GLOBAL STABILITY OF KELLER–SEGEL SYSTEMS IN CRITICAL LEBESGUE SPACES

JIE JIANG*

Wuhan Institute of Physics and Mathematics
Chinese Academy of Sciences
Wuhan 430071, HuBei Province, China

(Communicated by Michael Winkler)

ABSTRACT. This paper is concerned with the initial-boundary value problem for the classical Keller–Segel system
\[
\begin{aligned}
\rho_t - \Delta \rho &= -\nabla \cdot (\rho \nabla c), & x \in \Omega, & t > 0 \\
\gamma c_t - \Delta c + c &= \rho, & x \in \Omega, & t > 0
\end{aligned}
\]
\[\tag{1}\]
in a bounded domain $\Omega \subset \mathbb{R}^d$ with $d \geq 2$ under homogeneous Neumann boundary conditions, where $\gamma \geq 0$. We study the existence of non-trivial global classical solutions near the spatially homogeneous equilibria $\rho = c \equiv \mathcal{M} > 0$ with $\mathcal{M}$ being any given large constant which is an open problem proposed in [2, p. 1687]. More precisely, we prove that if $0 < \mathcal{M} < 1 + \lambda_1$ with $\lambda_1$ being the first positive eigenvalue of the Neumann Laplacian operator, one can find $\varepsilon_0 > 0$ such that for all suitable regular initial data $(\rho_0, \gamma c_0)$ satisfying
\[
\frac{1}{|\Omega|} \int_\Omega \rho_0 dx - \mathcal{M} = \gamma \left( \frac{1}{|\Omega|} \int_\Omega c_0 dx - \mathcal{M} \right) = 0 \tag{2}
\]
and
\[
\|\rho_0 - \mathcal{M}\|_{L^{d/2}(\Omega)} + \|\nabla c_0\|_{L^d(\Omega)} < \varepsilon_0, \tag{3}
\]
problem (1) possesses a unique global classical solution which is bounded and converges to the trivial state $(\mathcal{M}, \mathcal{M})$ exponentially as time goes to infinity. The key step of our proof lies in deriving certain delicate $L^p - L^q$ decay estimates for the semigroup associated with the corresponding linearized system of (1) around the constant steady states. It is well-known that classical solution to system (1) may blow up in finite or infinite time when the conserved total mass $m \equiv \int_\Omega \rho_0 dx$ exceeds some threshold number if $d = 2$ or for arbitrarily small mass if $d \geq 3$. In contrast, our results indicates that non-trivial classical solutions starting from initial data satisfying (2)-(3) with arbitrarily large total mass $m$ exists globally provided that $|\Omega|$ is large enough such that $m < (1 + \lambda_1)|\Omega|$. 

1. Introduction. In this paper, we study the initial-boundary value problem for the following classical Keller–Segel system of chemotaxis:
\[
\begin{aligned}
\rho_t - \Delta \rho &= -\nabla \cdot (\rho \nabla c), & x \in \Omega, & t > 0 \\
\gamma c_t - \Delta c + c &= \rho, & x \in \Omega, & t > 0 \\
\frac{\partial \rho}{\partial \nu} &= \frac{\partial c}{\partial \nu} = 0, & x \in \partial \Omega, & t > 0 \\
(\rho(x, 0) = \rho_0(x), \quad \gamma c(x, 0) = \gamma c_0(x), & x \in \Omega,
\end{aligned}
\]
\[\tag{4}\]

2010 Mathematics Subject Classification. Primary: 35K51, 35K59; Secondary: 35Q92.

Key words and phrases. Chemotaxis, Keller–Segel model, global solutions, global stability.

* Corresponding author.
where $\Omega \subset \mathbb{R}^d$ with $d \geq 2$ is a bounded domain with smooth boundary. Here, $\rho$ and $c$ denote the density of cells and the concentration of chemical signal, respectively. $\gamma \geq 0$ is a given constant; when $\gamma = 0$, (4) is reduced to a parabolic–elliptic system which is usually called a simplified Keller–Segel model in the existing literature.

A well-known fact of the Keller–Segel model (4) is that classical solutions with large initial data may blow up when dimension $d \geq 2$ (see [7, 18, 2, 16, 17] and references cited therein). In particular, a critical-mass phenomenon exists in the two-dimensional case. More precisely, if the conserved total mass of cells $m \equiv \int_\Omega \rho_0 dx$ is lower than certain number, then global classical solution exists and remains bounded for all times; otherwise, it may blow up in finite or infinite time [7, 19]. It was observed that if $\gamma = 0$, the threshold number is $4\pi$ for any bounded domain and is $8\pi$ for a disk of any radius.

On the other hand, since $\rho = c \equiv \mathcal{M}$ with any positive number $\mathcal{M}$ is a spatially homogeneous steady solution, an open problem proposed in a recent survey [2, p. 1687] is that for any given initial data $(\rho_0, c_0)$ sufficiently close to $\mathcal{M}$, whether we can get a non-trivial global classical solution which is bounded for all times. The present contribution is devoted to this problem and gives a partially affirmative answer. More precisely, we prove that if $0 < \mathcal{M} < 1 + \lambda_1$ with $\lambda_1$ being the first positive eigenvalue of the Neumann Laplacian operator, one can find $\varepsilon_0 > 0$ such that for all suitable regular initial data $(\rho_0, c_0)$ satisfying

\begin{equation}
\frac{1}{|\Omega|} \int_\Omega \rho_0 dx - \mathcal{M} = \gamma \left( \frac{1}{|\Omega|} \int_\Omega c_0 dx - \mathcal{M} \right) = 0 \tag{5}
\end{equation}

and $\|\rho_0 - \mathcal{M}\|_{L^{4/3}(\Omega)} + \|\nabla c_0\|_{L^6(\Omega)} < \varepsilon_0$, problem (4) possesses a unique global classical solution which is bounded and converges to $(\mathcal{M}, \mathcal{M})$ exponentially as time goes to infinity.

Observing that the conserved total mass $m = \int_\Omega \rho_0 dx = \mathcal{M}|\Omega| < (1 + \lambda_1)|\Omega|$, our result indicates a new observation that classical solution can be obtained globally starting from suitable initial data of arbitrarily large total mass $m$ provided that the area $|\Omega|$ is large, correspondingly. Note that due to the existing results, the threshold number is $8\pi$ when $\Omega = B$ being a disk in $\mathbb{R}^2$ for the case $\gamma = 0$, no matter how large the radius is. In this respect, we rigorously prove that globally bounded nontrivial classical solution exists with any over-$8\pi$ total mass for $\Omega = B$ if the radius is large enough. We also note that in [15], numerical evidence of existence of global classical solution with total mass above the critical mass $8\pi$ was showed for a simplified Keller–Segel-Stokes system with zero Dirichlet boundary conditions for the chemical concentration $c$, fluid velocity $u$ and Neumann boundary condition for the cell density $\rho$, respectively. However, there was no analytical proof available and hence it is unknown whether the presence of Stokes fluid plays an essential role in their example since fluid advection had already been conjectured to regularize singular nonlinear dynamics [13].

It is worth mentioning that in [19] (also [18]), when $d \geq 3$ and in the radial setting, Winkler proved that arbitrary small perturbations of any initial data may immediately produce blow-up when the considered topology is chosen in $L^p \times W^{1,2}$ with $p \in (1, \frac{2d}{d+2})$. Due to their result, for any $\varepsilon$, one can always find $(\rho_{0e}, c_{0e})$ satisfying $\|\rho_{0e} - \mathcal{M}\|_{L^p} + \|c_{0e} - \mathcal{M}\|_{W^{1,2}} \leq \varepsilon$ such that the solution starting from $(\rho_{0e}, c_{0e})$ blows up in finite time. We remark that there is no contradiction with our results since on the one hand we have the restriction (5) on the initial data and
on the other hand, the metric space $L^{d/2} \times W^{1,d}$ under consideration in our result is more regular and hence smaller than $L^p \times W^{1,2}$ when $d \geq 3$.

In addition, we would like to point out that the metric space $L^{d/2} \times W^{1,d}$ is a scaling-invariant space for the Keller–Segel system. To see this point, we observe that system (4) with the second equation replaced by $\gamma c_t - \Delta c = \rho$ has the following property (taking $\Omega = \mathbb{R}^d$): if $(\rho, c)$ is a solution to (4), then the pair $(\rho_\lambda, c_\lambda)$ given by

$$\rho_\lambda(x, t) = \lambda^2 \rho(\lambda x, \lambda^2 t), \quad c_\lambda(x, t) = c(\lambda x, \lambda^2 t), \quad \forall \lambda > 0,$$

is also a solution. Then we easily verify that the norm of $(\rho_\lambda, c_\lambda)$ in $L^\infty(0, T; L^{d/2}(\Omega)) \times L^\infty(0, T; W^{1,d}(\Omega))$ is scaling-invariant and thus we call $L^{d/2}(\Omega) \times W^{1,d}(\Omega)$ a scaling-invariant space (or a critical space). Some discussions on blow-up criteria in super-critical spaces for Keller–Segel models can be found in [2, 11].

To formulate our results, we need to introduce some notion and notations. $L^q(\Omega)$, $1 \leq q \leq \infty$ denotes the usual Lebesgue space and $\| \cdot \|_{L^q(\Omega)}$ denotes its norm. When $q = 2$, we denote the norms $\| \cdot \|_{L^2(\Omega)}$ by $\| \cdot \|$ for simplicity. Similarly, $W^{k,p}(\Omega)$, $k \in \mathbb{N}$, $1 \leq q \leq \infty$ denotes the usual Sobolev space with norm $\| \cdot \|_{W^{k,p}(\Omega)}$. When $q = 2$, we simply denote $W^{k,2}(\Omega)$ by $H^k(\Omega)$. Throughout this paper, $C, K, C_i$ and $K_i$ ($i \in \mathbb{N}$) stand for positive constants which may vary from line to line and special dependence will be pointed out in the text if necessary. For any $M \geq 0$, denote $L^p_M(\Omega)$ ($1 \leq p < \infty$) the closed convex subset of $L^p(\Omega)$ satisfying $\frac{1}{|\Omega|} \int_\Omega w dx = M$ with $w \in L^p(\Omega)$. Note that if $M = 0$, $L^p_0(\Omega)$ is a Banach spaces and the following Poincaré’s inequality holds:

$$\|w\|_{L^p(\Omega)} \leq C\|\nabla w\|_{L^p(\Omega)}, \quad \text{for all } w \in L^p_0(\Omega)$$

and we denote $\lambda_1$ the first positive eigenvalue of the Neumann Laplacian operator such that

$$\lambda_1\|w\|_{L^2(\Omega)}^2 \leq \|\nabla w\|_{L^2(\Omega)}^2, \quad \text{for all } w \in L^2_0(\Omega). \quad (6)$$

Now we are in a position to state our main results. The first one is concerned with global stability for the parabolic–elliptic Keller–Segel system with $\gamma = 0$.

**Theorem 1.1.** Let $d \geq 2$. For any given constants $0 < M < 1 + \lambda_1$ and $\lambda' < \mu_0 \triangleq \lambda_1(1 - \frac{M}{1 + \lambda_1})$, there exists $\varepsilon_0 > 0$ depending on $\lambda'$ and $\Omega$ such that for any non-negative initial datum $\rho_0 \in C(\overline{\Omega}) \cap L^1_M(\Omega)$ satisfying $\|\rho_0 - M\|_{L^d(\Omega)} \leq \varepsilon_0$, system (4) has a unique global classical solution such that

$$\|\rho(\cdot, t) - M\|_{L^\infty(\Omega)} + \|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \leq Ce^{-\lambda_1 t} \quad \text{for all } t \geq 1 \quad (7)$$

with some $C > 0$.

The other result is about the doubly parabolic case with $\gamma = 1$ which is given below.

**Theorem 1.2.** Let $d \geq 2$. For any given constants $0 < M < 1 + \lambda_1$ and $\mu' < \mu_1 \triangleq \lambda_1 - \frac{1}{2}\left(\sqrt{4\lambda_1 M + 1} - 1\right) > 0$, there exists $\varepsilon_0 > 0$ depending on $M, \mu'$ and $\Omega$ such that for any non-negative initial data $(\rho_0, c_0) \in (C(\overline{\Omega}) \cap L^1_M(\Omega)) \times (C^1(\overline{\Omega}) \cap L^1_M(\Omega))$ satisfying $\partial_t c_0 = 0$ on $\partial \Omega$ and $\|\rho_0 - M\|_{L^d(\Omega)} + \|\nabla c_0\|_{L^d(\Omega)} \leq \varepsilon_0$, system (4) has a unique global classical solution such that

$$\|\rho(\cdot, t) - M\|_{L^\infty(\Omega)} + \|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \leq Ce^{-\mu' t} \quad \text{for all } t \geq 1 \quad (8)$$

with some $C > 0$. 


The case $\mathcal{M} = 0$ corresponds to the small-initial data results established for the Keller–Segel system in bounded domains in [18, 3] and for the Keller–Segel–Navier–Stokes system in [14, 20]. The former two papers exploited a one-step contradiction argument while in the other two it was proved based on the implicit function theorem and fixed point theory, respectively. The classical $L^p - L^q$ decay estimates for the heat semigroup $e^{t\Delta}$ plays an essential role since the first equation of (4) can be regarded as a heat equation for $\rho$ with a quadratic perturbation $-\nabla \cdot (\rho \nabla c)$. With small initial data, the solution to the nonlinear equation should behave like a solution to the heat equation with higher order perturbations.

In the case $\mathcal{M} > 0$, the key step consists in the analysis of the corresponding linearized system around $(\mathcal{M}, \mathcal{M})$ and there comes certain new difficulties. To see this point, we introduce the system for the reduced quantities, i.e., for any given $\mathcal{M} > 0$, let $u = \rho - \mathcal{M}$ and $v = c - \mathcal{M}$ be the reduced cell density and chemical concentration, respectively. It follows that

$$
\begin{align*}
\left\{ 
\begin{array}{l}
u_t - \Delta u + \mathcal{M} \Delta v = -\nabla \cdot (u \nabla v), \\
\gamma v_t - \Delta v + v = u, \\
\partial u / \partial \nu = \partial v / \partial \nu = 0, \\
u(x, 0) = u_0(x) = \rho_0 - \mathcal{M}, \quad \gamma v(x, 0) = \gamma v_0(x) = \gamma (c_0 - \mathcal{M}),
\end{array} \right. \\
(x \in \Omega, t > 0) \quad (x \in \Omega, t > 0) \quad (x \in \partial \Omega, t > 0) \quad (x \in \Omega).
\end{align*}
$$

(9)

Now our problem transforms to the existence of global solutions with small initial data for system (9) which can be regarded as a quadratic perturbation $-\nabla \cdot (u \nabla v)$ of its corresponding linearized system which reads:

$$
\begin{align*}
\left\{ 
\begin{array}{l}
u_t - \Delta \tilde{u} + \mathcal{M} \Delta \tilde{v} = 0, \\
\gamma \tilde{v}_t - \Delta \tilde{v} + \tilde{v} = \tilde{u}, \\
\partial \tilde{u} / \partial \nu = \partial \tilde{v} / \partial \nu = 0,
\end{array} \right. \\
(x \in \Omega) \quad (x \in \Omega, t > 0) \quad (x \in \partial \Omega, t > 0).
\end{align*}
$$

(10)

The analysis of the decay behavior of solutions to the above linearized system becomes quite important and constitutes the major part of the present paper.

