HOMOLOGY HANDLES WITH TRIVIAL ALEXANDER POLYNOMIAL

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Abstract. Using Freedman and Quinn’s result for $\mathbb{Z}$-homology 3-spheres, we show that a 3-dimensional homology handle with trivial Alexander polynomial bounds a homology $S^1 \times D^3$. As a consequence, a distinguished homology handle with trivial Alexander polynomial is topologically null $\tilde{H}$-cobordant.

1. Introduction

In 1976, Kawauchi introduced the smooth $\tilde{H}$-cobordism group $\Omega(S^1 \times S^2)$, whose elements are equivalence classes of distinguished homology handles. A distinguished homology handle is a pair $(M, \alpha)$ of a compact, oriented 3-manifold $M$ having the homology of $S^1 \times S^2$, and a specified generator $\alpha$ of $H_1(M; \mathbb{Z}) \cong \mathbb{Z}$. The equivalence relation is $\tilde{H}$-cobordism, which means that two distinguished homology handles $(M_0, \alpha_0)$ and $(M_1, \alpha_1)$ are $\tilde{H}$-cobordant if there is a pair $(W, \varphi)$ of a smooth connected 4-dimensional cobordism $W$ between $M_0$ and $M_1$, and a first cohomology class $\varphi \in H^1(W; \mathbb{Z})$ such that

1. $\varphi|_{M_i}$ are dual to $\alpha_i$ for $i = 0, 1$,
2. $H_*(\widetilde{W}_\varphi; \mathbb{Q})$ is finitely generated over $\mathbb{Q}$ for each $*$, where $\widetilde{W}_\varphi$ is the infinite cyclic covering of $W$ associated with $\varphi$.

In this case, $(W, \varphi)$ (or simply $W$) is called a smooth $\tilde{H}$-cobordism between $(M_0, \alpha_0)$ and $(M_1, \alpha_1)$ (or between $M_0$ and $M_1$). If $(M, \alpha)$ is $\tilde{H}$-cobordant to $(S^1 \times S^2, \alpha_{S^1 \times S^2})$, where $\alpha_{S^1 \times S^2}$ is the homology class of $S^1 \times *$ with a fixed orientation, then we say that $(M, \alpha)$ is null $\tilde{H}$-cobordant, and $(W^+, \varphi)$ (or $W^+$) is a null $\tilde{H}$-cobordism of $(M, \alpha)$ (or of $M$). Equivalently, there is a smooth $\tilde{H}$-cobordism $(W^+, \varphi)$ with $\partial W^+ = M$. Under a sum operation $\bigcirc$ called the circle union, $\Omega(S^1 \times S^2)$ is an abelian group, and $[(S^1 \times S^2, \alpha_{S^1 \times S^2})]$ plays the role of the identity. Furthermore, the inverse $-[(M, \alpha)]$ of $[(M, \alpha)]$ is $[-(M, \alpha)]$, where $-M$ is $M$ with a reversed orientation. For details, see [8] and [10].

Likewise, we can define the topological $\tilde{H}$-cobordism group $\Omega^{top}(S^1 \times S^2)$ in the topological category by using topological 4-manifolds in the definition of

2010 Mathematics Subject Classification. 57N70, 57M10.

Key words and phrases. Homology handles, $\tilde{H}$-cobordism.
There is a natural surjective homomorphism $\psi: \Omega(S^1 \times S^2) \to \Omega^{\text{top}}(S^1 \times S^2)$ by forgetting smooth structures.

Results in knot concordance motivate a number of questions on the $\tilde{H}$-cobordism groups $\Omega(S^1 \times S^2)$ and $\Omega^{\text{top}}(S^1 \times S^2)$.

In the knot concordance group $C$, let $C_\Delta$ be the subgroup generated by knots with trivial Alexander polynomial, and $C_T$ the subgroup generated by topologically slice knots. Using Donaldson’s diagonalization theorem [2], Casson observed that there are knots with trivial Alexander polynomial but which are not smoothly slice (appearing in [1]). After Donaldson’s result, Freedman proved that a knot with trivial Alexander polynomial is topologically slice [3], [4]. Thus $C_\Delta \subset C_T$ and $C_T$ is non trivial, i.e., the map $C \to C^{\text{top}}$ is not injective, where $C^{\text{top}}$ is the topologically flat knot concordance group.

One can expect similar results in the $\tilde{H}$-cobordism groups. Let $\Omega_\Delta$ be the subgroup generated by distinguished homology handles with trivial Alexander polynomial, and $\Omega_T$ the kernel of the map $\psi: \Omega(S^1 \times S^2) \to \Omega^{\text{top}}(S^1 \times S^2)$.

