Effects of microscopic dynamics on Brownian coagulation

Guolong Li
University of Cambridge

Abstract

We consider two different models for colloidal particles. In the first model, we consider their free motion to be diffusion while in the second model we take it to be integrated Ornstein-Uhlenbeck process. In both models, we derived collision estimates for pairs of particles. In particular, we found that these estimates would be different to the Brownian case even when the particles’ free motion is Brownian at macroscopic scales. As a consequence, the coagulation kernel and diffusivity in the coagulation-diffusion equations would also be affected accordingly. We then proved that there exists a unique solution to the coagulation-diffusion equations in these cases under physically reasonable assumptions.

1 Introduction

In this paper, we investigate the dynamics for large clouds of colloidal particles whose free motion is Brownian at macroscopic scales, but is not Brownian on the scale of the particles themselves. The parameters of our processes will be chosen so that each particle follows a free path for a time of order one between collisions. The macroscopic free path may therefore be considered as Brownian. However the actual collisions, being determined by microscopic dynamics, will be strongly affected by the departure of the free motion from the Brownian case.

We make two new contributions. The first is to prove collision estimates for pairs of particles. Having in mind eventually a system of $N$ particles, where $N$ is large, we choose the scale of the particle radius so that any given pair meets in order one time with probability of order $1/N$. The mass and radius of each particle will also affect the characteristics of its free motion, in a way we shall take as given, based on some physical arguments.
We prove collision estimates in two cases. In the first case, the particle, in addition to its basic molecular diffusivity, is considered as suspended in an incompressible fluid, though which it acquires a drift, which we shall take to be periodic. In the second case, following the derivation of physical Brownian motion from particle dynamics, we suppose that the free motion is an integrated Ornstein–Uhlenbeck process. In both cases, it is well understood that, under appropriate scalings, the macroscopic motion is Brownian, with diffusivity depending on the size of the particles. Our analysis shows how the small-scale motion, in both cases, leads to strong departures from the Brownian case for the collision probabilities. More precisely, consider two particles in $\mathbb{R}^d$ with $d \geq 3$, having radius $r$, evolving under dynamics which is approximately Brownian with constant diffusivity $a$, but where a departure from Brownian behaviour is visible on a length scale of order $\lambda$. We investigate the collision event in the limit $r, \lambda \to 0$. It is intuitive to expect that for two particles starting from $x_1, x_2$ colliding at $X$ at time $T$,

$$\mathbb{P}(T \in dt, X \in dx) \sim p(t, x_1, x)p(t, x_2, x)k(r, \lambda) dtdx,$$

where $p$ is the transition density of a standard Brownian motions and $k(r, \lambda)$ for small $r, \lambda$ is to be determined.

In [8], Norris has considered the case where the particles’ dynamics are exactly Brownian and showed that $k(r, 0) = c_d ar^{d-2}$ for some constant $c_d$. Therefore, in the case where $\lambda \gg r$, we would expect that $k(r, \lambda) \approx c_d ar^{d-2}$. However, when $\lambda \ll r$ we will give two cases where we can show that the non-Brownian microscopic dynamics leads to different rates for coagulation.

We will first look at the case of diffusion at rate $a$ enhanced by a $\lambda$-periodic drift $b^\lambda(x)$. In [1], Fannjiang and Papanicolaou showed that when $b^\lambda(x) = b(x/\lambda)$ for some $1$-periodic (i.e. $b(x + x') = b(x)$ for any integer point $x'$) divergence-free zero-mean $b$ then the underlying motion converges weakly to a Brownian motion with diffusivity $\bar{a}$, which in general does not equal to $a$. We will see that when $\bar{a}$ and $a$ are both scalars

$$\lim_{r \to 0} \lim_{\lambda \to 0} k(r, \lambda)r^{2-d} = c_d\bar{a}$$

while

$$\lim_{\lambda \to 0} \lim_{r \to 0} k(r, \lambda)r^{2-d} = c_da.$$
Then we will look at the case where the particles’ motions are modelled by integrated Ornstein-Uhlenbeck processes, and make appropriate scaling so that their motions converge to Brownian motions. We will see that when $r \ll \lambda$, $k(r, \lambda) \sim f(\lambda) r^{d-1}$ for some function $f$. So, we will have that $k(r, \lambda) \ll k(r, 0)$ in this case. Intuitively this is because, in the Ornstein-Uhlenbeck case, when two particles come close to each other, they are likely to get far away again with almost constant speed so that their trajectories are almost straight lines, while in the Brownian case, the particles are likely to move back and forth more before they go away from each other and this results more chance for them to collide.

Our second contribution is to the theory of coagulation-diffusion equations. This was motivated by the mass-dependent diffusivities and collision probabilities emerging in the first part of the paper, to which prior work on coagulation-diffusion equations did not apply. We show in Section 5 and 6 an existence and uniqueness result which does apply for the diffusivity and collision probabilities associated with the Ornstein–Uhlenbeck case.

Consider a large cloud of colloidal particles in which when two particles collide, they coagulate and continue the random motion as a larger particle. As the number of particles becomes large, the distribution of the particles is expected to converge to the solution of the coagulation-diffusion equations

$$\dot{\mu}(x, dy) = \frac{1}{2} a(y) \Delta_x \mu_t(x, dy) + K^+(\mu)(x, dy) - K^-(\mu)(x, dy),$$

(2)

where

$$K^+(\mu)(x, A) = \frac{1}{2} \int_0^\infty \int_0^\infty 1_{y+y' \in A} K(y, y') \mu(x, dy) \mu(x, dy'),$$

$$K^-(\mu)(x, A) = \int_{y' \in A} \int_0^\infty K(y, y') \mu(x, dy) \mu(x, dy').$$

Here, $y$ represents the mass of the particles and we assume their macroscopic free motion is approximately a Brownian motion with diffusivity $a(y)$. Further, $\mu_t$ is a kernel on $\mathbb{R}^d \times (0, \infty)$ with $d \geq 3$. In this context, for a measurable set $A \subseteq (0, \infty)$, $\mu_t(x, A)$ represents the density of particles of mass within the set $A$ at position $x$ at time $t$. So, $\frac{1}{2} a(y) \Delta_x \mu_t(x, dy)$ represents the rate of change of $\mu$ due to diffusion. Moreover, the coagulation kernel $K : (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$ is a measurable function. Intuitively, we can think $K(y, y')$ as the rate at which a particle of mass $y$ and a particle of mass $y'$ coagulate and form a particle of mass $y + y'$ when the two particles are at same position. Thus, $K^+$ represents the rate at which new particles are created due to coagulation and
Consider a Brownian particle with diffusivity \( \alpha \) and density \( K \). The particle is at \( x \), \( y \) of mass \( f \). (1) can be generalized in the case when two particles are different. For two particles, the problem in general. See [13], [14] and [25] for related work. We now explain the connection in a heuristic level. (1) can be generalized in the case when two particles are different. For two particles of mass \( y_1 \) and \( y_2 \) starting at \( x_1 \) and \( x_2 \) respectively, colliding at \( X \) at time \( T \)

\[
\mathbb{P}(T \in dt, X \in dx) \sim p_1(t, x_1, X)p_2(t, x_2, X)K(y_1, y_2)dt dx,
\]

where \( p_1 \) and \( p_2 \) are the transition densities of the particles. We now give an interpretation of this. For a small region \( dx \), but still large relative to the size of the particles, we take time interval \( dt \) sufficiently small so that when the particles have been in \( dx \) during \( dt \), they are almost certain to be in \( dx \) during the entire \( dt \). Then the probability that the particles collide in \( dx \) during \( dt \) is the probability they are both in \( dx \) during \( dt \) multiplied by \( Kdt/dx \). Or, we can say that when the two particles are in \( dx \) during \( dt \), they have probability \( Kdt/dx \) to collide. Now, for large \( N \), we scale the particle mass to be \( f^N(y) \) for some functions \( f^N \) so that \( K(f^N(y_1), f^N(y_2)) = K(y_1, y_2)/N \). We also approximate the number of particles of mass \( f^N(y) \) in \( dx \) at time \( t \) to be \( N\mu_t(x, y)dx \), which is still large. Then for a particle of mass \( f^N(y_1) \) in \( dx \) during \( dt \), the total probability it collides there with a particle of mass \( f^N(y_2) \) will be \( N\mu_t(x, y_2)K(f^N(y_1), f^N(y_2))dt = \mu_t(x, y_2)K(y_1, y_2)dt \). Therefore, the expected total number of coagulations between particles of mass \( y_1 \) and \( y_2 \) there will be \( N\mu_t(x, y_1)\mu_t(x, y_2)K(y_1, y_2)dx \). As \( N \to \infty \), by central limit theorem, we can approximate the number of these coagulations to be \( N\mu_t(x, y_1)\mu_t(x, y_2)K(y_1, y_2)dx + o(Ndx) \) and this means the coagulations contribute to a loss of \( \mu_t(x, y_1)\mu_t(x, y_2)K(y_1, y_2)dx + o(dx) \) to \( \mu(x, y_1)dx \) and \( \mu(x, y_2)dx \) and a gain of the same amount to \( \mu(x, y_1 + y_2)dx \) during \( dt \). Integrating over \( y_1 \) and \( y_2 \) explains the form \( K^+ \) and \( K^- \) in the coagulation equation.

Note that (2) only makes sense if \( \mu_t(x, y) \) is twice differentiable in \( x \). On [7], Norris reformulated the equations so that it makes sense without prior assumptions on \( \mu_t \). He defined \( p^{t,x',x}(y) = (2\pi a(y)t)^{-d/2}e^{-|x'-x|^2/2a(y)t} \) and

\[
P_t\mu(x, dy) = \int_{\mathbb{R}^d} \mu(x', dy)p^{t,x',x}(y)dx'.
\]

Consider a Brownian particle with diffusivity \( a(y) \) starting at \( x' \), then \( p^{t,x',x}(y) \) is the probability density that the particle is at \( x \) at time \( t \). Norris then reformulated the Smoluchowski equations
Many of the prior works considered function solutions in the discrete case, i.e. \( \mu_t(x, y) = \sum_{m=1}^{\infty} f^m_t(x) \delta_m(dy) \), see [18, 19, 20, 21, 22]. We will restrict our review on the existence to works addressing the continuous case. In [15], Amann proved local existence and uniqueness in a general setting, assuming uniform bounds on diffusivity and coagulation rates and uniform positivity of the diffusivity. Later, in [16], Amann and Walker proved global existence for small initial data under similar hypotheses. In [11], Laurençot and Mischler showed the global existence when \( a \frac{1}{a} \) and \( K \) are all bounded on compacts and the coagulation kernel satisfies the Galkin-Tupchiev monotonicity condition

\[
K(y_1, y_2) \leq K(y_1 + y_2, y_1)
\]

along with the growth bound

\[
\lim_{y' \to \infty} \sup_{y \leq R} \frac{K(y, y')}{y'} = 0.
\]

If we assume further that mass of all particles are uniformly positive, then Mischler and Rodriguez Richard shoved in [17] that the monotonicity condition can be weakened by

\[
K(y_1, y_2) \leq K(y_1 + y_2, y_1) + K(y_1 + y_2, y_2)
\]

in the context of coagulation-diffusion in a bounded domain in \( \mathbb{R}^3 \).

In [23], Ball and Carr noted that in the spatially homogeneous setting, the questions of uniqueness and mass conservation for coagulation equations are related to the existence of moment bounds for solutions. In [12] and [26], Rezakhanlou and Hammond obtained suitable moment bounds for solutions under assumptions including that the diffusivity \( a \) is positive, uniformly bounded and non-increasing, and that the coagulation kernel \( K \) satisfies

\[
\sup_{y, y'} \frac{K(y, y')}{yy'} < \infty
\]

and

\[
\lim_{y + y' \to \infty} \frac{K(y, y')}{(y + y')(a(y) + a(y'))} \to 0.
\]

In [27], Rezakhanlou has shown that the non-increasing condition on the diffusivity can be relaxed
to some extent. In\[7\], Norris assumed that $K(y, y') \leq w(y)w(y')$ for some sublinear function $w$ and gave a proof for the existence and uniqueness in the case requiring $a^{-d/2}w$ to be sublinear and allowing the diffusivity and coagulation kernels to be unbounded for particles of small mass.

If we assume the microscopic free motions of the particles are Ornstein-Uhlenbeck processes in $\mathbb{R}^3$ satisfying the Einstein-Stokes relation. Then we can show that under certain scaling limit, an appropriate choice for $K$ would be

$$K(y_1; y_2) = (y_1^{1/3} + y_2^{1/3})^2 \sqrt{\frac{1}{y_1} + \frac{1}{y_2}},$$

and the diffusivity $a(y) = y^{-1/3}$. If we use the result from \[10\] instead of Einstein-Stokes relation, we will get the same coagulation kernel but $a(y) = y^{-2/3}$. If we fix $y'$, we see that $K(y) \sim y^{2/3}$ for large $y$. Therefore, in either of the cases, $a^{-d/2}w$ cannot be sublinear and $\frac{K(y, y')}{(y + y')(a(y) + a(y'))}$ doesn’t converge to 0. So, we can’t directly apply the results in \[12\, 7\, 26\, 27\] to obtain the existence and uniqueness of the solution. In this paper, we will give criteria for the existence and uniqueness of solution to the Smoluchowski coagulation equations which works in these two cases.

2 Main result

Consider two particles in $\mathbb{R}^d$ of radii $r_1N^{-1/(d-2)}$ and $r_2N^{-1/(d-2)}$ starting at $x_1$ and $x_2$. The reason for the scaling term $N^{-1/(d-2)}$ is that if we consider a system of $N$ particles, this scaling makes the number of collisions happening per unit time to be of order $N$. For $i = 1, 2$, let $X^i_t$, the position of particle $i$, satisfy

$$dX^i_t = \sqrt{a_i(X^i)}dB^i_t + b_i(X^i_t)dt,$$

with $a_i$ being bounded Hölder continuous scalar functions, $b_i$ being bounded measurable functions and $B^i$ being independent standard Brownian motions. Let $p_1$ and $p_2$ be the transition density of the two particles respectively. Now, set $T$ to be the first time when the two particles collide and $X_T$ be the centre of mass of the two particles at time $T$. In \[8\], Norris has proved that if $b_i = 0$ and $a_i$ are constants then for any uniformly continuous bounded function $g$ supported on $[\delta, R) \times \mathbb{R}^d$ with $\delta, R > 0$ we have

$$N\mathbb{E}[g(T, X_T)1_{T<R}] \to K \int_0^R \int_{\mathbb{R}^d} p_1(0, x_1; s, z)p_2(0, x_2; s, z)g(s, z)dzds,$$
as $N \to \infty$, where $K = c_d(a_1 + a_2)(r_1 + r_2)^{d-2}$ and

$$\frac{1}{c_d} = \int_0^\infty \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{1}{4t}} dt.$$ 

We will generalize this result into the following theorem.

**Theorem 2.1.** For all $d \geq 3$ and $R \in [1, \infty)$ there is a constant $C$ depending only on $d$ and $R$ with the following property. Let $N \in (0, \infty)$, $x_i \in \mathbb{R}^d$ and $y_i, r_i \in [R^{-1}, R]$, $i = 1, 2$, be given. For $i = 1, 2$, let $a_i : \mathbb{R}^d \to [R^{-1}, R]$ be Hölder continuous functions and $b_i : \mathbb{R}^d \to \mathbb{R}^d$ be measurable with $|b_i(x)| \leq R$ for all $x \in \mathbb{R}^d$. Set $a(x) = a_1(x) + a_2(x)$, $r = r_1 + r_2$ and $K(x) = c_d a(x) r^{d-2}$ with

$$\frac{1}{c_d} = \int_0^\infty \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{1}{4t}} dt.$$ 

For $i = 1, 2$, let $X^i$ be a diffusion in $\mathbb{R}$ satisfying

$$dX^i_t = \sqrt{a_i(X^i)} dB^i_t + b_i(X^i_t) dt,$$

with $B^1, B^2$ independent standard Brownian motions and $x_1 \neq x_2$. Set

$$T = \inf\{t \geq 0 : |X^1_t - X^2_t| \leq r N^{-1/(d-2)}\}, \quad X_T = (y_1 X^1_T + y_2 X^2_T)/(y_1 + y_2).$$

For $i = 1, 2$, for $s, t \in \mathbb{R}$ and $x, z \in \mathbb{R}^d$, let $p_i(s, x; t, z)$ be the transition density of $X^i$. Let $\delta \geq 0$ and $1 \geq \epsilon \geq 2r N^{-1/(d-2)}$ be given and let $g$ be a bounded measurable function on $[0, \infty) \times \mathbb{R}^d$, supported on $[\delta^2, R] \times \mathbb{R}^d$. Write $\|g\|$ for the uniform norm and set

$$\phi_g(\epsilon) = \sup_{|s-t| \leq \epsilon^2, |x-z| \leq \epsilon} |g(s, z) - g(t, x)|.$$ 

Then

$$|\text{NE}(g(T, X_T)1_{T<R}) - \int_0^R \int_{\mathbb{R}^d} K(z)p_1(0, x_1; s, z)p_2(0, x_2; s, z)g(s, z)dzds|$$

$$\leq C[\epsilon^{2-d}\|g\|/N + \epsilon^2 + \phi_g(\epsilon)](\delta \vee |x_1 - x_2|)^{2-d}.$$
As an application, we will investigate how Brownian particles coagulate under a periodic drift. We let the particles’ motions satisfy

\[ dX^\lambda_i(t) = \sqrt{\alpha_i}dB_i(t) + b^\lambda_i X^\lambda_i(t)dt, \]

where \( b^\lambda_i(x) = b_i(x/\lambda) \) for some 1-periodic divergence-free zero-mean \( b \). In [1], [2] and [6], they have shown that the underlying motion converges weakly to a Brownian motion with diffusivity \( \bar{\alpha}_i \) as \( \lambda \to 0 \) for some \( \bar{\alpha}_i \). We will assume that \( a_i \) and \( b_i \) are chosen such that both \( a_i \) and \( \bar{\alpha}_i \) are scalars.

