MONOTONE QUANTITIES INVOLVING A WEIGHTED $\sigma_k$ INTEGRAL ALONG INVERSE CURVATURE FLOWS

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Abstract. We give a family of monotone quantities along smooth solutions to the inverse curvature flows in Euclidean spaces. We also derive a related geometric inequality for closed hypersurfaces with positive $k$-th mean curvature.

1. Introduction

The aim of this paper is to introduce some monotone quantities involving a weighted $\sigma_k$ integral along inverse curvature flows in the Euclidean space $\mathbb{R}^{m+1}$.

Given a smooth closed hypersurface $\Sigma \subset \mathbb{R}^{m+1}$, let $\{\kappa_1, \cdots, \kappa_m\}$ be the principal curvatures of $\Sigma$. For any $1 \leq k \leq m$, define the $k$-th mean curvature $H_k$ and the normalized $k$-th mean curvature $\sigma_k$ of $\Sigma$ by

$$H_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} \kappa_{i_1} \cdots \kappa_{i_k} \quad \text{and} \quad \sigma_k = \frac{H_k}{\binom{m}{k}},$$

respectively (the notation $\sigma_k$ here follows that of Reilly in [11, 13]). When $k = 0$, define $H_0 = \sigma_0 = 1$. A family of closed hypersurfaces $\{\Sigma_t\}_{t \in I}$, given by a smooth map

$$X : \Sigma \times I \rightarrow \mathbb{R}^{m+1}$$

where $I$ is an open interval and $\Sigma_t = X(\Sigma, t)$, is said to evolve according to an inverse curvature flow if

$$\frac{\partial X}{\partial t} = \frac{\sigma_{k-1}}{\sigma_k} \nu$$

for some $1 \leq k \leq m$. Here $\nu$ is the unit outward normal to $\Sigma_t$ and $\sigma_{k-1}, \sigma_k$ are computed on $\Sigma_t$.

Our main result is

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Theorem 1. Suppose \( \{ \Sigma_t \}_{t \in I} \) is a smooth solution to an inverse curvature flow (1) for some \( 2 \leq k \leq m \). Given any point \( O \in \mathbb{R}^{m+1} \), let \( r \) be the Euclidean distance to \( O \). Then the function

\[
Q_k(\Sigma_t) = \left( \int_{\Sigma_t} \sigma_{k-1} d\mu \right)^{-\frac{m-k}{m+1-k}} \left( \int_{\Sigma_t} \sigma_k r^2 d\mu - \int_{\Sigma_t} \sigma_{k-2} d\mu \right)
\]

is monotone decreasing and \( Q_k(t) \) is a constant function if and only if \( \Sigma_t \) is a round sphere for each \( t \). Here \( d\mu \) is the volume form on \( \Sigma_t \).

Here are some remarks concerning Theorem 1.

Remark 1. The long time existence of smooth solutions to (1) was established by Gerhardt in [5] and by Urbas in [16] when the initial surface \( \Sigma \) is star-shaped with \( \sigma_k > 0 \). Moreover, they proved that the rescaled hypersurfaces \( \{ \tilde{\Sigma}_t \} \), parametrized by \( \tilde{X}(\cdot, t) = e^{-t}X(\cdot, t) \), converge to a sphere in the \( C^\infty \) topology as \( t \to \infty \).

Remark 2. Theorem 1 does not address the case \( k = 1 \), i.e. when the flow is the inverse mean curvature flow. In that case, if one defines \( \sigma_{-1} = \langle Y, \nu \rangle \) where \( Y \) is a position vector field in \( \mathbb{R}^{m+1} \), it was proved in [10] that

\[
Q_1(\Sigma_t) = |\Sigma_t|^{-\frac{m-1}{m}} \left[ \frac{1}{m} \int_{\Sigma_t} H r^2 d\mu - (m+1)\text{Vol}(\Omega_t) \right]
\]

is monotone decreasing along the inverse mean curvature flow \( \frac{\partial X}{\partial t} = -H \nu \). Here \( |\Sigma_t| \) is the area of \( \Sigma_t \), \( \text{Vol}(\Omega_t) \) is the volume of the region \( \Omega_t \) enclosed by \( \Sigma_t \) and \( H = \sum_{i=1}^m \kappa_i \) is the mean curvature of \( \Sigma_t \). Results in [10] were motivated by the work of Brendle, Hung and Wang in [3].

