Research Article

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Marangoni convection in layers of water-based nanofluids under the effect of rotation

https://doi.org/10.1515/math-2021-0073
received April 22, 2021; accepted July 7, 2021

Abstract: A linear stability analysis is performed for the onset of Marangoni convection in a horizontal layer of a nanofluid heated from below and affected by rotation. The top boundary of the layer is assumed to be impenetrable to nanoparticles with their distribution being determined from a conservation condition while the bottom boundary is assumed to be a rigid surface with fixed temperature. The motion of the nanoparticles is characterized by the effects of thermophoresis and Brownian diffusion. A modification model is used in which the effects of Brownian diffusion and thermophoresis are taken into consideration by new expressions in the nanoparticle mass flux. Also, material properties of the nanofluid are modelled by non-constant constitutive expressions depending on nanoparticle volume fraction. The steady-state solution is shown to be well approximated by an exponential distribution of the nanoparticle volume fraction. The Chebyshev-Tau method is used to obtain the critical thermal and nanoparticle Marangoni numbers. Different stability boundaries are obtained using the modified model and the rotation.

Keywords: linear stability, Brownian motion, rotation, thermophoresis, Chebyshev method

MSC 2020: 76E06, 76E09, 76E25

1 Introduction

The last decade witnessed great interest in nanofluids due to their wide applications in science and engineering because of their thermal and mechanical properties. This type of fluid appears to increase the heat transfer performance significantly in comparison to ordinary fluids. Nanofluids are not obtained naturally but are synthesized in laboratories and consist of base fluids, such as water and organic solvents, in addition to metallic or metallic oxide nanoparticles with a maximum diameter of 100 nm.

Buongiorno [1] proposed a model for convective transport in nanofluids, which combines the effect of Brownian motion and thermophoresis. His model is used by many researchers to study the onset of thermal instability of a nanofluid layer. For example, Tzou [2,3] showed that the presence of nanoparticles in a base fluid could reduce the threshold of the Rayleigh-Benard instability, and Nield and Kuzentsov [4,5] studied the onset of convection in a horizontal nanofluid layer under several effects including the effect of porous medium. The onset of Rayleigh-Benard convection in nanofluids under the effect of rotation is studied by Yadav et al. [6]. Gupta et al. [7] studied the onset of Rayleigh-Benard convection in nanofluids under the effect of magnetic field. Shivakumara and Dhananjaya [8] studied the penetrative Brinkman convection in an anisotropic porous layer saturated by a nanofluid and Abdullah and Lindsay [9,10] investigated the onset of Marangoni convection in a layer of nanofluid.

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The onset of Marangoni convection in a base fluid is induced by the dependence of the surface tension on temperature. Pearson [11] studied the stability of Marangoni convection of a layer of base fluid heated from below and showed that instability is caused by surface tension and not by the buoyancy forces. Nield [12] demonstrated that the effect of buoyancy forces can be neglected if the fluid layer depth is less than 1 mm. The results of Pearson [11] and Nield [12] have been expanded and refined by many researchers for deformable and non-deformable surfaces (e.g., Takashima [13,14], Benguria and Depassier [15], Wilson [16], Shivakumara et al. [17], Hashim and Arifin [18], and Shivakumara et al. and [19]).

Previous studies in the thermal stability of nanofluid layers assume that material properties such as effective thermal conductivity and effective viscosity behave as constant functions across the nanofluid layer. Abdullah et al. [20] studied the stability of Marangoni convection in a layer of nanofluid using a modification model of the nanoparticle mass flux in which the effects of Brownian diffusion and thermophoresis are taken into consideration. Also, they assumed that material properties of the nanofluid layer such as effective thermal conductivity and effective viscosity are modelled by non-constant constitutive expressions which depend on temperature and nanoparticle volume fraction.

The literature enriched also by the study of Brownian motions of a particle in a fluid (see e.g. [21–23]) revealed that no research has been conducted regarding the stability of Marangoni convection in a nano-fluid layer under the effect of rotation. Therefore, the object of the present study is to investigate this problem for layers of distilled water (DW)/alumina and DW/cupric oxide nanofluids using a modified model with material properties which are modelled by non-constant constitutive expressions that depend on nanoparticle volume fraction.

The numerical calculations will adopt the models of Khanafar and Vafai [24] for the effective dynamic viscosity and effective thermal conductivity of these nanofluids. Other models will be adopted for comparison such as Hamilton and Crosser’s model [25] for the effective thermal conductivity and Brinkman’s model [26] for the effective dynamic viscosity.

The work is set out as follows. Section 2 formulates the problem, introduces the field equations that describe the model, constructs the steady state solution and produces the non-dimensional linearized equations and boundary conditions. Section 3 establishes the eigenvalue problem to be solved. Section 4 formulates the normal mode analysis. Section 5 describes the numerical procedure used to treat the eigenvalue problem. Section 6 presents results and compares stability boundaries based on the classical model of convection under the effect of rotation and the general model of convection under the effect of rotation in which constitutive functions are assumed to be non-constants. Section 7 concludes.

2 Field equations

Following the work of Tzou [2,3] and Nield and Kuznetsov [4,5], for the motion of a nanofluid in the absence of chemical reactions leads to the development of the following field equations:

\[ \frac{\partial \psi^i}{\partial x_j} = 0, \quad (1) \]

\[ \rho_{\text{eff}}(\psi^* \left( \frac{\partial \psi^i}{\partial t^*} + v_k^* \frac{\partial \psi^i}{\partial x_k^*} \right) = - \frac{\partial P^*}{\partial x_j^*} + \frac{\partial}{\partial x_j^*} \left[ \mu_{\text{eff}}(\psi^*) \left( \frac{\partial \psi^i}{\partial x_j^*} + \frac{\partial \psi^j}{\partial x_i^*} \right) \right] + 2(\psi^* \times \Omega), \quad (2) \]

\[ \frac{\partial \psi^*}{\partial t^*} + v_k^* \frac{\partial \psi^*}{\partial x_k^*} = - \frac{1}{\rho_p} \frac{\partial f_i^*}{\partial x_i^*}, \quad (3) \]