Note that cross diffusion appears when $\mathcal{M} > 0$ even in the linearized system (10), while when $\mathcal{M} = 0$, the corresponding linearized system is much simpler since it contains a decoupled heat equation for $\tilde{u}$ (or $\rho$) that is why the proof in [18, 3, 14] strongly relies on the decay estimates for $e^{t\Delta}$. However, this is insufficient in our case due to the presence of cross diffusion coupling which brings some severe difficulties in the analysis especially for the doubly parabolic case $\gamma = 1$. In order to study the decay behavior of solutions to (10), we first use perturbation theory for semigroups and some delicate energy estimates in Hilbert spaces to derive certain exponentially decay estimates for $(\tilde{u}, \nabla \tilde{v})$ in $L^2_0(\Omega) \times L^2_0(\Omega)$ with explicit decay rates (see Lemma 3.1 and Lemma 3.4).

Since we aim to establish global stability in the scaling-invariant Lebesgue spaces which as we mentioned above is $L^{d/2}(\Omega) \times L^d(\Omega)$ for $(\rho, \nabla c)$, we need next is deriving similar $L^p - L^q$ decay estimates for the semigroup associated with the linearized system (10), denoted in the sequel by $e^{t\mathcal{L}}$ for $\gamma = 0$ and $e^{t\mathcal{L}}$ for $\gamma = 1$, respectively. More precisely, we need decay estimate of $e^{t\mathcal{L}}(u_0, v_0)$ in $L^p$ in terms of $\|u_0\|_{L^{d/2}} + \|v_0\|_{L^d}$. The proof is nontrivial due to the cross diffusion coupling as well as the difference in the critical Lebesgue exponents between $u_0$ and $\nabla v_0$. Exploiting the Gronwall inequalities, the well-known $L^p - L^q$ estimates for the Neumann heat semigroups, the obtained exponentially decay properties and more importantly, by a delicate successive iteration argument carried out simultaneously
with respect to time and Lebesgue exponents, we successfully establish the desired
decay estimates (see Lemma 3.5 and Lemma 3.6). With this at hand, we can finish
our proof either by adapting the one-step contradiction argument in [18, 3] or by
implicit function theorem as done in [14]. Since the local well-posedness and blow-
up criteria in super-critical spaces for classical solutions is well-known (Lemma 2.1),
we find it more convenient to discuss in the former way.

We point out that assumption $0 < M < 1 + \lambda_1$ or equivalently, $0 < m = M|\Omega| <
(1 + \lambda_1)|\Omega|$ is necessary for the exponentially decay property of solutions for (10)
in $L^2_0(\Omega) \times H^1(\Omega) \cap L^2_0(\Omega)$ which could be easily verified by similar linear stability
analysis as done in [12]. For the case $\Omega = B$ being a disk of any radius in $\mathbb{R}^2$,
it can be calculated that $\lambda_1 |B| \approx 1.84118^2\pi \approx 3.39\pi < 8\pi$ (cf. [1]). As is well
known, for $\gamma = 0$ and a disk with any radius, any initial datum with total mass
less than $8\pi$ should develop a globally bounded classical solution. However due
to our result, we need the radius to be large when the total mass is greater than
$\lambda_1 |B| \approx 3.39\pi$. The reason is that our argument strongly relies on the exponentially
decay property of the linearized system (also the solution obtained for the original
system has an exponentially decay property) and as we mentioned above, a strong
condition $0 < M < 1 + \lambda_1$ is then needed. Nevertheless, this smallness condition
links the dynamics of solutions explicitly with the geometric quantity $|\Omega|$ which is
seemingly new and of some interest. Roughly speaking, our result implies that with
any total mass there will be no overcrowding of cells provided that they initially
distribute almost homogeneous spatially and the bounded domain is large enough,
while the existing critical-mass theory is independent of the volume of the region.

At last, we would like to mention that our stability result can be easily gener-
ialized to the chemo-repulsion Keller–Segel system recently studied in [4] under no
smallness assumption on $M$ with slight modification of the proof. The reason is
that its corresponding linearized system exhibits an exponentially decay property
with no smallness restriction on $M$ (see Remark 1). Then one can remove the
smallness assumption in the $L^p - L^q$ decay estimates derived in Lemma 3.5–Lemma
3.6 and hence prove the stability result for the chemo-repulsion case. Recently in
[10], the author proposes a new method which strongly relies on the stability of
constant solutions in scaling-invariant spaces to study the eventual smoothness and
exponential stabilization of the weak solutions for the chemo-repulsion system in
higher dimensional settings.

The rest of this paper is organized as follows. In Section 2, we introduce some
useful lemmas which are needed in the subsequent proof. In Section 3, we analyze
the decay properties of the corresponding linearized systems in both $\gamma = 0$ and
$\gamma = 1$. Delicate $L^p - L^q$ estimates are given which are crucial in our proof. In the
last section, we prove our main results using the one-step contradiction argument
borrowed from [18, 3] with slight modification.

2. Preliminaries. First, we introduce the result on existence and uniqueness of
local classical solutions to the Keller–Segel system (4). The proof for the doubly
parabolic case $\gamma = 1$ can be found in [2, 8] and the simplified case $\gamma = 0$ can be
done in almost the same way except when dealing with the second elliptic equation,
classical elliptic theory together with Sobolev embeddings are needed.

Lemma 2.1 (Local well-posedness). Assume $\rho_0 \in C(\overline{\Omega})$ and $\gamma c_0 \in W^{1,\sigma}(\Omega)$ are
non-negative with $\sigma > d$. Then there exists $T_{\text{max}} > 0$ such that (4) possesses a
unique classical solution $(\rho, c) \in (C([0, T_{\text{max}}) \times \overline{\Omega}) \cap C^2,1(\overline{\Omega} \times (0, T_{\text{max}})))^2$ which is
non-negative. Moreover, $T_{\text{max}} < \infty$ if and only if
\[
\limsup_{t \nearrow T_{\text{max}}} \| \rho(\cdot, t) \|_{L^\infty(\Omega)} = \infty.
\]
For any $\theta > \frac{d}{4}$, if the solution of (4) satisfies
\[
\| \rho(\cdot, t) \|_{L^\theta(\Omega)} < \infty \quad \text{for all } t \in (0, T_{\text{max}}),
\]
then $T_{\text{max}} = \infty$, and there holds
\[
\sup_{t > 0} (\| \rho(\cdot, t) \|_{L^\infty(\Omega)} + \| c(\cdot, t) \|_{W^{1, \infty}(\Omega)}) < \infty.
\]

The following result shows that a lower-order perturbation to a sectorial operator is still a sectorial operator [21, 5].

**Lemma 2.2.** Suppose that $A$ is a sectorial operator and $B$ is a linear operator with $D(A) \subset D(B)$ such that for any $x \in D(A)$, there holds
\[
\| Bx \| \leq \varepsilon \| Ax \| + K_\varepsilon \| x \|
\]
where $\varepsilon > 0$ is an arbitrary small constant and $K_\varepsilon$ is a positive constant depending on $\varepsilon$. Then $A + B$ is sectorial.

The following lemma presents an estimate for frequently used integrals throughout this paper, the proof of which can be found in [18, 9].

**Lemma 2.3.** Suppose $0 < \alpha < 1$, $0 < \beta < 1$, $\gamma > 0$, $\delta > 0$ and $\gamma \neq \delta$. Then there holds
\[
\int_0^t (1 + (t - s)^{-\alpha}) e^{-\gamma(t-s)} (1 + s^{-\beta}) e^{-\delta s} ds \leq C(\alpha, \beta, \delta, \gamma)(1 + t^{\min(0, 1-\alpha-\beta)}) e^{-\min(\gamma, \delta) t}
\]
for all $t > 0$, where $C(\alpha, \beta, \delta, \gamma) = C \cdot \left( \frac{1}{|\delta - \gamma|} + \frac{1}{1-\alpha} + \frac{1}{1-\beta} \right)$ with $C > 0$ being a generic constant when $0 < t \leq 1$ or when $t > 1$ and $\alpha + \beta \geq 1$, while when $t > 1$ and $\alpha + \beta < 1$, the constant $C$ may also depend on $\left( \frac{2(1-\alpha-\beta)}{\alpha(\delta - \gamma)} \right)^{\frac{1-\alpha}{\alpha}}$.

Last, we recall the important $L^p-L^q$ estimates for the Neumann heat semigroup on bounded domains (see e.g., [3, 18]).

**Lemma 2.4.** Suppose $\{e^{t\Delta} \}_{t \geq 0}$ is the Neumann heat semigroup in $\Omega$, and $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in $\Omega$ under Neumann boundary conditions. Then there exist $k_1, \ldots, k_4 > 0$ which only depend on $\Omega$ such that the following properties hold:

(i) If $1 \leq q \leq p \leq \infty$, then
\[
\| e^{t\Delta} w \|_{L^p(\Omega)} \leq k_1 (1 + t^{-\frac{d}{2}(\frac{1}{q} - \frac{1}{p})}) e^{-\lambda_1 t} \| w \|_{L^q(\Omega)} \quad \text{for all } t > 0
\]
for all $w \in L^q_0(\Omega)$;

(ii) If $1 \leq q \leq p \leq \infty$, then
\[
\| \nabla e^{t\Delta} w \|_{L^p(\Omega)} \leq k_2 (1 + t^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{p})}) e^{-\lambda_1 t} \| w \|_{L^q(\Omega)} \quad \text{for all } t > 0
\]
for each $w \in L^q(\Omega)$;

(iii) If $2 \leq q \leq p < \infty$, then
\[
\| \nabla e^{t\Delta} w \|_{L^p(\Omega)} \leq k_3 e^{-\lambda_1 t} (1 + t^{-\frac{d}{2}(\frac{1}{q} - \frac{1}{p})}) \| \nabla w \|_{L^q(\Omega)} \quad \text{for all } t > 0
\]
for all $w \in W^{1,p}(\Omega)$;

(iv) If $1 < q < p \leq \infty$, then
\[
\| e^{t\Delta} w \|_{L^p(\Omega)} \leq k_4 e^{-\lambda_1 t} (1 + t^{-\frac{d}{2}(\frac{1}{q} - \frac{1}{p})}) \| w \|_{L^q(\Omega)} \quad \text{for all } t > 0
\]
for all $w \in L^q(\Omega)$.
(iv) If $1 < q \leq p \leq \infty$, then
\[
\|e^{t\Delta} \nabla \cdot w\|_{L^p(\Omega)} \leq k_4(1 + t^{-\frac{1}{2} - \frac{1}{q} + \frac{1}{q} - \frac{1}{p}})e^{-\lambda_1 t}\|w\|_{L^q(\Omega)} \quad \text{for all } t > 0 \tag{14}
\]
for any $w \in (W^{1,p}(\Omega))^d$.

3. Decay properties of the linearized systems. In this section, we try to establish $L^p - L^q$ estimates for the associated semigroups of linearized system (10).

3.1. The parabolic–elliptic case: $\gamma = 0$. In this part, we consider the simplified parabolic–elliptic case $\gamma = 0$. Now, the linearized problem reads
\[
\begin{aligned}
\dot{u}_t - \Delta \dot{u} + M\Delta \dot{v} &= 0 & x \in \Omega, \ t > 0 \\
-\Delta \dot{v} + \ddot{v} &= \ddot{u} & x \in \Omega, \ t > 0 \\
\frac{\partial \ddot{u}}{\partial \nu} &= 0, & x \in \partial \Omega, \ t > 0 \\
\ddot{u}(x, 0) &= u_0(x) & x \in \Omega,
\end{aligned}
\tag{15}
\]
or equivalently,
\[
\begin{aligned}
\ddot{u}_t - \Delta \ddot{u} + M\Delta (I - \Delta)^{-1}\ddot{u} &= 0 & x \in \Omega, \ t > 0 \\
\frac{\partial \ddot{u}}{\partial \nu} &= 0, & x \in \partial \Omega, \ t > 0 \\
\ddot{u}(x, 0) &= u_0(x) & x \in \Omega,
\end{aligned}
\]
since $I - \Delta$ is invertible on $L^p_0(\Omega)$ for any $p > 1$.

Denote $\mathcal{L} = \Delta - M\Delta (I - \Delta)^{-1}$ with domain $D(\mathcal{L}) = W^{2,p}_N(\Omega) \cap L^p_0(\Omega)$, where
\[
W^{2,p}_N(\Omega) := \{ w \in W^{2,p}(\Omega), \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial \Omega. \}
\]
Denoting $B = \Delta (I - \Delta)^{-1}$, then one easily checks that $B$ is a bounded operator on $L^p_0(\Omega)$. Indeed, since $\Delta$ generates a $C_0$-semigroup of contraction on $L^p_0(\Omega)$, we infer by the Hill-Yosida theorem that $\| (I - \Delta)^{-1} \|_{L^p} \leq 1$. It follows that
\[
\| \Delta (I - \Delta)^{-1} \|_{L^p} = \| I - (I - \Delta)^{-1} \|_{L^p} \leq 2.
\]
Then by Lemma 2.2, $\mathcal{L}$ is also a sectorial operator and thus generates an analytic semigroup on $L^p_0(\Omega)$, which is denoted by $e^{t\mathcal{L}}$ in the sequel.

Next, we show that the following exponentially decay estimates of $e^{t\mathcal{L}}$ in $L^2_0(\Omega)$.

Lemma 3.1. Suppose $0 < M < 1 + \lambda_1$. Then for any $u_0 \in L^2_0(\Omega)$, there holds
\[
\|e^{t\mathcal{L}} u_0\|_{L^2(\Omega)} \leq e^{-\mu_0 t}\|u_0\|_{L^2(\Omega)}, \quad \forall t \geq 0, \tag{16}
\]
where $\mu_0 \triangleq \lambda_1(1 - \frac{\lambda_1}{1 + \lambda_1})$.

