ON A LOGARITHMIC STABILITY ESTIMATE FOR AN INVERSE HEAT CONDUCTION PROBLEM

AYMEN JBALIA*

Department of Mathematics, Faculty of Sciences of Bizerte
7021 Jarzouna Bizerte, Tunisia

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Abstract. We are concerned with an inverse problem arising in thermal imaging in a bounded domain \( \Omega \subset \mathbb{R}^n, n = 2, 3 \). This inverse problem consists in the determination of the heat exchange coefficient \( q(x) \) appearing in the boundary of a heat equation with Robin boundary condition.

1. Introduction. This paper investigates an inverse boundary coefficient problem in thermal imaging. The inverse problem under consideration consists in recovering the unknown heat exchange (heat loss) coefficient \( q(x) \) appearing in the heat equation with Robin boundary condition. This recovery may be obtained from boundary temperature measurements. In practice, this kind of inverse problem can be used to model the damage localization or corrosion detection in an inaccessible portion of some material object [23, 28] and heat loss as well [8, 17, 32, 34, 33].

In this paper we consider a \( C^3 \)-smooth bounded domain \( \Omega \) of \( \mathbb{R}^n (n = 2, 3) \) with boundary \( \Gamma := \partial \Omega \). We assume that there exist two subsets of \( \Gamma \) disjoint, \( \Gamma_a \) and \( \Gamma_i \), with nonzero surface measure such that

\[
\Gamma := \Gamma_a \cup \Gamma_i; \quad \Gamma_a \neq \emptyset, \quad \text{and} \quad \Gamma_i \neq \emptyset, \tag{1.1}
\]

where \( \Gamma_a \) denotes the “known” accessible portion of \( \partial \Omega \). We impose on the portion \( \Gamma_i \) (may be inaccessible) a condition modeling the heat loss. The support of the applied heat flux \( g \) (stationary or time dependent) is contained in \( \Gamma_a \).

The temperature distribution, denoted by \( u \), satisfies the following initial-boundary value problem

\[
\begin{aligned}
\partial_t u(x, t) &= \Delta u(x, t), \quad (x, t) \in \Omega \times (0, \infty), \\
\partial_n u(x, t) &= g, \quad (x, t) \in \Gamma_a \times (0, \infty), \\
\partial_n u(x, t) + q(x)u(x, t) &= 0, \quad (x, t) \in \Gamma_i \times (0, \infty), \\
u(x, 0) &= u_0(x), \quad x \in \Omega,
\end{aligned}
\tag{1.2}
\]

where \( \partial_n \) denotes the derivative in the direction of the exterior unit normal vector \( \nu \) to \( \Gamma \).

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* Corresponding author: Aymen Jbalia.
In engineering applications, the stationary heat flux \( g \) corresponds to an uniform heating of the outer surface. Typically, this is the case when heat or flash lamps are used to provide the input flux \( g \). In this paper, we study separately the two cases: stationary and time-dependent heat flux. Here, it is worth noticing that the time-periodic heat flux problem was studied in [6].

Besides, for an initial condition \( u_0 \in L^2(\Omega) \), one can prove that the problem (1.2) has a unique solution \( u \in C([0, \infty]; L^2(\Omega)) \cap \cap C([0, \infty]; H^2(\Omega)) \cap L^\infty((0, \infty); H^1(\Omega)) \) and to be able to get some estimations of the solution, we assume that for a fixed positive constant \( R_0 \), we have

\[
\|u_0\|_{L^2(\Omega)} \leq R_0. \tag{1.3}
\]

In this paper, we restrict our attention to the Robin boundary condition \( \partial_x u + q(x)u = 0 \) which, according to [7], corresponds to a Newton-cooling type of heat loss on the boundary with ambient temperature scaled to zero. Now, using the same notions as in [14], we introduce the vector space

\[ B_{s,r}(\mathbb{R}^n) = \{ y \in S'(\mathbb{R}^n); (1 + |\xi|^2)^{s/2} \hat{y} \in L^r(\mathbb{R}^n) \}, \quad \text{for } s \in \mathbb{R} \text{ and } 1 \leq r \leq \infty, \]

equipped with the norm

\[
\|y\|_{B_{s,r}(\mathbb{R}^n)} = \left\| \left(1 + |\xi|^2\right)^{s/2} \hat{y} \right\|_{L^r(\mathbb{R}^n)},
\]

where \( S'(\mathbb{R}^n) \) is the space of temperate distributions on \( \mathbb{R}^n \), \( \hat{y} \) is the Fourier transform of the function \( y \) and \( B_{s,r}(\mathbb{R}^n) \) is a Besov space (see Chapter 10 in [22]). The Besov spaces \( B_{s,r}(\mathbb{R}^n) \) play an important role in generalizing many classic functional spaces. Moreover, the space \( B_{s,2}(\mathbb{R}^n) \) is the Sobolev space \( H^s(\mathbb{R}^n) \). In addition, if \( s \in (0, 1) \) and \( r = \infty \), we have \( B_{s,\infty}(\mathbb{R}^n) = C^s(\mathbb{R}^n) \) where \( C^s \) is the Hölder space (see the Appendix of [21]). Using local charts and partition of unity, \( B_{s,r}(\partial \Omega) \) is defined from \( B_{s,r}(\mathbb{R}^{n-1}) \) in the same way as \( H^s(\mathbb{R}^n) \) is built from \( H^s(\mathbb{R}^{n-1}) \).

