Integration Formulas and Exact Calculations in the Calogero-Sutherland Model

P.J. Forrester*
Department of Mathematics
University of Melbourne
Parkville, Victoria 3052
Australia

Abstract

Some integration formulas which either occur or are implicit in Ha’s recent exact calculation of some correlations in the Calogero-Sutherland model are discussed. These integration formulas include the calculation of the inner product $\langle 0 | \rho(0) | \kappa \rangle$ between the density operator acting on an excited state and the ground state, and a generalization of the Selberg integral due to Dotsenko and Fateev.

1 Introduction

The Calogero-Sutherland model refers to quantum particles on a line interacting via the $1/r^2$ pair potential. This model has recently been identified with one-dimensional non-interacting anyons [1-3] and the excitations have been shown to exhibit fractional statistics [2,4,5]. Furthermore the model exhibits some remarkable solvability properties which illustrate the latter: the ground state dynamical density-density correlation and retarded single particle Green’s function can be calculated exactly at all rational couplings [4-10]. The calculation of these quantities, which is due in its full generality to Ha [4], uses the theory of Jack symmetric polynomials introduced by the present author [6-9] to calculate the static correlations and retarded single particle Green’s function at integer couplings.

In this paper we will discuss some integration formulas which either occur or are implicit in Ha’s [4] calculation (all references to Ha below refer to [4]).

2 Calculation of the ground state dynamical density-density correlation $D(x, t)$ in the finite system

2.1 Eigenfunctions of $H$

In periodic boundary conditions, the $1/r^2$ quantum many body Hamiltonian is given by

$$H := -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2\lambda(\lambda - 1) \left( \frac{\pi}{L} \right)^2 \sum_{1 \leq j < k \leq N} \frac{1}{\sin^2 \pi(x_k - x_j)/L}$$

(2.1)

The corresponding ground state wave function is [11]

$$| 0 > := \psi_0(x_1, \ldots, x_N) := \prod_{1 \leq j < k \leq N} \left( \sin \pi(x_k - x_j)/L \right)^\lambda,$$

(2.2)
and the excited states are labelled by partitions

\[ \kappa := (\kappa_1, \ldots, \kappa_N), \quad \text{where} \quad \kappa_1 \geq \ldots \geq \kappa_N, \quad \text{and} \quad \kappa_j \in \mathbb{Z} \quad (j = 1, \ldots, N) \quad (2.3) \]

with a unique excited state for each partition [11]. The exact excited states are, up to normalization, [8]

\[ | \kappa > := \psi_0(x_1, \ldots, x_N) \times \sum_{\lambda \in \mathbb{Z}} \frac{s^{\lambda}_{\kappa}(e^{2\pi i x_1/L}, \ldots, e^{2\pi i x_N/L})}{\prod_{j=1}^{N} e^{2\pi i \kappa_j x_j/L} e^{2\pi i E_\kappa/L}}, \quad \kappa_N \geq 0 \quad (2.4) \]

where \( s^{\lambda}_{\kappa} \) is a Jack symmetric polynomial with normalization chosen so that the coefficient of the monomial symmetric function

\[ \sum_{\text{symmetric combination}} e^{2\pi i \kappa_1 x_1/L} e^{2\pi i \kappa_2 x_2/L} \ldots e^{2\pi i \kappa_N x_N/L} \quad (2.5) \]

in its power series expansion is unity [12]. (In [8] we used the notation \( C^{(1/\lambda)}_\kappa \) of Kaneko [13] for the Jack polynomial, which has a different normalization to \( s^{\lambda}_{\kappa} \); for our purposes below the latter is more convenient.) Also

\[ \kappa - \kappa_N := (\kappa_1 - \kappa_N, \kappa_2 - \kappa_N, \ldots, \kappa_N - \kappa_N) \quad (2.6) \]

2.2 Eigenfunction expansion of \( D(x, t) \)

To calculate the ground state dynamical density-density correlation \( D(x, t) \) Ha introduces the eigenstates \( | \kappa > \) as a complete set:

\[ D(x, t) := \langle 0 | \rho(x, t) \rho(0, 0) | 0 > - \rho^2 = \sum_{\kappa, \kappa \neq 0} \frac{\langle 0 | \rho(x) | \kappa > | \kappa | \rho(0) | 0 >}{\langle \kappa | \kappa > | 0 >} e^{-i(E_\kappa - E_0)t/\hbar} = \sum_{\kappa, \kappa \neq 0} | < 0 | \rho(0) | \kappa > |^2 e^{2\pi i |\kappa| x/L - i(E_\kappa - E_0)t/\hbar} \quad (2.7) \]

