An Introduction to the Quantum Supermembrane

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Abstract: We review aspects of quantisation of the 11-dimensional supermembrane world volume theory. We explicitly construct vertex operators for the massless states and study interactions of supermembranes. The open supermembrane and its vertex operators are discussed. We show how our results have direct applications to Matrix theory by appropriate regularisation of the supermembrane.

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1 Introduction

As we review the progress over the last few decades we find that there are unexplored avenues in the 11 dimensional supermembrane theory which can lead to a better understanding of quantum M-theory and hence non-perturbative string/quantum gravity. The supermembrane is a 2+1 dimensional object moving in 11 dimensional space, with a world volume theory \[1\] which when quantised will give us a glimpse of the fundamental degrees of freedom of M-theory. Attempts to quantise the world volume theory in analogy to 10 dimensional world sheet string theory have revealed many interesting features which distinguish it from the 10 dimensional string theory, and also make it difficult to solve. The main source of difficulties lies in the fact that, unlike the string, the 2+1 dimensional world volume theory is interacting. Moreover it does not have conformal invariance. Certain features of the supermembrane are understood, namely, it can be regularized to yield a supersymmetric Matrix theory \[2\], and it’s spectrum is continuous \[3, 4\]. These lead to a multiparticle interpretation of the spectrum \[5\], and the need for a second quantised description of the membrane. We still do not have a complete understanding of this, nor is the existence of a normalisable ground state confirmed. However, there is evidence from Matrix theory that such a state indeed exists and it contains massless states corresponding to the massless multiplet of 11 dimensional supergravity\[6\].

Given this situation, we can look for further quantities to describe the supermembrane which will hopefully shed some more light on the underlying quantum theory. The interactions of the massless sector is one interesting avenue to explore. There are computations of \(d = 11\) scattering amplitudes for the massless sector \[7, 8, 9\] using the quantised superparticle. However the calculated amplitudes diverge, and the coefficients can be fixed only by duality with 10 dimensional string theory. As we know, in the case of the superstring \[10\] as well as the superparticle \[7, 9\] scattering amplitudes are evaluated by determining the vertex operators and inserting them into path-integral amplitudes. The vertex operators for the supermembrane derived in \[11\], precisely seek to achieve the same for a supermembrane scattering amplitude \[12, 13\]. The operators are determined uniquely and provide the first step towards understanding massless interactions. Both the superstring and the superparticle vertex operators are contained in the supermembrane vertex operators. By reducing the two spatial dimensions of the supermembrane world volume to zero, the superparticle is obtained. (This we call
the particle limit). On the other hand, wrapping one of the directions of the membrane along a compact direction gives us a superstring in \( d = 10 \). This ‘double dimensional reduction (DDR)’ of the \( d = 11 \) supermembrane vertex operators gives us all the superstring vertex operators. Also, by virtue of the fact that Matrix theory is recovered after appropriate regularisation, these vertex operators give linearised Matrix action in weak \( d = 11 \) supergravity background.

We also discuss the vertex operators for the open membrane as it is closely related to heterotic string theory \([14]\) and in recent times has assumed importance due to its relation to non-commutative theories in presence of a background constant three form gauge field strength \([13]\). Many other topics regarding the quantum supermembrane have been discussed in detail in previous reviews \([16, 17]\).

The first section is an introduction to supermembrane basics, mainly to fix notations. We also review the relation of supermembrane to Matrix theory, and then the multiparticle interpretation of the supermembrane spectrum. In the second section we give the vertex operators for the massless sector of the theory. The third section deals with scattering amplitude calculations with the help of vertex operators.
2 Supermembranes

In analogy with the particle and the string, the action for the membrane is given by the 2+1 dimensional world volume swept out in the target space. The supermembrane is an extension obtained by adding fermionic target space ‘coordinates’. A supersymmetric action involving the bosonic and the fermionic coordinates can be written down consistently only in special dimensions, of which d=11 is the maximal value. This action in the background of \( d = 11 \) supergravity takes the form

\[
S = T \int d^3 \xi \left[ -\frac{1}{2} \sqrt{-g} g^{ij} \Pi_a \Pi_b \eta_{ab} + \frac{1}{2} \sqrt{-g} - \epsilon^{ijk} \Pi_i A \Pi_j B \Pi_k C B_{CBA} \right]. \tag{1}
\]

Superspace notation has been used where space comprises of both the bosonic and fermionic coordinates. Explicitly

\[
\Pi_i^A = \partial_i Z^M E_M^A \quad A = (a, \alpha); \quad a = 0, \ldots, 10 \quad \alpha = 1, \ldots, 32
\]

\[
Z^M = (X^\mu, \theta^\alpha) \quad M = (\mu, \dot{\alpha}) \quad \mu = 0, \ldots, 10 \quad \dot{\alpha} = 1, \ldots, 32
\]

where \( Z^M \) are the coordinates of the membrane in the target superspace, \( E_M^A \) is the super-elfbein, and \( \xi^i, (i = 0, 1, 2) \) denote the 2+1 dimensional world volume coordinates. Tangent space indices are denoted by \( A \) and world indices by \( M \). \( g_{ij} \) is the world volume metric \( (g = \det g_{ij}) \) which is considered independent here, and \( B_{CBA} \), the three form of \( d = 11 \) superspace. \( T \) is the tension of the supermembrane, and it is the sole length scale in the theory. Unlike the superstring coupling constant \( g_s \) there is no analogous supermembrane coupling constant. This precludes a perturbative expansion of supermembrane interactions making the supermembrane theory non-perturbative from the very outset. We shall set \( T = 1 \) for convenience henceforth.

The action has redundant degrees of freedom due to the presence of the local reparametrisation freedom of the world volume and fermionic \( \kappa \) symmetry of the action [3]. The \( \kappa \) symmetry of the action requires one crucial fact: the component fields of the graviton \( h_{\mu \nu} \), gravitino \( \Psi_{\mu} \) and the three form \( C_{\mu \nu \rho} \) obey \( d = 11 \) supergravity equations of motion [22]. This is similar to the GS formulation of the superstring, but quite different from the NSR superstring, where one has to resort to \( \beta \) function evaluation to show that the background fields obey the supergravity equations of motion. However,\(^2\)

\(^2\)There are other formulations of the action using twistors [20].
one aspect of the action (1) is that though it gives the coupling of the supermembrane to superspace background fields, it is very difficult to extract the coupling in terms of the component fields (i.e. $h_{\mu\nu}, \Psi_\mu, C_{\mu\nu\rho}$). The method of gauge completion used in [19], has succeeded in determining the couplings only to order $\theta^3$ of the target space fermions. There are other methods of evaluating the action in components as in [21], but, as we show in the next section to linear order in the fields, the couplings can be determined exactly, through the evaluation of vertex operators.

