Inverse problem of the limit shape for convex lattice polygonal lines

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Abstract

It is known that random convex polygonal lines on $\mathbb{Z}_+^2$ (with the endpoints fixed at $0 = (0, 0)$ and $n = (n_1, n_2) \to \infty$) have a limit shape with respect to the uniform probability measure, identified as the parabola arc $\sqrt{c(1 - x_1)} + \sqrt{x_2} = \sqrt{c}$, where $n_2/n_1 \to c$. The present paper is concerned with the inverse problem of the limit shape. We show that for any strictly convex, $C^3$-smooth arc $\gamma \subset \mathbb{R}_+^2$ starting at the origin, there is a probability measure $P_n$ on convex polygonal lines, under which the curve $\gamma$ is their limit shape.

Key words: Convex lattice polygonal lines; Limit shape; Inverse problem of limit shape; Local limit theorem

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1. Introduction

Consider a convex lattice polygonal line $\Gamma$ with vertices on $\mathbb{Z}_+^2 := \{(i, j) \in \mathbb{Z}^2 : i, j \geq 0\}$, starting at the origin and such that the slope of each of its edges is nonnegative and does not exceed the angle of 90°. Convexity means that the slope of consecutive edges is strictly increasing. Let $\Pi$ be the set of all such polygonal lines with finitely many edges, and by $\Pi_n$ the subset of polygonal lines $\Gamma \in \Pi$ with the right endpoint fixed at $n = (n_1, n_2) \in \mathbb{Z}_+^2$.

The limit shape, with respect to a probability measure $P_n$ on $\Pi_n$ as $n \to \infty$, is understood as a planar curve $\gamma^*$ such that, for any $\varepsilon > 0$,

$$\lim_{n \to \infty} P_n\{\Gamma \in \Pi_n : d(\tilde{\Gamma}_n, \gamma^*) \leq \varepsilon\} = 1,$$

where $\tilde{\Gamma}_n = S_n(\Gamma)$, subject to a suitable scaling $S_n : \mathbb{R}^2 \to \mathbb{R}^2$, and $d(\cdot, \cdot)$ is some metric on the path space, e.g., induced by the Hausdorff distance between compact sets,

$$d_H(A, B) := \max \left\{ \max_{x \in A} \min_{y \in B} |x - y|, \max_{y \in B} \min_{x \in A} |x - y| \right\}.$$

Of course, the limit shape and its very existence may depend on the probability law $P_n$. With respect to the uniform distribution on $\Pi_n$, the problem was solved independently by

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Vershik [14], Bárány [2] and Sinai [12], who showed that if \( n_1, n_2 \to \infty \) so that \( n_2/n_1 \to c \in (0, \infty) \), then under the scaling \( S_n : (x_1, x_2) \mapsto (x_1/n_1, x_2/n_2) \) limit (1.1) holds with respect to the Hausdorff metric \( d_H \) and with the limit shape \( \gamma^* \) identified as a parabola arc

\[
\sqrt{c(1 - x_1)} + \sqrt{x_2} = \sqrt{c}, \quad 0 \leq x_1, x_2 \leq 1. \tag{1.3}
\]

Recently, Bogachev and Zarbaliev \([7, 8]\) proved that the same limit shape (1.3) appears for a large class of measures \( P_n \) of the form

\[
P_n(\Gamma) := B_n^{-1} \prod_{e_i \in \Gamma} b_{k_i}, \quad B_n := \sum_{\Gamma \in \Pi_n} \prod_{e_i \in \Gamma} b_{k_i} \quad (\Gamma \in \Pi_n), \tag{1.4}
\]

where the product is taken over all edges \( e_i \) of \( \Gamma \in \Pi_n \), \( k_i \) is the number of lattice points on the edge \( e_i \) except its left endpoint, and

\[
b_k := \binom{r + k - 1}{k} = \frac{(r + 1) \cdots (r + k - 1)}{k!}, \quad k = 0, 1, 2, \ldots. \tag{1.5}
\]

This result has provided first evidence in support of a conjecture on the limit shape universality, put forward independently by Vershik [14, p. 20] and Prokhorov [11]. The class of probability measures (1.4) with coefficients (1.5) belongs to a general meta-type of decomposable combinatorial structures known as multisets (see [1] §2.2]). Bogachev [4] has extended the universality result to a much wider class of multiplicative probability measures (1.4) including the analogues of two other well-known meta-types of decomposable structures — selections and assemblies (cf. [1] §2.2]; for example, this class includes the uniform distribution on the subset of “simple” polygonal lines (i.e., those with no lattice points apart from vertices).

However, universality of the limit shape \( \gamma^* \) given by (1.3) has its boundaries; indeed, in the present paper we consider the inverse problem and show that any \( C^3 \)-smooth, strictly convex arc \( \gamma \in \mathbb{R}_+^2 \) (started at the origin) may appear as the limit shape with respect to a suitable probability measure \( P_n^* \) on \( \Pi_n \), as \( n \to \infty \). For early drafts of this result (treated in terms of approximation of convex curves by random polygonal lines), see \([5, 6]\).

Like in \([4, 7]\), our construction employs an elegant probabilistic approach based on randomization and conditioning (see [1]), first used in the polygonal context by Sinai [12]. The idea is to represent a given measure \( P_n \) on \( \Pi_n \) as the conditional distribution, \( P_n(\Gamma) = Q(\Gamma \mid \Pi_n) \), induced by a suitable “global” probability measure \( Q \) defined on the space \( \Pi = \cup_n \Pi_n \) of convex lattice polygonal lines with a free right endpoint. In turn, the measure \( Q = Q_\nu \) depending on a two-dimensional parameter \( z = (z_1, z_2) \) is constructed as the distribution of a suitable integer-valued random field \( \nu = \nu(\cdot) \) with mutually independent components, defined on the subset \( \mathcal{X} \subset \mathbb{Z}_+^2 \) consisting of points \( x = (x_1, x_2) \in \mathbb{Z}_+^2 \) with co-prime coordinates. Note that a polygonal line \( \Gamma \in \Pi \) is easily retrieved from a configuration \( \{\nu(x)\}_{x \in \mathcal{X}} \) using the collection of the corresponding edges \( \{x\nu(x)\}_{x \in \mathcal{X}} \) and the convexity property.

It turns out, however, that in order to fit a given curve \( \gamma \) the parameter \( z = (z_1, z_2) \) needs to allow for a dependence on \( x \in \mathcal{X} \). We derive suitable parameter functions \( z_1(x) \) and \( z_2(x) \), assuming that they depend on \( x \) through the ratio \( x_2/x_1 \) only, which is particularly convenient in conjunction with the parameterization of the curve \( \gamma \) using its tangent slope. As one would anticipate, if \( \gamma = \gamma^* \) then the functions \( z_1(x), z_2(x) \) are reduced to constants and our method recovers the uniform distribution on \( \Pi_n \).

To summarize, our main result is as follows.

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Theorem 1.1. Let $\gamma \subset \mathbb{R}^2_+$ be a strictly convex, $C^3$-smooth arc, with the endpoints $0$ and $(1, c_\gamma)$ and with the curvature bounded from below by a positive constant. Suppose that $n_2/n_1 \to c_\gamma$, and set $\Gamma_n := n_1^{-1} \Gamma$. Then there is a probability measure $P_n^\gamma$ on $\Pi_n$ such that, for any $\varepsilon > 0$,
\[
\lim_{n \to \infty} P_n^\gamma \{ \Gamma \in \Pi_n : d_H(\Gamma_n, \gamma) \leq \varepsilon \} = 1. \tag{1.6}
\]

Remark 1.1. It will be more convenient to use another metric on the space of convex paths (denoted by $d_L$), based on the tangential parameterization of paths and a sup-distance between the corresponding arc lengths. However, the metrics $d_L$ and $d_H$ are equivalent.

Remark 1.2. It is interesting to try and relax the $C^3$-smoothness condition on $\gamma$ (e.g., by permitting “change-points” or corners), as well as to allow for degeneration of the curvature (e.g., through possible flat segments). We will address these issues elsewhere.

The rest of the paper is organized as follows. In Section 2, we introduce the space of convex paths on the plane and endow it with a suitable metric. In Section 3, the measures $Q_z^\gamma$ and $P_n^\gamma$ are constructed for a given convex curve $\gamma$. In Section 4, the parameter vector-function $z(x)$ is chosen to guarantee the convergence of “expected” scaled polygonal lines $\tilde{\Gamma}_n = n_1^{-1} \Gamma$ to the target curve $\gamma$ (Theorem 4.2). Refined first-order moment asymptotics are obtained in Section 5, while higher-order moment sums are analyzed in Section 6. Section 7 is devoted to the proof of a local central limit theorem (Theorem 7.1). Finally, the limit shape result, with respect to both $Q_z$ and $P_n^\gamma$, is proved in Section 8 (Theorems 8.1 and 8.2 respectively).

Some notations. For a row-vector $x = (x_1, x_2) \in \mathbb{R}^2$, its Euclidean norm (length) is denoted by $|x| := (x_1^2 + x_2^2)^{1/2}$, and $\langle x, y \rangle := x^\top y = x_1y_1 + x_2y_2$ is the corresponding inner product of vectors $x, y \in \mathbb{R}^2$. We denote $\mathbb{Z}_+ := \{ k \in \mathbb{Z} : k \geq 0 \}$, $\mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$, and similarly $\mathbb{R}_+ := \{ x \in \mathbb{R} : x \geq 0 \}$, $\mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R}_+$. For $z = (z_1, z_2) \in \mathbb{R}_+^2$ and $x = (x_1, x_2) \in \mathbb{Z}_+^2$, we use the multi-index notation $z^x := z_1^{x_1} z_2^{x_2}$. The notation $a_n \asymp b_n$ as $n \to \infty$ means that
\[
0 < \lim \inf_{n \to \infty} \frac{x_n}{y_n} \leq \lim \sup_{n \to \infty} \frac{x_n}{y_n} < \infty.
\]
We also use the standard notation $x_n \sim y_n$ for $x_n/y_n \to 1$ as $n_1, n_2 \to \infty$.