Proof. We only perform the formal energy estimates here which could be easily justified by density argument. First, we note that
\[
\int_{\Omega} \dot{u} dx = \int_{\Omega} \dot{v} dx = 0.
\]
Now, multiplying the second equation of (15) by $-\Delta \ddot{v}$ and integrating by parts, we get
\[
\|\Delta \dot{v}\|_{L^2(\Omega)}^2 + \|\nabla \dot{v}\|_{L^2(\Omega)}^2 = \int_{\Omega} \nabla \ddot{u} \cdot \nabla \ddot{v} dx \leq \|\nabla \ddot{u}\|_{L^2(\Omega)} \|\nabla \ddot{v}\|_{L^2(\Omega)}.
\]
Then observing that by Poincaré’s inequality (6),
\[
\|\Delta \ddot{v}\|_{L^2(\Omega)}^2 \geq \lambda_1 \|\nabla \ddot{v}\|_{L^2(\Omega)}^2,
\]
we infer that
\[ \|\nabla \tilde{v}\|_{L^2(\Omega)} \leq \frac{1}{1 + \lambda_1} \|\nabla \tilde{u}\|_{L^2(\Omega)} \]
and hence
\[ \left| \int_{\Omega} \Delta \tilde{v} \, dx \right| \leq \frac{1}{1 + \lambda_1} \|\nabla \tilde{u}\|_{L^2(\Omega)}^2. \]
On the other hand, a multiplication of the first equation by \(\tilde{u}\) and an integration over \(\Omega\) yields that
\[ \frac{1}{2} \frac{d}{dt} \|\tilde{u}\|_{L^2(\Omega)}^2 + \|\nabla \tilde{u}\|_{L^2(\Omega)}^2 = -\mathcal{M} \int_{\Omega} \Delta \tilde{v} \, dx \]
\[ \leq \mathcal{M} \frac{1}{1 + \lambda_1} \|\nabla \tilde{u}\|_{L^2(\Omega)}^2. \]
Therefore, by Poincaré's inequality again, we have
\[ \frac{1}{2} \frac{d}{dt} \|\tilde{u}\|_{L^2(\Omega)}^2 + \lambda_1 (1 - \mathcal{M} \frac{1}{1 + \lambda_1}) \|\tilde{u}\|_{L^2(\Omega)}^2 \leq 0 \]
which yields
\[ \|\tilde{u}(t)\|_{L^2(\Omega)}^2 \leq \|u_0\|_{L^2(\Omega)}^2 e^{-2\mu_0 t}, \quad \forall t \geq 0 \quad (17) \]
with \(\mu_0 = \lambda_1 (1 - \mathcal{M} \frac{1}{1 + \lambda_1})\). This completes the proof.

Thanks to the exponentially decay property in \(L_0^p(\Omega)\), we are able to prove the following

**Lemma 3.2.** Suppose \(0 < \mathcal{M} < 1 + \lambda_1\). For any \(p > 1\), \(1 \leq q \leq p \leq \infty\) and \(u_0 \in L_0^p(\Omega)\), there holds
\[ \|e^{t\mathcal{L}} u_0\|_{L^p(\Omega)} \leq K_1 (1 + t^{-\frac{q}{2}}(\frac{1}{4} - \frac{1}{p})) e^{-\mu_0 t} \|u_0\|_{L^q(\Omega)}, \quad \forall t > 0, \quad (18) \]
where \(K_1 > 0\) depends only on \(d\) and \(\Omega\).

**Proof.** The proof consists of two parts. First, we derive for \(t \leq 1\), there exists \(K > 0\) depending only on \(d\) and \(\Omega\) such that
\[ \|e^{t\mathcal{L}} u_0\|_{L^p(\Omega)} \leq K t^{-\frac{q}{2}}(\frac{1}{4} - \frac{1}{p}) \|u_0\|_{L^q(\Omega)}. \]
Then for \(t \geq 1\), arguing in the same way as done in [18], invoking Lemma 3.1, we prove that
\[ \|e^{t\mathcal{L}} u_0\|_{L^p(\Omega)} \leq K e^{-\mu_0 t} \|u_0\|_{L^q(\Omega)} \]
with \(K > 0\) depending only on \(d\) and \(\Omega\). Then our assertion follows by combining the above two estimates.

By the variation-of-constants formula, we observe that
\[ \tilde{u}(t) = e^{t\mathcal{L}} u_0 - \mathcal{M} \int_0^t e^{(t-s)\mathcal{L}} \Delta \tilde{v}(s) \, ds. \]
Denoting \(\tilde{u}(t) = e^{t\mathcal{L}} u_0\), it follows from above and Lemma 2.4 that for \(t \leq 1\),
\[ \|e^{t\mathcal{L}} u_0\|_{L^p(\Omega)} = \left\| e^{t\mathcal{L}} u_0 - \mathcal{M} \int_0^t e^{(t-s)\mathcal{L}} \Delta \tilde{v}(s) \, ds \right\|_{L^p(\Omega)} \]
\[ \leq \|e^{t\mathcal{L}} u_0\|_{L^p(\Omega)} + k_1 \mathcal{M} \int_0^t e^{-\lambda_1 (t-s)} \|\Delta \tilde{v}(s)\|_{L^p(\Omega)} \, ds \]
\[ \leq k_1 t^{-\frac{q}{2}}(\frac{1}{4} - \frac{1}{p}) \|u_0\|_{L^q(\Omega)} + 2k_1 \mathcal{M} \int_0^t e^{-\lambda_1 (t-s)} \|\tilde{u}(s)\|_{L^p(\Omega)} \, ds, \]
since \( \Delta \hat{v} = \Delta (I - \Delta)^{-1} \hat{u} \) and \( \| \Delta (I - \Delta)^{-1} \|_{L^p(\Omega)} \leq 2 \) for any \( 1 < p < \infty \) which also holds true for \( p = \infty \). Indeed, noting that \( \| \hat{v} \|_{L^\infty(\Omega)} \leq \| \hat{u} \|_{L^\infty(\Omega)} \) by standard energy estimates, we infer that \( \| \Delta \hat{v} \|_{L^\infty(\Omega)} \leq \| \hat{u} \|_{L^\infty(\Omega)} + \| \hat{v} \|_{L^\infty(\Omega)} \leq 2 \| \hat{u} \|_{L^\infty(\Omega)} \).

Let \( y(t) = t^{\frac{d}{2} - \frac{1}{p}} \| \hat{u}(t) \|_{L^p(\Omega)} \). Then we derive that for \( t \leq 1 \)

\[
y(t) \leq k_1 \| u_0 \|_{L^q(\Omega)} + 2k_1 M \int_0^t e^{-\lambda_1 (t-s)} s^{-\frac{d}{2} \left( \frac{1}{n} - \frac{1}{p} \right)} y(s) ds
\leq k_1 \| u_0 \|_{L^q(\Omega)} + 2k_1 M \int_0^1 s^{-\frac{d}{2} \left( \frac{1}{n} - \frac{1}{p} \right)} y(s) ds.
\]

An application of Gronwall’s inequality yields that

\[
y(t) \leq k_1 \| u_0 \|_{L^q(\Omega)} \exp \left\{ 2k_1 M \int_0^t s^{-\frac{d}{2} \left( \frac{1}{n} - \frac{1}{p} \right)} ds \right\}
\leq k_1 \| u_0 \|_{L^q(\Omega)} \exp \left\{ \frac{2k_1 M}{1 - \frac{d}{2} \left( \frac{1}{n} - \frac{1}{p} \right)} t - \frac{d}{2} \left( \frac{1}{n} - \frac{1}{p} \right) \right\}
\leq k_1 \| u_0 \|_{L^q(\Omega)} e^{4k_1 M}
\]

provided that \( \frac{d}{q} > \frac{1}{d} - \frac{1}{p} \). Therefore, for \( t \leq 1 \) and \( \frac{1}{d} > \frac{1}{q} - \frac{1}{p} \),

\[
\| e^{tL} u_0 \|_{L^p(\Omega)} \leq K_2 t^{-\frac{d}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} \| u_0 \|_{L^q(\Omega)},
\]

where \( K_2 = k_1 e^{4k_1 M} < k_1 e^{4k_1 (1 + \lambda_1)} \).

Similarly, if \( \frac{1}{q} > \frac{1}{d} - \frac{1}{p} \), for some \( 2 \leq N \in \mathbb{N} \), we may find \( \{ q_j \}_{j=0}^{N-1} \) between \( q = q_N \) and \( p = q_0 \) such that \( \frac{1}{d} > \frac{1}{q_j+1} - \frac{1}{q_j} \) and hence

\[
\| e^{tL} u_0 \|_{L^p(\Omega)} \leq K_2 \left( \frac{t}{N} \right)^{-\frac{d}{2} \left( \frac{1}{q_j+1} - \frac{1}{q_j} \right)} \| e^{\frac{N-1}{2} t \frac{N}{q_j} L} u_0 \|_{L^{q_j}(\Omega)}
\leq K_2^2 \left( \frac{t}{N} \right)^{-\frac{d}{2} \left( \frac{1}{q_j+1} - \frac{1}{q_j} \right) \left( \frac{t}{N} \right) - \frac{d}{2} \left( \frac{1}{q_j+1} - \frac{1}{q_j} \right)} \| u_0 \|_{L^{q_j}(\Omega)}
\leq\ldots
\leq K_2^N \left( \frac{t}{N} \right)^{-\frac{d}{2} \left( \frac{1}{q_j+1} - \frac{1}{q_j} \right) \ldots \left( \frac{t}{N} \right) - \frac{d}{2} \left( \frac{1}{q_j+1} - \frac{1}{q_j} \right)} \| u_0 \|_{L^{q_j}(\Omega)}
\leq K_3 t^{-\frac{d}{2} \left( \frac{1}{q_j+1} - \frac{1}{q_j} \right)} \| u_0 \|_{L^{q_j}(\Omega)}, \tag{19}
\]

where \( K_3 = (K_2 \sqrt{N})^N \) since \( \frac{d}{2} \left( \frac{1}{q_j+1} - \frac{1}{q_j} \right) < \frac{N}{2} \).

Obviously, \( N \leq d + 1 \). Thus, there is \( K_3 > 0 \) depending on \( d \) and \( \Omega \) only such that for all \( p > 1 \) and \( 1 \leq q \leq p \leq \infty \), there holds for \( t \leq 1 \)

\[
\| e^{tL} u_0 \|_{L^p(\Omega)} \leq K_4 t^{-\frac{d}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} \| u_0 \|_{L^q(\Omega)} \tag{20}
\]

Now, for \( t \geq 1 \) and \( p \geq 2 \), we derive by (20) and Lemma 3.1 that

\[
\| e^{tL} u_0 \|_{L^p(\Omega)} \leq K_4 2^{\frac{d}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} \| e^{(t-\frac{1}{p})L} u_0 \|_{L^2(\Omega)}
\leq K_4 2^{\frac{d}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} e^{-\mu_0 (t-1)} \| e^{\frac{1}{2} L} u_0 \|_{L^2(\Omega)}
\leq K_4^2 2^{\frac{d}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} e^{-\mu_0 (t-1)} \times \max \{ 2^{\frac{d}{2} \left( \frac{1}{q} - \frac{1}{p} \right)}, |\Omega|^{\frac{d}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} \} \| u_0 \|_{L^q(\Omega)} \tag{21}
\]
and for $p < 2$, we have
\[
\|e^{t\mathcal{L}}u_0\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{2} - \frac{1}{p}} \|e^{t\mathcal{L}}u_0\|_{L^2(\Omega)} \\
\leq |\Omega|^{\frac{1}{2} - \frac{1}{p}} e^{-\mu_0(t-\frac{1}{2})} \|e^{\frac{t}{2}\mathcal{L}}u_0\|_{L^2(\Omega)} \\
\leq \text{K}_4 \|e^{\frac{t}{2}(\frac{1}{2} - \frac{1}{p})}|\Omega|^{\frac{1}{2} - \frac{1}{p}} e^{-\mu_0(t-\frac{1}{2})}\|u_0\|_{L^p(\Omega)}. \quad (22)
\]

Finally, a combination of (20), (21) and (22) completes the proof. \(\square\)

With minor modification and in the same manner as done in proof of Lemma 3.2, we can prove the following result for the special case when $u_0 = \nabla \cdot w$.

**Lemma 3.3.** Suppose $0 < \mathcal{M} < 1 + \lambda_1$. For any $1 < q \leq p \leq \infty$ and $u_0 = \nabla \cdot w$, there holds
\[
\|e^{t\mathcal{L}}u_0\|_{L^p(\Omega)} \leq \text{K}_5 (1 + e^{-\frac{1}{2} - \frac{1}{q}}) e^{-\mu_0 t} \|w\|_{L^q(\Omega)},
\]
where $\text{K}_5$ depends only on $d$ and $\Omega$.

### 3.2. The fully parabolic case: $\gamma = 1$.

In this part, we consider the case $\gamma = 1$. Similar as before, denote $(\tilde{u}, \tilde{v})$ the solution to the corresponding linearized system to (9). Then, $(\tilde{u}, \tilde{v})$ satisfies
\[
\begin{aligned}
\tilde{u}_t - \Delta \tilde{u} + \mathcal{M} \Delta \tilde{v} = 0, & \quad x \in \Omega, \ t > 0 \\
\tilde{v}_t - \Delta \tilde{v} + \tilde{v} = \tilde{u}, & \quad x \in \Omega, \ t > 0 \\
\frac{\partial \tilde{u}}{\partial \nu} = \frac{\partial \tilde{v}}{\partial \nu} = 0, & \quad x \in \partial \Omega, \ t > 0 \\
\tilde{u}(x, 0) = u_0(x), & \quad \tilde{v}(x, 0) = v_0(x) \quad x \in \Omega.
\end{aligned}
\]

Denote $\Delta$ the usual Laplacian operator with homogeneous Neumann boundary condition. Since $-\Delta$ is analytic on $L^p_0(\Omega)$ with domain $D_p(\Delta) = W^{2,p}_N(\Omega) \cap L^p_0(\Omega)$, we can define the power $(-\Delta)^s$ of $-\Delta$ for any $s \in \mathbb{R}$ and we denote the domain of $(-\Delta)^s$ in $L^p_0(\Omega)$ by $D_p((-\Delta)^s)$.

