1 Introduction

The Boolean Satisfiability problem, or SAT, is of central importance in the computer science community, both from a practical and from a theoretical point of view. An instance of SAT is a formula in propositional logic with Boolean variables, and the problem asks whether there exists an assignment of truth and false values to the variables such that the given formula evaluates to the truth value. We always assume that the input formula to SAT is in CNF form. The challenge is in designing a provably optimal algorithm for SAT.

From the time that Cook, Levin, Karp, etc. [1, 2, 3] defined the class NP of decision problems, and showed that SAT and many interesting combinatorial problems are in fact complete in this class, many efforts were dedicated to proving other seemingly hard problems NP-complete. Many important problems were indeed shown as such, see [4] for early and classic reductions.

Given a decision problem $A$, there is often two parallel ways to advance an NP-hardness proof. The first is to transform $A$ to a problem $A'$ such that we can infer the hardness of $A$ from that of $A'$, and moreover, $A'$ is naturally more similar to a satisfiability problem. Examples of this kind of NP-hardness proof are widespread. The second is to restrict SAT to a subclass, or otherwise change the problem SAT into NP-Compete SAT' such that the problem $A$ is naturally more similar to SAT'. In this case, we might be able to directly reduce SAT' to A. Instances of the latter approach are encountered specially in geometric situations. For instance, PLANAR 1-in-3-SAT has been used to obtain a difficult NP-hardness proof for the Minimum Weight Triangulation problem [5]. PLANAR-SAT has been used repeatedly in connection with problems of plane geometry. We refer to [6] for review and definition of many special cases of PLANAR-SAT and related problems. Therefore, special forms of SAT might be important tools for reducing complexity of NP-hardness proofs.

Our problem can be described as a special case of SAT, that we call CONTIGUOUS SAT. Although the definition, soon to be given, is very simple and natural, we have not seen this problem defined earlier. This could be because of the lack of motivation for it. Our motivation, and the starting point of our research, is given in Section 1.1 below. The problem CONTIGUOUS SAT is SAT restricted to input formulas in which i) an ordering of the clauses is fixed (say the usual left-to-right written ordering) ii) clauses containing a fixed literal appear contiguously in this ordering. The formula below is an input to CONTIGUOUS SAT with the left-to-right ordering of the clauses.

$$(x_1 \land x_2 \land x_3) \lor (x_1 \land x_2 \land \neg x_3) \lor (x_1 \land \neg x_3 \land \neg x_1)$$

But the following is not a valid input.

$$(x_1 \land x_2 \land x_3) \lor (\neg x_1 \land x_2 \land \neg x_3) \lor (x_1 \land \neg x_3 \land \neg x_1)$$

1.1 Motivation

The incentive behind our definition of CONTIGUOUS SAT has been a problem in visibility computation. Two points in a planar (or higher dimensional) configuration are visible from each other if and only if the line
segment between them does no cross any "obstacle". What constitutes an obstacle varies by the problem. For instance, for visibility inside a polygon, the polygon (boundary) itself is the obstacle. There exists a sizable literature on computing visibility in different configurations of points and obstacles in the Euclidean plane. These include visibility inside a polygon [7, 8, 9, 10], a point inside a region with obstacles [11, 12], etc. We refer to [13] for more on visibility in the plane. Recently, there has been some attention to situations wherein there is uncertainty in the eye location or the obstacle positions. Then, two points are visible to each other with a certain probability. Examples of these kinds of problems can be found in [14, 15].

A different avenue where uncertainty enters the visibility problem is by assuming that each obstacle can appear in different places according to a probability distribution. The simplest case is where the obstacles are of the simplest non-trivial shape, namely line segments in the plane, and each segment has two possible places. A primitive question in this setting is the following. Given a point \( p \) and a line segment \( s \) in the plane, determine whether there exists a non-zero probability that the point \( p \) can see the entire segment \( s \), see Figure 1. We define a problem called SEGMENT COVER to capture the simplest situation.

**Definition (SEGMENT COVER).** An uncertain segment is a pair of closed intervals on the unit interval, \( s = \{I_1, I_2\} \), \( I_i \subset I = [0,1], \ i = 1,2 \). Let \( s_j = \{I_{j1}, I_{j2}\}, j = 1, \ldots, n \) be \( n \) given uncertain segments. Decide whether there exists a choice of intervals \( I_{ij} \in s_j, j \in \{1, \ldots, n\}, i_j \in \{1, 2\}, \) such that

\[
I \subset \bigcup_{j=1}^{n} I_{ij}.
\]

The following is easy to observe.

