The $\rho$ and $A$ mesons in strong abelian magnetic field in $SU(2)$ lattice gauge theory

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Abstract

We calculated correlators of vector, axial and pseudoscalar currents in external strong abelian magnetic field according to $SU(2)$ gluodynamics. The masses of neutral $\rho$ and $A$ mesons with various spin projections to the axis parallel to the external magnetic field $B$ have been calculated. We found that the masses of neutral mesons with zero spin $s = 0$ decrease in increasing magnetic field, while the masses of the $\rho$ and $A$ mesons with spin $s = \pm 1$ increase in the mentioned field. Also we performed extrapolation and renormalization of masses on the lattice.

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1. Introduction

Quantum Chromodynamics in external magnetic field is an area that presents enormous interest for physicists. Strong magnetic fields could be associated with formation of the early Universe. It is assumed that magnetic fields $\sim 2$ GeV existed in the Universe during the electroweak phase transition [1]. It is known that cosmic space objects called magnetars or neutron stars possess magnetic field in their cores equal to $\sim 1$ MeV.

Recently there was a number of amazing effects experimentally discovered and theoretically approved. For instance, STAR Collaboration detected chiral magnetic effect during non-central...
collisions of gold ions based on data provided by the RHIC [2–4]. Later this effect was also observed in ALICE experiment at the LHC.

The values of magnetic fields in non-central heavy-ion collisions can reach up to \(15m^2_\pi \sim 0.27 \text{ GeV}^2\) [5], energy of hadronic scale. This magnetic field is caused by motion of ions, and it creates charge asymmetry of particles emitted from different sides of the reaction plane. Such strong magnetic field can be created in terrestrial laboratories, which makes it possible to explore quark – hadronic matter under extreme conditions.

Strong enough magnetic fields lead also to modifications of the QCD phase diagram. Phenomenological models show that the critical temperature of transition between phases of confinement and deconfinement varies with the increase of external magnetic field, and the phase transition becomes the one of the first order [6]. The increase of \(T_c\) is also predicted in Nambu–Jona-Lasinio models: NJL, EPNJL, PNJL [7] and PNJL8 [8], Gross–Neveu model [9, 10].

The first prediction of lattice simulations with two flavours in QCD regarding behavior of the deconfinement temperature and the chiral symmetry restoration in increasing magnetic field was made in [11] showing the increase of the deconfinement and chiral restoring temperatures. However lattice simulations with \(N_f = 2 + 1\) [12] revealed that \(T_c\) decreases under increasing field value. As a matter of fact, the chiral perturbation theory predicts the decrease of the transition temperature under increasing magnetic field value as well [13].

These effects were studied in the past but only for the case of a chromomagnetic (not usual abelian magnetic) field [14–16]. External magnetic fields lead to the enhancement of the chiral symmetry breaking [17–21]. The analytical calculations [22–24] predict a linear increase of the chiral condensate under increasing magnetic field in the leading order. Lattice simulations in \(SU(2)\) represent a clear qualitative evidence that background magnetic field enhances spontaneous breaking of chiral symmetry in a non-abelian gauge theory [25]. Numerical simulations in \(SU(3)\) theory show that the chiral condensate depends on the strength of applied field as exponent function with \(n = 1.6 \pm 0.2\) [26]. It was found from lattice simulations of \(N_f = 2\) QCD that the chiral condensate increases with the magnetic field [27]. In \(SU(2)\) lattice gauge theory with the \(N_f = 4\) flavours the chiral symmetry breaking enhances with the increase of the magnetic field value [28]. The deconfinement transition temperature \(T_c\) decreases with rising magnetic field at small \(eB\) and increases at large \(eB\) [29]. AdS/CFT approach shows the quadratic behavior [30].

Numerical calculations in QCD with \(N_f = 2\) and \(N_f = 2 + 1\) also showed that strongly interacting matter in the presence of the external magnetic field behaves like a paramagnetic both above and below the deconfinement transition [31–33]. The influence of an external magnetic field on the equation of state of the quark–gluon plasma was also explored in [34].

In the framework of the Nambu–Jona-Lasinio model it was shown that QCD vacuum becomes a superconductor [35, 36] along the direction of the magnetic field in the presence of sufficiently strong magnetic fields \((B_c = m^2_\rho/e \simeq 10^{16} \text{ Tl})\). This transition to superconducting phase is accompanied by condensation of charged \(\rho\) mesons. Calculations on the lattice [37] indicate existence of the superconducting phase as well. We have investigated the behavior of masses of neutral \(\rho\) and \(A\) mesons with spin projection \(s = 0, \pm 1\) to the axis of the magnetic field in \(SU(2)\) lattice gauge theory. Such explorations have not performed yet. The study of the \(SU(2)\) theory is a prerequisite for studies of the more realistic theory with \(SU(3)\) gauge group. At the qualitative level the physical properties of mesons in \(SU(2)\) and \(SU(3)\) gluodynamics are believed to be the same. Overlap Dirac operator is numerically expensive and calculations of the spectrum in the \(SU(3)\) theory are substantially more time consuming for big lattices. So we consider \(SU(2)\) theory and study physical effects which are similar to \(SU(3)\).
In QCD condensation of neutral $\rho$ and $A$ mesons in external magnetic field would become an evidence of the superfluid phase existence. In [38] they calculated the mass of neutral vector $\rho$ meson according to the relativistic quark–antiquark model and found out that in the phase of confinement the mass of this neutral $\rho$ meson with zero spin increases under raising magnetic field which contradicts the results presented in [35]. This paper is organized in the following way. In Section 2 we describe technical and numerical specifications of our simulations. In Section 3 we discuss measured observables. Section 4 is devoted to the methods of calculations. The results of our calculations are presented in Sections 6 and 7.

