CLIFFORD ALGEBRA, ISOPARAMETRIC FOLIATION AND RELATED GEOMETRIC CONSTRUCTIONS

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Dedicated to Professor Chiakuei Peng on the Occasion of His 75th Birthday

Abstract. Based on representation theory of Clifford algebra, Ferus, Karcher and Münzner constructed a series of isoparametric foliations. In this paper, we will survey recent studies on isoparametric hypersurfaces of OT-FKM type and investigate related geometric constructions with mean curvature flow.

1. Introduction

Let \( N \) be a connected complete Riemannian manifold. A non-constant smooth function \( f \) on \( N \) is called transnormal, if there exists a smooth function \( b : \mathbb{R} \to \mathbb{R} \) such that the gradient of \( f \) satisfies \( |\nabla f|^2 = b(f) \). Moreover, if there exists another function \( a : \mathbb{R} \to \mathbb{R} \) so that the Laplacian of \( f \) satisfies \( \Delta f = a(f) \), then \( f \) is said to be isoparametric. Each regular level hypersurface of \( f \) is then called an isoparametric hypersurface. It was proved by Wang (see [Wa87]) that each singular level set is also a smooth submanifold (not necessarily connected), the so-called focal submanifold. The whole family of isoparametric hypersurfaces together with the focal submanifolds form a singular Riemannian foliation, which is called the isoparametric foliation. For recent study of isoparametric functions on general Riemannian manifolds, especially on exotic spheres, see [GT13], [GT14] and [QT15].

E. Cartan firstly gave a systematic study on isoparametric hypersurfaces in real space forms and proved that an isoparametric hypersurface is exactly a hypersurface with constant principal curvatures in these cases. For the spherical case (the most interesting and complicated case), Cartan obtained the classification result under the assumption that the number of the distinct principal curvatures is at most 3. Later, H. F. Münzner [Mü80] extended widely Cartan’s work. Precisely, given an isoparametric hypersurface \( M^n \) in \( S^{n+1}(1) \), let \( \xi \) be a unit normal vector field along \( M^n \) in \( S^{n+1}(1) \), \( g \) the number of distinct principal curvatures of \( M \),

\[
\cot \theta_\alpha \ (\alpha = 1, \cdots, g \ ; 
0 < \theta_1 < \cdots < \theta_g < \pi)
\]

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the principal curvatures with respect to $\xi$ and $m_\alpha$ the multiplicity of $\cot \theta_\alpha$. Münzner proved that $m_\alpha = m_{\alpha+2}$ (indices mod $g$), $\theta_\alpha = \theta_1 + \frac{\alpha-1}{g} \pi$ ($\alpha = 1, \ldots, g$), and there exists a homogeneous polynomial $F : \mathbb{R}^{n+2} \to \mathbb{R}$ of degree $g$, the so-called Cartan-Münzner polynomial, satisfying

\[
\begin{aligned}
|\tilde{\nabla} F|^2 &= g^2 r^{2g-2}, \\
\tilde{\Delta} F &= \frac{m_2-m_1}{2} g^2 r^{g-2},
\end{aligned}
\]

where $r = |x|$, $m_1$ and $m_2$ are the two multiplicities, and $\tilde{\nabla}, \tilde{\Delta}$ are Euclidean gradient and Laplacian, respectively. Moreover, Münzner obtained the remarkable result that $g$ must be 1, 2, 3, 4 or 6 (see a new simplified proof by Fang [Fa17]). Since then, the classification of isoparametric hypersurfaces with $g = 4$ or 6 in a unit sphere has been one of the most challenging problems in differential geometry.

Recently, due to the classification theorem of Chi (see [CCJ07], [Im08], [Ch11], [Ch13] and [Ch16]), an isoparametric hypersurface with $g = 4$ in a unit sphere must be homogeneous or of OT-FKM type (see below). For $g = 6$, R. Miyaoka [Mi13], [Mi16] completed the classification by showing that isoparametric hypersurfaces in this case are always homogeneous.

