Bohr’s absolute convergence problem for \(H_p\)-Dirichlet series in Banach spaces

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Abstract

The Bohr-Bohnenblust-Hille Theorem states that the width of the strip in the complex plane on which an ordinary Dirichlet series \(\sum_n a_n n^{-s}\) converges uniformly but not absolutely is less than or equal to 1/2, and this estimate is optimal. Equivalently, the supremum of the absolute convergence abscissas of all Dirichlet series in the Hardy space \(\mathcal{H}_\infty\) equals 1/2. By a surprising fact of Bayart the same result holds true if \(\mathcal{H}_\infty\) is replaced by any Hardy space \(\mathcal{H}_p, 1 \leq p < \infty\), of Dirichlet series. For Dirichlet series with coefficients in a Banach space \(X\) the maximal width of Bohr’s strips depend on the geometry of \(X\); Defant, García, Maestre and Pérez-García proved that such maximal width equal \(1 – 1/\text{Cot}(X)\), where \(\text{Cot}(X)\) denotes the maximal cotype of \(X\). Equivalently, the supremum over the absolute convergence abscissas of all Dirichlet series in the vector-valued Hardy space \(\mathcal{H}_\infty(X)\) equals \(1 – 1/\text{Cot}(X)\). In this article we show that this result remains true if \(\mathcal{H}_\infty(X)\) is replaced by the larger class \(\mathcal{H}_p(X), 1 \leq p < \infty\).

1 Main result and its motivation

Given a Banach space \(X\), an ordinary Dirichlet series in \(X\) is a series of the form \(D = \sum_n a_n n^{-s}\), where the coefficients \(a_n\) are vectors in \(X\) and \(s\) is a complex variable. Maximal domains where such Dirichlet series converge conditionally, uniformly or absolutely are half planes \([\text{Re} > \sigma]\), where \(\sigma = \sigma_c, \sigma_u\) or \(\sigma_a\) are called the abscissa of conditional, uniform or absolute convergence, respectively. More precisely, \(\sigma_\alpha(D)\) is the infimum of all \(r \in \mathbb{R}\) such that on \([\text{Re} > r]\) we have convergence of \(D\) of the requested type \(\alpha = c, u\) or \(a\). Clearly, we have \(\sigma_c(D) \leq \sigma_u(D) \leq \sigma_a(D)\), and it can be easily shown that \(\sup \sigma_a(D) – \sigma_c(D) = 1\), where the supremum is taken over all Dirichlet series \(D\) with coefficients in \(X\). To determine the maximal width of the strip on which a Dirichlet series in \(X\) converges uniformly

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but not absolutely, is more complicated. The main result of [8] states, with the notation given below, that

\[ S(X) := \sup \sigma_a(D) - \sigma_u(D) = 1 - \frac{1}{\text{Cot}(X)}. \]  

(1)

Recall that a Banach space \( X \) is of cotype \( q \), \( 2 \leq q < \infty \) whenever there is a constant \( C \geq 0 \) such that for each choice of finitely many vectors \( x_1, \ldots, x_N \in X \) we have

\[ \left( \sum_{k=1}^{N} \| x_k \|_X^q \right)^{1/q} \leq \frac{C}{2} \left( \int_{T^N} \left\| \sum_{k=1}^{N} x_k z_k \right\|_X^2 \, dz \right)^{1/2}, \]

(2)

where \( T := \{ z \in \mathbb{C} \mid |z| = 1 \} \) and \( T^N \) is endowed with \( N \)th product of the normalized Lebesgue measure on \( T \). We denote by \( C_r(X) \) the best of such constants \( C \). As usual we write

\[ \text{Cot}(X) := \inf \left\{ 2 \leq q < \infty \mid X \text{ cotype } q \right\}, \]

and (although this infimum in general is not attained) we call it the optimal cotype of \( X \). If there is no \( 2 \leq q < \infty \) for which \( X \) has cotype \( q \), then \( X \) is said to have no finite cotype, and we put \( \text{Cot}(X) = \infty \). To see an example,

\[ \text{Cot}(X)(\ell_q) = \begin{cases} q & \text{for } 2 \leq q \leq \infty \\ 2 & \text{for } 1 \leq q \leq 2. \end{cases} \]

The scalar case \( X = \mathbb{C} \) in (1) was first studied by Bohr and Bohnenblust-Hille: In 1913 Bohr in [4] proved that \( S(\mathbb{C}) \leq \frac{1}{2} \), and in 1931 Bohnenblust and Hille in [3] that \( S(\mathbb{C}) \geq \frac{1}{2} \). Clearly, the equality

\[ S(\mathbb{C}) = \frac{1}{2}, \]

(3)

nowadays called Bohr-Bohnenblust-Hille Theorem, fits with (1). Let us give a second formulation of (1). Define the vector space \( \mathcal{H}_\infty(X) \) of all Dirichlet series \( D = \sum_n a_n n^{-s} \) in \( X \) such that

\( \bullet \) \( \sigma_c(D) \leq 0 \),

\( \bullet \) the function \( D(s) = \sum_n a_n \frac{1}{n^s} \) on \( \text{Re } s > 0 \) is bounded.

