Kinetic equations for systems with long-range interactions: a unified description

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Received 17 February 2010
Accepted 29 April 2010
Published 28 May 2010

Online at stacks.iop.org/JSTAT/2010/P05019
doi:10.1088/1742-5468/2010/05/P05019

Abstract. We complete the existing literature on the kinetic theory of systems with long-range interactions. Starting from the BBGKY hierarchy, or using projection operator technics or a quasilinear theory, a general kinetic equation can be derived when collective effects are neglected. This equation (which is not well known) applies to possibly spatially inhomogeneous systems, which is specific to systems with long-range interactions. Interestingly, the structure of this kinetic equation bears a clear physical meaning in terms of generalized Kubo relations. Furthermore, this equation takes a very similar form for stellar systems and two-dimensional point vortices, providing therefore a unified description of the kinetic theory of these systems. If we assume that the system is spatially homogeneous (or axisymmetric for point vortices), this equation can be simplified and reduces to the Landau equation (or its counterpart for point vortices). Our formalism thus offers a simple derivation of Landau-type equations.

We also use this general formalism to derive a kinetic equation, written in angle-action variables, describing spatially inhomogeneous systems with long-range interactions. This new derivation solves the shortcomings of our previous derivation (Chavanis 2007 Physica A 377 469). Finally, we consider a test particle approach and derive general expressions for the diffusion and friction (or drift) coefficients of a test particle evolving in a bath of field particles. We make contact with the expressions previously obtained in the literature. As an application of the kinetic theory, we argue that, for one-dimensional systems and two-dimensional point vortices, the relaxation time is shorter for inhomogeneous (or non-axisymmetric) distributions than for homogeneous (or axisymmetric) distributions because there are potentially more resonances. We compare this prediction with existing...
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numerical results. For the HMF model, we argue that the relaxation time scales like $N$ for inhomogeneous distributions and like $e^N$ for permanently homogeneous distributions. Phase-space structures can reduce the relaxation time by creating some inhomogeneities and resonances. Similar results are expected for 2D point vortices. For systems with higher dimension, the relaxation time scales like $N$. The relaxation time of a test particle in a bath also scales like $N$ in any dimension.

**Keywords:** stochastic particle dynamics (theory), dynamical processes (theory), kinetic theory of gases and liquids

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1. Introduction

Kinetic theories of systems with long-range interactions are important to understand the dynamical evolution of the system and to study transport properties. They can be used to determine the timescale of ‘collisional’ relaxation towards the Boltzmann distribution, in particular its dependence on the number of particles $N$. Furthermore, they can tell whether or not the system will truly relax towards statistical equilibrium. Indeed, it is not obvious that complex systems (like systems with long-range interactions) do eventually relax towards Boltzmann equilibrium because the ergodicity assumption which sustains the statistical theory can break down. Different kinetic theories have been developed over the years.

*Dilute gases.* The first kinetic theory of many-particles systems was elaborated by Boltzmann [1] for a dilute gas. In that case, the particles do not interact except during strong collisions. Boltzmann derived his famous equation from which he proved that
the entropy increases (H-theorem) and that the system ultimately relaxes towards the Maxwell distribution of statistical equilibrium.

Coulombian plasmas. Landau [2] derived a kinetic equation for a Coulombian plasma by starting from the Boltzmann equation and considering a weak deflection approximation. Indeed, for a Coulombian potential of interaction slowly decreasing with distance like \( r^{-1} \), weak collisions are the most frequent ones. Each encounter induces a small change in the velocity of a particle, but the cumulated effect of these encounters leads to a macroscopic process of diffusion in velocity space. The treatment of Landau, which assumes that the particles follow linear trajectories with constant velocity in a first approximation, yields a logarithmic divergence of the diffusion coefficient for both small and large impact parameters, but the equation can still be used successfully if appropriate cut-offs are introduced. A natural lower cut-off, which is called the Landau length, corresponds to an impact parameter leading to a deflection at 90°. On the other hand, in a neutral plasma, the potential is screened on a distance corresponding to the Debye length. Phenomenologically, the Debye length provides an upper cut-off. Later on, Lenard [3] and Balescu [4] developed a more precise kinetic theory that could take into account collective effects. This gives rise to the inclusion of the dielectric function \( |\epsilon(k, k \cdot v)|^2 \) in the denominator of the potential of interaction appearing in the kinetic equation. Physically, this means that the particles are ‘dressed’ by a polarization cloud. The original Landau equation, which ignores collective effects, is recovered from the Lenard–Balescu equation when \( |\epsilon(k, k \cdot v)|^2 = 1 \). However, with this additional term, it is found that the logarithmic divergence at large scales is now removed and that the Debye length is indeed the natural upper lengthscale to consider. At about the same time, Rostoker and Rosenbluth [5] and Hubbard [6] developed a test particle approach and derived a Fokker–Planck equation describing the relaxation of a test particle in a bath of field particles. They calculated the diffusion and friction coefficients by evaluating the first and second moments of the velocity deflection and took into account collective effects. It is interesting to note, for historical reasons, that Hubbard [6] was apparently not aware of the works of Lenard [3] and Balescu [4] at that time and developed his approach independently. However, the three approaches are closely related. Indeed, if we substitute the diffusion and friction coefficients found by Hubbard in the general form of the Fokker–Planck equation and perform minor transformations (a substitution that Hubbard has not explicitly made), one obtains the Lenard–Balescu equation!

Stellar systems. In stellar dynamics, Chandrasekhar [7]–[9] developed a kinetic theory of stars in order to determine the timescale of collisional relaxation and the rate of escape of stars from globular clusters. To simplify the kinetic theory, he considered an infinite and homogeneous system. He started from the general Fokker–Planck equation and determined the diffusion coefficient and the friction force (second and first moments of the velocity increments) by considering the mean effect of a succession of two-body encounters\(^1\). Since his approach can take into account large deflections, there is no divergence at small impact parameters and the gravitational analog of the Landau length appears naturally in the treatment of Chandrasekhar. However, his approach leads to a

\(^1\) Later, Cohen et al [10], Gasiorowicz et al [11] and Rosenbluth et al [12] proposed a simplified derivation of the coefficients of diffusion and friction for stellar systems and for plasmas (without collective effects). The friction force has also been calculated by Marochnik [13], Kalnajs [14], Kandrup [15], Tremaine and Weinberg [16], Bekenstein and Maoz [17], Maoz [18], Nelson and Tremaine [19] and Chavanis [20] using different approaches.
logarithmic divergence at large scales that is more difficult to remove in stellar dynamics than in plasma physics because of the absence of Debye shielding for the gravitational force. In a series of papers, Chandrasekhar and von Neumann [21] developed a completely stochastic formalism of gravitational fluctuations and showed that the fluctuations of the gravitational force are given by the Holtsmark distribution (a particular Lévy law) in which the nearest neighbor plays a prevalent role. From these results, they argued that the logarithmic divergence has to be cut-off at the interparticle distance. However, since the interparticle distance is smaller than the Debye length, the same arguments should also apply in plasma physics, which is not the case. Therefore, the conclusions of Chandrasekhar and von Neumann are usually taken with circumspection. In particular, Cohen et al [10] argue that the logarithmic divergence should be cut-off at the physical size $R$ of the cluster (or at the Jeans length) since the Jeans length is the presumable analog of the Debye length in the present context. These kinetic theories lead to a relaxation time of the form $t_R \sim (N/\ln N)t_D$, where $t_D$ is the dynamical time. Chandrasekhar [22] also developed a Brownian theory of stellar dynamics and showed that, from a qualitative point of view, the results of kinetic theory can be understood very simply in that framework. In particular, he showed that the dynamical friction is necessary to reproduce the Maxwell–Boltzmann distribution at equilibrium and that the coefficients of diffusion and friction are related to each other by an Einstein relation. This relation is confirmed by his more precise kinetic theory. It is important to emphasize, however, that Chandrasekhar did not derive the kinetic equation for the evolution of the system as a whole. Indeed, he considered the Brownian motion of a test star in a fixed distribution of field stars (bath) and derived the corresponding Fokker–Planck equation$^2$. This equation has been used by Chandrasekhar [9], Spitzer and Härm [23], Michie [24], King [25] and Lemou and Chavanis [26] to study the evaporation of stars from globular clusters. King [27] noted that, if we were to describe the dynamical evolution of the cluster as a whole, the distribution of the field particles must evolve in time in a self-consistent manner, so the kinetic equation must be an integro-differential equation. The kinetic equation obtained by King [27] is equivalent to the Landau equation$^3$. It is interesting to note, for historical reasons, that none of the previous authors seemed to be aware of the work of Landau [2] at that time$^4$. There is, however, an important difference between stellar dynamics and plasma physics. Neutral plasmas are usually spatially homogeneous due to Debye shielding. By contrast, stellar systems are spatially inhomogeneous. The above-mentioned kinetic theories developed for an infinite homogeneous system can be applied to an inhomogeneous system only if we make a local approximation. In that case, the collision term is calculated as if the system were spatially homogeneous or as if the collisions could be treated as local. Then, the effect of spatial inhomogeneity is only retained in the advective (Vlasov) term which describes the evolution of the system due

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$^2$ Indeed, Chandrasekhar [8, 9, 22] models the evolution of globular clusters by the Kramers equation which has a fixed temperature (canonical description) while a more relevant kinetic equation would be the Landau equation which conserves the energy (microcanonical description).

$^3$ This is not obvious in the form given by King [27], which is deduced from the work of Rosenbluth et al [12]. However, the connection with the Landau equation can easily be made by using simple integration by parts; see Chavanis [28] for a recent discussion of that issue.

$^4$ To our knowledge, the first explicit reference to the Landau equation in astrophysics literature appeared much later, in the paper of Kandrup [29]. There is also a strange comment related to the work of Landau in the paper of Cohen et al [10].
to mean field effects\(^5\). This leads to the Vlasov–Landau–Poisson system, which is the standard kinetic equation of stellar dynamics. To our knowledge, this equation was first written about and studied by Hénon [35]. Hénon also exploited the timescale separation between the dynamical time \(t_D\) and the relaxation time \(t_R \gg t_D\) to derive a simplified kinetic equation for \(f(\epsilon, t)\), where \(\epsilon = \frac{v^2}{2} + \Phi(r, t)\) is the individual energy, called the orbit-averaged-Fokker–Planck equation. In this approach, the distribution function \(f(r, v, t)\), averaged over a short timescale, is a steady state of the Vlasov equation of the form \(f(\epsilon, t)\), which slowly evolves in time (on a long timescale) due to the development of ‘collisions’ (i.e. correlations due to finite \(N\) effects or graininess). Cohn [36] numerically solved this equation to describe the collisional evolution of star clusters and investigate the gravothermal catastrophe that was predicted by Antonov [37] and Lynden-Bell and Wood [38] on the basis of statistical mechanics. The local approximation, which is a crucial step in the kinetic theory, is supported by the stochastic approach of Chandrasekhar and von Neumann [21] showing the preponderance of the nearest neighbor. However, this remains a simplifying assumption which is not easily controllable. In particular, as we have already indicated, the local approximation leads to a logarithmic divergence at large scales that is difficult to remove. This divergence would not have occurred if a full account of spatial inhomogeneity had been given since the start. On the other hand, the Fokker–Planck equation is based on a Markov assumption and this assumption is not clearly justified for stellar systems. Memory effects can be important for self-gravitating systems because, as shown by Chandrasekhar [39], the temporal correlation function of the force decreases algebraically like \(1/t\) instead of exponentially. This slow decay results in a logarithmic divergence of the diffusion coefficient for large times when one considers the Kubo formula. As shown by Lee [40], the spatial and temporal logarithmic divergences are equivalent, i.e. they are two manifestations of the same phenomenon. The effect of spatial and temporal delocalization was investigated by Hénon [41], Prigogine and Severne [42], Ostriker and Davidsen [43], Gilbert [44], Lerche [45], Severne and Haggerty [46], Parisot and Severne [47], Kandrup [29] and Chavanis [20, 48]. In particular, Kandrup derived a generalized Landau equation by using projection operator techniques [29]. Recently, Chavanis obtained this equation in a simpler manner from the BBGKY hierarchy [48] or from a quasilinear theory [20] as an expansion in \(1/N\) in a proper thermodynamic limit. This generalized kinetic equation is interesting because it can take into account the effects of spatial inhomogeneity and memory which are neglected in the previous approaches. It clearly shows which approximations are needed in order to recover the Landau equation. However, the generalized Landau equation remains extremely complicated for practical applications.

Two-dimensional point vortices and non-neutral plasmas. It is well known that a non-neutral plasma under a strong magnetic field is isomorphic to a system of point vortices in 2D hydrodynamics [49]. These systems are described by Hamiltonian equations in which the coordinates \(x\) and \(y\) of the particles are canonically conjugate [50]. The particles (charges or vortices) interact through a long-range potential, a role that is played by the electric potential in a plasma or by the stream function in 2D hydrodynamics. The

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\(^5\) The equation now called the ‘Vlasov equation’ was introduced by Jeans [30] in stellar dynamics and by Vlasov [31] in plasma physics. See Hénon [32] for interesting comments about the name that one should give to that equation. The Vlasov–Poisson system describes collisionless plasmas [33]. It also describes collisionless stellar systems, such as elliptical galaxies, whose ages are much less than the Chandrasekhar relaxation time [34].
statistical mechanics of 2D point vortices was pioneered by Onsager [51] and further developed by Montgomery and Joyce [52] and Lundgren and Pointin [53] in a mean field approximation (recently, Eyink and Sreenivasan [54] have discovered that Onsager had first performed this mean field theory in unpublished notes). Kinetic theories of these systems were developed much later. Dubin and O’Neil [55] derived a kinetic equation for a 2D guiding center plasma by starting from the Klimontovich equation and using a quasilinear theory. Independently, Chavanis [56] derived a kinetic equation for 2D point vortices by using projection operator techniques and, more recently [57], by using a BBGKY-like hierarchy or a quasilinear theory. The kinetic equation derived by Dubin and O’Neil takes into account collective effects and can be considered as the counterpart of the Lenard–Balescu equation in plasma physics. The kinetic equation derived by Chavanis ignores collective effects and can be considered as the counterpart of the Landau equation in plasma physics and stellar dynamics. However, the approach of Chavanis is formulated for an arbitrary distribution of point vortices while the approach of Dubin and O’Neil is restricted to axisymmetric distributions. These kinetic equations are valid at the order O(1/N) in a proper thermodynamic limit, so that they describe the evolution of the system on a timescale of order NtD. For N → +∞, we get the 2D Euler equation which describes collisionless systems. At the order 1/N, the collision operator describes the development of correlations between point vortices. However, when we consider axisymmetric distributions, it is found that the collision operator cancels out when the profile of angular velocity becomes monotonic. In that case, the evolution stops even if the system has not reached statistical equilibrium [56]–[58]. This implies that the relaxation time of the system as a whole is larger than NtD, where tD is the dynamical time, so that higher order correlations (three-body, four-body …) must be advocated [56]. It is also possible that the point vortex gas does not reach the Boltzmann distribution predicted by statistical mechanics. Indeed, the dynamics may be non-ergodic, as discussed by Khanin [59]. On the other hand, using an analogy with stellar dynamics and Brownian theory, Chavanis [56, 57, 60] derived a Fokker–Planck equation describing the relaxation of a test vortex in a bath of field vortices at statistical equilibrium. The relaxation is due to the competition between a diffusion term and a drift term. The systematic drift [60], which is the counterpart of the dynamical friction [22] in stellar dynamics, is necessary to obtain the Boltzmann distribution at statistical equilibrium. The diffusion coefficient and the drift term are related to each other by an appropriate Einstein relation and they are inversely proportional to the local shear created by the field vortices [56, 57, 60]. The properties of this Fokker–Planck equation have been studied by Chavanis and Lemou [58]. On the other hand, Chavanis and Sire [61] have studied the statistics of the velocity fluctuations arising from a random distribution of point vortices by using an approach similar to the one developed by Chandrasekhar and von Neumann [21] for the gravitational force, but leading to different results due to the lower dimension of space. The numerous analogies between the statistical mechanics and the kinetic theories of stellar systems and 2D vortices are discussed by Chavanis [62]. Other interesting kinetic theories of 2D point vortices. As reported by Eyink and Sreenivasan [54], Onsager first pointed out analogies between stellar systems and 2D vortices. In a letter to Lin he wrote: ‘at negative temperatures, the appropriate statistical methods have analogs not in the theory of electrolytes, but in the statistics of stars…’. However, Onsager did not develop this matter further.

doi:10.1088/1742-5468/2010/05/P05019
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vortices have been developed by Nazarenko and Zakharov [63], Marmanis [64], Sire and Chavanis [65] and Newton and Mezic [66].