**Question 1.** Is $\psi: \Omega(S^1 \times S^2) \to \Omega^{\text{top}}(S^1 \times S^2)$ injective?

**Question 2.** Is $\Omega_\Delta \subset \Omega_T$?

In this paper, using the work of Freedman and Quinn, we prove the following theorem, which is the positive answer to Question 2.

**Theorem 1.** A distinguished homology handle with trivial Alexander polynomial is topologically null $\tilde{H}$-cobordant.

We expect a negative answer to Question 1. Then one can also ask about $\Omega_T/\Omega_\Delta$.

**Question 3.** How big is the gap between two groups if $\Omega_T/\Omega_\Delta$ is non-trivial?

In the knot concordance group $C$, Hedden, Livingston, and Ruberman showed that $C_T/C_\Delta$ contains a $\mathbb{Z}^\infty$-subgroup [5], and Hedden, Kim and Livingston showed that it also has a $\mathbb{Z}_2^\infty$-subgroup [6].

**Acknowledgement**

I would like to thank my advisor Matthew Hedden for an enlightening email discussion.

**2. Alexander polynomial of homology handles**

In this section, we review Alexander polynomial of homology handles. We refer the reader to [8], [9], and [11] for more details.

Let $\tilde{M}$ be an oriented homology handle. Then we have the infinite cyclic covering $\tilde{M}$ of $M$ associated with the abelianization map $\pi_1(M) \to H_1(M) \cong \mathbb{Z}$. Let $t$ be a generator of the deck transformation group $\mathbb{Z}$ of the covering space.
Since $M$ is compact and triangulable, it admits a finite CW-complex, and thus the chain complex $C_i(\tilde{M}; \mathbb{Z})$ can be considered as a free and finitely generated module over the group ring $\Lambda = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$, with one generator for each $i$-cell of $M$. Since the group ring $\Lambda$ is Noetherian, one can see that the homology $H_i(\tilde{M}; \mathbb{Z})$ is a finitely generated module over $\Lambda$. For an exact sequence $E \to F \to H_1(\tilde{M}; \mathbb{Z}) \to 0$ of $\Lambda$-modules with $E$ and $F$ free modules of finite rank, a presentation matrix $P$ is a matrix representing the homomorphism $E \to F$. If the rank of $F$ is $r \geq 1$, then the first elementary ideal $\mathcal{E}$ of $P$ is the ideal over $\Lambda$ generated by all the $r \times r$ minors of $P$. If there are no $r \times r$ minors, then we have $\mathcal{E} = 0$, and if $r = 0$, then we set $\mathcal{E} = \Lambda$. The Alexander polynomial of $M$ is defined to be any generator $\Delta_M(t)$ of the smallest principal ideal over $\Lambda$ containing $\mathcal{E}$.

Another description. Let $\mu$ be a smoothly embedded simple closed oriented curve in $M$ representing a generator of $H_1(M; \mathbb{Z})$. Let $T(\mu)$ be a tubular neighborhood of $\mu$. We choose simple closed oriented smooth curves $K$ and $l$ in $\partial T(\mu)$ intersecting in a single point so that $l$ is homologous to $\mu$ in $T(\mu)$, and $K$ bounds a disk in $T(\mu)$ with $\text{lk}(\mu, K) = +1$. Note that the choice of a curve $l$ is not unique. Choose a diffeomorphism $h : S^1 \times S^1 \to \partial T(\mu)$ such that $h(S^1 \times 0) = l$ and $h(0 \times S^1) = K$. Let $Y = (M \setminus \text{Int} T(\mu)) \cup_h (D^2 \times S^1)$. Then $Y$ is a $\mathbb{Z}$-homology 3-sphere, and $K$ is a knot in $Y$. The Alexander polynomial $\Delta_M(t)$ of $M$ is defined to be the Alexander polynomial $\Delta_K(t)$ of $K$ in $Y$.

Both definitions agree with the following: Let $A$ be a Seifert matrix for a knot $K$ in $Y$. We know that $tA - A^T$ is a presentation matrix for the $\Lambda$-module $H_1(\widetilde{X(K)}, \mathbb{Z})$, where $X(K)$ is a knot exterior of $K$ in $Y$, and $\widetilde{X(K)}$ is the infinite cyclic coverings of $X(K)$. Let $Y_0(K)$ be the 3-manifold obtained from $Y$ by 0-surgery along $K$ in $Y$. Then we have a canonical isomorphism $H_1(\widetilde{X(K)}; \mathbb{Z}) \cong H_1(Y_0(K); \mathbb{Z})$. In fact, $Y_0(K) \cong M$ as the two surgeries along $\mu$ and $K$ are dual to each other. So, $tA - A^T$ is also a presentation matrix for the $\Lambda$-module $H_1(\tilde{M}; \mathbb{Z})$. The matrix $tA - A^T$ is a square matrix, so by definition $\Delta_M(t) = \det(tA - A^T) = \Delta_K(t)$.

