Bayesian nonparametric estimation in the current status continuous mark model

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November 26, 2019

Abstract

In this paper we consider the current status continuous mark model where, if the event takes place before an inspection time $T$ a “continuous mark” variable is observed as well. A Bayesian nonparametric method is introduced for estimating the distribution function of the joint distribution of the event time ($X$) and mark ($Y$). We consider a prior that is obtained by assigning a distribution on heights of cells, where cells are obtained from a partition of the support of the density of $(X,Y)$. As distribution on cell heights we consider both a Dirichlet prior and a prior based on the graph-Laplacian on the specified partition. Our main result shows that under appropriate conditions, the posterior distribution function contracts pointwisely at rate $(n/\log n)^{-\rho}$, where $\rho$ is the Hölder smoothness of the true density. In addition to our theoretical results we provide computational methods for drawing from the posterior using probabilistic programming. The performance of our computational methods is illustrated in two examples.

1 Introduction

1.1 Problem formulation

Survival analysis is concerned with statistical modelling of the time until a particular event occurs. The event may for example be the onset of a disease or failure of equipment. Rather than observing the time of event exactly, censoring is common in practice. If the event time is only observed when it occurs prior to a specific (censoring) time, one speaks of right censoring. In case it is only known whether the event took place before a censoring time or not, one speaks of current status censoring. The resulting data are then called current status data.

In this paper we consider the current status continuous mark model where, if the event takes place before an inspection time $T$, a “continuous mark” variable is observed as well. More specifically, denote the event time by $X$ and the mark by $Y$. Independent of $(X,Y)$, there is an inspection time $T$ with density function $g$ on $[0,\infty)$. Instead of observing each $(X,Y)$ directly, we observe inspection time $T$ together with the information whether the event occurred before time $T$ or not. If it did so, the additional mark random variable $Y$ is also observed, for which we assume $P(Y = 0) = 0$. Hence, an observation of this experiment can be denoted by $W = (T,Z) = (T,\Delta \cdot Y)$ where $\Delta = 1_{\{X \leq T\}}$ (note that, equivalently, $\Delta = 1_{\{Z > 0\}}$). This experiment is repeated $n$ times independently, leading to the observation set $\mathcal{D}_n = \{W_i, i = 1, \ldots, n\}$. We are interested in estimating the joint distribution function $F_0$ of $(X,Y)$ nonparametrically, based on $\mathcal{D}_n$.

An application of this model is the HIV vaccine trial studied by Hudgens, Maathuis & Gilbert (2007). Here, the mark is a specifically defined viral distance that is only observed if a participant to the trial got HIV infected before the moment of inspection.
1.2 Related literature

In this section we review earlier research efforts on models closely related to that considered here.

Survival analysis with a continuous mark can be viewed as the continuous version of the classical competing risks model. In the latter model, failure is due to either of \( K \) competing risks (with \( K \) fixed) leading to a mark value that is of categorical type. As the mark variable encodes the cause of failure it is only observed if failure has occurred before inspection. These “cause events” are known as competing risks. Groeneboom, Maathuis & Wellner (2008) study nonparametric estimation for current status data with competing risks. In that paper, they show that the nonparametric maximum likelihood estimator (NPMLE) is consistent and converges globally and locally at rate \( n^{1/3} \).

Huang & Louis (1998) consider the continuous mark model under right-censoring, which is more informative compared to the current-status case because the exact event time is observed for noncensored data. For the nonparametric maximum likelihood estimator of the joint distribution function of \((X,Y)\) at a fixed point, asymptotic normality is shown.

Hudgens, Maathuis & Gilbert (2007) consider interval censoring case \( k, k = 1 \) being the specific setting of current-status data considered here. In this paper the authors show that both the NPMLE and a newly introduced estimator termed “midpoint imputation MLE” are inconsistent. However, coarsening the mark variable (i.e. making it discrete, turning the setting to that of the competing risks model), leads to a consistent NPMLE. This is in agreement with the results in Maathuis & Wellner (2008).

Groeneboom, Jongbloed & Witte (2011) and Groeneboom, Jongbloed & Witte (2012) consider the exact setting of this paper using frequentist estimation methods. In Groeneboom, Jongbloed & Witte (2011) two plug-in inverse estimators are proposed. They prove that these estimators are consistent and derive the pointwise asymptotic distribution of both estimators. Groeneboom, Jongbloed & Witte (2012) define a nonparametric estimator for the distribution function at a fixed point by finding the maximiser of a smoothed version of the log-likelihood. Pointwise consistency of the estimator is established. In both papers numerical illustrations are included.

1.3 Contribution

In this paper, we consider Bayesian nonparametric estimation of the bivariate distribution function \( F_0 \) in the current status continuous mark model. This approach has not been adopted before, neither from a theoretical nor computational perspective (within the Bayesian setting). Whereas consistent nonparametric estimators exist within frequentist inference, convergence rates are unknown. We prove consistency and derive Bayesian contraction rates for the bivariate distribution function of \((X,Y)\) using a prior on the joint density \( f \) of \((X,Y)\) that is piecewise constant. For the values on the bins we consider two different prior specifications. Our main result shows that under appropriate conditions, the posterior distribution function contracts pointwisely at rate \((n/\log n)^{-\frac{\rho^2}{\rho^2+2}}\), where \( \rho \) is the Hölder smoothness of the true density.

The proof is based on general results from Ghosal & Van der Vaart (2017) for obtaining Bayesian contraction rates. Essentially, it requires the derivation of suitable test functions and proving that the prior puts sufficient mass in a neighbourhood of the “true” bivariate distribution. The latter is proved by exploiting the specific structure of our prior. In addition to our theoretical results, we provide computational methods for drawing from the posterior using probabilistic programming in the Turing Language under Julia (see Bezanson et al. (2017), Ge, Xu & Ghahramani (2018)). The performance of our computational methods is illustrated in two examples.

1.4 Outline

The outline of this paper is as follows. In section 2 we introduce further notation for the current status continuous mark model and detail the two priors considered. Subsequently, we derive posterior
contraction rates under some assumptions on the underlying bivariate distribution in section 3. The proof is given in section 4. Section 5 contains numerical illustrations.

1.5 Notation

For two sequences \( \{a_n\} \) and \( \{b_n\} \) of positive real numbers, the notation \( a_n \lesssim b_n \) (or \( b_n \gtrsim a_n \)) means that there exists a constant \( C > 0 \), independent of \( n \), such that \( a_n \leq C b_n \). We write \( a_n \asymp b_n \) if both \( a_n \lesssim b_n \) and \( a_n \gtrsim b_n \) hold. We denote by \( F \) and \( F_0 \) the cumulative distribution functions corresponding to the probability densities \( f \) and \( f_0 \) respectively. The Hellinger distance between two densities \( f, g \) is written as \( h^2(f, g) = \frac{1}{2} \int (f^{1/2} - g^{1/2})^2 \). The Kullback-Leibler divergence of \( f \) and \( g \) and the \( L_2 \)-norm of \( \log(f/g) \) (under \( f \)) by

\[
KL(f, g) = \int f \log \frac{f}{g}, \quad V(f, g) = \int f \left( \log \frac{f}{g} \right)^2.
\]

2 Likelihood and prior specification

2.1 Likelihood

In this section we derive the likelihood for the joint density \( f \) based on data \( D_n \). As \( W_1, \ldots, W_n \) are independent and identically distributed, it suffices to derive the joint density of \( W_1 = (T_1, Z_1) \) (with respect to an appropriate dominating measure). Recall that \( f \) denotes the density of \( (X, Y) \). Let \( F \) denote the corresponding distribution function of \( (X, Y) \). The marginal distribution function of \( X \) is given by \( F_X(t) = \int_0^t f(u, s) \, ds \, du \). Define the measure \( \mu \) on \([0, \infty]^2\) by

\[
\mu(B) = \mu_2(B) + \mu_1(\{x \in [0, \infty) : (x, 0) \in B\}), \quad B \in \mathcal{B}
\]

where \( \mathcal{B} \) is the Borel \( \sigma \)-algebra on \([0, \infty)^2\) and \( \mu_i \) is Lebesgue measure on \( \mathbb{R}^i \). The density of the law of \( W_1 \) with respect to \( \mu \) is then given by

\[
s_f(t, z) = g(t) \left( 1_{\{z > 0\}} \partial_2 F(t, z) + 1_{\{z = 0\}} (1 - F_X(t)) \right), \quad (\text{1})
\]

where \( \partial_2 F(t, z) = \frac{\partial}{\partial z} F(t, z) = \int_0^t f(u, z) \, du \). By independence the likelihood of \( f \) based on \( D_n \) is given by

\[
t(f) = \prod_{i=1}^n s_f(T_i, Z_i).
\]