To begin with, one restricts oneself to quantising the supermembrane propagating in flat target space. The coordinates in $d = 11$ flat space are $Z^M = (X^\mu, \theta^\alpha)$, $\theta$ is a Majorana spinor. The superelfbein can be determined [1], and we have

\[
\Pi^i_\mu = \partial_i X^\mu - i \bar{\theta} \Gamma^\mu \partial_i \theta, \quad \Pi^\alpha_i = \partial_i \theta^\alpha,
\]

\[
B_{\mu\nu\rho} = 0, \quad B_{\mu\nu\alpha} = -\frac{i}{6} (\Gamma^\mu \theta)_{(\alpha} (\Gamma^\nu \theta)_{\beta)}, \quad B_{\mu\alpha\beta} = -\frac{1}{6} (\Gamma^\mu \theta)_{(\alpha} (\Gamma^\nu \theta)_{\beta} (\Gamma^\rho \theta)_{\gamma)}.
\]

In addition, we use the equation of motion for the world volume metric $g_{ij} = \eta_{ab} \Pi^a_i \Pi^b_j$. Plugging this into (1) gives the supermembrane in flat space

\[
S = -\int d^3 \xi \left\{ \sqrt{-g(X, \theta)} - \epsilon^{ijk} \bar{\theta} \Gamma_{\mu\nu} \partial_i \theta \right\}
\]

\[
\left[ \frac{1}{2} \partial_j X^\mu \left( \partial_k X^\nu + \bar{\theta} \Gamma^\nu \partial_k \theta + \frac{1}{6} \bar{\theta} \Gamma^\mu \partial_j \theta \Gamma^\nu \partial_k \theta \right) \right].
\]

The reparametrisation of the world volume can be used to fix the bosonic degrees of freedom to eight. (E.g. one can go to static gauge by identifying $X^{0,1,2} = \xi^{0,1,2}$ leaving 8 transverse degrees). The $\kappa$ symmetry of the action is then used to reduce the number of fermionic degrees of freedom. The most efficient way of dealing with the above action and its symmetries however is to go to the lightcone gauge [1, 2]. The light cone directions $X^{\pm} = \frac{1}{\sqrt{2}} (X^0 \pm X^{10})$ are singled out, and $X^-$ is identified with the time direction of the world volume, which we denote by $\tau$ here (hereafter, we denote the spatial directions $\xi^{1,2} \equiv \sigma^{1,2}$). The $\kappa$ symmetry also allows to set 1/2 of the fermionic degrees of freedom to zero. The gauge conditions thus read:

\[
X^+ = X^+_0 + \tau, \quad (4)
\]

\[
\Gamma^+ \theta = 0, \quad (5)
\]
where $\Gamma^\pm = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^{10})$. One is left with 9 transverse degrees of freedom and the fermionic coordinate $\theta$ is reduced to having 16 degrees of freedom in this gauge. In addition one has to solve for $X^-$ which gives an extra constraint reducing the total number of bosonic degrees of freedom to 8. The details of this derivation can be found in [1, 2].

The Hamiltonian density derived from the gauge fixed lagrangian has the following form ($P^+ = \delta L/\delta \partial_\tau X^-$; $\vec{P} = \delta L/\delta \partial_\tau \vec{X}$):

$$H = \frac{1}{2P^+} \left( \vec{P}^2 + \bar{g} \right) - \epsilon^{rs} \partial_r X^a \gamma^a \partial_s \theta$$

where $P^+$ can fixed as a constant of motion, $\vec{P} = \partial_\tau \vec{X}$, $\bar{g} = (\epsilon^{rs} \partial_r X \partial_s X)^2$ and $a, b, \ldots$ denote the transverse directions. We also use $SO(9)$ $\gamma$ matrices henceforth. It is noteworthy that the Hamiltonian does not depend on the centre of mass coordinates $\vec{X}_0$ and $\theta_0$. The supermembrane mass formula

$$M^2 = 2P^+ P^- - \vec{P}_0^2$$

is also independent of the centre of mass coordinates. The Hamiltonian is accompanied by a constraint (obtained by taking curl of the equation $\partial_r X^- = \partial_r X^a \partial_r X_a - i\theta \partial_r \theta$)

$$\phi = \epsilon^{rs} \left( \partial_r \vec{P} \partial_s \vec{X} - i\partial_r \theta \partial_s \theta \right) \approx 0$$

The potential in the Hamiltonian

$$\bar{g} = \det (\partial_r \vec{X} \partial_s \vec{X}) = (\epsilon^{rs} \partial_r X \partial_s X)^2$$

vanishes for configurations where one of the directions is independent of one of the membrane world volume coordinates. In other words there are valleys in the potential signalling degenerate surfaces. The membrane surface can be made arbitrarily narrow in one direction, and hence a single membrane can be deformed into arbitrary number of membranes without any cost of energy. This is interpreted as the non-existence of a definite membrane number, and no definite membrane topology. A path integral approach to membrane quantisation like in string theory (à la Polyakov), where there is a sum over different topologies for the world sheet, cannot be achieved.

The light cone Hamiltonian is still invariant under a class of diffeomorphisms, called area-preserving diffeomorphisms (APD), acting as [22, 9]:

$$\delta X^\mu = -\xi^r \partial_r X^\mu \quad \delta \theta = -\xi^r \partial_r \theta$$
with \( \xi^r = \epsilon^s \partial_s \xi \), such that \( \partial_s \xi^r = 0 \) (\( \xi \) is a scalar parameter), and hence area of the two surface is preserved under this transformation. The area preserving diffeomorphisms as defined above can be attributed with the following Lie bracket structure:

\[
\delta A = \{ \xi, A \} = \epsilon^r \partial_r \xi \partial_s A.
\] (11)

Thus \( \{ A, B \} = \epsilon^r \partial_r A \partial_s B \) defines a Lie bracket for any two functions \( A, B \).

