EXPONENTIAL CONVERGENCE TO EQUILIBRIUM FOR SUBCRITICAL SOLUTIONS OF THE BECKER-DÖRING EQUATIONS

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ABSTRACT. We prove that any subcritical solution to the Becker-Döring equations converges exponentially fast to the unique steady state with same mass. Our convergence result is quantitative and we show that the rate of exponential decay is governed by the spectral gap for the linearized equation, for which several bounds are provided. This improves the known convergence result by Jabin & Nethammer [17]. Our approach is based on a careful spectral analysis of the linearized Becker-Döring equation (which is new to our knowledge) in both a Hilbert setting and in certain weighted \( \ell^1 \) spaces. This spectral analysis is then combined with uniform exponential moment bounds of solutions in order to obtain a convergence result for the nonlinear equation.

KEYWORDS: Becker-Döring equation, exponential trend to equilibrium, spectral gap, decay of semigroups.

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1. Introduction

1.1. The Becker-Döring Equations. The Becker-Döring equations are a model for the kinetics of first-order phase transitions, applicable to a wide variety of phenomena such as crystallization, vapor condensation, aggregation of lipids or phase separation in alloys. They give the time evolution of the size distribution of clusters of a certain substance through the following infinite system of ordinary differential equations:

\[
\begin{align*}
\frac{d}{dt} c_i(t) &= W_{i-1}(t) - W_i(t), \quad i \geq 2, \\
\frac{d}{dt} c_1(t) &= -W_1(t) - \sum_{k=1}^{\infty} W_k(t),
\end{align*}
\]

where

\[W_i(t) := a_i c_1(t) c_i(t) - b_{i+1} c_{i+1}(t) \quad i \geq 1.\]  

Here the unknowns are the real functions \(c_i(t)\) for \(i \geq 1\) an integer, and represent the density of clusters of size \(i\) at time \(t \geq 0\) (this is, clusters composed of \(i\) individual particles). They give the size distribution of clusters of the phase which is assumed to have a small total concentration inside a large ambient phase — be it clusters of crystals, lipids, or droplets of water forming in vapor. The numbers \(a_i, b_i\) (for \(i \geq 1\)) are the coagulation and fragmentation coefficients, respectively, and we always assume them to be strictly positive, i.e.,

\[a_i > 0 \quad \text{for integer } i \geq 1, \quad b_i > 0 \quad \text{for integer } i \geq 2.\]

We set \(b_1 = 0\) for convenience in writing some of the equations. If we represent symbolically by \(\{i\}\) the species of clusters of size \(i\), then (1.1) is based on the assumption that the only (relevant) chemical reactions that take place are

\[\{i\} + \{1\} \rightleftharpoons \{i+1\}\]

for integer \(i \geq 1\). These reactions all take place both ways; the rate at which \(\{i\} + \{1\} \rightarrow \{i+1\}\) occurs is \(a_i c_1 c_i\), while the rate at which \(\{i+1\} \rightarrow \{i\} + \{1\}\) occurs is \(b_{i+1} c_{i+1}\). These rates are proportional to the concentrations of the species appearing in the left hand side, in agreement with the law of mass action from chemistry. Note that \(b_i\) appears in the system (1.1) only for \(i \geq 2\), since single particles cannot break any further. The quantity \(W_i(t)\) defined in (1.2) thus represents the net rate of the reaction \(\{i\} + \{1\} \rightarrow \{i+1\}\). The sum

\[g := \sum_{i=1}^{\infty} ic_i(0) = \sum_{i=1}^{\infty} ic_i(t) \quad \text{for all } t \geq 0.\]

is usually called the density or mass of the solution, and is a conserved quantity of the evolution.

The system (1.1) was originally proposed in [6], though with a variation: the monomer density \(c_1\) was assumed constant and hence equation (1.1b) did not appear. The present version, where the total density is conserved, was first discussed in [7] and [29], and is a widely used model in classical nucleation theory. We refer to these works, as well as the more recent reviews [30, 26] for a background on the physics and applications of the Becker-Döring equations.

We define the detailed balance coefficients \(Q_i\) recursively by

\[Q_1 = 1, \quad Q_{i+1} = \frac{a_i}{b_{i+1}} Q_i \quad \text{for } i \geq 1.\]
An equilibrium of equation (1.1) is a constant-in-time solution (with finite mass). For a given $z \geq 0$, the sequence $c_i := Q_i z^i$ for $i \geq 1$ is formally an equilibrium of (1.1), since all of the $W_i$ vanish. However, some of these sequences do not have finite mass (and hence are not equilibria). The largest possible number $z_s \geq 0$ (possibly $+\infty$) for which $\sum_i iQ_i z^i < +\infty$ for all $0 \leq z < z_s$ is called the critical monomer density (i.e., $z_s$ is the radius of convergence of the power series with coefficients $iQ_i$). The quantity $z_s$ is also called monomer saturation density, hence the subscript. The critical mass (or, again, saturation mass) is then defined by

$$g_s := \sum_{i=1}^{\infty} iQ_i z_s^i \in [0, +\infty].$$

We emphasize that both $g_s$ and $z_s$ are completely determined by the coefficients $a_i$, $b_i$. It is clear from this that the sequences $\{Q_i z^i\}$ are equilibria for $z < z_s$, and $\{Q_i z_s^i\}$ is also an equilibrium if $g_s < +\infty$. These are in fact the only finite-mass equilibria [3]. Since $z \mapsto \sum_{i=1}^{\infty} iQ_i z^i$ is continuous and strictly increasing for $z < z_s$ (and up to $z = z_s$ if $g_s < +\infty$), one sees that for any finite mass $\theta \leq g_s$ there is a unique equilibrium with mass $\theta$. We call a solution subcritical when it has mass $\theta < g_s$, critical when its mass is exactly $g_s$, and supercritical when it has mass $\theta > g_s$ (the two latter cases only make sense provided $g_s < +\infty$).

From the above discussion one sees that for a subcritical or critical solution there exists an equilibrium with the same mass as the solution; this is, there exists $z \leq z_s$ (which must be unique) for which

$$\sum_{i=1}^{\infty} iQ_i z^i = \sum_{i=1}^{\infty} ic_i(t) = \sum_{i=1}^{\infty} ic_i(0) = \theta \quad \text{for } t \geq 0. \tag{1.7}$$

When talking about a subcritical or critical solution we will often denote by $z$ the unique number satisfying (1.7), and refer to it as the equilibrium monomer density. On the other hand, for supercritical solutions there is no equilibrium with the same mass as the solution.

The critical density $g_s$ marks a difference in the behavior of solutions: above the critical density a phase transition phenomenon takes place, reflected in the fact that the excess density $\theta - g_s$ is concentrated in larger and larger clusters as time passes. On the other hand, below or at the critical density a stationary state of the same mass as the solution is eventually reached. Since we are concerned with the study of subcritical solutions we will always assume that

$$z_s > 0 \quad \text{(equivalently, } g_s > 0). \tag{1.8}$$

A fundamental quantity related to (1.1) is the free energy $H(c)$, defined (at least formally) for any sequence $c = (c_i)_{i \geq 1}$ by

$$H(c) := \sum_{i=1}^{\infty} c_i \left( \log \frac{c_i}{Q_i} - 1 \right), \tag{1.9}$$

which decreases along solutions of (1.1) (see [3]): for a (strictly positive, suitably decaying for large $i$) solution $c = c(t) = (c_i(t))_{i \geq 1}$ of (1.1) we have

$$\frac{d}{dt} H(c(t)) = -D(c(t)) := \sum_{i=1}^{\infty} a_i Q_i \left( \frac{c_i c_{i+1}}{Q_i Q_{i+1}} - \frac{c_i}{Q_i} \right) \left( \log \frac{c_i c_{i+1}}{Q_i Q_{i+1}} - \log \frac{c_i}{Q_i} \right) \leq 0 \tag{1.10}$$
for $t \geq 0$. Since mass is conserved, if one fixes $z \leq z_s$ then the same identity is true of the functional $F_z = F_z(c)$ given by

$$F_z(c) := \sum_{i=1}^{\infty} c_i \left( \log \frac{c_i}{z^i Q_i} - 1 \right) + \sum_{i=1}^{\infty} z^i Q_i = H(c) - \log z \sum_{i=1}^{\infty} i c_i + \sum_{i=1}^{\infty} z^i Q_i,$$

(1.11)

this is,

$$\frac{d}{dt} F_z(c(t)) = -D(c(t)) \quad \text{for } t \geq 0.$$  

(1.12)

It is readily checked that $F_z(z^i Q_i) = 0$, i.e., $F_z$ vanishes at the equilibrium $(z^i Q_i)_{i \geq 1}$.

1.2. Typical coefficients. Remarkable model coefficients appearing in the theory of density-conserving phase transitions (see [27, 25]) are given by

$$a_i = i^\alpha, \quad b_i = a_i \left( z_s + \frac{q_i}{1-\mu} \right) \quad \text{for all } i \geq 1,$$

(1.13)

for some $0 < \alpha \leq 1$, $z_s > 0$, $q > 0$ and $0 < \mu < 1$. These coefficients may be derived from simple assumptions on the mechanism of the reactions taking place; we take particular values from [25]:

$$\alpha = 1/3, \quad \mu = 2/3 \quad \text{(diffusion-limited kinetics in 3-D)},$$

$$\alpha = 0, \quad \mu = 1/2 \quad \text{(diffusion-limited kinetics in 2-D)},$$

$$\alpha = 2/3, \quad \mu = 2/3 \quad \text{(interface-reaction-limited kinetics in 3-D)},$$

$$\alpha = 1/2, \quad \mu = 1/2 \quad \text{(interface-reaction-limited kinetics in 2-D)}.$$  

(1.14)

One obtains from (1.13) and (1.5) that in this case

$$Q_i = \frac{a_1 a_2 \ldots a_{i-1}}{b_2 b_3 \ldots b_i} = i^{-\alpha} \prod_{j=2}^{i} \left( z_s + \frac{q}{j^{1-\mu}} \right)^{-1} \quad \text{for all } i \geq 1,$$

where the product is understood to be equal to 1 for $i = 1$ (so $Q_1 = 1$). One can deduce from this that the critical monomer density is indeed the quantity $z_s$ appearing in (1.13), and that $\varrho_s$ is a finite positive quantity.

Another kind of reasoning that leads to a similar set of coefficients is the following: while one may determine the coagulation coefficients through an understanding of the aggregation mechanism and estimate that

$$a_i = i^\alpha,$$

(1.15)

for some $0 < \alpha \leq 1$ as above, the equilibrium state of a system may be obtained from statistical mechanics considerations: it should be

$$c_i = c^i \exp(-\epsilon_i),$$

where $\epsilon_i$ is the binding energy of a cluster of size $i$ (the energy required to assemble a cluster of size $i$ from $i$ monomers, in units of Boltzmann’s constant times the temperature). Depending on the kind of aggregates considered, this binding energy can be estimated (at least for large $i$) as

$$\epsilon_i = -\beta(i-1) + \sigma(i-1)^\mu,$$

where $\beta$ is the energy released when adding one particle to a cluster, and $\sigma$ is related to the surface tension of the aggregates. The values of $\mu$ and $\alpha$ for various situations are still those in (1.14). From this we may deduce that

$$Q_i = \exp \left( \beta(i-1) - \sigma(i-1)^\mu \right) = z_s^{1-i} \exp \left( -\sigma(i-1)^\mu \right),$$

(1.16)
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where we define $z_s := \exp(-\beta)$. By (1.5) and (1.15) we finally have

$$ a_i = i^\alpha, \quad b_i = z_s(i - 1)^\alpha \exp \left( \sigma i^\mu - \sigma(i - 1)^\mu \right) \quad \text{for all } i \geq 1, \tag{1.17} $$

for some $0 \leq \alpha \leq 1$, $z_s > 0$ and $0 < \mu < 1$. In this case $z_s$ is still consistent with our definition of the critical monomer density. Observe that for large $i$ we have $i^\mu - (i - 1)^\mu \sim \mu i^{\mu - 1}$, so the fragmentation coefficients become roughly

$$ b_i \sim z_s a_i \exp \left( \sigma \mu i^{\mu - 1} \right) \sim a_i \left( z_s + \frac{z_s \sigma \mu}{i^{1-\mu}} \right), \tag{1.18} $$

which is like (1.13) with $q = z_s \sigma \mu$.

Both example coefficients (1.17) and (1.13) are often used in the literature, so we always state explicitly the consequences of our results for them.

1.3. **Main results.** The mathematical theory of the Becker-Döring equations (1.1) has been developed in a number of papers since the first rigorous works on the topic [3, 2]. Existence and uniqueness of a solution for all times is ensured [3, Theorems 2.2 and 3.6] provided that

$$ a_i = O(i), \quad b_i = O(i), \quad \sum_{i=1}^\infty i^2 c_i(0) < +\infty. \tag{1.19} $$

As expected, the unique solution conserves mass (this is, (1.4) holds rigorously). Weaker, or different, conditions may be imposed for this to hold, but (1.19) will be enough for our purposes in the present paper. Since we use some results from [17], we often also require the following:

$$ \inf_{k \geq 1} a_k := q > 0, \tag{1.20a} $$

$$ 0 < z_s < +\infty, \tag{1.20b} $$

$$ \lim_{i \to +\infty} \frac{Q_{i+1}}{Q_i} = \frac{1}{z_s}, \tag{1.20c} $$

$$ \lim_{i \to +\infty} \frac{a_{i+1}}{a_i} = 1. \tag{1.20d} $$

Regarding (1.20c), we recall that $z_s$ is the radius of convergence of the power series with coefficients $iQ_i$; hence, if the limit of $Q_{i+1}/Q_i$ exists, it can only be $z_s$. We notice that the conditions in [17] are slightly different from the ones above. However, the differences are not essential and all arguments can be carried out with the ones here, which are more convenient and are satisfied by the example coefficients (1.13) and (1.17). Our study of the spectral properties of the linearized operator, and in particular the proof that it is self-adjoint in Proposition 2.10, seems to require the above limiting behavior of the ratios $Q_{i+1}/Q_i$ and $a_{i+1}/a_i$ (compare to the conditions in [17]). It may be possible to modify our proofs to accommodate slightly more general conditions, but we avoid this in order to simplify the main argument.

The relationship between the long-time behavior of supercritical solutions and late-stage coarsening theories is especially interesting, and has been studied in [28, 31, 9, 25]; in particular, in [25] it was rigorously shown that a suitable rescaling of supercritical solutions approximates a solution of the Lifshitz-Slyozov equations [19]. Asymptotic approximations of solutions have been developed in [24, 23, 12, 13].

For subcritical solutions (the regime which we study in the present paper) it was proved in [3, 2] that a stationary state with the same mass as the solution is approached as time passes. The rate of this convergence was studied in [17] and was found to be at least like $\exp(-Ct^{1/3})$ for some constant $C$. Our purpose here is to improve this by showing that in fact the speed of convergence
is exponential (this is, like \( \exp(-\lambda t) \) for some \( \lambda > 0 \)). We give upper and lower bounds of the constant \( \lambda \), and study its behavior as the density \( \varrho \) of the solution approaches (from below) the critical density \( \varrho_0 \). An important quantity which appears in our analysis is

\[
B := \sup_{k \geq 1} \left( \sum_{j=k+1}^{\infty} Q_j z^j \right)^{1 \over z} \left( \sum_{j=1}^{k} {1 \over a_j Q_j z^j} \right) \in (0, +\infty].
\] (1.21)

Observe that \( B \) depends only on the coefficients \( a_i \) (i \( \geq 1 \)) and \( b_i \) (i \( \geq 2 \)). We are able to show that subcritical solutions of (1.1) converge exponentially to equilibrium and \( 1/B \) is a good measure of the speed of convergence whenever \( B \) is finite:

**Theorem 1.1.** Assume (1.19) and (1.20). Let \( c = (c_i) \) be a nonzero subcritical solution to equation (1.1) with initial condition \( c(0) \) such that, for some \( \nu > 0 \),

\[
M := \sum_{i=1}^{\infty} \exp (\nu i) c_i(0) < +\infty,
\] (1.22)

and define the equilibrium monomer density \( z > 0 \) by (1.7). Then there exist explicit \( \bar{\nu} \in (0, \nu) \) and \( \lambda_* > 0 \) such that, for any \( \eta \in (0, \bar{\nu}) \), there is a number \( C > 0 \) which depends only on \( \varrho, \eta, M \) and \( \sum_{i=1}^{\infty} e^{\eta i} z^i Q_i \) such that

\[
\sum_{i=1}^{\infty} \exp (\eta i) |c_i(t) - z^i Q_i| \leq C \exp (-\lambda_* t) \quad \text{for all } t \geq 0,
\] (1.23)

Under these conditions \( B \) is finite, and if \( \lim_{i \to +\infty} a_i = +\infty \) we may take \( \lambda_* = 1/B \).

For the particular case of the example coefficients (1.13) or (1.17), if \( \alpha < 2(1 - \mu) \) we show that \( B \) can be bounded by

\[
C_1 \left( \log \frac{z_{\alpha}}{z} \right)^{-2 + \frac{2(1 - \mu)}{\alpha}} \leq B \leq C_2 \left( \log \frac{z_{\alpha}}{z} \right)^{-2 + \frac{2(1 - \mu)}{\alpha}}
\]

for \( z < z_{\alpha} \), where \( C_1 \) and \( C_2 \) depend only on the coefficients \( (a_i)_i, (b_i)_i \) (and in particular are independent of \( z \)). In the case \( \alpha \geq 2(1 - \mu) \) we have

\[
C_1 \leq B \leq C_2
\]

for \( z \leq z_{\alpha} \) and some (other) constants \( C_1, C_2 \) that again depend only on the coefficients. We observe that if \( z = z_{\alpha} \) then \( B \) is finite if and only if \( \alpha \geq 2(1 - \mu) \).

