Exponential Dichotomy for Hyperbolic Systems with Periodic Boundary Conditions

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Abstract

We investigate evolution families generated by general linear first-order hyperbolic systems in one space dimension with periodic boundary conditions. We state explicit conditions on the coefficient functions that are sufficient for the existence of exponential dichotomies on $\mathbb{R}$ in the space of continuous periodic functions.

1 Introduction

1.1 Problem setting and the main results

We consider linear homogeneous first-order hyperbolic systems in one space variable

$$\partial_t u_j + a_j(x, t)\partial_x u_j + \sum_{k=1}^n b_{jk}(x, t)u_k = 0, \quad (x, t) \in \mathbb{R}^2, \quad j \leq n,$$

$$\text{(1.1)}$$
with periodic boundary conditions
\[ u_j(x + 1, t) = u_j(x, t), \quad (x, t) \in \mathbb{R}^2, \quad j \leq n, \] (1.2)
and initial conditions
\[ u_j(x, s) = u_j^*(x), \quad x \in \mathbb{R}, \quad j \leq n. \] (1.3)

Here \( s \in \mathbb{R} \) is an arbitrary fixed initial time. Throughout the paper, we suppose that the coefficient functions \( a_j, b_{jk} : \mathbb{R}^2 \to \mathbb{R} \) are bounded, continuous, and 1-periodic with respect to the space variable \( x \). Furthermore, the initial data \( u_j^* : \mathbb{R} \to \mathbb{R} \) are supposed to be continuous and 1-periodic. Finally, we suppose that the leading order coefficients \( a_j \) have bounded and continuous partial derivatives in \( x \) and \( t \), and that the following condition is fulfilled:
\[ \inf \{|a_j(x,t)| : (x,t) \in \mathbb{R}^2, j \leq n\} > 0. \] (1.4)

Our goal is to state conditions on the coefficients \( a_j \) and \( b_{jk} \) such that the evolution family generated by the initial-boundary value problem (1.1)–(1.3) has an exponential dichotomy on \( \mathbb{R} \). In particular, we will prove that the following conditions are sufficient for the existence of an exponential dichotomy:

- \( b_{jj}(x,t) \neq 0 \) for all \( j, x \) and \( t \), while for all \( j \neq k \) the functions \( |b_{jk}| \) are uniformly small (in terms of the coefficients \( a_j \) and \( b_{jj} \), see Theorem 1.2).

- \( a_j(x,t)b_{jj}(x,t) < 0 \) for all \( j, x \) and \( t \), while for all \( j \neq k \), \( x \), \( t \) it holds \( a_j(x,t) \neq a_k(x,t) \) and \( b_{jk}(x,t) \to 0 \) as \( t \to \pm \infty \) uniformly in \( x \) (cf. Theorem 1.5).

To formulate our results more precisely, let us introduce the evolution family generated by (1.1)–(1.3), whose existence is stated in Theorem 1.1 below. Recall that, for \( j \leq n \) and \((x,t) \in \mathbb{R}^2\), the \( j \)-th characteristic of the system (1.1) through the point \((x,t)\) is defined as the solution \( \tau_j(\xi, x, t) \) of the initial value problem
\[ \partial_\tau \xi_j(\tau, x, t) = a_j(\xi_j(\tau, x, t), \tau), \quad \xi_j(t, x, t) = x. \]

It is easy to show by integration along characteristics that, if \( u = (u_1, ..., u_n) \) is a classical solution to (1.1)–(1.3), then
\[
\begin{align*}
u_j(x,t) &= \exp \left(-\int_s^t b_{jj}(\xi_j(\tau, x, t), \tau)d\tau\right) u_j^*(s, x, t)) \\quad - \int_s^t \exp \left(-\int_\tau^t b_{jj}(\xi_j(\tau, x, t), r)d\tau\right) \sum_{k \neq j} b_{jk}(\xi_j(\tau, x, t), \tau) u_k(\xi_j(\tau, x, t), \tau)d\tau \end{align*}
\]
for all \((x,t) \in \mathbb{R}^2\) and all \( j \leq n \). Vice versa, if the initial functions \( u_j^* \) are \( C^1 \)-smooth, then any solution \( u \) to (1.5), which is 1-periodic in \( x \), is a classical solution to (1.1)–(1.3).
We will work in the Banach space
\[ C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) := \{ u \in C(\mathbb{R}; \mathbb{R}^n) : u(x + 1) = u(x) \text{ for all } x \in \mathbb{R} \} \]

normed by
\[ \| u \| := \sup \{|u_j(x)| : x \in \mathbb{R}, j \leq n\}. \tag{1.6} \]

As usual, the space of all linear bounded operators \( A : C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \rightarrow C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \) will be denoted by \( \mathcal{L}(C_{\text{per}}(\mathbb{R}; \mathbb{R}^n)) \), and the operator norm will be defined by
\[ \| A \| := \sup \{ \| Au \| : u \in C_{\text{per}}(\mathbb{R}; \mathbb{R}^n), \| u \| \leq 1 \}. \]

The following result states that the initial-boundary value problem (1.1)–(1.3) is well-posed.

**Theorem 1.1** Suppose (1.4) holds. Then, given \( s \in \mathbb{R} \), for every \( u^s \in C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \) there exists exactly one continuous function \( u : \mathbb{R}^2 \rightarrow \mathbb{R}^n \) satisfying (1.2) and (1.5). Moreover, the map
\[ u^s \mapsto U(t, s)u^s := u(\cdot, t) \]
from \( C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \) to itself defines a strongly continuous, exponentially bounded evolution family of invertible operators \( U(t, s) \in \mathcal{L}(C_{\text{per}}(\mathbb{R}; \mathbb{R}^n)) \), which means that
- \( U(t,t) = I \) and \( U(t,s) = U(t,r)U(r,s) \) for all \( t, r, s \in \mathbb{R} \),
- the map \( (t,s) \in \mathbb{R}^2 \mapsto U(t,s)u \in C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \) is continuous for each \( u \in C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \),
- there exist \( K \geq 1 \) and \( \omega \in \mathbb{R} \) such that
\[ \| U(t, s) \| \leq Ke^{\omega(t-s)} \text{ for all } t \geq s. \tag{1.7} \]

In order to formulate our main results, let us introduce the following notation:
\[ \alpha_j^- := \inf \{ a_{j}(x,t) : (x,t) \in \mathbb{R}^2 \}, \]
\[ \alpha_j^+ := \sup \{ a_{j}(x,t) : (x,t) \in \mathbb{R}^2 \}, \]
\[ \beta_j^- := \inf \{ b_{jj}(x,t) : (x,t) \in \mathbb{R}^2 \}, \]
\[ \beta_j^+ := \sup \{ b_{jj}(x,t) : (x,t) \in \mathbb{R}^2 \}, \]
\[ \beta_j := \sup \left\{ \sum_{k \neq j} |b_{jk}(x,t)| : (x,t) \in \mathbb{R}^2 \right\}. \]

By the assumption (1.4), either \( \alpha_j^- > 0 \) or \( \alpha_j^+ < 0 \).

We are now prepared to formulate our first sufficient condition for the existence of an exponential dichotomy for (1.1)–(1.3) on \( \mathbb{R} \).
Theorem 1.2 Suppose (1.4) holds. Moreover, suppose that
\[
\inf\{|b_{jj}(x,t)| : (x,t) \in \mathbb{R}^2, j \leq n\} > 0 \tag{1.8}
\]
and the following inequalities are true for all \( j \leq n \):
\[
\beta_j < \beta_j^+ \frac{\alpha_j^-}{\alpha_j^+} \quad \text{if} \quad \alpha_j^- > 0, \beta_j^+ > 0, \tag{1.9}
\]
\[
\beta_j < -\beta_j^+ \frac{\alpha_j^+}{\alpha_j^-} \left(1 - e^{-\beta_j^+/\alpha_j^-}\right) \quad \text{if} \quad \alpha_j^+ < 0, \beta_j^- < 0, \tag{1.10}
\]
\[
\beta_j < \beta_j^+ \frac{1 - e^{\beta_j^-/\alpha_j^+}}{1 - e^{-\beta_j^+/\alpha_j^-}} \left(e^{\beta_j^+/\alpha_j^-} - e^{\beta_j^-/\alpha_j^+} + 1\right)^{-1} \quad \text{if} \quad \alpha_j^- > 0, \beta_j^+ < 0, \tag{1.11}
\]
\[
\beta_j < -\beta_j^- \frac{1 - e^{\beta_j^+/\alpha_j^-}}{1 - e^{-\beta_j^-/\alpha_j^+}} \left(e^{\beta_j^-/\alpha_j^+} - e^{\beta_j^+/\alpha_j^-} + 1\right)^{-1} \quad \text{if} \quad \alpha_j^+ < 0, \beta_j^- > 0. \tag{1.12}
\]
Then the evolution family \( U(t,s) \) has an exponential dichotomy on \( \mathbb{R} \), which means that there exist a projection \( P = P^2 \in \mathcal{L}(C_{\text{per}}(\mathbb{R}; \mathbb{R}^n)) \) and positive constants \( M \) and \( \omega \) such that
\[
\|U(t,0)PU(0,s)\| + \|U(s,0)(I - P)U(0,t)\| \leq Me^{-\omega(t-s)} \quad \text{for all} \quad t \geq s.
\]

Remark 1.3 Roughly speaking, Theorem 1.2 claims the following: Given \( a_j \) and \( b_{jj} \) satisfying (1.4) and (1.8), the evolution family \( U(t,s) \) has an exponential dichotomy on \( \mathbb{R} \) if \( b_{jk} \) with \( j \neq k \) are sufficiently small, in the sense of the inequalities (1.9)–(1.12).

Remark 1.4 If the coefficients \( a_j \) and \( b_{jj} \) are constants, then
\[
\alpha_j^+ = \alpha_j^- = a_j, \beta_j^+ = \beta_j^- = b_{jj}
\]
and, hence, (1.8)–(1.12) is equivalent to
\[
\beta_j < |b_{jj}| \quad \text{if} \quad a_j b_{jj} > 0, \quad \beta_j < \frac{|b_{jj}|}{2e^{b_{jj}/a_j} - 1} \quad \text{if} \quad a_j b_{jj} < 0.
\]
In particular, if \( n = 2 \), then these conditions read
\[
\begin{align*}
|b_{12}| &< |b_{11}| \quad \text{if} \quad a_1 b_{11} > 0, \quad |b_{21}| < |b_{22}| \quad \text{if} \quad a_2 b_{22} > 0, \\
|b_{12}| &< \frac{|b_{11}|}{2e^{-b_{11}/a_1} - 1} \quad \text{if} \quad a_1 b_{11} < 0, \quad |b_{21}| < \frac{|b_{22}|}{2e^{-b_{22}/a_2} - 1} \quad \text{if} \quad a_2 b_{22} < 0.
\end{align*}
\]
\( \tag{1.13} \)
Now we formulate our second sufficient condition for the existence of an exponential dichotomy for (1.1)–(1.3) on \( \mathbb{R} \).

