Solvable points on higher genus curves

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Abstract

It is known that for a curve defined over \( \mathbb{Q} \) of genus \( g \leq 4 \), there exists a point on the curve defined over a solvable extension of \( \mathbb{Q} \). We relate points on curves of genus \( g \geq 5 \) over solvable extensions to the Bombieri-Lang conjecture. Specifically, it shows that varieties parameterising points defined over extensions with a fixed solvable Galois group are of general type. Moreover, we show the existence of certain subvarieties in these varieties imply the existence of solvable morphisms from the curve.

1 Introduction

For curves of genus at least 2, Faltings’ theorem states there are only finitely many points defined over any (fixed) number field. On the other hand, points defined over the algebraic closure of \( \mathbb{Q} \) are Zariski dense. Therefore, it is natural to ask about points defined over some class of number fields. One such class of classical interest is the solvable number fields, prompting the following.

Question. Given a curve \( C \) defined over a number field \( K \), does there exist a number field \( F/K \) with solvable Galois group, such that \( C(F) \neq \emptyset \)?

Pál proved that for any curve of genus 0, 2, 3, or 4 defined over \( \mathbb{Q} \), there is at least one solvable point \([\text{Pál}04]\), and Wiles & Çiperani proved the same for genus 1 curves \([\text{WC}08]\). It is true for all \( C \), when \( K \) is instead a finite extension of \( \mathbb{Q}_p \), since there are \( \mathbb{Q}_p \) points, and all finite extensions of \( \mathbb{Q}_p \) are solvable. Pál \([\text{Pál}04]\) constructed counterexamples for almost all \( g \geq 5 \) over local fields where the absolute Galois group of the residue field has quotients isomorphic to \( S_5 \), \( \text{PSL}_3(\mathbb{F}_2) \) and \( \text{PSL}_3(\mathbb{F}_3) \). The question remains open for curves defined over number fields. We will show there are “not too many” points for a curve of genus \( g \geq 5 \) defined over number fields having a given solvable Galois group, assuming the following conjecture of Bombieri and, independently, Lang \([\text{Lan}86]\). All varieties in this paper will be defined over a fixed number field \( K \), and rational will mean \( K \)-rational.

Conjecture (Bombieri-Lang). Let \( V \) be a variety of general type, then rational points on \( V \) are not Zariski dense.
We first recall the definition of general type:

**Definition 1.** A smooth variety is of general type if its Kodaira dimension is equal to the dimension of the variety

Unwinding the definition, gives the following.

**Definition 2.** Let \( V \) be an \( n \)-dimensional variety, then \( V \) is of general type if for large enough \( r \), the image of the map associated to \( rK_V \) (where \( K_V \) is the canonical divisor) is \( n \)-dimensional.

The associated map in the above definition is the map (unique up to isomorphisms of \( \mathbb{P}^k \)) to \( \mathbb{P}^k \) where each coordinate is an element of a basis of \( \mathcal{L}(rK_V) = \{ f \in k(V) | \text{Div}(f) + rK_V \geq 0 \} \). For example, a curve is of general type if and only if it has genus \( g \geq 2 \). In this case, the Bombieri-Lang conjecture is simply Faltings' theorem. In this paper, it will be necessary to study singular varieties and so a notion of general type for such varieties is needed. It is known that Kodaira dimension is a birational invariant, and so the definition of general type can be extended as follows.

**Definition 3.** A variety is of general type if a desingularisation is of general type.

We return to the problem of points with a given Galois group. Fix a transitive subgroup \( G \leq S_n \), then \( G \) acts on \( C^n \) by permuting the factors. As \( G \) is finite and its orbits are contained in affine patches, the quotient variety, \( C^n/G \), exists. Points on the variety will be denoted as \( [(P_1, ..., P_n)] \), where \( (P_1, ..., P_n) \) is a representative of the equivalence class. Rational points on such varieties are closely related to points on the original curve with given Galois group. More precisely:

**Proposition 1.** A rational point on \( C^n/G \) is a union of Galois orbits of points on \( C \) defined over fields with Galois group contained in \( G \).

