ALL AROUND
THE SPINLESS SALPETER EQUATION

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ABSTRACT

We review some important topics related to the semirelativistic description of bound states by the spinless Salpeter equation: the special case of the Coulomb interaction, numerical approximation methods, and a way to avoid the problematic square-root operator of the relativistic kinetic energy.
1. Introduction

The “spinless Salpeter equation” represents a well-defined standard approximation to the Bethe–Salpeter formalism for the description of bound states within relativistic quantum field theories. It may be derived from the Bethe–Salpeter equation by eliminating, in full accordance with the spirit of an instantaneous interaction, any dependence on timelike variables, which leads to the Salpeter equation, and by neglecting any reference to the spin degrees of freedom of the involved bound-state constituents and restricting exclusively to positive-energy solutions.

The Hamiltonian governing the dynamics of the quantum system under consideration

- incorporates relativistic kinematics by involving the square-root operator of the relativistic kinetic energy, \( \sqrt{p^2 + m^2} \) for particles of mass \( m \) and momentum \( p \), but

- describes the forces acting between the bound-state constituents by an arbitrary coordinate-dependent static interaction potential \( V(x) \).

For the case of bound states consisting of two particles of equal mass \( m \), the generic Hamiltonian \( H \) in the center-of-momentum frame of these constituents, expressed in terms of the relative momentum \( p \) and the relative coordinate \( x \), reads

\[
H = 2\sqrt{p^2 + m^2} + V(x) .
\]

(1)

The equation of motion involving this type of Hamiltonian,

\[
H|\psi\rangle = E|\psi\rangle ,
\]

with energy eigenvalues \( E \) and corresponding Hilbert-space eigenstates \( |\psi\rangle \), has been widely used, e. g., for the (semi-)relativistic description of hadrons as bound states of quarks within the framework of potential models.

2. The Spinless Relativistic Coulomb Problem

Without doubt, a central rôle in physics is played by the Coulomb potential \( V_C(r) \). This Coulomb potential is a spherically symmetric potential, i. e., one which depends only on the radial coordinate \( r \equiv |x| \); its interaction strength is parametrized by some coupling constant \( \kappa \):

\[
V(x) = V_C(r) = -\frac{\kappa}{r} , \quad \kappa > 0 .
\]

(2)

The bound-state problem defined by the semirelativistic Hamiltonian of Eq. (1) with the Coulomb potential (2) is what we call the “spinless relativistic Coulomb problem.”

2.1. (Mostly) Analytic Results

Over the past years, the spinless relativistic Coulomb problem has been the subject of intense study. We summarize, in chronological order, the knowledge gained so far:
Herbst, in a rigorous mathematical discussion, developed the complete spectral theory of the one-particle counterpart of the operator $H_1$, $H_2$—from which one may directly deduce for the two-particle semirelativistic Coulombic Hamiltonian under consideration its essential self-adjointness for $\kappa \leq 1$ and the existence of its Friedrichs extension up to some critical value, viz.,

$$\kappa_{cr} = \frac{4}{\pi},$$

of the coupling constant—and derived some strict lower bound on the energy $E_0$ of the ground state which translates in the two-particle case to

$$E_0 \geq 2 m \sqrt{1 - \left(\frac{\pi \kappa}{4}\right)^2} \quad \text{for } \kappa < \frac{4}{\pi}.$$

Durand and Durand presented in an involved analysis of the spinless relativistic Coulomb problem for the particular case of vanishing orbital angular momentum of the bound-state constituents the explicit construction of the analytic solution for the corresponding wave function.

Castorina et al. generalized the behaviour of this wave function near the origin, that is, for small relative distances of the bound-state constituents, to arbitrary values of the orbital angular momentum.

Hardekopf and Sucher carried out a comprehensive numerical investigation of one- and two-particle relativistic wave equations for both spin-0 as well as spin-$\frac{1}{2}$ particles. Their numerical results for the ground-state energy $E_0$ of the spinless relativistic Coulomb problem are consistent with the lower bound of Herbst and provide no evidence that this ground-state energy indeed approaches zero as the coupling constant $\kappa$ rises to its critical value $\kappa_{cr} = 4/\pi$, that is,

$$E_0(\kappa = \kappa_{cr}) \neq 0.$$

Martin and Roy improved the lower bound of Herbst somewhat to

$$E_0 \geq 2 m \sqrt{1 + \sqrt{1 - \kappa^2}/2} \quad \text{for } \kappa < 1.$$

Raynal et al. made use of a generalized version of the “local-energy” theorem. Assume

1. that the Fourier transform $\hat{V}(p)$ of the interaction potential $V(x)$ is strictly negative, except at infinity, as is the case for the Coulomb potential $E$,
2. that the spectrum of the Hamiltonian under consideration, $H$, is discrete, and
3. that the ground state of the Hamiltonian exists.

