How to Achieve the Capacity of Asymmetric Channels

Marco Mondelli, S. Hamed Hassani, and Rüdiger Urbanke

Abstract

We discuss coding techniques that allow reliable transmission up to the capacity of a discrete memoryless asymmetric channel. Some of the techniques are well-known and go back sixty years ago, some are recent, and others are new. We take the point of view of modern coding theory and we discuss how recent advances in coding for symmetric channels help in providing more efficient solutions also for the asymmetric case. In more detail, we consider three basic approaches. The first one is Gallager's scheme that consists of concatenating a linear code with a non-linear mapper so that the input distribution can be appropriately shaped. The second one is an integrated approach in which the coding scheme is used both for source coding, in order to create codewords distributed according to the capacity-achieving input distribution, and for channel coding, in order to provide error protection. Such a technique has been recently introduced in the context of polar codes, and we show how to apply it also to the design of sparse graph codes. The third approach is new and separates completely the two tasks of source coding and channel coding by “chaining” together several codewords. In this way, we can combine any suitable source code with any suitable channel code in order to provide optimal schemes for asymmetric channels. All the variants that we discuss in this paper yield low-complexity coding solutions and, with one exception, are provably capacity-achieving.

I. INTRODUCTION

In this paper, our goal is to describe coding techniques, some old and well-known, some recent, and some novel, which allow reliable transmission up to the capacity of discrete memoryless, asymmetric channels. As we will see, there are at least three very different approaches to this problem, and each of those has many variants. Before describing the asymmetric coding techniques, let us quickly review how to solve the problem in the symmetric scenario.

Symmetric Channel Coding: A Review. Let \( W \) be a symmetric binary-input, discrete memoryless channel (B-DMC) and denote by \( I(W) \) its capacity. Two relatively recent schemes that are capable of achieving the capacity for this class of channels are polar codes \([1]\) and spatially coupled codes \([2]\).

Polar codes are closely related to Reed-Muller (RM) codes, since, in both cases, the rows of the generator matrix of a code of block length \( n = 2^m \) are chosen from the rows of the matrix \( G_n = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \otimes m \), where \( \otimes \) denotes the Kronecker product. The crucial difference between polar and RM codes lies in the choice of the rows. The RM rule always consists in picking the rows of largest Hamming weight, while the polar rule depends on the channel. To explain how this last rule works, let us recall that polar codes are decoded with a successive cancellation (SC) algorithm which makes decisions on the bits one-by-one in a pre-chosen order. In particular, in order to decode the \( i \)-th bit, the algorithm uses the values of all the previous \( i - 1 \) decisions. Therefore, the polar rule aims at choosing the rows of \( G_n \) which correspond to the synthetic channels which are the most reliable when decoded in this successive manner. The complexity of encoding and decoding scales as \( O(n \log n) \). If we fix a rate \( R < I(W) \) and let \( n \to \infty \), then the error probability \( P_e \) scales as \( O(2^{-\sqrt{n}}) \) \([3]\). If we fix any error probability \( P_e \) and we let \( n \to \infty \), then the gap to capacity \( I(W) - R \) scales as \( O(n^{-1/\mu}) \). The scaling exponent \( \mu \) is lower bounded by 3.579 and upper bounded by 5.77 \([4], [5]\). It is conjectured that the lower bound on \( \mu \) can be increased up to 3.627, which is the value for the binary erasure channel (BEC).

Spatially coupled LDPC (SC-LDPC) codes are constructed by connecting together \( L \) standard LDPC codes in a chain-like fashion and by terminating this chain so that the decoding task becomes easier towards the boundary. The decoding is done using the belief propagation (BP) algorithm which has linear complexity in the block length.
a desired gap to capacity $I(W) - R$, one can choose an appropriate degree distribution for the component codes, as well as an appropriate connection pattern of the chain and length of the code, so that the resulting ensemble allows reliable transmission over any B-DMC [2]. For a discussion on the scaling behavior of spatially coupled codes we refer the reader to [6].

These schemes have been generalized to arbitrary symmetric DMCs. Channel polarization for $q$-ary input alphabets is first discussed in [7] and more general constructions based on arbitrary kernels are described in [8]. Furthermore, polar codes have been built exploiting various algebraic structures on the input alphabet [9]–[11]. Results concerning the iterative decoding threshold on the BEC for non-binary SC-LDPC code ensembles are presented in [12], [13], and the corresponding threshold saturation is proved in [14]. The threshold analysis under windowed decoding is provided in [15].

**Asymmetric Channel Coding: Existing and Novel Techniques.** Consider now transmission over an asymmetric DMC. The basic problem which one faces is that linear codes impose a uniform input distribution, while the capacity-achieving input distribution for an asymmetric channel is, in general, not the uniform one. This mismatch in the input distribution bounds the achievable transmission rate away from capacity.

The classical solution to the problem of coding over asymmetric channels goes back to Gallager and consists of concatenating a linear code with a non-linear mapper so that the input distribution becomes biased [19]. In principle, by making the input distribution arbitrarily close to the capacity-achieving one, this solution allows one to approach capacity arbitrarily closely. However, the complexity is in general considerably increased by the non-linear mapper. Indeed, in order to induce an input distribution of the form $a/b$, with $a, b \in \mathbb{N}$, the mapper has to work on blocks of size $b$ and the complexity increases exponentially in $n$. As pointed out in [20], this problem is somewhat mitigated by the fact that the capacity is relatively insensitive to a slight mismatch in the input distribution and, therefore, this approach works quite well, especially if combined with iterative coding techniques.

More recently, polar codes have been used to achieve the capacity of binary-input asymmetric DMCs. In particular, in [21] the authors propose a solution that makes use of the concatenation of two polar codes: one of them is used to solve a source coding problem, in order to have codewords distributed according to the capacity-achieving input distribution; the other is used to solve a channel coding problem, in order to provide error correction. However, such a scheme requires polarization for both the inner and the outer codes, and, therefore, the error probability scales roughly as $(1/2)^{n/4}$. Thus, in order to obtain the same performance as standard polar codes, the square of their block length is required. A very simple and more efficient solution for transmission over asymmetric channels is presented in [22]. In this scheme, in order to transmit over channels whose optimal input distribution is not uniform, the polar indices are partitioned into three groups: some are used for information transmission; some are used to ensure that the input distribution is properly biased; some carry random bits shared between the transmitter and the receiver. The purpose of this shared randomness is to facilitate the performance analysis. Indeed, as in the case of LDPC code ensembles, the error probability is obtained by averaging over the randomness of the ensemble.

In short, the methods in [21], [22] exploit the fact that polar codes are well suited not only for channel coding but also for lossless source coding [23]. Clearly, this is not a prerogative only of polar codes, as, for example, sparse graph codes have been successfully used for both channel coding and source coding purposes [24]. Motivated by this fact, we describe a scheme based on SC-LDPC codes which achieves the capacity of asymmetric DMCs by solving both a source coding and a channel coding problem at the same time.

We add one more possibility to this list of solutions. As we will discuss in more detail later, by “chaining” together several codewords we can decouple the problem of source coding (creating a biased codeword from unbiased bits) from the problem of channel coding (providing error correction). By doing so, one can use any suitable source coding solution and combine it with any suitable channel coding solution in order to reliably transmit over an asymmetric channel arbitrarily close to its capacity.

The motivation for this paper is twofold. On the one hand, it is instructive to investigate to what degree recent advances in coding for symmetric channels have made it possible to construct efficient schemes also for the asymmetric case. This part of the paper is more tutorial in nature. On the other hand, we introduce new schemes to the mix of possible solutions. In this way, we show that what perhaps once was considered as a difficult problem is in fact

---

1It was shown in [16] that the optimal distribution for binary-input channels is not too far from the uniform one, in the sense that the capacity-achieving input distribution always has a marginal in the interval $(1/e, 1 - 1/e)$ (for a generalization of the upper bound to any finite-input DMC, see [17]). In addition, at most $1 - e^{-5.8}$ percent of capacity is lost if we use the uniform input distribution instead of the optimal input distribution [16]. This result was later strengthened in [18], where the Z-channel is proved to be extremal in this sense.
quite easy to solve with existing “primitives”. By presenting all the schemes in a unified manner, we hope to have simplified the task for anyone who is interested in comparing the various possibilities. There is no clear winner and what solution is best for a particular applications will depend very much on the details of the situation.

The rest of the paper is organized as follows. Section II discusses two related and simpler problems, namely how to achieve the symmetric capacity of an asymmetric channel and how to perform error correction using biased codewords. The proposed solutions to these problems will be later used as “primitives” to solve the transmission problem over asymmetric channels. Then, we describe in three consecutive sections the three main coding approaches and some of their possible variants for the problem of achieving the capacity of an asymmetric DMC: Gallager’s mapping in Section III, the integrated scheme in Section IV, and the chaining construction in Section V. Concluding remarks are provided in Section VI.

II. WARM UP: TWO RELATED PROBLEMS

After establishing the notation and reviewing some known concepts, in this preliminary section we discuss two how many parity checks one needs to share between transmitter and receiver in order to transmit reliably a biased binary codeword.

A. Notation and Prerequisites

Throughout this paper, we consider the transmission over a DMC $W : \mathcal{X} \rightarrow \mathcal{Y}$ with input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}$. If the channel is binary-input, we usually take $\mathcal{X} = \mathcal{F}_2 = \{0,1\}$ and we say that $X$ is a Bernoulli$(\alpha)$ random variable if $\mathbb{P}(X = 1) = \alpha$ for some $\alpha \in [0,1]$. However, for the analysis of LDPC ensembles, it is convenient to consider the standard mapping $0 \leftrightarrow 1$ and $1 \leftrightarrow -1$. It will be clear from the context whether the input alphabet is $\{-1,1\}$ or $\{0,1\}$. The probability of the output being $y$ given an input $x$ is denoted by $W(y | x)$ and the probability of the input being $x$ given an output $y$ is denoted by $p_X(y | x)$. We write $I(W)$ and $I_s(W)$ to indicate the capacity and the symmetric capacity of $W$, respectively. Given the scalar components $X(i), \ldots, X(j)$ and $X_i, \ldots, X_j$, we use $X^{i:j}$ as a shorthand for the column vector $(X(i), \ldots, X(j))^T$ and, similarly, $X_{i:j}$ as a shorthand for the column vector $(X_i, \ldots, X_j)^T$ with $i \leq j$. The index set $\{1, \ldots, n\}$ is abbreviated as $[n]$ and, given a set $A \subseteq [n]$, we denote by $A^c$ its complement. We denote by $\log$ and $\ln$ the logarithm in base 2 and base $e$, respectively. For any $x \in [0,1]$, we define $\bar{x} = 1 - x$. The binary entropy function is given by $h_2(x) = -x \log x - \bar{x} \log \bar{x}$. When discussing sparse graph coding schemes, the parity check matrix is denoted by $P$. We do not use $H$ to denote the parity check matrix, as it is done more frequently, since the symbol $H(\cdot)$ already indicates the entropy of a random variable. When discussing polar coding schemes, we assume that the block length $n$ is a power of 2, say $n = 2^m$ for $m \in \mathbb{N}$, and we denote by $G_n$ the polar matrix given by $G_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \end{bmatrix}^\otimes m$, where $\otimes$ denotes the Kronecker product.