Let us now recall the isoparametric hypersurfaces of OT-FKM type (c.f. [FKM81]). Given a symmetric Clifford system \{P_0, \ldots, P_m\} on $\mathbb{R}^2l$, i.e., $P_0, \ldots, P_m$ are symmetric matrices satisfying $P_\alpha P_\beta + P_\beta P_\alpha = 2 \delta_{\alpha\beta} l_{2l}$, Ferus, Karcher and Münzner defined a polynomial $F : \mathbb{R}^2l \to \mathbb{R}$ by

\[
F(x) = |x|^4 - 2 \sum_{\alpha=0}^{m} \langle P_\alpha x, x \rangle^2.
\]

They verified that $f = F|_{S^{2l-1}(1)}$ is an isoparametric function on $S^{2l-1}(1)$ and each level hypersurface of $f$ has 4 distinct constant principal curvatures with $(m_1, m_2) = (m, l - m - 1)$, provided $m > 0$ and $l - m - 1 > 0$, where $l = k\delta(m)$ ($k = 1, 2, 3, \ldots$) and $\delta(m)$ is the dimension of an irreducible module of the Clifford algebra $C_{m-1}$. As usual, for OT-FKM type, we denote the two focal submanifolds by $M_+ = f^{-1}(1)$ and $M_- = f^{-1}(-1)$, which have codimensions $m_1 + 1$ and $m_2 + 1$ in $S^{2l-1}(1)$, respectively.

2. Generalizations of OT-FKM construction

In this section, we will discuss an interesting construction in [QT16]. Inspired by the OT-FKM construction, for a symmetric Clifford system \{P_0, \ldots, P_m\} on $\mathbb{R}^2l$ with the Euclidean metric $\langle \cdot, \cdot \rangle$, we define for $0 \leq i \leq m$

\[
M_i := \{ x \in S^{2l-1}(1) \mid \langle P_0 x, x \rangle = \langle P_1 x, x \rangle = \cdots = \langle P_i x, x \rangle = 0 \},
\]

and then we have a sequence

\[M_m = M_+ \subset M_{m-1} \subset \cdots \subset M_0 \subset S^{2l-1}(1).\]
For $0 \leq i \leq m - 1$, it is natural to define a function $f_i : M_i \to \mathbb{R}$ by $f_i(x) = \langle P_{i+1}x, x \rangle$ for $x \in M_i$ (see also [TY17]).

Similarly, by defining for $1 \leq i \leq m$,

$$N_i := \{ x \in S^{2l-1}(1) \mid \langle P_{0}x, x \rangle^2 + \langle P_{1}x, x \rangle^2 + \cdots + \langle P_{i}x, x \rangle^2 = 1 \},$$

we construct another sequence (to understand the relation of inclusion, see [QT16])

$$N_1 \subset N_2 \subset \cdots \subset N_m = M_- \subset S^{2l-1}(1).$$

And for $2 \leq i \leq m$, we define a function $g_i : N_i \to \mathbb{R}$ by $g_i(x) = \langle P_{i}x, x \rangle$ for $x \in N_i$.

**Theorem 1.** ([QT16]) Assume the notations as above.

1. For $0 \leq i \leq m - 1$, the function $f_i : M_i \to \mathbb{R}$ with $\text{Im}(f_i) = [-1, 1]$ is an isoparametric function satisfying

$$|\nabla f_i|^2 = 4(1 - f_i^2), \quad \Delta f_i = -4(l - i - 1)f_i.$$

For any $c \in (-1, 1)$, the regular level set $U_c = f^{-1}_i(c)$ has 3 distinct principal curvatures

$$\sqrt{\frac{1-c}{1+c}}, \quad 0, \quad \sqrt{\frac{1+c}{1-c}}$$

with multiplicities $l - i - 2$, $i + 1$, and $l - i - 2$ respectively, w.r.t. the unit normal $\xi = \frac{\nabla f_i}{|\nabla f_i|}$. For $c = \pm 1$, the two focal submanifolds $U_{\pm 1} = f^{-1}_i(\pm 1)$ are both isometric to $S^{l-1}(1)$ and are totally geodesic in $M_i$.

Particularly, we have a minimal isoparametric sequence

$$M_m \subset M_{m-1} \subset \cdots \subset M_0 \subset S^{2l-1}(1),$$

i.e., each $M_{i+1}$ is a minimal isoparametric hypersurface in $M_i$ for $0 \leq i \leq m - 1$. Moreover, $M_{i+j}$ is minimal in $M_i$.