It shares all the requisite properties of Lie bracket, namely antisymmetry, associativity and satisfies the Jacobi identity. The Hamiltonian density can thus be rewritten as:

\[
\mathcal{H} = \left( \vec{P} \cdot \vec{P} - \frac{1}{2} \left\{ X^a, X^b \right\}^2 + \theta \gamma^a \left\{ X^a, \theta \right\} \right)
\] (12)

In fact, with the above knowledge, one can start with a lagrangian invariant under APD, to yield (12) as the Hamiltonian. The APD invariant Lagrangian can be written in a compact form by introducing an auxiliary field \( \omega \), which transforms as a gauge field under APD transformations:

\[
\delta \omega = \partial_r \omega + \{ \xi, \omega \}
\] (13)

By defining a covariant derivative,

\[
DX^a = \partial_r X^a - \{ \omega, X^a \},
\]

the lagrangian becomes [2]:

\[
\mathcal{L} = \frac{1}{2} (DX^a)^2 - \frac{1}{4} \left\{ X^a, X^b \right\}^2 - i \theta \gamma^a \{ X^a, \theta \}
\] (14)

The lagrangian is also invariant under the target space supersymmetries, 16 of which are linearly realised with parameter \( \eta \), and due to the gauge fixing, another 16 of which are non-linearly realised with parameter \( \epsilon \). They are of the form:

\[
\delta X^a = -2 \epsilon \gamma^a \theta \quad \delta \omega = -2 \epsilon \theta
\]

\[
\delta \theta = i DX^a \gamma_a \epsilon - \frac{i}{2} \left\{ X^a, X^b \right\} \gamma_{ab} \epsilon + \eta
\] (15)

Note that the action is invariant under the above transformations up to total derivatives, and for closed supermembranes the total derivatives do not make any contribution (The fields are assumed to vanish at \( \tau = \pm \infty \)).
However, for open supermembranes, where there are boundaries at the end of spatial directions of the world volume, Dirichlet or Neumann boundary conditions have to be imposed to ensure that the boundary terms vanish. One finds that to ensure invariance under supersymmetry, it is necessary that the supermembrane ends only on 1, 5 or 9-dimensional hypersurfaces. This set of conditions have been derived earlier in \cite{24, 23} and by demanding invariance under $\kappa$ symmetry of the covariant action \cite{25}.

To impose the boundary conditions on the ends of the supermembrane, we define the normal and tangential derivatives on the boundary:

\begin{align}
\partial_n X^\mu &\equiv n^r \partial_r X^\mu \\
\partial_t X^\mu &\equiv \epsilon^{rs} n_r \partial_s X^\mu
\end{align}

where $n^r$ is the unit normal on the boundary and $\epsilon^{rs} n_r$ the unit tangential vector to the boundary. For the supermembrane ending on a $p$ dimensional hypersurface the following boundary conditions are required:

\begin{align}
\partial_n X^M &= 0 &\text{for } M = 2, \ldots, p &\text{(Neumann)} \\
\partial_t X^m &= 0 &\text{for } m = p + 1, \ldots, 10 &\text{(Dirichlet)}
\end{align}

To check for invariance under supersymmetry transformations \cite{13}, we vary the action and find the following boundary terms:

\begin{align}
\delta S &= - \int d\tau \int d\sigma \eta \gamma^a \theta \partial_t X^a + \\
&\quad + \int d\tau \int d\sigma \epsilon \gamma^d \left( \gamma \cdot DX - \frac{1}{2} \gamma^{ab} \{X^a, X^b\} \right) \theta \partial_t X^d.
\end{align}

On imposing the (13), we find to get the terms to vanish along the Neumann directions, additional conditions on $\theta$ must be imposed. These translate as

\begin{align}
\eta \gamma^M \theta = \epsilon \gamma^M \gamma^N \theta = \epsilon \gamma^M \gamma^m \gamma^N \theta = 0.
\end{align}

Defining $P_\pm = 1/2(1 \pm \gamma^{p+1} \ldots \gamma^{10})$, which act as projection operators for dimensions $p = 1, 2, 5, 6, 9$, we find that the following conditions:

\begin{align}
P_- \theta = 0 &\quad P_+ \epsilon = 0 &\quad P_- \eta = 0
\end{align}

are required so that (21) is obeyed, which restrict $p = 1, 5, 9$. It is interesting to see that (22) results in the fermionic degrees of freedom being reduced to
8 on the boundary. For \( p = 9 \), \( P_- \) coincides with the chiral operator for the boundary theory. This is the first sign that the boundary theory, which is essentially a string theory induced by the membrane has a heterotic structure. In fact, by looking at the equations of motion obeyed by the membrane on the boundary, we find that they are ‘free’ equation of motion for a string. For simplicity, we discuss the \( p = 9 \) case, and its relation to heterotic Matrix theory [24]. The \( p = 5 \) case also has many interesting applications [27], especially in the light of non-commutative open membrane theories proposed and discussed in [13]. The \( p = 1 \) case is yet to be investigated.

Once we have ensured that the boundary terms vanish, the bulk equation of motion for the fields are:

\[
D^2 X^a - \left\{ \left\{ X^a, X^b \right\}, X^b \right\} - i \left\{ \theta, \gamma^a \theta \right\} = 0 \quad (23)
\]
\[
D\theta + \{ \gamma \cdot X, \theta \} = 0 \quad (24)
\]
\[
\{ DX^a, X^a \} - i \{ \theta, \theta \} = 0 \quad (25)
\]

The last of these, the equation of motion for the auxiliary gauge field \( \omega \) is same as the constraint (8). What is interesting is how these equations reduce on the boundary. Taking \( a = m \) in (23) on the boundary, the normal derivative becomes:

\[
\partial_n X^m = \text{constant} \quad (26)
\]

Using this, we define \( \bar{\gamma} = \sum_{p+1} \partial_n X^m \gamma_m = \text{constant matrix} \). The equations obtained by putting \( a = M \) in (23, 24) reduce to the following linear wave equations on the boundary:

\[
(\partial^2 - \partial^2_\tau) X^M = 0 \quad (27)
\]
\[
(\partial_\tau - \bar{\gamma} \partial_\tau) \theta = 0 \quad (28)
\]

(recall that \( \partial_\tau \) is the tangential derivative along the spatial boundary of the membrane). In this way we see that the supermembrane equations of motion with the above boundary conditions induce a superstring theory on its boundary. The restrictions on the value of \( p \) can then be easily understood, because only for these values is it possible to match bosonic and fermionic degrees of freedom on the boundary. In particular, for \( p = 9 \), we obtain the world sheet equations for the heterotic string, and \( \bar{\gamma} = \gamma^{11} \) becomes a chirality matrix on the \((9+1)\)-dimensional brane. By (22), only one of the chiralities of \( \theta \) survives on the boundary, leaving an appropriate equation of
motion for the chiral coordinate. We had to put $\partial_n \theta = 0$ to obtain the free heterotic string equations of motion.

