The Sasa–Satsuma (complex mKdV II) and the complex sine-Gordon II equation revisited: recursion operators, nonlocal symmetries, and more

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We present a new symplectic structure and a hereditary recursion operator for the Sasa–Satsuma equation which is widely used in nonlinear optics. Using an integro-differential substitution relating this equation to a third-order symmetry flow of the complex sine-Gordon II equation enabled us to find a hereditary recursion operator and higher Hamiltonian structures for the latter equation.

We also show that both the Sasa–Satsuma equation and the third-order symmetry flow for the complex sine-Gordon II equation are bi-Hamiltonian systems, and we construct several hierarchies of local and nonlocal symmetries for these systems.

I Introduction

Finding a recursion operator for a system of PDEs is of paramount importance, as the whole integrable hierarchy for the system in question is then readily generated by the repeated application of the recursion operator to a suitably chosen seed symmetry, see e.g. [7, 32, 11] and references therein. Moreover, using formal adjoint of the recursion operator enables us to produce infinitely many conserved quantities for our hierarchy (see e.g. [7, 11] for further details).
The recursion operators of multicomponent systems often have a richer structure of nonlocalities than in the one-component case, which makes such operators more difficult to find. For instance, there are two-component integrable generalizations, Eqs. (2) and (3), of the sine-Gordon equation and of the mKdV equation respectively. No recursion operator was found for Eqs. (2) and (3) so far even though the corresponding Lax pairs [36, 17, 38] and bilinear representations [17, 20, 18] are well known. The goal of the present paper is to fill this gap. Namely, we find and study the recursion operators for the so-called complex sine-Gordon II equation (2) and for the Sasa–Satsuma equation (3).

The Sasa–Satsuma equation [36] (see also [46, 44]) for a complex function $U$ has the form

$$U_t = U_{xxx} + 6U\bar{U}U_x + 3U(\bar{U}U)_x. \quad (1)$$

Here and below the bar refers to the complex conjugate.

It is natural to refer to this equation as to the complex mKdV II as (1) is one of the two integrable complexifications of the famous modified KdV equation, the other complexification (complex mKdV I) being simply $U_t = U_{xxx} + 6U\bar{U}U_x$.

In turn, the complex sine-Gordon II equation [17, 38] is a hyperbolic PDE for a complex function $\psi$ of the form

$$\psi_{xy} = \frac{\bar{\psi}\psi_x\psi_y}{\psi\bar{\psi} + c} + (2\psi\bar{\psi} + c)(\psi\bar{\psi} + c)k\psi,$$

where $c$ and $k$ are arbitrary constants. The usual sine-Gordon equation $\phi_{xy} = c^2k\sin(\phi)$ is recovered upon setting $\psi = \bar{\psi} = \sqrt{-c}\sin(\phi/4)$.

The Sasa–Satsuma and the complex sine-Gordon II equation are of considerable interest for applications. The Sasa–Satsuma equation is widely used in nonlinear optics, see e.g. [24, 34] and references therein, because the integrable cases of the so-called higher order nonlinear Schrödinger equation [25] describing the propagation of short pulses in optical fibers are related through a gauge transformation either to the Sasa–Satsuma equation [36] or to the so-called Hirota equation [19].

The complex sine-Gordon II equation, along with the Pohlmeyer–Lund–Regge model [33, 26, 16, 30] also known as the complex sine-Gordon I, define integrable perturbations of conformal field theories [12, 2, 8, 3]; see e.g. [35, 37] for other applications. Moreover, the complex sine-Gordon I and II are the only equations for one complex field in the plane for which the (multi)vortex solutions are found in closed form [31, 5, 6].

Following [21] we set $u = \psi$ and $v = \bar{\psi}$ and write the complex sine-Gordon II equation along with its complex conjugate as a system for $u$ and $v$:

$$u_{xy} = \frac{uv_xu_y}{uv + c} + (2uv + c)(uv + c)ku, \quad v_{xy} = \frac{uv_xv_y}{uv + c} + (2uv + c)(uv + c)kv. \quad (2)$$
Likewise, upon setting \( p = U \) and \( q = \bar{U} \) in the Sasa–Satsuma equation, proceeding in the same fashion as above and writing out \((pq)_x\) as \( pq_x + p_x q \), we obtain

\[
p_t = p_{xxx} + 9pq p_x + 3p^2 q_x, \quad q_t = q_{xxx} + 9pq q_x + 3q^2 p_x. \tag{3}
\]

From now on we shall treat \( u, v, p \) and \( q \) as independent variables that can be real or complex and consider systems (2) and (3) that are more general than the original complex sine-Gordon II and Sasa–Satsuma (complex mKdV II) equations which can be recovered under the reductions \( v = \bar{u} \) and \( q = \bar{p} \) respectively. In what follows we shall refer to (2) as to the complex sine-Gordon II system and to (3) as to the Sasa–Satsuma system.

