POWERFREE SUMS OF PROPER DIVISORS

BY

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Abstract. Let $s(n) := \sum_{d \mid n, d < n} d$ denote the sum of the proper divisors of $n$. It is natural to conjecture that for each integer $k \geq 2$, the equivalence $n$ is $k$th powerfree $\iff s(n)$ is $k$th powerfree holds almost always (meaning, on a set of asymptotic density 1). We prove this for $k \geq 4$.

1. Introduction. A 19th century theorem of Gegenbauer asserts that for each fixed $k$, the set of positive integers not divisible by the $k$th power of an integer larger than 1 has asymptotic density $\zeta(k)^{-1}$, where $\zeta(s)$ is the familiar Riemann zeta function. Recall that the asymptotic density of a set $\mathcal{A}$ of positive integers is the limiting proportion of the elements of $\mathcal{A}$ up to $x$, more precisely the limit as $x \to \infty$ of the quantity $\frac{1}{x} \# \{a \leq x : a \in \mathcal{A} \}$, subject to existence.

Call an integer $k$th powerfree, or $k$-free for short, when it is not divisible by the $k$th power of an integer larger than 1. In this note we investigate the frequency with which the sum-of-proper-divisors function $s(n) := \sum_{d \mid n, d < n} d$ assumes $k$-free values. As we proceed to explain, there is a natural guess to make here, formulated below as Conjecture 1.1.

Fix $k \geq 2$. If $n$ is not $k$-free, then $p^k \mid n$ for some prime $p$. Moreover, if $y = y(x)$ is any function tending to infinity, then the upper density of $n$ divisible by $p^k$ for some $p > y^{1/k}$ is at most $\sum_{p > y^{1/k}} p^{-k} = o(1)$. Hence, almost always a non-$k$-free number $n$ is divisible by $p^k$ for some $p^k \leq y$. To be precise, when we say that a statement about positive integers $n$ holds almost always, we mean that it holds for all $n \leq x$ with $o(x)$ exceptions, as $x \to \infty$. (Importantly, we allow the statement itself to involve the growing upper bound $x$.)

It was noticed by Alaoglu and Erdős [AE44] that whenever $y = y(x)$ tends to infinity with $x$ slowly enough, $\sigma(n)$ is divisible by all of the integers

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in $[1, y]$ almost always. (We give a proof below with $y := (\log \log x)^{1-\epsilon}$; see Lemma 2.2.) Hence, almost always $n$ and $s(n) = \sigma(n) - n$ share the same set of divisors up to $y$. Putting this together with the observations of the last paragraph, we see that if $n$ is not $k$-free, then $s(n)$ is not $k$-free, almost always. The same reasoning shows that if $n$ is $k$-free, then $s(n)$ is not divisible by $p^k$ for any $p \leq y^{1/k}$, almost always. Thus, if it could be shown that almost always $s(n)$ is not divisible by $p^k$ for any prime $p > y^{1/k}$, then we would have established the following conjecture.

**Conjecture 1.1.** Fix $k \geq 2$. On a set of integers $n$ of asymptotic density 1,

$$n \text{ is } k\text{-free} \iff s(n) \text{ is } k\text{-free}.$$  

The case $k = 2$ of Conjecture 1.1 is alluded to by Luca and Pomerance in [LP15] (see Lemma 2.2 there and the discussion following). Their arguments show that $s(n)$ is squarefree on a set of positive lower density (in fact, of lower density at least $\zeta(2)^{-1} \log 2$). Conjecture 1.1, for every $k \geq 2$, would then be a consequence of the following general conjecture of Erdős–Granville–Pomerance–Spiro [EGPS90] (see Remark 3.5 below).

**Conjecture 1.2.** If $A$ is a set of natural numbers of positive upper density, then $s(A) := \{s(n) : n \in A\}$ also has positive upper density.

We recall that the upper and lower densities of a set of positive integers are defined in the exact same way as the asymptotic density, but with $\limsup$ and $\liminf$ replacing the limit respectively (so that these always exist).

Our result is as follows.

**Theorem 1.3.** Conjecture 1.1 holds for each $k \geq 4$.

To prove Conjecture 1.1 for a given $k$, it is enough (by the above discussion) to show that almost always $s(n)$ is not divisible by $p^k$ for any $p^k > (\log \log x)^{0.9}$. The range $p \leq x^{o(1)}$ can be treated quickly by using familiar arguments (versions of which appear, e.g., in [Pol14]). The main innovation in our argument—and the source of the restriction to $k \geq 4$—is the handling of larger $p$ using a theorem of Wirsing [Wir59] that bounds the “popularity” of values of the function $\sigma(n)/n$.

