THE HOROBOUNDARY AND ISOMETRY GROUP OF THURSTON’S LIPSCHITZ METRIC

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Abstract. We show that the horofunction boundary of Teichmüller space with Thurston’s Lipschitz metric is the same as the Thurston boundary. We use this to determine the isometry group of the Lipschitz metric, apart from in some exceptional cases. We also show that the Teichmüller spaces of different surfaces, when endowed with this metric, are not isometric, again with some possible exceptions of low genus.

1. Introduction

Consider a connected oriented surface $S$ of negative Euler characteristic. A hyperbolic metric on $S$ is a Riemannian metric of constant curvature $-1$. The Teichmüller space $\mathcal{T}(S)$ of $S$ is the space of complete finite-area hyperbolic metrics on $S$ up to isotopy.

In [27], Thurston defined an asymmetric metric on Teichmüller space:

$$L(x, y) := \log \inf_{\phi \cong \text{Id}} \sup_{p \neq q} \frac{d_x(\phi(p), \phi(q))}{d_y(p, q)}, \quad \text{for } x, y \in \mathcal{T}(S).$$

In other words, the distance from $x$ to $y$ is the logarithm of the smallest Lipschitz constant over all homeomorphisms from $x$ to $y$ that are isotopic to the identity. Thurston showed that this is indeed a metric, although a non-symmetric one. In the same paper, he showed that this distance can be written

$$L(x, y) = \log \sup_{\alpha \in \mathcal{S}} \frac{\ell_y(\alpha)}{\ell_x(\alpha)},$$

where $\mathcal{S}$ is the set of isotopy classes of non-peripheral simple closed curves on $S$, and $\ell_x(\alpha)$ denotes the shortest length in the metric $x$ of a curve isotopic to $\alpha$.

Thurston’s Lipschitz metric has not been as intensively studied as the Teichmüller metric or the Weil–Petersson metric. The literature includes [16, 23, 18, 24, 19, 25, 5, 20]; most of it is very recent.

In this chapter, we determine the horofunction boundary of the Lipschitz metric. The horofunction boundary of a metric space was introduced by Gromov in [9], and has applications in studying isometry groups [14], random walks [12], quantum metric spaces [21], and is the right setting for Patterson–Sullivan measures [4].

We show that the compactification of Teichmüller space by horofunctions is isomorphic to the Thurston compactification, and we give an explicit expression for the horofunctions.

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Theorem 3.6. A sequence $x_n$ in $\mathcal{T}(S)$ converges in the Thurston compactification if and only if it converges in the horofunction compactification. If the limit in the Thurston compactification is the projective class $[\mu] \in \mathcal{P}\mathcal{M}\mathcal{L}$, then the limiting horofunction is

$$\Psi_\mu(x) = \log \left( \sup_{\eta \in \mathcal{M}\mathcal{L}} \frac{i(\mu, \eta)}{\ell_x(\eta)} \right) / \sup_{\eta \in \mathcal{M}\mathcal{L}} \frac{i(\mu, \eta)}{\ell_b(\eta)}.$$  

Here, $b$ is a base-point in $\mathcal{T}(S)$, and $i(\cdot, \cdot)$ denotes the geometric intersection number. Recall that the latter is defined for pairs of curve classes $(\alpha, \beta) \in \mathcal{S} \times \mathcal{S}$ to be the minimum number of transverse intersection points of curves $\alpha'$ and $\beta'$ with $\alpha' \in \alpha$ and $\beta' \in \beta$. This minimum is realised if $\alpha'$ and $\beta'$ are closed geodesics. The geometric intersection number extends to a continuous symmetric function on $\mathcal{M}\mathcal{L} \times \mathcal{M}\mathcal{L}$.

It is known that geodesics always converge to a point in the horofunction boundary; see Section 2. Hence, an immediate consequence of the above theorem is the following.

Corollary 1.1. Every geodesic of Thurston’s Lipschitz metric converges in the forward direction to a point in the Thurston boundary.

This generalises a result of Papadopoulos [16], which states that every member of a special class of geodesics, the stretch lines, converges in the forward direction to a point in the Thurston boundary.

The action of the isometry group of a metric space extends continuously to an action by homeomorphisms on the horofunction boundary. Thus, the horofunction boundary is useful for studying groups of isometries of metric spaces. One of the tools it provides is the detour cost, which is a kind of metric on the boundary. We calculate this in Section 6.

Denote by Mod$_S$ the extended mapping class group of $S$, that is, the group of isotopy classes of homeomorphisms of $S$. It is easy to see that Mod$_S$ acts by isometries on $\mathcal{T}(S)$ with the Lipschitz metric. We use the detour cost to prove the following.

Theorem 7.9. If $S$ is not a sphere with four or fewer punctures, nor a torus with two or fewer punctures, then every isometry of $\mathcal{T}(S)$ with Thurston’s Lipschitz metric is an element of the extended mapping class group Mod$_S$.

This answers a question in [17, §4].

It is well known that the subgroup of elements of Mod$_S$ acting trivially on $\mathcal{T}(S)$ is of order two if $S$ is the closed surface of genus two, and is just the identity element in the other cases considered here.

Theorem 7.9 is an analogue of Royden’s theorem concerning the Teichmüller metric, which was proved by Royden [22] in the case of compact surfaces and analytic automorphisms of $\mathcal{T}(S)$, and extended to the general case by Earle and Kra [7]. Our proof is inspired by Ivanov’s proof of Royden’s theorem, which was global and geometric in nature, as opposed to the original, which was local and analytic.

The following theorem shows that distinct surfaces give rise to distinct Teichmüller spaces, except possibly in certain cases. Denote by $S_{g,n}$ a surface of genus $g$ with $n$ punctures.
Theorem [7.10] Let $S_{g,n}$ and $S_{g',n'}$ be surfaces of negative Euler characteristic. Assume $\{(g,n),(g',n')\}$ is different from each of the three sets
\[\{(1,1),(0,4)\}, \{(1,2),(0,5)\}, \text{ and } \{(2,0),(0,6)\}.\]
If $(g,n)$ and $(g',n')$ are distinct, then the Teichmüller spaces $T(S_{g,n})$ and $T(S_{g',n'})$ with their respective Lipschitz metrics are not isometric.

This is an analogue of a theorem of Patterson [20]. In the case of the Teichmüller metric it is known that one has the following three isometries:
\[T(S_{1,1}) \cong T(S_{0,4}), T(S_{1,2}) \cong T(S_{0,5}), \text{ and } T(S_{2,0}) \cong T(S_{0,6}).\]
It would be interesting to know if these equivalences still hold when one takes instead the Lipschitz metric.

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2. The horofunction boundary

Let $(X,d)$ be a possibly non-symmetric metric space, in other words, $d$ has all the properties of a metric except that it is not necessarily symmetric.

We endow $X$ with the topology induced by the symmetrised metric $d_{sym}(x,y) := d(x,y) + d(y,x)$. Note that for Thurston’s Lipschitz metric, this topology is just the usual one on $T(S)$; see [15].

The horofunction boundary of $(X,d)$ is defined as follows. One assigns to each point $z \in X$ the function $\psi_z : X \to \mathbb{R}$,
\[\psi_z(x) := d(x,z) - d(b,z),\]
where $b$ is some base-point. Consider the map $\psi : X \to C(X), z \mapsto \psi_z$ from $X$ into $C(X)$, the space of continuous real-valued functions on $X$ endowed with the topology of uniform convergence on bounded sets of $d_{sym}$.

Proposition 2.1 ([2]). The map $\psi$ is injective and continuous.

Proof. The triangle inequality implies that $\psi_x(\cdot) - \psi_y(\cdot) \leq d(y,x) + d(x,y)$, for all $x$ and $y$ in $X$. The continuity of $\psi$ follows.

Let $x$ and $y$ be distinct points in $X$, and relabel them such that $d(b,x) \geq d(b,y)$.

We have
\[\psi_y(x) - \psi_x(x) = d(x,y) - d(b,y) - d(x,x) + d(b,x) \geq d(x,y),\]
which shows that $\psi_x$ and $\psi_y$ are distinct. \qed
The horofunction boundary is defined to be

\[
X(\infty) := \text{cl}\{\psi_z \mid z \in X\} \setminus \{\psi_z \mid z \in X\},
\]

where \(\text{cl}\) denotes the closure of a set. The elements of \(X(\infty)\) are called horofunctions. This definition first appeared, for the case of symmetric metrics, in [9]. For more information, see [2], [21], and [1].

One may check that if one changes to an alternative base-point \(b'\), then the new function assigned to a point \(z\) is related to the old by \(\psi'_z(\cdot) = \psi_z(\cdot) - \psi_z(b')\). It follows that the horofunction boundary obtained using \(b'\) is homeomorphic to that obtained using \(b\), and the horofunctions are related by \(\xi'(\cdot) = \xi(\cdot) - \xi(b')\).

Note that, if the metric \(d_{\text{sym}}\) is proper, meaning that closed balls are compact, then uniform convergence on bounded sets is equivalent uniform convergence on compact sets.

The functions \(\{\psi_z \mid z \in X\}\) satisfy \(\psi_z(x) \leq d(x,y) + \psi_z(y)\) for all \(x\) and \(y\). Hence, for all horofunctions \(\eta\),

\[
\eta(x) \leq d(x,y) + \eta(y), \quad \text{for all } x \text{ and } y \text{ in } X.
\]

(1)

It follows that all elements of \(\text{cl}\{\psi_z \mid z \in X\}\) are 1-Lipschitz with respect to the metric \(d_{\text{sym}}\). We conclude that, for functions in this set, uniform convergence on bounded sets is equivalent to pointwise convergence.

Moreover, if \(d_{\text{sym}}\) is proper, then the set \(\text{cl}\{\psi_z \mid z \in X\}\) is compact by the Ascoli–Arzelà Theorem, and we call it the horofunction compactification.

