Noncommutative Figà-Talamanca-Herz algebras for Schur multipliers

Cédric Arhancet

Abstract. In this work, we introduce a noncommutative analogue of the Figà-Talamanca-Herz algebra $A_p(G)$ on the natural predual of the operator space $\mathcal{M}_{p,cb}$ of completely bounded Schur multipliers on the Schatten space $S_p$. We determine the isometric Schur multipliers and prove that the space $\mathcal{M}_p$ of bounded Schur multipliers on the Schatten space $S_p$ is the closure in the weak operator topology of the span of isometric multipliers.

Mathematics Subject Classification (2010). Primary 46L51; Secondary, 46M35, 46L07.

Keywords. Figà-Talamanca-Herz algebra, noncommutative $L^p$-spaces, complex interpolation, Schur multipliers, operator spaces.

1. Introduction

The Fourier algebra $A(G)$ of a locally compact group $G$ was introduced by P. Eymard in \cite{Eymard}. The algebra $A(G)$ is the predual of the group von Neumann algebra $VN(G)$. If $G$ is abelian with dual group $\hat{G}$, then the Fourier transform induces an isometric isomorphism of $L^1(\hat{G})$ onto $A(G)$. In \cite{Figa}, A. Figà-Talamanca showed, if $G$ is abelian, that the natural predual of the Banach space of the bounded Fourier multipliers on $L_p(G)$ is isometrically isomorphic to a space $A_p(G)$ of continuous functions on $G$. Moreover $A_2(G) = A(G)$ isometrically. In \cite{Herz} and \cite{Figa}, C. Herz proved that the space $A_p(G)$ is a Banach algebra for the usual product of functions (see also \cite{Herz2}). Hence $A_p(G)$ is an $L_p$-analogue of the Fourier algebra $A(G)$. These algebras are called Figà-Talamanca-Herz algebras. In \cite{Runde}, V. Runde introduced an operator space analogue $OA_p(G)$ of the algebra $A_p(G)$. The underlying Banach space of $OA_p(G)$ is different from the Banach space $A_p(G)$. Moreover, it is possible to show (in using a suitable variant of \cite[Theorem 5.6.1]{Runde}) that $OA_p(G)$ is the natural predual of the operator space of the completely bounded Fourier multipliers. We refer to \cite{Arhancet}, \cite{Arhancet2}, \cite{Arhancet3} and \cite{Runde} for other operator space analogues of $A_p(G)$.

The purpose of this article is to introduce noncommutative analogues of these algebras in the context of completely bounded Schur multipliers on Schatten spaces $S_p$. Recall that a map $T: S_p \to S_p$ is completely bounded if $\text{Id}_{S_p} \otimes T$ is bounded on $S_p(S_p)$. If $1 \leq p < \infty$, the operator space $\text{CB}(S_p)$ of completely bounded maps from $S_p$ into itself is naturally a dual operator space. Indeed, we have a

\begin{footnote}
This work is partially supported by ANR 06-BLAN-0015.
\end{footnote}
Suppose that the tensor product is a maximal commutative subset of $CB(M)$. Moreover, we will prove that the subspace $M_{p,cb}$ of completely bounded Schur multipliers is isometrically isomorphic to the usual matricial product or the Schur product is a completely contractive Banach algebra.

Moreover, by definition, we have a completely contractive inclusion $M_{p,cb}$. Hence $M_{p,cb}$ is naturally a dual operator space with $M_{p,cb} = (S_p \hat{\otimes} S_p^*)$. If we denote by $\psi_p: S_p \hat{\otimes} S_p^* \to S_1$ the map $(A, B) \mapsto A \ast B$, where $\ast$ is the Schur product, we will show that $(M_{p,cb})_\perp = \ker \psi_p$. Now, we define the operator space $M_{p,cb}$ as the space $\text{Im} \psi_p$ equipped with the operator space structure of $S_p \hat{\otimes} S_p^*/\ker \psi_p$. We have completely isometrically $(M_{p,cb})^* = M_{p,cb}$. Moreover, by definition, we have a completely contractive inclusion $M_{p,cb} \subset S_1$. Recall that elements of $S_1$ can be regarded as infinite matrices. Our principal result is the following theorem.

**Theorem 1.1.** Suppose $1 \leq p < \infty$. The predual $M_{p,cb}$ of the operator space $M_{p,cb}$ equipped with the usual matricial product or the Schur product is a completely contractive Banach algebra.

In [27] and [17], R. S. Strichartz and S. K. Parott showed that if $1 \leq p \leq \infty$, $p \neq 2$ every isometric Fourier multiplier on $L_p(G)$ is a scalar multiple of an operator induced by a translation. In [10], A. Figà-Talamanca showed that the space of bounded Fourier multipliers is the closure in the weak operator topology of the span of these operators. We give noncommutative analogues of these two results.

**Theorem 1.2.**

1. Suppose $1 \leq p \leq \infty$. If $p \neq 2$, an isometric Schur multiplier on $S_p$ is defined by a matrix $[a_{ij}]$ with $a_{ij} \in \mathbb{T}$.

2. Suppose $1 \leq p < \infty$. The space $M_p$ of bounded Schur multipliers on $S_p$ is the closure of the span of isometric Schur multipliers in the weak operator topology.

The paper is organized as follows. In §2, we fix notations and we show that the natural preduals of $M_p$ and $M_{p,cb}$ admit concrete realizations as spaces of matrices. We give elementary properties of these spaces.

In §3, we show that the operator space $M_{p,cb}$ equipped with the matricial product is a completely contractive Banach algebra.

In §4, we turn to the Schur product. We observe that the natural predual $M_p$ of the Banach space $M_p$ of bounded Schur multipliers is a Banach algebra for the Schur product. Moreover, we show that the space $M_{p,cb}$ equipped with the Schur product is a completely contractive Banach algebra.

In §5, we determine the isometric Schur multipliers on $S_p$ and prove that the space $M_p$ is the closure in the weak operator topology of the span of isometric multipliers.

### 2. Preduals of spaces of Schur multipliers

Let us recall some basic notations. Let $T = \{z \in \mathbb{C} \mid |z| = 1\}$ and $\delta_{ij}$ the symbol of Kronecker.

If $E$ and $F$ are Banach spaces, $B(E,F)$ is the space of bounded linear maps between $E$ and $F$. We denote by $\otimes$, the Banach projective tensor product. If $E,F$, and $G$ are Banach spaces we have $(E \otimes F)^* = B(E,F)^*$ isometrically. In particular, if $E$ is a dual Banach space, $B(E)$ is also a dual Banach space. If $(E_0,E_1)$ is a compatible couple of Banach spaces we denote by $(E_0,E_1)_\theta$ the intermediate space obtained by complex interpolation between $E_0$ and $E_1$.

The readers are refereed to [3], [7], [18] and [23] for the details on operator spaces and completely bounded maps. We let $CB(E,F)$ for the space of all completely bounded maps endowed with the norm $\|T\|_{E \to F,cb} = \sup_{n \geq 1} \|Id_{M_n} \otimes u\|_{M_n(E) \to M_n(F)}$. 
Lemma 2.1. Suppose $E$ is completely contractive. Moreover, if $E$ is contractive maps. Then the map $T \mapsto T^*$ from $CB(E,F)$ into $CB(F^*,E^*)$ is a complete isometry.

If $I$ is a set, we denote by $C_I$ the operator space $B(\ell_2^I, \ell_2^I)$ and by $R_I$ the operator space $B(\ell_2^I, C)$. We have a complete isometry $B(\ell_2^I) = CB(C_I)$ (see [3, (1.14)]).

The complex interpolated space between two compatible operator spaces $E_0$ and $E_1$ is the usual Banach space $E_\theta$ with the matrix norms corresponding to the isometric identifications $M_n(E_\theta) = (M_n(E_0), M_n(E_1))_\theta$. Let $F_0, F_1$ be two other compatible operator spaces. Let $\varphi: E_0 + E_1 \to F_0 + F_1$ be a linear map. If $\varphi$ is completely bounded as a map from $E_0$ into $F_0$, and from $E_1$ into $F_1$, then, for any $0 \leq \theta \leq 1$, $\varphi$ is completely bounded from $E_0$ into $F_0$ with

$$\|\varphi\|_{cb,E_0\to F_0} \leq (\|\varphi\|_{cb,E_0\to F_0})^{1-\theta} (\|\varphi\|_{cb,E_1\to F_1})^{\theta}.$$ 

If $E_0 \cap E_1$ is dense in both $E_0$ and $E_1$, we have a completely contractive inclusion

$$(CB(E_0), CB(E_1))_\theta \subset CB(E_\theta)$$

(see [11, Lemma 0.2]).

We denote by $\otimes$ the operator space projective tensor product, by $\otimes_{\min}$ the operator space minimal tensor product, by $\otimes_h$ the Haagerup tensor product, by $\otimes_{\sigma h}$ the normal Haagerup tensor product, by $\otimes$ the normal spatial tensor product, by $\otimes_{wh}$ the weak$^*$ Haagerup tensor product and by $\otimes_{ch}$ the extended Haagerup tensor product (see [3, 8] and [26]). Suppose that $F,E,G$ and $H$ are operator spaces. If $\varphi: E \to F$ and $\psi: G \to H$ are completely bounded maps then the maps $\varphi \otimes \psi: E \otimes_h G \to F \otimes_h H$ and $\varphi \otimes \psi: E \otimes G \to F \otimes H$ are completely bounded and we have

$$\|\varphi \otimes \psi\|_{cb,E \otimes_h G \to F \otimes_h H} \leq \|\varphi\|_{cb,E \to F} \|\psi\|_{cb,G \to H}$$

and

$$\|\varphi \otimes \psi\|_{cb,E \otimes G \to F \otimes H} \leq \|\varphi\|_{cb,E \to F} \|\psi\|_{cb,G \to H}.$$ 

If $E, F$ are operator spaces, we have $E \otimes_h F \subset E \otimes_{wh} F$ completely isometrically (see [3] page 43).

If $E, F$ and $G$ are operator spaces, we denote by $CB(E \times F,G)$ the space of jointly completely bounded map. We have

$$CB(E \times F,G) = CB(E \otimes F,G) = CB(E, CB(F,G))$$

completely isometrically. Consequently, we have $(E \otimes F)^* = CB(E,F^*)$ completely isometrically. In particular, if $E$ is a dual operator space, $CB(E)$ is also a dual operator space.

At several times, we will use the next easy lemma left to the reader.

**Lemma 2.1.** Suppose $E$ and $F$ are operator spaces. Let $V: E \to F$ and $W: F \to E$ be any completely contractive maps. Then the map

$$\Theta_{V,W}: CB(E) \to CB(F)$$

is completely contractive. Moreover, if $E$ and $F$ are reflexive then this map is also $w^*$-continuous.
A Banach algebra $\mathcal{A}$ equipped with an operator space structure is called completely contractive if the algebra product $(a, b) \rightarrow ab$ from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$ is a jointly completely contractive bilinear map.

We equip $\ell_\infty^J$ with its natural operator space structure coming from its structure as a $C^*$-algebra and the Banach space $\ell_1^I$ with its natural operator space structure coming from its structure of predual of $\ell_\infty^J$.