2.2 Prior

In this section, we define a prior on the class of all bivariate density functions on \( \mathbb{R}^2 \), denote as

\[
\mathcal{F} = \left\{ f : \mathbb{R}^2 \rightarrow [0, \infty) : \int_{\mathbb{R}^2} f(x, y) \, dx \, dy = 1 \right\}.
\]

For any \( f \in \mathcal{F} \), if \( S \) denotes the support of \( f \) and \( \cup_j C_j, j = 1, \ldots, p_n \) is a partition of \( S \), we define a prior on \( \mathcal{F} \) by constructing

\[
\theta(x, y) = \sum_j \frac{\theta_j}{|C_j|} 1_{C_j}(x, y), \quad (x, y) \in \mathbb{R}^2
\]

where \( |C| = \mu_2(C) \) is the Lebesgue measure of the set \( C \). Let \( \theta \) denote the vector \( \theta = (\theta_1, \ldots, \theta_{p_n}) \). We require that all \( \theta_j \) are nonnegative and that \( \theta \) satisfies \( \sum_j \theta_j = 1 \). We consider two types of prior on \( \theta \).
1. **Dirichlet.** For a fixed parameter $\alpha = (\alpha_1, \ldots, \alpha_{pn})$ consider $\theta \sim \text{Dirichlet}(\alpha)$. This prior is attractive as draws from the posterior distribution can be obtained using a straightforward data-augmentation algorithm (Cf. Section 5).

2. **Normal with graph Laplacian covariance matrix.** For a positive-definite matrix $\Upsilon$, let $H \sim N_{pn}(0, \tau^{-1} \Upsilon^{-1})$, conditionally on $\tau$. Each element of $H$ corresponds to one value of $\theta$. Next, set

$$
\theta_j = \frac{\psi(H_j)}{\sum_j \psi(H_j)}, \quad \text{where} \quad \psi(x) = e^x/(1 + e^x).
$$

The matrix $\Upsilon$ is chosen as follows. The partition of $S$ induces a graph structure on the bins, where each bin corresponds to a node in the graph, and nodes are connected when bins are adjacent (meaning that they are either horizontal or vertical “neighbours”). Let $L$ denote the graph Laplacian of the graph obtained in this way. This is the $pn \times pn$ matrix given by

$$
L_{i,i'} = \begin{cases} 
\text{degree node } i & \text{if } i = i' \\
-1 & \text{if } i \neq i' \text{ and nodes } i \text{ and } i' \text{ are connected} \\
0 & \text{otherwise.}
\end{cases}
$$

Now we take

$$
\Upsilon = L + p_n^{-2}I.
$$

2.1 **Remark** A property of the Dirichlet prior is that values of $\theta_j$ in adjacent bins are a negatively correlated, preventing the density to capture smoothness. See more in numerical study section 5. The idea of the graph-Laplacian prior is to induce positive correlation on adjacent bins and thereby specify a prior that produces draws is smoother on the graph corresponding to the partition. As we will see, this comes at the cost of increased computational complexity.

2.2 **Remark** One can argue whether the presented prior specifications are truly nonparametric. It is not if one adopts as definition that the size of the parameter should be learned by the data. For that, a solution could be to put a prior on $pn$ as well. While possible, this would severely complicate drawing from the posterior. As an alternative, one can take large values of $pn$ (so that the model is high-dimensional), and let the data determine the amount of smoothing by incorporating flexibility in the prior. As the Dirichlet prior lacks smoothness properties, fixing large values of $pn$ will lead to overparametrisation, resulting in high variance estimates (under smoothing). On the contrary, as we will show in the numerical examples, for the graph Laplacian prior, this overparametrisation can be substantially balanced/regularised by equipping the parameter $\tau$ with a prior distribution. The idea of histogram type priors with positively correlated adjacent bins has recently been used successfully in other settings as well, see for instance Gugushvili et al. (2018), Gugushvili et al. (2019).

3 **Posterior contraction**

In this section we derive a contraction rate for the posterior distribution of $F_0$. Denote as $\Pi_n(\cdot|D_n)$ under the prior measure $\Pi_n$ described in section 2.2.

3.1 **Assumption** The underlying joint density of the event time and mark, $f_0$, has compact support given by $\mathcal{M} = [0, M_1] \times [0, M_2]$ and is $\rho$-Hölder continuous on $\mathcal{M}$ ($\rho \in (0, 1]$). That is, there exists a positive constant $L$ such that for any $(x_1, y_1)$ and $(x_2, y_2)$ in $\mathcal{M}$,

$$
|f_0(x_1, y_1) - f_0(x_2, y_2)| \leq L \|(x_1, y_1) - (x_2, y_2)\|^\rho.
$$

In addition, there exist positive constants $\bar{M}$ and $\underline{M}$ such that

$$
\bar{M}(\min(x, y)^\rho) \leq f_0(x, y) \leq \underline{M}, \quad \text{for all } (x, y) \in \mathcal{M}.
$$
3.2 Assumption The censoring density $g$ is bounded away from 0 and infinity on $(0, M_1)$. That is, there exist positive constants $K$ such that $0 < K g(t) < K < \infty$ for all $t \in (0, M_1)$.

3.3 Assumption Conditions for prior:

1. For the Dirichlet prior, parameter $\alpha = (\alpha_1, \ldots, \alpha_p)$ satisfies $\alpha_1 \leq 1$ for all $l = 1, \ldots, p$ and some constant $a \in \mathbb{R}^+$.

2. For the graph-Laplacian prior, the prior specification is completed by specifying a prior distribution for $\tau$ supported on the positive halfline. For computational convenience, we assign the Gamma($\beta, \gamma$) distribution prior for $\tau$ with density function $f_\tau(\tau) \propto \tau^{\beta - 1} e^{-\gamma \tau}$ which is a conjugate prior of normal distribution.

3.4 Theorem Fix $(x, y) \in [0, M_1] \times (0, M_2]$. Consider either of the priors defined in section 2.2 and hyper-parameters satisfy assumption 3.3. Define $\eta_n = (n/\log n)^{-\gamma/\rho}$ where $\rho$ denotes the Hölder parameter in assumption 3.1. If $f_0$ and $g$ satisfy assumptions 3.1 and 3.2 respectively, then for sufficiently large $C$

$$E_0 \Pi_n(f \in F : |F(x, y) - F_0(x, y)| > C\eta_n | \mathcal{D}_n) \to 0, \quad \text{as} \quad n \to \infty.$$ 

Before we give a proof of theorem 3.4, we state two lemmas which are sufficient to give the contraction rate in the theorem. Define $\varepsilon_n \leq (n/\log n)^{-\gamma/\rho}$ (more specifically (24)), note that $\varepsilon_n \leq \eta_n$ and $n\varepsilon_n^2 \to \infty$ as $n \to \infty$.

3.5 Lemma Fix $f_0$ and $g$ satisfying the conditions in assumption 3.1 and 3.2. Define

$$S_n = \{f \in \mathcal{F} : KL(s_{f_0}, s_f) \leq \varepsilon_n^2, V(s_{f_0}, s_f) \leq \varepsilon_n^2\}.$$ 

Then we have $\Pi_n(S_n) \geq e^{-c_n \varepsilon_n^2}$ for some constant $c > 0$.

