Nonconvex Matrix Factorization is Geodesically Convex: Global Landscape Analysis for Fixed-rank Matrix Optimization From a Riemannian Perspective

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Abstract

In this paper we study the landscape of a general matrix optimization problem with a fixed-rank positive semidefinite (PSD) constraint. We perform the Burer-Monteiro factorization, i.e., factorize a PSD matrix \(X\) as \(YY^T\), and consider a particular Riemannian quotient geometry in a search space that has a total space equipped with the Euclidean metric. When the original objective \(f\) satisfies standard restricted strong convexity and smoothness properties, we characterize the global landscape of the factorized objective under the Riemannian quotient geometry. In particular, we show that the entire search space can be divided into three regions: \((R_1)\) the region near the target parameter of interest, where the factorized objective is geodesically strongly convex and smooth; \((R_2)\) the region containing neighborhoods of all strict saddle points; \((R_3)\) the remaining regions, where the factorized objective has a large gradient. Our results cover both noisy and noiseless settings in applications of interest. To the best of our knowledge, this is the first global landscape analysis of the Burer-Monteiro factorized objective under the Riemannian quotient geometry. Our results provide a fully geometric explanation for the superior performance of vanilla gradient descent under the Burer-Monteiro factorization. When \(f\) satisfies a weaker restricted strict convexity property, we show there exists a neighborhood near local minimizers such that the factorized objective is geodesically convex. To prove our main results we provide a comprehensive landscape analysis of a matrix factorization problem with a least squares objective, which serves as a critical bridge in establishing the results in the general setting. Our conclusions are also based on a result of independent interest stating that the geodesic ball centered at \(Y\) with a radius one-third of the least singular value of \(Y\) is a geodesically convex set under the Riemannian quotient geometry, a result that, as a corollary, also implies a quantitative bound of the convexity radius in the Bures-Wasserstein space. The convexity radius obtained in this paper is sharp up to constants.

Keywords: Matrix factorization, global landscape analysis, Riemannian optimization, quotient geometry

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1 Introduction

In this paper, we consider the following optimization problem

\[
\min_{X \in \mathbb{S}^{p \times p} \geq 0, \text{rank}(X) = r} f(X), \quad 0 < r \leq p.
\]

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Without loss of generality, we assume \( f \) is symmetric in \( X \in \mathbb{R}^{p \times p} \), i.e., \( f(X) = f(X^T) \); otherwise, we can set \( \tilde{f}(X) = \frac{1}{2}(f(X) + f(X^T)) \) and have \( \tilde{f}(X) = f(X) \) for all \( X \geq 0 \) [BKS16]. In addition, we assume \( f \) is twice continuously differentiable (in the usual sense) with respect to \( X \). To accelerate the computation of (1) while coping with the rank constraint, a line of research has studied the following nonconvex factorization formulation (also dubbed as Burer-Monteiro factorization in the literature [BM05]):

\[
\min_{\mathcal{Y} \in \mathbb{R}^{p \times r}_{+}} \tilde{h}(\mathcal{Y}) := f(\mathcal{Y}\mathcal{Y}^T),
\]

i.e. \( X \) gets factorized as \( \mathcal{Y}\mathcal{Y}^T \) for a rectangular matrix \( \mathcal{Y} \). Here, \( \mathbb{R}^{p \times r}_{+} \) denotes the set of \( p \)-by-\( r \) matrices with full column rank.

The nonconvex factorization formulation (2) has been shown to be effective in many settings. For example, when \( f \) is convex and smooth, [BKS16] showed vanilla gradient descent (GD) on \( \mathcal{Y} \) in (2) converges locally to the global minimizer at the classic sublinear convergence rate. When \( f \) is well-conditioned, e.g., it satisfies certain restricted strong convexity and smoothness properties (see the forthcoming Definition 1), a number of works have demonstrated that GD in (2) or in its asymmetric version, i.e., when \( X = LR^T \) (\( L \in \mathbb{R}^{p \times r} \), \( R \in \mathbb{R}^{2 \times r} \)), achieves linear convergence provided the scheme is initialized closely enough to the global minimizer [BKS16, CW15, CLS15, LLSW19, MWCC19, SL15, TBS\textsuperscript{+}16, WZG17, ZWL15, ZL15, DC20, TMC20]. More surprisingly, perturbed GD or GD with random initialization has also been observed to have fast global convergence performance when carrying out the Burer-Monteiro factorization [CCFM19, YD21, BZL22].

The fact that the objective in (2) is nonconvex has led researchers to investigate the reasons behind the observed superior performance of vanilla GD. A body of works, for example, have shown that the factorization in (2) actually does not introduce spurious local minima when the objective \( f \) is well-conditioned [BNS16, CL19, GJZ17, LHT19, PKCS17, ZBL21, ZSL19, ZYW18, ZLTW18]. Similar benign landscape results have been proved for the Burer-Monteiro factorization in solving semidefinite programs [BVB20, JBAS10, MMO17, LXB19, WW20]. These works take a big step in demystifying the observed performance of the Burer-Monteiro factorization, but they are also restrictive in the sense that they mainly focus on the landscape analysis of the objective near the global minimum or at stationary points. Such a partial characterization is limited as it does not yield, in general, explicit guarantees on the rate of convergence to the global minimizer [GHJY15, LPP\textsuperscript{+}19, DJL\textsuperscript{+}17].

There have been a few attempts to study the global geometry of related optimization problems by characterizing their landscapes in the whole search space, rather than solely in regions near the global minimizer or near stationary points. For example, [GHJY15], [SQW18, CHLW22] and [SQW16] studied the global landscape geometries of tensor decomposition, phase retrieval and complete dictionary recovery problems, respectively. The global Euclidean geometries of (2) and their asymmetric versions were studied in [LLA\textsuperscript{+}19] and [ZLTW21], respectively. In particular, [ZLTW21] considered a general well-conditioned objective \( f \) and showed that the landscape of \( \tilde{h}(\mathcal{Y}) \) is benign and that the whole search space can be characterized as follows: when \( \mathcal{Y} \) is far from the global optimal, then either the magnitude of the gradient of \( \tilde{h}(\mathcal{Y}) \) is large, or the Hessian evaluated at \( \mathcal{Y} \) has a negative eigenvalue; when \( \mathcal{Y} \) is close to the global minimizer, then \( \tilde{h}(\mathcal{Y}) \) satisfies certain regularity condition which can facilitate local linear convergence of GD. [LLA\textsuperscript{+}19] considered a more restricted setting and assumed \( f \) is a least squares objective, but they obtained a stronger local geometric result: \( \tilde{h}(\mathcal{Y}) \) is strongly convex in certain directions near the global minimizer. These results are encouraging and provide some evidence on why vanilla GD enjoys fast convergence performance when \( f \) is well-conditioned. However, the description of local geometry near the global minimizer provided in [LLA\textsuperscript{+}19, ZLTW21] is not quite intuitive for two reasons: first,
it is unclear how to generalize the regularity condition proved in [ZLTW21] to other settings beyond the Burer-Monteiro factorization; second, it is not obvious how to interpret the restricted directions that make the Hessian positive in [LLA+19]. Based on the existing results in the literature, we ask the central question that we explore in this paper:

**Can we provide a more intuitive and geometric explanation for why the Burer-Monteiro factorization works?**

In addition, except for the landscape analysis for the special noisy matrix trace regression with a least squares objective in [LLA+19, Section IV.B], the analyses in [LLA+19, ZLTW21] mainly focus on the noiseless setting. Specifically, [LLA+19, ZLTW21] assume there exists a rank r parameter matrix of interest \( X^* \) which is the global minimizer of (1) and satisfies \( \nabla f(X^*) = 0 \). We note these assumptions often hold in applications in noiseless settings, while do not hold in the noisy case; see the upcoming application in Example 1. Moreover, the guarantee in [LLA+19, Section IV.B] is customized to the noisy matrix trace regression problem and does not apply in more generality. Thus, it is natural to ask:

**Can we analyze the global optimization landscape of (2) in the general noisy setting?**

Finally, the previous results for landscape analysis in the literature are restricted to the setting where \( f \) is well-conditioned, and thus we wonder:

**Can we study the landscape of (2) under a weaker assumption on \( f \)?**

In this work, we provide affirmative answers to the above three questions. First, we note that, under the Euclidean formulation, if \( Y \) is a stationary point/local minimizer of (2), then \( YO \) is also a stationary point/local minimizer for any \( O \in O_r \). This ambiguity makes \( h(Y) \) unavoidably nonconvex in any neighborhood of a stationary point [LLA+19, Proposition 2] and it becomes a fundamental hurdle in [LLA+19, ZLTW21] for providing a more intuitive landscape analysis. To tackle this difficulty, we resort to tools from Riemannian optimization [AMS09, Bou20] and consider a particular Riemannian quotient geometry on (2) [JBAS10]; see Section 1.4 for a brief introduction of Riemannian optimization on quotient manifolds. Specifically, we encode the invariance mapping, i.e., \( Y \mapsto YO \), in an abstract search space by defining the equivalence classes \( [Y] = \{YO : O \in O_r\} \). Since the invariance mapping is performed via the Lie group \( O_r \) smoothly, freely and properly, we have \( \mathcal{M}_p := \mathcal{M}_p/O_r \) is a quotient manifold of \( \mathcal{M}_p := \mathbb{R}^{p \times r} \) [Lee13, Theorem 21.10]. Moreover, we equip \( \mathcal{T}_r \mathcal{M}_p \) with the metric \( g_{\mathcal{T}_r \mathcal{M}_p}(\eta_Y, \theta_Y) = \text{tr}(\eta_Y^T \theta_Y) \) for any \( \eta_Y, \theta_Y \in \mathcal{T}_r \mathcal{M}_p \), where \( \mathcal{T}_r \mathcal{M}_p := \mathbb{R}^{p \times r} \) is the tangent space of \( \mathcal{M}_p \) at \( Y \). Since \( h(Y) \) is invariant along the equivalence classes of \( \mathcal{M}_p \), (2) induces the following optimization problem on the quotient manifold \( \mathcal{M}_p \):

\[
\min_{[Y] \in \mathcal{M}_p} h([Y]) := h(Y) \tag{3}
\]

Our main contribution is on providing a purely geometric and intuitive explanation of the success of vanilla GD in the Burer-Monteiro factorization via performing landscape analysis of (3) under the Riemannian quotient geometry. Our results cover various scenarios of \( f \) and allow noise as well. An informal statement of our main results (Theorems 2-6 and Corollary 1) is provided in the following Theorem 1.

**Theorem 1 (Informal Results).** (a) When \( f \) satisfies restricted strong convexity and smoothness properties (see Definition 1), and the noise level at the rank \( r \) matrix parameter of interest \( X^* \) is controlled, then the global landscape of (3) is benign in the following sense: given any \( Y \in \mathbb{R}_{p \times r}^* \), one of the following three properties holds:

(i) the magnitude of Riemannian gradient of \( h([Y]) \) is large;
(ii) the Riemannian Hessian of $h([Y])$ has a large negative eigenvalue and there is an explicit escaping direction;

(iii) when $YY^T$ is close to $X^*$, $h([Y])$ is geodesically strongly convex and smooth. Moreover, the distance between the global minimizer and $X^*$ is bounded by a quantity that depends on the noise level of the problem or equivalently on the magnitude of $\nabla f(X^*)$.

(b) When $f$ satisfies a weaker restricted strict convexity property (see Definition 2), there exists a neighborhood around a local minimizer such that $h([Y])$ is geodesically convex.

To the best of our knowledge, this is the first landscape analysis for the Burer-Monteiro factorization under the Riemannian quotient geometry. Thanks to this Riemannian formulation, in either setting of $f$ we have that there exists a local region in which the factorized objective is either geodesically strongly convex or geodesically convex. Such results are novel and can not be obtained under the frameworks of [LLA+19, ZLTW21] as $h(Y)$ is in essence nonconvex under the Euclidean geometry. Moreover, we also provide the global landscape analysis of (3) when $f$ is well-conditioned.

The set of three properties described in Theorem 1(a) is known as the robust strict saddle property in the literature [GHJY15, JGN+17, SQW15] and many algorithms are guaranteed to achieve fast global convergence when these properties hold [SFF19, CB19, GIJLY15, JGN+17, SQW18].

Finally, since the horizontal lift of the Riemannian gradient of $h([Y])$ is the same as the Euclidean gradient (see the forthcoming Lemma 3) of $\hat{h}(Y)$, the gradient descent algorithms under the Riemannian quotient geometry and the Euclidean geometry are exactly the same from a computational point of view. Thus, our geometric landscape results give a fully geometric and intuitive explanation for the superior performance of vanilla gradient descent under the Burer-Monteiro factorization.

To prove our main results, we first show a novel geodesic convexity property of the Riemannian quotient manifold $\mathcal{M}_{\Sigma^+}^r$ which is also of independent interest. Specifically, we show the geodesic convexity radius of $\mathcal{M}_{\Sigma^+}^r$ at $[Y] \in \mathcal{M}_{\Sigma^+}^r$ is of order at least $\sigma_r(Y)/3$; this radius is sharp up to the constant $1/3$. In addition, we base our global landscape analysis of (3) when $f$ is well-conditioned on the global landscape analysis of the following optimization problem:

$$
\min_{[Y] \in \mathcal{M}_{\Sigma^+}^r} H([Y]) := \frac{1}{2} \|YY^T - X^*\|_F^2,
$$

(4)

where $X^*$ is the rank $r$ parameter matrix of interest mentioned before. We show that the landscape of (4) is benign in the same sense as in Theorem 1(a). Moreover, the optimization landscape of (4) is preserved for the general low-rank matrix optimization (3) when $f$ satisfies the restricted strong convexity and smoothness properties. When $f$ satisfies a weaker restricted strict convexity property, we show that the Riemannian Hessian of $h([Y])$ is positive definite at a local minimizer, and there exists a neighborhood around the local minimizer such that $h([Y])$ is geodesically convex.

1.1 Additional Related Literature

Riemannian manifold optimization methods are powerful tools when solving optimization problems with geometric constraints [AMS09, Bou20]. A lot of progress in this topic was made for studying the convergence of Riemannian optimization algorithms when solving low-rank matrix estimation problems, including matrix completion [KOM09, BA11, Van13, DGHG22], robust PCA [ZY18], matrix trace regression [WCCL16, MBS11, LHLZ20], blind deconvolution/phase retrieval [HH18, LHLZ20] and general fixed-rank matrix optimization [MMBS14].

There are some precedents for the study of geometric landscape of an optimization problem under the Riemannian formulation. For example, [MZL19] and [AV22] provided landscape analyses
for robust subspace recovery and block Rayleigh quotient of symmetric or PSD matrices over the Grassmannian manifold. [Lin] studied the landscape of orthogonal group synchronization over the Stiefel manifold after performing the Burer-Monteiro factorization. Geometric landscape analysis of a quartic-quadratic optimization problem under a spherical constraint was examined in [ZMXY21]. The landscape analyses in these works were mainly performed at stationary points. Under the embedded geometry for the set of fixed-rank matrices, [UV20] proved that the landscape of (1) when f is quadratic and satisfies the restricted strong convexity and smoothness properties is benign. There again their focus is on the landscape at stationary points.

1.2 Organization of the Paper

The rest of this article is organized as follows. After a brief introduction of notation, we introduce Riemannian optimization and Riemannian optimization under the quotient geometry in Section 1.3. Geometric properties of \( M_{p,r}^q \) are provided in Section 2. The global landscape analysis of \( h([Y]) \) when f is well-conditioned is given in Section 3. The local geometry of \( h([Y]) \) when f satisfies restricted strict convexity property is given in Section 4. In Section 5, we present the geometric landscape analysis for \( H([Y]) \). Proofs for the main results are provided in Sections 6 and 7. Conclusion and future work are given in Section 8. Additional proofs and lemmas are presented in Appendices A-D.

1.3 Notation and Preliminaries

The following notation will be used throughout this article. We use \( \mathbb{R}^{p_1 \times p_2}, \mathbb{S}^{p \times p}, \) and \( \mathbb{R}^{p \times r} \) to denote the spaces of \( p_1 \)-by-\( p_2 \) real matrices, \( p \)-by-\( p \) real symmetric matrices, and \( p \)-by-\( r \) real full column rank matrices, respectively. Let \( \mathbb{O}_{p,r} \) be the set of \( p \)-by-\( r \) column orthogonal matrices and \( \mathbb{O}_r := \mathbb{O}_{r,r} \). Uppercase and lowercase letters (e.g., \( A, a \)), lowercase boldface letters (e.g., \( u \)), uppercase boldface letters (e.g., \( U \)) are often used to denote scalars, column vectors, and matrices, respectively. For any \( a, b \in \mathbb{R} \), let \( a \wedge b := \min\{a, b\} \), \( a \vee b := \max\{a, b\} \). For any matrix \( X \in \mathbb{R}^{p_1 \times p_2} \) with singular value decomposition (SVD) \( \sum_{i=1}^{\min\{p_1, p_2\}} \sigma_i(X) u_i v_i^\top \), where \( \sigma_1(X) \geq \sigma_2(X) \geq \ldots \geq \sigma_{\min\{p_1, p_2\}}(X) \), denote its Frobenius norm and spectral norm as \( \|X\|_F = \sqrt{\sum_{i=1}^{\min\{p_1, p_2\}} \sigma_i^2(X)} \) and \( \|X\|_2 = \sigma_1(X) \), respectively. Let \( X_{\text{max}(r)} = \sum_{i=1}^{r} \sigma_i(X) u_i v_i^\top \) be the best rank-\( r \) approximation of \( X \) in the Frobenius norm. Also, denote \( \text{tr}(X) \) and \( X^{-1} \) as the trace and inverse of \( X \), respectively. For any \( X \in \mathbb{S}^{p \times p} \) having eigendecomposition \( U \Sigma U^\top \) with non-increasing eigenvalues on the diagonal of \( \Sigma \), let \( \lambda_i(X) \) be the \( i \)-th largest eigenvalue of \( X \), \( \lambda_{\text{min}}(X) \) be the least eigenvalue of \( X \), and \( X^{1/2} = U \Sigma^{1/2} U^\top \). We write \( X \succeq 0 \) if \( X \) is a symmetric positive semidefinite (PSD) matrix. Throughout the paper, the SVD (or eigendecomposition) of a rank \( r \) matrix \( X \) (or symmetric matrix \( X \)) refers to its economic or reduced version. For any \( p \)-by-\( r \) column orthonormal matrix \( U \), let \( P_U = U U^\top \) represents the orthogonal projector onto the column space of \( U \); we also note \( U_\perp = U^{(p-r)} \) as an orthonormal complement of \( U \). Finally, suppose \( f : \mathbb{R}^{p_1 \times p_2} \to \mathbb{R} \) is a differentiable scalar function, let \( \nabla f(X) \) and \( \nabla^2 f(X) \) be its Euclidean gradient and Hessian, respectively. We define the bilinear form of the Euclidean Hessian of \( f \) as \( \nabla^2 f(X)[Z_1, Z_2] := \langle \nabla^2 f(X)Z_1, Z_2 \rangle \) for any \( Z_1, Z_2 \in \mathbb{R}^{p_1 \times p_2} \), where \( \langle \cdot, \cdot \rangle \) is the standard Euclidean inner product.