HMF model. In the past, the kinetic theories of systems with long-range interactions have been essentially developed for plasmas, stellar systems and 2D vortices. Recently, there was a renewed interest from some of the statistical mechanics community for systems with long-range interactions. In that context, a simple model of systems with long-range interactions, the so-called Hamiltonian mean field (HMF) model [67], has been extensively studied [68]. It consists of $N$ particles moving on a ring and interacting via a cosine potential. It can be viewed as a one-dimensional periodic system where the potential of interaction is truncated to one Fourier mode. This is certainly the simplest system with long-range interactions that we can imagine. However, it remains highly non trivial and, interestingly, it exhibits many features common to other systems with long-range interactions such as self-gravitating systems. The kinetic theory of the spatially homogeneous phase of the HMF model has been developed by Bouchet [69], Bouchet and Dauxois [70], Chavanis et al [71] and Chavanis [72] (Inagaki [73] first developed a kinetic theory of the HMF model, but he arrived at incorrect conclusions). They considered an expansion of the equations of the dynamics in powers of $1/N$ in a proper thermodynamic limit. For $N \to +\infty$, this leads to the Vlasov equation. At the order $O(1/N)$, this leads to the one-dimensional version of the Lenard–Balescu equation. However, the Lenard–Balescu collision operator is known to vanish in one dimension [74,75]. This implies that the relaxation time is larger than $Nt_D$, where $t_D$ is the dynamical time. This result of kinetic theory is in agreement with direct numerical simulations that lead to $t_R \sim N^{1.7}t_D$ [76] or even $t_R \sim e^{N}t_D$ [77] depending on the initial conditions. These authors also considered the relaxation of a test particle in a bath of field particles. It is described by a Fokker–Planck equation, involving a diffusion and a friction, which is the one-dimensional version of the Fokker–Planck equation derived by Hubbard [6] in plasma physics. It can also be deduced from the Lenard–Balescu equation by making a bath approximation. Bouchet and Dauxois [70] used this Fokker–Planck equation to show that the temporal correlation function of the velocity decreases algebraically with time leading to (weak or strong) anomalous diffusion (related results were previously obtained by Marksteiner et al [78] in the very different context of optical lattices). The theoretical results of Bouchet and Dauxois [70] are in very good agreement with the direct numerical simulations of Yamaguchi et al [79] in the situations considered. Chavanis and Lemou [80], developing the theory of Potapenko et al [81], used the Fokker–Planck equation to study the relaxation of the distribution function tail and show that it has a front structure moving very slowly (logarithmically) with time. Some features of the kinetic theory of the spatially inhomogeneous phase of the HMF model have been considered by Chavanis [82].

A first objective of the present paper is to develop a unified kinetic theory of systems with long-range interactions (see also the complementary paper [83]). In particular, we show that, when collective effects are neglected, a generalized kinetic equation can be given for stellar systems (see equation (2)), for the HMF model (see equation (2)) and

\[ \text{doi:10.1088/1742-5468/2010/05/P05019} \]
for 2D point vortices (see equation (52)). This equation, which is not well known, applies to spatially inhomogeneous systems (or non-axisymmetric distributions of point vortices) and bears a clear physical meaning in terms of generalized Kubo relations. Furthermore, it avoids the use of Fourier–Laplace transforms and remains in physical space. For homogeneous distributions (or for axisymmetric distributions in the case of point vortices), it reduces to the Landau equation (or to its counterpart in 2D hydrodynamics). Therefore, our paper offers a new derivation of Landau-type equations from a simple formalism and develops the calculations in detail. A drawback of our approach is to neglect collective effects that would lead to Lenard–Balescu-type equations in the homogeneous case. A second objective of the paper is to use this general formalism to derive a kinetic equation, written in angle-action variables, describing spatially inhomogeneous systems with long-range interactions like stellar systems and the HMF model. This new derivation solves the shortcomings of our previous derivation [84]. We also derive general expressions for the diffusion and friction (or drift) coefficients of a test particle evolving in a bath of field particles and make contact with the expressions previously obtained in the literature. Finally, we use this kinetic theory to argue that, for one-dimensional systems and two-dimensional point vortices, the relaxation time is shorter for inhomogeneous (or non-axisymmetric) distributions than for homogeneous (or axisymmetric) distributions because there are potentially more resonances. We compare this prediction with existing numerical results. For the HMF model, we argue that the relaxation time scales like $N$ for inhomogeneous distributions and like $e^N$ for permanently homogeneous distributions. Phase-space structures can reduce the relaxation time by creating some inhomogeneities and resonances. Similar results are expected for 2D point vortices. For systems with higher dimension, the relaxation time scales like $N$. The relaxation time of a test particle in a bath also scales like $N$ in any dimension.

2. Stellar systems and HMF model

2.1. The evolution of the system as a whole: the kinetic equation

Let us consider an isolated system of particles with long-range interactions described by the Hamiltonian equations

$$\begin{align*}
    m \frac{dr_i}{dt} &= \frac{\partial H}{\partial v_i}, \\
    m \frac{dv_i}{dt} &= -\frac{\partial H}{\partial r_i}, \\
    H &= \frac{1}{2} \sum_{i=1}^{N} mv_i^2 + m^2 \sum_{i<j} u(r_i - r_j),
\end{align*}$$

(1)

where $u(r - r')$ is a binary potential of interaction depending only on the absolute distance $|r - r'|$ between the particles. We consider the proper thermodynamic limit $N \to +\infty$ in such a way that the normalized energy $\epsilon = E/(u_\ast N^2 m^2)$ and the normalized inverse temperature $\eta = \beta N m^2 u_\ast$ are of order unity, where $u_\ast$ represents the typical value of the potential of interaction [85]. By a suitable normalization of the parameters, we can consider $N \to +\infty$ with $V \sim E/N \sim \beta \sim m \sim 1$ and $u_\ast \sim 1/N$. This is the usual Kac prescription which amounts to putting $1/N$ in front of the interaction energy (for rigorous mathematical results see [86,87]). This makes the system extensive. We can also consider $N \to +\infty$ with $V \sim E \sim \beta/N \sim u_\ast \sim 1$ and $m \sim 1/N$. When collective effects

doi:10.1088/1742-5468/2010/05/P05019
are ignored, the kinetic equation describing the evolution of the distribution function \( f(r, v, t) \) at the order \( O(1/N) \) is given by

\[
\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{N-1}{N} (F) \cdot \frac{\partial f}{\partial v} = \frac{\partial}{\partial \nu} \int_0^{+\infty} d\tau \int dr_1 d\nu_1 F^\nu(1 \to 0) G(t, t - \tau) \times \left[ F^\nu(1 \to 0) \frac{\partial}{\partial \nu} + F^\nu(0 \to 1) \frac{\partial}{\partial \nu_1} \right] f(r, v, t - \tau) \frac{f}{m}(r_1, v_1, t - \tau),
\]

where \( F(1 \to 0) \) is the force (by unit of mass) created by particle 1 (with position \( r_1 \) and velocity \( v_1 \)) on particle 0 (with position \( r \) and velocity \( v \)) and \( F^\nu(1 \to 0) = F(1 \to 0) - \langle F \rangle/N \) is the fluctuating force with respect to the average force \( \langle F \rangle(r, t) = -\nabla \Phi(r, t) \) experienced by particle 0 (here \( \Phi(r, t) = \int u(r - r') \rho(r', t) dr' \) is the potential). On the other hand, \( G(t, t - \tau) \) is the Greenian constructed with the mean flow in phase-space.

The kinetic equation (2) is not well known and this is why we try to publicize it in this paper. It was obtained by Kandrup \[29\] from a projection operator formalism and by Chavanis \[20,48\] from the BBGKY hierarchy and from a quasilinear theory. The BBGKY hierarchy is closed by neglecting the three-body correlation function, which is of order \( O(1/N^2) \). The structure of this equation bears a clear physical meaning. It involves a diffusion term and a friction term. The coefficients of diffusion and friction are given by generalized Kubo formulae, i.e. the time integral of the temporal correlations of the fluctuating force. The collision term \( C_N[f] \) (rhs) is valid at the order 1/\( N \), so it describes the ‘collisional’ evolution of the system (ignoring collective effects) on a timescale of order \( Nt_D \), where \( t_D \sim R/\sqrt{m\nu a_s} \) is the dynamical time that is of order unity in the thermodynamic limit. For \( N \to +\infty \), we obtain the Vlasov equation in which collisions (more properly correlations) are neglected (for rigorous mathematical results see \[88\]–[90]). When coupled to an attractive long-range potential (e.g., the gravitational potential), the Vlasov equation can generate a process of violent collisionless relaxation towards a quasi stationary state on a few dynamical times \( t_D \). A statistical theory of violent relaxation has been developed by Lynden-Bell \[91\] for stellar systems, but its domain of applicability is more general (for kinetic theories of violent relaxation in stellar systems, see \[20,75\], [92]–[94]). Equation (2) is a non-Markovian integro-differential equation. It can describe the evolution of stellar systems, or the evolution of the HMF model, taking into account delocalizations in space and time (i.e. spatial inhomogeneity and memory effects). If we make a Markovian approximation and extend the time integral to \( +\infty \), we obtain

\[
\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{N-1}{N} \langle F \rangle \cdot \frac{\partial f}{\partial v} = \frac{\partial}{\partial \nu} \int_0^{+\infty} d\tau \int dr_1 d\nu_1 F^\nu(1 \to 0) G(t, t - \tau) \times G(t, t - \tau) \left[ F^\nu(1 \to 0) \frac{\partial}{\partial \nu} + F^\nu(0 \to 1) \frac{\partial}{\partial \nu_1} \right] f(r, v, t) \frac{f}{m}(r_1, v_1, t).
\]

The Markov approximation is not rigorously justified for self-gravitating systems because the force auto-correlation function decreases algebraically like \( 1/t \) \[39\], instead of exponentially. The Markov approximation is also expected to be incorrect for the HMF model and other systems with long-range interactions if we are close to the critical point. However, except for these situations, the Markovian approximation should be justified in the \( N \to +\infty \) limit because the timescale \( \sim Nt_D \) on which \( f(r, v, t) \) changes is long compared to the timescale \( \tau_{\text{corr}} \) on which the integrand in equation (2) has significant support. The Markovian equation (3) applies to possibly spatially inhomogeneous
distribution functions, which is a specificity of systems with long-range interactions. Now, for spatially homogeneous distribution functions $f(v, t)$, the kinetic equation reduces to

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \int_0^{+\infty} d\tau \int dr_1 dv_1 F^\mu(1 \to 0) G(t, t - \tau)$$

$$\times \left[ F^\nu(1 \to 0) \frac{\partial}{\partial v^\nu} + F^\nu(0 \to 1) \frac{\partial}{\partial v_1^\nu} \right] f(v, t) \frac{f}{m}(v_1, t).$$

(4)

This is valid as long as the homogeneous distribution is Vlasov stable. In fact, the distribution $f(v, t)$ will change due to the development of correlations and graininess effects (i.e. the rhs of the kinetic equation) and, at some point of the evolution, the distribution may become Vlasov unstable. In that case, a dynamical phase transition from a homogeneous state to an inhomogeneous state can take place. Such a transition has been illustrated numerically for the HMF model by Campa et al [95]. Using the symmetry of the force $F(0 \to 1) = -F(1 \to 0)$, and the equations of the unperturbed trajectories

$$v(t - \tau) = v(t) = v, \quad r(t - \tau) = r(t) - v(t)\tau = r - v\tau,$$

(5)

corresponding to $\langle F \rangle = 0$, the kinetic equation can be written

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \int dv_1 K^{\mu\nu} \left( \frac{\partial}{\partial v^\nu} - \frac{\partial}{\partial v_1^\nu} \right) f(v, t) \frac{f}{m}(v_1, t),$$

(6)

with

$$K^{\mu\nu} = \int_0^{+\infty} d\tau \int dr_1 F^\mu(1 \to 0, t) F^\nu(1 \to 0, t - \tau).$$

(7)

Now, the force (by unit of mass) created by particle 1 on particle 0 is given by

$$F(1 \to 0) = -m \frac{\partial u}{\partial r}(r - r_1).$$

(8)

Introducing the Fourier transform of the potential

$$u(x) = \int e^{ik \cdot x} \tilde{u}(k) \, dk, \quad \tilde{u}(k) = \int e^{-ik \cdot x} u(x) \, dx \frac{1}{(2\pi)^d},$$

(9)

we get

$$F(1 \to 0, s) = -im \int k e^{ik \cdot (r(s) - r_1(s))} \tilde{u}(k) \, dk.$$  

(10)

Using the equations of motion (5), and introducing $x = r - r_1$ and $w = v - v_1$, we obtain

$$F(1 \to 0, t - \tau) = -im \int k e^{ik \cdot (x - w\tau)} \tilde{u}(k) \, dk.$$  

(11)

Therefore

$$K^{\mu\nu} = -m^2 \int_0^{+\infty} d\tau \int dx \int dk \int dk' k^{\mu} k'^{\nu} e^{i(k + k') \cdot x} \frac{1}{m} e^{-ik' \cdot w\tau} \tilde{u}(k) \tilde{u}(k').$$

(12)
Using the identity
\[ \delta(x) = \int \frac{e^{ik \cdot x}}{(2\pi)^d} \, dk, \]
and integrating on \( x \), then on \( k' \), we find that
\[ K^{\mu \nu} = (2\pi)^d m^2 \int_{-\infty}^{+\infty} d\tau \int dk k^\mu k^\nu e^{i k \cdot (-k)} \hat{u}(k) \hat{u}(-k). \]
(14)

Since \( u(x) \) is real, we have \( \hat{u}(-k) = \hat{u}(k)^* \). Furthermore, since \( u(-x) = u(x) \), we have \( \hat{u}(k)^* = \hat{u}(k) \). Therefore, \( \hat{u}(k) \) is real and \( \hat{u}(-k) = \hat{u}(k) \). Therefore,
\[ K^{\mu \nu} = (2\pi)^d m^2 \int_{0}^{+\infty} d\tau \int dk k^\mu k^\nu e^{i k \cdot \tau} \hat{u}(k)^2. \]
(15)

Making the transformation \( \tau \to -\tau \), then \( k \to -k \), and adding the resulting expression to equation (15), we get
\[ K^{\mu \nu} = \frac{1}{2} (2\pi)^d m^2 \int_{-\infty}^{+\infty} d\tau \int dk k^\mu k^\nu e^{i k \cdot \tau} \hat{u}(k)^2. \]
(16)

Using the identity (13), we finally obtain
\[ K^{\mu \nu} = \pi (2\pi)^d m^2 \int dk k^\mu k^\nu \delta(k \cdot w) \hat{u}(k)^2. \]
(17)

