An analysis of the logic of Riesz Spaces with strong unit

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Abstract

We study Łukasiewicz logic enriched with a scalar multiplication with scalars taken in $[0,1]$. Its algebraic models, called Riesz MV-algebras, are, up to isomorphism, unit intervals of Riesz spaces with a strong unit endowed with an appropriate structure. When only rational scalars are considered, one gets the class of DMV-algebras and a corresponding logical system. Our research follows two objectives. The first one is to deepen the connections between functional analysis and the logic of Riesz MV-algebras. The second one is to study the finitely presented MV-algebras, DMV-algebras and Riesz MV-algebras, connecting them from logical, algebraic and geometric perspective.

Keywords: Łukasiewicz logic, MV-algebra, Riesz MV-algebra, DMV-algebra, limit, norm-completion, finitely presented, tensor product, polyhedron.

Introduction

In this paper we present the logical system $\mathcal{RL}$ which extends the infinitely valued Łukasiewicz logic with a family of unary operators that are semantically interpreted as scalar multiplication with scalars from the real interval $[0,1]$. The category of the corresponding algebraic structures is equivalent with the category of Riesz spaces with strong unit.

Recall that Łukasiewicz logic $L$ is the system that has $\{\rightarrow, \neg\}$ as basic connectives and whose axioms are L1-L4 below:

\begin{align*}
\text{(L1)}\ & \varphi \rightarrow (\psi \rightarrow \varphi) \\
\text{(L2)}\ & (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)) \\
\text{(L3)}\ & (\varphi \lor \psi) \rightarrow (\psi \lor \varphi) \\
\text{(L4)}\ & (\neg \psi \rightarrow \neg \varphi) \rightarrow (\varphi \rightarrow \psi).
\end{align*}

The only deduction rule is modus ponens.

The corresponding algebraic structures, MV-algebras, were defined by C.C. Chang in 1958 [7]. Chang’s definition was inspired by the theory of lattice
ordered groups, consequently MV-algebras are structures \((A, \oplus, \neg, 0)\) satisfying some appropriate axioms, where \(x \to y = \neg x \oplus y\) for any \(x, y\). The connection between MV-algebras and Abelian lattice ordered groups was fully investigated by D. Mundici [28] who proved the fundamental result that MV-algebras are categorically equivalent with Abelian lattice-ordered groups with strong unit.

Since the standard model of Ł is the real interval \([0, 1]\) endowed with the Łukasiewicz negation \(\neg x = 1 - x\) and the Łukasiewicz implication \(x \to y = \min(1 - x + y, 1)\), a natural problem was to study Łukasiewicz logic enriched with a product operation, semantically interpreted in the real product on \([0, 1]\). This line of research led to the definition of PMV-algebras, which are MV-algebras endowed with an internal binary operation, but in this case the standard model only generates a proper subvariety. Through an adaptation of Mundici’s equivalences, PMV-algebras and their logic are connected with the theory of lattice-ordered rings with strong unit.

A different approach is presented in [12, 16], where the real product on \([0, 1]\) is interpreted as scalar multiplication, with scalars taken in \([0, 1]\). The system \(\mathcal{RL}\) further developed in this paper is a relatively simple extension of \(\mathcal{L}\) which is obtained by adding to the infinitely valued Łukasiewicz logic a family of unary operators \(\{\nabla_r\}_{r \in [0,1]}\), whose dual operators are semantically interpreted in a scalar multiplication. Consequently, Riesz MV-algebras – the corresponding algebraic structures – are categorically equivalent with Riesz spaces with a strong unit. Note that our results are not the first connection between Łukasiewicz logic and the theory of Riesz spaces, one can see [2, 18] for previous investigations, while the seminal idea of a connection between Riesz Spaces and a subcategory of MV-algebras was given in [14].

In this paper, we establish connections between the system \(\mathcal{RL}\) and elements of functional analysis, where Riesz spaces are fundamental structures. After some needed preliminaries, in Section 2 we define \(\mathcal{RL}\) and prove some logic-related results, the main being a syntactical characterization of the uniform convergence in a Riesz Space. Using this concept of limit of formulas, in Theorem 2.12 we describe any formula in \(\mathcal{RL}\) as a sequence of formulas in the Rational Łukasiewicz logic [17] and in Section 2.4 we characterize two norm-completions of the Lindenbaum-Tarski algebra of \(\mathcal{R}\).

The central part of the paper deals with the problem of finding under which conditions we can establish categorical equivalences between subcategories of finitely presented MV-algebras, DMV-algebras and Riesz MV-algebras. This investigation is carried in Sections 3.1-3.3 using three different points of view, that take advantage of state-of-the-art techniques on MV-algebras: polyhedra, tensor product and the fresh approach via categories of presentations. Finally, we link all results together in Section 3.4, where we use the syntactical notion of limit to fully describe those theories of \(\mathcal{RL}\) that are axiomatized by formulas of \(\mathcal{L}\).

1 Preliminaries on algebraic structures

MV-algebras are the algebraic counterpart of Łukasiewicz propositional calculus. They are structures \((A, \oplus, \cdot, 0)\) of type \((2, 1, 0)\) such that, for any \(x, y \in A\),

\[
\begin{align*}
(A, \oplus, 0) & \text{ is an Abelian monoid, } (x^*)^* = x, \\
(x^* \oplus y)^* \oplus y & = (y^* \oplus x)^* \oplus x, \quad 0^* \oplus x = 0^*.
\end{align*}
\]
Further operations are defined as follows: 1 is 0', the Łukasiewicz implication is \( x \rightarrow y = x^* \oplus y \), the Łukasiewicz conjunction is \( x \land y = (x^* \oplus y^*)^* \) and Chang's distance is \( d(x, y) = (x^* \land y^*) - (x^* \land y^*) \), for any \( x, y \in A \). If \( x \lor y = x \oplus (y \land x^*) \) and \( x \land y = (x^* \lor y^*)^* \) then \( (A, \lor, \land, 1, 0) \) is a bounded distributive lattice. If \( x \in A \) and \( n \in \mathbb{N} \) then \( nx = 0 \) and \( (n + 1)x = (nx) \oplus x \).

The variety of MV-algebras is generated by the standard structure
\[
[0,1]_{\text{MV}} = ([0,1], \lor, *, 0),
\]
where \([0,1]\) is the real unit interval, \( x^* = 1 - x \) and \( x \sqcup y = \min(1, x + y) \) for any \( x, y \in [0,1] \). We urge the interested reader to consult [9, 15] for a basic introduction to MV-algebras and [31] for advanced topics.

A fruitful research direction has arisen from the idea of endowing MV-algebras with a product operation. In particular, when we consider a scalar multiplication, we obtain the notion of Riesz MV-algebras.

A Riesz MV-algebra [16] is a structure
\[
(R, \oplus, *, 0, \{r \mid r \in [0,1]\})
\]
such that \((R, \oplus, *, 0)\) is an MV-algebra and \(\{r \mid r \in [0,1]\}\) is a family of unary operations such that the following properties hold for any \(x, y \in A\) and \(r, q \in [0,1]\):
\[
r(x \oplus y^*) = (rx) \circ (ry)^*, \quad (r \circ q^*) \cdot x = (rx) \circ (qx)^*,
\]
\[
r(qx) = (rq)x,
\]
\[
1x = x.
\]
The variety of Riesz MV-algebras is generated by
\[
[0,1]_{\text{R}} = ([0,1], \lor, *, \{\alpha \mid \alpha \in [0,1]\}, 0),
\]
where \(([0,1], \lor, *, 0)\) is the standard MV-algebra and \(x \mapsto \alpha x\) is the real product of \(\alpha\) and \(x\), for \(\alpha, x \in [0,1]\).

An intermediate class between MV-algebras and Riesz MV-algebras is the class of DMV-algebras [17]. By [21] Corollary 2.1, DMV-algebras can be seen as MV-algebras endowed with a scalar multiplication with scalars taken in \([0,1]_Q = [0,1] \cap \mathbb{Q}\). In this case the standard model is \([0,1]_Q = ([0,1]_Q, \oplus, *, \{q \mid q \in [0,1]_Q\})\), where \(x \mapsto qx\) is the product of \(q\) and \(x\), for any \(q \in [0,1]_Q\), and it generates the variety of DMV-algebras. Endowed with suitable morphisms, DMV-algebras and Riesz MV-algebras form categories that we shall denote by DMV and RMV respectively.

A fundamental result in the theory of MV-algebras is their categorical equivalence with Abelian lattice-ordered groups with a strong unit [23]. Such groups have compatible group and lattice structures, and a distinguished Archimedean element, called strong unit [5]. If \((G, u)\) is a lattice-ordered group with strong unit, then \(([0,u], \oplus, *, 0)\) is an MV-algebra, where \(x \oplus y = (x + y) \land u\) and \(x^* = u - x\) for any \(x, y \in [0,u]\). A similar correspondence is proved for Riesz MV-algebras and Riesz spaces with strong unit [16], as well as for DMV-algebras and divisible \(\ell u\)-groups [17]. Moreover, in [21], the authors prove that DMV-algebras are categorical equivalent with \(\mathbb{Q}\)-vector lattices with a strong unit, that is lattice-ordered linear spaces over \(\mathbb{Q}\).

Finally, we mention that the structures considered in this paper are semisimple. Recall that semisimple MV-algebras are subdirect products of MV-subalgebras of \([0,1]\) and they enjoy an convenient functional representation. Indeed, given
a semisimple MV-algebra $A$, this isomorphic with a separating MV-subalgebra of $C(X)$, where $X$ is a suitable compact Hausdorff space [9, Corollary 3.6.8]. A DMV-algebra or a Riesz MV-algebra is semisimple if its MV-algebra reduct is semisimple.

2 The logic $\mathcal{RL}$ of Riesz MV-algebras

In this section we present the system $\mathcal{RL}$ [16], a conservative extension of Łukasiewicz logic $\mathcal{L}$, that has Riesz MV-algebras as models.

Since Riesz MV-algebras are categorically equivalent with Riesz spaces with strong unit, $\mathcal{RL}$ can be seen as the logical counterpart of the latter class of structures.

