A Quantum Mechanical Approach To A System of Self-Gravitating Particles And The Problem Of Gravitational Collapse

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Abstract

By making an intuitive choice for the single-particle density of a system of \( N \) self-gravitating particles, without any source for the radiation of energy, we have been able to calculate the binding energy of the system by treating these particles as fermions. Our expression for the ground state energy of the system shows a dependence of \( N^{7/3} \) on the particle number, which is in agreement with the results obtained by other workers. We also arrive at a compact expression for the radius of a star following which we correctly reproduce the nucleon number to be found in a typical star. Using this value, we obtain the well-known result for the limiting value of the mass, \( M \), of a neutron star \((M \simeq 3.12M_\odot, M_\odot \) being the solar mass) beyond which the black hole formation should take place. Generalizing the present calculation to the case of white dwarfs, we have been able to obtain the so-called Chandrasekhar limit for the mass, \( M_{Ch} \), \((M_{Ch} \simeq 1.44M_\odot)\) below which the stars are expected to go over to the white dwarf state. We reproduce this by introducing a radius, equivalent to Schwarzschild radius, at the interface of the neutron stars and white dwarfs. This is justified by considering the fact that it gives rise to the correct value for the degree of ionization \( \mu_e (\mu_e \approx 2) \) for heavy nuclei.

Subject headings: Self-gravitating particles, Neutron star, Blackhole, White dwarf, Chandrasekhar limit
1. Introduction

It was first pointed out by Chandrasekhar\textsuperscript{1} and then, independently, by Landau\textsuperscript{2}, long back that a degenerate system composed of a large number of self gravitating particles will necessarily undergo gravitational collapse if the particle number exceeds certain critical value. This happens after the stars finish up their nuclear fuel. Soon after this, Chandrasekhar\textsuperscript{3} made the momentous discovery regarding the life history of certain stars, according to which the stars with masses $M$ less than $\approx 1.4M_{\odot}$, $M_{\odot}$ being the solar mass, evolve in the same way as the sun after the nuclear power in their cores gets exhausted. When this happens, they contract to white dwarfs. In a white dwarf star, the assembly of the free electrons within the star, which usually forms a degenerate Fermi gas exerts sufficient outward pressure to counteract the inward gravitational pull. A star like our sun is said to lie on the main sequence of the Hertzsprung-Russel (HR) diagram\textsuperscript{4}, since it has still the source at its core for the generation of energy. In the distant future, it is also supposed to evolve to become a red giant and then finally to a white dwarf. Coming to the case of stars having masses less than about three times the solar mass, they may condense even more as they collapse such that their density becomes comparable to that of the nucleons inside the atomic nuclei. At this stage, the electrons and protons react by inverse $\beta$-decay and form neutrons. This is how the neutron stars are formed. These are the compact objects having a dominance of neutrons in their interiors. As such, in them, the outward pressure arises from the degenerate neutrons. Lastly, one comes accross the most interesting case of stars that are having masses more than three times the solar mass. In such cases, the collapse is complete and they lead to the formation of the so called black holes. As the name implies, the black holes trap light and material particles falling on them and also prevent these from getting out of them. This is due to the fact that gravity is very strong inside the black holes. Mathematically, when the
radius of a neutron star becomes less than a certain limit called the Schwarzschild radius 
\[ R_s = \left( \frac{2GM}{c^2} \right) \], \( M \) being the mass of the star, \( G \) Universal Gravitational Constant and \( c \), the velocity of light, then only it can become a black hole. Since, it is the gravitational attraction among the particles in a gravitating system which makes it to collapse, this amounts to an enormous increase in density and the temperature at its central region. For a star becoming a black hole (\( M \geq 3M_\odot \)), the whole star enters the horizon and ends up as a singularity at the centre. That is, the centre of a black hole is considered to be a mathematical singularity where the matter is supposed to have infinite density.