Therefore, the kinetic equation takes the form
\[ \frac{\partial f}{\partial t} = \pi (2\pi)^d m \frac{\partial}{\partial \nu} \int k^\mu k^\nu \delta(k \cdot w) \hat{u}(k)^2 \left( f_1 \frac{\partial f}{\partial \nu} - f \frac{\partial f_1}{\partial \nu} \right) \, dv_1 \, dk, \]
(18)

where we have denoted \( f = f(v, t) \) and \( f_1 = f(v_1, t) \). The calculation of \( K^{\mu \nu} \) is detailed in appendix A. Finally, the kinetic equation can be written as
\[ \frac{\partial f}{\partial t} = K_2 \frac{\partial}{\partial \nu} \int w^2 \delta^{\mu \nu} - w^\mu w^\nu \left( f_1 \frac{\partial f}{\partial \nu} - f \frac{\partial f_1}{\partial \nu} \right) \, dv_1, \]
(19)

with
\[ K_2 = 8\pi^5 m \int_0^{+\infty} k^3 \hat{u}(k)^2 \, dk, \quad K_2 = 8\pi^3 m \int_0^{+\infty} k^2 \hat{u}(k)^2 \, dk. \]
(20)

This is the original form given by Landau [2] for the Coulombian interaction (in \( d = 3 \)). Using \( (2\pi)^3 \hat{u}(k) = 4\pi e^2 / (m^2 k^2) \), we get \( K_3 = 2\pi e^4 / m^3 \ln \Lambda \) where \( \ln \Lambda = \int_0^{+\infty} dk / k \) is the usual Coulomb logarithm that has to be regularized with appropriate cut-offs. The larger cut-off is the Debye length \( \lambda_D^3 \sim k_BT/(ne^2) \) and the lower cut-off is the Landau length \( L_{\text{min}} \sim e^2/(m v^2) \sim e^2/(k_BT) \sim 1/(n \lambda_D^3) \). Then, we get \( \ln \Lambda \sim \ln(\lambda_D/L_{\text{min}}) \sim \ln(n \lambda_D^3) \), where \( \Lambda = n \lambda_D^3 \) is the plasma parameter. It gives the number of electrons in the Debye sphere and it usually satisfies \( \Lambda \gg 1 \) (for plasmas, the \( 1/N \) expansion is replaced by a \( 1/\Lambda \) expansion) [33]. The Landau equation also applies to stellar systems if we make a local approximation. In that case, the distribution functions \( f = f(v, t) \) and \( f_1 = f(v_1, t) \) must be replaced by \( f = f(r, v, t) \) and \( f_1 = f(r, v_1, t) \) and we must restore the advective (Vlasov) term in the lhs where the gravitational field is determined by the Poisson

\[ \text{doi:10.1088/1742-5468/2010/05/P05019} \]

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Kinetic equations for systems with long-range interactions: a unified description

equation. Using \((2\pi)^d \delta (k) = -4\pi G/k^2\), this yields the Vlasov–Landau–Poisson equation

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = K \frac{\partial}{\partial v} \int \frac{w^2 \delta_{w w'} - w^\mu w'^\mu}{w^3} \left( f_1 \frac{\partial f}{\partial v^\nu} - f \frac{\partial f_1}{\partial v_1^\nu} \right) dv_1,
\]

(21)

\[
\Delta \Phi = 4\pi G \int f(r,v,t) dv,
\]

(22)

where \(K = 2\pi m G^2 \ln \Lambda\) and \(\ln \Lambda = \int_0^{+\infty} dk/k\) is the usual Coulomb logarithm that has to be regularized with appropriate cut-offs. The larger cut-off is the Jeans length \(\lambda_3^J \sim k_B T/(G m^2 n)\), which is of the order of the system size \(R\) (using a virial type argument \(v^2 \sim GM/R\) we get \(R^2 \sim v^2/(G m) \sim k_B T/(G m^2) \sim \lambda_3^J\)). The lower cut-off is the gravitational Landau length \(L_{\text{min}} \sim G m/v^2 \sim G m^2/(k_B T) \sim 1/(n \lambda_3^J)\). Then we get \(\ln \Lambda \sim \ln(\lambda_3^J/L_{\text{min}}) \sim \ln(n \lambda_3^J) \sim \ln N\) where \(N \sim n \lambda_3^J \sim n R^3\) is the number of stars in the cluster. In a globular cluster and in a galaxy, we have \(N \gg 1\) [34].

The kinetic equation generalizing the Landau equation (19) by taking into account collective effects is the Lenard–Balescu equation [3, 4]:

\[
\frac{\partial f}{\partial t} = \pi (2\pi)^d m \frac{\partial}{\partial \omega} \int k^\mu k^\nu \delta (k \cdot w) \frac{|\hat{u}(k)|^2}{|\epsilon(k,k \cdot v)|^2} \left( f_1 \frac{\partial f}{\partial v^\nu} - f \frac{\partial f_1}{\partial v_1^\nu} \right) dv_1 dk,
\]

(23)

where \(\epsilon(k,\omega)\) is the dielectric function

\[
\epsilon(k,\omega) = 1 + (2\pi)^d \hat{u}(k) \int \frac{k \cdot (\partial f/\partial v)}{\omega - k \cdot v} dv.
\]

(24)

The Landau equation is recovered by taking \(|\epsilon(k,k \cdot v)|^2 = 1\). In plasma physics, it has been shown that collective effects regularize at large scales the logarithmic divergence that appears in the Landau equation. This treatment precisely shows that the Coulomb logarithm must be cut-off at the Debye length \(\Lambda_D\). In stellar dynamics, when we make the local approximation, the dielectric function tends to zero as we approach the Jeans wavenumber, leading to a strong divergence. This is of course a manifestation of the Jeans instability for a homogeneous distribution. This shows that the local approximation is valid well below the Jeans length \(\lambda_J\). In that case, collective effects can be neglected and the Coulomb logarithm must be cut-off at the Jeans length \(\lambda_J\), i.e. the system size. Note that the divergences at large scale are due to the assumption of spatial homogeneity and to the linear trajectory approximation, and are not present in the generalized Landau equations (2). The Landau and the Lenard–Balescu equations conserve mass \(M = \int f dv\) and energy \(E = \int f(v^2/2) dv\) (reducing to the kinetic energy \(E = \int f(v^2/2) dv\) for a spatially homogeneous system) and monotonically increase the Boltzmann entropy \(S = -\int (f/m) \ln(f/m) dv\) \((H\text{-theorem})\). The collisional evolution is due to a condition of resonance between the orbits of the particles. For spatially homogeneous systems, the condition of resonance encapsulated in the \(\delta\)-function corresponds to \(k \cdot v = k \cdot v_1\) with \(v \neq v_1\). For \(d > 1\), the only stationary solution of the Landau and Lenard–Balescu equations is the Maxwell distribution \(f = A e^{-\beta m v^2/2}\) and these equations relax towards the Maxwell distribution for \(t \to +\infty\). Since the collision term is valid at the order \(O(1/N)\), the relaxation time scales like

\[
t_R \sim N t_D, \quad (d > 1).
\]

(25)

doi:10.1088/1742-5468/2010/05/P05019
In a plasma, we get from equation (19) an estimate of the collision frequency \( \nu \sim ne^4/(m^2v^3) \ln \Lambda \), where \( mv^2 \sim k_B T \). The relaxation time can be estimated as \( t_R \sim \nu^{-1} \sim (\Lambda/\ln \Lambda)\omega_p^{-1} \), where \( \omega_p \sim (ne^2/m)^{1/2} \sim v/\lambda_D \) is the plasma pulsation. In a stellar system, we get from equation (21) an estimate of the relaxation time \( t_R \sim v^3/(nm^2G^2 \ln N) \) leading to \( t_R \sim (N/\ln N)t_D \), where \( t_D \sim \lambda_D/v \sim R/v \) is the dynamical time. This is the Chandrasekhar relaxation time. On the other hand, the scaling (25), predicted in [72], has been confirmed for a two-dimensional plasma in [28,96]. For one-dimensional systems, like the HMF model, the Lenard–Balescu kinetic equation (23) reduces to

\[
\frac{\partial f}{\partial t} = 2\pi^2 m \frac{\partial}{\partial v} \int dv_1 dk |k| \frac{\hat{u}(k)^2}{|\epsilon(k,kv)|^2} \delta(v-v_1) \left( f_1 \frac{\partial f}{\partial v} - f \frac{\partial f_1}{\partial v_1} \right) = 0, \tag{26}
\]

where we have used the identity \( \delta(\lambda x) = (1/|\lambda|)\delta(x) \). Therefore, the collision term \( C_N[f] \) vanishes at the order 1/N because there is no resonance. The kinetic equation reduces to \( \partial f/\partial t = 0 \), so the distribution function does not evolve at all on a timescale \( \sim Nt_D \). This result has been known for a long time in plasma physics and was discovered by Eldridge and Feix [74] (see also [75]) for a one-dimensional plasma: ‘to first order in the small parameter \( g = (n\lambda_D)^{-1} \), the detailed balance between drag and diffusion is valid not only at thermal equilibrium but for any stable function \( f(v) \). So the Maxwellization is, at least, a second order effect in \( g \) and consequently a very slow process’. This result has been rediscovered recently for the HMF model by Bouchet and Dauxois [70], Chavanis et al [71] and Chavanis [72]. This implies that, for one-dimensional homogeneous systems, the relaxation time towards statistical equilibrium is larger than \( Nt_D \). We therefore expect that

\[
t_R > Nt_D, \quad (d = 1). \tag{27}
\]

For 1D neutral plasmas, the relaxation time has been estimated numerically to scale like \( t_R \sim (n\lambda_D)^2\omega_p^{-1} \) [97,98] (which is the next order term in the expansion in \( g = 1/\Lambda \)), but a more precise numerical study would be useful to check the exact value of the exponent. For the HMF model, it has been found numerically that \( t_R \sim N^{1.7}t_D \) [76] or \( t_R \sim e^{Nt_D} \) [77], depending on the type of initial conditions. For 1D inhomogeneous distributions, the relaxation time can be reduced and approach the natural scaling \( Nt_D \) associated with the generalized kinetic equation (2) (see the discussion in section 5).

### 2.2. Test particle in a thermal bath: the Fokker–Planck equation

We now consider a ‘test’ particle (tagged particle) evolving in a steady distribution of ‘field’ particles. The test particle has a stochastic motion. We assume the system to be spatially homogeneous. Let us call \( P(v,t) \) the probability density of finding the test particle with velocity \( v \) at time \( t \). The evolution of \( P(v,t) \) can be obtained from the Lenard–Balescu equation (23) by considering that the distribution \( f_1 \) of the field particles is fixed. Thus, we replace \( f = f(v,t) \) by \( P = P(v,t) \) and \( f_1 = f(v_1,t) \) by \( f_1 = f(v_1) \), where \( f(v) \) is any stable stationary solution of the Vlasov equation. This procedure [72] transforms the integro-differential equation (23) into the differential equation

\[
\frac{\partial P}{\partial t} = \pi(2\pi)^d m \frac{\partial}{\partial v^\mu} \int k^\mu k^\nu \delta(\mathbf{k} \cdot \mathbf{w}) \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k},\mathbf{k} \cdot \mathbf{v})|^2} \left( f_1 \frac{\partial P}{\partial v^\mu} - P \frac{\partial f_1}{\partial v^\nu} \right) \, dv_1 \, dk, \tag{28}
\]

\[\text{doi:10.1088/1742-5468/2010/05/P05019}\]

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where $\epsilon(\mathbf{k}, \omega)$ is the dielectric function corresponding to the fixed distribution function $f(\mathbf{v})$. Equation (28) can be written in the form of a Fokker–Planck equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v_\mu}\left(D^{\mu\nu} \frac{\partial P}{\partial v_\nu} - P \eta^\mu\right),$$  \hspace{1cm} (29)$$

involving a diffusion term

$$D^{\mu\nu} = \pi(2\pi)^d m \int k^{\mu} k^{\nu} \delta(\mathbf{k} \cdot \mathbf{w}) \frac{\hat{u}(\mathbf{k})^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} f_1 \, d\mathbf{v}_1 \, d\mathbf{k},$$  \hspace{1cm} (30)$$

and a friction term due to the polarization

$$\eta^\mu = F^\mu_{\text{pol}} = \pi(2\pi)^d m \int k^{\mu} k^{\nu} \delta(\mathbf{k} \cdot \mathbf{w}) \frac{\hat{u}(\mathbf{k})^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} \frac{\partial f_1}{\partial v_1^\nu} \, d\mathbf{v}_1 \, d\mathbf{k}. \hspace{1cm} (31)$$

The diffusion term can be directly derived from the Kubo formula, and the friction force due to the polarization can be directly obtained from a linear response theory, as shown by Kandrup [15] and Chavanis [20] in the general case of a possibly inhomogeneous distribution. Since the diffusion coefficient depends on the velocity $\mathbf{v}$ of the test particle, it is useful to rewrite equation (29) in a form that is fully consistent with the general Fokker–Planck equation

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial v^\mu \partial v^\nu} \left(P \frac{\langle \Delta v^\mu \Delta v^\nu \rangle}{\Delta t}\right) - \frac{\partial}{\partial v^\mu} \left(P \frac{\langle \Delta v^\mu \rangle}{\Delta t}\right),$$  \hspace{1cm} (32)$$

with

$$\frac{\langle \Delta v^\mu \Delta v^\nu \rangle}{2\Delta t} = D^{\mu\nu}, \hspace{1cm} \frac{\langle \Delta v^\mu \rangle}{\Delta t} = \frac{\partial D^{\mu\nu}}{\partial v^\nu} + \eta^\mu \equiv F^\mu_{\text{friction}}. \hspace{1cm} (33)$$

Substituting equations (30) and (31) into equation (33), and using an integration by parts, we find that the diffusion and friction coefficients are given by

$$\frac{\langle \Delta v^\mu \Delta v^\nu \rangle}{2\Delta t} = \pi(2\pi)^d m \int k^{\mu} k^{\nu} \delta(\mathbf{k} \cdot \mathbf{w}) \frac{\hat{u}(\mathbf{k})^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} f_1 \, d\mathbf{v}_1 \, d\mathbf{k},$$  \hspace{1cm} (34)$$

$$\frac{\langle \Delta v^\mu \rangle}{\Delta t} = \pi(2\pi)^d m \int k^{\mu} k^{\nu} f_1 \left(\frac{\partial}{\partial v^\nu} - \frac{\partial}{\partial v^\nu_1}\right) \delta(\mathbf{k} \cdot \mathbf{w}) \frac{\hat{u}(\mathbf{k})^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} \, d\mathbf{v}_1 \, d\mathbf{k}. \hspace{1cm} (35)$$

The two forms (29) and (32) of the Fokker–Planck equation have their own interest. The expression (32) where the diffusion coefficient is placed after the two derivatives $\partial^2 (DP)$ involves the total friction force $F^\mu_{\text{friction}} = \langle \Delta v^\mu \rangle/\Delta t$ and the expression (29) where the diffusion coefficient is placed between the derivatives $\partial D \partial P$ isolates the part of the friction $\eta^\mu = F^\mu_{\text{pol}}$ due to the polarization [20].

In stellar dynamics, the coefficients of diffusion and friction were first obtained by Chandrasekhar [7, 8] by considering the mean effect of a succession of binary encounters. They were also computed by Cohen et al [10], Gasiorowicz et al [11] and Rosenbluth et al [12] in a plasma physics context by neglecting collective effects. This leads to expressions that are consistent with equations (34) and (35) with $|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2 = 1$ (see, e.g., [28] for a discussion of this link). These expressions can also be obtained directly from the Hamiltonian equations of motion by making a systematic expansion of the trajectory of the particles in powers of $1/N$ in the limit $N \to +\infty$, as shown in appendix A of [20]. On the
other hand, the expressions (34) and (35) for the diffusion and the friction, properly taking into account collective effects, were first derived by Hubbard [6] in plasma physics by directly evaluating $\langle \Delta v^\mu \Delta v^\nu \rangle$ and $\langle \Delta v^\mu \rangle$. Collective effects encapsulated in the dielectric function remove the logarithmic divergence that otherwise appears at large scales. The work of Hubbard [6] was done independently from the works of Lenard [3] and Balescu [4] that were published at about the same time, but they are clearly connected since the equations (34) and (35) of Hubbard [6] can be derived from the Lenard–Balescu equation, and vice versa, as explained above (see also discussion in [20]).