If $I$ is an index set and if $E$ is a vector space, we write $M_f(E)$ for the space of the $I \times I$ matrices with entries in $E$. We denote by $M^n_f(E)$ the subspace of matrices with a finite number of non null entries. For $I = \{1, \ldots, n\}$, we simplify the notations, we let $M_n(E)$ for $M_{(1,\ldots,n)}(E)$. We write $M_\fin$ for $M^n_\fin(\mathbb{C})$. We use the inclusion $M_I \otimes M_I \subset M_{I \times I}$ with the identification $[A \otimes B]_{(i,r),(u,s)} = a_{iu}b_{rs}$. For all $i, j, k, l \in I$, the tensor $\epsilon_{ij} \otimes \epsilon_{kl}$ identifies to the matrix $[\delta_{il}\delta_{jr}\delta_{is}\delta_{kt}]_{(i,s),(u,t)} \in I \times I$ (see [24] page 5 for more information on these identifications).

Given a set $I$, the set $\mathcal{P}_f(I)$ of all finite subsets of $I$ is directed with respect to set inclusion. For $J \in \mathcal{P}_f(I)$ and $A \in M_J$, we write $T_J(A)$ for the matrix obtained from $A$ by setting each entry to zero if its row and column index are not both in $J$. We call $(T_J(A))_{J \in \mathcal{P}_f(I)}$ the net of finite submatrices of $A$.

The Schatten-von Neumann class $S_p^I$, $1 \leq p < \infty$, is the space of those compact operators $A$ from $\ell_p^J$ into $\ell_p^J$ such that $\|A\|_{S_p^I} = (\text{Tr}(A^*A)^{1/p})^{p} < \infty$. The space $S_\infty^I$ of compact operators from $\ell_2^J$ into $\ell_2^J$ is equipped with the operator norm. For $I = \mathbb{N}$, we simplify the notations, we let $S_p$ for $S_p^\mathbb{N}$. The space $S_\infty^I(S_p^K)$ of compact operators from $\ell_2^J \otimes \ell_2^K$ into $\ell_2^J \otimes \ell_2^K$ is equipped with the operator norm. If $1 \leq p < \infty$, the space $S_p^I(S_p^K)$ is the space of those compact operators $C$ from $\ell_2^J \otimes \ell_2^K$ into $\ell_2^J \otimes \ell_2^K$ such that $\|C\|_{S_p^I(S_p^K)} = ((\text{Tr} \otimes \text{Tr})(C^*C)^{1/p})^{p} < \infty$.

Elements of $S_p^I$ are regarded as matrices $A = [a_{ij}]_{i,j \in I}$ of $M_J$. If $A \in S_p^I$ we denote by $A^T$ the operator of $S_p^I$ whose the matrix is the matrix transpose of $A$. If $1 \leq p \leq \infty$, $A \in S_p^I$ and $B \in S_p^J$, the operator $AB^T$ belongs to $S_p^I$. We let $(A, B)_{S_p^I(S_p^K)} = \text{Tr}(AB^T)$. We have $(A, B)_{S_p^I(S_p^K)} = \lim_I \sum_{i,j \in I} a_{ij}b_{ij}$.

We equip $S_p^I$ with its natural operator space structure coming from its structure as a $C^*$-algebra. We equip $S_\infty^I$ with its natural operator space structure coming from its structure as dual of $S_\infty^I$. If $1 < p < \infty$, we give on $S_p^I$ the operator space structure defined by $S_p^I = (S_\infty^I, S_p^I)$ completely isometrically (see [24] page 140 for interesting remarks on this definition). By the same way, we define an operator space structure on $S_p^I(S_p^K)$. We have completely isometrically $S_p^I(S_p^K) = S_p^I(S_p^K)$ and similar operator space structures. G. Pisier showed that a map $T: S_p^I \rightarrow S_p^I$ is completely bounded if $Id_{S_p^I} \otimes T$ is bounded on $S_p^I(S_p^K)$ (see [21] Lemma 1.7). The readers are refereed to [21] for the details on operator space structures on the Schatten-von Neumann class.

We denote by $*$ the Schur (Hadamard) product: if $A = [a_{ij}]_{i,j \in I}$ and $B = [b_{ij}]_{i,j \in I}$ are matrices of $M_I$ we have $A*B = [a_{ij}b_{ij}]_{i,j \in I}$. We recall that a matrix $A$ of $M_I$ defines a Schur multiplier $M_A$ on $S_p^I$ if for any $B \in S_p^I$, the matrix $M_A(B) = A*B$ represents an element of $S_p^I$. In this case, by the closed graph theorem, the linear map $B \mapsto M_A(B)$ is bounded on $S_p^I$. The notation $M_p$ stands for the algebra of all bounded Schur multipliers on the Schatten space $S_p^I$. We denote by $M_{p,cb}$ the space of completely bounded Schur multipliers on $S_p^I$. We give the space $M_{p,cb}$ the operator space structure induced by $CB(S_p^I)$. For $I = \mathbb{N}$, we simplify the notations, we let $M_p$ for $M_p^\mathbb{N}$ and $M_{p,cb}$ for $M_p^\mathbb{N}_{p,cb}$. Recall that if $A \in S_p^I$, we have $M_A \in M_p$ (see [3] page 225).
If $M_C \in \mathcal{M}_p^I$, we have $M_C \in \mathcal{M}_p^I$. Moreover, if $A \in S_p^I$ and $B \in S_p^I$, we have
\[
\langle MC(A), B \rangle_{S_p^I, S_p^I} = \langle A, MC(B) \rangle_{S_p^I, S_p^I}.
\]
If $1 \leq p \leq \infty$, the Banach spaces $\mathcal{M}_p^I$ and $\mathcal{M}_p^I$ are isometric and the operator spaces $\mathcal{M}_{p,cb}^I$ and $\mathcal{M}_{p,cb}^I$ are completely isometric. We have $\mathcal{M}_\infty^I = \mathcal{M}_{\infty,cb}^I$ isometrically (see e.g. [10, Remark 2.2] and [13, Lemma 2]). Moreover, we have $\mathcal{M}_{p,cb}^I = \ell_\infty^I \otimes \delta \ell_p^I$ completely isometrically (see e.g. [26, Theorem 3.1]) and $\mathcal{M}_1^I = \ell_1^I$ isometrically.

If $M_A \in \mathcal{M}_p^I$ is a Schur multiplier, we have $\|M_{T_j(A)}\|_{B(S_p^I)} \leq \|M_A\|_{B(S_p^I)}$ for any finite subset $J$ of $I$. Moreover, if $M_A \in \mathcal{M}_{p,cb}^I$, we have for any finite subset $J$ of $I$ the inequality $\|M_{T_j(A)}\|_{CB(S_p^I)} \leq \|M_A\|_{CB(S_p^I)}$.

It is well-known that the map $(A, B) \mapsto A \ast B$ from $S_p^I \times S_p^I$ into $S_p^I$ is contractive. In order to study the preduals of $\mathcal{M}_p^I$ and $\mathcal{M}_{p,cb}^I$, we need to show that this map is jointly completely contractive.

**Proposition 2.2.** Suppose $1 \leq p \leq \infty$. The bilinear map
\[
S_p^I \times S_p^I \longrightarrow S_p^I
\]
\[
(A, B) \longmapsto A \ast B
\]
is jointly completely contractive.

**Proof.** We denote $\beta : \ell_p^I \to \ell_\infty^I$ the canonical contractive map. We have
\[
\|\beta\|_{cb, C_l \to \ell_\infty} = \|\beta\|_{\ell_p^I \to \ell_\infty} \leq 1 \quad \text{and} \quad \|\beta\|_{cb, R_l \to \ell_\infty} = \|\beta\|_{\ell_p^I \to \ell_\infty} \leq 1
\]
(see [3, (1.10)]). Then by tensoring, the map $C_l \otimes_k R_l \to \ell_\infty^I \otimes_k \ell_p^I$ is completely contractive. Now recall that we have a completely isometric canonical map $\ell_\infty^I \otimes_k \ell_p^I \to \mathcal{M}_p^I$ and a completely isometric map $T \mapsto T^*$ from $CB(S_p^I)$ into $CB(S_p^I)$. Then the map
\[
S_p^I = C_l \otimes_k R_l \Longrightarrow \ell_\infty^I \otimes_k \ell_p^I \longrightarrow \mathcal{M}_p^I \longrightarrow CB(S_p^I)
\]
is completely contractive. This means that the map $A \mapsto M_A$ from $S_p^I$ into $CB(S_p^I)$ is completely contractive. Then the map $(A, B) \mapsto A \ast B$ from $S_p^I \times S_p^I$ into $S_p^I$ is completely jointly contractive. By the commutativity of $\ast$ and $\otimes$, the map from $S_p^I \times S_p^I$ into $S_p^I$ is also completely jointly contractive. Finally, we obtain the result by bilinear interpolation (see [23, page 57] and [2] page 94).

Then we can define the completely contractive map
\[
\psi_p^I : S_p^I \otimes S_p^I \longrightarrow S_p^I
\]
\[
A \otimes B \mapsto A \ast B.
\]
As $S_p^I \otimes S_p^I$ embeds contractively into $S_p^I \otimes S_p^I$, the map $\psi_p^I$ induces a contraction from $S_p^I \otimes S_p^I$ into $S_p^I$, which we denote by $\varphi_p^I$. We let $\psi_p^I = \psi_p^I$. The following theorem (and the comments which follow) is a noncommutative version of a theorem of Figà-Talamanca [10]. This latter theorem states that the natural predual of the space of bounded Fourier multipliers admits a concrete realization as a space $A_p(G)$ of continuous functions on $G$. In the sequel, we consider the dual pairs $CB(S_p^I)$, $S_p^I \otimes S_p^I$, and $B(S_p^I)$, $S_p^I \otimes S_p^I$, where $1 \leq p < \infty$. 

Noncommutative Figà-Talamanca-Herz algebras 5

[10]
**Theorem 2.3.** Suppose $1 \leq p < \infty$.

1. The pre-annihilator $(\mathfrak{M}_{p,cb}^I)^\perp$ of the space $\mathfrak{M}_{p,cb}^I$ of completely bounded Schur multipliers on $S_p^I$ is equal to $\text{Ker } \psi_p^I$. We have a complete isometry $\mathfrak{M}_{p,cb}^I = (S_p^I \otimes S_p^I / \text{Ker } \psi_p^I)^*$.

2. The pre-annihilator $(\mathfrak{M}_p^I)^\perp$ of the space $\mathfrak{M}_p^I$ of bounded Schur multipliers on $S_p^I$ is equal to $\text{Ker } \varphi_p^I$. We have an isometry $\mathfrak{M}_p^I = (S_p^I \otimes \gamma S_p^I / \text{Ker } \varphi_p^I)^*$.

**Proof.** We will only prove the part 1. The proof of part 2 is similar. Let $C = \sum_{k=1}^i A_k \otimes B_k \in S_p^I \otimes S_p^I$. Suppose $\psi_p^I$ is an element of $\mathfrak{M}_p^I$. We have an isometry $M$ such that $\psi_p^I = \psi_p^I \otimes \gamma S_p^I / \text{Ker } \varphi_p^I$.