1.4 Riemannian Optimization Under Quotient Geometries

In this section, we first give a brief introduction to Riemannian optimization and then discuss how to perform Riemannian optimization under quotient geometries.
Riemannian optimization concerns optimizing a real-valued function $f$ defined on a Riemannian manifold $\mathcal{M}$. The calculations of Riemannian gradients and Riemannian Hessians are key ingredients to perform continuous optimization over the Riemannian manifold. Suppose $X \in \mathcal{M}$, $g_X(\cdot, \cdot)$ is the Riemannian metric, and $T_X\mathcal{M}$ is the tangent space of $\mathcal{M}$ at $X$. Then the Riemannian gradient of a smooth function $f : \mathcal{M} \to \mathbb{R}$ at $X$ is defined as the unique tangent vector, $\nabla f(X) \in T_X\mathcal{M}$, such that $g_X(\nabla f(X), \xi_X) = Df(X)[\xi_X]$ for all $\xi_X \in T_X\mathcal{M}$. Let $p \in \mathcal{M}$. The Riemannian Hessian of $f$ at $X \in \mathcal{M}$ is a linear mapping $\nabla^2 f(X) : T_X\mathcal{M} \to T_X\mathcal{M}$ defined as

$$\nabla^2 f(X)[\xi_X] = \nabla_{\xi_X} \nabla f(X), \quad \forall \xi_X \in T_X\mathcal{M},$$

where $\nabla$ is the Riemannian connection on $\mathcal{M}$, which is a generalization of the directional derivative along a vector field to Riemannian manifolds [AMS09, Section 5.3]. We say $X \in \mathcal{M}$ is a Riemannian first-order stationary point (FOSP) of $f$ if $\nabla f(X) = 0$ and call a Riemannian FOSP a strict saddle if the Riemannian Hessian evaluated at this point has a strict negative eigenvalue. Given a subset $S$ of $\mathcal{M}$, we call $f : S \to \mathbb{R}$ geodesically convex if $S$ is geodesically convex (i.e. any two points in $S$ can be connected by a geodesic that is completely contained in $S$) and $\nabla^2 f(X) \preceq 0$ for all $X \in S$; we call $f : S \to \mathbb{R}$ $\mu$-geodesically strongly convex if $S$ is geodesically convex and $\nabla^2 f(X) \succeq \mu \Id$ for all $X \in S$.

Next, we provide more details on how to perform Riemannian optimization on quotient manifolds. Quotient manifolds are often defined via an equivalence relation, which contains that leave the equivalence class unchanged. The equivalence class (or fiber) of $\mathcal{M}$ at a given point $X$ is defined by the set $[X] = \{X_1 \in \mathcal{M} : X_1 \sim X\}$. The set $\mathcal{M} := \mathcal{M}/\sim := \{[X] : X \in \mathcal{M}\}$ is called a quotient of $\mathcal{M}$ by $\sim$. The mapping $\pi : \mathcal{M} \to \mathcal{M}/\sim$, $X \mapsto [X]$ is called the quotient map or canonical projection and the set $\mathcal{M}$ is called the total space of the quotient $\mathcal{M}/\sim$. If $\mathcal{M}$ further admits a smooth manifold structure and $\pi$ is a smooth submersion, then we call $\mathcal{M}$ a quotient manifold of $\mathcal{M}$.

Due to the abstractness of equivalence classes, the tangent space $T_[X]\mathcal{M}$ of $\mathcal{M}$ at $[X]$ calls for a representation in the tangent space $T_X\mathcal{M}$ of the total space $\mathcal{M}$. By the equivalence relation $\sim$, the representation of elements in $T_[X]\mathcal{M}$ should be restricted to the directions in $T_X\mathcal{M}$ without inducing displacement along the equivalence class $[X]$. This can be achieved by decomposing $T_X\mathcal{M}$ into complementary spaces $T_X\mathcal{M} = V_X\mathcal{M} \oplus H_X\mathcal{M}$, where $\oplus$ is the direct sum. Here, $V_X\mathcal{M}$ is called the vertical space, which contains that leave the equivalence class $[X]$ unchanged. $H_X\mathcal{M}$ is called the horizontal space of $T_X\mathcal{M}$, which is complementary to $V_X\mathcal{M}$ and provides a proper representation of the abstract tangent space $T_[X]\mathcal{M}$ [AMS09, Section 3.5.8]. Once $\mathcal{M}$ is endowed with $H_X\mathcal{M}$, a given tangent vector $\eta_{[X]} \in T_[X]\mathcal{M}$ at $[X]$ is uniquely represented by a horizontal tangent vector $\eta_X \in H_X\mathcal{M}$ that satisfies $D\pi(X)[\eta_X] = \eta_{[X]}$ [AMS09, Section 3.5.8]. The tangent vector $\eta_X \in H_X\mathcal{M}$ is also called the horizontal lift of $\eta_{[X]}$ at $X$.

Next, we introduce the notion of Riemannian quotient manifolds. Suppose the total space $\mathcal{M}$ is endowed with a Riemannian metric $\bar{g}_\mathcal{M}$, and for every $[X] \in \mathcal{M}$ and every $\eta_{[X]}, \theta_{[X]} \in T_[X]\mathcal{M}$, the expression $\bar{g}_\mathcal{M}(\eta_{[X]}, \theta_{[X]})$, i.e., the inner product of the horizontal lifts of $\eta_{[X]}, \theta_{[X]}$ at $X$, does not depend on the choice of the representative $\mathcal{X}$. Then the metric $\bar{g}_\mathcal{M}$ in the total space induces a metric $g_{[X]}$ on the quotient space, i.e., $g_{[X]}(\eta_{[X]}, \theta_{[X]}) := \bar{g}_\mathcal{M}(\eta_X, \theta_X)$. The quotient manifold $\mathcal{M}$ endowed with $g_{[X]}$ is called a Riemannian quotient manifold of $\mathcal{M}$ [AMS09, Section 3.6.2]. Optimization on Riemannian quotient manifolds is particularly convenient because computation of representatives of Riemannian gradients and Hessians in the abstract quotient space can be directly performed by means of their analogues in the total space. To be specific, suppose $f : \mathcal{M} \to \mathbb{R}$ is an
objective function in the total space that is invariant along the fibers of \( \overline{M} \), i.e., \( f(X_1) = f(X_2) \) whenever \( X_1 \sim X_2 \). Then \( f \) induces a function \( f : \mathcal{M} \rightarrow \mathbb{R} \) on the quotient space. Furthermore, if the horizontal space is canonically chosen (as we do in this paper), i.e., \( \mathcal{H}_X \overline{M} \) is the orthogonal complement of \( \mathcal{V}_X \overline{M} \) in \( T_X \overline{M} \) with respect to \( g_X \), then the horizontal lift of the Riemannian gradient of \( f \) is \( \text{grad} f([X]) = \text{grad} f(X) \) \cite[Section 3.6.2]{AMS09}, where \( \text{grad} f(X) \) denotes the Riemannian gradient of \( f \) at \( X \) in the total space.

Finally, the Riemannian connection on the Riemannian quotient manifold \( \mathcal{M} \) can also be uniquely represented by the Riemannian connection in the total space \( \overline{M} \). Suppose \( \eta, \theta \) are two vector fields on \( \mathcal{M} \) and \( \eta_X \) and \( \theta_X \) are the horizontal lifts of \( \eta[X] \) and \( \theta[X] \) in \( \mathcal{H}_X \overline{M} \). Then the horizontal lift of \( \overline{\nabla}_{\theta[X]} \eta \) on the quotient manifold is given by \( \overline{\nabla}_{\theta[X]} \eta = \overline{P} \nabla_X (\overline{\nabla}_{\theta_X} \overline{\eta}) \), where \( \overline{\eta} \) denotes the horizontal lift of the vector field \( \eta \) and \( \overline{\nabla}_{\theta_X} \overline{\eta} \) is the Riemannian connection in the total space \cite[Proposition 5.3.3]{AMS09}. We also define the bilinear form of the horizontal lift of the Riemannian Hessian as \( \text{Hess} f([X])|\theta_X, \eta_X] = \overline{g}_X \left( \text{Hess} f([X])|\theta_X[X], \eta_X \right) \) for any \( \theta_X, \eta_X \in \mathcal{H}_X \overline{M} \). Then, by recalling the definition of the Riemannian metric \( g_{[X]} \) in the quotient space, we have

\[
\text{Hess} f([X])|\theta_X, \eta_X] = g_X \left( \text{Hess} f([X])|\theta_X[X], \eta_X \right) = g_{[X]} \left( \text{Hess} f([X])|\theta_X[X], \eta_X[X] \right).
\]

So \( \text{Hess} f([X]) \) is completely characterized by \( \text{Hess} f([X]) \) in the lifted horizontal space.

## 2 Geometric Properties and Geodesic Convexity of Balls in \( \mathcal{M}^q_{r+} \)

Recall the quotient manifold we are working with is \( \mathcal{M}^q_{r+} := \overline{M}^q_{r+} / \mathcal{Q} \), and we equip the tangent space \( T_Y \overline{M}^q_{r+} \) with the metric \( \overline{g}_Y (\eta_Y, \theta_Y) = \text{tr}(\eta_Y \theta_Y) \). The following Lemma 1 provides the corresponding vertical and horizontal spaces of \( T_Y \overline{M}^q_{r+} \), and shows \( \mathcal{M}^q_{r+} \) is a Riemannian quotient manifold endowed with the Riemannian metric \( g_{[Y]} \) induced from \( \overline{g}_Y \).

**Lemma 1** \cite{JBAS10, MA20}. Given \( U \in \text{St}(r,p) \) that spans the top \( r \) eigenspace of \( YY^\top \), the vertical and horizontal spaces of \( T_Y \overline{M}^q_{r+} \) are given as follows:

\[
\mathcal{V}_Y \overline{M}^q_{r+} = \{ \theta_Y : \theta_Y = Y \Omega, \Omega = -\Omega^\top \in \mathbb{R}^{p \times r} \},
\]

\[
\mathcal{H}_Y \overline{M}^q_{r+} = \{ \theta_Y : \theta_Y = Y (Y^\top Y)^{-1} S + U_\perp D, S \in \mathbb{S}^{r \times r}, D \in \mathbb{R}^{(p-r) \times r} \}.
\]

The dimensions of \( \mathcal{V}_Y \overline{M}^q_{r+} \) and \( \mathcal{H}_Y \overline{M}^q_{r+} \) are \((r^2 - r)/2 \) and \((pr - (r^2 - r))/2 \), respectively and \( \mathcal{V}_Y \overline{M}^q_{r+} \) is orthogonal to \( \mathcal{H}_Y \overline{M}^q_{r+} \) with respect to \( \overline{g}_Y \). Finally, \( \mathcal{M}^q_{r+} \) is a Riemannian quotient manifold endowed with the metric \( g_{[Y]} \) induced from \( \overline{g}_Y \).

Next, we provide geodesics on \( \mathcal{M}^q_{r+} \), which have been studied in \cite{MA20}.

**Lemma 2.** Let \( Y_1, Y_2 \in \mathbb{R}^{p \times r} \), and \( Q_U \Sigma U \Sigma U^\top \) be the SVD of \( Y_1^\top Y_2 \). Denote \( Q^* = Q_Y Q_U^\top \). Then

- \( Y_2 Q^* - Y_1 \in \mathcal{H}_Y \overline{M}^q_{r+} \), \( Q^* \) is one of the best orthogonal matrices aligning \( Y_1 \) and \( Y_2 \), i.e., \( Q^* \in \arg \min_{Q \in \mathbb{O}_q} \| Y_2 Q - Y_1 \|_F \) and the geodesic distance between \([Y_1]\) and \([Y_2]\) is \( d([Y_1], [Y_2]) = \| Y_2 Q^* - Y_1 \|_F \);

- if \( Y_1^\top Y_2 \) is nonsingular, then \( Q^* \) is unique and the Riemannian logarithm \( \text{Log}_{[Y_1]} [Y_2] \) is uniquely defined and its horizontal lift at \( Y_1 \) is given by \( \text{Log}_{[Y_1]} [Y_2] = Y_2 Q^* - Y_1 \); moreover, the unique minimizing geodesic from \([Y_1]\) to \([Y_2]\) is \([Y_1 + t(Y_2 Q^* - Y_1)]\) for \( t \in [0, 1] \).


Proof. First, by Lemma 1, to guarantee \( Y_2 Q^* - Y_1 \in \mathcal{H}_{Y_i, \mathcal{M}^+_p} \), it is enough to show \( Y_1^T (Y_2 Q^* - Y_1) \in S_t^{p \times r} \). This holds as \( Y_1^T (Y_2 Q^* - Y_1) = Q_{U} \Sigma Q_{U}^T Y_1 = Q_{U} \Sigma Q_{U}^T - Y_1^T Y_1 \in S_t^{p \times r} \).

The rest of the results in the lemma can be found in [MA20, Proposition 5.1, Theorem 4.7 and Proposition 4.4].

Given any \( Y \in \mathbb{R}^{p \times r} \) and \( x > 0 \), let \( B_x([Y]) := \{ [Y]_1 : d([Y]_1, [Y]) < x \} \) be the geodesic ball centered at \([Y]\) with radius \( x \). It is known that for any Riemannian manifold there exists a convex geodesic ball at every point [DCFF92, Chapter 3.4]. However, it is often unclear how large this convex geodesic ball can be in different examples. In the next result, we want to quantify the convexity radius around a point \([Y]\) in the manifold \( \mathcal{M}^+_p \).

**Theorem 2.** Given any \( Y \in \mathbb{R}^{p \times r} \), the geodesic ball centered at \([Y]\) with radius \( x \leq r_Y := \sigma_r(Y)/3 \) is geodesically convex. In fact, for any two points \([Y]_1, [Y]_2 \in B_x([Y])\), there is a unique shortest geodesic joining them, which is entirely contained in \( B_x([Y]) \).

It has been shown in [MA20, Theorem 6.3] that the injectivity radius of \( \mathcal{M}^+_p \) at \([Y]\) is \( \sigma_r(Y) \), and since the convexity radius is smaller than the injectivity radius, the geodesic convexity radius we proved in Theorem 2 is optimal up to a universal constant. Moreover, when \( r = p \), the geometry we considered for \( \mathcal{M}^+_p \) is also known as the Bures-Wasserstein geometry on the set of symmetric positive definite matrices, i.e., \( S_{pp}^+ := \{ X : X \in S^{p \times p} > 0, \text{rank}(X) = p \} \) [MMP18, BJL19, vO20]. Distinct from the common Log-Euclidean metric [AFPA07] or the affine invariant metric [Moa05] on \( S_{pp}^+ \), the set \( S_{pp}^+ \) under the Bures-Wasserstein geometry is not complete. So, to the best of our knowledge, Theorem 2 also provides the first explicit geodesic convexity radius for \( S_{pp}^+ \) under the Bures-Wasserstein geometry.

### 3 Global Landscape Analysis of (3) When \( f \) satisfies Restricted Strong Convexity and Smoothness Properties

In this section, we consider \( f \) satisfies the following \((2r, 4r)\)-restricted strong convexity (RSC) and smoothness (RSM) properties:

**Definition 1.** We say \( f : \mathbb{R}^{p \times p} \to \mathbb{R} \) satisfies the \((2r, 4r)\)-restricted strong convexity and smoothness properties with parameter \( 0 \leq \delta < 1 \) if for any \( X, G \in \mathbb{R}^{p \times p} \) with \( \text{rank}(X) \leq 2r \) and \( \text{rank}(G) \leq 4r \), the Euclidean Hessian of \( f \) satisfies

\[
(1 - \delta) \| G \|_F^2 \leq \nabla^2 f(X)[G, G] \leq (1 + \delta) \| G \|_F^2. \tag{6}
\]

The RSC and RSM properties are satisfied in a number of examples and have been studied in [WZG17, ZLTW18, ZLTW21, LZT19]. Next, we provide the stylized PSD matrix trace regression for an illustration.