2. Preliminaries: convex polygonal paths

Let $g$ be a bounded function defined on some interval $[0, a]$, such that $g(0) = 0$, and suppose that $g$ is non-decreasing and convex on $[0, a]$. Convexity means that the function’s epigraph $\{(u, v) : 0 \leq u \leq a, g(u) \leq v\}$ is a convex set on the plane. Furthermore, assume that $g$ is continuous on $[0, a]$ and piecewise differentiable, with the derivative $g'$ continuous everywhere except at a finite set of points (we allow $g'(a)$ to be infinite, $g'(a) \leq +\infty$). It follows that the function $t = g'(u)$ is nonnegative and non-decreasing in its domain, and in particular $0 \leq t_0 \leq g'(u) \leq t_1 \leq \infty$ ($0 \leq a \leq a$), where
\[
t_0 := \inf_{0 \leq u \leq a} g'(u), \quad t_1 := \sup_{0 \leq u \leq a} g'(u). \tag{2.1}
\]

Denote by $\gamma_g \equiv \gamma$ the graph of a function $g$ with the above properties, and let $\mathcal{G}$ be the set of all such curves. For the spaces $\Pi_n, \Pi$ of convex polygonal lines introduced above, we have
with the convention that on $[0, \infty)$ then function $g$ and hence our assumptions imply that every curve $\gamma \in \mathfrak{G}$, where $\gamma(t) = u(t)$ is given explicitly by the Hausdorff metric $\Pi$ (see (2.1)) then $u_\gamma(t) \equiv 0$ for all $t < t_0$ and $u_\gamma(t) = a$ for all $t \geq t_1$.

Denote by $\ell_\gamma(t)$ the length of the part of $\gamma$ where the tangent slope does not exceed $t$, $\ell_\gamma(t) = \int_0^{u_\gamma(t)} \sqrt{1 + g'_\gamma(u)^2} \, du, \quad 0 \leq t \leq \infty. \quad (2.3)$

Our assumptions imply that every curve $\gamma \in \mathfrak{G}$ is rectifiable, since $\ell_\gamma(\infty) = \int_0^{u_\gamma(\infty)} \sqrt{1 + g'_\gamma(u)^2} \, du \leq \int_0^{a} (1 + g'_\gamma(u)) \, du = a + g_\gamma(a) < \infty.

Finally, we define the function $d_\mathcal{L} : \mathfrak{G} \times \mathfrak{G} \to \mathbb{R}_+ \cup \{+\infty\}$ by setting $d_\mathcal{L}(\gamma_1, \gamma_2) := \sup_{0 \leq t \leq \infty} |\ell_\gamma(t) - \ell_{\gamma_2}(t)|, \quad \gamma_1, \gamma_2 \in \mathfrak{G}. \quad (2.4)$

**Proposition 2.1.** The function $d_\mathcal{L}(\cdot, \cdot)$ satisfies all properties of a distance.

**Proof.** Clearly, $d_\mathcal{L}(\gamma_1, \gamma_2) = d_\mathcal{L}(\gamma_2, \gamma_1)$ and $d_\mathcal{L}(\gamma, \gamma) = 0$. The triangle axiom is also obvious. So it remains to verify that if $d_\mathcal{L}(\gamma_1, \gamma_2) = 0$ then $\gamma_1 = \gamma_2$. To this end, approximating $\gamma_1, \gamma_2 \in \mathfrak{G}$ by $C^2$-smooth strictly convex curves $\gamma_k^1, \gamma_k^2$, respectively, we reduce the problem to checking that if $\gamma_k^1, \gamma_k^2$ are close to each other in the sense of $d_\mathcal{L}$, then they are also close in the Hausdorff metric $d_H$; that is, if $d_\mathcal{L}(\gamma_k^1, \gamma_k^2) \to 0$ then $d_H(\gamma_k^1, \gamma_k^2) \to 0$ ($k \to \infty$).

Next, for a strictly convex, increasing function $g_\gamma \in C^2[0, a]$, the function $u_\gamma(t)$ defined in (2.2) is given explicitly by $u_\gamma(t) = \begin{cases} 0, & 0 \leq t \leq t_0, \\ (g'_\gamma)^{-1}(t), & t_0 \leq t \leq t_1, \\ a, & t_1 \leq t \leq \infty, \end{cases} \quad (2.5)$

where $(g'_\gamma)^{-1}(t)$ is the (ordinary) inverse of the derivative $g'_\gamma(u)$. In particular, the equations $u = u_\gamma(t), \quad v = g_\gamma(u_\gamma(t))$ determine a parameterization of the curve $\gamma$ via the derivative $t = g'_\gamma(u)$. Differentiating formula (2.3) with respect to $t$, we find $\frac{du_\gamma}{dt} = \frac{1}{\sqrt{1 + t^2}} \frac{d\ell_\gamma}{dt}, \quad u_\gamma(0) = 0, \quad (2.6)$

and hence $\frac{dv_\gamma}{dt} = \frac{dg_\gamma}{du} \cdot \frac{du_\gamma}{dt} = \frac{t}{\sqrt{1 + t^2}} \frac{d\ell_\gamma}{dt}, \quad v_\gamma(0) = 0. \quad (2.7)
Integrating equations (2.6), (2.7) by parts, we obtain
\[
\begin{align*}
\dot{u}_t(t) &= \frac{\ell_t(t)}{\sqrt{1 + t^2}} + \int_0^t \frac{s \ell_s(s)}{(1 + s^2)^{3/2}} \, ds, \\
\dot{v}_t(t) &= \frac{\ell_v(t)}{\sqrt{1 + t^2}} - \int_0^t \frac{\ell_v(s)}{(1 + s^2)^{3/2}} \, ds.
\end{align*}
\]  
(2.8)

Note that these equations are linear in \(\ell_t\). Hence, setting for \(\gamma_1^k, \gamma_2^k\)
\[
\Delta u_k(t) := u_{\gamma_1^k}(t) - u_{\gamma_2^k}(t), \quad \Delta v_k(t) := v_{\gamma_1^k}(t) - v_{\gamma_2^k}(t),
\]
from (2.8) we get
\[
\begin{align*}
\Delta u_k(t) &= \frac{\Delta \ell_k(t)}{\sqrt{1 + t^2}} + \int_0^t \frac{s \Delta \ell_s(s)}{(1 + s^2)^{3/2}} \, ds, \\
\Delta v_k(t) &= \frac{\Delta \ell_v(t)}{\sqrt{1 + t^2}} - \int_0^t \frac{\Delta \ell_v(s)}{(1 + s^2)^{3/2}} \, ds.
\end{align*}
\]
This implies that if \(\Delta \ell_k(t) \to 0\), uniformly in \(t \in [0, \infty]\), then \(\Delta u_k(t) \to 0\), \(\Delta v_k(t) \to 0\), also uniformly on \([0, \infty]\) \((k \to \infty)\). This completes the proof.

From the proof of Proposition 2.1 one can see that the following result holds.

**Corollary 2.2.** The metrics \(d_L\) and \(d_H\) are equivalent; in particular, \(d_L(\gamma_k, \gamma) \to 0\) if and only if \(d_H(\gamma_k, \gamma) \to 0\).

Consider a fixed convex curve \(\gamma \in \mathcal{S}\), represented as the graph of an increasing convex function \(g_\gamma\), which for definiteness is assumed to be defined on the interval \([0, 1]\). We will be working under the following assumption.

**Assumption 2.1.** The function \(g_\gamma\) is strictly increasing and strictly convex on \([0, 1]\), and \(g_\gamma \in C^2[0, 1]\). In particular, \(g_\gamma'(u) \geq 0\), \(g_\gamma''(u) \geq 0\) for all \(u \in [0, 1]\). Moreover, the curvature \(\kappa_\gamma\) of the curve \(\gamma\), given by the formula
\[
\kappa_\gamma(u) = \frac{g_\gamma''(u)}{(1 + g_\gamma'(u)^2)^{3/2}}, \quad 0 \leq u \leq 1,
\]  
(2.9)
is uniformly bounded from below,
\[
\inf_{u \in [0,1]} \kappa_\gamma(u) \geq K_0 > 0.
\]  
(2.10)

As was mentioned in the proof of Proposition 2.1, the graph \(\gamma\) of the function \(g_\gamma\) can be parameterized by the derivative \(t = g_\gamma'(u)\) via the equations \(u = u_\gamma(t), v = g_\gamma(u_\gamma(t))\), where \(u_\gamma(t)\) is given by (2.5). Expression (2.9) for the curvature is then reduced to
\[
\kappa_\gamma(t) = \frac{g_\gamma''(u_\gamma(t))}{(1 + t^2)^{3/2}}, \quad t_0 \leq t \leq t_1,
\]  
(2.11)
where \(t_0 = \inf_u g_\gamma'(u), t_1 = \sup_u g_\gamma'(u)\) (see (2.1)).
3. Construction of the measures $Q_z^\gamma$ and $P_n^\gamma$

Consider the set $\mathcal{X} \subset \mathbb{Z}_+^2$ of all pairs of co-prime nonnegative integers,

$$\mathcal{X} := \{x = (x_1, x_2) \in \mathbb{Z}_+^2 : \gcd(x_1, x_2) = 1\},$$  \hspace{1cm} (3.1)

where “gcd” stands for “greatest common divisor”. Denote by $\tau(x) := x_2/x_1 \in [0, +\infty]$ the slope of the vector $x = (x_1, x_2) \in \mathcal{X}$. Let $\Phi := (\mathbb{Z}_+)^\mathcal{X}$ be the space of functions on $\mathcal{X}$ with nonnegative integer values, and consider the subspace of functions with finite support, $\Phi_0 := \{\nu \in \Phi : \#(\text{supp} \nu) < \infty\}$, where $\text{supp} \nu := \{x \in \mathcal{X} : \nu(x) > 0\}$. It is easy to see that the space $\Phi_0$ is in one-to-one correspondence with the space $\Pi = \bigcup_{n \in \mathbb{Z}_+^2} \Pi_n$ of all (finite) convex lattice polygonal lines, whereby each $x \in \mathcal{X}$ determines the direction of a potential edge, only utilized if $x \in \text{supp} \nu$, in which case the value $\nu(x) > 0$ specifies the scaling factor, altogether yielding a vector edge $x \nu(x)$; finally, assembling all such edges into a polygonal line is uniquely determined by the fixation of the starting point (at the origin) and the convexity property. Note that $\nu(x) \equiv 0$ formally corresponds to the “trivial” polygonal line with coinciding endpoints. In what follows, we identify the spaces $\Pi$ and $\Phi_0$.

