IRREDUCIBILITY OF PERFECT REPRESENTATIONS
OF DOUBLE AFFINE HECKE ALGEBRAS

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In the paper, we prove that the quotient of the polynomial representation of the double affine Hecke algebra (DAHA) by the radical of the duality pairing is always irreducible (apart from the roots of unity) provided that it is finite dimensional. We also find necessary and sufficient conditions for the radical to be zero, which is a $q$-generalization of Opdam’s formula for the singular $k$-parameters with the multiple zero-eigenvalue of the corresponding Dunkl operators.

Concerning the terminology, perfect modules in the paper are finite dimensional possessing a non-degenerate duality pairing. The latter induces the canonical duality anti-involution of DAHA. Actually, it suffices to assume that the pairing is perfect, i.e. identifies the module with its dual as a vector space, but we will stick to the finite dimensional case.

We also assume that perfect modules are spherical, i.e., quotients of the polynomial representation of DAHA, and invariant under the projective action of $PSL(2, \mathbb{Z})$. We do not impose the semisimplicity in

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contrast to [C3]. The irreducibility theorem from this paper is stronger and at the same time the proof is simpler than those in [C3].

The irreducibility follows from the projective $PSL(2, \mathbb{Z})$-action which readily results from the $\tau_-$-invariance. The latter always holds if $q$ is not a root of unity. At roots of unity, it is true for special $k$ only. We do not give in the paper necessary and sufficient conditions for the $\tau_-$-invariance as $q$ is a root of unity. Generally, it is not difficult to check (if it is true).

The polynomial representation has the canonical duality pairing. It is defined in terms of the difference-trigonometric Dunkl operators, similar to the rational case where the differential-rational operators are used, and involves the evaluation at $q^{-\rho_k}$ instead of the value at zero. The quotient of the polynomial representation by the radical $Rad$ of this pairing is a universal quasi-perfect representation. By the latter, we mean a DAHA-module with a non-degenerate but maybe non-perfect duality paring.

The polynomial representation, denoted by $V$ in the paper, is quasi-perfect and irreducible for generic values of the DAHA-parameters $q, t$. It is also $Y$-semisimple, i.e., there exists a basis of eigenvectors of the $Y$-operators, and has the simple $Y$-spectrum for generic $q, t$.

The radical $Rad$ is nonzero when $q$ is a root of unity or as $t = \zeta q^k$ for special fractional $k$ and proper roots of unity $\zeta$.

We give an example of reducible $V$ which has no radical ($B_n$). The complete list will be presented in the next paper.

**Semisimplicity.** Typical examples of $Y$-semisimple perfect representations are the nonsymmetric Verlinde algebras, generalizing the Verlinde algebras. The latter describe the fusion of the integrable representations of the Kac-Moody algebras, and, equivalently, the reduced category of representations of quantum groups at roots of unity. The third interpretation is via factors/subfactors. Generally, these algebras appear in terms of the vertex operators (coinvariants) associated with Kac-Moody or Virasoro-type algebras.

There are at least two important reasons to drop the semisimplicity constraint:

First, it was found recently that the fusion procedure for a certain Virasoro-type algebra leads to a non-semisimple variant of the Verlinde algebra. As a matter of fact, there are no general reasons to expect semisimplicity in the massless conformal field theory. The positive
definite inner product in the Verlinde algebra, which guarantees the semisimplicity, is given in terms of the masses of the points/particles.

Second, non-semisimple representations of DAHA are expected to appear when the whole category of representations of Lusztig’s quantum group at roots of unity is considered. Generally, non-spherical representations could be necessary. However the anti-spherical (Steinberg-type) representations, which are spherical constructed for $t^{-1}$ in place of $t$, are expected to play an important role.

The simplest non-semisimple example at roots of unity ($A_1$) is considered at the end of the paper in detail.

Concerning the necessary and sufficient condition for the radical of $V$ to be nonzero, it readily follows from the evaluation formula for the nonsymmetric Macdonald polynomials [C2]. This approach does require the $q,t$-setting because the evaluation formula collapses in the limit. Cf. [DO], Section 3.2.

The method from [O2] (see also [DJO] and [J]) based on the shift operator is also possible, and even becomes simpler with $q,t$ than in the rational/trigonometric case. It will be demonstrated in the next paper. The definition of the radical of the polynomial representation is due to Opdam in the rational case. See, e.g., [DO]. In the $q,t$-case, the radical was introduced in [C1, C2].

Rational limit. Interestingly, the quotient of $V$ by the radical is always irreducible for the rational DAHA. The justification is immediate and goes as follows.

This quotient has the zero-eigenvalue (no other eigenvalues appear in the rational setting) of multiplicity one. Any its proper submodule will generate at least one additional zero-eigenvector, which is impossible.

The DAHA and its rational degeneration are connected by exp-log maps of some kind [C4], but these maps are of analytic nature in the infinite dimensional case and cannot be directly applied to the polynomial representation.

Generally, the $q,t$-methods are simpler in many aspects than those in the rational degeneration thanks to the existence of the Macdonald polynomials and their analytic counterparts. It is somewhat similar to the usage of the unitary invariant scalar product in the theory of compact Lie groups vs. the abstract theory of Lie algebras. The $q,t$-generalization of Opdam’s formula for singular $k$ and the theory of perfect representations are typical examples in favor of the $q,t$-setting.
However, with the irreducibility of the universal quasi-perfect quotient of the polynomial representation, it is the other way round.

My guess is that it happens because the $q, t$-polynomial representation contains more information than could be seen after the rational degeneration. I mean mainly the semisimplicity which do not exist in the rational theory and can be incorporated only if the rational DAHA is extended by the ”first jet” towards $q$ (not published).

It must be mentioned here that the rational theory is for complex reflection groups. The $q, t$-theory is mainly about the crystallographic groups. Not all complex reflection groups have affine and double affine extensions.

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1. Affine Weyl groups

Let $R = \{\alpha\} \subset \mathbb{R}^n$ be a root system of type $A, B, ..., F, G$ with respect to a euclidean form $(z, z')$ on $\mathbb{R}^n \ni z, z'$, $W$ the Weyl group generated by the reflections $s_\alpha$, $R_+ \subset R$ the set of positive roots ($R_- = -R_+$), corresponding to (fixed) simple roots roots $\alpha_1, ..., \alpha_n$, $\Gamma$ the Dynkin diagram with $\{\alpha_i, 1 \leq i \leq n\}$ as the vertices.

We will also use the dual roots (coroots) and the dual root system:

$$R^\vee = \{\alpha^\vee = 2\alpha/(\alpha, \alpha)\}.$$  

The root lattice and the weight lattice are:

$$Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i \subset P = \bigoplus_{i=1}^n \mathbb{Z}\omega_i,$$

where $\{\omega_i\}$ are fundamental weights: $(\omega_i, \alpha^\vee_j) = \delta_{ij}$ for the simple coroots $\alpha^\vee_i$. 

Replacing $\mathbb{Z}$ by $\mathbb{Z}_\pm = \{m \in \mathbb{Z}, \pm m \geq 0\}$ we obtain $Q_\pm, P_\pm$. Note that $Q \cap P_+ \subset Q_+$. Moreover, each $\omega_j$ has all nonzero coefficients (sometimes rational) when expressed in terms of $\{\alpha_i\}$. Here and further see [B].