In [6], Guan and Li proved that

\[
\left( \int_{\Sigma_t} \sigma_{k-1} d\mu \right)^{\frac{1}{m+1-k}} \left( \int_{\Sigma_t} \sigma_k d\mu \right)
\]

is monotone decreasing along the inverse curvature flow (1). Using this together with the result of Gerhardt and Urbas, Guan and Li derived the quermassintegral inequalities

\[
\left( \frac{1}{\omega_m} \int_{\Sigma} \sigma_{k-1} d\mu \right)^{\frac{1}{m+1-k}} \leq \left( \frac{1}{\omega_m} \int_{\Sigma} \sigma_k d\mu \right)^{\frac{1}{m-k}}
\]

for any star-shaped \( \Sigma \) with \( \sigma_k > 0 \). Here \( \omega_m \) is the volume of the \( m \)-dimensional unit sphere in \( \mathbb{R}^{m+1} \). Unlike the quantity in [3], \( Q_k(\Sigma_t) \) in (2) is not scaling invariant (it has a unit of length square when scaled with respect to \( O \)). However, it is still interesting to know if \( Q_k(\Sigma_t) \) always has a fixed sign. We answer this question in the next theorem.
Theorem 2. Let $\Sigma \subset \mathbb{R}^{m+1}$ be a smooth closed hypersurface. Suppose $\sigma_k > 0$ on $\Sigma$ for some $1 \leq k \leq m$. Given any point $O \in \mathbb{R}^{m+1}$, let $r$ be the Euclidean distance to $O$. Then

$$
\int_{\Sigma} \sigma_l r^p \, d\mu \leq \int_{\Sigma} \sigma_k r^{p+k-l} \, d\mu
$$

for any integer $0 \leq l < k$ and any real number $p \geq 0$. Moreover, the equality holds if and only if $\Sigma$ is a round sphere centered at $O$.

Let $l = k - 2$ and $p = 0$, (4) becomes

$$
\int_{\Sigma} \sigma_{k-2} \, d\mu \leq \int_{\Sigma} \sigma_k r^2 \, d\mu.
$$

Therefore, Theorem 2 implies that $Q_k(t)$ is always nonnegative along the inverse curvature flow (1).

This paper is organized as follows. In Section 2, we review some basics facts about $\sigma_k$. In Section 3, we derive the monotonicity of $Q_k(\Sigma_t)$. In Section 4, we prove the inequality (4).

2. Notations and preliminaries

Given a smooth closed hypersurface $\Sigma \subset \mathbb{R}^{m+1}$, let $\nabla$ and $\nabla$ denote the connections on $\mathbb{R}^{m+1}$ and $\Sigma$ respectively. Let $\nu$ be the outward unit normal to $\Sigma$. The shape operator of $\Sigma$ with respect to $\nu$ is defined by

$$
A(\cdot) = \nabla (\cdot) \nu : T\Sigma \to T\Sigma,
$$

where $T\Sigma$ is the tangent bundle of $\Sigma$. Given any local frame $\{e_i\}_{i=1}^m$ on $\Sigma$, define $\{A_i^j\}$ by $A(e_i) = A_i^j e_j$, where $i, j \in \{1, \ldots, m\}$ and Einstein summation convention is applied. Recall that $H_0 = \sigma_0 = 1$ and

$$
H_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} \kappa_{i_1} \cdots \kappa_{i_k} \quad \text{and} \quad \sigma_k = \frac{H_k}{(m)_k}
$$

for $1 \leq k \leq m$, where $\{\kappa_i\}_{i=1}^m$ are the eigenvalues of $A$. In terms of $\{A_i^j\}$, $H_k$ can be computed by

$$
H_k = \frac{1}{k!} \sum_{1 \leq i_1 < \cdots < i_k \leq m} \delta^{i_1 \cdots i_k}_{j_1 \cdots j_k} A_{i_1}^{j_1} \cdots A_{i_k}^{j_k},
$$

where $\delta^{i_1 \cdots i_k}_{j_1 \cdots j_k} = 0$ if $i_p = i_q$ or $j_p = j_q$ for some $p \neq q$, or if the two sets $\{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\}$; otherwise $\delta^{i_1 \cdots i_k}_{j_1 \cdots j_k}$ is defined as the sign of the permutation $(i_1, \ldots, i_k) \mapsto (j_1, \ldots, j_k)$. Because of (5), we define $H_k = 0$ for any $k > m$. 