2. Similarly, for $2 \leq i \leq m$, the function $g_i : N_i \to \mathbb{R}$ with $\text{Im}(g_i) = [-1, 1]$ is an isoparametric function satisfying

$$|\nabla g_i|^2 = 4(1 - g_i^2), \quad \Delta g_i = -4ig_i.$$

For any $c \in (-1, 1)$, the regular level set $V_c = g^{-1}_i(c)$ has 3 distinct principal curvatures

$$\sqrt{\frac{1-c}{1+c}}, \quad 0, \quad \sqrt{\frac{1+c}{1-c}}$$

with multiplicities $i - 1$, $l - i$, and $i - 1$ respectively, w.r.t. the unit normal $\eta = \frac{\nabla g_i}{|\nabla g_i|}$. For $c = \pm 1$, the two focal submanifolds $V_{\pm 1} = g^{-1}_i(\pm 1)$ are both isometric to $S^{l-1}(1)$ and are totally geodesic in $N_i$.

In particular, we get another minimal isoparametric sequence

$$N_1 \subset N_2 \subset \cdots \subset N_m \subset S^{2l-1}(1),$$
i.e., each $N_{i-1}$ is a minimal isoparametric hypersurface in $N_i$ for $2 \leq i \leq m$. Moreover, $N_i$ is minimal in $N_{i+j}$.

**Remark 2.** Using representations of Clifford algebras, M. Radeschi in [Ra14] generalized isoparametric foliations of OT-FKM type and constructed indecomposable singular Riemannian foliations of higher codimension on round spheres, most of which are non-homogeneous.

Next we turn to eigenvalues of Laplacian. Given an $n$-dimensional closed Riemannian manifold $M^n$, recall that the Laplace-Beltrami operator acting on smooth functions on $M$ is an elliptic operator and has a discrete spectrum

$$\{0 = \lambda_0(M) < \lambda_1(M) \leq \cdots \leq \lambda_k(M) \leq \cdots, k \uparrow \infty\},$$

with each eigenvalue counted with its multiplicity. Following the way in [TY13] and [TXY14], the construction of isoparametric functions in Theorem 1 implies the following result on eigenvalue estimates.

**Theorem 3.** ([QT16]) Let $\{P_0, \cdots, P_m\}$ be a symmetric Clifford system on $\mathbb{R}^{2l}$.

1. For the sequence $M_m \subset M_{m-1} \subset \cdots \subset M_0 \subset S^{2l-1}(1)$, the following inequalities hold
   a). $\lambda_k(M_i) \leq \frac{l-i-2}{l-i-3} \lambda_k(M_{i+1})$ provided that $0 \leq i \leq m-1$ and $l-i-3 > 0$;
   b). $\lambda_k(M_{i+1}) \leq 2 \lambda_k(S^{l-1}(1))$ provided that $0 \leq i \leq m-1$.

2. For the sequence $N_1 \subset N_2 \subset \cdots \subset N_m \subset S^{2l-1}(1)$, the following inequalities hold
   a). $\lambda_k(N_i) \leq \frac{i-1}{i-2} \lambda_k(N_{i-1})$ provided that $3 \leq i \leq m$;
   b). $\lambda_k(N_{i-1}) \leq 2 \lambda_k(S^{l-1}(1))$ provided that $2 \leq i \leq m$.

As an unexpected phenomenon, the relations between the focal maps of isoparametric foliations constructed in Theorem 1 and harmonic maps were found. To be more precise, let $M$ and $N$ be closed Riemannian manifolds, and $f$ a smooth map from $M$ to $N$. The energy functional $E(f)$ is defined by $E(f) = \frac{1}{2} \int_M |df|^2 dV_M$. The map $f$ is called harmonic if it is a critical point of the energy functional $E$. We refer to [EL78] and [EL88] for the background and development of this topic. For $N = S^n(1)$, a map $\varphi : M \to S^n(1)$ is called an eigenmap ([Ta01]) if the $\mathbb{R}^{n+1}$-components are eigenfunctions of the Laplacian of $M$ and all have the same eigenvalue. In particular, $\varphi$ is a harmonic map. In 1980, Eells and Lemaire (See p. 70 of [EL83]) posed the following

**Problem 4.** Characterize those compact manifolds $M$ for which there is an eigenmap $\varphi : M \to S^n(1)$ with $\dim(M) \geq n$?

In 1993, Eells and Ratto (See p. 132 of [ER93]) emphasized again that it is quite natural to study the eigenmaps to $S^n(1)$. Another application of the construction in Theorem 1 is the following
Theorem 5. (QT10) Let \( \{P_0, \cdots, P_m\} \) be a symmetric Clifford system on \( \mathbb{R}^l \).

