THE EXISTENCE OF A NONTRIVIAL WEAK SOLUTION TO A DOUBLE CRITICAL PROBLEM INVOLVING A FRACTIONAL LAPLACIAN IN $\mathbb{R}^N$ WITH A HARDY TERM

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Abstract In this paper, we consider the existence of nontrivial weak solutions to a double critical problem involving a fractional Laplacian with a Hardy term:

$$(-\Delta)^s u - \gamma \frac{u^*}{|x|^{2s}} = \left[\frac{|u|^{2^*_s(\beta)-2} u}{|x|^2} + \mu I*F_\alpha(\cdot, u)\right](x)f_\alpha(x, u), \quad u \in \dot{H}^s(\mathbb{R}^n),$$

(0.1)

where $s \in (0, 1), 0 \leq \alpha, \beta < 2s < n, \mu \in (0, n), \gamma < \gamma_H$, $I_\mu(x) = |x|^{-\mu}, F_\alpha(x, u) = \frac{|u(x)|^{2^*_s(\alpha)}}{|x|^{2^*_s(\alpha)}}$, $f_\alpha(x, u) = \frac{|u(x)|^{2^*_s(\alpha)-2} u(x)}{|x|^{2s}}$, $2^*_s(\alpha) = (1 - \frac{\alpha}{2s}) \cdot 2^*_s(\alpha)$, $\delta_\mu(\alpha) = (1 - \frac{\mu}{2n}) \cdot 2^*_s(\alpha)$, $2^*_s(\alpha) = \frac{2(n-\alpha)}{n-2s}$ and $\gamma_H = 4^{\frac{2^*_s(\alpha)}{2^*_s(\alpha)-2}}$. We show that problem (0.1) admits at least a weak solution under some conditions.

To prove the main result, we develop some useful tools based on a weighted Morrey space. To be precise, we discover the embeddings

$$\dot{H}^s(\mathbb{R}^n) \hookrightarrow L^{2^*_s(\alpha)}(\mathbb{R}^n, |x|^{-\alpha}) \hookrightarrow L^p \frac{\alpha+2s}{r} \frac{\alpha}{2^*_s(\alpha)}(\mathbb{R}^n, |x|^{-\alpha}),$$

(0.2)

where $s \in (0, 1), 0 < \alpha < 2s < n, p \in [1, 2^*_s(\alpha))$ and $r = \frac{\alpha}{2^*_s(\alpha)}$. We also establish an improved Sobolev inequality,

$$\left(\int_{\mathbb{R}^n} \frac{|u(y)|^{2^*_s(\alpha)}}{|y|^{2^*_s(\alpha)_s}} dy\right)^{2^*_s(\alpha)_s} \leq C\|u\|^\theta_{\dot{H}^s(\mathbb{R}^n)}\|u\|^{1-\theta}_{L^p \frac{\alpha+2s}{r} \frac{\alpha}{2^*_s(\alpha)}(\mathbb{R}^n, |x|^{-\alpha})}, \quad \forall u \in \dot{H}^s(\mathbb{R}^n),$$

(0.3)

where $s \in (0, 1), 0 < \alpha < 2s < n, p \in [1, 2^*_s(\alpha))$, $r = \frac{\alpha}{2^*_s(\alpha)}, 0 < \max\{\frac{2}{2^*_s(\alpha)}, \frac{2^*_s-1}{2^*_s(\alpha)}\} < \theta < 1$, $2^*_s = \frac{2n}{2s}$ and $C = C(n, s, \alpha) > 0$ is a constant. Inequality (0.3) is a more general form of Theorem 1 in Palatucci, Pisante [1].

By using the mountain pass lemma along with (0.2) and (0.3), we obtain a nontrivial

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weak solution to problem (0.1) in a direct way. It is worth pointing out that (0.2) and (0.3) could be applied to simplify the proof of the existence results in [2] and [3].

**Key words**  existence of a weak solution; fractional Laplacian; double critical exponents; Hardy term; weighted Morrey space; improved Sobolev inequality

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1 Introduction

In this paper, we consider the existence of nontrivial weak solutions to a double critical problem involving a fractional Laplacian with a Hardy term:

$$(-\Delta)^s u - \gamma \frac{u}{|x|^{2s}} = \frac{|u|^{2^*(\beta)-2}u}{|x|^{\beta}} + \left[I_\mu * F_\alpha(\cdot, u)\right](x) f_\alpha(x, u), \ u \in \dot{H}^s(\mathbb{R}^n),$$  \hspace{1cm} (1.1)

where $s \in (0, 1)$, $0 \leq \alpha, \beta < 2$, $n \in (0, n)$, $\gamma < \gamma_H$, $I_\mu(x) = |x|^{-\mu}$, $F_\alpha(x, u) = \frac{|u(x)|^{2^*_{\mu}(\alpha)}}{|x|^{|\mu|_{\alpha}}}$, $f_\alpha(x, u) = \frac{|u(x)|^{2^*_{\mu}(\alpha)}-2u(x)}{|x|^{\mu}}$, $\gamma_H = \frac{4^{4/s}}{\pi^2}$ (see Lemmas 2.1–2.2). It is worth pointing out that (1.1) is
equivalent to problem (0.1) in a direct way. It is worth pointing out that (0.2) and (0.3)

We note that $2^*_{\mu}(\alpha)$ is the critical fractional Hardy-Sobolev exponent and $\gamma_H$ is the best fractional Hardy constant on $\mathbb{R}^n$ (see Lemmas 2.1–2.2). It is worth pointing out that $(2^*_{\mu}(\alpha), \delta_{\mu}(\alpha))$ is a pair of critical exponents in the sense of Fractional Hardy-Sobolev inequality and Hardy-Littlewood-Sobolev inequality, which can be seen later in (2.6). The fractional Laplacian $(-\Delta)^s$ is defined on the Schwartz class (space of rapidly decaying $C^\infty$ functions in $\mathbb{R}^n$) through the Fourier transform

$$\hat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \hat{u}(\xi), \forall \xi \in \mathbb{R}^n,$$

where $\hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi x} u(x) dx$ is the Fourier transform of $u$.

Throughout this paper, we denote the norm of $L^p(\mathbb{R}^n, |y|^{-\lambda})$ by

$$||u||_{L^p(\mathbb{R}^n, |y|^{-\lambda})} := \left(\int_{\mathbb{R}^n} \frac{|u(y)|^p}{|y|^\lambda} dy\right)^{1/p}$$

for any $0 \leq \lambda < n$ and $p \in [1, +\infty)$. The homogeneous fractional Sobolev space of order $s \in (0, 1)$ is defined as $\dot{H}^s(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) : |\xi|^s \hat{u}(\xi) \in L^2(\mathbb{R}^n)\}$, which is in fact the completion of $C^\infty_0(\mathbb{R}^n)$ under the norm

$$||u||^2_{\dot{H}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 dx.$$

The dual space of $\dot{H}^s(\mathbb{R}^n)$ is denoted by $\dot{H}^s(\mathbb{R}^n)'$. See [4] and references therein for the basics on the fractional Laplacian.

We say that $u \in \dot{H}^s(\mathbb{R}^n)$ is a weak solution to (1.1) if

$$\int_{\mathbb{R}^n} \left[(-\Delta)^s u(-\Delta)^s \phi - \frac{u \phi}{|x|^{2s}}\right] dx$$

$$= \int_{\mathbb{R}^n} \frac{|u|^{2^*(\beta)-2} u \phi}{|x|^{\beta}} dx + \int_{\mathbb{R}^n} \left[I_\mu * F_\alpha(\cdot, u)\right](x) f_\alpha(x, u) \phi(x) dx.$$
By using truncation skills, the authors of [3] showed the existence of minimizers for

$$B_\alpha(u, v) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)|^{2^*_p(\alpha)}}{|x|^{\delta_\mu(\alpha)}} |v(y)|^{2^*_\mu(\alpha)} \, dx \, dy, \quad \forall u, v \in \dot{H}^s(\mathbb{R}^n),$$  \hspace{1cm} (1.2)

where $s \in (0, 1), 0 < \alpha < 2s < n, \mu \in (0, n), 2^*_\mu(\alpha) = (1 - \frac{\alpha}{2n}) 2^*_\mu(\alpha)$ and $\delta_\mu(\alpha) = (1 - \frac{\alpha}{2n}) \alpha$. In particular, $2^*_\mu := 2^*_\mu(0) = \frac{2n}{n-2\mu}$ and $2^* := 2^*(0) = \frac{2n}{n-2}$. Setting $\tilde{u}_t(x) = t^{\frac{n-2}{2s}} u(tx)$ and $\tilde{v}_t(y) = t^{\frac{n-2}{2s}} v(ty), t > 0$, we get that $B_\alpha(\tilde{u}_t, \tilde{v}_t) = B_\alpha(u, v)$. The energy functional associated to (1.1) is defined as

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left[ (-\Delta) \tau u \right]^2 - \gamma u^2 \left[ \frac{u}{|x|^{2s}} \right] \, dx - \frac{1}{2^*_\mu(\beta)} \int_{\mathbb{R}^n} \left| u \right|^{2^*_\mu(\beta)} \, dx - \frac{1}{2} \cdot 2^*_\mu(\alpha) B_\alpha(u, u).$$

A nontrivial critical point of $I$ is a nontrivial weak solution to equation (1.1).

The problem of multiple critical exponents has been extensively studied by scholars, see [2, 3, 5–15]. Dating back to [3], Filippucci et al. studied the following double critical equation

$$\begin{aligned}
-\Delta \rho u - \kappa \frac{u^{p-1}}{|x|^p} &= u^{p^* - 1} + u^{p^* - 1} \left( \frac{p^*}{n} \right) \quad \text{in} \quad \mathbb{R}^n, u \geq 0, \quad u \in D^{1, p}(\mathbb{R}^n),
\end{aligned} \hspace{1cm} (1.3)$$

where $n \geq 2, p \in (1, n), \alpha \in (0, p), p^* = \frac{np}{n-p}$, $p^*(\alpha) = \frac{p(n-\alpha)}{n-p}, 0 \leq \kappa < \tilde{\kappa} = \left( \frac{n-2}{p} \right)^p$ and $\Delta\rho u := \text{div}(|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian of $u$. Their work space $D^{1, p}(\mathbb{R}^n)$ is defined as the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm $||u||_{D^{1, p}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\nabla u|^p dx \right)^\frac{1}{p}$, i.e.,

$$D^{1, p}(\mathbb{R}^n) := \{ u \in L^p(\mathbb{R}^n) : \nabla u \in L^p(\mathbb{R}^n) \}.$$

By using truncation skills, the authors of [3] showed the existence of minimizers for

$$\begin{aligned}
\Lambda(n, \kappa, \alpha) &= \inf_{u \in D^{1, p}(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^p dx - \kappa \int_{\mathbb{R}^n} \frac{|u|^p}{|x|^{2s}} dx}{\left( \int_{\mathbb{R}^n} \frac{|u|^{p^* - 1}}{|x|^\alpha} dx \right)^\frac{p^*}{p^* - 1}},
\end{aligned}$$

provided $\alpha \in (0, p)$ and $\kappa < \tilde{\kappa}$ or $\alpha = 0$ and $0 \leq \kappa < \tilde{\kappa}$. Then they obtained a nontrivial weak solution to problem (1.3) by using mountain pass lemma and a careful analysis of the concentration of the corresponding (PS) sequence.

In [7], Ghoussoub and Robert considered the Dirichlet boundary value problem:

$$\begin{aligned}
-\Delta u - \gamma \frac{u}{|x|^2} &= \lambda u + \frac{u^{2^* - 1}}{|x|^{\alpha - \alpha}} \quad \text{on} \quad \Omega, \quad u > 0 \quad \text{on} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega,
\end{aligned}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$ such that $0 \in \Omega, n \geq 3, \gamma < \frac{(n-2)^2}{4}, 0 \leq \alpha < 2, 2^*(\alpha) = \frac{2(n-\alpha)}{n-2}, 0 \leq \lambda < \lambda_1(\Omega)$ and $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta - \frac{\gamma}{|x|^2}$ on $H^1_0(\Omega) \setminus \{0\}$. Fruitful achievements have been made on the basis of their work; before long, Ghoussoub et al. [8] extended the results in [7] to the nonlocal case.