The form will be normalized by the condition $(\alpha, \alpha) = 2$ for the short roots. Thus,

$$\nu_\alpha \overset{\text{def}}{=} (\alpha, \alpha)/2$$ is either 1, or $\{1, 2\}$, or $\{1, 3\}$.

We will use the notation $\nu_{\text{long}}$ for the long roots ($\nu_{\text{shft}} = 1$).
Let \( \vartheta \in R^\vee \) be the \textbf{maximal positive coroot}. Considered as a root (it belongs to \( R \) because of the choice of normalization) it is maximal among all short positive roots of \( R \).

Setting \( \nu_i = \nu_{\alpha_i}, \nu_R = \{ \nu_{\alpha}, \alpha \in R \} \), one has

\[
(1.1) \quad \rho_\nu \overset{\text{def}}{=} \frac{1}{2} \sum_{\nu_i = \nu} \alpha = \sum_{\nu_i = \nu} \omega_i, \quad \text{where} \quad \alpha \in R_+, \nu \in \nu_R.
\]

Note that \((\rho_\nu, \alpha^\vee_i) = 1\) as \( \nu_i = \nu \). We will call \( \rho_\nu \) partial \( \rho \).

**Affine roots.** The vectors \( \tilde{\alpha} = [\alpha, \nu_{\alpha j}] \in \mathbb{R}^n \times \mathbb{R} \subset \mathbb{R}^{n+1} \) for \( \alpha \in R, j \in \mathbb{Z} \) form the \textbf{affine root system} \( \tilde{R} \supset R \) (\( z \in \mathbb{R}^n \) are identified with \([z, 0]\)). We add \( \alpha_0 \overset{\text{def}}{=} [-\vartheta, 1] \) to the simple roots for the maximal short root \( \vartheta \). The corresponding set \( \tilde{R} \) of positive roots coincides with \( R_+ \cup \{ [\alpha, \nu_{\alpha j}], \alpha \in R, j > 0 \} \).

We complete the Dynkin diagram \( \Gamma \) of \( R \) by \( \alpha_0 \) (by \( -\vartheta \) to be more exact). The notation is \( \tilde{\Gamma} \). One can obtain it from the completed Dynkin diagram for \( R^\vee \) from \([B]\) reversing the arrows. The number of laces between \( \alpha_i \) and \( \alpha_j \) in \( \tilde{\Gamma} \) is denoted by \( m_{ij} \).

The set of the indices of the images of \( \alpha_0 \) by all the automorphisms of \( \tilde{\Gamma} \) will be denoted by \( O \) (\( O = \{ 0 \} \) for \( E_8, F_4, G_2 \)). Let \( O' = r \in O, r \neq 0 \). The elements \( \omega_r \) for \( r \in O' \) are the so-called minuscule weights: \( (\omega_r, \alpha^\vee) \leq 1 \) for \( \alpha \in R_+ \).

Given \( \tilde{\alpha} = [\alpha, \nu_{\alpha j}] \in \tilde{R}, \ b \in B \), let

\[
(1.2) \quad s_{\tilde{\alpha}}(\tilde{z}) = \tilde{z} - (z, \alpha^\vee)\tilde{\alpha}, \ b'(\tilde{z}) = [z, \zeta - (z, b)]
\]

for \( \tilde{z} = [z, \zeta] \in \mathbb{R}^{n+1} \).

The \textbf{affine Weyl group} \( \widetilde{W} \) is generated by all \( s_{\tilde{\alpha}} \) (we write \( \widetilde{W} = \langle s_{\tilde{\alpha}}, \tilde{\alpha} \in \tilde{R}_+ \rangle \)). One can take the simple reflections \( s_i = s_{\alpha_i} \) \((0 \leq i \leq n)\) as its generators and introduce the corresponding notion of the length. This group is the semidirect product \( W \ltimes Q' \) of its subgroups \( W = \langle s_{\alpha}, \alpha \in \tilde{R}_+ \rangle \) and \( Q' = \{ a', a \in Q \} \), where

\[
(1.3) \quad \alpha' = s_{\alpha} s_{[\alpha, \nu_{\alpha}]} = s_{[-\alpha, \nu_{\alpha}]} s_{\alpha} \quad \text{for} \quad \alpha \in R.
\]

The \textbf{extended Weyl group} \( \widehat{W} \) generated by \( W \) and \( P' \) (instead of \( Q' \)) is isomorphic to \( W \ltimes P' \):

\[
(1.4) \quad (wb')(\tilde{z}) = [w(z), \zeta - (z, b)] \quad \text{for} \quad w \in W, b \in P.
\]

From now on, \( b \) and \( b' \), \( P \) and \( P' \) will be identified.
Given $b \in P_+$, let $w_0^b$ be the longest element in the subgroup $W_0^b \subset W$ of the elements preserving $b$. This subgroup is generated by simple reflections. We set

\[(1.5) \quad u_b = w_0 w_0^b \in W, \; \pi_b = b(u_b)^{-1} \in \hat{W}, \; u_i = u_{\omega_i}, \pi_i = \pi_{\omega_i}, \]

where $w_0$ is the longest element in $W$, $1 \leq i \leq n$.

The elements $\pi_r \overset{\text{def}}{=} \pi_{\omega_r}, r \in O'$ and $\pi_0 = \text{id}$ leave $\hat{\Gamma}$ invariant and form a group denoted by $\Pi$, which is isomorphic to $P/Q$ by the natural projection $\{\omega_r \mapsto \pi_r\}$. As to $\{u_r\}$, they preserve the set $\{-\vartheta, \alpha_i, i > 0\}$. The relations $\pi_r(\alpha_0) = \alpha_r = (u_r)^{-1}(-\vartheta)$ distinguish the indices $r \in O'$.

Moreover (see e.g., [C3]):

\[(1.6) \quad \hat{W} = \Pi \times \hat{\Pi}, \text{ where } \pi_r s_i \pi_r^{-1} = s_j \text{ if } \pi_r(\alpha_i) = \alpha_j, \; 0 \leq j \leq n.\]

Setting $\hat{w} = \pi_r \hat{w} \in \hat{W}$, $\pi_r \in \Pi, \hat{w} \in \hat{W}$, the length $l(\hat{w})$ is by definition the length of the reduced decomposition $\hat{w} = s_{i_1}...s_{i_2}s_{i_1}$ in terms of the simple reflections $s_i, 0 \leq i \leq n$.

The length can be also defined as the cardinality $|\lambda(\hat{w})|$ of

$$\lambda(\hat{w}) \overset{\text{def}}{=} \hat{R}_+ \cap \hat{w}^{-1}(\hat{R}_-) = \{\hat{\alpha} \in \hat{R}_+, \; \hat{w}(\hat{\alpha}) \in \hat{R}_-\}, \; \hat{w} \in \hat{W}.\]

Reduction modulo $W$. The following proposition is from [C2]. It generalizes the construction of the elements $\pi_b$ for $b \in P_+$.