2. Details of the calculations

To generate $SU(2)$ gauge field configurations we use the tadpole improved Symanzik action

$$S = \beta_{\text{imp}} \sum_{\text{pl}} S_{\text{pl}} - \frac{\beta_{\text{imp}}}{2u_0^2} \sum_{\text{rt}} S_{\text{rt}},$$

(1)

where $S_{\text{pl,rt}} = (1/2) \text{Tr}(1 - U_{\text{pl,rt}})$ is the plaquette (denoted by pl) or $1 \times 2$ rectangular loop term (rt), $u_0 = (W_{1 \times 1})^{1/4} = (1/2 \text{Tr} U_{\text{pl}})^{1/4}$ is the input tadpole factor computed at zero temperature [39]. This action suppresses ultraviolet dislocations, leading to non-physical near-zero modes of the Wilson–Dirac operator.

We calculated fermionic spectrum in the presence of $SU(2)$ gauge fields using the chiral-invariant overlap operator proposed by Neuberger [40]. This operator allows to explore the theory without chiral symmetry breaking and can be written as follows:

$$D_{\text{ov}} = \frac{\rho}{a} \left(1 + D_W / \sqrt{D_W^\dagger D_W}\right),$$

(2)

where $D_W = M - \rho/a$ is the Wilson–Dirac operator with the negative mass term $\rho/a$, $a$ is the lattice spacing in physical units, $M$ is the Wilson hopping term with $r = 1$. Fermionic fields comply with periodic boundary conditions imposed on space and anti-periodic boundary conditions imposed on time. The sign function is calculated by the min–max polynomial approximation

$$D_W / \sqrt{D_W^\dagger D_W} = \gamma_5 \text{sign}(H_W),$$

(3)

$H_W = \gamma_5 D_W$ is the hermitian Wilson–Dirac operator. To compute the sign function we use the 50 lowest Wilson–Dirac eigenmodes.

The Dirac operator in continuous space is $D = \gamma^\mu (\partial_\mu - iA_\mu)$ and the corresponding Dirac equation looks as follows:

$$D\psi_k = i\lambda_k \psi_k.$$  

(4)

The Neuberger overlap operator allows to calculate eigenfunctions $\psi_k$ and eigenvalues $\lambda_k$ for the test quark in external gauge field $A_\mu$. $A_\mu$ is the sum of $SU(2)$ gauge field and the external abelian uniform magnetic field. The eigenmodes of the Dirac operator make it possible to construct operators and correlators presented in Section 3.

Abelian gauge fields interact only with quarks. To include abelian magnetic field in the simulations we perform the following substitution:

$$A_{\mu ij} \rightarrow A_{\mu ij} + A_{\mu}^B \delta_{ij},$$

(5)

$$A_{\mu}^B(x) = \frac{B}{2} (x_1 \delta_{\mu,2} - x_2 \delta_{\mu,1}).$$

(6)
To match this exchange with lattice boundary conditions the twisted boundary conditions for fermions have been used [41].

Magnetic field in our calculations is directed along the third axes, its value is quantized

\[ qB = \frac{2\pi k}{(aL)^2}, \quad k \in \mathbb{Z}, \]  

where \( q = -1/3e \) is the electric charge of the \( d \)-quark. There is one type of fermions in the theory. The quantization condition imposes a limit on the minimal value of the magnetic field. It equals to \( \sqrt{eB} = 476.13 \text{ MeV} \) for the lattice volume \( 18^4 \) and lattice spacing \( a = 0.0998 \text{ fm} \). We are far from the saturation regime in magnetic field when \( k/(L^2) \) is not small because we use the values of \( k \) in the interval 0–6 for \( 18^4 \) lattice volume and \( k = 0–14 \) for \( 14^4 \) and \( 16^4 \) lattice volumes. For inversion of the mentioned operator we use the Gaussian source (with the radius \( r = 1.0 \) in lattice units in both spatial and time directions) and the point sink (a quark position smeared over the Gaussian profile).

Our calculations were performed on symmetric lattices with the lattice volumes \( 14^4, 16^6, 18^4 \) and lattice spacings \( a = 0.0681, 0.0998 \) and 0.1383 fm. We use statistically independent configurations of the gluon field for each value of the quark mass in the interval \( mqa = 0.01–0.8 \). For quark masses larger than 0.01 the inversion works well.

