathbb{K})$, then the points $c$ collinear to both $a$ and $b$ form a generalised quadrangle $Q(4, \mathbb{K})$; those for which either $ac$ or $bc$ are hexagon lines form two lines in $Q(4, \mathbb{K})$. Hence there are plenty of points $c$ in $Q(4, \mathbb{K})$ for which $ac$ and $bc$ are ideal. As a consequence, we only have to show that triangles, quadrangles and pentagons are null homotopic. This will be done in lemmas 1.6, 1.7, and 1.8.

Lemma 4.5
Every triangle is null homotopic.

Proof. If the triangle is geometric, i.e., contained in an ideal plane, then this is trivial. So we may assume that we have a triangle $a, b, c$ in a hexagonal plane. Let $\pi'$ be an ideal plane on some ideal line of $\langle a, b, c \rangle$ not containing $a$, $b$ or $c$. Then, by Section 2.4 item (iv), the ideal centres of the ideal lines of $\pi'$ form an ideal plane $\pi$. Then span $H := \langle \pi, \pi' \rangle$ is a hyperplane of $PG(6, \mathbb{K})$ meeting $Q$ in a nondegenerate hyperbolic quadric $Q^\perp$. Moreover, none of the points $a, b, c$ is incident with $\pi \cup \pi'$, and $\pi$ contains the pole of the plane $\langle a, b, c \rangle$. By Section 2.1 item (iv) the hexagonal planes in $H$ are those that share a point of $\pi$ or $\pi'$ and a line of $\pi'$ or $\pi$, respectively.

Now we apply the Klein correspondence to view the situation in a 3-space $PG(3, \mathbb{K})$. To be precise, we map $\pi$ to a point $x$ and hence $\pi'$ to a plane $\alpha$ off that point. Translated to the space $PG(3, \mathbb{K})$, the hexagonal planes in $H$ correspond with the points of $\alpha$ and with the planes through $x$. The plane $\langle a, b, c \rangle$ corresponds to a point $z$ in $\alpha$, and the points $a, b, c$ correspond to lines $A, B, C$, respectively, through $z$, but not contained in $\alpha$ and not incident with $x$. Also, none of the planes $\langle A, B \rangle$, $\langle A, C \rangle$, $\langle B, C \rangle$ contain $x$. Let $l$ be a line in $\alpha$ not through $z$ and let $\beta$ be a plane through $l$, different from $\alpha$, and not incident with $x$. Then the intersections $L_C := \beta \cap \langle A, B \rangle$, $L_B := \beta \cap \langle A, C \rangle$ and $L_A := \beta \cap \langle B, C \rangle$ form a triangle which corresponds with a null-homotopic triangle in $\Gamma$. By construction also the triangles $A, B, L_C$ and $A, C, L_B$ and $B, C, L_A$ correspond with null-homotopic triangles in $\Gamma$, and likewise so do the triangles $L_A, L_B, C$ and $L_A, L_C, B$ and...
Lemma 4.6
Every quadrangle $a, b, c, d$ with $a \not\in \{c, d\}$ and $b \not\in d$, is null homotopic.

Proof. Let $x$ be the ideal centre of $ab$ and let $y$ be the ideal centre of $cd$. It is easy to see that our assumptions imply that $x$ and $y$ are not collinear in $H(K)$. Suppose now first that the planes $\langle x, a, b \rangle$ and $\langle y, c, d \rangle$ are disjoint. This, as $x, y$ are noncollinear in $H(K)$ is equivalent to $x$ being opposite $y$ in $H(K)$. Let $X_1$ be the set of points of $H(K)$ collinear with $x$ and not opposite $y$, and let likewise $Y_1$ be the set of points of $H(K)$ not opposite $x$ and collinear with $y$. Note that $a \in X_1$ if and only if $d \notin Y_1$ (and similarly, $b \in X_1$ if and only if $c \notin Y_1$). For assume $a \in X_1$, then clearly, the only point of $Y_1$ collinear in $Q(6, K)$ with $a$ is also collinear with $a$ in $H(K)$, thus not collinear with $a$ in $\Gamma$, and hence $d \notin Y_1$. The remaining implications follow identically. In view of the above, either $b \notin X_1$ or $c \notin Y_1$. We may assume that $b \notin X_1$. So $c \in Y_1$ and we are left with the following cases.