Ghoussoub and Shakerian [2] generalized the results in [3] to the $(-\Delta)^s$ operator and considered the problem

$$\begin{aligned}
(-\Delta)^s u - \gamma \frac{u}{|x|^{2s}} &= |u|^{2^* - 2} u + \frac{|u|^{2^* - 2} u}{|x|^{\alpha - \alpha}}, \quad u \in \dot{H}^s(\mathbb{R}^n)
\end{aligned} \hspace{1cm} (1.4)$$
for \( s \in (0, 1), 0 < \alpha < 2s < n \) and \( 0 \leq \gamma < \gamma_H \). By the weighted harmonic extension for the fractional Laplacian obtained by Caffarelli and Silvestre in [16], they showed the existence of a nontrivial weak solution \( w \in X^s(\mathbb{R}^n_{+}+1) \) to

\[
\begin{aligned}
-\text{div}(y^{1-2s}\nabla w) &= 0, & \text{in } \mathbb{R}^{n+1}_+ \\
\frac{\partial w}{\partial u^s} &= \gamma \frac{w(x, 0)}{|x|^{2s}} + |w(x, 0)|^{2s-2}w(x, 0) + \frac{|w(x, 0)|^{2s(\alpha)-2}w(x, 0)}{|x|^\alpha} & \text{on } \mathbb{R}^n,
\end{aligned}
\]

where \( \frac{\partial w}{\partial u^s} := -k_s \lim_{y \to 0^+} y^{1-2s} \frac{\partial w(x, y)}{\partial y} \) and the space \( X^s(\mathbb{R}^{n+1}_+) \) is defined as the closure of \( C^\infty_0(\mathbb{R}^{n+1}_+) \) under the norm

\[
||w||_{X^s(\mathbb{R}^{n+1}_+)} := \left( k_s \int_{\mathbb{R}^{n+1}_+} y^{1-2s} |\nabla w|^2 \, dx \, dy \right)^{\frac{1}{2}},
\]

with \( k_s = \frac{\Gamma(s)}{2^{s-1} \Gamma(1-s)} \). Denoting the trace of \( w(x, y) \in X^s(\mathbb{R}^{n+1}_+) \) on \( \mathbb{R}^n \times \{ y = 0 \} \) by \( w(x, 0) \), we get that \( w(x) = w(x, 0) \) is in \( \dot{H}^s(\mathbb{R}^n) \) and is a weak solution to equation (1.4).

In [5], Yang and Wu studied

\[
(-\Delta)^su - \gamma \frac{u}{|x|^{2s}} = |u|^{2^*_s(\beta)-2u} + (I_\mu * |u|^{2^s}) |u|^{2^s-2}u, \quad u \in \dot{H}^s(\mathbb{R}^n),
\]

where \( s \in (0, 1), 0 < \beta < 2s < n, \mu \in (n - 2s, n), \gamma < \gamma_H \), \( I_\mu(x) = |x|^{-\mu} \), \( 2^*_s = \frac{2n-n+2s}{n-2s} \) and \( 2^*_s(\beta) = \frac{2(n-\beta)}{n-2s} \). By using the Nehari manifold method, they proved that equation (1.5) has a nontrivial weak solution if \( 0 < \gamma < \gamma_H \). For the cases of the standard Laplacian, biharmonic operator and p-biharmonic operator, the interested reader can refer to [6, 7, 9, 17–21].

Motivated by the above papers, we consider the existence of nontrivial weak solutions to problem (1.1). To the best of our knowledge, (1.1) has not been studied before.

Our main result is as follows:

**Theorem 1.1** The problem (1.1) possesses at least a nontrivial weak solution provided that either

(I) \( s \in (0, 1), 0 < \alpha, \beta < 2s < n, \mu \in (0, n) \) and \( \gamma < \gamma_H \),

or

(II) \( s \in (0, 1), 0 \leq \alpha, \beta < 2s < n \) while \( \alpha \cdot \beta = 0, \mu \in (0, n) \) and \( 0 \leq \gamma < \gamma_H \).

**Remark 1.2** Theorem 1.1 indicates that we can relax the lower bound of \( \gamma \) in equation (1.1) provided that \( \alpha, \beta > 0 \), which is different from equations (1.3)–(1.5). In the meantime, Theorem 1.1 relaxes the order \( \mu \) in \( I_\mu(x) = |x|^{-\mu} \) because equation (1.5) only allows \( \mu \in (n - 2s, n) \). Moreover, equation (1.5) is a special case of equation (1.1), with \( \alpha = 0 \).

There are three main difficulties in the proof of Theorem (1.1). First, truncation skills used in [3] and [2] do not work if we choose \( \dot{H}^s(\mathbb{R}^n) \) as the work space, since \( (-\Delta)^s \) is a nonlocal operator. Although the weighted harmonic extension can overcome the difficulty of the nonlocality of \( (-\Delta)^s \) if we work in \( X^s(\mathbb{R}^{n+1}_+) \), the appearance of the convolution term in (1.1) still prevents us from using truncation skills. Secondly, the compactness of the corresponding (PS) sequence may not hold, since equation (1.1) has two critical nonlinearities. For the equation with a single critical nonlinearity,

\[
-\Delta u + \lambda u = |u|^{2^*-2}u \quad \text{in } \Omega,
\]
where $\Omega$ is a bounded domain of $\mathbb{R}^n$, $n \geq 3$, $-\lambda_1(\Omega) < \lambda < 0$ and $2^* = \frac{2n}{n-2}$, Brézis and Nirenberg in [22] used the Brézis-Lieb lemma to prove the compactness of the (PS)$_{c}$ sequence if $\tilde{c} < c$ where $\tilde{c} = \frac{1}{n} S^*_{\alpha}$ and $S = \inf_{u \in D^{1,2}(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dx}{\int_{\mathbb{R}^n} |u|^{2^*} dx}$. It seems that the method of [22] does not apply to (1.1), as the Brézis-Lieb type lemma would lead to a system of inequalities which does not have explicit nontrivial solutions. Thirdly, there is an asymptotic competition between the energy carried by the two critical nonlinearities, so we have trouble in ruling out the “vanishing” of the (PS) sequence. Naturally, we would hope to overcome this difficulty by using the Nehari manifold method as in [5], but unfortunately, the corresponding limit equation does not exist, since

$$(-\Delta)^{s} v = \left( \int_{\mathbb{R}^n} \frac{|v(y)|^{2^*} \alpha}{|x-y|^{n+2s}} dy \right) \frac{|v(x)|^{2^*} - 2 v(x)}{|x|^{n \alpha}}$$  \quad (1.7)

is not translation invariant. To see this, let us go back to equation (1.5):

$$(-\Delta)^{s} u - \gamma \frac{u}{|x|^{\alpha}} = \frac{|u|^{2^*_s(\beta)-2} u}{|x|^{\beta}} + (I_{\mu} * |u|^{2^*}) |u|^{2^* - 2} u,$$

which is similar to equation (1.1). Here, the authors in [5] obtained a nontrivial weak solution to (1.5) by using the Nehari manifold method. The key step was to rule out the “vanishing” of the corresponding (PS) sequence by using the limit equation of (1.5). As can be seen in Section 3 in [5], there exists a (PS) sequence \{\nu_k\} for the energy functional corresponding to (1.5) such that $\nu_k \rightharpoonup \nu$ in $\dot{H}^s(\mathbb{R}^n)$ with $\nu$ solving (1.5). It may occur that $\nu \equiv 0$. Taking $v_k(x) = \lambda_{\nu_k} \nu_k(\lambda_{\nu_k} x + x_k)$ where $\lambda_{\nu_k} > 0$, $x_k \in \mathbb{R}^n$ and $\lambda_{\nu_k} \to \infty$ as $k \to +\infty$, they derived

$$\int_{\mathbb{R}^n} \frac{v_k \phi}{|x + \frac{x_k}{\lambda_{\nu_k}}|^{2s}} \to 0, \quad \int_{\mathbb{R}^n} \frac{|v_k|^{2^*_s(\beta)-2} v_k \phi}{|x + \frac{x_k}{\lambda_{\nu_k}}|^{\beta}} \to 0 \quad \text{as} \quad k \to +\infty$$

for any $\phi \in C_0^\infty(\mathbb{R}^n)$. Then $\nu$ weakly solves

$$(-\Delta)^s \nu = (I_{\mu} * |\nu|^{2^*}) |\nu|^{2^* - 2} \nu.$$  \quad (1.8)

Using the limit equation (1.8), they ruled out the “vanishing” of the (PS) sequence for the energy functional corresponding to (1.5). Clearly, this method does not apply to (1.1), since (1.7) is not translation invariant.

For these reasons, we use a direct way of proving Theorem 1.1. The crucial point is the utilization of the embeddings (see Section 3)

$$\dot{H}^s(\mathbb{R}^n) \hookrightarrow L^{2^*_s(\alpha)}(\mathbb{R}^n, |y|^{-\alpha}) \hookrightarrow L^{p,\frac{n-2s}{n-2s}p+r}(\mathbb{R}^n, |y|^{-p})$$  \quad (1.9)

for $s \in (0,1)$, $0 < \alpha < 2s < n$, $2^*_s(\alpha) = \frac{2n-\alpha}{n-2s}$, $p \in [1,2^*_s(\alpha))$ and $r = \frac{\alpha}{2^*_s(\alpha)}$, and the following improved Sobolev inequalities:

**Proposition 1.3** Let $s \in (0,1)$ and $0 < \alpha < 2s < n$. Then there exists $C = C(n, s, \alpha) > 0$ such that for any $\theta \in (0,1)$ and for any $p \in [1,2^*_s(\alpha))$, it holds that

$$\left( \int_{\mathbb{R}^n} \frac{|u(y)|^{2^*_s(\alpha)}}{|y|^{\alpha}} dy \right)^{\frac{1}{2^*_s(\alpha)}} \leq C ||u||_{H^s(\mathbb{R}^n)} ||u||_{L^p,\frac{n-2s}{n-2s}p+r}(\mathbb{R}^n, |y|^{-p})^{1-\theta}, \quad \forall u \in \dot{H}^s(\mathbb{R}^n),$$  \quad (1.10)

where $\tilde{\theta} = \max\{\frac{2}{2^*_s(\alpha)}, \frac{2^*_s-1}{2^*_s(\alpha)}\} > 0$ and $r = \frac{\alpha}{2^*_s(\alpha)}$. 

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Corollary 1.4 Let \( n \geq 2, 1 < p < n \) and \( 0 < \alpha < p \). Then there exists \( C = C(n, p, \alpha) > 0 \) such that for any \( \theta \in (\tilde{\theta}, 1) \) and for any \( m \in [1, p^*(\alpha)) \), it holds that

\[
\left( \int_{\mathbb{R}^n} \frac{|u(y)|^{p^*(\alpha)}}{|y|^m} \, dy \right)^{\frac{1}{m}} \leq C \|u\|_{D^{1,p}(\mathbb{R}^n)}^{\theta} \|u\|_{L^m,\mathbb{R}^n,|y|^{-m}}^{1-\theta}, \quad \forall u \in D^{1,p}(\mathbb{R}^n),
\]

where \( \tilde{\theta} = \max\{p^*(\alpha), \frac{n}{p^*(\alpha)}\} > 0 \) and \( r = \frac{\alpha}{p^*(\alpha)} \).

Remark 1.5 Proposition 1.3 and Corollary 1.4 are more general than Theorems 1–2 in Palatucci, Pisante in [1]; the detailed proof of this will be given in Section 3.

Now, we give the outline of the proof for Theorem 1.1. We use the mountain pass lemma to find critical points of \( I(u) \) on \( H^s(\mathbb{R}^n) \), which correspond to weak solutions for equation (1.1). Since problem (1.1) includes double critical exponents, we require the mountain pass level \( c < c^* \) for some suitable threshold value \( c^* \). This is crucial in ruling out the “vanishing” of the corresponding (PS) sequence. To this end, we introduce the minimization problems

\[
S_\mu(n, s, \gamma, \alpha) = \inf_{u \in H^s(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 \, dx - \gamma \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^{2s}} \, dx}{B_\alpha(u, u)^{\frac{n}{2s}}} \tag{1.12}
\]

and

\[
\Lambda(n, s, \gamma, \alpha) = \inf_{u \in H^s(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 \, dx - \gamma \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^{2s}} \, dx}{\int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|^{2s}} \, dx^{\frac{n}{2s}}} \tag{1.13}
\]

where \( B_\alpha(\cdot, \cdot) \) was defined in (1.2). Using the minimizers of \( S_\mu(n, s, \gamma, \alpha) \) and \( \Lambda(n, s, \gamma, \alpha) \), we can prove the mountain pass level \( c < c^* \) where

\[
c^* := \min \left\{ \frac{2\#(\alpha)}{2 \cdot 2}\bigg( S_\mu(n, s, \gamma, \alpha)^{\frac{n}{2s}} - \frac{s - 2}{2(n - 2s)} \Lambda(n, s, \gamma, \alpha)^{\frac{n}{2s}} \right\}.
\]

Then, the Mountain Pass Lemma gives a (PS)\(_c\) sequence \( \{u_k\}_{k=1}^{+\infty} \) for \( I(\cdot) \) at level \( c > 0 \), i.e.,

\[
\lim_{k \to +\infty} I(u_k) = c \quad \text{and} \quad \lim_{k \to +\infty} I'(u_k) = 0 \quad \text{strongly in} \quad H^s(\mathbb{R}^n)'.
\tag{1.14}
\]

Clearly, \( \{u_k\} \) is bounded, so we may assume that \( u_k \rightharpoonup u \) in \( H^s(\mathbb{R}^n) \) for some \( u \in H^s(\mathbb{R}^n) \). However, it may occur that \( u \equiv 0 \). Denote

\[
d_1 = \lim_{k \to +\infty} \int_{\mathbb{R}^n} \frac{|u_k|^{2^*_s(\beta)}}{|x|^\beta} \, dx, \quad d_2 = \lim_{k \to +\infty} B_\alpha(u_k, u_k).
\]