In our current study, some smoothness properties of the solution to the problem (1.2) are needed. In order to give sufficient conditions on data guaranteeing these smoothness properties, we introduce the following sets of boundary coefficients:

\[
D = \{ q \in B_{n-1/2,1}(\Gamma); q \geq 0, q \neq 0 \},
\]

\[
D_M = \{ q \in D; \|q\|_{B_{n-1/2,1}} \leq M \},
\]

and

\[
D_M^0 = \{ q \in D_M, \text{supp}(q) \subset \Gamma_i \},
\]

where \( M > 0 \) is a given constant. Let us recall that the function \( q(x) \) (heat exchange coefficient) in (1.2) is known as the Robin coefficient with a support in \( \Gamma_i \). So, the introduction of the space \( D_M^0 \) is suitable and it will be useful in the rest of the paper. The inverse Robin problem has been investigated, theoretically and numerically by several authors (see e.g. [1, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 18, 23, 25, 26, 27, 29, 32, 34]) where various type of stability estimates are given.

In this work, we firstly established a double-logarithmic stability estimate for recovering the boundary coefficient. Our result is considerably different from those already established in [3, 12] in two or three dimensional spaces where \( \Omega \) is of class \( C^\infty \). In this paper, \( \Omega \) is just of class \( C^3 \). Moreover in [3, 12] the error \( q - \tilde{q} \) is only estimated in a compact subset of \( \{ x \in \Gamma_i; u_q(x) \neq 0 \} \) by a simple logarithmic stability estimates. In this paper, the error \( q - \tilde{q} \) is obtained in the whole \( \Gamma_i \) by a double-logarithmic stability. This means that the stability decreases less rapidly.
near the points where the solution to the problem (1.2) vanishes. To the best of
our knowledge, our results generalize the majority of previous works.

The paper is organized as follows. In Section 2, we first describe the considered
inverse problem and we state the logarithmic stability estimate results. Then, we
prove rigorously and separately some important results by considering some hy-
pothesis introduced in Section 1. Finally, we give a stability result for the case of a
time dependent heat flux.

2. Stability of the determination of Robin coefficient. In this section, we
establish a double logarithmic stability estimate for the determination of a boundary
coefficient appearing in a boundary value problem for the heat equation with Robin
boundary condition. Firstly, we consider the case where the heat flux \( g \) is stationary
\( g := g(x) \). Then, the time-dependent case \( g := h(x,t) \) can be deduced using the
same techniques.

Now, we introduce \( \gamma \times (0,\infty) \) as a subset of the accessible sub-boundary \( \Gamma_a \times (0,\infty) \). We assume that \( \gamma \) does not meet \( \text{supp}(g) \) and the following condition holds
true:
\[
\gamma \times (0,\infty) \subset (\Gamma_a \setminus \text{supp}(g)) \times (0,\infty).
\]
(2.1)

The inverse problem associated to the problem (1.2) can be formulated as follows:

**Inverse problem.** Determine \( q \), supported on \( \Gamma_i \), from the boundary measure-
ments
\[
 u_q|_{\gamma \times (0,\infty)}(x,t) = \psi, \quad (x,t) \in \gamma \times (0,\infty),
\]
(2.2)
where \( \Gamma_i \) is assumed to be a priori known and \( u_q \) is the solution to the problem (1.2)
with coefficient \( q \).

The uniqueness results of the determination of \( \Gamma_i \) in (1.2)-(2.2) can be inspired
from [7] and [6]. More results for the stationary case can be found in the literature,
where different kinds of methods are used to establish uniqueness, stability and
numerical algorithms ( see e.g, [1, 2, 9, 12, 13, 15] and references therein).

2.1. Double logarithmic stability estimate. In this subsection, we establish a double logarithmic stability estimate for the inverse problem in thermal imag-
ing described above. Here, we require that the heat flux on a part of the accessible
boundary \( \Gamma_a \) remains the same at every time. Moreover, we assume that the bound-
ary function \( g \) introduced in the problem (1.2) only depends on the space variable
\( x \). A similar result can be deduced for the case of a time dependent heat flux. This
latter case will be detailed later. Now, let us consider the following assumption:
\[
(A_1) \quad \tilde{g} = \chi_{\Gamma_a} g \in H^{k+\frac{1}{2}}(\partial\Omega),
\]
where \( k \geq 0 \) is an integer and \( \chi_{\Gamma_a} \) is the characteristic function related to \( \Gamma_a \). Note
that, \( g \) is not identically equal to zero. Thus, let us assume that we have (1.1),
(2.2), (2.1) and (A1). By applying stationary heat flux, we get the following result.

**Theorem 2.1.** Let the functions \( u_q \) and \( u_{\tilde{q}} \) be solutions to (1.2) with coefficients
\( q \) and \( \tilde{q} \), respectively. Let the constants \( \alpha \in (0,1) \) and \( \sigma \in (0,1) \). Then, there
exist two positive constants \( A = A(\sigma,\Omega,\Gamma,\gamma) \) and \( B = B(\Omega,\gamma) \) such that for any
\( q, \tilde{q} \in D_0^0 \cap C^\alpha(\Gamma) \), we have:
\[
\|q - \tilde{q}\|_{L^\infty(\Gamma_i)} \leq \frac{A}{\ln \ln (B \|u_q - u_{\tilde{q}}\|_{L^\infty((0,\infty);L^2(\gamma))})}^\sigma,
\]
where $A$ and $B$ are independent of $q$ and $\tilde{q}$.

