FINITENESS OF MORDELL–WEIL GROUPS OF GENERIC ABELIAN VARIETIES

BY ALICE SILVERBERG

In a series of papers in the 1960s Shimura studied analytic families of abelian varieties with fixed polarization, endomorphism, and level structure. The isomorphism classes of abelian varieties in such a family are in one-to-one correspondence with the points of \( D/\Gamma \), where \( D \) is a symmetric domain and \( \Gamma \) is a discontinuous group of transformations of \( D \). Shimura constructed a fibre system \((V,W)\) where the base \( V \) is analytically isomorphic to \( D/\Gamma \), the fibres are the abelian varieties in the family, and \( V \) and \( W \) are quasi-projective varieties. The fibre \( A \) over the generic point of \( V \) is an abelian variety defined over the function field \( K \) of \( V \). The main result of this announcement is that, under certain conditions on the endomorphism algebra structure, the group of points of \( A \) defined over \( K \) is finite. Using completely different techniques, Shioda \[8\] proved this result in the case in which \( D \) is the complex upper half-plane and \( \Gamma \) is a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \).

The results in this note are an extension of part of the author’s Ph.D. thesis \[9\]. Details will appear elsewhere. I would like to express my sincere thanks to my thesis advisor, Professor Goro Shimura.

1. Let \( F \) be an arbitrary totally real number field of degree \( g \) over the rational number field \( \mathbb{Q} \). Let \( L \) be either (a) the field \( F \), (b) a totally indefinite quaternion algebra over \( F \) (and view \( L \) as embedded in \( M_2(\mathbb{R})^\theta \)), or (c) a totally imaginary quadratic extension \( K \) of \( F \). Let \( \Phi \) be a representation of \( L \) by complex matrices of degree \( n \) so that \( \Phi + \bar{\Phi} \) is equivalent to a rational representation of \( L \), and \( \Phi(1) = 1_n \) (writing \( 1_n \) for the identity matrix of size \( n \)). Assume that \([L : \mathbb{Q}]\) divides \( 2n \), and let \( m = 2n/[L : \mathbb{Q}] \). In (c), if \( \tau_1, \ldots, \tau_g, \bar{\tau}_1, \ldots, \bar{\tau}_g \) are the distinct embeddings of \( K \) in the complex number field \( \mathbb{C} \), write \( r_\nu \) and \( s_\nu \), respectively, for the multiplicities of \( \tau_\nu \) and \( \bar{\tau}_\nu \) in \( \Phi \) (then \( r_\nu + s_\nu = m \)). Suppose \( T \in M_m(L) \) satisfies \( {}^t T \rho = -T \), where \( {}^t \) is transpose on \( M_m(L) \), and \( \rho \) is complex conjugation on \( K \) and transpose on each factor of \( M_2(\mathbb{R})^\theta \). In (c), suppose \( {}^t T r_\nu \) has the same signature as

\[
\begin{pmatrix}
1 & 0 \\
0 & -1 &
\end{pmatrix}
\]

for every \( \nu \). Let \( \mathcal{M} \) be a lattice in \( L^m \), and let \( v_1, \ldots, v_s \) be elements of \( L^m \). Let \( \Omega \) denote the collection of data \((L, \Phi, \rho, T, \mathcal{M}, v_1, \ldots, v_s)\).

Suppose \( A \) is an abelian variety with a polarization \( C \), \( \theta \) is an embedding of \( L \) into \( \text{End}(A) \otimes \mathbb{Q} \), and \( t_1, \ldots, t_s \) are elements of \( A \) of finite order.

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DEFINITION. \((A, C, t_1, \ldots, t_s)\) is a polarized abelian variety of type \(\Omega\) if

(1) there is a holomorphic mapping \(\xi\) of \(\mathbb{C}^n\) onto \(A\) inducing an isomorphism of a complex torus \(\mathbb{C}^n / Y\) onto \(A\) satisfying \(\xi(\Phi(a)u) = \theta(a)\xi(u)\) for every \(u \in \mathbb{C}^n\) and \(a \in \theta^{-1}(\text{End}(A))\); (2) if \(\gamma\) is the involution of \(\text{End}(A) \otimes \mathbb{Q}\) determined by \(C\), then \(\theta(a)^{-1} = \theta(a^\gamma)\) for every \(a \in L\); (3) there is an \(R\)-linear isomorphism \(\eta\) of \((L \otimes_{\mathbb{Q}} R)^{m}\) onto \(\mathbb{C}^n\) such that \(\eta(M) = Y\), \(t_i = \xi(\eta(v_i))\) for \(i = 1, \ldots, s\), and \(\eta(ax) = \Phi(a)\eta(x)\) for every \(a \in L\) and \(x \in (L \otimes_{\mathbb{Q}} R)^{m}\); and (4) \(C\) determines a Riemann form \(R\) on \(\mathbb{C}^n / Y\) such that \(R(\eta(x), \eta(y)) = \text{tr}(xT^*y)\) for every \(x\) and \(y\) in \((L \otimes_{\mathbb{Q}} R)^{m}\).