After we have obtained the gauge fixed lagrangian, with the appropriate equation of motion for the world volume fields, what remains is the quantisation of the theory. Some attempts in this direction have been reviewed in [17]. Here we confine ourselves to brief comments and show how Matrix regularisation of the membrane leads to a interpretation of the spectrum. We describe the latter first.

2.1 In the light of Matrix theory

We now concentrate on the relation of the APD diffeomorphism transformations to SU(N) (or SO(N) for the open membrane) gauge transformations under suitable regularisations [2]. It was first discussed in [2], and led to the supermembranes’ relation to Matrix theory, and M theory [3]. In the case of open membranes, it leads to the relation to heterotic Matrix theory for membranes ending on 9-branes [14], and to other exotic theories [15].

Since this discussion appears in previous reviews [17], we just briefly give the relation here, with an emphasis on the regularisation of the open membrane. As stated earlier, the APD bracket has a Lie bracket structure. It is also easy to check that the commutator of two APDs leads to a third APD:

$$\{\xi_2, \xi_1\} = \xi_3$$

Given a basis of orthogonal functions $\{Y^A\}$ on the spatial manifold, we can expand the coordinates as $X^a(\sigma) = X_0^a + \sum_A X^a A Y_A(\sigma)$. The Lie bracket then assumes the following form:

$$\{Y_A, Y_B\} = g_{ABC} Y^C \quad Y^A = \eta^{AB} Y_B \quad \eta^{AB} \eta_{BC} = \delta^A_C$$

(30)

where

$$g_{ABC} = \int d^2 \sigma \epsilon^{rs} Y_A \partial_r Y_B \partial_s Y_C \quad \eta_{AB} = \int d^2 \sigma Y_A Y_B$$

(31)

Given the above, the basis which is infinite dimensional for a continuum manifold, can be restricted to some finite $A = 1, \ldots, \Lambda$ such that

$$\lim_{\Lambda \to \infty} f_{\Lambda}^{ABC} = g^{ABC}$$

(32)

with $f_{\Lambda}^{ABC}$ the structure constant of some finite dimensional group labeled by $\Lambda$. For closed membranes of arbitrary topology [2, 28] this finite dimensional
group is SU(N), and the Hamiltonian of the regulated membrane turns out to precisely coincide with the Hamiltonian of dimensionally reduced SU(N) super-Yang Mill’s theory. This can easily be seen by substituting the regularized coordinates in (12), and replacing the APDs by appropriate commutators. The non-zero $X_A$’s transform in the adjoint representation of the SU(N) group.

$$H = Tr \left( \frac{1}{2} + \frac{1}{4} [X^a, X^b]^2 + [X^a, \Theta] \gamma^a \Theta \right)$$

(33)

In the above $(X, \Theta)$ denote matrices. The same Hamiltonian was used in [5] to describe $N D_0$-branes in the infinite momentum frame, with their momenta along the 11th direction of $d = 11$ space. Hence supermembranes through Matrix theory are intimately related to $D$-branes and M-theory.

For open supermembranes, the matrix regularisation yields different finite dimensional groups, dependent on the topology of the continuum membrane: for the disc $D^2$, the cylinder, and the Möbius strip, we get the groups $SO(N)$, whereas for the projective plane we get $USp(2N)$ [29]. Moreover, depending whether the $X^a$ are Neumann or Dirichlet, they either transform in the symmetric or adjoint representation of the gauge group. We shall illustrate here with the example of the disc, the regularisation of the open membrane.

Consider $Y_A = Y_{lm}(\theta, \phi)$, or spherical harmonics, with $m \leq |l|$. The restriction that $l \leq N - 1$ for the spherical membrane leads to a basis with $\sum_{l=0}^{N-1} (2l+1) = N^2 - 1$ independent components, and the $X$ transform in the adjoint of the SU(N) gauge group. However, when the membrane is stuck on a hypersurface, it essentially corresponds to a disc topology, with boundary conditions imposed on the $X^m$ ($m = p + 1, \ldots 10$), this would translate as $K_{lm}^+ = Y_{lm} - (-)^{l+m} Y_{lm}$ for $l + m$ odd being the correct basis. This gives $\sum_{l=0}^{N-1} l = N(N-1)/2$ as the number of generators of the finite group. This as we know is the number of generators of $SO(N)$. For the Neumann directions (17), we get these directions to transform symmetric tensor representations of $SO(N)$ [23, 24]. In fact, the regularised action has the following form (in 11 dimensions):

$$S = \int Tr \left( DX^2 + DA_{10} + [A_{10}, X^i]^2 + [X^i, X^j]^2 - i\Theta^+D\Theta^+ - i\Theta^-D\Theta^- + 2i\Theta^+\Gamma^i[X^i, \Theta^-] \right)$$

(34)

Where we have distinguished $X_{10} = A_{10}$ as it transforms in the adjoint of the SO(N) group. Also the fermions are broken up as $\Theta^+, \Theta^-$ which transform in
the adjoint and symmetric representaion of the gauge group. The $X^i$ transform in the symmetric traceless representation of $SO(N)$. The above matrix regularisation is the heterotic Matrix theory [24], but without the twisted sector fields expected to yield the additional $E_8 \otimes E_8$ degrees of freedom [14]. These twisted fields appear at the boundaries of the membranes, and live only on the 9-branes. A proper membrane origin of these fields is yet to be determined. For the case of the membrane ending on a five brane, there are many interesting possibilities [27], but much remains to be done.

The resultant theory is yet to be quantised fully. However, from the nature of the Hamiltonian, it can be seen that the supermembrane spectrum is continuous and there is no mass gap. This points towards a multiparticle interpretation of the spectrum, and hence a second quantised picture of the supermembrane.

