VERTEX ALGEBRAS ASSOCIATED WITH HYPERTORIC VARIETIES

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ABSTRACT. We construct a family of vertex algebras associated with a family of symplectic singularity/resolution, called hypertoric varieties. While the hypertoric varieties are constructed by a certain Hamiltonian reduction associated with a torus action, our vertex algebras are constructed by (semi-infinite) BRST reduction. The construction works algebro-geometrically and we construct sheaves of \( \hbar \)-adic vertex algebras over hypertoric varieties which localize the vertex algebras. We show when the vertex algebras are vertex operator algebras by giving explicit conformal vectors. We also show that the Zhu algebras of the vertex algebras, associative \( \mathbb{C} \)-algebras associated with non-negatively graded vertex algebras, gives a certain family of filtered quantizations of the coordinate rings of the hypertoric varieties.

1. Introduction

Hypertoric varieties are a family of symplectic singularities and their symplectic resolutions. They are constructed by Hamiltonian reduction of a symplectic vector space by the action of a torus, and were originally studied as hyperkähler manifolds by R. Bielawski and A. S. Dancer in [BD]. It is well known that a hypertoric variety \( X \) has the universal family of \( \mathbb{C}^\times \)-equivariant Poisson deformations \( \tilde{X} \) over the vector space \( g^* \) where \( g^* \) is the dual of the Lie algebra of the torus of the Hamiltonian reduction constructing the hypertoric variety \( X \) (See [KV], [L2]). By using quantum Hamiltonian reduction, I. Musson and M. Van den Bergh in [MV] constructed a quantization of the hypertoric varieties, which we call quantized hypertoric algebras or hypertoric enveloping algebras, and studied its representation theory. This construction admits a certain localization as discussed in [BeKu] and [BLPW]. That is, we may construct a sheaf of noncommutative \( \mathbb{C}[\hbar] \)-algebras over the hypertoric variety whose algebra of global sections can be identified with the quantized hypertoric algebra. Moreover, the quantum Hamiltonian reduction can be interpreted as a certain BRST reduction as studied in [K]. In [L1] and [L2], I. Losev studied the isomorphism classes of filtered quantizations of the coordinate ring \( \mathbb{C}[X] \) of the hypertoric variety \( X \) and showed that there existed a universal family of filtered quantizations of \( \mathbb{C}[X] \) by using the result of [BeKa]. Each quantized hypertoric algebra is obtained as a fiber of the universal family of filtered quantizations.

Affine \( W \)-algebras are a family of vertex algebras which generalizes affine vertex algebras associated with affine Lie algebras and the Virasoro vertex algebra. The affine \( W \)-algebras were constructed by quantized Drinfel’d-Sokolov reduction in [FF2] and [FKW]. The construction can be interpreted as a certain quantization of Hamiltonian reduction of infinite-dimensional manifolds. Such a quantization of infinite-dimensional Hamiltonian reduction is called semi-infinite reduction or (semi-infinite) BRST reduction/cohomologies. Properties of the BRST cohomologies associated with the quantized Drinfel’d-Sokolov reduction, including the

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vanishing of higher cohomologies, were extensively studied by T. Arakawa in [A1], [A2]. In [AKM], T. Arakawa, F. Malikov and the author introduced the BRST reduction for sheaves of $\hbar$-adic vertex algebras over Poisson varieties and showed that the affine $W$-algebras at critical level admitted localization as sheaves over the corresponding Slodowy varieties. The resulting sheaves of $\hbar$-adic vertex algebras can be understood as quantization of sheaves of vertex Poisson algebras called jet bundles over the Slodowy varieties.

In this paper, we construct a new family of vertex algebras and study their structure. Our construction is based on a semi-infinite BRST reduction associated with the Hamiltonian reduction constructing the hypertoric varieties. Moreover, our construction also works for sheaves of $\hbar$-adic vertex algebras, and our vertex algebras admits a certain localization. Namely, we construct a sheaf of $\hbar$-adic vertex algebras over the universal family of Poisson deformations $\tilde{X}$ of the hypertoric variety $X$ by using the BRST reduction, and then the vertex algebra of its global sections coincides with our vertex algebra associated with $X$. As a corollary of the sheaf-theoretic construction, we describe the vertex algebra by a certain affine local coordinate of $\tilde{X}$, and show that the sheaf of $\hbar$-adic vertex algebras is locally isomorphic to the tensor product of a $\beta\gamma$-system and a Heisenberg vertex algebra (Proposition 6.1). By this isomorphism, we have a free field realization of our vertex algebra, which is an analog of the Wakimoto realization for affine vertex operator algebras (Proposition 7.3). The vertex algebra may or may not be a vertex operator algebra. We determine when the vertex algebra is a vertex operator algebra by constructing a conformal vector when it is a vertex algebra (Proposition 8.6).

The Zhu algebra of a $\mathbb{Z}_{\geq 0}$-graded vertex algebra is an associative algebra introduced by Y. Zhu in [Z] whose representation theory reflects fundamental aspects of the representation theory of the original vertex algebra. We show that the Zhu algebra of our vertex algebra is a certain family of filtered quantizations of the coordinate ring $\mathbb{C}[X]$, which include the universal family of quantizations (Proposition 9.7).

We summarize the content of each section. In Section 3, we summarize the definition and fundamental properties of the hypertoric varieties. We explicitly construct certain local coordinates which trivialize the Hamiltonian reduction in Section 3.4 and Section 3.5. In Section 4, vertex algebras, vertex Poisson algebras and $\hbar$-adic vertex algebras are introduced. In Section 5, we introduce the main object of this paper, the semi-infinite BRST reduction associated with the hypertoric varieties. In Section 5.1, we review the Clifford vertex superalgebras, an ingredient of the BRST cohomology. In Sections 5.2–5.4, we construct the jet bundle over a hypertoric variety by the BRST reduction. The results in these sections are used in the following sections. In Section 5.5, we construct a sheaf of $\hbar$-adic vertex algebra over the hypertoric variety by the BRST reduction. The cochain complex of the BRST cohomology. In Section 6, we study the local structure of the resulting sheaf of $\hbar$-adic vertex algebras by using the local coordinates in Section 5.4 and Section 5.5. In Section 7, we construct a vertex algebra from the $\hbar$-adic vertex algebra of global sections of our sheaf by using a certain symmetry of equivariant torus action on the $\hbar$-adic vertex algebras. We call the obtained vertex algebra a hypertoric vertex algebra. We also construct an analog of Wakimoto realization in Section 7. In Section 8, we determine when the hypertoric vertex algebra is a vertex operator algebra and construct its conformal vector if it is. Finally, in Section 9, we consider the Zhu algebra of the hypertoric vertex algebra.
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2. Preliminaries

Let $G$ be a torus and $V$ be a $G$-module. We denote the subset of all $G$-invariant elements of $V$ by $V^G$. For a character $\theta : G \to \mathbb{C}^*$, we denote the subset of all $G$-semi-invariant elements of weight $\theta$ by $V^{G,\theta}$. For a fractional character $\theta \in \text{Hom}(G, \mathbb{C}^*) \otimes \mathbb{Q}$ we also consider the space $V^{G,\theta}$ but it is zero unless $\theta \text{Hom}(G, \mathbb{C}^*)$. For an element $v \in V$, let $G_v = \{ g \in G \mid g \cdot v = v \}$ be the stabilizer of $v$.

For a commutative algebra $A$ over $\mathbb{C}$, let $\text{Spec} \ A$ be the affine scheme associated with $A$. For a commutative graded algebra $A = \bigoplus_{n \in \mathbb{Z}} A_n$, let $\text{Proj} \ A$ be the projective scheme over $\text{Spec} \ A_0$, which is associated with $A$. Throughout the paper, we only consider integral, separated and reduced schemes over $\mathbb{C}$. We call them varieties.

Let $X$ be a variety over $\mathbb{C}$. For a sheaf $\mathcal{F}$ on $X$ and an open subset $U \subset X$, we denote the set of local sections of $\mathcal{F}$ on $U$ by $\mathcal{F}(U)$ or $\Gamma(U, \mathcal{F})$. We denote the structure sheaf of $X$ by $\mathcal{O}_X$ and the coordinate ring of $X$ by $\mathbb{C}[X] = \mathcal{O}_X(X)$.

3. Hypertoric varieties

In this section, we recall the definition and fundamental properties of hypertoric varieties. The definition is given by Hamiltonian reduction by an action of a torus on a symplectic vector space. We will follow the algebraic presentation given in [BS]. We consider the same setting as one in [BeKn], and refer it for detail of our setting.

3.1. Hamiltonian torus action. Fix positive integers $1 \leq M < N$. Let $V = \mathbb{C}^N$ be an $N$-dimensional vector space, and let $G = (\mathbb{C}^*)^M$ be a $M$-dimensional torus. We consider that $G$ acts algebraically on $V$ and take a basis of $V$ such that the corresponding coordinate functions $x_1, \ldots, x_N \in V^* \subset \mathbb{C}[V]$ are weight vectors with respect to the action of $G$. Then, the action of $G$ is given by a $M \times N$ integer-valued matrix $\Delta = (\Delta_{ij})_{1 \leq i \leq M, 1 \leq j \leq N}$ as $(t_1, \ldots, t_M) \cdot x_j = t_1^{\Delta_{j1}} \cdots t_M^{\Delta_{Mj}} x_j$ for $(t_1, \ldots, t_M) \in G = (\mathbb{C}^*)^M$. Setting $\Delta_j = (\Delta_{i1})_{i=1,\ldots,M}$, the $j$-th column of the matrix $\Delta$, $\Delta_j$ is the weight of $x_j$ with respect to the $G$-action. We assume that $M \times M$ minors of $\Delta$ are relatively prime. This ensures that the map $\mathbb{Z}^N \times \Delta \to \mathbb{Z}^M$ is surjective and hence the stabilizer of a generic point is trivial.

The action of $G$ on $V$ induces an action on its cotangent bundle $T^* V = V \oplus V^*$. Set $\Delta^\pm = (\Delta, -\Delta)$, a $M \times 2N$ matrix, and let $y_1, \ldots, y_N \in V \subset \mathbb{C}[V^*]$ be dual to $x_1, \ldots, x_N$. Then, the action of $G$ on $T^* V$ is given by the matrix $\Delta^\pm$ as the action on $V$ is given by $\Delta$. We consider that $T^* V$ is a symplectic vector space with the standard symplectic form $\omega = dx_1 \wedge dy_1 + \cdots + dx_N \wedge dy_N$. Then, the action of $G$ on $T^* V$ is Hamiltonian and we have a moment map $\mu : T^* V \to \mathfrak{g}^*$ given by

\[ \mu((x_1, \ldots, x_N, y_1, \ldots, y_N)) = \left( \sum_{j=1}^N \Delta_{ij} x_j \cdot y_j \right)_{1 \leq i \leq M}, \]
where \( g = \text{Lie} G = \mathbb{C}^M \) is the Lie algebra of \( G \). Let \( A_1, \ldots, A_M \) be the standard basis of \( g = \mathbb{C}^M \). The moment map \( \mu \) induces a linear map

\[
\mu^* : g = \bigoplus_{i=1}^M \mathbb{C} A_i \longrightarrow \mathbb{C}[T^*V], \quad A_i \mapsto - \sum_{j=1}^N \Delta_{ij} x_j y_j,
\]

which we call the comoment map. By using the Poisson bracket \( \{-, -\} \) on the structure sheaf \( \mathcal{O}_{T^*V} \) of the symplectic space \( T^*V \), the induced \( g \)-action on \( \mathcal{O}_{T^*V} \) is described by the comoment map; namely, an element \( A \in g \) acts on \( \mathcal{O}_{T^*V} \) by \( A \mapsto \{\mu^*(A), -\} \).

### 3.2. Stability condition.

We identify \( Q^M \) with the space of fractional characters \( \text{Hom}(G, \mathbb{C}^\times) \otimes_{\mathbb{Z}} \mathbb{Q} \) of the torus \( G = (\mathbb{C}^\times)^M \). We fix \( \delta \in Q^M \) which we call a stability parameter.

Let \( S \subseteq T^*V \) be a subvariety of \( T^*V \) which is closed under the action of \( G \). A point \( p \in S \) is called \( \delta \)-semistable if there exists an \( m \in \mathbb{Z}_{>0} \) such that we have a function \( f \in \mathbb{C}[S]^G, m\delta \) with \( f(p) \neq 0 \). A point \( p \in S \) is called \( \delta \)-stable if, in addition, its stabilizer \( G_p \) is finite. We denote the subset of all \( \delta \)-semistable points \( S^\delta_{\text{ss}} \) or simply \( S_{\text{ss}} \). Also the set of all \( \delta \)-stable points are denoted \( S^\delta_{\text{st}} \) or simply \( S^\delta \).

The stability parameter \( \delta \) is said to be effective if \( S^\delta_{\text{ss}} \neq \emptyset \). We say that two effective \( \delta \)-semistable points \( \delta_1, \delta_2 \) such that \( S^\delta_{\text{ss}} \cap S^{\delta_2}_{\text{ss}} \neq \emptyset \) are equivalent if \( S^\delta_{\text{ss}} = S^{\delta_2}_{\text{ss}} \). In the above situation, we have a rational polyhedral fan \( \Delta(G, S) \) in \( Q^M \), called the G.I.T. fan, whose support is the set of all effective parameters \( \delta \) such that \( S^\delta_{\text{ss}} \neq \emptyset \) and whose walls are given by all stability parameters \( \delta \) such that \( S^\delta_{\text{ss}} \neq S^{\delta_2}_{\text{ss}} \). Under our assumption on the matrix \( \Delta \), the maximal cones of \( \Delta(G, T^*V) \) are \( M \)-dimensional.

We call such cones \( M \)-cones. The matrix \( \Delta \) is said to be unimodular if every \( M \times M \) minor of \( \Delta \) takes values in \( \{-1, 0, 1\} \).

### 3.3. Definition of hypertoric varieties.

Now we define hypertoric varieties. Fix an effective stability parameter \( \delta \in Q^M \), and let \( \mathfrak{X} = (T^*V)^{\delta}_{\text{ss}} \subseteq T^*V \) be the subset of all \( \delta \)-semistable points of \( T^*V \). For any \( \chi \in \mathfrak{g}^\times \), the level set \( \mu^{-1}(\chi) \) of level \( \chi \) with respect to the moment map \( \mu : T^*V \longrightarrow \mathfrak{g}^\times \) is closed under the action of \( G \).

For a subset \( S \subseteq T^*V \), two points \( p, q \in S \) are said to be \( S \)-equivalent if the closed \( G \)-orbits \( G \cdot p \) and \( G \cdot q \) intersect in \( S \).

Then, we define a hypertoric variety associated with the action of \( G \) and the stability parameter \( \delta \) as follows:

**Definition 3.1.** A hypertoric variety \( X_\delta \) associated with the action of \( G \) on \( T^*V \) and the stability parameter \( \delta \) is given by the quotient space

\[
X_\delta = (\mu^{-1}(0) \cap \mathfrak{X})/\sim
\]

where \( \sim \) is the \( S \)-equivalence.

Recall that the \( G \)-action on \( T^*V \) induces an action of \( G \) on the structure sheaf \( \mathcal{O}_{T^*V} \). By the fundamental fact of the geometric invariant theory, the hypertoric variety \( X_\delta \) is constructed as a projective scheme over \( X_0 = \text{Spec} \mathbb{C}[\mu^{-1}(0)]^G \);

\[
(2) \quad X_\delta \simeq \text{Proj} \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{C}[\mu^{-1}(0)]^{G,m\delta}.
\]

In the following, we summarize fundamental properties of hypertoric varieties.

**Proposition 3.2** (HN, Proposition 6.2; see also BeKu, Corollary 4.13). If \( \delta \) is in the interior of a \( M \)-cone of \( \Delta(G, \mu^{-1}(0)) \) then the hypertoric variety \( X_\delta \) is an orbifold. It is smooth if and only if \( \delta \) is in the interior of a \( M \)-cone of \( \Delta(G, \mu^{-1}(0)) \) and \( \Delta \) is unimodular. Moreover, the walls of the G.I.T. fan are \( \sum_{j \in J} Q \Delta_j \), where \( J \subseteq \{1, \ldots, N\} \) is any subset such that \( \dim_{\mathbb{Q}}(\sum_{j \in J} Q \Delta_j) = M - 1 \).
Lemma 3.3 ([BCKu], Lemma 4.7). The moment map $\mu$ is flat and $\mu^{-1}(0)$ is a reduced complete intersection in $T^*V$.

In the rest of the paper, we fix a unimodular matrix $\Delta$ and an effective stability parameter $\delta$ which lies in the interior of a $M$-cone of $\Delta(G,\mu^{-1}(0))$. By Proposition 3.2 for such $\Delta$ and $\delta$, we have a resolution of singularity $X_\delta \to X_0$. We also denote $X_0$ simply $X$. We denote the morphism of the resolution $\pi$; i.e. $\pi : X \to X_0$. Note that the symplectic structure on $T^*V$ induces a symplectic structure on $X$. It also induces a Poisson structure on $X_0$ and the morphism $\pi$ preserves these Poisson structures; i.e. we have a homomorphism of Poisson algebras $\mathcal{O}_{X_0} \to \mathcal{O}_X$.

Now we consider a certain basic fact for the semistable locus $X = (T^*V)_{\delta}^s$ with respect to the stability condition $\delta \in \mathbb{Q}^M$.

In the rest, we identify the space of fractional parameters $\text{Hom}(G,\mathbb{C}^*) \otimes \mathbb{Q}$, its dual space and $\mathbb{Q}^M$. We also identify the natural pairing between these spaces and the standard inner product of $\mathbb{Q}^M$, and denote them $(\cdot,\cdot)$. We denote the set of common zeros of the polynomials $f_1, \ldots, f_r \in T^*V$ by $T^*V[f_1, \ldots, f_r]$.

Lemma 3.4 ([BCKu], Lemma 4.3). For a point $p \in T^*V$, we set the subsets of indices $J_1 = \{j \mid x_j(p) \neq 0\}$ and $J_2 = \{j \mid y_j(p) \neq 0\}$. Then, $p \in (T^*V)^s_\delta$ if and only if $\delta \in \sum_{j \in J_1} \mathbb{Q}_{\geq 0} \Delta_j + \sum_{j \in J_2} \mathbb{Q}_{\leq 0} \Delta_j$.

Proposition 3.5. For a semistable point $p \in X = (T^*V)^s_{\delta}$, there exists a subset of indices $\{j_1, \ldots, j_m\} \subset \{1, \ldots, N\}$ such that $x_{j_i}(p) \neq 0$ or $y_{j_i}(p) \neq 0$ for any $i = 1, \ldots, M$ and $\det(\Delta_{j_1}, \ldots, \Delta_{j_m}) = \pm 1$.

Proof. Set $J_1 = \{j \mid x_j(p) \neq 0\}$, $J_2 = \{j \mid y_j(p) \neq 0\}$ and $J = J_1 \cup J_2$. By Lemma 3.4, $p$ is $\delta$-semistable if and only if $\delta \in \sum_{j \in J_1} \mathbb{Q}_{\geq 0} \Delta_j + \sum_{j \in J_2} \mathbb{Q}_{\leq 0} \Delta_j$. Thus, we have $\delta \in \sum_{j \in J} \mathbb{Q} \Delta_j \subset \mathbb{Q}^M$. Since we assume that the hypertoric variety $X_\delta$ is smooth, we have $\sum_{j \in J} \mathbb{Q} \Delta_j = \mathbb{Q}^M$ by Proposition 3.2. Take $j_1, \ldots, j_m \in J$ such that $\Delta_{j_1}, \ldots, \Delta_{j_m}$ are linearly independent. Then, $\det(\Delta_{j_1}, \ldots, \Delta_{j_m}) = \pm 1$ because the matrix $\Delta$ is unimodular.

3.4. Local trivialization of Hamiltonian reduction. Now we construct an affine open covering of $X$ which trivializes the Hamiltonian reduction with respect to $G$.