Define the local energy
\[ \mathcal{E}(p) \equiv T(p) + \frac{\int d^3q \, \tilde{V}(p-q) \, \phi(q)}{\phi(p)}; \]
here \( T(p) \) represents the kinetic energy, which in our case is, of course, given by
\[ T(p) = 2\sqrt{p^2 + m^2}, \]
and \( \phi(p) \) denotes some suitably chosen, positive trial function,
\[ \phi(p) > 0. \]

Then the lowest-lying eigenvalue of the Hamiltonian \( H, E_0 \), is bounded by
\[ \inf_p \mathcal{E}(p) \leq E_0 < \sup_p \mathcal{E}(p). \]

With the help of this theorem, Raynal et al. succeeded in restricting numerically the ground-state energy eigenvalue of the semirelativistic Hamiltonian \( [\mathcal{H}, \mathcal{B}] \), \( \mathcal{B} \), considered as a function of the coupling strength \( \kappa \), to some remarkably narrow band. In particular, at the critical value \( \kappa_{\text{cr}} = 4/\pi \) of this coupling constant \( \kappa \) they found
\[ 0.4825 < \frac{E_0(\kappa = \kappa_{\text{cr}})}{2m} < 0.4843, \]
which proved the nonvanishing of the ground-state energy eigenvalue \( E_0 \) at the critical coupling constant.

- Le Yaouanc et al. gave the systematic series expansion of the eigenvalues of the Hamiltonian \( [\mathcal{H}, \mathcal{B}] \), \( \mathcal{B} \) in powers of the coupling constant \( \kappa \) up to and including the order \( O(\kappa^6) \) for arbitrary states with vanishing orbital angular momentum. For the ground-state energy \( E_0 \) this expansion reads
\[ \frac{E_0}{2m} = 1 - \frac{\kappa^2}{8} - \frac{5\kappa^4}{128} + \frac{\kappa^5}{12\pi} + \frac{\kappa^6}{64} \ln \kappa + O(\kappa^6). \]

Higher orders of this expansion in the form given by Le Yaouanc et al., however, involve explicitly the corresponding wave functions and their derivatives, which renders difficult to derive any profit from it. Up to order \( O(\kappa^4) \) the above result coincides with the improved lower bound of Martin and Roy.

The presence of non-analytic \( \ln \kappa \) terms in the expansion quoted by Le Yaouanc et al. gives a clear hint at the non-analytic nature to be expected for the energy eigenvalues of the spinless relativistic Coulomb problem as functions of the coupling strength \( \kappa \).
2.2. Analytic Upper Bounds

In view of the fact that exact solutions to the spinless relativistic Coulomb problem are still lacking, we seek for an analytic upper bound on the ground-state energy level of the semirelativistic Coulombic Hamiltonian (1), (2). To this end, we make use of a rather standard variational technique, which proceeds along the steps of the following, extremely simple recipe:

1. Choose a suitable set of trial states \{|\lambda\rangle\}. The different members of this set are distinguished from each other by some sort of variational parameter \(\lambda\).

2. Compute the set of expectation values of the Hamiltonian under consideration, \(H\), with respect to these trial states \(|\lambda\rangle\) in order to obtain

\[ E(\lambda) \equiv \langle \lambda | H | \lambda \rangle . \]

3. Determine, from the first derivative, that value \(\lambda_{\text{min}}\) of the variational parameter \(\lambda\) which minimizes the resulting, \(\lambda\)-dependent expression \(E(\lambda)\).

4. Compute \(E(\lambda)\) at the point of the minimum \(\lambda_{\text{min}}\) to find in this way the minimal expectation value \(E(\lambda_{\text{min}})\) of the Hamiltonian \(H\) in the Hilbert-space subsector of the chosen trial states \(|\lambda\rangle\).

