PRYM–TYURIN VARIETIES COMING FROM CORRESPONDENCES WITH FIXED POINTS.

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Abstract. Our main theorem is an improvement of the Criterion of Kanev about Prym–Tyurin varieties induced by correspondences, which includes correspondences with fixed points. We give some examples of Prym–Tyurin varieties using this criterion.

1. Introduction

Let $C$ be a smooth projective curve over $\mathbb{C}$ and let $J = JC$ be the Jacobian of $C$ with canonical principal polarization $\Theta$. Consider $P \hookrightarrow J$ an abelian subvariety with induced polarization $i^*\Theta$. We say that $P$ is a Prym–Tyurin variety of exponent $q$ for the curve $C$ if it exists a principal polarization $\Xi \subset P$ such that

$$i^*\Theta \equiv q\Xi.$$ 

In [3] Kanev shows the following sufficient condition for a subvariety of $J$ to be a Prym–Tyurin variety for $C$. If $\gamma$ is an endomorphism of $J$ induced by an effective fixed point free symmetric correspondence which satisfies the equation

$$(1 - \gamma)(\gamma + q - 1) = 0,$$

then $P = \text{Im}(1 - \gamma)$ is a Prym–Tyurin variety of exponent $q$. However, this criterion does not include the case of a Prym variety (of exponent 2) associated to a double cover over $C$ with two branch points (cf. [7]). In this work we give some conditions to extend this criterion to a correspondence with fixed points. A correspondence is a line bundle $L$ on $C \times C$ (alternatively a divisor on $C \times C$); we say that the correspondence is effective if it is defined by an effective divisor on $C \times C$. Throughout this paper we will consider effective correspondences. For any point $p \in C$ define the line bundle on $C$

$$L(p) := L|_{(p) \times C}.$$ 

A point of $p \in C$, with $L(p) = \mathcal{O}_C(D)$, is a fixed point of $L$ if $D - p$ is an effective divisor. We prove the following

Theorem 1.1. Let $L$ be an effective symmetric correspondence on $C \times C$ of bidegree $(d, d)$ with $2n$ fixed points and $\gamma_L$ the endomorphism of the Jacobian $J$ induced by $L$. Suppose $n \leq d$ and

a) $\gamma_L$ satisfies the equation 1.1.

b) There are $n$ distinct fixed points $p_1, \ldots, p_n \in C$ such that

$$p_1, \ldots, p_i \in D_i, \quad p_i \notin D_i - p_i, \quad i = 1, \ldots, n.$$
where \( L(p_i) = \mathcal{O}_C(D_i) \) with \( D_i \) effective divisors.

Then \( P := \text{Im}(1 - \gamma_L) \) is a Prym–Tyurin variety of exponent \( q \) for the curve \( C \). Moreover, there exist theta divisors \( \Theta \) and \( \Xi \) on \( J \) and \( P \) respectively such that \( i^*\Theta = q\Xi \).

**Remark 1.2.** If \( L = \mathcal{O}_{C \times C}(D) \), the condition \( p_i \notin D_i - p_i \) for \( i = 1, \ldots, n \) of the Theorem 1.1 is equivalent to say that \( D \) intersects transversally to the diagonal \( \Delta \) in \( C \times C \) at the points \( p_1, \ldots, p_n \).

En particular, when \( d = n = 1 \) this criterion recovers the example of Prym varieties associated to double covers with two ramification points. We will use this criterion to construct new examples of Prym–Tyurin varieties.

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## 2. Subvarieties of the Jacobian

Let us recall some generalities about correspondences and abelian varieties. Let \( g \) be the genus of \( C \). We will consider correspondences on \( C \) as line bundles on \( C \times C \) which are defined by effective divisors on \( C \times C \). The bidegree \( (d_1, d_2) \) of a correspondence \( L \in \text{Pic}(C \times C) \) is defined by \( d_1 = \deg L|_{C \times \{s\}} \) and \( d_2 = \deg L|_{\{t\} \times C} \), and it is independent of the points \( s, t \in C \). Two correspondences \( L \) and \( L' \) are equivalents if there are line bundles \( L_1 \) and \( L_2 \) on \( C \) such that

\[
L' = L \otimes \pi_1^*L_1 \otimes \pi_2^*L_2,
\]

where \( \pi_1 \) and \( \pi_2 \) are the canonical projections of \( C \times C \). A correspondence \( L \) induces an endomorphism of \( \text{Pic}(C) \) given by

\[
\gamma_L : \mathcal{O}_C(\sum r_ip_i) \mapsto L(p_1)^{r_1} \otimes \cdots \otimes L(p_n)^{r_n},
\]

with \( p_i \in C \) and \( r_i \in \mathbb{Z} \). For all \( N \in \text{Pic}(C) \) we have \( \deg \gamma_L(N) = d_2 \deg(N) \); in particular, a correspondence induces an endomorphism of the Jacobian \( \gamma_L : J \to J \), which does not depend on the class of equivalence of \( L \) and every endomorphism of \( J \) is obtained in this way (Theorem 11.5.1. [2]). An endomorphism \( \gamma_L \) associated to \( L \) is symmetric if and only if \( \tau^*L = L \) where \( \tau : (x, y) \mapsto (y, x) \) on \( C \times C \), and then \( d_1 = d_2 = d \). Consider a symmetric endomorphism \( \gamma \in \text{End}(J) \) such that \( 1 - \gamma \) is a primitive endomorphism satisfying

\[
(1 - \gamma)^2 = q(1 - \gamma),
\]

for some positive integer \( q \). Define

\[
P := \text{Im}(1 - \gamma),
\]

which is an abelian subvariety of \( J \) of exponent \( q \). Let \( P \hookrightarrow JC \) be the inclusion and \( i^*\Theta \) the induced polarization on \( P \).
Proposition 2.1. (Prop. 1.6 [3], Prop. 12.1.8 [2]) There exist a principally polarized abelian variety \((P_0, \Xi_0)\), homomorphisms \(\sigma, j\) and an isogeny \(\mu\) in the diagram

\[
P_0 \xrightarrow{\mu} P \xrightarrow{i} J
\]

such that they verify the following:

a) \(j^* \Theta \equiv q \Xi_0\)

b) \(i \circ \mu = j; \quad j \circ \sigma = 1 - \gamma; \quad \sigma \circ j = q_{P_0}; \quad \sigma \circ \gamma = (1 - q)\sigma.\)

