On the computation of the straight lines contained in a rational surface.

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Abstract

In this paper we present an algorithm to compute the real and complex straight lines contained in a rational surface, defined by a rational parametrization. The algorithm relies on the well-known theorem of Differential Geometry that characterizes real straight lines contained in a surface as curves that are simultaneously asymptotic lines, and geodesics. We also report on an implementation carried out in Maple 18. Examples and timings show the efficiency of the algorithm for moderate degrees, compared with a brute-force approach.

1. Introduction

Straight lines are certainly notable curves in an algebraic surface. Probably the most famous result on algebraic surfaces containing straight lines is related to cubic surfaces: G. Salmon [19], after correspondence with A. Cayley, proved that projective smooth cubic surfaces contain exactly 27 (projective, complex and real) straight lines, some of them at infinity. Schläfli [20] proved later that the number of real straight lines must be 3, 7, 15 or 27. If the cubic is singular [2], the number of straight lines goes down to 21.

Projective nonsingular cubic surfaces happen to be rational surfaces. If a parametrization of a rational cubic surface is known, one can compute the straight lines contained in the surface from the base points of the parametrization [2, 3]. However, unlike cubics, surfaces of degree higher than 3 do not necessarily contain straight lines (see Theorem 1.27 in [22]). Furthermore, in the affirmative case, up to our knowledge there is no known algorithm other than the brute-force approach to find them.

Computing the straight lines in a surface can be interesting on its own right, but it provides, additionally, useful information on the surface. If the surface

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contains some real line, then the surface is non-compact. Furthermore, knowing the (complex or real) straight lines contained in a surface helps to find the symmetry center, symmetry planes and symmetry axes, if any, of the surface, since any of these symmetries maps the straight lines onto each other. Also, this information can help to identify whether or not two given surfaces are similar, i.e. equal up to position and scaling, since two similar surfaces contain the same number of straight lines, that must be mapped to each other by the similarity.

In this paper we approach the problem of determining the straight lines contained in a surface defined by a rational parametrization of any degree. In order to do this, we exploit the well-known result in Differential Geometry that characterizes real non-singular straight lines contained in a surface, as curves that are simultaneously asymptotic lines, and geodesics. This characterization provides differential conditions to find the straight lines contained in the surface, that we transform into algebraic conditions; this way, we can take advantage of classical methods in polynomial algebra to solve the problem. Other special straight lines, in particular the ones contained in the singular part of the parametrization, can also be found. Additionally, the same strategy also allows to compute the complex straight lines contained in the surface.

The structure of the paper is the following. In Section 2 we provide some known and preliminary results. In Section 3 we present the algorithm. We prove that the algorithm works properly in Section 4. Section 5 reports on practical results, timings, etc. of an implementation of the algorithm in the computer algebra system Maple 18, that can be downloaded from [24]. Our conclusions are presented in Section 6. In Appendix I, we list the parametrizations used in the examples of Section 5.

In the paper, sometimes we will use the term “line” to refer to a curve contained in the surface we are analyzing. Hence, not every “line” is a “straight line”, although the converse statement is, obviously, true.

2. Notation, terminology, hypotheses and known results.

In this section we fix the notation, terminology and hypotheses to be used throughout the paper; we refer the interested reader to [7, 13] or [23] for further reading on the concepts and results mentioned here, mostly coming from the field of Differential Geometry.

In this paper we work with a rational surface $S \subset \mathbb{R}^3$, not a plane, parametrized by

\[
\mathbf{x}(t, s) = (x(t, s), y(t, s), z(t, s)),
\]

where $x(t, s), y(t, s), z(t, s)$ are real rational functions. We assume that this parametrization is proper, i.e. injective for almost all points of $S$, or equivalently that the mapping

\[
\mathbb{C}^2 \rightarrow S \\
(t, s) \rightarrow \mathbf{x}(t, s)
\]

is birational. One can check properness by using the algorithms in [14, 15]; for reparametrization questions one can see [16, 17]. We will consider the Euclidean
space $\mathbb{R}^3$ furnished with the usual dot product $\langle \cdot, \cdot \rangle$, and the usual Euclidean norm $\| \cdot \|$. Additionally, we use the notation $\bullet_t$ (resp. $\bullet_s$) for the partial derivative of $\bullet$ with respect to $t$ (resp. $s$); similarly $\bullet_{tt}, \bullet_{ts}, \bullet_{ss}$ represent second partial derivatives of $\bullet$.

We say that $x(t, s)$ is regular at a point $P_0 = x(t_0, s_0)$, if $(x_t \times x_s)(t_0, s_0) \neq 0$. At such a point we can define the normal vector $N$,

$$N = \frac{x_t \times x_s}{\|x_t \times x_s\|}. \quad (1)$$

If $(x_t \times x_s)(t_0, s_0) = 0$, in which case $N$ is not defined, we say that $x(t_0, s_0)$ is a singular point of the parametrization $x(t, s)$; we denote by $\text{Sing}_x$ the set consisting of all the singular points of $x(t, s)$. Furthermore, we represent the matrices defining the first fundamental form and second fundamental form of $S$ by $I, II$, respectively; recall that

$$I = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \langle x_t, x_t \rangle & \langle x_t, x_s \rangle \\ \langle x_s, x_t \rangle & \langle x_s, x_s \rangle \end{bmatrix}, \quad II = \begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} \langle x_{tt}, N \rangle & \langle x_{ts}, N \rangle \\ \langle x_{st}, N \rangle & \langle x_{ss}, N \rangle \end{bmatrix}. \quad (2)$$

We also introduce the following notation:

$$e^* = \langle x_{tt}, x_t \times x_s \rangle, \quad f^* = \langle x_{ts}, x_t \times x_s \rangle, \quad g^* = \langle x_{ss}, x_t \times x_s \rangle. \quad (3)$$

Notice that $e^*, f^*, g^*$ are the result of multiplying $e, f, g$ by $\|x_t \times x_s\|$. Since $x(t, s)$ is rational, $e^*, f^*, g^*$ are rational functions. The following result is well-known.

**Theorem 1.** Let $C \subset S$ be a real curve contained in a surface $S$ parametrized by $x(t, s)$, and assume that $C \not\subset \text{Sing}_x$. Then $C$ is a straight line iff it is simultaneously an asymptotic line, and a geodesic line.