Then \( \mathcal{H}_\infty(X) \) together with the norm

\[ \| D \|_{\mathcal{H}_\infty(X)} = \sup_{\text{Re } s > 0} \left\| \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \right\|_X \]

forms a Banach space. For any Dirichlet series \( D \) in \( X \) we have

\[ \sigma_u(D) = \inf \left\{ \sigma \in \mathbb{R} \mid \sum_n a_n \frac{1}{n^{\sigma}} \in \mathcal{H}_\infty(X) \right\}. \]

(4)

In the scalar case \( X = \mathbb{C} \), this is (what we call) Bohr’s fundamental theorem from [5], and for Dirichlet series in arbitrary Banach spaces the proof follows similarly. Together with (4) a simply translation argument gives the following reformulation of (1):

\[ S(X) = \sup_{D \in \mathcal{H}_\infty(X)} \sigma_a(D) = 1 - \frac{1}{\text{Cot}(X)}. \]

(5)
Following an ingenious idea of Bohr each Dirichlet series may be identified with a power series in infinitely many variables. More precisely, fix a Banach space $X$ and denote by $\mathfrak P(X)$ the vector space of all formal power series $\sum c_\alpha z^\alpha$ in $X$ and by $\mathfrak D(X)$ the vector space of all Dirichlet series $\sum_n a_n n^{-s}$ in $X$. Let us as usual $(p_n)_n$ be the sequence of prime numbers. Since each integer has a unique prime number decomposition $n = p_1^{a_1} \cdots p_k^{a_k}$ with $a_j \in \mathbb N_0$, $1 \leq j \leq k$, the linear mapping

$$\mathfrak B_X : \mathfrak P(X) \to \mathfrak D(X), \quad \sum_{\alpha \in \mathbb N_0^{|\mathbb N|}} c_\alpha z^\alpha \mapsto \sum_{n=1}^{\infty} a_n n^{-s}, \text{ where } a_{p^n} = c_\alpha$$

is bijective; we call $\mathfrak B_X$ the Bohr transform in $X$. As discovered by Bayart in [1] this (a priori very) formal identification allows to develop a theory of Hardy spaces of scalar–valued Dirichlet series.

Similarly we now define Hardy spaces of $X$–valued Dirichlet series. Denote by $dw$ the normalized Lebesgue measure on the infinite dimensional polytorus $\mathbb T^\infty = \prod_{k=1}^{\infty} \mathbb T$, e.g. the countable product measure of the normalized Lebesgue measure on $\mathbb T$. For any multi index $\alpha = (\alpha_1, \ldots, \alpha_n, 0, \ldots) \in \mathbb Z^{[\mathbb N]}$ (all finite sequences in $\mathbb Z$) the $\alpha$th Fourier coefficient $\hat f(\alpha)$ of $f \in L_1(\mathbb T^\infty, X)$ is given by

$$\hat f(\alpha) = \int_{\mathbb T^\infty} f(w) w^{-\alpha} dw,$$

where we as usual write $w^\alpha$ for the monomial $w_1^{\alpha_1} \cdots w_n^{\alpha_n}$. Then, given $1 \leq p < \infty$, the $X$-valued Hardy space on $\mathbb T^\infty$ is the subspace of $L_p(\mathbb T^\infty, X)$ defined as

$$H_p(\mathbb T^\infty, X) = \left\{ f \in L_p(\mathbb T^\infty, X) \mid \hat f(\alpha) = 0, \forall \alpha \in \mathbb Z^{[\mathbb N]} \setminus \mathbb N_0^{[\mathbb N]} \right\}. \quad (7)$$

Assigning to each $f \in H_p(\mathbb T^\infty, X)$ its unique formal power series $\sum \hat f(\alpha) z^\alpha$ we may consider $H_p(\mathbb T^\infty, X)$ as a subspace of $\mathfrak P(X)$. We denote the image of this subspace under the Bohr transform $\mathfrak B_X$ by

$$\mathcal{H}_p(X).$$

This vector space of all (so-called) $\mathcal{H}_p(X)$-Dirichlet series $D$ together with the norm

$$\|D\|_{\mathcal{H}_p(X)} = \|\mathfrak B_X^{-1}(D)\|_{H_p(\mathbb T^\infty, X)}$$

forms a Banach space; in other words, through Bohr’s transform $\mathfrak B_X$ from (6) we by definition identify

$$\mathcal{H}_p(X) = H_p(\mathbb T^\infty, X), 1 \leq p < \infty.$$ 

For $p = \infty$ we this way of course could also define a Banach space $\mathcal{H}_\infty(X)$, and it turns out that at least in the scalar case $X = \mathbb C$ this definition then coincides with the one given above; but we remark that these two $\mathcal{H}_\infty(X)$’s are different for arbitrary $X$. It is important to note that by the Birkhoff-Khinchine ergodic theorem the following internal description of the $\mathcal{H}_p(X)$-norm for finite Dirichlet polynomials $D = \sum_{k=1}^{n} a_k n^{-s}$ holds:

$$\|D\|_{\mathcal{H}_p(X)} = \lim_{T \to \infty} \left( \frac{1}{2T} \int_{-T}^{T} \left\| \sum_{k=1}^{n} a_k \frac{1}{n^{it}} \right\|_X^p \, dt \right)^{1/p}$$

(see e.g. Bayart [1] for the scalar case, and the vector-valued case follows exactly the same way).
Motivated by (4) we define for $D \in \mathcal{D}(X)$ and $1 \leq p < \infty$

$$\sigma_{\mathcal{H}_p(X)}(D) := \inf \left\{ \sigma \in \mathbb{R} \mid \sum_n \frac{a_n}{n^\sigma} \in \mathcal{H}_p(X) \right\},$$

and motivated by (5) we define

$$S_p(X) := \sup_{D \in \mathcal{D}(X)} \sigma_a(D) - \sigma_{\mathcal{H}_p(X)}(D) = \sup_{D \in \mathcal{H}_p(X)} \sigma_a(D)$$

(for the second equality use again a simple translation argument). A result of Bayart [1] shows that for every $1 \leq p < \infty$

$$S_p(\mathbb{C}) = \frac{1}{2}, \quad (8)$$

which according to Helson [13] is a bit surprising since $\mathcal{H}_\infty(\mathbb{C})$ is much smaller than $\mathcal{H}_p(\mathbb{C})$.

The following theorem unifies and generalizes (1), (3) as well as (8), and it is our main result.

**Theorem 1.1.** For every $1 \leq p \leq \infty$ and every Banach space $X$ we have

$$S_p(X) = 1 - \frac{1}{\operatorname{Cot}(X)}.$$

The proof will be given in section 4. But before we start let us give an interesting reformulation in terms of the monomial convergence of $X$-valued $H_p$-functions on $\mathbb{T}^\infty$.

