Introduction.– The interplay of crystalline order and Bose-Einstein condensation encompasses relevant physical phenomena from the realm of superconductors, where it rules the motion of Cooper pairs of fermions through crystals [5], to the field of ultracold atomic gases, where laser-induced lattices modulate the superfluid flow of condensed bosons [6], and also to the supersolid phase [7–9]. Supersolidity, by means of which crystalline structures are endowed with superfluid properties, was first envisaged to take place in solid Helium at very low temperature and very high pressure [10]. However, only recently, supersolid properties have been realized in ultracold-gas systems by means of spin-orbit coupling [11], by using Bose-Einstein condensate (BEC) to optical-cavity coupling [11, 12], and also as a result of competing contact and dipolar interactions [13, 14]. These diverse methods have been used to simultaneously achieve a phase coherence, capable of providing the superfluid property, and a modulated particle density, reflecting the translational symmetry breaking of crystals.

The accomplishment of spin-dependent synthetic gauges in atomic gases that are electrically neutral has represented a breakthrough for the simulation of quantum systems [15]. The spin-orbit-coupled configuration arranged in Ref. [1] to observe the stripe phase is a relevant instance, where the spin-dependent gauge, in combination with the linear coupling of a pseudo-spin-1/2 BEC, simulates the dynamics of a superconducting junction in the presence of an external magnetic field. There, Josephson currents are expected to flow due to the relative phases induced by the vector potential (the gauge). In the regime where the penetration of the magnetic field in the junction (the Josephson penetration depth) is smaller than the junction extension, the junction response mimics a weak-superconductor bulk and it is only threaded by singular flux lines that define Josephson vortices [10]. The junction response, with and without external magnetic field, was analytically calculated in one-dimensional (1D) junctions by Owen and Scalapino in 1967 as solutions to the sine-Gordon equation [16]. The Josephson junction of the analogue pseudo-spin-1/2 BEC is equivalently long, the ultracold-gas response to a gauge field should follow the simulated superconducting system. In the rest of the paper we show that this is indeed the case, and the bosonic junction become threaded by an array of vortices whose positions mark the minima of the density stripes in the bulk condensate. The vortices can equally emerge in the absence of gauge field whenever a chiral current density flows around the junction.

Pseudo-spin-1/2 BEC.– The BEC dynamics will be modeled by the Gross-Pitaevskii (GP) equation for the pseudo-spin-1/2 order parameter $\Psi = [\Psi_\uparrow, \Psi_\downarrow]^T$:

$$i\hbar \frac{\partial \Psi}{\partial t} = \left( \frac{\hat{p}^2}{2m} + g |\Psi_\uparrow|^2 - \frac{\omega_\Omega}{2} \right) \Psi_\downarrow - \frac{\omega_\Omega}{2} \Psi_\uparrow + \mathcal{O} \left( |\Psi_{\downarrow}|^2 \right)$$

(1)

where $\hat{p} = -i\hbar \partial_x$ is the momentum operator, and the strength of the contact, repulsive interparticle interaction $g = 4\pi \hbar^2 a_s/m$ is measured by the positive $s$-wave scattering length $a_s > 0$. In the presence of spin-orbit-coupling the momentum operator transforms as $\hat{p} \rightarrow \hat{p} + \sigma_z \hbar k_\ell$, by adding the momentum contribution of the gauge field $\hbar k_\ell$ through the Pauli matrix $\sigma_z$. For the sake of simplification of the analytical treatment, we will
assume periodic boundary conditions in the spatial coordinate \( \Psi_{\sigma}(x, t) = \Psi_{\sigma}(x + L, t) \), with \( \sigma = \uparrow, \downarrow \), that is, a 1D ring geometry of length \( L \). The normalization is fixed by the number of particles \( N = \int dx (|\Psi_{\uparrow}|^2 + |\Psi_{\downarrow}|^2) \), which is a conserved quantity derived from the continuity equations \( \partial_t |\Psi_{\uparrow}|^2 + \partial_x J_{\uparrow} = - (\partial_t |\Psi_{\downarrow}|^2 + \partial_x J_{\downarrow}) = J_{\Omega} \). Here \( J_{\sigma} = |\Psi_{\sigma}|^2 v_{\sigma} \) are the axial particle current densities in the condensates, with superfluid velocities \( v_{\sigma} = (\hbar/m) \partial_x \arg \Psi_{\sigma} \), whereas \( J_{\Omega} = \Omega |\Psi_{\uparrow}| |\Psi_{\downarrow}| \sin \varphi \) is the Josephson current of tunneling particles across the junction, which is modulated by the relative phase \( \varphi(x) = \arg \Psi_{\uparrow} - \arg \Psi_{\downarrow} \). In the presence of spin-orbit coupling the continuity equations hold with shifted superfluid velocities \( v = (\hbar/m) (\partial_x \arg \Psi + \sigma x k_{\ell}) \).

