GEOMETRY AND TOPOLOGY OF SYMMETRIC POINT ARRANGEMENTS

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Abstract. We investigate point arrangements \( v_i \in \mathbb{R}^d, i \in \{1, \ldots, n\} \) with certain prescribed symmetries. The arrangement space of \( v \) is the column span of the matrix in which the \( v_i \) are the rows. We characterize properties of \( v \) in terms of the arrangement space, e.g., we characterize whether an arrangement possesses certain symmetries or whether it can be continuously deformed into another arrangement while preserving symmetry in the process. We show that whether a symmetric arrangement can be continuously deformed into its mirror image depends non-trivially on several factors, e.g., the decomposition of its representation into irreducible constituents, and whether we are in even or odd dimensions.

1. Introduction

By point arrangement (or just arrangement) we mean a finite family of points \( v_i \in \mathbb{R}^d, i \in \mathbb{N} := \{1, \ldots, n\} \). An arrangement is symmetric w.r.t. some permutation group \( \Gamma \subseteq \text{Sym}(\mathbb{N}) \), if any permutation \( \phi \in \Gamma \) can be realized on the points via some orthogonal transformation of \( \mathbb{R}^d \). Central to our treatment of arrangements is the notion of arrangement spaces\(^1\). For this, let \( M \) be the matrix

\[
M := \begin{pmatrix}
- & v_1^\top \\
- & \\
- & v_n^\top
\end{pmatrix} \in \mathbb{R}^{n \times d}
\]

with the \( v_i \) as rows. We call \( M \) the arrangement matrix (or just matrix) of \( v \). The arrangement space \( U := \text{span} \ M \subseteq \mathbb{R}^n \) is then the column span of \( M \). Arrangements with the same arrangement space will be called equivalent.

The definition of arrangement space is motivated by a recurring idea in geometry, and despite its simplicity has some interesting and non-trivial applications. To our knowledge, no common name (or no name at all) was introduced for this concept so far. We shall list a few of its applications.

The notion of equivalence (i.e., having the same arrangement space) has a direct geometric interpretation: two arrangements of the same dimension are equivalent, if and only if they are related by an invertible linear transformation (see Theorem 2.1). Arrangement spaces are hence of interest when one mainly cares about linear, affine or convex dependencies between points, as e.g. in the study of point configurations.

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\(^1\)Not to be confused with configuration spaces and realization spaces. These notions are not immediately related.
and oriented matroids (see e.g. [1, Section 6.3]). Thus, we can think of a subspace $U \subseteq \mathbb{R}^n$ as defining an equivalence class of point arrangements. A representative of this class is obtained as follows: choose vectors $u_1, ..., u_d \in U$ that span $U$, and define the matrix $M := (u_1, ..., u_d) \in \mathbb{R}^{n \times d}$ with the $u_i$ as its columns. The rows of $M$ are then a point arrangement with matrix $M$ and arrangement space $U$.

Properties which are invariant under invertible linear transformations are determined by the arrangement space. For example, the rank of an arrangement $\operatorname{rank} v := \operatorname{rank}\{v_1, ..., v_n\}$ equals $\dim U$, where $U$ is its arrangement space. In particular, there is always a full-dimensional arrangement $v$ (i.e., $\operatorname{rank} v = d$) with arrangement space $U$. Also, whether an arrangement is a linear transformation of a rational arrangement $v_i \in \mathbb{Q}^d$ or a 01-arrangement $v_i \in \{0, 1\}^d$ is determined by the arrangement space.

For studying point configurations and polytopes in higher dimensions, there exists the notion of Gale duality (see [1, Section 6]). The construction of the Gale dual is usually considered as quite artificial and technical. The following diagram gives a natural construction via arrangement spaces:

\[
\begin{array}{ccc}
v & \xrightarrow{\text{Gale}} & \bar{v} \\
\downarrow & & \downarrow \\
U & \longrightarrow & U^\perp
\end{array}
\]

In other words: $v$ and $\bar{v}$ are Gale duals if and only if their arrangement spaces are orthogonal complements of each other. Many properties are shared between $U$ and its orthogonal complement (e.g., possessing a rational basis, being invariant w.r.t. some orthogonal representation, etc.), and so it is clear that many properties are expressed by $v$ and its Gale dual likewise.

We close with an application to linear matroids (see, e.g. [2] for a general reference, and Section 6 in particular). The linear matroid of an arrangement $v$ is given by its independent sets, that is, subsets $I \subseteq N$ for which $\{v_i \in \mathbb{R}^n \mid i \in I\}$ is linearly independent. The same matroid can be obtained from the arrangement space $U$ of $v$. Choose as independent sets the $I \subseteq N$ for which $\operatorname{span}\{e_i \in \mathbb{R}^n \mid i \in I\}$ has trivial intersection with $U^\perp$ ($e_i \in \mathbb{R}^n$ is the $i$-th standard basis vector). Let us call this the matroid of $U$. There exists the notion of a dual matroid, and it is not hard to see that the dual of the matroid of $U$ is the matroid of $U^\perp$. Clearly, this is now the linear matroid of the Gale dual of $v$. This gives a short proof of the fact that the dual of a linear matroid is linear.

In this paper, we apply the idea of arrangement spaces to symmetric point arrangements. While the “symmetry” in “symmetric arrangement” is easily defined with the language of representation theory, we demonstrate that the study of these contains some intricacies that are not obvious from just studying group representations. Our treatment of the subject has to be distinguished from a list of similarly flavored ideas, as e.g. orbit polytopes [3, 4] or symmetric bar-joint-frameworks [5–7].

To our knowledge, symmetric arrangements are rarely discussed for their own sake, but are usually equipped with an additional structure as the one of a polytope or a graph. The notion discussed here is more general, as we neither require all points to lie on their convex hull, nor do we require symmetry for any distances constraints between points.

We give a quick overview of the content of this paper. Section 2 contains a short proof of the mentioned geometric characterization of equivalence, and a version of
Schur’s lemma. From Section 3 on our investigations focus on so-called normalized arrangements. We show that for these, the arrangement space also determines metric properties. Section 4 gives a formal definition of symmetric arrangements, or more precisely, $\Gamma$-arrangements for some $\Gamma \subseteq \text{Sym}(N)$. We show that the arrangement spaces of symmetric arrangements are invariant subspaces of $\mathbb{R}^n$, and that this property characterizes symmetric arrangements. In Section 5 we apply the developed techniques to answer questions of the following kind: given two $\Gamma$-arrangements $v, \bar{v}$, is it possible to continuously deform $v$ into $\bar{v}$ while preserving the symmetry in the process? As will turn out, the arrangements space plays a role in answering these question. For example, we shall obtain the following result: if $v$ and $\bar{v}$ are irreducible arrangements of odd dimension and their arrangement spaces are non-orthogonal, then $v$ can be deformed into $\bar{v}$ in the sense explained above.

The concept of arrangement spaces comes up naturally in the process of studying symmetries in spectral graph theory. In particular, given a graph $G = (N, E)$, one might consider arrangements for which the arrangement space is an eigenspace of $G$, that is, an eigenspace of its adjacency matrix. This class of arrangements shows interesting geometric and algebraic properties and provides a common language for certain ideas in graph drawings [8], balanced arrangements [9], sphere packings [10], polytopes (so-called eigenpolytopes) [11,12], etc. Some of these issues will be discussed in a follow up paper, that is currently under preparation (see Section 6).

2. Preliminary facts

We shortly list two preliminary facts. We first proof the geometric characterization of equivalence for arrangements.

**Theorem 2.1.** Two $d$-dimensional arrangements $v, \bar{v}$ are equivalent, if and only if they are related by an invertible linear transformations.

**Proof.** Let $M, \bar{M}$ be the matrices of $v$ and $\bar{v}$ respectively.

Assume $\bar{v} = T v$ for some $T \in \text{GL}(\mathbb{R}^d)$. We can express this in terms of the arrangements matrices: $\bar{M} = MT^\top$. Clearly, the column span of $M$ and $\bar{M}$ are then identical and the arrangements are equivalent.