The proof of Theorem 2.1 is postponed to subsection 2.2. So, following [15] and [14], we modulate the problem of detecting corrosion damage by electric measurements. To this end, we consider the following boundary value problem

$$\begin{cases}
\Delta v(x) = 0, & x \in \Omega, \\
\partial_n v(x) = g(x), & x \in \Gamma_a, \\
\partial_n v(x) + q(x)v(x) = 0, & x \in \Gamma_1.
\end{cases} \tag{2.3}$$

Using Theorem 2.3 of [16] and the fact that $B_{n-1/2,1}(\Gamma)$ is continuously embedded in $B_{n-3/2,1}(\Gamma)$, we obtain that (for any $q \in D$) the problem (2.3) has a unique solution $v_q \in H^\alpha(\Omega)$ ($n = 2, 3$). Moreover, we have

$$\|v_q\|_{H^\alpha(\Omega)} \leq R_1, \quad \text{for all } q \in D^0_M, \tag{2.4}$$

where $R_1 = R_\delta(\Omega, g, M)$ denotes a positive constant. In addition, we introduce the auxiliary initial-boundary value problem

$$\begin{cases}
\partial_t w(x, t) = \Delta w(x, t), & x \in \Omega, t > 0, \\
\partial_n w(x, t) = 0, & x \in \Gamma_a, t > 0, \\
\partial_n w(x, t) + q(x)w(x, t) = 0, & x \in \Gamma_1, t > 0, \\
w(x, 0) = w_q(x, 0) - v_q, & x \in \Omega.
\end{cases} \tag{2.5}$$

Next, let us denote by $u_q$ the solution to the problem (1.2) and let us decompose this solution into the sum

$$u_q = v_q + w_q, \tag{2.6}$$

where $v_q$ is the solution to (2.3) and $w_q$ is the solution to (2.5).

2.2. Proof of the stability estimate. To prove Theorem 2.1, we need the following technical lemmas. The first lemma, was proved in [3] and is based on the classical theory of analytic semigroups (see e.g. [30, 31]).

**Lemma 2.1.** Let $\Gamma$ be given by (1.1). Assume that (1.3) and (2.4) hold. Let $w_q$ be the solution to (2.5). Then, there exist two positive constants $C$ and $\mu$ such that

$$\|w_q(\cdot, t)\|_{H^\alpha(\Omega)} \leq Ce^{-\mu t}/\sqrt{t}, \text{ for all } t > 0,$$

where $C$ depending on $\Omega$, $\Gamma$, $R_0$, and $R_1$ but independent of $q$.

Note that by virtue of Lemma 4.2 of [4], there exist $\delta^* > 0$ and $0 < r^* \leq \text{diam}(\partial\Omega)$ such that for any $0 < \delta < \delta^*$ and $\tilde{x} \in \partial\Omega$, we have

$$\{x \in B(\tilde{x}, r^*) \cap \Gamma, |v_q(x)| \geq \delta\} \neq \emptyset, \tag{2.7}$$

where $v_q$ is the solution to (2.3) and $B(\tilde{x}, r^*)$ means the ball, in $\mathbb{R}^n$, with center $\tilde{x}$ and with radius $r^*$.

To state the second technical lemma, we first consider an arbitrary function $f \in C^\alpha(\Gamma)$ such that

$$[f]_\alpha := \sup\{|f(x) - f(y)|; |x - y|^{-\alpha}; x, y \in \Gamma, x \neq y\} \leq M_0, \tag{2.8}$$

where $M_0 > 0$ is a given constant and $[f]_\alpha$ denotes the infimum over all constants $L$ such that

$$|f(x) - f(y)| \leq L|x - y|^\alpha, \quad x, y \in \Gamma \tag{2.9}$$
is well satisfied. Let us recall that $f \in C^\alpha(\Gamma)$ if there exists $L \geq 0$ such that (2.9) holds. We have the following result.

**Lemma 2.2.** Assume that (1.1) holds. Let $0 < \alpha < 1$. Then, there exist three positive constants $\delta^*, s^* := |\ln \delta^*|$ and $c = c(\Omega, \Gamma)$ such that for any $0 < \delta < \delta^*$ satisfying (2.7), $s > s^*$ and for a given $f \in C^\alpha(\Gamma)$ satisfying (2.8) we have:

$$
\|f\|_{L^\infty(\Gamma)} \leq \frac{c}{s^\alpha} + e^s \|f v_q\|_{L^\infty(\Gamma)},
$$

(2.10)

where $v_q$ is the solution to (2.3).

