Semiconformal curvature tensor and perfect fluid spacetimes in general relativity

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ABSTRACT
The semiconformal curvature tensor and its divergence have been studied for perfect fluid spacetimes. It is seen, apart from other results, that the perfect fluid spacetimes with divergence-free semiconformal curvature tensor either satisfy the vacuum-like equation of state or represent FLRW cosmological model.

1. Introduction and preliminaries
Let the Einstein’s field equations be

\[ R_{ab} - \frac{1}{2} g_{ab} R = - k T_{ab}. \]  

(1)

Contraction of this equation by \( g^{ab} \) leads to

\[ T = k R. \]  

(2)

From Equation (2), Equation (1) becomes

\[ R_{ab} = - k \left( T_{ab} - \frac{1}{2} T g_{ab} \right). \]  

(3)

Ishii [1], introduced the set of conharmonic transformations as a subgroup of the conformal group of transformations. These conharmonic transformations are given by

\[ g_{bc} \rightarrow e^{2\beta} g_{bc}, \]  

(4) and satisfying the condition

\[ \nabla_b \beta^b + \nabla^b \nabla_b \beta^b = 0, \]  

(5)

where \( g_{bc} \) and \( \tilde{g}_{bc} \) are the metric tensors for Riemannian spaces \( V \) and \( \tilde{V} \), respectively, and \( \beta \) is a real scalar function.

For n-dimensional Riemannian differentiable manifold \( (M^n, g) \) \( (n \geq 4) \), a rank four tensor \( L_{bcd}^h \) which is invariant under conharmonic transformation, is given by [2]

\[ L_{bcd}^h = R_{bcd}^h + \frac{1}{n-2} (\delta^h_c R_{bd} - \delta^h_d R_{bc} + g_{bd} R^h_c - g_{bc} R^h_d), \]  

(6)

where \( R_{bcd}^h \) and \( R_{bd} \) are Riemann and Ricci curvature tensors respectively. The geometrical significance of conharmonic curvature tensor has been discussed by Shaikh and Kumar [3] while the physical significance of this tensor has been investigated by Abdussattar and Dwivedi [4] and Ahsan and Ali [5]. Kim [6,7] introduced a curvature like tensor such that it remains invariant under condition (5). He calls this new tensor as semiconformal curvature tensor and denotes it by \( P_{bcd}^h \). For a Riemannian manifold \( (M^n, g) \) this tensor is defined as

\[ P_{bcd}^h = - (n-2) B C_{bcd}^h + [A + (n-2) B] L_{bcd}^h, \]  

(7)

provided the constants A and B are non-zero. Weyl conformal curvature tensor [8], is expressed as

\[ C_{bcd}^h = R_{bcd}^h + \frac{1}{n-2} (\delta^h_c R_{bd} - \delta^h_d R_{bc} + g_{bd} R^h_c - g_{bc} R^h_d) + \frac{R}{(n-1)(n-2)} (\delta^h_c g_{bd} - \delta^h_d g_{bc}), \]  

(8)

where \( R_{ab} \) is Ricci tensor and \( R \) is the scalar curvature tensor. For \( A = 1 \) and \( B = -(1/(n-2)) \), the semiconformal curvature tensor reduces to conformal curvature tensor, while for \( A = 1 \) and \( B = 0 \), it reduces to conharmonic curvature tensor. The Covariant form of the
semiconformal curvature tensor can be written as

\[ P_{bcd} = -(n-2)BCh_{bcd} + [A + (n-2)B]L_{bcd}. \]

It may be noted that the semiconformal curvature tensor \( P_{bcd} \) satisfies the following symmetry property

\[ P_{bcd} = -P_{bhc} = -P_{hbc} = P_{cdhb} \]

and

\[ P_{bcd} + P_{cbhd} + P_{chbd} = 0. \]