(i) In case $a \notin X_1$, the intersection $a^1 \cap b^1 \cap \langle y, c, d \rangle$ is a singleton $\{u\}$. Our assumptions imply that $u \notin \{c, d, y\}$. If $u$ belonged to the (hexagon) line $yc$, then, since $c \in b^1$, also $y \in b^1$, a contradiction. So $u \notin yc$ and likewise $u \notin yd$. Moreover, neither $au$ nor $bu$ are hexagon lines, as neither $a$ nor $b$ belong to $X_1$. Hence $u$ is collinear in $\Gamma$ with all of $a, b, c, d$ and we can subdivide the quadrangle $a, b, c, d$ in the triangles $a, b, u$ and $b, c, u$ and $c, d, u$ and $d, a, u$.

(ii) In case $a \in X_1$, we have $d \notin Y_1$. Pick any point $v$ on the ideal line $ab$, different from $a$ and $b$. Our assumptions imply easily that there is a unique point $w$ on $cd$ collinear in $\Gamma$ with $v$. Since $v \notin X_1$, we know that $vw$ is an ideal line. Also, since $c$ is collinear with $b$ in $\Gamma$, and $w$ cannot be, we deduce that $w \neq c$. Similarly $w \neq d$. By the previous arguments the quadrangles $a, v, w, d$ and $v, b, c, w$ are null homotopic, and they subdivide $a, b, c, d$. Hence also in this case $a, b, c, d$ is null homotopic. Note that the situation considered here does not occur when $|K| = 2$.

Suppose now secondly that the planes $\langle a, b, x \rangle$ and $\langle c, d, y \rangle$ meet in a point $z$. Our assumptions imply that $z$ is distinct from the intersection point $x_1$ of $ab$ with $xz$, and also from the intersection point $y_1$ of $cd$ with $yz$. Now clearly the ideal centres of $ad$ and $bc$ are opposite $z$, which is the ideal centre of $x_1y_1$. Hence, by the previous arguments (with $ab$ replaced by $wd$ and $vb$, respectively), the quadrangles $a, x_1, y_1, d$ and $b, x_1, y_1, d$ are null homotopic. Since they subdivide $a, b, c, d$, the lemma follows.

Lemma 4.7
Every quadrangle $a, b, c, d$ is null homotopic.

Proof. By Lemma 4.6 we may assume that $a$ and $c$ are collinear on $Q(6, K)$. If $ac$ is ideal, then we are done by the fact that all triangles are null homotopic, see Lemma 4.6. Hence we may assume that $ac$ is a hexagon line. Clearly, we may assume that $b$ and $d$ are not collinear on the quadric as otherwise $a, b, c, d$ lie in a plane of the quadric and then $ad$ meets $bc$ in some point $e$. The triangles $a, b, e$ and $c, d, e$ are null homotopic by Lemma 4.6, hence the result.

The plane $\langle a, b, c \rangle$ is degenerate since it contains a hexagon line. Considering any other plane $\pi$ through $bc$, it follows that $\pi$ is an ideal plane. Since $d \not\in \pi$, the line $l = \pi \cap d^1$ does not coincide with $bc$ and hence contains at least two points $u, v$ off $bc$. If both $du$ and $dv$ were hexagon lines, then also $dc$ must be a hexagon line, a contradiction. So we may assume that $du$ is ideal. But now we have subdivided the quadrangle $a, b, c, d$ into the circuits $c, d, u$ and $c, u, b$ and $u, b, a, d$. The two former are null homotopic by Lemma 4.5 while the latter is null homotopic by Lemma 4.6 noting that $u$ is not collinear with $a$ on the quadric.
Lemma 4.8
Every pentagon \( a_0, a_1, a_2, a_3, a_4 \) is null homotopic.

