New Core-Guided and Hitting Set Algorithms for Multi-Objective Combinatorial Optimization

João Cortes
INESC-ID - Instituto Superior Técnico, Universidade de Lisboa, Portugal

Inês Lynce
INESC-ID - Instituto Superior Técnico, Universidade de Lisboa, Portugal

Vasco Manquinho
INESC-ID - Instituto Superior Técnico, Universidade de Lisboa, Portugal

Abstract

In the last decade, a plethora of algorithms for single-objective Boolean optimization has been proposed that rely on the iterative usage of a highly effective Propositional Satisfiability (SAT) solver. But the use of SAT solvers in Multi-Objective Combinatorial Optimization (MOCO) algorithms is still scarce. Due to this shortage of efficient tools for MOCO, many real-world applications formulated as multi-objective are simplified to single-objective, using either a linear combination or a lexicographic ordering of the objective functions to optimize.

In this paper, we extend the state of the art of MOCO solvers with two novel unsatisfiability-based algorithms. The first is a core-guided MOCO solver. The second is a hitting set-based MOCO solver. Experimental results obtained in a wide range of benchmark instances show that our new unsatisfiability-based algorithms can outperform state-of-the-art SAT-based algorithms for MOCO.

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1 Introduction

It is ubiquitous in real-world problems to try to optimize several objectives simultaneously. For instance, when making a vacation plan with multiple destinations, one wants to minimize the time spent in airports as well as the total amount spent on the plane tickets. However, in situations with multiple objectives, one can rarely obtain a solution that minimizes all objective functions. It is usually the case that decreasing the value of an objective function results in increasing the value of another objective function. This occurs in many application domains [13, 15, 25].

One way to deal with multi-objective problems is to transform them into single-objective. For example, this can be achieved by defining a linear combination of the objective functions. However, the weight of each objective function is hard to determine. Another option is to define a lexicographic order of the functions [17], but this might result in an unbalanced solution where the first function is minimized while the remaining ones have a high value.

Another approach is to determine the Pareto front of the multi-objective problem. In this case, we are interested in finding all Pareto-optimal solutions, i.e. all solutions for which one cannot decrease the value of any of the objective functions without increasing the value of another. After determining the Pareto front, one can select a representative subset of solutions to present the user [9].

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Several frameworks based on stochastic search have been developed to approximate the Pareto front of Multi-Objective Combinatorial Optimization (MOCO) problems [6, 26]. There are also several exact algorithms based on iterative calls to a satisfiability checker, such as the Opportunistic Improvement Algorithm [8]. Additionally, the Guided-Improvement Algorithm (GIA) [19], is implemented in the optimization engine of Satisfiability Modulo Theories (SMT) solver Z3 for finding Pareto optimal solutions of SMT formulas. More recently, new algorithms based on the enumeration of Minimal Correction Subsets (MCSs) [23] or \( P \)-minimal models [21] have been proposed. A common thread to these iterative and enumeration algorithms is that they follow a SAT-UNSAT approach.

In this paper, we propose two new UNSAT-SAT algorithms for MOCO. In the first algorithm, an unsatisfiable core-guided approach is used that relies on encodings of the objective functions to cut effectively the search space in each SAT call. Additionally, we also propose a hitting set based approach for MOCO where the previous core-guided algorithm is used to enumerate a multi-objective hitting set. Experimental results show that the proposed core-guided approach is complementary to the existing SAT-based algorithms for MOCO, thus extending the state of the art tools for MOCO based on SAT technology.

The paper is organized as follows. Section 2 defines the Multi-Objective Combinatorial Optimization problem, as well as common notation used in the remainder of the paper. Next, sections 3 and 4 describe the new core-guided and hitting set-based algorithms for MOCO, as well as proofs of correction. Experimental results and comparison with other SAT-based algorithms are provided in section 5. Finally, conclusions are presented in section 6.

2 Preliminaries

We will start with the definitions that fall in the SAT domain. Then, we introduce the definitions specific to the problem at hand, namely, solving MOCO problems.

Definition 1 (Boolean Satisfiability problem (SAT)). Consider a set of Boolean variables \( V = \{x_1, \ldots, x_n\} \). A literal is either a variable \( x_i \in V \) or its negation \( \neg x_i \equiv \bar{x}_i \). A clause is a set of literals, and a unary clause contains just one literal. A Conjunctive Normal Form (CNF) formula \( \phi \) is a set of clauses. A model \( \nu \) is a set of literals, such that if \( x_i \in \nu \), then \( \bar{x}_i \notin \nu \) and vice versa.

The truth value of \( \nu(x_i) \), denoted by \( \nu(x_i) \), is a function of \( \nu \), and is defined recursively by the following rules. First, the truth value of all literals is covered by

\[
\nu(x_i) = \top, \text{ if } x_i \in \nu, \\
\nu(x_i) = \bot, \text{ if } \bar{x}_i \in \nu, \\
\nu(\neg x_i) = \neg \nu(x_i).
\]

(We say \( \nu \) assigns the value \( \nu(x_i) \) to the variable \( x_i \) and \( \neg \nu(x_i) \) to \( \bar{x}_i \).)

Secondly, a clause \( c \) is true iff it contains at least one literal assigned to true. Finally, formula \( \phi \) is true iff it contains only true clauses,

\[
\nu(\phi) \equiv \bigwedge_{c \in \phi} \nu(c), \quad \nu(c) \equiv \bigvee_{l \in c} \nu(l).
\]

The model \( \nu \) satisfies the formula \( \phi \) iff \( \nu(\phi) \) is true. In that case, \( \nu \) is \( (\phi) \) feasible. A set of models is feasible iff all its elements are feasible.