Fix a subset of indices $J = \{j_1, \ldots, j_m\} \subset \{1, \ldots, N\}$ such that the minor $\det(\Delta_{j_1}, \ldots, \Delta_{j_m}) = \pm 1$. We set

$$
\mathcal{U}_J = \{p \in X \mid x_j(p) \neq 0 \text{ or } y_j(p) \neq 0 \text{ for any } j \in J\}.
$$

By Proposition 3.5, we have $X = \bigcup_J \mathcal{U}_J$. The stability parameter $\delta$ can be written in a linear combination of $\{\Delta_j\}_{j \in J}$: $\delta = \sum_{j \in J} \alpha_j \Delta_j$, where $\alpha_j \in \mathbb{Q}$. Note that $\alpha_j \neq 0$ for all $j \in J$ since otherwise $\delta$ lies on the G.I.T. walls by Proposition 3.2. Set $J_1 = \{j \mid \alpha_j > 0\}$ and $J_2 = \{j \mid \alpha_j < 0\}$. Then, by Lemma 3.4, we have $x_j(p) \neq 0$ for $j \in J_1$, $y_j(p) \neq 0$ for $j \in J_2$ and $J = J_1 \cup J_2$. Thus we have the following finer description:

(3) $$
\mathcal{U}_J = \{p \in X \mid x_j(p) \neq 0 \text{ for } j \in J_1, \ y_j(p) \neq 0 \text{ for } j \in J_2\}.
$$

We show that the Hamiltonian reduction with respect to the $G$-action is trivialized locally on each open subset $\mathcal{U}_J$. By multiplying a certain positive integer $m \in \mathbb{Z}_{>0}$ to $\delta = \sum_{j \in J} \alpha_j \Delta_j$, we have $m\delta = \sum_{j \in J}(m\alpha_j)\Delta_j$ so that $m\alpha_j \in \mathbb{Z}$. Since the weight of $x_j$ with respect to the $G$-action is $\Delta_j$, we have a polynomial

$$
f_J = \prod_{j \in J_1} x_j^{m\alpha_j} \prod_{j \in J_2} y_j^{-m\alpha_j} \in \mathbb{C}[X]^G \mathcal{O}_{\mathcal{U}_J}^G.
$$
of weight $m\delta$ such that $f_j(p) \neq 0$ for any $p \in U_J$. Note that $x_j^{-1} = f_j^{-1}(f_jx_j^{-1}) \in \mathcal{O}_X(U_J)$ (resp. $y_j^{-1} = f_j^{-1}(f_j y_j^{-1}) \in \mathcal{O}_X(U_J)$) for $j \in J_1$ (resp. $j \in J_2$). Since $\det(\Delta_j)_{j \in J} = \pm 1$, for each $i = 1, \ldots, M$, there exist $\lambda_{ij} \in \mathbb{Z}$ for $j \in J$ such that $\sum_{j \in J} \lambda_{ij} \Delta_j = e_i$, where $e_i$ is the $i$-th standard basis of $\mathbb{Z}^M$. Set

$$T_i^j = \prod_{j \in J_1} x_j^{\lambda_{ij}} \prod_{j \in J_2} y_j^{-\lambda_{ij}} \in \mathcal{O}_X(U_J).$$

Then, $T_i^j$ is a local section of weight $e_i$ with respect to the $G$-action and it is invertible in $\mathcal{O}_X(U_J)$ for $i = 1, \ldots, M$. In the following, we also write simply $T_i$ when there is no chance to confuse. For each $j \notin J$, we have $G$-invariant local sections

$$a_j^* = x_jT_1^{-\Delta_{1j}} \cdots T_M^{-\Delta_{Mj}}, \quad a_j' = y_jT_1^{\Delta_{1j}} \cdots T_M^{\Delta_{Mj}} \in \mathcal{O}_X(U_J).$$

Again we also write simply $a_{ij}^*$, $a_{ij}'$ instead of $a_j^*_{ij}$, $a_j'^{ij}$ when there is no confusion. Note that $\{j\}$ for $j \notin J$, $J_1$ and $J_2$ are disjoint with one another, and hence $T_1, \ldots, T_M$ contain at most one from each symplectic pair $(x_k, y_k)$ for $k = 1, \ldots, N$. Thus we have $\{a_j, a_j'\} = \{y_j, x_j\} = 1$ for $j \notin J$ and $\{T_i, a_j^*\} = \{T_i, a_j'\} = \{T_i, T_j\} = 0$ for $i, i' = 1, \ldots, M$ and $j \notin J$.

Now we describe the trivialization of the Hamiltonian reduction locally on $\mathfrak{U}_J$. For $i = 1, \ldots, M$, put $\gamma_i = \mu^*(A_i) = \sum_{j = 1}^N \Delta_{ij} x_j y_j \in \mathcal{O}_X(\mathfrak{X})$. Then we have an identity

$$\mathcal{O}_X(U_J) = \mathbb{C}[a_j^*, a_j' \mid j \notin J] \otimes \mathbb{C}[T_1^{\pm}, \ldots, T_M^{\pm}] \otimes \mathbb{C}[\gamma_1, \ldots, \gamma_M].$$

Indeed, we can describe the generators of $\mathcal{O}_X(U_J)$ as polynomials of the generators in the right hand side. For $j \notin J$, we have

$$x_j = a_j^* T_1^{-\Delta_{1j}} \cdots T_M^{-\Delta_{Mj}}, \quad y_j = a_j' T_1^{\Delta_{1j}} \cdots T_M^{\Delta_{Mj}}.$$

Note that $a_j^* a_j' = x_j y_j$ for $j \notin J$. Since the matrix $(\Delta_{ij})_{i,j=1,\ldots,M,j \notin J}$ is invertible with the inverse matrix $(\lambda_{ij})_{i,j=1,\ldots,M,j \notin J}$, from the identity $\gamma_i - \sum_{j \notin J} \Delta_{ij} a_j^* a_j' = \sum_{j \notin J} \Delta_{ij} a_j x_j y_j$, we obtain $x_j y_j = \sum_{i = 1}^M \lambda_{ij} (\gamma_i - \sum_{k \notin J} \Delta_{ik} a_k^* a_k)$ for $j \in J$. Thus, for $j \in J$, we have

$$x_j = T_1^{-\Delta_{1j}} \cdots T_M^{-\Delta_{Mj}}, \quad y_j = T_1^{\Delta_{1j}} \cdots T_M^{\Delta_{Mj}} \sum_{i = 1}^M \lambda_{ij} (\gamma_i - \sum_{k \notin J} \Delta_{ik} a_k^* a_k).$$

It is clear that a similar identity holds for $j \in J_2$. This implies that the identity (4) holds.

Note that we have $\{a_j^*, a_j'\} = \{\gamma_i, a_j\} = 0$ for $i = 1, \ldots, M, j \notin J$, and $\{\gamma_i, T_j\} = T_i$ for $i, j = 1, \ldots, M$ by the construction. We regard $\gamma_1, \ldots, \gamma_M$ as a linear basis of the Lie algebra $\mathfrak{g}$ through the homomorphism $\mu^*$. Then, the identity (4) gives an isomorphism of Poisson algebras

$$\mathcal{O}_X(U_J) \simeq \mathbb{C}[T^* \mathfrak{C}^{N-M}] \otimes \mathbb{C}[G] \otimes \mathbb{C}[\mathfrak{g}^*] \simeq \mathbb{C}[T^* \mathfrak{C}^{N-M}] \otimes \mathbb{C}[T^* \mathfrak{G}]$$

and thus we have the trivialization $\mathfrak{U}_J \simeq T^* \mathfrak{C}^{N-M} \times T^* \mathfrak{G}$. Set $U_J = (\mu^{-1}(0) \cap \mathfrak{U}_J)/G \subset X$. Since the $G$-action and the moment map $\mu$ are trivialized, we have $U_J \simeq T^* \mathfrak{C}^{N-M}$ as symplectic manifolds. Then, we have an affine open covering $X = \bigcup J U_J$ with Darboux coordinate $(a_j^*, a_j')_{j \notin J}$ for each $J$.

We denote the trivialization $\nu_J : U_J \longrightarrow U_J \times T^* \mathfrak{G}$ and the corresponding isomorphism $\nu_J^* : \mathcal{O}_{U_J} \otimes \mathcal{O}_{T^* \mathfrak{G}} \longrightarrow \mathcal{O}_{U_J \mathfrak{U}_J}$. For $I$ and $J$, we denote the coordinate transformation $\varphi_{IJ} = \nu_I \circ \nu_J^{-1} : U_I \times T^* \mathfrak{G} \longrightarrow U_J \times T^* \mathfrak{G}$ and the corresponding isomorphism $\varphi_{IJ}^* = (\nu_J^*)^{-1} \circ \nu_I^* : \mathcal{O}_{U_I} \otimes \mathcal{O}_{T^* \mathfrak{G}} \longrightarrow \mathcal{O}_{U_J} \otimes \mathcal{O}_{T^* \mathfrak{G}}$ on $U_I \cap U_J$. This induces the coordinate transformation $\varphi_{IJ} : U_J \times \mathfrak{G} \longrightarrow U_I \times \mathfrak{G}$ and the corresponding isomorphism $\varphi_{IJ}^* : \mathcal{O}_{U_I} \otimes \mathcal{O}_{\mathfrak{G}} \longrightarrow \mathcal{O}_{U_J} \otimes \mathcal{O}_{\mathfrak{G}}$ because $\gamma_1, \ldots, \gamma_M$ are
induced from the obvious isomorphism $c \colon G \to O$ coordinate transformation between $\Phi$ and $G$. Since it is the coordinate translation of the $G$-torsor $\mu^{-1}(0) \cap X \to X$.

3.5. Symplectic deformation of the hypertoric variety $X$. For the symplectic variety $X$, it is known that there exists a universal family of filtered Poisson deformations of the symplectic structure of $X$, which explicitly given as follows.

Set $\widetilde{X} = \mathbf{X} \times g^*$. We regard $\widetilde{X}$ as a smooth algebraic Poisson variety where $g^*$ is equipped with the trivial Poisson structure. We extend the moment map $\mu$ to $\widetilde{\mu} : \widetilde{X} \to g^*$ such that the corresponding comoment map $\widetilde{\mu}^* : g \to \mathbb{C}[\mathbf{X}] = \mathbb{C}[X] \otimes \mathbb{C}[g^*]$ is given by $\widetilde{\mu}^*(A_i) = \mu^*(A_i) - c_i$ where we denote the standard basis of $g \subset \mathbb{C}[g^*]$ by $c_1, \ldots, c_M$ instead of $A_1, \ldots, A_M$ in order to avoid confusion. Clearly the torus $G$ acts on $\mathbf{X}$ freely and the $G$-action preserves the preimage $\widetilde{\mu}^{-1}(0)$. Then, we define the Poisson manifold $\widetilde{X} = \mathbf{X}_G = \widetilde{\mu}^{-1}(0)/G \simeq \mathbf{X}/G$. Here the last isomorphism is induced from the obvious isomorphism $\widetilde{\mu}^{-1}(0) \simeq \mathbf{X}$ which identifies $c_i$ with $\mu^*(A_i)$ for $i = 1, \ldots, M$. By the second projection $\rho : \widetilde{X} = \mathbf{X} \times g^* \to g^*$ induces the morphism $\rho : \tilde{X} \to \tilde{g}$ of Poisson schemes, and we have $\rho^{-1}(0) \simeq X$. Note that $\tilde{X}$ is a symplectic scheme over $g^*$ and the isomorphism $\rho^{-1}(0) \simeq X$ is an isomorphism of holomorphic symplectic manifold. It is known that $\tilde{X}$ is a universal family of filtered Poisson deformations of $X$ over $\tilde{g}^* \simeq H^2(X, \mathbb{C})$, namely, the structure sheaf $\mathcal{O}_X$ is a universal family of filtered Poisson deformations of the sheaf of Poisson algebras $\mathcal{O}_X$. Moreover, the family is equivariant with respect to an action of a torus $\mathbb{G} = \mathbb{C}^\times$ which we discuss in Section 3.4. Refer [L1] for the universality of the above $\mathbb{G}$-equivariant Poisson deformations, which is based on results of [KV].

While the hypertoric varieties $X$ and $\widetilde{X}$ are constructed by Hamiltonian reduction by the action of the torus $G$, their structure sheaves can be constructed also by Hamiltonian reduction of algebras. Namely, The structure sheaf of $\widetilde{X}$ is given by the following (dual) Hamiltonian reduction

$$\mathcal{O}_X \simeq \left( p_*(\mathcal{O}_{\widetilde{X}} / \sum_{i=1}^M \mathcal{O}_{\widetilde{X}} \mu^*(A_i)) \right)^G = \left( p_*(\mathcal{O}_{\widetilde{X}} / \sum_{i=1}^M \mathcal{O}_{\widetilde{X}} (\mu^*(A_i) - c_i)) \right)^G,$$

where $\rho : \mathbf{X} \to \mathbf{X}$ is the projection. It is an algebra over $\mathbb{C}[c_1, \ldots, c_M] = \mathbb{C}[g^*]$. The hypertoric variety $X$ is the fiber of $\tilde{X} \to \tilde{g}^*$ at $c_1 = \cdots = c_M = 0$, and we have

$$\mathcal{O}_X = \left( p_*(\mathcal{O}_{\mathbf{X}} / \sum_{i=1}^M \mathcal{O}_{\mathbf{X}} \mu^*(A_i)) \right)^G.$$

We consider local trivialization of the Hamiltonian reduction of $\mathbf{X}$ by the $G$-action. Recall the affine open covering $\mathbf{X} = \bigcup_J \mathcal{U}_J$ which trivializes the Hamiltonian reduction in Section 3.4. Set $\tilde{\mathcal{U}}_J = \mathcal{U}_J \times g^* \subset \widetilde{\mathbf{X}}$ for each $J$. Then, we have an affine open covering $\mathbf{X} = \bigcup_J \mathcal{U}_J$. Since the $G$-action preserves $\mathcal{U}_J$ and it acts on $g^*$ trivially, $\tilde{\mathcal{U}}_J$ is also preserved by the $G$-action. We set $\tilde{\mathcal{U}}_J = \tilde{\mu}^{-1}(0) \cap \tilde{\mathcal{U}}_J/G$, and then we have an open covering $\tilde{\mathbf{X}} = \bigcup_J \tilde{\mathcal{U}}_J$.

By the trivialization of the Hamiltonian reduction on $\mathcal{U}_J$ discussed in Section 3.4 we have an isomorphism $\tilde{\mathcal{U}}_J \simeq T^* \mathbb{C}^{N-M} \times G \times g^* \times g^*$. The isomorphism is given by the following description of the algebra of local sections $\mathcal{O}_X(\tilde{\mathcal{U}}_J)$:

(6) \[
\mathcal{O}_X(\tilde{\mathcal{U}}_J) \simeq \mathbb{C}[a_j^*, a_j \mid j \notin J] \otimes \mathbb{C}[T_1, \ldots, T_M] \otimes \mathbb{C}[\gamma_1, \ldots, \gamma_M] \otimes \mathbb{C}[c_1, \ldots, c_M]
\]

where the local sections $a_j^*, a_j, T_i, \gamma_i$ are defined in Section 3.4. In the above local coordinate, the comoment map $\tilde{\mu}^*$ is given by $\tilde{\mu}^*(A_i) = \gamma_i - c_i$ for $i = 1, \ldots, M$. Moreover, since the $G$-action on $\mathcal{O}_X(\tilde{\mathcal{U}}_J)$ corresponds to the $g$-action $A_i \mapsto \rho A_i$. 

of vertex Poisson algebras and a sheaf of trivially, and $T_i$ has weight $\varepsilon_i$ with respect to the $G$-action for $i = 1, \ldots, M$. Therefore, we have

$$O_{\bar{\mu}^{-1}(0)}(\tilde{U}_J) \simeq \mathbb{C}[a_{i_j}, a_j | j \notin J] \otimes_{\mathbb{C}} \mathbb{C}[T^i_1, \ldots, T^i_M] \otimes_{\mathbb{C}} \mathbb{C}[c_1, \ldots, c_M].$$

and

$$O_Z(\tilde{U}_J) = O_{\bar{\mu}^{-1}(0)}(\tilde{U}_J)^G \simeq \mathbb{C}[a_{i_j}, a_j | j \notin J] \otimes_{\mathbb{C}} \mathbb{C}[c_1, \ldots, c_M].$$

It induces the isomorphism $\tilde{U}_J \simeq T^* \mathbb{C}^{N-M} \times \mathbb{C}^M$, and hence the open covering $\tilde{X} = \bigcup_J \tilde{U}_J$ is an affine open covering. Let $\tilde{\nu}_J : \tilde{U}_J \rightarrow \tilde{U}_J \times G \times g^*$ be the above trivialization, and we denote the corresponding algebra isomorphism $\tilde{\nu}_J^* : O_{\tilde{U}_J} \otimes_{\mathbb{C}} O_G \otimes_{\mathbb{C}} O_{g^*} \rightarrow O_{\tilde{U}_J}$. Then we have the coordinate transformation over $\tilde{U}_I \cap \tilde{U}_J$ for $I, J$, $\tilde{\phi}_{IJ} = \tilde{\nu}_J \circ \tilde{\nu}_I^{-1} : \tilde{U}_J \times G \times g^* \rightarrow \tilde{U}_I \times G \times g^*$, and the algebra isomorphism $\tilde{\phi}_{IJ}^* : O_{\tilde{U}_J} \otimes_{\mathbb{C}} O_G \otimes_{\mathbb{C}} O_{g^*} \rightarrow O_{\tilde{U}_I} \otimes_{\mathbb{C}} O_G \otimes_{\mathbb{C}} O_{g^*}$. This coordinate transformation induces the coordinate translation of $G$-torsor $\tilde{\phi}_{IJ} : \tilde{U}_J \times G \rightarrow \tilde{U}_I \times G$ over $\tilde{U}_I \cap \tilde{U}_J$, and the coordinate translation of local coordinates of $\tilde{X}$, $\tilde{\phi}_{IJ} : \tilde{U}_J \rightarrow \tilde{U}_I$. The corresponding algebra isomorphisms are also denoted $\tilde{\phi}_{IJ}^*$.

4. Sheaves of $h$-adic vertex algebras

In this section, we review the definitions of vertex algebras and $h$-adic vertex algebras, and we introduce certain sheaves of vertex Poisson algebras and certain sheaves of $h$-adic vertex algebras. Based on these sheaves, we will construct a sheaf of vertex Poisson algebras and a sheaf of $h$-adic vertex algebras in the next section.

4.1. Vertex algebras and $h$-adic vertex algebras. A vertex algebra $V$ is a vector space over $\mathbb{C}$ equipped with the following structure; the vacuum vector $1 \in V$, the translation operator $\partial : V \rightarrow V$ and the vertex operator $Y(a, z) = \sum_{n \in \mathbb{Z}} a(-n-1)z^n \in \text{End}_\mathbb{C}(V)[[z, z^{-1}]]$ for each $a \in V$ subject to the following axioms:

1. $Y(a, z)$ is linear with respect to $a \in V$,
2. $Y(a, z)$ is a field, i.e. $a(n)b = 0$ for any $a, b \in V$ if $n \gg 0$,
3. $Y(1, z) = Id_V$,
4. $Y(a, z)1 \in V[[z]]$ and $Y(a, z)1|_{z=0} = a$ for any $a \in V$,
5. $[\partial, Y(a, z)] = \partial Y(a, z)$ for any $a \in V$, and $\partial 1 = 0$,
6. for any $a, b \in V$, the vertex operators $Y(a, z)$ and $Y(b, w)$ are mutually local; namely, there exists $N \in \mathbb{Z}_{\geq 0}$ such that

$$(z - w)N [Y(a, z), Y(b, w)] = 0.$$ 

It is well-known that fundamental identities for vertex algebras such as $\partial a = a(-2)1$, $Y(\partial a, z) = \partial Y(a, z)$ and the operator product expansion (or so called Borcherds’ identity) follow from the above axioms. We say that the vertex algebra $V$ is commutative if $a(n) = 0$ on $V$ for any $a$ and $n \in \mathbb{Z}_{\geq 0}$.

A vertex Poisson algebra $V$ is a tuple $(V, \mathbf{1}, \partial, Y_-, (\mathbf{a}, z), Y_+((\mathbf{a}, z))$ where $Y_-(\mathbf{a}, z), Y_+((\mathbf{a}, z)) : V \rightarrow \text{End}_\mathbb{C}(V)[[z, z^{-1}]]$ are fields on $V$,

$$Y_-(a, z) = \sum_{n \in \mathbb{Z}_{\geq 0}} a(-n-1)z^n, \quad Y_+(a, z) = \sum_{n \in \mathbb{Z}_{<0}} a(-n-1)z^n$$

such that $(V, \mathbf{1}, \partial, Y_-(\mathbf{a}, z))$ is a commutative vertex algebra, and $(V, \partial, Y_+(\mathbf{a}, z))$ is a vertex Lie algebra; namely the operators $a(n)$ satisfy the following relations:

1. $a(n)b = (-1)^{n+1} \sum_{j \geq 0} (-1)^j \partial^j (b(n+j)a)/j!$,
2. $a(m)b(n)c - b(n)a(m)c = \sum_{j \geq 0} \binom{m}{j} a((j)b)(m+n-j)c$,
3. $[\partial, Y_+(a, z)] = \partial Y_+(a, z),$ and
(4) \(a(n)\) is a derivation with respect to the product \((-1)\), for any \(a, b, c \in V\) and \(m, n \in \mathbb{Z}_{\geq 0}\).

Let \(h\) be an indeterminate, which commutes with any other operators. An \(h\)-adic vertex algebra \(V\) is a tuple \((V, 1, \partial, Y(\_, \_))\) such that \(V\) is a flat \(\mathbb{C}[\![h]\!]\)-module complete in \(h\)-adic topology, the vacuum vector \(1 \in V\) and \(\mathbb{C}[\![h]\!]\)-linear map \(\partial : V \to V\) satisfy the same axiom with the above, and \(Y(\_, \_) : V \to \text{End}_{\mathbb{C}}(V)[[z, z^{-1}]]\) is \(\mathbb{C}[\![h]\!]\)-linear map such that the products \(\{a(n)\}_n\) are continuous with respect to \(h\)-adic topology, and \((V/h^N V, 1, \partial, Y(\_, \_))\) is a vertex algebra for each \(N \in \mathbb{Z}_{\geq 1}\). Note that a \(h\)-adic vertex algebra is not a vertex algebra over \(\mathbb{C}[\![h]\!]\) since \(Y(a, z)\) is not a field on \(V\). Namely for any \(N \in \mathbb{Z}_{\geq 1}\), \(Y(a, z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}\) satisfies \(a(n)b \equiv 0\) modulo \(h^N\) if \(n \gg 0\), but not \(a(n)b = 0\).