For cardinality reasons and since $x$ parametrizes a surface, if $L \subset S$ is a straight line covered by the parametrization $x$, i.e. such that almost all points of $L$ are the image of some point $(t, s)$ in the parameter space, the Zariski closure of $x^{-1}(L)$ has dimension 1. In other words, there is an algebraic curve $p(t, s) = 0$ in the $(t, s)$ plane whose image under $x$ is $L$. Conversely, consider a curve $C \subset S$, described as the set of points $x(t, s)$ with $p(t, s) = 0$, where $p(t, s)$ is polynomial and square-free. Since $p(t, s)$ does not have multiple components, there exists a point $(t_0, s_0)$ where $p(t_0, s_0) = 0$ and some partial derivative $p_t, p_s$ is nonzero. Applying the Implicit Function Theorem at $(t_0, s_0)$, we have that $C$ can, at least locally, be parametrized as either $x(t, s(t))$ or $x(t_0, s)$. In the first case, observe that if $C$, parametrized by $x(t, s(t))$, is not contained in $\text{Sing}_x$ and $s(t)$ is a real function, then $C$ is an asymptotic line of $S$ iff $s(t)$ satisfies

$$e^* + 2f^* \cdot \frac{ds}{dt} + g^* \cdot \left(\frac{ds}{dt}\right)^2 = 0. \quad (4)$$

Additionally, in the same conditions $C$ is a geodesic line of $S$ iff $s(t)$ satisfies

$$I \cdot \frac{d^2s}{dt^2} = \Gamma_{22} \cdot \left(\frac{ds}{dt}\right)^3 + \left(2\Gamma_{12}^2 - \Gamma_{22}^2\right) \left(\frac{ds}{dt}\right)^2 + \left(\Gamma_{11}^2 - 2\Gamma_{12}^2\right) \frac{ds}{dt} - \Gamma_{11}^2. \quad (5)$$
where $\hat{\Gamma}^{i}_{jk} = \Gamma^{i}_{jk} \cdot I$, with $I = EG - F^2$ (the determinant of the first fundamental form), and where the $\Gamma^{i}_{jk}$ are the Christoffel symbols of $x(t, s)$. One can find explicit formulae for the Christoffel symbols, for instance, in page 268 of [13]. Since $x(t, s)$ is rational, the $\hat{\Gamma}^{i}_{jk}$ are rational functions.

As for curves $x(c, s)$, with $c$ a constant, considering analogous formulae to (4) and (5) in terms of $dt$ instead of $ds$, one can see that such a curve is an asymptotic line iff $g^{*} = 0$, and is a geodesic line iff $\hat{\Gamma}^{1}_{22} = 0$. Therefore, we have the following result, which will be useful in Section $3$; here we denote

$$\eta(t, s) = \gcd\left(\text{num}\left(g^{*}(t, s)\right), \text{num}\left(\hat{\Gamma}^{1}_{22}(t, s)\right)\right),$$

where $\text{num}(\bullet)$ represents the numerator of $\bullet$.

**Lemma 2.** The curve $x(c, s)$, where $c \in \mathbb{R}$, is a straight line iff $t - c$ divides $\eta(t, s)$.

Finally, we recall that a surface $S$ is said to be ruled if at every point $P \in S$ there exists a (real or complex) straight line $L_P$ through $P$ contained in $S$.

3. Computation of the straight lines.

In this section we describe the algorithm for computing the straight lines contained in the surface $S$ that are covered by the parametrization $x(t, s)$. Results on the computation of the curves contained in a rational surface not covered by the parametrization, or methods for surjectively reparametrizing rational surfaces, can be found in [1, 4] for special kinds of surfaces, and also in [18, 21]. Observe that in any case, the set of points not covered by the parametrization $x(t, s)$ has at most dimension 1.

**Overview of the section:**

First, in Subsection $3.1$ we consider the computation of the real straight lines not contained in $\text{Sing}_x$, that can be locally parametrized as $x(t, s(t))$, where $s(t)$ satisfies both Eq. (4) and (5) (see Section $2$). We address this computation under the most general conditions, leaving special situations for a later subsection. The main idea is as follows: Eq. (4) and Eq. (5) provide conditions involving $t, s, \frac{ds}{dt}, \frac{d^2s}{dt^2}$ for the function $s(t)$. Our strategy then is to move from these differential conditions to algebraic conditions. In order to do this, we replace $\frac{ds}{dt}, \frac{d^2s}{dt^2}$ in Eq. (4) and Eq. (5) by two new variables $\omega := \frac{ds}{dt}, \gamma := \frac{d^2s}{dt^2}$, linked by the relationship $\gamma = \frac{d\omega}{dt}$. An extra condition involving $\omega, \gamma$ is obtained by differentiating Eq. (4) with respect to $t$, and using that $\gamma = \frac{d\omega}{dt}$. Then we employ algebraic methods, namely resultants, to eliminate the variables $\omega, \gamma$, which leads to a polynomial condition $\xi(t, s) = 0$. The curves, contained in $S$, parametrized as

$$x(t, s), \mu_i(t, s) = 0,$$
where $\mu_i(t, s)$ is a real factor of $\xi(t, s)$ are then potential candidates for straight lines contained in $S$. However, not all these curves are straight lines. A criterion to distinguish those ones which are straight lines from those ones which are not, is given in Subsection 3.2.

In Subsection 3.3 we address the computation of the real straight lines corresponding to some special situations, left aside in Subsection 3.1. In particular, here we include straight lines contained in $\text{Sing}_x$, as well as straight lines parametrized as $x(c, s)$, with $c$ a constant. The whole algorithm is given in Subsection 3.4. Finally, in Subsection 3.5 we prove why the same strategy also allows to compute the complex straight lines contained in the surface.

3.1. Development of the method.

Let us replace $\omega := \frac{ds}{dt}$ in Eq. (4); thus we get
$$e^* + 2f^* \cdot \omega + g^* \cdot \omega^2 = 0,$$
where $e^*, f^*, g^*$ are given in Eq. (3), and are evaluated at $(t, s(t))$, where the function $s = s(t)$ is unknown; for simplicity, in the sequel we will write $s$ instead of $s(t)$. Differentiating Eq. (8) with respect to the variable $t$, and introducing $\gamma := \frac{d\omega}{dt}$, we get
$$A(t, s, \omega) \cdot \gamma + B(t, s, \omega) = 0,$$
where
$$A(t, s, \omega) = 2(f^* + g^* \omega)$$
and
$$B(t, s, \omega) = g^* \cdot \omega^3 + (2f^* + g^*) \omega^2 + (2f^* + e^*) \cdot \omega + e^*.$$  
Moreover, if $C$ is also a geodesic of $S$ then from Eq. (5) and using $\omega := \frac{ds}{dt}$, $\gamma := \frac{d^2s}{dt^2}$, we have
$$I \cdot \gamma = \tilde{\Gamma}^1_{22} \omega^3 + (2\tilde{\Gamma}^1_{12} - \tilde{\Gamma}^2_{22}) \omega^2 + (\tilde{\Gamma}^1_{11} - 2\tilde{\Gamma}^2_{12}) \omega - \tilde{\Gamma}^2_{11}.$$  
If $f^*$ and $g^*$ are not both identically 0 (the special case when both $f^*, g^*$ are zero will be treated later), $f^* + g^* \omega$, seen as a polynomial in $\omega$, is not identically 0 either. Then from Eq. (9) we can write $\gamma = -B(t, s, \omega)/A(t, s, \omega)$. Substituting this expression into Eq. (12) and clearing denominators we get a polynomial relationship between $t, s, \omega$,
$$\tilde{N}(t, s, \omega) = 0.$$  
Let $\tilde{M}(t, s, \omega)$ be the polynomial obtained by clearing denominators in Eq. (8), and let $M(t, s, \omega), N(t, s, \omega)$ be the primitive parts of $\tilde{M}(t, s, \omega)$ and $\tilde{N}(t, s, \omega)$ with respect to $\omega$. Finally, let $\tilde{\xi}(t, s)$ be the resultant of $M(t, s, \omega)$ and $N(t, s, \omega)$ with respect to $\omega$, and let $\xi(t, s)$ be the result of removing from $\tilde{\xi}(t, s)$ the
denominators of the components of \( x(t, s) \) and the denominators of the rational functions in Eq. (8), (9), (12), if any.

