LINEARITY ON ORDINARY SIEGEL MODULI SCHEMES AND JOINT UNLIKELY ALMOST INTERSECTIONS

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Abstract. The goal of this paper is to study a $p$-adic analog of the joint of the conjectures of André–Oort and André–Pink. More precisely, on a product of ordinary Siegel formal moduli schemes, we study the distribution of points whose components are either CM points or points in Hecke orbits. We use linearity of formal subschemes of the product as the $p$-adic analog of geodesicness over complex numbers. Moreover, we relax the usual incidence relations by using $p$-adic distance. We also study a $p$-adic formal scheme theoretic analog of the Ax–Lindemann theorem.

1. Introduction

1.1. Joint unlikely intersections. In the theory of unlikely intersections, a prototype conjecture is due to Manin and Mumford, which states that an irreducible closed subvariety of an abelian variety containing infinitely many torsion points is a translation of an abelian subvariety. It was proved by Raynaud [33] using a $p$-adic method. The Mordell–Lang conjecture [20] is a vast generalization of the Manin–Mumford conjecture, which replaces torsion points by the division points of a lattice. The Mordelic part of this conjecture [19], where one takes the same statements but only with lattice points, was proved by Faltings in his fundamental works [11, 12] on Diophantine approximation, i.e., comparison of local and global heights. The whole conjecture was later settled by McQuillan [24]. Poonen [30] and Zhang [47] generalized the Mordell–Lang conjecture, replacing division points by points in their small neighborhoods, with the distance between any two points defined as the Néron-Tate height of their difference under the group structure.$^{[1]}$

$^{[1]}$This generalization was motivated by the Bogomolov conjecture (a Theorem of Ullmo [42] and Zhang [46]), which is the generalization of the Manin–Mumford conjecture in this style.
Now we turn to Shimura varieties and we have two conjectures on unlikely intersections. The André–Oort conjecture [1, 35] is the analog of the Manin–Mumford conjecture for Shimura varieties. It asserts that an irreducible closed subvariety of a Shimura variety with a Zariski dense subset of CM points is an irreducible component of a Hecke translation of a Shimura subvariety (in particular, totally geodesic). It is now a theorem of Pila, Shankar and Tsimerman [28]. The André–Pink conjecture [1, 29] is partially modeled on the Mordell–Lang conjecture. It asserts that an irreducible closed subvariety of a Shimura variety that has Zariski dense intersection with a Hecke orbit is totally geodesic. The conjecture has been proved by Richard and Yafaev [34] for certain Shimura varieties, including the ones of abelian type. In these proofs, a fundamental result, the Ax–Lindemann theorem, plays an essential role.

Since the André–Pink conjecture and the André–Oort conjecture do not imply each other, we propose the following joint conjecture, as a plain generalization of both conjectures.

**Conjecture 1.1.1.** Let $S_1, S_2$ be two Shimura varieties over $\mathbb{C}$ and $V \subset S_1 \times S_2$ an irreducible closed subvariety. Let $O \subset S_1(\mathbb{C})$ be a Hecke orbit, and $CM \subset S_2(\mathbb{C})$ the set of CM points. Assume that $V(\mathbb{C}) \cap (O \times CM)$ is Zariski dense in $V$. Then $V$ is totally geodesic.

We also want an analog of the result of Poonen and Zhang for Shimura varieties. However, due to the lack of a global group structure on Shimura varieties, such an analog does not exist [3]. Instead, we consider the $p$-adic analog of Conjecture 1.1.1 for ordinary Siegel formal moduli schemes where the analog of geodesicness is linearity, as suggested by Moonen [25, 26]. We in fact conjecture a generalization of this $p$-adic analog using $p$-adic distance (generalizing $p$-adic local height). We prove cases of this conjecture. We also conjecture a $p$-adic formal scheme theoretic analog of the Ax–Lindemann theorem (see [43, Theorem 1.1-1.4]), and prove cases of this conjecture.

Below, we describe the conjectures and theorems of this paper in more detail.

1.2. **Linearity.** Let $k$ be an algebraic closure of $\mathbb{F}_p$, and $F$ a complete discrete valuation field of characteristic 0 with residue field $k$, equipped with the natural $p$-adic valuation. The valuation extends naturally to the algebraic closure $\overline{F}$. Let $F^o \subset F$ and $\overline{F^o} \subset \overline{F}$ be the valuation rings. By a formal torus over $F^o$, we mean a self-product of the completion of the multiplicative group over $F^o$ along the unit. By a translated formal subtorus of a formal torus over $F^o$, we always mean the translation of a nontrivial formal subtorus over $F^o$ by an $\overline{F^o}$-point, regarding this $\overline{F^o}$-point as a closed formal $F^o$-subscheme of the torus.

Let $\mathcal{G}$ be the Siegel formal moduli scheme over $\mathbb{Z}_p$ of abelian schemes of relative dimension $g$ with ordinary reduction of a certain level outside $p$. Abusing notation, we simply denote $\mathcal{G}_{F^o}$ by $\mathcal{G}$ in this introduction. By a formal subscheme of $\mathcal{G}$, we always mean a locally closed formal subscheme of $\mathcal{G}$. (A formal subscheme may not extend to a closed formal subscheme.) For a formal subscheme $\mathfrak{V}$ of $\mathcal{G}$ and $x \in \mathfrak{V}(k)$, let $\mathfrak{V}_x$ be the formal completion of $\mathfrak{V}$ at $x$. Since $\mathcal{G}_x$ is naturally a formal torus over $\overline{F^o}$ by the Serre–Tate theory (see 2.3), we shall call it a formal

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[2]Conjecture 1.1.1 does not formally follow from the André–Oort conjecture and André–Pink conjecture. However, as pointed out to us by Gao, combining the aforementioned proofs of both conjectures via the method of Pila and Zannier [31], one may obtain a proof of Conjecture 1.1.1, for Shimura varieties of abelian type.

[3]The Bogomolov conjecture (equivalently) generalizes the Manin–Mumford conjecture replacing torsion points by points of small heights. However, CM points on Shimura varieties not necessary have small heights. Thus an analog of the Bogomolov conjecture in this form does not exist either.
residue torus to emphasize the toric structure. We call \( S \) weakly linear at \( x \) if \( S_x \) is a union of translated formal subtori of \( S_x \). We call \( S \) weakly linear if so it is at every \( k \)-point. See 4.2 for more discussion about this notion.

Regarding the formal residue tori as uniformizing spaces of \( S \), we propose a \( p \)-adic formal scheme theoretic analog of the Ax–Lindemann theorem (see [43, Theorem 1.1-1.4]).

**Conjecture 1.2.1.** Assume that \( S \) is a closed formal subscheme of an open formal subscheme \( \mathcal{U} \) of \( S \), \( x \in S(k) \) and \( T \) a translated formal subtorus of \( S_x \). If \( S \) is the schematic image of \( T \rightarrow \mathcal{U} \), i.e. the minimal closed formal subscheme of \( \mathcal{U} \) through which this morphism factors, then \( S \) is weakly linear.

The first main results of this paper are

**Theorem 1.2.2.** Conjecture 1.2.1 holds if \( S_k \) is unibranch without embedded points and \( T \) contains a torsion point.

**Theorem 1.2.3.** Let \( S \) be a connected formal subscheme of \( S \) flat over \( F^0 \) such that \( S_k \) is unibranch. Let \( x \in S(k) \).

1. If \( S_x \) contains a translated formal subtorus of \( S_x \) of the same dimension with \( S_x \), then for every \( z \in S(k) \), \( S_z \) contains a translated formal subtorus of \( S_z \) of the same dimension with \( S_z \).

2. Assume that \( S_k \) has no embedded points. If \( S \) is weakly linear at \( x \), then it is weakly linear.

The analog of Theorem 1.2.3 over \( k \) is a result of Chai [7, (5.3)]. We give a new proof (Proposition 3.3.3). Chai [7, (5.3.1)] also conjectured that the unibranch assumption in Proposition 3.3.3 could be removed. Assuming this conjecture, we can remove the unibranch assumption in both Theorem 1.2.2 and Theorem 1.2.3.

1.3. **Joint unlikely almost intersections.** In view of the discussion below Conjecture 1.1.1, we propose the following analog of the result of Poonen [30] and Zhang [47].

**Conjecture 1.3.1.** Let \( CM \subset S(F^0) \) be the set of CM points in \( S(F^0) \). Let \( O \) be a Hecke orbit in \( S(F^0) \). Let \( S \) be a formal subscheme of \( S^2 \). Let \( Z \subset S_k \) be a closed subscheme contained in the Zariski closure (with reduced induced structure) of the reduction of \( S \cap (O \times CM) \) for all \( \epsilon > 0 \). Then \( S \) contains a union of weakly linear closed formal subschemes that contains \( Z \).

Here \( S \subset S^2(F^0) \) is the subset of points of \( p \)-adic distance \( \leq \epsilon \) to \( S \), see 5.3. Note that \( Z \) could be larger than the Zariski closure of the reduction of \( S(F^0) \cap (O \times CM) \).

Let us justify our formulation of Conjecture 1.3.1. Define \( (O \times CM)_\epsilon \subset S^2(F^0) \) to be the subset of points of \( p \)-adic distance \( \leq \epsilon \) to some point in \( O \times CM \). Then \( S(F^0) \cap (O \times CM)_\epsilon \subset S \cap (O \times CM) \). Conjecture 1.3.1 would be a more strict analog of Poonen [30] and Zhang [47] if we replace “the reduction of \( S \cap (O \times CM) \)” by “\( S(F^0) \cap (O \times CM)_\epsilon \)”, and consider its schematic closure. However, even the schematic density of \( S(F^0) \cap (O \times CM) \) in \( S \) may not imply the weak linearity of \( S \). Indeed, the naive generalization of the Mordell–Lang conjecture for formal tori is wrong, see [40, 2.3].

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[4] As pointed out by Liu, it could be complicated to define nonprime-to-\( p \) Hecke action using moduli schemes with levels at \( p \). A convenient definition is by using the usual Hecke action over \( F \) which preserves ordinarity. However, our Hecke–Frobenius orbits below are always defined using moduli schemes.
Assuming Conjecture 1.2.1, to prove Conjecture 1.3.1, it is enough to show that for some $x \in Z(k)$, $\mathcal{Y}_x$ contains a union of translated formal subtori of $\mathcal{S}^{2d}$ that contains $Z_x$ as a closed formal subscheme. Another main result of this paper (Theorem 1.3.2) is toward this weaker form of Conjecture 1.3.1. To state it, we need some more notions. By the “Frobenius” on $\mathcal{S}$, we mean the “canonical lifting” of the (relative) Frobenius endomorphism on $\mathfrak{S}_k$, see 4.1. The Frobenius action is a special $p$-primary Hecke action. By a Hecke–Frobenius orbit on $\mathcal{S}$, we mean an orbit under the prime-to-$p$ Hecke action and (forward and backward) Frobenius action, see 5.1. A Hecke–Frobenius orbit is contained in a unique Hecke orbit. If $g = 1$, then a Hecke–Frobenius orbit is the same as a Hecke orbit.

**Theorem 1.3.2.** Let $d, d'$ be non-negative integers and $\mathcal{Y}$ a closed formal subscheme of $\mathcal{S}^{d+d'}$. For $i = 1, \ldots, d$, let $O_i$ be a Hecke–Frobenius orbit in $\mathcal{S}(\mathcal{F}^0)$.

1. Let $e$ be a positive integer. Let $Z \subset \mathcal{Y}_k$ be a closed subscheme contained in the Zariski closure of the reduction of $\mathcal{Y}(\mathbb{F}^0) \cap \bigcup_{E: F \leq e} \mathcal{S}(E^0)$ for all $\epsilon > 0$. Then for every $x \in Z(k)$, $\mathcal{Y}_x$ contains a union of translated formal subtori of $\mathcal{S}^{n+d'}$ that contains $Z_x$ as a closed formal subscheme.

2. Assume that the reduction of $\mathcal{Y}_k \cap \left( \prod_{i=1}^d O_i \times CM^d \right)$ is Zariski dense in (the underlying topological space of) an irreducible component $\mathcal{Y}_k$ for all $\epsilon > 0$. There is a nonempty open subscheme of $\mathcal{Y}_k$ such that for every $x$ of its $k$-points, $\mathcal{Y}_x$ contains a translated formal subtorus of $\mathcal{S}^{d+d'}$.

Besides the result of Poonen [30] and Zhang [47], another motivation for Theorem 1.3.2 is our previous work [32], where we proved that CM points outside a formal subscheme can not be $p$-adically too close to it. Theorem 1.3.2 is an attempt to include Hecke orbits.

1.4. **Structure and strategy.** In Section 2, we introduce two Igusa (formal) schemes with natural projections $\mathcal{I} \to \mathcal{I}_0 \to \mathcal{S}$. There is a toric action on $\mathcal{I}_0$.

In Section 3, we use the toric action on $\mathcal{I}_0$ to study linearity. We prove Theorem 1.2.3. We also prepare some technical lemmas for later use.

In Section 4, we introduce the Frobenii on these formal schemes. Then, we further discuss linearity. We end this section by proving Theorem 1.2.2 using a result of de Jong [9] that relates Frobenius-invariance and linearity for formal tori.

In Section 5, we define more notations used in Theorem 1.3.2 and its proof. In particular, we introduce the adic generic fiber of $\mathcal{I}$, which is a perfectoid space.

Theorem 1.3.2 (1) is proved at the end of Section 6. The main strategy for the proof is to use the toric action on $\mathcal{I}_0$ to reduce Theorem 1.3.2 (1) to an analog for $\mathcal{I}_0$ and canonical liftings, which says that for a set of canonical liftings approaching a formal subscheme $\mathcal{W} \subset \mathcal{I}_0$, density of the set of their reductions implies local Frobenius-invariance of $\mathcal{W}$. (Here and below, canonical liftings are units of the formal residue tori of $\mathcal{I}_0$ (and $\mathcal{S}$), and are CM points.) In 6.1, we apply

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[5] Letting $d' = 1$, it is not hard to see that we get an equivalent statement using embeddings of Siegel moduli schemes. The current statement is more convenient for our purpose. However, the case $d' = 1$ is weaker than the general case in which we have Frobenii on all components, not just the simultaneous Frobenius.
some perfectoid technique in our previous work [32] (originated from the work of Xie [45]) to the adic generic fiber of \( J \) to prove this analog on \( J_0 \). It in fact holds for a power of \( J_0 \). In 6.2, using this result for \( J_0^{d+d'} \) and the toric action on the first \( d \) factors, which commutes with the prime-to-\( p \) Hecke action, we prove a weaker version of Theorem 1.3.2 (1), where each \( O_i \) is a prime-to-\( p \) Hecke orbit and \( \text{CM} \) is replaced by the set of canonical liftings. In 6.3, using the fact that forward Frobenius action shrinks the formal residue tori toward canonical liftings and backward Frobenius action increases ramification, we reduce Theorem 1.3.2 (1) to the weaker version.

In Section 7, we prove Theorem 1.3.2 (2) by “globalizing” a trick of Boxall [5], used by Serban [39] [40] for formal groups, to include all ramified points. The “globalization” is achieved using the Igusa scheme \( J_0 \) and some lemmas about \( J_0 \) prepared in Section 3.

Acknowledgements. The author thanks Ziyang Gao and Yifeng Liu for their helpful comments on the paper. He thanks Brian Conrad and Vlad Serban for the inspiring discussions. The author was motivated by Daniel Kriz’s talks in the spring of 2021 to study Igusa schemes. The research is partially supported by the NSF grant DMS-2000533.

2. Igusa schemes

2.1. Siegel moduli schemes. Let \( G_m \) be the multiplicative group over \( \mathbb{Z}_p \) and \( \widehat{G}_m \) the formal completion of \( G_m \) along the unit. We identify \( \widehat{G}_m \) with its \( p \)-divisible group via the Serre–Tate theorem. For an abelian scheme \( A \), let \( \hat{A} \) be the formal completion of \( A \) along the unit.

Now we define an ordinary Siegel formal moduli scheme. Let \( \text{Nilp}_{\mathbb{Z}_p}^{op} \) be the opposite category of \( \mathbb{Z}_p \)-algebras with \( p \) nilpotent. For \( R \in \text{Nilp}_{\mathbb{Z}_p}^{op} \), an abelian \( R \)-scheme \( A \) is ordinary if \( \hat{A}[p^\infty] \simeq \widehat{G}_m^g \) and \( A[p^\infty]/\hat{A}[p^\infty] \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^g \). Let \( N \geq 3 \) be an integer prime to \( p \). For a principally polarized \( g \)-dimensional abelian scheme \( A \) over a \( \mathbb{Z}_p \)-scheme, a level-\( N \)-structure is an isomorphism of finite group schemes \( (\mathbb{Z}/N\mathbb{Z})^g \simeq A[N] \) where the skew-symmetric form on \( (\mathbb{Z}/N\mathbb{Z})^g \) is the standard one, and on \( A[N] \) is the one induced from the Weil pairing. The functor assigning to \( R \in \text{Nilp}_{\mathbb{Z}_p}^{op} \) the set of isomorphism classes of principally polarized ordinary abelian \( R \)-schemes with level-\( N \)-structure is representable by a formal scheme \( \mathcal{S} = \mathcal{S}_N \) topologically of finite type and flat over \( \mathbb{Z}_p \). (One may use quotient to obtain more general level structures.)

We relate \( \mathcal{S} \) to the usual Siegel moduli schemes. Let \( S = S_N \) over \( \text{Spec } \mathbb{Z}_p \) be the moduli space of principally polarized \( g \)-dimensional abelian schemes over \( \mathbb{Z}_p \)-schemes with level-\( N \)-structure. Remove the nonordinary locus in the special fiber of \( S \). Then \( \mathcal{S} \) is the \( p \)-adic completion of the resulted scheme.

2.2. Igusa schemes. Consider the \( p \)-divisible group \( X = \widehat{G}_m^g \times (\mathbb{Q}_p/\mathbb{Z}_p)^g \) over \( \mathbb{Z}_p \), with its connected part \( \hat{X} = \widehat{G}_m^g \) and étale part \( X_{\text{ét}} = (\mathbb{Q}_p/\mathbb{Z}_p)^g \).

Fix the canonical polarization \( X \simeq X^\vee, \hat{X} \simeq X^\vee_{\text{ét}} \) and \( X_{\text{ét}} \simeq \hat{X}^\vee \). For an integer \( i \geq 0 \) or \( i = \infty \), let \( \mathcal{S}_i/\text{Spf } \mathbb{Z}_p \) be the functor assigning to \( R \in \text{Nilp}_{\mathbb{Z}_p}^{op} \) the set of isomorphism classes of triples

\[
\mathcal{S}_i = \{(A, \varepsilon_0, \varepsilon_1) : A \in \mathcal{S}(R), \varepsilon_0 : \hat{A}[p^i] \simeq \hat{X}_R[p^i], \varepsilon_1 : A[p^i]/\hat{A}[p^i] \simeq X_{\text{ét}, R}[p^i]\}/\sim,
\]

where the isomorphisms \( \varepsilon_0 \) and \( \varepsilon_1 \) are required to preserve the filtered polarizations \( \hat{A}[p^i] \to (A[p^i]/\hat{A}[p^i])^\vee \) and \( A[p^i]/\hat{A}[p^i] \to \hat{A}[p^i]^\vee \) up to a common scalar. By [14], the functor \( \mathcal{S}_i \) is
represented by a formal scheme (which we still denote by $S_i$ so that $S_0 = \mathcal{E}$). For $j \geq i$, we have the natural projection $f_{j,i} : \mathcal{S}_j \to \mathcal{S}_i$, which is surjective finite étale if $j < \infty$. And $J_0 := \mathcal{S}_\infty$ is the inverse limit of $S_i$'s (in the category of formal $\mathbb{Z}_p$-schemes). Below, we only use $f_{j,i}$ for $j < \infty$, and 

$$
\pi_i := f_{\infty,i} : J_0 \to S_i.
$$

Let $\mathcal{J}/\text{Spf } \mathbb{Z}_p$ be the functor assigning to $R \in \text{Nilp}_{\mathbb{Z}_p}^{\text{op}}$ the set of isomorphism classes of pairs

$$
\mathcal{J} = \{(A, \varepsilon) : A \in \mathcal{S}(R), \varepsilon : A[p^\infty] \simeq X_R\}/\sim,
$$

where the isomorphism $\varepsilon$ is required to preserve the polarization up to a scalar. Then we have natural projections

$$(2.2) \quad \mathcal{J} \twoheadrightarrow J_0 \xrightarrow{\pi_0} \mathcal{S}.$$

**Proposition 2.2.1** ([6, Proposition 4.3.3, Corollary 4.3.5, Lemma 4.3.10]). (1) The functor $\mathcal{J}$ is representable by a formal scheme in the sense of [37, 2.2] (which we still denote by $\mathcal{J}$).