Note that, for all integers $k$, we have $M_k \in \mathfrak{M}_p^I$. If $i, j$ are elements of $I$ we have

$$\langle M_{e_{ij}}, C \rangle_{CB(S_p^I), S_p^I \otimes S_p^I} = \left( M_{e_{ij}} \sum_{k=1}^i A_k \otimes B_k \right)_{CB(S_p^I), S_p^I \otimes S_p^I}$$

$$= \sum_{k=1}^i \langle e_{ij} \ast A_k, B_k \rangle_{S_p^I \otimes S_p^I}$$

$$= \sum_{k=1}^i \langle e_{ij}, A_k \ast B_k \rangle_{S_p^I \otimes S_p^I}$$

$$= \left( e_{ij} \sum_{k=1}^i A_k \ast B_k \right)$$

$$= \left[ \psi_p^I(C) \right]_{ij}.$$ 

By continuity, if $C \in S_p^I \otimes S_p^I$, we have $\langle M_{e_{ij}}, C \rangle_{CB(S_p^I), S_p^I \otimes S_p^I} = \left[ \psi_p^I(C) \right]_{ij}$. We deduce that, if $C \in \text{Ker } \psi_p^I$ and $M_D \in \mathfrak{M}_{p,cb}^I$, we have for all $J \in \mathcal{P}_f(I)$

$$\langle M_T(A), C \rangle_{CB(S_p^I), S_p^I \otimes S_p^I} = 0.$$ 

Now, it is easy to see that we have $M_T(A) \rightarrow M_D$ in $CB(S_p^I)$ (i.e., for all $A \in S_p^I$, we have $M_T(D) \rightarrow M_D$). Then $M_T(A) \rightarrow M_D$ in $CB(S_p^I)$. Moreover, recall that, for all $J \in \mathcal{P}_f(I)$, we have $\|M_T(A)\|_{\mathfrak{M}_p^I} \leq \|M_D\|_{\mathfrak{M}_p^I}$. Thus $M_T(A) \rightarrow M_D$. Consequently, if $C \in \text{Ker } \psi_p^I$ and $M_D \in \mathfrak{M}_{p,cb}^I$ we have

$$\langle M_D, C \rangle_{CB(S_p^I), S_p^I \otimes S_p^I} = \lim_{J} \langle M_T(A), C \rangle_{CB(S_p^I), S_p^I \otimes S_p^I} = 0.$$ 

Thus we have $\text{Ker } \psi_p^I \subset (\mathfrak{M}_{p,cb}^I)^\perp$. Now we will show that $(\text{Ker } \psi_p^I)^\perp \subset \mathfrak{M}_{p,cb}^I$. Suppose that $T \in (\text{Ker } \psi_p^I)^\perp$. If $i, j, k, l$ are elements of $I$ such that $(i, j) \neq (k, l)$, the tensor $e_{ij} \otimes e_{kl}$ belongs to $\text{Ker } \psi_p^I$. Therefore we have

$$\langle T(e_{ij}), e_{kl} \rangle_{S_p^I \otimes S_p^I} = \langle T, e_{ij} \otimes e_{kl} \rangle_{CB(S_p^I), S_p^I \otimes S_p^I} = 0.$$
Hence $T$ is a Schur multiplier. We conclude that $(\text{Ker } \psi_p^I)^\perp \subset \mathfrak{M}_{p,cb}^I$. Since $\text{Ker } \psi_p^I$ is norm-closed in $S_p^I \hat{\otimes} S_p^I$, we deduce that
\[
(\mathfrak{M}_{p,cb}^I)^\perp \subset \left( (\text{Ker } \psi_p^I)^\perp \right)^\perp = \text{Ker } \psi_p^I.
\]
Then the first claim of part 1 of the theorem is proved.

Now, we will show that $\mathfrak{M}_{p,cb}^I$ is a maximal contractive inclusion. Let $T : S_p^I \to S_p^I$ be a bounded map which commutes with all Schur multipliers $M_{e_{ij}} : S_p^I \to S_p^I$ where $i, j \in I$. Then, for all $i, j, k, l \in I$ such that $(i, j) \neq (k, l)$ we have
\[
\langle T(e_{ij}), e_{kl} \rangle_{S_p^I,S_p^I} = (TM_{e_{ij}}(e_{ij}), e_{kl})_{S_p^I,S_p^I} = (M_{e_{ij}}T(e_{ij}), e_{kl})_{S_p^I,S_p^I} = \langle T(e_{ij}), M_{e_{ij}}(e_{kl}) \rangle_{S_p^I,S_p^I} = 0.
\]
Hence $T$ is a Schur multiplier. This proves the claim. Then $\mathfrak{M}_{p,cb}^I$ is weak* closed in $CB(S_p^I)$. We immediately deduce the second claim of part 1 of the theorem. \hfill \Box

If $1 \leq p < \infty$, we define the operator space $\mathfrak{M}_{p,cb}^I$ as the space $\text{Im } \psi_p^I$ equipped with the operator space structure of $S_p^I \hat{\otimes} S_p^I / \text{Ker } \psi_p^I$. We let $\mathfrak{N}_{p,cb} = \mathfrak{M}_{p,cb}^I$. We have completely isometrically $\mathfrak{M}_{p,cb}^I = \mathfrak{M}_{p,cb}^I \subset S_p^I$. By definition, we have a completely contractive inclusion $\mathfrak{M}_{p,cb}^I \subset S_p^I$. We define the Banach space $\mathfrak{M}_{p,cb}^I$ as the space $\text{Im } \varphi_p^I$ equipped with the norm of $S_p^I \hat{\otimes} S_p^I / \text{Ker } \varphi_p^I$. We let $\mathfrak{N} = \mathfrak{M}_{p,cb}^I$. We can isometrically \((\mathfrak{M}_{p,cb}^I)^* = \mathfrak{M}_{p,cb}^I\).

By duality, well-known results on $\mathfrak{M}_{p,cb}^I$ and $\mathfrak{M}_{p,cb}^I$ translate immediately into results on $\mathfrak{N}_{p,cb}^I$ and $\mathfrak{M}_{p,cb}^I$. If $1 \leq p < \infty$, there is a contractive inclusion $\mathfrak{M}_{p,cb}^I \subset \mathfrak{M}_{p,cb}^I$. If $1 < p < \infty$, the Banach spaces $\mathfrak{M}_{p,cb}^I$ and $\mathfrak{M}_{p,cb}^I$ are isometric and the operator spaces $\mathfrak{M}_{p,cb}^I$ and $\mathfrak{M}_{p,cb}^I$ are completely isometric. We have a completely isometric isomorphism
\[
\ell_1^I \hat{\otimes} \ell_1^I \longrightarrow \mathfrak{N}_{p,cb}^I,
\]
and isometric isomorphisms
\[
\ell_1^I \otimes \ell_1^I \longrightarrow \mathfrak{M}_{p,cb}^I \quad \text{and} \quad \ell_1^I \otimes \ell_1^I \longrightarrow \mathfrak{M}_{p,cb}^I = \mathfrak{M}_{p,cb}^I.
\]
Suppose $1 \leq p \leq q \leq 2$, we have injective contractive maps
\[
\mathfrak{M}_{p}^I \subset \mathfrak{M}_{p,cb}^I \subset \mathfrak{M}_{q}^I \subset \mathfrak{M}_{2}^I \quad \text{and} \quad \mathfrak{N}_{p,cb}^I \subset \mathfrak{M}_{p,cb}^I \subset \mathfrak{M}_{q,cb}^I \subset \mathfrak{M}_{2,cb}^I
\]
(see [11] page 219). One more time, by duality, we deduce that we have injective contractive inclusions
\[
\mathfrak{M}_{2}^I \subset \mathfrak{M}_{q}^I \subset \mathfrak{M}_{p}^I \quad \text{and} \quad \mathfrak{N}_{2,cb}^I \subset \mathfrak{M}_{q,cb}^I \subset \mathfrak{M}_{p,cb}^I \subset \mathfrak{N}_{1,cb}^I.
\]
Actually, the last inclusions are completely contractive. It is a part of Proposition 2.7.

Suppose $1 \leq p < \infty$. By a well-known property of the Banach projective tensor product, an element $C$ in $S_p^I$ belongs to $\mathfrak{M}_{p}^I$ if and only if there exists two sequences $(A_n)_{n \geq 1} \subset S_p^I$ and $(B_n)_{n \geq 1} \subset \mathfrak{M}_{p,cb}^I$. Noncommutative Figà-Talamanca-Herz algebras 7
\[ S^I_p \] such that the series \( \sum_{n=1}^{+\infty} A_n \otimes B_n \) converges absolutely in \( S^I_p \otimes S^I_p \), and \( C = \sum_{n=1}^{+\infty} A_n \ast B_n \) in \( S^I_p \). Moreover, we have
\[
\|C\|_{\mathcal{M}^I_p} = \inf \left\{ \sum_{n=1}^{+\infty} \|A_n\|_{S^I_p} \|B_n\|_{S^I_p} \mid C = \sum_{n=1}^{+\infty} A_n \ast B_n \right\}
\]
(2.2)
where the infimum is taken over all possible ways to represent \( C \) as before. We observe that we have an inclusion \( M^\text{fin}_I \subset \mathcal{M}^I_p \). It is clear that \( M^\text{fin}_I \) is dense in \( \mathcal{M}^I_p \).

**Remark 2.4.** The Banach spaces \( \mathcal{M}^I_p \) and \( \mathcal{M}^I_p,cb \) contain the space \( \ell^I_{p,\infty} \). We deduce that, if \( I \) is infinite, the Banach spaces \( \mathcal{M}^I_p, \mathcal{M}^I_p,cb, \mathcal{R}^I_p \) and \( \mathcal{R}^I_p,cb \) are not reflexive.

Now we make precise the duality between the operator spaces \( \mathcal{M}^I_p,cb \) and \( \mathcal{R}^I_p,cb \) on the one hand and the Banach spaces \( \mathcal{M}^I_p \) and \( \mathcal{R}^I_p \) on the other hand. Moreover, the next lemma specifies the density of \( M^\text{fin}_I \) in \( \mathcal{R}^I_p \) and \( \mathcal{R}^I_p,cb \).

**Lemma 2.5.** Suppose \( 1 \leq p < \infty \).

1. If \( J \) is a finite subset of \( I \), the truncation map \( T_J : \mathcal{R}^I_p,cb \to \mathcal{M}^I_p,cb \) is completely contractive. Moreover, if \( A \in \mathcal{R}^I_p,cb \), we have in \( \mathcal{M}^I_p,cb \)
\[
T_J(A) \to A.
\]
(2.3)
2. For any completely bounded Schur multiplier \( M_A \in \mathcal{M}^I_p,cb \) and any \( B \in \mathcal{R}^I_p,cb \), we have
\[
\langle M_A, B \rangle_{\mathcal{M}^I_p,cb,\mathcal{R}^I_p,cb} = \lim_{J} \sum_{i,j \in J} a_{ij} b_{ij}.
\]
(2.4)
3. If \( J \) is a finite subset of \( I \), the truncation map \( T_J : \mathcal{R}^I_p \to \mathcal{R}^I_p \) is contractive. Moreover, if \( A \in \mathcal{R}^I_p \), we have \( T_J(A) \to A \) in \( \mathcal{R}^I_p \).
4. For any bounded Schur multiplier \( M_A \in \mathcal{M}^I_p \) and any \( B \in \mathcal{R}^I_p \), we have
\[
\langle M_A, B \rangle_{\mathcal{M}^I_p,\mathcal{R}^I_p} = \lim_{J} \sum_{i,j \in J} a_{ij} b_{ij}.
\]

**Proof.** We only prove the assertions for the operator space \( \mathcal{R}^I_p,cb \). If \( i,j \) are elements of \( I \) and \( M_A \in \mathcal{M}^I_p,cb \), we have
\[
\langle M_A, e_{ij} \rangle_{\mathcal{M}^I_p,cb,\mathcal{R}^I_p} = \langle M_A, e_{ij} \ast e_{ij} \rangle_{\mathcal{M}^I_p,cb,\mathcal{M}^I_p,cb} = \langle M_A(e_{ij}), e_{ij} \rangle_{S^I_p,S^I_p} = a_{ij}.
\]
Then we deduce that, for all \( M_A \in \mathcal{M}^I_p,cb \) and all \( B \in \mathcal{M}^\text{fin}_I \), we have \( \langle M_A, B \rangle_{\mathcal{M}^I_p,cb,\mathcal{M}^I_p,cb} = \sum_{i,j \in J} a_{ij} b_{ij} \). Now, it is not difficult to see that, for any finite subset \( J \) of \( I \), the truncation map \( T_J : S^I_p \to S^I_p \) is completely contractive. Then, it follows easily that the truncation map \( T_J : \mathcal{M}^I_p,cb \to \mathcal{M}^I_p,cb \) is completely contractive. Hence, by duality and by using the density of \( M^\text{fin}_I \) in \( \mathcal{R}^I_p,cb \), we deduce that the truncation map \( T_J : \mathcal{R}^I_p,cb \to \mathcal{R}^I_p,cb \) is completely contractive. Furthermore, by density of \( M^\text{fin}_I \) in \( \mathcal{R}^I_p,cb \), it is not difficult to prove the assertion (2.3). Finally, the equality (2.4) is now immediate. \( \square \)
Finally, we end the section by giving supplementary properties of these operator spaces. For that, we need the following proposition inspired by [16, Proposition 2.4]. If $x,y \in \mathbb{R}$, we denote by $M_{x,y}: S_p^I \to S_p^I$ the Schur multiplier associated with the matrix $[e^{i\alpha x}e^{-iy}]_{r,s \in I}$ of $M_I$ and by $\overline{M}_{x,y}: S_p^I \to S_p^I$ the Schur multiplier associated with the matrix $[e^{-i\alpha x}e^{-iy}]_{r,s \in I}$ of $M_I$. It is easy to see that, for all $x,y \in \mathbb{R}$, the maps $M_{x,y}: S_p^I \to S_p^I$ and $\overline{M}_{x,y}: S_p^I \to S_p^I$ are completely contractive. We denote by $dx$ the normalized measure on $[0,2\pi]$.