Fix a Banach space $X$ and $1 \leq p \leq \infty$, and define the set of monomial convergence of $H_p(\mathbb{T}^\infty, X)$:

$$\text{mon}_{H_p}(\mathbb{T}^\infty, X) = \left\{ z \in B_{c_0} \mid \sum_{\alpha} \| \hat{f}(\alpha) z^\alpha \|_X < \infty \text{ for all } f \in H_p(\mathbb{T}^\infty, X) \right\}.$$

Philosophically, this is the largest set $M$ on which for each $f \in H_p(\mathbb{T}^\infty, X)$ the definition $g(z) = \sum_{\alpha} \hat{f}(\alpha) z^\alpha$, $z \in M$ leads to an extension of $f$ from the distinguished boundary $\mathbb{T}^\infty$ to its “interior” $B_{c_0}$ (the open unit ball of the Banach space $c_0$ of all null sequences). For a detailed study of sets of monomial convergence in the scalar case $X = \mathbb{C}$ see [9], and in the vector-valued case [10].

We later need the following two basic properties of monomial domains (in the scalar case see [8, p.550] and [7, Lemma 4.3], and in the vector-valued case the proofs follow similar lines).

**Remark 1.2.**

1. Let $z \in \text{mon}_{H_p}(\mathbb{T}^\infty, X)$. Then $u = (z_{\sigma(n)})_{n} \in \text{mon}_{H_p}(\mathbb{T}^\infty, X)$ for every permutation $\sigma$ of $\mathbb{N}$.

2. Let $z \in \text{mon}_{H_p}(\mathbb{T}^\infty, X)$ and $x = (x_n)_{n} \in \mathbb{D}^\infty$ be such that $|x_n| \leq |z_n|$ for all but finitely many $n$‘s. Then $x \in \text{mon}_{H_p}(\mathbb{T}^\infty, X)$. 

4
Given $1 \leq p \leq \infty$ and a Banach space $X$, the following number measures the size of $\text{mon } \ell_p(\mathbb{T}^\infty, X)$ within the scale of $\ell_r$-spaces:

$$M_p(X) = \sup \left\{ 1 \leq r \leq \infty \mid \ell_r \cap B_{c_0} \subset \text{mon } \ell_p(\mathbb{T}^\infty, X) \right\}.$$ 

The following result is a reformulation of Theorem 1.1 in terms of vector-valued $H_p$-functions on $\mathbb{T}^\infty$ through Bohr’s transform $\mathcal{B}_X$. The proof is modelled along ideas from Bohr’s seminal article [4, Satz IX].

**Corollary 1.3.** For each Banach space $X$ and $1 \leq p \leq \infty$ we have

$$M_p(X) = \frac{\text{Cot}(X)}{\text{Cot}(X) - 1}.$$ 

**Proof.** We are going to prove that $S_p(X) = 1/M_p(X)$, and as a consequence the conclusion follows from Theorem 1.1. We begin by showing that $S_p(X) \leq 1/M_p(X)$. Fix $q < M_p(X)$ and $r > 1/q$; then we have that $(\frac{1}{p_n})_n \in \ell_q \cap B_{c_0}$ and, by the very definition of $M_p(X)$, $\sum a \|\hat{f}(\alpha)(\frac{1}{p_n})^\alpha\|_X < \infty$ converges absolutely for every $f \in H_p(\mathbb{T}^\infty, X)$. We choose now an arbitrary Dirichlet series $D = \mathfrak{B}_X f = \sum_n a_n \frac{1}{n^r} \in \mathcal{H}_p(X)$ with $f \in H_p(\mathbb{T}^\infty, X)$.

Then

$$\sum_n \|a_n\|_X \frac{1}{n^r} = \sum_a \|a_{p^\alpha}\|_X \left(\frac{1}{p^\alpha}\right)^r = \sum_a \|\hat{f}(\alpha)\|_X \left(\frac{1}{p^\alpha}\right)^r < \infty.$$ 

Clearly, this implies that $S_p(X) \leq r$. Since this holds for each $r > 1/q$, we get that $S_p(X) \leq 1/q$, and since this now holds for each $q < M_p(X)$, we have $S_p(X) \leq 1/M_p(X)$. Conversely, let us take some $q > M_p(X)$; then there is $z \in \ell_q \cap B_{c_0}$ and $f \in H_{\infty}(\mathbb{T}^\infty, X)$ such that $\sum a \hat{f}(\alpha) z^\alpha$ does not converge absolutely. By Remark 1.2 we may assume that $z$ is decreasing, and hence $(z_n n^{1/q})_n$ is bounded. We choose now $r > q$ and define $w_n = \frac{1}{p_n^{1/r}}$. By the Prime Number Theorem we know that there is a universal constant $C > 0$ such that

$$0 < \frac{z_n}{w_n} = z_n p_n^{1/r} = z_n n^{1/q} \frac{p_n^{1/r}}{\log n} \leq C z_n n^{\frac{1}{q} (1 - 1/r)}.$$ 

The last term tends to 0 as $n \to \infty$; hence $z_n \leq w_n$ but for a finite number of $n$’s. By Remark 1.2 this implies that $\sum_a \hat{f}(\alpha) w^\alpha$ does not converge absolutely. But then $D = \mathfrak{B}_X f = \sum_n a_n n^{-r} \in \mathcal{H}_p(X)$ satisfies

$$\sum_n \|a_n\|_X \frac{1}{n^{1/r}} = \sum_a \|a_{p^\alpha}\|_X \left(\frac{1}{p^{1/r}}\right)^\alpha = \sum_a \|\hat{f}(\alpha)\|_X w^\alpha = \infty.$$ 

This gives that $\sigma_a(D) \geq 1/r$ for every $r > q$, hence $\sigma_a(D) \geq 1/q$. Since this holds for every $q > M_p(X)$, we finally have $S_p(X) \geq 1/M_p(X)$. 