Corollary 2.2. We will use same notation as in Theorem 2.1. Let \( \bar{a} = \bar{a}_1 + \bar{a}_2 \), then for any bounded continuous measurable function \( g \) on \([0, \infty) \times \mathbb{R}^d\), supported on \([\delta^2, R) \times \mathbb{R}^d\), we have

\[
\lim_{N \to \infty} \lim_{\lambda \to 0} |N\mathbb{E}(g(T, X^\lambda_T)1_{T<R}) - \int_0^R \int_{\mathbb{R}^d} \bar{K}p_1(0, x_1; s, z)p_2(0, x_2; s, z)g(s, z)dzds| \to 0
\]

and

\[
\lim_{\lambda \to 0} \lim_{N \to \infty} |N\mathbb{E}(g(T, X^\lambda_T)1_{T<R}) - \int_0^R \int_{\mathbb{R}^d} Kp_1(0, x_1; s, z)p_2(0, x_2; s, z)g(s, z)dzds| \to 0,
\]

where \( K = c_d ar^{-d-2} \) and \( \bar{K} = c_d \bar{a} r^{-d-2} \) and \( p_1 \) and \( p_2 \) are the transition densities of Brownian motions with diffusivity \( \bar{\alpha}_1 \) and \( \bar{\alpha}_2 \) respectively.

Then, we will show analogous results for Ornstein-Uhlenbeck particles.

Theorem 2.3. For \( d \geq 3 \) and \( i = 1, 2 \), let \( x_i \in \mathbb{R}^d \) and \( y_i, \tau_i, b_i > 0 \) be given. Assume \( x_1 \neq x_2 \). Further, for natural number \( N \), let \( V^N_i, X^N_i \) be Ornstein-Uhlenbeck velocity-position processes satisfying

\[
dV^N_i(t) = Nb_idB^1_i - N\tau_i V^N_i dt,
\]

\[
dx^N_i(t) = V^N_i(t)dt,
\]

\[
V^N_i(0) = 0,
\]

\[
X^N_i(0) = x_i,
\]

with \( B^1, B^2 \) independent standard Brownian motions. Let \( r_N \) denote the sum of the radii of the
two particles. Set

\[ T = \inf \{ t \geq 0 : |X_1^N(t) - X_2^N(t)| \leq r_N, \ X(T) = (y_1X_1^N(T) + y_2X_2^N(T))/(y_1 + y_2) \}. \]

Suppose \( r_N < rN^{-\alpha} \) for some \( \alpha > \frac{1}{2} \) and \( r > 0 \). Let \( g \) be a uniformly continuous and bounded measurable function on \( [0, \infty) \times \mathbb{R}^d \), supported on \( [t_0, t_1] \times \mathbb{R}^d \) with \( 0 < t_0 < t_1 \). Then

\[
|N^{-\frac{1}{2}}(r_N)^{1-d}E[g(T, X(T))]| - c_d \sqrt{\frac{b_1^2}{\tau_1} + \frac{b_2^2}{\tau_2}} \int_{t_0}^{t_1} \int_{\mathbb{R}^d} q_1(0, x_1; t, z)q_2(0, x_2; t, z)g(t, z)dtdz| \to 0,
\]

where \( q_i \) is the transition density for the \( d \)-dimensional Brownian motion with diffusivity \( a_i = \left( \frac{b_i}{\tau_i} \right)^2 \) and \( c_d \) is the product of the volume of a unit ball in \( \mathbb{R}^{d-1} \) and the expected norm of a standard normal vector in \( \mathbb{R}^d \).

**Theorem 2.4.** Under the same setting as Theorem 2.3, but suppose now that \( r_N > N^{-\alpha} \) for some \( \alpha < \frac{1}{2} \) and \( r_N \to 0 \) as \( N \to \infty \). Let \( g \) be a uniformly continuous and bounded measurable function on \( [0, \infty) \times \mathbb{R}^d \), supported on \( [t_0, t_1] \times \mathbb{R}^d \) with \( 0 < t_0 < t_1 \). Then

\[
|(r_N)^{2-d}E[g(T, X(T))]| - c_d \left( \frac{b_1}{\tau_1} \right)^2 + \left( \frac{b_2}{\tau_2} \right)^2 \int_{t_0}^{t_1} \int_{\mathbb{R}^d} q_1(0, x_1; t, z)q_2(0, x_2; t, z)g(t, z)dtdz| \to 0,
\]

where \( q_i \) is the transition density for the \( d \)-dimensional Brownian motion with diffusivity \( a_i = \left( \frac{b_i}{\tau_i} \right)^2 \) and

\[
\frac{1}{c_d} = \int_0^\infty \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{1}{2}\pi t} dt.
\]

As we can see, the form of \( K \) is different under different microscopic dynamics of the particles and this will also change the properties of Smoluchowski coagulation equations. Now, we assume the particles have same density, i.e. their mass \( y \sim r^3 \). Then in the Brownian case, Einstein-Stokes relation suggests that \( a(y) \sim \frac{1}{y^{1/3}} \). So, we have \( K(y_1 + y_2) = c_d(a_1 + a_2)(r_1 + r_2)^{d-2} \sim (y_1^{1/3} + y_2^{1/3})(\frac{1}{y_1^{1/3}} + \frac{1}{y_2^{1/3}}) \). In [7], Norris proved that (3) has a unique solution when \( K(y, y') \leq w(y)w(y') \) for some sublinear function \( w \) such that \( a^{-\frac{1}{2}}w \) is also sublinear. So, in the Brownian case, this result applies when we pick \( w(y) = c(y^{1/3} + y^{-1/3}) \) for some constant \( c \). If we assume the particle is making diffusion under periodic drift, then Corollary 2.2 suggests that under certain scaling limit, we should take \( K(y_1 + y_2) = c_d(\bar{a}_1 + \bar{a}_2)(r_1 + r_2)^{d-2} \sim (y_1^{1/3} + y_2^{1/3})(\frac{1}{y_1^{1/3}} + \frac{1}{y_2^{1/3}}) \). When the streamline of the drift is closed and \( a \) is relatively small compared to the scale of the drift, [28]
suggests that $\bar{a}$ is approximately $c\sqrt{a}$, and we can still apply Norris’s work in this case.

However, in the Ornstein-Uhlenbeck case, Theorem 2.3 suggests that $K \sim \sqrt{\frac{\nu_1}{\tau_1} + \frac{\nu_2}{\tau_2} (y_1^{-1/3} + y_2^{-1/3})^2}$ and the effective diffusivity of the two particles are $\frac{\nu_1}{\tau_1}$ and $\frac{\nu_2}{\tau_2}$ respectively. For $i = 1, 2$, in [10], it is assumed that the drag force on a particle is caused by the particle being hit by random particles of much smaller size and higher speed and it has been shown that under certain scaling limit it is appropriate to choose $\tau_i = y_i^{-1/3}$ and $b_i = y_i^{-2/3}$. On the other hand, according to Einstein relation, where it is assumed that the drag force is caused by friction, the appropriate choice would be $\tau_i = y_i^{-2/3}$ and $b_i = y_i^{-5/6}$. In both cases, we have

$$K \sim (y_1^{1/3} + y_2^{1/3})^2 \left( \frac{1}{y_1} + \frac{1}{y_2} \right),$$

and the effective diffusivity of a particle with mass $y$ would be $y^{-2/3}$ according to [10] and $y^{-1/3}$ according to Einstein relation. In both cases, we cannot directly apply prior results to obtain existence and uniqueness of the solution. Therefore, we will investigate alternative approaches to the well-posedness of (3).

In order to make sense of (3), we need to first show that both $K^+$ and $K^-$ are kernels. Let $\nu(x, \cdot)$ denote the product measure of $\mu(x, \cdot)$ and $\mu(x, \cdot)$. Then as $K : (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$ is a measurable function, $K\nu(x, \cdot)$ is also a measure. As $f : (y, y') \rightarrow y + y'$ is a measurable function from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}^d$, $K^+(\mu)(x, \cdot)$ is the image measure of $K\nu(x, \cdot)$ induced by $f$. Also, $g : (y, y') \rightarrow y'$ is a measurable function, and thus $K^-(\mu)(x, \cdot)$ is the image measure of $K\nu(x, \cdot)$ induced by $g$.

Further, we assume the following conditions throughout this paper

(i) $K(y, z) \leq w(y)w(z)$ with $w : (0, \infty) \rightarrow (0, \infty)$ a non-decreasing sublinear function.

(ii) For some $\delta > 0$, $\mu_0 1_{y < \delta} = 0$.

(iii) The diffusivity $a$ is strictly positive and measurable.

Write $\mathcal{M}[0, T]$ for the set of kernels

$$\mu : [0, T] \times \mathbb{R}^d \times B(0, \infty) \rightarrow [0, \infty].$$

We will also use the notation $\langle f, \mu_t \rangle(x) = \int_0^\infty f(y)\mu_t(x, dy)$ for $f : (0, \infty) \rightarrow (0, \infty)$. We call a
process \( \mu_t \in \mathcal{M}[0,T] \) a solution of (3) if it satisfies (3) for \( t \leq T \) and
\[
\sup_{t \leq T} \| \langle y, \mu_t \rangle \|_1 < \infty.
\]
This notion of solutions will also be used throughout this paper for other pdes. It has been shown that
\[
\| \langle y, \mu_t \rangle \|_1 \leq \| \langle y, \mu_0 \rangle \|_1,
\]
provided both sides are finite, see [11]. We will now state our main results.

**Theorem 2.5.** Assume conditions (i), (ii) and (iii) hold. Let \( (\mu_1^t)_{t \leq T} \) and \( (\mu_2^t)_{t \leq T} \) be solutions of (3) such that for \( i = 1, 2 \), \( \sup_{t \leq T} \| \langle w^2, \mu_i^t \rangle \|_\infty < \infty \). Then \( \mu_1 = \mu_2 \).

In [12], Hammond and Rezakhanlou proved that when the mass \( y \) takes integer values, there is at most one solution \( \mu \) such that \( \sup_{t \leq T} \| \langle w^2, \mu_t \rangle \|_\infty < \infty \). Our result works in the case when \( y \) can take values in positive real numbers, and we will see that the method we used gives a natural iteration scheme which can prove the existence result under certain conditions. Moreover this theorem works for a wide range of situations. There is no explicit requirement for the diffusivity and the condition \( \sup_{t \leq T} \| \langle w^2, \mu_t \rangle \|_\infty < \infty \) looks reasonable.

**Theorem 2.6.** We assume that the function \( w \) can be chosen so that for some constant \( C \)
\[
w^2(y + y')p(y + y') - w^2(y)p(y) - w^2(y')p(y') \leq C[w(y)w(y')p(y) + w(y)w(y')p(y')].
\]
(4)
If in addition, the initial kernel \( \mu_0 \) satisfies \( \sup_{t > 0} \| \langle w^2, \mu_t \rangle \|_\infty < \infty \) and \( \| \langle y, \mu_0 \rangle \|_1 < \infty \), then there exists \( T > 0 \) such that there exists a unique solution to our PDEs up to time \( T \).

Note that (4) is satisfied if \( w(y) = c_1 y^u \) and \( a(y) = c_2 y^{-v} \) with \( 0 < u \leq 1 \) and \( c_1, c_2, v > 0 \). To see this, we note that \( \frac{p(y)}{p(y + y')} \geq \left( \frac{y}{y + y'} \right)^{vd/2} \). By dividing both side of (4) by \( p(y + y') \), it suffices to show that
\[
(y + y')^{2a} - (y)^{2a} \left( \frac{y}{y + y'} \right)^{vd/2} - (y')^{2a} \left( \frac{y'}{y + y'} \right)^{vd/2} \leq C[(yy')^u \left( \frac{y}{y + y'} \right)^{vd/2} + (yy')^u \left( \frac{y'}{y + y'} \right)^{vd/2}]\]
As this inequality is homogeneous and symmetric in \( y \) and \( y' \), we can assume without loss of
generality that \( y \geq y' = 1 \). Then it suffices to show that

\[
(y + 1)^{2u+vd/2} - y^{2u+vd/2} \leq Cy^{u+vd/2}
\]

for some \( C > 0 \). This is true because \( \frac{(y+1)^{2u+vd/2} - y^{2u+vd/2}}{y^{u+vd/2}} \) is continuous on \( y \geq 1 \) and

\[
\limsup_{y \to \infty} \frac{(y + 1)^{2u+vd/2} - y^{2u+vd/2}}{y^{u+vd/2}} < \infty.
\]

**Corollary 2.7.** When \( K(y_1, y_2) = (y_1^{1/3} + y_2^{1/3})^2 \sqrt{\frac{1}{y_1} + \frac{1}{y_2}} \), and when \( a(y) = y^{-\frac{1}{3}} \) or \( a(y) = y^{-2/3} \), there exists a \( T > 0 \) such that there exists a unique solution to our PDEs up to time \( T \).

**Proof.** Note that

\[
K(y_1, y_2) \leq 2(y_1^{2/3} + y_2^{2/3})(y_1^{-1/2} + y_2^{-1/2})
\]

Since \( \mu_0 1_{y < \delta} = 0 \), we only need to care about the case when \( y_1, y_2 \geq \delta \). So, we can pick \( w(y) = 4\delta^{-7/6}y^{2/3} \).

Now, we will give two cases where we can show the global existence of the solutions.

**Theorem 2.8.** If all conditions in Theorem 2.6 are satisfied, then there exists a unique global solution to our PDEs in the following two cases:

(a) \( K(y_1, y_2) \leq w(y)v(y') + w(y')v(y) \) for some \( v \) such that \( wvp \) is sublinear.

(b) \( \sup_{t>0}(1+t)^{1+\epsilon}\|\langle w^2, P_t(\mu_0)\rangle\|_\infty < c \), for some \( \epsilon > 0 \) and sufficiently small \( c > 0 \) depending on \( \epsilon \) and \( C \).

Taking \( v(y) = y^{-\frac{1}{2}} \) and \( w(y) = 4\sqrt{2}y^{2/3} \), the condition (a) is satisfied for our case where

\[
K(y_1, y_2) = (y_1^{1/3} + y_2^{1/3})^2 \sqrt{\frac{1}{y_1} + \frac{1}{y_2}}
\]

and the diffusivity \( a(y) = y^{-\frac{1}{3}} \). Condition (b) is satisfied if, for example,

\[
\int_{\mathbb{R}^d} \int_0^\infty \mu_0(x, dy) w^2(y)(1 + a(y)^{-\frac{d}{2}})dx < h,
\]

for sufficiently small \( h \). We can now conclude the following result.
Corollary 2.9. Assume $K(y_1, y_2) = (y_1^{1/3} + y_2^{1/3})^2 \sqrt{\frac{1}{y_1} + \frac{1}{y_2}}$. If $a(y) = y^{-\frac{1}{3}}$, then there exists a unique global solution. If $a = y^{-\frac{2}{3}}$ and (b) is satisfied, then there also exists a unique global solution.

3 Estimate for diffusion particles

In [8], Norris proved Theorem 2.1 in the case $X_i$ is a Brownian motion. Intuitively it is unlikely for two particles to collide at $x$ and at time $t$ unless they are both close to $x$ at a time $s$ slightly before $t$. If the two particles are near $x$ at time $s$, then we can approximate their behaviour during $(s, t)$ as Brownian motion with diffusivity $a(x)$. In this section, we will use this idea to prove Theorem 2.1. We first list the tools we are going to use. To start with, we want to have some idea about the behaviour of $p_i$ and we will use the following result from [3].

Theorem 3.1. Using same notation as in the above theorem, there exists a constant $C$ depending only on $d$ and $R$ such that for all $x, y \in \mathbb{R}^d$,

$$C^{-1}t^{-d/2}\exp\{-C|y - x|^2/t\}e^{-Ct} \leq p_i(0, x; t, y) \leq Ct^{-d/2}\exp\{-|y - x|^2/Ct\}e^{Ct}.$$

Moreover, $p_i(0, x; t, y)$ is locally H"older continuous in $t > 0$ and $y$.

Next, we note that it is intuitive to believe that $\int_{t_0}^{t_1} \int_{\mathbb{R}^d} p_1(0, x_1; s, z)p_2(0, x_2; s, z)dzds$ measures the expected amount of time when the two particles are “close”, and more precisely, we would expect

$$\int_{t_0}^{t_1} \int_{\mathbb{R}^d} p_1(0, x_1; s, z)p_2(0, x_2; s, z)dzds = \lim_{h \to 0} V(h)^{-1}\mathbb{E}\left(\int_{t_0}^{t_1} 1_{|X_1 - X_2| < h}ds\right),$$

where $V(h)$ denotes the volume of the $d$-dimensional sphere with radius $h$. Actually, using the above theorem, we can prove the following more general result.