The Boolean Satisfiability problem, known as Satisfiability (SAT) problem, reads as follows: Given a CNF formula \( \phi \), decide if there is any model \( \nu \) that satisfies it. In other words, decide if \( \exists \nu : \nu(\phi) \). In that case \( \phi \) is a satisfiable formula. Otherwise, it is unsatisfiable.
Our algorithms require a SAT solver, to be used as an Oracle. As they run, they place queries to the Oracle, and act accordingly to its replies. Note that a true SAT Oracle only answers yes or no. Our oracle, if the problem is unsatisfiable, replies with an explanation of why it is so, called a core (Definition 3). The following interface suffices for our intended use.

Definition 2 (SAT solver). Let $\phi, \alpha$ be CNF formulas. We call $\phi$ the main formula and $\alpha$ the assumptions. A SAT solver solves the CNF instance of the working formula $\omega = \phi \cup \alpha$, i.e. decides on the satisfiability of $\omega$.

A query to the solver is denoted by $\phi$-SAT($\alpha$). The value returned is a pair ($\nu, \kappa$), containing a feasible model $\nu$ and a core of assumptions $\kappa$, i.e. a subset of the assumptions $\alpha$ contained in some core of $\omega$. If the working formula $\omega$ is not satisfiable, $\nu$ does not exist, and the call returns ($\emptyset$, $\bullet$). If $\omega$ is satisfiable, the call returns ($\bullet$, $\emptyset$).

Definition 3 (core $\kappa$). Given a CNF formula $\phi$, we say a formula $\kappa$ is an unsatisfiable core of $\phi$ iff $\kappa \subseteq \phi$ and $\kappa \models \bot$.

As long as a core $\kappa$ of formula $\phi$ is not broken, by removing some of its clauses from $\phi$ the formula will remain unsatisfiable. Breaking it will not necessarily make the formula satisfiable, though.

Definition 4 (Minimal Correction subset (MCS)). Let $\mu \subseteq \phi$ for some unsatisfiable CNF formula $\phi$. If $\phi \setminus \mu$ is satisfiable, then $\mu$ is called a correction subset of $\phi$. If there is no other correction subset $\mu' \subset \mu$, then $\mu$ is a Minimal Correction subset (MCS).

Correcting a formula by dropping some clauses is a particular case of a relaxation. If the dropped set is a correction subset, then the obtained formula is necessarily satisfiable.

Definition 5 (relaxing/tightening a formula). Given $\phi$ we call a formula $\psi$ a relaxation of $\phi$ iff $\phi \models \psi$. We also say $\psi$ relaxes $\phi$. Conversely, $\phi$ tightens $\psi$.

Now we will review the definitions related directly to the problem we want to solve. It is called Multi-Objective Combinatorial Optimization (MOCO) (Definition 11), and it is a generalization of Pseudo-Boolean Optimization (PBO) (Definition 7). The objective functions are PB (Definition 6), and so are the clauses. Note that the PB clauses generalize the clauses of propositional logic.

Definition 6 (Pseudo-Boolean function, clause, formula (PB)). To any linear function $\{0,1\}^n \to \mathbb{N}$, given by

$$g(x) = g(x_1 \ldots x_n) = \sum_i w_i x_i, \quad w_i \in \mathbb{N},$$

we call an (integer linear) PB function. Expressions like

$$g(x) \propto k, \quad \propto \in \{\leq, \geq, =\},$$

are called PB clauses. A PB formula is a set of PB clauses. Let $x$ be the Boolean tuple $\nu(V) \equiv (\nu(x_1), \ldots, \nu(x_n))$. A model $\nu$ satisfies a clause $c$ if $c(x)$ is $\top$. Given

\[\footnote{We may use a Pseudo-Boolean (PB) formula (Definition 6). In that case, we assume the solver first translates it into CNF.}\]
a formula $\phi$, a model $\nu$ is said $\phi$-feasible if it satisfies every clause in $\phi$. If a feasible model $\nu$ exists, then $\phi$ is satisfiable, and $\nu$ satisfies $\phi$. The set of Boolean tuples $Z = \{x \in \{0,1\}^n : \exists \nu (\phi [\nu].x = \nu(V)\}$ is called feasible space of the formula $\phi$, and its elements $x$ are called feasible points. Any subset of the feasible space is called a $\phi$-feasible set.

Note that there is a one-to-one relation between the feasible points of a formula, as defined above, and the set of feasible models. It is given by $x = \nu(V)$. This redundancy is meant to help combine the notations.

▶ Definition 7 (Pseudo-Boolean Optimization, PBO). Let $\phi$ be a PB formula, and $f$ be a PB function. Then, minimize the value of the objective over the feasible space $Z$ of the formula. That is,

$$\begin{align*}
\text{find } \arg \min_{x \in Z} f.
\end{align*}$$

We will generalize this problem to the multi-objective case, but first let us introduce what we mean by optimizing several objectives at once.

Multi-objective optimization builds upon a criterion of comparison (or order) of tuples of numbers. The most celebrated one is called Pareto order or dominance (Definition 8).

▶ Definition 8 (Pareto partial order $\preceq$). Let $Y$ be some subset of $\mathbb{N}^n$. For any $y, y' \in Y$,

$$\begin{align*}
y \preceq y' & \iff \forall i, y_i \leq y'_i, \\
y < y' & \iff y \preceq y' \land y \neq y', \\
y \succ y' & \iff y \preceq y' \land y \neq y', \\
y \asymp y' & \iff y \succ y' \land y \neq y'.
\end{align*}$$

We say $y$ dominates $y'$ iff $y \preceq y'$. We say $y$ strictly-dominates $y'$ iff $y \succ y'$.

Given a tuple of objective functions sharing a common domain $X$, we can compare two elements $x, x' \in X$ by comparing the corresponding tuples in the objective space.

