Efficient Adaptive Regret Minimization

Zhou Lu∗† Elad Hazan∗†

August 16, 2022

Abstract

In online convex optimization the player aims to minimize her regret against a fixed comparator over the entire repeated game. Algorithms that minimize standard regret may converge to a fixed decision, which is undesirable in changing or dynamic environments. This motivates the stronger metric of adaptive regret, or the maximum regret over any continuous sub-interval in time.

Existing adaptive regret algorithms suffer from a computational penalty - typically on the order of a multiplicative factor that grows logarithmically in the number of game iterations. In this paper we show how to reduce this computational penalty to be doubly logarithmic in the number of game iterations, and with minimal degradation to the optimal attainable adaptive regret bounds.

1 Introduction

Online convex optimization is a standard framework for iterative decision making that has been extensively studied and applied to numerous learning settings. In this setting, a player iteratively chooses a point from a convex decision set, and receives loss from an adversarially chosen loss function. Her aim is to minimize her regret, or the difference between her accumulated loss and that of the best fixed comparator in hindsight. It was previously argued by [7] that in changing environments, regret is not the correct metric, as it incentivizes static behavior. The alternative proposed in the same paper is adaptive regret - or the maximum regret over any continuous sub-interval in time.

Numerous studies have investigated adaptive regret algorithms for online convex optimization, and at this time tight bounds are known on the adaptive regret in a variety of settings. However, without exception, all previous approaches introduced a significant computational overhead to derive adaptive regret bounds. The technical reasoning is that all previous approaches follows the method of reduction, from regret to adaptive regret via expert algorithms, introduced in [7]. The number of experts required to maintain optimal adaptive regret, was bounded by Θ(log T).

In this work we study the computational complexity of adaptive regret algorithms. We prove that O(log log T) experts are sufficient to obtain near-optimal adaptive regret for online convex optimization. The implication of this result is a decrease in the computational overhead of adaptive regret minimization algorithms from Θ(log T) to O(log log T).

1.1 Summary of Results

Our starting point is the approach of of [7] for minimizing adaptive regret: an experts algorithm is applied such that every expert is a (standard) regret minimizing algorithm, whose starting point in

∗Google AI Princeton
†Princeton University
time differentiates it from the other experts. Instead of restarting an expert every single iteration, previous approaches retain a set of active experts, and update only these.

The fundamental property that we study in this paper is how to maintain this set of active experts. Previous approaches require a set size that is logarithmic in the total number of iterations.

We investigate a trade-off between the regret bound and the number of experts needed. By reducing the number of active experts to $O(\log \log T/\epsilon)$, we give an algorithm with an $\tilde{O}(|I|^{1+\epsilon})$ adaptive regret.

| Algorithm | Regret over $I = [s,t]$ | Computational Overhead |
|-----------|------------------------|------------------------|
| [7]       | $O(\sqrt{T})$          | $\Theta(\log T)$      |
| [3, 9]    | $O(\sqrt{|I|})$        | $\Theta(\log T)$      |
| [2]       | $\tilde{O}(\sqrt{\sum_{t=s}^t \|\nabla_t\|^2})$ | $\Theta(\log T)$      |
| [10]      | $\tilde{O}(\min_H \sqrt{\sum_{t=s}^t \|\nabla_t\|^2_H})$ | $\Theta(\log T)$      |
| This paper | $O(\sqrt{|I|^{1+\epsilon}})$ | $O(\log \log T/\epsilon)$ |

Table 1: Comparison of results. We evaluate the regret performance of the algorithms on any interval $I = [s,t]$, and hide irrelevant parameters.

1.2 Related Works

For an in depth treatment of the framework of online convex optimization including adaptive regret minimization see [6].

Online learning with shifting experts were studied in the seminal work of [8], and later [11]. In this setting, the comparator is allowed to shift $k$ times between the experts, and the regret is no longer with respect to a static expert, but to a $k$-partition of $[1, T]$ in which each segment has its own expert. The algorithm Fixed-Share proposed by [8] is a variant of the Hedge algorithm [5]. On top of the multiplicative updates, it adds a uniform exploration term to avoid the weight of any expert from becoming too small. This provably allows a regret bound that tracks the best expert in any interval. [11] improved this method by only mixing only with the past posteriors instead of all experts.