Let us now consider particular cases.

- If $d = 1$, the expressions of the diffusion and friction simplify to

$$D(v) = 4\pi^2 m f(v) \int_0^{+\infty} \frac{k\hat{u}(k)^2}{|\epsilon(k, kv)|^2} \, dk,$$

$$\eta(v) = 4\pi^2 m f'(v) \int_0^{+\infty} \frac{k\hat{u}(k)^2}{|\epsilon(k, kv)|^2} \, dk = D(v) \frac{d\ln f}{dv},$$

and the Fokker–Planck equation (28) reduces to

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v} \left[ D(v) \left( \frac{\partial P}{\partial v} - P \frac{d\ln f}{dv} \right) \right].$$

The diffusion and the friction coefficients were derived by Eldridge and Feix [74] for one-dimensional plasmas and by Bouchet and Dauxois [70] for the HMF model by directly evaluating $\langle (\Delta v)^2 \rangle$ and $\langle \Delta v \rangle$ (see [68] for detailed calculations). Alternatively, Chavanis [72] (see also [20, 28, 71]) obtains equations (36)–(39) as a particular case of the general expressions (28)–(35) valid in any dimension of space.

- If we neglect collective effects, the diffusion and friction coefficients reduce to

$$D^{\mu\nu} = \pi (2\pi)^d m \int k^{\mu} k^{\nu} \delta(k \cdot w) \hat{u}(k)^2 f_1 \, dv_1 \, dk,$$

$$\eta^{\mu} = F^{\mu}_{\text{pol}} = \pi (2\pi)^d m \int k^{\mu} k^{\nu} \delta(k \cdot w) \hat{u}(k)^2 \frac{\partial f_1}{\partial v_1^{\nu}} \, dv_1 \, dk,$$

$$F^{\mu}_{\text{friction}} = \pi (2\pi)^d m \int k^{\mu} k^{\nu} f_1 \left( \frac{\partial}{\partial v^{\nu}} - \frac{\partial}{\partial v_1^{\nu}} \right) \delta(k \cdot w) \hat{u}(k)^2 \, dv_1 \, dk.$$

After a series of elementary transformations, we obtain

$$\frac{\partial D^{\mu\nu}}{\partial v^{\nu}} = \pi (2\pi)^d m \int k^{\mu} k^{\nu} \left[ \frac{\partial}{\partial v^{\nu}} \delta(k \cdot w) \right] \hat{u}(k)^2 f_1 \, dv_1 \, dk,$$

$$= -\pi (2\pi)^d m \int k^{\mu} k^{\nu} \left[ \frac{\partial}{\partial v_1^{\nu}} \delta(k \cdot w) \right] \hat{u}(k)^2 f_1 \, dv_1 \, dk,$$

$$= \pi (2\pi)^d m \int k^{\mu} k^{\nu} \delta(k \cdot w) \hat{u}(k)^2 \frac{\partial f_1}{\partial v_1^{\nu}} \, dv_1 \, dk = \eta^{\mu}. \quad (43)$$
Combining equation (33) with (43), we find that
\[ \frac{\langle \Delta v \rangle}{\Delta t} = 2\eta^\mu, \text{ i.e. } F_{\text{friction}} = 2F_{\text{pol}}, \] (44)
so that the friction force \( F_{\text{friction}} \) is equal to twice the friction due to the polarization \( F_{\text{pol}} \).

This explains the difference of factor 2 in the calculation of \( F_{\text{friction}} \) by Chandrasekhar [8] and the calculations of \( F_{\text{pol}} \) by Kalnajs [14], Kandrup [15] and Chavanis [20]. Some simplified forms of the diffusion coefficient and friction force are given in [28] in different dimensions of space \( d \).

- For a thermal bath, the field particles have the Maxwell distribution of statistical equilibrium
\[ f(v_1) = \left( \frac{\beta m}{2\pi} \right)^{d/2} \rho e^{-\beta m(v_1^2/2)}. \] (45)
Substituting
\[ \frac{\partial f_1}{\partial v_1^\nu} = -\beta m f_1 v_1^\nu, \] (46)
in equation (31), we obtain
\[ \eta^\mu = -\beta \pi (2\pi)^d m^2 \int k^\mu(k \cdot v_1) \delta[k \cdot (v - v_1)] \frac{\hat{u}(k)^2}{|\epsilon(k, k \cdot v)|^2} f_1 \, dv_1 \, dk. \] (47)
Using the \( \delta \)-function to replace \( k \cdot v_1 \) by \( k \cdot v \), we get
\[ \eta^\mu = -\beta \pi (2\pi)^d m^2 \nu^\nu \int k^\mu k^\nu \delta[k \cdot (v - v_1)] \frac{\hat{u}(k)^2}{|\epsilon(k, k \cdot v)|^2} f_1 \, dv_1 \, dk. \] (48)
Comparing the resulting expression with equation (30), we find that
\[ \eta^\mu = -\beta m D^{\mu\nu} \nu^\nu, \] (49)
which is the appropriate Einstein relation for our problem. Note that it is valid for the friction force due to the polarization \( F_{\text{pol}} \), not for the total friction force \( F_{\text{friction}} \) [20]. The Fokker–Planck equation (28) takes the form
\[ \frac{\partial P}{\partial t} = \frac{\partial}{\partial v^\mu} \left[ D^{\mu\nu}(v) \left( \frac{\partial P}{\partial v^\nu} + \beta m P v^\nu \right) \right], \] (50)
where the diffusion coefficient is given by equation (30) with equation (45). This is similar to the Kramers equation except that the diffusion coefficient is a tensor and that it depends on the velocity. Some simplified forms of the diffusion coefficient are given in [72] in different dimensions of space \( d \).

We emphasize that, according to equation (28), a test particle relaxes towards an equilibrium state [28] on a typical time \( t_R \sim N t_D \) in any dimension of space (for stellar systems we have \( t_R \sim (N/\ln N) t_D \)). In particular, in \( d = 1 \) dimension, the distribution \( P(v,t) \) of a test particle relaxes towards the distribution of the bath \( f(v) \) (that can be different from the Maxwellian as long as it is a stable steady state of the Vlasov equation) on a timescale of order \( N t_D \) while the overall distribution of the system \( f(v,t) \) does not change at all on this timescale [98].
3. Two-dimensional point vortices

3.1. The evolution of the system as a whole: the kinetic equation

We consider a system of point vortices described by the Hamiltonian equations

$$\gamma \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \gamma \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i},$$

$$H = \gamma^2 \sum_{i<j} u(r_i - r_j).$$

(51)

The usual interaction corresponds to $u(r - r') = -(1/2\pi) \ln |r - r'|$ but we shall let $u$ arbitrary in order to treat more general cases. We consider the proper thermodynamic limit $N \to +\infty$ in such a way that $V \sim E \sim \beta/N \sim 1$ and $\gamma \sim 1/N$ [85] (for rigorous mathematical results see [99]-[103]). When collective effects are ignored, the kinetic equation describing the evolution of the smooth vorticity profile $\omega(r, t)$ at the order $O(1/N)$ can be written

$$\frac{\partial \omega}{\partial t} + \frac{N-1}{N} \langle V \rangle \cdot \frac{\partial \omega}{\partial r} = \frac{\partial}{\partial r^\mu} \int_0^t d\tau \int dr_1 V^\nu(1 \to 0) G(t, t - \tau)$$

$$\times \left[ V^\nu(1 \to 0) \frac{\partial}{\partial r^\nu} + V^\nu(0 \to 1) \frac{\partial}{\partial r^\nu} \right] \omega(r, t - \tau) \frac{\omega(r_1, t - \tau)}{\gamma} \omega(r_1, t - \tau),$$

(52)

where $V(1 \to 0)$ is the velocity created by point vortex 1 (located in $r_1$) on point vortex 0 (located in $r$) and $V^\nu(1 \to 0) = V(1 \to 0) - \langle V \rangle/N$ is the fluctuating velocity with respect to the average velocity $\langle V \rangle(r, t) = -z \times \nabla \psi(r, t)$ at the location of point vortex 0 (here $\psi(r, t) = \int u(r - r')\omega(r', t) dr'$ is the stream function). On the other hand, $G(t, t - \tau)$ is the Greenian constructed with the mean flow. This kinetic equation has been obtained by Chavanis [56, 57] from a projection operator formalism, a BBGKY-like hierarchy and a quasilinear theory. The BBGKY hierarchy is closed by neglecting the three-body correlation function which is of order $O(1/N^2)$. The structure of this equation bears a clear physical meaning. It involves a diffusion term and a drift term. Furthermore, the coefficients of diffusion and drift are given by generalized Kubo formulæ, i.e. the time integral of the temporal correlations of the fluctuating velocity. The collision term $C_N[\omega]$ (rhs) is valid at the order $1/N$ so that it describes the ‘collisional’ evolution of the point vortex gas (ignoring collective effects) on a timescale of order $Nt_D$, where $t_D \sim R^2/(N\gamma) \sim 1/\omega$ is the dynamical time that is of order unity in the thermodynamic limit. For $N \to +\infty$, we obtain the 2D Euler equation in which collisions (more properly correlations) are neglected. The 2D Euler equation is the counterpart of the Vlasov equation in plasma physics and stellar dynamics. The 2D Euler–Poisson system can generate a process of violent relaxation towards a quasi stationary state, as described by Miller [104] and Robert and Sommeria [105]. This is the hydrodynamical counterpart of the Lynden-Bell [91] theory of violent relaxation in stellar dynamics (see, e.g. [62, 93], for a description of this analogy and [57, 93, 106, 107] for kinetic theories of violent relaxation in 2D turbulence). Equation (52) is a non-Markovian integro-differential equation. It can describe the dynamics of point vortices taking into account delocalizations in space and time (i.e. non-axisymmetry and memory effects). This kinetic equation is the vortex analog of the generalized Landau equation (2) for stellar systems. If we make the
Markovian approximation and extend the time integral to $+\infty$, we obtain
\[
\frac{\partial \omega}{\partial t} + \frac{N - 1}{N} \langle V \rangle \cdot \frac{\partial \omega}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \int_0^{+\infty} d\tau \int dr_1 \, V'(1 \to 0)G(t, t - \tau) \times \left[ \frac{\partial}{\partial r} V'(1 \to 0) + \frac{\partial}{\partial r_1} V'(0 \to 1) \right] \omega(r, t) \frac{\omega}{\gamma}(r_1, t).
\]
(53)

The Markovian approximation may not be justified in every situation since it has been shown numerically that point vortices can exhibit long jumps (Lévy flights) and correlations (see section 5 for a more detailed discussion). However, for $N \to +\infty$, the Markovian approximation should be justified in general because the timescale $\sim N t_D$ on which $\omega(r, t)$ changes is long compared to the timescale $\tau_{\text{corr}}$ on which the integrand in equation (52) has significant support. The Markovian equation (53) still applies to possibly non-axisymmetric distributions, which is the norm for 2D flows. Now, for axisymmetric distributions $\omega(r, t)$, the kinetic equation reduces to
\[
\frac{\partial \omega}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \int_0^{+\infty} d\tau \int dr_1 \, V'(1 \to 0)G(t, t - \tau) \times \left[ V'(1 \to 0) \frac{\partial}{\partial r} + V'(0 \to 1) \frac{\partial}{\partial r_1} \right] \omega(r, t) \frac{\omega}{\gamma}(r, t).
\]
(54)

Using $V'(0 \to 1) = -(r/r_1)V'(1 \to 0)$ (see below) and the equations of the unperturbed trajectories
\[
r(t - \tau) = r(t) = r, \quad \theta(t - \tau) = \theta(t) - \Omega(r, t)\tau = \theta - \Omega(r, t)\tau,
\]
(55)
corresponding to $\langle V \rangle = V(r, t)e_\theta$ with $V(r, t) = \Omega(r, t)r$ (where $\Omega$ is the angular velocity), the kinetic equation can be written
\[
\frac{\partial \omega}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \int_0^{+\infty} r_1 \, dr_1 K \left( \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r_1} \frac{\partial}{\partial r_1} \right) \omega(r, t) \frac{\omega}{\gamma}(r, t),
\]
(56)
with
\[
K = r^2 \int_0^{+\infty} d\tau \int_0^{2\pi} d\theta_1 V'(1 \to 0, t)V'(1 \to 0, t - \tau).
\]
(57)

In consistency with the Markov approximation, we have assumed that the angular velocity $\Omega(r, t)$ does not change on the timescale $\tau_{\text{corr}}$ on which the velocity correlations have their support. The velocity created by point vortex 1 on point vortex 0 is given by $V'(1 \to 0) = -\gamma z \times \nabla u(r - r_1)$, leading to
\[
V'(1 \to 0) = \gamma \frac{\partial u}{r} (r - r_1).
\]
(58)

The potential of interaction is of the form $u(r - r_1) = u(r, r_1, \theta - \theta_1)$. It depends on $\cos(\theta - \theta_1)$ and is invariant with respect to the interchange of 0 and 1. Introducing the Fourier transform with respect to the angles
\[
u(r, r_1, \phi) = \sum_m e^{im\phi} \hat{u}_m(r, r_1), \quad \hat{u}_m(r, r_1) = \frac{1}{2\pi} \int_0^{2\pi} u(r, r_1, \phi) \cos(m\phi) d\phi,
\]
(59)
we get
\[ V_r(1 \to 0, s) = \frac{i}{r(s)} \sum_m m e^{i m(\theta(s) - \theta_1(s))} u_m(r(s), r_1(s)). \] (60)

Using the equations of motion (55), and introducing \( \phi = \theta - \theta_1 \) and \( \Delta \Omega = \Omega(r, t) - \Omega(r_1, t) \), we obtain
\[ V_r(1 \to 0, t - \tau) = \frac{i}{r} \sum_m m e^{i m(\phi - \Delta \Omega \tau)} u_m(r, r_1). \] (61)

Therefore
\[ K = -\gamma^2 \int_{0}^{+\infty} d\tau \int_{0}^{2\pi} d\phi \sum_{mn} m^2 e^{i n \phi} e^{i m(\phi - \Delta \Omega \tau)} u_m(r, r_1) u_n(r, r_1). \] (62)

Using the identity
\[ \delta_{n,0} = \int_{0}^{2\pi} e^{i n \theta} \frac{d\theta}{2\pi}, \] (63)
and integrating on \( \phi \), then summing on \( n \), we find that
\[ K = 2\pi \gamma^2 \int_{0}^{+\infty} d\tau \sum_m m^2 e^{i m \Delta \Omega \tau} u_m(r, r_1). \] (64)
Since \( u_m(r_1, t) = \hat{u}_m(r_1, r_1) \), the foregoing equation can be rewritten
\[ K = 2\pi \gamma^2 \int_{0}^{+\infty} d\tau \sum_m m^2 e^{i m \Delta \Omega \tau} \hat{u}_m(r, r_1)^2. \] (65)

Making the transformation \( \tau \to -\tau \), then \( m \to -m \), and adding the resulting expression to equation (65), we get
\[ K = \pi \gamma^2 \int_{-\infty}^{+\infty} d\tau \sum_m m^2 e^{i m \Delta \Omega \tau} \hat{u}_m(r, r_1)^2. \] (66)