**Proof.** Let \( [(P_1, ..., P_n)] \) be a rational point on \( C^n/G \), then for all \( \sigma \in \text{Gal}(\overline{K}/K) \), \( \sigma[(P_1, ..., P_n)] = [(\sigma P_1, ..., \sigma P_n)] \). The Galois action is the coordinate-wise Galois action, and so \( [(P_1, ..., P_n)] \) is a finite Galois orbit, then the homomorphism \( \text{Gal}(K(P_{i_1}, ..., P_{i_m})/K) \to S_n \) must have image contained in \( G \). Repeating this for each Galois orbit gives the desired statement.

This motivates the study of these varieties to understand points on the curve with given Galois group. The rest of the paper is dedicated to the structure of these varieties. To this end, we determine when these varieties are of general type.

**Theorem 1.** Suppose \( G \) contains exactly \( m \) transpositions of the form \((1, i)\). Then \( C^n/G \) is of general type if and only if \( g(C) > m + 1 \).
This is proved in Section 2.

**Corollary 1.** If $G$ is a solvable group and $g(C) \geq 5$, $C^n/G$ is of general type

**Proof.** Suppose $G$ contains exactly $m$ transpositions of the form $(1, i)$. Then $G$ contains a subgroup isomorphic to $S_{m+1}$. As $G$ is solvable, $m + 1 \leq 4$. Then $g(C) > m + 1$, and so by Theorem 1, $C^n/G$ is of general type. \[ \blacksquare \]

Combining the corollary with the Bombieri-Lang conjecture, we get the following theorem.

**Theorem 2.** Assume the Bombieri-Lang conjecture. Let $G$ be a solvable group, and $C$ be a smooth curve of genus at least 5. Then for any transitive embedding $G \hookrightarrow S_n$, the rational points on $C^n/G$ are not Zariski dense.

For curves of genus 2 and 3, we can easily compute the Kodaira dimension of $C^n/G$, where $G$ is the Galois group of points defined by Pál's method. For $g(C) = 2$, $C$ is hyperelliptic, and so pulling rational points of $\mathbb{P}^1$ back through the double covering generically gives points defined over number fields with Galois group $S_2$. The variety $C^2/S_2$ is birational to the Jacobian, and has Kodaira dimension 2. When $g(C) = 3$, $C$ is generically a plane quartic. Intersecting the quartic with lines in $\mathbb{P}^2$ gives points with Galois groups $S_4$ generically. The quotient $C^4/S_4$ is uniruled over the Jacobian, and so has Kodaira dimension $-\infty$. In particular, neither of these varieties are of general type.

As well as this conditional result, we also have an unconditional restriction on the rational points of these varieties.

**Definition 4.** Fix a curve $C$. A rational curve $D$ in $\text{Sym}^n C$ is of fibre type if $D$ generically intersects the image of $\{P\} \times C^{n-1}$ in a single point (as $P$ varies over $C$). A curve in $C^n/G$ is of fibre type if it maps injectively onto its image in $\text{Sym}^n C$ and its image is of fibre type.

We prove the following theorem about curves of fibre type in Section 3.

**Theorem 3.** Let $G$ be a transitive subgroup of $S_n$, and suppose there exists a curve of fibre type in $C^n/G$ with Zariski dense rational points, then $C$ has a morphism to $\mathbb{P}^1$ with Galois group $G$.

Combining this with the geometry of generic high genus curves gives the following

**Corollary 2.** Let $C$ be a sufficiently generic curve of genus $\geq 7$, and $G$ a solvable, transitive subgroup of $S_n$ for some $n$. No curve of fibre type in $C^n/G$ has Zariski dense rational points.

**Proof.** A theorem of Zariski [Zar26] states that for a sufficiently generic curve of genus $\geq 7$, the Galois group of any morphism $C \rightarrow \mathbb{P}^1$ is not solvable. By Theorem 3, the existence of a curve of fibre type with dense rational points would imply the existence of a morphism from $C$ with Galois group $G$, contradicting Zariski's theorem. Thus, there are no rational curves of fibre type with dense rational points. \[ \blacksquare \]
As this holds for any number field containing $K$, the corollary strengthens to no curves in $C^n/G$ are of fibre type, for $G$ solvable. The fibre type condition is quite restrictive, but for $n$ near the gonality of $C$, the dimensions of the fibres of $\text{Sym}^n C$ over $\text{Jac}(C)$ are small enough that this behaviour is typical in the symmetric power.

2 When Quotients are of General Type

2.1 The no transpositions case

First, we prove the following special case,

**Theorem 4.** Let $C$ be a smooth curve of genus at least 2, and let $G \subset S_n$ be a subgroup that contains no transpositions. Then $C^n/G$ is of general type.