Let us briefly address the relationship of the Sasa–Satsuma system (3) with other integrable systems. First of all, Eq.(3) can be obtained \[46\] as a reduction of the four-component Yajima–Oikawa system (8) from \[46\].

On the other hand, consider the vector modified KdV equation

\[
V_t = V_{xxx} + \langle V, V \rangle V_x + \langle V, V_x \rangle V \tag{4}
\]

studied by Svinolupov and Sokolov \[42\] (see also \[43, 45\]). Here \( V = (V^1, \ldots, V^n)^T \) is an \( n \)-component vector, \( \langle \cdot, \cdot \rangle \) stands for the usual Euclidean scalar product of two vectors, and the superscript \( T \) here and below indicates the transposed matrix. A simple linear change of variables \( V^1 = \sqrt{3}(p + q)/\sqrt{2}, V^2 = i\sqrt{3}(p - q)/\sqrt{2} \) takes (3) into (4) with \( n = 2 \).

Setting \( r = \sqrt{3}p \) and \( s = \sqrt{3}q \) turns (3) into the system

\[
\begin{align*}
  r_t &= r_{xxx} + r(3sr_x + rs_x), \\
  s_t &= s_{xxx} + s(3rs_x + sr_x),
\end{align*}
\tag{5}
\]

studied by Foursov \[15\], who found a Hamiltonian structure for (5) of the form

\[
\tilde{P} = \begin{pmatrix}
  -\frac{1}{3}rD^{-1}_x \circ r & D_x + \frac{1}{3}rD^{-1}_x \circ s \\
  D_x + \frac{1}{3}sD^{-1}_x \circ r & -\frac{1}{3}sD^{-1}_x \circ s
\end{pmatrix}
\tag{6}
\]

where \( D_x \) is the operator of total \( x \)-derivative, see e.g. \[7, 11, 32\] for details and for the background on the recursion operators, Hamiltonian and symplectic structures. The corresponding Hamiltonian density is \[15\] \((2/3)r^2s^2 - r_x s_x\). This immediately yields a Hamiltonian structure (8) for (3). Note that in \[14, 15\] there seems to be a misprint in \( \tilde{P} \), and in (6) we corrected this misprint.

In \[14, p.89\] Foursov claims to have found some skew-symmetric operators that are likely to provide higher Hamiltonian structures for (5) but he failed to verify that these operators are indeed
Hamiltonian. The explicit form of these operators was not presented in [15] or [14], so apparently it was never proved in the literature that (5) (and hence (3)) are bi-Hamiltonian systems. We establish the bi-Hamiltonian nature of (3) in Theorem 1 below.

Now turn to the complex sine-Gordon II system (2). Recall that Eq.(2) can be obtained (see e.g. [17, 21, 28]) as the Euler–Lagrange equation for the functional

$$S = \int L \, dx \, dy,$$

where

$$L = \frac{1}{2} \frac{u_x v_y + u_y v_x}{uv + c} + k(uv + c)uv.$$  

A few conservation laws and generalized symmetries for (2) can be readily found e.g. using computer algebra [21, 28]. In particular [21], Eq.(2) is compatible with

$$u_t = u_{xxx} - \frac{3uv_x u_{xx}}{uv + c} - \frac{9u_x^2 v_x}{uv + c} + \frac{3u_x^2 v_x^2 u_x}{(uv + c)^2},$$

$$v_t = v_{xxx} - \frac{3vu_x v_{xx}}{uv + c} - \frac{9v_x^2 u_x}{uv + c} + \frac{3v_x^2 u_x^2 v_x}{(uv + c)^2}.$$  

The compatibility here means that the flow (7) commutes with the nonlocal flow (14) associated with (2) or, equivalently, the right-hand sides of (7) constitute the characteristic of a third-order generalized symmetry for (2), see e.g. [32] for general background on symmetries.