From (1.12), (1.13) and (1.14), we have

\[
d_1^{\frac{1}{2^*_s(\beta)}} A_1 \leq d_2, \quad d_2^{\frac{1}{2^*_s(\alpha)}} A_2 \leq d_1,
\tag{1.15}
\]

where \( A_1 = \Lambda(n, s, \gamma, \beta) - \frac{2(n-\beta)}{2n} c \frac{2^*_s(\beta)}{2^*_s(\alpha)} \) and \( A_2 = S_\mu(n, s, \gamma, \alpha) - \frac{2(n-\beta)}{2n} c \frac{2^*_s(\beta)}{2^*_s(\alpha)} \). Since \( c < c^* \), we derive that \( A_1 > 0 \) and \( A_2 > 0 \). Thus (1.15) implies that \( d_1 \geq \varepsilon_0 > 0 \) and \( d_2 \geq \varepsilon_0 > 0 \) (if \( d_1 = 0 \) and \( d_2 = 0 \), then \( c = 0 \), which is a contradiction), i.e.,

\[
\lim_{k \to +\infty} \int_{\mathbb{R}^n} \frac{|u_k|^{2^*_s(\beta)}}{|x|^\beta} \, dx \geq \varepsilon_0 > 0, \quad \lim_{k \to +\infty} B_\alpha(u_k, u_k) \geq \varepsilon_0 > 0,
\]

so the embeddings (1.9) and the improved Sobolev inequality (1.10) imply that

\[
0 < C \leq \|u_k\|_{L^\infty,\mathbb{R}^n,|y|^{-2r}} \leq C \quad \text{for any} \quad k \geq K \quad \text{large},
\]

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where \( r = \frac{\alpha}{2^*(\alpha)} \) and \( C > 0 \) is a constant. For any \( k \geq K \) large, we may find \( \lambda_k > 0 \) and \( x_k \in \mathbb{R}^n \) such that
\[
\lambda_k^{-2s+2r} \int_{B_{\lambda_k}(x_k)} \frac{|u_k(y)|^2}{|y|^{2r}} dy > \| u_k \|_{L^{n, \infty-2s+2r}(\mathbb{R}^n, |y|^{-2r})}^2 - \frac{C}{2k} > C_1 > 0.
\]
Letting \( v_k(x) = \lambda_k^{n-2s} u_k(\lambda_k x) \), we have \( v_k \rightharpoonup v \neq 0 \) in \( \dot{H}^s(\mathbb{R}^n) \). In fact, we can prove that \( \{ \tilde{x}_k = \frac{x_k}{\lambda_k} \} \) is bounded and that
\[
\int_{B_1(\tilde{x}_k)} \frac{|v_k(x)|^2}{|x|^{2r}} dx \geq C_1 > 0.
\]
From (1.16), we have \( \int_{B_R(0)} \frac{|v(x)|^2}{|x|^{2r}} dx \geq C_1 > 0 \), since \( r < s \). Moreover, we can check that \( \{ v_k \} \) is a new (PS) sequence for \( I(\cdot) \) at the same energy level \( c \), so \( v \neq 0 \) solves (1.1).

It remains to deal with the minimization problems (1.12)–(1.13). To this end, we need some kind of compactness. When \( \alpha = 0 \), we use the method introduced by Filippucci et al. in [3] or Dipierro et al. in [23] to prove the existence of minimizers for \( S_\mu(n, s, \gamma, 0) \). Next, we focus on the case of \( \alpha > 0 \). Both Ghoussoub et al. in [2] and Filippucci et al. in [3] use truncation skills and a careful analysis of concentration to eliminate the “vanishing” of the corresponding minimizing sequence. This would inevitably lead to tedious and complex calculations. In addition, the authors in [2] and [11] had to work in the extension space \( X^*(\mathbb{R}_+^{n+1}) \) to deal with the non-local operator \( (-\Delta)^s \). If \( \alpha > 0 \), the embeddings (1.9) and the inequality (1.10) allow us to adopt a direct but easier way to prove the existence of minimizers for \( S_\mu(n, s, \gamma, \alpha) \) and \( \Lambda(n, s, \gamma, \alpha) \) in \( \dot{H}^s(\mathbb{R}^n) \). Moreover, (1.9) and (1.10) are very useful to rule out the “vanishing” of the corresponding (PS) sequence. As far as we know, the strategy we adopt is new; we do not use truncation skills nor do we work in measurable.

“Palais-Smale” as (PS) for short. \( \mathbb{N} = \{1, 2, \cdots \} \) is the set of natural numbers. \( \mathbb{R} \) and \( \mathbb{C} \) denote the sets of real and complex numbers respectively. By saying that a function is “measurable”, we always mean that the function is “Lebesgue” measurable. “\( \wedge \)” denotes the Fourier transform and “\( \vee \)” denotes the inverse Fourier transform.
Moreover, if $\gamma < \gamma_F$ from Lemma 2.1, we have

$$\|u\|_{\Omega} = \left\| \int_{\Omega} u(x)^2 dx \right\|$$

equivalent norm on $\dot{H}^s(\mathbb{R}^n)$, $\forall u \in \dot{H}^s(\mathbb{R}^n)$,

(2.1)

where $\gamma_H := 4^s \frac{\Gamma^2(\frac{n+2s}{n})}{\Gamma^2(\frac{n}{n})}$ is the best constant in the above inequality on $\mathbb{R}^n$.

Lemma 2.2 (fractional Hardy-Sobolev inequalities: Lemma 2.1 of [2]) Let $s \in (0, 1)$ and $0 \leq \alpha \leq 2s < n$. Then there exist positive constants $c$ and $C$ such that

$$\left( \int_{\mathbb{R}^n} \frac{|u|^{2^*_s(\alpha)}}{|x|^\alpha} dx \right)^{\frac{1}{2^*_s(\alpha)}} \leq C \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 dx - \gamma \int_{\mathbb{R}^n} \frac{u^2}{|x|^{2s}} dx, \quad \forall u \in \dot{H}^s(\mathbb{R}^n).$$

(2.2)

Moreover, if $\gamma < \gamma_H = 4^s \frac{\Gamma^2(\frac{n+2s}{n})}{\Gamma^2(\frac{n}{n})}$, then

$$C \left( \int_{\mathbb{R}^n} \frac{|u|^{2^*_s(\alpha)}}{|x|^\alpha} dx \right)^{\frac{1}{2^*_s(\alpha)}} \leq \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 dx - \gamma \int_{\mathbb{R}^n} \frac{u^2}{|x|^{2s}} dx, \quad \forall u \in \dot{H}^s(\mathbb{R}^n).$$

(2.3)

From Lemma 2.1, the following inequality holds for all $\gamma < \gamma_H$ and any $u \in \dot{H}^s(\mathbb{R}^n)$:

$$\left( 1 - \frac{\gamma}{\gamma_H} \right) \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 dx \leq \|u\|^2 \leq \left( 1 + \frac{\gamma}{\gamma_H} \right) \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 dx,$$

(2.4)

where $\|u\| = \left( \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 dx - \gamma \int_{\mathbb{R}^n} \frac{u^2}{|x|^{2s}} dx \right)^{\frac{1}{2}}$ and $\gamma_{\pm} = \max\{\pm \gamma, 0\}$. We define an equivalent norm on $\dot{H}^s(\mathbb{R}^n)$ by $\| \cdot \|$ and denote the inner product of $u, v \in \dot{H}^s(\mathbb{R}^n)$ by

$$\langle u, v \rangle = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}} v dx - \gamma \int_{\mathbb{R}^n} \frac{uv}{|x|^{2s}} dx.$$

Lemma 2.3 Let $s \in (0, 1)$ and $0 < r < s < \frac{n}{2}$. If $\{u_k\}$ is a bounded sequence in $\dot{H}^s(\mathbb{R}^n)$ and $u_k \rightharpoonup u$ in $\dot{H}^s(\mathbb{R}^n)$, then

$$\frac{u_k}{|x|^r} \rightharpoonup \frac{u}{|x|^r} \quad \text{in} \quad L^q_{\text{loc}}(\mathbb{R}^n).$$

Proof Since $u_k \rightharpoonup u$ in $\dot{H}^s(\mathbb{R}^n)$, by Corollary 7.2 in [4], we have

$$u_k \rightarrow u \quad \text{in} \quad L^q_{\text{loc}}(\mathbb{R}^n) \quad (1 \leq q < 2^*_s) \quad \text{and} \quad u_k \rightharpoonup u \quad \text{a.e.} \quad \text{on} \quad \mathbb{R}^n.$$

From Lemma 2.1, we have $\int_{\Omega} \frac{|u_k|^2}{|x|^{2q}} dx \leq C_{s,n} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u_k|^2 dx \leq \tilde{C}$. For any compact set $\Omega \subset \mathbb{R}^n$, using Hölder’s inequality, we have

$$\int_{\Omega} \frac{|u_k - u|^2}{|x|^{2s}} dx \leq \left( \int_{\Omega} \frac{|u_k - u|^2}{|x|^{2s}} dx \right)^{\frac{s}{2}} \left( \int_{\Omega} |u_k - u|^2 dx \right)^{\left(1 - \frac{s}{2}\right)} \leq C \left( \int_{\Omega} |u_k - u|^2 dx \right)^{\left(1 - \frac{s}{2}\right)} \rightarrow 0.$$

□
Proposition 2.4 (Hardy-Littlewood-Sobolev inequality, Theorem 4.3 in [27]) Let $t, r > 1$ and $\mu \in (0, n)$ with $\frac{1}{r} + \frac{\mu}{n} + \frac{1}{t} = 2$, $f \in L^t(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$. There exists a sharp constant $C(t, n, \mu, r)$, independent of $f$, $h$ such that
\[
\left\| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x) h(y)}{|x - y|^\mu} \, dx \, dy \right\| \leq C(t, n, \mu, r) \| f \|_{L^t(\mathbb{R}^n)} \| h \|_{L^r(\mathbb{R}^n)}. \tag{2.5}
\]
If $t = r = \frac{2n}{2n - \mu}$, then $C(t, n, \mu, r) = C(n, \mu) = \pi^\frac{\mu}{2} \left( \frac{\Gamma(\frac{\mu}{2})}{\Gamma(\frac{n - \mu}{2})} \right)^\frac{1}{r}$. In this case, there is equality in (2.5) if and only if $f \equiv \text{constant} h$ and $h(x) = A(\varepsilon^2 + |x - a|^2)^{-\frac{2n - \mu}{4}}$ for some $A \in \mathbb{C}$, $\neq 0$ and $\varepsilon \in \mathbb{R}$ and $a \in \mathbb{R}^n$.

Let $s \in (0, 1)$, $0 \leq \alpha < 2s < n$, $\mu \in (0, n)$. $\forall \varepsilon \in \mathbb{H}^s(\mathbb{R}^n)$, and take $t = r = \frac{2n}{2n - \mu}$ > 1 and $f(\cdot) = h(\cdot) = \frac{|u(\cdot)|^{2\beta(\cdot)}}{\int_{\mathbb{R}^n} |u(\cdot)|^{2\beta(\cdot)} \, dx}$ in (2.5). Then Lemma 2.2 implies that $f, h \in L^\frac{2n}{2n - \mu}(\mathbb{R}^n)$, and for the $B_\alpha(h, \cdot)$ introduced in (1.2), we have
\[
B_\alpha(u, u) \leq C(n, \mu) \left( \int_{\mathbb{R}^n} |u|^{2\beta(\cdot)} \, dx \right)^{\frac{2n - \mu}{n}} \leq C\| u \|_{H^s(\mathbb{R}^n)}, \quad \forall u \in \mathbb{H}^s(\mathbb{R}^n). \tag{2.6}
\]

Lemma 2.5 (A variant of Brezis-Lieb lemma) Let $r > 1$, $q \in [1, r]$ and $\delta \in [0, nq/r)$. Assume that $\{u_k\}$ is a bounded sequence in $L^r(\mathbb{R}^n, |x|^{-\delta r/q})$ and $u_k \rightharpoonup w$ a.e. on $\mathbb{R}^n$. Then,
\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \left| \frac{|u|^q}{|x|^{\delta}} - \frac{|u_k - w|^q}{|x|^{\delta}} - \frac{|u|^q}{|x|^{\delta}} \right|^{\frac{q}{q - 1}} = 0,
\]
\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \left| \frac{|w_k|^q}{|x|^{\delta}} - \frac{|w_k - w|^q}{|x|^{\delta}} - \frac{|u|^q}{|x|^{\delta}} \right|^{\frac{q}{q - 1}} = 0.
\]