Let \((V, 1, \partial, Y(\_, \_))\) be an \(h\)-adic vertex algebra. Assume that \(V/hV\) is commutative. Then, \(Y_\circ(\_, \_):= h^{-1}Y(\_, \_)\) modulo \(h\) satisfies the axiom of vertex Lie algebras. Thus, \((V/hV, 1, \partial, Y(\_, \_))\) modulo \(h\) is a vertex Poisson algebra.

### 4.2. Jet bundles.

Let \(X\) be a scheme over \(\mathbb{C}\). Let \(J_\infty X\) be the corresponding \(\infty\)-jet scheme; i.e. \(J_\infty X\) is a scheme defined by \(\text{Hom}(\text{Spec } R, J_\infty X) = \text{Hom}(\text{Spec } R[[t]], X)\) for any \(\mathbb{C}\)-algebra \(R\). A point of \(J_\infty X\) represents an \(\infty\)-jet \(p(t) = \sum_{n=0}^\infty p_n t^n\) \((p_n \in X)\) on \(X\). A canonical morphism \(\pi_\infty : J_\infty X \to X\) is given by \(p(t) \mapsto p(0) = p_0\). We consider the direct image of the structure sheaf \(\mathcal{O}_{J_\infty X}\) of the \(\infty\)-jet scheme \(J_\infty X\) by the morphism \(\pi_\infty\). The obtained sheaf on \(X\) is denoted \(\mathcal{O}_{J_\infty X}\) by abuse of notation, and call it the jet bundle on \(X\). The corresponding homomorphism between their structure sheaves \(\pi_\infty^* : \mathcal{O}_X \to \mathcal{O}_{J_\infty X}\) is an injective homomorphism of commutative algebras. The derivation with respect to \(t\) on \(R[[t]]\) induces a derivation \(\partial\) on the jet bundle \(\mathcal{O}_{J_\infty X}\). Thus, the jet bundle \(\mathcal{O}_{J_\infty X}\) is a sheaf of commutative vertex algebras on \(X\). Moreover, when \(X\) is a Poisson scheme, the Poisson bracket \(\{\_, \_\}\) on \(\mathcal{O}_X\) induces a structure of vertex Poisson algebras on \(\mathcal{O}_{J_\infty X}\) satisfying \(f(0)g = \{f, g\}\) and \(f(n)g \equiv 0\) for \(f, g \in \mathcal{O}_X \subset \mathcal{O}_{J_\infty X}\) and \(n \in \mathbb{Z}_{\geq 1}\). For detail of the construction of vertex Poisson algebra structure, see [AKM, Lemma 2.1.3.1].

In the present paper, we consider a smooth symplectic manifold \(X\). Assume that a local Darboux coordinate \((U; x_1, \ldots, x_r, y_1, \ldots, y_r)\) is given. Then, the algebra of local sections of the structure sheaf \(\mathcal{O}_X(U)\) is the polynomial ring \(\mathbb{C}[x_1, \ldots, x_r, y_1, \ldots, y_r]\) and the Poisson bracket is given by \(\{y_i, x_j\} = \delta_{ij}, x_1, x_i = y_i, y_j = 0\). The jet bundle looks like

\[
\mathcal{O}_{J_\infty X}(U) = \mathbb{C}[x_1, (-n), \ldots, x_r, (-n), y_1, (-n), \ldots, y_r, (-n) \mid n \in \mathbb{Z}_{\geq 1}],
\]

so that we identify \(x_i = x_i(-1), y_i = y_i(-1)\) under the embedding \(\mathcal{O}_X \to \mathcal{O}_{J_\infty X}\). The derivation \(\partial\) on \(\mathcal{O}_{J_\infty X}\) is given by \(\partial(a(-n)) = na(-n-1)\) for \(a = x_i, y_i (i = 1, \ldots, r)\) and \(n \in \mathbb{Z}_{\geq 1}\). Finally, the vertex Poisson algebra structure on \(\mathcal{O}_{J_\infty X}(U)\) is given by

\[
Y_+(x_i(-1), z) = -\sum_{n \geq 1} \frac{\partial}{\partial y_i(-n)} z^{-n}, \quad Y_+(y_i(-1), z) = \sum_{n \geq 1} \frac{\partial}{\partial x_i(-n)} z^{-n}.
\]

### 4.3. \(h\)-adic \(\beta\gamma\)-systems and \(h\)-adic Heisenberg vertex algebras.

Let \(x_1, \ldots, x_N, y_1, \ldots, y_N\) be the standard coordinate functions on \(T^* \mathbb{C}^N = \mathbb{C}^{2N}\). We consider that they are Darboux coordinates with respect to the standard symplectic form. The \(h\)-adic \(\beta\gamma\)-system on \(\mathbb{C}^{2N} = T^* \mathbb{C}^N\) is an \(h\)-adic vertex algebra \(\mathcal{D}^{\text{ch}}(\mathbb{C}^{2N})_h\) such that \(\mathcal{D}^{\text{ch}}(\mathbb{C}^{2N})_h\) is isomorphic

\[
\mathcal{D}^{\text{ch}}(\mathbb{C}^{2N})_h = \mathbb{C}[\![h]\!][x_1, (-n), \ldots, x_N, (-n), y_1, (-n), \ldots, y_N, (-n) \mid n \in \mathbb{Z}_{\geq 1}]1
\]
as a $\mathbb{C}[[h]]$-module, and its OPEs are given by $x_i(z)y_j(w) \sim -h/(z-w)$, and $x_i(z)x_j(w) \sim y_i(z)y_j(w) \sim 0$ for $i, j = 1, \ldots, N$, where we denote $x_i(z) = Y(x_i(-1), 1, z)$ and $y_i(z) = Y(y_i(-1), 1, z)$. Clearly it is an $h$-adic analogue of the vertex algebra $\beta\gamma$-system.

In [AKM], we discussed localization of algebras of chiral differential operators (CDOs), including the $\beta\gamma$-system, as sheaves of $h$-adic vertex algebras on cotangent bundles; i.e. the above $h$-adic $\beta\gamma$-system gives a sheaf of $h$-adic vertex algebras on $\mathbb{C}^{2N} = T^*\mathbb{C}^N$ as follows: For the $h$-adic $\beta\gamma$-system, OPEs (and hence $(n)$-products) between vertex operators are determined by the Wick formula and thus they turn out to be bi-differential operators in the variables $x_i(-n), y_i(-n)$. Therefore, even for rational functions in $x_i(-n), y_i(-n)$, the same bi-differential operators give well-defined OPEs ($(n)$-products) between them. Therefore, we have a sheaf of $h$-adic vertex algebras $D_{T^*\mathbb{C}^N}^{h\beta\gamma}$ on $T^*\mathbb{C}^N$. See Lemma 2.2.8.1 and Theorem 2.2.10.1 in [AKM] for the detail of the above discussion.

As we discussed in the previous section, the jet bundle $O_{Jh_{T^*\mathbb{C}^N}}$ on the symplectic vector space $T^*\mathbb{C}^N$ is equipped with the vertex Poisson algebra structure. The $h$-adic $\beta\gamma$-system $D_{T^*\mathbb{C}^N}^{h\beta\gamma}$ is a quantization of $O_{Jh_{T^*\mathbb{C}^N}}$; namely, the quotient $D_{T^*\mathbb{C}^N}^{h\beta\gamma}/hD_{T^*\mathbb{C}^N}^{h\beta\gamma}$ is isomorphic to $O_{Jh_{T^*\mathbb{C}^N}}$ as vertex Poisson algebras.

Similarly we define an $h$-adic Heisenberg vertex algebra. Let $W = \bigoplus_{i=1}^M \mathbb{C}c_i$ be a vector space with a symmetric inner product $\langle \cdot, \cdot \rangle$. Consider the $h$-adic vertex algebra which is defined as $\mathbb{C}[[h]]$-module

$$V(\cdot)h(W) = \mathbb{C}[[h]] [c_1(-n), \ldots, c_M(-n) \mid n \in \mathbb{Z}_{\geq 1}],$$

and OPEs are given by $c_i(z)c_j(w) \sim h^2(c_i, c_j)/(z-w)^2$ for $i, j = 1, \ldots, M$. Clearly, it is a natural $h$-adic analogue of the usual Heisenberg vertex algebra defined by $(W, \langle \cdot, \cdot \rangle)$. This implies that the Wick formula holds for the OPEs between vertex operators of $V(\cdot)h(W)$ and hence the OPEs are defined as bi-differential operators in the variables $c_i(-n)$ for $i = 1, \ldots, M, n \in \mathbb{Z}_{\geq 1}$. Thus, by the same argument for the $\beta\gamma$-system, the $h$-adic vertex algebra induces a sheaf of $h$-adic vertex algebras on the vector space $W$. We denote the sheaf $V_{\beta\gamma}W(\cdot)$.

5. Semi-infinite BRST reduction

Now we construct a sheaf of $h$-adic vertex algebras on the hypertoric variety $\tilde{X}$ in this section. Our construction is based on a vertex algebra analog of the Hamiltonian reduction, which we call (semi-infinite) BRST reduction or BRST cohomology.

In Section 5.1 we introduce an $h$-adic variant of a fermionic vertex superalgebra called the Clifford vertex superalgebra or the free field of colored fermions. To establish fundamental properties of the BRST reduction, we first need to consider the corresponding reduction for a sheaf of vertex Poisson algebras, the jet bundle on $\tilde{X}$. In Sections 5.2–5.3 we introduce the BRST reduction for vertex Poisson algebras and study its structure. The BRST reduction for a sheaf of $h$-adic vertex algebras is defined in Section 5.4 and we show that the structure of such a sheaf of $h$-adic vertex algebras can be studied by using a certain double complex in Section 5.6.

5.1. Clifford $h$-adic vertex superalgebra. In this subsection, we introduce the Clifford $h$-adic vertex superalgebra $Cl^{vert}(\mathfrak{g} \oplus \mathfrak{g}^*)$ associated with the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ with the standard inner product $\langle \cdot, \cdot \rangle$.

We fix a basis $\mathfrak{g} = \bigoplus_{i=1}^M \mathbb{C}a_i$ and its dual basis $\mathfrak{g}^* = \bigoplus_{i=1}^M \mathbb{C}a_i^*$ with respect to $\langle \cdot, \cdot \rangle$ as previous sections. Let $\Pi \mathfrak{g}$ (resp. $\Pi \mathfrak{g}^*$) be the odd vector space corresponding to the even vector space $\mathfrak{g}$ (resp. $\mathfrak{g}^*$), and let $\Pi \mathfrak{g} = \bigoplus_{i=1}^M \mathbb{C}\psi_i$ (resp.
Πg* = ⊕_{i=1}^{M} Cψ_i^* be the odd basis corresponding to the even basis g = ⊕_{i=1}^{M} C A_i (resp. g* = ⊕_{i=1}^{M} C A_i^*). We identify the coordinate rings C[Πg] = Λ(g*), C[Πg*] = Λ(g) and C[T Πg] = Λ(g ⊗ g*) where T(W) is the exterior algebra of a vector space W. Note that the inner product ⟨·, ·⟩ on g ⊗ g* gives a Poisson superalgebra structure on Λ(g ⊗ g*): ⟨ψ_i, ψ_j⟩ = δ_{ij}, ⟨ψ_i, ψ_j⟩ = 0 = ⟨ψ_i^*, ψ_j^*⟩ for i, j = 1, ..., M. A vertex Poisson superalgebra analogue of Λ(g ⊗ g*) ("the jet bundle" over the super-manifold T Πg) is naturally constructed as follows: Define Λ^{vert}(g ⊗ g*) as an anti-commutative algebra

\[ \Lambda^{vert}(g ⊗ g*) = \bigwedge_{n ≥ 1} Cψ_i^* \wedge \bigwedge_{n ≥ 1} Cψ_i(-n)1, \]

and the Poisson structure is defined by ψ_i(m)(ψ_j(-n-1)) = δ_{m-n, 0}δ_{ij} and ψ_i(m)(ψ_j(-n-1)) = 0 = ψ_i^*(m)(ψ_j(-n-1)) for i, j = 1, ..., M and n, m ∈ Z≥0. Then Λ^{vert}(g ⊗ g*) is a vertex Poisson superalgebra. Identifying ψ_i = ψ_i(-1)1, ψ_i^* = ψ_i^*(-1)1 for i = 1, ..., M, the exterior algebra Λ(g ⊗ g*) is a subalgebra of Λ^{vert}(g ⊗ g*).

Now we consider the Clifford h-adic vertex superalgebra, a quantization of the vertex Poisson superalgebra Λ^{vert}(g ⊗ g*). Define the h-adic vertex superalgebra C^{vert}(g ⊗ g*) as a C[[h]]-module,

\[ C^{vert}(g ⊗ g*) = \bigwedge_{n ≥ 1} C[[h]]ψ_i^* \wedge \bigwedge_{n ≥ 1} C[[h]]ψ_i(-n)1, \]

where the completion of the tensor product with respect to the h-adic topology. We denote the vertex operators ψ_i(z) = Y(ψ_i, z) and ψ_i^*(z) = Y(ψ_i^*, z). Then the defining OPEs are given by

\[ ψ_i(z)ψ_j^*(w) \sim \frac{hδ_{ij}}{z - w}, \quad ψ_i(z)ψ_j(w) \sim 0 \sim ψ_i^*(z)ψ_j^*(w) \]

for i, j = 1, ..., M. These OPEs give the structure of h-adic vertex algebra on C^{vert}(g ⊗ g*), which we call the Clifford (h-adic) vertex superalgebra. Clearly we have C^{vert}(g ⊗ g*)/⟨h⟩ ≃ Λ^{vert}(g ⊗ g*) and thus the Clifford vertex superalgebra C^{vert}(g ⊗ g*) is a quantization of Λ^{vert}(g ⊗ g*).

Note that the vertex Poisson superalgebra Λ^{vert}(g ⊗ g*) and the Clifford vertex algebra C^{vert}(g ⊗ g*) are Z-graded by the degree deg(ψ_i(-n)) = −1, deg(ψ_i^*(-n)) = 1 and deg(1) = 0 for i = 1, ..., M and n ∈ Z. Let Λ^{vert,n}(g ⊗ g*) and C^{vert,n}(g ⊗ g*) be the homogeneous subspaces of degree n. Moreover we have the following decomposition of Λ^{vert,n}(g ⊗ g*) (resp. C^{vert,n}(g ⊗ g*)) as a C-vector space (resp. a C[[h]]-module)

\[ \Lambda^{vert,n}(g ⊗ g*) = \bigoplus_{p+q=n} A_{C^{vert,p}}(g*) \otimes A_{C^{vert,q}}(g), \]

\[ C^{vert,n}(g ⊗ g*) = \bigoplus_{p+q=n} A^{\otimes}_{C[[h]]^{p}}(g*) \otimes A^{\otimes}_{C[[h]]^{q}}(g), \]

where \( \bigoplus \) (resp. \( \otimes \)) is the completion of the direct sum (resp. the tensor product) with respect to the h-adic topology, and

\[ \Lambda_R^{vert}(g) = \bigwedge_{i=1}^{M} Rψ_i(-n), \quad \Lambda_R^{vert}(g*) = \bigwedge_{i=1}^{M} Rψ_i^*(-n) \]

for a commutative algebra R, and Λ^{vert,n}(g), Λ^{vert,n}(g*) are the homogeneous subspaces of degree n.
5.2. Poisson BRST reduction. In Sections 5.2 and 5.3 we construct the jet bundle of the hypertoric variety $\tilde{X}$ in terms of BRST reduction. The construction is based on the construction of jet bundles of Slodowy varieties by the BRST reduction in [AKM].

Recall that we have the moment map $\mu : T^* V \to g^*$ and semistable locus $\bar{X} \subset T^* V$ associated with the torus $G = (\mathbb{C}^*)^M$-action on the symplectic vector space $T^* V = T^* \mathbb{C}^N$. Here we took the stability parameter $\delta$ such that the Hamiltonian reduction $X = X_\delta$ is a smooth symplectic manifold. Set $\tilde{X} = \bar{X} \times g^*$ and $\tilde{\mu} : \tilde{X} \to g^*$ as in Section 5.2. Also, let $\tilde{X}$ be the hypertoric variety as we introduced in Section 5.2. The jet bundle $\mathcal{O}_{J^\infty \tilde{X}}$ on $\tilde{X}$ is a sheaf of vertex Poisson algebras. By applying the jet scheme functor $J^\infty$ to the moment map $\tilde{\mu} : \tilde{X} \to g^*$, we have a morphism $\tilde{\mu}^\infty : J^\infty \tilde{X} \to J^\infty g^*$ and hence a homomorphism of vertex Poisson algebras

$$\tilde{\mu}^\infty : S(g \otimes \mathbb{C}[t^{-1}] t^{-1}) \to \mathcal{O}_{J^\infty \tilde{X}}(\tilde{X})$$

where the symmetric algebra $S(g \otimes \mathbb{C}[t^{-1}] t^{-1})$ has trivial Poisson structure $\lambda_{(-z, z)}$. The homomorphism $\tilde{\mu}^\infty$ is explicitly given by $\tilde{\mu}^\infty(A_i) = \sum_{j=1}^N \Delta_{ij} x_j(-1) y_i - c_i$ for $i = 1, \ldots, M$, where $\Delta = (\Delta_{ij})$ is the matrix defined in Section 5.1.

Consider the sheaf of tensor product vertex Poisson algebras $C_{VPA} = \mathcal{O}_{J^\infty \tilde{X}} \otimes_\mathbb{C} \Lambda^{vert}(g \oplus g^*)$. The $Z$-grading of $\Lambda^{vert}(g \oplus g^*)$ induces a $Z$-grading on $C_{VPA}$

$$C_{VPA} = \bigoplus_{n \in \mathbb{Z}} C_{VPA}^n, \quad C_{VPA}^n = \mathcal{O}_{J^\infty \tilde{X}} \otimes_\mathbb{C} \Lambda^{vert,n}(g \oplus g^*).$$

Set $Q_{VPA} = \sum_{i=1}^M \tilde{\mu}^\infty(A_i)(-1)^i \psi^*_i \in C_{VPA}^1(\tilde{X})$, an odd element of degree +1 in $C_{VPA}$. Let $d_{VPA} = Q_{VPA}(0) = \sum_{i=1}^M \sum_{n \in \mathbb{Z}} \tilde{\mu}^\infty(A_i)(-n-1)^i \psi^*_i(0)$ be an operator on $C_{VPA}$. By definition, the operator $d_{VPA}$ is a derivation on $C_{VPA}$.

**Proposition 5.1.** We have $(d_{VPA})^2 = 0$, and hence, for any open subset $\tilde{U} \subset \tilde{X}$, $(C_{VPA}(\tilde{U}) = \mathcal{O}_{J^\infty \tilde{X}}(\tilde{U}) \otimes_\mathbb{C} \Lambda^{vert}(g \oplus g^*), d_{VPA})$ is a cochain complex.

**Proof.** Since $\tilde{\mu}^\infty$ is a homomorphism of vertex Poisson algebras and the vertex Poisson algebra $S(g \otimes \mathbb{C}[t^{-1}] t^{-1})$ has trivial Poisson structure, we have $\tilde{\mu}^\infty(A_i)(-n)(\tilde{\mu}^\infty(A_j)) = 0$ for any $n \geq 0$ and $i, j = 1, \ldots, M$. Thus, we have $Q_{VPA}(0) \cdot Q_{VPA} = 0$. Then, by the axiom of vertex Poisson algebras, we have $Q_{VPA}(0) = (1/2)(Q_{VPA}(0)Q_{VPA})(0) = 0$.

Now we define the notion of the BRST cohomologies for vertex Poisson algebras. Take an open subset $\tilde{U} \subset \tilde{X}$, we consider the cochain complex $(C_{VPA}(\tilde{U}), d_{VPA})$, called (Poisson) BRST complex. Then, we denote its cohomology group

$$H^{\infty/2+*}_{VPA}(\tilde{g}, \mathcal{O}_{J^\infty \tilde{X}}(\tilde{U})) = H^* (C_{VPA}(\tilde{U}), d_{VPA}),$$

and call it (Poisson) BRST cohomology groups.

Note that we have $\partial \circ d_{VPA} = d_{VPA} \circ \partial$ following from $[\partial, Y_\mu(Q_{VPA}, \ast)] = \partial Y_\mu(Q_{VPA}, \ast)$. This implies that translation operator $\partial$ preserves the subspaces $\ker d_{VPA}$ and $\im d_{VPA} \subset C_{VPA}(\tilde{U})$. Moreover, by the axiom of vertex Poisson algebras, the coboundary operator $d_{VPA} = Q_{VPA}(0)$ is a derivation with respect to $(n)$-products for all $n \in \mathbb{Z}$. Hence, the 0-th BRST cohomology $H^{\infty/2+0}_{VPA}(\tilde{g}, \mathcal{O}_{J^\infty \tilde{X}}(\tilde{U})) = H^0(C_{VPA}(\tilde{U}), d_{VPA})$ is again a vertex Poisson algebra.