For the converse, assume that $U := \text{span } M = \text{span } \bar{M}$, and $\delta := \dim U$. Choose a basis $u_1, ..., u_\delta \in U$ and consider two liftings $w_1, ..., w_\delta \in \mathbb{R}^d$ and $\bar{w}_1, ..., \bar{w}_\delta \in \mathbb{R}^d$ thereof, one w.r.t. $M$, and one w.r.t. $\bar{M}$. A desired transformation that maps $v$ onto $\bar{v}$ is then given by any $T \in \text{GL}(\mathbb{R}^d)$ for which its transpose $T^\top$ maps $\bar{w}_i \mapsto w_i$ and maps $\ker \bar{M}$ bijectively onto $\ker M$. We see this as follows: the $u_i$ together with $\ker \bar{M}$ span all of $\mathbb{R}^d$. Furthermore, $MT^\top$ and $\bar{M}$ act identically on these spanning vectors of $\mathbb{R}^d$:

$$MT^\top \bar{w}_i = Mw_i = u_i = \bar{M} \bar{w}_i, \quad \text{for all } i \in \{1, ..., \delta\},$$

and $MT^\top x = 0 = \bar{M}x$ for any $x \in \ker \bar{M}$. Hence $MT^\top = \bar{M}$. \hfill $\Box$

This result will not be used as such in this paper but it paves the way to the stronger version **Theorem 3.1** which applies to normalized arrangements.

The second preliminary fact is a form of Schur’s lemma for orthogonal representations. For that, recall that for two $\Gamma$-representations $T, \bar{T}$, a map $R \in \mathbb{R}^{d \times d}$ with $RT_\phi = \bar{T}_\phi R$ for all $\phi \in \Gamma$ is called equivariant.
Theorem 2.2 (Schur’s lemma). Given two linear representations $T, \bar{T}: \Gamma \rightarrow O(\mathbb{R}^d)$. If at least one of these is irreducible, then every equivariant map between them is of the form $\alpha R$ for some orthogonal $R \in O(\mathbb{R}^d)$.

It is convenient to briefly repeat the proof for this specific form. For re-usability, we first prove the following lemma:

Lemma 2.3. If $T: \Gamma \rightarrow O(\mathbb{R}^d)$ is an irreducible representation and $R \in \mathbb{R}^{d \times d}$ is symmetric and commutes with $T_\phi$ for all $\phi \in \Gamma$, then $R = \alpha Id$ for some $\alpha \in \mathbb{R}$.

Proof. Since $R$ commutes with $T_\phi$ for all $\phi \in \Gamma$, each eigenspace of $R$ is preserved by each $T_\phi$. So, each such preserved eigenspace is an invariant subspace of $T$. But since $T$ is irreducible, $T$ has no non-trivial invariant subspace. Hence, $R$ must have a single eigenspace to eigenvalue, say, $\alpha \in \mathbb{R}$. And since $R$ is symmetric, it is diagonalizable, and we have $R = \alpha Id$. □

We proceed with the proof of Schur’s lemma:

Proof of Schur’s lemma (Theorem 2.2). Without loss of generality, assume that $T$ is irreducible. Let then $R$ be an equivariant map between $T$ and $\bar{T}$, that is, $RT_\phi = \bar{T}_\phi R$, or $R = T_\phi RT_\phi^\top$ for all $\phi \in \Gamma$. We compute $R^\top R$:

$$R^\top R = T_\phi R^\top T_\phi RT_\phi^\top = T_\phi R^\top R T_\phi^\top = \alpha \Id.$$

Rearranging shows that the symmetric matrix $R^\top R$ commutes with $T_\phi$ for all $\phi \in \Gamma$, and by Lemma 2.3 we can conclude $R^\top R = \alpha \Id$ for some $\alpha \in \mathbb{R}$. Since $R^\top R$ is positive semi-definite, we have $\alpha \geq 0$, and we can choose $\alpha \in \{ \pm \alpha^{1/2} \}$ so that $\alpha^{-1} R$ is orthogonal. □

Recall, that two representations are said to be isomorphic, if there is an invertible equivariant map between them. We immediately obtain the following corollary:

Corollary 2.4. Given two linear representations $T, \bar{T}: \Gamma \rightarrow O(\mathbb{R}^d)$. If at least one of them is irreducible and there exists a non-zero equivariant map between them, then they are isomorphic.

Isomorphic representations are considered identical for all practical purposes, as they are related by a simple change of basis. In particular, one is irreducible if and only if the other one is.

3. Normalized Arrangements

We want to study arrangements up to orthogonal transformations, i.e., we do not care about changes in orientation, but we do care about more general transformations. We can use the “ignorance” of the arrangement space for our purpose by restricting to a certain class of arrangements.

Let $v$ be an arrangement with matrix $M$. Then $v$ is called spherical, if $M^\top M = \alpha \Id$ for some non-zero $\alpha \in \mathbb{R}$. If $\alpha = 1$, it is called a normalized arrangement. The term “spherical” stems from the interpretation of $M^\top M$ as (a scaled version of) the covariance matrix:

$$\text{Cov}(v) := \frac{1}{n - 1} \sum_{i=1}^{n} v_i v_i^\top \implies M^\top M = (n - 1) \text{Cov}(v).$$
Spherical arrangements are always full-dimensional:
\[ \text{rank } v = \text{rank } M = \text{rank}(M^\top M) = \text{rank}(\alpha \text{Id}) = d. \]
The columns of the matrix \( M \) of a normalized arrangement form an orthonormal system (ONS) in \( \mathbb{R}^n \), and \( MM^\top \) acts as ortho-projector onto the arrangement space of \( v \). Every spherical arrangement can be normalized by simple multiplication with a scalar. Further, given any non-zero subspace \( U \subseteq \mathbb{R}^n \), we can obtain a normalized arrangement \( v \) with arrangement space \( U \) from the matrix \( M = (u_1, ..., u_d) \), where the \( u_i \) are an orthonormal basis (ONB) of \( U \).

We prove a version of Theorem 2.1 for normalized arrangements:

**Theorem 3.1.** Two \( d \)-dimensional normalized arrangements \( v, \bar{v} \) are equivalent if and only if they are related by an orthogonal transformation \( T \in O(\mathbb{R}^d) \). In particular, the transformation is given by
\[ T = \bar{M}^\top M, \]
where \( M, \bar{M} \) are the matrices of \( v \) and \( \bar{v} \) respectively.

**Proof.** Let \( U, \bar{U} \subseteq \mathbb{R}^n \) be the arrangement spaces of \( v \) and \( \bar{v} \) respectively. One direction of the proof can be taken verbatim from the non-normalized version Theorem 2.1.

For the other direction, assume that \( v \) and \( \bar{v} \) are equivalent, that is, \( U = \bar{U} \). We have to check that \( T := \bar{M}^\top M \) satisfies \( MT^\top = \bar{M} \):
\[ MT^\top = MM^\top \bar{M} \overset{(*)}{=} \bar{M}, \]
where in \( (*) \) we used that \( MM^\top \) acts as ortho-projector onto \( U \). Since the columns of \( M \) are in \( \bar{U} = U \), \( MM^\top \) acts as identity. It remains to show that \( T \) is orthogonal:
\[ TT^\top = \bar{M}^\top MM^\top \bar{M} = \bar{M}^\top \bar{M} = \text{Id}, \]
where \( MM^\top \) was dropped for the same reason as above.

\[ \square \]

By Theorem 3.1 above, metric properties of normalized arrangements are completely determined by their arrangement space. For example, define the radius \( r(v) \) of an arrangement by
\[ [r(v)]^2 := \frac{1}{n} \sum_{i \in \mathcal{N}} \|v_i\|^2 = \frac{1}{n} \text{tr}(M^\top M). \]
If \( v \) is normalized, that is, \( M^\top M = \text{Id} \), we obtain \([r(v)]^2 = d/n \), where \( d = \text{dim } U \) can be read from the arrangement space. The radius \( r(v) \) can best be visualized for arrangements for which all points lie on a common sphere. In that case, \( r(v) \) is the radius of that sphere.