▶ Definition 9 (Pareto Dominance $\preceq$). Let $F : X \to Y \subseteq \mathbb{N}^n$ be a multi-objective function, mapping the decision space $X$ into the objective space $Y$. For any $x, x' \in X$,

$$\begin{align*}
x \preceq x' & \iff F(x) \preceq F(x'), \\
x \preceq x' & \iff F(x) \preceq F(x'), \\
x > x' & \iff x \preceq x', \\
x \asymp x' & \iff x \succ x'.
\end{align*}$$

We say $x$ dominates $x'$ iff $x \preceq x'$. We say $x$ strictly-dominates $x'$ iff $x \preceq x'$.

One consequence of this choice of comparison criterion is that most such optimizations have many different good solutions mapped to different points in the objective space, contrary to what happens in the single-objective case. Therefore, the solution of the problem is actually a set, traditionally called Pareto front. Its elements are good in the sense that for each one there is no other that can, in conscience, vouch for its removal.

▶ Definition 10 (Fronts). Given a a multi-objective function $F : X \to Y$ and a feasible space $Z \subseteq X$, the Pareto front of $Z$ is a subset $\mathcal{P} \subseteq Z$ containing all elements that are not strictly-dominated,

$$\mathcal{P} = \{x \in Z : \nexists x' : x \preceq x' \}.$$
We call \( \text{img-front} \) to the subset \( Y \subseteq Y \) which is the image of \( P \) by \( F \),

\[
\text{img-front} = \{ y \in Y : y = F(x), x \in P \}.
\]

Finally, we call \( \text{arg-front} \) of \( Z \), or simply \( \text{front} \) of \( Z \), to any subset \( Z \) of the Pareto Front \( P \) that is mapped by \( F \) into \( \text{img-front} \), in a one-to-one fashion. We will use the notation \( \text{front} = \text{arg-front} \).

The problem we want to solve is the following multi-objective generalization of PBO (Definition 7).

\[\text{Definition 11 (MOCO).} \quad \text{Let } F : X \rightarrow Y \subseteq \mathbb{N}^n \text{ be a multi-objective PB function, mapping the decision space } X \subseteq \{0, 1\}^n \text{ into the objective space } Y. \text{ Let } Z \subseteq X \text{ be the feasible space of some PB formula } \phi, \text{ with variables in } V. \text{ Then, find } \text{front} Z(\phi) F. \quad (5)\]

An instance will be denoted by the triple \( \langle Z, V, F \rangle \).

Because the solutions of the problems are sets, bounds are now bound sets (Definition 14). In the single objective case, a bound is a value \( l \) such that \( \forall y = f(x) : l \leq y \), or equivalently, \( \exists y = f(x) : l > y \). This equivalence is broken by the generalization. Each of the above defining properties of a lower bound give rise to a differently flavoured comparison of sets (Definitions 12 and 13). Therefore, the next definition generalizes correctly the notion of lower bound.

\[\text{Definition 12 (set coverage).} \quad \text{Let } A \text{ and } B \text{ be subsets of some decision space } X, \text{ equipped with a multi-objective function } F. \text{ Then, } A \text{ covers } B \text{ iff every element of } B \text{ is dominated by some element of } A, \text{ i.e. } \forall b \in B, \exists a \in A : a \subseteq b, \text{ and } A \text{ strictly covers } B \text{ iff } \forall b \in B, \exists a \in A : a \prec b.\]

\[\text{Definition 13 (set non-inferiority).} \quad \text{Let } A \text{ and } B \text{ be subsets of some decision space } X, \text{ equipped with a multi-objective function } F. \text{ Then } A \text{ is non-inferior to } B \text{ iff there is no element of } B \text{ that strictly-dominates an element of } A, \text{ i.e. } \forall a \in A, b \in B : \neg(a \prec b), \text{ and } A \text{ is strictly non-inferior to } B \text{ iff } \forall a \in A, b \in B : \neg(a \geq b).\]

The term non-inferior is admittedly a poor name choice. The only favoring argument is that the stronger candidate non-dominated is usually reserved for the fronts themselves.

Note that in the single objective case non-inferiority and coverage are the same thing. Therefore, the next definition generalizes correctly the notion of lower bound.

\[\text{Definition 14 (bound sets).} \quad L \subseteq X \text{ is a (strictly) lower bound set of } Z \subseteq X \text{ iff } L \text{ (strictly) covers and is (strictly) non-inferior to } Z. \text{ If } L \text{ is a lower bound set of } Z \text{ we say } L \subseteq Z. \text{ If it is a strictly lower bound set, we say } L \prec Z.\]

One way to generate a lower bound set of some Pareto front is to solve a related problem, where the formula is replaced by a relaxed version (Definition 5). In order to guide the search we will need to embed dominance relations into CNF formulas. For instance, we are interested in ensuring we do not spend time looking for solutions that are dominated by some other known feasible solution. But how can we express it on a SAT query? We need to translate the requirement into a CNF formula. A particular example of such a translator is called an unary counter. In particular, they have been used to implement efficient PB satisfiability solvers that simply forward PB queries into a SAT solver after translating them.
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Point is
- optimal;
- feasible, not generated;
- non-feasible fence bound.

Region is
- fenced;
- dominated;
- fenced and dominated.