The rest of paper is organized as follows. In section II we present symplectic structure and recursion operator for the Sasa–Satsuma system (3) and show that (3) is a bi-Hamiltonian system. In section III we employ a nonlocal change of variables relating systems (7) and (3) in order to construct recursion operator, Hamiltonian and symplectic structures for (2) and (7) from those of (3), and we show that (7) is a bi-Hamiltonian system. Finally, in section IV we discuss the hierarchies of local and nonlocal symmetries for (2), (3) and (7).

II Recursion operator and symplectic structure for the Sasa–Satsuma system

A straightforward but tedious computation proves the following assertion.

Theorem 1 The Sasa–Satsuma system (3) possesses a Hamiltonian structure

$$P = \begin{pmatrix} -pD_x^{-1} \circ p & D_x + pD_x^{-1} \circ q \\ D_x + qD_x^{-1} \circ p & -qD_x^{-1} \circ q \end{pmatrix},$$

a symplectic structure

$$J = \begin{pmatrix} 3pD_x^{-1} \circ p & D_x + 5pD_x^{-1} \circ q \\ D_x + 5qD_x^{-1} \circ p & 3qD_x^{-1} \circ q \end{pmatrix},$$
and a hereditary recursion operator \( R = \mathcal{P} \circ \mathcal{J} \) that can be written as

\[
R = \begin{pmatrix}
D_x^2 + 6pq + q_x D_x^{-1} \circ p + b D_x^{-1} \circ p + 3p_x D_x^{-1} \circ b & 2p^2 - 2z_1 p D_x^{-1} \circ q - 3p_x D_x^{-1} \circ a \\
2q^2 - 2z_2 q D_x^{-1} \circ p + 3q_x D_x^{-1} \circ b & D_x^2 + 6pq + p_x D_x^{-1} \circ q + a D_x^{-1} \circ q - 3q_x D_x^{-1} \circ a
\end{pmatrix}
\] (10)

where \( a = p_x + 2z_1 q, \ b = q_x + 2z_2 p; \ z_1 = D_x^{-1}(p^2) \) and \( z_2 = D_x^{-1}(q^2) \) are potentials for the following conservation laws of (3):

\[
D_t(p^2) = D_x(2pp_{xx} - p_x^2 + 6p^3 q), \quad D_t(q^2) = D_x(2qq_{xx} - q_x^2 + 6pq^2).
\]

Hence (3) has an infinite hierarchy of compatible Hamiltonian structures \( \mathcal{P}_k = R^k \circ \mathcal{P}, \ k = 0, 1, 2, \ldots, \) \( \mathcal{P}_0 \equiv \mathcal{P}, \) an infinite hierarchy of symplectic structures \( \mathcal{J}_k = \mathcal{J} \circ R^k, \ k = 0, 1, 2, \ldots, \) and an infinite hierarchy of commuting symmetries of the form \( \mathcal{K}_i = R^i(\mathcal{K}_0), \ i = 0, 1, 2, \ldots, \) where \( \mathcal{K}_0 = (p_x, q_x)^T. \)

The Sasa–Satsuma system (3) is bi-Hamiltonian with respect to \( \mathcal{P}_0 \) and \( \mathcal{P}_1: \)

\[
\begin{pmatrix} p_t \\ q_t \end{pmatrix} = \mathcal{P}_0 \begin{pmatrix} \delta \mathcal{H}_1/\delta p \\ \delta \mathcal{H}_1/\delta q \end{pmatrix} = \mathcal{P}_1 \begin{pmatrix} \delta \mathcal{H}_0/\delta p \\ \delta \mathcal{H}_0/\delta q \end{pmatrix},
\]

where \( \mathcal{H}_i = \int H_i dx, i = 0, 1, \ H_0 = pq, \ H_1 = 2p^2 q^2 - p_x q_x. \)

Here \( \delta/\delta p \) and \( \delta/\delta q \) denote variational derivatives with respect to \( p \) and \( q. \)

Note that the Hamiltonian structure \( \mathcal{P} \) above is nothing but the second Hamiltonian structure of the AKNS system, see e.g. [27]. This Hamiltonian structure (more precisely, its counterpart (6) for Eq.(5)) has already appeared in [14, 15].