It had been long since pointed out by Fisher and Ruelle\(^5\) that to have a rigorous treatment of an infinite system of \( N \) interacting particles using statistical mechanics, it is necessary that the relevant forces must be of a saturating character. In that case, the total energy of the finite system which is to be an extensive quantity ought to possess a lower bound. That is, it is to be proportional to the number of particle in the system. If, on the other hand, the forces are not of saturating character, then the binding energy per particle increases indefinitely with the number of particles \( N \), so that it obviously becomes impossible to define the usual thermodynamic variables for such infinite systems. In a pioneering work Fisher and Reulle\(^5\) have given the general criteria to describe the saturation property for systems governed by not too singular forces. An example of a non-saturating force is the well known gravitational force which happens to be so, because of its long range nature and attractive character. Taking all the facts into consideration, Levy-Leblond\(^6\) has succeeded in deriving both an upper and a lower bound for the ground state energy of a nonrelativistic quantum mechanical system of \( N \) particles interacting through gravitational forces. By treating these particles as bosons he has shown that the binding energy per particle goes as \( N^2 \), whereas for a system of fermions, it varies as \( N^{4/3} \). However, by extending this approach to a system consisting of \( N \) negative light
fermions with mass $m$ and $N$ positive heavy fermions with mass $m_p$, ($m_p \gg m$) and treating the entire system semirelativistically, the author\textsuperscript{6} has found that the binding energy of the system also increases faster than $N$. It is further shown by him that above a critical number of particles, $N_r$, the Hamiltonian is no longer bounded from below and the system faces an unescapable collapse. As an illustration of this calculation to the case of white dwarf stars, he finds that the pressure of the degenerate electron gas cannot balance the gravitational pull if the total number of particles in a star is greater than the number $N_r = A \left( \frac{2\hbar c}{G m^2} \right)^{3/2}$, where $A$ is some numerical coefficient which is to be adjusted taking into consideration of the physics of the problem. The limiting value of the mass of the star $M_r$ ($M_r = N_r m_n$, $m_n$ being the neutron mass) is being identified with the so-called Chandrasekhar limit. Unfortunately, this very approach of Levy-Leblond\textsuperscript{6} cannot be generalized to the case when heavy particles, such as neutrons, alone form a degenerate Fermi gas. Because, in that case, this would need a full relativistic treatment of the gravitational interaction. In a latter work, Ruffini and Bonazzola\textsuperscript{7}, without using the equation of state approach, could succeed in doing a full relativistic calculation of the binding energy of a system of $N$ self-gravitating particles each of mass $m_p$, $m_p$ being very heavy, following the self consistent field method based on the general theory of relativity. By this they were able to obtain a critical value for the particle number beyond which instability was found to set in within the system. However, for the number of particles of such a high order of magnitude, it was shown by them that the Newtonian treatment of such a system led to an utter failure.

We, in the present work, have tried to calculate the binding energy of a self-gravitating system of particles by treating them as fermions. As far as the evaluation of the total kinetic energy of the system is concerned, it is done within the Thomas-Fermi approximation (TF)\textsuperscript{8}. The potential energy of the system is being evaluated within the
socalled Hartree approximation. The form of the single-particle density for the system used by us in the present calculation is such that it has a singularity at the origin. Unlike the earlier calculations, the method used here is a nonrelativistic quantum mechanical derivation based on Newtonian mechanics. The most interesting result of the present theory is that it gives rise to a compact expression for the radius of a star, following which we are able to obtain a limiting value for the critical mass of a neutron star in a natural way beyond which the black hole formation takes place. A further generalization of the present work to the case of white dwarfs enables us to derive the socalled Chandrasekhar limit. In sec.2 of this paper we have presented the mathematical formulation of our theory. Sec.3 is devoted to the various situations which lead to the formation of neutron stars, and black holes and the derivation of the well-known Chandrasekhar limit for the white dwarfs. In sec.4 a brief discussion of the results of the present theory is given.