where

\[ |\kappa| := \sum_{j=1}^{N} \kappa_j \quad (2.8a) \]

\[ E_\kappa - E_0 = \left( \frac{2\pi}{L} \right)^2 \sum_{j=1}^{N} (\kappa_j^2 + \lambda \kappa_j (N + 1 - 2j)) \quad (2.8b) \]

and

\[ \rho(x, t) := e^{-iHt/\hbar} \rho(x) e^{iHt/\hbar} \quad (2.8c) \]

with

\[ \rho(x) = \sum_{j=1}^{N} \delta(x - x_j) \quad (2.8d) \]

Ha then quotes the value of \( < \kappa | \kappa > \), for \( \kappa_N \geq 0 \), from Macdonald [14]. With our choice of normalization for the Jack symmetric polynomial, from Kadell [12], for \( \kappa_N \geq 0 \)

\[ < \kappa | \kappa > = L^N N! \frac{f^\lambda_N(\kappa)}{f^\lambda_N(\kappa)} \quad (2.9) \]
where
\[ f_N^{(\lambda)}(\kappa) = \prod_{1 \leq i < j \leq N} ((j - i)\lambda + \kappa_i - \kappa_j) \lambda \] (2.10a)
and
\[ \bar{f}_N^{(\lambda)}(\kappa) = \prod_{1 \leq i < j \leq N} (1 - \lambda + (j - i)\lambda + \kappa_i - \kappa_j) \lambda \] (2.10b)
with
\[ (a)_\lambda := \frac{\Gamma(a + \lambda)}{\Gamma(a)} \] (2.10c)

The cases \( \kappa_N < 0 \) are not mentioned by Ha. In fact provided the choice of normalization implied by \( s_\lambda^{(\kappa)} \) is made, the following result is valid.

**Proposition 1**

For \( \kappa_N < 0 \)
\[ \langle \kappa | \kappa > = | \hat{\kappa} | \hat{\kappa} > \] (2.11a)
where
\[ \hat{\kappa} = (-\kappa_N, -\kappa_{N-1}, \ldots, -\kappa_1) \] (2.11b)

**Proof**

By definition
\[ \langle \kappa | \kappa > := \left( \prod_{l=1}^{N} \int_{0}^{L} dx_l \right) | s_\lambda^{(\kappa)}(e^{2\pi ix_1}/L, \ldots, e^{2\pi ix_N}/L) |^2 \]
\[ \times \prod_{1 \leq j < k \leq N} |e^{2\pi ix_j}/L - e^{2\pi ix_k}/L|^{2\lambda} \] (2.12)

But [12]
\[ s_\lambda^{(\kappa)}(e^{2\pi ix_1}/L, \ldots, e^{2\pi ix_N}/L) = s_\lambda^{(\hat{\kappa})}(e^{-2\pi ix_1}/L, \ldots, e^{-2\pi ix_N}/L) \] (2.13)
where the convention, in the case \( \kappa_1 > 0 \),
\[ s_\lambda^{(\hat{\kappa})}(t_1, \ldots, t_N) := \prod_{l=1}^{N} t_l^{\kappa_1} s_\lambda^{(\kappa_1)}(t_1, \ldots, t_N) \] (2.14)
is to be adopted (the formula (2.13) is special to the particular normalization of \( s_\lambda^{(\kappa)} \)). The result (2.11a) follows immediately.

**Remark**: For \( \kappa_N < 0 \), from (2.4), it is generally true that
\[ \langle \kappa | \kappa > = \langle \kappa - \kappa_N | \kappa - \kappa_N > \]

Proposition 1 is useful in the cases when both \( \kappa_N < 0 \) and \( \kappa_1 \leq 0 \). In such cases all parts of \( \hat{\kappa} \) are non-negative so there is an equality between the norms of states \( | \hat{\kappa} > \) labelled by partitions with non-negative parts, and states \( | \kappa > \) labelled by partitions with non-positive parts.