We shall use standard notation and notions from Banach space theory, as presented, e.g. in [3, 4]. For everything needed on polynomials in Banach spaces see e.g. [11] and [12].
2 Relevant inequalities

The main aim here is to prove a sort of polynomial extension of the notion of cotype. Recall the definition of $C_q(X)$ from (2). Moreover, from Kahane’s inequality we know that, given $1 \leq q < \infty$, there is a (best) constant $K \geq 1$ such that for each Banach space $X$ and each choice finitely many vectors $x_1, \ldots, x_N \in X$

$$\left( \int_{T^N} \left\| \sum_{k=1}^N x_k z_k \right\|_X^2 \, dz \right)^{1/2} \leq K \int_{T^N} \left\| \sum_{k=1}^N x_k z_k \right\|_X \, dz.$$ 

As usual we write $|\alpha| = \alpha_1 + \ldots + \alpha_N$ and $\alpha! = \alpha_1! \ldots \alpha_N!$ for every multi index $\alpha \in \mathbb{N}_0^N$.

**Proposition 2.1.** Let $X$ be a Banach space of cotype $q$, $2 \leq q < \infty$, and

$$P : \mathbb{C}^N \to X, \quad P(z) = \sum_{\alpha \in \mathbb{N}_0^N, |\alpha|=m} c_\alpha z^\alpha$$

be an $m$-homogeneous polynomial. Let

$$T : \mathbb{C}^N \times \ldots \times \mathbb{C}^N \to X, \quad T(z^{(1)}, \ldots, z^{(m)}) = \sum_{i_1, \ldots, i_m=1}^N a_{i_1, \ldots, i_m} z_i^{(1)} \ldots z_i^{(m)}$$

be the unique $m$-linear symmetrization of $P$. Then

$$\left( \sum_{i_1, \ldots, i_m} \left\| a_{i_1, \ldots, i_m} \right\|_X^q \right)^{1/q} \leq (C_q(X) K)^m \frac{m^m}{m!} \int_{T^N} \| P(z) \|_X \, dz.$$ 

Before we give the proof let us note that [?, Theorem 3.2] is an $m$-linear result that, combined with polarization gives (with the previous notation)

$$\left( \sum_{i_1, \ldots, i_m} \left\| a_{i_1, \ldots, i_m} \right\|_X^q \right)^{1/q} \leq C_q(X) m^m \frac{m^m}{m!} \sup_{z \in D^N} \| P(z) \|.$$ 

Our result allows to replace (up to the constant $K$) the $\| \|_\infty$ norm with the smaller norm $\| \|_1$. We prepare the proof of Proposition 2.1 with three lemmas.

**Lemma 2.2.** Let $X$ be a Banach space of cotype $q$, $2 \leq q < \infty$. Then for every $m$-linear form

$$T : \mathbb{C}^N \times \ldots \times \mathbb{C}^N \to X, \quad T(z^{(1)}, \ldots, z^{(m)}) = \sum_{i_1, \ldots, i_m=1}^N a_{i_1, \ldots, i_m} z_i^{(1)} \ldots z_i^{(m)}$$

we have

$$\left( \sum_{i_1, \ldots, i_m=1}^N \left\| a_{i_1, \ldots, i_m} \right\|_X^q \right)^{1/q} \leq (C_q(X) K)^m \int_{T^N} \ldots \int_{T^N} \| T(z^{(1)}, \ldots, z^{(m)}) \|_X \, dz^{(1)} \ldots dz^{(m)}.$$ 

**Proof.** We prove this result by induction on the degree $m$. For $m = 1$ the result is an immediate consequence of the definition of cotype $q$ and Kahane’s inequality. Assume that
the result holds for \( m - 1 \). By the continuous Minkowski inequality we then conclude that for every choice of finitely many vectors \( a_{i_1, \ldots, i_m} \in X \) with \( 1 \leq i_j \leq N, 1 \leq j \leq m \) we have

\[
\sum_{i_1, \ldots, i_m} \| a_{i_1, \ldots, i_m} \|_X^q = \sum_{i_1, \ldots, i_{m-1}} \sum_{i_m} \| a_{i_1, \ldots, i_m} \|_X^q \\
\leq C_q(X)^q K^q \left( \sum_{i_1, \ldots, i_{m-1}} \left( \int_{T \cap \mathbb{R}} \| \sum_{i_m} a_{i_1, \ldots, i_m} z_{i_m}^{(m)} \|_X^q dz^{(m)} \right)^{q/2} \right) \\
\leq C_q(X)^q K^q \left( \sum_{i_1, \ldots, i_{m-1}} \| a_{i_1, \ldots, i_m} \|_X^q \left( \int_{T \cap \mathbb{R}} dz^{(m)} \right)^{q/2} \right) \\
\leq C_q(X)^q K^q \left( \int_{T \cap \mathbb{R}} \sum_{i_1, \ldots, i_{m-1}} a_{i_1, \ldots, i_m} z_{i_1}^{(1)} \ldots, z_{i_{m-1}}^{(m-1)} \|_X dz^{(1)} \ldots dz^{(m-1)} \right)^{q/2},
\]

which is the conclusion.

The following two lemmas are needed to produce a polynomial analog of the preceding result.