Next, we define the BRST cohomologies as a sheaf on the hypertoric variety $\tilde{X}$. For an open subset $\tilde{U} \subset \tilde{X}$, let $\tilde{U}$ be an open subset of $\tilde{X}$ such that $\tilde{U}$ is closed under the $G$-action and $\rho^{-1}(\tilde{U}) = \tilde{U} \cap \tilde{\mu}^{-1}(0)$. The following lemma asserts that the BRST cohomology $H^{\infty/2+*}_{VPA}(\tilde{g}, \mathcal{O}_{J^\infty \tilde{X}}(\tilde{U}))$ is supported on $\tilde{\mu}^{-1}(0) \cap \tilde{U}$ and it does not depend
on the choice of $\tilde{U}$. Then, we define a sheaf $\mathcal{H}_{VPA}^{\infty/2+}\bullet (g, \mathcal{O}_{J_{\infty}})$ over the hypertoric variety $\tilde{X}$ as the sheaf associated with the presheaf $\tilde{U} \mapsto \mathcal{H}_{VPA}^{\infty/2+}\bullet (g, \mathcal{O}_{J_{\infty}}(\tilde{U}))$ for $\bullet \in \mathbb{Z}$.

**Lemma 5.2 (AKM, Theorem 2.3.2.1).** The presheaf $\tilde{U} \mapsto \mathcal{H}_{VPA}^{\infty/2+}\bullet (g, \mathcal{O}_{J_{\infty}}(\tilde{U}))$ over $\tilde{X}$ is supported on $\tilde{\mu}^{-1}(0)$ and hence it does not depend on the choice of $\tilde{U}$.

The lemma will be proved in Section 5.3.

### 5.3. Double complex associated with the BRST complex

The BRST cochain complex can be decomposed into a double cochain complex as follows.

Set $C_{VPA}^{p,q} = \mathcal{O}_{J_{\infty}} \otimes \Lambda^\text{vert,p} g^* \otimes \Lambda^\text{vert,q} g$ for $p, q \in \mathbb{Z}$. Then, we have $C_{VPA}^n = \bigoplus_{p+q=n} C_{VPA}^{p,q}$ for any $n \in \mathbb{Z}$. Note that we have $\Lambda^\text{vert,p} g^* = 0$ unless $p \geq 0$ and $\Lambda^\text{vert,q} g = 0$ unless $q \leq 0$. Consider the operators $d_{VPA}^p = \sum_{i=1}^M \sum_{n \geq 0} \tilde{\mu}_{-i} (A_i) (n) \psi_{i,n}^*$ on $C_{VPA}$. Then, $d_{VPA}$ maps from $C_{VPA}^{p,q}$ to $C_{VPA}^{p+1,q}$, $d_{VPA}$ maps from $C_{VPA}^{p,q}$ to $C_{VPA}^{p,q+1}$, and we have $d_{VPA} = d_{VPA}^+ + d_{VPA}^-$. $d_{VPA} \circ d_{VPA} = -d_{VPA} \circ d_{VPA}$. Thus, we have a double complex $(C_{VPA}, d_{VPA}^+, d_{VPA}^-)$ whose total complex is the BRST complex $(C_{VPA}, d_{VPA})$.

Fix an arbitrary $p \in \mathbb{Z}_{\geq 0}$ and an open subset $\tilde{U} \subset \tilde{X}$. Consider the complex $(C_{VPA}^{p,*}(\tilde{U}), d_{VPA}^+)$. By the explicit description $d_{VPA}^+ = \sum_{i=1}^M \sum_{n \geq 0} \tilde{\mu}_{-i} (A_i) (-n-1) \psi_{i,n}^*$ of the coboundary operator, the complex $(C_{VPA}^{p,*}(\tilde{U}), d_{VPA}^+)$ coincides with the Koszul complex of $\mathcal{O}_{J_{\infty}}(\tilde{U})$ with respect to the sequence $\{\tilde{\mu}_{-i} (A_i) (-n-1)\}_{i=1,...,M, n=0,1,\ldots}$ (with reversing the degree of the complex). Clearly the sequence $\{\tilde{\mu}^*(A_i)\}_{i=1,...,M}$ is a regular sequence in $\mathcal{O}_{J_{\infty}}(\tilde{U})$. Then, by the same argument of the proof of [AKM Theorem 2.3.3.1], $\{\tilde{\mu}^*(A_i) (-n-1)\}_{i,n}$ is also a regular sequence in $\mathcal{O}_{J_{\infty}}(\tilde{U})$. This implies that the cohomology $H^i(C_{VPA}^{p,*}(\tilde{U}), d_{VPA}^+)$ vanishes if $q \neq 0$. Moreover, when $\tilde{U}$ is affine, we have $H^0(C_{VPA}^{p,*}(\tilde{U}), d_{VPA}^+) \simeq \mathcal{O}_{J_{\infty}}(\tilde{U})$ if $\tilde{\mu}^{-1}(0) \cap \tilde{U} \neq \emptyset$, and zero otherwise for any $p \geq 0$.

Consider the column filtration $\tau_{C_{VPA}}(\tilde{U})$: i.e. for $p \in \mathbb{Z}_{\geq 0}$, $\tau_p C_{VPA}(\tilde{U}) = \bigoplus_{k \geq p, q \leq 0} C_{VPA}^{k,q} (\tilde{U})$. We consider the spectral sequence $\tau E_{p,q}^\bullet (\tilde{U})$ associated with the column filtration. Then we have $\tau E_2^{p,q} (\tilde{U}) = H^p(H^q(C_{VPA}(\tilde{U}), d_{VPA}^-), d_{VPA}^+)$.

**Lemma 5.3.** The spectral sequence $\tau E_{p,q}^\bullet (\tilde{U})$ converges to the total cohomology $\tau E_{p,q}^\infty (\tilde{U}) = H^p(H^q(C_{VPA}(\tilde{U}), d_{VPA}^-), d_{VPA}^+)$.

**Proof.** To prove the convergence, we consider subcomplexes which are bounded both above and below. For $m \in \mathbb{Z}_{\geq 0}$, let $(C_{VPA})_m(\tilde{U}) = \partial^m(\mathcal{O}_{\tilde{X}}(\tilde{U}) \otimes \Lambda^m g^* + \Lambda^{m-1} g^*)$. where we consider $\mathcal{O}_{\tilde{X}}(\tilde{U})$ (resp. $\Lambda^m g^*$) as a subalgebra of $\mathcal{O}_{J_{\infty}}(\tilde{U})$ (resp. $\Lambda^{m-1} g^*$). Set $(C_{VPA})_m(\tilde{U}) = (C_{VPA})(\tilde{U}) \cap C_{VPA}^m(\tilde{U})$. Then, we have $C_{VPA}^m = \bigoplus_{m \geq 0} (C_{VPA})_m$. By direct computation, for $a \in \mathcal{O}_{\tilde{X}}(\tilde{U})$ and $\varphi \in \Lambda^m g^*$, we have

$$d_{VPA}(a \otimes \varphi) = \sum_{i=1}^M \tilde{\mu}^*(A_i) a \otimes \psi_{i,n}^* \varphi + \sum_{i=1}^M \tilde{\mu}^*(A_i) a \otimes \psi_{i,n}^{*-1} \varphi,$$

and hence $d_{VPA}$ preserves the subspace $(C_{VPA})_0(\tilde{U})$. Since $d_{VPA} = Q_{VPA}(0)$ commutes with the translation operator $\partial$ by the axiom of vertex Poisson algebras, $d_{VPA}$ also preserves $(C_{VPA})_m(\tilde{U})$ for any $m \in \mathbb{Z}_{\geq 0}$. Therefore, $((C_{VPA})_m(\tilde{U}), d_{VPA}^+, d_{VPA}^-)$ is a double subcomplex of $(C_{VPA}(\tilde{U}), d_{VPA}^+, d_{VPA}^-)$. Consider the spectral sequence $(E_{p,q})_m(\tilde{U})$ associated with the double complex $((C_{VPA})_m(\tilde{U}), d_{VPA}^+, d_{VPA}^-)$. Since
we have an isomorphism the coordinate transformation of \( \tilde{\eta} \) induced in Section 3.5, on which the vanishing of the negative BRST cohomologies.

Lemma 5.4. The BRST cohomology \( H_{\text{VPA}}^{\infty/2+n}(g, O_{J_n \tilde{X}}(\tilde{U})) = H^n(C_{\text{VPA}}(\tilde{U}), d_{\text{VPA}}) \) vanishes if \( n < 0 \) for any open subset \( U \subset \tilde{X} \).

5.4. Zeroth Poisson BRST cohomology. Now we determine the 0-th BRST cohomology \( H_{\text{VPA}}^{\infty/2+0}(g, O_{J_n \tilde{X}}) \). We consider the affine open subset \( U \subset \tilde{X} \) introduced in Section 5.3 on which the \( G \)-torsor \( \tilde{\mu}^{-1}(0) \cap \tilde{U}_J \rightarrow \tilde{U}_J \) is trivial. Namely, we have an isomorphism \( \tilde{\mu}^{-1}(0) \cap \tilde{U}_J \simeq \tilde{U}_J \times G \times g^* \) given by the explicit local coordinate \( \tilde{U}_J \). By applying the functor \( J_\infty \) to \( U \), we have

\[
O_{\tilde{A}_\infty}^{\infty}(\tilde{U}_J) = C[\{a_j^{j(n)} \mid j \in \mathbb{Z}_{\geq 1}\}] \otimes C[T_{i=1} \ldots T_{i=M}\{c_i \mid i = 1, \ldots, M\}] \setminus \{1\}
\]

because \( \tilde{\mu}_\infty^{-1}(0) \simeq J_\infty(\tilde{\mu}^{-1}(0)) \). The action of \( \tilde{\mu}_\infty^{-1}(A_\infty)(n) \) for \( i = 1, \ldots, M, n \in \mathbb{Z}_{\geq 0} \), in the above local coordinate is explicitly given by \( \tilde{\mu}_\infty^{-1}(A_\infty)(n) = \sum_{i=1}^n T_i \frac{\partial}{\partial T_i} \) by direct calculation. Note that this action coincides with the action of \( g[t] \) induced from the regular representation of \( J_\infty G \) on \( \mathbb{C}[J_\infty G] \subset O_{\tilde{\mu}_\infty^{-1}(0)}(\tilde{U}_J) \). Since \( \tau E_1^{\rho} \tilde{U}_J \simeq O_{\tilde{\mu}_\infty^{-1}(0)}(\tilde{U}_J) \) if \( q = 0 \) and zero otherwise, we have \( \tau E_2^{0,0}(\tilde{U}_J) = \ker d_{\text{VPA}}^r \) where

\[
d_{\text{VPA}}^r = \sum_{n=1}^M \sum_{k=0}^n \psi^{p,q}_{i(n-1)} T_i \frac{\partial}{\partial T_i} T_i \frac{\partial}{\partial T_i}
\]
in the above local coordinate. Thus, we have

\[
\tau E_2^{0,0}(\tilde{U}_J) \simeq C[\{a_j^{j(n)} \mid j \in \mathbb{Z}_{\geq 1}\}] \otimes C[\{c_i \mid i = 1, \ldots, M\}] \setminus \{1\} \simeq O_{J_\infty \tilde{X}}(\tilde{U}_J)
\]

and \( \tau E_r^{0,0}(\tilde{U}_J) \) collapses at \( r = 2 \). Therefore, we have

\[
H_{\text{VPA}}^{\infty/2+0}(g, O_{J_n \tilde{X}}(\tilde{U}_J)) \simeq O_{J_\infty \tilde{X}}(\tilde{U}_J)
\]
by Lemma 5.3.

We have the affine covering \( \tilde{X} = \bigcup_{J} \tilde{U}_J \); For each indices \( I \) and \( J \), we have the coordinate transformation of \( \tilde{\varphi}_{IJ}^J : O_{\tilde{U}_J} \rightarrow O_{\tilde{U}_J} \) introduced in Section 5.3. Its restriction gives the coordinate transformation \( \tilde{\varphi}_{IJ}^J : O_{\tilde{U}_J} \rightarrow O_{\tilde{U}_J} \). Applying the jet scheme functor \( J_\infty \), we have the isomorphisms \( J_\infty \tilde{\varphi}_{IJ}^J : O_{J_\infty \tilde{U}_J} \rightarrow O_{J_\infty \tilde{U}_J} \) and \( J_\infty \tilde{\varphi}_{IJ}^J : O_{J_\infty \tilde{U}_J} \rightarrow O_{J_\infty \tilde{U}_J} \). These coordinate transformations are compatible with the isomorphism \( \tilde{\varphi}_{IJ}^J \), and thus we have the following isomorphism of sheaves of Poisson algebras:

\[
H_{\text{VPA}}^{\infty/2+0}(g, O_{J_n \tilde{X}}(\tilde{X})) \simeq O_{J_\infty \tilde{X}}
\]
by gluing up \( O_{J_\infty \tilde{U}_J} \) with \( J_\infty \tilde{\varphi}_{IJ}^J \).

In the rest of this section, we discuss the BRST reduction \( H_{\text{VPA}}^{\infty/2+0}(g, O_{J_n \tilde{X}}(\tilde{X})) \) of the coordinate ring \( O_{J_n \tilde{X}}(\tilde{X}) \simeq \mathbb{C}[J_\infty(T^*C^n \times g^*)] \). Recall the decomposition of the BRST complex, which is introduced in the proof of Lemma 5.3 \( C_{\text{VPA}} = \bigoplus_{m \geq 0}(C_{\text{VPA}})_m, d_{\text{VPA}} \) where \( (C_{\text{VPA}})_0 = O_{\tilde{X}} \otimes \mathbb{C} \Lambda(g \oplus g^*) \) and \( (C_{\text{VPA}})_m = \)
\[ \partial^m(C_{VPA})_0 \] for \( m \geq 1 \). The subcomplex \((C_{VPA})_0 = \mathcal{O}_\tilde{X} \otimes \mathbb{C} \Lambda_i g, d_{VPA} \) coincides with the Poisson BRST complex of the Poisson algebra \( \mathcal{O}_\tilde{X} \) by the commutant map \( \mu^* : g \to \mathcal{O}_\tilde{X} \). For the detail of the fundamental properties of BRST cohomology of associative algebras, refer [K]. By similar arguments to the above (see also [K Section 6.3]), we have \( H^0((C_{VPA})_0(\tilde{U}), d_{VPA}) \simeq (\mathcal{O}_\tilde{X}(\tilde{U})/\sum_{i=1}^M \mathcal{O}_\tilde{X}(\tilde{U})\mu_i(A_i))^G \) for any open subset \( \tilde{U} \subseteq \tilde{X} \). Let \( \mathcal{H}^0((C_{VPA})_0, d_{VPA}) \) be the sheaf over \( \tilde{X} \) associated with the presheaf \( \mathcal{H} \to H^0((C_{VPA})_0(\tilde{U}), d_{VPA}) \) where we take \( \tilde{U} \) an open subset of \( \tilde{X} \) which is preserved by the action of \( G \) and \((\mu^{-1}(0) \cap \tilde{U})/G = \tilde{U} \). Then, we have \( \mathcal{H}^0((C_{VPA})_0, d_{VPA}) \simeq \mathcal{O}_\tilde{X} \) and \( \Gamma(\tilde{X}, H^0((C_{VPA})_0, d_{VPA})) \simeq \mathcal{O}_\tilde{X}(\tilde{X}) \simeq H^0((\mathcal{O}_\tilde{X}(\tilde{X}), d_{VPA}) \) because \( \tilde{X} \to \tilde{X}_0 = \mu^{-1}(0)/G \simeq \text{Spec} H^0((\mathcal{O}_\tilde{X}(\tilde{X}), d_{VPA}) \) is a resolution of normal singularity. Since the translation operator \( \partial \) commutes with the coboundary operator \( d_{VPA} \), we have \( H^0((C_{VPA})_m(\tilde{X}), d_{VPA}) = \partial^m H^0((C_{VPA})_0(\tilde{X}), d_{VPA}) \simeq \partial^m \mathcal{O}_\tilde{X}(\tilde{X}) \) for any \( m \in \mathbb{Z}_{\geq 0} \). Therefore we have the following proposition.

**Proposition 5.5.** We have
\[ H^0_{VPA}(\tilde{U}, \mathcal{O}_{\tilde{X}}(\tilde{X})) = \mathcal{H}^0((C_{VPA}(\tilde{X}), d_{VPA}) \simeq \mathcal{O}_{\tilde{X}}(\tilde{X}). \]
That is, the Poisson BRST reduction commutes with the global section functor \( \Gamma \).

**5.5. BRST cohomologies.** Let \( D_{T^*V}^\hbar \) be the sheaf of \( \hbar \)-adic \( \beta\gamma \)-system sheaf over the symplectic vector space \( T^*V \) which we defined in Section 4.3. By restriction, we define \( D_{VPA}^\hbar = D_{T^*V}^\hbar |_{\tilde{X}}, \) the sheaf of \( \hbar \)-adic vertex algebras over \( \tilde{X} \). Let \( V_{\tilde{g}}(\cdot, \hbar) \) be a Heisenberg vertex algebra generated by elements \( c_1, \ldots, c_M \in \tilde{g} \) with the inner product given by \( \langle c_i, c_j \rangle = \sum_{k=1}^N \Delta_{ik} \Delta_{jk} \) for \( i, j = 1, \ldots, M \). That is, it is the localization over \( \tilde{g}^* \) of the \( \hbar \)-adic vertex algebra \( V_{\tilde{g}}(\cdot, \hbar)(\tilde{g}^*) \) given by
\[ V_{\tilde{g}}(\cdot, \hbar)(\tilde{g}^*) = \mathbb{C}[[\hbar]]/\mathfrak{c}_1(\cdot, \hbar), \ldots, c_M(\cdot, \hbar) \mid n \in \mathbb{Z}_{\geq 1}, \]
and as a \( \mathbb{C}[[\hbar]] \)-module, and \( c_1, \ldots, c_M \) are bosonic elements whose OPEs are given by \( c_i(z) c_j(w) \sim \hbar^2 \langle c_i, c_j \rangle / (z - w)^2 \) where \( \langle c_i, c_j \rangle = Y(c_{i-1}, z) ; z \). Set \( \tilde{D}_{VPA}^\hbar = \tilde{D}_{VPA}^\hbar \circ V_{\tilde{g}}(\cdot, \hbar), \) a sheaf of \( \hbar \)-adic vertex algebras over \( \tilde{X} = \tilde{X} \times \tilde{g}^* \). Here \( \circ \) is the completion of the tensor product \( \otimes_{\mathbb{C}[[\hbar]]} \) with respect to the \( \hbar \)-adic topology as in Section 5.2.

To construct the BRST reduction for \( \tilde{D}_{VPA}^\hbar \), we need to introduce a quantization of the commutant map \( \tilde{\mu}_\hbar \). Consider a commutative vertex algebra \( V_0(\hbar) = \mathbb{C}[A_{(-n)} \mid n \geq 1] \). Define a \( \mathbb{C}[\beta]-module \) homomorphism
\[ \mu_{\hbar} : V_0(\hbar) \to \tilde{D}_{VPA}^\hbar, \quad \mu_{\hbar}(A_i) = \sum_{j=1}^N \Delta_{ij} x_{j(-1)} y_j - c_i. \]

**Lemma 5.6.** The above map \( \mu_{\hbar} \) preserves the OPEs; i.e. we have \( \mu_{\hbar}(A_i)(z) \mu_{\hbar}(A_j)(w) \sim 0 \) for \( i, j = 1, \ldots, M \).

This lemma is obviously checked by direct computation. We call the map \( \mu_{\hbar} \) a chiral commutant map with respect to the \( G \)-action on \( \tilde{X} \).

Consider the sheaf of \( \hbar \)-adic vertex superalgebras \( C_{VSA} = \tilde{D}_{VPA}^\hbar \otimes \text{Clifford}(g \otimes g^*) \)
where \( \text{Clifford}(g \otimes g^*) \) is the Clifford \( \hbar \)-adic vertex superalgebra defined in Section 5.3.

The \( \mathbb{Z} \)-grading of \( \text{Clifford}(g \otimes g^*) \) induces a \( \mathbb{Z} \)-grading on \( C_{VSA} = \bigoplus_{n \in \mathbb{Z}} C_{VSA}^n \) where \( C_{VSA}^n = \tilde{D}_{VPA}^\hbar \otimes \text{Clifford}(g \otimes g^*) \) and \( \bigoplus^n \) is the completion of the direct sum with respect to the \( \hbar \)-adic topology. Consider an odd element \( Q_{VSA} = \sum_{i=1}^M \mu_{\hbar}(A_i)(-1) \psi^*_i \) of degree +1 in \( C_{VSA} \). Note that the image of \( Q_{VSA}(0) \) lies in \( \hbar C_{VSA} \). Let \( d_{VSA} = h^{-1}Q_{VSA}(0) = h^{-1} \sum_{i=1}^M \sum_{n \in \mathbb{Z}} \mu_{\hbar}(A_i)(-n-1) \psi^*_i \) be a derivation on \( C_{VSA} \) homogeneous of degree +1.
Proposition 5.7. We have \((d_{\text{AVA}})^2 = 0\), and hence, for any open subset \(\tilde{U} \subset \tilde{X}\), \((C_{\text{AVA}}(\tilde{U})) = \tilde{D}_{X,\hbar}^{ch}(\tilde{U}) \oplus C^{\text{vert}}(g \oplus g^*)\), \(d_{\text{AVA}}\) is a cochain complex.