3.6 Lemma Fix $(x, y) \in [0, M_1] \times (0, M_2]$. Define $U_n(x, y) := \{f \in \mathcal{F} : |F(x, y) - F_0(x, y)| > C\eta_n\}$. There exists a sequence of test functions $\Phi_n$ such that

$$E_0(\Phi_n) = o(1),$$

$$\sup_{f \in U_n(t, z)} E_f(1 - \Phi_n) \leq c_1 e^{-c_2 C^2 n \varepsilon_n^2},$$

for some positive constants $c_1, c_2$ and $C$ appeared in theorem 3.4.

Proof of Theorem 3.4 The proof follows from the general idea in Ghosal, Ghosh & Van der Vaart (2000). Fix $(x, y) \in [0, M_1] \times (0, M_2]$, define $U_n(x, y) := \{f \in \mathcal{F} : |F(x, y) - F_0(x, y)| > C\eta_n\}$. Write the posterior mass on the set $U_n(x, y)$ as

$$\Pi_n(U_n(x, y) | \mathcal{D}_n) = D_n^{-1} \int_U \prod_{i=1}^n \frac{s_f(W_i)}{s_{f_0}(W_i)} d\Pi_n(f),$$

where

$$D_n = \int \prod_{i=1}^n \frac{s_f(W_i)}{s_{f_0}(W_i)} d\Pi_n(f).$$

Lemma 3.5 implies that (see lemma 8.1 in Ghosal, Ghosh & Van der Vaart (2000))

$$P_0(D_n \leq \exp(-(c + 1)n\varepsilon_n^2)) \to 0, \quad \text{as} \quad n \to \infty.$$
Then we can only consider on event \( \{ D_n \geq \exp(-c + 1) \gamma^2_n \} \). Using the test sequence in lemma 3.6 the posterior mass of \( U_n(x, y) \) satisfies

\[
\mathbb{E}_0 \Pi_n(U_n(x, y) \mid D_n) = \mathbb{E}_0 \Pi_n(U_n(x, y) \mid D_n) \Phi_n + \mathbb{E}_0 \Pi_n(U_n(x, y) \mid D_n)(1 - \Phi_n)
\]

\[
\leq \mathbb{E}_0 \Phi_n + e^{(c+1)\gamma^2_n} \mathbb{E}_0 \int_{U_n(x, y)} \prod_{i=1}^{n} \frac{s_f(W_i)}{s_{f_0}(W_i)} (1 - \Phi_n) d\Pi_n(f)
\]

\[
= \mathbb{E}_0 \Phi_n + e^{(c+1)\gamma^2_n} \int_{U_n(x, y)} \mathbb{E}_f(1 - \Phi_n) d\Pi_n(f)
\]

\[
\leq o(1) + c_1e^{(c+1)\gamma^2_n}e^{-c_2C^2\gamma^2_n} \to 0.
\]

The final step follows by taking \( C \) (appeared in theorem 3.4) large enough such that \( c_2C^2 > c + 1 \).

\[ \square \]

### 4 Proof of Lemmas

#### 4.1 Proof of lemma 3.5

**Proof** To give a lower bound for \( \Pi_n(S_n) \), we construct a subset \( \Omega_n \) of \( S_n \) and derive a lower bound of \( \Pi_n(\Omega_n) \) for both priors considered in section 2.2.

We first give a sequence of approximations for \( f_0 \). Let \( \delta_n = (n/\log n)^{-\frac{1}{c+2}} \). Denote \( A_{n,j} = ((j-1)\delta_n, j\delta_n] \), \( B_{n,k} = ((k-1)\delta_n, k\delta_n] \) for \( j = 1, 2, \ldots, J_n - 1, k = 1, 2, \ldots, K_n - 1 \) and \( A_{n,j} = ((J_n - 1)\delta_n, M_1], B_{n,k} = ((K_n - 1)\delta_n, M_2], J_n = [M_1\delta_n^{-1}], K_n = [M_2\delta_n^{-1}] \). Then \( \cup_{j,k}(A_{n,j} \times B_{n,k}) \) is a regular partition on \( M \). Let \( f_{0,n} \) be the piecewise constant density function defined by

\[
f_{0,n}(t, z) = \sum_{j=1}^{J_n} \sum_{k=1}^{K_n} \frac{w_{0,j,k}}{|A_{n,j} \times B_{n,k}|} 1_{A_{n,j} \times B_{n,k}}(t, z), \tag{8}\]

where \( w_{0,j,k} = \int_{A_{n,j} \cap B_{n,k}} f_0(u, v) \, dv \, du \). That is, we approximate \( f_0 \) by averaging it on each bin. Note that \( f_{0,n} \) has support \( M \). Define the set

\[
\Omega_n := \left\{ f \in \mathcal{F} : ||f - f_{0,n}||_{\infty} \leq \frac{1}{6} M \delta_n^6, \supp(f) \supseteq M \right\}. \tag{9}\]

By Lemma A.1 in appendix, we know that \( \Omega_n \subseteq S_n \). Now we give a lower bound for \( \Pi_n(\Omega_n) \), for the two type of priors.

Let \( p_n = J_n K_n \) denote the total number of bins. According to the prior specifications in section 2.2, for any \( f \in \mathcal{F} \), we parameterize

\[
f_{\theta}(x, y) = \sum_{j,k} \frac{\theta_{j,k}}{|A_{n,j} \times B_{n,k}|} 1_{A_{n,j} \times B_{n,k}}(x, y), \quad (x, y) \in \mathbb{R}^2,
\]

where \( \theta \) denotes the vector obtained by stacking all coefficients \( \{ \theta_{j,k}, j = 1, \ldots, J_n, k = 1, \ldots, K_n \} \).

Recall that \( f_{0,n} \) is defined by the local averages \( \{ w_{0,j,k}, j, k \geq 1 \} \). For any \( (t, z) \in A_{n,j} \times B_{n,k}, j, k \geq 1 \), we have

\[
| f_{\theta}(t, z) - f_{0,n}(t, z) | = | A_{n,j} \times B_{n,k} |^{-1} | \theta_{j,k} - w_{0,j,k} | \leq \delta_n^{-2} \max_{j,k} | \theta_{j,k} - w_{0,j,k} |.
\]

In the second step we use \( | A_{n,j} \times B_{n,k} | \geq \delta_n^2 \) for all \( j, k \). Hence

\[
\left\{ f_{\theta} \in \mathcal{F} : \max_{j,k} | \theta_{j,k} - w_{0,j,k} | \leq \frac{1}{6} M \delta_n^2 \right\} \subseteq \Omega_n. \tag{10}\]
Consider the two type of priors defined in section 2.2.

- Endowing prior $\theta \sim \text{Dirichlet}(\alpha)$, fixed $\alpha = (\alpha_1, \ldots, \alpha_{p_n})$, $ap_n^{-1} \leq \alpha_l \leq 1$ for all $l = 1, \ldots, p_n$.

By Lemma 6.1 in Ghosal, Ghosh & Van der Vaart (2000), we have

$$
\Pi_n(\Omega_n) \geq \Pi_n \left( \max_{j,k} |\theta_{j,k} - w_{0,j,k}| \leq \frac{1}{6} M \delta_n^{p+2} \right)
$$

$$
\geq \Gamma \left( \sum_{l=1}^{p_n} \alpha_l \right) \left( \frac{1}{6} M \delta_n^{p+2} \right)^{p_n} \prod_{l=1}^{p_n} \alpha_l
$$

$$
\geq \exp \left( \log \Gamma(a) + p_n \log \left( \frac{1}{6} M \delta_n^{p+2} \right) + p_n \log(ap_n^{-1}) \right)
$$

$$
\geq \exp(-C_1 \delta_n^{-2} \log n) = \exp(-c n^{\varepsilon})
$$

for some constant $C_1, c > 0$. This finishes the proof for the Dirichlet prior.