Proof For the case of $\delta = 0$, one can refer to Lemma 2.3 in [28]: we focus on the case of $\delta > 0$. Fixing $\varepsilon > 0$, small, there exists $C(\varepsilon) > 0$ such that for all $a, b \in \mathbb{R}$, we have
\[
|a + b|^q - |a|^q \leq \varepsilon |a|^q + C(\varepsilon)|b|^q, \quad |a + b|^{q - 1}(a + b) - |a|^{q - 1}a \leq \varepsilon |a|^q + C(\varepsilon)|b|^q.
\]
Using the inequality $(a + b)^p \leq 2^{p - 1}(a^p + b^p)$ for $a, b \geq 0$ and $p \geq 1$, we obtain
\[
|a + b|^q - |a|^q \leq \left( \frac{\varepsilon |a|^q + C(\varepsilon)|b|^q}{|b|^q} \right)^\frac{1}{q} \leq \varepsilon |a|^{r'} + \tilde{C}(\varepsilon)|b|^{r'}, \tag{2.7}
\]
and
\[
|a + b|^{q - 1}(a + b) - |a|^{q - 1}a \leq \left( \frac{\varepsilon |a|^q + C(\varepsilon)|b|^q}{|b|^q} \right)^\frac{1}{q - 1} \leq \varepsilon |a|^{r'} + \tilde{C}(\varepsilon)|b|^{r'}, \tag{2.8}
\]
where $\varepsilon = 2^{q - 1} - q^q$ and $\tilde{C}(\varepsilon) = 2^{q - 1} - C(\varepsilon)|b|^{q - 1}$, taking $a = \frac{u_k - w}{|u_k - w|^p}$, $b = w$ in (2.7) and (2.8), respectively. The rest is similar to the proof of Lemma 2.3 in [28], so we omit the details. \hfill \Box

Lemma 2.6 (Weak Young inequality, Section 4.3 in [27]) Let $n \in \mathbb{N}$, $\mu \in (0, n)$, $\hat{p}, \hat{r} > 1$ and $\frac{1}{\hat{r}} + \frac{\mu}{n} + \frac{1}{\hat{p}} = 1$. If $v \in L^\hat{r}(\mathbb{R}^n)$, then $I_\mu v \in L^\hat{r}(\mathbb{R}^n)$ and
\[
\left( \int_{\mathbb{R}^n} |I_\mu v|^r \right)^{\frac{1}{r}} \leq C(n, \mu, \hat{p}) \left( \int_{\mathbb{R}^n} |v|^p \right)^{\frac{1}{p}}, \tag{2.9}
\]
where $I_\mu(x) = |x|^{-\mu}$. In particular, we can set $\hat{r} = \frac{nq}{n - (\frac{n}{p})\mu}$ for $\hat{p} \in (1, \frac{n}{\mu} - 1)$.\hfill \Box

Lemma 2.7 (Brezis-Lieb type lemma, Lemma 2.4 in [29]) Let $n \in \mathbb{N}$, $\mu \in (0, n)$, $\frac{2n - \mu}{2n} \leq p < \infty$ and let $\{u_k\}_{k \in \mathbb{N}}$ be a bounded sequence in $L^\frac{2n}{2n - \mu}(\mathbb{R}^n)$. If $u_k \rightharpoonup u$ a.e. on $\mathbb{R}^n$ as $k \to \infty$, then
\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \left[ (I_\mu |u_k|^p) |u_k|^p - (I_\mu |u_k - u|^p) |u_k - u|^p \right] = \int_{\mathbb{R}^n} (I_\mu |u|^p) |u|^p. \tag{2.10}
\]
Lemma 2.8 Let $s \in (0,1)$, $0 \leq \alpha < 2s < n$ and $\mu \in (0,n)$. If $\{u_k\}_{k \in \mathbb{N}}$ is a bounded sequence in $\dot{H}^s(\mathbb{R}^n)$ and $u_k \to u$ in $\dot{H}^s(\mathbb{R}^n)$, then we have
\[
\lim_{k \to \infty} B_\alpha(u_k, u_k) = \lim_{k \to \infty} B_\alpha(u_k - u, u_k - u) + B_\alpha(u, u),
\]
where $B_\alpha(\cdot, \cdot)$ was defined in (1.2).

**Proof** For $s \in (0,1)$, $0 \leq \alpha < 2s < n$ and $\mu \in (0,n)$, we can check that $\frac{2n - n}{2^\#(\alpha)} < 1 < 2^\#(\alpha)$. Therefore, taking $p = 2^\#(\alpha)$ in Lemma 2.7, we have $\frac{2np}{2n - \mu} = 2^*(\alpha)$. Since $u_k \in \dot{H}^s(\mathbb{R}^n)$ and $u_k \to u$ in $\dot{H}^s(\mathbb{R}^n)$, the embedding $\dot{H}^s(\mathbb{R}^n) \hookrightarrow L^{2^*(\alpha)}(\mathbb{R}^n, |x|^{-\alpha})$ in Lemma 2.2 implies that
\[
\frac{u_k}{|x|^{\frac{\alpha}{2^\#(\alpha)}}} \to \frac{u}{|x|^{\frac{\alpha}{2^*(\alpha)}}} \quad \text{a.e. on } \mathbb{R}^n.
\]
Consequently, Lemma 2.7 gives the desired equality. \hfill \Box

Lemma 2.9 Let $s \in (0,1)$, $0 \leq \alpha < 2s < n$, $\mu \in (0,n)$ and let $\{u_k\}_{k \in \mathbb{N}}$ be a bounded sequence in $L^{2^*(\alpha)}(\mathbb{R}^n, |x|^{-\alpha})$. If $u_k \to u$ a.e. on $\mathbb{R}^n$ as $k \to \infty$, then for any $\phi \in L^{2^*(\alpha)}(\mathbb{R}^n, |x|^{-\alpha})$, we have
\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} [I_\mu * F_\alpha(\cdot, u_k)](x)f_\alpha(x, u_k)\phi(x)dx = \int_{\mathbb{R}^n} [I_\mu * F_\alpha(\cdot, u)](x)f_\alpha(x, u)\phi(x)dx,
\]
where $F_\alpha$ and $f_\alpha$ were introduced in (1.1).

**Proof** Since $\phi = \phi^+ - \phi^-$, we just consider $\phi \geq 0$. For $n \in \mathbb{N}$, we denote $\tilde{u}_k = u_k - u$ and rewrite the left hand side of (2.11) as
\[
\int_{\mathbb{R}^n} [I_\mu * F_\alpha(\cdot, u_k)](x)f_\alpha(x, u_k)\phi(x)dx \\
= \int_{\mathbb{R}^n} [I_\mu * (F_\alpha(\cdot, u_k) - F_\alpha(\cdot, \tilde{u}_k))](x)f_\alpha(x, u_k)\phi(x)dx \\
+ \int_{\mathbb{R}^n} [I_\mu * (f_\alpha(\cdot, u_k)\phi - f_\alpha(\cdot, \tilde{u}_k)\phi)](x)F_\alpha(x, \tilde{u}_k)dx \\
+ \int_{\mathbb{R}^n} [I_\mu * F_\alpha(\cdot, \tilde{u}_k)](x)f_\alpha(x, \tilde{u}_k)\phi(x)dx := \tilde{B}_1 + \tilde{B}_2 + \tilde{B}_3.
\]
Denote $p = 2^\#(\alpha)$ in this Lemma. Apply Lemma 2.5 with $(r, q, \delta) = (\frac{2np}{2n - \mu}, p, \delta_{\mu}(\alpha))$ by taking $(w_n, w) = (u_n, u)$ and then $(w_n, w) = (u_n, u^*_{\frac{\alpha}{2^\#(\alpha)}})$, and Lemma 2.6 with $\hat{p} = \frac{2n}{2n - \mu}$. We can then complete the proof by imitating the argument of Lemma 2.4 in [28]. \hfill \Box

3 Proof of Proposition 1.3 and Corollary 1.4

In this section, we give some basic properties of a weighted Morrey space and then prove Proposition 1.3 and Corollary 1.4.

The Morrey spaces were introduced by Morrey in 1938 [30] to investigate the local behavior of solutions to some partial differential equations. Nowadays the Morrey spaces are extended to more general cases (see [1, 31, 32]). Letting $p \in [1, +\infty)$ and $\gamma \in (0, n)$, the usual homogeneous Morrey space
\[
L^{p, \gamma}(\mathbb{R}^n) = \{ u : \|u\|_{L^{p, \gamma}(\mathbb{R}^n)} < +\infty \}.
\]
was introduced in [1] with the norm
\[
||u||_{L^p,\gamma}(\mathbb{R}^n) = \sup_{R > 0, x \in \mathbb{R}^n} \left\{ R^{\gamma - n} \int_{B_R(x)} |u(y)|^{\frac{p}{\gamma}} dy \right\}^{\frac{1}{p}}.
\]

One can see that if \( \gamma = n \), then \( L^{p,\gamma}(\mathbb{R}^n) \) coincides with \( L^p(\mathbb{R}^n) \) for any \( p \geq 1 \); similarly, \( L^{p,0}(\mathbb{R}^n) \) coincides with \( L^\infty(\mathbb{R}^n) \).

Here we mainly state a special weighted Morrey space \( L^{p,\gamma + \lambda}(\mathbb{R}^n, |y|^{-\lambda}) \), which was used in [31] and [32]. For \( p \in [1, +\infty) \), \( \gamma, \lambda > 0 \) and \( \gamma + \lambda \in (0, n) \), we say that a Lebesgue measurable function \( u : \mathbb{R}^n \to \mathbb{R} \) belongs to \( L^{p,\gamma + \lambda}(\mathbb{R}^n, |y|^{-\lambda}) \) if
\[
||u||_{L^{p,\gamma + \lambda}(\mathbb{R}^n, |y|^{-\lambda})} = \sup_{R > 0, x \in \mathbb{R}^n} \left\{ R^{\gamma + \lambda - n} \int_{B_R(x)} \frac{|u(y)|^{\frac{p}{\gamma}}}{|y|^{\lambda}} dy \right\}^{\frac{1}{p}} < +\infty.
\]

Then the following fundamental properties (1)–(5) hold via Hölder’s inequality:

1. If \( L^{p,\gamma}(\mathbb{R}^n, |y|^{-\rho \lambda}) \subseteq L^{p,\gamma + \lambda}(\mathbb{R}^n, |y|^{-\lambda}) \) for \( \rho = \frac{\gamma}{\gamma + \lambda} > 1 \);
2. For any \( p \in (1, +\infty) \), we have \( L^{p,\gamma + \lambda}(\mathbb{R}^n, |y|^{-\lambda}) \subseteq L^{1,\gamma + \lambda}(\mathbb{R}^n, |y|^{-\lambda}) \);
3. Take \( \gamma + \lambda = n \), we get \( L^{p}(\mathbb{R}^n, |y|^{-\lambda}) \).

Moreover, if we assume that \( s \in (0, 1) \) and \( 0 < \alpha < 2s < n \), then we have
4. For any \( p \in [1, 2^*_s(\alpha)) \), \( \dot{H}^s(\mathbb{R}^n) \subseteq L^{2^*_s(\alpha)}(\mathbb{R}^n, |y|^{-\alpha}) \subseteq L^{p,\frac{n-2s}{n-\alpha}+\alpha}(\mathbb{R}^n, |y|^{-\alpha}) \) with \( r = \frac{n}{n-\alpha} \), and the three norms in these spaces share the same dilatation invariance;
5. For any \( p \in [1, 2^*_s) \), \( \dot{H}^s(\mathbb{R}^n) \subseteq L^{2^*_s}(\mathbb{R}^n) \subseteq L^{p,\frac{n-2s}{n-\alpha}-\alpha}(\mathbb{R}^n) \), refer to page 815 in [1].