We review already known results, we add some new insight and we deepen the connection with notions from functional analysis. The system $\mathcal{RL}$ has uncountable syntax, but the compactness theorem still holds. We further define the truth degree and the provability degree of a formula in $\mathcal{RL}$ and we prove Pavelka-style completeness. The system allows us to define a syntactic notion of limit. Consequently, we express any formula of $\mathcal{RL}$ as a syntactic limit of formulas in $\mathcal{QL}$. Moreover, in Section 2.4 we show that the Lindenbaum-Tarski algebra of $\mathcal{RL}$ naturally becomes a metric space and we analyze its Cauchy completeness.

We recall that the logical system $\mathcal{RL}$, is obtained from Łukasiewicz logic by adding a connective $\nabla_r$ for any $r \in [0, 1]$. Hence the logical connectives are \{$\neg, \to$\} $\cup$ \{$\nabla_r \mid r \in [0, 1]$\} and the axioms are the following:

(L) the axioms (L1)-(L4) of Łukasiewicz logic,
(R1) $\nabla_r (\phi \to \psi) \leftrightarrow (\nabla_r \phi \to \nabla_r \psi)$
(R2) $\nabla_{(r \circ q')}(\phi) \leftrightarrow (\nabla_{q'} \phi \to \nabla_{r} \phi)$
(R3) $\nabla_r (\nabla_{q'} \phi) \leftrightarrow \nabla_{r \circ q'} \phi$
(R4) $\nabla_1 \phi \leftrightarrow \phi$.

The only deduction rule is Modus Ponens. Note that “from $\phi$, we prove $\nabla_r \phi$ for any $r \in [0, 1]$” is a derivative rule.

The set of formulas of $\mathcal{RL}$ is denoted $\text{Form}_{\mathcal{RL}}$. For any $r \in [0, 1]$ and $\phi \in \text{Form}_{\mathcal{RL}}$ we set $\Delta_r \phi = \neg \nabla_r \neg \phi$.

If we only consider $\{\nabla_q \mid q \in [0, 1] \cap \mathbb{Q}\}$, we obtain the system $\mathcal{QL}$ that has DMV-algebras as models [17, 21]. We refer to [16] and [17, 21] for a more detailed account on the logics $\mathcal{RL}$ and $\mathcal{QL}$ respectively.

Remark 2.1. [Completeness results] Łukasiewicz logic $\mathcal{L}$ is complete with respect to its standard model $[0, 1]_{\text{MV}}$ [8], the logic $\mathcal{RL}$ is complete with respect to $[0, 1]_{\mathbb{R}}$ [16] and $\mathcal{QL}$ is complete with respect to $[0, 1]_{\mathbb{Q}}$ [17].

Note that both $\mathcal{RL}$ and $\mathcal{QL}$ are conservative extensions of $\mathcal{L}$.

As usual, a set of formulas $\Theta \subseteq \text{Form}_{\mathcal{RL}}$ is consistent if there exists a formula $\phi$ such that $\Theta \nvdash \phi$. Moreover, $\Theta$ is consistent if and only if $e(\Theta) = \{1\}$ for some evaluation $e : \text{Form}_{\mathcal{RL}} \to [0, 1]$.

Remark 2.2. [Compactness] If $\Theta$ is a set of formulas such that for any finite $\Gamma \subseteq \Theta$ there exists a $[0, 1]$-valuation $e_{\Gamma}$ such that $e_{\Gamma}(\Gamma) = \{1\}$, then there exists a $[0, 1]$-valuation $e$ such that $e(\Theta) = \{1\}$. The proof is a direct consequence of the fact that every finite subset of $\Theta$ is consistent.
For any formula $\varphi$ with $n$ variables we define the corresponding term function $f_\varphi : [0, 1]^n \to [0, 1]$ in the usual manner.

Recall that a unital piecewise linear function (PWL$_n$-function) with real (or integer, or rational) coefficients is a continuous function $f : [0, 1]^n \to [0, 1]$ such that there exist $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$, $f_i = c_{i1}x_1 + \ldots + c_{in}x_n + b_i$ with $c_{ij}, b_i \in \mathbb{R}$ (or $c_{ij}, b_i \in \mathbb{Z}$, or $c_{ij}, b_i \in \mathbb{Q}$) and for any $x = (x_1, \ldots, x_n) \in [0, 1]^n$, there exists $i$ such that $f(x) = f_i(x)$.

One can easily see that $f_\varphi$ is a PWL$_n$-function with integer coefficients if $\varphi$ is a formula of $L$, $f_\varphi$ is a PWL$_n$-function with rational coefficients if $\varphi$ is a formula of $QL$ and $f_\varphi$ is a PWL$_n$-function with real coefficients if $\varphi$ is a formula of $RL$.

In [27, 16, 17] the free MV-algebra, free DMV-algebra and free Riesz MV-algebra are defined as algebras of $[0, 1]$-valued functions. By [27, Theorem 1 and Theorem 2], $MV_n$ is the MV-algebra of PWL$_n$-functions in $n$ variables and integer coefficients; by [16, Corollary 7], $RMV_n$ is the Riesz MV-algebra of PWL$_n$-functions in $n$ variables and real coefficients; by [17, Theorem 4.5], $DMV_n$ is the DMV-algebra of PWL$_n$-functions in $n$ variables and rational coefficients.

The Lindenbaum-Tarski algebras of $L$, $QL$ and $RL$ are denoted by $L$, $QL$ and $RL$, respectively. If $[\varphi]$ is the equivalence class of a formula $\varphi$, then, in the Lindenbaum-Tarski algebras $QL$ and $RL$, we define $r[\varphi] = [\Delta_v \varphi]$ for any appropriate $r$. When only $n$ variables are considered, the Lindenbaum-Tarski algebras $L_n$, $QL_n$ and $RL_n$ are isomorphic to $MV_n$, $DMV_n$ and $RMV_n$, respectively.

2.1 Truth degree and provability degree

In this section we define the truth degree and the provability degree of a formula, inspired by similar results in [19]. We also note that our system can be studied in the general framework developed in [32].

Let $\eta$ is a fixed theorem of $RL$. For any $r \in [0, 1]$ we denote by $\eta_r$ the formula $\Delta_v \eta$. Thus, in the Lindenbaum-Tarski algebra of $RL$, $[\eta_r] = [\Delta_v \eta] = r[\eta] = r1$.

In the next proposition we will prove that this approach does not depend upon the choice of $\eta$.

Proposition 2.3. The following properties hold for any $r, q \in [0, 1]$:

(a) If $\vdash \tau$ and $\tau_r = \Delta_v \tau$, then $\vdash \tau_r \leftrightarrow \eta_r$,
(b) If $\vdash \tau$ and $\tau_r = \Delta_v \tau$, then $\Theta \vdash \Delta_v \eta \rightarrow \varphi$ if and only if $\Theta \vdash \Delta_v \tau \rightarrow \varphi$,
(c) $\vdash \eta_r \leftrightarrow \eta_{r\cdot}$,
(d) $\vdash (\eta_r \rightarrow \eta_q) \leftrightarrow \eta_{r\cdot q}$,
(e) $\vdash \Delta_v \eta_q \leftrightarrow \eta_{r\cdot q}$,
(f) $\varepsilon(\Delta_v \varphi) = r\varepsilon(\varphi)$ where $\varphi$ is a formula and $r$ is an arbitrary evaluation,
(g) $\varepsilon(\eta_r) = r$, where $r$ is an arbitrary $[0, 1]$-evaluation,
(h) $r \leq q$ if $\vdash \eta_r \rightarrow \eta_q$.

Proof. It is straightforward by completeness of $RL$.

Remark 2.4. When $r$ and $q$ are chosen in $[0, 1] \cap \mathbb{Q}$, Proposition 2.3(c)-(d) are the bookkeeping axioms of Rational Pavelka logic [19, 3.3]. Proposition 2.3(e) can be also considered a bookkeeping axiom, since it shows that the logical constants act naturally with respect to the scalar operation. Thus, we can think of $RL$ as an axiomatic extension of a Pavelka propositional calculus.
Definition 2.5. Let \( \varphi \) be an arbitrary formula of RL, we define:

1. the truth degree of \( \varphi \), by
   \[ \| \varphi \| = \min \{ e(\varphi) \mid e \text{ is a } [0,1]-\text{evaluation} \}, \]
2. the provability degree of \( \varphi \), by
   \[ | \varphi | = \max \{ r \in [0,1] \mid \vdash \eta_r \rightarrow \varphi \}. \]

Note that, in general, provability degree and truth degree are defined by infimum and supremum. In this case they are indeed minimum and maximum.

Proposition 2.6 (Pavelka completeness). If \( \varphi \) is an formula of RL, then

\[ | \varphi | = \| \varphi \|. \]

Proof. We recall that if \( \varphi \) is a formula in the variables \( v_1, \ldots, v_n \), there exists a continuous PWL\(_n\)-function with real coefficients such that, for any evaluation \( e \), \( e(\varphi) = f_\varphi(e(v_1), \ldots, e(v_n)) \). Since \( f_\varphi \) is a continuous function over a compact set, Weierstrass’s extreme values theorem ensures the existence of the minimum value. This entails the existence of a minimal truth-value \( e_m(\varphi) \), and \( | \varphi | \) is indeed a minimum.

On the other direction, by completeness of RL, for any \( r \) such that \( \vdash \eta_r \rightarrow \varphi \), we have \( r \leq e(\varphi) \) for any evaluation \( e \). In particular we have \( r \leq e_m(\varphi) \). Being \( e_m \) the minimal evaluation, \( e_m(\varphi) \leq e(\varphi) \) for any other evaluation \( e \), that is \( \vdash \eta_{e_m(\varphi)} \rightarrow \varphi \) by completeness, and \( | \varphi | \) is a maximum.

2.2 A logical approach to limits

Since any formula \( \varphi \) of RL corresponds to a continuous piecewise linear function with real coefficients \( f_\varphi : [0,1]^n \rightarrow [0,1] \), we shall now express the uniform convergence of functions as a syntactic notion in RL.

We firstly prove the following.