### 4. Symmetric arrangements

We now consider point arrangements with certain geometric symmetries. The symmetries will be prescribed by some permutation group \( \Gamma \subseteq \text{Sym}(N) \). To get an idea for what this means, consider the following: if there were a permutation \( \phi \in \Gamma \) that exchanges 1 and 2, we then would require the existence of an orthogonal transformation in \( \mathbb{R}^d \) that exchanges points \( v_1 \) and \( v_2 \) (it can happen that \( v_1 = v_2 \), and then any transformations achieves this; but that is the boring case).
We will investigate how point arrangements with such symmetries can be constructed and classified. In fact, we will see that symmetric arrangements can be fully characterized by their arrangement space.

**Definition 4.1.** A point arrangements \( v \) is called \( \Gamma \)-arrangement, if there exists a representation \( T: \Gamma \to O(\mathbb{R}^d) \) with

\[
(4.1) \quad T_\phi v_i = v_{\phi(i)}, \quad \text{for all } i \in \mathbb{N} \text{ and } \phi \in \Gamma.
\]

\( T \) is called a representation of \( v \).

Condition (4.1) in above definition can be stated using the matrix \( M \) of \( v \):

\[
(4.2) \quad P_\phi M = M T_\phi, \quad \text{for all } \phi \in \Gamma,
\]

where \( P_\phi \in \text{Perm}(n) \) is the permutation matrix that permutes the components of a vector in \( \mathbb{R}^n \) according to \( \phi \). This statement is equivalent to \( P_\phi^T M = M T_\phi^T \) (which can be easily verified in this form by writing out the matrix \( M \)), but we will mostly apply it in the form (4.2). We also see that \( M \) and \( M^T \) are now equivariant maps between \( T \) and the permutation matrix representation \( \phi \mapsto P_\phi \) in \( \mathbb{R}^n \).

The representation of a \( \Gamma \)-arrangement does not have to be uniquely determined by its set of points. If, however, \( v \) is full-dimensional, the representation is indeed unique: if \( \text{span}\{v_1, \ldots, v_n\} = \mathbb{R}^d \), we can find a basis of \( \mathbb{R}^d \) of points, say, \( v_{i_1}, \ldots, v_{i_d} \in \mathbb{R}^d \). For each \( \phi \in \Gamma \), the transformation \( T_\phi \in O(\mathbb{R}^d) \) is then already uniquely determined by

\[
T_\phi v_{i_j} = v_{\phi(i_j)}, \quad \text{for all } j \in \{1, \ldots, d\}.
\]

We will therefore often speak of the representation of \( v \).

**Lemma 4.2.** If \( v, \bar{v} \) are two \( \Gamma \)-arrangements with matrices \( M, \bar{M} \), then \( \bar{M}^T M \) is an equivariant map between the representations of \( v \) and \( \bar{v} \).

**Proof.** This follows from (4.2) and its equivalent form: let \( T, \bar{T} \) be representations of \( v \) and \( \bar{v} \), then

\[
\bar{T}_\phi \bar{M}^T M = \bar{M}^T P_\phi M = \bar{M}^T M T_\phi.
\]

At the moment, we still discuss general symmetric arrangements, that is, not necessarily spherical or normalized. However, recall that if normalized arrangements \( v \) and \( \bar{v} \) are related by an orthogonal transformation, then Theorem 3.1 states that this transformations is \( \bar{M}^T M \). This transformation is then invertible (because orthogonal). So, if \( v, \bar{v} \) are \( \Gamma \)-arrangements, the same transformation \( \bar{M}^T M \) is also an equivariant map between the representations \( T \) and \( \bar{T} \), and we can conclude that the representations are isomorphic.

**Corollary 4.3.** Equivalent normalized \( \Gamma \)-arrangements have isomorphic representations.

The converse of Corollary 4.3 is not true: if two (normalized) arrangements have isomorphic representations, they might still be non-equivalent. We will discuss this further in Section 5. For now, note the freedom of choice for \( v_1 \) in the following construction, which might result in non-equivalent arrangements:
Figure 1. Left: a $\Gamma$-arrangement with group $\Gamma \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ acting on four points via the permutations \{id, (12)(34), (13)(24), (14)(23)\}. The arrangement is reducible, as it can be projected onto $W$ and $W^\perp$ without loosing symmetry (note that $90^\circ$ rotations are not a prescribed symmetry). Right: an irreducible $\Gamma$-arrangement with $\Gamma \simeq D_6$ (symmetry group of the regular hexagon) on six points. This arrangement cannot be projected to a lower dimension while keeping rotational symmetry.

Construction 4.4. We construct a $\Gamma$-arrangement with prescribed representation, that acts transitively on its point set.

Choose a transitive permutation group $\Gamma \subseteq \text{Sym}(N)$ and a representation $T : \Gamma \to O(\mathbb{R}^d)$ of it. Let $\Gamma_1 \subset \Gamma$ be the stabilizer of $1 \in N$. Consider the following subspace:

$$\text{Fix}(T, \Gamma_1) := \{ x \in \mathbb{R}^d \mid T_\phi x = x, \text{ for all } \phi \in \Gamma_1 \}. $$

Choose a $v_1 \in \text{Fix}(T, \Gamma_1)$. Since $\Gamma$ is transitive, for any $i \in N$ there is a $\phi_i \in \Gamma$ with $\phi_i(1) = i$. We then define the desired arrangement as follows:

$$v_i := T_{\phi_i} v_1, \quad \text{for all } i \in N.$$ 

This will turn out to be a $\Gamma$-arrangement with representation $T$.

If $\phi \in \Gamma$ maps $i \in N$ to $j$, then $\psi := \phi_j^{-1} \circ \phi \circ \phi_i \in \Gamma_1$, and

$$T_\phi v_i = T_{\phi_j} T_\psi v_1 = T_{\phi_j} v_1 = v_j,$$

since $v_1 \in \text{Fix}(T, \Gamma_1)$ and thus is fixed by $T_\psi$. This shows that, in particular, the construction is independent of the choice of the $\phi_i \in \Gamma$. We then have that $T$ is a $\Gamma$-representation as needed in Definition 4.1, and thus, that $v$ is a $\Gamma$-arrangement with representation $T$.

In above construction, a necessary condition for the existence of non-zero arrangements with representation $T$ is $\text{Fix}(T, \Gamma_1) \neq \{0\}$. A necessary condition for the existence of multiple non-equivalent $\Gamma$-arrangements with the prescribed representation is $\dim \text{Fix}(T, \Gamma_1) \geq 2$. However, this is not sufficient.

Symmetric arrangements inherit a notion of irreducibility from their representations: a $\Gamma$-arrangement shall be called irreducible if its representation is irreducible. It shall be called reducible otherwise. There are equivalent ways to characterize reducibility in geometric terms (see Lemma 4.6 and Figure 1). To describe these, we require the following construction:
Definition 4.5. Given arrangements \( v \) and \( \bar{v} \) of dimension \( d \) and \( \bar{d} \) respectively, the product arrangement \( v \times \bar{v} \) is defined by

\[
(v \times \bar{v})_i := (v_i, \bar{v}_i) \in \mathbb{R}^{d+\bar{d}}, \quad \text{for all } i \in N.
\]

If \( M, \bar{M} \) are the matrices of the arrangements \( v \) and \( \bar{v} \) respectively, the product arrangement has the matrix \((M, \bar{M})\), i.e., the columns of \( M \) and \( \bar{M} \) are joined to a single matrix. If \( U, \bar{U} \subseteq \mathbb{R}^n \) are the arrangement spaces of \( v \) and \( \bar{v} \), then the product has the arrangement space \( U + \bar{U} \).

Product arrangement can be used to obtain new \( \Gamma \)-arrangements from old ones: it is not hard to see that products of \( \Gamma \)-arrangements are again \( \Gamma \)-arrangements. If \( T, \bar{T} \) are representations of \( v \) and \( \bar{v} \), then \( v \times \bar{v} \) has representation

\[
(4.3) \quad \phi \mapsto (T \times \bar{T})_\phi := \begin{pmatrix} T_\phi & \bar{T}_\phi \end{pmatrix}.
\]

We list some geometric characterizations of being reducible:

Lemma 4.6. Let \( v \) be a non-zero \( \Gamma \)-arrangement. The following are equivalent:

(i) The arrangement \( v \) is reducible.