Let us now introduce on $\Phi_0(\mathcal{X})$ a probability measure $Q_z^\gamma$ by setting

$$Q_z^\gamma(\Gamma) := \prod_{x \in \mathcal{X}} z^{\nu(x)}(1 - z^x), \quad \Pi \ni \Gamma \leftrightarrow \nu \in \Phi_0(\mathcal{X}),$$  \hspace{1cm} (3.2)

where $z \equiv z(x) = (z_1(x), z_2(x))$ is a parameter (vector-)function such that $0 \leq z_j(x) < 1$ ($x \in \mathcal{X}$). Its explicit form, determined by a given curve $\gamma \in \mathcal{G}$, will be specified later on. So far, we only assume that

$$Z := \prod_{x \in \mathcal{X}} (1 - z^x) > 0,$$  \hspace{1cm} (3.3)

which guarantees that (3.2) is well defined. Definition (3.2) implies that the random variables $\{\nu(x)\}_{x \in \mathcal{X}}$ are mutually independent and have geometric distribution with parameter $z^x$,

$$Q_z^\gamma(\nu(x) = k) = z^{kx}(1 - z^x), \quad k \in \mathbb{Z}_+;$$  \hspace{1cm} (3.4)

in particular, the corresponding expected value and variance are given by [9] \text{§XI.2, p. 269]

$$E_z^\gamma[\nu(x)] = \frac{z^x}{1 - z^x} = \sum_{k=1}^\infty z^{kx}, \quad \text{Var}[\nu(x)] = \frac{z^x}{(1 - z^x)^2} = \sum_{k=1}^\infty k z^{kx}. \hspace{1cm} (3.5)$$

Note that $Q_z^\gamma$ can be extended in a standard way to a measure on the space $\Phi(\mathcal{X})$ of all nonnegative integer-valued functions on $\mathcal{X}$. However, $Q_z^\gamma$ is in fact concentrated on the subset $\Phi_0(\mathcal{X}) \subset \Phi(\mathcal{X})$ consisting of all finite configurations $\nu(\cdot)$.

**Lemma 3.1.** Condition (3.3) is necessary and sufficient in order that $Q_z^\gamma(\nu \in \Phi_0(\mathcal{X})) = 1$.

**Proof.** According to (3.4),

$$\sum_{x \in \mathcal{X}} Q_z^\gamma(\nu(x) > 0) = \sum_{x \in \mathcal{X}} z^x < \infty$$

whenever the infinite product in (3.3) is convergent. By Borel–Cantelli’s lemma, this implies that only finitely many events $\{\nu(x) > 0\}$ may occur ($Q_z^\gamma$-a.s), and the lemma is proved. \hfill \square

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As a result, with $Q^{\gamma}_2$-probability 1 a realization of the random field $\nu(\cdot)$ determines a (random) convex polygonal line $\Gamma \in \Pi$. Denote by $\xi_\Gamma = (\xi_1, \xi_2)$ the right endpoint of $\Gamma$, so that $\Pi_n = \{\Gamma \in \Pi : \xi_\Gamma = n\}$. Accordingly, $Q^{\gamma}_2$ induces the conditional distribution $P^{\gamma}_n$ on $\Pi_n$, 

$$
P_n^{\gamma}(\Gamma) := Q_n^{\gamma}\{\Gamma | \xi_\Gamma = n\} = \frac{Q^{\gamma}_2(\Gamma)}{Q^{\gamma}_2(\xi_\Gamma = n)}, \quad \Gamma \in \Pi_n.
$$

(3.6)

4. The choice of the parameter function $z(x)$

In the above construction, the measure $P^{\gamma}_n$ depends on the vector parameter $\{z(x)\}_{x \in \mathcal{X}}$. So far, this function was only assumed to guarantee the convergence of the infinite product in (3.3). Let us now adjust it to a given curve $\gamma \in \mathcal{G}$.

Let $\Gamma(t)$ denote the part of the polygonal line $\Gamma \in \Pi$ where the slope of edges does not exceed $t \in [0, \infty]$. Set $\mathcal{X}(t) := \{x \in \mathcal{X} : \tau(x) \leq t\}$. Recalling the association $\Gamma \leftrightarrow \nu$ described in Section 3, the polygonal line $\Gamma(t)$ is determined by the truncated configuration $\nu(x)\mathbf{1}_{\mathcal{X}(t)}(x)$. Denote by $\xi(t) = (\xi_1(t), \xi_2(t))$ the right endpoint of $\Gamma(t)$,

$$
\xi_j(t) = \sum_{x \in \mathcal{X}(t)} x_j \nu(x), \quad (j = 1, 2),
$$

(4.1)

and by $\ell_\Gamma(t)$ its length,

$$
\ell_\Gamma(t) = \sum_{x \in \mathcal{X}(t)} |x| \nu(x).
$$

(4.2)

Let us impose the following calibration condition,

$$
\lim_{n \to \infty} n^{-1}E^{\gamma}_2[\ell_\Gamma(t)] = \ell_\gamma(t), \quad 0 \leq t \leq \infty,
$$

(4.3)

where $E^{\gamma}_2$ stands for the expectation with respect to the measure $Q^{\gamma}_2$ and $\ell_\gamma(t)$ is the corresponding length function associated with a given curve $\gamma$. More specifically, denote $\rho_n := c_\gamma / c_n \to 1$ and set $\tilde{x} := (x_1, \rho_n x_2)$ for $x = (x_1, x_2)$. We will seek $z_1(x), z_2(x)$ in the form

$$
z_j(x) = \exp\{-\alpha_n \delta_j(\tau(\tilde{x}))\} \quad (j = 1, 2),
$$

(4.4)

where

$$
\alpha_n := (\rho_n n_1)^{-1/3} \to 0, \quad \tau(\tilde{x}) = \tilde{x}_2 / \tilde{x}_1 = \rho_n x_2 / x_1,
$$

(4.5)

and $\delta(t) = (\delta_1(t), \delta_2(t))$ is a function on $[0, \infty]$ such that

$$
\inf_{0 \leq t \leq \infty} \delta_j(t) \geq \delta_\star > 0 \quad (j = 1, 2).
$$

(4.6)

Remark 4.1. Note that the right endpoint of the scaled polygonal line $\tilde{\Gamma}_n := n_1^{-1} \Gamma (\Gamma \in \Pi)$ has the coordinates $(1, c_n)$, where $c_n := n_2 / n_1$, whereas the right endpoint of the arc $\gamma$ lies at the point $(1, c_\gamma)$, where $c_\gamma := g_\gamma(1)$ $(0 < c_\gamma < \infty)$. Hence, in order for relation (1.6) to be true, it is natural to pass to the limit $n \to \infty$ in such a way that $c_n \to c_\gamma$. In what follows, we will always be assuming that this condition is fulfilled.
According to (3.5) and (4.2), and using notation (4.4), we have
\[ E_\gamma^{\ell} \left[ \ell_{\mathcal{Y}}(t) \right] = \sum_{x \in \mathcal{X}(t)} |x| \sum_{k=1}^{\infty} z^{\ell x} = \sum_{k=1}^{\infty} \sum_{x \in \mathcal{X}(t)} |x| e^{-\alpha_n k(x, \delta(\tau(x)))} . \] (4.7)

To deal with sums over the sets \( \mathcal{X}(t) \subset \mathcal{X} \), the following lemma will be instrumental. Recall that the Möbius function \( \mu(m) \) (\( m \in \mathbb{N} \)) is defined as follows: \( \mu(1) := 1 \), \( \mu(m) := (-1)^d \) if \( m \) is a product of \( d \) different prime numbers, and \( \mu(m) := 0 \) otherwise (see [10, §16.3, p. 234]); in particular, \( |\mu(m)| \leq 1 \) for all \( m \in \mathbb{N} \).

**Lemma 4.1.** Let \( f : \mathbb{R}^2_+ \to \mathbb{R} \) be a function such that \( f(0, 0) = 0 \) and
\[ \sum_{k=1}^{\infty} \sum_{x \in \mathbb{Z}^2_+} |f(hkx)| < \infty, \quad h > 0. \] (4.8)

For \( h > 0 \), consider the functions
\[ F(h) := \sum_{m=1}^{\infty} \sum_{x \in \mathcal{X}} f(hmx), \quad F^2(h) := \sum_{x \in \mathcal{X}} f(hx). \] (4.9)

Then the following identities hold for all \( h > 0 \)
\[ F(h) = \sum_{x \in \mathbb{Z}^2_+} f(hx), \quad F^2(h) = \sum_{m=1}^{\infty} \mu(m) F(hm). \] (4.10)

**Proof.** Recalling definition (3.1) of the set \( \mathcal{X} \), observe that \( \mathbb{Z}^2_+ = \bigcup_{m=0}^{\infty} m \mathcal{X} \); hence, definition (4.9) of \( F(\cdot) \) is reduced to (4.10). Representation (4.10) for \( F^2(\cdot) \) follows from the Möbius inversion formula (see [10, Theorem 270, p. 237]), provided that \( \sum_{k,m} |F^2(hkm)| < \infty \). To verify the last condition, using (4.9) we obtain
\[ \sum_{k,m=1}^{\infty} |F^2(kmh)| \leq \sum_{k=1}^{\infty} \left( \sum_{m=1}^{\infty} \sum_{x \in \mathcal{X}} |f(hmx)| \right) = \sum_{k=1}^{\infty} \sum_{x \in \mathbb{Z}^2_+} |f(hkx)| < \infty, \]
according to (4.8). \( \Box \)

**Theorem 4.2.** Suppose that the functions \( \delta_1(t), \delta_2(t) \) satisfy condition (4.6). Then, in order that equation (4.3) be fulfilled for all \( t \in [0, \infty) \), it is necessary and sufficient that
\[ \delta_j(t) \equiv +\infty \quad (j = 1, 2), \quad t < t_0, \quad t > t_1, \] (4.11)
\[ \delta_1(t) + t \delta_2(t) = \kappa g_{\gamma}^u(u_{\gamma}(t))^{1/3}, \quad 0 < t < t_1, \] (4.12)
where \( \kappa := (2\zeta(3)/\zeta(2))^{1/3} \), \( \zeta(s) := \sum_{k=1}^{\infty} 1/k^s \) is the Riemann zeta function, and the function \( u_{\gamma}(t) \) is given by (2.5).