Towards an attempt of obtaining the exact solution of Einstein field equations, the symmetry assumptions on spacetime help to achieve the target. These symmetries are defined through the vanishing of the Lie derivatives of certain tensor with respect to a spacelike, timelike or a null vector. The symmetries of spacetimes are also called as collineations. The notion of semiconformal curvature collineation, defined in terms of other curvature tensors, has been introduced by Pundeer et al. [9], who obtained the necessary and sufficient conditions under which a spacetime consisting electromagnetic fields may admit related collineations. Einstein’s gravity is well connected with the Maxwell’s electrodynamics, so the symmetries of spacetime related to electromagnetic fields can be coupled with the Maxwell’s equations. Recently El-Salam [10] has used the technique of fractional calculus for the study of Maxwell’s equations. This lucid technique may further be used for the symmetries of spacetime and the semiconformal curvature collineation may be analysed in this light.

Abduussattar and Babita Dwivedi in their paper [4] have given a detailed study on the divergence of conharmonic curvature tensor in perfect fluid spacetime. Also, the divergence of a \( W \)-curvature tensor in perfect fluid spacetime is studied by Ahsan and Ali [5], who have shown the representation of perfect fluid as Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmological model under certain conditions. Motivated by these works on the divergence of curvature tensors we choose the semiconformal curvature tensor which is invariant under conharmonic transformation.

The present paper is yet another effort to study the semiconformal curvature tensor. In terms of the energy-momentum and Ricci tensors, the divergence of semiconformal curvature tensor has been expressed in Section 2. It may be noted that the divergence of semiconformal curvature tensor vanishes for Einstein spaces under certain conditions. Finally, in the last section a divergence-free semiconformal curvature tensor is considered for a perfect fluid spacetime and we found that if the semiconformal curvature tensor has zero divergence and the energy momentum tensor is of Codazzi type [11], then \( (\mu + p) \) is constant. Furthermore, if the semiconformal curvature tensor is divergence-free then either \( \mu + p = 0 \), i.e. the vacuum - like equation of state is satisfied by the spacetime or the fluid is acceleration-free, vorticity-free, shear-free and represent a Friedmann–Lemaitre–Robertson–Walker (FLRW) cosmological model with \( \mu - 3p = \text{constant} \).

2. Spacetime with divergence of semiconformal curvature tensor

In four dimensional manifold, from Equation (7) the semiconformal curvature tensor is given by

\[ P_{bcd}^h = -2BCh_{bcd} + [A + 2B]L_{bcd}^h. \]  

From Equations (6) and (8) for \( n = 4 \) the conharmonic and conformal curvature tensors may be written in the following form

\[ L_{bcd}^h = R_{bcd}^h + \frac{1}{2}(\delta^h_cR_{bd} - \delta^h_bR_{cd} + g_{bd}R_{ch}^h - g_{bc}R_{dh}^h), \]

and

\[ C_{bcd}^h = R_{bcd}^h + \frac{1}{2}(\delta^h_cR_{bd} - \delta^h_bR_{cd} + g_{bd}R_{ch}^h - g_{bc}R_{dh}^h) \]

\[ + \frac{R}{6}(\delta^h_dg_{bc} - \delta^h_cg_{bd}). \]

Now we write

**Theorem 2.1:** Semiconformal curvature tensor is proportional to conharmonic curvature tensor, if the scalar curvature is zero in the spacetime.

**Proof:** The semiconformal curvature tensor in space of dimension \( n = 4 \), with the help of Equations (10) and (11), is given by

\[ P_{bcd}^h = A[R_{bcd}^h + \frac{1}{2}(\delta^h_cR_{bd} - \delta^h_bR_{cd} + g_{bd}R_{ch}^h - g_{bc}R_{dh}^h)] \]

\[ - \frac{BR}{3}(\delta^h_dg_{bc} - \delta^h_cg_{bd}). \]

while its contraction is given by

\[ P_{bc} = -\left(\frac{A + 2B}{2}\right)R_{gbc}. \]

which is also invariant under condition in Equation (5).