\[ (\rho c)_{\text{eff}}(\psi^*) \left( \frac{\partial T^*}{\partial t^*} + v_k^* \frac{\partial T^*}{\partial x_k^*} \right) = - \frac{\partial q_k^*}{\partial x_k^*} + h_p(T^*) \frac{\partial f_i^*}{\partial x_i^*}, \quad (4) \]
where \( P^* = p + \frac{1}{2}(\Omega \times \nu^*)^2 \) is the sum of hydrostatic pressure and centrifugal forces, \( v^*_j \) is the velocity, \( J^*_k \) is the mass flux of nanoparticles within the nanofluid, \( q^*_k \) is the heat flux, \( \mu^*_j \) is the effective dynamic viscosity and the quantities \( \rho^*_j \) and \( \rho^*_f \) are the effective specific heats and density of nanofluid. The density and specific heat of the nanoparticle material, \( (\rho^*_p, c^*_p) \), and the base fluid, \( (\rho^*_f, c^*_f) \), are related to \( \rho^*_j \) and \( \rho^*_f \), by the expressions

\[
\rho^*_j = \varphi^* \rho^*_p + (1 - \varphi^*) \rho^*_f,
\]

\[
(pc)_j = \varphi^* \rho^*_p c^*_p + (1 - \varphi^*) \rho^*_f c^*_f.
\]

Equations (1)–(4) are affected by Buongiorno’s assumption, which resides in the specification of the nanofluid mass flux with constitutive formula

\[
J^*_k = -\rho^*_p \left( C_B T^* \frac{\partial \varphi^*}{\partial x^*_k} + \varphi^* \frac{C_T}{T^*} \frac{\partial T^*}{\partial x^*_k} \right),
\]

where the first component describes Brownian diffusion of nanoparticles and the second component describes the influence of thermophoresis. \( C_B \) and \( C_T \) are two parameters that have the constitutive specifications

\[
C_B = \frac{k}{3 \pi \mu d_p}, \quad C_T = \frac{H_f}{\rho_p \left( \frac{0.26 \kappa_f}{2 \kappa_f + \kappa_p} \right)},
\]

where \( k \) is the Boltzmann constant, \( d_p \) is the nanoparticle diameter, \( \mu_f \) is the dynamic viscosity of the base fluid and \( \kappa_f \) and \( \kappa_p \) are, respectively, the thermal conductivities of the base fluid and the nanoparticle material. The heat flux vector \( q^*_k = q^*_k \mathbf{e}_k \) in equation (4) has the constitutive expression:

\[
q^*_k = -\kappa^*_k \left( \frac{\partial T^*}{\partial x^*_k} + h_p(T^*) J^*_k \right),
\]

where \( \kappa^*_k \) is the effective thermal conductivity of nanofluid and \( h_p(T^*) \) is the specific enthalpy of nanoparticle material. Substituting this expression into the energy equation, (4), we get

\[
(pc) \left( \frac{\partial T^*}{\partial t^*} + v^*_k \frac{\partial T^*}{\partial x^*_k} \right) = \frac{\partial}{\partial x^*_k} \left( \kappa^*_k \left( \frac{\partial T^*}{\partial x^*_k} \right) \right) - c^*_p \frac{\partial T^*}{\partial x^*_k} J^*_k,
\]

where \( c^*_p = \frac{\partial h^*_p}{\partial T^*} \).

### 2.1 Boundary conditions

According to the problem formulation, the nanofluid is assumed to rest on a rigid boundary, which is maintained at a constant temperature \( T_0 \) at \( x^*_3 = 0 \), whereas on \( x^*_3 = d \), the surface is non-deformable and loses heat by convection to an environment at temperature \( T_{ext} \). These conditions are embodied in the equations:

\[
T^*(t^*; x^*_0, 0) = T_0,
\]

\[
q^*_k(t^*; x^*_0, d) = R(T^*(t^*; x^*_0, d) - T_{ext}),
\]

where \( R \) is a constant that represents a heat transfer coefficient and \( \beta \) takes the values 1 and 2. Mechanical conditions require the fluid velocity on the boundary \( x^*_3 = 0 \) to be zero and taking into account the influence of surface tension; we assume the boundary \( x^*_3 = d \) to be flat and free of shear stress. These conditions are brought together into the conditions

\[
v^*_k(t^*; x^*_3, 0) = 0; \quad k = (1, 2, 3),
\]
\[ \nu_j(t; x_\beta, d) = 0, \]
\[ \sigma_{\beta 3}(t; x_\beta, d) = \frac{\partial y}{\partial x_\beta}, \]
where \( \sigma_{31} \) and \( \sigma_{32} \) are the shear components of nanofluid stress in the plane \( x_3 = d \). At the boundary \( x_3 = d \), the condition (14) can simplify the surface tension boundary condition (15) to
\[ \rho_\text{eff} \left( \frac{\partial \nu_j(t; x_\beta, d)}{\partial x_3} \right) = \frac{\partial \nu_j}{\partial x_3} + \frac{\partial T^*}{\partial x_3}, \]
where \( \varphi^* \) and \( T^* \) are evaluated on \( x_3 = d \). The no slip condition on a rigid boundary implies that the horizontal components of the fluid velocity and all of \( x_1, x_2 \) partial derivatives of each component of the fluid velocity vanish, i.e., \( \nu_1^* = \nu_2^* = 0 \) and from equation (1) it can be obtained that
\[ \frac{\partial \nu_i^*}{\partial x_i} = 0. \]
If we introduce \( \xi_i^* = \varepsilon_{ijk} v_{k,i}^* \) to be the fluid vorticity, then
\[ \xi_j^*(t; x_\beta, 0) = 0, \quad \frac{\partial \xi_j^*}{\partial x_3} = 0. \]
Finally, the confining boundaries \( x_1^* = 0 \) and \( x_3^* = d \) are assumed to be impervious to the flow of nanoparticles and so the appropriate boundary conditions require that fluxes of nanoparticles crossing both boundaries are zero. Consequently,
\[ \int_{x_0^*}^{x_1^*} (t^*; x_\beta, 0) = \int_{x_0^*}^{x_1^*} (t^*; x_\beta, d) = 0, \]
since the constraint that
\[ \lim_{|x_1^*| \to \infty} \frac{1}{\mathcal{A}} \int_{\mathcal{A}} \left( \frac{1}{d} \int_0^d \varphi^*\left( t^*; x_\beta, x_3^* \right) dx_3^* \right) dx_1^* = \varphi, \]
where \( \varphi \) is the average volume fraction of nanoparticles across the layer and \( \mathcal{A} \) is an arbitrary region of the \((x_1^*, x_3^*)\) plane enclosing area \(|\mathcal{A}|\).