We first observe that as $G$ is a subgroup of $S_n$, there is a map $\pi_2 : C^n/G \to \text{Sym}^n C$ so that $\pi_1 : C^n \to \text{Sym}^n C$ factors through $C^n \to C^n/G$. The variety $C^n/G$ may be singular, and so it needs to be replaced by a desingularisation, $X$ which has a birational morphism $X \to C^n/G$. Composing this map with $\pi_2$ gives a map $\phi : X \to \text{Sym}^n C$. This is summarised in the following commutative diagram,

$$
\begin{array}{ccc}
C^n & \xrightarrow{\pi_1} & C^n/G \\
\downarrow & & \downarrow \\
\text{Sym}^n C & \xleftarrow{\phi} & X \\
\end{array}
$$

The variety $C^n/G$ can only be singular where a point of $C^n$ is fixed by some non-trivial $g \in G$, as elsewhere $C^n/G$ is locally isomorphic to patches of $C^n$. As $G$ contains no transpositions, if $(P_1, \ldots, P_n)$ is fixed by a non-trivial element of $G$, either at least two pairs of $P_i$ are equal, or at least three $P_i$ are equal. Therefore the singular locus is contained in a subvariety of codimension at least 2.

To prove the theorem, some results are needed about the structure of the maps $\pi_i$.

**Lemma 1.** The ramification locus of $\pi_1$ is the set $\Delta = \{(P_1, \ldots, P_n)|P_i = P_j$ for some $i \neq j\}$, and under the hypotheses of the theorem, the ramification locus of $\pi_2$, away from the singular points of $C^n/G$, is the image of $\Delta$ under the quotient map.

**Proof.** We start with $\pi_1$. The pre-image, $\pi_1^{-1}(P_1 + \ldots + P_n)$, generically consists of $n!$ points, $(P_{\sigma(1)}, \ldots, P_{\sigma(n)})$ as $\sigma$ ranges over $S_n$. There is, therefore, ramification precisely when two of these images collide, when $P_i = P_j$ for some $i \neq j$. 

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As \( \pi_1 \) factors through \( C^n/G \), \( \pi_2 \) is unramified outside of the image of \( \Delta \). The ramification index of \( \pi_1 \) at a point of \( \Delta \), \( R \), under \( \pi_1 \) is equal to the size of its stabiliser in \( S_n \). Similarly, the ramification index for the map to \( C^n/G \) at \( R \) is the size of its stabiliser in \( G \). Away from the singular locus, this stabiliser is trivial. Therefore, as ramification indices are multiplicative under composition, the ramification index of \( \pi_2 \) at the image of \( R \) must be equal to that of \( \pi_1 \) at \( R \). In particular, the map is ramified there.

And now to prove Theorem 4.

**Proof.** As being of general type (for smooth varieties) is a statement about canonical divisors, the first step is to relate \( K_{C^n} \) and \( K_X \). We will obtain such a relation by considering the two maps to \( \text{Sym}^n C \).

There is a generalisation of the Riemann-Hurwitz formula for a generically finite morphism of smooth varieties \( f : X \to Y \), which states \( K_X \cong f^*K_Y + R + E \) where \( R \) is an effective divisor supported on the ramification locus, and \( E \) is an effective divisor supported on the exceptional locus [Deb01]. Applying this to \( \pi_1 \) gives

\[
K_{C^n} \cong \pi_1^*K_{\text{Sym}^n C} + \Delta
\]

This can be pushed forward to yield

\[
n!K_{\text{Sym}^n C} \cong \pi_1_*K_{C^n} - \pi_1_*\Delta
\]

By the preceding lemma, \( \pi_2 \) is ramified in the image of \( \Delta \), except for the codimension 2 or smaller set where \( C^n/G \) is possibly singular. Therefore \( \phi \) must also be ramified in the image of \( \Delta \). Applying the same formula to \( \phi \) gives

\[
K_X \cong \phi^*K_{\text{Sym}^n C} + \Delta' + R' + E
\]

Where \( \Delta' \) is the image of \( \Delta \) in \( X \), and \( R' \) and \( E \) are both effective divisors.

