Scalar Polynomial Singularities in Power-Law Spacetimes

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(Dated: September 5, 2018)

Recently, Helliwell and Konkowski (gr-qc/0701149) have examined the quantum “healing” of some classical singularities in certain power-law spacetimes. Here I further examine classical properties of these spacetimes and show that some of them contain naked strong curvature singularities.

I. POWER-LAW SPACETIMES

The purpose of this study is to examine scalar polynomial singularities associated with power-law spacetimes that can be given in the form

$$ds^2 = r^\alpha dt^2 - r^\beta dr^2 - \frac{r^{\gamma}}{C^2} d\theta^2 - r^\delta (dz + A d\theta)^2,$$  \hfill (1)

where $\alpha, \beta, \gamma$ and $\delta$ are (real) constants and $A$ and $C$ are functions of $\theta$ only. Helliwell and Konkowski [1] have recently considered the quantum “healing” of classical singularities in the spacetimes (1) (for $C$ constant and $A = 0$).

An elementary transformation in $r$ can be used to simplify (1) into two cases and in the notation of Helliwell and Konkowski these are: $\alpha = \beta$ for $\alpha \neq \beta + 2$ (Type I) and $\alpha = \beta + 2$ (Type II). The present discussion is organized as follows: first we locate all polynomial singularities in these spacetimes, then we examine the affine distance to the singularities along timelike and null geodesics. Further, we examine details in the $t - r$ subspaces, including the construction of double-null coordinates and finally the focusing conditions associated with the singularities.

II. SINGULARITIES IN TYPE I

Consider the Newman-Penrose tetrad [2]

$$l_a = r^\delta [\sqrt{r^\beta - \delta}, 0, -A, -1],$$ \hfill (2)

$$n_a = \frac{1}{2} [r^{\beta - \delta}, 0, A, 1],$$ \hfill (3)

$$m_a = \frac{1}{\sqrt{2}} [0, -\sqrt{r^\beta}, -i \sqrt{r^\gamma} C, 0]$$ \hfill (4)

and

$$\bar{m}_a = \frac{1}{\sqrt{2}} [0, -\sqrt{r^\beta}, i \sqrt{r^\gamma} C, 0].$$ \hfill (5)

This tetrad generates the Type I spacetime

$$ds^2 = r^\delta (dt^2 - dr^2) - \frac{r^{\gamma}}{C^2} d\theta^2 - r^\delta (dz + A d\theta)^2.$$ \hfill (6)

The non-vanishing tetrad components of the trace-free Ricci tensor are given by

$$\Phi_{00} = \frac{r^\delta (\delta + \gamma - 2) (\beta - \delta)}{8r^\beta r^2},$$ \hfill (7)

$$\Phi_{02} = \frac{\beta \gamma - \delta^2 + \delta \gamma + \beta \delta + 2 \beta + 2 \delta}{16r^\beta r^2},$$ \hfill (8)

where $\alpha, \beta, \gamma$ and $\delta$ are (real) constants and $A$ and $C$ are functions of $\theta$ only.
\[ \Phi_{11} = \frac{-\gamma^2 + \beta \gamma + \beta \delta + 2 \gamma}{16 r^\beta r^2} \]  
(9)

and

\[ \Phi_{22} = \frac{(\beta - \delta)(\delta + \gamma - 2)}{32 r^\beta r^2} . \]  
(10)

The non-vanishing tetrad components of the Weyl tensor are given by

\[ \Psi_{0} = \frac{-r^\delta(-\delta + \gamma + 2)(\beta - \delta)}{8 r^\beta r^2} , \]  
(11)

\[ \Psi_{2} = \frac{-3 \beta \delta + \delta^2 + 3 \beta \gamma + \delta \gamma - 2 \gamma^2 + 4 \gamma - 2 \beta - 2 \delta}{48 r^\beta r^2} \]  
(12)

and

\[ \Psi_{4} = \frac{(-\delta + \gamma + 2)(\beta - \delta)}{32 r^\beta r^\delta r^2} . \]  
(13)

Finally, the Ricci scalar is given by

\[ R = \frac{-\delta \gamma - \delta^2 + 2 \delta + 2 \gamma + 2 \beta - \gamma^2}{2 r^\beta r^2} . \]  
(14)

In general these spacetimes are of Petrov type I but reduce to Petrov type D (and then O) for the following specializations: \( \gamma = \delta \) (\( \beta = \delta \)), \( \gamma = \delta + 2 \) (\( \beta = \delta \)), \( \gamma = \beta \) (\( \beta = \delta, \beta = \delta - 2 \)), \( \gamma = 2 \beta + 2 - \delta \) (\( \beta = \delta, \beta = \delta - 2 \)).