**Proof.** Let $\delta > 0$ be defined as in (2.7), and let $\tilde{x} \in \Gamma$. For any function $f \in C^\alpha(\Gamma)$, one can distinguish two cases:

- Either $|v_q(\tilde{x})| \geq \delta$, so

$$
1 \leq \frac{1}{\delta} |v_q(\tilde{x})|.
$$

Consequently,

$$
|f(\tilde{x})| \leq \frac{1}{\delta} |f(\tilde{x})v_q(\tilde{x})|.
$$

(2.11)

- Either $|v_q(\tilde{x})| < \delta$, then set $r := \sup\{0 < \rho; |v_q| < \delta, \text{ on } B(\tilde{x}, \rho) \cap \Gamma\}$ which by using (2.7) implies that

$$
r \leq r^* \text{ and } \partial B(\tilde{x}, r) \cap \{x \in B(\tilde{x}, r^*) \cap \Gamma; |v_q(x)| \geq \delta\} \neq \emptyset.
$$

Now, we can find $\hat{x} \in \partial B(\tilde{x}, r)$ such that $|v_q(\hat{x})| \geq \delta$. Then, using relations (2.8) and (2.11) we immediately get

$$
|f(\tilde{x})| \leq |f(\hat{x}) - f(\tilde{x})| + |f(\tilde{x})| \leq |f|_{\alpha} |\tilde{x} - \hat{x}|^{\alpha} + \frac{1}{\delta} |f(\tilde{x})v_q(\tilde{x})|.
$$

Consequently, it follows that

$$
\|f\|_{L^\infty(\Gamma)} \leq M_0 r^\alpha + \frac{1}{\delta} \|f v_q\|_{L^\infty(\Gamma)},
$$

(2.12)

where the constant $M_0$ is given by (2.8).

Since $|v_q| \leq \delta$ in $B(\tilde{x}, r) \cap \Gamma$, one can use Corollary 2.6 of [14] to justify the existence of a constant $c_1 > 0$ such that

$$
e^{-\frac{c_1}{\ln(\delta)}} \leq \delta |B(\tilde{x}, r) \cap \Gamma| \leq \delta |\Gamma|.
$$

Applying the function $\ln$ to the previous inequality, we obtain

$$
r \leq \frac{c}{\ln(\delta)},
$$

where $c$ is a positive constant depending on $c_1$ and $\Gamma$.

Thus, (2.12) implies that

$$
\|f\|_{L^\infty(\Gamma)} \leq \frac{c}{\ln(\delta)} + \frac{1}{\delta} \|f v_q\|_{L^\infty(\Gamma)}, \text{ for all } 0 < \delta < \delta^*.
$$

(2.13)

Now, by setting $\delta = e^{-s} > 0$, we may rewrite relation (2.13) as follows

$$
\|f\|_{L^\infty(\Gamma)} \leq \frac{c}{s^\alpha} + e^s \|f v_q\|_{L^\infty(\Gamma)}, \text{ for all } s > s^* = |\ln \delta^*|,
$$

which achieves the proof. \(\square\)

We also need the following Theorem [[5], Theorem 2].
Besides, let us define $\tilde{v}_{\infty}$, we get
\[ c \leq v \leq C \]
we have:
\[ C \]
Now, we are ready to give the main result of the current subsection.

**Proof of the Theorem 2.1.** Let $\gamma \subset \Gamma_a \subset \Gamma$ be defined as in (2.1). Let $u_q$ (resp. $u_q$) be the solution to the problem (1.2) with the coefficient $q$ (resp. with the coefficient $\tilde{q}$). Analogously, we can define the functions $v_q$ and $v_{\tilde{q}}$ . Using relation (2.6), Lemma 2.1 and the trace Theorem, we can find a constant $\lambda > 0$ such that for all $t > 0$
\[ \|v_q - v_{\tilde{q}}\|_{L^2(\gamma)} \leq \|u_q - u_{\tilde{q}}\|_{L^\infty((0,\infty);L^2(\gamma))}. \]

Besides, let us define $\tilde{v} = v_q - v_{\tilde{q}}$ satisfying $\Delta \tilde{v} = 0$. From the interpolation inequality and (2.4), we get:
\[ \|\tilde{v}\|_{L^2(\Gamma)} + \|\partial_n \tilde{v}\|_{L^2(\Gamma)} \leq C_4 \|\tilde{v}\|_{H^1(\Omega)}^{1/2}, \]

where $C_1, C_2, C_3$ are positive constants. Hence, by the trace Theorem, we get
\[ \|\tilde{v}\|_{L^2(\Gamma)} + \|\partial_n \tilde{v}\|_{L^2(\Gamma)} \leq C_4 \|\tilde{v}\|_{H^1(\Omega)}^{1/2}, \]

where $C_4$ is a positive constant.

Now, let $\gamma_0 \subset\subset \gamma$ and $P\tilde{v} = \Delta \tilde{v} = 0$. From Theorem 2.2, we have
\[ \|\tilde{v}\|_{H^1(\Omega)} \leq e^{\epsilon/\epsilon} \left( \|\tilde{v}\|_{H^1(\gamma_0)} + \|\partial_n \tilde{v}\|_{L^2(\gamma_0)} \right) + \epsilon \|\tilde{v}\|_{H^2(\Omega)}. \]

As done previously, we use the interpolation inequality, to obtain
\[ \|\tilde{v}\|_{H^1(\gamma_0)} \leq C_5 \|\tilde{v}\|_{H^2(\gamma_0)} \|\tilde{v}\|_{L^2(\gamma)}^{1/3}. \]

By the trace Theorem, relations (2.4) and (2.14), we obtain
\[ \|\tilde{v}\|_{H^1(\gamma_0)} \leq C_6 \|u_q - u_{\tilde{q}}\|_{L^\infty((0,\infty);L^2(\gamma))}, \]
where $C_5, C_6$ are positive constants. Consequently, combining the fact that $\partial_\nu \bar{v} = 0$ on $\gamma$ together with inequality (2.17) and (2.16), we get

$$\|\bar{v}\|_{L^1(\Omega)} \leq C_6 e^{c/\epsilon} \|u_q - u_q\|_{L^1((0,\infty);L^2(\gamma))} + \epsilon^\tau \|\bar{v}\|_{L^1(\Omega)}.$$  