1.4. **Method of proof.** Our proof is based on a study of the linearization of the Becker-Döring equations around the equilibrium, for which we show the existence of a spectral gap whose size is well estimated by \( 1/B \) (see Theorem 2.15 for details). This implies exponential convergence to equilibrium for the linearized system, which can be extended to the nonlinear equations by means of techniques developed in the literature on kinetic equations, and particularly on the Boltzmann equation [21, 16].

We observe that the improvement with respect to [17] comes from the use of a different method. The main tool in [17] is an inequality between the free energy (or entropy) \( H \) defined by (1.9) and its production rate \( D \) (see (1.10)) in the spirit of the ones available for the Boltzmann equation [10]. As pointed out in [17], an inequality like \( H \leq CD \) for some constant \( C > 0 \), which would directly imply an exponential convergence to equilibrium, is roughly analogous to a functional log-Sobolev inequality, which is known not to hold for a measure with an exponential tail. Since
this is the case for the stationary solutions of the Becker-Döring equation, it is believed (though, to our knowledge, not proved) that this inequality does not hold in general for this equation; hence, the following weaker inequality (this is, weaker for small $H$) is proved in [17]:

$$\frac{H}{|\log H|^2} \leq CD,$$

implying a convergence like $\exp(-Ct^{1/3})$. This obstacle has a parallel in the Boltzmann equation, for which the corresponding inequality (known as Cercignani's conjecture) has been proved not to hold in general, and can be substituted by inequalities like $H^{1+\epsilon} \leq CD$ for $\epsilon > 0$ (we refer to the recent review [10] for the history of the conjecture and a detailed bibliography). However, just as for the space homogeneous [21] and the full Boltzmann equation [16] this can be complemented by the study of the linearized equation in order to show full exponential convergence. By following a parallel reasoning for the Becker-Döring system we can upgrade the convergence rate to exponential.

Hence, our analysis is built around a study of the linearized Becker-Döring equation, which is new to our knowledge. We prove here the existence of a positive spectral gap of the operator $L$, defined in Section 2 as a suitable linearization of Eq. (1.1) around the equilibrium $(Q_i)_{i \geq 1} = (z^i Q_i)_{i \geq 1}$, in different spaces:

1. We provide first a spectral description of $L$ in a Hilbert space setting. Namely, we shall investigate the spectral properties of the operator $L$ in the weighted space $\mathcal{H} = \ell^2(Q)$. This analysis is carried out with two (complementary) techniques: on the one hand, under reasonable conditions on the coefficients, one can show that $L$ is self-adjoint in $\mathcal{H}$ and, resorting to a compactness argument, the existence of a non-constructive spectral gap can be shown. On the other hand, using a discrete version of the weighted Hardy's inequality, the positivity of the spectral gap is completely characterized in terms of necessary and sufficient conditions on the coefficients. Moreover, and more importantly, quantitative estimates of this spectral gap are given.

2. Unfortunately, as it occurs classically for kinetic models, the Hilbert space setting which provides good estimates for the linearized equation is usually not suitable for the nonlinear equation. Thus, inspired by previous results on Navier-Stokes and Boltzmann equation [15, 21], we derive the spectral properties of the linearized operator in a larger weighted $\ell^1$ space. We use for this an abstract result (see Theorem 3.1) allowing to enlarge the functional space in which the exponential decay of a semigroup holds. This follows recent techniques developed in [16], though we give a self-contained proof simplified in our setting. The application of this theoretical result requires some important technical efforts, see Theorem 3.5.

It is worth pointing out that our techniques parallel the historical development of the study of the exponential decay of the homogeneous Boltzmann equation. We first show by non-constructive methods based on Weyl's theorem that the linearized operator $L$ has a positive spectral gap. An exposition of similar techniques for the linearized homogeneous Boltzmann equation can be found in [8]. Explicit estimates for this spectral gap were given in [5], and similar techniques for the extension of the spectral gap were devised in [21] and developed in [16], and used to study the rate of convergence to equilibrium of the homogeneous Boltzmann equation.
In Section 2 we carry out the plan in point (1) above, and in Section 3 we carry out point (2). The application to the nonlinear equation and the proofs of our main results are then given in Section 4.

2. The linearized Becker-Döring equations

2.1. The linearized operator. For the whole of Section 2 we assume the following:

**Hypothesis 2.1.** We take \((a_i)_{i \geq 1}\) and \((b_i)_{i \geq 1}\) satisfying (1.3), define \((Q_i)_{i \geq 1}\) by (1.5), and assume (1.8). We also take \(0 < z \leq z_a\) and set

\[
Q_i = Q_i z^i, \quad i \geq 1.
\]

We also assume that

\[
A := \sum_{i=1}^{\infty} i^2(1 + a_i + b_i)^2 Q_i < +\infty.
\]

We remark that, when \(z < z_a\), the sum in (2.2) is indeed finite under “reasonable” conditions on the coefficients (such as (1.19)). Hence condition (2.2) is important mainly for the case \(z = z_a\). Also, when \(z = z_a\), condition (2.2) ensures that \(\bar{\theta}_0 < +\infty\).

The choice of \(0 < z \leq z_a\) corresponds to the choice of a mass \(0 < \theta \leq \theta_b\) given by (1.7). The unique equilibrium with mass \(\theta\) is precisely \((Q_i)_{i \geq 1}\), given by (2.1). Notice that (1.5) implies that

\[
a_i Q_i Q_i = b_{i+1} Q_{i+1}, \quad i \geq 1.
\]

Consider a solution \((c_i)_{i \geq 1}\) of (1.1) with mass \(\theta\) (this is, a subcritical or critical solution). In order to linearize equation (1.1) around the steady state \((Q_i)\), we define the fluctuation \(h = (h_i)_{i \geq 1}\) by

\[
c_i = Q_i (1 + h_i)
\]

where the components of \(h\) may have any sign. Let us carry out some formal computations, and leave the precise definition of the linearized operator for Section 2.2. Notice that, in order for (1.7) to be satisfied, it is necessary that

\[
\sum_{i=1}^{\infty} i Q_i h_i(t) = 0 \quad \text{for all } t \geq 0.
\]

The weak form of (1.1) reads as follows:

\[
\sum_{i=1}^{\infty} \phi_i \frac{d}{dt} c_i = \sum_{i=1}^{\infty} (a_i c_i(t) c_{i+1}(t) - b_{i+1} c_{i+1}(t)) (\phi_{i+1} - \phi_i - \phi_1)
\]

for any sequence \((\phi_i)_{i \geq 1}\). Plugging into this the ansatz (2.4) yields, for any sequence \((\phi_i)\),

\[
\sum_{i=1}^{\infty} \frac{d}{dt} h_i(t) Q_i \phi_i
\]

\[
= \sum_{i=1}^{\infty} \left( a_i Q_i Q_i (1 + h_i(t)) (1 + h_1(t)) - b_{i+1} Q_{i+1} (1 + h_{i+1}(t)) \right) (\phi_{i+1} - \phi_i - \phi_1).
\]

Using (2.3) one sees that, for any \(i, j \geq 1\),

\[
a_i Q_i Q_j (1 + h_i(t)) (1 + h_1(t)) - b_{i+1} Q_{i+1} (1 + h_{i+1}(t))
\]

\[
= a_i Q_i Q_j (h_i(t) + h_1(t) - h_{i+1}(t)) + a_i Q_i h_i(t) h_1(t).
\]
This means that the fluctuation \( h(t) = (h_i(t))_i \) satisfies
\[
\frac{d}{dt} h_i(t) = L_i(h(t)) + \Gamma_i(h(t), h(t))
\]  
(2.8)
where the linear operator \( L \) is given, in weak form, by
\[
\sum_{i=1}^{\infty} L_i(h) Q_i \phi_i = \sum_{i \geq 1} W^L_i (\phi_{i+1} - \phi_i - \phi_1) \\
= \sum_{i=1}^{\infty} a_i Q_i Q_1 (h_i + h_1 - h_{i+1} + h_{i+1})(\phi_{i+1} - \phi_i - \phi_1)
\]  
(2.9)
for any sequences \( h = (h_i)_i, (\phi_i)_i \), where
\[
W^L_i := a_i Q_i Q_1 (h_i + h_1) - b_{i+1} Q_{i+1} h_{i+1} = a_i Q_i Q_1 (h_i + h_1 - h_{i+1}).
\]  
(2.10)
Alternatively, we may write \( L \) in strong form as
\[
L_1(h) = -\frac{1}{Q_1} \left( W^L_1 + \sum_{i=1}^{\infty} W^L_i \right), \quad L_i(h) = \frac{1}{Q_i} (W^L_{i-1} - W^L_i) \quad (i \geq 2).
\]  
(2.11)
The bilinear operator \( \Gamma(f, g) \) is defined in weak form by
\[
\sum_{i=1}^{\infty} \Gamma_i(f, g) Q_i \phi_i = \frac{1}{2} \sum_{i \geq 1} a_i Q_i Q_1 (f_i g_1 + f_1 g_i) (\phi_{i+1} - \phi_1 - \phi_1)
\]  
for any sequences \( f = (f_i)_i, g = (g_i)_i \) and \( (\phi_i)_i \). Alternatively, the strong form of \( \Gamma_i(f, g) \) can be written as
\[
\Gamma_1(f, g) = -a_1 Q_i f_1 g_1 - \frac{1}{2} \sum_{i=1}^{\infty} a_i Q_i (f_i g_1 + f_1 g_i)
\]  
(2.12)
and
\[
\Gamma_i(f, g) = \frac{1}{2Q_i} \left( a_{i-1} Q_{i-1} Q_1 (f_{i-1} g_1 + f_1 g_{i-1}) - a_i Q_i Q_1 (f_i g_1 + f_1 g_i) \right) \quad i \geq 2.
\]  
(2.13)
Neglecting in (2.8) the quadratic term \( \Gamma(h, h) \), one is faced with the linearized problem:
\[
\frac{d}{dt} h(t) = L(h(t)), \quad h(0) = h^0,
\]  
(2.14)
which should be understood as the linear approximation of equation (1.1) close to the equilibrium \( (Q_i)_{i \geq 1} \). Our purpose in the rest of this section is to study the operator \( L \) and its spectral properties, thus obtaining the asymptotic behavior of equation (2.14).

### 2.2. Study of the linearized operator and proof of existence of a spectral gap

The first thing we need to do in order to define rigorously the operator \( L \) is to give its domain. We take expression (2.9) as a starting point, and we denote by \( \ell_{00} \) the set of compactly supported sequences \( h = (h_i)_i \) (this is, the sequences for which there exists \( N > 0 \) such that \( h_i = 0 \) for all \( i \geq N \)).

**Definition 2.2** (The operator \( L \)). Assume Hypothesis 2.1. For a compactly supported sequence \( h = (h_i)_i \in \ell_{00} \) we define \( L(h) \) by the expression (2.11) (or, equivalently, (2.9) for \( h, \phi \in \ell_{00} \)).
Notice that the only infinite sum in (2.11) converges by (2.2). (Actually, the slightly weaker condition \( \sum_i a_i Q_i < +\infty \) would be enough for the definition, but we keep (2.2) for simplicity).

One can give a more compact expression of \( L \) by direct inspection, using (2.3) repeatedly:

**Lemma 2.3.** Assume Hypothesis 2.1. For any compactly supported sequence \( h \),

\[
L_i(h) = -\sigma_i h_i + \sum_{j=1}^{\infty} \xi_{i,j} h_j \quad \text{for } i \geq 1
\]  

(2.15)

where \( \sigma_i \) are defined by

\[
\sigma_1 = 3a_1 Q_1 + \sum_{i=1}^{\infty} a_i Q_i, \quad \sigma_i = a_i Q_1 + b_i \quad \text{for } i \geq 2,
\]  

(2.16)

\( \xi_{i,j} \) are defined by

\[
\begin{align*}
Q_1 \xi_{1,2} &= Q_2 \xi_{2,1} = 2b_2 Q_2 - a_2 Q_1 Q_2, \\
Q_i \xi_{1,i} &= Q_1 \xi_{i,1}, \quad b_i Q_i - a_i Q_1 Q_i \quad \text{for } i > 2, \\
Q_i \xi_{i-1,i} &= Q_{i-1} \xi_{i-1,i} = b_i Q_i \quad \text{for } i > 2,
\end{align*}
\]  

(2.17)  

(2.18)  

(2.19)

and \( \xi_{i,j} = 0 \) for \( j \notin \{1, i-1, i+1\} \).

The numbers \( \xi_{i,j} \) represent the nondiagonal entries of the infinite-dimensional matrix that defines \( L \). One can see that the only nonzero entries of this matrix are in the diagonal and in the first line and column. In addition, the numbers \( \xi_{i,j} \) have the important property that \( (Q_i \xi_{i,j})_{i,j \geq 1} \) is a symmetric matrix, which suggests considering the inner product with weight \( Q_i \). There is another reason why this inner product appears naturally: since the nonlinear equation (1.1) has a Lyapunov functional (see (1.9)–(1.12)), one may look at the second-order approximation of this functional close to the equilibrium \( (Q_i)_{i \geq 1} \), which happens to be

\[
\|h\|^2_H := \frac{1}{2} \sum_{i=1}^{\infty} Q_i h_i^2.
\]  

(2.20)

It becomes then natural to study the linearized operator \( L \) in the Hilbert space \( H := \ell^2(Q) \) defined as

\[
H = \{ h = (h_i)_{i \geq 1} \mid \|h\|_H < \infty \}
\]

with inner product denoted by \( \langle ., . \rangle \). As remarked above, \( L \) becomes a symmetric operator in \( H \). Another crucial property of \( L \) in this Hilbert space is that it is dissipative; this is,

\[
\langle Lh, h \rangle \leq 0 \quad \text{for all } h \in \ell_{00},
\]

as can be readily seen from (2.9). Though \( L \) is in general not continuous on \( H \), it is easy to give a dense subspace of \( H \) in which it is bounded:

**Lemma 2.4.** Assume Hypothesis 2.1 and (1.20c). Then, there exists \( C > 0 \) (depending only on \( z \) and the coefficients \((a_i)_{i \geq 1}, (b_i)_{i \geq 1}\)) such that

\[
\|L(h)\|_H \leq C \|h\|_{H_2}
\]

(2.21)

for any compactly supported sequence \( h \), where

\[
H_2 = \ell^2(Q(1 + \sigma^2)) := \left\{ h \in H \mid \|h\|^2_{H_2} := \sum_{i=1}^{\infty} Q_i(1 + \sigma_i^2) h_i^2 < +\infty \right\}
\]  

(2.22)
and \((\sigma_i)_i\) was defined in (2.16).

**Remark 2.5.** In fact, instead of condition (1.20c) it is enough to have that \(Q_{i+1}/Q_i\) is uniformly bounded in \(i\), as can be seen from the proof. Notice that the assumptions (H1)–(H4) of [17] imply the boundedness of \((Q_{i+1}/Q_i)_i\).

**Proof.** Instead of (2.21) we will show, equivalently, that

\[
\langle Lh, \phi \rangle \leq C \|h\|_{\mathcal{H}} \|\phi\|_{\mathcal{H}} \quad \text{for all } \phi \in \mathcal{H}.
\]  
(2.23)

In this proof \(C\) denotes any positive quantity depending only on \(z\) and the coefficients \((a_i, b_i)_{i \geq 1}\), possibly changing from line to line. From (2.9) we have, using Cauchy-Schwarz’s inequality,

\[
\langle Lh, \phi \rangle = \sum_{i=1}^{\infty} a_i Q_i (h_i + h_1 - h_{i+1}) (\phi_{i+1} - \phi_i - \phi_1)
\]

\[
\leq Q_1 \left( \sum_{i=1}^{\infty} a_i^2 Q_i (h_i + h_1 - h_{i+1})^2 \right)^{1/2} \left( \sum_{i=1}^{\infty} Q_i (\phi_{i+1} - \phi_i - \phi_1)^2 \right)^{1/2}.
\]  
(2.24)

The term inside the first parentheses is bounded by

\[
\sum_{i=1}^{\infty} a_i^2 Q_i (h_i + h_1 - h_{i+1})^2 \leq 3 \sum_{i=1}^{\infty} a_i^2 Q_i (h_i^2 + h_1^2 + h_{i+1}^2)
\]

\[
\leq 3 \sum_{i=1}^{\infty} \sigma_i^2 Q_i h_i^2 + 3Ah_1^2 + 3 \sum_{i=1}^{\infty} a_i^2 Q_i h_{i+1}^2
\]

\[
\leq C \|h\|^2_{\mathcal{H}_2} + \sum_{i=1}^{\infty} \frac{Q_{i+1}}{Q_i} b_i^2 Q_{i+1} h_{i+1}^2 \leq C \|h\|^2_{\mathcal{H}_2},
\]

where we used (2.3) and the fact that \(Q_{i+1}/Q_i\) is bounded uniformly in \(i\) thanks to (1.20c). A similar reasoning shows that the second parenthesis in (2.24) is bounded by

\[
\sum_{i=1}^{\infty} Q_i (\phi_{i+1} - \phi_i - \phi_1)^2 \leq C \|\phi\|^2_{\mathcal{H}},
\]

which shows (2.23). \(\square\)

The operator \(L\) is defined only on compactly supported sequences \(h\). We will now extend it to a larger domain, so that it becomes a closed operator, and study its spectrum. We will consider two ways to do this, later shown to lead to the same result: one of them is to look at the closure \(L\) of \(L\), and the other one is to look at an associated quadratic form, which will lead to an extension denoted by \(\mathcal{L}\). Let us first describe the latter.

