Regularizable cycles associated with a Selberg type integral under some resonance condition

Katsuhisa Mimachi and Masaaki Yoshida

Abstract. We study the twisted homology group attached to a Selberg type integral under some resonance condition, which naturally appears in the \( su_2 \)-conformal field theory and the representation of the Iwahori-Hecke algebra. We determine the dimension of the space of the regularizable cycles. The dimension-formula is given in terms of the generalized hypergeometric series \( _3F_2 \).

1 Introduction and Main results

A Selberg type integral

\[
\int \prod_{1 \leq i < j \leq m} (x_i - x_j)^g \prod_{1 \leq j \leq m, 1 \leq k \leq n} (x_i - z_k)^\lambda_k \, dx_1 \cdots dx_m
\]  

(1.1)

is used to express a conformal block in conformal field theory \cite{5} \cite{6} \cite{7} \cite{15} and to represent the hypergeometric function due to Heckman and Opdam of type BC \cite{9} \cite{11}. The integral (1.1) can be thought of the pairing between the de Rham cohomology group and the twisted homology group (the homology group with coefficients in the local system), which are studied by many authors, e.g. \cite{2} \cite{3} \cite{10}. The study under resonance conditions, however, is not well pursued \cite{4} \cite{13}. The purpose of this article is to study the twisted homology group associated with (1.1) under some resonance condition. The dimension of the space of the regularizable cycles is determined.

Let \( \text{Conf}_n(\mathbb{C}) \) denote the configuration space of \( n \) distinct points of \( \mathbb{C} \):

\[
\text{Conf}_n(\mathbb{C}) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n ; z_i \neq z_j \text{ if } i \neq j \}.
\]

For each point \( z = (z_1, \ldots, z_n) \in \text{Conf}_n(\mathbb{C}) \), let \( \Phi_{g, \lambda}^{m, n} \) be the function

\[
\Phi_{g, \lambda}^{m, n}(x; z) = \Phi_{g, \lambda_1, \ldots, \lambda_n}^{m, n}(x_1, \ldots, x_m; z_1, \ldots, z_n)
\]

\[
= \prod_{1 \leq i < j \leq m} (x_i - x_j)^g \prod_{1 \leq j \leq m} \prod_{1 \leq k \leq n} (x_i - z_k)^{\lambda_k}
\]

on the complex manifold.
\[ X_{m,n}(x_1, \ldots, x_m; z_1, \ldots, z_n) = \left\{ x \in \mathbb{C}^m \mid \prod_{1 \leq i < j \leq m} (x_i - x_j) \prod_{1 \leq i \leq m} \prod_{1 \leq k \leq n} (x_i - z_k) \neq 0 \right\}. \]

Let \( \mathcal{L} \) be the local system determined by \( \Phi_{m,n}^{\lambda} \). Compactly supported twisted (loaded) chains and locally finite twisted (loaded) chains, together with the natural boundary operators, define the twisted homology group \( H_m(X_{m,n}, \mathcal{L}) \) and the locally finite twisted homology group \( H^\ell_m(X_{m,n}, \mathcal{L}) \), respectively.

The symmetric group \( S_m \) acts on \( X_{m,n} \) through the coordinates \( x = (x_1, \ldots, x_m) \). The action of \( S_m \) on \( X_{m,n} \) defines the invariant parts of \( H_m(X_{m,n}, \mathcal{L}) \) and \( H^\ell_m(X_{m,n}, \mathcal{L}) \), which will be denoted by \( H_m(X_{m,n}, \mathcal{L})^{S_m} \) and \( H^\ell_m(X_{m,n}, \mathcal{L})^{S_m} \). There is a natural map

\[ \iota : H_m(X_{m,n}, \mathcal{L})^{S_m} \rightarrow H^\ell_m(X_{m,n}, \mathcal{L})^{S_m}. \]

A cycle in the image \( \text{Im} \iota \) is called a regularizable cycle, and a preimage of a regularizable cycle is called a regularization of it.