Lemma 2.7. Let \( A \) be a compact topological space, \( f_n : A \rightarrow \mathbb{R}, n \in \mathbb{N} \), be a sequence such that there exists a decreasing sequence \( (g_n)_n \) of continuous functions such that \( \bigwedge_n g_n = 0 \) (where \( \bigwedge \) denotes the pointwise infimum) and

\[ |f(x) - f_n(x)| \leq g_n(x) \]

for every \( n \in \mathbb{N} \) and \( x \in A \). Then \( (f_n)_n \) uniformly converges to \( f \).

Proof. Since the sequence \( (g_n)_n \) is decreasing and all \( g_n \)'s are continuous, the sequence converges pointwise to the zero function. Being \( A \) compact, by the Dini’s theorem, the convergence is uniform. Thus, for every \( \varepsilon > 0 \) there is \( k \in \mathbb{N} \) with \( 0 \leq g_n(x) \leq \varepsilon \) for every \( n \geq k \) and \( x \in A \). By hypothesis, from this it follows that \( 0 \leq |f(x) - f_n(x)| \leq g_n(x) \leq \varepsilon \) for every \( n \geq k \) and \( x \in A \) and the claim is settled.

Now we are ready to prove our main result.

Theorem 2.8. Let \( (\varphi_n)_n \subseteq \text{Form}_\text{RL} \) and \( \varphi \in \text{Form}_\text{RL} \). Then the following are equivalent:

(i) \( (f_{\varphi_n})_n \) uniformly converges to \( f_\varphi \),
(ii) for any \( r \in [0,1] \) there exists \( k \) such that \( \vdash \eta_r \rightarrow (\varphi \leftrightarrow \varphi_n) \) for any \( n \geq k \),
(iii) there is a decreasing sequence \( (|\psi_n|)_n \) in RL such that \( \bigwedge_n [\psi_n] = 0 \) (pointwise) and \( d([\varphi_n], [\varphi]) \leq |\psi_n| \) for any \( n \).
(iv) there exists an increasing sequence \((r_n)_n\) in \([0,1]\) such that \(\bigvee_n r_n = 1\) and \(\vdash \eta_n \to (\varphi \leftrightarrow \varphi_n)\) for any \(n\).

Proof. (i)\(\Leftrightarrow\)(ii) Using the correspondence between functions and formulas, the condition \(\vdash \eta_n \to (\varphi \leftrightarrow \varphi_n)\) becomes \(|f_{\varphi_n} - f_\varphi| \leq f_\eta^*\), which means \(|f_{\varphi_n} - f_\varphi| \leq 1 - r\). The equivalence is obvious.

(i)\(\Leftrightarrow\)(iii) Recall that \(\bigwedge_n [\psi_n] = 0\) is the pointwise convergence to 0. In one direction the result follows from the fact that uniform convergence implies order convergence – see for example [23, Theorem 16.2] – and the functions associated to \(\eta_n\) are constant. In the other direction, we have that \(\bigwedge_n f_{\psi_n} = 0\) and \(|f_\varphi(x) - f_{\psi_n}(x)| = d(f_\varphi, f_{\psi_n})(x) \leq f_{\psi_n}(x)\) for any \(x\). Thus, the result follows by Lemma 2.7.

(iv) \(\Rightarrow\) (iii) Choosing \(s_n\) be sup\(\{f_{\psi_n}(x) \mid x \in \supp(f_{\psi_n})\}\). By Dini’s Theorem \([[\psi_n]]_n\) uniformly converges to 0, and since the uniform convergence implies the order convergence, \(\bigwedge s_n = 0\) and \(d(f_\varphi, f_{\psi_n})(x) \leq f_{\psi_n} - s_n\). If we take \(r_n = s_n\) then \(\bigvee r_n = 1\) and \(r_n \leq f_{(\varphi_n \leftrightarrow \varphi_\eta)}\), and the conclusion follows.  

\[\square\]

Remark 2.9. [Uniform convergence and order convergence] Condition (iii) from the above proposition is similar to the property of order-convergence\footnote{Cfr [23, Theorem 16.1] for the analogous definition in Riesz Spaces.} in the Lindenbaum-Tarski algebra \(RL\), but it is stronger. Indeed, when one deals with functions in Riesz Spaces, the infimum of the functions does not need to coincide with the pointwise infimum. Nonetheless, it is easily seen that if \(\inf_n g_n = g\), then \(g(x) \leq \bigwedge g_n(x)\) for any \(x\). Thus, \(\bigwedge g_n(x) = 0\) for any \(x\) forces \(g = 0\).

Similarly, \(\bigvee_n g_n(x) = 1\) for any \(x\) forces \(\sup_n g_n = 1\). In [23, Exercise 18.14(i)] one can find an example of a sequence of functions in \(C([0,1])\) for which the supremum function is identically 1, but the pointwise supremum is not. Finally, we remark that in general the order-convergence and the uniform convergence do not coincide. An example for \(C([0,1])\) can be obtained from [23, Example 16.18]. Whence, our condition is, so to speak, the best possible to have a logical definition of uniform convergence.

Definition 2.10. If \((\varphi_n)_n \subseteq \text{Form}_{RL}\) and \(\varphi \in \text{Form}_{RL}\), then \(\varphi\) is the limit of the sequence \((\varphi_n)_n\), in symbols \(\varphi = \lim_n \varphi_n\), if the equivalent conditions from Proposition 2.8 hold.

2.3 From Rational Łukasiewicz logic to \(RL\) via limits

We recall that a formula \(\varphi\) of \(RL\) is a formula of \(QL\) if contains only symbols \(\bigvee_q\) with \(q \in [0,1] \cap \mathbb{Q}\). In the following, we shall call such formulas rational.

A sequence \((\varphi_n)_n\) of formulas is increasing (resp. decreasing) if \(\vdash \varphi_n \to \varphi_{n+1}\) (resp. \(\vdash \varphi_{n-1} \to \varphi_n\)). We shall write \(g_n \uparrow g\) if \(g\) is the limit of \(g_n\)’s and the sequence is increasing and \(g_n \downarrow g\) if \(g\) is the limit of \(g_n\)’s and the sequence is decreasing, where the \(g_n\)’s are piecewise linear functions.

Lemma 2.11. For any function \(f\) in \(RMV_n\) there exist an increasing sequence of functions \(\{g_n\}_{n \in \mathbb{N}}\) and a decreasing sequence of functions \(\{h_n\}_{n \in \mathbb{N}}\), both in \(DMV_n\), such that \(g_n \uparrow f\) and \(h_n \downarrow f\).
Proof. Since \( R M V_n \) is isomorphic with the algebra of term functions of the logic \( \mathcal{RL} \), we shall prove the result by structural induction. All the technical properties used for uniform convergence can be found in [23].

If \( f = \pi_i \), one of the projection functions, it is enough to take \( g_n = h_n = \pi_i \) for any \( n \in \mathbb{N} \).

If \( f = l^* \), then there exists sequences \( \{t_n\}_{n \in \mathbb{N}} \) and \( \{s_n\}_{n \in \mathbb{N}} \) such that \( t_n \uparrow l \) and \( s_n \downarrow l \). Since \( l^* = 1 - l \), it is easily seen that \( 1 - t_n \downarrow 1 - l \) and \( 1 - s_n \downarrow 1 - l \) and the claim is settled. If \( f = f_1 \oplus f_2 \), that there exist \( \{t^1_n\}_{n \in \mathbb{N}} \) and \( \{s^1_n\}_{n \in \mathbb{N}} \) such that \( t^1_n \uparrow f_1 \) and \( s^1_n \downarrow f_1 \) and \( \{t^2_n\}_{n \in \mathbb{N}} \) and \( \{s^2_n\}_{n \in \mathbb{N}} \) such that \( t^2_n \uparrow f_2 \) and \( s^2_n \downarrow f_2 \). Thus, \( t^1_n \oplus t^2_n \uparrow f_1 \oplus f_2 \) and \( s^1_n \oplus s^2_n \downarrow f_1 \oplus f_2 \), which settles the claim.

Finally, if \( f = \Delta_r g \), there exist \( \{h_n\}_{n \in \mathbb{N}} \) and \( \{l_n\}_{n \in \mathbb{N}} \) such that \( h_n \uparrow g \) and \( l_n \downarrow g \). Moreover, it is possible to find an increasing sequence and a decreasing sequence of rational numbers \( \{t_n\}_{n \in \mathbb{N}} \) and \( \{s_n\}_{n \in \mathbb{N}} \) such that \( t_n \uparrow r \) and \( s_n \downarrow r \). Being \( R M V_n \) a semisimple algebra, \( \Delta_r, h_n \uparrow \Delta_r g \) and \( \Delta_r l_n \downarrow \Delta_r g \). By [21], functions of the type \( \Delta_r g \), with \( q \) rational and \( g \in DMV_m \), are term functions for \( \mathcal{QL} \) and the claim is settled.

\[ \square \]

**Theorem 2.12.** For any formula \( \varphi \) of \( \mathcal{RL} \) there exist an increasing sequence of rational formulas \( (\alpha_n) \) and a decreasing sequence rational formulas \( (\beta_n) \) such that \( \varphi = \lim_n \alpha_n = \lim_n \beta_n \).

**Proof.** It is a consequence of Theorem [48] and Lemma [41]. \[ \square \]

### 2.4 Completions of the Lindenbaum-Tarski algebra

In the sequel we show that the Lindenbaum-Tarski algebra \( RL_n \) becomes a normed space in a natural way and we analyse two different norm completions.

We note that this problem can be also studied for \( MV_n \) and \( DMV_n \), but in the context of Riesz MV-algebras it is intimately connected with the theory of Banach lattices, as shown in the final remarks of this section.

**Definition 2.13.** For any \( [\varphi] \) in the Lindenbaum-Tarski algebra \( RL_n \) we define: 

1. \( \|\varphi\| = \sup\{f_\varphi(x)|x \in [0,1]^n\} \).
2. \( I(\varphi) = \int f_\varphi(x)dx \),

where \( f_\varphi : [0,1]^n \to [0,1] \) is the piecewise linear function associated to \( \varphi \).

In any Riesz MV-algebra \( R \) it is possible to define the unit seminorm \( \|\cdot\| : R \to [0,1] \) by \( \|x\| = \inf\{r \in [0,1] | x \leq r1\} \) for any \( x \in R \) [15]. One can easily see that \( \|\varphi\|_u \) from (un) is the unit seminorm on \( RL_n \), which, in this case, is actually a norm.

