Magnetic Susceptibility and Landau Diamagnetism of Quantum Collisional Maxwellian Plasmas

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With the use of correct expression of the electric conductivity of quantum collisional plasmas (A. V. Latyshev and A. A. Yushkanov, Transverse electrical conductivity of a quantum collisional plasma in the Mermin approach. - Theor. and Math. Phys., 175:(1), 559–569 (2013)) the kinetic description of a magnetic susceptibility is obtained and the formula for calculation of Landau diamagnetism is deduced.

Key words: magnetic susceptibility, transverse electric conductivity, Maxwellian collisional plasma, Landau diamagnetism.

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Introduction

Magnetisation of electron gas in a weak magnetic fields compounds of two independent parts (see, for example, [1]): from the paramagnetic magnetisation connected with own (spin) magnetic momentum of electrons (Pauli’s paramagnetism, W. Pauli, 1927) and from the diamagnetic magnetisation connected with quantization of orbital movement of electrons in a magnetic field (Landau diamagnetism, L. D. Landau, 1930).

Landau diamagnetism was considered till now for a gas of the free electrons. It has been thus shown, that together with original approach
developed by Landau, expression for diamagnetism of electron gas can be obtained on the basis of the kinetic approach [2].

The kinetic method gives opportunity to calculate the transverse dielectric permeability. On the basis of this quantity it's possible to obtain the value of the diamagnetic response.

However such calculations till now were carried out only for collisional-less case. The matter is that correct expression for the transverse dielectric permeability of quantum plasma existed till now only for gas of the free electrons. Expression known till now for the transverse dielectric permeability in a collisional case gave incorrect transition to the classical case [3]. So this expression were accordingly incorrect.

Central result from [4] connects the mean orbital magnetic moment, a thermodynamic property, with the electrical resistivity, which characterizes transport properties of material. In this work was discussed the important problem of dissipation (collisions) influence on Landau diamagnetism. The analysis of this problem is given with use of exact expression of transverse conductivity of quantum plasma.

In work [5] is shown that a classical system of charged particles moving on a finite but unbounded surface (of a sphere) has a nonzero orbital diamagnetic moment which can be large. Here is considered a non-degenerate system with the degeneracy temperature much smaller than the room temperature, as in the case of a doped high-mobility semiconductor.

In work [6] for the first time the expression for the quantum transverse dielectric permeability of collisional plasma has been derived. The obtained in [6] expression for transverse dielectric permeability satisfies to the necessary requirements of compatibility.

In the present work for the first time with use of correct expression for the transverse conductivity [6] the kinetic description of a magnetic susceptibility of quantum collisional Maxwellian plasmas is given. The formula for calculation of Landau diamagnetism for Maxwellian collisional
plasmas is deduced.

The graphic analysis of a magnetic susceptibility and comparison of a magnetic susceptibility Maxwellian and degenerate plasmas is made.

In work [7] the kinetic description has been considered magnetic susceptibility and Landau diamagnetism of the quantum collisional Maxwellian plasmas.

2. Magnetic susceptibility of quantum Maxwellian plasmas

Magnetization vector $\mathbf{M}$ of electron plasma is connected with current density $\mathbf{j}$ by the following expression [8]

$$\mathbf{j} = c \mathbf{rot} \mathbf{M},$$

where $c$ is the light velocity.

Magnetization vector $\mathbf{M}$ and a magnetic field strength $\mathbf{H} = \mathbf{rot} \mathbf{A}$ are connected by the expression

$$\mathbf{M} = \chi \mathbf{H} = \chi \mathbf{rot} \mathbf{A},$$

where $\chi$ is the magnetic susceptibility, $\mathbf{A}$ is the vector potential.

From these two equalities for current density we have

$$\mathbf{j} = c \mathbf{rot} \mathbf{M} = c \chi \mathbf{rot} (\mathbf{rot} \mathbf{A}) = c \chi [\nabla (\nabla \cdot \mathbf{A}) - \Delta \mathbf{A}].$$

Here $\Delta$ is the Laplace operator.