Corollary 3.2. Let $X = X_1 - X_2$, for $0 \leq t_0 < t_1$ and $x_1 \neq x_2$ we have for all bounded uniformly continuous function $f$,

$$\int_{t_0}^{t_1} \int_{\mathbb{R}^d} p_1(0, x_1; s, z)p_2(0, x_2; s, z)f(z)dzds = \lim_{h \to 0} V(h)^{-1}\mathbb{E}\left(\int_{t_0}^{t_1} 1_{|X_1 - X_2| < h}f(X_2^2)ds\right).$$
Proof. Let $S_n = [\max\{\frac{1}{n}, t_0\}, t_1] \times \{x \in \mathbb{R}^d : |x| \leq n\}$. Note that

$$V(h)^{-1}\mathbb{E}\left[\int_{t_0}^{t_1} 1_{|X_s| < h} f(X_s^2)\right] = V(h)^{-1} \int_{t_0}^{t_1} \int_{\mathbb{R}^d} \int_{|y-z| \leq h} p_1(0, x_1; s, y)p_2(0, x_2; s, z)f(z)dydzdt. \tag{5}$$

By continuity of $p$ we know that

$$\lim_{h \to 0} V(h)^{-1} \int_{|y-z| \leq h} p_1(0, x_1; s, y)p_2(0, x_2; s, z)f(z)dsdydz = p_1(0, x_1; s, z)p_2(0, x_2; s, z)f(z).$$

So, if we let $h \to 0$ in (5) and justify changing the order of limit and integral on the right hand side, we would get the desired result. Using the Hölder continuity result, we could change the order of limit and integral if the integral domain was $S_n$ instead of $[t_0, t_1] \times \mathbb{R}^d$. We will show that we can approximate the integral by integrating over $S_n$. Using the above theorem, we deduce that

$$V(h)^{-1} \int_{S_n} \int_{|y-z| \leq h} p_1(0, x_1; s, y)p_2(0, x_2; s, z)f(z)dzdydz \leq C^2 s^{-d} \exp\left\{-\frac{|x_1 - y|^2 + |x_2 - z|^2}{Cs}\right\}.$$

Now, if we let $h < \frac{|x_1 - x_2|}{2}$ and assume $|y - z| \leq h$ then we have by triangle inequality that

$$|x_1 - y| + |x_2 - z| \geq \frac{|x_1 - x_2|}{2},$$

and thus

$$|x_1 - y|^2 + |x_2 - z|^2 \geq \frac{|x_1 - x_2|^2}{8}.$$

Therefore, we can further deduce that

$$V(h)^{-1} e^{2Ct_1} \int_{S_n} \int_{|y-z| \leq h} C^2 s^{-d} \exp\left\{-\frac{|x_1 - y|^2 + |x_2 - z|^2}{Cs}\right\}dzdydz \leq C^2 e^{2Ct_1} \int_{S_n} s^{-d} \min\left\{e^{-\frac{|x_1 - y|^2}{8Cs}}, e^{-\frac{|x_1 - y|^2}{Cs}}\right\}dsdy.$$

Note that for $m > 0$ sufficiently large we have for all $|y| > m$ and $0 < s \leq t_1$

$$s^{-d} e^{-\frac{|x_1 - y|^2}{8Cs}} \leq t_1^{-d} e^{-\frac{|x_1 - y|^2}{Ct_1}}.$$
Then we obtain
\[
\int_{|y|>m} \int_0^{t_1} s^{-d} \min\{e^{-\frac{|x_1-x_2|^2}{8cs}}, e^{-\frac{|x_1-y|^2}{Cs}}\} ds dy \\
\leq t_1 \int_{|y|>m} t_1^{-d} e^{-\frac{|x_1+y|^2}{ct^1}} dy < \infty,
\]
and
\[
\int_{|y|\leq m} \int_0^{t_1} s^{-d} \min\{e^{-\frac{|x_1-x_2|^2}{8cs}}, e^{-\frac{|x_1-y|^2}{Cs}}\} ds dy \\
\leq V(m) \int_0^{t_1} s^{-d} e^{-\frac{|x_1-x_2|^2}{Cs}} dy < \infty.
\]
Summing up, we have
\[
\int_{\mathbb{R}^d} \int_0^{t_1} s^{-d} \min\{e^{-\frac{|x_1-x_2|^2}{8cs}}, e^{-\frac{|x_1-y|^2}{Cs}}\} ds dy < \infty.
\]
Thus,
\[
C^2 e^{2Ct_1} \int_{(S_n)^c} s^{-d} \min\{e^{-\frac{|x_1-x_2|^2}{8cs}}, e^{-\frac{|x_1-y|^2}{Cs}}\} ds dy \to 0
\]
and the convergence is uniform in \(h\). Now, we use Hölder continuity to obtain
\[
\lim_{h \to 0} V(h)^{-1} \mathbb{E}\left( \int_{t_0}^{t_1} 1_{|X_s|<h} f(X_s^2) ds \right) \\
= \lim_{h \to 0} V(h)^{-1} \int_{\mathbb{R}^d} \int_{|y-z|\leq h} \int_{t_0}^{t_1} p_1(0, x_1; s, y)p_2(0, x_2; s, z)f(z) ds dy dz \\
= \lim_{h \to 0} \lim_{n \to \infty} V(h)^{-1} \int_{(S_n)^c} \int_{|y-z|\leq h} \int_{t_0}^{t_1} p_1(0, x_1; s, y)p_2(0, x_2; s, z)f(z) ds dy dz \\
= \lim_{n \to \infty} \lim_{h \to 0} V(h)^{-1} \int_{(S_n)^c} \int_{|y-z|\leq h} \int_{t_0}^{t_1} p_1(0, x_1; s, y)p_2(0, x_2; s, z)f(z) ds dy dz \\
= \lim_{n \to \infty} \int_{(S_n)^c} p_1(0, x_1; s, y)p_2(0, x_2; s, y)f(y) dy ds \\
= \int_{t_0}^{t_1} \int_{\mathbb{R}^d} p_1(0, x_1; s, z)p_2(0, x_2; s, z)f(z)dz ds.
\]
We could swap the order of limit in the third line to the fourth line because we have uniform convergence of the integral and we used Hölder continuity to go from the fourth line to the fifth line.
We will also use the following theorem:

**Theorem 3.3.** Consider

\[ dX_t = b_t dt + dB_t, \]

\[ X_0 = x, \]

where \( B_t \) is a standard Brownian motion in \( \mathbb{R}^d \) and \( b_t \) is an \( \mathcal{F}_t \) adapted process and \( |b_i| < C \) for all \( t \) and \( i = 1, 2, \ldots, d \). Then for all \( t > 0 \) and all \( y \in \mathbb{R}^d \), the random variable \( X_t \) has a density function \( \rho \) such that

\[
\frac{1}{(2\pi t)^{d/2}} \prod_{i=1}^{d} \left( \int_{|x^i - y^i|/\sqrt{t}}^{\infty} z e^{-(z+C\sqrt{t})^2/2} dz \right) \leq \rho(t, y),
\]

and

\[
\frac{1}{(2\pi t)^{d/2}} \prod_{i=1}^{d} \left( \int_{|x^i - y^i|/\sqrt{t}}^{\infty} z e^{-(z-C\sqrt{t})^2/2} dz \right) \geq \rho(t, y).
\]

Moreover, \( \rho \) can be chosen to be locally Hölder continuous.

**Proof.** In [9], they have shown that the above inequalities are true in the case when \( b_t \) is a function of \( X_t \). But, their method also works if \( b_t \) is any \( \mathcal{F}_t \) adapted process. They have shown that the two bounds are attained respectively when \( b_i^t = C\text{sgn}(X_i^t - y^i) \) and when \( b_i^t = C\text{sgn}(y^i - X_i^t) \). Let \( p_y^+(0, x; t, z) \) denote the transition density when \( b_i^t = C\text{sgn}(y^i - X_i^t) \). Then, for any \( b \) adapted to \( \mathcal{F}_t \), by Cameron-Martin formula, we can let \( \rho \) be such that

\[
\frac{\rho(t, y)}{p_y^+(0, x; t, y)} = \mathbb{E}_{x,y} \left\{ \prod_{i=1}^{d} \exp \left[ \int_0^t \left( b_i^s - C\text{sgn}(y^i - W_i^s) \right) dB_i^s - \frac{1}{2} \int_0^t \left( b_i^s - C\text{sgn}(y^i - W_i^s) \right)^2 ds \right] \right\},
\]

where \( W \) is the motion satisfying

\[ dW_i^s = C\text{sgn}(y^i - W_i^s) ds + dB_i^s \]

and \( \mathbb{E}_{x,y} \) denotes the expectation conditioning on \( W_0 = x \) and \( W_t = y \). The conditional expectation \( \mathbb{E}_{x,y} \) can be shown to be well defined using Gaussian heat kernel estimation. In [5], they have shown that for \( s < t \), we can represent \( B_s \) by

\[
B_i^t = B_i^s + \int_0^s \frac{\partial}{\partial W_i^s} \log[p_y^+(s, W_s; t, y)] ds,
\]
for some unconditional Brownian motion $B'$. Also note that

$$\left[ b_s^i - C \text{sgn}(y^i - W_s^i) \right] \frac{\partial}{\partial W_t} \log[p_y^+(s, W_s; t, y)] ds \leq 0.$$ 

So, we can plug these in and obtain

$$\frac{\rho(t, y)}{p_y^+(0, x; t, y)} = E_{x,y} \left\{ \prod_{i=1}^d \exp \left\{ \int_0^t (b_s^i - C \text{sgn}(y^i - W_s^i)) dB_t^i - \frac{1}{2} \int_0^t (b_s^i - C \text{sgn}(y^i - W_s^i))^2 ds \right. \right.$$

$$\left. + \int_0^t (b_s^i - C \text{sgn}(y^i - W_s^i)) \frac{\partial}{\partial W_t} \log[p_y^+(s, W_s; t, y)] ds \right\}$$

$$\leq E_{x,y} \left\{ \prod_{i=1}^d \exp \left[ \int_0^t (b_s^i - C \text{sgn}(y^i - W_s^i)) dB_t^i - \frac{1}{2} \int_0^t (b_s^i - C \text{sgn}(y^i - W_s^i))^2 ds \right] \right.$$

$$\left. \right\} = 1.$$ 

This proves the upper bound for the density and the lower bound can be obtained similarly.

Now, we will use these bounds to prove the H"older continuity of the density. Let $s > 0$ be fixed, and let $Y_t = X_{t+s}$ and $G_t = F_{t+s}$, then $Y$ is a $G_t$ adapted process satisfying the conditions in the theorem. So, the random variable $Y_t$ has a density function $\rho_{s,u}(t, y)$ satisfying

$$\rho_{s,u}(t, y) \leq \frac{1}{(2\pi t)^{\frac{d}{2}}} \prod_{i=1}^d \left( \int_{|u^i - y^i|/\sqrt{t}}^\infty \right) w e^{-(w-C\sqrt{t})^2/2} \, dw$$

and

$$\rho_{s,u}(t, y) \geq \frac{1}{(2\pi t)^{\frac{d}{2}}} \prod_{i=1}^d \left( \int_{|u^i - y^i|/\sqrt{t}}^\infty \right) w e^{-(w+C\sqrt{t})^2/2} \, dw,$$

where $u$ denotes the position $X_s$. Now, let $|y - z| < \epsilon$ for some sufficiently small positive $\epsilon$, we have for $s < t$,

$$\rho(t, y) = \int_{\mathbb{R}^d} \rho(s, u) \rho_{s,u}(t - s, y) \, du,$$

and

$$\rho(t, y) - \rho(t, z) = \int_{\mathbb{R}^d} \rho(s, u) [\rho_{s,u}(t - s, y) - \rho_{s,u}(t - s, z)] \, du.$$
If we let $s = t - \epsilon$, then we have

\[
\tilde{\rho}_{s,u}(t-s,y) \leq \frac{1}{(2\pi\epsilon)^{d/2}} \prod_{i=1}^{d} \left( \int_{|u_i-y_i|/\sqrt{C\epsilon}}^{\infty} e^{-(w-C\epsilon)^2/2dw} \right)
\]

\[
= \frac{1}{(2\pi\epsilon)^{d/2}} \prod_{i=1}^{d} \left( \int_{|u_i-y_i|/\sqrt{C\epsilon}}^{\infty} (w+C\epsilon)e^{-w^2/2dw} \right)
\]

\[
= \frac{1}{(2\pi\epsilon)^{d/2}} \prod_{i=1}^{d} \left[ e^{-\frac{|u_i-y_i|^2}{2}/\sqrt{C\epsilon}} \left( \int_{|u_i-y_i|/\sqrt{C\epsilon}}^{\infty} e^{-w^2/2dw} \right) \right]
\]

\[
\leq \frac{1}{(2\pi\epsilon)^{d/2}} \prod_{i=1}^{d} \left[ e^{-\frac{|u_i-y_i|^2}{2}/\sqrt{C\epsilon}} \left( 1 + \frac{C\epsilon}{\int_{|u_i-y_i|/\sqrt{C\epsilon}}^{\infty} e^{-w^2/2dw} } \right) \right]
\]

\[
\leq \frac{1}{(2\pi\epsilon)^{d/2}} \prod_{i=1}^{d} \left[ e^{-\frac{|u_i-y_i|^2}{2}/\sqrt{C\epsilon}} \left( 1 + c\sqrt{\epsilon} \right) \right].
\]

for some constant $c$ which doesn’t depend on $u$. Now we will denote $c$ as a constant depending only on $C$ and $d$ and its value might change from line to line. Similarly, we would have

\[
\tilde{\rho}_{s,u}(t-s,z) \geq \frac{1}{(2\pi\epsilon)^{d/2}} \prod_{i=1}^{d} \left[ e^{-\frac{|u_i-z_i|^2}{2}/\sqrt{C\epsilon}} \left( 1 - c\sqrt{\epsilon} \right) \right]
\]

So, if $|u - y| \leq \epsilon^{1/4}$, then we have

\[
\frac{\tilde{\rho}_{s,u}(t-s,y)}{\tilde{\rho}_{s,u}(t-s,z)} \leq \frac{1 + c\sqrt{\epsilon}}{1 - c\sqrt{\epsilon}} \prod_{i=1}^{d} \left[ e^{-\frac{|u_i-y_i|^2}{2}/\sqrt{C\epsilon}} e^{\frac{|u_i-z_i|^2}{2}/\sqrt{C\epsilon}} \right]
\]

\[
\leq \frac{1 + c\sqrt{\epsilon}}{1 - c\sqrt{\epsilon}} \prod_{i=1}^{d} \left[ e^{C(2|u_i-y_i^i|+\epsilon)} + \epsilon^\frac{1}{2} \right]
\]

\[
\leq 1 + c\epsilon^{\frac{1}{4}}.
\]

If $|u - y| \geq \epsilon^{1/4}$, then we simply have

\[
\tilde{\rho}_{s,u}(t-s,y) \leq ce^{-\frac{d}{2}e^{-\frac{\epsilon}{\sqrt{C\epsilon}}} \leq \epsilon},
\]

18
say. So, for general \( u \in \mathbb{R}^d \), we would have

\[
\bar{\rho}_{s,u}(t - s, y) - \bar{\rho}_{s,u}(t - s, z) \leq c \varepsilon^{1/4} \bar{\rho}_{s,u}(t - s, z) + \varepsilon.
\]

Thus, we have

\[
\rho(t, y) - \rho(t, z) = \int_{\mathbb{R}^d} \rho(s, u)[\bar{\rho}_{s,u}(t - s, y) - \bar{\rho}_{s,u}(t - s, z)]du
\leq \int_{\mathbb{R}^d} \rho(s, u)[c \varepsilon^{1/4} \bar{\rho}_{s,u}(t - s, z) + \varepsilon]du
= c \varepsilon^{1/4} \rho(t, z) + \varepsilon.
\]

Similarly, we would have

\[
\rho(t, z) - \rho(t, y) \leq c \varepsilon^{1/4} \rho(t, y) + \varepsilon,
\]

which gives the Hölder continuity in \( y \).

Now, we will use the same idea to obtain the Hölder continuity in \( t \). Let \( 0 < t_2 - t_1 \leq \varepsilon \) and let \( t_1 - s = \sqrt{\varepsilon} \). Then we have

\[
\bar{\rho}_{s,u}(t_1 - s, y) \leq \frac{1}{(2\pi \sqrt{\epsilon})^{d/2}} \prod_{i=1}^{d} [e^{-\frac{|u^i - y^i|}{2} - C \varepsilon^{1/4}(1 + \varepsilon^{1/4})}](1 - c \varepsilon^{1/4})
\]

and

\[
\bar{\rho}_{s,u}(t_2 - s, y) \geq \frac{1}{[2\pi(\sqrt{\epsilon} + \varepsilon)]^{d/2}} \prod_{i=1}^{d} [e^{-\frac{|u^i - y^i|}{2} + C(\sqrt{\varepsilon} + \varepsilon)}}(1 - c \varepsilon^{1/4})].
\]

When \( |u - y| \leq \varepsilon^{1/4} \) we can also have

\[
\frac{1}{[2\pi(\sqrt{\epsilon} + \varepsilon)]^{d/2}} \prod_{i=1}^{d} [e^{-\frac{|u^i - y^i|}{2} + C(\sqrt{\varepsilon} + \varepsilon)}}(1 - c \varepsilon^{1/4})] \geq \frac{1}{(2\pi \sqrt{\epsilon})^{d/2}} \prod_{i=1}^{d} [e^{-\frac{|u^i - y^i|}{2} + C \varepsilon^{1/4}(1) + c \varepsilon^{1/4}}(1 - c \varepsilon^{1/4})].
\]

This gives us

\[
\frac{\bar{\rho}_{s,u}(t_1 - s, y)}{\bar{\rho}_{s,u}(t_2 - s, y)} \leq 1 + c \varepsilon^{1/4}.
\]
Again, when $|u - y| \geq \epsilon^\frac{1}{8}$ we have

$$\bar{\rho}_{s,u}(t_1 - s, y) \leq c\epsilon^{-\frac{d}{4}}e^{-c\epsilon^{-\frac{1}{4}}} \leq \epsilon,$$

So, for general $u \in \mathbb{R}^d$, we would have

$$\bar{\rho}_{s,u}(t_1 - s, y) - \bar{\rho}_{s,u}(t_2 - s, y) \leq c\epsilon^{1/4}\bar{\rho}_{s,u}(t_2 - s, y) + \epsilon.$$

Thus, we have

$$\rho(t_1, y) - \rho(t_2, y) = \int_{\mathbb{R}^d} \rho(s, u)[\bar{\rho}_{s,u}(t_1 - s, y) - \bar{\rho}_{s,u}(t_2 - s, y)]du$$

$$\leq \int_{\mathbb{R}^d} \rho(s, u)[c\epsilon^{1/4}\bar{\rho}_{s,u}(t_2 - s, y) + \epsilon]du$$

$$= c\epsilon^{1/4}\rho(t_2, y) + \epsilon.$$

Similarly, we would have

$$\rho(t_2, y) - \rho(t_1, y) \leq c\epsilon^{1/4}\rho(t_2, y) + \epsilon$$

and this proves that the transition density is locally H"older continuous in $t$. \qed

Now, we can start proving Theorem \ref{thm:holder}. \hfill \null

\textbf{Proof.} From now on, we shall write $C$ as a constant depending only on $d$ and $R$, and the value of $C$ might change from line to line. We set $X(t) = X^1(t) - X^2(t)$. As $a_i$ are scalars, we would like to use Dubins-Schwarz theorem to relate $X$ with a Brownian motion with drift. Let

$$A(t) = \int_0^t [a_1(X^1(s)) + a_2(X^2(s))]ds.$$

So, $A(t)$ is the quadratic variation process of $X(t)$. Set $\tau_t$ be the stopping time such that $A(\tau_t) = t$, then we have

$$dt = dA(\tau_t) = [a_1(X^1(\tau_t)) + a_2(X^2(\tau_t))]d\tau_t$$

Let $Y(t) = X(\tau_t)$ and

$$B_t = \int_0^{\tau_t} \sqrt{a_1(X^1(s))}dB^1_s - \int_0^{\tau_t} \sqrt{a_2(X^2(s))}dB^2_s.$$
Then we have, by Dubins-Schwarz theorem, that $B$ is a Brownian motion. Note that $A(t)$ is continuous and strictly increasing and goes from 0 to infinity, we have $\tau_{A(t)} = t$ and $Y(A(t)) = X(\tau_{A(t)}) = X(t)$. Moreover, let

$$b(t) = \frac{b_1(X_{\tau_t}^1) - b_2(X_{\tau_t}^2)}{a_1(X_{\tau_t}^1) + a_2(X_{\tau_t}^1)},$$

we have

$$dY(t) = dX^1(\tau_t) - dX^2(\tau_t)
= \sqrt{a_1(X_{\tau_t}^1)}dB^1(\tau_t) + b_1(X_{\tau_t}^1)d(\tau_t) - \sqrt{a_2(X_{\tau_t}^2)}dB^2(\tau_t) - b_2(X_{\tau_t}^2)d(\tau_t)
= dB_t + [b_1(X_{\tau_t}^1) - b_2(X_{\tau_t}^2)][a_1(X_{\tau_t}^1) + a_2(X_{\tau_t}^1)]^{-1}dt
= dB_t + b(t)dt.$$

Because $b_i$’s are bounded above by $R$ and $a_i$’s are bounded below by $R^{-1}$, we have $|b|$ is bounded by some constant depending on $R$.

