Conﬁned states in the tight-binding model on the hexagonal golden-mean tiling

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Abstract. We study the tight-binding model with two distinct hoppings \((t_L, t_S)\) on the two-dimensional hexagonal golden-mean tiling and examine the conﬁned states with \(E = 0\), where \(E\) is the eigenenergy. Some conﬁned states found in the case \(t_L = t_S\) are exact eigenstates even for the system with \(t_L \neq t_S\), where their amplitudes are smoothly changed. By contrast, the other states are no longer eigenstates of the system with \(t_L \neq t_S\). This may imply the existence of macroscopically degenerate states which are characteristic of the system with \(t_L = t_S\), and that a discontinuity appears in the number of the conﬁned states in the thermodynamic limit.

1. Introduction
Quasicrystals have attracted much interest since the discovery of the quasicrystalline phase of Al-Mn alloy [1]. Among them, electron correlations in quasicrystals have actively been discussed after the observation of quantum critical behavior in Au-Al-Yb [2]. Recently, long-range correlations have been observed—such as superconductivity in the Al-Zn-Mg quasicrystal [3] and the ferromagnetically ordered states in the Au-Ga-Gd and Au-Ga-Tb quasicrystals [4]. These experiments have necessarily stimulated further theoretical investigations on electron correlations in quasiperiodic tilings [5–14].

A simple example of such investigations is the study of magnetically ordered states in the Hubbard model on the bipartite quasiperiodic tilings, i.e., the Penrose [15, 16], Ammann-Beenker [17–19], and Socolar dodecagonal tilings [20]. Non-interacting systems on the above tilings have a common feature in the density of states, namely, macroscopically degenerate states at \(E = 0\), so-called conﬁned states. These play an essential role for stabilizing the magnetically ordered states, in particular, in the weak coupling regime. Therefore, to understand magnetic properties of quasiperiodic tilings, it is important to examine their conﬁned states under the tight-binding model [16, 19–23].

Recently, the hexagonal golden-mean tiling has been introduced [24], with a section of the tiling and its constituent tiles shown in Fig. 1. This tiling is composed of large rhombuses, parallelograms, and small rhombuses, and, one of its important features is the existence of two length scales which is in contrast to the Penrose, Ammann-Beenker, and Socolar dodecagonal tilings. In our previous paper [25], we have considered the vertex model on the hexagonal golden-mean tiling, where the hopping integral on each edge is assumed to be equivalent. However, it
Figure 1. (a) Hexagonal golden-mean tiling [24]. Blue thin and red bold lines indicate the hopping integrals $t_L$ and $t_S$ defined on the long and short length edges. Solid circles indicate the $F_0$, $F_4$, and $C_0$ vertices. (b) Large rhombus, parallelogram, and small rhombus.

is also instructive to clarify confined state properties in the tight-binding model with distinct hoppings.

The paper is organized as follows: in Sec. 2, we introduce the tight-binding model on the hexagonal golden-mean tiling and show the density of states. In Sec. 3, we discuss confined state properties in the model and compare our results to our previous work and the Penrose tiling. A summary is given in the last section.

2. Tight-binding model on the hexagonal golden-mean tiling

Here we briefly summarise relevant properties of the hexagonal golden-mean tiling as introduced in the original paper [24]. The tiling can be generated using deflation rules for eight distinct directed tiles, and there are 32 allowed vertex configurations. Important for further discussion are the $F_0$, $F_4$, and $C_0$ vertices. The $F_0$ vertex is located at the centre of six adjacent large rhombuses and locally has 6-fold rotational symmetry, while the $F_4$ and $C_0$ vertices locally have 3-fold rotational symmetry.

As the tiling is multi-length-scale, we can introduce two kinds of hopping integrals in the tight-binding model. The tight-binding model with two distinct hoppings is given as

$$H = -t_L \sum_{(ij)} (c_i^\dagger c_j + \text{H.c.}) - t_S \sum_{(ij)} (c_i^\dagger c_j + \text{H.c.}),$$

where $c_i$ ($c_i^\dagger$) annihilates (creates) an electron at the $i$th site. $t_L(t_S)$ denotes the transfer integral
between the nearest neighbour pairs on the long (short) bonds, which are shown as the thin (bold) lines in Fig. 1(a). Then, we examine the density of states as

\[ D(E) = \frac{1}{N} \sum_i \delta(E - \epsilon_i), \]

where \( \epsilon_i \) is the \( i \)th eigenvalue of the Hamiltonian eq. (1) and \( N \) is the number of sites. Figures 2(a), (b), and (c) show the density of states for the tight-binding models with \((t_L, t_S) = (1, 0.5), (1, 1), \) and \((0.5, 1). \) (d) shows the integrated densities of states

\[ I(E) = \int_{-\infty}^{E} D(\epsilon) d\epsilon. \]

Figure 2. Density of states for the tight-binding model on the hexagonal golden-mean tiling when \((t_L, t_S) = (a) (1,0.5), (b) (1,1), \) and \((c) (0.5,1). \) (d) shows the integrated densities of states \( I(E) = \int_{-\infty}^{E} D(\epsilon) d\epsilon. \)
3. Constrained state properties
The macroscopically degenerate states at $E = 0$ can be described in a simple form by considering their appropriate linear combination. Some constrained states for the case $t_L = t_S$ have explicitly been shown in the previous paper [25], where each constrained state has amplitudes in the finite region of real space. Since a certain finite region is repeated quite regularly in the tiling, the number of constrained states is infinite, leading to the delta-function-like peak in the density of states.