The optimal bounds for shifting experts apply to high dimensional continuous sets and structured decision problems, they do not necessarily yield efficient algorithms. This is the motivation for adaptive regret algorithms for online convex optimization, which was introduced and advocated for in [7], who gave an algorithm called Follow-the-Leading-History with $O(\log^2 T)$ adaptive regret for strongly convex online convex optimization, based on the construction of experts with exponential look-back. However, their bound on the adaptive regret for general convex cost functions was $O(\sqrt{T \log T})$. Later, [8] followed this idea and generalized adaptive regret to an universal bound for any sub-interval with the same length. They obtained an improved $O(\sqrt{|I| \log T})$ regret bound for any interval $I$. This bound was further improved to $O(\sqrt{|I| \log T})$ by [9] using a coin-betting technique. Recently, [2] achieved a more refined second-order bound $\tilde{O}(\sqrt{\sum_{t \in I} \|\nabla_t\|^2})$, and [10] further improved it to $\tilde{O}(\min_H \sqrt{\sum_{t \in I} \|\nabla_t\|^2_H})$, which matches the regret of Adagrad [4]. However, these algorithms are all based on the initial exponential-lookback technique of [7], and requires $\Theta(\log T)$ experts per round, increasing the computational complexity of the base algorithm in their reduction by this factor.

Besides adaptive regret, other notions of adaptivity have also been considered in online learning. One natural way is to consider dynamic regret introduced by [18], which allows the comparator to be time-varying with a bounded path length. The work of [18] gave an algorithm with an $O(\sqrt{TP})$
dynamic regret bound where $P$ denotes the total sequential distance of the moving predictors, also called the “path length”. Though this bound is already optimal in general, recently there have been works further improving the dynamic regret bound under further assumptions [17,15]. Another line of works explore the relationship between these two concepts, that adaptive regret implies dynamic regret [16], and algorithms that achieve both adaptive and dynamic regrets simultaneously [14].

Related to adaptivity, an important building block in adaptive algorithms to attain tighter bounds are parameter-free online learning initiated in [12]. Later parameter-free methods [11,13,2] attain the optimal $\tilde{O}(G D \sqrt{T})$ regret for online convex optimization without knowing any constants ahead of time, and without the usual logarithmic penalty that is a consequence of the doubling trick.

1.3 Paper Outline
In section 2 we formally define the online convex optimization framework and the basic assumptions we need. In section 3 we present our algorithm and show a simplified analysis that leads to an $\tilde{O}(\frac{1}{k})$ adaptive regret bound with doubly-logarithmic number of experts. We generalize this analysis and give our main theoretical guarantee in Section 4.

2 Setting
We consider the online convex optimization (OCO) problem. At each round $t$ the player $\mathcal{A}$ chooses $x_t \in \mathcal{K}$ where $\mathcal{K} \subset \mathbb{R}^d$ is some convex domain, then the adversary reveals loss function $\ell_t(x)$ and player suffers loss $\ell_t(x_t)$. The goal is to minimize regret:

$$\text{Regret}(\mathcal{A}) = \sum_{t=1}^{T} \ell_t(x_t) - \min_{x \in \mathcal{K}} \sum_{t=1}^{T} \ell_t(x).$$

A more subtle goal is to minimize the regret over different sub-intervals of $[1,T]$ at the same time, corresponding to a potential changing environment, which is captured by the notion of adaptive regret introduced by [7]. [3] extended this notion to depend on the length of sub-intervals, and provided an algorithm that achieves an $\tilde{O}(\sqrt{|I|})$ regret bound for all sub-intervals $I$. In particular, they define strongly adaptive regret as follows:

$$\text{SA-Regret}(\mathcal{A}, k) = \max_{I=[s,t], t-s=k} \left( \sum_{\tau=s}^{t} \ell_{\tau}(x_{\tau}) - \min_{x \in \mathcal{K}} \sum_{\tau=s}^{t} \ell_{\tau}(x) \right).$$

We make the following assumption on the loss $\ell_t$, which is standard in literature [6].

**Assumption 1.** The loss $\ell_t$ is convex, $G$-Lipschitz, non-negative and upper bounded by 1. The domain $\mathcal{K}$ has diameter $D$. For simplicity, we assume $G, D \geq 1$.

3 A More Efficient Adaptive Regret Algorithm

The work of [7] proposed an algorithm achieving an $\tilde{O}(\sqrt{T})$ adaptive regret, by initiating different OCO algorithms at each time step, then treating them as experts and running a multiplicative weight algorithm on top of them. It’s easy to further achieve an $\tilde{O}(\sqrt{|I|})$ adaptive regret bound by using parameter-free OCO algorithms as experts, or using a different approach of setting the $\eta$ in the multiplicative weight algorithm as in [3].
Algorithm 1 Efficient Follow-the-Leading-History (EFLH) - Basic Version

1: Input: OCO algorithm $\mathcal{A}$, active expert set $S_t$
2: Let $\mathcal{A}_t$ be an instance of $\mathcal{A}$ in initialized at time $t$. Initialize: $S_1 = \{1\}$, $w_1^{(1)} = \frac{1}{2}$.
3: Pruning rule: for $k \geq 1$, the lifespan $l_t$ of integer $t = r2^{2k} - 1$ is $2^{2k+1} (= 4$ if $2 \nmid t$), where $2^{2k} - 1 \not\mid r$.