Proof. For \( i = 0, 1, \ldots, 4 \), let \( l_i = a_i a_{i+1} \) be the ideal line through \( a_i \) and \( a_{i+1} \) (indices being computed modulo 5). Suppose first that \( l_{i+2} \not\subset a_i^\perp \) for some \( i \). For instance, \( l_2 \not\subset a_0^\perp \). As the line \( l_2 \) is ideal, all singular planes on \( l_2 \) but one are ideal. So, we can choose an ideal plane \( \beta \) on \( l_2 \) such that the plane \( \alpha = \langle a_0^\perp \cap \beta, a_0 \rangle \) is different from the unique hexagonal plane containing all hexagonal lines through \( a_0 \). Consequently, at most one of the lines of \( \alpha \) through \( a_0 \) is hexagonal.

Given an ideal line \( l \) of \( \alpha \) through \( a_0 \), let \( c = l \cap (\alpha \cap \beta) \). Then \( c \) is collinear with both \( a_2 \) and \( a_3 \) in \( \Gamma_3 \), since \( \beta \) is an ideal plane. Thus, we can split \( a_0, a_1, a_2, a_3, a_4 \) into \( a_0, a_1, a_2, c \) and \( c, a_2, a_3, a_4 \).

Suppose now that \( l_{i+2} \subset a_i^\perp \) for every \( i = 0, 1, \ldots, 4 \). Then \( a_0, a_1, a_2, a_3, a_4 \) is contained in a singular plane. Put \( b := l_0 \cap l_2 \) and \( c := l_4 \cap l_2 \). Then \( a_0, a_1, a_2, a_3, a_4 \) splits into the sum of \( a_0, b, c \) and \( b, a_1, a_2, b \) and \( c, a_3, a_4 \). \( \square \)

A GAP code

Listing 1: gamma1.gap

```gap
# Setup
Read("util.gap");
F := GF(q);
z := Z(q);
G2q := G2(q);
vorb := [0,0,1,0,1] * z^0;
Display("Verifying group size...");
Assert(0, Size(G2q) = q^6 * (q^6-1) * (q^2-1));

# The second (bigger) geometry, with the second plane orbit,
# is determined by the following elements:
u_space := [
    [1,0,0,0,0,0],
    [1,0,0,0,0,0],
    [1,0,0,0,1,0],
    [0,0,1,0,1,0],
    [0,0,0,1,0,0],
] * z^0;
u_stab := []; u_index := [
    (q^6-1)/(q-1),
    q^4*(1+q^2+q^4),
    (1+q)*(1+q^2+q^4)*(q^4-q^3)
]; u_size := List(u_index, x -> Size(G2q) / x);
Read("build_stabs.gap");
Read("build_amalgam2.gap");

Display("Coset enumeration...");
A := FG / rels;
U := Subgroup(A, GeneratorsOfGroup(A){stabgens [1]});
idx := Index(A, U);
Print("Expected index", Index(G2q, U1), "\n");
Print("Actual index", idx, "\n");
Print("Resulting covering factor", idx/Size(G2q, U1), "\n");
```

Listing 2: gamma2.gap

```gap
# Setup
Read("util.gap");
F := GF(q);
z := Z(q);
```
G2q := G2(q);
voorb := [0, 0, 1, 0, 1, 1] * z^0;
Display("Verifying group size ...");
Assert(0, Size(G2q) = q^6*(q^6-1)*(q^2-1));
#
# The first (smaller) geometry, with the first plane orbit,
# is determined by the following elements:
uspace := [  
  [1, 0, 0, 0, 0, 0],
  [1, 0, 0, 0, 0, 0],
  [1, 0, 0, 0, 0, 0],
  [1, 0, 0, 0, 0, 0],
] * z^0;
ustab := [ ];
usize := [  
  (q^6-1)/(q-1),
  q^4*(1+q^2+q^4),
  (1+q)*(1+q^2+q^4)*(q^3+q^2),
  (1+q)*(1+q^2+q^4)*(q^3+q^2)  
];
usize := List(usize, x -> Size(G2q) / x);
Read("build_stabs.gap");
Read("build_amalgam.gap");