**Generalized sine-Gordon equation.** The symmetry of Eq. (1) suggests the search for stationary states with sharing density profiles \( |\Psi_{\uparrow}|^2 = |\Psi_{\downarrow}|^2 = n(x) \) and opposite superfluid velocities \( v_{\uparrow} = - v_{\downarrow} \). In this case, by subtracting the two continuity equations one gets a single equation for the relative phase that rules the particle currents:

\[
\partial_x \left[ n(x) \frac{\partial \varphi}{\partial x} \right] = \frac{n(x)}{\lambda_{\Omega}^2} \sin \varphi \tag{2}
\]

where \( \lambda_{\Omega} = \sqrt{\hbar/2m\Omega} \) is the characteristic length determined by the linear coupling. The square parenthesis is intrinsically the chiral density current \( J_{\uparrow} - J_{\downarrow} = (\hbar/m) n(x) \partial_x \varphi \), which provides the superfluid relative velocity \( \mathbf{v}_{\uparrow} - \mathbf{v}_{\downarrow} = (J_{\uparrow} - J_{\downarrow})/n(x) \). The presence of spin-orbit coupling transforms just the chiral density current in Eq. (2) such that \( J_{\uparrow} - J_{\downarrow} = (\hbar/m) n(x) (\partial_x \varphi + 2k_{\ell}) \).

By making use of the gauge invariance property, one can transform the phase \( \varphi \to \varphi' = \varphi + \chi \) and simultaneously the gauge field \( 2k_{\ell} \to 2k_{\ell}' = 2k_{\ell} - \partial_x \chi \), so that the particle currents remain unchanged. In particular, by using \( \chi = 2k_{\ell} x \) the gauge field is moved from the chiral current to the Josephson current, which in the new gauge becomes \( J_{\Omega} = \Omega |\Psi_{\uparrow}| |\Psi_{\downarrow}| \sin(\varphi' - 2k_{\ell} x) \). This alternative gauge, and correspondingly the alternative Eq. (2), will be convenient for the discussion of stationary states in the presence of spin-orbit coupling.

Equation (2) reduces to the stationary sine-Gordon equation when the condensate density is constant \( n(x) = \bar{n} \), which is the general assumption inside the superconducting junctions. There, the sine-Gordon equation is obtained from the electrodynamic relation between the junction current \( J_J \) and the total (externally applied plus current-induced) magnetic field \( \partial_x H_y = J_J \), where, for definiteness, we have assumed a \( y \)-oriented field in a planar \( xy \) junction with long axis \( x \) and unit length across. This Maxwell equation is accompanied by the relation between the variation of the gauge-invariant phase \( \varphi_{sc} \) and the magnetic flux, whereby \( \partial_x \varphi_{sc} = (2\pi d/\Phi_0) H_y \), where \( d \) is a characteristic transverse length and \( \Phi_0 \) is the flux quantum \( \Phi_0 = \hbar/2m \). From this latter expression, and inspection of Eq. (2), it is clear that, apart from constants, the chiral current (here the relative velocity) in the neutral gas plays the role of the total magnetic field threading the superconducting junction, and therefore it determines the junction dynamics. In particular, from comparison of both mathematical models, by writing \( \langle m/\hbar \rangle (J_{\uparrow} - J_{\downarrow})/\tilde{n} = \alpha \partial_x \varphi_{sc} \), and \( (n(x)/\bar{n}) \sin \varphi = \alpha \sin \varphi_{sc} \), with \( \bar{n} \) a constant density and \( \alpha \) a non-dimensional constant, one recovers an equivalent sine-Gordon equation in the new relative phase \( \varphi_{sc} \). Although it is not apparent that this mapping can be realized, as we show next, this is indeed the case in the absence of spin-orbit coupling. From direct integration, Owen and Scalapino found two types of general solutions to the 1D sine-Gordon equations [1], namely \( \varphi_{sc,0}(x) = \arcsin[2 \sin(kx, m) \text{dn}(kx, m)] \), and \( \varphi_{sc,1}(x) = \arcsin[2 \sin(kx, m) \text{cn}(kx, m)] \), where \( \text{sn}, \text{cn}, \text{dn} \) stands for the Jacobi elliptic functions of parameter \( m \in [0, 1] \), and "wavenumber" \( k = 2jK(m)/L \), with \( j = 2, 4, 6... \) (in a system with periodic boundary conditions), and \( K(m) \) is the complete elliptic integral of the first kind. By mapping these solutions into the neutral gas model, and despite the fact that the corresponding phases \( \varphi_0(x) \) and \( \varphi_1(x) \) depend also on yet unknown condensate densities \( n_0(x) \) and \( n_1(x) \), one gets general expressions for the axial current densities and the particle tunneling across the junction:

\[
\varphi_0(x) = \arcsin \left[ 2 \bar{n} \alpha \frac{\text{sn}(kx, m) \text{dn}(kx, m)}{n_0(x)} \right], \tag{3}
\]

\[
J_{0,\uparrow} = -J_{0,\downarrow} = \frac{\hbar k}{m} \bar{n} \alpha \text{cn}(kx, m), \tag{4}
\]

\[
J_{0,\Omega} = 2 \bar{n} \alpha \sin(kx, m) \text{dn}(kx, m), \tag{5}
\]

and

\[
\varphi_1(x) = \arcsin \left[ 2 \bar{n} \alpha \frac{\text{sn}(kx, m) \text{cn}(kx, m)}{n_1(x)} \right], \tag{6}
\]

\[
J_{1,\uparrow} = -J_{1,\downarrow} = \frac{\hbar k}{m} \bar{n} \alpha \text{sn}(kx, m), \tag{7}
\]

\[
J_{1,\Omega} = 2 \bar{n} \alpha \sin(kx, m) \text{cn}(kx, m), \tag{8}
\]

where due to the constraints posed by the continuity equations \( \hbar k^2/2m = \Omega \) for \( \varphi_0 \), and \( \hbar k^2/2m = \Omega \) for \( \varphi_1 \).

It turns out that neither the condensate current densities nor the Josephson currents depend locally on the density. As can be seen in the top panels of Fig. 1(a), both group of solutions describe junction-vortex arrays composed of \( j = kL/4K(m) \) vortices (\( j/2 \) vortex pairs due to the ring geometry), which are either counter-rotating, in Eqs. (4) and (5), or co-rotating, in Eqs. (7) and (8). In superconductors, these alternative configurations correspond, respectively, to a junction dynamics dominated by superconducting currents, or dominated by the applied magnetic field. In the neutral ultracold gas, the same picture essentially holds. When the gauge is absent, the particle currents (4) and (7) belong to dynamically stable (see below) excited states of the bosonic system.

**Analytical solutions.** The condensate densities \( n_0(x) \) and \( n_1(x) \) underlying the above junction-vortex arrays complement the given particle currents in order to fulfill the GP Eq. (1). We have found the corresponding
analytical solutions to this equation that produce such densities in the absence of gauge, namely

$$\Psi_0(x) = \sqrt{\bar{n}} \left[ \left( \frac{1}{\alpha} \right) \text{sn}(kx, \mathbf{m}) + \alpha \left( \frac{i}{\sqrt{1 - \alpha^2}} \right) \text{dn}(kx, \mathbf{m}) \right], \quad (9)$$

where $\alpha = \sqrt{1 - \frac{2\hbar\Omega}{\sqrt{g} n}}$, and $\bar{n} = N/[2L[\alpha^2 + (1 - \alpha^2)f(m)]$, with $f(m) = [K(m) - E(m)]/[mK(m)]$ and $E(m)$ is the complete elliptic integral of second kind; and

$$\Psi_1(x) = \sqrt{\bar{n}} \left[ \left( \frac{1}{i} \right) \text{sn}(kx, \mathbf{m}) + \alpha \left( \frac{1}{\sqrt{1 - \alpha^2}} \right) \text{cn}(kx, \mathbf{m}) \right], \quad (10)$$