4: for $t = 1, \ldots, T$ do
5: Let $W_t = \sum_{j \in S_t} w_t^{(j)}$.
6: Play $x_t = \sum_{j \in S_t} \frac{w_t^{(j)} x_t^{(j)}}{W_t}$, where $x_t^{(j)}$ is the prediction of $\mathcal{A}_j$.
7: for $j \in S_t$ do
8: \[
     w_{t+1}^{(j)} = w_t^{(j)} \left( 1 + \min \left\{ \frac{1}{2}, \sqrt{\frac{\log T}{l_j}} \right\} (\ell_t(x_t) - \ell_t(x_t^{(j)})) \right)
\]
9: end for
10: Update $S_t$ according to the pruning rule and add $t + 1$ to get $S_{t+1}$. Initialize
\[
     w_{t+1}^{t+1} = \min \left\{ \frac{1}{2}, \sqrt{\frac{\log T}{l_{t+1}}} \right\}
\]
11: end for

However, although these algorithms can achieve a near-optimal $\tilde{O}(\sqrt{|I|})$ adaptive regret, they have to use $\Theta(\log T)$ number of experts per round. Keeping $\Theta(\log T)$ number of experts can be expensive in practice, and it would be very efficient if we can improve it to $O(\log \log T)$. To this end, we propose a more efficient algorithm which achieves vanishing regret and uses only $O(\log \log T)$ number of experts at the same time.

The intuition stems from [7], in which the experts' lifespan is of form $2^k$. By lifespan we denote the length of the interval that this expert is run on (it also chooses its parameters optimally according to its lifespan). This leads to $\Theta(\log T)$ number of active experts per round, and we could potentially improve it to $O(\log \log T)$ if we change the lifespan to be $2^{2k}$. This in turn, might lead to a worse regret bound because there are fewer experts available now, but after solving the recursion formula of regret bounds we found an $\tilde{O}(\sqrt{|I|})$ vanishing regret bound is achievable. The formal regret guarantee is given below.

**Theorem 1.** Given an OCO algorithm $\mathcal{A}$ with regret bound $cGD\sqrt{T}$ for some constant $c \geq 1$, the adaptive regret of Algorithm 1 is upper bounded by $30cGD\sqrt{T|I|^{\frac{1}{2}}}$ for any interval $I \subset [1, T]$. The number of active experts per round is $O(\log \log T)$.

The existence of such OCO algorithms is well-known, for example [18]. The sub-optimal $\tilde{O}(\sqrt{|I|^{\frac{1}{2}}})$ rate can be improved to be closer to the optimal $\tilde{O}(|I|^{\frac{1}{2}})$ rate while still using only $O(\log \log T)$ number of experts. We will discuss this extension in the next section.

### 3.1 Proof of Theorem 1

Without loss of generality we only need to consider intervals with length at least 8, because for any interval with length smaller the claim is apparently true. The proof idea is to derive a recursion of regret bounds, then make an induction on the length of intervals. The key observation is that, due to the double-exponential construction of interval lengths, for any interval $[s, t]$, it’s guaranteed that
Lemma 4. For any interval \( I = [s,t] \), there exists an integer \( i \in [s, t - \sqrt{t-s}/2] \), such that \( A_i \) is alive throughout \([i,t]\).

Proof. Assume \( 2^{2k} \leq t - s \leq 2^{2k+1} \), then \( \sqrt{t-s}/2 \leq 2^{2k-1} \). Notice that \( t \geq 2^{2k} + 1 \). Assume \( r \geq 2 \) is the largest integer such that \( r2^{2k-1} \leq t \), then one of \( i = (r-1)2^{2k-1} \) and \( i = (r-2)2^{2k-1} \) is satisfactory because its lifespan is \( 2^{2k+1} \). When \( r \geq 2 \), one of \( r - 2 \) and \( r - 1 \) is odd and we use that to guarantee the lifespan isn’t strictly larger than \( 2^{2k+1} \).

In fact, Lemma 2 implies an even stronger argument for the coverage of \([t - \sqrt{t-s}/2, t]\), that is \( 2^{2k-1} \leq t - i \leq 2^{2k+1} \), and as a result \( \eta = \frac{1}{\sqrt{2^{2k+1}}} \) is (nearly) optimal for this chosen expert. This property means that we don’t need to tune \( \eta \) optimally for the length \( \sqrt{t-s}/2 \), but only need to tune \( \eta \) with respect to the lifespan of the expert itself. For example, the OGD algorithm \( A_i \) achieves (nearly) optimal regret on \([i, t]\) as well because the optimal learning rate for \([i, t]\) is the same as that for \([i, i + l_i]\) up to a constant factor of 2. To see this, notice that \( l_i \geq t - i \) and \( t - i \geq \frac{t}{4} \).

