On coupled Lane-Emden equations arising in dusty fluid models

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Abstract. We investigate the existence of solutions for mixed boundary value problems in coupled Lane-Emden equations. Such problems arise e.g. in the study of multicomponent diffusion and reaction processes inside catalyst particles under spherical symmetry. In contrast to the frequently applied decomposition method of Adomian, we employ the geometric theory of ODEs to show that this boundary value problem can be transformed into a terminal value problem. To this end, we determine the integral manifold on which solutions of the boundary value problem necessarily lie. This will be done in suitable warped and blown-up coordinates. Moreover, we comment on the numerical implementation.

1. Introduction

1.1. Some history
In astrophysics, the Lane-Emden equation is Poisson’s equation for the gravitational potential of a self-gravitating, spherically symmetric and polytropic fluid at hydrostatic equilibrium:

$$
\frac{d}{dr} \left( r^2 \frac{d\rho}{dr} \right) + r^2 \rho^n = 0, \quad \rho(0) = 1, \quad \frac{d\rho}{dr}(0+) = 0.
$$

(1.1)

Here, the pressure $P$ is assumed to be proportional to the power $\frac{n+1}{n}$ of the ‘density’ $\delta = \text{const} \cdot \rho^n$ via the polytropic relation $P = \text{const} \cdot \delta^{1+1/n} = \text{const} \cdot \rho^{n+1}$ of index $n$. So the solution of (1.1) provides the dynamics of a scaled density and hence of a scaled pressure.

Jonathan Homer Lane (1819-1880), an American astrophysicist, was the first to perform a mathematical analysis of the sun as a gaseous body. With his work *On the theoretical temperature of the sun under the hypothesis of a gaseous mass maintaining its volume by its internal heat and depending on the laws of gases known from terrestrial experiments* from 1870 (cf. [1]) he initiated the theory of stellar evolution. Jacob Robert Emden (1862-1940), a Swiss astrophysicist and meteorologist, provided a mathematical model as a basis of stellar structure by his work *Gas balls: Applications of the mechanical heat theory to cosmological and meteorological problems* in 1907 (cf. [2]). In a series of papers, Ralph Howard Fowler (1889-1944) considered a generalization of (1.1) of the form

$$
\frac{d}{dr} \left( r^\alpha \frac{d\rho}{dr} \right) + r^\alpha f(\rho) = 0
$$

(1.2)
in particular for \( f(\rho) \) equal to \( \rho^n \) or \( e^{\rho n} \) (see e.g. [3], [4]). A simple coordinate change \( r \rightarrow \theta \) entails \( u_{\theta\theta} + \gamma f(u) = 0 \) for some \( \gamma \) (cf. Chapter 7 of [5] or [6]). By now, there is a vast literature on the generalized Emden-Fowler equation

\[
\frac{d}{dr} \left( p(r) \frac{dp}{dr} \right) + q(r) f(p) = 0,
\]

(1.3)

The review article [7] of J.S.W.Wong offers a comprehensive survey for the years up to 1975. For more recent theoretical and numerical work on generalized Emden-Fowler equations we refer to [8], [9], [10], [11], [12], [13], [14], [15] and [16], [17], [18], [19], [20], [21], [22] and the references therein. Often, numerical computations are based on Adomian’s decomposition method (see e.g. [16] or [20]).

1.2. Lane-Emden equation in chemical engineering applications

One of the important fields of application of the Lane-Emden equation is the analysis of the diffusive transport and chemical reaction of species inside a porous catalyst particle. Aris [23] and Metha & Aris [24] were among the first who solved this problem as boundary value problem, assuming power-law kinetics for a single chemical reaction taking place inside a spherical catalyst particle - the most relevant practical catalyst geometry:

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dc}{dr} \right) = \phi^2 c^n, \quad c(0) = 1, \quad \frac{dc}{dr}(0+) = 0
\]

(1.4a)

where the so called Thiele modulus \( \phi \) is a constant parameter. The derivation of equation (1.4a) is based on the assumption that the diffusion of the considered species inside the porous catalyst obeys Fick’s law with a constant, i.e. concentration-independent, diffusivity.

