ON LOOP EQUATIONS IN KdV
EXACTLY SOLVABLE STRING THEORY

Simon Dalley
Joseph Henry Laboratories, Princeton University,
Princeton, New Jersey 08544, U.S.A.

Abstract
The non-perturbative behaviour of macroscopic loop amplitudes in the exactly solvable string theories based on the KdV hierarchies is considered. Loop equations are presented for the real non-perturbative solutions living on the spectral half-line, allowed by the most general string equation $[\hat{P}, Q] = Q$, where $\hat{P}$ generates scale transformations. In general the end of the half-line (the ‘wall’) is a non-perturbative parameter whose rôle is that of boundary cosmological constant. The properties are compared with the perturbative behaviour and solutions of $[P, Q] = 1$. Detailed arguments are given for the $(2, 2m - 1)$ models while generalisation to the other $(p, q)$ minimal models and $c = 1$ is briefly addressed.
1 Introduction

Soon after the seminal works on non-perturbative 2D gravity appeared [1], a number of authors discussed the behaviour of macroscopic string amplitudes in those theories [2, 3, 4]. In genus perturbation theory these are the surfaces with boundaries whose lengths scale in the continuum limit. It is possible however to discuss such correlators independently of this weak coupling expansion. Since the latter is only asymptotic and typically not Borel resummable one must input some non-perturbative information in order to even begin discussing such questions. The mathematical possibilities are of course in principle endless. It is not so easy however to come up with a systematic formalism. In what follows the extra information that is required will be taken to be the principle that the KdV structure organising genus perturbation theory is present non-perturbatively.

This point of view was taken to its logical conclusion in refs. [5, 6] where it was found that new non-perturbative solutions were allowed that had not been previously considered. It is expedient to briefly review the main points of that work. Consider the $(2, 2m-1)$ minimal models, for which one has the KdV flows
$$
\partial_t u = -\partial_{\theta} R_{k+1}[u] \quad \text{(notation is hopefully standard)},
$$
which define the isospectral symmetries $u \rightarrow u + \epsilon \partial_{\theta} u$ leaving invariant the spectrum of the hamiltonian operator $-\partial_{\theta}^2 + u$. Once the string susceptibility $u$ has been determined these flows yield correlators of the local operators with couplings $t_k$. The minimal physical requirement that one can impose on $u$ is its scaling equation under a change of the length scale in the theory. For the set of parameters $\{u, \{t_k\}\}$ the KdV flows determine this equation to be [3]:
$$
\frac{L}{R} = 0 \quad \text{(1)}
$$
where
$$
R = \sum_{k \geq 0} (k+1/2)t_k R_k = t_0 - \sum_{k \geq 1} (k+1/2)t_k (D^{-1} L)^{k-1} u \quad \text{(2)}
$$
$$
D = \nu \frac{\partial}{\partial \theta_0}, \quad L = -\frac{1}{4} D^3 + \frac{1}{2} u D + \frac{1}{2} D u \quad \text{(3)}
$$
and $\nu$ is the renormalised value of $1/N$. The string equation (1) may be written as the canonical commutation relation on the half-line, $[P, Q] = Q$, for co-

\footnote{This choice clearly has aesthetic and computational qualities but is nevertheless not, at present, a scientific argument.}
ordinate (Lax) operator $Q = D^2 - u$ and $\hat{P} = \sum_{k \geq 0} t_k Q_+^{k+1/2}$ the generator of scale transformations of these co-ordinates, in a double-scaled Dyson gas with polynomial potential.

To reproduce the $\nu$-perturbative results as specified by the matrix models one must choose a solution which is asymptotically a solution of $R = 0$ at $t_0 \to -\infty$. One possibility is to choose an exact solution of $R = 0$, for which there are two types of real solution $u$. Those accumulating poles as $t_0 \to \infty$ (type 1), and solutions of $k$ odd critical points and non-singular flows around those points (type 2), for which $u$ has a real asymptotic expansion at $t_0 \to \infty$.