Therefore, the components of the space curve defined by \( x(t, s) \), where \( \xi(t, s) = 0 \), are potential candidates for straight lines contained in \( S \). Each of these components can be described as

\[
x(t, s), \; \mu_i(t, s) = 0, \tag{14}
\]

where \( \mu_i(t, s) \) is a real factor of \( \xi(t, s) \). Notice that in general it is not enough to consider the factors over \( \mathbb{Q} \), since the equations of the straight lines contained in \( S \) are not necessarily rational.

**Remark 1.** There is a number of papers where the problem of factoring over the reals is addressed, see for instance [5, 6, 8, 10, 11]. In our case, we used the command `AFactors` of Maple 18, which works finely for polynomials of moderate and medium degrees.

### 3.2. Checking candidates.

Not every curve (14) gives rise to a straight line contained in \( S \). This is not contradictory, since in the process to compute the polynomial \( \xi(t, s) \) we treat \( \omega \) as an independent variable, without taking into account the (differential) relationship \( \omega = \frac{ds}{dt} \). Therefore, we need a criterion to distinguish those real factors of \( \xi(t, s) \) which give rise to straight lines contained in \( S \), from those which do not. In order to do this, we use the following result.

**Proposition 3.** Let \( C \) be the set of points of \( \mathbb{R}^3 \) parametrized by \( x(t, s) \) with \( \alpha(t, s) = 0 \), where \( \alpha(t, s) \) is irreducible over the reals and depends explicitly on \( s \). Then \( C \) corresponds to a straight line if and only if there exists a rational function \( \lambda(t, s) \) and a constant vector \( u \) such that the following equality

\[
(x_t \alpha_s - x_s \alpha_t, y_t \alpha_s - y_s \alpha_t, z_t \alpha_s - z_s \alpha_t) = \lambda(t, s) \cdot u \tag{15}
\]

holds modulo \( \alpha(t, s) \).

**Proof.** If \( \alpha(t, s) \) depends on \( s \), then from the Implicit Function Theorem \( \alpha(t, s) = 0 \) (locally) defines \( s = s(t) \). Furthermore, \( \frac{ds}{dt} = -\frac{\alpha_t}{\alpha_s} \). Using this, the tangent vector to a point of \( C \) is parallel to

\[
(x_t \alpha_s - x_s \alpha_t, y_t \alpha_s - y_s \alpha_t, z_t \alpha_s - z_s \alpha_t).
\]

Then the condition in the statement of the lemma is equivalent to the unitary tangent to \( C \) being constant.

In order to check condition (15), one proceeds in the following way for each real factor \( \mu_i(t, s) \) of \( \xi(t, s) \), not a common factor of the components of the left hand-side of (15), taking \( \alpha(t, s) \equiv \mu_i(t, s) \) for each value of \( i \):

\[\text{In such a case, the space curve defined by } x(t, s) \text{ with } \mu_i(t, s) = 0 \text{ degenerates into a point.}\]
1. Let \( t = a \in \mathbb{Z} \). Then \((a,b), m(b) = \alpha(a,b) = 0,\) is a point of the curve \( \alpha(t,s) = 0.\)

2. Let \( w(t,s) = (x_t \alpha_s - x_s \alpha_t, y_t \alpha_s - y_s \alpha_t, z_t \alpha_s - z_s \alpha_t), \) and let \( w_0 \) be the result of evaluating \( w(t,s) \) at the point \( t = a, s = b. \) If \( w_0 \) is zero, we choose a different \((a,b).\)

3. \( C \) is a straight line iff all the components of \( w(t,s) \times w_0 \) are divisible by \( \alpha(t,s). \) Furthermore, in the affirmative case \( C \) is parallel to \( w_0. \)

3.3. Special lines.

We need to pursue now some additional straight lines that may have been missed in the process described in Subsection 3.1. Assume first, as we did in Subsection 3.1 that \( f^*, g^* \) are not identically zero. We also need to examine the lines parametrized by \( x(t,s) \) with \( \delta_j(t,s) = 0, j = 1, \ldots, 4, \) where:

1. \( \delta_1(t,s) \) is the gcd of the coefficients of the numerator of \( A(t,s,\omega), \) seen as a polynomial of degree 1 in \( \omega. \)
2. \( \delta_2(t,s) \) is the product of the contents of \( M(t,s,\omega) \) and \( N(t,s,\omega) \) with respect to \( \omega; \) notice that these contents were eliminated when we moved from \( \tilde{M}, \tilde{N} \) to \( M, N. \)
3. In order to compute the straight lines, if any, consisting of points \( x(t,s) \) where \( f^* + g^* \omega = 0, \) we observe that in this case we have
\[
\omega = -\frac{f^*}{g^*}.
\]
Substituting this expression in Eq. (8) and clearing the denominator yields the condition
\[
\delta_3(t,s) = e^* g^* - (f^*)^2 = 0.
\]
Notice that \( \delta_3(t,s) \) is the determinant of the second fundamental form multiplied by \( \|N\|^2 \) (see Eq. (2)).
4. \( \delta_4(t,s) \) is the numerator of \( \|x_t \times x_s\|^2. \) This is necessary in order to also capture the straight lines contained in \( S \) consisting of points where \( x(t,s) \) is singular, i.e. where \( \|x_t \times x_s\|^2 = 0. \)

For each curve \( x(t,s), \) with \( \delta_i(t,s) = 0, i = 1, \ldots, 4, \) we can also use Proposition 3 in order to check whether or not the curve corresponds to a straight line. Additionally, in the case when \( f^*, g^* \) are not both identically zero we also need to compute the straight lines parametrized as \( x(c,s), \) where \( c \) is a constant. These straight lines are found by applying Lemma 2.

Finally, we address the special case when \( f^*, g^* \) are both identically 0. In this situation \( e^* \) cannot be identically 0, because otherwise \( S \) is a plane (see pg. 147 of [7]), which is excluded by hypothesis. Hence (8) reduces to \( e^*(t,s) = 0. \) If \( e^*(t,s) = e^*(t) \) then we just need to look for lines that can be parametrized as \( x(c,s) \) with \( c \) a root of \( e^*(t), \) which can be done by applying Lemma 2.
Otherwise, $e^*(t, s) = 0$ implicitly defines $s = s(t)$; differentiating $e^*(t, s) = 0$ twice with respect to $t$, and using $\omega := \frac{ds}{dt}$; $\gamma := \frac{d\omega}{dt}$ as before, we get an expression like Eq. (9), with

$$A(t, s, \omega) = e^*_s, \quad B(t, s, \omega) = e^*_s \cdot \omega^2 + 2e^*_s \cdot \omega + e^*_t.$$  

(16)

Now writing $\gamma = -B(t, s, \omega)/A(t, s, \omega)$, we can proceed as in Subsection 3.1 to find an expression like Eq. (13). From this moment on the situation is analogous to that in Subsection 3.1 and a polynomial $\xi(t, s)$ is computed. In this case we also have to examine the lines parametrized as $x(t, s)$, $\delta_i(t, s) = 0$, where $i = 1, \ldots, 4$ and the polynomials $\delta_1(t, s)$, $\delta_2(t, s)$ and $\delta_4(t, s)$ are computed as in the general case. No polynomial analogous to $\delta_3(t, s)$ appears in this case, though, so in this case we can define $\delta_3(t, s) := 1$.