(2) The special fiber $\mathcal{J}_{\mathbb{F}_p}$ is the perfection of $\mathcal{J}_{0,\mathbb{F}_p}$.

(3) The formal scheme $\mathcal{J}$ is the Witt vector scheme of $\mathcal{J}_{\mathbb{F}_p}$, i.e., it is the unique lift of $\mathcal{J}_{\mathbb{F}_p}$ to a flat $p$-adic formal scheme over $\mathbb{Z}_p$.

**Lemma 2.2.2.** For $y \in J_0(k)$ and a positive integer $n$, the local ring $\mathcal{O}_{J_0/p^n,y}$ is noetherian.

Proof. Note that $J_{0,p^n}$ is a scheme and is the inverse limit of $S_i/p^n$'s. The proof is the same as the proof of the noetherianness in [23, Proposition 2.4].

2.3. **Local toric structure.** Let $k$ be an algebraic extension of $\mathbb{F}_p$. Let $W$ be the ring of Witt vectors of $k$. Let $\mathfrak{M}/\text{Spf } W$ be the height-0 component of the Rapoport-Zink space of $X_k$, i.e., for $R \in \text{Nilp}_{W}^{\text{op}}$, $\mathfrak{M}(R)$ is the set of isomorphism classes of pairs $(X, \varepsilon')$ where $X$ is a principally polarized $p$-divisible group over $R$, and $\varepsilon : X_{R/p} \to X_{R/p}$ is a polarization-preserving (up to a scalar) quasi-isogeny of height 0. Then $\mathfrak{M}$ is representable by a formal scheme locally formally of finite type over $W$ (see [37, Theorem 6.1.2]). In particular, $\mathfrak{M}$ is determined by its restriction to the subcategory $\text{Art}_{W}^{\text{op}}$ of $\text{Nil}_{W}^{\text{op}}$ of artinian local $W$-algebras with residue field $k$. Then by the Serre–Tate theory [18] [8], $\mathfrak{M}/\text{Spf } W$ coincides with the polarized deformation space of $X_k$ over $\text{Art}_{W}^{\text{op}}$. By [8, 2.18, 2.19] or [18, p 150], $\mathfrak{M}(R)$ equivalently classifies equivalence classes of polarized extensions

$$(2.3) \quad 0 \to \hat{X}_R \to G \to X_{\text{ét},R} \to 0.
$$

In particular we have the universal object

$$(2.4) \quad 0 \to \hat{X}_{\mathfrak{M}} \to G^{\text{univ}} \to X_{\text{ét},\mathfrak{M}} \to 0.
$$

There is a formal torus structure on $\mathfrak{M}$ defined by Baer sum (see [8, Theorem 2.19, Corollary 2.24]):

$$
\mathfrak{M} \cong \hat{G}_{m,W}^{g(g+1)/2}.
$$

The unit of $\mathfrak{M}$ corresponds to $X_W$ with the canonical identification $(X_W)_{W/p} = X_{W/p}$. 
For \( x \in \mathcal{S}(k) \), \( \pi^{-1}_0(\{x\}) \subset \mathcal{I}_{0,k} \) is a union of \( k \)-points. If \( x \) corresponds to a polarized ordinary abelian variety \( A_x \) over \( k, y \in \pi^{-1}_0(\{x\}) \) corresponds to the choice of a pair polarization-preserving isomorphisms \( \hat{A}_x \simeq \hat{X}_k \) and \( A_x/\hat{A}_x \simeq \hat{X}_{\text{ét},k} \). Then we have an isomorphism
\[
a_y : \hat{M}_x \simeq \hat{\mathcal{M}}
\]
that sends \( A \in \mathcal{S}_x(R) \), \( R \in \text{Art}^\text{op}_W \), to a polarized extension
\[
0 \to \hat{X}_R \to A[p^\infty] \to X_{\text{ét},R} \to 0
\]
determined by the above two isomorphisms over \( k \) (see \([8, 2.18, 2.19] \) or \([18, \text{p} 150] \)). For a different \( y, a_y \) differs by an automorphism of \( \hat{\mathcal{M}} \). (The automorphism could be made explicit. However, we will not discuss it here.) In particular, we have a natural formal torus structure on \( \hat{\mathcal{S}}_x \) via \( a_y \), independent of the choice of \( y \).

For \( y \in \mathcal{I}_0(k) \), let \( \mathcal{I}_{0,y} \) be the formal completion of \( \mathcal{I}_{0,W} \) at \( y \). Then we have natural isomorphisms of formal schemes
\[
b_y : \mathcal{I}_{0,y} \simeq \mathcal{S}_{\pi_0(y)}, \ c_y : \mathcal{I}_{0,y} \simeq \hat{\mathcal{M}}
\]
defined as follows. By Lemma 2.2.2, we only need to define the morphisms on \( R \)-points for \( R \in \text{Art}^\text{op}_W \). By the discussion in the last paragraph, a point \( \tilde{y} \in \mathcal{I}_0(R) \) with reduction \( y \) corresponds to a pair: \( A \in \mathcal{S}_{\pi_0(y)}(R) \) and an extension \( (2.5) \). Let \( b_y(\tilde{y}) = A \) (i.e., \( b_y = \pi_0|_{\mathcal{I}_{0,y}} \)), and let \( c_y(\tilde{y}) \) be \( (2.5) \). Then by the discussion in the last paragraph, \( b_y \) and \( c_y \) are isomorphisms. It is also easy to check that \( c_y \circ b_y^{-1} = a_y \). In particular, we have a natural formal torus structure on \( \mathcal{I}_0 \) and \( \hat{\mathcal{S}} \) from now on.

2.4. Toric action. We define an action of \( \hat{\mathcal{M}} \) on \( \mathcal{I}_{0,W} \):
\[
\hat{\mathcal{M}} \times \mathcal{I}_{0,W} \to \mathcal{I}_{0,W}, \ (h, x) \mapsto hx
\]
following [21, Proposition 2.3.5]. The proof in [21, Proposition 2.3.5] is not rigorous (see also \([16, 1.2.10] \)), and we modify it as follows. For \( R \in \text{Nilp}^\text{op}_W \) (not necessarily in \( \text{Art}^\text{op}_W \)) and \( h \in \hat{\mathcal{M}}(R) \), we have a polarized extension \( (2.3) \) for \( G \) coming from base change of the universal object \( (2.4) \). For \( x \in \mathcal{I}_0(R) \), let \( x \) correspond to \( A \in \mathcal{S}(R) \) and a polarized extension \( (2.5) \). The Baer sum of \( (2.3) \) and \( (2.5) \) is a polarized extension
\[
0 \to \hat{X}_R \to G' \to X_{\text{ét},R} \to 0.
\]
From the polarization-preserving quasi-isogeny \( G_{R/p} \to X_{R/p} \) of height 0, we have a polarization-preserving quasi-isogeny \( G'_{R/p} \to A[p^\infty]_{R/p} \) of height 0. It lifts uniquely to a polarization-preserving quasi-isogeny \( G' \to A[p^\infty] \), by the rigidity theorem for \( p \)-divisible groups up to isogeny \([18, \text{Lemma 1.1.3}] \). Then the Serre–Tate theory \([18, \text{Theorem 1.2.1}] \), there is a unique \( A' \in \mathcal{S}(R) \), with a \( p \)-primary quasi-isogeny to \( A \) and the restriction of the \( p \)-primary quasi-isogeny to the \( p \)-divisible groups is identified with \( G' \to A[p^\infty] \). Combined with \( (2.7) \), we get an element in \( \mathcal{I}_0(R) \), which we denote by \( hx \).

The following lemma is easy to check by definition.

Lemma 2.4.1. The action of \( \hat{\mathcal{M}} \) stabilizes \( \mathcal{I}_{0,y} \). Via \( c_y \), the action of \( \hat{\mathcal{M}} \) on \( \mathcal{I}_{0,y} \) is identified with the multiplication on \( \hat{\mathcal{M}} \), equivalently on \( \mathcal{I}_{0,y} \), i.e., \( hx = c_y^{-1}(h) + x \).
Remark 2.4.2. For $g = 1$, such an action was also defined by Howe [16] for a slightly different Igusa scheme. A subtlety is that the $GL_2(A_f^0)$-action in [16] differs from our $GL_2(A_f^0)$-action (see 5.1 below) by a suitable twist. This was shown in an earlier version of [16].

3. Toric action and linearity

In the rest of this paper, let $k$ be an algebraic closure of $F_p$. Let $F$ be a finite extension of $\text{Frac}W$. Let $\varpi$ be a uniformizer of $F^0$.

In this section, we apply the action of $\mathcal{M}$ on the Igusa scheme $I_0$ in 2.4 to study the linearity of subschemes of $S_k$ and $S$. The first two subsections are preparations. Then we prove Theorem 1.2.3. In the final subsection, we prepare some technical lemmas for later use.

3.1. Preliminaries. We need some preliminaries on commutative algebra.

In order to deal with formal schemes that are not necessary noetherian, we clarify the definitions. We will only deal with affine formal schemes in the non-noetherian case. A closed formal subscheme of an affine formal scheme $\text{Spf}A$ is of the form $\text{Spf} A/I$ where $I$ is a closed ideal so that $A/I$ is complete and separated with the quotient topology (see [22, 23.B, 23.D]). Note that every ideal is closed if $A$ is noetherian, see [22, 24.B]. For a morphism $f : \text{Spf} B \to \text{Spf} A$, corresponding to a ring homomorphism $f^\#: A \to B$, define the pullback or schematic preimage

$$f^{-1}(\text{Spf} A/I) = \text{Spf} B/IB,$$

where the overline denotes taking closure (in $B$), and define the schematic image

$$f(\text{Spf} B) = \text{Spf} A/\ker f^\#.$$

Let $\Lambda$ be an index set, and for $\lambda \in \Lambda$, let $f_\lambda : \text{Spf} B \to \text{Spf} A$ be a morphism and regarded as a $B$-point of $\text{Spf} A$. Define schematic closure of $\{f_\lambda\}_{\lambda \in \Lambda}$ to be the schematic image of

$$\prod_{\lambda \in \Lambda} f_\lambda : \prod_{\lambda \in \Lambda} \text{Spf} B \to A.$$  

We say that $\{f_\lambda\}_{\lambda \in \Lambda}$ is schematically dense if the schematic image is $\text{Spf} A$.

For a morphism $f : \mathcal{Y} \to \mathcal{X}$ of locally noetherian formal schemes, pullback of a formal scheme over $\mathcal{X}$ is defined as the fiber product. The schematic image $f(\mathcal{Y})$ of $f$ is the minimal closed formal subscheme of $\mathcal{X}$ through which $f$ factors. (The uniqueness is clear and the existence is [17, Lemma 2.8].) We call $f$ schematically surjective if $f(\mathcal{Y}) = \mathcal{X}$. In the affine case, schematic image is given by the discussion in the last paragraph. Understanding affine or locally noetherian schemes as formal schemes (with discrete topology on the rings), the above definition coincides with the usual one [4, 2.5][41, Section 01R5].

The formation of schematic image under a proper morphism is compatible with flat pullback. More precisely, we have the following lemma.

Lemma 3.1.1 ([17, Proposition 2.10]). Let $f : \mathcal{Y} \to \mathcal{X}$ and $g : \mathcal{X}' \to \mathcal{X}$ be morphisms of locally noetherian formal schemes, where $f$ is proper and $g$ is flat. Let $\mathcal{Y}' = \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ with induced morphism $f : \mathcal{Y}' \to \mathcal{X}'$. Then $f(\mathcal{Y}') = g^{-1}(f(\mathcal{Y}))$.

For a locally noetherian formal scheme $\mathcal{X}$, the dimension $\text{dim}(\mathcal{X})$ of $\mathcal{X}$ is defined as the supremum of the Krull dimensions of its local rings. If $\mathcal{X}$ is affine, then $\text{dim}(\mathcal{X})$ is equal to the Krull dimension of $\mathcal{O}_{\mathcal{X}}(\mathcal{X})$. 
Lemma 3.1.2. Let $f : \mathcal{Y} \to \mathcal{X}$ be a schematically surjective finite morphism of noetherian formal schemes.

(1) For a closed subscheme $\mathfrak{Z} \subset \mathcal{X}$ (defined by an open ideal), $f_{\restriction f^{-1}(\mathfrak{Z})} : f^{-1}(\mathfrak{Z}) \to \mathfrak{Z}$ is set theoretically surjective.

(2) We have $\dim \mathcal{Y} = \dim \mathcal{X}$.

(3) For a closed point $x$ of $\mathcal{X}$, $f : \mathcal{Y}_{f^{-1}(x)} \to \mathcal{X}_{x}$ is a schematically surjective finite morphism. (Note that $f^{-1}(x)$ is a finite union of closed points of $\mathcal{Y}$, and $\mathcal{Y}_{f^{-1}(x)}$ is defined in the obvious way.) In particular, $\dim \mathcal{Y}_{f^{-1}(x)} = \dim \mathcal{X}_{x}$.

Proof. It is enough to consider the affine case and assumes that $\mathcal{X}, \mathcal{Y}$ are schemes. Since a finite morphism is quasi-compact and closed, by [41, Lemma 01R8], $f$ is set theoretically surjective. Then so is $f_{\restriction f^{-1}(\mathfrak{Z})}$. (2) is [41, Lemma 0ECG]. Since completion preserves finiteness [41, Lemma 0315 (4)] and injectivity (assuming noetherianness) by the Artin-Rees lemma, (3) follows. □

Corollary 3.1.3. Let $f : \mathcal{Y} \to \mathcal{X}$ be a schematically surjective finite adic morphism of noetherian formal $F^\mathfrak{o}$-schemes.

(1) The induced map $\mathcal{Y}(F^\mathfrak{o}) \to \mathcal{X}(F^\mathfrak{o})$ is surjective.

(2) Assume $\mathcal{Y} = \text{Spf} \mathcal{B}$ where $\mathcal{B}$ is a complete local domain over $F^\mathfrak{o}$ with maximal ideal $\mathfrak{m}$ such that the structure morphism $F^\mathfrak{o} \to \mathcal{B}$ is injective and $F^\mathfrak{o}/(F^\mathfrak{o} \cap \mathfrak{m}) \to \mathcal{B}/\mathfrak{m}$ is an isomorphism. Then $\mathcal{Y}(F^\mathfrak{o}) \neq \emptyset$.

Proof. (1) It is enough to consider the affine case so that $\mathcal{X} = \text{Spf} \mathcal{A}$ and $\mathcal{Y} = \text{Spf} \mathcal{B}$. Any $x \in \mathcal{X}(F^\mathfrak{o})$ is given by $A/I \simeq E^\mathfrak{o}$ where $E/F$ is a finite extension. Regard $x = \text{Spec} E^\mathfrak{o}$ as a closed subscheme of $\text{Spec} \mathcal{A}$. By Lemma 3.1.2 (1), $\text{Spec} B/IB \to \text{Spec} E^\mathfrak{o}$ is set theoretically surjective. Since $B/IB$ is finite over $E^\mathfrak{o}$, the underlying reduced subscheme of an irreducible component of $\text{Spec} B/IB$ that is surjective to $\text{Spec} E^\mathfrak{o}$ is of the form $\text{Spec} K^\mathfrak{o}$ where $K/E$ is a finite extension. Since $A \to B$ is adic, it is easy to check that $B \to K^\mathfrak{o}$ is continuous. This gives a $K^\mathfrak{o}$-point of $\mathcal{Y}$ whose image is $x$.

(2) follows from (1) and the Noether normalization lemma for complete local domain [41, Lemma 032D]. Note that the Noether normalization morphism being adic is shown in the proof of [41, Lemma 032D]. □

We prepare a lemma on (adic) completion.

Lemma 3.1.4. Let $R$ be a ring and $M, N$ two $R$-modules.

(1) Let $I \subset R$ be an ideals. A surjection $M \to N$ between $R$-modules with kernel $K$ induces a surjection $\widehat{M}_I \to \widehat{N}_I$ between the $I$-adic completions, whose kernel is the closure of the image of $K$ in $\widehat{M}_I$.

(2) Assume that $N$ is finitely presented. Let $\{M_n\}_{n=1}^\infty$ be submodules of $M$ such that $M_{n+1} \subset M_n$. Let $\widehat{M} = \varprojlim_n M/M_n$ and $M \otimes_R N = \varprojlim_n M \otimes_R N/[M_n \otimes_R N]$. Here for a submodule $M' \subset M$, $[M' \otimes_R N]$ is the image of $M' \otimes_R N$ in $M \otimes_R N$. Then the natural morphism $\widehat{M} \otimes_R N \to M \otimes_R N$ is an isomorphism.

Proof. (1) is [22, 23.I, Poposition] (see also [41, Lemma 0315 (2)]). Now we prove (2). By the right exactness of tensor, for a submodule $M' \subset M$, $M/M' \otimes_R N \simeq M \otimes_R N/[M' \otimes_R N]$. Thus we only need to show that $\widehat{M} \otimes_R N \simeq \varprojlim_n (M/M_n \otimes_R N)$. Let $N$ be the cokernel of a morphism $\Phi : R^i \to R^j$. From $\Phi$, we have induced morphisms $\phi : M^i \to \widehat{M}^j$ and $\phi_n : (M/M_n)^i \to$
Below, we denote \( (M/M_n)^{\hat{\cdot}} \). We need to prove the isomorphism between the cokernel of \( \phi \) and the inverse limit of the cokernels of \( \phi_n \)'s. By the surjectivity of \( M/M_n \to M/M_n \) for \( m \geq n \), we have the surjectivity of the morphism from the image of \( \phi_m \) to the image of \( \phi_n \). The expected isomorphism follows from the Mittag–Leffler lemma. 

We prepare an important lemma.

**Lemma 3.1.5.** Let \( R \) be a ring, \( \alpha \in R \) primary (i.e., \( \alpha R \) is a primary ideal) and not a zero divisor.

1. For every positive integer \( n \), \( \alpha^n \) is primary.
2. Assume that \( R \) is noetherian. Let \( I \subset R \) be an ideal containing \( \alpha \), \( \hat{R}_I \) the corresponding \( I \)-adic completion. The natural map \( R/\alpha^n \to \hat{R}_I/\alpha^n \) is injective for every \( n \).

*Proof.* (1) We do induction on \( n \). The case \( n = 1 \) is trivial. For a general \( n > 1 \), assume that \( \alpha^n \rvert ab \) and \( \alpha^n \nmid a \), we will show that \( \alpha^n \) divides a power of \( b \). Since \( \alpha \) is primary, if \( \alpha \nmid a \), \( \alpha \) divides a power of \( b \). Taking \( n \)-th power, we are done. Otherwise, \( \alpha \rvert a \) and write \( a = \alpha a_1 \). Then \( \alpha^n | a_1 b \). Since \( \alpha \) is not a zero divisor, \( \alpha^{n-1} | a_1 b \). Since \( \alpha^n \nmid a = \alpha a_1 \), \( \alpha^{n-1} \nmid a_1 \). By the induction hypothesis, \( \alpha^{n-1} \) divides a power of \( b \). So \( \alpha^{n(n-1)} \) divides a power of \( b \). Since \( \alpha^n | \alpha^{n(n-1)} \), we are done.

(2) Let \( x \in R \) such that the image of \( x \) in \( \hat{R}_I = \lim_{\leftarrow m} R/I^m \) is in \( \alpha^n \hat{R}_I \). Then \( x \in \alpha^n R + I^m \) for every positive integer \( m \), i.e., \( [x] \in [I]^m \) for the images \( [x], [I] \) of \( x, I \) in \( R/\alpha^n \). Since \( \alpha^n \) is primary, by [44, p 144, Theorem 2] (which requires \( R \) to be noetherian), \( \bigcap_m [I]^m = \{0\} \). So \( [x] = 0 \). (2) follows.

### 3.2. Pullback to \( \mathcal{X}_0 \)

Let \( \mathfrak{Y} = \mathfrak{Y}_0 \subset \mathcal{G}_{F_0} \) be a formal subscheme. In Lemma 3.2.5 below, we want to study the pullback of \( \mathfrak{Y} \) to \( \mathcal{X}_0 \) which is non-noetherian. Before that, we consider an easier analog of Lemma 3.2.5 in the noetherian case.