(1). For \( 0 \leq i \leq m - 1 \), both of the focal maps \( \phi_{\pm} : M_{i+1} \to U_{\pm} \cong S^{l-1}(1) \) defined by
\[
\phi_{\pm}(x) = \frac{1}{\sqrt{2}}(x \pm P_{i+1}x), \; x \in M_{i+1},
\]
are submersive eigenmaps with the same eigenvalue \( 2l - i - 3 \).

(2). For \( 2 \leq i \leq m \), both of the focal maps \( \psi_{\pm} : N_{i-1} \to V_{\pm} \cong S^{l-1}(1) \) defined by
\[
\psi_{\pm}(x) = \frac{1}{\sqrt{2}}(x \pm P_ix), \; x \in N_i,
\]
are submersive eigenmaps with the same eigenvalue \( l + i - 2 \).

We conclude this section with talking about progress of two conjectures on minimal submanifolds. Let \( W^n \) be a closed Riemannian manifold minimally immersed in \( S^{n+p}(1) \). Let \( B \) be the second fundamental form and define an extrinsic quantity
\[
\sigma(W) = \max \{ |B(X,X)|^2 \mid X \in TM, \; |X| = 1 \}.
\]

In 1986, H. Gauchman [Ga86] established a well known rigidity theorem which states that if \( \sigma(W) < 1/3 \), then the submanifold \( W \) must be totally geodesic. When the dimension \( n \) of \( W \) is even, the rigidity theorem above is optimal. As presented in [Ga86], there exist minimal submanifolds in unit spheres which are not totally geodesic, with \( |B(X,X)|^2 \equiv 1/3 \) for any unit tangent vector \( X \). When the dimension \( n \) of \( W \) is odd and \( p > 1 \), the conclusion still holds under a weaker assumption \( \sigma(W) \leq \frac{n+1}{n-1} \).

In 1991, P. F. Leung [Le91] proved that if \( n \) is odd, a closed minimally immersed submanifold \( W^n \) with \( \sigma(W) \leq \frac{n}{n-1} \) is totally geodesic provided that the normal connection is flat. Based on this fact, he proposed the following

**Conjecture 6.** If \( n \) is odd, \( W^n \) is minimally immersed in \( S^{n+p}(1) \) with \( \sigma(W) \leq \frac{n}{n-1} \), then \( W \) is homeomorphic to \( S^n \).

By investigating the second fundamental form of the Clifford minimal hypersurfaces in unit spheres, Leung also posed the following stronger

**Conjecture 7.** If \( n \) is odd and \( W^n \) is minimally immersed in \( S^{n+p}(1) \) with \( \sigma(W) < \frac{n+1}{n-1} \), then \( W \) is homeomorphic to \( S^n \).

For minimal submanifolds in unit spheres with flat normal connections, Conjecture 7 was proved by T. Hasanis and T. Vlachos [HV01]. In fact, they showed that the condition \( \text{Ric}(W) > \frac{n(n-3)}{n-1} \) is equivalent to the inequality \( \sigma(W) < \frac{n+1}{n-1} \). Thus in the case that the normal connection is flat, Conjecture 7 follows from Theorem B in [HV01].

Recall that the examples with even dimensions and \( \sigma(W) = 1/3 \) given in [Ga86] originated from the Veronese embeddings of the projective planes \( \mathbb{R}P^2, \mathbb{C}P^2, \mathbb{H}P^2 \) and...
\[\mathbb{O}P^2 \text{ in } S^4(1), S^7(1), S^{13}(1) \text{ and } S^{25}(1), \text{ respectively.} \] Observe that those Veronese submanifolds are just the focal submanifolds of isoparametric hypersurfaces in unit spheres with \( g = 3 \). Hence, it is quite natural for us to consider the case with \( g = 4 \).

**Theorem 8.** ([Q16]) Let \( M^n \) be an isoparametric hypersurface in \( S^{n+1}(1) \) with \( g = 4 \) and multiplicities \((m_1, m_2)\), and denote by \( M_+ \) and \( M_- \) the focal submanifolds of \( M^n \) in \( S^{n+1}(1) \) with dimension \( m_1 + 2m_2 \) and \( 2m_1 + m_2 \) respectively. Then \( M_{\pm} \) are minimal in \( S^{n+1}(1) \) with \( \sigma(M_{\pm}) = 1 \). However, \( M_{\pm} \) are not homeomorphic to the spheres.

**Remark 9.** If \( m_1 \) is odd, \( M_+ \subset S^{n+1}(1) \) in Theorem 8 is a counterexample to Conjecture 6 and Conjecture 7. Similarly, if \( m_2 \) is odd, \( M_- \subset S^{n+1}(1) \) is also a counterexample to both of the conjectures.