Let $\mathcal{X} = L^p_0(\Omega) \times D_p((-\Delta)^{\frac{1}{2}})$ for $1 < p < \infty$ with norm
\[
\|(u, v)\|_{\mathcal{X}} = \|u\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(\Omega)}
\]
and define
\[
\mathcal{A} = \begin{pmatrix}
\Delta & -\mathcal{M} \Delta \\
0 & \Delta - 1
\end{pmatrix}
\]
with domain $D(\mathcal{A}) = D_p(\Delta) \times D_p((-\Delta)^{\frac{1}{2}})$. We observe that
\[
\mathcal{A} = \begin{pmatrix}
\Delta & 0 \\
0 & \Delta - 1
\end{pmatrix} + \begin{pmatrix}
0 & -\mathcal{M} \Delta \\
1 & 0
\end{pmatrix} = \Lambda + \mathcal{U},
\]
where $\Lambda$ is a sectorial operator on $\mathcal{X}$. Moreover, one easily verifies that $D(\mathcal{A}) = D(\Lambda) \subset D(\mathcal{U}) = D_p((-\Delta)^{\frac{1}{2}}) \times D_p(\Delta)$ and for any $(u, v) \in D(\Lambda)$, by interpolation, there holds
\[
\mathcal{M} \|\Delta v\|_{L^p} + \|\nabla u\|_{L^p} \leq \varepsilon \left(\|\Delta u\|_{L^p} + \|\nabla \Delta v - \nabla v\|_{L^p}\right) + \text{K}_6 (\|u\|_{L^p} + \|\nabla v\|_{L^p}). \quad (25)
\]

Then Lemma 2.2 indicates that $\mathcal{A}$ is a sectorial operator as well. For the sake of convenience, we denote
\[
\begin{pmatrix}
\tilde{u}(t) \\
\tilde{v}(t)
\end{pmatrix} = e^{t\mathcal{A}} \begin{pmatrix}
u_0 \\
v_0
\end{pmatrix} = \Phi^t(u_0, v_0).
\]

Now, similar as before, we first prove the exponentially decay property for the semigroup $e^{t\mathcal{A}}$ in the Hilbert space $L^2_0 \times (H^1 \cap L^2_0)$. 

Lemma 3.4. Assume $0 < M < 1 + \lambda_1$. For any given initial data $u_0 \in L^2_0(\Omega)$ and $v_0 \in H^1(\Omega) \cap L^2_0(\Omega)$, the solution of (24) satisfies the following exponentially decay estimate
\[
\|\tilde{u}\|_{L^2(\Omega)}^2 + M\|\nabla \tilde{v}\|_{L^2(\Omega)}^2 \leq e^{-2\mu_1 t}(\|u_0\|_{L^2(\Omega)}^2 + M\|\nabla v_0\|_{L^2(\Omega)}^2) \quad \text{for all } t \geq 0,
\]
where $\mu_1 \triangleq \lambda_1 - \frac{1}{2}\left(\sqrt{4\lambda_1 M + 1} - 1\right) > 0$.

Proof. Here we only perform formal energy estimates which could be rigorously justified by density arguments. First, we observe that
\[
\int_\Omega \tilde{u}(t) dx = \int_\Omega \hat{\nu}(t) dx = 0.
\]
Multiplying the first equation by $\tilde{u}$ and the second equation by $-M\Delta \tilde{v}$, integrating by parts and adding the resultant up, we obtain that
\[
\frac{1}{2} \frac{d}{dt} \left(\|\tilde{u}\|_{L^2(\Omega)}^2 + M\|\nabla \tilde{v}\|_{L^2(\Omega)}^2\right) + M\left(\|\nabla \tilde{v}\|_{L^2(\Omega)}^2 + \|\Delta \tilde{v}\|_{L^2(\Omega)}^2\right) + \|\nabla \tilde{u}\|_{L^2(\Omega)}^2
\]
\[
= 2M \int_\Omega \Delta \tilde{u} \cdot \nabla \tilde{v} dx \leq \delta \|\tilde{u}\|_{L^2(\Omega)}^2 + \frac{M^2}{\delta} \|\nabla \tilde{v}\|_{L^2(\Omega)}^2.
\]
In views of Poincaré’s lemma, we infer that
\[
\frac{1}{2} \frac{d}{dt} \left(\|\tilde{u}\|_{L^2(\Omega)}^2 + M\|\nabla \tilde{v}\|_{L^2(\Omega)}^2\right) + M(\lambda_1 + 1 - \frac{M}{\delta})\|\nabla \tilde{v}\|_{L^2(\Omega)}^2 + \lambda_1(1 - \delta)\|\tilde{u}\|_{L^2(\Omega)}^2 \leq 0.
\]
Picking $\delta = \delta_0 \triangleq \frac{1}{2}\left(\frac{\sqrt{4\lambda_1 M + 1}}{\lambda_1} - \frac{1}{\lambda_1}\right)$, we get
\[
\lambda_1 + 1 - \frac{M}{\delta} = \lambda_1(1 - \delta) \equiv \mu_1 > 0
\]
whenever $M < 1 + \lambda_1$ holds. Here, $\delta$ is picked such that $\lambda_1 + 1 - \frac{M}{\delta} = \lambda_1(1 - \delta)$ holds and attains the maximum at $\delta_0$. Then the conclusion follows from solving an ordinary differential inequality.

Remark 1. In the chemo-repulsion case, i.e., when the first equation in (4) is replaced by $\rho_t - \Delta \rho = \nabla \cdot (\rho \nabla c)$. The corresponding linearized system reads
\[
\begin{align*}
\tilde{u}_t - \Delta \tilde{u} - M\Delta \tilde{v} &= 0, \quad x \in \Omega, \ t > 0 \\
\tilde{v}_t - \Delta \tilde{v} + \tilde{v} &= \tilde{u}, \quad x \in \Omega, \ t > 0.
\end{align*}
\]
Then, for any given initial data $u_0 \in L^2_0(\Omega)$ and $v_0 \in H^1(\Omega)$, the solution of (26) satisfies the following exponentially decay estimate
\[
\|\tilde{u}\|_{L^2(\Omega)}^2 + M\|\nabla \tilde{v}\|_{L^2(\Omega)}^2 \leq e^{-2\lambda_1 t}(\|u_0\|_{L^2(\Omega)}^2 + M\|\nabla v_0\|_{L^2(\Omega)}^2) \quad \text{for all } t \geq 0.
\]

Proof. Multiplying the first equation of (26) by $\tilde{u}$ and the second equation by $-M\Delta \tilde{v}$, integrating by parts and adding the resultants up, we obtain that
\[
\frac{1}{2} \frac{d}{dt} \left(\|\tilde{u}\|_{L^2(\Omega)}^2 + M\|\nabla \tilde{v}\|_{L^2(\Omega)}^2\right) + M\left(\|\nabla \tilde{v}\|_{L^2(\Omega)}^2 + \|\Delta \tilde{v}\|_{L^2(\Omega)}^2\right) + \|\nabla \tilde{u}\|_{L^2(\Omega)}^2 = 0.
\]
In view of Poincaré’s inequality (6), we infer that
\[ \| \nabla \tilde{v} \|_{L^2(\Omega)}^2 = \int\nabla (\tilde{v} - \overline{v}) \cdot \nabla \tilde{v} \, dx \]
\[ = - \int (\tilde{v} - \overline{v}) \Delta \tilde{v} \, dx \]
\[ \leq \|\tilde{v} - \overline{v}\|_{L^2(\Omega)} \|\Delta \tilde{v}\|_{L^2(\Omega)} \]
\[ \leq 1/\sqrt{\lambda_1} \|\nabla \tilde{v}\|_{L^2(\Omega)} \|\Delta \tilde{v}\|_{L^2(\Omega)} \]
which indicates that \( \lambda_1 \|\nabla \tilde{v}\|_{L^2(\Omega)}^2 \leq \|\Delta \tilde{v}\|_{L^2(\Omega)}^2 \) and hence
\[ \frac{1}{2} \frac{d}{dt} \left( \|\tilde{u}\|_{L^2(\Omega)}^2 + \mathcal{M} \|\nabla \tilde{v}\|_{L^2(\Omega)}^2 \right) + \mathcal{M} \|\nabla \tilde{v}\|_{L^2(\Omega)}^2 \left( 1 + \lambda_1 \right) + \lambda_1 \|\tilde{u}\|_{L^2(\Omega)}^2 \leq 0. \]

Then the conclusion follows by solving an ordinary differential inequality. \( \square \)

The next \( L^p - L^q \) estimates for \( e^{tA} \) plays a key role in the proof for the case \( \gamma = 1 \), which is established based on Lemma 2.4 and Lemma 3.4 by similar arguments as we done in the previous part. However, a coupling (linear) system is now under consideration. The presence of the cross diffusion and the restrictions on the parameter \( q \) in Lemma 2.4-(iii,iv) bring a lot of difficulties and hence the calculations here are more involved.

**Lemma 3.5.** Assume \( d \geq 2 \) and \( 0 < \mathcal{M} < 1 + \lambda_1 \). Then for any \( u_0 \in C(\overline{\Omega}) \cap L^1_0(\Omega) \) and \( v_0 \in C^1(\overline{\Omega}) \cap L^2_0(\Omega) \) satisfying \( \partial_r v_0 = 0 \) on \( \partial \Omega \), there holds
\[ \|\tilde{u}(t)\|_{L^p(\Omega)} \leq K_6 e^{-\mu_1 t} \left( 1 + t^{\frac{d}{2} - \frac{3}{4}} \right) \left( \|u_0\|_{L^{4/2}(\Omega)} + \|\nabla v_0\|_{L^q(\Omega)} \right) \quad \forall t > 0 \]
for any \( p > 1 \) satisfying \( \frac{d}{2} \leq p < \infty \), and
\[ \|\nabla \tilde{v}(t)\|_{L^p(\Omega)} \leq K_7 p e^{-\mu_1 t} \left( 1 + t^{\frac{d}{2} - \frac{3}{4}} \right) \left( \|u_0\|_{L^{4/2}(\Omega)} + \|\nabla v_0\|_{L^{4/2}(\Omega)} \right) \quad \forall t > 0 \]
for any \( p \) satisfying \( d \leq p < \infty \), where \( K_6, K_7 > 0 \) depend on \( d, \mathcal{M} \) and \( \Omega \) only.

**Proof.** The proof consists of several steps.

**Step 1.** We derive from variation-of-constants formula the following expressions for solutions of (24) for all \( t > 0 \)
\[ \tilde{u}(t) = e^{tA} u_0 - \mathcal{M} \int_0^t e^{(t-s)A} \Delta \tilde{v}(s) \, ds, \]
and
\[ \tilde{\tilde{\tilde{v}}}(t) = e^{t(A-1)} v_0 + \int_0^t e^{(t-s)(A-1)} \tilde{u}(s) \, ds. \]
Substituting \( \tilde{\tilde{\tilde{v}}} \) into the expression of \( \tilde{u} \) leads to
\[ \tilde{u}(t) = e^{tA} u_0 - \mathcal{M} \int_0^t e^{(t-s)A} \Delta e^{s(A-1)} v_0 \, ds - \mathcal{M} \int_0^t e^{(t-s)A} \frac{d}{dt} \int_0^s e^{(s-\tau)(A-1)} \tilde{u}(\tau) \, d\tau \, ds. \]

**Step 2.** We claim that when \( t \leq 1 \), for any \( 1 < p < \infty \) and any \( r \geq 2 \) satisfying \( \frac{1}{r} - \frac{1}{p} < \frac{1}{2} \), there holds
\[ \| \int_0^t e^{(t-s)A} \Delta e^{s(A-1)} v_0 \, ds \|_{L^p(\Omega)} \leq K \|\nabla v_0\|_{L^r(\Omega)} \]
with \( K \) depending on \( \Omega \) only.
In fact, if \(2 \leq r \leq p < \infty\), by Lemma 2.4 and Lemma 2.3, we infer that
\[
\| \int_0^t e^{(t-s)\Delta} \Delta e^{s(\Delta-1)} v_0 ds \|_{L^p(\Omega)}
\leq k_1 \int_0^t e^{-\lambda_1(t-s)} (1 + (t-s)^{-\frac{1}{2}}) \| \nabla e^{s(\Delta-1)} v_0 \|_{L^p(\Omega)} ds
\leq k_1 k_4 \int_0^t e^{-\lambda_1(t-s)} (1 + (t-s)^{-\frac{1}{2}}) e^{-s(\lambda_1+1)} (1 + s^{-\frac{1}{p}(\frac{1}{2}-\frac{1}{r}))} \| \nabla v_0 \|_{L^r(\Omega)} ds
\leq K \| \nabla v_0 \|_{L^r(\Omega)} / (1 - \frac{d}{2} (\frac{1}{r} - \frac{1}{p}))
\leq K \| \nabla v_0 \|_{L^r(\Omega)},
\]
where \(K > 0\) depends on \(\Omega\) only provided that \(0 \leq \frac{1}{2} - \frac{1}{p} < \frac{1}{r}\) (when \(r = p\), we calculate directly without using Lemma 2.3). On the other hand, if \(p < r\), applying Hölder’s inequality, we have
\[
\| \int_0^t e^{(t-s)\Delta} \Delta e^{s(\Delta-1)} v_0 ds \|_{L^p(\Omega)}
\leq |\Omega|^{\frac{1}{p} - \frac{1}{2}} \int_0^t \| e^{(t-s)\Delta} \Delta e^{s(\Delta-1)} v_0 \|_{L^r(\Omega)} ds
\leq k_3 k_4 |\Omega|^{\frac{1}{p} - \frac{1}{2}} \int_0^t e^{-\lambda_1(t-s)} (1 + (t-s)^{-\frac{1}{2}}) e^{-s(\lambda_1+1)} \| \nabla v_0 \|_{L^r(\Omega)} ds
\leq K \| \nabla v_0 \|_{L^r(\Omega)}.
\]

**Step 3.** Now, due to Lemma 2.4, Lemma 2.3 and (27), we infer for any \(p > 1\), \(1 \leq q \leq p < \infty\), \(r \geq 2\) and \(t \leq 1\) that
\[
\| \tilde{u}(t) \|_{L^p(\Omega)}
\leq \| e^{t\Delta} u_0 \|_{L^p(\Omega)} + M \int_0^t \| e^{(t-s)\Delta} \Delta e^{s(\Delta-1)} v_0 \|_{L^p(\Omega)} ds
+ M \int_0^t \| e^{(t-s)\Delta} \nabla \left( \nabla \int_0^s e^{(s-\tau)(\Delta-1)} \tilde{u}(\tau) d\tau \right) \|_{L^p(\Omega)} ds
\leq k_1 (1 + t^{-\frac{1}{2}} (\frac{1}{2} - \frac{1}{p})) \| u_0 \|_{L^q(\Omega)} + c \| \nabla v_0 \|_{L^r(\Omega)} + k_2 k_4 M J_1
\leq 2 k_1 t^{-\frac{1}{2}} (\frac{1}{2} - \frac{1}{p}) \| u_0 \|_{L^q(\Omega)} + c \| \nabla v_0 \|_{L^r(\Omega)} + k_2 k_4 M J_1
\]
provided that \(\frac{1}{2} - \frac{1}{p} < \frac{1}{r}\), where
\[
J_1(t) \triangleq \int_0^t \int_0^s e^{-\lambda_1(t-s)} (1 + (t-s)^{-\frac{1}{2}}) e^{-(\lambda_1+1)(s-\tau)} (1 + (s-\tau)^{-\frac{1}{2}}) \| \tilde{u}(\tau) \|_{L^r} d\tau ds.
\]