3. The Proof of the Criterion

The idea of the proof of the Kanev’s Criterion is to find divisors \(\Theta\) on \(J\) and \(\Xi\) on \(P\) such that we have the equality of divisors \(i^* \Theta = q \Xi\). Our contribution to the original proof resides in the following proposition, proved in the context of a correspondence which fixed points verify the condition b) of the Theorem 1.1.

Proposition 3.1. There exists theta divisors \(\Theta\) and \(\Xi_0\) on \(J\) and \(P_0\) such that \(j^*(\Theta) = q \Xi_0\).

The proof of the criterion follows from this proposition.

Proof of the Theorem 1.1: If \(\Theta\) is the divisor of the previous proposition the divisor \(i^* \Theta\) is well defined, since by the Proposition 2.1 \(j = i \circ \mu\) and \(\mu\) is an isogeny. Put \(i^* \Theta = \sum_k r_k D_k\) with \(r_k > 0\) and \(\mu^* D_k = \sum_l \Xi_{kl}\) with pairwise different irreducible divisors \(D_k\) on \(P\) and \(\Xi_{kl}\) on \(P_0\). Hence \(j^* \Theta = \mu^* i^* \Theta = \sum_k r_k \Xi_{kl}\). According to the Proposition 3.1 \(r_k = q\) for all \(k\) and \(\Xi_0 = \sum_k \Xi_{kl}\). We define \(\Xi = \sum_k D_k\). Since \(\mu^* \Xi = \Xi_0\) defines a principal polarization, the isogeny \(\mu\) is an isomorphism and \(\Xi\) defines also a principal polarization. Therefore \(i^* \Theta = q \Xi\).

Let \(\alpha_c : C \to J\) be the Abel–Jacobi morphism with base point \(c \in C\) and \(L = \mathcal{O}_{C \times C}(\mathcal{D})\) an effective correspondence.

Lemma 3.1. There exists \(\epsilon \in \text{Pic}(C)\) such that for all \(y \in P_0\) and for all \(c \in C\)

\[
\alpha_c^* \sigma^* \mathcal{O}_{P_0}(t_y^* \Xi_0) = j(y)^{-1} \otimes \mathcal{O}_C(c) \otimes L(c)^{-1} \otimes \epsilon.
\]

Proof: See [2] Lemma 12.9.4.

We apply the previous lemma to the correspondence \(\tau^* L\) to obtain a line bundle \(\eta \in \text{Pic}(C)\).

Lemma 3.2. The lines bundle \(\epsilon\) and \(\eta\) satisfy

\[
\epsilon \otimes \eta = \mathcal{W}_C \otimes \mathcal{O}_C(\Delta.D),
\]

where \(\Delta.D\) is the intersection of the correspondence (as divisor in \(C \times C\)) with the diagonal in \(C \times C\) and \(\mathcal{W}_C\) is the canonical divisor.
Proof: See [4] Theorem 7.6. (p. 211).

In particular, when $L$ is symmetric we have
\[ \varepsilon^{\circ 2} = W_C \otimes O_C(\Delta.D), \]
and hence, $\deg \varepsilon = g + n - 1$ where $2n$ is the number of fixed points of $L$ with multiplicity.

We shall adapt the Kanev’s proof to admit fixed points in the criterion. Let $L \in \text{Pic}(C \times C)$ be an effective symmetric correspondence with fixed points $p_1, \ldots, p_{2n} \in C$. Suppose that $p_1, \ldots, p_n$ satisfy the assumptions of the Theorem 1.1.

Proof of the Proposition 3.1: Let $\beta := O_C(p_1 + \cdots + p_n)$ and $\varepsilon' = \varepsilon \otimes \beta^{-1}$, where $\varepsilon$ is given by the Lemma 3.1. Observe that $\varepsilon'$ is a line bundle of degree $g - 1$. Let $\Theta = W_{g-1} - \varepsilon'$. We can assume that the divisor $\Xi_0$ of the Proposition 2.1 is symmetric.

Firstly, we shall prove that $j^{-1}(\Theta) \subset \text{Supp} \Xi_0$. Let $y \in P_0 - \Xi_0$. We have to show that $j(y) \notin \Theta$, i.e., $h^0(j(y) \otimes \varepsilon') = 0$. According to the lemma 3.1, we have the following expression for $j(y) \otimes \varepsilon'$
\[ M := j(y) \otimes \varepsilon' = \alpha^* \sigma^* O_{F_0}(t^* y \Xi_0) \otimes O_C(-c) \otimes L(c) \otimes \beta^{-1}. \]
Observe that $M$ does not depend on $c$. Since $\Xi_0$ is symmetric and $y \notin \Xi_0$, $c$ is not in the divisor $\alpha^* \sigma^*(t^* y \Xi_0)$ for every $c \in C$. Hence, any $c$ in the open set $U = C - \{p_1, \ldots, p_{2n}\}$ is not a base point of the line bundle $M \otimes O_C(c) \otimes \beta \in \text{Pic}^g(C)$, since $c$ is not a base point of $L(c)$. Then
\[ h^0(M \otimes \beta) = h^0(M \otimes \beta \otimes O_C(c)) - 1, \]
for all $c \in U$. We claim that $h^0(M \otimes \beta) = n$. Suppose that $h^0(M \otimes \beta) \geq n + 1$. By Riemann–Roch we have
\[ h^0(W_C \otimes M^{-1} \otimes \beta^{-1} \otimes O_C(-c)) = h^0(M \otimes \beta \otimes O_C(c)) - n - 1 = h^0(M \otimes \beta) - n. \]
On the other hand
\[ h^0(W_C \otimes M^{-1} \otimes \beta^{-1}) = h^0(M \otimes \beta) - n, \]
hence every $c \in U$ is a base point of $W_C \otimes M^{-1} \otimes \beta^{-1}$ if $h^0(M \otimes \beta) \geq n + 1$, which is impossible. We conclude that $h^0(M \otimes \beta) = n$.