**Lemma 3.2.1.** Assume that \( \mathfrak{Y} \) is affine, \( \mathfrak{Y}_k \) is irreducible and has no embedded points. For a closed formal subscheme \( \mathfrak{X} \subset \mathfrak{Y} \) and \( x \in \mathfrak{X}(k) \) (then \( \mathfrak{X}_x \) is a closed formal subscheme of \( \mathfrak{Y}_x \) by Lemma 3.1.4 (1)), if \( \mathfrak{X}_x = \mathfrak{Y}_x \), then \( \mathfrak{X} = \mathfrak{Y} \).

*Proof.* Claim: the natural map \( \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}) \to \mathcal{O}_{\mathfrak{Y}_x} \) is injective. (Here \( \mathcal{O}_{\mathfrak{Y}_x} \) is the complete local ring at \( x \).) Thus the composition of the diagram of natural morphisms

\[
\mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}) \to \mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) \to \mathcal{O}_{\mathfrak{X}_x} \simeq \mathcal{O}_{\mathfrak{Y}_x}
\]

is injective (the condition in the lemma gives the last isomorphism). Then the first morphism in this diagram is injective (also surjective by definition), and thus an isomorphism. So \( \mathfrak{X} = \mathfrak{Y} \).

Now we prove the claim. Since \( \mathfrak{Y} \) is flat over \( F_0 \), \( \varpi \in \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}) \) is not a zero-divisor. Since \( \mathfrak{Y}_k \) is irreducible and has no embedded points, \( \varpi \in \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}) \) is primary. By Lemma 3.1.5 (2) (with \( R = \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}) \), \( \alpha = \varpi \) and \( I \) the ideal of \( \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}) \) defining \( x \)), \( \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y})/\varpi^n \to \mathcal{O}_{\mathfrak{Y}_x}/\varpi^n \) is injective. Since \( \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}) \) and \( \mathcal{O}_{\mathfrak{Y}_x} \) are \( \varpi \)-adically complete (see [41, Lemma 090T]) and taking limits of inverse systems is left exact, the claim follows. □

Now we consider the pullback of \( \mathfrak{Y} \) to \( \mathcal{X}_0 \). Let \( f_{j,i} : \mathfrak{G}_j \to \mathfrak{S}_i \) (for \( j \geq i \geq 0 \)) be the surjective finite étale morphism as in 2.2. Let \( \mathfrak{Y}_0 = \mathfrak{Y} \) and assume it to be connected. Let \( \mathfrak{Y}_{i+1} \) be a connected component of \( f_{i+1,i}^{-1}(\mathfrak{Y}_i) \) inductively. Let \( \mathfrak{Y} \) be the inverse limit of \( \mathfrak{Y}_i \)'s, i.e., \( \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}) \)
is the $\varpi$-adic completion of $\varprojlim_j O_{\mathfrak{g}_j}(\mathfrak{g}_i)$. Then $\mathfrak{Y}$ is naturally a closed formal subscheme of $\pi_0^{-1}(\mathfrak{g})$ where $\pi_i : \mathfrak{g}_0 \to \mathfrak{S}_i$ (and $\mathfrak{g}_0 = \mathfrak{S}$) is also defined in as in 2.2. Moreover $\mathfrak{Y}_k$ is the inverse limit of $\mathfrak{S}_{i,k}$’s, i.e., $O_{\mathfrak{g}_i}(\mathfrak{Y})/\varpi$ is $\varprojlim_j O_{\mathfrak{g}_j}(\mathfrak{g}_i)/\varpi$. Since a finite étale morphism is open and closed, every $f_{j,i}|\mathfrak{g}_{j,k}$ is surjective to $\mathfrak{g}_{i,k}$. So $\pi_0|_{\mathfrak{g}_k}$ is surjective to $\mathfrak{Y}_k$.

**Lemma 3.2.2.** For $x \in \mathfrak{g}(k)$, let $y \in \pi_0^{-1}\{(x)\} \cap \mathfrak{Y}$. Then the isomorphism $\pi_0|_{\mathfrak{g}_0,y} : \mathfrak{g}_0,y \simeq \mathfrak{S}_x$ (see (2.6) and the discussion below it) induces an isomorphism

$$\hat{O}_{\mathfrak{g}_0,y} \simeq \hat{O}_{\mathfrak{g}_x}.$$  

**Proof.** Let $x_i = \pi_i(y)$. Since $f_{j,i} : \mathfrak{S}_j \to \mathfrak{S}_i$ is étale, the composition $f_{j,i}|\mathfrak{g}_j : \mathfrak{g}_j \to f_{j,i}^{-1}(\mathfrak{g}_i) \to f_{j,i}(\mathfrak{g}_i)$ is étale, where the first morphism is an open embedding. Since $k$ is algebraically closed, $f_{j,i}|\mathfrak{g}_j$ induces an isomorphism between the quotients by powers of maximal ideals $O_{\mathfrak{g}_j}/m_{x_j}^n \simeq O_{\mathfrak{g}_i}/m_{x_i}^n$. The rest follows routinely from definition. □

**Assumption 3.2.3.** Here are some assumptions for later use:

1. $\mathfrak{g}_k$ is connected and unibranch;
2. $\mathfrak{g}$ is affine (for the sake of considering closed formal subschemes in the non-noetherian case, see 3.1);
3. $\mathfrak{g}$ is flat over $F^\infty$;
4. $\mathfrak{g}_k$ contains no embedded points.

**Lemma 3.2.4.** (1) Under Assumption 3.2.3 (1), $\mathfrak{g}_{i,k}$ is unibranch and irreducible.

(2) Under Assumption 3.2.3 (4), $\mathfrak{g}_{i,k}$ has no embedded points.

**Proof.** (1) Since $f_{j,i}$ is étale, $\mathfrak{g}_{i,k}$ is unibranch by definition. By Lemma B.0.5, the spectra of local rings of $\mathfrak{g}_{i,k}$ are irreducible. So $\mathfrak{g}_{i,k}$ is irreducible.

(2) Since a flat ring homomorphism preserves non-zero divisors, a flat morphism for noetherian schemes maps an associated point to an associated point. By [41, Lemma 05AL], which says that embedded points of noetherian schemes are non-generic associated points, (2) follows. (Or one may prove (2) by interpreting Assumption 3.2.3 (4) as Serre’s condition $S_1$, which is preserved by an étale morphism.) □

**Lemma 3.2.5.** Under Assumption 3.2.3, for a closed formal subscheme $\mathfrak{y} \subset \mathfrak{Y}$ and $y \in \mathfrak{y}(k)$, if $\mathfrak{g}_y = \hat{\mathfrak{Y}}_y$, then $\mathfrak{y} = \hat{\mathfrak{Y}}$.

**Proof.** Claim: the natural map $O_{\mathfrak{g}}(\hat{\mathfrak{Y}}) \to O_{\mathfrak{g},y}$ is injective. Here $O_{\mathfrak{g},y}$ is the complete local ring at $y$. Thus the composition of the diagram of natural morphisms

$$O_{\mathfrak{g}}(\mathfrak{Y}) \to O_{\mathfrak{g}}(\mathfrak{y}) \to O_{\mathfrak{g},y} \simeq \hat{O}_{\mathfrak{g},y}$$

is injective (the condition in the lemma gives the last isomorphism). Then the first morphism in this diagram is injective (also surjective by definition), and thus an isomorphism. So $\mathfrak{Y} = \hat{\mathfrak{Y}}$.

We prove the claim. Since $O_{\mathfrak{g}}(\mathfrak{Y})$ is the $\varpi$-adic completion of $\varprojlim \mathfrak{g}_i$ and $O_{\mathfrak{g},x}$ is $\varpi$-adically complete (see [41, Lemma 090T]), we only need to show that the natural map

$$O_{\mathfrak{g}_i}(\mathfrak{g}_i)/\varpi^n \to \hat{O}_{\mathfrak{g},y}/\varpi^n$$
is injective for every positive integer \( n \). Recall that we have an isomorphism \( \hat{\mathcal{O}}_{\mathfrak{m},y} \cong \hat{\mathcal{O}}_{\mathfrak{m}, \pi_i(y)} \) induced by \( \pi_i \) as in the case \( i = 0 \) discussed above. Then the case \( i = 0 \) is proved in Lemma 3.2.1. For the general \( i \), the proof is the same. \( \square \)

Recall the action of \( \mathcal{M} \) on \( \mathfrak{I}_{0,y} \) defined in 2.4. Here and below, we understand an \( E^o \)-point \( P \) of \( \mathcal{M}_{F^o} \) as a closed formal \( F^o \)-subscheme of \( \mathcal{M}_{F^o} \), i.e. its schematic image.

**Corollary 3.2.6.** Under Assumption 3.2.3, for a finite extension \( E/F \) and \( y \in \mathfrak{Y}(k) \), \( P \in \mathcal{M}_{F^o}(E^o) \) stabilizes \( \hat{\mathfrak{V}}_y \) if and only if it stabilizes \( \hat{\mathfrak{V}} \).

**Proof.** Then \( P \) stabilizes \( \hat{\mathfrak{V}} \), by definition, if and only if the schematic preimage of \( \hat{\mathfrak{V}} \) under the multiplication morphism \( P \times \hat{\mathfrak{V}} \to \mathfrak{I}_{0,F^o} \) is \( P \times \hat{\mathfrak{V}} \); \( P \) stabilizes \( \hat{\mathfrak{V}}_y \) if and only if the schematic preimage of \( \hat{\mathfrak{V}}_y \) under the multiplication morphism \( P \times \hat{\mathfrak{V}}_y \to \mathfrak{I}_{0,F^o,y} \) is \( P \times \hat{\mathfrak{V}}_y \). The corollary follows from Lemma 3.2.5, applied to \( P \times \hat{\mathfrak{V}} \cong \hat{\mathfrak{V}}_{F^o} \). \( \square \)

3.3. Linearity over \( k \). We need some preliminaries.

Replacing \( F \) by \( k[[t]] \) in the last subsection (so that \( \mathfrak{G}_{k[[t]]} \) is defined via the natural morphism \( k \to k[[t]] \) and so on), we have the analogous discussion. In particular, as a special case of the analog of Corollary 3.2.6, we have the following lemma. (Indeed, we may replace \( k[[t]] \) by \( k((t)) \) in the lemma. But we do not need it.)

**Lemma 3.3.1.** Under Assumption 3.2.3 with (3) replaced by \( \mathfrak{V} = \mathfrak{V}_k \), a point in \( \mathcal{M}_k(k[[t]]) \) stabilizes \( \hat{\mathfrak{V}}_y \) for some \( y \in \mathfrak{Y}(k) \) if and only if it stabilizes \( \hat{\mathfrak{V}} \).

We need the following simple lemma.

**Lemma 3.3.2.** Let \( f \in k[[x_1, \ldots, x_n]] \) and \( f \neq 0 \). There exists a continuous homomorphism \( k[[x_1, \ldots, x_n]] \to k[[t]] \) such that the image of \( f \) is nonzero.

**Proof.** Let \( g \) be the sum of the (finitely many) monomials in \( f \) with the minimal total degrees \( m \). Consider the continuous homomorphism \( k[[x_1, \ldots, x_n]] \to k[[t]] \) with \( x_i, i = 1, \ldots, n \) sent to \( a_it \) where \( a_i \in k \) is to be determined. Then \( g \) is sent to \( h(a_1, \ldots, a_n)t^m \). Choose \( a_i \)'s such that \( h(a_1, \ldots, a_n) \neq 0 \) and the lemma follows. \( \square \)

Now we discuss linearity over \( k \). We call a subscheme \( Z \) of \( \mathfrak{G}_k \) (resp. formal subscheme \( \mathfrak{G} \) of \( \mathfrak{G}_{F^o} \)) linear at \( x \in Z(k) \) (resp. \( \mathfrak{G}(k) \)) if its formal completion at \( x \) is a formal subtorus of \( \mathfrak{G}_{k,x} \) (resp. \( \mathfrak{G}_{x,F^o} \)). It is called linear if it is linear everywhere.

We will extensively use two facts: first, for \( y \in \mathfrak{Y}(k) \), \( \pi_0|_{\mathfrak{I}_{0,y}} : \mathfrak{I}_{0,y} \cong \mathfrak{G}_{\pi_0(x)} \) is an isomorphism of formal tori (see (2.6) and the discussion below it); second, the action of \( \mathcal{M} \) on \( \mathfrak{I}_{0,y} \) becomes multiplication after a natural isomorphism \( \mathfrak{I}_{0,y} \cong \mathcal{M} \), see Lemma 2.4.1.

**Proposition 3.3.3.** Under Assumption 3.2.3 with (3) replaced by \( \mathfrak{V} = \mathfrak{V}_k \), if \( \mathfrak{V} \) is linear at some \( x \in \mathfrak{Y}(k) \), then it is linear.

**Proof.** Claim: for every \( z \in \mathfrak{Y}(k) \), \( \mathfrak{V}_z \) is stable by all \( k[[t]] \)-points of a closed formal subgroup \( \mathfrak{G} \) of the formal torus \( \mathfrak{G}_{k,z} = \mathfrak{G}_{z,k} \) that is isomorphic to \( \mathfrak{V}_z \). We will prove the claim later. Now we prove that \( \mathfrak{V}_z = \mathfrak{G} \) which gives the proposition. Since \( \mathfrak{V} \) is linear at \( x \), \( \mathfrak{G}_{\mathfrak{I}_{0,x}} \) is reduced. By the injectivity of \( \hat{\mathcal{O}}_{\mathfrak{I}_{0,x}} \to \hat{\mathfrak{G}}_{\mathfrak{I}_{0,x}} \) (Krull intersection theorem), \( \mathfrak{V} \) is reduced at \( x \). Thus \( \mathfrak{V} \) is generically reduced by the openness of the reduced locus. Then since \( \mathfrak{V} \) has no embedded
components, it is reduced. Then by the excellence of $\mathfrak{V}$, $\mathfrak{V}_x$ is reduced. Let $\mathfrak{C} \subset \mathfrak{V}_x$ be a formal branch (see Definition B.0.4) which is integral by definition. Since a $k[[t]]$-point of $\mathfrak{G}$, understood as the schematic image of $P$, contains the unit of $\mathfrak{G}$, the schematic image $\mathfrak{C}'$ of the multiplication morphism $P \times \mathfrak{C} \to \mathfrak{V}_x$ contains $\mathfrak{C}$. Since $R[[t]]$ is integral for an integral domain $R$, $\mathcal{O}_{P \times \mathfrak{C}}(P \times \mathfrak{C})$ is integral. So by the definition of schematic image, $\mathfrak{C}'$ is integral, and thus equals $\mathfrak{C}$ by dimension reason. Since $\mathfrak{C}$ contains the unit of $\mathfrak{G}_z$, $\mathfrak{C} = \mathfrak{C}'$ contains $P$, i.e., $\mathfrak{C}$ contains $\mathfrak{G}(k[[t]])$. By Lemma 3.3.2, $\mathfrak{G}(k[[t]])$ is schematically dense in $\mathfrak{G}$. So $\mathfrak{C}$ contains $\mathfrak{G}$. Since $\mathfrak{V}$ is integral and of finite type over $k$, all complete local rings have the same dimension. So $\mathfrak{C} = \mathfrak{G}$. Thus $\mathfrak{V}_x = \mathfrak{G}$.

Now we prove the claim. Let $y \in \pi_0^{-1}(\{x\}) \cap \widetilde{V}$. By $\hat{\mathcal{O}}_{\mathfrak{V}_x,y} \simeq \hat{\mathcal{O}}_{\mathfrak{V}_x}$ (see (3.1)) and $\pi_0|_{\mathfrak{V}_y} : \mathfrak{I}_{0,y} \simeq \mathfrak{G}_x$ of formal tori, $\mathfrak{V}_y$ is stable under the multiplication on $\mathfrak{I}_{0,y,k}$ by all $k[[t]]$-points of a closed formal subgroup of $\mathfrak{I}_{0,y,k}$ isomorphic to $\mathfrak{V}_x$. By Lemma 2.4.1, $\mathfrak{V}_y$ is stable, now under the action of $\mathfrak{M}_k$, by all $k[[t]]$-points of a closed formal subgroup of $\mathfrak{M}_k$ isomorphic to $\mathfrak{V}_x$. By the “only if” part of Lemma 3.3.1, $\mathfrak{V}$ is stable by all $k[[t]]$-points of a closed formal subgroup of $\mathfrak{M}_k$ isomorphic to $\mathfrak{V}_x$. Now we reverse the reasoning for $w \in \pi_0^{-1}(\{z\}) \cap \widetilde{V}$. By the “if” part of Lemma 3.3.1, $\mathfrak{V}_w$ is stable by all $k[[t]]$-points of a closed formal subgroup of $\mathfrak{M}_k$ isomorphic to $\mathfrak{V}_x$. By Lemma 2.4.1, $\mathfrak{V}_w$ is stable, now under the multiplication on $\mathfrak{I}_{0,w,k}$ by all $k[[t]]$-points of a closed formal subgroup of $\mathfrak{M}_k$ isomorphic to $\mathfrak{V}_x$. By (3.1) with $x,y$ replaced by $z,w$ which gives $\mathfrak{V}_w \simeq \mathfrak{V}_z$, and $\pi_0|_{\mathfrak{V}_w} : \mathfrak{I}_{0,w} \simeq \mathfrak{G}_z$ of formal tori, the claim follows.

Remark 3.3.4. (1) Proposition 3.3.3 is a result of Chai [7, (5.3)].

(2) Clearly, one may use the same idea to deal with linearity over $F^0$. Indeed, then one will get Corollary 3.4.3 below. However, we will give another proof.

3.4. Lift linearity. The following result of Chai is important for us.

Lemma 3.4.1 ([7, (5.5)]). Let $Z \subset \mathfrak{G}_k$ be a linear subscheme. There is a unique linear formal subscheme $Z^{\text{can}}$ of $\mathfrak{G}_{F^0}$ lifting $Z$.

Remark 3.4.2. Note that the endomorphism ring of $\mathfrak{G}_m^n$, over $F^0$ or $k$, is $\mathbb{Z}_p$. (One can see this easily by identifying $\mathfrak{G}_m^n$ with its $p$-divisible group.) Then one easily obtains a classification of subtori. In particular, the linear lifting of a formal subtorus of $\mathfrak{G}_m^n$ to $\mathfrak{G}_m^n_{F^0}$ exists and is unique. Lemma 3.4.1 is the globalization this fact.

Corollary 3.4.3. Under Assumption 2.3.2, if $\mathfrak{V}$ is linear at $x \in \mathfrak{V}(k)$, then it is linear.

Proof. We can write $\mathfrak{V}_k$ as a union of affine opens such that each affine open contains $x$. Thus we may assume that $\mathfrak{V}$ is affine. Let $Z$ be the underlying reduced subscheme of $\mathfrak{V}_k$, which is linear by Proposition 3.3.3. Let $Z^{\text{can}}$ be as in Lemma 3.4.1. By Remark 3.4.2, $Z^{\text{can}} = \mathfrak{V}_x$. By Lemma 3.2.1, $Z^{\text{can}} = \mathfrak{V}$.

Remark 3.4.4. Note that Corollary 3.4.3 is a special case of Theorem 1.2.2.

Our following proof of Theorem 1.2.3 is based on the same idea as the proof of Corollary 3.4.3, but worked out on $\mathfrak{I}_0$ rather than on $\mathfrak{G}$.