If the exponent of the irreducible component of the divisor \( \tilde{D} = \pi^{-1}(D) \), where \( \pi : (\mathbb{P}^1(\mathbb{C}))^m \rightarrow (\mathbb{P}^1(\mathbb{C}))^m \) is the minimal blow-up along the non-normally crossing loci of \( D \), is an integer, the irreducible component or the exponent itself is said to be resonant. The resonance condition on the exponents makes the kernel of the map \( \iota \) non-trivial. Hence describing the kernel and the image of the map \( \iota \) under a resonance condition is a fundamental problem. In this paper, especially, we determine the dimension of the space of the regularizable cycles under the following resonance condition:

\[ 2\lambda_j + g \in \mathbb{Z} \quad (1 \leq j \leq r) \tag{1.2} \]

for each \( r \) such that \( 0 \leq r \leq n \). For the other divisors, we assume the non-resonance condition:

\[ 2\lambda_j + g \notin \mathbb{Z} \quad (r + 1 \leq j \leq n), \tag{1.3} \]

\[ k\lambda_j + \left(\begin{array}{c} k \\ 2 \end{array}\right) g \notin \mathbb{Z} \quad (1 \leq j \leq n, \ k = 1 \text{ and } 3 \leq k \leq m) \tag{1.4} \]

and

\[ k\lambda_\infty + \left(\begin{array}{c} k \\ 2 \end{array}\right) g \notin \mathbb{Z} \quad (1 \leq k \leq m), \quad \left(\begin{array}{c} k \\ 2 \end{array}\right) g \notin \mathbb{Z} \quad (2 \leq k \leq m), \tag{1.5} \]
where
\[
\lambda_\infty = - \sum_{1 \leq i \leq n} \lambda_i - (m - 1)g \quad \text{and} \quad \binom{1}{2} = 0;
\]
this is our assumption on the exponents \(\lambda_i\)’s and \(g\), throughout this paper.

In what follows, to indicate the dependence on the number \(r\), we denote by \(L(r)\) the local system determined by \(\Phi_{g,\lambda}^{m,n}\) with the resonance condition \((1.2)\) for \(r\) such that \(0 \leq r \leq n\). It follows from Theorem 2 of [3] with the Poincaré duality that, even under this resonance condition, the rank of \(H_{m}(X_{m,n}, L(r))^{S_m}\) and the rank of \(H_{m}(X_{m,n}, L(r))^{S_m}\) remain to be \(D_{m,n} = \binom{n+m-2}{m}\).

Let \(I_{m,n}(r)\) denote the dimension of \(\text{Im } \iota \subset H_{m}(X_{m,n}, L_{m,n}(r))^{S_m}\), and \(K_{m,n}(r)\) the dimension of \(\ker \iota \subset H_{m}(X_{m,n}, L_{m,n}(r))^{S_m}\). Then
\[
K_{m,n}(0) = 0, \quad K_{1,n}(r) = 0,
\]
and
\[
D_{m,n} = K_{m,n}(r) + I_{m,n}(r). \quad (1.6)
\]

We have the following.

**Theorem 1.** Suppose \((1.2), (1.3), (1.4)\) and \((1.5)\). Then we have
\[
I_{m,n}(r) = \binom{n+m-2}{m} \, _3F_2 \left( \binom{-r, -m, -m+1}{-n-m+2, -n-m+3}; 1 \right), \quad (1.7)
\]

where
\[
_3F_2 \left( \begin{array}{c} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{array} ; x \right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k(\alpha_3)_k}{(\beta_1)_k(\beta_2)_k k!} x^k
\]
with \((a)_k = a(a+1)\cdots(a+k-1)\).

In the special cases \(r = n\) and \(r = n-1\), \((1.7)\) becomes simpler.

**Theorem 2.** Under the same condition as in Theorem 1, we have
\[
(1) \quad I_{m,n}(n) = \binom{n}{m} - \binom{n}{m-1}, \quad \text{and} \quad (2) \quad I_{m,n}(n-1) = \binom{n-1}{m-1}.
\]

The dimension of the irreducible representation of the Iwahori-Hecke algebra \(H(S_n)\) parametrized by the Young diagram \((n-m, m)\) is \(\binom{n}{m} - \binom{n}{m-1}\), and the representation is realized in terms of the regularizable cycles of \(H_{m}(X_{m,n}, L_{m,n}(n))^{S_m}\) in [12]. Therefore, Theorem 2 shows that the representation space is exactly equal to \(\text{Im } \iota\).
2 Proof of Theorem 1