### 3. Observables

The following observables were calculated

\[ \left\langle \psi^\dagger(x)O_1\psi(x)\psi^\dagger(y)O_2\psi(y) \right\rangle_A, \]  

where \( O_1, O_2 = \gamma_5, \gamma_5\gamma_\mu, \gamma_\mu \) are Dirac gamma matrices, \( \mu, \nu = 1, \ldots, 4 \) are Lorentz indices. In the Euclidean space \( \psi^\dagger = \overline{\psi} \) [42]. In presence of gauge field being the sum of gluonic fields and abelian constant magnetic field these current–current correlators in a meson channel can be expressed via Dirac propagators.

In the continuous field theory (8) can be written as follows:

\[ \int DA_\mu e^{-\mathcal{S}_M[A_\mu]} \left[ \text{Tr} \left( \frac{1}{D + m} O_1 \right) \text{Tr} \left( \frac{1}{D + m} O_2 \right) \right] - \int DA_\mu e^{-\mathcal{S}_M[A_\mu]} \left[ \text{Tr} \left( \frac{1}{D + m} O_1 \frac{1}{D + m} O_2 \right) \right]. \]  

The first term in the numerator of (9) is the disconnected part, while the second one is connected. We checked that the disconnected part makes rather little relative contribution to correlators in comparison with the connected part in our model. Also this disconnected part does not effect the values of meson masses. Therefore we calculate only the connected parts of correlators (9).

Correlators (9) are defined by Dirac propagators. To calculate these correlators we should first determine the inverse matrix for the massive Dirac operator. For the \( M \) lowest eigenstates of the Dirac operator this matrix is represented by the sum:

\[ \frac{1}{D + m}(x, y) = \sum_{k < M} \frac{\psi_k(x)\psi_k^\dagger(y)}{i\lambda_k + m}. \]
In our calculations we used $M = 50$. Observables (8) have the following form

$$
\langle \bar{\psi} O_1 \psi \bar{\psi} O_2 \psi \rangle_A = \sum_{k, p < M} \frac{\langle k | O_1 | k \rangle \langle p | O_2 | p \rangle - \langle p | O_1 | k \rangle \langle k | O_2 | p \rangle}{(i\lambda_k + m)(i\lambda_p + m)}.
$$

The first term in (9) is not essential as we mentioned above. Magnetic and gluonic fields are considered in the way described above.

The mass of neutral $\rho$ meson was extracted from the correlator of vector currents $\langle j_V^\mu (x) \times j_V^\nu (y) \rangle_A$, where $j_V^\mu (x) = \bar{\psi} (x) \gamma^\mu \psi (x)$. We calculated the mass that has magnetic fieldwise projection of its spin equal to zero and it corresponds to $O_1, O_2 = \gamma_3$ in the expression (11). The mass of a neutral $A$ meson can be found from the correlator of axial currents $\langle j_A^\mu (x) j_A^\nu (y) \rangle_A$, where $j_A^\mu (x) = \bar{\psi} (x) \gamma^5 \gamma^\mu \psi (x)$. The correlator $\langle j_{PS}^\mu (x) j_{PS}^\nu (y) \rangle_A$ enables us to compute the mass of $\pi$ meson, where $j_{PS}^\mu = \bar{\psi} (x) \gamma^5 \psi (x)$ is the pseudoscalar current.

### 4. Methods

To calculate masses we apply two methods. The first one is based on spectral expansion of the lattice correlation function

$$
C(n_t) = \langle \bar{\psi} (0, n_t) O_1 \psi (0, n_t) \bar{\psi} (0, 0) O_2 \psi (0, 0) \rangle_A
$$

$$
= \sum_k \langle 0 | O_1 | k \rangle \langle k | O_2^\dagger | 0 \rangle e^{-n_t a E_k},
$$

$$
C(n_t) = A_0 e^{-n_t a E_0} + A_1 e^{-n_t a E_1} + \ldots,
$$

(12)

(13)

$A$ is some constant value, $E_0$ is the energy of the lowest state. For a particle with average momentum equal to zero $\langle \vec{p} \rangle = 0$ this energy is equal to its mass $E_0 = m_0$. $E_1$ is the energy of the first excited state, $a$ is the lattice spacing, $n_t$ is the number of a lattice site in the line of time direction. From expansion (13) one can see that for large values $n_t$ the main contribution comes from the ground state energy.

Due to periodic boundary conditions the contribution of the ground state into meson propagator has the form

$$
C_{0t}(n_t) = A_0 e^{-n_t a E_0} + A_0 e^{-(N_T - n_t) a E_0} = 2A_0 e^{-N_T a E_0/2} \cosh((N_T - n_t) a E_0).
$$

(14)

The value of the ground state mass can be obtained by fitting the function (14) to the lattice correlator (12). To minimize errors we take various $n_t$ values from the interval $4 < n_t < N_T - 4$.