Figure 1 Illustration of a run of Core-Guided (Algorithm 1) in the objective space. The img-front is the set \{1, 2, 3\}. The fence bound \(\lambda\) gets updated at each iteration of the while cycle at line 6, starting at \(A\) and ending at \(\Omega\). The arrows are guided by the core \(\kappa\) (line 19). The green shading represents the evolution of the fence. Darker regions have been fenced for longer. The blue regions are blocked by optimal points. Darker regions are dominated by more points. We will be done in 7 iterations. After verifying that \(A\) is not feasible, we are instructed by the cores \(k\) to move along the diagonal twice. We find point 1 fenced. Therefore the associated \(x\) is copied into \(I\) and the dominated region is blocked. We extend \(\lambda\) twice, and find point 2. After moving once more, we find part of the fence blocked, and the point branded with \(i\) is never generated. The next movement stations \(\lambda\) at \(\Omega\). Point 3 is found. The Oracle acknowledges we are done, by returning \(\kappa = \emptyset\) (line 15):

\[ f_i(x) \geq k \Rightarrow o_{i,k}, \quad x = \nu(V). \]  

Definition 15 (Unary Counter). Let \(f_i : \{0, 1\}^m \rightarrow \mathbb{N}\) be a \(\text{PB}\) function and set \(V\) be an ordered set of variables that parametrize the domain of \(f_i\),

\[ V = \{x_1, \ldots, x_m\}, \quad f_i(x) = f_i(x_1, \ldots, x_m). \]  

Consider the \(\text{CNF}\) formula \(\theta\) with variables \(V \cup O\), where \(O \cap V = \emptyset\) and \(O\) contains one variable \(o_{i,k}\) for each value \(k \in \mathbb{N}\) : \(\exists x : k = f_i(x)\). The elements of \(O\) are the order variables. We call the tuple \((f_i, V, O, \theta)\) an unary counter of \(f_i\) iff all feasible models \(\nu\) of \(\theta\) satisfy

3 Unsatisfiability-based Algorithm

Although core-guided algorithms for single-objective problems such as Maximum Satisfiability \cite{7, 16, 14, 3, 2} have been initially proposed more than one decade ago, to the best of our knowledge, there is no such algorithm for MOCO. The main goal of our algorithm is to take advantage of unsatisfiable cores identified by a SAT solver in order to lazily expand the allowed search space.

3.1 Algorithm Description

Algorithm \[\text{Algorithm 1}\] presents the pseudo-code for an exact core-guided SAT algorithm for MOCO. Figure \[\text{Figure 1}\] illustrates an abstract execution of the algorithm.

Let \((\phi, V, F)\) be a MOCO instance. Recall that \(\phi\) denotes the set of \(\text{PB}\) constraints, \(V\) is the set of variables and \(F\) denotes the list of \(m\) objective functions.
Input: \(\langle \phi, V, F \rangle\) // MOCO instance
Output: front \(\phi\) // one arg-front

1. \(m \leftarrow |F|\) // one arg-front
2. \(I \leftarrow \emptyset\)
3. \(\Theta \leftarrow \emptyset\)
4. \(\lambda \leftarrow \{0, 0, \ldots, 0\}\) // init fence upper bound
5. while \(\text{true}\) do
6.   \(x \leftarrow \nu(V)\)
7.   \(I \leftarrow I \cup \{x' \in I : x \preceq x'\} \cup \{x\}\)
8.   \(\Theta \leftarrow \Theta \cup \{\bigvee_{i=1}^{m} \neg o_i f_i(x)\}\) // block region dominated by \(x\)
9.   \(\nu, \kappa \leftarrow \Theta - \text{SAT}(\alpha)\)
10. if \(\kappa = \emptyset\) then
11.   return \(I\) // arg-front found
12. else
13.   foreach \(\{\neg o, k\} \in \kappa\) do
14.     \(\lambda_i \leftarrow k\) // expand fence, as suggested by \(\kappa\)
15.   end
16. end

Algorithm 1 Core-Guided MOCO solver Core-Guided

First, the algorithm starts by building a working formula with the problem constraints and an unary counter for each objective function (lines 3-4). Next, a vector \(\lambda\) of size \(m\) is initialized with the lower bound of each objective function (line 5), assumed to be 0.

At each iteration of the main cycle the assumptions \(\alpha\) are assembled from order variables \(o\), chosen with the value of \(\lambda\) in mind (line 7). The call to \(\text{next}(i, \lambda)\) returns the next smallest value belonging to the image of the objective \(i\). Given the semantics of the order variables \(o, k\) (Definition 15), the tuple \(\lambda\) fences the search space, i.e. \(\nu\) satisfies \(\alpha\) only if the corresponding tuple \(x\) satisfies \(F(x) \preceq \lambda\). If the SAT call (line 10) returns a solution (i.e. \(\nu \neq \emptyset\)), \(x\) is stored in and all dominated solutions are removed from \(I\) (line 11). Moreover, one can readily block all feasible solutions dominated by \(x\) using a single clause (line 12).

Usually there are several feasible fenced solutions. This is so because the algorithm may increase multiple entries of \(\lambda\) at once. In any case, the inner while loop (lines 9-13) collects all such solutions.

When the working formula [7] becomes unsatisfiable, the SAT solver provides an unsatisfiable core \(\kappa\). If \(\kappa\) is empty, then the unsatisfiability does not depend on the assumptions, i.e. it does not depend on the bounds imposed on the objective functions. At that point, we can conclude that no more solutions exist that are both satisfiable and not dominated by

2 May be replaced by \(\lambda_i + 1\).
3 This technique had already been proposed when enumerating minimal models [21].
an element of \( I \). As a result, the algorithm can safely terminate (line 16). Otherwise, the
literals in \( \kappa \) denote a subset of the fence walls \( \lambda \) that may be too restrictive, in the sense
that unless we increment them (line 19) no new non-dominated solutions can be found.

### 3.2 Algorithm Properties

**Lemma 16.** The img-front \( F \) of \( I \), \( Z \), \( \emptyset \) (Definition 10) is an invariant of the inner loop
in lines 9-13.