**Theorem 1.5** Suppose that (1.4) is true and that either

\[
\sup \{ b_{jj}(x,t) : (x,t) \in \mathbb{R}^2, j \leq n \} < 0 \tag{1.14}
\]

or

\[
\inf \{ b_{jj}(x,t) : (x,t) \in \mathbb{R}^2, j \leq n \} > 0. \tag{1.15}
\]

Moreover, suppose the following:

for all \( 1 \leq j \neq k \leq n \) and \( \varepsilon > 0 \) there exists \( c > 0 \) such that

\[
|b_{jk}(x,t)| < \varepsilon \text{ for all } x \in \mathbb{R} \text{ and } t \in \mathbb{R} \setminus [-c,c] \tag{1.16}
\]

and

for all \( 1 \leq j \neq k \leq n \) there exists \( \tilde{b}_{jk} \in C^1(\mathbb{R}^2) \) such that

\[
b_{jk}(x,t) = \tilde{b}_{jk}(x,t)(a_j(x,t) - a_k(x,t)) \text{ for all } (x,t) \in \mathbb{R}^2. \tag{1.17}
\]

Then the evolution family \( U(t,s) \) has an exponential dichotomy on \( \mathbb{R} \).

**Remark 1.6** The condition (1.16) implies that \( b_{jk}(x,t) \to 0 \) as \( t \to \pm \infty \) uniformly in \( x \in \mathbb{R} \) for each \( j \neq k \). In particular, (1.16) is satisfied for time-constant or time-periodic "non-diagonal" coefficients if and only if they are identically zero. This is a disadvantage of Theorem 1.5. However, the advantage of Theorem 1.5 is that the "non-diagonal" coefficients have to be small only for large \( |t| \) rather than uniformly over \( x \) and \( t \) (like in Theorem 1.2).

Our approach to proving Theorems 1.2 and 1.5 is based on the following criterion [11, Theorem 1.1]:

**Theorem 1.7** A strongly continuous, exponentially bounded evolution family \( \{U(t,s)\}_{t \geq s} \) on a Banach space \( X \) has an exponential dichotomy on \( \mathbb{R} \) if and only if for every bounded and continuous map \( \tilde{f} : \mathbb{R} \to X \) there exists a unique bounded and continuous map \( \tilde{u} : \mathbb{R} \to X \) such that

\[
\tilde{u}(t) = U(t,s)u(s) + \int_s^t U(t,\tau)\tilde{f}(\tau)d\tau \text{ for all } t \geq s. \tag{1.18}
\]

For proving Theorems 1.2 and 1.5, we set \( X = C_{\text{per}}(\mathbb{R};\mathbb{R}^n) \) and apply Theorem 1.7 to the evolution family \( U(t,s) \) generated by the initial-boundary value problem (1.1)–(1.3) according to Theorem 1.1. We do this as explained below.
Let \( \tilde{f} : \mathbb{R} \to C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \) be a bounded and continuous map. We first show (cf. Lemma 4.1) that a bounded and continuous map \( \tilde{u} : \mathbb{R} \to C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \) is a solution to (1.18) if and only if \( f(x, t) := [\tilde{f}(t)](x) \) and \( u(x, t) := [\tilde{u}(t)](x) \) satisfy equations

\[
u_j(x, t) = c_j(0, x, t) \nu_j(1, \tau_j(0, x, t)) - \int_0^x d_j(\xi, x, t) \left( \sum_{k \neq j} b_{jk}(\xi, \tau_j(\xi, x, t)) \nu_k(\xi, \tau_j(\xi, x, t)) - f_j(\xi, \tau_j(\xi, x, t)) \right) d\xi \tag{1.19}
\]

for all \( j \leq n \), where

\[
c_j(\xi, x, t) := \exp \left( - \int_\xi^x \frac{b_{jj}(\eta, \tau_j(\eta, x, t))}{a_j(\eta, \tau_j(\eta, x, t))} d\eta \right), \quad d_j(\xi, x, t) := \frac{c_j(\xi, x, t)}{a_j(\xi, \tau_j(\xi, x, t))}. \tag{1.20}
\]

Here, for a given \((x, t) \in \mathbb{R}^2\), by \( \tau_j(\xi, x, t) \) we denote the solution to the initial value problem

\[
\frac{\partial \xi}{\partial \xi} \tau_j(\xi, x, t) = 1, \quad \tau_j(x, x, t) = t, \tag{1.21}
\]

i.e. \( \tau_j(\cdot, x, t) = (\xi_j(\cdot, x, t))^{-1} \).

Now, on the account of Theorem 1.7, the existence of an exponential dichotomy is reduced to the unique solvability of (1.19) for every \( f \). To prove the last fact, we rewrite the system of integral equations (1.19) in the operator form

\[
u = \mathcal{C} \nu + \mathcal{D} \nu + \mathcal{F} f
\]

with certain linear bounded operators \( \mathcal{C}, \mathcal{D} \) and \( \mathcal{F} \). Assumptions (1.4) and (1.8) of Theorem 1.2 imply that \( I - \mathcal{C} \) is invertible (cf. Lemma 4.2), while Assumptions (1.9)–(1.12) imply that

\[
\|\mathcal{D}\| < \frac{1}{\|I - \mathcal{C}\|^{-1}}
\]

(cf. Lemma 4.3 and Corollary 4.4). This, in its turn, gives the invertibility of \( I - \mathcal{C} - \mathcal{D} \), as desired. Assumptions of Theorem 1.5 ensure that the operator \( I - \mathcal{C} - \mathcal{D} \) is Fredholm of index zero and that it is injective, what immediately gives the desired bijectivity.

**Remark 1.8** The well-known relationship between the exponential dichotomy and the Green’s function (see e.g. the proof of [11, Theorem 1.1]) can be stated as follows. Suppose that the assumptions of Theorem 1.2 or Theorem 1.5 are fulfilled. Let \( U(t, s) \) be the evolution family on \( C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \) generated by the problem (1.1)–(1.3). Then for every bounded function \( f : \mathbb{R} \to C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \) the equation (1.18) has a unique bounded continuous solution \( u : \mathbb{R} \to C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \) given by the Green’s formula

\[
u(t) = \int_{-\infty}^{\infty} G(t, s) f(s) ds,
\]
where
\[ G(t, s) = \begin{cases} U(t, 0)PU(0, s) & \text{for } t > s, \\ -U(s, 0)(I - P)U(0, t) & \text{for } t < s. \end{cases} \]

The paper is organized as follows. Section 2 provides examples showing that the assumptions of Theorems 1.2 and 1.5 are essential. In Section 3 we give a proof of Theorem 1.1 about the existence of an evolution family. In Section 4.1 we establish an equivalence between the mild and weak continuous solution concepts. Theorems 1.2 and 1.5 are proved in Sections 4.2 and 4.3, respectively. Section 5 contains a concluding discussion and open problems.

2 Examples

Example 2.1 Consider the $2 \times 2$-hyperbolic system with non-zero constant coefficients
\[
\begin{align*}
\partial_t u_1 + a_1 \partial_x u_1 + b_{11} u_1 + b_{12} u_2 &= 0, \\
\partial_t u_2 + a_2 \partial_x u_2 + b_{21} u_1 + b_{22} u_2 &= 0
\end{align*}
\] (2.1)
subjected to the periodic conditions in the space variable
\[ u_1(x, t) = u_1(x + 1, t), \quad u_2(x, t) = u_2(x + 1, t). \] (2.2)

The problem (2.1)–(2.2) has constant nontrivial solutions (which obviously prevents an exponential dichotomy on $\mathbb{R}$) iff
\[ b_{11} b_{22} - b_{12} b_{21} = 0. \] (2.3)
On the other hand, the assumptions (1.9)–(1.12) of Theorem 1.2 for the problem (2.1)–(2.2) are equivalent to (1.13). This implies the inequalities
\[ |b_{12}| < |b_{11}| \text{ and } |b_{21}| < |b_{22}|, \] (2.4)
contradicting to (2.3). It follows that the assumptions (1.9)–(1.12) are essential for the statement of Theorem 1.2.

The problem (2.1)–(2.2) has $x$-independent non-constant time-periodic solutions (what, again, prevents an exponential dichotomy on $\mathbb{R}$) if the ODE system
\[ u'_1 + b_{11} u_1 + b_{12} u_2 = u'_2 + b_{21} u_1 + b_{22} u_2 = 0 \] (2.5)
has non-constant periodic solutions. The characteristic equation corresponding to this system reads
\[ \lambda^2 - \lambda(b_{11} + b_{22}) + (b_{11} b_{22} - b_{12} b_{21}) = 0 \] (2.6)
The system (2.5) has non-constant time-periodic solutions if and only if the equation (2.6) has nonzero purely complex solutions. The latter is true if
\[ b_{22} = -b_{11}, \quad b_{11}^2 < -b_{12}b_{21}, \]
which again contradicts to (2.4) and, hence, to the assumptions (1.8)–(1.12) of Theorem 1.2.

Note also that the problem (2.1)–(2.2) does not satisfy the condition (1.16) of Theorem 1.5.

**Example 2.2** Recall that a dichotomy system is exponentially stable if the dichotomy projection coincides with the identity operator. We now show that the assumptions of Theorems 1.2 and 1.5 do not necessarily imply the exponential stability.

Suppose that \( n = 2 \) and consider the decoupled system
\[
\begin{align*}
\partial_t u_1 + a_1(x, t)\partial_x u_1 + b_{11}(x, t)u_1 &= 0, \\
\partial_t u_2 + a_2(x, t)\partial_x u_2 + b_{22}(x, t)u_2 &= 0
\end{align*}
\]
with the conditions (1.2) and (1.3). Suppose that, in addition to the conditions (1.4) and (1.8), we have \( b_{11} > 0 \) and \( b_{22} < 0 \). The solution to (2.7), (1.2), (1.3) is given by the formulas
\[
\begin{align*}
u_1(x, t) &= \exp \left( -\int_{t}^{s} b_{11}(\xi_1(r, s, t), r)dr \right) u_1^s(\xi_1(s, x, t)), \\
\nu_2(x, t) &= \exp \left( -\int_{s}^{t} b_{22}(\xi_2(r, s, t), r)dr \right) u_2^s(\xi_2(s, x, t)).
\end{align*}
\]

It follows that \( \nu_1 \) exponentially decays as \( t \to \infty \), while \( \nu_2 \) exponentially decays as \( t \to -\infty \), for any \( u^s \in C_{\text{per}}(\mathbb{R}; \mathbb{R}^2) \). One can easily define the dichotomy projection as \( Pu = (u_1, 0) \), hence \( (I - P)u = (0, u_2) \). Since \( P \neq I \), the problem is not exponentially stable.