### 3. Pinkall-Thorbergsson Construction

In this section, we will recall the construction in [PT89] and find some interesting geometric properties.

Let \( \{E_1, E_2, ..., E_{m-1}\} \) be a set of orthogonal matrices on \( \mathbb{R}^l \) with the Euclidean metric, which satisfy \( E_\alpha E_\beta + E_\beta E_\alpha = -2\delta_{\alpha\beta} \text{Id} \) for \( 1 \leq \alpha, \beta \leq m - 1 \). Define

\[
P_0 := \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix}, \quad P_1 := \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix}, \quad P_\alpha := \begin{pmatrix} 0 & E_{\alpha-1} \\ -E_{\alpha-1} & 0 \end{pmatrix},
\]

for \( 2 \leq \alpha \leq m \). Then \( \{P_0, P_1, ..., P_m\} \) is a set of orthogonal matrices on \( \mathbb{R}^{2l} \) with the Euclidean metric, which satisfy \( P_\alpha P_\beta + P_\beta P_\alpha = 2\delta_{\alpha\beta} \text{Id} \), for \( 0 \leq \alpha, \beta \leq m \), i.e. \( \{P_0, P_1, ..., P_m\} \) is a symmetric Clifford system on \( \mathbb{R}^{2l} \). For any \( 0 \leq \alpha \leq m \), \( P_\alpha \) has eigenvalues \( \pm 1 \) of equal multiplicity \( l \). Denote the eigenspaces of \( \pm 1 \) for \( P_\alpha \) by \( E_{\pm}(P_\alpha) \).

According to [PT89], for \( 0 < t \leq \frac{\pi}{4} \), one can define

\[
M_t^+ := \{ z = (x, y) \in \mathbb{R}^l \oplus \mathbb{R}^l = \mathbb{R}^{2l} : |x| = \cos t, |y| = \sin t, \langle x, y \rangle = 0, \langle x, E_\alpha y \rangle = 0 \text{ for } 1 \leq \alpha \leq m - 1 \}.
\]

Clearly, \( M_t^+ \) is an embedded submanifold in \( S^{2l-1}(1) \subset \mathbb{R}^{2l} \) of dimension \( 2l - m - 2 \).

Write \( a := \tan t \) and \( b := \cot t \). Moreover, write \( Q_0 := \begin{pmatrix} a \text{Id} & 0 \\ 0 & -b \text{Id} \end{pmatrix} \), and \( Q_\alpha := P_\alpha \), for \( 1 \leq \alpha \leq m \). Then

\[
M_t^+ := \{ z = (x, y) \in \mathbb{R}^{2l} : |x|^2 + |y|^2 = 1, \langle z, Q_\alpha z \rangle = 0 \text{ for } 0 \leq \alpha \leq m \}.
\]

For \( Q_\alpha \), \( 0 \leq \alpha \leq m \), we have the following lemma, which will be useful later.
Lemma 10. For any $z \in M^l_+$ and $\alpha, \beta \in \{1, 2, \ldots, m\}$, the following identities hold:

$$
\langle Q_0 Q_0 z, Q_0 z \rangle = -2 \cot 2t,
\langle Q_0 Q_0 z, Q_0 z \rangle = 0,
\langle Q_0 Q_0 z, Q_0 z \rangle = 0,
\langle Q_0 Q_0 z, Q_0 z \rangle = 0,
\langle Q_0 Q_0 z, Q_0 z \rangle = 0.
$$

Moreover, $\langle Q_0 Q_0 z, Q_0 z \rangle = -2\delta_{\alpha\beta} \cot 2t$.

3.1 Extrinsic geometry. In this subsection, we investigate the extrinsic geometric properties of $M^l_+$ in $S^{2l-1}(1)$.

Proposition 11. For any given $z \in M^l_+$, one has the following statements:

1. The normal space $N_z M^l_+$ of $M^l_+$ in the unit sphere $S^{2l-1}(1) \subset \mathbb{R}^{2l}$ at $z$ is given by

$$
N_z M^l_+ = \text{Span}\{Q_0 z, Q_1 z, \ldots, Q_m z\}.
$$

The tangent space $T_z M^l_+$ of $M^l_+$ at $z$ is given by

$$
T_z M^l_+ = \{(u, v) \in \mathbb{R}^l \oplus \mathbb{R}^l = \mathbb{R}^{2l} \mid \langle (u, v), z \rangle = 0, \langle (u, v), Q_i z \rangle = 0 \text{ for } 0 \leq i \leq m\}.
$$

Moreover, $\mathbb{R}^{2l} = T_z M^l_+ \oplus N_z M^l_+ \oplus \text{Span}\{z\}$.