Let us recall some notation and facts concerning B-DMCs. We will be telegraphic and refer the reader to [25] for more details. First, consider a symmetric B-DMC with $\mathcal{X} = \{-1,1\}$. Assume that $X$ is transmitted, $Y$ is the received observation, and $L(Y)$ the corresponding log-likelihood ratio, namely for any $y \in \mathcal{Y}$,

$$L(y) = \ln \frac{W(y | 1)}{W(y | -1)}. \quad (1)$$

Let us denote by $a$ the density of $L(Y)$ assuming that $X = 1$ and let us call it an $L$-density.

We say that an $L$-density $a$ is symmetric if

$$a(y) = e^y a(-y). \quad (2)$$

Since the log-likelihood ratio constitutes a sufficient statistic for decoding, two symmetric DMCs are equivalent if they have the same $L$-density. A meaningful choice for the representative of each equivalence class is $W(y | 1) = a(y)$ and, by symmetry, $W(y | -1) = a(-y)$. Indeed, by using the assumption (2), one can show that this choice of $W(y | x)$ yields an $L$-density equal to $a(y)$ (see [25, Lemma 4.28]).

As a final reminder, the capacity $I(W)$ can be computed as a function of the $L$-density $a$ according to the following formula [25, Lemma 4.35],

$$I(W) = \int a(y) \left(1 - \log(1 + e^{-y})\right) dy. \quad (3)$$
B. How to Achieve the Symmetric Capacity of Asymmetric Channels

**Problem Statement.** Let $X$ be binary and uniformly distributed. The aim is to transmit $X$ over the (not necessarily symmetric) B-DMC $W$ with a rate close to $I_s(W)$.

**Design of the Scheme.** The original construction of polar codes directly allows to achieve the symmetric capacity of any B-DMC [1].

For sparse graph codes, some more analysis is required. Here, we will follow an approach inspired by [25] Section 5.2. An alternative path, which considers the average of the density evolution analysis with respect to each codeword, is considered in [26]. Both approaches lead to the same result.

The codebook of a block of length $n$ and rate $R$ with parity check matrix $P$ is given by the set of $x^{1:n} \in \mathbb{F}_2^n$ s.t. $P x^{1:n} = 0^{1:(1-R)n}$, where $0^{1:(1-R)n}$ denotes a column vector of $(1 - R)n$ zeros. In words, the transmitter and the receiver know that the results of the parity checks are all zeros. Let us consider a slightly different model in which the values of the parity checks are chosen uniformly at random and this randomness is shared between the transmitter and the receiver: first one picks the parity checks uniformly at random; then, one picks a codeword uniformly at random among those which satisfy the parity checks. Clearly, this is equivalent to picking directly one codeword chosen uniformly at random from the whole space $\mathbb{F}_2^n$. Indeed, the space $\mathbb{F}_2^n$ is partitioned into the cosets determined by the parity checks. Choosing uniformly first the coset and then the codeword inside the coset is equivalent to directly choosing the codeword uniformly at random. As a result, we can model the codeword as a sequence of $n$ uniform i.i.d. bits.

Since the channel can be asymmetric, we need to define two distinct $L$-densities according to the transmitted value. For simplicity, let us map the input alphabet $\mathbb{F}_2$ into $\{-1, 1\}$ and denote by $a^+(y)$ and $a^-(y)$ the $L$-density for the channel assuming that $X = 1$ and $X = -1$ is transmitted, respectively. Let us now flip the density associated to $-1$, namely, $a^-(y)$, so that positive values indicate “correct” messages. By the symmetry of the message-passing equations (see Definition 4.81 in [25]), the sign of all those messages which enter or exit the variable nodes with associated transmitted value $-1$ is flipped as well. Therefore, the density evolution analysis for a particular codeword is equivalent to that for the all-1 codeword provided that we initialize the variable nodes with associated value 1 and $-1$ to $a^+(y)$ and $a^-(y)$, respectively. Now, each transmitted bit is independent and uniformly distributed. Thus, we pick a variable node with $L$-density $a^+(y)$ with probability $1/2$ and with $L$-density $a^-(y)$ with probability $1/2$.

As a result, the density evolution equations for our asymmetric setting are the same as those for transmission over the “symmetrized channel” with $L$-density given by

$$a^s(y) = \frac{1}{2}(a^-(y) + a^+(y)). \quad (4)$$

This channel is, indeed, symmetric and its capacity equals the symmetric capacity of the actual channel $W$ over which transmission takes place. These two results are formalized by the propositions below which are proved in Appendix A.

**Proposition 1:** Consider transmission over a B-DMC and let $a^+(y)$ and $a^-(y)$ be the $L$-densities assuming that $X = 1$ and $X = -1$ is transmitted, respectively. Then, the $L$-density $a^s(y)$ given by (4) is symmetric.

**Proposition 2:** Consider transmission over a B-DMC $W$ with symmetric capacity $I_s(W)$ and let $a^+(y)$ and $a^-(y)$ be the $L$-densities assuming that $X = 1$ and $X = -1$ is transmitted, respectively. Define the $L$-density of the “symmetrized channel” $a^s(y)$ as in (4). Then,

$$I_s(W) = \int a^s(y) \left(1 - \log(1 + e^{-y})\right) dy. \quad (5)$$

Consequently, in order to achieve the symmetric capacity $I_s(W)$ of the (possibly asymmetric) channel $W$, it suffices to construct a code which achieves the capacity of the symmetric channel with $L$-density $a^s$. Indeed, the density evolution analysis for the transmission over $W$ is exactly the same as for transmission over the symmetrized channel. Further, the capacity of the symmetrized channel equals the symmetric capacity $I_s(W)$ of the original channel $W$. As a result, in order to solve the problem, we can employ, for instance, an $(1, x)$-regular SC-LDPC ensemble with sufficiently large degrees.

In short, the problem of achieving the symmetric capacity of any B-DMC can be solved by using codes (e.g., polar, spatially coupled) which are provably optimal for symmetric channels.
C. How to Transmit Biased Bits

Let us consider a generalization of the previous problem in which we are given the codeword to transmit and the bits of this codeword are biased, i.e., they are not chosen according to a uniform distribution.

Problem Statement. Let \( W \) be a B-DMC s.t. the capacity-achieving input distribution is \( p^*(0) = \bar{\alpha} \) and \( p^*(1) = \alpha \) for some \( \alpha \in [0, 1] \). Let \( X^{1:n} \) be a sequence of \( n \) i.i.d. Bernoulli(\( \alpha \)) random variables. The aim is to transmit \( X^{1:n} \) over \( W \) sharing with the receiver the value of roughly \( H(X^{1:n} | Y^{1:n}) = nH(X | Y) \) parity checks, where \( X \) and \( Y \) denote the input and the output of the channel, respectively.

Design of the Scheme. The codeword \( X^{1:n} \) contains at most \( nH(X) \) bits of information. As \( I(W) = H(X) - H(X | Y) \), we have that \( nH(X | Y) \) is the minimum amount of redundancy which guarantees a reliable transmission. As in the model of Section II-B, this redundancy needs to be shared between the transmitter and the receiver. However, differently from the previous model, the actual value of the parity checks depends here on the codeword itself. In other words, the codeword \( x^{1:n} \) is given to the transmitter, which computes \( s^{1:n}H(X | Y) = P x^{1:n} \), where \( P \) denotes the parity check matrix. Furthermore, we assume that \( s^{1:n}H(X | Y) \) is also known at the receiver. Therefore, solving this problem does not give directly a solution to the (more interesting) issue of achieving the capacity of asymmetric channels. However, the current scenario is an important primitive which will be used in Sections IV and V where we describe coding techniques that achieve the capacity of asymmetric channels.

For the sake of simplicity, let us map the input alphabet into \{\(-1, 1\}\}. Since the input distribution is not uniform, the belief propagation (BP) algorithm needs to take into account also the prior on \( X \) and it is no longer based on the log-likelihood ratio (L). Let \( L_p(y) \) denote the posterior log-likelihood ratio, defined as

\[
L_p(y) = \ln \frac{p_X(y|1)}{p_X(y|-1)}.
\]

Following the lead of Section II-B, let us define the densities of \( L_p(Y) \) assuming that \( X = 1 \) and \( X = -1 \) is transmitted and let us denote these posterior \( L \)-densities as \( a^+_p(y) \) and \( a^-_p(y) \), respectively. If we flip the density associated to \( X = -1 \), namely, \( a^-_p(-y) \), then, by the symmetry of the message-passing equations, the sign of the messages which enter or exit the variable nodes with associated transmitted value \( X = -1 \) is flipped as well.

Therefore, the density evolution analysis for a particular codeword is equivalent to that for the all-one codeword provided that we initialize the variable nodes with associated value \( 1 \) to \( a^+_p(y) \), and the variable nodes with value \(-1 \) to and \( a^-_p(-y) \), respectively. As \( P(X = -1) = \alpha \), the density evolution equations for our asymmetric setting are the same as those for transmission over the “symmetrized channel” with posterior \( L \)-density given by

\[
a^+_p(y) = \alpha a^-_p(-y) + \bar{\alpha} a^+_p(y).
\]

The propositions below show that this channel is, indeed, symmetric and establish the relation between the conditional entropy \( H(X | Y) \) and \( a^+_p(y) \). The proofs of these results can be found in Appendix B.

**Proposition 3:** Consider transmission over a B-DMC and let \( a^+_p(y) \) and \( a^-_p(y) \) be the posterior \( L \)-densities assuming that \( X = 1 \) and \( X = -1 \) is transmitted, respectively. Then, the posterior \( L \)-density \( a^+_p(y) \) given by (7) is symmetric.

**Proposition 4:** Consider transmission over a B-DMC \( W \) with capacity-achieving input distribution \( p^* \). Let \( X \sim p^* \) and \( Y \) be the input and the output of the channel. Denote by \( a^+_p(y) \) and \( a^-_p(y) \) the posterior \( L \)-densities assuming that \( X = 1 \) and \( X = -1 \) is transmitted. Define the posterior \( L \)-density of the “symmetrized channel” \( a^+_p(y) \) as in (7). Then,

\[
H(X | Y) = \int a^+_p(y) \log(1 + e^{-y})dy.
\]

Since the channel with posterior \( L \)-density \( a^+_p(y) \) is symmetric, its capacity is given by

\[
\int a^+_p(y) \left(1 - \log(1 + e^{-y})\right)dy = 1 - \int a^+_p(y) \log(1 + e^{-y})dy.
\]

In addition, the capacity-achieving input distribution for a symmetric channel is the uniform one. Therefore, the RHS of (8) represents the amount of redundancy per bit which has to be shared between transmitter and receiver. This amount of redundancy equals that required to solve our original problem of transmitting a biased codeword, because the density evolution analysis is the same for the actual channel \( W \) and for the symmetrized channel. As a result, in order to transmit reliably \( n \) biased bits by sharing roughly \( nH(X | Y) \) parity checks, it suffices to construct a code.
which achieves the capacity of the symmetric channel with posterior $L$-density $p^*_n(y)$. To do that, we can employ, for instance, an $(1,r)$-regular SC-LDPC ensemble with sufficiently large degrees.