## 3 Proof of Theorem 1.1

Assuming that the condition (1.4) is fulfilled, we have to prove that the problem (1.1)–(1.3) generates an exponentially bounded evolution family \( U(t, s) \) on \( C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \). For the proof we use [7, Theorem 2.1] stating that under the zero-order compatibility conditions between (1.2) and (1.3), which are automatically fulfilled for \( u^s \in C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \), the system (1.5) has a unique continuous solution. This means that there exists a unique strongly continuous evolution family \( U(t, s) \) on \( C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \) associated to (1.1)–(1.3). To prove that \( U(t, s) \) is exponentially bounded, we use the following a priori estimate derived in the proof of [7, Theorem 2.1]:
\[
\max_{j, x} \max_{s \leq r \leq t} |u_{j}| \leq (3 + 2n)^{\frac{t}{\tau}} \|u^s\| \quad \text{for all } t \geq s,
\]
where
\[
\theta = \min \left\{ \left( 2 \sup_{j,x,t} |a_j| \right)^{-1}, \left( 2n(n + 1) \sup_{j,k,x,t} |b_{jk}| \right)^{-1} \right\}.
\]
Since \( u = U(t, s)u^s \), then from (3.1) we get
\[
\|U(t, s)\| \leq (3 + 2n)^{\frac{t-s}{\theta}} \leq \exp \left\{ \frac{\log (3 + 2n)^{\theta}(t-s)}{\theta} \right\}.
\]
This means that the estimate (1.7) is true with \( K = 1 \) and \( \omega = \theta^{-1} \log (3 + 2n) \). Note that here we essentially use the boundedness of \( a_j \) and \( b_{jk} \).

Theorem 1.1 is therewith proved.

4 Existence of an exponential dichotomy on \( \mathbb{R} \)

4.1 Equivalence of the mild and weak solution concepts

Here we establish the equivalence between the mild and weak continuous solution concepts, i.e. the equivalence of the equations (1.18) and (1.19), respectively.

Let \( BC(\mathbb{R}; C_{\text{per}}(\mathbb{R}; \mathbb{R}^n)) \) be the Banach space of all bounded and continuous maps \( u : \mathbb{R} \to C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \), with the norm
\[
\|u\|_{\infty} := \sup_{t \in \mathbb{R}} \|u(t)\|,
\]
where \( \| \cdot \| \) is the norm in \( C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \) introduced in (1.6). As usual, we identify functions \( u \in BC(\mathbb{R}; C_{\text{per}}(\mathbb{R}; \mathbb{R}^n)) \) with functions \( \tilde{u} \in BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n) \) by means of \( \tilde{u}(x, t) = [u(t)](x) \).

Below we will use the same notation for the corresponding elements of the two spaces.

To shorten notation, we will write \( \tau_j(\xi) = \tau_j(\xi, x, t) \) and \( \xi_j(\tau) = \xi_j(\tau, x, t) \).

Lemma 4.1 Let \( f \in BC(\mathbb{R}; C_{\text{per}}(\mathbb{R}; \mathbb{R}^n)) \), the function \( u \in BC(\mathbb{R}; C_{\text{per}}(\mathbb{R}; \mathbb{R}^n)) \) satisfies (1.18) if and only if the corresponding function \( u \in BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n) \) satisfies (1.19) with the corresponding function \( f \in BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n) \).

Proof. The proof is divided into two claims.

Claim 1. Let \( f \in BC(\mathbb{R}; C_{\text{per}}(\mathbb{R}; \mathbb{R}^n)) \). A function \( u \in BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n) \) satisfies the system (1.19) if and only if it satisfies the system
\[
\begin{align*}
&u_j(x, t) = \exp \left( \int_t^s b_{jj}(\xi_j(r), r)dr \right) u^s_j(\xi_j(s)) \\
&- \int_s^t \exp \left( \int_t^r b_{jj}(\xi_j(r), r)dr \right) \left[ \sum_{k \neq j} b_{jk}(\xi_j(\tau), \tau) u_k(\xi_j(\tau), \tau) - f_j(\xi_j(\tau), \tau) \right] d\tau.
\end{align*}
\]
To prove Claim 1, note that (1.19) and (4.1) are two weak formulations of the problem

\[ \partial_t u_j + a_j(x,t)\partial_x u_j + \sum_{k=1}^n b_{jk}(x,t)u_k = f_j(x,t), \quad (x,t) \in \mathbb{R}^2, \quad j \leq n, \quad (4.2) \]
\[ u_j(x+1,t) = u_j(x,t), \quad (x,t) \in \mathbb{R}^2, \quad j \leq n, \quad (4.3) \]
both obtained by the integration along characteristic curves. Let us prove the sufficiency (the necessity is proved similarly). Suppose that \( u \) satisfies (4.1). One can easily compute the directional distributional derivative:

\[
(\partial_t + a_j(x,t)\partial_x)u_j(x,t) = -u_j^s(\xi_j(s))b_{jj}(x,t) \exp \int_t^s b_{jj}(\xi_j(r),r)dr - \sum_{k \neq j} b_{jk}(x,t)u_k(x,t) + f_j(x,t) \\
+ b_{jj}(x,t) \int_t^s \exp \int_t^r b_{jj}(\xi_j(r),r)dr \left[ \sum_{k \neq j} (b_{jk}u_k)(\xi_j(\tau),\tau) - f_j(\xi_j(\tau),\tau) \right] d\tau \\
= -\sum_{k=1}^n b_{jk}(x,t)u_k(x,t) + f_j(x,t), \quad j \leq n,
\]
the last equality being true due to (4.1). Here we used that

\[ (\partial_t + a_j(x,t)\partial_x)\xi_j(\tau, x, t) = 0. \]

Hence, the function \( u \) solves the problem (4.2)–(4.3) where the differential equations are fulfilled in a distributional sense. Without destroying the equalities in \( D' \), we rewrite the system (4.2) in the form

\[
(\partial_t + a_j(x,t)\partial_x) \left( c_j(0, x, t)^{-1}u_j \right) + c_j(0, x, t)^{-1} \sum_{k \neq j} b_{jk}(x,t)u_k = c_j(0, x, t)^{-1}f_j(x,t), \quad j \leq n.
\]

(4.4)

To prove that \( u \) satisfies (1.19) pointwise, we use (4.4) and the constancy theorem of distribution theory claiming that any distribution on an open set with zero generalized derivatives is a constant on any connected component of the set. As a consequence, the function

\[
G_j(x,t) = c_j(0, x, t)^{-1} \left[ u_j(x,t) + \int_0^x d_j(\xi,x,t) \left( \sum_{k \neq j} (b_{jk}u_k)(\xi,j_\tau(\xi)) - f_j(\xi,j_\tau(\xi)) \right) d\xi \right]
\]

(4.5)
is constant along the characteristic \( \tau_j(\xi, x, t) \) for all \( j \leq n \). Since \( G_j(x,t) \) is a continuous function, by the periodicity condition (4.3), we get

\[
G_j(x,t) = G_j(\xi, \tau_j(\xi)) = G_j(0, \tau_j(0)) = u_j(0, \tau_j(0)) = u_j(1, \tau_j(0)).
\]

(4.6)
Combining (4.5) with (4.6), we obtain (1.19), completing the proof of Claim 1.

Claim 2. Let \( f \in BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n) \). A function \( u \in BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n) \) satisfies (4.1) iff \( u \in BC(\mathbb{R}; C_{\text{per}}(\mathbb{R}; \mathbb{R}^n)) \) satisfies (1.18) with \( f \in BC(\mathbb{R}; C_{\text{per}}(\mathbb{R}; \mathbb{R}^n)) \).

To prove Claim 2, we introduce a two-parameter strongly continuous, exponentially bounded evolution family \( U_0(t, s) \in L(C_{\text{per}}(\mathbb{R}; \mathbb{R}^n)), t \geq s, \) by

\[
(U_0(t, s)u)(x) = \left[ \exp \left( \int_t^s b_{jj}(\xi_j(x, t, r), r)dr \right) u_j(\xi_j(s), s) \right]_{j=1}^n. \tag{4.7}
\]

Let \( B(t) : C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \to C_{\text{per}}(\mathbb{R}; \mathbb{R}^n) \) be a one-parameter family of linear operators defined by

\[
[B(t)v](x) = \left[ -\sum_{k \neq j} b_{jk}(x, t)v_k(x) \right]_{j=1}^n. \tag{4.8}
\]

In terms of \( U_0 \) the system (4.1) with \( f \equiv 0 \) reads

\[
u(t) = U_0(t, s)u(s) + \int_s^t U_0(t, \tau)B(\tau)u(\tau) \, d\tau.
\]

By the definition of the evolution operator \( U(t, s) \) we have

\[
U(t, s)u(s) = U_0(t, s)u(s) + \int_s^t U_0(t, \tau)B(\tau)U(\tau, s)u(s) \, d\tau,
\]

which gives us the following relation between \( U \) and \( U_0 \):

\[
U(t, s) = U_0(t, s) + \int_s^t U_0(t, \tau)B(\tau)U(\tau, s) \, d\tau. \tag{4.9}
\]

To prove the sufficiency part of Claim 2, assume that \( u \in BC(\mathbb{R}; C_{\text{per}}(\mathbb{R}; \mathbb{R}^n)) \) satisfies (1.18). By (4.9), the equation (1.18) can be written as

\[
u(t) = U(t, s)u(s) + \int_s^t U(t, \tau)f(\tau) \, d\tau
\]

\[
= U_0(t, s)u(s) + \int_s^t U_0(t, \tau)B(\tau)U(\tau, s)u(s) \, d\tau + \int_s^t \int_s^\tau U_0(t, \sigma)B(\sigma)U(\sigma, \tau)f(\tau) \, d\sigma \, d\tau. \tag{4.10}
\]

Plugging (1.18) into the second summand in the right-hand side of (4.10) yields

\[
u(t) = U_0(t, s)u(s) + \int_s^t U_0(t, \tau)B(\tau)u(\tau) \, d\tau - \int_s^t \int_s^\tau U_0(t, \tau)B(\tau)U(t, \sigma)f(\sigma) \, d\sigma \, d\tau
\]

\[
+ \int_s^t U_0(t, \tau)f(\tau) \, d\tau + \int_s^t \int_s^\tau U_0(t, \sigma)B(\sigma)U(\sigma, \tau)f(\tau) \, d\sigma \, d\tau. \tag{4.11}
\]
After the changing of the order of integration in the last summand, the third and the last
summands cancel out, and we get

\[ u(t) = U_0(t, s)u(s) + \int_s^t U_0(t, \tau)B(\tau)u(\tau) \, d\tau + \int_s^t U_0(t, \tau)f(\tau) \, d\tau. \]  

(4.12)

Taking into account (4.7) and (4.8), we see that (4.12) coincides with (4.1), as desired.