Observe that we can write
\[ M \otimes \beta = \alpha^* \sigma^* O_{F_0}(t^* y \Xi_0) \otimes O_C(-p_1) \otimes L(p_1) = \alpha^* \sigma^* O_{F_0}(t^* y \Xi_0) \otimes O_C(D_1 - p_1). \]
Since $p_1 \notin D_1 - p_1$ and $p_1$ is not a base point of $\alpha^* \sigma^* O_{F_0}(t^* y \Xi_0)$, $p_1$ is not a base point of $M \otimes \beta$. Hence, $h^0(M \otimes \beta \otimes O_C(-p_1)) = n - 1$.

In general, we have
\[ M \otimes \beta \otimes O_C(-p_1 - \cdots - p_i) = \alpha^* \sigma^* O_{F_0}(t^* y \Xi_0) \otimes O_C(-p_{i+1}) \otimes L(p_{i+1}) \otimes O_C(-p_1 - \cdots - p_i) = \alpha^* \sigma^* O_{F_0}(t^* y \Xi_0) \otimes O_C(D_{i+1} - (p_1 + \cdots + p_{i+1})), \]
for \( i = 1, \ldots, n - 1 \). By the assumptions on the fixed points, \( p_{i+1} \) is not a base point of \( M \otimes \beta \otimes \mathcal{O}_C(-p_1 - \cdots - p_i) = M \otimes \mathcal{O}_C(p_{i+1} + \cdots + p_n) \) and then \( h^0(M \otimes \mathcal{O}_C(p_i + \cdots + p_n)) - 1, \) for \( i = 1, \ldots, n - 1 \).

By induction on \( n \), we have \( h^0(M) = 0 \).

Now, since \( \Xi_0 \subset P_0 \) defines a principal polarization, it is of the form \( \Xi_0 = \sum \Xi_k \) where the \( \Xi_k \) are irreducible divisors linearly independent in \( NS(P_0) \). What we have just proved implies \( j^*(\Theta) = \sum r_k \Xi_k \), with \( r_k \geq 0 \). By the Proposition (2.1) \( j^*(\Theta) \equiv \sum q \Xi_k \), hence \( r_k = q \) for all \( k \). Therefore \( j^*\Theta = q \Xi_0 \) for \( \Theta = W_{g-1} - \varepsilon' \).

\[ \square \]

4. Examples

Coverings over hyperelliptic curves. In this section we consider a similar construction to the given in [6]. Let \( X \) be an hyperelliptic curve of genus \( g \geq 3 \), with hyperelliptic involution \( i \) and let \( h : X \to \mathbb{P}^1 \) be the map given by the linear system \( g_2^1 \). Consider \( f : \bar{X} \to X \) a covering of degree 3 with two ramification points, from a projective smooth irreducible curve \( \bar{X} \). Suppose that the branch locus of \( f \) does not contain ramification points of \( h \). We define a new curve \( C \) by using the following cartesian diagram

\[
\begin{array}{ccc}
C := (f^{(2)})^{-1}(g_2^1) & \xrightarrow{} & \bar{X}^{(2)} \\
\pi = f^{(2)}_C & \downarrow & \downarrow f^{(2)} \\
\mathbb{P}^1 \simeq g_2^1 & \xrightarrow{j} & X^{(2)}
\end{array}
\]

where \( f^{(2)} \) is the second symmetric product of \( f \) and \( \pi \) is of degree 9. Let us assume \( C \) smooth and irreducible. We will define the same correspondence on \( C \) as in [6].

Let \( s : \bar{X}^2 \to \bar{X}^{(2)} \) be the canonical map. We denote \( \tilde{C} := s^{-1}(C) \subset \bar{X}^2 \). Let \( p_1 : \tilde{C} \to \bar{X} \) denote the projection on the first factor. Define

\[ \tilde{D} := \{(a,b) \in \tilde{C} \times \bar{C} \mid p_1(a) = p_1(b)\}, \]

with reduced subscheme structure. This is an effective divisor on \( \tilde{C}^2 \) containing the diagonal \( \bar{\Delta} \). Define \( Y := \tilde{D} - \bar{\Delta} \). The divisor \( D := (s \times s)_*(Y) \) is an effective symmetric correspondence on \( C \) of bidegree \( (4,4) \).

Set-theoretically this correspondence is defined as follows. Given \( z \in \mathbb{P}^1 \), put \( h^{-1}(z) = x + ix \) and \( f^{-1}(x) = \{x_1, x_2, x_3\} \), \( f^{-1}(ix) = \{y_1, y_2, y_3\} \). We denote for \( i, j \in \{1, 2, 3\} \)

\[ P_{ij} = x_i + y_j \in C \subset \bar{X}^{(2)} \]

Then \( \pi^{-1} = \{P_{ij} \mid i, j = 1, 2, 3\} \). Let

\[ D(P_{ij}) = \sum_{l=1, l\neq j}^3 P_{il} + \sum_{k=1, k \neq i}^3 P_{kj}. \]

This define an effective symmetric correspondence of bidegree \( (4,4) \). The associated endomorphism of the Jacobian \( \gamma_D \) verifies the equation (cf. [6])

\[ \gamma_D^2 + \gamma_D - 2 = 0. \]