3.4. The whole algorithm.

From the ideas and results in the previous subsections we can derive the following algorithm, Algorithm STLines, to compute the straight lines contained in $S$. The algorithm requires that the surface $S$ defined by $x(t, s)$ is not ruled. Ruled surfaces of degree higher than two can be characterized by means of the Pick’s invariant of the surface, which is defined as $J = K - H$, where $K$ is the Gauss’ curvature of the surface and $H$ is the mean curvature of the surface. Ruled surfaces that are not quadrics are exactly those ones with vanishing $J$ (see pages 89, 90 of [12]). On the other hand, every quadric contains infinitely many complex lines, so every quadric can be regarded as a ruled surface. However, in Section 4 we will see that in fact we do not need to check whether or not $S$ is ruled in advance, since ruled surfaces can be recognized while we are running the algorithm.

Notice that the algorithm has two loops, in steps 14 and 19. The number of iterations in each of these loops equals the number of factors of the polynomials $\mu(t, s)$ and $\eta(t, s)$, computed in the steps 13 and 18, respectively. Therefore, in order to show that the algorithm works properly we need to show that under the considered hypotheses $\mu(t, s)$, $\eta(t, s)$ are not identically zero; otherwise the number of factors, and therefore of iterations in the loops of either step 14 or step 19, would be infinite. We will prove this in the next section.

3.5. Why we also get the complex straight lines.

In this subsection we show that, whenever we compute an absolute factorization of the polynomials $\mu(t, s)$ and $\eta(t, s)$ in steps 13 and 18 of Algorithm STLines, i.e. a factorization over the complex numbers, Algorithm STLines also provides the complex straight lines contained in $S$. This type of factorization can be computed, for instance, also using the command AFactors of Maple 18. The fact that Algorithm STLines also provides the complex straight lines contained in $S$ is not necessarily obvious: although these lines can be parametrized as $x(t, s(t))$ or $x(c, s)$, in this case $s(t)$, $c$ are complex, while the results in Section 2 assume that $s(t)$ or $t(s)$ are real functions. Therefore, we need to see that Eq. (4) and Eq. (5) also hold when $s(t), c$ are complex.
Algorithm 1 STLines

Require: A proper parametrization \( \mathbf{x}(t, s) \) of an algebraic surface \( S \), which is not ruled.

Ensure: The straight lines contained in \( S \) that are covered by the parametrization \( \mathbf{x}(t, s) \).

1: **Part I:** Straight lines not of the type \( \mathbf{x}(c, s) \), where \( c \) is a constant.
2: find the numerator \( \tilde{M}(t, s, \omega) \) of the left hand-side of (8).
3: if \( f^*, g^* \) are not both identically zero then
4: compute \( A(t, s, \omega), B(t, s, \omega) \) using Eq. (10) and Eq. (11).
5: else
6: compute \( A(t, s, \omega), B(t, s, \omega) \) using Eq. (16).
7: end if
8: substitute \( \gamma = -B(t, s, \omega)/A(t, s, \omega) \) into Eq. (12) and clear denominators to get \( \tilde{N}(t, s, \omega) \).
9: compute the primitive parts \( M(t, s, \omega), N(t, s, \omega) \) of \( \tilde{M}(t, s, \omega) \) and \( \tilde{N}(t, s, \omega) \) with respect to \( \omega \).
10: let \( \xi(t, s) = \text{Res}_\omega(M(t, s, \omega), N(t, s, \omega)) \).
11: let \( \zeta(t, s) \) be the result of removing from \( \xi(t, s) \) the denominators of the
   components of \( \mathbf{x}(t, s) \), and the denominators of the rational functions in Eqs. (8), (9), (12), if any.
12: compute the polynomials \( \delta_1(t, s), \delta_2(t, s), \delta_3(t, s), \delta_4(t, s) \) defined in Subsec. 3.3.
13: compute \( \mu(t, s) \), the square-free part of \( \mu^*(t, s) = \xi(t, s) \cdot \delta_1(t, s) \cdot \delta_2(t, s) \cdot \delta_3(t, s) \cdot \delta_4(t, s) \).
14: for each irreducible component of \( \mu(t, s) = 0 \), do
15: use Proposition 3 to check whether or not it corresponds to a straight line contained in \( S \).
16: end for
17: **Part II:** Straight lines of the type \( \mathbf{x}(c, s) \), where \( c \) is a constant.
18: let \( \eta(t, s) = \gcd\left(\text{num}(g^*(t, s)), \text{num}\left(\tilde{\Gamma}_{22}^1(t, s)\right)\right) \).
19: for each factor \( t - c_i \) of \( \eta(t, s) \), only depending on \( t \) do
20: compute the line \( \mathbf{x}(c_i, s) \), \( i = 1, \ldots, n \).
21: end for
22: return the list of straight lines computed in Part I, Part II, or the message no straight lines found.
In order to do this, we need to look closer at the notions of normal curvature and geodesic curvature, that we recall here for the convenience of the reader. Let \( C \subset S \) be a curve contained in \( S \), parametrized by \( y(t) \). Let \( \{t, n, b\} \) denote the Frenet frame of \( C \); recall that \( t \) is the unitary vector parallel to \( y' \). If \( s \) represents the arc-length, the curvature vector \( k \) of \( C \) is defined as \( k = \dot{t} \), where the dot means the derivative with respect to \( s \). Furthermore, the normal curvature, \( k_n \), and geodesic curvature, \( k_g \), are defined as

\[
k_n = \langle \dot{t}, N \rangle, \quad k_g = \langle \dot{t}, N \times t \rangle.
\]

(17)

Additionally, the normal curvature vector, \( k_n \), and the geodesic curvature vector, \( k_g \), are defined as

\[
k_n = k_n \cdot N, \quad k_g = k_g \cdot (N \times t),
\]

(18)

so that \( k = k_n + k_g \). Then we say that \( C \) is an asymptotic line of \( C \) if \( k_n = 0 \), and we say that \( C \) is a geodesic line if \( k_g = 0 \).

For the curves \( x(t, s(t)) \) or \( x(t(s), s) \), with \( s(t), t(s) \) real functions, the conditions \( k_n = 0 \) and \( k_g = 0 \) are equivalent to Eq. (4) and Eq. (5). However, when \( s(t), t(s) \) are complex, \( k_n, k_g \) are not necessarily well-defined. The reason is that \( \langle , \rangle \) is not an inner product in \( \mathbb{C}^3 \), since the positive-definiteness property does not hold anymore. Because of this, \( N \) is not necessarily well-defined even though \( x(t, s) \), and therefore \( x_t, x_s \), are well-defined, because \( \|x_t \times x_s\| \) can vanish although \( x_t \times x_s \) is nonzero. Also, it can happen that \( y'(t) \neq 0 \) but \( ||y'(t)|| = 0 \), in which case \( t \) is not well-defined either.

In order to see that complex straight lines satisfy Eq. (4) and Eq. (5) too, we need the following lemmata, the first of which can be easily proven. Here, \( \mathcal{L}(u, v) \) represents the linear variety spanned by the vectors \( u, v \).