**Proposition 1.1.** Given $b \in P$, there exists a unique decomposition $b = \pi_b u_b$, $u_b \in W$ satisfying one of the following equivalent conditions:

i) $l(\pi_b) + l(u_b) = l(b)$ and $l(u_b)$ is the greatest possible,

ii) $\lambda(\pi_b) \cap R = \emptyset$. Moreover, $u_0(b) \overset{\text{def}}{=} b_- \in P_- = -P_+$ is a unique element from $P_-\text{ which belongs to the orbit } W(b).$ \hfill \Box

For $\hat{\alpha} = [\alpha, \nu_\alpha] \in \hat{R}_+$, one has:

\[(1.7) \quad \lambda(b) = \{\hat{\alpha}, (b, \alpha^\vee) > j \geq 0 \text{ if } \alpha \in R_+, \]

\[\quad (b, \alpha^\vee) \geq j > 0 \text{ if } \alpha \in R_-\},\]

\[(1.8) \quad \lambda(\pi_b) = \{\hat{\alpha}, \alpha \in R_-, (b_-, \alpha^\vee) > j \geq 0 \text{ if } u_b^{-1}(\alpha) \in R_+, \]

\[\quad (b_-, \alpha^\vee) \geq j > 0 \text{ if } u_b^{-1}(\alpha) \in R_-\},\]

\[(1.9) \quad \lambda(u_b) = \{\alpha \in R_+, (b, \alpha^\vee) > 0\}.\]
2. Double Hecke algebras

By $m$, we denote the least natural number such that $(P, P) = (1/m)\mathbb{Z}$.

Thus $m = 2$ for $D_{2k}$, $m = 1$ for $B_{2k}$ and $C_k$, otherwise $m = |\Pi|$.

The double affine Hecke algebra depends on the parameters $q, t, \nu, \nu \in \{\nu_\alpha\}$. The definition ring is $\mathbb{Q}_{q,t} \overset{\text{def}}{=} \mathbb{Q}[q^{\pm 1/m}, t^{\pm 1/2}]$ formed by the polynomials in terms of $q^{\pm 1/m}$ and $\{t_\nu^{\pm 1/2}\}$. We set

$$t_\alpha = t_\nu = q^{\nu_\alpha}, \quad q_i = q^{\nu_\alpha_i},$$

(2.1) where $\bar{\alpha} = [\alpha, \nu_\alpha] \in \bar{R}$, $0 \leq i \leq n$.

It will be convenient to use the parameters $\{k_\nu\}$ together with $\{t_\nu\}$, setting

$$t_\alpha = t_\nu = q_\alpha^{k_\nu} \text{ for } \nu = \nu_\alpha, \quad \text{and } \rho = (1/2) \sum_{\alpha > 0} k_\alpha \alpha.$$

For pairwise commutative $X_1, \ldots, X_n$,

$$X_b = \prod_{i=1}^{n} X_i^{l_i} q^j \text{ if } \bar{b} = [b, j], \quad \hat{w}(X_b) = X_{\hat{w}(\bar{b})},$$

(2.2) where $b = \sum_{i=1}^{n} l_i \omega_i \in P$, $j \in \frac{1}{m}\mathbb{Z}$, $\hat{w} \in \hat{W}$.

We set $(\bar{b}, \bar{c}) = (b, c)$ ignoring the affine extensions.

Later $Y_b = Y_b q^{-j}$ will be needed. Note the negative sign of $j$.

**Definition 2.1.** The double affine Hecke algebra $\mathcal{H}$ is generated over $\mathbb{Q}_{q,t}$ by the elements $\{T_i, 0 \leq i \leq n\}$, pairwise commutative $\{X_b, b \in P\}$ satisfying (2.2), and the group $\Pi$, where the following relations are imposed:

1. $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0, \quad 0 \leq i \leq n$;
2. $T_i T_j T_i \ldots = T_j T_i T_j \ldots, \quad m_{ij}$ factors on each side;
3. $\pi_r T_i \pi_r^{-1} = T_j$ if $\pi_r(\alpha_i) = \alpha_j$;
4. $T_i X_b T_i = X_b X_{\alpha_i}^{-1}$ if $(b, \alpha_i^\vee) = 1, \quad 0 \leq i \leq n$;
5. $T_i X_b = X_b T_i$ if $(b, \alpha_i^\vee) = 0$ for $0 \leq i \leq n$;
6. $\pi_r X_b \pi_r^{-1} = X_{\pi_r(b)}$, $r \in O'$.
Given \( \tilde{w} \in \tilde{W}, r \in O \), the product
\[
T_{\pi_r \tilde{w}} \overset{\text{def}}{=} \pi_r \prod_{k=1}^{l} T_{i_k}, \quad \text{where} \quad \tilde{w} = \prod_{k=1}^{l} s_{i_k}, \quad l = l(\tilde{w}),
\]
does not depend on the choice of the reduced decomposition (because \( \{T\} \) satisfy the same “braid” relations as \( \{s\} \) do). Moreover,
\[
T_{b} T_{\tilde{w}} = T_{b \tilde{w}} \quad \text{whenever} \quad l(\tilde{v} \tilde{w}) = l(\tilde{v}) + l(\tilde{w}) \quad \text{for} \quad \tilde{v}, \tilde{w} \in \tilde{W}.
\]
In particular, we arrive at the pairwise commutative elements
\[
Y_{b} = \prod_{i=1}^{n} Y_{i}^{l_{i}} \quad \text{if} \quad b = \sum_{i=1}^{n} l_{i} \omega_{i} \in P, \quad \text{where} \quad Y_{i} \overset{\text{def}}{=} T_{\omega_{i}},
\]
satisfying the relations
\[
T_{i}^{-1} Y_{b} T_{i}^{-1} = Y_{b} T_{\alpha_{i}}^{-1} \quad \text{if} \quad (b, \alpha_{i}) = 1,
\]
\[
T_{i} Y_{b} = Y_{b} T_{i} \quad \text{if} \quad (b, \alpha_{i}) = 0, \quad 1 \leq i \leq n.
\]

The Demazure-Lusztig operators are defined as follows:
\[
T_{i} = t_{i}^{1/2} s_{i} + (t_{i}^{1/2} - t_{i}^{-1/2})(X_{\alpha_{i}} - 1)^{-1}(s_{i} - 1), \quad 0 \leq i \leq n,
\]
and obviously preserve \( \mathbb{Q}[q, t^{\pm1/2}][X] \). We note that only the formula for \( T_{0} \) involves \( q \):
\[
T_{0} = t_{0}^{1/2} s_{0} + (t_{0}^{1/2} - t_{0}^{-1/2})(qX_{\vartheta} - 1)^{-1}(s_{0} - 1),
\]
where \( s_{0}(X_{b}) = X_{b} X_{\vartheta}^{(b, \vartheta)} q^{(b, \vartheta)}, \quad \alpha_{0} = [-\vartheta, 1] \).

The map sending \( T_{j} \) to the formula in (2.7), and \( X_{b} \mapsto X_{b} \) (see (2.2)), \( \pi_{r} \mapsto \pi_{r} \) induces a \( \mathbb{Q}_{q,t} \)-linear homomorphism from \( \mathcal{H} \) to the algebra of linear endomorphisms of \( \mathbb{Q}_{q,t}[X] \). This \( \mathcal{H} \)-module, which will be called the polynomial representation, is faithful and remains faithful when \( q, t \) take any nonzero complex values assuming that \( q \) is not a root of unity.