**Proposition 2.6.** Suppose $1 \leq p < \infty$. The space $\mathcal{M}^I_{p,cb}$ of completely bounded Schur multipliers on $S_p^I$ is 1-completely complemented in the space $CB(S_p^I)$.

**Proof.** Let $T: S_p^I \to S_p^I$ be a completely bounded map. For any $A \in \mathcal{M}^I_{p,cb}$ the map

$$[0, 2\pi] \times [0, 2\pi] \to S_p^I \quad (x, y) \mapsto M_{x,y}T \overline{M}_{x,y}(A)$$

is continuous and we have

$$\left\| \int_0^{2\pi} \int_0^{2\pi} M_{x,y}T \overline{M}_{x,y}(A) dxdy \right\|_{S_p^I} \leq \int_0^{2\pi} \int_0^{2\pi} \left\| M_{x,y}T \overline{M}_{x,y}(A) \right\|_{S_p^I} dxdy$$

$$\leq \int_0^{2\pi} \int_0^{2\pi} \left\| M_{x,y} \right\|_{S_p^I \to S_p^I} \left\| T \right\|_{S_p^I \to S_p^I} \left\| \overline{M}_{x,y} \right\|_{S_p^I \to S_p^I} \left\| A \right\|_{S_p^I} dxdy$$

By the previous computation, we deduce that there exists a unique linear map $P(T): S_p^I \to S_p^I$ such that for all $A \in S_p^I$ we have

$$(P(T))(A) = \int_0^{2\pi} \int_0^{2\pi} M_{x,y}T \overline{M}_{x,y}(A) dxdy.$$ 

Moreover, for all $\sum_{k=1}^l A_k \otimes B_k \in \mathcal{M}_{dfn} \otimes S_p^I$, we have

$$\left\| \left( I_{S_p^I} \otimes P(T) \right) \left( \sum_{k=1}^l A_k \otimes B_k \right) \right\|_{S_p(S_p^I)} = \left\| \sum_{k=1}^l A_k \otimes \int_0^{2\pi} \int_0^{2\pi} M_{x,y}T \overline{M}_{x,y}(B_k) dxdy \right\|_{S_p(S_p^I)}$$

$$= \left\| \int_0^{2\pi} \int_0^{2\pi} \left( I_{S_p^I} \otimes M_{x,y} T \overline{M}_{x,y} \right) \left( \sum_{k=1}^l A_k \otimes B_k \right) dxdy \right\|_{S_p(S_p^I)}$$

$$\leq \left\| T \right\|_{cb,S_p^I \to S_p^I} \left\| \sum_{k=1}^l A_k \otimes B_k \right\|_{S_p(S_p^I)}.$$
Thus we see that the linear map $P(T)$ is actually completely bounded and that we have $\| P(T) \|_{cb,S_p^I \to S_p^I} \leq \| T \|_{cb,S_p^I \to S_p^I}$. Now, for all $r, s, k, l \in I$ we have

$$\langle P(T)e_{rs}, e_{kl} \rangle_{S_p^I \to S_p^I} = \int_0^{2\pi} \int_0^{2\pi} M_{x,y} T \overline{M_{x,y}} e_{rs}, e_{kl} \rangle_{S_p^I \to S_p^I} \, dx \, dy$$

$$= \int_0^{2\pi} \int_0^{2\pi} e^{-ixr} e^{-isy} \langle M_{x,y} T e_{rs}, e_{kl} \rangle_{S_p^I \to S_p^I} \, dx \, dy$$

$$= \left( \int_0^{2\pi} \int_0^{2\pi} e^{-ixr} e^{-isy} e^{ixl} e^{iym} \, dx \, dy \right) \langle T e_{rs}, e_{kl} \rangle_{S_p^I \to S_p^I}$$

$$= \delta_{rk} \delta_{sl} \langle T(e_{rs}), e_{kl} \rangle_{S_p^I \to S_p^I}.$$

Then the linear map $P(T) : S_p^I \to S_p^I$ is a Schur multiplier. Moreover, if $T : S_p^I \to S_p^I$ is a Schur multiplier, we have $P(T) = T$.

Now, if $T \in M_n(CB(S_p^I))$ and $A_{kl} \in M_m(S_p^I)$, with the notations of Lemma 2.1 we have

$$\left\| \left[ \int_0^{2\pi} \int_0^{2\pi} M_{x,y} T_{ij} \overline{M_{x,y}} (A_{kl}) \, dx \, dy \right]_{1 \leq i,j \leq n} \right\|_{M_m(S_p^I)} \leq \int_0^{2\pi} \int_0^{2\pi} \left\| [M_{x,y} T_{ij} \overline{M_{x,y}} (A_{kl})] \right\|_{M_m(CB(S_p^I))} \, dx \, dy$$

$$= \int_0^{2\pi} \int_0^{2\pi} \left\| \left( Id_{M_n} \otimes \Theta_{M_{x,y} \overline{M_{x,y}}} \right)(T) \right\|_{M_m(CB(S_p^I))} \, dx \, dy$$

$$\leq \| T \|_{M_m(CB(S_p^I))} \| A_{kl} \|_{M_m(S_p^I)} \quad \text{by Lemma 2.1}$$

Thus we obtain

$$\| (Id_{M_n} \otimes P)(T) \|_{M_m(CB(S_p^I))} = \| [P(T_{ij})]_{1 \leq i,j \leq n} \|_{M_m(CB(S_p^I))} \leq \| T \|_{M_m(CB(S_p^I))}.$$  

We deduce that the map $P : CB(S_p^I) \to \mathcal{M}_{2,cb}$ is completely contractive. The proof is complete. $\square$

**Proposition 2.7.**

1. We have completely isometric isomorphisms

$$\ell^1 \hat{\otimes} \ell^1 \quad e_i \otimes e_j \quad \mathcal{M}_{2,cb}$$

and

$$\ell^1 \times I \quad A \quad \mathcal{M}_{2,cb}.$$

2. Suppose $1 \leq p \leq q \leq 2$. We have injective completely contractive maps

$$\mathcal{M}_{1,cb}^I \subset \mathcal{M}_{p,cb}^I \subset \mathcal{M}_{q,cb}^I \subset \mathcal{M}_{2,cb}^I \quad \text{and} \quad \mathcal{R}_{2,cb}^I \subset \mathcal{R}_{q,cb}^I \subset \mathcal{R}_{p,cb}^I \subset \mathcal{R}_{1,cb}^I.$$
Proof. 1) By minimality, we have a completely contractive map $\mathcal{M}_{2,cb}^I \to \ell^I_{\infty}$. We will show that the inverse map is completely contractive. We have a complete isometry

$$
\ell^I_{\infty} \twoheadrightarrow B(S^I_2) = CB(C_{I\times I})
$$

Now we know that $(R_{I\times I})^* = C_{I\times I}$. Then we deduce a complete isometry

$$
\ell^I_{\infty} \twoheadrightarrow CB(C_{I\times I}) \twoheadrightarrow CB(R_{I\times I})
$$

By interpolation, we deduce a complete contraction

$$
\ell^I_{\infty} \twoheadrightarrow (CB(C_{I\times I}), CB(R_{I\times I}))^{\frac{1}{2}}.
$$

Recall that we have $(C_{I\times I}, R_{I\times I})^{\frac{1}{2}} = S^I_2$ completely isometrically (see [21] pages 137 and 140). Then we have a complete contraction

$$(CB(C_{I\times I}), CB(R_{I\times I}))^{\frac{1}{2}} \to CB(S^I_2).$$

Finally, we obtain a complete contraction $\ell^I_{\infty} \to CB(S^I_2)$. We obtain the other isomorphism by duality.

2) Let $1 \leq p \leq q \leq 2$. Recall that we have a contraction from $\mathcal{M}_{p,cb}^I$ into $\mathcal{M}_{2,cb}^I$ (see [11] page 219). Moreover we have $\mathcal{M}_{2,cb}^I = \ell^I_{\infty}$ completely isometrically. Thus we have a complete contraction $\mathcal{M}_{p,cb}^I \to \mathcal{M}_{2,cb}^I$. Now, there exists $0 \leq \theta \leq 1$ with $S^I_\theta = (S^I_p, S^I_q)_\theta$. Moreover, the identity mapping $\mathcal{M}_{p,cb}^I \to \mathcal{M}_{p,cb}^I$ is completely contractive. By interpolation, we obtain a complete contraction

$$
\mathcal{M}_{p,cb}^I \to (\mathcal{M}_{p,cb}^I, \mathcal{M}_{2,cb}^I)_\theta.
$$

On one hand, we know that we have a complete contraction

$$(CB(S^I_p), CB(S^I_q))_\theta \to CB((S^I_p, S^I_q)_\theta) = CB(S^I_q).$$

On the other hand, the space $\mathcal{M}_{p,cb}^I$ of completely bounded Schur multipliers is 1-completely complemented in the space $CB(S^I_p)$. Then we have a complete contraction $(\mathcal{M}_{p,cb}^I, \mathcal{M}_{2,cb}^I)_\theta \to \mathcal{M}_{q,cb}^I$. By composition, we deduce that we have a complete contraction $\mathcal{M}_{p,cb}^I \subset \mathcal{M}_{q,cb}^I$. We obtain the other completely contractive maps by duality.