Proof. We use the same argument in Proposition 5.1. By Lemma 5.6, we have \(\mu_{ch}(A_i)_{(m)}\mu_{ch}(A_j) = 0\) for all \(i, j = 1, \ldots, M\) and \(n \geq 0\). Thus, we have \(Q_{\text{AVA}(0)}Q_{\text{AVA}(0)} = 0\). By Borcherds’ identity, we have

\[
(Q_{\text{AVA}(0)})^2 = (1/2)|Q_{\text{AVA}(0)}, Q_{\text{AVA}(0)}| = (1/2)(Q_{\text{AVA}(0)}Q_{\text{AVA}(0)}(0)) = 0.
\]

\[\square\]

Now we define the notion of the chiral BRST cohomologies. Taking an open subset \(\tilde{U} \subset \tilde{X}\), we consider the cochain complex \((C_{\text{AVA}}(\tilde{U}), d_{\text{AVA}})\), called a BRST complex. Then, for \(n \in \mathbb{Z}\), we denote its cohomology group \(H_{\text{AVA}}^{n}(\tilde{U})\), and call it the \(n\)-th BRST cohomology. Note that we have \([\partial, Y(Q_{\text{AVA}})] = \partial . Y(Q_{\text{AVA}}, z)\) on \(C_{\text{AVA}}\) by the axiom of \(h\)-adic vertex superalgebras. By taking the coefficient of \(z^{-1}\), we obtain \(\partial \circ Q_{\text{AVA}} - Q_{\text{AVA}} \circ \partial = 0\). Thus, the translation operator \(\partial\) preserves the subspaces Ker \(d_{\text{AVA}}\) and Im \(d_{\text{AVA}}\).

Further, for any element \(a, b \in C_{\text{AVA}}\) and for any \(n \in \mathbb{Z}\), we have \(Q_{\text{AVA}(0)}(a_{(n)}b) = (-1)^{n}a_{(n)}Q_{\text{AVA}(0)}b\) by the Borcherds’ identity. By taking \(a, b\) from Ker \(d_{\text{AVA}}\), we conclude that \(Q_{\text{AVA}(0)}(a_{(n)}b) = 0\) and thus \(a_{(n)}b \in \text{Ker} d_{\text{AVA}}\). Also, by taking \(a \in \text{Ker} d_{\text{AVA}}\) and \(b \in C_{\text{AVA}}\), we have \(a_{(n)}Q_{\text{AVA}(0)}b = Q_{\text{AVA}(0)}(a_{(n)}b) \in \text{Im} d_{\text{AVA}}\). Therefore we conclude the following proposition.

Proposition 5.8. For an open subset \(\tilde{U} \subset \tilde{X}\), the 0-th BRST cohomology group \(H_{\text{AVA}}^{0}(\tilde{U}) = H^{0}(C_{\text{AVA}}(\tilde{U}), d_{\text{AVA}})\) is an \(h\)-adic vertex algebra.

Next, we define the BRST cohomology group as a sheaf on the hypertoric variety \(\tilde{X}\). For an open subset \(\tilde{U} \subset \tilde{X}\), let \(\tilde{U}\) be an open subset of \(\tilde{X}\) such that \(\tilde{U}\) is closed under the \(G\)-action and \(p^{-1}(\tilde{U}) = \tilde{U} \cap \tilde{\mu}^{-1}(0)\). The following lemma asserts that the BRST cohomology group \(H_{\text{AVA}}^{\infty/0}(\tilde{U})\), \(\tilde{D}_{X,\hbar}^{ch}(\tilde{U})\) is supported on \(\tilde{\mu}^{-1}(0)\) and it does not depend on the choice of \(\tilde{U}\). Then, we define a sheaf \(H_{\text{AVA}}^{\infty/\bullet}(\tilde{U}), \tilde{D}_{X,\hbar}^{ch}(\tilde{U})\) over the hypertoric variety \(\tilde{X}\) as the sheaf associated with the presheaf \(\tilde{U} \mapsto H_{\text{AVA}}^{\infty/\bullet}(\tilde{U})\).

Lemma 5.9 (cf. [AKM], Theorem 2.3.5.1). The presheaf \(\tilde{U} \mapsto H_{\text{AVA}}^{\infty/0}(\tilde{U})\) over \(\tilde{X}\) is supported on \(\tilde{\mu}^{-1}(0)\) and hence it does not depend on the choice of \(\tilde{U}\).

In the rest of this section, we prove Lemma 5.9. The coboundary operator of the BRST complex \(d_{\text{AVA}} = h^{-1}Q_{\text{AVA}(0)} = h^{-1}\sum_{i=1}^{M} \sum_{n \in \mathbb{Z}} \mu_{ch}(A_i)(-n-1)\psi^{\ast}_{i(n)}\) is separated into two parts \(d_{\text{AVA}}^{+}\) and \(d_{\text{AVA}}^{-}\); namely, putting

\[
d_{\text{AVA}}^{+} = h^{-1}\sum_{n \geq 0} n\psi_{i(-n-1)}\mu_{ch}(A_i)(n), \quad d_{\text{AVA}}^{-} = h^{-1}\sum_{n \geq 0} \mu_{ch}(A_i)(-n-1)\psi^{\ast}_{i(n)},
\]

we have \(d_{\text{AVA}} = d_{\text{AVA}}^{+} + d_{\text{AVA}}^{-}\). Moreover, we have \(d_{\text{AVA}}^{+} \circ d_{\text{AVA}}^{-} = -d_{\text{AVA}}^{-} \circ d_{\text{AVA}}^{-}\) because \(\mu_{ch}(A_i)(n)\mu_{ch}(A_j) = 0\) and \(\psi^{\ast}_{i(n)}\psi^{\ast}_{j(n)} = 0\) for any \(i, j = 1, \ldots, M\) and \(n \geq 0\). Thus, we have a double complex \((C_{\text{AVA}}, d_{\text{AVA}}^{+}, d_{\text{AVA}}^{-})\) where

\[
C_{p,q}^{\text{AVA}} = \tilde{D}_{X,\hbar}^{ch} \otimes \Lambda^{p,q}_{C,\hbar}(g) \otimes \Lambda^{p,q}_{C,\hbar}(g)\]

for \(p, q \in \mathbb{Z}\), is induced from the decomposition of the \(h\)-adic Clifford vertex algebra. Note that \(C_{p,q}^{\text{AVA}} = 0\) unless \(p \geq 0\) and \(q \leq 0\); that is, \((C_{\text{AVA}}, d_{\text{AVA}}^{+}, d_{\text{AVA}}^{-})\) is the fourth quadrant cochain double complex.
The BRST complex $C_{\text{bVA}} = \tilde{D}_{\mathcal{X}, h}^{b} \otimes C^{vert}(g \oplus g^*)$ is naturally equipped with a filtration $F_p C_{\text{bVA}}$ by powers of $h$: $F_p C_{\text{bVA}} = h^p C_{\text{bVA}}$ for $p \in \mathbb{Z}_{\geq 0}$. For each $p \in \mathbb{Z}_{\geq 0}$, the associated graded space is $\text{Gr}_p C_{\text{bVA}} = F_p C_{\text{bVA}}/F_{p+1} C_{\text{bVA}} \cong O_{J_\infty \mathcal{X}} \otimes \Lambda^{vert}(g \oplus g^*)$ as vertex Poisson superalgebras.

Consider the action of $d^+_\text{bVA}$ and $d^-_\text{bVA}$ on the vertex Poisson superalgebra $\text{Gr}_p C_{\text{bVA}} \cong O_{J_\infty \mathcal{X}} \otimes \Lambda^{vert}(g \oplus g^*)$. The operators $d^+_\text{bVA}$, $d^-_\text{bVA}$ act by

$$d^+_\text{bVA} = \sum_{i=1}^{M} \sum_{n \geq 0} \psi_i^{n-1} \mu_i^{n}(A_i)(n), \quad d^-_\text{bVA} = \sum_{i=1}^{M} \sum_{n \geq 0} \mu_i^{n}(A_i)(-n-1) \psi_i^{n}(n)$$

respectively on $O_{J_\infty \mathcal{X}} \otimes \Lambda^{vert}(g \oplus g^*)$.

Thus, for each $p \in \mathbb{Z}_{\geq 0}$, the double complex $(\text{Gr}_p C_{\text{bVA}}, d^+_\text{bVA}, d^-_\text{bVA})$ is isomorphic to the double complex $(C_{\text{bVA}}, d^+_\text{bVA}, d^-_\text{bVA})$ associated with the Poisson BRST complex which we discussed in Section 5.3. By Lemma 5.2, for an open subset $\tilde{U} \subset \mathcal{X}$ such that $\tilde{U} \cap \tilde{\mu}^{-1}(0) = \emptyset$, we have

$$H^*(\text{Gr}_p C_{\text{bVA}}(\tilde{U}), d^+_\text{bVA} + d^-_\text{bVA}) = H^*(C_{\text{bVA}}(\tilde{U}), d_{\text{bVA}}) = 0$$

for any $\bullet \in \mathbb{Z}$ and $p \in \mathbb{Z}_{\geq 0}$. Now we consider the spectral sequence $E^{p,q}_r$ associated with the filtered complex $(F_p C_{\text{bVA}}, d_{\text{bVA}})$. Then, we have $E^{p,q}_1(\tilde{U}) = 0$ for any $p$, $q \in \mathbb{Z}$ by the above and thus $E^{p,q}_\infty(\tilde{U})$ collapses at $r = 1$. Since the filtration $F_p C_{\text{bVA}}$ is bounded above and complete, $E^{p,q}_\infty(\tilde{U})$ converges to $\text{Gr}_p H^{p+q}(C_{\text{bVA}}(\tilde{U}), d_{\text{bVA}})$ by the complete convergence theorem [We, Theorem 5.5.10]. Therefore, we have the vanishing $H^*(C_{\text{bVA}}(\tilde{U}), d_{\text{bVA}}) = 0$ for an open subset $\tilde{U}$ which intersects trivially with $\tilde{\mu}^{-1}(0)$, and it proves Lemma 5.9.

By a similar argument, we obtain the vanishing of negative BRST cohomologies as follows.

**Proposition 5.10.** For $n < 0$, we have $H^{n/2+n}_{\text{bVA}}(\hat{D}^b_{\mathcal{X}, h}) = 0$.

**Proof.** For any $p \geq 0$, $n < 0$ and any open subset $\tilde{U} \subset \mathcal{X}$, we have

$$H^n(\text{Gr}_p C_{\text{bVA}}(\tilde{U}), d^+_\text{bVA} + d^-_\text{bVA}) = H^n(C_{\text{bVA}}(\tilde{U}), d_{\text{bVA}}) = 0$$

by Lemma 5.4. Again we consider the spectral sequence $E^{p,q}_r$ associated with the $h$-adic filtration. Then, for any $p$, $q$ such that $p + q < 0$, we have $E^{p,q}_\infty(\tilde{U}) = 0$. Since the filtration is complete and bounded above, and we have $E^{p,q}_1(\tilde{U}) = E^{p,q}_2(\tilde{U}) = \ldots$ for $p$, $q$ with $p + q < 0$, the spectral sequence $E^{p,q}_\infty(\tilde{U})$ converges to the cohomology $\text{Gr}_p H^{p+q}(C_{\text{bVA}}(\tilde{U}), d_{\text{bVA}})$ when $p + q < 0$. Thus the cohomology $H^n(C_{\text{bVA}}(\tilde{U}), d_{\text{bVA}})$ vanishes for negative $n$. \qed

**Definition 5.11.** We write the 0-th cohomology $H^{n/2+n} \otimes (g, \hat{D}^b_{\mathcal{X}, h})$ by $\hat{D}^b_{\mathcal{X}, h}$.

**5.6. Spectral sequence associated with the double complex.** For any fixed $k \in \mathbb{Z}$, we have the complex $(C_{\text{bVA}}, d_{\text{bVA}})$. Set a $C[[h]]$-submodule

$$C_{\text{bVA}}^k = \hat{D}^b_{\mathcal{X}, h} \otimes \Lambda^{vert,k}_{C[[h]]}(g) \subset \hat{D}^b_{\mathcal{X}, h} \otimes \Lambda^{vert,k}_{C[[h]]}(g) \otimes \Lambda^{vert,k}_{C[[h]]}(g^*) = C_{\text{bVA}}^k,$$

and we have a complex $(C_{\text{bVA}}, d_{\text{bVA}})$. Note that

$$H^n(C_{\text{bVA}}^k, d_{\text{bVA}}) \simeq H^n(C_{\text{bVA}}^k, d_{\text{bVA}}) \otimes \Lambda^{vert,k}_{C[[h]]}(g^*)$$

for any $k, n \in \mathbb{Z}$.

Consider the filtration of $C_{\text{bVA}}$ given by the powers of $h$, denoted $F_p C_{\text{bVA}} = h^p C_{\text{bVA}}$ ($p \geq 0$). Clearly, the coboundary operator $d_{\text{bVA}}^+$ preserves the filtration. Let $E^{p,q}_r$ be the spectral sequence associated with the filtration $F_p C_{\text{bVA}}$. Then, we have $E^{p,q}_r = F_p C_{\text{bVA}}^{p+q}/F_{p+1} C_{\text{bVA}}^{p+q} \simeq O_{J_\infty \mathcal{X}} \otimes \Lambda^{vert,p+q}_{C[[h]]}(g)$ on which the coboundary operator acts...
by \( \hat{d}_{\text{NVA}} = d_{\text{NVA}}^- \) as we see in the previous section. Take an open subset \( \tilde{\mathcal{U}} \subset \tilde{X} \). By the result of Section 5.3, we have

\[
E^r_{1,q}(\tilde{\mathcal{U}}) \simeq \begin{cases} O'_{\tilde{\mu}^{-1}}(0)(\tilde{\mathcal{U}}) & (p + q = 0) \\ 0 & (p + q \neq 0) \end{cases}
\]

where

\[
O'_{\tilde{\mu}^{-1}}(0)(\tilde{\mathcal{U}}) = \mathcal{O}_{J_{\infty \tilde{X}}}(\tilde{\mathcal{U}}) / \sum_{i=1}^{M} \sum_{n \geq 0} \tilde{\mu}^n(A_i)(-n-1) \mathcal{O}_{J_{\infty \tilde{X}}}(\tilde{\mathcal{U}}).
\]

Note that \( O'_{\tilde{\mu}^{-1}}(0)(\tilde{\mathcal{U}}) = O_{\tilde{\mu}^{-1}}(0)(\tilde{\mathcal{U}}) \) when \( \mathcal{U} \) is an affine open subset. The above implies that the spectral sequence \( E^p_{q}(\tilde{\mathcal{U}}) \) collapses at \( r = 1 \). Since the filtration \( F_r C_{\ast}(\tilde{\mathcal{U}}) \) is bounded above and complete, the collapse implies the convergence of the spectral sequence by the complete convergence theorem [We, Theorem 5.5.10].

**Lemma 5.12.** For \( p \geq 0 \) and an open subset \( \tilde{\mathcal{U}} \subset \tilde{X} \), we have an isomorphism

\[
H^q(C^p_{\text{NVA}}(\tilde{\mathcal{U}}), d^-_{\text{NVA}}) \simeq O'_{\tilde{\mu}^{-1}}(0)(\tilde{\mathcal{U}})[[h]] \otimes \Lambda^{\text{vert},p}[[g^*]]
\]

if \( q = 0 \), and zero otherwise.

Consider the double complex \((C_{\text{NVA}}(\tilde{\mathcal{U}}), d^-_{\text{NVA}}, d^+_{\text{NVA}})\). Consider the column filtration \( \tau_{\geq} C_{\text{NVA}} \) of the double complex \( C_{\text{NVA}}: \tau_{\geq} C_{\text{NVA}}(\tilde{\mathcal{U}}) = \bigoplus_{k \geq p, q \leq 0} C^k_{\text{NVA}}(\tilde{\mathcal{U}}) \) where \( \bigoplus^\wedge \) is the completion with respect to the \( h \)-adic topology. Let \( \tau^p_{\geq} E_{p,q}(\tilde{\mathcal{U}}) \) be the spectral sequence associated with the column filtration \( \tau_{\geq} C_{\text{NVA}} \). By Lemma 5.12, we have

\[
\tau^p_{\geq} E_{2,q}(\tilde{\mathcal{U}}) = H^p(H^q(C_{\text{NVA}}(\tilde{\mathcal{U}}), d^-_{\text{NVA}}), d^+_{\text{NVA}}) 
\]

\[
\simeq \begin{cases} H^p(O'_{\tilde{\mu}^{-1}}(0)(\tilde{\mathcal{U}})[[h]] \otimes \Lambda^{\text{vert},p}[[g^*]], d^+_{\text{NVA}}) & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}
\]

Thus, the spectral sequence \( \tau^p_{\geq} E_{p,q}(\tilde{\mathcal{U}}) \) collapses at \( r = 2 \). In the rest of this section we prove the following proposition.

**Proposition 5.13.** The spectral sequence \( \tau^p_{\geq} E_{p,q}(\tilde{\mathcal{U}}) \) converges to the total cohomology \( H^{p+q}(C_{\text{NVA}}(\tilde{\mathcal{U}}), d_{\text{NVA}}) \).

Consider the completion \( \hat{C}_{\text{NVA}}(\tilde{\mathcal{U}}) \) of the BRST complex \( C_{\text{NVA}}(\tilde{\mathcal{U}}) \) with respect to the column filtration \( \tau_{\geq} C_{\text{NVA}}(\tilde{\mathcal{U}}) \). Then, the column filtration \( \tau_{\geq} \hat{C}_{\text{NVA}}(\tilde{\mathcal{U}}) \) is bounded above, complete, and the spectral sequence \( \tau^p_{\geq} E_{p,q}(\tilde{\mathcal{U}}) \) collapses at \( r = 2 \). Thus, the spectral sequence \( \tau^p_{\geq} E_{p,q}(\tilde{\mathcal{U}}) \) converges to the total cohomology group \( H^{p+q}(\hat{C}_{\text{NVA}}(\tilde{\mathcal{U}}), d_{\text{NVA}}) \) of the completed complex by the complete convergence theorem [We, Theorem 5.5.10].

For \( p, q \in \mathbb{Z} \), set

\[
A^p_{\infty} = \{ \alpha \in \tau_{\geq p} C^p_{\text{NVA}}(\tilde{\mathcal{U}}) \mid \text{d}_{\text{NVA}} \alpha = 0 \}, \\
\tilde{A}^p_{\infty} = \{ \alpha \in \tau_{\geq p} \hat{C}^p_{\text{NVA}}(\tilde{\mathcal{U}}) \mid \text{d}_{\text{NVA}} \alpha = 0 \}.
\]

To prove the convergence of the spectral sequence \( \tau^p_{\geq} E_{p,q}(\tilde{\mathcal{U}}) \) to \( H^{p+q}(C_{\text{NVA}}(\tilde{\mathcal{U}}), d_{\text{NVA}}) \), it is sufficient to show \( A^p_{\infty} = \tilde{A}^p_{\infty} \) and \( \bigcap_{p \geq 0} \tau_{\geq p} H^n(C_{\text{NVA}}(\tilde{\mathcal{U}}), d_{\text{NVA}}) = 0 \) for any \( n \).

**Lemma 5.14.** We have \( A^p_{\infty} = \tilde{A}^p_{\infty} \) for any \( p, q \in \mathbb{Z} \).
Proof. Take arbitrary \( \alpha = \sum_{k=0}^{\infty} \alpha_k \in \tilde{A}_q^\infty \) where \( \alpha_k \in C_{hVA}^{p+k,q-k} \). We show that \( \alpha \) belongs to \( A_{q}^\infty \). By the condition, we have \( d^{+}_{hVA} \alpha_0 = 0 \) and \( d^{+}_{hVA} \alpha_k = -d^{+}_{hVA} \alpha_{k+1} \) for \( k \geq 0 \). Since the coboundary operators \( d^{+}_{hVA}, d^{-}_{hVA} \) of the double complex preserve the \( \mathbb{h} \)-adic filtration \( F_{hVA}(\tilde{U}) \), we may assume that \( \alpha_k \in F_{hVA}(\tilde{U}) \) implies \( \alpha_{k+1} \in F_{hVA}(\tilde{U}) \) for any \( k \geq 0 \). Considering modulo \( F_{hVA}(\tilde{U}) \), we have the isomorphism of double complexes \((C_{VA}/F_{hVA}, d^{+}_{hVA}, d^{-}_{hVA}) \cong (C_{VPA}, d^{+}_{VPA}, d^{-}_{VPA}) \), which we showed in Section 5.5. The double complex \( C_{VPA}(\tilde{U}) \) is decomposed into a direct sum of bounded double complexes, and thus the associated spectral sequence converges as Lemma 5.3. This implies that there exists an integer \( k_1 \geq 0 \) such that \( \alpha_l \in F_{hVA}(\tilde{U}) \) for all \( l \geq k_1 \).

Assume that, for an integer \( s \geq 0 \), there exists \( k_s \geq 0 \) such that \( \alpha_l \in F_{s}C_{hVA}(\tilde{U}) \) for \( l \geq k_s \). By the condition, we have \( d^{+}_{hVA} \alpha_k = -d^{+}_{hVA} \alpha_{k+1} \) for \( k \geq k_s \), where \( \alpha_k \) is the image of \( \alpha_k \) in \( F_{s}C_{hVA}(\tilde{U})/F_{s+1}C_{hVA}(\tilde{U}) \cong C_{hVA}(\tilde{U}) \). By the above equalities, \( \overline{\alpha_k} \) for \( k \geq k_s \) belong to the same bounded double subcomplex, and hence we have \( \overline{\alpha_{k+1}} = \overline{\alpha_k} = 0 \) for some \( k_s+1 \geq 0 \). Thus, inductively on \( s \geq 0 \), we have an integer \( k_s \geq 0 \) such that \( \alpha_l \in F_{s}C_{hVA}(\tilde{U}) \) for any \( l \geq k_s \). This implies that \( \alpha \in C_{hVA}(\tilde{U}) \), and thus \( \alpha \in A_{q}^\infty \). \( \square \)

The above lemma asserts that the spectral sequence \( \tau E_{p,q}^r(\tilde{U}) \) weakly converges to the total cohomology \( H^{p+q}(C_{VA}(\tilde{U}), d_{hVA}) \); that is we have

\[
\tau E_{p,q}^r(\tilde{U}) = \cdots = \tau E_{p,q}^\infty(\tilde{U}) \cong \text{Gr}_p H^{p+q}(C_{VA}(\tilde{U}), d_{hVA})
\]

for any \( p, q \in \mathbb{Z} \).