**Lemma 2.3.** Let \( X \) be a Banach space, and \( f : \mathbb{C} \to X \) a holomorphic function. Then for \( R_1, R_2, R \geq 0 \) with \( R_1 + R_2 \leq R \) we have

\[
\int_T \int_T \| f(R_1 z_1 + R_2 z_2) \|_X dz_1 dz_2 \leq \int_T \| f(Rz) \|_X dz.
\]

**Proof.** By the rotation invariance of the normalized Lebesgue measure on \( T \) we get

\[
\int_T \int_T \| f(R_1 z_1 + R_2 z_2) \|_X dz_1 dz_2 = \int_T \int_T \| f(R_1 z_1 + R_2 z_2) \|_X dz_1 d z_2 \\
= \int_T \int_T \| f(z_2(R_1 z_1 + R_2) \|_X dz_1 d z_2 = \int_T \int_T \| f(z_2|R_1 z_1 + R_2|) \|_X dz_2 d z_1 \\
= \int_T \int_T \| f(z_2 r(z_1) R) \|_X dz_2 d z_1 = \int_0^{2\pi} \int_0^{2\pi} \| f(r e^{i\theta} R e^{i\phi}) \|_X \frac{dt}{2\pi} \frac{ds}{2\pi}.
\]

where \( r(z) = \frac{1}{2\pi} |R_1 z + R_2|, z \in T \). We know that for each holomorphic function \( h : \mathbb{C} \to X \) we have

\[
\int_T \| h(z) \|_X dz = \sup_{0 \leq r \leq 1} \int_0^{2\pi} \| h(re^{i\theta}) \|_X \frac{dt}{2\pi}
\]

(see e.g. Blasco and Xu [2, p. 338]). Define now \( h(z) = f(Rz) \), and note that \( 0 \leq r(z) \leq 1 \) for all \( z \in T \). Then

\[
\int_T \int_T \| f(R_1 z_1 + R_2 z_2) \|_X dz_1 dz_2 = \int_0^{2\pi} \int_0^{2\pi} \| h(re^{i\theta} e^{i\phi}) \|_X \frac{dt}{2\pi} \frac{ds}{2\pi} \\
\leq \int_0^{2\pi} \int_T \| h(z) \|_X d z \frac{ds}{2\pi} = \int_T \| f(Rz) \|_X dz.
\]

This completes the proof. \( \square \)

A sort of iteration of the preceding result leads to the next
Lemma 2.4. Let $X$ be a Banach space, and $f : \mathbb{C}^N \to X$ a holomorphic function. Then for every $m$

$$
\int_{T^N} \cdots \int_{T^N} \| f(z^{(1)} + \ldots + z^{(m)}) \|_X \, dz^{(1)} \ldots dz^{(m)} \leq \int_{T^N} \| f(m z) \|_X \, dz.
$$

Proof. We fix some $m$, and do induction with respect to $N$. For $N = 1$ we obtain from Lemma 2.3 that

$$
\int_{T} \cdots \int_{T} \int_{T} \int_{T} \| f(z^{(1)} + \ldots + z^{(m-2)} + z^{(m-1)} + z^{(m)}) \|_X \, dz^{(m-1)} \, dz^{(m)} \, dz^{(1)} \ldots dz^{(m-2)} = \int_{T} \cdots \int_{T} \int_{T} \int_{T} \| f(z^{(1)} + \ldots + z^{(m-2)} + 2w) \|_X \, dw \, dz^{(1)} \ldots dz^{(m-2)}
$$

$$
= \int_{T} \cdots \int_{T} \int_{T} \int_{T} \| f(z^{(1)} + \ldots + z^{(m-3)} + 3w) \|_X \, dz^{(1)} \ldots dz^{(m-3)} \, dw
$$

$$
\leq \ldots \leq \int_{T} \| f(m z) \|_X \, dz.
$$

We now assume that the conclusion holds for $N - 1$ and write each $z \in T^N$ as $z = (u, w)$, with $u \in T^{N-1}$ and $w \in T$. Then, using the case $N = 1$ in the first inequality and the inductive hypothesis in the second, we have

$$
\int_{T^N} \cdots \int_{T^N} \| f(z^{(1)} + \ldots + z^{(m)}) \|_X \, dz^{(1)} \ldots dz^{(m)}
$$

$$
= \int_{T^N} \cdots \int_{T^N} \left( \int_{T} \cdots \int_{T} \| f(u^{(1)}, w_1) + \ldots + (u^{(m)}, w_m) \|_X \, dw_1 \ldots dw_N \right) \, du^{(1)} \ldots du^{(m)}
$$

$$
\leq \int_{T^N} \cdots \int_{T^N} \left( \int_{T} \cdots \int_{T} \| f((u^{(1)}, mw) + \ldots + (u^{(m)}, mw)) \|_X \, du \right) \, du^{(1)} \ldots du^{(m)}
$$

$$
= \int_{T} \left( \int_{T} \cdots \int_{T} \| f((u^{(1)}, mw) + \ldots + (u^{(m)}, mw)) \|_X \, du \right) \, dw
$$

$$
= \int_{T^N} \| f(mz) \|_X \, dz,
$$

as desired. $\square$

We are now ready to give the proof of the inequality from Proposition 2.1. By the polarization formula we know that for every choice of $z^{(1)}_1, \ldots, z^{(m)}_m \in T^N$ we have