Using relation (2.4) once again, we infer that

$$\|\bar{v}\|_{H^1(\Omega)} \leq C_6 e^{c/\epsilon} \|u_q - u_q\|_{L^1((0,\infty);L^2(\gamma))} + \epsilon^\tau R_1.$$  

(2.18)

Denoting $f(\epsilon) = C_6 e^{c/\epsilon} \|u_q - u_q\|_{L^1((0,\infty);L^2(\gamma))} + \epsilon^\tau R_1$ for $\epsilon > 0$, the minimizer $\epsilon_{\text{min}}$ of $f$ solves

$$f(\epsilon_{\text{min}}) = \frac{R_1}{\|u_q - u_q\|_{L^1((0,\infty);L^2(\gamma))}}$$

$$f(\epsilon) = \frac{c e^{c/\epsilon}}{\tau \epsilon^{1+1}}.$$  

The function $f$ is non increasing with $f(0^+) = +\infty$ and $f(+\infty) = 0$, so that the above equation has a unique solution $\epsilon_{\text{min}}$.

- If $\epsilon_0 > \epsilon_{\text{min}}$, and by choosing $\epsilon_{\text{min}} = \epsilon$ in (2.18) we have

$$\|\bar{v}\|_{H^1(\Omega)} \leq \left( \frac{\tau}{c} \epsilon_0 + 1 \right) R_1 \epsilon_{\text{min}} = CR_1 \epsilon_{\text{min}},$$  

(2.19)

where $\epsilon_{\text{min}}$ is sufficiently small satisfying for some $c' > c$,

$$f(\epsilon_{\text{min}}) = \frac{R_1}{\|u_q - u_q\|_{L^1((0,\infty);L^2(\gamma))}} \leq e^{c'/\epsilon_{\text{min}}}.$$

Then, $\epsilon_{\text{min}} \leq c/\log(R_1/\|u_q - u_q\|_{L^1((0,\infty);L^2(\gamma))})$. By plugging this estimate into (2.19), we obtain the existence of two positive constants $A_1$ and $B_1$ satisfying

$$\|\bar{v}\|_{H^1(\Omega)} \leq \frac{A_1}{\ln \left( B_1 \|u_q - u_q\|_{L^1((0,\infty);L^2(\gamma))} \right)}.$$

- If $\epsilon_0 \leq \epsilon_{\text{min}}$, we obtain $f(\epsilon_0) \geq R_1/\|u_q - u_q\|_{L^1((0,\infty);L^2(\gamma))}$, and thus

$$\|\bar{v}\|_{H^1(\Omega)} \leq R_1 \leq f(\epsilon_0) \|u_q - u_q\|_{L^1((0,\infty);L^2(\gamma))} = \frac{R_1}{\|u_q - u_q\|_{L^1((0,\infty);L^2(\gamma))}},$$

where $C$ is independent of $\bar{v}$, $R_1$ and $\|u_q - u_q\|_{L^1((0,\infty);L^2(\gamma))}$. Since,

$$R_1/\|u_q - u_q\|_{L^1((0,\infty);L^2(\gamma))} \geq (\log(R_1/\|u_q - u_q\|_{L^1((0,\infty);L^2(\gamma))}))^{1/\tau},$$

we obtain again the existence of two positive constants $A_1$ and $B_1$ such that

$$\|\bar{v}\|_{H^1(\Omega)} \leq \frac{A_1}{\ln \left( B_1 \|u_q - u_q\|_{L^1((0,\infty);L^2(\gamma))} \right)}.$$

Then, for all $\epsilon \in [0, \epsilon_0]$, we deduce the existence of two positive constants $A_1$ and $B_1$ satisfying

$$\|\bar{v}\|_{H^1(\Omega)} \leq \frac{A_1}{\ln \left( B_1 \|u_q - u_q\|_{L^1((0,\infty);L^2(\gamma))} \right)}.$$  

(2.20)

Based on (2.15), we get from (2.20) the existence of two positive constants $A_2$ and $B_2$ such that

$$\|\bar{v}\|_{L^2(\Gamma)} + \|\partial_\nu \bar{v}\|_{L^2(\Gamma)} \leq \frac{A_2}{\ln \left( B_2 \|u_q - u_q\|_{L^1((0,\infty);L^2(\gamma))} \right)^{1/2}}.$$  

(2.21)
Now, let us introduce $\psi = (q - \tilde{q})v_q$. We fix a parameter $\theta$ satisfying $2/3 < \theta < 1$ if $d = 2$ and $3/5 < \theta < 1$ if $d = 3$, and set $l = 3\theta/2$ for $d = 2$ and $l = 5\theta/2$ for $d = 3$. For these choices of $l$, we get that $H^l(\Gamma)$ is continuously embedded in $L^\infty(\Gamma)$. Therefore, using the interpolation inequalities, we get

$$
\|\psi\|_{H^l(\Gamma)} \leq C \|\psi\|_{H^{3/2}(\Gamma)} \|\psi\|_{L^2(\Gamma)}^{1-\theta}, \quad \text{if } d = 2,
$$

$$
\|\psi\|_{H^l(\Gamma)} \leq C \|\psi\|_{H^{5/2}(\Gamma)} \|\psi\|_{L^2(\Gamma)}^{1-\theta}, \quad \text{if } d = 3.
$$