A basic tool in the study of $H_k$ is the $(k-1)$-th Newton transformation $T_{k-1} : T\Sigma \to T\Sigma$ (cf. [11, 12]). If we write $T_{k-1}(e_j) = (T_{k-1})^i_j e_i$, then $\{(T_{k-1})^i_j\}$ are given by

$$(T_{k-1})^i_j = \frac{1}{(k-1)!} \sum_{1 \leq i_1, \ldots, i_{k-1} \leq m; 1 \leq j_1, \ldots, j_{k-1} \leq m} \delta^{i_1 \ldots i_{k-1}}_{j_1 \ldots j_{k-1}} A_{i_1 \ldots i_k} A_{j_1 \ldots j_k}.$$ 

A more geometric way to understand $T_{k-1}$ is that if $\{e_i\}_{i=1}^m$ consist of eigenvectors of $A$ with $A(e_j) = \kappa_j e_j$, then $T_{k-1}(e_j) = \Lambda_j e_j$, where

$$\Lambda_j = \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq m, \ j \not\in \{i_1, \ldots, i_{k-1}\}} \kappa_{i_1} \cdots \kappa_{i_{k-1}}.$$ 

When $k = 1$, one defines $T_0 = \text{Id}$, the identity map. It follows from (6) that

$$\text{tr}(T_{k-1}) = [m - (k - 1)]H_{k-1},$$

$$\text{tr}(T_{k-1} \circ A) = kH_k$$

and

$$\text{tr}(T_{k-1} \circ A \circ A) = H_k H_1 - (k + 1)H_{k+1}.$$ 

Another useful property of $T_{k-1}$ is that it is divergence free (cf. [12]),

$$\text{div} T_{k-1} = 0.$$ 

Here tr$(\cdot)$ and div$(\cdot)$ denote the trace and the divergence taken on $\Sigma$ respectively.

Suppose $\{\Sigma_t\}$ is a family of evolving hypersurfaces given by a smooth map $X : \Sigma \times I \to \mathbb{R}^{m+1}$ with

$$\frac{\partial X}{\partial t} = F\nu$$

where $\nu$ is the outward unit normal to $\Sigma_t = X(\Sigma, t)$ and $F$ denotes the speed of the flow which may depend on $X$, the principal curvatures of $\Sigma_t$ and time $t$. The following evolution equation of $H_k$ is standard (see, e.g., [6, Proposition 4] or [11, Lemma A]),

$$H'_k = \text{tr}(T_{k-1} \circ A')$$

$$= (T_{k-1})^i_j \left[ - (\nabla^2 F)^i_j - F(A \circ A)^i_j \right].$$

Here “’” denotes the derivative taken with respect to $t$. 
Proposition 1. Let $f$ be an arbitrary function on $\mathbb{R}^{m+1}$. Along the flow (11),
\begin{equation}
\left( \int_{\Sigma_t} H_t f d\mu \right)' = \int_{\Sigma_t} \left\{ -(T_{l-1})^i_j \left( \nabla^2 f \right)_j^i + (l + 1) H_l \langle \nabla f, \nu \rangle \right. \\
+ (l + 1) H_{l+1} f \} F d\mu
\end{equation}
for any $1 \leq l \leq m$.

Proof. Direct calculation gives
\begin{equation}
\left( \int_{\Sigma_t} H_t f d\mu \right)' = \int_{\Sigma_t} \left( H_t^i f + H_l \langle \nabla f, F\nu \rangle + H_l F H_1 \right) d\mu \\
= \int_{\Sigma_t} \left\{ (T_{l-1})^i_j \left[ -\left( \nabla^2 F \right)_j^i - F(A \circ A)_j^i \right] f \\
+ H_l F \langle \nabla f, \nu \rangle + H_l H_1 f \right\} d\mu \\
= \int_{\Sigma_t} \left\{ -(T_{l-1})^i_j \left( \nabla^2 f \right)_j^i + (l + 1) H_{l+1} f \\
+ H_l \langle \nabla f, \nu \rangle \} F d\mu,
\end{equation}
where we used (12), (10) and (9).

Note that the two Hessians $(\nabla^2 f)_i^j$ and $(\nabla^2 f)_i^j$ are related by
\begin{equation}
(\nabla^2 f)_i^j = (\nabla^2 f)_i^j + \frac{\partial f}{\partial \nu} A_{ij},
\end{equation}
where $A_{ij} = g_{il} A_j^l$ is the second fundamental form of $\Sigma_t$ and $g$ denotes the induced metric on $\Sigma_t$.