**Proof.** Consider some particular iteration of the internal loop. Line 11 and line 12 remove all
elements of \( I \), \( Z \), \( \emptyset \) that are dominated by the feasible point \( x \). Line 11 filters the explicit
set \( I \), line 12 filters the implicit set \( Z \), \( \emptyset \). Solutions that are strictly dominated by \( x \) cannot
be mapped into an element of \( X \). The other solutions \( x' \) that are filtered out must attain
the same objective vector attained by \( x \), \( F(x') = F(x) \). Because \( x \) is also inserted at line 11,
removing \( x' \) will not disturb \( F \).

**Lemma 17.** At the start of each iteration of the external loop (lines 9-22), every solution
in \( I \) is optimal, and no two elements of \( I \) attain the same objective vector.

**Proof.** We prove this by contradiction. Assume that there is a non-optimal solution \( x \in I \)
at the start of the external loop (line 6). In the first iteration, this does not occur because \( I \) is empty. Hence, this can only occur if the inner loop (lines 9-13) finishes with a non-optimal
solution \( x \in I \).

The inner loop (lines 9-13) enumerates solutions inside the fence defined by \( \lambda \). We know
that \( F(x) < \lambda \) because it is inside the fence and the values in \( \lambda \) never decrease. If \( x \) is non-optimal,
then there must be an optimal solution \( x' \) such that \( F(x') < F(x) < \lambda \). Hence, \( x' \) is also inside the fence. As a result, \( x' \) must be found before the inner loop finishes, since
at each iteration only dominated solutions are blocked (line 12). If \( x \) is found before \( x' \),
then \( x \) is excluded from \( I \) (line 11) when \( x' \) is found. Otherwise, if \( x' \) is found first, then \( x \) is not
found by the SAT solver (blocked at line 12) because it is dominated by \( x' \). Therefore, we
cannot have a non-optimal solution \( x \in I \) at the end of the inner loop or at the start of each
iteration of the external loop (lines 9-22).

Furthermore, no two elements of \( I \) attain the same objective vector since when a solution
\( x \) is found, all other solutions \( x' \) such that \( F(x) = F(x') \) are also blocked (line 12).

**Lemma 17** establishes a weaker form of anytime optimality. The elements of the incumbent
list \( I \) are not necessarily optimal at anytime, but they are optimal immediately after
completing the inner loop. It is easy enough to make it anytime optimal. This could be
achieved if the algorithm refrains from adding solutions directly to \( I \) in the inner loop and
maintain a secondary list, where it stores the solutions that are still not necessarily optimal.
This list takes the role of \( I \) inside the inner loop. After completing the inner loop, all elements
of the secondary list are optimal, and can be safely transferred to the main list \( I \).

**Lemma 18.** Algorithm \( I \) is sound.

**Proof.** If the algorithm returns, \( Z(\emptyset \land \alpha) = \emptyset \). Because \( \kappa \) is empty, no core of the unsatisfiable
formula \( \emptyset \land \alpha \) intersects \( \alpha \), and \( \emptyset \) is also unsatisfiable, \( Z(\emptyset) = \emptyset \). Using Lemma 16 both at
the end and at the start of the course of the algorithm, the img-front of \( I \) is the img-front of
\( Z(\emptyset) \), with \( I \) given by line 4. Because the order variables are only restricted by the unitary
counter formula, the img-front of \( Z(\emptyset) \) is the img-front of \( Z(\emptyset) \). Therefore \( I \) must contain
an arg-front of the problem. Using Lemma 17, every element of \( I \) is optimal, and there is no pair
\( x, x' \in I \) such that \( F(x) = F(x') \). Therefore, \( I \) is an arg-front of the \( \text{MOCO} \) instance.
Input: $(\phi, V, F)$ // MOCO instance
Output: $\text{front}_F$ // one arg-front
1 $\psi \leftarrow \emptyset$ // relaxed formula $\psi$ is initially empty
2 while true do
3 $\Delta \leftarrow \emptyset$
4 foreach $x \in \text{front}_\psi F$ // use auxiliary solver
5 $\alpha_x \leftarrow \{\{l\}, l \in \nu(V) = x\}$
6 $(\bullet, \kappa) \leftarrow \phi\text{-SAT}(\alpha_x)$
7 if $\kappa \neq \emptyset$ then
8 $\Delta \leftarrow \kappa \cup \Delta$
9 end
10 end
11 if $\Delta = \emptyset$ then // if $T$ is $\phi$-feasible
12 return $T$ // arg-front found
13 end
14 foreach $\kappa \in \Delta$ do
15 $\psi \leftarrow \psi \cup \{\neg l, \{l\} \in \kappa\}$ // Increment $\psi$, adding $\neg\kappa$
16 end
17 end
Algorithm 2 Hitting-Sets based MOCO solver Hitting-Sets

Lemma 19. Algorithm 1 is complete.

Proof. The inner loop will always come to fruition, because in the worst case it will generate every feasible solution dominated by the current $\lambda$ once, and the feasible space is finite.

If the algorithm does not return for some particular instance, then $\kappa$ is never empty. In that case, every iteration of the external loop starting at line 3 will increase at least one of the entries of $\lambda$. Eventually, one entry $i$ must achieve the upper limit of $f_i$, and the order variable retrieved by $o_{i, \lambda_{i+1}}$ will not exist. Because the evolution of $\lambda_i$ is monotonous, the assumptions will contain at most $m - 1$ variables, from that point on. By the same token, the assumptions $\alpha$ will eventually be empty, and so must be $\kappa \subseteq \alpha$, contradicting the assumption that the algorithm never terminates. ◀

4 Hitting Set-based Algorithm

In this section we propose a MOCO solver based on the enumeration of hitting sets. We briefly motivate the algorithm, and prove its correctness and soundness.