In multicomponent mixtures, the transport of species in porous catalysts should not be described by Fick’s law, but by the Maxwell-Stefan equations. The application of the latter equations to mass transport phenomena in porous solids leads to the so called Dusty Gas Model [25]. In this model, the solid catalyst is treated as ‘dust’, i.e. an ensemble of ‘huge’ motionless molecules which undergo exchange of momentum with the fluid species moving inside the porous catalyst structure. This approach accounts for three different kinds of transport contributions: bulk and Knudsen diffusion, surface diffusion and viscous Poiseuille-type flow. The Dusty Gas Model was originally formulated for ideal gas mixtures. Later it was generalized (named Dusty Fluid Model) to make it also applicable to nonideal multicomponent fluid mixtures, see e.g. [26], [27].

Combining the Dusty Fluid Model with the mass balances of species being transported inside a porous spherical catalyst body yields [28]:

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 D(x) \frac{dx}{dr} \right) = N R(x),
\]

(1.4b)

where \( x \) stands for the vector of mole fractions of chemical species and \( D(x) \) for the concentration-dependent matrix of diffusion coefficients. Note that, based on the theory of irreversible thermodynamics, it can be shown that \( D \) is the product of two positive definite matrices (cf. [29]) and thus similar to a diagonal matrix with positive diagonal elements (cf. [30], Thm.7.6.3). On the right-hand side of eq. (1.4b), \( N \) represents the stoichiometry matrix of the considered reaction network, and \( R(x) \) represents the vector of reaction kinetic expressions. Assuming a constant \( D \)-matrix, one gets the following simplified catalyst diffusion model:

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dx}{dr} \right) = f(x), \quad f(x) = D^{-1} N R(x).
\]

(1.4c)
Among all possible rate expressions, linear, bilinear and quadratic functions $R(x)$ are of highest relevance in chemical kinetics, due to the fact that chemical reactions are mostly based on mono-molecular and/or bi-molecular events. Focusing on this class of reaction kinetics, one can identify a sub-vector $z$ containing the concentrations of all species which only appear in linear and bilinear rate expressions, and another sub-vector $y$ containing the concentrations of the species which also appear in the quadratic terms. A practical example for this situation is the acid-catalysed dimerization of C4-olefines, a complex reaction accompanied by the formation of butane trimers, tetramers and isomers [31]. In this example the main reactant isobutene is the only species which appears in quadratic kinetic terms.

The just discussed class of catalyst diffusion problems can be reformulated as quadratic Lane-Emden boundary value problem (with $\beta = \frac{d}{dr}$):

$$y'' + \frac{2}{r} y' = L_1(y) \left[ K_{11} y + K_{12} z \right], \quad y'(0+) = 0, \quad y(1) = \beta_1,$$

$$z'' + \frac{2}{r} z' = L_2(y) \left[ K_{21} y + K_{22} z \right], \quad z'(0+) = 0, \quad z(1) = \beta_2,$$

for $x = (y^T, z^T)^T$ and $r \in (0, 1]$ where the $L_j(y)$ are matrices, linear in $y$, and the $K_{ij}$ are constant matrices, all of appropriate sizes. In a more compact notation, the problem can be given as follows:

$$x'' + \frac{2}{r} x' = L(y) K x, \quad x'(0+) = 0, \quad x(1) = \beta,$$

for $x = (y^T, z^T)^T$ and $r \in (0, 1]$.

1.3. Outlook and summary

In Section 2 we treat the scalar boundary value problem

$$\begin{align*}
(r^2 x')' - k r^2 x'' &= 0, \\
x'(0+) &= 0, \quad x(1) = \beta > 0
\end{align*}$$

of Lane-Emden-Fowler type (with $\beta = \frac{d}{dr}$). Based on the geometric theory of ordinary differential equations, we first determine the integral manifold $S$ in $(r, x, x')$-space of the (1.6a)-solutions which satisfy the boundary condition $x'(0+) = 0$ and thus reveal the underlying geometric structure. In the scalar case, the incoming manifold $S$ will be shown to be, globally, the graph of a smooth function $x' = y_b(r, x)$ so that, in a second step, the boundary value problem $(1.6a) \& (1.6b)$ can be converted to a terminal value problem (with $x'(1) = y_b(1, \beta)$). We establish in Section 3 that this geometric approach can also be taken for coupled Lane-Emden equations as (1.5a).

The idea of converting boundary value into initial value problems has a long history. Here, we just refer to [6], where W.F. Ames and E. Adams have used group methods for the Emden-Fowler equations with boundary conditions that are different from ours.