Let $T$ be a tubular neighbourhood of $D = \bigcup_{1 \leq i < j \leq m} \{x_i - x_j = 0\} \cup \{x_i - x_i = 0\} \cup \{x_i = \infty\}$ in $(\mathbb{P}^1(\mathbb{C}))^m$, and put $T^\circ = T - D$. The inclusion $T^\circ \subset X_{m,n} = (\mathbb{P}^1(\mathbb{C}))^m \setminus D$ leads to the exact sequence

$$\cdots \rightarrow H_{m+1}(X_{m,n}, T^\circ, \mathcal{L}_{m,n}(r))^S_m \rightarrow H_m(T^\circ, \mathcal{L}_{m,n}(r))^S_m \rightarrow \cdots$$

$$\rightarrow H_m(X_{m,n}, \mathcal{L}_{m,n}(r))^S_m \rightarrow H_m(X_{m,n}, T^\circ, \mathcal{L}_{m,n}(r))^S_m \rightarrow \cdots .$$

Since the relative homology group $H_k(X_{m,n}, T^\circ, \mathcal{L}_{m,n}(r))^S_m$ can be canonically identified with $H_m^H(X_{m,n}, \mathcal{L}_{m,n}(r))^S_m$, we have the exact sequence

$$H_{m+1}^H(X_{m,n}, \mathcal{L}_{m,n}(r))^S_m \rightarrow H_m(T^\circ, \mathcal{L}_{m,n}(r))^S_m$$

$$\rightarrow H_m(X_{m,n}, \mathcal{L}_{m,n}(r))^S_m \rightarrow H_m^H(X_{m,n}, \mathcal{L}_{m,n}(r))^S_m,$$

where the last arrow is the natural map $\iota$. Theorem 2 in [13] and the Poincaré duality imply $H_{m+1}^H(X_{m,n}, \mathcal{L}_{m,n}(r))^S_m = 0$. So our task is to study $H_m(T^\circ, \mathcal{L}_{m,n}(r))^S_m$.

We first blow up $(\mathbb{P}^1(\mathbb{C}))^m$. Let $\pi : (\mathbb{P}^1(\mathbb{C}))^m \rightarrow (\mathbb{P}^1(\mathbb{C}))^m$ be the minimal blow-up of $(\mathbb{P}^1(\mathbb{C}))^m$ along the non-normally crossing loci of $D$ (cf. [13]). The exponent of

$$\pi^{-1}\{x_{\sigma(1)} = x_{\sigma(2)} = \cdots = x_{\sigma(k)} = z_j\} \quad (\sigma \in S_m, \quad 2 \leq k \leq m, \quad 1 \leq j \leq n)$$

is $k\lambda_j + (k^2)g$, that of

$$\pi^{-1}\{x_{\sigma(1)} = x_{\sigma(2)} = \cdots = x_{\sigma(k)} = \infty\} \quad (\sigma \in S_m, \quad 2 \leq k \leq m)$$

is $k\lambda_\infty + (k^2)g$, and that of

$$\pi^{-1}\{x_{\sigma(1)} = x_{\sigma(2)} = \cdots = x_{\sigma(k)}\} \quad (3 \leq k \leq m, \quad \sigma \in S_m)$$

is $(k^2)g$. The condition [12] means that $\pi^{-1}\{x_{\sigma(1)} = x_{\sigma(2)} = z_j\}$ for $\sigma \in S_m, 1 \leq j \leq r$ is a resonant divisor.

If we denote $\tilde{D} = \pi^{-1}(D)$ by $\sum \tilde{D}_j$, where each $\tilde{D}_j$ is the irreducible divisor (which might be an exceptional one), a neighbourhood of $\tilde{D}_j$ with small radius is set to be $N(\tilde{D}_j)$, and

$$N^\circ(\tilde{D}_j) = N(\tilde{D}_j) \setminus \tilde{D} \subset \tilde{X}_{m,n}.$$
Then we have $\tilde{T}^\circ = \sum_j N^\circ(\tilde{D}_j) \subset \tilde{X}_{m,n}$.

If $\tilde{D}_i$ is an irreducible non-resonant divisor, we have

$$H_k(N^\circ(\tilde{D}_i), \pi^*\mathcal{L}_{m,n}(r)) = 0 \quad (0 \leq k \leq 2m).$$

If $\tilde{D}_i$ is an irreducible non-resonant divisor, and if $\tilde{D}_j$ is an irreducible resonant divisor, we have

$$H_k(N^\circ(\tilde{D}_i) \cap N^\circ(\tilde{D}_j), \pi^*\mathcal{L}_{m,n}(r)) = 0 \quad (0 \leq k \leq 2m).$$

This is because $N^\circ(\tilde{D}_i)$ and $N^\circ(\tilde{D}_i) \cap N^\circ(\tilde{D}_j)$ have a punctured 1-disc, as a direct summand, carrying a non-resonant exponent.