Using the identity (13), we finally obtain
\[ K = 2\pi^2 \gamma^2 \sum_m m^2 \delta(\Omega) \delta(\Omega_1) \hat{u}_m(r, r_1)^2. \] (67)

Therefore, the kinetic equation takes the form
\[ \frac{\partial \omega}{\partial t} = 2\pi^2 \gamma \left( \frac{1}{r} \frac{\partial}{\partial r} \int_{0}^{+\infty} r_1 dr_1 \chi(r, r_1) \delta(\Omega - \Omega_1) \left( \frac{1}{r} \frac{\partial \omega}{\partial r} - \frac{1}{r_1} \frac{\partial \omega}{\partial r_1} \right) \right), \] (68)
with
\[ \chi(r, r_1) = \sum_{m=-\infty}^{+\infty} |m| \hat{u}_m(r, r_1)^2, \] (69)
where we have used the identity \( \delta(\lambda x) = (1/|\lambda|) \delta(x) \) and noted \( \omega = \omega(r, t), \omega_1 = \omega(r_1, t), \Omega = \Omega(r, t) \) and \( \Omega_1 = \Omega(r_1, t) \). The calculation of \( \chi(r, r_1) \) is detailed in appendix B for the ordinary potential \( u(r - r') = -(1/2\pi) \ln |r - r'| \). Equation (68) has been
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It is the vortex analog of the Landau equation (18). A more general kinetic equation, taking into account collective effects, has been derived by Dubin and O’Neil [55] by a different method. It is the vortex analog of the Lenard–Balescu equation (23). The kinetic equation (68) conserves circulation $\Gamma = \int \omega \, dr$, energy $E = \frac{1}{2} \int \omega \psi \, dr$ and angular momentum $L = \int \omega r^2 \, dr$, and monotonically increases the Boltzmann entropy $S = -\int (\omega / \gamma) \ln (\omega / \gamma) \, dr$ ($H$-theorem). The collisional evolution is due to a condition of resonance between distant orbits of the point vortices. For axisymmetric systems, the condition of resonance encapsulated in the $\delta$-function corresponds to $\Omega(r, t) = \Omega(r_1, t)$ with $r \neq r_1$. This describes ‘long-range collisions’: there is a vorticity current in $r$ if and only if there exist point vortices in $r_1 \neq r$ rotating with the same angular velocity as vortices in $r$. The evolution stops when the profile of angular velocity becomes monotonic (so that there is no resonance) even if the system has not reached the statistical equilibrium state given by the Boltzmann distribution [51]–[53]. This ‘kinetic blocking’ has been illustrated numerically in [58]. Indeed, the Boltzmann distribution $\omega = Ae^{-\beta(\psi + \frac{1}{2} \Omega L r^2)}$ is not the unique steady state of the kinetic equation (68): any vorticity distribution associated with a monotonic profile of angular velocity is a steady state of equation (68). In particular, if the profile of angular velocity is initially monotonic, the collision term $C_N[\omega]$ vanishes at the order $1/N$ because there is no resonance [56]. The kinetic equation reduces to $\partial \omega / \partial t = 0$, so the vorticity does not evolve at all on a timescale $\sim Nt_D$. This implies that, for axisymmetric distributions of point vortices, the relaxation time towards statistical equilibrium is larger than $Nt_D$. We therefore expect that

$$t_R > Nt_D \quad \text{(axisymmetric flows).}$$

For non-axisymmetric distributions, the relaxation time can be reduced and approach the natural scaling $Nt_D$ associated with the generalized kinetic equation (52) (see discussion in section 5). In any case, for $N \gg 1$, the relaxation towards the Boltzmann distribution is a very slow process. The possible slow timescale of mixing was pointed out by Onsager in a letter to Lin [54]: ‘I still have to find out whether the processes anticipated by these considerations are rapid enough to play a dominant role in the evolution.’ and it is now confirmed by the kinetic theory (as far as we know, the process of violent relaxation [91, 104, 105] was not foreseen by Onsager, who focused on the ordinary statistical equilibrium state). In fact, the scaling of the relaxation time with $N$ is still not known for axisymmetric distributions and it would be interesting to obtain it numerically. It is not even clear whether the system of point vortices truly relaxes towards the Boltzmann distribution predicted by statistical mechanics [51]–[53]. Indeed, as discussed by Khanin [59], the dynamics may be non-ergodic. This was also a concern of Onsager: ‘We inquire about the ergodic motion of the system’, Onsager wrote to Lin [54].

3.2. Test vortex in a thermal bath: the Fokker–Planck equation

We now consider the relaxation of a ‘test’ vortex (tagged particle) evolving in a steady distribution of ‘field’ vortices. The test vortex has a stochastic motion. Let us call $P(\mathbf{r}, t)$ the probability density of finding the test vortex at position $\mathbf{r}$ at time $t$. For axisymmetric
distributions, the evolution of \( P(r, t) \) can be obtained from the kinetic equation (68) by considering that the distribution of the field vortices \( \omega_1 \) is fixed. Thus, we replace \( \omega = \omega(r, t) \) by \( P = P(r, t) \) and \( \omega_1 = \omega(r_1, t) \) by \( \omega_1 = \omega(r_1) \), where \( \omega(r) \) is any stable stationary solution of the 2D Euler equation. This procedure [56]–[58] transforms the integro-differential equation (68) into the differential equation

\[
\frac{\partial P}{\partial t} = 2\pi^2 \gamma \frac{1}{r} \frac{\partial}{\partial r} \int_0^{+\infty} r_1 dr_1 \chi(r, r_1) \delta(\Omega - \Omega_1) \left( \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r_1} \frac{\partial}{\partial r_1} \right) P(r, t) \omega(r_1).
\]

This is the vortex analog of equation (28). Equation (71) can be written in the form of a Fokker–Planck equation

\[
\frac{\partial P}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[ r \left( \frac{\partial P}{\partial r} - P \eta \right) \right],
\]

involving a diffusion term

\[
D = \frac{2\pi^2 \gamma}{r^2} \int_0^{+\infty} r_1 dr_1 \chi(r, r_1) \delta(\Omega - \Omega_1) \omega(r_1),
\]

and a drift term due to the polarization

\[
\eta \equiv V_r^{\text{pol}} = \frac{2\pi^2 \gamma}{r} \int_0^{+\infty} dr_1 \chi(r, r_1) \delta(\Omega - \Omega_1) \frac{d\omega}{dr}(r_1).
\]

The diffusion coefficient can be directly derived from the Kubo formula and the drift velocity due to the polarization can be directly derived from a linear response theory, as shown by Chavanis [56, 57, 60] in the general case of a possibly non-axisymmetric distribution. Since the diffusion coefficient depends on the position \( r \) of the test vortex, it is useful to rewrite equation (71) in a form that is fully consistent with the general Fokker–Planck equation

\[
\frac{\partial P}{\partial t} = \frac{1}{2r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \left( \frac{\langle(\Delta r)^2 \rangle}{\Delta t} P \right) \right) - \frac{1}{r} \frac{\partial}{\partial r} \left( rP \frac{\langle \Delta r \rangle}{\Delta t} \right),
\]

with

\[
\frac{\langle(\Delta r)^2 \rangle}{2\Delta t} = D,
\]

\[
\frac{\langle \Delta r \rangle}{\Delta t} = \frac{\partial D}{\partial r} + \eta \equiv V_r^{\text{drift}}.
\]

Substituting equations (73) and (74) in equation (76), and using an integration by parts, we find that the diffusion and drift coefficients are given by

\[
\frac{\langle(\Delta r)^2 \rangle}{2\Delta t} = \frac{2\pi^2 \gamma}{r^2} \int_0^{+\infty} r_1 dr_1 \chi(r, r_1) \delta(\Omega - \Omega_1) \omega(r_1),
\]

\[
\frac{\langle \Delta r \rangle}{\Delta t} = 2\pi^2 \gamma \int_0^{+\infty} r dr_1 \omega_1 \left( \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r_1} \frac{\partial}{\partial r_1} \right) \chi(r, r_1) \delta(\Omega - \Omega_1) \frac{1}{r^2}.
\]

The two forms (72) and (75) of the Fokker–Planck equation are interesting to consider as they involve respectively the drift due to the polarization \( V_r^{\text{pol}} \) or the total drift \( V_r^{\text{drift}} \) [57]. Expressions (77) and (78) for the diffusion coefficient and the drift term can be obtained directly from the Hamiltonian equations, by making a systematic expansion

doi:10.1088/1742-5468/2010/05/P05019
of the trajectories of the point vortices in powers of $1/N$ in the limit $N \to +\infty$, as shown in appendix C of [58].

Let us now consider particular cases.

- If the profile of angular velocity of the field vortices $\Omega(r)$ is monotonic, using the identity $\delta(\Omega - \Omega_1) = \delta(r - r_1)/|\Omega'(r)|$, we find that the expressions of the diffusion and drift simplify into

$$
D(r) = 2\pi^2 \gamma \frac{\chi(r, r)}{|\Sigma(r)|} \omega(r),
$$

(79)

$$
\eta = 2\pi^2 \gamma \frac{\chi(r, r)}{|\Sigma(r)|} \omega'(r) = D(r) \frac{d\ln \omega}{dr},
$$

(80)

$$
\frac{\langle \Delta r \rangle}{\Delta t} = D'(r) + D(r) \frac{d\ln \omega}{dr},
$$

(81)

where $\Sigma(r) = r\Omega'(r)$ is the local shear. The Fokker–Planck equation (71) can then be written

$$
\frac{\partial P}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[ rD(r) \left( \frac{\partial P}{\partial r} - P \frac{d\ln \omega}{dr} \right) \right],
$$

(82)

with a diffusion coefficient given by equation (79). Equation (82) is the vortex analog of equation (39). The expressions (79) and (80) of the diffusion coefficient and of the drift, that are inversely proportional to the shear, were first derived for a thermal bath in [60] and for an arbitrary distribution of the field vortices in [56] (see remark before equation (123)). For the usual potential of interaction (see appendix B), we have $\chi(r, r) = (1/8\pi^2) \ln \Lambda$, where $\ln \Lambda = \sum_{m=1}^{\infty} (1/m)$ is a Coulombian logarithm that has to be regularized appropriately. It scales like $\ln \Lambda \sim \frac{1}{2} \ln N$ (see, e.g., [58]).

- For a thermal bath, the field vortices have the Boltzmann distribution of statistical equilibrium

$$
\omega(r_1) = A \gamma e^{-\beta \gamma \psi'(r_1)},
$$

(83)

where $\psi'(r_1) = \psi(r_1) + (\Omega_L/2)r_1^2$ is the relative stream function taking into account the invariance by rotation [58]. We have

$$
\frac{d\omega_1}{dr_1} = -\beta \gamma \omega(r_1) \frac{d\psi'}{dr_1} = \beta \gamma \omega(r_1)(\Omega(r_1) - \Omega_L)r_1,
$$

(84)

where we have used $\Omega(r) = V(r)/r = -(1/r) d\psi/dr$. Substituting this relation in equation (74), we obtain

$$
\eta = \beta \frac{2\pi^2 \gamma^2}{r} \int_0^{+\infty} dr_1 \chi(r, r_1) \delta(\Omega - \Omega_1) \omega_1(\Omega_1 - \Omega_L)r_1.
$$

(85)

Using the $\delta$-function to replace $\Omega_1$ by $\Omega$, then using $\Omega(r) - \Omega_L = -(1/r)d\psi'/dr$, and comparing the resulting expression with equation (73), we finally find that

$$
\eta = -D \beta \gamma \frac{d\psi'}{dr}.
$$

(86)

The drift is perpendicular to the relative mean field velocity $\langle \mathbf{V'} \rangle = -(d\psi'/dr)\mathbf{e}_\theta$ and the drift coefficient satisfies an Einstein relation $\xi = D \beta \gamma$ [60]. We stress that the Einstein

doi:10.1088/1742-5468/2010/05/P05019
relation is valid for the drift $V^{\text{pol}}_r = \eta$ due to the polarization, not for the total drift $V^{\text{drift}}_r$ [57]. For a thermal bath, using equation (86), the Fokker–Planck equation (72) takes the form

\[ \frac{\partial P}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[ rD(r) \left( \frac{\partial P}{\partial r} + \beta \gamma P \frac{d\psi'}{dr} \right) \right], \]  

(87)

where $D(r)$ is given by equation (73) with equation (83). Of course, if the profile of angular velocity of the Boltzmann distribution is monotonic, we find that equation (80) with equation (83) returns equation (86) with a diffusion coefficient given by equation (79) with equation (83). Note that the systematic drift $V^{\text{pol}}_r = -D \beta \gamma \nabla \psi'$ of a point vortex [60] is the counterpart of the dynamical friction $F^{\text{pol}} = -D \beta m v$ of a star [8] and the Fokker–Planck equation (87) is the counterpart of the Fokker–Planck equation (50). We have presented the results for axisymmetric distributions, but similar results can be obtained for unidirectional flows [56,60].

We emphasize that, according to equation (71), a test vortex relaxes towards an equilibrium state [58] on a typical time $t_R \sim (N/\ln N) t_D$ for any flow. In particular, for axisymmetric flows, the distribution $P(r,t)$ of a test particle relaxes towards the distribution of the bath $\omega(r)$ (which can be different from the Boltzmann distribution as long as it is a stable steady state of the 2D Euler equation) on a timescale of order $(N/\ln N) t_D$, while the overall distribution of the system $\omega(r,t)$ does not change at all on this timescale [56]–[58].

4. Kinetic equation with angle-action variables

4.1. The evolution of the system as a whole: the kinetic equation

The kinetic equation (18) derived in section 2 is valid for spatially homogeneous systems. A manner to describe spatially inhomogeneous systems is to use angle-action variables $(\phi, J)$. A kinetic equation for $f(J,t)$ has been derived by Chavanis [84] by using two different strategies: (i) by incorporating in a Fokker–Planck equation the expressions of the coefficients of diffusion and friction obtained by Valageas [108] or, (ii) by starting from the Klimontovich equation, using a quasilinear theory, and solving the equations with Laplace–Fourier transforms. However, the second derivation presents some shortcomings that were explicitly mentioned in [84]. In particular, a factorization hypothesis $A_{n,n'}(J,J') = A_n(J) A_{n'}(J')$ is made without precise justification. However, if we neglect collective effects at the end of the calculations, only the product $A_n(J) A_{n'}(J')$ appears, so it can be replaced by $A_{n,n'}(J,J')$ and the final equation is well-posed (while the intermediate equations are not!). Here, we show that, if collective effects are neglected right from the beginning, there is no need to make an (illicit) factorization hypothesis. Indeed, the kinetic equation in angle-action variables can be derived from the general equation

\[ \frac{\partial f}{\partial t} = \frac{\partial}{\partial J} \int_0^{+\infty} d\tau \int d\phi_1 dJ_1 F(1 \rightarrow 0) G(t, t - \tau) \times \left[ F(1 \rightarrow 0) \frac{\partial}{\partial J} + F(0 \rightarrow 1) \frac{\partial}{\partial J_1} \right] f(J,t) \frac{f}{m}(J_1,t), \]  

(88)
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with

\[ F(1 \rightarrow 0) = -m \frac{\partial u}{\partial \phi}(x(J, \phi) - x(J_1, \phi_1)), \]  

(89)

which is the analog of the kinetic equations (3) and (53). Not only does the present approach solve the problems of our former approach [84], but it also shows a nice unity with the kinetic theories developed in sections 2 and 3. This equation is valid at the order \( O(1/N) \), so it describes the evolution of the system on a timescale \( \sim Nt_D \), where \( t_D \) is the dynamical time. For simplicity, we consider one-dimensional systems, but we shall indicate at the end how our results can be generalized in \( d \) dimensions. Introducing the Fourier transform of the potential with respect to the angles

\[ u(x(J, \phi) - x(J_1, \phi_1)) = \sum_{n,n_1} A_{n,n_1}(J, J_1) e^{i(n\phi - n_1\phi_1)}, \]  