The following assumptions will be useful. They hold for the Lipschitz metric; see [18].

**Assumption A.** The metric \(d_{\text{sym}}\) is proper.

A geodesic in a possibly non-symmetric metric space \((X,d)\) is a map \(\gamma\) from a closed interval of \(\mathbb{R}\) to \(X\) such that

\[
d(\gamma(s),\gamma(t)) = t - s,
\]

for all \(s\) and \(t\) in the domain, with \(s < t\).

**Assumption B.** Between any pair of points in \(X\), there exists a geodesic with respect to \(d\).

**Assumption C.** For any point \(x\) and sequence \(x_n\) in \(X\), we have \(d(x_n,x) \to 0\) if and only if \(d(x,x_n) \to 0\).

**Proposition 2.2 ([3]).** Assume [21], [2], and [C] hold. Then, \(\psi\) is an embedding of \(X\) into \(C(X)\), in other words, is a homeomorphism from \(X\) to its image.

**Proof.** That \(\psi\) is injective and continuous was proved in Proposition 2.1.

Let \(z_n\) be a sequence in \(X\) escaping to infinity, that is, eventually leaving and never returning to every compact set. We wish to show that no subsequence of \(\psi_{z_n}\) converges to a function \(\psi_y\) with \(y \in X\). Without loss of generality, assume that \(\psi_{z_n}\) converges to \(\xi \in \text{cl}\{\psi_z \mid z \in X\}\).

Since \(d_{\text{sym}}\) is proper, \(d_{\text{sym}}(y, z_n)\) must converge to infinity. For each \(n \in \mathbb{N}\), let \(\gamma_n\) be a geodesic segment with respect to \(d\) from \(y\) to \(z_n\). Choose \(r > d(b,y) + \xi(y)\). It follows from assumption [C] that the function \(t \mapsto d_{\text{sym}}(y, \gamma_n(t))\) is continuous for each \(n \in \mathbb{N}\). Note that this function is defined on a closed interval and takes the value 0 at one endpoint and \(d_{\text{sym}}(y, z_n)\) at the other. Therefore, for \(n\) large enough, we may find \(t_n \in \mathbb{R}_+\) such that \(d_{\text{sym}}(y, x_n) = r\), where \(x_n := \gamma(t_n)\).
Since \( d_{\text{sym}} \) is proper, and all the \( x_n \) lie in a closed ball of radius \( r \), we may assume, by taking a subsequence if necessary, that \( x_n \) converges to some point \( x \in X \).

Observe that \( \psi_{x_n}(x_n) = \psi_y(y) - d(y, x_n) \), for all \( n \in \mathbb{N} \). Since the \( \psi_{x_n} \) are 1-Lipschitz with respect to \( d_{\text{sym}} \), we may take limits and get \( \xi(x) = \psi_y(y) - d(y, x) \). On the other hand, \( \psi_y(x) = d(x, y) - d(b, y) \). So \( \psi_y(x) - \xi(x) = r - d(b, y) - \xi(y) > 0 \). This shows that \( \xi \) is distinct from \( \psi_y \).

Now let \( p_n \) be a sequence in \( X \) such that \( \psi_{p_n} \) converges to \( \psi_p \) in \( \psi(X) \). From what we have just shown, \( p_n \) can not have any subsequence escaping to infinity. Therefore \( p_n \) is bounded in the \( d_{\text{sym}} \) metric. It then follows from the compactness of closed balls and the continuity and injectivity of \( \psi \) that \( p_n \) converges to \( p \). \( \square \)

We henceforth identify \( X \) with its image.

**Proposition 2.3.** Assume \([A] \) and \([B] \) hold. Let \( x_n \) be a sequence in \( X \) converging to a horofunction. Then, only finitely many points of \( x_n \) lie in any closed ball of \( d_{\text{sym}} \).

**Proof.** Suppose \( x_n \) is a sequence in \( X \) such that some subsequence of \( d_{\text{sym}}(b, x_n) \) is bounded. By taking a further subsequence if necessary, we may assume that \( x_n \) converges to a point \( x \) in \( X \). By Proposition \([2.2] \) \( \psi_{x_n} \) converges to \( \psi_x \), and so \( x_n \) does not converge to a horofunction. \( \square \)

A path \( \gamma : \mathbb{R} \rightarrow X \) is called an almost-geodesic if, for each \( \epsilon > 0 \),

\[
|d(\gamma(0), \gamma(s)) + d(\gamma(s), \gamma(t)) - t| < \epsilon, \quad \text{for } s \text{ and } t \text{ large enough, with } s \leq t.
\]

Rieffel \([21] \) proved that every almost-geodesic converges to a limit in \( X(\infty) \). A horofunction is called a Busemann point if there exists an almost-geodesic converging to it. We denote by \( X_B(\infty) \) the set of all Busemann points in \( X(\infty) \).

Isometries between possibly non-symmetric metric spaces extend continuously to homeomorphisms between the horofunction compactifications.

**Proposition 2.4.** Assume that \( f \) is an isometry from one possibly non-symmetric metric space \((X, d)\) to another \((X', d')\), with base-points \( b \) and \( b' \), respectively. Then, for every horofunction \( \xi \) and point \( x \in X \), \n
\[
f \cdot \xi(x) = \xi(f^{-1}(x)) - \xi(f^{-1}(b')).
\]

**Proof.** Let \( x_n \) be a sequence in \( X \) converging to \( x \). We have \n
\[
f \cdot \xi(x) = \lim_{n \to \infty} d'(x, f(x_n)) - d'(b', f(x_n))
\]

\[
= \lim_{n \to \infty} \left( d(f^{-1}(x), x_n) - d(b, x_n) + d(b, x_n) - d(f^{-1}(b'), x_n) \right)
\]

\[
= \xi(f^{-1}(x)) - \xi(f^{-1}(b')). \quad \square
\]

3. **The horoboundary of Thurston’s Lipschitz metric**

We start with a general lemma relating joint continuity to uniform convergence on compact sets.

**Lemma 3.1.** Let \( X \) and \( Y \) be two topological spaces and let \( i : X \times Y \rightarrow \mathbb{R} \) be a continuous function. Let \((x_n)_n\) be a sequence in \( X \) converging to \( x \in X \). Then, \( i(x_n, \cdot) \) converges to \( i(x, \cdot) \) uniformly on compact sets of \( Y \).
Proof. Take any \( \epsilon > 0 \), and let \( K \) be a compact subset of \( Y \). The function \( i(\cdot, \cdot) \) is continuous, and so, for any \( y \in Y \), there exists an open neighbourhood \( U_y \subset X \) of \( x \) and an open neighbourhood \( V_y \subset Y \) of \( y \) such that \( |i(x', y') - i(x, y)| < \epsilon \) for all \( x' \in U_y \) and \( y' \in V_y \). Since \( K \) is covered by \( \{ V_y \mid y \in K \} \), there exists a finite sub-covering \( \{ V_{y_1}, \ldots, V_{y_n} \} \). Define \( U := \bigcap_i U_{y_i} \). This is an open neighbourhood of \( x \), and \( |i(x', y) - i(x, y)| < \epsilon \) for all \( y \in K \) and \( x' \in U \). \( \Box \)

We use Bonahon’s theory of geodesic currents \([8]\). The space of geodesic currents is a completion of the space of homotopy classes of curves on \( S \), not necessarily simple, equipped with positive weights.

More formally, let \( G \) be the space of geodesics on the universal cover of \( S \), endowed with the compact-open topology. A geodesic current is a positive measure on \( G \) that is invariant under the action of the fundamental group of \( S \).

It is convenient to work with the space of geodesic currents because both Teichmüller space \( T(S) \) and the space \( \mathcal{ML} \) of compactly supported measured geodesic laminations are embedded into it in a very natural way. Furthermore, there is a continuous symmetric bilinear form \( i(x, y) \) on this space that restricts to the usual intersection form when \( x \) and \( y \) are in \( \mathcal{ML} \), and takes the value \( i(x, y) = \ell_x(y) \) when \( x \in T(S) \) and \( y \in \mathcal{ML} \).

We denote by \( \mathcal{PML} \) the projective space of \( \mathcal{ML} \), that is, the quotient of \( \mathcal{ML} \) by the multiplicative action of the positive real numbers. We use \( [\mu] \) to denote the equivalence class of \( \mu \in \mathcal{ML} \) in \( \mathcal{PML} \). We may identify \( \mathcal{PML} \) with the cross-section \( P := \{ \mu \in \mathcal{ML} \mid \ell_b(\mu) = 1 \} \). We have the following two formulas for the Lipschitz metric.

\[
L(x, y) = \log \sup_{\eta \in \mathcal{ML}} \frac{\ell_y(\eta)}{\ell_x(\eta)} = \log \sup_{\eta \in P} \frac{\ell_y(\eta)}{\ell_x(\eta)}, \tag{2}
\]

The second is very useful because the supremum is taken over a compact set, and is therefore attained.

Recall that we have chosen a base-point \( b \) in \( T(S) \). Define, for any geodesic current \( x \),

\[
Q(x) := \sup_{\eta \in \mathcal{ML}} \frac{i(x, \eta)}{\ell_b(\eta)} = \sup_{\eta \in P} \frac{i(x, \eta)}{\ell_b(\eta)}
\]

and

\[
\mathcal{L}_x : \mathcal{ML} \to \mathbb{R}_+ : \mu \mapsto \frac{i(x, \mu)}{Q(x)}
\]

Let \( T^T(S) := T(S) \cup \mathcal{PML} \) be the Thurston compactification of Teichmüller space.

Identify \( \mathcal{PML} \) with \( P \), and consider a sequence \( x_n \) in \( T^T(S) \). Then, \( x_n \) converges to a point \( x \) in the Thurston compactification if and only if there is a sequence \( \lambda_n \) of positive real numbers such that \( \lambda_n x_n \) converges to \( x \) as a geodesic current. One can take \( \lambda_n \) to be identically 1 if \( x \in T(S) \).

