The Poincaré conjecture for stellar manifolds

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Abstract This paper proves that any closed, simply connected, connected, compact stellar manifold is a stellar sphere. That implies the Poincaré conjecture.

1 Introduction

We prove that a closed, compact, connected and simply connected stellar manifold is a stellar sphere. As a corollary we obtain the famous Poincaré conjecture:

Every simply connected closed 3-manifold is homeomorphic to the 3-sphere.

That was stated by Henri Poincaré in 1904 [1]. Analogues of this hypothesis were successfully proved in dimensions higher than 3, see [2], [5], [12], [14], [16].

We prove this conjecture for stellar manifolds. Since every 3-dimensional manifold can be triangulated and any two stellar equivalent manifolds are piecewise linearly homeomorphic ([4], [7], [8]), our result does imply the famous Poincaré conjecture.

2 Main result

A stellar $n$-manifold $M$ can be identified with the sum of its $n$-dimensional simplexes ($n$-simplexes):

$$M = \sum_{i=1}^{n} g_i$$

with coefficients from $\mathbb{Z}_2$. We will call \( \{g_i\}_{i=1}^{n} \) generators of $M$. 

All vertices in $M$ can be enumerated and any $n$-simplex $s$ from $M$ corresponds to the set of its vertices

$$s = (i_1 \ i_2 \ldots \ i_{n+1}),$$

where $i_1 \ i_2 \ldots \ i_{n+1}$ are integers.

The boundary operator $\partial$ is defined on a simplex as

$$\partial(i_1 \ i_2 \ldots \ i_{n+1}) = (i_1 \ i_2 \ldots \ i_n) + (i_1 \ i_2 \ldots \ i_{n-1} \ i_{n+1}) + \ldots + (i_2 \ i_3 \ldots \ i_{n+1})$$

and linearly extended to any complex, i.e.

$$\partial M = \sum_{i=1}^{n} \partial g_i.$$ 

A manifold is called closed if $\partial M = 0$.

If two simplexes $(i_1 \ i_2 \ldots \ i_m)$ and $(j_1 \ j_2 \ldots \ j_n)$ do not have common vertices then one can define their join

$$(i_1 \ i_2 \ldots \ i_m) \star (j_1 \ j_2 \ldots \ j_n)$$

as the union

$$(i_1 \ i_2 \ldots \ i_m) \cup (j_1 \ j_2 \ldots \ j_n).$$

If two complexes $K = \sum \ q_i$ and $L = \sum \ p_j$ do not have common vertices then their join is defined as

$$K \star L = \sum_{i,j} q_i \star p_j.$$ 

If $A$ is a simplex in a complex $K$ then we can introduce its link:

$$lk(A, K) = \{B \in K \ ; \ A \star B \in K\}.$$ 

The star of $A$ in $K$ is $A \star lk(A, K)$. Thus,

$$K = A \star lk(A, K) + Q(A, K),$$

where the complex $Q(A, K)$ is composed of all the generators of $K$ that do not contain $A$. A complex with generators of the same dimension is called a uniform complex.

**Definition 1 (Subdivision)** Let $A$ be a simplex of a complex $K$. Then any integer $a$ which is not a vertex of $K$ defines starring of

$$K = A \star lk(A, K) + Q(A, K)$$

at $a$ as

$$\hat{K} = a \star A \star lk(A, K) + Q(A, K).$$

This is denoted as

$$\hat{K} = (A \ a) K.$$
The next operation is the inverse of subdivision. It is called a stellar weld and defined as follows.

**Definition 2 (Weld)** Consider a complex

\[ \hat{K} = a \star lk(a, \hat{K}) + Q(a, \hat{K}), \]

with \( lk(a, \hat{K}) = \partial A \star B \) where \( B \) is a subcomplex in \( \hat{K} \), \( A \) is a simplex and \( A \not\in \hat{K} \). Then the (stellar) weld \( (A a)^{-1} \hat{K} \) is defined as

\[ (A a)^{-1} \hat{K} = A \star B + Q(a, \hat{K}). \]

A stellar move is one of the following operations: subdivision, weld, enumeration change on the set of vertices. Two complexes \( M \) and \( L \) are called stellar equivalent if one is obtained from the other by a finite sequence of stellar moves. It is denoted as \( M \sim L \).

If a complex \( L \) is stellar equivalent to \( (1\ 2\ \ldots\ n+1) \) then \( L \) is called a stellar \( n \)-ball. On the other hand, if \( K \sim \partial(1\ 2\ \ldots\ n+2) \) then \( K \) is a stellar \( n \)-sphere.

**Definition 3 (Stellar manifold)** Let \( M \) be a complex. If, for every vertex \( i \) of \( M \), the link \( lk(i, M) \) is either a stellar \((n-1)\)-ball or a stellar \((n-1)\)-sphere, then \( M \) is a stellar \( n \)-dimensional manifold \((n\text{-manifold})\).

If \( i \) is a vertex of \( M \) then

\[ M = i \star lk(i, M) + Q(i, M). \]

If \( \partial M = 0 \), then \( Q(i, M) \) is a stellar manifold.

Indeed, consider an arbitrary vertex \( j \) of \( Q(i, M) \). Then

\[ lk(i, M) = j \star lk(j, lk(i, M)) + Q(j, lk(i, M)) \]

and

\[ Q(i, M) = j \star lk(j, Q(i, M)) + Q(j, Q(i, M)). \]

Since \( M \) is a stellar manifold and \( \partial M = 0 \)

\[ i \star lk(j, lk(i, M)) + lk(j, Q(i, M)) \]

is a stellar sphere. Hence, it follows from \([3]\) that \( lk(j, Q(i, M)) \) is either a stellar ball or a stellar sphere.