Now, we have enough tools to start our proof. We define for each $s \in (0, \infty)$ and $z \in \mathbb{R}^d$ the process

$$M_t = 1_{t < s}p_1(t, X_{\tau_t}^1; s, z)p_2(t, X_{\tau_t}^2; s, z), t \geq 0.$$

Then $M$ is continuous almost surely, $(M_t)_{t < s}$ is a martingale, $M_t = 0$ for all $t \geq s$ and $M_t$ is uniformly bounded up to $T$. Hence, by optional stopping and bounded convergence,

$$p_1(0, x_1; s, z)p_2(0, x_2; s, z) = \mathbb{E}[p_1(T, X_T^1; s, z)p_2(T, X_T^2; s, z)].$$

On multiplying by $g(s, z)K(z)$ and integrating over $(0, R) \times \mathbb{R}^d$ we obtain

$$\int_0^R \int_{\mathbb{R}^d} p_1(0, x_1; s, z)p_2(0, x_2; s, z)g(s, z)K(z)dzds = \mathbb{E}[\int_0^R \int_{\mathbb{R}^d} p_1(T, X_T^1; s, z)p_2(T, X_T^2; s, z)g(s, z)K(z)dzds]. \quad (7)$$

We will now estimate the right hand side of the above equation. We will show that the contribution when $s$ is "far" from $T$ or when $z$ is "far" from $X_T$ to the integral inside the expectation is small.
so that we can approximate it by

\[ g(T, X_T) \int_0^R \int_{\mathbb{R}^d} p_1(T, X^1_T; s, z)p_2(T, X^2_T; s, z)K(z)dzds. \]

Using Corollary 3.2 we can relate the above integral with the expectation of a functional of \( X \). Then, we use the relation \( Y(A(t)) = X(t) \) to write it as a functional of \( Y \) and use the same idea as in Corollary 3.2 again to approximate it as an integral involving \( \rho \) instead of \( p_i \). Finally, when \( s \) is ”close” to \( T \), we will use Theorem 3.3 to approximate \( \rho \) as the transition density of a standard Brownian motion which we can then evaluate. We first claim that

\[ \int_{\epsilon^2}^R \int_{\mathbb{R}^d} p_1(0, x_1; s, z)p_2(0, x_2; s, z)a(z)dzds \leq C\epsilon^{2-d}. \]  

Let \( \rho \) be the density function for \( Y \), and recall that \( X(t) = Y(A(t)) \). Using the same argument as in Corollary 3.2 we have

\[
\int_{\epsilon^2}^R \int_{\mathbb{R}^d} p_1(0, x_1; s, z)p_2(0, x_2; s, z)a(z)dzds \\
= \lim_{h \to 0} V(h)^{-1}\mathbb{E}\left[ \int_{\epsilon^2}^R \mathbb{1}_{|X_s|<h}a(X^2_s)ds \right] \\
= \lim_{h \to 0} V(h)^{-1}\mathbb{E}\left[ \int_{\epsilon^2}^R \mathbb{1}_{|X_s|<h}(a_1(X^1_s) + a_2(X^2_s))ds \right] \\
= \lim_{h \to 0} V(h)^{-1}\mathbb{E}\left[ \int_{\epsilon^2}^R \mathbb{1}_{|Y_A(s)|<h}(a_1(X^1_s) + a_2(X^2_s))ds \right] \\
= \lim_{h \to 0} V(h)^{-1}\mathbb{E}\left( \int_{\epsilon^2}^R \mathbb{1}_{|Y_A(s)|<h}dA(s) \right) \\
\leq \lim_{h \to 0} V(h)^{-1}\mathbb{E}\left( \int_{R^{-1}\epsilon^2}^{R^2} \mathbb{1}_{|Y_s|<h}ds \right) \\
= \int_{R^{-1}\epsilon^2}^{R^2} \rho(s, 0)ds.
\]
Now, using the result from Theorem 3.3, we have for $R^{-1} \epsilon^2 < s < R^2$,

$$
\rho(s, 0) \leq \frac{1}{(2\pi s)^{\frac{d}{2}}} \prod_{i=1}^{d} \left( \int_{|x_1 - x_2|/\sqrt{s}}^{\infty} z e^{-(z-C\sqrt{s})^2/2} dz \right) $$

$$
\leq C s^{-\frac{d}{2}} \prod_{i=1}^{d} \left( \int_{0}^{\infty} z e^{-(z-CR)^2/2} dz \right) $$

$$
\leq C s^{-\frac{d}{2}}.
$$

So, we have

$$
\int_{R^{-1} \epsilon^2}^{R^2} \rho(s, 0) ds \leq C \int_{R^{-1} \epsilon^2}^{R^2} s^{-\frac{d}{2}} \leq C \epsilon^{2-d},
$$

as required.

Now, we claim that when $|x_1 - x| \leq \frac{\epsilon}{2}$ and $|x_2 - x| \leq \frac{\epsilon}{2}$, then

$$
\int_{0}^{R} \int_{|z-x|>\epsilon} p_1(0, x_1; s, z)p_2(0, x_2; s, z) dz ds \leq C \epsilon^{2-d}.
$$

For this, we can just apply Theorem 3.1 and obtain

$$
\int_{0}^{R} \int_{|z-x|>\epsilon} p_1(0, x_1; s, z)p_2(0, x_2; s, z) dz ds
\leq \int_{0}^{R} \int_{|z-x|>\epsilon} C^2 s^{-d} \exp\left\{ -\frac{|x_1 - z|^2 + |x_2 - z|^2}{C s} \right\} e^{2Cs} dz ds
\leq \int_{0}^{R} \int_{|z-x|>\epsilon} C^2 e^{2CRs^{-d}} \exp\left\{ -\frac{|x_1 - z|^2 + |x_2 - z|^2}{C s} \right\} dz ds
\leq C \int_{|z-x|>\epsilon} \int_{0}^{R} s^{-d} \exp\left\{ -\frac{|x_1 - z|^2 + |x_2 - z|^2}{C s} \right\} ds dz
\leq C \int_{|z-x|>\epsilon} (|x_1 - z|^2 + |x_2 - z|^2)^{-d+1} \int_{0}^{\infty} s^{-d} \exp\left\{ -\frac{1}{C s} \right\} ds dz
\leq C \int_{|z-x|>\epsilon} ((z - x) - \frac{\epsilon}{2})^{-d+1} dz
\leq C \int_{\frac{\epsilon}{2}}^{\infty} r^{d-1}(r^2)^{-d+1} dr
\leq C \epsilon^{2-d},
$$

as required.
Combine (8) and (9) we can estimate the integral on the right hand side of (7) by

\[
\left| \int_0^R \int_{\mathbb{R}^d} p_1(T, X_1^1; s, z)p_2(T, X_2^2; s, z)g(s, z)K(z)dzds \right|
\]

\[
- g(T, X_T) \int_0^R \int_{\mathbb{R}^d} p_1(T, X_1^1; s, z)p_2(T, X_2^2; s, z)K(z)dzds \right|
\]

\[
\leq C\epsilon^{2-d} + \phi_g(\epsilon) \int_0^R \int_{\mathbb{R}^d} p_1(T, X_1^1; s, z)p_2(T, X_2^2; s, z)K(z)dzds,
\]

(10)

provided \( T < R \). So, we will now aim to estimate the value of

\[
\int_0^R \int_{\mathbb{R}^d} p_1(T, X_1^1; s, z)p_2(T, X_2^2; s, z)K(z)dzd.
\]

By same argument as in the proof of (8), we have

\[
\int_0^\epsilon \int_{\mathbb{R}^d} p_1(0, x_1; s, z)p_2(0, x_2; s, z)a(z)dzds
\]

\[
= \lim_{h \to 0} V(h)^{-1}\mathbb{E}\left( \int_0^{\epsilon^2} \mathbf{1}_{|Y_A(s)| < h} dA(s) \right)
\]

\[
\geq \int_0^{R^{-1}\epsilon^2} \rho(s, 0)ds.
\]

Next, we will use Theorem [3.3] to approximate \( \rho(t, 0) \) for small \( t \). Let \( q \) be the transition density of a standard Brownian motion in \( \mathbb{R}^d \), and suppose \( 2|x_1 - x_2| < \epsilon \). Then for \( t < R^{-1}\epsilon^2 \) we can use similar derivation as in (6) to obtain

\[
\rho(t, 0) \geq \frac{1}{(2\pi t)^{\frac{d}{2}}} \prod_{i=1}^d \left( e^{-\frac{|x_1^i - x_2^i|^2}{2t}} (1 - C\sqrt{t}) \right)
\]

\[
\geq \frac{1}{(2\pi t)^{\frac{d}{2}}} \prod_{i=1}^d \left( e^{-\frac{|x_1^i - x_2^i|^2}{2t}} (1 - C\epsilon) \right)
\]

\[
\geq q(0, x_1 - x_2; t, 0)(1 - C\epsilon),
\]

where \( q \) is the transition density for a standard Brownian motion. Now we note that

\[
\int_0^\infty q(0, x_1 - x_2; s, 0)ds = \int_0^\infty \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x_1^i - x_2^i|^2}{2t}} dt. = \frac{1}{c_d}|x_1 - x_2|^{2-d}
\]

24
We also know that
\[
\int_{R^{-1}e^{2}}^{\infty} q(0, x_1 - x_2; s, 0)ds \leq \int_{R^{-1}e^{2}}^{\infty} \frac{1}{(2\pi s)^{\frac{d}{2}}}ds \leq C\epsilon^{2-d},
\]
and thus we have
\[
|x_1 - x_2|^{2-d} - cd \int_{0}^{R} \int_{\mathbb{R}^{d}} p_1(0, x_1; s, z)p_2(0, x_2; s, z)a(z)dzds
\]
\[
\leq |x_1 - x_2|^{2-d} - cd \int_{0}^{R^{-1}\epsilon} \rho(s, 0)ds
\]
\[
\leq |x_1 - x_2|^{2-d} - (1 - C\epsilon^{2})cd \int_{0}^{R^{-1}\epsilon} q(0, x_1 - x_2; s, 0)ds
\]
\[
\leq C\epsilon^{2-d} + C\epsilon|x_1 - x_2|^{2-d}.
\]

By similar method we can obtain
\[
cd \int_{0}^{R} \int_{\mathbb{R}^{d}} p_1(0, x_1; s, z)p_2(0, x_2; s, z)a(z)dzds - |x_1 - x_2|^{2-d} \leq C\epsilon^{2-d} + C\epsilon|x_1 - x_2|^{2-d}.
\]

From these combined with (8) and (9), we will have that if \(|x_1 - x| \leq \frac{\epsilon}{2}\) and \(|x_2 - x| \leq \frac{\epsilon}{2}\), then
\[
|g(0, x)||x_1 - x_2|^{2-d} - cd \int_{0}^{R} \int_{\mathbb{R}^{d}} p_1(0, x_1; s, z)p_2(0, x_2; s, z)a(z)g(s, z)dzds|
\]
\[
\leq |g(0, x)||(|x_1 - x_2|^{2-d} - cd \int_{0}^{R} \int_{\mathbb{R}^{d}} p_1(0, x_1; s, z)p_2(0, x_2; s, z)a(z)dzds)|
\]
\[
+ |cd \int_{0}^{R} \int_{\mathbb{R}^{d}} p_1(0, x_1; s, z)p_2(0, x_2; s, z)a(z)|g(s, z) - g(0, x)|dzds|
\]
\[
\leq \|g\|(C\epsilon^{2-d} + C\epsilon|x_1 - x_2|^{2-d}) + \phi_g(\epsilon)|x_1 - x_2|^{2-d}.
\]

Hence, on the event \(\{T < R\}\) we have
\[
|Ng(T, X_T) - \int_{0}^{\infty} \int_{\mathbb{R}^{d}} K(z)p_1(T, X^{1}_T; s, z)p_2(T, X^{2}_T; s, z)g(s, z)dzds| \leq C[\epsilon^{2-d}\|g\| + N(\phi_g(\epsilon) + \epsilon)].
\]
Recall (7), we have

\[
|\mathbb{E}(g(T, X_T)1_{T < R}) - \int_0^R \int_{\mathbb{R}^d} K(z)p_1(0, x_1; s, z)p_2(0, x_2; s, z)g(s, z)dzds|
\]

\[
\leq C[\varepsilon^{2-d}\|g\| + N(\varepsilon + \phi_g(\varepsilon))]
\mathbb{P}(\delta^2 \leq T \leq R).
\]

On the other hand, we also have

\[
\int_{\delta^2}^{2R} \int_{\mathbb{R}^d} p_1(0, x_1; s, z)p_2(0, x_2; s, z)a(z)dzds
\]

\[
= \mathbb{E}[\int_{\delta^2}^{2R} \int_{\mathbb{R}^d} p_1(T, X_T^1; s, z)p_2(T, X_T^2; s, z)a(z)dzds]
\]

\[
\geq \mathbb{E}[\mathbf{1}_{\delta^2 < T < R} \int_T^{T+R} \int_{\mathbb{R}^d} p_1(T, X_T^1; s, z)p_2(T, X_T^2; s, z)a(z)dzds].
\]

Using the result from Theorem 3.1, we would have

\[
\int_T^{T+R} p_1(T, X_T^1; s, z)p_2(T, X_T^2; s, z)a(z)dzds
\]

\[
\geq \frac{1}{C} \int_{\mathbb{R}^d} \int_0^R s^{-d} \exp\{-C|X_T^1 - z|^2 + |X_T^2 - z|^2\}dsdz
\]

\[
\geq \frac{(rN^{-1/(d-2)})^{2-d}}{C} \geq \frac{N}{C}.
\]

Therefore, we have

\[
N\mathbb{P}(\delta^2 \leq T \leq R) \leq C \int_{\delta^2}^{2R} \int_{\mathbb{R}^d} p_1(0, x_1; s, z)p_2(0, x_2; s, z)a(z)dzds.
\]

Using Theorem 3.1 again we get

\[
\int_{\delta^2}^{2R} \int_{\mathbb{R}^d} p_1(0, x_1; s, z)p_2(0, x_2; s, z)a(z)dzds \leq C(\delta \vee |x_1 - x_2|)^{2-d}.
\]

So, we get

\[
|\mathbb{E}(g(T, X_T)1_{T < R}) - \int_0^R \int_{\mathbb{R}^d} K(z)p_1(T, X_T^1; s, z)p_2(T, X_T^2; s, z)g(s, z)dzds|
\]

\[
\leq C[\varepsilon^{2-d}\|g\|/N + \varepsilon + \phi_g(\varepsilon)](\delta \vee |x_1 - x_2|)^{2-d},
\]
as desired.

As an application, we now prove Corollary 2.2.

Proof. For $i = 1, 2$, let $p_i^\lambda$ be the transition density of $X_i^\lambda$. It is known that we can choose $p_i^\lambda(0, x; s, z)$ to be continuous in $z$, uniformly in $\lambda$. We now prove that

$$p_i^\lambda(0, x; s, z) \to p_i(0, x; s, z)$$

pointwise. Suppose that is not the case, then $\exists \epsilon > 0$ such that $\forall \delta > 0$ we can find $\lambda < \delta$ such that $|p_i^\lambda(0, x; s, z) - p_i(0, x; s, z)| > \epsilon$. Also there exists $\gamma > 0$ such that for all $|x - z| < \gamma$, $|p_i(0, x; s, z) - p_i(0, x; s, x)| < \frac{1}{3}\epsilon$ and $|p_i^\lambda(0, x; s, z) - p_i^\lambda(0, x; s, x)| < \frac{1}{3}\epsilon$. As a result, we have $\exists \epsilon' > 0$ such that $\forall \delta > 0$ we can find $\lambda < \delta$ such that $|\mathbb{P}(|X_i^\lambda(s) - z| < \gamma) - \mathbb{P}(|X_i - z| < \gamma)| > \epsilon'$. But we know that $X_i^\lambda$ converges weakly to $X_i$, so we have a contradiction.

Next, we prove that

$$\int_0^R \int_{\mathbb{R}^d} p_1^\lambda(0, x; s, z)p_2^\lambda(0, x; s, z)g(s, z)\,dz\,ds \to \int_0^R \int_{\mathbb{R}^d} p_1(0, x; s, z)p_2(0, x; s, z)g(s, z)\,dz\,ds.$$ 

Using results [6], we have that

$$p_i^\lambda(0, x; t, y) \leq Ct^{-d/2} \exp\{-|y - x|^2/Ct\}e^{Ct},$$

for some constant $C$ independent of $\lambda$. Then with similar computation used in Corollary 3.2, we can show that

$$\int_0^R \int_{\mathbb{R}^d} C^2s^{-d} \exp\{-|z - x_1|^2 - |z - x_2|^2/Cs\}e^{2Cs}\,dz\,ds < \infty.$$ 

Then by dominated convergence theorem we obtain

$$\int_0^R \int_{\mathbb{R}^d} p_1^\lambda(0, x; s, z)p_2^\lambda(0, x; s, z)g(s, z)\,dz\,ds \to \int_0^R \int_{\mathbb{R}^d} p_1(0, x; s, z)p_2(0, x; s, z)g(s, z)\,dz\,ds.$$ 

Now, fix $\lambda$, Theorem 2.1 tells us that

$$\lim_{N \to \infty} \mathbb{E}(g(T, X_T^\lambda)1_{T < R}) - \int_0^R \int_{\mathbb{R}^d} Kp_1^\lambda(0, x; s, z)p_2^\lambda(0, x; s, z)g(s, z)\,dz\,ds \to 0,$$
and taking $\lambda \to 0$ we have

$$\lim_{\lambda \to 0} \lim_{N \to \infty} |N \mathbb{E}(g(T, X_1^\lambda) 1_{T < R}) - \int_0^R \int_{\mathbb{R}^d} Kp_1(0, x_1; s, z)p_2(0, x_2; s, z)g(s, z)dzds| \to 0,$$

as desired. Note that $g(T, X_1^\lambda)$ is a bounded almost everywhere continuous functional on $X_1^\lambda$ and $X_2^\lambda$ and thus by weak convergence we have

$$\lim_{\lambda \to 0} \mathbb{E}(g(T, X_1^\lambda)) = \mathbb{E}(\bar{T}, X_{\bar{T}})$$

where $\bar{T}$ is the collision time of two Brownian particles with diffusivity $\bar{a}_1$ and $\bar{a}_2$ respectively and $X_{\bar{T}}$ is their centre of mass at time $\bar{T}$. Using Theorem 2.1 again we have

$$\lim_{N \to \infty} \lim_{\lambda \to 0} |N \mathbb{E}(g(T, X_1^\lambda) 1_{T < R}) - \int_0^R \int_{\mathbb{R}^d} Kp_1(0, x_1; s, z)p_2(0, x_2; s, z)g(s, z)dzds| \to 0,$$

as required. \(\square\)

## 4 Estimate for Ornstein-Uhlenbeck particles

We consider in this section two particles starting from $x_1$ and $x_2$. For $i = 1, 2$ let $V_i^N$ and $X_i^N$ be their velocity and position respectively and we suppose they are modelled by the Ornstein-Uhlenbeck process satisfying

$$dV_i^N(t) = Nb_i dB_i^t - N\tau_i V_i^N dt,$$

$$dX_i^N(t) = V_i^N(t) dt.$$

As $N$ tends to infinity, the position $X_i^N$ converges weakly to a Brownian motion and the rate of the convergence is fixed. So, if we let the radii of the particles decrease slowly enough with $N$, the collision will happen in similar way as in the Brownian case. However, if we consider the case when the radii of the particles decrease sufficiently fast then when the two particles come close they are likely to move away from each other with almost constant velocities. But, in the Brownian case, when two particles come close to each other, they are likely to stay around for a bit longer and will thus have more chance to collide. As a result, the scale of the collision rate will be smaller in the Ornstein-Uhlenbeck case. The aim of this section is to investigate the collision distribution for the
Ornstein-Uhlenbeck particles and compare the result with the Brownian case.