(ii) The arrangement \( v \) is (equivalent to) the product of two \( \Gamma \)-arrangements of smaller dimension.

(iii) There exists a proper subspace \( W \subseteq \mathbb{R}^n \), so that the projections \( \pi_W(v) \) and \( \pi_{W^\perp}(v) \) are \( \Gamma \)-arrangements with the same representation as \( v \).

Proof. Assume (i), that is, that \( v \) is reducible. By this, \( T \) is reducible, and there exists a proper non-zero \( T \)-invariant subspace \( W \subseteq \mathbb{R}^d \). Consider the projection \( \bar{v} := \pi_W(v) \) onto \( W \). The ortho-projector \( \pi_W \) commutes with \( T_\phi \) for all \( \phi \in \Gamma \), hence \( \bar{v} \) is a \( \Gamma \)-arrangement with representation \( T \):

\[
T_\phi \bar{v}_i = T_\phi \pi_W(v_i) = \pi_W(T_\phi v_i) = \pi_W(v_{\phi(i)}) = \bar{v}_{\phi(i)}.
\]

The orthogonal complement \( W^\perp \) is \( T \)-invariant as well and the same reasoning applies to the projection \( \pi_{W^\perp}(v) \). This proves (iii).

We now assume (iii). By equivalence, we can assume

\[
W = \text{span}\{e_1, \ldots, e_\delta\} \quad \text{and} \quad W^\perp = \text{span}\{e_{\delta+1}, \ldots, e_d\}
\]

(\text{where } e_i \text{ denote the } i\text{-th standard basis vectors}). If \( M = (u_1, \ldots, u_d) \) is the matrix of \( v \), the projected arrangements have matrices

\[
M_W = (u_1, \ldots, u_\delta) \quad \text{and} \quad M_{W^\perp} = (u_{\delta+1}, \ldots, u_d).
\]

We can consider these projections as arrangements in \( \mathbb{R}^\delta \) resp. \( \mathbb{R}^{d-\delta} \). The join of the matrices is obviously \( M \), and thus the product of the projections is \( v \). This proves (ii).

Finally, assume (ii), that is, \( v = v^{(1)} \times v^{(2)} \) is a \( d \)-dimensional \( \Gamma \)-arrangement with representation \( T \), and decomposes into \( d_\delta \)-dimensional \( \Gamma \)-arrangements \( v^{(i)} \) with representation \( T^{(i)} \). By (4.3) we have \( T = T^{(1)} \times T^{(2)} \). Then \( W := \text{span}\{e_1, \ldots, e_{d_\delta}\} \) is \( T \)-invariant, and hence \( T \) is reducible and (i) holds.

\[\Box\]

The irreducible \( \Gamma \)-arrangements are the elementary building blocks of general \( \Gamma \)-arrangements: each \( \Gamma \)-arrangement \( v \) can be written as a product

\[
v = v^{(1)} \times \cdots \times v^{(K)}
\]
with irreducible $\Gamma$-arrangements $v^{(k)}$, for $k \in \{1, \ldots, K\}$. This decomposition into irreducible constituents might not be unique.

As in the section before, we are especially interested in spherical and normalized arrangements. General $\Gamma$-arrangements can be far from spherical, but there is no concern in the case of irreducible $\Gamma$-arrangements:

**Proposition 4.7.** Irreducible $\Gamma$-arrangements are spherical.

**Proof.** Let $v$ be an irreducible $\Gamma$-arrangement with representation $T$. By Lemma 4.2 (with setting $v = \bar{v}$), we see that $M^\top M$ is an equivariant map between $T$ and $T$, i.e., it commutes with $T$. Then, $M^\top M = \alpha \text{Id}$ follows from Lemma 2.3. □

In the reducible case, consider two normalized $\Gamma$-arrangements $v$ and $\bar{v}$. Then $\alpha v \times \beta \bar{v}$ is a $\Gamma$-arrangement, but to be spherical we necessarily need $|\alpha| = |\beta|$ (this is not sufficient, though).

The following two theorems give a characterization of $\Gamma$-arrangements in terms of their arrangement spaces. For this, note that $\Gamma$ acts on $\mathbb{R}^n$ via $\phi \mapsto P_\phi$, hence we can speak of $\Gamma$-invariant subspaces of $\mathbb{R}^n$. With the following results we answer the question of classification of symmetric arrangements (by shifting the problem to a classification of $\Gamma$-invariant subspaces).

**Theorem 4.8.** The arrangement space of a $\Gamma$-arrangement is $\Gamma$-invariant.

**Proof.** Let $v$ be a $\Gamma$-arrangement with matrix $M$, arrangement space $U \subseteq \mathbb{R}^n$ and representation $T$. For every $\phi \in \Gamma$ holds

$$P_\phi U = \text{span}(P_\phi M) = \text{span}(MT_\phi) = \text{span} M = U.$$ 

Hence $U$ is $\Gamma$-invariant. □

The converse of Theorem 4.8 is true for spherical arrangements:

**Theorem 4.9.** A spherical arrangement with $\Gamma$-invariant arrangement space is a $\Gamma$-arrangement. Moreover, its representation can be chosen as

$$T_\phi := M^\top P_\phi M, \quad \text{for all } \phi \in \Gamma,$$

where $M$ is the matrix of the arrangement.

**Proof.** Let $v$ be a spherical arrangement, $M$ its matrix, and $U \subseteq \mathbb{R}^n$ its $\Gamma$-invariant arrangement space. The arrangement $v$ is a $\Gamma$-arrangement if and only $\alpha v$ is, and both have the same arrangement space. Hence, we can assume that $v$ is normalized.

We show that $v$ is a $\Gamma$-arrangement by proving that (4.4) is indeed the desired representation. Let $\phi, \psi \in \Gamma$ and observe

$$T_\phi T_\psi = (M^\top P_\phi M)(M^\top P_\psi M) = (M^\top P_\phi)(MM^\top)(P_\psi M).$$

Since $v$ is normalized, $MM^\top$ acts as ortho-projector onto $U$. Since span $M = U$ is invariant w.r.t. $P_\psi$, the columns of $P_\psi M$ are in $U$ again. Thus, $MM^\top$ acts as identity to its right and can be dropped. We obtain

$$T_\phi T_\psi = M^\top P_\phi P_\psi M = M^\top P_{\phi \psi} M = T_{\phi \psi}.$$

This shows, that $T$ is a linear representation. Equivalently, one shows that $T_\psi^\top = T_{\psi^{-1}}$, i.e., that $T_\phi$ is orthogonal.
It remains to prove $T_\phi M^\top = M^\top P_\phi$ for all $\phi \in \Gamma$:

$$T_\phi M^\top = (M^\top P_\phi M)M^\top = (MM^\top (P_{\phi^{-1}} M))^\top.$$ 

Again, $MM^\top$ acts to its right (on the columns of $P_{\phi^{-1}} M$) as ortho-projector, hence as identity, and can be dropped. We obtain $T_\phi M^\top = (P_{\phi^{-1}} M)^\top M^\top = M^\top P_\phi$ and we are done. □

The preceding theorem is not necessarily true for non-spherical arrangements. E.g. the vertices of a rectangle resp. square share the same arrangement space (they are linear transformations of each other), but only one is a $\Gamma$-arrangement for $\Gamma \simeq D_4$ (the symmetry group of the square).

Theorem 4.8 and Theorem 4.9 together make clear that studying spherical $\Gamma$-arrangements is essentially just studying $\Gamma$-invariant subspaces of $\mathbb{R}^n$.

**Corollary 4.10.** A spherical arrangement is a $\Gamma$-arrangement if and only if its arrangement space is a $\Gamma$-invariant subspace of $\mathbb{R}^n$.

It is in general a non-trivial task to determine the $\Gamma$-invariant subspaces of $\mathbb{R}^n$, and hence it is non-trivial to determine the $\Gamma$-arrangements. But as soon as a $\Gamma$-invariant subspace is known, we can obtain a $\Gamma$-arrangement as a representative of the equivalence class described by this subspace.