**Proof.** Let us set
\[ f(x) := |x| e^{-\alpha_n (\xi, \delta(\tau(\xi)))} 1_{\mathcal{X}(t)}(x), \quad x \in \mathbb{R}^2, \] (4.13)
suppressing for simplicity the dependence on $t$ and $n$. Following notations (4.9) of Lemma 4.1, equation (4.7) is rewritten in the form

$$E^*_2[\ell_\tau(t)] = \sum_{k=1}^{\infty} k^{-1} F^*(k),$$

(4.14)

whereas from (4.10) we have

$$F(h) = \sum_{x_1=0}^{\infty} \sum_{0 \leq x_2 \leq 2x_1} h|x| e^{-\alpha_n h(x, \delta(\tau))}$$

(4.15)

$$\leq h \sum_{x_1, x_2=0}^{\infty} (x_1 + x_2) e^{-\alpha_n h(x_1 + x_2)/2} = h \sum_{y=0}^{\infty} y^2 e^{-\alpha_n h\delta y/2}$$

$$= h \frac{e^{-\alpha_n h\delta/2} + e^{-\alpha_n h\delta}}{(1 - e^{-\alpha_n h\delta/2})^2} = O(1) \alpha_n^{-3} h^{-2}.$$  

(4.16)

In particular, this gives $F(hk) = O(k^{-2})$, uniformly in $k \in \mathbb{N}$, and it follows that condition (4.8) of Lemma 4.1 is satisfied. Hence, using (4.10) and (4.15) and recalling that $n_1^{-1} = \rho_n \alpha_n^3$, from (4.14) with $h = k$ we obtain

$$n_1^{-1} E^*_2[\ell_\tau(t)] = \rho_n \alpha_n^3 \sum_{k,m=1}^{\infty} m\mu(m) F(km)$$

$$= \rho_n \sum_{k,m=1}^{\infty} m\mu(m) \sum_{x_1=1}^{\infty} \sum_{0 \leq x_2 \leq 2x_1} \alpha_n^3 |x| e^{-km\alpha_n(x, \delta(\tau))}.$$  

(4.17)

Taking into account estimate (4.16), we see that the general term in the double sum over $k, m$ in (4.17) admits a uniform bound of the form $O(1) k^{-3} m^{-2}$, which is a term of a convergent series. Therefore, we can apply Lebesgue’s dominated convergence theorem to pass to the limit in (4.17) termwise, as $n \to \infty$ (i.e., $\alpha_n \to 0$). In order to find this limit, note that the internal double series over $x_1, x_2$ in (4.17) is a Riemann sum for the integral

$$\int \int_{0 \leq x_2 \leq 2x_1} \sqrt{x^2_1 + x^2_2} e^{-km\alpha_n(x_1\delta_1(x_2/x_1) + x_2\delta_2(x_2/x_1))} dx_1 dx_2.$$  

(4.18)

Moreover, this sum does converge to integral (4.18) as $\alpha_n \to 0$, since the integrand function in (4.18) is directly Riemann integrable, as follows from an estimation similar to (4.16).

By the change of variables $y_1 = u$, $y_2 = us$ integral (4.18) is reduced to

$$\int_0^t \sqrt{1 + s^2} \left( \int_0^\infty u^2 e^{-km\alpha_n(\delta_1(s) + s\delta_2(s))} du \right) ds = \frac{2}{(km)^3} \int_0^t \frac{\sqrt{1 + s^2}}{\left(\delta_1(s) + s\delta_2(s)\right)^3} ds.$$  

Substituting this into (4.17) we get

$$\lim_{n \to \infty} n_1^{-1} E^*_2[\ell_\tau(t)] = \frac{2}{\zeta(3)} \sum_{k=1}^{\infty} \frac{1}{k^3} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} \int_0^t \frac{\sqrt{1 + s^2}}{\left(\delta_1(s) + s\delta_2(s)\right)^3} ds$$

$$= \frac{2\zeta(3)}{\zeta(2)} \int_0^t \frac{\sqrt{1 + s^2}}{\left(\delta_1(s) + s\delta_2(s)\right)^3} ds.$$  

(4.19)
where we used the identity $\sum_{m=1}^{\infty} m^{-2} \mu(m) = \zeta(2)^{-1}$, which readily follows by the M"{o}bius inversion formula (4.10) applied to $F^2(h) = h^{-2}$, $F(h) = \sum_{m=1}^{\infty} (hm)^{-2} = h^{-2} \zeta(2)$ (cf. [10, §17.5, Theorem 287, p. 250]). Recalling the notation $\kappa$ introduced in Theorem 4.2 and using condition (4.3), from (4.19) we obtain

$$\kappa^{3} \int_{0}^{t} \frac{\sqrt{1 + s^2}}{\left( \delta_1(s) + s \delta_2(s) \right)} \, ds = \ell_\gamma(t), \quad 0 \leq t \leq \infty. \quad (4.20)$$

According to definitions (2.2) and (2.3), we have $\ell_\gamma(t) \equiv 0$ for $t \in [0, t_0)$ and $\ell_\gamma(t) \equiv \ell_\gamma(\infty)$ for $t \in (t_1, \infty)$, while for $t \in (t_0, t_1)$ the derivative $\ell'_\gamma(t)$ is determined by formula (2.6). Hence, differentiating identity (4.20) with respect to $t$, we obtain (4.11) and (4.12). \qed

**Proposition 4.3.** For $t \in [t_0, t_1]$ let set

$$\delta_1(t) := \kappa \varkappa(t)^{1/3} \frac{c_\gamma \sqrt{1 + t^2}}{c_\gamma + t}, \quad \delta_2(t) := \frac{\delta_1(t)}{c_\gamma}, \quad (4.21)$$

where $c_\gamma = g_\gamma(1)$ and the curvature $\varkappa(t)$ is given by (2.11). Then the functions $\delta_1(t), \delta_2(t)$ satisfy assumption (4.6) and equation (4.12).

**Proof.** It is straightforward to verify that equation (4.12) is satisfied. A lower bound of the form (4.6) follows from assumption (2.10). \qed

**Remark 4.2.** In the “classical” case, where the curve $\gamma = \gamma^*$ is determined by equation (1.3), it is easy to check that the corresponding curvature (see (2.9)) is given by

$$\varkappa_{\gamma^*}(t) = \frac{c(1 + t/c)^3}{2(1 + t^2)^{3/2}}, \quad 0 \leq t \leq \infty.$$ 

Hence, expressions (4.21) are reduced to the constants $\delta_1 = \kappa(c/2)^{1/3}$, $\delta_2 = \delta_1/c$ (cf. [7]).

**Assumption 4.1.** Throughout the rest of the paper, we assume that the parameters $z_1(x), z_2(x)$ ($x \in X$) are chosen according to formulas (4.4) with the functions $\delta_1(t), \delta_2(t)$ given by (4.11), (4.21). In particular, the measure $Q_z^2$ becomes dependent on $n = (n_1, n_2)$, as well as the $Q_z^2$-probabilities and the corresponding expected values.

### 5. Asymptotics of the expectation

In this section, we derive a few corollaries from the above choice of $z_1(x), z_2(x)$, assuming throughout that Assumptions 2.1 and 4.1 are satisfied.

**Theorem 5.1.** The convergence in (4.3) is uniform in $t \in [0, \infty]$,

$$\lim_{n \to \infty} \sup_{0 \leq t \leq \infty} \left| n_1^{-1} E_z^2[\ell_{\ell}(t)] - \ell_\gamma(t) \right| = 0. \quad (5.1)$$

We will use the following simple criterion (see [7, Lemma 4.3]).

**Lemma 5.2.** Let $\{f_n(t)\}$ be a sequence of non-decreasing functions on a finite interval $[a, b]$, such that, for each $t \in [a, b]$, $\lim_{n \to \infty} f_n(t) = f(t)$, where $f(t)$ is a continuous (non-decreasing) function on $[a, b]$. Then $f_n(t) \to f(t)$ uniformly on $[a, b]$. 