**Proposition 3.2.** A sequence \( (x_n)_n \) in \( T^T(S) \) converges to a point \( x \in T^T(S) \) if and only if \( \mathcal{L}_{x_n} \) converges to \( \mathcal{L}_x \) uniformly on compact sets of \( \mathcal{ML} \).

**Proof.** Assume that \( x_n \) converges in the Thurston compactification to a point \( x \in T^T(S) \). This implies that, for some sequence \( (\lambda_n)_n \) of positive real numbers, \( \lambda_n x_n \) converges to \( x \) as a geodesic current. We now apply Lemma 3.1 to Bonahon’s
intersection function to get that \( i(\lambda_n x_n, \cdot) \) converges uniformly on compact sets of \( \mathcal{ML} \) to \( i(x, \cdot) \). Therefore, since \( \mathcal{P} \) is compact, \( Q(\lambda_n x_n) \) converges to \( Q(x) \), which is a positive real number. So, \( \mathcal{L}_{x_n}(\cdot) = i(\lambda_n x_n, \cdot)/Q(\lambda_n x_n) \) converges to \( \mathcal{L}_x(\cdot) \) uniformly on compact sets of \( \mathcal{ML} \).

Now assume that \( \mathcal{L}_{x_n} \) converges to \( \mathcal{L}_x \) uniformly on compact sets of \( \mathcal{ML} \). Let \( y_n \) be a subsequence of \( x_n \) converging in \( \mathcal{T}^T(S) \) to a point \( y \). As before, we have that \( \mathcal{L}_{y_n} \) converges to \( \mathcal{L}_y \) uniformly on compact sets of \( \mathcal{ML} \). Combining this with our assumption, we get that \( \mathcal{L}_y \) and \( \mathcal{L}_x \) agree. Therefore, \( i(y, \cdot) = \lambda i(x, \cdot) \) for some \( \lambda > 0 \). It follows that \( x \) and \( y \) are the same point in \( \mathcal{T}^T(S) \). We have shown that every convergent subsequence of \( x_n \) converges to \( x \), which implies that \( x_n \) converges to \( x \).

For each \( z \in \mathcal{T}^T(S) \), define the map

\[
\Psi_z(x) := \log \sup_{\eta \in \mathcal{ML}} \frac{\mathcal{L}_z(\eta)}{\ell_x(\eta)}, \quad \text{for all } x \in \mathcal{T}(S).
\]

Note that, if \( z \in \mathcal{T}(S) \), then \( \Psi_z(x) = L(x, z) - L(b, z) \) for all \( x \in \mathcal{T}(S) \).

For \( x \in \mathcal{T}(S) \) and \( y \in \mathcal{T}(S) \cup \mathcal{P} \), let \( R_{xy} \) be the set of elements \( \eta \) of \( \mathcal{P} \mathcal{ML} (= \mathcal{P}) \) where \( \mathcal{L}_\eta(\eta)/\mathcal{L}_x(\eta) \) is maximal.

**Lemma 3.3.** Let \( x_n \) be a sequence of points in \( \mathcal{T}(S) \) converging to a point \([\mu]\) in the Thurston boundary. Let \( y \) be a point in \( \mathcal{T}^T(S) \) satisfying \( i(y, \mu) \neq 0 \), and let \( \nu_n \) be a sequence in \( \mathcal{P} \) such that \( \nu_n \in R_{x_n y} \) for all \( n \in \mathbb{N} \). Then, any limit point \( \nu \) of \( \nu_n \) satisfies \( i(\mu, \nu) = 0 \).

**Proof.** Consider the sequence of functions \( F_n(\eta) := \mathcal{L}_y(\eta)/\mathcal{L}_{x_n}(\eta) \). By Proposition 3.2, \( \mathcal{L}_{x_n} \) converges to \( \mathcal{L}_\mu \) uniformly on compact sets. Therefore, for any sequence \( \eta_n \) in \( \mathcal{P} \) converging to a limit \( \eta \), we have that \( F_n(\eta_n) \) converges to \( \mathcal{L}_y(\eta)/\mathcal{L}_\mu(\eta) \) provided \( \mathcal{L}_y(\eta) \) and \( \mathcal{L}_\mu(\eta) \) are not both zero. So, by evaluating on a sequence \( \eta_n \) converging to \( \mu \), we see that \( \sup_{\mathcal{P}} F_n \) converges to \( +\infty \). On the other hand, for any sequence \( \eta_n \) converging to some \( \eta \) satisfying \( i(\mu, \eta) > 0 \), we get that \( F_n(\eta_n) \) converges to something finite, and so \( \eta_n \not\in R_{x_n y} \) for \( n \) large enough. The conclusion follows.

A measured lamination is **maximal** if its support is not properly contained in the support of any other measured lamination. It is **uniquely-ergodic** if every measured lamination with the same support is in the same projective class. Recall that, if \( \mu \) is maximal and uniquely-ergodic, and \( \eta \in \mathcal{ML} \) satisfies \( i(\mu, \eta) = 0 \), then \( \mu \) and \( \eta \) are proportional [6, Lemma 2.1].

**Lemma 3.4.** The map \( \Psi : \mathcal{T}^T(S) \to C(\mathcal{T}(S)) : z \mapsto \Psi_z \) is injective.

**Proof.** Let \( x \) and \( y \) be distinct elements of \( \mathcal{T}^T(S) \). By Proposition 3.2, \( \mathcal{L}_x \) and \( \mathcal{L}_y \) are distinct. So, by exchanging \( x \) and \( y \) if necessary, we have \( \mathcal{L}_x(\mu) < \mathcal{L}_y(\mu) \) for some \( \mu \in \mathcal{P} \). Since \( \mathcal{L}_x \) and \( \mathcal{L}_y \) are continuous, we may choose a neighbourhood \( N \) of \( \mu \) in \( \mathcal{P} \) small enough that there are real numbers \( u \) and \( v \) such that

\[
\mathcal{L}_x(\eta) \leq u < v \leq \mathcal{L}_y(\eta), \quad \text{for all } \eta \in N.
\]

Since the set of maximal uniquely-ergodic measured laminations is dense in \( \mathcal{P} \), we can find such a measured lamination \( \mu' \) in \( N \) that is not proportional to \( x \). Let \( \rho_n \) be a sequence of points in \( \mathcal{T}(S) \) converging to \([\mu']\) in the Thurston boundary, and let \( \nu_n \) be a sequence in \( \mathcal{P} \) such that \( \nu_n \in R_{\rho_n x} \) for all \( n \in \mathbb{N} \).
Since \( \mu' \) is maximal and uniquely ergodic, \( i(\mu', \eta) \neq 0 \) for all \( \eta \in M\mathcal{L} \) not proportional to \( \mu' \). So \( i(\mu', x) \neq 0 \), whether \( x \) is in \( \mathcal{T}(S) \) or in the Thurston boundary.

By Lemma 3.3, any limit point \( \nu \in \mathcal{P} \) of \( \{\nu_n\}_n \) satisfies \( i(\mu', \nu) = 0 \), and hence equals \( \mu' \). So, \( \nu_n \in N \) for large \( n \). Therefore, by taking \( n \) large enough, we can find a point \( p \) in \( \mathcal{T}(S) \) such that the supremum of \( \mathcal{L}_x(\cdot)/\ell_p(\cdot) \) is attained in the set \( N \).

Putting all this together, we have
\[
\sup_p \frac{\mathcal{L}_x(\cdot)}{\ell_p(\cdot)} = \sup_N \frac{\mathcal{L}_x(\cdot)}{\ell_p(\cdot)} \leq \sup_N \frac{u}{\ell_p(\cdot)} < \sup_N \frac{\nu}{\ell_p(\cdot)} \leq \sup_p \frac{\mathcal{L}_y(\cdot)}{\ell_p(\cdot)}.
\]

Thus, \( \Psi_x(p) < \Psi_y(p) \), which implies that \( \Psi_x \) and \( \Psi_y \) differ. \( \Box \)

**Lemma 3.5.** The map \( \Psi : \mathcal{T}^T(S) \to C(\mathcal{T}(S)) : z \mapsto \Psi_z \) is continuous.

**Proof.** Let \( x_n \) be a sequence in \( \mathcal{T}^T(S) \) converging to a point \( x \) also in \( \mathcal{T}^T(S) \). By Proposition 3.2, \( \mathcal{L}_{x_n} \) converges uniformly on compact sets to \( \mathcal{L}_x \). For all \( y \in \mathcal{T}(S) \), the function \( \ell_y \) is bounded away from zero on \( \mathcal{P} \). We conclude that \( \mathcal{L}_{x_n}(\cdot)/\ell_y(\cdot) \) converges uniformly on \( \mathcal{P} \) to \( \mathcal{L}_x(\cdot)/\ell_y(\cdot) \), for all \( y \in \mathcal{T}(S) \). It follows that \( \Psi_{x_n} \) converges pointwise to \( \Psi_x \). As noted before, this implies that \( \Psi_{x_n} \) converges to \( \Psi_x \) uniformly on bounded sets of \( \mathcal{T}(S) \). \( \Box \)

**Theorem 3.6.** The map \( \Psi \) is a homeomorphism between the Thurston compactification and the horofunction compactification of \( \mathcal{T}(S) \).

**Proof.** The injectivity of \( \Psi \) was proved in Lemma 3.4 and so \( \Psi \) is a bijection from \( \mathcal{T}^T(S) \) to its image. The map \( \Psi \) is continuous by Lemma 3.5. As a continuous bijection from a compact space to a Hausdorff one, \( \Psi \) must be a homeomorphism from \( \mathcal{T}^T(S) \) to its image. So \( \Psi(\mathcal{T}^T(S)) \) is compact and therefore closed. Using the continuity again, we get \( \Psi(\mathcal{T}(S)) \subset \Psi(\mathcal{T}^T(S)) \subset \text{cl} \Psi(\mathcal{T}(S)) \). Taking closures, we get \( \Psi(\mathcal{T}^T(S)) = \text{cl} \Psi(\mathcal{T}(S)) \), which is the horocompactification. \( \Box \)

4. Horocyclic foliations and stretch lines

Our goal in this section is to show that every horofunction of the Lipschitz metric is Busemann. This will be achieved by showing that every horofunction is the limit of a particular type of geodesic introduced by Thurston [27], called a stretch line.