where $\alpha = \sqrt{1 - \frac{2\hbar\Omega}{\sqrt{g} n}}$, and $\bar{n} = N/[2L[\alpha^2 + (1 - \alpha^2)f(m)]$. A relevant difference between the two types of vortex arrays is the absence, for counter-rotating vortices
phase constant-density states with zero momentum $U_0^*$ show the asymptotic tendencies $\text{s}_n(kx)$ labeled in panel (a) with $\hbar\Omega = 1.0 \, gN/L$. The families of bound dark soliton states (b) corresponding to point $b$ in Fig. 1(b). They can be described, at low linear coupling, as bound dark solitons characterized by two close axial $\pi$-phase jumps, and, at higher coupling, as bright solitons (BS) featured by flat axial phase profiles, in spite of the repulsive interparticle interaction. An important difference between the non-current states (bDS-BS) and the arrays of Josephson vortices is the dynamical stability of the latter (see the Appendix for details).

When the gauge is present the co-rotating Josephson vortices continue existing with the same Josephson current (10) obtained in the absence of gauge (for equal number of particles). As a result, the chiral currents are simply shifted from Eqs. (7) in a constant that depends on the gauge momentum $\hbar k_x$. As can be seen in the compared numerical solutions of Fig. 1(d), the relative-phase gradient and the particle density are also modulated in a similar way around the same average values $\sim N/L$ and $\sim 2\pi j/L$, respectively, where $j$ is the number of vortex pairs; just the modulation amplitudes vary, decreasing along with the difference $\Delta = 2k_x - 2\pi j/L$ [see the path traced by states A, B, C in Fig. 1(c)] for increasing gauge momentum. Notice, that in the presence of gauge the relative superfluid velocity adds the term $2k_x$ to the relative phase gradient, and hence the relative velocities (and with them the system energy) decrease with $\Delta$. This effect may be better understood with the gauge field entering the Josephson current as $J_0 \propto \sin(\varphi' - 2k_x x)$. It is then clear that if the relative phase is locked by the local coupling phase $\varphi' \sim 2k_x x$ the particle tunneling is canceled $J_0 \sim 0$, and the corresponding coupling energy $E_\Omega \sim -\hbar\Omega|\tilde{\Psi}_1\psi_1|$ minimizes the system energy. In this gauge, by following the same steps that lead to Eqs. (7) and (8), one obtains the relative phase expression $\varphi(x') = 2k_x x + \arcsin[2\hbar\text{s}\text{sn}(kx, m)\text{cn}(kx, m)/n(x)]$.

Since the vortex-array is energetically favored by the gauge field, it eventually becomes the system ground state for a particular range of the spin-orbit-coupling strength, around the value $k_\ell = 2\pi j/L$ [point C in Fig. 1(c)-(d)]. At this point, the chiral current density is shifted by $-2\hbar k_\ell \text{s}\text{sn}(kx, m)$ with respect to the value given by Eq. (7), and shows oscillations around a zero average value. This scenario is not dissimilar from the dynamical response of a bulk superfluid subject to an external rotation, which leads to an Abrikosov vortex lattice. Equivalently, the emerging Josephson vortices are the junction of in-phase (DS) and out of phase (DS*) overlapped dark solitons (replicating the soliton trains in scalar condensates [19]) are linked by families of co-rotating junction-vortex arrays (JV$_k$) described by Eq. (10); the link is apparent when the energy-coupling graph is extended into the negative coupling domain. On the other hand, the families of counter-rotating junction-vortex arrays (JV$_0$) described by Eq. (9) connect, at lower energy, the uniform, constant-density states (U$^*_0$) with the family of in-phase dark solitons. It is worth pointing out that this latter connection is also made at higher energy by a family of non-current states (bDS-BS) composed by trains of staggered solitons (typical examples are shown in Fig. 2(b), (c)).

![Figure 2](image-url)

**FIG. 2.** (a) Energy per particle $E/N$ versus linear coupling $\hbar\Omega$ of stationary states for fixed number of particles so that $gN/L = \hbar^2(2\pi/L)^2/m$. The labels correspond to out of phase dark solitons DS*, in-phase dark solitons DS, out of phase constant-density states with zero momentum $U_0^*$, out of phase constant-density states with minimum non-zero momentum $U_1^*$, co-rotating Josephson vortices JV$_1$, counter-rotating Josephson vortices JV$_0$ and bound-dark solitons along with bright solitons bDS-BS. (b) The density profile of bound dark soliton states (bDS), corresponding to point $b$ labeled in panel (a) with $\hbar\Omega = 0.4 \, gN/L$. (c) The density profile of bright soliton states (BS), corresponding to point $c$ labeled in panel (a) with $\hbar\Omega = 1.0 \, gN/L$.