We close this section by proving that a full-dimensional $\Gamma$-arrangement is irreducible if and only if its arrangement space is irreducible as a $\Gamma$-invariant subspace of $\mathbb{R}^n$. The same is not true for $\Gamma$-arrangements that are not of full dimension. Such arrangements are always reducible, regardless of their arrangement space, since $\text{span} v = \text{span} \{v_1, ..., v_n\}$ is an invariant subspace of $T$.

**Proposition 4.11.** A full-dimensional $\Gamma$-arrangement $v$ is irreducible if and only if its arrangement space $U \subseteq \mathbb{R}^n$ is irreducible as a $\Gamma$-invariant subspace of $\mathbb{R}^n$.

**Proof.** Let $M$ be the matrix of $v$, and $T$ its representation. Since $v$ is full-dimensional, we can interpret $M$ as an invertible linear map $M : \mathbb{R}^d \to U$.

By Theorem 4.8, the map $\phi \mapsto P_\phi|_U$ is a $\Gamma$-representation on $U$ (since $U$ is $\Gamma$-invariant). By (4.2) $M$ is then an invertible equivariant map between $T$ and $\phi \mapsto P_\phi|_U$. Hence, these representations are isomorphic, and, in particular, one is irreducible if and only if the other one is.

The arrangement $v$ is irreducible if and only if $T$ is. The subspace $U$ is irreducible if and only of $\phi \mapsto P_\phi|_U$ is. These statements about irreducibility are hence linked since $T$ and $\phi \mapsto P_\phi|_U$ are isomorphic. □

5. **Topology of symmetric arrangements**

In this section, we study the topology of symmetric point arrangements. Observe, that the set of all $d$-dimensional arrangements indexed by $N$ carries the structure of an $nd$-dimensional vector space, and hence is equipped with a natural topology.

We shall be concerned with the following type of questions: suppose we have two (normalized) $\Gamma$-arrangements $v$ and $\bar{v}$. We try to continuously deform $v$ into $\bar{v}$. This is certainly possible if we are allowed to move the points freely. We therefore restrict to deformations, in which every intermediate step is itself a $\Gamma$-arrangement, i.e., the symmetry is preserved. There are still trivial solution, as every arrangement can be continuously deformed to the zero-arrangement (which is always symmetric) via $t \mapsto tv$. For that reason, we additionally impose the constraint that the arrangement
stays normalized during the deformation process. See Figure 2 for a visualization of such a deformation.

We can achieve “deformations” in trivial cases, e.g., when \( \bar{v} = Tv \) for some orientation preserving orthogonal transformation \( T \in SO(\mathbb{R}^d) \). We use the fact that \( SO(\mathbb{R}^d) \) is path-connected in \( \mathbb{R}^{d \times d} \) to construct a continuous transition between \( v \) and \( Tv \). Of course, this is not really a deformation in the intuitive sense, but more a continuous re-orientation. The situation is less obvious if we consider orientation reversing transformations, i.e., \( \bar{v} = Tv \) with \( T \in O(\mathbb{R}^d) \setminus SO(\mathbb{R}^d) \). In fact, the answer for when an arrangement can be continuously deformed into its mirror image depends on the dimension of the arrangement.

We finally consider “proper deformations” between non-equivalent \( \Gamma \)-arrangements, i.e., where the arrangements are not just re-orientations of each other. In particular, we ask for which \( \Gamma \)-arrangements such a non-trivial deformation is possible at all. This gives a notion of rigidity for \( \Gamma \)-arrangements. We characterize rigidity of an arrangement in terms of its arrangement space.

As motivated above, we consider deformations that preserve symmetry and normalization. For that matter, for each \( \Gamma \subset \text{Sym}(N) \), we consider the set

\[
\mathcal{S}_d(\Gamma) := \{ \text{d-dimensional normalized } \Gamma\text{-arrangements on } N \},
\]
equipped with the subspace topology.

**Definition 5.1.** A d-dimensional \( \Gamma \)-deformation (or just deformation) is a curve (that is, a continuous function) \( v(\cdot) : [0, 1] \to \mathcal{S}_d(\Gamma) \).

\( \Gamma \)-arrangements \( v, \bar{v} \in \mathcal{S}_d(\Gamma) \) are deformation equivalent, if there exists a \( \Gamma \)-deformation \( v(\cdot) \) with \( v(0) = v \) and \( v(1) = \bar{v} \). We then also say more intuitively, that \( v \) can be deformed into \( \bar{v} \).

We already reasoned, that for our question it will be important to distinguish between orientation preserving and orientation reversing transformations. We therefore define the following more specific versions of equivalence and representation-isomorphy.
Figure 3. Implications between the five relations defined on $\mathcal{S}_d(\Gamma)$. The black arrows indicate unconditional implications. The gray arrows indicate conditional implications, and the conditions are written next to the arrow. The conditions are necessary. The conditions “rigid/flexible” mean that at least one (and then both) of the involved arrangements is rigid/flexible. The conditions “odd/even” mean that the arrangements live in odd/even dimensional space.

- Two arrangements are *positively equivalent* if they are related by a linear transformation of positive determinant.
- Two representations are *positively isomorphic* if there exists an equivariant map between these with positive determinant.

In total, we now defined five relations on $\mathcal{S}_d(\Gamma)$, each of which is easily seen to be an equivalence relation: equivalence (eq), positive equivalence (eq$^+$), representation-isomorphy (iso), positive representation-isomorphy (iso$^+$) and deformation equivalence (d-eq).

A major goal of this section is to prove the eleven implications presented in Figure 3\textsuperscript{2}. Some of the implications are conditional. For these, we also prove that the conditions are necessary. We will focus here on the irreducible arrangements in $\mathcal{S}_d(\Gamma)$. Most results can be modified to work for reducible arrangements as well, but we will not pursue this. One major difference in that case is that instead of odd/even dimensional arrangements, one has to speak of arrangements with and without even/odd dimensional irreducible constituents.

The following implications in Figure 3 are trivially seen to be true, or where already discussed in the introduction to this section:

$$\text{eq}^+ \to \text{eq}, \quad \text{iso}^+ \to \text{iso}, \quad \text{eq} \to \text{d-eq}.$$ 

Further, if normalized $\Gamma$-arrangements $v, \bar{v}$ are equivalent, then the relating transformation (which is $\bar{M}^T M$) is simultaneously an invertible equivariant map between their representations (see also Corollary 4.3). This proves

$$\text{eq} \to \text{iso}, \quad \text{eq}^+ \to \text{iso}^+.$$ 

We next aim to prove a necessary condition for two arrangements to be deformation equivalent. The corresponding statement can be found in Figure 3 in the form of the unconditional implication d-eq $\to$ iso$^+$. In other words: to be deformation equivalent...

\textsuperscript{2}Keep this figure at hand, as we refer back to it quite frequently. You may mark the already proven arrows to keep track of the progress.
equivalent, it is necessary to have positively isomorphic representations. This seems to be quite natural: the set of (pair-wise non-isomorphic) representations of $\Gamma$ is finite. The discrete nature of this set does not work well with the continuity of deformations. We make this precise: for a $\Gamma$-representation $T: \Gamma \to O(\mathbb{R}^d)$ let

$$S^+_d(T) := \{ v \in S_d(\Gamma) \mid v \text{ has a representation positively isomorphic to } T \}.$$ 

The sets $S^+_d(T)$ are the equivalence classes in $S_d(\Gamma)$ under the relation of positive representation-isomorphy. We prove the following topological result:

**Theorem 5.2.** $S^+_d(T)$ is an open subset of $S_d(\Gamma)$.

For what follows, it is convenient to introduce the following function:

$$\det(v, \bar{v}) := \det(\bar{M}^\top M),$$

where $v, \bar{v}$ are arrangements, and $M, \bar{M}$ their matrices. We list some of its properties:

**Lemma 5.3 (Properties of $\det(\cdot, \cdot)$).** The following holds:

(i) $\det(\cdot, \cdot)$ is continuous in both arguments.

(ii) If normalized arrangements $v, \bar{v}$ are equivalent, then they have $\det(v, \bar{v}) = \pm 1$, and if they are positively equivalent, then they have $\det(v, \bar{v}) = 1$.

(iii) If two arrangements $v$ and $\bar{v}$ have orthogonal arrangement spaces, then they also have $\det(v, \bar{v}) = 0$.