2. For $1 \leq \alpha \leq m$, the shape operator $A_\alpha$ of $M^l_+$ with respect to $Q_\alpha z$ has principal curvatures $1, 0, -1$ of multiplicities $l - m - 1, m$, and $l - m - 1$, respectively. Moreover, $T_z M^l_+ = E_+(Q_\alpha z) \oplus E_0(Q_\alpha z) \oplus E_-(Q_\alpha z)$, where

$$
E_+(Q_\alpha z) = E_-(Q_\alpha) \cap T_z M^l_+,
E_0(Q_\alpha z) = \text{Span}\{Q_\alpha Q_\beta z \mid 0 \leq \beta \leq m, \beta \neq \alpha\},
$$

and

$$
E_-(Q_\alpha z) = E_+(Q_\alpha) \cap T_z M^l_+
$$

are principal distributions of $1, 0, \text{ and } -1$, respectively.

3. Let $E_+(Q_0)$ and $E_-(Q_0)$ be eigenspaces of $Q_0$ with eigenvalues $a$ and $-b$, respectively. For the unit normal vector $Q_0 z$, the shape operator $A_0$ has principal curvatures $\cot t, 0, -\tan t$ of multiplicities $l - m - 1, m$, and $l - m - 1$, respectively. Moreover, $T_z M^l_+ = E_+(Q_0 z) \oplus E_0(Q_0 z) \oplus E_-(Q_0 z)$, where

$$
E_+(Q_0 z) = E_-(Q_0) \cap T_z M^l_+,
$$
\[ E_0(Q_0 z) = \text{Span}\{Q_0^{-1}Q_\alpha z \mid 1 \leq \alpha \leq m\}, \]

and

\[ E_-(Q_0 z) = E_+(Q_0) \cap T_z M^I_+ \]

are principal distributions of \( \cot t \), \( 0 \), and \( -\tan t \), respectively.

**Proof.** (1) It follows directly from the definition of \( M^I_+ \).

(2) Observe that for any tangent vector \( X \in T_z M^I_+ \), \( A_\alpha X = -(P_\alpha X)^T \), where \( (P_\alpha X)^T \) is the tangent projection of \( P_\alpha X \). Define

\[ D_+ = E_-(Q_\alpha) \cap T_z M^I_+, \]

\[ D_0 = \text{Span}\{Q_\alpha Q_\beta z \mid 0 \leq \beta \leq m, \beta \neq \alpha\}, \]

and

\[ D_- = E_+(Q_\alpha) \cap T_z M^I_+. \]

Clearly, \( D_\pm \) are subspaces of \( T_z M^I_+ \). Moreover, it follows from Lemma 10 that \( D_0 \) is also a subspace of \( T_z M^I_+ \). Then for any \( X \in D_+ \), we have \( A_\alpha X = X \); For any \( X \in D_0 \), \( A_\alpha X = 0 \); For any \( X \in D_- \), \( A_\alpha X = -X \). Since

\[ D_+ = E_-(Q_\alpha) \cap T_z M^I_+ \]

\[ = \{(u, v) \in \mathbb{R}^2 \mid P_\alpha(u, v) = -(u, v), \langle(u, v), z\rangle = 0, \langle(u, v), Q_\beta z\rangle = 0 \text{ for } 0 \leq \beta \leq m, \beta \neq \alpha\}, \]

we have \( \dim D_+ \geq l - m - 1 \). Similarly, \( \dim D_- \geq l - m - 1 \). Thus \( D_+, D_0 \) and \( D_- \) are mutually orthogonal subspaces of \( T_z M^I_+ \) and

\[ \dim D_+ + \dim D_0 + \dim D_- \geq l - m - 1 + m + l - m - 1 = 2l - m - 2. \]

Then (2) follows easily.