Therefore, it follows from (14) and (15) that
\begin{equation}
\left( \int_{\Sigma_t} H_t f d\mu \right)' = \int_{\Sigma_t} \left\{ -(T_{l-1})^i_j \left[ -\left( \nabla^2 f \right)_j^i - \langle \nabla f, \nu \rangle A_j^i \right] \\
+ (l + 1) H_{l+1} f + H_l \langle \nabla f, \nu \rangle \right\} F d\mu \\
= \int_{\Sigma_t} \left\{ -(T_{l-1})^i_j \left( \nabla^2 f \right)_j^i + (l + 1) H_l \langle \nabla f, \nu \rangle \\
+ (l + 1) H_{l+1} f \} F d\mu,
\end{equation}
where we also used (8). \qed

We end this section by noting the following fact regarding $\sigma_k > 0$.

Lemma 1. For a closed hypersurface $\Sigma$ in $\mathbb{R}^{m+1}$, the condition $\sigma_k > 0$ implies $\sigma_l > 0$, $\forall \ 1 \leq l \leq k$. 
This follows from the characterization of the Garding’s cone
\[ \Gamma^+_k = \{ \kappa \in \mathbb{R}^m \mid \sigma_l(\kappa) > 0, \ \forall \ 1 \leq l \leq k \} \]
as the connected component of the set \( \{ \kappa \in \mathbb{R}^m \mid \sigma_k(\kappa) > 0 \} \) containing the point \((1, 1, \cdots, 1)\) (cf. [3, Proposition 2.6] or [2, Proposition 3.2]) and the fact that \( \Sigma \) always has a point at which \( \kappa_i > 0, \ \forall \ 1 \leq i \leq m \).

3. Derivation of monotone quantities

Given any point \( O \in \mathbb{R}^{m+1} \), let \( r \) be the Euclidean distance to \( O \).

Consider the function \( f = \frac{1}{2} r^2 \), which satisfies
\[ \nabla^2 f = g, \text{ where } g \text{ is the Euclidean metric on } \mathbb{R}^{m+1} \]
\[ \nabla f = Y, \text{ where } Y \text{ is the position vector starting from } O. \]

With such a choice of \( f \), it follows from Proposition 1, (7) and the definition \( \sigma_l = H_l / \binom{m}{l} \) that
\[
\left( \int_{\Sigma_t} \sigma_l f d\mu \right)' = \int_{\Sigma_t} \left[ -l\sigma_{l-1} + (l + 1)\sigma_l(Y, \nu) + (m - l)\sigma_{l+1} f \right] F d\mu
\]
along the flow (11), \( \forall \ 1 \leq l \leq m. \)

**Proof of Theorem 1** Suppose the flow speed \( F \) in (11) is given by
\[ F = \frac{\sigma_{k-1}}{\sigma_k} \]
for some \( 2 \leq k \leq m \). Then (16) gives
\[
\left( \int_{\Sigma_t} \sigma_k f d\mu \right)' = \int_{\Sigma_t} \left[ -k\sigma_{k-1} + (k + 1)\sigma_k(Y, \nu) + (m - k)\sigma_{k+1} f \right] \frac{\sigma_{k-1}}{\sigma_k} d\mu.
\]
Choose \( l = k \) in (17), we have
\[
\left( \int_{\Sigma_t} \sigma_k f d\mu \right)' = \int_{\Sigma_t} \left[ -k\sigma_{k-1}^2 + (k + 1)\sigma_k(Y, \nu) + (m - k)\sigma_{k+1} f \right] \frac{\sigma_{k-1}}{\sigma_k} d\mu
\]
\[ \leq \int_{\Sigma_t} \left[ -k\sigma_{k-2} + (k + 1)\sigma_{k-2} + (m - k)\sigma_k f \right] d\mu
\]
\[ = \int_{\Sigma_t} \sigma_{k-2} d\mu + (m - k) \int_{\Sigma_t} \sigma_k f d\mu, \]
where we used the fact \( \sigma_k > 0 \), the Newton inequality
\[ \sigma_{i-1}\sigma_{i+1} \leq \sigma_i^2 \]
for any $1 \leq i \leq m - 1$, and the Hsiung-Minkowski formula ([7])

$$\int_{\Sigma} \sigma_{j-1} d\mu = \int_{\Sigma} \sigma_j \langle Y, \nu \rangle d\mu$$

for any $1 \leq j \leq m$.