Let \( \mu \) be a complete geodesic lamination. In other words, \( \mu \) is not strictly contained within another geodesic lamination, or equivalently, the complementary regions of \( \mu \) are all isometric to open ideal triangles in hyperbolic space. Note that if the surface \( S \) has punctures, then \( \mu \) has leaves going out to the cusps.

We foliate each of the complementary triangles of \( \mu \) with horocyclic arcs as shown in Figure 1. The horocyclic arcs meet the boundary of each triangle perpendicularly. Note that there is a non-foliated region at the center of each triangle, which is bounded by three horocyclic arcs meeting tangentially. So, the foliation obtained is actually a partial foliation. This partial foliation on \( S \setminus \mu \) may be extended continuously to a partial foliation on the whole of \( S \). Given a hyperbolic structure \( g \) on \( S \), we define a transverse measure on the partial foliation on \( S \) by requiring the measure of every sub-arc of \( \mu \) to be its length in the metric \( g \). The partial foliation with this transverse measure is called the horocyclic foliation, and is denoted \( F_\mu(g) \).
Collapsing all non-foliated regions, we obtain a well-defined element $F_\mu(g)$ of $\mathcal{MF}$, the space of measured foliations on $S$ up to Whitehead equivalence. Recall that two measured foliations are said to be Whitehead equivalent if one may be deformed to the other by isotopies, deformations that collapse to points arcs joining a pair of singularities, and the inverses of such maps.

Note that the horocyclic foliation has around each puncture an annulus of infinite width foliated by closed leaves parallel to the puncture. Such a foliation is said to be trivial around punctures. A measured foliation is said to be totally transverse to a geodesic lamination if it is transverse to the lamination and trivial around punctures. A measured foliation class is said to be totally transverse to a geodesic lamination if it has a representative that is totally transverse. Let $\mathcal{MF}(\mu)$ be the set of measure classes of measured foliations that are totally transverse to $\mu$. The horocyclic foliation is clearly in $\mathcal{MF}(\mu)$. Thurston proved that the map $\phi_\mu: \mathcal{T}(S) \to \mathcal{MF}(\mu): g \mapsto F_\mu(g)$ is in fact a homeomorphism.

The horocyclic foliation gives us a way of deforming the hyperbolic structure by stretching along $\mu$. Define the stretch line directed by $\mu$ and passing through $x \in \mathcal{T}(S)$ to be

$$\Gamma_{\mu,x}(t) := \phi_\mu^{-1}(e^t F_\mu(x)), \quad \text{for all } t \in \mathbb{R}. $$

Stretch lines are geodesics for Thurston’s Lipschitz metric, that is,

$$L(\Gamma_{\mu,x}(s), \Gamma_{\mu,x}(t)) = t - s, \quad \text{for all } s \text{ and } t \in \mathbb{R} \text{ with } s < t,$$

for $\mu$ does not just consist of geodesics converging at both ends to punctures.

The stump of a geodesic lamination is its largest sub-lamination on which there can be put a compactly-supported transverse measure. Théret showed that a measured foliation class is totally transverse to a complete geodesic lamination $\mu$ if and only if its associated measured lamination in $\mathcal{ML}$ transversely meets every component of the stump of $\mu$.

**Theorem 4.1.** Every point of the horofunction boundary of Thurston’s Lipschitz metric is a Busemann point.

![Figure 1. An ideal triangle foliated by horocycles.](image-url)
Proof. By Theorem 3.6 the horofunction and Thurston boundaries coincide. Let [ν] ∈ PML be any point of the Thurston boundary. Choose a maximal uniquely-ergodic element µ of MŁ so that [µ] is different from [ν]. So, i(ν, µ) > 0. Take a completion ℏ of µ. The stump of ℏ contains the support of µ, and so must equal this support, since µ is maximal. Since µ is uniquely-ergodic, it has only one component, which meets ν transversely. Let F denote the element of MF associated to ν.

By [24, Lemma 1.8], F is totally transverse to ℏ. Therefore the map t ↦→ φ−1 ℏ(e tF) is a stretch line directed by ℏ. It was shown in [16] that such a stretch line converges in the positive direction to [ν] in the Thurston compactification. Since a stretch line is a geodesic, we conclude that the horofunction Ψν corresponding to [ν] is a Busemann point.

□

5. The detour cost

Let (X, d) be a possibly non-symmetric metric space with base-point b. We define the detour cost for any two horofunctions ξ and η in X(∞) to be

\[ H(ξ, η) = \sup_{W ∋ ξ} \inf_{x ∈ W \cap X} \left( d(b, x) + η(x) \right) , \]

where the supremum is taken over all neighbourhoods W of ξ in X ∪ X(∞). This concept appears in [1]. An equivalent definition is

\[ H(ξ, η) = \inf_{t \to \infty} \lim_{t \to \infty} \left( d(b, γ(t)) + η(γ(t)) \right) , \]

(3)

where the infimum is taken over all paths γ : \( \mathbb{R}_+ \to X \) converging to ξ.

Lemma 5.1. Let ξ and η be horofunctions. Then,

\[ η(x) ≤ ξ(x) + H(ξ, η) \]

for all x in X.

Proof. By [1],

\[ η(x) ≤ \left( d(x, z) − d(b, z) \right) + \left( d(b, z) + η(z) \right) \]

for all x and z in X.

Note that there is always a path γ converging to ξ such that

\[ \lim_{t \to \infty} \left( d(b, γ(t)) + η(γ(t)) \right) = H(ξ, η) . \]

Taking the limit as z moves along such a path gives the result.

The following was proved in [28, Lemma 3.3].

Lemma 5.2. Let γ be an almost-geodesic converging to a Busemann point ξ, and let y ∈ X. Then,

\[ \lim_{t \to \infty} d(y, γ(t)) + ξ(γ(t)) \]

Moreover, for any horofunction η,

\[ H(ξ, η) = \lim_{t \to \infty} d(b, γ(t)) + η(γ(t)) . \]

Proof. Let ε > 0. Putting s = t in the definition of almost-geodesic, we see that

\[ |d(γ(0), γ(t)) − t| < ε, \]

for t large enough.

Using this and again the fact that γ is an almost-geodesic, we get

\[ |d(γ(0), γ(s)) + d(γ(s), γ(t)) − d(γ(0), γ(t))| < 2ε, \]

for sufficiently large t.
for $s$ and $t$ large enough, with $s \leq t$. Letting $t$ tend to infinity gives

$$|d(\gamma(0), \gamma(s)) + \xi(\gamma(s)) - \xi(\gamma(0))| \leq 2\epsilon, \quad \text{for } s \text{ large enough.}$$

But, since $\gamma$ converges to $\xi$,

$$|d(y, \gamma(s)) - d(\gamma(0), \gamma(s)) - \xi(y) + \xi(\gamma(0))| < \epsilon, \quad \text{for } s \text{ large enough.}$$

Combining these, we deduce the first statement of the lemma.

By Lemma 5.1, $\eta(x) \leq \xi(x) + H(\xi, \eta)$, for all $x$ in $X$. Evaluating at $x = \gamma(s)$, adding $d(b, \gamma(s))$ to both sides, and using the first part of the lemma with $y = b$, we get

$$\limsup_{s \to \infty} d(b, \gamma(s)) + \eta(\gamma(s)) \leq H(\xi, \eta).$$

On the other hand, from (3),

$$H(\xi, \eta) \leq \liminf_{s \to \infty} d(b, \gamma(s)) + \eta(\gamma(s)).$$

This establishes the second statement of the lemma. \Box

**Proposition 5.3.** Let $f$ be an isometry from one possibly non-symmetric metric space $(X, d)$ to another $(X', d')$, with base-points $b$ and $b'$ respectively. Then, the detour costs in $X$ and $X'$ are related by

$$H'(f \cdot \xi, f \cdot \eta) = \xi(f^{-1}(b')) + H(\xi, \eta) - \eta(f^{-1}(b')), \quad \text{for all } \xi, \eta \in X(\infty).$$

In particular, every isometry preserves finiteness of the detour cost.

**Proof.** Let $\xi$ and $\eta$ be in $X(\infty)$. By Proposition 2.4 the horofunction $\eta$ is mapped by $f$ to $f \cdot \eta(\cdot) = \eta(f^{-1}(\cdot)) - \eta(f^{-1}(b'))$. We have

$$H'(f \cdot \xi, f \cdot \eta) = \inf_{\gamma} \lim_{t \to \infty} \inf \left( d'(b', f(\gamma(t))) + \eta(\gamma(t)) - \eta(f^{-1}(b')) \right)$$

$$= \inf_{\gamma} \lim_{t \to \infty} \left( d(f^{-1}(b'), \gamma(t)) - d(b, \gamma(t)) \right)$$

$$+ \left( d(b, \gamma(t)) + \eta(\gamma(t)) \right) - \eta(f^{-1}(b'))$$

$$= \xi(f^{-1}(b')) + H(\xi, \eta) - \eta(f^{-1}(b')),$$

where each time the infimum is taken over all paths in $X$ converging to $\xi$. \Box

Note that we can take $(X, d)$ and $(X', d')$ to be identical and the isometry to be the identity map, in which case the proposition says how the detour cost depends on the base-point:

$$H'(\xi, \eta) = \xi(b') + H(\xi, \eta) - \eta(b').$$

For the next proposition, we will need the following assumption.