We recall that a state defined on an MV-algebra \( A \) is a function \( s : A \to [0,1] \) such that \( s(1) = 1 \) and \( s(x \oplus y) = s(x) + s(y) \) for all \( x, y \in A \) with \( x \oplus y = 0 \). A state of \( A \) is said to be faithful if \( s(x) = 0 \) implies \( x = 0 \). States were defined in [29] and they are generalizations of finitely additive probability measures on boolean algebras. If \( R \) is a Riesz MV-algebra, \( s : R \to [0,1] \) is a state of \( R \) iff it is a state of its MV-algebra reduct. Moreover, \( (R,s) \) is a normed space. If we define \( \rho_s(x,y) = s((x \oplus y^*) \oplus (x^* \circ y)) \) for any \( x, y \in R \), then \( \rho_s \) is a pseudometric on \( R \). We say that \( (R,s) \) is state-complete if \( (R,\rho_s) \) is a complete metric space. From [29] Theorem 3.4, we infer that \( I \) is a faithful state on \( RL_n \) which attains rational values on rational formulas.

Consequently, \( (RL_n, \|\cdot\|_u) \) and \( (RL_n, I) \) are normed spaces.
Proposition 2.14. Assume \((\varphi_n)_n \subseteq \text{Form}_{\text{RL}}\) and \(\varphi \in \text{Form}_{\text{RL}}\). Then:

\(\text{(a)}\) \(\lim_n \varphi_n = \varphi \iff \lim_n \|d(\varphi_n, \varphi)\|_u = 0\),

\(\text{(b)}\) if \(\lim_n \varphi_n = \varphi\) then \(\lim_n I(\varphi_n) = I(\varphi)\)

\(\text{(c)}\) \(\vdash \neg \varphi \iff I(f_\varphi) = 0\).

Proof. (a) and (b) are straightforward from the fact that the domain of the corresponding term functions is \([0,1]^n\) and the convergence of \((f_\varphi_n)_n\) to \(f_\varphi\) is uniform.

(c) On one hand, we have that \(\vdash \neg \varphi\) implies \(e(\varphi) = 0\) for any evaluation, which entails \(f_\varphi = 0\) and \(I(f_\varphi) = 0\). On the other hand, being \(f_\varphi\) a non-negative and continuous function, \(\int f_\varphi(x)dx = 0\) implies \(f_\varphi = 0\). Since \(e(\varphi) = f_\varphi(e(v_1), \ldots, e(v_n))\), we get \(e(\varphi) = 0\) for any evaluation, which is equivalent to \(\vdash \neg \varphi\) by completeness.

In the following we characterize the Cauchy completions of the normed spaces \((RL_n, \| \cdot \|_u)\) and \((RL_n, I)\).

Theorem 2.15. The norm-completion of the normed space \((RL_n, \| \cdot \|_u)\) is isometrically isomorphic with \((C([0,1]^n), \| \cdot \|_\infty)\).

Proof. The maximal ideal space \(\text{Max}(RL_n)\) of \(RL_n\) coincide with the maximal ideal space of its MV-algebra reduct therefore, by \([31\text{ Theorem 4.16(iv)}]\), it is homeomorphic with \([0,1]^n\). We now use the fact that \(RL_n\) is dense in \(C([0,1]^n)\), as proven in \([25\text{ Lemma 7.4)}\].

Corollary 2.16. For any continuous function \(f : [0,1]^m \rightarrow [0,1]\) there exists a sequence of formulas \(\{\varphi_n\}_{n \in \mathbb{N}} \subseteq \text{Form}_{\text{RL}}\) such that \(f = \lim_n f_{\varphi_n}\).

Remark 2.17. If \(MV_n\) is the Lindenbaum-Tarski algebra of Łukasiewicz logic, then \((MV_n, I)\) is a normed space. Let \(\mu\) be the Lebesgue measure associated to \(I\), assume \(L^1(\mu)_u\) is the algebra of \([0,1]\)-valued integrable functions on \([0,1]^n\) and define \(s_\mu(f) = I(f)\) for any \(f \in L^1(\mu)_u\) - note that \(f\) is the class of \(f\), provided we identify two functions that are equal \(\mu\)-almost everywhere. Then \((L^1(\mu)_u, s_\mu)\) is the Cauchy completion of \((MV_n, I)\) by \([31\text{ Theorem 16.7)}\], where the author mention that \(L^1(\mu)_u\) is not just an MV-algebra, but it is equipped with a far richer structure than \((MV_n, I)\). Moreover, being \(\mu\) finite and being each function of \(L^1(\mu)_u\) bounded and measurable, \(L^1(\mu)_u\) coincide with the unit interval of \(L^\infty(\mu)\), but the latter is endowed with a different norm.

Theorem 2.18. The normed space \((L^1(\mu)_u, s_\mu)\) from Remark 2.17 is a state-complete Riesz MV-algebra. Moreover, \((L^1(\mu)_u, s_\mu)\) is, up to isomorphism, the norm-completion of the normed space \((RL_n, I)\).

Proof. By \([23\text{ Theorem 4 and Proposition 1)}\], \((L^1(\mu)_u, s_\mu)\) is a norm-complete Riesz MV-algebra. We can safely identify \(MV_n\) with an MV-subalgebra of \(RL_n\) and we note that \((MV_n, I)\) is a normed subspace of \((RL_n, I)\). By \([31\text{ Theorem 16.7)}\], the algebra \((L^1(\mu)_u, s_\mu)\) defined above is the Cauchy completion of \((MV_n, I)\). Since \(f_\varphi\) is an integrable function for any formula \(\varphi\) of \(RL\), we infer that the Cauchy completion of \((MV_n, I)\) coincide with the Cauchy completion of \((RL_n, I)\).

Since Riesz spaces play a central rôle in functional analysis, the theory of Riesz MV-algebras has been connected with \(C^*\)-algebras, \(M\)-spaces and \(L\)-spaces. In particular, the analogues of Kakutani’s representation theorems are

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known for Riesz MV-algebras. The interested reader is referred to [16, 22] for further details. Following these connections and using the functor $\Gamma_\mathcal{R}$, the norm-completion of $(\mathbb{R}L_n, \| \cdot \|_u)$ is, up to isomorphism, the unit interval of an M-space. Similarly, the norm-completion of $(\mathbb{R}L_n, I)$ is, up to isomorphism, the unit interval of an L-space.

3 Dualities and adjunctions for finitely presented algebras

We recall that an ideal for an MV-algebra is a downward closed set, which is also closed with respect to the sum $\oplus$. Ideals in a Riesz MV-algebra (or a DMV-algebra) coincide with ideals of its MV-algebra reduct. A principal ideal is an ideal generated by one element of the algebra. If $A$ is an MV-algebra and $a \in A$ then the principal ideal generated by $a$ is

$$(a) = \{ x \in A \mid x \leq na \text{ for some } n \in \mathbb{N} \}.$$

An MV-algebra (a Riesz MV-algebra, a DMV-algebra) is finitely generated if it is generated by a finite set of elements, and it is finitely presented if it is the quotient of a free finitely generated MV-algebra (Riesz MV-algebra, DMV-algebra) by a principal ideal. Note that free algebras and finitely presented algebras are semisimple, see [9] for the case of MV-algebras, [21] for the case of DMV-algebras and [13] for the case of Riesz MV-algebras.

Finitely presented structures are intimately connected with logic since, up to isomorphism, they are Lindenbaum-Tarski algebras corresponding to a finitely axiomatizable theories with finitely many variables [31, Lemma 3.19 and Theorem 6.3].

In this section we focus on the algebraic approach, which provides more tools for our investigations towards finding equivalences, dualities or adjunction between subclasses of finitely presented MV-algebras, Riesz MV-algebras and DMV-algebras. We present three different approaches and, in the last section of the chapter, we interpret our results within the logical systems studied in the previous section.

3.1 The tensor product and the finitely presented structures

In the sequel we denote by $\text{MV}_{ss}$, $\text{DMV}_{ss}$ and $\text{RMV}_{ss}$ the full subcategories of semisimple algebras. Our main tool is the tensor product of semisimple MV-algebras, defined by D. Mundici in [30], which allows us to complete the above diagram with adjoint functors. Let us now present few significant results.

Let $\otimes$ be the tensor product of semisimple MV-algebras. In [20] is proved that $(\mathcal{U}_\otimes, T_\otimes)$ is an adjoint pair, where $\mathcal{U}_\otimes : \text{RMV}_{ss} \rightarrow \text{MV}_{ss}$ is the forgetful functor and $T_\otimes : \text{MV}_{ss} \rightarrow \text{RMV}_{ss}$ is defined on objects by $T_\otimes(A) = [0, 1] \otimes A$ and on morphisms through a suitable universal property: if $\iota_A : A \rightarrow T_\otimes(A)$ is the standard embedding, for any $A \in \text{MV}_{ss}$, then for any $f : A \rightarrow B$, $T_\otimes(f)$ is defined as the unique map that satisfies $\iota_B \circ f = T_\otimes(f) \circ \iota_A$. An analogous result is proved in [21] for MV-algebras but, in this case, the left adjoint functor $D_\otimes : \text{MV}_{ss} \rightarrow \text{DMV}_{ss}$ is defined by $D_\otimes(A) = ([0, 1] \cap \mathbb{Q}) \otimes A$.