Hence, the Mayer-Vietoris sequence

$$\cdots \to H_m(U \cap V, \pi^*\mathcal{L}_{m,n}(r)) \to H_m(U, \pi^*\mathcal{L}_{m,n}(r)) \oplus H_m(V, \pi^*\mathcal{L}_{m,n}(r)) \to H_m(U \cup V, \pi^*\mathcal{L}_{m,n}(r)) \to \cdots,$$

for $U = N^\circ(\tilde{D}_i)$ and $V = N^\circ(\tilde{D}_j)$ stated above, implies

$$H_m(U \cup V, \pi^*\mathcal{L}_{m,n}(r)) \cong H_m(V, \pi^*\mathcal{L}_{m,n}(r)).$$

Applying this equivalence repeatedly, we have

$$H_m(T^\circ, \mathcal{L}_{m,n}(r)) \cong H_m(\tilde{T}^\circ, \pi^*\mathcal{L}_{m,n}(r))$$

$$\cong H_m(\bigcup_{1 \leq k \leq r} \bigcup_{1 \leq i < j \leq m} N^\circ(\pi^{-1}\{x_i = x_j = z_k\}), \pi^*\mathcal{L}_{m,n}(r)). \quad (2.1)$$

Thus

$$H_m(T^\circ, \mathcal{L}_{m,n}(r))^S_m \cong H_m(\tilde{T}^\circ, \pi^*\mathcal{L}_{m,n}(r))^S_m$$

$$\cong H_m(\bigcup_{1 \leq k \leq r} \bigcup_{1 \leq i < j \leq m} N^\circ(\pi^{-1}\{x_i = x_j = z_k\}), \pi^*\mathcal{L}_{m,n}(r))^S_m. \quad (2.2)$$

At this stage, we have the following.

**Lemma 1.** For $m \geq 2$,

$$K_{m,n}(r) = D_{m-2,n} + K_{m,n}(r-1) - K_{m-2,n}(r-1),$$

where $D_{0,n} = 0.$
Proof. Set
\[ U = \bigcup_{k=1}^{r-1} \bigcup_{\sigma \in S_m} N^\sigma(\pi^{-1}\{x_{\sigma(1)} = x_{\sigma(2)} = z_k\}) \]
and
\[ V = \bigcup_{\sigma \in S_m} N^\sigma(\pi^{-1}\{x_{\sigma(1)} = x_{\sigma(2)} = z_r\}). \]

Then
\[
0 \rightarrow H_m(U \cap V, \pi^*L_{m,n}(r))^S_m \rightarrow H_m(U, \pi^*L_{m,n}(r))^S_m \oplus H_m(V, \pi^*L_{m,n}(r))^S_m \rightarrow H_m(U \cup V, \pi^*L_{m,n}(r))^S_m \rightarrow 0, \tag{2.3}
\]

since
\[ H_k(U \cap V, \pi^*L_{m,n}(r)) = 0 \quad \text{for} \quad k \neq m. \]

By definition, we have
\[ \dim H_m(U \cup V, \pi^*L_{m,n}(r))^S_m = K_{m,n}(r), \tag{2.4} \]
and
\[ \dim H_m(U, \pi^*L_{m,n}(r))^S_m = K_{m,n}(r-1). \tag{2.5} \]

Lemma 3 stated in Section 4 reads
\[ H_m(V, \pi^*L_{m,n}(r))^S_m \cong H_{m-2}(X_{m-2,n}, L'_{m-2,n}(r-1))^S_{m-2}, \]

where \( L'_{m-2,n}(r-1) = L_{m-2,n}|_{\lambda_r \rightarrow \lambda_r + 2g(r-1)} \) is the local system on \( X_{m-2,n} \) determined by
\[ \Phi^{g,\lambda}_{m-2,n}(x; z) \prod_{i=1}^{m-2} (x_i - z_r)^{2g}. \]

Hence, we have
\[ \dim H_m(V, \pi^*L_{m,n}(r))^S_m = D_{m-2,n}. \tag{2.6} \]
On the other hand, we have

\[ U \cap V = \bigcup_{\sigma \in S_m} \left\{ \left( \bigcup_{k=1}^{r-1} N^o \left( \pi^{-1}\{x_{\sigma(1)} = x_{\sigma(2)} = z_k\} \right) \right) \cap N^o \left( \pi^{-1}\{x_{\sigma(3)} = x_{\sigma(4)} = z_r\} \right) \right\} . \]

Hence the same argument for proving Lemma 3 shows

\[ H_m(U \cap V, \pi^*L_{m,n}(r))^{S_m} = H_m(U \cap V, \pi^*L_{m,n}(r))^{S_m} = H_{m-2}(U \cap V, \pi^*L_{m-2,n}(r-1))^{S_{m-2}}, \]

where \( \pi \) in the righthand-side is the restriction of \( \pi : (\mathbb{P}^1(\mathbb{C}))^m \to (\mathbb{P}^1(\mathbb{C}))^m \) on the \( m-2 \) space: \( x_{m-1} = x_{m} = 0 \).