- Let $\theta_{j,k} = \frac{\psi(H_{j,k})}{\sum_{j\neq j, k \neq k} \psi(H_{j,k})}$ as defined in (2) and let $\tau \sim \Gamma(\beta, \gamma)$, $H_0 \sim \text{Dirichlet}(\beta, \gamma)$, $\Psi \sim \Gamma(\beta, \gamma)$, $\pi \sim \text{Dirichlet}(\beta, \gamma)$, $\Upsilon = L + p_n^{-2} I$

and each element of the vector $H$ has exactly same order with $\theta$.

For the fixed values $w_{0,j,k}, 1 \leq j \leq J_n, 1 \leq k \leq K_n$, there exists a matrix $H_0$ such that

$$
w_{0,j,k} = \frac{\psi(H_{0,j,k})}{\sum_{j\neq j, k \neq k} \psi(H_{j,k})}.
$$

For the ease of exposition, we choose $H_0$ such that it satisfies $\sum_{j,k} \psi(H_{0,j,k}) = 1$, then $w_{0,j,k} = \psi(H_{0,j,k})$ for all $j, k$.

Denote $x = (x_1, \ldots, x_m), x \in \mathbb{R}^m$ for some $m \in \mathbb{N}$. Define the function $\zeta(x) = \frac{\psi(x)}{\sum_{j=1}^{m} \psi(x_j)}$. Using the inequality $\frac{ab}{(a+b)^2} \leq \frac{1}{4}$ for $a, b \geq 0$, the partial derivatives of $\zeta$ satisfy

$$
\left| \frac{\partial \zeta(x)}{\partial x_1} \right| = \frac{\psi(x_1)(\sum_{j>1} \psi(x_j))}{(\sum_{j=1}^{m} \psi(x_j))^2(1 + e^{x_1})} \leq \frac{1}{4},
$$

$$
\left| \frac{\partial \zeta(x)}{\partial x_l} \right| = \frac{\psi(x_1)\psi(x_l)}{(\sum_{j=1}^{m} \psi(x_j))^2(1 + e^{x_l})} \leq \frac{1}{4}, \quad l = 2, \ldots, m.
$$

Then we have for any $x = (x_1, \ldots, x_m)$ and $x^0_l = (x_1, \ldots, x^0_l, \ldots, x_m) \in \mathbb{R}^m, l = 1, \ldots, m$

$$
|\zeta(x) - \zeta(x^0_l)| \leq \frac{1}{4} |x_l - x^0_l|.
$$

Hence for any $x, x^0 = (x_1^0, \ldots, x_m^0) \in \mathbb{R}^m$,

$$
|\zeta(x) - \zeta(x^0)| \leq |\zeta(x) - \zeta(x^0)| + |\zeta(x^0) - \zeta(x^0)|
$$

$$
+ \cdots + |\zeta(x^0_m) - \zeta(x^0, x^0, \ldots, x^0_m)|
$$

$$
\leq \frac{1}{4} \sum_{l=1}^{m} |x_l - x^0_l|.
$$

Let $m = p_n$ and $x$ correspond to the vector $H$. Then we have for $j, k \geq 1$,

$$
|\theta_{j,k} - w_{0,j,k}| \leq \frac{1}{4} \sum_{j,k} |H_{j,k} - H_{0,j,k}|.
$$
Combining this with (10), we have

$$\Pi_n(\Omega_n) \geq \Pi_n(\{f_H \in F : H \in B_n\}), \text{ where } B_n = \left\{ H : |H_{j,k} - H_{0,j,k}| \leq \frac{2}{3} M \delta_n^{p+1} p_n^{-1}, \text{ for all } j, k \right\}.$$ 

It is therefore sufficient to give a lower bound for the prior probability on $\{f_H : H \in B_n\}$. Note that

$$\Pi_n(\{f_H : H \in B_n\}) = \frac{\beta}{\Gamma(\alpha)} \int_0^\infty \tau^{\beta - 1} e^{-\gamma}(2\pi)^{-\frac{1}{2}} \tau^{pn/2} |\gamma|^{-1/2} \int_{B_n} \exp \left( -\frac{1}{2} \tau H^T \gamma H \right) \, dH \, d\tau.$$

In order to calculate the integral $\int_{B_n} \exp(-\frac{1}{2} \tau H^T \gamma H) \, dH$ at the right hand side for $\tau$ fixed, we first note the following facts. Denote the eigenvalues of $\gamma$ by $0 < \lambda_1 < \cdots < \lambda_{p_n}$. Then $\gamma$ has the following properties:

$$|\gamma| = \lambda_1 \cdots \lambda_{p_n} \leq (\lambda_{p_n})^{p_n},$$

$$\text{tr}(\gamma) = \sum_{l=1}^{p_n} \lambda_l = \sum_{l=1}^{p_n} (L_{l,l} + p_n^{-2}) = p_n^{-1} + \sum_{l=1}^{p_n} L_{l,l},$$

$$x^T \gamma x \leq \lambda_{p_n} x^T x, \text{ for any } p_n \text{ dim vector } x,$$

where $|\gamma|$ denotes the determinant of $\gamma$. By definition of the Laplacian matrix $L$, (3), we know

$$\sum_{l=1}^{p_n} L_{l,l} = 2 \cdot 4 + 3(2(J_n - 2) + 2(K_n - 2)) + 4(J_n - 2)(K_n - 2) < 4p_n,$$

the first term denotes we have 4 nodes of 2 connections (corners), the second item denotes $2(J_n - 2) + 2(K_n - 2)$ of 3 connections (edges) and the final term counts $(J_n - 2)(K_n - 2)$ of full 4 connections (inside). Using (12), we know

$$\lambda_{p_n} \leq \sum_{l=1}^{p_n} \lambda_l = p_n^{-1} + \sum_{l=1}^{p_n} L_{l,l} \leq 4p_n + p_n^{-1}$$

Using (13), we have

$$\int_{B_n} \exp \left( -\frac{1}{2} \tau H^T \gamma H \right) \, dH \geq \int_{B_n} \exp \left( -\frac{1}{2} \tau \lambda_{p_n} H^T H \right) \, dH.$$ 

We give an upper bound for $H^T H$. By assumption (5) again, we have $M \delta_n^{p+2} \leq w_{0,j,k} = \psi(H_{0,j,k}) \leq 4M \delta_n^2$, then we can bound $H_{0,j,k}$ by

$$\log(M \delta_n^{p+2}) \leq \log \left( \frac{M \delta_n^{p+2}}{1 - M \delta_n^{p+2}} \right) \leq H_{0,j,k} \leq \log \left( \frac{4M \delta_n^2}{1 - 4M \delta_n^2} \right) \leq \log(8M \delta_n^2) < 0.$$ 

Then for any $H \in B_n$,

$$H_{j,k} \leq H_{0,j,k} + 2M \delta_n^{p+2} p_n^{-1} \leq \log(8M \delta_n^2) + M \delta_n^{p+2} p_n^{-1} < 0.$$ 