### 2.2 Models to be investigated

This problem will be investigated using two models:

**Model I.** It is the usual classical model in which the effective thermal conductivity and effective dynamic viscosity behave as parameters and the nanoparticle flux has the form:
\[ \Re = -\rho_\text{f} \left( C_{\text{eff}} \frac{\partial \varphi^*}{\partial x_3} + \varphi^* \frac{C_\varphi}{T_0} \frac{\partial T^*}{\partial x_3} \right). \]

**Model II.** In which the effective dynamic viscosity and effective thermal conductivity are non-constant functions of nanoparticle volume fraction, \( \varphi^* \), and the nanoparticle mass flux has the form:
\[ \int_{x_0^*}^{x_1^*} (t^*; x_\beta, 0) = \int_{x_0^*}^{x_1^*} (t^*; x_\beta, d) = 0, \]
2.3 Steady state

Equations (1)–(3) and (10) have a steady state solution in which the nanofluid is at rest, the diffusion mass flux is zero, the temperature is \( T(x_3^*) \) and the volume fraction of nanoparticles is \( \Phi(x_3^*) \).

Model I. In this case, we can show that

\[
T(x_3^*) = T_0 + \frac{\text{Nu}(T_{\text{ext}} - T_0)}{\text{Nu} + (\kappa_{\text{eff}}/\kappa_f)} \frac{x_3^*}{d}, \quad \Phi(x_3^*) = \Phi_0 + \frac{\eta \Phi}{T_0} \frac{\text{Nu}(T_0 - T_{\text{ext}})}{\text{Nu} + (\kappa_{\text{eff}}/\kappa_f)} \frac{x_3^*}{d},
\]

where \( \eta = C_\gamma/(C_{\text{top}}) \), \( \Phi_0 \) is the volume fraction of nanoparticle on the boundary \( x_3^* = 0 \) with value determined from conservation of nanoparticles and the Nusselt number is \( \text{Nu} = Rd/\kappa_f \). In this model, the temperature gradient and the gradient of the volume fraction of nanoparticles are constants and lead to the distributions described in equations (23). These equations show that both temperature and nanoparticle volume fraction are well described by linear functions of depth. Hence, in this model the steady-state solution is well approximated by a linear distribution of temperature and nanoparticle volume fraction.

Model II. The solution of temperature and the volume fraction of nanoparticles in the steady state assume that \( \kappa_{\text{eff}} \) is a function of \( \Phi \). The steady state volume fraction \( \Phi(x_3^*) \) and temperature \( T(x_3^*) \) satisfy the equations

\[
J_3 = -\rho_j \left( C_p T \frac{d\Phi}{dx_3^*} + \Phi \frac{C_p T}{T} \frac{dT}{dx_3^*} \right) = 0, \quad \kappa_{\text{eff}}(\Phi) \frac{dT}{dx_3^*} = R(T_d - T_{\text{ext}}) = 0
\]

with the boundary condition \( T(0) = T_0, T_d = T(d) \) along with the constraint condition

\[
\frac{1}{d} \int_0^d \Phi dx_3^* = \bar{\Phi}.
\]

Using the non-dimensional spatial variable, \( x_3 = x_3^*/d \), we can show that

\[
\frac{\kappa_{\text{eff}}(\Phi)}{\kappa_f} \frac{dT}{dx_3} + \text{Nu}(T_d - T_{\text{ext}}) = 0,
\]

\[
\Phi = \Phi_0 \exp \left( \frac{\eta(T_0 - T)}{T} \right).
\]

Equation (27) shows that the nanoparticle volume fraction is well approximated by an exponential distribution. Abdullah et al. [20] solved equations (26) and (27) numerically for \( T(x_3^*) \) and \( \Phi(x_3^*) \) and proved that the linear temperature gradient across the layer is a robust assumption, but this is not the case for the nanoparticle volume fraction in this model.

2.4 Linearized equations

Equations (1)–(3) and (10) and their associated boundary conditions are linearized by introducing the perturbations

\[
\nu_j = \epsilon \tilde{\nu}_j, \quad \text{P} = \text{P}(x_3) + \epsilon \tilde{P}, \quad T^* = T(x_3) + \epsilon \tilde{T}, \quad \Phi^* = \Phi(x_3) + \epsilon \tilde{\Phi}, \quad J_j^* = \epsilon \tilde{J}_j \quad \text{and} \quad \xi_j^* = \epsilon \tilde{\xi}_j,
\]

Thus, the linearized equations have the form:

\[
\frac{\partial \Phi_j}{\partial x_3^*} = 0,
\]
\[
\rho_{\text{eff}}(\Phi) \frac{\partial \tilde{v}_j}{\partial t^*} = -\frac{\partial \tilde{\Phi}}{\partial x_j} + \frac{\partial \mu_{\text{eff}}(\Phi) \left( \frac{\partial \tilde{v}_k}{\partial x_k} + \frac{\partial \tilde{v}_k}{\partial x_j} \right)}{\partial x_j} + 2(\tilde{\phi} \times \Omega)_j,
\]
(30)

\[
(\rho c)_{\text{eff}}(\Phi) \left( \frac{\partial \tilde{T}}{\partial t^*} + \tilde{v}_j \frac{\partial \tilde{T}}{\partial x_j} \right) = \frac{\partial}{\partial x_j} \left( \kappa_{\text{eff}}(\Phi) \frac{\partial \tilde{T}}{\partial x_j} \right) + \frac{\partial}{\partial x_j} \left( \frac{\partial \tilde{v}_k}{\partial x_j} \frac{\partial \mu_{\text{eff}}(\Phi)}{\partial x_k} \frac{\partial \tilde{T}}{\partial x_j} \right) - \frac{\partial \tilde{T}}{\partial x_j} f_j.
\]
(31)