By construction, \( D \) is a correspondence with fixed points coming from the ramification points of \( f \). More precisely, if \( x \in X \) is a branch point of \( f \) with \( x_1 = x_2 \) then
\( \pi^{-1}(h(x)) = \{2(x_1+y_1), 2(x_1+y_2), 2(x_1+y_3), x_3+y_1, x_3+y_2, x_3+y_3\} \) and \( P_{11}, P_{12}, P_{13} \) are the three fixed points on the fiber over \( h(x) \). Moreover, these points verify

\[
\begin{align*}
P_{11} &\in D(P_{11}), \\
P_{11}, P_{12} &\in D(P_{12}), \\
P_{11}, P_{12}, P_{13} &\in D(P_{13}),
\end{align*}
\]

and \( P_{1i} \notin D(P_{1i}) - P_{1i} \) for \( i = 1, 2, 3 \), which are the conditions of the Theorem 1.1.

Applying the criterion we obtain that \( P := \text{Im}(1 - \gamma D) \) is a Prym–Tyurin variety of exponent 3 for the curve \( C \). In order to compute his dimension we need to calculate the degree of the ramification divisor of \( \pi \). On the fiber over a Weierstrass point we have 3 simple ramification points i.e. with index of ramification 2. If \( h(x) \in \mathbb{P}^1 \) is the image of a branch point of \( f \) then the fiber \( \pi^{-1}(h(x)) \) contains 3 simple ramification points. Therefore, the degree of the ramification divisor of \( \pi \) is \( w_{\pi} = 3(2g + 2) + 3(2) \).

Using the Riemann–Hurwitz formula we get \( g_C = 3g - 2 \). By the Corollary 5.3.10 and Proposition 11.5.2 ([2])

\[
\text{exp}(P) \dim P = \frac{1}{2} \text{Tr}_r(1 - \gamma D) = (g_C - d + \frac{1}{2}(\Delta.D)),
\]

where \( \text{exp}(P) \) is the exponent of \( P \) as subvariety of \( JC \), \( \text{Tr}_r \) is the rational trace and \( (\Delta.D) \) denotes the number of fixed points of the correspondence. Hence,

\[
\dim P = \frac{1}{3}(g_C - 4 + 3) = g - 1,
\]

Thus we have obtained a \( 2g + 1 \)-dimensional family of Prym–Tyurin varieties of dimension \( g - 1 \) and exponent 3. We must to show that the curve \( C \) is irreducible and smooth. With a slight modification on the argument of [6], using a suitable classifying homomorphism of the covering \( h \circ f \), is not difficult to prove the irreducibility of \( C \).

**Lemma 4.1.** The curve \( C \) given by the diagram 4.1 is smooth.

**Proof:** The curve \( C \subset \tilde{X}^{(2)} \) is smooth in a point \( \tilde{E} \) if and only if the Zariski tangent space \( T_{\tilde{E}}C \) is of dimension 1. If \( E := \pi(\tilde{E}) \), then the diagram 4.1 yields a diagram

\[
\begin{array}{ccc}
T_{\tilde{E}}C & \xrightarrow{\tilde{d}} & T_{\tilde{E}}\tilde{X}^{(2)} \\
\downarrow \text{d} & & \downarrow \text{df}^{(2)} \\
T_{\tilde{E}}\mathbb{P}^1 & \xrightarrow{\tilde{d}_j} & T_{\tilde{E}}X^{(2)}
\end{array}
\]

Since \( \tilde{X}^{(2)} \) and \( X^{(2)} \) are smooth of dimension 2, hence

\[
\dim T_{\tilde{E}}C = \dim \text{Ker } df^{(2)} + \dim (\text{Im } dj \cap \text{Im } df^{(2)}) = 2 - \dim df^{(2)} + \dim (\text{Im } dj \cap \text{Im } df^{(2)}) = 1 + (2 - \dim (\text{Im } dj + \text{Im } df^{(2)})).
\]

Then \( C \) is smooth in \( \tilde{E} \) if and only if \( \text{Im } dj + \text{Im } df^{(2)} = T_{\tilde{E}}X^{(2)} \) and this happens if and only if the composition

\[
T_{\tilde{E}}\mathbb{P}^1 \xrightarrow{\tilde{d}_j} T_{\tilde{E}}X^{(2)} \to \text{Coker } df^{(2)}
\]
is surjective. Recall the canonical isomorphisms (cf. [1] Ex.IV B-2)

\[ T_E X^{(2)} \simeq H^0(X, \mathcal{O}_E(E)), \]
\[ T_{\tilde{E}} \tilde{X}^{(2)} \simeq H^0(\tilde{X}, \mathcal{O}_{\tilde{E}}(\tilde{E})), \]
\[ T_E \mathbb{P}^1 \simeq H^0(X, \mathcal{O}_X(E))/H^0(X, \mathcal{O}_X). \]

The sheaf \( \mathcal{O}_X(E) \) may be considered as a subsheaf of the sheaf of rational functions on \( X \) by setting \( \mathcal{O}_X(E)_q = m_{X,q}^{-\nu_q(E)} \) for all \( q \in X \), where \( m_{X,q}^{-\nu_q(E)} \) is the ideal of rational functions with at most poles of degree \( \nu_q(E) \) at \( q \). Then \( H^0(X, \mathcal{O}_X(E)) \) is a subspace of the function field \( K(X) \) and \( \mathcal{O}_X(E)_q = m_{X,q}^{-\nu_q(E)}/\mathcal{O}_{X,q} \).

