Semiquandles and flat virtual knots

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Abstract

We introduce an algebraic structure we call semiquandles whose axioms are derived from flat Reidemeister moves. Finite semiquandles have associated counting invariants and enhanced invariants defined for flat virtual knots and links. We also introduce singular semiquandles and virtual singular semiquandles which define invariants of flat singular virtual knots and links. As an application, we use semiquandle invariants to compare two Vassiliev invariants.

Keywords: Flat knots and links, virtual knots and links, singular knots and links, semiquandles, Vassiliev invariants

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1 Introduction

Recent works such as [10] take a combinatorial approach to knot theory in which knots and links are regarded as equivalence classes of knot and link diagrams. New types of combinatorial knots and links can then be defined by introducing new types of crossings and Reidemeister-style moves that govern their interactions. These new combinatorial classes of knots and links have various topological and geometric interpretations relating to simple closed curves in 3-manifolds, rigid vertex isotopy of graphs, etc.

A flat crossing is a classical crossing in which we ignore the over/under information. A flat knot or link is a planar projection or shadow of a knot or link on the surface on which the knot or link diagram is drawn. Every classical knot diagram may be regarded as a decorated or lift of a flat knot, and conversely every classical knot diagram has a corresponding flat shadow.

At first glance, flat knots might seem uninteresting since flattening classical crossings apparently throws away the information which defines knotting. However, a little thought reveals potential applications of flat crossings: invariants of links with classical intercomponent crossings and flat intracomponent crossings are related to link homotopy and Milnor invariants of ordinary classical links, for example.

Another place where flat crossings prove useful is in virtual knot theory. Every purely flat knot is trivial, i.e., reducible by flat Reidemeister moves to the unknot. However, flat virtual knots and links (i.e., diagrams with virtual and flat crossings) are generally non-trivial. Non-triviality of a flat virtual says that no choice of classical crossing information for the flat crossings yields a classical knot. Hence, flat crossings are useful in the study of non-classicality in virtual knots.

A singular crossing is a crossing where two strands are fused together. Singular knots and links may be understood as rigid vertex isotopy classes of knotted and linked graphs, and they play a role in the study of Vassiliev invariants of classical knots and links.

In this paper, we define an algebraic structure we call a semiquandle which yields counting invariants for flat virtual knots. The paper is organized as follows. In section 2 we define flat, singular and virtual knots and links. In section 3 we define semiquandles and give some examples. In section 4 we define singular semiquandles by including operations at singular crossings. In section 5 we define virtual semiquandles and virtual singular semiquandles by including an operation at virtual crossings. In section 6 we give examples to show that the counting invariants with respect to finite semiquandles can distinguish flat virtual knots and links. In section 7 we give an application
to computing Vassiliev invariants of virtual knots. In section 8 we collect some questions for further research.

2 Flat knots, virtual knots and singular knots

Let us introduce several types of knots we will discuss in this paper. We assume that all knots are oriented unless otherwise specified. The simplest type of knot of those we consider, a flat knot, is an immersion of $S^1$ in $\mathbb{R}^2$. Alternatively, a flat knot can be described as an equivalence class of knot diagrams where under/over strand information at each crossing is unspecified. The equivalence relation is given by flat versions of the Reidemeister moves. Here, we illustrate the flat Reidemeister moves.

It is an easy exercise to show that any flat knot is related to the trivial flat knot (i.e. the flat knot with no crossings) by a sequence of flat Reidemeister moves. While the theory of flat knots appears uninteresting, if we consider the analogous theory of flat virtual knots, we enter a highly non-trivial category.

A flat virtual knot is a decorated immersion of $S^1$ in $\mathbb{R}^2$, where each crossing is decorated to indicate that it is either flat or virtual. (Virtual crossings are pictured by an encircled flat crossing.) Once again, we may also describe a flat virtual knot as an equivalence class of virtual knot diagrams where under/over strand information at each classical crossing is unspecified. The corresponding equivalence relation is given by the flat versions of the virtual Reidemeister moves in addition to the flat versions of the ordinary Reidemeister moves.

