A CONSTRUCTION OF BALANCED DEGREE-MAGIC GRAPHS

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Abstract. A graph \( G \) is called degree-magic if it admits a labelling of the edges by integers \( 1, 2, \ldots, |E(G)| \) such that the sum of the labels of the edges incident with any vertex \( v \) is equal to \( (1 + |E(G)|) \deg(v)/2 \). Degree-magic graphs extend supermagic regular graphs. In this paper, a new construction of balanced degree-magic graphs is introduced.

Keywords: supermagic graphs; degree-magic graphs; cycle graphs.

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1. INTRODUCTION

The finite simple graphs and multigraphs without loops and isolated vertices are considered. If \( G \) is a graph, then \( V(G) \) and \( E(G) \) stand for the vertex set and the edge set of \( G \), respectively. Cardinalities of these sets are called the order and the size of \( G \). For any integers \( p \) and \( q \), the set of all integers \( z \) satisfying \( p \leq z \leq q \) is indicated by \([p, q]\).

Let a graph \( G \) and a mapping \( f \) from \( E(G) \) into the set of positive integers be given. The index-mapping of \( f \) is the mapping \( f^* \) from \( V(G) \) into the set of positive integers defined by

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\[ f^*(v) = \sum_{e \in E(G)} \eta(v, e) f(e) \quad \text{for every} \quad v \in V(G), \]

where \( \eta(v, e) \) is equal to 1 when \( e \) is an edge incident with a vertex \( v \), and 0 otherwise. An injective mapping \( f \) from \( E(G) \) into the set of positive integers is called a \textit{magic labelling} of \( G \) for an \textit{index} \( \lambda \) if its index-mapping \( f^* \) satisfies

\[ f^*(v) = \lambda \quad \text{for all} \quad v \in V(G). \]

A magic labelling \( f \) of \( G \) is called a \textit{supermagic labelling} if the set \( \{ f(e) : e \in E(G) \} \) consists of consecutive positive integers. A graph \( G \) is said to be \textit{supermagic} (magic) whenever there exists a supermagic (magic) labelling of \( G \).

A bijective mapping \( f \) from \( E(G) \) into \([1, |E(G)|]\) is called a \textit{degree-magic labelling} (or only \textit{d-magic labelling}) of a graph \( G \) if its index-mapping \( f^* \) satisfies

\[ f^*(v) = \frac{1 + |E(G)|}{2} \deg(v) \quad \text{for all} \quad v \in V(G). \]

A \textit{d-magic} labelling \( f \) of \( G \) is called \textit{balanced} if for all \( v \in V(G) \) it holds

\[ |\{e \in E(G) : \eta(v, e) = 1, f(e) \leq |E(G)|/2\}| = |\{e \in E(G) : \eta(v, e) = 1, f(e) > |E(G)|/2\}|. \]

A graph \( G \) is said to be \textit{degree-magic} (balanced \textit{degree-magic}) (or only \textit{d-magic}) when a \textit{d}-magic (balanced \textit{d}-magic) labelling of \( G \) exists.

The concept of magic graphs was put forward by Sedláček [10]. Later, supermagic graphs were introduced by Stewart [11]. Besides, a new constructuion of supermagic complements of some graphs was recommended [9]. Moreover, the notion of degree-magic graphs was then suggested by Bezegová and Ivančo [1] as an extension of supermagic regular graphs. Recently, numerous papers are published on degree-magic and supermagic graphs, see [2, 3, 4, 5, 6, 7, 8] for more complehensive references.

Let one recall the basic properties of \textit{d}-magic graphs that will be used in the next.

\textbf{Theorem 1.1.} [1] Let \( G \) be a regular graph. Then \( G \) is supermagic if and only if it is \textit{d}-magic.
Theorem 1.2. [1] Let $H_1$ and $H_2$ be edge-disjoint subgraphs of a graph $G$ which form its decomposition. If $H_1$ is $d$-magic and $H_2$ is balanced $d$-magic, then $G$ is a $d$-magic graph. Moreover, if $H_1$ and $H_2$ are both balanced $d$-magic, then $G$ is a balanced $d$-magic graph.