The second method we use is the Maximal Entropy Method (MEM) [43]. Euclidean correlator of the imaginary time $G(\tau, \vec{p}) = \int d^3 x \langle O(\tau, \vec{x}) O^\dagger (0, \vec{0}) \rangle e^{-i\vec{p} \vec{x}}$ corresponds to the spectral function $\rho(\omega, \vec{p})$ as follows:

$$
G(\tau, \vec{p}) = \frac{1}{2\pi} \int_0^\infty \frac{d \omega}{2\pi} K(\tau, \omega) \rho(\omega, \vec{p}).
$$

(15)

In general case $\rho(\omega, \vec{p})$ contains all the properties of mesons and hadrons having desired quantum numbers. We presume $\langle \vec{p} \rangle = 0$ and do not consider any functions from it. The first peak in the spectral function corresponds to the energy of the ground state. The kernel in (15) can be
expressed as follows:

$$K(\tau, \omega) = \frac{\cosh[\omega(\tau - 1/2T)]}{\sinh(\omega/2T)}, \quad (16)$$

where $T$ is the temperature, $\tau$ is the Euclidean time, $\omega$ is the frequency. To extract the spectral function we should perform an inversion of Eq. (15).

This problem on the lattice is ill-defined as the correlator $G(\tau)$ can be calculated only numerically at discrete points $\tau_i = \tau_{\text{min}} + (i - 1)a$, $i = 1, ..., N_\tau$, and $N_\tau \sim 10–50$. The integral was approximated by the discrete sum at points $\omega_n = n \Delta \omega$, $n = 1, ..., N_\omega$ and $N_\omega$ is usually $\sim O(10^3)$. We cut the integral (15) off at some $\omega_{\text{max}}$. Nevertheless this inversion turns out to be impossible. Yet the ideas of Bayesian probability theory make it possible to overcome this problem.

The most probable spectral function $\rho(\omega)$ can be computed provided that we find the maximum of conditional probability $P[\rho|DH\alpha m]$, where $D$ is the data, $H$ is our hypothesis, $\alpha$ is a real and positive parameter, $m = m(\omega)$ is a default model. This procedure is equivalent to the maximization of free energy $F = L - \alpha S$, where $S$ is Shannon entropy

$$S = \int_0^\infty d\omega \left[ \rho(\omega) - m(\omega) - \rho(\omega) \ln \frac{\rho(\omega)}{m(\omega)} \right], \quad (17)$$

$L$ is the standard likelihood function. Detailed explanation of how to make a corresponding discretization on the lattice is offered in [43]. We take $\alpha \in [\alpha_{\text{min}}, \alpha_{\text{max}}]$ and average the data within this interval. This interval was chosen in such a way that the results may vary slightly (by approximately 10%).

The kernel (16) contains divergence at $\omega = 0$ leading to the unstable behavior of the procedure under small energies. The Bryan key idea was to redefine the kernel and the spectral function

$$\tilde{K}(\omega, \tau) = \frac{\omega}{2T} K(\omega, \tau), \quad \tilde{\rho}(\omega) = \frac{2T}{\omega} \rho(\omega), \quad (18)$$

so that $K(\omega, \tau)\rho(\omega) = \tilde{K}(\omega, \tau)\tilde{\rho}(\omega)$ and apply the SVD theorem to the modified discretized kernel $\tilde{K}(\omega_n, \tau_i)$, see [44]. We use this modified algorithm to determine the spectral function in the following form

$$\tilde{\rho}(\omega) = \tilde{m}(\omega) \exp \sum_{i=1}^N \tilde{c}_i \tilde{u}_i(\omega).$$

The column vectors $u_i$ ($i = 1, ..., N$) are normalized

$$\langle u_i | u_j \rangle = \sum_{n=1}^{N_\omega} u_i(\omega_n) u_j(\omega_n) = \delta_{ij}, \quad (20)$$

c_i are the coefficients and we consider $\tilde{K}(0, \tau) = 1$.

Therefore, to reconstruct spectral function $\rho(\omega)$ we have to choose default model $\tilde{m}(\omega)$ correctly. Such a default model should describe high and low energy behaviors of the spectral function correctly. According to the analysis [45] we choose the following form of the model:

$$\tilde{m}(\omega) = m_a \omega + m_b, \quad m_a = \frac{G(N_\tau/2)}{T^2}, \quad m_b = \frac{aH}{8\pi^2} \frac{3}{8\pi^2}, \quad (21)$$
Fig. 1. The squared mass of the neutral $\pi$ meson calculated via the pseudoscalar correlator $C^{PSPS}(n_{t})$ versus the bare lattice quark mass for $12^4$, $14^4$, $16^4$, $18^4$ lattice volumes, $0.1383$ fm lattice spacing, $\beta = 3.1000$ and zero external magnetic field.

where $a_{H} = 1$ for scalar and pseudoscalar channels, $a_{H} = 2$ for vector and axial vector channels [46]. We also try to apply other default models (constant function, $\sim \omega^2$, vary the $m_{a}$ and $m_{b}$), but the choice (21) gives the best convergence for the MEM.

5. Vector meson mass at $B = 0$

In this section we present the calculation of neutral $\rho$ meson mass in $SU(2)$ gluodynamics under zero magnetic field. We compare our results to the previous ones to make sure that our method works properly. We measure the correlator of pseudoscalar currents $C^{PSPS}(n_{t}) = \langle j^{PS}(0, n_{t}) j^{PS}(0, 0) \rangle_{A}$, where $j^{PS}(0, n_{t}) = \bar{\psi}(0, n_{t})\gamma_{5}\psi(0, n_{t})$ and calculate the mass of the lowest energy state of neutral $\pi$ meson for different quark masses, volumes and lattice spacings.