The symmetries \( \mathcal{K}_i \) commute because \( \mathcal{R} \) is hereditary. Applying \( \mathcal{R} \) to an obvious symmetry \( \mathcal{K}_0 = (p_x, q_x)^T \) of (3) yields the symmetry \( \mathcal{K}_1 = \mathcal{R}(\mathcal{K}_0) = (p_{xxx} + 9pq p_x + 3p^2 q_x, q_{xxx} + 9pq q_x + 3q^2 p_x)^T, \) i.e., the right-hand side of (3). In turn, \( \mathcal{R}(\mathcal{K}_1) \) is a local fifth-order symmetry for (3). We guess that \( \mathcal{K}_i \) are local for all natural \( i \) but so far we were unable to provide a rigorous proof of this.

Unlike the overwhelming majority of the hitherto known recursion operators, see e.g. discussion in [39] and references therein, the nonlocal variables appear explicitly in the coefficients of \( \mathcal{R}. \) Perhaps this is the very reason why \( \mathcal{R} \) was not found earlier. The nonlocal variables in the coefficients of \( \mathcal{R} \) are abelian pseudopotentials as in [23] and unlike e.g. the nonlocalities in the recursion operator discovered by Karasu et al. [22] and later rewritten in [40]: the nonlocalities in the operator from [22, 40] are nonabelian pseudopotentials.
### III Recursion operator

for the complex sine-Gordon II system

There is a well-known (see e.g. [1] for discussion and references) transformation $z = \sqrt{2/3} g \rho$ relating the symmetry flow $g_t = g_{xxx} + g_x^3/2$ of the sine-Gordon equation $g_{xy} = \sin(g)$ and the mKdV equation $z_t = z_{xxx} + z^2 z_x$. Moreover, there exists [1] a nonlocal generalization of this transformation that sends the third-order symmetry flow

$$u_t = u_{xxx} - 3 \frac{u v_x v_x}{u + c} + 3 \frac{-u v_x - c u_x + u^2 v_x}{(u + c)^2} u v_x,$$

$$v_t = v_{xxx} - 3 \frac{u v_x v_x}{u + c} - 3 \frac{-v^2 u_x + u v v_x + c v_x}{(u + c)^2} u v_x$$

of the complex sine-Gordon I equation (see e.g. [16, 21, 28])

$$u_{xy} = \frac{v u_x u_y}{u + c} + (u + c) k u, \quad v_{xy} = \frac{u v_x v_y}{u + c} + (u + c) k v,$$  \(11\)

into the two-component generalization of the mKdV equation

$$p_t = p_{xxx} + 6 p q p_x, \quad q_t = q_{xxx} + 6 p q q_x$$

that belongs to the hierarchy of the well-known AKNS system

$$p_t = p_{xx} + p^2 q, \quad q_t = -q_{xx} - q^2 p.$$  \(12\)

Note that this nonlocal transformation also sends [1] the second-order symmetry flow

$$u_t = u_{xx} - 2 \frac{u u_x v_x}{u + c}, \quad v_t = -v_{xx} + 2 \frac{v u_x v_x}{u + c}$$

of (11) into the AKNS system (12).

It turns out that upon a suitable redefinition of nonlocal variables the nonlocal transformation in question in combination with a suitable rescaling of dependent variables $p$ and $q$ also sends the third-order symmetry flow (7) of the complex sine-Gordon II equation (2) into the Sasa–Satsuma system (3). Namely, we have the following result.

**Theorem 2** The substitution

$$p = \frac{i \sqrt{2} u_x \exp\left(-\frac{1}{2} w_1\right)}{u v + c}, \quad q = \frac{i \sqrt{2} v_x \exp\left(\frac{1}{2} w_1\right)}{u v + c},$$

where $i = \sqrt{-1}$, $w_1 = D^{-1}_x(\rho_1)$ is a potential for the conservation law $D_t(\rho_1) = D_x(\sigma_1)$ of (7),

$$\rho_1 = \frac{u v_x - u u_x}{u v + c}, \quad \sigma_1 = \frac{u v_{xxx} - u u_{xxx}}{u v + c} + \frac{(3 u v + 2 c)(v u_{xx} - u_x v_{xx})}{(u v + c)^2}$$

$$+ \frac{(12 u v + 11 c)(v u_x - u v_x) u_x v_x}{(u v + c)^3},$$

takes Eq.(7) into the Sasa–Satsuma system (3).
Remark 1 Let
\[ \tilde{p} = \frac{i\sqrt{2}u_x \exp(-\frac{1}{2}w_1)}{\sqrt{uv + c}}, \quad \tilde{q} = \frac{i\sqrt{2}v_x \exp(\frac{1}{2}w_1)}{\sqrt{uv + c}}, \]
where \( w_1 = D_x^{-1}(\rho_1) \) is now a potential for the conservation law \( D_y(\rho_1) = D_x(\theta_1) \) of (2), \( \rho_1 \) is given above and
\[ \theta_1 = \frac{vu_y - uv_y}{uv + c}. \]