**Assumption D.** For every sequence $x_n$ in $X$, if $d_{sym}(b, x_n)$ converges to infinity, then so does $d(b, x_n)$.

This assumption is satisfied by the Lipschitz metric [18].

**Proposition 5.4.** Assume that $(X, d)$ satisfies assumptions [A], [B], [C] and [D]. If $\xi$ is a horofunction, then $H(\xi, \xi) = 0$ if and only if $\xi$ is Busemann.
Proposition 5.5. For all horofunctions $\xi$, $\eta$, and $\nu$,

(i) $H(\xi, \eta) \geq 0$;
(ii) $H(\xi, \nu) \leq H(\xi, \eta) + H(\eta, \nu)$.

Proof. From (i), we get that $d(b, y) + \eta(y) \geq 0$, for all $y$ in $X$. We conclude that $H(\xi, \eta)$ is non-negative.

(ii) By Lemma 5.1
$$d(b, x) + \nu(x) \leq d(b, x) + \eta(x) + H(\eta, \nu), \quad \text{for all } x \in X.$$
It follows from this that $H(\xi, \nu) \leq H(\xi, \eta) + H(\eta, \nu)$. □

By symmetrising the detour cost, the set of Busemann points can be equipped with a metric. For $\xi$ and $\eta$ in $X_B(\infty)$, let

$$\delta(\xi, \eta) := H(\xi, \eta) + H(\eta, \xi).$$ (4)

We call $\delta$ the detour metric. This construction appears in [1, Remark 5.2].

**Proposition 5.6.** The function $\delta: X_B(\infty) \times X_B(\infty) \to [0, \infty]$ is a (possibly $\infty$-valued) metric.

**Proof.** The only metric space axiom that does not follow from Propositions 5.4 and 5.5, and the symmetry of the definition of $\delta$ is that if $\delta(\xi, \eta) = 0$ for Busemann points $\xi$ and $\eta$, then these two points are identical. So, assume this equation holds. By (i) of Proposition 5.5, both $H(\xi, \eta)$ and $H(\eta, \xi)$ are zero. Applying Lemma 5.1 twice, we get that $\xi(x) = \eta(x)$ for all $x \in X$. □

The following proposition shows that each isometry of $X$ induces an isometry on $X_B(\infty)$ endowed with the detour metric. The independence of the base-point was observed in [1, Remark 5.2].

**Proposition 5.7.** Let $f$ be an isometry from one possibly non-symmetric metric space $(X,d)$ to another $(X',d')$, with base-points $b$ and $b'$ respectively. Then, the detour metrics in $X$ and $X'$ are related by

$$\delta'(f \cdot \xi, f \cdot \eta) = \delta(\xi, \eta),$$

for all $\xi, \eta \in X(\infty)$. In particular, the detour metric does not depend on the base-point.

**Proof.** The first part follows from Proposition 5.3. The second part then follows by taking $X = X'$ and $d = d'$, with $f$ the identity map. □

6. The Detour Cost for the Lipschitz Metric

We will now calculate the detour cost for Thurston’s Lipschitz metric. This result will be crucial for our study of the isometry group in the next section.

If $\xi_j$ are a finite set of measured laminations pairwise having zero intersection number, then we define their sum $\sum_j \xi_j$ to be the measured lamination obtained by taking the union of the supports and endowing it with the sum of the transverse measures. A measured lamination is said to be ergodic if it is non-trivial and can not be written as a sum of projectively-distinct non-trivial measured laminations. Each measured lamination $\xi$ can be written in one way as the sum of a finite set of projectively-distinct ergodic measured laminations. We call these laminations the ergodic components of $\xi$.

Let $\mu \in \mathcal{ML}$ be expressed as $\mu = \sum_j \mu_j$ in terms of its ergodic components. For $\nu \in \mathcal{ML}$, we write $\nu \ll \mu$ if $\nu$ can be expressed as $\nu = \sum_j f_j \mu_j$, where each $f_j$ is in $\mathbb{R}_+$.

Recall that $\Psi_\mu$ denotes the horofunction associated to the projective class of the measured lamination $\mu$.

**Theorem 6.1.** Let $\nu$ and $\mu$ be measured laminations. If $\nu \ll \mu$, then

$$H(\Psi_\mu, \Psi_\nu) = \log \sup_{\eta \in \mathcal{ML}} \frac{i(\mu, \eta)}{\ell_b(\eta)} + \log \max_j (f_j) - \log \sup_{\eta \in \mathcal{ML}} \frac{i(\nu, \eta)}{\ell_b(\eta)},$$
where $\nu$ is expressed as $\nu = \sum_j f_j \mu_j$ in terms of the ergodic components $\mu_j$ of $\mu$.
If $\nu \not\ll \mu$, then $H(\Psi_\mu, \Psi_\nu) = +\infty$.

**Remark.** Here, and in similar situations, we interpret the supremum to be over the set where the ratio is well defined, that is, excluding values of $\eta$ for which both the numerator and the denominator are zero.

The proof of this theorem will require several lemmas.

Consider a measured foliation $(F, \mu)$. Each connected component of the complement in $S$ of the union of the compact leaves of $F$ joining singularities is either an annulus swept out by closed leaves or a minimal component in which all leaves are dense. We call the latter components the *minimal domain* of $F$.

Denote by Sing $F$ the set of singularities of $F$. A curve $\alpha$ is quasi-transverse to $F$ if each connected component of $\alpha \setminus$ Sing $F$ is either a leaf or is transverse to $F$, and, in a neighbourhood of each singularity, no transverse arc lies in a sector adjacent to an arc contained in a leaf. Quasi-transverse curves minimise the total variation of the transverse measure in their homotopy class, in other words, $\mu(\alpha) = i((F, \mu), \alpha)$, for every curve $\alpha$ quasi-transverse to $(F, \mu)$; see [8].

**Lemma 6.2.** Let $\epsilon > 0$, and let $x_1$ and $x_2$ be two points on the boundary of a minimal domain $D$ of a measured foliation $(F, \mu)$. Then, there exists a curve segment $\sigma$ going from $x_1$ to $x_2$ that is contained in $D$ and quasi-transverse to $F$, such that $\mu(\sigma) < \epsilon$. Moreover, $\sigma$ can be chosen to have a non-trivial initial segment and terminal segment that are transverse to $F$.

**Proof.** Let $\tau_1$ and $\tau_2$ be two non-intersecting transverse arcs in $D$ starting at $x_1$ and $x_2$ respectively, parameterised so that $\mu(\tau_1[0, s]) = \mu(\tau_2[0, s]) = s$, for all $s$. We also require that the lengths of $\tau_1$ and $\tau_2$ with respect to $\mu$ are less than $\epsilon/2$.

Choose a point on $\tau_1$ and a direction, either left or right, such that the chosen half-leaf $\gamma$ is infinite, that is, does not hit a singularity. Since $D$ is a minimal component, $\gamma[0, \infty)$ is dense in $D$. Let $t_2$ be the first time $\gamma$ intersects $\tau_2$, and let $t_1 = \tau_1$ be the last time before $t_2$ that $\gamma$ intersects $\tau_1$.

Let $s_1$ and $s_2$ be such that $\tau_1(s_1) = \gamma(t_1)$ and $\tau_2(s_2) = \gamma(t_2)$. We may assume that $s_1$ is different from $s_2$, for otherwise, continue along $\gamma$ until the next time $t_3$ it intersects $\tau_1[0, s_1] \cup \tau_2[0, s_2]$. If the intersection is with $\tau_1[0, s_1]$, then we take the leaf segment $\gamma[t_2, t_3]$ with the reverse orientation. If the intersection is with $\tau_2[0, s_2]$, then we take the leaf segment $\gamma[t_1, t_3]$. In either case, after redefining $s_1$ and $s_2$ so that our chosen leaf segment starts at $\tau_1(s_1)$ and ends at $\tau_2(s_2)$, we get that $s_1 \neq s_2$.

If $\tau_1[0, s_1]$ and $\tau_2[0, s_2]$ leave $\gamma$ on opposite sides, then, by perturbing the concatenation $\tau_1[0, s_1] \star \gamma[t_1, t_2] \star \tau_2[0, s_2]$, we can find a curve segment $\sigma$ in $D$ that passes through $x_1$ and $x_2$ and is transverse to $F$ with weight $\mu(\sigma) = s_1 + s_2 < \epsilon$; see Figure 2.

If $\tau_1[0, s_1]$ and $\tau_2[0, s_2]$ leave $\gamma$ on the same side, then we apply [8] Theorem 5.4 to get an arc $\gamma'$ parallel to $\gamma$, contained in a union of a finite number of leaves and singularities, with end points $\gamma'(t_1) = \tau_1(s'_1)$ and $\gamma'(t_2) = \tau_2(s'_2)$ contained in $\tau_1[0, s_1]$ and $\tau_2[0, s_2]$, respectively (see Figure 3). Since $s_1 \neq s_2$, the points $x_1$, $x_2$, $\tau_2(s_2)$, and $\tau_1(s_1)$ do not form a rectangle foliated by leaves. Hence the endpoints of $\gamma'$ are not $x_1$ and $x_2$. As before, by perturbing $\tau_1[0, s'_1] \star \gamma'[t_1, t_2] \star \tau_2[0, s'_2]$, it is easy to construct a curve with the required properties. 

\[ \square \]
**Lemma 6.3.** Let $\xi_j; j \in \{0, \ldots, J\}$ be a finite set of ergodic measured laminations that pairwise have zero intersection number, such that no two are in the same projective class, and let $C > 0$. Then, there exists a curve $\alpha \in \mathcal{S}$ such that $i(\xi_0, \alpha) > C i(\xi_j, \alpha)$ for all $j \in \{1, \ldots, J\}$.