Moreover,

$$
\|\psi\|_{L^\infty(\Gamma)} \leq C \|\psi\|_{H^{3/2}(\Gamma)} \|\psi\|_{L^2(\Gamma)}^{1-\theta}, \quad \text{if } d = 2,
$$

$$
\|\psi\|_{L^\infty(\Gamma)} \leq C \|\psi\|_{H^{5/2}(\Gamma)} \|\psi\|_{L^2(\Gamma)}^{1-\theta}, \quad \text{if } d = 3.
$$

Since $q, \tilde{q} \in D_M$, then by using (2.4), we immediately get

$$
\|\psi\|_{H^{-1/2}(\Gamma)} \leq \|(q - \tilde{q})v_q\|_{H^{-1/2}(\Gamma)} \leq C \|q - \tilde{q}\|_{B_{n-1/2,1}(\Gamma)} \|v_q\|_{H^{n-1/2}(\Gamma)},
$$

where $n = 2, 3$. Consequently,

$$
\|(q - \tilde{q})v_q\|_{L^\infty(\Gamma)} \leq C_\gamma \|(q - \tilde{q})v_q\|_{L^2(\Gamma)}^{1-\theta}, \quad (2.22)
$$

where $C_\gamma$ is a positive constant. Returning to the definition of $\tilde{v}$, we get

$$
(q - \tilde{q})v_q = -\partial_\nu \tilde{v} - \tilde{q}\tilde{v}. \quad (2.23)
$$

Combining (2.21), (2.22) and (2.23) we obtain the existence of two positive constants $A_3$ and $B_3$ such that

$$
\|(q - \tilde{q})v_q\|_{L^\infty(\Gamma)} \leq \|(q - \tilde{q})v_q\|_{L^\infty(\Gamma)} \leq \frac{A_3}{\ln \left( B_3 \|u_q - u_{\tilde{q}}\|_{L^\infty((0,\infty);L^2(\gamma))} \right)^{(1-\theta)\tau/2}}. \quad (2.24)
$$

On the other hand, recall that $q, \tilde{q} \in D_M \cap C^\alpha(\Gamma)$. Then, Lemma 2.2 (by putting $f = q - \tilde{q}$) and relation (2.24) ensure the existence of two positive constants $A_4$ and $C_8$ satisfying

$$
\|q - \tilde{q}\|_{L^\infty(\Gamma)} \leq \frac{C_8}{s^\alpha} + e^s \frac{A_4}{\ln \left( B \|u_q - u_{\tilde{q}}\|_{L^\infty((0,\infty);L^2(\gamma))} \right)^{(1-\theta)\tau/2}}, \quad s \geq s^* = |\ln \delta^*|.
$$

Here, the parameter $\theta$ can be determined explicitly in terms of the given constant $\sigma$ through the relation $\sigma = (1-\theta)\tau/2$. In the same way, as done in [4], we minimize on $s$ and we obtain

$$
\|q - \tilde{q}\|_{L^\infty(\Gamma)} \leq \frac{A}{\ln \left( B \|u_q - u_{\tilde{q}}\|_{L^\infty((0,\infty);L^2(\gamma))} \right)^\sigma},
$$

where $A, B$ are positive constants. This achieves the proof of the theorem.
2.3. Stability estimate for the case of time dependent heat flux. In what
follows, we use the same notations as in the previous subsection. Let \( u_h \in C \left( [0, \infty[, L^2(\Omega) \right) \cap C([0, \infty[, H^2(\Omega)) \cap L^\infty((0, \infty), H^1(\Omega)) \) be the solution to the fol-
lowering problem

\[
\begin{aligned}
\partial_t u_h(x, t) &= \Delta u_h(x, t), \quad (x, t) \in \Omega \times (0, \infty), \\
\partial_n u_h(x, t) &= h(x, t), \quad (x, t) \in \Gamma_a \times (0, \infty), \\
\partial_{\nu_\Gamma} u_h(x, t) + q(x) u_h(x, t) &= 0, \quad (x, t) \in \Gamma_i \times (0, \infty), \\
u_h(x, 0) &= u_0(x), \quad x \in \Omega.
\end{aligned}
\]  

(2.25)

We assume that \( u_h(.)\), (0 \( \in L^2(\Omega) \) and \( \| u_h(\cdot, 0) \|_{L^2(\Omega)} \leq R_0 \), where \( R_0 \) is the same
positive constant given by (1.3). Then, the inverse problem associated to the prob-
lem (2.25) can be set as follows:

“Find the coefficient \( q \),

such that the solution \( u_h \) to (2.25) satisfies the following measurement

\[
u|_{\gamma \times (0, \infty)} = \zeta.
\]  

(2.26)

In this subsection, we suppose that we have the following assumptions

(A_2) \( \tilde{h} = \chi_{\gamma}, h \in C([0, \infty), H^{3/2}(\partial \Omega)), \) and \( \partial_{\nu_\Gamma} \tilde{h} \in C([0, \infty), H^{1/2}(\partial \Omega)). \)

(A_3) There exist \( g \in H^1(\Gamma_a) \) and \( \mu > 0 \) such that

\[
\lim_{t \to +\infty} \left( \| h(\cdot, t) - g \|_{H^{1/2}(\Gamma_a)} + \left[ \int_0^t e^{-2\mu(t-s)} \| \partial_{\nu_\Gamma} h(\cdot, s) \|_{H^{3/2}(\Gamma_a)}^2 ds \right]^{1/2} \right) = 0.
\]

Now, let us assume that (1.1), (2.1), (2.2), (A_1), (A_2) and (A_3) hold. As we have
done in the previous subsection, we can deduce the following result.