In short, suppose that one manages to share with the receiver the value of roughly $nH(X \mid Y)$ parity checks which depend on the codeword to be transmitted. Then, the problem of sending $n$ biased bits over a B-DMC can be solved by using sparse graph codes which are provably optimal for a suitable symmetric channel.

To sum up, so far we have discussed how to achieve the symmetric capacity of a B-DMC and how to transmit biased bits. Now, let us move to the main topic of this paper which consists in describing three approaches to achieve the actual capacity of any DMC. As we will see, the solutions to the two problems of this section will be regarded as useful primitives.

### III. Approach I: Gallager’s Mapping

The solution proposed by Gallager [19, pg. 208] consists of using a standard linear code and to apply a non-linear mapper to the encoded bits in such a way that the resulting input distribution is appropriately biased. Before moving to a general description of the scheme, let us start with an example to convey the main ideas.

#### A. A Concrete Example

Let $\mathcal{X} = \{0, 1, 2\}$ and suppose that we want to transmit over a channel $W : \mathcal{X} \to \mathcal{Y}$ with a capacity-achieving input distribution of the following form: $p'(0) = 3/8$, $p'(1) = 3/8$, $p'(2) = 2/8$.

Let $\mathcal{V} = \{0, 1, \cdots, 7\}$ and consider the function $f : \mathcal{V} \to \mathcal{X}$ which maps three elements of $\mathcal{V}$ into $0 \in \mathcal{X}$, three other elements of $\mathcal{V}$ into $1 \in \mathcal{X}$, and the remaining two elements of $\mathcal{V}$ into $2 \in \mathcal{X}$. In this way, the uniform distribution over $\mathcal{V}$ induces the capacity achieving distribution over $\mathcal{X}$. Define the channel $W' : \mathcal{V} \to \mathcal{Y}$ as

$$W'(y \mid v) = W(y \mid f(v)). \quad (9)$$

Take a code which achieves the symmetric capacity of $W'$. Then, we can use this code to achieve the capacity of $W$ via the mapping $f$.

The above scheme works under the assumption that we are able to construct codes which achieve the symmetric capacity for any given input alphabet size. Sometimes it is more convenient to achieve this goal indirectly by using only binary codes. Indeed, suppose that the channel and, therefore, the optimal input distribution change for some reason. In this case, we might have to change the alphabet $\mathcal{V}$, and, if we code directly on $\mathcal{V}$, we will have to change the code itself. If the code needs to be implemented in hardware, this might not be convenient. However, if we manage to use the same binary code and only modify some preprocessing steps, then it is easy to accomplish any required change in the input distribution. This approach can be detailed as follows.

Pick $\mathcal{U} = \{0, 1\}^3$ and consider the function $g : \mathcal{U} \to \mathcal{X}$ which maps three elements of $\mathcal{U}$ into $0 \in \mathcal{X}$, three other elements of $\mathcal{U}$ into $1 \in \mathcal{X}$, and the remaining two elements of $\mathcal{U}$ into $2 \in \mathcal{X}$. In this way, the uniform distribution over $\mathcal{U}$ induces the capacity achieving distribution over $\mathcal{X}$. The set $\mathcal{U}$ contains binary triplets and $u \in \mathcal{U}$ can be written as $u = (u_1, u_2, u_3)$, where $u_i \in \{0, 1\}$ for $i \in \{1, 2, 3\}$. Define the channels $W''_1 : \{0, 1\} \to \mathcal{Y}$, $W''_2 : \{0, 1\} \to \mathcal{Y} \times \{0, 1\}$, $W''_3 : \{0, 1\} \to \mathcal{Y} \times \{0, 1\} \times \{0, 1\}$ as

$$W''_1(y \mid u_1) = \frac{1}{4} \sum_{u_2, u_3} W(y \mid g(u_1, u_2, u_3)),
$$

$$W''_2(y, u_1 \mid u_2) = \frac{1}{4} \sum_{u_3} W(y \mid g(u_1, u_2, u_3)), \quad (10)$$

$$W''_3(y, u_1, u_2 \mid u_3) = \frac{1}{4} W(y \mid g(u_1, u_2, u_3)).$$

Take three binary codes which achieve the symmetric capacities of $W''_1$, $W''_2$, and $W''_3$. Then, we can use these codes to achieve the capacity of $W$ via the mapping $g$. 

B. Description of the General Scheme

Problem Statement. Let $W$ be a DMC s.t. the capacity-achieving input distribution is $\{p^*(x)\}_{x \in \mathcal{X}}$. The aim is to transmit over $W$ with rate close to $I(W)$.

Design of the Scheme. Pick $\delta > 0$ and find a rational approximation $\hat{p}(x)$ which differs from $p^*(x)$ by at most $\delta$ in total variation distance. In formulars, take $\hat{p}(x) = px/dx$ with $nx, dx \in \mathbb{N}$ for any $x \in \mathcal{X}$ s.t.

$$\frac{1}{2} \sum_{x \in \mathcal{X}} |p^*(x) - \hat{p}(x)| < \delta.$$ 

Let $d_{\text{gcd}}$ be the least common divisor of $\{dx\}_{x \in \mathcal{X}}$. Take an extended alphabet $\mathcal{V}$ with cardinality equal to $d_{\text{gcd}}$ and consider the function $f : \mathcal{V} \rightarrow \mathcal{X}$ which maps $d_{\text{gcd}}$ elements of $\mathcal{V}$ into $x \in \mathcal{X}$. Define the channel $W^g : \mathcal{V} \rightarrow \mathcal{Y}$ as in (9). Denote by $X$ and $Y$ the input and the output of the channel $W$, respectively. Let $V$ be uniformly distributed over $\mathcal{V}$ and set $X = f(V)$. Since the uniform distribution over $\mathcal{V}$ induces the input distribution $\hat{p}(x)$ over $\mathcal{X}$, we have that $X \sim \hat{p}(x)$. Construct a code $\mathcal{C}$ which achieves the symmetric capacity of $W^g$. Therefore, by using the code $\mathcal{C}$ and the mapping $f$, we can transmit at rate $R$ arbitrarily close to

$$I_s(W') = I(V; Y) = I(X; Y) \lesssim I(W).$$

As $\delta$ goes to 0, the distribution $\hat{p}$ tends to $p^*$ and $I(X; Y)$ approaches $I(W)$.

If one wants to restrict to binary codes, select a rational approximation of the form $\hat{p}(x) = nx/2^t$ for $t, nx \in \mathbb{N}$. Pick $\mathcal{U} = \{0,1\}^t$ and consider the function $g : \mathcal{U} \rightarrow \mathcal{X}$ which maps $nx$ elements of $\mathcal{U}$ into $x \in \mathcal{X}$. The set $\mathcal{U}$ contains binary vectors of length $t$ which can be written in the form $u_{1:t} = (u_{1}, \ldots, u_{t})^T$, where $u_j \in \{0,1\}$ for $j \in \{1, \ldots, t\}$. Define the synthetic channels $W_j^g : \{0,1\} \rightarrow \mathcal{Y} \times \{0,1\}^{j-1}$, similarly to (10), i.e.,

$$W_j^g(y, u_{1:t-1} | u_j) = \frac{1}{2^{t-1}} \sum_{u_{t+1:t} \in \mathcal{U}} W(y | g(u_{1:t})).$$

Let $U_{1:n}$ be a sequence of $n$ i.i.d. random variables uniform over $\{0,1\}$. Set $X = g(U_{1:t})$. Since the uniform distribution over $\mathcal{U}$ induces the input distribution $\hat{p}(x)$ on $\mathcal{X}$, we have that $X \sim \hat{p}(x)$. Construct $t$ codes $\mathcal{C}_1, \ldots, \mathcal{C}_t$ s.t. $\mathcal{C}_j$ has rate $R_j$ which is arbitrarily close to the symmetric capacity of the channel $W_j^g$. Therefore, by using these codes and the mapping $g$, we can transmit at rate $R$ arbitrarily close to

$$\sum_{j=1}^{t} I_s(U_j; Y | U_{1:j-1}) = I(X; Y) \lesssim I(W),$$

where the first equality comes from the chain rule and $I(X; Y)$ approaches $I(W)$ as $\delta$ goes to 0.

Let us now explain formally how the encoding and decoding operations are done for the schemes mentioned above (see also Figure 1). Then, we will consider the performance of this approach relating the gap $I(W) - I(X; Y)$ to $\delta$ and to the cardinalities of the input and the output alphabets.

Encoding. First, consider the scheme based on a single non-binary code. Let $M$ be the information message which can be thought as a binary string of length $nR$ and let $E$ be the encoder of the code $\mathcal{C}$. The output of the encoder is $v_{1:n} = (v^{(1)}, \ldots, v^{(n)})^T$, where $v^{(i)} \in \mathcal{V}$ for $i \in \{1, \ldots, n\}$. Then, $v_{1:n}$ is mapped component-wise by the function $f$ into $x_{1:n} = (x^{(1)}, \ldots, x^{(n)})^T$, with $x^{(i)} \in \mathcal{X}$ s.t. $x^{(i)} = f(v^{(i)})$.

Second, consider the scheme based on $t$ binary codes. Let $M = (M_1, \ldots, M_t)$ be the information message divided into $t$ parts so that $M_j$ can be thought as a binary string of length $nR_j$ for $j \in \{1, \ldots, t\}$. Let $E_j$ be the encoder of the code $\mathcal{C}_j$ which maps $M_j$ into $u_{1:j} = (u_{1}^{(j)}, \ldots, u_{n}^{(j)})^T$, where $u_{i}^{(j)} \in \{0,1\}$ for $i \in \{1, \ldots, n\}$. Then, $u_{1:t}^n$ is mapped component-wise by the function $g$ into $x_{1:n} = (x^{(1)}, \ldots, x^{(n)})^T$, with $x^{(i)} \in \mathcal{X}$ s.t. $x^{(i)} = g(u_{i}^{(j)})$.

Finally, the sequence $x_{1:n}$ is transmitted over the channel $W$.

Decoding. First, consider the scheme based on a single non-binary code. Let $D$ be the decoder of the code $\mathcal{C}$, which accepts as input the channel output $y_{1:n}^m$ and outputs the estimate $\hat{M}$.

Second, consider the scheme based on $t$ binary codes. Let $D_j$ be the decoder of the code $\mathcal{C}_j$. It accepts as input the channel output $y_{1:n}^m$ and the previous re-encoded estimates $(\hat{u}_{1}^{1:n}, \ldots, \hat{u}_{j-1}^{1:n})$. It outputs the current estimate $\hat{M}_j$. To make the use of the previous estimates possible, the decoding occurs successively, i.e. the decoders $D_1, \ldots, D_t$ are activated in series.
The situation is schematically represented in Figure 1.