2. Mathematical Formulation of the Theory

The Hamiltonian for the system of \( N \) gravitating particles each of mass \( m \) interacting through a sum of pair-wise gravitational interactions is written as

\[
H = \sum_{i=1}^{N} -\frac{\hbar^2 \nabla_i^2}{2m} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} v(|X_i - X_j|), \tag{1}
\]

where \( v(|X_i - X_j|) = -\frac{g^2}{|X_i - X_j|} \), with \( g^2 = Gm^2 \), \( G \) being Newton’s Universal gravitational constant. Using this, the ground state energy of the system at zero temperature is given as\(^8\)

\[
E_0 = \langle H \rangle = \langle KE \rangle + \langle PE \rangle, \tag{2a}
\]

Assuming the particles to be fermions, the total kinetic energy of the system \( \langle KE \rangle \) has been evaluated within the Thomas-Fermi approximation, whose expression is given
as
\[
<K E> = \left(\frac{3h^2}{10m}\right)(3\pi^2)^{2/3} \int d\vec{X} \rho(\vec{X})^{5/3}, \tag{2b}
\]
and the total potential energy \(<PE>\) is written as
\[
<PE> = -\left(\frac{g^2}{2}\right) \int d\vec{X} d\vec{X}' \frac{1}{|\vec{X} - \vec{X}'|} \rho(\vec{X})\rho(\vec{X}') \tag{2c}
\]
In order to evaluate the above integrals, we assume that all the particles within the system (which are identical in nature) are described by some kind of distribution. The trial single-particle density \(\rho(\vec{X})\) we choose is of the form:
\[
\rho(\vec{X}) = A [\exp(-x^\alpha)]/x^{3\alpha}, \tag{3a}
\]
where \(x = (r/\lambda), r = |\vec{X}|\) and \(A\) is the normalization constant, such that
\[
\int \rho(\vec{X})d\vec{X} = N \tag{3b}
\]
The index ‘\(\alpha\)’ has been adjusted in order to bring the expression for the binding energy to have the correct dependance with the particle number. Besides, the convergence of the integrals is also to be satisfied. As one can notice from above, \(\rho(\vec{X})\) is singular at \(r = 0\). The existence of a singularity in the single-particle density at the origin of the coordinate system need not be unphysical. In case of a black hole it simply means that the centre of a black hole is a mathematical singularity where matter has infinite density. As far as the universe is concerned, a singularity in the particle density at origin is thought to be related to the so called Big Bang theory, which is being assumed to be the most important current theory for the origin of the Universe. There have been a few most important advances in this direction by Hawking and Penrose\(^4\) who have shown that any model of the Universe which has the observed characteristics of approximate homogeneity and isotropy must start from a singularity. Even, Einstein’s General Theory of Relativity...
(GTR) which when applied to cosmology accounts for such an initial singularity of the Universe.

Evaluations of the integrals in Eq.(2) have been made taking a set of values for $\alpha$ like $\alpha = 4, 3, 2, 1$ and $\alpha = \frac{1}{4}, \frac{1}{3}$ and $\frac{1}{2}$. It is to be noted that the value $\alpha = (1/2)$ proves to be the most appropriate choice for the single particle density of the system. This can be seen from the results we shall be discussing later. With help of the above choice for $\rho(r)$, the expression for the total energy, $E_0(\lambda)$, of the system of $N$ self-gravitating particles is obtained as

$$E(\lambda) = \left(\frac{12}{25\pi}\right)\left(\frac{h^2}{m}\right)\left(\frac{3\pi N}{16}\right)^{5/3} \frac{1}{\lambda^2} - \left(\frac{g^2 N^2}{16}\right)\frac{1}{\lambda}$$

(4)

Minimizing this with respect to $\lambda$, it is found that the minimum occurs at

$$\lambda = \lambda_0 \simeq \left(\frac{h^2}{mg^2}\right) \times (2.023764)/N^{1/3}$$

(5)

Evaluating Eq.(4) at $\lambda = \lambda_0$, the total binding energy of the system is found as

$$E_0 \simeq -(0.015442)N^{7/3}(\frac{mg^4}{h^2})$$

(6)