Expression (2.9) suggests the introduction of a symmetric form \(\mathcal{E}\) in \(\mathcal{H}\) by setting

\[
\mathcal{E}(h, g) = -\langle h, Lg \rangle = \sum_{i \geq 1} a_i Q_i Q_1 (h_{i+1} - h_i - h_1) (g_{i+1} - g_i - g_1) \quad \text{for } h, g \in \ell_{00},
\]

which can be naturally extended to the domain

\[
\mathcal{D}(\mathcal{E}) = \{ h \in \mathcal{H} \; ; \; \mathcal{E}(h, h) < \infty \}.
\]
by setting
\[ E(h, g) = \sum_{i \geq 1} a_i Q_i Q_1 \left( h_{i+1} - h_i - h_1 \right) \left( g_{i+1} - g_i - g_1 \right) \quad \text{for } h, g \in D(E). \tag{2.25} \]

Then, one has the following

**Proposition 2.6.** Assume Hypothesis 2.1. Define the space \( H_1 \) by
\[ H_1 = \left\{ h \in H : \|h\|_{H_1}^2 := \sum_{i=1}^{\infty} (1 + \sigma_i) Q_i h_i^2 < \infty \right\}, \]
where we recall that \( \sigma_i \) was defined in (2.16).

The form \( E \) with domain \( D(E) \) defined by (2.25) is a closed symmetric form on \( H \). Its domain \( D(E) \) contains \( H_1 \) and there exists \( C > 0 \) (depending only on the coefficients \((a_i)_{i \geq 1}\) and \((b_i)_{i \geq 2}\)) such that
\[ |E(h, g)| \leq C \|h\|_{H_1} \|g\|_{H_1} \quad \text{for all } h, g \in H_1. \tag{2.26} \]

**Proof.** The fact that \( E \) is symmetric and bilinear is clear. Moreover, one proves as in [14, Example 1.2.4] that \((E, D(E))\) is closed thanks to Fatou’s Lemma. Let us prove (2.26), which in turn shows that \( H_1 \subseteq D(E) \). By a usual depolarization argument it is enough to show (2.26) for \( g = h \). We have
\[ E(h, h) = \sum_{i=1}^{\infty} a_i Q_i Q_1 \left( h_{i+1} + h_1 - h_i \right)^2 \leq 3 \sum_{i=1}^{\infty} a_i Q_i Q_1 \left( h_i^2 + h_i^2 + h_i^2 + h_i^2 \right) \]
\[ = 3Ah_1^2 + 3 \sum_{i=1}^{\infty} a_i Q_i Q_1 h_i^2 + 3 \sum_{i=2}^{\infty} b_i Q_i h_i^2 \leq C \|h\|_{H_1}^2, \]
where \( A \) is the quantity in (2.2). This proves the desired bound. \( \square \)

Due to the previous Proposition, according to [14, Theorem 1.3.1 & Corollary 1.3.1], there exists a unique non-positive definite self-adjoint operator \( L \) on \( H \) such that \( D(L) \subset D(E) \) and
\[ E(h, g) = -\langle Lh, g \rangle \quad \text{for all } h \in D(L), \ g \in D(E). \tag{2.27} \]

More precisely, \( D(E) = D(\sqrt{-L}) \). It is clear that the linear operator \((L, D(L))\) extends the above linear operator \( L \), defined on \( \ell_{00} \), and that \( \ell_{00} \) is a core for \( L \). It is also easy to see, under the conditions of Lemma 2.4, that the domain of \( L \) must include the space \( H_2 \) (since \( L \) is a closed operator that extends \( L_2 \)), and that the expression of \( L \) in \( H_2 \) is still given by (2.11) (since each of the sums converges absolutely in this space, as deduced from the proof of Lemma 2.4) or alternatively by (2.9) (for any \( \phi \in \ell_{00} \)). It is also easy to see that 0 is an eigenvalue of \( L \) with explicit eigen-space:

**Lemma 2.7.** Assume Hypothesis 2.1 and (1.20c). Then 0 is an eigenvalue of \( L \), with a one-dimensional associated eigenspace spanned by the sequence defined by \( h_i = i \) for \( i \geq 1 \).

**Proof.** Under the condition (2.2) one sees that \( h = (i)_{i \geq 1} \in H_2 \subseteq D(L) \) (notice that the latter inclusion holds due to the previous discussion and Lemma 2.4). It is clear then from (2.27) and (2.25) that \( \langle Lh, g \rangle = 0 \) for all \( g \in D(E) \), so \( Lh = 0 \). On the other hand, if there is any \( \tilde{h} = (\tilde{h}_i)_{i \geq 1} \in D(L) \) such that \( L(\tilde{h}) = 0 \), then \( E(h, \tilde{h}) = 0 \) and consequently \( \tilde{h}_{i+1} = h_i + h_1 \) for all \( i \geq 1 \). This implies that \( h_i = ih_1 \) for \( i \geq 1 \), so \( \tilde{h} \) is a multiple of \( h \). \( \square \)
Remark 2.8. Again, Lemma 2.7 holds with the milder condition that $Q_{i+1}/Q_i$ is uniformly bounded instead of (1.20c).

It is well-known [18, 14] that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ (or, equivalently, the operator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$), generates a strongly continuous semigroup of contractions $\{S_t, t \geq 0\}$ in $\mathcal{H} = \ell^2(\mathcal{Q})$. In particular, for any $h^0 = (h^0_i)_{i \geq 1} \in \mathcal{H}$ the linearized problem
\[
\frac{d}{dt} h_i(t) = L_i(h(t)), \quad h_i(0) = h^0_i,
\]
(2.28)

admits a unique mild solution $h(t) = (h_i(t))_{i \geq 1}$ given by $h(t) = S_t h^0$ satisfying
\[
\|h(t)\| \leq \|h^0\| \quad \text{for all } t \geq 0.
\]
Moreover, $S_t(\mathcal{H}) \subset \mathcal{D}(\mathcal{E})$ for any $t > 0$ and, if $h^0 = (h^0_i)_{i \geq 1} \in \mathcal{D}(\mathcal{L})$ then, for any $t \geq 0$,
\[
\sup_{s \in [0, t]} \|h(s)\|_{\mathcal{D}(\mathcal{E})} < \infty \quad \text{(we refer to [14, Chapter 1] for details). In fact, for } h^0 \in \mathcal{D}(\mathcal{L}) \text{ we have that } h \text{ is } C^1 \text{ in time, } h(t) \in \mathcal{D}(\mathcal{L}) \text{ for all } t \geq 0 \text{ and (2.28) is satisfied pointwise in time. Hence, for } h^0 \in \mathcal{D}(\mathcal{L}) \text{ we have}
\]
\[
\frac{d}{dt} \|h(t)\|^2 = (h, Lh) = -\mathcal{E}(h, h) = -\sum_{i \geq 1} \alpha_i Q_i Q_1 (h_{i+1} - h_i - h_1)^2 \quad \text{for all } t \geq 0.
\]

The advantage of the approach involving the quadratic form is that we obtain, in a simple way, a self-adjoint extension of $L$ on which we have a lot of information. However, it is not easy to give explicitly the domain $\mathcal{D}(L)$ of the operator $L$. We consider also a different approach based on the closure of $L$. Since $L$ is symmetric, it is closable and we define $L$ as its closure:

**Definition 2.9 (The operator $L$).** Assume Hypothesis 2.1. We define the linear operator $L$, with domain $\mathcal{D}(L)$, as the closure in $\mathcal{H}$ of the linear operator $L$ (with domain $\ell^2_0$).

The following proposition gives a certain structure to $L$ that will allow us to identify its domain, prove it has a spectral gap (in a non-constructive way, using Weyl’s theorem) and show that in fact $L = \mathcal{L}$:

**Proposition 2.10.** Assume Hypothesis 2.1 and conditions (1.20a), (1.20c) and (1.20d). Assume also that $s < s_\alpha$. Then the linear operator $L$ is self-adjoint with domain $\mathcal{D}(L) = \mathcal{H}_2$ (as defined in (2.22)) and, given $\delta > 0$, it can be written as
\[
L(h) = L^C(h) + L^M(h) \quad \text{for all } h \in \mathcal{D}(L)
\]
(2.29)
such that
1. $L^C$ is a compact operator on $\mathcal{H}$.
2. $L^M$ is a self-adjoint operator with domain $\mathcal{H}_2$, and with spectrum
\[
\mathcal{S}(L^M) \subseteq (-\infty, -\lambda_M]
\]
where
\[
\lambda_M := \sigma \left(1 - 2 \frac{\sqrt{\ell}}{1 + \ell}\right) - \delta, \quad \ell := s_\alpha/z
\]
and
\[
\sigma := \inf_{k \geq 1} \sigma_k,
\]
which is strictly positive due to (1.20a).
Proof. Take an integer $N \geq 1$, to be fixed later. We define, for all $i \geq 1$,

$$L_i^C(h) := \sum_{j=1}^{\infty} \chi_{\{\min(i,j) \leq N\}} \xi_{i,j} h_j \quad \text{for } h \in H,$$

$$L_i^M(h) := -\sigma_i h_i + \sum_{j=1}^{\infty} \chi_{\{\min(i,j) > N\}} \xi_{i,j} h_j \quad \text{for } h \in H_2,$$

(2.30)

(2.31)

We recall that the notation $\sigma_i$ and $\xi_{i,j}$ was defined in (2.16)–(2.19). The notation $\chi_{\{\ldots\}}$ represents a function which is equal to 1 when the condition in the brackets is satisfied, 0 otherwise. In a more explicit way,

$$\begin{cases}
L_1^C(h) = \sum_{j=2}^{\infty} \xi_{1,j} h_j, & L_2^C(h) = \xi_{2,1} h_1 + a_2 Q_1 h_3 \\
L_i^C(h) = \xi_{i,1} h_1 + b_i h_{i-1} \chi_{\{i \leq N+1\}} + a_i Q_1 h_{i+1} \chi_{\{i \leq N\}} & \text{for } i > 2;
\end{cases}$$

(2.32)

while

$$L_i^M(h) = -\sigma_i h_i + b_i h_{i-1} \chi_{\{i > N+1\}} + a_i Q_1 h_{i+1} \chi_{\{i > N\}} \quad \text{for } h \in H_2, \quad (i \geq 1).$$

(2.33)

With calculations similar to those in the proof of Lemma 2.4 one sees that the sums in (2.30)–(2.31) converge, that $L_i^C : H \to H$ is bounded and that there exists $C > 0$ such that $\|L_i^M h\|_H \leq C\|h\|_H$ for any $h \in H_2$. Moreover, both $L_i^C$ and $L_i^M$ are easily seen to be symmetric operators. One sees that $L_i^C : H \to H$ is a finite-rank operator, since except for its first component $L_1^C$ it only depends on a finite number of components of $h$. In particular, it is a compact operator. Since it is also symmetric, $L_i^C$ is thus self-adjoint.

Let us investigate now the remaining part $L_i^M$. We will prove that, for $N > 1$ large enough, $L_i^M$ is self-adjoint thanks to Kato-Rellich’s theorem [18, Theorem 4.3, p. 287]. To do so, define the multiplication operator $T_i$ with domain $H_2$ by

$$T_i(h) = -\sigma_i h_i \quad \text{for } h \in H_2, \quad i \geq 1.$$  

Being a multiplication operator, $T_i$ is self-adjoint in $H$. Define the operator $S_i$, also with domain $H_2$, as the remaining part in (2.33):

$$S_i(h) = b_i h_{i-1} \chi_{\{i > N+1\}} + a_i Q_1 h_{i+1} \chi_{\{i > N\}} \quad \text{for } h \in H_2, \quad i \geq 1,$$

so that $L_i^M = T_i + S_i$. Clearly, $S_i$ is symmetric. Let us now prove that $S_i$ is $T_i$-bounded with relative bound smaller than one. To do so, we compute

$$\|T(h)\|_H = \left( \sum_{i=1}^{\infty} \sigma_i^2 Q_i h_i^2 \right)^{1/2},$$

while

$$\|S(h)\|_H = \left( \sum_{i=1}^{\infty} |S_i(h)|^2 Q_i \right)^{1/2} \leq \left( \sum_{i>N+1} b_i^2 h_{i-1}^2 Q_i \right)^{1/2} + \left( \sum_{i>N} (a_i Q_1)^2 h_{i+1}^2 Q_i \right)^{1/2} \leq \left( \sum_{i>N+1} b_i^2 Q_i h_{i-1}^2 \right)^{1/2} + \left( \sum_{i>N} a_i^2 Q_1^2 Q_i h_{i+1}^2 \right)^{1/2} \leq \left( \sum_{i>N} h_i^2 \right)^{1/2} + \left( \sum_{i>N+1} Q_i h_i^2 \right)^{1/2} \leq \left( \sum_{i>N} \mu_i Q_i h_i^2 \right)^{1/2} + \left( \sum_{i>N+1} \nu_i Q_i h_i^2 \right)^{1/2}$$

(2.34)
where
\[ \nu_i := a^2_{i-1} Q_1^2 \frac{Q_i}{Q_i}, \quad \nu_i = a^2_i Q_1^2 \frac{Q_i}{Q_i+1} = b^2_i Q_i+1. \]

Observe that, since \( \sigma_i = a_i Q_1 + b_i = Q_1 (a_i + a_{i-1} \frac{Q_i}{Q_i}) \), and calling \( \ell := z_s/\bar{z_s} \),
\[
\frac{\nu_i}{\sigma_i^2} = \frac{a^2_{i-1}}{a_i + a_{i-1} \frac{Q_i}{Q_i}} \frac{Q_i}{Q_i} \quad \to \quad \frac{\ell}{(1+\ell)^2} \quad \text{as } i \to +\infty,
\]
due to (1.5), (1.20c) (which implies \( Q_i/\ell \to \ell \) as \( i \to +\infty \)) and (1.20d). Similarly,
\[
\frac{\nu_i-1}{\sigma_i^2} \to \frac{\ell}{(1+\ell)^2} \quad \text{as } i \to +\infty.
\]

Hence, from (2.34) we see that for any \( \epsilon > 0 \) we can find \( N > 0 \) such that
\[
\|S(h)\|_H \leq (1+\epsilon) \frac{2\sqrt{\ell}}{1+\ell} \left( \sum_{i>N} \sigma_i^2 Q_i h_i^2 \right)^{1/2} \leq (1+\epsilon) \frac{2\sqrt{\ell}}{1+\ell} \|T(h)\|_H =: \theta \|T(h)\|_H. \tag{2.35}
\]

Since we are assuming \( z < z_0 \), we have \( \ell > 1 \) and it is possible to choose \( \epsilon > 0 \) such that \( \theta = 2(1+\epsilon)\sqrt{\ell}/(1+\ell) \) < 1. In other words, \( S \) is \( T \)-bounded with relative bound \( \theta \) strictly smaller than 1. According to Kato-Rellich's Theorem \( L^M = S + T \) with domain \( \mathcal{D}(L^M) = \mathcal{D}(T) = H_2 \) is self-adjoint.

Let us now show that the spectrum of \( L^M \) is to the left of \( \lambda_{M} \). Since \( \sigma = \inf_{i \geq 1} \sigma_i > 0 \), one has \( \langle T(h), h \rangle \leq -\sigma \|h\|^2 \) for any \( h \in \mathcal{D}(T) \). With the terminology of [18], this exactly means that the multiplication operator \( T \) is bounded from above with upper bound \( \gamma_T = -\sigma \). Combining inequality (2.35) with [18, Theorem 4.11, p. 291], \( L^M \) is bounded from above with upper bound \( \gamma_M \) such that \( \gamma_M \geq \gamma_T - \theta |\gamma_T| = -(1-\theta)\sigma \), i.e.
\[
\langle L^M h, h \rangle \leq -(1-\theta)\sigma \|h\|^2 \quad \text{for all } h \in \mathcal{D}(T).
\]

This proves in particular point (2) of the Proposition with \( \lambda_M = 1 - \theta > 0 \). Notice that we can take \( \epsilon > 0 \) arbitrarily small, so \( \lambda_M \) can be arbitrarily close to \( 2\sqrt{\ell}/(1+\ell) \).

Now the identity \( L(h) = L^C(h) + L^M(h) \) is obviously true for \( h \in \ell_{00} \). Then, it is clear from the bounds above that the closure of \( L \) has domain \( H_2 \) and is equal to \( L^C + L^M \), which shows (2.29). Since \( L \) is a compact self-adjoint perturbation of \( L^M \), Weyl's Theorem ensures that \( L \) is self-adjoint. \( \square \)

**Corollary 2.11.** Assume the conditions of Proposition 2.10. There exists \( \lambda_0 > 0 \) (defined in a non-constructive way) such that
\[
\langle L h, h \rangle \leq -\lambda_0 \|h\|^2 \quad \text{for all } h \in H_2 \text{ such that } \sum_{i=1}^\infty i h_i Q_i = 0. \tag{2.36}
\]

In addition, the operators \( L \) and \( L \) are equal. Finally, \( L \) is the generator of a \( C_0 \)-semigroup \( (S_t)_{t \geq 0} \) in \( H \) such that
\[
\|S_t h\|_H \leq e^{-\lambda_0 t} \|h\|_H \quad \forall h \in H \text{ such that } \sum_{i=1}^\infty i h_i Q_i = 0, \quad t \geq 0.
\]
Proof: Since $L^C$ is a compact perturbation of $L^M$, Weyl’s Theorem ensures that the operator $L$ is self-adjoint and its essential spectrum $\mathcal{S}_{\text{ess}}(L)$ coincides with that of $L^M$, in particular $\mathcal{S}_{\text{ess}}(L) \subseteq (-\infty, -\lambda_M]$. Consequently, the part of $\mathcal{S}(L)$ contained in $(-\lambda_M, +\infty)$ is a set of isolated eigenvalues of finite multiplicity. As we also know that $L$ is non-positive, we have $\mathcal{S}(L) \subseteq (-\infty, 0]$, having only a finite number of eigenvalues in $(-\lambda_M, 0]$. From Lemma 2.7 we know that $0$ is an eigenvalue of $L$ with eigenspace spanned by $(i)_{i \geq 1}$. Since $\lambda_M > 0$, $L$ must have a strictly positive spectral gap $\lambda_0$, which gives (2.36).