There is one other type of real solution to (1) with correct asymptotic behaviour as $t_0 \to -\infty$ (type 3). This has no poles and has $u \to 0$ as $t_0 \to \infty$ ($R \to t_0$).

It is realised by a Dyson gas restricted to lie on the spectral half-line i.e. there is an infinite potential ‘wall’ at the critical edge of the charge distribution. In the following section this representation will be used to derive Dyson-Schwinger loop equations. Figure 1 shows the numerical result for the type 3 susceptibility for pure gravity found in ref. [5].

## 2 Type 3 Loop equations

The scaling symmetry that led to (1) is one of the non-isospectral symmetries of the KdV hierarchy (see e.g. [9]). Together with the isospectral symmetries (KdV flows) it implies a family of constraint equations by applying the recursion operator $L D^{-1}$:

$$(L D^{-1})^n L R = 0, \quad n \geq 0$$

These take the form of Virasoro constraints $L_n \tau = 0, \quad u = -2 D^2 \log \tau$ [10].

There is an additional non-isospectral symmetry which, if applied, constrains the solution $u$ further. Invariance under this transformation is the $L_{-1}$ constraint $D R = 0$, which is the hermitian matrix model string equation. This is the Galilean transformation: A translation of the Dyson gas potential, given in infinitesimal form by:

$$u \quad \to \quad \tilde{u} = u - \epsilon$$

$$t_k \quad \to \quad \tilde{t}_k = t_k + \epsilon(k + 3/2)t_{k+1}$$

is an invariance of $R$ but is not respected by (1).

---

2 The pole-free nature of the solutions was verified for the $k = 1, 2$ critical points in ref. [5] and has also been checked for $k = 3$ and the Ising model [9].
In the case of type 3 there is a further parameter in the theory which has so far been neglected \([6]\). For the solution on the half-line one should allow the infinite potential wall to be at some arbitrary scaled position, \(\sigma\) say. This modifies the canonical position and momentum to \(Q + \sigma\) and \(\tilde{P} + \sigma P\), where \(P = \sum_{k \geq 1} t_k Q_k^{k-1/2}\) is the translation operator. In this way the string equation is modified to \([6]\):

\[
[\tilde{P} + \sigma P, Q + \sigma] = Q + \sigma
\]  

equivalently

\[
LR - \sigma DR = 0
\]  

Since \(\sigma\) has the same dimension as \(u\) and \([6]\) is the scaling equation one identifies the evolution with respect to \(\sigma\): \( \nu \partial_\tau u = -DR\). This is the differentiated form of the \(L_{-1}\) condition in the present case: \(L_{-1} \tau = \partial_\sigma \tau\). The Virasoro constraints are the expression of diffeomorphism invariance of the spectral line and the presence of a ‘wall’ has induced a boundary term on the right hand side. Similar boundary terms appear in the other constraints when \(\sigma \neq 0\), since varying the boundary as \(\sigma \to \sigma + \epsilon \sigma^{n+1}\) implies that

\[
L_n \tau = \sigma^{n+1} \frac{\partial \tau}{\partial \sigma}
\]  

The Galilean and higher KdV symmetries are now respected when the correct transformation properties of \(\sigma\) are taken into account. In particular to \([4]\) one must add \(\sigma \to \sigma - \epsilon\). \([7]\) becomes an equation for \(\tilde{u}\) in terms of \(\tilde{t}\), independent of \(\sigma\). The constraints \([8]\) may be rewritten in the old form for \(n \geq 0\) by using the invariance of the Virasoro algebra under

\[
L_n \to \tilde{L}_n = e^{-\sigma L_{-1}} L_n e^{\sigma L_{-1}}
\]  

Given \(L_{-1} \tau = \partial_\sigma \tau\) the equations \([8]\) are easily seen to be equivalent to \(\tilde{L}_n \tau = 0\).

The transformation \([4]\) describes a finite Galilean transformation by \(\sigma\), and the possibility of making redefinitions of this sort can be viewed as another reason for introducing \(\sigma\) in the most general framework.