This minimum \(E(\lambda_{\text{min}})\) provides, of course, only an upper bound to the proper energy eigenvalue \(E\) of the Hamiltonian \(H\):

\[ E \leq E(\lambda_{\text{min}}) . \]

Let us apply the above simple recipe to the case of the Coulomb potential, Eq. (2). For the Coulomb potential, the most reasonable choice of trial states is obviously the one for which the coordinate-space representation \(\psi(x)\) of the states \(|\lambda\rangle\) for vanishing radial and orbital angular momentum quantum numbers is given by the hydrogen-like trial functions

\[ \psi(x) = \sqrt{\frac{\lambda^3}{\pi}} \exp(-\lambda r), \quad \lambda > 0 . \]

For this particular set of trial functions we obtain for the expectation values we shall be interested in, namely, the ones of the square of the momentum \(p\) and of the inverse of the radial coordinate \(r\), respectively, evaluated with respect to the trial states \(|\lambda\rangle\):

\[ \langle \lambda | p^2 | \lambda \rangle = \lambda^2 \]

and

\[ \langle \lambda | \frac{1}{r} | \lambda \rangle = \lambda . \]

Now, as an immediate consequence of the fundamental postulates of any quantum theory, the expectation value of some given Hamiltonian \(\hat{H}\) taken with respect to any normalized Hilbert-space state and, therefore, in particular, taken with respect to any
of the above trial states must necessarily be larger than or equal to that eigenvalue $E_0$ of the Hamiltonian $H$ which corresponds to its ground state:

$$E_0 \leq E(\lambda) \equiv \langle \lambda | H | \lambda \rangle .$$

The application to the semirelativistic Hamiltonian of Eq. (1) yields for the right-hand side of this inequality

$$E(\lambda) = 2 \langle \lambda | \sqrt{\mathbf{p}^2 + m^2} | \lambda \rangle + \langle \lambda | V(\mathbf{x}) | \lambda \rangle . \quad (3)$$

In order to obtain a first crude estimate, we may take advantage of some trivial but nevertheless fundamental inequality. This inequality relates the expectation values of both the first and second powers of any self-adjoint but otherwise arbitrary operator $O = O^\dagger$ taken with respect to arbitrary Hilbert-space vectors $|$ in the domain of $O$ normalized to unity; it reads

$$|\langle O \rangle| \leq \sqrt{\langle O^2 \rangle} .$$

Applying this inequality to the kinetic-energy part of the above expression for $E(\lambda)$, we may replace, in turn, $E(\lambda)$ by an upper bound which can be evaluated much easier than $E(\lambda)$ itself:

$$E(\lambda) \leq 2 \sqrt{\langle \lambda | \mathbf{p}^2 | \lambda \rangle + m^2 + \langle \lambda | V(\mathbf{x}) | \lambda \rangle} .$$

Identifying in this—as far as its evaluation is concerned, simplified—upper bound the until-now general potential $V(\mathbf{x})$ with the Coulomb potential (2) and inserting both of the $\lambda$-dependent expectation values given above implies

$$E(\lambda) \leq 2 \sqrt{\lambda^2 + m^2 - \kappa \lambda} .$$

From this intermediate result, upon inspection of the limit $\lambda \to \infty$, we may state already at this rather early stage that, for the semirelativistic Hamiltonian (1), (2) to be bounded from below at all, the Coulombic coupling strength $\kappa$ has to stay below a certain critical value:

$$\kappa \leq 2 .$$

The value of the variational parameter $\lambda$ which minimizes the latter upper bound may be determined from the first derivative of this expression with respect to $\lambda$:

$$\lambda_{\text{min}} = \frac{m \kappa}{2 \sqrt{1 - \frac{\kappa^2}{4}}} .$$

For this particular value of $\lambda$, by shuffling together all our previous inequalities, we find that the energy eigenvalue corresponding to the ground state of the semirelativistic Hamiltonian (1) with the Coulomb potential (2), $E_0$, is bounded from above by

$$E_0 \leq 2 m \sqrt{1 - \frac{\kappa^2}{4}} .$$
The reality of this upper bound requires again $\kappa \leq 2$.

It is straightforward to improve this crude estimate by direct evaluation of Eq. (3). For our choice of trial functions, the resulting upper bound on the ground-state energy $E_0$ may be expressed in terms of the hypergeometric function $F$:

$$E_0 \leq \left[ \frac{128}{15 \pi} F \left( -\frac{1}{2}, 2; \frac{7}{2}; 1 - \frac{m^2}{\lambda^2} \right) - \kappa \right] \lambda .$$

Quite obviously, in this case the minimizing value of the variational parameter $\lambda$ must be determined numerically. As before, the limit $\lambda \to \infty$ tell us that now

$$\kappa \leq \frac{16}{3 \pi} .$$

3. Numerical Approximation Methods

For an arbitrary interaction potential, it is, in general, not possible to find a closed solution to the spinless Salpeter equation. Therefore, several numerical approximation methods for the (numerical) solution of this equation have been introduced.

All of these numerical schemes aim at the conversion of the spinless Salpeter equation into an equivalent matrix eigenvalue problem, and confine themselves to the case of a spherically symmetric potential $V(r)$. Here, we briefly sketch the maybe most efficient among them, namely, the “semianalytical matrix method” developed by Lucha et al.