Letting \(y(t) = t^{\frac{1}{2} (\frac{1}{q} - \frac{1}{p})} \| \tilde{u}(t) \|_{L^p(\Omega)}\) and noticing by changing the order in integrations that
\[
\tilde{J}_1
\triangleq \int_0^t \int_0^s e^{-\lambda_1(t-s)} e^{-(\lambda_1+1)(s-\tau)} (1 + (t-s)^{-\frac{1}{2}}) (1 + (s-\tau)^{-\frac{1}{2}}) \tau^{-\frac{1}{2}} (\frac{1}{2} - \frac{1}{p}) y(\tau) d\tau ds
= \int_0^t \left[ \int_{\tau}^t e^{-\lambda_1(t-s)} e^{-(\lambda_1+1)(s-\tau)} (1 + (t-s)^{-\frac{1}{2}}) (1 + (s-\tau)^{-\frac{1}{2}}) ds \right] d\tau
\]
\[ \times \tau^{-\frac{q}{2}(\frac{1}{q}-\frac{1}{p})} y(\tau) \, d\tau \]
\[ \leq C \int_0^t e^{-\lambda_1(t-\tau)} \tau^{-\frac{q}{2}(\frac{1}{q}-\frac{1}{p})} y(\tau) \, d\tau \]
\[ \leq C \int_0^t \tau^{-\frac{q}{2}(\frac{1}{q}-\frac{1}{p})} y(\tau) \, d\tau, \]
we derive that for \( t \leq 1 \),
\[ y(t) \leq K \left( \| u_0 \|_{L^p(\Omega)} + K \| \nabla v_0 \|_{L^r(\Omega)} \right) + K M J_1 \]
\[ \leq K \left( \| u_0 \|_{L^p(\Omega)} + \| \nabla v_0 \|_{L^r(\Omega)} \right) + K M \int_0^t \tau^{-\frac{q}{2}(\frac{1}{q}-\frac{1}{p})} y(\tau) \, d\tau. \]

Thus, by Gronwall’s inequality we infer that for \( 0 \leq \frac{1}{q} - \frac{1}{p} < \frac{2}{d} \) and for \( t \leq 1 \),
\[ y(t) \leq K \left( \| u_0 \|_{L^p(\Omega)} + \| \nabla v_0 \|_{L^r(\Omega)} \right) \exp \{ K M \int_0^t \tau^{-\frac{q}{2}(\frac{1}{q}-\frac{1}{p})} \, d\tau \} \]
\[ \leq K \left( \| u_0 \|_{L^p(\Omega)} + \| \nabla v_0 \|_{L^r(\Omega)} \right) \exp \left\{ \frac{K M}{1 - \frac{q}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} \right\} \]

with \( K > 0 \) depending on \( \Omega \) only.

**Step 4.** Choosing \( r = q = l \geq 2 \) in (28), then for any \( l \leq p < \infty \) satisfying \( 0 \leq \frac{1}{l} - \frac{1}{p} < \frac{1}{2d} \), we deduce that for \( t \leq 1 \),
\[ \| \tilde{v}(t) \|_{L^p(\Omega)} \leq K t^{-\frac{q}{2}(\frac{1}{q}-\frac{1}{p})} \left( \| u_0 \|_{L^l(\Omega)} + \| \nabla v_0 \|_{L^l(\Omega)} \right) \]
with \( K \) depending on \( \Omega \) only. As a consequence, we deduce for \( t \leq 1 \) and \( 2 \leq l \leq p < \infty \) satisfying \( \frac{1}{l} - \frac{1}{p} < \frac{1}{2d} \) that
\[ \| \nabla v(t) \|_{L^p(\Omega)} \]
\[ \leq \| \nabla e^{(\Delta-1)s}v_0 \|_{L^p(\Omega)} + \int_0^t \| \nabla e^{(t-s)(\Delta-1)} \tilde{u}(s) \|_{L^p(\Omega)} \, ds \]
\[ \leq 2k_3 t^{-\frac{q}{2}(\frac{1}{q}-\frac{1}{p})} \| \nabla v_0 \|_{L^l(\Omega)} + k_2 \int_0^t e^{-(\lambda_1+1)(t-s)}(1 + (t-s)^{-\frac{1}{2}}) \| \tilde{u}(s) \|_{L^p(\Omega)} \, ds \]
\[ \leq 2k_3 t^{-\frac{q}{2}(\frac{1}{q}-\frac{1}{p})} \| \nabla v_0 \|_{L^l(\Omega)} \]
\[ + k_2 K \left( \| u_0 \|_{L^l(\Omega)} + \| \nabla v_0 \|_{L^l(\Omega)} \right) \int_0^t e^{-(\lambda_1+1)(t-s)}(1 + (t-s)^{-\frac{1}{2}}) s^{-\frac{q}{2}(\frac{1}{q}-\frac{1}{p})} \, ds \]
\[ \leq K t^{-\frac{q}{2}(\frac{1}{q}-\frac{1}{p})} \left( \| u_0 \|_{L^l(\Omega)} + \| \nabla v_0 \|_{L^l(\Omega)} \right) \]
with \( K \) depending on \( \Omega \) only, since by Lemma 2.3 again, when \( t \leq 1 \) and \( 0 \leq \frac{1}{l} - \frac{1}{p} < \frac{1}{2d} \),
\[ \int_0^t e^{-(\lambda_1+1)(t-s)}(1 + (t-s)^{-\frac{1}{2}}) s^{-\frac{q}{2}(\frac{1}{q}-\frac{1}{p})} \, ds \]
\[ \leq e^{\lambda_1} \int_0^t e^{-(\lambda_1+1)(t-s)}(1 + (t-s)^{-\frac{1}{2}}) s^{-\frac{q}{2}(\frac{1}{q}-\frac{1}{p})} e^{-\lambda_1 s} \, ds \]
\[ \leq K(1 + t \min \left( \frac{1}{l} - \frac{1}{p}, 0 \right) ) \]
\[ \leq K \leq K t^{-\frac{q}{2}(\frac{1}{q}-\frac{1}{p})}. \]
Summing up, we have for $t \leq 1$ and any $2 \leq l \leq p < \infty$ satisfying $0 \leq \frac{1}{r} - \frac{1}{p} < \frac{1}{2^d}$ that

$$
\|\ddot{u}(t)\|_{L^p(\Omega)} + \|\nabla \ddot{v}(t)\|_{L^p(\Omega)} \leq K t^{-\frac{d}{2}(\frac{1}{r} - \frac{1}{p})} \left(\|u_0\|_{L^1(\Omega)} + \|\nabla v_0\|_{L^1(\Omega)}\right)
$$

(29)

with $K$ depending on $\Omega$ only.

On the other hand, for any $2 \leq l \leq p < \infty$ such that $\frac{1}{r} - \frac{1}{p} \geq \frac{1}{2^d}$, we may split $(\frac{1}{p}, \frac{1}{r})$ and $(0, t)$ evenly into $N$-intervals, respectively, with $N = d + 1$. Denoting the end-points for $N$-intervals of $(\frac{1}{p}, \frac{1}{r})$ by $\frac{1}{p} = \frac{1}{t_0} < \frac{1}{t_1} < \ldots < \frac{1}{t_N} = \frac{1}{r}$, then $\frac{1}{r_{j+1}} - \frac{1}{r_j} < \frac{1}{2^d}$ and by iteration, we deduce that for $t \leq 1$

$$
\|\ddot{u}(t)\|_{L^p(\Omega)} + \|\nabla \ddot{v}(t)\|_{L^p(\Omega)}
\leq K\left(\frac{t}{N}\right)^{-\frac{d}{2}(\frac{1}{r} - \frac{1}{p})} \left(\|\ddot{u}(\frac{N-1}{N} t)\|_{L^1(\Omega)} + \|\nabla \ddot{v}(\frac{N-1}{N} t)\|_{L^1(\Omega)}\right)
\leq \ldots
\leq K^N \left(\frac{t}{N}\right)^{-\frac{d}{2}(\frac{1}{r} - \frac{1}{p})} (\|u_0\|_{L^1(\Omega)} + \|\nabla v_0\|_{L^1(\Omega)})
\leq K t^{-\frac{d}{2}(\frac{1}{r} - \frac{1}{p})} (\|u_0\|_{L^1(\Omega)} + \|\nabla v_0\|_{L^1(\Omega)})
$$

(30)

with $K$ depending on $d$ and $\Omega$ only.

**Step 5.** Now taking $q = \frac{d}{2}$ and $r = d$ in (28), then for any $p > 1$ and $\frac{d}{2} \leq p \leq d$, we deduce from (28) that

$$
\|\ddot{u}(t)\|_{L^p(\Omega)} \leq K t^{-\frac{d}{2}(\frac{3}{2} - \frac{1}{\lambda})} (\|u_0\|_{L^{d/2}(\Omega)} + \|\nabla v_0\|_{L^\lambda(\Omega)}) \quad \text{for } t \leq 1
$$

(31)

where $K$ depends on $\Omega$. It follows that for $t \leq 1$,

$$
\|\nabla v(t)\|_{L^\lambda(\Omega)}
\leq \|\nabla e^{(\lambda-1)t} v_0\|_{L^\lambda(\Omega)} + \int_0^t \|\nabla e^{(\lambda-t-s)(\lambda-1)} \ddot{u}(s)\|_{L^\lambda(\Omega)} ds
\leq 2k_3 \|v_0\|_{L^\lambda(\Omega)} + k_2 \int_0^t e^{-(\lambda-1)(t-s)} (1 + (t-s)^{-\frac{1}{2}}) \|\ddot{u}(s)\|_{L^\lambda(\Omega)} ds
\leq 2k_3 \|v_0\|_{L^\lambda(\Omega)}
+ k_2 K (\|u_0\|_{L^{d/2}(\Omega)} + \|\nabla v_0\|_{L^{d}(\Omega)}) \int_0^t e^{-(\lambda-1)(t-s)} (1 + (t-s)^{-\frac{1}{2}}) s^{-\frac{1}{2}} ds
\leq K (\|u_0\|_{L^{d/2}(\Omega)} + \|\nabla v_0\|_{L^\lambda(\Omega)}).
$$

(32)

When $d < p < \infty$, thanks to (29), (30), (31) and (32), we infer that

$$
\|\ddot{u}(t)\|_{L^p(\Omega)} \leq K (t/2)^{-\frac{d}{2}(\frac{1}{r} - \frac{1}{p})} (\|\ddot{u}(t/2)\|_{L^\lambda(\Omega)} + \|\nabla \ddot{v}(t/2)\|_{L^\lambda(\Omega)})
\leq K (t/2)^{-\frac{d}{2}(\frac{1}{r} - \frac{1}{p})} (t/2)^{-\frac{d}{2}(\frac{3}{2} - \frac{1}{\lambda})} (\|u_0\|_{L^{d/2}(\Omega)} + \|\nabla v_0\|_{L^\lambda(\Omega)})
\leq K t^{-\frac{d}{2}(\frac{3}{2} - \frac{1}{\lambda})} (\|u_0\|_{L^{d/2}(\Omega)} + \|\nabla v_0\|_{L^\lambda(\Omega)}) \quad \text{for } t \leq 1
$$

with $K$ depending on $d$ and $\Omega$. Summing up, we conclude that (31) holds for any $p > 1$ satisfying $\frac{d}{2} \leq p < \infty$ with $K > 0$ depending on $\Omega$ and $d$ at most.
As a consequence, for $t \leq 1$ and any $d \leq p < \infty$,
\[
\|\nabla \tilde{v}(t)\|_{L^p(\Omega)} \\
\leq \|\nabla e^{t(\Delta-1)}v_0\|_{L^p(\Omega)} + \int_0^t \|\nabla e^{(t-s)(\Delta-1)}\tilde{u}(s)\|_{L^p(\Omega)} ds \\
\leq 2k_3t^{-\frac{d-2}{2}}\|\nabla v_0\|_{L^d(\Omega)} + k_2 \int_0^t e^{-(\lambda_1+1)(t-s)}(1 + (t-s)^{-\frac{1}{2}})\|\tilde{u}(s)\|_{L^p(\Omega)} ds \\
\leq 2k_3t^{-\frac{d-2}{2}}\|\nabla v_0\|_{L^d(\Omega)} \\
+ k_2K(\|u_0\|_{L^{4d}(\Omega)} + \|\nabla v_0\|_{L^d}) \int_0^t e^{-(\lambda_1+1)(t-s)}(1 + (t-s)^{-\frac{1}{2}})s^{-\frac{d-2}{2}}(\frac{d}{2} - \frac{1}{2}) ds \\
\leq K(1 + p)t^{-\frac{d-2}{2}}(\|u_0\|_{L^{4d}(\Omega)} + \|\nabla v_0\|_{L^d(\Omega)})
\] (33)

with $K$ depending on $\Omega$ and $d$, since by Lemma 2.3, when $t \leq 1$
\[
\int_0^t e^{-(\lambda_1+1)(t-s)}(1 + (t-s)^{-\frac{1}{2}})s^{-\frac{d-2}{2}}(\frac{d}{2} - \frac{1}{2}) ds \\
\leq e^{\lambda_1} \int_0^t e^{-(\lambda_1+1)(t-s)}(1 + (t-s)^{-\frac{1}{2}})s^{-\frac{d-2}{2}}(\frac{d}{2} - \frac{1}{2}) e^{\lambda_1} ds \\
\leq K\frac{p}{d} \leq K\frac{p}{d} t^{-\frac{d-2}{2}}(\frac{d}{2} - \frac{1}{2}).
\]