The parameter \(\sigma\) transforms in the same way as \(t_{m-1}/t_m\) under \([4]\), where \(t_k = 0, k > m\) and \(t_m\) is invariant in this case. Following the reasoning of ref.\([11]\) it suggests that its rôle is that of boundary cosmological constant. Indeed if one takes the usual expression for the renormalised macroscopic loop wavefunction \([2]\):

\[
<w(l)> = \int_{-\infty}^{\infty} <z|\exp(\nu^2 \partial_z^2 - u)|z> dz
\]  

4
a finite Galilean transformation by $\sigma$ exhibits the $e^{-\sigma l}$ dependence of the loop. The net effect of the transformation is to define away the non-exponential dependence of $<w(l)>$ upon $\sigma$. In this case one can now say that $\sigma$ couples to the boundary operator $[11]$ in the sense that

$$\left( \frac{\partial}{\partial \sigma} \right)_i < \prod_i w(l_i) > = \sum_i l_i < \prod_i w(l_i) >$$

(11)

where

$$\left( \frac{\partial}{\partial \sigma} \right)_i = \left( \frac{\partial}{\partial \sigma} \right)_t - \sum_{k \geq 1} (k + 1/2)t_k \frac{\partial}{\partial t_{k-1}}$$

(12)

$$\left( \frac{\partial^\nu u}{\partial \sigma^n} \right)_i = \delta_{n1}$$

(13)

It is worth noting that there is a certain amount of ambiguity in the definition of the loop wavefunction, the choice (10) being perhaps the simplest. Two reasonable requirements might be that it agree with the results of Liouville theory and have a local (KdV) operator expansion at small $l$. The former is meaningful perturbatively in $\nu$ and the latter perturbatively in $l$. This does not fix contributions non-perturbative in both of these parameters however. In this letter the definition (10) will always be assumed.

The arguments up to now have been somewhat heuristic so it is instructive to give a more careful treatment of some of the points. First a proof of the fact that (7) is the string equation of a Dyson gas on $[\sigma, \infty)$ will be given. To agree with the conventions for non-universal constants adopted implicitly earlier, the following calculation is performed with an even polynomial action $NV/\Lambda$ on the interval $[-2, 2]$, the ends of the charge distribution coinciding with infinite walls at $\pm 2$ (in un-scaled variables). The scaling regions around $\pm 2$ thus furnish identical copies of the system on the half-line. Introducing an infinitesimal cutoff $\delta$, let the scaled positions of the walls be $\pm 2 \mp \sigma \delta^2$. Further renormalised parameters are defined by;

$$\Lambda = 1 + t_0 \delta^{2m}$$

$$\frac{\Lambda_n}{N} = 1 - z \delta^{2m}$$

$$R_n = 1 - u(z) \delta^2, \ n \sim N$$

(14)

for the neighbourhood of the $m$th critical point. The equations of motion in the orthogonal polynomial formalism [12] are;

$$\frac{(2n + 1)\Lambda}{N} - <n|\lambda V'|n> = \frac{2\Lambda}{N}(2 - \sigma \delta^2)P_n^2$$

(15)
The notation \(P_n\) is shorthand for \(P_n(2)\exp(-NV(2)/2\Lambda)\) where \(P_n(\lambda)\) are polynomials orthogonal on \([-2, 2]\). Terms involving \(P_n\) in (13)(16) are boundary terms picked up when one integrates by parts standard identities to derive the equations of motion. The left hand side of (15) is:

\[
\frac{(2n + 1)\Lambda}{N} - \sqrt{R_{n+1}} <n|V'|n+1> - \sqrt{R_n} <n-1|V'|n> = \frac{2\Lambda}{N} \sqrt{R_n} P_n P_{n-1} \tag{18}
\]

using (16). From the work on polynomial potentials [1]:

\[
\frac{n\Lambda}{N} - <n-1|V'|n> \sqrt{R_n} = \delta^{2m}(R(z) + O(\delta^2)) \tag{19}
\]