Let the scalar potential is equal to zero. Vector potential we take orthogonal to the direction of a wave vector $\mathbf{q}$ ($\mathbf{q} \mathbf{A} = 0$) in the form of a harmonic wave

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0 e^{i(\mathbf{q} \mathbf{r} - \omega t)}.$$

Such vector field is solenoidal

$$\text{div} \mathbf{A} = \nabla \mathbf{A} = 0.$$

Hence, for current density we receive equality

$$\mathbf{j} = -c \chi \Delta \mathbf{A} = c \chi q^2 \mathbf{A}.$$
On the other hand, the connection of electric field \( E \) and vector potential \( A \) is as follows

\[
E = -\frac{1}{c} \frac{\partial A}{\partial t} = \frac{i\omega}{c} A.
\]

It is leads to the relation

\[
j = \sigma_{tr} E = \sigma_{tr} \frac{i\omega}{c} A,
\]

where \( \sigma_{tr} \) is the transverse electric conductivity.

For our case from (1.1) and (1.2) we obtain following expression for the magnetic susceptibility

\[
\chi = \frac{i\omega}{c^2 q^2} \sigma_{tr}. \tag{1.1}
\]

Expression of transversal conductivity of Maxwellian collisional plasmas it is defined by the general formula [6]:

\[
\sigma_{tr}(q, \omega, \nu) = \sigma_0 \frac{i\nu}{\omega} \left( 1 + \frac{\omega B(q, \omega + i\nu) + i\nu B(q, 0)}{\omega + i\nu} \right), \tag{1.2}
\]

where \( \sigma_0 \) is the static conductivity, \( \sigma_0 = e^2 N/m \nu \), \( N \) is the concentration (number density) of plasmas particles, \( e \) and \( m \) is the electron charge and mass, \( \nu \) is the effective collisional frequency of plasmas particles,

\[
B(q, 0) = \frac{\hbar^2}{8\pi^3 m N} \int \frac{f_k - f_{k-q} \varepsilon_{k-q} \varepsilon_{k}}{\varepsilon_k - \varepsilon_{k-q} - \hbar(\omega + i\nu)} k^2 d^3k,
\]

\[
B(q, \omega + i\nu) = \frac{\hbar^2}{8\pi^3 m N} \int \frac{f_k - f_{k-q} \varepsilon_{k-q} \varepsilon_{k}}{\varepsilon_k - \varepsilon_{k-q} - \hbar(\omega + i\nu)} k^2 d^3k,
\]

\[
f_k = \frac{4\pi^{3/2}\hbar^3}{m^3 v_T^3} \exp \left( -\frac{\varepsilon_k}{k_B T} \right), \quad \varepsilon_k = \frac{\hbar^2 k^2}{2m},
\]

\[
\varepsilon_T = \frac{mv_T^2}{2} = \frac{\hbar^2 k_T^2}{2m} = k_B T,
\]

\( \varepsilon_k \) is the electrons energy, \( \varepsilon_T \) is the heat electrons energy, \( k_B \) is the Boltzmann's constant, \( v_T = 1/\sqrt{\beta} \) is the heat electrons velocity, \( \beta = m/2k_B T \), \( \hbar \) is the Planck’s constant.
\[ k_{\perp}^2 = k^2 - \left(\frac{kq}{q}\right)^2. \]

According to (1.1) and (1.2) magnetic susceptibility of the quantum collisional Maxwellian plasmas it is equal

\[ \chi(q, \omega, \nu) = -\frac{e^2 N}{mc^2 q^2} \left( 1 + \frac{\omega B(q, \omega + i\nu) + i\nu B(q, 0)}{\omega + i\nu} \right). \] (1.3)

From the formula (1.3) it is visible, that at \( \omega = 0 \) frequency of collisions plasma particles \( \nu \) drops out of the formula (1.3). Hence, the magnetic susceptibility in a static limit does not depend from frequencies of collisions of plasma and the following form also has:

\[ \chi(q, 0, \nu) = -\frac{e^2 N}{mc^2 q^2} \left[ 1 + \frac{\hbar^2}{8\pi^3 mN} \int \frac{f_k - f_{k-q}}{\mathcal{E}_k - \mathcal{E}_{k-q}} k_{\perp}^2 d^3k \right]. \] (1.4)