$$
T(z^{(1)}_1, \ldots, z^{(m)}_m) = \frac{1}{2^m m!} \sum_{\varepsilon_i = \pm 1} \varepsilon_i \ldots \varepsilon_m p \left( \sum_{i=1}^N \varepsilon_i z^{(i)}_i \right)
$$
(see e.g [11] or [12]). Hence we deduce from Lemma 2.4
\[
\int_{\mathbb{T}^N} \cdots \int_{\mathbb{T}^N} \left\| T(z^{(1)}, \ldots, z^{(m)}) \right\|_X d\tilde{z}^{(1)} \cdots d\tilde{z}^{(m)} \\
\leq \frac{1}{2^m m!} \sum_{\ell_i=\pm 1} \int_{\mathbb{T}^N} \cdots \int_{\mathbb{T}^N} \left\| \sum_{i=1}^N \varepsilon_i z^{(i)} \right\|_X d\tilde{z}^{(1)} \cdots d\tilde{z}^{(m)} \\
= \frac{1}{2^m m!} \sum_{\ell_i=\pm 1} \int_{\mathbb{T}^N} \cdots \int_{\mathbb{T}^N} \left\| \sum_{i=1}^N z^{(i)} \right\|_X d\tilde{z}^{(1)} \cdots d\tilde{z}^{(m)} \\
= \frac{1}{m!} \int_{\mathbb{T}^N} \cdots \int_{\mathbb{T}^N} \left\| \sum_{i=1}^N z^{(i)} \right\|_X d\tilde{z}^{(1)} \cdots d\tilde{z}^{(m)} \\
\leq \frac{1}{m!} \int_{\mathbb{T}^N} \left\| P(mz) \right\|_X d\tilde{z} = \frac{m^m}{m!} \int_{\mathbb{T}^N} \left\| P(z) \right\|_X d\tilde{z}.
\]
Then by Lemma 2.2 we obtain
\[
\left( \sum_{i_1, \ldots, i_m} \left\| a_{i_1, \ldots, i_m} \right\|_X^q \right)^{1/q} \leq \left( C_q(X) K \right)^m \int_{\mathbb{T}^N} \cdots \int_{\mathbb{T}^N} \left\| T(z^{(1)}, \ldots, z^{(m)}) \right\|_X d\tilde{z}^{(1)} \cdots d\tilde{z}^{(m)} \\
= \left( C_q(X) K \right)^m \frac{m^m}{m!} \int_{\mathbb{T}^N} \left\| P(z) \right\|_X d\tilde{z},
\]
which completes the proof of Proposition 2.1. □

A second proposition is needed which allows to reduce the proof of our main result 1.1 to the homogeneous case. It is a vector-valued version of a result of [6, Theorem 9.2] with a similar proof (here only given for the sake of completeness).

**Proposition 2.5.** There is a contractive projection
\[
\Phi_m : H_p(\mathbb{T}^N, X) \to H_p(\mathbb{T}^N, X), \ f \mapsto \hat{f}_m,
\]
such for all \( f \in H_p(\mathbb{T}^N, X) \)
\[
\hat{f}(\alpha) = \hat{f}_m(\alpha) \text{ for all } \alpha \in \mathbb{N}_0^N \text{ with } |\alpha| = m.
\]

**Proof.** Let \( \mathcal{D}(\mathbb{C}^N, X) \subset H_p(\mathbb{T}^N, X) \) be the subspace all finite polynomials \( f = \sum_{\alpha \in \Lambda} c_{\alpha} z^{\alpha} \); here \( \Lambda \) is a finite set of multi indices in \( \mathbb{N}_0^N \) and the coefficients \( c_{\alpha} \in X \). Define the linear projection \( \Phi_m^0 \) on \( \mathcal{D}(\mathbb{C}^N, X) \) by
\[
\Phi_m^0(f)(z) = f_m(z) = \sum_{\alpha \in \Lambda, |\alpha| = m} \hat{f}(\alpha) z^{\alpha};
\]
clearly, we have (9). In order to show that \( \Phi_m^0 \) is a contraction on \( \mathcal{D}(\mathbb{C}^N, X), \| \cdot \|_p \) fix some function \( f \in \mathcal{D}(\mathbb{C}^N, X) \) and \( z \in \mathbb{T}^N \), and define
\[
f(z) : \mathbb{T} \to X, \ w \mapsto f(zw).
\]
Clearly, we have
\[
f(zw) = \sum_k f_k(z) w^k,
\]
and hence

\[ f_m(z) = \int_T f(zw) w^{-m} \, dw. \]

Integration, the continuous Minkowski inequality and the rotation invariance of the normalized Lebesgue measure on \( T^N \) give

\[
\int_{T^N} \left\| f_m(z) \right\|^p_X \, dz = \int_{T^N} \left\| f(zw) w^{-m} \right\|^p_X \, dw \, dz \\
\leq \int_{T^N} \left( \int_T \left\| f(zw) \right\|^p_X \, dw \right)^p \, dz \leq \int_T \int_{T^N} \left\| f(zw) \right\|^p_X \, dz \, dw = \int_{T^N} \left\| f(z) \right\|^p_X \, dz,
\]

which proves that \( \Phi_m^0 \) is a contraction on \( (\mathcal{P}(\mathbb{C}^N, X), \| \cdot \|_p) \). By Fejer’s theorem (vector-valued) we know that \( \mathcal{P}(\mathbb{C}^N, X) \) is a dense subspace of \( H_p(T^N, X) \). Hence \( \Phi_m^0 \) extends to a contractive projection \( \Phi_m \) on \( H_p(T^N, X) \). This extension \( \Phi_m \) still satisfies (9) since for each multi index \( \alpha \) the mapping \( H_p(T^N, X) \to X, f \mapsto \hat{f}(\alpha) \) is continuous.

\[ \square \]

3 Proof of the main result

We are now ready to prove Theorem 1.1. Let \( 1 \leq p < \infty \), and recall from (1) that

\[ 1 - \frac{1}{\text{Cot}(X)} = S_\infty(X) \leq S_p(X); \]

see Remark 3.1 for a direct argument. Hence it suffices to concentrate on the upper estimate in Theorem 1.1: Since we obviously have \( S_p(X) \leq S_1(X) \), we are going to prove that

\[ S_1(X) \leq 1 - \frac{1}{\text{Cot}(X)}. \quad (10) \]

Suppose first that \( X \) has no finite cotype. For \( D = \sum_n a_n n^{-s} \in \mathcal{H}_1(X) \) we take \( f \in H_1(T^\infty, X) \) with \( D = \mathfrak{B}_X f \). Note that

\[ |\hat{f}(\alpha)| \leq \int_{T^\infty} |f(w) w^{-\alpha}| \, dw = \| f \|_{L_1(T^\infty, X)} < \infty \]

and, by the definition of \( \mathfrak{B}_x \), the coefficients of \( D \) are also bounded by \( \| f \|_{L_1(T^\infty, X)} \). As a consequence,

\[ \sum_{n=1}^{\infty} \| a_n \|_X \, \frac{1}{n^s} \leq \sum_{n=1}^{\infty} \| f \|_{L_1(T^\infty, X)} \, \frac{1}{n^s} < \infty \]

whenever \( \text{Re} \, s > 1 \). This means that \( S_1(X) \leq 1 \) and gives (10) for \( \text{Cot}(X) = \infty \).