Therefore, we have

\[ \dim H_m(U \cap V, \pi^*L_{m,n}(r))^{S_m} = K_{m-2,n}(r-1). \tag{2.7} \]

Combination of (2.2) and (2.3–6) implies the required result.

Furthermore we have

**Proposition 1.** For \( m \geq 2 \),

\[ K_{m,n}(r) = rD_{m-2,n} - K_{m-2,n}(1) - K_{m-2,n}(2) - \cdots - K_{m-2,n}(r-1). \]

**Proof.** Summing up the equalities

\[ K_{m,n}(r) = D_{m-2,n} + K_{m,n}(r-1) - K_{m-2,n}(r-1), \]
\[ K_{m,n}(r-1) = D_{m-2,n} + K_{m,n}(r-2) - K_{m-2,n}(r-2), \]
\[ \vdots \]
\[ K_{m,n}(3) = D_{m-2,n} + K_{m,n}(2) - K_{m-2,n}(2), \]
\[ K_{m,n}(2) = D_{m-2,n} + K_{m,n}(1) - K_{m-2,n}(1), \]

each of which is a special case of the equality in Lemma 1, we have the required result, since \( K_{m,n}(1) = K_{m-2,n}(1) = 0 \).
Theorem 1.3 of [13] implies $K_{2,n}(r) = r$. Proposition 1 implies
\[ K_{3,n}(r) = rD_{1,n}, \tag{2.8} \]
since $K_{1,n}(1) = \cdots = K_{1,n}(r-1) = 0$. Generally we have the following.

**Proposition 2.** For $m \geq 2$ and $r \geq 0$,
\[ K_{m,n}(r) = \binom{r}{1} D_{m-2,n} - \binom{r}{2} D_{m-4,n} + \cdots + (-1)^{s-1} \binom{r}{s} D_{m-2s,n} + \cdots \]
\[ = \sum_{s \geq 1} (-1)^{s-1} \binom{r}{s} D_{m-2s,n}, \tag{2.9} \]
where $\binom{s}{s} = 0$ for $s > r$.

Proof. Suppose the equality in case $m$. Then Proposition 1 shows
\[ K_{m+2,n}(r) = rD_{m,n} - K_{m,n}(1) - K_{m,n}(2) - \cdots - K_{m,n}(r-1) \]
\[ = rD_{m,n} - \sum_{1 \leq t \leq r-1} \sum_{1 \leq s \leq t} (-1)^{s-1} \binom{t}{s} D_{m-2s,n} \]
\[ = rD_{m,n} - \sum_{1 \leq s \leq t \leq r-1} \binom{t}{s} D_{m-2s,n} \]
\[ = rD_{m,n} - \sum_{1 \leq s \leq r} (-1)^{s-1} \binom{r}{s+1} D_{m-2s,n} \]
\[ = \sum_{s \geq 1} (-1)^{s-1} \binom{r}{s} D_{m+2-2s,n}. \]

For the fourth equality, we used Lemma 2 below. The induction on $m$ leads to the required result. \[\square\]

**Lemma 2.**
\[ 1 + \binom{s+1}{s} + \binom{s+2}{s} + \cdots + \binom{r-1}{s} = \binom{r}{s+1}. \]

Proof.
\[
\binom{r}{s+1} = \binom{r-1}{s+1} + \binom{r-1}{s} \\
= \{ \binom{r-2}{s+1} + \binom{r-2}{s} \} + \binom{r-1}{s} \\
= \{ \binom{r-3}{s+1} + \binom{r-3}{s} \} + \binom{r-2}{s} + \binom{r-1}{s} \\
\cdots \\
= \{ \binom{s+1}{s+1} + \binom{s+1}{s} \} + \binom{s+2}{s} + \cdots + \binom{r-1}{s} \\
= 1 + \binom{s+1}{s} + \binom{s+2}{s} + \cdots + \binom{r-1}{s}.
\]

By Proposition 2 and (1.6), we obtain

\[
I_{m,n}(r) = D_{m,n} - K_{m,n}(r) \\
= \sum_{s=0}^{[m/2]} (-1)^s \binom{r}{s} D_{m-2s,n},
\]

(2.10)

where \([x]\) denotes the largest integer not exceeding \(x\).