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Proof of Theorem 5.1. Note that for each \( n \) the function
\[
f_n(t) := n^{-1} E^2_\gamma[\ell(t)] = \frac{1}{n} \sum_{x \in \mathcal{X}(t)} |x| E^2_\gamma[\nu(x)]
\]
is non-decreasing in \( t \) and the limiting function \( f(t) := \ell_\gamma(t) \) given by (2.3) is continuous on \([0, \infty)\). Hence, by Lemma 5.2 the convergence in (5.1) is uniform in \( t \) on every finite interval \([0, t^*]\). To complete the proof, it suffices to check that for any \( \varepsilon > 0 \) and for large enough \( n \), there exists \( t^* < \infty \) such that for all \( t \geq t^* \)
\[
n^{-1} E^2_\gamma[\ell(t) - \ell(t)] \leq \varepsilon.
\] (5.2)

Using (4.7), similarly to (4.16) we can write
\[
E^2_\gamma[\ell(\infty) - \ell(t)] = \sum_{k=1}^{\infty} \sum_{x \in \Lambda \setminus \mathcal{X}(t)} |x| e^{-\alpha_n k \langle \tilde{x}, \delta(\tau(\tilde{x})) \rangle}
\]
\[
\leq \sum_{k=1}^{\infty} \sum_{x_1=1}^{\infty} \sum_{x_2 > tx_1} (x_1 + x_2) e^{-\alpha_n k \delta(x_1 + x_2)/2}.
\] (5.3)

Note that the number of integer pairs \((x_1, x_2)\) (with \( x_1 \geq 1, x_2 \geq 0 \)) satisfying the conditions \( x_1 + x_2 = y \) and \( x_2 > tx_1 \) does not exceed \( y/(t+1) \). Hence, again using estimate (4.16), we see that the right-hand side of (5.3) is bounded from above by
\[
\sum_{k=1}^{\infty} \sum_{y=1}^{\infty} \frac{y^2}{t+1} e^{-\alpha_n k \delta y/2} = \frac{1}{t+1} \sum_{k=1}^{\infty} \frac{O(1)}{O(1)} = \frac{O(1)}{O(1)},
\]
Recalling that \( \alpha_3^n = 1/(\rho_n n_1) \sim n^{-1} \), this implies estimate (5.2) for all \( t \) large enough.

**Theorem 5.3.** Uniformly in \( t \in [0, \infty) \) we have
\[
\lim_{n \to \infty} n^{-1} E^2_\gamma[\xi_1(t)] = u_\gamma(t), \quad \lim_{n \to \infty} n^{-1} E^2_\gamma[\xi_2(t)] = g_\gamma(u_\gamma(t)).
\] (5.4)

In particular, for \( t = \infty \) this yields
\[
\lim_{n \to \infty} n^{-1} E^2_\gamma(\xi_1) = 1, \quad \lim_{n \to \infty} n^{-1} E^2_\gamma(\xi_2) = c_\gamma.
\] (5.5)

**Proof.** Similarly to representation (4.17), one can show that
\[
n^{-1} E^2_\gamma[\xi_1(t)] = \rho_n \sum_{k,m=1}^{\infty} m \mu(m) \sum_{x_1=1}^{\infty} \sum_{1 \leq x_2 \leq tx_1} \alpha_n^3 x_1 e^{-k \mu_n \langle \tilde{x}, \delta(\tau(\tilde{x})) \rangle}.
\] (5.6)

Assuming that \( t_0 \leq t \leq t_1 \) and passing to the limit similarly as in the proof of Theorem 4.2
we obtain, using (4.12) and making the substitution $x_2 = sx_1$, that

\[
\lim_{n \to \infty} n^{-1} E^\gamma_z[\xi_1(t)] = \sum_{k,m=1}^{\infty} m \mu(m) \int_{0 \leq x_2 \leq tx_1} x_1 e^{-km(x, \delta(x))} \, dx_1 \, dx_2
\]

\[
= \sum_{k,m=1}^{\infty} m \mu(m) \frac{2}{(km)^3} \int_{t_0}^{t} \frac{ds}{(\delta_1(s) + s\delta_2(s))^3}
\]

\[
= 2 \sum_{k=1}^{\infty} \frac{\mu(m)}{m^2} \int_{t_0}^{t} \frac{ds}{\kappa^3 g'_\gamma(u(s))}
\]

\[
= 2 \zeta(3) \int_{0}^{u_\gamma(t)} \frac{d g'_\gamma(u)}{g''_\gamma(u)} = u_\gamma(t).
\]

Similarly,

\[
\lim_{n \to \infty} n^{-1} E^\gamma_z[\xi_2(t)] = \sum_{k,m=1}^{\infty} m \mu(m) \int_{0 \leq x_2 \leq tx_1} x_2 e^{-km(x, \delta(x))} \, dx_1 \, dx_2
\]

\[
= \sum_{k,m=1}^{\infty} m \mu(m) \frac{2}{(km)^3} \int_{t_0}^{t} \frac{ds}{(\delta_1(s) + s\delta_2(s))^3}
\]

\[
= 2 \zeta(3) \int_{0}^{u_\gamma(t)} \frac{d g'_\gamma(u)}{g''_\gamma(u)} = g_\gamma(u_\gamma(t)).
\]

Finally, the uniform convergence in (5.4) can be proved similarly as in Theorem 5.1.

For the future applications, we need to estimate the rate of convergence in (5.5) with sufficient accuracy. To this end, we require some more smoothness of the function $g_\gamma$.

**Assumption 5.1.** In addition to Assumptions 2.1 and 4.1 we now suppose that $g_\gamma \in C^3([0, 1])$.

**Theorem 5.4.** Under Assumption 5.1 $E^\gamma_z(\xi_j) - n_j = O(n_1^{2/3})$ as $n \to \infty$ ($j = 1, 2$).

**Proof.** Consider $\xi_1$ (the case $\xi_2$ is handled similarly). From (5.6) with $t = \infty$ we have

\[
E^\gamma_z(\xi_1) = \sum_{k,m=1}^{\infty} \frac{\mu(m)}{k^2 \alpha_n} F_1(km\alpha_n),
\]

where

\[
F_1(h) := \sum_{x_1=1}^{\infty} \sum_{x_2=0}^{\infty} f_1(hx_1, hx_2), \quad f_1(x_1, x_2) := x_1 e^{-\langle \bar{x}, \delta(\bar{x}) \rangle}.
\]

(5.8)

Repeating the calculations as in (5.7), we note that

\[
\int_{\mathbb{R}^2_+} f_1(hx_1, hx_2) \, dx_1 \, dx_2 = \frac{2}{\rho_n h^2 \kappa^3},
\]

\[
\zeta(3) \kappa^3 \int_{0}^{u_\gamma(t)} \frac{d g'_\gamma(u)}{g''_\gamma(u)} = u_\gamma(t).
\]
so that
\[
\sum_{k,m=1}^{\infty} \frac{\mu(m)}{\alpha_n k^3} \left( \int_{\mathbb{R}_+^2} f_1(hx_1, hx_2) \, dx_1 \, dx_2 \right) \bigg|_{h=\alpha_n km} = \frac{2}{\rho_n \alpha_n^2 k^3} \sum_{k,m=1}^{\infty} \frac{\mu(m)}{k^3 m^2} = \frac{1}{\rho_n \alpha_n^3} = n_1.
\]

Hence, we obtain the representation
\[
E_1^\alpha \left[ \xi_1 \right] - n_1 = \sum_{k,m=1}^{\infty} \frac{\mu(m)}{\alpha_n k} \Delta_1(\alpha_n km),
\]
where
\[
\Delta_1(h) := F_1(h) - \int_{\mathbb{R}_+^2} f_1(hx_1, hx_2) \, dx_1 \, dx_2.
\]

Using that \( \delta_1(t) \geq \delta_s > 0 \) and \( \rho_n \leq 1/2 \), we have
\[
F_1(h) \leq \sum_{x_1=1}^{\infty} \sum_{x_2=0}^{\infty} hx_1 e^{-h(x_1 + x_2)\delta_s/2} = \frac{he^{-h\delta_s/2}}{1 - e^{-h\delta_s/2}}.
\]

Hence, \( F_1(h) = O(h^{-2}) \) as \( h \to 0 \) and \( F_1(h) = O(h^{-\beta}) \) for any \( \beta > 0 \) as \( h \to +\infty \).

Therefore, the function \( F_1(h) \) is well defined for all \( h > 0 \) and its Mellin transform
\[
M_1(s) := \int_0^{\infty} h^{s-1} F_1(h) \, dh
\]
(see, e.g., [16, Ch. VI, § 9]) is a regular function for \( \Re{s} > 2 \). From a two-dimensional version of the Müntz formula (see [7, Lemma 5.1]), it follows that \( M_1(s) \) is meromorphic in the half-plane \( \Re{s} > 1 \) and has the single (simple) pole at point \( s = 2 \). Moreover, for all \( 1 < \Re{s} < 2 \)
\[
M_1(s) = \int_0^{\infty} h^{s-1} \Delta_1(h) \, dh.
\]

The inversion formula for the Mellin transform [16, Theorem 9a, pp. 246–247] yields
\[
\Delta_1(h) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} h^{-s} M_1(s) \, ds, \quad 1 < c < 2.
\]

In order to make use of formula (5.13), we need to find explicitly the analytic continuation of function (5.11) to the strip \( 1 < \Re{s} < 2 \). Let us use the Euler–Maclaurin summation formula
\[
\sum_{x=0}^{\infty} f(x) = \int_0^{\infty} f(x) \, dx + \frac{1}{2} f(0) + \int_0^{\infty} B_1(x) f'(x) \, dx,
\]
where \( B_1(x) := x - [x] - 1/2 \) and \([x]\) is the integer part of \( x \). In view of Assumption 5.1 and equations (2.11), (4.21), we can apply this formula to the sum over \( x_2 \) in (5.8). Using the substitution \( x_2 = tx_1/\rho_n \), we obtain
\[
F_1(h) = \sum_{x_1=1}^{\infty} hx_1 \int_0^{\infty} e^{-h(x_1, \delta(t))} \, dx_2 + \frac{1}{2} \sum_{x_1=1}^{\infty} hx_1 e^{-h(x_1\delta(0))} + O(1) \frac{e^{-\text{const}\cdot h}}{h}.
\]

Hence, the function \( F_1(h) \) has the single pole at point \( s = 1 \).

\[
F_1(h) = \frac{h}{\rho_n} \sum_{x_1=1}^{\infty} \sum_{x_2=0}^{\infty} e^{-hx_1\psi(t)} \, dt + O(1) \frac{e^{-\text{const}\cdot h}}{h},
\]

(5.15)
where (cf. (4.12))

$$\psi(t) := \delta_1(t) + t\delta_2(t) \equiv \kappa g''(u_\gamma(t))^{1/3}. \quad (5.16)$$