Proof of Theorem 1.2.3. We may assume that $\mathfrak{V}$ is affine. We may enlarge $F$ so that the given translated formal subtorus in $\mathfrak{V}_x$ are translations by $F^0$-points.
(1) Let $Z$ be the underlying reduced subscheme of $\mathfrak{N}_k$, which is linear by Proposition 3.3.3. Let $Z^{\text{can}}$ be as in Lemma 3.4.1. By Remark 3.4.2, $\mathfrak{N}_k$ contains $P + Z_x^{\text{can}}$ for some $P \in \mathfrak{O}_x(F^o)$. Let $Z^{\text{can}}$ be the restriction of $\pi_0^{-1}(Z^{\text{can}})$ to $\mathfrak{N}_k$. By (3.1), we have $\tilde{O}_{z,y} \simeq O_{z,x}$ and analogously $\tilde{O}_{Z^{\text{can}},z} \simeq \tilde{O}_{Z^{\text{can}},x}$. Then since $\pi_0|_{\mathfrak{N}_k} : \mathfrak{N}_k \simeq \mathfrak{O}_x$, is an isomorphism of formal tori, by Lemma 2.4.1 (about the action of $\mathfrak{M}$ on $\mathfrak{N}_0$, $\mathfrak{N}_k$ contains $h(\tilde{Z}^{\text{can}})$ for some $h \in \mathfrak{M}(F^o)$. Applying Lemma 3.2.5 with $\mathfrak{M}$ replaced by $\tilde{Z}^{\text{can}}$ and $\mathfrak{N}$ replaced by $h^{-1}\tilde{\mathfrak{M}} \cap Z^{\text{can}}$, we have $\tilde{Z}^{\text{can}} \subset h^{-1}\tilde{\mathfrak{M}}$ i.e., $h\tilde{Z}^{\text{can}} \subset \tilde{\mathfrak{M}}$ (here we used that $h$ is an $F^o$-point but not for some field extension, see the convention above Corollary 3.2.6). Let $w \in \pi_0^{-1}({\{z\}}) \cap \tilde{\mathfrak{N}}$. Then (1) follows by reversing the reasoning, i.e. applying Lemma 2.4.1, the isomorphism $\pi_0|_{\mathfrak{N}_k} : \mathfrak{N}_k \simeq \mathfrak{O}_x$ of formal tori and (3.1) with $x, y$ replaced by $z, w$ which gives $\tilde{\mathfrak{N}}_w \simeq \mathfrak{N}_z$. 

(2) The proof is almost the same as (1). Instead of $\tilde{\mathfrak{N}}_y$ contains $h(\tilde{Z}^{\text{can}})$ for some $h \in \mathfrak{M}(F^o)$ now $\tilde{\mathfrak{N}}_y$ is the union of $h_i\tilde{Z}^{\text{can}}$’s where $i$ lies in a finite index set and $h_i \in \mathfrak{M}(F^o)$. So $\tilde{\mathfrak{N}}$ contains $\bigcup h_i\tilde{Z}^{\text{can}}$. By Lemma 3.2.5 (with $\mathfrak{M} = \tilde{\mathfrak{M}} \cap (\bigcup h_i\tilde{Z}^{\text{can}}) \subset \tilde{\mathfrak{M}}$, here we use the condition that $\mathfrak{N}_k$ has no embedded points), $\tilde{\mathfrak{N}}$ is contained in $\bigcup h_i\tilde{Z}^{\text{can}}$. Then (2) follows. □

3.5. Technical results. Now we want to prepare a result (Corollary 3.5.4) that will only be used in 7.2 (so that the reader may skip this subsection for the moment).

Lemma 3.5.1. Under Assumption 3.2.3 (1/2), further assume that $\mathfrak{N}_k$ is reduced. Then for a closed formal subscheme $\mathfrak{P} \subseteq \tilde{\mathfrak{N}}$, there exists a positive integer $s$, a closed formal subscheme $\mathfrak{P}_1 \subseteq \mathfrak{P}$ and a closed subscheme $Y \subset \mathfrak{N}_k$ of $\dim Y < \dim \mathfrak{N}_k$ such that $\mathfrak{P} \subset \mathfrak{P}_1 \subseteq \mathfrak{N}_k / \varpi^s$ and $\mathfrak{P}_1 \cap \mathfrak{N}_k < \pi_0^{-1}(Y)$. In particular, for $w \in \mathfrak{P}(k) \backslash \pi_0^{-1}(Y)$, $\mathfrak{N}_w \subset \tilde{\mathfrak{N}}_w / \varpi^s$. 

Proof. Let $I \subset O_{\mathfrak{P}}(\tilde{\mathfrak{N}})$ be the defining ideal of $\mathfrak{P}$. Let $s$ be the maximal non-negative integer such that $I \subset \varpi^sO_{\mathfrak{P}}(\tilde{\mathfrak{N}})$ (whose existence is guaranteed by the $\varpi$-adic separatedness of $O_{\mathfrak{P}}(\tilde{\mathfrak{N}})$). Let $J := \{a \in O_{\mathfrak{P}}(\tilde{\mathfrak{N}}) : \varpi^s a \in I\}$. Then $I = \varpi^s J$ and thus $J \subset \varpi O_{\mathfrak{P}}(\tilde{\mathfrak{N}})$. Then $J / \varpi$ contains a nonzero element of $O_{\mathfrak{P}}(\tilde{\mathfrak{N}}) / \varpi = \varprojlim O_{\mathfrak{P}}(\mathfrak{N}_i) / \varpi$. So this nonzero element comes from some $O_{\mathfrak{P}}(\mathfrak{N}_i) / \varpi$, and defines a closed subscheme $Y' \subset \mathfrak{N}_i$. Since $\mathfrak{N}_{i,k}$ is irreducible by Lemma 3.2.4 and reduced of the same dimension with $\mathfrak{N}_k$ (as an étale morphism preserves reducedness and dimension), $\dim Y' < \dim \mathfrak{N}_{i,k} = \dim \mathfrak{N}_i$. Then we take $Y = f_{i,0}(Y')$. □

Lemma 3.5.2. Under Assumption 3.2.3, further assume that there is a connected affine flat formal $F^o$-scheme $\mathfrak{W}'$ with reduced unibranch special fiber $\mathfrak{W}'_k$ and a finite schematically surjective morphism $f : \mathfrak{W}' \rightarrow \tilde{\mathfrak{N}}$. For a closed formal subscheme $\mathfrak{P} \subseteq \tilde{\mathfrak{N}}$, there exists a positive integer $s$ and a closed subscheme $Y \subset \mathfrak{N}_k$ of $\dim Y < \dim \mathfrak{N}_k$ such that for $w \in \mathfrak{P}(k) \backslash \pi_0^{-1}(Y)$ and $z = \pi_0(w)$, $f^{-1}(\pi_0|_{\mathfrak{N}_w}(\mathfrak{P}_w)) \subset \mathfrak{W}'_{f^{-1}(z) / \varpi^s}$.

Proof. Let $\mathfrak{W}'$ be the inverse limit of $\mathfrak{W}_i \times_{\mathfrak{P}_i} \mathfrak{W}_i$’s. Explicitly, $O_{\mathfrak{W}'}(\mathfrak{W}')$ is the $\varpi$-adic completion of $\varprojlim \mathfrak{W}_i \otimes_{O_{\mathfrak{W}_i}} O_{\mathfrak{W}_i}(\mathfrak{W}')$. By Lemma 3.1.4 (2), we have $O_{\mathfrak{W}'}(\mathfrak{W}') \simeq O_{\tilde{\mathfrak{N}}}(\tilde{\mathfrak{N}}) \otimes_{O_{\mathfrak{N}}(\mathfrak{W})} O_{\mathfrak{W}'}(\mathfrak{W}')$. Let $\tilde{f} : \mathfrak{W}' \rightarrow \mathfrak{N}$ be the natural morphism. For $y \in \mathfrak{N}(k)$ and $x = \pi_0(y)$, by Lemma 3.1.4 (2) again (for the first and third isomorphisms in (3.3)) and (3.1) (for the second isomorphism),

\[ O_{\tilde{f}^{-1}(y)} \simeq O_{\tilde{f}^{-1}(x)} \otimes_{O_{\mathfrak{W'}}(\mathfrak{W})} O_{\mathfrak{W}'}(\mathfrak{W'}) \simeq O_{\tilde{f}^{-1}(y)} \otimes_{O_{\mathfrak{W}_i}(\mathfrak{W}_i)} O_{\mathfrak{W}_i}(\mathfrak{W}_i) \simeq O_{\mathfrak{W}_i,f^{-1}(x)} \]
Indeed, since the ideal defining \( y \) contains \( \varpi \), the ideal \( \mathcal{O}_{\tilde{\mathcal{Y}}}(\tilde{\mathcal{Y}}') \) contains \( \varpi \), and it thus is open and closed. This is the closed ideal defining \( \tilde{f}^{-1}(y) \). So Lemma 3.1.4 (2) can be applied to get the first isomorphism.

Let \( \mathcal{Y}' = f^{-1}(\mathcal{Y}) \subset \tilde{\mathcal{Y}} \). Claim 1: \( \mathcal{Y}' \neq \tilde{\mathcal{Y}} \). The claim will be proved later. Since, by definition, \( \mathcal{O}_{\tilde{\mathcal{Y}}}(\tilde{\mathcal{Y}}')/\varpi \) is the direct limit of \( \mathcal{O}_{\tilde{\mathcal{Y}}}(\mathcal{Y}))/\varpi \mathcal{O}_{\tilde{\mathcal{Y}}}(\mathcal{Y}')/\varpi s \), which are finite étale over \( \mathcal{O}_{\mathcal{Y}}(\mathcal{Y}')/\varpi \), we can apply the proof of Lemma 3.5.1 to \( \mathcal{Y}' \subset \tilde{\mathcal{Y}} \) (here we need the unbranchness of \( \mathcal{Y}_k' \)) to find a positive integer \( s \), and a closed subscheme \( Y' \subset \mathcal{Y}_k' \) of \( \dim Y' < \dim \mathcal{Y}_k' \) such that for \( y' \in \mathcal{Y}'(k) \), \( \mathcal{Y}'_{y'} \subset \mathcal{Y}_k' / \varpi s \) unless \( \pi_0(y') \in Y' \). By Lemma 3.1.2 (2) and the flatness, \( \dim \mathcal{Y}_k = \dim \mathcal{Y} - 1 = \dim \mathcal{Y}' - 1 = \dim \mathcal{Y}_k' \). Claim 2: for \( y \in \mathcal{Y}(k) \) and \( x = \pi_0(y) \), \( (3.4) \quad \mathcal{Y}'_{\tilde{f}^{-1}(y)} \simeq f^{-1}(\pi_0|_{\mathcal{Y}_{y}}(\mathcal{Y}')) \).

Under (3.3). Then the lemma follows with \( Y = f(Y') \) whose dimension is \( \dim Y' < \dim \mathcal{Y}_k \).

To prove the claims, we use the following commutative diagram, where the two isomorphisms are (3.1) and (3.3), and the others are the natural ones:

\[
\begin{array}{ccc}
\mathcal{O}_{\bar{\mathcal{Y}}} & \longrightarrow & \mathcal{O}_{\bar{\mathcal{Y}},y} \\
\downarrow & & \downarrow \\
\mathcal{O}_{\tilde{\mathcal{Y}}}(\mathcal{Y}') & \longrightarrow & \mathcal{O}_{\tilde{\mathcal{Y}},\tilde{f}^{-1}(y)} \\
\end{array}
\]

Since \( \mathcal{O}_{\mathcal{Y}}(\mathcal{Y}) \to \mathcal{O}_{\mathcal{Y}}(\tilde{\mathcal{Y}}') \) is injective by the definition of schematic surjectivity, the last vertical morphism is injective by the Artin-Rees lemma. So the middle vertical morphism is injective.

We first prove Claim 1. Let \( I \subset \mathcal{O}_{\mathcal{Y}}(\mathcal{Y}) \) be the ideal defining \( \mathcal{Y} \). By Lemma 3.1.4 (1) and Lemma 3.2.5, the image of \( I \) in \( \mathcal{O}_{\mathcal{Y},I}(\tilde{Y}) \) is nonzero. Thus the image of \( I \) in \( \mathcal{O}_{\mathcal{Y},\tilde{f}^{-1}(y)} \) is nonzero. So the image of \( I \) in \( \mathcal{O}_{\mathcal{Y},\tilde{f}^{-1}(y)} \) is nonzero. Claim 1 is proved.

Now we prove Claim 2. Indeed, by Lemma 3.1.4 (1) (and the definition of pullback), the left (resp. right) hand side of (3.4) is defined by the closure of \( I \mathcal{O}_{\mathcal{Y},\tilde{f}^{-1}(y)} \) in \( \mathcal{O}_{\mathcal{Y},\tilde{f}^{-1}(y)} \) (resp. the closure of \( I \mathcal{O}_{\mathcal{Y},\tilde{f}^{-1}(x)} \) in \( \mathcal{O}_{\mathcal{Y},\tilde{f}^{-1}(x)} \)), where \( \mathcal{O}_{\mathcal{Y},\tilde{f}^{-1}(y)} \) (resp. \( \mathcal{O}_{\mathcal{Y},\tilde{f}^{-1}(x)} \)) becomes an \( \mathcal{O}_{\mathcal{Y}}(\mathcal{Y}) \)-module through the lower left (resp. upper right) part of the diagram.

Remark 3.5.3. We may remove Assumption 3.2.3 (4) and instead assume \( \mathcal{Y}_y \subset \bar{\mathcal{Y}}_y \) form some \( y \in \mathcal{Y}(k) \) in Lemma 3.5.2. Then the conclusion still holds by the same proof, except that Lemma 3.2.5 is not needed anymore.

Corollary 3.5.4. Let \( f: \mathcal{Y}' \to \mathcal{Y} \) be as is in Lemma 3.5.2. Assume that for some \( x \in \mathcal{Y}(k) \),

\( \mathcal{Y}_x \cap (\mathcal{Y}_x + (\mathcal{G}_x[p^r]\{\mathcal{G}_x[p^r-1]\})) \neq \mathcal{Y}_x \)

for some positive integer \( r \). There exists a positive integer \( s \) and a closed subscheme \( Y \subset \mathcal{Y}_k \) of \( \dim Y < \dim \mathcal{Y}_k \) such that for every \( z \in \mathcal{Y}(k) \setminus Y(k) \),

\( f^{-1}(\mathcal{Y}_z \cap (\mathcal{Y}_z + (\mathcal{G}_z[p^r]\{\mathcal{G}_z[p^r-1]\}))) \subset \mathcal{Y}'_{\tilde{f}^{-1}(z)}/\varpi^s \).

Proof. Up to enlarging \( F^c \), we may assume that \( F \) contains all \( p^r \)-th roots of unity. Let \( \mathcal{M}[p^r]\bar{\mathcal{Y}} \) be the union of \( h\bar{\mathcal{Y}} \)'s, \( h \in \mathcal{M}[p^r](F^c) \). By the assumption in the corollary, (3.1), and the two
facts recalled above Proposition 3.3.3, we can let \( \mathfrak{V} = \hat{\mathfrak{V}} \cap \mathfrak{M}[p^e] \hat{\mathfrak{V}} \) in Lemma 3.5.2. With \( s, Y \) as in Lemma 3.5.2, by (3.1) with \( x, y \) replaced by \( z, w \) which gives \( \hat{\mathcal{O}}_{\mathfrak{V}, w} \simeq \hat{\mathcal{O}}_{\mathfrak{V}, z} \), and the two facts recalled above Proposition 3.3.3 with \( y \) replaced by \( w \), the corollary follows.

\[ \square \]

Remark 3.5.5. By Remark 3.5.3, Assumption 3.2.3 (4) is not needed.

4. Frobenii and linearity

We introduce Frobenii on the Siegel and Igusa formal schemes. Then, we relate linearity and Shimura subvarieties, and discuss a new notion of linearity. We finally prove Theorem 1.2.2.

4.1. Lifts of Frobenii. Let \( \mathfrak{F} : X \to X \) be the canonical lifting of the endomorphism defined by the absolute Frobenius. Simply, on \( \hat{X} \), \( \mathfrak{F} \) is the \( p \) power morphism, and on \( \mathbb{X}_{\mathfrak{U}} \), \( \mathfrak{F} \) is the identity morphism. Then \( \ker \mathfrak{F} = \hat{X}[p] \), and we have the isomorphism \( X/\hat{X}[p] \simeq X \).

Let \( \phi_{\mathfrak{S}} : \mathfrak{S} \to \mathfrak{S} \) be the morphism given by the natural transformation \( A \mapsto A/\hat{A}[p] \) (on \( R \)-points for \( R \in \text{Nilp}_{\mathfrak{F}_p}^{\text{op}} \)). Then the restriction of \( \phi_{\mathfrak{S}} \) to \( \mathfrak{S}_{\mathfrak{F}_p} \) is the absolute Frobenius on \( \mathfrak{S}_{\mathfrak{F}_p} \).

Let \( \phi_\mathfrak{I} : \mathfrak{I} \to \mathfrak{I} \) be the morphism given by the natural transformation \( (A, \varepsilon) \mapsto (A/\hat{A}[p], \varepsilon') \) where \( \varepsilon' : A/\hat{A}[p] \simeq \mathbb{X}_{\mathfrak{R}}/\hat{\mathbb{X}}_{\mathfrak{R}}[p] \simeq \mathbb{X}_{\mathfrak{R}} \) is the induced isomorphism. Then the restriction of \( \phi_\mathfrak{I} \) to \( \mathfrak{I}_{\mathfrak{F}_p} \) is the absolute Frobenius on \( \mathfrak{I}_{\mathfrak{F}_p} \).

Similarly, we have \( \phi_{\mathfrak{S}, \mathfrak{I}} : \mathfrak{S}_0 \to \mathfrak{I}_0 \) that lifts the absolute Frobenius on \( \mathfrak{I}_{\mathfrak{F}_p} \). Then the natural projections \( \mathfrak{I} \to \mathfrak{S} \) are compatible with the liftings of the Frobenii in an obvious way.

Below, we will often use base changes of the Frobenii \( \phi_{\mathfrak{S}}, \phi_{\mathfrak{I}}, \) and \( \phi_{\mathfrak{I}_0} \). If there is no confusion, we will use \( \phi_{\mathfrak{S}}, \phi_{\mathfrak{I}}, \) and \( \phi_{\mathfrak{I}_0} \) to denote their base changes to lighten the notations. (We also recall that the base change of the absolute Frobenius on a \( \mathbb{F}_p \)-scheme is the relative Frobeni.)

For \( x \in \mathfrak{S}(k), \phi_{\mathfrak{S}}|_{\mathfrak{S}_x} \) factors through \( \mathfrak{S}_{\phi_{\mathfrak{S}}(x)} \), and we always understand \( \phi_{\mathfrak{S}}|_{\mathfrak{S}_x} \) as from \( \mathfrak{S}_x \) to \( \mathfrak{S}_{\phi_{\mathfrak{S}}(x)} \). Similarly, For \( y \in \mathfrak{I}_0(k), \phi_{\mathfrak{I}_0}|_{\mathfrak{I}_0,y} \) is to \( \mathfrak{I}_0,\phi_{\mathfrak{I}_0}(y) \).

Lemma 4.1.1 ([18, LEMMA 4.1.2 and p 171]). (1) For \( y \in \mathfrak{I}_0(k), c_{\phi_{\mathfrak{I}}(y)} \circ \phi_{\mathfrak{I}_0} \circ c_{y}^{-1} \) is the \( p \)-power endomorphism of the formal torus \( \mathfrak{M} \).

(2) For \( x \in \mathfrak{S}(k) \) and \( y \in \pi_0^{-1}(\{x\}) \), \( a_{\phi_{\mathfrak{S}}(y)} \circ \phi_{\mathfrak{S}} \circ c_{y}^{-1} \) is the \( p \)-power endomorphism of the formal torus \( \mathfrak{M} \).

For a positive integer \( m \) and an open formal subscheme \( \mathfrak{U} \) of \( \mathfrak{S}_{\mathfrak{F}_0} \), let \( \mathfrak{U}^{(m)} \) be the open formal subscheme of \( \mathfrak{S}_{\mathfrak{F}_0} \) supported on the topological image of \( \mathfrak{U}_k \) by \( \phi_{\mathfrak{S}}^m \). Then \( \phi_{\mathfrak{S}}^m|_{\mathfrak{U}} \) factors through \( \mathfrak{U}^{(m)} \). Below, we always understand \( \phi_{\mathfrak{S}}^m|_{\mathfrak{U}} \) as from \( \mathfrak{U} \) to \( \mathfrak{U}^{(m)} \).

Corollary 4.1.2. Let \( \mathfrak{V} \) be a closed formal subscheme of an open formal subscheme \( \mathfrak{U} \) of \( \mathfrak{S}_{\mathfrak{F}_0} \).

(1) For a positive integer \( m \), the formal completion of \( \phi_{\mathfrak{S}}^m|_{\mathfrak{U}}(\mathfrak{V}) \) at \( \phi_{\mathfrak{S}}^m(x) \) is \( \phi_{\mathfrak{S}}^m|_{\mathfrak{S}_x}(\mathfrak{V}_x) \).

(2) If the formal completion in (1) has a formal branch (see Definition B.0.4) that is a translated formal subtorus, then \( \mathfrak{V}_x \) has a formal branch that is a translated formal subtorus whose schematic image by \( \phi_{\mathfrak{S}}^m|_{\mathfrak{S}_x,\mathfrak{F}_0} \) is the former one.