Suppose \( E = \sum_i n_i q_i \) with \( q_i \) distinct points in \( X \) and \( n_i \) positive integers. For \( h \in H^0(\mathcal{O}_X(E)) \) the \( i \)-th component of \( d_j(h + H^0(\mathcal{O}_X)) \in \bigoplus m_{X,q_i}^{-n_i} = H^0(\mathcal{O}_E(E)) \), is the image of \( h \) in \( m_{X,q_i}^{-n_i}/\mathcal{O}_{X,q_i} \). Denote by \( q^1, q^2, q^3 \in \tilde{X} \) the points over \( q \). Then, \( \tilde{E} = \sum_i n_i^1 q_i^1 + \sum_i n_i^2 q_i^2 + \sum_i n_i^3 q_i^3 \), with \( n_i^1 + n_i^2 + n_i^3 = n_i \) and we can assume \( n_i^1 \geq n_i^2 \geq n_i^3 \). Hence,

\[ H^0(\mathcal{O}_{\tilde{E}}(\tilde{E})) = \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{3} m_{X,q_i}^{-n_i}/\mathcal{O}_{X,q_i} \].

Let \( t_i \) be a local parameters of \( X \) at \( q_i \) and \( t_i^j \) local parameters of \( \tilde{X} \) at \( q_i^j \) for \( j = 1, 2, 3 \). Then \( d(f^{(2)}((t_i^j)^\nu)) = (t_i^j)^\nu \) for \( j = 1, 2, 3 \) and for all \( \nu \in \mathbb{Z} \). Thus, \( C \) is smooth in \( \tilde{E} \) if and only if the map

\[ H^0(\mathcal{O}_X(E)) \to \bigoplus_i m_{X,q_i}^{-n_i}/m_{X,q_i}^{-n_1} \]  \hspace{1cm} (4.4)

is surjective. If \( E = \pi(\tilde{E}) \) is smooth or \( E = \pi(2q_1) = 2q \), the right hand of 4.4 is zero. If \( \tilde{E} = q_1 + q_2 \), then \( \pi(\tilde{E}) = 2q \) and the right hand of 4.4 is \( m_{X,q}^{-2}/m_{X,q}^{-1} \). Let \( h \in H^0(\mathcal{O}_X(E)) \) with corresponding divisor \( E_h \neq E \) in \( g_1^1 \). Then \( E_h = \text{Div}(h) + E \), hence \( \nu_q(E_h) = 0 = 2 + \nu_q(h) \), i.e., \( \nu_q(h) = -2 \). Therefore, the image of \( h \) is a generator of \( m_{X,q}^{-2}/m_{X,q}^{-1} \) and \( C \) is smooth in \( E \).

\[ \Box \]

**Coverings of \( \mathbb{P}^1 \).** Let \( X \) be a smooth projective curve of genus \( g_X \) and consider a covering \( f : X \to \mathbb{P}^1 \) of degree \( n \geq 2 \). Let \( g_{n+2}^1 \) be the linear system defining the covering and \( B_f \subset \mathbb{P}^1 \) the branch points. Assume that the \( g_{n+2}^1 \) is complete, then \( n \leq g_X - 1 \). The degree of the ramification divisor, given by the Riemann–Hurwitz formula, is

\[ w_f = 2g_X + 2n + 2. \]

Let us to define set-theoretically the curve

\[ C := \{ E \in X^{(n)} \mid \vert g_{n+2}^1 - E \vert \neq \emptyset \}. \]

For \( z \in \mathbb{P}^1 \), we denote \( f^{-1}(z) = \{ P_1, \ldots, P_{n+2} \} \) and by \( P_{i_1, \ldots, i_n} = P_{i_1} + \cdots + P_{i_n} \) a point in \( C \), with \( \{ i_1, \ldots, i_n \} \subset \{ 1, \ldots, n + 2 \} \). The curve \( C \) comes with a map \( h : C \to \mathbb{P}^1 \) given by

\[ P_{i_1, \ldots, i_n} \mapsto f(P_{i_1}) = \cdots = f(P_{i_n}), \]

which is of degree \( \binom{n+2}{n} \). In fact, \( h \) corresponds to the composition of the classifying map \( \pi_1(X - B_f, z_q) \to S_{n+2} \) with the monomorphism \( S_{n+2} \to S_N \) (with \( N = \binom{n+2}{n} \)), which is given by the action of \( S_{n+2} \) on the cosets of the subgroup \( S_n \times \tilde{S}_2 \), which has a transitive image. The proof of the fact that \( C \) is
smooth and irreducible is a slight generalization of Lemma 12.7.1 ([2]).

Let \( z \in D^1 \) be a branch point above which the ramification is \( P_1 = P_2 \), then the ramification of \( h \) above \( z \) is given by

\[
P_{i_1, \ldots, i_{n-1}} = P_{2i_1, \ldots, 2i_{n-1}} \quad \{i_1, \ldots, i_{n-1}\} \subset \{3, \ldots, i_{n+2}\}.
\]

Hence the fibers of \( f \) with only one ramification point induce \( \binom{n}{n-1} = n \) simple ramification points for \( h \). If above \( z \) the covering \( f \) has more than one simple ramification, the ramification of \( h \) above \( z \) is a more complicated but not difficult to compute for special cases. Using this information we can compute the degree of the ramification divisor \( w_h \) of \( h \) and the genus of \( C \). For example, if \( f \) has no more than one simple ramification, we obtain

\[
w_h = n \cdot w_f, \quad g_C = n \cdot g_x + \frac{n(n+1)}{2}.
\]

Let us to define an effective symmetric correspondence on \( C \) as follows

\[
D : P_{i_1, \ldots, i_n} \mapsto \sum P_{j_1, \ldots, j_n},
\]

where the sum is over the subsets of \( \{1, \ldots, n\} \) such that \( |\{i_1, \ldots, i_n\} \cap \{j_1, \ldots, j_n\}| = n-2 \). This is a correspondence of bidegree \( (d, d) = (\frac{n(n-1)}{2}, \frac{n(n-1)}{2}) \). In order to \( P_1, \ldots, n \) be a fixed point we must have

\[
P_{1, \ldots, n} = P_{i_1, \ldots, i_{n-2}, n+1, n+2},
\]

for some \( \{i_1, \ldots, i_{n-2}\} \subset \{1, \ldots, n\} \), then \( P_{n+1} = P_{j_1} \) and \( P_{n+2} = P_{j_2} \) with \( \{j_1, j_2\} \subset \{1, \ldots, n\} \). The correspondence has no fixed points on the fiber \( h^{-1}(z) \) if and only if \( f^{-1}(z) \) admits at most one ramification point of index \( \leq 3 \). In order to apply our criterion to this correspondence we must verify that the endomorphism induced by \( D \), denoted by \( \gamma_D \), satisfies the quadratic equation 1.1.