Since $L$, the closure of $L_0$, is self-adjoint, the operator $L$ is by definition essentially self-adjoint. As such, it only has one possible self-adjoint extension. Since $L$ is another such extension, we have $L = L$. The final part of the result is a classical consequence of (2.36). \hfill \Box

Proposition 2.10 does not apply in the case $z = z_\nu$. In order to include that case we need to obtain more delicate estimates:

Lemma 2.12. Assume the hypotheses of Proposition 2.10, but take $z = z_\nu$. Define then

$$\delta_k = \frac{b_k}{a_k Q_1} - 1 = \frac{a_{k-1} Q_{k-1}}{a_k Q_k} - 1 \quad \text{for } k \geq 1,$$

and assume that

$$\liminf_{k \to \infty} \delta_k \sqrt{a_k} > 0. \quad (2.37)$$

Then the conclusions of Proposition 2.10 still hold true for $L$, for some $\lambda_M > 0$. As a consequence, the conclusions of Corollary 2.11 are true (i.e., $L$ has a positive spectral gap and $L = L$).

Proof: The proof follows the same steps as that of Proposition 2.10 and is based upon the splitting of $L$ as $L = L^C + L^M$ with $L^C$ and $L^M$ defined by (2.32) and (2.33) respectively. It will consist in proving that, under the supplementary assumption (2.37), one can choose $N > 1$ large enough such that

$$\langle L^M(h), h \rangle \leq -\lambda_M ||h||_H^2 \quad \text{for all } h \in H_2 \quad (2.38)$$

holds true for some positive $\lambda_M > 0$. Once we have this, the rest of the proofs of Proposition 2.10 and Corollary 2.11 are still valid. Indeed, arguing as in the proof of Proposition 2.10, this implies the existence of a positive spectral gap (defined in a non-constructive way) for $L$ thanks to Weyl’s theorem. One computes, for some $N > 1$ to be fixed later,

$$\langle L^M(h), h \rangle = -\sum_{i=1}^{\infty} \sigma_i Q_i h_i^2 + \sum_{i > N+1} b_i Q_i h_i h_{i-1} + \sum_{i > N} a_i Q_i Q_i h_i h_{i+1}$$

$$= -\sum_{i=1}^{\infty} \sigma_i h_i^2 + 2 \sum_{i > N} a_i Q_i Q_i h_i h_{i+1} \quad (2.39)$$

where we used (1.5). For each $i \geq 1$ we take $r_i > 1$ (also to be fixed later), and use Young’s inequality to deduce that

$$2 \sum_{i > N} a_i Q_i Q_i h_i h_{i+1} \leq \sum_{i > N+1} r_i a_i Q_i h_i^2 + \sum_{i > N+1} \frac{1}{r_i} a_i Q_i h_i^2$$

$$= \sum_{i > N+1} r_i a_i Q_i h_i^2 Q_i + \sum_{i > N+2} \frac{1}{r_i} b_i h_i^2 Q_j,$$
again by (1.5). Since \( b_i = (a_i Q_1)(1 + \delta_i) \) and \( \sigma_i = b_i + a_i Q_1 = (a_i Q_1)(2 + \delta_i) \) for any \( i > 1 \), one gets

\[
\langle L^M h, h \rangle \leq -\sum_{i=1}^{\infty} \sigma_i Q_i h_i^2 + \sum_{i>N+1} a_i Q_1 \left( r_i + \frac{1+\delta_i}{r_i} \right) h_i^2 Q_i \\
\leq -\sum_{i=1}^{N+1} \sigma_i Q_i h_i^2 + \sum_{i>N+1} a_i Q_1 \left( r_i + \frac{1+\delta_i}{r_i} - 2 - \delta_i \right) Q_i h_i^2.
\]

Take \( N \) large enough so that for \( \delta_i > 0 \) for all \( i > N + 1 \). We make the choice

\[
r_i = 1 + \delta_i/2 \quad \text{for } i > N + 1,
\]

to obtain

\[
\langle L^M h, h \rangle \leq -\sum_{i=1}^{N+1} \sigma_i Q_i h_i^2 - Q_1 \sum_{i>N+1} a_i \delta_i^2 \left( \frac{2}{4 + 2\delta_i} \right) Q_i h_i^2.
\]

According to (2.37),

\[
C := Q_1 \lim inf_i \frac{a_i \delta_i^2}{4 + 2\delta_i} > 0,
\]

so choosing \( N \) large enough,

\[
\langle L^M h, h \rangle \leq -\sum_{i=1}^{N+1} \sigma_i Q_i h_i^2 - \frac{C}{2} \sum_{i>N+1} Q_i h_i^2 \leq -\sigma \sum_{i=1}^{N+1} Q_i h_i^2 - \frac{C}{2} \sum_{i>N+1} Q_i h_i^2,
\]

which yields (2.38) with \( \lambda_M = \min\{\sigma, \frac{C}{2}\} > 0 \). \( \square \)

Remark 2.13. For the typical coefficients given by (1.13), with \( z = z_0 \), one sees that

\[
\delta_k = \frac{q}{z_0 \rho^{k+1}} \quad k \geq 1.
\]

Since \( a_k = k^\alpha \) with \( \alpha > 0 \), one sees that (2.37) holds true if and only if \( \alpha \geq 2(1-\mu) \). In other words, \( L \) has a positive spectral gap whenever \( \alpha \geq 2(1-\mu) \). We will prove this later on in a different way, by a constructive argument related to Hardy’s inequality which gives explicit estimates of the size of the gap. We will also show that, for \( \alpha < 2(1-\mu) \), \( L \) does not have a positive spectral gap (see Lemma 2.23).

2.3. Explicit spectral gap estimates in \( \ell^2(Q) \). Let us now study conditions ensuring that \( L \) admits a positive spectral gap in the space \( \mathcal{H} = \ell^2(Q) \). Since \( L \) is a self-adjoint operator in this space, it having a spectral gap of size \( \lambda_0 \) is equivalent to the functional dissipativity inequality

\[
\langle h, Lh \rangle \leq -\lambda_0 \|h\|_{\mathcal{H}}^2 \quad \text{for all } h \in \ell_{00} \text{ such that } \sum_{i=1}^{\infty} i Q_i h_i = 0 \quad (2.40)
\]

this is,

\[
\lambda_0 \sum_{i=1}^{\infty} Q_i h_i^2 \leq \sum_{i=1}^{\infty} a_i Q_i Q_1 (h_{i+1} - h_i - h_1)^2 \quad (2.41)
\]

for all \( h \in \ell_{00} \) such that \( \sum_{i=1}^{\infty} i Q_i h_i = 0 \). Notice that it is enough to have the inequality for compactly supported sequences \( h \), since \( \ell_{00} \) is a core for \( L \). On the other hand, if (2.41) holds for all \( h \in \ell_{00} \) then it must actually hold for all sequences \( h \), with the understanding that either or both sides of the inequality may be infinite.
We already saw in Proposition 2.10 (see also Lemma 2.12) sufficient conditions ensuring the existence of a positive spectral gap. However, this existence result was based on a compactness argument and, consequently, does not provide any quantitative information about the size spectral gap. In this section we adopt a different viewpoint by studying the inequality (2.40) directly. This will result in useful estimates on the spectral gap, explicit in terms of the coefficients \((a_k)_{k \geq 1}\) and \((b_k)_{k \geq 1}\).

**Definition 2.14.** Assume Hypothesis 2.1. We call \(\lambda_0\) the size of the spectral gap in \(\mathcal{H}\) of the operator \(L\); equivalently, \(\lambda_0\) is the largest nonnegative constant such that (2.40) (or equivalently, (2.41)) holds. Notice that \(\lambda_0 = 0\) whenever \(L\) has no spectral gap.

Our main estimate of \(\lambda_0\) is the following:

**Theorem 2.15 (Bound of the spectral gap for the linearized Becker-Döring operator).** Assume Hypothesis 2.1. Recall the definition of \(B\) given in (1.21):

\[
B = \sup_{k \geq 1} \left( \sum_{j=k+1}^{\infty} Q_j \right) \left( \sum_{j=1}^{k} \frac{1}{a_j Q_j} \right).
\]

It holds that

\[
\frac{1}{4B} \leq \lambda_0, \quad (2.42)
\]

which should be understood as saying that \(B = +\infty\) if \(\lambda_0 = 0\). If we additionally assume that

\[
M_3 := \sum_{i=1}^{\infty} \frac{1}{a_i Q_i} \left( \sum_{j=i+1}^{\infty} j Q_j \right)^2 < +\infty
\]

(2.43)

and call

\[
M_2 := \sum_{i=1}^{\infty} i^2 Q_i
\]

(2.44)

(which is finite due to (2.2)), then we also have an upper bound of \(\lambda_0\):

\[
B - \frac{M_3}{M_2} \leq \frac{1}{\lambda_0} \leq 4B,
\]

which should be understood as saying that \(B = +\infty\) if and only if \(\lambda_0 = 0\).

Using this it is easy to give quite general conditions on the coefficients \(a_i, b_i\) such that the spectral gap \(\lambda_0\) is strictly positive whenever \(z < z_s\). In particular, the following lemma applies to the “typical” coefficients (1.13) and (1.17). Notice that the following result has already been obtained in Prop. 2.10 by a completely different argument; however, the following Corollary is constructive, relying on the above Theorem:

**Corollary 2.16.** Assume Hypothesis 2.1 and conditions (1.20a), (1.20c) and (1.20d). Then for any \(0 < z < z_n\) the spectral gap \(\lambda_0\) of \(L\) in \(\mathcal{H}\) is strictly positive.

**Proof.** Due to (2.42) it is enough to show that \(B < +\infty\). Let us call \(Q_k := z^k Q_k\) as usual, and

\[
m_k := \sum_{j=k+1}^{\infty} Q_j, \quad n_k := \sum_{j=1}^{k} \frac{1}{a_j Q_j} \quad \text{for all } k \geq 1.
\]
We will show that
\[
m_k \xrightarrow{k \to +\infty} \left(\frac{z_k}{z} - 1\right)^{-1} Q_k
\]
(2.45)
\[
n_k \xrightarrow{k \to +\infty} \left(\frac{z_k}{z} - 1\right)^{-1} \frac{1}{a_k Q_k},
\]
(2.46)

which then implies that
\[
m_k n_k = \frac{1}{a_k Q_k} (n_k a_k Q_k) \leq \frac{1}{a_k Q_k} (n_k a_k Q_k) < C \quad \text{for all } k \geq 1
\]
for some $C > 0$, due to the lower bound on (1.20a) and the limits (2.45) and (2.46). This implies that $B = \sup_{k \geq 1} (m_k n_k) < C$, so we just have to prove (2.45) and (2.46).

We prove them by using the Stolz-Cesàro theorem. Notice that
\[
\frac{m_{k+1} - m_k}{Q_{k+1} - Q_k} = -\frac{Q_{k+1}}{Q_{k+1} - Q_k} = \left(\frac{Q_k}{Q_{k+1}} - 1\right)^{-1} \left(\frac{z_k}{z} - 1\right)^{-1},
\]
due to (1.20c). Since $m_k$ is strictly decreasing and tends to 0 as $k \to +\infty$, this implies (2.45).

In a similar way, due to (1.20c) and (1.20d),
\[
\frac{n_{k+1} - n_k}{(a_{k+1} Q_{k+1})^{-1} - (a_k Q_k)^{-1}} = \left(1 - \frac{a_{k+1} Q_{k+1}}{a_k Q_k}\right)^{-1} \to \left(1 - \frac{z}{z_0}\right)^{-1},
\]
which, since $n_k$ is strictly increasing and unbounded, implies (2.46).

Let us begin the proof of Theorem 2.15. We will prove it through an inequality which is stronger than (2.40), and equivalent in some cases, as we will show immediately afterwards:

**Lemma 2.17.** Assume Hypothesis 2.1. Define $\lambda_1$ as the largest nonnegative constant such that
\[
\lambda_1 \sum_{i=1}^{\infty} Q_i (h_i - h_{i+1})^2 \leq \sum_{i=1}^{\infty} a_i Q_i (h_{i+1} - h_i - h_1)^2 \quad \text{for all } h = (h_i)_{i \geq 1}.
\]
(2.47)

Then
\[
\frac{1}{4B} \leq \lambda_1 \leq \frac{1}{B},
\]
(2.48)

which again should be understood as saying that $\lambda_1 = 0$ if and only if $B = +\infty$.

**Proof.** We call
\[
\mu_i := Q_{i+1}, \quad \nu_i = a_i Q_i \quad \text{for } i \geq 1.
\]

Hardy’s inequality from Appendix A says that $B$ is finite if and only if the following inequality holds for some $\lambda_1 > 0$:
\[
\lambda_1 \sum_{i=1}^{\infty} \mu_i \left(\sum_{j=1}^{i} f_j\right)^2 \leq \sum_{i=1}^{\infty} \nu_i f_i^2 \quad \text{for all } f = (f_i)_{i \geq 1}.
\]
(2.49)

In addition, if any of these conditions hold, we have (2.48). Hence it is enough to show that (2.47) is equivalent to (2.49).

For one implication, assume that (2.49) holds. If we take any sequence $h = (h_i)_{i \geq 1}$ and call
\[
f_i := h_{i+1} - h_i - h_1 \quad \text{for } i \geq 1
\]
then (2.49) is just (2.47): notice that the term on the right hand side of (2.49) is
\[
\sum_{i=1}^{\infty} \nu_i f_i^2 = \sum_{i=1}^{\infty} a_i Q_i (h_{i+1} - h_i - h_1)^2
\]
(2.50)
and the term to the left of (2.49) is
\[
\lambda_1 \sum_{i=1}^{\infty} \mu_i \left( \sum_{j=1}^{i} f_j \right)^2 = \lambda_1 \sum_{i=1}^{\infty} \mu_i \left( \sum_{j=1}^{i} (h_{j+1} - h_j - h_1) \right)^2
\]
(2.51)
which is finite. (We recall that \( M_2 \) and \( M_3 \) were defined in (2.43)–(2.44).)

\[\frac{1}{\lambda_0} \leq \frac{1}{\lambda_1} \leq \frac{1}{\lambda_0} + \frac{M_3}{M_2}, \quad (2.52)\]

where for the right inequality we need to assume additionally that both \( M_3 \) is finite. (We recall that \( M_2 \) and \( M_3 \) were defined in (2.43)–(2.44)).

\textbf{Proof.} Denote the right-hand side of (2.47) by \( D(h) \) for convenience (it is equal to \( \mathcal{E}(h, h) = - \langle \mathcal{L} h, h \rangle \) for any \( h \in \mathcal{D}(\mathcal{L}) \), but we rename it to include any sequence \( h \)). Let us show the first inequality. For any sequence \( h = (h_i)_{i \geq 1} \in \ell_{00} \) with \( \sum_i i Q_i h_i = 0 \) we have
\[
\sum_{i=1}^{\infty} Q_i (h_i - i h_1)^2 = \sum_{i=1}^{\infty} Q_i h_i^2 + h_1^2 \sum_{i=1}^{\infty} i^2 Q_i - 2 h_1 \sum_{i=1}^{\infty} i Q_i h_i \geq \sum_{i=1}^{\infty} Q_i h_i^2 = \|h\|_{H}^2,
\]
since the mixed term obtained when expanding the square vanishes due to the orthogonality condition on \( h \). We have, using (2.47),
\[
\lambda_1 \|h\|_{H}^2 \leq \lambda_1 \sum_{i=1}^{\infty} Q_i (h_i - i h_1)^2 \leq D(h).
\]
This shows the first inequality.

For the second inequality we assume that \( \lambda_0 > 0 \) (since otherwise it is a trivial statement). Take any sequence \( h = (h_i)_{i \geq 1} \in \ell_{00}. \) Calling
\[
M_2 := \sum_{i=1}^{\infty} i^2 Q_i, \quad \overline{h} := \frac{1}{M_2} \sum_{i=1}^{\infty} i Q_i h_i
\]
we have
\[
\sum_{i=1}^{\infty} Q_i (h_i - i h_1)^2 = \sum_{i=1}^{\infty} Q_i (h_i - i \overline{h})^2 + \sum_{i=1}^{\infty} Q_i (i \overline{h} - i h_1)^2, \quad (2.53)
\]
since the mixed term obtained from the square is zero due to the definition of \( \mathcal{T} \). If we call \( g_i := h_i - i \mathcal{T} \) we see that \( \sum_{i=1}^{\infty} iQ_i g_i = 0 \) and we may use (2.40) to obtain

\[
\sum_{i=1}^{\infty} Q_i (h_i - i \mathcal{T})^2 \leq \lambda_\infty^{-1} D(g) = \lambda_\infty^{-1} D(h). \tag{2.54}
\]

This deals with the first term in (2.53). For the second term, use that

\[
M_2 |\mathcal{T} - h_1| = \left| \sum_{i=1}^{\infty} iQ_i (h_i - ih_1) \right| = \left| \sum_{i=1}^{\infty} iQ_i \sum_{j=1}^{i-1} (h_{j+1} - h_j - h_1) \right|
\leq \sum_{i=1}^{\infty} \sum_{j=1}^{i-1} iQ_i |h_{j+1} - h_j - h_1| = \sum_{j=1}^{\infty} |h_{j+1} - h_j - h_1| \sum_{i=j+1}^{\infty} iQ_i
\leq \left( \sum_{j=1}^{\infty} a_j Q_i (h_{j+1} - h_j - h_1)^2 \right)^{1/2} \left( \sum_{j=1}^{\infty} \frac{1}{a_j Q_j} \left( \sum_{i=j+1}^{\infty} iQ_i \right)^2 \right)^{1/2} = M_3^{1/2} D(h)^{1/2}
\]

to obtain

\[
\sum_{i=1}^{\infty} Q_i (h_i - i h_1)^2 = (\mathcal{T} - h_1)^2 M_2 \leq M_3^2 D(h). \tag{2.55}
\]

Using (2.54) and (2.55) in (2.53) we finally obtain

\[
\sum_{i=1}^{\infty} Q_i (h_i - i h_1)^2 \leq \left( \frac{1}{\lambda_0} + \frac{M_3}{M_2} \right) D(h)
\]

for every \( h \in \ell_{00} \). It is easy to see that this implies the inequality for all sequences \( h \), hence proving the second inequality in (2.52). \( \square \)

2.4. Estimates of the spectral gap size near the critical density. In many cases of interest one can estimate the quantity \( B \) as \( z \) approaches \( z_* \). This implies an estimate on the size of the spectral gap for the linearized Becker-Döring equations, as stated in Theorem 2.15. We remark that our main estimate for \( B \), Lemma 2.23 below, applies to the coefficients (1.13) as well as (1.17).