**Performance**. As the codes $C$ and $C_j$ can be used to transmit reliably at rates $R$ and $R_j$, then with high probability $\hat{M} = M$ and $\hat{M}_j = M_j$ ($j \in \{1, \ldots, t\}$). As a result, we can transmit over $W$ at rates close to $I(X; Y)$, where the input distribution is $\tilde{p}(x)$. Also, since the mutual information is a smooth function of the input distribution, if $\delta$ gets small, then $I(X; Y)$ approaches $I(W)$. This statement is made precise by the following proposition, which is proved in Appendix C.

**Proposition 5**: Consider transmission over the channel $W : \mathcal{X} \rightarrow \mathcal{Y}$ and let $I(p)$ be the mutual information between the input and the output of the channel when the input distribution is $p$. Let $p$ and $p^x$ be input distributions s.t. their total variation distance is upper bounded by $\delta$, i.e., $\frac{1}{2} \sum_{x \in \mathcal{X}} |p^x(x) - p(x)| < \delta$, for $\delta \in (0, 1/8)$. Then,

\begin{align}
|I(p^x) - I(p)| < 3\delta \log |\mathcal{Y}| + h_2(\delta), \\
|I(p^x) - I(p)| < 7\delta \log |\mathcal{X}| + h_2(\delta) + h_2(4\delta).
\end{align}

Note that the bounds (13) and (14) depend separately on the input and the output alphabet. Therefore, we can conclude that under the hypotheses of Proposition 5 we have

$$|I(p^x) - I(p)| = O \left( \delta \log \left( \frac{\min(|\mathcal{X}|, |\mathcal{Y}|)}{\delta} \right) \right).$$

The main issue of the approach based on Gallager’s mapping lies in the considerable increase in complexity due to the non-linear function $f$ (or $g$). Indeed, the complexity of the non-linear mapper grows exponentially in $d_{\text{lcd}}$ (or exponentially in $2^t$). Hence, in the following sections we consider two alternative solutions.
IV. APPROACH 2: INTEGRATED SCHEME

The basic idea of this approach is to use a coding scheme that is simultaneously good for lossless source coding as well as channel coding. The source coding part is needed to create a biased input distribution from uniform bits, whereas the channel coding part provides reliability for transmission over the channel. A provably capacity-achieving scheme was first proposed in [22] in the context of polar codes. Such a scheme is reviewed in Section IV-A and we refer the interested reader to [27] for a more detailed illustration. We also describe how to extend the idea to sparse graph codes in Section IV-B.

Problem Statement. Let $W$ be a DMC s.t. the capacity-achieving input distribution is $\{p^*(x)\}_{x \in X}$ and $|X| = 2$. Let $X \sim p^*$ and $p^*(1) = \alpha$ for some $\alpha \in [0, 1]$. The aim is to transmit $X$ over $W$ with rate close to $I(W)$.

A. Polar Codes – [22]

Design of the Scheme. Let $(U^{1:n})^T = (X^{1:n})^TG_n$, where $X^{1:n}$ is a column vector of $n$ i.i.d. components drawn according to the capacity-achieving input distribution $p^*$.

Let us start from the source coding part of the scheme. Consider the sets $\mathcal{H}_X$ and $\mathcal{L}_X$ defined as follows: for $i \in \mathcal{H}_X$, $U^{(i)}$ is approximately uniformly distributed and independent of $U^{1:i-1}$; for $i \in \mathcal{L}_X$, $U^{(i)}$ is approximately a deterministic function of the past $U^{1:i-1}$. In formulas,

$$\mathcal{H}_X = \{i \in [n] : Z(U^{(i)} \mid U^{1:i-1}) \geq 1 - \delta_n\},$$

$$\mathcal{L}_X = \{i \in [n] : Z(U^{(i)} \mid U^{1:i-1}) \leq \delta_n\},$$

where $\delta_n = 2^{-n^3}$ for $\beta \in (0, 1/2)$ and $Z(\cdot \mid \cdot)$ denotes the Bhattacharyya parameter. More formally, given $(T, V) \sim p_{T,V}$, where $T$ is binary and $V$ takes values in an arbitrary discrete alphabet $\mathcal{Y}$, we define

$$Z(T \mid V) = 2 \sum_{v \in \mathcal{Y}} \mathbb{P}_V(v) \sqrt{\mathbb{P}_{T \mid V}(0 \mid v) \mathbb{P}_{T \mid V}(1 \mid v)}.$$

It can be shown that the Bhattacharyya parameter $Z(T \mid V)$ is close to 0 or 1 if and only if the conditional entropy $H(T \mid V)$ is close to 0 or 1 (see Proposition 2 of [23]). Consequently, if $Z(T \mid V)$ is close to zero, then $T$ is approximately a deterministic function of $V$, and, if $Z(T \mid V)$ is close to 1, then $T$ is approximately uniformly distributed and independent of $V$. Furthermore, $\mathcal{H}_X$ and $\mathcal{L}_X$ contain, respectively, a fraction of positions which approaches $H(X)$ and $1 - H(X)$, namely,

$$\lim_{n \to \infty} \frac{1}{n} |\mathcal{H}_X| = H(X),$$

$$\lim_{n \to \infty} \frac{1}{n} |\mathcal{L}_X| = 1 - H(X).$$

This is a remarkable and highly non-trivial property of the matrix $G_n$, which is polarizing in the sense that with high probability either $U^{(i)}$ is independent from $U^{1:i-1}$ or it is a deterministic function of it (for a detailed proof of polarization, see [1]). Consequently, we can compress $X^{1:n}$ into the sequence $\{U^{(i)}\}_{i \in \mathcal{L}_X}$, which has size roughly $nH(X)$. Indeed, given $\{U^{(i)}\}_{i \in \mathcal{L}_X}$, we can recover the whole vector $U^{1:n}$ in a successive manner, since $U^{(i)}$ is approximately a deterministic function of $U^{1:i-1}$ for $i \in \mathcal{L}_X$.

Now, let us consider the channel coding part of the scheme. Assume that the channel output $Y^{1:n}$ is given, and interpret this as side information for $X^{1:n}$. Consider the sets $\mathcal{H}_X \mid Y$ and $\mathcal{L}_X \mid Y$ which contain, respectively, the positions $i \in [n]$ in which $U^{(i)}$ is approximately uniform and independent of $(U^{1:i-1}, Y^{1:n})$ and approximately a deterministic function of $(U^{1:i-1}, Y^{1:n})$. In formulas,

$$\mathcal{H}_X \mid Y = \{i \in [n] : Z(U^{(i)} \mid U^{1:i-1}, Y^{1:n}) \geq 1 - \delta_n\},$$

$$\mathcal{L}_X \mid Y = \{i \in [n] : Z(U^{(i)} \mid U^{1:i-1}, Y^{1:n}) \leq \delta_n\}.$$

Similarly to the case of lossless compression, $\mathcal{H}_X \mid Y$ and $\mathcal{L}_X \mid Y$ contain, respectively, a fraction of positions which approaches $H(X \mid Y)$ and $1 - H(X \mid Y)$. Consequently, given the channel output $Y^{1:n}$ and the sequence $\{U^{(i)}\}_{i \in \mathcal{L}_X \mid Y}$, we can recover the whole vector $U^{1:n}$ in a successive manner.
and the receiver. For positions with bits chosen uniformly at random, and this randomness is assumed to be shared between the transmitter and the receiver, finally the elements of \( I \) over the channel.

There are not sufficiently independent given the previous bits and the prior distribution on \( X \). Nevertheless, not all bits in \( \mathcal{H}_X \) have a very strong bias, as the inclusion \( \mathcal{H}_X \supset \mathcal{L}_X \) is strict. In particular, from (17) we only know to a deterministic rule.

The encoder first places the information bits into \( \mathcal{F}_r = \mathcal{H}_X \cap \mathcal{L}_X \cap Y \). The top left square represents the subdivision of indices which yields the source coding part of the scheme. The top right square represents the subdivision of indices which yields the channel coding part of the scheme. As a result, the set of indices \([n]\) can be partitioned into three subsets (bottom image): the information indices \( \mathcal{I} = \mathcal{H}_X \cap \mathcal{L}_X \cap Y \); the frozen indices \( \mathcal{F}_r = \mathcal{H}_X \cap \mathcal{L}_X \cap Y \) filled with uniformly random bits; the frozen indices \( \mathcal{F}_d = \mathcal{H}_X \) chosen according to a deterministic rule.

To construct a polar code for the channel \( W \) we proceed as follows. We place the information in the positions indexed by \( \mathcal{I} = \mathcal{H}_X \cap \mathcal{L}_X \cap Y \). Since \( \mathcal{H}_X \supset \mathcal{L}_X \), it follows that

\[
\lim_{n \to \infty} \frac{1}{n} |\mathcal{I}| = H(X) - H(X \mid Y) = I(X; Y).
\]

Hence, our requirement on the transmission rate is met.

The remaining positions are frozen. More precisely, they are divided into two subsets, namely \( \mathcal{F}_r = \mathcal{H}_X \cap \mathcal{L}_X \cap Y \) and \( \mathcal{F}_d = \mathcal{H}_X \). For \( i \in \mathcal{F}_r \), \( U^{(i)} \) is independent of \( U^{1:i-1} \), but it cannot be reliably decoded using \( Y^{1:n} \). We fill these positions with bits chosen uniformly at random, and this randomness is assumed to be shared between the transmitter and the receiver. For \( i \in \mathcal{F}_d \), the value of \( U^{(i)} \) has to be chosen in a particular way. This is because almost all these positions are in \( \mathcal{L}_X \) and, hence, \( U^{(i)} \) is approximately a deterministic function of \( U^{1:i-1} \). The situation is schematically represented in Figure 2.

**Encoding.** The encoder first places the information bits into \( \{u^{(i)}\}_{i \in \mathcal{I}} \). Then, \( \{u^{(i)}\}_{i \in \mathcal{F}_r} \) is filled with a random sequence which is shared between the transmitter and the receiver. Finally, the elements of \( \{u^{(i)}\}_{i \in \mathcal{F}_d} \) are computed in successive order and, for \( i \in \mathcal{F}_d \), \( u^{(i)} \) is set to \( \arg \max_{u \in \{0,1\}} P_{U^{(i)} \mid U^{1:i-1}}(u \mid u^{1:i-1}) \). These probabilities can be computed recursively with complexity \( \Theta(n \log n) \). Since \( G_n = G_n^{(-1)} \), the vector \( (x^{1:n})^T = (u^{1:n})^T G_n \) is transmitted over the channel.

**Decoding.** The decoder receives \( y^{1:n} \) and computes the estimate \( \hat{u}^{1:n} \) of \( u^{1:n} \) according to the rule

\[
\hat{u}^{(i)} = \begin{cases} 
    u^{(i)}, & \text{if } i \in \mathcal{F}_r \\
    \arg \max_{u \in \{0,1\}} P_{U^{(i)} \mid U^{1:i-1}}(u \mid u^{1:i-1}), & \text{if } i \in \mathcal{F}_d \\
    \arg \max_{u \in \{0,1\}} P_{U^{(i)} \mid U^{1:i-1}, Y^{1:n}}(u \mid u^{1:i-1}, y^{1:n}), & \text{if } i \in \mathcal{I}
\end{cases}
\]

where \( P_{U^{(i)} \mid U^{1:i-1}, Y^{1:n}}(u \mid u^{1:i-1}, y^{1:n}) \) can be computed recursively with complexity \( \Theta(n \log n) \).