For a numerical implementation, the Taylor polynomials – near $x' = 0$ – for the incoming manifold $S$ can be easily computed in a recursive algorithm. The backward flow of such a local approximation of the manifold then generates an approximation of the global incoming manifold. Its intersection with the terminal plane $\{(r, x, x') : r = 1, x = \beta\}$ gives an approximation for the initial value of the desired solution of (1.5a). In this 2-dimensional case, it is still open whether the incoming manifold $S$ will be the graph of a smooth function $x' = f(r, x)$ over a sufficiently large domain containing $(r, x) = (1, \beta)$.

Generalizations to positive $K = K(x)$ or even to positive $K = K(r, x)$ are obvious since the constant matrix $K$ in (1.5b) is not crucial for the existence of the above-mentioned incoming
manifolds and thus not essential for the geometry inherent in these Lane-Emden type equations. One might as well consider differential operators \( \frac{d}{dr} \left( r^p \frac{d}{dr} \right) \) with \( p \neq 2 \).

2. Scalar Lane-Emden boundary value problems

2.1. The model for spherical porous catalyst pellets

We consider the boundary value problem

\[
x'' + \frac{2}{r} x' = k x^n, \quad x'(0+) = 0, \quad x(1) = \beta > 0,
\]

for \( x \in \mathbb{R} \), \( r \in (0,1] \), \( k > 0 \). Since the linear cases for \( n = 0 \) and \( n = 1 \) are readily solved – with the solutions \( x(r) = \beta + k(r^2 - 1)/6 \) and \( x(r) = \beta \sinh(\sqrt{k}r)/(r \sinh\sqrt{k}) \) resp. – we take \( n \geq 2 \) in the following discussions. Of course, the second order system (2.1) can equivalently be written as the first order non-autonomous system

\[
x' = y, \quad y' = k x^n - \frac{2}{r} y, \quad y(0+) = 0, \quad x(1) = \beta
\]

in \( \mathbb{R}^2 \). System (2.1) was investigated in [32] and [33] by means of Adomian’s decomposition method.

We introduce a ‘time’ variable \( t \) via \( r = e^{-t} \) and employ the notation \( \dot{\cdot} = \frac{d}{dt} \). So the bijection \( t \in [0, \infty) \leftrightarrow r = e^{-t} \in [0,1] \) can be thought of as the solution of the initial value problem \( \dot{r} = -r, \ r(0) = 1 \). Hereby, the second order system (2.1) can equivalently be written as the first order autonomous system

\[
\dot{r} = -r, \quad \dot{x} = -ry, \quad \dot{y} = 2y - kr x^n, \quad r(0) = 1, \ x(0) = \beta, \ y(\infty) = 0
\]

in \( \mathbb{R}^3 \). The differential equation in (2.2b) possesses the \( x \)-axis \( \mathcal{X} = \{(r, x, y) = (0, x, 0) : x \in \mathbb{R}\} \) as a line of equilibria. Moreover, the \( r \)-axis is invariant for the differential equation in (2.2b).

The linearizations at equilibria \((0, x_0, 0)\) are given by

\[
J(x_0) = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
-k x_0^n & 0 & 2
\end{pmatrix}
\]

possessing the eigenvalues \(-1, 0, 2\) with corresponding eigenspaces spanned by \((1,0,0)^T\), \((0,1,0)^T\) and \((0,0,1)^T\) respectively. Thus, near such points \((0, x_0, 0)\), there exist local center-stable manifolds – tangent to the center-stable eigenspaces – of the form \( y = y_{cs}^0(r, x) \) with \( y_{cs}^0(0, x) \equiv 0 \) and \( y_{cs}^0(r, 0) \equiv 0 \).

These local center-stable manifolds form the stable manifold \( \mathcal{M}_s \) for compact parts \( \mathcal{X}^0 = \{(r, x, y) = (0, x, 0) : x \in [0, x_0]\} \). It allows – locally near \( r = 0 \) – the representation \( y = y_s(r, x) \) with

\[
y_s(r, 0) = 0 \quad \text{and} \quad y_s(0, x) = 0.
\]

The flow on \( \mathcal{M}_s \) is then determined by

\[
\dot{r} = -r, \quad \dot{x} = -ry_s(r, x) \quad \text{or} \quad \dot{x}' = y_s(r, x)
\]
with \((x, x') = (x_0, 0)\) for \(r = 0^+\) and \(x_0 \in [0, x_0^0]\).

