IS IT POSSIBLE TO UNIFY THE QCD EVOLUTION OF STRUCTURE FUNCTIONS IN $X$ AND $Q^2$?

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Abstract

We start from the two existing QCD evolution equations for structure functions, the BFKL and DGLAP equations, and discuss the theoretical hints for a unifying picture of the evolution in $x$ and $Q^2$. The main difficulty is due to the property of angular ordering of the gluon radiation driving the evolution and the cancellation of the related collinear singularities. At the leading log $1/x$ and leading log $Q^2$ accuracy, we find a unified set of equations satisfying the constraints.

1 The two QCD evolution equations and the unification problem

There is a well-known method to obtain the QCD-theoretical predictions for quark and gluon structure functions measured in Deep-Inelastic Scattering: the resummation of leading logarithmic contributions at all orders of the perturbative theory; indeed, the existence of collinear and infrared singularities in the evaluation of radiation corrections to the point like lepton-parton scattering leads to effective coupling constants of order unity, and thus to the need of resummation techniques. Two types of resummation do exist.
At large $Q^2 = -q^2$, where $q$ is the quadri-momentum transferred to the target-proton, the \textit{collinear} singularities lead to effective coupling constants of order $\alpha_s(Q^2) \ln \left( Q^2/\Lambda_{QCD}^2 \right)$. The resummation to all orders of the leading logarithms ($LLQ^2$) leads to the well-known "Altarelli-Parisi" (or DGLAP) evolution equations\textsuperscript{1,2).}

At small values of the Bjorken variable $x = Q^2/2p.q$ ($p$ is the quadri-momentum of the target), a similar problem appears with the soft part of the gluon radiation, namely an effective coupling constant of order $\alpha_s \ln 1/x$. It is thus necessary to resum the corresponding leading logarithms ($LL1/x$) to all orders. This non-straightforward resummation has been first performed by L. Lipatov and collaborators (BFKL) and leads to a singular behaviour\textsuperscript{3)} of structure functions at small-$x$. This result has been recently revived by Hera results on the quark structure function in the proton at very small-$x$. They have revealed a behaviour in qualitative agreement with this QCD prediction\textsuperscript{4).}

For both phenomenological and theoretical reasons, it is interesting to address the problem of unifying the two mentioned equations into a single scheme. Since Hera experiments\textsuperscript{5)} cover a very large range in $x$ and $Q^2$, it is quite important to have a unified description of the QCD-evolution of structure functions in the whole $x-$range. Moreover, it could solve the dependence on initial conditions for the evolution equations, which has to do with the unknown non-perturbative regime of QCD.

On a more theoretical ground which is of concern in the present paper, it is to be remarked that the unification problem has already been suggested and discussed in the past. It has first been noticed that both $LL1/x$ and $LLQ^2$ can be formally taken into account by a suitable combination\textsuperscript{5)} of the evolution kernels. On a more rigorous basis, it has been shown that a uniform description of the gluon radiation responsible for the evolution of structure functions in the whole $x$-range is possible due to the property of \textit{angular ordering}\textsuperscript{6}. Within that picture, it can be shown that the collinear singularities present in all gluon production amplitudes contribute only in regions satisfying the following kinematical property:

\begin{equation}
Q/x \gg \ldots \gg \theta_i \gg \theta_{i-1} \gg \ldots \gg \theta_1
\end{equation}

where $\theta_i \approx (q_T)_i/x_i$ are the angles of the emitted gluon with respect to the direction of the first emitted gluon momentum. One can separate two cases:
i) at fixed $x$, corresponding to finite (non strongly ordered) $x_i$, one recovers the well-known $q_t$-ordering of the $LLQ^2$ resummation technique\(^2\).

ii) at small $x$, the gluon momentum fractions $x_i$ are necessarily strongly ordered ($\frac{x_i}{x_{i-1}} \ll 1$) and thus $q_t$-ordering is not implied by the relations (1) and angular ordering is expected to contribute to $LL1/x$ singularities, together with the infrared singularities.

However, it remains an important constraint to be fulfilled by any unification scheme based on angular ordering. The Lipatov equation is only recovered provided that these $LL1/x$ singularities related to angular ordering exactly cancel in the evolution of structure functions \(^6,7\). This cancellation is not valid for other observables such as multiplicities, average transverse momentum etc... As we shall see now, this stringent constraint leads to non-trivial consequences on the unified evolution equations.