The two junction-vortex families (9) and (10) have a simple interpretation as excited states of the pseudospinor system, since they connect, by varying the linear coupling, distinct solutions of the GP Eq. (1) that lack Josephson currents. To make the picture clearer, Fig. 2(a) depicts these connections between families of stationary states in an energy-coupling chart for a system with fixed average density $gN/L = \hbar^2(2\pi/L)^2/m$ (see the Appendix for the energy calculations). The families of in-phase (DS) and out of phase (DS*) overlapped dark solitons (replicating the soliton trains in scalar condensates [19]) are linked by families of co-rotating junction-vortex arrays (JV$_k$) described by Eq. (10); the link is apparent when the energy-coupling graph is extended into the negative coupling domain. On the other hand, the families of counter-rotating junction-vortex arrays (JV$_0$) described by Eq. (9) connect, at lower energy, the uniform, constant-density states (U$^*_0$) with the family of in-phase dark solitons. It is worth pointing out that this latter connection is also made at higher energy by a family of non-current states (bDS-BS) composed by trains of staggered solitons (typical examples are shown in Fig. 2(b), (c)). They can be described, at low linear coupling, as bound dark solitons (bDS) characterized by two close axial $\pi$-phase jumps, and, at higher coupling, as bright solitons (BS) featured by flat axial phase profiles, in spite of the repulsive interparticle interaction. An important difference between the non-current states (bDS-BS) and the arrays of Josephson vortices is the dynamical stability of the latter (see the Appendix for details).

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response to a chiral current, which produces particle density depletions around the vortex cores. It is precisely this effect, the emerging density modulation or striped density, which has been highlighted as the fingerprint of the regime dominated by the gauge field.

**Experimental prospects.**—The direct observation in ultracold atoms of junction-vortex arrays and their induced current seems a feasible goal. Experiments in two elongated condensates of ultracold bosonic gases loaded in double-well potentials have already observed sine-Gordon solitons in the relative phase of the coupled atomic clouds [20], not too long after the first observation of Josephson-vortex cores in superconducting junctions [21]. As can be inferred from the above discussion, the observed solitons are the signature of existing chiral currents and Josephson vortices in the system. This experimental achievement, which reflects the ability of tuning the linear coupling and directly imaging the relative phase (and particle density), presents a promising scenario for the manipulation of junction vortices, as already happens in superconductors [22], thus for the control of the associated interference patterns [16]. Although the presence of spin-orbit coupling can ensure the appearance of junction vortices at low linear coupling, it is not, as we have shown, a necessary ingredient. Alternatively, the excitation of relative currents between spin components (a chiral density current, e.g. in a similar setting as the toroidal spinor of Ref. [23]) could also trigger the emergence of the junction-vortex arrays (see also [21] for a similar discussion in superconductors on the role of magnetic fields and relative superconducting currents).

**APPENDIX**

**Linear stability**

The linear stability of stationary states $\Psi$ can be found from the Bogoliubov equations for the excitation modes $|u_j(x), v_j(x)|$ with energy $\mu + \hbar \omega$ [24]:

$$
\begin{align*}
\left(-\frac{\hbar^2 \partial_x^2}{2m} + 2g|\Psi_\sigma|^2 - \mu\right) u_\sigma + g\Psi_\sigma^2 v_\sigma - \frac{\hbar \Omega}{2} u_\sigma &= \hbar \omega u_\sigma, \\
\left(-\frac{\hbar^2 \partial_x^2}{2m} - 2g|\Psi_\sigma|^2 + \mu\right) v_\sigma - g(\Psi_\sigma^*)^2 u_\sigma + \frac{\hbar \Omega}{2} v_\sigma &= \hbar \omega v_\sigma,
\end{align*}
$$

(11)

where the subindexes $\{\sigma, \bar{\sigma}\}$ stand for $\{\uparrow, \downarrow\}$ and vice versa. Figure 3 reports our numerical results for the frequencies of unstable modes (hence with complex-frequency eigenvalues in Eq. 11) for the families of stationary states with striped density profile (red lines) presented in Fig. 2. As can be seen, only the families of junction-vortex arrays $JV_0$ and $JV_1$ contain dynamically stable configurations, showing the key role played by the Josephson currents as a stabilization feature in pseudo-spinor condensates. Even though, there also exist unstable states within these families, close to the connection points with other unstable stationary states, as overlapped dark solitons (DS) and out of phase staggered solitons (DS) and out of phase uniform states ($U^*_0$).