Figure 1 shows the stable manifold \(M_s\) in the phase portrait of \((2.2b)\) for \(n = 2\) and \(k = 1\). The blue curve, connecting the vertical line \(L = \{(r, x, y) = (1, \beta, y) : y \in \mathbb{R}\}\) and the horizontal line \(X\), represents the sought boundary value solution within the stable manifold of the equilibrium manifold \(X^0\).

We will derive this stable manifold in the following subsection. To this end, we will define a suitable change of coordinates \((r, x, y) \rightarrow (r, \xi, \eta)\) such that the origin of the \((\xi, \eta)\)-system (cf. \((2.9b)\) below) will have a unique smooth stable manifold \(\eta = S(\xi)\) giving rise to the stable manifold of \(X^0\) for system \((2.2b)\) (cf. in particular Remark 2.1).

![Geometry of (2.2b) for n = 2.](image)

\section{2.2. Analysis of the model}
Suppose \(x(r)\) is the solution of \((2.1)\) – satisfying \((2.4b)\) for \(r\) near 0 – and define for positive \(\sigma\) the scalings
\[
u(t) := u(r) \quad \text{with the ‘time’ } t \geq 0.\]

We present the analysis of system \((2.1)\) in these new coordinates.

(a) We compute the resulting boundary value problem for \(v = v(t)\) and \(w := \dot{v}(t)\) and note that \(\sigma = \frac{2}{n-1} \in (0, 2]\)

\[
sigma = \frac{2}{n-1} \in (0, 2]\tag{2.6}\]

generates the autonomous system
\[
\begin{align*}
\dot{v} &= w, \\
\dot{w} &= \sigma(1-\sigma)v + (1-2\sigma)w + kv^n,
\end{align*}\tag{2.7a}

with
\[
x'(0+) = 0 \iff \lim_{t \to -\infty} e^{(\sigma+1)t}[\sigma v(t) + w(t)] = 0. \tag{2.7b}\]
(b) The origin \((0,0)\) is the only equilibrium of \((2.7a)\) in the right half-plane \(R\) for \(n \geq 3\). For \(n \in [2,3)\), there’s a second equilibrium – of saddle type – in the right half-plane \(R\). The eigenvalues of the linearization of \((2.7a)\) at the origin are \(-\sigma\) and \(1 - \sigma\). The eigenspace corresponding to the smaller eigenvalue \(-\sigma\) is given by \(\sigma v + w = 0\). The origin is exponentially stable for \(n \in [2,3)\) and a saddle for \(n > 3\). For \(n = 3\), it is a weak saddle with a stable manifold and a ‘weakly unstable’ center manifold. The condition \((2.7b)\) reveals that the desired solution of the boundary value problem has to lie on the strongly stable manifold of the origin.

Moreover, the case \(n = 5\) is special in the sense that the vector field in \((2.7a)\) will be divergence free. So, system \((2.7a)\) will be integrable for \(n = 5\). A first integral for \((2.9b)\) will be given – in terms of new coordinates – in \((2.9d)\) below.

(c) We employ the linear change
\[
(v, w) \mapsto (v, z) = (v, \sigma v + w)
\]  
leading to
\[
\dot{v} = -\sigma v + z, \quad \dot{z} = (1 - \sigma)z + kv^n, 
\]
such that the (strong) stable eigenspace of the linearized system at the origin is given by the \(v\)-axis.

(d) Finally, to single out the strongly stable direction, we use the blow-up
\[
(v, z) \mapsto (\xi, \eta) = \left(v^{1/\sigma}, \frac{z}{v}\right)
\]
and arrive at the following autonomous blown-up quadratic planar system on the right half-plane \(H = \{(\xi, \eta) : \xi \geq 0\}:
\[
\dot{\xi} = \xi \left[\frac{\eta}{\sigma} - 1\right], \quad \dot{\eta} = \eta - \eta^2 + k \xi^2.
\]  
The origin \((\xi, \eta) = (0,0)\) now is a saddle for \((2.9b)\) with a unique stable and a unique unstable manifold. The unstable manifold is contained in the \(\eta\)-axis. The local stable manifold \(\eta = S(\xi)\) is tangent to the \(\xi\)-axis and thus of the form \(S(\xi) = S_2\xi^2 + h.o.t.\). Moreover, the global stable manifold is a strictly decreasing function \(S(\xi)\) over \(\xi \geq 0\) with
\[
0 > S(\xi) \geq -2k\xi \left[1 + \sqrt{1 + 4k\xi}\right]^{-1/2} \quad \text{for} \quad \xi > 0
\]  
where the lower bound comes from the nullcline of \(\eta\). The reduced flow on \(\eta = S(\xi)\) is determined by \(\xi = \xi \left[-1 + S(\xi)/\sigma\right].\)