(iv) If $v, \bar{v}$ are irreducible $\Gamma$-arrangements with $\det(v, \bar{v}) = 0$, then their arrangement spaces are orthogonal.

(v) If $\Gamma$-arrangements $v, \bar{v}$ have $\det(v, \bar{v}) \neq 0$, then they have isomorphic representations, and if $\det(v, \bar{v}) > 0$, then they have positively isomorphic representations.

**Proof.** Statement (i) is directly seen from the definition.

In the following, let $v, \bar{v}$ be arrangements with arrangement matrices $M$ and $\bar{M}$ respectively.

If $v$ and $\bar{v}$ are normalized and equivalent, then $\bar{M}^\top M$ is the orthogonal transformation between these. This already shows

$$\det(v, \bar{v}) = \det(\bar{M}^\top M) = \pm 1.$$ 

If the arrangements are positively isomorphic, then the transformation $\bar{M}^\top M$ has positive determinant. This proves (ii).

If the arrangement spaces span $M$ and span $\bar{M}$ are orthogonal, then each column of $M$ is orthogonal on each column of $\bar{M}$. This then shows $\bar{M}^\top M = 0$, in particular $\det(\bar{M}^\top M) = 0$, and hence (iii).

For the rest of the proof, assume that $v$ and $\bar{v}$ are $\Gamma$-arrangements. Recall that then $\bar{M}^\top M$ is an equivariant map between any representations of $v$ and $\bar{v}$.

To prove (iv), assume that $v, \bar{v}$ are irreducible. By Schur’s lemma $\bar{M}^\top M = \alpha R$ for some $R \in O(\mathbb{R}^d)$. Hence, if $\det(v, \bar{v}) = \det(\bar{M}^\top M) = 0$, then only because $\alpha = 0 \Rightarrow \bar{M}^\top M = 0$, i.e., span $\bar{M} \perp$ span $M$.

On the other hand, to show (v), observe that $\det(M^\top M) \neq 0$ implies that $\bar{M}^\top M$ is invertible. Since there is then an invertible equivariant map, the representations are isomorphic. If $\det(\bar{M}^\top M) > 0$, they are positively isomorphic.

Note the following immediate consequence of above properties, by simply combining (iv) with (v):
**Corollary 5.4.** If irreducible arrangements $v, \bar{v} \in S_d(\Gamma)$ have non-orthogonal arrangement spaces, then they have isomorphic representations.

We proceed with the proof of the theorem.

**Proof of Theorem 5.2.** We show that every $v \in S_d^+(T)$ has an open neighborhood in $S_d(\Gamma)$ which is completely contained in $S_d^+(T)$.

Fix an arrangement $v \in S_d^+(T)$, and consider the set

$$S := \{\bar{v} \in S_d(\Gamma) \mid \det(v, \bar{v}) > 0\}.$$

$S$ is the pre-image of the open set $\mathbb{R}_{>0}$ under a continuous function $\det(v, \cdot)$, hence open. Also, $v \in S$, and so $S$ is an open neighborhood of $v$ in $S_d(\Gamma)$.

It remains to show that $S \subseteq S_d^+(T)$. Choose a $\bar{v} \in S$, in particular, $\det(v, \bar{v}) > 0$. By Lemma 5.3, the representations of $v$ and $\bar{v}$ are positively isomorphic, and $\bar{v} \in S_d^+(T)$. \hfill $\square$

We now see that the equivalence classes of $S_d(\Gamma)$ under $\text{iso}^+$ provide a decomposition of $S_d(\Gamma)$ into disjoint open sets. It is well-known that points (in our case, arrangements) in distinct open components cannot be connected by a continuous path. This shows $\text{d-eq} \rightarrow \text{iso}^+$.

**Corollary 5.5.** Deformation equivalent arrangements $v, \bar{v} \in S_d(\Gamma)$ have positively isomorphic representations.

This now motivates question whether the $S_d^+(T)$ are already the path-connected components of $S_d(\Gamma)$, i.e., whether positive isomorphy of the representations already suffices to imply deformation equivalence.

Surprisingly, the only case that we cannot immediately deal with, is, when the arrangements are equivalent. We start by constructing a deformation between non-equivalent arrangements. From now on, we restrict to irreducible arrangements.

**Proposition 5.6.** If two non-equivalent irreducible arrangements $v, \bar{v} \in S_d(\Gamma)$ have positively isomorphic representations, then they are deformation equivalent.

**Proof.** The idea is as follows: first re-orient $\bar{v}$ so that the representations of $v$ and $\bar{v}$ become identical. Second, make a linear transition between the arrangements. Finally, make sure that all intermediate arrangements are normalized $\Gamma$-arrangements.

If $T, \bar{T}: \Gamma \rightarrow O(\mathbb{R}^d)$ are positively isomorphic representations of $v$ and $\bar{v}$, we then can choose an orthogonal equivariant map $R \in \text{SO}(\mathbb{R}^d)$ of positive determinant between these, i.e., $T_{\phi} = R\bar{T}_{\phi}R^T$ for all $\phi \in \Gamma$. Then, $\bar{v}$ and $\bar{v} := R\bar{v}$ are positively equivalent, hence deformation equivalent by the already shown implication $\text{eq}^+ \rightarrow \text{d-eq}$. The arrangement $\bar{v}$ is a $\Gamma$-arrangement with representation $T$:

$$T_{\phi} \bar{v}_i = R\bar{T}_{\phi}R^T\bar{v}_i = R\bar{T}_{\phi}\bar{v}_i = R\bar{v}_{\phi(i)} = \hat{v}_{\phi(i)}.$$

It remains to construct a deformation from $v$ to $\bar{v}$. Concatenating this with the deformation between $\bar{v}$ and $\bar{v}$ then proves the statement.

We define the continuous map (not necessarily a deformation, because not necessarily normalized)

$$t \mapsto v'(t) := (1 - t)v + t\bar{v}.$$

Remark, that $v'(t) \neq 0$ for all $t \in [0, 1]$. This is clear for $t \in \{0, 1\}$. If there is a $t \in (0, 1)$ with $v'(t) = 0$, then we can rearrange this to $v = t/(1-t)\cdot\bar{v}$. But then, $v$ is
just a scaled version of $\vec{v}$, in particular, equivalent to $\vec{v}$, and by this also equivalent to $\vec{v}$. This is in contradiction to the assumptions, that $v$ and $\vec{v}$ are non-equivalent.

Since both $v$ and $\vec{v}$ are $\Gamma$-arrangements with representation $T$, so is $v'(t)$ for all $t \in [0, 1]$. In particular, $v'(t)$ is irreducible, and by Proposition 4.7 $v'(t)$ is spherical. This means, that if $M'(t)$ is the matrix of $v'(t)$, then $M'(t)^T M'(t) = \alpha(t) \text{Id}$ for some continuous function $\alpha: [0, 1] \rightarrow \mathbb{R} \setminus \{0\}$. We finally define

$$t \mapsto v(t) := \alpha(t)^{-1/2} \cdot v'(t),$$

which is continuous. Then all $v(t)$ are normalized, and $v(\cdot)$ is the desired deformation between $v$ and $\vec{v}$. \hfill \Box

The remaining case of equivalent arrangements with positively isomorphic representations turns out to be surprisingly non-trivial. In fact, we will observe, that the answer depends on the parity of the dimension: positive representation-isomorphy suffices in even dimensions to imply the existence of a deformation, but not always in odd dimension. The following lemma explains where the differences arise.

**Lemma 5.7.** Let $T, \bar{T} : \Gamma \rightarrow O(\mathbb{R}^d)$ be two isomorphic irreducible representations. Depending on the parity of $d$, the following holds:

(i) if $d$ is odd, then $T$ and $\bar{T}$ are already positively isomorphic.

(ii) if $d$ is even, then either all equivariant maps between $T$ and $\bar{T}$ have non-positive determinant, or all of them have non-negative determinant.