**Lemma 5.15.** For any \( n \in \mathbb{Z} \), we have \( \bigcap_{p \geq 0} \tau_{p} H^n(C_{VA}(\tilde{U}), d_{hVA}) = 0 \).

**Proof.** Take arbitrary \( \alpha \in \bigcap_{p \geq 0} \tau_{p} H^n(C_{VA}(\tilde{U}), d_{hVA}) \). Let \( \sum_{k=0}^{\infty} \alpha_k \) where \( \alpha_k \in C_{VA}^{p+k,q-k}(\tilde{U}) \) be a cocycle which represent \( \alpha \). By the convergence of \( \tau E_{p,q}^r(\tilde{U}) \), we have \( \bigcap_{p \geq 0} \tau_{p} H^n(C_{VA}(\tilde{U}), d_{hVA}) = 0 \), and thus we have \( \alpha' = \sum_{k=0}^{\infty} \alpha_k \in C_{VA}(\tilde{U}) \) with \( \alpha_k' \in C_{VA}^{p+k,q-k}(\tilde{U}) \) such that \( d_{hVA} \alpha_k' = \sum_{k=0}^{\infty} \alpha_k \). We have \( d^{+}_{hVA} \alpha_k' = 0 \) and \( d^{+}_{hVA} \alpha_k' + d^{-}_{hVA} \alpha_{k+1}' = \alpha_k + \alpha_{k+1} \) for \( k \geq 0 \). Let \( \overline{\alpha_k} \) (resp. \( \overline{\alpha_k} \)) be the image of \( \alpha_k \) (resp. \( \alpha_k' \)) in \( C_{hVA}(\tilde{U}) \). Then, we have \( d^{+}_{hVA} \alpha_0 = 0 \) and \( d^{+}_{hVA} \alpha_k + d^{-}_{hVA} \alpha_{k+1} = \overline{\alpha_k} \) for \( k \geq 0 \). Note that we have \( \overline{\alpha_k} = 0 \) for \( k \geq k_s \) and \( \overline{\alpha_k} = \overline{\alpha_k'} \) for \( k \geq k_1 \). By the above equalities, \( \overline{\alpha_k} \) for \( k \geq 0 \) also belongs to the same bounded double subcomplex. Thus, there exists \( k_1 \geq 0 \) such that \( \overline{\alpha_k} = 0 \) i.e. \( \alpha_k' \in F_{s}C_{hVA}(\tilde{U}) \) for \( l \geq k_1 \). By the same argument of the proof of Lemma 5.13 inductively on \( s \), there exists \( k_s \geq 0 \) such that \( \alpha_l' \in F_{s}C_{hVA}(\tilde{U}) \) for \( l \geq k_s \). Therefore, \( \alpha' = \sum_{k=0}^{\infty} \alpha_k' \in C_{hVA}(\tilde{U}) \), and hence \( \alpha = d_{hVA} \alpha' = 0 \) in \( H^n(C_{VA}(\tilde{U}), d_{hVA}) \). \( \square \)

Lemma 5.15 together with Lemma 5.13 gives the convergence of the spectral sequence \( \tau E_{p,q}^r(\tilde{U}) \) to the total cohomology \( H^{p+q}(C_{VA}(\tilde{U}), d_{hVA}) \) (Proposition 5.14).

6. Local structure of BRST reduction

In the previous sections, we defined the sheaf of \( \mathbb{h} \)-adic vertex algebras \( \tilde{D}_{hVA}^\infty \) over the hypertoric variety \( \tilde{X} \). Now we describe the local structure of the BRST reduction \( D_{hVA}(\tilde{U}_j) \) over the affine open subset \( \tilde{U}_j \subseteq \tilde{X} \) using with the local coordinate which we defined in Section 5.3.

Consider the affine open subset \( \tilde{U}_j \subseteq \tilde{X} \) defined in Section 5.3 and recall the local coordinate functions in \( \mathcal{O}_{\tilde{X}}(\tilde{U}_j) \) of [1]. We identify these coordinate functions with their image in \( \mathcal{O}_{\tilde{X}}(\tilde{U}_j) \) and their lifts on \( \tilde{D}_{hVA}^\infty(\tilde{U}_j) \). Then, consider
the cochains $a^J_j = a^J_{j-1} 1, a^J_j = a^J_{j-1} 1 \in C^0_{\text{NVA}}(\tilde{U}_j)$ for $j \notin J$. Since $a^J_j$ and $a^J_{j-1}$ have none of the factors $x_{k-1}y_k$ for $k = 1, \ldots, N$ and they are $G$-invariant, we have $\mu_{ch}(A_i(a) ) a^J_j = \mu_{ch}(A_i(a))a^J_{j-1} = 0$ for any $i = 1, \ldots, M, j \notin J$ and $n \in \mathbb{Z}_{\geq 0}$. It implies that $d_{\text{NVA}}a^J_j = d_{\text{NVA}}a^J_{j-1} = 0$ and thus $a^J_j, a^J_{j-1}$ define elements in $H^0(C^0_{\text{NVA}}(\tilde{U}_j), d_{\text{NVA}}) = \mathcal{D}^b_{\psi}(\tilde{U}_j)$. We denote these elements the same notation $a^J_j$ and $a^J_{j-1} \in \mathcal{D}^b_{\psi}(\tilde{U}_j)$. For $i = 1, \ldots, M$, we have a cochain $c_i = c_{i-1} 1 \in C^0_{\text{NVA}}(\tilde{U}_j)$. By direct calculation, we have $d_{\text{NVA}}c_i = -h \sum_{j=1}^M (c_i, c_j) \psi^*_j(1) 1$, which is not necessarily zero. Note that, on $\tilde{U}_j$, we have a cochain $\partial \log T^j_I = T^j_I(\log T^j_I)^{-1} 1 \in C^0_{\text{NVA}}(\tilde{U}_j)$ for $j = 1, \ldots, M$. Again since $T^j_I$ has none of the factors $x_{k-1}y_k$ for $k = 1, \ldots, N$, and $T^j_I$ is of weight $e_j$ with respect to the $G$-action, we have $\mu_{ch}(A_i(a) ) \partial \log T^j_I = e_{j1}d_j h 1$ for $i = 1, \ldots, M$ and $n \in \mathbb{Z}_{\geq 0}$. For $i = 1, \ldots, M$, set a cochain locally defined on $\tilde{U}_j$,

\begin{equation}
\left(12\right)
b^j_i = c_i + h \sum_{j=1}^M (c_i, c_j) \partial \log T^j_I \in C^0_{\text{NVA}}(\tilde{U}_j).
\end{equation}

Then, we have $d_{\text{NVA}}b^j_i = 0$, and thus $b^j_i$ defines an element of $b^j_i \in H^0(C^0_{\text{NVA}}(\tilde{U}_j), d_{\text{NVA}}) = \mathcal{D}^b_{\psi}(\tilde{U}_j)$ for $i = 1, \ldots, M$. By Proposition \textcolor{red}{5.8} and Lemma \textcolor{red}{5.12}, we have $H^0(C^0_{\text{NVA}}(\tilde{U}_j), d_{\text{NVA}}) \simeq \mathcal{D}^b_{\psi}(\tilde{U}_j)$ is an $h$-adic vertex algebra. Thus, we have\n
\begin{equation}
\text{C}[h][a^J_{j-1}, a^J_{j-1}, b^j_i] \mid j \notin J, i = 1, \ldots, M, n \in \mathbb{Z}_{\geq 1} \subset H^0(C_h(\tilde{U}_j), d_{\text{NVA}}).
\end{equation}

By Proposition \textcolor{red}{5.13} and Lemma \textcolor{red}{5.12}, we have $H^0(C^0_{\text{NVA}}(\tilde{U}_j), d_{\text{NVA}}) = \mu_{\text{ch}}^b(\tilde{U}_j) = H^0(C^+(\tilde{U}_j), d_{\text{NVA}}^+) \subset \mathcal{O}_{\tilde{X}}(\tilde{U}_j)[[h]] \otimes \Lambda_{\text{ch}}^*([[h]]).$ Since $d_{\text{NVA}} = d_{\text{VPA}}$ on $\text{Gr} C^+$, we have an embedding

\begin{equation}
H^0(C^+(\tilde{U}_j), d_{\text{NVA}}^+) = \text{Ker} d_{\text{NVA}}^+ \subset \mathcal{O}_{\tilde{X}}(\tilde{U}_j)[[h]].
\end{equation}

By \textcolor{red}{(13)}, we have

\begin{equation}
\mathcal{O}_{\tilde{X}}(\tilde{U}_j)[[h]] = \mathcal{O}_{\tilde{X}}(\tilde{U}_j)[[h]]\langle a^J_{j-1}, a^J_{j-1}, c_{i-1} \rangle \mid j \notin J, i = 1, \ldots, M, n \in \mathbb{Z}_{\geq 1},
\end{equation}

and thus the $h$-adic vertex subalgebra of \textcolor{red}{(13)} coincides with $H^0(C^0_{\text{NVA}}(\tilde{U}_j), d_{\text{NVA}})$. Here note that the elements $a^J_{j-1}, a^J_{j-1}$ and $b^j_i$ for $j \notin J, i = 1, \ldots, M$ and $n \in \mathbb{Z}_{\geq 1}$ are algebraically independent because their images $a^J_{j-1}, a^J_{j-1}, c_{i-1}$ in $\mathcal{O}_{\tilde{X}}(\tilde{U}_j)[[h]]/h \mathcal{O}_{\tilde{X}}(\tilde{U}_j)[[h]]$ are algebraically independent.

**Proposition 6.1.** For the affine open subset $\tilde{U}_j \subset \tilde{X}$ defined in Section 3.3, we have\n
\begin{equation}
\mathcal{D}^b_{\psi}(\tilde{U}_j) = H^0(C^0_{\text{NVA}}(\tilde{U}_j), d_{\text{NVA}}) = \mathcal{O}_{\tilde{X}}(\tilde{U}_j)[[h]]\langle a^J_{j-1}, a^J_{j-1}, b^j_i \rangle \mid j \notin J, i = 1, \ldots, M, n \in \mathbb{Z}_{\geq 1}\n\end{equation}

\begin{equation}
\simeq \mathcal{D}^b(T^*\mathbb{C}^{-M})_h \otimes V_{\psi}(\mathbb{C}_h(\mathbb{G})).
\end{equation}

**Proof.** The isomorphism as $\mathbb{C}[[[h]]]$-modules follows from the above discussion. We consider the structure as an $h$-adic vertex algebra. Note that, by the explicit construction in Section 3.3, $a^J_j, a^J_{j-1}$, and $b^j_i$ contain no pair $(x_k, y_k)$ for $k = 1, \ldots, N$ except that $a^J_j$ and $a^J_{j-1}$ contain a pair $(x_j, y_j)$. Thus, by direct easy calculation, we obtain OPEs $a^J_j(z) a^J_j(w) \sim h d_j^j/(z-w), b^j_i(z) b^j_i(w) \sim h^2 (c_i, c_i)/(z-w)^2$, and all other combinations have trivial OPEs. Thus, we have the isomorphism of $h$-adic vertex algebras of the statement. \hfill \qed
7. Equivariant torus action and vertex algebra of global sections

In the previous sections, we defined the sheaf of $h$-adic vertex algebras $\tilde{D}_{X,h}^h$ over the hypertoric variety $\tilde{X}$, and studied its structure. The space of global sections, $\tilde{D}_{X,h}^h(\tilde{X})$ is naturally equipped with the structure of $h$-adic vertex algebra. We also have an $h$-adic vertex algebra constructed by the global BRST reduction $H^{\infty/2+0}(g, \tilde{D}_{X,h}^h(\tilde{X}))$. In this section, we construct the vertex algebras from these $h$-adic vertex algebras using a certain equivariant torus action, which reflect the essential structure of the original $h$-adic vertex algebras.

Consider an action of one-dimensional torus $\mathbb{S} = \mathbb{C}^\times$ on $\tilde{X}$ which induces an action on the structure sheaf $\mathcal{O}_X = \mathcal{O}_\tilde{X} \otimes_{\mathbb{C}} \mathcal{O}_g$, such that the weights of the generators with respect to the action is given by $\mathbb{S}$-wt$(x_k) = \mathbb{S}$-wt$(y_k) = 1/2$, $\mathbb{S}$-wt$(c_i) = 1$ for $k = 1, \ldots, N$ and $i = 1, \ldots, M$. Note that, with respect to this action, the Poisson bracket on $\mathcal{O}_\tilde{X}$ is homogeneous of weight $-1$. Since the $\mathbb{S}$-action commutes with the $G$-action, we have the induced $\mathbb{S}$-action on the hypertoric variety $\tilde{X}$.

Moreover, we have the equivariant $\mathbb{S}$-action on the sheaf $\tilde{D}_{X,h}^h(\tilde{X})$ over $\mathbb{C}$ such that the weights of the generators given by $\mathbb{S}$-wt$(x_k(-n)) = \mathbb{S}$-wt$(y_k(-n)) = 1/2$, $\mathbb{S}$-wt$(c_i(-n)) = 1$, $\mathbb{S}$-wt$(h) = 1$ and $\mathbb{S}$-wt$(1) = 0$ for $k = 1, \ldots, N$, $i = 1, \ldots, M$ and $n \in \mathbb{Z}_{\geq 1}$. Note that the OPEs of $\tilde{D}_{X,h}^h$ are homogeneous with respect to the $\mathbb{S}$-action. Extend this action onto the BRST complex $C_{NVA}$ by $\mathbb{S}$-wt$(\psi^0_{i(-n)}) = 0$, $\mathbb{S}$-wt$(\psi_{i(-n)}) = 1$ for $i = 1, \ldots, M$ and $n \in \mathbb{Z}_{\geq 1}$. Then, the element $Q_{NVA} \in C_{NVA}$ is homogeneous of weight $\mathbb{S}$-wt$(Q_{NVA}) = 1$, and hence the coboundary operator $d_{NVA} = h^{-1}Q_{NVA(0)}$ is a homogeneous operator of weight $0$ on the complex $C_{NVA}$.

This implies that the BRST cohomology sheaf $H^{\infty/2+0}_{NVA}(g, \tilde{D}_{X,h}^h(\tilde{X}))$ is also equipped with the induced equivariant $\mathbb{S}$-action over $\tilde{X}$. In particular, the space of global sections $H^{\infty/2+0}_{NVA}(g, \tilde{D}_{X,h}^h(\tilde{X}))$ is a $\mathbb{C}[[h]]$-module with an $\mathbb{S}$-action over $\mathbb{C}$.

Recall the affine open covering $\tilde{X} = \bigcup_j \tilde{U}_j$. For any $J$, the open subset $\tilde{U}_J$ is closed under the $\mathbb{S}$-action, and $C_{NVA}(\tilde{U}_J)$ is decomposed into the direct product of weight spaces because the coordinate functions of $\tilde{U}_J$ are all homogeneous. Since the coboundary operators $d_{NVA}$ is homogeneous of weight $0$, the $0$-th cohomology group $\tilde{D}_{X,h}^h(\tilde{U}_J) = H^0(C_{NVA}(\tilde{U}_J), d_{NVA})$ is also a direct product of weight spaces. Therefore, the $h$-adic vertex algebra of global sections can be decomposed into a direct product of weight spaces: $\tilde{D}_{X,h}^h(\tilde{X}) = \prod_{m \geq 0} \tilde{D}_{X,h}^h(\tilde{X})^\mathbb{Z}_m$. Note that the weights $m \in \frac{1}{2} \mathbb{Z}_{\geq 0}$ are non-negative and we have $\tilde{D}_{X,h}^h(\tilde{X})^{\mathbb{Z}_0} = \mathbb{C}$. Consider the subspace $\tilde{D}_{X,h}^h(\tilde{X})_{fin} = \bigoplus_{m \in \frac{1}{2} \mathbb{Z}_{\geq 0}} \tilde{D}_{X,h}^h(\tilde{X})^{\mathbb{Z}_m}$. This subspace is a $\mathbb{C}[[h]]$-module since the weights are non-negative and $\mathbb{S}$-wt$(h) = 1$. Moreover, since the OPEs preserve the $\mathbb{S}$-weight, they also preserve the subspace. Now we set

$$\tilde{D}^h(\tilde{X}) = \tilde{D}_{X,h}^h(\tilde{X})_{fin} / (h - 1),$$

the quotient space by the ideal generated by $h - 1$. It is a $\mathbb{C}$-vector space equipped with OPEs induced from ones on $\tilde{D}_{X,h}^h(\tilde{X})$. Since all the identities between the vertex operators of $\tilde{D}_{X,h}^h(\tilde{X})$ are satisfied by the vertex operators of $\tilde{D}^h(\tilde{X})$, the $\mathbb{C}$-vector space $\tilde{D}^h(\tilde{X})$ is a vertex algebra.

Similarly, considering the $h$-adic vertex subalgebra $H^{\infty/2+0}_{NVA}(g, \tilde{D}_{X,h}^h(\tilde{X})) \subset \tilde{D}_{X,h}^h(\tilde{X})$, we have a $\mathbb{C}[[h]]$-submodule

$$H^{\infty/2+0}_{NVA}(g, \tilde{D}_{X,h}^h(\tilde{X}))_{fin} = \bigoplus_{m \geq 0} H^{\infty/2+0}_{NVA}(g, \tilde{D}_{X,h}^h(\tilde{X}))^{\mathbb{Z}_m} \subset H^{\infty/2+0}_{NVA}(g, \tilde{D}_{X,h}^h(\tilde{X})).$$
We define a vertex algebra over $\mathbb{C}$ by

$$D^{ch}(\tilde{X}) = H^{2+\infty}_{\text{VA}}(0, \tilde{D}^{ch}_{X,h}(\tilde{X}))_{fin} / (h - 1)$$

**Definition 7.1.** We call the vertex algebras $\tilde{D}^{ch}(\tilde{X})$, $D^{ch}(\tilde{X})$ defined by (14), (15) hypertoric vertex algebras.

**Remark 7.2.** Later in Proposition 7.4 we prove that the two vertex algebras $\tilde{D}^{ch}(\tilde{X})$ and $D^{ch}(\tilde{X})$ coincide.

By the result of the previous section, the sheaf of $h$-adic vertex algebra $\tilde{D}^{ch}_{X,h}$ is isomorphic to the tensor product of a $\beta\gamma$-system and a Heisenberg vertex algebra. It gives an analog of Wakimoto realization (free field realization) of the hypertoric vertex algebra $D^{ch}(\tilde{X})$ (and $\tilde{D}^{ch}(\tilde{X})$). (cf. [Wa], [FF1])

For the affine open subset $\tilde{U}_J \subset \tilde{X}$, we have the restriction homomorphism $\tilde{D}^{ch}_{X,h}(\tilde{X}) \to \tilde{D}^{ch}_{X,h}(\tilde{U}_J)$ between $h$-adic vertex algebras. By Proposition 3.1 we have

$$\tilde{D}^{ch}_{X,h}(\tilde{U}_J) = \mathbb{C}[h][a_{j,-n}, a_{j,-n}^*, b_{j,-n}^* | j \notin J, i = 1, \ldots, M, n \in \mathbb{Z}_{\geq 1}]$$

Then, the image of the $\mathbb{C}[h]$-submodule $\tilde{D}^{ch}_{X,h}$ under the homomorphism is included in the $\mathbb{C}[h][a_{j,-n}^*, a_{j,-n}^*] \ast \mathbb{C}[b_{j,-n}^* i, j, n]$. Thus, we have the following $\mathbb{C}$-linear map

$$D^{ch}(\tilde{X}) \to \tilde{D}^{ch}(\tilde{X}) \to \mathbb{C}[a_{j,-n}, a_{j,-n}^*, b_{j,-n}^* | j \notin J, i = 1, \ldots, M] \otimes \mathbb{C}[b_{j,-n}^* i, j, n] \cong D^{ch}(\mathbb{C}^{(N-M)}) \otimes \mathbb{C}[V_J(\mathfrak{g})]$$

by taking quotients by $(h - 1)$ where $D^{ch}(\mathbb{C}^{(N-M)})$ is a $\beta\gamma$-system and $V_J(\mathfrak{g})$ is a Heisenberg vertex algebra. Clearly, this is a homomorphism between vertex algebras over $\mathbb{C}$.

For $\lambda \in \mathfrak{g}^*$, let $\pi_\lambda$ is the Heisenberg Fock space of highest weight $\lambda$; i.e. $\pi_\lambda$ is an irreducible highest weight module with a highest weight vector $|\lambda\rangle \in \pi_\lambda$ on which the action is given by $b_{j,0}^*|\lambda\rangle = (\lambda(\alpha))|\lambda\rangle$ and $b_{i,n}^*|\lambda\rangle = 0$ for $i = 1, \ldots, M$ and $n > 0$.