From (15)(17) and (19) one has;

\[
\delta^{2m} Y^2 = \frac{2\Lambda}{N} (2 - \sigma \delta^2) P_n^2 \tag{20}
\]

\[
Y = (R(z) + R(z + \nu \delta) + O(\delta^2))^{1/2} \tag{21}
\]

and eliminating \(P_n, P_{n-1}\) from (18) gives the string equation at the first non-trivial order in \(\delta\):

\[
\frac{\nu^2}{2} RR'' - \frac{\nu^2}{4} (R')^2 - (u - \sigma) R^2 = 0 \tag{22}
\]

Differentiating once yields (7). The Dyson gas also supplies the appropriate boundary conditions \(u \to \sigma\) as \(t_0 \to \infty\) since \(u\) marks the edge of the charge distribution in the leading WKB approximation. In fact a BIPZ analysis [14] shows that in this approximation the charge density acquires a square-root divergence at the wall \((t_0 > 0)\). More generally at \(t_0 < 0\) this divergence has exponentially small residue, as can be most easily seen from the conventional form of the Dyson-Schwinger equations. These can be derived in the usual way [13] since the type 3 solutions have a path integral representation, and should correspond to the Virasoro constraints described earlier when Taylor expanded along the lines of refs. [10]. Starting from the partition function on \([-\Sigma, \Sigma]\) say, in unscaled variables;

\[
Z = \int^{\Sigma}_{-\Sigma} \prod_{i=1}^{N} d\lambda_i \Delta^2(\lambda)e^{-N \sum_{i} V(\lambda_i)} \tag{23}
\]
and defining the loop generating function and its Laplace transform (marked loops are used for convenience);
\[
W(L) = \frac{1}{N} \sum_i e^{L\lambda_i}
\]
(24)
\[
\tilde{W}(T) = \frac{1}{N} \sum_i \frac{1}{T - \lambda_i}
\]
(25)

one may perform an infinitesimal change of variables \(\lambda_i \rightarrow \lambda_i + \epsilon/(T-\lambda_i)\) in (23).
The only new contribution to the standard analysis \[13\] is from the variation of the boundaries, giving
\[
\frac{\partial Z}{\partial \Sigma} \left( \frac{1}{T - \Sigma} - \frac{1}{T + \Sigma} \right)
\]
(26)
The first loop equation is then
\[
V'(T) <\tilde{W}(T)> + \Pi(T) = <\tilde{W}(T)\tilde{W}(T)> + \frac{2\Sigma}{N^2(T^2 - \Sigma^2)} \frac{\partial Z}{\partial \Sigma}
\]
(27)
where \(\Pi(T)\) is a linear combination of \(<W'(0)>, <W'(0)>, \ldots\). In terms of \(W(L)\) one has the equation of motion (introducing connected correlators);
\[
V' \left( \frac{\partial}{\partial L} \right) <W(L)>_c = \int_0^L dL' \left( \frac{1}{N^2} <W(L')W(L - L')>_c 
+ <W(L')>_c <W(L - L')>_c - \frac{2\sinh\Sigma \partial \log Z}{N^2} \right)
\]
(28)
showing a new source-like term for the loop generating function.