The images of the \( Y_{b} \) are called the difference Dunkl operators. To be more exact, they must be called difference-trigonometric Dunkl operators, because there are also difference-rational Dunkl operators.

The polynomial representation is the \( \mathcal{H} \)-module induced from the one dimensional representation \( T_{i} \mapsto t_{i}^{1/2}, \quad Y_{i} \mapsto Y_{i}^{1/2} \) of the affine Hecke subalgebra \( \mathcal{H}_{Y} = \langle T, Y \rangle \). Here the PBW-Theorem is used: for arbitrary
nonzero \( q, t \), any element \( H \in \mathcal{H} \) has a unique decomposition in the form
\[
H = \sum_{w \in W} g_w f_w T_w, \quad g_w \in \mathbb{Q}_q, t[X], \quad f_w \in \mathbb{Q}_q, t[Y].
\]

The definition of DAHA and the polynomial representation are compatible with the **intermediate subalgebras** \( \mathcal{H}^\flat \subset \mathcal{H} \) with \( P \) replaced by any lattice \( B \ni b \) between \( Q \) and \( P \). Respectively, \( \Pi \) is changed to the image \( \Pi^\flat \) of \( B/Q \) in \( \Pi \).

From now on, we take \( X_a, Y_b \) with the indices \( a, b \in B \).

We will continue using the notation \( V \) for the \( B \)-polynomial representation:
\[
V = \mathbb{Q}_q, t[X] = \mathbb{Q}_q, t[X, b \in B].
\]

We also set \( \hat{W}^\flat = B \cdot W \subset \hat{W} \), and replace \( m \) by the least \( \hat{m} \in \mathbb{N} \) such that \( \hat{m}(B, B) \subset \mathbb{Z} \) in the definition of the \( \mathbb{Q}_q, t \).

**Automorphisms.** The following duality anti-involution is of key importance for the various duality statements:
\[
(2.10) \quad \phi : \ X_b \mapsto Y_b^{-1}, \ T_i \mapsto T_i \ (1 \leq i \leq n),
\]
It preserves \( q, t, \) and their fractional powers.

We will also need the automorphisms of \( \mathcal{H}^\flat \) (see [C2], [C3]):
\[
(2.11) \quad \tau_+: \ X_b \mapsto X_b, \ Y_r \mapsto X_r Y_r q^{(\omega_r, \omega_r)/(2m)}, \ \pi_r \mapsto q^{-(\omega_r, \omega_r)} X_r \pi_r,
\]
\[
(2.12) \quad \tau_- \overset{\text{def}}{=} \phi \tau_+ \phi, \ \sigma \overset{\text{def}}{=} \tau_+ \tau_- \tau_+ = \tau_- \tau_+ \tau_-,
\]
where \( r \in O' \). They fix \( T_i \ (i \geq 1) \), \( t, q \) and fractional powers of \( t, q \).

Note that \( \tau_- = \sigma \tau_+ \sigma^{-1} \).

In the definition of \( \tau_\pm \) and \( \sigma \), we need to add \( q^{\pm 1/(2m)} \) to \( \mathbb{Q}_q, t \).

The automorphism \( \tau_- \) acts trivially on \( \{ T_i \ (i \geq 0), \ \pi_r, Y_b \} \). Hence it naturally acts in the polynomial representation \( V \). The automorphism \( \tau_+ \) and therefore \( \sigma \) do not act in \( V \). The automorphism \( \sigma \) sends \( X_b \) to \( Y_b^{-1} \) and is associated with the Fourier transform in the DAHA theory.

Actually, all these automorphisms act in the central extension of the **elliptic braid group** defined by the relations of \( \mathcal{H} \), where the quadratic relation is dropped. The central extension is by the fractional powers of \( q \).

The elements \( \tau_\pm \) generate the projective \( PSL(2, \mathbb{Z}) \), which is isomorphic to the braid group \( B_3 \) due to Steinberg.
3. Macdonald polynomials

This definition is due to Macdonald (for \( k_{sht} = k_{	ext{ing}} \in \mathbb{Z}_+ \)), who extended in [M] Opdam’s nonsymmetric polynomials introduced in the differential case in [O1] (Opdam mentions Heckman’s contribution in [O1]). The general case was considered in [C2].

We continue using the same notation \( X, Y, T \) for these operators acting in the polynomial representation. The parameters \( q, t \) are generic in the following definition.

**Definition 3.1.** The nonsymmetric Macdonald polynomials \( \{ E_b, b \in P \} \) are unique (up to proportionality) eigenfunctions of the operators \( \{ L_f \} \) acting in \( \mathbb{Q}[X] \):

\[
L_f(E_b) = f(q^{-b_1})E_b, \quad \text{where} \quad b_1 \overset{\text{def}}{=} b - u_b^{-1}(\rho),
\]

where \( u_b \) is from Proposition [1.1].

They satisfy

\[
E_b - X_b \in \bigoplus_{c > b} \mathbb{Q}(q,t)X_c, \quad \langle E_b, X_c \rangle_0 = 0 \quad \text{for} \quad P \ni c > b,
\]

where we set \( c > b \) if

\[
c_+ - b_+ \in B \cap Q_+ \quad \text{or} \quad c_+ = b_+ \quad \text{and} \quad c - b \in B \cap Q_+.
\]

The following **intertwiners** are the key in the theory:

\[
\Psi_i = \tau_+(T_i) + \frac{t_i^{1/2} - t_i^{-1/2}}{Y_{\alpha_i} - 1}, \quad i \geq 0, \quad P_r = \tau_+(\pi_r), \quad r \in O',
\]

\[
\Psi_{\tilde{w}} = P_r \Psi_i \cdots \Psi_i \text{ for reduced decompositions } \tilde{w} = \pi_r s_{i_1} \cdots s_{i_l}.
\]

Note the formulas

\[
\tau_+(T_0) = X_0^{-1}T_0^{-1}, \quad X_0 = qX_0^{-1}, \quad \tau_+(\pi_r) = q^{-(\omega_r, \omega_r)/2}X_r \pi_r.
\]

The products \( \Psi_{\tilde{w}} \) do not depend on the choice of the reduced decomposition, intertwine \( Y_b \), and transform the \( E \)-polynomials correspondingly. Namely, for \( \tilde{w} \in \tilde{W} \),

\[
\Psi_{\tilde{w}}Y_b = Y_{\tilde{w}(b)}\Psi_{\tilde{w}}, \quad \text{where} \quad Y_{[b,j]} \overset{\text{def}}{=} Y_b q^{-j},
\]

\[
E_b = \text{Const } \Psi_{\tilde{w}}(E_c) \quad \text{for} \quad \text{Const } \neq 0, \quad b = \tilde{w}(c),
\]

provided that \( \pi_b = \tilde{w} \pi_c \) and \( l(\pi_b) = l(\tilde{w}) + l(\pi_c) \).
Here we use the affine action of $\hat{W}$ on $z \in \mathbb{R}^n$:

$$\begin{align*}
(wb)(z) &= w(b + z), \ w \in W, b \in P, \\
s_\alpha(z) &= z - ((z, \alpha) + j)\alpha, \ \alpha = [\alpha, \nu, j] \in \tilde{R}.
\end{align*}$$

The definition of the $E$-polynomials and the action of the intertwiners are compatible with the transfer to the intermediate subalgebras $\mathcal{H}^b$. Recall that the $B$-polynomial representation is

$$V = \mathbb{Q}_{q,t}[X_b] \overset{\text{def}}{=} \mathbb{Q}_{q,t}[X_b, b \in B].$$

We note that the $\Psi$-intertwiners were introduced by Knop and Sahi in the case of $GL_n$.

