Charmonium Mass Spectrum with Spin-Dependent Interaction in Momentum-Helicity Space

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(Dated: July 22, 2014)

In this paper we have solved the nonrelativistic form of the Lippmann-Schwinger equation in the momentum-helicity space by inserting a spin-dependent quark-antiquark potential model numerically. To this end, we have used the momentum-helicity basis states for describing a nonrelativistic reduction of one gluon exchange potential. Then we have calculated the mass spectrum of the charmonium ψ(c ¯c), and finally we have compared the results with the other theoretical results and experimental data.

1. INTRODUCTION

During the past years, several models and methodological approaches based on solving the relativistic and nonrelativistic form of the Schrödinger or Lippmann-Schwinger equation have been developed for studying the light and heavy mesons in the coordinate and momentum spaces respectively.

Recently, the three-dimensional approach based on momentum-helicity basis states for studying the Nucleon-Nucleon scattering and deuteron state has been developed [1, 2]. We extend this approach to particle physics problems by solving the nonrelativistic form of the Lippmann-Schwinger equation to obtain the mass spectrum of the heavy mesons using the nonrelativistic quark-antiquark interaction in terms of a linear confinement, a Coulomb, and various spin-dependent pieces.

In the heavy-quark (c,b) mesons the differences between energy levels are small compared to the particle masses. Hence, the nonrelativistic Lippmann-Schwinger equation can be used to study their quantum behavior. To this end, we have used the nonrelativistic form of the Lippmann-Schwinger equation in the momentum-helicity representation to study the charmonium as a heavy meson. For this purpose, we have used a nonrelativistic quark-antiquark potential based on one-gluon exchange in the momentum-helicity representation.

This article is organized as follows. In Sect. 2, the nonrelativistic Lippmann-Schwinger equation in the momentum-helicity basis states which leads to coupled and uncoupled integral equations for various quantum numbers is presented briefly. In Sect. 3, a spin dependent quark-antiquark potential model is described in the momentum-helicity basis states. The details of the numerical calculations and the results obtained for the charmonium are presented in Sect. 4. Finally, a summary and an outlook are provided in Sect. 5.

2. LIPPMANN-SCHWINGER EQUATION IN MOMENTUM-HELICITY BASIS STATES

The nonrelativistic form of the homogenous Lippmann-Schwinger equation for describing the heavy meson bound state is given by:

\[ |\Phi_j^{M_j}\rangle = \frac{1}{E - \frac{p^2}{m}} V |\Phi_j^{M_j}\rangle, \]

where \( V \) denotes the quark-antiquark interaction, \( m \) is mass of the quark or antiquark and \( |\Phi_j^{M_j}\rangle \) is the meson bound state with the total angular momentum \( j \). \( M_j \) is projection of the total angular momentum \( j \) along the quantization axis. The integral form of this equation in the momentum-helicity basis states is written as [3]:

\[ \Phi_{S_j}^{M_j}(p) = \frac{2\pi}{E - \frac{p^2}{m}} \sum_{\Lambda} \int_0^\infty dp' p' V_{M_j \Lambda}(p, p') \Phi_{S_j}^{M_j}(p'), \]

with:

\[ V_{M_j \Lambda}(p, p') = \int_{-1}^{1} d\cos \theta' V_{M_j \Lambda}(p, p', \theta') d_{M_j \Lambda}(\theta'), \]

where \( p \) is the magnitude of the relative momentum of the quark and antiquark, \( S \) is the total spin of meson, \( \Lambda \) is the spin projection along the relative momentum and \( d_{M_j \Lambda}(\theta') \) are the rotation matrices. For an arbitrary total angular
two coupled equations for channels

\[ \Phi_{0j}^M(p) = \frac{2\pi}{E - \frac{p^2}{m}} \int_0^\infty dp' p'^2 V_{M0}^0(p,p') \Phi_{0j}^0(p'). \]  

(4)

Also for \( j = 0 \) and triplet case of the total spin state, Eq. (2) leads to one equation as:

\[ \Phi_{1j}^0(p) = \frac{2\pi}{E - \frac{p^2}{m}} \int_0^\infty dp' p'^2 V_{00}^1(p,p') \Phi_{1j}^0(p'). \]  