**Example 1** (Matrix Trace Regression). In PSD matrix trace regression, the goal is to recovery a rank \( r \) \( p \)-by-\( p \) PSD matrix \( X^* \) from the observation \( y = \mathcal{A}(X^*) + \epsilon \), where \( \mathcal{A} \in \mathbb{R}^{p \times p} \to \mathbb{R}^n \) is a known linear map and \( \epsilon \in \mathbb{R}^n \) is the observational noise. The objective is

\[
\min_{X \in \mathbb{R}^{p \times p}, X \succ 0, \text{rank}(X) = r} f(X) := \frac{1}{2} \| \mathcal{A}(X) - y \|_2^2.
\]

So the Euclidean gradient of \( f(X) \) is \( \nabla f(X) = \mathcal{A}^\top (\mathcal{A}(X) - y) \), and the quadratic form of the Euclidean Hessian satisfies

\[
\nabla^2 f(X)[D, D] = \| \mathcal{A}(D) \|_2^2, \quad \forall D \in \mathbb{R}^{p \times p}.
\]
Here we use the notation $A^\top$ to denote the adjoint of the linear map $A$. Note, the evaluation of the Euclidean gradient at $X^*$ is $\nabla f(X^*) = -A^\top(e)$, which is in general non-zero when $e \neq 0$.

If the linear map $A$ satisfies the $4r$-restricted isometry property (4r-RIP) [CP11], i.e., $(1 - R_{4r})\|Z\|_F^2 \leq \|A(Z)\|_F^2 \leq (1 + R_{4r})\|Z\|_F^2$ holds for all $Z$ of rank at most $4r$ with parameter $0 \leq R_{4r} < 1$, then the RSC and RSM in Definition 1 hold for $f$ with $\delta = R_{4r}$.

Next, we provide expressions of Riemannian gradient and Hessian of (3) under the Riemannian quotient geometry.

**Lemma 3.** (Riemannian Gradient and Hessian of (3) [LLZ21b, Proposition 1]) Suppose $Y \in \mathbb{R}^{p \times r}_*$ and $\theta_Y \in H_{Y} \mathcal{M}^+$. Then, recalling that $f$ has been assumed to be symmetric, we have

$$\nabla h([Y]) = 2\nabla f(YY^\top)Y,$$

$$\text{Hess} h([Y]) \theta_Y, \theta_Y = \nabla^2 f(YY^\top)[YY^\top + \theta_Y Y^\top, Y\theta_Y + \theta_Y Y^\top] + 2\nabla f(YY^\top), \theta_Y \theta_Y^\top.$$  

Suppose the rank $r$ matrix of interest $X^*$ has eigendecomposition $U^*\Sigma^* U^{*\top}$. Let $Y^* = U^*\Sigma^{1/2} \in \mathbb{R}^{p \times r}_*$ and $\kappa^* = \sigma_1(Y^*)/\sigma_r(Y^*)$ be the condition number of $Y^*$. We are ready to present our main results, where we describe the global landscape of (3) when $f$ satisfies the RSC and RSM properties. We will split the landscape of $h([Y])$ into the following five regions (not necessarily non-overlapping): for $\mu, \alpha, \beta, \gamma \geq 0$, define

\begin{align*}
\mathcal{R}_1 &:= \{ Y \in \mathbb{R}^n_{p \times r} | d([Y],[Y^*]) \leq \mu \sigma_r(Y^*)/\kappa^* \}, \\
\mathcal{R}_2 &:= \left\{ Y \in \mathbb{R}^n_{p \times r} \mid \|Y\| \leq \beta \|Y^*\|, \|YY^\top\|_F \leq \gamma \|Y^*Y^*\|_F \right\}, \\
\mathcal{R}_3 &:= \{ Y \in \mathbb{R}^n_{p \times r} \mid \text{Hess} H([Y]) \|F \geq \alpha \mu \sigma_r(Y^*)/(4\kappa^*) \} \|YY^\top\|_F \leq \gamma \|Y^*Y^*\|_F \}, \\
\mathcal{R}_3' &:= \{ Y \in \mathbb{R}^n_{p \times r} \mid \|Y\| > \beta \|Y^*\|, \|YY^\top\|_F \leq \gamma \|Y^*Y^*\|_F \}, \\
\mathcal{R}_3'' &:= \{ Y \in \mathbb{R}^n_{p \times r} \mid YY^\top \geq \gamma \|Y^*Y^*\|_F \}, \\
\mathcal{R}_3''' &:= \{ Y \in \mathbb{R}^n_{p \times r} \mid d([Y],[Y^*]) > \mu \sigma_r(Y^*)/\kappa^* \}
\end{align*}

where $H([Y])$ is given in (4). As we will see later, the use of $\|Y^*\|, \|Y^*Y^*\|_F$ in the definitions of $\mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_3'$ and $\mathcal{R}_3''$ is motivated by the connection between the gradients and Hessians of (3) and (4) provided in Proposition 1.

Since $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_3' \cup \mathcal{R}_3'' = \mathbb{R}^n_{p \times r}$, we can easily check the following lemma holds.

**Lemma 4.** $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_3' \cup \mathcal{R}_3'' = \mathbb{R}^n_{p \times r}$.

In the following Theorem 3, we show $h([Y])$ is geodesically strongly convex and smooth in $\mathcal{R}_1$ for proper choices of $\mu$ and $\delta$.

**Theorem 3** (Local Geodesic Strong Convexity of (3)). Suppose $0 \leq \mu \leq 1/3$. For any $Y \in \mathcal{R}_1$, we have

$$\lambda_{\min}(\text{Hess} h([Y])) \geq 2 (1 - \mu/\kappa^*)^2 - 14\mu/3)\sigma_r^2(Y^*)$$

$$- (4\delta(\sigma_1(Y^*) + \mu \sigma_r(Y^*)/\kappa^*)^2 + 14\mu \sigma_r^2(Y^*)/3 + 2\|\nabla f(X^*)\|_{\max(r)}\|_F),$$

$$\lambda_{\max}(\text{Hess} h([Y])) \leq 4 (\sigma_1(Y^*) + \mu \sigma_r(Y^*)/\kappa^*)^2 + 14\mu \sigma_r^2(Y^*)/3$$

$$+ (4\delta(\sigma_1(Y^*) + \mu \sigma_r(Y^*)/\kappa^*)^2 + 14\mu \sigma_r^2(Y^*)/3 + 2\|\nabla f(X^*)\|_{\max(r)}\|_F),$$

where $\kappa^* := \sigma_1(Y^*)/\sigma_r(Y^*)$ is the condition number of $Y^*$.  

number of $Y^*$.  

In particular, if $\mu$ is further chosen such that $(1 - \mu/\kappa)^2 - 7\mu/3 > 0$ and 
\[
\delta \leq \frac{(1 - \mu/\kappa)^2 - 7\mu/3}{4(2(\kappa + \mu/\kappa)^2 + 7\mu/3)} \quad \text{and} \quad \|\nabla f(X^*)\|_{\text{F}} \leq ((1 - \mu/\kappa)^2 - 7\mu/3) \sigma_r^2(Y^*)/4,
\]
then 
\[
\lambda_{\min}(\text{Hess } h([Y])) \geq ((1 - \mu/\kappa)^2 - 7\mu/3) \sigma_r^2(Y^*) > 0.
\]

Recall $(\nabla f(X^*))_{\text{max}(r)}$ denotes the best rank $r$ approximation of $\nabla f(X^*)$. Thus $h([Y])$ is geodesically strongly convex and smooth in $\mathcal{R}_1$.

Moreover, if there is a Riemannian FOSP $\hat{Y}$ in $\mathcal{R}_1$, then it is the unique local minima in $\mathcal{R}_1$ and it satisfies:
\[
d([\hat{Y}, [Y^*]) \leq \frac{2}{((1 - \mu/\kappa)^2 - 7\mu/3) \sigma_r^2(Y^*)} \|\nabla f(Y^*Y^*^\top)Y^*\|_F
\leq \frac{2\|Y^*\|}{((1 - \mu/\kappa)^2 - 7\mu/3) \sigma_r^2(Y^*)} \|\nabla f(X^*)\|_{\text{max}(r)}\|_F.
\]

**Remark 1.** Similar to the recent work [ZZ20] studying the threshold of $\delta$ guaranteeing the absence of spurious local minimizers in a local region around $X^*$, here we also provide an explicit dependence of the radius of $\mathcal{R}_1$ on $\delta$. Recall that here $\mathcal{R}_1$ is the region where we can guarantee the geodesic strong convexity of $h([Y])$. We notice the upper bound for $\delta$ decreases as $\mu$ increases. This suggests that a stronger requirement on $\delta$ is needed if we desire a larger radius for the region where we can guarantee geodesic strong convexity.

Next, we show $\text{Hess } h([Y])$ has at least one negative eigenvalue for any $[Y] \in \mathcal{R}_2$ under proper assumptions. Moreover, we can explicitly find a direction for escaping the strict saddle points.

**Theorem 4** (Region with Negative Eigenvalue in Riemannian Hessian of (3)). Given any $[Y] \in \mathcal{R}_2$, let $\theta_Y = Y - Y^*Q$, where $Q \in \mathbb{O}_r$ is the best orthogonal matrix aligning $Y^*$ and $Y$. Then we have
\[
\text{Hess } h([Y])\theta_Y, \theta_Y \leq \left((\alpha - 2(\sqrt{2} - 1))\sigma_r^2(Y^*)
\right.
\quad + 2\delta \left(2\beta^2\|Y^*\|^2 + (1 + \gamma)\|Y^*Y^*^\top\|_F\right) + 2\|\nabla f(X^*)\|_{\text{max}(r)}\|_F \right)\|\theta_Y\|_F^2.
\]

In particular, if $\alpha < 2(\sqrt{2} - 1)$,
\[
\delta \leq \frac{2(\sqrt{2} - 1) - \alpha}{8(2\beta^2\|Y^*\|^2 + (1 + \gamma)\|Y^*Y^*^\top\|_F)} \sigma_r^2(Y^*) \quad \text{and} \quad \|\nabla f(X^*)\|_{\text{max}(r)}\|_F \leq \frac{2(\sqrt{2} - 1) - \alpha}{8} \sigma_r^2(Y^*),
\]
then we have
\[
\text{Hess } h([Y])\theta_Y, \theta_Y \leq \frac{\alpha - 2(\sqrt{2} - 1)}{2} \sigma_r^2(Y^*)\|\theta_Y\|_F^2 < 0.
\]

So $\text{Hess } h([Y])$ has at least one negative eigenvalue and $\theta_Y = Y - Y^*Q$ is an escaping direction.

We note the escaping direction $\theta_Y$ has already been identified in [GJZ17] when studying the landscape of (2) under the Euclidean geometry. Here we show that this is still an escaping direction under the Riemannian quotient geometry.

Finally, we show the Riemannian gradient of $h([Y])$ has large magnitude in all three regions $\mathcal{R}', \mathcal{R}_2$ and $\mathcal{R}_3''$. 

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Theorem 5 (Regions with Large Riemannian Gradient of (3)). (i) Given any $Y \in R'_3$, we have
\[
\|\text{grad} h(Y)\|_F \geq \alpha \mu \sigma_r^2(Y^*)/(4\kappa^*) - (2\delta(1 + \gamma)) \|Y^*\| \|Y^* Y^T\|_F + 2\beta \|Y^*\| \|f(Y^*)\|_{\max(f)} \|Y^* Y^T\|_F;
\]
(ii) given any $Y \in R''_3$, we have
\[
\|\text{grad} h(Y)\|_F > (2\beta - \beta) \|Y^*\|^2 - (2\delta(1 + \gamma)) \|Y^*\| \|Y^* Y^T\|_F + 2\|Y^*\| \|f(Y^*)\|_{\max(f)} \|Y^* Y^T\|_F;
\]
(iii) given any $Y \in R'''_3$, we have
\[
\|\text{grad} h(Y)\|_F > (2(\gamma - 1) - 2\delta(\gamma + 1)) \gamma^{1/2} \|Y^* Y^T\|^{3/2}_F/\sqrt{r} - 2\gamma^{1/2} \|Y^* Y^T\|^{1/2}_F \|f(Y^*)\|_{\max(f)} \|Y^* Y^T\|_F/\sqrt{r};
\]
In particular, if $\beta > 1$, $\gamma > 1$, $\delta \leq \delta_{\min}$ and $\|f(Y^*)\|_{\max(f)} \leq \Psi$, where
\[
\delta_{\min} = \frac{\alpha \mu}{32\kappa^* \beta(1 + \gamma)} \left( \|Y^* Y^T\|_F \right)^2 \land \frac{\beta^2 - 1}{4(1 + \gamma)} \left( \|Y^*\| \|Y^* Y^T\|_F \right)^2 \land \frac{\gamma - 1}{4(1 + \gamma)};
\]
and
\[
\Psi = \frac{\alpha \mu}{32\kappa^* \beta} \sigma_r^2(Y^*) \land \frac{\beta^2 - 1}{4} \|Y^* Y^T\|_F \land \frac{\gamma - 1}{4} \|Y^* Y^T\|_F;
\]
we have the gradient norm $\|\text{grad} h(Y)\|_F$ in regions $R'_3, R''_3$ and $R'''_3$ are lower bounded by strict positive quantities given as follows
\[
R'_3: \|\text{grad} h(Y)\|_F > \alpha \mu \sigma_r^2(Y^*)/(8\kappa^*),
\]
\[
R''_3: \|\text{grad} h(Y)\|_F > (\beta^3 - \beta) \|Y^*\|^3,
\]
\[
R'''_3: \|\text{grad} h(Y)\|_F > (\gamma - 1) \gamma^{1/2} \|Y^* Y^T\|^{3/2}_F/\sqrt{r}.
\]
A direct corollary from Theorems 2, 3, 4 and 5 is given below.

Corollary 1 (Benign Landscape of (3) When $f$ Satisfies RSC and RSM). Suppose $\mu, \alpha, \beta, \gamma \geq 0$ in the definitions of $R_1, R_2, R'_3, R''_3$ and $R'''_3$ satisfy $\mu \leq 1/3$, $(1 - \mu/\kappa^*)^2 - 7\mu/3 > 0$, $\alpha < 2(\sqrt{2} - 1)$, $\beta > 1$ and $\gamma > 1$. Then if
\[
\delta \leq \frac{(1 - \mu/\kappa^*)^2 - 7\mu/3}{4(2(\kappa^* + \mu/\kappa^*)^2 + 7\mu/3)} \land \frac{(2(\sqrt{2} - 1) - \alpha) \sigma_r^2(Y^*)}{8(2\beta^2 \|Y^*\|^2 + (1 + \gamma) \|Y^* Y^T\|_F)} \land \delta_{\min}
\]
and
\[
\|f(Y^*)\|_{\max(f)} \leq \left( (1 - \mu/\kappa^*)^2 - 7\mu/3 \right) \sigma_r^2(Y^*)/4 \land \frac{2(\sqrt{2} - 1) - \alpha}{8} \sigma_r^2(Y^*) \land \Psi,
\]
where $\delta_{\min}$ and $\Psi$ are defined in (9) and (10), respectively, we have the global geometric landscape of (3) is benign in the following sense:

- in $R_1$, which is a geodesically convex set by Theorem 2, $h(Y)$ is geodesically strongly convex and smooth;
- in $R_2$, $\text{Hess} h(Y)$ has a negative eigenvalue and there exists an explicit escaping direction;
- in $R_3 := R'_3 \cup R''_3 \cup R'''_3$, $h(Y)$ has large gradient.
In addition, if there exists a Riemannian FOSP $\hat{Y}$ in $\mathcal{R}_1$, then $[\hat{Y}]$ is the unique global minimizer of (3) and the bound of the distance between $[\hat{Y}]$ and $[Y^*]$ is provided in (8).

**Remark 2.** Corollary 1 provides the first global landscape analysis for Buruer-Monteiro factorized matrix optimization objective under the Riemannian quotient geometry. Different from the previous global landscape analysis for matrix factorization under the Euclidean geometry [LLA+19, ZLTW21], we are able to show $h([Y])$ is actually geodesically strongly convex and smooth in $\mathcal{R}_1$ while $h(Y)$ is nowhere convex in $\mathcal{R}_1$ under the Euclidean geometry. Since the gradient descents under the Riemannian quotient geometry and the Euclidean geometry are exactly the same given the same stepsize, our geometric landscape analysis results give a fully geometric and intuitive explanation of the success of vanilla gradient descent under the Buruer-Monteiro factorization. Moreover, our results cover the setting $\nabla f(X^*)$ is non-zero but with a relatively small magnitude as well.

Under the embedded geometry of the set of fixed-rank matrices, [UV20] analyzed the landscape of (1) at stationary points when $f$ is quadratic and satisfies restricted strong convexity and smoothness properties. Comparing to their results, we provide a global geometric landscape analysis of (3) under the Riemannian quotient geometry and our results hold for a general $f$ satisfying RSC and RSM.