(3) As in the proof of (2), for any tangent vector \( X \in T_z M^I_+ \), \( A_\alpha X = -(Q_0 X)^T \). Define

\[ D_+ = E_-(Q_0) \cap T_z M^I_+, \]

\[ D_0 = \text{Span}\{Q_0^{-1}Q_\alpha z \mid 1 \leq \alpha \leq m\}, \]

and

\[ D_- = E_+(Q_0) \cap T_z M^I_+. \]

It is clear that \( D_\pm \) are subspaces of \( T_z M^I_+ \). By a direct computation, we have

\[ \langle Q_0^{-1}Q_\alpha z, z\rangle = 0, \langle Q_0^{-1}Q_\alpha z, Q_\beta z\rangle = 0, \text{ for } 1 \leq \alpha \leq m, 0 \leq \beta \leq m. \]

It follows that \( D_0 \) is also a subspace of \( T_z M^I_+ \). Moreover,

\[ \langle Q_0^{-1}Q_\alpha z, Q_0^{-1}Q_\beta z\rangle = \delta_{\alpha\beta} \]
for $1 \leq \alpha, \beta \leq m$. Then for any $X \in D_+$, we have $A_0 X = b X$; For any $X \in D_0, A_0 X = 0$; For any $X \in D_-, A_0 X = -a X$. Meanwhile, it is easy to verify that $D_+, D_0, D_-$ are mutually orthogonal subspaces of $T_z M_+$. Since

$$D_+ = E_-(Q_0) \cap T_z M_+^t = \{(0, v) \in \mathbb{R}^{2l} \mid \langle(0, v), z \rangle = 0, \langle(0, v), P_\alpha z \rangle = 0 \text{ for } 1 \leq \alpha \leq m\}.$$  

we have $\dim D_+ \geq l - m - 1$. Similarly, $\dim D_- \geq l - m - 1$. Thus $D_+, D_0$ and $D_-$ are mutually orthogonal subspaces of $T_z M_+^t$ and

$$\dim D_+ + \dim D_0 + \dim D_- \geq l - m - 1 + m + l - m - 1 = 2l - m - 2.$$  

With these arguments, we can prove (3) easily. □

### 3.2 Scalar curvature

Let $M^n$ be a submanifold in $S^{n+p}(1)$, and $B$ the second fundamental form as before. Around each point $z \in M^n$, we can choose an adapted moving frame $e_1, \ldots, e_{n+p}$. Restricted on $M$, $\{e_1 | 1 \leq i \leq n\}$ is a local orthonormal basis of $TM$, and $\{e_1 | n + 1 \leq \alpha \leq n + p\}$ is a local orthonormal basis of $NM$. Denote

$$A_\alpha e_i = \sum_{j=1}^n h_{ij}^\alpha e_j,$$

$$H^\alpha = \text{Tr}(A_\alpha) = \sum_{i=1}^n h_{ii}^\alpha,$$

$$H = \sum_\alpha H^\alpha e_\alpha, \quad |H|^2 = \sum_\alpha (H^\alpha)^2, \quad |B|^2 = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2.$$  

The following lemma follows from the Gauss lemma.

**Lemma 12.** The scalar curvature $S$ of $M^n$ with the induced metric in $S^{n+p}(1)$ is given by

$$S = n(n - 1) + |H|^2 - |B|^2.$$  

**Proposition 13.** The scalar curvature $S^t$ of $M_+^t$ with the induced metric in $S^{2l-1}(1)$ is $S^t = (2l - m - 2)(2l - m - 3) - 2(l - m - 1)(l - 1) + (l - m - 1)(l - m - 2)(\tan^2 t + \cot^2 t)$. In particular, the scalar curvature $S^t$ of $M_+^t$ is a positive constant for $0 < t \leq \frac{\pi}{4}$, and $S^t \geq S^\frac{\pi}{4}$.

**Proof.** According to Proposition $\Box$ $H^0 = 2(l - m - 1) \cot 2t, \ H^\alpha = 0$ for $1 \leq \alpha \leq m$, and

$$|B|^2 = 2m(l - m - 1) + (l - m - 1)(\tan^2 t + \cot^2 t).$$  

For $t = \frac{\pi}{4}$, the scalar curvature of $M_+^\frac{\pi}{4}$ with the induced metric in $S^{2l-1}(1)$ is given by $S^\frac{\pi}{4} = 4(l - m - 1)(l - m - 2) + 2m(l - m - 2) + m(m + 1) > 0$.

Then the result follows from the lemma above. □
Remark 14. (1) For any $0 < t < \frac{\pi}{4}$, both $|H|^2$ and $|B|^2$ of $M^l_+ \subset S^{2l-1}(1)$ are positive constants.