Display("Coset enumeration ...");
A := FG / rels;
U := Subgroup(A, GeneratorsOfGroup(A){stabgens[1]});
idx := Index(A, U);
Print("Expected index = ", Index(G2q, U1), "\n");
Print("Actual index = ", idx, "\n");
Print("Resulting covering factor = ", idx/Index(G2q, U1), "\n");

for i in [1..3] do
  Print("Computing stabilizer number ", i, "\n");
  if q=2 then
    ustab[i] := Stabilizer(G2q, uspace[i], OnSubspacesByCanonicalBasis);
  else
    # Computing the full stabilizer of uspace[i] directly is very expensive.
    # So instead, we use a trick: We compute the stabilizer of uspace[i] inside two small subgroups, and then take the group generated by these two stabilizers. Empirically, this gives us the desired full stabilizer. We verify that we obtained the correct subgroup by checking the size of the resulting group.
    if q = 4 then
      tmp1 := Group(G2q.1, G2q.2);
    else
      tmp1 := Group(G2q.1, G2q.3);
    fi;
    tmp2 := Group(G2q.2, G2q.3);
    tmp1 := Stabilizer(tmp1, uspace[i], OnSubspacesByCanonicalBasis);
    tmp2 := Stabilizer(tmp2, uspace[i], OnSubspacesByCanonicalBasis);
    ustab[i] := Group(Union(List([tmp1, tmp2], GeneratorsOfGroup)));
    sx := SizeViaOrbit(ustab[i], vorb);
    Assert(0, sx = usize[i]);
    SetSize(ustab[i], usize[i]);
  fi;
  if Size(usize) > 3 then
Display("Computing the second plane stabilizer");
# We know that U3 and U4 are conjugated by the following element
# in the line stabilizer:
g := [
[0,0,0,1,0,0],
[0,0,0,0,0,1],
[0,0,0,0,1,0],
[1,0,0,0,0,0],
[0,0,1,0,0,0],
[0,1,0,0,0,0]]^z^0;
U4 := U3^g;

Listing 4: build_amalgam.gap

# U3 and U4 are conjugated by this element in U2 (which
# exchanges the two plane classes, and preserves lines).
g := [
[0,0,0,1,0,0],
[0,0,0,0,0,1],
[0,0,0,0,1,0],
[1,0,0,0,0,0],
[0,0,1,0,0,0],
[0,1,0,0,0,0]]^z^0;

# Compute the stabilizer intersections

Display("Computing the double stabilizers ...");
U12 := Stabilizer(U2, u_space[1], OnSubspacesByCanonicalBasis);
U13 := Stabilizer(U3, u_space[1], OnSubspacesByCanonicalBasis);
U23 := Stabilizer(U3, u_space[2], OnSubspacesByCanonicalBasis);
U123 := Stabilizer(U23, u_space[1], OnSubspacesByCanonicalBasis);
U14 := Stabilizer(U4, u_space[1], OnSubspacesByCanonicalBasis);
U24 := Stabilizer(U4, u_space[2], OnSubspacesByCanonicalBasis);
U124 := Stabilizer(U24, u_space[1], OnSubspacesByCanonicalBasis);

Display("Finding generators suitable for amalgamation ...");
repeat
  x := PseudoRandom(U12);
until ClosureGroup(U123,x)=U12;
repeat
  y := PseudoRandom(U23);
until ClosureGroup(U123,y)=U23;
repeat
  a := PseudoRandom(U13);
until ClosureGroup(U123,a)=U13 and ClosureGroup(U23,a)=U3;
repeat
  c := PseudoRandom(U14);
until ClosureGroup(U124,c)=U14;
repeat
  h := PseudoRandom(U124)^((g^-1);
until U124 = ClosureGroup(U123^g,h^g);

# Verify "nice" generator properties — namely, that the intersections of
# the groups are generated by the intersections of their generating sets.
Display("Verifying ...");
A GAP CODE

Listing 5: build_amalgam2.gap

# Compute the stabilizer intersections
Display("Computing the double stabilizers ...");
U12 := Stabilizer(U2, u_space[1], OnSubspacesByCanonicalBasis);
U13 := Stabilizer(U3, u_space[1], OnSubspacesByCanonicalBasis);
U23 := Stabilizer(U3, u_space[2], OnSubspacesByCanonicalBasis);
U123 := Stabilizer(U23, u_space[1], OnSubspacesByCanonicalBasis);