**Proof.** Let $R, \bar{R}$ be two invertible equivariant maps between $T$ and $\bar{T}$.

Assume that $d$ is odd. As a multiple of the identity, $-\text{Id}$ commutes with $T_\phi$ for all $\phi \in \Gamma$. Thus, also $-R$ is equivariant:

$$-RT_\phi = -\text{Id} \cdot RT_\phi = -\text{Id} \cdot T_\phi R = T_\phi (-\text{Id}) R = \bar{T}_\phi (-R).$$

Because $d$ is odd, we have $\det(-R) = -\det(R)$, and either $R$ or $-R$ is an equivariant map of positive determinant, which proves (i).

Now assume that $d$ is even. We observe that $R^T \bar{R}$ commutes with $T_\phi$ for all permutations $\phi \in \Gamma$:

$$R^T \bar{R} T_\phi = R^T \bar{T}_\phi R = T_\phi R^T \bar{R}.$$ By Lemma 2.3, $R^T \bar{R} = \alpha \text{Id}$ for some $\alpha \in \mathbb{R}$. But since $d = 2\delta$ is even, and $R$ and $\bar{R}$ are invertible, we have that

$$\det(R) \det(\bar{R}) = \det(R^T \bar{R}) = \det(\alpha \text{Id}) = \alpha^{2\delta}$$

is positive, no matter the sign of $\alpha$. Thus, $\det(R)$ and $\det(\bar{R})$ have the same sign, which proves (ii). \hfill \Box

In particular, above lemma proves the implication $\text{iso} \rightarrow \text{iso}^+$ under the assumption of odd dimension, and also shows that there are counterexamples in even dimensions.

We now show that in even dimensions and for irreducible $T$, the $S^+_d(T)$ are indeed the path-connected components of $S_d(\Gamma)$.

**Theorem 5.8.** Let $d$ be even. Irreducible arrangements $v, \bar{v} \in S_d(\Gamma)$ are deformation equivalent if and only if their representations are positively isomorphic.
Proof. We have already shown one direction (see Corollary 5.5). It remains to show that $S_d(T)$ is path-connected.

If $v, \bar{v} \in S_d(T)$ are non-equivalent, then a deformation exists by Proposition 5.6. We therefore can assume that they are equivalent. Then they are related by the orthogonal transformation $\bar{M}^\top M \in O(R^d)$, where $M, \bar{M}$ are the matrices of $v$ and $\bar{v}$. We know that $\bar{M}^\top M$ is also an equivariant map between the representations of $v$ and $\bar{v}$. Since the representations are positively isomorphic, and $d$ is even, by Lemma 5.7 all equivariant maps between the representations must have positive determinant. In particular, $\det(\bar{M}^\top M) > 0$, and $v$ and $\bar{v}$ are positively equivalent.

We then apply the already proven implication $\text{eq}^+ \rightarrow \text{d-eq}$.

This proves one part of the implication $\text{iso}^+ \rightarrow \text{d-eq}$ in Figure 3 (the one under the assumption of even dimension). The other part will be proven when we discuss odd dimensions.

For the remainder of this section, let $\tau \in O(R^d) \setminus SO(R^d)$ be some orientation reversing transformation, i.e., $\det(\tau) = -1$. We call $\tau$ a reflection or mirror operation.

In even dimensions, reflections cannot be reversed by a continuous deformation:

Corollary 5.9. If $d$ is even and $v \in S_d(\Gamma)$ an irreducible arrangement, then $v$ and $\tau v$ are not deformation equivalent.

Proof. Since $\tau$ is the transformation between $v$ and $\tau v$, it is also an equivariant map between their representations. Since $d$ is even, by Lemma 5.7 all equivariant maps between the representations must have a negative determinant. Thus, the representations are not positively isomorphic, and $v$ and $\tau v$ are not deformation equivalent by Theorem 5.8.

In odd dimensions, the situation is more diverse. In particular, whether $S_d^+(T)$ is path-connected depends on the group $\Gamma$ and the particular representation $T$. The conditions can be stated in terms of rigidity:

Definition 5.10. A $\Gamma$-arrangements $v \in S_d(\Gamma)$ is rigid if it cannot be deformed to any non-equivalent arrangement. Otherwise it is called flexible.

The following implication in Figure 3 is equivalent to above definition: $\text{d-eq} \rightarrow \text{eq}^+$ under the assumption of rigidity.

There is a straightforward way to characterize rigidity in terms of arrangement spaces as follows: an arrangement $v$ is rigid, if any deformation starting in $v$ has constant arrangement space during the deformation. This simply follows from Theorem 3.1.

We list further characterizations of rigidity:

Theorem 5.11. Let $v \in S_d(\Gamma)$ be irreducible with arrangement space $U \subseteq R^n$ and representation $T$. The following are equivalent:

(i) The arrangement $v$ is flexible.

(ii) There exists a non-equivalent arrangement $\bar{v} \in S_d(\Gamma)$ with a representation isomorphic to $T$.

(iii) There exists a $\Gamma$-invariant subspace $\bar{U} \subseteq R^n$ different from $U$, that is non-orthogonal to $U$.

Proof. The main tools of the proof will be the properties of $\det(\cdot, \cdot)$ (see Lemma 5.3). We will prove

$(i) \implies (iii) \implies (ii) \implies (i)$. 

Assume \((i)\), that is, \(v\) is flexible, and there is a deformation \(v(\cdot)\) with \(v(0) = v\), and \(v(1)\) some non-equivalent arrangement. If \(\det(v, v(1)) \neq 0\), then \((iii)\) already follows from Lemma 5.3. We therefore assume \(\det(v, v(1)) = 0\). Now, consider the function
\[
\gamma(t) := \det(v(t), v), \quad \text{for } t \in [0, 1],
\]
which is continuous in \(t\), and satisfies \(\gamma(0) = 1\) and \(\gamma(1) = 0\). By the intermediate value theorem, there is a \(t \in (0, 1)\) with, say, \(\gamma(t) = \det(v, v(t)) = 1/2\). Since then \(\det(v, v(t)) \neq 0\), we have that the arrangement spaces of \(v\) and \(v(t)\) are non-orthogonal by Lemma 5.3, and because of \(\det(v, v(t)) < 1\) it follows from the same result that the arrangement spaces are also distinct. Lastly, the arrangement space of \(v(t)\) is \(\Gamma\)-invariant by Theorem 4.8, and hence we proved \((iii)\).

Now, assume \((iii)\), and choose a representative \(\bar{v} \in \mathcal{S}_d(\Gamma)\) with arrangement space \(\bar{U}\). Since \(U\) and \(\bar{U}\) are non-orthogonal, by Corollary 5.4 \(v\) and \(\bar{v}\) have isomorphic representations. Since their arrangement spaces are also distinct, they are not equivalent, and we obtain \((ii)\).

Finally, assume \((ii)\), and let \(\bar{v} \in \mathcal{S}_d(\Gamma)\) be the non-equivalent arrangement with isomorphic representation. In particular, \(\bar{v}\) is irreducible. Either \(\bar{v}\) or \(\tau \bar{v}\) (which is also non-equivalent to \(v\)) has a representation that is positively isomorphic to \(T\), hence we can assume that this holds for \(\bar{v}\). Then, there is a deformation between \(v\) and \(\bar{v}\) by Proposition 5.6. Since \(v\) and \(\bar{v}\) are non-equivalent, we see that \(v\) must be flexible, which gives \((i)\).

From this result we learn, that in order for a certain permutation group \(\Gamma \subseteq \text{Sym}(N)\) to allow for flexible \(\Gamma\)-arrangements, it is necessary that the \(\Gamma\)-representation \(\phi \mapsto P_{\phi}\) has an irreducible constituent of multiplicity at least two. Such a constituents gives rise to several non-equivalent \(\Gamma\)-arrangements with isomorphic representations, hence flexibility by Theorem 5.11. Conversely, if all irreducible constituents are distinct, then all \(\Gamma\)-arrangements are rigid.

We now go back to the question of when \(\mathcal{S}_d(T)\) is path-connected for \(d\) odd. While in even dimensions \(v\) and \(\tau v\) are not deformation-equivalent because they cannot have positively isomorphic representations, the same reasoning does not hold up in odd dimensions. Here, isomorphy already implies positive isomorphy by Lemma 5.7.

We will now see, that surprisingly, an odd-dimensional arrangement can be deformed into its mirror image if and only if it can be deformed at all, i.e., if it is not rigid.