Let us set the notation. We call

\[
g_k := -\log Q_k - k \log z \quad \text{for } k \geq 1, \quad w := \log \frac{z}{z_*}
\]
so that

\[
Q_k = Q_k z^k = \exp (-g_k - kw) \quad \text{for } k \geq 1.
\]

**Lemma 2.19.** Assume that there exists \( k_0 > 0 \) such that:

1. The sequence \( (g_k)_k \) is increasing for \( k \geq k_0 \).
2. The sequence \( (g_k/k)_k \) is decreasing for \( k \geq k_0 \).

Assume also that:

3. There are constants \( C > 0 \) and \( 0 < \mu < 1 \) such that

\[
g_{k+1} - g_k \sim C k^{\mu-1}. \tag{2.56}
\]

4. There are constants \( C, \alpha > 0 \) such that

\[
a_k \sim C k^\alpha \quad \text{as } k \to +\infty. \tag{2.57}
\]
Then for some numbers \( C_1 < C_2 \) depending only on the coefficients \((a_k)_k\) and \((b_k)_k\), we have that for all \( z_s/2 < z < z_s \),
\[
\frac{C_1 Q_k}{w + k^{\mu - 1}} \leq \sum_{j=k}^{\infty} Q_j \leq \frac{C_2 Q_k}{w + k^{\mu - 1}} \quad \text{for } k \geq 1.
\] (2.58)

and
\[
\frac{C_1}{k^{\alpha} Q_k(w + k^{\alpha - 1})} \leq \sum_{j=1}^{k} \frac{1}{a_j Q_j} \leq \frac{C_2}{k^{\alpha} Q_k(w + k^{\alpha - 1})} \quad \text{for } k \geq 1.
\] (2.59)

Remark 2.20. We give the above estimates only for \( z > z_s/2 \) for simplicity, and since we are interested in the behavior for \( z \) close to \( z_s \). They can easily be derived for \( z > \delta \), for any \( \delta > 0 \) (with constants \( C_1, C_2 \) depending on \( \delta \)), but are very poor estimates for small \( z \).

Remark 2.21. Both example coefficients (1.13) and (1.17) satisfy all hypotheses of Lemma 2.19. Notice that for (1.17) the sequence \( g_k \) is
\[
g_k = -\log z_s + \sigma(k - 1)^\mu \quad \text{for } k \geq 1
\]
while for (1.13) we have
\[
g_k = \alpha \log k - \log z_s + \sum_{j=2}^{k} \log \left(1 + \frac{q/z_s}{j^{1-\mu}}\right) \quad \text{for } k \geq 1.
\]
From this it is easy to check that all the requirements of Lemma 2.19 are met.

Remark 2.22. Notice that, for any sequence \((g_k)_k\) satisfying the assumptions of Lemma 2.19, the following hold:

(i) There is a constant \( C > 0 \) such that
\[
g_k/k \leq C k^{\mu - 1} \quad \text{for } k \geq 1.
\] (2.60)
This is a simple consequence of (2.56). In particular, \( g_k/k \) tends to 0 as \( k \to +\infty \).

(ii) There exists \( C > 0 \) such that
\[
g_k \sim (C/\mu) k^\mu \quad \text{as } k \to +\infty.
\] (2.61)
This is easily deduced from (2.56) by using the Stolz-Cesàro Theorem. In particular, \( g_k \) diverges to \(+\infty\).

(iii) Points (1), (2) together with (2.60) and (2.61), show that there exists \( k_1 > 0 \) such that, for all \( k \geq k_1 \),
\[
g_k \geq g_j \quad \text{for all } j \leq k,
\] (2.62)
\[
g_k/k \leq g_j/j \quad \text{for all } j \leq k.
\] (2.63)

All this properties will be used repeatedly in the proof of Lemma 2.19.

Proof of Lemma 2.19. We will use in this proof a simple bound for the following geometric sum: for any \( \xi > 0 \)
\[
\frac{\exp(-k\xi)}{\xi} \leq \sum_{j=k}^{\infty} \exp(-k\xi) \leq \frac{\exp(-k\xi)}{\xi} \quad \text{for } k \geq 0.
\] (2.64)
Also, we will denote by $C$ any constant that only depends on the coefficients $(a_k)_k$ and $(b_k)_k$, and which may change from one line to the next. We will show \((2.58)\) and \((2.59)\) for $k \geq k_1$. Note that this is enough, since for $k$ bounded there are always constants $C_1, C_2$ which satisfy \((2.58)\)–\((2.59)\).

**Step 1: Proof of the upper bound in \((2.58)\).** We use two different estimates to prove this upper bound. The first one is

$$
\sum_{j=k}^{\infty} Q_j \leq \sum_{j=k}^{\infty} \exp(-g_j - jw) \leq \exp(-g_k) \sum_{j=k}^{\infty} \exp(-kw) \leq Q_k \frac{\exp(w)}{w} \leq C \frac{Q_k}{w},
$$

(2.65)

using that $g_k$ is increasing, and also Eq. \((2.64)\). The second one is

$$
\sum_{j=k}^{\infty} Q_j = \sum_{j=k}^{\infty} \exp(-g_j - jw) \leq \exp(-kw) \sum_{j=k}^{\infty} \exp(-g_j) \leq C Q_k k^{1-\mu},
$$

(2.66)

due to Lemma B. 3 (see also the Remark B. 1 immediately afterwards). We deduce from \((2.65)\) and \((2.66)\) that \(\sum_{j=k}^{\infty} Q_j\) is bounded by the harmonic mean of both right hand sides, this is,

$$
\sum_{j=k}^{\infty} Q_j \leq \frac{C Q_k}{w + k^{\mu-1}}.
$$

(2.67)

**Step 2: Proof of the lower bound in \((2.58)\).**

Using that \((g_k/k)_k\) is decreasing we have

$$
\sum_{j=k}^{\infty} \exp(-g_j - jw) = \sum_{j=k}^{\infty} \exp(-j(g_j/j + w)) \geq \sum_{j=k}^{\infty} \exp(-j(g_k/k + w)) \geq \frac{Q_k}{w + g_k/k} \geq \frac{C Q_k}{w + k^{\mu-1}}.
$$

(2.68)

Here we have also used \((2.60)\).

**Step 3: Proof of the upper bound in \((2.59)\).**

We use two different bounds, in a similar way as in Step 1. The first one is

$$
\sum_{j=1}^{k} Q_j^{-1} a_j^{-1} = \sum_{j=1}^{k} \exp(g_j + jw) j^{-\alpha} \leq \exp(g_k) \sum_{j=1}^{k} \exp(jw) j^{-\alpha}
$$

where we have used \((2.62)\). Now, since \(\exp(jw) \leq \exp((y + 1)w)\) and \(y^{-\alpha} \geq j^{-\alpha}\) for any \(y \in (j-1, j)\), one has

$$
\sum_{j=1}^{k} \exp(jw) j^{-\alpha} \leq \sum_{j=1}^{k} \int_{j-1}^{j} \exp((y + 1)w) y^{-\alpha} dy = \int_{0}^{k} \exp((y + 1)w) y^{-\alpha} dy.
$$

Using then Lemma B. 2 in Appendix B (with \(\mu = 1\)), we get \(\sum_{j=1}^{k} \exp(jw) j^{-\alpha} \leq C \exp(kw) k^{-\alpha}\) from which we deduce that

$$
\sum_{j=1}^{k} (Q_j a_j)^{-1} \leq \frac{C}{w Q_k k^\alpha} \quad k \geq 1.
$$
The second bound is

$$\sum_{j=1}^{k} \exp(g_j + jw) j^{-\alpha} \leq \exp(kw) \sum_{j=1}^{k} \exp(g_j) j^{-\alpha} \leq C \, Q_k^{-1} k^{1-\mu-\alpha},$$

where we used Lemma B. 3 again. Taking the harmonic mean of these two bounds gives the desired one as before.

**Step 4: Proof of the lower bound in (2.59).**

Similarly to (2.68) we have

$$\sum_{j=1}^{k} \frac{1}{Q_j^{-1} a_j^{-1}} = \sum_{j=1}^{k} \exp(g_j + jw) j^{-\alpha} \geq k^{-\alpha} \sum_{j=1}^{k} \exp(j(g_k/k + w))$$

$$\geq k^{-\alpha} \frac{1}{w + g_k/k} \geq k^{-\alpha} \frac{C}{w + k^{\mu-1}},$$

using again (2.60) and (2.63). □

As a consequence of the previous lemma we have an estimate of the size of the quantity $B$ in (1.21), useful mainly because it gives its blow-up rate as $z$ approaches $z_s$, making more precise the above Lemma 2.12:

**Lemma 2.23.** Assume the conditions in Lemma 2.19. If $\alpha > 2(1-\mu)$ there are numbers $C_1, C_2 > 0$ such that

$$C_1 \left( \frac{z_s}{z} \right)^{-2+\frac{\alpha}{1-\mu}} \leq B \leq C_2 \left( \frac{z_s}{z} \right)^{-2+\frac{\alpha}{1-\mu}}$$

for $z < z_s$. In the case $\alpha \geq 2(1-\mu)$ the quantity $B$ is bounded uniformly up to $z = z_s$: there exist numbers $C_1, C_2 > 0$ such that

$$C_1 \leq B \leq C_2$$

for $z \leq z_s$.

**Proof:** Recall the definition of $B$ given in (1.21):

$$B = \sup_{k \geq 1} \left( \sum_{j=k+1}^{\infty} Q_j \right) \left( \sum_{j=1}^{k} \frac{1}{a_j Q_j} \right).$$

Using the bounds in Lemma 2.19 and the fact that $Q_{k+1}/Q_k$ is bounded from below and above uniformly with respect to $w$, we have that, for some numbers $C_1, C_2$,

$$C_1 \sup_{k \geq 1} \frac{1}{k^\alpha (w + k^{\mu-1})} \leq B \leq C_2 \sup_{k \geq 1} \frac{1}{k^\alpha (w + k^{\mu-1})^2}.$$  \hfill (2.69)

In the case $\alpha < 2(1-\mu)$, the minimum of the function $x \mapsto x^\alpha (w + x^{\mu-1})^2$ occurs at $x = C_{\alpha,\mu} w^{-1/(1-\mu)}$, for $C_{\alpha,\mu}$ some quantity depending only on $\alpha$ and $\mu$. Hence we obtain

$$C_1 w^{-2+\frac{\alpha}{1-\mu}} \leq B \leq C_2 w^{-2+\frac{\alpha}{1-\mu}},$$

which gives the result when $\alpha < 2(1-\mu)$ (recall that $w = \log(z_s/z)$).

On the other hand, if $\alpha \geq 2(1-\mu)$, then the supremum in (2.69) is bounded for any value of $w \geq 0$, which shows the second part of the result. □
3. Spectral gap in weighted $\ell^1$ spaces

We now extend the above spectral gap result to the larger functional space $X = \ell^1(e^{\eta_i}Q_i)$ given by
\[
X := \{ h = (h_i), \quad \|h\|_X := \sum_{i=1}^{\infty} \exp(\eta_i) Q_i |h_i| < +\infty \},
\]
for some $0 < \eta < \frac{1}{2} \log \frac{\lambda_1}{2}$. The choice of $\eta$ ensures that this space is actually larger than the Hilbert space $H = \ell^2(\mathcal{Q})$ used in the previous section, since
\[
\|h\|_X \leq \left( \sum_{i=1}^{\infty} h_i^2 Q_i \right)^{1/2} \left( \sum_{i=1}^{\infty} Q_i \exp(2\eta_i) \right)^{1/2} = \sqrt{2} \left( \sum_{i=1}^{\infty} Q_i \exp(2\eta_i) \right)^{1/2} \|h\|_H,
\]
and the latter parenthesis is finite for $\eta < \frac{1}{2} \log \frac{\lambda_1}{2}$. In order to see this notice that, since $Q_i = z^i Q_z$, $\sum_{i=1}^{\infty} Q_i \exp(2\eta_i) = \sum_{i=1}^{\infty} Q_i r^i$ where $r := \exp(2\eta + \log z)$.

Since the radius of convergence of the power series with coefficients $i Q_i$ is, by definition, $z_s$, the above sum is finite whenever $r < z_s$, this is, when $\eta < \frac{1}{2} \log \frac{\lambda_1}{2}$.

Our approach to show that $L$ can be extended to an unbounded operator on $X$ that has a positive spectral gap uses recent techniques [16] based upon a suitable decomposition of the linearized operator into a dissipative part and a “regularizing” part. We use the following result, which is a slight improvement over some of the consequences of [16]:

**Theorem 3.1.** Let $\mathcal{Y}, \mathcal{Z}$ be two Banach spaces with norms $\| \cdot \|_\mathcal{Y}, \| \cdot \|_\mathcal{Z}$ such that $\mathcal{Y} \subseteq \mathcal{Z}$ and
\[
\| h \|_\mathcal{Z} \leq C_\mathcal{Y} \| h \|_\mathcal{Y}, \quad h \in \mathcal{Y}
\]
for some $C_\mathcal{Y} > 0$. Let $L_\mathcal{Y} : \mathcal{D}(L_\mathcal{Y}) \to \mathcal{Y}$ and $L_\mathcal{Z} : \mathcal{D}(L_\mathcal{Z}) \to \mathcal{Z}$ be unbounded operators in $\mathcal{Y}$ and $\mathcal{Z}$, respectively, and such that $L_\mathcal{Z}$ is an extension of $L_\mathcal{Y}$ (i.e., $\mathcal{D}(L_\mathcal{Y}) \subseteq \mathcal{D}(L_\mathcal{Z})$ and $L_\mathcal{Z}|_{\mathcal{D}(L_\mathcal{Y})} = L_\mathcal{Y}$). Assume that:

1. $L_\mathcal{Y}$ generates a $C_0$-semigroup $(S_t)_{t \geq 0}$ on $\mathcal{Y}$.
2. It holds that
\[
\| S_t h_0 \|_\mathcal{Y} \leq C_1 \exp(-\lambda_1 t) \| h_0 \|_\mathcal{Y}, \quad h_0 \in \mathcal{Y}, \quad t \geq 0
\]
for some $C_1 > 0$, $\lambda_1 \in \mathbb{R}$.
3. $L_\mathcal{Z} = A + B$, where $A, B$ are two linear operators on $\mathcal{Z}$ such that:
   a. $A : \mathcal{Z} \to \mathcal{Y}$ is bounded, i.e.,
\[
\| Ah \|_\mathcal{Y} \leq C_A \| h \|_\mathcal{Z}, \quad h \in \mathcal{Z},
\]
for some $C_A > 0$.
   b. $B$ generates a $C_0$-semigroup $(S_t^B)_{t \geq 0}$ on $\mathcal{Z}$ which satisfies
\[
\| S_t^B h_0 \|_\mathcal{Z} \leq C_2 \exp(-\lambda_2 t) \| h_0 \|_\mathcal{Z}, \quad h_0 \in \mathcal{Z}, \quad t \geq 0
\]
for some $C_2 > 0$, and some $\lambda_2 > \lambda_1$.

Then $L_\mathcal{Z}$ generates a $C_0$-semigroup $(V_t)_{t \geq 0}$ on $\mathcal{Z}$ (an extension of $(S_t)_{t \geq 0}$) which satisfies
\[
\| V_t h_0 \|_\mathcal{Z} \leq C \exp(-\lambda_1 t) \| h_0 \|_\mathcal{Z}, \quad h_0 \in \mathcal{Z}, \quad t \geq 0
\]
for $C = C_2 + C_\mathcal{Y} C_1 C_2 C_A (\lambda_2 - \lambda_1)^{-1}$.
This theorem is directly inspired from the results in [16]. However, while the proofs in [16] are based on estimates of the resolvents of \( L^Y \) and \( L_Z \), we present a completely different one based on the study of the dynamics generated by \( L^Y \) and \( L_Z \). In addition to being remarkably simple, this proof allows us to state the result in general for any two Banach spaces (not necessarily one of them Hilbert) and to reach an exponential decay like \( \lambda_1 \) in (3.6). The resolvent methods in [16] are able to give more precise estimates on the behavior of the semigroup on \( Z \), but here (3.6) will suffice.