Considering the case of the two-particle system ($N=2$), from Eq.(6), we find

$$E_0 = -(0.077823)(\frac{mg^4}{h^2})$$

This is seen to be quite high compared to the actual binding energy of the two-body system whose value is $(0.25)\ (\frac{mg^4}{h^2})$. Comparing the two results, one should not consider Eq.(6) to be a drawback of the present theory, because it is supposed to be very accurate for very large $N$. Looking at Eq.(6), we find that $E_0$ varies as $N^{7/3}$ where $N$ is the particle number. Such a dependence of the binding energy for the system on $N$ was also found by Levy-Leblond\textsuperscript{6} by assuming the particles to be fermions and looking at the...
distribution of N-points on a cubic lattice. By this, he was able to obtain both an upper and a lower bound for the binding energy of the system which, for large N, were given as

\[-(0.5)N^{7/3}\left(\frac{mg^4}{\hbar^2}\right) \leq E_0 \leq -(0.001055)N^{7/3}\left(\frac{mg^4}{\hbar^2}\right)\]  

(7)

Anyway, comparing our result, as shown in Eq.(6), with Eq.(7), we find that it does not violate the inequalities established by Levy-Leblond.

3. Formation of Compact Objects
3.1. Neutron Stars and Black Holes

Before we go to make an estimate of the critical mass of a neutron star beyond which black hole formation should take place, we have to first know about the radius of a star. It must be noted that the size of any compact object (either an atom or a star) is not well defined in quantum theory. The justification regarding the identification of the radius \( R_0 \) of a star with \( 2\lambda_0 \) follows from the consideration of the so-called tunneling effects used in quantum mechanics. Classically, it is known that a particle has a turning point where the potential energy becomes equal to the total energy. Since the kinetic energy and therefore the velocity are equal to zero at such a point, the classical particle is expected to be turned around or reflected by the potential barrier. For example, considering the case of an electron in the hydrogen atom ground state such classical turning point occurs where the potential \( V(r) = -e^2/r = E_{\text{total}} = -e^2/2a_0 \); that is at \( r = 2a_0 \). Quantum mechanically, the probability distribution \( r^2\rho(r) \) has a non-zero value for \( r > 2a_0 \); that is, the electron has access to the region \( r > 2a_0 \) which is forbidden by classical theory. Such penetration or tunneling into or through the potential energy barriers is typical of quantum theory results. If the electron had a value of \( r > 2a_0 \), then its kinetic energy would have to be negative to satisfy the condition \( E_{\text{total}} = T + V \), with \( V > E_{\text{total}} \). Since negative kinetic energy is physically absurd, \( r = 2a_0 \) is to be identified as the classical
radius. Using the above idea, from the present theory one can easily see that at \( \lambda = 2\lambda_0 \), the potential energy of the system becomes equal to the total energy, thereby proving that the radius of the star \( R_0 = 2\lambda_0 \).

In order that a neutron star, after it finishes up all its nuclear fuel at the centre, would form a black hole, one must have

\[
R_0 \leq R_s, \tag{8}
\]

where

\[
R_s = \left(\frac{2GM}{c^2}\right), \tag{9}
\]
is the Schwarzschild radius\(^4\) of the corresponding black hole. Following Eqs.(5) and (8), one finds that the number of nucleons \( N \) in the star satisfies the inequality

\[
N \geq 1.696758N_1, \tag{10a}
\]

where

\[
N_1 = \left(\frac{\hbar c}{Gm_n^2}\right)^{3/2}, \tag{10b}
\]

\( m_n \) being the mass of a neutron. The equality sign in Eq.(10 a) refers to the critical value, denoted by \( N_{C_1} \), for the number of particles in a neutron star beyond which black hole formation takes place. A numerical estimation of \( N_{C_1} \) gives

\[
N_{C_1} \approx 1.70N_1 \approx 3.73 \times 10^{57} \tag{11}
\]

From Eq.(10 b) it follows that

\[
N_1^{2/3} = \left(\frac{\hbar c}{Gm_n^2}\right) = \left(\frac{\text{Plankmass}}{\text{nucleonmass}}\right)^2 \tag{12}
\]