**Lemma 3.1** (Theorem 1 in [33], or Theorem D in [34]) Suppose that \( 0 < \tilde{s} < n, 1 < \tilde{p}' \leq \tilde{q} < +\infty, \tilde{p}' = \frac{\tilde{p}}{\tilde{p}-1} \) and that \( V \) and \( W \) are nonnegative measurable functions on \( \mathbb{R}^n \), \( n \geq 1 \). If, for some \( \sigma > 1 \),
\[
|Q|^{\frac{1}{\tilde{q}} + \frac{1}{\tilde{p}'}} \left( \frac{1}{|Q|} \int_Q V^\sigma dy \right)^{\frac{1}{\sigma \tilde{q}}} \left( \frac{1}{|Q|} \int_Q W^{(1-\tilde{p})\sigma} dy \right)^{\frac{1}{\sigma \tilde{p}'}} \leq C_\sigma \tag{3.1}
\]
for all cubes \( Q \subset \mathbb{R}^n \), then for any function \( f \in L^{\tilde{p}}(\mathbb{R}^n, W(y)) \), we have
\[
\left( \int_{\mathbb{R}^n} |\ell_{\tilde{s}} f(y)|^{\tilde{p}} W(y) dy \right)^{\frac{1}{\tilde{p}}} \leq C C_\sigma \left( \int_{\mathbb{R}^n} |f(y)|^{\tilde{p}} W(y) dy \right)^{\frac{1}{\tilde{p}}}, \tag{3.2}
\]
where \( C = C(\tilde{p}, \tilde{q}, n) \) and \( \ell_{\tilde{s}} f \) denotes the Riesz potential of order \( \tilde{s} \), namely
\[
\ell_{\tilde{s}} f(y) = \int_{\mathbb{R}^n} \frac{f(z)}{|y-z|^{n-\tilde{s}}} dz. \tag{3.3}
\]

**Remark 3.2** One can refer to [1] for more information about the Riesz potential.

**Proof of Proposition 1.3** For \( u \in \dot{H}^s(\mathbb{R}^n) \), we have \( \dot{g}(\xi) := |\xi|^s\hat{u}(\xi) \in L^2(\mathbb{R}^n) \) and
\[
||u||_{\dot{H}^s(\mathbb{R}^n)} = ||g||_{L^2(\mathbb{R}^n)},
\]
by Plancherel’s theorem. Thus, \( u(x) = (\frac{1}{|\xi|^s})^\gamma \ast g(x) = \ell_\gamma g(x) \), where
\[
\ell_\gamma g(x) = \int_{\mathbb{R}^n} \frac{g(z)}{|x-z|^{n-\gamma}} dz.
\]

First, take \( \tilde{s} = s, \tilde{p} = 2, \max\{2, 2^*_s - 1\} < \tilde{q} < 2^*_s(\alpha) \), \( W(y) \equiv 1 \), \( V(y) = \frac{|u(y)|^{\frac{\sigma}{2^*_s(\alpha)-\tilde{q}}}}{|y|^{\alpha}} \) and \( \sigma = \frac{1}{2^*_s(\alpha)-1} \) in Lemma 3.1. Then (3.1) becomes
\[
|Q|^{\frac{1}{\tilde{q}} + \frac{1}{\tilde{p}'}} \left( \frac{1}{|Q|} \int_Q V^\sigma dy \right)^{\frac{1}{\sigma \tilde{q}}} \leq C_\sigma. \tag{3.4}
\]
Secondly, we verify condition (3.1). For any fixed $x \in \mathbb{R}^n$, replacing $Q$ by ball $B_R(x)$, since $0 < [2s^*(\alpha) - \hat{q}]\sigma < 1$ and $\frac{t\sigma}{1 - [2s^*(\alpha) - \hat{q}]\sigma} < n$, we deduce, by Hörmander’s inequality, that

\[ R^{-n} \int_{B_R(x)} V^\sigma \, dy = R^{-n} \int_{B_R(x)} \frac{|u|^{2s^*(\alpha)-\hat{q}\sigma}}{|y|^{t\sigma}} \, dy = R^{-n} \int_{B_R(x)} \frac{1}{|y|^{(1-t)\sigma}} \left( \int_{B_R(x)} |u|^{2s^*(\alpha)-\hat{q}\sigma} \, dy \right) \]

\[ \leq R^{-n} \left( \int_{B_R(0)} \frac{dy}{|y|^{1-[2s^*(\alpha)-\hat{q}]\sigma}} \right)^{1-[2s^*(\alpha)-\hat{q}]\sigma} \left( \int_{B_R(x)} \frac{|u|^{2s^*(\alpha)-\hat{q}\sigma}}{|y|^{(1-t)\sigma}} \, dy \right) \]

\[ \leq CR^{-t\sigma-n[2s^*(\alpha)-\hat{q}\sigma]} \left( \int_{B_R(x)} \frac{|u|}{|y|^p} \, dy \right)^{2s^*(\alpha)-\hat{q}\sigma}, \]

where $t := \frac{\hat{q}}{2s^*(\alpha)}$ and $r := \frac{(1-t)\alpha}{2s^*(\alpha) - \hat{q}} = \frac{\alpha}{2s^*(\alpha)}$. Therefore,

\[ R^{s+\frac{n-\alpha}{2s^*(\alpha)-\hat{q}}} \left( R^{-n} \int_{B_R(x)} V^\sigma \, dy \right)^{\frac{1}{s}} \]

\[ \leq R^{s+\frac{n-\alpha}{2s^*(\alpha)-\hat{q}}} \left\{ CR^{-t\sigma-n[2s^*(\alpha)-\hat{q}\sigma]} \left( \int_{B_R(x)} \frac{|u|}{|y|^r} \, dy \right)^{2s^*(\alpha)-\hat{q}\sigma} \right\}^{\frac{1}{s}} \]

\[ \leq C \left\{ R^{s+\frac{n-2\alpha}{2s^*(\alpha)-\hat{q}}} R^{-n} \int_{B_R(x)} \frac{|u|}{|y|^r} \, dy \right\}^{\frac{2s^*(\alpha)-\hat{q}}{s}} \leq C||u||_{L^1_{\frac{n-2\alpha}{2s^*(\alpha)-\hat{q}}}(\mathbb{R}^n, |y|^{-r})} := C\sigma. \]

Since $u = \ell_s g$, and by Lemma 1.1,

\[ \int_{\mathbb{R}^n} \frac{|u(y)|^{2s^*(\alpha)}}{|y|^{\sigma}} \, dy = \int_{\mathbb{R}^n} \ell_s g(y)^2 V(y) \, dy \leq (CC_2)^2 ||g||_{L^2} \]

\[ \leq C||u||_{H^s(\mathbb{R}^n)} ||u||_{L^1_{\frac{n-2\alpha}{2s^*(\alpha)-\hat{q}}}(\mathbb{R}^n, |y|^{-r})}^{\theta}. \]

Then, for any $\theta = \frac{\hat{q}}{2s^*(\alpha)}$ satisfying $\max\{\frac{2}{2s^*(\alpha)}, \frac{2s^*(\alpha)-1}{2s^*(\alpha)}\} < \theta < 1$ and any $p \in [1, 2s^*(\alpha))$, we have

\[ \left( \int_{\mathbb{R}^n} \frac{|u(y)|^{2s^*(\alpha)}}{|y|^{\sigma}} \, dy \right)^{\frac{1}{2s^*(\alpha)}} \leq C||u||_{H^s(\mathbb{R}^n)} ||u||_{L^1_{\frac{n-2\alpha}{2s^*(\alpha)-\hat{q}}}(\mathbb{R}^n, |y|^{-r})}^{\frac{\theta}{1-\theta}}. \]

\[ \square \]

**Proof of Corollary 1.4**  For $n \geq 3$ and any $u \in C_0^\infty(\mathbb{R}^n)$, we have

\[ u(x) = \Delta^{-1} \Delta u = C_1 \int_{\mathbb{R}^n} \frac{\Delta u(y)}{|x - y|^{n-2}} \, dy = C_2 \int_{\mathbb{R}^n} \frac{(x - y) \nabla u(y)}{|x - y|^{n}} \, dy. \]

Thus

\[ |u(x)| \leq |C_2| \int_{\mathbb{R}^n} \frac{|
abla u(y)|}{|x - y|^{n-1}} \, dy \leq C\ell_1(|\nabla u|)(x), \]

where $C_1 = C_1(n)$, $C_2 = C_2(n)$ and $C = C(n) > 0$ are different constants. These inequalities hold for $n = 2$ via the logarithmic kernel (see [1]). By the density of $C_0^\infty(\mathbb{R}^n)$ in $D^{1,p}(\mathbb{R}^n)$, this is also true for any $u \in D^{1,p}(\mathbb{R}^n)(n \geq 2)$.

Take $\tilde{s} = 1$, $\tilde{p} = p > 1$, max$\{p, p^* - 1\} < \tilde{q} < p^*(\alpha)$, $W(y) \equiv 1$, $V(y) = \frac{|u(y)|^{p^*(\alpha)-\hat{q}}}{|y|^\hat{q}}$ and $\sigma = \frac{1}{p^* - \hat{q}} > 1$ in Lemma 3.1. The remaining argument is similar to the case in $H^s(\mathbb{R}^n)$. \[ \square \]
Lemma 3.3 (Theorem 1 in [1]) Let $s \in (0,1)$, $n > 2s$ and $2^* = \frac{2n}{n - 2s}$. Then there exists $C = C(n, s) > 0$ such that for any $\theta \in \left(\frac{2}{2^*} \cdot 1 - \frac{1}{2} \right)$, we have
\[
\|u\|_{L^2(\mathbb{R}^n)} \leq C \|u\|_{\dot{H}^{\theta}(\mathbb{R}^n)}^{\theta} \|u\|_{L^{2n/(n-2s)}(\mathbb{R}^n)}^{1-\theta}, \quad \forall u \in \dot{H}^{\theta}(\mathbb{R}^n). \tag{3.5}
\]

Remark 3.4 If $\alpha = 0$ in Proposition 1.3, then inequality (1.10) becomes inequality (3.5).

4 Solving the Minimization Problems (1.12)–(1.13)

In this section, we solve the minimization problems (1.12)–(1.13). Using the embeddings (1.9) and the inequality (1.10), we can prove the existence of minimizers for
\[
S_\mu(n, s, \gamma, \alpha) = \inf_{u \in \dot{H}^s(\mathbb{R}^n) \setminus \{0\}} \frac{||u||^2}{B_\alpha(u, u)^{2/(\gamma \alpha)}}
\]
and
\[
\Lambda(n, s, \gamma, \alpha) = \inf_{u \in \dot{H}^s(\mathbb{R}^n) \setminus \{0\}} \left( \frac{||u||^2}{\int_{\mathbb{R}^n} |u|^{2\alpha} \frac{dx}{|x|^\alpha}} \right)^{1/(\gamma \alpha)},
\]
where $B_\alpha(\cdot, \cdot)$ was defined in (1.2). We can derive the following results:

Proposition 4.1 Let $s \in (0,1)$. Then,

1. If $0 < \alpha < 2s < n$, $\mu \in (0, n)$ and $\gamma < \gamma_H$, $S_\mu(n, s, \gamma, \alpha)$ is attained in $\dot{H}^s(\mathbb{R}^n)$;
2. If $n > 2s$, $\mu \in (0, n)$ and $0 < \gamma < \gamma_H$, $S_\mu(n, s, \gamma, 0)$ is attained in $\dot{H}^s(\mathbb{R}^n)$;
3. If $0 < \alpha < 2s < n$ and $\gamma < \gamma_H$, $\Lambda(n, s, \gamma, \alpha)$ is attained in $\dot{H}^s(\mathbb{R}^n)$;
4. If $n > 2s$ and $0 < \gamma < \gamma_H$, $\Lambda(n, s, \gamma, 0)$ is attained in $\dot{H}^s(\mathbb{R}^n)$.

Remark 4.2 We only prove (1)–(2) in this section, since the strategy can be applied to prove (3)–(4); although (3) has been proved in [2], our method is more direct and effective, as we can derive $S_\mu(n, s, \gamma, \alpha) \geq \frac{\Lambda(n, s, \gamma, \alpha)}{C(n, \mu)^{2/(\gamma \alpha)}}$ and $S_\mu(n, s, 0, 0) = \frac{\Lambda(n, s, 0, 0)}{C(n, \mu)^{2/(\gamma \alpha)}}$ from (2.6).

Proof of Proposition 4.1 (1) If $0 < \alpha < 2s < n$ and $\gamma < \gamma_H$, let $\{u_k\}$ be a minimizing sequence of $S_\mu(n, s, \gamma, \alpha)$, that is,
\[
B_\alpha(u_k, u_k) = 1, \quad \|u_k\|^2 \to S_\mu(n, s, \gamma, \alpha).
\]
Then the embeddings (1.9), the improved Sobolev inequality (1.10), and (2.6) imply that there exists $C > 0$ such that
\[
0 < C \leq \|u_k\|_{L^{2n/(n-2s)}(\mathbb{R}^n)} \leq C^{-1},
\]
where $r = \frac{\alpha}{2\gamma(\alpha)}$. For any $k \geq 1$, we may find $\lambda_k > 0$ and $x_k \in \mathbb{R}^n$ such that
\[
\lambda_k^{-2s+2r} \int_{B_\lambda_k(x_k)} \frac{|u_k(y)|^2}{|y|^{2r}} dy \geq \|u_k\|_{L^{2n/(n-2s)}(\mathbb{R}^n)}^2 \geq \frac{C}{2k} \geq C_1 > 0.
\]
Let $v_k(x) = \frac{\lambda_k^{-s+2r}}{\lambda_k} u_k(\lambda_k x)$ and $\tilde{x}_k = \frac{x_k}{\lambda_k}$. Then
\[
\int_{B_1(\tilde{x}_k)} \frac{|v_k(x)|^2}{|x|^{2r}} dx \geq C_1 > 0. \tag{4.1}
\]
Since $S_\mu(n, s, \gamma, \alpha)$ is invariant under the previous dilation given by $\lambda_k$, we have
\[
B_\alpha(v_k, v_k) = 1, \quad \|v_k\|^2 \to S_\mu(n, s, \gamma, \alpha).
\]