Our strategy for proving Theorem 2.3 is to first consider the case where the particles just continue their free motions after they collide and allow them to recollide later. We will divide the time $[t_0, t_1]$ into little time intervals, so that in each interval the velocities of the particles are unlikely to change much. In this case, we can make good predictions about whether and where the particles are going to collide in a time interval based on their positions and velocities at the start of the interval. Also, We know the distribution of positions and velocities of the particles at any time, so we can estimate the distribution of the time and place where the particles collide. Then, we will show that allowing the particles to recollide won’t change our estimation by much because the particles are very unlikely to collide more than once anyway. This is because after the particles collide, they are likely to continue their free motions with almost constant velocities for a small amount of time and this time turns out to be enough for them to get far away from each other so that they are unlikely to collide again. Now, we will start our proof.

Proof. We shall write $C$ as a constant, and $C^N$ as a sequence of constants such that $C^N \to 0$ as $N \to \infty$. We allow the values of $C$ and $C^N$ to change from line to line. We know that when $t_1 \geq t \geq t_0$, $(V_i^N, X_i^N)$ are bivariate normal distributed with

$$|\text{Var}(V_i^N) - \frac{N b_i^2}{2 \tau_i}| \leq \frac{C}{N},$$

$$|\text{Var}(X_i^N) - \frac{b_i^2 t}{\tau_i^2}| \leq \frac{C}{N},$$

$$|\text{Cov}(X_i^N, V_i^N, i) - \frac{b_i^2}{2 \tau_i^2}| \leq \frac{C}{N},$$

Now, we choose a constant $\epsilon > 0$ depending on $\alpha$ which is sufficiently small for all needs in the remaining of the proof. Let $k = \frac{1}{2} - \epsilon$, $\beta = \frac{1}{2} - 2\epsilon$ and $m = \frac{1}{2} + \epsilon$. So we can choose a constant $\lambda$ such that

$$2(k - 1) > \lambda > \frac{2}{9}(m - 2\alpha - 4).$$

Let $h_N = \frac{(t_1 - t_0)}{[N^\beta]^{1/2}}$. Let $t_i^N = ih_N - h_N$ and we subdivide $(t_0, t_1]$ into $S_1^N, S_2^N, ..., S_{[N^\beta]}^N$ where $S_i^N = (t_i^N, t_{i+1}^N]$. Let $\tau_i^N$ be the event that $|X_1^N(t) - X_2^N(t)| \leq r_N$ for some $t \in S_i^N$ but $|X_1^N(t_i^N) - X_2^N(t_i^N)| \leq \frac{C}{N}$. This is because after the particles collide, they are likely to continue their free motions with almost constant velocities for a small amount of time and this time turns out to be enough for them to get far away from each other so that they are unlikely to collide again. Now, we will start our proof.
$X_2^N(t_i^N) > r_N$ and let $B_i^N$ be the following event

\[
B_i^N = \{ |V_1^N(t_i^N) - V_2^N(t_i^N)| > N^k \} \cap \{ \max \{ |V_1^N(t_i^N)|, |V_2^N(t_i^N)| \} < N^m \} \\
\cap \{ |X_1^N(t_i^N) - X_2^N(t_i^N)| > r_N \} \\
\cap \{ \exists 0 \leq t \leq h_N : |X_1^N(t_i^N) - X_2^N(t_i^N) + t(V_1^N(t_i^N) - V_2^N(t_i^N))| \leq r_N \}.
\]

We want to use $B_i^N$ to approximate $A_i^N$ and estimate the probability of $B_i^N$ happening. Informally, for technical reasons, as the typical speeds of the particles are of order $\sqrt{N}$, we want to ignore the probability that either $\{ |V_1^N(t_i^N) - V_2^N(t_i^N)| > N^k \}$ or max\{ $|V_1^N(t_i^N)|, |V_2^N(t_i^N)|$ \} $< N^m$ happens.

Moreover, $S_i^N$ is a small time interval during which the velocities of the particles are unlikely to change much, and thus we want to approximate $A_i^N$ by

\[
\{ |X_1^N(t_i^N) - X_2^N(t_i^N)| > r_N \} \cap \{ \exists 0 \leq t \leq h_N : |X_1^N(t_i^N) - X_2^N(t_i^N) + t(V_1^N(t_i^N) - V_2^N(t_i^N))| \leq r_N \}
\]

We will start by estimating the probability that $B_i^N$ happens. For $v \in \mathbb{R}^d$, let

\[
D^N(v) = \{ x \in \mathbb{R}^d : |x| \geq r_N \} \cap \{ \exists 0 \leq t \leq h_N : |x + tv| \leq r_N \}
\]

Note that

\[
Vol(D^N(v)) = |v|(r_N)^{d-1}Vol(S_{d-1})h_N,
\]

where $S_{d-1}$ is the $d - 1$ dimensional sphere with radius 1. Also we have

\[
\sup_{|v| < N^m} (\sup_{x \in D^N_v} |x|) \leq CN^{m-\beta}.
\]

Now let $\tilde{V}_i^N = N^{-\frac{1}{2}} V_i^N$, and let $p_i^N$ be the transition density of $(\tilde{V}_i^N, X_i^N)$. So we have $(\tilde{V}_i^N, X_i^N)$ is bivariate normal distributed with

\[
|\text{Var}(\tilde{V}_i^N) - \frac{b_i^2}{2t_i}| \leq \frac{C}{N}, \quad |\text{Var}(X_i^N) - \frac{b_i^2t}{\tau_i^2}| \leq \frac{C}{N}, \quad |\text{Cov}(X_i^N, \tilde{V}_i^N)| \leq \frac{C}{\sqrt{N}}.
\]
Let $H^N = \{ v, u \in \mathbb{R}^d : N^{k-\frac{1}{2}} < |v-u|; |v|, |u| < N^{m-\frac{1}{2}} \}$, we have the following

$$
P(B^N) = \int_{H^N} \int_{\mathbb{R}^d} \int_{y-z \in D^N(N^{\frac{k}{2}}(u-v))} p_1^N(0,0,x_1; t_1^N, u, y)p_2^N(0,0,x_2; t_1^N, v, z) dy dz dv du. \quad (13)$$

Let $f_i$ denote the probability density function of normal independent random variables in $\mathbb{R}^d$ with mean zero and variance $\frac{b^2_i}{2\tau_i}$ respectively and let $f$ be that of their difference. Then we have

$$(1 + C^N)q_2(0, x_2; t_1^N(1 + C^N), z)f_2\left(\frac{v}{1 + C^N}\right)$$
$$\geq p_2^N(0,0,x_2; t_1^N, v, z)$$
$$\geq (1 - C^N)q_2(0, x_2; t_1^N(1 - C^N), z)f_2\left(\frac{v}{1 - C^N}\right).$$

Now, assume without loss of generality that $x_1 = 0$, then for $y-z \in D^N(N^{\frac{k}{2}}(u-v))$ we have

$$(1 + C^N)q_1(0,0; t_1^N(1 + C^N), z)(\frac{|z| - C^N}{|z|})f_1\left(\frac{u}{1 + C^N}\right)$$
$$\geq p_1^N(0,0,0; t_1^N, u, y)$$
$$\geq (1 - C^N)q_1(0,0; t_1^N(1 - C^N), z)(\frac{|z| + C^N}{|z|})f_1\left(\frac{u}{1 - C^N}\right).$$

Combining these inequalities with (12) and (13), we have

$$(r_N)^{-d}N^{-\frac{3}{2}} + \beta P(B^N) \leq Vol(S_{d-1})(t_1 - t_0) \int_{H^N} |u-v|f_1\left(\frac{u}{1 + C^N}\right)f_2\left(\frac{v}{1 + C^N}\right) dv du$$
$$\cdot \int_{\mathbb{R}^d} (1 + C^N)q_2(0, x_2; t_1^N(1 + C^N), z)q_1(0,0; t_1^N(1 - C^N), z)(\frac{|z| - C^N}{|z|}) dz. \quad (14)$$

Now, note that for $N$ sufficiently large we have

$$1_{(u,v) \in H^N}|u-v|f_1\left(\frac{u}{1 + C^N}\right)f_2\left(\frac{v}{1 + C^N}\right) \leq |u-v|f_1\left(\frac{u}{2}\right)f_2\left(\frac{v}{2}\right),$$

which is integrable over $\mathbb{R}^d \times \mathbb{R}^d$ and also

$$1_{(u,v) \in H^N}|u-v|f_1\left(\frac{u}{1 + C^N}\right)f_2\left(\frac{v}{1 + C^N}\right) \to |u-v|f_1(u)f_2(v).$$
pointwise. Thus, by dominated convergence theorem we have

\begin{equation}
\int_{H^N} |u - v|f_1(\frac{u}{1 + C^N})f_2(\frac{v}{1 + C^N})dvdu
\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u - v|f_1(u)f_2(v)dvdu + C^N
\leq \int_{\mathbb{R}^d} |v|f(v)dvdu + C^N.
\tag{15}
\end{equation}

On the other hand, for \( t_0 \leq t \leq t_1 \), we have

\[ (1 + C^N)q_2(0, x_2; t(1 + C^N), z)q_1(0, 0; t(1 + C^N), z \frac{|z| - C^N}{|z|}) \to q_2(0, x_2; t, z)q_1(0, 0; t, z), \]

uniformly over \( t \) and \( z \). Moreover, there exists a constant \( c \) such that whenever \( |z| > c \), we have

\[ (1 + C^N)q_2(0, x_2; t(1 + C^N), z)q_1(0, 0; t(1 + C^N), z \frac{|z| - C^N}{|z|}) - q_2(0, x_2; t, z)q_1(0, 0; t, z) \]
\[ \leq (1 + C^N)q_2(0, x_2; t_1(1 + C^N), z)q_1(0, 0; t_1(1 + C^N), z \frac{|z| - C^N}{|z|}) - q_2(0, x_2; t_1, z)q_1(0, 0; t_1, z). \]

Again by dominated convergence theorem, we have

\[ (1 + C^N)q_2(0, x_2; t_1(1 + C^N), z)q_1(0, 0; t_1(1 + C^N), z \frac{|z| - C^N}{|z|}) - q_2(0, x_2; t_1, z)q_1(0, 0; t_1, z) \to 0, \]

and

\begin{equation}
\int_{|z| > c} (1 + C^N)q_2(0, x_2; t_1^N(1 + C^N), z)q_1(0, 0; t_1^N(1 + C^N), z \frac{|z| - C^N}{|z|})dz
\leq q_2(0, x_2; t_1^N, z)q_1(0, 0; t_1^N, z)dz + C^N.
\tag{16}
\end{equation}

By uniform convergence, we have

\begin{equation}
\int_{|z| \leq c} (1 + C^N)q_2(0, x_2; t_1^N(1 + C^N), z)q_1(0, 0; t_1^N(1 + C^N), z \frac{|z| - C^N}{|z|})dz
\leq q_2(0, x_2; t_1^N, z)q_1(0, 0; t_1^N, z)dz + C^N.
\tag{17}
\end{equation}
So, with (14), (15), (16) and (17) we can deduce.

\[(r_N)^{-d} N^{-\frac{1}{2} + \beta} \mathbb{P}(B_i^N) \leq \text{Vol}(S_{d-1})(t_1 - t_0)(\int_{\mathbb{R}^d} |v| f(v) dv du + C^N) (\int_{\mathbb{R}^d} q_2(0, x_2; t^N_i, z) q_1(0, 0; t^N_i, z) dz + C^N) \]

\[\leq c_d(t_1 - t_0) \sqrt{\frac{b_1^2}{\tau_1} + \frac{b_2^2}{\tau_2}} \int_{\mathbb{R}^d} q_2(0, x_2; t^N_i, z) q_1(0, 0; t^N_i, z) dz + C^N.\]

Now, for general \(x_1\), we would have

\[(r_N)^{-d} N^{-\frac{1}{2} + \beta} \mathbb{P}(B_i^N) \leq c_d(t_1 - t_0) \sqrt{\frac{b_1^2}{\tau_1} + \frac{b_2^2}{\tau_2}} \int_{\mathbb{R}^d} q_2(0, x_2; t^N_i, z) q_1(0, 0; t^N_i, z) dz + C^N. \tag{18}\]

Similarly we can show that

\[(r_N)^{-d} N^{-\frac{1}{2} + \beta} \mathbb{P}(B_i^N) \geq c_d(t_1 - t_0) \sqrt{\frac{b_1^2}{\tau_1} + \frac{b_2^2}{\tau_2}} \int_{\mathbb{R}^d} q_2(0, x_2; t^N_i, z) q_1(0, 0; t^N_i, z) dz - C^N.\]

Next, we want to show that the probability one of \(\tau^N_i\) and \(B^N_i\) happens but the other doesn’t happen is bounded by \(C^N(r_N)^d N^{-\beta+\frac{1}{2}}\). Note that for \(0 < s < h_N\) and \(j = 1, 2\) we have,

\[V_j^N(t^N_i + s) = V_j^N(t^N_i) e^{-N\tau_j s} + \int_{t^N_i}^{t^N_i + s} e^{-N\tau_j (t^N_i + s - s')} N B_j dB_j^{s'} \]

Let

\[U_s = \int_{t^N_i}^{t^N_i + s} e^{-N\tau_j (t^N_i + s - s')} N B_j dB_j^{s'} = e^{-N\tau_j s} \int_0^s e^{N\tau_j s'} N B_j dB_j^{t^N_i + s'} \]

and \(M(s) = \int_0^s [e^{N\tau_j s'} N B_j]^2 ds'\), we have, by Dubins-Schwarz,

\[U_s = e^{-N\tau_j s} W_{M(s)},\]

for some Brownian motion \(W\). By standard Doob’s martingale inequality applied on the exponential of Brownian motion we obtain

\[\mathbb{P}(\sup_{0 < s < h_N} |W_{M(s)}| \geq N h_N^k) \leq 2 e^{-N^2 h_N^{2k}}.\]
Note that $M(h_N) \leq C^N N^{2-\beta} r_N$. Thus $2e^{-\frac{N^2 h_N}{2M(h_N)}} \leq C^N (r_N)^d N^{-\beta + \frac{1}{2}}$ and

$$\mathbb{P}(\sup_{0 < s < h_N} |V_j^N(t^N_i + s) - V_j^N(t^N_i)| \geq Nh_N^k) \leq C^N (r_N)^d N^{-\beta + \frac{1}{2}}. \tag{19}$$

As a result

$$\mathbb{P}(\sup_{0 < s < h_N} |X_j^N(t^N_i + s) - X_j^N(t^N_i)| - V_j^N(t^N_i) \int_0^s e^{-N\tau_j s'} ds' \geq Nh_N^k) \leq C^N (r_N)^d N^{-\beta + \frac{1}{2}}.$$

So, we can further condition on the event

$$\sup_{0 < s < h_N} |X_j^N(t^N_i + s) - X_j^N(t^N_i)| - V_j^N(t^N_i) \int_0^s e^{-N\tau_j s'} ds' < Nh_N^k$$

for $j = 1, 2$. Note that $\frac{Nh_N^{1+k}}{N^\alpha} \rightarrow 0$ and $e^{-N\tau_j h_N} \rightarrow 1$ as $N \rightarrow \infty$. We can now consider the following events

$$C_i^N = \{|X_1^N(t^N_i) - X_2^N(t^N_i)| > r_N\}$$

$$\cap \{\exists 0 \leq t \leq h_N : |X_1^N(t^N_i) - X_2^N(t^N_i) + V_1^N(t^N_i) \int_0^t e^{-N\tau_1 s'} ds' - V_2^N(t^N_i) \int_0^t e^{-N\tau_2 s'} ds'| \leq r_N - 2Nh_N^{1+k}\}$$

and

$$G_i^N = \{|X_1^N(t^N_i) - X_2^N(t^N_i)| > r_N\}$$

$$\cap \{\exists 0 \leq t \leq h_N : |X_1^N(t^N_i) - X_2^N(t^N_i) + V_1^N(t^N_i) \int_0^t e^{-N\tau_1 s'} ds' - V_2^N(t^N_i) \int_0^t e^{-N\tau_2 s'} ds'| \leq r_N + 2Nh_N^{1+k}\}.$$

Then under the conditions we had, we obtain $C_i^N \subseteq A_i^N \subseteq G_i^N$. Moreover, using the same approximation method we used before we have that the probability that $B_i^N$ happens but $C_i^N$ doesn’t is bounded above by $C^N (r_N)^d N^{-\beta + \frac{1}{2}}$ and also the probability that $G_i^N$ happens but $B_i^N$ doesn’t is bounded by $C^N (r_N)^d N^{-\beta + \frac{1}{2}}$. Thus we have the probability one of $\tau_i^N$ and $B_i^N$ happens but the other doesn’t happen is $C^N (r_N)^d N^{-\beta + \frac{1}{2}}$. 