To prove the necessity, let \( u \in BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n) \) satisfy (4.1). In terms of (4.7) and (4.8), the equation (4.1) coincides with (4.12) or, the same, with (4.11). Applying the formula (4.9) to (4.11), we get

\[ u(t) = U_0(t, s)u(s) + \int_s^t U_0(t, \tau)B(\tau)u(\tau) \, d\tau + \int_s^t U_0(t, \tau)f(\tau) \, d\tau \]

(4.13)

Set

\[ v(\tau) = u(\tau) - \int_s^\tau U(\tau, \rho)f(\rho) \, d\rho. \]  

(4.14)

Now (4.13) reads

\[ v(t) = U_0(t, s)u(s) + \int_s^t U_0(t, \tau)B(\tau)v(\tau) \, d\tau. \]  

(4.15)

Combining (4.15) with (4.9), we conclude that \( v(t) = U(t, s)u(s) \). On the account of (4.14), the equation (4.13) admits the representation

\[ u(t) = U_0(t, s)u(s) + \int_s^t U_0(t, \tau)B(\tau)U(\tau, s)u(s) \, d\tau + \int_s^t U(t, \tau)f(\tau) \, d\tau \]

\[ = U(t, s)u(s) + \int_s^t U(t, \tau)f(\tau) \, d\tau, \]

what finishes the proof of Claim 2.

Lemma 4.1 readily follows from Claims 1 and 2.
4.2 Proof of Theorem 1.2

Assuming that the assumptions (1.4), (1.8) and (1.9)–(1.12) are fulfilled, we have to prove that the evolution family of the original problem has an exponential dichotomy on \( \mathbb{R} \).

Let us introduce operators \( C, D, F \in \mathcal{L}(BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n)) \) by

\[
(Cu)_j(x,t) := \exp \left( -\int_0^x \frac{b_j(\eta, \tau_j(\eta, x, t))}{a_j(\eta, \tau_j(\eta, x, t))} d\eta \right) u_j(1, \tau_j(0, x, t)),
\]

\[
(Du)_j(x,t) := -\int_0^x d_j(\xi, x, t) \sum_{k \neq j} b_{jk}(\xi, \tau_j(\xi, x, t)) u_k(\xi, \tau_j(\xi, x, t)) d\xi,
\]

\[
(Ff)_j(x,t) := \int_0^x d_j(\xi, x, t) f_j(\xi, \tau_j(\xi, x, t)) d\xi.
\]

Then the equation (1.19) reads

\[ u = Cu + Du + Ff. \]

On the account of Theorem 1.7 and Lemma 4.1, we are reduced to show that, given \( f \in BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n) \), the system (1.19) has a unique solution in \( BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n) \). In other words, we have to prove that the operator \( I - C - D \in \mathcal{L}(BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n)) \) is bijective.

The proof will be divided into two lemmas.

**Lemma 4.2** Suppose that the conditions (1.4) and (1.8) are fulfilled. Then the operator \( I - C \) is bijective, and for all \( j \leq n \) and \( \|u\| = 1 \) it holds

\[
\left\| [(I - C)^{-1}u]_j \right\| \leq \frac{1}{1 - e^{-j/\alpha_j}} \quad \text{if} \quad \alpha_j^- > 0, \beta_j^- > 0, \quad (4.16)
\]

\[
\left\| [(I - C)^{-1}u]_j \right\| \leq \frac{1}{1 - e^{-j/\alpha_j}} \quad \text{if} \quad \alpha_j^+ < 0, \beta_j^+ < 0, \quad (4.17)
\]

\[
\left\| [(I - C)^{-1}u]_j \right\| \leq \frac{e^{j/\alpha_j^-} - e^{-j/\alpha_j^-}}{1 - e^{j/\alpha_j^-}} + 1 \quad \text{if} \quad \alpha_j^- > 0, \beta_j^+ < 0, \quad (4.18)
\]

\[
\left\| [(I - C)^{-1}u]_j \right\| \leq \frac{e^{j/\alpha_j^+} - e^{-j/\alpha_j^+}}{1 - e^{j/\alpha_j^+}} + 1 \quad \text{if} \quad \alpha_j^+ < 0, \beta_j^- > 0. \quad (4.19)
\]

**Proof.** For all \( j \leq n \) and \( (x, t) \in \mathbb{R}^2 \) we have

\[
-\frac{b_{jj}(x, t)}{a_j(x, t)} \leq -\frac{\beta_j^-}{\alpha_j^-} < 0 \quad \text{if} \quad \alpha_j^- > 0, \beta_j^- > 0,
\]

\[
-\frac{b_{jj}(x, t)}{a_j(x, t)} \leq -\frac{\beta_j^+}{\alpha_j^+} < 0 \quad \text{if} \quad \alpha_j^+ < 0, \beta_j^+ < 0,
\]
\[0 < -\frac{b_{jj}(x, t)}{a_j(x, t)} \leq -\frac{\beta_j^-}{\alpha_j^-} \quad \text{if} \quad \alpha_j^- > 0, \quad \beta_j^- < 0,
\]
\[0 < -\frac{b_{jj}(x, t)}{a_j(x, t)} \leq -\frac{\beta_j^+}{\alpha_j^+} \quad \text{if} \quad \alpha_j^+ < 0, \quad \beta_j^- > 0.
\]

Hence, for all \( j \leq n \) and \( \|u\| = 1 \) it holds
\[
\|(Cu)_j\| \leq e^{-\beta_j^-/\alpha_j^-} < 1 \quad \text{if} \quad \alpha_j^- > 0, \quad \beta_j^- > 0, \quad (4.20)
\]
\[
\|(Cu)_j\| \leq e^{-\beta_j^+/\alpha_j^+} < 1 \quad \text{if} \quad \alpha_j^+ < 0, \quad \beta_j^+ < 0, \quad (4.21)
\]
\[
\|(Cu)_j\| \leq e^{-\beta_j^-/\alpha_j^-} \quad \text{if} \quad \alpha_j^- > 0, \quad \beta_j^+ < 0, \quad (4.22)
\]
\[
\|(Cu)_j\| \leq e^{-\beta_j^+/\alpha_j^+} \quad \text{if} \quad \alpha_j^+ < 0, \quad \beta_j^- > 0. \quad (4.23)
\]

Now the bounds (4.16) and (4.17) easily follow from (4.20) and (4.21), respectively.

To prove (4.18) and (4.19), for an arbitrary fixed \( f \in BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n) \) consider the equation
\[ u = Cu + f \text{ with respect to } u \in BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n). \]
Then for all \( j \leq n \) and \( x, t \in \mathbb{R} \) we have
\[
u_j(x, t) = \exp \left( -\int_0^x \frac{b_{jj}(\eta, \tau_j(\eta, x, t))}{a_j(\eta, \tau_j(\eta, x, t))} d\eta \right) \nu_j(1, \tau_j(0, x, t)) + f_j(x, t). \quad (4.24)
\]

In particular,
\[
u_j(1, t) = \exp \left( -\int_0^1 \frac{b_{jj}(\eta, \tau_j(\eta, 1, t))}{a_j(\eta, \tau_j(\eta, 1, t))} d\eta \right) \nu_j(1, \tau_j(0, 1, t)) + f_j(1, t). \quad (4.25)
\]

Introduce operators \( \tilde{C}_j \in \mathcal{L}(BC(\mathbb{R})) \) by
\[
(\tilde{C}_j v)(t) := \exp \left( -\int_0^1 \frac{b_{jj}(\eta, \tau_j(\eta, 1, t))}{a_j(\eta, \tau_j(\eta, 1, t))} d\eta \right) v(\tau_j(0, 1, t)).
\]

We see at once that the operators \( \tilde{C}_j \) are bijective and
\[
\|\tilde{C}_j^{-1}\| \leq e^{\beta_j^+/\alpha_j^+} < 1 \quad \text{if} \quad \alpha_j^- > 0, \quad \beta_j^+ < 0, \quad (4.26)
\]
\[
\|\tilde{C}_j^{-1}\| \leq e^{\beta_j^-/\alpha_j^-} < 1 \quad \text{if} \quad \alpha_j^+ < 0, \quad \beta_j^- > 0. \quad (4.27)
\]

Hence, (4.25) is uniquely solvable with respect to \( \bar{u}(t) = u(1, t) \). Moreover,
\[
\bar{u}_j = - (I - \tilde{C}_j^{-1})^{-1} \tilde{C}_j^{-1} \bar{f}_j;
\]
where \( \bar{f}_j = f_j(1, t) \). Inserting this into (4.24) and letting \( C_j u_j := (Cu)_j \), we get
\[
u_j = - C_j (I - \tilde{C}_j^{-1})^{-1} \tilde{C}_j^{-1} \bar{f}_j + f_j,
\]
what entails that
\[
\|(I - C_j)^{-1}\| \leq \|C_j\| (I - \tilde{C}_j^{-1})^{-1} \|\tilde{C}_j^{-1}\| + 1.
\]
Now, (4.18) and (4.19) follow from (4.22), (4.23), (4.26) and (4.27). \(\square\)
By Lemma 4.2, the bijectivity of $I - C - D \in \mathcal{L}(BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n))$ is equivalent to the bijectivity of $I - (I - C)^{-1}D \in \mathcal{L}(BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n))$.

