FUNDAMENTAL ASPECTS OF THE ISM FRACTALITY

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Abstract. The ubiquitous clumpy state of the ISM raises a fundamental and open problem of physics, which is the correct statistical treatment of systems dominated by long range interactions. A simple solvable hierarchical model is presented which explains why systems dominated by gravity prefer to adopt a fractal dimension around 2 or less, like the cold ISM and large scale structures. This has direct relation with the general transparency, or blackness, of the Universe.

1. Introduction: Clumpiness in Astrophysical Gases

The extreme inhomogeneity, the supersonic turbulence, and multi-phase nature of the interstellar medium (ISM), meaning an extreme departure from thermodynamical equilibrium, was well recognised many decades ago (e.g. von Weizsäcker 1951). Over the years, new wavelengths and higher angular resolution have revealed several times that the interstellar gas was more clumpy than accepted before. Often, it was argued that the smallest clouds had been resolved, to be disproved later on (typically they span “three times the beamwidth”, Verschuur 1993).

Clearly, the radically opposite behaviours of cosmic and terrestrial gases must be understood. Since the ISM covers many decades of density and temperature, very different physical processes, such as radiation transfer, chemistry, magnetic fields, and gravity are simultaneously involved, confusing the issue about the relative importance of each one. Star formation can even be viewed as an extreme prolongation of the ISM inhomogeneity. The concurrent interplay of so many factors makes the study of the ISM formidable. Moreover, different kinds of devices and skills are needed to unveil its fullness, increasing the communication problem across specialties.
With respect to terrestrial gas, the new factor in the ISM and larger scale structures is, of course, self-gravity. Although there are discussions whether the molecular clouds are bounded by gravity or just marginally so (e.g. Blitz 1993), in any case gravity pervades cosmic structures from the largest galaxy superclusters down to stars and planets. By continuity its rôle must be expected to be generally dominant throughout the scales.

2. Homogeneity in the Classical Gas

Before discussing the inhomogeneity of cosmic gas, one should clearly understand why in terrestrial conditions gas tends toward homogeneity. In terrestrial situations a classical gas is confined by exterior factors (walls, etc.), which means that in the Lagrangian of the \( N \) gas particles,

\[
L(q_i, \dot{q}_i) = T(\dot{q}_i) - U(q_i), \quad \text{and} \quad U(q_i)/T(\dot{q}_i) \to 0, \quad (i = 1, \ldots N),
\]

the kinetic energy term \( T(\dot{q}_i) \) dominates the interaction term \( U(q_i) \). The motion of individual particles being highly chaotic, sensitive to perturbations, no integrals beside the classical integrals does exist, and the system can be assumed ergodic in good approximation. Typically, perturbations ("the walls") reshuffle the particles in phase space while preserving in average the scalar energy integral. This allows the average time invariance of the system. But the global vector integrals (angular momentum, centers of velocity and mass) must vanish in the reference frame of the box. Since the system (1) is locally dominated by the kinetic term, it must be translationally invariant in space. As well known to this symmetry corresponds spatial homogeneity. At scales much larger than the interparticle distances the gas must tend to be smooth, i.e. differentiable. The use of hydrodynamics and other differential equations is then justified.

In such a situation the total energy of the system is proportional to the volume, i.e. extensive. In order to be able to take the large \( N \) limit, the extensivity of energy is an additional requisite in usual statistical mechanics.

Thus, the a priori complex behaviour of a gas can be statistically simple when a global symmetry, such as translation invariance, exists. However, when the interaction potential becomes non-negligible at lower temperature this symmetry may be broken. Often a phase transition reveals then another symmetry (e.g. when crystallisation occurs). But if the interactions are short range, then energy is still extensive in good approximation.

In contrast, when the interactions are long range and attractive, as for gravity, they can never be neglected in the large \( N \) limit, and energy is no longer extensive: the ground to expect a statistical homogeneous state, even locally, is lost. Curiously, the inapplicability of thermodynamics to gravitating systems is well known by thermodynamicists (e.g. Jaynes 1957; Prigogine 1962), but often ignored in astrophysics.
The association of thermodynamics with gravity leads immediately to paradoxes, such as the occurrence of negative specific heats (Lynden-Bell & Lynden-Bell 1977), implying that a thermal equilibrium is impossible!