**Remark 3 (Conditions).** Suppose $\kappa^* = O(1)$ and $r = O(1)$, the conditions in (12) and (13) for $\delta$ and $\|\nabla f(X^*)\|_{\max(r)}$ can be summarized as $\delta \leq c_1$ and $\|\nabla f(X^*)\|_{\max(r)} \leq c_2\sigma_r(X^*)$ for some small universal positive constants $c_1, c_2$. The condition for $\delta$ is not sharp compared to the recent attempts on establishing a sharp threshold of $\delta$ to guarantee the absence of spurious local minimizers under the factorization formulation [ZSL19, ZBL21, MS22]. On the other hand, our geometric landscape results are much stronger and finer than theirs. In the meanwhile, we also note the condition for $\|\nabla f(X^*)\|_{\max(r)}$ is weak. This is because in typical statistical applications, e.g., the matrix trace regression in Example 1, $\|\nabla f(X^*)\|_{\max(r)}$ often matches the information-theoretic lower bound for estimating $X^*$ and $O(\sigma_r(X^*))$ is the initialization requirement for common local algorithms to converge [CW15, LHLZ20], so we often have $\|\nabla f(X^*)\|_{\max(r)} \leq \sigma_r(X^*)$ in the standard low-rank matrix recovery literature. Moreover, this condition automatically holds when $\nabla f(X^*) = 0$, which appears in typical noiseless applications. Finally, we note the error bound between the local minimizer in $\mathcal{R}_1$ and $Y^*$ provided in (8) also matches the information-theoretic lower bound in common applications [CW15, Section 4].

The landscape results in Corollary 1 imply that perturbed gradient descent is guaranteed to converge to the global minima in polynomial time under some weak assumptions for $f$ [SFF19, CB19, GHIJY15, JGN+17]. Moreover, the local geodesic strong convexity and smoothness immediately suggest that if there exists a Riemannian FOSP $\hat{Y}$ in $\mathcal{R}_1$, then vanilla GD initialized in $\mathcal{R}_1$ will stay in $\mathcal{R}_1$ and converge linearly to $[\hat{Y}]$ following the proof of [Bou20, Theorem 11.29].

### 4 Local Landscape Analysis of (3) When $f$ Satisfies Restricted Strict Convexity Property

In this section we describe the landscape of (3) under a weaker assumption on $f$. In particular, we assume $f$ satisfies the following restricted strict convexity property.

**Definition 2.** We say $f : \mathbb{R}^{p \times p} \rightarrow \mathbb{R}$ satisfies the $(r, 2r)$-restricted strict convexity property if for any $X, G \in \mathbb{R}^{p \times p}$ with $\text{rank}(X) \leq r$ and $\text{rank}(G) \leq 2r$, the Euclidean Hessian of $f$ satisfies $\nabla^2 f(X)[G, G] > 0$. 


It is clear that if $f$ satisfies $(2r, 4r)$-restricted strong convexity and smoothness property (for some $\delta \in [0, 1)$), then it satisfies $(r, 2r)$-restricted strict convexity property. Under this weaker assumption on $f$, we prove the following local geometric landscape results for $h([Y])$.

**Theorem 6.** (Local Landscape of $h([Y])$) Suppose $f$ satisfies the $(r, 2r)$-restricted strict convexity property. Consider $\hat{Y} \in \mathbb{R}^{p \times r}$ such that $\hat{Y} \hat{Y}^\top$ is a local minimizer of $\min_{X \in \mathbb{R}^{p \times r}, \text{rank}(X) \leq 2r} f(X)$ with rank $r$. Then there exists a neighborhood around $[\hat{Y}]$ on which $h([Y])$ is geodesically convex.

We note that compared to the results in Section 3, the landscape analysis in Theorem 6 is local and it is challenging to work out the explicit local geodesic convexity radius in this setting.

**5 Global Landscape Analysis of $H([Y])$ in (4)**

In this section, we provide the global landscape analysis of (4) under the Riemannian quotient geometry. This result is critical in establishing the global landscape analysis of (3) when $f$ satisfies RSC and RSM.

Recall $X^*$ is of rank $r$, $X^* = Y^* Y^* \top$ and $\kappa^* = \sigma_1(Y^*)/\sigma_r(Y^*)$ is the condition number of $Y^*$. First, it is clear $[Y^*]$ is the unique global minimizer of (4). Next, by Lemma 3, we have the following expressions of Riemannian gradient and Hessian of $H([Y])$:

\[
\begin{align*}
\text{grad} H([Y]) &= 2(YY^\top - X^*)Y, \\
\text{Hess} H([Y])[\theta_Y, \theta_Y] &= \|Y\theta^\top_Y + \theta_Y Y^\top\|_F^2 + 2\langle YY^\top - X^*, \theta_Y \theta^\top_Y \rangle. 
\end{align*}
\]  

(14)

Next, we show for the special objective (4), there is only one stationary point, which is $[Y^*]$.

**Theorem 7.** $[Y^*]$ is the unique Riemannian FOSP of (4).

The proof of Theorem 7 is presented in Appendix B.

Next, we show that the optimization problem (4) is geodesically strongly convex and smooth in a neighborhood of $Y^*$.

**Theorem 8** (Local Geodesic Strong Convexity of (4)). Suppose $0 \leq \mu \leq 1/3$. Then, for any $Y \in \mathbb{R}_1$,

\[
\begin{align*}
\lambda_{\min}(\text{Hess} H([Y])) &\geq (2 - \mu/\kappa^*)^2 - (14/3)\mu)\sigma_r^2(Y^*), \\
\lambda_{\max}(\text{Hess} H([Y])) &\leq 4(\sigma_1(Y^*) + \mu\sigma_r(Y^*)/\kappa^*)^2 + 14\mu\sigma_r^2(Y^*)/3.
\end{align*}
\]

In particular, if $\mu$ is further chosen such that $2(1 - \mu/\kappa^*)^2 - (14/3)\mu > 0$, we have $H([Y])$ is geodesically strongly convex and smooth in $\mathbb{R}_1$.

In the next two theorems, we show that, for $Y \notin \mathbb{R}_1$, either the Riemannian Hessian evaluated at $Y$ has a large negative eigenvalue, or the norm of the Riemannian gradient is large.

**Theorem 9** (Region with Negative Eigenvalue in the Riemannian Hessian of (4)). Given any $Y \in \mathbb{R}_2$, let $\theta_Y = Y - Y^* Q$, where $Q \in \mathbb{O}_r$ is the best orthogonal matrix aligning $Y^*$ and $Y$. Then

\[
\text{Hess} H([Y])[\theta_Y, \theta_Y] \leq (\alpha - 2(\sqrt{2} - 1))\sigma_r^2(Y^*)\|\theta_Y\|_F^2.
\]

In particular, if $\alpha < 2(\sqrt{2} - 1)$, we have $\text{Hess} H([Y])$ has at least one negative eigenvalue and $\theta_Y$ is an escaping direction.
Theorem 10 (Regions with Large Riemannian Gradient of (4)).

(i) \[\|\text{grad} \, H([Y])\|_F > \alpha \mu \sigma_r^3([Y^*])/(4\kappa^*), \quad \forall [Y] \in \mathcal{R}_g^r;\]

(ii) \[\|\text{grad} \, H([Y])\|_F \geq 2(\|Y\|^3 - \|Y\|\|Y^*\|^2) > 2(\beta^3 - \beta)\|Y^*\|^3, \quad \forall [Y] \in \mathcal{R}_g^r;\]

(iii) \[\langle \text{grad} \, H([Y]), Y\rangle > 2(1 - 1/\gamma)\|YY^\top\|_F^2, \quad \forall [Y] \in \mathcal{R}_g^r.\]

In particular, if \(\beta > 1\) and \(\gamma > 1\), we have the Riemannian gradient of \(H([Y])\) has large magnitude in all regions \(\mathcal{R}_g^r, \mathcal{R}_g^r,\) and \(\mathcal{R}_g^r\).

Remark 4 (Comparison of Radii for the Positive Definiteness of Riemannian Hessians of (4) under Other Geometries). We note (4) can also be formulated as an optimization problem under the embedded geometry,

\[
\min_{X \in \mathcal{M}_T^g} \tilde{H}(X) := \frac{1}{2}\|X - X^*\|_F^2,
\]

where \(\mathcal{M}_T^g = \{X \in \mathbb{S}^{p \times p}: \text{rank}(X) = r, X > 0\}\) is an embedded submanifold in \(\mathbb{S}^{p \times p}\) [VAV09]. In Lemma 5 below, whose proof is presented in Appendix B.5, we quantify the radius for the positive definiteness of the Riemannian Hessian under the embedded geometry.

Lemma 5. (Radius for Positive Definiteness of Hess \(\tilde{H}(X)\) under the Embedded Geometry) Define \(\mathcal{R}_1' := \{X \in \mathcal{M}_T^g : \|X - X^*\|_F \leq \mu'\sigma_r(X^*)\}\), then for any \(X \in \mathcal{R}_1'\), we have

\[\text{Hess} \, \tilde{H}(X)[\xi, \xi] \geq (1 - \frac{2\mu'}{1 - \mu})\|\xi\|_F^2, \quad \forall [\xi] \in \mathcal{T}_X \mathcal{M}_T^g.\]

Suppose \(X\) and \(X^*\) have decompositions \(YY^\top\) and \(Y^*Y^*\), respectively. The condition \(\|X - X^*\|_F \leq \mu'\sigma_r(X^*)\) in Lemma 5 implies \(d([Y], [Y^*]) \leq \sqrt{\frac{1}{2(\sqrt{2} - 1)}\mu'\sigma_r(Y^*)}\) by Lemma 12 Eq. (38).

So, compared with the radius of \(\mathcal{R}_1\), the radius for the positive definiteness of Riemannian Hessian under the embedded geometry is in general bigger by a factor of the condition number of \(Y^*\).

More generally, since the spectra of Riemannian Hessians of an optimization problem under two different geometries are sandwiched by each other at Riemannian FOSPs [LLZ21b], the positive definiteness of the Riemannian Hessian at \(X^*\) under one geometry implies the positive definiteness of the Riemannian Hessian under any another geometry. Moreover, because there always exists a convex geodesic ball at every point for any Riemannian manifold [DCFF92, Chapter 3.4], we have that under any geometry for fixed-rank PSD matrices, there exists a neighborhood around \(X^*\) such that the optimization problem (15) is geodesically strongly convex. In Theorem 8, we provide the geodesic strong convexity radius under \(\mathcal{M}_T^g\), it is interesting to figure out the radius under other common geometries, such as the embedded one, and explore under which geometry, the geodesic strong convexity radius of (15) achieves its maximum.

6 Proofs in Section 3

We first present a result where we compare the Riemannian gradients and Hessians of \(H([Y])\) in (4) and \(h([Y])\) in (3) when \(f\) satisfies RSC and RSM properties.

Proposition 1. Suppose \(f\) satisfies the \((2\mu, 4\mu)\)-restricted strong convexity and smoothness properties with parameter \(\delta\) given in Definition 1. For any \(Y \in \mathbb{R}^{p \times p}\), we have

\[
\|\text{grad} \, H([Y]) - \text{grad} \, h([Y])\|_F \leq 2\delta\|Y\|\|YY^\top - X^*\|_F + 2\|Y\|\|\nabla f(X^*)\|_{\max(r)}\|_F.
\]
Moreover, for any $\theta_Y \in \mathcal{H}_Y \overline{M}_r^\theta$, we have

$$\left| \text{Hess} H([\mathbf{Y}])[\theta_Y, \theta_Y] - \text{Hess} h([\mathbf{Y}])[\theta_Y, \theta_Y] \right| \leq \delta \|Y \theta_Y^T + \theta_Y Y^T\|_F^2 + 2 \delta \|YY^T - X^*\|_F \|\theta_Y \theta_Y^T\|_F + 2 \|\nabla f(X^*)\|_{\max(r)} \|\theta_Y \theta_Y^T\|_F. \tag{17}$$

Proof. First,

$$\|\text{grad} H([\mathbf{Y}]) - \text{grad} h([\mathbf{Y}])\|_F \leq \frac{\text{max}_{\Delta, |\Delta| = 1} 2 \langle (\nabla f(YY^T) - (YY^T - X^*)), \Delta \rangle}{\|Y \theta_Y + \theta_Y Y^T\|_F^2}$$

Lemma 3, (14)

$$\text{max}_{\Delta, |\Delta| = 1} 2 \langle (\nabla f(YY^T) - (YY^T - X^*)), \Delta \rangle$$

$$\leq \frac{\text{max}_{\Delta, |\Delta| = 1} 2 \langle (\nabla f(YY^T) - (YY^T - X^*)), \Delta Y^T \rangle}{\|Y \theta_Y + \theta_Y Y^T\|_F^2}$$

Lemma 14, 15

$$\leq \delta \|Y \theta_Y + \theta_Y Y^T\|_F^2 + 2 \delta \|YY^T - X^*\|_F \|\theta_Y \theta_Y^T\|_F + 2 \|\nabla f(X^*)\|_{\max(r)} \|\theta_Y \theta_Y^T\|_F$$

Second,

$$\left| \text{Hess} H([\mathbf{Y}])[\theta_Y, \theta_Y] - \text{Hess} h([\mathbf{Y}])[\theta_Y, \theta_Y] \right|$$

Lemma 3, (14)

$$\leq \|\nabla^2 f(YY^T)[Y \theta_Y + \theta_Y Y^T, Y \theta_Y + \theta_Y Y^T] - \|Y \theta_Y + \theta_Y Y^T\|_F^2 \|\nabla f(YY^T)[Y \theta_Y + \theta_Y Y^T, Y \theta_Y + \theta_Y Y^T] \|$$

$$+ 2 \langle (\nabla f(YY^T) - (YY^T - X^*)), \theta_Y \theta_Y^T \rangle\rangle$$

Lemma 14, 15

$$\leq \delta \|Y \theta_Y + \theta_Y Y^T\|_F^2 + 2 \delta \|YY^T - X^*\|_F \|\theta_Y \theta_Y^T\|_F + 2 \|\nabla f(X^*)\|_{\max(r)} \|\theta_Y \theta_Y^T\|_F$$

Here (a) is because $f$ satisfies the $(2r, 4r)$-restricted strong convexity and smoothness properties with parameter $\delta$ and $\text{rank}(YY^T) = r$, $\text{rank}(Y \theta_Y + \theta_Y Y^T) \leq 2r$. This finishes the proof of this proposition.

Next, we present the proofs for Theorems 3, 4 and 5.

6.1 Proof of Theorem 3

Suppose the best orthogonal matrix that aligns $\mathbf{Y}$ and $\mathbf{Y}^*$ is $\mathbf{Q}$. Then by definition, $\mathbf{Y} \in \mathcal{R}_1$ implies

$$\|Y - Y^* \mathbf{Q}\|_F \leq d([Y], [Y^*]) \leq \mu \sigma_r(Y^*)/\kappa^*.$$

Thus

$$\sigma_r(Y) = \sigma_r(Y - Y^* \mathbf{Q} + Y^* \mathbf{Q}) \geq \sigma_r(Y^*) - \|Y - Y^* \mathbf{Q}\| \geq (1 - \mu/\kappa^*) \sigma_r(Y^*), \tag{18}$$

$$\sigma_1(Y) = \sigma_1(Y - Y^* \mathbf{Q} + Y^* \mathbf{Q}) \leq \sigma_1(Y^*) + \|Y - Y^* \mathbf{Q}\| \leq \sigma_1(Y^*) + \mu \sigma_r(Y^*)/\kappa^*,$$

(19)
By Proposition 1, for any \( \theta_Y \in \mathcal{H}_Y \mathcal{M}_2^p \), we have

\[
\begin{align*}
&\left| \text{Hess } H([Y]) [\theta_Y, \theta_Y] - \text{Hess } h([Y]) [\theta_Y, \theta_Y] \right| \\
\leq &\delta \| \theta_Y^\top + \theta_Y Y^\top \|_F^2 + 2\delta \| Y Y^\top - X^* \|_F \| \theta_Y \theta_Y^\top \|_F + 2 \| \nabla f(X^*) \|_{\max(r)} \| \theta_Y \theta_Y^\top \|_F \\
\overset{\text{(a)}}{\leq} &4\delta \| \theta_Y \|_F^2 + 14\delta \| Y \|_F \| d([Y], [Y^*]) \|_F^2 / 3 + 2 \| \nabla f(X^*) \|_{\max(r)} \| \theta_Y \|_F^2.
\end{align*}
\]

(20)

here (a) is by Lemma 13 and Lemma 12 Eq. (40). Thus,

\[
\text{Hess } h([Y]) [\theta_Y, \theta_Y] \\
\geq \text{Hess } H([Y]) [\theta_Y, \theta_Y] - \| \text{Hess } H([Y]) [\theta_Y, \theta_Y] - \text{Hess } h([Y]) [\theta_Y, \theta_Y] \|.
\]

Theorem 8

\[
\begin{align*}
\begin{align*}
&\left( 2(1 - \mu / \kappa^*)^2 - 14\mu / 3 \right) \sigma_r^2(Y^*) \\
&- (4\delta(1 + \mu \sigma_r(Y^*) / \kappa^*)^2 + 14\delta \mu \sigma_r^2(Y^*) / 3 + 2 \| \nabla f(X^*) \|_{\max(r)} \| \theta_Y \|_F^2
\end{align*}
\end{align*}
\]

and

\[
\begin{align*}
&\| \text{Hess } h([Y]) [\theta_Y, \theta_Y] \| \\
\leq &\text{Hess } H([Y]) [\theta_Y, \theta_Y] + \| \text{Hess } H([Y]) [\theta_Y, \theta_Y] - \text{Hess } h([Y]) [\theta_Y, \theta_Y] \|.
\end{align*}
\]

Theorem 8

\[
\begin{align*}
&\left( 4(1 + \mu \sigma_r(Y^*) / \kappa^*)^2 + 14\mu \sigma_r^2(Y^*) / 3 \\
&+ (4\delta(1 + \mu \sigma_r(Y^*) / \kappa^*)^2 + 14\delta \mu \sigma_r^2(Y^*) / 3 + 2 \| \nabla f(X^*) \|_{\max(r)} \| \theta_Y \|_F^2 \right) \| \theta_Y \|_F^2.
\end{align*}
\]

If \( \mu \) is chosen such that \((1 - \mu / \kappa^*)^2 - 7\mu / 3 > 0 \) and

\[
\delta \leq \frac{(1 - \mu / \kappa^*)^2 - 7\mu / 3}{4(2(1 + \mu / \kappa^*)^2 + 7\mu / 3)}
\]

and \( \| \nabla f(X^*) \|_{\max(r)} \| F \leq \left( (1 - \mu / \kappa^*)^2 - 7\mu / 3 \right) \sigma_r^2(Y^*) / 4 \),

then

\[
\lambda_{\min}(\text{Hess } h([Y])) \geq (1 - \mu / \kappa^*)^2 - 7\mu / 3 \sigma_r^2(Y^*).
\]

Therefore, \( h([Y]) \) is geodesically strongly convex in \( \mathcal{R}_1 \) since \( \mathcal{R}_1 \) is geodesically convex by Theorem 2.