Then for \( k = 0 \) we have
\[ D_y(\tilde{p}) = 0, \quad D_y(\tilde{q}) = 0, \]
where \( D_y \) stands for the total \( y \)-derivative. In other words, if \( k = 0 \) then \( \tilde{p} \) and \( \tilde{q} \) provide nonlocal \( y \)-integrals for (2), and \( D_x \ln(\tilde{p}) \) and \( D_x \ln(\tilde{q}) \) are local \( y \)-integrals of (2). Therefore, the complex sine-Gordon II system (2) for \( k = 0 \) is Liouvillean and \( C \)-integrable, see [9, 10] for the construction of symmetries of such systems using the integrals thereof. Using the above local integrals enables us to find the general solution for (2) with \( k = 0 \) along the lines of [47].

Passing from \( p \) and \( q \) to \( u \) and \( v \) yields from \( \mathcal{R} \) a recursion operator \( \mathfrak{R}_0 \) for (7). The operator \( \mathfrak{R}_0 \) is hereditary because so is \( \mathcal{R} \), see [13]. It is easily seen that \( \mathfrak{R}_0 \) is a recursion operator for (2) as well. Upon removing an inessential overall constant factor in \( \mathfrak{R}_0 \) we obtain the following result:

Theorem 3 The complex sine-Gordon II system (2) has a hereditary recursion operator
\[
\mathfrak{R} = \left( \begin{array}{c}
D_x^2 - \frac{2uv_x}{uv + c} + \frac{uv_{xx} - 12uv_x v_x}{(uv + c)^2} + \frac{cu_x v_x}{(uv + c)^2} - \frac{2uu_{xx} + 6u_x^2}{uv + c} + \frac{4u^2 u_x v_x}{(uv + c)^2} \\
-2uv_{xx} + 6v_x^2 + \frac{4u^2 u_x v_x}{(uv + c)^2}
\end{array} \right)
\]
\[
+ \left( \begin{array}{c}
\sum_{a=1}^{5} Q^1_{\alpha} D_x^{-1} \circ \gamma_{\alpha,1} - \frac{5}{\alpha} \sum_{a=1}^{5} Q^1_{\alpha} D_x^{-1} \circ \gamma_{\alpha,2} \\
\sum_{a=1}^{5} Q^2_{\alpha} D_x^{-1} \circ \gamma_{\alpha,1} - \frac{5}{\alpha} \sum_{a=1}^{5} Q^2_{\alpha} D_x^{-1} \circ \gamma_{\alpha,2}
\end{array} \right),
\]
where \( w_1 \) is as in Remark 1; \( y_1 = D_x^{-1} ((u_x^2 \exp(-w_1))/(uv + c)) \) and \( y_2 = D_x^{-1} ((v_x^2 \exp(w_1))/(uv + c)) \) are potentials for the nonlocal conservation laws \( D_y(\zeta_i) = D_x(\zeta_i), i = 1, 2, \) of (2),
\[ \zeta_1 = \frac{u_x^2 \exp(-w_1)}{uv + c}, \quad \zeta_1 = ku^2(uv + c) \exp(-w_1), \]
\[ \zeta_2 = \frac{v_x^2 \exp(w_1)}{uv + c}, \quad \zeta_2 = kv^2(uv + c) \exp(w_1); \]
are symmetries for (2), and
\[ Q_1 = \begin{pmatrix} y_1u \\ u_x \exp(-w_1) - vy_1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} v_x \exp(w_1) - uy_2 \\ y_2v \end{pmatrix}, \]
\[ Q_3 = \begin{pmatrix} u \\ -v \end{pmatrix}, \quad Q_4 = c \begin{pmatrix} -u_{xx} + \frac{2u_{xx}v_x}{w + c} + 4\exp(w_1)y_1v_x - 4uy_2y_1 \\ v_{xx} - \frac{2u_{xx}v_x}{w + c} - 4\exp(w_1)y_2u_x + 4vy_2y_1 \end{pmatrix}, \quad Q_5 = \begin{pmatrix} u_x \\ v_x \end{pmatrix} \]
are cosymmetries for (2).