3. Proof of Theorem \[1\]

Throughout this section, homologies are all over $\mathbb{Z}$.

Let $M$ be an oriented homology handle, so that its homology groups are isomorphic to those of $S^1 \times S^2$, and suppose that $\Delta_M(t) = 1$. We will use the same notation as Section \[2\]. By attaching a 2-handle $D^2 \times D^2$ to the boundary $M \times 0$ of $M \times [0, 1]$ along $\mu$ with a framing determined by the curve $l$, we obtain a cobordism $X = (M \times [0, 1]) \cup_{l\text{-framing}} (D^2 \times D^2)$ between $Y$ and $M$. Then $K$ is a knot in $Y$ with Alexander polynomial $\Delta_K(t) = \Delta_M(t) = 1$. By \[1\] 11.7B Theorem, there is a pair $(W', D)$, where $W'$ is a contractible topological 4-manifold, and $D$ is a locally flat 2-disk properly embedded in $W'$ such that $\partial(W', D) = (Y, K)$. By
stacking \( X \) to \( W' \) along \( Y \), we obtain a topological 4-manifold \( W'' = X \cup_Y W' \), which \( M \) bounds. Furthermore, we obtain a locally flat 2-sphere \( S \) in \( W'' \) from the union of the cocore of the 2-handle and the locally flat 2-disc \( D \).

**Lemma 3.1.** The 4-manifold \( W'' \) has the homology of \( D^2 \times S^2 \).

**Proof.** First, we compute the homology of \( X \), which is obtained from \( M \times [0, 1] \) by attaching a 2-handle \( D^2 \times D^2 \). The attaching region is a tubular neighborhood of \( \mu \), and is homeomorphic to \( S^1 \times D^2 \). From the Mayer-Vietoris sequence, we have the following:

(1) \[ \cdots \to H_i(S^1 \times D^2) \to H_i(M \times [0, 1]) \oplus H_i(D^2 \times D^2) \to H_i(X) \]
\[ \to H_{i-1}(S^1 \times D^2) \to \cdots \to H_0(X) \to 0. \]

Note that \( H_i(S^1 \times D^2) \to H_1(M \times [0, 1]) \) is an isomorphism and \( H_0(S^1 \times D^2) \to H_0(M \times [0, 1]) \oplus H_0(D^2 \times D^2) \) is injective. Then it is easy to find that \( H_i(X) \cong \mathbb{Z} \) if \( i = 0, 2, 3 \), and trivial otherwise.

Next, we compute the homology of \( W'' \) using the Mayer-Vietoris sequence as follows. Since the intersection between \( X \) and \( W' \) is \( Y \), we have the following:

\[ \cdots \to H_i(Y) \to H_i(X) \oplus H_i(W') \to H_i(W'') \]
\[ \to H_{i-1}(Y) \to \cdots \to H_0(W'') \to 0. \]

Note that \( H_i(Y) \cong H_i(S^3) \) and \( H_i(W') \cong H_i(B^4) \). Since the map \( H_0(Y) \to H_0(X) \oplus H_0(W') \) is injective, we have \( H_0(W'') \cong \mathbb{Z} \), \( H_1(W'') \cong 0 \), and \( H_2(W'') \cong \mathbb{Z} \). Considering the maps \( H_3(Y) \to H_3(X) \leftrightarrow H_3(M) \) induced by inclusions, the right map is an isomorphism from the long exact sequence (1), and the images of two maps are homologous in \( H_3(X) \). Then the left map is also an isomorphism, and thus \( H_3(W'') \cong H_4(W'') \cong 0 \).

\[ \square \]

**Lemma 3.2.** The locally flat 2-sphere \( S \) represents a generator of \( H_2(W'') \), and its self-intersection \( S \cdot S \) is 0.