The tetrad components of the trace-free Ricci and Weyl tensors vanish identically in four cases: (i) \( \beta = \gamma = \delta = 0 \), (ii) \( \beta = \delta = 0, \gamma = 2 \), (iii) \( \beta = \gamma = 0, \delta = 2 \) and (iv) \( \beta = -2, \delta = \gamma = -2 \). In cases (i), (ii) and (iii) the spacetime is flat. In case (iv) the Ricci scalar reduces to 12.

Invariants of any spacetime consist of the Ricci scalar, the invariant of lowest degree, and invariants of higher degree constructed from appropriate products of the tetrad components of the trace-free Ricci and Weyl tensors [3] [4]. From (14) it follows that \( R \) diverges like \( 1/r^\beta r^2 \) as \( r \to 0 \) unless

\[ \beta = \gamma^2/2 - \gamma + \delta^2/2 - \delta + \delta \gamma/2 . \]  
(15)

Substitution of \( \beta \) from (15) into the Ricci invariant of next degree shows that the invariant diverges like \( (1/r^\beta r^2)^2 \) as \( r \to 0 \) unless

\[ (\delta - 2 + \gamma)^2(\delta^2 - 2\delta + 4 + \gamma \delta - 2\gamma + \gamma^2) = 0 . \]  
(16)

Substitution of \( \beta \) from (15) and \( \delta = 2 - \gamma \) from (16) into the first Weyl invariant shows that this invariant diverges like \( (1/r^\beta r^2)^2 \) as \( r \to 0 \) unless

\[ \gamma^2(\gamma - 2)^2 = 0 . \]  
(17)

The case \( \gamma = 0 \) gives \( \delta = 2 \) and \( \beta = 0 \), that is, the flat case (iii). The case \( \gamma = 2 \) gives \( \delta = 0 \) and \( \beta = 0 \), that is the flat case (ii). Substitution of \( \beta \) from (15) and \( \delta^2 - 2\delta + 4 + \gamma \delta - 2\gamma + \gamma^2 = 0 \) from (16) into the first Weyl invariant shows that this invariant diverges as \( (1/r^\beta r^2)^2 \) as \( r \to 0 \) for all real \( \gamma \) and \( \delta \). Finally, note that both factors in (16) cannot vanish simultaneously for real \( \gamma \) and \( \delta \).

It follows that for Type I power-law spacetimes, except for the four cases (i) through (iv), the spacetimes contain a scalar polynomial singularity at \( r = 0 \) for all \( \beta + 2 > 0 \).
III. SINGULARITIES IN TYPE II

Now consider the Newman-Penrose tetrad consisting of (2) and (3), both with \( \beta \) replaced by \( \beta + 2 \), along with (4) and (5) unchanged. This tetrad generates the Type II spacetime

\[
ds^2 = r^{\beta+2}(dt^2 - \frac{dr^2}{r^2} - \frac{r^\gamma}{C^2}d\theta^2 - r^\delta(dz + A\theta)^2).
\]  

(18)

The non-vanishing tetrad components of the trace-free Ricci tensor are now given by

\[
\Phi_{00} = \frac{r^\delta (\beta - \delta + 2) (\delta + \gamma)}{8r^\beta r^2},
\]

(19)

\[
\Phi_{02} = \frac{\beta \gamma - \delta^2 + \delta \gamma + \beta \delta + 2 \gamma + 2 \delta}{16r^\beta r^2},
\]

(20)

\[
\Phi_{11} = \frac{-\gamma^2 + \beta \gamma + \beta \delta + 2 \gamma + 2 \delta}{16r^\beta r^2}
\]