To calculate the inner product \( \langle 0 | \rho(0) | \kappa > \), Ha uses an expansion of \( \rho(x) \) in terms of Jack symmetric polynomials, and a fundamental integration formula for Jack symmetric polynomials (see e.g. [12]). Again only the cases with \( \kappa_N \geq 0 \) were considered. Let us give an alternative derivation of the evaluation of the inner product, applicable for all \( | \kappa > \).
Proposition 2

With $\kappa = -c + \mu$, ($\mu \neq c$) $\mu_N \geq 0$, $c \in \mathbb{Z}_{\geq 0}$

$$<0|\rho(0)|\kappa> = N L^{N-1} \left( \prod_{l=1}^{N} \int_{0}^{1/2} d\theta_l e^{-2\pi i \theta_l c} |1 - e^{2\pi i \theta_l}|^{2\lambda} \right) s_\mu^\lambda(1, e^{2\pi i \theta_2}, \ldots, e^{2\pi i \theta_N})$$

$$\times \prod_{1 \leq j < k \leq N} |e^{2\pi i \theta_k} - e^{2\pi i \theta_j}|^{2\lambda}$$

$$= N L^{N-1} (-1)^{Nc} (Nc - |\mu|) N! f_N^\lambda(\mu) \prod_{j=1}^{N} \Gamma((N - j)\lambda + 1)$$

$$\times \lim_{\epsilon \to 0} \frac{1}{\epsilon} \prod_{j=1}^{N} \Gamma(-c - \epsilon + 1 + (N - j)\lambda + \mu_j) \Gamma(c + \epsilon + 1 + (j - 1)\lambda - \mu_j)$$

(2.15)

Proof

We start with the fundamental integration formula for Jack polynomials, written in the form of ref.[12] (with extension to real $a, b, \lambda$ following [15])

$$\left( \prod_{l=1}^{N} \int_{-1/2}^{1/2} d\theta_l e^{\pi i \theta_l (a - b)} |1 - e^{2\pi i \theta_l}|^{(a + b)} \right) s_\mu^\lambda(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_N})$$

$$\times \prod_{1 \leq j < k \leq N} |e^{2\pi i \theta_k} - e^{2\pi i \theta_j}|^{2\lambda}$$

$$= N! f_N^\lambda(\mu) (-1)^{\mu} \prod_{j=1}^{N} \frac{\Gamma(a + b + 1 + (N - j)\lambda)}{\Gamma(a + b + 1 + (N - j)\lambda + \mu_j)} \frac{\Gamma(b + 1 + (j - 1)\lambda - \mu_j)}{\Gamma(b + 1 + (j - 1)\lambda - \mu_j)}$$

(2.16)

We make the substitutions $a = -b$, then $b = c + \epsilon$, where $c \in \mathbb{Z}_{\geq 0}$, and next take the limit $\epsilon \to 0$. On the l.h.s. we expand the exponentials involving $c$:

$$\prod_{l=1}^{N} e^{-2\pi i \theta_l c} = 1 - 2\pi i \epsilon \sum_{l=1}^{N} \theta_l + O(\epsilon^2)$$

and thus obtain

$$<0|\kappa> = -2\pi i \epsilon \left( \prod_{l=1}^{N} \int_{-1/2}^{1/2} d\theta_l e^{-2\pi i \theta_l c} \right) \left( \sum_{l=1}^{N} \theta_l \right) s_\mu^\lambda(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_N})$$

$$\times \prod_{1 \leq j < k \leq N} |e^{2\pi i \theta_k} - e^{2\pi i \theta_j}|^{2\lambda} + O(\epsilon^2)$$

(2.17)

Since we are assuming $\kappa \neq 0$,

$$<0|\kappa> = 0,$$

(2.18)

by orthogonality of the ground state with the excited states. In the $O(\epsilon)$ term of (2.17), we can use the symmetry of the integrand to replace $\sum_{l=1}^{N} \theta_l$ by $N\theta_1$, and then use the subsequent periodicity of the integrand in $\theta_2, \ldots, \theta_N$ to change variables $\theta_j \to \theta_j + \theta_1$ ($j = 2, \ldots, N$). Noting

$$s_\mu^\lambda(e^{2\pi i \theta_1}, e^{2\pi i (\theta_2 + \theta_1)}, \ldots, e^{2\pi i (\theta_N + \theta_1)}) = e^{2\pi i \theta_1 |\mu|} s_\mu^\lambda(1, e^{2\pi i \theta_2}, \ldots, e^{2\pi i \theta_N})$$

the l.h.s. of (2.16) then reads

$$-2\pi i \epsilon \left( \int_{-1/2}^{1/2} d\theta_1 e^{-2\pi i \theta_1 (Nc - |\mu|)} \right) \left( \prod_{l=2}^{N} \int_{-1/2}^{1/2} d\theta_l e^{-2\pi i \theta_l c} |1 - e^{2\pi i \theta_l}|^{2\lambda} \right)$$

4
\[ x \tilde{s}_\lambda^\kappa(1, e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_N}) \prod_{2 \leq j < k \leq N} |e^{2\pi i \theta_k} - e^{2\pi i \theta_j}|^{2\lambda} + O(e^2) \]  
\quad (2.19)

After computing the integral over \( \theta_1 \), and applying the same operations to the r.h.s. of (2.16), the result (2.15) then follows.