**Performance.** The block error probability \( P_e \) can be upper bounded by

\[
P_e \leq \sum_{i \in \mathcal{I}} Z(U^i \mid U^{1:i-1}, Y^{1:n}) = O(2^{-n^\beta}), \ \forall \beta \in (0,1/2).
\]
that the set \( \mathcal{H}_X \setminus \mathcal{L}_X \) is of size \( o(n) \) and hence might contain a non-zero, but vanishing, fraction of the whole set of indices. Hence, one might worry whether the transmitter and receiver are capable of reconstructing all these positions in exactly the same way. To see that indeed they can, note that both the encoder and the decoder perform identical computations. In particular at the receiver this computation is not based on the observation \( Y^{1:n} \) (which contains randomness) but solely on the previously decoded components of \( U \) and on the bias of \( X \).

B. Sparse Graph Codes

**Design of the Scheme.** Consider a linear code with parity check matrix \( P \) with \( nH(X) = nh_2(\alpha) \) rows and \( n \) columns, namely \( P \in \mathbb{F}_2^{nh_2(\alpha) \times n} \). Let \( X^{1:n} \in \mathbb{F}_2^n \) be a codeword to be transmitted over the channel and let \( S^{1:nh_2(\alpha)} \in \mathbb{F}_2^{nh_2(\alpha)} \) be the vector of “syndromes” s.t. \( S^{1:nh_2(\alpha)} = PX^{1:n} \).

Recall that in the integrated scheme we need to achieve the source coding and the channel coding part at the same time. To do so, we will divide \( S^{1:nh_2(\alpha)} \) into two parts, i.e., \( S^{1:nh_2(\alpha)} = (S_1^{1:nI(W)}, S_2^{1:nH(X|Y)T}) \). The first part, namely \( S_1^{1:nI(W)} \), is designed to contain directly the information bits. This is quite different from what happens in a standard parity check code, in which the values of the parity checks are fixed and shared between the encoder and the decoder. On the other hand, in the proposed scheme, the value of the parity checks contains information about the message that is transmitted. The second part, namely \( S_2^{1:nH(X|Y)} \), contains bits chosen uniformly at random, and this randomness is assumed to be shared between the transmitter and the receiver. Note that \( S_2^{1:nH(X|Y)} \) does not depend on the information bits. Also the parity check matrix \( P \) is divided into two parts, i.e. \( P = [P_1, P_2]^T \), where \( P_1 \in \mathbb{F}_2^{nI(W)\times n} \) and \( P_2 \in \mathbb{F}_2^{nH(X|Y)\times n} \).

The choice of the parity check matrix \( P_2 \) concerns the channel coding part of the scheme. Recall the problem considered in Section II-C given a sequence \( X^{1:n} \) of \( n \) i.i.d. Bernoulli(\( \alpha \)) random variables, the number of parity checks to be shared between the transmitter and the receiver is roughly \( nH(X|Y) \). Further, the parity check matrix can be chosen to be that of a code which achieves the capacity of the “symmetrized channel” with posterior \( L \)-density given by (7). Now, assume that \( X^{1:n} \) has the correct distribution and assume that it fulfills the \( nH(X|Y) \) parity checks \( S_2^{1:nH(X|Y)} \). Observe that the parity checks \( S_2^{1:nH(X|Y)} \) are shared with the receiver. Then, by the argument of Section II-C if \( P_2 \) is the parity check matrix of an \((1,x)\)-regular SC-LDPC ensemble with sufficiently large degrees, \( X^{1:n} \) can be decoded reliably with high probability. This suffices to solve the channel coding part of the problem.

The choice of the parity check matrix \( P_1 \) concerns the source coding part of the scheme. In particular, we need to fulfill the assumptions that we required for the channel coding part: \( X^{1:n} \) needs to have the correct distribution and \( P_2X^{1:n} = S_2^{1:nH(X|Y)} \). In addition, we also require that \( P_1X^{1:n} = S_1^{1:nI(W)} \), where \( S_1^{1:nI(W)} \) contains the information bits. In short, we need to satisfy all the parity check equations, namely \( PX^{1:n} = S^{1:nh_2(\alpha)} \), where \( S^{1:nh_2(\alpha)} \) can be modeled as a vector of \( nh_2(\alpha) \) i.i.d. random variables taken uniformly at random.

Before moving on with the description of the scheme, let us make a brief detour to recall how lossless source coding works and to present a solution based on sparse graph codes. The aim is to compress the vector \( X^{1:n} \) of \( n \) i.i.d. Bernoulli(\( \alpha \)) random variables into a binary sequence of size roughly \( nh_2(\alpha) \). We want to solve the problem using the parity check matrix \( \hat{P} \) of a LDPC code as the linear compressor and the BP decoder as the decompressor, respectively [24]. More specifically, given the codeword \( x^{1:n} \) to be compressed, the encoder computes \( s^{1:nh_2(\alpha)} = \hat{P}x^{1:n} \). The task of the decoder can be summarized as follows:

**Task 1.** Recover \( x^{1:n} \) given \( s^{1:nh_2(\alpha)} \) using the BP algorithm.

Now, consider transmission over the BSC(\( \alpha \)). The encoder sends a codeword \( e^{1:n} \) s.t. \( \hat{P}e^{1:n} = 0^{1:nh_2(\alpha)} \). Then, the decoder receives \( y^{1:n} \) and computes the syndrome \( \hat{P}y^{1:n} = \hat{P}e^{1:n} \), where \( e(i) = 1 \) if the \( i \)-th bit was flipped by the channel and 0 otherwise. Consider the following two tasks:

**Task 2.** Recover \( e^{1:n} \) given the syndrome \( \hat{P}y^{1:n} \) using the BP algorithm.

**Task 3.** Recover \( e^{1:n} \) given that \( \hat{P}e^{1:n} = 0^{1:nh_2(\alpha)} \) using the BP algorithm.

After some thinking, one realizes that **Task 1** and **Task 2** coincide. Then, by writing down the message-passing equations for **Task 2**, one obtains that **Task 2** succeeds if and only if **Task 3** succeeds (see equations (1)-(3) in [24]). This suffices to prove that one can use LDPC codes to perform source coding.

Let us come back to our original problem of achieving the capacity of a B-DMC. The source coding part of our approach is basically the inverse of source coding. Indeed, we start from the parity checks \( S^{1:nh_2(\alpha)} \) which are chosen uniformly at random and we want to obtain a biased codeword \( X^{1:n} \). In other words, we perform the task of the decoder, but from the side of the encoder. In this case, the standard BP algorithm is not suitable and it will not be
In order to solve this issue, we combine the message-passing algorithm with decimation steps [28]: after every $t$ iterations of the BP algorithm, for some fixed $t \in \mathbb{N}$, we decimate a small fraction of the codeword bits. More precisely, this means that we set each decimated bit to its most likely value. The decimated bits are now fixed and they will not change during the next iterations of the algorithm, in the sense that the modulus of their log-likelihood ratio is set to $+\infty$. The algorithm ends when all the bits have been decimated. At this point, we count how many parity checks are not fulfilled. If $P_1$ (and, consequently, $P$) is the parity check matrix of a regular SC-LDPC ensemble with sufficiently large degrees, then the fraction of unfulfilled parity checks will tend to 0 as $n$ goes large. To cope with this negligible fraction of unfulfilled parity checks, we can precode the vector of parity checks with a negligible loss in rate. As a result, also the source coding part of the problem is solved.

Note that the decimation steps allow the BP algorithm to converge, but at the same time make its analysis difficult. Indeed, the analysis of this sort of algorithms remains an open problem.

From numerical simulations, we also note that the first event occurs with vanishing probability. However, this empirical observation is not supported by a theoretical analysis because of the decimation steps introduced in the BP algorithm.

In short, the standard BP algorithm does not work because we are “on the wrong side of capacity”. Note that the decimation steps allow the BP algorithm to converge, but at the same time make its analysis difficult. Indeed, the analysis of this sort of algorithms remains an open problem. Consequently, the proposed scheme based on sparse graph codes works well in numerical simulations, but we currently have no theoretical guarantees on its properties.

**Encoding.** The encoder first places the information bits into $s_1^{1:nI(W)}$. Then, $s_2^{1:nH(X|Y)}$ is filled with a sequence chosen uniformly at random and shared between the transmitter and the receiver. From the syndrome $s_2^{1:nH(X)} = (s_1^{1:nI(W)}, s_2^{1:nH(X|Y)})^T$, the encoder deduces the codeword $x^{1:n}$ using the BP algorithm with decimation steps. The vector $x^{1:n}$ is transmitted over the channel.

**Decoding.** The decoder receives the sequence $y^{1:n}$. Then, it runs the BP algorithm using the $nH(X|Y)$ parity checks $s_2^{1:nH(X|Y)}$ shared with encoder. Let $\hat{x}^{1:n}$ be the output of the BP algorithm. Thus, we obtain the estimate $s_1^{1:nI(W)} = P_1 \hat{x}^{1:n}$ of the information vector.

**Performance.** There are two possible types of errors. First, the encoder may fail to produce a codeword $x^{1:n}$ with the correct distribution (namely, with roughly $n\alpha$ 1’s), given the vector of parity checks $s_2^{1:nh_2(\alpha)}$. Second, the decoder may not estimate correctly the transmitted vector, i.e., $\hat{x}^{1:n} \neq x^{1:n}$, which implies that the information vector is not correctly recovered at the receiver, i.e., $s_1^{1:nI(W)} \neq s_1^{1:nI(W)}$.

The second event occurs with vanishing probability and this result is provable following the argument of Section II-C. From numerical simulations, we also note that the first event occurs with vanishing probability. However, this empirical observation is not supported by a theoretical analysis because of the decimation steps introduced in the BP algorithm.

**V. APPROACH 3: CHAINING CONSTRUCTION**

Whereas in the integrated approach discussed in the preceding section the idea was to use a single code to accomplish both the source and the channel coding part, the chaining construction allows to separate these two tasks completely. In this way, one can combine any solution to the source coding part with any solution to the channel coding part.

**Problem Statement.** Let $W$ be a DMC s.t. the capacity-achieving input distribution is $\{p^*(x)\}_{x \in \mathcal{X}}$ and $|\mathcal{X}| = 2$. Let $X \sim p^*$ and $p^*(1) = \alpha$ for some $\alpha \in [0, 1]$. The aim is to transmit $X$ over $W$ with rate close to $I(W)$.

**Design of the Scheme.** First, consider the source coding part of the scheme. This is an inverse source coding problem and in the previous section we have described a solution which makes use of LDPC codes. Let us now consider this task from a more general point of view.