Write \(H_r\) for \(\{Z \in M_r(\mathbb{C})|^t Z = Z; \text{Im}(Z)\) is positive symmetric\} and \(H_{r,s}\) for \(\{Z|\text{complex matrix with } r \text{ rows and } s \text{ columns; } 1 - Z^t Z \text{ is positive hermitian}\}\). Let \(D\) be \(H_{m/2}^0\) in (a), \(H_m^0\) in (b), and \(H_{r_1, s_1} \times \cdots \times H_{r_s, s_g}\) in (c).

The isomorphism classes of polarized abelian varieties of type \(\Omega\) are in one-to-one correspondence with the points of \(D/T\), where \(T\) is a suitably defined group of transformations on \(D\) (see [3] and [4]). In [5] Shimura showed that for each \(\Omega\), one can construct a fibre system \(\mathcal{F}\) in which the base \(V\) is analytically isomorphic to \(D/T\) and the fibres are the polarized abelian varieties of type \(\Omega\).

**Theorem 1.** If \(\dim(V) \geq 1\) then the group of points of the generic fibre defined over the function field of \(V\) is finite.

The remainder of this paper is a sketch of the proof of Theorem 1.

2. The Mordell-Weil group of Theorem 1 is isomorphic to the group of rationally defined algebraic sections from the base \(V\) to the fibre variety \(W\). If \(V\) is one-dimensional, one sees easily that these sections extend to global holomorphic sections. For higher dimensions we have the following result, which is a consequence of a result of Igusa (Theorem 6 of [1]) when the base variety \(V\) is compact.

**Proposition.** Let \(f\) be a rational section from \(V\) to \(W\). Then \(f\) is defined at every point of \(V\) so that \(f\) gives a holomorphic section from \(V\) to \(W\).

When \(\dim(V) = 1\) and \(V\) is compact, the second derivative of a holomorphic section is an automorphic form of weight three with respect to \(\Gamma\). The Eichler-Shimura cohomology isomorphism (Theorem 8.4 of [7]) can be used to show these automorphic forms are zero, and this then restricts the number of holomorphic sections. When \(D\) is \(H^1_\Gamma\) or \(H^1_{\Gamma,1}\) with \(r > 1\), the use of the Eichler-Shimura cohomology isomorphism is replaced by the application of a theorem of Matsushima and Shimura (Theorem 3.1 of [2]), which says there are no automorphic forms of mixed weight with at least one nonpositive weight.

3. The cases of Theorem 1 discussed in §2 can be used to prove the theorem in the remaining cases. We select a large collection of embeddings of base varieties \(V'\), for which the theorem is known, into a variety \(V\) for which we want to prove the theorem. A section \(f\) over \(V\) may be pulled back to sections over the varieties \(V'\). Since every section over every \(V'\) is of finite order, we can obtain a dense set of points of \(V\) which map via \(f\) to points of finite order.
in the fibres over \( V \). To show \( f \) is torsion, we must show these orders are bounded. We do this by proving a theorem giving a uniform bound for orders of torsion points on fibres with complex multiplication (Theorem 2 below). The finiteness of the Mordell-Weil group of the generic fibre then follows.

For \( u \) in \( V \), write \( Q_u = (A_u, C_u, \theta_u, t_1(u), \ldots, t_s(u)) \) for the fibre over \( u \). The fibre system \( \mathcal{F} \) is defined over a number field \( k_\Omega \) of finite degree such that for every \( u \in V \), \( k_\Omega(u) \) is the field of moduli of \( Q_u \) (see [5]). Call \( Q_u \) a \textit{"CM-fibre"} if \( A_u \) is isogenous to \( A_1 \times \cdots \times A_t \), where \( A_i \) has complex multiplication by a \( CM \)-field of degree \( 2 \cdot \dim(A_i) \), for \( i = 1, \ldots, t \) (thus, \( A_u \) has \( CM \) in the sense of [6]).

**THEOREM 2.** Let \( k \) be any subfield of \( \mathbb{C} \) which is finitely generated over \( \mathbb{Q} \) and contains \( k_\Omega \). There is a constant \( B \), depending only on the field \( k \) and the fibre system \( \mathcal{F} \), and independent of the choice of \( CM \)-fibre \( Q_u \), so that \( |A_u(k(u))_{\text{torsion}}| \leq B \).

The proof of Theorem 2 requires Shimura's Main Theorem of Complex Multiplication.

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