From expression (1.3) it is visible, that a magnetic susceptibility in collisionless quantum Maxwellian plasma is equal:

\[ \chi(q, \omega, 0) = -\frac{e^2 N}{mc^2 q^2} \left[ 1 + \frac{\hbar^2}{8\pi^3 mN} \int \frac{f_k - f_{k-q}}{\mathcal{E}_k - \mathcal{E}_{k-q} - \hbar\omega} k_{\perp}^2 d^3k \right]. \] (1.5)

At \( \omega \to 0 \) the formula (1.5) passes in the formula (1.4).

Let’s deduce the formula for calculation of a magnetic susceptibility of quantum collisional Maxwellian plasmas.

After obvious linear replacement of variables the formula for integral \( B(q, \omega + i\nu) \) will be transformed to the form

\[ B(q, \omega + i\nu) = \frac{\hbar^2}{8\pi^3 mN} \times \]

\[ \times \int \frac{(\mathcal{E}_{k+q} + \mathcal{E}_{k-q} - 2\mathcal{E}_k)f_k k_{\perp}^2 d^3k}{[\mathcal{E}_k - \mathcal{E}_{k-q} - \hbar(\omega + i\nu)][\mathcal{E}_{k+q} - \mathcal{E}_k - \hbar(\omega + i\nu)]}. \] (1.6)

Let’s enter dimensionless variables

\[ z = \frac{\omega + i\nu}{k_T v_T} = x + iy, \quad x = \frac{\omega}{k_T v_T}, \quad y = \frac{\nu}{k_T v_T}, \quad Q = \frac{q}{k_T}. \]
Let’s pass to integration on the vector $\mathbf{K} = \mathbf{k}/k_T$, where $k_T = p_T/\hbar = mv_T/\hbar$ is the thermal wave number. Vectors $\mathbf{K}, \mathbf{k}$ we will direct along an axis $x$, believing $\mathbf{K} = K_x(1,0,0)$ and $\mathbf{k} = k(1,0,0)$.

Then

$$E_k = \frac{\hbar^2 k_T^2}{2m} K^2 = \mathcal{E}_T K^2,$$

$$E_k - E_{k-q} - \hbar(\omega + i\nu) = 2\mathcal{E}_T Q\left(K_x - \frac{z}{Q} - \frac{Q}{2}\right),$$

$$E_{k+q} - E_k - \hbar(\omega + i\nu) = 2\mathcal{E}_T Q\left(K_x - \frac{z}{Q} + \frac{Q}{2}\right),$$

$$E_{k+q} + E_{k-q} - 2E_k = 2\mathcal{E}_T Q^2,$$

$$f_k = \frac{4\pi^{3/2}}{k_T^3} \exp\left(-\frac{E_k}{\mathcal{E}_T}\right) = \frac{4\pi^{3/2}}{k_T^3} \exp(-K^2).$$

The magnetic susceptibility in dimensionless variables is equal

$$\chi(Q, x, y) = -\frac{e^2 N}{mc^2 k_T^2 Q^2} \left(1 + \frac{x}{z} B(Q, z) + \frac{i y}{z} B(Q, 0)\right). \quad (1.7)$$

Here according to (1.6)

$$B(Q, z) = \frac{1}{2\pi^{3/2}} \int \frac{e^{-K_x^2} K_z^2 d^k}{(K_x - z/Q)^2 - (Q/2)^2},$$

$$B(Q, 0) = \frac{1}{2\pi^{3/2}} \int \frac{e^{-K_x^2} K_z^2 d^k}{K_x^2 - (Q/2)^2}.$$

The internal double integral is equal

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-K_x^2 - K_y^2} K_z^2 (K_y^2 + K_z^2) dK_y dK_z = \pi.$$ 