The coefficients of the Macdonald polynomials are rational functions in terms of $q^{(\nu)}, t^{(\nu)}$. The following evaluation formula holds:

$$E_b(q^{-\rho_k}) = q^{(\rho_k, b)} \prod_{[\alpha, j] \in \lambda'(\pi_b)} \frac{1 - q^{t(j)}X_{\alpha}(q^{\rho_k})}{1 - q^{a_j}X_{\alpha}(q^{\rho_k})},$$

(3.19) $\lambda'(\pi_b) = \{[\alpha, j] \mid [\alpha, \nu, j] \in \lambda(\pi_b)\}.$

Explicitly, (see (1.8)),

$$\lambda'(\pi_b) = \{[\alpha, j] \mid \alpha \in R_+, \quad (b_-, \alpha') > j > 0 \text{ if } u^{-1}(\alpha) \in R_-, \quad (b_-, \alpha') \geq j > 0 \text{ if } u^{-1}(\alpha) \in R_+\}.$$

Formula (3.19) is the Macdonald evaluation conjecture in the non-symmetric variant from [C2]. Note that one has to consider only long $\alpha$ (resp., short) if $k_{\text{sh.t}} = 0$ (resp., $k_{\text{lng}} = 0$) in the $\lambda$-set.

We have the following duality formula for $b, c \in P$:

$$E_b(q^{e})E_c(q^{-\rho_k}) = E_c(q^{h})E_b(q^{-\rho_k}), \ b_z = b - u^{-1}(\rho_k).$$

(3.22) See [C2]. The proof is based on the anti-involution $\phi$ from (2.10).

The action of $\tau_-$. The automorphism $\tau_+$ is a formal conjugation by the Gaussian $\gamma(q^z) = q^{(z,z)/2}$ where we set $X_b(q^z) = q^{(b,z)}$. We treat $\gamma$ as an element in a completion of the polynomial representation with
the extended action of $\mathcal{H}$. Actually, only the $W$-invariance of $\gamma$ and the relations

$$\omega_j(\gamma) = q^{(\omega_i, \omega_j)/2} X_i^{-1} \gamma$$

for $j = 1, \ldots, n$ are needed here. For instance, one can (formally) take

$$\gamma_x \overset{\text{def}}{=} \sum_{b \in B} q^{-(b, b)/2} X_b.$$  \hfill (3.23)

Applying $\sigma$ and using that $\tau^{-1} = \sigma \tau^{-1} \sigma^{-1}$, we obtain that the automorphism $\tau^{-1}$ in $V$ is proportional to the multiplication by

$$\gamma_y \overset{\text{def}}{=} \sum_{b \in B} q^{(b, b)/2} Y_b$$  \hfill (3.24)

provided that $|q| < 1$. We use that $V$ is a union of finite dimensional spaces preserved by the $Y$-operators. This observation is convenient, although not absolutely necessary, to check the following proposition.

**Proposition 3.2.**

i) For generic $q, t$ or for any $q, t$ provided that the polynomial $E_b$ for $b \in B$ is well-defined,

$$\tau_-(E_b) = q^{-(b, b)/2} - (b, \rho_k) E_b \quad \text{for} \quad P_- \ni b_- \in W(b).$$  \hfill (3.25)

ii) For arbitrary $q, t,$

$$\tau_-(T_i) = T_i, \quad \tau_-(\Psi_i) = \Psi_i \quad \text{for} \quad i > 0,$$

$$\tau(\tau_+(\pi_r)) = q^{(\omega_r, \omega_r)/2} Y_r \tau_+(\pi_r), \quad \tau_-(\tau_+(T_0)) = \tau_+(T_0)^{-1} Y_0,$$

$$\tau_-(\Psi_0) = \Psi_0 Y_0 = Y_0^{-1} \Psi_0, \quad Y_0 = q^{-1} Y_0^{-1}.$$

iii) If $q$ is not a root of unity and $t_\nu$ are arbitrary, then $\tau_-$ preserves any $Y$-submodule of $V$.

**Proof.** The first two claims are straightforward. As for (iii), since $q$ is generic one can assume that $0 < q < 1$ and define $\tilde{\tau}_-$ as the operator of multiplication by $C^{-1} \gamma_y$ using (3.24) and taking

$$C = \sum_{b \in B} q^{(b, b)/2} Y_b(1) = \sum_{b \in B} q^{(b, b)/2} q^{(b, \rho_k)}.$$  

Then $\tilde{\tau}_-$ coincides with $\tau_-$ for generic $k$, when all $E$-polynomials exist and the $X$-spectrum of $V$ is simple, due to (i). This gives the coincidence for any $k$. \qed
4. The radical

Following [C1, C2], we set

\[(4.1) \quad \{f, g\} = \{L_{\iota(f)}(g(X))\} \quad \text{for} \quad f, g \in \mathcal{V},\]

\[\iota(X_b) = X_{-b} = X^{-1}, \quad \iota(z) = z \quad \text{for} \quad z \in \mathbb{Q}_{q,t},\]

where \(L_f\) is from Definition 3.1. It induces the \(\mathbb{Q}_{q,t}\)-linear anti-involution \(\phi\) of \(\mathcal{H}_t^b\) from (2.10).

**Lemma 4.1.** For arbitrary nonzero \(q, t_{shl}, t_{lsh}\),

\[(4.2) \quad \{f, g\} = \{g, f\} \quad \text{and} \quad \{H(f), g\} = \{f, H^\phi(g)\}, \quad H \in \mathcal{H}_t^b.\]

The quotient \(\mathcal{V}'\) of \(\mathcal{V}\) by the radical \(\text{Rad} \overset{\text{def}}{=} \text{Rad}\{,\,\}\) of the pairing \(\{,\,\}\) is an \(\mathcal{H}_t^b\)-module such that

a) all \(Y\)-eigenspaces of \(\mathcal{V}'\) are zero or one-dimensional,

b) \(E(q^{-\rho k}) \neq 0\) if the image \(E'\) of \(E\) in \(\mathcal{V}'\) is a nonzero \(Y\)-eigenvector.

The radical \(\text{Rad}\) is the greatest \(\mathcal{H}_t^b\)-submodule in the kernel of the map \(f \mapsto \{f, 1\} = f(q^{-\rho k})\).