2. Balanced Degree-Magic Graphs

An injective mapping $f$ from $E(G)$ into the set of positive integers is called a single-consecutive labelling (SC-labelling) of a graph $G$ if its index-mapping $f^*$ satisfies

$$f^*(V(G)) = [a, a + |V(G)| - 1] \text{ for some integer } a.$$ 

Let $f_i, i \in \{1, 2\},$ be a SC-labelling of a graph $G_i$. The labellings $f_1$ and $f_2$ are called complementary if $f_1(E(G_1)) \cap f_2(E(G_2)) = \emptyset$ and $f_1(E(G_1)) \cup f_2(E(G_2)) = [1, m]$, where $m = |E(G_1)| + |E(G_2)|$. The complementary labellings $f_1$ and $f_2$ are called 

balanced if all pairs of vertices $u \in V(G_1), v \in V(G_2)$ satisfy

$$|\{e \in E(G_1) : \eta(u, e) = 1, f_1(e) \leq \lfloor m/2 \rfloor\}|$$

$$+ |\{e \in E(G_2) : \eta(v, e) = 1, f_2(e) \leq \lfloor m/2 \rfloor\}|$$

$$= |\{e \in E(G_1) : \eta(u, e) = 1, f_1(e) > \lfloor m/2 \rfloor\}|$$

$$+ |\{e \in E(G_2) : \eta(v, e) = 1, f_2(e) > \lfloor m/2 \rfloor\}|.$$ 

Now, one is able to prove the following Proposition.

Proposition 2.1. Let $H_1$ and $H_2$ be spanning subgraphs of a graph $G$ which form its decomposition with vertices $v_1, v_2, \ldots, v_n$. Let $f$ be a SC-labelling of $H_1$ such that $f^*(v_1) < f^*(v_2) < \cdots < f^*(v_n)$ and let $g$ be a SC-labelling of $H_2$ such that $g^*(v_1) > g^*(v_2) > \cdots > g^*(v_n)$. If $f$ and $g$ are complementary, then $G$ is a supermagic graph.

Proof. Since $f$ is a SC-labelling of $H_1$ such that $f^*(v_i) = f^*(v_1) + (i - 1)$ and $g$ is a SC-labelling of $H_2$ such that $g^*(v_i) = g^*(v_1) - (i - 1)$ for all $i \in [1, n], f^*(v_i) + g^*(v_i) = f^*(v_1) + g^*(v_1)$. Now, consider a mapping $\varphi$ from $E(G)$ into the set of positive integers defined by

$$\varphi(v_i v_j) = \begin{cases} 
  f(v_i v_j) & : \, v_i v_j \in E(H_1), \\
  g(v_i v_j) & : \, v_i v_j \in E(H_2). 
\end{cases}$$
Obviously, \( \varphi^*(v_i) = f^*(v_i) + g^*(v_i) = f^*(v_1) + g^*(v_1) \). Since \( \varphi(E(G)) = f(E(H_1)) \cup g(E(H_2)) \) and the labellings \( f \) and \( g \) are complementary, \( \varphi \) is a supermagic labelling of \( G \). Therefore, \( G \) is a desired graph.

If the graph \( G \) in Proposition 2.1 is regular, then \( G \) is \( d \)-magic by Therorem 1.1. For balanced \( d \)-magic graphs, one can show the following assertion.

**Proposition 2.2.** Let \( H_1 \) and \( H_2 \) be spanning subgraphs of a regular graph \( G \) which form its decomposition with vertices \( v_1, v_2, \ldots, v_n \). Let \( f \) be a SC-labelling of \( H_1 \) such that \( f^*(v_1) < f^*(v_2) < \cdots < f^*(v_n) \) and let \( g \) be a SC-labelling of \( H_2 \) such that \( g^*(v_1) > g^*(v_2) > \cdots > g^*(v_n) \). If \( f \) and \( g \) are (balanced) complementary, then \( G \) is a (balanced) \( d \)-magic graph.

**Proof.** By using the same proof as Proposition 2.1, \( G \) is a supermagic graph. Because \( G \) is regular, \( G \) is \( d \)-magic by Theorem 1.1. Since \( f \) and \( g \) are balanced complementary, for each vertex \( v_i, i \in [1, n] \), of \( G \) it holds

\[
|\{ e \in E(G) : \eta(v_i, e) = 1, \varphi(e) \leq \lfloor |E(G)|/2 \rfloor \}| \\
= |\{ e \in E(H_1) : \eta(v_i, e) = 1, f(e) \leq \lfloor |E(G)|/2 \rfloor \}| \\
+ |\{ e \in E(H_2) : \eta(v_i, e) = 1, g(e) \leq \lfloor |E(G)|/2 \rfloor \}| \\
= |\{ e \in E(H_1) : \eta(v_i, e) = 1, f(e) > \lfloor |E(G)|/2 \rfloor \}| \\
+ |\{ e \in E(H_2) : \eta(v_i, e) = 1, g(e) > \lfloor |E(G)|/2 \rfloor \}| \\
= |\{ e \in E(G) : \eta(v_i, e) = 1, \varphi(e) > \lfloor |E(G)|/2 \rfloor \}|.
\]

Thus, \( \varphi \) is a balanced \( d \)-magic labelling of \( G \). That is, \( G \) is an expected graph.