Put \( D_j := D(P_{j_1, \ldots, j_{n-2}, n+1, n+2}) \), where \( j = \{j_1, \ldots, j_{n-2}\} \subset \{1, \ldots, n\} \). We want to compute

\[
D^2(P_{1, \ldots, n}) = \sum D_j.
\]

Observe that \( P_{1, \ldots, n} \) appears in \( D_j \) for any \( j \), then \( P_{1, \ldots, n} \) appears in \( D^2(P_{1, \ldots, n}) \) with coefficient \( \frac{n(n-1)}{2} \). Let us divide the elements of the fiber \( h^{-1}(z) \) in two types, which are of the form \( P_{k_1, \ldots, k_{n-2}, n+1, n+2} \) and \( P_{k_1, \ldots, k_{n-1}, n+1} \) (or \( P_{k_1, \ldots, k_{n-1}, n+2} \)). Observe that \( P_{k_1, \ldots, k_{n-2}, n+1, n+2} \) appears in \( D_j \) if and only if

\[
|\{k_1, \ldots, k_{n-2}\} \cap \{j_1, \ldots, j_{n-2}\}| = n-4,
\]

hence it is in as many \( D_j \)‘s as subsets of \( n-4 \) elements of \( \{k_1, \ldots, k_{n-2}\} \), that is, a point of the form \( P_{k_1, \ldots, k_{n-2}, n+1, n+2} \) appears in \( D^2(P_{1, \ldots, n}) \) with coefficient \( \frac{(n-2)}{2} \).

Similarly, \( P_{k_1, \ldots, k_{n-1}, n+1} \) appears in \( D_j \) if and only if

\[
|\{k_1, \ldots, k_{n-1}\} \cap \{j_1, \ldots, j_{n-2}\}| = n-3,
\]

hence \( P_{k_1, \ldots, k_{n-1}, n+1} \) appears in \( D^2(P_{1, \ldots, n}) \) with coefficient \( \frac{(n-1)}{2} \). We can write

\[
D^2(P_{1, \ldots, n}) = \frac{n(n-1)}{2} P_{1, \ldots, n} + \frac{(n-2)(n-3)}{2} \sum_{\{j_1, \ldots, j_{n-2}\} \subset \{1, \ldots, n\}} P_{j_1, \ldots, j_{n-2}, n+1, n+2}
\]

\[
+ \frac{(n-1)(n-2)}{2} \left( \sum_{\{j_1, \ldots, j_{n-1}\} \subset \{1, \ldots, n\}} P_{j_1, \ldots, j_{n-1}, n+1} + \sum_{\{j_1, \ldots, j_{n-1}\} \subset \{1, \ldots, n\}} P_{j_1, \ldots, j_{n-1}, n+2} \right).
\]
Since \( \frac{(n-1)(n-2)}{2} - \frac{(n-2)(n-3)}{2} = n - 2 \) and \( \frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} = n - 1 \) we have

\[
D^2(P_1, \ldots, n) = (n-1)P_1, \ldots, n - (n-2)D(P_1, \ldots, n) + \frac{(n-1)(n-2)}{2}h^* (h(P_1, \ldots, n)).
\]

If \( \gamma_D \) denotes the endomorphism of \( JC \) induced by \( D \), then we obtain the following equation in the Jacobian

\[
\gamma_D^2 + (n - 2)\gamma_D - (n-1) = 0.
\]

We have defined an effective symmetric correspondence on \( C \) which verify the equation 1.1.

Suppose that \( D \) is free of fixed points. According to the Kanev’s Criterion, the abelian variety \( P := \text{Im}(1 - \gamma_D) \) is a Prym–Tyurin variety of exponent \( n \) for the curve \( C \).

Using the formula 4.2 and 4.5 to compute its dimension we obtain \( \text{dim} P = g_X \).

Applying the Theorem 1.1 we have that \( P := \text{Im}(1 - \gamma_D) \subset JC \) is a Prym–Tyurin variety of exponent 2. We use 4.2 to compute its dimension

\[
\text{dim} P = \frac{1}{2} (g_C - 1 + 1) = g_X.
\]

**Case n = 2.** Suppose that we have a covering \( f : X \to \mathbb{P}^1 \) of degree 4 with two simple ramification points on two fibers and no more than one simple ramification point on the others fibers. The covering \( h : C \to \mathbb{P}^1 \) is of degree 6. In this case we have a correspondence on \( C \times C \) of bidegree (1, 1), that is, an involution on the curve \( C \), which sends \( P_{1,2} \) to \( P_{3,4} \). We get a fixed point of the correspondence for every fiber with two simple ramification points, then the involution has two fixed points. Computing the degree of ramification divisor of \( w_f \) and \( w_h \) we obtain the genus of \( C \) as follows

\[
w_f = 2g_X + 6, \quad w_h = 2(w_f - 4) + 2(3) = 2g_X - 2, \quad g_C = -6 + \frac{w_h}{2} + 1 = 2g_X.
\]

Applying the Theorem 1.1 we have that \( P := \text{Im}(1 - \gamma_D) \subset JC \) is a Prym–Tyurin variety of exponent 2. We use 4.2 to compute its dimension

\[
\text{dim} P = \frac{1}{2} (g_C - 1 + 1) = g_X.
\]