Due to the factor \( 1/(\epsilon \prod_{j=1}^{N} \Gamma(\epsilon + 1 + (j-1)\gamma - \mu_j)) \) on the r.h.s. of (2.15), we immediately observe from Proposition 1

Corollary 1 (Ha)
For \( c = 0 \) and \( \lambda = p/q \) (\( p \) and \( q \) relatively prime positive integers), \( < 0|\rho(0)|\kappa > \) is non-zero if and only if

\[ \kappa = (\alpha_1, \ldots, \alpha_q, p_1, \ldots, p, 1, \ldots, 1, 0, \ldots, 0) \]  
\quad (2.20)

where

\[ \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_q \quad \text{and} \quad q + \sum_{j=1}^{p} \beta_j \leq N \]  
\quad (2.21)

Remark: The integers \( \alpha_1, \ldots, \alpha_q \) are said to label quasi-particle excitations while the integers \( \sum_{j=1}^{k} \beta_j \ (k = 1, \ldots, p) \) are said to label quasi-hole excitations.

In the cases \( c > 0 \), we can deduce from Proposition 1 a similar result.

Corollary 2
(i) With \( \kappa = -c + \mu, c \in \mathbb{Z}^+, \kappa_N = 0 \) and with \( \lambda \) as in Corollary 1, for \( < 0|\rho(0)|\kappa > \) to be non-zero \( \mu \) must be of the form

\[ \mu = (c, \ldots, c, c-1, \ldots, c-1, \ldots, c-p, \ldots, c-p, \alpha_{q-1}, \ldots, \alpha_1, 0) \]  
\quad (2.22)

(ii) In the cases (2.22),

\[ < 0|\rho(0)|\kappa >= < 0|\rho(0)|\hat{\kappa} > \]  
\quad (2.23)

where

\[ \hat{\kappa} = (c, c - \alpha_1, \ldots, c - \alpha_{q-1}, p_1, \ldots, p, 1, \ldots, 1, 0, \ldots, 0) \]  
\quad (2.24)

Proof
(i) Using the condition \( \kappa_N = 0 \) and noting that

\[ \lim_{\epsilon \to 0} \frac{1}{\epsilon \Gamma(-c - \epsilon + 1)} = (-1)^c \Gamma(c) \]

the r.h.s. of (2.15) can be written

\[ NL^{N-1}(-1)^{(N+1)c} \Gamma(c)(Nc - |\mu|)N! f_N^\lambda(\mu) \prod_{j=1}^{N} \Gamma(1 + (N - j)\lambda - \mu_j) \prod_{j=1}^{N-1} \frac{1}{\Gamma(-c + 1 + (N - j)\lambda + \mu_j)} \]  
\quad (2.25)

We see immediately from the factor \( 1/\Gamma(c + 1 - \mu_1) \) that for (2.25) to be non-zero we require

\[ \kappa_1 \leq c \quad \text{and thus} \quad 0 \leq \kappa_j \leq c \ (j = 2, \ldots, N - 1) \]  
\quad (2.26)

Assuming this condition, we see from the second product that (2.25) is non-zero provided

\[ -c + 1 + (N - j)\lambda + \mu_j \neq 0, -1, -2, \ldots (j = 2, \ldots, N - 1) \]  
\quad (2.27)
With \( \lambda = p/q \) we see that (2.27) holds for \( j = N - 1, \ldots, N - q + 1 \) so we can choose
\[
\mu_{N-1} = \alpha_1, \ldots, \mu_{N-q+1} = \alpha_{q-1},
\]
(2.28)
the only constraint being that
\[
0 \leq \alpha_1 \leq \ldots \leq \alpha_{q-1} \leq b
\]
(2.29)
In the next case, \( j = N - q \), (2.27) reads
\[
-c + 1 + p + \kappa_{N-q} \neq 0, -1, \ldots
\]
(2.30)
Hence we must have
\[
c \geq \kappa_{N-q} \geq c - p
\]
(2.31)
Assuming this condition and (2.26), (2.27) then holds for all remaining \( j \). The partitions giving a non-zero value to \( < 0|\rho(0)|\kappa > \) are therefore given by (2.20).