Following ref.\[4\] it is a simple matter to take the continuum limit by introducing, in the neighbourhood of the \(m\)th critical point, renormalised parameters;
\[
\Sigma = \Sigma_c - \sigma \delta^2, \quad T = -\Sigma_c - \tau \delta^2
\]
(29)
\[
<\tilde{W}(T)>_c = \frac{1}{2} V'(T) + \delta^{2m-1} <\tilde{w}(\tau)>_c
\]
\[
<\prod_{i=1}^M \tilde{W}(T_i)>_c = \delta^{4m+2-M(2m+3)} <\prod_{i=1}^M \tilde{w}(\tau_i)>_c
\]
Equation (27) becomes
\[
<w(\tau)>_c^2 + \nu^2 <\tilde{w}(\tau)\tilde{w}(\tau)>_c = \pi(\tau) + \frac{\nu^2}{\sigma + \tau} \frac{\partial \log Z}{\partial \sigma}
\]
(30)
For completeness the higher loop equations are
\[
2 <\tilde{w}(\tau)>_c <\tilde{w}(\tau) \prod_{i=1}^M \tilde{w}(\tau_i)>_c + \nu^2 <\tilde{w}(\tau)\tilde{w}(\tau) \prod_{i=1}^M \tilde{w}(\tau_i)>_c
\]
$$\begin{align*}
+ & \sum_{I,J} \langle \tilde{w}(\tau) \prod_{i \in I} \tilde{w}(\tau_i) \rangle_c < \tilde{w}(\tau) \prod_{j \in J} \tilde{w}(\tau_j) \rangle_c \\
+ & \sum_{i=1}^{M} \langle \tilde{w}(\tau_i) \ldots \frac{\partial}{\partial \tau_i} \tilde{w}(\tau_i) - \tilde{w}(\tau) \rangle_c \frac{\tau_i - \tau}{\tau_i} \ldots \tilde{w}(\tau_M) \rangle_c \\
= & <O_0 \prod_{i=1}^{M} \tilde{w}(\tau_i) \rangle_c + \frac{\nu^2}{\sigma + \tau} \frac{\partial}{\partial \sigma} < \prod_{i=1}^{M} \tilde{w}(\tau_i) \rangle_c \\
\end{align*}$$

(31)

where $O_0$ is the (bulk) puncture operator and the index sets $I, J$ are such that $I \cup J = \{1, \ldots, M\}$: $I, J \neq \emptyset = I \cap J$. $\pi(\tau)$ is a polynomial in $\tau$, of degree $2m - 1$ at the $m$th critical point, which determines the structure of the charge density in the scaling region, given by $\frac{1}{2\pi i} \text{Disc} < \tilde{w}(\tau) \rangle_{\nu=0}$ in the saddle-point approximation. The last term in (30) should be $\frac{1}{\sigma + \tau} \int \mathcal{R} dt_0$ from the discussion following (5), which is exponentially small in $\nu$ at $t_0 \to -\infty$ but is $O(1)$ for $t_0 \to \infty$. The physical meaning of the divergence as $\tau + \sigma \to 0^+$, is that large loops become unsuppressed, since $\tilde{w}(\tau)$ is the Laplace transform w.r.t. $\tau$ of the renormalised macroscopic loop wavefunction (10). Because of the $e^{-\sigma l}$ dependence of (10) the Laplace transform only exists for $\tau > -\sigma$. As indicated in the discussion following (22), a concomitant divergence appears in the charge density as $\tau + \sigma \to 0^-$.  

3 Discussion

The previous calculations have shown that type 3 solutions yield very simple $e^{-\sigma l}$ behaviour for macroscopic loops. This exponential behaviour is much like the $\nu$-perturbative result. At each order of a WKB expansion the charge density has support on the half-line $(-\infty, -u]$ of the real $\tau$ axis, specified in the leading approximation by the (single) cut in $\sqrt{\pi(\tau)}$. The rest of the real $\tau$ axis describes the loop function $\tilde{w}(\tau)$ which on inverse Laplace transformation is seen to have $e^{-ul}$ times power law behaviour. At genus zero $w(l)$ contains universal terms with inverse powers of $l$, as emphasised in refs.[15], since $\pi(\tau)$ is polynomial:

$$<w(l)> = \frac{1}{\nu} \sum_{k \geq 0} k! t_k l^{-k-1/2} + O(l^{1/2})$$

(32)

The Laplace transform does not exist because of these terms, but one can proceed by differentiating w.r.t. $\tau$ a sufficient number of times. It is in this sense

---

Footnote 3: In fact there are also negative powers of $l$ in $w(l)$ preventing naive Laplace transformation, but these come from low genus and can be systematically isolated.
that $w(l)$ and $\tilde{w}(\tau)$ are transforms of one another. Alternatively one can work at fixed ‘area’ instead of fixed bulk cosmological constant. The offending terms are analytic in the latter and so do not contribute to an inverse transform to fixed ‘area’. They correspond to finite loops spanned by infinitesimal surfaces.