3. Non commutative Figà-Talamanca-Herz algebras

We begin with the cases $p = 1$ and $p = 2$. Recall that we have a completely isometric isomorphism $\mathcal{M}_{1,cb}^I = \ell^I_1 \otimes \ell^I_1$ (see [2,11]) and a completely contractive inclusion $\mathcal{M}_{1,cb}^I \subset S^I_1$. Hence, the trace on $S^I_1$ induces a completely contractive functional

$$
\text{Tr} : \quad \ell^I_1 \otimes \ell^I_1 \twoheadrightarrow \mathbb{C}
$$

$$
e_i \otimes e_j \twoheadrightarrow \delta_{ij}.
$$

By tensoring, we deduce a completely contractive map

$$
Id_{\ell^I_1} \otimes \text{Tr} \otimes Id_{\ell^I_1} : \ell^I_1 \otimes h \ell^I_1 \otimes \ell^I_1 \otimes h \ell^I_1 \to \ell^I_1 \otimes h \ell^I_1.
$$

By composition with the canonical completely contractive map

$$
(\ell^I_1 \otimes h \ell^I_1) \otimes (\ell^I_1 \otimes h \ell^I_1) \to \ell^I_1 \otimes h \ell^I_1 \otimes h \ell^I_1
$$

$$
\ell^I_1 \otimes h \ell^I_1 \otimes h \ell^I_1
$$

Noncommutative Figà-Talamanca-Herz algebras 11
we obtain a completely contractive map
\[ Id_{\ell^1} \otimes \text{Tr} \otimes Id_{\ell^1} : (\ell^1 \otimes_h \ell^1) \otimes (\ell^1 \otimes_h \ell^1) \to \ell^1 \otimes_h \ell^1. \]
With the identification \( \mathcal{R}^I_{1,cb} = \ell^1 \otimes_h \ell^1 \), we obtain the completely contractive map
\[ \mathcal{R}^I_{1,cb} \otimes \mathcal{R}^I_{1,cb} \quad \mapsto \quad \mathcal{R}^I_{1,cb} \quad \mapsto \quad AB. \]
This means that the space \( \mathcal{R}^I_{1,cb} \) equipped with the matricial product is a completely contractive Banach algebra. Now, recall that we have \( \mathcal{R}^I_{2,cb} = \ell^1 \otimes \ell^1 \) completely isometrically. Then, by a similar argument, \( \mathcal{R}^I_{2,cb} \) equipped with the matricial product is also a completely contractive Banach algebra. For other values of \( p \), the proof is more complicated since we do not have any explicit description of \( \mathcal{R}^I_{p,cb} \).

In the following proposition, we give a link between \( \mathcal{R}^I_{p,cb} \) and \( \mathcal{R}^I_{p,cb} \times I \).

**Proposition 3.1.** Suppose \( 1 \leq p < \infty \). Then there exists a canonical complete contraction
\[ \mathcal{R}^I_{p,cb} \otimes \mathcal{R}^I_{p,cb} \quad \mapsto \quad \mathcal{R}^I_{p,cb} \quad \mapsto \quad A \otimes B. \]

**Proof.** The identity mapping on \( S^I_p \otimes S^I_p \) extends to a complete contraction \( S^I_p \otimes S^I_p \to S^I_p (S^I_p) \). Hence by tensoring, we obtain a completely contractive map
\[ \beta : S^I_p \otimes S^I_p \otimes S^I_p \otimes S^I_p \to S^I_p (S^I_p) \circ (S^I_p). \]
The map \( \psi^I_p : S^I_p \otimes S^I_p \to \mathcal{R}^I_{p,cb} \) is a complete quotient map. By [7, Proposition 7.1.7], we obtain a complete quotient map
\[ \psi^I_p \otimes \psi^I_p : S^I_p \otimes S^I_p \otimes S^I_p \otimes S^I_p \to \mathcal{R}^I_{p,cb} \circ \mathcal{R}^I_{p,cb}. \]
Finally, by the commutativity of \( \otimes \), the map
\[ \alpha : S^I_p \otimes S^I_p \otimes S^I_p \otimes S^I_p \quad \mapsto \quad S^I_p \otimes S^I_p \otimes S^I_p \otimes S^I_p \]
is completely isometric. We will prove that there exists a unique linear map such that the following diagram is commutative and that this map is completely contractive.

\[
\begin{array}{c}
S^I_p \otimes S^I_p \otimes S^I_p \otimes S^I_p \quad \alpha \quad S^I_p \otimes S^I_p \otimes S^I_p \otimes S^I_p \quad \beta \quad S^I_p \otimes S^I_p \otimes S^I_p \otimes S^I_p \\
\psi^I_p \otimes \psi^I_p \quad \mapsto \quad \psi^I_p \otimes \psi^I_p \quad \mapsto \quad \psi^I_p \otimes \psi^I_p \\
\mathcal{R}^I_{p,cb} \otimes \mathcal{R}^I_{p,cb} \quad \mapsto \quad \mathcal{R}^I_{p,cb} \otimes \mathcal{R}^I_{p,cb} \quad \mapsto \quad \mathcal{R}^I_{p,cb} \otimes \mathcal{R}^I_{p,cb}
\end{array}
\]
We have \( \mathcal{R}^I_{p,cb} \otimes \mathcal{R}^I_{p,cb} = (S^I_p \otimes S^I_p \otimes S^I_p \otimes S^I_p) / \text{Ker} (\psi^I_p \otimes \psi^I_p) \) completely isometrically. It suffices to show that \( \text{Ker} (\psi^I_p \otimes \psi^I_p) \subset \text{Ker} (\psi^I_p \otimes I \beta \alpha) \). By [7, Proposition 7.1.7], we have the equality
\[ \text{Ker} (\psi^I_p \otimes \psi^I_p) = \text{closure} \left( \text{Ker} (\psi^I_p) \otimes S^I_p \otimes S^I_p + S^I_p \otimes S^I_p \otimes \text{Ker} (\psi^I_p) \right). \]
Since the space \( \ker (\psi_p^{I \times I} \beta \alpha) \) is closed in \( S_p^I \otimes S_p^I \otimes S_p^I \), it suffices to show that
\[
\ker (\psi_p^I) \otimes S_p^I \otimes S_p^I + S_p^I \otimes S_p^I \otimes \ker (\psi_p^I) \subset \ker (\psi_p^{I \times I} \beta \alpha).
\]
Let \( E \in \ker (\psi_p^I) \otimes S_p^I \otimes S_p^I \). There exists integers \( n_i, m_j \), matrices \( A_{k,i}, C_{l,j} \in S_p^I \) and \( B_{k,i}, D_{l,j} \in S_p^I \) such that the sequences
\[
\left( \sum_{k=1}^{n_i} A_{k,i} \otimes B_{k,i} \right)_{i \geq 1} \quad \text{and} \quad \left( \sum_{l=1}^{m_j} C_{l,j} \otimes D_{l,j} \right)_{j \geq 1}
\]
are convergent in \( S_p^I \otimes S_p^I \),
\[
E = \left( \lim_{i \to +\infty} \sum_{k=1}^{n_i} A_{k,i} \otimes B_{k,i} \right) \otimes \left( \lim_{j \to +\infty} \sum_{l=1}^{m_j} C_{l,j} \otimes D_{l,j} \right)
\]
and
\[
\psi_p^I \left( \lim_{i \to +\infty} \sum_{k=1}^{n_i} A_{k,i} \otimes B_{k,i} \right) = 0.
\]
Then, in the space \( S_p^I \), we have
\[
\sum_{k=1}^{n_i} A_{k,i} * B_{k,i} \xrightarrow{i \to +\infty} 0.
\]
Moreover, note that, by continuity of the map \( \psi_p^I : S_p^I \otimes S_p^I \to S_p^I \), the sequence \( \left( \sum_{l=1}^{m_j} C_{l,j} * D_{l,j} \right)_{j \geq 1} \) is convergent. Now, we have
\[
\psi_p^{I \times I} \beta \alpha (E) = \psi_p^{I \times I} \beta \alpha \left( \left( \lim_{i \to +\infty} \sum_{k=1}^{n_i} A_{k,i} \otimes B_{k,i} \right) \otimes \left( \lim_{j \to +\infty} \sum_{l=1}^{m_j} C_{l,j} \otimes D_{l,j} \right) \right)
\]
\[
= \lim_{i \to +\infty} \lim_{j \to +\infty} \sum_{k=1}^{n_i} \sum_{l=1}^{m_j} \psi_p^{I \times I} \beta \alpha \left( A_{k,i} \otimes B_{k,i} \otimes C_{l,j} \otimes D_{l,j} \right)
\]
\[
= \lim_{i \to +\infty} \lim_{j \to +\infty} \sum_{k=1}^{n_i} \sum_{l=1}^{m_j} \psi_p^{I \times I} \left( A_{k,i} \otimes C_{l,j} \otimes B_{k,i} \otimes D_{l,j} \right)
\]
\[
= \lim_{i \to +\infty} \lim_{j \to +\infty} \sum_{k=1}^{n_i} \sum_{l=1}^{m_j} \left( A_{k,i} * C_{l,j} \right) \otimes \left( B_{k,i} \otimes D_{l,j} \right)
\]
\[
= \lim_{i \to +\infty} \lim_{j \to +\infty} \sum_{k=1}^{n_i} \sum_{l=1}^{m_j} \left( A_{k,i} * B_{k,i} \right) \otimes \left( C_{l,j} * D_{l,j} \right)
\]
\[
= \left( \lim_{i \to +\infty} \sum_{k=1}^{n_i} A_{k,i} * B_{k,i} \right) \otimes \left( \lim_{j \to +\infty} \sum_{l=1}^{m_j} C_{l,j} * D_{l,j} \right)
\]
\[
= 0 \quad \text{by (3.1)}.
\]
We prove that \( S_p^I \otimes S_p^I \otimes \ker (\psi_p^I) \subset \ker (\psi_p^{I \times I} \beta \alpha) \) by a similar computation. The proof is complete. \( \square \)
Now, we define the map $V: M_{ij}^{\text{fin}} \otimes M_{ij}^{\text{fin}} \to M_{ij}^{\text{fin}} \otimes M_{ij}^{\text{fin}}$ by $V(e_{ij} \otimes e_{kl}) = \delta_{kl} e_{ik} \otimes e_{kj}$.

**Proposition 3.2.** With respect to trace duality, the map $W: M_{ij}^{\text{fin}} \otimes M_{ij}^{\text{fin}} \to M_{ij}^{\text{fin}} \otimes M_{ij}^{\text{fin}}$ defined by

$$W(e_{ij} \otimes e_{kl}) = \delta_{jk} e_{il} \otimes e_{lj}$$

is the dual map of $V$. Moreover, the map $V$ induces a partial isometry $V: S_2^f \otimes S_2^f \to S_2^f \otimes S_2^f$.

**Proof.** For all $i, j, k, l, r, s, t, u \in I$, we have

$$\text{Tr} \left( V(e_{ij} \otimes e_{kl})(e_{rs} \otimes e_{tu})^T \right) = \delta_{kl} \text{Tr} \left( (e_{ik} \otimes e_{kj})(e_{rs}^T \otimes e_{tu}^T) \right)$$

and

$$\text{Tr} \left( (e_{ij} \otimes e_{kl})(W(e_{rs} \otimes e_{tu})^T) \right) = \delta_{st} \text{Tr} \left( (e_{ij} \otimes e_{kl})(e_{rs} \otimes e_{aa})^T \right)$$

We conclude that $W$ is the dual map of $V$. The fact that $V$ induces a partial isometry is clear. 

**Proposition 3.3.** Suppose $1 \leq p \leq \infty$. The maps $V: M_{ij}^{\text{fin}} \otimes M_{ij}^{\text{fin}} \to M_{ij}^{\text{fin}} \otimes M_{ij}^{\text{fin}}$ and $W: M_{ij}^{\text{fin}} \otimes M_{ij}^{\text{fin}} \to M_{ij}^{\text{fin}} \otimes M_{ij}^{\text{fin}}$ admit completely contractive extensions $V: S_p^f(S_p^f) \to S_p^f(S_p^f)$ and $W: S_p^f(S_p^f) \to S_p^f(S_p^f)$.