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By Hölder’s inequality,

\[ 0 < C_1 \leq \int_{B_1(\tilde{x}_k)} \frac{|v_k(x)|^2}{|x|^{2r}} \, dx \leq \left( \int_{B_1(\tilde{x}_k)} \frac{1}{|x|^{2r}} \right)^{1-\frac{\alpha}{2^*_r}} \left( \int_{B_1(\tilde{x}_k)} \frac{|v_k(x)|^{2^*_r(\alpha)}}{|x|^\alpha} \, dx \right)^{\frac{2}{2^*_r}}.
\]

Therefore,

\[ \int_{B_1(\tilde{x}_k)} \frac{|v_k(x)|^{2^*_r(\alpha)}}{|x|^\alpha} \, dx \geq C > 0. \quad (4.2) \]

We claim that \( \{\tilde{x}_k\} \) is bounded. If, on the contrary, \( |\tilde{x}_k| \to +\infty \), then for any \( x \in B_1(\tilde{x}_k) \), \( |x| \geq |\tilde{x}_k| - 1 \) for \( k \) large. Therefore,

\[ \int_{B_1(\tilde{x}_k)} \frac{|v_k(x)|^{2^*_r(\alpha)}}{|x|^\alpha} \, dx \leq \frac{1}{(|\tilde{x}_k| - 1)^\alpha} \int_{B_1(\tilde{x}_k)} |v_k(x)|^{2^*_r(\alpha)} \, dx \]

\[ \leq \frac{C}{(|\tilde{x}_k| - 1)^\alpha} \left( \int_{B_1(\tilde{x}_k)} |v_k(x)|^{2^*_r} \, dx \right)^{\frac{\alpha}{2^*_r}} \]

\[ \leq \frac{C}{(|\tilde{x}_k| - 1)^\alpha} \left\| v_k \right\|_{H^s(\mathbb{R}^n)}^{2^*_r(\alpha)} \leq \frac{\tilde{C}}{(|\tilde{x}_k| - 1)^\alpha} \to 0, \]

as \( k \to +\infty \), which contradicts (4.2). Hence, \( \{\tilde{x}_k\} \) is bounded, and from (4.1) we may find \( R > 0 \) such that

\[ \int_{B_R(0)} \frac{|v_k(x)|^2}{|x|^{2r}} \, dx \geq C_1 > 0. \quad (4.3) \]

Since \( \left\| v_k \right\| = \left\| u_k \right\| \leq C \), there exists a \( v \in \dot{H}^s(\mathbb{R}^n) \) such that

\[ v_k \rightharpoonup v \quad \text{in} \quad \dot{H}^s(\mathbb{R}^n), \quad v_k \to v \quad \text{a.e.} \quad \text{on} \quad \mathbb{R}^n, \quad \text{up to subsequences.} \]

According to Lemma 2.3, we have \( \frac{\alpha}{|x|^r} \to \frac{\alpha}{|x|^s} \) in \( L^2_{\text{loc}}(\mathbb{R}^n) \), since \( r = \frac{\alpha}{2^*_r(\alpha)} < s \), therefore,

\[ \int_{B_R(0)} \frac{|v(x)|^2}{|x|^{2r}} \, dx \geq C_1 > 0, \]

and we deduce that \( v \neq 0 \). We may verify by Lemma 2.8 that

\[ 1 = B_\alpha(v_k, v_k) = B_\alpha(v_k - v, v_k - v) + B_\alpha(v, v) + o(1). \]

By the weak convergence \( v_k \rightharpoonup v \) in \( \dot{H}^s(\mathbb{R}^n) \),

\[ S_\mu(n, s, \gamma, \alpha) = \lim_{k \to \infty} \left| v_k \right|^2 = |v|^2 + \lim_{k \to \infty} \left| v_k - v \right|^2 \]

\[ \geq S_\mu(n, s, \gamma, \alpha) \left( B_\alpha(v, v) \right)^{\frac{1}{2^*_r(\alpha)}} \]

\[ + S_\mu(n, s, \gamma, \alpha) \left( \lim_{k \to \infty} B_\alpha(v_k - v, v_k - v) \right)^{\frac{1}{2^*_r(\alpha)}} \]

\[ \geq S_\mu(n, s, \gamma, \alpha) \left( B_\alpha(v, v) + \lim_{k \to \infty} B_\alpha(v_k - v, v_k - v) \right)^{\frac{1}{2^*_r(\alpha)}} \]

\[ = S_\mu(n, s, \gamma, \alpha). \]
Here we use the fact that $(a + b)\frac{1}{a+b} \leq a^{1/(a+b)} + b^{1/(a+b)}$, $\forall a \geq 0, b \geq 0$ and $2\#(\alpha) > 1$, so we have

$$B_\alpha(v, v) = 1, \quad \lim_{k \to \infty} B_\alpha(v_k - v, v_k - v) = 0,$$

since $v \not\equiv 0$. It turns out that

$$S_\mu(n, s, \gamma, \alpha) = ||v||^2, \quad \lim_{k \to \infty} ||v_k - v||^2 = 0.$$

By formula (A.11) in [35],

$$\int_{\mathbb{R}^n} |(-\Delta)^{s}|v|^2 dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v|^2 dx,$$

hence, $|v|$ is also a minimizer of $S_\mu(n, s, \gamma, \alpha)$, so we can assume that $v \geq 0$. Thus $S_\mu(n, s, \gamma, \alpha)$ is achieved if $0 < \alpha < 2s$ and $\gamma < \gamma_H$.

(2) If $\alpha = 0$ and $0 \leq \gamma < \gamma_H$, we are inspired by the method introduced by R. Filippucci et al. in [3] and S. Dipierro et al. in [23]. Let $\{u_k\}$ be a minimizing sequence of $S_\mu(n, s, \gamma, 0)$, that is,

$$B_0(u_k, u_k) = 1, \quad S_\mu(n, s, \gamma, 0) \leq ||u_k||^2 < S_\mu(n, s, \gamma, 0) + \frac{1}{k}.$$

From the fractional Polya-Szegö inequality in [36] and formula (A.11) in [35], we have

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} |u_k|^2 dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} |u_k|^2 dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u_k|^2 dx,$$

where $|u_k|^*$ is the symmetric decreasing rearrangement of $|u_k|$. Furthermore, it is clear (Theorem 3.4 in [27]) that

$$1 = B_0(|u_k|, |u_k|) \leq B_0(|u_k|^*, |u_k|^*), \quad \int_{\mathbb{R}^n} \frac{|u_k|^2}{|x|^{2s}} dx \leq \int_{\mathbb{R}^n} \frac{|u_k|^*|^2}{|x|^{2s}} dx.$$

Denoting $v_k := |u_k|^*$, $v_k$ is radial symmetric and decreasing. Since $0 \leq \gamma < \gamma_H$, we have that

$$S_\mu(n, s, \gamma, 0) \leq \frac{||v_k||^2}{B_0(v_k, v_k)^{\frac{2s}{n}}} \leq ||v_k||^2 \leq ||u_k||^2 < S_\mu(n, s, \gamma, 0) + \frac{1}{k}.$$

Therefore, $\{v_k\}$ is a minimizing sequence of $S_\mu(n, s, \gamma, 0)$ and $||v_k||$ is uniformly bounded. Noticing that $B_0(v_k, v_k) \geq 1$, the embeddings $H^s(\mathbb{R}^n) \hookrightarrow L^{2^*_s}(\mathbb{R}^n) \hookrightarrow L^{2, n-2s}(\mathbb{R}^n)$ (see Section 3), inequality (2.6), and Lemma 3.3 imply that there exists $C > 0$ such that

$$0 < C \leq ||v_k||_{L^{2, n-2s}(\mathbb{R}^n)} \leq C^{-1}.$$

Therefore we may find $\lambda_k > 0$ and $x_k \in \mathbb{R}^n$ such that

$$\lambda_k^{2s} \int_{B_{\lambda_k}(x_k)} |v_k(y)|^2 dy > ||v_k||_{L^{2, n-2s}(\mathbb{R}^n)}^2 - \frac{C}{2k} \geq C_1 > 0.$$

Letting $\tilde{v}_k(x) = \lambda_k^{-\frac{2s}{n}} v_k(\lambda_k x)$ and $\tilde{x}_k = \frac{x_k}{\lambda_k}$, we see that $\{\tilde{v}_k\}$ is also a minimizing sequence of $S_\mu(n, s, \gamma, 0)$ and satisfies

$$\int_{B_1(\tilde{x}_k)} |\tilde{v}_k(x)|^2 dx \geq C_1 > 0. \quad (4.5)$$

Since $||\tilde{v}_k|| = ||v_k|| \leq C$, there exists $\tilde{v} \in \dot{H}^s(\mathbb{R}^n)$ such that $\tilde{v}_k \to \tilde{v}$ in $\dot{H}^s(\mathbb{R}^n)$ up to subsequences, so we need to prove $\tilde{v} \not\equiv 0$. 

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When it comes to $\alpha > 0$, the energy functional associated to (1.1) is $4.1$ to prove the existence of a nontrivial weak solution for equation (1.1). Recall that the $0$ and $0$ $\alpha > 0$ $\leq 0$

**Proof of Theorem 1.1**

We shall now use the minimizers of $S_\mu(n, s, \gamma, 0)$, then we prove that $v_k = |u_k|^*$ is also a minimizing sequence of $S_\mu(n, s, \gamma, 0)$, since $0 \leq \gamma < \gamma_\mu$. Since $v_k$ is radial symmetric and decreasing, we can easily eliminate vanishing. If $\alpha > 0$ and $0 \leq \gamma < \gamma_\mu$, the same strategy can be applied to the proof of Proposition 4.1-(1). When it comes to $\alpha > 0$ and $\gamma < 0$, we fail to prove that $v_k = |u_k|^*$ is a minimizing sequence of $S_\mu(n, s, \gamma, \alpha)$, but (1.9) and (1.10) are very effective in this situation.

**5 Proof of Theorem 1.1**

We shall now use the minimizers of $S_\mu(n, s, \gamma, \alpha)$ and $\Lambda(n, s, \gamma, \beta)$ obtained in Proposition 4.1 to prove the existence of a nontrivial weak solution for equation (1.1). Recall that the energy functional associated to (1.1) is

$$I(u) = \frac{1}{2}||u||^2 - \frac{1}{2} \int_{\mathbb{R}^n} \frac{|u|^{2\beta}/\beta}{|x|^\beta} \, dx - \frac{1}{2} \cdot \frac{2^\#}{2} B_\alpha(u, u), \quad \forall u \in \dot{H}^s(\mathbb{R}^n),$$

(5.1)

where $B_\alpha(\cdot, \cdot)$ was defined in (1.2). Fractional Sobolev and Hardy-Sobolev inequalities yield that $I \in C^1(\dot{H}^s(\mathbb{R}^n), \mathbb{R})$ such that

$$\langle I'(u), \phi \rangle = \langle u, \phi \rangle - \int_{\mathbb{R}^n} \frac{|u|^{2\beta}/\beta - 2 \alpha}{|x|^\beta} u \phi \, dx - \int_{\mathbb{R}^n} \left[ I_\mu + F_\alpha(\cdot, u) \right](x) f_\alpha(x, u) \phi(x) \, dx.$$  

Note that a nontrivial critical point of $I$ is a nontrivial weak solution to equation (1.1).

**Remark 4.3** To prove Proposition 4.1-(2), firstly we choose a minimizing sequence $\{u_k\}$ of $S_\mu(n, s, \gamma, 0)$, then we prove that $v_k = |u_k|^*$ is a minimizing sequence of $S_\mu(n, s, \gamma, 0)$, since $0 \leq \gamma < \gamma_\mu$. Since $v_k$ is radial symmetric and decreasing, we can easily eliminate vanishing. If $\alpha > 0$ and $0 \leq \gamma < \gamma_\mu$, the same strategy can be applied to the proof of Proposition 4.1-(1). When it comes to $\alpha > 0$ and $\gamma < 0$, we fail to prove that $v_k = |u_k|^*$ is a minimizing sequence of $S_\mu(n, s, \gamma, \alpha)$, but (1.9) and (1.10) are very effective in this situation.

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(5.1)

where $B_\alpha(\cdot, \cdot)$ was defined in (1.2). Fractional Sobolev and Hardy-Sobolev inequalities yield that $I \in C^1(\dot{H}^s(\mathbb{R}^n), \mathbb{R})$ such that

$$\langle I'(u), \phi \rangle = \langle u, \phi \rangle - \int_{\mathbb{R}^n} \frac{|u|^{2\beta}/\beta - 2 \alpha}{|x|^\beta} u \phi \, dx - \int_{\mathbb{R}^n} \left[ I_\mu + F_\alpha(\cdot, u) \right](x) f_\alpha(x, u) \phi(x) \, dx.$$  

Note that a nontrivial critical point of $I$ is a nontrivial weak solution to equation (1.1).

**Lemma 5.1** (Mountain Pass Lemma, [37]) Let $(E, || \cdot ||)$ be a Banach space and let $I \in C^1(E, \mathbb{R})$ such that the following conditions are satisfied:

(1) $I(0) = 0$;
(2) There exist $\rho, r > 0$ such that $I(u) \geq \rho$ for all $u \in E$ with $||u|| = r$;

\[ \text{Springer} \]
(3) There exist \( v_0 \in E \) such that \( \lim_{t \to +\infty} \sup_{t \in [0,1]} I(tv_0) < 0 \).