Let \( \tau := (1 - \mu / \kappa^*)^2 - 7\mu / 3 \) \( \sigma_r^2(Y^*) \). Suppose \( \hat{Y} \) is a Riemannian FOSP in \( \mathcal{R}_1 \), then we have \( \hat{Y} \) is the unique Riemannian FOSP in \( \mathcal{R}_1 \). This is because if \( \hat{Y}' \) is another FOSP in \( \mathcal{R}_1 \) and \( Q' \) is the best orthogonal matrix that aligns \( \hat{Y} \) and \( \hat{Y}' \). Then by the geodesic strong convexity, we have [Bou20, Chapter 11]:

\[
\begin{align*}
h([\hat{Y}']) \geq & h([\hat{Y}]) + \langle \text{grad } h([\hat{Y}]), \hat{Y}' Q' - \hat{Y} \rangle + \frac{T}{2} \| \hat{Y}' Q' - \hat{Y} \|_F^2, \\
h([\hat{Y}]) \geq & h([\hat{Y}']) + \langle \text{grad } h([\hat{Y}']), \hat{Y} Q'^\top - \hat{Y}' \rangle + \frac{T}{2} \| \hat{Y}' Q' - \hat{Y} \|_F^2.
\end{align*}
\]

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Notice \( \text{grad} h(\hat{Y}) = \text{grad} h([\hat{Y}^\prime]) = 0 \) by assumption and sum over the above two equations yields

\[
\tau \| \hat{Y}^\prime Q^\prime - \hat{Y} \|^2_F < 0.
\]

Since \( \tau > 0 \), we have \( \hat{Y}^\prime Q^\prime = \hat{Y} \) and \( [\hat{Y}] = [\hat{Y}^\prime] \). Moreover, \( \hat{Y} \) is a local minimizer as for any other \( Y \in \mathcal{R}_1 \):

\[
h([Y]) \geq h([\hat{Y}]) + \langle \text{grad} h([\hat{Y}]), YQ'' - \hat{Y} \rangle + \frac{\tau}{2} \| YQ'' - \hat{Y} \|_F^2 = h([\hat{Y}]) + \frac{\tau}{2} \| YQ'' - \hat{Y} \|_F^2,
\]

where \( Q'' \) is the best orthogonal matrix that aligns \( \hat{Y} \) and \( Y \).

Now, let \( Q \) be the best orthogonal matrix that aligns \( \hat{Y} \) and \( Y^* \). By a similar argument as above we have

\[
h([Y^*]) \geq h([\hat{Y}]) + \langle \text{grad} h([\hat{Y}]), Y^*Q - \hat{Y} \rangle + \frac{\tau}{2} \| Y^*Q - \hat{Y} \|_F^2,
\]

\[
h([\hat{Y}]) \geq h([Y^*]) + \langle \text{grad} h([Y^*]), \hat{Y}Q^\top - Y^* \rangle + \frac{\tau}{2} \| Y^*Q - \hat{Y} \|_F^2.
\]

Notice \( \text{grad} h([\hat{Y}]) = 0 \) by assumption and sum over the above two equations yields

\[
\tau \| Y^*Q - \hat{Y} \|_F^2 \leq \langle \text{grad} h([Y^*]), Y^* - \hat{Y} Q^\top \rangle = \langle 2\nabla f(Y^*Y^\top)Y^*, Y^* - \hat{Y} Q^\top \rangle \leq 2 \| \nabla f(Y^*Y^\top)Y^* \|_F \| Y^* - \hat{Y} Q^\top \|_F.
\]

So (21) yields

\[
\| Y^*Q - \hat{Y} \|_F \leq \frac{2}{\tau} \| \nabla f(Y^*Y^\top)Y^* \|_F \leq \frac{2}{\tau} \max_{\| \Delta \|_F \leq 1} \langle \nabla f(Y^*Y^\top), \Delta Y^* \rangle
\]

\[
= \frac{2}{\tau} \max_{\| \Delta \|_F \leq 1} \langle \nabla f(Y^*Y^\top), \Delta Y^* \rangle
\]

\[
\leq \frac{2}{\tau} \| Y^* \|_F \| (\nabla f(Y^*))_{\text{max}(r)} \|_F.
\]

### 6.2 Proof of Theorem 4

First, we know \( \theta_Y \in \mathcal{H}_Y \mathcal{M}_{\| \cdot \|_F}^q \) by Lemma 2. By Theorem 9, we have \( \text{Hess} H([\hat{Y}])[\theta_Y, \theta_Y] \leq (\alpha - 2(\sqrt{2} - 1))\sigma^2_Y(Y^*)\|\theta_Y\|_F^2 \).

In addition, Proposition 1 implies

\[
\left| \text{Hess} H([\hat{Y}])[\theta_Y, \theta_Y] - \text{Hess} h([\hat{Y}])[\theta_Y, \theta_Y] \right|
\leq \delta \| Y_{\theta_Y}^\top + \theta_Y Y^\top \|_F^2 + 2 \delta \| YY^\top - X^* \|_F^2 \| \theta_Y \theta_Y^\top \|_F^2 + 2 \| (\nabla f(X^*))_{\text{max}(r)} \|_F \| \theta_Y \theta_Y^\top \|_F^2
\]

\[
\leq 4 \delta \| Y_{\theta_Y}^\top \|_F^2 + 2 \delta \| YY^\top - X^* \|_F^2 \| \theta_Y \theta_Y^\top \|_F^2 + 2 \| (\nabla f(X^*))_{\text{max}(r)} \|_F \| \theta_Y \theta_Y^\top \|_F^2
\]

\[
\leq 2 \delta \left( 2\beta^2 \| Y^* \|_F^2 + (1 + \gamma) \| Y^*Y^* \|_F^2 \right) \| \theta_Y \theta_Y^\top \|_F^2 + 2 \| (\nabla f(X^*))_{\text{max}(r)} \|_F \| \theta_Y \theta_Y^\top \|_F^2,
\]

where (a) is because \( Y \in \mathcal{R}_2 \). Thus,

\[
\text{Hess} h([\hat{Y}])[\theta_Y, \theta_Y] \leq \text{Hess} H([\hat{Y}])[\theta_Y, \theta_Y] + \left| \text{Hess} H([\hat{Y}])[\theta_Y, \theta_Y] - \text{Hess} h([\hat{Y}])[\theta_Y, \theta_Y] \right|
\leq (\alpha - 2(\sqrt{2} - 1))\sigma^2_Y(Y^*)\|\theta_Y\|_F^2
\]

\[
+ 2 \delta \left( 2\beta^2 \| Y^* \|_F^2 + (1 + \gamma) \| Y^*Y^* \|_F^2 \right) \| \theta_Y \theta_Y^\top \|_F^2 + 2 \| (\nabla f(X^*))_{\text{max}(r)} \|_F \| \theta_Y \theta_Y^\top \|_F^2.
\]
So if
\[ \delta \leq \frac{(2\sqrt{2} - 1 - \alpha)\sigma^2_r(Y^*)}{8(2\beta^2\|Y^*\|^2 + (1 + \gamma)\|Y^*Y^*^T\|_F)} \text{ and } \|(\nabla f(X^*))_{\max(r)}\|_F \leq \frac{2(\sqrt{2} - 1 - \alpha)\sigma^2_r(Y^*)}{8}, \]
then
\[ \left| \text{Hess } H([Y])[\theta_Y, \theta_Y] - \text{Hess } h([Y])[\theta_Y, \theta_Y] \right| \leq \frac{2(\sqrt{2} - 1 - \alpha)\sigma^2_r(Y^*)}{2} \|\theta_Y\|^2. \]
Thus,
\[ \text{Hess } h([Y])[\theta_Y, \theta_Y] \leq \text{Hess } H([Y])[\theta_Y, \theta_Y] + \left| \text{Hess } H([Y])[\theta_Y, \theta_Y] - \text{Hess } h([Y])[\theta_Y, \theta_Y] \right| \leq \frac{\alpha - 2(\sqrt{2} - 1)}{2}\sigma^2_r(Y^*)\|\theta_Y\|^2. \]

### 6.3 Proof of Theorem 5

We prove the results for the three regions separately.

**When** \( Y \in \mathcal{R}_3' \), **By Proposition 1,**
\[ \|\text{grad } H([Y]) - \text{grad } h([Y])\|_F \leq 2\delta\|Y\|Y^T - X^*\|_F + 2\|Y\|\|(\nabla f(X^*))_{\max(r)}\|_F \leq 2\delta(1 + \gamma)\|Y^*\||Y^*Y^*^T\|_F + 2\beta\|Y^*\||(\nabla f(X^*))_{\max(r)}\|_F. \]
Thus, combining the above result with Theorem 10, we have
\[ \|\text{grad } h([Y])\|_F \geq \|\text{grad } H([Y])\|_F - \|\text{grad } H([Y]) - \text{grad } h([Y])\|_F \geq \alpha\mu\sigma^3_r(Y^*)/(4\kappa^*) - (2\delta(1 + \gamma))\|Y^*\||Y^*Y^*^T\|_F + 2\beta\|Y^*\||(\nabla f(X^*))_{\max(r)}\|_F. \]
In particular, if \( \delta \leq \frac{\alpha\mu\sigma^2_r(Y^*)}{32\kappa^*\beta(1 + \gamma)}\|\theta_Y\|^2 \) and \( \|(\nabla f(X^*))_{\max(r)}\|_F \leq \frac{\alpha\mu\sigma^2_r(Y^*)}{32\kappa^*\beta}\sigma^2_r(Y^*), \) we have
\[ \|\text{grad } h([Y])\|_F \geq \alpha\mu\sigma^3_r(Y^*)/(8\kappa^*). \] (22)

**When** \( Y \in \mathcal{R}_3'' \), **By Proposition 1,**
\[ \|\text{grad } H([Y]) - \text{grad } h([Y])\|_F \leq 2\delta\|Y\|Y^T - X^*\|_F + 2\|Y\|\|(\nabla f(X^*))_{\max(r)}\|_F \leq 2\delta(1 + \gamma)\|Y\||Y^*Y^*^T\|_F + 2\|Y\|\|(\nabla f(X^*))_{\max(r)}\|_F. \]
Thus
\[ \|\text{grad } h([Y])\|_F \geq \|\text{grad } H([Y])\|_F - \|\text{grad } H([Y]) - \text{grad } h([Y])\|_F \geq 2\|\|Y^3 - \|Y\|^2\|^2\|_F - (2\delta(1 + \gamma))\|Y\||Y^*Y^*^T\|_F + 2\|Y\|\|(\nabla f(X^*))_{\max(r)}\|_F; \]
In particular, if \( \delta \leq \frac{\beta^2 - 1}{4(1 + \gamma)}\|Y^3\|^2 \leq \frac{1}{4(1 + \gamma)}\|\|Y^3\|^2\|^2\|_F \) and \( \|(\nabla f(X^*))_{\max(r)}\|_F \leq \frac{\beta^2 - 1}{4}\|Y^*\|^2 < \frac{\|Y^3\|^2}{4\|Y^*\|^2}, \) we have
\[ \|\text{grad } h([Y])\|_F \geq \|Y^3 - \|Y\|^2\|^2 > (\beta^3 - \beta)\|Y^*\|^3. \] (23)
When $Y \in \mathcal{R}_3^n$. We have

$$\left|\frac{\nabla H([Y]) - \nabla h([Y])}{\sqrt{r}}\right|_{F} \leq \left(2\frac{1}{\gamma} - \gamma\right)\left(2\delta + \frac{1}{\gamma}\right)\left\|\nabla f([X])\right\|_{\infty} \frac{\left\|\nabla f([X])\right\|_{\infty}^{1/2}}{\sqrt{r}}.$$

Moreover, since $\left\|\nabla f([X])\right\|_{\infty} \leq \frac{1}{\gamma} \left\|\nabla f([X])\right\|_{2}$, we have

$$\left\|\nabla h([Y])\right\|_{F} \leq \frac{1}{\gamma} \left\|\nabla f([X])\right\|_{\infty} \frac{\left\|\nabla f([X])\right\|_{\infty}^{1/2}}{\sqrt{r}}.$$
Next, we present a key proposition in proving Theorem 2 which is based on [DCFF92, Chapter 3.4, Lemma 4.1 and Proposition 4.2]. The result in [DCFF92], which is a classic result in Riemannian geometry, holds for generic Riemannian manifolds, but it is only when revisiting its proof that we can provide an explicit quantitative estimate for the radius of geodesic convexity around an arbitrary point in the manifold $\mathcal{M}_{r+}$.

We introduce the following notation. Given any $Y \in \mathbb{R}^{m \times r}$ and $x > 0$ we use $S_x([Y]) := \{[Y_1]: d([Y_1], [Y]) = x\}$ to denote the geodesic sphere of radius $x$ centered at $[Y]$.

**Proposition 2.** Given $Y \in \mathbb{R}^{m \times r}$, any geodesic in $\mathcal{M}_{r+}$ that is tangent at $[Y]$ to the geodesic sphere $S_\rho([Y])$ of radius $\rho$ with $\rho < c_Y := \sigma_r(Y)$ stays out of the geodesic ball $B_\rho([Y])$ for some neighborhood of $[Y]$.

**Proof.** Denote $T_1S_\rho([Y])$ as the unit tangent bundle restricted to geodesic sphere $S_\rho([Y])$, that is:

$$T_1S_\rho([Y]) = \{([Y'], \xi_{[Y']}) : [Y'] \in S_\rho([Y]), \xi_{[Y']} \in T_{[Y']}(\mathcal{M}_{r+}), g_{[Y']}([\xi_{[Y']}, \xi_{[Y']}) = 1\}.$$

Let $\gamma : I \times T_1S_\rho([Y]) \rightarrow \mathcal{M}_{r+}$, $I = (-\epsilon, \epsilon)$ for some small enough $\epsilon > 0$, be a differentiable map such that $t \mapsto \gamma(t, [Y'], \xi_{[Y']})$ is the geodesic that at the instant $t = 0$ passes through $[Y']$ with velocity $\xi_{[Y']}$, and $\|\xi_{[Y']}\|_F = 1$. Since $[Y'] \in S_\rho([Y])$ and $\rho < \sigma_r(Y)$, by Lemma 9, $Y^T Y$ is nonsingular, so we can define $u(t, [Y'], \xi_{[Y']}) = \log([Y' \gamma(t, [Y'], \xi_{[Y']}])$). In addition, let $F : I \times T_1S_\rho([Y]) \rightarrow \mathbb{R}$ be

$$F(t, [Y'], \xi_{[Y']}) = g_{[Y']}(u(t, [Y'], \xi_{[Y']}), u(t, [Y'], \xi_{[Y']}))$$

$$= g_{[Y']}\left(\log([Y' \gamma(t, [Y'], \xi_{[Y']}]), \log([Y' \gamma(t, [Y'], \xi_{[Y']}])\right)$$

$$= \|([Y' + t\xi_{[Y']}], \xi_{[Y']}Q_t - Y, (Y' + t\xi_{[Y']}Q_t - Y)\|_F^2$$

$$= t^2 + 2t\langle Y', \xi_{[Y']}\rangle - 2tr(\Sigma_t) + \|Y'\|_F^2 + \|Y\|_F^2,$$

where $Y^T (Y' + t\xi_{[Y']})$ has SVD $Q_{U_t}\Sigma_tQ_{V_t}^T$ and $Q_t = Q_{V_t}Q_{U_t}^T$. Geometrically, $F$ measures the square distance from $[Y]$ to points along the geodesic $\gamma$. We now discuss how the function $F$ behaves around $t = 0$.

We have both $u$ and $F$ are differentiable and $\frac{\partial F}{\partial t} = 2g_{[Y']}(\partial u/\partial t, u)$. If a geodesic $\gamma$ is tangent to the geodesic sphere $S_\rho([Y])$ at the point $[Y'] = \gamma(0, [Y'], \xi_{[Y']})$, then from the Gauss Lemma [DCFF92, Chapter 3, Lemma 3.5], we have

$$\frac{\partial F}{\partial t}(0, [Y'], \xi_{[Y']}) = 2g_{[Y']}\left(\frac{\partial u}{\partial t}(0, [Y'], \xi_{[Y']}), u(0, [Y'], \xi_{[Y']})\right) = 0.$$

This means $(0, [Y'], \xi_{[Y']})$ is a stationary point of $F$ at $t = 0$ for fixed $[Y']$ on $S_\rho([Y])$ and fixed $\xi_{[Y']}$. If we can show that we have $\frac{\partial F}{\partial t}(0, [Y'], \xi_{[Y']}) > 0$, then we would be able to conclude that $t = 0$ is a local minimizer of $F(\cdot, [Y'], \xi_{[Y']})$. Given the geometric interpretation of the function $F$, this would further imply that there exists a neighborhood of $[Y']$ such that the geodesic $\gamma$ stays out of the geodesic ball $B_\rho([Y])$. We thus focus on proving the positivity of the second derivative.