Keeping track of only the main term in (5.15) and writing dots for functions that are regular for \(\Re s > 1\), the Mellin transform of \(F_1(h)\) can be represented as follows

$$M_1(s) = \frac{1}{\rho_n} \int_0^\infty h^s \left( \sum_{x_1=1}^\infty \frac{a_1^2}{x_1} \int_0^\infty e^{-hx_1 \psi(t)} \, dt \right) \, dh + \cdots$$

$$= \frac{1}{\rho_n} \sum_{x_1=1}^\infty \frac{1}{x_1^s} \int_0^\infty \frac{1}{\psi(t)} \, \Gamma(s+1) \, dt + \cdots$$

$$= \frac{1}{\rho_n} \zeta(s-1) \Gamma(s+1) \Psi(s) + \cdots, \quad (5.17)$$

where

$$\Psi(s) := \int_0^\infty \frac{1}{\psi(t)^{s+1}} \, dt.$$ 

Recalling (2.9), function (5.16) may be rewritten in the form

$$\psi(t) = \kappa \infty(t)^{1/3} \sqrt{1 + t^2}, \quad t_0 \leq t \leq t_1,$$

and Assumption 2.1 implies that the function \(\Psi(s)\) is regular if \(\Re s > 0\). Furthermore, it is well known that the gamma function \(\Gamma(s)\) is analytic for \(\Re s > 0\) [13, §4.41, p. 148], whereas the zeta function \(\zeta(s)\) has a single pole at point \(s = 1\) [13, §4.43, p. 152]. It follows that the right-hand side of (5.17) is regular in the strip \(1 < \Re s < 2\) and hence provides the required analytic continuation of the function \(M_1(s)\) originally defined by (5.11).

Setting \(h = \alpha_n km\) and returning to formulas (5.10) and (5.13), we get for \(1 < c < 2\)

$$E_z^\gamma(\xi_1) - n_1 = \sum_{k,m=1}^{\infty} \frac{\mu(m)}{\alpha_n k} \frac{1}{2\pi i} \int_{c-\infty}^{c+\infty} \frac{M_1(s)}{(km\alpha_n)^s} \, ds$$

$$= \frac{1}{2\pi i} \int_{c-\infty}^{c+\infty} \frac{M_1(s)}{\alpha_n^{s+1} \zeta(s)} \, ds \quad (5.18)$$

Using that \(\zeta(s) \neq 0\) for \(\Re s \geq 1\), we can transform the contour of integration \(\Re s = c\) in (5.18) to the union of a small semi-circle \(s = 1 + re^{it}\) \((-\pi/2 \leq t \leq \pi/2)\) and two vertical lines, \(s = 1 \pm it\) \((t \geq r)\). Furthermore, studying resolution (5.17), one can show that \(M_1(1 \pm it) = O(|t|^{-2})\) as \(t \to \infty\). As a result, the right-hand side of (5.18) is bounded by \(O(\alpha_n^{-2})\). Thus, the proof of the theorem for \(\xi_1\) is complete.

6. Asymptotics of higher-order moments

Throughout this section, we suppose that Assumptions 2.1 and 4.1 hold.
6.1. Second-order moments

Let \( K_z := \text{Cov}(\xi, \xi) \) be the covariance matrix (with respect to the measure \( Q_z \)) of the random vector \( \xi = \sum_{x \in X} x \nu(x) \). Since the random variables \( \nu(x) \) are mutually independent, we see using (3.5) that the elements \( K_z(i, j) = \text{Cov}(\xi_i, \xi_j) \) \((i, j \in \{1, 2\})\) of \( K_z \) are given by

\[
K_z(i, j) = \sum_{x \in X} x_i x_j \text{Var}[\nu(x)] = \sum_{x \in X} x_i x_j \sum_{k=1}^{\infty} k^r z^k.
\]

**Theorem 6.1.** As \( n \to \infty \),

\[
K_z = 3\kappa^{-1} n_1^{1/3} (1 + o(1)) B,
\]

where the elements of the matrix \( B := (B_{ij}) \) are given by

\[
B_{11} = \int_0^1 \frac{du}{g_3^*(u)^{1/3}}, \quad B_{12} = B_{21} = \int_0^1 \frac{g_3^*(u)}{g_3^*(u)^{1/3}} \frac{du}{g_3^*(u)^{1/3}}, \quad B_{22} = \int_0^1 \frac{g_3^*(u)^2}{g_3^*(u)^{1/3}} \frac{du}{g_3^*(u)^{1/3}}.
\]

**Proof.** Let us consider \( K_z(1, 1) \) (the other elements of \( K_z \) are analyzed in a similar manner). Substituting (4.4) into (6.1), by the Möbius inversion formula (cf. (5.6)) we obtain

\[
K_z(1, 1) = \sum_{k=1}^{\infty} k x_1^2 e^{-\kappa n(\bar{x}, \delta(\bar{x}))} = \sum_{k,m=1}^{\infty} k m^2 \mu(m) \sum_{y_1=1}^{\infty} \sum_{y_2=0}^{\infty} y_1^2 e^{-k m n(\bar{y}, \delta(\bar{y}))}.
\]

Arguing as in the proof of Theorems 4.2 and 5.3, we obtain

\[
\lim_{n \to \infty} \alpha_n^4 \sum_{x_1=1}^{\infty} \sum_{x_2=0}^{\infty} x_1^2 e^{-k m n(\bar{x}, \delta(\bar{x}))} = \iint_{\mathbb{R}_+^2} x_1^2 e^{-k m n(\bar{x}, \delta(\bar{x}))} \, dx_1 \, dx_2
\]

\[
= \frac{6}{(km)^4} \int_0^{\delta_1} ds \int_0^{\delta_2} \frac{ds}{(\delta_1(s) + s \delta_2(s))^4}.
\]

Returning to (6.4) and using (4.11), (4.12), we get

\[
\lim_{n \to \infty} \alpha_n^4 K_z(1, 1) = \frac{6 \zeta(3)}{\zeta(2)} \int_0^{\delta_1} \frac{ds}{\kappa n^4 (g_3^*(u(s))^4)} = \frac{3}{\kappa} \int_0^1 \frac{du}{g_3^*(u)^{1/3}},
\]

and the first formula in (6.3) follows, since \( \alpha_n = (\rho_n n_1)^{-1/3} \) and \( \rho_n \to 1 \) as \( n \to \infty \). \( \square \)

**Lemma 6.2.** As \( n \to \infty \),

\[
\det K_z \sim \left( \frac{3}{\kappa} \right)^2 \left( \int_0^1 \frac{du}{g_3^*(u)^{1/3}} \int_0^1 \frac{g_3^*(u)^2}{g_3^*(u)^{1/3}} \frac{du}{g_3^*(u)^{1/3}} - \left( \int_0^1 \frac{g_3^*(u)}{g_3^*(u)^{1/3}} \frac{du}{g_3^*(u)^{1/3}} \right)^2 \right) n_1^{8/3}.
\]

**Proof.** The proof readily follows from Theorem 6.1. \( \square \)
From Theorem 6.1 and Lemma 6.2, it follows (e.g., using the Cauchy–Schwarz inequality) that the matrix $K_z$ is (asymptotically) positive definite; in particular, $\det K_z > 0$ and hence $K_z$ is invertible. Let $V_z = K_z^{-1/2}$ be the (unique) square root of $K_z^{-1}$, that is, a symmetric, positive definite matrix such that $V_z^2 = K_z^{-1}$. Recall that the matrix norm induced by the Euclidean vector norm $|\cdot|$ is defined by $\|A\| := \sup_{|x|=1} |xA|$. We need some general facts about this norm (see [7, §7.2, pp. 33–34] for simple proofs and bibliographic comments).

**Lemma 6.3.** If $A$ is a real matrix then $\|A^*A\| = \|A\|^2$.

**Lemma 6.4.** If $A = (a_{ij})$ is a real $d \times d$ matrix, then

$$\frac{1}{d} \sum_{i,j=1}^d a_{ij}^2 \leq \|A\|^2 \leq \sum_{i,j=1}^d a_{ij}^2. \tag{6.6}$$

**Lemma 6.5.** Let $A$ be a symmetric $2 \times 2$ matrix with $\det A \neq 0$. Then

$$\|A^{-1}\| = \frac{\|A\|}{\det A}. \tag{6.7}$$

We can now prove the following estimates for the norms of $K_z$ and $V_z$.

**Lemma 6.6.** As $n \to \infty$, we have

$$\|K_z\| \asymp n_1^{4/3}, \quad \|V_z\| \asymp n_1^{-2/3}. \tag{6.8}$$

**Proof.** Using Theorem 6.1 and the upper bound in Lemma 6.4, we get

$$\|K_z\|^2 \leq K_z(1,1)^2 + 2K_z(1,2) + K_z(2,2)^2 = O(n_1^{8/3}). \tag{6.9}$$

On the other hand, by Theorem 6.1 and the lower bound in Lemma 6.4

$$\|K_n\|^2 \geq \frac{1}{2} \left( K_z(1,1)^2 + K_z(2,2)^2 \right)$$

$$\geq K_z(1,1) K_z(2,2) \sim \left( \frac{3}{\kappa} \right)^2 n_1^{8/3} \int_0^1 \frac{du}{g_u^\gamma(u)^{1/3}} \int_0^1 g_u^\gamma(u)^2 \frac{du}{g_u^\gamma(u)^{1/3}}. \tag{6.10}$$

Combining (6.9) and (6.10) we obtain the first estimate in (6.8).

Further, Lemma 6.3 implies that $\|V_z\|^2 = \|K_z^{-1}\|$. In turn, Lemma 6.5 yields $\|K_z^{-1}\| = \|K_z\|/\det K_z$, and it remains to use Lemmas 6.2 and 6.6 to obtain the second part of (6.8).