**Energy of junction-vortex arrays**

The energy functional for the pseudo-spin-1/2 system in a state $\Psi = \int dx |\sum_{\sigma}(h^2(-i\partial_x + \sigma k_x)\Psi_\sigma^2/2m + g|\Psi_\sigma|^4/2) - \hbar \Omega \text{Re}(\Psi^* \Psi)|$. Using this functional for the counter-rotating junction-vortex state (9) at $k_f = 0$, the system mean energy is obtained as

$$
E_0 = g\hbar^2 L \left[ \alpha^4 + \frac{1}{3m} \frac{\text{m} \alpha^2}{m} \left( 1 + 2 \alpha^2 \right) \right] + f(m) \frac{(1 - \text{m} \alpha^2)^2}{3m} \left( m - \frac{1}{2} \right),
$$

(12)

and the corresponding chemical potential is $\mu = h^2 k^2 (1 + \text{m})/2m + g\hbar^2 \alpha^2 - \hbar \Omega/2$. This family exists for $\alpha \in [0, 1/\sqrt{\text{m}}]$ and linear coupling values inside the range $h \Omega \in [-g\hbar^2 /2m, g\hbar^2 /2m]$. For $\alpha = 0$, at the maximum linear coupling $\hbar \Omega = g\hbar^2 /2m$, the particle tunneling is suppressed, and the co-rotating vortex states merge with the family of dark-soliton trains, whose functional form is $\Psi_{\text{BS}} \propto \sin(kx, m)$ [19, 25]. For $0 < \alpha^2 \leq 1/\text{m}$ the minimum linear coupling is reached in the finite system when $\Omega = h(2\pi L)^2/2m$, and the co-rotating vortices transform into a uniform density state $|\Psi_\sigma|^2 = N/2L$ without currents.

Analogously, for the co-rotating vortex arrays [10], the...
mean energy is given by
\[
E_1 = \frac{g n^2 L}{2} \left[ \frac{\alpha^2 (1 + \alpha^2)}{2} + \frac{(1 - \alpha^2)}{3m} \left( 1 + 2\alpha^2 + (1 - \alpha^2) \frac{(2 - m) f(m)}{2} \right) \right],
\]
and the chemical potential is \( \mu = \hbar^2 k^2 (1 - m) / 2m + g n - \hbar \Omega / 2 \). The family exists for \( \alpha \in [0, 1] \), and in the range of linear coupling \( \Omega \in [-g n / 2, g n / 2] \). For vanishing linear coupling \( \Omega = 0 \), at \( \alpha = 1 \), the system splits into two separated condensates with constant density \( n = N / 2L \). For \( \alpha = 0 \), the co-rotating vortex states merge also with the family of dark-soliton trains.

**Static states: staggered soliton trains**

Striped-density profiles in the pseudo-spin-1/2 system can be trivially obtained by means of overlapped dark-soliton trains, with either in- or out-of-phase spin profiles, that reproduce the corresponding solutions to the scalar GP equation [19, 25]. However, there are additional states specific of the pseudo-spinor system, made of static staggered solitons trains with \( \pi \)-phase shift between spin components (labeled as bDS-BS in Fig. 2(a)) that also produce particle density modulations in the absence of spin-orbit coupling. Both bound-dark-soliton (bDS) and bright-soliton (BS) trains are alternative configurations (see Ref. 25 about the latter configuration).

A simple ansatz with constant real parameter \( \gamma \in [0, 1] \) can be given by \( \Psi_{\uparrow, \downarrow} = \sqrt{n} \left[ \pm \gamma - (1 - \gamma) \sin(k \gamma, m) \right] \), which just interpolates between the family of in-phase dark solitons, for \( \gamma = 0 \), and the family of uniform out of phase states, for \( \gamma = 1 \). The wavefunction nodes vanish for values \( \gamma \geq 0.5 \), and so the axial particle density acquires a striped profile without axial phase variation, as can be seen in Fig. 2(c).