Proof. We notice first that \( A : Z \rightarrow Z \) is a bounded operator. Since \( B \) is the generator of a \( C_0 \)-semigroup \( (S^B_t)_{t \geq 0} \) in \( Z \), by the bounded perturbation theorem, the operator \( L_Z = A + B \) generates a \( C_0 \)-semigroup \( (V_t)_{t \geq 0} \) in \( Z \) which, additionally, is given by the Duhamel’s formula [4, Theorem 4.9]:

\[
V_t h_0 = S^B_t h_0 + \int_0^t V_{t-s} (A S^B_s h_0) \, ds \quad \forall h_0 \in Z, \ t \geq 0.
\]

Thus, for fixed \( h_0 \in Z \) and \( t \geq 0 \), we have

\[
\|V_t h_0\|_Z \leq \|S^B_t h_0\|_Z + \int_0^t \|V_{t-s} (A S^B_s h_0)\|_Z \, ds
\leq C_2 \exp(-\lambda_2 t) \|h_0\|_Z + \int_0^t \|V_{t-s} (A S^B_s h_0)\|_Z \, ds, \tag{3.7}
\]

where we used (3.5) in the last inequality. Now, for any \( s \in (0, t) \), \( (A S^B_s h_0) \in Y \) since \( A \) maps \( Z \) to \( Y \). Then, since \( L_Z \) is an extension of \( L^Y \), it is clear that \((V_t)_{t \geq 0}\) is an extension of \((S_t)\), and

\[
V_{t-s} (A S^B_s h_0) = S_{t-s} (A S^B_s h_0) \in Y \quad \forall s \in (0, t).
\]

In particular, thanks to (3.3),

\[
\int_0^t \|V_{t-s} (A S^B_s h_0)\|_Z \, ds \leq C_Y \int_0^t \|S_{t-s} (A S^B_s h_0)\|_Y \, ds
\leq C_Y C_1 \int_0^t \exp(-\lambda_1 (t-s)) \|A S^B_s h_0\|_Y \, ds
\leq C_Y C_1 C_A \int_0^t \exp(-\lambda_1 (t-s)) \|S^B_s h_0\|_Z \, ds.
\]

Using again (3.5) we deduce from (3.7)

\[
\|V_t h_0\|_Z \leq C_2 \exp(-\lambda_2 t) \|h_0\|_Z + C_3 C_1 C_2 C_A e^{-\lambda_1 t} \|h_0\|_Z \int_0^t \exp(-(\lambda_2 - \lambda_1) s) \, ds
\leq C_2 \exp(-\lambda_2 t) \|h_0\|_Z + \frac{C_Y C_1 C_2 C_A}{\lambda_2 - \lambda_1} \exp(-\lambda_1 t) \|h_0\|_Z
\]

which, since \( \lambda_2 > \lambda_1 \) reads

\[
\|V_t h_0\|_Z \leq \left( \frac{C_Y C_1 C_2 C_A}{\lambda_2 - \lambda_1} + C_2 \right) \exp(-\lambda_1 t) \|h_0\|_Z, \quad t \geq 0
\]

and yields the desired result. \( \square \)
Our motivation for studying the spectral properties of the linearized operator in the larger space $X$ is that in this larger space we have useful bounds of the nonlinear remainder term $\Gamma(h, h)$ defined in (2.13):

**Proposition 3.2.** Assume Hypothesis 2.1 and take $0 < \eta < \frac{1}{2} \log \frac{z}{2}$. There is a constant $C > 0$ depending only on $(a_i)_{i \geq 1}$, $(b_i)_{i \geq 1}$ and $z$ such that

$$\|\Gamma(h, h)\|_X \leq C \|h\|_X \|h\|_{X_1}, \quad \forall h \in X_1$$

where $X_1 = \ell_1((1 + \sigma_i) \exp(\eta i) \mathcal{Q}_i)$, i.e.

$$X_1 := \left\{ h = (h_i) \mid \|h\|_X := \sum_i (1 + \sigma_i) \exp(\eta i) \mathcal{Q}_i |h_i| < +\infty \right\}$$

and we recall that $(\sigma_i)_{i \geq 1}$ was defined in (2.16).

**Proof:** Recall that the bilinear operator $\Gamma(f, g)$ is defined in weak form by

$$\sum_{i=1}^{\infty} \Gamma_i(f, g) \mathcal{Q}_i \phi_i = \frac{1}{2} \sum_{i \geq 1} a_i \mathcal{Q}_i \mathcal{Q}_1 (f_i g_1 + f_1 g_i) \left( \phi_{i+1} - \phi_i - \phi_{i-1} \right)$$

while its strong form is given in (2.12)-(2.13). Let us first treat the term $\Gamma_1(h, h)$ (the first component of $\Gamma(h, h)$). From (2.12) we have

$$\mathcal{Q}_1 |\Gamma_1(h, h)| = |a_1 \mathcal{Q}_1 h_1^2 + \sum_{i=1}^{\infty} a_i \mathcal{Q}_i h_1 h_i| \leq a_1 \mathcal{Q}_1^2 h_1^2 + |h_1| \sum_{i=1}^{\infty} a_i \mathcal{Q}_i |h_i| \leq \sigma_1 \mathcal{Q}_1 h_1^2 + |h_1| \sum_{i=1}^{\infty} \sigma_i \mathcal{Q}_i |h_i| \leq \frac{2}{Q_1} \|h\|_X \|h\|_{X_1}$$

which bounds $\Gamma_1(h, h)$. For the terms with $i \geq 2$, $\Gamma_i(h, h)$ is given by (2.13):

$$\Gamma_i(h, h) = \Gamma_i^+(h, h) - \Gamma_i^-(h, h),$$

with

$$\Gamma_i^+(h, h) = \frac{a_{i-1} \mathcal{Q}_{i-1} \mathcal{Q}_1}{Q_i} h_{i-1} h_1 = b_i h_{i-1} h_1 \quad \text{and} \quad \Gamma_i^-(h, h) = a_i \mathcal{Q}_i h_i \quad (i \geq 2).$$

We bound these terms separately. For $\Gamma^+(h, h) = (\Gamma_i^+(h, h))_{i \geq 2}$ we have, using (1.5),

$$\|\Gamma^+(h, h)\|_X = \sum_{i=2}^{\infty} \exp(\eta i) b_i \mathcal{Q}_i |h_1| |h_{i-1}| = |h_1| \sum_{i=1}^{\infty} \exp(\eta (i + 1)) b_{i+1} \mathcal{Q}_{i+1} |h_i| = \exp(\eta) |h_1| \sum_{i=1}^{\infty} a_i \mathcal{Q}_i \exp(\eta i) \mathcal{Q}_i |h_i| \leq \frac{\exp(\eta)}{Q_1} \|h\|_X \|h\|_{X_1}.$$\(\Box\)

The proof is even simpler for $\Gamma^-(h, h) = (\Gamma_i^-(h, h))_{i \geq 2}$ since

$$\|\Gamma^-(h, h)\|_X = |h_1| \sum_{i=2}^{\infty} a_i \mathcal{Q}_1 \exp(\eta i) \mathcal{Q}_i |h_i| \leq |h_1| \sum_{i=2}^{\infty} \sigma_i \exp(\eta i) \mathcal{Q}_i |h_i| = |h_1| \|h\|_{X_1} \leq \frac{1}{Q_1} \|h\|_X \|h\|_{X_1},$$

which finishes the result.
The following result shows that we may extend the linearized operator $L$ from Definition 2.2 to an operator on $X$ with domain $X_1$. It can be proved by direct estimates on the expression of $L$ given in (2.11) which are similar to those in the proof of the previous lemma, and we omit its proof:

**Lemma 3.3.** Assume Hypothesis 2.1 and (1.20c). There is a constant $C > 0$ depending only on $(a_i)_{i \geq 1}$, $(b_i)_{i \geq 1}$ and $z$ such that

$$
\|L(h)\|_X \leq C \|h\|_{X_1} \quad \text{for all } h \in \ell_{00}.
$$

Similarly, there is another constant $C > 0$ depending only on $(a_i)_{i \geq 1}$, $(b_i)_{i \geq 1}$, $z$ and $\eta$ such that

$$
\sum_{i=1}^{\infty} Q_i \frac{\exp(\eta i)}{\sigma_i} |L_i(h)| \leq C \|h\|_X \quad \text{for all } h \in \ell_{00}.
$$

(3.9)

The above lemma allows us to extend $L$ to $X_1$:

**Definition 3.4.** We denote by $\Lambda$ the extension of the linearized operator $L$ to the domain $X_1$.

Recall that the operator $L$ is given in strong form by

$$
L_1(h) = -\frac{1}{Q_1} \left( W^L_1 + \sum_{k=1}^{\infty} W^L_k \right), \quad L_i(h) = \frac{1}{Q_i} \left( W^L_{i-1} - W^L_i \right) \quad (i \geq 2),
$$

where $W^L_i := a_i Q_i Q_1 (h_i + h_1 - h_{i+1})$. Then, one has

**Theorem 3.5 (Extension of the spectral gap).** Assume Hypothesis 2.1 and conditions (1.20a), (1.20c) and (1.20d), and assume also that $z < z_0$. Take $0 < \eta < \frac{1}{2} \log \frac{z_0}{z}$ Then, the operator $\Lambda$ generates a strongly continuous semigroup $(\exp(t \Lambda))_{t \geq 0}$ on $X$ and there exists $0 < \lambda_* \leq \lambda_0$ such that, for some $C > 0$,

$$
\|\exp(t \Lambda)g\|_X \leq C \exp(-\lambda_* t) \|g\|_X \quad \text{for all } t \geq 0 \text{ and any } g \in X \text{ such that } \sum_{i=1}^{\infty} i Q_i g_i = 0.
$$

(We recall that $\lambda_0$ is the size of the spectral gap of $L$ in $\mathcal{H}$, as given by Definition 2.14). In addition, if we assume

$$
\lim_{i \to +\infty} a_i = +\infty \quad \text{(3.10)}
$$

then we may take $\lambda_* = \lambda_0$.

**Proof.** We apply Theorem 3.1 to the Banach spaces

$$
\mathcal{H}^\perp := \{ h \in \mathcal{H} \mid \sum_{i=1}^{\infty} i Q_i h_i = 0 \},
$$
a subspace of $\mathcal{H} = \{ h = (h_i)_i \mid \|h\|_\mathcal{H} < \infty \}$ (where we recall that $\|h\|^2_\mathcal{H} := \frac{1}{2} \sum_{i=1}^{\infty} Q_i h_i^2$) and

$$
X^\perp := \{ h \in X \mid \sum_{i=1}^{\infty} i Q_i h_i = 0 \},
$$
a subspace of $X = \{ h = (h_i)_i \mid \|h\|_X := \sum_i \exp(\eta i) Q_i |h_i| < +\infty \}$. Note that under our conditions the map

$$
M : h \mapsto M(h) := \sum_{i=1}^{\infty} i Q_i h_i
$$
is continuous both in $\mathcal{H}$ (as a consequence of (2.2)) and in $X$; hence, $\mathcal{H}^\perp$ and $X^\perp$ are well-defined closed subspaces of $\mathcal{H}$ and $X$, respectively. We define

$$
\mathcal{H}_2^\perp := \mathcal{H}^\perp \cap \mathcal{H}_2, \quad X_1^\perp := X^\perp \cap X_1
$$

where $\mathcal{H}_2$ is defined in (2.22) and $X_1$ is given by (3.8). Let us briefly explain how one can project any sequence $g \in X$ to $X^\perp$. To do so, for any sequence $g = (g_i)_i$, let us introduce the sequence

$$
\mathcal{M}_i(g) = \frac{1}{g} M(g) \quad \forall i \geq 1 \quad \text{where} \quad g := \sum_{k=1}^{\infty} k Q_k.
$$

(3.11)

Since $\sum_{i \geq 1} \exp(\eta i) Q_i < \infty$, one checks that $\mathcal{M} : X \to X$ is a bounded operator and for any $g \in X$ one easily checks that

$$
g - \mathcal{M} g \in X^\perp.
$$

In the same way, the operator $\mathcal{M} : \mathcal{H} \to \mathcal{H}$ is bounded and $h - \mathcal{M} h \in \mathcal{H}^\perp$ for any $h \in \mathcal{H}$.

To play the role of the linear operators $L_2$ and $L_\varphi$ in Theorem 3.1 we take $L^\perp := L|_{\mathcal{H}^\perp}$ with domain $\mathcal{D}(L^\perp) = \mathcal{H}_2^\perp$ and $\Lambda^\perp := \Lambda|_{X^\perp}$ with domain $\mathcal{D}(\Lambda^\perp) = X_1^\perp$. From (2.9) it is clear that the image of $L$ is contained in $\mathcal{H}^\perp$, and that of $\Lambda$ is contained in $X^\perp$, so $L^\perp$ and $\Lambda^\perp$ are unbounded operators on $\mathcal{H}^\perp$ and $X^\perp$, respectively.

Let us check the hypotheses of Theorem 3.1. As remarked in (3.2), $\mathcal{H}^\perp$ is contained in $X^\perp$. Moreover, $\mathcal{H}^\perp$ is dense in $X^\perp$, since the set of compactly supported sequences with $\sum_{i=1}^{\infty} i Q_i h_i = 0$ is contained in both of them. Points (1) and (2) in Theorem 3.1 are established in Corollary 2.11 where it is shown that one may take $C_1 = 1$ and $\lambda_1 = \lambda_0$ in Theorem 3.1, where $\lambda_0$ is the spectral gap estimated in Section 2.3.

It remains to show point (3) of Theorem 3.1.

**Step 1: the splitting $\Lambda^\perp = A + B$.** Take an integer $N \geq 1$, to be fixed later. We first split $\Lambda$ as $A + B$ in a similar way to Proposition 2.10, and we then write $\Lambda^\perp = A^\perp + B^\perp$ by projecting to $X^\perp$. We define $L^C$ and $L^M$ similarly to (2.30)–(2.31), but this time in $X$: we choose $N > 2$ and set, for all $i \geq 1$,

$$
\Lambda^C_i(h) := \sum_{j=1}^{\infty} \chi_{\{\min\{i,j\} \leq N\}} \xi_{i,j} h_j - \sigma_i h_i \chi_{\{i \leq N\}} \quad \text{for } h \in X,
$$

(3.12)

$$
\Lambda^M_i(h) := -\sigma_i h_i \chi_{\{i > N\}} + \sum_{j=1}^{\infty} \chi_{\{\min\{i,j\} > N\}} \xi_{i,j} h_j \quad \text{for } h \in X_1.
$$

(3.13)

We recall that the notation $\sigma_i$ and $\xi_{i,j}$ was defined in (2.16)–(2.19). Remember in particular that

$$
\sigma_i = a_i Q_1 + b_i \quad \text{for } i \geq 2.
$$

(3.14)

We will see below that the sums converge absolutely for $h$ in the corresponding domain, so the definition is correct. We also define the truncated identity operator $I^N_i = (I^N_i)_i > 1$ as

$$
I^N_i(h) := h_i \chi_{\{i \leq N\}}
$$

and then set, for $R > 0$ to be fixed later,

$$
A = \Lambda^C + R I^N_i, \quad h \in X
$$

$$
B = \Lambda^M - R I^N_i, \quad h \in X_1.
$$
so that $\Lambda = A + B$. We finally project this splitting to $X'\perp$ thanks to the operator $M$ introduced in (3.11). Namely, let
\[ A^{\perp}h = Ah - M(Ah), \quad h \in X^{\perp} \]
\[ B^{\perp}h = Bh - M(Bh), \quad h \in X'_1, \]

**Step 2:** $A$ is “regularizing”. We show now that $Ah \in H$ for all $h \in X$ (which, since $M: H \rightarrow H$ is bounded will imply then that $A^{\perp}(h) \in H^{\perp}$ for all $h \in X^{\perp}$). Notice that one may write, more explicitly,
\[
\begin{cases}
\Lambda_1^C(h) = \sum_{j=2}^{\infty} \xi_{1,j} h_j - \sigma_1 h_1 \\
\Lambda_2^C(h) = \xi_{2,1} h_1 + a_2 Q_1 h_3 - \sigma_2 h_2 \\
\Lambda_3^C(h) = \xi_{i,1} h_1 + b_i h_{i-1} X_{\{i \leq N+1}\} + a_i Q_1 h_{i+1} X_{\{i \leq N\}} - \sigma_i h_i X_{\{i \leq N\}} & \text{for } i > 2.
\end{cases}
\]

The only infinite sum in the definition of $A$ (the one in $\Lambda_1^C$) converges since, from (2.17)–(2.18),
\[
\left| \sum_{j=2}^{\infty} \xi_{1,j} h_j \right| \leq \frac{2}{Q_1} \sum_{j=2}^{\infty} b_j Q_j |h_j| + \sum_{j=2}^{\infty} a_j Q_j |h_j| \leq \frac{2}{Q_1} |h|_X.
\]

In order to see that $Ah \in H$ it is enough to show that $\sum_{i=N+2}^{\infty} Q_i A_i(h)^2 \leq C |h|_X^2$ for some $C > 0$ (we omit the first $N + 1$ terms, for which a similar bound is obviously true). But this is true since for $i \geq N + 2$ the only term in $A_i(h)$ is the first one in the expression of $\Lambda_1^C(h)$ in (3.15) and
\[
\sum_{i=N+2}^{\infty} Q_i A_i(h)^2 = h_1^2 \sum_{i=N+2}^{\infty} Q_i \xi_{i,1}^2 = h_2^2 \sum_{i=N+2}^{\infty} Q_i (b_i - a_i Q_1)^2 \leq A h_1^2 \leq \frac{A}{Q_1^2} |h|_X^2
\]

where $A$ is defined in (2.2). Thus, $A : X \rightarrow H$ is bounded, and hence $A^{\perp} : X'\perp \rightarrow H^{\perp}$ also is.