Hence, last two integrals are reduced to the one-dimensional integrals

$$B(Q, z) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\tau^2} d\tau}{(\tau - z/Q)^2 - (Q/2)^2},$$

$$B(Q, 0) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\tau^2} d\tau}{\tau^2 - (Q/2)^2}.$$
2. Landau diamagnetism of quantum Maxwellian collisionless plasmas

Landau diamagnetism in collisionless plasma is usually defined as a magnetic susceptibility in a static limit for a homogeneous external magnetic field. Thus the diamagnetism value can be found by means of (1.1) through two non-commutative limits

$$\chi_L = \lim_{q \to 0} \left[ \lim_{\omega \to 0} \chi(q, \omega, \nu = 0) \right]. \quad (2.1)$$

Into collisionless plasma this expression (2.1) should lead to the known formula of Landau’s diamagnetism of quantum Maxwellian plasmas [9]

$$\chi_L = -\frac{e^2 N}{6mc^2k_B^2} = -\frac{e^2 Nk_BTH^2}{3m^4c^2}. \quad (2.2)$$

So the relative magnetic susceptibility by means of (2.2) is equal:

$$\frac{\chi(Q, x, y)}{\chi_L} = \frac{6}{Q^2} \left( 1 + \frac{x}{z} B(Q, z) + \frac{iy}{z} B(Q, 0) \right). \quad (2.3)$$

Let’s deduce by means of expression (2.3) formula for Landau diamagnetism. At $z = 0$ from the formula (2.3) for magnetic susceptibility the quantum collisionless Maxwellian plasmas we obtain the following expression

$$\frac{\chi(Q)}{\chi_L} = \frac{6}{Q^2} \left[ 1 + \frac{1}{2\sqrt{\pi}Q} \int_{-\infty}^{\infty} \frac{e^{-\tau^2} - e^{-(\tau - Q)^2}}{\tau - Q/2} d\tau \right], \quad (2.4)$$

or

$$\frac{\chi(Q)}{\chi_L} = \frac{6}{Q^2} \left[ 1 + \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\tau^2} d\tau}{\tau^2 - (Q/2)^2} \right].$$

Let’s prove with use (2.4), that

$$\frac{\chi(0)}{\chi_L} = \lim_{Q \to 0} \frac{6}{Q^2} \left[ 1 - \frac{1}{2\sqrt{\pi}Q} \int_{-\infty}^{\infty} \frac{e^{-(\tau - Q)^2} - e^{-\tau^2}}{\tau - Q/2} d\tau \right] = 1. \quad (2.5)$$
This relation also will prove equality (2.2). Really, we will spread out function
\[ \varphi(Q) = \frac{e^{-(\tau-Q)^2} - e^{-\tau^2}}{\tau - Q} \]
in series on degrees \( Q \) near to a point \( Q = 0 \)
\[ \varphi(Q) = 2e^{-\tau^2}Q + 2\tau e^{-\tau^2}Q^2 + \left(\frac{4}{3}\tau^2 - 1\right)Q^3 + \cdots. \]

Now according to (2.5) we obtain
\[ \frac{\chi(0)}{\chi_L} = \lim_{Q \to 0} \frac{6}{Q^2} \left\{ 1 - \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\tau^2} \left[ 2 + 2\tau e^{-\tau^2}Q + \left(\frac{4}{3}\tau^2 - 1\right)Q^2 + \cdots \right] d\tau \right\} = \]
\[ = -\frac{3}{Q^2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\tau^2} \left(\frac{4}{3}\tau^2 - 1\right) d\tau = 1, \]
as was to be shown.

3. Analysis of results

For graphic research of the magnetic susceptibility we will be to use the formula (2.4).

On fig. 1 comparison of a magnetic susceptibility of Maxwellian and degenerate plasmas as functions of the dimensionless wave number for a case of collisionless plasmas in static limit is presented.

From fig. 1 it is clear, that there is a value of dimensionless wave number \( Q^* \), in which values of magnetic susceptibility of degenerate and Maxwellian plasmas are equal each other. So at \( 0 \leq Q < Q^* \) (\( Q > Q^* \)) values of a magnetic susceptibility of degenerate plasma it is more (less) than values of the magnetic susceptibility of Maxwellian plasmas.