The $\nu$-non-perturbative exponential behaviour in $l$ of loops for type 1 and type 2 solutions is more complicated. For type 1 the discreteness of the spectrum of $-D^2 + u$ implies that $<w(l)>$ behaves like an infinite sum of exponentials with different arguments \[2\]. Only in the $l \to \infty$ limit does one recover a simple $e^{-e_0 l}$ behaviour, where $e_0$ is the lowest eigenvalue. It is sometimes suggested that a solution satisfying a loop equation such as \[30\], derived from a path integral representation, is ‘physical’ and that, by implication, one that does not is ‘unphysical’. There is no known path integral formula for type 1 solutions. This is not (presently) known to contradict any physical principle however. The hermitian matrix model is the path integral representation of type 2 solutions, and shows that the loop expectation is always diverging as $l \to \infty$ for such solutions \[3\]. This markedly different behaviour is due to the fact that the charge density has support on the whole spectral line, in particular it has an exponential tail. For a tail $\rho(e) \sim \exp(-|e|^p)$ as $e \to -\infty$ a dimensional argument shows that $<w(l)> \sim \exp^{l + 1/(p-1)}$ as $l \to \infty$. This means that one cannot define the loop function $\tilde{w}(\tau)$ in this case.

To conclude one may note some possible generalisations of the results of this letter to other models with $c \leq 1$. For the $(p,q)$ minimal models described by the generalised KdV hierarchy \[16\], the string equation (scaling equation) $[\hat{P}, Q] = Q$ will provide new solutions analoging those of type 3. Generally one only knows how to treat macroscopic loops embedded at a single point in the line of $q-1$ points. By the same argument \[11\], shifting the non-derivative part of $Q$, one can identify a parameter coupling to a boundary operator. Perturbatively it appears that only one parameter can be generated in this way (e.g. for the Ising model $(4,3)$ it is the boundary magnetic field), which led the authors of ref.\[11\] to conclude that certain operators could not be expressed in the KdV formalism. However in the case of type 3, non-isospectral evolution equations also play a role in defining couplings and one can imagine restricting the $q-1$ types of charge to different half-lines i.e. the wall becomes ‘time’-dependent. The necessary argument in terms of the W-constraints to confirm or deny the validity of this naive picture is a little involved. A detailed account of the $[\hat{P}, Q] = Q$ version of other minimal models will be given by the authors of ref.\[7\]. At $c = 1$ the
picture is similar. By placing a wall in the scaling region given by the inverse quadratic potential \[17\] one has a stable quantum mechanical system providing a non-perturbative definition of the theory. More particularly the macroscopic loop amplitude at time $t$ is non-perturbatively well-defined;

$$< w(l, t) > = \int_\sigma^\infty d\lambda \psi^\dagger(\lambda, t)e^{-\lambda l}\psi(\lambda, t)$$

(33)

$$\psi(\lambda, t) = \int_{E_F}^{E_P} dE e^{iEt}\Psi(E, \lambda)$$

(34)

where the wavefunctions $\Psi$ vanish at the wall $\lambda = \sigma$ and the matrix model corresponds to the continuation to euclidean time. It exhibits a dependence $\exp - l\sigma$, at the expense of introducing a linear term in the potential. The extra parameter $\sigma$ more generally has a continuous argument since again it may be time-dependent. Understanding of the possible flow structure at $c = 1$ is still hazy. Is stabilisation by a simple wall, possibly fluctuating in position, the only quantum mechanical problem non-perturbatively compatible with a flow structure organising perturbation theory, akin to $c < 1$?

**Acknowledgements:** The influence of Tim Morris on the early stages of this work cannot be overstated. I am also grateful to Clifford Johnson for helpful information, and to Tim Morris and Andrea Pasquinnuci for comments on the manuscript. This work was supported by an SERC studentship and post-doctoral fellowship RFO/B/91/9033.