Let \( t_0 > 0 \) be such that \( \|tv_0\| > r \) and \( I(t_0v_0) < 0 \), and define

\[
c := \inf_{g \in \Gamma} \sup_{t \in [0,1]} I(g(t)),
\]

where

\[
\Gamma := \left\{ g \in C^{0}([0,1], E) : g(0) = 0, g(1) = t_0v_0 \right\}.
\]

Then \( c \geq \rho > 0 \), and there exists a (PS) sequence \( \{u_k\} \subset E \) for \( I \) at level \( c \), i.e.,

\[
\lim_{k \to +\infty} I(u_k) = c \quad \text{and} \quad \lim_{k \to +\infty} I'(u_k) = 0 \quad \text{strongly in} \quad E'.
\]

We now use Lemma 5.1 to prove the following Propositions:

**Proposition 5.2** Let \( s \in (0, 1), 0 < \alpha, \beta < 2s < n, \mu \in (0, n) \) and \( \gamma < \gamma_H \). Consider the functional \( I \) defined in (5.1) on the Banach space \( H^s(\mathbb{R}^n) \). Then there exists a (PS) sequence \( \{u_k\} \subset H^s(\mathbb{R}^n) \) for \( I \) at some \( c \in (0, c^*) \), i.e.,

\[
\lim_{k \to +\infty} I(u_k) = c \quad \text{and} \quad \lim_{k \to +\infty} I'(u_k) = 0 \quad \text{strongly in} \quad H^s(\mathbb{R}^n)',
\]

where

\[
c^* := \min \left\{ \frac{2^\#(\alpha) - 1}{2 \cdot 2^\#(\alpha)} S(\mu, s, \gamma, \alpha)^{\frac{2^\#(\alpha)}{2s^*(\alpha)} - 1}, \frac{2s - \beta}{2(n - \beta)} \Lambda(n, s, \gamma, \beta)^{\frac{s - \alpha}{n - \beta}} \right\}.
\]

**Proof** We now verify the conditions of Lemma 5.1. For any \( u \in H^s(\mathbb{R}^n) \),

\[
I(u) = \frac{1}{2} \|u\|^2 - \frac{1}{2} \frac{2^*(\beta)}{\mu(\beta)} \int_{\mathbb{R}^n} \frac{|u|^{2^*(\beta)}}{|x|^\beta} \, dx - \frac{1}{2} \frac{2^\#(\alpha)}{\mu(\alpha)} B(\alpha, u, u)
\]

\[
\geq \frac{1}{2} \|u\|^2 - C_1 \|u\|^{2s^*(\beta)} - C_2 \|u\|^{2\mu^*(\alpha)}.
\]

Since \( s \in (0, 1), 0 < \alpha, \beta < 2s < n \) and \( \mu \in (0, n) \), we have that \( 2s^*(\beta) > 2 \) and \( 2\mu^*(\alpha) > 2s^*(\alpha) > 2 \). Therefore, there exists \( r > 0 \) small enough such that

\[
\inf_{\|u\|=r} I(u) > 0 = I(0),
\]

so (1) and (2) of Lemma 5.1 are satisfied.

From

\[
I(tu) = \frac{t^2}{2} \|u\|^2 - \frac{t^{2s^*(\beta)}}{\mu^*(\beta)} \int_{\mathbb{R}^n} \frac{|u|^{2s^*(\beta)}}{|x|^{\beta}} \, dx - \frac{t^{2\mu^*(\alpha)}}{\mu^*(\alpha)} B(\alpha, u, u),
\]

we derive that \( \lim_{t \to +\infty} I(tu) = -\infty \) for any \( u \in H^s(\mathbb{R}^n) \). Consequently, for any fixed \( v_0 \in H^s(\mathbb{R}^n) \), there exists \( t_{v_0} > 0 \) such that \( \|tv_0\| > r \) and \( I(tv_0v_0) < 0 \). Thus, (3) of Lemma 5.1 is satisfied.

Using (1) and (3) in Proposition 4.1, we obtain a minimizer \( U_{\gamma, \alpha} \in H^s(\mathbb{R}^n) \) for \( S(\mu, s, \gamma, \alpha) \) and \( V_{\gamma, \beta} \in H^s(\mathbb{R}^n) \) for \( \Lambda(n, s, \gamma, \beta) \), respectively. Thus there exist

\[
v_0 := \begin{cases} U_{\gamma, \alpha}, & \text{if} \quad \frac{2^\#(\alpha) - 1}{2 \cdot 2^\#(\alpha)} S(\mu, s, \gamma, \alpha)^{\frac{2^\#(\alpha)}{2s^*(\alpha)} - 1} \leq \frac{2s - \beta}{2(n - \beta)} \Lambda(n, s, \gamma, \beta)^{\frac{s - \alpha}{n - \beta}}; \\ V_{\gamma, \beta}, & \text{if} \quad \frac{2^\#(\alpha) - 1}{2 \cdot 2^\#(\alpha)} S(\mu, s, \gamma, \alpha)^{\frac{2^\#(\alpha)}{2s^*(\alpha)} - 1} > \frac{2s - \beta}{2(n - \beta)} \Lambda(n, s, \gamma, \beta)^{\frac{s - \alpha}{n - \beta}}. \end{cases}
\]
and \( t_0 > 0 \) such that \( ||t_0v_0|| > r \) and \( I(t_0v_0) < 0 \). We can define

\[
c := \inf_{g \in \Gamma} \sup_{t \in [0, 1]} I(g(t)),
\]

where

\[
\Gamma := \left\{ g \in C^0([0, 1], \dot{H}^s(\mathbb{R}^n)) : g(0) = 0, g(1) = t_0v_0 \right\}.
\]

Clearly, we have that \( c > 0 \). For the case of \( v_0 = U_{\gamma, \alpha} \), we can derive that

\[
0 < c < \frac{2^\#(\alpha) - 1}{2 \cdot 2^\#(\alpha)} S_\mu(n, s, \gamma, \alpha) \frac{2^\#(\alpha)}{2^\#(\alpha) - 1}.
\]

In fact, \( \forall t \geq 0 \), so we have that

\[
I(tU_{\gamma, \alpha}) \leq f_1(t) := \frac{t^2}{2} ||U_{\gamma, \alpha}||^2 - \frac{t^2 2^\#(\alpha)}{2 \cdot 2^\#(\alpha)} B_\alpha(U_{\gamma, \alpha}, U_{\gamma, \alpha}).
\]

Straightforward computations yield that \( f_1(t) \) attains its maximum at the point

\[
\hat{t} = \left( \frac{||U_{\gamma, \alpha}||^2}{B_\alpha(U_{\gamma, \alpha}, U_{\gamma, \alpha})} \right)^{\frac{1}{2[2^\#(\alpha) - 1]}}.
\]

and that

\[
\sup_{t \geq 0} f_1(t) = \frac{2^\#(\alpha) - 1}{2 \cdot 2^\#(\alpha)} S_\mu(n, s, \gamma, \alpha) \frac{2^\#(\alpha)}{2^\#(\alpha) - 1}.
\]

We obtain that

\[
\sup_{t \geq 0} I(tU_{\gamma, \alpha}) \leq \sup_{t \geq 0} f_1(t) = \frac{2^\#(\alpha) - 1}{2 \cdot 2^\#(\alpha)} S_\mu(n, s, \gamma, \alpha) \frac{2^\#(\alpha)}{2^\#(\alpha) - 1}. \tag{5.3}
\]

The equality does not hold in (5.3), otherwise, we would have that \( \sup_{t \geq 0} I(tU_{\gamma, \alpha}) = \sup_{t \geq 0} f_1(t) \).

Let \( t_1 > 0 \), where \( \sup_{t \geq 0} I(tU_{\gamma, \alpha}) \) is attained. We have

\[
f_1(t_1) - \frac{t_1^2}{2} \int_{\mathbb{R}^n} \frac{|U_{\gamma, \alpha}|^{2^*(\beta)}}{|x|^{2^*(\beta)}} \, dx = f_1(\hat{t}),
\]

which means that \( f_1(t_1) > f_1(\hat{t}) \), since \( t_1 > 0 \). This contradicts the fact that \( \hat{t} \) is the unique maximum point of \( f_1(t) \). Thus,

\[
\sup_{t \geq 0} I(tU_{\gamma, \alpha}) < \sup_{t \geq 0} f_1(t) = \frac{2^\#(\alpha) - 1}{2 \cdot 2^\#(\alpha)} S_\mu(n, s, \gamma, \alpha) \frac{2^\#(\alpha)}{2^\#(\alpha) - 1}. \tag{5.4}
\]

Similarly, for the case of \( v_0 = V_{\gamma, \beta} \), we can verify that

\[
\sup_{t \geq 0} I(tV_{\gamma, \beta}) < \frac{2s - \beta}{2(n - \beta)} \Lambda(n, s, \gamma, \beta)^\frac{2s - \beta}{2(n - \beta)}, \tag{5.5}
\]

and thus that \( 0 < c < \frac{2s - \beta}{2(n - \beta)} \Lambda(n, s, \gamma, \beta)^\frac{2s - \beta}{2(n - \beta)}. \)

From (5.4) and (5.5), we have

\[
0 < c < c^* := \min \left\{ \frac{2^\#(\alpha) - 1}{2 \cdot 2^\#(\alpha)} S_\mu(n, s, \gamma, \alpha) \frac{2^\#(\alpha)}{2^\#(\alpha) - 1}, \frac{2s - \beta}{2(n - \beta)} \Lambda(n, s, \gamma, \beta)^\frac{2s - \beta}{2(n - \beta)} \right\}.
\]

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Since (1)–(3) of Lemma 5.1 are satisfied, there exists a sequence \( \{u_k\} \subset \dot{H}^s(\mathbb{R}^n) \) such that
\[
\lim_{k \to +\infty} I(u_k) = c \quad \text{and} \quad \lim_{k \to +\infty} I'(u_k) = 0 \quad \text{strongly in} \quad \dot{H}^s(\mathbb{R}^n)' .
\]

**Proposition 5.3** Let \( s \in (0,1), n > 2s, \alpha = 0 < \beta < 2s \) or \( \beta = 0 < \alpha < 2s, \mu \in (0,n) \) and \( 0 \leq \gamma < \gamma_H \). Consider the functional \( I \) defined in (5.1) on the Banach space \( \dot{H}^s(\mathbb{R}^n) \). Then there exists a (PS) sequence \( \{u_k\} \subset \dot{H}^s(\mathbb{R}^n) \) for \( I \) at some \( c \in (0, c^*) \), i.e.,
\[
\lim_{k \to +\infty} I(u_k) = c \quad \text{and} \quad \lim_{k \to +\infty} I'(u_k) = 0 \quad \text{strongly in} \quad \dot{H}^s(\mathbb{R}^n)' ,
\]
where
\[
c^* := \min \left\{ \frac{2^\#(\alpha)}{2 \cdot 2^\#(\alpha)} S_\mu(n,s,\gamma,\alpha)^{\frac{2^\#(\alpha)-1}{2}}, \frac{2s-\beta}{2(n-\beta)} \Lambda(n,s,\gamma,\beta)^{\frac{n-\beta}{2s-\beta}} \right\} .
\]

**Proof** Imitate the proof of Proposition 5.2. Since \( 0 \leq \gamma < \gamma_H \), using (2) and (4) in Proposition 4.1, we obtain a minimizer \( u_\gamma \in \dot{H}^s(\mathbb{R}^n) \) for \( S_\mu(n,s,\gamma,0) \) and \( V_\gamma \in \dot{H}^s(\mathbb{R}^n) \) for \( \Lambda(n,s,\gamma,0) \). The rest is standard.