**Proposition 5.12.** If \(d\) is odd and \(v \in \mathcal{S}_d(\Gamma)\) an irreducible arrangement, then \(v\) and \(\tau v\) are deformation equivalent if and only if \(v\) is flexible.

**Proof.** If there exists a deformation \(v(\cdot)\) between \(v\) and \(\tau v\), then \(\gamma(t) := \det(v, v(t))\) is continuous and satisfies \(\gamma(0) = 1\) as well as \(\gamma(1) = -1\). By the intermediate value theorem, there is a \(t \in (0, 1)\), so that, say, \(\gamma(t) = 1/2\). By Lemma 5.3, the arrangement spaces of \(v\) and \(v(t)\) are then non-orthogonal, and by Theorem 5.11, \(v\) is flexible.

The other way around, assume that \(v\) is flexible. By Theorem 5.11, there exists a non-equivalent arrangement \(\bar{v}\), so that \(v\) and \(\bar{v}\) have isomorphic representations. In fact, since \(v\) and \(\tau v\) have the same arrangement space, and \(d\) is odd, it follows from Lemma 5.3 and Lemma 5.7 that the representation of \(\bar{v}\) must be positively...
isomorphic to both, the representations of $v$ and $\tau v$. With Proposition 5.6, we then obtain a deformation $v \rightarrow \bar{v} \rightarrow \tau v$.

Hence, if $v \in S^+_{d}(T)$ is rigid, the open set $S^+_{d}(T)$ cannot be path-connected in odd dimensions.

The following theorem is the equivalent to Theorem 5.8 in odd dimensions:

**Theorem 5.13.** Let $d$ be odd, and $v, \bar{v} \in S_d(\Gamma)$ irreducible and flexible arrangements. Then $v, \bar{v}$ are deformation equivalent if and only if they have isomorphic representations.

**Proof.** Again, one direction was shown in Corollary 5.5, and it remains to show that $S_d(\Gamma)$ is path-connected.

If $v, \bar{v}$ have isomorphic representations, then since $d$ is odd, Lemma 5.7 tells us that they also have positively isomorphic representations. If $v$ and $\bar{v}$ are non-equivalent, they are then deformation equivalent by Proposition 5.6. If they are positively equivalent, we can already conclude deformation equivalence by the proven implication $\text{eq}^+ \rightarrow d\text{-eq}$. If, however, $v$ and $\bar{v}$ are equivalent, but not positively equivalent, then $\tau v$ and $\bar{v}$ are positively equivalent, and by $\text{eq}^+ \rightarrow d\text{-eq}$ and Proposition 5.12 there exists a chain of deformations $v \rightarrow \tau v \rightarrow \bar{v}$. Hence, $v$ and $\bar{v}$ are deformation equivalent. □

The promised statement from the introduction follows as a corollary of above theorem, together with Corollary 5.4.

**Corollary 5.14.** If $d$ is odd and $v, \bar{v} \in S_d(\Gamma)$ are irreducible arrangements with non-orthogonal arrangement spaces, then they are deformation equivalent.

With the help of all of these results we can now verify the remaining implications in Figure 3, and show that the conditions are necessary.

We already know that $\text{iso}^+ \rightarrow d\text{-eq}$ holds under the assumption of even dimension. Theorem 5.13 now explains that the same implication is possible by just assuming flexibility (independent of the parity of the dimension). On the other hand, if $v$ is rigid of odd dimension, then $v$ and $\tau v$ form a counterexample.

The implication $\text{iso}^+ \rightarrow \text{eq}^+$ (assuming even dimension and rigidity) is obtained by taking a detour over $d\text{-eq}$:

$$\text{iso}^+ \rightarrow d\text{-eq} \rightarrow \text{eq}^+,$$

where we need even dimension for the first implication, and rigidity for the second. The counterexample for odd dimension is, as above, $v$ and $\tau v$. The counterexample for flexible $v$ is obtain by taking any other non-equivalent arrangement $\bar{v}$ that is obtained by deforming $v$.

Finally, the implication $\text{iso} \rightarrow \text{eq}$ assuming rigidity is proven as follows: if $v$ and $\bar{v}$ have isomorphic representations, then the representation of $v$ is positively isomorphic to either a representation of $\bar{v}$ or $\tau \bar{v}$. Now, either $v, \bar{v}$ are equivalent, or by Proposition 5.6 $v$ can be deformed into $\bar{v}$ or $\tau \bar{v}$. But since $v$ is rigid, $v$ must already be equivalent to one (and then both) of $\bar{v}$ and $\tau \bar{v}$. Obvious counterexamples are obtained by taking $v$ and a non-equivalent arrangement $\bar{v}$ with isomorphic representation, which exists when $v$ is flexible.

As a last result let us mention, that we can upper bound the number of equivalence classes w.r.t. some of the relations in Figure 3:
Theorem 5.15. For some \( d \geq 1 \), denote by \( S_d \) the set of normalized irreducible \( \Gamma \)-arrangements of dimension at least \( d \). It holds

(i) There are at most \( n/d \) arrangements in \( S_d \), so that any two have no isomorphic representations.

(ii) There are at most \( n/d \) pair-wise non-equivalent rigid arrangements in \( S_d \).

Proof. Let \( v^{(i)} \in S_i, i \in I \) be a family of arrangements, and let \( U_i \subseteq \mathbb{R}^n \) be their arrangement spaces. Recall, that \( \dim U_i \) equals the dimension of \( v^{(i)} \) (normalized arrangements are full-dimensional), hence \( \dim U_i \geq d \).

If \( U_i \perp U_j \), then \( v^{(i)} \) and \( v^{(j)} \) have isomorphic representations by Corollary 5.4.

In conclusion, for (i) we need to assume \( U_i \perp U_j \) for all distinct \( i, j \in I \). We obtain

\[
\bigoplus_{i \in I} U_i \subseteq \mathbb{R}^n \implies d \cdot |I| \leq \sum_{i \in I} \dim U_i \leq n.
\]

Rearranging gives the desired inequality \( |I| \leq n/d \) of (i).

If now the \( v^{(i)} \) are rigid and non-equivalent, we again obtain that their arrangement spaces are necessarily pair-wise orthogonal by Theorem 5.11, and above reasoning applies to show (ii).

6. Conclusion and outlook

In this paper we applied the concept of arrangement spaces in the context of symmetric point arrangements. We have shown that symmetric arrangements are in a certain one-to-one correspondence with invariant subspaces of \( \mathbb{R}^n \). We applied this to show how arrangement spaces can be used to make statements about rigidity and continuous deformations of arrangements.

Given some \( \Gamma \subseteq \text{Sym}(N) \), we already noted that it is in general a non-trivial task to determine the \( \Gamma \)-invariant subspaces of \( \mathbb{R}^n \). It is therefore non-trivial to obtain symmetric arrangements in this way. However, if the point set is equipped with the addition structure of a graph, we have easy access to some of the invariant subspaces. In fact, let \( G = (N, E) \) be a graph on \( N \), and \( A \) its adjacency matrix. We now can consider arrangements for which their arrangements space is an eigenspace of \( A \). We currently prepare a follow-up paper, in which we investigate the following implications of such a choice:

- If \( \Gamma \subseteq \text{Aut}(G) \) is a subgroup of the automorphism group of \( G \), then each eigenspace is already \( \Gamma \)-invariant. We therefore obtain an efficient tool to construct symmetric arrangements (or better, symmetric graph realizations), as eigenspaces are comparatively easy to compute and usually come already equipped with a basis – an ONB of eigenvectors.

- The points of such an arrangement are in a specific balanced configuration. In fact, each vertex \( i \in N \) is in the span of the barycenter of its neighbors \( N_i := \{ j \in N \mid j \sim i \} \) (if the barycenter is non-zero).

Quite some work was done for such arrangements, e.g. in the context of distance-regular and strongly regular graphs. Such graphs do not necessarily have a lot of symmetries, and hence, these classes have to be distinguished from arc- and distance-transitive graphs. In a future paper, we investigate the algebraic and geometric properties of arrangements constructed from highly symmetric graphs, and their relation to symmetric polytopes via eigenpolytopes.
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