**Lemma 1.** The problems SEGMENT COVER and CONTIGUOUS SAT are equivalent with linear-time reductions.

![Figure 1: The projection of uncertain segments on s according to q defines four uncertain intervals.](image)

We remark that the above geometric view to CONTIGUOUS SAT is indeed of great benefit in proving our results. Hence, the lack of this point of view might justify the lack of attention and results concerning CONTIGUOUS SAT in the literature.

### 1.2 Results

The main result of this paper is the NP-hardness proof for SEGMENT COVER.

**Theorem 1.** The problem SEGMENT COVER is NP-Complete. Hence, so is CONTIGUOUS SAT.

The proof will occupy the Section 2 of the paper. This result seems interesting especially when at the first sight the restriction on CONTIGUOUS SAT seems severe.
We then try to compute an approximate solution. Indeed, very naturally the MAX-SAT problem can be used to find an approximate covering of the unit interval. In the other direction, we prove that there is a ratio $c < 1$ such that there can be no algorithm computing a $c$-approximation of the optimal covering in polynomial time, unless $P=NP$, see Theorem 3. In addition, we define a second type of approximation in which we require that the covered interval returned by the approximation algorithm be a contiguous interval. We show that this approximation problem is not meaningful for general input, see Theorem 4.

We also consider the case where the given intervals are all of equal length. This case can be seen to be related to 1-dimensional analog of the problem BCU of [16]. We show in Theorem 2 that the SEGMENT COVER remains NP-Complete with this restriction. We then deduce that 1-dimensional BCU is NP-hard. This strengthens the results of [16] considerably. See Related Work for the definition of BCU.

1.3 Related Work

There are many restrictions of SAT that are NP-complete. They include $k$-SAT, $k \geq 3$, NAE-SAT, 1-IN-3 SAT [17], PLANAR 3-SAT [18], PLANAR 1-IN-3 SAT [19], MONOTONE PLANAR CUBIC 1-IN-3 SAT [20], 4-BOUNDED PLANAR 3-CONNECTED 3-SAT [21], etc, to give a taste of the wide range of the restrictions on SAT in the literature.

We are interested in this paper in SAT instances where each variable is restricted to a few clauses. Let us denote by $(r, s)$-SAT the SAT problem restricted to clauses containing exactly $r$ variables, and each variable appearing in at most $s$ clauses. Tovey [22] has shown that $(3, 3)$-SAT is always satisfiable and $(3, 4)$-SAT is NP-Complete. In addition, it is proved in [23] that 3-SAT restricted to instances where each variable appears at most three times is NP-Complete. A stronger result, proved in Dahlhaus et al. [24], states that PLANAR 3-SAT in which each variables appears exactly three times, and twice with one literal, a third time as the other literal, is still NP-Complete. These results will be used in Section 2.

The problems around the visibility concept have been an active research area since the beginning of computational geometry. Point and edge visibility[7, 8, 9, 10], the art gallery problem [25], the watchman route problem [26, 27], visibility graphs and their recognition [13, 28] are among the topics of interest in this field. It seems that, there has not been much emphasize on visibility computations in presence of uncertainty, in spite of the fact that, uncertainty is very natural in applications and indeed has been studied from a more practical viewpoint, like robot motion planning [29, 30, 31, 32, 33]. This lack of interest seems reasonable given that one of the most basic operations needed for visibility computations is NP-hard, i.e., our problem SEGMENT COVER. Visibility with uncertain obstacles have been studied previously in [15]. They show that, in a setting similar to SEGMENT COVER, if each uncertain segment has one position, but exists with a certain probability, then there is an efficient algorithm to compute the probability that the segments cover the interval $I$. On the other hand, if each uncertain segment has a finite set of possibilities, instead of two as in SEGMENT COVER, computing the same probability is #P-Complete. It is easy to see that in this case the problem is harder than computing the permanent of a 0-1 matrix.