The main idea is to compute a sequence of relaxations $\psi$ of the formula [2] and solve the corresponding problems. The front $\emptyset$ of the relaxed problem gets incrementally closer to the desired front $Z$ and will eventually reach it in a finite amount of time.

4.1 Algorithm Description

Algorithm 2 contains the pseudo-code for our hitting set-based algorithm for MOCO. Figure 2 illustrates an abstract execution of the algorithm.

The algorithm starts by setting the relaxed formula $\psi$ to the empty one (line 1). The main cycle that starts at line 2 will hone the relaxation until we get the desired result. At each
Figure 2 Illustration of a run of the Hitting-Sets (Algorithm 2) in the objective space. The Pareto front is the set \{1, 2, 3\}, and the feasible solutions are marked by \(\circ\). For each iteration of the main while cycle at line 2 we get a narrower lower bound \(\mathbb{T}\) (line 4), culminating in the solution. We are done in 3 iterations, marked by A, B and \(\triangle\). The shading represents the number of iterations whose freshly found points dominate the region. The lighter tone was painted by A, the darker one by all three. We start with the empty formula (line 1), and we get A. Because the only point in A is not feasible, we tighten the relaxation (line 16). Iteration B generates one feasible point, 1, which is therefore optimal. Note that the region dominated by 1 can be pruned from now on. The other point is used to tighten the formula once more. Lastly, the lower bound contains the feasible points 2 and 3 in addition to 1 which was already found, and the algorithm stops.

iteration, we solve the current relaxed formula \(\psi\) at line 3. This will be accomplished by using some \textsc{MOCO} solver. Because this amounts to computing a lower bound set, the Core-Guided algorithm, previously described, is a good choice for the task. We anticipate that it performs well for problems whose front is in the vicinity of the origin, given that by construction the focus of its search is biased to that region. Notice that the first relaxation’s arg-front is the set that contains the origin only (assuming all literals in the objective functions are positive). We expect that the first few relaxations will stay close to it.

Next, for each element \(x\) in \(\mathbb{T}\) (the Pareto-front of \(\psi\)), we check the \(\mathcal{F}\)-feasibility of \(\nu : \nu(V) = x\), using the assumptions mechanism, and returns a (possibly empty) core of assumptions \(\kappa\). The assumptions \(\alpha_{x}\) built at line 6 are a set of unitary clauses whose polarity is inherited from \(\nu\),

\[
\nu(x_i) \implies x_i \in \alpha, \quad \neg\nu(x_i) \implies \neg x_i \in \alpha. \quad (8)
\]

Given the interface of our \textsc{SAT} solver (Definition 2), the returned core \(\kappa\) will be void iff \(\alpha_{x} \land \mathcal{F}\) is satisfiable. In this case, \(x\) corresponds to an optimal solution.

The diagnosis \(\Delta\) is central for the algorithm. Intuitively, it reports if and why the relaxed problem’s solution is different from the true Pareto solution. We add every non-empty \(\kappa\) to the diagnosis \(\Delta\) (line 9). In the end, \(\Delta\) is empty iff every element of the relaxed front \(\mathbb{T}\) is \(\mathcal{F}\)-feasible. At that point, we have found a \(\mathcal{T}\)-feasible lower bound set. All such sets are arg-fronts, and so the algorithm terminates at line 14. Otherwise, if \(\Delta\) is not empty, then the found cores are added to the relaxed formula \(\psi\) (line 16). This step ensures all tentative solutions found in line 4 hit the unsatisfiable cores found and that the algorithm advances in a monotonous fashion towards the solution. This will be further discussed in Lemma 22.
4.2 Algorithm Properties

Given a MOCO instance \( \langle \varphi, V, F \rangle \), the formula \( \varphi \) encodes the feasible space \( Z \) implicitly, which in turn defines the desired front \( Z \). This is a many to one correspondence, in the sense that there are many different values of \( \varphi \) that encode the same Pareto front. It may happen that some of the counterpart instances are easier to solve than the original one, which begs the question: given \( \varphi \), can we effectively find a simpler formula \( \psi \) with the same Pareto front? This is the motto of the proposed algorithm. It is done by iteratively honing a relaxed formula (Definition 5).

The main idea is to compute a sequence of relaxations that get incrementally tighter. In that case, the corresponding front \( T \) gets incrementally closer to the desired front \( Z \),

\[
\varphi \Rightarrow \psi_1 \Rightarrow \ldots \Rightarrow \psi_n \Rightarrow \ldots \Rightarrow \psi_1 \Rightarrow T_1 \Rightarrow \ldots \Rightarrow T_n
\]

where \( Z \) is one of the desired arg-fronts, and \( T_i \) is an arg-front of \( \psi_i \).

\[\blacktriangleright\text{Lemma 20.}\]

Consider some multi-objective function \( F : X \rightarrow Y \). Let \( Z, T \) be subsets of \( X \), such that \( T \subseteq Z \). Then, any arg-front of \( T \) is a lower bound set of any arg-front of \( Z \) (Definition 14), i.e. \( T \subseteq Z \Rightarrow \exists T \subseteq Z \).

Lemma 20 is true because optimizing over a superset of some feasible space always returns a (non-strict) lower bound set. In a sense, the optimization can only be more extreme when applied to the superset. In particular, the feasible space of a relaxed formula is a superset of the original one. This is why the chain of \( \subseteq \) relations in Equation (10) is correct.

\[\blacktriangleright\text{Lemma 21.}\]

Let \( \varphi \) be a formula, \( Z \subseteq X \) be its feasible space and \( F : X \rightarrow Y \) be some multi-objective function. Let \( L \) be a lower bound set of the Pareto front of \( Z \). Then, any element \( x \in L \) that is feasible belongs to the Pareto front, \( L \subseteq \mathcal{P} \). If all elements \( x \in L \) are feasible, then \( L \) is an arg-front.