Finally, we express (2.10) in terms of the generalized hypergeometric series. The equalities

\[
(n + m - 2)! = 2^{2s}(-n - m + 2)/2_s((-n - m + 3)/2)_s(n + m - 2 - 2s)!
\]

and

\[
m! = 2^{2s}(-m/2)_s((-m + 1)/2)_s(m - 2s)!
\]

lead to

\[
\binom{n + m - 2 - 2s}{m - 2s} = \binom{n + m - 2}{m} \frac{(-m/2)_s((-m + 1)/2)_s}{((-n + m + 2)/2)_s((-n - m + 3)/2)_s}.
\]

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On the other hand,
\[
\binom{r}{s} = (-1)^s \frac{(-r)^s}{s!}.
\]

Hence, we reach the desired expression
\[
I_{m,n}(r) = \binom{n+m-2}{m} \, _3F_2\left(\begin{array}{c}
-n, -m/2, (-m+1)/2 \\
(n-m+2)/2, (-n-m+3)/2
\end{array} ; 1\right).
\]

This completes the proof of Theorem 1.

3 Proof of Theorem 2

We first derive (1) of Theorem 2. Note that, in case \( m \) is even \( 2j \),
\[
I_{m,n}(n) = \binom{n+2j-2}{2j} \, _3F_2\left(\begin{array}{c}
-n, -j, 1/2 - j \\
-j+1 - n/2, -j - 3/2 - n/2
\end{array} ; 1\right),
\]
and, in case \( m \) is odd \( 2j+1 \),
\[
I_{m,n}(n) = \binom{n+2j-1}{2j+1} \, _3F_2\left(\begin{array}{c}
-n, -j - 1/2, -j \\
-n+1 - j/2 - n/2, 1 - j - n/2
\end{array} ; 1\right).
\]

Pfaff-Saalschütz’s theorem (Theorem 2.2.6 in [1])
\[
_3F_2\left(\begin{array}{c}
a, b, -j \\
c, 1 + a + b - c - j
\end{array} ; 1\right) = \frac{(c-a)_j(c-b)_j}{(c)_j(c-a-b)_j} \quad \text{for} \quad j \in \mathbb{Z}_{\geq 0} \quad (3.1)
\]
and a contiguity relation
\[
(b-a) \, _3F_2\left(\begin{array}{c}
a, b, -j \\
c, a+b-c+2-j
\end{array} ; 1\right) + a \, _3F_2\left(\begin{array}{c}
a+1, b, -j \\
c, a+b-c+2-j
\end{array} ; 1\right)
\]
\[- b \, _3F_2\left(\begin{array}{c}
a, b+1, -j \\
c, a+b-c+2-j
\end{array} ; 1\right) = 0,
\]
which follows from the identity
\[ a(a + 1)_k(b)_k - b(a)_k(b + 1)_k = (a - b)(a)(b)_k, \]

imply
\[
3F_2\left( \begin{array}{c} a, b, -j \\ c, a + b - c + 2 - j \\ \end{array} : 1 \right) \\
= \frac{a}{a - b} \frac{(c - a - 1)_j(c - b)_j}{(c)_j(c - a - b - 1)_j} + \frac{b}{b - a} \frac{(c - a - 1)_j(c - b - 1)_j}{(c)_j(c - a - b - 1)_j}.
\]

Hence we have
\[
3F_2\left( \begin{array}{c} -n, -j, \frac{1}{2} - j \\
-1 + \frac{1}{2}, -j + \frac{3 - n}{2} \\ \end{array} : 1 \right) \\
= \frac{-n}{-n + j - \frac{1}{2}} \frac{(1 - \frac{n}{2})_j(n - \frac{1}{2})_j}{(\frac{n}{2})_j(n - \frac{1}{2})_j} + \frac{-j + \frac{1}{2}}{n - j + \frac{1}{2}} \frac{(\frac{n}{2})_j(-\frac{1 + n}{2})_j}{(\frac{n}{2})_j(n - \frac{1}{2})_j}.
\]

which turns out to be
\[
\frac{1}{2}(-1 + 4j - n) \frac{(-\frac{n}{2})_j(\frac{1 - n}{2})_j^{j-1}}{(\frac{n}{2})_j(n - \frac{1}{2})_j} \\
= (n + 1 - 4j) \frac{n(n - 1) \cdots (n - 2j + 2)}{(2j)!} \frac{1}{(n + 2j - 2)(n + 2j - 3) \cdots n(n - 1)}.
\]