**Case n = 3.** Let \( f : X \to \mathbb{P}^1 \) be a covering of degree 5 with ramifications as in the case \( n = 2 \). The associated covering \( h : C \to \mathbb{P}^1 \) is of degree 10 and the correspondence \( D \) on \( C \) is of bidegree (3, 3). The fibers of \( f \) with two simple ramification points, contribute with 3 ramification points the index 4, 2, 2 on the corresponding fiber of \( h \), indeed, if \( P_1 = P_2 \) and \( P_3 = P_4 \) are the ramification points on one of these fibers, we have on \( C \)

\[
P_{135} = P_{145} = P_{235} = P_{245}, \quad P_{123} = P_{124}, \quad P_{134} = P_{234}.
\]

Since \( D(P_{135}) = P_{245} + P_{124} + P_{234} \), \( P_{135} \) is the only fixed point of \( D \) on this fiber. Then \( D \) has two fixed points. We have that

\[
w_f = 2g_X + 8, \quad w_h = 3(w_f - 4) + 2(5) = 2g_X + 2, \quad g_C = -6 + \frac{w_h}{2} + 1 = 3g_X + 2.
\]
By the Theorem 1.1 we have that \( P \coloneqq \text{Im}(1 - \gamma D) \) is a Prym–Tyurin variety of exponent 3. As before, we compute the dimension of \( P \) and we obtain \( \dim P = \frac{1}{3}(3g_X + 2 - 3 + 1) = g_X \).

**Case n=4.** Consider a covering \( f : X \to \mathbb{P}^1 \) of degree 6 having two fibers with three simple ramifications on each one and the others fibers with no more than one simple ramification point. The associated covering \( h : C \to \mathbb{P}^1 \) is of degree 15 and the correspondence \( D \) on \( C \) is of bidegree \( (6, 6) \). Let \( P_1 = P_2, P_3 = P_4, P_5 = P_6 \) be the three ramification points on one fiber of \( f \). They contribute, on the corresponding fiber of \( h \), with 3 ramification points of index 4 as follows

\[
\begin{align*}
P_1 &= P_{1235} = P_{1245} = P_{1236} = P_{1246} \\
P_2 &= P_{1345} = P_{1346} = P_{2345} = P_{2346} \\
P_3 &= P_{1356} = P_{1456} = P_{2356} = P_{2456}.
\end{align*}
\]

Observe that \( P_1, P_2, P_3 \) are the three fixed points of \( D \) on the fiber and they verify the conditions b) of the Theorem 1.1. Applying the criterion we get a Prym–Tyurin variety \( P \) of exponent 4. Observe that

\[
\begin{align*}
w_f &= 2g_X + 10 \\
w_h &= 4(w_f - 6) + 2(9) \\
g_C &= 4g_X + 5.
\end{align*}
\]

Hence the variety \( P \) is of dimension \( g_X \).

We shall show that in these examples \( P \cong JX \) as principally polarized abelian varieties. Let \( Q_0 \in X \) and let \( \alpha_n : X^{(n)} \to JX \) be the map

\[
E \mapsto \mathcal{O}_X(E - nQ_0),
\]

for all \( E \in X^{(n)} \). Let \( \varphi = \alpha_n|_C \) be the restriction to \( C \).

**Lemma 4.2.** There exists a constant \( b \in JX \) such that for all \( Q \in C \) it verifies

\[
\varphi(D(Q)) = (1 - n)\varphi(Q) + b.
\]

**Proof :** Let

\[
b = \frac{n(n - 1)}{2}(f^*(t) - (n + 2)Q_0),
\]

with \( t \in \mathbb{P}^1 \), then \( b \) is a constant in \( JX \) for any \( t \). A straightforward computation shows that

\[
\varphi(D(P_{1\ldots n})) + (n - 1)\varphi(P_{1\ldots n}) = \frac{1}{2}n(n - 1)(P_{1\ldots n+2} - (n + 2)Q_0).
\]

By the Universal Property of the Jacobian there exist a unique map \( \tilde{\varphi} \) such that for all \( c \in C \) the following diagram commutes

\[
\begin{array}{ccc}
C & \xrightarrow{\varphi} & JX \\
\downarrow{\alpha_c} & & \downarrow{\tilde{\varphi}(c)} \\
JC & \xrightarrow{\tilde{\varphi}} & JX
\end{array}
\]
The Lemma 4.2 tell us that the map $\tilde{\varphi}$ factorize by $P \subset JC$ since $\tilde{\varphi} \circ (\gamma_D + n - 1) = 0$. Then there exists a map $\psi$ making the following diagram commutative

$$
\begin{array}{ccc}
C & \xrightarrow{\varphi} & JX \\
\pi_c & \downarrow & \downarrow \psi_c \\
P & \rightarrow & JX \\
\end{array}
$$

(4.8)

where $\pi_c = (1 - \gamma_D) \circ \alpha_c$ is the Abel–Prym map.

**Proposition 4.1.** Let $P = \text{Im}(1 - \gamma_D) \subset JC$ with $\gamma_D$ the endomorphism induced by 4.6. If $\dim P = \dim JX$ then $\psi$ is an isomorphism of polarized abelian varieties.

**Proof:** Let $g = g_X = \dim P$. Observe that $\varphi(C)$ generate $JX$ as abelian variety and since $Z$ and $JX$ have the same dimension $\psi : P \to JX$ is an isogeny. According to Welters’ Criterion (Theorem 12.2.2 [2]) we have

$$
\pi_*[C] = \frac{n}{(g - 1)!} \wedge^{g-1} [\Xi] \quad \text{in} \quad H^{2g-2}(P, \mathbb{Z}).
$$

Suppose that have proved that

$$
\varphi_*[C] = \frac{n}{(g - 1)!} \wedge^{g-1} [\Theta] \quad \text{in} \quad H^{2g-2}(JX, \mathbb{Z}),
$$

(4.9)

then $\psi_* \wedge^{g-1} [\Xi] = \wedge^{g-1}[\Theta]$ and by the Lemma 12.2.3. [2] $\psi$ is an isomorphism.