The ideal model of the gravitating perfect gas enclosed in a spherical box at a fixed temperature shows itself the limit of the association (see Binney & Tremaine 1987, Fig. [8-1]). The isothermal sphere equilibrium does exist only above a critical temperature, which corresponds to the Jeans’ critical temperature. At sufficiently high temperature the system energy is positive and a box is required to confine the system, as for usual terrestrial gas. At some lower temperature the system energy is negative, so is bound by gravity, but a thermal equilibrium still exists: this is the regime of the stars. However, below the Jeans’ critical temperature, no equilibrium solution does exist which fulfills the local homogeneity assumption. The self-gravity then dominates. The open problem is then to characterise the asymptotic statistical state of the perfect gas below the Jeans’ critical temperature. Clearly, this concerns many astrophysical systems, from most of the ISM to the large-scale cosmological structures.

3. Statistical Equilibrium with Long-Range Interactions

Before introducing any additional physics relevant to the ISM, the statistical behaviour resulting just from the interplay between gravity and dynamics should be characterised. This partial problem is a highly idealised simplification of the full complexity of the ISM, but no progress can be expected if this fundamental aspect remains obscure.

The classical gas is based on a very rough model of particles without interaction. Yet, more important it includes the symmetry of translational invariance. In the ISM and large scale structures, the dominant symmetry is scale-invariance. The Lagrangian of the gas particles is

\[ L(q_i, \dot{q}_i) = T(\dot{q}_i) - U(q_i), \quad \text{with} \quad U(\alpha q_i) = \alpha^p U(q_i), \quad p = -1. \quad (2) \]

The idea is to represent this empirical fact by a hierarchical system statistically scale-invariant. In many cases of solid state physics, systems in phase transition build also long-range correlations with universal scaling laws; interestingly, even surprisingly crude models, such as the Ising model, reproduce remarkably well the measured scaling laws (e.g. Stanley 1995). Important is therefore to include the right symmetry in the model.

Here we suppose that the system is made of hierarchical mass clumps, each clump being made, in average, of \( N \) sub-clumps, recursively over a number \( L \) of levels above the ground level 0 (see Pfenniger & Combes 1994, PC94). A clump is characterised by a finite mass \( M \) and a finite length scale \( r \). The mass distribution, to be scale-invariant, scales as a fixed power
$D$, which defines the mass fractal dimension. Thus,

$$M_L = N M_{L-1} = N^L M_0, \quad \text{and} \quad M_L = M_0 \left( \frac{r_L}{r_0} \right)^D. \quad (3)$$

In real physical systems a lower and upper levels must exist, which define, analogously to walls in the classical gas, the boundary conditions. The boundary conditions are convenient to abstract the complications coming from the external world. For example: 1) the lowest level of fragmentation is reached in the ISM when the heat transfer time exceeds the dynamical time (see PC94); 2) the upper level in cosmological structures is given by the time-dependence introduced at the largest scale by the universal expansion, which proceeds at a slower pace than the small-scale clustering.

The second hypothesis concerns the particle interactions. Generalising the gravitating case, the potential $\Phi$ is supposed to be a power law, i.e. $\Phi = G M r^p / p$ ($p = -1$ for gravity, and $G > 0$ for attracting interactions). Suggested by the system hierarchical organisation, we approximate the potential energy $U_L$ at level $L$ by

$$U_L = N U_{L-1} + \frac{G}{p} M_L^2 r_L^p \left( 1 + \alpha \frac{r_{L-1}}{r_L} \right). \quad (4)$$

The term with $\alpha$ represents tidal interactions. Since the scale ratio is constant, this term is constant and can be absorbed in a new coupling constant $G'$. We will see that the main results do not depend on the value of $G$ (indeed scale invariance effects depend on the power law exponent $p$).

For parameters suited to the ISM ($D \approx 1 - 2$, $N = 5 - 8$, Scalo 1990) the approximation (4) has been checked to be accurate at the percent level.

The third hypothesis is of statistical character. Thermodynamics can not be used since it excludes long-range interactions. The only remaining statistical tool is the virial theorem. If a scale invariant system finds a statistical equilibrium, at each level the virial theorem must hold,

$$2 T_L - p U_L = 3 P_{L+1} V_L, \quad (5)$$

where $T_L$ is the total kinetic energy cumulated up level $L$, $P_{L+1}$ is the outer pressure, and $V_L \sim r_L^3$ the clump volume. Here the outer pressure is purely kinetic: it is given by $2/3$ of the kinetic energy density outside the sub-clumps. Approximately,

$$P_L = \frac{2 T_L - N T_{L-1}}{3 V_L - NV_{L-1}}. \quad (6)$$

which neglects the superpositions of clumps during collisions. For the sake of simplicity, in PC94 the pressure term was neglected, yet the clump “collisions” were determined to be frequent for the typical parameters of the
ISM. Clump collisions may well disrupt or merge them, but the supposed Jeans unstable medium (due e.g. to fast cooling) also fragments clouds fast, reforming the clumps before a crossing time. Therefore, in a statistical equilibrium the average $N$ should be constant, and at any moment the fraction of colliding clumps should be small. 