**Proof.** Formulas (4.2) are from Theorem 2.2 of [C2]. Concerning the rest, let us recall the argument from [C3]. Since \(\text{Rad}\{,\,\}\) is a submodule, the form \(\{,\,\}\) is well defined and nondegenerate on \(\mathcal{V}'\). For any pullback \(E \in \mathcal{V}\) of \(E' \in \mathcal{V}'\), \(E(q^{-\rho k}) = \{E, 1\} = \{E', 1'\}\). If \(E'\) is a \(Y\)-eigenvector in \(\mathcal{V}'\) and \(E(q^{-\rho k})\) vanishes, then

\[\{Q_{q,t}[Y_b](E'), \mathcal{H}_t^b(1')\} = 0 = \{E', \mathcal{V} \cdot \mathcal{H}_t^b(1')\}\]

Therefore \(\{E', \mathcal{V}'\} = 0\), which is impossible. \(\Box\)

In the following lemma, \(q\) is generic, but \(t_{\nu}\) are not supposed generic. The Macdonald polynomials \(E_b\) always exist for \(b = b^o\), satisfying the conditions

\[(4.3) \quad q^{-a_i} \neq q^{-b^o_i} \quad \text{for all} \quad a \succ b^o.\]

We call such \(b^o\) primary. Sufficiently big \(b\) are primary.

**Lemma 4.2.** i) A \(Y\)-eigenvector \(E \in \mathcal{V}\) belongs to \(\text{Rad}\) if and only if \(E(q^{-\rho k}) = 0\). The equality \(E(q^{-\rho k}) = 0\) automatically results in the equalities

\[(4.4) \quad E(q^{-b^o_i}) = 0 \quad \text{for all} \quad b^o \in B^\times \overset{\text{def}}{=} \{b^o \in B \mid E_{b^o}(q^{-\rho k}) \neq 0\}.\]
ll) Let us assume that the radical is nonzero. Then for any constant \( C > 0 \) (1 \( \leq \) \( i \) \( \leq \) \( n \)), there exists primary \( b^o \) such that \( (\alpha_i, b^o) > C \) and \( E_{b^o}(q^{-\rho_k}) = 0 \), i.e., \( E_{b^o} \in \text{Rad} \).

Proof. The first claim follows from Lemma 4.1. If \( E \in \text{Rad} \) and there is no such \( b^o \) for certain \( C \), then the number of common zeros of the translations \( c(E) \) of \( E \) for any number of \( c \in B \) is infinite, which is impossible because the degree of \( E \) is finite. \( \square \)

We come to the following theorem generalizing the description of singular \( k \) from [O2].

**Theorem 4.3.** Assuming that \( q \) is generic, the radical vanishes if and only if \( E_{b^o}(q^{-\rho_k}) \neq 0 \) for all sufficiently big primary \( b^o \), i.e., if the product in the right-hand side of (3.19) is nonzero for all \( b \in B \) with sufficiently big \( (b, \alpha_i) \) for \( i > 0 \). \( \square \)

We can define **quasi-perfect representations** as \( \mathcal{H}^\flat \)-modules which have a nondegenerate form \( \{ , \} \) satisfying (4.2). Then the greatest quasi-perfect quotient of the polynomial representation is \( V/\text{Rad} \). Indeed, any quasi-perfect quotient \( V \) of \( V \) supplies it with a form \( \{ f, g \}_V = \{ f', g' \} \) for the images \( f', g' \) of \( f, g \) in \( V \). Then a proper linear combination \( \{ , \}_o \) of \( \{ , \} \) and \( \{ , \}_V \) will satisfy \( \{ 1, 1 \}_o = 0 \), which immediately makes it zero identically.

5. **The irreducibility**

In this section \( q, t \) are arbitrary nonzero, including roots of unity.

**Theorem 5.1.** i) If the quotient \( V' \) of the polynomial representation \( V \) by the radical \( \text{Rad}\{ , \} \) is finite dimensional and \( \tau_- \)-invariant, then it is an irreducible \( \mathcal{H}^\flat \)-module. The radical is always \( \tau_- \)-invariant if \( q \) is not a root of unity.

ii) At roots of unity, the \( \tau_- \)-invariance holds when the radical is \( \mathcal{H}^\flat \)-generated by linear combinations \( \sum c_b E_b \) (provided that \( E_b \) exist) over \( b \) with coinciding \( q^{-(b, b_-)/2-(b, \rho_k)} \) from (3.25).

Proof. Using \( \phi \tau_+ \phi = \tau_+ \), the relation

\[
\{ \tau_+ f, g \} = \{ f, \tau_- g \} \quad \text{for} \quad f, g \in V'
\]

defines the action of \( \tau_+ \) in \( V' \) and therefore the action of \( \sigma \) there satisfying

\[
\tau_+ \tau_-^{-1} \tau_+ = \sigma = \tau_-^{-1} \tau_+ \tau_-^{-1}.
\]
The pairing \( \{f, g\}_\sigma \overset{\text{def}}{=} \{\sigma f, g\} = \{f, \sigma^{-1}g\} \) corresponds to the anti-
involution \( \varpi = \sigma \cdot \phi = \phi \cdot \sigma^{-1} \) of \( \mathcal{H}^b \), sending
\[
(5.1) \quad \varpi : T_i \mapsto T_i, \pi_r \mapsto \pi_r, Y_b \mapsto Y_b, X_b \mapsto T_{w_0}^{-1}X_{c(b)}T_{w_0}
\]
for \( 0 \leq i \leq n, b \in B \).

It holds in either direction, from \( f \) to \( g \) and the other way round, but
the form \( \{f, g\}_\sigma \), generally speaking, could be non-symmetric. Actually
it is symmetric, but we do not need it for the proof.

Using this non-degenerate pairing, we proceed as follows. Any proper
\( \mathcal{H}^b \)-submodule \( \mathcal{V}' \) of \( \mathcal{V}' \) contains at least one \( Y \)-eigenvector \( e'' \), so
we can assume that \( \mathcal{V}' = \mathcal{H}^b e'' \). The corresponding eigenvalue cannot
coincide with that of 1 thanks to the previous lemma. Therefore
\( \{1', \mathcal{V}' \}_\sigma = 0 \) for the image \( 1' \) of 1 in \( \mathcal{V}' \), and the orthogonal comple-
ment of \( \mathcal{V}' \) in \( \mathcal{V}' \) is a proper \( \mathcal{H}^b \)-submodule of \( \mathcal{V}' \) containing \( 1' \), which
is impossible.

Using Proposition 3.2, we obtain (ii).

Let us check that the pairing \( \{f, g\}_\sigma \) is symmetric. First of all,
\[
\{\tau_+(1'), 1'\} = \{1', \tau_-(1')\} = \{1', 1'\} = 1 \Rightarrow \\
\{1', 1'\}_\sigma = \{\sigma(1'), 1'\} = \{\tau_+(1'), \tau_-(1')\} = \{\tau_+(1'), 1'\} = 1.
\]
Then, \( \{1', f\}_\sigma - \{f, 1'\}_\sigma = \{\sigma(1') - \sigma^{-1}(1'), f\} = \{(1 - \sigma^{-2})(1'), f\}_\sigma \). However \( \sigma^{-2} \)
coincides with \( T_{w_0}^{-1} \) up to proportionality in irreducible
\( \mathcal{H}^b \)-modules where \( \sigma \) acts (see \[C3\]). Thus \( (1 - \sigma^{-2})(1') \) is proportional
to \( 1' \) and must be zero in \( \mathcal{V}' \) due to the calculation above. We obtain
that \( 1' \) is in the radical of the pairing \( \{f, g\}_\sigma - \{g, f\}_\sigma \), which makes
this difference identically zero since \( 1' \) is a generator.