Here \( Q^i_\alpha \) and \( \gamma_{a,j} \) denote \( i \)th component of \( Q_\alpha \) and \( j \)th component of \( \gamma_\alpha \) respectively, i.e.,
\[ Q_\alpha \equiv \begin{pmatrix} Q^1_\alpha \\ Q^2_\alpha \end{pmatrix}, \quad \gamma_\alpha \equiv (\gamma_{a,1}, \gamma_{a,2}). \]

Note that if we use the tensorial notation (see e.g. [28]), we can rewrite the operator
\[ \sum_{\alpha=1}^{5} Q^{1}_\alpha D^{-1}_x \circ \gamma_{a,1} + \sum_{\alpha=1}^{5} Q^{1}_\alpha D^{-1}_x \circ \gamma_{a,2} \]
\[ + \sum_{\alpha=1}^{5} Q^{2}_\alpha D^{-1}_x \circ \gamma_{a,1} + \sum_{\alpha=1}^{5} Q^{2}_\alpha D^{-1}_x \circ \gamma_{a,2} \]
in a more concise form, namely \( \sum_{\alpha=1}^{5} Q_\alpha \otimes D^{-1}_x \circ \gamma_\alpha. \)
Remark 2 Eq. (7) has a recursion operator of precisely the same form as the $\mathcal{R}$ given above, but in this case the nonlocal variables should be defined in a slightly different way: $w_1$ should be as in Theorem 2, and

$$y_1 = D_x^{-1}((u_x^2 \exp(-w_1))/(uv + c)),$$

and

$$y_2 = D_x^{-1}((v_x^2 \exp(w_1))/(uv + c))$$

should now be potentials for the nonlocal conservation laws $D_i(\zeta_i) = D_x(\chi_i), i = 1, 2$, of (7), where $\zeta_i$ are as in Theorem 1 and

$$\chi_1 = \exp(-w_1)
\begin{pmatrix}
\frac{2u_x u_{xxx}}{uv + c} - \frac{u_{xx}^2}{(uv + c)^2} - \frac{2u u_{x} v_{xx}}{uv + c} - \frac{2u_{x}^2 v_{xx}}{(uv + c)^2} - \frac{2(6u + 7c)u_{x}^3 v}{(uv + c)^3} + \frac{3u_{x}^2 v_{x}^2}{(uv + c)^3} \\
\frac{1}{uv + c}D_x - \frac{u v_{x}}{(uv + c)^2} & 0
\end{pmatrix},$$

$$\chi_2 = \exp(w_1)
\begin{pmatrix}
\frac{2v_x v_{xxx}}{uv + c} - \frac{v_{xx}^2}{(uv + c)^2} - \frac{2v_{x}^2 u_{xx}}{uv + c} - \frac{2v_{x} v u_{xx}}{uv + c} + \frac{3v_{x}^2 u_{x}^2 v_{x}}{(uv + c)^3} - \frac{2(6u + 7c)u_{x} v_{x}^3}{(uv + c)^3}
\end{pmatrix}.$$

Recall that Eq. (2) possesses [28, 29] a local symplectic structure

$$\mathcal{J} = \begin{pmatrix} 0 & \frac{1}{uv + c}D_x - \frac{u v_{x}}{(uv + c)^2} \\
\frac{1}{uv + c}D_x - \frac{u v_{x}}{(uv + c)^2} & 0 \end{pmatrix},$$

i.e., Eq. (2) can be written in the form $\mathcal{J} \mathcal{H}_u = k(2uv + c)(v, u)^T = \delta \mathcal{H}/\delta u$, where $u = (u, v)^T$ and $\mathcal{H} = \int k(uv + c)uv dx$. The symplectic structure $\mathcal{J}$ is readily seen to be shared by Eq. (7).

Inverting $\mathcal{J}$ yields a nonlocal Hamiltonian structure for (7) of the form

$$\mathcal{P}_n = \mathcal{R}^n \circ \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_1 \circ \mathcal{R}^n,$$

where $\mathcal{R}$ is hereditary, and hence all of these structures are compatible. However, the structures $\mathcal{P}_1$ and $\mathcal{J}_1$ are already very cumbersome, so we do not display them here. Nevertheless, it is readily checked that the following assertion holds.