**Proof.** We first prove that $V$ and $W$ admit completely contractive extensions from $S_\infty^f(S_\infty^f)$ into $S_\infty^f(S_\infty^f)$. Suppose that $B = \sum_{i,j,k,l,J} b_{ijkl} \otimes e_{ij} \otimes e_{kl} \in M_{\text{fin}} \otimes M_{\text{fin}} \otimes M_{\text{fin}}$ with $J \in \mathcal{P}_f(I)$ and $b_{ijkl} \in M_{\text{fin}}$ for all $i, j, k, l \in J$. Note that the matrix $U = \sum_{r,s,J} e_{rs} \otimes e_{sr}$ of $S_\infty^f(S_\infty^f)$ is unitary. Then we have

$$\left\|(Id_{S_\infty^f} \otimes V)(B)\right\|_{S_\infty^f(S_\infty^f)} = \left\| \sum_{i,j,k,l,J} b_{ijkl} \otimes e_{ik} \otimes e_{kj} \right\|_{S_\infty^f(S_\infty^f)}$$

$$= \left\| \left( I_{S_\infty^f} \otimes \left( \sum_{r,s,J} e_{rs} \otimes e_{sr} \right) \right) \left( \sum_{i,j,k,l,J} b_{ijkl} \otimes e_{ik} \otimes e_{kj} \right) \right\|_{S_\infty^f(S_\infty^f)}$$

$$= \left\| \sum_{r,s,i,j,k,l,J} b_{ijkl} \otimes e_{rs}e_{ik} \otimes e_{sr}e_{kj} \right\|_{S_\infty^f(S_\infty^f)}$$

$$= \left\| \sum_{i,j,k,l,J} b_{ijkl} \otimes e_{ik} \otimes e_{ij} \right\|_{S_\infty^f(S_\infty^f)}$$

$$= \left\| \sum_{k,l,J} e_{kk} \otimes \left( \sum_{i,j,l,J} b_{ijkl} \otimes e_{ij} \right) \right\|_{S_\infty^f(S_\infty^f)}$$

$$= \max_{k,l,J} \left\| \sum_{i,j,l,J} b_{ijkl} \otimes e_{ij} \right\|_{S_\infty^f(S_\infty^f)}$$

$$\leq \|B\|_{S_\infty^f(S_\infty^f)} \quad \text{(submatrices)}$$
and
\[
\left\| (\text{Id}_{Lp} \otimes W)(B) \right\|_{S_{\infty}(S^L(Lp))} = \left\| \sum_{i,j \in J} b_{ij} \otimes e_{il} \otimes e_{jj} \right\|_{S_{\infty}(S^L(Lp))}
\]
\[
= \left\| (\text{Id}_{Lp} \otimes U) \left( \sum_{i,j \in J} b_{ij} \otimes e_{il} \otimes e_{jj} \right) (\text{Id}_{Lp} \otimes U) \right\|_{S_{\infty}(S^L(Lp))}
\]
\[
= \left\| \sum_{i,j \in J} b_{ij} \otimes e_{il} \otimes e_{jj} \right\|_{S_{\infty}(S^L(Lp))}
\]
\[
= \max_{j \in J} \left\| \sum_{i \in J} b_{ij} \otimes e_{ii} \right\|_{S_{\infty}(S^L(Lp))}
\]
\[
\leq \left\| \sum_{i,j,k,l \in J} b_{ijkl} \otimes e_{kj} \otimes e_{il} \right\|_{S_{\infty}(S^L(Lp))}
\]
\[
= \left\| (\text{Id}_{Lp} \otimes e_{rs} \otimes e_{sr}) \left( \sum_{i,j,k,l \in J} b_{ijkl} \otimes e_{kj} \otimes e_{il} \right) \right\|_{S_{\infty}(S^L(Lp))}
\]
\[
= \left\| \sum_{i,j,k,l \in J} b_{ijkl} \otimes e_{rs} e_{kj} \otimes e_{sr} e_{il} \right\|_{S_{\infty}(S^L(Lp))}
\]
\[
= \| B \|_{S_{\infty}(S^L(Lp))}
\]

Then we deduce the claim. Hence, by duality, the maps \( V^* : S^L_p(S^L_p) \to S^L_p(S^L_p) \) and \( W^* : S^L_p(S^L_p) \to S^L_p(S^L_p) \) are completely contractive. Moreover, we know that \( W = V^* \). By interpolation between \( p = 1 \) and \( p = \infty \), we obtain that the maps \( V : S^L_p(S^L_p) \to S^L_p(S^L_p) \) and \( W : S^L_p(S^L_p) \to S^L_p(S^L_p) \) are completely contractive.

Now, we define the linear map
\[
\Delta : \begin{array}{c}
\mathbb{M}_J \\
A
\end{array} \rightarrow \begin{array}{c}
\mathbb{M}_{I \times I} \\
[a_{ts} \delta_{ur}](t,r),(u,s) \in I \times I.
\end{array}
\]

**Proposition 3.4.** Let \( 1 \leq p \leq \infty \). Suppose that \( M_A : S^L_p \to S^L_p \) is a completely bounded Schur multiplier on \( S^L_p \) associated with a matrix \( A \) of \( M_I \). Then the map \( V(M_A \otimes \text{Id}_{S^L_p})W \) is a bounded Schur multiplier on \( S^L_p(S^L_p) \). Its associated matrix is \( \Delta(A) \).
Proof. If $i, j, k, l \in I$ and $M_A \in \mathfrak{M}_{p,cb}^I$, we have

\[
M_{\Delta(A)}(e_{ij} \otimes e_{kl}) = \left( [a_{ts} \delta_{ur}[t,r],(u,s) \in I \times I] \right) \ast \left( [\delta_{it} \delta_{ju} \delta_{kr} \delta_{ls},(t,r),(u,s) \in I \times I] \right)
\]

\[
= \delta_{jk} a_{il} \left( [\delta_{it} \delta_{ju} \delta_{kr} \delta_{ls},(t,r),(u,s) \in I \times I] \right)
\]

\[
= \delta_{jk} a_{il} e_{ik} \otimes e_{kl}
\]

and

\[
V(M_A \otimes Id_{S^p_J})W(e_{ij} \otimes e_{kl}) = \delta_{jk} V(M_A \otimes Id_{S^p_J})(e_{il} \otimes e_{jj})
\]

\[
= \delta_{jk} a_{il} V(e_{il} \otimes e_{kk})
\]

\[
= \delta_{jk} a_{il} e_{ik} \otimes e_{kl}
\]

□

Recall that, for all operator spaces $E$ and $F$, the map $R \otimes T \mapsto R \otimes T$ is completely contractive from $CB(E) \hat{\otimes} CB(F)$ into $CB(E \otimes \min F)$ and from $CB(E) \hat{\otimes} CB(F)$ into $CB(E \hat{\otimes} F)$ (see [4, Proposition 5.11]).

Proposition 3.5. Suppose $1 \leq p \leq \infty$. Let $I,J$ be any sets. The map

\[
\begin{align*}
CB(S^I_p) & \to CB(S^J_p(S^I_p)) \\
T & \mapsto T \otimes Id_{S^J_p}
\end{align*}
\]

is a complete contraction.

Proof. By definition, we have $S^I_p(S^J_p) = S^J_p \otimes \min S^I_p$ and $S^I_p(S^J_p) = S^J_p \hat{\otimes} S^I_p$ completely isometrically. Then we obtain two complete contractions

\[
\begin{align*}
CB(S^I_p) & \to CB(S^J_p \hat{\otimes} CB(S^I_p)) \\
T & \mapsto Id_{S^J_p} \otimes T \mapsto Id_{S^J_p} \otimes T
\end{align*}
\]

and

\[
\begin{align*}
CB(S^I_p) & \to CB(S^J_p(S^I_p)) \\
T & \mapsto Id_{S^J_p} \otimes T \mapsto Id_{S^J_p} \otimes T.
\end{align*}
\]

By interpolation, we obtain a completely contractive map

\[
CB(S^I_p) \to \left( CB(S^J_p(S^I_p)), CB(S^J_p(S^I_p)) \right)_{\frac{1}{p}}.
\]

We conclude by composing with the complete contraction

\[
\left( CB(S^J_p(S^I_p)), CB(S^J_p(S^I_p)) \right)_{\frac{1}{p}} \to CB(S^J_p(S^I_p))
\]

and by using the Fubini’s theorem (see [21, Theorem 1.9]).

□

Remark 3.6. It is easy to see that this map is completely isometric.

The next theorem is the principal result of this paper.
**Theorem 3.7.** Suppose $1 \leq p < \infty$. The space $\mathcal{R}_{p,cb}^I$ equipped with the usual matricial product is a completely contractive Banach algebra. More precisely, if $A$ and $B$ are matrices of $\mathcal{R}_{p,cb}^I$ and $i, j \in I$, the limit $\lim_{J} \sum_{k \in J} a_{ik} b_{kj}$ exists. Moreover, the matrix $AB$ of $M_I$ defined by $\lim_{J} \sum_{k \in J} a_{ik} b_{kj}$ belongs to $\mathcal{R}_{p,cb}^I$. Finally, the map $\mathcal{R}_{p,cb}^I \hat{\otimes} \mathcal{R}_{p,cb}^I \to \mathcal{R}_{p,cb}^I$ is completely contractive.

**Proof.** We have already seen that it suffices to prove the theorem with $1 < p < \infty$. If $M_A \in M_{I}^{\circ}$, by Proposition 3.4, we have the following commutative diagram

$$
\begin{array}{ccc}
S_p^I(S_p^I) & \xrightarrow{M_{\Delta(A)}} & S_p^I(S_p^I) \\
\downarrow W & & \downarrow V \\
S_p^I(S_p^I) & \xrightarrow{M_{\Delta Id_{S_p^I}}} & S_p^I(S_p^I).
\end{array}
$$

By Proposition 3.5, the map $M_A \mapsto M_A \otimes Id_{S_p^I}$ is completely contractive from $M_{I}^{\circ}$ into $M_{I}^{\circ} \otimes M_{I}^{\circ}$. Moreover it is easy to see that this map is w*-continuous. Since $S_{p}^{I}(S_{p}^{I})$ is reflexive, by Lemma 2.1 and by composition, the map $M_A \mapsto M_{\Delta(A)}$ from $M_{I}^{\circ}$ into $M_{I}^{\circ} \otimes M_{I}^{\circ}$ is a complete contraction and is w*-continuous. We denote by $\Delta_\ast : M_{I}^{\times I} \to \mathcal{R}_{p,cb}^I$ its preadjoint. Now, by Lemma 2.5, we have for all $i, j \in I$ and for all matrices $A, B$ of $M_{I}^{\times I}$

$$
[\Delta_\ast(A \otimes B)]_{ij} = \left\langle M_{e_{ij}}, \Delta_\ast(A \otimes B) \right\rangle_{\mathcal{R}_{p,cb}^I, \mathcal{R}_{p,cb}^I} = \left\langle M_{\Delta(e_{ij})}, A \otimes B \right\rangle_{\mathcal{R}_{p,cb}^{I \times I}, \mathcal{R}_{p,cb}^{I \times I}} = \lim_{J} \sum_{k \in J} a_{ik} b_{kj} = \Delta_{[A,B]}_{ij}.
$$

Thus we conclude that, if $A, B \in M_{I}^{\times I}$, we have $\Delta_\ast(A \otimes B) = AB$. By Proposition 3.1 and by density of $M_{I}^{\times I}$ in $\mathcal{R}_{p,cb}^I \hat{\otimes} \mathcal{R}_{p,cb}^I$, we deduce that the map $M_{I}^{\times I} \otimes M_{I}^{\times I} \to \mathcal{R}_{p,cb}^I$ is completely contractive. Finally, we complete the proof by a straightforward approximation argument using Lemma 2.5. □
Remark 3.8. We do not know if the space $\mathcal{R}_p^I$ equipped with the usual matricial product is a Banach algebra. The Banach space analogue of Proposition 3.5 is false. It is the reason which explains that the method does not work for $\mathcal{R}_p^I$. However, note that if $\mathcal{M}_p^I = \mathcal{M}_p^{I,cb}$ isometrically we have $\mathcal{R}_p^I = \mathcal{R}_p^{I,cb}$ isometrically. For $1 < p < \infty$, $p \neq 2$ the equality $\mathcal{M}_p^I = \mathcal{M}_p^{I,cb}$ is a classical open question.