In the sequel it is convenient to consider an equivalence relation on the set of vertices of a stellar manifold. Among all possible such equivalence relations we are mostly interested in those that meet certain regularity properties underlined by the following definition.

**Definition 4 (Regular equivalence)** Given a stellar manifold \( M \), an equivalence relation \( "\simeq" \) on the set of vertices from \( M \) is called regular if it meets the following conditions:
(i) No generator \( g \in M \) has two vertices that are equivalent to each other.

(ii) For any generator \( g \in M \) there might exist not more than one generator \( p \in M \setminus g \) such that any vertex of \( g \) is either equal or equivalent to some vertex of \( p \). We call such two generators equivalent, \( g \simeq p \).

Our proof of the Poincaré conjecture is based on the following result.

**Theorem 2.1** A connected stellar 3-manifold \( M \) with a finite number of generators admits a triangulation

\[
N = a \ast (S/\simeq),
\]

where \( a \notin S \) is a vertex, \( S \) is a stellar 2-sphere and "\( \simeq \)" is a regular equivalence relation. Moreover, if \( M \) is closed then for any generator \( g \in S \) there exists exactly one generator \( p \in S \setminus g \) such that \( g \simeq p \).

**Proof.** Let us choose an arbitrary generator \( g \in M \) and an integer \( a \) that is not a vertex of \( M \). Then

\[
M \sim (g \ a)M \text{ and } (g \ a)M = a \ast \partial g + M \setminus g,
\]

where \( M \setminus g \) is defined by all the generators of \( M \) excluding \( g \). We construct \( N \) in finite number of steps. Let \( N_0 = (g \ a)M \). Suppose we constructed already \( N_k \) and there exists a generator \( p \in Q(a, N_k) \) that has at least one common 2-simplex with \( \text{lk}(a, N_k) \). Without loss of generality, we can assume that

\[
p = (1 \ 2 \ 3 \ 4).
\]

and \((1 \ 2 \ 3)\) belongs to \( \text{lk}(a, N_k) \). If the vertex \((4)\) does not belong to \( \text{lk}(a, N_k) \) then

\[
N_{k+1} = ((a \ 4) \ b)^{-1}((1 \ 2 \ 3) \ b)N_k.
\]

If the vertex \((4)\) belongs to \( \text{lk}(a, N_k) \) then after introducing a new vertex \( d \notin N_k \) we take

\[
L = ((a \ d) \ b)^{-1}((1 \ 2 \ 3) \ b)(N_k \setminus p + (1 \ 2 \ 3 \ d)),
\]

where \( b \notin (N_k \setminus p + (1 \ 2 \ 3 \ d)) \), and

\[
N_{k+1} = a \ast (\text{lk}(a, L)/\simeq) + Q(a, L)
\]

endowed with the equivalence \( d \simeq (4) \).

By construction

\[
N_{k+1} = a \ast (\text{lk}(a, N_k) + \partial p) + Q(a, N_k) \setminus p
\]

if \( (4) \notin \text{lk}(a, N_k) \). Otherwise,

\[
N_{k+1} = a \ast ((\text{lk}(a, N_k) + \partial g)/\simeq) + Q(a, N_k) \setminus p,
\]

where \( g = (1 \ 2 \ 3 \ d), \ d \simeq (4). \)
Since $M$ is connected and has a finite number of generators there exists a natural number $m$ such that

$$N_m = a \star (S/ \simeq)$$

where $S$ is a stellar 2-sphere and "$\simeq"$ is a regular equivalence relation. If $M$ is closed, then $\partial N_m = 0$, and therefore, for any generator $g \in S$ there exists exactly one generator $p \in S \setminus g$ such that $g \simeq p$.

Q.E.D.

It is known [3], [4], [7] that any two 3-dimensional manifolds admitting stellar equivalent triangulations are piecewise linearly homeomorphic. On the other hand, every compact 3-dimensional manifold admits a stellar triangulation with a finite number of generators [3]. Hence, the Poincaré conjecture follows from the following statement.

**Theorem 2.2 (the Poincaré Conjecture)** A simply connected, connected closed stellar 3-manifold $M$ with a finite number of generators is homeomorphic to the 3-sphere, $\partial(1 2 3 4 5)$.

**Proof.** Let $M$ be a closed, connected and simply connected 3-dimensional stellar manifold with a finite number of generators. By Theorem 2.1 $M$ admits a triangulation

$$N = a \star (S/ \simeq)$$

where $a \notin S$ is a vertex, $S$ is a stellar 2-sphere and "$\simeq"$ is a regular equivalence relation. Moreover, for any generator $g \in S$ there exists exactly one generator $p \in S \setminus g$ such that $g \simeq p$. Let us show that $a \star (S/ \simeq)$ is homeomorphic to the 3-sphere. If $g$ is a generator of $S$, then

$$a \star (S \setminus g)/ \simeq$$

is connected and, by Seifert – Van Kampen theorem (see e.g. [3]), it is simply connected.

The barycentric subdivision $Br(a \star (S \setminus g)/ \simeq)$ of $a \star (S \setminus g)/ \simeq$ is geometrically collapsible (or $Br(a \star (S \setminus g)/ \simeq) \searrow 0$). Thus by Whitehead’s result on regular neighborhoods (see [3], [15]) $Br(a \star (S \setminus g)/ \simeq)$ is a combinatorial 3-ball. Therefore $N = a \star (S/ \simeq)$, two 3-balls identified along their boundaries, and $N$ is homeomorphic to the 3-sphere. Thus, $M$ is homeomorphic to the 3-sphere. Q.E.D.

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