Let $Q_{U_t} = [q_{u11}, \ldots, q_{ur}]$, $Q_{V_t} = [q_{v11}, \ldots, q_{v}]$, $\Sigma_t = \text{diag}(\sigma_1, \ldots, \sigma_r)$. From (26), for any
we know

we provide a simple formula for the second derivatives of $D_{05}$ we

in this way some results in the literature, e.g. \[ \text{Lemma 8} \]

Moreover, in Lemma 8 we give a sharp bound on the condition number of the orthogonal Procrustes problem, complementing

pr denoted by $S$.

Therefore, \( B_{rV}([Y]) \) is a totally normal neighborhood of \([Y]\), so for any \([Y_1], [Y_2] \in B_{rV}([Y])\), there exists a unique minimizing geodesic $\gamma$ joining them, and its
length is less than $2r_Y$. Let us now consider $[Y_1], [Y_2] \in B_\varepsilon([Y]) \subseteq B_r([Y])$. We show that $\gamma$ is contained in $B_r([Y])$.

Notice that for any point $[Y']$ on $\gamma$, $\min(d([Y'], [Y_1]), d([Y'], [Y_2])) < r_Y$. Then we have
\[
d([Y'], [Y]) \leq \min(d([Y'], [Y_1]) + d([Y_1], [Y]), d([Y'], [Y_2]) + d([Y_2], [Y])) < 2r_Y = \gamma_r := 2\sigma_r(Y)/3.
\] (29)

Let $[\tilde{Y}]$ be the point in $\gamma$ such that the maximum distance from $[Y]$ to $\gamma$ is attained, and denote this distance by $\rho$. If $[\tilde{Y}]$ is either $[Y_1]$ or $[Y_2]$, then we are done. If not, we have that the points of $\gamma$ in any neighborhood of $[\tilde{Y}]$ remain in the closure of $B_\rho([Y])$ while $\gamma$ is tangential to $S_\rho([Y])$ at $[\tilde{Y}]$. Since $\rho < c_Y := \sigma_r(Y)$ by (29), this contradicts Proposition 2. This finishes the proof. \qed

8 Conclusion and Discussion

In this paper, we have studied the optimization landscape of the Burer-Monteiro factorized objective for a general fixed-rank PSD matrix optimization problem under the Riemannian quotient geometry. When $f$ satisfies the restricted strong convexity and smoothness properties, we show the landscape of the factorized objective is benign by characterizing its geometry in the entire domain. When $f$ satisfies a weaker restricted strict convexity property, we show there exists a neighborhood near the local minimizer such that the factorized objective is geodesically convex.

There are many interesting extensions to the results in this paper to be explored in the future. First, the current requirement on $\delta$ to guarantee the benign global landscape of (3) in Corollary 1 may not be sharp and it would be interesting to explore whether we can establish similar landscape results with a sharper dependence on $\delta$. Second, it is well known that geometry plays a central role in Riemannian optimization. Picking a proper metric that maximizes the geodesic convexity radius of the objective is favorable. As we have mentioned in Remark 4, it would be interesting to explore under what geometries the geodesic strong convexity radii of (3) and (4) are maximized. Third, the landscape of the Burer-Monteiro factorized objective can be more complicated when $f$ does not satisfy RSC and RSM, see [YZLS22]. An interesting future research direction is to explore under what assumptions on $f$ will the Burer-Monteiro factorization continue to work for efficient optimization. Finally, in this work we have mainly focused on the exact-parameterization setting, i.e., the number of columns of $Y$ is equal to the rank of the parameter of interest $X^*$. It would be very interesting to try to characterize the landscape of (3) in the over-parameterized setting, i.e., when the number of columns of $Y$ is greater than the rank of $X^*$. In that case, the optimality conditions for $X^*$ need to be carefully developed as $X^*$ is merely a boundary point of the working manifold.

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A Proofs in Section 4

A.1 Proof of Theorem 6

First, given any non-zero \( \theta \in \mathcal{H}_Y \mathcal{M}_{r,+,} \), we have \( \theta \in \mathcal{M}_{r,+,} \) is a tangent vector of the set \( \{ X \in \mathbb{S}^{p \times p} \succ 0, \text{rank}(X) \leq 2r \} \) [RW09, Chapter 6]. Since \( \hat{Y} \hat{Y}^T \) is a rank \( r \) local minimizer of \( \min_{X \in \mathbb{S}^{p \times p} \succ 0, \text{rank}(X) \leq 2r} f(X) \), the first-order optimality condition implies that \( \langle \nabla f(\hat{Y} \hat{Y}^T), C \rangle \geq 0 \) holds for any tangent vector \( C \). Thus, we have \( \langle \nabla f(\hat{Y} \hat{Y}^T), \theta \hat{Y} \hat{Y}^T \rangle \geq 0 \) by setting \( C = \theta \hat{Y} \hat{Y}^T \).

Thus, by Lemma 3, we have

\[
\text{Hess} h([\hat{Y}],[\theta, \hat{Y}]) = \nabla^2 f(\hat{Y} \hat{Y}^T)[\hat{Y} \theta \hat{Y}^T + \theta \hat{Y} \hat{Y}^T, \hat{Y} \theta \hat{Y}^T + \theta \hat{Y} \hat{Y}^T] + 2\langle \nabla f(\hat{Y} \hat{Y}^T), \theta \hat{Y} \hat{Y}^T \rangle \\
\geq \nabla^2 f(\hat{Y} \hat{Y}^T)[\hat{Y} \theta \hat{Y}^T + \theta \hat{Y} \hat{Y}^T, \hat{Y} \theta \hat{Y}^T + \theta \hat{Y} \hat{Y}^T] > 0,
\]

where the last inequality is because \( f \) satisfies the \((r,2r)\)-restricted strict convexity property and because \( \hat{Y} \theta \hat{Y}^T + \theta \hat{Y} \hat{Y}^T \) is nonzero by virtue of Lemma 13. This implies that there is a neighborhood \( V \) around \([\hat{Y}]\) such that \( \text{Hess} h([\hat{Y}]) \succ 0 \) for any \([\hat{Y}] \in V \). The result follows by combining the above with Theorem 2.

B Proofs in Section 5

B.1 Proof of Theorem 7

Recall \( X^* \) has eigendecomposition \( U^* \Sigma^* U^{*T} \), where recall by this we mean the economic/reduced version of the eigendecomposition. For any FOSP \( Y \in \mathbb{R}^{p \times p}_+ \), let \( B = U^* Y \) and \( W = Y - P_{U^*} Y \).

Then \( Y = U^* B + W \). So

\[
(YY^T - X^*)Y = ((U^* B + W)(U^* B + W)^T - X^*) (U^* B + W) \\
= ((U^* B + W)(U^* B + W)^T - X^*) U^* B + ((U^* B + W)(U^* B + W)^T - X^*) W \\
W^T U^* = 0 (U^* B + W)B^T B - U^* \Sigma^* B + (U^* B + W) W^T W. \tag{30}
\]

Since \( Y \) is a FOSP, we have \( (YY^T - X^*)Y = 0 \). We split the analysis into the following three cases.

Case 1: \( B = 0 \). In this case, \( (YY^T - X^*)Y = 0 \) and (30) imply \( WW^T W = 0 \), which further implies \( W^T W W^T W = 0 \). Since a PSD matrix has a unique principal square root [JOR01], we have \( W^T W = 0 \). Thus \( W = 0 \). This shows \( Y = U^* B + W = 0 \), and thus \( Y \) cannot belong to \( \mathbb{R}^{p \times p}_+ \). It follows that \( B = 0 \) can not occur at FOSPs.

Case 2: \( W = 0 \). In this case, \( Y = U^* B \in \mathbb{R}^{p \times p}_+ \), so \( B \) is invertible and \( Y \) lies in the column span of \( U^* \). The fact that \( (YY^T - X^*)Y = 0 \) and (30) imply

\[
U^*(BB^T - \Sigma^*) B = 0 \implies BB^T - \Sigma^* = 0.
\]

Thus, \( YY^T = U^* BB^T U^{*T} = U^* \Sigma^* U^{*T} = X^* \), which implies \([Y] = [Y^*] \) by Lemma 11.

Case 3: \( B \neq 0, W \neq 0 \). Since \( W \) lies in the column space of \( U^* \), \( (YY^T - X^*)Y = 0 \) and (30) imply

\[
P_{U^*}(YY^T - X^*)Y = 0 \implies W(B^TB + W^TW) = 0.
\]
Since $Y^T Y = B^T B + W^T W$, we have $WY^T Y = 0$. As $Y \in \mathbb{R}_+^{p \times t}$, $Y^T Y$ is positive definite, this implies $W = 0$ and contradicts with the assumption. So $B \neq 0$, $W \neq 0$ cannot happen at FOSPs.

From the above, we deduce that only Case 2 can happen at FOSPs and thus $[Y^*]$ is the unique FOSP in (4).

**B.2 Proof of Theorem 8**

Denote by $Q$ the best orthogonal matrix that aligns $Y$ and $Y^*$. Then by the assumption on $Y$ we have

$$||Y - Y^*Q|| \leq ||Y - Y^*Q||_F = d([Y], [Y^*]) \leq \mu \sigma_r(Y^*)/\kappa^*.$$  

(31)

Thus

$$\sigma_r(Y) = \sigma_r(Y - Y^*Q + Y^*Q) \geq \sigma_r(Y^*) - ||Y - Y^*Q|| \geq (1 - \mu/\kappa^*) \sigma_r(Y^*),$$  

(32)

$$\sigma_1(Y) = \sigma_1(Y - Y^*Q + Y^*Q) \leq \sigma_1(Y^*) + ||Y - Y^*Q|| \leq \sigma_1(Y^*) + \mu \sigma_r(Y^*)/\kappa^*,$$

where the first inequality in each line follows from Weyl’s theorem [Ste98, Theorem 4.29].

To provide bounds for the spectrum of $\nabla^2 H([Y])$, we just need to compute lower and upper bounds for $\nabla^2 H([Y])[\theta_Y, \theta_Y]$ for any $\theta_Y \in \mathcal{H}_Y \mathcal{M}_{r+}^q$. First,

$$\nabla^2 H([Y])[\theta_Y, \theta_Y] = (14) \frac{1}{2} ||Y \theta_Y + \theta_Y Y^T||_F^2 + 2 \langle YY^T - X^*, \theta_Y \theta_Y \rangle$$

Lemma 13 Eq. (40)

$$\geq (2 \sigma_1(Y) \sigma_r(Y^*) \theta_Y, \theta_Y \theta_Y)$$

$$\geq 2 \sigma_1^2(Y) \sigma_r(Y^*) ||\theta_Y||_F^2 - 2 \langle YY^T - X^*, \theta_Y \theta_Y \rangle$$

(32), Lemma 12 Eq. (40)

$$\geq (2 - \mu/\kappa^*)^2 \sigma_1^2(Y^*) ||\theta_Y||_F^2 - 2 \langle YY^T - X^*, \theta_Y \theta_Y \rangle$$

$$\geq (2 - \mu/\kappa^*)^2 - (14/3) \mu \sigma_r(Y^*) \sigma_1(Y^*) ||\theta_Y||_F^2.$$

Likewise,

$$\nabla^2 H([Y])[\theta_Y, \theta_Y] = (14) \frac{1}{2} ||Y \theta_Y + \theta_Y Y^T||_F^2 + 2 \langle YY^T - X^*, \theta_Y \theta_Y \rangle$$

Lemma 13 Eq. (40)

$$\leq (4 \sigma_1(Y) \sigma_r(Y^*) \theta_Y, \theta_Y \theta_Y)$$

(32), Lemma 12 Eq. (40)

$$\leq (4 \sigma_1(Y^*) + \mu \sigma_r(Y^*) \kappa^*)^2 + 14 \mu \sigma_r^2(Y^*)/3 \||\theta_Y||_F^2.$$  

From the above we conclude that when $\mu$ is chosen such that $2 (1 - \mu/\kappa^*)^2 - (14/3) \mu > 0$, we have $H([Y])$ in (4) is geodesically strongly convex and smooth in $\mathcal{R}_1$ as $\mathcal{R}_1$ is a geodesically convex set by Theorem 2.

**B.3 Proof of Theorem 9**

First, notice $\theta_Y \in \mathcal{H}_Y \mathcal{M}_{r+}$ and $\||\theta_Y||_F = d([Y], [Y^*])$ by Lemma 2. In addition, a simple calculation yields

$$YY^T - X^* + \theta_Y \theta_Y^T = Y \theta_Y^T + \theta_Y Y^T.$$  

(33)

Then (14) implies

$$\langle \nabla H([Y]), \theta_Y \rangle = \langle 2(YY^T - X^*)Y, \theta_Y \rangle$$

$$= \langle YY^T - X^*, \theta_Y Y^T + Y \theta_Y^T \rangle = \langle YY^T - X^*, \theta_Y \theta_Y^T + YY - X^* \rangle,$$

(34)
and

$$
\text{Hess} H(\hat{Y})[\theta_Y, \theta_Y] = \|Y \theta^T_Y + \theta_Y Y^T\|^2_F + 2\langle YY^T - X^*, \theta_Y \theta_Y \rangle
$$

$$
\overset{(33)}{=} \|YY^T - X^* + \theta_Y \theta_Y\|^2_F + 2\langle YY^T - X^*, \theta_Y \theta_Y \rangle
$$

$$
- \|\theta_Y \theta_Y\|^2_F + \|YY^T - X^*\|^2_F + 4\langle YY^T - X^*, \theta_Y \theta_Y \rangle
$$

$$
\overset{(34)}{=} \|\theta_Y \theta_Y\|^2_F - 3\|YY^T - X^*\|^2_F + 4\langle \text{grad} H(\hat{Y}), \theta_Y \rangle
$$

Lemma 12 Eq. (39)

$$
\leq - \|YY^T - X^*\|^2_F + 4\|\text{grad} H(\hat{Y})\|_F \|\theta_Y\|_F
$$

Lemma 12 Eq. (38)

$$
\leq - (2(\sqrt{2} - 1)) \sigma^2_2(Y^*) \|\theta_Y\|_F^2,
$$

here (a) is because $\|\text{grad} H(\hat{Y})\|_F \leq \alpha \mu \sigma^2_2(Y^*)/(4\alpha)$ implies $\|\text{grad} H(\hat{Y})\|_F \leq \alpha \|\text{grad} H(\hat{Y})\|_F / \sigma^2_2(Y^*)/4 = \alpha \|\theta_Y\|_F / \sigma^2_2(Y^*)/4$.

This finishes the proof of this theorem.

B.4 Proof of Theorem 10

First, if $Y \in \mathcal{R}'_3$, the fact that $\|\text{grad} H(\hat{Y})\|_F$ is large holds by definition. Next, we show that the norm of the gradient is large when $Y \in \mathcal{R}''_3$ and when $Y \in \mathcal{R}'''_3$.

(When $Y \in \mathcal{R}'_3$): Suppose $U \Sigma V^T$ and $U^* \Sigma^* V^T$ are the SVDs of $Y$ and $Y^*$, respectively. Then

$$
\|\text{grad} H(\hat{Y})\|_F \overset{(14)}{=} 2\|YY^T - X^*Y^*\|_F
$$

$$
= 2\|U \Sigma^3 V^T - U^* \Sigma^2 U^T U \Sigma V^T\|_F
$$

$$
= 2\|U \Sigma^3 - U^* \Sigma^2 U^T U \Sigma\|_F
$$

$$
\geq \|P_U(U \Sigma^3 - U^* \Sigma^2 U^T U \Sigma)\|_F
$$

$$
\geq 2\|\Sigma^3 - U^T U^* \Sigma^2 U^T U \Sigma\|_F
$$

$$
\geq 2\|\Sigma^3 - U^T U^* \Sigma^2 U^T U \Sigma\|_F
$$

$$
\geq 2\|\Sigma^3 - \|X\|_F\|\Sigma^2\|_F (\beta^3 - \beta)\|\Sigma^3\|_F
$$

$$
\geq 2\|\Sigma^3 - \|X\|_F^2\|\Sigma^2\|_F (\beta^3 - \beta)\|\Sigma^3\|_F
$$

here (a) is because $Y \in \mathcal{R}''_3$, and the subscript $[1, 1]$ indicates the entry in the first row and first column of the corresponding matrix.

(When $Y \in \mathcal{R}'''_3$): In this case, we have

$$
\langle \text{grad} H(\hat{Y}), Y \rangle \overset{(14)}{=} \langle 2YY^T - X^*, Y, Y \rangle
$$

$$
= \langle 2YY^T - X^*, YY^T \rangle
$$

$$
\geq 2\|YY^T\|_F^2 - 2\|X^*, YY^T\|_F \geq 2\|YY^T\|_F^2 - 2\|YY^T\|_F \|Y^*Y^*\|_F
$$

$$
\geq 2(1 - 1/\gamma) \|YY^T\|_F^2.
$$
B.5 Proof of Lemma 5

Suppose \( X \in \mathcal{M}_{r+}^f \) has eigendecomposition \( U \Sigma U^T, \xi \) \( = [U \ U_\perp] \begin{bmatrix} S & D^T \\ D & 0 \end{bmatrix} [U \ U_\perp]^T \in T_x \mathcal{M}_{r+}^f \).