Therefore, we have
\[
I_{2j,n}(n) = \frac{n(n - 1) \cdots (n - 2j + 2)}{(2j)!} \frac{1}{(n + 1 - 4j)}.
\]

In the same way, we have
\[\begin{align*}
3F_2\left( \begin{array}{c}
-n, -j - \frac{1}{2} - j \\
-n + 1 - j, 1 - j - \frac{n}{2}
\end{array} ; 1 \right) &= \frac{(n + 1 - 4j)\frac{(n - 2)(n - 3)\cdots(n - 2j + 1)}{(n + 2j - 1)(n + 2j - 3)\cdots(n + 1)}}{2j + 1!} (n - 4j - 1).
\end{align*}\]

This completes the proof of (1).

Next we derive (2) of Theorem 2. Note that

\[I_{2j+1,n}(n) = \frac{n(n - 1)\cdots(n - 2j + 1)}{(2j + 1)!} (n - 4j - 1).\]

Pfaff-Saalschütz’s theorem (3.1) implies

\[\begin{align*}
3F_2\left( \begin{array}{c}
-n + 1, -j - \frac{1}{2} - j \\
-n + 1 - \frac{n}{2}, -j - \frac{3 - n}{2}
\end{array} ; 1 \right) &= \frac{(n - 2j)(1 - n)_{2j} \cdot \cdots \cdot (n - 2j)(1 - n)_{2j}}{(1 - n - j)(\frac{1}{2} - n)_{2j}}.
\end{align*}\]
Therefore, we obtain
\[ I_{2j,n}(n-1) = \frac{(1-n)_{2j}}{(1)_2} = \binom{n-1}{2j} \]
and
\[ I_{2j+1,n}(n-1) = \frac{(1-n)_{2j+1}}{(1)_{2j+1}} = \binom{n-1}{2j+1}. \]
This completes the proof of (2).

4 A Key Lemma

In this section, we fix \( z_1 \) to be 0 for simplicity, and we study the homological structure around the subvariety \( \{ (x_1, \ldots, x_m) \in \mathbb{C}^m \mid x_i = x_j = 0 \} \) of codimension two for \( 1 \leq i < j \leq m \).

Choose an appropriate open set \( N^o(x_i = x_j = 0) \subset X_{m,n} \) such that \( H_m(N^o(x_i = x_j = 0), L_{m,n}(r)) \cong H_m(N^o(\pi^{-1}\{ x_i = x_j = 0 \}), \pi^*L_{m,n}(r)) \).

For example, we take
\[ N^o(x_i = x_j = 0) = \{ (x_1, \ldots, x_m) \in X_{m,n} \mid |x_i|, |x_j| < \varepsilon(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_m) \}, \]
where
\[ \varepsilon(x_3, \ldots, x_m) = \min\{ \varepsilon|x_3|, \ldots, \varepsilon|x_m|, \varepsilon^2 \}, \]
and \( \varepsilon \) is a positive number satisfying \( \varepsilon < \min\{ 1, |z_2|, \ldots, |z_n| \} \).

Lemma 3. Suppose \( \text{Lem} \). Then we have the isomorphism
\[ H_m(\cup_{1 \leq i < j \leq m} N^o(x_i = x_j = 0), L_{m,n}(r))^{S_m} \]
\[ \cong H_{m-2}(X_{m-2,n}, L'_{m-2,n}(r-1))^{S_{m-2}}, \]
where the local system $\mathcal{L}_{m-2,n}(r-1)$ is determined by

$$
\prod_{1 \leq i < j \leq m-2} (x_i - x_j)^g \prod_{1 \leq i \leq m-2} \left\{ x_i^{\lambda_1 + 2g} \prod_{2 \leq k \leq n} (x_i - z_k)^{\lambda_k} \right\}.
$$

**Proof.** The function $\Phi_{m,n}^{g,\lambda}$ defining the local system $\mathcal{L}_{m,n}(r)$ restricted on $N^\circ (x_1 = x_2 = 0)$ is reduced to

$$
\prod_{1 \leq i < j \leq m} (x_i - x_j)^g \prod_{1 \leq i \leq m} x_i^{\lambda_1} \prod_{3 \leq i \leq m} \prod_{2 \leq k \leq n} (x_i - z_k)^{\lambda_k}.
$$

