A note on large bounding and non-bounding finite group-actions on surfaces of small genus

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Abstract. The classification of finite group-actions on closed surfaces of small genus is well-known. In the present paper we are interested in the question of which of these group-actions are bounding (extend to a compact 3-manifold with the surface as its unique boundary component, e.g. to a handlebody) or geometrically bounding (extend to a hyperbolic 3-manifold with totally geodesic boundary) concentrating, as a typical case, on large group-actions on surfaces of genus 3.

1. Introduction

All finite group-actions in the present paper will be orientation-preserving and faithful, all manifolds will be orientable. We are interested in large group-actions of a finite group $G$ on a closed hyperbolic surface $\Sigma$ of genus $g \geq 2$. By choosing a hyperbolic structure on a quotient-orbifold $\Sigma/G$ and lifting the structure to $\Sigma$, we can assume that $G$ acts by isometries, for some hyperbolic structure on $\Sigma$; then the group of all lifts of elements of $G$ to the universal covering $\mathbb{H}^2$ of $\Sigma$ is a Fuchsian group $F$, and we have an exact sequence

$$1 \to K \hookrightarrow F \to G \to 1$$

where $K \cong \pi_1(\Sigma)$ denotes the universal covering group. We denote a Fuchsian group by its signature (which is also the signature of the quotient-orbifold $\Sigma/G$); for example, by $(p, q, r)$ we denote the triangle group (of genus 0 which we omit) of orientation-preserving elements in the group generated by the reflections in the sides of a hyperbolic triangle with angles $2\pi/p$, $2\pi/q$ and $2\pi/r$, and a quadrangle group $(p, q, r, s)$ is defined analogously.

We say that a finite $G$-action on a surface $\Sigma$ bounds if the $G$-action on $\Sigma$ extends to a $G$-action on a compact 3-manifold $M$ with $\partial M = \Sigma$ (e.g., to a handlebody); it bounds geometrically if $M$ can be chosen as a compact hyperbolic 3-manifold $M$ with totally geodesic boundary. In this second case, by an application of Mostow rigidity one can assume that $G$ acts by isometries also on $M$ (and then also $\Sigma = \partial M$ achieves a hyperbolic structure on which $G$ acts by isometries).
In [WZ] the authors determine which finite group-actions on a surface of genus two extend to a 3-manifold (i.e., bound), and in particular also to the 3-sphere, for some embedding of the surface into $S^3$ (in order to give explicit geometric descriptions or “visualizations of these actions in the familiar 3-space). In the present paper we consider the case $g = 3$ but concentrate on possible extensions to handlebodies and hyperbolic 3-manifolds with totally geodesic boundary instead of $S^3$.

We refer to Broughton [B] for the classification of the finite group-actions on surfaces of genus 3. We will represent finite group-actions on surfaces by surjections $F \to G$, always assumed to have torsionfree kernel, of a Fuchsian group $F$ onto a finite group $G$; the kernel of such a surjection is a torsionfree Fuchsian group $K$ (a surface group), and $G \cong F/K$ acts (by isometries) on the hyperbolic surface $\mathbb{H}^2/K$. In the following theorem, we list the largest group-actions on a surface of genus 3 in decreasing order and determine which of these actions bound, bound a handlebody or bound geometrically, representing the actions by a surjection $F \to G$ of a Fuchsian group $F$ to a finite group $G$. We denote by $\mathbb{D}_n$ the dihedral group of order $2n$, by $A_4$ and $S_4$ the alternating and symmetric groups of orders 12 and 24; for the groups $D_{2,8,5}$ and $D_{2,12,5}$ we refer to [B].