Fig. 1 demonstrates the linear dependence of the squared $\pi$ meson mass versus the bare quark mass. The chiral perturbation theory predicts linear dependence of squared pion mass on the renormalized quark mass $m_{q}^{ren}$, and this dependence can be expressed as follows:

$$ f_{\pi}^{2}m_{\pi}^{2} = m_{q}^{ren}\langle \bar{\psi}\psi \rangle, $$

(22)

where $f_{\pi}$ is the pion decay constant, $\langle \bar{\psi}\psi \rangle$ is the chiral condensate. Provided that the mass of quark tends to zero, the pions are massless. Fig. 1 shows that mentioned functions are slightly shifted relative to the origin of the coordinates and that the shift of quark mass from zero corresponds to the renormalization of this quark mass on the lattice.

We calculate the mass of neutral vector $\rho$ meson under zero magnetic field via the correlators of vector currents. The symmetry between the different spatial directions was taken into account which improved the statistics significantly (thrice). The quark mass renormalization was not considered because its value is very small at zero field. The extrapolation of $\rho$ meson mass to the limit of infinite physical volume was performed for several values of quark mass and two lattice spacings $0.1383$ fm and $0.1155$ fm. In Fig. 2 we see the mass dependence from inverse physical volume for lattice spacing $0.1155$ fm.

Numerical uncertainties in the values of the meson masses correspond to statistical uncertainty of our calculation.
Fig. 2. The extrapolation of the neutral $\rho$ meson mass to the infinite physical volume under zero external magnetic field. Masses were calculated via the vector correlator $C^{VV}(n_t)$ for several bare quark masses, lattice volumes, lattice spacing 0.1155 fm, $\beta = 3.2000$, and $V_{phys} = (aL)^3$ is the physical volume.

Fig. 3. The mass of the neutral vector $\rho$ meson for various quark masses and two lattice spacings. The extrapolation was performed to the physical mass of pion $m_\pi = 135$ MeV. The results were obtained under zero external magnetic field by the fit (14).

Having calculated $m_\rho$ for different quark masses $m_q$ we extrapolated our results to the quark mass $m_{q0}$ corresponding to the $\pi$ meson mass equal to 135 MeV (Fig. 3). For the extrapolation we used the linear function $m(\rho) = A + Bm_q$ and found the coefficient $A$ and its error by $\chi^2$ method. We obtained $m_\rho \simeq 980 \pm 30$ MeV for the lattice spacing $a = 0.1338$ fm and 1020 ± 20 MeV for $a = 0.1155$ fm in $SU(2)$ gluodynamics. Experimental value of the mass of neutral $\rho$ meson equals to $\sim 775.26 \pm 0.25$ MeV [47] so the quenching effects amount $\sim 20$–$30\%$ in this case.

6. The masses of mesons at $B \neq 0$

In Fig. 4 the squared pion mass is depicted for 16$^4$ lattice volume, various bare quark masses and different values of the magnetic field $H = \sqrt{eB}$. The observed linear dependence is legitimate for the chiral perturbation theory.
Fig. 4. The squared mass of the neutral $\pi$ meson calculated via pseudoscalar correlator $C^{PSPS}(n_f)$ versus bare lattice quark mass for the lattice volume $16^4$, lattice spacing $0.1155$ fm, $\beta = 3.2000$ and diverse values of the external magnetic field.

Fig. 5. The mass of the neutral $\pi$ meson obtained via the $C^{PSPS}(n_f)$ versus the squared value of the magnetic field for renormalized and nonrenormalized quark masses.

Fig. 5 represents the $\pi$ meson mass versus the value of the squared magnetic field. The mass depends on the lattice volume not significantly. For the renormalized quark mass and magnetic fields less than 1 GeV we have obtained linear dependence of the mass on the magnetic field value; the slope of the graph of this function is negative, which qualitatively agrees with results of A. Smilga obtained in the chiral perturbation theory [24]. The angle of the slope differs from ChPT because we explore the $SU(2)$ gauge theory without dynamical quarks.

If the direction of external magnetic field is parallel to the 3rd coordinate axis then meson masses having magnetic fieldwise projections of their spins equal to zero are calculated from the expression (11) with $O_1, O_2 = \gamma_3$. The diagonal components of the correlators are not equal to zero while the nondiagonal ones equal to zero within the error bars.

Fig. 6 shows the mass of neutral vector meson with zero spin calculated in accordance with the Maximal Entropy Method for different lattice volumes, spacings and bare quark masses. We have not performed mass extrapolations and renormalizations though. We found that the mass
Fig. 6. Dependence of the mass of the neutral vector $\rho$ meson with zero spin $s = 0$ on the value of external magnetic field for the lattice volumes $14^4, 16^4, 18^4$ and lattice spacings $a = 0.0998$ fm, $0.1338$ fm obtained by use of the Maximal Entropy Method.

Fig. 7. The mass of the neutral axial $A$ meson with zero spin $s = 0$ versus the value of external magnetic field for the lattice volumes $14^4, 16^4$ and lattice spacings $a = 0.0998$ fm, $0.1338$ fm obtained by use of the Maximal Entropy Method.

decreases under raising magnetic field for all the sets of data. We use the ensemble of $O(10)$ results of MEM procedure and calculate average and standard deviations for the best set of MEM parameters. To calculate errors we took the $\omega$-discretization into account as well.