The associated linearized boundary conditions on \(x^*_j = 0\) are

\[
\tilde{v}_j(t^*; x^*_j, 0) = \frac{\partial \tilde{v}_j(t^*; x^*_j, 0)}{\partial x_j^*} = \tilde{v}_j(t^*; x^*_j, 0) = \tilde{T}(t^*; x^*_j, 0) = \tilde{f}_j(t^*; x^*_j, 0) = 0,
\]
(33)

and the associated linearized boundary conditions on \(x^*_j = d\) are

\[
\tilde{v}_j(t^*; x^*_j, d) = \frac{\partial \tilde{v}_j(t^*; x^*_j, d)}{\partial x_j^*} = \tilde{f}_j(t^*; x^*_j, d) = 0,
\]
(34)

\[
\tilde{\phi}(t^*; x^*_j, d) = \frac{\partial \tilde{\phi}(t^*; x^*_j, d)}{\partial x_j^*} = \tilde{T}(t^*; x^*_j, d) = 0.
\]
(35)

The linearized constrained condition requires that

\[
\lim_{|x^*| \to \infty} \frac{1}{|x'\xi|} \left( \frac{d}{d x_j^*} \int_0^d \phi(t^*; x^*_j, x_j^*) dx_j^* \right) = 0.
\]
(37)

In Model I, the functions \(\rho_{\text{eff}}(\Phi), \mu_{\text{eff}}(\Phi), (\rho c)_{\text{eff}}(\Phi)\) and \(\kappa_{\text{eff}}(\Phi)\) behave as parameters and the following expression in equations (32) and (36) vanishes, i.e.,

\[
\phi \frac{\partial \kappa_{\text{eff}}(\Phi)}{\partial x_j^*} \frac{\partial \tilde{T}}{\partial x_j^*} = 0,
\]
(38)

whereas in Model II these functions are functions of \(x_j^*\).

### 2.5 Non-dimensional equations

Following standard procedures, equations (29)–(32) are non-dimensionalized using the scale length \(d\) and time \(t^* = \frac{d^2}{a} t\) and replacing \(\tilde{v}_k, \tilde{P}, \tilde{T}, \tilde{\phi} \) and \(\tilde{f}_k\) by the following expressions:

\[
\tilde{v}_k = \frac{a}{d} v_k, \quad \tilde{P} = \frac{a \mu}{d^2} p, \quad \tilde{T} = |T_d - T_0| \theta, \quad \tilde{\phi} = |\Phi_d - \Phi_0| \psi, \quad \tilde{f}_k = \frac{1}{d^3} \rho_{\text{eff}}(\Phi) \Phi_d - \Phi_0 \Omega k,
\]
(39)

where \(a = \kappa_{t}/(\rho c)_t\) is the diffusivity. The final form of the non-dimensional linearized equations becomes:

\[
\frac{\partial \tilde{v}_j}{\partial x_j} = 0,
\]
(40)

\[
\frac{f_1}{Pr} \frac{\partial \tilde{v}_j}{\partial t} = -\frac{\partial \tilde{P}}{\partial x_j} + \frac{\partial}{\partial x_k} \left( f_2 \left( \frac{\partial \tilde{v}_l}{\partial x_k} + \frac{\partial \tilde{v}_k}{\partial x_l} \right) \right) + \sqrt{T_u} \varepsilon \Omega_k \delta_{ij},
\]
(41)
\[
\begin{align*}
\text{Le} \left( \frac{\partial \psi}{\partial t} + \frac{v_3}{|\Phi_d - \Phi_0|} \frac{d\Phi}{dx} \right) &= -\frac{f_2}{\alpha k}, \\
\frac{f_1}{\alpha k} \frac{\partial \theta}{\partial t} + \frac{v_3}{|T_d - T_0|} \frac{dT}{dx} &= \frac{\partial}{\partial x} \left( f_2 \frac{\partial \theta}{\partial x} \right) + \frac{1}{|T_d - T_0|} \frac{dT}{dx} f_3 \theta - \frac{N_B}{\text{Le}} \frac{1}{|T_d - T_0|} \frac{dT}{dx} f_3,
\end{align*}
\]

\begin{equation}
\tag{42}
\end{equation}

\begin{equation}
\tag{43}
\end{equation}

where \( f_1, \ldots, f_5 \) are functions with expressions

\begin{equation}
\begin{align*}
f_1(x_3) &= \frac{\rho \text{eff} (\Phi)}{\rho_f}, \\
f_2(x_3) &= \frac{\mu \text{eff} (\Phi)}{\mu_f}, \\
f_3(x_3) &= \frac{(\rho c) \text{eff} (\Phi)}{\rho_f c_f}, \\
f_4(x_3) &= \frac{k \text{eff} (\Phi)}{k_f},
\end{align*}
\end{equation}

\begin{equation}
\tag{44}
\end{equation}

where \( \text{Pr} = \frac{\mu_f}{\nu_f} \) is the Prandtl number, \( T_a = \frac{\Delta d_{\alpha \mu}}{\mu_f} \) is the Taylor number, \( \text{Le} = \frac{x}{\alpha k} \) is the Lewis number and \( N_B = -\frac{(pc) \text{eff} (\Phi)}{\rho_f} |\Phi_d - \Phi_0| \) is the modified particle density increment. In Model II, the functions \( f_1, \ldots, f_5 \) depend on \( x_3 \) whereas, in Model I these functions behave as parameters. The associated non-dimensional forms of the lower boundary conditions at \( x_3 = 0 \) are

\[
\begin{align*}
v_3(t; \beta_0, 0) &= \frac{\partial \psi(t; \beta_0, 0)}{\partial x_3} = \xi(t; \beta_0, 0) = \theta(t; \beta_0, 0) = f_2(t; \beta_0, 0) = 0,
\end{align*}
\]

\begin{equation}
\tag{45}
\end{equation}

and on the upper boundary, at \( x_3 = 1 \), the non-dimensional boundary conditions are