**Step 3:** $B$ is strictly dissipative. We prove that
\[
\langle \text{sign}(h), Bh \rangle_{X'\perp, X} \leq -\lambda_3 |h|_X \quad \text{for all } h \in X_1, \tag{3.16}
\]

where $\langle \cdot, \cdot \rangle_{X'\perp, X}$ denotes the duality pairing between $X$ and its dual $X'$, and $\lambda_3$ will be any number larger than $\lambda_0$ when (3.10) holds, and some positive number otherwise. One sees that (3.13) may be rewritten as
\[
\Lambda^M(h) = -\sigma_1 h_i X_{\{i > N\}} + b_i h_{i-1} X_{\{i > N+1\}} + a_i Q_1 h_{i+1} X_{\{i > N\}} & \text{for } i > 1.
\]

For any $h \in X_1$ one has
\[
\langle \text{sign}(h), Bh \rangle_{X'\perp, X} = \sum_{i=1}^{\infty} \exp(\eta i) Q_i \text{sign}(h_i) B_i(h) = -\sum_{i=N+1}^{\infty} \exp(\eta i) \sigma_i Q_i |h_i| - \sum_{i=1}^{\infty} \exp(\eta i) Q_1 |h_i| + \sum_{i=N+1}^{\infty} \exp(\eta i) Q_i h_{i-1} + \sum_{i=N}^{\infty} \exp(\eta i) Q_1 \text{sign}(h_i) a_i h_{i+1}.
\]

The last two sums can be bounded by
\[
\sum_{i=N+1}^{\infty} \exp(\eta i) Q_i h_{i-1} + \sum_{i=N}^{\infty} \exp(\eta i) Q_i a_i h_{i+1} \leq \sum_{i=N}^{\infty} \left( \exp(\eta) \frac{Q_{i+1}}{\sigma_i Q_i} h_{i-1} + \exp(-\eta) \frac{Q_i Q_{i-1}}{\sigma_i Q_i} a_{i-1} \right) \exp(\eta i) \sigma_i Q_i |h_i| = \sum_{i=N}^{\infty} \mu_i \exp(\eta i) \sigma_i Q_i |h_i|
\]
Choosing $\lambda$ hence, this is not obvious, since the property (3.14), $\mu_i$ has a limit as $i \to +\infty$:

$$
\frac{Q_1 a_i}{\sigma_i} = \frac{Q_1 a_i}{Q_1 a_i + b_i} = \frac{a_i}{a_i + a_{i-1} Q_i} \to \frac{1}{1 + \ell}
$$

where $\ell := z_a/z \geq 1$, while

$$
\frac{b_i}{\sigma_i} = \frac{1}{1 + 2a_i Q_i} = \left(1 + \frac{a_i Q_i}{a_{i-1} Q_i}ight)^{-1} \to \frac{\ell}{1 + \ell}.
$$

Hence,

$$
\mu_i \to \frac{\exp(\eta) + \exp(-\eta) \ell}{1 + \ell} \quad \text{as} \quad i \to +\infty.
$$

One easily checks that this limit is strictly less than 1 if and only if $\exp(\eta) < \ell$. Since we are assuming $\exp(\eta) < \sqrt{\ell} \leq \ell$, for $N$ large enough we can find $\mu < 1$ such that

$$
\langle \text{sign}(h), B h \rangle_{X', X} \leq -(1 - \mu) \sum_{i=N+1}^{\infty} \exp(\eta) \sigma_i Q_i [h_i] - R \sum_{i=1}^{N} \exp(\eta) Q_i [h_i].
$$

Choosing $\lambda_3 = (1 - \mu)\sigma$ and $R = \lambda_3$ we obtain

$$
\langle \text{sign}(h), B h \rangle_{X', X} \leq -\lambda_3 \sum_{i=1}^{\infty} \exp(\eta) Q_i [h_i].
$$

This is nothing but (3.16).

If we assume additionally that (3.10) holds (i.e., $\lim_{j \to \infty} a_j = +\infty$) then take any $\lambda_3 > \lambda_0$. One can choose $N > 1$ large enough so that $\sigma_i \geq \frac{\lambda_3}{1 - \mu}$ for any $i \geq N$ and then, by picking $R \geq \lambda_3$ we get

$$
\langle \text{sign}(h), B^\perp h \rangle_{X', X} \leq -\lambda_3 \sum_{i=1}^{\infty} e^{\eta i} Q_i [h_i],
$$

which proves (3.16) with any $\lambda_3 > \lambda_0$.

**Step 4:** $B^\perp$ is strictly dissipative. We now show that if we take any $\lambda_2 < \lambda_3$ (with the $\lambda_3$ from Step 3) then one can additionally choose $N$ in (3.12)–(3.13) such that

$$
\langle \text{sign}(h), B^\perp h \rangle_{X', X} \leq -\lambda_2 \|h\|_X \quad \text{for all} \quad h \in X_1^\perp.
$$

This is not obvious, since the property (3.16) of $B$ is not necessarily shared by its projection $B^\perp$. However, in this case and for $N$ large enough, $B^\perp$ happens to be a small perturbation of $B$ on $X_1^\perp$. We need to estimate, for $h \in X_1^\perp$, the quantity

$$
\mathbf{M}(B h) = \sum_{i=N+1}^{\infty} i Q_i A_i^M(h) - R \sum_{i=1}^{N} i Q_i h_i.
$$

We have, with very similar calculations as those needed to obtain (3.9), that

$$
\sum_{i=N+1}^{\infty} i Q_i |A_i^M(h)| \leq \varepsilon_1(N) \sum_{i=N+1}^{\infty} Q_i \exp(\eta i) \sum_{i=N+1}^{\infty} Q_i A_i^M(h) \leq C_\varepsilon_1(N) \|h\|_X
$$

(3.20)
with

$$\epsilon_1(N) := \sup_{i \geq N+1} \frac{i \sigma_i}{\exp(\eta_i)},$$

which tends to 0 as $N \to +\infty$ (as implied by (1.20c) and (1.20d) by considering the ratio of two consecutive terms in the sequence). On the other hand, since $\sum_{i=1}^{\infty} iQ_i h_i = 0$,

$$R \sum_{i=1}^{N} iQ_i h_i = R \sum_{i=N+1}^{\infty} iQ_i h_i \leq \frac{R}{\sigma} \epsilon_1(N) \|h\|_X$$

(3.21)

where we recall that $\sigma = \inf_{i \geq 1} \sigma_i$. From (3.20), (3.21), (3.11) and (3.16) we have

$$\langle \text{sign}(h), B^\perp h \rangle_{X', X} \leq -\lambda_3 \|h\|_X + \sum_{i=1}^{\infty} Q_{i} \exp(\eta_i) \left( \frac{R}{\sigma} + C \right) \epsilon_1(N) \|h\|_X$$

for all $h \in X^+_1$. Choosing $N$ large enough proves (3.19), since $\epsilon_1(N) \to 0$ as $N \to +\infty$ (notice that the choice of $R$ from Step 3 is not affected).

**Step 5:** $B$ generates a semigroup on $X$. To prove that $B$ is the generator of a $C_0$-semigroup on $X$ we shall invoke Miyadera perturbation Theorem [4]. Notice that $B$ may be written as

$$B = T + C$$

where $T$ is the multiplication operator (with domain $\mathcal{D}(T) = \mathcal{D}(B) = X_1$) given by

$$T_i(h) = -(\sigma_i + R) \chi_{\{i \geq N\}} h_i \quad \forall i \geq 1, \ h \in \mathcal{D}(B)$$

while $C$ is defined by

$$C_i(h) = b_i h_{i-1} \chi_{\{i \geq N+1\}} + a_i Q_i h_{i+1} \chi_{\{i \geq N\}}, \quad i \geq 1.$$

It is clear that $T$ is the generator of a $C_0$-semigroup $(U^t)_{t \geq 0}$ on $X$ given by

$$U^t_i(h) = \exp \left(-t (\sigma_i + R) \chi_{\{i \geq N\}} \right) h_i \quad i \geq 1, \ t \geq 0, \ h \in X.$$  

Moreover, $C : X_1 \to X$ is bounded. Indeed, it is not difficult to check, as we did in the proof of (3.16) that

$$\|C(h)\|_X \leq \sum_{i=N+1}^{\infty} \mu_i \sigma_i \exp(\eta_i) Q_i |h_i| \leq \|h\|_X \sup_{i \geq N+1} \mu_i \quad \forall h = (h_i)_{i \geq 1} \in X_1$$

(3.22)

where $\mu_i$ is defined by (3.18). Since the sequence $(\mu_i)_i$ is bounded, this shows in particular, that $C$ is $T$-bounded. One deduces then from (3.22) that

$$\|CU^t h\|_X \leq \sum_{i=N+1}^{\infty} \mu_i \sigma_i \exp(-t(\sigma_i + R)) \exp(\eta_i) Q_i |h_i| \quad h \in X_1, \ t \geq 0$$

which readily yields

$$\int_0^1 \|CU^t h\|_X \, dt \leq \sum_{i=N+1}^{\infty} \mu_i \frac{\sigma_i}{\sigma_i + R} \exp(\eta_i) Q_i |h_i|.$$  

In particular, we see that

$$\int_0^1 \|CU^t h\|_X \, dt \leq \mu \|h\|_X \quad \forall h \in X^+_1$$
where \( \mu := \sup_{i \geq N+1} \mu_i \). We already saw in Step 3 that \( N \) can be chosen large enough so that \( \mu < 1 \). Then, the above inequality exactly means that \( C \) is a Miyadera perturbation of \( T \) (see [4, Section 4.4, p. 127-128]) and that \( B = T + C \) is the generator of a \( C_0 \)-semigroup \( (S_t^B)_{t \geq 0} \) on \( X \). Notice then that (3.16) exactly means that \( (\lambda_3 + B) \) is the generator of a contraction semigroups in \( X \).

**Step 6:** \( (\lambda_2 + B^\perp) \) generates a \( C_0 \)-semigroup of contractions in \( X^\perp \). Notice that the approach used in the previous step seems difficult to apply directly to \( B^\perp \), the reason being essentially that, in the above splitting \( B = T + C \), the operators \( T \) and \( C \) do not map their respective domains to \( X^\perp \). To prove that \( (\lambda_2 + B^\perp) \) generates a \( C_0 \)-semigroup of contractions in \( X^\perp \), we adopt another more indirect way. Since \( A : X \to \mathcal{H} \) is bounded, it is clear that \( A : X \to X \) is a bounded operator. Therefore, by the bounded perturbation theorem, \( \Lambda = A + B \) is the generator of a \( C_0 \) semigroup \( (T_t)_{t \geq 0} \) in \( X \). Moreover, since \( \Lambda(x_1) \subset X^\perp \), the closed subspace \( X^\perp \subset X \) is invariant under \( (T_t)_{t \geq 0} \). There, as well-known [11, Chapter II.2.3, p. 60-61], the restriction \( \Lambda^\perp = \Lambda|_{X^\perp} \) with domain \( \mathcal{D}(\Lambda^\perp) = \mathcal{D}(\Lambda) \cap X^\perp = X^\perp \) is the generator of a \( C_0 \)-semigroup in \( X^\perp \). We already saw that \( \Lambda^\perp = \Lambda_2 + B^\perp \) where \( B^\perp \) is a bounded operator with domain \( X^\perp_1 \) and \( \Lambda_2 : X^\perp_1 \to \mathcal{H}^\perp \) is bounded. In particular, \( \Lambda^\perp : X \to X \) is also a bounded operator and then,

\[
B^\perp = -\Lambda_2 + \Lambda^\perp
\]

is the bounded perturbation of the generator \( \Lambda^\perp \). Thus, \( B^\perp \) is the generator of a \( C_0 \)-semigroup \( (S_t^{B^\perp})_t \) in \( X^\perp \). Moreover, according to Step 4, inequality (3.19) implies that \( \lambda_2 + B^\perp \) is a dissipative operator in \( X^\perp \) (in the sense of [11, Chapter II.3.b, p. 82]). Since \( \lambda_2 + B^\perp \) generates a \( C_0 \)-semigroup in \( X^\perp \), according to the Lumer-Phillips Theorem, this semigroup is a contraction semigroup, or equivalently

\[
\| S_t^{B^\perp} h_0 \|_X \leq \exp(-\lambda_2 t) \| h_0 \|_X \quad \forall h_0 \in X^\perp, \quad t \geq 0
\]

i.e. (3.5) holds with \( C_2 = 1 \) and \( \lambda_2 \) provided by (3.19).

**Step 7:** Conclusion. To obtain the desired conclusion, we only have to apply Theorem 3.1 with \( \lambda_1 = \lambda_0 \) being the spectral gap of \( L \) while \( 0 < \lambda_1, \lambda_2 \) is any positive number strictly smaller than \( \min(\lambda_0, \lambda_3) \) where \( \lambda_3 \) is constructed in Step 3. We just recall here that, if \( \lim_{i \to \infty} a_i = +\infty \), the number \( \lambda_3 \) is any arbitrary real number strictly larger than \( \lambda_0 \), and thus in this case any \( \lambda_* \in (0, \lambda_0) \) would yield the conclusion. \( \square \)

4. **Exponential convergence for the Becker-Döring equations**

4.1. **Local exponential convergence.** In this section we prove a local version of Theorem 1.1.

**Theorem 4.1 (Local exponential convergence for the Becker-Döring equations).** Assume (1.19) and (1.20). Let \( c = (c_i) \) be a nonzero subcritical solution to equation (1.1) (i.e., with density \( r < r_o \)) with initial condition \( c(0) \) such that (1.22) holds for some \( \nu > 0 \). Take \( z > 0 \) satisfying (1.7).

Then there exist some \( 0 < \eta < \nu \), some \( C, \epsilon > 0 \) and some \( \lambda_* > 0 \) (all explicit) such that if

\[
\sum_{i=1}^{\infty} \exp(\eta i) |c_i(0) - Q_i| < \epsilon
\]

then

\[
\sum_{i=1}^{\infty} \exp(\eta i) |c_i(t) - Q_i| \leq C \exp(-\lambda_* t) \quad \text{for all } t \geq 0.
\]
In addition, \( \lambda_* \) can be taken equal to \( \lambda_0 \) (the size of the spectral gap of the linear operator \( \mathcal{L} \)) if \( \lim_{i \to +\infty} a_i = +\infty \).

We remark that all the constants in the above result can be explicitly estimated. The proof of the above local convergence results relies on two crucial estimates. The first one was proved in Prop. 3.2, a bound of the nonlinear term \( \Gamma(h, h) \); the second one is a uniform–in–time bound of exponential moments for the Becker-Döring equations, which we take from [17]:

**Proposition 4.2.** Assume (1.19) and the conditions (1.20). Let \( c = (c_i) \), be a nonzero subcritical solution to equation (1.1) with initial condition \( c(0) \) such that (1.22) holds for some \( \nu > 0 \). Then there exists \( 0 < \nu < \nu \) and \( K_1 > 0 \) such that

\[
\sum_{j=1}^{\infty} \exp(\nu j) c_j(t) \leq K_1,
\]

where \( c = (c_i(t))_i \) is the unique solution to (1.1) with initial datum \( c(0) = (c_i(0))_{i \geq 1} \).

We notice that (1.19) and (1.20), together with the definition of \( Q_i \) through (1.5), imply conditions (H1)–(H4) of [17]. In particular, since \( z_i \) is defined as the radius of convergence of the series \( \sum z^i Q_i \), it is clear that

\[
\lim_{i \to +\infty} Q_i^{1/i} = \frac{1}{z_i},
\]

which is (H3) of [17].

We are ready then to prove Theorem 4.1:

**Proof of Theorem 4.1.** Let \( c \) be a solution of (1.1) and let \( h = (h_i) \) be the fluctuation around the equilibrium, defined as in (2.4). Since Proposition 4.2 holds under our conditions, we can find \( 0 < \nu < \nu \) such that (4.2) holds. We take any \( \eta < \min\{\nu, \frac{1}{2} \log \frac{1}{\nu} \} \), and consider \( X \) the vector space defined in (3.1). Note that Proposition 3.2 and Theorem 3.5 are applicable under these conditions, and consider the quantity \( \lambda_* \) appearing in Theorem 3.5.

The fluctuation \( h \) satisfies equation (2.8) in \( X \):

\[
\frac{d}{dt} h = \Lambda[h] + \Gamma(h, h),
\]

so, if \( (S_t)_{t \geq 0} \) denotes the evolution semigroup generated by \( \Lambda \), then

\[
h(t) = S_t h(0) + \int_0^t S_{t-s} \Gamma(h(s), h(s)) \, ds.
\]

Recall that \( \sum_{i \geq 1} i h_i(t) Q_i = 0 = \sum_{i \geq 1} i \Gamma_i(h(t), h(t)) Q_i \) for any \( t \geq 0 \) so that, according to Proposition 3.2 and Theorem 3.5, we have (for some constant \( C > 0 \)),

\[
\|h(t)\|_X \leq \|S_t h(0)\|_X + \int_0^t \|S_{t-s} \Gamma(h(s), h(s))\|_X \, ds
\]

\[
\leq C \|h(0)\|_X \exp(-\lambda_* t) + C \exp(-\lambda_* t) \int_0^t \exp(\lambda_* s) \|\Gamma(h(s), h(s))\|_X \, ds
\]

\[
\leq C \|h(0)\|_X \exp(-\lambda_* t) + C \exp(-\lambda_* t) \int_0^t \exp(\lambda_* s) \|h(s)\|_X \|h(s)\|_X \, ds \quad (4.3)
\]
where we recall that $X_1$ is defined by (3.8). For any $\delta \in (0, \nu - \eta)$, we have from Cauchy-Schwarz's inequality that

$$
\|h(t)\|_{X_1} = \sum_i |h_i(t)| \eta_i \exp(\eta i) \\
\leq \left( \sum_{i=1}^{\infty} |h_i(t)| \eta_i \exp(\eta i - \delta i) \right)^{1/2} \left( \sum_{i=1}^{\infty} |h_i(t)| \eta_i \exp(\eta i + \delta i) \right)^{1/2}.
$$

Now, using that

$$
i \exp(-\delta i) \leq C, \quad i \exp(\eta i + \delta i) \leq C \exp(\nu i)
$$

for all $i \geq 1$, we deduce that

$$
\|h(t)\|_{X_1} \leq C \|h(t)\|_{X_1}^{1/2} \left( \sum_{i=1}^{\infty} |h_i(t)| \eta_i \exp(\nu i) \right)^{1/2} \leq CK_1^{1/2} \|h(t)\|_{X_1}^{1/2}
$$

for all $t \geq 0$.