With the above notations, define $T_\otimes^0 : \text{DMV}_{ss} \rightarrow \text{RMV}_{ss}$ by
Theorem 3.3. Let $A$ be a finitely presented MV-algebra, and let $\mathcal{M}_0$ of $\mathcal{M}$. We recall that by [20, Proposition 5.1], $\mathcal{M}(\mathcal{U}) \simeq \mathcal{M}_0$. Moreover, it is well defined: for $f \in \mathcal{M}$ and trivially we prove that $\alpha \circ h \in \mathcal{M}_0$ if and only if $h \in \mathcal{M}_0$. Since $P \otimes \mathcal{M}_0$ is either a Riesz MV-algebra or a DMV-algebra, by [12] Lemma 3.19 and Proposition 3.10 we get $d(\alpha(h), \alpha(f)) = \alpha(d(h, f))$. By definition $h \in [f]$ implies $d(h, f) \in \mathcal{I}(J)$, therefore $\alpha(d(h, f)) = \alpha(\mathcal{I}(J))$. We have $\alpha \circ h \in \mathcal{M}_0$ if and only if $h \in \mathcal{M}_0$. By universal property of the semisimple tensor product, there exists a homomorphism of MV-algebras $\lambda$ $\lambda : P \otimes \mathcal{M}_0 \to (P \otimes \mathcal{M}_0) / \mathcal{I}(J)$ such that $\lambda(\alpha \circ f) = \alpha(f)$. By [16] Corollary 2 (or [21] Lemma 3.1) in the case of DMV-algebras, $\lambda$ is a homomorphism of Riesz MV-algebras (or DMV-algebras). On the other side, consider the bimorphism.

Remark 3.1. The functors $\mathcal{T}_\otimes \circ \mathcal{D}_\otimes$ and $\mathcal{T}_\otimes$ are naturally isomorphic, since they are adjoints to the same forgetful functor.

In the following we explore the behaviour of the tensor functors $\mathcal{D}_\otimes$ and $\mathcal{T}_\otimes$ with respect to finitely presented algebras.

We start with the simple result that an homomorphism of MV-algebras preserves principal ideals.

Lemma 3.2. Let $I = (a)$ be a principal ideal in an MV-algebra $A$. For any homomorphism of MV-algebras $f : A \to B$, $(f(I)] = (f(a)]$.

Proof. It is a routine application of the definitions of MV-homomorphisms and principal ideals. $\square$

Let $P$ be either $[0, 1]$ or $[0, 1] \cap \mathbb{Q}$ and $\iota_A : A \to P \otimes A$ be the embedding in the semisimple tensor product. We recall that by [20, Proposition 5.1], $[0, 1] \otimes \mathcal{M}_0 \simeq \mathcal{M}_0$ and by [21, Theorem 3.3], $[0, 1] \otimes \mathcal{M}_0 \simeq \mathcal{M}_0$.

Theorem 3.3. Let $A$ be a finitely presented MV-algebra, and let $J$ be an ideal of $\mathcal{M}_0$ such that $A \simeq \mathcal{M}_0 / J$. Then

$$P \otimes (\mathcal{M}_0 / J) \simeq (P \otimes \mathcal{M}_0) / \mathcal{I}(J).$$

Proof. In the sequel, we will denote the embedding $\iota_A : A \to P \otimes A$ with $\iota$. We define the map $\beta$

$\beta : P \otimes \mathcal{M}_0 / J \to (P \otimes \mathcal{M}_0) / \mathcal{I}(J)$ by

$\beta([\alpha \otimes h]) = [\alpha \otimes f]_{\iota(J)}$. By definition of $\otimes$, $\beta$ is trivially a bimorphism. Moreover, it is well defined: for $h \in [f]_J$ we prove that $[\alpha \otimes h]_{\iota(J)} = [\alpha \otimes f]_{\iota(J)}$. We have 

$\alpha \circ h \in (\mathcal{M}_0 / J)$ if and only if $d(\alpha \circ h, \alpha \circ f) \in \mathcal{I}(J)$ if and only if $\alpha(d(h, f)) \in \mathcal{I}(J)$. Since $P \otimes \mathcal{M}_0$ is either a Riesz MV-algebra or a DMV-algebra, by [12] Lemma 3.19 and Proposition 3.10 we get 

$d(\alpha(h), \alpha(f)) = d(h, f) \in (\mathcal{I}(J))$. By definition $h \in [f]$ implies $d(h, f) \in \mathcal{I}(J)$, therefore $\alpha(d(h, f)) \in \mathcal{I}(J)$. We have $\alpha \circ h \in \mathcal{M}_0$ if and only if $h \in \mathcal{M}_0$. By universal property of the semisimple tensor product, there exists a homomorphism of MV-algebras $\lambda$ $\lambda : P \otimes \mathcal{M}_0 / J \to (P \otimes \mathcal{M}_0) / \mathcal{I}(J)$ such that $\lambda(\alpha \circ f) = \alpha(f)$. By [16] Corollary 2 (or [21] Lemma 3.1) in the case of DMV-algebras, $\lambda$ is a homomorphism of Riesz MV-algebras (or DMV-algebras). On the other side, consider the bimorphism.
\[ \beta_2 : P \times MV_n \to P \otimes MV_n / J \] defined by \( \beta_2(\alpha, f) = \alpha \otimes [f]_J \).

By universal property there exists a map \( \Omega : P \otimes MV_n \to P \otimes MV_n / J \) such that \( \Omega(\alpha \otimes f) = \alpha \otimes [f]_J \).

By Lemma 3.2, \( (\iota(J)) = (\iota(g)) = (1 \otimes g) \), therefore if \( h \in (1 \otimes g) \) there exists \( n \in \mathbb{N} \) such that \( h \leq n(1 \otimes g) \). Then \( \Omega(h) \leq n\Omega(1 \otimes g) = 1 \otimes [g]_J \); since \( g \in J \) the latter equals to 0, therefore \( h \in \text{Ker}(\Omega) \) and \( (\iota(J)) \subseteq \text{Ker}(\Omega) \). By the general homomorphism theorems – which hold for MV-algebras – there exists a map \( \gamma : P \otimes MV_n / (\iota(J)) \to P \otimes MV_n / J \) such that \( \gamma((\alpha \otimes f) \iota(J)) = \Omega(\alpha \otimes f) = \alpha \otimes [f]_J \).

On generators we have
\[ (\gamma \circ \lambda)(\alpha \otimes [f]_J) = \lambda((\alpha \otimes f) \iota(J)) = \alpha \otimes [f]_J = \mathbf{I}(\alpha \otimes [f]_J), \]
and by universal property \( \gamma \circ \lambda = \mathbf{I} \). Denoted by \( \pi \) the canonical epimorphism \( \pi : P \otimes MV_n \to P \otimes MV_n / (\iota(J)) \), we get
\[ (\lambda \circ \gamma \circ \pi)(\alpha \otimes f) = \lambda(\alpha \otimes [f]_J) = [\alpha \otimes f]_{\iota(J)} = (\mathbf{I} \circ \pi)(\alpha \otimes f) \]
and again by universal property, \( \lambda \circ \gamma \circ \pi = \mathbf{I} \circ \pi \). Being \( \pi \) is surjective, it follows that \( \lambda \circ \gamma = \mathbf{I} \), and the claim is settled.

If \( A \) is a finitely presented MV-algebra, Theorem 3.3 ensures that \( \mathcal{T}_\otimes(A) \) is a finitely presented Riesz MV-algebra and \( \mathcal{D}_\otimes(A) \) is a finitely presented DMV-algebra. We can therefore restrict the tensor functors to the full subcategories whose objects are finitely presented structures.

**Remark 3.4.** Let \( I \) be a principal ideal in \( DMV_n \), generated by \( a \in DMV_n \setminus MV_n \). By [11, Remark 2.4] and [21, Theorem 33] there exists a natural number \( k \) and \( a_1, \ldots, a_k \in MV_n \) such that \( a = \frac{1}{k}a_1 \oplus \cdots \oplus \frac{1}{k}a_k \). Let \( b = a_1 \oplus \cdots \oplus a_k \). Since \( \frac{1}{k}a_i \leq a_i \), we have \( a \leq b \) therefore the ideal generated by \( a \) is included in the ideal generated by \( b \), in symbols \( (a) \subseteq (b) \). For the other inclusion, \( \delta_k(a_i) = \frac{1}{k}a_i \leq a \) for any \( i \), therefore \( a_i = k\delta_k(a_i) \in (a) \). Hence \( b \in (a) \), and \( (a) = (b) \).

This means that we can replace the generator of a principal ideal in \( DMV_n \) by an element of \( MV_n \).

We denote \( \mathcal{D}_{\otimes}^{fp} \) the restriction and co-restriction of \( \mathcal{D}_\otimes \) to full subcategories of finitely presented structures.

**Theorem 3.5.** The functor \( \mathcal{D}_{\otimes}^{fp} : MV_{fp} \to DMV_{fp} \) is faithful and essentially surjective.

**Proof.** Let \( D \in DMV_{fp} \). There exists \( n \) such that \( A \simeq DMV_n / I \) where \( I \subseteq DMV_n \) is a principal ideal. By Remark 3.4 there exists \( b \in MV_n \) such that \( I = (b) \). By Theorem 3.3 \( DMV_n / (b)_{DMV} \simeq [0,1]_Q \otimes MV_n / (b)_{MV} \), hence \( A \simeq \mathcal{D}_{\otimes}^{fp}(MV_n / (b)_{MV}) \) and \( \mathcal{D}_{\otimes}^{fp} \) is essentially surjective. Let \( \sigma, \tau : A \to B \) arrows in \( MV_{fp} \) such that \( \mathcal{D}_{\otimes}^{fp}(\sigma) = \mathcal{D}_{\otimes}^{fp}(\tau) \). In particular \( \mathcal{D}_{\otimes}^{fp}(\sigma)(\iota_A(a)) = \mathcal{D}_{\otimes}^{fp}(\tau)(\iota_A(a)) \), and by definition of \( \mathcal{D}_{\otimes}^{fp} \) on arrows, \( \iota_B(\sigma(a)) = \iota_B(\tau(a)) \), for any \( a \in A \). Being \( \iota_B \) injective, we infer \( \sigma = \tau \). □

To prove a similar result for finitely presented Riesz MV-algebras we need extra conditions of objects. Indeed, the next proposition shows that we cannot prove the analogue of Remark 3.4 in the case of Riesz MV-algebras. We refer
to Section 3.3 for the notion of a polyhedron and for suggestions for further readings.

**Proposition 3.6.** There exists at least a principal ideal in $RMV_n$ that is not generated by an element of $MV_n$.

**Proof.** Let $P \subseteq [0, 1]^n$ be a non-rational polyhedron. By [13] Theorem 3.3, there exists $f \in RMV_n$ such that $P = f^{-1}(0)$. Consider now $I = (f)$, and assume that we can find $g \in I$ such that $g \in MV_n$ and $I = (g)$. Trivially, since $f \leq kg$ and $g \leq mf$ for appropriate $m, k \in \mathbb{N}$, $f(x) = 0$ iff $g(x) = 0$, for any $x \in [0, 1]^n$. This means $P = g^{-1}(0)$, which contradicts [24] Proposition 2.1, that is the fact that rational polyhedra are exactly zerosets of piecewise linear functions with integer coefficients (see also [25] Lemma 2.5).