Lemma 21 implies that every lower bound set with only feasible elements must be itself an arg-front (this is an exact analogy with the single-objective case, where lower bound set is replaced by infimum and arg-front by arg-min). By construction of the diagnosis \( \Delta \), this is equivalent to the condition used in Algorithm 3 to decide if it can terminate.

To ensure the sequence gets to \( Z \) in a finite number of steps, we need more than a string of relaxations. Each entry \( \psi_i \) must be strictly tighter than the predecessor \( \psi_{i-1} \).

\[\blacktriangleright\text{Lemma 22.}\]

Consider Algorithm 3. Let \( \psi \) be the relaxed formula at some iteration, and \( \psi' \) be the relaxed formula at the next iteration. Then,

1. \( \psi \) relaxes \( \psi' \), i.e. \( \psi \models \psi' \);
2. Both \( \psi \) and \( \psi' \) relax \( \varphi \), i.e. \( \varphi \models \psi, \varphi \models \psi' \);
3. \( \psi' \) does not relax \( \psi \), i.e. \( \psi \models \psi' \);

Proof. Each statement will be proven in turn.

The first is true because \( \psi \subseteq \psi' \), by construction (line 16).

We prove the second by induction on the number of iterations. Initially \( \psi \) is empty. Therefore, \( \psi \) relaxes any formula, in particular \( \varphi \). Assume \( \psi \models \psi \) for some iteration. Consider one of the clauses \( \neg \kappa \) added at line 16. We know that \( \varphi \land \kappa \) is unsatisfiable. Therefore, \( \varphi \land \kappa \models \bot \Rightarrow \models \varphi \models \neg \kappa \). Given the assumption \( \models \psi \), we get \( \models \psi \land \neg \kappa \). Repeating the process for the other added clauses \( \neg \kappa_i \), we get \( \models \psi \land \neg \kappa_1 \ldots \land \neg \kappa_n \equiv \psi' \).
Assume $\psi'$ is a relaxation of $\psi$. Then, any $\psi$-feasible model $\nu$ is also $\psi'$-feasible. We will prove there is at least one model that violates this. To start, note that it only makes sense to consider $\psi'$ if there is some non-empty core $\kappa$ in the diagnosis $\Delta$, otherwise the algorithm would have terminated before updating $\psi$ into $\psi'$. Let $\kappa$ be one element of $\Delta$, generated at line 7 while $\psi$ is current. Consider the Boolean tuple $x$ used to build the assumptions of the query that generated $\kappa$. Let $\nu : \nu(V) = x$. The model $\nu$ is $\psi$-feasible, because it is part of the arg-front of $\psi$. The model $\nu$ satisfies $\kappa$ because $\kappa \subseteq \alpha_x$ and the way $\alpha_x$ is constructed (line 6, Equation (8)). Therefore, $\nu$ does not satisfy $\neg \kappa$. Because $\neg \kappa \subseteq \psi'$, $\nu$ cannot satisfy $\psi'$, i.e. there is at least one $\psi$-feasible model that is not $\psi'$-feasible.

Proposition 23. Algorithm 2 is sound.

Proof. By Lemma 22, $\psi$ relaxes $\phi$ and therefore solves a relaxation of the original problem. By Lemma 20, it is a lower bound set of $Z$. When the algorithm returns, all elements of $T$ are feasible. By Lemma 21, $T$ must be an arg-front.

Proposition 24. Algorithm 2 is complete.

Proof. Assume Algorithm 2 never ends, implying $T$ is never completely feasible (i.e. $T \not\subseteq Z$). The number of relaxed feasible spaces $T$ is finite. If Algorithm 2 does not end, it will enumerate all of them, never repeating any: at any iteration, the updated relaxed formula effectively blocks the reappearance of any feasible space seen before, because by Lemma 22 the updated value $\psi'$ strictly tightens $\psi$. Then, this sequence is necessarily finite, and so must be the number of iterations. But in that case, Algorithm 2 must end, and we have a contradiction.

Consider the sequence whose entries are the value of $F(T)$ computed at the beginning of each iteration of the main cycle at line 2. The last element of this sequence is the solution. It may happen that for some $i$, the entries indexed by $i$ and $i + 1$ are the same. Therefore, the sequence may include blocks of contiguous entries that share the same value. In the worst case scenario, there are many different arg-fronts for the same img-front, and the algorithm ends up enumerating all of them without any movement in the objective space. We expect the algorithm will be effective whenever a few of the relaxed problems are enough to get to the full solution. Otherwise, we can end up solving an exponential number of problems.

5 Experimental Results

This section evaluates the performance of the algorithms proposed in Sections 3 and 4. These new algorithms are compared against other SAT-based solvers for MOCO.

5.1 Algorithms and Implementation

The Core-Guided algorithm proposed in Algorithm 1 uses the selection delimiter encoding [11] that has been shown to be more compact. Next, the selection delimiter encoding is extended such that an unary encoding is produced for each objective function. Additionally, an order encoding [22] is also used on the unary representation of each objective function. We refer the interested reader to the literature for further details on this and other encodings [20, 10, 11, 12]. Observe that any unary encoding from PB into CNF can be used.

The Core-Guided-Strat algorithm is an alternative implementation of Algorithm 1 that uses the well-known technique of stratification [11, 13, 24]. In this case, literals in each objective function are split into partitions according to their weights. Core-Guided-Strat
first solves the MOCO instance considering only the literals in the partition with highest coefficients. Next, after finding an approximation of the Pareto front using only part of the objective functions, the next partition with highest coefficients for each objective function is uncovered. This process is repeated until all literals in all objective functions are represented.

The **Hitting-Sets** algorithm implements Algorithm 2. This hitting set based approach uses Algorithm 1 in order to find the relaxed arg-front (line 4 of Algorithm 2).