Now if \( X \) has finite cotype, take \( q > \text{Cot}(X) \) and \( \epsilon > 0 \), and put \( s = (1 - \frac{1}{q})(1 + 2\epsilon) \).

Choose an integer \( k_0 \) such \( p_{k_0}^{\epsilon/q} > eC_q(X) K \sum_{j=1}^{\infty} \frac{1}{p_j^{1+\epsilon}} \), and define

\[ \bar{p} = (p_0, \ldots, p_{k_0}, p_{k_0+1}, p_{k_0+2}, \ldots). \]
We are going to show that there is a constant \( C(q, X, \varepsilon) > 0 \) such that for every \( f \in H_1(\mathbb{T}^\infty, X) \) we have
\[
\sum_{\alpha \in \mathbb{N}_0^q} \| \hat{f}(\alpha) \|_X \frac{1}{\beta^{\varepsilon\alpha}} \leq C(q, X, \varepsilon) \| f \|_{H_1(\mathbb{T}^\infty, X)}.
\] (11)

This finishes the argument: By Remark 1.2 the sequence \( 1/p^s \in \text{mon} H_1(\mathbb{T}^\infty, X) \). But in view of Bohr's transform from (6), this means that for every Dirichlet series \( D = \sum_n a_n n^{-s} = \mathcal{B}_X f \in \mathcal{H}_1(X) \) with \( f \in H_1(\mathbb{T}^\infty, X) \) we have
\[
\sum_{n=1}^\infty \| a_n \|_X \frac{1}{n^s} = \sum_{\alpha \in \mathbb{N}_0^q} \| \hat{f}(\alpha) \|_X \frac{1}{\beta^{\varepsilon\alpha}} < \infty.
\]

Therefore \( \sigma_q(D) \leq (1 - \frac{1}{q}) (1 + 2 \varepsilon) \) for each such \( D \) which, since \( \varepsilon > 0 \) was arbitrary, is what we wanted to prove.

It remains to check (11); the idea is to show first that (11) holds for all \( X \)-valued \( H_1 \)-functions which only depend on \( N \) variables: There is a constant \( C(q, X, \varepsilon) > 0 \) such that for all \( N \) and every \( f \in H_1(\mathbb{T}^N, X) \) we have
\[
\sum_{\alpha \in \mathbb{N}_0^q} \| \hat{f}(\alpha) \|_X \frac{1}{\beta^{\varepsilon\alpha}} \leq C(q, X, \varepsilon) \| f \|_{H_1(\mathbb{T}^\infty, X)}.
\] (12)

In order to understand that (12) implies (11) (and hence the conclusion), assume that (12) holds and take some \( f \in H_1(\mathbb{T}^\infty, X) \). Given an arbitrary \( N \), define
\[
f_N : \mathbb{T}^N \to X, \quad f_N(u) = \int_{\mathbb{T}^\infty} f(w, \bar{w}) d\bar{w}.
\]
Then it can be easily shown that \( f_N \in L_1(\mathbb{T}^N, X), \| f_N \|_1 \leq \| f \|_1 \), and \( \hat{f}_N(\alpha) = \hat{f}(\alpha) \) for all \( \alpha \in \mathbb{Z}^N \). If we now apply (12) to this \( f_N \), we get
\[
\sum_{\alpha \in \mathbb{N}_0^q} \| \hat{f}(\alpha) \|_X \frac{1}{\beta^{\varepsilon\alpha}} \leq C(q, X, \varepsilon) \| f \|_{H_1(\mathbb{T}^\infty, X)},
\]
which, after taking the supremum over all possible \( N \) on the left side, leads to (11).

We turn to the proof of (12), and here in a first step will show the following: For every \( N \), every \( m \)-homogeneous polynomial \( P : \mathbb{C}^N \to X \) and every \( u \in \ell_q \), we have
\[
\sum_{\alpha \in \mathbb{N}_0^q, |\alpha| = m} \| \hat{P}(\alpha) u^\alpha \|_X \leq (e C_q(X) K)^m \int_{\mathbb{T}^N} \| P(z) \|_X dz \left( \sum_{j=1}^\infty |u_j|^q \right)^{m/q}.
\] (13)

Indeed, take such a polynomial \( P(z) = \sum_{\alpha \in \mathbb{N}_0^q, |\alpha| = m} \hat{P}(\alpha) z^\alpha, \ z \in \mathbb{T}^N \), and look at its unique \( m \)-linear symmetrization
\[
T : \mathbb{C}^N \times \ldots \times \mathbb{C}^N \to X, \ T(z^{(1)}, \ldots, z^{(m)}) = \sum_{i_1, \ldots, i_m=1}^N a_{i_1, \ldots, i_m} z^{(1)}_{i_1} \ldots z^{(m)}_{i_m}.
\]
Then we know from Proposition 2.1 that
\[
\left( \sum_{i_1,\ldots,i_m} a_{i_1,\ldots,i_m} \right)^{1/q} \leq \left( eC_q(X) K \right)^m \int_{T^N} \| P(z) \|_X \, dz.
\]
Hence (13) follows by Hölder’s inequality:
\[
\sum_{\alpha \in \mathbb{N}_0^N, |\alpha| = m} \| \hat{P}(\alpha) u^\alpha \|_X = \sum_{i_1,\ldots,i_m=1}^N \| a_{i_1,\ldots,i_m} \|_X |u_{i_1} \cdots u_{i_N}|
\leq \left( eC_q(X) K \right)^m \int_{T^N} \| P(z) \|_X \, dz \left( \sum_{j=1}^\infty |u_j|^q \right)^{1/q}.
\]