Proof. (1) Let \( i : \mathfrak{V} \to \mathfrak{U} \) (resp. \( j : \mathfrak{U}_x \to \mathfrak{U}_x \)) be the natural morphism which is proper (resp. flat). Apply Lemma 3.1.1 with \( \mathfrak{Y} = \mathfrak{V}, \mathfrak{X} = \mathfrak{U}, \mathfrak{X}' = \mathfrak{U}_x, f = \phi_{\mathfrak{S}}^m|_{\mathfrak{U}} \circ i \) and \( g = j \) and note that \( \phi_{\mathfrak{S}_k} \) is a homeomorphism.

(2) By Lemma 4.1.1 (2), the schematic preimage of a translated formal subtorus by \( \phi_{\mathfrak{S}}^m|_{\mathfrak{S}_x,\mathfrak{F}_0} \) is a union of translated formal subtori of the same dimension. Then (2) follows from (1) and Lemma 3.1.2 (3). \[ \square \]
4.2. More on linearity.

4.2.1. Linearity and speciality. The notion “weakly linear” was used by Chai [7, (5.3)] over $k$. For our purpose, it should be understood as modeled on the (equivalent) alternative “weakly special” of “geodesic” in the complex algebraic setting, see [25] for the definition and equivalence. And weakly special subvarieties of $S_C$ are in fact defined over $\overline{Q}$.

Let us recall two stronger notions. First, for a formal subscheme $V$ of $S_{F^o}$ and $x \in V(k)$, we call $V$ quasi-linear at $x$ if $V_x \subset S_{F^o,x}$ is a union of translations by torsion points of formal subtori of $S_{F^o,x}$ (following Moonen [26]). We call $V$ quasi-linear if $V_x$ is quasi-linear for every $x \in V(k)$. Second, a subvariety of $S_{\overline{Q}}$ is special if it is an irreducible component of a Hecke translation of a Shimura subvariety.

It is natural to ask about the relation between linearity, which is an analytic notion, and speciality, which is an algebraic notion. Assume that $V$ come from a closed subvariety $V$ of $S_{\overline{Q}}$, i.e., $V$ is an open formal subscheme of the $p$-adic formal completion of the Zariski closure of $V$ in $S_{F^o}$ if $V$ is defined over a subfield of $F$. If $V$ is weakly special, then $V$ is quasi-linear. This is a result of Noot [27]. Conversely, if for some $x \in V(k)$, $V_x$ is quasi-linear, then $V$ is special. This is a result of Moonen [26]. Two natural (expected) generalizations of this result would be: (1) replace “quasi-linear” by “weakly linear”; (2) the algebraicity of quasi-linear/weakly linear formal subschemes. See [7] for a partial result about (2) on a product of Hilbert modular schemes. We hope to return to these generalizations in the future.

4.2.2. Quasi-linearity. We have a “quasi-linear” version of Conjecture 1.2.1.

**Conjecture 4.2.1.** Assume that $V$ is a closed formal subscheme of an open formal subscheme $U$ of $S_{F^o}$, $x \in V(k)$ and $T \subset S_{F^o,x}$ a formal subtorus translated by a torsion point. If $V$ is the schematic image of $T \to U$, then $V$ is quasi-linear.

Theorem 1.2.2 follows from the following theorem on Conjecture 4.2.1.

**Theorem 4.2.2.** Conjecture 4.2.1 holds if $V$ is unibranch without embedded points and $T$ contains a torsion point.

The following analog of Theorem 1.2.3 (2) for quasi-linearity is a special case of Theorem 4.2.2.

**Corollary 4.2.3.** Let $V$ be a weakly linear formal subscheme of $S_{F^o}$ such that $V$ is unibranch and has no embedded points. If $V$ is quasi-linear at $x$, then it is quasi-linear.

Recall that we also have the notion “linear” in 3.3. Now we discuss the relation between the notions “linear”, “quasi-linear” and “weakly linear”.

**Proposition 4.2.4.** Let $V$ be a closed formal subscheme of an open formal subscheme $U$ of $S_{F^o}$. Assume that $V$ is reduced and flat over $F^o$, and $V_k$ has no embedded points. Assume that there exists a positive integer $m$ such that $V := \phi_{U}^{[m]}(V)$ is linear, then $V$ is quasi-linear.

*Proof.* Let $Z$ be the underlying reduced subscheme of $V_k$. We show that $Z$ is linear as follows. By the excellence and reducedness of $Z$, for $x \in Z(k)$, $Z_x$ is reduced. Since the relative Frobenius induces (a homeomorphism on $S_k$ and) homeomorphisms on the spectra of the completions of $[6]$As pointed out by Gao, [43, Theorem 1.4] implies the equivalence without algebraicity.
the local rings of $\mathcal{S}_k$, $Z_x$ has the same number of formal branches with $\mathcal{W}_k, \phi_{\mathcal{E}}(x)$, which is 1 by the linearity of $\mathcal{W}_k'$ at $\phi_{\mathcal{E}}(x)$. Then by Lemma 4.1.1 (2), the linearity of $\mathcal{W}_k'$ at $\phi_{\mathcal{E}}(x)$ and dimension reason (see Lemma 3.1.2 (3)), $Z$ is linear.

We may assume that $\mathcal{V}$ is connected. As in the proof of Corollary 3.4.3, we may assume that $\mathcal{V}$ is affine. By Lemma 3.1.1 and Remark 3.4.2, we may enlarge $F$ and assume that $F$ contains all $p^n$-th roots of unity. Now we can apply the discussion in 3.2. We have $\mathcal{V}$ and the analogs $\mathcal{W}'$ and $\mathcal{Z}_{\text{can}}$ in $\mathcal{J}_0,F_0$ (where $\mathcal{Z}_{\text{can}}$ is as in Lemma 3.4.1) which we can choose so that $\mathcal{W}, \mathcal{Z}_{\text{can}} \subset \phi_{\mathcal{J}_0}^{-m}(\mathcal{V}')$, since the natural projections $\pi_0$ is compatible with the liftings of the Frobenii. By Lemma 2.4.1 and Lemma 4.1.1 (1), $\phi_{\mathcal{J}_0}^{-m}(\mathcal{V}')$ is the union of $h\mathcal{Z}_{\text{can}}, h \in m,F_0$. Claim: $\mathcal{V}$ is the union of some $h\mathcal{Z}_{\text{can}}$’s. Then the lemma follows from Lemma 2.4.1 (about the action of $m,F_0$ on $\mathcal{J}_0$), $\mathcal{O}_{\mathcal{V},x} \simeq \mathcal{O}_{\mathcal{V},x}$ (see (3.1)) and $\pi_0|_{\mathcal{J}_0,y} \simeq \mathcal{S}_x$ of formal tori for all $x \in \mathcal{W}(k)$ and $y \in \pi_0^{-1}(\{x\}) \cap \mathcal{V}$.

To prove the claim, instead of the the non-noetherian formal schemes, we work on their completions which are noetherian. Take $x \in \mathcal{W}(k)$ and $y \in \pi_0^{-1}(\{x\}) \cap \mathcal{V}$. Then the excellence of $\mathcal{V}, \mathcal{V}_y \simeq \mathcal{V}_x$ (see (3.1)) is reduced. Thus $\mathcal{V}_y$ is the union of its formal branches. And the formal branches are flat over $F^\circ$ (by the flatness of $\mathcal{V}_y$ and that Spf $F^\circ$ is integral and regular of dimension 1, see [13, III.9.7]). For $h \in m,F_0$, $\mathcal{J}_h = (\mathcal{V} \cap h\mathcal{Z}_{\text{can}})_y$ is a union of some formal branches of $\mathcal{V}_y$. We only need to prove that $\mathcal{J}_h \neq \emptyset, \mathcal{J}_h = h\mathcal{Z}_{\text{can}}$. (Then by Lemma 3.2.5, $\mathcal{V}$ is the union of some $h\mathcal{Z}_{\text{can}}$’s, and thus quasi-linear.) Assume that $\mathcal{J}_h \neq h\mathcal{Z}_{\text{can}}$, by the flatness, $\mathcal{J}_{h,k} \neq (h\mathcal{Z}_{\text{can}})_k = Z_{\text{can}}$. Then $(\mathcal{V} \cap h\mathcal{Z}_{\text{can}})_k \neq Z_{\text{can}}$. Since $Z_k$ is the inverse limit of reduced irreducible schemes (see 3.2), the image of $(\mathcal{V} \cap h\mathcal{Z}_{\text{can}})_k$ in $Z_k$ is a strictly smaller closed subscheme. An irreducible component of this strictly smaller closed subscheme gives an embedded component of $\mathcal{V}_k$, contradiction. □

Remark 4.2.5. (1) The proof indeed gives us stronger linearity of $\mathcal{V}$. First, $\mathcal{W}_x \subset \mathcal{G}_{F^\circ,x}$ is a union of translations by a set $T_x$ torsion points of a single formal subtorus of $\mathcal{G}_{F^\circ,x}$. Moreover, for $x, x'$, there exists an isomorphism $\mathcal{G}_{F^\circ,x} \simeq \mathcal{G}_{F^\circ,x'}$ such that $T_x \simeq T_{x'}$ under this isomorphism.

(2) In Theorem 4.2.2, Proposition 4.2.4, Corollary 4.2.3 and Corollary 4.2.6, we also have this stronger linearity of $\mathcal{V}$ of (1).

The following corollary is an analog of the fact that a weakly special subvariety is special if and only if it contains a special (i.e., CM) point. It will not to be used later. Its proof is close to the proof of Theorem 4.2.2 (in the end of this section), and thus omitted.

Corollary 4.2.6. Let $\mathcal{W}$ be a weakly linear reduced formal subscheme of $\mathcal{G}_{F^\circ}$ such that $\mathcal{W}_k$ is unibranch and has no embedded points. If for some $x \in \mathcal{W}(k), \mathcal{W}_x$ contains a torsion point of $\mathcal{G}_{F^\circ,x}$, then $\mathcal{W}$ is quasi-linear.

Remark 4.2.7. If the unibranch assumption in Proposition 3.3.3 is removable, so is the one in Theorem 4.2.2, Proposition 4.2.4, Corollary 4.2.3 and Corollary 4.2.6.

4.3. Proof of Theorem 1.2.2. We need an extra lemma.

Lemma 4.3.1. Let $\mathcal{F} \subset \mathcal{G}_{m,F^\circ}^n$ be a reduced closed formal subscheme that is flat over $F^\circ$ and stabilized by the $p^n$-power endomorphism of $\mathcal{G}_{m,F^\circ}^n$, which we denote by $p^n$ for short, for some
positive integer $c$. Then $\mathfrak{Z}$ contains a union of formal subtori of $\hat{G}^n_{m,F^o}$ that contains $\mathfrak{Z}'_k$, the underlying reduced formal subscheme of $\mathfrak{Z}_k$.

**Proof.** First, we find a formal subtorus in $\mathfrak{Z}$. Since $p^c$ induces the relative Frobenius on $\text{Spec} \mathcal{O}_{\mathfrak{Z}_k}(\mathfrak{Z}_k)$ which is a homeomorphism, it permutes the branches of $\mathfrak{Z}_k$. Then a branch $\mathcal{C}$ of $\mathfrak{Z}_k$ is stabilized by $p^{c_1}$ for some positive integer $c_1$. Let $\hat{\mathcal{C}}$ be a formal branch of $\mathfrak{Z}$ such that $\mathcal{C} \subset \mathcal{C}_k$. The (finite) union of the schematic images of $\hat{\mathcal{C}}$ by non-negative powers of $p^{c_1}$ is stabilized by $p^{c_1}$. Then one of them, say $\mathcal{B}$, satisfies $p^{c_1}\mathfrak{Z}^1(\mathcal{B}) \subset \mathfrak{B}$ for a positive integer $c_2$. A result of de Jong [9] implies that $\mathcal{B}$ is a formal subtorus. Note that $\mathcal{C} \subset \mathcal{B}_k$. The lemma follows. \hfill $\square$

Now we prove Theorem 4.2.2, and thus Theorem 1.2.

**Proof of Theorem 4.2.2.** Let $m$ be a positive integer such that $\phi^m_{\mathfrak{E}|x,F^o}(\mathfrak{Z})$ is a formal subtorus of $\mathfrak{G}_{\phi^m_{\mathfrak{E}}(x),F^o}$ (which is possible by Lemma 4.1.1). Let $\mathfrak{W}'$ be the schematic image of $\phi^m_{\mathfrak{E}|x,F^o}(\mathfrak{Z}) \to \mathfrak{W}^m(\mathfrak{Z})$ (as defined below Lemma 4.1.1). Since $\mathfrak{W} \subset \phi^m_{\mathfrak{E}|x,F^o}(\mathfrak{Z})$ and $\phi^m_{\mathfrak{E}|x,F^o}(\mathfrak{Z}) \supset \mathfrak{W}'$ by the definition of schematic image, $\phi^m_{\mathfrak{E}|x,F^o}(\mathfrak{Z}) = \mathfrak{W}'$. Similarly, let $n$ be a positive integer such that $\phi^n_{\mathfrak{E}}$ stabilizes $\phi^n_{\mathfrak{E}}(x)$ so that $\phi^n_{\mathfrak{E}|x,F^o}(\mathfrak{Z})$ stabilizes $\phi^n_{\mathfrak{E}}(x)$. Then $\phi^n_{\mathfrak{E}}$ stabilizes $\mathfrak{W}'$.

Let $Z$ be the underlying reduced subscheme of $\phi^m_{\mathfrak{E}|x,F^o}(\mathfrak{Z}_k)$, equivalently, of $\mathfrak{W}'_k$ (see Lemma 3.1.2 (1)). By the excellence of $Z$, for $z \in Z(k)$, $Z_z$ is reduced. Since the relative Frobenius induces a homeomorphism on $\mathfrak{S}_k$ and homeomorphisms on the spectra of the completions of the local rings of $\mathfrak{S}_k$, $Z_z$ has only one formal branch, and thus is integral. Claim: $Z$ is linear and $Z^\text{can} = \mathfrak{W}'$, where $Z^\text{can}$ is as in Lemma 3.4.1. Then since $\mathfrak{W}$ is reduced by the definition of schematic image and the reducedness of $\mathfrak{Z}$, the theorem follows from Proposition 4.2.4.

Now we prove the claim. By the definition of schematic image, $\phi^m_{\mathfrak{E}|x,F^o}(\mathfrak{Z})$ is reduced, and then $\mathfrak{W}'$ is reduced. By the excellence of $\mathfrak{W}'$, $\mathfrak{W}'_k$ is reduced for every $z \in \mathfrak{W}'(k)$. For $z \in \mathfrak{W}'(k)$ stabilized by $\phi^m_{\mathfrak{E}}$, $\mathfrak{W}'_z$ is reduced for every $z \in \mathfrak{W}'(k)$. By Lemma 4.3.1 and that $Z_z$ is integral, $Z$ is linear at $z$ and $\mathfrak{W}'_z$ contains the unique formal subtorus of $\mathfrak{G}_{F^o,z}$ whose reduction is $Z_z$ (see Remark 3.4.2). Let $Z^\text{can}$ be as in Lemma 3.4.1. Then $Z^\text{can}_z$ is this unique formal subtorus. So $Z^\text{can} \subset \mathfrak{W}'$. Since $\phi^n_{\mathfrak{E}|x,F^o}(\mathfrak{Z}_k)$ is contained in the formal completion of $\phi^m_{\mathfrak{E}|x,F^o}(\mathfrak{Z}_k)$ at $\phi^m_{\mathfrak{E}}(x)$, it is contained $Z^\text{can}_z$. So $\phi^n_{\mathfrak{E}|x,F^o}(\mathfrak{Z}) \subset Z^\text{can}_z$. So $\mathfrak{W}' \subset Z^\text{can}$. \hfill $\square$

5. More notions

5.1. Hecke action. Let $\mathbb{A}_f$ be the ring of finite-outside-$p$ adeles of $\mathbb{Q}$. For a principally polarized $g$-dimensional abelian scheme $A$ over a connected $\mathbb{Z}_p$-scheme, a similitude infinite prime-to-$p$ level structure is a similitude symplectic isomorphism

$$\mathbb{A}_f^{\log} \simeq \lim_{\substack{\longrightarrow \cr N, p \nmid N}} A[N] \otimes \mathbb{Q}$$

fixed by the fundamental group of the $\mathbb{Z}_p$-scheme (this condition is independent of the choice of the base point for defining the fundamental group). The functor assigning to $R \in \text{Nilp}_{\mathbb{Z}_p}^\text{op}$ the set of prime-to-$p$ isogeny classes of principally polarized ordinary abelian $R$-schemes with infinite level structure is representable by

$$\mathfrak{S} = \lim_{\substack{\longrightarrow \cr N, p \nmid N}} \mathfrak{S}_N.$$
It is equipped with a natural $\text{GSp}_{2g}(\mathbb{A}_f^p)$-action. Similarly, we have the infinite prime-to-$p$ level version $\tilde{\mathcal{I}}_0$ of $\mathcal{I}_0$ with a $\text{GSp}_{2g}(\mathbb{A}_f^p)$-action. (We do not need the one of $\mathcal{I}$.) It is easy to check that the natural projection $\pi_0 : \tilde{\mathcal{I}}_0 \to \tilde{\mathcal{G}}$ is $\text{GSp}_{2g}(\mathbb{A}_f^p)$-equivariant. Moreover, we have the following commutative (in fact Cartesian) diagram

$$\begin{array}{ccc}
\tilde{\mathcal{I}}_0 & \xrightarrow{\pi_0} & \tilde{\mathcal{G}} \\
pr_{\mathcal{I}_0} \downarrow & & \downarrow pr_\mathcal{G} \\
\mathcal{I}_0 & \xrightarrow{\pi_0} & \mathcal{G}
\end{array}$$

where the vertical morphisms are natural projections.

The action of the formal deformation torus $\mathcal{M}$ on $\mathcal{I}_0$, defined in 2.4, obviously lifts to an action of $\mathcal{M}$ on $\tilde{\mathcal{I}}_0$. It is direct to check that the action of $\mathcal{M}$ on $\tilde{\mathcal{I}}_0$ commutes with the $\text{GSp}_{2g}(\mathbb{A}_f^p)$-action, and the actions of $\mathcal{M}$ on $\tilde{\mathcal{I}}_0$ and $\mathcal{I}_0$ commute with the natural projection $pr_{\mathcal{I}_0}$. Thus for $h \in \mathcal{M}(W)$ and $T \in \text{GSp}_{2g}(\mathbb{A}_f^p)$, we have

$$\pi_0 \circ h \circ pr_{\mathcal{I}_0} \circ T = \pi_0 \circ pr_{\mathcal{I}_0} \circ h \circ T = pr_\mathcal{G} \circ \pi_0 \circ T \circ h = pr_\mathcal{G} \circ T \circ \pi_0 \circ h.$$ (It can be displayed in a $3 \times 3$ commutative diagram, which is left to the reader.) Replacing $W$ by $F^o$, (5.2) still holds after base change.

For $z \in \tilde{\mathcal{I}}_0(k)$ or $\tilde{\mathcal{G}}(k)$, similar to (2.6) and the discussion below it, we have a formal torus structure on the formal residue disc at $z$. It is directly to check that $pr_\mathcal{G}$, $pr_{\mathcal{I}_0}$ (thus all morphisms in (5.1)) are isomorphisms of formal tori when restricted to each formal residue torus.

For $z \in \tilde{\mathcal{I}}_0(k)$, $\tilde{\mathcal{G}}(k)$, $\mathcal{I}_0(k)$ or $\mathcal{G}(k)$, we use $z^{\text{can}}$ to denote the canonical lifting of $z$, which is by definition the unit of the formal residue torus at $z$. Then all morphisms in (5.1) commutes with taking canonical lifting. It is also direct to check that the $\text{GSp}_{2g}(\mathbb{A}_f^p)$-action commutes with taking canonical lifting.

Finally, we define the notion “Hecke–Frobenius orbit”. We first define “prime-to-$p$ Hecke orbit”. For $x \in \mathcal{G}(F^o)$, $pr_{\mathcal{G}}^{-1}(\{x\})$ is a union of $F^o$-points, since $pr_{\mathcal{G}}$ is an isomorphism when restricted to each formal residue disc. Define the prime-to-$p$ Hecke orbit of $x$ in $\mathcal{G}(F^o)$ to be the image by $pr_{\mathcal{G}}$ of the $\text{GSp}_{2g}(\mathbb{A}_f^p)$-orbit of some $F^o$-point $\tilde{x} \in pr_{\mathcal{G}}^{-1}(\{x\})$ in $\tilde{\mathcal{G}}(F^o)$. Clearly, this definition does not depend on the choice of the lift $\tilde{x}$. Then the Hecke–Frobenius orbit of $x$ is the union of the images of its prime-to-$p$ Hecke orbit by $\phi_{\mathcal{G}}^n$, $n \in \mathbb{Z}$. The same discussion applies to $pr_{\mathcal{I}_0}$.