**Proof.** Since the $\xi_j$ do not intersect, we may combine them to form a measured lamination $\xi := \sum_j \xi_j$.

Consider first the case where $\xi_0$ is not a curve. Take a representative $(F, \mu)$ of the element of $\mathcal{MF}$ corresponding to $\xi$ by the well known bijection between $\mathcal{ML}$ and $\mathcal{MF}$. The decomposition of $\xi$ into a sum of ergodic measured laminations corresponds to the decomposition of $\mu$ into a sum of partial measured foliations $(F, \mu_j)$. Each $\mu_j$ is supported on either an annulus of closed leaves of $F$ (if $\xi_j$ is a curve), or a minimal domain. For each $j$, let $F_j := (F, \mu_j)$.
Let \( I' \) be a transverse arc contained in the interior of the minimal domain on which \( \mu_0 \) is supported. Write \( \mu^c := \sum_{j=1}^{J} \mu_j = \mu - \mu_0 \) and \( F^c := (F, \mu^c) \). The measures \( \{\mu_j\} \) are mutually singular, and so there is a Borel subset \( X \) of \( I' \) such that \( \mu_0(X) = \mu_0(I') \) and \( \mu^c(X) = 0 \). But \( X \) may be approximated from above by open sets of \( I' \):

\[
\mu^c(X) = \inf \{\mu^c(U) \mid X \subset U \subset I' \text{ and } U \text{ is open} \}.
\]

Therefore, we can find an open set \( U \) such that \( X \subset U \subset I' \) and \( \mu^c(U) < \mu_0(I')/C \). Since \( U \) is open, it is the disjoint union of a countable collection of open intervals. Since \( \mu_0(U) = \mu_0(I') > \mu^c(U)/C \), at least one of these intervals \( I \) must satisfy \( \mu_0(I) > \mu^c(I)/C \). Choose \( \epsilon > 0 \) such that \( \mu_0(I) - \epsilon > \mu^c(I)/C \).

For \( x \) and \( y \) in \( I \), we denote by \([x,y]\) the closed sub-arc of \( I \) connecting \( x \) and \( y \). Open and half-open sub-arcs are denoted in an analogous way. Let \( x_0 \) and \( x_1 \) be the endpoints of \( I \).

Choose a point \( x \in I \) such that \( \mu[x_0, x] < \epsilon/3 \) and there is an infinite half-leaf \( \gamma \) of \( F \) starting from \( \gamma(0) = x \). So, we may go along \( \gamma \) until we reach a point \( y := \gamma(t) \) in \( I \) such that \( \mu[y, x_1] < \epsilon/3 \).

Let \( B \) be the set of intervals \([p, q]\) in \( I \) such that the finite leaf segment \( \gamma[0, t] \) crosses \( I \) at \( p \) and at \( q \), the two crossings are in the same direction, and \( \gamma \) does not cross \( I \) in the interval \([p, q]\).

Consider an element \([p, q]\) of \( B \). Assume that \( \gamma \) passes through \( p \) before it passes through \( q \), that is, \( \gamma(t_p) = p \) and \( \gamma(t_q) = q \) for some \( t_p \) and \( t_q \) in \([0, t]\), with \( t_p < t_q \). (The other case is handled similarly.) Let \( \beta \) be the closed curve consisting of \( \gamma[t_p, t_q] \) concatenated with the sub-interval \([p, q]\) of \( I \). Since \( \gamma \) contains no singular point of \( F \), there exists a narrow rectangular neighbourhood of \( \gamma[t_p, t_q] \) not containing any singular point of \( F \), and so we may perturb \( \beta \) to get a closed curve \( \beta' \) that is transverse to \( F \) (see Figure 5). We have

\[
i(\beta, F_j) = i(\beta', F_j) = \mu_j(\beta') = \mu_j[p, q],
\]

for all \( j \in \{0, \ldots, J\} \). The second equality uses the fact that \( \beta' \) is transverse to \( F_j \), and hence quasi-transverse. Let \( Z \) be the set of curves \( \beta \) obtained in this way from the elements \([p, q]\) of \( B \).

The set \([x, y]\) \union \( B \) is composed of a finite number of intervals of the form \([r, s]\), where \( \gamma[0, t] \) crosses \( I \) at \( r \) and \( s \) in different directions and does not cross the

\[\text{Figure 5. Diagram for the proof of Lemma 6.3.}\]
interval \((r,s)\). From \(\gamma(t)\), we continue along \(\gamma\) until the first time \(t'\) that \(\gamma\) crosses one of the intervals \([r,s]\) comprising \([x,y]\) \(\cup B\). The direction of this crossing will be the same as either that at \(r\) or that at \(s\). We assume the former case; the other case is similar. As before, we have that the curve \(\beta\) formed from the segment of \(\gamma\) going from \(r\) to \(\gamma(t')\), concatenated with the sub-arc \([r,\gamma(t')]\) of \(I\) satisfies \(i(\beta,F_j) = \mu_j(r,\gamma(t'))\), for all \(j\). We add \([r,\gamma(t')]\) to the set \(B\), and \(\beta\) to the set \(Z\).

Observe that \([x,y]\) \(\cup B\) remains composed of the same number of intervals of the same form, only now one of them is shorter.

We continue in this manner, adding intervals to \(B\) and curves to \(Z\). Since \(\gamma\) crosses \(I\) on a dense subset of \(I\), the maximum \(\mu\)-measure of the component intervals of \([x,y]\) \(\cup B\) can be made as small as we wish. But there are a fixed number of these components, and so we can make \(\mu([x,y]\cup B)\) as small as we wish. We make it smaller than \(\epsilon/3\).

We have

\[
\max_{\beta \in Z} i(\beta,F_0) \geq \frac{\sum_{\beta \in Z} i(\beta,F_0)}{\sum_{\beta \in Z} i(\beta,F^c)} = \frac{\mu_0(\cup B)}{\mu^c(\cup B)} \geq \frac{\mu_0(I) - \epsilon}{\mu^c(I)} > C.
\]

So, some curve \(\beta \in Z\) satisfies \(i(\beta,\xi_0) = i(\beta,F_0) > C i(\beta,F^c) \geq C i(\beta,\xi_j)\), for all \(j \in \{1,\ldots,J\}\).

Now consider the case where \(\xi_0\) is a curve. A slight adaption of the proof of [8, Proposition 3.17] shows that there is a curve \(\alpha \in S\) having positive intersection number with \(\xi_0\) and zero intersection number with every other curve in the support of \(\xi\).

By [8, Proposition 5.9], there exists a measured foliation \((F,\mu)\) representing the element of \(\mathcal{M}F\) associated to \(\xi\) such that \(\alpha\) is transverse to \(F\) and avoids its singularities.

Again we use the decomposition of \(\mu\) into a sum of mutually-singular partial measured foliations \((F,\mu_j)\), corresponding to the \(\xi_j\). Since \(\alpha\) is transverse to \(F\), we have, for each \(j\), that \(i(\alpha,\xi_j) = \mu_j(\alpha)\), where \(\mu_j(\alpha)\) denotes the total mass of \(\alpha\) with respect to the transverse measure \(\mu_j\). It follows that \(\alpha\) crosses the annulus \(A\) associated to \(\xi_0\) at least once, but never enters any of the annuli associated to the other curves in the support of \(\xi\).

Consider the following directed graph. We take a vertex for every minimal domain of \(F\) through which \(\alpha\) passes, and for every time \(\alpha\) crosses \(A\). So, there is at most one vertex associated to each minimal domain but there may be more than one associated to \(A\). As we move along the curve \(\alpha\), we get a cyclic sequence of these vertices. We draw a directed edge between each vertex of this cyclic sequence and the succeeding one, and label it with the point of \(S\) where \(\alpha\) leaves the minimal domain or annulus and enters the next. There may be more than one directed edge between a pair of vertices, but each will have a different label.

The curve \(\alpha\) induces a circuit in this directed graph. Choose a simple sub-circuit \(c\) that passes through at least one vertex associated to a crossing of \(A\). Here, simple means that no vertex is visited more than once.
Construct a curve in $S$ as follows. For each vertex passed through by $c$ associated to a crossing of $A$, take the associated segment of $\alpha$. For each vertex passed through by $c$ associated to a minimal domain, choose $\epsilon$ to be less than the height of $A$ divided by $C$, and take the curve segment given by Lemma 6.2 passing through the minimal domain, joining the points labeling the incoming and outgoing directed edges. When we concatenate all these curve segments, we get a curve $\alpha'$ that passes through $A$, and that is quasi-transverse to $F$. Furthermore, $i(\xi_j, \alpha') < \epsilon$ for each non-curve component $\xi_j$ of $\xi$, and $i(\xi_j, \alpha') = 0$ for each curve component different from $\xi_0$. The conclusion follows.

**Lemma 6.4.** Let $\nu$ and $\mu$ be measured laminations. If $\nu \ll \mu$, then

$$\sup_{\eta \in \mathcal{ML}} \frac{i(\nu, \eta)}{i(\mu, \eta)} = \max(f_j),$$

where $\nu$ is expressed as $\nu = \sum_j f_j \mu_j$ in terms of the ergodic components $\mu_j$ of $\mu$. If $\nu \not\ll \mu$, then the supremum is $+\infty$.

**Proof.** In the case where $i(\nu, \mu) > 0$, we take $\eta := \mu$ to get that the supremum is $\infty$.

So assume that $i(\nu, \mu) = 0$. In this case, we can write $\nu = \sum_j f_j \xi_j$ and $\mu = \sum_j g_j \xi_j$, where the $\xi_j$'s are a finite set of ergodic measured laminations pairwise having zero intersection number, and the $f_j$ and $g_j$ are non-negative coefficients such that, for all $j$, either $f_j$ or $g_j$ is positive. Relabel the indexes so that $\max_j (f_j/g_j) = f_0/g_0$. We have

$$I(\eta) := \frac{i(\nu, \eta)}{i(\mu, \eta)} = \frac{\sum_j f_j i(\xi_j, \eta)}{\sum_j g_j i(\xi_j, \eta)}.$$

Simple algebra establishes that $I(\eta) \leq f_0/g_0$ for all $\eta$.