Moreover, use of Equation (10) in (12), gives rise to

\[ P_{bcd}^h = AL_{bcd}^h - \frac{BR}{3}(\delta^h_dg_{bc} - \delta^h_cg_{bd}), \]

which from Equation (13) leads to

\[ P_{bcd}^h - C(\delta^h_bP_{bc} - \delta^h_cP_{bd}) \]

\[ = AL_{bcd}^h, \quad \text{where} \quad C = \frac{2B}{3(A + 2B)}. \]

Thus if scalar curvature is zero then from Equation (14), semiconformal and conharmonic curvature tensors are proportional to each other.

**Corollary 2.1:** If we consider a spacetime with zero scalar curvature and \( A = 1 \) in Equation (14), the two tensors \( P_{bcd}^h \) and \( L_{bcd}^h \) become identical.
Using Equations (13) and (15), we write

\[ p_{bcd}^h = \text{AR}_{bcd}^h + \left( \frac{3A + 4B}{12} \right) (\delta_c^h R_{bd} - \delta_d^h R_{bc}) \quad (15) \]

Using Equations (13) and (15), we write

\[ p_{bcd}^h = \text{AR}_{bcd}^h - D(s^h_{bc} p_{bd} - s^h_{bd} p_{bc}) \quad \text{where} \]

\[ D = \left( \frac{3A + 4B}{6A + 12B} \right) \quad (16) \]

The second term of the right side of Equation (16) will vanish as the scalar curvature is zero. This completes the proof.

Further, from Equation (16), we may state

**Corollary 2.2:** For Einstein spaces with zero scalar curvature the semiconformal and Riemannian curvature tensors become identical if \( A = 1 \).

**Theorem 2.3:** The semiconformal curvature tensor is divergence-free if the Codazzi type energy-momentum tensor is divergence-free.

**Proof:** Taking covariant derivative on both sides of Equation (12), we get

\[
\nabla_\epsilon p_{bcd}^h = A[\nabla_\epsilon R_{bcd}^h + \frac{1}{2} (s_c^h \nabla_\epsilon R_{bd} - s_d^h \nabla_\epsilon R_{bc} + g_{bd} \nabla_\epsilon R_{c}^h - g_{bc} \nabla_\epsilon R_{d}^h)] \\
- \frac{B}{3} (s_d^h g_{bd} - s_c^h g_{bc}),
\]

which on contracting over \( h \) and \( e \) leads to

\[
\nabla_h p_{bcd}^h = A[\nabla_h R_{bcd}^h + \frac{1}{2} (\nabla_c R_{bd} - \nabla_d R_{bc}) + g_{bd} \nabla_c R - g_{bc} \nabla_d R)] \\
- \frac{B}{3} (g_{bc} \nabla_d R - g_{bd} \nabla_c R). \quad (17)
\]

We know that the Bianchi identity is given by

\[
\nabla_h R_{bcd}^h + \nabla_c R_{bdh}^h + \nabla_d R_{bhc}^h = 0,
\]

or

\[
\nabla_h R_{bcd}^h = \nabla_d R_{bc} - \nabla_b R_{cd}. \quad (18)
\]

From Equations (17) and (18), we have

\[
\nabla_h p_{bcd}^h = A \left[ \nabla_d \left( R_{bd} - \frac{1}{2} R_{gd} \right) - \nabla_c \left( R_{bd} - \frac{1}{2} R_{bg} \right) \right] \\
- \frac{B}{3} (g_{bc} \nabla_d R - g_{bd} \nabla_c R), \quad (19)
\]

which on using Equations (2) and (3), may be expressed as

\[
\nabla_h P_{bcd}^h = -k \left\{ A(\nabla_d T_{bc} - \nabla_c T_{bd}) \\
+ \frac{B}{3} (g_{bc} \nabla_h T_{bd}^0 - g_{bd} \nabla_h T_{bc}^0) \right\}. \quad (20)
\]

If the energy-momentum, \( T_{ab} \), is Codazzi type \([11]\), that is, it satisfies Yang’s equation,

\[
\nabla_d T_{bc} = \nabla_c T_{bd}, \quad (21)
\]

then from Equations (21) and (20), the proof follows.