The above two Propositions describe methods to construct supermagic graphs and \( d \)-magic graphs by using SC-labellings respectively. In order to use Proposition 2.2, one needs reasonable SC-labellings of some graphs.

**Lemma 2.3.** Let \( G \) be a cycle graph of order 4 with vertices \( v_1, v_2, v_3, v_4 \) and let \( k, h \) be positive integers. Then there are a SC-labelling \( f \) of \( G \) such that \( f(E(G)) = \{k, k+1, k+4, k+6\} \) and \( f^*(v_1) < f^*(v_2) < f^*(v_3) < f^*(v_4) \) and a SC-labelling \( g \) of \( G \) such that \( g(E(G)) = \{h, h+1, h+3, h+5\} \) and \( g^*(v_1) > g^*(v_2) > g^*(v_3) > g^*(v_4) \). Moreover, if \( k = 1 \) and \( h = 3 \), then the SC-labellings \( f \) and \( g \) are balanced complementary.
Proof. Consider a mapping $f$ from $E(G)$ into the set of positive integers given by

$$f(e) = \begin{cases} 
  k & : e = v_1v_3, \\
  k + 6 & : e = v_3v_4, \\
  k + 1 & : e = v_4v_2, \\
  k + 4 & : e = v_2v_1.
\end{cases}$$

It is easy to see that $f(E(G)) = \{k, k + 1, k + 4, k + 6\}$ and $f^*(v_1) < f^*(v_2) < f^*(v_3) < f^*(v_4)$. Hence, $f$ is a desired SC-labelling of $G$. Moreover, consider a mapping $g$ from $E(G)$ into the set of positive integers defined by

$$g(e) = \begin{cases} 
  h + 1 & : e = v_1v_3, \\
  h + 3 & : e = v_3v_4, \\
  h & : e = v_4v_2, \\
  h + 5 & : e = v_2v_1.
\end{cases}$$

One can see that $g(E(G)) = \{h, h + 1, h + 3, h + 5\}$ and $g^*(v_1) > g^*(v_2) > g^*(v_3) > g^*(v_4)$. Thus, $g$ is a required SC-labelling of $G$. Now, consider the case $k = 1$ and $h = 3$, one then has

$$f(e) = \begin{cases} 
  1 & : e = v_1v_3, \\
  7 & : e = v_3v_4, \\
  2 & : e = v_4v_2, \\
  5 & : e = v_2v_1,
\end{cases}$$

and

$$g(e) = \begin{cases} 
  4 & : e = v_1v_3, \\
  6 & : e = v_3v_4, \\
  3 & : e = v_4v_2, \\
  8 & : e = v_2v_1.
\end{cases}$$

Clearly, $f$ and $g$ are balanced complementary labellings. $\square$

**Lemma 2.4.** Let $G$ be a cycle graph of odd order $n \geq 3$ with vertices $v_1, v_2, \ldots, v_n$ and let $k, h$ be positive integers. Then there exist a SC-labelling $f$ of $G$ such that $f(E(G)) = [k, k + n - 1]$ and
\(f^*(v_1) < f^*(v_2) < \cdots < f^*(v_n)\) and a SC-labelling \(g\) of \(G\) such that \(g(E(G)) = [h, h + n - 1]\) and \(g^*(v_1) > g^*(v_2) > \cdots > g^*(v_n)\). Moreover, if \(k = 1\) and \(h = n + 1\), then the SC-labellings \(f\) and \(g\) are balanced complementary.