**Proposition 7.3.** For each $J$ and $\lambda \in \mathfrak{g}^*$, we have an action of the hypertoric vertex algebra $D^{ch}(\tilde{X})$ on $\mathcal{F}_{\beta\gamma} \otimes \pi_\lambda$ where $\mathcal{F}_{\beta\gamma} = \mathbb{C}[a_{j,-n}^*, a_{j,-n}^* | j \notin J, i = 1, \ldots, M]$ is the Fock space of the $\beta\gamma$-system and $\pi_\lambda$ is the Fock space of the Heisenberg vertex algebra of highest weight $\lambda$.

8. Conformal vectors

In this section, we construct the conformal vector explicitly by an analog of the Segal-Sugawara construction.

First assume that the symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ is degenerate. In such a case, we have an element $\zeta = \sum_{i=1}^{M} a_i c_i \in \mathfrak{g}$ ($a_i \in \mathbb{C}$ for $i = 1, \ldots, M$) such that $\langle \zeta, c_i \rangle = 0$ for any $i = 0, \ldots, M$. Then, $\zeta = \zeta(-1) \mathbb{1} \in C^0_{\text{VA}}(\tilde{X})$ satisfies $\zeta(\alpha) = 0$ for any $\alpha \in C_{\text{VA}}$ and $n \geq 0$. In particular, we have $d_{\text{VA}} \zeta = -1 Q_{\text{VA}(0)} \zeta = 0$. Clearly, $\zeta$ does not lie in $\text{Im} d_{\text{VA}}$ and thus $\zeta$ defines a nonzero central vector in $H^0(C_{\text{VA}}(\tilde{X}), d_{\text{VA}})$ and in $D^{ch}(\tilde{X})$. Therefore, the vertex algebra $D^{ch}(\tilde{X})$ has nontrivial center and hence it is not a vertex operator algebra.

Now, assume that the symmetric bilinear form $\langle \cdot, \cdot \rangle$ is nondegenerate. Let $\{ e_i \}_{i=1,\ldots,M} \subset \mathfrak{g}$ be the dual basis of the basis $\{ c_i \}_{i=1,\ldots,M}$ with respect to the bilinear form. Set $\omega_H = (1/2) \sum_{i=1}^{M} c_i (-1) e_i \in V_{\phi,(-1),h}(\mathfrak{g}^*) \subset C_{\text{VA}}(\tilde{X})$. The following lemma is the standard fact.
Lemma 8.1. For $i = 1, \ldots, M$ and $m, n \in \mathbb{Z}$, we have $[\omega_{H(m+1), c_i(n)}] = -\hbar^2 n c_i(m+n)$. In particular, one obtains $\mu_{ch}(A_i(n)) \omega_H = -\hbar^2 \delta n c_i$ for $i = 1, \ldots, M$ and $n \geq 0$.

Lemma 8.2. We have the OPE
\[
\omega_H(z) \omega_H(w) \sim \frac{\hbar^4}{(z-w)^4} M + \frac{\hbar^2}{(z-w)^2} 2\omega_H(w) + \frac{\hbar^2}{z-w} \partial_w \omega_H(w).
\]

Proof. It is direct and standard calculation using Lemma 8.1. \qed

Let $\kappa \in \mathbb{C}$ be a parameter. For $j = 1, \ldots, N$ and $\kappa$, let
\[
\omega_{\kappa,j} = \kappa x_{j(-1)} y_{j(-1)} - (\kappa - 1) x_{j(-1)} y_{j(-2)} \in D^{ch}_{\mathcal{X}, H}(\mathcal{X}) \subset C^0_{\text{VVA}}(\tilde{\mathcal{X}}).
\]

Lemma 8.3. For $j = 1, \ldots, N$, and $m, n \in \mathbb{Z}$, we have
\[
[\omega_{\kappa,j(m+1), x_{k(n)}}] = -\hbar (n + \kappa) x_{j(m+n)}, \quad [\omega_{\kappa,j(m+1), y_{k(n)}}] = -\hbar (n - \kappa + 1) y_{j(m+n)}.
\]

In particular, we have
\[
\mu_{ch}(A_i(n)) \omega_{\kappa,j} = \begin{cases} \hbar \Delta_{ij} x_{j(-1)} y_j & (n = 1) \\ \hbar^2 \Delta_{ij} (1 - 2\kappa) & (n = 2) \\ 0 & \text{(otherwise.)} \end{cases}
\]
for $i = 1, \ldots, M$ and $n \geq 0$.

Lemma 8.4. We have the OPE
\[
\omega_{\kappa,j}(z) \omega_{\kappa,j}(w) \sim \frac{\hbar^4}{(z-w)^4} - \frac{1}{2} + \frac{\hbar^2}{(z-w)^2} 2\omega_{\kappa,j}(w) + \frac{\hbar^2}{z-w} \partial_w \omega_{\kappa,j}(w)
\]
for $k = 1, \ldots, N$ and for any $\kappa \in \mathbb{C}$.

Set $\omega_F = \sum_{i=1}^M \psi_{i(-2)} \psi_i \in C^0_{\text{VVA}}(\mathcal{X})$. By direct calculation, we have the following lemma.

Lemma 8.5. We have the commutation relations $[\omega_{F(m+1), \psi_i(n)}] = \hbar n \psi_i(m+n)$, $[\omega_{F(m+1), \psi_i(n)}] = \hbar n \psi_i(m+n)$ for $i = 1, \ldots, M$ and $m, n \in \mathbb{Z}$. In particular, we have $d_{\text{VVA}} \omega_F = \sum_{i=1}^M \mu_{ch}(A_i(-1)) \psi_{i(-2)}$. Moreover, we have the following OPE
\[
\omega_F(z) \omega_F(w) \sim \frac{\hbar^4}{(z-w)^4} - M - N - 2\omega_F(w) + \frac{\hbar^2}{z-w} \partial_w \omega_F(w).
\]

Now we set $\omega = \hbar \sum_{k=1}^N \omega_{1/2,k} + \omega_H + \hbar \omega_F \in C^0_{\text{VVA}}(\tilde{\mathcal{X}})$. Then the following proposition is obvious from the above lemmas.

Proposition 8.6. We have $d_{\text{VVA}}(\omega) = 0$, and thus $\omega \in C^0_{\text{VVA}}(\tilde{\mathcal{X}})$ defines an element in $H^0(Ch_{\text{VVA}}(\tilde{\mathcal{X}}), d_{\text{VVA}})$ and in $D^{ch}(\tilde{\mathcal{X}})$ which we also write $\omega$. Moreover, the element $\omega$ has the OPE
\[
\omega(z) \omega(w) \sim \frac{\hbar^4}{(z-w)^4} - M - N - 2\omega(w) + \frac{\hbar^2}{z-w} \partial_w \omega(w).
\]
Namely, $\omega \in D^{ch}(\tilde{\mathcal{X}})$ is a conformal vector.

The operator $\omega(1)$ gives a non-negative grading on $C_{\text{VVA}}(\tilde{\mathcal{X}})$; conf-wt($x_k$) = conf-wt($y_k$) = 1/2, conf-wt($c_i$) = 1, conf-wt($\psi_i$) = 0 and conf-wt($\psi_i$) = 1 for $k = 1, \ldots, N$, $i = 1, \ldots, M$. The vertex algebra $D^{ch}(\tilde{\mathcal{X}})$ is 1/2-$\mathbb{Z}_{\geq 0}$-graded by the action of $\omega(1)$ such that any element of conformal weight 0 is proportional to the vacuum $1$. Therefore, $D^{ch}(\tilde{\mathcal{X}})$ is a vertex operator algebra.
Moreover, take $\lambda = (\lambda_k)_{k=1,\ldots,N} \in \mathbb{R}^N$, an orthogonal vector with all row vectors $\Delta^i = (\Delta_{ij})_{j=1,\ldots,N}$ of the matrix $\Delta$ for $i = 1, \ldots, M$. Then, the vector

$$\omega_\lambda = \omega + h \sum_{k=1}^N \lambda_k (x_{k(-2)} y_{k(-1)} + x_{k(-1)} y_{k(-2)}) = h \sum_{k=1}^N \omega_{(1/2+\lambda_k),k} + \omega_H + h\omega_F$$

is also a conformal vector in $\mathcal{C}_{\text{AVA}}(\tilde{X})$. Since $\lambda$ is orthogonal with $\Delta^i = (\Delta_{ij})_{j=1,\ldots,N}$ for $i = 1, \ldots, M$, we have

$$\mu_{\text{ch}}(A_i)(n) \sum_{k=1}^N \lambda_k (x_{k(-2)} y_{k(-1)} + x_{k(-1)} y_{k(-2)}) = \delta_{n2} h^2 \sum_{k=1}^N \Delta_{ik} (-2\lambda_k) = 0$$

for all $i = 1, \ldots, N$ and $n \geq 0$. Thus, we have $\omega_\lambda \in \text{Ker} d_{\text{AVA}}$, and hence $\omega_\lambda$ induces a conformal vector in $\mathcal{D}^{\text{ch}}(\tilde{X})$. Note that $\omega_{\lambda(1)}$ also gives a grading on $\mathcal{D}^{\text{ch}}(\tilde{X})$ but the grading may be negative in contrast to the standard one $\omega_{(1)}$.

9. Zhu algebras

In this section, we discuss the Zhu algebra of the hypertoric vertex algebra, an associative algebra which reflects fundamental aspects of the representation theory of the corresponding vertex operator algebra. Our goal is to show that the Zhu algebra of $\mathcal{D}^{\text{ch}}(\tilde{X})$ coincides with the universal family of quantization of the Poisson algebra $\mathbb{C}[X]$.

9.1. The definition of Zhu algebras. Let $V = \bigoplus_{d \geq 0} V_d$ be a vertex algebra with $\mathbb{Z}_{\geq 0}$-grading. For a homogeneous element $a \in V_d$, we denote its grading $d_a = d$. For a homogeneous element $a \in V_{d_a}$, an element $b \in V$ and positive integer $m \in \mathbb{Z}_{>0}$, we define

$$a *_m b = \sum_{j \geq 0} \binom{d_a}{j} a_{(-m+j)} b \in V,$$

and extend it on $V$ linearly. We simply denote $* = *_1$, $\circ = *_2$ for $m = 1, 2$. Let $A(V) = V / V \circ V$ be the quotient vector space where $V \circ V = \{a \circ b \mid a, b \in V\}$. As proved in [Z], the product $* = *_1$ is a linear associative product on the vector space $A(V)$ with the unit $1$. The associative algebra $A(V)$ is called the Zhu algebra of the vertex algebra $V$.

Besides the Zhu algebra $A(V)$, we also have a Poisson algebra corresponding to the vertex algebra $V$. Consider the vector space $\overline{A}(V) = V / V_{(-2)} V$ where $V_{(-2)} V = \{a_{(-2)} b \mid a, b \in V\}$. The vertex algebra operator $(-1)$ gives a commutative associative product on $\overline{A}(V)$, and moreover, $\overline{A}(V)$ is a Poisson algebra with the Poisson bracket $\{a, b\} = a_{(0)} b$ modulo $V_{(-2)} V$. We call the Poisson algebra $\overline{A}(A)$ the $C_2$ Poisson algebra of the vertex algebra $V$. Note that, while the definition of Zhu algebra $A(V)$ requires the $\mathbb{Z}_{\geq 0}$-grading on the vertex algebra $V$, the grading is not needed to define $C_2$ Poisson algebra $\overline{A}(V)$. In some known cases, the Zhu algebra gives a quantization of the $C_2$ Poisson algebra; e.g. the affine vertex operator algebra associated with the simple Lie algebra, Virasoro vertex algebra and $\gamma \gamma$ systems.

Also for an $h$-adic vertex algebra $V_h$, we define $\overline{A}(V_h) = V_h / V_{h(-2)} V_h$, a commutative $\mathbb{C}[[h]]$-algebra. For the sheaf of $h$-adic vertex algebras $\mathcal{D}^{\text{ch}}_{X,h}$ over $\tilde{X}$, we define the sheaf of $\mathbb{C}[[h]]$-algebras $\overline{A}(\mathcal{D}^{\text{ch}}_{X,h})$ as the quotient sheaf $\overline{A}(\mathcal{D}^{\text{ch}}_{X,h}) = \mathcal{D}^{\text{ch}}_{X,h} / (\mathcal{D}^{\text{ch}}_{X,h(-2)} \mathcal{D}^{\text{ch}}_{X,h})$. Namely, it is the sheaf associated with the presheaf $U \mapsto \overline{A}(\mathcal{D}^{\text{ch}}_{X,h}(U))$ for an open subset $U \subset \tilde{X}$.
9.2. Quantization of the hypertoric variety. The associative algebra quantizing the hypertoric variety $X$ was first introduced by I. Musson and M. Van den Bergh in [MV].

Let $D(V)$ be the Weyl algebra on the affine space $V = \mathbb{C}^N$, that is, the algebra of differential operators with polynomial coefficients. We denote the standard coordinate functions on $V$ by $x_1, \ldots, x_N$ as in Section 3 and the corresponding differential operators $\partial_k = \partial/\partial x_k$ for $k = 1, \ldots, N$. Then the Weyl algebra $D(V)$ is isomorphic to $\mathbb{C}[x_k, \partial_k \mid k = 1, \ldots, N]$ as a $\mathbb{C}$-vector space. The action of the torus $G = (\mathbb{C}^*)^M$ on $V$ induces an action on the algebra $D(V)$. Define a map $\mu_D : \mathfrak{g} \to D(V)$ by $A_i \mapsto \mu_D(A_i) = \sum_{k=1}^N \Delta_{ik} x_k \partial_k$ for $i = 1, \ldots, M$. Clearly, this map quantizes the commutant map $\mu^*$ and we call $\mu_D$ a quantized commutant map. Set $\tilde{D}(V) = D(V) \otimes_{\mathbb{C}} \mathbb{C}[c_1, \ldots, c_M]$, and extend the action of the torus $G$ onto $\tilde{D}(V)$ so that $G$ acts on $\mathbb{C}[c_1, \ldots, c_M]$ trivially. Define the associative algebra $D(\tilde{X})$ by quantum Hamiltonian reduction as follows:

\[
\tilde{D}(X) = \left( \frac{D(V)}{\bigoplus_{i=1}^M D(V)(\mu_D(A_i) - c_i)} \right)^G = \frac{D(V)^G}{\bigoplus_{i=1}^M D(V)^G(\mu_D(A_i) - c_i)}. \tag{16}
\]

It is not difficult to examine that $D(\tilde{X})$ is an associative algebra, and its associated graded algebra with respect to the Berenstein filtration, i.e., the filtration induced from $\deg x_k = \deg \partial_k = 1$ and $\deg c_i = 0$, coincides with $\mathbb{C}[\tilde{X}]$ as Poisson algebras. The algebra $D(\tilde{X})$ is an algebra over $\mathbb{C}^M = \text{Spec} \mathbb{C}[c_1, \ldots, c_M]$, and it is a family of filtered quantizations of the Poisson algebra $\mathbb{C}[X]$, while the Poisson algebra $\mathbb{C}[\tilde{X}]$ is a family of Poisson deformations of $\mathbb{C}[X]$ over $\mathbb{C}^M$ in the sense of [L1], [L2].

The algebra $D(\tilde{X})$ was introduced in [MV], and it is called a quantized hypertoric algebra or a hypertoric enveloping algebra. One can construct a sheaf of associative $\mathbb{C}[\hbar]$-algebras on $\tilde{X}$ whose algebra of global sections coincides with $D(\tilde{X})$. See [BoKu] and [BLPW]. Moreover we can describe the above quantum Hamiltonian reduction by a certain BRST cohomology, which is analogous to the BRST cohomology in this paper. See [IK].

Consider the action of the $N$-dimensional abelian Lie algebra $\mathbb{C}^N = \bigoplus_{i=1}^N \mathbb{C} x_k \partial_k$ on $D(V)$ by the commutation $[x_k \partial_k, \cdot] = 0$ for $k = 1, \ldots, N$. The action corresponds to the natural action on $\mathbb{C}^N$. The algebra $D(V)$ is decomposed into the direct sum of weight spaces with respect to this action: $D(V) = \bigoplus_{\xi \in \mathbb{Z}^N} D(V)^{\mathbb{C}^N \cdot \xi}$. Consider the sublattice $\bigoplus_{i=1}^M \mathbb{Z} \Delta_i \subset \mathbb{Z}^N$ where $\Delta_i = (\Delta_{ij})_{i=1,\ldots,N}$. It can be identified with the weight lattice of the torus $G$ and its Lie algebra $\mathfrak{g} = \bigoplus_{i=1}^M \mathbb{C} A_i$ because $A_i \in \mathfrak{g}$ acts on $D(V)$ by $\mu_D(A_i) = \sum_{j=1}^N \Delta_{ij} x_j \partial_j$. Take the orthogonal sublattice $\Lambda_0 \subset \mathbb{Z}^N$ of $\bigoplus_{i=1}^M \mathbb{Z} \Delta_i$. Then, we have $D(V)^G = \bigoplus_{\xi \in \Lambda_0} D(V)^{\mathbb{C}^N \cdot \xi}$ and it induces the weight decomposition of the quantized hypertoric algebra: $D(\tilde{X}) = \bigoplus_{\xi \in \Lambda_0} D(\tilde{X})^{\mathbb{C}^N \cdot \xi}$. The following lemma is obvious.

**Lemma 9.1.** The weight space $D(\tilde{X})^\mathbb{C} \subset D(\tilde{X})$ of weight 0 is given by

\[
D(\tilde{X})^\mathbb{C} = \mathbb{C}[x_1 \partial_1, \ldots, x_N \partial_N] \otimes_{\mathbb{C}} \mathbb{C}[c_1, \ldots, c_M].
\]

Setting $P_\zeta = \prod_{k: \zeta_k > 0} x_k^{\zeta_k} \prod_{k: \zeta_k < 0} \partial_k^{-\zeta_k}$ for $\zeta = (\zeta_1, \ldots, \zeta_N) \in \Lambda_0$, the weight space $D(\tilde{X})^{\mathbb{C} \cdot \zeta}$ is a $D(\tilde{X})^{\mathbb{C}}$-module generated by $P_\zeta$.

Clearly, the associated graded algebra $\mathbb{C}[\tilde{X}]$ has also the same weight decomposition: $\mathbb{C}[\tilde{X}] = \bigoplus_{\zeta \in \Lambda_0} \mathbb{C}[\tilde{X}]^{\mathbb{C} \cdot \zeta}$. For each $\zeta \in \Lambda_0$, we have the same description for $\mathbb{C}[\tilde{X}]^{\mathbb{C} \cdot \zeta}$ as Lemma 9.1 that is, $\mathbb{C}[\tilde{X}]^\mathbb{C} = \mathbb{C}[x_1 y_1, \ldots, x_N y_N, c_1, \ldots, c_M]$ for $\zeta = 0$.
and $\mathbb{C}[\tilde{X}]^{T,\zeta}$ is a $\mathbb{C}[\tilde{X}]^T$-module generated by $P_\zeta$, where we identify $P_\zeta \in D(\tilde{X})^{T,\zeta}$ with its image $P_\zeta = \prod_{k: \zeta_k > 0} x_k^{\zeta_k} \prod_{k: \zeta_k < 0} y_k^{-\zeta_k} \in \mathbb{C}[\tilde{X}]^{T,\zeta}$.

9.3. Weyl group symmetries. The Weyl algebra $D(V)$ has natural automorphisms in $(\mathbb{Z}/2\mathbb{Z})^N \ltimes \mathfrak{S}_N$, generated by permutations $\sigma \in \mathfrak{S}_N$, $\sigma(x_k) = x_{\sigma(k)}$, $\sigma(\partial_k) = \partial_{\sigma(k)}$, and Fourier transformations $x_k \mapsto -y_k$, $y_k \mapsto x_k$, for each $k = 1, \ldots, N$. It naturally induces an action on the weight lattice $\mathbb{Z}^N$. Let $\mathcal{W}$ be the subgroup of all elements in $(\mathbb{Z}/2\mathbb{Z})^N \ltimes \mathfrak{S}_N$ which fix the sublattice $\Lambda_0$ pointwise. Since $\Delta^1, \ldots, \Delta^M$ span the sublattice which is orthogonal to $\Lambda_0$, an element $\sigma \in \mathcal{W}$ maps $\mu_D(A_i)$ to a linear combination $\sum_{j=1}^M \lambda_j \mu_D(A_j)$, $\lambda_j \in \mathbb{Z}$. Then, the action of $\mathcal{W}$ on $D(V)$ is extended onto $\tilde{D}(V) = D(V) \otimes_{\mathbb{C}} \mathbb{C}[c_1, \ldots, c_M]$ by $\sigma(c_i) = \sum_{j=1}^M \lambda_j c_j$. By the definition [10], the action $\mathcal{W}$ on $\tilde{D}(V)$ induces automorphisms of the quantized hypertoric algebra $D(\tilde{X})$. It also induces automorphisms of the Poisson algebra $\mathbb{C}[\tilde{X}]$. The algebras $D(\tilde{X})$ and $\mathbb{C}[\tilde{X}]$ also have other automorphisms which fix the parameters $c_1, \ldots, c_M$, denoted $\mathcal{W}$ in [BLPW Section 8.1], but we will ignore such automorphisms. Now consider the $\mathcal{W}$-invariant subalgebras $D(\tilde{X})^\mathcal{W}$ and $\mathbb{C}[\tilde{X}]^\mathcal{W}$. The algebra $D(\tilde{X})^\mathcal{W}$ (resp. $\mathbb{C}[\tilde{X}]^\mathcal{W}$) is also a family of filtered quantizations (resp. Poisson deformations) of the Poisson algebra $\mathbb{C}[X]$ over the space $\mathcal{M}/\mathcal{W}$. By Corollary 2.13 and Proposition 3.5 in [L2], $D(\tilde{X})^\mathcal{W}$ (resp. $\mathbb{C}[\tilde{X}]^\mathcal{W}$) is characterized as the universal family of filtered quantizations (resp. Poisson deformations) of the Poisson algebra $\mathbb{C}[X]$.