(ii) To prove this statement we substitute \( \mu \) as given by (2.22) in the r.h.s. of (2.15), and then take the complex conjugate of the integrand, which we can do without changing the value of the integral since the latter is real. The stated result then follows from (2.14) and the fact that \( s_{\mu}^{\lambda} \) is a homogeneous function.

Remark: One consequence of Corollaries 1 and 2 is that the non-zero terms in (2.7) have \( \kappa \) with all parts non-negative or all parts non-positive. Furthermore Corollary 2, together with Proposition 1, (2.8a) and (2.8b) show that in (2.7) a term having \( \kappa \) with all parts non-positive gives the same contribution as the term corresponding to \( \hat{\kappa} \), which has all parts non-negative, except that \( \exp(2\pi i|\kappa| x/L) \) needs to be replaced by \( \exp(-2\pi i|\kappa| x/L) \).

3 Normalization of the retarded single particle Green’s function

3.1 An exact result of Ha

From the theory of the above section, with \( \lambda = p/q \), Ha has reported that an exact closed form expression for \( D(x,t) \) can be obtained in the thermodynamic limit. This exact expression is in the form of a \( p + q \) dimensional integral. Ha has also announced a similar evaluation of the retarded single particle Green’s function describing hole propagation (for \( \lambda = p \) this formula has also been independently derived by the present author [9]):
\[
\lim_{N,L \to \infty \atop N/L = p} < 0|\hat{\psi}(x,t)\psi(0,0)|0 >
= \rho De^{-i\pi\lambda\rho x} \left( \prod_{i=1}^{q-1} \int_{0}^{\infty} dx_i \right) \left( \prod_{j=1}^{p} \int_{0}^{1} dy_j \right) F(q - 1, p, \lambda|x_i, y_j|) \times e^{i(Q_{p,q-1}x-E_{p,q-1}t)}
\]
(3.1a)
where the momentum \( Q \) and the energy \( E \) variables are given by
\[
Q_{p,q} := 2\pi \rho \left( \sum_{i=1}^{q} x_i + \sum_{j=1}^{p} y_j \right)
\]
\[
E_{p,q} := (2\pi \rho)^2 \left( \sum_{i=1}^{q} \epsilon_p(x_i) + \sum_{j=1}^{p} \epsilon_H(y_j) \right)
\]
(3.1b)
with
\[ \epsilon_P(x) = x(x + \lambda) \]
\[ \epsilon_H(y) = \lambda y(1 - y), \]  
(3.1c)

the form factor \( F \) is given by
\[ F(q, p, \lambda|x_i, y_j\rangle) = \prod_{i=1}^{q} \prod_{j=1}^{p} (x_i + \lambda y_j)^{-2} \prod_{i<j} |x_i - x'_i|^{2\lambda} \prod_{j<j'} |y_j - y'_j|^{2/\lambda} \]
\[ \prod_{i=1}^{q} (\epsilon_P(x_i))^{1-\lambda} \prod_{j=1}^{p} (\epsilon_H(y_j))^{1-1/\lambda} \]  
(3.1d)

and the normalization \( D \) is given by
\[ D = \frac{\lambda^{2p(q-1)} \Gamma(1 + \lambda)}{\lambda^{2p+1}(q-1)!p!} A(q - 1, p, \lambda) \]  
(3.1e)

with
\[ A(m, n, \lambda) := \frac{\Gamma^m(\lambda)\Gamma^n(1/\lambda)}{\prod_{i=1}^{m} \Gamma^2(p - \lambda(i - 1)) \prod_{j=1}^{n} \Gamma^2(1 - (j - 1)/\lambda)} \]  
(3.1f)

In this section we focus attention on a corollary of (3.1), following from the normalization condition
\[ <0|\psi^1(x, t)\psi(0, 0)|0 > = \rho, \]  
(3.2)

which is the exact evaluation of the multidimensional integral
\[ \left( \prod_{i=1}^{q} \int_0^\infty dx_i \right) \left( \prod_{j=1}^{p} \int_0^1 dy_j \right) F(q - 1, p, \lambda|x_i, y_j\rangle) = 1/D \]  
(3.3)

With \( q = 1 \) (3.3) is a special case of the Selberg integral [16] (see e.g. [12] for a more accessible reference). However for \( q > 1 \) the value of the integral is not well known. In the next subsection we will show that in this case (3.3) is a special case of an integral implicit in the work of Dotsenko and Fateev [17].