Looking at the result given in Eq.(11), one finds that this is in fantastic agreement with the well known result for the number of nucleons in a typical star, as estimated earlier\(^{10}\). Using this, one also finds that

\[
M_{C_1} = N_{C_1}m_n \approx 3.122134M_\odot, \tag{13}
\]
where $M_\odot = 2 \times 10^{33}g$, is the mass of the sun. Thus, we find that for neutron stars more massive than $\approx 3M_\odot$, the collapse is complete and these are the stars which lead to the black holes. Now, corresponding to $N = N_{C_1}$, we calculate the radius of a neutron star, which gives

$$R_0 = 2\lambda_0 \leq 3.39352\left(\frac{\hbar}{m_n c}\right)\left(\frac{\hbar c}{Gm_n^2}\right)^{1/2}$$

This is the same result as found earlier by Shapiro and Teukolsky (ST). A numerical estimate of Eq.(14) gives $9.25 \times 10^5cm$ compared to the value of $3 \times 10^5cm$ quoted by ST

3.2. White Dwarfs and The Derivation of the Chandrasekhar limit

In view of the result shown in Eq.(13), it is apparent that if the mass of a star is less than $\approx 3M_\odot$, but not too low, it must remain as a neutron star. At this stage, one is likely to ask, is there any lower bound on the mass of a neutron star? In order to answer this question, we imagine of a radius, denoted by $R'_s$, equivalent to the Schwarzschild radius, upto which the neutron star is likely to exist. Above this $R'_s$, one no longer talks of a neutron star. Rather, one has to speak of a white dwarf, provided the mass of the star is less than the Chandrasekher limit at the time when its nuclear fuel gets exhausted. Mathematically, we write down the expression for the $R'_s$ as

$$R'_s = \frac{2GM}{<\bar{v}^2>}$$

As one can see from above, $R'_s$ has been written in a fashion similar to the Schwarzschild radius except for the fact that the $c^2$ factor in the Schwarzschild radius has been replaced by the average of the velocity square $<\bar{v}^2>$. The quantity $<\bar{v}^2>$ is to be here understood as the escape velocity of a particle from a neutron star. Quantitatively, we
choose $< \vec{v}^2 >$ as

$$< \vec{v}^2 >= c^2 \left( \frac{m_e}{m_n} \right)^\eta,$$  \hspace{1cm} (16)

where the value of the exponent $\eta$ in the above equation is to be adjusted inorder to reproduce the value 2 for the degree of ionization for heavy nuclei. In doing this, the so-called Chandrasekhar limit\textsuperscript{3} for the mass of a white dwarf ($M \approx 1.44 M_\odot$) is obtained in a natural way. In order to show this, we now consider the following inequality,

$$R_s < R_0 < R'_s = \left[ \frac{2GM}{c^2} \right] \left( \frac{m_n}{m_e} \right)^\eta,$$  \hspace{1cm} (17)

Analysing Eq.(17), for the case $R_0 < R'_s$, we obtain

$$N = N_{C_2} \geq 1.696757 \left( \frac{\hbar c}{G m_n^2} \right)^{3/2} \left[ \left( \frac{m_e}{m_n} \right)^{3/4} \right]^\eta \hspace{1cm} (18)$$

Following this, we write

$$M_{C_2} = m_n N_{C_2} \geq 3.126 M_\odot \left[ 3.5613 \times 10^{-3} \right]^\eta \hspace{1cm} (19)$$

Using the above equation, we now go on varying $\eta$. For each value of $\eta$, we try to calculate the degree of ionization $\mu_e$ using the relation\textsuperscript{12}

$$\mu_e^2 = 5.83 \left( \frac{M_\odot}{M_{C_2}} \right) \hspace{1cm} (20)$$

It can be easily seen that only when $\eta = 0.137271$, $\mu_e$ becomes 2.01. For heavy nuclei, it has been known that $\mu_e$, which is being interpreted as the degree of ionization has a value close to 2. Now, corresponding to the above $\eta$, we find that

$$M_{C_2} \simeq 1.44 M_\odot, \hspace{1cm} (21)$$

the well known Chandrasekhar limit\textsuperscript{2}. A further justification regarding our above choice of $R'_s$ is given in sec.4. Thus, the mass of a neutron star happens to be such that
\( M_{Ch} \leq M^{NS} \leq 3.12M_{\odot} \). For a star having masses \( M < M_{Ch} \), the formation of white dwarfs should take place after such a star finishes up all its nuclear energy.