In the traditional lossless source coding setting, the input is a sequence $X^{1:n}$ of $n$ i.i.d. Bernoulli$(\alpha)$ random variables and the encoder consists of a map from the set of source outputs, i.e., $\{0,1\}^n$, to the set $\{0,1\}^*$ of finite-length binary strings. Let $f : \{0,1\}^n \to \{0,1\}^*$ be the encoding mapping, so that $U = f(X^{1:n})$ is the compact description of the source output $X^{1:n}$. For a good source code, the expected binary length of $U$ is close to the entropy
of the source, which in our case equals \( nh_2(\alpha) \). In addition, there must be a decoding function \( g : \{0, 1\}^* \rightarrow \{0, 1\}^n \) s.t. \( X^{1:n} = g(f(X^{1:n})) \). This is required to hold with high probability with respect to the choice of \( X^{1:n} \). A large number of solutions to this problem has been proposed to date, such as Huffman coding, arithmetic coding, Lempel-Ziv compression [29], polar codes [23], and LDPC codes [24], just to name a few.

In our setting, we do not consider finite-length binary strings in \( \{0, 1\}^* \) with expected binary length close to the entropy of the source. On the other hand, we consider mappings from \( \{0, 1\}^n \) to \( \{0, 1\}^m \) (and vice versa), where \( m \) is an integer of size roughly \( nh_2(\alpha) \). In addition, we do not start with the biased vector \( X^{1:n} \) but with the compressed vector \( U^{1:m} \) which is a sequence of \( m \) i.i.d. uniformly distributed random variables. More precisely, the encoder and the decoder implement the maps \( g : \{0, 1\}^m \rightarrow \{0, 1\}^n \) and \( f : \{0, 1\}^n \rightarrow \{0, 1\}^m \), respectively. With high probability with respect to the choice of \( U^{1:m} \), we need to ensure that \( X^{1:n} = g(U^{1:m}) \) is a vector of \( n \) i.i.d. Bernoulli(\( \alpha \)) random variables and that \( U^{1:m} = f(g(U^{1:m})) \). Note that not all schemes that are good for the traditional source coding setting are also suitable for this inverse source coding problem. For example, if we wish to use LDPC codes as in the previous section, the standard message-passing algorithm needs to be combined with decimation steps, which makes it hard to rigorously analyze the whole procedure. On the other hand, some solutions to the standard source coding setting immediately yield an algorithm for the inverse problem. As an example, consider the decimation steps, which makes it hard to rigorously analyze the whole procedure. On the other hand, we consider mappings from \( \{0, 1\}^n \) to \( \{0, 1\}^m \) and share with the receiver the positions indexed by \( g(X) \) and the previously decoded positions, since \( nH(X | Y) \) parity checks. An alternative solution is based on polar coding techniques. Multiply \( X^{1:n} \) with the polarizing matrix \( G_n \) and share with the receiver the positions in \( \mathcal{L}^C_{X^1:n} \), which are roughly \( nH(X | Y) \). Indeed, the remaining positions can be deduced from the channel output \( Y^{1:m} \) and the previously decoded bits by running the SC algorithm.

Here, the issue lies in the fact that the value of the redundant bits depends on the codeword itself and, therefore, the encoder and the decoder cannot agree on it before transmission starts. Hence, we need to bind in some way the source and the channel coding parts of the scheme so that the receiver has access to the redundancy of the message when it needs to decode it. To do so, we draw inspiration from the “chaining” construction introduced in [30]. Consider the transmission of \( k \) blocks of information and use a part of the current block to store the redundancy of the previous block. More specifically, in block 1 fill \( U^{1:m} \) with information bits, compute \( X^{1:n} = g(U^{1:m}) \) and the associated redundant bits. In block \( j \) (\( j \in \{2, \cdots, k-1\} \)) fill \( U^{1:m} \) with the redundancy coming from the previous block, store the actual information bits in the remaining positions, compute \( X^{1:n} = g(U^{1:m}) \) and the associated redundant bits. In block \( k \) transmit only the redundancy coming from the previous block at rate \( I_s(W) \) using a code which achieves the symmetric capacity of the channel \( W \) (see Section II-B for polar coding and sparse graph coding schemes which achieve such a goal).

In the generic block \( j \) (\( j \in \{2, \cdots, k-1\} \)) we have at our disposal roughly \( nh_2(\alpha) \) positions, since \( m \) is an integer of size roughly \( nh_2(\alpha) \). We use \( nH(X | Y) \) of them to store the redundancy of the previous block and the remaining positions contain the actual information bits. Therefore, the rate of transmission of the actual information is roughly \( I(W) = h_2(\alpha) - H(X | Y) \), as requested. Note that we lose some rate in the last block since we are limited to a rate of \( I_s(W) < I(W) \). However, this rate loss decays like \( 1/k \) and, by choosing \( k \) large, we can achieve a rate as close to \( I(W) \) as desired.

At the receiver, the decoding has to be performed “backwards”, starting with block \( k \) and ending with block 1. Indeed, block \( k \) can be easily decoded since the underlying code achieves the symmetric capacity of the channel. For block \( j \) (\( j \in \{k-1, \cdots, 1\} \)), the decoder can recover \( X^{1:n} \) using the redundancy coming from the next block which has been already decoded. Finally, from \( X^{1:n} \) the decoder deduces \( U^{1:m} = f(X^{1:n}) \).

**Encoding.** To fix the ideas, assume that the channel coding part of block \( j \) (\( j \in \{1, \cdots, k-1\} \)) is performed by means of a regular SC-LDPC code of rate \( I(W) \) with sufficiently large degrees. Let \( P \) be its parity check matrix. Assume also that \( C_s \) is a code which achieves the symmetric capacity of the channel \( W \). Denote by \( E_s \) the encoder of the code \( C_s \).
Let us start from block 1. The transmitter picks \( nh_2(\alpha) \) bits of information, puts them into \( u_{(1)}^{1:nh_2(\alpha)} \), computes the codeword \( x_{(1)}^{1:n} = g(u_{(1)}^{1:nh_2(\alpha)}) \) and the vector of parity checks \( s_{(1)}^{1:nH(X|Y)} = Px_{(1)}^{1:n} \). The codeword \( x_{(1)}^{1:n} \) is going to be transmitted over the channel \( W \) and the vector of parity checks \( s_{(1)}^{1:nH(X|Y)} \) is going to be stored in the next block. For block \( j \) (\( j \in \{2, \ldots, k-1\} \)), the transmitter puts the vector of parity checks \( s_{(j-1)}^{1:nH(X|Y)} \) coming from the previous block plus \( nI(W) \) bits of new information into \( u_{(j)}^{1:nh_2(\alpha)} \), computes the codeword \( x_{(j)}^{1:n} = g(u_{(j)}^{1:nh_2(\alpha)}) \), which is going to be transmitted over the channel \( W \), and the vector of parity checks \( s_{(j)}^{1:nH(X|Y)} = Px_{(j)}^{1:n} \), which is going to be stored in the next block. For block \( k \), the transmitter fills \( u_{(k)}^{1:nh_2(\alpha)} \) with the vector of parity checks \( s_{(k-1)}^{1:nH(X|Y)} \) coming from the previous block and transmits over the channel \( W \) the codeword obtained from \( E_s \). Note that this codeword has length roughly \( nH(X|Y)/I_s(W) \). The overall rate of communication is given by

\[
R = \frac{nh_2(\alpha) + n(k-2)I(W)}{n(k-1) + nH(X|Y)/I_s(W)},
\]

which, as \( k \) goes large, tends to the required rate \( I(W) \).

**Decoding.** The receiver operates “backwards” after the reception of all the \( k \) blocks. Denote by \( D_s \) the decoder of the code \( C_s \) and by \( D_{BP} \) the decoder which implements the BP algorithm in order to decode the SC-LDPC code with parity check matrix \( P \).

Let us start from block \( k \). The received message \( y_{(k)}^{1:nH(X|Y)/I_s(W)} \) is fed to the decoder \( D_s \) of the code \( C_s \), in order to obtain the estimate \( \hat{x}_{(k)}^{1:nH(X|Y)} \) on the payload of block \( k \). This immediately yields the estimate \( \hat{s}_{(k-1)}^{1:nH(X|Y)} \) on the vector of parity checks of block \( k-1 \). For block \( j \) (\( j \in \{k-1, \ldots, 2\} \)), the decoder \( D_{BP} \) runs the BP algorithm using the received message \( y_{(j)}^{1:n} \) together with the estimate \( \hat{s}_{(j)}^{1:nH(X|Y)} \) on the vector of parity checks. The output of the BP decoder is \( \hat{x}_{(j)}^{1:n} \) and the receiver computes \( \hat{u}_{(j)}^{1:nh_2(\alpha)} = f(\hat{x}_{(j)}^{1:n}) \), which yields an estimate on the information bits transmitted in block \( j \) and on the vector of syndromes \( \hat{s}_{(j-1)}^{1:nH(X|Y)} \) of block \( j-1 \). For block 1 the decoding process is the same as that for block \( j \) (\( j \in \{k-1, \ldots, 2\} \)). The only difference consists in the fact that \( \hat{u}_{(1)}^{1:nh_2(\alpha)} \) contains solely an estimate on information bits.

The situation is schematically represented in Figure 3.

**Performance.** There are four possible types of errors.

Figure 3. Coding over asymmetric channels via the chaining construction. We consider the transmission of \( k = 3 \) blocks and we store in block \( j \) (\( j \in \{2, 3\} \)) the vector of parity checks of block \( j-1 \).
1) In block \( j \) (\( j \in \{1, \ldots, k-1\} \)), the encoder may fail to produce a codeword \( x_{(j)}^{1:n} \) with the correct distribution (namely, with roughly \( n/2 \) 1’s).

2) In block \( j \) (\( j \in \{1, \ldots, k-1\} \)), we may have that \( \hat{x}_{(j)}^{1:n} \neq x_{(j)}^{1:n} \).

3) In block \( k \), we may have that \( \hat{x}_{(k)}^{1:n} \neq x_{(k)}^{1:n} \).

4) In block \( j \) (\( j \in \{1, \ldots, k-1\} \)), given that \( \hat{x}_{(j)}^{1:n} = x_{(j)}^{1:n} \), we may have that \( \hat{u}_{(j)}^{1:m_2(\alpha)} \neq u_{(j)}^{1:m_2(\alpha)} \).

Recall that the source coding part of the scheme needs to ensure that with high probability \( X^{1:n} = g(U^{1:m}) \) is a vector of \( n \) i.i.d. Bernoulli(\( \alpha \)) random variables and that \( U^{1:m} = f(g(U^{1:m})) \). Hence, the first and the last event occur with negligible probability. By the argument of Section II-C in order to reliably decode the biased codeword \( X^{1:n} \), it suffices to share with the receiver the value of roughly \( nH(X | Y) \) parity checks computed via the parity check matrix of an SC-LDPC code with sufficiently large degrees. Hence, the second event occurs with negligible probability. Since the code \( C_s \) achieves the symmetric capacity of the channel \( W \), also the third event occurs with negligible probability. As a result, the overall block error probability of the proposed scheme tends to 0, as \( n \) goes large.