The striped density profile of these nonlinear states shares clear similarities with the patterns of standing waves in a linear system, where the boundary conditions select the permitted solutions to a wave equation. In fact, in the limit \( m \rightarrow 0 \) the Jacobi sn function tends to the trigonometric sine function, and the previous ansatz becomes an exact solution to Eqs. (1) without nonlinearity, that is to the coupled Schrödinger equations. For high interaction, the staggered-soliton family becomes disconnected from the branch of out of phase dark solitons. In that regime, the staggered-soliton state at zero coupling \( \Omega = 0 \) corresponds to bound dark solitons per spin-component that present opposite \( \pi \)-phase jumps.

[1] J.-R. Li, J. Lee, W. Huang, S. Burchesy, B. Shteynas, F. C. Top, A. O. Jamison, and W. Ketterle, Nature 543, 91 (2017).
[2] T.-L. Ho and S. Zhang, Phys. Rev. Lett. 107, 150403 (2011).
[3] Y. Li, G. I. Martone, and S. Stringari, “Annual review of cold atoms and molecules.” (2015) Chap. Spin-orbit-coupled Bose-Einstein Condensates, p. 201.
[4] C. Owen and D. Scalapino, Physical Review 164, 538 (1967).
[5] N. M. Plakida, High-temperature superconductivity: experiment and theory (Springer Science & Business Media, 2012).
[6] O. Morsch and M. K. Oberthaler, Rev. Mod. Phys. 78, 179 (2006).
[7] A. Andreiev and I. Lifshitz, Zh. Eksp. Teor. Fiz. 56, 2057 (1969).
[8] G. V. Chester, Phys. Rev. A 2, 256 (1970).
[9] A. J. Leggett, Phys. Rev. Lett. 25, 1543 (1970).
[10] M. H.-W. Chan, R. Hallock, and L. Reatto, Journal of Low Temperature Physics 172, 317 (2013).
[11] J. Léonard, A. Morales, P. Zupancic, T. Esslinger, and T. Donner, Nature 543, 87 (2017).
[12] J. Léonard, A. Morales, P. Zupancic, T. Donner, and T. Esslinger, Science 358, 1415 (2017).
[13] F. Böttcher, J.-N. Schmidt, M. Wenzel, J. Hertkorn, M. Guo, T. Langen, and T. Pfau, Physical Review X 9, 011051 (2019).
[14] L. Chomaz, D. Petter, P. Ilzhöfer, G. Natale, A. Trautmann, C. Politi, G. Durastante, R. Van Bijnen, A. Patscheider, M. Suhnen, et al., Physical Review X 9, 021012 (2019).
[15] Y.-J. Lin, K. Jiménez-García, and I. B. Spielman, Nature 471, 83 (2011).
[16] A. Barone and G. Paterno, Physics and Applications of the Josephson Effect (Wiley and Sons Inc., 1982).
[17] P. Anderson, in Progress in Low Temperature Physics, Vol. 5, edited by C. Gorter (Elsevier, 1967) pp. 1 – 43.
[18] V. M. Kaurov and A. B. Kuklov, Phys. Rev. A 71, 011601(R) (2005).
[19] L. D. Carr, C. W. Clark, and W. P. Reinhardt, Physical Review A 62, 063610 (2000).
[20] T. Schweigler, V. Kasper, S. Erne, I. Mazets, B. Rauer, F. Cataldini, T. Langen, T. Gasenzer, J. Berges, and J. Schmiedmayer, Nature 545, 323 (2017).
[21] D. Roditchev, C. Brun, L. Serrier-Garcia, J. C. Cuevas, V. H. L. Bessa, M. V. Milosevic, F. Debontridder, V. Stolyarov, and T. Cren, Nat. Phys. 11, 323 (2015).
[22] V. V. Dremov, S. Y. Grebennik, A. G. Shishkin, D. S. Baranov, R. A. Hovhannisyan, O. V. Skryabin, N. Lebedev, I. A. Golovichanskiy, V. I. Chichkov, C. Brun, et al., Nature communications 10, 1 (2019).
[23] S. Beattie, S. Moulder, R. J. Fletcher, and Z. Hadzibabic, Phys. Rev. Lett. 110, 025301 (2013).
[24] P. Pitaevskii and S. Stringari, Bose-Einstein Condensation (Oxford University Press, 2003).
[25] R. Kanamoto, L. D. Carr, and M. Ueda, Phys. Rev. A 79, 063616 (2009).
[26] D. L. Zhang, H. B. Qiu, and A. Muñoz Mateo, Phys. Rev. A 101, 063623 (2020).