(5)

For \( S = 1 \) and \( j > 0 \) it is more complicated. For example for \( j = 1 \), Eq. (2) leads to one equation for channel \( P \) and two coupled equations for channels \( S \) and \( D \) as follows:

\[ \Psi_{111}(p) = \frac{2\pi}{E - \frac{p^2}{m}} \int_0^\infty dp' p'^2 \left( V_{11}^1(p,p') - V_{-11}^1(p,p') \right) \Psi_{111}(p'), \]  

(6)

\[ \Psi_{011}(p) = \frac{2\pi}{E - \frac{p^2}{m}} \frac{1}{3} \int_0^\infty dp' p'^2 \left\{ \left[ 2V_{11}^1(p,p') + 2V_{01}^1(p,p') + V_{00}^1(p,p') + 2V_{10}^1(p,p') + 2V_{11}^1(p,p') \right] \Psi_{011}(p') \\
+ \sqrt{2} \left[ V_{11}^1(p,p') + V_{01}^1(p,p') - V_{00}^1(p,p') - 2V_{10}^1(p,p') + V_{11}^1(p,p') \right] \right\} \Psi_{211}(p'), \]  

(7)

\[ \Psi_{211}(p) = \frac{2\pi}{E - \frac{p^2}{m}} \frac{1}{3} \int_0^\infty dp' p'^2 \left\{ \sqrt{2} \left[ V_{11}^1(p,p') - 2V_{01}^1(p,p') - V_{00}^1(p,p') + V_{10}^1(p,p') + V_{11}^1(p,p') \right] \Psi_{011}(p') \\
+ \left[ V_{11}^1(p,p') - 2V_{01}^1(p,p') + 2V_{00}^1(p,p') - 2V_{10}^1(p,p') + V_{11}^1(p,p') \right] \right\} \Psi_{211}(p'), \]  

(8)

where \( \Psi_{jS}(p) \) is the partial wave component of the wave function which is connected to the momentum-helicity component of the wave function as

\[ \Phi_{jS}^\Lambda(p) = \sum_i \sqrt{\frac{2l+1}{4\pi}} C(lsj; 0\Lambda\Lambda) \Psi_{jS}(p). \]  

(9)

The inverse relation is written as:

\[ \Psi_{jS}(p) = \sqrt{\frac{4\pi(2l+1)}{2j+1}} \sum_{\Lambda} C(lsj; 0\Lambda\Lambda) \Phi_{jS}^\Lambda(p). \]  

(10)

3. QUARK-ANTIQUARK POTENTIAL IN MOMENTUM-HELCITY BASIS STATES

The spin dependent potential model that we have used in our calculations is sum of the Linear and a simple nonrelativistic reduction of an effective one gluon exchange potential without retardation. This potential in the coordinate space is given in terms of

\[ V(r, p) = \sigma r + f_c \alpha_s \left\{ \frac{1}{r} - \frac{\pi}{m^2} \delta(r) + \frac{1}{m^2} \frac{p \cdot p}{r} - \frac{3}{4m^2} \frac{L \cdot (\sigma_1 + \sigma_2)}{r^3} \right\} \\
- \frac{2\pi}{3m^2} \delta(r)(\sigma_1 \cdot \sigma_2) - \frac{1}{4m^2} \frac{3(\sigma_1 \cdot \hat{r})(\sigma_2 \cdot \hat{r}) - (\sigma_1 \cdot \sigma_2)}{r^3} \right\}, \]  

(11)

where \( \sigma \) is the string tension, \( \alpha_s \) is the strong-interaction fine-structure constant, \( f_c \) is the color factor which is -4/3 for quark-antiquark and -2/3 for quark-quark, \( \sigma_1 \) and \( \sigma_2 \) are the Pauli matrices and \( L \) is the total orbital angular
momentum operator. Fourier transformation of this potential to momentum space yields:

\[ \langle p|V|p' \rangle = \sigma \left[ \delta(q) r_c + \frac{1}{2\pi^2 q^4} \left( 2 \cos(q r_c) - 2 + q r_c \sin(q r_c) \right) \right] \\
+ f_c \alpha_s e^{-\lambda^2 q^2} \left\{ \frac{\delta(q)}{r_c} + \frac{1}{2\pi^2 q^2} \left( 1 - \frac{\sin(q r_c)}{q r_c} \right) \left( 1 + \frac{p^2}{m^2} \right) - \frac{1}{8\pi^2 m^2} \right\} \]

\[ + \frac{3}{8\pi^2 m^2} \lambda^2 q^2 \left[ 6 \frac{p' p}{q^2} \gamma S(S + 1) - 2 \Lambda S(S + 1) - 2 \Lambda^2 - 2 \Lambda^2 \Lambda' \right] \]

where \( q = p' - p \) is the momentum transfer. The kernels of integral equations have singularity. To overcome this problem we have used the regularized form of linear confining and Coulomb parts of the potential \[5\]. Details of Fourier transformation of regularized parts of the potential are given in Appendix A. Also we have used a Gaussian form factor, \( \exp(-\lambda^2 q^2) \) at the quark-gluon vertex as in Ref. \[6\] to remove singularity of the kernels due to existence of one gluon exchange potential. The variable \( \lambda \) can be interpreted as size of the quark. In Ref. \[6\] the pointlike quark-gluon vertex is replaced by a form factor, \( 1/(q^2 + \beta^2) \) in which \( \beta^{-1} \) is the effective quark size to eliminate the singularity. In this work we have used both regularized form and Gaussian form factor for coulomb and \( f_c \alpha_s p^2/(m^2 r) \) parts of the potential which cause the convergence of numerical results faster. Therefore, the final form of the potential in the momentum-helicity space is written as:

\[ V_{AA'}^{S}(p, p') \equiv \langle p|SA'|p'SA' \rangle = \sigma \left[ \delta(q) r_c + \frac{1}{2\pi^2 q^4} \left( 2 \cos(q r_c) - 2 + q r_c \sin(q r_c) \right) \right] \\
+ f_c \alpha_s e^{-\lambda^2 q^2} \left\{ \frac{\delta(q)}{r_c} + \frac{1}{2\pi^2 q^2} \left( 1 - \frac{\sin(q r_c)}{q r_c} \right) \left( 1 + \frac{p^2}{m^2} \right) - \frac{1}{8\pi^2 m^2} \right\} \]

\[ + \frac{3}{8\pi^2 m^2} \lambda^2 q^2 \left[ 6 \frac{p' p}{q^2} \gamma S(S + 1) - 2 \Lambda S(S + 1) - 2 \Lambda^2 - 2 \Lambda^2 \Lambda' \right] \]

\[ - p p' \gamma S(S + 1) - 3 \frac{p p'}{\gamma} (S(S + 1) - 2 \Lambda^2 - 2 \Lambda^2 \Lambda) \right\}, \]

where \( \gamma = p' \cdot \hat{p} = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi') \) and \( |p; pSA\rangle \) is the momentum-helicity basis state which is eigenstate of the helicity operator \( S \cdot \hat{p} \) as:

\[ S \cdot \hat{p}|p; pSA\rangle = \Lambda|p; pSA\rangle. \]

Also we have \[6\]:

\[ \langle pSA|p'SA' \rangle = \sum_{N=-S}^{S} \exp[iN(\varphi - \varphi')] d_{N\Lambda}(\theta) d_{N\Lambda'}(\theta'). \]

If the vector \( p \) is along z-direction, it is clear that the Eq. (15) is reduced to:

\[ \langle pSA|p'SA' \rangle = \exp[-i\Lambda\varphi'] d_{N\Lambda}(\theta'). \]

For numerical calculations we need the matrix elements of the potential \( V_{AA'}^{S}(p, p', \theta) \). These matrix elements is related to the matrix elements Eq. (13) as follows:

\[ V_{AA'}^{S}(p, p', \theta) = \exp[i\Lambda\varphi'] \langle p z|SA|p'; p'SA' \rangle. \]

By considering Eqs. (13), (16) and (17), the final form of the matrix elements of the potential which is inserted in the numerical calculations is written as:

\[ V_{AA'}^{S}(p, p', \theta') = \sigma d_{N\Lambda}(\theta') \left[ \delta(q) r_c + \frac{1}{2\pi^2 q^4} \left( 2 \cos(q r_c) - 2 + q r_c \sin(q r_c) \right) \right] \]
$$\eta$$ is so weak. I show the mixed charmonium states in Table II by their dominant partial wave. Points N

existence of the tensor term in the potential mix. They are compared with the experimental data and another theoretical work. From Eqs. (7) and (8) it is clear that

accuracy.

variable. In our calculations we have chosen N

P1

points to achieve an acceptable

accuracy.