The **P-Minimal** algorithm implements a SAT-UNSAT based approach based on the enumeration of P-Minimal models [21]. This algorithm is implemented with the same PB to CNF encoding as the **Core-Guided**. Finally, the **ParetoMCS** is based on the stratified enumeration of Minimal Correction Subsets. We used the publicly available implementation of ParetoMCS.

### 5.2 Benchmark Sets

The following MOCO problems were considered: the multi-objective Development Assurance Level (DAL) Problem [5], the multi-objective Flying Tourist Problem (FTP) [15], and the multi-objective Set Covering (SC) Problem [4, 21].

The DAL benchmark set encodes different levels of rigor of the development of a software or hardware component of an aircraft. The development assurance level defines the assurance activities aimed at eliminating design and coding errors that could affect the safety of an aircraft. The goal is to allocate the smallest DAL to functions in order to decrease the development costs [5].

The FTP benchmark instances encode the problem of a tourist that is looking for a flight travel route to visit \( n \) cities. The tourist defines its home city, the start and end of the route. She specifies the number of days \( d_i \) to be spent on each city \( c_i \) \((1 \leq i \leq n)\) and also a time window for the complete trip. The problem is to find the route that minimizes the time spent on flights and the sum of the prices of the tickets.

The SC benchmark is a generalization of the classical set covering problem. Let \( X \) be some ground set and \( A \) a cover of \( X \). Each element in \( A \) has an associated cost tuple. The goal is to find a cover of \( X \) contained in \( A \) that Pareto-optimizes the overall cost.

### 5.3 Experimental Setup and Evaluation Metrics

All results were obtained on a Intel Xeon Silver 4110 CPU @ 2.10GHz, with 64 GB of RAM. Each tool was executed on each instance with a time out of 1 hour and 10 GB of memory.

Finding the whole Pareto front is extremely hard for most problem instances. All tested algorithms are exact, but in most cases only an approximation of the Pareto front could be found within the time limit. In order to evaluate the quality of the approximations provided by each tool, we use the Hypervolume (HV) [27] indicator. Since the HV indicator needs a reference front, for each benchmark instance we combined the approximations produced by all algorithms to build the reference front. HV is a metric that measures the volume of the objective space dominated by an approximation, up to a given reference point, and thus larger values are preferred. A normalization procedure is carried out so that the values of HV are always between 0 and 1.

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4. [https://gitlab.ow2.org/sat4j/moco](https://gitlab.ow2.org/sat4j/moco)

5. The benchmark instances are available online at [https://www.lifl.fr/LION9/challenge.php](https://www.lifl.fr/LION9/challenge.php). Although they define a lexicographic order to the objective functions, in the context of this paper, we ignore this order and compute the Pareto front.
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Figure 3 Comparison of the HV results for the DAL instances. Each series is sorted independently, smaller values first. Vertical scale is logarithmical.

Figure 4 Comparison of the HV results for the FTP instances. Vertical scale is logarithmical.

5.4 Results and Analysis

Figure 3 shows a cactus plot of all tools on the DAL benchmark set. In this case, the SAT-UNSAT approach from P-Minimal is the overall best performing algorithm. The size of the encoding of the objective functions is not very large. The Core-Guided-Strat and Hitting-Sets algorithms take more iterations to find the set of solutions in the Pareto front. This is more notorious in Hitting-Sets, as many hitting sets have to be enumerated. Nevertheless, in some instances, the number of MCS is large and the ParetoMCS was unable to find MCSs in the Pareto front. Overall, its performance is similar to our algorithms.

The results for the FTP benchmark set are provided in Figure 4. In this case, the use of stratification shows to be crucial. Observe that both Core-Guided-Strat and ParetoMCS use this technique. Note that ParetoMCS does not have an explicit representation of the objective functions. However, the representation used in Core-Guided-Strat is still effective for these instances, despite some large coefficients in the objective functions. As a result, these algorithms are able to quickly find a solution close to the Pareto front, providing better overall performance than the other algorithms.

Figure 5 shows the results for the SC benchmark set. In these instances, the Core-Guided-Strat was able to outperform all other algorithms, since it does not need to relax all variables to find solutions in the Pareto front. Note that when Hitting-Sets is able
to find solutions, these are in the Pareto front, providing higher HV values in these cases. However, a common feature of the Hitting-Sets is the need to enumerate many hitting sets before being able to find a feasible solution. Despite using stratification, ParetoMCS is unable to perform as well as Core-Guided-Strat.

6 Conclusions

This paper proposes two new algorithms for Multi-Objective Combinatorial Optimization (MOCO). The first is a core-guided approach, while the second is based on the enumeration of hitting sets. These are the first algorithms for MOCO that follow an UNSAT-SAT strategy.

Experimental results on three different sets of benchmark instances show that the new core-guided approach results in a robust algorithm that is competitive or outperforms other SAT-based algorithms for MOCO. Using unary counters to express Pareto dominance in CNF shows to be an effective way to harness the power of SAT solvers in solving MOCO. The ability to express concepts related to dominance makes the algorithms conceptually simple, and is therefore a useful tool in developing other MOCO solvers based on SAT Oracles.

The algorithm based on the hitting set approach uses the core-guided MOCO algorithm to incrementally enumerate the hitting sets. The current implementation of the hitting set algorithm is not competitive in some benchmark instances due to having to enumerate a large number of hitting sets. The usage of the well-known stratification technique has a significant impact on the performance of the core-guided algorithm in several sets of instances. This technique is yet to be implemented in the hitting set approach and we expect that including stratification will allow to cut a large number of the algorithm’s iterations.

Overall, the new algorithms extend the state of the art algorithms for MOCO based on calls to a SAT solver by complementing the existing tools.

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