Lemma 3. \( |S_t| = O(\log \log T) \).

Proof. At any time up to \( T \), there can only be \( O(\log \log T) \) different lifespans sizes by the algorithm definition. Notice that for any \( k \), the number of active experts with lifespan of \( 2^{2k+1} \) is at most 4.

Lemma 3 already proves the efficiency claim of Theorem 1. To bound the regret we make an induction on the length of interval \(|I|\). Let \( 2^{2k} \leq |I| \leq 2^{2k+1} \), we will prove by induction on \(|I|\). We need the following technical lemma.

Lemma 4. For any \( x \geq 1 \), we have that

\[
6x^{\frac{3}{2}} \geq 6(x - x^{\frac{1}{2}}/2)^{\frac{3}{2}} + (x^{\frac{1}{2}}/2)^{\frac{3}{2}}
\]

Proof. Let \( y = (x^{\frac{1}{2}}/2)^{\frac{1}{2}} \), after simplification the above inequality becomes

\[
6x^{\frac{3}{2}} \geq 6(x - x^{\frac{1}{2}}/2)^{\frac{3}{2}} + (x^{\frac{1}{2}}/2)^{\frac{3}{2}}\]

\[\iff 12\sqrt{2}y^3 \geq 6(4y^4 - y^2)^{\frac{3}{2}} + y \]

\[\iff (12\sqrt{2}y^3 - y)^4 \geq 6^4(4y^4 - y^2)^3 \]

\[\iff (12\sqrt{2}y^2 - 1)^4 \geq 1296y^2(4y^2 - 1)^3 \]

\[\iff (62208 - 13824\sqrt{2})y^6 - 13824y^4 + (1296 - 48\sqrt{2})y^2 + 1 \geq 0 \]

The derivative of the LHS is non-negative because \( y \geq 1 \) and \( 62208 - 13824\sqrt{2} \geq 13824 \). This proves the LHS is monotonically increasing in \( y \), and we only need to prove its non-negativity when \( y = 1 \), which can be verified.

We first need to derive a regret bound on the sub-interval \([i, t]\) which is covered by a single expert \( A_i \). The regret on \([i, t]\) can be decomposed as the sum of the expert regret and the multiplicative weight regret to choose that best expert in the interval. The expert regret is upper bounded by \( 2cGD\sqrt{t-i} \) due to the optimality of \( A_i \). The multiplicative weight regret can be upper bounded by \( O(\sqrt{\log T(t-i)}) \) in the same way as \( \text{[3]} \) and we leave the proof to appendix.
Lemma 5. For the i and \(A_i\) chosen in Lemma 2, the regret of Algorithm 1 over the sub-interval \([i, t]\) is upper bounded by 
\[2cGD\sqrt{t-i} + 3\sqrt{\log T(t-i)}\].

Now we have gathered all the pieces we need to prove our induction.

**Base case:** for \(|I| = 1\), the regret is upper bounded by 
\[1 \leq 30cGD\sqrt{\log T} \cdot 1^{\frac{3}{4}}\].

**Induction step:** suppose for any \(|I| < m\) we have the regret bound in the statement of theorem. Consider now 
\[t - s = m\], from Lemma 2 we know there exists an integer \(i \in [s, t - \sqrt{t-s}/2]\), such that \(A_i\) is alive throughout \([i, t]\). Algorithm 1 guarantees an 
\[2cGD\sqrt{t-i} + 3\sqrt{\log T(t-i)} \leq 5cGD\sqrt{\log T(t-i)}^{\frac{1}{2}}\]
regret over \([i, t]\) by Lemma 5, and by induction the regret over \([s, i]\) is upper bounded by 
\[30cGD\sqrt{\log T} \cdot 1^{\frac{3}{4}}\]. By the monotonicity of the function
\[f(y) = 6(x-y)^\frac{3}{4} + \sqrt{y}\] when the variable \(y \geq \sqrt{x}/2\), we reach the desired conclusion by Lemma 4. To see the monotonicity, we use the fact
\[y \geq \sqrt{x}/2\] to see that
\[f'(y) = \frac{1}{2\sqrt{y}} - \frac{9}{2(x-y)^\frac{1}{4}}\]
\[\leq \frac{1}{2\sqrt{y}} - \frac{9}{2x^{\frac{3}{4}}}\]
\[\leq \frac{1}{2\sqrt{y}} - \frac{9}{2\sqrt{2}y}\]
\[\leq 0\]