The reader interested in other work on powerfree values of arithmetic functions may consult [Pap03, PSS03, BL05, BP06] as well as the survey [Pap05].

**Notation and conventions.** We reserve the letters $p, q, P$, with or without subscripts, for primes and we write $\log_k$ for the $k$th iterate of the natural logarithm. We write $P^+(n)$ and $P^-(n)$ for the largest and the smallest prime factors of $n$, with the conventions that $P^+(1) = 1$ and $P^-(1) = \infty$. We
adopt the Landau–Bachmann–Vinogradov notation from asymptotic analysis, with all implied constants being absolute unless specified otherwise.

2. Preliminaries. The following lemma is due to Pomerance (see [Pom77, Theorem 2]).

**Lemma 2.1.** Let $a, k$ be integers with $\gcd(a, k) = 1$ and $k > 0$. Let $x \geq 3$. The number of $n \leq x$ for which there does not exist a prime $p \equiv a \pmod{k}$ such that $p \parallel n$ is $O(x(\log x)^{-1/\phi(k)})$.

The next lemma justifies the claim in the introduction that $\sigma(n)$ is usually divisible by all small primes. It is well-known (see, e.g., [LP15, Lemma 2.1]) but we include the short proof.

**Lemma 2.2.** Fix $\epsilon > 0$. Almost always, the number $\sigma(n)$ is divisible by every positive integer $d \leq (\log_2 x)^{1-\epsilon}$.

**Proof.** Notice that $d \parallel \sigma(n)$ whenever there is a prime $p \equiv -1 \pmod{d}$ such that $p \parallel n$. For each $d \leq (\log_2 x)^{1-\epsilon}$, the number of $n \leq x$ for which there is no such $p$ is $O(x \exp(-\log_2 x^\epsilon))$, by Lemma 2.1. Now sum on $d \leq (\log_2 x)^{1-\epsilon}$. ■

Our next lemma bounds the number of $n \leq x$ for which $n$ and $\sigma(n)$ possess a large common prime divisor. In what follows, we say that a positive integer $a$ is *squarefull* if no prime appears only to the first power in $a$; or, in other words, if $p^2$ divides $a$ for every prime $p$ dividing $a$. By the *squarefull part* of a natural number, we shall mean its largest squarefull divisor.

**Lemma 2.3.** Almost always, the greatest common divisor of $n$ and $\sigma(n)$ has no prime divisor exceeding $\log_2 x$.

With more effort, it could be shown that $\gcd(n, \sigma(n))$ is almost always the largest divisor of $n$ supported on primes not exceeding $\log_2 x$. Compare with [ELP08, Theorem 8], which is the corresponding assertion with $\sigma(n)$ replaced by $\phi(n)$.

**Proof of Lemma 2.3.** Put $y := \log_2 x$. We start by removing those $n \leq x$ with squarefull part exceeding $\frac{1}{2}y$. The number of these $n$ is $O(xy^{-1/2})$, which is $o(x)$ and hence negligible.

Suppose that $n$ survives and there is a prime $p > y$ dividing $n$ and $\sigma(n)$. Since $p \parallel \sigma(n)$, we can choose a prime power $q^e \parallel n$ for which $p \parallel \sigma(q^e)$. Then $y < p \leq \sigma(q^e) < 2q^e$, forcing $e = 1$. Hence, $p \parallel \sigma(q) = q + 1$ and $q \equiv -1 \pmod{p}$. Since $pq \parallel n$, we deduce that the number of $n$ belonging to this
case is at most
\[
\sum_{p>y} \sum_{q \leq x \equiv -1 \pmod{p}} \frac{x}{pq} \ll x \sum_{p>y} \frac{1}{p} \sum_{q \leq x \equiv -1 \pmod{p}} \frac{1}{q} \ll x \log x \sum_{p>y} \frac{1}{p^2}.
\]
which is again \(o(x)\). Here the sum on \(q\) has been estimated by the Brun–Titchmarsh inequality (see, e.g., [Ten15, Theorem 4.16, p. 83]) and partial summation.

The next lemma bounds the number of \(n \leq x\) with two large prime factors that are multiplicatively close.