### 5.2. Perfectoid Igusa space

Let $\mathcal{C}_p$ be the $p$-adic completion of $\mathcal{T}$ and $\mathcal{C}_p^t$ the $t$-adic completion of an algebraic closure of $k((t))$. Then $\mathcal{C}_p$ and $\mathcal{C}_p^t$ are perfectoid fields, and $\mathcal{C}_p^t$ is a tilt of $\mathcal{C}_p$ in the sense of Scholze [36]. We have the tilting equivalence [36] between the category of perfectoid spaces over $\text{Spa}(\mathcal{C}_p, \mathcal{C}_p^t)$ and $\text{Spa}(\mathcal{C}_p^t, \mathcal{C}_p^t)$. The image of an object or a morphism over $\text{Spa}(\mathcal{C}_p, \mathcal{C}_p^t)$ under the tilting equivalence is called its tilt.

Let $\mathcal{I}/\text{Spa}(\mathcal{C}_p, \mathcal{C}_p^t)$ be the adic generic fiber of $\mathcal{I}_{\mathcal{C}_p^t}$ in the sense of [37], as in [6]. By Proposition 2.2.1, $\mathcal{I}$ is a perfectoid space whose tilt $\mathcal{I}^t$ over $\text{Spa}(\mathcal{C}_p^t, \mathcal{C}_p^t)$ is the adic generic fiber of $\mathcal{I}_{\mathcal{C}_p^t}$. Then we have the tilting bijection (A.1):

$$\rho : \mathcal{I}(\mathcal{C}_p, \mathcal{C}_p^t) \simeq \mathcal{I}^t(\mathcal{C}_p^t, \mathcal{C}_p^t).$$
Let $\phi_I$ be the endomorphism of $I$ induced by the base change of $\phi_\mathcal{Z}$ to $\mathbb{C}_p$. Let $\phi_{I^o}$ be the endomorphism of $I^o$ induced by the base change of $\phi_\mathcal{Z}$ to $\mathbb{C}_p^o$ (along $\mathbb{Z}_p \to \mathbb{F}_p \to \mathbb{C}_p^o$). Then similar to (5.4), one can check that $\phi_I$ is the tilt of $\phi_{I^o}$. By [32, Lemma 2.3.1], we have

$$\rho \circ \phi_I = \phi_{I^o} \circ \rho.$$ 

(5.3)

For $x \in \mathcal{S}(k)$, let $A_{x,can} \in \mathcal{S}(W(k))$ corresponding to $x_{can}$. Fix an isomorphism $\varepsilon_{x,can} : A_{x,can}[p^\infty] \simeq X_W(k)$ with reduction $\varepsilon_x : A_x[p^\infty] \simeq X_k$. Then $(A_{x,can}, \varepsilon_{x,can}) \in \mathcal{I}(W(k))$ and $(A_x, \varepsilon_x) \in \mathcal{I}(k)$. We still use $(A_{x,can}, \varepsilon_{x,can})$ to denote the corresponding point in $\mathcal{I}(\mathbb{C}_p, \mathbb{C}_p^o)$ and use $(A_x, \varepsilon_x)$ to denote the corresponding point in $\mathcal{I}^o(\mathbb{C}_p^o, \mathbb{C}_p^o)$. Then by [36, Theorem 5.2] (explicated in [32, Lemma 2.3.2]), we have

$$\rho((A_{x,can}, \varepsilon_{x,can})) = (A_x, \varepsilon_x).$$

(5.4)

5.3. $p$-adic distance. Let $| \cdot |$ be the $p$-adic norm on $\overline{F}$. Let $\mathcal{X}$ be a formal scheme over $F^o$. For a formal subscheme $\mathfrak{Z} \subset \mathcal{X}$ and a point $x \in \mathcal{X}(\overline{F}^o)$, define

$$d(x, \mathfrak{Z}) = d_{\mathcal{X}}(x, \mathfrak{Z}) = \inf \{ |\delta| : \delta \in \overline{F}^o \text{ such that } (x \mod \delta) \in \mathfrak{Z} / \overline{F}^o / \delta \},$$

Here $\inf \emptyset := \infty$ so that $d(x, \mathfrak{Z}) < \infty$ if and only if the reduction of $x$ is in $\mathfrak{Z}(k)$. If $\mathfrak{Z}$ is closed in $\mathcal{X}$, the distance can be defined using the ideal defining $\mathfrak{Z}$, see for example [32, 2.2]. Define

$$\mathfrak{Z}_\epsilon = \{ x \in \mathcal{X}(\overline{F}^o) : d(x, \mathfrak{Z}) \leq \epsilon \}.$$ 

(5.5)

Directly from the definition, we have the following lemmas.

**Lemma 5.3.1.** Assume that $\mathfrak{Z} \subset \mathfrak{U} \subset \mathcal{X}$ as formal schemes, then for $x \in \mathfrak{U}(\overline{F}^o)$, $d_{\mathfrak{U}}(x, \mathfrak{Z}) = d_{\mathcal{X}}(x, \mathfrak{Z})$.

**Lemma 5.3.2.** Let $E/F$ be a finite extension. Then $d(x, \mathfrak{Z}_{E^o}) = d(x, \mathfrak{Z})$.

**Lemma 5.3.3.** Let $\mathfrak{Z}'$ be a formal subscheme of $\mathfrak{Z}$. Then $d(x, \mathfrak{Z}) \leq d(x, \mathfrak{Z}')$.

**Lemma 5.3.4.** Let $\mathfrak{Z} = \mathfrak{U} \cup \mathfrak{V}$, where $\mathfrak{U}, \mathfrak{V}$ are closed formal subschemes of $\mathfrak{Z}$. Assume that there exists $\delta \in F^o - \{0\}$ vanishing on $\mathfrak{V}$. Then $d(x, \mathfrak{U}) \leq d(x, \mathfrak{Z}) / |\delta|$.

**Lemma 5.3.5.** Let $\mathfrak{Z} = \mathfrak{U} \cap \mathfrak{V}$, where $\mathfrak{U}, \mathfrak{V}$ are closed formal subschemes of $\mathcal{X}$. Then $\mathfrak{U} \cap \mathfrak{V}_\epsilon = \mathfrak{Z}_\epsilon$.

**Lemma 5.3.6.** For a morphism $f : \mathcal{X} \to \mathfrak{Y}$ of formal scheme over $F^o$, and a formal subscheme $\mathfrak{Y}$ of $\mathfrak{Y}$, we have $d_{\mathfrak{Y}}(f(x), \mathfrak{Y}) = d_{\mathcal{X}}(x, f^{-1}(\mathfrak{Y}))$.

Assume that $f : \mathcal{X} \to \mathfrak{Y}$ is a morphism of locally noetherian formal schemes, so that the schematic image $f(\mathfrak{Z})$ is defined. Note that $\mathfrak{Z}$ is a formal subscheme of $f^{-1}(f(\mathfrak{Z}))$. Then Lemma 5.3.3 and Lemma 5.3.6 imply the following corollary.

**Corollary 5.3.7.** We have $d(f(x), f(\mathfrak{Z})) \leq d(x, \mathfrak{Z})$.

6. Bounded ramification

We prove Theorem 1.3.2 (1). The reader may refer to 1.4 for an overview of the proof.
6.1. Canonical liftings on $\mathcal{I}_0$.

**Proposition 6.1.1.** Let $\mathfrak{M}$ be a formal subscheme of $\mathcal{I}_{0,F^\omega}$ and $\{y_n\}_{n=1}^\infty$ a sequence in $\mathfrak{M}_0(k)$ such that $d(y_n^\text{can}, \mathfrak{M}) \to 0$. Let $Y \subset \mathfrak{M}_k$ be a closed subscheme such that

1. for every integer $N > 0$, $Y$ is contained in the Zariski closure of $\{y_n\}_{n=1}^\infty$;
2. there exists $y \in Y(k)$ and a positive integer $c$ such that the formal completion $Y_y$ is reduced and satisfies $\phi^c_{\mathcal{I}_0}(Y_y) \subset Y_y$ (in particular, $\phi^c_{\mathcal{I}_0}(y) = y$).

Then $\mathfrak{M}_y$ contains a union of formal subtori of $\mathcal{I}_{0,y}$ that contains $Y_y$ as a closed formal subscheme.

**Proof.** Let $\pi : \mathcal{I} \to \mathcal{I}_0$ be the natural morphism as in (2.2), and $\pi^\flat$ the base change of $\pi$ to $\mathcal{C}_p^\text{co}$ (along $\mathbb{Z}_p \to \mathbb{F}_p \to \mathcal{C}_p^\text{co}$). Still use $\pi$ to denote the corresponding morphism between $\mathcal{I}$ and the adic generic fiber of $\mathcal{I}_0$. Since $d(y_n^\text{can}, \mathfrak{M}) \to 0$, by (5.4) and Lemma A.3.2 (with Assumption A.3.1 locally assured by Proposition 2.2.1), $\pi \left( (\pi^\flat \circ \rho)^{-1}(Y(\mathcal{C}_p^\text{co})) \right)$ consists of adic generic fibers of points in $\mathfrak{M}(\mathcal{C}_p^\text{co})$. Note that taking adic generic fibers induces a bijection $\mathcal{I}(\mathcal{C}_p^\text{co}) \simeq \mathcal{I}(\mathcal{C}_p^\text{co})$ (see [32, Lemma 2.1.3]). Let $Y_y^\sharp \subset \mathfrak{M}_y(\mathcal{C}_p^\text{co})$ correspond to $\pi \left( (\pi^\flat \circ \rho)^{-1}(Y_y(\mathcal{C}_p^\text{co})) \right)$.

Recall that $\pi^\flat$ is compatible with the Frobenii and $\pi$ is compatible with the liftings of the Frobenii. Then by condition (2) on $y$ and (5.3) (which says $\rho$ is compatible with the $\phi_I, \phi^\dagger$), we have $\phi^c_{\mathcal{I}_0}(Y_y^\sharp) \subset Y_y^\sharp$. Let $\mathfrak{Y} \subset \mathcal{I}_{0,y,F^\omega}$ be the schematic closure of $Y_y^\sharp$. Then $\mathfrak{Y}$ is reduced and flat over $\mathcal{C}_p^\text{co}$, and $\phi^c_{\mathcal{I}_0}(\mathfrak{Y}) \subset \mathfrak{Y}$. Then by Lemma 4.3.1, it is enough to show that $Y_y \subset \mathfrak{Y}$.

Now we show that $Y_y \subset \mathfrak{Y}$. By the construction of $\mathfrak{Y}$, the image of the reduction map $Y_y(\mathcal{C}_p^\text{co}) \to Y_y(\mathcal{C}_p^\text{co}/t)$ is contained in the image of $\mathfrak{Y}(\mathcal{C}_p^\text{co}/\pi) = \mathfrak{Y}(\mathcal{C}_p^\text{co}/t)$. We only need to show that the image of the reduction map $Y_y(\mathcal{C}_p^\text{co}) \to Y_y(\mathcal{C}_p^\text{co}/t)$ is schematically dense in $Y_y$, i.e., not contained in a closed formal subscheme of $Y_y$ strictly smaller than $Y_y$. Since $Y_y$ is reduced by assumption, it is enough to show this for each formal branch of $Y_y$, and thus we may assume that $Y_y$ is integral. By the Noether normalization lemma for complete local domains [41, Lemma 032D], $Y_y = \text{Spf } R'$ such that $R'$ is finite over a subring $R \simeq k[[x_1, \ldots, x_n]]$. We only need to show that the image of the reduction map $\text{Spf } R(\mathcal{C}_p^\text{co}) \to \text{Spf } R(\mathcal{C}_p^\text{co}/t)$ is schematically dense in $\text{Spf } R$. This to show this, let $f \in R$ and $f \neq 0$. There exists a continuous homomorphism $R \to \mathcal{C}_p^\text{co}$ such that the image of $f$ is nonzero (see Lemma 3.3.2). Then for some large positive integer $m$, $\phi^{1/p^m}(f) \notin \mathcal{C}_p^\text{co}/t$ Point $\phi^{1/p^m}/t$ is not in the zero locus of $f$. □

**Remark 6.1.2.** In the proof, we can not take schematic closure in $\mathcal{I}_{0,F^\omega}$ due to its non-noetherianness.

6.2. Unramified points on Siegel moduli schemes. We first deal with prime-to-$p$ Hecke orbits.

**Lemma 6.2.1.** Let $\mathfrak{Y}$ be a formal subscheme of $\mathfrak{S}_{F^\omega}$. Let $\{x_1\}_{n=1}^\infty$ be a sequence in the prime-to-$p$ Hecke orbit of $x_1 \in \mathfrak{S}(F^\omega)$ such that the reduction $\overline{x_n}$ of $x_n$ is in $\mathfrak{M}(k)$ and $d(x_n, \mathfrak{Y}) \to 0$. Let $Z$ be the Zariski closure of $\{\overline{x_n}\}_{n=1}^\infty$ in $\mathfrak{M}_k$. For $x \in Z(k)$, $\mathfrak{M}_x$ contains a union of translated formal subtori that contains $Z_x$ as a closed formal subscheme.

**Proof.** Up to replacing $\{x_n\}_{n=1}^\infty$ by a subsequence, we may assume that $Z$ is the Zariski closure of $\{\overline{x_n}\}_{n=1}^\infty$. We want apply Proposition 6.1.1 to prove the lemma. We have four steps.
First, we construct \( \{y_n\}_{n=1}^{\infty} \) and \( \mathfrak{M} \). Choose \( y_1 \in \mathfrak{I}(k) \) such that \( \pi_0(y_1) = \overline{\eta}_1 \). By the isomorphism \( \pi_0|_{\mathfrak{I}_0(y_1)} : \mathfrak{I}_0(y_1) \cong \mathcal{G}_1 \) of formal tori and the \( \mathfrak{M} \)-action on \( \mathfrak{I}_0(y_1) \) described in Lemma 2.4.1, there exists \( h \in \mathfrak{M}(F^c) \) such that \( \pi_0(hy_1^{\text{can}}) = x_1 \). Let \( \overline{y}_1 \in \text{pr}_0^{-1}(y_1) \). Then \( \text{pr}_0(\overline{y}_1^{\text{can}}) = y_1^{\text{can}} \), since \( \text{pr}_0 \) commutes with taking canonical lifting. Let \( \overline{\eta}_1 = \overline{\pi_0(hy_1^{\text{can}})} \). By (6.1) and that the actions of \( \mathfrak{M} \) on \( \mathfrak{I}_0 \) and \( \mathfrak{I}_0(y_1) \) commute with the natural projection \( \text{pr}_0 \), we have
\[
\text{pr}_0(\overline{\eta}_1) = \pi_0 \circ \text{pr}_0(hy_1^{\text{can}}) = \pi_0 \circ (h \text{pr}_0(\overline{y}_1^{\text{can}})) = \pi_0(hy_1^{\text{can}}) = x_1.
\]
Thus, there exists \( T_n \in \text{GSp}_{2g}(K^p_f) \) such that \( x_n = \text{pr}_0(T_n \overline{\eta}_1) \). Let \( y_n = \text{pr}_0(T_n \overline{y}_1) \). Then
\[
y_n^{\text{can}} = \text{pr}_0((T_n \overline{y}_1)^{\text{can}}) = \text{pr}_0(T_n \overline{y}_1^{\text{can}})
\]
since \( \text{pr}_0 \) and \( T_n \) commute with taking canonical lifting. Then by (5.2), we have
\[
\pi_0(hy_n^{\text{can}}) = \pi_0 \circ h \circ \text{pr}_0 \circ T_n(\overline{y}_1^{\text{can}}) = \text{pr}_0 \circ T_n \circ \overline{\pi_0(hy_1^{\text{can}})} = \text{pr}_0 \circ T_n(\overline{\eta}_1) = x_n.
\]
Let \( \mathfrak{M}' = \pi_0^{-1}(\mathfrak{M}) \). Then \( d(hy_n^{\text{can}}, \mathfrak{M}') = d(x_n, \mathfrak{M}) \rightarrow 0 \). Let \( \mathfrak{M}' = h^{-1} \mathfrak{M}' \). Then \( d(y_n, \mathfrak{M}) \rightarrow 0 \).

Second, we construct \( Y \). Recall \( f_{i,j} : \mathfrak{G}_j \rightarrow \mathfrak{G}_i \) and \( \pi_1 : \mathfrak{I}_0 \rightarrow \mathfrak{G}_i \) defined in 2.2. Claim: there exists a sequence \( \{Z_i\}_{i=0}^{\infty} \) such that \( Z_0 \) is an irreducible component of \( Z \), and \( Z_{i+1} \) is an irreducible component of \( f_{i+1,i}(Z_i) \) containing a Zariski dense subset of points in \( \{\pi_i(y_n)\}_{n=1}^{\infty} \).

Assuming the claim, let \( Y = \lim_{n \geq 0} Z_i \). Then condition (1) in Proposition 6.1.1 on \( Y \) follows easily from Lemma B.0.1. The proof of the claim is inductive and as follows. Assume that we have \( Z_i \) as desired. The image of
\[
\{\pi_{i+1}(y_n)\}_{n=1}^{\infty} \cap f_{i+1,i}^{-1}(Z_i).
\]
by \( f_{i+1,i} \) is \( \{\pi_i(y_n)\}_{n=1}^{\infty} \cap Z_i \), and Zariski dense in \( Z_i \) by inductive hypothesis. Thus the Zariski closure of (6.1) contains an irreducible component of \( f_{i+1,i}^{-1}(Z_i) \), which we choose to be \( Z_{i+1} \).

Third, for \( y \in \pi_0^{-1}(\{x\}) \cap Y \), condition (2) in Proposition 6.1.1 holds. Indeed, by the first part of Proposition B.0.9, \( \pi_0|_{\mathfrak{G}_y} \) is an isomorphism to a union \( Z'_i \) of formal branches of \( Z_x \). Then since \( Z \) is of finite type over \( k \), there exists a positive integer such that \( Z \) is defined over the finite field \( F_{p^\ell} \) and \( x \in Z(F_{p^\ell}) \). Then \( \phi_{\mathfrak{G}_y}^\ell(Z_x) \subset Z_x \). By the excellence and reducedness of \( Z \), \( Z_x \) is reduced. So \( Y_y \) is reduced. Moreover, by a similar (and simpler) argument as in the proof of Lemma 4.3.1, \( Z'_i \) is stabilized by a power of \( \phi_{\mathfrak{G}_y}^\ell \). Since \( \psi_0 \) is compatible with the liftings of Frobeni, \( Y_y \) is stabilized by the same power \( \phi_{\mathfrak{G}}^\ell \).

Finally, by the second part of Proposition B.0.9, we can choose finitely many \( y \)'s as in the third step, such that every formal branch of \( Z_x \) is contained in some \( \pi_0|_{\mathfrak{G}_y} \). The lemma now follows from Proposition 6.1.1. \( \square \)

Now we slightly improve Lemma 6.2.1, by including canonical liftings, as follows. On a power of \( \mathfrak{G} \), we can define prime-to-\( p \) Hecke action and canonical liftings in the obvious way.

**Lemma 6.2.2.** For non-negative integers \( a, b \), let \( \mathfrak{G}^{a+b} = \mathfrak{G}^a \times \mathfrak{G}^b \). Let \( \mathfrak{M} \) be a formal subscheme of \( \mathfrak{G}^{a+b} \). Let \( \{u_n\}_{n=1}^{\infty} \) be a sequence in the prime-to-\( p \) Hecke orbit of \( u_1 \in \mathfrak{G}^{a+c}(F^c) \) and \( \{v_n\}_{n=1}^{\infty} \) be a sequence of canonical liftings in \( \mathfrak{G}^b \). Assume that the reduction \( \overline{x}_n \) of \( x_n \) is in \( \mathfrak{M}(k) \) and \( d(x_n, \mathfrak{M}) \rightarrow 0 \). Let \( Z \) be the Zariski closure of \( \{\overline{x}_n\}_{n=1}^{\infty} \) in \( \mathfrak{M}_k \). For \( x \in Z(k) \), \( \mathfrak{M}_x \) contains a union of translated formal subtori that contains \( Z_x \) as a closed formal subscheme.
Proof. The analog of Proposition 6.1.1 holds for \( Z_6^{a+b} \) by the same proof. Then the proof of the lemma is essentially identical with the proof of Lemma 6.2.1 (with the toric action only on the first \( a \) factors). We omit the details. \( \square \)

6.3. Forward Frobenius action. The following Theorem 6.3.3 is equivalent to Theorem 1.3.2 (1) (the implication will be given below, and the inverse implication is easy and left to the reader). And Theorem 6.3.3 has a form closer to Lemma 6.2.2, and easier to prove.