**Lemma 4.** Let \( C \subset S \) be parametrized by \( y(t) = x(t, s(t)) \) for some real function \( s(t) \).

1. The condition \( k_n = 0 \) is equivalent to \( y'' \in \mathcal{L}(x_t, x_s) \), where \( y'' \) is evaluated at \( t \), and \( x_t, x_s \) at \( (t, s(t)) \).
2. The condition \( k_g = 0 \) is equivalent to \( \langle y'', (x_t \times x_s) \times y' \rangle = 0 \), where \( y', y'' \) are evaluated at \( t \), and \( x_t, x_s \) at \( (t, s(t)) \).

**Lemma 5.** Let \( y(t) = x(t, s(t)) \) be a complex straight line (i.e. \( s(t) \) is a complex function such that the imaginary part of \( y(t) \) is nonzero) contained in \( S \). Then, \( y(t) \) satisfies that: (1) \( y'' \in \mathcal{L}(x_t, x_s) \); (2) \( y'', (x_t \times x_s) \times y' \) is zero.

**Proof.** (1) \( y(t) = x(t, s(t)) \) is a straight line iff \( y'(t), y''(t) \) are linearly dependent for all \( t \). Since \( y' \in \mathcal{L}(x_t, x_s) \), it follows that \( y'' \in \mathcal{L}(x_t, x_s) \). Therefore, condition (1) holds. (2) The triple product \( \langle y'', (x_t \times x_s) \times y' \rangle \) can be written as a determinant, where the first and last rows correspond to the coordinates of \( y'' \) and \( y' \) respectively. Since these rows are proportional, the value of the determinant is zero. \( \square \)
Remark 2. One can prove a completely analogous lemma for curves \( \mathbf{x}(t(s), s) \subset S \), with \( t(s) \) a complex function. In particular, curves \( \mathbf{x}(c, s) \) with \( c \in \mathbb{C} \) are of this type. From here one can conclude that Lemma 2 also holds when \( c \in \mathbb{C} \).

The conditions of Lemma 5 are exactly the conditions appearing in the statements (1) and (2) of Lemma 4. But these conditions imply Eq. (4) and Eq. (5).

Corollary 6. If \( \mathbf{x}(t, s(t)) \) is a real or complex straight line contained in \( S \), then \( s(t) \) satisfies Eq. (4) and Eq. (5).

Interestingly enough, and unlike the real case, Eq. (4) and Eq. (5) are necessary conditions for non-singular complex straight lines, but they are not sufficient. Consider for instance the surface \( \mathbb{S} \). Consider for instance the surface \( \mathbb{S} \) parametrized by
\[
\mathbf{x}(t, s) = \left( \frac{1}{2}t(s^3 + 3s) + \frac{1}{2}s^2, t, t + s \right),
\]
and let \( s(t) = i \), where \( i^2 = -1 \). Then \( \mathbf{y}(t) = \mathbf{x}(t, i) = \left( it - \frac{1}{2}, t^2, t + i \right) \) is a complex parabola contained in \( S \). One can check that \( s(t) = i \) satisfies Eq. (4) and Eq. (5); however, clearly \( \mathbf{y}(t) \) does not define a straight line. The next result sheds some light on the kind of curves contained in \( S \) which satisfy Eq. (4) and Eq. (5), but are not straight lines.

Lemma 7. Let \( \mathbf{y}(t) = \mathbf{x}(t, s(t)) \), with \( s(t) \) a complex function, parametrize a complex curve \( \mathbb{C} \subset S \). If \( \mathbf{y}(t) \) satisfies Eq. (4) and Eq. (5) but is not a straight line, then \( \| \mathbf{x}_t \times \mathbf{x}_s \| \) identically vanishes over the points \((t, s(t))\).

Proof. Eq. (4) and Eq. (5) are equivalent to conditions (1) and (2) in Lemma 5. Since we are assuming that Eq. (4) and Eq. (5) hold, \( \langle \mathbf{y}'', (\mathbf{x}_t \times \mathbf{x}_s) \rangle = 0 \). Since \( \langle \mathbf{y}'', (\mathbf{x}_t \times \mathbf{x}_s) \times \mathbf{y}' \rangle \) can be computed as a determinant consisting of the components of \( \mathbf{y}'', \mathbf{x}_t \times \mathbf{x}_s, \mathbf{y}' \), we have \( \mathbf{y}'' = a(\mathbf{x}_t \times \mathbf{x}_s) + b\mathbf{y}' \) for some \( a, b \in \mathbb{C} \). On the other hand, since \( \mathbf{y}'' \in \mathcal{L}(\mathbf{x}_t, \mathbf{x}_s) \), and \( \mathbf{y}' \in \mathcal{L}(\mathbf{x}_t, \mathbf{x}_s) \), we deduce that \( \mathbf{x}_t \times \mathbf{x}_s \in \mathcal{L}(\mathbf{x}_t, \mathbf{x}_s) \). Hence, there exist \( c, d \in \mathbb{C} \) such that \( \mathbf{x}_t \times \mathbf{x}_s = c\mathbf{x}_t + d\mathbf{x}_s \). Therefore, we have
\[
\langle \mathbf{x}_t \times \mathbf{x}_s, \mathbf{x}_t \times \mathbf{x}_s \rangle = c\langle \mathbf{x}_t, \mathbf{x}_t \times \mathbf{x}_s \rangle + d\langle \mathbf{x}_s, \mathbf{x}_t \times \mathbf{x}_s \rangle.
\]

Now \( \langle \mathbf{x}_t, \mathbf{x}_t \times \mathbf{x}_s \rangle \) can be computed as a determinant whose rows are the components of \( \mathbf{x}_t, \mathbf{x}_t \) and \( \mathbf{x}_s \), respectively. Since the first and third rows of the determinant are equal, we get \( \langle \mathbf{x}_t, \mathbf{x}_t \times \mathbf{x}_s \rangle = 0 \). Similarly, \( \langle \mathbf{x}_s, \mathbf{x}_t \times \mathbf{x}_s \rangle = 0 \). So from Eq. (19), \( \langle \mathbf{x}_t \times \mathbf{x}_s, \mathbf{x}_t \times \mathbf{x}_s \rangle = \| \mathbf{x}_t \times \mathbf{x}_s \|^2 = 0 \).

Remark 3. The same condition, i.e. \( \| \mathbf{x}_t \times \mathbf{x}_s \|^2 = 0 \) is also obtained for curves \( \mathbf{x}(t(s), s) \). Notice that in the complex case this condition does not mean that the point is singular, i.e. that \( \mathbf{x}_t \times \mathbf{x}_s = 0 \).

If \( \mathbb{C} \) is parametrized by \( \mathbf{y}(t) = \mathbf{x}(t, s(t)) \), with \( s(t) \) a complex function, and \( \mathbf{y}(t) \) satisfies Eq. (4) and Eq. (5) but is not a straight line, we say that it is a
special curve. Similarly for curves $\mathbf{x}(t(s), s)$, with $t(s)$ a complex function. We represent by $\mathcal{V}$ the union of all the special curves contained in $S$. Since $\mathbf{x}(t, s)$ parametrizes a surface $\|\mathbf{x}_t \times \mathbf{x}_s\|^2$ is not identically zero; therefore we get the following result, that is needed in Section 4.

Lemma 8. The dimension of $\mathcal{V}$ is at most 1.

Finally, observe that Proposition 3 also holds when $C$ is a complex irreducible curve. Therefore, we can use Proposition 3 as well when working over $\mathbb{C}$, in order to check what complex candidate curves are in fact straight lines.