### 6.2. Asymptotics of the moment sums

Denote $\nu_0(x) := \nu(x) - E_\gamma^z[\nu(x)]$ $(x \in X)$, and for $q \in \mathbb{N}$ set

$$m_q(x) := E_\gamma^z[\nu(x)^q], \quad \mu_q(x) := E_\gamma^z[\nu_0(x)^q] \tag{6.11}$$

(for notational simplicity, we suppress the dependence on $\gamma$ and $z$).

The following two-sided estimate of $\mu_q(x)$ can be easily proved using Newton’s binomial formula and Lyapunov’s inequality (cf. [7, Lemmas 6.2 and 6.6]).
Lemma 6.7. For each $q \in \mathbb{N}$ and all $x \in X$,
\[ \mu_2(x)^{q/2} \leq \mu_q(x) \leq 2^q m_q(x). \] (6.12)

Next, we need a general upper bound for the moments of geometric random variables proved in [7, Lemma 6.3].

Lemma 6.8. For each $q \in \mathbb{N}$, there exists a constant $C_q > 0$ such that, for all $x \in X$,
\[ m_q(x) \leq \frac{C_q x^q}{(1 - x)^q}. \] (6.13)

Using estimate (6.13) and repeating the calculations in the proof of Lemma 6.4 in [7], one obtains the following asymptotic bound.

Lemma 6.9. For each $q \in \mathbb{N}$,
\[ \sum_{x \in X} |x|^q m_q(x) = O(1) n_1^{(q+2)/3}, \quad n \to \infty. \]

Lemma 6.9, together with bounds (6.12) and Theorem 6.1, implies the following asymptotic estimate (cf. [7, Lemma 6.6]).

Lemma 6.10. For any integer $q \geq 2$,
\[ \sum_{x \in X} |x|^q \mu_q(x) \asymp n_1^{(q+2)/3}, \quad n \to \infty. \]

Using Lemma 6.10 and the lower bound $\delta_j(t) \geq \delta^*$ (see (4.6)), the next asymptotic bound is obtained by a straightforward adaptation of the proof of a similar result in [7, Lemma 6.7].

Lemma 6.11. For each $q \in \mathbb{N}$,
\[ E_z[\ell_I^q - E_z^\gamma[\ell_I^q]] = O(n_1^{2q/3}), \quad n \to \infty. \]

Finally, let us consider the Lyapunov coefficient
\[ L_z := ||V_z||^3 \sum_{x \in X} |x|^3 \mu_3(x), \] (6.14)

The next asymptotic estimate is an immediate consequence of Lemmas 6.6 and 6.10.

Lemma 6.12. As $n \to \infty$, one has $L_z \asymp n_1^{-1/3}$.

7. Local limit theorem

The role of a local limit theorem in our approach is to yield the asymptotics of the probability $Q_z^\gamma\{ \xi = n \} \equiv Q_z^\gamma(\Pi_n)$ appearing in the representation of the measure $P_n^\gamma$ as a conditional distribution, $P_n^\gamma(\cdot) = Q_z^\gamma(\cdot | \Pi_n) = Q_z^\gamma(\cdot) / Q_z^\gamma(\Pi_n)$.

As before, we denote $a_z := E_z^\gamma(\xi)$ and $K_z := \text{Cov}(\xi, \xi) = E_z^\gamma$. Let $f_{0,1}(\cdot)$ be the density of a standard two-dimensional normal distribution $\mathcal{N}(0, I)$ (i.e., with zero mean and identity covariance matrix),
\[ f_{0,1}(x) = \frac{1}{2\pi} e^{-|x|^2/2}, \quad x \in \mathbb{R}^2. \]

Then the density of the normal distribution $\mathcal{N}(a_z, K_z)$ is given by
\[ f_{a_z,K_z}(x) = (\det K_z)^{-1/2} f_{0,1}(x - a_z) V_z^t, \quad x \in \mathbb{R}^2. \] (7.1)
Theorem 7.1 (Local limit theorem). Under Assumptions 2.1 and 4.1 uniformly in \( m \in \mathbb{Z}^2_+ \)

\[
Q^*_Z \{ \xi = m \} = f_{a_1, K_1}(m) + O(n^{-5/3}). \tag{7.2}
\]

Let us make some preparations for the proof. Recall that the random variables \( \{ \nu(x) \}_{x \in \mathcal{X}} \) are mutually independent and have geometric distribution with parameter \( z^x \), respectively. In particular, the characteristic function \( \varphi_\nu(t) := E^*_Z(e^{it\nu}) \) of \( \nu(x) \) is given by

\[
\varphi_\nu(t; x) = \frac{1 - z^x}{1 - z^x e^{it}}; \tag{7.3}
\]

hence, the characteristic function \( \varphi_\xi(\lambda) := E^*_Z(e^{i\lambda\xi}) \) of the vector \( \xi = \sum_{x \in \mathcal{X}} x \nu(x) \) reads

\[
\varphi_\xi(\lambda) = \prod_{x \in \mathcal{X}} \varphi_\nu(\langle x, \lambda \rangle; x) = \prod_{x \in \mathcal{X}} \frac{1 - z^x}{1 - z^x e^{i\langle x, \lambda \rangle}}. \tag{7.4}
\]

Let us start with a general absolute estimate for the characteristic function of a centered random variable (for a proof, see [7] Lemma 7.10).

Lemma 7.2. Let \( \varphi_{\nu_0}(t; x) := E^*_Z(e^{i\mu(x)}) \) be the characteristic function of the random variable \( \nu_0(x) := \nu(x) - E^*_Z[\nu(x)] \). Then

\[
|\varphi_{\nu_0}(t; x)| \leq \exp\left\{-\frac{1}{2} \mu_2(x) t^2 + \frac{1}{3} \mu_3(x) |t|^3\right\}, \quad t \in \mathbb{R}. \tag{7.5}
\]

The next lemma provides two estimates (proved in [7] Lemmas 7.11 and 7.12) for the characteristic function \( \varphi_{\xi_0}(\lambda) := E^*_Z(e^{i\lambda, \xi_0}) \) of the centered vector \( \xi_0 := \xi - a_z = \sum_{x \in \mathcal{X}} x \nu_0(x) \). Recall that the Lyapunov coefficient \( L_z \) is defined in (6.14), and \( V_z := K_z^{-1/2} \).

Lemma 7.3. (a) For all \( \lambda \in \mathbb{R}^2 \),

\[
|\varphi_{\xi_0}(\lambda V_z)| \leq \exp\left\{-\frac{1}{2} \lambda^2 + \frac{1}{3} L_z |\lambda|^3\right\}. \tag{7.6}
\]

(b) If \( |\lambda| \leq L_z^{-1} \) then

\[
|\varphi_{\xi_0}(\lambda V_z) - e^{-|\lambda|^2/2}| \leq 16 L_z |\lambda|^2 e^{-|\lambda|^2/6}. \tag{7.7}
\]

The next global bound is obtained by repeating the proof of Lemma 7.14 in [7].

Lemma 7.4. For all \( \lambda \in \mathbb{R}^2 \),

\[
|\varphi_{\xi_0}(\lambda)| \leq e^{-J_n(\lambda)}, \tag{7.8}
\]

where

\[
J_n(\lambda) := \frac{1}{4} \sum_{x \in \mathcal{X}} e^{-\alpha_n(\delta, x)} (1 - \cos \langle \lambda, x \rangle) \geq 0. \tag{7.9}
\]

We can now proceed to the proof of Theorem 7.1

**Proof of Theorem 7.1.** By the Fourier inversion formula, we can write

\[
Q^*_Z \{ \xi = m \} = \frac{1}{4\pi^2} \int_{T^2} e^{-i\langle \lambda, m-a_z \rangle} \varphi_{\xi_0}(\lambda) \, d\lambda, \quad m \in \mathbb{Z}^2_+, \tag{7.10}
\]
where $T^2 := \{ \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 : |\lambda_1| \leq \pi, |\lambda_2| \leq \pi \}$. On the other hand, the characteristic function corresponding to the normal probability density $f_{\alpha, K_z}(x)$ (see (7.1)) is given by

$$\varphi_{\alpha, K_z}(\lambda) = e^{i(\lambda, x) - |\lambda V_z^{-1}|^2/2}, \quad \lambda \in \mathbb{R}^2,$$

so by the Fourier inversion formula

$$f_{\alpha, K_z}(m) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-i(\lambda, m-a)} - |\lambda V_z^{-1}|^2/2 \, d\lambda, \quad m \in \mathbb{Z}_+^2. \quad (7.11)$$

Note that if $|\lambda V_z^{-1}| \leq L_z^{-1}$ then, according to Lemmas 6.6 and 6.12

$$|\lambda| \leq |\lambda V_z^{-1}| \cdot ||V_z|| \leq L_z^{-1} ||V_z|| = O(n_1^{-1/3}) = o(1),$$

which of course implies that $\lambda \in T^2$. Using this observation and subtracting (7.11) from (7.10), we get, uniformly in $m \in \mathbb{Z}_+^2$,

$$|Q^2_z\{ \xi = m \} - f_{\alpha, K_z}(m)| \leq I_1 + I_2 + I_3, \quad (7.12)$$

where

$$I_1 := \frac{1}{4\pi^2} \int_{\{ \lambda : |\lambda V_z^{-1}| \leq L_z^{-1} \}} \left| \varphi_{\alpha_0}(\lambda) - e^{-|\lambda V_z^{-1}|^2/2} \right| \, d\lambda,$$

$$I_2 := \frac{1}{4\pi^2} \int_{\{ \lambda : |\lambda V_z^{-1}| > L_z^{-1} \}} e^{-|\lambda V_z^{-1}|^2/2} \, d\lambda,$$

$$I_3 := \frac{1}{4\pi^2} \int_{T^2 \cap \{ \lambda : |\lambda V_z^{-1}| > L_z^{-1} \}} \left| \varphi_{\alpha_0}(\lambda) \right| \, d\lambda.$$