For \(n = 5\), the system \((2.9b)\) possesses the first integral
\[
I(\xi, \eta) = \frac{k}{3} \xi^3 + (\eta - \eta^2)\xi. 
\]  
We show the phase portrait of equation \((2.9b)\) for \(n = 2\), i.e. \(\sigma = 2\), and \(k = 1\) in figure 2 and for \(n = 4 > 3\), i.e. \(\sigma \leq 1\), and \(k = 1\) in figure 3 – the blue curve \(\eta = \eta(\xi)\) represents the stable manifold of the origin of equation \((2.9b)\) and corresponds to the blue solution of the boundary value problem \((2.2b)\) shown in figure 1.
Remark 2.1 (Alternative approach)

One may compute the stable manifold \( y = y_s(r,x) \) of the equilibrium line \( X^0 \) of the 3D-system (2.2b) directly in a particular form. First note that the stable manifold \( \eta = S(\xi) = S_2 \xi^2 + \text{h.o.t.} \) of the origin in (2.9b) corresponds to an invariant manifold, the stable manifold of \( X^0 \), for (2.2b) given by

\[
ry + S(rx^{1/\sigma})x = 0.
\]

Hence, the Ansatz

\[
y = y_s(r,x) := -\frac{x}{r} S(\xi) \quad \text{with} \quad \xi := rx^{1/\sigma},
\]

(2.10a)

\( S(\xi) = O(\xi^2) \) and \( 1/\sigma = \frac{n-1}{2} \geq 1/2 \) for the stable manifold \( M_\sigma \) of the equilibrium line \( X^0 \) will lead to a determining equation for \( S \) that is exactly the determining equation for \( \eta = S(\xi) \) being the stable manifold of the origin with respect to equation (2.9b), i.e.

\[
\xi \left[ \frac{2S}{n-1} - 1 \right] S_\xi = S - S^2 + k\xi^2, \quad S(0) = 0 \quad S_\xi(0) = 0.
\]

(2.10b)

Equation (2.10a) shows a special dependence of the invariant manifold on \( (r,x) \) and thus reveals the special ‘symmetry’ of the invariant manifold.
We would like to add that $M_r$ restricted to $(r, x) \in [0, 1] \times [0, x_0]$ is positive invariant with respect to (2.2b). This is a consequence of (2.4a) and (2.9c).

We summarize the above analysis:

**Theorem 2.2 (1D Lane-Emden boundary value problem)**
The scalar Lane-Emden boundary value problem (2.1) has a unique positive solution $x = x(r)$. Its terminal value at $r = 1$ is given by

$$x(1) = \beta > 0, \quad x'(1) = -\beta S(\beta^{(n-1)/2}) > 0,$$

where the function $\eta = S(\xi)$, $\xi \geq 0$, represents the unique stable manifold of the origin of (2.9b) within the right half-plane $H = \{ (\xi, \eta) : \xi \geq 0 \}$ (cf. (2.10b)). The solution of (2.1) can be computed from the reduced equation (cf. (2.4b))

$$x' = -\frac{x}{r} S(r x^{(n-1)/2}) \geq 0 \quad x(1) = \beta. $$

(2.12)

We close this section with a few computational and technical remarks.

**Remark 2.3**
(a) The Taylor polynomials at $\xi = 0$ for $\Sigma$ can be easily computed in a recursive algorithm. The backward flow of such a local approximation of the stable manifold then generates an approximation of the global stable manifold over $[0, \beta]$. Its intersection with the line $\{ (\xi, \eta) : \xi = \beta \}$ determines the (approximate) initial value for the desired solution of (2.9b).
(b) Instead of the blow-up (2.9a) one could use

$$(v, z) \mapsto (\xi, \eta) = (v, z/v)$$

(2.13a)
and arrive at

$$\dot{\xi} = \xi [\eta - \sigma], \quad \dot{\eta} = \eta - \eta^2 + k \xi^{n-1}. $$