Note that the following move is forbidden.
As with ordinary virtual knots, flat virtual knot diagrams have a geometric interpretation as flat knot diagrams on surfaces. In this case, the virtual crossings are interpreted as artifacts of a projection of the knot diagram on the surface to a knot diagram in the plane [10].

Finally, we’d like to consider flat knots and flat virtual knots that have singularities. These singularities should be thought of as rigid vertices, or places where the knot is actually glued to itself. Thus, flat singular knots are simply equivalence classes of flat knots where some crossings are decorated to indicate that they are singular. The Reidemeister moves corresponding to flat singular equivalence are the ordinary flat equivalence moves together with the following two moves.

Similarly, flat virtual singular knots are equivalence classes of flat virtual knots where some of the crossings may be designated as singular. Hence, there are three types of crossings that may be contained in a diagram of a flat virtual singular knot. The equivalence relation is given by all of the previous flat, virtual, and singular moves together with the following move.

We call the simplest non-trivial flat virtual singular knot that contains all three types of crossings the Triple Crazy Trefoil. This is the knot pictured below.

3 Semiquandles

A number of algebraic structures have been defined in recent years with axioms derived from variations on the Reidemeister moves. The earliest of these is the quandle (see [9, 13]) in which we have generators corresponding to arcs in a link diagram and an invertible binary operation at crossings.
Subsequent papers have generalized this idea in various ways. In \[5\], ambient isotopy is replaced with framed isotopy to define \textit{racks}. In \[12, 6\], arcs in an oriented knot diagram are replaced with \textit{semiarcs} to define \textit{biquandles}. In \[11\], an operation at virtual crossings is included in the biquandle definition to yield \textit{virtual biquandles}.

\textbf{Definition 1} A \textit{semiquandle} is a set \(X\) with two binary operations \((x, y) \mapsto x^y, x_y\) such that for all \(x, y, z \in X\) we have

\begin{enumerate}[label=(\roman*)]
  \item for all \(x, y \in X\) there are unique \(w, z \in X\) with \(x = w^y\) and \(x = z_y\),
  \item \(x^y = y^x\) iff \(y^x = x\),
  \item \((x^y)^{y^x} = x\) and \((x^y)_{(y^x)} = x\), and
  \item \((x^y)^z = (x^{z^y})^{y^z}, (y_x)^{z^y} = (y^z)_{x^{z^y}}\), and \((z_x)_{y^z} = (z^y)_x\).
\end{enumerate}

Axiom (0) says the actions \(x \mapsto x^y\) and \(x \mapsto x_y\) are invertible. The unique \(z, w\) in axiom (0) will be denoted \(z = x^y x^{-1}\) and \(w = x y^x\). These axioms come from dividing an oriented flat knot into semiarcs, i.e. edges between vertices in the flat diagram regarded as a graph, and then translating the flat Reidemeister moves into algebraic axioms.

\begin{figure}
  \includegraphics[width=0.5\textwidth]{semiquandle_diagram.png}
  \caption{Diagram illustrating semiquandle axioms.}
\end{figure}

In the first Reidemeister move, right-invertibility guarantees the uniqueness of \(y\) given \(x\), and the relationship between \(x\) and \(y\) becomes axiom (i).

\begin{figure}
  \includegraphics[width=0.5\textwidth]{reidemeister_diagram.png}
  \caption{Diagram illustrating the first Reidemeister move.}
\end{figure}

The direct II move, in which both strands are oriented in the same direction, give us axiom (ii). Given axiom (0), the reverse II move yields the same relationship between \(x\) and \(y\) where the uncrossed strands are labeled \(x\) and \(y^x\).
Reidemeister move III yields the three equations in axiom (iii).

**Definition 2** For any flat virtual link $L$, the fundamental semiquandle $FSQ(L)$ of $L$ is the set of equivalence classes of semiquandle words in a set of generators corresponding to semiarcs in a diagram $D$ of $L$, i.e. edges in the graph obtained from $D$ by considering flat crossings as vertices, under the equivalence relation generated by the semiquandle axioms and the relations at the crossings. As with the knot quandle, fundamental rack and knot biquandle, we can express the fundamental semiquandle with a presentation read from a diagram.