We need the following simple lemma.

Lemma 6.3.1. Let \( R \) be a countable ring (e.g., \( R \) is a finitely generated algebra over a countable field) and \( V = \text{Spec} \ R \). For every non-negative integer \( i \), let \( X_i \subset V \) be a subset.

(1) Assume that no \( X_i \) is Zariski dense in \( V \) and \( \bigcup_{i=0}^{\infty} X_i \) is Zariski dense in \( V \). Then there exists an increasing sequence \( \{i_m\}_{m=0}^{\infty} \) of non-negative integers and \( x_{i_m} \in X_{i_m} \) such that \( \{x_{i_m}\}_{m=0}^{\infty} \) is Zariski dense in \( V \).

(2) Assume that every \( X_i \) is Zariski dense in \( V \). Then there exists \( x_i \in X_i \) such that \( \{x_i\}_{i=0}^{\infty} \) is Zariski dense in \( V \).

Proof. Let \( \{f_m\}_{m=1}^{\infty} \) be an enumeration of elements in \( R \setminus R^\times \). Let \( D(f_m) \subset V \) be the associated open subset. For (1), by the Zariski density of the union, there exists \( t_1 \) and \( x_{i_1} \in X_{i_1} \cap D(f_1) \). Since the union of \( X_1, X_2, \ldots, X_{i_1} \) is not Zariski dense in \( V \), the union of the rest of \( X_i \)'s is still Zariski dense in \( V \). Then there exists \( i_2 > i_1 \) and \( x_{i_2} \in X_{i_2} \cap D(f_2) \). Continuing this process, the lemma follows. For (2), choose \( x_i \in X_i \cap D(f_i) \). \( \square \)

For \( z \in \mathcal{S}(k) \) and a non-negative integer \( m \), a quasi-canonical lifting of \( z \) (of order \( p^m \)) is a torsion (of order \( p^m \)) of the formal residue torus at \( z \).

Lemma 6.3.2 ([10]). CM points in \( \mathcal{S}(\overline{F}^p) \) are the same as quasi-canonical liftings.

For a non-negative integer \( d \) and \( I = (i^{(1)}, \ldots, i^{(d)}) \in \mathbb{Z}^d \), let \( \phi^I = \left( \phi_{\mathcal{S}}^{i^{(1)}}, \ldots, \phi_{\mathcal{S}}^{i^{(d)}} \right) \).

Theorem 6.3.3. Let \( d, d', m \) be non-negative integers. Let \( \mathcal{U} \) be a formal subscheme of \( \mathcal{S}_{\overline{F}^p}^{d+d'} \). Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence in the prime-to-\( p \) Hecke orbit of \( x_1 \in \mathcal{S}^{d}(\overline{F}^p) \), and \( \{y_n\}_{n=1}^{\infty} \) a sequence of quasi-canonical liftings in \( \mathcal{S}^{d'}(\overline{F}^p) \) of order dividing \( p^m \). For every \( n \), let \( I_n = (i_n^{(1)}, \ldots, i_n^{(d)}) \in \mathbb{Z}_{\geq -m}^d \) and let \( w_n \in \phi_{\mathcal{S}}^{I_n} \left( \{x_n\} \times \{y_n\} \right) \). Assume that the reduction \( \overline{w_n} \) of \( w_n \) is in \( \mathcal{U}(k) \) and \( d(\overline{w_n}, \mathcal{U}) \to 0 \). Let \( Z \) be the Zariski closure of \( \overline{\{w_n\}}_{n=1}^{\infty} \) in \( \mathcal{U}_k \). For \( x \in \mathcal{S}(k) \), \( \mathcal{U}_x \) contains a union of translated formal subtori that contains \( Z_x \) as a closed formal subscheme.

Proof. Let \( \mathcal{U} \) be an open formal subscheme of \( \mathcal{S}_{\overline{F}^p}^{d+d'} \) that contains \( \mathcal{U} \) as a closed formal subscheme.

Lemma 6.3.4. If Theorem 6.3.3 holds for \( m = 0 \), then it holds for every \( m \).

Proof. Let \( \phi^m = (\phi_{\mathcal{S}}^{m}, \ldots, \phi_{\mathcal{S}}^{m})|_{\mathcal{U}} \) with \( d+d' \)-terms (see the paragraph above Corollary 4.1.2 (2) for the restriction. By Corollary 5.3.7, \( d(\phi^m(w_n), \phi^m(\mathcal{U})) \to 0 \). By the relation between \( \phi_{\mathcal{S}} \) and the \( p \) power endomorphisms of formal residue tori in Lemma 4.1.1 (2), we can apply the case \( m = 0 \) to obtain that for a general \( m \), the formal completion \( \phi^m(\mathcal{U})_{\phi^m(x)} \) contains a translated subtorus that contains \( \phi^m(Z)_{\phi^m(x)} \). Then the lemma follows from Corollary 4.1.2 (2). \( \square \)
We continue the proof of the theorem. By Lemma 6.3.4, we may assume that \( m = 0 \). Then \( w_n = (\phi^{l_n}(x_n), y_n) \).

First, we prove the theorem in the case \( d' = 0 \). We use Lemma 6.3.1 to simplify the problem as follows. Up to replacing \( \{ w_n = \phi^{l_n}(x_n) \}_{n=1}^{\infty} \) by a subsequence, we may assume that the set of reductions \( \{ \overline{w_n} \}_{n=1}^{\infty} \) is contained in an affine open \( V \subset \mathbb{Z} \). For a non-negative integer \( i \), let \( X_i = \{ \overline{w_n} : i_n^{(1)} = i \} \). Then \( \bigcup_{i=0}^{\infty} X_i = \{ \overline{w_n} \}_{n=1}^{\infty} \), which is Zariski dense in \( V \) by assumption. By Lemma 6.3.1 (1), up to replacing \( \{ w_n \}_{n=1}^{\infty} \) by a subsequence, we may assume one of the following two conditions:

- there exists a non-negative integer \( i^{(1)} \) such that \( X_{i^{(1)}} \) is Zariski dense in \( V \) and \( X_i = \emptyset \) for \( i \neq i^{(1)} \);
- \( i_n^{(1)} \to \infty \) as \( n \to \infty \).

Continue this process for \( i_n^{(2)}, \ldots, i_n^{(d)} \) and after permuting \( 1, \ldots, d \), we arrive at a decomposition \( d = a + b \) and a sequence \( \{ w_n \}_{n=1}^{\infty} \) with dense reduction in \( Z \) such that

- for \( j = 1, \ldots, a \), \( i_n^{(j)} = i_1^{(j)} \) (thus does not depend on \( n \));
- for \( j = a + 1, \ldots, a + b \), \( i_n^{(j)} \to \infty \) as \( n \to \infty \).

Now let us continue the proof of the theorem in the case \( d' = 0 \). According to \( \mathcal{G}^{a+b} = \mathcal{G}^a \times \mathcal{G}^b \), we write \( w_n = (u_n, v_n) \). Since the prime-to-\( p \) Hecke action and \( \phi_{\mathcal{G}} \) commute, \( \{ u_n \}_{n=1}^{\infty} \) is a sequence in the prime-to-\( p \) Hecke orbit of \( u_1 = \phi^{I_1^{(a)}}(x_1^{(a)}) \in \mathcal{G}^a(F^0) \), where the superscript \( "(a)" \) means the first \( a \)-component, i.e., \( I_1^{(a)} = (i_1^{(1)}, \ldots, i_1^{(a)}) \). Note that \( p \)-power endomorphism on a formal torus over \( F^0 \) shrinks the torus toward the unit. Since \( v_n^{(j)} \to \infty \) as \( n \to \infty \) for \( j = a + 1, \ldots, a + b \), by the relation between \( \phi_{\mathcal{G}} \) and the \( p \)-power endomorphisms of formal residue tori in Lemma 4.1.1 (2), \( d(\overline{v_n^{\text{can}}}, v_n) \to 0 \) in \( \mathcal{G}^b \). Here \( \overline{v_n} \) is the reduction of \( v_n \). Let \( z_n = (u_n, \overline{v_n^{\text{can}}}) \). Then \( d(z_n, \mathcal{Y}) \to 0 \). Now the theorem follows from Lemma 6.2.2.

For a general \( d' \), by the same process, we arrive at \( z_n = (u_n, \overline{v_n^{\text{can}}}, y_n) \) (recall that \( y_n \) is a canonical lifting) such that \( d(z_n, \mathcal{Y}) \to 0 \). The theorem follows from Lemma 6.2.2.

\( \Box \)

**Lemma 6.3.5.** For a non-negative integer \( m \), let \( F_m \) be \( F \) adjoining all \( p^m \)-th roots of unity. For \( x \in F^0 \) such that \( 0 < |x - 1| < 1 \), there exists a positive integer \( m_0 \) such that for every positive integer \( m \) and \( x_m \in p^{-m}(\{x\}) \), \( p^{-m} \leq |F(x_m) : F|, |F_m : F| \leq p^m \).

**Proof.** The second “\( \leq \)” is immediate. For the first “\( \leq \)”, inspecting the Newton polygon, the ramification index of \( F(x_m) \) over \( F \) is \( \geq p^{m-m_0} \) for some non-negative integer \( m_0 \). The same inequality for \( F_m \) is well-known.

\( \Box \)

**Proof of Theorem 1.3.2 (1).** We have a reduction step that will be used for both (1) and (2) of Theorem 1.3.2. After enlarging \( F \), we may assume that each \( O_i \) is the Hecke–Frobenius orbit of a point \( o_i \in \mathcal{G}(F^0) \). Since the Hecke orbit of a CM point consists of CM points, we may assume that each \( o_i \) is not a CM point (in particular, not a canonical lifting). For a non-negative integer \( m \), define

\[
O_i^{(m)} := \bigcup_{I \in \mathbb{Z}_2^{d-m}} \phi^{I}(O_i),
\]
and $CM^{(m)} \subset CM$ the subset of quasi-canonical lifting of order dividing $p^m$. By the description of prime-to-$p$ Hecke orbits in 6.2, every element in the prime-to-$p$ Hecke orbit of $o_i$ have the same $F^0$-coordinate with $o_i$ under a suitable coordinate system (by which we mean an isomorphism to $\mathbb{G}_m^{(g+1)/2}$) of the corresponding formal residue torus over $F^0$. By the relation between $\phi_\mathbb{G}$ and the $p$ power endomorphisms of formal residue tori in Lemma 4.1.1 (2), we can apply the first "≤" of Lemma 6.3.5 to conclude that for $m$ large enough, depending on $e$ and $o_i$'s,

\[
\left(\prod_{i=1}^{d} O_i \times CM^{d'}\right) \cap \bigcup_{[E:F] \leq e} \mathfrak{S}(E^\sigma) \subset \prod_{i=1}^{d} O_i^{(m)} \times (CM^{(m)})^{d'}.
\]

Now we can prove Theorem 1.3.2 (1) using Theorem 6.3.3. By assumption there exists a sequence $\{\epsilon_n\}_{n=0}^{\infty}$ of positive real numbers with $\epsilon_n \to 0$, such that the Zariski closure $Z_n$ of the reduction of the intersection of $\mathfrak{M}_{\epsilon_n}$ with (6.2), which is in $\mathfrak{M}_k$, contains $Z$. By the noetherianness of $\mathfrak{M}_k$, $Z_n$ stabilizes for $i$ large enough. Thus we may assume that $Z_n = Z$ for all $n$. By Lemma 6.3.1 (2) (applied with $V = Z$ and $X_i$ the reduction of the intersection of $\mathfrak{M}_{\epsilon+i}$ with (6.2)), Theorem 6.3.3 implies Theorem 1.3.2 (1).

7. Backward Frobenius action

7.1. Boxall’s trick. We recall a result of Serban [39], which has its origin in Boxall’s study [5] of unlikely intersection for abelian varieties over finite fields.

Let $\mathfrak{G}$ be a formal torus over $F^0$. For a non-negative integer $r$, let $F_{\text{cycl},r}$ be $F$ adjoining all $p^r$-th roots of unity. Then $F_{\text{cycl},r} = F \left(\mathfrak{G}(\overline{F})[p^r]\right)$. Let

\[
\mathfrak{G}(F_{\text{cycl},r})^{\text{div}} = \{P \in \mathfrak{G}(\overline{F}) : p^n P \in \mathfrak{G}(F_{\text{cycl},r}) \text{ for some } n \in \mathbb{Z}_{\geq 0}\}.
\]

Lemma 7.1.1 ([39, Lemma 2.5]). For $r > 1$ and $P \in \mathfrak{G}(F_{\text{cycl},r})^{\text{div}}$ such that $p^{r-1} P \not\in \mathfrak{G}(F_{\text{cycl},r})$, there exists $\sigma \in \text{Gal} (\overline{F}/F_{\text{cycl},r})$ such that $\sigma(P) - P \in \mathfrak{G}(F_{\text{cycl},r})[p^r]\mathfrak{G}(F_{\text{cycl},r})[p^{r-1}].$

Remark 7.1.2. In [39, Lemma 2.5], it is required that $P$ is a torsion point. However, this requirement does not play a role in the proof of [39, Lemma 2.5]. See [40, Lemma 2.2].

Corollary 7.1.3. Let $r > 1$, $\Gamma \subset \mathfrak{G}(F_{\text{cycl},r})^{\text{div}}$ such that $p^{r-1}(\Gamma) \not\subset \mathfrak{G}(F_{\text{cycl},r})$. Then for every closed formal subscheme $\mathfrak{Z} \subset \mathfrak{G}$ and $\epsilon > 0$,

\[
\mathfrak{Z}_\epsilon \cap \Gamma \subset \bigcup_{Q \in \mathfrak{G}(F_{\text{cycl},r})[p^r]\mathfrak{G}(F_{\text{cycl},r})[p^{r-1}]} \left(\mathfrak{Z}_\epsilon \cap \left(\mathfrak{Z} + \left(\mathfrak{G}[p^r]\mathfrak{G}[p^{r-1}]\right)\right)_\epsilon\right).
\]

Proof. For $\sigma \in \text{Gal}(\mathbb{C}_p/F)$, $d(P, \mathfrak{Z}) = d(\sigma(P), \mathfrak{Z}) = d(P, \mathfrak{Z} - (\sigma(P) - P))$. Then Lemma 7.1.1 implies the first “⊂”. The second “⊂” follows by applying Lemma 5.3.2, Lemma 5.3.3 and Lemma 5.3.5.

The following result will be useful in order to apply Corollary 7.1.3.

Lemma 7.1.4. Let $\mathfrak{Z}_1, \mathfrak{Z}_2 \subset \mathfrak{G}$ be integral closed formal subschemes such that $\dim \mathfrak{Z}_1 \leq \dim \mathfrak{Z}_2$. Assume that and $\mathfrak{Z}_1$ contains no translated formal subtori. Then for $r$ large enough,

\[
\dim \mathfrak{Z}_1 \cap \left(\mathfrak{Z}_2 + \left(\mathfrak{G}[p^r]\mathfrak{G}[p^{r-1}]\right)\right) < \dim \mathfrak{Z}_2.
\]
Proof. We prove by contradiction. Assume that for infinitely many \( r \), there exists \( Q_r \in \mathcal{G}(F^o_{\text{cycl},r})[p^r]\) such that \( \dim \left( \mathcal{G}_{3,1,F^o_{\text{cycl},r}} \cap (3_2,F^o_{\text{cycl},r} + Q_r) \right) = \dim 3_2 \). Then there exist closed formal subschemes \( 3_i,F^o_{\text{cycl},r} \) of \( 3_2,F^o_{\text{cycl},r} \) such that \( \mathcal{G}_{3_i,F^o_{\text{cycl},r}} = \mathcal{G}_{3_2,F^o_{\text{cycl},r}} + Q_r \). For \( P \in \mathcal{G}_{3_2,F^o_{\text{cycl},r}} \) (which exists by Corollary 3.1.3), we want to find a translation of \( P \) in \( 3_1(F^o) \). By Galois descent [4, p 139, Example B ] (applied to the affine scheme corresponding to the union of Galois conjugates of \( 3_2,F^o_{\text{cycl},r} \)), the union of Galois conjugates of \( 3_2,F^o_{\text{cycl},r} \) is \( 3_2,F^o_{\text{cycl},r} \). Then for some \( \sigma \in \text{Gal}(F^o_{\text{cycl},r}/F) \), \( P \in \sigma(3_2,F^o_{\text{cycl},r}) \). So \( P + \sigma(Q_r) \in 3_1(F^o) \). By [38, Theorem 1.3 (1)] and the infinitude of such \( r \)'s, \( 3_1 \) contains a translated formal subtorus and this is a contradiction. \( \Box \)

7.2. Proof of Theorem 1.3.2 (2). First, we have a reduction step about our formal subscheme \( \mathfrak{U} \) as follows. By Lemma 5.3.4, we may assume that \( \mathfrak{U} \) is flat over \( F^o \) after replacing \( \mathfrak{U} \) by its maximal closed subscheme that is flat over \( F^o \). By the reduced fiber theorem [3], after replacing \( F \) by a finite extension and \( \mathfrak{U} \) by a nonempty open formal subscheme which does not affect the truth of Theorem 1.3.2 (2) by Lemma 5.3.1 and Lemma 5.3.2, we may assume that

- \( \mathfrak{U}_k \) is connected, unibranch and has no embedded points;
- there is a flat formal \( F^o \)-scheme \( \mathfrak{U}' \) with reduced unibranch special fiber \( \mathfrak{U}'_k \) and a finite schematically surjective morphism \( f : \mathfrak{U}' \to \mathfrak{U} \).

These assumptions will be used when we apply Corollary 3.5.4 later.

Second, we have the reduction step about the set of points as in the beginning of the proof of Theorem 1.3.2 (1) (in the end of last section).

Now assume that for some \( x \in \mathfrak{U}(k) \), \( \mathfrak{U}_{k,x} \) contains no translated formal subtori of \( \mathfrak{G}_x^{d + d'} \), we will show that the reduction of \( \mathfrak{U}_k \cap \left( \prod_{i=1}^d O_i \times CM^{d'} \right) \) is not Zariski dense in of \( \mathfrak{U}_k \) for \( \epsilon \) small enough. This is a contradiction and Theorem 1.3.2 (2) follows.

By Lemma 7.1.4, we can choose \( r \) large enough such that

\[
\dim \mathfrak{U}_x \cap (\mathfrak{U}_x + (\mathfrak{G}_x[p^r]\setminus\mathfrak{G}_x[p^{r-1}])) < \dim \mathfrak{U}_x.
\]

Let \( e \) be a positive integer that is to be determined. Let

\[
\Gamma = \left( \prod_{i=1}^d O_i \times CM^{d'} \right) \setminus \bigcup_{[E:F] \leq \epsilon} \mathfrak{G}(E^o).
\]

By the second “\( \leq \)” of Lemma 6.3.5, we may choose (and do choose) \( e \) large enough, depending on \( r \), such that for every \( z \in \mathfrak{G}(k) \),

\[
p^{r-1}(\Gamma \cap \mathfrak{G}_z(F^o)) \not\subseteq \mathfrak{G}_z(F^o_{\text{cycl},r}).
\]

Then by Lemma 5.3.6 (for \( \mathfrak{G}_z \to \mathfrak{G} \)) and Corollary 7.1.3, for \( \epsilon > 0 \) and \( P \in \mathfrak{U}_k \cap \Gamma \),

\[
d_{\mathfrak{G}_z}(P,\mathfrak{U}_z \cap (\mathfrak{G}_z[p^r]\setminus\mathfrak{G}_z[p^{r-1}])) \leq \epsilon,
\]

where \( z \) is the reduction of \( P \). By Corollary 3.1.3 (1), there exists \( Q \in \mathfrak{U}(F^o) \) such that \( f(Q) = P \). Then by Corollary 3.5.4 and (7.1), Lemma 5.3.3 and Lemma 5.3.6, there exists a positive integer \( s \) and a closed subscheme \( Y \subset \mathfrak{U}_k \) (independent of \( P, Q \)) of \( \dim Y < \dim \mathfrak{U}_k \)
such that if \( z \not\in Y(k) \), then
\[
d_{\mathcal{G}_z}(P, \mathcal{G}_z \cap (\mathcal{G}_z + (\mathcal{G}_x[p^r] \setminus \mathcal{G}_z[p^{r-1}])))
d_{\mathcal{U}_{\mathcal{F}_1}(x)}(Q, f^{-1}(\mathcal{G}_z \cap (\mathcal{G}_z + (\mathcal{G}_x[p^r] \setminus \mathcal{G}_z[p^{r-1}]))) \geq d_{\mathcal{U}_{\mathcal{F}_1}(x)}(Q, \mathcal{G}_f/\omega^*) = |\omega^*|.
\]
Thus for \( \epsilon < |\omega^*| \), the reduction of \( \mathcal{U}_\epsilon \cap \Gamma \) is contained in \( Y \).