Hence

$$H_{j,k}^2 \leq (H_{0,j,k} - 2M \delta_n^{p+2} p_n^{-1})^2 \leq (\log(M \delta_n^{p+2}) - 2M \delta_n^{p+2} p_n^{-1})^2 \leq C_2(\log n)^2$$

for some constant $C_2 > 0$. Then we have

$$H^T H \leq C_2 p_n(\log n)^2.$$
Using this and the fact that $B_n$ is a hyper-rectangle in $\mathbb{R}^p$,
\[
\int_{B_n} e^{-\frac{1}{2} \tau^T H^T \tau H} \, d\tau \geq \exp \left( -\frac{1}{2} C_2 \tau \lambda_p p_n (\log n)^2 \right) \int_{B_n} 1 \cdot d\tau
\]
\[
= \left( \frac{4}{3} M \delta_p^{-2} \right)^{p_n} \exp \left( -\frac{1}{2} C_2 \tau \lambda_p p_n (\log n)^2 \right).
\]
Hence we have
\[
\Pi_n(\{f_H : H \in B_n\}) \geq \frac{\gamma^\beta}{\Gamma(\alpha)} (2\pi)^{-p/2} 3^{-p_n} (4M \delta_p^{-2} p_n^{-1})^{p_n} |\Psi|^{-1/2}
\]
\[
\times \int_0^\infty \tau^\beta + \frac{1}{2} \tau^{-1} \exp \left( -\frac{1}{2} C_2 \tau \lambda_p p_n (\log n)^2 + \gamma \right) d\tau
\]
\[
= \frac{\gamma^\beta}{\Gamma(\alpha)} (4M (12\pi)^{-\frac{1}{2}} \delta_p^{-2} p_n^{-1})^{p_n} |\Psi|^{-1/2} \frac{\Gamma(\beta + p_n/2)}{(\frac{1}{2} C_2 \lambda_p p_n (\log n)^2 + \gamma)^{\beta + \frac{p_n}{2}}} \geq \frac{\gamma^\beta}{\Gamma(\alpha)} (4M (12\pi)^{-\frac{1}{2}} \delta_p^{-2} p_n^{-1})^{p_n} \left( \frac{\beta + p_n/2}{2 c C_2 \lambda_p p_n (\log n)^2 + \gamma} \right)^{\beta + \frac{p_n}{2}} (\beta + p_n/2)^{-1/2}.
\]
In the final step we use (11), $|\Psi|^{-1/2} = |\Psi|^{-\frac{1}{2}} \geq (\lambda_p^{-1})^{-\frac{1}{2}} p_n$ and $\Gamma(x) \asymp (x/e)^x x^{-1/2}$ when $x$ is large enough. By the inequality (14), we further have
\[
\Pi_n(\{f_H : H \in B_n\}) \gtrsim \exp \left( p_n \log(4M (12\pi)^{-\frac{1}{2}} \delta_p^{-2} p_n^{-1}) (4p_n + p_n^{-1} - \frac{1}{2}) \right.
\]
\[
+ \left( \frac{\beta + p_n/2}{2} \right) \log(C_3 p_n^{-1} (\log n)^{-2}) - \frac{1}{2} \log(\beta + p_n/2) \right)
\]
\[
\gtrsim \exp(-C_4 \delta_p^{-2} \log n) = \exp(-c n^{3/2})
\]
for some positive constants $C_3, C_4, c$. In the last step we use (24).

For both types of prior, we derived $\Pi(\Omega_n) \gtrsim \exp(-c n^{3/2})$, finishing the proof. $\square$

4.2 Proof of lemma 3.6

Proof Recall that $\eta_n = (n/\log n)^{-\frac{4}{5+2\beta}}$. Note that $\eta_n \asymp \varepsilon_n^{2/3}$. For $(t, z) \in [0, M_1] \times (0, M_2)$, define sets
\[
U_n,1(t, z) = \{ f : F(t, z) > F_0(t, z) + C \eta_n \},
\]
\[
U_n,2(t, z) = \{ f : F(t, z) < F_0(t, z) - C \eta_n \}.
\]
Then $U_n(t, z) = U_n,1(t, z) \cup U_n,2(t, z)$. We consider different test functions in different regimes of $t$:
\[ t \in (0, M_1) \] and \[ t \in (0, M_2) \].

Now fix $(t, z) \in (0, M_1) \times (0, M_2)$. Define test sequences
\[
\Phi_n^+(t, z) = 1 \left\{ \frac{1}{n} \sum_{i=1}^{n} \kappa_n^+ (t, z; T_i, Z_i) - \int_{t+h_n}^{t+h_n} g(x) F_0(x, z) \, dx > \varepsilon_n/2 \right\},
\]
\[
\Phi_n^-(t, z) = 1 \left\{ \frac{1}{n} \sum_{i=1}^{n} \kappa_n^- (t, z; T_i, Z_i) - \int_{t-h_n}^{t} g(x) F_0(x, z) \, dx < -\varepsilon_n/2 \right\},
\]
where
\[
\kappa_n^+ (t, z; T, Z) = 1_{[t+h_n]}(T) 1_{(0, z]}(Z),
\]
\[
\kappa_n^- (t, z; T, Z) = 1_{[t-h_n,t]}(T) 1_{(0, z]}(Z).
\]
and let
\[ h_n = (2M_2)^{-1}C\eta_n \min(1, M^{-1}) \quad \text{and} \quad e_n = \frac{1}{2}CK\eta_nh_n \]
be two sequences tending to zero. Recall that \( C \) is defined in theorem 3.4. By assumption 3.2, we have \( K < g < K \). Then for any bivariate density function \( f \),

\[
\mathbb{E}_f(\kappa_n^+(t, z; T, Z)) = \int_0^{t+h_n} \int_{(a, z)}(u) \mu(x, u) d\mu(x, u)
\]

\[
= \int_t^{t+h_n} \int_z^u g(x) \partial_2F(x, u) d\mu_2(x, u)
\]

\[
= \int_t^{t+h_n} g(x) F(x, z) dx
\]

\[
\leq \int_t^{t+h_n} g(x) dx \leq K h_n
\]

where \( s_f \) is the density function of \( (T, Z) \) defined in (1). The same upper bound holds for \( \mathbb{E}_f(\kappa_n^-(t, z; T, Z)) \). By Bernstein’s inequality (Van der Vaart (1998), lemma 19.32),

\[
\mathbb{E}_0(\max(\Phi_n^+(t, z), \Phi_n^-(t, z))) \leq 2 \exp\left(-\frac{1}{16} \frac{n\epsilon_n^2}{K h_n + e_n/2}\right) = o(1).
\]

When \( f \in U_{n,1}(t, z) \), by the monotonicity of \( F \) and \( f_0 \leq M \), we have

\[
F(x, z) - F_0(x, z) \geq F(t, z) - F_0(t, z) - (F_0(x, z) - F_0(t, z))
\]

\[
\geq C\eta_n - MM_2h_n \geq C\eta_n/2.
\]

Then it follows

\[
\int_t^{t+h_n} g(x)(F(x, z) - F_0(x, z)) dx \geq \frac{C\eta_n}{2} \int_t^{t+h_n} g(x) dx \geq \frac{CK}{2} \eta_nh_n = e_n.
\]

Hence, for \( f \in U_{n,1} \) we have

\[
\mathbb{E}_f(1 - \Phi_n^+(t, z)) = \mathbb{P}_f\left(\frac{1}{n} \sum_{i=1}^n \kappa_n^+(t, z|T_i, Z_i) - \int_t^{t+h_n} g(x) F_0(x, z) dx < e_n/2\right)
\]

\[
\leq \mathbb{P}_f\left(\frac{1}{n} \sum_{i=1}^n \kappa_n^+(t, z|T_i, Z_i) - \int_t^{t+h_n} g(x) F(x, z) dx \leq -e_n/2\right).
\]

Further, Bernstein’s inequality gives

\[
\mathbb{E}_f(1 - \Phi_n^+(t, z)) \leq 2 \exp\left(-\frac{1}{16} \frac{n\epsilon_n^2}{K h_n + e_n/2}\right) \leq c_1 e^{-c_2 n\epsilon_n^2}
\]

for some constants \( c_1, c_2 > 0 \).