**Lemma 4.3** Suppose that the conditions (1.4) and (1.8) are fulfilled. Then for all $j \leq n$ and $\|u\| = 1$ it holds

$$\left\| (I - C)^{-1}Du_j \right\| \leq \frac{\beta_j \alpha_j^+}{\beta_j^+ \alpha_j}$$

if $\alpha_j^- > 0$, $\beta_j^{-} > 0$,

$$\left\| (I - C)^{-1}Du_j \right\| \leq -\frac{\beta_j \alpha_j^-}{\beta_j^+ \alpha_j^-} \left(1 - e^{-\beta_j^+ / \alpha_j^-}\right)$$

if $\alpha_j^+ < 0$, $\beta_j^+ < 0$,

$$\left\| (I - C)^{-1}Du_j \right\| \leq \frac{\beta_j}{\beta_j^+} \frac{1 - e^{-\beta_j^- / \alpha_j^+}}{1 - e^{-\beta_j^- / \alpha_j^-}} \left(e^{\beta_j^- / \alpha_j^-} - e^{-\beta_j^- / \alpha_j^-} + 1\right)$$

if $\alpha_j^- > 0$, $\beta_j^+ < 0$,

$$\left\| (I - C)^{-1}Du_j \right\| \leq -\frac{\beta_j}{\beta_j^+} \frac{1 - e^{-\beta_j^- / \alpha_j^+}}{1 - e^{-\beta_j^- / \alpha_j^-}} \left(e^{\beta_j^- / \alpha_j^-} - e^{-\beta_j^- / \alpha_j^-} + 1\right)$$

if $\alpha_j^+ < 0$, $\beta_j^- > 0$.

**Proof.** We see at once that

$$|(Du)_j(x,t)| \leq \beta \int_0^x \frac{1}{|a_j(\xi, \tau_j(\xi, x, t))|} \exp\left(-\int_\xi^x \frac{b_{jj}(\eta, \tau_j(\eta, x, t))}{a_j(\eta, \tau_j(\eta, x, t))} d\eta\right) d\xi.$$ 

Moreover, for $x \geq \xi$ it holds

$$\exp\left(-\int_\xi^x \frac{b_{jj}(\eta, \tau_j(\eta, x, t))}{a_j(\eta, \tau_j(\eta, x, t))} d\eta\right) \leq \exp\left(-\frac{\beta_j^-}{\alpha_j^-}(x - \xi)\right)$$

if $\alpha_j^- > 0$, $\beta_j^- > 0$,

$$\exp\left(-\int_\xi^x \frac{b_{jj}(\eta, \tau_j(\eta, x, t))}{a_j(\eta, \tau_j(\eta, x, t))} d\eta\right) \leq \exp\left(-\frac{\beta_j^+}{\alpha_j^+}(x - \xi)\right)$$

if $\alpha_j^+ < 0$, $\beta_j^+ < 0$,

$$\exp\left(-\int_\xi^x \frac{b_{jj}(\eta, \tau_j(\eta, x, t))}{a_j(\eta, \tau_j(\eta, x, t))} d\eta\right) \leq \exp\left(-\frac{\beta_j^-}{\alpha_j^-}(x - \xi)\right)$$

if $\alpha_j^- > 0$, $\beta_j^+ < 0$,

$$\exp\left(-\int_\xi^x \frac{b_{jj}(\eta, \tau_j(\eta, x, t))}{a_j(\eta, \tau_j(\eta, x, t))} d\eta\right) \leq \exp\left(-\frac{\beta_j^+}{\alpha_j^+}(x - \xi)\right)$$

if $\alpha_j^+ < 0$, $\beta_j^- > 0$.

Therefore,

$$|(Du)_j(x,t)| \leq \frac{\beta_j}{\alpha_j} \int_0^x e^{-\beta_j^- (x - \xi) / \alpha_j} d\xi = \frac{\beta_j \alpha_j^+}{\beta_j^+ \alpha_j} \left(1 - e^{-\beta_j^- / \alpha_j^+}\right)$$

if $\alpha_j^- > 0$, $\beta_j^- > 0$,

$$|(Du)_j(x,t)| \leq -\frac{\beta_j}{\alpha_j^+} \int_0^x e^{-\beta_j^+ (x - \xi) / \alpha_j} d\xi = -\frac{\beta_j \alpha_j^-}{\beta_j^+ \alpha_j^+} \left(1 - e^{-\beta_j^+ / \alpha_j^-}\right)$$

if $\alpha_j^+ < 0$, $\beta_j^+ < 0$. 

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\[ |(Du)_j(x, t)| \leq \frac{\beta_j}{\alpha_j} \int_0^x e^{-\beta_j (x-\xi)/\alpha_j} \, d\xi = \frac{\beta_j}{\beta_j^-} \left( 1 - e^{-\beta_j^- / \alpha_j^-} \right) \text{ if } \alpha_j^- > 0, \beta_j^+ < 0, \]

\[ |(Du)_j(x, t)| \leq -\frac{\beta_j}{\alpha_j} \int_0^x e^{-\beta_j^+ (x-\xi)/\alpha_j^+} \, d\xi = -\frac{\beta_j}{\beta_j^+} \left( 1 - e^{-\beta_j^+ / \alpha_j^+} \right) \text{ if } \alpha_j^+ < 0, \beta_j^- > 0. \]

Taking into account that \[ [(I - C)^{-1}Du]_j = (I - C_j)^{-1}(Du)_j \] and combining the obtained bounds with Lemma 4.2, we get the desired assertion.

\[ \square \]

**Corollary 4.4** Under the assumptions of Theorem 1.2,

\[ \|(I - C)^{-1}D\| < 1. \]

Consequently, the operator \( I - C - D \in \mathcal{L}(BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n)) \) is bijective, what completes the proof of Theorem 1.2.

### 4.3 Proof of Theorem 1.5

On the account of Lemma 4.1, we have to prove the bijectivity of the operator \( I - C - D \in \mathcal{L}(BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n)) \). This will follow from Theorems 4.5 and 4.8 below.

#### 4.3.1 Fredholm alternative

Here we prove that the operator \( I - C - D : BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n) \rightarrow BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n) \) is Fredholm of index zero.

**Theorem 4.5** Suppose that the conditions (1.4), (1.8), (1.16) and (1.17) are fulfilled. Let \( \mathcal{K} \) denote the vector space of all bounded continuous solutions to (1.19) with \( f \equiv 0 \). Then

(i) \( \dim \mathcal{K} < \infty \) and the vector space of all \( f \in BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n) \) such that there exists a bounded continuous solution to (1.19) is a closed subspace of codimension \( \dim \mathcal{K} \) in \( BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n) \).

(ii) If \( \dim \mathcal{K} = 0 \), then for any \( f \in BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n) \) there exists a unique bounded continuous solution \( u \) to (1.19).

The proof extends the ideas of [8, 9], where the Fredholm alternative is proved for time-periodic solutions to boundary value hyperbolic problems.

One of the technical tools we intend to employ is a generalized Arzela-Ascoli compactness criteria for unbounded domains, see [12]. To formulate it, we need a corresponding notion of equicontinuity.

**Definition 4.6** A family \( \Phi \subset BC_{\text{per}}(\mathbb{R}^2; \mathbb{R}^n) \) is called equicontinuous on \( [0, 1] \times \mathbb{R} \) if

- \( \Phi \) is equicontinuous on any compact set in \( [0, 1] \times \mathbb{R} \), and
for any $\varepsilon > 0$ there exists $T > 0$ such that

$$|u(x', t') - u(x'', t'')| < \varepsilon \quad (4.28)$$

for all $x', x'' \in [0, 1]$, all $t', t'' \in \mathbb{R} \setminus [-T, T]$, and all $u \in \Phi$.

**Theorem 4.7** (a generalized Arzela-Ascoli theorem). A family $\Phi \subset \text{BC}\text{per}(\mathbb{R}^2; \mathbb{R}^n)$ is pre-compact in $\text{BC}\text{per}(\mathbb{R}^2; \mathbb{R}^n)$ if and only if $\Phi$ is bounded in $\text{BC}\text{per}(\mathbb{R}^2; \mathbb{R}^n)$ and equicontinuous on $[0, 1] \times \mathbb{R}$.

**Proof. 4.5.** Lemma 4.2 states that the operator $I - C : \text{BC}\text{per}(\mathbb{R}^2; \mathbb{R}^n) \to \text{BC}\text{per}(\mathbb{R}^2; \mathbb{R}^n)$ is bijective. Then the operator $I - C - D$ is Fredholm of index zero if and only if

$$I - (I - C)^{-1}D : \text{BC}\text{per}(\mathbb{R}^2; \mathbb{R}^n) \to \text{BC}\text{per}(\mathbb{R}^2; \mathbb{R}^n) \text{ is Fredholm of index zero.} \quad (4.29)$$

Nikolsky’s criterion [6, Theorem XIII.5.2] says that an operator $I + K$ on a Banach space is Fredholm of index zero whenever $K^2$ is compact. Hence, we are done with (4.29) if we show that the operator $[(I - C)^{-1}D]^2 : \text{BC}\text{per}(\mathbb{R}^2; \mathbb{R}^n) \to \text{BC}\text{per}(\mathbb{R}^2; \mathbb{R}^n)$ is compact. As the composition of a compact and a bounded operator is a compact operator, it is enough to show that

$$D(I - C)^{-1}D : \text{BC}\text{per}(\mathbb{R}^2; \mathbb{R}^n) \to \text{BC}\text{per}(\mathbb{R}^2; \mathbb{R}^n) \text{ is compact.}$$

Since $D(I - C)^{-1}D = D^2 + DC(I - C)^{-1}D$ and $(I - C)^{-1}D$ is bounded, it is sufficient to prove that

$$D^2, DC : \text{BC}\text{per}(\mathbb{R}^2; \mathbb{R}^n) \to \text{BC}\text{per}(\mathbb{R}^2; \mathbb{R}^n) \text{ are compact.} \quad (4.30)$$

To show (4.30), we use Theorem 4.7. Given $T > 0$, set $\Pi(T) = \{(x, t) \in \mathbb{R}^2 : 0 \leq x \leq 1, -T \leq t \leq T\}$. Fix an arbitrary bounded set $Y \subset \text{BC}\text{per}(\mathbb{R}^2; \mathbb{R}^n)$. For (4.30) it is sufficient to prove the following two statements:

$$\text{D}^2Y \text{ and } DCY \text{ are equicontinuous on } \Pi(T) \text{ for an arbitrary fixed } T > 0 \quad (4.31)$$

and

given $\varepsilon > 0$, there exists $T > 0$ such that (4.28) is fulfilled for all $x', x'' \in [0, 1]$, $t', t'' \in \mathbb{R} \setminus [-T, T]$, $u \in D^2Y$ and $u \in DCY$. \quad (4.32)

Let us start with (4.31). Denote by $C(\Pi(T))$ (respectively, $C^1(\Pi(T))$) the Banach space of continuous (respectively, continuously differentiable) vector functions $u$ on $\Pi(T)$ such that $u(0, t) = u(1, t)$. As $C^1(\Pi(T))$ is compactly embedded into $C(\Pi(T))$ (due to the Arzela-Ascoli theorem), it is sufficient to show that

$$\|D^2u|_{\Pi(T)}\|_{C^1(\Pi(T))} + \|DCu|_{\Pi(T)}\|_{C^1(\Pi(T))} = O(\|u\|) \quad \text{for all } u \in Y. \quad (4.33)$$