**Proof.** Consider a mapping \(f\) from \(E(G)\) into the set of positive integers given by

\[
f(e) = \begin{cases} 
  k + (n-1)/2 & : e = v_nv_1, \\
  k & : e = v_1v_2, \\
  k + (n-1)/2 + 1 & : e = v_2v_3, \\
  k + 1 & : e = v_3v_4, \\
  k + (n-1)/2 + 2 & : e = v_4v_5, \\
  k + 2 & : e = v_5v_6, \\
  \vdots \\
  k + (n-1)/2 - 1 & : e = v_{n-2}v_{n-1}, \\
  k + n - 1 & : e = v_{n-1}v_n.
\end{cases}
\]

One is able to check that \(f(E(G)) = [k, k + n - 1]\) and \(f^*(v_1) < f^*(v_2) < \cdots < f^*(v_n)\). Thus, \(f\) is a desired SC-labelling of \(G\). Besides, consider a mapping \(g\) from \(E(G)\) into the set of positive integers defined by

\[
g(e) = \begin{cases} 
  h + (n-1)/2 & : e = v_1v_n, \\
  h & : e = v_nv_{n-1}, \\
  h + (n-1)/2 + 1 & : e = v_{n-1}v_{n-2}, \\
  h + 1 & : e = v_{n-2}v_{n-3}, \\
  h + (n-1)/2 + 2 & : e = v_{n-3}v_{n-4}, \\
  h + 2 & : e = v_{n-4}v_{n-5}, \\
  \vdots \\
  h + (n-1)/2 - 1 & : e = v_3v_2, \\
  h + n - 1 & : e = v_2v_1.
\end{cases}
\]

One can get that \(g(E(G)) = [h, h + n - 1]\) and \(g^*(v_1) > g^*(v_2) > \cdots > g^*(v_n)\). Hence, \(g\) is a required SC-labelling of \(G\). Now, consider the case \(k = 1\) and \(h = n + 1\), one then gets
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\[
f(e) = \begin{cases} 
1 + (n-1)/2 : e = v_nv_1, \\
1 : e = v_1v_2, \\
2 + (n-1)/2 : e = v_2v_3, \\
2 : e = v_3v_4, \\
3 + (n-1)/2 : e = v_4v_5, \\
3 : e = v_5v_6, \\
\vdots \\
(n-1)/2 : e = v_{n-2}v_{n-1}, \\
n : e = v_{n-1}v_n,
\end{cases}
\]

and

\[
g(e) = \begin{cases} 
n + 1 + (n-1)/2 : e = v_1v_n, \\
n + 1 : e = v_nv_{n-1}, \\
n + 2 + (n-1)/2 : e = v_{n-1}v_{n-2}, \\
n + 2 : e = v_{n-2}v_{n-3}, \\
n + 3 + (n-1)/2 : e = v_{n-3}v_{n-4}, \\
n + 3 : e = v_{n-4}v_{n-5}, \\
\vdots \\
n + (n-1)/2 : e = v_3v_2, \\
2n : e = v_2v_1.
\end{cases}
\]

Evidently, f and g are balanced complementary labellings. \(\square\)

In the next results, one is able to prove some sufficient conditions for balanced \(d\)-magic graphs.

**Theorem 2.5.** Let \(G\) be a graph which can be decomposable into two spanning cycle subgraphs of order 4. Then \(G\) is a balanced \(d\)-magic graph.

**Proof.** Suppose that two spanning cycle subgraphs of \(G\) have vertices \(v_1, v_2, v_3, v_4\). Thus, by Lemma 2.3, there are two balanced complementary SC-labellings \(f, g\) of these cycles such that \(f^*(v_1) < f^*(v_2) < f^*(v_3) < f^*(v_4)\) and \(g^*(v_1) > g^*(v_2) > g^*(v_3) > g^*(v_4)\). Since these two
cycles are regular and form its decomposition, $G$ is a regular graph. Therefore, according to Proposition 2.2, $G$ is a balanced $d$-magic graph. □

Combining Theorem 1.2 and Theorem 2.5, one immediately has

**Corollary 2.6.** For any positive integer $k$, if a graph $G$ can be decomposable into $2k$ spanning cycle subgraphs of order 4, then $G$ is a balanced $d$-magic graph.

Joining Theorem 1.1 and Corollary 2.6, one absolutely has

**Corollary 2.7.** For any positive integer $k$, if a graph $G$ can be decomposable into $2k$ spanning cycle subgraphs of order 4, then $G$ is a supermagic graph.

**Theorem 2.8.** Let $G$ be a graph which can be decomposable into two spanning cycle subgraphs of odd order $n \geq 3$. Then $G$ is a balanced $d$-magic graph.