**Example 1** The pictured flat Kishino knot has the listed fundamental semiquandle presentation.

\[ FSQ(K) = \langle a, b, c, d, e, f, g, h \mid a^c = b, c^a = d, b^d = e, d_b = c, e^g = f, g^e = h, f^h = g, h_f = a \rangle \]

**Remark 2** An alternative definition for the fundamental semiquandle of a flat virtual knot is that $FSQ(L)$ is the quotient of the (strong) knot biquandle of any lift of $L$ (i.e., choice of classical crossing type for the flat crossings of $L$) under the equivalence relation generated by setting $a \sim a$ and $b \sim b$ for all $a, b \in B(L)$. Indeed, this operation yields a “flattening” functor $SQ : \mathcal{B} \rightarrow \mathcal{S}$ from the category of strong biquandles to the category of semiquandles.

**Example 3** For any set $X$ and bijection $\sigma : X \rightarrow X$, the operations $x^y = \sigma(x)$ and $x_y = \sigma^{-1}(x)$ define a semiquandle structure on $X$. We call this type of semiquandle a constant action semiquandle since the actions of $y$ on $x$ is constant as $y$ varies.

As is the case with quandles and biquandles (see [8, 15]), for a finite semiquandle $X = \{x_1, \ldots, x_n\}$ we can conveniently express the semiquandle structure with a block matrix $M_X = [U|L]$ where $U_{i,j} = k$ and $L_{i,j} = l$ for $x_k = (x_i)^{(x_j)}$ and $x_l = (x_i)^{(x_j)}$. This matrix notation enables us to do computations with semiquandles without the need for formulas for $x^y$ and $x_y$.

**Example 4** The constant action semiquandle on $X = \{1, 2, 3\}$ with $\sigma = (132)$ has semiquandle matrix

\[
M_X = \begin{bmatrix}
3 & 3 & 3 & 2 & 2 & 2 \\
1 & 1 & 1 & 3 & 3 & 3 \\
2 & 2 & 2 & 1 & 1 & 1
\end{bmatrix}.
\]

**Example 5** Any (strong) biquandle in which $a^a = a_b$ and $a^a = a^b$ is a semiquandle. Indeed, an alternative name for semiquandles might be symmetric biquandles. An example of a non-constant action semiquandle found in [15] is

\[
M_T = \begin{bmatrix}
1 & 4 & 2 & 3 & 1 & 3 & 4 & 2 \\
2 & 3 & 1 & 4 & 3 & 1 & 2 & 4 \\
4 & 1 & 3 & 2 & 2 & 4 & 3 & 1 \\
3 & 2 & 4 & 1 & 4 & 2 & 1 & 3
\end{bmatrix}.
\]
4 Singular semiquandles

Let us now consider what happens to our algebraic structure when we allow singular crossings in an oriented flat virtual knot. As with flat crossings, we define two binary operations at a singular crossing. One notable difference is that unlike flat crossings, singular crossings are permanent – there are no moves which either introduce or remove singular crossings. Indeed, the number of singular crossings is an invariant of singular knot type. In particular, we do not need right-invertibility for our operations at singular crossings.

Definition 3 Let $X$ be a semiquandle. A singular semiquandle structure on $X$ is a pair of binary operations on $X$ denoted $(x, y) \rightarrow x^y, x_y$ satisfying for all $x, y, z \in X$

(i) $(y^x)^y = (y^x)^x$ and $(z^y)^{(y^z)} = (z^y)^{(y^z)}$

(ii) $(x^y)^z = (x^y)^{z^x}$ and $(y^x)^{z^x} = (y^x)^{z^x}$

We call axioms (i) and (ii) the hat axioms for the obvious reason. These axioms come from the subset of the oriented singular flat Reidemeister moves pictured below.

To see that the two pictured oriented singular moves are sufficient to give us all of the oriented flat singular moves, we note the following key lemmas.

Lemma 1 The move follows from the flat Reidemeister moves and the two pictured moves.

Proof.
Similar move sequences yield the other oriented flat/singular type III moves.

**Lemma 2** The reverse oriented singular II move follows from the flat moves and the moves pictured above.

**Proof.**
Starting with one side of a reverse singular II move, we can use flat moves to get a symmetrical diagram in which we can apply a direct singular II move.

Reversing the process gives the other side of the reverse singular II move.