For numerical calculations as a first step we have used the Gaussian quadrature grid points to discretize the momentum and the angle variables. The integration interval for the momentum is covered by two different hyperbolic and linear mappings of the Gauss-Legendre points from the interval [-1, +1] to the intervals [0, 1] and [0, 3] respectively. N

P1

are the number of grid points for momentum and angle variables in Table IV. N

P2

= 200 grid points for to achieve an acceptable

accuracy.

Then we have calculated the matrix elements of the potential $V_{AA'}(p, p', \theta')$, from Eq. (18). According to the Eq. (3) integration over the spherical angle variable $\theta'$, has been done independently. Finally, we have solved the integral equations (4)-(8) as eigenvalue equations. The integration over momentum variable is cut off at $q_{max}$ = 10 GeV. This selection is carried out so that the numerical results do not depend on this choice. The typical values for $p_1$ and $p_2$ are 1 GeV and 3 GeV, respectively. These selections are done till the total number of grid points for momentum intervals are decreased. Other selections can be done but by different grid points for momentum variables.

The parameters of the potential model which are shown in Table I are fixed by a fit to the masses of the states $\eta_c$, $J/\psi$ and $h_c$, similar to what is done in Ref. 9. The results of charmonium mass spectrum are shown in Table II. They are compared with the experimental data and another theoretical work. From Eqs. (7) and (8) it is clear that existence of the tensor term in the potential mix $S$- and $D$- partial waves but this mixed as it is shown in Table III is so weak. I show the mixed charmonium states in Table II by their dominant partial wave.

As a test of our numerical calculations we have shown convergence of the results as a function of number of grid points $N_{p1}$, $N_{p2}$ and $N_{\theta}$ for the momentum and angle variables in Table IV. $N_{p1}$, $N_{p2}$ are the number of grid points for the intervals $[0, p_2]$ and $[p_2, p_{max}]$ respectively. $N_{\theta}$ is corresponding to number of grid points for spherical angle variable. In our calculations we have chosen $N_{p1}=100$, $N_{p2}=100$ $N_{\theta}=200$ grid points for to achieve an acceptable accuracy.

5. SUMMARY AND OUTLOOK

In this paper we have extended an approach based on momentum-helicity basis states for calculation of mass spectrum of heavy mesons by solving nonrelativistic form of the Lippmann-Schwinger equation. As an application we have used this approach to obtain the mass spectrum of charmonium. The advantage of working with helicity states is that states are the eigenstates of the helicity operator appearing in the quark-antiquark potential. Thus, using the helicity representation is less complicated than using the spin representation with a fixed quantization axis for representation of spin dependent potentials. This work is the first step toward for studying single, double, and triple heavy-flavor baryons in the framework of the nonrelativistic quark model by formulation of the Faddeev equation in the 3D momentum-helicity representation. Furthermore, we can apply this formalism straightforwardly for investigation of heavy pentaquark systems, which can be considered as two-body (heavy meson, baryon) systems with meson-nucleon potentials which is underway.
TABLE I: Parameters of the model.

| Parameter | Value |
|-----------|-------|
| \( \sigma \) [GeV/fm] | 1.222 |
| \( \lambda \) [GeV\(^{-1}\)] | 0.3154 |
| \( m \) [GeV] | 1.269 |
| \( \alpha_s \) | 0.2863 |
| \( r_c \) [fm] | 10 |

TABLE II: Comparison of the obtained charmonium mass spectrum with the experimental data and another work.