**Proof.** Let \( \alpha \) be the 2-disc obtained from the union of \( \mu \times [0, 1] \) and the core of the 2-handle. Then its boundary is \( \mu \) in \( \partial W'' = M \), and it intersects \( S \) in a single point. Thus, to show that \( S \) represents a generator of \( H_2(W'') \), it suffices that \( \alpha \) represents a generator of a \( \mathbb{Z} \)-summand of \( H_2(W'', \partial W'') \). Note that \( H_2(M) \cong H_2(M \times [0, 1]) \cong H_2(X) \cong H_2(W'') \) from long exact sequences in the proof of Lemma 3.1. We consider the long exact sequence of the pair \( (W'', \partial W'') \):

\[ H_2(M) \to H_2(W'') \xrightarrow{[A]} H_2(W'', M) \xrightarrow{\partial} H_1(M) \to 0. \]

It is well-known that the map \( [A] \) is represented by an intersection form \( A \) of \( H_2(W'') \) with respect to some basis since \( \partial W'' \neq \emptyset \) and \( H_1(W'') \) is trivial, see [17, §3]. Because the first map is an isomorphism, the intersection form \( A \) is trivial.
and $H_2(W'', M) \cong H_1(M) \cong \mathbb{Z}$. Since $\partial[\alpha] = [\mu]$ and $[\mu]$ is a generator of $H_1(M)$, $\alpha$ represents a generator of $H_2(W'', M)$. \hfill \square

Since $S$ is locally flat, it has a normal bundle by [4, §9.3], and hence it has a tubular neighborhood $T(S)$ in $W''$. The normal bundle over $S$ is determined by its Euler number, which equals the algebraic intersection number between the 0-section and any other section transverse to it. By Lemma 3.2, $S \cdot S = 0$. Thus, the normal bundle is trivial, and $T(S)$ is homeomorphic to $S^2 \times D^2$. Let $\psi : S^2 \times D^2 \to W''$ be an embedding of the tubular neighborhood of $S$. Let $X(S)$ be the exterior of the sphere $S$, i.e., $X(S) = W'' \setminus \text{Int}T(S)$. Let $W$ be the 4-manifold obtained from $X(S)$ by gluing in $D^3 \times S^1$ back along the boundary $S^2 \times S^1$ of $X(S)$. That is, $W = X(S) \cup_{\psi|_{S^2 \times S^1}} (D^3 \times S^1)$.

**Lemma 3.3.** The 4-manifold $W$ has the homology of $D^3 \times S^1$.

**Proof.** The homology exact sequence of the pair $(W'', X(S))$ yields:

$$\cdots \to H_i(X(S)) \to H_i(W'') \to H_i(W'', X(S)) \to H_{i-1}(X(S)) \to \cdots \to H_0(W'', X(S)) \to 0.$$  

Via excision, $H_i(W'', X(S)) \cong H_i(T(S), \partial T(S)) \cong \mathbb{Z}$ for $i = 2, 4$, and trivial otherwise. Then we can easily obtain $H_0(X(S)) \cong H_3(X(S)) \cong \mathbb{Z}$ and $H_2(X(S)) \cong 0$. For $i = 1, 2$, we have the following sequence:

$$0 \to H_2(X(S)) \to H_2(W'') \to H_2(W'', X(S)) \to H_1(X(S)) \to 0,$$

where $[S]$ is mapped to 0 under $H_2(W'') \to H_2(W'', X(S))$. Thus, $H_1(X(S)) \cong H_2(X(S)) \cong \mathbb{Z}$.

Now, we compute the homology of $W$ using the Mayer-Vietoris sequence for the pair $(X(S), D^3 \times S^1)$. In the long exact sequence

$$\cdots \to H_i(S^2 \times S^1) \xrightarrow{\Phi_i} H_i(X(S)) \oplus H_i(D^3 \times S^1) \to H_i(W) \to H_{i-1}(S^2 \times S^1) \to \cdots \to H_0(W) \to 0,$$

$\Phi_i$ is injective for $i = 0, 1$, and bijective for $i = 2, 3$, which implies that $W$ has a homology of $D^3 \times S^1$. \hfill \square

**Proof of Theorem 1.** Let $(M, \alpha)$ be a distinguished homology handle with trivial Alexander polynomial. Then by above lemmas, there is a topological 4-dimensional manifold $W$ whose homology is isomorphic to that of $S^1 \times D^3$, and whose boundary is $M$. Choose a cohomology class $\varphi \in H^1(W)$ whose restriction to $M$ is dual to $\alpha$. By [11, Assertion 5], the infinite cyclic covering $\tilde{W}_\varphi$ has finitely generated homology groups over $\mathbb{Q}$ since $W$ has the homology of the circle. Thus the pair $(W, \varphi)$ is a null $\tilde{H}$-cobordism of $(M, \alpha)$. \hfill \square
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