4 Approaching the Optimal Rate

The basic approach given in the previous section achieves vanishing adaptive regret with only 
\(O(\log \log T)\) number of experts, improving the efficiency of previous works [7, 3]. However, the \(\tilde{O}(|I|^{\frac{1}{2}})\) rate is not satisfying, and it’s desirable to improve it to \(\tilde{O}(|I|^{\frac{1}{2}})\) while preserving the expert efficiency. In this section, we extend the basic version of Algorithm 1 and show how to achieve an \(\tilde{O}(|I|^{\frac{1}{2} + \epsilon})\) adaptive regret bound with 
\(O(\log \log T/\epsilon)\) number of experts.

The intuition stems from the recursion of regret bounds. Suppose the construction of our experts guarantees that for any interval with length \(x\), there exists a sub-interval with length \(\Theta(x^\alpha)\) in the end which is covered by some expert with the same initial time, for some constant \(\alpha \geq 0\). Then similarly, we need to solve the recursion of a regret bound function \(g\) such that
\[g(x) \geq g(x - x^\alpha) + x^{\frac{\alpha}{2}}\]

which approximately gives the solution of \(g(x) = \Theta(x^{1-\frac{\alpha}{2}})\) ([1] and Theorem 1 correspond to \(\alpha = 1, \frac{1}{2}\) respectively). To approach the optimal rate we set \(\alpha = 1 - \epsilon\), giving an \(\tilde{O}(|I|^{\frac{1}{2} + \epsilon})\) regret bound. It’s left to decide which construction guarantees such covering with \(\alpha = 1 - \epsilon\).

Suppose our construction contains experts with lifespan of the form \(f(n)\), then it’s equivalent to require that \(f(n+1)^{1-\epsilon} \sim f(n)\) which is approximately \(f(n+1) \sim f(n)^{1+\epsilon}\). Initializing \(f(1) = 2\), for example, gives an alternative choice of double-exponential lifespan \(2^{(1+\epsilon)k}\).

We also need to slightly modify how we define the experts and the pruning rule, since \(2^{(1+\epsilon)k}\) isn’t necessarily an integer now. Define \(l_k = 2^{(1+\epsilon)k}/2 + 1\), we initialize an expert with lifespan \(4l_k\) for
Lemma 8. The proof is essentially the same as that of Theorem 1, the main new step is to derive a generalized version of Lemma 4.

4.1 Proof of Theorem 6

Algorithm 2 Efficient Follow-the-Leading-History (EFLH) - Full Version

1: Input: OCO algorithm $\mathcal{A}$, active expert set $S_t$, horizon $T$ and constant $\epsilon > 0$.
2: Let $\mathcal{A}_{t,k}$ be an instance of $\mathcal{A}$ initialized at $t$ with lifespan $4l_k = 4[2^{(1+\epsilon)k}/2] + 4$, for $2^{(1+\epsilon)k}/2 \leq T$.
3: Initialize: $S_1 = \{(1,1), (1,2), \ldots\}$, $w_1^{(1,k)} = \min \left\{ \frac{1}{2}, \sqrt{\frac{\log T}{l_k}} \right\}$.
4: for $t = 1, \ldots, T$ do
5: Let $W_t = \sum_{(j,k) \in S_t} w_t^{(j,k)}$.
6: Play $x_t = \sum_{(j,k) \in S_t} \frac{w_t^{(j,k)}}{W_t} x_t^{(j,k)}$, where $x_t^{(j,k)}$ is the prediction of $\mathcal{A}_{(j,k)}$.
7: Perform multiplicative weight update to get $w_{t+1}$. For $(j,k) \in S_t$
\[ w_{t+1}^{(j,k)} = w_t^{(j,k)} \left( 1 + \min \left\{ \frac{1}{2}, \sqrt{\frac{\log T}{l_k}} \right\} \left( \ell_t(x_t) - \ell_t(x_t^{(j,k)}) \right) \right) \]
8: Update $S_t$ according to the pruning rule. Initialize
\[ w_{t+1}^{(t+1,k)} = \min \left\{ \frac{1}{2}, \sqrt{\frac{\log T}{l_k}} \right\} \]
if $(t+1,k)$ is added to $S_{t+1}$ (when $l_k | t - 1$).
9: end for

Every $k$ satisfying $2^{(1+\epsilon)k}/2 \leq T$. Additionally, we initialize an expert with lifespan $4l_k$ at time $t$ if $l_k | (t-1)$. We have the following regret guarantee for Algorithm 2.