The above recurrences for $M_L$, $V_L$, $U_L$, and $T_L$ in Eqs. \([3-6]\) can be solved exactly in finite terms. With $x \equiv r_L/r_{L-1} = N^{1/D} > 1$, we find, 

$$
\frac{M_L}{M_0} = x^{DL}, \quad \frac{V_L}{V_0} = x^{3L}, \quad \frac{U_L/M_L}{U_0/M_0} = \frac{x^{(D+p)(L+1)-1}}{x^{(D+p)-1}}, \quad \frac{T_L/V_L}{T_0/V_0} = 1 - \beta \left[ \frac{1}{1-x^{2(D+p)-2}} + \frac{x^{L(D-3)}}{x^{(D+p)-1}} \left( 1 - x^{(L+1)(D+p)} \right)^{-1} \right]^{-1} \quad (7)
$$

where $\beta \equiv pU_0/2T_0$ is the ground virial ratio ($0 \leq \beta \leq 1$). The free parameters are $D$, $N$, $p$, and $\beta$. Although not very illuminating at first sight, the functional properties of the solution \((7)\) are very interesting. We just summarise here the most important features. More detail will be presented elsewhere (Pfenniger 1996).

First, not any combinations of parameters lead to a physical solution. Not only the kinetic energy $T_L$ must positive, but also the pressure, which is proportional to the velocity dispersion $v_L$ squared, $\frac{1}{2}M_Lv_L^2 = T_L - NT_{L-1}$.

In the “thermodynamical limit” $L \rightarrow \infty$ a striking phenomenon occurs. The range of physical solutions shrinks on a subspace of the parameter space, leading to a new constraint, 

$$
D = \frac{3 - p}{2 - \ln(1 - \beta)/\ln N}. \quad (8)
$$

So, for $p = -1$, $D < 2$ in any case. For systems not too confined by the outer pressure ($1/2 < \beta < 1$) and $N > 5$, we have $D < 1.7$, while systems with more outer pressure (“pressure confined clouds”) increase $D$ up to 2. Therefore, we conclude that hierarchical gravitating systems in statistical equilibrium can indeed exist, but with $D < 2$.

The scale-velocity dispersion relation takes the exact scale-free form $v \propto \tau^\kappa$, where $\kappa \equiv (D + p)/2$. The result for $p = -1$ in PC94 remains therefore unchanged in spite of the inclusion of the outer pressure. For fractal ISM cold clouds with substantial ambient pressure (Blitz 1993), $\beta \approx 1/2$, and with $N = 5-10$ (Scalo 1990), we expect $D \approx 1.7$ and $\kappa \approx 0.35$, which is comparable to Larson’s (1981) size-linewidth relationship.

Finally, the short to long-range interaction transition occurs at $p = -3$. For $p < -3$ the typical feasible state is non-longer hierarchical, but with a single level, large outer pressure ($\beta \ll 1/2$), large $N$ and a dimension $D$ close to 3: the classical homogeneous gas is recovered.
4. Final Remarks: Universality of the Fractal Structure

With a simple model we have motivated here the proposition that strongly gravitating systems in statistical average tend to lower their dimensionality below 2, so to adopt a very inhomogeneous structure, as manifest in many cases. The property $D < 2$ is well documented for the cold ISM (Scalo 1990), and for the large scale structures (Coleman & Pietronero 1992). Actually, the general transparency, or blackness, of the Universe at most of the wavelengths, extending the De Chéseaux-Olbers paradox to non-stellar objects, reflects for a good part the widespread $D < 2$ fractality.

A deceptive effect occurs when observing fractals with $D < 2$, such as cold ISM clouds: their orthogonal projections have the same $D < 2$ (Falconer 1990), so less than a surface. Practically it means that over the more scales the gas does indeed behave as a fractal, the smaller is the sky fraction over which 50% of the mass projects. The bias is then obvious: when sampling the sky, most of it appears at low column density so mass looks well sampled. In fact the mass is poorly sampled because most of it resides in small regions of the sky at high column density, therefore likely to be optically thick. Not only increasingly higher angular resolution is required to resolve the smallest structures, but also more wavelength types for piercing all the decades of column densities. In addition, the velocity dispersion (or temperature) decreases at smaller scales, so the smallest structures are both the coldest and the densest ones, so often hard to detect.

If the fractal state in fast cooling cosmic gas is indeed universal, we have little reason not to expect that the outer HI galactic disks are just the warmer “atmosphere” of a fractal which extends down to very small scale. The lowest temperature in the Universe being 2.73 K, it is then natural to assume that much more gas mass can be hidden in very cold molecular gas, making a sizable fraction of the galactic dark matter (PC94).

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