For states of definite orbital angular momentum $\ell$ described by some wave function $\psi(x)$, we define the reduced radial wave function $u(r)$ by

$$\psi(x) = u(r) \frac{r}{m} Y_{\ell m}(\theta, \phi) ;$$

here $Y_{\ell m}(\theta, \phi)$ are the spherical harmonics of angular momentum $\ell$ and projection $m$. For this reduced radial wave function $u(r)$, an integral representation of the spinless Salpeter equation may be found. Furthermore, since we consider only the case of equal (nonvanishing) masses $m$ of the bound-state constituents, we may eliminate from the kinetic-energy part of this integral equation any dependence on the mass $m$ by scaling the radial variable $r$ like

$$x := m r .$$

Introducing the scaled reduced radial wave function

$$\tilde{u}(x) := u \left( \frac{x}{m} \right) = u(r)$$

as well as the dimensionless energy eigenvalue

$$\tilde{E} := \frac{E}{m} .$$
and the dimensionless interaction potential

\[ \tilde{V}(x) := \frac{V\left(\frac{x}{m}\right)}{m} = \frac{V(r)}{m}, \]

this particular integral representation of the spinless Salpeter equation becomes

\[ \left[ \tilde{E} - \tilde{V}(x) \right] \tilde{u}(x) = \frac{2}{\pi} \int_0^\infty dy G_\ell(x, y) \left[ -\frac{d^2}{dy^2} + \frac{\ell(\ell + 1)}{y^2} + 1 \right] \tilde{u}(y), \quad (4) \]

where the kernel \( G_\ell \) is defined by

\[ G_\ell(x, y) = 2^\ell \eta^{\ell+1} \left( \frac{1}{z} \frac{\partial}{\partial z} \right)^\ell \frac{1}{z} \left[ (s-z)^{\ell/2} K_\ell\left(\sqrt{s-z}\right) - (s+z)^{\ell/2} K_\ell\left(\sqrt{s+z}\right) \right], \]

with

\[ s \equiv x^2 + y^2, \]
\[ z \equiv 2xy. \]

Here \( K_\ell \) is the modified Bessel function\(^9\) of the second kind of order \( \ell \).

In the free case, i. e., for

\[ V(r) \equiv 0, \]

as well as for potentials which are less singular than the Coulomb potential, i. e.,

\[ V(r) \propto \frac{1}{r^\eta} \quad \text{with} \quad \eta < 1, \]

the reduced radial wave function \( u(r) \) behaves for small \( r, r \to 0 \), asymptotically like

\[ u(r) \propto r^{\ell+1}. \]

Accordingly, we make the ansatz

\[ \tilde{u}(x) = x^{\ell+1} w(x). \]

With this substitution, the integro-differential equation \((4)\) is equivalent to

\[ \left[ \tilde{E} - \tilde{V}(x) \right] x^{\ell+1} w(x) = \frac{2}{\pi} \int_0^\infty dy G_\ell(x, y) y^{\ell+1} \left[ 1 - \frac{d^2}{dy^2} - \frac{2(\ell + 1)}{y} \frac{d}{dy} \right] w(y). \]

In order to solve this scaled form of the spinless Salpeter equation, we expand the solution \( w(x) \) of this integro-differential equation into a complete orthonormal system \( \{f_n(x), \ n = 0, 1, 2, \ldots\} \) of basis functions for \( L_2(R^3) \):

\[ w(x) = \sum_{n=0}^N \lambda_n f_n(x), \]
with some set of real coefficients $\lambda_n$. For finite $N$, this expansion represents, of course, only an approximation to the exact solution $w(x)$. The crucial point of the present approach is our rather sophisticated choice of the basis functions $f_n$ which, at least in principle, allows for a thorough analytical treatment of the spinless Salpeter equation:

$$f_n(x) := \sqrt{2} \exp(-x) L_n(2x),$$

where $L_n(x)$ are the Laguerre polynomials.