\[34\]
Now, we let $T' = \min\{ t_i^N : B_i^N \text{happens}\}$, then we claim that the probability that $B_i^N$ happens but $T' \neq t_i^N$ is bounded by $C^N(r_N)^dN^{-\beta + \frac{1}{2}}$. Let $P_{ij}^N$ be the probability that $B_i^N$ and $B_j^N$ both happen and we want to show that
\[
\sum_{j < i} P_{ij}^N \leq C^N(r_N)^dN^{-\beta + \frac{1}{2}}.
\]

For $j = i - 1$, we can use similar argument as above to say that the probability that $B_j^N$ happens and $(V_1^N(t_j^N) - V_2^N(t_j^N)) \cdot (X_1^N(t_j^N) - X_2^N(t_j^N)) \leq 0$ is bounded above by $C^N(r_N)^dN^{-\beta + \frac{1}{2}}$. So, the probability that both $B_j^N$ and $B_i^N$ happens is bounded by $C^N(r_N)^dN^{-\beta + \frac{1}{2}}$. Now, we will show that for all $j < i - 1$ and $t_i^N - t_j^N \leq N^{\lambda}$, $P_{ij}^N \leq C^N(r_N)^dN^{-2\beta + \frac{1}{2}}$. We condition on $B_j^N$ happens and $F_i$. It suffices to show that the probability $B_i^N$ happens is bounded by $C^Nh_N$. Let $s = t_i^N - t_j^N$, $X^N = X_1^N - X_2^N$ and $V^N = V_1^N - V_2^N$. Then $X^N(t_i^N)$ is normal distributed with mean
\[
V_1^N(t_j^N) \int_0^s e^{-N\tau_1 s'} ds' - V_2^N(t_j^N) \int_0^s e^{-N\tau_2 s'} ds'
\]
and variance between $\frac{1}{e}N^2s^3$ and $CN^2s^3$ and $V^N(t_i^N)$ is normal distributed with mean
\[
V_1^N(t_j^N)e^{-N\tau_1 s} - V_2^N(t_j^N)e^{-N\tau_2 s}
\]
and variance between $\frac{1}{e}N^2s$ and $CN^2s$. Also, their correlation is between $\frac{1}{e}$ and $1 - \frac{1}{e}$. Note that in order to make $B_i^N$ happen, we either need
\[
|X^N(t_i^N) - V_1^N(t_j^N) \int_0^s e^{-N\tau_1 s'} ds' + V_2^N(t_j^N) \int_0^s e^{-N\tau_2 s'} ds'| > \frac{1}{3}sN^k,
\]
(20)
or
\[
|V^N(t_i^N) - V_1^N(t_j^N)e^{-N\tau_1 s} + V_2^N(t_j^N)e^{-N\tau_2 s}| > \frac{1}{2}N^k.
\]
(21)
First, we condition on the velocity $V^N(t_i^N)$ such that (21) is false. Then the conditional probability density function of $X^N(t_i^N)$ is bounded above by $N^{-m}$ whenever (20) is satisfied. Then by same calculation as earlier we obtain the probability that $B_i^N$ happens is bounded above by $C^Nh_N$.

Now, we condition on that (21) is true. Using same method as deriving (19), this happens with probability at most $C^NN^{-m}s^{3d}$ and the conditional probability density function of $X^N(t_i^N)$ is at

35
most \( \frac{C}{(N^2 s^3)^2} \). So, by same calculation as earlier again we can deduce that \( B^N_i \) happens is bounded above by \( C^N h_N \). This concludes that \( P^N_{ij} \leq C^N (r_N)^{d+1} N^{-2\beta+\frac{1}{2}} \).

Now, suppose \( N^{-1} > t_i^N - t_j^N \). We again condition on \( B^N_j \) happens and \( F_{ij} \). Then, we know that conditional on any \( V^N(t_i^N), X^N(t_i^N) \) will be normal distributed with variance at least \( \frac{N^{2+3\lambda}}{C} \).

Therefore, we have

\[
P^N_{ij} \leq C(r_N)^d N^{-\beta+\frac{1}{2}} N^{m-\beta}(r_N)^d N^{-\frac{d}{2}(2+3\lambda)}.
\]

Also

\[
m - \alpha(d - 1) - \frac{d}{2}(2 + 3\lambda) \leq m - 2\alpha - 3 - \frac{9\lambda}{2} < 1,
\]

we have \( P^N_{ij} \leq C^N (r_N)^{d+1} N^{-2\beta+\frac{1}{2}} \). Finally, for \( t_i^N - t_j^N > N^{-1} \) and condition on \( B^N_j \) happens and \( F_{ij} \), we know that conditional on any \( V^N(t_i^N), X^N(t_i^N) \) will be normal distributed with variance at least \( \frac{(t_i^N - t_j^N)^d}{C} \). Therefore, we have

\[
P^N_{ij} \leq C(r_N)^d N^{-\beta+\frac{1}{2}} N^{m-\beta}(r_N)^d (t_i^N - t_j^N)^{-\frac{d}{2}}
\]

and thus

\[
\sum_{j:t_i^N - t_j^N > N^{-1}} P^N_{ij} \leq C(r_N)^d N^{-\beta+\frac{1}{2}} N^{m-\beta}(r_N)^d \int_{N^{-1}}^{\infty} s^{-\frac{d}{2}} ds
\]

\[
\leq C(r_N)^d N^{-\beta+\frac{1}{2}} N^{m}(r_N)^{d-1} N^{\frac{d}{2}-1}
\]

\[
\leq C^N (r_N)^d N^{-\beta+\frac{1}{2}},
\]

where for the last inequality we use the fact that \( r_N < N^{-\alpha} \) and \( 2m - 1 - \frac{\alpha}{2} < 0 \). Now, if we sum up over \( j \) we can conclude that the probability that \( B^N_i \) happens but \( T' \neq t_i^N \) is bounded by \( C^N (r_N)^d N^{-\beta+\frac{1}{2}} \).

So far, we analysed the collision events during the time interval \([t_0, t_1]\), and the only place we used the lower bound \( t_0 \) is to make sure that for \( t > t_0 \) \((V_i^N, X_i^N)\) are bivariate normal distributed...
with

\[ |\text{Var}(V_i^N) - \frac{N_b^2}{2\tau_i^2}| \leq \frac{C}{N}, \]
\[ |\text{Var}(X_i^N) - \frac{b^2t_i}{\tau_i^2}| \leq \frac{C}{N}, \]
\[ |\text{Cov}(X_i^N, V_i^N, i) - \frac{b^2}{2\tau_i^2}| \leq \frac{C}{N}. \]

So, our analysis would still work if we replace \( t_0 \) by \( N-(1-\epsilon) \). By same method as earlier we could show that the probability that collision happens before \( N-(1-\epsilon) \) is at most \( C_N(r_N)^d-N^{-\beta+\frac{1}{2}} \). Also we could show that the probability \( T < N-(1-\epsilon) \) and \( B_i^N \) happens is bounded by \( C_N(r_N)^d-N^{-\beta+\frac{1}{2}} \).

Thus, we can conclude that the probability \( T < t_0 \) and \( B_i^N \) happens is bounded by \( C_N(r_N)^d-N^{-\beta+\frac{1}{2}} \).

Therefore, we obtain

\[ \mathbb{P}(T \in [t_0, t_1] \text{ or } T' \in [t_0, t_1] \text{ and } |T - T'| > h_N) \leq C_N(r_N)^dN^{-\beta+\frac{1}{2}}. \]

By similar argument as before, we have

\[ \mathbb{P}(T \in [t_0, t_1] \text{ or } T' \in [t_0, t_1] \text{ and } |X(T) - X_1^N(T')| > 3N^\alpha h_N) \leq C_N(r_N)^dN^{-\beta+\frac{1}{2}}. \]

So, because \( g \) is bounded and uniformly continuous, we would have

\[ |(r_N)^{-d}N^{-\frac{1}{2}}(\mathbb{E}(g(T, X(T)) - g(T', X_1^N(T'))))| \to 0. \]

We also have

\[ |(r_N)^{-d}N^{-\frac{1}{2}}\mathbb{E}[g(T', X_1^N(T')) - \sum_i 1_{B_i^N}g(t_i^N, X_1^N(t_i^N))]| \to 0. \]

By similar analyse as in deriving (18), we would get

\[ |\mathbb{E}[1_{B_i^N}g(t_i^N, X_1^N(t_i^N))]
- c_d(r_N)^dN^{-\beta+\frac{1}{2}} \sqrt{\frac{b_1^2}{\tau_1} + \frac{b_2^2}{\tau_2}} r^{d-1}(t_1 - t_0) \int_{\mathbb{R}^d} q_1(0, x_1; t_i^N, z)q_2(0, x_1; t_i^N, z)g(t_i^N, z)dz| \]
\[ \leq C_N(r_N)^dN^{-\beta+\frac{1}{2}}. \]
Now, by continuity of $\int_{\mathbb{R}^d} q_1(0, x_1; t_1^N, z)q_2(0, x_1; t_1^N, z)g(t_1^N, z)dz$, we get

$$|(r_N)^{1-d} N^{-\frac{d}{2}} \mathbb{E}(g(T, X(T))) - c_d \sqrt{\frac{b_1^2}{\tau_1} + \frac{b_2^2}{\tau_2}} \int_{t_0}^{t_1} \int_{\mathbb{R}^d} q_1(0, x_1; t, z)q_2(0, x_2; t, z)g(t, z)dtdz| \to 0,$$

as desired.

Next, we will look at what happens if the radius of the particles goes to zero slowly. More precisely, we will prove Theorem 2.4. We start our proof with the following lemma.

**Lemma 4.1.** For all $m > \frac{1}{2}$, $k > 0$ and $t_1 > 0$ we have that there exists a constant $C$ such that,

$$\mathbb{P}(\sup_{t < t_1} |V_{t_i}^N(t)| > N^m) < N^{-k},$$

for all $N$.

**Proof.** Again, we let $C$ be a constant whose value can change from line to line. We fix $\beta > \frac{1}{2}$ and let $h_N = \frac{t_1}{[N^\beta]}$. Let $t_i^N = ih_N - h_N$ and we subdivide $(0, t_1]$ into $S_1^N, S_2^N, ..., S_{[N^\beta]}^N$, where $S_i^N = (t_i^N, t_{i+1}^N]$. Choose any $m > m' > \frac{1}{2}$ and $k' > k + \beta > 0$. Because $V_{t_i}^N(t)$ is Gaussian distributed with

$$\text{Var}(V_{t_i}^N(t)) \leq \frac{N b_i^2}{2 \tau_1},$$

for all $t$, we have

$$\mathbb{P}(|V_{t_i}^N(t_j^N)| > N^{m'}) < C N^{-k'},$$

for any $0 < j \leq N_{[N^\beta]}$. Thus, the probability that there is any $t_j^N < t_1$ with $|V_{t_i}^N(t_j^N)| > N^{m'}$ is bounded above by $C N^{-k' + \beta}$ for sufficiently large $N$. Also, using same method as in the derivation of (19) we can show that condition on the event $|V_{t_i}^N(t_j^N)| > N^{m'}$,

$$\mathbb{P}(\sup_{0 < s < h_N} |V_{t_i}^N(t_j^N + s)| \geq N^m) \leq C N^{-k'}.$$

Summing up over $0 < j \leq N_{[N^\beta]}$ we can conclude that

$$\mathbb{P}(\sup_{t < t_1} |V_{t_i}^N(t)| > N^m) < N^{-k},$$

as desired. \qed
Now, we can start proving Theorem 2.4.

**Proof.** Again let $C$ be a constant whose value can change from line to line and let $W_i^N = X_i^N + \frac{1}{N^\alpha} V_i^N$, then $W_i^N$ is a $d$-dimensional Brownian motion with diffusivity $(\frac{b_i}{\tau_i})^2$. Let $m < 1 - \alpha$ and define the stopping time $T'$ by

$$T' = \inf\{t \geq 0 : |V_i^N(t)| > N^m\} \land T.$$

Then, by the above lemma we have $\mathbb{P}(T \neq T' \text{ and } T' < t_1) < CN^{-2d}$. We can now repeat the argument used in proving (7) to show that

$$ \int_{t_0}^{t_1} \int_{\mathbb{R}^d} q_1(0, x_1; t, z)q_2(0, x_2; t, z)g(t, z)dtdz = \int_{t_0}^{t_1} \int_{\mathbb{R}^d} q_1(T', W_1^N(T'); t, z)q_2(T', W_2^N(T'); t, z)g(t, z)dtdz.$$

Let $\epsilon \geq 2r_N$. Note that if $T < T'$, then

$$r_N - CN^{m-1} \leq ||W_1^N(T') - W_2^N(T')|| \leq r_N + CN^{m-1}.$$

By same method as in deriving (11) we would have that on the event $\{T = T'\}$

$$||W_1^N(T') - W_2^N(T')||^{2-d}g(T, X_T) - c_d[(\frac{b_1}{\tau_1})^2 + (\frac{b_2}{\tau_2})^2] \int_0^\infty \int_{\mathbb{R}^d} q_1(T', W_1^N(T'); s, z)q_2(T', W_2^N(T'); s, z)g(s, z)dzds| \leq C[\epsilon^{2-d}\|g\| + (r_N)^{2-d}(\phi_{\delta}(\epsilon) + \epsilon^2)].$$

Thus

$$|(r_N)^{2-d}g(T, X_T) - c_d[(\frac{b_1}{\tau_1})^2 + (\frac{b_2}{\tau_2})^2] \int_0^\infty \int_{\mathbb{R}^d} q_1(T', W_1^N(T'); s, z)q_2(T', W_2^N(T'); s, z)g(s, z)dzds| \leq C[(\epsilon^{2-d} + (r_N)^{1-d}N^{m-1})\|g\| + (r_N)^{2-d}(\phi_{\delta}(\epsilon) + \epsilon^2)].$$

39
When $T \neq T'$ we simply have
\[
|(r_N)^{2-d} g(T, X_T) - cd[(\frac{b_1}{T_1})^2 + (\frac{b_2}{T_2})^2] \int_0^\infty \int_{\mathbb{R}^d} q_1(T', W_1^N(T'); s, z)q_2(T', W_2^N(T'); s, z)g(s, z)dzds| \leq C(r_N)^{2-d}.
\]

Also using same method as at the end of the proof of Theorem 2.1 we could get
\[
P(T' < t_1) \leq C(r_N)^{d-2}.
\]
Thus,
\[
\left|(r_N)^{2-d}[\mathbb{E}[g(T, X(T))]) - cd[(\frac{b_1}{T_1})^2 + (\frac{b_2}{T_2})^2] \int_0^{t_1} \int_{\mathbb{R}^d} q_1(0, x_1; t, z)q_2(0, x_2; t, z)g(t, z)dtdz| \right.
\]
\[
\leq C[\epsilon^{2-d} + (r_N)^{1-d}N^{m-1}]\|g\| + (r_N)^{2-d}(\phi_g(\epsilon) + \epsilon^2)](r_N)^{d-2} + C P(T \neq T' \text{ and } T' < t_1)(r_N)^{2-d}.
\]

By letting $\epsilon \to 0$ as $N \to \infty$ we have the that the right hand side of the above inequality converges to zero as desired.

5 Uniqueness proof

As explained in the introduction, we are interested in the well-posedness of Smoluchowski coagulation equations. We first explain a heuristic argument for Theorem 2.5. Suppose we have two solutions $\mu_1$ and $\mu_2$ and let $\mu = \mu_2 - \mu_1$. By Hahn decomposition theorem, for each $t$ and $x$, we can decompose $\mathbb{R}^d$ into a positive set $P(t, x)$ and a negative set $N(t, x)$ such that for all $A \subseteq P(t, x)$, $\mu(A) \geq 0$ and for all $A \subseteq N(t, x)$, $\mu(A) \leq 0$, and this decomposition is essentially unique. Define
\[
|\mu_t|(x, A) = \int_A \mu_t(x, dy)1_{y \in P} - \int_A \mu_t(x, dy)1_{y \in N}
\]
and consider $\|\langle w, |\mu_t| \rangle\|_1$. Suppose at time $s$ and position $x$, there are more particles of mass $y$ in $\mu_2$ than in $\mu_1$. We look at what further difference would this cause. Since those extra particles can coagulate with particles of mass $y'$, this will decrease $|\mu|$ at position $x$ and mass $y$, and in the worst case increase $|\mu|$ for mass $y'$ and $y + y'$. So, the total rate of increase of $\langle w, |\mu_t| \rangle$ at time $s$
due to those extra particles will be at most
\[
\mu(x, dy) \int_0^\infty K(y, y')(\mu_1^s(x, dy') + \mu_2^s(x, dy')) \left[w(y + y') + w(y') - w(y)\right]
\leq \mu(x, dy) \int_0^\infty K(y, y')(\mu_1^s(x, dy') + \mu_2^s(x, dy'))(2w(y'))
\leq 2\mu(x, dy)w(y)\|\langle w^2, \mu_1^s + \mu_2^s \rangle\|_\infty.
\]

Further the Brownian motion of the particles won’t increase \(\|\langle w, |\mu_t| \rangle\|_1\). So, we can integrate the above inequality over \(s, x\) and \(y\) and obtain
\[
\|\langle w, |\mu_t| \rangle\|_1 \leq 2\sup_{s \leq t} (\|\langle w^2, \mu_1^s + \mu_2^s \rangle\|_\infty) \int_0^t \|\langle w, |\mu_s| \rangle\|_1 ds.
\]

Then we can use Gronwall to show that \(\mu_1^t = \mu_2^t\) provided \(\sup_{s \leq t} (\|\langle w^2, \mu_1^s + \mu_2^s \rangle\|_\infty) < \infty\). So, this argument indicates that \(\|\langle w, |\mu_t| \rangle\|_1\) is the right norm to look at.