**Example 6** Let \( X \) be a semiquandle. Then clearly setting \( x\hat{y} = x^y \) and \( x\tilde{y} = x_y \) for all \( x, y \in X \) defines a compatible singular structure, which we call the flat singular structure, \((X, X)\).

**Example 7** Let \( X \) be a semiquandle. Then \( a\hat{b} = a\tilde{b} = b \) is a compatible singular structure, since we have

\[
(y_x)(\hat{z} y) = x^y = (y_x)(\tilde{z} y), \quad (y_x)(y\hat{z}) = y_x = (x\hat{y})(y\tilde{z})
\]

\[
(x^y)\hat{y} = z = (z_y)\hat{y} = (x\tilde{z} y)^y, \quad (y_x)\tilde{z}^y = (y_x)x^y = y = (y^z)\tilde{y} = (y^z)x\hat{y}
\]

and

\[
(y\tilde{y})y_x = (x^y)y_x = x = (z_y)\hat{y}.
\]

Let us call this singular structure the operator singular structure on \( X \), denoted \((X, O)\).

As with the flat virtual case, for any flat singular virtual link \( L \) there is an associated fundamental singular semiquandle \( \text{FSSQ}(L) \) with presentation readable from the diagram. Elements of \( \text{FSSQ}(L) \) are equivalence classes of singular semiquandle words in generators corresponding to semiarcs in the diagram (here we divide the diagram at both flat and singular crossing points, but not at virtual crossings) under the equivalence relation generated by the axioms (0), (i), (ii), (iii), (hi) and (hii).

**Example 8** The *triple crazy trefoil* pictured below has the listed fundamental singular semiquandle presentation.

\[
\langle a, b, c, d \mid a\hat{c} = b, \ c\hat{a} = d, \ d\hat{a} = a, \ b\hat{d} = c\rangle.
\]

As with the semiquandle structure, we can represent the singular operations in a finite singular semiquandle with matrices encoding the operation tables. Indeed, it seems convenient to combine these matrices with the semiquandle operation matrices into a single block matrix of the form

\[
M_T = \begin{bmatrix}
  i^j & i^j \\
  i^j & i^j
\end{bmatrix}.
\]
Example 9 The constant action semiquandle $X = \{1, 2, 3\}$ with $\sigma = (132)$ and operator singular structure $a^b = a_2 = b$ has block matrix

$$M_{(X,O)} = \begin{bmatrix}
3 & 3 & 3 & 2 & 2 \\
1 & 1 & 1 & 3 & 3 \\
2 & 2 & 2 & 1 & 1 \\
1 & 2 & 3 & 1 & 2 \\
1 & 2 & 3 & 1 & 2 \\
\end{bmatrix}.$$ 

5 Virtual semiquandles and virtual singular semiquandles

As with singular crossings, we can further generalize semiquandles by adding an operation at virtual crossings. The simplest way to do this is to use a unary operation at each virtual crossing defined by applying a bijection $v$ when going through a virtual crossing from right to left when looking in the direction of the strand being crossed and applying $v^{-1}$ when going through a virtual crossing from left to right when looking in the direction of the strand being crossed.

As noted in [11], this setup ensures that the virtual I, II and III moves are respected by the virtual operation.

The interaction of the virtual crossings with the flat and singular crossings given by the Reidemeister moves tell us how the virtual operation should interact with the semiquandle and singular semiquandle structures – namely, $v$ must be an automorphism of both structures.

Definition 4 A virtual semiquandle is a semiquandle $S$ with a choice of automorphism $v : S \rightarrow S$. A virtual singular semiquandle is a singular semiquandle with a semiquandle automorphism $v : S \rightarrow S$ which is also an automorphism of the singular structure. That is, $v : S \rightarrow S$ is a bijection satisfying

$$v(x^y) = v(x)^v(y), \quad v(x_y) = v(x)_{v(y)}, \quad v(x^b) = v(x)^v(y), \quad \text{and} \quad v(x_2) = v(x)_{v(y)}.$$
Example 10 Every semiquandle is a virtual semiquandle with $v = \text{Id}_S$. More generally, the set of virtual semiquandle structures on a semiquandle $S$ corresponds to the set of conjugacy classes in the automorphism group $\text{Aut}(S)$ of the semiquandle $S$: let $v, v', \phi \in \text{Aut}(S)$ with $v' = \phi^{-1}v\phi$. Then $\phi(S,v) \rightarrow (S,v')$ is an isomorphism of virtual semiquandles.