(2.13b)
This system can be discussed in complete analogy to system (2.9b). Just note that the case $n = 2$ yields an additional linear term $k \xi$ so that the stable manifold of the origin is of the form $\eta = \Sigma(\xi) := -\frac{k}{3} \xi + \text{h.o.t.}$. In the original coordinates one then has the invariance of

$$r y + x \Sigma(r^\sigma x) = 0 \quad \text{with} \quad \Sigma(0) = 0. $$

(2.13c)
(c) Generalizations to positive $k = k(x)$ or even to positive $k = k(r, x)$ for $0 \leq r \leq 1$, $0 \leq x \leq x_0$ are obvious. One just has to ensure the existence of the stable integral manifold for the blown-up planar systems corresponding to (2.9b).
(d) Up to System (2.9b), the above analysis of System (2.1) is valid for negative $k$ too.
3. The 2D Lane-Emden boundary value problem from the dusty fluid approach

We consider the boundary value problem (1.5a) & (1.5b) in the form

\[ x''_1 + \frac{2}{r} x'_1 = x_1 [K_{11} x_1 + K_{12} x_2], \quad x'_1(0) = 0, \quad x_1(1) = \beta_1, \]
\[ x''_2 + \frac{2}{r} x'_2 = x_1 [K_{21} x_1 + K_{22} x_2], \quad x'_2(0) = 0, \quad x_2(1) = \beta_2. \]  \hspace{1cm} (3.1)

for \( x = (x_1, x_2)^T := (y, z)^T \in \mathbb{R}^2 \) and \( r \in (0, 1] \) with positive constants \( \beta_i \) and a nonzero matrix \( K \). We note that \( x_1 \equiv 0 \) is invariant for the differential equation in (3.1) whereas \( x_2 \equiv 0 \) is not in case \( K_{21} \neq 0 \).

We present the analysis of system (3.1) in new coordinates given by

\[ u_i(r) = r^2 x_i(r), \quad r = e^{-t}, \quad v_i(t) := u_i(r) \]  \hspace{1cm} (3.2)

with the ‘time’ \( t \geq 0 \) where the exponent 2 comes from the Fowler-transformation \( u_i = r^\sigma x_i \) with \( \sigma = \frac{2}{n-1} \) for the quadratic right-hand side of (3.1) \( (n = 2, \text{ cf. the scalings in (2.5)} ) \). Thereby we are left to the following autonomous Lane-Emden boundary value problem in \( \mathbb{R}^4 \) for \( v_i(t) = u_i(r) \):

\[ \dot{v}_1 = w_1 \]
\[ \dot{w}_1 = -2 v_1 - 3 w_1 + [K_{11} v_1 + K_{12} w_2] v_1 \]
\[ \dot{v}_2 = w_2 \]
\[ \dot{w}_2 = -2 v_2 - 3 w_2 + [K_{21} v_1 + K_{22} w_2] v_1 \]  \hspace{1cm} (3.3a)

with

\[ v(0) = \beta, \quad v(\infty) = 0 = w(\infty) \]  \hspace{1cm} (3.3b)

and

\[ x'_1(0+) = 0 \iff \lim_{t \to \infty} e^{3t} [2v_i(t) + w_i(t)] = 0. \]  \hspace{1cm} (3.3c)

The Jacobian blocks \( \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \) have the negative eigenvalues \(-1\) and \(-2\) with the eigenspaces \( v + w = 0 \) and \( 2v + w = 0 \) respectively. As in Section 2, the boundary conditions \( x'_1(0+) = 0 \) ask for the determination of the strong stable manifolds. We will proceed in three steps.

(a) First we note that a blow-up of the fast eigen-directions via

\[ \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} \rightarrow \begin{pmatrix} v_1 \\ z_1 = 2v_1 + w_1 \end{pmatrix} \rightarrow \begin{pmatrix} \xi_1 = v_1 \\ \eta_1 = z_1/v_1 \end{pmatrix} \]
\[ \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} \rightarrow \begin{pmatrix} v_2 \\ z_2 = 2v_2 + w_2 \end{pmatrix} \rightarrow \begin{pmatrix} \xi_2 = v_2 \\ \eta_2 = z_2/v_1 \end{pmatrix} \]  \hspace{1cm} (3.4a)

is adapted to the common factor \( x_1 \) on the right-hand side of (3.1) and hence to the common factor \( v_1 \) in the quadratic terms on the right-hand side of (3.3a). The transformation (3.4a) leads to the autonomous blown-up Lane-Emden equations in \( \mathbb{R}^4 \):