A concrete example is the following.

**Example 3.7.** Consider the function $f \in RMV_1$ defined by $f(x) = -\sqrt{2}x + 1$ for $x \leq \frac{\sqrt{2}}{2}$ and $f(x) = 0$ otherwise. Thus, $f^{-1}(0) = [\frac{\sqrt{2}}{2}, 1]$. Since $[\frac{\sqrt{2}}{2}, 1]$ has an irrational vertex, we can infer the desired conclusion as in Proposition 3.6.

At this stage, we can only prove the following.

**Lemma 3.8.** The functor $\mathcal{T}^{fp} : MV_{fp} \to RMV_{fp}$ is faithful.

**Proof.** It follows by the general properties of the tensor functor, as in the proof of Theorem 3.5.

The previous result suggests that $MV_{fp}$ is equivalent with a subcategory of $DMV_{fp}$ that has the same objects as $DMV_{fp}$, but fewer morphisms. A similar remark holds if we consider a subcategory of $RMV_{fp}$ instead of $DMV_{fp}$. Thus, we now define concrete subcategories of $DMV_{fp}$ and $RMV_{fp}$ that are equivalent to $MV_{fp}$.

**Lemma 3.9.** Let $A, B$ be MV-algebras and $i_A : A \to \mathcal{D}_\otimes(A)$, $i_B : B \to \mathcal{D}_\otimes(B)$ the canonical embeddings of the tensor product. For a morphism $f_\otimes : \mathcal{D}_\otimes(A) \to \mathcal{D}_\otimes(B)$ the following are equivalent:

(a) $f_\otimes(i_A(A)) \subseteq i_B(B)$,

(b) there exists a unique morphism $f : A \to B$ such that $f_\otimes \circ i_A = i_B \circ f$.

**Proof.** For the non-trivial implication, we set $f(a) = b$ if $a \in A$ and $b \in B$ such that $f_\otimes(i_A(a)) = i_B(b)$. The uniqueness follows from the universal property of the tensor product.

Let $DMV_{fp}'$ be the subcategory of $DMV_{fp}$ whose objects are algebras $D = \mathcal{D}_\otimes(A)$ where $A$ is a finitely presented MV-algebra and whose morphisms are $g : \mathcal{D}_\otimes(A) \to \mathcal{D}_\otimes(B)$ such that $g(i_A(A)) \subseteq i_B(B)$. Similarly, let $RMV_{fp}'$ be the subcategory of $RMV_{fp}$ whose objects are algebras $R = \mathcal{T}_\otimes(A)$ where $A$ is a finitely presented MV-algebra and whose morphisms are $g : \mathcal{T}_\otimes(A) \to \mathcal{T}_\otimes(B)$ such that $g(i_A(A)) \subseteq i_B(B)$.

**Proposition 3.10.** The categories $MV_{fp}$, $DMV_{fp}'$, and $RMV_{fp}'$ are equivalent.
Proof. One can easily see that the co-restrictions of $\mathcal{D}_\otimes$ and $\mathcal{T}_\otimes$ are full and essentially surjective. The faithfulness follows by Lemma 3.9 which is easily adapted to the case of Riesz MV-algebras.

Note that, as a consequence of Proposition 3.6, the functor $\mathcal{T}_{\otimes}^{fp} : \mathbf{MV}_{fp} \to \mathbf{RMV}_{fp}$ is not essentially surjective, while $\mathbf{RMV}'_{fp}$ has exactly those objects that make the functor essentially surjective.

Let $\mathbf{DMV}_{fp}$ be the full subcategory of $\mathbf{DMV}$ whose objects are algebras $D = D\otimes(A)$ with $A$ being a finitely presented MV-algebra. Hence $\mathbf{DMV}_{fp}$ is equivalent to $\mathbf{DMV}_{fp}$ and $\mathbf{DMV}'_{fp}$ is a subcategory equivalent to $\mathbf{MV}_{fp}$.

Note that $\mathbf{DMV}'_{fp}$ and $\mathbf{DMV}_{c\,fp}$ have the same objects, but $\mathbf{DMV}'_{fp}$ has fewer morphisms. We say that the category $\mathbf{DMV}_{c\,fp}$ is concrete since the objects are defined fixing a particular representation. An equivalent concrete category can be defined, by considering the full subcategory of $\mathbf{DMV}$ whose objects have the form $\mathbf{DMV}/I$, where $I$ is a principal ideal [6]. Another concrete category is defined in Section 3.3 using polyhedra.

Note that, in our setting, $A$ and $\iota_A(A)$ are isomorphic MV-algebras. Since $\iota_A(A)$ is an MV-subalgebra of an enriched structure - DMV-algebra or Riesz MV-algebra - we say that the elements of $\iota_A(A)$ are MV-elements. In order to get a category that is equivalent to $\mathbf{MV}_{fp}$ we have to select fewer morphisms, exactly those morphisms that preserves the MV-elements.

### 3.2 Categories of presentations

We proved that the category $\mathbf{MV}_{fp}$ of finitely presented algebras is equivalent to subcategories of DMV-algebras and Riesz MV-algebras whose objects are defined choosing a concrete representation. Fixing a representation might not be desirable when analysing abstract properties therefore, in this section, we present a different construction inspired by the categorical approach to presentations and theories in the general framework of institutions [10]. We analyse finitely presented structures from the point of view of their presentations. In other words, we define categories whose objects are theories determined by finite presentations.

We denote by $\mathbf{MV}_X$, $\mathbf{DMV}_X$ and $\mathbf{RMV}_X$ the Lindenbaum-Tarski algebras over the set $X$ of variables, corresponding to the systems $\mathbf{L}$, $\mathbf{QL}$ and $\mathbf{RL}$ from Section 2. Let $\iota_X$ be the unique morphism of MV-algebras $\iota_X : \mathbf{MV}_X \to \mathbf{DMV}_X$ defined by $\iota_X(x) = x$ for any $x \in X$. $\iota_X$ is an trivially an embedding. In the following, we will often identify $S \subseteq \mathbf{MV}_X$ with $\iota_X(S)$. In particular, $\mathbf{MV}_X$ will be identified with an MV-subalgebra of $\mathbf{DMV}_X$. Similarly, we shall identify $\mathbf{MV}_X$ with an MV-subalgebra of $\mathbf{RMV}_X$. In this approach, the elements of the isomorphic copy of $\mathbf{MV}_X$ are the MV-elements of $\mathbf{DMV}_X$ or $\mathbf{RMV}_X$.

The theories of $\mathbf{L}$ are in bijective correspondence with the filters of $\mathbf{MV}_X$ [15] Proposition 6.3.15] and, consequently, with the ideals of $\mathbf{MV}_X$. Thus, we represent a theory by a pair $(X,I)$, where $I$ is an ideal of $\mathbf{MV}_X$. In the following we consider only finitely generated theories, but the development can be generalized to arbitrary ones.

**Definition 3.11.** We define the category $\mathbf{ThMV}$ in the following way: the objects are pairs $(X,I)$ with $X$ a set and $I$ an finitely generated ideal of $\mathbf{MV}_X$.
the arrows are maps \( \sigma : (X, I) \rightarrow (Y, J) \) such that \( \sigma : M V_X \rightarrow M V_Y \) and \( \sigma(I) \subseteq J \).

**Remark 3.12.** (1) In the general theory of institutions, morphisms of theories are usually chosen to be maps from variables to variables, expressing the fact that the truth is invariant under renaming. We use substitutions as morphisms, instead of using renaming maps.

(2) If \((X, I)\) and \((Y, J)\) from \( \text{ThMV} \) are isomorphic then \( M V_X / I \) and \( M V_Y / J \) are isomorphic MV-algebras. Moreover, if \( M V_X / I \) and \( M V_Y / J \) are isomorphic MV-algebras then there are morphisms \( \sigma : (X, I) \rightarrow (Y, J) \) and \( \tau : (Y, J) \rightarrow (X, I) \) in \( \text{ThMV} \) such that \( \sigma(\tau(y))/J = y/J \) and \( \tau(\sigma(x))/I = x/I \) for any \( x \in X \) and \( y \in Y \).

(3) In \( \text{Th} \), a different category of presentations is defined, by considering morphisms between the quotients \( M V_X / I \) and \( M V_Y / J \). This category is equivalent with \( M V_{fp} \) and they are obviously equivalent with a quotient category of \( \text{ThMV} \).

We define \( \text{ThDMV} \) in a similar way and we denote \( \text{ThDMV}' \) the subcategory of \( \text{ThDMV} \) that has the same objects but fewer morphisms: an arrow \( \sigma : (X, I) \rightarrow (Y, J) \) from \( \text{ThDMV} \) is an arrow in \( \text{ThDMV}' \) if \( \sigma(X) \subseteq M V_Y \).

Note that we are asking for arrows that preserve MV-elements.

The following definition is essential for our development.

**Definition 3.13.** We say that an ideal \( J \) of \( DM V_X \) is MV-generated if \( J = (J \cap M V_X)_{DM V} \). We remark that the inclusion \( (J \cap M V_X)_{DM V} \subseteq J \) is always satisfied. The MV-generated ideals of \( R M V_X \) are similarly defined. Thus, an ideal is an MV-ideal if it is generated by its MV-elements and it is easily seen that a principal ideal of either \( DM V_X \) or \( RM V_X \) is MV-generated if and only if we can replace its generator with an element of \( M V_X \).

**Lemma 3.14.** Let \( I \) be a principal ideal of \( DM V_X \). Then \( I \) is MV-generated and \( \iota_X^{-1}(I \cap M V_X) \) is a principal ideal in \( M V_X \).

**Proof.** Let \( I = [g]_{DM V} \). By Remark 3.4 there exists \( g^* \in M V_X \) such that \( I = [g^*]_{DM V} \). Hence \( g^* \in I \cap M V_X \), \( I \subseteq (I \cap M V_X)_{DM V} \) and \( I \) is MV-generated.