**Proof of Theorem 1.1**

1. The case \( s \in (0,1), 0 < \alpha, \beta < 2s < n, \mu \in (0,n) \) and \( \gamma < \gamma_H \).

Let \( \{u_k\}_{k \in \mathbb{N}} \) be a (PS) sequence as in Proposition 5.2, i.e.,
\[
I(u_k) \to c, \quad I'(u_k) \to 0 \quad \text{strongly in} \quad \dot{H}^s(\mathbb{R}^n)' \quad \text{as} \quad k \to +\infty.
\]
Then
\[
I(u_k) = \frac{1}{2} ||u_k||^2 - \frac{1}{2} \frac{2^\#(\beta)}{2^\#(\alpha)} \int_{\mathbb{R}^n} \frac{|u_k|^{2^\#(\beta)}}{|x|^{\beta}} \, dx - \frac{1}{2} \frac{2^\#(\alpha)}{2^\#(\alpha)} B_\alpha(u_k, u_k) = c + o(1) \quad (5.6)
\]
and
\[
\langle I'(u_k), u_k \rangle = ||u_k||^2 - \frac{1}{2^\#(\beta)} \int_{\mathbb{R}^n} \frac{|u_k|^{2^\#(\beta)}}{|x|^{\beta}} \, dx - B_\alpha(u_k, u_k) = o(1). \quad (5.7)
\]

From (5.6) and (5.7), if \( 2 \cdot 2^\#(\alpha) \geq 2^\#(\beta) > 2 \), we have
\[
c + o(1)||u_k|| = I(u_k) - \frac{1}{2^\#(\beta)} \langle I'(u_k), u_k \rangle \geq \left( \frac{1}{2} - \frac{1}{2^\#(\beta)} \right) ||u_k||^2 .
\]
If \( 2^\#(\beta) > 2 \cdot 2^\#(\alpha) > 2 \), we have
\[
c + o(1)||u_k|| = I(u_k) - \frac{1}{2^\#(\alpha)} \langle I'(u_k), u_k \rangle \geq \left( \frac{1}{2} - \frac{1}{2^\#(\alpha)} \right) ||u_k||^2 .
\]
Thus, \( \{u_k\}_{k \in \mathbb{N}} \) is bounded in \( \dot{H}^s(\mathbb{R}^n) \), so from (5.7) there exists a subsequence, still denoted by \( \{u_k\} \), such that \( ||u_k|| \to b \), \( \int_{\mathbb{R}^n} \frac{|u_k|^{2^\#(\beta)}}{|x|^{\beta}} \, dx \to d_1 \), \( B_\alpha(u_k, u_k) \to d_2 \) and
\[
b = d_1 + d_2 .
\]
By the definitions of \( \Lambda(n,s,\gamma,\beta) \) and \( S_\mu(n,s,\gamma,\alpha) \), we get
\[
d_1^{\frac{2}{2^\#(\beta)}} \Lambda(n,s,\gamma,\beta) \leq b , \quad d_2^{\frac{1}{2^\#(\alpha)}} S_\mu(n,s,\gamma,\alpha) \leq b .
\]
Therefore
\[
d_1^{\frac{2}{2^\#(\beta)}} \Lambda(n,s,\gamma,\beta) \leq d_1 + d_2 , \quad d_2^{\frac{1}{2^\#(\alpha)}} S_\mu(n,s,\gamma,\alpha) \leq d_1 + d_2 .
\]
These inequalities lead to
\[
\frac{2^* - 2}{2^* + 2} \left( \Lambda(n, s, \gamma, \beta) - d_1 \right) \leq d_2, \quad \frac{1}{2} d_2^{2^* (\alpha)} \left( S_{\mu}(n, s, \gamma, \alpha) - d_2^{2^* (\alpha)} \right) \leq d_1. \tag{5.8}
\]

We claim that
\[
\Lambda(n, s, \gamma, \beta) - d_1 \geq 0, \quad S_{\mu}(n, s, \gamma, \alpha) - d_2^{2^* (\alpha)} > 0.
\]

In fact, since \(c + o(1)||u_k|| = I(u_k) - \frac{1}{2} (I'(u_k), u_k)\), we have
\[
\left(1 - \frac{1}{2^* (\beta)}\right) \int_{\mathbb{R}^n} \frac{|u_k|^{2^* (\beta)}}{|x|^{\beta}} \, dx + \left(1 - \frac{1}{2 \cdot 2^* (\alpha)}\right) B_\alpha(u_k, u_k) = c + o(1)||u_k||,
\]
i.e.,
\[
\left(\frac{1}{2} - \frac{1}{2^* (\beta)}\right) d_1 + \left(\frac{1}{2} - \frac{1}{2 \cdot 2^* (\alpha)}\right) d_2 = c, \tag{5.9}
\]

so
\[
d_1 \leq \frac{2(n - \beta)}{2s - \beta} c, \quad d_2 \leq \frac{2 \cdot 2^* (\alpha)}{2^* (\alpha) - 1} c.
\]

Using the upper bound of \(d_1, d_2\) and the fact that \(0 < c < c^*\), we have
\[
\Lambda(n, s, \gamma, \beta) - d_1 \geq A_1 > 0, \quad S_{\mu}(n, s, \gamma, \alpha) - d_2^{2^* (\alpha)} \geq A_2 > 0,
\]

where \(A_1 = \Lambda(n, s, \gamma, \beta) - \left[\frac{2(n - \beta)}{2s - \beta} c \right]^{2^* (\beta)}\) and \(A_2 = S_{\mu}(n, s, \gamma, \alpha) - \left[\frac{2 \cdot 2^* (\alpha)}{2^* (\alpha) - 1} c \right]^{2^* (\alpha)}\). Thus

(5.8) implies that
\[
d_1^{2^* (\alpha)} A_1 \leq d_2, \quad d_2^{\frac{1}{2^* (\alpha)}} A_2 \leq d_1.
\]

If \(d_1 = 0\) and \(d_2 = 0\), then (5.9) implies that \(c = 0\), which is in contradiction with \(c > 0\). Therefore \(d_1 > 0\) and \(d_2 > 0\), so we can choose \(\varepsilon_0 > 0\) such that \(d_1 \geq \varepsilon_0 > 0\) and \(d_2 \geq \varepsilon_0 > 0\), and there exists a \(K > 0\) such that \(k \geq K\) and
\[
\int_{\mathbb{R}^n} \frac{|u_k|^{2^* (\beta)}}{|x|^{\beta}} \, dx > \varepsilon_0/2, \quad B_{\alpha}(u_k, u_k) > \varepsilon_0/2.
\]

Then inequality (2.6), the embeddings (1.9), and improved Sobolev inequality (1.10) imply that there exists \(C > 0\) such that
\[
0 < C \leq ||u_k||_{L^{2s,n-2s+2r}(\mathbb{R}^n, |y|^{-2r})} \leq C^{-1},
\]

where \(r = \alpha^{2^*/(2^* - 1)}\). For any \(k > K\), we may find \(\lambda_k > 0\) and \(x_k \in \mathbb{R}^n\) such that
\[
\lambda_k^{-2s+2r} \int_{B_{\lambda_k}(x_k)} \frac{|u_k|^2 |y|^{2r}}{|y|^{2r}} \, dy > ||u_k||_{L^{2s,n-2s+2r}(\mathbb{R}^n, |y|^{-2r})}^2 - \frac{C}{2K} \geq C_1 > 0.
\]

Letting \(v_k(x) = \lambda_k^{n-2s} u_k(\lambda_k x)\), since \(||v_k|| = ||u_k|| \leq C\), there exists a \(v \in \dot{H}^s(\mathbb{R}^n)\) such that
\[
v_k \rightharpoonup v \quad \text{in} \quad \dot{H}^s(\mathbb{R}^n).
\]

In a fashion similar to the proof of Proposition 4.1-(1) in Section 4, we can prove that \(v \neq 0\).
In addition, the boundedness of \( \{v_k\} \) in \( \dot{H}^s(\mathbb{R}^n) \) implies that \( \{ |v_k|^{2^*_s(\beta)-2}v_k \} \) is bounded in \( L^{2^*_s(\beta)}(\mathbb{R}^n, |x|^{-\beta}) \) and

\[
|v_k|^{2^*_s(\beta)-2}v_k \rightharpoonup |v|^{2^*_s(\beta)-2}v \quad \text{in} \quad L^{2^*_s(\beta)}(\mathbb{R}^n, |x|^{-\beta}).
\]  

(5.10)

For any \( \phi \in L^{2^*_s(\alpha)}(\mathbb{R}^n, |x|^{-\alpha}) \), Lemma 2.9 implies that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} [I_\mu * F_\alpha(\cdot, v_k)](x)f_\alpha(x, v_k)\phi(x)dx = \int_{\mathbb{R}^n} [I_\mu * F_\alpha(\cdot, v)](x)f_\alpha(x, v)\phi(x)dx.
\]  

(5.11)

Since \( \dot{H}^s(\mathbb{R}^n) \hookrightarrow L^{2^*_s(\alpha)}(\mathbb{R}^n, |x|^{-\alpha}) \), (5.11) holds for any \( \phi \in \dot{H}^s(\mathbb{R}^n) \).

Finally, we need to check that \( \{v_k\}_{k \in \mathbb{N}} \) is also a (PS) sequence for \( I \) at energy level \( c \). Since the norms in \( \dot{H}^s(\mathbb{R}^n) \) and \( L^{2^*_s(\alpha)}(\mathbb{R}^n, |x|^{-\alpha}) \) are invariant under the special dilation \( v_k(x) = \lambda_k^{\frac{n}{2^*_s}} u_k(\lambda_k x) \), we have

\[
\lim_{k \to +\infty} I(v_k) = c.
\]

Moreover, \( \forall \phi \in \dot{H}^s(\mathbb{R}^n) \), so we have \( \phi_k(x) = \lambda_k^{\frac{n}{2^*_s}} \phi(\frac{x}{\lambda_k}) \in \dot{H}^s(\mathbb{R}^n) \). From \( I'(v_k) \to 0 \) in \( \dot{H}^s(\mathbb{R}^n)' \), we can derive that

\[
\lim_{k \to +\infty} \langle I'(v_k), \phi \rangle = 0.
\]

Thus (5.10) and (5.11) lead to

\[
\langle I'(v), \phi \rangle = \lim_{k \to +\infty} \langle I'(v_k), \phi \rangle = 0.
\]

Hence \( v \) is a nontrivial weak solution of (1.1).

(II) If is the case that \( s \in (0, 1), 0 \leq \alpha, \beta < 2s < n \), while \( \alpha \cdot \beta = 0, \mu \in (0, n) \) and \( 0 \leq \gamma < \gamma_H \).

Case (i) \( \alpha = 0 < \beta < 2s \) or \( \beta = 0 < \alpha < 2s \).

In this case, the embeddings (1.9) and inequality (1.10) are still effective. Since \( \alpha > 0 \) or \( \beta > 0 \), we get a nontrivial weak solution to (1.1), as above, by using (1.9), (1.10) and Proposition 5.3.

Case (ii) \( \alpha = 0 \) and \( \beta = 0 \).

In this case, (1.9) and (1.10) are useless. Since the limit equation for (1.1) is

\[
(-\Delta)^s v = |v(x)|^{2^*_s-2}v(x) + \left( \int_{\mathbb{R}^n} \frac{|v(y)|^{2^*_p}}{|x-y|^p} dy \right) |v(x)|^{2^*_p-2}v(x),
\]

by using the Nehari manifold method in [5], we can also get a non-trivial weak solution to (1.1), if \( 0 \leq \gamma < \gamma_H \).

\[\square\]

Remark 5.4 The method we adopt to prove Theorem 1.1 can be applied to prove a similar existence result for the \( p \)-Laplace type problem involving double critical exponents. To go further, we consider

\[
-\Delta_p u - \kappa \frac{|u|^{p-2}u}{|x|^p} = \sum_{i=1}^2 \left( \int_{\mathbb{R}^n} \frac{|u(y)|^{p^*_i(\alpha_i)}}{|x-y|^{n_i(\alpha_i)}} dy \right) \frac{|u(x)|^{p^*_i(\alpha_i)-2}u(x)}{|x|^{\delta_i(\alpha_i)}}, \quad x \in \mathbb{R}^n.
\]  

(5.12)

where \( n \geq 2 \) is an integer, \( p \in (1, n), \kappa < \kappa := [(n-p)/p]^p \), and \( \mu_i \in (0, n) \), while \( \alpha_i \in (0, p), p^*_i(\alpha_i) = (1 - \frac{\mu_i}{n}) \cdot p^*(\alpha_i), \delta_i(\alpha_i) = (1 - \frac{\mu_i}{2n})\alpha_i \) and \( p^*(\alpha_i) = p(n - \alpha_i)/(n - p) \) for \( i = 1, 2 \).
We say that \( u \in D^{1,p}(\mathbb{R}^n) \) is a weak solution to (5.12) if
\[
\int_{\mathbb{R}^n} \left[ |\nabla u|^{p-2} \nabla u \nabla \phi - \kappa |u|^{p-2} u \phi \right] = \sum_{i=1}^{2} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \frac{|u(y)|^{p_{\alpha_i}^*(\alpha_i)}}{|x-y|^{\mu_i} \delta(y)} dy \right) \frac{|u(x)|^{p_{\alpha_i}^*(\alpha_i)-2} u(x) \phi}{|x|^{p_{\alpha_i}^*(\alpha_i)}}
\]
for any \( \phi \in D^{1,p}(\mathbb{R}^n) \). The following main result holds:

**Theorem 5.5** The problem (5.12) possesses at least a nontrivial weak solution provided that either

(I) \( n \geq 2, \ p \in (1, n), \ 0 < \alpha_1, \alpha_2 < p, \ 0 < \mu_1, \mu_2 < n \) and \( \kappa < \bar{\kappa} \),

or

(II) \( n \geq 2, \ p \in (1, n), \ 0 \leq \alpha_1, \alpha_2 < p \), while \( \alpha_1 \cdot \alpha_2 = 0, \ 0 < \mu_1, \mu_2 < n \) and \( 0 \leq \kappa < \bar{\kappa} \).

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