Consider the sum map

$$
s : X^{(2)} \times X^{(n)} \to X^{(n+2)},
$$

and define $Z := s^{-1}(g^{n+2})$. Let $C' = \pi_1(Z) \subset X^{(2)}$ and $C = \pi_2(Z) \subset X^{(n)}$ be the projections of $Z$ over the factors. The curves $C$ and $C'$ are isomorphic because both are isomorphic to $Z$. Then we can describe $C'$ as follows

$$
C' = \{ p + q \in X^{(2)} \mid |g_{n+2} - p - q| \neq 0 \}.
$$

**Lemma 4.3.** $\alpha_{2*}[C'] = \frac{n}{(g - 1)!} \wedge^{(g-1)} [\Theta] \text{ in } H^{2g-2}(JX, \mathbb{Z})$.

**Proof:** It suffices to prove that $n[X] = \alpha_{2*}[C'] \text{ in } H^{2g-2}(JX, \mathbb{Z})$ since the class of $X$ in $JX$ is $\frac{1}{(g - 1)!} \wedge^{(g-1)}[\Theta]$. Let $q : X \times X \to X^{(2)}$ be the sum map and $\Delta_{P^1}$ respectively $\Delta_X$ the diagonals in $P^1$ respectively in $X$. Since $[\Delta_{P^1}] = [P^1 \times \{a\}] + [\{b\} \times P^1]$, $a, b \in P^1$, we have

$$
[\Delta_X] + q^*[C'] = (f \times f)^*([P^1 \times \{a\}] + [\{b\} \times P^1])
$$

$$
= [X \times D_a] + [D_b \times X]
$$

$$
= (n + 2)[X \times \{p\}] + (n + 2)[p] \times [X] \quad \text{in} \quad H^2(X^2, \mathbb{Z}),
$$

with $D_a, D_b \in g_{n+2}$ and for some $p \in X$. Let $\delta := q(\Delta_X)$. Applying $q_*$ we obtain

$$
[\delta] + 2[C'] = 2(n + 2)[q(X \times \{p\})] = 2(n + 2)[X + p],
$$

(4.10)

since $\Delta_X$ and $X \times \{p\}$ are isomorphic to their images in $X^{(2)}$. Recall that $\alpha_{2*}(\delta) = 2_*[X] = 4[X]$ where $2_*$ is the push forward homomorphism of the multiplication by 2 in $JX$ (cf. Theorem 12.7.2 [2]). Applying $\alpha_{2*}$ to the equation 4.10 we get

$$
4[X] + 2\alpha_{2*}[C'] = 2(n + 2)\alpha_{2*}[X + p] = 2(n + 2)[X] \quad \text{in} \quad H^2(X^2, \mathbb{Z}).
$$

Thus,

$$
\alpha_{2*}[C'] = n[X] \quad \text{in} \quad H^2(X^2, \mathbb{Z}).
$$

This completes the proof.
We denote \( \eta = \alpha_{n+2}(g_{n+2}^1) \in W_{n+2} \subset JX \), where \( W_n = \alpha_n(X^{(n)}) \). Therefore, we have the diagram

\[
\begin{array}{ccc}
Z \subset X^{(2)} \times X^{(n)} & \xrightarrow{s} & X^{(n+2)} \supset g_{n+2}^1 \\
\pi_1 & & \pi_2 \\
C' \subset X^{(2)} & \xrightarrow{\alpha_2} & W_n \\
\alpha_n & & \alpha_2(C') \subset W_2 \\
\end{array}
\]  

(4.11)

and then \((\eta - W_n).W_2 = \alpha_2(C')\) and \((\eta - W_2).W_n = \alpha_n(C)\). Hence

\[ \alpha_2(C') = \eta - \alpha_n(C), \]

and we conclude that \( \alpha_n(C) \) is algebraically equivalent to \( nX \). This shows the equality 4.9 and the proof of the Proposition 4.1 is completed.

\[ \square \]

**Remark 4.4.** The examples coming from coverings of \( \mathbb{P}^1 \) are generalizations of the Recillas’ construction [9]

**References**

[1] Arbarello, E.; Cornalba, M.; Griffiths, P.A.; Harris, J.: *Geometry of Algebraic Curves* Vol. 1, Grundlehren 267, Springer -Verlag, 1985.
[2] Birkenhake, Ch.; Lange, H: *Complex Abelian Varieties*, Grundlehren 302, Springer Verlag, 1982. Compositio Math. 39, 47-105, 1979.
[3] Kanev, V.: *Principal polarizations of Prym-Tjurin varieties*, Compositio Math. 64, no. 3, 243-270, 1987.
[4] Kanev, V.: *Theta divisors of generalized Prym varieties. I*. Algebraic geometry, Sitges (Barcelona), 1983, 166-215, Lecture Notes in Math., 1124, Springer, Berlin, 1985.
[5] Kanev, V.: *Spectral curves and Prym-Tjurin varieties I*. Proceedings of the Egloffstein conference 1993, 151-1998, de Gruyter 1995.
[6] Lange, H.; Recillas, S.; Rojas, A.M.: *A family of Prym-Tjurin varieties of exponent 3*, to appear in Journal of Algebra, 2004.
[7] Mumford, D.: *Prym varieties I*, Contributions to analysis, New York Academic Press, 1974.
[8] Mumford, D.: *Abelian Varieties*, Oxford University Press, 1970.
[9] Recillas, S.: *Jacobians of curves with \( g_i \)'s are the Prym's of trigonal curves*.

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