Since the proof that \( \iota_X^{-1}(I \cap M V_X) \) is an ideal of \( M V_X \) is routine, let us prove that it is a principal ideal. By Remark 3.3 it is enough to consider the case \( g \in I \cap M V_X \). We have \( \iota_X^{-1}(g) \in \iota_X^{-1}(I \cap M V_X) \), therefore \( \iota_X^{-1}(g)_{M V} \subseteq \iota_X^{-1}(I \cap M V_X) \). On the other side, for any \( a \in \iota_X^{-1}(I \cap M V_X) \), we have \( \iota_X(a) \in I \cap M V_X \subseteq I \), therefore there exists \( n \in \mathbb{N} \) such that \( \iota_X(a) \leq ng \). This entails that \( a \leq n \iota_X^{-1}(g) \) and \( \iota_X^{-1}(I \cap M V_X) \subseteq (\iota_X^{-1}(g))_{M V} \) and the claim is settled.

**Theorem 3.15.** We define a functor \( \mathcal{D} \mathcal{T} : \text{ThMV} \rightarrow \text{ThDMV}' \) as follows:

- **Objects:** \( \mathcal{D} \mathcal{T}(X, I) = (X, [I]_{DM V}) \), where \( [I]_{DM V} \) is the DMV-ideal generated by \( I \) in \( DM V_X \);

- **Arrows:** if \( \sigma : (X, I) \rightarrow (Y, J) \) is an arrow in \( \text{ThMV} \), \( \mathcal{D} \mathcal{T}(\sigma) \) is the unique morphism \( \sigma_D : (X, [I]_{DM V}) \rightarrow (Y, [J]_{DM V}) \) such that \( \sigma_D : DM V_X \rightarrow DM V_Y \) and \( \sigma_D(x) = \sigma(x) \) for any \( x \in X \).

The functor \( \mathcal{D} \mathcal{T} \) establishes a categorical equivalence between \( \text{ThMV} \) and \( \text{ThDMV}' \).
Proof. We will prove that $DT$ is full, faithful and essentially surjective.

$DT$ is faithful: for any $\tau, \sigma : (X, I) \to (Y, J)$, let $\tau_D = \sigma_D$. Then we have $i_Y(\tau(x)) = i_Y(\tau_D(x)) = i_Y(\sigma_D(x)) = i_Y(\sigma(x))$. Being $i_Y$ injective, we get $\tau(x) = \sigma(x)$ for any $x \in X$. Since $X$ is the generating set, $\tau = \sigma$.

$DT$ is full: let $\tau : (X, i_X(I)) \to (Y, i_Y(J))$ a morphism in ThDMV. Then $\tau : DMV_X \to DMV_Y$, $\tau(X) \subseteq MV_Y$ and $\tau(i_X(I)) \subseteq (i_Y(J))_D$. We define $\sigma : (X, I) \to (Y, J)$ as the unique map $\sigma : MV_X \to MV_Y$ such that $\sigma(x) = \tau(x) \in MV_Y$ for any $x \in X$. More precisely, $i_Y(\sigma(x)) = \tau(i_X(x))$, and since $X$ is the generating set, $\tau \circ \sigma = \tau \circ i_X$.

$$
\begin{array}{ccc}
MV_X & \xrightarrow{\sigma} & MV_Y \\
i_X & \downarrow & \downarrow i_Y \\
DMV_X & \xrightarrow{\tau} & DMV_Y
\end{array}
$$

We have to prove that $\sigma(I) \subseteq J$. By hypothesis we get $i_Y(\sigma(I)) = \tau(i_X(I)) \subseteq (i_Y(J))_D$. Let $a \in I$, then $\sigma(a) \in \sigma(I)$ and $i_Y(\sigma(a)) \in (i_Y(J))_D$. Whence there exist $a_1, \ldots, a_m \in \mathbb{N}$ and $b_1, \ldots, b_m \in J$ such that $i_Y(\sigma(a)) \leq \delta_{a_1}(i_Y(b_1)) + \ldots + \delta_{a_m}(i_Y(b_m)) \leq i_Y(b_1) + \ldots + i_Y(b_m)$, since $\delta_k(x) \leq x$ for any $k \in \mathbb{N}$. Then $i_Y(\sigma(a)) \leq i_Y(b_1 + \ldots + b_m)$ implies $\sigma(a) \leq b_1 + \ldots + b_m \in J$ so $\sigma(I) \subseteq J$ and $\tau = DT(\sigma)$.

$DT$ is essentially surjective: we have to prove that for any $(Y, J)$ in ThDMV there exists $(X, I)$ in ThMV such that $DT(X, I) = (Y, J)$. If we choose $I = i_X^{-1}(J \cap MV_X)$, the conclusion follows by Lemma 3.14.

The same approach works for theories in RL, but we have to restrict our attention to MV-generated ideals. Indeed, by Proposition 3.16 there are ideals in $RMV_X$ that are not MV-generated. Let ThRMV be the category whose objects are couples $(X, I)$, where $I \subseteq RMV_X$ is an ideal generated by $f \in DMV_X$ and whose morphisms $\tau : (X, I) \to (Y, J)$ are maps $\tau : RMV_X \to RMV_Y$ such that $\tau(I) \subseteq J$ and $\tau(X) \subseteq MV_Y$.

Proposition 3.16. The categories ThMV, ThDMV, and ThRMV are equivalent.

Proof. It is similar with the proof of Theorem 3.15.

Finally, we give the last point of view on the relations between finitely presented objects.

3.3 Polyhedra and finitely presented algebras

In order to give out last characterization we need some preliminary notions of polyhedral geometry. An $m$-simplex in $\mathbb{R}^n$ is the convex hull $C$ of $m + 1$ affinely independent points $\{v_0, \ldots, v_m\}$ in the euclidean space $\mathbb{R}^n$; the points $v_i$ are called vertexes and $C$ is rational if any vertex has rational coordinates. A (rational) polyhedron is the union of finitely many (rational) simplexes.
If $P \subseteq [0,1]^n$ and $Q \subseteq [0,1]^m$ are polyhedra, then a map $z : P \rightarrow Q$ is a $\mathbb{Z}$-map if $z = (z_1, \ldots, z_m)$ where all $z_i : [0,1]^n \rightarrow [0,1]$ are piecewise linear functions with integer coefficients. Note that $\mathbb{Q}$-maps and $\mathbb{R}$-maps are similarly defined, each corresponding to the given ring of coefficients.

Let us denote by $\text{Pol}^Z_{[0,1]}$ the category of polyhedra with $\mathbb{R}$-maps as morphisms, by $\text{RatPol}^Z_{[0,1]}$ the category of rational polyhedra with $\mathbb{Q}$-maps as morphisms and by $\text{RatPol}^Z_{[0,1]}$ the category of rational polyhedra with $\mathbb{Z}$-maps as morphisms, where all polyhedra are lying in a unit cube. Following the ideas of Baker and Beynon \[1, 4, 3\], in \[26, 9\] the authors proves that the categories $\text{RatPol}^Z_{[0,1]}\text{Pol}^Z_{[0,1]}$ and $\text{MV}_{\text{fp}}$ are dual. This result is generalized to DMV-algebras in \[21\], meaning that the categories $\text{RatPol}^Q_{[0,1]}$ and $\text{DMV}_{\text{fp}}$ are dual, while in \[13\] it is proved that $\text{Pol}^Z_{[0,1]}$ and $\text{RMV}_{\text{fp}}$ are dual.

We shall now further analyze these connections. Firstly, let us recall the definitions of the functors that define these dualities.

Let $A$ be a finitely presented Riesz MV-algebra and assume $A \simeq RMV_n / I$, where $I$ is a principal ideal generated by $f \in RMV_n$. If $P = f^{-1}(0)$, then $P$ is a polyhedron in $\text{Pol}^Z_{[0,1]}$ and $A \simeq RMV_n \mid_P$, where $RMV_n \mid_P = \{ g \mid_P \mid g \in RMV_n \}$. Thus, in the case of polyhedra with $\mathbb{R}$-maps and finitely presented Riesz MV-algebras, the functor $\mathcal{R} : \text{Pol}^Z_{[0,1]} \rightarrow \text{RMV}_{\text{fp}}$ is defined as follows:

\[
\mathcal{R}(P) = RMV_n \mid_P, \text{ for any polyhedron } P \subseteq [0,1]^n;
\]

\[
\mathcal{R}(\lambda) : \mathcal{R}(Q) \rightarrow \mathcal{R}(P) \text{ defined by } \mathcal{R}(\lambda)(f) = f \circ \lambda, \text{ for any } \lambda : P \rightarrow Q, \text{ with } P \subseteq [0,1]^n \text{ and } Q \subseteq [0,1]^m.
\]

The functors $\mathcal{M} : \text{RatPol}^Z_{[0,1]} \rightarrow \text{MV}_{\text{fp}}$ and $\mathcal{D} : \text{RatPol}^Z_{[0,1]} \rightarrow \text{DMV}_{\text{fp}}$ are defined similarly.

This is the second concrete category of finitely presented structures. The first concrete representation for was given at the end of Section 3.1.

**Lemma 3.17.** Let $\sigma : DMV_m \mid_P \rightarrow DMV_m \mid_Q$ be a map between finitely presented DMV-algebras, where $P, Q$ are rational polyhedra and assume $\lambda = (\lambda_1, \ldots, \lambda_n) : Q \rightarrow P$ such that $\mathcal{D}(\lambda) = \sigma$. Then $\sigma(\{\pi_1, \ldots, \pi_n\}) \subseteq MV_m$ iff $\lambda_i \in MV_m$.