Now, we will prove this result rigorously. We assume \(\mu_1^t\) and \(\mu_2^t\) are solutions and for \(i = 1, 2, \sup_{t \leq T} \|\langle w^2, \mu_i^t \rangle\|_\infty < \infty\). Now we formulate a differential equation describing the behaviour of individual particles in the solution. For kernels \(\nu\) and \(\mu\) and any measurable set \(A\), let
\[
K^{\nu_+}(\mu)(x, A) = \int_0^\infty \int_0^\infty 1_{y + y' \in A} y' y K(y, y') \nu(x, dy) \mu(x, dy'),
\]
\[
K^{\nu_-}(\mu)(x, A) = \int_A \mu(x, dy') \int_0^\infty K(y, y') \nu(x, dy),
\]
and \(K^{\nu}(\mu) = K^{\nu_+}(\mu) - K^{\nu_-}(\mu)\). By similar analysis as earlier, \(K^{\nu \pm}(\mu)\) are also kernels. Denote \(K^{i \pm} = K^{\mu_1 \pm}\) and \(K^i = K^{\mu_i}\). Consider the linear evolution equation
\[
q^i_t + \int_0^t P_{t-s} K^{i_+}_s (q^i_s) ds = P_t q_0 + \int_0^t P_{t-s} K^{i_+}_s (q^i_s) ds.
\] (22)

Let \(\mathcal{M}'\) be the set of \(q\) which can be written as \(q^+ - q^-\) with \(q^+, q^- \in \mathcal{M}\). We say \(q \in \mathcal{M}'\) is a solution to (22) up to time \(T\) if \(q\) satisfies (22) for \(t \leq T\) and
\[
\sup_{t \leq T} \|\langle y, |q_t| \rangle\|_1 < \infty.
\]
Also, let $S$ be the set of $\nu : \mathbb{R}^d \times \mathcal{B}(0, \infty) \to [-\infty, \infty]$ such that $\|\langle y, |\nu| \rangle\|_1 < \infty$.

**Proposition 5.1.** If we start at $q_0 = \mu_0$ then $q_t^i = \mu_t^i$ is a solution of (22).

**Proof.** Note that

$$K_s^{i-}(\mu_s^i) = K_s^{-}(\mu_s^i),$$

and

$$K_s^{i+}(\mu_s^i)(x, A) = \int_0^\infty \int_0^\infty 1_{y + y' \in A} \frac{y'}{y + y'} K(y, y') \mu_t^i(x, dy) \mu_t^i(x, dy')
\quad = \frac{1}{2} \int_0^\infty \int_0^\infty 1_{y + y' \in A} \frac{y'}{y + y'} K(y, y') \mu_t^i(x, dy) \mu_t^i(x, dy')
\quad + \int_0^\infty \int_0^\infty 1_{y + y' \in A} \frac{y}{y + y'} K(y, y') \mu_t^i(x, dy) \mu_t^i(x, dy')
\quad = K_s^{i+}(\mu_s^i)(x, A).$$

Plugging these into (3), we have

$$\mu_t^i + \int_0^t P_{t-s} K_s^{i-}(\mu_s^i) ds = P_t \mu_0 + \int_0^t P_{t-s} K_s^{i+}(\mu_s^i) ds,$$

and thus $\mu_t^i$ is a solution of (22). \qed

We now look at the heuristic meaning of the above equation. Suppose a particle with initial dis-
tribution $yq_0$ and makes Brownian motion and coagulates with other particles distributed according
to $\mu^i$ then at time $t$ its distribution is $yq_t^i$.

**Proposition 5.2.** Assume $\|\langle y, q_0 \rangle\|_1 < \infty$ then equation (22) has at a unique solution in $\mathcal{M}'$.

Moreover, if $q_0$ is non-negative, then $q_t$ is also non-negative.

**Proof.** Let $c_s^i(x, y) = \int_0^\infty K(y, y') \mu_s^i(x, dy')$ and (22) becomes

$$q_t^i + \int_0^t P_{t-s} c_s^i(q_s^i) ds = P_t q_0 + \int_0^t P_{t-s} K_s^{i+}(q_s^i) ds.$$

Suppose $q_0 \geq 0$ and consider first the equation

$$\lambda_t^i + \int_0^t P_{t-s} c_s^i \lambda_s^i ds = P_t q_0.$$

(23)
Let $B^{x,x',a,t}$ be the conditional Brownian motion with diffusivity $a$ being at $x$ at time 0 and at $x'$ at $t$. Then by Feynman-Kac formula, we have

$$\lambda^i_t(x,dy) = \int_{\mathbb{R}^d} q_0(z,dy) \mathbb{E}\left[ \exp\left( -\int_0^t c^i(B^z_{s,x,a}(y),t) \, ds \right) \right] \nu^{z,x}(y) \, dz$$

is a solution of this equation and in particular, $\lambda^i_t$ is non-negative. Then, we want to show that this is the unique solution of (23). By linearity, it satisfies to show that zero solution is the unique solution of

$$\lambda^i_t + \int_0^t P_{t-s} c_s \lambda^i_s \, ds = 0.$$

For $z > 0$, we have

$$\|\langle 1_{y \leq z}, |\lambda^i_t| \rangle\|_1 \leq \|\langle 1_{y \leq z}, \int_0^t P_{t-s} c_s |\lambda^i_s| \, ds \rangle\|_1$$

$$\leq \|\langle 1, 1_{y \leq z}, \int_0^t c_s |\lambda^i_s| \, ds \rangle\|_1.$$

Recall that $\sup_{t \leq T} \|\langle w^2, \mu^i_t \rangle\|_\infty < \infty$, we have $c_s 1_{y \leq z}$ is bounded. Thus, we can apply Gronwall inequality to show $\|\langle 1_{y \leq z}, |\lambda^i_t| \rangle\|_1 = 0$ and as this works for all $z > 0$, we have $\lambda^i_t = 0$.

Now, we want to show that

$$q^i_t + \int_0^t P_{t-s} c_s q^i_s \, ds = P_t q_0 + \int_0^t P_{t-s} K^{i+}_s(q^i_s) \, ds.$$

has a unique solution. Again, we assume without loss of generality that $q_0 = 0$. Let $\lambda^i_t$ be the unique solution of (23). Note that, $1_{y < 2\delta} K^{i+}_s(q^i_s) = 0$, we have $1_{y < 2\delta} q^i_s = 1_{y < 2\delta} \lambda^i_s = 0$. Then, we have $1_{2\delta \leq y < 3\delta} K^{i+}_s(q^i_s) = 0$ and thus $1_{2\delta \leq y < 3\delta} q^i_s = 0$, and we can keep going. This proves the uniqueness of $q^i_t$.

Now, as we showed equation (23) has a unique solution, similarly, we can show

$$\lambda^i_t + \int_s^t P_{t-s} c_{t'} \lambda^i_{t'} \, dt' = P_{t-s} \lambda_s,$$

has a unique solution too. Thus for $t \geq s$, we define $f_{s,t} : S \to S$ such that if $(\lambda_{t'})_{s \leq t' \leq t}$ solves (24)
then $f_{s,t}(\lambda_s) = \lambda_t$. Now, we will show a version of variation of constant formula

$$q^i_t = \lambda^i_t + \int_0^t f_{s,t}(K^i_s + (q^i_s)) ds. \tag{25}$$

Suppose first $q^i$ is a solution of (25), we want to show that then $q^i$ is indeed a solution of (22). By subtracting (23) from (22), it suffices to show

$$q^i_t - \lambda^i_t + \int_0^t P_{t-s} K^i_s (q^i_s - \lambda^i_s) ds = \int_0^t P_{t-s} K^i_s (q^i_s) ds.$$

Plugging (25) into the left hand side of this equation, we have

$$\begin{align*}
q^i_t - \lambda^i_t + \int_0^t P_{t-s} K^i_s (q^i_s - \lambda^i_s) ds
&= \int_0^t f_{s,t}(K^i_s + (q^i_s)) ds + \int_0^t P_{t-s} K^i_s (\int_0^s f_{s',s}(K^i_{s'} + (q^i_{s'}))) ds' ds
&= \int_0^t f_{s,t}(K^i_s + (q^i_s)) ds + \int_0^t P_{t-s} K^i_s (f_{s,s'}(K^i_{s'} + (q^i_{s'}))) ds' ds
&= \int_0^t [f_{s,t}(K^i_s + (q^i_s)) + \int_s^t P_{t-s} K^i_s (f_{s,s'}(K^i_{s'} + (q^i_{s'}))) ds'] ds
&= \int_0^t P_{t-s} K^i_s (q^i_s) ds,
\end{align*}$$

where we used (24) for the last step. This shows that any solution of (25) is also a solution of (22).

Now, we construct a solution to (25). For natural number $n$, We note that $1_{n\delta \leq y < (n+1)\delta} q^i_s$ depends only on $1_{n\delta \leq y < (n+1)\delta} (q^i_s)$. So, we can inductively give a solution of (25) by letting

$$1_{y < 2\delta} q^i_s = 1_{y < 2\delta} \lambda^i_s$$

and for $n > 1$

$$1_{n\delta \leq y < (n+1)\delta} q^i_s = 1_{n\delta \leq y < (n+1)\delta} [\lambda^i_s + \int_0^t f_{s,t}(K^i_s + (1_{y < n\delta} q^i_s)) ds].$$

Moreover, if $q_0$ is non-negative, then $\lambda^i$ is non-negative too, and using the above construction, we can see by induction that $q_t$ is non-negative too.

Proposition 5.3. For a solution $q^i$ of (22), we have, for $s \leq t$,

$$\|\langle w, q^i_t \rangle \|_1 \leq \|\langle w, q^i_s \rangle \|_1$$

44
Proof. Assume first $q_0 \geq 0$, then for any $z > 0$, we have

$$\sup_{s \leq t} \| \langle 1_{y \leq z} w, K_s^{i+} (q_s) \rangle \|_1 \leq \sup_{s \leq t} [\| \langle 1_{y \leq z} w^2, q_s \rangle \|_1 \langle w, \mu_s \|_\infty \rangle ] < \infty.$$ 

So, we have

$$\| \langle 1_{y \leq z} w, q_t \rangle \|_1 = \| \langle 1_{y \leq z} w, q_0 \rangle \|_1 + \int_0^t \| \langle 1_{y \leq z} w, K_s^{i+} (q_s) - K_s^{i-} (q_s) \rangle \|_1 \rangle ds,$$

where

$$\langle 1_{y \leq z} w, K_s^{i+} (q_s) - K_s^{i-} (q_s) \rangle$$

$$\leq \int_0^z w(y) \int_0^y \frac{y'}{y} K(y', y - y') q_s(dy') \mu_s(dy) - w(y) q_s(dy) \int_0^\infty K(y, y') \mu_s(dy')$$

$$\leq \int_0^z \int_0^\infty K(y, y') \mu_s(dy') \langle 1_{y \leq z} w, q_s(dy) \rangle [w(y + y') \frac{y}{y + y'} - w(y)]$$

$$\leq 0.$$

Therefore,

$$\| \langle 1_{y \leq z} w, q_t \rangle \|_1 \leq \| \langle 1_{y \leq z} w, q_0 \rangle \|_1,$$

for all $z \geq 1$. Let $z \to \infty$, we conclude that

$$\| \langle w, q_t \rangle \|_1 \leq \| \langle w, q_0 \rangle \|_1$$

when $q_0 \geq 0$. By linearity, we can extend this to

$$\| \langle w, q_t \rangle \|_1 \leq \| \langle w, q_0 \rangle \|_1,$$

without the condition $q_0 \geq 0$. Similarly, we also have

$$\| \langle w, q_s \rangle \|_1 \leq \| \langle w, q_t \rangle \|_1,$$

whenever $s \leq t$. □
Now, for \( t \geq s \) let \( \Phi_{s,t}^i : S \to S \) be the map such that if \((q_i^t)_{s \leq t' \leq t}\) solves

\[
q_i^{t'} + \int_s^{t'} P_{t' - t''} K_{t''}^i (q_i^{t''}) dt'' = P_{t' - t} q_i^{s} + \int_s^{t'} P_{t' - t''} K_{t''}^i (q_i^{t''}) dt'',
\]

then \( \Phi_{s,t}^i(q_i^t) = q_i^t \). We want to verify that we can apply the variation of constant formula in the following way.

**Proposition 5.4.**

\[
\mu_1^2 - \mu_1^1 = \int_0^t \Phi_{s,t}^1[(K_s^2 - K_s^1)(\mu_s^2)] ds.
\]

**Proof.** We will start with showing that there exists \( \nu \in M' \) such that

\[
\nu - \mu_1^1 = \int_0^t \Phi_{s,t}^1[(K_s^2 - K_s^1)(\nu_s)] ds.
\]

For any \( z > 0 \), we have from earlier result that

\[
\|(1_{y \leq z}w, \Phi_{s,t}^1[(K_s^2 - K_s^1)(\nu_s)])\|_1 \leq \|(1_{y \leq z}w, (K_s^2 - K_s^1)(\nu_s))\|_1.
\]

For \( i = 1, 2 \), we have

\[
\|(1_{y \leq z}w, K_s^i(\nu_s))\|_1 \leq \int_0^z \int_0^\infty K(y, y') \mu_s^i(dy') \left( |\nu_s(dy)w(y) + |\nu_s(dy)w(y + y')| \right) \|_1
\]

\[
\leq \int_0^z \int_0^\infty w(y')w(y) \mu_s^i(dy') \left( |\nu_s(dy)w(y) + |\nu_s(dy)w(y + y')| \right) \|_1
\]

\[
\leq 2\|(w, \mu_s^1)\|_\infty w(2z)\|(1_{y \leq z}w, (\nu_s))\|_1.
\]

So, we conclude that

\[
\|(1_{y \leq z}w, \Phi_{s,t}^1[(K_s^2 - K_s^1)(\nu_s)])\|_1 \leq C\|(1_{y \leq z}w, (\nu_s))\|_1
\]

for some constant \( C \). As this works for all \( z > 0 \), we can use iteration scheme to show that such \( \nu \) exists. Then we can use similar analysis as in the proof of Proposition 5.2 to show \( \nu = \mu^2 \). \( \square \)

Now, we have enough tools to prove Theorem 2.5.
Proof. We have
\[\|\langle w, |\mu_t^2 - \mu_t^1| \rangle\|_1 = \|\langle w, |\int_0^t \Phi_{s,t}([K_s^2 - K_s^1](\mu_s^2))| ds \rangle\|_1\]
\[\leq \|\langle w, \int_0^t |\Phi_{s,t}([K_s^2 - K_s^1](\mu_s^2))| ds \rangle\|_1\]
\[\leq \|\langle w, \int_0^t (K_s^2 - K_s^1)(\mu_s^2) ds \rangle\|_1.\]

Now, we also have
\[\|\langle w, (K_s^2 - K_s^1)(\mu_s^2) ds \rangle\|_1\]
\[\leq \|\langle w, \int_0^\infty \int_0^\infty (\mu_s^2 - \mu_s^1)(dy)\mu_s^2(dy')K(y, y')(w(y') + \frac{y'}{y + y'}w(y + y'))\|_1\]
\[\leq 2\|\langle w, \int_0^\infty \int_0^\infty (\mu_s^2 - \mu_s^1)(dy)\mu_s^2(dy')K(y, y')w(y')\|_1\]
\[\leq 2\|\langle w, \int_0^\infty \int_0^\infty (\mu_s^2 - \mu_s^1)(dy)\mu_s^2(dy')w(y)w(y')^2\|_1\]
\[\leq 2\|\langle w, |\mu_s^2 - \mu_s^1| \rangle\|_1\|\langle w^2, \mu_s^2 \rangle\|_\infty.\]

Because we assumed \(\|\langle w^2, \mu_t^2 \rangle\|_\infty < C\) for some constant \(C\), we have
\[\|\langle w, |\mu_t^2 - \mu_t^1| \rangle\|_1 \leq 2C \int_0^t \|\langle w, |\mu_s^2 - \mu_s^1| \rangle\|_1 ds.\]

Also by definition of solutions, we know that \(\|\langle w, |\mu_t^2 - \mu_t^1| \rangle\|_1 < \infty\). So, we can apply Gronwall inequality and obtain \(\mu_t^1 = \mu_t^2\) almost surely. This concludes the proof of Theorem 2.5. \(\Box\)

Now, we will show the uniqueness part of Theorem 2.6.

Proof. By Theorem 2.5, it satisfies to show that for some \(T > 0\), if \(\mu\) is a solution to (3), then \(\sup_{t \leq T} \|\langle w^2, \mu_t \rangle\|_\infty < \infty\). For this, we will use a similar approach as in Section 5 of [7]. For any \(z > 0\), apply \(P_z\) to equation (3), multiply by \(1_{y \leq z}w^2\) and integrate over \((0, \infty)\) to obtain, for all \(s, t \geq 0\),
\[\langle 1_{y \leq z}w^2, P_s\mu_t \rangle \leq \langle 1_{y \leq z}w^2, P_{s+t}\mu_0 \rangle + \int_0^t \langle 1_{y \leq z}w^2, P_{s+t-r}\mu_r \rangle dr.\]
Using the condition

\[ w^2(y + y')p(y + y') - w^2(y)p(y) - w^2(y')p(y') \leq C[w(y)w(y')p(y) + w(y)w(y')p(y')], \]

we obtain

\[
\langle 1_{y \leq z} w^2, P_{s+t-r} \mu_r \rangle \\
\leq C \int_{\mathbb{R}^d} \int_0^z \int_0^z w(y)w(y')p^{s+t-r,x,x'}(y')K(y, y')\mu_r(x', dy)\mu_r(x', dy')dx'
\leq C \int_{\mathbb{R}^d} \int_0^z w(y)^2w(y')^2p^{s+t-r,x,x'}(y')\mu_r(x', dy)\mu_r(x', dy')dx'
\leq C \| \langle 1_{y \leq z} w^2, \mu_r \rangle \|_\infty \langle 1_{y \leq z} w^2, P_{s+t-r} \mu_r \rangle (x)
\]

Now, set \( h(t) = \sup_{s \geq 0} \| \langle 1_{y \leq z} w^2, P_{s} \mu_t \rangle \|_\infty \). We then obtain

\[
h(t) \leq \| \langle 1_{y \leq z} w^2, P_{s+t} \mu_0 \rangle \|_\infty + C \int_0^\infty h(s)^2 ds,
\]

and this implies

\[
h(t) \leq \left[ \| \langle 1_{y \leq z} w^2, P_{s+t} \mu_0 \rangle \|_\infty - Ct \right]^{-1}.
\]

As this is true for all \( z \), we can set \( T = \frac{1}{c} \| \langle 1_{y \leq z} w^2, P_{s+t} \mu_0 \rangle \|_\infty \) and conclude \( \sup_{t \leq T} \| \langle w^2, \mu_t \rangle \|_\infty < \infty \) as desired.

Now, we will use same strategy to prove the uniqueness part of Theorem 2.8.