The quotient \( \mathcal{V}' \) is not \( \tau_- \)-invariant if \( q \) is a root of unity and \( k \) are
generic. In this case (see \[C2, C3\]), all \( \mathcal{E}_b \) and \( \mathcal{E}_b = E_b / E_b(q^{-\rho_b}) \) are
well defined. The radical is linearly generated by the differences \( \mathcal{E}_b - \mathcal{E}_c \)
when
\[
u_b = u_c, \quad b_\equiv c_\mod NA \cap B \text{ for } (A, B) = Z, \quad q^N = 1.
\]
The polynomials \( E_b \) and \( \mathcal{E}_b \) are \( \tau_- \)-eigenvectors. Their eigenvalues are
\( q^{-(b_-, b_-)/2} - (b_-, \rho_b) \). Therefore \( \tau_- \) does not preserve the radical.

An example of reducible \( \mathcal{V}' \). For the root system \( B_n(n > 2) \), let
\[
n \geq l > n/2 + 1, \quad r = 2(l - 1), \quad k_{\text{img}} = -\frac{s}{r}, \quad l, s \in N, \quad (s, r) = 1.
\]
We will assume that $k_{\text{sht}}$ is generic.

Then Theorem 4.3 readily gives that the radical is zero. Indeed, the numerator of the formula from (3.19) is nonzero for all $b$ because

$$q^j_\alpha t_\alpha X_\alpha(q^{\rho_k}) = q^{j+k\alpha+(\alpha^\vee,\rho_k)}_\alpha \neq 1 \text{ for any } \alpha \in \mathcal{R}_+, j > 0,$$

and the denominator is nonzero because

$$q^j_\alpha X_\alpha(q^{\rho_k}) = q^{j+(\alpha^\vee,\rho_k)}_\alpha \neq 1 \text{ for any } \alpha \in \mathcal{R}_+, j > 0.$$

We use that $(\alpha^\vee, \rho_k)$ involves $k_{\text{sht}}$ unless $\alpha$ belongs to the root subsystem $A_{n-1}$ formed by $\epsilon_l - \epsilon_m$ in the notation of [B].

Thus all Macdonald polynomials $E_b$ are well-defined and the $Y$-action in $\mathcal{V}$ is semisimple. The semisimplicity results from (5.3).

The following relation holds:

$$q^{j+1}_\alpha X_\alpha(q^{\rho_k}) = q^{j-k\alpha+(\alpha^\vee,\rho_k)}_\alpha = 1 \text{ for } \alpha = \epsilon_l, j = 2(l-1)s$$

in the notation from [B]. Indeed, $(\alpha^\vee, \rho_k) = k_{\text{sht}} + 2(l-1)\text{ling}$. Let

$$\tilde{\alpha}^\bullet = [-\alpha, \nu_\alpha, j] = [-\epsilon_l, 2(l-1)s].$$

Here $\alpha$ is short, so $\nu_\alpha = 1$.

**Proposition 5.2.** The polynomial representation has a proper submodule $\mathcal{V}^\bullet$ which is the linear span of $E_b$ for $b$ such that $\lambda(\pi_b)$ contains $\tilde{\alpha}^\bullet$. The quotient $\mathcal{V}/\mathcal{V}^\bullet$ is irreducible.

**Proof.** This statement follows from the Main Theorem of [C3]. It is easy to check it directly using the intertwiners from (3.17). Indeed, given $b$, the linear span $\sum_{\tilde{w}} \Psi_{\tilde{w}}(E_b)$ is an $\mathcal{H}^\phi$-submodule of $\mathcal{V}$ when all $\tilde{w} \in \tilde{W}$ are taken, not only the ones satisfying $l(\tilde{w}\pi_b) = l(\tilde{w}) + l(\pi_b)$. If $\pi_b$ contains $\tilde{\alpha}^\bullet$ but $\tilde{w}\pi_c$ does not, then $\Psi_{\tilde{w}}(E_b) = 0$ because the product $\Psi_{\tilde{w}} \Psi_{\tilde{\pi}_b}(1)$ can be transformed using the homogeneous Coxeter relations to get the combination

$$\cdots (\tau_+ (T_i) - t^{1/2}_i)(\tau_+(T_i) + t^{-1/2}_i) \cdots (1)$$

somewhere. This combination is identically zero. □
6. A NON-SEMISIMPLE EXAMPLE

Let us consider the case of $A_1$ assuming that $q^{1/2}$ is a primitive $2N$-th root of unity. We set $t = q^k$,

$$B = P = \mathbb{Z}, \quad Q = 2\mathbb{Z}, \quad X = X_{\omega_1}, \quad Y = Y_{\omega_1}, \quad T = T_1.$$ 

Thus the $E$-polynomials will be numbered by integers, and $Y(E_m) = q^{\lambda_m}E_m$ for

$$\lambda_m = -m_2, \quad m_2 \overset{\text{def}}{=} (m + \text{sgn}(m)k)/2, \quad \text{sgn}(0) = -1,$$

provided that $E_m$ exists. The $\lambda_m$ are called weights of $E_m$.

Note that $\pi = sp$ in the polynomial representation $\mathcal{V} = \mathbb{Q}_{q,t}[X, X^{-1}]$ for $s(f(X)) = f(X^{-1})$, $p(f(X)) \overset{\text{def}}{=} f(q^{1/2}X)$. The definition ring is $\mathbb{Q}_{q,t} = \mathbb{Q}[q^{\pm1/4}, t^{\pm1/2}]$, where $q^{1/4}$ is used to introduce of $\tau_\pm$. Otherwise $q^{1/2}$ is sufficient.

We will need the following lemma, which is similar to the considerations from [CO].

Let $\hat{V}_0 = \mathbb{Q}_{q,t}, \hat{V}_1 = \mathbb{Q}_{q,t}X, \ldots,$

$$\hat{V}_m = B_m\hat{V}_m, \quad \hat{V}_{m+1} = A_m\hat{V}_m, \ldots,$$

where $m > 0$, $A_m = q^{m/2}Xm$, $B_m$ is the restriction of the intertwiner $t^{1/2}(T + \frac{t^{1/2} - t^{-1/2}}{Y_{\omega_1} - 1})$ to $\hat{V}_m$ provided that $q^{2\lambda_m} \neq 1$ for $\lambda_m = -m/2 - k/2$.

If $q^{2\lambda_m} = 1$ and the denominator of $B_m$ becomes infinity, then we set $B_m = t^{1/2}T$, $\hat{V}_m = \hat{V}_m + T\hat{V}_m$.

**Lemma 6.1.** i) The space $\hat{V}_{\pm m}$ is one-dimensional or two-dimensional. In the latter case, it is the Jordan 2-block satisfying $(Y - q^{\pm\lambda_m})^2\hat{V}_{\pm m} = \{0\}$. If $\dim\hat{V}_m = 1$ then $\dim\hat{V}_{m+1} = 1$ and the generators are

$$E_m = B_{m-1} \cdots B_1A_0(1), \quad E_{m+1} = A_mE_m.$$ 

If $\dim\hat{V}_m = 2$, then $\dim\hat{V}_{m+1} = 2$ and the $E$-polynomials $E_m, E_{m+1}$ do not exist, although these spaces contain the $E$-polynomials of smaller degree.

ii) Let us assume that either $q^{2\lambda_m} = t$ or $q^{2\lambda_m} = t^{-1}$. Then $\dim\hat{V}_m = 1$ and this space is generated by $E_m$. If $\hat{V}_m$ is one-dimensional then respectively $(T + t^{-1/2})E_m = 0$ or $(T - t^{1/2})E_m = 0$. If $\dim\hat{V}_m = 2$, then respectively

$$(T + t^{-1/2})E_m \quad \text{or} \quad (T - t^{1/2})E_m$$
is nonzero and proportional to the (unique) \( E \) -polynomial which is con-
tained in the space \( \hat{V}_m \).