(90)

where

\[ A_{n,n_1}(J, J_1) = \frac{1}{(2\pi)^2} \int u(x(J, \phi) - x(J_1, \phi_1))e^{-i(n\phi - n_1\phi_1)} \, d\phi \, d\phi_1, \]  

(91)

we get

\[ F(1 \rightarrow 0) = -im \sum_{n,n_1} A_{n,n_1}(J, J_1)ne^{i(n\phi - n_1\phi_1)}. \]  

(92)

Substituting this expression in the kinetic equation (88), we obtain

\[
\frac{\partial f}{\partial \tau} = -m^2 \frac{\partial}{\partial J} \int_0^{+\infty} d\tau \int d\phi_1 \int dJ_1 \sum_{l,l_1} \sum_{n,n_1} A_{n,n_1}(J, J_1)ne^{i(n\phi - n_1\phi_1)}G(t, t - \tau) \\
\times \left[ A_{l,l_1}(J, J_1)le^{i(l\phi - l_1\phi_1)} \frac{\partial}{\partial J} + A_{l,l}(J_1, J)le^{i(l_1\phi - l\phi)} \frac{\partial}{\partial J_1} \right] f(t, J) \frac{f(J, t)}{m}.
\]  

(93)

With angle-action variables, the equations of motion take the very simple form

\[ J(t - \tau) = J(t) = J, \quad \phi(t - \tau) = \phi(t) - \Omega(J, t)\tau = \phi - \Omega(J, t)\tau, \]  

(94)

where \( 2\pi\Omega^{-1}(J, t) \) is the period of the orbit with action \( J \) [84]. Substituting these relations in equation (93) and making the transformations \( l \rightarrow -l \) and \( l_1 \rightarrow -l_1 \) in the second integral (friction term), we obtain successively

\[
\frac{\partial f}{\partial \tau} = -m^2 \frac{\partial}{\partial J} \int_0^{+\infty} d\tau \int d\phi_1 \int dJ_1 \sum_{l,l_1} \sum_{n,n_1} A_{n,n_1}(J, J_1)ne^{i(n\phi - n_1\phi_1)} \\
\times e^{i(l\phi(t-\tau) - l_1\phi_1(t-\tau))} \left[ A_{l,l_1}(J, J_1)l_1 \frac{\partial}{\partial J} - A_{-l_1,-l}(J_1, J)l_1 \frac{\partial}{\partial J_1} \right] f(t, J) \frac{f(J, t)}{m}.
\]  

(95)

doi:10.1088/1742-5468/2010/05/P05019
Integrating over $\phi$

Using the identity (13), we finally obtain the kinetic equation

$$\tau$$

Making the transformation $\phi$ and using the fact that the average distribution does not depend on $\phi$, we get

$$\frac{\partial f}{\partial t} = 2\pi m^2 \frac{\partial}{\partial J} \int_{-\infty}^{+\infty} \, dt \int dJ_1 \sum_{n,n_1} |A_{n,n_1}(J,J_1)|^2 n \times e^{i(n\Omega(J,t) - n_1 \Omega(J,t))} \left( n \frac{\partial}{\partial J} - n_1 \frac{\partial}{\partial J_1} \right) f(J,t) \frac{f}{m}(J_1,t).$$

Making the transformation $\tau \rightarrow -\tau$, then $(n,n_1) \rightarrow (-n,-n_1)$, and adding the resulting expression to equation (99), we obtain

$$\frac{\partial f}{\partial t} = \pi m^2 \frac{\partial}{\partial J} \int_{-\infty}^{+\infty} \, dt \int dJ_1 \sum_{n,n_1} |A_{n,n_1}(J,J_1)|^2 n \times e^{i(n\Omega(J,t) - n_1 \Omega(J,t))} \left( n \frac{\partial}{\partial J} - n_1 \frac{\partial}{\partial J_1} \right) f(J,t) \frac{f}{m}(J_1,t).$$

Using the identity (13), we finally obtain the kinetic equation

$$\frac{\partial f}{\partial t} = 2\pi^2 m \frac{\partial}{\partial J} \int dJ_1 \sum_{n,n_1} |A_{n,n_1}(J,J_1)|^2 n \times \delta(n\Omega(J,t) - n_1 \Omega(J_1,t)) \left( n f_1 \frac{\partial f}{\partial J} - n_1 f \frac{\partial f_1}{\partial J_1} \right),$$

which returns the result obtained in [84]. However, it is derived here in a more satisfactory manner. In $d$ dimensions, $n, J, \Omega$ are replaced by vectors and the coefficient $2\pi^2$ is replaced by $\pi(2\pi)^d$. This kinetic equation describes inhomogeneous stellar systems and the HMF model.

The kinetic equation (101) conserves mass $M = \int f \, dJ$ and energy $E = \int f \epsilon(J) \, dJ$ and monotonically increases the Boltzmann entropy $S = -\int f \ln f \, dJ$ (H-theorem) [84].

\[ \text{doi:10.1088/1742-5468/2010/05/P05019} \]
The collisional evolution is due to a condition of resonance encapsulated in the \( \delta \)-function. The evolution stops when this condition of resonance can no longer be satisfied even if the system has not reached the statistical equilibrium state given by the Boltzmann distribution \( f = A e^{-\beta \epsilon(J)} \). Indeed, the Boltzmann distribution is not the only steady state of equation (101): any distribution satisfying \( n \Omega(J, t) \neq n_1 \Omega(J_1, t) \) for any couples \((n, J) \neq (n_1, J_1)\) is a steady state. However, the important point that we want to make here (it will be discussed further in section 5) is that, for inhomogeneous systems, there generically exists many more resonances than for homogeneous systems. In particular, for one-dimensional homogeneous systems, there is no resonance at all (see section 2). We therefore expect that the relaxation time will be reduced for inhomogeneous 1D systems as compared to homogeneous 1D systems. In fact, we cannot conclude that equation (101) tends to the Boltzmann distribution (since it is not the only steady state), but, since entropy increases while energy and mass are conserved, the system is expected to approach the Boltzmann distribution on the natural timescale \( Nt_D \) on which equation (101) is valid, provided that there are enough resonances. Indeed, due to ‘collisions’ and resonances, the system becomes ‘more mixed’. These arguments will be further discussed in section 5 in the light of existing numerical results.

4.2. Test particle in a thermal bath: the Fokker–Planck equation

Implementing a test particle approach as in sections 2.2 and 3.2, we find that the equation for \( P(J, t) \) is

\[
\frac{dP}{dt} = 2\pi^2 m \frac{\partial}{\partial J} \int dJ_1 \sum_{n,n_1} |A_{n,n_1}(J, J_1)|^2 n \delta(n \Omega(J) - n_1 \Omega(J_1)) \left( n f_1 \frac{\partial P}{\partial J} - n_1 P \frac{df_1}{dJ_1} \right).
\]  

Equation (102) can be written in the form of a Fokker–Planck equation

\[
\frac{\partial P}{\partial t} = \frac{\partial}{\partial J} \left( D \frac{\partial P}{\partial J} - P \eta \right),
\]  

involving a diffusion term

\[
D = 2\pi^2 m \int dJ_1 \sum_{n,n_1} |A_{n,n_1}(J, J_1)|^2 n \delta(n \Omega(J) - n_1 \Omega(J_1))f_1,
\]  

and a friction term due to polarization

\[
\eta \equiv F^{pol} = 2\pi^2 m \int dJ_1 \sum_{n,n_1} |A_{n,n_1}(J, J_1)|^2 n \delta(n \Omega(J) - n_1 \Omega(J_1))n_1 \frac{df_1}{dJ_1}.
\]

The ordinary Fokker–Planck equation is

\[
\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial J^2} \left( \frac{\langle (\Delta J)^2 \rangle}{\Delta t} P \right) - \frac{\partial}{\partial J} \left( P \frac{\langle \Delta J \rangle}{\Delta t} \right),
\]  

with

\[
\frac{\langle (\Delta J)^2 \rangle}{2\Delta t} = D, \quad \frac{\langle \Delta J \rangle}{\Delta t} = \frac{dD}{dJ} + \eta \equiv F^{friction}.
\]
Substituting equations (104) and (105) in equation (107) and using an integration by parts, we find that the diffusion and friction coefficients are given by

$$\frac{\langle (\Delta J)^2 \rangle}{2\Delta t} = 2\pi^2 m \int dJ_1 f_1 \sum_{n,n_1} |A_{n,n_1}(J,J_1)|^2 n^2 \delta(n\Omega(J) - n_1\Omega(J_1)),$$

(108)

$$\frac{\langle \Delta J \rangle}{\Delta t} = 2\pi^2 m \int dJ_1 f_1 \sum_{n,n_1} \left( n \frac{\partial}{\partial J} - n_1 \frac{\partial}{\partial J_1} \right) |A_{n,n_1}(J,J_1)|^2 n \delta(n\Omega(J) - n_1\Omega(J_1)).$$

(109)

These expressions can be obtained directly from the Hamiltonian equations of motion by making a systematic expansion of the trajectory of the particles in powers of $1/N$ in the limit $N \to +\infty$ [108].

Let us consider particular cases.

• If we consider a distribution of field particles $f(J_1)$ such that there is no resonance: $n\Omega(J) \neq n_1\Omega(J_1)$ for any couple $(n,J) \neq (n_1,J_1)$, we first get

$$D = 2\pi^2 m \int dJ_1 \sum_n |A_{n,n_1}(J,J_1)|^2 n^2 \delta(n\Omega(J) - \Omega(J_1)) f_1.$$

(110)

Then, using the identity $\delta(n(\Omega - \Omega_1)) = \delta(J - J_1)/|n\Omega'(J)|$, we find that

$$D(J) = 2\pi^2 m f(J) \sum_n |n| \frac{|A_{n,n_1}(J,J)|^2}{|\Omega'(J)|}.$$

(111)

Similarly

$$\eta = 2\pi^2 m f'(J) \sum_n |n| \frac{|A_{n,n_1}(J,J)|^2}{|\Omega'(J)|} = D(J) \frac{d\ln f}{dJ}.$$

(112)

Using equation (107), we obtain

$$\frac{\langle \Delta J \rangle}{\Delta t} = D'(J) + D(J) \frac{d\ln f}{dJ}.$$

(113)

Finally, the Fokker–Planck equation (103) can be written

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial J} \left[ D(J) \left( \frac{\partial P}{\partial J} - P \frac{d\ln f}{dJ} \right) \right],$$

(114)

with a diffusion coefficient given by equation (111).

• If the field particles are at statistical equilibrium (thermal bath), their distribution is the Boltzmann distribution

$$f(J_1) = Ae^{-\beta m \epsilon(J_1)},$$

(115)

where $\epsilon(J)$ is the individual energy of the orbit with action $J$. Using $d\epsilon/dJ = \Omega(J)$, we find that

$$\frac{df_1}{dJ_1} = -\beta m f_1 \frac{d\epsilon_1}{dJ_1} = -\beta m f_1 \Omega(J_1).$$

(116)
Substituting this relation in equation (105), we obtain
\[ \eta = -2\pi^2 m^2 \beta \int dJ_1 \sum_{n,n_1} |A_{n,n_1}(J, J_1)|^2 n \delta(n\Omega(J) - n_1\Omega(J_1))n_1f_1\Omega(J_1). \] (117)

Using the \( \delta \)-function to replace \( n_1\Omega(J_1) \) by \( n\Omega(J) \) and comparing the resulting expression with equation (104), we finally get
\[ \eta = -D\beta m\Omega(J), \] (118)
which is the appropriate Einstein relation for our problem. For a thermal bath, using equation (118), the Fokker–Planck equation (103) can be written
\[ \frac{\partial P}{\partial t} = \frac{\partial}{\partial J} \left[ D(J) \left( \frac{\partial P}{\partial J} + \beta mP\Omega(J) \right) \right], \] (119)
where \( D(J) \) is given by equation (104) with equation (115).

5. Discussion

The kinetic theory of systems with long-range interactions leads to the following scenario for the evolution of the system:

(1) The first stage of the dynamical evolution is described by the Vlasov equation. This is a purely mean field equation ignoring correlations (‘collisions’) between particles. Due to violent relaxation and phase mixing, the coarse-grained distribution function \( f(r, v, t) \) converges towards a quasi stationary state \( \bar{f}(r, v) \) which is a dynamically stable (robust) steady state of the Vlasov equation. The convergence takes a few dynamical times \( t_D \) independent of \( N \). The fine-grained distribution \( f(r, v, t) \) develops filaments at smaller and smaller scales and does not achieve a steady state. This collisionless relaxation explains the rapid emergence of coherent structures in stellar dynamics (galaxies), two-dimensional turbulence (jets and vortices) and in the HMF model (quasi stationary states).

(2) If the system mixes efficiently, the QSS is given by the Lynden-Bell statistical theory of the Vlasov equation. This is the usual Boltzmann approach taking into account the specificities of the Vlasov equation (Casimir constraints). Therefore, the Lynden-Bell prediction depends on the details of the initial condition, not only on the mass and energy. However, violent relaxation can be incomplete: the system may not mix well and the collisionless relaxation may not be ergodic. In that case, other distributions can emerge. Among them, the Tsallis distributions (polytropes) seem to play a particular role in certain situations. However, they are not universal attractors. Other distributions can emerge as well, but they are difficult to predict since they depend on the dynamics and on the efficiency of mixing.

(3) On longer timescales, the system evolves under the effect of ‘collisions’, i.e. correlations due to graininess (finite \( N \)) effects. This gives rise to a collision term in the rhs of the Vlasov equation. In principle, the collisional relaxation is towards the Boltzmann statistical equilibrium state (for stellar systems, the relaxation towards Boltzmann equilibrium is hampered by the problems of evaporation and gravothermal catastrophe). The collisional relaxation time \( t_R(N) \) depends on the number of particles \( N \) and diverges rapidly as \( N \to +\infty \).
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(4) Due to the huge timescale separation between the dynamical time \( t_D \) and the relaxation time \( t_R(N) \), the distribution function \( f(r,v,t) \) passes by a sequence of quasi equilibrium states, which are quasi stationary states of the Vlasov equation, slowly evolving in time due to finite \( N \) effects (graininess). In stellar dynamics, this collisional evolution is described by the orbit-averaged-Fokker–Planck equation.

This scenario was put forward by Hénon [35,109] in the context of stellar dynamics, and further discussed and improved by many authors. It has become the ‘standard scenario’ of stellar dynamics [34,110]. A similar scenario was proposed by Chavanis [56,62] for two-dimensional point vortices, by developing the analogy between the dynamics of stellar systems and point vortices. Inspired by these results, this scenario was also considered by Yamaguchi et al [76] in relation to the HMF model. These authors understood many important aspects of the dynamics. However, we would like to complete some of their arguments in the light of more recent observations: (i) Yamaguchi et al [76] argue that, during the collisional evolution, the system remains always close to a stable steady state of the Vlasov equation. Recently, the numerical study of Campa et al [95] has demonstrated that, during the collisional evolution, the distribution function \( f(v,t) \) can be well-represented by a Tsallis [111] distribution (polytrope) with a time dependent index \( q(t) \) until it becomes Vlasov unstable and triggers a dynamical phase transition from the non magnetized (homogeneous) phase to the magnetized (inhomogeneous) phase. Therefore, in that case, the relaxation involves a dynamical instability [95]. In fact, at the verge of the dynamical phase transition, the imaginary part of the complex pulsation \( \omega_i \) becomes zero and it is possible that the system remains marginally stable in the subsequent evolution (this would be an interesting point to check). (ii) Yamaguchi et al [76] reject the possibility of Tsallis \( q \)-distributions [111], although they seem to play an important role in the HMF model, as discussed in [112]. (iii) They argue that the relaxation time of the HMF model scales algebraically like \( N^{1.7} \), although, as we shall see below, the picture is more complex.