When \( f \in U_{n,2}(t, z), x \in [t - h_n, t] \), we have

\[
F(x, z) - F_0(x, z) \leq F(t, z) - F_0(t, z) + F_0(t, z) - F_0(x, z)
\]

\[
\leq -C\eta_n + MM_2h_n \leq -C\eta_n/2
\]

and

\[
\int_{t-h_n}^t g(x)(F(x, z) - F_0(x, z)) dx \leq -\frac{CK}{2} \eta_nh_n = -e_n.
\]
Hence for $f \in U_{n,2}$, the type II error satisfies
\[
\mathbb{E}_f(1 - \Phi_n^{-}(t, z)) \leq P_f \left( \frac{1}{n} \sum_{i=1}^{n} \kappa_n^{-}(t, z|T_i, Z_i) - \int_{h_n}^{t} g(x) F(x, z) \, dx \geq e_n/2 \right).
\]
Using Bernstein’s inequality again, we have
\[
\mathbb{E}_f(1 - \Phi_n^{-}(t, z)) \leq c_1 e^{-c_2 C^2 n \varepsilon_n^2}, \quad \text{for some } c_1, c_2 > 0.
\]
For the boundary case $(t, z) \in [0, M_1] \times (0, M_2]$. With the similar idea, in order to give non-zero test sequences, we use $\kappa_n^{-}$ define $\Phi_n^+(0, z), \Phi_n^{-}(0, z)$ and $\kappa_n^{+}$ define $\Phi_n^-(M_1, z), \Phi_n^{-}(M_1, z)$. When $f \in U_{n,1}(0, z)$, using the tests sequence $\Phi_n^{+}(0, z)$ defined in case $t \in (0, M_1)$, we have
\[
\sup_{f \in U_{n,1}(0, z)} \mathbb{E}_f(1 - \Phi_n^{+}(0, z)) \leq c_1 e^{-c_2 C^2 n \varepsilon_n^2}.
\]
When $f \in U_{n,2}(M_1, z)$, using the tests sequence $\Phi_n^{-}(M_1, z)$ defined in case $t \in (0, M_1)$, we have
\[
\sup_{f \in U_{n,2}(M_1, z)} \mathbb{E}_f(1 - \Phi_n^{-}(M_1, z)) \leq c_1 e^{-c_2 C^2 n \varepsilon_n^2}.
\]
Note that for any $f \sim \Pi_n$ and $t \in A_{n,j}, j = 1, \ldots, J_n$,
\[
\int_0^{M_2} f(t, v) \, dv = |A_{n,j}|^{-1} \sum_{k=1}^{K_n} \theta_{j,k} \leq \delta_n^{-1} K_n = M_2. \tag{15}
\]
Here we use $\theta_{j,k} \leq 1$ and $|A_{n,j}| \geq \delta_n$. When $f \in U_{n,2}(0, z)$, for any $x \in [0, h_n]$, using (15) we have
\[
F(x, z) - F_0(x, z) \leq F(x, z) - F(0, z) + F(0, z) - F_0(0, z) \leq \int_0^{x} \int_0^{z} f(u, v) \, dv \, du - C\eta_n \leq M_2 h_n - C\eta_n \leq -C\eta_n/2
\]
and
\[
\int_0^{h_n} g(x)(F(x, z) - F_0(x, z)) \, dx \leq -e_n.
\]
Define tests sequence
\[
\Phi_n^{-}(0, z) = 1 \left\{ \frac{1}{n} \sum_{i=1}^{n} \kappa_n^{+}(0, z|T_i, Z_i) - \int_0^{h_n} g(x) F_0(x, z) \, dx < -e_n/2 \right\}.
\]
Hence by the Bernstein’s inequality,
\[
\mathbb{E}_f(1 - \Phi_n^{-}(0, z)) \leq c_1 e^{-c_2 C^2 n \varepsilon_n^2}.
\]
By the similar arguments as above, when $f \in U_{n,1}(M_1, z)$, for any $x \in [M_1 - h_n, M_1]$, using (15) we have
\[
F(x, z) - F_0(x, z) \geq F(x, z) - F(M_1, z) + F(M_1, z) - F_0(M_1, z) \geq C\eta_n - \int_{M_1 - h_n}^{M_1} \int_0^{z} f(u, v) \, dv \, du \geq C\eta_n - M_2 h_n \geq C\eta_n/2
\]
\[
\int_0^{h_n} g(x)(F(x,z) - F_0(x,z)) \, dx \geq -e_n.
\]

Define tests sequence
\[
\Phi^+_n(M_1, z) = \left\{ \frac{1}{n} \sum_{i=1}^n \kappa_n(M_1, z| T_i, Z_i) - \int_{M_1-h_n}^{M_1} g(x)F_0(x,z) \, dx > e_n/2 \right\},
\]
hence,
\[
\mathbb{E}_f(1 - \Phi^+_n(M_1, z)) \leq c_1 e^{-c_2 n^2 \varepsilon_n^2}.
\]

To conclude, take \( \Phi_n(t, z) = \max(\Phi^+_n(t, z), \Phi^-_n(t, z)) \), we derived
\[
\mathbb{E}_0 \Phi_n(t, z) = o(1),
\]
\[
\sup_{f \in U_n(t,z)} \mathbb{E}_f(1 - \Phi_n(t, z)) \leq c_1 e^{-c_2 n^2 \varepsilon_n^2}.
\]

\[\square\]

5 Computational study

In this section we present algorithms for drawing from the posterior distribution for both priors described in section 2.2.

5.1 Dirichlet prior

First, we consider the case where \( \{(X_i, Y_i), i = 1, \ldots, n\} \) is a sequence of independent random vectors, with common density \( f_0 \) that is piecewise constant on \( A_{n,j} \times B_{n,k} \) and compactly supported. This “no-censoring” model has likelihood

\[
l(\theta) = \prod_{j,k} \theta^{C_{j,k}},
\]

where \( C_{j,k} = \sum_{i} 1\{(X_i, Y_i) \in A_{n,j} \times B_{n,k}\} \) denotes the number of observations that fall in bin \( A_{n,i} \times B_{n,k} \). Clearly, the Dirichlet prior is conjugate for the likelihood, resulting in the posterior being of Dirichlet type as well and known in closed form. In case of censoring, draws from the posterior for the Dirichlet prior can be obtained by data-augmentation, where the following two steps are alternated

1. Given \( \theta \) and censored data, simulate the “full data”. This is tractable since the censoring scheme tells us in which collection of bins the actual observation can be located. Then one can renormalise the density \( f \) conditioned on these bins and select a specific bin accordingly and generate the “full data”. Cf. Figure 1.

2. Given the “full data”, draw samples for \( \theta \) from the posterior according to a Dirichlet distribution.

5.2 Graph Laplacian prior

For the graph-Laplacian prior, one could opt for a data-augmentation scheme as well, but its attractiveness is lost, since step (2) is not anymore of simple form. Therefore, we propose to bypass data-augmentation in this case and use a probabilistic programming language to draw from the posterior. In such a language, only the hierarchical scheme and sampling method need to be specified. From this, the likelihood and prior are computed. Subsequently generic implementations of sampling methods are called. An example of such a language is STAN, where Hamiltonian Monte Carlo (HMC),
or more specifically the No U-Turn Sampler (NUTS) (see for instance Robert et al. (2018), van de Meent et al. (2018), Betancourt (2018)), is the sampler used. More recently, an implementation in the Julia language (see Bezanson et al. (2017)) has been provided in the Turing package (see Ge, Xu & Ghahramani (2018)). In this paper we will use this package.

Unfortunately, there is presently no easy way to specify models with censored observations within the Turing-language. However, the model with censoring can easily be tweaked into a more familiar form that specifies the likelihood correctly. The only essential for a probabilistic programming language are the likelihood and hierarchical model specification. Specification of the prior is completely straightforward, while the likelihood can be specified by assuming a model with (conditionally independent) Bernoulli distributed random variables $Z_1, \ldots, Z_n$. For the $i$-th observation, let $I_i$ denote the set of indices corresponding to the shaded areas as in either left- or right-hand-side panel of Figure 1. Hence, the union of all boxes with indices in $I_i$ specifies the area where the $i$-th observation is located. The success probability of $Z_i$ is then given by inner product of the vector of shaded areas with the vector of corresponding probabilities $\theta_{j,k}$. Viewed in this way, the observation vector is simply a vector of length $n$ consisting of ones, corresponding to observations $z_1 = z_2 = \ldots = z_n = 1$. The actual amount of programming is modest (Cf. appendix B).

5.3 Numerical examples

In the following simulations, we use the DynamicNUTS sampler from Hoffman & Gelman (2014). For the Dirichlet prior we took 5,000 iterations of which the first half was discarded as burn-in. For the graph-Laplacian prior we took 2,000 iterations or which the first 100 iterations were discarded as burn-in.