**Proof.** In case (a), we would have

\[
\langle 1_{y \leq z} w^2, P_{s+t-r} \mu_r \rangle \\
\leq C \| \langle 1_{y \leq z} w^2, \mu_r \rangle \|_\infty \langle 1_{y \leq z} w^2, P_{s+t-r} \mu_r \rangle (x) + \| \langle 1_{y \leq z} w^2, \mu_r \rangle \|_\infty \langle 1_{y \leq z} w^2, P_{s+t-r} \mu_r \rangle (x).
\]

Note that,

\[
\langle wv, P_s \mu_t \rangle \leq \langle wv, P_{s+t} \mu_0 \rangle + \int_0^t \langle wv, P_{s+t-r} K(\mu_r) \rangle dr,
\]

and since \( wvp \) is sublinear, we will have

\[
\langle wv, P_s \mu_t \rangle \leq \langle wv, P_{s+t} \mu_0 \rangle < c.
\]
for some constant c. So, we obtain

\[ h(t) \leq \| \langle 1_{y \leq z} \omega^2, P_{s+t} \mu_0 \rangle \|_{\infty} + 2cC \int_0^t h(s) ds, \]

and we can apply Gronwall to conclude that \( \sup_{t \leq T} \| \langle \omega^2, \mu_t \rangle \|_{\infty} < \infty \) for any \( T > 0 \).

For case (b), we set \( h(t) = \sup_{s \geq 0} (1+s+t)^{1+\epsilon} \| \langle 1_{y \leq z} \omega^2, P_s \mu_t \rangle \|_{\infty} \). Then, by similar computation as earlier, we will have

\[ h(t) \leq c + C \int_0^t h(s)^2 (1+s)^{1+\epsilon} ds. \]

If, for example, \( c \) is small enough such that

\[ 4c^2 C \int_0^\infty \frac{1}{(1+s)^{1+\epsilon}} ds < c, \]

then we have \( h(t) < 2c \) for all \( t \) and thus we have uniqueness of the global solution. \( \Box \)

6 Existence

In this section we will prove Theorem 2.6 and Theorem 2.8. We consider the following linear PDE

\[ q_t = P_t q_0 + \int_0^t P_{t-s} K_s'(q_s) ds \quad (26) \]

for \( t \leq T \) with \( \nu_0 = \mu_0 = q_0 \) and \( \nu_s \) non-negative satisfying \( \sup_{s \leq T} \| \langle \omega^2, \nu_s \rangle \|_{\infty} \leq c \). Proposition 5.2 tells us the existence, uniqueness and non-negativity of \( q \).

Now, let \( G \) be the set of \( \tau \in \mathcal{M} \) such that \( \sup_{s \leq T} \| (y, \tau_s) \|_1 < \infty \) and \( H \) be the set of \( \tau \in G \) such that \( \sup_{s \leq T} \| (\omega^2, \tau_s) \|_{\infty} < \infty \). We can then define function \( f : H \rightarrow G \) so that for any \( \nu \in H \), \( f(\nu) = q \), where \( q \) is the solution of (26). We will now give a bound on \( \| \langle \omega^2, q_t \rangle \|_{\infty} \).

Proposition 6.1. Assume

\[ w^2(y + y')p(y + y') - w^2(y)p(y) - w^2(y')p(y') \leq C \{ w(y)w(y')p(y) + w(y)w(y')p(y') \}, \]

then we can find \( c > 0 \) and \( T > 0 \) so that \( \sup_{s \leq T} \| \langle \omega^2, f(\nu_s) \rangle \|_{\infty} \leq c \) if \( \sup_{s \leq T} \| \langle \omega^2, \nu_s \rangle \|_{\infty} \leq c \).
Proof. Again, we can copy the argument in [7]. For any \( z > 0 \), let

\[
h(t) = \sup_{s \geq 0} \| \langle 1_{y < z} w^2, P_s q_t \rangle \|_\infty.
\]

Then we have

\[
\langle 1_{y < z} w^2, P_s q_t \rangle \leq \langle 1_{y < z} w^2, P_{s+t} (q_0) \rangle + \int_0^t \langle w^2, P_{s+t-r} K_r (q_r) \rangle dr
\]

and

\[
\langle 1_{y < z} w^2, P_{s+t-r} K_r (q_r) \rangle
\]

\[
\leq \int_{\mathbb{R}^d} \int_0^2 \int_0^2 K(y, y') \nu_r (dy') q_r (dy) \left( \frac{y^2 (y + y')}{y + y'} p^{s+t-r,x'} (y + y') - p^{s+t-r,x'} (y) w^2 (y) \right) dx
\]

\[
\leq \int_{\mathbb{R}^d} \int_0^2 \int_0^2 K(y, y') \nu_r (dy') q_r (dy) + \nu_r (dy) q_r (dy')
\]

\[
\times \left( \frac{y^2 (y + y')}{y + y'} p^{s+t-r,x'} (y + y') - p^{s+t-r,x'} (y') w^2 (y') \right) dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^d} \int_0^2 \int_0^2 K(y, y') \nu_r (dy') q_r (dy) + \nu_r (dy) q_r (dy')
\]

\[
\times (w^2 (y + y') p^{s+t-r,x'} (y + y') - p^{s+t-r,x'} (y) w^2 (y) - p^{s+t-r,x'} (y') w^2 (y') dx
\]

\[
\leq C \int_{\mathbb{R}^d} \int_0^2 \int_0^2 K(y, y') \nu_r (dy') q_r (dy) + \nu_r (dy) q_r (dy')
\]

\[
\times (p^{s+t-r,x'} (y) w (y) w(y') + p^{s+t-r,x'} (y') w(y) w(y)) dx
\]

\[
\leq C \langle w^2, P_{s+t-r} \nu_r \rangle \| 1_{y < z} w^2, q_r \|_\infty + \langle 1_{y < z} w^2, P_{s+t-r} q_r \rangle \| w^2, \nu_r \|_\infty
\]

So we have

\[
h(t) \leq \| \langle w^2, q_0 \rangle \|_\infty + 2 \int_0^t c Ch(s) ds,
\]

for all \( t \leq T \). Then we can use Gronwall inequality to obtain

\[
h(t) \leq \| \langle w^2, q_0 \rangle \|_\infty e^{2cCt}.
\]
So, we can pick $c$ large and $T$ small such that

$$c \geq \|\langle w^2, q_0 \rangle \|_{\infty} e^{2cCT}.$$  

As this works for all $z > 0$, we conclude that if we have

$$\|\langle w^2, P_s \mu_t \rangle \|_{\infty} \leq c$$

for all $s$ and $t \leq T$, then we also have

$$\|\langle w^2, P_s f(\mu)_t \rangle \|_{\infty} \leq c$$

for all $s$ and $t \leq T$. \hfill $\Box$

Now, we will modify our argument for proving Theorem 2.5 to prove Theorem 2.6.

*Proof.* Suppose

$$\|\langle w^2, P_s \mu^i_t \rangle \|_{\infty} \leq c$$

for all $s \geq 0$ and $t \leq T$.

Let $q^i = f(\mu^i)$ and let $\Phi^i_{s,t}$ be the map mapping $q^i_s$ to $q^i_t$ as defined earlier. Also let $K^i = K^{\mu^i}$.

Then by variation of constants formula we have

$$q^2_t - q^1_t = \int_0^t \Phi^1_{s,t} (K^2_s - K^1_s)(q^2_s)ds.$$
Further we would have

\[
\|\langle w, |(K^2_s - K^1_s)(q^2_s)|ds \rangle_1 \|
\leq \int_{\mathbb{R}} \int_{0}^{\infty} \int_{0}^{\infty} |(\mu^2_s - \mu^1_s)(dy)|q^2_s(dy')K(y, y')(w(y') + \frac{y'}{y + y'}w(y + y')) \, ds
\]

\[
\leq 2 \int_{\mathbb{R}} \int_{0}^{\infty} \int_{0}^{\infty} |(\mu^2_s - \mu^1_s)(dy)|q^2_s(dy')K(y, y')w(y') \, ds
\]

\[
\leq 2 \int_{\mathbb{R}} \int_{0}^{\infty} \int_{0}^{\infty} |(\mu^2_s - \mu^1_s)(dy)|q^2_s(dy')w(y)w(y')^2 \, ds
\]

\[
\leq 2\|\langle w, |\mu^2_s - \mu^1_s| \rangle \|_1 \|\langle w^2, q^2_s \rangle \|_\infty
\]

\[
\leq 2c\|\langle w, |\mu^2_s - \mu^1_s| \rangle \|_1.
\]

Therefore, we have

\[
\|\langle w, |q^2_t - q^1_t| \rangle \|_1 \leq 2c \int_{0}^{t} \|\langle w, |\mu^2_s - \mu^1_s| \rangle \|_1 \, ds.
\]

So, for \( T \) sufficiently small we would have \( f \) is a contraction with respect to the metric \( d_T(\mu^1, \mu^2) = \sup_{s \leq T} \|\langle w, |\mu^1_s - \mu^2_s| \rangle \|_1 \) in the space of measures \( \mu \) with \( \|\langle w^2, P_s, \mu \rangle \|_\infty \leq c \). By contraction mapping theorem, \( f \) must have a fixed point and that fixed point is the solution we want. This finishes the proof of Theorem 2.6. \( \square \)

Now, we can modify the argument to proof Theorem 2.8

**Proof.** First, we assume (a) in Theorem 2.8 then we have

\[
\langle w, P_s q_t \rangle \leq \int_{\mathbb{R}} \int_{0}^{R} \int_{0}^{R} \langle w, P_{s+t}(q_0) \rangle + \int_{0}^{t} \langle w, P_{s+t-r}K^\nu_r(q_r) \rangle \, dr.
\]

Note that, when \( w \) is sublinear, \( \langle w, P_{s+t-r}K^\nu_r(q_r) \rangle \) is non-positive. So, we have

\[
\langle w, P_s q_t \rangle \leq \int_{\mathbb{R}} \int_{0}^{R} \int_{0}^{R} \langle w, P_{s+t}(q_0) \rangle \leq \sup_{t>0} \|\langle w^2, P_t(q_0) \rangle \|_\infty \leq c,
\]

for some constant \( c \). Moreover, the inequality

\[
\langle w^2, P_{s+t-r}K^\nu_r(q_r) \rangle
\]

\[
\leq C[\langle w^2, P_{s+t-r}\nu_r \rangle \|\langle w^2, \mu_r \rangle \|_\infty + \langle w^2, P_{s+t-r}q_r \rangle \|\langle w^2, \nu_r \rangle \|_\infty]
\]

52
becomes now

\[
\langle w^2, P_{s+t-r}K_r^\nu(q_r) \rangle \\
\leq C(\langle wv, P_{s+t}\nu_r \rangle \|w^2\|_\infty + \langle wv, P_{s+t-r}q_r \rangle \|w^2, \nu_r \|_\infty \\
+ \langle w^2, P_{s+t-r}q_r \rangle \|wv, \nu_r \|_\infty + \langle w^2, P_{s+t-r}q_r \rangle \|\langle wv, \nu_r \|_\infty \rangle \].
\]

Now, for constants \(a, b > 0\), if we have

\[
\|\langle w^2, P_s\nu_t \rangle\|_\infty \leq ae^{bt}
\]

for all \(s\) and \(t\) then we also have

\[
h(t) \leq c + 2Cc \int_0^t (h_s + ae^{bs})ds \leq c + 2\frac{aCc}{b}e^{bt} + 2Cc \int_0^t h_sds.
\]

By Grownwall, we have

\[
h(t) \leq [c + 2\frac{aCc}{b}e^{bt}]e^{ct}.
\]

So, for any \(T > 0\), we can choose \(b\) to be sufficiently large such that if

\[
\|\langle w^2, P_s\nu_t \rangle\|_\infty \leq ae^{bt}
\]

for all \(t \leq T\), then

\[
h(t) \leq ae^{bt}
\]

for all \(t \leq T\). Then we use same argument as earlier to show that there is some \(T'\) such that \(f\) is a contraction with respect to \(d'_T\) whenever \(\sup_{t > 0} \|\langle w^2, P_t\mu_0 \rangle\|_\infty \leq ae^{bt}\). So, we have existence of the solution up to time \(T'\) and then by same argument we can extend the solution to \(2T'\) and so on up to time \(T\). As it works for any \(T\), we have a global solution.

Now, if we assume instead \((b)\) in Theorem 2.8 then recall

\[
\langle w^2, P_sq_t \rangle \leq \langle w^2, P_{s+t}(q_0) \rangle + \int_0^t \langle w^2, P_{s+t-r}K_r^\nu(q_r) \rangle dr,
\]

53
and
\[ \langle w^2, P_{s+t-r}K_r'(q_r) \rangle \leq C \left[ \langle w^2, P_{s+t-r} \nu_r \rangle \| \langle w^2, q_r \rangle \|_\infty + \langle w^2, P_{s+t-r} \nu_r \rangle \| \langle w^2, \nu_r \rangle \|_\infty \right]. \]

Suppose \( \sup_{s \geq 0} (1 + s + t)^{1+\epsilon} \| \langle 1_{y \leq z} w^2, P_s \nu_t \rangle \|_\infty < 2c \) and set
\[ h(t) = \sup_{s \geq 0} (1 + s + t)^{1+\epsilon} \| \langle 1_{y \leq z} w^2, P_s \nu_t \rangle \|_\infty, \]

we obtain
\[ h(t) \leq c + C \int_0^t \frac{4c h(r)}{(1 + r)^{1+\epsilon}} \, dr. \]

So, if \( c \) is small enough such that \( 8c^2 Ch \int_0^\infty \frac{1}{(1 + r)^{1+\epsilon}} \, dr < c \), then we have \( h(t) < 2c \) for all \( t \). By similar argument as earlier, we obtain global existence of the solution. \( \square \)
References

[1] A. Fannjiang and G. Papanicolaou. *Convection enhanced diffusion for periodic flows*. SIAM J. Appl. Math. 54 (1994), no. 2, 333-408.

[2] P.H. Haynes, V. H. Hoang, J. R. Norris and K. C. Zygalakis. *Homogenization for advection-diffusion in a perforated domain*. Probability and mathematical genetics, 397-415, London Math. Soc. Lecture Note Ser., 378, Cambridge Univ. Press, Cambridge, 2010.

[3] E.B. Davies. *Heat Kernels and Spectral Theory*. Cambridge Tracts in Mathematics. Cambridge University Press, 1990.

[4] E.B. Fabes and D.W. Stroock. *A New Proof of Moser’s Parabolic Harnack Inequality Via the Old Ideas of Nash*. LIDS-P. Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, 1986.

[5] T.J. Lyons and W.A. Zheng. On conditional diffusion processes. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 115:243–255, 1990.

[6] J.R. Norris. Long-time behaviour of heat flow: Global estimates and exact asymptotics. *Archive for Rational Mechanics and Analysis*, 140:161–195, 1997.

[7] J.R. Norris. Measure solutions for the Smoluchowski coagulation-diffusion equation. 2014.

[8] J.R. Norris. Brownian Coagulation. *Comm. Math. Sci. Supplemental Issue*, No. 1, 93-101, 2004

[9] Zhongmin Qian and Weian Zheng. Sharp bounds for transition probability densities of a class of diffusions. *Comptes Rendus Mathematique*, 335(11):953 – 957, 2002.

[10] D. Dürr, S. Goldstein and J.L. Lebowitz A mechanical model of Brownian motion *Communications in Mathematical Physics*, 78(1980/81),no.4, 507-530.

[11] Laurencot, P. and Mischler, S. *The continuous coagulation-fragmentation equations with diffusion*. Archive for rational mechanics and analysis 162 (2002), no. 1, 45-99.

[12] Hammond, Alan and Rezakhanlou, Fraydoun. *Moment bounds for the Smoluchowski equation and their consequences*. Comm. Math. Phys. 276 (2007), no. 3, 645-670.
[13] Hammond, Alan and Rezakhanlou, Fraydoun. *The kinetic limit of a system of coagulating Brownian particles.* Arch. Ration. Mech. Anal. 185 (2007), no. 1, 1-67.

[14] Mohammad Yaghouti, Alan Hammond and Fraydoun Rezakhanlou. *Coagulation, diffusion and the continuous Smoluchowski equation.* Stochastic Process. Appl., 119 (2009), no. 9, 3042-3080.

[15] Herbert Amann. *Coagulation-fragmentation processes.* Arch. Ration. Mech. Anal, 151(2000), no.4, 339-366.

[16] Herbert Amann and Christoph Walker. *Local and global strong solutions to continuous coagulation-fragmentation equations with diffusion.* J. Differential Equations, 218(2005), no.1, 159-186.

[17] Mischler, S and Rodriguez Richard, M. *Existence globale pour l’équation de Smoluchowski continue non homogène et comportement asymptotique des solutions.* C. R. Math. Acad. Sci. Paris, 365(2003), no. 5, 407-412.

[18] J.A. Cañizo, L. Desvillettes, K. Fellner. *Regularity and mass conservation for discrete coagulation-fragmentation equations with diffusion.* Ann. Inst. H. Poincaré Anal. Non Linéaire, 27(2010), no. 2, 639-654.

[19] Laurençot, P. and Mischler, S. *Global existence for the discrete diffusive coagulation-fragmentation equations in L1*. Rev. Mat. Iberoamericana, 18 (2002), no. 3, 731-745.

[20] Dariusz Wrzosek. *Existence of solutions for the discrete coagulation-fragmentation model with diffusion.* Topol. Methods Nonlinear Anal., 9(1997), no. 2, 279-296.

[21] Dariusz Wrzosek. *Weak solutions to the Cauchy problem for the diffusive discrete coagulation-fragmentation system.* J. Math. Anal. Appl., 289(2004), no. 2, 405-418.

[22] Dariusz Wrzosek. *Mass-conserving solutions to the discrete coagulation-fragmentation model with diffusion.* Nonlinear Anal. 49(2002), no. 3, Ser. A: Theory Methods, 297-314.

[23] J. M. Ball and J. Carr. *The discrete coagulation-fragmentation equations: existence, uniqueness, and density conservation.* J. Statist. Phys., 61(1990), no. 1-2, 203-234.

[24] I. F. Bailleul. *Spatial coagulation with bounded coagulation rate.* J. Evol. Equ., 11(2011), no. 3, 675-686.
[25] J. R. Norris. *Cluster coagulation*. Comm. Math. Phys., 209(2000), no. 2, 407-435.

[26] Fraydoun Rezakhanlou. *Moment bounds for the solutions of the Smoluchowski equation with coagulation and fragmentation*. Proc. Roy. Soc. Edinburgh Sec. A, 140(2010), no. 5, 1041-059.

[27] Fraydoun Rezakhanlou. *Pointwise bounds for the solutions of the Smoluchowski equation with diffusion*. Arch. Ration. Mech. Anal., 212(2014), no. 3, 1011-1035.

[28] G.A. Pavliotis, A.M. Stuart, K.C. Zygalakis *Calculating effective diffusivities in the limit of vanishing molecular diffusion*. Journal of Computational Physics, 228(2009), no. 4, 1030-1055.