From fig. 1 it is obvious, that into quantum collisionless plasma (\( \nu = 0 \)) in static limit (\( \omega = 0 \)) the magnetic susceptibility is function of wave number, monotonously decreasing to zero both for degenerate, and for Maxwellian plasmas.
On fig. 2 and 3 dependence of magnetic susceptibility of collisionless plasmas as from wave number (fig. 2), and from dimensionless frequency of oscillations of an electromagnetic field $x$ (fig. 3) is presented.

From fig. 2 it is obvious, that the magnetic susceptibility is monotonously decreasing function of wave number at all values of frequency of oscillations of electromagnetic field. Thus for all $x < 1 \ (\omega < \omega_p)$ values of a magnetic susceptibility that more than the values of frequency of oscillations of the electromagnetic fields is more.

From fig. 3 it is clear, that a magnetic susceptibility as function of oscillation frequencies of electromagnetic field has the maximum near to frequency $\omega = Q\omega_p$ and with growth $Q$ moves to the right.

On fig. 4 and 5 dependence of real (fig. 4) and imaginary (fig. 5) parts of magnetic susceptibility from quantity of dimensionless frequency of oscillations of electromagnetic field in the case $Q = 0.5$ is presented.

From fig. 4 it is clear, that the real part has a maximum, which is
displaced to the right with growth of frequency of collisions of plasma particles. Irrespective of frequency of collisions of particles of plasma with growth of frequency of fluctuations of an electromagnetic field the quantity of the real part of a magnetic susceptibility leaves from above on the asymptotics

$$\lim_{x \to 0} \text{Re} \left( \frac{\chi(Q, x, y)}{\chi_L} \right) = \frac{6}{Q^2}. $$

Not resulting necessary graphics we will inform, that with decrising values of wave number a maximum of a magnetic susceptibility moves to the left and becomes sharp at small values of frequency collisions of particles of plasma. With growth of frequency of collisions the maximum starts to smooth out and vanishes.

From fig. 5 it is obvious, that an imaginary part of a magnetic susceptibility as function of dimensionless frequency of oscillations of an electromagnetic field has a minimum. This minimum moves to the left with growth frequency collisions of particles of plasma. With growth of the dimensionless frequencies of oscillations of an electromagnetic field an imaginary part of the magnetic susceptibilities leaves from below on the asymptotics $\text{Im} \left( \frac{\chi}{\chi_L} \right) = 0$. We will notice, that a minimum of an imaginary part not vanishes with growth as frequencies of collisions of particles of plasma, and dimensionless wave number.

Let’s notice, that the less frequency of collisions of particles of plasma, the it is more than value of the real and imaginary parts of the magnetic susceptibilities.

4. Conclusion

In the present work the kinetic description of magnetic susceptibilities of quantum collisional Maxwellian plasmas with use before deduced correct formulas for transversal electric conductivity of quantum plasma is given.
Influence of collisions of particles of plasma on the magnetic susceptibility is found out. Thereby the answer to a question on influence dissipations on Landau diamagnetism put in work [4] is given. For collisionless plasmas with the help the kinetic approach the known formula of Landau diamagnetism is deduced.
Fig 2. Magnetic susceptibility of collisionless plasma, curves 1, 2 and 3 correspond to parameter values $x = 0.01, 0.1$ and $x = 0.2$.

Fig 3. Magnetic susceptibility of collisionless plasma, curves 1, 2, 3 correspond to parameter values $Q = 0.45, 0.50, 0.55$. 
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Fig 4. Real part of magnetic susceptibility; $Q = 0.5$, curves 1, 2, 3 correspond to parameter values $y = 0.001, 0.05, 0.1$.

Fig 5. Imaginary part of magnetic susceptibility; $Q = 0.5$, curves 1, 2, 3 correspond to parameter values $y = 0.001, 0.05, 0.1$. 