In order to calculate the radius of a white dwarf star, one has to consider the fact that in these stars, the outward pressure is due to the degenerate electrons rather than due to the neutrons as is the case with the neutron stars. Therefore, in white dwarfs, it is this outward electron pressure which is counterbalanced by the inward gravitational pull arising out of the neutrons. While generalizing the present calculation to white dwarfs, we ignore the effect of the gravitational forces between the electrons and electrons and between electrons and neutrons, as these are negligibly small. This is justified considering the fact that the neutron mass is very high compared to the electron mass. Thus the mass \('m'\) that appears in the kinetic energy term in Eq.(1) should now represent the electron mass \( m_e \) and the symbol \( g^2 \) that appears in the interparticle potential term should, as before, be given as \( g^2 = Gm_n^2 \), \( m_n \) being the mass of a neutron. With these modifications, the expression for \( R_{WD}^{0} \) is obtained as

\[
R_{WD}^{0} \simeq \left( \frac{\hbar^2}{Gm_em_n^2} \right) 4.047528/N^{1/3} \tag{22}
\]

\( R_{WD}^{0} \) should be such that its value has to be greater than \( R'_s \). It can be easily verified that for masses \( M \leq 1.44M_{\odot} \), \( R_{WD}^{0} > R'_s \). For \( M = 1.0M_{\odot} \), we have calculated the radius of the white dwarf using Eq.(22). This gives \( R_{WD}^{0} \approx 2.49 \times 10^9 \text{ cm} \), which is in close agreement with the value estimated by others\(^{11}\). For this mass, \( R'_s \approx 0.832 \times 10^6 \text{ cm} \); thus showing that \( R_{WD}^{0} > R'_s \). Using the above value of \( R_{WD}^{0} \), we have estimated the mass density inside a white dwarf of mass \( M = 1.0M_{\odot} \). This gives \( \rho_{WD} \approx 3.1 \times 10^4 \text{ g/cm}^3 \), which is again of the right order of magnitude as reported by others\(^{13}\). Using the above value of \( \rho_{WD} \), the density of particles within a white dwarf star is found to be \( \approx 1.80 \times 10^{28} \text{ cm}^{-3} \). It is because of such a high value for the particle density, the effects of the pauli exclusion principle becomes important in such stars and hence, the matter in such
a state is considered to be quantum mechanically degenerate.

Since $R_{WD}^0$ is also supposed to be larger than $R_s$, this gives rise to the fact that

$$N \geq 1.696757 \left( \frac{\hbar c}{G m_n^2} \right) \left( \frac{m_n}{m_e} \right)^{3/4}$$

(23)

Following this, one obtains

$$R_0 \geq 0.5184 \left( \frac{\hbar}{m_e c} \right) \left( \frac{h c}{G m_n^2} \right)^{1/2}$$

(24)

This is the well known relation as obtained before\textsuperscript{11}. The expression in the right hand side of the above equation when evaluated gives $\approx 2.6 \times 10^8 cm$. For $M = 1.0 M_\odot$, the estimated value of $R_{WD}^0$ actually satisfies the above inequality. Now, consider the case of a neutron star of mass $M = 1.5 M_\odot$. Following Eq.(5), its radius becomes $R_0 = 1.18 \times 10^6$ cm. Using this, the matter density inside such a star is found to be $\rho^{NS} \approx 4.3 \times 10^{14} g/cm^3$. This being of the same order as the mass density within an atomic nucleus, one is justified to call them as neutron stars. In the black hole state ($M \approx 3.2 M_\odot$), the radius of the corresponding neutron star becomes $R_0 = 9.18 \times 10^5 cm$. Its Schwarzschild radius $R_s$ is found to be $3(\frac{M}{M_\odot}) km \approx 9.6 \times 10^5 cm$. Thus, one finds that for a black hole, $R_0 < R_s$. This is what is expected to happen for neutron stars having masses $M \geq 3.12 M_\odot$.

