Jeon, Bogwang
On the number of hyperbolic Dehn fillings of a given volume. (English) Zbl 1462.57023
Trans. Am. Math. Soc. 374, No. 6, 3947-3969 (2021).

By Thurston’s hyperbolic Dehn filling theorem, excluding finitely many fillings for each cusp of an n-cusped hyperbolic 3-manifold $M$, the manifold obtained by surgery with coefficients $p_i/q_i$ on the cusps is again hyperbolic; moreover the volumes of the manifolds obtained by Dehn filling on $M$ converge from below to the volume of $M$ when $|p_i| + |q_i| \to \infty$ ($1 \leq i \leq n$). In this context, the following question is still open: Does there exist a constant $c = c(M)$ such that, for the number $N_M(v)$ of hyperbolic Dehn fillings of $M$ with a given volume $v$, one has $N_M(v) < c$ for any $v$? (It is known instead that the number $N(v)$ of hyperbolic 3-manifolds with a given volume $v$ is finite but unbounded when $v \to \infty$.)

The main result of the present paper is the following partial solution for the case of 1-cusped hyperbolic 3-manifolds. Let $M$ be a 1-cusped hyperbolic 3-manifold whose cusp shape is quadratic; then there exists a constant $c$ such that $N_M(v) < c$ for any $v$. The proof of the main theorem is an interesting combination of different ideas from number theory, algebraic geometry, and model theory. By a result of W. D. Neumann and D. Zagier [Topology 24, 307–332 (1985; Zbl 0589.57015)], the volume of the manifold obtained by $p/q$-filling of the cusp of $M$ can be approximated from the volume of $M$ by using a quadratic form $Q_M(p, q)$. As the author notes, usual well-known techniques for counting rational or integral points over algebraic varieties in diophantine geometry do not work in this case (since the volumes are presumably not algebraic). “To overcome this technical difficulty, we employ results from the so-called “o-minimal theory” in logic. An o-minimal structure is simply a generalization of the class of semialgebraic sets, and an element of it, called a definable set, shares many common properties with a semialgebraic set.”

Reviewer: Bruno Zimmermann (Trieste)

MSC:
57K32 Hyperbolic 3-manifolds
57K31 Invariants of 3-manifolds (including skein modules, character varieties)

Keywords:
cusped hyperbolic 3-manifold; hyperbolic Dehn filling; hyperbolic volume

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