Theorem 6. Given an OCO algorithm $\mathcal{A}$ with regret bound $c\text{GD}\sqrt{T}$ for some constant $c \geq 1$, the adaptive regret of Algorithm 2 is upper bounded by $40c\text{GD}\sqrt{\log T | I | 1/2}$ for any interval $I \subset [1, T]$. The number of active experts per round is $O(\log \log T/\epsilon)$.

Remark 7. There are some minor differences between Algorithm 1 and Algorithm 2 due to the nice properties of $2^{2k}$. At each time step there is at most one expert initiated in Algorithm 1, while in Algorithm 2 there could be multiple experts. Besides, Algorithm 2 needs to know $T$ in advance but Algorithm 1 can adapt automatically. These make Algorithm 1 slightly more efficient and adaptive than Algorithm 2.

4.1 Proof of Theorem 6

The proof is essentially the same as that of Theorem 1, the main new step is to derive a generalized version of Lemma 4.

Lemma 8. For any $x \geq 1$, for $\epsilon < \frac{1}{2}$, we have that
\[ 8x^{1+\epsilon} \geq 8(x - x^{1-\epsilon}/2)^{1+\epsilon} + (x/2)^{1+\epsilon} \]

Proof. We would like to upper bound the term $(x - x^{1-\epsilon}/2)^{1+\epsilon}$. Notice that $0 < x^{-\epsilon} < 1$, we have that
\[ (1 - x^{-\epsilon}/2)^{1+\epsilon} = e^{\frac{1+\epsilon}{2} \log(1-x^{-\epsilon}/2)} \leq e^{-\frac{1+\epsilon}{4} x^{-\epsilon}} \leq 1 - \frac{1 + \epsilon}{8} x^{-\epsilon} \]
where the last step follows from $e^{-x} = 1 - \frac{x}{2}$ when $0 < x \leq 1$. The above estimation gives us
\[ x^{1+\epsilon} - (x - x^{1-\epsilon}/2)^{1+\epsilon} \geq \frac{1+\epsilon}{8}x^{1+\epsilon} \] which concludes our proof.
We go through the rest of the proof, and omit details which are the same as Theorem 1. The number of active experts per round is upper bounded by $4 \log_{1+\epsilon} \log_2 T = O(\log \log T/\epsilon)$, since at each time step there are at most 4 active experts with lifespan $4l_k$ for any $k$.

As for the regret bound, similarly we have the following property.

**Lemma 9.** For any interval $I = [s, t]$, there exists an integer $i \in [s, t - (t - s)^{1-\epsilon}/2]$, such that $A_i$ is alive throughout $[i, t]$.

And the choice of $\eta = \sqrt{\frac{1}{l_k}}$ is still optimal for each expert up to a constant factor of 2. A same analysis of Lemma 5 yields the following.

**Lemma 10.** For the $i$ and $A_{(i,j)}$ chosen in Lemma 9, the regret of Algorithm 2 over the sub-interval $[i, t]$ is upper bounded by $2cGD\sqrt{t-i} + 3\sqrt{\log T(t-i)}$.

We proceed to state our induction on $|I|$.  

**Base case:** for $|I| = 1$, the regret is upper bounded by $1 \leq 40cGD\sqrt{\log T} \cdot 1^{1+\epsilon}$.

**Induction step:** suppose for any $|I| < m$ we have the regret bound in the statement of theorem. Consider now $t - s = m$, from Lemma 9 we know there exists an integer $i \in [s, t - (t - s)^{1-\epsilon}/2]$ and $k$ satisfying $l_k \leq (t - s)^{1-\epsilon}/2 \leq 4l_k$, such that $A_{(i,k)}$ is alive throughout $[i, t]$. Algorithm 2 guarantees an 

$$2cGD\sqrt{t-i} + 3\sqrt{\log T(t-i)} \leq 5cGD\sqrt{\log T(t-i)}^{\frac{1}{2}}$$

regret over $[i, t]$ by Lemma 10 and by induction the regret over $[s, i]$ is upper bounded by $40cGD\sqrt{\log T(i-s)}^{\frac{1}{2}}$. By the monotonicity of the function $f(y) = 8(x - y)^{\frac{1}{1+\epsilon}} + \sqrt{y}$ when the variable $y \geq x^{1-\epsilon}/2$, we reach the desired conclusion by Lemma 8. To see the monotonicity, we use the fact $y \geq x^{1-\epsilon}/2$ to see that

$$f'(y) = \frac{1}{2\sqrt{y}} - \frac{4(1+\epsilon)}{(x-y)^{\frac{1}{1+\epsilon}}} \leq \frac{1}{2\sqrt{y}} - \frac{4(1+\epsilon)}{x^{\frac{1}{1+\epsilon}}} \leq \frac{1}{2\sqrt{y}} - \frac{4(1+\epsilon)}{\sqrt{2y}} \leq 0$$

5 **Conclusion**

In this paper, we propose more efficient algorithms for adaptive regret minimization. We apply a new construction of experts with double-exponential lifespans $2^{(1+\epsilon)^k}$, then obtain an $\tilde{O}(|I|^{\frac{1}{1+\epsilon}})$ adaptive regret bound with $O(\log \log T/\epsilon)$ number of experts. Our result characterizes the trade-off between regret and efficiency in minimizing adaptive regret in online learning, showing how to achieve near-optimal adaptive regret bounds with $O(\log \log T)$ number of experts.