**Lemma 2.4.** For all large \(x\), the number of \(n \leq x\) divisible by a pair of primes \(q_1, q_2\) with
\[
x^{1/10 \log_3 x} < q_1 \leq x \quad \text{and} \quad q_1 x^{-1/(\log_3 x)^2} \leq q_2 \leq q_1
\]
is \(O(x/\log_3 x)\).

**Proof.** The number of such \(n\) is at most
\[
x \sum_{x^{1/10 \log_3 x} < q_1 \leq x} \frac{1}{q_1} \sum_{q_1 x^{-1/(\log_3 x)^2} \leq q_2 \leq q_1} \frac{1}{q_2}.
\]
By Mertens’ theorem, the inner sum is
\[
\ll \log \left( \frac{\log q_1}{\log(q_1 x^{-1/(\log_3 x)^2})} \right) + \frac{1}{\log(q_1 x^{-1/(\log_3 x)^2})} \ll \frac{\log x}{(\log q_1)(\log_3 x)^2},
\]
leading to an upper bound for our count of \(n\) of
\[
\ll \frac{x \log x}{(\log_3 x)^2} \sum_{x^{1/10 \log_3 x} < q_1 \leq x} \frac{1}{q_1 \log q_1} \ll \frac{x \log x}{(\log_3 x)^2} \cdot \frac{\log_3 x}{\log x} = \frac{x}{\log_3 x}.
\]
Here the final sum has been estimated by the prime number theorem and partial summation.

We conclude this section by quoting the main result of [Wir59].

**Lemma 2.5 (Wirsing).** There exists an absolute constant \(\lambda_0 > 0\) such that
\[
\# \left\{ m \leq x : \frac{\sigma(m)}{m} = \alpha \right\} \leq \exp\left( \lambda_0 \frac{\log x}{\log_2 x} \right)
\]
for all \(x \geq 3\) and all real numbers \(\alpha\).

### 3. Proof of Theorem 1.3

As discussed in the introduction, it is enough to establish the following proposition. From now on, \(y := (\log_2 x)^{0.9}\).
Proposition 3.1. Fix $k \geq 4$. Almost always, $s(n)$ is not divisible by $p^k$ for any $p^k > y$.

We split the proof of Proposition 3.1 into two parts, according to the size of $p$.

3.1. . . . when $y < p^k \leq x^{1/2 \log_3 x}$. The following is a weakened form of [Pol14, Lemma 2.8].

Lemma 3.2. For all large $x$, there is a set $\mathcal{E}(x)$ having size $o(x)$, as $x \to \infty$, such that, for all $d \leq x^{1/2 \log_3 x}$, the number of $n \leq x$ not belonging to $\mathcal{E}(x)$ for which $d \mid s(n)$ is $O(x/d^{0.9})$.

Summing the bound of Lemma 3.2 over $d = p^k$ with $y < p^k \leq x^{1/2 \log_3 x}$ gives $o(x)$. It follows that almost always, $s(n)$ is not divisible by $p^k$ for any $p^k \in (y, x^{1/2 \log_3 x}]$.

3.2. . . . when $p^k > x^{1/2 \log_3 x}$. The treatment of this range of $p$ is based on the following result, which may be of independent interest.

Theorem 3.3. For all large $x$, there is a set $\mathcal{E}(x)$ having size $o(x)$, as $x \to \infty$, such that the number of $n \leq x$ not belonging to $\mathcal{E}(x)$ for which $d \mid s(n)$ is

$$\ll \frac{x}{d^{1/4 \log x}}$$

uniformly for positive integers $d > x^{1/2 \log_3 x}$ satisfying $P^-(d) > \log_2 x$.

The crucial advantage of Theorem 3.3 over Lemma 3.2 is the lack of any restriction on the size of $d$. Since $k \geq 4$, when we sum the bound of Theorem 3.3 over $d = p^k$ with $x^{1/2 \log_3 x} < p^k < x^2$, the result is $O(x \log_2 x/\log x)$, which is $o(x)$. So the proof of Theorem 1.3 will be completed once Theorem 3.3 is established.

Turning to the proof of Theorem 3.3, let $\mathcal{E}(x)$ denote the collection of $n \leq x$ for which at least one of the following fails:

1. $n > x/\log x$,
2. the largest squarefull divisor of $n$ is at most $\log_2 x$,
3. $P^+(n) > x^{1/10 \log_3 x}$,
4. $P^+(n)^2 \nmid n$,
5. $P^+(\gcd(n, \sigma(n))) \leq \log_2 x$,
6. $P^+(n) > P_2^+(n)x^{1/(\log_3 x)^2}$, where $P_2^+(n) := P^+(n/P^+(n))$ is the second-largest prime factor of $n$.