**Theorem 2.4:** For Einstein Spaces, the semiconformal curvature tensor is divergence-free.

**Proof:** The covariant derivative of \( p_{bcd}^h \) (from Equation (15)) is

\[
\nabla_\epsilon p_{bcd}^h = A \nabla_\epsilon p_{bcd}^h \\
+ \left( \frac{3A + 4B}{12} \right) (\delta_c^h \nabla_\epsilon R_{bd} - \delta_d^h \nabla_\epsilon R_{bc}).
\]

Now contracting the above equation over \( h \) and \( e \), and then using Equation (18), we get

\[
\nabla_h p_{bcd}^h = A(\nabla_d R_{bc} - \nabla_c R_{bd}) \\
+ \left( \frac{3A + 4B}{12} \right) (\nabla_c R_{bd} - \nabla_d R_{bc}). \quad (22)
\]

It is known that a \( n \)-dimensional Riemannian space is an Einstein space if its Ricci tensor \( R_{bc} \) is proportional to the metric tensor, i.e. \( R_{bc} = \rho g_{bc} \). In 4-dimensional spacetime of general relativity \( R_{bc} = R/4g_{bc} \). Thus use of \( R_{bc} = R/4g_{bc} \) in Equation (22) completes the proof. Note that in this case, its scalar curvature \( R \) is constant \([12]\), i.e. \( \nabla_d R = 0 \) is greater than or equal to 2.

Therefore, from Equation (22), using the conditions of Einstein spacetime, we get divergence zero of the semiconformal curvature tensor.

**Theorem 2.5:** For the spaces of constant curvature, the divergence of semiconformal curvature tensor vanishes if the Ricci tensor is of Codazzi type and divergence-free.

**Proof:** The divergence of semiconformal curvature tensor, from Equation (22), may be expressed as

\[
\nabla_h p_{bcd}^h = A(\nabla_d R_{bc} - \nabla_c R_{bd}) \\
+ \left( \frac{3A + 4B}{12} \right) (\nabla_h R_{bd}^0 g_{bc} - \nabla_h R_{bc}^0 g_{bd}). \quad (23)
\]

We get immediate proof of the theorem, if the Ricci curvature tensor is of Codazzi type (see Equation (21)) and divergence-free.
3. Perfect fluid spacetime with divergence-free semiconformal curvature tensor

**Theorem 3.1:** If in the perfect fluid spacetime, the semiconformal curvature tensor is divergence-free and the energy-momentum tensor is Codazzi type, then \((\mu - 3p)\) is constant.

**Proof:** For a perfect fluid spacetime, the energy momentum tensor is given by

\[
T_{ab} = (\mu + p)u_au_b + pg_{ab},
\]

where \(p\) is the isotropic pressure, \(\mu\) is the energy density and the fluid four-velocity vector is denoted by \(u_a\).

Multiplying Equation (24) by \(g^{ab}\) and using (timelike vector) \(u_au^a = -1\), we get

\[
T = -\mu + 3p.
\]  

(25)

Now, if the energy-momentum tensor is Codazzi type and the divergence of semiconformal curvature tensor vanishes, then Equation (22) implies

\[
\left(\frac{3A - 4B}{12}\right)(\nabla_c T_{bd} - \nabla_d T_{cb}) = 0.
\]  

(26)

Use of Equation (25) in Equation (26) leads to

\[
\left(\frac{4B - 3A}{12}\right)(\nabla_c (\mu - 3p)g_{bd} - \nabla_d (\mu - 3p)g_{bc}) = 0,
\]

which leads to

\[
\nabla_c (\mu - 3p) = 0.
\]  

(27)

This equation implies that \((\mu - 3p)\) is constant and the proof is completed. ■

**Theorem 3.2:** If for a perfect fluid the semiconformal curvature tensor is divergence-free, then either \((\mu + p) = 0\) (i.e. the perfect fluid spacetime satisfies the vacuum-like equation of state) or the spacetime is Friedmann–Lemaître–Robertson–Walker cosmological model satisfying \((\mu - 3p) = constant\).