Then by [LLZ21a, Proposition 2], we have

\[
\text{Hess} \, \tilde{H}(X)[\xi_x, \xi_x] = \| \xi_x \|^2_F + 2 \langle X - X^*, U_\perp \Sigma^{-1} D^T U_\perp \rangle.
\]

Moreover,

\[
2 \langle X - X^*, U_\perp \Sigma^{-1} D^T U_\perp \rangle = 2 \langle X - X^*, U_\perp D U^T U \Sigma^{-1} U^T D U_\perp \rangle = 2 \langle X - X^*, P_{U_\perp} \xi_x P_{U} X^{-1} P_{U} \xi_x P_{U_\perp} \rangle.
\]

Thus

\[
|2 \langle X - X^*, U_\perp \Sigma^{-1} D^T U_\perp \rangle| \\
\leq 2\| X - X^* \|_F \| P_{U_\perp} \xi_x P_{U} \|^2 \sigma_1(X^{-1}) = 2 \| X - X^* \|_F \| \xi_x \|_F^2 / \sigma_r(X).
\]

Finally, \( \| X - X^* \|_F \leq \mu' \sigma_r(X^*) \) implies \( \sigma_r(X) \geq \sigma_r(X^*) - \| X - X^* \| / (1 - \mu' \sigma_r(X^*)) \). Thus

\[
\text{Hess} \, \tilde{H}(X)[\xi_x, \xi_x] \geq \| \xi_x \|^2_F - 2 \| X - X^* \|_F \| \xi_x \|_F^2 / \sigma_r(X)
\]

\[
\geq \| \xi_x \|^2_F - 2 \| X - X^* \|_F \| \xi_x \|_F^2 / ((1 - \mu' \sigma_r(X^*)) \geq (1 - \frac{2 \mu'}{1 - \mu'}) \| \xi_x \|_F^2.
\]

C Auxiliary Lemmas for Theorem 2

Lemma 7. ([MA20, Theorem 6.3]) For any \( Y \in \mathbb{R}^{p \times r}_s \), the injectivity radius of \( \mathcal{M}^q_{r+} \) at \( Y \) is \( \sigma_r(Y) \).

Lemma 8. Suppose \( A(t) \in \mathbb{R}^{p_1 \times p_2} \) depends smoothly on a time variable \( t \), so that the singular value \( \sigma_i(t) = \sigma_i(A(t)) \) and left (right) singular vectors \( u_i(t) = u_i(A(t)) \) (\( v_i(t) = v_i(A(t)) \)) also depend smoothly on \( t \). Then for every \( i = 1, \ldots, p_1 \wedge p_2 \):

\[
\dot{\sigma}_i(t) = \langle u_i(t), \dot{A}(t) v_i(t) \rangle \quad \text{and} \quad \ddot{\sigma}_i(t) = \langle u_i(t), \ddot{A}(t) v_i(t) \rangle + \left\langle \dot{A}(t), \frac{d(u_i(t) v_i(t)^T)}{dt} \right\rangle.
\]

Here for a smooth function \( \phi \) of \( t \), \( \dot{\phi}(t) := \frac{d\phi}{dt}, \ddot{\phi}(t) := \frac{d^2\phi}{dt^2} \).

Proof. First notice \( \langle u_i(t), A(t) v_i(t) \rangle = \sigma_i(t) \). Differentiating with respect to \( t \) on both sides yields

\[
\dot{\sigma}_i(t) = \langle \dot{u}_i(t), A(t) v_i(t) \rangle + \langle u_i(t), \dot{A}(t) v_i(t) \rangle + \langle u_i(t), A(t) \dot{v}_i(t) \rangle
\]

\[
\implies \dot{\sigma}_i(t) = \sigma_i(t) \langle \dot{u}_i(t), u_i(t) \rangle + \langle u_i(t), \dot{A}(t) v_i(t) \rangle + \sigma_i(t) \langle v_i(t), \dot{v}_i(t) \rangle \tag{35}
\]

\[
\implies \dot{\sigma}_i(t) = \langle u_i(t), A(t) v_i(t) \rangle.
\]

Here (a) is because \( \langle u_i(t), u_i(t) \rangle = 1 \), so \( \langle u_i(t), u_i(t) \rangle = 0 \). Similarly we have \( \langle v_i(t), \dot{v}_i(t) \rangle = 0 \).

Differentiating with respect to \( t \) on both sides of \( \dot{\sigma}_i(t) = \langle u_i(t), A(t) v_i(t) \rangle \), we get

\[
\ddot{\sigma}_i(t) = \langle u_i(t), \ddot{A}(t) v_i(t) \rangle + \langle \dot{u}_i(t), \dot{A}(t) v_i(t) \rangle + \langle u_i(t), \dot{A}(t) \dot{v}_i(t) \rangle
\]

\[
\implies \ddot{\sigma}_i(t) = \langle u_i(t), \ddot{A}(t) v_i(t) \rangle + \left\langle \dot{A}(t), \frac{d(u_i(t) v_i(t)^T)}{dt} \right\rangle.
\]

\( \square \)
Lemma 9. Given any \( Y, Y' \in \mathbb{R}^{p \times r} \) such that \( d([Y'], [Y]) < \sigma_r(Y) \). Let \( O' = \arg \min_{O \in O_r} \|Y' - Y\|_F \). Then \( Y^T(Y + t(Y' - Y)) \) is nonsingular and \( \det(Y^T(Y + t(Y' - Y))) > 0 \) for all \( t \in [0, 1] \).

Proof. Suppose \( U \) spans the top \( r \) eigenspace of \( YY^T \). By Lemma 2, we know \( Y' - Y \in H_Y \mathcal{M}_r \), so we can represent \( Y' - Y \) by \( Y(Y^TY)^{-1}S + U_\perp D \) for some \( S \in \mathbb{R}^{r \times r}, D \in \mathbb{R}^{(p-r) \times r} \) by Lemma 1.

For the sake of contradiction, suppose there exists a \( t^* \in [0, 1] \) such that \( Y^T(Y + t^*(Y' - Y)) \) is singular. We show next that this implies \( d([Y'], [Y]) = \|Y' - Y\|_F \geq \sigma_r(Y) \), thus contradicting our assumption.

Since \( Y^T(Y + t^*(Y' - Y)) \) is singular, the matrix \( Y(Y^TY)^{-1}Y^T(Y + t^*(Y' - Y)) = Y + t^*Y(Y^TY)^{-1}S \) is singular also. It follows that

\[
\|Y' - Y\|_F^2 = \|Y(Y^TY)^{-1}S\|_F^2 + \|U_\perp D\|_F^2 \geq \|t^*Y(Y^TY)^{-1}S\|_F^2 \geq \min_{Y' \in \mathbb{R}^{p \times r}, \|Y'\| \leq \|Y\|} \|\tilde{Y} - Y\|_F \overset{(a)}{=} \sigma_r(Y).
\]

Here (a) is by the Schmidt-Mirsky theorem, see [Ste98, Chapter 1, Theorem 4.32]. We thus conclude that \( Y^T(Y + t(Y' - Y)) \) is nonsingular for all \( t \in [0, 1] \).

Using now the continuity of \( \det(Y^T(Y + t(Y' - Y))) \) with respect to \( t \), the fact that \( Y^T(Y + t(Y' - Y)) \) is nonsingular for all \( t \in [0, 1] \), and the fact that at \( t = 0 \) we have \( \det(Y^TY) > 0 \), we deduce that \( \det(Y^T(Y + t(Y' - Y))) > 0 \) for all \( t \in [0, 1] \). This completes the proof.

The fact that the matrix \( Y^T(Y + t(Y' - Y)) \) is nonsingular for all \( t \in [0, 1] \) has been proved in [MA20, Proposition 6.2]. Here we have presented the proof again for completeness, emphasizing the fact that the determinant does not change sign. This fact will be needed in the next lemma.

Lemma 10 (Condition Number for the Perturbation of Orthogonal Procrustes Problem). Let \( Y, Y' \in \mathbb{R}^{p \times r} \) such that \( d([Y'], [Y]) < \sigma_r(Y) \), and let \( \Delta_{Y'} \) and \( \Delta_{Y} \), be two \( \mathbb{R}^{p \times r} \) matrices. Let \( O_t = \arg \min_{O \in O_r} \|Y' + t\Delta_{Y'} - Y + t\Delta_{Y}\|_F \) for \( t \geq 0 \). Then

\[
\left\| \frac{dO_t}{dt} \right\|_{F, t=0} \leq \sqrt{2} \left( \frac{\|\Delta_{Y'}\|_F}{(\sigma_r(Y)^2 + \sigma_{r-1}^2(Y))^{1/2}} + \frac{\|\Delta_{Y}\|_F}{(\sigma_r(Y)^2 + \sigma_{r-1}^2(Y))^{1/2}} \right) - d([Y'], [Y]) \right).
\]

Proof. Let us introduce another alignment matrix \( O'_t \):

\[
O'_t = \arg \min_{O \in O_r} \|Y' + t\Delta_{Y'} - O - (Y + t\Delta_{Y})\|_F.
\]

It is easy to see that \( O_0O'_0 = O_t \) and \( O'_0 = I_r \). By Lemma 9, we know that when \( d([Y'], [Y]) < \sigma_r(Y) \), then \( \det(Y^TY'O_0) > 0 \), and thus \( \det((Y + t\Delta_{Y})^T(Y + t\Delta_{Y'})O_0) > 0 \) for all small enough \( t \). By Lemma 2, we have \( O'_t = V_{t}U_{t}^T \), where \( (Y + t\Delta_{Y})^T(Y + t\Delta_{Y'})O_0 \) has SVD \( U_{t} \sum_{t} V_{t}^T \), so for small enough \( t \) we have \( \text{sign}(\det(O'_t)) = \text{sign}(\det((Y + t\Delta_{Y})^T(Y + t\Delta_{Y'})O_0)). \)

By the continuity of the determinant, we have \( \det(O'_t) > 0 \) for small enough \( t \) because \( \text{sign}(\det(O'_0)) = \text{sign}(\det(Y^TY'O_0)) > 0 \). This implies that the best alignment matrix between \( Y + t\Delta_{Y} \) and \( (Y' + t\Delta_{Y'})O_0 \) is a rotation matrix, i.e., it has determinant 1, for small enough \( t \). Then by [Sõd93, Corollary 2.1], we have

\[
\|O'_t - O_0\|_F \leq \sqrt{2}t \left( \frac{\|\Delta_{Y'}O_0\|_F}{(\sigma_r(Y)^2 + \sigma_{r-1}^2(Y))^{1/2}} + \frac{\|\Delta_{Y}\|_F}{(\sigma_r(Y)^2 + \sigma_{r-1}^2(Y))^{1/2}} \right) + O(t^3) + \sqrt{2}t \left( \frac{\|\Delta_{Y'}\|_F}{(\sigma_r(Y)^2 + \sigma_{r-1}^2(Y))^{1/2}} + \frac{\|\Delta_{Y}\|_F}{(\sigma_r(Y)^2 + \sigma_{r-1}^2(Y))^{1/2}} \right) + O(t^3).
\]
Finally, we have
\[
\frac{dO_t}{dt}_{t=0} = \lim_{t \to 0} \frac{O_t - O_0}{t} = \lim_{t \to 0} \frac{O_t O_t^T - O_0 O_0^T}{t} = O_0 O_t^T - O_0 O_0^T,
\]
where the last inequality follows from Weyl’s theorem \(Y\).

This implies
\[
\Delta Y' = \sigma_r(Y)^2 + \sigma_{r-1}(Y)\frac{1}{2} > \sigma_r(Y)^2 - \sigma_{r-1}(Y)\frac{1}{2} - d([Y'], [Y]),
\]
where the last inequality follows from (36).

\[\Box\]

**Remark 5.** Lemma 10 provides a generalization of Corollary 2.1 in [Söd93]. There, \(O_t\) is defined as a minimizer among rotation matrices, i.e. elements in \(\mathbb{O}_r\) with determinant one, whereas here we are interested in arbitrary orthogonal matrices. In our proof, however, we show that we can reduce our setting to that in [Söd93].

**C.1 Proof of Lemma 6**

For given \([Y'] \in B_x([Y])\), let \(Q\) be defined as in Lemma 2, i.e. \(Q\) is the best matrix in \(\mathbb{O}_r\) aligning \(Y\) and \(Y'\). Then
\[
\sigma_r(Y') = \sigma_r(Y'Q) = \sigma_r(Y'Q - Y + Y) \geq \sigma_r(Y) - \|Y'Q - Y\|_F \geq \sigma_r(Y) - \|Y'Q - Y\|_F
\]
\[
\geq \sigma_y(Y) - x,
\]
where the first inequality follows from Weyl’s theorem [Ste98, Theorem 4.29]. Combining with Lemma 7, (37) implies that the injectivity radius at \([Y']\) is no smaller than \(\sigma_r(Y) - x\).

Moreover, for any other \([Y''] \in B_x([Y])\), we have
\[
d([Y''], [Y']) \leq d([Y''], [Y]) + d([Y'], [Y]) < \frac{2}{3} \sigma_r(Y) \leq \sigma_r(Y) - x.
\]
This implies \(B_{\sigma_r(Y) - x}([Y']) \supset B_x([Y])\) and finishes the proof of this lemma.

**D Additional Lemmas**

The next lemma establishes a one-to-one correspondence between PSD matrices with rank \(r\) and the space \(M^q_r\).

**Lemma 11.** ([MA20, Proposition A.1]) Let \(Y_1, Y_2 \in \mathbb{R}_+^{p \times r}\). Then \(Y_1Y_1^T = Y_2Y_2^T\) if and only if \(Y_2 = Y_1O\) for some \(O \in \mathbb{O}_r\). Moreover, \(\{X : X \in \mathbb{S}^{p \times p} \succcurlyeq 0, \text{rank}(X) = r\} = \{Y Y^T : Y \in \mathbb{R}_+^{p \times r}\}\).

The next lemma states that this correspondence is a locally bi-Lipschitz map and quantifies the corresponding local Lipschitz constants.

**Lemma 12.** For any \(Y_1, Y_2 \in \mathbb{R}_+^{p \times r}\), we have
\[
d^2([Y_1], [Y_2]) \leq \frac{1}{2(\sqrt{2} - 1)\sigma_r^2(Y_2)} \|Y_1 Y_1^T - Y_2 Y_2^T\|_F^2,
\]
(38)
and
\[ \| (Y_1 - Y_2 Q)(Y_1 - Y_2 Q)^\top \|_F^2 \leq 2 \| Y_1 Y_1^\top - Y_2 Y_2^\top \|_F^2, \]  
(39)
where \( Q = \arg \min_{O \in \mathbb{O}_r} \| Y_1 - Y_2 O \|_F \).

In addition, for any \( Y_1, Y_2 \in \mathbb{R}^{p \times r}_+ \) obeying \( d([Y_1], [Y_2]) \leq \frac{1}{3} \sigma_r(Y_2) \), we have
\[ \| Y_1 Y_1^\top - Y_2 Y_2^\top \|_F \leq \frac{7}{3} \| Y_2 d([Y_1], [Y_2]) \|. \]  
(40)

**Proof.** The first statement is from [TBS+16, Lemma 5.4], the second statement is from [GJZ17, Lemma 6] and the third statement is a slight modification of [TBS+16, Lemma 5.3]. \( \square \)

**Lemma 13.** ([LLZ21b, Proposition 2]) Let \( Y \in \mathbb{R}^p_{++} \), and let \( X = YY^\top \). Then \( 2\sigma_r^2(Y)\|\theta_Y\|_F^2 \leq \|Y\theta_Y^\top + \theta_Y Y^\top\|_F^2 \leq 4\sigma_r^2(Y)\|\theta_Y\|_F^2 \) holds for all \( \theta_Y \in \mathcal{H}_Y \mathcal{M}_r^q \).

**Remark 6.** The previous result can be interpreted as a quantitative bound for the metric distortion of the differential of the correspondence between \( X \) and \( \{Y\} \). Indeed, the tangent plane of \( \{X : X \in \mathbb{S}(p) \ni X \neq 0, \text{rank}(X) = r\} \) at \( X = YY^\top \) consists of matrices of the form \( (Y\theta_Y^\top + \theta_Y Y^\top) \) for \( \theta_Y \in \mathcal{H}_Y \mathcal{M}_r^q \), as has been discussed in [LLZ21b]. We observe that from Lemma 12 one could directly obtain the bound \( 2(\sqrt{2} - 1)\sigma_r^2(Y)\|\theta_Y\|_F^2 \leq \|Y\theta_Y^\top + \theta_Y Y^\top\|_F^2 \), while Lemma 13 provides a better constant for this inequality. The upper bound \( \|Y\theta_Y^\top + \theta_Y Y^\top\|_F^2 \leq 4\sigma_r^2(Y)\|\theta_Y\|_F^2 \) follows from straightforward linear algebra considerations.

**Lemma 14.** ([ZLTW21, Lemma 10]) Suppose \( f \) satisfies \( (2r, 4r) \)-restricted strong convexity and smoothness properties with parameter \( 0 \leq \delta < 1 \) as in Definition 1. Then for any \( p \)-by-\( p \) real-valued matrices \( C, D, H \) with \( \text{rank}(C), \text{rank}(D) \leq r \) and \( \text{rank}(H) \leq 2r \), we have
\[ \| \nabla f(C) - \nabla f(D) - (C - D), H \| \leq \delta \| C - D \|_F \| H \|_F. \]

**Lemma 15.** ([LHZ21, Lemma 3]) Let \( X \) be a \( p_1 \)-by-\( p_2 \) real-valued matrix. For any non-negative integer \( r \leq p_1 \wedge p_2 \), we have
\[ \| X_{\text{max}(r)} \|_F = \sup_{\| B \|_F \leq 1, \text{rank}(B) \leq r} \langle B, X \rangle \]  
(41)
If \( \text{rank}(X) \leq r \), then
\[ \| X \|_F = \sup_{\| B \|_F \leq 1, \text{rank}(B) \leq r} \langle B, X \rangle. \]  
(42)

**References**

[AFPA07] Vincent Arsigny, Pierre Fillard, Xavier Pennec, and Nicholas Ayache. Geometric means in a novel vector space structure on symmetric positive-definite matrices. *SIAM journal on matrix analysis and applications*, 29(1):328–347, 2007.