(21)

and

\[
\Phi_{22} = \frac{(\beta - \delta + 2) (\delta + \gamma)}{32r^\beta r^2}.
\]

(22)

The non-vanishing tetrad components of the Weyl tensor are now given by

\[
\Psi_0 = -\frac{r^\delta (-\delta + \beta + 2)(\gamma - \delta)}{8r^\beta r^2},
\]

(23)

\[
\Psi_2 = -\frac{(\gamma - \delta) (-\delta - 2\gamma + 6 + 3\beta)}{48r^\beta r^2},
\]

(24)

and

\[
\Psi_4 = -\frac{(-\delta + \beta + 2)(\gamma - \delta)}{32r^\beta r^2}.
\]

(25)

Finally, the Ricci scalar is now given by

\[
R = \frac{\delta^2 + \delta \gamma + \gamma^2}{2r^\beta r^2}.
\]

(26)

In general these spacetimes are of Petrov type I but reduce to Petrov type O for \( \gamma = \delta \), and Petrov type D (and then O) for the following specializations: \( \gamma = \beta + 2 \) (\( \beta = \delta - 2 \)), \( \gamma = 2\beta + 4 - \delta \) (\( \beta = \delta - 2 \)).

The tetrad components of the trace-free Ricci and Weyl tensors now vanish identically only for \( \delta = \gamma = 0 \) in which case the spacetime is flat. Except in this case the Ricci scalar itself diverges like \( 1/r^\beta r^2 \) as \( r \to 0 \).

It follows that for Type II power-law spacetimes, except for the flat case \( \delta = \gamma = 0 \), the spacetimes contain a scalar polynomial singularity at \( r = 0 \) for all \( \beta + 2 > 0 \).

IV. AFFINE DISTANCE TO THE SINGULARITIES

Geodesics of the spacetimes (11) satisfy

\[
r^\beta r^2 = \frac{c^2}{r^\alpha} - \frac{r^\gamma \dot{\theta}^2}{C^2} - \frac{\dot{\theta}^2}{r^\delta} - 2\mathcal{L}
\]

(27)
where $2L = 0$ in the null case and $2L = 1$ in the timelike case, $c_1$ and $c_2$ are constants of the motion and $\dot{x} \equiv d/d\lambda$ where $\lambda$ is an affine parameter. It follows from (27) that

$$\lambda_* - \lambda_0 \geq \int_0^{r_*} r^{(\alpha + \beta)/2} dr$$

(28)

where $\lambda_*$ and $r_*$ are finite and non-zero. As a result, the singularities at $r = 0$ are at finite affine distance if and only if

$$\alpha + \beta + 2 > 0.$$  

(29)

Whereas condition (29) offers no further restriction on $\beta$ for singularities in Type II spacetimes (we again have $\beta + 2 > 0$), for Type I spacetimes we obtain the refined condition $\beta + 1 > 0$ for singularities, assuming, as usual, that they must be at a finite affine distance.

V. THE $t-r$ SUBSPACES $\Sigma$

In what follows we examine in further detail the $t-r$ subspaces (designated by $\Sigma$) of the spacetimes (1). Since we are primarily interested in the structure of the singularities at $r = 0$ we first explore the conditions under which null geodesics can terminate there. Now transforming from $\theta$ to $\tilde{\theta}$ where $d\theta/C(\theta) = d\tilde{\theta}$ we have a constant of the motion $c_3$ and the null geodesic equation can be given as

$$r^\beta \dot{r}^2 = \frac{1}{r^\alpha} - \frac{c_2^2}{r^\delta} - \frac{c_3^2}{r^\gamma}$$

(30)

where, without loss in generality, we have set $c_1^2 = 1$. There are four cases of interest: (i) If $c_2 = c_3 = 0$ then clearly there are no turning points $\dot{r} = 0$ at finite $r$. (ii) If $c_2 = 0, c_3 \neq 0$ then there is a minimum $r$ at $r_0 > 0$ for $\alpha \geq \gamma$. (iii) If $c_2 = 0, c_3 \neq 0$ then there is a minimum $r$ at $r_0 > 0$ for $\alpha \geq \delta$. (iv) If $c_2 \neq 0$ and $c_3 \neq 0$ then there is a minimum $r$ at $r_0$ where

$$\frac{1}{r_0^\alpha} = \frac{c_2^2}{r_0^\delta} + \frac{c_3^2}{r_0^\gamma}$$

(31)

as long as the constants of the motion satisfy

$$c_2^2(\delta - \gamma) \leq (\alpha - \gamma)r_0^{\delta - \alpha}$$

(32)

and

$$c_3^2(\gamma - \delta) \leq (\alpha - \delta)r_0^{\gamma - \alpha}.$$  

(33)