### 3.2 A generalization of the Selberg integral

Dotsenko and Fateev [17] consider the multidimensional integral
\[ J_{nm}(\alpha, \beta; \hat{\rho}) := \left( \prod_{i=1}^{n} \int_{C_i} dt_i \right) \left( \prod_{j=1}^{m} \int_{S_j} d\tau_j \right) f_{nm}(\{t_i\}, \{\tau_j\}, \alpha, \beta; \hat{\rho}) \]  
(3.4a)

where
\[ f_{nm}(\{t_i\}, \{\tau_j\}, \alpha, \beta; \hat{\rho}) := \prod_{i=1}^{n} t_i^{\alpha'} (1 - t_i)^{\beta'} \prod_{j=1}^{m} \tau_j^{\alpha} (1 - \tau_j)^{\beta} \frac{\prod_{i<j'} (t_i - t_{i'})^{2\beta} \prod_{j<j'} (\tau_j - \tau_{j'})^{2\hat{\rho}}}{\prod_{i=1}^{n} \prod_{j=1}^{m} (\tau_j - t_i)^2}, \]  
(3.4b)

the multivalued product \( \prod_{i<j'} (t_i - t_{i'})^{2\beta} \) is defined so that when all the variables \( t_i \) are real and ordered \( t_1 > t_2 > \ldots > t_n \), the phases of the product are zero (the product \( \prod_{j<j'} (\tau_j - \tau_{j'})^{2\hat{\rho}} \) is defined similarly), and the contours \( C_i, S_j \) are as in figure 1a. The parameters \( \alpha, \alpha', \beta, \beta' \) and \( \hat{\rho}, \hat{\rho}' \) are subject to the relations
\[ \alpha' = -\hat{\rho} \alpha \quad \beta' = -\hat{\rho}' \beta \quad \hat{\rho}' = 1/\hat{\rho} \]  
(3.5)
Interpolating between the formulas (A.15) and (A.16) of [17] we can read off that

\[ J_{nm}(\alpha, \beta; \rho) = \prod_{j=0}^{n-1} \frac{\sin \pi(2 - 2m + \alpha' + \beta' + \rho'(n - 1 + j))}{\sin \pi(1 + \alpha' + j\rho')} \times \tilde{J}_{nm}^{(n)}(\alpha, \beta; \tilde{\rho}) \]  

(3.6a)

where

\[ \tilde{J}_{nm}^{(n)}(\alpha, \beta; \tilde{\rho}) := \left( \prod_{i=1}^{n} \int_{K_i} dt_i \right) \left( \prod_{j=1}^{m} \int_{S_j} d\tau_j \right) \tilde{f}_{nm}(t_i, \tau_j, \alpha, \beta; \tilde{\rho}) \]  

(3.6b)

with \( \tilde{f}_{nm} \) defined as in (3.4b) except that each factor \((1 - t_i)^{-l} \) is to be replaced by \((t_i - 1)^{-l} \), and the contours \( K_i \) are as in figure 1b. Furthermore, all the contours on the l.h.s. and r.h.s. of (3.6a) can be collapsed onto the real axis. Extra "phase factors", due to the products

\[ \prod_{i<j} (t_i - t_j)^{2\tilde{\rho}} \quad \text{and} \quad \prod_{j<j'} (\tau_j - \tau_{j'})^{2\tilde{\rho}} \]

result, but are the same for both sides of the equation and so cancel. We thus have

\[ \tilde{I}_{nm}(\alpha, \beta; \tilde{\rho}) = \prod_{j=0}^{n-1} \frac{\sin \pi(1 + \alpha' + j\tilde{\rho}')}{\sin \pi(2 - 2m + \alpha' + \beta' + \tilde{\rho}'(n - 1 + j))} I_{nm}(\alpha, \beta; \tilde{\rho}) \]  

(3.7a)

where

\[ \tilde{I}_{nm}(\alpha, \beta; \tilde{\rho}) := \left( \prod_{i=1}^{n} \int_{1}^{\infty} dt_i \right) \left( \prod_{j=1}^{m} \int_{0}^{1} d\tau_j \right) |f_{nm}(t_i, \tau_j, \alpha, \beta; \tilde{\rho})| \]  

(3.7b)

and

\[ I_{nm}(\alpha, \beta; \tilde{\rho}) := P \left( \prod_{i=1}^{n} \int_{0}^{1} dt_i \right) \left( \prod_{j=1}^{m} \int_{0}^{1} d\tau_j \right) |f_{nm}(t_i, \tau_j, \alpha, \beta; \tilde{\rho})| \]  