4. Schur product

In this section, we replace the matricial product by the Schur product. First, it is easy to show the following proposition.

**Proposition 4.1.** Suppose $1 \leq p < \infty$. The Banach space $\mathcal{R}_p^I$ equipped with the Schur product is a commutative Banach algebra.

**Proof.** It suffices to use the equality (2.2) and the fact that $\mathcal{S}_p^I$ equipped with the Schur product is a Banach algebra (see [3] page 225). \qed

Now we will show the completely bounded analogue of this proposition. We define the pointwise product

$$P : \ell_1^I \otimes \ell_1^I \rightarrow \ell_1 \quad e_i \otimes e_j \mapsto \delta_{ij} e_i.$$ 

This map is well-defined and is completely contractive (see [3] page 211). Then, by tensoring, we obtain a completely contractive map

$$P \otimes P : (\ell_1^I \otimes \ell_1^I) \otimes (\ell_1^I \otimes \ell_1^I) \rightarrow \ell_1 \otimes \ell_1^I.$$ \hspace{1cm} (4.1)

By [5, Theorem 6.1], the map

$$\left(\ell_1^I \otimes \ell_1^I\right) \otimes_{\sigma_h} \left(\ell_1^I \otimes \ell_1^I\right) \rightarrow \left(\ell_1^I \otimes \ell_1^I\right) \otimes_{\sigma_h} \left(\ell_1^I \otimes \ell_1^I\right) \quad a \otimes b \otimes c \otimes d \mapsto a \otimes c \otimes b \otimes d$$

is completely contractive. Moreover, by [5] (5.23), we have the following commutative diagram

$$\begin{array}{ccc}
\left(\ell_1^I \otimes \ell_1^I\right) \otimes_{\sigma_h} \left(\ell_1^I \otimes \ell_1^I\right) & \rightarrow & \left(\ell_1^I \otimes \ell_1^I\right) \otimes_{\sigma_h} \left(\ell_1^I \otimes \ell_1^I\right) \\
\uparrow & & \downarrow \\
\left(\ell_1^I \otimes \ell_1^I\right) \otimes_{\sigma_h} \left(\ell_1^I \otimes \ell_1^I\right) & \rightarrow & \left(\ell_1^I \otimes \ell_1^I\right) \otimes_{\sigma_h} \left(\ell_1^I \otimes \ell_1^I\right) \\
\end{array}$$

By [5, Theorem 4.2], [5] Theorem 5.3 and by duality, we deduce that the map

$$\left(\ell_1^I \otimes \ell_1^I\right) \otimes_{\sigma_h} \left(\ell_1^I \otimes \ell_1^I\right) \rightarrow \left(\ell_1^I \otimes \ell_1^I\right) \otimes_{\sigma_h} \left(\ell_1^I \otimes \ell_1^I\right) \quad a \otimes b \otimes c \otimes d \mapsto a \otimes c \otimes b \otimes d$$

is well-defined and completely contractive. Composing this map and (4.1), we deduce a completely contractive map

$$\left(\ell_1^I \otimes \ell_1^I\right) \otimes_{\sigma_h} \left(\ell_1^I \otimes \ell_1^I\right) \rightarrow \ell_1^I \otimes \ell_1^I \quad a \otimes b \otimes c \otimes d \mapsto P(a \otimes c) \otimes P(b \otimes d).$$
With the identification $\mathcal{R}^I_{p,cb} = \ell^I_p \otimes \ell^I_p$, we obtain a completely contractive map

$$\mathcal{R}^I_{1,cb} \otimes \mathcal{R}^I_{1,cb} \longrightarrow \mathcal{R}^I_{1,cb}, \quad A \otimes B \longmapsto A \ast B.$$ 

This means that $\mathcal{R}^I_{1,cb}$ equipped with the Schur product is a completely contractive Banach algebra.

Now, recall that we have $\mathcal{R}^I_{2,cb} = \ell^I_1 \otimes \ell^I_1$ completely isometrically. Then, by a similar argument, $\mathcal{R}^I_{2,cb}$ equipped with the Schur product is also a completely contractive Banach algebra. We will use a strategy similar to that used in the proof of Theorem 3.3 for other values of $p$.

We start by defining the Schur multiplier $M_E : S^I_p(S^I_p) \to S^I_p(S^I_p)$ associated with the matrix $E = [\delta_{rt}\delta_{su}](t,r),(u,s) \in I \times I$ of $M_{I \times I}$. It is not difficult to see that $M_E$ is a completely positive contraction. Note that, for all $i, j, k, l \in I$, we have

$$M_E(e_{ij} \otimes e_{kl}) = \left(\delta_{rt}\delta_{su} \right)(t,r),(u,s) \in I \times I \ast \left(\delta_{rt}\delta_{su} \delta_{kl}\delta_{ij} \right)(t,r),(u,s) \in I \times I$$

$$= \delta_{ik}\delta_{jl}e_{ij} \otimes e_{kl}.$$ 

Now, we define the linear map

$$\eta : M_{I} \longrightarrow M_{I \times I}, A \longmapsto \left[a_{rs}\delta_{rt}\delta_{su} \right](t,r),(u,s) \in I \times I.$$

**Proposition 4.2.** Let $1 \leq p \leq \infty$. Suppose that $M_A : S^I_p \to S^I_p$ is a completely bounded Schur multiplier on $S^I_p$ associated with a matrix $A$. Then the map $M_E(M_A \otimes \text{Id}_{S^I_p})M_E$ is a bounded Schur multiplier on $S^I_p(S^I_p)$. Its associated matrix is $\eta(A)$.

**Proof.** If $i, j, k, l \in I$ and $M_A \in \mathcal{M}^I_{p,cb}$, we have

$$M_{\eta(A)}(e_{ij} \otimes e_{kl}) = \left[a_{rs}\delta_{rt}\delta_{su} \right](t,r),(u,s) \in I \times I \ast \left(\delta_{rt}\delta_{su} \delta_{kl}\delta_{ij} \right)(t,r),(u,s) \in I \times I$$

$$= \delta_{ik}\delta_{jl}a_{ij}e_{ij} \otimes e_{kl} \quad (4.2)$$

and

$$M_E(M_A \otimes \text{Id}_{S^I_p})M_E(e_{ij} \otimes e_{kl}) = \delta_{ik}\delta_{jl}M_E(M_A \otimes \text{Id}_{S^I_p})(e_{ij} \otimes e_{kl})$$

$$= \delta_{ik}\delta_{jl}a_{ij}e_{ij} \otimes e_{kl}. \quad \Box$$

**Theorem 4.3.** Suppose $1 \leq p < \infty$. The space $\mathcal{R}^I_{p,cb}$ equipped with the Schur product is a commutative completely contractive Banach algebra.
Proof. We have already seen that it suffices to prove the theorem with \(1 < p < \infty\). If \(M_A \in \mathcal{M}_{p,cb}^I\), by Proposition 4.2, we have the following commutative diagram

\[
\begin{array}{ccc}
S_p^I(S_p^I) & \xrightarrow{M_A} & S_p^I(S_p^I) \\
M_E & & M_E \\
S_p^I(S_p^I) & \xleftarrow{M_A \otimes I d_{S_p^I}} & S_p^I(S_p^I).
\end{array}
\]

We have already seen that the map \(M_A \mapsto M_A \otimes I d_{S_p^I}\) is completely contractive from \(\mathcal{M}_{p,cb}^I\) into \(\mathcal{M}_{p,cb}^{I \times I}\) and \(w^*\)-continuous. Since \(S_p^I(S_p^I)\) is reflexive, by Lemma 2.1 and by composition, the map \(M_A \mapsto M_0(A)\) from \(\mathcal{M}_{p,cb}^I\) into \(\mathcal{M}_{p,cb}^{I \times I}\) is a complete contraction and is \(w^*\)-continuous.

We denote by \(\eta^*: \mathcal{R}_{p,cb}^{I \times I} \rightarrow \mathcal{R}_{p,cb}^I\) its preadjoint. Now, by Lemma 2.5, we have for all \(i,j \in I\) and for all matrices \(A,B\) of \(\mathcal{M}_{p,cb}^{I \times I}\)

\[
\begin{align*}
&\left[\eta_*(A \otimes B)\right]_{ij} = \left\langle M_{e_{ij}}, \eta_*(A \otimes B)\right\rangle_{\mathcal{M}_{p,cb}^I,\mathcal{M}_{p,cb}^I} \\
&= \left\langle M_{\eta(e_{ij})}, A \otimes B\right\rangle_{\mathcal{M}_{p,cb}^{I \times I},\mathcal{M}_{p,cb}^{I \times I}} \\
&= \left\langle M_{[\delta_{ir}\delta_{js}\delta_{rt}\delta_{su}]_{(t,r),(u,s) \in I \times I}}, [a_{tu}b_{rs}]_{(t,r),(u,s) \in I \times I}\right\rangle_{\mathcal{M}_{p,cb}^{I \times I},\mathcal{M}_{p,cb}^{I \times I}} \\
&= a_{ij}b_{ij} \\
&= [A \ast B]_{ij}.
\end{align*}
\]

Thus we conclude that if \(A, B \in \mathcal{M}_{p,cb}^{I \times I}\) we have \(\eta_*(A \otimes B) = A \ast B\). By Proposition 3.1 and by density of \(\mathcal{M}_{p,cb}^{I \times I} \otimes \mathcal{M}_{p,cb}^{I \times I}\) in \(\mathcal{M}_{p,cb}^{I \times I} \otimes \mathcal{M}_{p,cb}^{I \times I}\), we deduce that the map

\[
\begin{align*}
\mathcal{M}_{p,cb}^{I \times I} & \xrightarrow{\otimes \otimes} \mathcal{M}_{p,cb}^{I \times I} \xrightarrow{\eta_*} \mathcal{R}_{p,cb}^{I} \\
A \otimes B & \mapsto A \ast B
\end{align*}
\]

admits a unique bounded extension from \(\mathcal{R}_{p,cb}^{I} \otimes \mathcal{R}_{p,cb}^{I}\) into \(\mathcal{R}_{p,cb}^{I}\). Moreover, this map is completely contractive. Finally, we complete the proof by a straightforward approximation argument with Lemma 2.5. \(\square\)

Now, we will give a more simple proof of this theorem. It is easy to see that \(\eta\) induces a completely isometric map \(\eta: S_p^I \rightarrow S_p^I(S_p^I)\). Moreover, by the computation (4.2), its range is clearly 1-completely complemented by \(M_E: S_p^I(S_p^I) \rightarrow S_p^I(S_p^I)\). We denote by \(\eta^{-1}: \eta(S_p^I(S_p^I)) \rightarrow S_p^I\) the inverse map of \(\eta\).
For all $B \in \eta(S^I_p(S^I_p))$, we have $\eta^{-1}(B) = [b_{(t,r),(s,s)}]_{r,s \in I}$. Finally, for all $i,j,k,l \in I$ we have

$$
\eta M A \eta^{-1} M_E (e_{ij} \otimes e_{kl}) = \delta_{ik} \delta_{j_l} \eta M A \eta^{-1} (e_{ij} \otimes e_{kl})
$$

$$
= \delta_{ik} \delta_{j_l} \eta M A (\delta_{ir} \delta_{js} \delta_{kr} \delta_{ls})_{r,s \in I}
$$

$$
= \delta_{ik} \delta_{j_l} \eta M \eta(A) (e_{ij} \otimes e_{kl})
$$

where we have used the computation (4.2) in the last equality.