If there are minima in $r$ along null geodesics then a discussion in the $t-r$ subspace of the spacetimes (1) below the minima is adequate for these geodesics in the sense that they do not populate that region of the $t-r$ subspace. However, since there are geodesics for which $c_2$ and $c_3$ do not vanish simultaneously and which terminate at $r = 0$, the $t-r$ subspaces are not totally geodesic and discussions restricted to these subspaces are not complete. However, an examination of these subspaces is, though incomplete, very instructive. We now construct double-null coordinates and examine focusing conditions.

A. Double Null Coordinates

It is clear from (4) and (18) that the trajectories $r = r_0 = constant > 0$ are timelike for $r > 0$. However, the nature of the singularities at $r = 0$ is not clear. Here we construct double-null coordinates to clarify the nature of the singularities. By double-null coordinates we mean coordinates which label affinely parameterized null geodesics of the full spacetime. The associated spacetime diagrams that we draw are, however, restricted to the $t-r$ subspaces.
1. Type I

On $\Sigma$ we have

$$ds^2_\Sigma = r^\beta (dt - dr)(dt + dr). \quad (34)$$

Introducing null coordinates $u$ and $v$ where $u = \text{constant}$ along $dt = dr$ and $v = \text{constant}$ along $dt = -dr$ and writing $r = f(u,v)$ and $t = g(u,v)$ it follows that

$$f(u,v) = g(u,v) + F(u) = -g(u,v) + H(v). \quad (35)$$

With the choices

$$F(u) = 2u, \quad H(v) = 2v, \quad (36)$$

we have

$$r = u + v, \quad t = v - u, \quad (37)$$

so that

$$ds^2_\Sigma = -4(u + v)\beta dudv \quad (38)$$

where we have orientated the future so that $dv > 0$ but $du < 0$. We have the following tangents to null geodesics in the full spacetime subject to the transformations:\[37\]: along $v = v_0 = \text{constant}$

$$k^a = -\frac{\delta^a_u}{(u + v_0)^\beta} \quad (39)$$

so that $k^b \nabla_b k^a = 0$ and along $u = u_0 = \text{constant}$

$$l^a = \frac{\delta^a_v}{(v + u_0)^\beta} \quad (40)$$

so that $l^b \nabla_b l^a = 0$. As a result, the singularities at $r = 0$ are at finite affine distance only for $\beta + 1 > 0$ as discussed above. The trajectories $r = \text{constant}$ are straight vertical lines and timelike throughout the $u-v$ diagram. Trajectories of constant $t$ are straight horizontal lines and spacelike throughout the $u-v$ diagram. A diagram is shown below in Figure 1.

FIG. 1: Representation of the $t-r$ subspace for Type I power-law spacetimes in the double-null coordinates $u$ and $v$. The singularity at $r = 0$ is timelike.
2. Type II

On $\Sigma$ we now have

$$ds^2 = r^{\beta+2}\left(dt - \frac{dr}{r}\right)(dt + \frac{dr}{r}). \quad (41)$$

Introducing null coordinates $u$ and $v$ where now $u = constant$ along $dt = \frac{dr}{r}$ and $v = constant$ along $dt = -\frac{dr}{r}$ and again writing $r = f(u, v)$ and $t = g(u, v)$ it now follows that

$$f(u, v) = F(u)e^{g(u,v)} = H(v)e^{-g(u,v)}. \quad (42)$$

With the choices

$$F(u) = u^2, \quad H(v) = v^2, \quad (43)$$

so that $u$ and $v$ are not simultaneously zero, we have

$$r = uv, \quad t = \ln\left(\frac{v}{u}\right), \quad (44)$$

so that

$$ds^2 = -4(uv)^{\beta+1}dudv \quad (45)$$

where again we have orientated the future so that $dv > 0$ but $du < 0$. We now have tangents to null geodesics in the full spacetime subject to the transformations (44) so that