(3.7c)

where \( P \) stands for the principal value integral. The integral (3.7c) is evaluated by (A.35) of [17] (there is a minor misprint in this equation: in the final two products the lower terminals should be 0 instead of 1), so inserting its value in (3.7a) and using the formula

\[ \Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z} \]

we conclude

\[ \tilde{I}_{nm}(\alpha, \beta; \tilde{\rho}) = m!n!\tilde{\rho}^{2nm} \prod_{l=0}^{n-1} \prod_{j=1}^{m} \frac{\Gamma(l\tilde{\rho})}{\Gamma(\rho')} \frac{\Gamma(j\tilde{\rho} - n)}{\Gamma(\tilde{\rho})} \times \prod_{l=0}^{n-1} \frac{\Gamma(1 + \beta' + l\tilde{\rho}')\Gamma(-1 + 2m - \alpha' - \beta' - (n - 1 + l)\tilde{\rho}')}{\Gamma(-\alpha' - l\tilde{\rho}')} \times \prod_{j=0}^{m-1} \frac{\Gamma(1 - n + \alpha + j\tilde{\rho})\Gamma(1 - n + \beta + (m - 1 + j)\tilde{\rho})}{\Gamma(2 - n + \alpha + \beta + (m - 1 + j)\tilde{\rho})} \]  

(3.8)

This is the sought generalization of the Selberg integral.

It is straightforward to express the l.h.s. of (3.3) in terms of this integral:

\[ \left( \prod_{i=1}^{q-1} \int_{0}^{\infty} dx_i \right) \left( \prod_{j=1}^{p} \int_{0}^{1} dy_j \right) F(q - 1, p, \lambda; \{x_i, y_j\}) = \frac{1}{\lambda^{pq-1}} \tilde{I}_{q-1,p}(q/p - 1, q/p - 1, q/p) \]  

(3.9)

That the evaluations given by (3.3) and (3.8) agree in (3.9) is demonstrated in the appendix.
4 Conjecture for the static two-particle distribution function

As our final result we will combine our knowledge of the integrals (2.15) and (3.8) with a conjecture of Ha to conjecture an integral formula for the static two particle distribution function

\[
\rho^{(2)}(x) := TL \frac{< 0| \sum_{j \neq k} \delta(x - x_j) \delta(0 - x_k)|0>}{<0|0>},
\]

for rational coupling, where TL denotes the thermodynamic limit. The conjecture states that the so called form factor, which for our purpose is the function \( F \) in the integrand of (3.1a), is given by (3.1d) for any two point correlation function in which the intermediate states involve only \( q \) quasi-particles and \( p \) quasi-holes (recall the remark after Corollary 1).

To apply the conjecture, we note that for a system of \( N + 2 \) particles

\[
< 0| \sum_{j \neq k} \delta(x - x_j) \delta(0 - x_k)|0>
\]

\[
= c_{NL}(\lambda) \sin \pi x/L |2^\lambda \left( \prod_{l=1}^{N} \int_{0}^{1} dx_l |1 - e^{2\pi i x_l/L}|2^\lambda |1 - e^{2\pi i (x_l - x)/L}|2^\lambda \right) \times \prod_{1 \leq j < k \leq N} |e^{2\pi i x_k/L} - e^{2\pi i x_j/L}|^\lambda
\]

\[
= c_{NL}(\lambda) \sin \pi x/L |2^\lambda \sum_{\kappa} | < \kappa | \Phi > |^2 e^{2\pi i x|\kappa|/L}
\]

where

\[
|\Phi > := \prod_{l=1}^{N} |1 - e^{2\pi i x_l/L}|2^\lambda \prod_{1 \leq j < k \leq N} |e^{2\pi i x_k/L} - e^{2\pi i x_j/L}|^\lambda
\]

From Corollary 1 we see that, with \( \lambda = p/q < \kappa | \Phi > \) is non-zero only if \( \kappa \) consists of \( 2(q - 1) \) quasi-particle and \( 2p \) quasi-hole labels. For example, if \( \lambda = 1/2 \), the states giving a non-zero contribution to (4.2) are, in the cases \( \kappa_N \geq 0 \), of the form

\[
\kappa = (q_1, q_2, 2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0)
\]

\[
\text{with } q_1 \geq q_2 \quad \text{and} \quad p_2 + p_1 \leq N - 2.
\]

The conjecture of Ha can now be applied to predict

\[
\rho^{(2)}(x) = \rho^2 C(\lambda)(\rho x)^D \left( \prod_{i=1}^{2(q-1)} \int_{0}^{\infty} dx_i \right) \left( \prod_{j=1}^{2p} \int_{0}^{1} dy_j \right) F(2(q - 1), 2p, \lambda |\{x_i, y_j\}| \cos(Q_{2p,2(q-1)}x)
\]