**Step 6.** Now, for $t \geq 1$, when $p < 2$, using Hölder’s inequality and Lemma 3.4, we find
\[
\|\tilde{u}(t)\|_{L^p(\Omega)} + \sqrt{M}\|\nabla \tilde{v}(t)\|_{L^p(\Omega)} \\
\leq K|\Omega|^\frac{1}{p} - \frac{1}{2} \left(\|\tilde{u}(t)\|_{L^2(\Omega)} + \sqrt{M}\|\nabla \tilde{v}(t)\|_{L^2(\Omega)}\right) \\
\leq K|\Omega|^\frac{1}{p} - \frac{1}{2} e^{-\mu_1(t-\frac{1}{2})} \left(\|\tilde{u}(\frac{1}{2})\|_{L^2(\Omega)} + \sqrt{M}\|\nabla \tilde{v}(\frac{1}{2})\|_{L^2(\Omega)}\right) \\
\leq K|\Omega|^\frac{1}{p} - \frac{1}{2} |\Omega|^\frac{1}{2} - \frac{1}{2} e^{-\mu_1(t-\frac{1}{2})} \left(\|\tilde{u}(\frac{1}{2})\|_{L^d(\Omega)} + \sqrt{M}\|\nabla \tilde{v}(\frac{1}{2})\|_{L^d(\Omega)}\right) \\
\leq K(1 + \sqrt{M})e^{-\mu_1 t} \left(\|u_0\|_{L^{4d}(\Omega)} + \|\nabla v_0\|_{L^d(\Omega)}\right)
\] (34)

with $K$ depending on $d$ and $\Omega$ at most, and when $2 \leq p < \infty$, we derive from (30) and Lemma 3.4 that
\[
\|\tilde{u}(t)\|_{L^p(\Omega)} + \|\nabla \tilde{v}(t)\|_{L^p(\Omega)} \\
\leq K \left(\|\tilde{u}(t-\frac{1}{2})\|_{L^2(\Omega)} + \|\nabla \tilde{v}(t-\frac{1}{2})\|_{L^2(\Omega)}\right) \\
\leq \frac{K}{\sqrt{M}} \left(\|\tilde{u}(t-\frac{1}{2})\|_{L^2(\Omega)} + \sqrt{M}\|\nabla \tilde{v}(t-\frac{1}{2})\|_{L^2(\Omega)}\right) \\
\leq \frac{K}{\sqrt{M}} e^{-\mu_1(t-1)} \left(\|\tilde{u}(\frac{1}{2})\|_{L^2(\Omega)} + \sqrt{M}\|\nabla \tilde{v}(\frac{1}{2})\|_{L^2(\Omega)}\right) \\
\leq \frac{K}{\sqrt{M}} |\Omega|^\frac{1}{2} - \frac{1}{2} e^{-\mu_1(t-1)} \left(\|\tilde{u}(\frac{1}{2})\|_{L^d(\Omega)} + \sqrt{M}\|\nabla \tilde{v}(\frac{1}{2})\|_{L^d(\Omega)}\right)
\]
for any $t$ depends on $\mathcal{M}$ and $\Omega$ only. Finally, we may conclude the proof by combining (31), (33), (34) and (35). □

Remark 2. Under the assumption of Lemma 3.5, for any $2 \leq l \leq p < \infty$, there holds

$$
\|\tilde{u}(t)\|_{L^p(\Omega)} + \|\nabla \tilde{v}(t)\|_{L^p(\Omega)} \leq K e^{-\mu_1 t} (1 + t^{-\frac{d}{2} + \frac{1}{q} - \frac{1}{p}}) (\|u_0\|_{L^1(\Omega)} + \|\nabla v_0\|_{L^1(\Omega)})
$$

(36)

with $K$ depending on $d$ and $\Omega$ only.

For the special case $u_0 = \nabla \cdot w$ and $v_0 = 0$, we have the following

Lemma 3.6. Assume $d \geq 2$ and $0 < \mathcal{M} < 1 + \lambda_1$. Suppose $u_0 = \nabla \cdot w$ and $v_0 = 0$. Then there holds

$$
\|\tilde{u}(t)\|_{L^p(\Omega)} \leq K_8 e^{-\mu_1 t} (1 + t^{-\frac{1}{2} - \frac{d}{q} + \frac{1}{q} - \frac{1}{p}}) \|w\|_{L^q(\Omega)}
$$

for any $\frac{d}{q} < q < p < \infty$, where $K_8 > 0$ depends on $d$, $\mathcal{M}$ and $\Omega$ if $d \geq 3$ and also depends on $1/|q-1|$ if $d = 2$.

Proof. Under our assumption, for any $t \leq 1$, in the same way as before, we infer for any $1 < q \leq p < \infty$ that

$$
\|\tilde{u}(t)\|_{L^p(\Omega)} \leq 2k_4 t^{-\frac{1}{2} - \frac{d}{q} + \frac{1}{q} - \frac{1}{p}} \|w\|_{L^q(\Omega)} + k_2 k_4 \mathcal{M} J_2
$$

(37)

where

$$
J_2 \triangleq \int_0^t \int_0^s e^{-\lambda_1 (t-s)} (1 + (s-t)^{-\frac{1}{2}}) e^{-(\lambda_1 + 1)(s-t)} (1 + (s-t)^{-\frac{1}{2}}) (1 + (s-t)^{-\frac{1}{2}}) \|\tilde{u}(\tau)\|_{L^p(\Omega)} d\tau ds.
$$

Letting $y(t) = t^{\frac{1}{2} + \frac{d}{2} + \frac{1}{q} - \frac{1}{p}} \|\tilde{u}(t)\|_{L^q(\Omega)}$ and noticing by Lemma 2.3 that

$$
\tilde{J}_2(t) \triangleq \int_0^t \int_0^s e^{-\lambda_1 (t-s)} e^{-(\lambda_1 + 1)(s-t)} (1 + (s-t)^{-\frac{1}{2}}) (1 + (s-t)^{-\frac{1}{2}}) \|\tilde{u}(\tau)\|_{L^q(\Omega)} d\tau ds
$$

$$
= \int_0^t \left[ \int_\tau^t e^{-\lambda_1 (t-s)} e^{-(\lambda_1 + 1)(s-t)} (1 + (t-s)^{-\frac{1}{2}}) (1 + (s-t)^{-\frac{1}{2}}) (1 + (s-t)^{-\frac{1}{2}}) ds \right]
$$

$$
\times \tau^{-\frac{1}{2} - \frac{d}{2} + \frac{1}{q} - \frac{1}{p}} y(\tau) d\tau
$$

$$
\leq C \int_0^t e^{-\lambda_1 (t-\tau)} \tau^{-\frac{1}{2} - \frac{d}{2} + \frac{1}{q} - \frac{1}{p}} y(\tau) d\tau,
$$

we find that for $t \leq 1$,

$$
y(t) \leq K \|w\|_{L^q(\Omega)} + K \mathcal{M} \tilde{J}_2 \leq K \|w\|_{L^q(\Omega)} + K \mathcal{M} \int_0^1 \tau^{-\frac{1}{2} - \frac{d}{q} - \frac{1}{p}} y(\tau) d\tau.
$$

(38)

Observe that for $\frac{1}{q} - \frac{1}{p} < \frac{1}{2}$,

$$
\int_0^1 \tau^{-\frac{1}{2} - \frac{d}{q} - \frac{1}{p}} d\tau \leq \frac{K}{\frac{1}{2} - \frac{d}{q} - \frac{1}{p}}.
$$
with $K$ depending on $\Omega$ only. As a result, we deduce from Gronwall’s inequality that for $t \leq 1$ and $\frac{1}{q} - \frac{1}{p} < \frac{1}{2d}$.

$$\|\tilde{u}(t)\|_{L^p(\Omega)} \leq Kt^{-\frac{1}{2} - \frac{d}{2} \left(\frac{1}{q} - \frac{1}{p}\right)} \|w\|_{L^q(\Omega)} \tag{39}$$

with $K$ depending on $\Omega$. It follows that for $t \leq 1$ and $\frac{1}{q} - \frac{1}{p} < \frac{1}{2d}$

$$\|\nabla \tilde{v}(t)\|_{L^p(\Omega)} \leq \int_0^t \|\nabla e^{(t-s)(\Delta - 1)} \tilde{u}(s)\|_{L^p(\Omega)} ds$$

$$\leq k_2 \int_0^t e^{-(\lambda_1 + 1)(t-s)}(1 + (t-s)^{-\frac{1}{2}})\|\tilde{u}(s)\|_{L^p(\Omega)} ds$$

$$\leq k_2 K \|w\|_{L^q(\Omega)} \int_0^t e^{-(\lambda_1 + 1)(t-s)}(1 + (t-s)^{-\frac{1}{2}})s^{-\frac{1}{2} - \frac{d}{2} \left(\frac{1}{q} - \frac{1}{p}\right)} ds$$

$$\leq K t^{-\frac{1}{2} - \frac{d}{2} \left(\frac{1}{q} - \frac{1}{p}\right)} \|w\|_{L^q(\Omega)}, \tag{40}$$

since for $t \leq 1$, there holds

$$\int_0^t e^{-(\lambda_1 + 1)(t-s)}(1 + (t-s)^{-\frac{1}{2}})s^{-\frac{1}{2} - \frac{d}{2} \left(\frac{1}{q} - \frac{1}{p}\right)} ds$$

$$\leq e^{\lambda_1} \int_0^t e^{-(\lambda_1 + 1)(t-s)}(1 + (t-s)^{-\frac{1}{2}})s^{-\frac{1}{2} - \frac{d}{2} \left(\frac{1}{q} - \frac{1}{p}\right)} e^{-\lambda_1 s} ds$$

$$\leq K(1 + t^{-\frac{d}{2} \left(\frac{1}{q} - \frac{1}{p}\right)})$$

$$\leq K t^{-\frac{1}{2} - \frac{d}{2} \left(\frac{1}{q} - \frac{1}{p}\right)}.$$

On the other hand, note that our assumption ensures $0 \leq \frac{1}{q} - \frac{1}{p} < \frac{2}{3}$. Therefore, for the case $\frac{1}{q} - \frac{1}{p} \geq \frac{1}{2d}$, we may split $(\frac{1}{p}, \frac{1}{q})$ into four parts evenly with endpoints denoted by $\frac{1}{q} = \frac{1}{r_0} < \frac{1}{r_1} < \ldots < \frac{1}{r_d} = \frac{1}{q}$ such that $0 < \frac{1}{r_{i+1}} - \frac{1}{r_i} < \frac{1}{2d}$.

Now, we divide our discussion into three cases regarding the dimensions. First, if $d \geq 4$, there holds $2 < q \leq p < \infty$. Then we may use (30) to derive that

$$\|\tilde{u}(t)\|_{L^p(\Omega)} + \|\nabla \tilde{v}(t)\|_{L^p(\Omega)} \leq K(t/2)^{-\frac{d}{2} \left(\frac{1}{q} - \frac{1}{p}\right)}(\|\tilde{u}(t/2)\|_{L^q(\Omega)} + \|\nabla \tilde{v}(t/2)\|_{L^q(\Omega)})$$

$$\leq K(t/2)^{-\frac{d}{2} \left(\frac{1}{q} - \frac{1}{p}\right)}(t/2)^{-\frac{1}{2} - \frac{d}{2} \left(\frac{1}{q} - \frac{1}{p}\right)} \|w\|_{L^q(\Omega)}$$

$$\leq K t^{-\frac{1}{2} - \frac{d}{2} \left(\frac{1}{q} - \frac{1}{p}\right)} \|w\|_{L^q(\Omega)}.$$

Secondly, if $d = 3$ and $\frac{1}{q} - \frac{1}{p} \geq \frac{1}{2d}$, we must have $p > 2$ since $q > \frac{d}{2} = \frac{3}{2}$. If $2 < q \leq p$, the proof is the same as above. If $1 < q \leq 2 < p$, we infer by the fact $\frac{1}{q} - \frac{1}{2} < \frac{3}{4} - \frac{1}{2} = \frac{1}{4} = \frac{1}{2d}$ that

$$\|\tilde{u}(t)\|_{L^p(\Omega)} + \|\nabla \tilde{v}(t)\|_{L^p(\Omega)} \leq K(t/2)^{-\frac{d}{2} \left(\frac{1}{q} - \frac{1}{p}\right)}(\|\tilde{u}(t/2)\|_{L^q(\Omega)} + \|\nabla \tilde{v}(t/2)\|_{L^q(\Omega)})$$

$$\leq K t^{-\frac{1}{2} - \frac{d}{2} \left(\frac{1}{q} - \frac{1}{p}\right)}(t/2)^{-\frac{1}{2} - \frac{d}{2} \left(\frac{1}{q} - \frac{1}{p}\right)} \|w\|_{L^q(\Omega)}$$

$$\leq K t^{-\frac{1}{2} - \frac{d}{2} \left(\frac{1}{q} - \frac{1}{p}\right)} \|w\|_{L^q(\Omega)}.$$

Lastly, if $d = 2$, we have $1 < q \leq p < \infty$. The cases $2 < q \leq p$ and $\frac{3}{4} < q \leq 2 < p$ can be dealt with in the same way as above. It remains to consider the case $\frac{1}{2} < \frac{1}{p} < \frac{3}{4} < \frac{1}{q} < 1$ and the case $\frac{1}{p} \leq \frac{1}{2} < \frac{3}{4} < \frac{1}{q} < 1$. In the former case, we infer from
Proof of Theorem 1.1 and Theorem 1.2. With the key $L^p - L^q$ decay estimates established in Lemmas 3.2–3.3 and Lemmas 3.5–3.6, it remains to complete our proof by an adaptation of the one-step contradiction argument from [18, 3] or using the implicit function theory as done in [14]. In this paper, we choose the former way for convenience since we already have Lemma 2.1 at hand. The main idea is to compute the difference between the solution of the nonlinear problem and the one to the corresponding linearized problem. Since the nonlinear problem can be regarded as its linearized one with a quadratic perturbation, with small initial data, the difference between their solutions should also be of a quadratic order. The major difference is now $\mathcal{M} > 0$, we have to compare the associated nonlinear semigroup with the linearized ones $e^{t\mathcal{C}}$ or $e^{t\mathcal{A}}$ while when $\mathcal{M} = 0$, we only have to compute its difference with $e^{t\mathcal{A}}$ whose decay behavior is well-known as shown in Lemma 2.4.

Now, we report the proof in detail as follows.

**Proposition 1.** Suppose $d \geq 2$ and $0 < \mathcal{M} < 1 + \lambda_1$. For any fixed $q_0 \in (\frac{d}{2}, d)$ and $\lambda' < \mu_0$, there exists $\varepsilon_0 > 0$ depending on $d$, $\Omega$, $q_0$ and $\lambda'$ such that for any initial datum $u_0 \in C(\overline{\Omega}) \cap L^0_0(\Omega)$ satisfying $u_0 + \mathcal{M} \geq 0$ and $\|u_0\|_{L^{d/2}(\Omega)} \leq \varepsilon$ for
some $\varepsilon < \varepsilon_0$, problem (9) with $\gamma = 0$ has global classical solution which is globally bounded and satisfies
\[
\|u(t) - e^{t\mathcal{L}}u_0\|_{L^p(\Omega)} \leq \varepsilon e^{-\lambda't}(1 + t^{-1 + \frac{d}{p'}}), \quad \forall t > 0,
\] (43)
for all $\theta \in [q_0, \infty]$.