After some straightforward algebra, the scaled spinless Salpeter equation may be cast into the form of a matrix eigenvalue equation for the coefficient vector $\lambda \equiv \{\lambda_n\}$ in the above expansion of the solution $w(x)$. In self-explanatory matrix notation, this eigenvalue equation is given by

$$\tilde{E} \lambda = \left( P^{(\ell)} \right)^{-1} \left[ \left( T^{(\ell)} \right)^T + V^{(\ell)} \right] \lambda,$$

where the “power matrix” $P^{(\ell)}_{nm}$ and the “potential matrix” $V^{(\ell)}_{nm}$ are defined by

$$P^{(\ell)}_{nm} := \int_0^\infty dx \ x^{\ell+1} f_n(x) f_m(x) = P^{(\ell)}_{mn},$$

$$V^{(\ell)}_{nm} := \int_0^\infty dx \ x^{\ell+1} \tilde{V}(x) f_n(x) f_m(x) = V^{(\ell)}_{mn},$$

and the “kinetic matrix” $T^{(\ell)}_{nm}$ represents the action of the kinetic term in the spinless Salpeter equation on the vector of basis functions $f_n$. In this way, the solution of the spinless Salpeter equation can be reduced to a simple matrix eigenvalue problem. The eigenvalues of this equation are the energies $E$ of the bound state under consideration. The corresponding eigenvectors $\{\lambda_n\}$ give the radial wave functions $u(r)$ according to

$$u(r) = (mr)^{\ell+1} \sum_{n=0}^N \lambda_n f_n(mr).$$

For $N = \infty$ this treatment would be exact. For $N < \infty$ it provides an approximation to the exact solution of increasing accuracy with increasing $N$, that is, with increasing size of the involved matrices. As is evident from the above construction, this procedure works analytically, at least for all potentials of the type “power times exponential,” i. e., for all those potentials which involve only terms of the form

$$r^n \exp(-b_n r),$$

with (maybe vanishing) constants $b_n$.

Let us illustrate this simple prescription for the case of bound states with vanishing orbital angular momentum $\ell$ of their constituents, that is, $\ell = 0$. All we need are the
explicit expressions of the kinetic matrix \( T^{(0)} \) and of the inverse of the power matrix \( P^{(0)} \) for \( \ell = 0 \). The first few entries \( T_{nm}^{(0)} \) in the kinetic matrix \( T^{(0)} \) read

\[
T_{nm}^{(0)} = \frac{2}{\pi} \frac{8}{(2n+2m+3)!!} S_{nm},
\]

where the matrix \( S \) is given by

\[
S = \begin{pmatrix}
1 & -3 & -5 & -21 & \cdots \\
-1 & 81 & -375 & -1029 & \cdots \\
-1 & -225 & 13125 & -77175 & \cdots \\
-3 & -441 & -55125 & 3565485 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

(Note the range of the indices \( n \) and \( m \): \( n, m = 0, 1, \ldots, N \).) The elements \( P_{nm}^{(0)} \) of the power matrix \( P^{(0)} \) read explicitly

\[
P_{nm}^{(0)} = \frac{1}{2} \int_0^\infty dx \, x \exp(-x) L_n(x) L_m(x)
\]

\[
= \frac{1}{2} \begin{cases}
2n+1 & \text{for } m = n \\
-m & \text{for } m = n+1 \\
-n & \text{for } m = n-1 \\
0 & \text{else}
\end{cases}
\]

\[
= \frac{1}{2} \begin{pmatrix}
1 & -1 & 0 & 0 & \cdots \\
-1 & 3 & -2 & 0 & \cdots \\
0 & -2 & 5 & -3 & \cdots \\
0 & 0 & -3 & 7 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

The inverse of this matrix, required for the matrix eigenvalue equation (3), depends explicitly on the size \( N \) of the involved matrices and is given by

\[
(P^{(0)})^{-1}_{nm} = \sum_{k=n}^{N} \frac{2}{k+1}
\]

for \( n = 0, 1, \ldots, N \),

\[
(P^{(0)})^{-1}_{nm} = (P^{(0)})^{-1}_{mn} = (P^{(0)})^{-1}_{nn}
\]

for \( n = 0, 1, \ldots, N, \ m = 0, 1, \ldots, n-1 \).