Every flat singular virtual knot or link has a fundamental virtual singular semiquandle obtained by dividing the knot or link into semiarcs at flat, singular and virtual crossings; then $VFSSQ(L)$ has generators corresponding to semiarcs and relations at the crossings as determined by crossing type in addition to relations coming from the virtual singular semiquandle axioms.

6 Counting invariants of flat singular virtuals

As with finite groups, quandles and biquandles, finite semiquandles can be used to define computable invariants of flat virtual knots and links by counting homomorphisms.

Definition 5 Let $L$ be a flat virtual link and $T$ a finite semiquandle. The semiquandle counting invariant of $L$ with respect to $T$ is the cardinality

$$sc(L,T) = |\text{Hom}(FSQ(L),T)|$$

of the set of semiquandle homomorphisms $f : FSQ(L) \rightarrow T$ from the fundamental semiquandle of $L$ to $T$ (i.e., maps such that $f(x\cdot y) = f(x)\cdot f(y)$ and $f(x/y) = f(x)/f(y)$ for all $x, y \in FSQ(L)$).

Remark 11 A semiquandle homomorphism $f : FSQ(L) \rightarrow T$ can be pictured as a “coloring” of a diagram $D$ of $L$ by $T$, i.e., an assignment of an element of $T$ to each semiarc in $D$ such that the colors satisfy the semiquandle operation conditions at every crossing.

Example 12 The semiquandle counting invariant with respect to the semiquandle $T$ in example 5 distinguishes the flat Kishino knot $FK$ from from the flat unknot $FU$ with $sc(FK,T) = 16$ and $sc(FU,T) = 4$. This same semiquandle also distinguishes the flat virtual knot $K$ from below from both the unknot and the flat Kishino, with $sc(K,T) = 2$.

\[
\includegraphics[width=0.2\textwidth]{knot.png}
\]

We can enhance the semiquandle counting invariant by taking note of the cardinality of the image subsemiquandles $\text{Im}(f)$ for each homomorphism to obtain a multiset-valued invariant, which we can also express in a polynomial form by converting multiset elements to exponents of a dummy variable $z$ and multiplicities to coefficients. Note that specializing $z = 1$ in the enhanced invariant yields the original counting invariant.

Definition 6 Let $L$ be a flat virtual link and $T$ a finite semiquandle. The enhanced semiquandle counting multiset is the multiset

$$sqcm(L,T) = \{\text{Im}(f) \mid f \in \text{Hom}(FSQ(L),T)\}$$

and the enhanced semiquandle polynomial is

$$sqp(L,T) = \sum_{f \in \text{Hom}(FSQ(L),T)} z^{\text{Im}(f)}.$$
For singular semiquandles, we also have counting invariants and polynomial enhanced invariants.

**Definition 7** Let $L$ be a flat singular virtual link and $(T, S)$ a finite singular semiquandle. Then we have the singular semiquandle counting invariant

$$ssc(L, (T, S)) = |\text{Hom}(FSSQ(L), (T, S))|,$$

the enhanced singular semiquandle counting multiset

$$ssqcm(L, (T, S)) = \{\text{Im}(f) \mid f \in \text{Hom}(FSSQ(L), (T, S))\},$$

and the enhanced singular semiquandle polynomial

$$ssq(L, (T, S)) = \sum_{f \in \text{Hom}(FSSQ(L), (T, S))} 2^{\text{Im}(f)}.$$

**Example 13** The constant action semiquandle $(X_{(132)}, O)$ with operator singular structure distinguishes the triple crazy trefoil $TCT$ from the singular knot with one singular crossing and no other crossings $SU_1$: 

$$ssq(TCT, (X_{(132)}, O)) = 0 \quad ssq(SU_1, (X_{(132)}, O)) = 9z^3$$

Finally, we have counting invariants for flat singular virtual knots and links defined analogously using finite virtual singular semiquandles.