Proposition 2.1. Let the functions \( u_{q_1} \) and \( u_{q_2} \) be solutions to the problem (2.25)
with coefficients \( q_1 \) and \( q_2 \), respectively. Let the constants \( \alpha \in (0, 1) \) and \( \alpha' \in (0, 1) \). Then, there exist two positive constants \( A' = A'(\alpha', \Omega, \gamma \times (0, \infty)) \) and \( B' = B'(\Omega, \gamma \times (0, \infty)) \) such that for any \( q_1 \) and \( q_2 \) in \( \Sigma = D^0_\mathcal{M} \cap C^\alpha(\Gamma) \) we have:

\[
\| q_1 - q_2 \|_{L^\infty(\Gamma_i)} \leq \frac{A'}{\ln \left( \ln \left( B' \| u_{q_1} - u_{q_2} \|_{L^\infty((0, \infty), H^2(\Omega))} \right) \right)^{\alpha'}}.
\]

Proof. Let \( u_h \) be the solution to the problem (2.25) and denote by \( v \) the solution
to the problem (2.3) where \( g \) is given by (A_3). As done in (2.6), we may write
\( u_h = v + w_h \), where \( w_h \) is the solution to the following problem

\[
\begin{aligned}
\partial_t w(x, t) &= \Delta w(x, t), \quad x \in \Omega, t > 0, \\
\partial_n w(x, t) &= h(x, t) - g(x), \quad x \in \Gamma_a, t > 0, \\
\partial_{\nu_\Gamma} w(x, t) + q(x) w(x, t) &= 0, \quad x \in \Gamma_i, t > 0, \\
w(x, 0) &= u_0(x) - v(x), \quad x \in \Omega.
\end{aligned}
\]  

(2.27)

Besides, let us define \( u_h^0 \) be the solution to the problem (2.25) with boundary con-
dition

\[
\partial_t u_h^0(x, t) = h(x, t) - g(x), \quad x \in \Gamma_a, t > 0,
\]

and with initial condition \( u_h^0(\cdot, 0) \equiv 0 \) in \( \Omega \).
So, the function $w_h$ can rewritten as follows: $w_h = u_h^0 + \tilde{w}_h$. Clearly $\tilde{w}_h$ solves the following initial-boundary value problem

$$
\begin{aligned}
\partial_t \tilde{w}_h(x,t) &= \Delta \tilde{w}_h(x,t), & x \in \Omega, t > 0, \\
\partial_n \tilde{w}_h(x,t) &= 0, & x \in \Gamma_a, t > 0, \\
\partial_n \tilde{w}_h(x,t) + q(x) \tilde{w}_h(x,t) &= 0, & x \in \Gamma, t > 0, \\
\tilde{w}_h(x,0) &= u_0(x) - v(x), & x \in \Omega.
\end{aligned}
$$

(2.28)

According to Lemma 2.1, we see that $\tilde{w}_h$ satisfies

$$
\|\tilde{w}_h(\cdot,t)\|_{H^1(\Omega)} \leq C_0 \frac{e^{-\mu t}}{\sqrt{t}}, \quad t > 0,
$$

(2.29)

where $C_0 = C_0(R_0, R_1)$ is a positive constant. Now, getting back to the problem (2.25), with replacing $u_h$ by $u_h^0$ (with initial condition $u_h^0(x,0) = 0$, $x \in \Omega$), it is not hard to find by classical decompositions techniques, the following estimation

$$
\|u_h^0(\cdot,t)\|_{H^1(\Omega)} \leq C\|h(\cdot,t)\|_{H^{1/2}(\Gamma_a)} + e^{-\mu t}\|h(\cdot,0)\|_{H^{1/2}(\Gamma_a)} + \int_0^t e^{-2\mu(t-s)}\|\partial_t h(\cdot,s)\|^2_{H^{1/2}(\Gamma_a)} ds^{1/2},
$$

(2.30)

where $C$ is a given positive constant.

Combining relations (2.29) and (2.30), we get

$$
\|w_h(\cdot,t)\|_{H^1(\Omega)} \leq C_0 \frac{e^{-\mu t}}{\sqrt{t}} + C(\|h(\cdot,t)\|_{H^{1/2}(\Gamma_a)} + e^{-\mu t}\|h(\cdot,0)\|_{H^{1/2}(\Gamma_a)}) + \int_0^t e^{-2\mu(t-s)}\|\partial_t h(\cdot,s)\|^2_{H^{1/2}(\Gamma_a)} ds^{1/2}).
$$

Finally, by (A3) and $e^{-\mu t}\|h(\cdot,0)\|_{H^{1/2}(\Gamma_a)} \to 0$ as $t \to +\infty$, one can proceed as done in the proof of Theorem 2.1 to get the announced result. \qed

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REFERENCES

[1] G. Alessandrini, L. Del Piero and L. Rondi, Stable determination of corrosion by a single electrostatic boundary measurement, Inverse Probl., 19 (2003), 973–984.

[2] G. Alessandrini and E. Sincich, Solving elliptic Cauchy problems and the identification of nonlinear corrosion, J. Comput. Appl. Math., 198 (2007), 307–320.