For instance, for \( N = 3 \) this inverse reads

\[
(P^{(0)})^{-1}_{4x4} = \frac{1}{6} \begin{pmatrix}
25 & 13 & 7 & 3 \\
13 & 13 & 7 & 3 \\
7 & 7 & 7 & 3 \\
3 & 3 & 3 & 3
\end{pmatrix}.
\]
Consider, for example, the so-called funnel potential \( V_F(r) \), which depends on just two parameters, namely, on the Coulomb coupling constant \( \kappa \) and on the slope \( a \) of the linear term: \( V(r) = V_F(r) \equiv -\kappa/r + a \cdot r \). This funnel-shaped potential represents the prototype of all “realistic,” that is, phenomenologically acceptable, “QCD-inspired” static interquark potentials proposed for the description of hadrons as bound states of (constituent) quarks in the framework of potential models, the quarks inside a hadron being bound by the strong interactions arising from quantum chromodynamics. Now, according to the above definition, the scaled form of the funnel potential, \( \tilde{V}_F(x) \), reads
\[
\tilde{V}_F(x) = -\frac{\kappa}{x} + \frac{a}{m^2} x ,
\]
which entails for the corresponding potential matrix
\[
V^{(\ell)}_F = -\kappa P^{(-1)} + \frac{a}{m^2} P^{(\ell+1)} .
\]
In particular, specializing again to the case \( \ell = 0 \), the matrix elements of the funnel potential, taken with respect to \( \ell = 0 \) states, are given by
\[
V^{(0)}_F = -\kappa + \frac{a}{m^2} P^{(1)} .
\]
Since, according to the orthonormalization condition for the basis functions \( \{f_n(x)\} \), the power matrix \( P^{(-1)} \) is identical to the unit matrix,
\[
P^{(-1)} = 1_{N \times N} ,
\]
we have
\[
V^{(0)}_F = -\kappa + \frac{a}{m^2} P^{(1)} .
\]
Working out in detail the \( \ell = 1 \) power matrix \( P^{(1)}_{nm} \), we obtain
\[
\begin{align*}
P^{(1)}_{nn} &= \frac{3 n (n + 1) + 1}{2} \quad \text{for } n = 0, 1, \ldots, N , \\
P^{(1)}_{n,n+1} &= -n (n + 2) - 1 \quad \text{for } n = 0, 1, \ldots, N - 1 , \\
P^{(1)}_{n,n+2} &= \frac{n (n + 3) + 2}{4} \quad \text{for } n = 0, 1, \ldots, N - 2 , \\
P^{(1)}_{nm} &= 0 \quad \text{else} .
\end{align*}
\]
Explicitly, the first entries of \( P^{(1)} \) read
\[
P^{(1)} = \frac{1}{2} \begin{pmatrix}
1 & -2 & 1 & 0 & \cdots \\
-2 & 7 & -8 & 3 & \cdots \\
1 & -8 & 19 & -18 & \cdots \\
0 & 3 & -18 & 37 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} .
\]
In summary, shuffling together the above results
• for the kinetic matrix $T^{(0)}$,
• for the inverse of the power matrix $P^{(0)}$, and
• for the potential matrix $V^{(0)}$ of the potential under consideration

one ends up with a well-defined matrix eigenvalue equation, which represents the $\ell = 0$ special case of the general matrix form (5) of the spinless Salpeter equation.

4. Effectively Semirelativistic Hamiltonians of Nonrelativistic Form

Almost all of the troubles which one encounters when trying to apply the spinless Salpeter equation are obviously brought about by the nonlocality of the “square-root” operator of the relativistic kinetic energy, $\sqrt{\mathbf{p}^2 + m^2}$, in the Hamiltonian $H$, Eq. (1). In contrast to the nonrelativistic limit, obtained from the expansion of the square root up to the lowest $\mathbf{p}^2$-dependent order, $\sqrt{\mathbf{p}^2 + m^2} = m + \mathbf{p}^2/(2m) + \ldots$, the presence of this relativistic kinetic-energy operator prevents, in general, a thoroughly analytic discussion; one is forced to rely on numerical solutions of the problem. This (intrinsic) difficulty of any (semi-)relativistic formalism may be circumvented by approximating the Hamiltonian (1) by the corresponding “effectively semirelativistic” Hamiltonian, which retains apparently the easier-to-handle nonrelativistic kinematics but resembles its relativistic counterpart to the utmost possible extent by replacing some of its basic parameters by effective ones which depend, in a well-defined manner, on the square of the relevant momentum $\mathbf{p}$. The main idea of the construction of these “effectively semirelativistic” Hamiltonians is as follows.

Applying again the above-mentioned fundamental inequality

$$|\langle O \rangle| \leq \sqrt{\langle O^2 \rangle},$$

which holds for any self-adjoint operator $O = O^\dagger$ and arbitrary Hilbert-space vectors $|\rangle$ (in the domain of $O$) normalized to unity, to the relativistic kinetic-energy operator $\sqrt{\mathbf{p}^2 + m^2}$ yields

$$\langle \sqrt{\mathbf{p}^2 + m^2} \rangle \leq \sqrt{\langle \mathbf{p}^2 \rangle + m^2}.$$