We also emphasize that the above scenario is not the only possibility and that, as reviewed in [48,85,113], the evolution can be more complex with the formation of phase-space holes, periodic solutions, oscillations, vortex crystals etc.

We would like now to use the kinetic theory presented in the present paper to interpret the results of numerical simulations that have been obtained in different contexts. We briefly review these numerical results, then provide new interpretations that can open interesting directions of research.

Stellar systems. The scenario proposed by Hénon [35,109] has been studied by several authors and a good understanding of stellar dynamics has been reached by astrophysicists for a long time. In a first regime, a self-gravitating system, initially out-of-mechanical equilibrium, undergoes a process of violent collisionless relaxation towards a virialized state. This collisionless regime is appropriate to understand the actual structure of elliptical galaxies. In this regime, the dynamical evolution of the cluster is described by the Vlasov–Poisson system. The theory of violent relaxation developed by Lynden-Bell [91], which predicts a distribution function of the form \( f = Ae^{-\beta \epsilon} \), where \( \epsilon = v^2/2 + \Phi(r) \) is the individual energy, explains the isotropic isothermal core of elliptical galaxies without recourse to collisions that operate on a much longer timescale. However, his theory does not explain the structure of the halo that results from incomplete relaxation. The
velocity distribution in the halo is radially anisotropic and the density profile decreases like $r^{-4}$ (instead of the $r^{-2}$ isothermal law). Theories of incomplete relaxation have been elaborated by Bertin and Stiavelli [114], Stiavelli and Bertin [115], Hjorth and Madsen [116], Chavanis et al [93] and Chavanis [20,94]. N-body numerical simulations of violent relaxation have been made by van Albada [117], who compared his results with the phenomenological de Vaucouleurs $R^{1/4}$ law for the surface brightness of elliptical galaxies. He showed that clumpy and sufficiently cold (low values of the virial ratio $2K/|W|$) initial conditions yield distributions that reproduce the de Vaucouleurs law.

The more recent numerical simulations of Londrillo et al [118] and Trenti et al [119] confirm these results and show that the numerically obtained profiles are in remarkable agreement with the models of incomplete violent relaxation proposed by Bertin and Stiavelli [114] and Stiavelli and Bertin [115]. The first model leads to a distribution function of the form $f_{\infty} = A(-\epsilon)^{3/2} \exp(-\alpha \epsilon - c j^2/2)$, where $j = r \times v$ is the individual angular momentum. The second model, which is based on the existence of a third quasi-invariant $Q$ (in addition to mass $M$ and energy $E$), leads to distribution functions of the form $f^{(r)} = A \exp[-\alpha \epsilon - d j^2/|\epsilon|^{3/2} r^{2}]$. In the simulations [118,119], the initial condition needs to be sufficiently clumpy and cold to generate enough mixing required for a successful application of the Bertin–Stiavelli statistical theory of violent relaxation. Numerical simulations starting from homogeneous spheres [120]–[122] show less angular momentum mixing and lead to different results. In particular, they display a larger amount of mass loss (evaporation) than simulations starting from clumpy initial conditions. Clumps thus help the system to reach a ‘universal’ final state from a variety of initial conditions, which can explain the similarity of the density profiles observed in elliptical galaxies.

On longer timescales, encounters between stars must be taken into account and the dynamical evolution of the cluster is governed by the Vlasov–Landau–Poisson system. This collisional regime is appropriate to understand the actual structure of globular clusters. In the collisional regime, the evolution is first governed by the evaporation of high energy stars. Numerical simulations [110] show that the system reaches a quasi-stationary state, close to the Michie–King distribution $f = A(e^{-\beta \epsilon} - e^{-\beta \epsilon_{m}})e^{-\beta j^2}/(2\pi^2)$, which slowly evolves in amplitude due to evaporation as the system loses mass and energy. The density follows the isothermal law $r^{-2}$ in the central region (with a core of almost uniform density) and decreases like $r^{-7/2}$ in the halo (which has an anisotropic velocity distribution). Due to the evaporation, the halo expands while the core shrinks, as required by energy conservation. At some point of the evolution, when the density contrast becomes sufficiently high, the system undergoes a thermodynamical instability related to the Antonov [37] instability, and the gravothermal catastrophe [38] takes place. This instability is due to the negative specific heat of the inner system, which evolves by losing energy to the profit of the halo and thereby grows hotter, perpetuating this process. This leads to core collapse. Core collapse has been studied numerically by Cohn [36] using the orbit-averaged-Fokker–Planck equation. He finds that the collapse is self-similar, that the central density becomes infinite in a finite time, and that the density decreases like $r^{-2}$-23. In reality, core collapse is arrested by the formation of binary stars. These binaries can release sufficient energy to stop the collapse [35] and even drive a re-expansion of the cluster in a post-collapse regime [123]. Then, in principle, a series of gravothermal oscillations should follow [124]. When the system is confined in a box, instead of being
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free to expand, Taruya and Sakagami [125] observe numerically that the evolution follows a sequence of $q$-distributions (polytropes), with a time dependent index $q(t)$, until an instability develops. They interpret this instability as a thermodynamical instability (in connection to Tsallis generalized thermodynamics), whereas we interpret it as a Vlasov dynamical instability for polytropes [95,126].

Point vortices. The scenario proposed by Chavanis [56,62], by analogy with stellar dynamics, has been tested numerically by Kawahara and Nakanishi [127,128] in two very interesting papers. We would like to complete their discussion in the light of the kinetic theory presented in this paper. The kinetic theory developed in [56,57,62] is valid at the order $O(1/N)$ and, consequently, the natural timescale is $Nt_D$. This is precisely the relaxation time that Kawahara and Nakanishi observe [128]. This seems consistent at first sight with the kinetic theory. However, if we consider things carefully, the situation appears to be more complicated. Indeed, we have seen in section 3.1 that the collision term can vanish, leading to a relaxation time larger than $Nt_D$. However, we have indicated that this cancelation occurs only for axisymmetric systems with monotonic profile of angular velocity. Now, in the simulations of Kawahara and Nakanishi [128] the system is non-axisymmetric. Indeed, in most cases, the phase of violent relaxation leads to a vortex crystal (see their figures 2(b) and 7). Therefore, we cannot use the axisymmetric kinetic equation (68) and we must come back to the more general kinetic equation (52). Little is known concerning this complicated equation but we expect that it increases the entropy and conserves energy and circulation [57]. It is not clear whether this equation rigorously converges towards the Boltzmann distribution but, since the entropy increases, we deduce that the system has the tendency to approach the Boltzmann distribution (it becomes more mixed). This is because there are additional resonances with respect to a purely axisymmetric evolution. Since the proper timescale of this kinetic equation is $Nt_D$, the kinetic theory of non-axisymmetric flows may explain the numerical observations of Kawahara and Nakanishi [128]. There remains, however, a difficulty. Indeed, in their figure 2(a), violent relaxation leads to an apparently axisymmetric distribution with a presumably monotonically decreasing profile of angular velocity. In that case, there should be no resonance and the relaxation time should be larger than $Nt_D$, contrary to what is observed numerically. We speculate that, in this case, the system is not really axisymmetric but that there exists ‘structures’ in the flow (analogous to Dawson’s vortices in neutral plasmas under a strong magnetic field [129]). This claim is corroborated by the fact that Kawahara and Nakanishi [128] observe anomalous diffusion, which means that the evolution of the flow is more complex than it seems. These results have been confirmed recently by Yoshida [130], who showed that the mean square displacements exhibit a universal time dependence $\langle [r(t)-r(0)]^2 \rangle \sim t^\mu$ with $\mu = 1.75 \pm 0.1$. These authors attribute anomalous diffusion to occasional long jumps of the particles convected by long living large vortices. If there exists such ‘structures’, they may induce new resonances and explain the relaxation towards Boltzmann equilibrium on a timescale $Nt_D$. This may also imply that there is no contradiction with the kinetic theory concerning the diffusion process. The diffusion coefficient (79) and the Fokker–Planck equation (82) derived in [56,62] are only valid for axisymmetric flows without ‘structures’. In more general cases, the diffusion equation is given by equation (142) of [57], which involves delocalizations in space and time. It is plausible that these non ideal effects lead to anomalous diffusion for short times. Normal diffusion should be recovered for larger times.
(anomalous diffusion could also be a finite size effect and it could disappear for large $N$). As indicated by Kawahara and Nakanishi [128], this is precisely what happens for neutral plasmas under a strong magnetic field in relation to the Taylor–McNamara [131] diffusion. Of course, this scenario demands confirmation, but it seems plausible enough to reconcile theory and observations. More work remains to be done in order to better understand these results. In this respect, we may note the recent study of Sano [132] on the kinetic theory of point vortices. He confirmed the results of Chavanis [56, 57] (leading to Landau-type kinetic equations) and derived a kinetic equation corresponding to the Lenard–Balescu equation for plasmas using the theory of the Fredholm integral equation. We may also note that the exponential tail of the velocity distribution observed by Kawahara and Nakanishi [128] is consistent with the results reported by Chavanis and Sire [133] when the vortices have a finite core, which is equivalent to putting a restriction on the point vortex distance in the initial state. Therefore, this observation (exponential tail) is not inconsistent with kinetic theory.

**HMF model.** The scenario proposed by Yamaguchi *et al.* [76] (see also Chavanis *et al.* [71]) has been checked numerically by these authors and by Campa *et al.* [77, 95]. It is also consistent with previous simulations by Latora *et al.* [134], although these authors did not interpret QSSs in terms of the Vlasov equation. The relaxation time has been discussed by Bouchet and Dauxois [70] and Chavanis [48] in the light of the kinetic theory. These authors argue that, for spatially homogeneous systems, the relaxation time is larger than $Nt_D$ since there is no resonance. Bouchet and Dauxois [70], referring to the simulations of Yamaguchi *et al.* [76], argue that the relaxation time of homogeneous systems scales like $N^{1.7}$ while Chavanis [48], referring to the simulations of Campa *et al.* [77], argues that the relaxation time of homogeneous systems scales like $e^N$. We would like to reconsider and complete their discussions. In the simulation of Yamaguchi *et al.* [76], which is performed for subcritical energies $U < U_c$, the system does not remain spatially homogeneous, therefore the homogeneous kinetic theory cannot be advocated until the end. Alternatively, Campa *et al.* [77] perform supercritical simulations $U > U_c$, in which the system remains permanently spatially homogeneous, and find that the relaxation time scales like $e^N$. Therefore, when the system is homogeneous the relaxation time is exponentially long! This is in agreement with the kinetic theory, but this shows that the relaxation is not due to just three-body, four-body, $n$-body . . . correlations (as we could naively believe), which would imply a relaxation time scaling like $N^2, N^3, N^{n-1} . . .$. All the correlation functions seem to matter! Therefore, the kinetic theory of spatially homogeneous systems cannot apparently be truncated at some power of $1/N$. If we assume that the point vortex model behaves like the HMF model, we may conjecture that the relaxation time for an axisymmetric distribution of point vortices with monotonic profile of angular velocity scales like $e^N$ (however, the possibility that such distributions never achieve statistical equilibrium, or on the contrary, that they achieve statistical equilibrium on a timescale $N^2$ like 1D plasmas [97, 98], cannot be rejected). When the system is spatially inhomogeneous, the relaxation time is reduced with respect to $e^N$. This leads to a reinterpretation of the numerical results. Yamaguchi *et al.* [76] performed two types of simulations at subcritical energies $U < U_c$: (i) the first one, corresponding to unstable stationary Vlasov states ($U < U_* < U_c$), leads to a spatially inhomogeneous evolution right from the beginning. In that case, the collisional relaxation time scales like $N$, which is the proper scaling of the kinetic theory for spatially inhomogeneous
systems. Indeed, there exist many resonances that ‘push’ the system towards the Boltzmann distribution (see section 4). The question whether the system really reaches the Boltzmann distribution, or a distribution close to it, remains open, since we cannot prove that the kinetic equations (2) and (101) rigorously converge towards the Boltzmann distribution. It has to be noted that Jain et al [135], starting from unstable stationary Vlasov states, argue that the relaxation time scales like \( \ln N \). However, it is not clear in their simulations whether the system has reached the Boltzmann distribution of statistical equilibrium or a QSS that is situated close to the Boltzmann distribution. The latter may be reached on longer time, scaling like \( N \), as discussed in section 6 of [76].

(ii) The second simulation, which starts from Vlasov stable stationary states \((U^* < U < U_c)\), leads to spatially homogeneous structures whose velocity profile slowly changes due to the effect of correlations (finite \( N \) effects). In that case, there is no resonance and we should expect a relaxation time scaling like \( e^N \). However, Campa et al [95] argue that the relaxation towards Boltzmann equilibrium involves a dynamical phase transition that is triggered when the homogeneous phase becomes Vlasov unstable. In that case, the system becomes spatially inhomogeneous and resonances appear. This considerably accelerates the relaxation with respect to the exponential scaling and leads to a non trivial (but probably non universal) scaling \( N^{1.7} \). Therefore, in that case, the timescale cannot be solely understood in terms of the kinetic theory of homogeneous systems, although this theory explains why the scaling is larger than \( N \). Morita and Kaneko [136], considering an unsteady initial condition with \( U < U_c \) and \( M_0 = 1 \), found a relaxation time \( t_R \sim N t_D \). In their simulations, the system is always spatially inhomogeneous (the magnetization has an oscillatory behavior) and the relaxation time is consistent with the natural scaling of the kinetic theory for inhomogeneous distributions. Their results may be consistent with the general kinetic equations (2) and (101) although this is of course difficult to check.

Let us now consider the early simulation of Latora et al [134], which starts from an out-of-equilibrium initial state with magnetization \( M_0 = 1 \). Violent relaxation leads the system to a spatially homogeneous QSS which is followed by a collisional relaxation towards the Boltzmann distribution. They find a relaxation time scaling like \( N t_D \). This timescale seems to be inconsistent with the kinetic theory of spatially homogeneous systems. However, as in the case of 2D point vortices, we speculate that there exists ‘phase-space structures’ that make the system spatially inhomogeneous and induce additional resonances. These inhomogeneities may reduce the timescale of relaxation to the natural timescale \( N t_D \). These ‘phase-space structures’ are indeed present in the simulations of Latora et al [134] and they seem to be responsible for anomalous diffusion in a way similar to what happens in 2D point vortex dynamics [128,130]. These phase-space structures may also lead to incomplete relaxation and to QSSs that are different from the Lynden-Bell distribution [112,134]. Again, this scenario demands confirmation, but it seems plausible enough to reconcile theory and observations. We must, however, be very careful because these striking features (phase-space structures and anomalous diffusion) may be due to finite size effects [137,138] and disappear for \( N \to +\infty \). Also, diffusion may be anomalous for short times and normal for large times. This may be a way to reconcile the approaches of Rapisarda and Pluchino [139] and Bouchet and Dauxois [70], who studied temporal correlation functions and anomalous diffusion. Indeed, their studies are based on very different arguments: Bouchet and Dauxois [70] assume that the background distribution is spatially homogeneous and use standard kinetic theory based on Fokker–
Planck equations. Alternatively, Rapisarda and Pluchino [139] assume that anomalous diffusion is due to the presence of phase-space structures and use nonlinear Fokker–Planck equations. We believe that these two approaches are not antagonistic but that they apply to different regimes or, maybe, different numbers of particles. Finally, we consider a situation that seems to be in conflict with the scenario that we have developed previously. Campa et al [95] have performed a simulation at supercritical energy $U > U_c$ starting from a Vlasov unstable initial condition. In a first stage, the magnetization rapidly increases until the system reaches an inhomogeneous QSS. Since the statistical equilibrium state is spatially homogeneous, the magnetization ultimately decreases and tends to zero for $t \to +\infty$. However, Campa et al [95] find that the relaxation time is extremely long, scaling presumably like $e^N$. This is not in agreement with the $\ln N$ relaxation time [135]. This is also not in agreement with the picture that we have developed above, since the system is spatially inhomogeneous and should therefore relax on the ‘natural’ timescale $Nt_D$. This shows that the picture may be more complex than originally thought. For example, the QSS could be in a state in which there are very few resonances so that the collisional evolution is slowed down. On the other hand, since the system becomes homogeneous for large times, the collisional relaxation is less and less efficient (since there are less and less resonances) and this could explain the long relaxation time. It would be interesting to study this case in more detail.