For each $C > 0$, we apply Lemma 6.3 to get a curve $\alpha_C$ such that $i(\xi_0, \alpha_C) > C i(\xi_j, \alpha_C)$ for all $j \in \{1, \ldots, J\}$. By choosing $C$ large enough, we can make $I(\alpha_C)$ as close as we like to $f_0/g_0$.

We conclude that $\sup_\eta I(\eta) = f_0/g_0$. □

**proof of Theorem 6.1** Let $F$ be the measured foliation corresponding to $\mu$. So, $i(F, \alpha) = i(\mu, \alpha)$, for all $\alpha \in \mathcal{S}$. As in the proof of Theorem 4.1, we may find a complete geodesic lamination $\mu'$ that is totally transverse to $F$. Consider the stretch line $\gamma(t) := \phi_{\mu'^{-1}}^t(F)$ directed by $\mu'$ and passing through $\phi_{\mu'^{-1}}^{-1}(F)$.

From the results in [16], $\gamma$ converges in the positive direction to $[\mu]$ in the Thurston boundary. So, by Theorem 3.6 $\gamma$ converges to $\Psi_\mu$ in the horofunction boundary. Therefore, since $\gamma$ is a geodesic,

$$H(\Psi_\mu, \Psi_\nu) = \lim_{t \to \infty} \left( L(b, \gamma(t)) + h_\nu(\gamma(t)) \right) = \lim_{t \to \infty} \left( \log \sup_{\eta \in \mathcal{ML}} \frac{\ell_\eta(\gamma(t))}{\ell_b(\eta)} + \log \sup_{\eta \in \mathcal{ML}} \frac{i(\nu, \eta)}{\ell_\eta(\gamma(t))} \right) - \log \sup_{\eta \in \mathcal{ML}} \frac{i(\nu, \eta)}{\ell_b(\eta)}.$$

From [23] Cor. 2], for every $\eta \in \mathcal{ML}$, there exists a constant $C_\eta$ such that

$$i(\phi_{\mu'}(\gamma(t)), \eta) \leq \ell_{\gamma(t)}(\eta) \leq i(\phi_{\mu'}(\gamma(t)), \eta) + C_\eta, \quad \text{for all } t \geq 0.$$

So,

$$i(F, \eta) \leq e^{-t} \ell_{\gamma(t)}(\eta) \leq i(F, \eta) + e^{-t} C_\eta, \quad \text{(5)}$$
for all $\eta \in \mathcal{ML}$ and $t \geq 0$.

So, $e^{-tL_{\gamma(t)}}$ converges pointwise to $i(F, \cdot) = i(\mu, \cdot)$ on $\mathcal{ML}$. Since $\gamma(t)$ converges to $[\mu]$, we get, by Proposition $3.2$ that $L_{\gamma(t)}$ converges to $L_{\mu}$ uniformly on compact sets. Combining this with the convergence of $e^{-tL_{\gamma(t)}}$, and evaluating at any measured lamination, we see that $e^{-tQ(\gamma(t))}$ converges to $Q(\mu)$. Using again the convergence of $e^{-tL_{\gamma(t)}}$, we conclude that $e^{-tL_{\gamma(t)}}$ converges to $i(\mu, \cdot)$ uniformly on compact sets. So,

$$\lim_{t \to \infty} \sup_{\eta \in \mathcal{ML}} \frac{e^{-tL_{\gamma(t)}(\eta)}}{\ell_b(\eta)} = \sup_{\eta \in \mathcal{ML}} \frac{i(\mu, \eta)}{\ell_b(\eta)}.$$ 

From the left-hand inequality of (5), we get

$$\sup_{\eta \in \mathcal{ML}} \frac{i(\nu, \eta)}{e^{-tL_{\gamma(t)}(\eta)}} \leq \sup_{\eta \in \mathcal{ML}} \frac{i(\nu, \eta)}{i(\mu, \eta)}$$

for $t \geq 0$.

But the limit of a supremum is trivially greater than or equal to the supremum of the limits. We conclude that

$$\lim_{t \to \infty} \sup_{\eta \in \mathcal{ML}} \frac{i(\nu, \eta)}{e^{-tL_{\gamma(t)}(\eta)}} = \sup_{\eta \in \mathcal{ML}} \frac{i(\nu, \eta)}{i(\mu, \eta)}$$

The result now follows on applying Lemma $6.4$. □

**Corollary 6.5.** If $\nu$ and $\mu$ in $\mathcal{ML}$ can be written in the form $\nu = \sum_j f_j \eta_j$ and $\mu = \sum_j g_j \eta_j$, where the $\eta_j$ are ergodic elements of $\mathcal{ML}$ that pairwise have zero intersection number, and the $f_j$ and $g_j$ are positive coefficients, then the detour metric between $\Psi_\nu$ and $\Psi_\mu$ is

$$\delta(\Psi_\nu, \Psi_\mu) = \log \max_j \frac{f_j}{g_j} + \log \max_j \frac{g_j}{f_j}.$$ 

If $\nu$ and $\mu$ can not be simultaneously written in this form, then $\delta(\Psi_\nu, \Psi_\mu) = +\infty$.

7. ISOMETRIES

In this section, we prove Theorems 7.9 and 7.10.

Recall that the curve complex $\mathcal{C}(S)$ is the simplicial complex having vertex set $S$, and where a set of vertices form a simplex when they have pairwise disjoint representatives. The automorphisms of the curve complex were characterised by Ivanov [10], Korkmaz [13], and Luo [15].

**Theorem 7.1** (Ivanov-Korkmaz-Luo). Assume that $S$ is not a sphere with four or fewer punctures, nor a torus with two or fewer punctures. Then all automorphisms of $\mathcal{C}(S)$ are given by elements of $\text{Mod}_S$.

We will also need the following theorem contained in [11], which was stated there for measured foliations rather than measured laminations. For each $\mu \in \mathcal{ML}$, we define the set $\mu^\perp := \{ \nu \in \mathcal{ML} | i(\nu, \mu) = 0 \}$.

**Theorem 7.2** (Ivanov). Assume that $S$ is not a sphere with four or fewer punctures, nor a torus with one or fewer punctures. Then the co-dimension of the set $\mu^\perp$ in $\mathcal{ML}$ is equal to 1 if and only if $\mu$ is a real multiple of a simple closed curve.

By Theorem 6.4, the horoboundary can be identified with the Thurston boundary. So, the homeomorphism induced on the horoboundary by an isometry of $\mathcal{T}(S)$ may be thought of as a map from $\mathcal{PML}$ to itself.
Lemma 7.3. Let $f$ be an isometry of the Lipschitz metric. For all $[\mu_1]$ and $[\mu_2]$ in $\mathcal{PML}$, we have $i(f[\mu_1], f[\mu_2]) = 0$ if and only if $i([\mu_1], [\mu_2]) = 0$.

Proof. For any two elements $[\mu]$ and $[\eta]$ of $\mathcal{PML}$, we have, by Proposition 5.3, that $H(\Psi_{[\eta]}, \Psi_{[\mu]})$ is finite if and only if $H(\Psi_{f[\eta]}, \Psi_{f[\mu]})$ is finite. Also, from Theorem 6.1, $H(\Psi_{[\eta]}, \Psi_{[\mu]})$ is finite if and only if $[\mu] \ll [\eta]$. It follows that $f$ preserves the relation $\ll$ on $\mathcal{PML}$.

Two elements $[\mu_1]$ and $[\mu_2]$ of $\mathcal{PML}$ satisfy $i([\mu_1], [\mu_2]) = 0$ if and only if there is some projective measured lamination $[\eta] \in \mathcal{PML}$ such that $[\mu_1] \ll [\eta]$ and $[\mu_2] \ll [\eta]$.

We conclude that $[\mu_1]$ and $[\mu_2]$ have zero intersection number if and only if $f[\mu_1]$ and $f[\mu_2]$ have zero intersection number. \qed

Lemma 7.4. Let $f$ be an isometry of $(\mathcal{T}(S), L)$. Then, there is an extended mapping class that agrees with $f$ on $\mathcal{PML}$.

Proof. By Theorem 8.6, the horoboundary can be identified with the Thurston boundary. So, the map induced by $f$ on the horoboundary is a homeomorphism of $\mathcal{PML}$ to itself.

We identify $\mathcal{PML}$ with the level set $\{\mu \in \mathcal{ML} \mid \ell_0(\mu) = 1\}$. There is then a unique way of extending the map $f : \mathcal{PML} \to \mathcal{PML}$ to a positively homogeneous map $f_* : \mathcal{ML} \to \mathcal{ML}$. Evidently, $f_*$ is also a homeomorphism.

From Lemma 7.3 we get that $(f_* \mu)^\perp = f_*(\mu^\perp)$ for all $\mu \in \mathcal{ML}$. By Theorem 7.2, real multiples of simple closed curves can be characterised in $\mathcal{ML}$ as those elements $\mu$ such that the co-dimension of the set $\mu^\perp$ equals to 1. Since $f_*$ is a homeomorphism on $\mathcal{ML}$, it preserves the co-dimension of sets. We conclude that $f_*$ maps elements of $\mathcal{ML}$ of the form $\lambda \alpha$, where $\lambda > 0$ and $\alpha \in \mathcal{S}$, to elements of the same form.

So, $f$ induces a bijective map on the vertices of the curve complex. From Lemma 7.3, it is clear that there is an edge between two curves $\alpha_1$ and $\alpha_2$ in $\mathcal{C}(S)$ if and only if there is an edge between $f_0 \alpha_1$ and $f_0 \alpha_2$. Using the fact that a set of vertices span a simplex exactly when every pair of vertices in the set have an edge connecting them, we see that $f$ acts as an automorphism on the curve complex.