It remains to calculate the normalization \( C(\lambda) \). For this purpose we recall from our previous work [6,eq.(5.23b)] the small-\( x \) result

\[
\lim_{x \to 0^+} \frac{\rho^{(2)}(x)}{\rho^2(\rho x)^{2\lambda}} = A(\lambda)
\]

where

\[
A(\lambda) := (2\pi \lambda)^{2\lambda} \frac{(\lambda!)^3}{(2\lambda)!(3\lambda)!}
\]
Furthermore

\[
\left(\prod_{i=1}^{2(q-1)} \int_0^\infty dx_i \right) \left(\prod_{j=1}^{2p} \int_0^1 dy_j \right) F(2(q - 1), 2p, \lambda \{x_i, y_j\})
\]

\[= \lambda^{-4pq + 2p/q} \tilde{I}_{2(q-1), 2p}(q/p - 1, q/p - 1, q/p) \quad (4.7)
\]

where \(\tilde{I}\) is defined by (3.7b) and evaluated by (3.8). Hence the normalization is given explicitly by

\[C(\lambda) = \frac{A(\lambda)}{\lambda^{-4pq + 2p/q} \tilde{I}_{2(q-1), 2p}(q/p - 1, q/p - 1, q/p)} \quad (4.8)
\]

In the special case \(\lambda = p\) the conjecture (4.5) agrees with the known exact evaluation [7].

**Acknowledgements**

Some of this work was done during a visit to the Laboratoire de Physique Théorique at Orsay organised by B. Jancovici, whom I thank for his efforts. Financial support was provided by the CNRS and the ARC. Also I thank T. Miwa for a critical reading of the manuscript.
Appendix

According to (3.3) and (3.8), (3.9) says that for $\lambda = p/q$,

$$
\frac{\lambda^{p-q+1}(q-1)!p! \prod_{l=1}^{q-1} l \prod_{j=1}^{p} j \prod_{i=1}^{q-1} (1 - (j - 1)/\lambda)}{\lambda^{2p(q-1)} \Gamma(1 + \lambda) \Gamma^{q-1}(\lambda) \Gamma^{p}/(1/\lambda)}
$$

$$
= (q-1)!p!\lambda^{-2(q-1)p} \prod_{l=1}^{q-1} \frac{\Gamma(l)\lambda l}{\Gamma(1/\lambda)} \prod_{j=1}^{p} \frac{\Gamma(j/\lambda - (q-1))}{\Gamma(1/\lambda)}
$$

$$
\times \prod_{l=0}^{q-2} \frac{\Gamma((l+1)\lambda)\Gamma(1+p-l\lambda)}{\Gamma(1-(l+1)\lambda)} \prod_{j=0}^{p-1} \frac{\Gamma(1-q+j+1)/\lambda)}{\Gamma(1+j+1/\lambda)}
$$

(A1)

Here we will show how to reduce the r.h.s. of this identity to the l.h.s. First, by replacing $l \to l+1$ and $j \to j+1$ in the second last and last products, the r.h.s reads

$$
(q-1)!p!\lambda^{-2(q-1)p} \frac{\Gamma(1+p)}{\Gamma(1+p-(q-1)\lambda)} \prod_{l=1}^{q-1} \frac{\Gamma^{2}(l)\lambda l\gamma(1+p-l\lambda)}{\Gamma(1-l\lambda)\Gamma(1-\lambda)} \prod_{j=1}^{p} \frac{\Gamma^{3}(1-q+j+1/\lambda)}{\Gamma(1+j+1/\lambda)(1/\lambda)}
$$

(A2)

Now

$$
\prod_{l=1}^{q-1} \frac{\Gamma(1+p-l\lambda)}{\Gamma(1-l\lambda)} = \prod_{l=1}^{q-1} (p-l\lambda)(p-1-l\lambda) \ldots (1-l\lambda)
$$

$$
= \lambda^{p(q-1)} \prod_{j=1}^{p} \frac{\Gamma(j/\lambda)}{\Gamma(j/\lambda - (q-1))}
$$

(A3)

and

$$
\prod_{j=1}^{p} \frac{1}{\Gamma(1+j/\lambda)} = \frac{\lambda^{p} \prod_{j=1}^{p} \frac{1}{\Gamma(j/\lambda)}}{p!}
$$

(A4)

Substituting (A3) and (A4) in (A2) reproduces the l.h.s. of (A1), as required.
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