By the definition of \( \Gamma \) and (6.2),
\[
(7.2) \quad \mathcal{U}_\epsilon \cap \left( \prod_{i=1}^d O_i \times CM^{d'} \right) \setminus \Gamma \subset \mathcal{U}_\epsilon \cap \left( \prod_{i=1}^d O_i \times CM^{d'} \right) \bigcap \bigcup_{[E:F] \leq \epsilon} \mathcal{S}(E^\circ).
\]
Since \( \mathcal{U}_{k,x} \) contains no translated formal subtori of \( \mathcal{S}_x^{d+d'} \), by Theorem 1.3.2 (1), for \( \epsilon \) small enough, the reduction of the right hand side, and thus the left hand side, of (7.2) is not Zariski dense in of \( \mathcal{U}_k \) (and is in fact finite). Thus the reduction of
\[
\mathcal{U}_\epsilon \cap \left( \prod_{i=1}^d O_i \times CM^{d'} \right) = (\mathcal{U}_\epsilon \cap \Gamma) \cup \left( \mathcal{U}_\epsilon \cap \left( \prod_{i=1}^d O_i \times CM^{d'} \right) \setminus \Gamma \right)
\]
is not Zariski dense in of \( \mathcal{U}_k \) for \( \epsilon \) small enough.

**Appendix A. Approximation on perfectoid spaces**

We recall some applications of Scholze’s approximation lemma on perfectoid spaces [36] (see [32] and [45] for more applications).

**A.1. Approximation lemma.** Let \( (R, R^+) \) be a perfectoid affinoid \((\mathbb{C}_p, \mathbb{C}_p^\circ)\)-algebra with tilt the \((\mathbb{C}_p, \mathbb{C}_p^\circ)\)-algebra \((R^\circ, R^{\circ+})\). In particular, we have the \( \sharp \)-map [36, Proposition 5.17]
\[
R^\circ \rightarrow R, \ f \mapsto f^\sharp.
\]
We remind the reader that this \( \sharp \)-map is not a ring homomorphism, and we do not use the explicit definition of this map. The only important fact is (A.2) and Lemma A.1.1 below.

Let \( \mathcal{X} = \text{Spa}(R, R^+) \) with tilt \( \mathcal{X}^\circ = \text{Spa}(R^\circ, R^{\circ+}) \). Every \( x \in \text{Spa}(R, R^+) \) is an equivalence class valuations on \( R \). Then the tilting bijection [36, Definition 6.16]
\[
(\mathbb{A}.1) \quad \rho : \mathcal{X}(\mathbb{C}_p, \mathbb{C}_p^\circ) \simeq \mathcal{X}^\circ(\mathbb{C}_p^\circ, \mathbb{C}_p^\circ),
\]
that is defined by the following equation for all \( g \in R^\circ \):
\[
(\mathbb{A}.2) \quad |g(\rho(x))| = |g^\sharp(x)|.
\]
Below, we always choose a representative \( |\cdot| \) in the equivalence class of \( x \) such that \( |u(x)| = |u| \) for every \( u \in \mathbb{C}_p \). We make the same choice for \((\mathbb{C}_p, \mathbb{C}_p^\circ)\)-algebras.

**Lemma A.1.1** ([36, Corollary 6.7 (1)]). Let \( f \in R^+ \). Then for every \( 0 < \epsilon < 1 \), there exists \( g \in R^{\circ+} \) such that for every \( x \in \mathcal{X}(\mathbb{C}_p, \mathbb{C}_p^\circ) \), we have
\[
(\mathbb{A}.3) \quad |f(x) - g^\sharp(x)| \leq \frac{1}{2} \max\{|f(x)|, \epsilon\} = \frac{1}{2} \max\{|g^\sharp(x)|, \epsilon\}.
\]
A.2. Algebraic setting. Assume that \( \mathbb{C}_p^0 \) is the closure of an algebraic extension of \( k((t)) \) in \( \mathbb{C}_p^0 \). Also assume that there exists a \( k \)-algebra \( S \), such that \( R^0+ \) is the \( t \)-adic completion of \( S \otimes K^{\text{c}} \). We have the following corollary of Lemma A.1.1.

**Corollary A.2.1.** For \( u \in \mathbb{C}_p^0 \) with \( |u| < 1 \), let \( m_u := \min\{m \in \mathbb{Z} : |u|^m \leq \epsilon \} \). Then \( g \in R^0+ \), exists \( u \in \mathbb{C}_p^0 \) with \( |u| < 1 \), and

\[
g_c = \sum_{i \in \mathbb{Z} \cap (0, m_u)} g_{c, i} u^i
\]

with \( g_{c, i} \in S \), such that (A.4)

\[
g - g_c \in u^{|m_u|} R^0+.
\]

**Proof.** Let \( n \) be a positive integer such that \( |t|^n \leq \epsilon \). By the assumptions on \( \mathbb{C}_p^0 \) and \( S \), we can choose a finite extension \( E = k((u)) \) of \( k((t)) \) with \( |u| < 1 \), and (a finite sum) \( g' = \sum s_j a_j \in S \otimes E \), where \( s_j \in S \) and \( a_j \in E \), such that \( g - g' \in t^n R^0+ \). For \( a \in E \), there exist \( \alpha_h \in k \) such that

\[
a - \sum_{h \in \mathbb{Z} \cap (0, m_u)} \alpha_h u^h \in u^{|m_u|} E^0.
\]

Applying this to the finitely many \( a_j \)'s, the corollary follows. \( \square \)

A.3. Frobenius descent.

**Assumption A.3.1.** There is a sequence \( \{S_n \rightarrow S_{n+1}\}_{n=0}^{\infty} \) of morphisms of \( k \)-algebras such that

1. \( S = \lim_{n \rightarrow \infty} S_n \);
2. the absolute Frobenius map \( S_n \rightarrow S_n \), \( x \mapsto x^p \), factors through the image of \( S_{n-1} \) for \( n \geq 1 \).

Let \( X_0 \) be the adic generic fiber of \( \text{Spec} \, S_0 \otimes \mathbb{C}_p^0 \). Then we have a natural morphism \( \text{pr}^0 : X^0 \rightarrow X_0 \). Let \( \Lambda \subset \text{Spec} \, S_0(k) \), and \( \Lambda^{\text{zar}}_0 \) the Zariski closure of \( \Lambda \) in \( \text{Spec} \, S_0 \). We have the following maps and inclusions:

\[
\mathcal{X}(\mathbb{C}_p, \mathbb{C}_p^0) \xhookrightarrow{\text{pr}^0} \mathcal{X}^0(\mathbb{C}_p, \mathbb{C}_p^0) \xrightarrow{\text{pr}^0} \mathcal{X}_0(\mathbb{C}_p, \mathbb{C}_p^0) \supset \Lambda^{\text{zar}}(\mathbb{C}_p^0) \supset \Lambda.
\]

**Lemma A.3.2.** Assume Assumption A.3.1. Let \( f \in R^+ \). For \( 0 \leq \epsilon < 1 \), let \( \Xi_{\epsilon} := \{x \in \mathcal{X}(\mathbb{C}_p, \mathbb{C}_p^0) : |f(x)| \leq \epsilon \} \). Assume that \( \Lambda \subset \text{pr}^0(\rho(\Xi_{\epsilon})) \). Then \( (\text{pr}^0 \circ \rho)^{-1} (\Lambda^{\text{zar}}(\mathbb{C}_p^0)) \subset \Xi_{\epsilon} \).

**Proof.** First, assume that \( 0 < \epsilon < 1 \). Choose

\[
g_c = \sum_{i \in \mathbb{Z} \cap (0, m_u)} g_{c, i} u^i
\]

as in Corollary A.2.1 where \( g_{c, i} \in S \) for all \( i \). By the assumption on \( \Lambda \), every element \( x \in \Lambda \) can be written as \( \text{pr}^0 \circ \rho(y) \) where \( y \in \Xi_{\epsilon} \). By (A.2), (A.3) and (A.4), \( |g_c(\rho(y))| \leq \epsilon \). By the finiteness of the sum, we can choose a positive integer \( n \) such that \( g_{c, i} \in S_n \) for all \( i \). Then \( g^{p^n}_c \in S_0 \) and \( |g^{p^n}_c(x)| \leq \epsilon p^n \). Since

\[
g^{p^n}_c = \sum_{i \in \mathbb{Z} \cap (0, m_u)} g^{p^n}_{c, i} u^{p^n i},
\]
and
\begin{equation}
|g^{p_n}_{e,i}(x)| = 0 \text{ or } 1,
\end{equation}
the condition \( i \in [0, m_a) \) implies that \( g^{p_n}_{e,i}(x) = 0 \). Thus \( g^{p_n}_{e,i} \) lies in the ideal defining \( \Lambda^{zar} \). So for every \( x \in \Lambda^{zar}(\mathbb{C}_p^{\text{an}}) \), \( g^{p_n}_{e,i}(x) = 0 \). Thus \( g^p_e(x) = 0 \), and then \( g_e(x) = 0 \). By (A.2), (A.3) and (A.4), for every \( x \in \Lambda^{zar}(\mathbb{C}_p^{\text{an}}) \), we have
\begin{equation}
|f \left( \rho^{-1} \left( \text{pr}^{\text{h},-1}(x) \right) \right) | \leq \epsilon.
\end{equation}

Second, the case \( \epsilon = 0 \) follows by letting \( 0 < \epsilon < 1 \) and \( \epsilon \to 0 \). \( \square \)

**Appendix B. Inverse limit of schemes**

Let \( \{X_i\}_{i=0}^\infty \) be an inverse system of schemes with affine transition morphisms \( f_{j,i} : X_j \to X_i \), \( j \geq i \), so that the inverse limit \( \widetilde{X} := \lim \leftarrow X_i \) exists. Let \( \pi_i : \widetilde{X} \to X_i \) be the natural morphism.

**Lemma B.0.1.** Let \( \Lambda \subset \widetilde{X} \) be a subset and \( \Lambda_i = \pi_i(\Lambda) \subset X_i \). We have the following relation between Zariski closures:
\begin{equation}
\Lambda^{zar} = \bigcap_{i=0}^\infty \pi_i^{-1}(\Lambda_i^{zar}).
\end{equation}

**Proof.** It is enough to consider the affine case: \( X_i = \text{Spec} \, B_i \) and \( X = \text{Spec} \, B \), where \( B \) is the direct limit of \( B_i \)'s. The ideal \( I \subset B \) defining \( \Lambda^{zar} \), with reduced induced structure as a closed subscheme, is generated by the union of the images \( I_i \) in \( B \), where \( I_i \subset B_i \) is the ideal of elements whose image in \( B \) vanishes on \( \Lambda^{zar} \). By the definition of \( \Lambda_i \), \( I_i \) is the ideal defining \( \Lambda_i^{zar} \). Then (B.1) follows. \( \square \)

The rest of this appendix is devoted to proving Proposition B.0.9 about formal branches in the inverse limit with finite étale morphisms. We start with is a simple lemma.

**Lemma B.0.2.** Let \( X_i = \text{Spec} \, B_i \) where \( B_i \) is a noetherian local ring. Assume that every \( f_{j,i} \) is a surjective étale local morphism and induces an isomorphism between residue fields. Then \( \pi_i \) induces to an isomorphism between the formal completions of \( \widetilde{X} \) and \( X_i \).

**Proof.** By the assumption, every \( f_{j,i}^\sharp : B_i \to B_j \) is faithfully flat and thus injective. Thus we understand \( B_i \)'s, with maximal ideal \( p_i \)'s, as increasing subrings of \( B \) whose union is \( B \) with maximal ideal the union \( p \) of \( p_i \)'s. It is enough to show that \( B_i/p_i^n \simeq B/p^n \) for all \( n \). By the assumption, \( f_{j,i} \) induces an isomorphism between completions. In particular, \( B_i/p_i^n \to B_j/p_j^n \) is an isomorphism so that \( B_i \to B/p^n \) is surjective. The kernel is the union of \( p_j^n \cap B_i = p_i^n \) for all \( j \geq i \), is \( p_i^n \). \( \square \)

**Remark B.0.3.** The analog of the lemma (and the the analog of Proposition B.0.9 below) for henselizations instead of formal completions also holds. However, we will use formal completions in the main body of the paper. For \( X \) that may not be noetherian, passing from henselization to formal completion could be troublesome (compare with [41, Lemma 06LJ]). Thus we stick to formal completions in this appendix.

We need some definitions and lemmas about (formal) branches.
Definition B.0.4 ([41, Definition 0BQ2, Definition 0C38, Lemma 00ET, Lemma 0C37]).

1. A branch of a local ring is an irreducible component of the spectrum of its henselization.
2. Let X be a scheme and \( x \in X \). A branch of X at \( x \) is a branch of \( \mathcal{O}_{X,x} \). Let \( b(X,x) \) be the number of branches of X at \( x \).
3. We say that X is unibranch at \( x \) if \( b(X,x) = 1 \). It is unibranch if it is unibranch everywhere.
4. Let \( X = \text{Spf} \, A \), where A is a complete noetherian local ring. A formal branch of X is a closed formal subscheme \( \text{Spf} \, A/p \) for a minimal prime ideal \( p \) of A.

Lemma B.0.5 ([41, Lemma 0E20]). We have \( b(X,x) = \sum_i b(X_i,x) \), where \( X_i \)'s are irreducible components of X passing through \( x \).

Lemma B.0.6 ([41, Lemma 0C2E]). Assume that X is noetherian and excellent, and \( x \in X \) is a closed point. Then \( b(X,x) \) is the number of formal branches of the formal completion \( X_x \).

Lemma B.0.7. Let X be the spectrum of a reduced excellent local ring R, \( x \) the closed point, \( Z \subset X \) an irreducible component so that the formal completion \( Z_x \) is a closed formal subscheme of \( X_x \) by Lemma 3.1.4 (1). Then \( Z_x \) is a union of formal branches of \( X_x \).

Proof. We follow the proof of Lemma B.0.5 (i.e., [41, Lemma 0E20]). Let \( \nu_Z : Z' \to Z \) and \( \nu_X : X' \to X \) be the normalization morphisms. Then by [41, Lemma 02LX], \( Z' \) is naturally a closed subscheme of \( X' \), and moreover is a union of connected components of \( X' \). Let \( Z'_x \) and \( X'_x \) be the corresponding completions along \( \nu_X^{-1}(x) \) and \( \nu_X^{-1}(x) \). Then \( Z'_x \) is a union of connected components of \( X'_x \). By the excellence and [41, Lemma 0C23], it is not hard to show that the vertical (natural) morphisms in the following commutative diagram

\[
\begin{array}{ccc}
Z'_x & \longrightarrow & X'_x \\
\downarrow & & \downarrow \\
Z_x & \longrightarrow & X_x
\end{array}
\]

are normalization morphisms. Thus the left (resp. right) vertical morphism is surjective and maps one connected component of \( Z'_x \) (resp. \( X'_x \)) to exactly one branch of \( Z_x \) (resp. \( X_x \)) (see [41, Lemma 0C24] and note that \( Z_x, X_x \) are henselian). The lemma follows. \( \square \)

Proposition B.0.8. Assume that \( f : Y \to X \) is a proper étale (=finite étale) morphism between reduced excellent schemes and \( x \in X \) such that for every \( y \in f^{-1}(x) \), f induces an isomorphism between residue fields of \( y \) and \( x \). Let \( Z \subset Y \) be an irreducible component. Then \( f |_{Z_y} \) is an isomorphism to a union of branches of \( X_x \). And every branch of \( X_x \) is contained in \( f(Z_y) \) for some \( y \in f^{-1}(x) \).

Proof. By [41, Lemma 00ET] (more precise, its proof), Spec \( \mathcal{O}_{Z,y} \) is an irreducible component of Spec \( \mathcal{O}_{Y,y} \). By the assumptions, f induces an isomorphism \( Y_y \simeq X_x \) between completions. Then by Lemma B.0.7, the first part follows. Now we prove the second part. Let \( \nu_Z : Z' \to Z \) and
\[ \nu_X : X' \to X \] be the normalization morphisms. Consider the natural commutative diagram
\[ (B.2) \quad \prod_{y \in f^{-1}(x)} \prod_{y' \in \nu_{X_y}^{-1}(y)} Z_{y'} \to \prod_{x' \in \nu_X^{-1}(x)} X_{x'} \to X_x. \]

By the excellence and [41, Lemma 0C23], the vertical morphisms are normalization morphisms. Then the lemma follows, as in Lemma B.0.7, as long as every \( x' \) is the image of some \( y' \) in (B.2). But this is true since the properness of \( f \) the normalization morphisms implies that the natural dominant morphism \( Z' \to X' \) is proper and thus surjective. \( \square \)

Now we start to discuss the pro-finite analog of Proposition B.0.8. Let notations be as the beginning of this appendix. Assume that every \( X_i \) is noetherian, every \( f_{j,i} \) is étale. Assume that \( X_0 \) is irreducible. Let \( Z_0 = X_0 \) and let \( Z_{i+1} \subset f_{i+1,i}^{-1}(Z_i) \) be an irreducible component inductively. It is plain to check by definition that \( Z_j \) is an irreducible component of \( f_{j,i}^{-1}(Z_i) \) for any \( i \) (in particular, \( Z_j \) is an irreducible components of \( X_j \)). Let \( Y = \varprojlim Z_i \subset \bar{X} \). Let \( y \in Y \) and \( x_i = \pi_i(y) \in Z_i \) so that \( f_{j,i}(x_j) = x_i \). If \( f_{j,i} \) induces an isomorphism between residue fields of \( x_j \) and \( x_i \), by the étaleness, \( f_{j,i} \) induces an isomorphism between the henselizations of \( \mathcal{O}_{f_{j,i}^{-1}(Z_i),x_j} \) and \( \mathcal{O}_{Z_i,x_i} \). Then by Lemma B.0.5,
\[ (B.3) \quad b(Z_j, x_j) \leq b(Z_i, x_i). \]

And if \( b(Z_j, x_j) = b(Z_i, x_i) \), then \( z_j \) lies in only one irreducible component of \( f_{j,i}^{-1}(Z_i) \), which is \( Z_j \), so that \( f_{j,i}|_{Z_i,x_j} \) is an isomorphism to \( Z_i,x_i \). Moreover, if this is the case for all \( i,j \), then by Lemma B.0.2, \( \pi_0|^y \) is an isomorphism to the formal completion \( Z_{0,x_0} \)

**Proposition B.0.9.** Assume that every \( X_i \) is further excellent and reduced, every \( f_{j,i} \) is further proper. Then for \( x_0 \in Z_0 \) with separably closed residue field and \( y \in \pi_0^{-1}(x_0) \cap Y \), \( \pi_0|^y \) is an isomorphism to a union of formal branches of \( Z_{0,x_0} \). Moreover, there exists finitely many \( y \in \pi_0^{-1}(x_0) \) such that every branch of \( Z_{0,x_0} \) is contained in \( \pi_0(Y_y) \) for one of these \( y \)’s

**Proof.** If for all \( i \), the finite subscheme \( \{ x_i \in f_{i,0}^{-1}(x_0) : b(Z_i, x_i) = b(Z_0, x_0) \} \) of \( X_i \) is nonempty, by (B.3), these finite subschemes form an inverse system. Choose \( y \) in the inverse limit of these finite subschemes. We are done by the discussion above the proposition.

To deal with the general case, we do induction on \( b(Z_0, x_0) \). If \( b(Z_0, x_0) = 1 \), we are done by (B.3) and the last paragraph. For a general \( b(Z_0, x_0) \), if for some \( i \), \( b(Z_i, x_i) < b(Z_0, x_0) \) for every \( x_i \in f_{i,0}^{-1}(x_0) \), by Proposition B.0.8, we can replace 0 by \( i \), and conclude the proposition by the induction hypothesis. Otherwise, we are done by the last paragraph. \( \square \)

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