By employing this inequality, we obtain for an arbitrary expectation value $\langle H \rangle$ of the semirelativistic Hamiltonian $H$, Eq. (1),

$$\langle H \rangle = 2 \langle \sqrt{\mathbf{p}^2 + m^2} \rangle + \langle V \rangle$$

$$\leq 2 \sqrt{\langle \mathbf{p}^2 \rangle + m^2} + \langle V \rangle$$

$$= 2 \frac{\langle \mathbf{p}^2 \rangle + m^2}{\sqrt{\langle \mathbf{p}^2 \rangle + m^2}} + \langle V \rangle$$

$$= \left\langle 2 \frac{\mathbf{p}^2 + m^2}{\sqrt{\langle \mathbf{p}^2 \rangle + m^2}} + V \right\rangle. \quad (6)$$

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From now on we specify the generic Hilbert-space vectors in all expectation values to be the eigenstates of our Hamiltonian \( H \). In this case the expectation value of \( H \), \( \langle H \rangle \), as appearing, e. g., in Eq. (3), becomes the corresponding semirelativistic energy eigenvalue \( E \), i. e.,

\[
E \equiv \langle H \rangle ,
\]

and the inequality (3) tells us that this energy eigenvalue is bounded from above by

\[
E \leq \left\langle 2 \frac{p^2 + m^2}{\sqrt{\langle p^2 \rangle + m^2}} + V \right\rangle .
\]

The operator within brackets on the right-hand side of this inequality may be regarded as some “effectively semirelativistic” Hamiltonian \( H_{\text{eff}} \) which possesses, quite formally, the structure of a nonrelativistic Hamiltonian

\[
H_{\text{eff}} \equiv 2 \frac{p^2 + m^2}{\sqrt{\langle p^2 \rangle + m^2}} + V
\]

but involves, however, the effective mass

\[
\hat{m} = \frac{1}{2} \sqrt{\langle p^2 \rangle + m^2}
\]

as well as the effective nonrelativistic potential

\[
V_{\text{eff}} = \frac{2m^2}{\sqrt{\langle p^2 \rangle + m^2}} - \sqrt{\langle p^2 \rangle + m^2} + V
\]

\[
= 2\hat{m} - \frac{\langle p^2 \rangle}{\hat{m}} + V.
\]

The effective mass \( \hat{m} \) given by Eq. (8) and the constant, i. e., coordinate-independent, term in the effective potential \( V_{\text{eff}} \) of Eq. (9),

\[
2\hat{m} - \frac{\langle p^2 \rangle}{\hat{m}} ,
\]

obviously depend on the expectation value of the square of the momentum \( p \), that is, on \( \langle p^2 \rangle \), and will therefore differ for different energy eigenstates.

Motivated by our above considerations, we propose to approximate the true energy eigenvalues \( E \) of the semirelativistic Hamiltonian \( H \) of Eq. (1) by the corresponding “effective” energy eigenvalues \( E_{\text{eff}} \), defined as the expectation values of some effective Hamiltonian \( \tilde{H}_{\text{eff}} \) taken with respect to the eigenstates \( |\rangle_{\text{eff}} \) of its own,

\[
E_{\text{eff}} = \langle \tilde{H}_{\text{eff}} \rangle_{\text{eff}} .
\]
where the effective Hamiltonian $\tilde{H}_{\text{eff}}$, as far as its structure is concerned, is given by Eqs. (7) through (9) but is implicitly understood to involve the expectation values of $p^2$ with respect to the effective eigenstates $|\rangle_{\text{eff}}$ (that is, $\langle p^2 \rangle_{\text{eff}}$ in place of $\langle p^2 \rangle$):

$$\tilde{H}_{\text{eff}} = 4 \tilde{m} + \frac{p^2 - \langle p^2 \rangle_{\text{eff}}}{\tilde{m}} + V ,$$

with

$$\tilde{m} = \frac{1}{2} \sqrt{\langle p^2 \rangle_{\text{eff}} + m^2} .$$

Accordingly, the effective energy eigenvalues $E_{\text{eff}}$ are given by a rather simple formal expression, viz., by

$$E_{\text{eff}} = 4 \tilde{m} + \langle V \rangle_{\text{eff}} . \quad (10)$$

We intend to elaborate our general prescription for the construction of effectively semirelativistic Hamiltonians $\tilde{H}_{\text{eff}}$ in more detail for the particular case of power-law potentials depending only on the radial coordinate $r \equiv |x|$, i.e., for potentials of the form

$$V(r) = a r^n$$

with some constant $a$. The reason for this restriction is twofold:

1. On the one hand, for power-law potentials the general virial theorem\cite{virial} in its nonrelativistic form\cite{virial} appropriate for the present case,

$$\langle \frac{p^2}{\tilde{m}} \rangle_{\text{eff}} = \frac{1}{2} \langle r \frac{dV(r)}{dr} \rangle_{\text{eff}} ,$$

enables us to replace the expectation value of the potential in (10) immediately by a well-defined function of the expectation value of the squared momentum:

$$a \langle r^n \rangle_{\text{eff}} = \frac{2}{n} \langle \frac{p^2}{\tilde{m}} \rangle_{\text{eff}} .$$