Coming back to the case of a star in the white dwarf stage with a mass $M \approx 1.0 M_\odot$, we have estimated the mean temperature throughout the body of such a star by requiring that the thermal kinetic energy of the star be equal to its gravitational potential energy. Using the present theory, we have calculated the total binding energy of a white dwarf star of mass $M = 1 M_\odot$, using the expression

$$| E_0 | = 0.015442 N^{7/3} \frac{G^2 m_e m_n^4}{\hbar^2},$$

(25)
This gives

\[ | E_0 | \approx 0.67 \times 10^{49} \text{ ergs} \]

Comparing our result with those estimated earlier\textsuperscript{14}, we find that the agreement is extremely good. Leaving aside the details of the composition, as far as the star like the sun is concerned, which is at present a star on the main sequence, it can be considered to resemble with a white dwarf after its nuclear fuel gets exhausted. Using virial theorem, which tells that the sum of the potential energy and twice the kinetic energy of a self-gravitating system is zero\textsuperscript{4}, we obtain

\[ | E_{pot} | \approx 1.34 \times 10^{49} \text{ ergs}, \]

Following Eq.(26), we now calculate the value of the potential energy per gramme, and then equate it with the mean thermal kinetic energy, \( \frac{1}{2} v^2 \), per gramme of a particle (Hydrogen atom) inside the white dwarf. This gives the mean thermal velocity \( v \) of the particle as \( v \approx 1.6 \times 10^3 \text{km/sec} \), and hence, the corresponding mean temperature becomes \( \sim 5.5 \times 10^7 \text{K} \). The central temperature of a white dwarf has to be much more than the above value. As far as the sun is concerned, since its binding is found to be less than that of a white dwarf of the same mass\textsuperscript{14}, it is expected that the mean temperature of the sun has to be less than that of a white dwarf. The same is true for the central temperature which, for the sun, has a value \( \sim 2 \times 10^7 \text{K} \).

4. Discussion

As seen before, in order that a star can go to white dwarf state after its nuclear fuel at the core gets exhausted, it must have a mass less than \( \approx 1.44M_\odot \), the well known Chandrasekhar limit. To arrive at this result, we have introduced a radius \( R'_s \), equivalent to the Schwarzschild radius \( R_s \), such that \( R_0 < R'_s \), where we have defined \( R'_s = \frac{2GM}{\mu^2} \), having \( < \vec{v}^2 > = c^2 (\frac{m_e}{m_n})^\eta \), which is being interpreted as the escape velocity of a particle.
from the surface of the neutron star. In order that the above inequality is to be satisfied, one must have $M < M_{Ch} \approx (1.44)M_{\odot}$, which corresponds to an $\eta = 0.137271$. For such an $\eta$, we reproduce a value for the escape velocity $v(v \approx 0.62c)$ of a particle from the surface of the neutron star which is found to be of the right order of magnitude\textsuperscript{15}. This also gives the correct result for the degree of ionization, $(\mu_e \approx 2)$

To conclude, we, in this work, have succeeded in obtaining a nonrelativistic quantum mechanical derivation for the ground state binding energy of a system of self-gravitating particles by making a suitable choice for the single particle density. This has enabled us to arrive at a compact expression for the radius of astronomical objects like stars. Using the present theory, we have been able to estimate the critical mass of a neutron star beyond which black hole formation takes place. The present derivation of the Chandrasekhar limit for the white dwarf formation is based on introducing a radius equivalent to the Schwarzschild radius at the region of interface between the white dwarfs and neutron stars. From all these successes we feel that our choice for the single particle density of a system of self-gravitating particle is correct. Further investigation on various other properties of the system are under progress.
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