\[ \dot{\xi}_1 = -2 \xi_1 + \xi_1 \eta_1 \]
\[ \dot{\xi}_2 = -2 \xi_2 + \xi_1 \eta_2 \]
\[ \dot{\eta}_1 = K_{11} \xi_1 + K_{12} \xi_2 + \eta_1 - \eta_1^2 \]
\[ \dot{\eta}_2 = K_{21} \xi_1 + K_{22} \xi_2 + \eta_2 - \eta_1 \eta_2. \]  \hspace{1cm} (3.4b)
We note that \( \xi_1 = 0 \) implies \( \dot{\xi}_1 = 0 \) so that
\[
\{ \xi_1 = 0 \} \text{ is invariant.} \tag{3.4c}
\]
(b) This blown-up system (3.4b) has the origin as a saddle with Jacobian
\[
\begin{pmatrix}
-2 & I \\
K & I
\end{pmatrix}
\]
with eigenvalues +1 and −2. The corresponding (unique) 2D stable manifold of the origin for the blown-up system (3.4b) corresponds to the strong stable manifold of system (3.3a). The stable manifold of (3.4b) is locally of the form
\[
\eta = \Sigma(\xi) = -\frac{1}{3} K \xi + \text{h.o.t.} \tag{3.5b}
\]
So, for positive \( K_{ij} \), \( \Sigma(\xi) \) is negative for small positive \( \xi_1 \) and \( \xi_2 \) (cf. Remark 2.3(b)).
(c) The computations are done via two reduced 2D-systems. The first is describing the invariance of the stable manifold given by \( \eta = \Sigma(\xi) \), i.e.
\[
\dot{\eta} = \Sigma \dot{\xi} \quad \text{on} \quad \eta = \Sigma(\xi), \Sigma(0) = 0. \tag{3.6a}
\]
This 2-dimensional partial differential system (3.6a) reads
\[
K_{11} \xi_1 + K_{12} \xi_2 + \Sigma_1 - \Sigma_1^2 = (\Sigma_1)_{\xi_1} [\Sigma_1 \xi_1 - 2 \xi_1] + (\Sigma_1)_{\xi_2} [\Sigma_2 \xi_1 - 2 \xi_2],
\]
\[
K_{21} \xi_1 + K_{22} \xi_2 + \Sigma_2 - \Sigma_1 \Sigma_2 = (\Sigma_2)_{\xi_1} [\Sigma_1 \xi_1 - 2 \xi_1] + (\Sigma_2)_{\xi_2} [\Sigma_2 \xi_1 - 2 \xi_2]. \tag{3.6b}
\]
The Taylor polynomials for the components \( \Sigma_j \) of \( \Sigma \) can be easily computed in a recursive algorithm. The backward flow of a local approximation of the stable manifold then generates an approximation of the global stable manifold. Its intersection, if a singleton, with the plane \( \{ (\xi, \eta) : \xi = \beta \} \) determines the (approximate) initial value for the desired solution. In case the global stable manifold is of the form \( \eta = \Sigma_{\text{glob}}(\xi) \) over a sufficiently large domain in \( \xi \)-space (containing \( \beta \)), the second 2D-system determines the flow within the stable manifold via
\[
\dot{\xi} = \xi_1 \Sigma_{\text{glob}}(\xi) - 2 \xi \quad \text{with} \quad \xi(0) = \beta \tag{3.6c}
\]
with 2-dimensional \( \xi \).

We summarize the above analysis:

**Theorem 3.1 (2D Lane-Emden boundary value problem)**

The solution of the two-dimensional Lane-Emden boundary value problem (3.1) satisfies for small \( r > 0 \)
\[
x'(r) = -\frac{x_1}{r} \Sigma(r^2 x) \tag{3.7}
\]
where the right-hand side is given in terms of the function \( \eta = \Sigma(\xi) \) representing the unique local stable manifold \( \mathcal{M}_{\text{loc}} \) of system (3.4b) (cf. (3.5b)).
The intersection, if a singleton, of the corresponding global stable manifold \( \mathcal{M}_{\text{glob}} \) of system (3.4b) with the plane \( \{ (\xi, \eta) : \xi = \beta \} \) determines the terminal value \( x'(1) \). Moreover, the first solution component \( x_1(r) \) is positive on \( [0, 1] \).
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