Fig. 7 shows behavior of the neutral $A$ meson mass with zero spin depending on the external magnetic field. The mass decreases as well but under $eB \sim 0.7$ GeV$^2$ we observe the peak which might be a lattice artifact. In Tables 1 and 2 the masses of $\rho(s = 0)$ and $A(s = 0)$ mesons and their uncertainties are presented with the appropriate values of the lattice volume, lattice spacing, lattice quark mass and magnetic field.

The lattice artefacts are not significant yet we are restricted by the small lattice spacing which is not fine enough and cannot explore the masses under strong magnetic fields. As we know the radius of the lowest Landau level

$$r_L = \frac{1}{\sqrt{eB}}.$$  

(23)
Table 1
The masses of the neutral vector $\rho$ meson with zero spin and their uncertainties for various values of the lattice volume, lattice spacing, lattice quark mass and magnetic field obtained by use of MEM and by fit (14).

| $V_{latt}$ | $a$ (fm) | $am_q$ | $eB$ (GeV) | $m_{(s=0)}^{(\text{MEM})}$ (GeV) | $m_{(s=0)}^{(\text{fit (14)})}$ (GeV) |
|-----------|----------|--------|------------|---------------------------------|---------------------------------|
| $14^4$    | 0.0998   | 0.01   | 0.000      | 1.39 ± 0.06                     | 1.33 ± 0.04                     |
| $14^4$    | 0.0998   | 0.01   | 0.612      | 1.13 ± 0.05                     | 1.08 ± 0.06                     |
| $14^4$    | 0.0998   | 0.01   | 0.866      | 0.99 ± 0.08                     | 1.08 ± 0.06                     |
| $14^4$    | 0.0998   | 0.01   | 1.060      | 0.86 ± 0.06                     | 0.85 ± 0.04                     |
| $14^4$    | 0.0998   | 0.01   | 1.224      | 0.70 ± 0.10                     | 0.74 ± 0.02                     |
| $14^4$    | 0.0998   | 0.01   | 1.369      | 0.66 ± 0.08                     | 0.68 ± 0.03                     |
| $14^4$    | 0.0998   | 0.01   | 1.499      | 0.53 ± 0.07                     | 0.63 ± 0.06                     |
| $14^4$    | 0.0998   | 0.01   | 1.620      | 0.40 ± 0.07                     | 0.55 ± 0.02                     |
| $14^4$    | 0.0998   | 0.01   | 1.731      | 0.50 ± 0.08                     | 0.55 ± 0.06                     |
| $14^4$    | 0.0998   | 0.01   | 1.836      | 0.40 ± 0.10                     | 0.45 ± 0.02                     |
| $14^4$    | 0.0998   | 0.01   | 1.936      | 0.40 ± 0.09                     | 0.41 ± 0.02                     |
| $14^4$    | 0.0998   | 0.01   | 2.121      | 0.29 ± 0.14                     | 0.33 ± 0.03                     |
| $14^4$    | 0.0998   | 0.01   | 2.290      | 0.40 ± 0.20                     | 0.29 ± 0.02                     |
| $16^4$    | 0.0998   | 0.01   | 0          | 1.13 ± 0.07                     | 1.13 ± 0.04                     |
| $16^4$    | 0.0998   | 0.01   | 0.758      | 0.79 ± 0.07                     | 0.82 ± 0.10                     |
| $16^4$    | 0.0998   | 0.01   | 0.928      | 0.60 ± 0.07                     | 0.60 ± 0.01                     |
| $16^4$    | 0.0998   | 0.01   | 1.071      | 0.46 ± 0.07                     | 0.48 ± 0.05                     |
| $16^4$    | 0.0998   | 0.01   | 1.312      | 0.40 ± 0.07                     | 0.43 ± 0.10                     |
| $16^4$    | 0.0998   | 0.02   | 0          | 1.13 ± 0.005                    | 1.17 ± 0.10                     |
| $16^4$    | 0.0998   | 0.02   | 0.536      | 1.06 ± 0.005                    | 1.07 ± 0.09                     |
| $16^4$    | 0.0998   | 0.02   | 0.928      | 0.74 ± 0.05                     | 0.75 ± 0.02                     |
| $16^4$    | 0.0998   | 0.02   | 1.312      | 0.51 ± 0.02                     | 0.54 ± 0.01                     |
| $16^4$    | 0.0998   | 0.02   | 1.694      | 0.40 ± 0.03                     | 0.44 ± 0.001                    |
| $18^4$    | 0.0998   | 0.02   | 0          | 1.12 ± 0.03                     | 1.17 ± 0.04                     |
| $18^4$    | 0.0998   | 0.02   | 0.476      | 1.04 ± 0.05                     | 1.07 ± 0.03                     |
| $18^4$    | 0.0998   | 0.02   | 1.166      | 0.50 ± 0.03                     | 0.54 ± 0.01                     |
| $14^4$    | 0.1383   | 0.01   | 0          | 1.07 ± 0.04                     | 1.18 ± 0.09                     |
| $14^4$    | 0.1383   | 0.01   | 0.442      | 0.91 ± 0.07                     | 0.99 ± 0.04                     |
| $14^4$    | 0.1383   | 0.01   | 0.625      | 0.91 ± 0.04                     | 0.86 ± 0.04                     |
| $14^4$    | 0.1383   | 0.01   | 0.883      | 0.58 ± 0.10                     | 0.59 ± 0.02                     |
| $14^4$    | 0.1383   | 0.01   | 1.082      | 0.50 ± 0.18                     | 0.46 ± 0.03                     |
| $14^4$    | 0.1383   | 0.01   | 1.249      | 0.40 ± 0.14                     | 0.40 ± 0.01                     |
| $16^4$    | 0.1383   | 0.02   | 0          | 0.97 ± 0.01                     | 1.13 ± 0.02                     |
| $16^4$    | 0.1383   | 0.02   | 0.386      | 0.87 ± 0.01                     | 1.01 ± 0.03                     |
| $16^4$    | 0.1383   | 0.02   | 0.669      | 0.80 ± 0.01                     | 0.72 ± 0.07                     |
| $16^4$    | 0.1383   | 0.02   | 0.946      | 0.47 ± 0.04                     | 0.47 ± 0.02                     |
| $16^4$    | 0.1383   | 0.02   | 1.222      | 0.36 ± 0.04                     |                                  |