It should be noted that for all sufficiently large $T$ the functions $D^2u$ and $DCu$ restricted to $\Pi(T)$ depend only on $u$ restricted to $\Pi(2T)$.
We will use the following formulas

\[
\partial_x \tau_j(\xi) = -\frac{1}{a_j(x,t)} \exp \int_\xi^x \left( \frac{\partial_x a_j}{a_j^2} \right) (\eta, \tau_j(\eta)) d\eta,
\]

\[
\partial_t \tau_j(\xi) = \exp \int_\xi^x \left( \frac{\partial_x a_j}{a_j^2} \right) (\eta, \tau_j(\eta)) d\eta,
\]

being true for all \( j \leq n \), all \( \xi, x \in [0,1] \), and all \( t \in \mathbb{R} \). Here and below by \( \partial_i \) we denote the partial derivative with respect to the \( i \)-th argument. Then for all sufficiently large \( T > 0 \) the partial derivatives \( \partial_x D^2 u, \partial_t D^2 u, \partial_x DC u, \) and \( \partial_t DC u \) on \( \Pi(T) \) exist and are continuous for all \( u \in C^1(\Pi(2T)) \). Since \( C^1(\Pi(2T)) \) is dense in \( C(\Pi(2T)) \), then the desired property (4.33) will follow from the bound

\[
\|D^2 u\|_{C^1(\Pi(T))} + \|DC u\|_{C^1(\Pi(T))} = O(\|u\|_{C(\Pi(2T))}) \quad \text{for all} \quad u \in C^1(\Pi(2T)).
\]

This bound is proved similarly to [9, Lemma 4.2]:

We start with the estimate

\[
\|D^2 u\|_{C^1(\Pi(T))} = O(\|u\|_{C(\Pi(2T))}) \quad \text{for all} \quad u \in C^1(\Pi(2T)).
\]

Given \( j \leq n \) and \( u \in C^1(\Pi(2T)) \), let us consider the following representation for \((D^2 u)_j(x,t)\) obtained after the application of the Fubini’s theorem:

\[
(D^2 u)_j(x,t) = \sum_{k \neq j} \sum_{l \neq k} \int_0^x \int_\tau \frac{d_{jkl} (\xi, \eta, x, t) b_{jkl} (\xi, \tau_j(\xi)) u_l (\eta, \tau_k (\eta, \xi, \tau_j(\xi)))}{d \xi d \eta},
\]

where

\[
d_{jkl}(\xi, \eta, x, t) = d_j (\xi, x, t) d_k (\eta, \xi, \tau_j(\xi)) b_{kl} (\eta, \tau_k (\eta, \xi, \tau_j(\xi))).
\]

Since

\[
(\partial_x + a_j (x,t) \partial_x) \varphi (\tau_j (\xi, x, t)) = 0
\]

for all \( j \leq n \), \( \varphi \in C^1(\mathbb{R}) \), \( x, \xi \in [0,1] \), and \( t \in \mathbb{R} \), one can easily check that

\[
\|[(\partial_x + a_j (x,t) \partial_x) (D^2 u)]_j\|_{C(\Pi(T))} = O(\|u\|_{C(\Pi(2T))}) \quad \text{for all} \quad j \leq n \quad \text{and} \quad u \in C^1(\Pi(2T)).
\]

Hence the estimate \( \|\partial_x D^2 u\|_{C(\Pi(T))} = O(\|u\|_{C(\Pi(2T))}) \) will follow from the following one:

\[
\|\partial_t D^2 u\|_{C(\Pi(T))} = O(\|u\|_{C(\Pi(2T))}).
\]
We are therefore reduced to prove (4.39). To this end, we start with the following consequence of (4.37):

\[
\partial_t[(D^2u)_j(x,t)] = \sum_{k \neq j} \sum_{l \neq k} \int_0^x \int_\eta^x \frac{d}{dt} \left[ d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \tau_j(\xi)) \right] u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi))) d\xi d\eta
\]

\[
+ \sum_{k \neq j} \sum_{l \neq k} \int_0^x \int_\eta^x d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \tau_j(\xi))
\]

\[
\times \partial_t \tau_k(\eta, \xi, \tau_j(\xi)) \partial_t \tau_j(\xi) \partial_\xi u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi))) d\xi d\eta.
\]

Let us transform the second summand. Using (1.21), (4.34), and (4.35), we get

\[
d \frac{d}{d\xi} u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi)) = \left[ \partial_x \tau_k(\eta, \xi, \tau_j(\xi)) + \partial_t \tau_k(\eta, \xi, \tau_j(\xi)) \partial_\xi \tau_j(\xi) \right] \partial_\xi u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi)))
\]

\[
= \left( \frac{1}{a_j(\xi, \tau_j(\xi))} - \frac{1}{a_k(\xi, \tau_j(\xi))} \right) \partial_\xi \tau_k(\eta, \xi, \tau_j(\xi)) \partial_\xi u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi))).
\]

Therefore,

\[
b_{jk}(\xi, \tau_j(\xi)) \partial_t \tau_k(\eta, \xi, \tau_j(\xi)) \partial_\xi u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi)))
\]

\[
= a_j(\xi, \tau_j(\xi)) a_k(\xi, \tau_j(\xi)) \tilde{b}_{jk}(\xi, \tau_j(\xi)) \frac{d}{d\xi} u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi))),
\]

where the functions \( \tilde{b}_{jk} \in BC_{per}(\mathbb{R}^2; \mathbb{R}) \) are fixed to satisfy (1.17). Note that \( \tilde{b}_{jk} \) are not uniquely defined by (1.17) for \((x, t) \) with \( a_j(x, t) = a_k(x, t) \). Nevertheless, as it follows from (4.40), the right-hand side (and, hence, the left-hand side of (4.41)) do not depend on the choice of \( \tilde{b}_{jk} \), since \( \frac{d}{d\xi} u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi))) = 0 \) if \( a_j(x, t) = a_k(x, t) \).

Write

\[
\tilde{d}_{jkl}(\xi, \eta, x, t) = d_{jkl}(\xi, \eta, x, t) \partial_t \tau_j(\xi) a_k(\xi, \tau_j(\xi)) a_j(\xi, \tau_j(\xi)) \tilde{b}_{jk}(\xi, \tau_j(\xi)),
\]

where \( d_{jkl} \) are introduced by (4.38) and (1.20). Using (1.21) and (4.34), we see that the function \( \tilde{d}_{jkl}(\xi, \eta, x, t) \) is \( C^1 \)-smooth in \( \xi \) due to the regularity assumption (1.17). Similarly, using (4.35), we see that the functions \( d_{jkl}(\xi, \eta, x, t) \) and \( b_{jk}(\xi, \tau_j(\xi)) \) are \( C^1 \)-smooth in \( t \).

By (4.41) we have

\[
(\partial_t D^2u)_j(x,t) = \sum_{k \neq j} \sum_{l \neq k} \int_0^x \int_\eta^x \frac{d}{dt} [d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \tau_j(\xi))] u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi))) d\xi d\eta
\]
\[
+ \sum_{k \neq j} \sum_{l \neq k} \int_0^x \int_\eta^x \tilde{d}_{jkl}(\xi, \eta, x, t) \frac{d}{d\xi} u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi))) d\xi d\eta \\
= \sum_{k \neq j} \sum_{l \neq k} \int_0^x \int_\eta^x \frac{d}{dt} [d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \tau_j(\xi))] u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi))) d\xi d\eta \\
- \sum_{k \neq j} \sum_{l \neq k} \int_0^x \int_\eta^x \partial_\xi \tilde{d}_{jkl}(\xi, \eta, x, t) u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi))) d\xi d\eta \\
+ \sum_{k \neq j} \sum_{l \neq k} \int_0^x \left[ \tilde{d}_{jkl}(\xi, \eta, x, t) u_l(\eta, \tau_k(\eta, \xi, \tau_j(\xi))) \right]_{\xi=x}^{\xi=\eta} d\eta.
\] (4.42)

The desired estimate (4.39) now easily follows from the assumptions (1.4), (1.8) and (1.17) and the equations (4.37) and (4.42).

To finish with (4.33), it remains to show that
\[
\|\partial_t DCu|_{\Pi(T)}\|_{C(\Pi(T))} = O(\|u\|_{C(\Pi(2T))}) \text{ for all } u \in C^1(\Pi(2T)),
\] (4.43)
as the estimate for \(\partial_x DCu\) is obtained similarly to the case of \(\partial_x D^2u\). In order to prove (4.43), we consider an arbitrary integral contributing into \(DCu\), namely
\[
\int_0^x e_{jk}(\xi, x, t) b_{jk}(\xi, \tau_j(\xi)) u_k(1, \tau_k(0, \xi, \tau_j(\xi))) d\xi,
\] (4.44)
where
\[
e_{jk}(\xi, x, t) = d_j(\xi, x, t)c_k(0, \xi, \tau_j(\xi))
\] and \(j \leq n\) and \(k \leq n\) are arbitrary fixed. Differentiating (4.44) in \(t\), we get
\[
\int_0^x \frac{d}{dt} \left[ e_{jk}(\xi, x, t) b_{jk}(\xi, \tau_j(\xi)) \right] u_k(1, \tau_k(0, \xi, \tau_j(\xi))) d\xi \\
\int_0^x e_{jk}(\xi, x, t) b_{jk}(\xi, \tau_j(\xi)) \partial_t \tau_k(0, \xi, \tau_j(\xi)) \partial_t \tau_j(\xi) \partial^2 u_k(1, \tau_k(0, \xi, \tau_j(\xi))) d\xi.
\] (4.45)
Let us estimate the second integral; for the first one the desired estimate is obvious. Similarly to the above, we use (1.21), (4.34), and (4.35) to obtain
\[
\frac{d}{d\xi} u_k(1, \tau_k(0, \xi, \tau_j(\xi))) \\
= \left[ \partial_x \tau_k(0, \xi, \tau_j(\xi)) + \partial_t \tau_k(0, \xi, \tau_j(\xi)) \partial_\xi \tau_j(\xi) \right] \partial_2 u_k(1, \omega_k(0, \xi, \tau_j(\xi))) \\
= \left( \frac{1}{a_j(\xi, \tau_j(\xi))} - \frac{1}{a_k(\xi, \tau_j(\xi))} \right) \partial_t \tau_k(0, \xi, \tau_j(\xi)) \partial_2 u_k(1, \tau_k(0, \xi, \tau_j(\xi))).
\]
Taking into account (1.17), the last expression reads
\begin{align*}
b_{jk}(\xi, \tau_j(\xi))\partial_t \tau_k(0, \xi, \tau_j(\xi))\partial_2 u_k(1, \tau_k(0, \xi, \tau_j(\xi))) \\
= a_j(\xi, \tau_j(\xi)) a_k(\xi, \tau_j(\xi)) \frac{d}{d\xi} u_k(1, \tau_k(0, \xi, \tau_j(\xi))). \tag{4.46}
\end{align*}