**Note Added:** After this letter was typed a preprint appeared \[18\] where quantum mechanics on the half-line is discussed with regard to the requirement of unitarity of tachyon scattering at $c = 1$. 
Figure Caption

Figure 1: This shows the (unique) numerical solution of type 3 for pure gravity. The Hamiltonian $-D^2 + u$ has continuous spectrum down to the value of the left asymptote of the potential $u$ (zero in the figure shown). Those eigenvalues are more-or-less directly related to the positions of the Dyson gas charges on the spectral line, which should lie on the positive halfline. Note that although there is a small well in $u$, the previous identification indicates that it is too shallow to support bound states below the continuum. A rough numerical estimate using the proportions of the figure confirms this. As is explained in section 2, more generally the left asymptote can be $u = \sigma$ if the Dyson gas is restricted to $[\sigma, \infty)$. It is possible that non-perturbative subleties arise as $\sigma$ turns negative e.g. through an instability in $u$ similar to that found in ref.\textsuperscript{8}. 
References

[1] E.Brézin and V.Kazakov, Phys. Lett. B\textbf{236} (1990) 144;
D.Gross and A.Migdal, Phys. Rev. Lett. \textbf{64} (1990) 127;
M.Douglas and S.Shenker, Nucl. Phys. B\textbf{335} (1990) 135.

[2] T.Banks, M.Douglas, N.Seiberg and S.Shenker, Phys. Lett. B\textbf{238} (1990) 279;
D.Gross and A.Migdal, Nucl. Phys. B\textbf{340} (1990) 333.

[3] E.Brézin, E.Marinari and G.Parisi, Phys. Lett. B\textbf{242} (1990) 35.

[4] F.David, Mod. Phys. Lett. A\textbf{5} (1990) 1019.

[5] S.Dalley, C.Johnson and T.Morris, preprints SHEP 90/91-16 and 28 (both to appear in Nucl. Phys. B).

[6] S.Dalley, C.Johnson and T.Morris, SHEP 90/91-35 in proceedings of workshop on Random Surfaces and 2D Quantum Gravity, Barcelona 10-14 June 1991.

[7] C.Johnson, T.Morris and B.Spence, in preparation.

[8] M.Douglas, N.Seiberg and S.Shenker, Phys. Lett. B\textbf{244} (1990) 381.

[9] H.La, Mod. Phys. Lett. A\textbf{6} (1991) 573.

[10] M.Fukuma, H.Kawai and R.Nakyama, Int. J. Mod. Phys. A\textbf{6} (1991) 1385;
R.Dijkgraaf, E.Verlinde and H.Verlinde, Nucl. Phys. B\textbf{348} (1990) 435.

[11] E.Martinec, G.Moore and N.Seiberg, Phys. Lett. B\textbf{263} (1991) 190.

[12] D.Bessis, C.Itzykson and J-B.Zuber, Adv. Appl. Math. \textbf{1} (1980) 109.

[13] S.Wadia, Phys. Rev. D\textbf{24} (1981) 970;
A.Migdal, Phys. Rep. C\textbf{102} (1983) 199;
V.Kazakov, Mod. Phys. Lett. A\textbf{4} (1989) 2125.

[14] E.Brézin, C.Itzykson, G.Parisi and J-B.Zuber, Comm. Math. Phys. \textbf{59} (1978) 35.

[15] G.Moore, N.Seiberg and M.Staudacher, Nucl. Phys. B\textbf{362} (1991) 665;
N.Seiberg, in proceedings of workshop on Random Surfaces, Quantum Gravity and Strings, Cargèse 28 May - 1 June 1990.
[16] M.Douglas, Phys. Lett. B238 (1990) 176.

[17] D.Gross and N.Miljkovic, Phys. Lett. B238 (1990) 217;
     G.Parisi, Phys. Lett. B238 (1990) 209;
     P.Ginsparg and J.Zinn-Justin, Phys. Lett. B240 (1990) 333;
     E.Brézin, V.Kazakov and A.Zamolodchikov, Nucl. Phys. B333 (1990) 673.
     G.Moore, Rutgers preprint RU-91-12.

[18] G.Moore, M.Ronen Plesser and S.Ramgoolan, Yale preprint YCTP-P35-91.