VI. CONCLUDING REMARKS

This paper discusses three different approaches (and their variants) in order to achieve the capacity of asymmetric DMCs.

The first approach is based on Gallager’s mapping. The idea was first described in [19] and it consists of employing a non-linear function in order to make the input distribution match the capacity-achieving one. In this way, we can achieve the capacity of an asymmetric DMC by using either \( q \)-ary or binary codes which achieve the symmetric capacity of suitable channels. Note that the complexity of the system is considerably increased by the introduction of the non-linear mapper.

The second approach consists in an integrated scheme, which simultaneously performs the tasks of lossless source coding and of channel coding. The idea was first presented in the context of polar coding in [22] and, here, we extend it to spatially coupled codes. Indeed, sparse graph codes can be effectively used to create biased codewords from uniform bits (source coding part) and to provide error correction (channel coding part). Note that the solution based on spatially coupled codes is not completely provable. In order to generate the codeword with the capacity-achieving distribution from uniform bits, we combine the BP algorithm with decimation steps. This technique works well in numerical simulations, but the theoretical analysis of its properties is still an open problem.

The third approach is based on a chaining construction and it is new. The idea is to consider the transmission of \( k \) blocks and use a part of the current block to store the redundancy coming from the previous block. In this way, we decouple completely the lossless source coding from the channel coding task. This decoupling makes it possible to use an optimal scheme to reach each of these two objectives separately and many combinations are possible: for example, we can use spatially-coupled codes or polar codes for the channel coding part, and then polar codes or Huffman codes for the source coding part. Differently from the integrated scheme, no shared randomness is needed as long as the information bits are assumed to be uniformly distributed. On the downside, the chaining construction introduces extra delay, since the information to be transmitted is now coupled over \( k \) coding blocks. In addition, the value of \( k \) cannot be made too small, because we incur in a rate loss proportional to \( 1/k \) (see formula (22)).

As concerns the second and the third approach (integrated scheme and chaining construction), we restrict our discussion to the case of binary-input channels. In order to extend the results to arbitrary finite input alphabets, one needs schemes that solve the source coding and the channel coding tasks for the non-binary case. As concerns the source coding part, several works have focused on the construction of polar codes for \( q \)-ary alphabets [7]–[11]. Further, the solution based on Huffman coding is also quickly generalized to non-binary alphabets. As concerns the channel coding part, recall that in Section III we converted a non-binary channel into several binary channels by using the standard chain rule of mutual information (see formula (12)). Here, the same idea can be applied as well. Alternatively, one can use directly non-binary spatially coupled codes [12]–[15].

ACKNOWLEDGMENT

The authors would like to thank E. Telatar for helpful discussion. This work was supported by grant No. 200020_146832/1 of the Swiss National Science Foundation.
A. Proof of Propositions in Section II-B

Proof of Proposition 1: By definition of $L$-density, we have that

\[
a^+(y)\Delta y \approx \int_{t \in L^{-1}([y, y+\Delta y])} W(t \mid 1) \, dt = \int_{t \in L^{-1}([y, y+\Delta y])} e^{L(t)} W(t \mid -1) \, dt \\
\approx e^y \int_{t \in L^{-1}([y, y+\Delta y])} W(t \mid 1) \, dt = e^y a^-(y) \Delta y,
\]

where $L^{-1}$ is the inverse of the log-likelihood ratio defined in (1). By taking $\Delta y \to 0$, we obtain that

\[
a^+(y) = e^y a^-(y).
\] (23)

With the change of variable $y \to -y$, one also obtains that

\[
a^-(y) = e^y a^+(y).
\] (24)

As a result, condition (2) is fulfilled for the $L$-density $a^y(y)$ defined in (4) and the statement follows.

Proof of Proposition 2: Since the log-likelihood ratio constitutes a sufficient statistic, two B-DMCs are equivalent if they have the same $L$-densities given that $X = \pm 1$ is transmitted. As a representative for the equivalence class, we can take

\[
W(y \mid 1) = a^+(y), \\
W(y \mid -1) = a^-(y).
\] (25)

By definition of log-likelihood ratio and by using (23), we have

\[
L(y) = \ln \frac{W(y \mid 1)}{W(y \mid -1)} = \ln \frac{a^+(y)}{a^-(y)} = y.
\]

Therefore,

\[
\lim_{\Delta y \to 0} \frac{\mathbb{P}(L(Y) \in [y, y+\Delta y] \mid X = \pm 1)}{\Delta y} = a^\pm(y),
\]

which means that (25) is a valid choice.

Let $X$ be uniformly distributed. Then, after some calculations we have that

\[
I_s(W) = H(Y) - H(Y \mid X) = \frac{1}{2} \int W(y \mid 1) \log \frac{2W(y \mid 1)}{W(y \mid 1) + W(y \mid -1)} \, dy \\
+ \frac{1}{2} \int W(y \mid -1) \log \frac{2W(y \mid -1)}{W(y \mid 1) + W(y \mid -1)} \, dy.
\] (26)

By applying (24) and (23), the first integral simplifies to

\[
\frac{1}{2} \int a^+(y) \left(1 - \log(1 + e^{-y})\right) \, dy.
\] (27)

By applying (25), doing the change of variables $y \to -y$ and using (24), the second integral simplifies to

\[
\frac{1}{2} \int a^-(y) \left(1 - \log(1 + e^{-y})\right) \, dy.
\] (28)

By combining (26), (27), and (28), the result follows.
B. Proof of Propositions in Section II-C

Proof of Proposition 3: By definition of posterior \( L \)-density, we have that

\[
\bar{a} a^+(y) \Delta y \approx \int_{t \in \mathcal{L}^{-1}(y, y + \Delta y)} \bar{a} W(t \mid 1) \, dt = \int_{t \in \mathcal{L}^{-1}(y, y + \Delta y)} \alpha e^{L_p(t)} W(t \mid -1) \, dt \\
\approx e^y \int_{t \in \mathcal{L}^{-1}(y, y + \Delta y)} \alpha W(t \mid -1) \, dt = e^y \bar{a} a^-(y) \Delta y,
\]

where \( \mathcal{L}^{-1} \) is the inverse of the posterior log-likelihood ratio defined in (6). By taking \( \Delta y \to 0 \), we obtain that

\[
\bar{a} a^+(y) = e^y \bar{a} a^-(y).
\]

(29)

With the change of variable \( y \to -y \), one also obtains that

\[
\alpha a^-(y) = e^y \alpha a^+(y).
\]

(30)

As a result, condition (2) is fulfilled for the posterior \( L \)-density as \( p(y) \) and the statement follows.

Proof of Proposition 4: Since the log-likelihood ratio constitutes a sufficient statistic, two B-DMCs with non-uniform input distributions are equivalent if they have the same posterior \( L \)-densities given that \( X = \pm 1 \) is transmitted. As a representative for the equivalence class, we can take

\[
W(y \mid 1) = a^+(y), \\
W(y \mid -1) = a^-(y).
\]

(31)

By definition of posterior log-likelihood ratio and by using (29), we have

\[
L_p(y) = \ln \frac{p_X(Y \mid 1 \mid y)}{p_X(Y \mid -1 \mid y)} = \ln \frac{\bar{a} a^+(y)}{\alpha a^- (y)} = y.
\]

Therefore,

\[
\lim_{\Delta y \to 0} \frac{\mathbb{P}(L_p(Y) \in [y, y + \Delta y] \mid X = \pm 1)}{\Delta y} = a^\pm (y),
\]

which means that (31) is a valid choice.

Let \( X \in \{-1, 1\} \) be s.t. \( \mathbb{P}(X = -1) = \alpha \). Then, after some calculations we have that

\[
H(X \mid Y) = -\int \bar{a} W(y \mid 1) \log \frac{\bar{a} W(y \mid 1)}{\alpha W(y \mid 1) + \alpha W(y \mid -1)} \, dy - \int \alpha W(y \mid -1) \log \frac{\alpha W(y \mid -1)}{\alpha W(y \mid 1) + \alpha W(y \mid -1)} \, dy.
\]

By applying (31) and (29), the first integral simplifies to

\[
\int \bar{a} a^+(y) \left(1 - \log(1 + e^{-y})\right) \, dy.
\]

(33)

By applying (31), doing the change of variables \( y \to -y \) and using (30), the second integral simplifies to

\[
\int \alpha a^-(y) \left(1 - \log(1 + e^{-y})\right) \, dy.
\]

(34)

By combining (32), (33), and (34), the result follows.
C. Proof of Proposition [5]

Before starting with the proof of the proposition, let us state the following useful result [31], which is a refinement of [32] Lemma 2.7.

Lemma 1: Consider two distributions \( p \) and \( p^* \) over the alphabet \( \mathcal{X} \) s.t. their total variation distance is equal to \( \delta \), i.e., \( \frac{1}{2} \sum_{x \in \mathcal{X}} |p^*(x) - p(x)| = \delta \). Take \( X \sim p \) and \( X^* \sim p^* \). Then,

\[
|H(X^*) - H(X)| \leq \delta \log(|\mathcal{X}| - 1) + h_2(\delta) \leq \delta \log |\mathcal{X}| + h_2(\delta).
\]  

(35)

Proof of Proposition [5] Let \( X \sim p \), \( X^* \sim p^* \) and denote by \( Y \sim p_Y \) and \( Y^* \sim p_Y^* \) the outputs of the channel when the input is \( X \) and \( X^* \), respectively. Denote by \( W(y \mid x) \) the probability distribution associated to the channel \( W \). In order to prove \((43)\), we write

\[
|I(p^*) - I(p)| \leq |H(Y^*) - H(Y)| + |H(Y^* \mid X^*) - H(Y \mid X)|,
\]  

(36)

and we bound both terms as functions of \( \delta \) and \( |\mathcal{Y}| \). For the first term, observe that

\[
\frac{1}{2} \sum_{y \in \mathcal{Y}} |p_Y^*(y) - p_Y(y)| \leq \frac{1}{2} \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} W(y \mid x) p_Y^*(x) - p(x) | < \delta,
\]

(37)

where we used the fact that \( \sum_{y \in \mathcal{Y}} W(y \mid x) = 1 \) for any \( x \in \mathcal{X} \). Then, using Lemma [11] and the fact that \( h_2(\delta) \) is increasing for any \( \delta \in (0, 1/2) \), we obtain that

\[
|H(Y^*) - H(Y)| < \delta \log |\mathcal{Y}| + h_2(\delta).
\]  

(38)

For the second term, observe that the conditional distribution of \( Y^* \) given \( X^* = x \) and the conditional distribution of \( Y \) given \( X = x \) are both equal to \( W(y \mid x) \). Therefore,

\[
H(Y \mid X = x) = H(Y^* \mid X^* = x) \leq \log |\mathcal{Y}|.
\]

Consequently,

\[
|H(Y^* \mid X^*) - H(Y \mid X)| \leq \sum_{x \in \mathcal{X}} p_Y^*(x) - p(x)|H(Y \mid X = x) < 2\delta \log |\mathcal{Y}|.
\]

(39)

By combining \(36\) with \(38\) and \(39\), we obtain the desired result.