2.2 Approaching quantisation

The matrix regularisation of the supermembrane proves very useful, as one can use matrix quantum mechanics and interpret the $N \to \infty$ as the quantum supermembrane. However, the $N \to \infty$ limit is very subtle (membranes with different topology and hence different APD are approximated by the same $SU(N)$ regularisation). But the multiparticle interpretation of the supermembrane spectrum comes entirely from its relation to Matrix theory. It was shown in [3] that the supermembrane spectrum is continuous and there is no mass gap. The continuous spectrum of the supermembrane was interpreted initially as a signature of instability. However, now due to its relation to matrix theory, this is attributed to the presence of multi-particle states. In the original conjecture of [5], the diagonal elements of $SU(N)$ matrices are positions of $D0$ branes, and in the case of block diagonal matrices, each block corresponds to $N_i$ coincident $D0$-branes, where $N_i$ is the dimension of the $i$th block. Each of these can be thought as separate entities, and in the infinite $N$ limit, as separate membranes linked by thin tubes. This multiparticle interpretation of the supermembrane spectrum implies, that we should essentially treat the supermembrane world volume as a second quantised theory.

However, one crucial question still remains: can one find a normalisable ground state for the theory? And if so, does the spectrum have massless states? By just looking at the zero mode sector of the theory, one can build states which transform as $44 \oplus 88$ of $SO(9)$ in the bosonic sector and $128$ of $SO(9)$ in the fermionic sector. Since the Hamiltonian and hence mass (6)
Figure 2: Matrix-membrane correspondence

does not depend on the zero modes, this should give the massless sector of
the theory, provided the groundstate corresponding to the non-zero modes
transforms as a SO(9) singlet. Attempts to prove this have not seen much
success up to now. However, in Matrix theory, considerable progress has been
made in efforts to prove the existence of normalisable ground states [6] in the
case of SU(2) or SU(3). Further information can be found in the review [17]
and we refrain from giving the details here. In the next section, we discuss
vertex operators for the supermembrane, which is one of the directions which
can shed more light on this side of M-theory.

3 Vertex Operators

Though it still remains to be proven that the supermembrane has massless
states, there is sufficient evidence that they exist, and as discussed in previous
sections, should belong to the $d = 11$ supergravity multiplet, obeying appro-
priate onshell conditions. This then leads to the question whether there exists
vertex operators for the supermembrane in analogy with the superstring ver-
tex operators. The superstring vertex operators prove to be extremely useful
in the calculation of string interactions due to a number of reasons. The
superstring is a first quantised theory, and hence vertex operators which are
local operators act as ‘creation’ operators. Moreover, the conformal symme-
try of the string allows the asymptotic states of the spectrum to be mapped to local operators at finite points on the world sheet. The vertex operators can then be inserted in the path integral, and of scattering amplitudes evaluated using them. Further, conformal invariance also restricts the nature of the vertex operators which can are uniquely determined.

Vertex operators by definition are of the following form:

$$V_h = \int d^3 \xi \ h \cdot O_h [X, \theta] e^{i \vec{k} \cdot x}$$  \hspace{1cm} (35)

where \( h \) denotes the polarisation of the state, and \( O \) the local operator corresponding to that state which has a momentum \( \vec{k} \). Given the fact that supermembrane world volume theory does not have conformal invariance we have to look elsewhere to determine \( O \). However, the requirement that the vertex operators transform into one another under under supersymmetry transformation completely determines their structure. In addition the technique of double dimensional reduction to give the superstring in lower dimensions gives us an additional check for the supermembrane vertex operators. The superparticle vertex operators for \( d = 11 \) were determined in [7], and also serve as a useful guide in our calculations.

Thus for the supermembrane the vertex operator for the graviton should be of the form (35)

$$V_h = \int d\tau d^2 \sigma O^{ab}_h [X^a(\tau, \sigma_i), \theta(\tau, \sigma_i)] e^{i \vec{k} \cdot x}$$  \hspace{1cm} (36)

\( h_{ab} \) denotes the polarisation tensor of the graviton, and \( O^{ab} \) is a local operator of the supermembrane light cone coordinates. \( \vec{k} \) denotes the momentum of the graviton, and this operator in the superstring case creates a graviton state when acting on the string ground state. Using the fact that under supersymmetry transformations

$$\delta V_h = V_{\delta \psi(h)}$$  \hspace{1cm} (37)

where \( \delta \psi_h \) is the transformed gravitino polarisation vector, we can solve for \( V_h \). And \( V_\psi, V_C \) which have the following transformations under supersymmetry:

$$\delta V_C = V_{\delta \psi(h)} \quad \delta V_\psi = V_{\delta h} + V_{\delta C}$$  \hspace{1cm} (38)

Since we work in light cone gauge the polarisations are separated as transverse (with indices \( a, b \)) and longitudinal (with indices \( +, - \)). Further the
The gravitino is split up into two SO(9) spinors $\psi_a, \tilde{\psi}_b$. To solve for the above vertex operators, we have to resort to $d = 11$ onshell conditions. Note that these conditions are ensured by the $\kappa$ invariance of the supermembrane action. These conditions are given by

$$
\begin{align*}
&k^a h_{ab} = 0 = k^a C_{abc} \\
&\gamma^a \tilde{\psi}_a = 0 = k^a \psi_a \\
&\gamma^a \psi_a = \tilde{\psi}_a \\
&k^b \gamma_b \psi_a = k^b \tilde{\psi}_a \\
&\gamma^a h_{a} = 0 = \gamma^a h^{a} \\
&k^b \gamma_b h_c \gamma^{b} = 0 = \gamma^a h^{a} C_{abc} \\
&k^c \gamma_c \psi_a = \gamma^b k^b \tilde{\psi}_a = \gamma^b \tilde{\psi}_a = 0 = \gamma^b \tilde{\psi}_a \\
&\gamma^a C_{abc} = \gamma^b \gamma^c \psi_a = \gamma^b \psi_a = \gamma^b \psi_a \\
&k^b \gamma_b \psi_a = k^b \tilde{\psi}_a = k^b \tilde{\psi}_a
\end{align*}
$$

(39)