\[
\begin{align*}
v_3(t; \beta_1, 1) &= \frac{\partial \psi(t; \beta_1, 1)}{\partial x_3} = f_2(t; \beta_1, 1) = 0, \\
\frac{f_2}{\partial x_3^2} v_3(t; \beta_1, 1) - M_T \Delta_2 \theta(t; \beta_1, 1) - M_D \Delta_1 \psi(t; \beta_1, 1) &= 0, \\
\frac{f_2}{\partial x_3^2} \frac{\partial \theta(t; \beta_1, 1)}{\partial x_3} + \text{Nu} \frac{\partial \theta(t; \beta_1, 1)}{\partial x_3} + \frac{1}{|T_d - T_0|} \frac{dT}{dx_3} f_3 \psi(t; \beta_1, 1) &= 0,
\end{align*}
\]

\begin{equation}
\tag{46}
\end{equation}

where \( \Delta_2 \) is the Laplacian in two dimensions and \( M_T \) and \( M_D \) are, respectively, the temperature and nanoparticle Marangoni numbers which have the following definitions:

\[
\begin{align*}
M_T &= -\frac{|T_d - T_0| d \partial \theta}{\alpha \mu_f}, \\
M_D &= -\frac{|\Phi_d - \Phi_0| d \partial \psi}{\alpha \mu_f}.
\end{align*}
\]

\begin{equation}
\tag{47}
\end{equation}

The non-dimensional forms of the nanoparticles mass flux \( \hat{\mathbf{j}}_k \) with respect to Model I and Model II are as follows:

\[
\begin{align*}
\text{Model I.} \quad \hat{J}_k &= \left( \frac{\partial \psi}{\partial x_k} + \frac{N_A \partial \theta}{\partial x_k} \right), \\
\text{Model II.} \quad \hat{J}_k &= \left( -\frac{T}{T_0} \frac{\partial \psi}{\partial x_k} + \frac{N_A}{T} \frac{T_0}{T} \frac{\partial \psi}{\partial x_k} + \frac{2N_A}{\eta} \frac{d}{dx_3} \left( \frac{\Phi}{\Phi} \frac{\partial \psi}{\partial x_k} + \frac{\eta}{T} \frac{d}{dx_3} \psi \right) \delta_{k3} \right),
\end{align*}
\]

\begin{equation}
\tag{48}
\end{equation}

where \( N_A = \eta \left| \frac{T_d - T_0}{T_0} \right| \frac{\psi}{\theta} \) is the modified diffusivity ratio.

## 3 Eigenvalue problem

The eigenvalue problem is constructed by taking the curl operator of equation (41). Thus, the third component of the resulting equation has the form:

\[
\frac{\hat{J}_k}{\text{Pr}} \frac{\partial \xi}{\partial t} = f_2 \Delta_2 \xi + \sqrt{T_a} \frac{dT}{dx_3}, \quad \text{in Model I.}
\]

\begin{equation}
\tag{49}
\end{equation}
\[
\frac{f_1 \partial \xi}{Pr \partial t} = \frac{df_2}{dx_3} \frac{d\xi}{dx_3} + f_2 \Delta \xi + \sqrt{T_a} \frac{dw}{dx_3}, \quad \text{in Model II.} \tag{50}
\]

Taking the curl once again, the third component of the resulting equation has the form:

\[
\frac{f_1 \partial \Delta \xi}{Pr \partial t} = f_2 \Delta^2 \xi - \sqrt{T_a} \frac{\partial \xi}{\partial x_3}, \quad \text{in Model I.} \tag{51}
\]

\[
\frac{1}{Pr} \left( f_1 \Delta \left( \frac{\partial w}{\partial t} \right) + \frac{df_1}{dx_3} \frac{\partial}{\partial x_3} \left( \frac{\partial w}{\partial t} \right) \right) = f_2 \Delta^2 w + 2 \frac{df_2}{dx_3} \frac{\partial (\Delta w)}{\partial x_3} + \frac{df^2}{dx_3^2} \left( \frac{\partial^2 w}{\partial x_3^2} - \Delta w \right) - \sqrt{T_a} \frac{\partial \xi}{\partial x_3}, \quad \text{in Model II.} \tag{52}
\]

in which \( w, \xi \) represent the third components of the velocity field and vorticity, respectively, and \( \Delta_3 \) is the Laplacian in three dimensions. The third and fourth equations that complete the eigenvalue problem are derived from equations (42) and (43) where the expression of \( f_k \) is determined by the choice of the model. Thus, in Model I, the third components of equations (42) and (43) have the form:

\[
\text{Le} \left( \frac{\partial \psi}{\partial t} + Hv \right) = \Delta_3 \psi + N_0 \Delta_3 \theta, \quad \text{in Model I.} \tag{53}
\]

\[
f_3 \left( \frac{\partial \theta}{\partial t} - Hv \right) = f_4 \Delta \theta - \frac{HN_0}{\text{Le}} \left( \frac{\partial \psi}{\partial x_3} + N_0 \frac{\partial \theta}{\partial x_3} \right), \quad \text{in Model II.} \tag{54}
\]

For Model II, the third components of equations (42) and (43) have the form:

\[
\text{Le} \left( \frac{\partial \psi}{\partial t} + Hv \Phi \right) = \frac{T}{T_0} \Delta_3 \psi - H \frac{|T_a - T_d|}{T_0} \left( \frac{\partial \psi}{\partial x_3} - \frac{|T_a - T_d|^2}{T^2} \psi \right)
\]

\[
+ N_0 \frac{\Phi}{\phi} \frac{T_0}{T} \Delta_3 \theta + N_0 H \frac{\Phi}{\phi} \frac{T_a - T_d}{T_0} \left( 2 + \frac{T_d^2}{T^2} + \frac{T_0}{T} \right) \frac{\partial \theta}{\partial x_3} + 2N_0 \frac{\Phi}{\phi} \frac{T_d - T_0^2}{T_0^2} \theta, \quad \text{in Model II.} \tag{55}
\]

\[
f_3 \left( \frac{\partial \theta}{\partial t} + H \frac{\psi}{\phi} \frac{T_0}{T} \frac{dT}{dx_3} \right) = \frac{\partial}{\partial x_3} \left( f_4 \frac{\partial \theta}{\partial x_3} \right) - H \frac{\partial}{\partial x_3} \left( \psi \frac{df_2}{dx_3} \frac{d\Phi}{d\phi} \right)
\]

\[
- \frac{H N_0}{\text{Le}} \frac{T_0}{T} \frac{\partial \psi}{\partial x_3} + N_0 \frac{\Phi}{\phi} \frac{T_0}{T} \frac{\partial \theta}{\partial x_3} + 2N_0 H \frac{T_d - T_0^2}{T_0^2} \frac{\Phi}{\phi} \theta + \eta \frac{dT}{dx_3} \psi. \tag{56}
\]