Combining both formulae,

\[
n!K_X \geq \phi^*\pi_1_*K_{C^n} - \phi^*\pi_1_*\Delta + n!\Delta' + n!E
\]

The divisor \( \phi^*\pi_1_*\Delta \) is the locus of points \( \{(P_1, ..., P_n)\} \) where \( P_i = P_j \) for some \( i, j \), counted with multiplicity \( n! \) (the pullback contributes to multiplicity 2 due to ramifying with degree 2, and the pushforward contributes \( \frac{n!}{2} \)), as well as possibly some exceptional components. The non-exceptional part is exactly \( n!\Delta' \), and the exceptional part will be contained within \( n!E \). Therefore

\[
n!K_X \geq \phi^*\pi_1_*K_{C^n}
\]

Pullbacks only increase the number of global sections. Thus to show \( C^n/G \) is of general type it is enough to check that \( \pi_1_*K_{C^n} \) has \( n \) algebraically independent sections.

As \( C \) is of general type, \( K_C \) has a non-trivial section \( f \). Define \( f_i \) on \( C^n \) by \( f_i(P_1, ..., P_n) = f(P_i) \), this is a section of \( K_{C^n} \). Taking the elementary symmetric polynomials in \( f_i \) give sections of \( K_{C^n} \) which are \( G \)-invariant, and are algebraically independent. These are therefore sections of \( \pi_1_*K_{C^n} \), and the result follows.

\[\square\]
This establishes the main theorem in the case \( m = 0 \).

### 2.2 Proof of Theorem [T]

The general case will be proven by factoring the quotient map through symmetric powers, before using a result on symmetric powers of higher dimensional varieties to conclude. Before this, we need some lemmas on the structure of transitive subgroups of \( S_n \). The first of which is a slight generalisation of a result in Clark’s book [Cla84].

**Lemma 2.** Let \( G \subset S_n \) be a transitive subgroup, and suppose it contains exactly \( m \) transpositions of the form \((1, i)\), then \( m + 1 \mid n \).

**Proof.** Define an equivalence relation on \( \{1, \ldots, n\} \) by \( i \sim j \) if \( i = j \) or \((i, j)\) \( \in G \). Reflexitivity and symmetry are obvious from the definition. Suppose \( i \sim j \) and \( j \sim k \), then \((i, j)\), \((j, k)\) \( \in G \). As \( G \) is a subgroup, \((i, k) = (j, k)(i, j)\) \( \in G \), and so \( i \sim k \). Therefore \( \sim \) is transitive, and it defines an equivalence relation.

Consider an equivalence class \([i]\). By transitivity of \( G \), there is a \( \sigma \in G \) such that \( \sigma(1) = i \). If \( j \in [i] \), \((i, j)\) \( \in G \), and so \((1, \sigma^{-1}(j)) = \sigma^{-1}(i, j)\sigma \in G \), and \( \sigma^{-1}(j) \in [1] \). Therefore \( |[i]| \leq |[1]| \), and by symmetry, the inequality holds in the opposite direction, so all equivalence classes have the same size. \( |[1]| = m + 1 \) as it contains 1 along with an element for each transposition containing 1. Therefore \( \{1, \ldots, n\} \) can be partitioned into classes of size \( m + 1 \), and the result holds. \( \square \)

The second lemma describes a quotient of transitive subgroups.

**Lemma 3.** Let \( G \subset S_n \) and \( m \) be as before, let \( n' = \frac{n}{m+1} \) and define \( H \) be the subgroup of \( G \) generated by all transpositions. Then \( H \) is a normal subgroup, \( H \cong S_{m+1} \times \ldots \times S_{m+1} \) and there is a natural inclusion \( G/H \hookrightarrow S_{n'} \).

**Proof.** Group the transpositions according to the equivalence classes of the previous lemma. Transpositions in the same class will generate a symmetric group on the size of the class, \( m + 1 \), and transpositions from different classes commute. Therefore \( H \cong S_{m+1} \times \ldots \times S_{m+1} \), with the product running over equivalence classes.

Consider the transpositions corresponding to the first equivalence class; these generate a subgroup \( H' \). The conjugates of \( H' \) are the groups generated by the other equivalence classes. Let \( g \in G \) be an element fixing all the conjugates of \( H' \) under conjugation. As \( g(i, j)g^{-1} = (g(i), g(j)) \), this shows \( g(i) \sim i \) for all \( i \), therefore \( g \in H \). Conversely, all elements of \( H \) preserve \( H' \) and its conjugates. Therefore the homomorphism \( G \to S_{n'} \) induced by the conjugation action has kernel \( H \). \( \square \)

One final result is needed, describing the geometry of symmetric powers. This result does not seem to appear in the literature, although similar results do. We therefore give it here, along with the proof provided by MathOverflow user, Olivier Benoist [Ben13].
Lemma 4. Let $C$ be a curve of genus $g$. The variety $\text{Sym}^n C$ is of general type for $n < g$, (birational to) an abelian variety for $n = g$ and uniruled for $n > g$.