We may assume that lattice artefacts might appear under $r_L = a$. Therefore we make simple estimations and obtain the maximal value of magnetic field $eB = 3.9 \text{ GeV}^2$ for the spacing $a = 0.0998 \text{ fm}$, $2.9 \text{ GeV}^2$ for the $a = 0.1155 \text{ fm}$ and $2.0 \text{ GeV}^2$ for $0.1338 \text{ fm}$ lattice spacing.
Table 2
The masses of the neutral vector $A$ meson with zero spin and their uncertainties for various values of the lattice volume, lattice spacing, lattice quark mass and magnetic field obtained by use of MEM and by fit (14).

| $V_{latt}$ | $a$ (fm) | $am_q$ | $eB$ (GeV) | $m_A^{(s=0)}$ (GeV) | $m_A^{(s=0)}$ (GeV) (fit (14)) |
|------------|----------|--------|-------------|-----------------|------------------------------|
| $14^4$     | 0.0998   | 0.01   | 0           | 1.66 ± 0.60     | 1.68 ± 0.07                  |
| $14^4$     | 0.0998   | 0.01   | 0.612       | 1.80 ± 0.10     | 1.79 ± 0.08                  |
| $14^4$     | 0.0998   | 0.01   | 0.866       | 2.00 ± 0.10     | 2.00 ± 0.20                  |
| $14^4$     | 0.0998   | 0.01   | 1.060       | 1.72 ± 0.07     | 1.80 ± 0.10                  |
| $14^4$     | 0.0998   | 0.01   | 1.499       | 1.13 ± 0.09     | 1.20 ± 0.10                  |
| $16^4$     | 0.0998   | 0.01   | 0           | 1.59 ± 0.07     | 1.65 ± 0.17                  |
| $16^4$     | 0.0998   | 0.01   | 0.536       | 1.72 ± 0.07     | 1.80 ± 0.50                  |
| $16^4$     | 0.0998   | 0.01   | 0.758       | 1.99 ± 0.08     | 2.11 ± 0.08                  |
| $16^4$     | 0.0998   | 0.01   | 0.928       | 1.70 ± 0.09     | 1.80 ± 0.10                  |
| $16^4$     | 0.0998   | 0.01   | 1.071       | 1.10 ± 0.10     | 1.20 ± 0.30                  |
| $16^4$     | 0.0998   | 0.01   | 1.312       | 1.00 ± 0.10     | 0.80 ± 0.20                  |
| $16^4$     | 0.0998   | 0.01   | 1.515       | 0.60 ± 0.20     | 0.50 ± 0.10                  |
| $14^4$     | 0.1383   | 0.01   | 0           | 1.30 ± 0.07     | 1.30 ± 0.20                  |
| $14^4$     | 0.1383   | 0.01   | 0.442       | 1.48 ± 0.05     | 1.50 ± 0.10                  |
| $14^4$     | 0.1383   | 0.01   | 0.625       | 1.47 ± 0.06     | 1.39 ± 0.06                  |
| $14^4$     | 0.1383   | 0.01   | 0.883       | 1.48 ± 0.05     | 1.25 ± 0.09                  |
| $14^4$     | 0.1383   | 0.01   | 1.082       | 1.10 ± 0.10     | 1.04 ± 0.09                  |
| $14^4$     | 0.1383   | 0.01   | 1.249       | 1.00 ± 0.20     | 0.90 ± 0.08                  |

Fig. 8. Dependence of the mass of the neutral vector $\rho$ meson with spin $s = 0$ on the value of external magnetic field for the lattice volumes $16^4, 18^4$ and lattice spacings $a = 0.0998, 0.1155$ fm.