Because the polarisations are given in light cone gauge, the supersymmetry transformations (37, 38) are separated into 16 linearly realised ($\eta$) and 16 non-linearly realised ($\epsilon$). However for the sake of simplicity, we shall here record only on the linear supersymmetry transformations listed below

$$
\begin{align*}
&\delta h_{ab} = -\tilde{\psi}_a \gamma_b \eta, &\delta h_{a+} = -\frac{1}{\sqrt{2}} \psi_a \eta \\
&\delta C_{abc} = \frac{3}{2} \tilde{\psi}_a \gamma_{bc} \eta, &\delta C_{ab+} = \sqrt{2} \psi_a \gamma_{bc} \eta \\
&\delta \psi_a = k_b h_{ca} \gamma_{bc} + \frac{1}{\sqrt{2}} (\gamma^a \gamma_{bcd} F_{bcde} - 8 \gamma^a \gamma_{bcd} F_{abcd}) \eta \\
&\delta \tilde{\psi}_a = -\frac{\sqrt{2}}{2} \gamma^{bcd} \eta F_{bcde} \psi_a \\
&\delta \tilde{\psi}_+ = \delta \tilde{\psi}_a = 0 = \delta h_{++}
\end{align*}
$$

(40)

We now state the vertex operators, and shall explicitly check for the linear supersymmetry of the graviton vertex operator ($R^{abc} = \frac{1}{12} \theta \gamma^{abc} \theta$, $R^{ab} = \frac{1}{4} \theta \gamma^{ab} \theta$).

$$
\begin{align*}
V_h &= h_{ab} \left[ DX^a \ DX^b - \{ X^a, X^c \} \{ X^b, X^c \} - i \theta \gamma^a \{ X^b, \theta \} \\
&\quad - 2 DX^a R^{bc} k_c - 6 \{ X^a, X^c \} R^{bcd} k_d + 2 R^{ac} R^{bd} k_c k_d \right] e^{-ik \cdot X} (43) \\
V_{h+} &= -2 h_{a+} (DX^a - R^{ab} k_b) e^{-ik \cdot X} \\
V_{h++} &= h_{++} e^{-ik \cdot X} (44)
\end{align*}
$$

$$
\begin{align*}
V_C &= -C_{abc} DX^a \{ X^b, X^c \} e^{-ik \cdot X} + F_{abcd} \left[ (DX^a - \frac{2}{3} R^{ae} k_e) R^{bcd} \\
&\quad - \frac{1}{2} \{ X^a, X^b \} R^{cd} - \frac{1}{96} \{ X^e, X^f \} \theta \gamma^{abcd} \theta \right] e^{-ik \cdot X} (46) \\
V_{C+} &= C_{ab+} \{ X^a, X^b \} + 3 R^{abc} k_c \right] e^{-ik \cdot X} (47)
\end{align*}
$$

$$
\begin{align*}
V_\psi &= \psi_a \left[ (DX^a - 2 R^{ab} k_b + \gamma_c \{ X^c, X^a \} ) \theta \right] e^{-ik \cdot X} \\
&\quad + \tilde{\psi}_a \left[ \gamma \cdot DX \left( DX^a - 2 R^{ab} k_b + \gamma_c \{ X^c, X^a \} \right) \theta \right]
\end{align*}
$$

15
\[
\begin{align*}
&+\frac{1}{2}\gamma_{bc}\{X^b, X^c\} (DX^a - \{X^a, X^d\} \gamma^d)\theta + 8\gamma_b\theta \{X^b, X^c\} R^{cad} k_d \\
&+\frac{2}{3}\gamma_{bc}\{X^b, X^c\} R^{ad} k_d + \frac{4}{3}\gamma_{bc}\theta \left(\{X^a, X^b\} R^{cd} + \{X^c, X^d\} R^{ab}\right) k_d \\
&+\frac{2}{3}i \left(\gamma_b\theta \{X^a, \theta\} \gamma^b\theta - \theta \{X^a, \theta\} \theta \right) + \frac{8}{3}\gamma_b\theta R^{ac} R^{bd} k_c k_d \right] e^{-ik} \quad (48)
\end{align*}
\]

To see that the linear supersymmetry is realised, we implement the following transformation in the graviton vertex (43):
\[
\delta X^a = \delta \omega = 0, \delta \theta = \eta.
\]
In other words only the terms proportional to \(\theta\) shall contribute to the variation, which is written thus:
\[
\delta V = - \left[\psi_+ \theta + \tilde{\psi}_+ \left(\gamma_a DX^a + \frac{1}{2} \gamma^{ab} \{X^a, X^b\}\right) \theta\right] e^{-ik} X
\]

The terms in the first line can be grouped together to yield the gravitino polarisation as per (43). The terms in the second line have to vanish clearly, and by a partial integration cancel each other. Note that, in the case of the open membrane we have to be careful in order to ensure the vanishing of the above. In fact, we find that the additional condition of \(\partial_t \omega \bigr|_{\partial G} = 0, h_m M \bigr|_{\partial G} = 0\) has to be imposed to ensure the vanishing of the boundary terms. The condition can be understood easily as the residual symmetry of the string worldsheet is just a constant shift in the coordinates and there is no analogous APD gauge transformation. The second one implies that there are no \(R \otimes R\) one forms on the boundary string theory. In case of the membrane ending on 9-branes this is easy to understand as the boundary string theory is heterotic string theory.

Similarly linear supersymmetry transformations can be performed on the rest of the vertex operators to see that they respect the algebra. Details can be found in \([11]\). The non-linear supersymmetry transformations are more complicated and difficult to check, and an attempt in that direction is given in \([11]\). However, our vertex operators are further confirmed by a series of other consistency checks, namely their invariance under space-time gauge transformations and double dimensional reduction to yield superstring vertex operators. in \([11]\). To check for the invariance under \(\delta h_{ab} = k_{(a} \xi_{b)}\) of the graviton vertex, we find
\[
\delta V = \left[D(k \cdot X) \left(D(\xi \cdot X) - R^{ab} \xi_a k_b\right) - \{k \cdot X, X^c\}\{\xi \cdot X, X^c\} + 3\{k \cdot X, X^c\} R^{abc} \xi_a k_b - \frac{i}{2} \theta \xi_a \gamma^a \{k \cdot X, \theta\} - \frac{i}{2} \theta \{\xi \cdot X, \theta\}\right] e^{-ik} X
\]
By partially integrating the first two terms, and using the equations of motion for $X^a$ and $\theta$ \cite{23,24}, the terms completely cancel each other. Note however for the open super membrane, at the boundaries we have to impose $k_m \xi_M = 0$, which is perfectly consistent with the earlier condition $h_{mM} = 0$. For details of the other vertex operators, we refer to \cite{11}. We review the reduction to superstring vertex operators for the graviton briefly here (we denote the string coordinates as $X^i$). The double dimensional reduction consists of the identification of one of the supermembrane directions say $X^9 = \sigma^2$ as compact, to get a string in 10 dimensions. In the first approximation this means that $X^i$ are independent of $\sigma^2$, and hence $\{X^a, X^b\} \neq 0$ only when $a$ or $b = 9$. Thus:

$$\{X^a, X^b\} = \partial_1 X^{[a} \delta^b_9$$  \hspace{1cm} (52)