**Theorem.** The bounding and non-bounding finite group-actions on a surface of genus 3, of order $\geq 24$, are the following.

i) The two largest group-actions, of orders 168 and 96 are represented by surjections

$$(2, 3, 7) \to \text{PSL}_2(7) \quad \text{and} \quad (2, 3, 8) \to \mathbb{D}_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$$

and do not bound.

ii) Two actions of order 48 associated to surjections

$$(3, 3, 4) \to \mathbb{Z}_3 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4) \quad \text{and} \quad (2, 4, 6) \to \mathbb{Z}_2 \times S_4;$$

the first one is a subgroup of index 2 of the group of order 96 in i) and does not bound, the second one is the largest bounding group-action on a surface of genus 3; it bounds geometrically but does not extend to a handlebody.

iii) Two non-bounding actions of order 32 associated to surjections

$$(2, 4, 8) \to \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8) \quad \text{and} \quad (2, 4, 8) \to \mathbb{Z}_2 \times D_{2,8,5}.$$ 

iv) Two non-bounding actions of order 24 associated to surjections

$$(3, 3, 6) \to \text{SL}_2(3) \quad \text{and} \quad (2, 4, 12) \to D_{2,12,5}.$$
v) Three bounding actions of order 24 associated to surjections

\[ (2, 6, 6) \rightarrow \mathbb{Z}_2 \times A_4, \quad (3, 4, 4) \rightarrow S_4 \quad \text{and} \quad (2, 2, 2, 3) \rightarrow S_4; \]

these three actions are subgroups of index 2 of the geometrically bounding action of order 48 in ii). The last one is also the largest action on a surface of genus 3 which extends to a handlebody (in fact $S_4$ is the unique maximal handlebody group of genus 3, of maximal possible order $12(g - 1)$).

Finally, for each of the non-bounding actions in i) - iv) there is already a cyclic subgroup which does not bound.

**Corollary.** The largest bounding finite group-action on a surface of genus 3 is an action of $\mathbb{Z}_2 \times S_4$ of order 48 which bounds geometrically but does not extend to a handlebody. The largest finite group-action in genus 3 which extends to a handlebody is the action of the subgroup $S_4$ which bounds also geometrically.

The largest finite group-action on a surface of genus 4 is an action of the symmetric group $S_5$ associated to a surjection $(2, 4, 5) \rightarrow S_5$ (cf. [C] and its references); using similar methods, the following holds.

**Proposition.** The largest group-action on a surface of genus 4, of type $(2, 4, 5) \rightarrow S_5$, bounds geometrically. The largest action in genus 4 which extends to a handlebody is of type $(2, 2, 2, 3) \rightarrow D_3 \times D_3$.

See also [Z4] for a discussion of various aspects of finite group-actions on surfaces.

2. **Proof of the Theorem**

i) The Hurwitz-action of $\text{PSL}_2(7)$ on Klein’s quartic $\Sigma_3$ of genus 3 does not bound, i.e. does not extend to any compact 3-manifold $M$ with exactly one boundary component $\partial M = \Sigma_3$: the quotient orbifold $\Sigma_3/\text{PSL}_2(7)$ is the 2-sphere with three branch points of orders 2, 3 and 7 which does not occur as the unique boundary component of a compact 3-orbifold since a singular axis starting in the boundary point of order 7 can end only in a dihedral point of dihedral type $D_7$ but $\text{PSL}_2(7)$ has no dihedral subgroup $D_7$. A cyclic subgroup which does not bound is of type $(7, 7, 7) \rightarrow \mathbb{Z}_7$.

Similarly, the group $D_3 \rtimes (\mathbb{Z}_4 \rtimes \mathbb{Z}_4)$ acting on Fermat’s quadric of genus 3 has no subgroup $D_8$ and hence does not bound, a cyclic non-bounding subgroup is of type $(4, 8, 8) \rightarrow \mathbb{Z}_8$.