Let us summarize the different regimes discussed above and unveil new regimes yet to be explored. Let us first assume that $U_* < U_c$, where $U_c = 3/4$ is the critical energy below which the statistical equilibrium state is inhomogeneous ($M \neq 0$) and $U_*$ is the critical energy below which the initial condition (assumed spatially homogeneous) is Vlasov unstable. Case 1-a: if we start from an unstable steady state ($U < U_* < U_c$), the magnetization rapidly increases and the system reaches a QSS on a timescale $\ln N$ [135] before finally relaxing towards the Boltzmann distribution on a timescale $N$. Case 1-b: if we start from a stable steady state with subcritical energy $U_*, U < U_c$, the magnetization remains zero until the system experiences a dynamical phase transition achieved when the QSS, evolving due to finite $N$ effects, becomes marginally stable [95]. At that point, the magnetization rapidly increases and the system finally relaxes towards the Boltzmann distribution on a timescale $N^{1/2}t_D$ [76]. Case 1-c: if we start from a stable steady state with supercritical energy $U > U_c > U_*$, the magnetization remains permanently zero and the system relaxes towards the Boltzmann distribution on a timescale $e^N t_D$ [77]. Let us now assume that $U_*, U > U_c$, a situation that has been less considered. Case 2-a: if $U < U_* < U_c$, the situation should be similar to Case 1-a. Case 2-b: if we start from an unstable steady state with supercritical energy $U_* < U < U_c$, the magnetization rapidly increases and the system reaches an inhomogeneous QSS. Then, the magnetization decreases as the system relaxes towards the homogeneous Boltzmann distribution. The relaxation time is extremely long and scales like $e^N$ [95]. Case 2-c: if $U > U_* > U_c$, the situation should be similar to Case 1-c. Interestingly, if we consider as initial conditions homogeneous polytropes (Tsallis distributions), the case $U_* < U_c$ corresponds to ($q > 1$, $n > 1/2$, $1 < \gamma < 3$), i.e. distributions with a compact support (e.g. waterbag $q = +\infty$, semi-ellipse $q = 3$) and the case $U_* > U_c$ corresponds to ($1/3 < q < 1$, $n < -1$, $0 < \gamma < 1$), i.e. distributions with a power-law decay [71]. Case 3: if we start from an unsteady initial condition, the system undergoes violent relaxation towards (i) the Lynden-Bell distribution if mixing is efficient [138, 140], or (ii) a QSS.
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different from the Lynden-Bell distribution (e.g. a Tsallis distribution) if mixing is not
efficient enough and relaxation is incomplete [67, 77, 112, 134]. Case 4: It is also possible
that the system does not reach a QSS but develops long-lived oscillations [136]. Case 5:
finally, if we start from a metastable state (local entropy maximum), the system is expected
to remain in that state for a time scaling like $e^N$ [141, 142]. Other cases: there may be
other cases that have not been described above. For example, we could imagine a situation
where the system is initially homogeneous ($M = 0$), then undergoes a dynamical phase
transition leading to an inhomogeneous QSS ($M \neq 0$) like in Case 1-b, and finally relaxes
towards a homogeneous Boltzmann equilibrium ($M = 0$). Using the arguments of [95],
this situation may not be possible for the HMF model (at least when the distribution
function remains close to a polytrope). However, for a system in contact with a heat bath
(the so-called BMF model [71]), such a situation (‘pulse’) has been observed recently [143].

6. Conclusion

In conclusion, we have developed a kinetic theory of systems with long-range interactions
based on the generalized (and not well known) kinetic equations (2), (52) and (88). A
specificity of our approach is to remain in physical space and take into account spatial
inhomogeneity, which is the norm for systems with long-range interactions. This is the
main difference with kinetic theories of systems with short-range interactions and neutral
plasmas that are spatially homogeneous. Interestingly, the homogeneous kinetic equation
can be easily derived from the generalized kinetic equation with simple calculations. This
provides a new derivation of Landau-type equations and also offers a unified description
of the kinetic theory of stellar systems, 2D point vortices and the HMF model. As
an application, we have argued that the relaxation time of spatially inhomogeneous
1D systems (or non-axisymmetric 2D flows) should scale like $N t_D$ which is the natural
timescale of the kinetic theory. Indeed, spatially inhomogeneous systems involve many
resonances that allow the relaxation (or the approach) of the system towards Boltzmann
equilibrium. By contrast, for one-dimensional homogeneous systems (or axisymmetric
flows with a monotonic profile of angular velocity), the resonances disappear and the
relaxation time is considerably increased. For 1D plasmas, it scales like $N^2 t_D$ [97, 98].
For the HMF model, when the system remains permanently homogeneous, it scales like
$e^N t_D$. This corresponds to supercritical energies considered by Campa et al [77]. If
the QSS resulting from violent relaxation is homogeneous but, later evolving under the
development of correlations, undergoes at some point a dynamical phase transition (Vlasov
instability) from the homogeneous phase to the inhomogeneous phase [95], the relaxation
time should be intermediate between $N t_D$ and $e^N t_D$. For example, it can scale algebraically
like $N^\delta t_D$. This corresponds to the subcritical energies considered by Yamaguchi et al
[76]. They find $\delta = 1.7$, but this scaling may not be universal. Finally, for systems
with dimension $d > 1$, there always exist resonances, whether the system is spatially
inhomogeneous or spatially homogeneous, and the relaxation time scales like $N$ (with
logarithmic corrections in the case of stellar systems). In summary: (i) For $d > 1$, the
relaxation time of the system as a whole scales like $N$ whether the system is spatially
homogeneous or not. (ii) For $d = 1$, the relaxation time of the system as a whole scales
like $N$ if the system is permanently inhomogeneous and like $N^2$ [97, 98] or $e^N$ [77] if the
system is permanently homogeneous. When the system is homogeneous, then becomes
inhomogeneous, the relaxation time scales like $N^\delta$ with $\delta > 1$ (e.g. $\delta = 1.7$) [76]. (iii) The relaxation time of a test particle in a bath scales like $N$ in any dimension. We have motivated our arguments by a detailed comparison with existing numerical simulations and found interesting agreement (but also some discrepancies). This shows that kinetic theory can be a good approach to understanding the dynamics and thermodynamics of systems with long-range interactions but that much work remains to be done to obtain a complete description of spatially inhomogeneous systems.

Appendix A. Calculation of $K^{\mu\nu}$

If $d = 3$, we introduce a spherical system of coordinates where the $z$ axis is taken in the direction of $w$. Then

$$K^{\mu\nu} = \pi(2\pi)^3 m^2 \int_0^{+\infty} k^2 \, dk \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} d\phi \, k^\mu k^\nu \delta(k w \cos \theta) \hat{u}(k)^2.$$  (A.1)

Using $k_z = k \sin \theta \cos \phi$, $k_y = k \sin \theta \sin \phi$ and $k_z = k \cos \theta$, it is easy to see that only $K^{xx}$, $K^{yy}$ and $K^{zz}$ can be non-zero. The other components of the matrix $K^{\mu\nu}$ vanish by symmetry. Furthermore

$$K^{xx} = K^{yy} = \pi^2(2\pi)^3 m^2 \int_0^{+\infty} k^2 \, dk \int_0^\pi \sin \theta \, d\theta k^2 \delta(k w \cos \theta) \hat{u}(k)^2 \sin^2 \theta.$$  (A.2)

Using the identity $\delta(\lambda x) = (1/|\lambda|)\delta(x)$, we get

$$K^{xx} = K^{yy} = \pi^2(2\pi)^3 m^2 \int_0^{+\infty} k^2 \, dk \int_0^\pi \sin \theta \, d\theta k^2 \delta(k w \cos \theta) \hat{u}(k)^2 \sin^2 \theta.$$  (A.3)

With the change of variables $t = \cos \theta$, we obtain

$$K^{xx} = K^{yy} = \pi^2(2\pi)^3 m^2 \int_0^{+\infty} k^2 \, dk \int_0^\pi \sin \theta \, d\theta k^2 \delta(k w \cos \theta) \hat{u}(k)^2 \sin^2 \theta.$$  (A.4)

so that finally

$$K^{xx} = K^{yy} = 8\pi^5 m^2 \int_0^{+\infty} k^3 \, dk.$$  (A.5)

On the other hand

$$K^{zz} = 2\pi^2(2\pi)^3 m^2 \int_0^{+\infty} k^2 \, dk \int_0^\pi \sin \theta \, d\theta k^2 \delta(k w \cos \theta) \hat{u}(k)^2 = 0.$$  (A.6)

In conclusion, we obtain

$$K^{\mu\nu} = \frac{K}{w} \left( \delta^{\mu\nu} - \frac{w^\mu w^\nu}{w^2} \right),$$  (A.7)

with

$$K = 8\pi^5 m^2 \int_0^{+\infty} k^3 \, dk.$$  (A.8)
If \( d = 2 \), we introduce a polar system of coordinates where the \( x \) axis is taken in the direction of \( \mathbf{w} \). Then

\[
K_{\mu\nu} = \pi (2\pi)^2 m^2 \int_0^{+\infty} k \, dk \int_0^{2\pi} d\theta \, k^\mu k^\nu \delta(kw \cos \theta) \hat{u}(k)^2.
\]  

(A.9)

Using \( k_x = k \cos \theta, k_y = k \sin \theta \), it is easy to see that only \( K^{xx} \) and \( K^{yy} \) can be non-zero. The other components of the matrix \( K^{\mu\nu} \) vanish by symmetry. Furthermore

\[
K^{yy} = \pi (2\pi)^2 m^2 \int_0^{+\infty} k \, dk \int_0^{2\pi} d\theta \, k^2 \delta(kw \cos \theta) \hat{u}(k)^2 \sin^2 \theta,
\]

(A.10)

or, equivalently,

\[
K^{yy} = 2\pi (2\pi)^2 m^2 \frac{1}{w} \int_0^{+\infty} k^2 \, dk \hat{u}(k)^2 \int_0^{\pi} d\theta \, \delta(\cos \theta) \sin^2 \theta.
\]

(A.11)

With the change of variables \( t = \cos \theta \), we obtain

\[
K^{yy} = 2\pi (2\pi)^2 m^2 \frac{1}{w} \int_0^{+\infty} k^2 \, dk \hat{u}(k)^2 \int_{-1}^{1} \sqrt{1 - t^2} \delta(t) \, dt,
\]

(A.12)

so that finally

\[
K^{yy} = 8\pi^3 m^2 \frac{1}{w} \int_0^{+\infty} k^2 \hat{u}(k)^2 \, dk.
\]

(A.13)

On the other hand

\[
K^{xx} = \pi (2\pi)^2 m^2 \frac{1}{w} \int_0^{+\infty} k^2 \, dk \hat{u}(k)^2 \int_0^{2\pi} d\theta \, \delta(\cos \theta) \cos^2 \theta = 0.
\]

(A.14)

In conclusion, we obtain

\[
K^{\mu\nu} = \frac{K}{w} \left( \delta^{\mu\nu} - \frac{u^\mu u^\nu}{w^2} \right),
\]

(A.15)

with

\[
K = 8\pi^3 m^2 \int_0^{+\infty} k^2 \hat{u}(k)^2 \, dk.
\]

(A.16)

**Appendix B. Calculation of \( \chi(r, r_1) \)**

For the ordinary potential \( u(|r - r_1|) = -(1/2\pi) \ln |r - r_1| \), we have

\[
u(r, r_1, \phi) = -\frac{1}{4\pi} \ln(r^2 + r_1^2 - 2rr_1 \cos \phi),\]

(B.1)

where \( \phi = \theta - \theta_1 \). Its Fourier transform with respect to the angles can be written

\[
\hat{u}_m(r, r_1) = -\frac{1}{4\pi^2} \int_0^{2\pi} \ln(1 - \lambda \cos \phi) \cos(m\phi) \, d\phi - \frac{\delta_{m0}}{4\pi} \ln(r^2 + r_1^2),
\]

(B.2)
with $\lambda = 2rr_1/(r^2 + r_1^2)$. For $m \neq 0$, we have the identity

$$\int_0^\pi \ln (1 - \lambda \cos \phi) \cos m\phi \, d\phi = -\frac{\pi}{|m|} \left( \frac{1}{\lambda} - \frac{1}{\sqrt{\lambda^2 - 1}} \right)^{|m|}. \quad (B.3)$$

Therefore

$$\hat{u}_m(r, r_1) = \frac{1}{4\pi |m|} \left( \frac{r^2 + r_1^2 - |r^2 - r_1^2|}{2rr_1} \right)^{|m|} = \frac{1}{4\pi |m|} \left( \frac{r_<}{r_>} \right)^{|m|}, \quad (B.4)$$

where $r_<$ (resp. $r_>$) is the min (resp. max) of $r$ and $r_1$. For $m = 0$, we have the identity

$$\int_0^\pi \ln (1 - \lambda \cos \phi) \, d\phi = \pi \ln(1 + \sqrt{1 - \lambda^2}) - \pi \ln 2. \quad (B.5)$$

Therefore

$$\hat{u}_0(r, r_1) = -\frac{1}{4\pi} \ln \left( \frac{r^2 + r_1^2 + |r^2 - r_1^2|}{2} \right) = -\frac{1}{2\pi} \ln r_>. \quad (B.6)$$

Combining the previous results, the potential of interaction can be written

$$u(r, r_1, \phi) = -\frac{1}{2\pi} \ln r_> + \frac{1}{4\pi} \sum_{m \neq 0} \frac{1}{|m|} \left( \frac{r_<}{r_>} \right)^{|m|} e^{im\phi}. \quad (B.7)$$

The function defined by equation (69) takes the form

$$\chi(r, r_1) = \frac{1}{8\pi^2} \sum_{m=1}^{+\infty} \frac{1}{m} \left( \frac{r_<}{r_>} \right)^{2m} = -\frac{1}{8\pi^2} \ln \left[ 1 - \left( \frac{r_<}{r_>} \right)^2 \right]. \quad (B.8)$$

In particular

$$\chi(r, r) = \frac{1}{8\pi^2} \sum_{m=1}^{+\infty} \frac{1}{m} = \frac{1}{8\pi^2} \ln \Lambda, \quad (B.9)$$

where $\ln \Lambda$ is a Coulombian logarithm that has to be regularized appropriately (see, e.g. [58]).

Note added. After submission of this paper, a recent numerical study by Joyce and Worrakitpoonpon [144] has demonstrated that the relaxation time of a one-dimensional self-gravitating system scales like $N_{12}$. Since the system is one-dimensional but remains permanently spatially inhomogeneous, this result agrees with the prediction of kinetic theory made in the present paper. Joyce and Worrakitpoonpon [144] also show that the system first undergoes a phase of violent relaxation towards a QSS before relaxing towards the Boltzmann distribution, confirming the general scenario of section 5. As was shown previously by several authors [145]–[147], the QSS is generally not described by the statistical theory of Lynden-Bell due to various causes of incomplete relaxation [113]. Finally, the dynamics of self-gravitating systems in two dimensions has been studied recently by Teles et al [148] who reached similar conclusions.

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