**Definition 8** Let $L$ be a flat singular virtual link and $(T, S, v)$ a finite virtual singular semiquandle. Then we have the virtual singular semiquandle counting invariant

$$vssc(L, (T, S, v)) = |\text{Hom}(FVSSQ(L), (T, S, v))|,$$

the enhanced virtual singular semiquandle counting multiset

$$vssqcm(L, (T, S, v)) = \{\text{Im}(f) \mid f \in \text{Hom}(FVSSQ(L), (T, S, v))\},$$

and the enhanced virtual singular semiquandle polynomial

$$vssq(L, (T, S, v)) = \sum_{f \in \text{Hom}(FVSSQ(L), (T, S, v))} 2^{\text{Im}(f)}.$$

**Example 14** The flat virtual Hopf link $fH$ below is distinguished from the flat unlink of two components by the counting invariants with respect to the listed virtual semiquandle. Note that we can regard $T$ as a flat singular virtual semiquandle with trivial singular operations $x^{\bar{y}} = x \bar{x}$. 

$$M_{T, S} = \begin{bmatrix} 1 & 3 & 1 & 1 & 3 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 3 & 1 \end{bmatrix}, \quad v = (13)$$

$$vssq(fvH, (T, S, v)) = q + 4z^2 \quad vssq(U_2, (T, S, v)) = q + 4z^2 + 4z^3$$
Remark 15 We note that a virtual semiquandle is a virtual singular semiquandle with trivial singular structure, i.e. \( x^y = x^y = x \), a singular semiquandle is a virtual singular semiquandle with trivial virtual operation, i.e. \( v = \text{Id} \), and a semiquandle is a virtual singular semiquandle with trivial virtual and singular structures.

7 Application to Vassiliev invariants

In [7], we find several degree one Vassiliev invariants for virtual knots. One invariant, \( S \), takes its values in the free abelian group on the set of two-component flat virtual links. Another invariant, \( G \), takes its values in the free abelian group on the set of flat virtual singular knots with one singularity. It is easy to show that \( G \) is at least as strong as \( S \), but somewhat difficult to show that \( G \) is strictly stronger than \( S \). Here, we give the definitions of these invariants and provide an alternative proof that \( G \) is strictly stronger than \( S \).

Definition 9 Let \( K \) be a virtual knot with diagram \( \tilde{K} \). Let \( \tilde{K}^d_{\text{smooth}} \) be the flat virtual link obtained by smoothing \( \tilde{K} \) at the crossing \( d \) and projecting onto the associated flat virtual link. Furthermore, let \( \tilde{K}^0_{\text{link}} \) be the flat virtual link obtained by taking the flat projection of \( \tilde{K} \) disjoint union with the unknot. If \( [L] \) represents the generator of the free abelian group on the set of two-component flat virtual links associated to the link \( L \), then

\[
S(K) = \sum_d \text{sign}(d)([\tilde{K}^d_{\text{smooth}}] - [\tilde{K}^0_{\text{link}}]).
\]

Here, the sum ranges over all classical crossings in \( \tilde{K} \), and \( \text{sign}(d) \) is the local writhe.

Since this “smoothing” invariant has values involving flat virtual links, it is clear that semiquandles may be of use in computing \( S \) for pairs of virtual knots. Moreover, singular semiquandles can be put to use when computing the following invariant.

Definition 10 Let \( K \) be a virtual knot with diagram \( \tilde{K} \). Let \( \tilde{K}^d_{\text{glue}} \) be the flat virtual singular knot obtained by gluing \( \tilde{K} \) at the crossing \( d \) and projecting onto the associated flat virtual singular knot. Let \( \tilde{K}^0_{\text{sing}} \) be the flat virtual singular knot obtained by taking the flat projection of \( \tilde{K} \), introducing a kink via the flat Reidemeister 1 move, and gluing at the resulting crossing. If \( [L] \) represents the generator of the free abelian group on the set of flat virtual singular knots associated to the knot \( L \), then

\[
G(K) = \sum_d \text{sign}(d)([\tilde{K}^d_{\text{glue}}] - [\tilde{K}^0_{\text{sing}}]).
\]

Here again, the sum ranges over all classical crossings in \( \tilde{K} \), and \( \text{sign}(d) \) is the local writhe.