(4.1)

The pullback of (4.1) by the morphism

$$
\psi: (s_1, s_2, x_3, \ldots, x_m) \to (x_1, \ldots, x_m),
$$

where

$$
x_1 = s_1 x_3 \cdots x_m, \quad x_2 = s_2 x_3 \cdots x_m
$$

is expressed by

$$
(s_1 s_2)^{\lambda_1} (s_1 - s_2)^g \times \prod_{3 \leq i < j \leq m} (x_i - x_j)^g \prod_{3 \leq i \leq m} \left\{ x_i^{3\lambda_1 + 3g} \left( 1 - s_1 \frac{x_3 \cdots x_m}{x_i} \right)^g \left( 1 - s_2 \frac{x_3 \cdots x_m}{x_i} \right)^g \right\} \times \prod_{3 \leq i \leq m} \prod_{2 \leq k \leq n} (x_i - z_k)^{\lambda_k}.
$$

On the other hand, the condition $|x_1| < \varepsilon |x_i|$ and $|x_2| < \varepsilon |x_i|$ on $N^\circ (x_1 = x_2 = 0)$ implies

$$
\left| \frac{x_3 \cdots x_m}{x_i} \right| < \varepsilon \quad \text{and} \quad \left| \frac{x_3 \cdots x_m}{x_i} \right| < \varepsilon
$$

on $\psi^{-1} N^\circ (x_1 = x_2 = 0)$, for $3 \leq i \leq m$. This shows that the contribution of the factors

$$
\prod_{3 \leq i \leq m} \left( 1 - s_1 \frac{x_3 \cdots x_m}{x_i} \right)^g \left( 1 - s_2 \frac{x_3 \cdots x_m}{x_i} \right)^g
$$

to the local sytem $\psi^* \mathcal{L}_{m,n}(r)$ is trivial.
Thus, the local system $\psi^*\mathcal{L}_{m,n}(r)$ on $\psi^{-1}N^\circ(x_1 = x_2 = 0)$ can be considered as that determined by

$$(s_1s_2)^{\lambda_1}(s_1 - s_2)^g \times \prod_{3 \leq i < j \leq m} (x_i - x_j)^g \prod_{3 \leq i \leq m} x_i^{\lambda_1 + 2g} \prod_{3 \leq i \leq m} \prod_{2 \leq k \leq n} (x_i - z_k)^{\lambda_k},$$

(Note that $2\lambda_1 + g \in \mathbb{Z}$). This fact leads to the isomorphism

$$H_m(N^\circ(x_1 = x_2 = 0), \mathcal{L}_{m,n}(r))$$

$$\cong H_2(X_{2,1}(x_1, x_2; 0), \mathcal{L}_{2,1}(1)) \otimes H_{m-2}(X_{m-2,n}(x_3, \ldots, x_m; z), \mathcal{L}_{m-2,n}'(r - 1)).$$

Since rank $H_2(X_{2,1}, \mathcal{L}_{2,1}(1)) = 1$ from Theorem 1.3 of [13], we fix a generator of $H_2(X_{2,1}, \mathcal{L}_{2,1}(1))$ and identify $H_2(X_{2,1}, \mathcal{L}_{2,1}(1))$ with $\mathbb{C}$. Then we have

$$H_m(N^\circ(x_i = x_j = 0), \mathcal{L}_{m,n}(r))$$

$$\cong H_2(X_{2,1}(x_i, x_j; 0), \mathcal{L}_{2,1}(1))$$

$$\otimes H_{m-2}(X_{m-2,n}(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_m; z), \mathcal{L}_{m-2,n}'(r - 1))$$

$$\cong H_{m-2}(X_{m-2,n}(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_m; z), \mathcal{L}_{m-2,n}'(r - 1))$$

for $1 \leq i < j \leq m$. Henceforce,

$$H_m(\bigcup_{1 \leq i < j \leq m} N^\circ(x_i = x_j = 0), \mathcal{L}_{m,n}(r))^{S_m}$$

$$\cong H_{m-2}(X_{m-2,n}, \mathcal{L}'_{m-2,n}(r - 1))^{S_{m-2}}.$$

This completes the proof. \qed

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Katsuhisa Mimachi
Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro-ku, Tokyo 152-8551
Japan
mimachi@math.titech.ac.jp

Masaaki Yoshida
Department of Mathematics
Kyushu University
Ropponmatsu, Fukuoka 810-8560
Japan
myoshida@math.kyushu-u.ac.jp