### 4 Normal mode analysis

A normal mode analysis is used to investigate the stability of the boundary value problem of each model by writing the perturbed quantities in the form:

\[
\chi(x, t) = \chi(x_3) \exp(\sigma t + i(n_1 x_1 + m_2 x_2)), \tag{57}
\]

where \( \chi = (w, \theta, \psi, \xi) \), \( n, m \) are wave numbers along \( e_1 \) and \( e_2 \) directions, respectively, and \( \sigma \) is the growth rate. Thus, the final forms of the eigenvalue problem in each model are written as follows:

**Model I.**

\[
\frac{\sigma}{Pr} f_1 \xi = f_2(D^2 - a^2) \xi + \sqrt{T_a} Dw, \tag{58}
\]

\[
\frac{\sigma}{Pr} f_1(D^2 - a^2) w = f_2(D^2 - a^2)^2 w - \sqrt{T_a} D \xi, \tag{59}
\]

\[
\text{Le} \psi = -Le Hw + (D^2 - a^2) \psi + N_0(D^2 - a^2) \theta, \tag{60}
\]

\[
f_3 \sigma \theta = f_4 Hw + f_4(D^2 - a^2) \theta - H \frac{N_0}{\text{Le}} (D \psi + N_0 D \theta). \tag{61}
\]
The corresponding boundary conditions on \( x_3 = 0 \) have the form:

\[
(w(0) = Dw(0) = \xi(0) = \theta(0) = D\phi(0) + N_\eta D\theta(0) = 0,
\]

and on \( x_3 = 1 \)

\[
\theta(1) = w(1) = D\xi(1) = D\psi(1) + N_\eta D\theta(1) = 0,
\]

\[
f_2(1)D^2 w(1) + a^2 M_\theta \theta(1) + a^2 M_\psi \psi(1) = 0,
\]

\[
f_2(1)D\theta(1) + Nu\theta(1) = 0.
\]

**Model II.**

\[
\frac{\sigma}{Pr} f_1 \xi = DF_2 D\xi + f_2 (D^2 - a^2)\xi + \sqrt{T_a} Dw,
\]

\[
\frac{\sigma}{Pr} f_1 (D^2 - a^2)w + DF_1 Dw = f_2 (D^2 - a^2)w + 2DF_2 (D^2 - a^2) Dw + D^2 f_2 (D^2 + a^2)w - \sqrt{T_a} D\xi,
\]

\[
Le \psi = -Le H \Phi \psi + \frac{T}{T_0} (D^2 - a^2) \psi - H \frac{T_0 - T_1}{T_0} \left(1 + \frac{\eta T_0}{T} \right) D\psi - \frac{\eta |T_0 - T_1|^2}{T^2} \psi \\
+ N_\eta H \frac{T_0}{\Phi T_0} (D^2 - a^2) \theta + N_\eta H \frac{T_0 - T_1}{T_0} \left(2 + \frac{T_0^2}{T^2} \right) D\theta + 2N_\eta \frac{\Phi |T_0 - T_1|^2}{T_0^2} \theta,
\]

\[
f_3 \sigma \theta = -f_3 \frac{w}{|T_0 - T_1| \frac{dT}{dx}} + f_2 (D^2 - a^2) \theta + DF_1 D\theta - HD \left(\psi \frac{df_2}{d\phi} \right) \\
- H \frac{N_\eta}{Le} \left(\frac{T}{T_0} D\psi + N_\eta H \frac{T_0}{\Phi T_0} \theta + \frac{\eta dT}{T_0 dx_3} \psi \right).
\]

The associated boundary conditions at \( x_3 = 0 \) are

\[
w(0) = Dw(0) = \xi(0) = \theta(0) = 0,
\]

\[
D\psi(0) + N_\eta \frac{\Phi_0}{\Phi} D\theta(0) + \frac{\eta dT}{T_0 dx_3} \psi(0) = 0.
\]

While on \( x_3 = 1 \), the boundary conditions have the form:

\[
w(1) = D\xi(1) = 0,
\]

\[
f_2(1)D^2 w(1) + a^2 M_\theta \theta(1) + a^2 M_\psi \psi(1) = 0,
\]

\[
f_2(1)D\theta(1) + Nu\theta(1) - HF_2(1) \psi(1) = 0,
\]

\[
D\psi(1) + N_\eta \left(\frac{T_0}{T_1} \right)^2 \frac{\Phi_0}{\Phi} D\theta(1) + \frac{\eta T_0}{T_1^2} \frac{dT(1)}{dx_3} \psi(1) + 2N_\eta T_0 \frac{d}{T_0 dx_3} \left(\frac{\Phi}{\Phi} \right) \theta(1) = 0,
\]

where \( D = \frac{d}{dx_3} \) and \( a = \sqrt{n^2 + m^2} \) is the wave number.