It is proven in [7] that both \( S \) and \( G \) are degree one Vassiliev invariants and \( G \) is at least as strong as \( S \). To show that \( G \) is stronger than \( S \), consider the following pair of virtual knots.
Let us call the first of these knots \( K_1 \) and the second \( K_2 \). Since the only difference between the two knots is the signs of the crossings labelled \( a \) and \( b \), we see that

\[
S(K_2) - S(K_1) = 2(\tilde{K}_a^{\text{smooth}} - \tilde{K}_b^{\text{smooth}})
\]

and

\[
G(K_2) - G(K_1) = 2(\tilde{K}_a^{\text{glue}} - \tilde{K}_b^{\text{glue}}).
\]

Now \( \tilde{K}_a^{\text{smooth}} \) is the same as \( \tilde{K}_b^{\text{smooth}} \). They are both the flat virtual link pictured below.

![Image](image1.png)

It follows that \( S(K_1) = S(K_2) \). On the other hand, we can show using singular semiquandles that \( \tilde{K}_a^{\text{glue}} \) and \( \tilde{K}_b^{\text{glue}} \), as pictured below, are distinct.

![Image](image2.png)

Consider the following singular semiquandle, \( T \), given in terms of its matrix \( M \).

\[
M = \begin{bmatrix}
1 & 4 & 2 & 3 & 1 & 3 & 4 & 2 \\
2 & 3 & 1 & 4 & 3 & 1 & 2 & 4 \\
4 & 1 & 3 & 2 & 2 & 4 & 3 & 1 \\
3 & 2 & 4 & 1 & 4 & 2 & 1 & 3 \\
1 & 1 & 4 & 4 & 1 & 2 & 2 & 1 \\
1 & 1 & 4 & 4 & 4 & 3 & 3 & 4 \\
2 & 2 & 3 & 3 & 4 & 3 & 3 & 4 \\
2 & 2 & 3 & 3 & 1 & 2 & 2 & 1
\end{bmatrix}
\]

The enhanced singular semiquandle polynomial for \( \tilde{K}_a^{\text{glue}} \) is \( \text{ssqp}(\tilde{K}_a^{\text{glue}}, T) = 2z \) while the polynomial for \( \tilde{K}_b^{\text{glue}} \) is \( \text{ssqp}(\tilde{K}_b^{\text{glue}}, T) = 2z + 2z^4 \). Hence, the two flat virtual singular knots are distinct and, thus, \( G(K_1) \neq G(K_2) \).

8 Questions

In this section, we collect questions for future research.

Singular semiquandles bear a certain resemblance to virtual biquandles, in which a biquandle is augmented with operations at virtual crossings. Given a biquandle \( B \), the set of virtual biquandle
structures on $B$ forms a group isomorphic to the automorphism group of $B$. What is the structure of the set of singular semiquandle structures on a semiquandle $X$?

Our algebra-agnostic approach to computation of our various semiquandle-based invariants works well for small-cardinality semiquandles and link diagrams with small crossing numbers. However, for links with higher crossing numbers and larger coloring semiquandles we will need more algebraic descriptions. We have given a few examples of classes of semiquandle structures, e.g. constant action semiquandles and operator singular structures. What are some examples of group-based or module-based semiquandle and singular semiquandle structures akin to Alexander biquandles? (Note that the only Alexander biquandles which are semiquandles are constant action Alexander biquandles).

Enhancement techniques for biquandle counting invariants which should extend to semiquandles include \textit{semiquandle cohomology} which is the special case of Yang-Baxter cohomology described in \cite{CarterElhamdadiSaito2004} and the flattened case of $S$-cohomology as described in \cite{CenicerosNelson2007}. Similarly, we might define \textit{semiquandle polynomials} and the resulting enhancements of the counting invariants as in \cite{Kauffman2008}. What other enhancements of semiquandle counting invariants are there?

What is the relationship, if any, between semiquandle invariants and quaternionic biquandle invariants described in \cite{BartholomewFenn2008}?

Our \texttt{python} code for computing semiquandle-based invariants is available from the second listed author’s website at \url{www.esotericka.org}.

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