[AMS09] P-A Absil, Robert Mahony, and Rodolphe Sepulchre. *Optimization algorithms on matrix manifolds*. Princeton University Press, 2009.

[AV22] Foivos Alimisis and Bart Vandereycken. Geodesic convexity of the symmetric eigenvalue problem and convergence of Riemannian steepest descent. *arXiv preprint arXiv:2209.03480*, 2022.
[BA11] Nicolas Boumal and P-A Absil. Rtrmc: A Riemannian trust-region method for low-rank matrix completion. In Advances in neural information processing systems, pages 406–414, 2011.

[BJL19] Rajendra Bhatia, Tanvi Jain, and Yongdo Lim. On the Bures–Wasserstein distance between positive definite matrices. Expositiones Mathematicae, 37(2):165–191, 2019.

[BKS16] Srinadh Bhojanapalli, Anastasios Kyrillidis, and Sujay Sanghavi. Dropping convexity for faster semi-definite optimization. In Conference on Learning Theory, pages 530–582, 2016.

[BM05] Samuel Burer and Renato DC Monteiro. Local minima and convergence in low-rank semidefinite programming. Mathematical Programming, 103(3):427–444, 2005.

[BNS16] Srinadh Bhojanapalli, Behnam Neyshabur, and Nati Srebro. Global optimality of local search for low rank matrix recovery. In Advances in Neural Information Processing Systems, pages 3873–3881, 2016.

[Bou20] Nicolas Boumal. An introduction to optimization on smooth manifolds. http://sma.epfl.ch/ nboumal/#book, 2020.

[BVB20] Nicolas Boumal, Vladislav Voroninski, and Afonso S Bandeira. Deterministic guarantees for Burer-Monteiro factorizations of smooth semidefinite programs. Communications on Pure and Applied Mathematics, 73(3):581–608, 2020.

[BZL22] Yingjie Bi, Haixiang Zhang, and Javad Lavaei. Local and global linear convergence of general low-rank matrix recovery problems. 36th AAAI Conference on Artificial Intelligence, 2022.

[CB19] Chris Criscitiello and Nicolas Boumal. Efficiently escaping saddle points on manifolds. Advances in Neural Information Processing Systems, 32, 2019.

[CCFM19] Yuxin Chen, Yuejie Chi, Jianqing Fan, and Cong Ma. Gradient descent with random initialization: Fast global convergence for nonconvex phase retrieval. Mathematical Programming, 176(1-2):5–37, 2019.

[CHLW22] Jian-Feng Cai, Meng Huang, Dong Li, and Yang Wang. Nearly optimal bounds for the global geometric landscape of phase retrieval. arXiv preprint arXiv:2204.09416, 2022.

[CL19] Ji Chen and Xiaodong Li. Model-free nonconvex matrix completion: Local minima analysis and applications in memory-efficient kernel PCA. Journal of Machine Learning Research, 20(142):1–39, 2019.

[CLS15] Emmanuel J Candès, Xiaodong Li, and Mahdi Soltanolkotabi. Phase retrieval via Wirtinger flow: Theory and algorithms. IEEE Transactions on Information Theory, 61(4):1985–2007, 2015.

[CP11] Emmanuel J Candes and Yaniv Plan. Tight oracle inequalities for low-rank matrix recovery from a minimal number of noisy random measurements. IEEE Transactions on Information Theory, 57(4):2342–2359, 2011.
[CW15] Yudong Chen and Martin J Wainwright. Fast low-rank estimation by projected gradient descent: General statistical and algorithmic guarantees. arXiv preprint arXiv:1509.03025, 2015.

[DC20] Lijun Ding and Yudong Chen. Leave-one-out approach for matrix completion: Primal and dual analysis. IEEE Transactions on Information Theory, 66(11):7274–7301, 2020.

[DCFF92] Manfredo Perdigao Do Carmo and J Flaherty Francis. Riemannian geometry, volume 6. Springer, 1992.

[DGHG22] Shuyu Dong, Bin Gao, Wen Huang, and Kyle A Gallivan. On the analysis of optimization with fixed-rank matrices: a quotient geometric view. arXiv preprint arXiv:2203.06765, 2022.

[DJL+17] Simon S Du, Chi Jin, Jason D Lee, Michael I Jordan, Aarti Singh, and Barnabas Poczos. Gradient descent can take exponential time to escape saddle points. Advances in neural information processing systems, 30, 2017.

[Dor05] Leo Dorst. First order error propagation of the Procrustes method for 3d attitude estimation. IEEE transactions on pattern analysis and machine intelligence, 27(2):221–229, 2005.

[EAS98] Alan Edelman, Tomás A Arias, and Steven T Smith. The geometry of algorithms with orthogonality constraints. SIAM journal on Matrix Analysis and Applications, 20(2):303–353, 1998.

[GHJY15] Rong Ge, Furong Huang, Chi Jin, and Yang Yuan. Escaping from saddle points-online stochastic gradient for tensor decomposition. In Conference on learning theory, pages 797–842. PMLR, 2015.

[GJZ17] Rong Ge, Chi Jin, and Yi Zheng. No spurious local minima in nonconvex low rank problems: A unified geometric analysis. In Proceedings of the 34th International Conference on Machine Learning-Volume 70, pages 1233–1242. JMLR. org, 2017.

[HH18] Wen Huang and Paul Hand. Blind deconvolution by a steepest descent algorithm on a quotient manifold. SIAM Journal on Imaging Sciences, 11(4):2757–2785, 2018.

[JBAS10] Michel Journée, Francis Bach, P-A Absil, and Rodolphe Sepulchre. Low-rank optimization on the cone of positive semidefinite matrices. SIAM Journal on Optimization, 20(5):2327–2351, 2010.

[JGN+17] Chi Jin, Rong Ge, Praneeth Netrapalli, Sham M Kakade, and Michael I Jordan. How to escape saddle points efficiently. In Proceedings of the 34th International Conference on Machine Learning-Volume 70, pages 1724–1732. JMLR. org, 2017.

[JOR01] Charles R Johnson, Kazuyoshi Okubo, and Robert Reams. Uniqueness of matrix square roots and an application. Linear Algebra and its applications, 323(1-3):51–60, 2001.

[KOM09] Raghunandan H Keshavan, Sewoong Oh, and Andrea Montanari. Matrix completion from a few entries. In 2009 IEEE International Symposium on Information Theory, pages 324–328. IEEE, 2009.
[Lee13] John M Lee. Smooth manifolds. In Introduction to Smooth Manifolds, pages 1–31. Springer, 2013.

[LHLZ20] Yuetian Luo, Wen Huang, Xudong Li, and Anru R Zhang. Recursive importance sketching for rank constrained least squares: Algorithms and high-order convergence. arXiv preprint arXiv:2011.08360, 2020.

[LHZ21] Yuetian Luo, Rungang Han, and Anru R Zhang. A schatten-q low-rank matrix perturbation analysis via perturbation projection error bound. Linear Algebra and its Applications, 630:225–240, 2021.

[Linar] Shuyang Ling. Solving orthogonal group synchronization via convex and low-rank optimization: Tightness and landscape analysis. Mathematical Programming, To appear.

[LLA+19] Xingguo Li, Junwei Lu, Raman Arora, Jarvis Haupt, Han Liu, Zhaoran Wang, and Tuo Zhao. Symmetry, saddle points, and global optimization landscape of nonconvex matrix factorization. IEEE Transactions on Information Theory, 65(6):3489–3514, 2019.

[LLSW19] Xiaodong Li, Shuyang Ling, Thomas Strohmer, and Ke Wei. Rapid, robust, and reliable blind deconvolution via nonconvex optimization. Applied and computational harmonic analysis, 47(3):893–934, 2019.

[LLZ21a] Yuetian Luo, Xudong Li, and Anru R Zhang. Nonconvex factorization and manifold formulations are almost equivalent in low-rank matrix optimization. arXiv preprint arXiv:2108.01772, 2021.

[LLZ21b] Yuetian Luo, Xudong Li, and Anru R Zhang. On geometric connections of embedded and quotient geometries in Riemannian fixed-rank matrix optimization. arXiv preprint arXiv:2110.12121, 2021.

[LPP+19] Jason D Lee, Ioannis Panageas, Georgios Piliouras, Max Simchowitz, Michael I Jordan, and Benjamin Recht. First-order methods almost always avoid strict saddle points. Mathematical programming, 176(1-2):311–337, 2019.

[LXB19] Shuyang Ling, Ruitu Xu, and Afonso S Bandeira. On the landscape of synchronization networks: A perspective from nonconvex optimization. SIAM Journal on Optimization, 29(3):1879–1907, 2019.

[LZT19] Qiuwei Li, Zhihui Zhu, and Gongguo Tang. The non-convex geometry of low-rank matrix optimization. Information and Inference: A Journal of the IMA, 8(1):51–96, 2019.

[MA20] Estelle Massart and P-A Absil. Quotient geometry with simple geodesics for the manifold of fixed-rank positive-semidefinite matrices. SIAM Journal on Matrix Analysis and Applications, 41(1):171–198, 2020.

[MBS11] Gilles Meyer, Silvere Bonnabel, and Rodolphe Sepulchre. Linear regression under fixed-rank constraints: a Riemannian approach. In Proceedings of the 28th international conference on machine learning, 2011.
[MMBS14] Bamdev Mishra, Gilles Meyer, Silvère Bonnabel, and Rodolphe Sepulchre. Fixed-rank matrix factorizations and Riemannian low-rank optimization. *Computational Statistics*, 29(3-4):591–621, 2014.

[MMMO17] Song Mei, Theodor Misiakiewicz, Andrea Montanari, and Roberto Imbuzeiro Oliveira. Solving sdps for synchronization and maxcut problems via the grothendieck inequality. In *Conference on learning theory*, pages 1476–1515. PMLR, 2017.

[MMP18] Luigi Malagò, Luigi Montrucchio, and Giovanni Pistone. Wasserstein Riemannian geometry of gaussian densities. *Information Geometry*, 1(2):137–179, 2018.

[Moa05] Maher Moakher. A differential geometric approach to the geometric mean of symmetric positive-definite matrices. *SIAM journal on matrix analysis and applications*, 26(3):735–747, 2005.

[MS22] Ziye Ma and Somayeh Sojoudi. Noisy low-rank matrix optimization: Geometry of local minima and convergence rate. *arXiv preprint arXiv:2203.03899*, 2022.

[MWCC19] Cong Ma, Kaizheng Wang, Yuejie Chi, and Yuxin Chen. Implicit regularization in nonconvex statistical estimation: Gradient descent converges linearly for phase retrieval, matrix completion, and blind deconvolution. *Foundations of Computational Mathematics*, pages 1–182, 2019.

[MZL19] Tyler Maunu, Teng Zhang, and Gilad Lerman. A well-tempered landscape for non-convex robust subspace recovery. *Journal of Machine Learning Research*, 20(37), 2019.

[PKCS17] Dohyung Park, Anastasios Kyrillidis, Constantine Carmanis, and Sujay Sanghavi. Non-square matrix sensing without spurious local minima via the Burer-Monteiro approach. In *Artificial Intelligence and Statistics*, pages 65–74. PMLR, 2017.

[RW09] R Tyrrell Rockafellar and Roger J-B Wets. *Variational analysis*, volume 317. Springer Science & Business Media, 2009.

[SFF19] Yue Sun, Nicolas Flammarion, and Maryam Fazel. Escaping from saddle points on Riemannian manifolds. In *Advances in Neural Information Processing Systems*, volume 32, 2019.

[SL15] Ruoyu Sun and Zhi-Quan Luo. Guaranteed matrix completion via nonconvex factorization. In *Foundations of Computer Science (FOCS), 2015 IEEE 56th Annual Symposium on*, pages 270–289. IEEE, 2015.

[Söd93] Inge Söderkvist. Perturbation analysis of the orthogonal Procrustes problem. *BIT Numerical Mathematics*, 33(4):687–694, 1993.

[SQW15] Ju Sun, Qing Qu, and John Wright. When are nonconvex problems not scary? *arXiv preprint arXiv:1510.06096*, 2015.

[SQW16] Ju Sun, Qing Qu, and John Wright. Complete dictionary recovery over the sphere i: Overview and the geometric picture. *IEEE Transactions on Information Theory*, 63(2):853–884, 2016.

[SQW18] Ju Sun, Qing Qu, and John Wright. A geometric analysis of phase retrieval. *Foundations of Computational Mathematics*, 18(5):1131–1198, 2018.

33
[Ste98] Gilbert W Stewart. *Matrix algorithms: volume 1: basic decompositions*. SIAM, 1998.

[TBS+16] Stephen Tu, Ross Boczar, Max Simchowitz, Mahdi Soltanolkotabi, and Benjamin Recht. Low-rank solutions of linear matrix equations via Procrustes flow. In *International Conference on Machine Learning*, pages 964–973, 2016.

[TMC20] Tian Tong, Cong Ma, and Yuejie Chi. Accelerating ill-conditioned low-rank matrix estimation via scaled gradient descent. *Journal of Machine Learning Research*, to appear, 2020.

[UV20] André Uschmajew and Bart Vandereycken. On critical points of quadratic low-rank matrix optimization problems. *IMA Journal of Numerical Analysis*, 40(4):2626–2651, 2020.

[Van13] Bart Vandereycken. Low-rank matrix completion by Riemannian optimization. *SIAM Journal on Optimization*, 23(2):1214–1236, 2013.

[VAV09] Bart Vandereycken, P-A Absil, and Stefan Vandewalle. Embedded geometry of the set of symmetric positive semidefinite matrices of fixed rank. In *2009 IEEE/SP 15th Workshop on Statistical Signal Processing*, pages 389–392. IEEE, 2009.

[vO20] Jesse van Oostrum. Bures-Wasserstein geometry. *arXiv:2001.08056*, 2020.

[WCCL16] Ke Wei, Jian-Feng Cai, Tony F Chan, and Shingyu Leung. Guarantees of Riemannian optimization for low rank matrix recovery. *SIAM Journal on Matrix Analysis and Applications*, 37(3):1198–1222, 2016.

[WW20] Irene Waldspurger and Alden Waters. Rank optimality for the Burer–Monteiro factorization. *SIAM Journal on Optimization*, 30(3):2577–2602, 2020.

[WZG17] Lingxiao Wang, Xiao Zhang, and Quanquan Gu. A unified computational and statistical framework for nonconvex low-rank matrix estimation. In *Artificial Intelligence and Statistics*, pages 981–990, 2017.

[YD21] Tian Ye and Simon S Du. Global convergence of gradient descent for asymmetric low-rank matrix factorization. *Advances in Neural Information Processing Systems*, 34:1429–1439, 2021.

[YZLS22] Baturalp Yalcin, Haixiang Zhang, Javad Lavaei, and Somayeh Sojoudi. Factorization approach for low-complexity matrix completion problems: Exponential number of spurious solutions and failure of gradient methods. In *International Conference on Artificial Intelligence and Statistics*, pages 319–341. PMLR, 2022.

[ZBL21] Haixiang Zhang, Yingjie Bi, and Javad Lavaei. General low-rank matrix optimization: Geometric analysis and sharper bounds. *Advances in Neural Information Processing Systems*, 34:27369–27380, 2021.

[ZL15] Qingqing Zheng and John Lafferty. A convergent gradient descent algorithm for rank minimization and semidefinite programming from random linear measurements. In *Advances in Neural Information Processing Systems*, pages 109–117, 2015.

[ZLTW18] Zhihui Zhu, Qiuwei Li, Gongguo Tang, and Michael B Wakin. Global optimality in low-rank matrix optimization. *IEEE Transactions on Signal Processing*, 66(13):3614–3628, 2018.
[ZLTW21] Zhihui Zhu, Qiuwei Li, Gongguo Tang, and Michael B Wakin. The global optimization geometry of low-rank matrix optimization. *IEEE Transactions on Information Theory*, 67(2):1308–1331, 2021.

[ZMWY21] Haixiang Zhang, Andre Milzarek, Zaiwen Wen, and Wotao Yin. On the geometric analysis of a quartic–quadratic optimization problem under a spherical constraint. *Mathematical Programming*, pages 1–53, 2021.

[ZSL19] Richard Y Zhang, Somayeh Sojoudi, and Javad Lavaei. Sharp restricted isometry bounds for the inexistence of spurious local minima in nonconvex matrix recovery. *Journal of Machine Learning Research*, 20(114):1–34, 2019.

[ZWL15] Tuo Zhao, Zhaoran Wang, and Han Liu. A nonconvex optimization framework for low rank matrix estimation. In *Advances in Neural Information Processing Systems*, pages 559–567, 2015.

[ZWYG18] Xiao Zhang, Lingxiao Wang, Yaodong Yu, and Quanquan Gu. A primal-dual analysis of global optimality in nonconvex low-rank matrix recovery. In *International conference on machine learning*, pages 5862–5871, 2018.

[ZY18] Teng Zhang and Yi Yang. Robust PCA by manifold optimization. *The Journal of Machine Learning Research*, 19(1):3101–3139, 2018.

[ZZ20] Jialun Zhang and Richard Zhang. How many samples is a good initial point worth in low-rank matrix recovery? *Advances in Neural Information Processing Systems*, 33, 2020.