Let us first write the Altarelli-Parisi equations in a suitable form for unification. For this sake, we will restrict ourselves to the case of a fixed coupling constant $\bar{\alpha}_S$ (as for the BFKL derivation) and consider the double inverse Mellin transform of the singlet ($F_S$) and gluon ($F_G$) structure functions:

$$F_{S,G}(x, Q^2) = \int \frac{d\gamma}{2i\pi} e^{\gamma \ln Q^2/\Lambda^2} \int \frac{dj}{2i\pi} e^{(j-1)\ln 1/x} \varphi_{S,G}(j, \gamma).$$

The Altarelli-Parisi equations (for fixed $\bar{\alpha}_S$) can be written in matrix form for $\varphi_S$ and $\varphi_G$ as follows.

$$\begin{pmatrix} \varphi_S \\ \varphi_G \end{pmatrix} \equiv \begin{pmatrix} \varphi_S^{(0)} \\ \varphi_G^{(0)} \end{pmatrix} + \frac{\bar{\alpha}_S}{4\pi \gamma} \begin{pmatrix} \nu_F & 2n_F \phi_F^F \\ \phi_F^G & \nu_G \end{pmatrix} \begin{pmatrix} \varphi_S \\ \varphi_G \end{pmatrix},$$

where $\{\nu_G, \nu_F, \phi_F^G, \phi_F^F\}$ are the usual ($j$-dependent) Altarelli-Parisi weights\(^2\), and $\varphi_{S,G}^{(0)}$ are the initial conditions.

Now, one has to modify the equation (3) to take into account the BFKL contribution in the gluon sector. The dominant contribution can be expressed \(^3,5\) as a singularity in the $j$-plane situated at the value

$$j_L = 1 + \frac{\bar{\alpha}_S N_C}{\pi} \chi(\gamma)$$

where

$$\chi(\gamma) \equiv 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma); \ \psi(\gamma) \equiv \frac{d \ln \Gamma(\gamma)}{d \gamma}.$$
is the eigenvalue-function of the BFKL kernel. Indeed, assuming a simple pole singularity $\varphi_{S,G} \propto (j-j_L)^{-1}$ and inserting it in the Mellin transform (2), one gets:

$$F_{S,G}(x,q^2) = \int \frac{d\gamma}{2i\pi} e^{\gamma \ln Q^2/\Lambda^2} e^{\bar{\alpha}_S \chi(\gamma) \ln 1/x} \approx \left( \frac{Q^2}{\Lambda^2} \right)^{1/2} x^{-\bar{\alpha}_S} \bar{\alpha}_S^{1/2} \ln 2, \quad (4)$$

where one has made use of a saddle point method to integrate around the point $\gamma_c = 1/2$, $\chi(\gamma_c) = 4 \ln 2$. Equation (5) corresponds exactly to the leading BFKL behaviour$^3$, up to logarithmic corrections (which would be determined by the nature of the singularity in the $j$-plane).

A consistent modification of equation (3) in order to complement the singular behaviour (4) is the following$^8$; let us replace the gluon contribution to the anomalous dimensions:

$$\nu_G(j) \longrightarrow \nu_{G}^* = \gamma \chi(\gamma) \{ \nu_G + \Psi \} - \Psi, \quad (5)$$

where $\Psi$ is an arbitrary function holomorphic in the $j$-plane near $j = 1$ and below. Such a modification inserted in equation (3) provides a formal unification of the Altarelli-Parisi and Lipatov kernels. Indeed, inverting the relation (3), after the replacement $\nu_G \longrightarrow \nu_{G}^*$, one gets

$$\begin{pmatrix} \varphi_G \\ \varphi_S \end{pmatrix} = \begin{pmatrix} 1 & \frac{\bar{\alpha}_S}{4\pi \gamma} \nu_F \\ \frac{\bar{\alpha}_S}{4\pi \gamma} 2n_F \phi_F^G & 1 - \frac{\bar{\alpha}_S}{4\pi \gamma} \nu_G^* \end{pmatrix} \begin{pmatrix} \varphi_{G}^{(0)} \\ \varphi_{S}^{(0)} \end{pmatrix},$$

with

$$D(j,\gamma) = 1 - \frac{\bar{\alpha}}{4\pi \gamma} (\nu_{G}^* + \nu_F) + \left( \frac{\bar{\alpha}}{4\pi \gamma} \right)^2 (\nu_{G}^* \nu_{G}^* - 2n_F \phi_F^G \phi_F^G). \quad (6)$$

Now, the solutions of equation (6) depend on the region in the complex $j$-plane involved in the Mellin transform (2), and thus on the region in $\ln 1/x$ one is investigating; two cases appear:

i) when $x$ is not small, $\bar{\alpha}_S \ln 1/x \ll 1$, the modification (5) has no effect, since the zeroes of $D(j,\gamma)$ are obtained for small values of $\gamma$ (of order $\bar{\alpha}_S$).