Proof. The statements for $n \geq g$ follow from Riemann-Roch and the definition of the Jacobian variety, so it remains to prove it for $n < g$.

The Abel-Jacobi map gives a birational map $\text{Sym}^n C \to J(C)$, and so it is enough to prove the image, $W_n$, is of general type. If it were not, by a theorem of Ueno [Uen73], it would contain an abelian variety $A$ such that $A + W_n = W_n$. In particular, as $W_{g-1}$ is expressible as the sum of elements from $W_n$, $W_{g-1}$ is invariant under addition by $A$. Therefore for any $x \not\in W_{g-1}$, the locus $A + x$ is positive dimensional and disjoint from $W_{g-1}$, but $W_{g-1}$ is an ample divisor and so must intersect any positive dimensional subvariety.

Finally, a proof of the main theorem

Proof of Theorem 1. Let $G$ be a solvable, transitive subgroup of $S_n$. If $m = 0$, then the previous section gives the desired result. Otherwise, $m > 0$, and $H$ (as defined in lemma 3) is non-trivial. Therefore the map $C^n \to C^n/G$ factors through $C^n/H$, and this is $(\text{Sym}^{m+1} C)^n$. Let $V = \text{Sym}^{m+1} C$, then $V$ is of general type if and only if $g > m + 1$ by the preceding lemma. It remains to understand $V'/((G/H)$, but this maps surjectively onto $\text{Sym}^n V$ through a finite map. The symmetric power of a variety of dimension greater than 1 is of general type if and only if the variety is of general type [AA03]. If $g > m + 1$, then by pulling back the canonical on (a desingularisation of) $\text{Sym}^n V$ to $V'/((G/H)$, shows $V'/((G/H)$ is of general type. If $g \leq m + 1$, then $V$ is not of general type, so $V'$ is not, and so $V'/((G/H)$ is not either.

3 Rational Curves on $C^n/G$

We start by considering rational curves in $\text{Sym}^n C$.

Lemma 5. Suppose $\text{Sym}^n C$ contains a curve of fibre type, then $C$ has a morphism, $f$, of degree at most $n$ to $\mathbb{P}^1$. Moreover, the fibres of this map as divisors are the points of the rational curve, up to fixed points

Proof. Let $\iota : \mathbb{P}^1 \to \text{Sym}^n C$ be the rational curve. By the theory of Jacobians, the composite map $\mathbb{P}^1 \to \text{Sym}^n C \to \text{Jac}(C)$ is trivial. Therefore all the divisors in the image of $\iota$ are linearly equivalent. Fix a point on the curve, $D_0$. Then every other point of the curve is $D_0 + D$, where $D$ is a principal divisor. This induces a map $\iota : \mathbb{P}^1 \to \mathbb{P}(\mathcal{L}(D_0))$, where $\text{div}(\iota(t)) = \iota(t) + D_0$. Fix a basis $1, f_1, \ldots, f_m$ for $\mathcal{L}(D_0)$, then $\iota(t) = \sum a_i(t)f_i$, up to scaling. If $\iota(t_0) = [(P_1, \ldots, P_n)]$ for some $t_0$, then $\iota(t_0)(P_1) = 0$. By the fibre type property, $\iota(t)(P) = 0$ for exactly one value of $t$ (for generic $P$), and so $\sum a_i(t)f_i(P)$ must be linear in $t$. Therefore $t$ can be expressed as a rational function on $C$, and the fibres are as required.
We observe that if rational points are dense on the rational curve, then the resulting map can be chosen to be defined over the rationals.

We can now prove Theorem 3 that curves of fibre type in $C^n/G$ imply the existence of rational maps on $C$ with Galois group $G$.