By the substitution $\lambda = yV_z$, the integral $I_1$ is reduced to

$$I_1 = \frac{|\det V_z|}{4\pi^2} \int_{|y| \leq L_z^{-1}} \left| \varphi_{\alpha_0}(yV_z) - e^{-|y|^2/2} \right| \, dy = O(1) (\det K_z)^{-1/2} L_z \int_{\mathbb{R}^2} |y|^3 e^{-|y|^2/6} \, dy = O(n_1^{-5/3}), \quad (7.13)$$

on account of Lemmas 6.2, 6.12 and 7.3(b). Similarly, again putting $\lambda = yV_z$ and passing to the polar coordinates, we get, due to Lemmas 6.2 and 6.12

$$I_2 = \frac{|\det V_z|}{2\pi} \int_{L_z^{-1}} \int_{L_z^{-1}} r e^{-r^2/2} \, dr = O(n_1^{-4/3}) e^{-L_z^{-2}/2} = o(n_1^{-5/3}). \quad (7.14)$$

Finally, let us turn to $I_3$. Using Lemma 7.4, we obtain

$$I_3 = O(1) \int_{T^2 \cap \{ |\lambda V_z^{-1}| > L_z^{-1} \}} e^{-J_n(\lambda)} \, d\lambda, \quad (7.15)$$

where $J_n(\lambda)$ is given by (7.9). The condition $|\lambda V_z^{-1}| > L_z^{-1}$ implies that $|\lambda| > \sqrt{2} \eta \alpha_n$ and hence $\max\{ |\lambda_1|, |\lambda_2| \} > \eta \alpha_n$, where $\eta > 0$ is suitable (small enough) constant. Indeed, assuming the contrary, from (4.5) and Lemmas 6.6 and 6.12 it would follow

$$1 < L_z |\lambda V_z^{-1}| \leq L_z \eta \alpha_n \| K_z \|^{1/2} = O(\eta) \to 0 \quad \text{as} \quad \eta \downarrow 0,$$

in a manner consistent with the notation $O(\eta)$.
which is a contradiction. Hence, estimate (7.15) is reduced to

\[ I_3 = O(1) \left( \int_{|\lambda_1| > \eta \alpha_n} + \int_{|\lambda_2| > \eta \alpha_n} \right) e^{-J_n(\lambda)} d\lambda. \] (7.16)

Note that, by Assumption 2.1 and formulas (4.21), the functions \( \delta_1(t), \delta_2(t) \) are bounded above, \( \sup_t \delta_j(t) \leq \delta^* < \infty \). Hence, (7.9) implies

\[ J_n(\lambda) \geq \sum_{x \in X} e^{-\alpha_n \delta^*(x_1 + x_2)} \left( 1 - \cos(\lambda, x) \right). \] (7.17)

To estimate the first integral in (7.16), by keeping in summation (7.17) only pairs of the form \( x = (x_1, 1), \ x_1 \in \mathbb{Z}_+ \), we obtain

\[ J_n(\lambda) e^{\alpha_n \delta^*} \geq \sum_{x_1=0}^{\infty} e^{-\alpha_n \delta^* x_1} \left( 1 - \Re e^{i(\lambda_1 x_1 + \lambda_2)} \right) = \frac{1}{1 - e^{-\alpha_n}} - \Re \left( \frac{e^{i\lambda_2}}{1 - e^{-\alpha_n + i\lambda_1}} \right) \geq \frac{1}{1 - e^{-\alpha_n}} - \frac{1}{|1 - e^{-\alpha_n + i\lambda_1}|} \] (7.18)

because \( \Re u \leq |u| \) for any \( u \in \mathbb{C} \). Since \( \eta \alpha_n \leq |\lambda_1| \leq \pi \), we have

\[ |1 - e^{-\alpha_n + i\lambda_1}| \geq |1 - e^{-\alpha_n + i\eta \alpha_n}| \sim \alpha_n (1 + \eta^2)^{1/2} \quad (\alpha_n \to 0). \]

Substituting this estimate into (7.18), we conclude that \( J_n(\lambda) \) is asymptotically bounded from below by \( C(\eta) \alpha_n^{-1} \approx n_1^{1/3} \) (with some constant \( C(\eta) > 0 \)), uniformly in \( \lambda \) such that \( \eta \alpha_n \leq |\lambda_1| \leq \pi \). Thus, the first integral in (7.16) is bounded by \( O(1) \exp\left( -\text{const} \cdot n_1^{1/3} \right) = o(n_1^{-5/3}) \).

Similarly, the second integral in (7.16) is estimated by reducing the summation in (7.9) to that over \( x = (1, x_2) \) only. As a result, \( I_3 = o(n_1^{-5/3}) \). Substituting this estimate, together with (7.13) and (7.14), into (7.12) we get (7.2), and so the theorem is proved. \( \square \)

**Corollary 7.5.** In addition to the conditions of Theorem 7.1 suppose that Assumption 5.1 holds. Then, as \( n \to \infty \),

\[ Q_n^\gamma \{ \xi = n \} \sim n_1^{-4/3}. \] (7.19)

**Proof.** By Theorem 5.4 \( a_z = E_z(\xi) = n + O(n_1^{2/3}) \). Together with Lemma 6.4 this implies \( |(n - a_z) V_z| \leq |n - a_z| \cdot ||V_z|| = O(1) \). Hence, by Lemma 6.2 we get

\[ f_{a_z, K_z}(n) = \frac{1}{2\pi} (\det K_z)^{-1/2} e^{-|n-a_z|V_z|^2/2} \approx n_1^{-4/3}, \]

and (7.19) now readily follows from (7.2). \( \square \)

**8. Limit shape**

Throughout this section we work under Assumptions 2.1 4.1 and 5.1. Let us first establish that a given curve \( \gamma \in \mathcal{G} \) is indeed the limit shape of polygonal lines \( \Gamma \in \Pi \) with respect to the measure \( Q_n^\gamma \) (under the scaling \( \Gamma \mapsto n_1^{-1} \Gamma \)).
Theorem 8.1. For any $\varepsilon > 0$,
\[
\lim_{n \to \infty} Q_z^n \left\{ \Gamma \in \Pi : d_L(n^{-1}_1 \Gamma, \gamma) \leq \varepsilon \right\} = 1.
\]

Proof. In view of Theorem 5.1 we only need to check that for each $\varepsilon > 0$
\[
\lim_{n \to \infty} Q_z^n \left\{ \frac{1}{n_1} \sup_{0 \leq t \leq \infty} |\ell(t) - E_z^\gamma[\ell(t)]| > \varepsilon \right\} = 0.
\]

Note that the random process
\[
\ell(t) := \ell(t) - E_z^\gamma[\ell(t)] \quad (0 \leq t \leq \infty)
\]
has independent increments and zero mean, hence it is a martingale with respect to the filtration
\[
\mathcal{F}_t := \sigma(\nu(x), x \in \mathcal{X}(t), t \in [0, \infty]).
\]
From the definition of $\ell(t)$ (see (4.2)), it is also clear that $\ell(t)$ is càdlàg (i.e., its paths are everywhere right-continuous and have left limits).

Therefore, Kolmogorov–Doob’s submartingale inequality (see, e.g., [17, Corollary 2.1]) gives
\[
Q_z^n \left\{ \sup_{0 \leq t \leq \infty} |\ell(t)| > n_1\varepsilon \right\} \leq \frac{1}{(n_1\varepsilon)^2} \sup_{0 \leq t \leq \infty} \text{Var}[\ell(t)] \leq \frac{1}{n_1^2\varepsilon^2} \text{Var}(\ell).
\]
Furthermore, using decomposition (4.2) and Theorem 6.1, we have
\[
\text{Var}(\ell) = \sum_{x \in \mathcal{X}} |x| \text{Var}[\nu(x)] \leq \sum_{x \in \mathcal{X}} (x_1 + x_2) \text{Var}[\nu(x)]
\]
\[
= \text{Var}(\xi_1) + \text{Var}(\xi_2) = O(n_1^{4/3}).
\]
Finally, substituting (8.4) into (8.3), we see that the probability on the left-hand side is bounded
by $O(n_1^{-2/3}) \to 0$, which proves (8.1).

Let us now prove a limit shape result under the measure $P_n^n$ (cf. Theorem 1.1).

Theorem 8.2. For any $\varepsilon > 0$
\[
\lim_{n \to \infty} P_n^n \left\{ \Gamma \in \Pi_n : d_L(n^{-1}_1 \Gamma, \gamma) \leq \varepsilon \right\} = 1.
\]

Proof. Similarly to the proof of Theorem 8.1 it suffices to show that for each $\varepsilon > 0$
\[
\lim_{n \to \infty} P_n^n \left\{ \sup_{0 \leq t \leq \infty} |n_1^{-1}\ell(t)| > \varepsilon \right\} = 0,
\]
where the random process $\ell(t)$ is defined in (8.2). Recalling formula (3.6), we obtain
\[
P_n^n \left\{ \sup_{0 \leq t \leq \infty} |\ell(t)| > \varepsilon n_1 \right\} \leq \frac{Q_z^n \left\{ \sup_{0 \leq t \leq \infty} |\ell(t)| > \varepsilon n_1 \right\}}{Q_z^n \{ \xi = n \}}.
\]
To estimate the numerator in (8.6), similarly to the proof of Theorem 8.1 we use Kolmogorov–
Doob’s submartingale inequality, but now with the sixth order central moment. Combining this with Lemma 6.11 (with $q = 3$), we obtain
\[
Q_z^n \left\{ \sup_{0 \leq t \leq \infty} |\ell(t)| > n_1\varepsilon \right\} \leq \frac{1}{n_1^6} E_z^\gamma[|\ell(t)| - E_z^\gamma(\ell)|^6] = O(n_1^{-2}).
\]
On the other hand, by Corollary 7.5 the denominator in (8.6) decays no faster than at order
$n^{-4/3}$. Together with (8.7), this implies that the right-hand side of (8.6) admits an asymptotic
bound $O(n_1^{-2/3}) \to 0$. Hence, Theorem 8.2 is proved.
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