**Acknowledgement**

We thank Ohad Shamir, Yuanyu Wan and Xinyi Chen for helpful comments and suggestions.
References

[1] Olivier Bousquet and Manfred K Warmuth. Tracking a small set of experts by mixing past posteriors. Journal of Machine Learning Research, 3(Nov):363–396, 2002.

[2] Ashok Cutkosky. Parameter-free, dynamic, and strongly-adaptive online learning. In International Conference on Machine Learning, pages 2250–2259. PMLR, 2020.

[3] Amit Daniely, Alon Gonen, and Shai Shalev-Shwartz. Strongly adaptive online learning. In International Conference on Machine Learning, pages 1405–1411. PMLR, 2015.

[4] John Duchi, Elad Hazan, and Yoram Singer. Adaptive subgradient methods for online learning and stochastic optimization. Journal of machine learning research, 12(7), 2011.

[5] Yoav Freund and Robert E Schapire. A decision-theoretic generalization of on-line learning and an application to boosting. Journal of computer and system sciences, 55(1):119–139, 1997.

[6] Elad Hazan et al. Introduction to online convex optimization. Foundations and Trends® in Optimization, 2(3-4):157–325, 2016.

[7] Elad Hazan and Comandur Seshadhri. Efficient learning algorithms for changing environments. In Proceedings of the 26th annual international conference on machine learning, pages 393–400, 2009.

[8] Mark Herbster and Manfred K Warmuth. Tracking the best expert. Machine learning, 32(2):151–178, 1998.

[9] Kwang-Sung Jun, Francesco Orabona, Stephen Wright, and Rebecca Willett. Improved strongly adaptive online learning using coin betting. In Artificial Intelligence and Statistics, pages 943–951. PMLR, 2017.

[10] Zhou Lu, Wenhan Xia, Sanjeev Arora, and Elad Hazan. Adaptive gradient methods with local guarantees. arXiv preprint arXiv:2203.01400, 2022.

[11] Haipeng Luo and Robert E Schapire. Achieving all with no parameters: Adanormalhedge. In Conference on Learning Theory, pages 1286–1304. PMLR, 2015.

[12] H Brendan McMahan and Matthew Streeter. Adaptive bound optimization for online convex optimization. arXiv preprint arXiv:1002.4908, 2010.

[13] Francesco Orabona and Dávid Pál. Coin betting and parameter-free online learning. Advances in Neural Information Processing Systems, 29, 2016.

[14] Lijun Zhang, Shiyin Lu, and Tianbao Yang. Minimizing dynamic regret and adaptive regret simultaneously. In International Conference on Artificial Intelligence and Statistics, pages 309–319. PMLR, 2020.

[15] Lijun Zhang, Tianbao Yang, Jinfeng Yi, Rong Jin, and Zhi-Hua Zhou. Improved dynamic regret for non-degenerate functions. Advances in Neural Information Processing Systems, 30, 2017.

[16] Lijun Zhang, Tianbao Yang, Zhi-Hua Zhou, et al. Dynamic regret of strongly adaptive methods. In International conference on machine learning, pages 5882–5891. PMLR, 2018.

[17] Peng Zhao, Yu-Jie Zhang, Lijun Zhang, and Zhi-Hua Zhou. Dynamic regret of convex and smooth functions. Advances in Neural Information Processing Systems, 33:12510–12520, 2020.
[18] Martin Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. In Proceedings of the 20th international conference on machine learning (icml-03), pages 928–936, 2003.
A Proof of Lemma 5

Proof. The expert regret is upper bounded by $2cGD\sqrt{t-i}$ due to the optimality of $A_i$, and the choice of $\eta$ is optimal up to a constant factor of 2. We only need to upper bound the regret of the multiplicative weight algorithm. We focus on the case that $\sqrt{\frac{\log T}{T}} \leq \frac{1}{2}$, because in the other case the length $t-i$ of the sub-interval is $O(\log T)$, and its regret is upper bounded by $t-i = O(\sqrt{\log(t-i)})$, and the conclusion follows directly.