| \( n^{2S+1}L_J \) | Candidate | Exp. [8] | Ref. [9] | Mass [MeV] |
|-----------------|-----------|---------|---------|-----------|
| \( 1^1S_0 \)    | \( \eta_c \) | 2980.4 ± 1.2 | 2980 | 2980.4 |
| \( 1^3S_1 \)    | \( J/\psi \) | 3096.916 ± 0.011 | 3097 | 3096.9 |
| \( 1^1P_1 \)    | \( h_c \) | 3526.21 ± 0.25 | 3527 | 3526.2 |
| \( 1^3P_0 \)    | \( \chi_{c0} \) | 3415.16 ± 0.35 | 3430 | 3397.4 |
| \( 1^3P_1 \)    | \( \chi_{c1} \) | 3510.59 ± 0.10 | 3503 | 3503.5 |
| \( 2^1S_0 \)    | \( \eta_c' \) | 3638 ± 5 | 3674 | 3683.1 |
| \( 2^3S_1 \)    | \( \psi' \) | 3686.093 ± 0.034 | 3765 | 3760.8 |
| \( 1^3D_1 \)    | \( \psi'' \) | 3770 ± 2.4 | 3855 | 3850.6 |
| \( 3^3S_1 \)    | \( \psi''' \) | 4040 ± 10 | 4291 | 4285.4 |

TABLE III: Percent of each partial wave in mixed charmonium states.

| \( n^{2S+1}L_J \) | \( c\bar{c} \) | \( P_S \% \) | \( P_D \% \) |
|-----------------|-----------|---------|---------|
| \( 1^3S_1 (1^3S_1 - 1^3D_1) \) | \( J/\psi \) | 99.93 | 0.07 |
| \( 2^3S_1 (2^3S_1 - 2^3D_1) \) | \( \psi' \) | 99.90 | 0.10 |
| \( 3^3S_1 (3^3S_1 - 3^3D_1) \) | \( \psi''' \) | 99.88 | 0.12 |
| \( 1^3D_1 (1^3D_1 - 1^3S_1) \) | \( \psi'' \) | 99.88 | 0.12 |

TABLE IV: The calculated charmonium mass spectrum as a function of the number of grid points \( N_{P_X}, N_{P_Y} \) and \( N_{\theta} \).

A table with values for different grid sizes and corresponding mass spectra is provided, showing how the mass spectrum changes with the number of grid points.
Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Appendix A: Fourier transformation of the regularized linear confining and Coulomb parts of the potential

The three-dimensional Fourier transformation of the potential \( V(r) \) is defined as:

\[
V(p, p') = \frac{1}{2\pi^2 q^2} \int_0^\infty dr V(r) \sin qr,
\]

where \( q = |p - p'| \). Fourier transformation of the regularized linear confining and Coulomb parts of the quark-antiquark potential is written as:

\[
V(p, p') = \frac{1}{2\pi^2 q^2} \left\{ \int_0^{r_c} dr V(r) \sin qr + V(r_c) \int_{r_c}^{\infty} dr \sin qr \right\}
\]

\[
= \frac{1}{2\pi^2 q^2} \left\{ \int_0^{r_c} dr V(r) \sin qr + V(r_c) \int_{r_c}^{\infty} dr \sin qr - V(r_c) \int_0^{r_c} dr \sin qr \right\}
\]

\[
= \frac{1}{2\pi^2 q^2} \left\{ \int_0^{r_c} dr V(r) \sin qr + V(r_c) \delta(q) - V(r_c) \int_0^{r_c} dr \sin qr \right\},
\]

where potential is kept fixed at cutoff \( r_c \). Therefore inserting the linear \( V(r) = \sigma r \), and Coulomb \( V(r) = f_c \alpha_s/r \), parts of quark-antiquark potential in above equation and calculation of corresponding integrals analytically, yields:

\[
V(p, p') = \sigma \left[ \delta(q) r_c + \frac{1}{2\pi^2 q^4} (2 \cos(q r_c) - 2 + q r_c \sin(q r_c)) \right], \quad \text{(A3)}
\]

\[
V(p, p') = f_c \alpha_s \left[ \delta(q) \frac{r_c}{r_c} + \frac{1}{2\pi^2 q^4} (1 - \sin(q r_c)) \right]. \quad \text{(A4)}
\]

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