This implies for the effective energy eigenvalues

$$E_{\text{eff}} = 4 \tilde{m} + \frac{2}{n} \langle \frac{p^2}{\tilde{m}} \rangle_{\text{eff}} . \quad (11)$$

2. On the other hand, we take advantage of the fact that for power-law potentials it is possible to pass, without change of the fundamental commutation relations between coordinate variables and their canonically conjugated momenta, from the dimensional phase-space variables adopted at present to new, dimensionless phase-space variables and to rewrite the Hamiltonian in form of a Hamiltonian which involves only these dimensionless phase-space variables. The eigenvalues
Applying this procedure, we find for the effective energy eigenvalues

\[ E_{\text{eff}} - 4 \tilde{m} + \langle p^2 \rangle_{\text{eff}} \tilde{m} = \left( \frac{\tilde{p}^2}{\tilde{m}} + a r^n \right)_{\text{eff}} = \left( \frac{a^2}{\tilde{m}^{n+1}} \right) \epsilon. \]

Combining both of the above expressions for \( E_{\text{eff}} \), we obtain a relation which allows us to determine \( \langle p^2 \rangle_{\text{eff}} \) unambiguously in terms of the dimensionless energy eigenvalues \( \epsilon \):

\[ \langle p^2 \rangle_{\text{eff}}^{2+n} = \frac{1}{4} \left( \frac{n}{2+n} \right)^{2+n} a^2 \epsilon^{2+n} \left( \langle p^2 \rangle_{\text{eff}} + m^2 \right). \] (12)

For a given power \( n \) this equation may be solved for \( \langle p^2 \rangle_{\text{eff}} \). Insertion of the resulting expression into Eq. (11) then yields the corresponding eigenvalue \( E_{\text{eff}} \) of the effectively semirelativistic Hamiltonian \( \tilde{H}_{\text{eff}} \).

Bound-state solutions are usually characterized by some radial quantum number \( n_r \) and orbital angular-momentum quantum number \( \ell \). For instance, for the harmonic oscillator, i.e., for \( n = 2 \), the dimensionless energy eigenvalues \( \epsilon \) are given by

\[ \epsilon = 2 N \]

with

\[ N = 2 n_r + \ell + \frac{3}{2}. \]

For the Coulomb potential \( V(r) = -\kappa/r \), that is, for \( n = -1 \), Eq. (12) reduces to a linear equation for the expectation value \( \langle p^2 \rangle_{\text{eff}} \). Inserting the well-known expression for the dimensionless energy eigenvalues \( \epsilon \) of the (nonrelativistic) Coulomb problem,

\[ \epsilon = -\frac{1}{(2 N)^2} \]

with

\[ N = n_r + \ell + 1, \]

we obtain from this linear equation for \( \langle p^2 \rangle_{\text{eff}} \)

\[ \langle p^2 \rangle_{\text{eff}} = \frac{\kappa^2 m^2}{16 N^2 - \kappa^2}, \]

and, after inserting this expression into Eq. (11), for the effective energy eigenvalue

\[ E_{\text{eff}} = \frac{m}{N} \frac{8 N^2 - \kappa^2}{\sqrt{16 N^2 - \kappa^2}}. \]
The capability of this effective formalism to imitate the semirelativistic treatment becomes apparent by inspecting, for instance, the behaviour of the energy eigenvalues $E_{\text{eff}}$ for large orbital angular momenta $\ell$. For the linear potential $V(r) = a r$, that is, for $n = 1$, the effective formalism yields (in the ultrarelativistic limit, i.e., for $m = 0$) linear Regge trajectories:

$$E_{\text{eff}}^2 = 9 a \ell .$$

This fits very well to the exact semirelativistic result,

$$E_{\text{SR}}^2 = 8 a \ell ,$$

but is in clear contrast to the nonrelativistic approach, which gives,

$$E_{\text{NR}}^2 = 9 \left( \frac{a^2}{4m} \right) \frac{2}{3} \ell^{4/3} .$$

5. Summary and Conclusion

In spite of the regrettable fact that, at present, not even for the Coulomb potential the exact solution for the ground-state energy is known, the spinless Salpeter equation is certainly a useful tool for the semirelativistic description of bound states consisting of scalar bosons only as well as of the spin-averaged spectra of bound states consisting of fermionic constituents.

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