Using Lemma 9.1, we have description of the $\mathcal{W}$-invariant subalgebra $D(\tilde{X})^\mathcal{W}$ as follows: By the orthogonal decomposition $\bigoplus_{i=1}^M \mathbb{Z}{\Delta^i} \oplus \Lambda_0$, for $k = 1, \ldots, N$, we have the decomposition $x_k \partial_k = \sum_{i=1}^M \beta_i \mu_D(A_i) + z$ where $\beta_i \in \mathbb{C}$ and $z \in \bigoplus_{k=1}^N \mathbb{C} x_k \partial_k$ is an element which is orthogonal to $\mu_D(A_i)$ for all $i = 1, \ldots, M$. Set

$$H_k = x_k \partial_k - \sum_{i=1}^M \beta_i c_i \in D(\tilde{X})$$

for $k = 1, \ldots, N$. Since $H_k = z$ in $D(\tilde{X})$ and the group $\mathcal{W}$ fixes $\Lambda_0$ pointwise, $H_k$ is invariant under the action of $\mathcal{W}$ on $D(\tilde{X})$. Next, consider the element $P_\zeta \in D(\tilde{X})^{T,\zeta}$ in Lemma 9.1. Since $\sigma \in \mathcal{W}$ fixes the sublattice $\Lambda_0$ pointwise, $\sigma(P_\zeta)$ is again an element of $D(\tilde{X})^{T,\zeta}$. Moreover, we have $\sigma(P_\zeta) = P_\zeta$ since $P_\zeta$ is the only element which has none of the factors $x_k \partial_k$ for any $k = 1, \ldots, N$. Therefore, $P_\zeta$ is a $\mathcal{W}$-invariant element in $D(\tilde{X})^{T,\zeta}$.

**Lemma 9.2.** The set of polynomials $\{P_\zeta | \zeta \in \Lambda_0\} \cup \{H_k | k = 1, \ldots, N\}$ generates the $\mathcal{W}$-invariant subalgebra $\mathbb{C}[\tilde{X}]^\mathcal{W}$.

**Proof.** Let $R$ be a subalgebra of $\mathbb{C}[\tilde{X}]$ generated by the elements $\{P_\zeta | \zeta \in \Lambda_0\} \cup \{H_k | k = 1, \ldots, N\}$. Since the generators are $\mathcal{W}$-invariant and homogeneous, the subalgebra $R$ is a graded subalgebra of $\mathbb{C}[\tilde{X}]^\mathcal{W}$. Set $S = R \cap \mathbb{C}[g^*]^\mathcal{W}$. Then, we have $R \otimes_S \mathbb{C} \simeq \mathbb{C}[X]$ where $\mathbb{C}$ is an $S$-algebra induced from the specialization $c_i \mapsto 0$ for $i = 1, \ldots, M$. Thus, $R$ is a graded family of Poisson deformation of $\mathbb{C}[X]$ over $S$. By [L2 Proposition 2.12], we have a unique homomorphism $\mathbb{C}[g^*]^\mathcal{W} \rightarrow S$ which induces an isomorphism $\mathbb{C}[\tilde{X}]^\mathcal{W} \otimes_{\mathbb{C}[g^*]^\mathcal{W}} S \simeq R$ intertwining the isomorphisms $R \otimes_S \mathbb{C} \simeq \mathbb{C}[X] \simeq \mathbb{C}[\tilde{X}]^\mathcal{W} \otimes_{\mathbb{C}[g^*]^\mathcal{W}} \mathbb{C}$. By the definition of $R$, the embedding $R \hookrightarrow \mathbb{C}[\tilde{X}]^\mathcal{W}$ also intertwines the isomorphisms $R \otimes_S \mathbb{C} \simeq \mathbb{C}[X] \simeq \mathbb{C}[\tilde{X}]^\mathcal{W} \otimes_{\mathbb{C}[g^*]^\mathcal{W}} \mathbb{C}$. Consider the composition $\varphi : \mathbb{C}[\tilde{X}]^\mathcal{W} \rightarrow \mathbb{C}[\tilde{X}]^\mathcal{W}$ of the above homomorphisms $\mathbb{C}[\tilde{X}]^\mathcal{W} \rightarrow R$ and $R \hookrightarrow \mathbb{C}[\tilde{X}]^\mathcal{W}$. Then, $\varphi$ intertwines...
the isomorphisms $\mathbb{C}[\tilde{X}]^W \otimes_{\mathbb{C}[g]^W} \mathbb{C} \simeq \mathbb{C}[X] \simeq \mathbb{C}[\tilde{X}]^W \otimes_{\mathbb{C}[g]^W} \mathbb{C}$. Therefore, the homomorphism $\varphi$ is an isomorphism by the universality. This implies $R = \mathbb{C}[\tilde{X}]^W$.

9.4. The $C_2$ Poisson algebra. Now we determine the $C_2$ Poisson algebra $\mathcal{A}(D^{ch}(\tilde{X}))$ of the hypertoric vertex algebra $D^{ch}(\tilde{X})$. Consider the affine open covering $\tilde{X} = \bigcup_j \tilde{U}_j$, and we have an isomorphism of Proposition B.4

$$\tilde{D}^{ch}_{X,h}(\tilde{U}_j) = \mathbb{C}[\hbar][a_{j}^{(n)}, a_{j+1}^{(n)} \mid j \neq \infty] \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar][b_{i}^{(n)} \mid i = 1, \ldots, M]$$

$$\simeq D^{ch}(C^{2(N-M)})_{h} \otimes_{\mathbb{C}[\hbar]} V_{j_i, a_{i}}(g).$$

Thus, its $C_2$ Poisson algebra $\mathcal{A}(\tilde{D}^{ch}_{X,h}(\tilde{U}_j)) = \mathcal{A}(\tilde{D}^{ch}_{X,h}(\tilde{U}_j))$ for each affine open subset $\tilde{U}_j \subset \tilde{X}$ is given by

$$\mathcal{A}(\tilde{D}^{ch}_{X,h})(\tilde{U}_j) \simeq \mathbb{C}[\hbar][a_{j}^{(n)}, a_{j+1}^{(n)} \mid j \neq \infty] \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar][b_{i}^{(n)} \mid i = 1, \ldots, M] \simeq \mathcal{O}_{\tilde{X}}(\tilde{U}_j).$$

Moreover, the coordinate transformation of $D^{ch}_{X,h}$ on $\tilde{U}_j \cap \tilde{U}_j$ maps $b_{i}^{(n)}$ to $b_{i}^{(n)} - \sum_{i=1}^{M} (c_i, c_j) \partial \log(T_j^{1}/T_j^{1})$ for $i = 1, \ldots, M$ and the local sections $\partial \log(T_j^{1}/T_j^{1}) \equiv 0$ in the $C_2$ Poisson algebra $\mathcal{A}(\tilde{D}^{ch}_{X,h}(\tilde{U}_j \cap \tilde{U}_j))$. Thus, this coordinate transformation induces the coordinate transformation of $\mathcal{A}(\tilde{D}^{ch}_{X,h})$ such that $b_{i}^{(n)}$ is mapped to $b_{i}^{(n)}$ for each $i = 1, \ldots, M$ and each $\tilde{U}_j \cap \tilde{U}_j$.

Lemma 9.3. We have an isomorphism of sheaves of $\mathbb{C}[\hbar]$-algebras $\mathcal{O}_{\tilde{X}}[[\hbar]] \longrightarrow \mathcal{A}(\tilde{D}^{ch}_{X,h})$ which is locally given by

$$\mathcal{O}_{\tilde{X}}(\tilde{U}_j)[[\hbar]] \longrightarrow \mathcal{A}(\tilde{D}^{ch}_{X,h})(\tilde{U}_j)$$

$$a_{j}^{(i)} \mapsto a_{j}^{(i)}, \quad a_{j+1}^{(i)} \mapsto a_{j+1}^{(i)}, \quad c_i \mapsto b_{i}^{(i)}, \quad (j \neq \infty)$$

Since the global section functor $\Gamma(\tilde{X}, -)$ is left adjoint, $\mathcal{A}(\tilde{D}^{ch}_{X,h}(\tilde{X}))$ is a subalgebra of $\mathcal{A}(\tilde{D}^{ch}_{X,h}(\tilde{X}))$ which does not lie in the image of $H^0(C_{hVA}(\tilde{X}), d_{hVA})$. From this fact, we obtain the following fundamental fact for the hypertoric vertex algebra.

Proposition 9.4. We have $\tilde{D}^{ch}_{X,h}(\tilde{X}) = H^{0, 2+0}_{hVA}(g, \tilde{D}^{ch}_{X,h}(\tilde{X})) = H^0(C_{hVA}(\tilde{X}), d_{hVA})$, and hence $\tilde{D}^{ch}_{X,h}(\tilde{X}) = D^{ch}_{X,h}(\tilde{X})$.

Proof. If $H^0(C_{hVA}(\tilde{X}), d_{hVA}) \neq \tilde{D}^{ch}_{X,h}(\tilde{X})$, then clearly there exists an element of the $C_2$ Poisson algebra $\mathcal{A}(\tilde{D}^{ch}_{X,h}(\tilde{X}))$ which does not lie in the image of $H^0(C_{hVA}(\tilde{X}), d_{hVA}) \subset \tilde{D}^{ch}_{X,h}(\tilde{X})$. However, $\mathcal{A}(\tilde{D}^{ch}_{X,h}(\tilde{X}))$ is a subalgebra of $\mathcal{O}_{\tilde{X}}[[\hbar]]$ and any element of $\mathcal{O}_{\tilde{X}}[[\hbar]]$ is a subalgebra of $\mathcal{O}_{\tilde{X}}[[\hbar]]$ and any element of $\mathcal{O}_{\tilde{X}}[[\hbar]]$ is a subalgebra of $\mathcal{O}_{\tilde{X}}[[\hbar]]$ and any element of $\mathcal{O}_{\tilde{X}}[[\hbar]]$. Thus, we have no element in $\mathcal{A}(\tilde{D}^{ch}_{X,h}(\tilde{X}))$ which does not lie in the image of $H^0(C_{hVA}(\tilde{X}), d_{hVA})$. □

Recall the definition $D^{ch}_{X,h}(\tilde{X}) = \tilde{D}^{ch}_{X,h}(\tilde{X}) = \tilde{D}^{ch}_{X,h}(\tilde{X})_{fin}/(h - 1)$. By the isomorphism theorem, we have

$$\mathcal{A}(D^{ch}_{X,h}(\tilde{X})) \simeq \mathcal{A}(\tilde{D}^{ch}_{X,h}(\tilde{X})_{fin}/(h - 1))$$

$$\simeq \mathcal{A}(\tilde{D}^{ch}_{X,h}(\tilde{X})_{fin}/(h - 1)) \subset \mathcal{O}_{\tilde{X}}(\tilde{X})[\hbar]/(h - 1) \simeq \mathbb{C}[\tilde{X}].$$
Now recall the element \( P_\zeta = \prod_{k, \zeta_k > 0} x_k^{\zeta_k} \prod_{k, \zeta_k < 0} y_k^{\zeta_k} \) \( \in \mathcal{D}(\tilde{X}) \) for \( \zeta \in \Lambda_0 \) in Lemma 9.1. We consider the corresponding element

\[
\tilde{P}_\zeta = \prod_{k, \zeta_k > 0} x_k^{\zeta_k(-1)} \prod_{k, \zeta_k < 0} y_k^{\zeta_k(-1)} \mathbf{1} \in C^0_{\text{NVA}}(\tilde{X})
\]

of the BRST complex. Since \( \zeta \in \Lambda_0 \) is orthogonal to \( \Delta_i \) for \( i = 1, \ldots, M \) and the element \( \tilde{P}_\zeta \) has none of the factors \( x_{k(-1)} y_{k(-1)} \) for \( k = 1, \ldots, N \), we have \( d_{\text{NVA}}(\tilde{P}_\zeta) \equiv 0 \). Thus \( \tilde{P}_\zeta \) defines an element in \( \mathcal{D}^c(X) \), and in its \( C_2 \) Poisson algebra \( \overline{\mathcal{A}}(\mathcal{D}^c(X)) \). We denote these elements the same notation \( \tilde{P}_\zeta \). Next, recall the element \( H_k \) for \( k = 1, \ldots, N \) in (17). We define the corresponding element

\[
\tilde{H}_k = x_{k(-1)} y_k - \sum_{i=1}^M \beta_i c_i \in C^0_{\text{NVA}}(\tilde{X})
\]

for \( k = 1, \ldots, N \). Since \( H_k \equiv z \) is orthogonal to \( \mu_D(A_i) \) in \( \bigoplus_{i=1}^N \mathbb{C} x_i \partial_j \) for all \( i = 1, \ldots, M \), we have \( \mu_{\text{ch}}(A_j) c_i \tilde{H}_k = 0 \) for all \( n \geq 0 \), and hence \( d_{\text{NVA}}(\tilde{H}_k) = 0 \). We denote the corresponding element in \( \mathcal{D}^c(X) \) and \( \overline{\mathcal{A}}(\mathcal{D}^c(X)) \) the same notation \( \tilde{H}_k \). Clearly, \( \tilde{H}_1, \ldots, \tilde{H}_k \) together with the radical of the bilinear form \( \langle \cdot, \cdot \rangle \) on \( \bigoplus_{i=1}^M \mathbb{C} c_i \subset C^0_{\text{NVA}}(\tilde{X}) \) span the image of the space \( \bigoplus_{k=1}^N \mathbb{C} x_{k(-1)} y_k \oplus \bigoplus_{i=1}^M \mathbb{C} c_i \) in \( \mathcal{D}^c_{X,h}(X) \). By (13) and Lemma 9.2 we have the following proposition.

**Proposition 9.5.** The \( C_2 \) Poisson algebra \( \overline{\mathcal{A}}(\mathcal{D}^c(X)) \) of the hypertoric vertex algebra \( \mathcal{D}^c(X) \) is a subalgebra of \( \mathcal{A}(\tilde{X}) \) which contains the \( \mathcal{W} \)-invariant subalgebra \( \mathbb{C}[\tilde{X}]^W \), under the identification given by \( \tilde{H}_k \mapsto H_k \) for \( k = 1, \ldots, N \) and \( \tilde{P}_\zeta \mapsto P_\zeta \) for \( \zeta \in \Lambda_0 \).

### 9.5. Zhu algebra

As the final goal of the present paper, we determine the Zhu algebra \( A(\mathcal{D}^c(X)) \) of the hypertoric vertex algebra \( \mathcal{D}^c(X) \).

Consider a \( \frac{1}{2} \mathbb{Z}_{\geq 0} \)-graded vertex algebra structure on the BRST complex \( C_{\text{NVA}}(\tilde{X}) \), given by \( d_{x_i} = d_{y_i} = 1/2, d_{c_i} = 1, d_{v_i} = 0 \) and \( d_{v_i} = 1 \) for \( k = 1, \ldots, N \) and \( i = 1, \ldots, M \). This grading is compatible with the conformal weights on \( C_{\text{NVA}}(\tilde{X}) \) introduced in Section 5 when the bilinear form \( \langle \cdot, \cdot \rangle \) on \( \bigoplus_{i=1}^M \mathbb{C} c_i \) is nondegenerate. Thus, the coboundary operator \( d_{\text{NVA}} \) is homogeneous of degree 0, and hence \( \mathcal{D}^c_{X,h}(X) \) and \( \mathcal{D}^c(X) \) are also \( \frac{1}{2} \mathbb{Z}_{\geq 0} \)-graded. Using this grading, we define the Zhu algebra \( A(\mathcal{D}^c(X)) \) of the hypertoric vertex algebra \( \mathcal{D}^c(X) \).

First we characterize \( A(\mathcal{D}^c(X)) \) as a quantization of the \( C_2 \) Poisson algebra \( \overline{\mathcal{A}}(\mathcal{D}^c(X)) \). Recall that the hypertoric vertex algebra \( \mathcal{D}^c(X) = \mathcal{D}^c_{X,h}(X)_{\text{fin}}/(h - 1) \) is equipped with a filtration induced from the \( h \)-adic filtration on \( \mathcal{D}^c_{X,h}(X) \). The filtration induces a filtration of the associative algebra \( A(\mathcal{D}^c(X)) \).

**Proposition 9.6.** The Zhu algebra \( A(\mathcal{D}^c(X)) \) is a quantization of the \( C_2 \) Poisson algebra \( \overline{\mathcal{A}}(\mathcal{D}^c(X)) \). Namely, the associated graded algebra of \( A(\mathcal{D}^c(X)) \) with respect to the above filtration is isomorphic to \( \overline{\mathcal{A}}(\mathcal{D}^c(X)) \) as a Poisson algebra over \( \mathbb{C} \).

**Proof.** Note that the \( \mathbb{C}[[t]] \)-algebra \( A(\mathcal{D}^c_{X,h}(X)) \) is the Rees algebra of the filtered algebra \( A(\mathcal{D}^c(X)) \simeq A(\mathcal{D}^c_{X,h}(X)_{\text{fin}})/(h - 1) \). Thus, the associated graded algebra with respect to the filtration is given by \( \text{Gr} A(\mathcal{D}^c(X)) \simeq A(\mathcal{D}^c_{X,h}(X))/(h) \simeq A(\mathcal{D}^c_{X,h}(X))/(h) \). In the commutative vertex algebra \( \mathcal{D}^c_{X,h}(X)/(h) \), we have \( a \cdot b = a(-2)b + da a(-1)b \) and \( a \ast b = a(-1)b \) for \( a, b \in \mathcal{D}^c_{X,h}(X)/(h) \) where \( da \in \mathbb{R} \) is the
degree of $a$. Thus, the Zhu algebra $A(\mathcal{D}^a_{\mathcal{X},h}(\check{X})/(h))$ is isomorphic to the $C_2$ Poisson algebra $\mathcal{A}(\mathcal{D}^a_{\mathcal{X},h}(\check{X})/(h))$. By the isomorphism theorem, we have

$$\mathcal{A}(\mathcal{D}^a_{\mathcal{X},h}(\check{X})/(h)) = \mathcal{A}(\mathcal{D}^a_{\mathcal{X},h}(\check{X}))_{fin}/(h) \cong \mathcal{A}(\mathcal{D}^a_{\mathcal{X},h}(\check{X}))/(h) \cong \text{Gr} \mathcal{A}(\tilde{\mathcal{D}}^a_{\mathcal{X},h}(\check{X}))_{fin}/(h - 1) \cong \text{Gr} \mathcal{A}(\mathcal{D}^a(\check{X})) \cong \mathcal{A}(\mathcal{D}^a(\check{X})).$$

□

By [L2] Proposition 3.5, the $\mathcal{W}$-invariant subalgebra $\mathcal{D}(\check{X})^\mathcal{W}$ of the quantized hypertoric algebra $\mathcal{D}(\check{X})$ gives a universal family of filtered quantization of the Poisson algebra $\mathbb{C}[X]$, while $\mathbb{C}[X]^\mathcal{W}$ is the universal family of Poisson deformation of $\mathbb{C}[X]$. Let $S = \mathcal{A}(\mathcal{D}^a(\check{X})) \cap \mathbb{C}[g^\ast]$ be a Poisson-commutative subalgebra of $\mathcal{A}(\mathcal{D}^a(\check{X}))$. Then, by the universality of $\mathbb{C}[X]^\mathcal{W}$ ([L2] Proposition 2.12), we have a unique homomorphism $\mathbb{C}[g^\ast]^\mathcal{W} \to S$ and a unique isomorphism of Poisson algebras $\mathbb{C}[X]^\mathcal{W} \otimes_{\mathbb{C}[g^\ast]^\mathcal{W}} S \cong \mathcal{A}(\mathcal{D}^a(\check{X}))$. By Proposition 9.6, the Zhu algebra $A(\mathcal{D}^a(\check{X}))$ is a filtered quantization of $\mathcal{A}(\mathcal{D}^a(\check{X}))$ over $S$. Thus, by [L2] Proposition 3.5, we have a unique isomorphism $\mathcal{D}(\check{X})^\mathcal{W} \otimes_{\mathbb{C}[g^\ast]^\mathcal{W}} S \cong A(\mathcal{D}^a(\check{X}))$. Since we have the inclusions $\mathbb{C}[X]^\mathcal{W} \subset \mathcal{A}(\mathcal{D}^a(\check{X})) \subset \mathbb{C}[X]$, the above homomorphisms are compatible with $\mathbb{C}[g^\ast]^\mathcal{W} \to S \to \mathbb{C}[g^\ast]$. Thus we have the following proposition.

**Proposition 9.7.** The Zhu algebra $A(\mathcal{D}^a(\check{X}))$ of the hypertoric vertex algebra is a subalgebra of the quantized hypertoric algebra $\mathcal{D}(\check{X})$ which contains its $\mathcal{W}$-invariant subalgebra $\mathcal{D}(\check{X})^\mathcal{W}$.

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