**Proof.** According to Lemma 2.1, problem (4) with $\gamma = 0$ and nonnegative initial data $\rho_0 = u_0 + \mathcal{M}$ has a unique classical solution on $[0, T_{\text{max}}]$ and if $T_{\text{max}} < \infty$, we have $\limsup_{t \uparrow T_{\text{max}}} \|\rho(\cdot, t)\|_{L^\infty} = \infty$. Therefore, for problem (9) with $\gamma = 0$, we obtain a classical solution $(u, v) = (\rho - \mathcal{M}, c - \mathcal{M})$ on $[0, T_{\text{max}}]$ and if $T_{\text{max}} < \infty$, we have $\limsup_{t \uparrow T_{\text{max}}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$.

For fixed $\frac{d}{2} < q_0 < d$ and $d < \theta_0 < \frac{d q_0}{d - q_0}$, we set
\[
T_0 := \sup \left\{ T > 0 : \|u(t) - e^{t\mathcal{L}}u_0\|_{L^p(\Omega)} \leq \varepsilon e^{-\lambda't}(1 + t^{-1 + \frac{d}{p'}}), \right.\
\left. \text{for all } t \in [0, T) \text{ and all } \theta \in [q_0, \infty] \right\}.
\]
Then $T_0$ is well-defined and positive with $T_0 \leq T_{\text{max}}$, because both $u(t)$ and $e^{t\mathcal{L}}u_0$ are bounded near $t = 0$ due to Lemma 2.1 and Lemma 3.2, while on the other hand as $t \to 0^+$, $t^{-1 + \frac{d}{p'}} \geq t^{-1 + \frac{d}{p_0}} \to +\infty$ uniformly with respect to $\theta \in [q_0, \infty]$. Now we claim that when $\varepsilon_0$ is sufficiently small, we have $T_0 = \infty$. First, we observe that by Lemma 3.2,
\[
\|u(t)\|_{L^p(\Omega)} \leq \|u(t) - e^{t\mathcal{L}}u_0\|_{L^p(\Omega)} + \|e^{t\mathcal{L}}u_0\|_{L^p(\Omega)} \\
\leq \varepsilon (1 + t^{-1 + \frac{d}{p'}}) e^{-\lambda't} + K_1 (1 + t^{-1 + \frac{d}{p'}}) e^{-\mu t} \|u_0\|_{L^\theta(\Omega)} \\
\leq C_1 \varepsilon (1 + t^{-1 + \frac{d}{p'}}) e^{-\lambda't}
\]
holds for all $0 < t < T_0$ where $C_1$ depends on $d$ and $\Omega$. On the other hand, by Sobolev embedding theorem and the classical theory for elliptic equations, we have
\[
\|\nabla v(t)\|_{L^{q_2}(\Omega)} \leq C_2 \|v(t)\|_{W^{2,q_0}(\Omega)} \leq C_2 \|u(t)\|_{L^{q_0}(\Omega)}
\]
(44)
with $\frac{1}{q_0} = \frac{1}{2} + \frac{1}{q_2}$ and $C_2$ depending on $d$, $\Omega$ and $q_0$.

Due to the variation-of-constants formula, there holds
\[
u(t) - e^{t\mathcal{L}}u_0 = - \int_0^t e(t-s)\mathcal{L} \nabla \cdot (u(s) \nabla v(s)) ds.
\]
(45)
Now we estimate the term on right hand-side of (45). Invoking Lemma 3.3, for $\theta \in (\theta_0, \infty)$, there holds
\[
\|u(t) - e^{t\mathcal{L}}u_0\|_{L^\theta(\Omega)} \\
\leq \left\| \int_0^t e(t-s)\mathcal{L} \nabla \cdot (u(s) \nabla v(s)) ds \right\|_{L^\theta(\Omega)} \\
\leq K_5 \int_0^t e^{-\mu_0(t-s)} (1 + (t-s)^{-\frac{1}{2} - \frac{d}{2} (\frac{1}{p_0} - \frac{1}{\theta})}) \|u\nabla v\|_{L^{q_0}(\Omega)} ds \\
\leq K_5 \int_0^t e^{-\mu_0(t-s)} (1 + (t-s)^{-\frac{1}{2} - \frac{d}{2} (\frac{1}{p_0} - \frac{1}{\theta})}) \|u(s)\|_{L^{q_1}(\Omega)} \|\nabla v(s)\|_{L^{q_2}(\Omega)} ds
\]
\[ K_5 C_2 \int_0^t e^{-\mu_0(t-s)}(1 + (t-s)^{-\frac{1}{2} - \frac{\nu}{\nu_0} - \frac{1}{2}})\|u(s)\|_{L^{q_1}(\Omega)}\|u(s)\|_{L^{q_0}(\Omega)}ds \]
\[ \leq K_5 C_1 C_2 \int_0^t e^{-\mu_0(t-s)}(1 + (t-s)^{-\frac{1}{2} - \frac{\nu}{\nu_0} - \frac{1}{2}})\|\nabla u(s)\|_{L^{q_1}(\Omega)}\|\nabla v(s)\|_{L^{q_2}(\Omega)}ds \]
\[ \leq K_5 (1 + (t-s)^{-\frac{1}{2} - \frac{\nu}{\nu_0} - \frac{1}{2}})\|u(s)\|_{L^{q_1}(\Omega)}\|\nabla v(s)\|_{L^{q_2}(\Omega)}ds \]
\[ \leq K_5 C_1 C_2 e^{2\int_0^t e^{-\mu_0(t-s)}(1 + (t-s)^{-\frac{1}{2} - \frac{\nu}{\nu_0} - \frac{1}{2}})\|\nabla u(s)\|_{L^{q_1}(\Omega)}\|\nabla v(s)\|_{L^{q_2}(\Omega)}ds \]
\[ \leq K_5 C_1 C_2 e^{2\int_0^t e^{-\mu_0(t-s)}(1 + (t-s)^{-\frac{1}{2} - \frac{\nu}{\nu_0} - \frac{1}{2}})(1 + s^{-\frac{1}{2} + \frac{\nu}{\nu_0}})ds} \]
\[ \leq C_3 e^{2\lambda t}(1 + t^{-1 + \frac{\nu}{\nu_0}}) \]

for all \( t \in [0, T_0] \) where \( C_3 > 0 \) depends on \( d, \Omega, |\mu_0 - \lambda'|, q_0 \) and \( \theta_0 \), since under our assumption we can find \( q_1, q_2 \geq q_0 \) such that \( \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_0} + \frac{1}{d} \).

On the other hand, for \( \theta \in [q_0, \theta_0] \), we infer that

\[ \|u(t) - e^{t\mathcal{L}u_0}\|_{L^{\theta}(\Omega)} \]
\[ \leq \left\| \int_0^t e^{(t-s)\mathcal{L}} \nabla \cdot (u\nabla v)(s)ds \right\|_{L^{\theta}(\Omega)} \]
\[ \leq K_5 \int_0^t e^{-\mu_0(t-s)}(1 + (t-s)^{-\frac{1}{2} - \frac{\nu}{\nu_0} - \frac{1}{2}})\|\nabla u\|_{L^{q_1}(\Omega)}\|\nabla v\|_{L^{q_2}(\Omega)}ds \]
\[ \leq K_5 \int_0^t e^{-\mu_0(t-s)}(1 + (t-s)^{-\frac{1}{2} - \frac{\nu}{\nu_0} - \frac{1}{2}})\|\nabla u\|_{L^{q_1}(\Omega)}\|\nabla v\|_{L^{q_2}(\Omega)}ds \]
\[ \leq K_5 (1 + (t-s)^{-\frac{1}{2} - \frac{\nu}{\nu_0} - \frac{1}{2}})\|\nabla u\|_{L^{q_1}(\Omega)}\|\nabla v\|_{L^{q_2}(\Omega)}ds \]
\[ \leq K_5 C_1 C_2 e^{2\int_0^t e^{-\mu_0(t-s)}(1 + (t-s)^{-\frac{1}{2} - \frac{\nu}{\nu_0} - \frac{1}{2}})(1 + s^{-\frac{1}{2} + \frac{\nu}{\nu_0}})ds} \]
\[ \leq C_3 e^{\lambda t}(1 + t^{-1 + \frac{\nu}{\nu_0}}) \]

for all \( t \in [0, T_0] \).

As a result, we conclude that for all \( \theta \in [q_0, \infty] \), there holds

\[ \|u(t) - e^{t\mathcal{L}u_0}\|_{L^{\theta}(\Omega)} \leq C_3 e^{\lambda t}(1 + t^{-1 + \frac{\nu}{\nu_0}}) \]

for all \( t \in [0, T_0] \) where \( C_3 > 0 \) depends on \( d, \Omega, |\mu_0 - \lambda'|, q_0 \) and \( \theta_0 \), but is independent of \( T_0 \) or \( t \). Choosing \( \varepsilon_0 < \frac{1}{2C_3} \), we conclude that \( T_0 = \infty \) and hence \( T_{\max} = \infty \) as well which completes the proof.

Now we complete the proof of Theorem 1.1. We may fix \( q_0 \in (\frac{d}{2}, d) \), \( \theta_0 \in (d, \frac{d_{\max}}{d-q_0}) \) and \( \lambda' < \mu_0 \), then by Proposition 1, we get \( \varepsilon_0 > 0 \) such that \( (u, v) \) exists globally under the smallness assumptions. Recalling that \( (u, v) \) is just a reduction of \( \mathcal{M} \) from \( (\rho, c) \) we conclude that under the assumption of Theorem 1.1, problem (4) has a unique classical solution \( (\rho, c) \) that is globally bounded. Moreover, since \( \|e^{t\mathcal{L}u_0}\|_{L^\infty} \leq C e^{-\mu_0 t}\|u_0\|_{L^\infty} \) for \( t \geq 1 \) due to Lemma 3.2, we infer that for \( t \geq 1 \),

\[ \|u(t)\|_{L^\infty(\Omega)} \leq \|u(t) - e^{t\mathcal{L}u_0}\|_{L^\infty(\Omega)} + \|e^{t\mathcal{L}u_0}\|_{L^\infty(\Omega)} \leq C e^{-\lambda t} \]
which indicates that
\[
\|\rho(t) - \mathcal{M}\|_{L^\infty(\Omega)} + \|\nabla c(t)\|_{L^\infty(\Omega)} \leq Ce^{\lambda t}
\]
for \(t \geq 1\) with some \(C > 0\). This completes the proof of Theorem 1.1. \(\Box\)

Now, we prove Theorem 1.2. To this aim, we prove the following result for the reduced cell density and chemical concentration \((u, v)\) of \((9)\).

**Proposition 2.** Let \(d \geq 2\) and \(0 < \mathcal{M} < 1 + \lambda_1\). For any fixed \(q_0, \theta_0 > 0\) such that \(\frac{d}{2} < q_0 < d\) and \(d < \theta_0 < \frac{dq_0}{d - q_0}\), there exists \(\varepsilon_0 > 0\) depending on \(d, q_0, \mathcal{M}\) and \(\Omega\) such that for any initial data \((u_0, v_0) \in C(\bar{\Omega}) \cap L^1_0(\Omega) \times C^1(\bar{\Omega}) \cap L^1_0(\Omega)\) satisfying \(\partial_\Omega v_0 = 0\) on \(\partial \Omega\), \(u_0 \geq -\mathcal{M}\), \(v_0 \geq -\mathcal{M}\) and \(\|u_0\|_{L^1(\Omega)} + \|\nabla v_0\|_{L^1(\Omega)} \leq \varepsilon\) for some \(\varepsilon < \varepsilon_0\), system \((9)\) with \(\gamma = 1\) has global classical solution \((u, v)\) which is bounded such that
\[
\|u(t) - \bar{u}(t)\|_{L^1(\Omega)} \leq \varepsilon e^{-\mu t} (1 + t^{-1 + \frac{d}{2}}) \quad \text{for all} \ t > 0
\]
and for all \(\theta \in [q_0, \theta_0]\) with \(\mu' < \mu_1 = \lambda_1 - \frac{1}{2}\left(\sqrt{d\lambda_1 \mathcal{M} + 1} \right) > 0\).

**Proof.** According to Lemma 2.1, problem \((4)\) with nonnegative initial data \(\rho_0 = u_0 + \mathcal{M}\) and \(c_0 = v_0 + \mathcal{M}\) has a classical solution on \([0, T_{\text{max}}]\) and if \(T_{\text{max}} < \infty\), we have \(\limsup \|\rho(\cdot, t)\|_{L^\infty} = \infty\). Therefore, for problem \((9)\), we obtain a classical solution \((\rho - \mathcal{M}, c - \mathcal{M})\) on \([0, T_{\text{max}}]\) and if \(T_{\text{max}} < \infty\), we have \(\limsup \|u(\cdot, t)\|_{L^\infty} = \infty\). Denoting \((\bar{u}, \bar{v})\) the solution of linearized system \((24)\), we derive by variation-of-constants formula that for \(0 < t < T_{\text{max}}\),
\[
v(t) = e^{t(\Delta - 1)}v_0 + \int_0^t e^{(t-s)(\Delta - 1)}u(s)ds
\]
\[
= e^{t(\Delta - 1)}v_0 + \int_0^t e^{(t-s)(\Delta - 1)}\bar{u}(s)ds + \int_0^t e^{(t-s)(\Delta - 1)}(u(s) - \bar{u}(s))ds
\]
\[
= \bar{v}(t) + \int_0^t e^{(t-s)(\Delta - 1)}(u(s) - \bar{u}(s))ds.
\]

On the other hand, exploiting the semigroup \(e^{tA}\) and variation-of-constants formula again, we infer that
\[
\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = e^{tA} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t e^{(t-s)A} \begin{pmatrix} -\nabla \cdot (u(s)\nabla v(s)) \\ 0 \end{pmatrix} ds
\]
from which we represent \(u\) according to
\[
u(t) = \bar{u}(t) - \int_0^t \Phi_1^{t-s} (\nabla \cdot (u(s)\nabla v(s)), 0) ds.
\]

Now, like in [3] (see also [18]), fix some \(q_0 \in \left(\frac{d}{2}, d\right)\) and \(\theta_0 \in (d, \frac{dq_0}{d - q_0})\), one can choose \(q_1, q_2 > 0\) such that \(q_1 \in (q_0, \theta_0)\), \(q_2 \in (d, \frac{dq_0}{d - q_0})\) and \(\frac{1}{q_0} = \frac{1}{q_1} + \frac{1}{q_2}\). Define
\[
T_1 \triangleq \sup \left\{ T > 0 : \|u(t) - \bar{u}(t)\|_{L^1} \leq \varepsilon e^{-\mu' t} (1 + t^{-1 + \frac{d}{2}}) \right\}
\]
for all \(t \in [0, T]\) and all \(\theta \in [q_0, \theta_0]\).
Then $T_1$ is well-defined and positive with $T_1 \leq T_{\text{max}}$, since on the one hand, $\|u(t)\|_{L^\infty}$ and $\|\tilde{u}(t)\|_{L^\infty}$ are both bounded near $t = 0$ due to Lemma 2.1 and Remark 2. Hence $\|\tilde{u}(t)\|_{L^q}$ ≤ $|\Omega|^{1/q - 1/\beta_0} \|\tilde{u}(t)\|_{L^\infty}$ ≤ $\max\{\|\Omega\|^{1/\beta_0}, 1\} \|\tilde{u}(t)\|_{L^\infty}$ are uniformly bounded with respect to $\theta \in [\theta_0, \theta_0]$. On the other hand as $t \to 0^+$, $t^{-1 + \frac{1}{2\theta}} \geq t^{-1 + \frac{1}{2\theta}} \to +\infty$ uniformly with respect to $\theta \in [\theta_0, \theta_0]$. Now we claim that when $\varepsilon_0$ is sufficiently small, we have $T_1 = \infty$.