*Proof.* Assume that two spanning cycle subgraphs of $G$ of odd order $n \geq 3$ have vertices $v_1, v_2, \ldots, v_n$. Hence by Lemma 2.4, there are two balanced complementary SC-labellings $f, g$ of these cycles such that $f^*(v_1) < f^*(v_2) < \cdots < f^*(v_n)$ and $g^*(v_1) > g^*(v_2) > \cdots > g^*(v_n)$. It is clear that these two cycles are regular and they form its decomposition, so $G$ is a regular graph. Therefore, according to Proposition 2.2, $G$ is a balanced $d$-magic graph. □

Combining Theorem 1.2 and Theorem 2.8, one suddenly has

**Corollary 2.9.** For any positive integer $k$, if a graph $G$ can be decomposable into $2k$ spanning cycle subgraphs of odd order $n \geq 3$, then $G$ is a balanced $d$-magic graph.

Joining Theorem 1.1 and Corollary 2.9, one certainly has

**Corollary 2.10.** For any positive integer $k$, if a graph $G$ can be decomposable into $2k$ spanning cycle subgraphs of odd order $n \geq 3$, then $G$ is a supermagic graph.

Notice that there exist SC-labellings $f$ and $g$ of a cycle graph of order 8 with vertices $v_1, v_2, \ldots, v_8$ such that $f(E(G)) = [k, k + 3] \cup \{k + 8, k + 9, k + 11, k + 12\}$ and $g(E(G)) = [h, h + 3] \cup \{h + 6, h + 9, h + 10, h + 11\}$ for any positive integers $h, k$. Moreover, if $k = 1$ and...
$h = 5$, then the SC-labellings $f$ and $g$ are balanced complementary. These SC-labellings $f$ and $g$ are shown as follows.

$$f(e) = \begin{cases} 
  k & : e = v_1v_4, \\
  k+11 & : e = v_4v_6, \\
  k+2 & : e = v_6v_7, \\
  k+12 & : e = v_7v_8, \\
  k+3 & : e = v_8v_5, \\
  k+9 & : e = v_5v_3, \\
  k+1 & : e = v_3v_2, \\
  k+8 & : e = v_2v_1,
\end{cases}$$

and

$$g(e) = \begin{cases} 
  h+11 & : e = v_1v_4, \\
  h & : e = v_4v_6, \\
  h+9 & : e = v_6v_7, \\
  h+1 & : e = v_7v_8, \\
  h+6 & : e = v_8v_5, \\
  h+2 & : e = v_5v_3, \\
  h+10 & : e = v_3v_2, \\
  h+3 & : e = v_2v_1.
\end{cases}$$

One can prove that $f^*(v_1) < f^*(v_2) < \cdots < f^*(v_8)$ while $g^*(v_1) > g^*(v_2) > g^*(v_3) > g^*(v_4) > g^*(v_5) > g^*(v_6) > g^*(v_7) > g^*(v_8)$. Furthermore, consider the case $k = 1$ and $h = 5$, one then obtains
Given

\[
f(e) = \begin{cases} 
1 & : e = v_1v_4, \\
12 & : e = v_4v_6, \\
3 & : e = v_6v_7, \\
13 & : e = v_7v_8, \\
4 & : e = v_8v_5, \\
10 & : e = v_5v_3, \\
2 & : e = v_3v_2, \\
9 & : e = v_2v_1, 
\end{cases}
\]

and

\[
g(e) = \begin{cases} 
16 & : e = v_1v_4, \\
5 & : e = v_4v_6, \\
14 & : e = v_6v_7, \\
6 & : e = v_7v_8, \\
11 & : e = v_8v_5, \\
7 & : e = v_5v_3, \\
15 & : e = v_3v_2, \\
8 & : e = v_2v_1.
\end{cases}
\]

Obviously, \(f\) and \(g\) are balanced complementary labellings. However, by the method of Proposition 2.2, one cannot construct a balanced \(d\)-magic graph by using two balanced complementary labellings of a cycle subgraph of order 8 upwardly because the condition does not hold.

For the last result, two balanced complementary of SC-labellings of some cycle graphs and their associated balanced \(d\)-magic graphs are presented as follows.

**Figure 1.** Two balanced complementary SC-labellings of a cycle graph \(C_3\).
Figure 2. A balanced $d$-magic graph constructed by two spanning cycle subgraphs $C_3$.

Figure 3. Two balanced complementary SC-labellings of a cycle graph $C_4$.

Figure 4. A balanced $d$-magic graph constructed by two spanning cycle subgraphs $C_4$.

Figure 5. Two balanced complementary SC-labellings of a cycle graph $C_5$. 
Figure 6. A balanced $d$-magic graph constructed by two spanning cycle subgraphs $C_5$.

Figure 7. Two balanced complementary SC-labellings of a cycle graph $C_7$.

Figure 8. A balanced $d$-magic graph constructed by two spanning cycle subgraphs $C_7$. 
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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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