Theorem 4 Eq. (7) is a bi-Hamiltonian system:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \mathcal{P}_0 \begin{pmatrix} \delta \mathcal{H}_1/\delta u \\ \delta \mathcal{H}_1/\delta v \end{pmatrix} = \mathcal{P}_1 \begin{pmatrix} \delta \mathcal{H}_0/\delta u \\ \delta \mathcal{H}_0/\delta v \end{pmatrix},$$

where

$$\mathcal{H}_i = \int \tilde{H}_i dx, i = 0, 1,$$

$$\tilde{H}_0 = -\frac{u_x v_x}{uv + c}, \quad \tilde{H}_1 = \frac{u x x v_{xx}}{uv + c} - \frac{v u_x v_x u_{xx} + u u_x v_x v_{xx} - 4u^2 v_x^2}{(uv + c)^2} + \frac{u u_x v_x^2}{(uv + c)^3}.$$
It is straightforward to verify that the action of $\mathcal{R}$ on the obvious symmetry $u_x$ of (7) yields the right-hand side of (7), and $\mathcal{R}^2(u_x)$ is a fifth-order \textit{local} generalized symmetry for (2). We guess that the repeated application of $\mathcal{R}$ to $u_x$ yields a hierarchy of \textit{local} generalized symmetries for (2) but we were not able to prove this in full generality so far, as presence of the nonlocalities $y_i$ in the coefficients of $\mathcal{R}$ appears to render useless the hitherto known ways of proving locality for hierarchies of symmetries generated by the recursion operator, cf. e.g. [41] and references therein. Nevertheless, as $\mathcal{R}$ is a recursion operator for (7), Proposition 2 from [39] tells us that the only nonlocalities that could possibly appear in the hierarchy of the symmetries $\mathcal{R}^k(u_x)$, $k = 1, 2, \ldots$, are potentials of (possibly nonlocal) conservation laws for (2).

\section*{IV Nonlocal symmetries}

First of all, we can consider Eq.(2) as a nonlocal symmetry flow of Eq.(7). We have already noticed above that Eq.(2) can be written as

$$3u_y = \delta\mathcal{H}/\delta u,$$

(13)

where $\mathcal{H} = \int k(\nu w + c)uv dx$. Acting by $P = 3^{-1}$ on both sides of (13) we can formally rewrite the complex sine-Gordon II system (2) in the evolutionary form $u_y = P(\delta\mathcal{H}/\delta u)$, that is,

$$u_y = k \exp(-w_1/2)\sqrt{uv + c} \omega_2, \quad v_y = k \exp(w_1/2)\sqrt{uv + c} \omega_1,$$

(14)

where $\omega_1 = D_x^{-1} \left(v(2uv + c) \exp(-w_1/2)\sqrt{uv + c}\right)$ and $\omega_2 = D_x^{-1} \left(u(2uv + c) \exp(w_1/2)\sqrt{uv + c}\right)$ are potentials for the following nonlocal conservation laws of (7):

$$D_t \left(v(2uv + c) \exp(-w_1/2)\sqrt{uv + c}\right) = D_x \left(\exp(-w_1/2)\sqrt{uv + c} \left(\frac{4uv + 3c}{uv + c} v_{xx} + (4uv + c)v_{xx}\right) - 2v^2_x\right),$$

$$D_t \left(u(2uv + c) \exp(w_1/2)\sqrt{uv + c}\right) = D_x \left(\exp(w_1/2)\sqrt{uv + c} \left(4uv + c\right)u_{xx} + \frac{u^2(4uv + 3c)v_{xx}}{uv + c}\right) - 2v^2_x\right),$$

where

$$Q_{-1} = \begin{pmatrix} \exp(-w_1/2)\sqrt{uv + c} \omega_2 \\ \exp(w_1/2)\sqrt{uv + c} \omega_1 \end{pmatrix}.$$
is a nonlocal symmetry for (7). Moreover, we have $\mathcal{R}(Q_{-1}) = c^2 u_x$, i.e., the action of $\mathcal{R}$ on $Q_{-1}$ gives the ‘zeroth’ (obvious) symmetry $u_x$ up to a constant factor. Thus, Eq.(14) (and hence the complex sine-Gordon II system (2)) can be considered as a first negative flow in the hierarchy of (7).

Note that if we consider $\mathcal{R}$ as a recursion operator for Eq.(2), we have $\mathcal{R}(u_y) = k c^2 u_x$, in perfect agreement with the above result.