We will consider the following data generating settings for the joint distribution of $(X, Y)$:

1. $f(x, y) = (x + y)1_{[0,1] \times [0,1]}(x, y)$ (similar to example in Groeneboom, Jongbloed & Witte (2012));

2. the density of a Gaussian copula with correlation equal to $-0.7$.

In all cases we assume that $T \sim \sqrt{U}$ where $U$ is uniformly distributed on $[0,1]$. This implies that the density of $T$ is given by $t \mapsto 2t1_{[0,1]}(t)$. For the graph-Laplacian prior we took $\Upsilon = L + 0.01I$ where $L$ is defined in (3).
5.3.1 Experiment 1

Here we take density (1), sample size 100 and $J_n = K_n = 5$. In Figure 2 we show traceplots for the DynamicNUTS sampler. In the top row of Figure 3 we show for both priors a plot where each bin is coloured according to the deviation of the estimated posterior mean bin probability from the true bin probability. Clearly, the graph-Laplacian gives a much better fit. Moreover, the deviations visually appear to be smoother, in the sense that adjacent blocks tend to have similar colours.

Next, we repeat the experiment, though with a much finer grid specified by $J_n = K_n = 10$. Traceplots and a plot of the errors made are in figures 4 and the bottom panel of 3 respectively. Clearly, the errors are much smaller compared to $J_n = K_n = 5$. Moreover, the smoothing effect induced by the graph Laplacian prior is clearly visible. The sampler seems to have mixed after iteration 500 and for this reason the initial 500 samples were discarded as burnin samples. To compare the performance under both priors, we calculated the square root of the summed squared errors ($\sqrt{SSE}$). The results are as follows:

| Resolution / prior | Dirichlet | graph-Laplacian |
|--------------------|-----------|-----------------|
| $J_n = K_n = 5$    | 0.201     | 0.035           |
| $J_n = K_n = 10$   | 0.070     | 0.018           |

This confirms the superior performance of the graph-Laplacian prior for this example. As the true density is smooth, the latter is as expected.

5.3.2 Experiment 2

Here, we take the Gaussian copula, again with sample size $n = 100$. The setup of the experiment is the same as that of experiments 1. The results are displayed in figure 5. Again, we computed the square root of the summed squared errors ($\sqrt{SSE}$). The results are as follows:

| Resolution / prior | Dirichlet | graph-Laplacian |
|--------------------|-----------|-----------------|
| $J_n = K_n = 5$    | 0.225     | 0.147           |
| $J_n = K_n = 10$   | 0.160     | 0.080           |

As expected, the performance of the graph-Laplacian outperforms that of the Dirichlet.
Figure 3: Experiment 1: each bin is coloured according to the error within the bin, which is the estimated posterior mean of the bin probability minus the true bin probability. Left: Dirichlet prior. Right: graph-Laplacian prior. Note that the scale of colouring is the same in both figures. Top: $J_n = K_n = 5$. Bottom $J_n = K_n = 10$. 
A Technical proof

A.1 Lemma Define set

$$\Omega_n := \left\{ f \in \mathcal{F} : \|f - f_{0,n}\|_{\infty} \leq \frac{1}{6} M \delta_n, \text{supp}(f) \supseteq \mathcal{M} \right\},$$

where $f_{0,n}$ is defined in (8). Then $\Omega_n$ is a subset of $S_n$ (which is defined in (6)).

Proof By the definition of $f_{0,n}$ in (8), for any $(t,z) \in A_{n,j} \times B_{n,k},$

$$|f_{0,n}(t,z) - f_0(t,z)| = \left| A_{n,j} \times B_{n,k} \right|^{-1} \int_{A_{n,j}} \int_{B_{n,k}} f_0(u,v) dv du - f_0(t,z) \right|$$

$$\leq \left| A_{n,j} \times B_{n,k} \right|^{-1} \int_{A_{n,j}} \int_{B_{n,k}} |f_0(u,v) - f_0(t,z)| dv du$$

$$\leq \max_{(u,v) \in A_{n,j} \times B_{n,k}} |f_0(u,v) - f_0(t,z)|.$$

By assumption (4) on $f_0,$ we have

$$\max_{(u,v) \in A_{n,j} \times B_{n,k}} |f_0(u,v) - f_0(t,z)| \leq L \max_{(u,v) \in A_{n,j} \times B_{n,k}} \|(u,v) - (t,z)\| \leq L(2\sqrt{2} \delta_n)^p.$$

Hence

$$\|f_{0,n} - f_0\|_{\infty} = \max_{j,k} \max_{(t,z) \in A_{n,j} \times B_{n,k}} |f_{0,n}(t,z) - f_0(t,z)|$$

$$\leq \max_{j,k} \max_{(t,z) \in A_{n,j} \times B_{n,k}} L(2\sqrt{2} \delta_n)^p = L(2\sqrt{2} \delta_n)^p. \quad (16)$$
Figure 5: Experiment 2: each bin is coloured according to the error within the bin, which is the estimated posterior mean of the bin probability minus the true bin probability. Left: Dirichlet prior. Right: graph-Laplacian prior. Note that the scale of colouring is the same in both figures. Top: $J_n = K_n = 5$. Bottom $J_n = K_n = 10$. 

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When \( j \geq k \), we have

\[
|s_{f_1}(t, z) - s_{f_2}(t, z)| = |g(t) \left( \mathbf{1}_{\{z > 0\}} \int_0^t (f_1(u, z) - f_2(u, z)) \, du + \mathbf{1}_{\{z = 0\}} \int_t^M \int_0^{M_2} (f_1(u, v) - f_2(u, v)) \, dv \, du \right) |
\]

\[
\leq g(t) (M_1 ||f_1 - f_2||_\infty + M_1 M_2 ||f_1 - f_2||_\infty)
\]

\[
\leq M_1 (1 + M_2) g(t) ||f_1 - f_2||_\infty.
\]

Further, we have

\[
||s_{f_1} - s_{f_2}||_1 = \int_\mathcal{M} |s_{f_1} - s_{f_2}| \, d\mu \leq KM_1^2 M_2 (1 + M_2) ||f_1 - f_2||_\infty.
\]

By Lemma 8 of Ghosal & Van der Vaart (2007), we know

\[
KL(s_{f_0}, s_f) \lesssim h^2(s_{f_0}, s_f) (1 + \log ||s_{f_0} / s_f||_\infty),
\]

\[
V(s_{f_0}, s_f) \lesssim h^2(s_{f_0}, s_f) (1 + \log ||s_{f_0} / s_f||_\infty)^2.
\]

For any \( f \in \Omega_n \), we give upper bounds of \( h^2(s_{f_0}, s_f) \) and \( ||s_{f_0} / s_f||_\infty \). By (16) and (17), we know

\[
||s_{f_0} - s_f||_1 \leq K M_1^2 M_2 (1 + M_2) (||f_0 - f_{0,n}||_\infty + ||f_{0,n} - f||_\infty) \lesssim \delta^\rho_n.
\]

Using the inequality \( h^2(f_1, f_2) \leq \frac{1}{2} ||f_1 - f_2||_1 \), we then have

\[
h^2(s_{f_0}, s_f) \leq \frac{1}{2} ||s_{f_0} - s_f||_1 \lesssim \delta^\rho_n.
\]

We now give an upper bound on \( ||s_{f_0} / s_f||_\infty \), note that

\[
||s_{f_0} / s_f||_\infty \leq \max \left\{ \left\| \frac{\partial F_0}{\partial F} \right\|_\infty, \left\| \frac{1 - F_0}{1 - F_X} \right\|_\infty \right\} \leq \left\| \frac{f_0}{f} \right\|_\infty \leq \left\| \frac{f_{0,n}}{f_0} \right\|_\infty \left\| \frac{f_{0,n}}{f} \right\|_\infty.
\]