We need information on $\left( \int_{\Sigma_t} \sigma_{k-2} d\mu \right)'$. Setting $F = \frac{\sigma_{k-1}}{\sigma_k}$ and $f = 1$ in Proposition 1 gives

(19) $$\left( \int_{\Sigma_t} \sigma_t d\mu \right)' = (m - l) \int_{\Sigma_t} \frac{\sigma_{k-1}}{\sigma_k} d\mu.$$ Let $l = k - 2$ in (19), we have

(20) $$\left( \int_{\Sigma_t} \sigma_{k-2} d\mu \right)' = \left[ m - (k - 2) \right] \int_{\Sigma_t} \frac{\sigma_{k-1}}{\sigma_k} d\mu \geq \left[ m - (k - 2) \right] \int_{\Sigma_t} \sigma_{k-2} d\mu$$

where we again used the Newton inequality and the assumption $k \geq 2$.

Now it follows from (18) and (20) that

$$\left[ \int_{\Sigma_t} (\sigma_k r^2 - \sigma_{k-2}) d\mu \right]' \leq \left[(m - k) \left[ \int_{\Sigma_t} (\sigma_k r^2 - \sigma_{k-2}) d\mu \right] \right],$$

which then implies

(21) $$\left[ e^{-(m-k)t} \int_{\Sigma_t} (\sigma_k r^2 - \sigma_{k-2}) d\mu \right]' \leq 0.$$ On the other hand, setting $l = k - 1$ in (19) gives

(22) $$\left( \int_{\Sigma_t} \sigma_{k-1} d\mu \right)' = \left[ m - (k - 1) \right] \int_{\Sigma_t} \sigma_{k-1} d\mu.$$ By (21) and (22), we conclude that

$$\left[ \left( \int_{\Sigma_t} \sigma_{k-1} d\mu \right) - \frac{m-k}{m-(k-1)} \int_{\Sigma_t} (\sigma_k r^2 - \sigma_{k-2}) d\mu \right]' \leq 0.$$ If the derivative is zero at some time $t_0$, then

$$\kappa_1 = \cdots = \kappa_m$$

at $t = t_0$ by the equality case in the Newton inequality, which implies that $\Sigma_{t_0}$ is a round sphere. This completes the proof of Theorem 1. □
Remark 3. Let \( l = k \) in (19) and apply the Newton inequality, one has
\[
\left( \int_{\Sigma} \sigma_k \, d\mu \right)^{\prime} \leq (m - k) \int_{\Sigma} \sigma_k \, d\mu.
\] (23)

(23) and (22) imply the quantity in (3) is monotone decreasing, which
is the monotonicity of Guan and Li ([6]).

Remark 4. In (20), if \( k = 1 \), we do not have a point-wise inequality of
the form \( \frac{1}{\sigma_1} = \frac{m}{H} \geq \sigma_{-1} = \langle Y, \nu \rangle \). In this case, an analogue of (20) in
[10] was
\[
\text{Vol}(\Omega_t)^{\prime} \geq (m + 1)\text{Vol}(\Omega_t)
\]
which was derived using an inequality of Ros ([14])
\[
m \int_{\Sigma} \frac{1}{H} \, d\mu \geq (m + 1)\text{Vol}(\Omega).
\]

4. A RELATED INEQUALITY

In this section, we prove Theorem 2. First, we need a generalized
Hsiung-Minkowski formula (cf. [9, Theorem 2.1]). Similar formulas of
this type can be found in [1, 4, 11, 15].

Proposition 2. Let \( \Sigma \) be a smooth closed hypersurface in \( \mathbb{R}^{m+1} \) and \( f \)
be a smooth function on \( \Sigma \). Then
\[
\int_{\Sigma} f \sigma_l \, d\mu = \int_{\Sigma} f \sigma_{l+1} \langle Y, \nu \rangle \, d\mu
\]
\[
- \frac{1}{(m - l)(m)} \int_{\Sigma} \langle T_l(\nabla f), Y \parallel \rangle \, d\mu,
\]
for any \( 0 \leq l < m \). Here \( Y \) is a position vector field, \( Y \parallel \) denotes its
tangential component on \( \Sigma \), and \( \nu \) is the unit outward normal to \( \Sigma \).