Eqs.(3) and (2) possess nonlocal symmetries of the form

$$G_1 = \begin{pmatrix} q_x + z_2 p \\ -z_2 q \end{pmatrix}, \quad G_2 = \begin{pmatrix} -z_1 p \\ p_x + z_1 q \end{pmatrix},$$

and

$$Q_1 = \begin{pmatrix} y_1 u \\ u_x \exp(-w_1) - vy_1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} w_x \exp(w_1) - uy_2 \\ y_2 v \end{pmatrix},$$

respectively, where $z_1$ and $z_2$ were defined in Theorem 1. Therefore we have four hierarchies of nonlocal symmetries: $\mathcal{R}^i(G_j)$ for (3) and $\mathcal{R}^i(Q_j)$ for (2), where in both cases $j = 1, 2$ and $i = 0, 1, 2, \ldots$.

Two more hierarchies of nonlocal symmetries have the form $\mathcal{R}^i(G_0)$ for (3) and $\mathcal{R}^i(Q_0)$ for (2), where $i = 1, 2, \ldots$,

$$G_0 = \begin{pmatrix} p \\ -q \end{pmatrix}, \quad Q_0 = \begin{pmatrix} u \\ -v \end{pmatrix}.$$

Two somewhat more ‘usual’ nonlocal hierarchies of master symmetries (the latter are, roughly speaking, time-dependent symmetries such that repeatedly commuting them with a suitable time-independent symmetry yields an infinite hierarchy of time-independent commuting symmetries for the system in question, see e.g. [7] and references therein for further details), $\mathcal{R}^i(S_0)$ for (3) and $\mathcal{R}^i(S_0)$ for (7), where $i = 1, 2, \ldots$, originate from the scaling symmetries for (3) and (7),

$$S_0 = \begin{pmatrix} 3t(p_{xxx} + 9pqp_x + 3p^2 q_x) + xp_x + p \\ 3t(q_{xxx} + 9pqq_x + 3q^2 p_x) + qx + q \end{pmatrix}$$

and

$$S_0 = \begin{pmatrix} 3t \left( u_{xxx} - \frac{3uv_x u_{xx}}{uv + c} - \frac{9u_x^2 v_x}{uv + c} + \frac{3u^2 v_x^2 u_x}{(uv + c)^2} \right) + xu_x \\ 3t \left( v_{xxx} - \frac{3uv_x v_{xx}}{uv + c} - \frac{9v_x^2 u_x}{uv + c} + \frac{3v^2 u_x^2 v_x}{(uv + c)^2} \right) + xv_x \end{pmatrix},$$

respectively.

Finally, (7) has nonlocal symmetries

$$G_1 = \begin{pmatrix} \sqrt{uv + c} \exp(-w_1/2) \\ 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 \\ \sqrt{uv + c} \exp(w_1/2) \end{pmatrix},$$

11
obtained by differentiating $Q_{-1}$ with respect to $\omega_1$ and $\omega_2$, but these symmetries are annihilated by $\mathcal{R}$ and hence do not lead to new hierarchies of nonlocal symmetries.

It would be interesting to find out whether the systems in question possess nonlocal symmetries that do not belong to the above hierarchies, which is the form of solutions invariant under the nonlocal symmetries and whether these solutions could have any applications in nonlinear optics.

As a final remark, we note that because of the obvious symmetry of the complex sine-Gordon II system (2) under the interchange of $x$ and $y$, all of the above results concerning the recursion operator, Hamiltonian and symplectic structures, and hierarchies of symmetries for (2) remain valid if we replace all $x$-derivatives with $y$-derivatives and vice versa and swap the operators $D_x$ and $D_y$. Interestingly enough, the recursion operator $\tilde{\mathcal{R}}$ obtained from $\mathcal{R}$ upon such an interchange proves to be inverse to $\mathcal{R}$ on symmetries of (2) up to a constant factor. More precisely, for any symmetry $K$ of (2) we have

$$\tilde{\mathcal{R}}(K) = k^2 c^4 \mathcal{R}^{-1}(K).$$

Taking into account our earlier results we see that the ‘basic’ hierarchy of (2) can be represented by a diagram of the form

$$\cdots \rightarrow \mathcal{R}^{-2}(u_y) \rightarrow \mathcal{R}^{-1}(u_y) \rightarrow u_y \rightarrow u_x = \mathcal{R}(u_y)/kc^2 \rightarrow \mathcal{R}(u_x) \rightarrow \mathcal{R}^2(u_x) \rightarrow \cdots,$$

and we guess that all symmetries presented at this diagram are local, i.e., they do not involve nonlocal variables. This is a fairly common situation for hyperbolic PDEs, see e.g. the discussion in [4].

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