Let us show that only $o(x)$ integers $n \leq x$ fail one of (1)–(6). This is obvious for (1). The count of $n \leq x$ failing (2) is $\ll x \sum_{r > \log_2 x, r \text{ squarefull}} 1/r \ll x/\sqrt{\log_2 x}$, and thus is $o(x)$. That the count of $n \leq x$ failing (3) is $o(x)$ follows from standard bounds on the counting function of smooth (friable) numbers.
(e.g., [Ten15 Theorem 5.1, p. 512]), or Brun’s sieve. The set of \( n \leq x \) passing (3) but failing (4) has cardinality \( \ll x \sum_{r>x^{1/10} \log x} 1/r^2 = o(x) \). Condition (5) is handled by Lemma 2.3. That the count of \( n \leq x \) satisfying (1)–(5) and failing (6) is \( o(x) \) follows from Lemma 2.4.

Let \( d \) be as in Theorem 3.3. We separate the count of \( n \notin \mathcal{E}(x) \) for which \( d \mid s(n) \) according to whether \( P^+(n) < d^{1/4}(\log x)^2 \) or \( P^+(n) \geq d^{1/4}(\log x)^2 \).

We first consider \( n \neq \mathcal{E}(x) \) with \( P^+(n) \geq d^{1/4}(\log x)^2 \). Write \( n = mP \), where \( P := P^+(n) \). Then \( \gcd(m, P) = 1 \), and

\[
x/m \geq d^{1/4}(\log x)^2.
\]

We can rewrite the condition \( d \mid s(n) \) as

\[
Ps(m) \equiv -\sigma(m) \pmod{d}.
\]

For this congruence to have solutions, we must have \( \gcd(s(m)\sigma(m), d) = 1 \). Indeed, if there exists a prime \( q \) dividing both \( \sigma(m) \) and \( d \), then from \( q \mid d \), we have \( q > \log_2 x \), whereas since \( d \mid s(n) \), we also have \( q \mid s(n) \). But then the divisibility \( q \mid \sigma(m) \mid \sigma(n) \) leads to \( q \mid \gcd(n, \sigma(n)) \), contradicting condition (5) above. Since any common prime divisor of \( s(m) \) and \( d \) would, by the congruence, have to divide \( \sigma(m) \) as well, we must indeed have \( \gcd(s(m)\sigma(m), d) = 1 \).

As such, the above congruence condition on \( P \) places it in a unique coprime residue class modulo \( d \). Hence, given \( m \), the number of possible \( P \) (and hence possible \( n = mP \)) is

\[
\ll \frac{x}{md} + 1 \ll \frac{x}{md} + \frac{x}{md^{1/4}(\log x)^2},
\]

which when summed over \( m \leq x \) is \( \ll x/d^{1/4} \log x \), consistent with Theorem 3.3 (We use here the lower bound on \( d \)).

It remains to count \( n \leq x \), \( n \notin \mathcal{E}(x) \) where \( d \mid s(n) \) and \( P^+(n) < d^{1/4}(\log x)^2 \). For this case, we fix a constant

\[
\lambda > 2\lambda_0,
\]

where \( \lambda_0 \) is the constant appearing in Wirsing’s bound (Lemma 2.5). We will assume that \( d \leq x^{3/2} \), since \( s(n) \leq \sigma(n) < x^{3/2} \) for all \( n \leq x \), once \( x \) is sufficiently large (e.g., as a consequence of the bound \( \sigma(n) \ll n \log_2 (3n) \); see [HW08 Theorem 323]).

We write \( n = AB \), where \( A \) is the least unitary squarefree divisor of \( n/P^+(n) \) exceeding \( d^{1/4} \exp\left(\frac{\lambda \log x}{2 \log_2 x}\right) \). Such a divisor exists as \( n > x/\log x \) has maximal squarefull divisor at most \( \log_2 x \), whereupon its largest unitary squarefree divisor coprime to \( P^+(n) \) must be no less than

\[
\frac{1}{d^{1/4}(\log x)^2 \log_2 x} \cdot \frac{x}{\log x} \geq d^{1/4} \exp\left(\frac{\lambda}{2} \frac{\log x}{\log_2 x}\right).
\]
In order to have the sum of Theorem 3.3, and so also that of Theorem 1.3.