7. Lattice extrapolations at $B \neq 0$

In Figs. 8, 9 and 10 we present the masses with various spin projections that were received by fitting the cosh sinus function to correlators and further extrapolation of quark masses. The masses of the vector meson were calculated for various values of magnetic field.

We calculated these masses for nonzero magnetic field taking the quark mass renormalization $\delta m_{\text{ren}}^{\text{latt}}$ into account. On the basis of the mentioned calculation we can conclude that masses
of vector mesons depend on that renormalization not significantly. Afterwards we performed $\rho$ meson mass extrapolation on the mass of quark to its value under which the mass of $\pi$ meson is equal to 135 MeV. This procedure was done taking the renormalization of the quark mass into account. The extrapolation was done similarly to the case of zero magnetic field.

We calculate the mass of $\rho$ meson for several values of $m_q$ from the interval $m_q = 0.01–0.8$, perform fitting and find coefficients $a_i$ and $b_i$ in the equations

$$m_\rho(s = 0) = a_0 + a_1 m_q,$$

$$m_A(s = 0) = b_0 + b_1 m_q.$$  \hspace{1cm} (24)\hspace{1cm} (25)

Then we extrapolate $m_\rho(m_q)$ on physical values $m_\rho(m_{q_0})$ at $m_q = m_{q_0}$ using (24) and (25).

Various components of the vector currents correlators were computed, and it was found that diagonal components differ from zero essentially while nondiagonal ones are equal to zero within the error bars.
Correlators of the vector currents perpendicular to the magnetic field are \( C_{11}^{VV}(n_l) = \langle \mathbf{j}_1^V(\mathbf{0}, n_l) \mathbf{j}_1^V(\mathbf{0}, 0) \rangle_A \) and \( C_{22}^{VV}(n_l) = \langle \mathbf{j}_2^V(\mathbf{0}, n_l) \mathbf{j}_2^V(\mathbf{0}, 0) \rangle_A \), where \( j_1^V(\mathbf{0}, n_l) = \bar{\psi}(\mathbf{0}, n_l) \gamma_1 \psi(\mathbf{0}, n_l) \) and \( j_2^V(\mathbf{0}, n_l) = \bar{\psi}(\mathbf{0}, n_l) \gamma_2 \psi(\mathbf{0}, n_l) \). The masses with spin \( s = \pm 1 \) are found from the relations \( C^{VV}(s = 1) = (C_{11}^{VV} + iC_{22}^{VV})/\sqrt{2} \) and \( C^{VV}(s = -1) = -(C_{11}^{VV} - iC_{22}^{VV})/\sqrt{2} \).

In Fig. 8 one can see the dependence of the mass of the neutral vector \( \rho \) meson with zero spin on the value of the field after the mass renormalization and extrapolation for the lattice volumes \( 16^4 \) and \( 18^4 \) and lattice spacings \( a = 0.0998, 0.1155 \) fm. For the purposes of visualization we connected the points by splines. The mass of vector meson decreases nonlinearly under raising magnetic field for all the lattices. We observe a weak dependence of masses from lattice volumes and spacings, but the qualitative behavior for all the sets of data is the same.

Fig. 9 shows the dependence of the \( \rho \) meson mass with nonzero spin on the field value. Masses with spin \( s = \pm 1 \) increase under the raising field. These results were obtained after the quark mass extrapolation. The values of \( \rho \) meson masses with various spins after quark mass extrapolations are listed in Table 3.

In Fig. 10 we see the mass of the neutral axial meson with fieldwise spin projections \( s = 0, \pm 1 \). Calculation of the axial meson mass requires much more statistics than the calculation of the vector meson mass especially in cases with nonzero spin components. One can observe that the mass of \( A \) meson with zero spin decreases while the masses with \( s = \pm 1 \) increase slowly. The masses of \( A \) meson are represented in Table 4.

Unfortunately the quantum numbers of mesons on the lattice in the presence of magnetic field are not precise. The mixing takes place due to the interaction between photons and vector (axial) quark currents and can occur between neutral pion and neutral \( \rho \) (or \( A \)) meson with zero spin. Now there are no rigorous methods to disentangle these two states in magnetic field, this is the topic for the further work. However, we have clear signs of the increase of the vector and axial mesons masses with \( s = \pm 1 \) in our \( SU(2) \) theory.

8. Conclusions

In this work we explore the behavior of masses of the neutral pseudoscalar \( \pi \), vector \( \rho \) and axial \( A \) mesons in confinement phase in \( SU(2) \) quenched theory. This theory couples to overlap fermions in the presence of external magnetic field of hadronic scale.

We observe that the masses with fieldwise spin projection equal to zero differ from the masses with spin projection \( s = \pm 1 \). The masses with \( s = 0 \) decrease with the growth of the magnetic field while the masses with \( s = \pm 1 \) increase in the same conditions. We consider this phenomena to be the result of the anisotropy created by the strong magnetic field. We do not observe any
condensation of neutral mesons, so there are no evidences of superfluidity in the confinement phase. However, the presence of superconducting phase at high values of the magnetic field $B$ [35] in QCD is a hot topic for discussions. Condensation of charged $\rho$ mesons would be an evidence of the existence of superconductivity in QCD.

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