Set
\[
ed_{jk}(\xi, x, t) = e_{jk}(\xi, x, t) \partial_t \tau_j(\xi) a_k(\xi, \tau_j(\xi)) a_j(\xi, \tau_j(\xi)) \tilde{b}_{jk}(\xi, \tau_j(\xi)).
\]

Using (4.34) and (4.46), let us transform the second summand in (4.45) as
\begin{align*}
\int_0^x e_{jk}(\xi, x, t) b_{jk}(\xi, \tau_j(\xi)) \partial_t \tau_k(0, \xi, \tau_j(\xi)) \partial_t \tau_j(\xi) \partial_2 u_k(1, \tau_k(0, \xi, \tau_j(\xi))) d\xi \\
= \int_0^x \tilde{e}_{jk}(\xi, x, t) \frac{d}{d\xi} u_k(1, \omega_k(0, \xi, \tau_j(\xi))) d\xi \\
= \left[ \tilde{e}_{jk}(\xi, x, t) u_k(1, \tau_k(0, \xi, \tau_j(\xi))) \right]_{\xi=0}^{\xi=x} \\
- \int_0^x \partial_\xi \tilde{e}_{jk}(\xi, x, t) u_k(1, \tau_k(0, \xi, \tau_j(\xi))) d\xi. \tag{4.47}
\end{align*}

The bound (4.43) now easily follows from (4.45) and (4.47). This finishes the proof of the bound (4.36) and, hence the statement (4.31).

It remains to prove (4.32). Fix an arbitrary \(\varepsilon > 0\). We have to prove the estimates
\[
|(D^2 u)(x', t') - (D^2 u)(x'', t'')| < \varepsilon \tag{4.48}
\]
and
\[
|(DC u)(x', t') - (DC u)(x'', t'')| < \varepsilon \tag{4.49}
\]
for all \(u \in Y\) and all \(x', x'' \in [0, 1]\), \(t', t'' \in \mathbb{R} \setminus [-T, T]\) and some \(T > 0\).

Let us prove (4.48). By (4.37), given \(j \leq n\) and \(u \in Y\), we have
\[
|(D^2 u)_j(x', t') - (D^2 u)_j(x'', t'')| \leq |(D^2 u)_j(x', t')| + |(D^2 u)_j(x'', t'')| \\
= 2 \max_{j \leq n} \max_{x \in [0, 1]} \max_{t' \in \mathbb{R} \setminus [-T, T]} \left| \sum_{k \neq j} \sum_{l \neq k} \int_0^x \int_0^\infty \partial_{\eta} b_{jk}(\xi, \tau_j(\xi)) u_k(\eta, \tau_k(\eta, \xi, \tau_j(\xi))) d\xi d\eta \right| \\
\leq L \|u\| \max_{k \neq j} \max_{x \in [0, 1]} \max_{t' \in \mathbb{R} \setminus [-T, T]} \left| \int_0^x \int_0^\infty b_{jk}(\xi, \tau_j(\xi)) b_{kl}(\eta, \tau_k(\eta, \xi, \tau_j(\xi))) d\xi d\eta \right|, \tag{4.50}
\]
the constant \(L\) being dependent on \(n\), \(a_j\) and \(b_{jk}\) but not on \(u \in Y\) and \(b_{jk}\) with \(j \neq k\). Since \(\|u\|\) is bounded on \(Y\), the desired estimate (4.48) now straightforwardly follows from the estimate (4.50), the assumption (1.16), and the fact that \(\tau_j(\xi, x, t) \to \infty\) as \(t \to \pm \infty\).

The estimate (4.49) is obtained by the same argument, what finishes the proof of (4.32). The theorem is proved. \(\square\)

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4.3.2 Uniqueness of a bounded continuous solution

By the assumption (1.4), there exists \( m \in \{0, 1, \ldots, n\} \) such that for all \((x, t) \in \mathbb{R}^2\)

\[
a_j(x, t) > 0 \quad \text{for} \quad j \leq m \quad \text{and} \quad a_j(x, t) < 0 \quad \text{for} \quad m < j \leq n.
\]

Write

\[
x_j := \begin{cases} 
0 & \text{if } 1 \leq j \leq m, \\
1 & \text{if } m < j \leq n.
\end{cases}
\]

Theorem 4.8 Suppose that the conditions (1.4) and (1.16) are fulfilled. Moreover, assume that there is \( T > 0 \) such that either

\[
\inf_{t < -T} \int_{1-x_j}^{x_j} \left( \frac{b_{jj}}{a_j} \right) (\eta, \tau_j(\eta, 1-x_j, t)) d\eta < 0 \quad \text{for all } j \leq n \tag{4.51}
\]

or

\[
\sup_{t > T} \int_{1-x_j}^{x_j} \left( \frac{b_{jj}}{a_j} \right) (\eta, \tau_j(\eta, 1-x_j, t)) d\eta > 0 \quad \text{for all } j \leq n. \tag{4.52}
\]

Then a bounded continuous solution to (1.19) (if any) is unique.

Proof. Given \( T \in \mathbb{R} \), let \( Q_T = \{(x, t) \in \mathbb{R}^2 : 0 \leq x \leq 1, -\infty < t \leq T\} \) and \( Q_T = \{(x, t) \in \mathbb{R}^2 : 0 \leq x \leq 1, T \leq t < \infty\} \).

First assume that the condition (4.51) is fulfilled. By technical reasons, we will use an integral representation of the problem (1.1)–(1.2), which differs from (1.19), namely

\[
u_j(x, t) = c_j(x_j, x, t)u_j(1-x_j, \tau_j(x_j, x, t)) \\
- \int_{x_j}^{x} d_j(\xi, x, t) \left( \sum_{k \neq j} b_{jk}(\xi, \tau_j(\xi, x, t))u_k(\xi, \tau_j(\xi, x, t)) - f_j(\xi, \tau_j(\xi, x, t)) \right) d\xi. \tag{4.53}
\]

It is obtained by integration along the characteristic curves in \( x \), in the direction of time decreasing. Note that the integral equations (1.19) and (4.53) are equivalent in the sense that every continuous solution to (1.19) is a continuous solution to (4.53) and vice versa. The proof of this fact is similar to the proof of Lemma 4.1 (Claim 1). The equation (4.53) is more suitable for our purposes, since the right-hand side of (4.53) maps \( BC_{\text{per}}(Q^{-T}; \mathbb{R}^n) \) into itself for each \( T > 0 \), what is not the case for (1.19). Fix an arbitrary \( T > 0 \) and consider the equation (4.53) in \( Q^{-T} \). The latter can be written in the operator form

\[
u = \tilde{C}u + \tilde{D}u + \tilde{F}f \tag{4.54}
\]

with the operators \( \tilde{C}, \tilde{D}, \tilde{F} : BC_{\text{per}}(Q^{-T}; \mathbb{R}^n) \to BC_{\text{per}}(Q^{-T}; \mathbb{R}^n) \) given by

\[
(\tilde{C}v)_j(x, t) = c_j(x_j, x, t)v_j(1-x_j, \tau_j(x_j)), \quad j \leq n, \tag{4.55}
\]
\[(\hat{D}v)_j(x,t) = -\int_{x_j}^{x} d_j(\xi,x,t) \sum_{k \neq j} b_{jk}(\xi,\tau_j(\xi)) v_k(\xi,\tau_j(\xi)) d\xi, \quad j \leq n,\]

and

\[(\hat{F}f)_j(x,t) = \int_{x_j}^{x} d_j(\xi,x,t) f_j(\xi,\tau_j(\xi)) d\xi, \quad j \leq n,\]

respectively. Taking into account the definition of \(\hat{C}\), the notation (1.20), and the assumption (4.51), we get

\[\|\hat{C}\|_{L(BC_{per}(Q^{-T};\mathbb{R}^n))} \leq \exp \left\{ \max_{j \leq n} \inf_{t < -T} \int_{1-x_j}^{x_j} \left( b_{jj} \frac{1}{a_j} \right) (\eta, \tau_j(\eta, 1-x_j, t)) d\eta \right\} < 1.\]

It follows that the operator \(I - \hat{C} : BC_{per}(Q^{-T};\mathbb{R}^n) \to BC_{per}(Q^{-T};\mathbb{R}^n)\) is bijective and, hence, the operator equation (4.54) reads

\[u = (I - \hat{C})^{-1} \hat{D}u + (I - \hat{C})^{-1} \hat{F}f.\]

Using the assumption (1.16), fix \(T > 0\) so large that the norm of the operator \(\hat{D}\) is so small that

\[\|(I - \hat{C})^{-1} \hat{D}\|_{L(BC_{per}(Q^{-T};\mathbb{R}^n))} < 1.\]

By the Banach fixed-point theorem, there exists a unique function \(u \in BC_{per}(Q^{-T};\mathbb{R}^n)\) satisfying (4.53) in \(Q^{-T}\).

Now, let us consider (1.1)–(1.2) in the domain \(Q_{-T}\) and subject it to the initial condition

\[u_j|_{t = -T} = u_j^{-T}(x), \quad j \leq n.\]

Here \(u^{-T}(x)\) is the continuous solution to the problem (1.1)–(1.2) (or, the same, to (4.53)) in \(Q^{-T}\) at \(t = -T\). Due to the method of characteristics, the unknown \(u\) in \(Q_{-T}\) is given by the formula (4.53) if \(\tau_j(x_j, x, t) > -T\) and by the formula

\[u_j(x,t) = u_j^{-T}(\xi_j(-T, x, t)), \quad j \leq n,\]

otherwise. The existence and uniqueness of a continuous solution \(u\) to the problem (1.1), (1.2), (4.57) (or, the same, to (4.53), (4.58)) in \(Q_{-T}\) is proved in [7, Theorem 2.1].

We conclude that the system (1.19) in the strip \([0, 1] \times \mathbb{R}\) has a unique continuous solution bounded at \(-\infty\). This entails that a continuous solution to the system (1.19) (if any) is unique. The proof under the condition (4.51) is complete.