Using this in eq. (4.3) we have

$$
\|h(t)\|_X \leq C \|h(0)\|_X \exp(-\lambda_* t) + C \exp(-\lambda_* t) \int_0^t \exp(\lambda_* s) \|h(s)\|_X^{3/2} \ds.
$$

Then, the quantity $u(t) := \exp(\lambda_* t) \|h(t)\|_X$ satisfies

$$
u(t) \leq C \|h(0)\|_X + C \int_0^t u(s)^{3/2} \ds.
$$

We deduce then the conclusion from Gronwall's inequality. □

### 4.2. Global exponential convergence.

In order to prove Theorem 1.1 we complement the local result given in Theorem 4.1 with the following global result from [17]:

**Proposition 4.3.** Assume (1.19) and the conditions (1.20). Let $c = (c_i)$, be a nonzero subcritical solution to equation (1.1) with initial condition $c(0)$ such that (1.22) holds for some $\nu > 0$, and take $z > 0$ satisfying (1.7) as usual.

Then, there exist two explicit constants $C_1 > 0$ and $\kappa_1 > 0$ such that

$$
\sum_{j \geq 1} j |c_j(t) - Q_j| \leq C_1 \exp(-\kappa_1 t^{1/2}) \quad \forall t \geq 0.
$$

**Proof of Theorem 1.1.** Let $h = (h_i)$ be defined as in (2.4). Let us fix the same $\eta \in (0, \nu)$ as in Theorem 4.1 and denote by $X$ the $\ell^1$ space with weight $e^{\eta i}$ that was defined in (3.1). The idea is to use Proposition 4.3 to show that at a certain time $t_0$ we are close enough to the equilibrium to use Theorem 4.1. To do this we use a simple interpolation argument to estimate the norm

$$
\sum_{i \geq 1} |c_i(t) - Q_i| \exp(\eta i) = \|h(t)\|_X
$$
in terms of \( \sum_{i \geq 1} |c_i(t) - Q_i| \) and a slightly stronger norm. Namely, for any \( \delta \in (0, \eta - \nu) \), a simple interpolation argument yields

\[
\|h(t)\|_X = \sum_{i \geq 1} |h_i(t)| Q_i \exp(\eta i) \leq \left( \sum_{i \geq 1} i |h_i(t)| Q_i \right)^{\frac{1}{\eta}} \left( \sum_{i \geq 1} i^{-1} |h_i(t)| Q_i \exp((\eta + \delta)i) \right)^{\frac{\eta}{\eta + \delta}}.
\]

Therefore, according to (4.2) and (4.4), one obtains

\[
\|h(t)\|_X \leq C_{\nu, \eta} \exp\left( -\kappa_1 \frac{\delta}{\delta + \eta} t^+ \right) \quad \forall t \geq 0.
\] (4.5)

Consider the number \( \varepsilon > 0 \) given by Theorem 4.1. Then, there exists an explicit \( t_0 > 0 \) such that

\[
\|h(t_0)\|_X \leq \varepsilon
\]

and one deduces from (4.1) starting from \( t = t_0 \) that

\[
\|h(t)\|_X \leq C_{\mu} \|h(t_0)\|_X \exp(-\lambda t) \quad \forall t \geq t_0.
\]

Together with (4.5) for \( t < t_0 \), this concludes the proof. \( \square \)

**APPENDIX A. DISCRETE HARDY’S INEQUALITIES**

The following discrete version of Hardy’s inequality is used in the proof of Theorem 2.15:

**Theorem A. 1.** Consider two sequences of positive numbers \((\mu_i)\) and \((\nu_i)\). Then, the following are equivalent:

1. There exists a finite constant \( A \geq 0 \) such that

\[
\sum_{i=1}^{\infty} \mu_i \left( \sum_{j=1}^{i} f_j \right)^2 \leq A \sum_{i=1}^{\infty} \nu_i f_i^2
\] (A.1)

for any sequence \( f = (f_i) \).

2. The following holds

\[
B = \sup_{k \geq 1} \left( \sum_{j=k}^{\infty} \mu_j \right) \left( \sum_{i=1}^{k} \frac{1}{\nu_i} \right) < \infty.
\]

If these equivalent propositions hold true, then \( B \leq A \leq 4B \).

**Remark A. 2.** The above discrete version of Hardy’s inequality is not new [20] and we give a proof for the sake of completeness. It is a simple adaptation of the proof of the continuous version of Hardy’s inequality, originally due to Muckenhoupt [22], that can be found in [1, Chapitre 6].

**Proof.** Assume that there exists some finite \( A > 0 \) such that (A.1) hold true for any \( f = (f_i) \) and let us prove that \( A \geq B \). For any fixed \( k \geq 1 \), set

\[
f_i = \begin{cases} \frac{1}{\nu_i} & \text{for } i \leq k \\ 0 & \text{for } i > k. \end{cases}
\]
Then, one recognizes easily that $\sum_{i=1}^{\infty} \nu_i f_i^2 = \sum_{i=1}^{k} \frac{1}{\nu_i}$ while
$$\sum_{i=1}^{\infty} \mu_i \left( \sum_{j=1}^{i} f_j \right)^2 \geq \left( \sum_{i=k}^{\infty} \mu_i \right) \left( \sum_{j=1}^{k} \frac{1}{\nu_j} \right)^2.$$  

According to (A.1) one gets then
$$A \left( \sum_{i=1}^{k} \frac{1}{\nu_i} \right) \geq \left( \sum_{i=k}^{\infty} \mu_i \right) \left( \sum_{j=1}^{k} \frac{1}{\nu_j} \right)^2$$
or equivalently $A \geq \left( \sum_{i=k}^{\infty} \mu_i \right) \left( \sum_{j=1}^{k} \frac{1}{\nu_j} \right)^2$ for any $k \geq 1$. This proves that $B \leq A$.

Conversely, let us assume that $B < \infty$ and let us prove that $A \leq 4B$. Set
$$\Upsilon_i = \sum_{j=1}^{i} \frac{1}{\nu_j}, \quad \Gamma_i = \sum_{j=i}^{\infty} \mu_j, \quad \gamma_i = \sqrt{\Gamma_i} \quad \text{and} \quad \beta_i = \sqrt{\Upsilon_i}.$$  

Let $f = (f_i)_i$ be a given sequence. Thanks to Cauchy-Schwarz inequality
$$\left( \sum_{j=1}^{i} f_j^2 \right)^{1/2} \leq \left( \sum_{j=1}^{i} f_j^2 \nu_j \beta_j \right)^{1/2} \left( \sum_{j=1}^{i} \frac{1}{\nu_j \beta_j} \right)^{1/2}$$
Now, since $\frac{1}{\nu_j \beta_j} = \frac{1}{\nu_j} - \frac{1}{\nu_{j-1}}$, one gets
$$\frac{1}{\nu_j \beta_j} = \frac{1}{\nu_j \sqrt{\Upsilon_j}} = \frac{\sqrt{\Upsilon_j} - \sqrt{\Upsilon_{j-1}}}{\sqrt{\Upsilon_j}} \quad \forall j \geq 1.$$  

Using the general inequality
$$\frac{X - Y}{\sqrt{X}} \leq 2 \left( \sqrt{X} - \sqrt{Y} \right) \quad \forall X \geq Y > 0 \quad (A.2)$$
we get that
$$\left( \sum_{j=1}^{i} \frac{1}{\nu_j \beta_j} \right) \leq 2 \sum_{j=1}^{i} \left( \sqrt{\Upsilon_j} - \sqrt{\Upsilon_{j-1}} \right) = 2 \sqrt{\Upsilon_i} = 2 \beta_i$$
Therefore, the left-hand-side of (A.1) can be estimated as follows:
$$\sum_{i=1}^{\infty} \mu_i \left( \sum_{j=1}^{i} f_j \right)^2 \leq 2 \sum_{i=1}^{\infty} \mu_i \beta_i \left( \sum_{j=1}^{i} f_j^2 \nu_j \beta_j \right) = 2 \sum_{j=1}^{\infty} f_j^2 \nu_j \beta_j \left( \sum_{i=j}^{\infty} \mu_i \beta_i \right).$$

Then, one sees that to prove our claim, it suffices to prove that
$$\beta_j \left( \sum_{i=j}^{\infty} \mu_i \beta_i \right) \leq 2B \quad \forall j \geq 1. \quad (A.3)$$
One notices that, by definition of $B$, $\beta_i \gamma_i \leq \sqrt{B}$ for any $i \geq 1$. Therefore,
$$\sum_{i=j}^{\infty} \mu_i \beta_i \leq \sqrt{B} \sum_{i=j}^{\infty} \frac{\mu_i}{\gamma_i} = \sqrt{B} \sum_{i=j}^{\infty} \frac{\mu_i}{\sqrt{\Gamma_i}}.$$
Finally, by definition of $B$ and the above inequality reads

$$C_i \leq \frac{\sum_{i=1}^\infty \mu_i}{\sqrt{\Gamma_i}} \leq 2 \left( \sqrt{\Gamma_i} - \sqrt{\Gamma_{i+1}} \right)$$

and the above inequality reads

$$\sum_{i=1}^\infty \mu_i \beta_i \leq 2\sqrt{B} \sum_{i=1}^\infty \left( \sqrt{\Gamma_i} - \sqrt{\Gamma_{i+1}} \right) = 2\sqrt{B} \sqrt{\Gamma_j} = 2\sqrt{B} \gamma_j \quad \forall j \geq 1.$$

Finally, by definition of $B$, we get that

$$\beta_j \left( \sum_{i=1}^\infty \mu_i \beta_i \right) \leq 2\sqrt{B} \beta_j \gamma_j \leq 2B \quad \forall j \geq 1$$

which gives (A.3) and achieves the proof.

\[\square\]

**APPENDIX B. SOME ESTIMATES OF SPECIAL FUNCTIONS**

We collect here some technical estimates for special functions which turn useful for estimating the spectral gap size near the critical density (see Section 2.4).

**Lemma B.1.** Take $0 < \mu \leq 1$. For some quantity $C_\mu$ depending only on $\mu$,

$$\int_x^\infty \exp(-y^\mu) \, dy \leq C_\mu \exp(-x^\mu)x^{1-\mu} \quad \text{for all } x \geq 1. \quad (B.1)$$

**Proof.** Through the change of variables $u = y^\mu - x^\mu$ we obtain

$$\int_x^\infty \exp(-y^\mu) \, dy = \frac{1}{\mu} \exp(-x^\mu)x^{1-\mu} \int_0^\infty \exp(-u) \left( \frac{u}{x^\mu} + 1 \right)^{\frac{1-\mu}{\mu}} \, du$$

so the result holds with $C_\mu = \frac{1}{\mu} \int_0^\infty \exp(-u) \left( u + 1 \right)^{\frac{1-\mu}{\mu}} \, du$.

\[\square\]

**Lemma B.2.** Take $0 < \mu \leq 1$ and $0 \leq \alpha < 1$. For some quantity $C_{\mu,\alpha}$ depending only on $\mu$ and $\alpha$,

$$\int_0^x \exp(y^\mu)y^{-\alpha} \, dy \leq C_{\mu,\alpha} \exp(x^\mu)x^{1-\mu-\alpha} \quad \text{for all } x \geq 1. \quad (B.2)$$

**Proof.** We first break the integral in the intervals $(0, x/2]$ and $(x/2, x)$: for the first part,

$$\int_0^{x/2} \exp(y^\mu)y^{-\alpha} \, dy \leq \exp((x/2)^\mu) \int_0^{x/2} y^{-\alpha} \, dy = \frac{1}{1 - \alpha} \exp((x/2)^\mu) \left( \frac{x}{2} \right)^{1-\alpha} \leq A_{\mu,\alpha} \exp(x^\mu)x^{1-\mu-\alpha} \quad (B.3)$$

with

$$A_{\mu,\alpha} := \frac{2^{\alpha-1}}{1 - \alpha} \max \left\{ \exp((x/2)^\mu) - x^\mu \right\}.$$

For the second part, through the change of variables $u = x^\mu - y^\mu$ we obtain

$$\int_{x/2}^x \exp(y^\mu)y^{-\alpha} \, dy = \frac{1}{\mu} \exp(-x^\mu)x^{1-\mu-\alpha} \int_0^{x/2} \exp(-u) \left( 1 - \frac{u}{x^\mu} \right)^{\frac{1-\mu-\alpha}{\mu}} \, du$$

$$\leq B_{\mu,\alpha} \exp(-x^\mu)x^{1-\mu-\alpha} \int_0^{x/2} \exp(-u) \, du \leq B_{\mu,\alpha} \exp(-x^\mu)x^{1-\mu-\alpha} \quad (B.4)$$
with
\[ B_{\mu,\alpha} := \frac{1}{\mu} \max \left\{ 1, 2^{1-\alpha} \right\} \]

since \( \frac{1}{2^{\alpha}} \leq 1 - \frac{\mu}{2} \leq 1 \) on the region of integration. From (B.3) and (B.4) we conclude with \( C_{\mu,\alpha} := \max\{ A_{\mu,\alpha}, B_{\mu,\alpha} \} \).

We end this Appendix with an asymptotic result for special expansion series:

**Lemma B. 3.** Take \( \mu > 0 \) and a sequence \((g_k)_k\) such that, for some \( C_0 > 0 \),
\[ g_{k+1} - g_k \sim C_0 k^{\mu-1} \quad \text{as } k \to +\infty. \]  
(B.5)

Then
\[ \sum_{j=1}^{k} \exp(g_j) j^\nu \sim \frac{1}{C_0} \exp(g_k) k^{\nu+1-\mu} \quad \text{as } k \to +\infty \]  
(B.6)

and
\[ \sum_{j=k}^{\infty} \exp(-g_j) j^\nu \sim \frac{1}{C_0} \exp(-g_k) k^{\nu+1-\mu} \quad \text{as } k \to +\infty. \]  
(B.7)

**Remark B. 4.** One easily deduces that, in the conditions of the above lemma, there is a constant \( C \) which depends only on the sequence \((g_k)_k\) and \( \nu \) such that
\[ \sum_{j=1}^{k} \exp(g_j) j^\nu \leq C \exp(g_k) k^{\nu+1-\mu} \quad \text{and} \quad \sum_{j=k}^{\infty} \exp(-g_j) j^\nu \leq C \exp(-g_k) k^{\nu+1-\mu} \]
for all \( k \geq 1 \).

**Proof.** We prove (B.6) by using the Stolz-Cesàro theorem (which is a discrete analogue of l'Hôpital’s rule). Namely, setting
\[ U_k = \exp(g_k) k^{\nu+1-\mu} \quad \text{and} \quad V_k = \sum_{j=1}^{k} \exp(g_j) j^\nu \quad (k \geq 1), \]
in order to show that
\[ \lim_{k \to \infty} \frac{U_k}{V_k} = C_0, \]
we instead prove that
\[ \lim_{k \to \infty} \frac{U_{k+1} - U_k}{V_{k+1} - V_k} = C_0. \]  
(B.8)

One has
\[ \frac{U_{k+1} - U_k}{V_{k+1} - V_k} = \frac{(k+1)^{\nu+1-\mu} - \exp(g_k - g_{k+1}) k^{\nu+1-\mu}}{(k+1)^\nu} \]
\[ = \frac{[(k+1)^{\nu+1-\mu} - k^{\nu+1-\mu}] + k^{\nu+1-\mu} (1 - \exp(g_k - g_{k+1}))}{(k+1)^\nu}. \]

Since
\[ \left| (k+1)^{\nu+1-\mu} - k^{\nu+1-\mu} \right| \leq |\nu + 1 - \mu| \min\{ (k+1)^{\nu-\mu}, k^{\nu-\mu} \} \approx |\nu + 1 - \mu| k^{-\mu} \to 0 \]
as $k \to +\infty$ while
\[
\lim_{k \to \infty} \frac{k^{\nu+1-\mu} (1 - \exp(g_k - g_{k+1}))}{(k + 1)^\nu} = \lim_{k \to \infty} \frac{k^{\nu+1-\mu} (g_{k+1} - g_k) 1 - \exp(g_k - g_{k+1})}{g_{k+1} - g_k} = C_0,
\]
using $(1 - \exp(g_k - g_{k+1})/(g_{k+1} - g_k) \to 1$ together with (B.5), we obtain (B.8). We apply the Stolz-Cesàro theorem and obtain (B.6) (notice that, since (B.5) implies that $g_k \sim (C_0/\mu)k^\mu$, the sum $\sum_{j=1}^{\infty} \exp(g_j)j^\nu$ diverges and hence the hypotheses of the theorem are satisfied). Equation (B.7) can be proved by a very similar calculation. \qed

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