Display("Finding generators suitable for amalgamation ...");
# Find x in U12 such that U12 = <U123, x>
repeat
  x := PseudoRandom(U12);
until ClosureGroup(U12, x) = U12;

# Find y in U23 such that U23 = <U123, y>
repeat
  y := PseudoRandom(U23);
until ClosureGroup(U123, y) = U23;

# Find r in U13 such that U1 = <U12, r> and U3 = <U23, r>
repeat
  r := PseudoRandom(U13);
until ClosureGroup(U12, r) = U1 and ClosureGroup(U23, r) = U3;

# Verify "nice" generator properties — namely, that the intersections of
# the groups are generated by the intersections of their generating sets.
Display("Verifying...");
Assert(0, U12 = ClosureGroup(U123, x));
Assert(0, U23 = ClosureGroup(U123, y));
Assert(0, U13 = ClosureGroup(U123, r));

Display("Building the amalgam...");
g123 := ShallowCopy(SmallGeneratingSet(U123));
g123_names := List(1..Length(g123), n -> Concatenation("f", String(n)));
gens := Concatenation(g123, [x, y, r]);
gen_names := Concatenation(g123_names, ["x", "y", "r"]);
Assert(0, Size(gens) = Size(gen_names));

ugens := [];
ugens[1] := Concatenation(g123_names, ["x", "r"]);
ugens[2] := Concatenation(g123_names, ["x", "y"]);
ugens[3] := Concatenation(g123_names, ["y", "r"]);

# We provided lists of generators identified with elements in G2. Now
# we compute all relations between these.
Read("build_rels.gap");

Listing 6: build_rels.gap

Print("Constructing the amalgam...\n");
SetInfoLevel(InfoFpGroup, 3);

# FG := FreeGroup(Length(gens));
Assert(0, Length(gens) = Length(gen_names));
FG := FreeGroup(gen_names);

# Select the generators for each stabilizer
stab_gens := List(ugens, gs -> List(gs, g -> Position(gen_names, g)));

uresls := [];
rels := [];


stabs := List( stab_gens, t->Group( gens{t} ) );

# We now determine finite presentations for each stabilizer.
# To do this, we first find a permutation group isomorphic
# to that stabilizer, then use IsomorphismPGroupByGenerators
# to get the finite presentation.
#
# Finally, we form the union of the relations of the stabilizers.
#
for i in [1..Length(stab_gens)] do
  # Verify and set the size of the group.
  # We assume here that v_orb has been set to a vector with short orbit!
  Assert(0, SizeViaOrbit(stabs[i], v_orb) = u_size[i]);
  SetSize(stabs[i], u_size[i]);
  #
  Print("Generating relations for ", stab_gens[i], ";
  #
  # Determine a nice permutation presentation
  # We assume here that v_orb has been set to a vector with short orbit!
  phi := IsomorphismPermGroupViaOrbit(stabs[i], v_orb);
  H := Image(phi);
  tmp_FG_gens := GeneratorsOfGroup(FG){stab_gens[i]};
  u_rels[i] := GroupRelatorsViaOrbit(H, GeneratorsOfGroup(H), tmp_FG_gens);
  # Verify all relations actually hold in G
  Assert(0, ForAll( u_rels[i], r->MappedWord( r, tmp_FG_gens,
                      gens{stab_gens[i]} ) = One(stabs[i]) ));
  #
  Append(rels, u_rels[i]);
od;
rels := Set(rels);
Assert(0, ForAll(rels, r->MappedWord( r, GeneratorsOfGroup(FG), gens ) = gens[1]^0 ));

# We can efficiently compute a (lower bound on) the size of a matrix group G
# by finding a vector v with an orbit on which G acts (faithfully). The
# following method does just that. The caller is responsible for supplying
# a suitable vector v (the shorter the orbit, the faster the computations).
#
SizeViaOrbit := function (G, v)

  local orbit, phi, size;

  orbit := Orbit(G, v);
  orbit := ShallowCopy(orbit);; Sort(orbit);
  phi := ActionHomomorphism(G, orbit);
  size := Size(Image(phi));
  Unbind(phi); Unbind(orbit);
  return size;
end;