The SO(9) spinors also decompose into two SO(8) spinors ($S^\alpha, S^{\dot{\alpha}}$). Thus ($i, j = 1, \ldots, 8$ and $\Gamma^i$ are SO(8) matrices):

$$R^{ij} = \frac{1}{4} S \Gamma^{ij} S + \frac{1}{4} \tilde{S} \Gamma^{ij} \tilde{S} \hspace{1cm} R^{ij9} = \frac{1}{2} \tilde{S} \Gamma^{ij} S$$  \hspace{1cm} (53)

$$R^{ijk} = \frac{1}{6} S \Gamma^{ijk} \tilde{S} \hspace{1cm} R^{ij9} = \frac{1}{12} S \Gamma^{ij} S - \frac{1}{12} \tilde{S} \Gamma^{ij} \tilde{S}$$  \hspace{1cm} (54)

Using the above in the graviton vertex, one finds that the $h_{ij}$ should give the $NS \otimes NS$ vertex operator of type IIA superstring, while the $h_{i9}$ should give the $R \otimes R$ one form vertex in 10 dimensions. Despite the apparent differences in the structure of the vertex operators, we recover them perfectly starting from (43) and using (52, 53, 54). The vertex operator in 10 dimensions is:

$$(V_h)_{DDR} = h_{ij} \left[ \partial_0 X^i \partial_0 X^j - \partial_1 X^i \partial_1 X^j - \frac{1}{2} \partial_0 X^i (S \Gamma^{jm} S + \tilde{S} \Gamma^{jm} \tilde{S}) k_m ight. \\
+ \frac{1}{2} \partial_1 X^i (S \Gamma^{jm} S - \tilde{S} \Gamma^{jm} \tilde{S}) k_m + \frac{1}{4} S \Gamma^{jm} S \tilde{S} \Gamma^{jn} \tilde{S} k_m k_n \right] e^{-ik \cdot X}$$  \hspace{1cm} (55)

For the $h_{i9}$, we get:

$$h_{i9} \left[ -i \theta \gamma^i \partial_1 \theta + 2 \partial_0 X^i R^{m9} k_m + \partial_1 X^i R^{jim} k_m + 2 R^{im} R^{9mn} k_m k_n \right] e^{-ik \cdot X}$$  \hspace{1cm} (56)

Again the quartic terms are easily seen to agree. To get rid of the derivatives on $\theta$, which are absent in the superstring vertices, we make use of the superstring equations of motion $\partial_1 S = \partial_0 S$ and $\partial_1 \tilde{S} = -\partial_0 \tilde{S}$, and integrate the resulting expression by parts to find
\[ k_i h_{j9} \left[ S T^{ij} \Gamma^k \bar{S} \partial_- X^k - S T^k \Gamma^{ij} \bar{S} \partial_+ X^k \right] e^{-i k \cdot X} \]  

(57)

The same can be repeated for the three form vertex operator and the gravitino vertex operators. Thus, we have the complete supermembrane vertex operators both for closed as well as open supermembranes. It should be mentioned that in the case of open membranes ending on 9-branes, heterotic string vertex operators can be recovered on the boundary using the same techniques of double dimensional reduction. Except, the additional massless states of the $E_8 \times E_8$ gauge fields are not there. It shall be an interesting exercise to look for them in the open supermembrane spectrum. Also, the dimensional reduction for the open supermembrane leads to Type I' theory [26]. This is a manifestation of the web of dualities relating the various string theories.

### 3.1 Matrix theory vertex operators

Since the Matrix-regularisation of the supermembrane is a straightforward procedure described in section 2.2, the vertex operators can be easily applied to Matrix theory. The coordinates $X, \theta$ transform in the adjoint of $SU(N)$ and hence are $(N^2 - 1) \times (N^2 - 1)$ matrices (for the closed membrane). The continuum integral is replaced by a Trace operation. Hence the graviton vertex (written in configuration space) shall be of the form:

\[
V_h = Tr \left[ \dot{X} \dot{X} + [X^a, \dot{X}^b]^2 + \Theta \gamma^a \left[ X^b, \Theta \right] - 2 \dot{X}^a R^{bc} \frac{\partial}{\partial X^c} \right. \\
- \left. 6 [X^a, X^c] R^{bcd} \frac{\partial}{\partial X^d} + 2 (\Theta \gamma^{ac} \Theta) (\Theta \gamma^{bd} \Theta) \frac{\partial}{\partial X^c} \frac{\partial}{\partial X^d} \right] h_{ab}(X) 
\]

(58)

This agrees with previous calculations of linearised Matrix current in arbitrary backgrounds determined up to $O(\theta^2)$ [30]. Note that our vertex operators are known to all orders in $\theta$ and unlike as expected (i.e. terms up to $\theta^{32}$), they contain terms only up to $O(\theta^5)$. Note for the open-membrane also we can suitably regularise remembering to replace the Dirichlet direction by $SO(N)$ adjoint matrices and the rest by symmetric traceless ones. It remains now to implement the above in a scattering amplitude calculation.
4 Scattering Amplitudes

We now turn to the discussion of three point tree level scattering amplitudes. For this it is advantageous to work in the framework of the finite $N$ matrix theory. In order to define a tree level amplitude we first split off the center of mass degrees of freedom of the matrices by writing

$$X^a = x^a \mathbb{1} + \hat{X}^a, \quad \Theta^a = \theta^a \mathbb{1} + \hat{\Theta}^a$$

(59)

with traceless matrices $\hat{X}^a$ and $\hat{\Theta}^a$. An asymptotic 1-graviton state in matrix theory is then given by

$$|\text{IN}\rangle = |k_1, h_1\rangle_{x, \theta} \otimes |\text{GS}\rangle_{\text{SU}(N)\hat{X}, \hat{\Theta}}$$