### 5 Numerical method

Equations (58)–(61) of Model I together with the corresponding conditions (62) and (63) and equations (64)–(67) of Model II with the corresponding conditions (68) and (69) are solved numerically using the method of expansion of Chebyshev polynomials when the nanofluid layer is heated from below, i.e., \( H = 1 \). This method is known by its high accuracy and it allows stationary and over-stable modes to be treated simultaneously [27]. Let us define the variables \( y_1(z), \ldots, y_{10}(z) \) by the expressions

\[
\begin{align*}
 y_1 &= w, & y_2 &= Dw, & y_3 &= (D^2 - a^2)w, & y_4 &= (D^2 - a^2) Dw, & y_5 &= \psi, \\
 y_6 &= D\psi, & y_7 &= \theta, & y_8 &= D\theta, & y_9 &= \xi, & y_{10} &= D\xi
\end{align*}
\]
where the functions $y_1(z),...,y_6(z)$ are assigned the Chebyshev polynomial expansions of the form:

$$y_r = \sum_{k=0}^{N} a_k T_k(z), \quad 1 \leq r \leq 10,$$

(71)

where $N$ is a user-specified even integer and $T_k(z)$ are the Chebyshev polynomials of the first kind. The system of equations of Model I and Model II leads to the eigenvalue problem $AY = \sigma BY$, where $B$ is a singular matrix. The eigenvalues, $\sigma$, are calculated using a special routine. In Model I, the numerical solutions are obtained quite easily; however, in Model II this is not the case. In fact, the complexity in this case stems from the inhomogeneity of the eigenvalue problem, namely the multiplication of the non-constant functions $f_1(z),...,f_5(z)$ with the variables of the problem.

6 Results and discussion

Results are obtained for two types of nanofluids, DW/alumina and DW/cupric oxide for nanoparticles of size 30, 70 and 100 nm. Calculations will take advantage of the work of Khanafer and Vafai [24] who have proposed the following constitutive model for the effective thermal conductivity

$$\kappa_{\text{eff}}(\varphi) = 1.0 + \varphi \left(1.0112 + \frac{114.5625}{d_p} + 0.040457\kappa_f\right),$$

(72)

for these nanofluids at working temperatures close to room temperature (298°K). In expression (72), the volume fraction satisfies $\varphi^* \in [0.01, 0.10]$ and the nanoparticle diameter (expressed in nanometers) satisfies $d_p \in [11, 150]$. Also, the following effective dynamic viscosity expressions proposed by Khanafer and Vafai [24] will be used in the calculations

$$\mu_{\text{eff}}(\varphi^*) = a_0 + a_1 \varphi^* + a_2 \varphi^{*2} + a_3 \varphi^{*3} + a_4 \frac{\varphi^{*2}}{T^*} + a_5 \frac{\varphi^{*2}}{T^{*2}} + a_6 \frac{\varphi^{*3}}{T^*} + a_7 \frac{\varphi^{*3}}{T^{*2}} + a_8 \frac{\varphi^{*3}}{T^{*3}},$$

(73)

$$\mu_{\text{eff}}(\varphi^*) = b_0 + b_1 \varphi^* + b_2 \varphi^{*2} + b_3 \varphi^{*3} + b_4 \frac{\varphi^{*2}}{T^*} + b_5 \frac{\varphi^{*2}}{T^{*2}} + b_6 \frac{\varphi^{*3}}{T^*} + b_7 \frac{\varphi^{*3}}{T^{*2}} + b_8 \frac{\varphi^{*3}}{T^{*3}}.$$  (74)

In these expressions $\varphi^* \in [0.01, 0.09]$, Celsius temperature $T^* \in [20, 70]$ and in the case of the DW/alumina nanofluid, the nanoparticle diameter (expressed in nanometers) satisfies $d_p \in [13, 133]$. For comparison with the results obtained using the models proposed by Khanafer and Vafai [24], for the effective thermal conductivity and effective dynamic viscosity, another model proposed by Hamilton and Crosser [25] is used for the effective thermal conductivity of nanofluid, namely

$$\frac{\kappa(\varphi)}{\kappa_f} = \frac{(n-1)(1-\varphi)\kappa_f + (n\varphi + 1 - \varphi)\kappa_p}{\kappa_f(1-\varphi) + (n-1 + \varphi)\kappa_f},$$

(75)

and Brinkman’s model [26] for the effective dynamic viscosity is used, namely

$$\mu(\varphi) = \mu_f(1-\varphi)^{-52}.$$  (76)

The investigation of the stability boundaries is obtained based on the classical model of convection of a layer of nanofluid under the effect of rotation and compared with the general model of convection of a layer of nanofluid under the effect of rotation where the constitutive functions are assumed to be non-constants.

The main condition that must be satisfied is that all eigenvalues of the linear problem have negative real parts. Hence, the location of the stability boundary is determined by comparing the values of the true temperature Marangoni number, $M_T$, with its value on the stability boundary when the true value of the nanoparticle Marangoni number, $M_{\varphi}$, is known. If the true value of the temperature Marangoni number exceeds the value on the stability boundary, then the layer is unstable, otherwise it is stable. The true values
of the temperature Marangoni number and the nanoparticle Marangoni number are calculated using equations (47).

The results are obtained for a layer of 1 mm thickness, nanoparticles of size $d_p = 30$ nm and $\text{Nu} = 0.2$. The average nanoparticle volume fractions are ranging from 0.01 to 0.03 for DW/alumina nanofluid and from 0.01 to 0.06 for DW/cupric oxide nanofluid. The rotational effects are considered for $T_a = 0, 100, 1000$. The lower boundary of the layer has a constant temperature of 300°K and the atmospheric temperature at the upper boundary is 298°K.

The true and critical temperature Marangoni numbers for the layer of DW/alumina nanofluid, are evaluated for both Model I, Figure 1a, and Model II, Figure 1b, with different values of $T_a$ using the thermal conductivity and dynamic viscosity constitutive functions proposed by Brinkman-Hamilton-Crosser. The graphs of Model I in Figure 1a indicate that the layer of nanofluid is unstable for all values of $\bar{\phi} \in [0.01, 0.03]$ and for all values of $T_a$, whereas the graphs of Model II in Figure 1b indicate that the

Figure 1: The true and critical temperature Marangoni numbers for DW/alumina nanofluid with $d_p = 30$ nm, $\Delta T = 2^\circ$K for average volume fractions ranging from 0.01 to 0.03. Stability boundary curves for $M_T$ are calculated based on (a) Model I and (b) Model II, using the thermal conductivity and dynamic viscosity constitutive functions proposed by Brinkman-Hamilton-Crosser.
nano-fluid layer is unstable for all values of $\bar{\phi}$ when $T_a = 0, 100$. However when $T_a = 1000$, the nano-fluid layer is stable when $\bar{\phi} \geq 0.005$. Moreover, the graphs in Figure 1 indicate that the critical Marangoni number increases as the Taylor number increases showing that the rotation has a stabilizing effect. Also the graphs show that the critical Marangoni number in Model I are lower than the critical Marangoni number in Model II, which indicates that Model II is more stable than Model I. This result clarifies that the assumption of non-constant thermal conductivity and dynamic viscosity functions across the layer stabilizes the DW/alumina nano-fluid layer.