# Find a nice permutation presentation of a matrix group by its
# action on the orbit of a given vector.
#
IsomorphismPermGroupViaOrbit := function (G, v)

  local orbit, phi;

  Print("Computing orbit...
  orbit := Orbit( G, v );;
  orbit := ShallowCopy(orbit);; Sort(orbit);
  phi := ActionHomomorphism( G, orbit, "surjective" );;
  orbit := Orbit( G, v );;
  orbit := ShallowCopy(orbit);; Sort(orbit);
  Print("Computing permutation group...
  phi := ActionHomomorphism( G, orbit, "surjective" );;

Listing 7: util.gap
A GAP CODE

# Print("We are verifying that the perm group \$\text{is isomorphic to the stabilizer...\n"");
Assert(0, IsSurjective(\$\phi\$) and Size(Image(\$\phi\$)) = Size(G));
#
return \$\phi\$;
end;;

# Find relations of a finite presentation on \$FG\_\text{gens}\$ of the group \$H\$ generated
# by \$H\_\text{gens}\$, where each element of \$H\_\text{gens}\$ is mapped to the corresponding
# element of \$FG\_\text{gens}\$.
GroupRelatorsViaOrbit := function( H, H\_\text{gens}, FG\_\text{gens} )
local psi, Fgens, Frels;
Print("Determining finite presentation...\n");
psi := IsomorphismFPGroupByGenerators( H, H\_\text{gens} );
# Determine the relations
Fgens := FreeGeneratorsOfFPGroup(Image(psi));
Frels := RelationsOfFPGroup(Image(psi));
# Verify all relations actually hold in \$H\$
Assert(0, ForAll( Frels, r->MappedWord( r, Fgens, H\_\text{gens} ) = One(H) ));
return List( Frels, r->MappedWord( r, Fgens, FG\_\text{gens} ) );
end;;

# Auxiliary function used by G2 and Sp6 below. Computes a list of
# generators for G2 in the representation we need.
G2\_Sp\_helper := function( q, TMP)
local F, gens, g, g2, x, i, j, G;

Assert(0, IsEvenInt(q));
F := GF(q);
gens := [];
for g in GeneratorsOfGroup(TMP(3,F)) do
g2 := TransposedMat(g)^-1;
x := NullMat(6,6,F);
for i in [1..3] do
  for j in [1..3] do
    x[i][j] := g[i][j];
    x[3+i][3+j] := g2[i][j];
  od;
od;
Add(gens, x);

x := [1, 0, 0, 1, 0, 0],
  [0, 1, 0, 0, 0, 0],
  [0, 0, 1, 0, 0, 0],
  [0, 0, 0, 1, 0, 0],
  [0, 0, 1, 0, 1, 0],
  [0, -1, 0, 0, 0, 1] ] * One(F);
Add(gens, x);
return Group(gens);
end;;

# Return a suitable representation of the group G2(q),
# where q must be a power of two.
REFERENCES

G2 := function(q)
  local G;
  G := G2Sp_helper(q, SL);
  #SetSize(G, q^6*(q^6-1)*(q^2-1));
  return G;
end;
#
# Return a suitable representation of the group Sp(6,q),
# where q must be a power of two.
#
Sp6 := function(q)
  local G, x;
  G := G2Sp_helper(q, GL);
  if q = 2 then
    x := |
      [ 1, 0, 0, 1, 0, 0 ],
      [ 0, 1, 0, 0, 1, 0 ],
      [ 0, 0, 1, 0, 0, 1 ],
      [ 0, 0, 0, 1, 0, 0 ],
      [ 0, 0, 0, 0, 1, 0 ],
      [ 0, 0, 0, 0, 0, 1 ] |
    * One(GF(q));
    G := ClosureGroup(G, x);
  fi;
  #SetSize(G, q^6*(q^6-1)*(q^2-1));
  return G;
end;

References

[AS83] M. Aschbacher, S.D. Smith, Tits geometries over GF(2) defined by groups over GF(3), Comm. Algebra 11 (1983), 1675–1684.