We define the pseudo weight $\tilde{w}_i^{(j)} = \sqrt{l_j}w_i^{(j)}$ for $i \leq t \leq i+l_i$, and for $t > i+l_i$ we just set $\tilde{w}_t^{(j)} = \tilde{w}_{i+l_i}^{(j)}$. Let $\tilde{W}_t = \sum_{j \in S_t} \tilde{w}_t^{(j)}$, we are going to show the following inequality

$$\tilde{W}_t \leq t$$  \hspace{1cm} (1)

We prove this by induction. For $t = 1$ it follows from the fact that $\tilde{W}_1 = 1$. Now we assume it holds for all $t' \leq t$. We have

$$\tilde{W}_{t+1} = \sum_{j \in S_{t+1}} \tilde{w}_{t+1}^{(j)}$$

$$= \tilde{w}_{t+1}^{(t+1)} + \sum_{j \in S_{t+1}, j \leq t} \tilde{w}_{t+1}^{(j)}$$

$$\leq 1 + \sum_{j \in S_{t+1}, j \leq t} \tilde{w}_{t+1}^{(j)}$$

$$= 1 + \sum_{j \in S_{t+1}, j \leq t} \tilde{w}_t^{(j)} \left(1 + \sqrt{\frac{\log T}{l_j}(\ell_t(x_t) - \ell_t(x_t^{(j)}))}\right)$$

$$= 1 + \tilde{W}_t + \sum_{j \in S_t} \tilde{w}_t^{(j)} \sqrt{\frac{\log T}{l_j}(\ell_t(x_t) - \ell_t(x_t^{(j)}))}$$

$$= 1 + \tilde{W}_t + \sum_{j \in S_t} w_t^{(j)} (\ell_t(x_t) - \ell_t(x_t^{(j)}))$$

$$\leq t + 1 + \sum_{j \in S_t} w_t^{(j)} (\ell_t(x_t) - \ell_t(x_t^{(j)}))$$

We further show that $\sum_{j \in S_t} w_t^{(j)} (\ell_t(x_t) - \ell_t(x_t^{(j)})) \leq 0$:

$$\sum_{j \in S_t} w_t^{(j)} (\ell_t(x_t) - \ell_t(x_t^{(j)})) = W_t \sum_{j \in S_t} \frac{w_t^{(j)}}{W_t}(\ell_t(x_t) - \ell_t(x_t^{(j)}))$$

$$= W_t \sum_{j \in S_t} (\ell_t(x_t) - \ell_t(x_t))$$

$$= 0$$

which finishes the proof of induction.

By inequality (1) we have that

$$\tilde{w}_{t+1}^{(i)} \leq \tilde{W}_{t+1} \leq t + 1$$

Taking the logarithm of both sides, we have

$$\log(\tilde{w}_{t+1}^{(i)}) \leq \log(t + 1)$$
Recall the expression

\[ \tilde{w}_{t+1}^{(i)} = \prod_{\tau=i}^{t} \left( 1 + \sqrt{\frac{\log T}{l_i}} (\ell_{\tau}(x_\tau) - \ell_{\tau}(x_\tau)) \right) \]

By using the fact that \( \log(1 + x) \geq x - x^2, \forall x \geq -1/2 \) and

\[ \left| \sqrt{\frac{\log T}{l_i}} (\ell_{\tau}(x_\tau) - \ell_{\tau}(x_\tau)) \right| \leq 1/2 \]

we obtain

\[ \log(\tilde{w}_{t+1}^{(i)}) \geq \sum_{\tau=i}^{t} \sqrt{\frac{\log T}{l_i}} (\ell_{\tau}(x_\tau) - \ell_{\tau}(x_\tau)) - \sum_{\tau=i}^{t} \left[ \sqrt{\frac{\log T}{l_i}} (\ell_{\tau}(x_\tau) - \ell_{\tau}(x_\tau))^2 \right] \]

\[ \geq \sum_{\tau=i}^{t} \sqrt{\frac{\log T}{l_i}} (\ell_{\tau}(x_\tau) - \ell_{\tau}(x_\tau)) - \frac{\log T}{l_i} (t - i) \]

Combining this with \( \log(\tilde{w}_{t+1}^{(i)}) \leq \log(t + 1) \), we have that

\[ \sum_{\tau=i}^{t} (\ell_{\tau}(x_\tau) - \ell_{\tau}(x_\tau)) \leq \sqrt{\frac{\log T}{l_i}} (t - i) + \sqrt{\frac{l_i}{\log T}} \log(t + 1) \]

Notice that \( \frac{1}{4} l_i \leq t - i \leq l_i \) from Lemma 2 and 1 + 2 = 3, we conclude our proof.