In order to prove \((43)\), we write

\[
|I(p^*) - I(p)| \leq |H(X^*) - H(X)| + |H(X^* \mid Y^*) - H(X \mid Y)|,
\]

(40)

and we bound both terms with functions of \( \delta \) and \( |\mathcal{X}| \). The first term is easily bounded using Lemma [1] and the fact that \( h_2(\delta) \) is increasing for any \( \delta \in (0, 1/2) \),

\[
|H(X^*) - H(X)| < \delta \log |\mathcal{X}| + h_2(\delta).
\]

(41)

For the second term, consider the conditional distribution of \( X^* \) given \( Y^* = y \), i.e., \( p_{X^* \mid Y^*}(x \mid y) = p_X^*(x) W(y \mid x)/p_Y^*(y) \), and the conditional distribution of \( X \) given \( Y = y \), i.e., \( p_{X \mid Y}(x \mid y) = p(x) W(y \mid x)/p_Y(y) \). Then

\[
|H(X^* \mid Y^*) - H(X \mid Y)| = |\sum_{y \in \mathcal{Y}} p_{Y^*}(y) H(X^* \mid Y^* = y) - p_Y(y) H(X \mid Y = y)|
\]

\[
\leq \sum_{y \in \mathcal{Y}} p_{Y^*}(y) H(X^* \mid Y^* = y) - p_Y(y) H(X^* \mid Y^* = y) + \sum_{y \in \mathcal{Y}} p_Y(y) H(X^* \mid Y^* = y) - p_Y(y) H(X \mid Y = y)|.
\]

(42)

In order to bound the first term of \((42)\), observe that \( H(X^* \mid Y^* = y) \leq \log |\mathcal{X}| \) for any \( y \in \mathcal{Y} \). Therefore, using \(37\), we obtain

\[
|\sum_{y \in \mathcal{Y}} p_{Y^*}(y) H(X^* \mid Y^* = y) - p_Y(y) H(X^* \mid Y^* = y)| < 2\delta \log |\mathcal{X}|.
\]

(43)
For the second term of (42), let us denote by $d(y)$ the total variation distance between $p_{X|Y}^*(x|y)$ and $p_{X|Y}(x|y)$, namely,

$$d(y) = \frac{1}{2} \sum_{x \in \mathcal{X}} |p_{X|Y}^*(x|y) - p_{X|Y}(x|y)|.$$ 

Then, by Lemma [I]

$$|H(X^*|Y^* = y) - H(X|Y = y)| < d(y) \log |\mathcal{X}| + h_2(d(y)),$$

which implies that

$$| \sum_{y \in \mathcal{Y}} p_Y(y) H(X^*|Y^* = y) - p_Y(y) H(X|Y = y) | < \log |\mathcal{X}| \sum_{y \in \mathcal{Y}} p_Y(y) d(y) + \sum_{y \in \mathcal{Y}} p_Y(y) h_2(d(y)). \quad (44)$$

Now, let us focus on the quantity $\sum_{y \in \mathcal{Y}} p_Y(y) d(y)$:

$$\begin{align*}
\sum_{y \in \mathcal{Y}} p_Y(y) d(y) &= \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} |p_Y(y)p_{X|Y}^*(x|y) - p_Y(y)p_{X|Y}(x|y)| \\
&\leq \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} |p_Y(y)p_{X|Y}^*(x|y) - p_Y(y)p_{X|Y}^*(x|y)| + \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} |p_Y(y)p_{X|Y}^*(x|y) - p_Y(y)p_{X|Y}(x|y)| \\
&= \sum_{y \in \mathcal{Y}} |p_Y(y)| \sum_{x \in \mathcal{X}} p_{X|Y}^*(x|y) + \sum_{x \in \mathcal{X}} |p^*(x) - p(x)| \sum_{y \in \mathcal{Y}} W(y|x) < 4\delta.
\end{align*}$$

Observe that $h_2(t)$ is concave for any $t \in (0, 1)$ and increasing for $t \leq 1/2$. Then, as $\delta < 1/8$,

$$\sum_{y \in \mathcal{Y}} p(y) h_2(d(y)) \leq h_2(\sum_{y \in \mathcal{Y}} p(y)d(y)) < h_2(4\delta).$$

By combining (40) with (41), (42), (43), and (44), the result follows. ■

**References**

[1] E. Arıkan, “Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels,” *IEEE Trans. Inform. Theory*, vol. 55, no. 7, pp. 3051–3073, July 2009.

[2] S. Kudekar, T. Richardson, and R. Urbanke, “Spatially coupled ensembles universally achieve capacity under belief propagation,” *IEEE Trans. Inform. Theory*, vol. 59, no. 12, pp. 7761–7813, Dec. 2013.

[3] E. Arıkan and I. E. Telatar, “On the rate of channel polarization,” in *Proc. of the IEEE Int. Symposium on Inform. Theory*, Seoul, South Korea, July 2009, pp. 1493–1495.

[4] S. H. Hassani, K. Alishahi, and R. Urbanke, “Finite-length scaling for polar codes,” Apr. 2014, [Online]. Available: http://arxiv.org/pdf/1304.4778.pdf.

[5] D. Golfin and D. Burshtein, “Improved bounds on the finite length scaling of polar codes,” July 2013, [Online]. Available: http://arxiv.org/pdf/1307.5510v1.pdf.

[6] P. M. Olmos and R. Urbanke, “A scaling law to predict the finite-length performance of spatially-coupled LDPC codes,” Apr. 2014, [Online]. Available: http://arxiv.org/pdf/1404.5719v1.pdf.

[7] E. Şaşoğlu, I. E. Telatar, and E. Arıkan, “Polarization for arbitrary discrete memoryless channels,” in *Proc. of the IEEE Inform. Theory Workshop*, Taormina, Italy, Oct. 2009, pp. 144–148.

[8] R. Mori and T. Tanaka, “Channel polarization on q-ary discrete memoryless channels by arbitrary kernel,” in *Proc. of the IEEE Int. Symposium on Inform. Theory*, Austin, USA, June 2010, pp. 894–898.

[9] W. Park and A. Barg, “Polar codes for q-ary channels, $q = 2^r$,” *IEEE Trans. Inform. Theory*, vol. 59, no. 2, pp. 955–969, Feb. 2013.

[10] A. G. Sahebi and S. S. Pradhan, “Multilevel channel polarization for arbitrary discrete memoryless channels,” *IEEE Trans. on Inform. Theory*, vol. 59, no. 12, pp. 7839–7857, Dec. 2013.

[11] R. Nasser and I. E. Telatar, “Polar codes for arbitrary DMCs and arbitrary MACs,” Nov. 2013, [Online]. Available: http://arxiv.org/pdf/1311.3123v1.pdf.

[12] H. Uchikawa, K. Kasai, and K. Sakaniwa, “Design and performance of rate-compatible non-binary LDPC convolutional codes,” *IEICE Transactions*, pp. 2135–2143, 2011.

[13] A. Piemontese, A. G. Amat, and G. Colavolpe, “Nonbinary spatially-coupled LDPC codes on the binary erasure channel,” in *Proc. of the IEEE Int. Conf. Commun.*, Budapest, Hungary, June 2013, pp. 3270–3274.

[14] I. Andriyanova and A. G. Amat, “Threshold saturation for nonbinary SC-LDPC codes on the binary erasure channel,” Nov. 2013, [Online]. Available: http://arxiv.org/pdf/1311.2003v1.pdf.
[15] L. Wei, T. Koike-Akino, D. G. M. Mitchell, T. E. Fuja, and D. J. Costello Jr., “Threshold analysis of non-binary spatially-coupled LDPC codes with windowed decoding,” Mar. 2014, [Online]. Available: http://arxiv.org/pdf/1403.3583v1.pdf.

[16] E. E. Majani and H. Rumsey, Jr., “Two results on binary-input discrete memoryless channels,” in Proc. of the IEEE Int. Symposium on Inform. Theory, Budapest, Hungary, June 1991, p. 104.

[17] X.-B. Liang, “On a conjecture of Majani and Rumsey,” in Proc. of the IEEE Int. Symposium on Inform. Theory, Chicago, USA, June 2004, p. 62.

[18] N. Shulman and M. Feder, “The uniform distribution as a universal prior,” IEEE Trans. on Inform. Theory, vol. 50, no. 6, pp. 1356–1362, June 2004.

[19] R. G. Gallager, Information Theory and Reliable Communication. New York: Wiley, 1968.

[20] R. J. McEliece, “Are turbo-like codes effective on nonstandard channels?” IEEE Inform. Theory Soc. Newslett., vol. 51, no. 4, pp. 1–8, Dec. 2001.

[21] D. Sutter, J. M. Renes, F. Dupuis, and R. Renner, “Achieving the capacity of any DMC using only polar codes,” in Proc. of the IEEE Inform. Theory Workshop, Lausanne, Switzerland, Sept. 2012, pp. 114–118.

[22] J. Honda and H. Yamamoto, “Polar coding without alphabet extension for asymmetric models,” IEEE Trans. Inform. Theory, vol. 59, no. 12, pp. 7829–7838, Dec. 2013.

[23] E. Arikan, “Source polarization,” in Proc. of the IEEE Int. Symposium on Inform. Theory, Austin, USA, June 2010, pp. 899–903.

[24] G. Caire, S. Shamai, and S. Verdú, “A new data compression algorithm for sources with memory based on error correcting codes,” in Proc. of the IEEE Inform. Theory Workshop, Paris, France, Apr. 2003, pp. 291–295.

[25] T. Richardson and R. Urbanke, Modern Coding Theory. Cambridge University Press, 2008.

[26] C.-C. Wang, S. R. Kulkarni, and H. V. Poor, “Density evolution for asymmetric memoryless channels,” IEEE Trans. on Inform. Theory, vol. 51, no. 12, pp. 4216–4236, Dec. 2005.

[27] M. Mondelli, S. H. Hassani, I. Sason, and R. Urbanke, “Achieving Marton’s region for broadcast channels using polar codes,” Feb. 2014, [Online]. Available: http://arxiv.org/pdf/1401.6060.pdf.

[28] S. Ciliberti and M. Mézard, “The theoretical capacity of the parity source coder,” Journal of Statistical Mechanics: Theory and Experiment, no. 10, Oct. 2005.

[29] T. M. Cover and J. A. Thomas, Elements of Information Theory. New York: Wiley, 2006.

[30] S. H. Hassani and R. Urbanke, “Universal polar codes,” Dec. 2013, [Online]. Available: http://arxiv.org/pdf/1307.7223v2.pdf.

[31] Z. Zhang, “Estimating mutual information via Kolmogorov distance,” IEEE Trans. on Inform. Theory, vol. 53, no. 9, pp. 3280–3282, Sept. 2007.

[32] I. Csiszár and J. Körner, Information theory: Coding Theorems for Discrete Memoryless Systems. Cambridge University Press, 2011.