To prove the theorem under the condition (4.52), we again switch to a suitable integral representation which is equivalent to (1.19). The operator form of this integral representation in the domain \(Q_T\) for an arbitrary fixed \(T > 0\) reads

\[u = \tilde{C}u + \tilde{D}u + \tilde{F}f,\]

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where the operators $\tilde{C}, \tilde{D}, \tilde{F} : BC_{\text{per}}(Q_T; \mathbb{R}^n) \to BC_{\text{per}}(Q_T; \mathbb{R}^n)$ are defined by

$$(\tilde{C}v)_j(x, t) = c_j(1 - x_j, x, t)v_j(x_j, \tau_j(1 - x_j)), \quad j \leq n,$$

$$(\tilde{D}v)_j(x, t) = -\int_{1-x_j}^{x_j} d_j(\xi, x, t) \sum_{k \neq j} b_{jk}(\xi, \tau_j(\xi))v_k(\xi, \tau_j(\xi))d\xi, \quad j \leq n,$$

and

$$(\tilde{F}f)_j(x, t) = \int_{1-x_j}^{x_j} d_j(\xi, x, t)f_j(\xi, \tau_j(\xi))d\xi, \quad j \leq n.$$

Similarly to the above, the operator $I - \tilde{C} : BC_{\text{per}}(Q_T; \mathbb{R}^n) \to BC_{\text{per}}(Q_T; \mathbb{R}^n)$ is bijective by the assumption (4.52), and we have

$$u = (I - \tilde{C})^{-1}\tilde{D}u + (I - \tilde{C})^{-1}\tilde{F}f. \quad (4.60)$$

Due to (1.16), for sufficiently large $T$ it holds

$$\|(I - \tilde{C})^{-1}\tilde{D}|_{BC_{\text{per}}(Q_T; \mathbb{R}^n)}\| < 1.$$

This means that the equation (4.60) has a unique continuous solution in $Q_T$. Further we consider (1.1)–(1.2) in the domain $Q^T$ with the reverse time. The initial condition is posed at $t = T$, namely

$$u^T_j|_{t=T} = u^T_j(x), \quad j \leq n,$$

where $u^T(x)$ is the continuous solution to the problem (1.1)–(1.2) in $Q_T$ at $t = T$. By [7, Theorem 2.1], this problem has a unique continuous solution, what completes the proof. □

It follows that, under the conditions of Theorems 4.5 and 4.8, the system (1.19) has a unique solution $u \in BC(\mathbb{R}; C_{\text{per}}(R; R^n))$. Moreover, the conditions (1.14) and (1.15) entail both the condition (1.8) and one of the conditions (4.51) and (4.52). Hence, under the assumptions of Theorem 1.5, all assumptions of Theorems 4.5 and 4.8 are satisfied. Theorem 1.5 is therewith proved.

5 Discussion and open problems

5.1 Other boundary conditions

Despite the periodicity conditions (1.2) were essentially used in the proof of the main results, we believe that they are not necessary for the statements of Theorems 1.2 and 1.5. It would
be interesting to extend our approach to other types of boundary conditions. For instance, the boundary conditions of the reflection type, say,

\[ u_j(t, 0) = \sum_{k=1}^{n} (r_{jk}^0(t)u_k(t, 0) + r_{jk}^1(t)u_k(t, 1)) , \quad 1 \leq j \leq m, \]

\[ u_j(t, 1) = \sum_{k=1}^{n} (r_{jk}^1(t)u_k(t, 0) + r_{jk}^{11}(t)u_k(t, 1)) , \quad m < j \leq n, \]

are of a particular interest due to numerous applications in semiconductor laser modeling [13, 15, 16], boundary feedback control theory [1, 2, 14], chemotaxis problems [4].

### 5.2 Second-order hyperbolic equations

For the second-order hyperbolic equation without the zero-order term

\[ \partial_t^2 u - a^2(x, t)\partial_x^2 u + a_1(x, t)\partial_t u + a_2(x, t)\partial_x u = 0, \quad (x, t) \in \mathbb{R}^2, \quad (5.1) \]

with the periodic boundary condition

\[ u(x, t) = u(x + 1, t) \quad (5.2) \]

Theorems 1.2 and 1.5 provide sufficient conditions for the existence of an exponential trichotomy (see [5] for the definition). Indeed, in the new unknowns

\[ u_1 = \partial_t u + a(x, t)\partial_x u, \quad u_2 = \partial_t u - a(x, t)\partial_x u \quad (5.3) \]

the problem (5.1)–(5.2) reads as follows:

\[ \begin{align*}
\partial_t u_1 - a(x, t)\partial_x u_1 + b_{11}(x, t)u_1 + b_{12}(x, t)u_2 &= 0 \\
\partial_t u_2 + a(x, t)\partial_x u_2 + b_{21}(x, t)u_1 + b_{22}(x, t)u_2 &= 0,
\end{align*} \quad (5.4) \]

\[ u_j(x + 1, t) = u_j(x, t), \quad j = 1, 2, \quad (5.5) \]

where

\[ \begin{align*}
b_{11} &= \frac{a_1}{2} + \frac{a_2}{2a} + \frac{a\partial_x a - \partial_t a}{2a}, \\
b_{12} &= \frac{a_1}{2} - \frac{a_2}{2a} + \frac{a\partial_x a - \partial_t a}{2a}, \\
b_{21} &= \frac{a_1}{2} + \frac{a_2}{2a} + \frac{a\partial_x a + \partial_t a}{2a}, \\
b_{22} &= \frac{a_1}{2} - \frac{a_2}{2a} - \frac{a\partial_x a + \partial_t a}{2a}.
\end{align*} \]

One can easily see that the problems (5.1)–(5.2) and (5.4)–(5.5) are equivalent in the following sense: For any \( c \in \mathbb{R} \) the function \( u + c \) is a solution to (5.1)–(5.2) iff the function \( (u_1, u_2) \) given by (5.3) is a solution to (5.4)–(5.5). This means that, under the conditions ensuring the existence of the exponential dichotomy for (5.4)–(5.5), the problem (5.1)–(5.2)
has an exponential trichotomy. For instance, the assumptions (1.9)-(1.12) of Theorem 1.2 in the case of constant coefficients read

\[
|a_1 - a_2| < a_1 + a_2 \quad \text{if} \quad a_1 + a_2 > 0,
\]

\[
|a_1 + a_2| < a_2 - a_1 \quad \text{if} \quad a_2 - a_1 > 0,
\]

\[
|a_1 - a_2| < 2(-a_1 - a_2) \left( \exp^{\frac{-a_1-a_2}{2a_2}} - 1 \right) \quad \text{if} \quad a_1 + a_2 < 0,
\]

\[
|a_1 + a_2| < 2(a_1 - a_2) \left( \exp^{\frac{a_1-a_2}{2a_2}} - 1 \right) \quad \text{if} \quad a_1 - a_2 > 0.
\]

For the general second-order equation (with the zero-order term)

\[
\partial_t^2 u - a^2(x, t) \partial_x^2 u + a_1(x, t) \partial_t u + a_2(x, t) \partial_x u + a_3(x, t) u = 0, \quad (x, t) \in \mathbb{R}^2
\]

the first-order system reads

\[
\partial_t u_1 - a(x, t) \partial_x u_1 + b_{11}(x, t) u_1 + b_{12}(x, t) u_2 + a_3(x, t) u = 0
\]

\[
\partial_t u_2 + a(x, t) \partial_x u_2 + b_{21}(x, t) u_1 + b_{22}(x, t) u_2 + a_3(x, t) u = 0
\]

\[
\partial_t u + a(x, t) \partial_x u = u_1.
\]

Note that the last system fulfills neither the assumptions of Theorem 1.2 (as \(b_{33} = 0\), contradicting to (1.8)) nor of Theorem 1.5 (as \(a_1 = a_3\) and \(b_{13} \neq 0\), contradicting to (1.17)).

This shows that, in general, second-order hyperbolic equations require different techniques.

### 5.3 Robustness of exponential dichotomy

The question of robustness of an exponential dichotomy (stability property with respect to data perturbations) for hyperbolic PDEs seems to be a challenging open problem.

Nevertheless, our Theorems 1.2 and 1.5 give the following consequences.

**Corollary 5.1** Under the assumptions of Theorem 1.2, the exponential dichotomy is robust with respect to small perturbations of \(a_j\) and \(b_{jk}\). Specifically, there exists \(\varepsilon > 0\) such that the exponential dichotomy persists for all continuously differentiable functions \(\tilde{a}_j\) and \(\tilde{b}_{jk}\) that are 1-periodic in \(x\), satisfy the inequalities

\[
\sup_{j, x, t} |a_j - \tilde{a}_j| < \varepsilon \quad \text{and} \quad \sup_{j, k, x, t} |b_{jk} - \tilde{b}_{jk}| < \varepsilon,
\]

and fulfill the conditions (1.4), (1.8)–(1.12) with \(\tilde{a}_j\) and \(\tilde{b}_{jk}\) in place of \(a_j\) and \(b_{jk}\), respectively.

**Corollary 5.2** Suppose that the conditions (1.4), (1.16), and one of the conditions (1.14) and (1.15) are fulfilled.
1. If
\[ a_j \neq a_k \text{ for all } 1 \leq j \neq k \leq n \text{ and } (x, t) \in \mathbb{R}^2, \] (5.7)
then the exponential dichotomy is robust under small perturbations of \( a_j \) and \( b_{jk} \). Specifically, there exists \( \varepsilon > 0 \) such that the exponential dichotomy persists for all continuously differentiable functions \( \tilde{a}_j \) and \( \tilde{b}_{jk} \) that are 1-periodic in \( x \), satisfy the inequalities (5.6), and fulfill the conditions (1.4), (1.14), (5.7) with \( \tilde{a}_j \) and \( \tilde{b}_{jk} \) in place of \( a_j \) and \( b_{jk} \), respectively.

2. If
\[ b_{jk} \equiv 0 \text{ for all } 1 \leq j \neq k \leq n \]
in a neighborhood of the set \( \{(x, t) \in \overline{\Omega} : a_k(x, t) = a_j(x, t)\} \),
then the exponential dichotomy is robust under small perturbations of \( a_j \). Specifically, there exists \( \varepsilon > 0 \) such that the exponential dichotomy persists for all continuously differentiable functions \( \tilde{a}_j \) that are 1-periodic in \( x \), satisfy the first inequality in (5.6), and fulfill the conditions (1.4), (1.14), (1.16) with \( \tilde{a}_j \) in place of \( a_j \).

Henry [3, Theorem 7.6.10] established a general sufficient condition of the robustness of an exponential dichotomy for abstract evolution equations. Attempts to apply this approach to hyperbolic PDEs meet complications caused by loss of smoothness. In [10] these complications are overcome in the case of boundary conditions of the so-called smoothing type when the solutions of initial-boundary value problems become more regular than the initial data after some time. In the general case, the robustness issue for hyperbolic PDEs remains unexplored.

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