First, we apply $\nabla$ to both sides of (47) to deduce that

\[
\|\nabla v(t) - \nabla \tilde{u}(t)\|_{L^p(\Omega)} \\
\leq \int_0^t \|\nabla e^{(t-s)(\Delta^{-1})}(u(s) - \tilde{u}(s))\|_{L^p(\Omega)} ds \\
\leq \int_0^t k_2(1 + (t-s)^{-\frac{1}{2} - \frac{q}{2}}(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon})) e^{-(\lambda_1 + 1)(t-s)} \|u(s) - \tilde{u}(s)\|_{L^\infty(\Omega)} ds \\
\leq \int_0^t k_2(1 + (t-s)^{-\frac{1}{2} - \frac{q}{2}}(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon})) e^{-(\lambda_1 + 1)(t-s)} \varepsilon e^{-\mu t}(1 + s^{-1 + \frac{1}{2\theta}}) ds \\
\leq Ck_2 \varepsilon (1 + t^{-\frac{1}{2} + \frac{1}{2\theta}}) e^{-\mu t} \\
\leq C_4 \varepsilon (1 + t^{-\frac{1}{2} + \frac{1}{2\theta}}) e^{-\mu t}
\]

for all $t \in (0, T_1)$ and all $p \in [\theta_0, \frac{\log d}{\log \theta_0 + \varepsilon_0})$ where $C_4 > 0$ depends on $\theta_0, d$ and $\Omega$ only.

As a result, we infer by Lemma 3.5 that for $p \in [d, \frac{\log d}{\log \theta_0})$

\[
\|\nabla v(t)\|_{L^p(\Omega)} \leq \|\nabla \tilde{u}(t)\|_{L^p(\Omega)} + C_4 \varepsilon (1 + t^{-\frac{1}{2} + \frac{1}{2\theta}}) e^{-\mu t} \\
\leq \varepsilon \|\nabla v(t)\|_{L^p(\Omega)} + C_4 \varepsilon (1 + t^{-\frac{1}{2} + \frac{1}{2\theta}}) e^{-\mu t} \\
\leq C_4 \varepsilon (1 + t^{-\frac{1}{2} + \frac{1}{2\theta}}) e^{-\mu t}
\]

where $C$ depends on $d, M, \theta_0$ and $\Omega$. For $p \in [\theta_0, d)$, applying Hölder’s inequality and Lemma 3.5, there holds

\[
\|\nabla v(t)\|_{L^p(\Omega)} \leq \|\nabla \tilde{u}(t)\|_{L^q(\Omega)} + C_4 \varepsilon (1 + t^{-\frac{1}{2} + \frac{1}{2\theta}}) e^{-\mu t} \\
\leq K_7 \|\nabla \tilde{u}(t)\|_{L^q(\Omega)} + C_4 \varepsilon (1 + t^{-\frac{1}{2} + \frac{1}{2\theta}}) e^{-\mu t} \\
\leq C \varepsilon (1 + t^{-\frac{1}{2} + \frac{1}{2\theta}}) e^{-\mu t}
\]

with $C$ depending on $d, \theta_0, M$ and $\Omega$ only. Thus, we conclude from above that for all $p \in [\theta_0, \frac{\log d}{\log \theta_0})$

\[
\|\nabla v(t)\|_{L^p(\Omega)} \leq C_5 \varepsilon (1 + t^{-\frac{1}{2} + \frac{1}{2\theta}}) e^{-\mu t}
\]

(50)

with $C_5$ depending on $d, \theta_0, M$ and $\Omega$.

On the other hand, in views of the definition of $T_1$ and Lemma 3.5, we also have

\[
\|u(t)\|_{L^p(\Omega)} \leq \|\tilde{u}(t)\|_{L^p(\Omega)} + \varepsilon e^{-\mu t}(1 + t^{-1 + \frac{1}{2\theta}}) \\
\leq K_8 \varepsilon e^{-\mu t}(1 + t^{-1 + \frac{1}{2\theta}}) + \varepsilon e^{-\mu t}(1 + t^{-1 + \frac{1}{2\theta}}) \\
\leq C_6 \varepsilon e^{-\mu t}(1 + t^{-1 + \frac{1}{2\theta}})
\]

(51)

holds for all $p \in [\theta_0, \theta_0]$ and $0 < t < T_1$, where $C_6$ depends on $d$ and $\Omega$ only.
Now due to (49) and Lemma 3.6, we can finally estimate
\[ \|u(t) - \tilde{u}(t)\|_{L^q(\Omega)} \]
\[ \leq \int_0^t \|\Phi^{t-s}_1(\nabla \cdot (u(s)\nabla v(s)), 0)\|_{L^q(\Omega)} ds \]
\[ \leq K \int_0^t e^{-\mu_1(t-s)}(1 + (t-s)^{-\frac{1}{2}-\frac{d}{2q} + \frac{d}{2p}})\|u(s)\nabla v(s)\|_{L^{q_0}(\Omega)} ds \]
\[ \leq K \int_0^t e^{-\mu_1(t-s)}(1 + (t-s)^{-\frac{1}{2}-\frac{d}{2q} + \frac{d}{2p}})\|u(s)\|_{L^{q_1}(\Omega)} \|\nabla v(s)\|_{L^{q_2}(\Omega)} ds \]
\[ \leq K C_0 \int_0^t e^{-\mu_1(t-s)}(1 + (t-s)^{-\frac{1}{2}-\frac{d}{2q} + \frac{d}{2p}}) e^{-\mu_1'(s(1 + s^{-\frac{1}{2} + \frac{d}{2q}}))} ds \]
\[ \leq C_7 \varepsilon^2(1 + t^{-1 + \frac{d}{2p}}) e^{-\mu t} \]
for all \(0 < t < T_1\), where \(C_7\) depends on \(d, M, q_0\) and \(\Omega\), but is independent of \(T_1\) or \(t\). Now, choosing \(\varepsilon_0 < \frac{1}{2C_7}\), we conclude that \(T_1 = \infty\) which implies that \(T_{\max} = \infty\) as well, i.e., \((u, v)\) is global and bounded. \(\square\)

By the same argument as before, keeping in mind that \((u, v)\) is just a reduction of \(M\) from \((\rho, c)\), we obtain the existence of a unique global solution to problem (4) with \(\gamma = 1\) under the assumptions of Theorem 1.2. It remains to show the exponentially decay in \(L^\infty\)-norm. Invoking (50), the fact \(\|u(t)\|_{L^\infty} \leq C\) and Lemma 2.3, we deduce that
\[ \|u(t)\|_{L^\infty(\Omega)} \]
\[ \leq e^{\Delta t} u_0 + M \int_0^t e^{\Delta(t-s)} \Delta v(s)\|_{L^\infty(\Omega)} ds \]
\[ + \int_0^t \|e^{\Delta(t-s)} \nabla \cdot (u(s)\nabla v(s))\|_{L^\infty(\Omega)} ds \]
\[ \leq k_1 e^{-\lambda t} \|u_0\|_{L^\infty(\Omega)} + C \int_0^t (1 + (t-s)^{-\frac{1}{2} + \frac{d}{2q}}) e^{-\lambda_1(t-s)} \|\nabla v(s)\|_{L^p(\Omega)} ds \]
\[ + C \int_0^t (1 + (t-s)^{-\frac{1}{2} + \frac{d}{2p}}) e^{-\lambda_1(t-s)} \|u(s)\|_{L^p(\Omega)} \|\nabla v(s)\|_{L^p(\Omega)} ds \]
\[ \leq C e^{-\lambda t} + C \int_0^t (1 + (t-s)^{-\frac{1}{2} + \frac{d}{2p}}) e^{-\lambda_1(t-s)} \|\nabla v(s)\|_{L^p(\Omega)} ds \]
\[ \leq C e^{-\lambda t} + C \int_0^t (1 + (t-s)^{-\frac{1}{2} + \frac{d}{2p}}) e^{-\lambda_1(t-s)} (1 + s^{-\frac{1}{2} + \frac{d}{2p}}) e^{-\mu_1' s} ds \]
\[ \leq C e^{-\lambda t} + C e^{-\mu t} \]
\[ \leq C e^{-\mu t} \]
where \(p \in (d, \frac{d q_0}{d - d q_0})\) such that \(\frac{1}{2} + \frac{d}{2p} \in (0, 1)\) and \(\frac{1}{2} - \frac{d}{2q} \in (0, 1)\). As a consequence,
\[ \|\nabla v(t)\|_{L^\infty(\Omega)} \]
\[ \leq \|\nabla e^{((\Delta - 1) t)} v_0\|_{L^\infty(\Omega)} + \int_0^t \|\nabla e^{((\Delta - 1) (t-s))} u(s)\|_{L^\infty(\Omega)} ds \]
\[ \leq C e^{-(\lambda_1 + 1)} (1 + t^{-\frac{1}{2}}) \|v_0\|_{L^\infty(\Omega)} \]
+ C \int_0^t (1 + (t - s)^{-\frac{1}{2}} - \frac{d}{2\pi}) e^{-(\lambda_1 + 1)(t - s)} \|u(s)\|_{L^q(\Omega)} ds \leq C e^{-(\lambda_1 + 1)t} (1 + t^{-\frac{1}{2}}) + C \int_0^t (1 + (t - s)^{-\frac{1}{2}} - \frac{d}{2\pi}) e^{-(\lambda_1 + 1)(t - s)} e^{-\mu's} (1 + s^{-1} + \frac{d}{2\pi}) ds \\
\leq C e^{-(\lambda_1 + 1)t} (1 + t^{-\frac{1}{2}}) e^{-\mu't} (1 + t^{-\frac{1}{2}})

with some \( q > d \), which indicates that when \( t \geq 1 \),

\[ \|\nabla v(t)\|_{L^\infty(\Omega)} \leq Ce^{-\mu't}. \]

This completes the proof of Theorem 1.2. \( \square \)

REFERENCES

[1] M. Ashbaugh and A. Levine, Inequalities for Dirichlet and Neumann eigenvalues of the laplacian for domains on sphere, *Journées Équations Aux Dérivées Partielles*, 1997, 1–15.
[2] N. Bellomo, A. Belouquid, Y. Tao and M. Winkler, Toward a mathematical theory of Keller–Segel models of pattern formation in biology tissues, *Math. Mod. Meth. Appl. Sci.*, 25 (2015), 1663–1763.
[3] X. Cao, Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces, *Discrete Contin. Dynam. Syst. Ser. A*, 35 (2015), 1891–1904.
[4] T. Cieślak, P. Laurençot and C. Morales-Rodrigo, Global existence and convergence to steady states in a chemorepulsion system, *Banach Center Publ.*, Polish Acad. Sci., Warsaw, 81 (2008), 105–117.
[5] K.J. Engel and R. Nagel, *One-parameter Semigroup for Linear Evolution Equations*, GTM194, Springer, 2000.
[6] T. Hillen and K. J. Painter, A user’s guide to PDE models for chemotaxis, *J. Math. Biol.*, 58 (2009), 183–217.
[7] D. Horstmann and G.-F. Wang, Blow-up in a chemotaxis model without symmetry assumptions, *Euro. J. Appl. Math.*, 12 (2001), 159–177.
[8] D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system, *J. Different. Equ.*, 215 (2005), 52–107.
[9] J. Jiang, Global stability of homogeneous steady states in scaling-invariant spaces for a Keller–Segel–Navier–Stokes system, *J. Different. Equ.*, 267 (2019), 659–692.
[10] J. Jiang, Eventual smoothness and exponential stabilization of global weak solutions to some chemotaxis systems, preprint, submitted.
[11] J. Jiang, H. Wu and S. Zheng, Blow-up for a three dimensional Kelle–Segel model with consumption of chemoattractant, *J. Different. Equ.*, 264 (2018), 5432–5464.
[12] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.*, 26 (1970), 399–415.
[13] A. Kiselev and X. Xu, Suppression of chemotactic explosion by mixing, *Arch. Rational Mech. Anal.*, 222 (2016), 1077–1112.
[14] H. Kozono, M. Miura and Y. Sugiyma, Existence and uniqueness theorem on mild solutions to the Keller–Segel system coupled with the Navier–Stokes fluid, *J. Funct. Anal.*, 270 (2016), 1663–1683.
[15] A. Lorz, A coupled Keller–Segel–Stokes model: Global existence for small initial data and blow-up delay, *Commun. Math. Sci.*, 10 (2012), 555–574.
[16] T. Nagai and T. Senba, Global existence and blow-up of radial solutions to a parabolic-elliptic system of chemotaxis, *Adv. Math. Sci. Appl.*, 8 (1998), 145–156.
[17] T. Nagai, T. Senba and T. Suzuki, Chemotactic collapse in a parabolic system of mathematical biology, *Hiroshima Math. J.*, 30 (2000), 463–497.
[18] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller–Segel model, *J. Different. Equ.*, 248 (2010), 2889–2905.
[19] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller–Segel system, *J. Math. Pures Appl.*, 100 (2013), 748–767.
[20] H. Yu, W. Wang and S. Zheng, Global classical solutions to the Keller–Segel–Navier–Stokes system with matrix-valued sensitivity, *J. Math. Anal. Appl.*, 461 (2018), 1748–1770.
[21] S. Zheng. *Nonlinear Evolution Equations*, Chapman & Hall/CRC, Boca Raton, FL, 2004.

Received June 2019; revised July 2019.

E-mail address: jiang@uipm.ac.cn