By the lower bound in inequality (5), we have for any \( (t, z) \in A_{n,j} \times B_{n,k} \),

\[
f_{0,n}(t, z) = |A_{n,j} \times B_{n,k}|^{-1} w_{0,j,k} \geq M |A_{n,j} \times B_{n,k}|^{-1} \int_{A_{n,j} \times B_{n,k}} (\min(u, v))^\rho \, dv \, du.
\]

When \( \min(j, k) > 1 \),

\[
\int_{A_{n,j} \times B_{n,k}} (\min(u, v))^\rho \, dv \, du \geq \delta^\rho_n |A_{n,j} \times B_{n,k}|.
\]

When \( \min(j, k) = 1 \) and \( j \neq k \),

\[
\int_{A_{n,j} \times B_{n,k}} (\min(u, v))^\rho \, dv \, du = \frac{1}{\rho + 1} \delta^\rho_n |A_{n,j} \times B_{n,k}|.
\]

When \( j = k = 1 \),

\[
\int_{A_{n,j} \times B_{n,k}} (\min(u, v))^\rho \, dv \, du = 2 \int_0^{\delta_n} dv \int_0^v u^\rho \, du = \frac{2}{(\rho + 1)(\rho + 2)} \delta^\rho_n.
\]
Hence, in any of the cases, using $\rho \leq 1$, we obtain
\[
\int_{A_{n,j} \times B_{n,k}} (\min(u, v))^\rho \, dv \, du \geq \frac{1}{3} \delta_n^\rho |A_{n,j} \times B_{n,k}|.
\]
Then it follows that
\[
f_{0,n}(t, z) \geq \frac{M}{3} \delta_n^\rho.
\]
(21)

Combining with (16),
\[
\left\| \frac{f_0}{f_{0,n}} \right\|_{\infty} \leq 1 + \left\| \frac{f_0 - f_{0,n}}{f_{0,n}} \right\|_{\infty} \leq 1 + \frac{2^\rho 3L}{M}.
\]
(22)

Further, using (21) again, by definition of $\Omega_n$, for $f \in \Omega_n$,
\[
f(t, z) \geq f_{0,n}(t, z) - \frac{1}{6} M \delta_n^\rho \geq \frac{1}{2} f_{0,n}(t, z).
\]
Note that this implies if $f = 0$, then we have $f_{0,n} = 0$. Hence,
\[
\left\| \frac{f_{0,n}}{f} \right\|_{\infty} \leq 2.
\]
(23)

Substituting (22) and (23) into (20) gives that
\[
\left\| \frac{s_{f_0}}{s_f} \right\|_{\infty} \leq 2 \left( 1 + \frac{2^\rho 3L}{M} \right).
\]

Substituting this bound and (19) into (18) implies that there exists a $C_1 > 0$ such that
\[
KL(s_{f_0}, s_f) \leq C_1 \delta_n^\rho, \quad V(s_{f_0}, s_f) \leq C_1 \delta_n^\rho.
\]
Define
\[
\varepsilon_n = \sqrt{C_1 (n/\log n)^{-2(\rho+2)}} = \sqrt{C_1 \delta_n^\rho},
\]
(24)
then we have $\Omega_n \subseteq S_n$.

\section{B Programming details in the Turing language}

For each observation index $i \in \{1, \ldots, n\}$ the indices $I_i$ need to be computed and stored. Say that information is in the object $c_i$ (censoring information). Say that we define a function $bernpar$ that takes the full parameter vector $\theta$ and $c_i$ and outputs the corresponding success probability. Finally, if $z$ denotes the observation vector (taken to be a vector of length $n$ containing solely ones) and $L$ is the graph-Laplacian, then the model is specified as follows:

```turing
@model GraphLaplacianModel(z,ci,L) = begin
    tau ~ InverseGamma(.1,.1)
    H ~ MvNormalCanon(L•tau)
    theta = invlogit(H)
    for k in eachindex(z)
        z[k] ~ Bernoulli(bernpar(theta,ci[k]))
    end
end
```

Here, $\text{invlogit}$ refers to the function $\psi$ in (2).
References

Betancourt, M. (2018). A conceptual introduction to Hamiltonian Monte Carlo. arXiv:1701.02434.

Bezanson, J., Edelman, A., Karpinski, S. and Shah, V. B. (2017). Julia: A Fresh Approach to Numerical Computing. Society for Industrial and Applied Mathematics. 59, p. 65-98.

Chen, D., Sun, J. and Peace, K. E. (2013). Interval-Censored Time-to-Event Data: Methods and Applications. Chapman and Hall/CRC Biostatistics Series.

Choudhuri, N., and Ghosal, S. and Roy, A. (2007). Nonparametric binary regression using a Gaussian process prior. Statistical Methodology. 4, p. 227–243.

Ge, H., Xu, K., and Ghahramani, Z. (2018). Turing: A Language for Flexible Probabilistic Inference. Proceedings of the Twenty-First International Conference on Artificial Intelligence and Statistics, PMLR: 84, p. 1682-1690.

Ghosal, S., Ghosh, J.K. and Van der Vaart, A.W. (2000). Convergence rates of posterior distributions. Ann. Statist. 28, p. 500–531.

Ghosal, S., van der Vaart, A. (2007). Posterior convergence rates of Dirichlet mixtures at smooth densities. Ann. Statist. 2, p. 697–723.

Ghosal, S. and Van der Vaart, A.W. (2017). Fundamentals of Nonparametric Bayesian Inference. Cambridge University Press.

Groeneboom, P., Jongbloed, G. and Witte, B. I. (2011). Smooth Plug-in Inverse Estimators in the Current Status Mark Model. Scandinavian Journal of Statistics. 39, p. 15–33.

Groeneboom, P., Jongbloed, G. and Witte, B. I. (2012). A maximum smoothed likelihood estimator in the current status continuous mark model. J. Nonparametr. Stat. 24, p. 85–101.

Groeneboom, P., Maathuis, M. H. and Wellner, J. A. (2008). Current status data with competing risks: Consistency and rates of convergence of the MLE. Ann. Statist. 36, p. 1031–1063.

Gugushvili, S., van der Meulen, F. H., Schauer, M. R. and Spreij, P. (2019). Nonparametric Bayesian volatility estimation. (eds) 2017 MATRIX Annals, MATRIX Book Series. 2, p. 279–302.

Gugushvili, S., van der Meulen, F. H., Schauer, M. R. and Spreij, P. (2018). Bayesian wavelet de-noising with the Caravan prior. arXiv:1810.07668, to appear in ESAIM.

Hartog J. and van Zanten, H. (2017). Nonparametric Bayesian label prediction on a graph. arXiv:1612.01930 [stat.CO]

Hoffman, M. D. and Gelman, A. (2014). The No-U-turn Sampler: Adaptively Setting Path Lengths in Hamiltonian Monte Carlo. J. Mach. Learn. Res. 15, p. 1593–1623.

Huang, Y. and Louis, T. A. (1998). Nonparametric estimation of the joint distribution of survival time and mark variables. Biometrika. 85, p. 7856-7984.

Hudgens, M. G., Maathuis, M. H. and Gilbert, P. B. (2007). Nonparametric estimation of the joint distribution of a survival time subject to interval censoring and a continuous mark variable. Biometrics 63, p. 372-380.

Maathuis, M. H. and Wellner, J. A. (2008). Inconsistency of the MLE for the joint distribution of interval-censored survival times and continuous marks. Scand. J. Statist. 35, p. 83-103.

Moala, F. A. and O’Hagan, A. (2010). Elicitation of multivariate prior distributions: A nonparametric Bayesian approach. Journal of Statistical Planning and Inference. 140, p. 1635–3758.

Robert C. P., Elvira V., Tawn N., and Wu C. (2018). Accelerating MCMC algorithms Wiley Interdiscip. Rev. Comput. Stat. 10, e1435.
van de Meent, J. W., Paige, B., Yang, H. and Wood, F. (2018). An Introduction to Probabilistic Programming. arXiv:1809.10756.

Van der Vaart, A.W. (1998). *Asymptotic Statistics*. Cambridge University Press.