In that limit, one has from the very definition of $\chi(\gamma)$:

$$\chi(\gamma) \approx 1/\gamma + O(\gamma^2); \quad \nu_{G}^* \approx \nu_G + O(\bar{\alpha}_S). \quad (7)$$
One recovers the ordinary Altarelli-Parisi scheme\textsuperscript{3}) and the corresponding evolution equations (at fixed $\bar{\alpha}_S$).

ii) When $\bar{\alpha}_S \ln 1/x = \mathcal{O}(1)$, the singular structure of the BFKL kernel plays a role, driving the relevant domain of the Mellin integration over $\gamma$ near to the ”critical” value $\gamma_c = 1/2$.

In those conditions one recovers the singular behaviour compatible with the BFKL calculations. Taking the appropriate limit $j \to 1$, $\bar{\alpha}_S/(j-1) = \mathcal{O}(1)$:

$$D(j, \gamma) \propto 1 - \frac{\pi N_C \chi(\gamma)}{4\pi (j-1)} j \to 1 \approx 1 - \frac{\pi}{\pi} N_C \frac{4 \ln 2}{j-1}$$

(8)

At first sight, the DGLAP and BFKL evolution equations can be unified for an arbitrary regular function $\Psi(j)$ in eq.(6). For instance let us consider the combination:

$$\nu^*_G + \nu_F = \gamma \chi(\gamma) \{ \nu_G + \Psi \} + \nu_F - \Psi$$

appearing in $D(j, \gamma)$ at first order in $\bar{\alpha}_S$. Following the arguments of refs.\textsuperscript{6,7)} as we have stressed upon in our introductory discussion, the quark-loop contribution $\nu_F$, which is present at fixed value of $x$ as a result of collinear singularities, should be absent from the evolution equations for small value of $x$. More precisely, if not cancelled appropriately, it would bring a new $LL_1/x$ singularity, due to the angular-ordering property including emitted quarks. It is thus compelling to choose $\nu_F \approx \Psi$ when $j \to 1$ in order to obtain the desired cancellation. This is just the mechanism proposed in our paper\textsuperscript{8}). Indeed, the problem has been noticed to arise when one is to include "finite parts" into the evolution equations at small-$x$\textsuperscript{9}).

As a consequence, considering the proposed cancellation to be valid in the $j$-plane around the leading singularity $j_L$, one writes

$$\Psi(j_L) \approx \nu_F(j_L)$$

(10)

$$\nu_G(j_L) + \nu_F(j_L) = \left[ \frac{\pi N_C \log 2}{\pi} \right]^{-1}.$$  

As noticed in Ref.[8], the equations (10) lead to an appreciable modification of the location $j_L$ of the BFKL singularity endpoint, in better agreement with phenomenological determinations\textsuperscript{4}).

Among other interesting properties, the set of equations (10) ensures (at first order in $\bar{\alpha}_S$) that the quark loops do not contribute to the small-$x$ evolution.
Moreover, the expression of $D(j, \gamma)$ preserves the position of the saddle point in the $\gamma$ plane at the critical value $1/2$, as expected from the conformal properties of the BFKL kernel$^{10}$.

As a conclusion, the unification of the evolution equations for structure functions appears possible, at least in the leading logarithmic approximation and at fixed coupling constant $\alpha_s$. Despite stringent constraints due to the mismatch of $LL1/x$ and $LLQ^2$ perturbative resummations, a unified set of equations for the whole range in $x$ and $Q^2$ can be written and leads to non-trivial predictions. A number of interesting questions remain open for future investigation, let us list some of them:

1) Is it possible to implement unified evolution equations at the next-leading-order?

2) In the same context, how the result might be influenced by the running of $\alpha_s$?

3) What are the phenomenological consequences of unified equations?

We hope to be able to provide answers to these questions in the near future.

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