We now consider the tight-binding model eq. (1) with $t_L \neq t_S$. Some simple examples around $F_0$ vertices are shown in Fig. 3, where the $F_0$ vertex is located at the centre of six adjacent large rhombuses. Two constrained states $\Psi_1$ and $\Psi_2$ are located inside almost the same region. Since both

\[
\begin{align*}
\Psi_1 & \\
\Psi_2 & \\
\Psi_3 & \\
\Psi_4 & 
\end{align*}
\]

Figure 3. Four confined states around the $F_0$ vertices, shown as solid circles. The values at the vertices represent the amplitudes of the confined states with $s = t_S/t_L$.

states are reduced versions of the simple ones for $t_L = t_S$ obtained in the previous paper [25], we can say that both constrained states are not changed in their properties, and each fraction is thereby given as $1/(4\pi^s)$. Namely, these confined states have amplitudes in the distinct sublattices. This means that the staggered magnetization is induced by the infinitesimal Coulomb interactions if one considers the Hubbard model. Similar behavior appears in the confined states $\Psi_3$ and $\Psi_4$. However, each of the previously obtained confined states with $t_L = t_S$ are not necessarily shared in the model with $t_L \neq t_S$. Figure 4(a) shows the confined state around the $F_4$ vertex for the model with $t_L = t_S$, which was labelled as $\Phi_1$ in our paper [25]. When $t_L \neq t_S$, this state is not confined around the $F_4$ vertex; in fact, we can find no eigenstates with $E = 0$ in the circular
Figure 4. (a) Confined state $\Phi_1$ around the $F_4$ vertex, shown as a solid circle and (b) two confined states $\Phi_2$ and $\Phi_3$ around the $C_0$ vertex, shown as an open circle, for the tight-binding model with $t_L = t_S$ [25]. The numbers at the vertices colored by the red, blue, and green represent the amplitudes of the confined states $\Phi_1$, $\Phi_2$, $\Phi_3$, respectively.

region shown in Fig. 4(a) by numerically diagonalizing the Hamiltonian,

$$H' = H + \sum_i V_i c_i^\dagger c_i,$$

(3)

where $V_i$ is the potential in the outside of the circular region. We also examine the confined states around the $C_0$ vertex. In the case of $t_L = t_S$, there are five confined states (three $\Phi_2$ and two $\Phi_3$) in the circular region shown in Fig. 4(b). When the ratio is away from the condition $t_L = t_S$, the number of the confined states is changed from five to three. However, the three confined states are difficult to describe by the simple form (linear combination). We can therefore say that two confined states exist which are inherent in the special case with $t_L = t_S$, in contrast to the altered confined states discussed above. This is not inconsistent with the fact that, in the large cluster treatment, the eigenstates with $E = 0$ are not changed by varying $t_L/t_S$ since some eigenstates with $E = 0$ exist with amplitudes at the edges of the system. Therefore, magnetic properties in the weak coupling limit should depend on the ratio $t_L/t_S$.

We also compare our results to the confined state properties in the Penrose tiling. The Penrose tiling is composed of fat and skinny rhombuses with a unit length scale. However, we know from edge-matching rules that there are two kinds of edges in the deflation rule for the directed fat and skinny rhombuses, as shown in Fig. 5. This allows us to introduce distinct hoppings ($t_1$, $t_2$) in the tight-binding model on the Penrose tiling, where $t_1(t_2)$ is the hopping integral defined on the bond with a single (double) arrow. The vertex model on the Penrose tiling has been examined in detail and there are six types of confined states when $t_1 = t_2$ [21,22]. We confirm that six states are still confined in the same region, with some modifications in the amplitudes, as shown in Fig. 5. This means that the fraction of the confined states is not changed by the introduction of the distinct hopping integrals in the tight-binding model on the Penrose
Figure 5. (a) Deflation rule of fat and skinny rhombuses in the Penrose tiling. (b) Six types of the confined states in the tight-binding model on the vertices of the Penrose tiling. The numbers at the vertices represent the amplitudes of confined states with $s = t_2/t_1$.

This is in contrast to the case in the hexagonal golden-mean tiling as discussed above. We also note that the two hopping integrals cannot be introduced into the Ammann-Beenker and Socolar dodecagonal tilings since the edges of their tiles are not divided into two groups, due to their geometry. Therefore, we can say that the confined state properties in the hexagonal golden-mean tiling are distinct from those in the others.
4. Summary
We have investigated the tight-binding model with two distinct hoppings on the two-dimensional hexagonal golden-mean tiling. Examining the confined states, we have shown that some confined states found in the case \( t_L = t_S \) are also exact eigenstates for the system, even when \( t_L \neq t_S \). Similarly, some states are no longer eigenstates of the system when \( t_L \neq t_S \). This may imply the existence of macroscopically degenerate states which are characteristic of the system with \( t_L = t_S \). These properties are distinct from those in the tight-binding models on the conventional quasiperiodic tilings such as the Penrose, Ammann-Beenker, and Socolar dodecagonal tilings. It is an important problem to clarify magnetic properties in the Hubbard model since the confined states with \( E = 0 \) play an essential role for the weak coupling limit, which will be discussed in the future.

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