4. Why the algorithm recognizes ruled surfaces.

In this section, we will prove that polynomials $\mu(t, s)$ and $\eta(t, s)$ defined in steps 13 and 18 of the Algorithm STLines identically vanish if and only if $S$ is a ruled surface. Since Algorithm STLines requires $S$ not to be ruled, this proves that the loops in steps 14 and 19 do not run forever, so the algorithm works properly. Additionally, we observe that in fact we do not really need to check in advance whether or not $S$ is ruled: if while the algorithm is working we detect that either $\mu(t, s)$ or $\eta(t, s)$ identically vanishes, we give back the answer $S$ is a ruled surface, which in particular means that $S$ contains infinitely many (maybe complex) straight lines.

We need some previous results. The following lemma is perhaps known; however we could not find any appropriate reference in the literature, so we provide here a complete proof.

Lemma 9. Let $S$ be an algebraic surface. If $S$ contains infinitely many straight lines, then $S$ is a ruled surface.

Proof. Let $X$ be the algebraic set of all the straight lines $\ell$ contained in $\overline{S}$, the Zariski closure of $S$, which is contained in the Grassmannian $\mathbb{G}(1, 3)$ of lines in the complex projective space $\mathbb{P}^3$. Furthermore, let $I = \{ (x, \ell) \mid x \in \overline{S}, \ell \in X \}$. Now consider the incidence diagram:

$$
\begin{array}{c}
\mathcal{S} \\
p \searrow \nearrow q \\
I \\
\downarrow \\
X
\end{array}
$$

Since $S$ contains infinitely many straight lines, we have $\dim X \geq 1$. Additionally $p(I)$ is the union of the elements of $X$, considered as straight lines in $\mathbb{P}^3$. Since $p(I)$ is an infinite union of different straight lines it cannot be a curve, so $\dim(p(I)) \geq 2$. Therefore $p(I)$ is a closed surface contained in the irreducible surface $\overline{S}$. Hence $p(I) = \overline{S}$, so $p$ is surjective. Thus, for all $x \in S$ there is a straight line $\ell \in X$ passing through $x$ (equivalently, $\overline{S}$ is the union of its lines). Hence $\overline{S}$ is ruled, and so is $S$. \qed
Now we are ready to prove the result. We recall here that if \( \det(II) > 0 \) at a point \( P_0 \in S \), where \( II \) is the matrix defining the second fundamental form of \( x(t, s) \), then we say that \( P_0 \) is an \textit{elliptic point}; if \( \det(II) = 0 \), \( P_0 \) is a \textit{parabolic point}; if \( \det(II) < 0 \), \( P_0 \) is a \textit{hyperbolic point}.

**Theorem 10.** A rational surface \( S \) properly parametrized by \( x(t, s) \) is a ruled surface if and only some of the polynomials \( \mu(t, s), \eta(t, s) \) are identically zero.

**Proof.** (\( \Rightarrow \)) If \( S \) is ruled then \( S \) contains infinitely many straight lines. If \( \mu(t, s), \eta(t, s) \) are not identically zero then they have finitely many irreducible components. Therefore, there should be infinitely many straight lines not covered by the parametrization. But this cannot happen, because the set of points of \( S \) not covered by \( x(t, s) \) has dimension at most 1.

(\( \Leftarrow \)) If \( \eta(t, s) \) is identically zero then by Lemma \( \overline{[23]} \) any coordinate line \( x(c, s) \), with \( c \) a real constant, is a straight line. Since we have one coordinate line of this type through every point, then \( S \) is ruled. So assume that \( \mu(t, s) \) is identically zero, in which case either \( \xi(t, s) = 0 \) or \( \delta_3(t, s) \) is identically zero. If \( f^*, g^* \) are both identically zero \( \delta_3(t, s) \) then is a nonzero constant; otherwise \( \delta_3(t, s) = \det(II) \). In that case, if \( \delta_3(t, s) \) is identically zero then \( S \) is developable (see pg. 103 of \( \overline{[23]} \)), and therefore ruled. Therefore, let us assume that \( \xi(t, s) = 0 \) identically zero.

Now let \( \mathcal{M}, \mathcal{N} \) denote the algebraic surfaces in the \((t, s, \omega)\)-space defined by \( M(t, s, \omega) \) and \( N(t, s, \omega) \). If \( \xi(t, s) \) is identically zero, then \( \mathcal{M}, \mathcal{N} \) share a component \( P(t, s, \omega) = 0 \). We have three different cases, depending on whether \( M \) has degree two, degree one (when \( g^* = 0 \)) or degree zero (when \( g^* = f^* = 0 \)); we will address the case when \( M \) has degree two, which is the most difficult one; the other two cases are left to the reader.

Assuming that \( M \) has degree two in \( \omega \), and since \( M \) is primitive with respect to the variable \( \omega \) (because its content with respect to \( \omega \) was removed), \( M \) has at most two irreducible components, corresponding to

\[
\omega = -\frac{f^* + \sqrt{(f^*)^2 - 4e^*g^*}}{e^*}, \quad \omega = -\frac{f^* - \sqrt{(f^*)^2 - 4e^*g^*}}{e^*}.
\]

Without loss of generality, we will assume that \( P(t, s, \omega) = 0 \) contains the graph of

\[
\omega = -\frac{f^* + \sqrt{(f^*)^2 - 4e^*g^*}}{e^*}.
\]

We have now two possibilities, depending on whether there exists an open set \( \Omega \subset \mathbb{R}^2 \) where \((f^*)^2 - e^*g^* \geq 0 \), or not. We focus on the affirmative case; the negative case will be addressed later. Let \( \Omega \subset \mathbb{R}^2 \) be an open set where \((f^*)^2 - e^*g^* \geq 0, e^* \neq 0 \), and all the Christoffel’s symbols are well-defined. By Picard-Lindelöf’s Theorem, for any \((t_0, s_0) \in \Omega \) the initial value problem

\[
\begin{align*}
\frac{ds}{dt} &= -\frac{f^* + \sqrt{(f^*)^2 - 4e^*g^*}}{e^*}, \\
\frac{d}{dt}s(t_0) &= s_0,
\end{align*}
\]

where
has a unique, real, solution, \( s(t) \), defined over an interval \( I \) containing \( t_0 \). By construction, the space curve \( C_1 \) (in the \((t, s, \omega)\)-space) parametrized by

\[
\left( t, s(t), \frac{ds}{dt}(t) \right),
\]

with \( t \in I \subset \mathbb{R} \), \( t_0 \in I \), is contained in the surface \( P(t, s, \omega) = 0 \). Now let \( \omega_0 = \frac{ds}{dt}(t_0) \). Also by Picard-Lindelöf’s Theorem, the initial value problem

\[
\begin{cases}
\frac{dw}{dt} = \psi(t, s(t), \omega) \\
\omega(t_0) = \omega_0
\end{cases}
\]

where \( \psi(t, s(t), \omega) \) represents the right hand-side of Eq. (12) evaluated at \( s = s(t) \), has a unique, real, solution \( \omega(t) \). Again, by construction the space curve \( C_2 \) parametrized by

\[
(t, s(t), \omega(t)),
\]

with \( t \in J \), \( t_0 \in J \), is contained in the surface \( P(t, s, \omega) = 0 \). Now since \( C_1 \), \( C_2 \) are two analytic curves contained in the surface \( P(t, s, \omega) = 0 \), sharing the point \((t_0, s_0, \omega_0)\), and projecting onto the same curve \((t, s(t))\) of the \((t, s)\)-plane for \( t \in I \cap J \), then \( C_1 \) and \( C_2 \) must coincide. Since \( s(t) \) satisfies both Eq. (4) and Eq. (5) and \( s(t) \) is real, by Theorem 1 then \( x(t, s(t)) \) must be a straight line through the point \( P_0 = x(t_0, s_0) \). Since this construction is valid for \((t_0, s_0) \in \Omega \), we conclude that \( S \) contains infinitely many straight lines. But then the implication follows from Lemma 9.