$$k^a = -\frac{\delta^a_u}{u^{\beta+1}} \quad (46)$$

so that $k^a \nabla_b k^a = 0$ and

$$l^a = \frac{\delta^a_v}{v^{\beta+1}} \quad (47)$$

so that $l^a \nabla_b l^a = 0$. As a result, the singularities at $r = 0$ are at finite affine distance only for $\beta + 2 > 0$ also as shown above. The surfaces $r = constant > 0$ are now hyperbolae in the $u - v$ diagram and timelike. They become the degenerate null hyperbola for $r = 0$. Trajectories of constant $t$ are straight lines through the origin $u = v = 0$ which is a point of internal infinity. A diagram is shown below in Figure 2.

FIG. 2: Representation of the $t - r$ subspace for Type II power-law spacetimes in the double-null coordinates $u$ and $v$. The singularity at $r = 0$ is either past or future null. The origin $u = v = 0$ is a point of internal infinity.
B. Focusing conditions

In this section we consider a unified treatment of both types of spacetime. The vector

\[ k^a = (1/r^\alpha, \pm 1/r^{(\alpha+\beta)/2}, 0, 0) \]  

is tangent to a class of null geodesics of the spacetimes (1) and we use (48) to examine focusing conditions in the \( t-r \) subspaces. Define

\[ \Psi \equiv \lambda^n R_{ab}k^a k^b \]  

where \( R_{ab} \) is the Ricci tensor. We find

\[ \Psi = r^{(n/2-1)(\alpha+\beta+2)} \Delta \]  

where

\[ \Delta = 2\beta(\gamma + \delta) + \gamma(2 - \gamma) + \delta(2 - \delta) \equiv \Delta_I \]  

for Type I and

\[ \Delta = 2\beta(\gamma + \delta) + \gamma(4 - \gamma) + \delta(4 - \delta) \equiv \Delta_{II} \]  

for Type II. Clearly \( \Psi = 0 \) \( \forall \) \( n \) with \( \Delta = 0 \). For \( \Delta \neq 0 \), \( \Psi \rightarrow 0 \) as \( r \rightarrow 0 \) for \( n > 2 \) and \( |\Psi| \rightarrow \infty \) as \( r \rightarrow 0 \) for \( n < 2 \). For \( n = 2 \), \( \Psi = \Delta \) and for \( \Delta > 0 \) the singularities at \( r = 0 \) satisfy the strong curvature condition [6].

VI. CONCLUSIONS

Type I spacetimes can have strong curvature timelike naked singularities at \( r = 0 \) for \( \beta + 1 > 0 \) and \( \Delta_I > 0 \). Type II spacetimes can have strong curvature past-null naked singularities at \( r = 0 \) for \( \beta + 2 > 0 \) and \( \Delta_{II} > 0 \).

Acknowledgments

It is a pleasure to thank Thomas Helliwell and Deborah Konkowski for comments and in particular for suggesting a clarification of the Type II singularities. This work was supported by a grant from the Natural Sciences and Engineering Research Council of Canada and was made possible by use of GRTensorII [7].

[1] T. M. Helliwell and D. A. Konkowski gr-qc/0701149
[2] See, for example, H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, Exact Solutions of Einstein’s Field Equations (Cambridge University Press, Cambridge, 2003) chapter 7.
[3] See, for example, H. Stephani et al. [2] chapter 9.
[4] The Kretschmann scalar (the full contraction of the Riemann tensor) is not used here since it is a linear combination of the square of the Ricci scalar, the first Ricci invariant and the real component of the first Weyl invariant.
[5] We exclude factors which have no non-vanishing real roots.
[6] See, for example, C. J. S. Clarke, The Analysis of Space-Time Singularities (Cambridge University Press, Cambridge, 1993).
[7] This is a package which runs within Maple. It is entirely distinct from packages distributed with Maple and must be obtained independently. The GRTensorII software and documentation is distributed freely on the World-Wide-Web from the address http://grtensor.org