[3] M. Bellassoued, J. Cheng and M. Choulli, Stability estimate for an inverse boundary coefficient problem in thermal imaging, J. Math Anal. Appl., 343 (2008), 328–336.

[4] M. Bellassoued, M. Choulli and A. Jbalia, Stability of the determination of the surface impedance of an obstacle from the scattering amplitude, Math. Meth. Appl. Sci., 36 (2013), 2429–2448.

[5] L. Bourgeois, About stability and regularization of ill-posed elliptic Cauchy problems: the case of $C^{1,1}$ domains, Math.Model. Numer. Anal., 44 (2010), 715–735.

[6] K. Bryan and Jr. L. F. Caudill, An inverse problem in thermal imaging, SIAM J. Appl. Math., 56 (1996), 715–735.

[7] K. Bryan and Jr. L. F. Caudill, Uniqueness for a boundary identification problem in thermal imaging, in: J. Graef, R. Shivaji, B. Soni, Zhu (Eds.), Differential Equations and Computational Simulations III, in: Electron. J. Differ. Equ. Conf., 1 (1998), 23–39.

[8] S. Busenberg and W. Fang, Identification of semiconductor contact resistivity, Quar. J. Appl. Math., 49 (1991), 639–649.

[9] S. Chaabane, I. Fellah, M. Jaoua and J. Leblond, Logarithmic stability estimates for a Robin coefficient in two-dimensional Laplace inverse problems, Inverse Probl., 20, (2004), 47–59.
S. Chaabane, I. Feki and N. Mars, Numerical reconstruction of a piecewise constant Robin parameter in the two- or three-dimensional case, Inverse Probl., 28 (2012), 065016.

S. Chaabane and M. Jaoua, Identification of Robin coefficient by means of boundary measurements, Inverse Probl., 15 (1999), 1425–1438.

J. Cheng, M. Choulli and J. Lin, Stable determination of a boundary coefficient in an elliptic equation, Math Models Methods Appl Sci., 18 (2008), 107–123.

J. Cheng, M. Choulli and X. Yang, An iterative BEM for the inverse problem of detecting corrosion in a pipe, Numer. Math. J. Chinese Univ., 14 (2005), 252–266.

M. Choulli and A. Jbalia, The problem of detecting corrosion by electric measurements revisited, Discrete Contin. Dyn. Syst. Ser. S, 9 (2016), 643–650.

M. Choulli, An inverse problem in corrosion detection: Stability estimates, J. Inverse Ill-Posed Probl., 12 (2004), 349–367.

M. Choulli, Stability estimates for an inverse elliptic problem, J. Inverse Ill-Posed Probl., 10 (2002), 601–610.

W. Fang and E. Cumberbatch, Inverse problems for metal oxide semiconductor field-effect transistor contact resistivity, SIAM J. Appl. Math., 52 (1992), 699–709.

W. Fang and M. Lu, A fast collocation method for an inverse boundary value problem, Int. J. Numer. Methods Eng., 59 (2004), 1563–1585.

D. Fasino and G. Inglese, Stability of the solutions of an inverse problem for Laplace’s equation in a thin strip, Numer. Func. Anal. Opt., 22 (2001), 549–560.

D. Fujiwara, Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order, Proc. Japan Acad., 43 (1967), 82–86.

P. Germain, Thèse de doctorat: Solutions fortes, solutions faibles d’équations aux dérivées partielles d’évolution, Ecole polytechnique France, 2005.

L. Hörmander, The Analysis of Partial Differential Operators, 2, 2d ed: Springer-Verlag, Berlin, 1990.

G. Inglese, An inverse problem in corrosion detection, Inverse Probl., 13 (1997), 977–994.

M. Jaoua, S. Chaabane, C. Elhechmi, J. Leblond, M. Mahjoub and J. R. Partington, On some robust algorithms for the Robin inverse problem. International conference in honor of Claude Lobry, 2007.

B. Jin and X. Lu, Numerical identification for a Robin coefficient in parabolic problems, Math. Comp., 81 (2012), 1369–1398.

B. Jin and J. Zou, Numerical estimation of the Robin coefficient in a stationary diffusion equation, IMA J. Numer. Anal., 30 (2010), 677–701.

B. Jin and J. Zou, Numerical estimation of piecewise constant Robin coefficient, SIAM J. Control Optim., 48 (2009), 1977–2002.

P. G. Kaup, F. Santos and M. Vogelius, Method for imaging corrosion damage in thin plates from electrostatic data, Inverse Probl., 12 (1996), 279–293.

F. Lin and W. Fang, A linear integral equation approach to the Robin inverse problem, Inverse Probl., 21 (2005), 1757–1772.

A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983.

M. Renardy and R. C. Rogers, An Introduction to Partial Differential Equations, Springer-Verlag: New York, 1993.

Z. Sun, Y. Jiao, B. Jin and X. Lu, Numerical identification of a sparse Robin coefficient, Adv. Comput. Math., 41 (2015), 131–148.

F. M. White, Heat and Mass Transfer: Addison-Wesley, Reading, MA, 1988.

Y. Xu and J. Zou, Analysis of an adaptive finite element method for recovering the Robin coefficient, SIAM J. Control Optimiz., 53 (2015), 622–644.

F. Yang, L. Yan and T. Wei, The identification of a Robin coefficient by a conjugate gradient method, Int. J. Numer. Meth. Engng., 78 (2009), 800–816.

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E-mail address: jballia.aymen@yahoo.fr