**Proof:** Consider a perfect fluid spacetime with divergence-free semiconformal curvature tensor vanishes, i.e \(\nabla_b P_{bcd} = 0\) and by the virtue of Equation (24), Equation (20) gives

\[
A(\nabla_c T_{bd} - \nabla_d T_{cb}) = \frac{B}{3}(g_{bd} \nabla_c T^h_{\ell} - g_{bc} \nabla_c T^h_{\ell}).
\]  

(29)

Or

\[
A[\nabla_c ((\mu + p)u_b u_c + pg_{bc}) - \nabla_c ((\mu + p)u_b u_d + pg_{bd})] + \frac{B}{3}[g_{bc} \nabla_c ((\mu + p)u^h u_d + p^h_{\ell})] - g_{bd} \nabla_c ((\mu + p)u^h u_c + p^h_{\ell}) = 0,
\]

(30)

which on contraction by \(u^d\), leads to

\[
(\mu + p)\left[ Au_b u_c - \frac{B}{3} g_{bc} - \frac{B}{3} u_b u_c \right] + (\mu + p)\left[ Au_b u_c + Au_c u_b - \frac{B}{3} u_c u_b + A \nabla_c u_b \right] - (\mu + p)\left[ \frac{B}{3} g_{bc} + \frac{B}{3} u_b u_c \right] + A \nabla_c u_b + A \nabla_c (\mu + p)u_b - \frac{B}{3} \nabla_c p u_b = 0.
\]  

(31)

Here an overhead dot represents the covariant differential along the fluid flow vector and

\[
(\mu + p)\dot u^h = \nabla_c (\mu + p)u^h, \quad \dot u_b = \nabla_b u_b u^h, \quad \nabla_c u_b u^h = 0, \quad u^i_{\dot} = -1,
\]

and \(u_b\) denotes the acceleration of the congruence of the fluid flow and \(\dot{\theta} = \nabla_d u^d\) is the expansion scalar.

From the conservation equation of energy-momentum \(\nabla_b T^{cb} = 0\), and Equation (24), we get

\[
(\mu + p)u_c = -\nabla_c p - pu_c,
\]

(32)

and

\[
\dot{\mu} = - (\mu + p) \nabla_c u^c = (\mu + p) \dot{\theta}, \quad \text{Equations (32) and (33) are force and energy equations respectively.}
\]

Using these equations in Equation (31), we get

\[
A(\mu - p) u_b u_c + A \nabla_c (\mu - p) u_b - A \nabla_u p u_c + A \nabla_b u_b u^h = 0.
\]  

(34)

Contracting this equation by \(u^h\), we get

\[
\nabla_c (\mu - p) = - (\mu - p) u_c.
\]  

(35)

In view of force Equations (32), (35) takes the form

\[
(\mu + p)u_c = - \dot{\mu} u_c - \nabla_c \mu.
\]  

(36)

Also, with the help of Equations (35), (34) leads to

\[
A \nabla_b p u_c + A (\mu + p) \nabla_c u_b = 0,
\]

(37)

which on contracting with \(g^{bc}\) leads to

\[
\dot{\mu} = -\frac{1}{3} (\mu + p) \theta.
\]  

(38)

In view of energy Equations (33), (38) yields

\[
\dot{\mu} = 3p.
\]  

(39)

Equation (37) with the help of force Equation (32) may be written as

\[
(\mu + p)u_b u_c + \dot{p}(g_{bc} + u_b u_c) + (\mu + p) \nabla_c u_b = 0.
\]  

(40)
Now using Equation (38) in Equation (40), we get

\[(\mu + p)[u_b u_c - \frac{1}{3}(g_{bc} + u_b u_c) + \nabla_c u_b] = 0.\]

This equation indicates that either

\[(\mu + p) = 0,\]

or

\[\nabla_c u_b = \frac{1}{3}(g_{bc} + u_b u_c) - u_b u_c.\]

If \((\mu + p) = 0\) it may be noted either \(\mu = 0\) or \(p = 0\) (empty spacetime) or the perfect fluid expresses the vacuum-like equation of state [13].