The study of Chambers et al. [16] is relevant to ours. In that paper, the authors consider the following problem they call BCU. The centers of $n$ equal balls are to be chosen from $n$ uncertainty regions (in a $d$-dimensional Euclidean space). Find the minimum value of the radius for which the union of balls could be a connected region. In [16] it is shown that this problem is NP-hard in the plane, $d = 2$, even when each uncertainty region is a pair of points. Now, if we are on the real line, and we fix a radius $r$, then we must ask if the disks can be chosen such that their union is connected. This is a version of SEGMENT COVER in which all the lengths of segments are equal. Using the NP-Completeness of ALL-EQUAL SEGMENT COVER, we arrive at the stronger result that the problem BCU is NP-hard even on the real line, this will be proved rigorously in Section 3.
into two parts denoted $P, G$ defined by means of a complete bipartite graph denoted $a$ collection of uncertain intervals $S, variable uncertain segments corresponding literal. It is easy to see that the problem SEGMENT COVER is in NP. We reduce the NP-complete problem 3-SAT to it. When an interval is a union of sub-intervals such that the sub-intervals share at most endpoints with each other, by abuse of notation and for simplicity, we say that the interval is a disjoint union of the intervals. Let $\phi$ be the given 3-SAT formula and assume it has $s$ clauses, $C_1, \ldots, C_s$. We first divide the interval $I$ into $s$ disjoint intervals $B_j = [(j-1)/s, j/s], j = 1, \ldots, s$. The $B_j$ are in one-one correspondence with the clauses so that the intervals. In the sequel, we refer to the clauses sometimes by the name of the intervals $B_j$.

**Lemma 2.** For any choice of intervals from uncertain segments of $T_j$, at most two intervals among $B_{j1}, B_{j2}$ and $B_{j3}$ are covered.

A clause $C_j$ has three literals. We make a one-one association between the literals of the clause and the intervals $B_{j1}, B_{j2}$ or $B_{j3}$, in an arbitrary way. Thus we define the sub-interval of $B_j$ associated to a literal. Let $x_1, \ldots, x_m$ be the variables of the given formula. We shall construct a collection of uncertain intervals $T_i$ for the variable $x_i$. For each variable, these uncertain intervals are defined by means of a complete bipartite graph denoted $G_i, i = 1, \ldots, m$. The vertices of $G_i$ are divided into two parts denoted $P_i$ and $N_i$. The graph $G_i$ has a vertex for each sub-interval of a clause interval associated to a literal. Let $P_i$ be those sub-intervals that are associated to positive literals of $x_i$, and, let $N_i$ be sub-intervals associated to negative literals of $x_i$. This finishes the definition of $G_i$.

After constructing $G_i$ we define the set $S_i$ as follows. Let $J$ be an interval in $P_i$ or $N_i$, and let $d = d(J)$ be its degree in the graph. Partition the interval $J$ into $d$ disjoint (sharing only endpoints) intervals, and make a correspondence between the edges incident on $J$ and these sub-intervals. Perform this subdivision for all the intervals $J \in P_i$ and $J \in N_i$. Now the uncertain intervals $S_i$ are defined by the edges of the graph $G_i$ and their corresponding sub-intervals. In more detail, let $e = (J_P, J_N), J_P \in P_i, J_N \in N_i$ be an edge of the graph $G_i$. Then $e$ determines an interval inside $J_P$, and one inside $J_N$. The segment $s_e \in S_i$ is defined as the uncertain interval containing these two sub-intervals.

**2.1 Proof of Correctness**

Let the input to the SEGMENT COVER problem be the set of uncertain intervals $S = \bigcup_{i=1}^{m} S_i \cup \bigcup_{i=1}^{n} T_i$. In this section, we show that there is a covering of the unit interval with the uncertain segments $S$ if and
only if the given sentence $\phi$ is satisfiable.

Assume that $\phi$ is satisfiable. Observe that each uncertain segment $s \in S_i \subset S$, has a positive interval and a negative interval. Namely, the positive interval is that which corresponds to the incidence of the edge to the vertex in the positive part of $P_i$, i.e., $P_i$, hence we can write $s = \{s_p, s_n\}$. Now assume $x_i$ takes the value 1 (=true) in an assignment that satisfies $\phi$. Then we choose $s_p$, otherwise we choose $s_n$, for all $s \in S_i$.

**Lemma 3.** In the graph $G_i$, an interval $B_{jk}$, for some $j \in \{1, \ldots, n\}$, $k = \{1, 2, 3\}$, is associated to a vertex in $P_i$ if and only if $x_i$ has a positive literal in $B_j$. Analogously, an interval $B_{jk}$ belongs to $N_i$ if and only if $x_i$ has a negative literal in the $B_j$.