Furthermore, $$P^+(A) \leq P_2^+(n) < P^+(n)x^{-1/(\log_3 x)^2}$$
$$< d^{1/4}(\log x)^2x^{-1/(\log_3 x)^2} < d^{1/4}x^{-\lambda/\log_2 x}.$$

Since $A/P^+(A)$ is a unitary squarefree divisor of $n/P^+(n)$, to avoid contradicting the choice of $A$, we must have $A \leq d^{1/2} \exp(-\frac{\lambda}{2} \log x/\log_2 x)$. Then $\sigma(A) \ll A \log_2 A \ll A \log_2 x$, so that (for large $x$) $\sigma(A) < d^{1/2}$.

For each $B$ as above, we bound the number of corresponding $A$. First of all, since $\gcd(A, B) = 1$, the divisibility $d \mid s(n)$ translates to the congruence $\sigma(A) \sigma(B) \equiv AB \pmod{d}$. Now, we claim that $\gcd(A \sigma(B), d) = 1$. Indeed, for any prime $q$ dividing both $A$ and $d$, on the one hand, we must have $q \geq P^-(d) > \log_2 x$, while on the other, $q \mid d \mid s(n)$ and $q \mid A \mid n$ imply $q \mid \gcd(n, \sigma(n))$. This contradicts (5). It follows by an analogous argument that $\gcd(\sigma(B), d) = 1$, thus proving our claim. Consequently, the above congruence may be rewritten as

$$\frac{\sigma(A)}{A} \equiv \frac{B}{\sigma(B)} \pmod{d}.$$

Now for some $B$, consider any pair of squarefree integers $A_1, A_2$ coprime to $d$ and satisfying the above congruence along with $\sigma(A_1), \sigma(A_2) < d^{1/2}$. Then $\sigma(A_1)/A_1 \equiv \sigma(A_2)/A_2 \pmod{d}$, leading to $\sigma(A_1)A_2 \equiv A_1\sigma(A_2) \pmod{d}$. But also

$$\vert \sigma(A_1)A_2 - A_1\sigma(A_2) \vert \leq \max \{\sigma(A_1)A_2, A_1\sigma(A_2) \} < d,$$

thereby forcing $\sigma(A_1)/A_1 = \sigma(A_2)/A_2$. This shows that for each $B$, all corresponding $A$ have $\sigma(A)/A$ assume the same value, whereupon Lemma 2.5 bounds the number of possible $A$ by $\exp(\lambda_0 \log x/\log_2 x)$. Keeping in mind the upper bound (3.1) on $B$, we deduce that the number of $n$ falling into this case is at most

$$\frac{x}{d^{1/4}} \exp\left( -\frac{\lambda}{2} \log x/\log_2 x \right) \cdot \exp\left( \lambda_0 \log x/\log_2 x \right) = \frac{x}{d^{1/4}} \exp\left( \left( \lambda_0 - \frac{\lambda}{2} \right) \log x/\log_2 x \right).$$

Since $\lambda > 2\lambda_0$, this final quantity is $\ll x/d^{1/4}\log x$. This completes the proof of Theorem 3.3 and so also that of Theorem 1.3.

Remark 3.4. It is to be noted that one needs the condition $k \geq 4$ in order to have the sum

$$\sum_{x^{1/2}\log_3 x < \rho^k < x^2} \frac{x}{\rho^{k/4}\log x},$$
which arises from summing our upper bound in Theorem 3.3 over all \( d := p^k > x^{1/2 \log_3 x} \), be \( o(x) \). Indeed, for \( k \leq 3 \), this sum would be \( \gg x^{7/6} / (\log x)^2 \).

**Remark 3.5.** The conjecture of Erdős, Granville, Pomerance, and Spiro (quoted above as Conjecture 1.2) can be restated as saying that \( s^{-1}(A) \) has density 0 whenever \( A \) has density 0. If this holds, then the conclusion of Proposition 3.1 follows for each \( k \geq 2 \): take
\[
A = \{ n \text{ divisible by } p^k \text{ for some } p^k > \log_3(100n) \}.
\]

Unfortunately, very little is known in the direction of the EGPS conjecture. The record result (still quite weak) seems to be that of [PPT18], where it is shown that \( s^{-1}(A) \) has density 0 whenever \( A \) has counting function bounded by \( x^{1/2 + o(1)} \), as \( x \to \infty \).

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