ii) Concerning the first case, an axis of order 4 starting in a singular point of order 4 in the quotient 2-orbifold of type $(3,3,4)$ can only end in a singular point of type $D_4$ or $S_4$ but $\mathbb{Z}_3 \rtimes (\mathbb{Z}_4 \rtimes \mathbb{Z}_4)$ has no such subgroups; a cyclic non-bounding subgroup is of type $(4, 4, 4, 4) \rightarrow \mathbb{Z}_4$.
We show that the action of $\mathbb{Z}_2 \times S_4$ bounds geometrically. We consider a hyperbolic tetrahedron $T$, truncated by an orthogonal hyperplane at a vertex of type (2,4,6) where three edges of orders 2, 4 and 6 meet (with dihedral angles $\pi/2$, $\pi/4$ and $\pi/6$); the edge opposite to the edge of singular order 6 has order 3, all other edges have singular order 2. Let $T$ denote the associated tetrahedral group (the orientation-preserving subgroup of index 2 in the Coxeter group generated by the reflections in the four faces of the tetrahedron), with a triangle subgroup of type (2,4,6) generated by the rotations at the truncated vertex. Then it is easy to see that a surjection \((2, 4, 6) \to \mathbb{Z}_2 \times S_4\) defining the action on the surface of genus 3 extends to a surjection \(T \to \mathbb{Z}_2 \times S_4\). The covering of the 3-orbifold $T$ associated to the kernel of this surjection is a hyperbolic 3-manifold with totally geodesic boundary (a surface of genus 3), with an isometric action of $\mathbb{Z}_2 \times S_4$; this restricts to an isometric action of $\mathbb{Z}_2 \times S_4$ on the boundary which realizes the action in part ii) of the Theorem.

More explicitly, the rotational generators of the tetrahedral group $T$, suitably oriented, can be mapped to the following elements of $\mathbb{Z}_2 \times S_4$ where $c$ denotes the generator of $\mathbb{Z}_2$: the order 4 rotation is mapped to $c(1234)$, the order 6 rotation to $c(143)$ and the order 3 rotation to $(142)$, and this determines also the images of the order 2 rotations (see [GZ] or [Z2] for similar constructions).

The action of $\mathbb{Z}_2 \times S_4$ does not extend to a handlebody. More generally, any action \((p, q, r) \to G\) associated to a triangle group \((p, q, r)\) does not extend to a handlebody, see [Z1].

iii) and iv) are similar to i). The cyclic non-bounding subgroups in iii) are both of type \((4, 8, 8) \to \mathbb{Z}_8\), in iv) of types \((2, 3, 3, 6) \to \mathbb{Z}_6\) and \((2, 12, 12) \to \mathbb{Z}_{12}\). Concerning v), it is well-known that $S_4$ is the unique maximal handlebody group of genus 3 (of order $12(g - 1) = 24$), see [Z3].

The Proof of the Proposition is similar to the geometrically bounding case of part ii) of the Theorem, with the following choices. We consider a hyperbolic tetrahedron, truncated by an orthogonal hyperplane at a vertex of type (2,4,5) where three edges of orders 2, 4 and 5 meet; an edge of order 3 connects the edges of orders 4 and 5, all other edges have order 2. The order 4 rotation is mapped to (2345), the order 5 rotation to (12345) (oriented such that (12345) (12) = (2345)), the order 3 rotation to (135) (such that (12)(34) (135) = (12345)). With these choices, the proof is the same as that of part ii) of the Theorem.

Finally, $\mathbb{D}_3 \times \mathbb{D}_3$ is the unique maximal handlebody group of genus 4, of maximal possible order $12(g - 1)$ (see [Z3]).
References

[B] S.A. Broughton, *Classifying finite group actions on surfaces of low genus*. J. Pure Appl. Alg. 69 (1990), 233-270

[C] M.D.E. Conder, *Large group actions on surfaces*. Contemp. Math. 629 (2014), 77-98

[GZ] M. Gradolato, B. Zimmermann, *Extending finite group actions on surfaces to hyperbolic 3-manifolds*. Math. Proc. Cambridge Phil. Soc. 117 (1995), 137-151

[WZ] C. Wang, S. Wang, Y. Zhang, B. Zimmermann, *Finite group actions on the genus-2 surface, geometric generators and extendability*. Rend. Istit. Mat. Univ. Trieste 52 (2020), 513-524 (electronic version under http://rendiconti.dmi.units.it)

[Z1] B. Zimmermann, *Über Abbildungsklassen von Henkelkörpern*. Arch. Math. 33 (1979), 379-382

[Z2] B. Zimmermann, *Hurwitz groups and finite group actions on hyperbolic 3-manifolds*. J. London Math. Soc. 52 (1995), 199-208

[Z3] B. Zimmermann, *Genus actions of finite groups on 3-manifolds*. Michigan Math. J. 43 (1996), 593-610

[Z4] B. Zimmermann, *Hurwitz groups, maximal reducible groups and maximal handlebody groups*. arXiv:2110.11050