[BSS01] B. Baumeister, S.V. Shpectorov, G. Stroth, Flag-transitive affine dual polar spaces of $O_{2n+1}(q)$-type, unpublished manuscript, 2001.

[BH] R. Blok, C. Hoffman, A quasi Curtis-Tits-Phan theorem for the symplectic group, J. Algebra, to appear.

[BC] F. Buekenhout, A. M. Cohen, Diagram Geometry, in preparation
http://www.win.tue.nl/~amc/buek

[Cuy94] H. Cuypers, Symplectic geometries, transvection groups and modules, J. Combin. Theory Ser. A 65 (1994), 39–59.

[DHWHVM05] A. De Wispelaere, J. Huizinga, H. Van Maldeghem, Ovoids and spreads of the generalized hexagon $H(3)$, Discrete Math. 305 (2005), 299–311.

[Gra04] R. Gramlich, On the hyperbolic symplectic geometry, J. Combin. Theory Ser. A 105 (2004), 97–110.

[GVM06] R. Gramlich, H. Van Maldeghem, Intransitive geometries, Proc. London Math. Soc. 93 (2006), 666–692.

[GPP] R. Gramlich, A. Pasini, H. Pralle, On the graph of the incident point-hyperplane pairs of $PG(n,F)$, in preparation.

[Hal88] J. I. Hall, The hyperbolic lines of finite symplectic spaces, J. Combin. Theory Ser. A 47 (1988), 284–298.

[HS05] C. Hoffman, S. Shpectorov, New geometric presentations for $Aut(G_2(3))$ and $G_2(3)$, Bull. Belg. Math. Soc. Simon Stevin 12 (2005), 813–826.
REFERENCES

[Kan85] W. Kantor, Some exceptional 2-adic buildings, J. Algebra 92 (1985), 208–223.

[Pas85] A. Pasini, Some remarks on covers and apartments, in: Finite geometries (edited by C.A. Baker, L.M. Batten), Dekker, New York 1985, 223–250.

[Pas94] A. Pasini, Diagram geometries, Clarendon Press, New York 1994.

[Ron80] M. Ronan, A geometric characterization of Moufang hexagons, Invent. Math. 57 (1980), 227–262.

[ST34] H. Seifert, W. Threlfall, Lehrbuch der Topologie, Chelsea Publishing Company, New York 1934.

[Shp06] S.V. Shpectorov, Simple connectedness of hyperplane complements in dual polar spaces, preprint, 2006.

[Str93] M. Stroppel, A categorical glimpse at the reconstruction of geometries, Geom. Dedicata 46 (1993), 47–60.

[Tit74] J. Tits, Buildings of spherical type and finite BN-pairs, Lecture Notes in Mathematics 386, Springer, Berlin 1974.

[Tit86] J. Tits, Ensembles ordonnés, immeubles et sommes amalgamées, Bull. Soc. Math. Belg. Sér. A 38 (1986), 367–387.

[VMal98] H. Van Maldeghem, Generalized polygons, Birkhäuser, Basel 1998.

[WWW] M. Horn, WWW homepage, http://www.mathematik.tu-darmstadt.de/~mhorn.

Authors’ Addresses:

Ralf Gramlich, Max Horn
TU Darmstadt
FB Mathematik / AG 5
Schloßgartenstraße 7
64289 Darmstadt
Germany
gramlich@mathematik.tu-darmstadt.de
mhorn@mathematik.tu-darmstadt.de

Antonio Pasini
Università di Siena
Scienze Matematiche e Informatiche
Pian dei Mantellini 44
53100 Siena
Italy
pasini@unisi.it

Hendrik Van Maldeghem
Ghent University
Pure Mathematics and Computer Algebra
Krijgshaven 281, S22
9000 Gent
Belgium
hvm@cage.ugent.be

First author’s alternative address:
Ralf Gramlich
The University of Birmingham
School of Mathematics
Birmingham
B15 2TT
United Kingdom
ralfg@maths.bham.ac.uk