(60)

where $|k_1, h_1\rangle$ is the graviton state of the superparticle [8, 7] and $|\text{GS}\rangle$ denotes the exact SU($N$) normalized zero energy groundstate, whose explicit form is unknown but is believed to exist [6]. The tree level three point amplitude is then defined by

$$A_{3\text{-point}} = \langle \langle 1 | V_2 | 3 \rangle \rangle$$

(61)

where one inserts the graviton vertex operator

$$V_2 = h_{ab}^{(2)} \frac{1}{N} \text{Str} \left[ p^a p^b + 2 p^a \hat{P}^b + \hat{P}^a \hat{P}^b + [\hat{X}^a, \hat{X}^c] [\hat{X}^b, \hat{X}^c] \right] e^{ik \cdot \hat{X}} e^{ik \cdot x} + \text{fermions}$$

(62)

One may wonder how one could ever evaluate (61) without the knowledge of $|\text{GS}\rangle$. The first contribution to (61) takes the form

$$\langle k_1, h_1| p_a p_b e^{ik \cdot x} |k_3, h_3\rangle h_{ab}^{(2)} \langle \text{GS}| \text{Str} e^{ik \cdot \hat{X}} |\text{GS}\rangle$$

(63)

Now by SO(9) covariance $\langle \text{GS}| \text{Str} e^{ik \cdot \hat{X}} |\text{GS}\rangle = N$, as the only SO(9) scalar it could depend on would be $k^2$ which vanishes on shell. It must then be a constant which is fixed to be $N$ by considering the $k^a \to 0$ limit. Remarkably the remaining two terms in (61) upon inserting (62) vanish by a combination of SO(9) covariance and on-shell arguments:

$$h_{ab} \langle \text{GS}| \text{Str} \hat{P}^b e^{ik \cdot \hat{X}} |\text{GS}\rangle \sim k^b h_{ab} = 0$$

(64)

$$h_{ab} \langle \text{GS}| \text{Str} \left[ (\hat{P}^a \hat{P}^b + [\hat{X}^a, \hat{X}^c] [\hat{X}^b, \hat{X}^c]) e^{ik \cdot \hat{X}} \right] |\text{GS}\rangle \sim (k^a k^b + c \delta^{ab}) h_{ab} = 0$$
Figure 3: The Veneziano amplitude for Strings and Membranes

But as the first correlator in (63) is nothing but the bosonic contribution to the 3-point $d = 11$ superparticle amplitude [7] and as the fermionic terms work out in a similar fashion we see that our 3-point tree level amplitude

$$\langle\langle 1 | V_2 | 3 \rangle\rangle = \langle 1 | V_2 | 3 \rangle_{x,\theta} \langle GS | GS \rangle$$

agrees with the 3-point amplitude of $d = 11$ supergravity!

Clearly the next step would be to study $n$-point tree level amplitudes which should be given by

$$A_{n\text{-point}} = \langle\langle 1 | V_2 \Delta V_3 \Delta \ldots \Delta V_{n-1} | n \rangle\rangle$$

where $\Delta$ denotes the propagator $1/(\frac{1}{2} p_0^2 + \hat{H})$ built from the interacting membrane Hamiltonian $\hat{H}$. However, now we expect the details of the groundstate $| GS \rangle$ to enter the computation. Developing some perturbative scheme for calculating (66) would be highly desirable, but is conceivably very complicated as it must involve an expansion in both the propagator $\Delta$ and the groundstate $| GS \rangle$. In other words, even though we now have the vertex operators for the massless states, we are not able with present techniques to calculate the (super)membrane analogue of the Veneziano amplitude. The complications are easy to understand from the figure: whereas for the string, the intermediate states are just the massive string states, and therefore completely under control, we do not know how to interpret (and to manipulate) the states that propagate between two vertices in the case of the membrane. On the other hand, if we could master this calculation we might be able to obtain the full result to all orders “in one go”: there would be no need for unitarity corrections and the like in this membrane amplitude!

Instead we shall briefly comment on attempts to compute loop amplitudes within this scenario. Here, led by the formalism in light cone superstring
and superparticle theory, we propose to define a membrane $n$-point one-loop amplitude by the expression

$$
\mathcal{A}_{\text{1-loop, } n\text{-point}} = \int d^{11} p_0 \text{Tr}(\Delta V_1 \Delta V_2 \Delta V_3 \Delta V_4 \ldots \Delta V_n)
$$

(67)

where the trace is over the Hilbert space of $\hat{H}$. Again this appears as a daunting task, however the zero mode sector of (67) already yields some amount of information. In particular the trace over the fermionic zero mode $\theta$ of (59) tells us that all 2 and 3 particle amplitudes vanish at one loop, as at least four vertex operators ($\leq 16 \theta$’s) are needed to saturate the fermion zero mode trace

$$
\text{Tr}(\theta_{\alpha_1} \ldots \theta_{\alpha_N}) = \delta_{N,16} \epsilon^{\alpha_1 \ldots \alpha_{16}}
$$

(68)

In the pure graviton sector the first non-vanishing amplitude is then the 4-graviton amplitude whose leading momentum dependence is given by

$$
\mathcal{A}_{4h} = \epsilon^{\alpha_1 \ldots \alpha_{16}} \gamma^a_{\alpha_1 \alpha_2} \ldots \gamma^a_{\alpha_{15} \alpha_{16}} R^{(1)}_{a_1 a_2 a_3 a_4} \ldots R^{(4)}_{a_{13} a_{14} a_{15} a_{16}} \int d^{11} p_0 \text{Tr'} \Delta^4.
$$

(69)

We thus see the emergence of the expected $R^4$ term \[1, 11\] in the kinematical sector. For the remaining trace it is useful to introduce a Schwinger time parametrisation of the propagator $\Delta = \int_0^{\infty} dt \exp\left[-t(p_0^2/2 + \hat{H})\right]$. Then the remaining trace factorizes into the product of the classical bosonic partition function and the interacting contribution of the quantum fluctuations. For an evaluation of the classical contribution to this trace see \[13\].

Note added: After this paper was submitted for publication, progress has been made in reduction of supermembrane to Matrix string theory \[32\], determination of Matrix theory currents \[33\] and towards covariant quantisation of the bosonic \[34\] and supersymmetric \[35\] membrane.

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