Figure 2 illustrates the results of the true and critical temperature Marangoni numbers for DW/alumina nano-fluid using the thermal conductivity and dynamic viscosity constitutive functions calculated by Khanafer and Vafai. Stability boundaries based on Model I in Figure 2a show that the layer is unstable for all values of $\bar{\phi} \in [0.01, 0.03]$. In contrast, the layer in Model II is stable for all values of the Taylor number, $T_a$.

![Figure 2](image-url)
when $\phi$ exceeds specific values. In fact, when $T_a = 0, 100, 1000$, the specific values of $\phi$ at which stability ensue are 0.02, 0.007 and 0.0025, respectively, as shown in Figure 2b. The aforementioned results of Model II clarify that the assumption of non-constant thermal conductivity and dynamic viscosity functions across the layer stabilizes the DW/alumina nanofluid layer. This stability increases as the Taylor number increases showing that rotation has a stabilizing effect.

The evaluations of the true and critical temperature Marangoni numbers for the layer of DW/cupric oxide are plotted in Figure 3 using the thermal conductivity and dynamic viscosity constitutive functions proposed by Brinkman-Hamilton-Crosser and in Figure 4 using the thermal conductivity and dynamic viscosity constitutive functions proposed by Khanafer and Vafai. The graphs of Model I and Model II in these figures show that the nanofluid layer is stable for all values of $T_a$ when $\phi$ exceeds specific values. A significant result is seen in Figure 3b when $T_a = 1000$ in which the nanofluid layer is stable for all values of $\phi$.

Figure 3: The true and critical temperature Marangoni numbers for DW/cupric oxide nanofluid with $d_p = 30$ nm, $\Delta T = 2^\circ K$ for average volume fractions ranging from 0.01 to 0.06. Stability boundary curves for the temperature Marangoni number are calculated based on (a) Model I and (b) Model II, using thermal conductivity and dynamic viscosity constitutive functions proposed by Brinkman-Hamilton-Crosser.
As in Figure 1a and b, rotation and the assumption of non-constant thermal conductivity and dynamic viscosity functions across the layer stabilize the nanofluid layer.

Figure 5 represents a general comparison between the stability boundaries calculated based on Model I (Figure 5a) and those based on Model II (Figure 5b). For Models I and II, the figure shows that for the DW/alumina nanofluid the model of Khanafer and Vafai makes the nanofluid more stable compared to the model of Brinkman-Hamilton-Crosser. Moreover, for the DW/cupric oxide nanofluid the model of Brinkman-Hamilton-Crosser makes the nanofluid more stable compared to the model of Khanafer and Vafai if $\bar{\phi}$ within a specific range.

Finally, Figure 6 shows a comparison between Model I and Model II by calculating the critical temperature Marangoni number, $M_{T_{\text{crit}}}$, for a range of values of the average volume function of nanoparticles, $\bar{\phi}$, when $T_a = 1000$. Clearly, Model II is more stable than Model I for both types of nanofluids (DW/cupric oxide nanofluid and DW/alumina nanofluid).
Conclusions

This work investigated the linear stability of the onset of Marangoni convection for horizontal layers of DW/alumina and DW/cupric oxide nanofluids rotated about the $x_3$-axis, heated from below and losing heat from their upper surfaces by convection to a region at constant temperature. The lower boundaries of the layers are assumed to be rigid with fixed temperature and impermeable to nanoparticles while the top boundaries are assumed to be non-deformable free surfaces at which surface tension variations due to nanoparticle volume fraction and temperature are allowed.

Figure 5: The stability boundaries of critical temperature Marangoni numbers for DW/alumina and DW/cupric oxide nanofluids for $T_0 = 1000$, using the thermal conductivity and dynamic viscosity constitutive functions proposed by Brinkman-Hamilton-Crosser and Khanafer and Vafai based on (a) Model I and (b) Model II.

oxide and DW/alumina) and for both models of Brinkman-Hamilton-Crosser (Figure 6a) and Khanafer and Vafai (Figure 6b).
The stability of these nano fluids is investigated using two models. These models are distinguished by the condition that Model I assumes that constitutive properties are uniform across the layer of nano fluid, whereas in Model II the same properties are allowed to depend on local volume fraction of nanoparticles and satisfy constitutive expressions developed by Hamilton-Crosser-Brinkman and Khanafer and Vafai based on experimental evidence. Moreover, in Model I the classical expression of the nanoparticle mass flux is used in which the temperature and nanoparticle volume fraction are constants, whereas in Model II the temperature and nanoparticle volume fraction are assumed to be non-constants. Therefore, these changes contributed extra terms to the linear stability analysis for Model II that are not present in the same analysis based on Model I. The presence of these extra terms largely increases the stability boundary in Model II for both types of nano fluids and for both types of constitutive expressions developed by Hamilton-Crosser-Brinkman and Khanafer and Vafai. Findings also indicate that the effect of rotation plays a significant role in increasing the stability boundary in Model I and Model II for both types of nano fluids and for both models developed by Hamilton-Crosser-Brinkman and Khanafer and Vafai.

**Figure 6:** The stability boundaries of critical temperature Marangoni numbers for DW/alumina and DW/cupric oxide nano fluids based on Model I and Model II with $T_0 = 1,000$, using the models proposed by (a) Brinkman-Hamilton-Crosser and (b) Khanafer and Vafai.
The numerical results were obtained using spectral eigenfunction expansions in terms of Chebyshev polynomials. Different stability boundaries were obtained and several comparisons were made for Model I and Model II to clarify which of the two nanofluids. Different values of the nanoparticle volume fraction and on the type of the constitutive expressions used.

**Conflict of interest:** Authors state no conflict of interest.

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