Now, it is known that [2,14] the covariant derivative of the fluid flow vector may be expressed as

\[\nabla_c u_b = \frac{1}{3}(g_{bc} + u_b u_c) - u_b u_c + \sigma_{bc} + \omega_{bc},\]

where \(\sigma_{bc}\) and \(\omega_{bc}\) are the shear tensor and the rotation or vorticity tensors, respectively.

Now comparing Equation (41) with Equation (3) we get

\[\sigma_{bc} + \omega_{bc} = 0,\]

which implies that \(\sigma_{bc} = \omega_{bc} = 0\).

Using Equations (36), (38) and (3) in Equation (30), we get

\[
A \nabla_d \mu u_b u_c + A \nabla_d p u_b u_c - A \rho u_c g_{bd} \\
+ A \rho u_c u_d u_b + A \nabla_d p g_{bc} \\
- A \nabla_c \mu u_b u_d - A \nabla_c p u_b u_d + A \rho u_d g_{bc} \\
- A \rho u_d u_b u_c - A \nabla_c p g_{bd} \\
- \frac{B}{3}(\mu + p) u_d g_{bc} + \frac{B}{3}(\mu + p) u_d g_{bd} \\
- \frac{B}{3} \nabla_d \mu g_{bc} \\
+ \frac{B}{3} \nabla_d \mu g_{bd}
\]

Contracting this equation by \(g^{bc}\) and using Equation (32), we get

\[3A(\mu + p) u_d = 0,\]

Since \(3A(\mu + p) \neq 0\), Equation (44) yields

\[u_d = 0.\]

From Equations (45), (32) and (36) takes the form

\[\nabla_c p = -\rho u_c \quad \text{and} \quad \nabla_c \mu = -\mu u_c,\]

which on using Equation (39) leads to

\[\nabla_c \mu = 3 \nabla_c p.\]

From Equations (39) and (47) we obtain

\[(\mu - 3p) = \text{constant}\]

Thus if \(\mu + p \neq 0\), then from Equations (42), (45) and (48) the perfect fluid is shear-free, acceleration-free, vorticity-free and the energy density and pressure are constant over the spacelike hypersurface orthogonal to the fluid flow four velocity vector. These are the conditions for a spacetime to be Friedmann–Lemaitre–Robertson–Walker cosmological model.

On the contrary, if \(\mu = 0, p = 0\), then divergence zero is obvious for the semiconformal curvature tensor of the spacetime. Even though the fluid is shear-free, vorticity-free, acceleration-free, and energy density and pressure are constant over the spacelike hypersurface orthogonal to the fluid flow vector, now from Equation (24) we have

\[
A(\nabla_d T_{bc} - \nabla_c T_{bd}) + \frac{B}{3}(g_{bc} \nabla_h T_{d}^h - g_{bd} \nabla_h T_{c}^h)
\]

\[= A[\nabla_d((\mu + p) u_b u_c + p g_{bc}) \\
- \nabla_c((\mu + p) u_b u_d + p g_{bd})] \\
\times \frac{B}{3}(g_{bc} \nabla_h((\mu + p) u^h u_d + p s^h_d) \\
- g_{bd} \nabla_h((\mu + p) u^h u_c + p s^h_c)].\]

Further, using Equations (3), (45), (46) in (49), we get Equation (29) that is the semiconformal curvature is divergence-free.

This completes the proof of the theorem. \[\square\]

4. Conclusion

We have clarified the idea of divergence of semiconformal curvature tensor and the results are obtained on the divergence of semiconformal curvature tensor in general and special spaces. We have also obtained the necessary and sufficient conditions for the semiconformal curvature tensor to be divergence-free in perfect fluid spacetimes. Moreover, the divergence-free semiconformal curvature tensors in perfect fluid give rise to the Friedmann–Lemaitre–Robertson–Walker (FLRW) cosmological model of the universe.

Note

1. \(\nabla_e\) indicate the covariant derivative with respect to \(e\).

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