Proof of Theorem 3. As $C^n/G$ contains a curve of fibre type with Zariski dense rational points, $C$ has a rational map $f$ to $\mathbb{P}^1$. With respect to $f$, the function field of $C$ can be written as $K(t,x)$, where $K(t)$ is the function field for $\mathbb{P}^1$. As $f$ is degree $n$, $x$ satisfies $x^n + a_{n-1}(t)x^{n-1} + \ldots + a_0(t) = 0$, for some rational functions $a_i$. There exists a set of algebraic functions, $g_1(t), \ldots, g_m(t)$, which are all rational exactly when the polynomial has a Galois group contained in $G$. The $g_i$ are the resolvents for $G$. The function field $K(t, g_1, \ldots, g_m)$ corresponds to a curve $D$ mapping to $\mathbb{P}^1$, and the image of $D(K)$ is the rational values of $t$ for which $x^n + a_{n-1}(t)x^{n-1} + \ldots + a_0(t) = 0$ has Galois group contained in $G$. Since the curve of fibre type maps injectively onto its image, the rational points are the same, and so the fibres of $f$ over rational points are rational points on $C^n/G$. In particular, the Galois group of any fibre of $f$ above a rational point is contained in $G$, and so $D(K)$ maps onto $\mathbb{P}^1(K)$. As the latter set has density 1, it is not a thin set, and the map from $D$ to $\mathbb{P}^1$ must be an isomorphism. Therefore $g_1, \ldots, g_m \in K(t)$, and the Galois closure of $K(x,t)$ has Galois group contained in $G$.

We illustrate the utility of this result with the following

**Theorem 5.** There are at most finitely many cubic number fields $K$ with Galois group $C_3$ such that $X_0(34)(K) \neq X_0(34)(\mathbb{Q})$, where $X_0(34)$ denotes the modular curve of level $\Gamma_0(34)$.

Proof. Ozman and Siksek list $X_0(34)$ as a non-hyperelliptic genus 3 curve, where the Jacobian has rank 0 [OS18].

There is a sequence of maps $X_0(34)^3/C_3 \rightarrow \text{Sym}^3 X_0(34) \rightarrow J_0(34)$. There are only finitely many rational points on the right hand side, therefore the rational points on $\text{Sym}^3 X_0(34)$ are contained in a finite collection of rational curves, except for finitely many exceptions. These curves are necessarily of fibre type, since $X_0(34)$ is not hyperelliptic and so the Riemann-Roch space of a degree 3 divisor is at most 2 dimensional. Pulling back each rational curve to $X_0(34)^3/C_3$ give one of two cases, a pair of rational curves or an irreducible double cover of $\mathbb{P}^1$. In the former case, each must map injectively onto their image, and would be of fibre type. This would imply $X_0(34)$ has a map to $\mathbb{P}^1$ with Galois group $C_3$. In particular, $X_0(34)$ would have an automorphism of order 3, which it does not have. Therefore the pre-image of any rational curve in the symmetric power is a double cover. The double cover ramifies over points of the form $2P + Q$, and these are smooth points unless they are of the form $3P$. Suppose the double cover was not smooth in at least 2 points, then there are divisors $3P$ and $3Q$ in the same curve on the symmetric power, and so are linearly equivalent. As $P \neq Q$, $3P$ has a non-trivial section, and so by Riemann-Roch, $K_{X_0(34)} - 3P$ is effective. Therefore $K_{X_0(34)} \sim 3P + R$, and similarly,
$K_{X_0(34)} \sim 3Q + S$ for some points $R, S \in X_0(34)$. Comparing these, shows $R \sim S$, and so $R = S$. For the canonical embedding, the canonical divisor is a hyperplane section, so $P$ and $Q$ are flexes, and their tangents meet at $R$. Using an explicit model for $X_0(34)$ over the rationals, the 24 flexes and where their tangents meet the curve can be computed. The flexes are all Galois equivalent, and so are their tangents. The fourth intersection point of the tangent and the curve has coordinates which satisfy a degree 24 irreducible polynomial, and so the tangents never meet the curve at the same point, as there are 24 such points. The pullback of a rational curve on $\text{Sym}^3 X_0(34)$ therefore has at most 1 non-smooth ramification point. A triple cover of $\mathbb{P}^1$ by a genus 3 curve has total ramification degree 10, and so the pullback of a rational curve ramifies over its image in either 8 or 10 smooth points, corresponding to geometric genus 3 or 4 respectively. There can be at most finitely many rational points on such curves, and so there are finitely many points on $X_0(34)^3/C_3$.

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