Proof. \( Y \) being a position vector field implies
\[
(\nabla Y \parallel)_i^j = (\nabla Y)_i^j - A_i^j \langle Y, \nu \rangle = \delta_i^j - A_i^j \langle Y, \nu \rangle.
\]
Therefore, on \( \Sigma \)
\[
\text{div}(fT_l(Y \parallel)) = \langle \nabla f, T_l(Y \parallel) \rangle + f(T_l)_j^i (\nabla Y \parallel)_i^j
\]
\[
= \langle T_l(\nabla f), Y \parallel \rangle + f[\text{tr}(T_l) - \text{tr}(T_l \circ A) \langle Y, \nu \rangle],
\]
where we used (10), (25) and the fact \( T_l \) is self-adjoint. It follows from
(7), (8) and (26) that
\[
\text{div}(fT_l(Y \parallel)) = \langle T_l(\nabla f), Y \parallel \rangle + f[(m - l)H_l - (l + 1)H_{l+1} \langle Y, \nu \rangle].
\]
Integrating (27) over \( \Sigma \) gives (24). \(\square\)
Next, we need a result concerning the positivity of the Newton transformation $T_l$ in [2, Proposition 3.2].

**Proposition 3** ([2]). For a closed hypersurface $\Sigma \subset \mathbb{R}^{m+1}$, if $\sigma_k > 0$ for some $1 \leq k \leq m$, then the quadratic form associated to $T_l$ is positive definite for any $0 \leq l < k$.

**Proof of Theorem 2**. We first assume $O \notin \Sigma$. In this case, $r$ is a smooth positive function when restricted to $\Sigma$. Choose $f = r^p$ in Proposition 2, we have

$$\int_{\Sigma} r^p \sigma_l d\mu = \int_{\Sigma} r^p \sigma_{l+1} \langle Y, \nu \rangle d\mu$$

(28)

$$- \frac{1}{(m-l)^{\binom{m}{l}}} \int_{\Sigma} \langle T_l(\nabla r^p), Y|| \rangle d\mu,$$

where

$$\nabla r^p = pr^{p-1}\nabla r = pr^{p-2}Y||.$$

By Proposition 3,

$$\langle T_l(Y||), Y|| \rangle \geq 0$$

(30)

and $\langle T_l(Y||), Y|| \rangle = 0$ if and only if $Y|| = 0$. Thus it follows from (28), (29), (30) and the assumption $p \geq 0$ that

$$\int_{\Sigma} \sigma_l r^p d\mu \leq \int_{\Sigma} \sigma_{l+1} r^{p+1} d\mu.$$  

(31)

If $l+1 < k$, applying (31) repeatedly with $(l, p)$ replaced by $(l+1, p+1)$, $\ldots$, $(k-1, p+k-l-1)$ respectively gives

$$\int_{\Sigma} \sigma_{l+1} r^{p+1} d\mu \leq \int_{\Sigma} \sigma_{l+2} r^{p+2} d\mu \leq \cdots \leq \int_{\Sigma} \sigma_k r^{p+k-l} d\mu.$$

This proves the inequality (4). When the equality in (4) holds, the equality in (31) must hold, which implies $Y|| = 0$. Hence $\Sigma$ is a round sphere centered at $O$.

Next suppose $O \in \Sigma$. Let $B_\varepsilon \subset \Sigma$ be a small geodesic ball with geodesic radius $\varepsilon$. Integrating (27) over $\Sigma \setminus B_\varepsilon$ with $f = r^p$, we have

$$\int_{\Sigma \setminus B_\varepsilon} r^p \sigma_l d\mu = \int_{\partial B_\varepsilon} r^p \langle T_l(Y||), n \rangle d\tau + \int_{\Sigma \setminus B_\varepsilon} r^p \sigma_{l+1} \langle Y, \nu \rangle d\mu$$

(32)

$$- \frac{1}{(m-l)^{\binom{m}{l}}} \int_{\Sigma \setminus B_\varepsilon} \langle T_l(\nabla r^p), Y|| \rangle d\mu,$$
where $n$ is the inward unit normal to $\partial B_\epsilon$ in $\Sigma$ and $d\tau$ is the volume form on $\partial B_\epsilon$. It is clear that

$$\int_{\partial B_\epsilon} r^p \langle T_l(Y^\parallel), n \rangle d\tau \to 0, \quad \text{as } \epsilon \to 0.$$  

Therefore (32) implies

$$\int_{\Sigma} r^p \sigma_l d\mu = \int_{\Sigma} r^p \sigma_{l+1} \langle Y, \nu \rangle d\mu - \frac{1}{(m-l)(m)} \int_{\Sigma \setminus \{O\}} \langle T_l(\nabla r^p), Y^\parallel \rangle d\mu,$$

from which the inequality (4) and its equality case follow as in the previous case. \hfill \Box

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