(2) If $l - m - 1 = 1$, then $S^t = S^\frac{\pi}{4}$ for any $0 < t \leq \frac{\pi}{4}$.

3.3 Mean curvature flow. By Proposition 11, for $0 < t < \frac{\pi}{4}$, $M^l_+$ is not a minimal submanifold in $S^{2l-1}(1)$. Hence, it is interesting to consider the behavior of $M^l_+$ under mean curvature flow.

Proposition 15. We have a mean curvature flow as follows

(1) For the initial value $F(\cdot, 0) : M_+ \to S^{2l-1}(1),$
$$F(x, y; 0) = (\sqrt{2}\cos \beta(0)x, \sqrt{2}\sin \beta(0)y)$$
with $0 < \beta(0) < \frac{\pi}{4}$, the mean curvature flow of $F(\cdot, 0)$ is given by
$$F : M_+ \times (-\infty, T) \to S^{2l-1}(1), F(x, y; t) = (\sqrt{2}\cos \beta(t)x, \sqrt{2}\sin \beta(t)y),$$
where $\cos 2\beta(t) = \cos 2\beta(0)e^{4(l-m-1)t}$ and $1 = \cos 2\beta(0)e^{4(l-m-1)t}.$

(2) The mean curvature flow $F(M_+, t)$ has type I singularity at $T$. More precisely, there exists a constant $C > 0$ such that
$$\sup_{F(M_+, t)} |B|^2 \leq \frac{C}{T-t} \forall t \in [0, T).$$

(3) As $t \to T$, $F(M_+, t)$ converges to $S^{l-1}(1) = \{(x, 0) \in \mathbb{R}^{2l} \mid |x| = 1\}$.

Proof. (1). Consider the map
$$F : M_+ \times (-\infty, T) \to S^{2l-1}(1)$$
by $F(x, y; t) = (\sqrt{2}\cos \beta(t)x, \sqrt{2}\sin \beta(t)y)$, where the function $\beta(t)$ is given by
$$\cos 2\beta(t) = \cos 2\beta(0)e^{4(l-m-1)t}, \ 0 < \beta(t) < \pi/4.$$ It is clear that the image of $F$ is included in $M^\beta(t)_+$. By a direct computation,
$$\frac{\partial F}{\partial t} = -\sqrt{2}\beta'(t)(\sin \beta(t)x, -\cos \beta(t)y).$$

On the other hand, by Proposition 11, the mean curvature vector of $M^\beta(t)_+$ at the point $(\sqrt{2}\cos \beta(t)x, \sqrt{2}\sin \beta(t)y)$ is equal to
$$H|_{F(x, y; t)} = (l - m - 1)(\cot \beta(t) - \tan \beta(t))Q_0 F(x, y; t) = 2\sqrt{2}(l - m - 1)\cot 2\beta(t)(\sin \beta(t)x, -\cos \beta(t)y).$$ At last, the definition $\beta(t)$ yields the equality
$$\beta(t) = -2(l - m - 1)\cot 2\beta(t),$$ and hence the equation of the mean curvature flow $\frac{\partial F}{\partial t} = H|_{F(x, y; t)}$. 

(2). According to Proposition \[11\]
\[
\sup_{F(M+; t)} |B|^2 = 2m(l - m - 1) + (l - m - 1)(\tan^2 \beta(t) + \cot^2 \beta(t)).
\]

By (1),
\[
\sup_{F(M+; t)} |B|^2 = 2m(l - m - 1) + (l - m - 1)(\frac{1 - \cos 2\beta(t)}{1 + \cos 2\beta(t)} + \frac{1 + \cos 2\beta(t)}{1 - \cos 2\beta(t)}).
\]

Thus
\[
\lim_{t \to T} \sup_{F(M+; t)} |B|^2(T - t) = \lim_{t \to T} (l - m - 1) \frac{e^{4l(m-1)(T-t)} + 1}{e^{4l(m-1)(T-t)} - 1} (T - t)
\]
\[
= \frac{1}{2}.
\]

Then (2) follows easily.

(3). For any \((x, y) \in M_+\), \(\lim_{t \to T} F(x, y; t) = (\sqrt{2}x, 0)\). \(\square\)

**Remark 16.** (1). Roughly speaking, the family \(M_+^t\) in \(S^{2l-1}(1)\) constitutes a mean curvature flow.

(2). From Proposition \[15\] (1), \(F(M_+, t)\) converges to \(M_+\) as \(t \to -\infty\).

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