We now apply Theorem 7.4 of Ivanov-Korkmaz-Luo to deduce that there is some extended mapping class $h \in \text{Mod}_g$ that agrees with $f$ on $\mathcal{S}$, considered as a subset of $\mathcal{PML}$. But $\mathcal{S}$ is dense in $\mathcal{PML}$, and the actions of $h$ and $f$ on $\mathcal{PML}$ are both continuous. Therefore, $h$ and $f$ agree on all of $\mathcal{PML}$. \qed

Recall that $R_{xy}$ is the subset of $\mathcal{PML}$ where $\ell_y(\cdot)/\ell_x(\cdot)$ attains its maximum.

Lemma 7.5. Let $x$ and $y$ be distinct points in $\mathcal{T}(S)$. Then, $L(x, y) + L(y, z) = L(x, z)$ if and only if $R_{xy}$ and $R_{yz}$ have an element in common, in which case $R_{xy} \cap R_{yz} = R_{xz}$.

Proof. Assume $R_{xy} \cap R_{yz}$ contains an element $[\nu]$. So,

$$
\frac{\ell_y(\nu)}{\ell_x(\nu)} \geq \frac{\ell_y(\nu_1)}{\ell_x(\nu_1)} \frac{\ell_y(\nu_2)}{\ell_x(\nu_2)} \quad \text{for all } \nu_1 \text{ and } \nu_2 \text{ in } \mathcal{ML}.
$$

(6)

We conclude that

$$
\frac{\ell_z(\nu)}{\ell_x(\nu)} \geq \sup_{\nu_1 \in \mathcal{ML}} \frac{\ell_y(\nu_1)}{\ell_x(\nu_1)} \sup_{\nu_2 \in \mathcal{ML}} \frac{\ell_y(\nu_2)}{\ell_x(\nu_2)}.
$$
which implies $L(x, z) \geq L(x, y) + L(y, z)$. The opposite inequality is just the triangle inequality. Taking $\nu_2 = \nu_1$ in equation (10) yields

$$\frac{\ell_z(\nu)}{\ell_x(\nu)} \geq \sup_{\nu_1 \in ML} \frac{\ell_z(\nu_1)}{\ell_x(\nu_1)},$$

which implies that $[\nu] \in R_{xz}$.

Now assume $L(x, y) + L(y, z) = L(x, z)$. For any element $[\nu]$ of $R_{xz}$, we have $\ell_z(\nu)/\ell_x(\nu) = \exp(L(x, z))$. We also have $\ell_z(\nu)/\ell_y(\nu) \leq \exp(L(y, z))$. Therefore,

$$\frac{\ell_y(\nu)}{\ell_x(\nu)} = \frac{\ell_z(\nu)}{\ell_x(\nu)} \geq \frac{\exp(L(x, z))}{\exp(L(y, z))} = \exp(L(x, y)),$$

and so $[\nu] \in R_{xy}$. A similar argument shows that $[\nu] \in R_{yz}$. \hfill \Box

**Lemma 7.6.** Let $\gamma$ be a geodesic converging in the negative direction to a maximal uniquely-ergodic $[\mu] \in PLML$. Then, $R_{\gamma(s)\gamma(t)} = \{[\mu]\}$, for all $s$ and $t$ with $s < t$.

**Proof.** For all $u < t$, write $R_u := R_{\gamma(u)\gamma(t)}$. By Lemma 7.5, $R_u \subset R_v$ for all $u$ and $v$ satisfying $u < v < t$. But $R_u$ is compact and non-empty for all $u$. Therefore, there is some $[\nu] \in PLML$ such that $[\nu] \in R_u$, for all $u < t$. Applying Lemma 7.5 we get that $i(\mu, \nu) = 0$. Since $[\mu]$ is maximal and uniquely ergodic, this implies that $[\nu] = [\mu]$. We have proved that $[\mu] \in R_s$. Thurston showed [27] that there is a geodesic lamination associated to any given pair of hyperbolic structures that contains every maximally stretched lamination. Therefore, every pair of elements of $R_s$ has zero intersection number. We conclude that $R_s = \{[\mu]\}$. \hfill \Box

**Lemma 7.7.** Let $\mu$ be a complete geodesic lamination having maximal uniquely-ergodic stump $\mu_0$, and let $x$ be a point of $\mathcal{T}(S)$. Then, the stretch line $\Gamma_{\mu,x}$ is the unique geodesic converging in the negative direction to $[\mu_0]$ and in the positive direction to $[F_{\mu}(x)]$.

**Proof.** That $\Gamma_{\mu,x}$ is a geodesic was proved in [27], and that it has the required convergence properties was proved in [10] and [24].

Let $\gamma$ be any geodesic converging as in the statement of the lemma. Choose $s$ and $t$ in $\mathbb{R}$ such that $s < t$. By Lemma 7.4, $R_{\gamma(s)\gamma(t)} = \{[\mu_0]\}$. Therefore, by [27] Theorem 8.5, there exists a geodesic from $\gamma(s)$ to $\gamma(t)$ consisting of a concatenation of stretch lines, each of which stretches along some complete geodesic lamination containing $\mu_0$. However, the only complete geodesic lamination containing $\mu_0$ is $\mu$, and so the constructed geodesic is just a segment of the stretch line $\Gamma_{\mu,\gamma(s)}$. So $\Gamma_{\mu,\gamma(s)}$ passes through $\gamma(t)$. We conclude that $\Gamma_{\mu,\gamma(s)}$ equals $\Gamma_{\mu,\gamma(t)}$ up to reparametrisation. Since $s$ and $t$ are arbitrary, we get that $\gamma$ is just a stretch line directed by $\mu$. There is only one stretch line directed by $\mu$ converging to $[F_{\mu}(x)]$ in the forward direction, namely $\Gamma_{\mu,x}$. \hfill \Box

**Lemma 7.8.** Let $f$ and $h$ be two isometries of $(\mathcal{T}(S), L)$. If the extensions of $f$ and $h$ coincide on the boundary $PLML$, then $f = h$.

**Proof.** Let $x \in \mathcal{T}(S)$. Choose a complete geodesic lamination $\mu$ such that its stump $\mu_0$ is maximal and uniquely ergodic. Associated to $\mu$ and $x$ is the horocyclic foliation $F_{\mu}(x)$. The stretch line $\Gamma_{\mu,x}$ is geodesic in the Lipschitz metric $L$, and converges in the negative direction to $[\mu_0]$ and in the positive direction to $[F_{\mu}(x)]$; see [10]. By Lemma 7.7, it is the only geodesic to do so. The map $h^{-1} \circ f$ is an isometry on $(\mathcal{T}(S), L)$, and so maps geodesics to geodesics. It extends continuously
We conclude that it leaves $\Gamma_{\mu,x}$ invariant as a set. Clearly, it acts by translation along $\Gamma_{\mu,x}$.

Now let $\mu'$ be a complete geodesic lamination whose stump $\mu'_0$ is maximal and uniquely ergodic, and distinct from both $\mu_0$ and the support of the measured lamination associated to $F_{\mu}(x)$. By reasoning similar to that above, $h^{-1} \circ f$ also acts by translation along the stretch line $\Gamma_{\mu',x}$. If the translation distance along $\Gamma_{\mu',x}$ is non-zero, then either the iterates $(h^{-1} \circ f)^n(x)$ or the iterates $(h^{-1} \circ f)^{-n}(x)$ of the inverse map converge to $[\mu'_0]$. But this is impossible because $[\mu'_0]$ is not a limit point of $\Gamma_{\mu,x}$. Hence, the translation distance is zero and $x$ is fixed by $h^{-1} \circ f$. We have proved that $h(x) = f(x)$ for any $x \in T(S)$.

**Theorem 7.9.** If $S$ is not a sphere with four or fewer punctures, nor a torus with two or fewer punctures, then every isometry of $T(S)$ with Thurston’s Lipschitz metric is an element of the extended mapping class group $\text{Mod}_S$.

**Proof.** This is a consequence of Lemmas 7.4 and 7.8.

**Theorem 7.10.** Let $S_{g,n}$ and $S'_{g',n'}$ be surfaces of negative Euler characteristic. Assume $\{(g,n),(g',n')\}$ is different from each of the three sets

\[
\{(1,1),(0,4)\}, \quad \{(1,2),(0,5)\}, \quad \text{and} \quad \{(2,0),(0,6)\}.
\]

If $(g,n)$ and $(g',n')$ are distinct, then the Teichmüller spaces $T(S_{g,n})$ and $T(S'_{g',n'})$ with their respective Lipschitz metrics are not isometric.

**Proof.** Let $S_{g,n}$ and $S'_{g',n'}$ be two surfaces of negative Euler characteristic, and let $f$ be an isometry between the associated Teichmüller spaces $T(S_{g,n})$ and $T(S'_{g',n'})$, each endowed with its respective Lipschitz metric.

Recall that the dimension of $T(S_{g,n})$ is $6g - 6 + 2n$. Consider the cases not covered by Ivanov’s Theorem 7.2, namely the sphere with three or four punctures, and the torus with one puncture. A comparison of dimension shows that $T(S_{0,3})$ is not isometric to any other Teichmüller space, and that $T(S_{0,4})$ and $T(S_{1,1})$ may be isometric to each another but not to any other Teichmüller space.

So assume that neither $T(S_{g,n})$ nor $T(S'_{g',n'})$ is one of these exceptional surfaces.

By the same reasoning as in the first part of the proof of Lemma 7.4 we conclude that $f$ induces an isomorphism from the curve complex of $S_{g,n}$ to that of $S'_{g',n'}$. However, according to [15] Lemma 2.1, the only isomorphisms between curve complexes are $\mathcal{C}(S_{1,2}) \cong \mathcal{C}(S_{0,5})$ and $\mathcal{C}(S_{2,0}) \cong \mathcal{C}(S_{0,6})$. The conclusion follows.

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