If \((f^*)^2 - e^* g^* < 0\) for all \((t, s) \in \mathbb{R}^2\), so that all the points of \( S \) are elliptic, we proceed as before replacing Picard-Lindelöf’s Theorem by the Complex Existence and Uniqueness Theorem for differential equations in the complex domain (see for instance Theorem 2.2.1 in [9]). This way, we construct infinitely many curves contained in \( S \), satisfying Eq. (4) and Eq. (5). Since by Lemma 8 only finitely many of them can be special curves, we also get infinitely many straight lines (complex, this time) contained in \( S \), so \( S \) is ruled.

Remark 4. It can happen that \( S \) contains infinitely many complex straight lines, but not infinitely many real straight lines. However, in that case \( S \) must be a quadric. Indeed, assume that through each point \( p \in S \) there exists a complex line \( L = p + tv \), where \( v \in \mathbb{C}^3 \), such that \( L \subset S \). Since \( S \) is defined by a real parametrization, \( S \) has a real implicit equation. Therefore, the conjugate line \( \overline{L} = \overline{p} + iv \) is also contained in \( S \). Hence, through every point of \( S \) there are two straight lines contained in \( S \), and therefore \( S \) is doubly ruled. But the only doubly ruled algebraic surfaces are planes and quadrics.

5. Experimentation.

In this section we provide a detailed example of the algorithm \textsc{STLines}, as well as timings corresponding to several other examples. Furthermore, we
compare these timings with the timings corresponding to a brute-force approach. All the examples have been run in an Intel Core computer, revving up at 2.90 GHz, with 8 Gb of RAM memory.

5.1. A detailed example.
Consider the surface $S$ parametrized by

$$x(t, s) = (-s^3 + 3st^2 + 3s, 3s^2t - t^3 + 3t, 3s^2 - 3t^2).$$

This is a minimal surface of degree 9, called the Enneper surface. Eq. (8) yields

$$(-54)(\omega - 1)(\omega + 1)(s^2 + t^2 + 1)^4 = 0,$$

while Eq. (13) yields

$$2(\omega^2 + 1)(-tw + s) = 0.$$

Hence $M(t, s, \omega) = \omega^2 - 1$, $N(t, s, \omega) = 2(\omega^2 + 1)(-tw + s)$, and then

$$\xi(t, s) = 16(-t + s)(t + s).$$

Furthermore, $\delta_1(t, s) = s^2 + t^2 + 1$ and $\delta_2(t, s) = (s^2 + t^2 + 1)^2$. Additionally, $\delta_3(t, s)$ is the determinant of the second fundamental form, which yields $\delta_3(t, s) = -2916(s^2 + t^2 + 1)^4$. Also,

$$\|x_t \times x_s\|^2 = 81(t^2 + s^2 + 1)^4,$$

so $\delta_4(t, s) = (t^2 + s^2 + 1)^4$. Therefore,

$$\mu(t, s) = (t + s) \cdot (t - s) \cdot (s^2 + t^2 + 1).$$

We observe that the curve $\mu(t, s) = 0$ has three irreducible components. Let us analyze each one.

1. The component $t + s = 0$ is obviously rational. Plugging $s = -t$ into $x(t, s)$, we get

$$x(t, -t) = (2s^3 + 3s, -2s^3 - 3s, 0),$$

which defines a straight line through $(0, 0, 0)$ parallel to the vector $(1, -1, 0)$. However, in order to illustrate the criterion obtained from Proposition 3, let us see how to use this criterion here. Now let $\alpha(t, s) = t - s$, and let $a = 1$, so that $m(b) = 1 - b$. Furthermore,

$$w(t, s) = (3s^2 - 6st - 3t^2 - 3, -3s^2 - 6st + 3t^2 - 3, -6s + 6t).$$

After evaluating $w(t, s)$ at $t = 1, s = 1$, we get

$$w_0 = w(1, 1) = (-9, -9, 0).$$

Hence,

$$w(t, s) \times w_0 = (54s - 54t, -54s + 54t, 54s^2 - 54t^2).$$

We observe that all the components of $w(t, s) \times w_0$ are divisible by $\alpha(t, s) = t - s$, so $\alpha(t, s) = 0$ certainly corresponds to a straight line contained in $S$. 


(2) The component \( t - s = 0 \) is again rational. Plugging \( s = t \) into \( \mathbf{x}(t, s) \), we get
\[
\mathbf{x}(t, t) = (2s^3 + 3s, 2s^3 + 3s, 0),
\]
which defines a straight line through \((0, 0, 0)\) parallel to the vector \((1, 1, 0)\).

(3) The last component corresponds to \( s^2 + t^2 + 1 = 0 \). The component is again rational, and can be parametrized by
\[
\left( \frac{\beta}{2} \left( z^2 + \frac{1}{z} \right), \frac{z^2 - 1}{2z} \right),
\]
where \( \beta^2 + 1 = 0 \). By plugging this parametrization into \( \mathbf{x}(t, s) \), we get
\[
\left( -\frac{z^4 - 2z^2 + 1)(z^2 - 1)}{2z^3}, \frac{\beta(z^2 + 1)(z^4 + 2z^2 + 1)}{2z^3}, \frac{3(z^4 + 1)}{2z^2} \right),
\]
which clearly does not correspond to a straight line. However, let us check what the criterion from Proposition 3 yields in this case. First, we have \( \alpha(t, s) = t^2 + s^2 + 1 \); furthermore, we pick \( t = 1 \), so that \( m(b) = b^2 + 2 \). Now
\[
\mathbf{w}(t, s) = (18s^2t - 6t^3 - 6t, 6s^3 - 18st^2 + 6s, -24st).
\]
Furthermore,
\[
\mathbf{w}_0 = \mathbf{w}(1, b) = (18b^2 - 12, 6b^3 - 12b, -24b).
\]
Therefore,
\[
\mathbf{w}(t, s) \times \mathbf{w}_0 = (144bs(s^2 - 3t^2 + 4t + 1), -1152st - 24b(18s^2t - 6t^3 - 6t),
-288s^3 + 864t^2s - 288s + 24b(18s^2t - 6t^3 - 6t)),
\]
and one can easily see that \( t^2 + s^2 + 1 \) does not, for instance, divide the first component. So we deduce that \( t^2 + s^2 + 1 = 0 \) does not give rise to any straight line of \( S \).

Finally, one can check that
\[
\eta(t, s) = \gcd \left( g^*(t, s), \num \left( \Gamma^1_{22}(t, s) \right) \right) = 1,
\]
and therefore there are no straight lines of the type \( \mathbf{x}(c, s) \), where \( c \) is a constant.

The whole computation takes 0.078 seconds. Figure 1 shows a picture of the surface, together with the two straight lines we have computed.