Hence we have the following commutative diagram

$$
\begin{array}{ccc}
S^I_p(S^I_p) & \xrightarrow{M_{\eta(A)}} & S^I_p(S^I_p) \\
\downarrow{M_E} & & \downarrow{\eta} \\
\eta(S^I_p(S^I_p)) & & \eta(S^I_p(S^I_p)) \\
\uparrow{\eta^{-1}} & & \uparrow{\eta^{-1}} \\
S^I_p & \xrightarrow{M_A} & S^I_p
\end{array}
$$

We conclude with an argument similar to that used in the proof of Theorem 4.3.

5. Isometric multipliers

The next result is the noncommutative version of a theorem of Parrott [17] and Strichartz [27] which states that every isometric Fourier multiplier on $L^p(G)$ for $1 \leq p \leq \infty$, $p \neq 2$, is a scalar multiple of an operator induced by a translation.

**Theorem 5.1.** Suppose $1 \leq p \leq \infty$, $p \neq 2$. An isometric Schur multiplier on $S^I_p$ is defined by a matrix $[a_{ij}, b_{ij}]$ with $a_{ij}, b_{ij} \in \mathbb{T}$.

**Proof.** Suppose that $M_C$ is an isometric Schur multiplier on the Banach space $S^I_p$ defined by a matrix $C$. First, we observe that $M_C$ is onto. Indeed, for all $i,j \in I$, we have $M_C(e_{ij}) = c_{ij} e_{ij}$. Then $c_{ij} \neq 0$ since $M_C$ is one-to-one. Consequently $e_{ij}$ belongs to the range of $M_C$. By density, we conclude that $M_C$ is onto.

Now we use the theorem of Arazy [1] which describes the onto isometries on $S^I_p$. Then there exists two unitaries $U = [u_{ij}]$ and $V = [v_{ij}]$ of $B(l^2_I)$ satisfying for all $A \in S^I_p$

$$
C \ast A = U A V \quad \text{or} \quad C \ast A = U A^T V.
$$
Examine the first case, we have for all \( k, l \in I \)

\[ U e_{kl} V = C * e_{kl}. \]

Hence, for all \( i, j \in I \), we have the equality

\[ [U e_{kl} V]_{ij} = [C * e_{kl}]_{ij}. \]

Since

\[ [U e_{kl} V]_{ij} = u_{ik} v_{lj} \]

we have

\[ u_{ik} v_{lj} = \begin{cases} c_{kl} & \text{if } i = k \text{ and } j = l \\ 0 & \text{if } i \neq k \text{ or if } j \neq l. \end{cases} \]

Then \( u_{kk} v_{ll} = c_{kl} \). Each \( c_{kl} \) is non null since the image of each \( e_{kl} \) by the map \( M_C \) cannot be null. Then, for all \( k \) and all \( l \), we have \( u_{kk} \neq 0 \) and \( v_{ll} \neq 0 \). And for \( i \neq k \), we have \( u_{ik} v_{lj} = 0 \). Then if \( i \neq k \), we have \( u_{ik} = 0 \). Now if \( j \neq l \), we have \( u_{kk} v_{lj} = 0 \). Then if \( j \neq l \), we have \( v_{lj} = 0 \). Finally, for all \( i, j \in I \), we define the complex numbers \( a_i = u_{ii} \) and \( b_j = v_{jj} \). Since the diagonal matrices \( U \) and \( V \) are unitaries, we have \( a_i, b_j \in \mathbb{T} \). Thus we have the required form.

Examine the second case. We have for all \( k, l \in I \)

\[ U e_{lk} V = C * e_{lk}. \]

We deduce that, for all \( i, j, k, l \in I \), we have

\[ [U e_{lk} V]_{ij} = [C * e_{lk}]_{ij}. \]

Since

\[ [U e_{lk} V]_{ij} = u_{il} v_{kj} \]

we obtain \( u_{il} v_{kj} = c_{kl} \) and \( u_{il} v_{kj} = 0 \) if \( i \neq k \) or if \( j \neq l \). Each \( c_{kl} \) is non null since the image of each \( e_{kl} \) by the map \( M_C \) cannot be null. Then for all \( k, l \) we have \( u_{kl} \neq 0 \) and \( v_{kl} \neq 0 \). Thus the second case is absurd (if \( \text{card}(I) > 1 \)).

The converse is straightforward.

**Remark 5.2.** It is easy to see that an isometric Schur multiplier on \( S^f_2 \) is defined by a matrix \([a_{ij}]\) with \( a_{ij} \in \mathbb{T} \).

The next result is the noncommutative version of a theorem of Figà-Talamanca [10] which states that the space of bounded Fourier multipliers is the closure in the weak operator topology of the span of translation operators.

**Theorem 5.3.** Suppose \( 1 \leq p < \infty \).

1. The space \( \mathfrak{M}^f_{p, cb} \) of completely bounded Schur multipliers on \( S^f_p \) is the closure of the span of isometric Schur multipliers in the weak* topology and in the weak operator topology.
2. The space \( \mathfrak{M}^f_p \) of bounded Schur multipliers on \( S^f_p \) is the closure of the span of isometric Schur multipliers in the weak* topology and in the weak operator topology.

**Proof.** We will only prove the part 1. The proof of the part 2 is similar.

It is easy to see that an isometric Schur multiplier on \( S^f_2 \) is completely isometric. This fact allows us to consider the span of isometric Schur multipliers in \( \mathfrak{M}^f_{p, cb} \). Let \( C \) be a matrix of \( \mathfrak{M}^f_{p, cb} \). Suppose
that $C$ belongs to the orthogonal of the set of isometric Schur multipliers. Thus, we have for any isometric multiplier $M_{[a,b]}$ (with $a_i, b_j \in \mathbb{T}$)

$$0 = \left\langle M_{[a,b]}, C \right\rangle_{\mathfrak{M}_{p,cb}^I} = \lim_j \sum_{i,j \in J} a_i b_j c_{ij}.$$ 

Let $i_0, j_0$ be elements of $I$. Now, we choose the $a_i$’s, $b_j$’s, $a'_i$’s and $b'_j$’s such that $a_i = b_j = 1$ for all $i, j \in I$, $a'_i = -1$ if $i \neq i_0$, $a'_{i_0} = 1$, $b'_j = -1$ if $j \neq j_0$ and $b'_{j_0} = 1$. Then, we have

$$0 = \lim_j \sum_{i,j \in J} a_i b_j c_{ij} + \lim_j \sum_{i,j \in J} a'_i b'_j c_{ij} + \lim_j \sum_{i,j \in J} a'_i b_j c_{ij} + \lim_j \sum_{i,j \in J} a_i b'_j c_{ij} = 4c_{i_0 j_0}.$$ 

Hence $c_{i_0 j_0} = 0$. It follows that $C = 0$. Then, we deduce that the space $\mathfrak{M}_{p,cb}^I$ of completely bounded Schur multipliers is the closure of the span of isometric Schur multipliers in the weak* topology. Moreover, this topology is more finer that the weak operator topology. Thus, we have proved the theorem. □

Acknowledgment
The author is grateful to his thesis adviser Christian Le Merdy for support and advice. He also thanks Jean-Christophe Bourin, Pierre Fima and Eric Ricard for fruitful discussions and the anonymous referee for useful comments.

References
[1] Arazy, J.: The isometries of $C_p$. Israel J. Math. 22, 247–256 (1975)
[2] Bergh, J., Löfström, J.: Interpolation spaces. Springer-Verlag, Berlin (1976)
[3] Blecher, D., Le Merdy, C.: Operator algebras and their modules-an operator space approach. Oxford University Press (2004)
[4] Blecher, D., Paulsen, V.: Tensor products of operator spaces. J. Funct. Anal. 99, 262–292 (1991)
[5] Daws, M.: $p$-Operator Spaces and Figà-Talamanca-Herz algebras. J. Operator Theory 63, 47–83 (2010)
[6] Daws, M.: Representing multipliers of the Fourier algebra on non-commutative $L^p$ spaces (2009).
[arXiv:0906.5128v2[math.FA]]
[7] Effros E., Ruan, Z.-J.: Operator spaces. Oxford University Press (2000)
[8] Effros, E., Ruan Z.-J.: Operator space tensor products and Hopf convolution algebras. J. Operator Theory 50, 131–156 (2003)
[9] Eymard, P.: Algèbres $A_p$ et convoluteurs de $L^p$. In Séminaire Bourbaki, vol. 1969/1970, Exposés 364–381. Springer-Verlag (1971)
[10] Figà-Talamanca, A.: Translation invariant operators in $L^p$. Duke math. 32, 495–501 (1965)
[11] Harcharras, A.: Fourier analysis, Schur multipliers on $S^p$ and noncommutative $\Lambda(p)$-sets. Studia math. 137, 203–260 (1999)
[12] Herz, C.: The theory of $p$-spaces with an application to convolution Operators. Trans. Amer. Math. Soc. 154, 69–82 (1971)
[13] Hladnik, M.: Compact Schur multipliers. Proc. Amer. Math. Soc. 128, 2585–2591 (2000)
[14] Lambert A., Neufang M., Runde, V.: Operator space structure and amenability for Figà-Talamanca-Herz algebras. J. Funct. Anal. 211, 245–269 (2004)
[15] Larsen, R.: An introduction to the theory of multipliers. Springer-Verlag (1971)
[16] Neuwirth, S.: Cycles and 1-unconditional matrices. Proc. London Math. Soc. 93, 761–790 (2006)
[17] Parott, S. K.: Isometric multipliers. Pacific J. Math. 25, 159–166 (1968)
[18] Paulsen, V.: Completely bounded maps and operator algebras. Cambridge University Press (2002)
[19] Pier, J.-P.: Amenable Locally Compact Groups. Wiley-Interscience (1984)
[20] Pisier, G.: The operator Hilbert space $OH$, complex interpolation and tensor norms. Mem. Amer. Math. Soc. 122 (1996)
[21] Pisier, G.: Non-commutative vector valued $L_p$-spaces and completely $p$-summing maps. Astérisque 247 (1998)
[22] Pisier, G.: Similarity problems and completely bounded maps. Lecture Notes in Mathematics 1618. Expanded edition, Springer-Verlag (2001)
[23] Pisier, G.: Introduction to operator space theory. Cambridge University Press (2003)
[24] Runde, V.: Operator Figà-Talamanca-Herz algebras. Studia Math. 155, 153–170 (2003)
[25] Runde, V.: Representations of locally compact groups on $QSL_p$-spaces and a $p$-analog of the Fourier-Stieltjes algebra. Pacific J. Math. 221, 379–397 (2005)
[26] Spronk, N.: Measurable Schur multipliers and completely bounded multipliers of the Fourier algebras. Proc. London Math. Soc. 89, 161–192 (2004)
[27] Strichartz, R. S.: Isomorphisms of group algebras. Proc. Amer. Math. Soc. 17, 858–862 (1966)
[28] Xu, Q.: Interpolation of Schur multiplier spaces. Math. Z. 235, 707–715 (2000)