We are going to apply the lemma to integral \( k \). In the range \( 0 < k < N/2 \), the corresponding perfect representation is \( Y \)-semisimple. Using the reduction modulo \( N \) (see [CO]), it suffices to consider the interval \(-N/2 \leq k < 0\).

**Proposition 6.2.**

i) For integral \( k \) such that \(-N/2 \leq k < 0\), the quotient \( V_{2N+4|k|} \) is an irreducible \( \mathcal{H} \)-module of dimension \( 2N + 4|k| \).

ii) The polynomials \( E_m \) exist and \( E_m(q^{-k/2}) \neq 0 \) for the sequences:

\[
\begin{align*}
m &= \{0, 1, -1, \ldots, -|k| + 1, |k|\}, \\
m &= \{-2|k|, 2|k| + 1, \ldots, -N + 1, N\}, \\
m &= \{-N, N + 1, \ldots, -N - |k| + 1, N + |k|\},
\end{align*}
\]

respectively with \( 2|k|, 2(N - 2|k|), \) and \( 2|k| \) elements. They do not exist for \( 2|k| + 2|k| \) indices

\[
\begin{align*}
m &= \{-|k|, |k| + 1, \ldots, -2|k| + 1, 2|k|\}, \\
m &= \{-N - |k|, N + |k| + 1, \ldots, -N - 2|k| + 1, N + 2|k|\}.
\end{align*}
\]

iii) The \( Y \)-semisimple component of \( V_{2N+4|k|} \) of dimension \( 2N - 4|k| \) is linearly generated by \( E_m \) for

\[
\{m = -2|k|, 2|k| + 1, -2|k| - 1, \ldots, -N + 1, N\}.
\]

The corresponding \( Y \)-weights are

\[
\lambda = \left\{ \frac{|k|}{2}, \frac{-|k| - 1}{2}, \frac{|k| + 1}{2}, \ldots, \frac{N - |k|}{2}, \frac{|k| - N}{2} \right\}.
\]

iv) The rest of \( V_{2N+4|k|} \) is the direct sum of \( 4|k| \) Jordan 2-blocks of the total dimension \( 8|k| \). There are two series of the corresponding (multiple) weights \( \lambda \):

\[
\left\{ \frac{-|k|}{2}, \frac{|k| - 1}{2}, \ldots, \frac{0}{2} \right\}, \left\{ \frac{N - |k|}{2}, \frac{|k| - N - 1}{2}, \ldots, \frac{N - 1}{2}, \frac{-N}{2} \right\}.
\]

**Proof.** We will use the chain of the spaces of generalized eigenvectors

\[
\hat{V}_0 = Q_{q,t}, \hat{V}_1 = Q_{q,t}X, \hat{V}_{-1}, \ldots, \hat{V}_m, \ldots
\]

from Lemma 6.1. Recall that \( m > 0 \). The following holds:
0) the spaces $\hat{V}_{\pm m}$ are all one-dimensional from 0 to $m = |k|$, i.e., in the sequence $V_0, \ldots, V_{|k|+1}, V_{|k|};$
1) the intertwiner $B_m$ becomes infinity at $m = |k|$ ($B_{|k|} = t^{1/2}T$) and $\dim \hat{V}_m = 2$ in the range $|k| < m \leq 2|k|;
2) the intertwiner $B_m$ kills $1 \in \hat{V}_m$ at $m = 2|k|$, and after this $\dim \hat{V}_m = 1$ for $2|k| < m \leq N;
3) B_m$ is proportional to $(T + t^{-1/2})$ at $m = N$, $E_{-N} = X^N + X^{-N}$, and $\dim \hat{V}_m = 1$ as $N < m \leq N + |k|;
4) the intertwiner $B_m$ becomes infinity again at $m = N + |k|$, and afterwards $\dim \hat{V}_m = 2$ when $N + |k| < m \leq N + 2|k|;
5) B_m$ kills $E_{-N}$ at $m = N + 2|k|$, and $B_m(\hat{V}_m)$ is generated by $E_{-N-2|k|}$ of same $Y$-eigenvalue as $E_N$.

Concerning step (5), the polynomials $E_{-N-2|k|}$ and $E_N$ both exist, there evaluations are nonzero, and the difference

$$E = E_N/E_N(q^{-k/2}) - E_{-N-2|k|}/E_{-N-2|k|}(q^{-k/2})$$

belongs to the radical $Rad$, i.e., becomes zero in $V_{2N+4|k|}$.

Note that $(T + t^{1/2})E = 0$, which is important to know to continue the decomposition of $\mathcal{V}$ further. It follows the same lines.

We see that step (5) is the first step which produces no new elements in $V_{2N+4|k|}$. Namely:

$$B_{N+2|k|}(\hat{V}_{N+2|k|}) = \mathbb{Q}_{q,t}E_N \text{ in } V_{2N+4|k|},$$

and we can stop here.

The lemma gives that between (2) and (3), the polynomials $E_m$ exist, their images linearly generate the $Y$-semisimple part of $\mathcal{V}$. It is equivalent to the inequalities $E_m(q^{-\rho k}) \neq 0$ because they have different $Y$-eigenvalues.

Apart from (2)-(3), there will be Jordan 2-blocks with respect to $Y$. Let us check it.

First, we obtain the 2-dimensional irreducible representation of $\mathcal{H}_Y = \langle T, Y, \pi \rangle$ in the corresponding $\hat{V}$-space at step (1). Then we apply invertible intertwiners to this space (the weights will go back) and eventually will obtain the two-dimensional $\hat{V}$-space for the starting weight $\lambda = -|k|/2$. Note that $E_0 = 1$ is not from the $Y$-semisimple component of $V_{2N+4|k|}$. It belongs to a Jordan 2-block.

Second, the intertwiner (2) makes the last space one-dimensional and $Y$-semisimple (the corresponding eigenvalue is simple in $V_{2N+4|k|}$). It
will remain one-dimensional until (3). After step (3), we obtain the Jordan blocks. The steps (4)-(5) are parallel to (1)-(2).

The above consideration readily results in the irreducibility of the module $V_{2N+4|k|}$. Indeed, Lemma $[\square]$ (ii) gives that if a submodule of $V_{2N+4|k|}$ contains at least one simple $Y$-eigenvector then it contains the image of 1 and the whole space. Step (5) guarantees that it is always the case, because we can obtain $E_N$ beginning with an arbitrary $Y$-eigenvector.

The irreducibility and the existence of the projective $PSL(2,\mathbb{Z})$-action in $V_{2N+4|k|}$ also follow from Theorem $[\square]$ (ii) because the radical is generated by $E$ which is a linear combination of the $E$-polynomials with the coinciding $\tau_-$-eigenvalues.

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