5.2. Implementation and timings.

In this subsection, we compare timings of the implementation of our algorithm, implemented in Maple 18, with the timings of a brute-force approach, for several rational surfaces; the code for rational surfaces can be downloaded.
For the brute-force approach, we: (1) compute the implicit equation $F(x, y, z) = 0$ of the surface; (2) substitute $x := a_1 t + b_1$, $y := a_2 t + b_2$, $z := a_3 t + b_3$, where $t$ is a parameter, into $F(x, y, z) = 0$; (3) we get a polynomial in the variable $t$ that must be identically zero: this condition leads to a polynomial system in $a_i, b_i$; (4) the (real or complex) solutions where some $a_i$ is nonzero, correspond to the straight lines contained in the surface.

We show the results in Table 1. The column labeled “Deg.” contains the degree of the rational parametrization, i.e. the maximum of the degrees of the numerators and denominators of the components of the parametrization. “Time 1” corresponds to the timing of our algorithm, while “Time 2” corresponds to the timing of the brute-force approach. “\(n_\ell\)” corresponds to the number of straight lines computed by our algorithm; notice that our algorithm computes only the straight lines covered by the parametrization. “\(\text{deg}(\mu)\)” is the degree of the polynomial $\mu(t, s)$. Finally, in “Obs.” we have included certain observations on the surface. In the columns corresponding to Time 1 and Time 2, we show the best timing in bold; furthermore, whenever the timing exceeds 600 seconds, we write *. The parametrizations used in the examples are listed in Appendix I.
| Example | Deg. | Time 1  | Time 2  | \( n_\ell \) | \( \text{deg}(\mu) \) | Obs.                        |
|---------|------|---------|---------|--------------|----------------|-----------------|
| \( S_1 \) | 3    | 0.078   | 0.327   | 2            | 4              | Enneper surface. |
| \( S_2 \) | 3    | 0.671   | 0.452   | 18           | 46             | Clebsch surface. |
| \( S_3 \) | 4    | 0.047   | 0.172   | \( \infty \) | –              | Ruled quartic.   |
| \( S_4 \) | 3    | 2.153   | *       | 2            | 72             | Lines with non-rational coeffs. |
| \( S_6 \) | 5    | 0.483   | 162.616 | 1            | 92             |                  |
| \( S_7 \) | 4    | 0.468   | *       | 1            | 49             |                  |
| \( S_8 \) | 5    | 0.062   | 0.078   | 0            | 16             |                  |
| \( S_9 \) | 3    | 0.500   | 0.795   | 0            | 45             |                  |
| \( S_{10} \) | 4    | 8.549   | *       | 0            | 96             | Random polynomials. |
| \( S_{11} \) | 5    | 0.343   | 0.062   | 1            | 7              | Toric surface.   |
| \( S_{13} \) | 4    | 17.690  | *       | 0            | 96             | Random polynomials. |
| \( S_{14} \) | 5    | 12.964  | *       | 0            | 113            | Random polynomials. |
| \( S_{15} \) | 4    | 11.762  | 0.125   | 3            | 19             | Two complex lines. |

**Table 1:** Examples.

In the case of the Clebsch surface, the algorithm detects 18 straight lines because there are 6 lines not covered by the parametrization, and 3 lines at infinity; so \( 18 + 6 + 3 = 27 \).

Basically, the brute-force approach beats our method in cases when the computation of the implicit equation is very fast. In a generic situation our method is better; in fact, in cases like \( S_4, S_7, S_{10}, S_{13}, S_{14} \) the brute-force approach is not able to provide an answer in a reasonable amount of time.

### 6. Conclusions.

We have presented an algorithm to compute the straight lines contained in an algebraic surface, defined by a rational parametrization. The algorithm is based on the well-known result in Differential Geometry characterizing real, non-singular straight lines contained in a surface as lines which are simultaneously asymptotic lines, and geodesic lines. Experiments conducted on non-trivial examples with moderate degrees, show that our method is generally better than the brute-force approach.

The method can certainly be generalized to the case of implicit algebraic surfaces. In that case, we need to use the Implicit Function Theorem to write the coefficients of the first and second fundamental forms and the Christoffel symbols in terms of \( x, y, z \). Afterwards, some auxiliary variables must be properly eliminated. However, the experiments we have conducted show that this approach is not better than the brute-force approach. For this reason, we have left the implicit case out of the paper.
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7. Appendix I: parametrizations used in the examples.

The rational parametrizations used in Subsection 5.2 are the following:

- **S₁**: \( x(t, s) = (-s^3 + 3t^2s + 3s, 3s^2t - t^3 + 3t, 3s^2 - 3t^2) \)
- **S₂**: \( x(t, s) = \left(\frac{(s - 1)(ts + t - 1)}{-t + t^2 + s - s^2}, \frac{-t^2s + t + s - 1}{-t + t^2 + s - s^2}, \frac{t(1 - t - s^2)}{-t + t^2 + s - s^2}\right) \)
- **S₃**: \( x(t, s) = (st, s^2(t - 1), s^2(t + 1)) \)
- **S₄**: \( x(t, s) = \left(t + s^3 + t^3 + 1, \frac{2st + s^3 + t^3 + 1}{s}, \frac{3t^2 + t + s^3 + t^3 + 1}{t}\right) \)
- **S₆**: \( x(t, s) = \left(t^3 + \frac{s}{t^2 + 1}, \frac{t + s}{1 + s}, t^5 + s\right) \)
- **S₇**: \( x(t, s) = (t^3 - s, ts^3, s^4 + t^3) \)
- **S₈**: \( x(t, s) = (t, s^2, t^5 + s) \)
- **S₉**: \( x(t, s) = \left(s, \frac{s^3 + t^2}{s + t}, t^3\right) \)
- **S₁₀**: \( x(t, s) = \left(\frac{t^4 + 2s^3 - st^2 - 2st}{-2s^4 + 2s^3t^2 + t^3 + s^2}, \frac{-2s^4 + 2s^3t^2 - 2t^4 + s^2 - t}{-2s^4 + 2s^3t^2 + t^3 + s^2}, \frac{-s^3t + st^3 + 2t^3 + 2s}{-2s^4 + 2s^3t^2 + t^3 + s^2}\right) \)
- **S₁₁**: \( x(t, s) = (t^3s^3, t^2s^4, s^5) \)
- **S₁₃**: \( x(t, s) = \left(\frac{t(s^2 - t^2 - s)}{q(t)}, \frac{s(-2t^3 + 2st - 2t^2 + s - 1)}{q(t)}, \frac{s(-2s^3 - 2st^3 - t^2 + 1)}{q(t)}\right) \),

where \( q(t) = -73s^4 + 97s^2t^2 - 62s^3 - 56s^2 + 87t \).
\[ S_{14}: \]
\[ x(t, s) = \left( \frac{t(-s^3t + 2st^2 + t^2 + 2s - t)}{q(t)}, \frac{s(s^3t + 2st^3 - 2s^3 + 2st^2)}{q(t)}, \frac{s(-2s^3t - 2s^2t^2 - st)}{q(t)} \right), \]
where \( q(t) = -10s^4 - 83s^2t^2 - 4st^3 - 73s^2 + 97t^2 - 62t. \)

\[ S_{15}: \]
\[ x(t, s) = (t, t^2(s^2 + 1), s^2 + s + 1). \]