**Proof.** The result follows from the fact that $\sigma = \mathcal{D}(\lambda)$ implies $\sigma(\pi_i) = \mathcal{D}(\lambda)(\pi_i) = \pi_i \circ \lambda$. That is, $\sigma(\pi_i) = \lambda_i$, which settles the claim in both directions. For more details, see \[21\] Theorem 42. \[\square\]

Consider now the subcategory $\text{DMV}_{\text{fp}}^{\pi}$ whose objects are algebras of the form $DMV_n \mid_P$ for some rational polyhedron $P$, and whose morphisms are maps $\sigma : DMV_n \mid_P \rightarrow DMV_m \mid_Q$ such that $\sigma(\{\pi_1, \ldots, \pi_n\}) \subseteq MV_m$. This category is equivalent to $\text{MV}_{\text{fp}}$, since the above condition entails that both categories are dual to $\text{RatPol}^Z_{[0,1]}$. Similarly, let $\text{RMV}_{\text{fp}}^{\pi}$ be a category that has objects of the form $RMV_m \mid_P$, with $P$ a rational polyhedron and arrows $\sigma : RMV_n \mid_P \rightarrow RMV_m \mid_Q$ such that $\sigma(\{\pi_1, \ldots, \pi_n\}) \subseteq MV_m$. Hence $\text{DMV}_{\text{fp}}^{\pi}$, the opposite of $\text{RatPol}^Z_{[0,1]}$ and $\text{MV}_{\text{fp}}$ are equivalent. In this setting the MV-elements are the elements of the MV-algebra generated by the restriction of the projection maps in $DMV_n \mid_P$ or $RMV_n \mid_P$. Once again, our maps are asked to preserves MV-elements.
3.4 Ł-generated logical theories

The approaches presented in the previous subsections were motivated by the fact that, in order to enlighten subcategories of finitely presented DMV-algebras and Riesz MV-algebras that are equivalent to finitely presented MV-algebras, one has to characterize the MV-elements of the algebra at hand. This is a relatively straightforward task when provided with a concrete representation of the categories, but it is difficult in an abstract setting. The characterization of MV-elements is important for defining the right kind of morphisms for the subcategories considered. Indeed, in every approach, we fundamentally used morphisms that preserve MV-elements.

On the level of objects, the situation was reflected on ideals: in order to obtain categorical equivalences for $MV_{fp}$, the ideal that presents the algebra has to be MV-generated, which turns into being generated by an element of the free MV-algebra. This can be said, of course, once we have fixed a presentation $DMV_n/I$ of the DMV-algebra considered and we have embedded $MV_n$ in $DMV_n$ (the same applies for Riesz MV-algebras). Remark 3.4 allowed us to prove that any principal ideal in $DMV_n$ is MV-generated (see Lemma 3.14), while this is no longer true for Riesz MV-algebras, due to Proposition 3.6.

This remark, together with the correspondence between theories in logic and filter in the Lindenbaum-Tarski algebra, justifies the notion of Ł-generated theory. If $\Theta$ be a set of formulas in Łukasiewicz logic, then the set $\text{Thm}(\Theta, L) = \{ \varphi \in \text{Form}_L \mid \Theta \vdash \varphi \}$ the theory determined by $\Theta$ in $L$. Similar definitions are considered for $QL$ and $RL$.

**Definition 3.18.** If $T$ is a theory in $QL$ then we say that $T$ is Ł-generated if there exists a set of formulas $\Theta \subseteq \text{Form}_L$ such that $T = \text{Thm}(\Theta, QL)$. The Ł-generated theories of $RL$ are defined similarly.

In [15] it is proved that deductive systems corresponds to filters (and then ideals) in the Lindenbaum-Tarski algebra. We can easily prove the same in the logics $QL$ and $RL$. Moreover, if $\Theta = \{ \varphi \}$, the ideal is the principal ideal generated by the function that corresponds to $\varphi$.

We recall that the correspondence is defined as follows:

(i) for any filter $F$ in $MV_n$, $\{ \varphi \mid f_\varphi \in F \}$ is a deductive system,

(ii) for any deductive system $\Theta$, the set $\{ f_\varphi \mid \varphi \in \Theta \}$ is a filter of $MV_n$, for an appropriate $n$.

Thus, the following proposition is straightforward.

**Proposition 3.19.** Let $\varphi$ be a formula of $QL$. There exists a formula $\beta$ of $L$ such that $\text{Thm}(\varphi, QL) = \text{Thm}(\beta, QL)$.

**Proof.** We shall assume that all formulas involved have at most $n$ variables. It follows from Remark 3.14 that, if $f_\varphi$ is the formula that correspond to $\varphi$, there exists a $g \in MV_n$ such that $(f_\varphi)_{DMV} = (g)_{DMV}$. Moreover, since $DMV_n$ is also isomorphic with the algebras of term functions of $QL$, there exists $\beta \in \text{Form}_Q$ such that $g = f_\beta$. Thus, we have the following

$$\text{Thm}(\varphi, QL) \leftrightarrow (f_\varphi)_{DMV} = (f_\beta)_{DMV} \leftrightarrow \text{Thm}(\beta, QL),$$

and the claim is settled. \qed

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As a consequence of the above result, any theory in $QŁ$ is $Ł$-generated. Note that we cannot prove an analogous of Proposition 3.19 for Riesz MV-algebras. A concrete counterexample is given by the formula $ϕ = ¬(Δ_2 \sqrt{2} \oplus Δ_2 \sqrt{2} v_1)$. Indeed, with the notation of Proposition 3.19, $f_ϕ$ is the function $f : [0, 1] \to [0, 1]$ from Example 3.7. Since $(f)_RMV$ cannot be generated by an element of $MV_1$ or $DMV_1$, there isn’t a $β ∈ Ł$ such that $Thm(ϕ, RŁ) = Thm(β, QŁ)$.

Whence, in order to obtain something in this vein for Riesz MV-algebras, one should be able to give a precise characterization of those ideals of the form $(f)_RMV$, where $f$ is a piecewise linear function with at least one irrational coefficient, in terms of piecewise linear functions with rational coefficients. One possible approach relies on the results on limits of formulas proved in Section 2.

Definition 3.20. (i) An ideal $I$ of $RMV_n$, $n ∈ N$, is said to be norm-closed if, whenever $f_1, f_2, \ldots, f_m, \ldots$ is a sequence of elements of $I$ and $\{f_m\}_m ∈ N$ uniformly converges to $f$, then $f ∈ I$. A similar definition can be given for filters.

(ii) A $σ$-ideal $I$ of $RMV_n$ is an ideal closed for all existing countable suprema. That is, if $\{f_m\}_m ∈ N ⊆ I$ and $f = sup_m f_m$ exists in $RMV_n$, then $f ∈ I$.

(iii) A $σ$-filter $F$ of $RMV_n$ is a filter closed for all existing countable infima. That is, if $\{f_m\}_m ∈ N ⊆ F$ and $f = inf_m f_m$ exists in $RMV_n$, then $f ∈ F$.

We recall that any $σ$-ideal is norm-closed, see [33, Theorem 83.22] and subsequent discussion.

The definition of a norm-closed ideal, well known in the theory of Riesz spaces, suggested us the following deduction rule.

\[(⋆) \quad \text{if } ϕ = lim_m ϕ_m \text{ then } \varphi_1, \varphi_2, \ldots, \varphi_m, \ldots \to ϕ\]

Remark 3.21. Since the syntactical definition of limit is equivalent with uniform convergence, the limit is continuous with respect to the MV-algebraic operations. Thus, if $lim_n ϕ_n = ϕ$, then $lim_n ¬ϕ_n = ¬ϕ$. This entails that we can analogously define norm-closed filters and they are again dual to norm-closed ideals. Moreover, it is easily seen that $σ$-filters and $σ$-ideals are dual to each other.

Definition 3.22. $RL^*$ is the logic obtained from $RL$ by adding the deduction rule $(⋆)$.

Proposition 3.23. The theories of $RL^*$ are in correspondence with norm-closed ideals of the Lindenbaum-Tarski algebra of $RL$.

Proof. We need to prove that, for any $T = Thm(Θ, RL^*)$ the set $\{f_ψ \mid ψ ∈ T\}$ is a norm closed filter, and for any norm closed filter $F$ the set $\{f_ϕ \mid f_ϕ ∈ F\}$ is a deductive system. The general correspondence between theories and filters of the Lindenbaum-Tarski algebra is well-known [15] Proposition 6.3.15. We only need to check that $(⋆)$ implies that the corresponding filter is norm-closed and viceversa. But this is straightforward by definition.

Proposition 3.24. Let $ϕ$ be a formula of $RL$. There exists a sequence of formulas $Θ = \{ϕ_n\}_n ∈ N ⊆ Form_QŁ$ such that $Thm(ϕ, RL^*) = Thm(Θ, RL^*)$. 


Proof. By Theorem 2.12 there exists a decreasing sequence \( \{ \varphi_n \}_{n \in \mathbb{N}} \) such that \( \varphi = \lim_n \varphi_n \). We set \( \Theta = \{ \varphi_n \mid n \in \mathbb{N} \} \). One inclusion is direct consequence of the definition of \( RL^* \). In order to prove the other inclusion, we note that \( f_{\varphi_n} \) converges uniformly to \( f_{\varphi} \) and \( f_{\varphi} = \inf_n f_{\varphi_n} \) (cfr. [23, Theorems 16.1 and 16.2]). Since the set \( F = \{ f_\psi \mid \psi \in Thm(\varphi, RL^*) \} \) is a filter and clearly \( f_{\varphi} \in F \), we get that \( f_{\varphi_n} \in F \) for any \( \varphi_n \in \Theta \), which implies that \( \varphi_n \) belongs to \( Thm(\varphi, RL^*) \) for any \( n \) and the claim is settled.

We analyzed the conditions that allow us to establish categorical equivalences between subcategories of finitely presented MV-algebras, DMV-algebras and Riesz MV-algebras. We have worked with hulls (via tensor product), categories of presentation (using ideas from the theory of institutions) and restriction of free algebras to polyhedra (following the Baker-Beynon duality). All these approaches are connected by one issue: the inability to characterize a peculiar MV-subalgebra of a finitely presentable DMV-algebra or Riesz MV-algebra. Such elements are called MV-elements, a clear description of them being possible only when we choose a concrete way to define the finitely presentable algebras. Consequently we have defined the MV-generated ideals, which are ideals in a DMV-algebra or Riesz MV-algebra that are generated by MV-elements. This leads us to an analysis of those logical theories in the expansions of Łukasiewicz logic \( L \) that are generated by formulas of \( L \). Finally, we characterize the \( L \)-generated theories of \( RL \) using the notion of syntactical limit.

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