We finally give the proof of (12): Take \( f \in H_1(T^N, X) \), and recall from Proposition 2.5 that for each integer \( m \) there is an \( m \)-homogeneous polynomial \( P_m : \mathbb{C}^N \to X \) such that \( \| P_m \|_{H_1(T^N, X)} \leq \| f \|_{H_1(T^N, X)} \) and \( \hat{P}_m(\alpha) = \hat{f}(\alpha) \) for all \( \alpha \in \mathbb{N}_0^N \) with \( |\alpha| = m \). Finally, from (13), the definition of \( s \), and the fact that \( \max(\{p_k, p_j \}) \leq \hat{p}_j \) for all \( j \) we conclude that
\[
\sum_{\alpha \in \mathbb{N}_0^N} \| \hat{f}(\alpha) \|_X \frac{1}{\hat{p}^{s\alpha}} = \sum_{m=1}^\infty \sum_{\alpha \in \mathbb{N}_0^N, |\alpha| = m} \| \hat{P}_m(\alpha) \|_X \frac{1}{\hat{p}^{s\alpha}}
\leq \sum_{m=1}^\infty \left( eC_q(X) K \right)^m \| P_m \|_{H_1(T^N, X)} \left( \sum_{j=1}^\infty \frac{1}{\hat{p}_j} \right)^{m/q'}
= \sum_{m=1}^\infty \left( eC_q(X) K \right)^m \| f \|_{H_1(T^N, X)} \left( \sum_{j=1}^\infty \frac{1}{\hat{p}_j^{1+\varepsilon}} \right)^{m/q'}
= \sum_{m=1}^\infty \left( eC_q(X) K \right)^m \| f \|_{H_1(T^N, X)} \left( \sum_{j=1}^\infty \frac{1}{\hat{p}_j^{1+\varepsilon}} \frac{1}{\hat{p}_j^{\varepsilon/q'}} \right)^{1/q'}
\leq \| f \|_{H_1(T^N, X)} \sum_{m=1}^\infty \left( \frac{eC_q(X) K \left( \sum_{j=1}^\infty \frac{1}{\hat{p}_j^{1+\varepsilon}}\right)^{1/q'}}{\hat{p}_k^{\varepsilon/q'}} \right)^{1/q'}.
\]

This completes the proof of Theorem 1.1. □

Remark 3.1. We end this note with a direct proof of the fact
\[ 1 - \frac{1}{\text{Cot}(X)} \leq S_p(X), \quad 1 \leq p < \infty \] (14)
in which we do not use the inequality
\[ 1 - \frac{1}{\text{Cot}(X)} \leq S_\infty(X) \] (15)
from [8] (here repeated in (1)). The proof of (15) given in [8] in a first step shows that
\[ 1 - \frac{1}{\Pi(X)} \leq S_\infty(X) \]
where
\[ \Pi(X) = \inf \{ r \geq 2 | \text{id}_X \text{ is } (r, 1) - \text{summing} \}. \]
and then, in a second step, applies a fundamental theorem of Maurey and Pisier stating that \( \Pi(X) = \operatorname{Cot}(X) \).

The following argument for (14) is very similar to the original one from [8] but does not use the Maurey-Pisier theorem (since we here consider \( \mathcal{H}_p(X), 1 \leq p < \infty \) instead of \( \mathcal{H}_\infty(X) \)): By the proof of Corollary 1.3, inequality (14) is equivalent to

\[
M_p(X) \leq \frac{\operatorname{Cot}(X)}{\operatorname{Cot}(X) - 1}.
\]

Take \( r < M_p(X) \), so that \( \ell_r \cap B_{\ell_r} \subset \text{mon} \mathcal{H}_p(\mathbb{T}^\infty, X) \). Let \( H^1_p(\mathbb{T}^\infty, X) \) be the subspace of \( \mathcal{H}_p(\mathbb{T}^\infty, X) \) formed by all 1-homogeneous polynomials (i.e., linear operators). We can define a bilinear operator \( \ell_r \times H^1_p(\mathbb{T}^\infty, X) \to \ell_1(X) \) by \( (z, f) \mapsto (z_j f(e_j))_j \) which, by a closed graph argument, is continuous. Therefore, there is a constant \( M \) such that for all \( z \in \ell_r \) and all \( f \in H^1_p(\mathbb{T}^\infty, X) \) we have

\[
\sum_j |z_j| \|f(e_j)\|_X \leq M \|z\|_{\ell_r} \|f\|_{\mathcal{H}_p(\mathbb{T}^\infty, X)}.
\]

Taking the supremum over all \( z \in B_{\ell_r} \), we obtain for all \( f \in H^1_p(\mathbb{T}^\infty, X) \)

\[
\left( \sum_j \|f(e_j)\|_X^{r'} \right)^{1/r'} \leq M \|f\|_{\mathcal{H}_p(\mathbb{T}^\infty, X)}.
\]

Now, take \( x_1, \ldots, x_N \in X \) and define \( f \in H^1_p(\mathbb{T}^\infty, X) \) by \( f(e_j) = x_j \) if \( 1 \leq j \leq N \), \( f(e_j) = 0 \) if \( j > N \) and extend it by linearity. By the previous inequality and Lemma 2.5 we have

\[
\left( \sum_{j=1}^N \|x_j\|_X^{r'} \right)^{1/r'} \leq M \left( \int_{\mathbb{T}^N} \left\| \sum_{j=1}^N x_j z_j \right\|_X^{r'} dz \right)^{1/r'}.
\]

By Kahane’s inequality, \( X \) has cotype \( r' \), which means that \( r' > \operatorname{Cot}(X) \) or, equivalently, \( r < \frac{\operatorname{Cot}(X)}{\operatorname{Cot}(X) - 1} \). Since \( r < M_p(X) \) was arbitrary, we obtain (14).

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