From the above lemma, whenever we choose the uncertain segments as above, since each clause is satisfied, each clause interval $B_j$ has a vertex (=literal) all of whose incident edges have chosen that vertex. Hence, the associated interval among $B_{j1}, B_{j2}$ and $B_{j3}$ is covered. It remains to cover the two remaining intervals. This is easily done by a suitable choice for the uncertain segments of the set $T_j$. This finishes one direction of the proof.

Consider now the other direction. We have to show that if there is a choice for each uncertain segment $s \in S$, such that the unit interval is covered, then, there is an assignment of 0 and 1 to the variables $x_i$ that satisfies the given formula $\phi$. Consider a clause $B_j = (\lambda_1 \lor \lambda_2 \lor \lambda_3)$, where $\lambda_i$ are literals. And let $x_{i_1}, x_{i_2}, x_{i_3}$ be the corresponding variables. The interval $B_j$ is covered by the chosen segments. Recall that the uncertain segments correspond to the edges of the graphs $G_i$ (other than elements of the $T_i$) and that a choice of an interval for an uncertain segment is equivalent to choosing one endpoint of the corresponding edge.

**Lemma 4.** Consider the graphs $G_{i_1}, G_{i_2}, G_{i_3}$. Let the literals $\lambda_1, \lambda_2, \lambda_3$ correspond to the vertices $v_1, v_2, v_3$, respectively, in these graphs. There exists at least one vertex $v$ among the $v_i, i = 1, 2, 3$, such that, each edge incident on $v$ has chosen $v$.

*Proof.* The uncertain segments in $T_j$ leave at least one of $B_{j1}, B_{j2}$ and $B_{j3}$ uncovered, let it be $B_{j1}$. The interval $B_{j1}$ has to be covered using the uncertain segments of $S$. Recall that $B_{j1}$ corresponds to one of the vertices $v_i$. All the edges incident on $v_1$ are required to choose $v_1$, otherwise, some part of the interval $B_{j1}$ would remain uncovered. 

To construct an assignment from the choices of uncertain segments $S$ we do as follows. Consider any clause $B_j$. Lemma 4 gives a $k \in \{1, 2, 3\}$ such that a vertex in $G_{i_k}$ is chosen by all its incident edges. If the vertex is in $P_{i_k}$, set $x_{i_k} = 1$, otherwise set $x_{i_k}$ to be 0.

First, we show we have defined a valid assignment. Let $B_p$ ($B_n$) be the clause in which $x$ appears as a positive (negative) literal and when considering that clause we have set $x$ to be 1 (0). Let $G_x$ be the graph of the variable $x$ with parts $P_x$ and $N_x$. Since we have set $x = 1$ in $B_p$, there is a vertex of $P_x$ in which all the edges have chosen that vertex. Similarly, there is a vertex in $N_x$ such that all the edges have chosen that vertex. But every vertex in $P_x$ is connected to every vertex in $N_x$, hence there has to be an edge where both ends are chosen, a contradiction.
Second, we show that the assignment satisfies all the clauses. We use the notation as above. Take a fixed but arbitrary $B_j$, and let $x$ be the variable returned by Lemma 4 and $v_x$ the vertex of $G_x$ all whose incident edges have chosen it. If $v_x$ is associated to a positive literal in $B_j$, $v_x$ is in the positive part of $G_x$. Hence our procedure setting $x = 1$, satisfies the clause. Let $x$ have a negative literal in $B_j$. Then the sub-interval associated to $v_x$ appears in the negative part of $G_x$. Hence setting $x = 0$ will satisfy $B_j$. This finishes the proof of the correctness of the reduction.

2.2 Complexity

In this section we bound the run-time of the reduction procedure. First, we assume that the given formula $\phi$ is an arbitrary 3-SAT instance. In linear time in number of clauses we construct the sets $T_j$ of uncertain intervals. Let variable $x_i$ appear in $p_i$ clauses as a positive literal and in $n_i$ clauses as a negative literal. Then the graph $G_i$ is $K_{p_i,n_i}$ and has $p_i n_i$ edges. Thus our reduction is of complexity $O(s + \sum_{i=1}^{m} p_i n_i)$. This finishes the proof of NP-hardness and hence that of Theorem 1.

If we start by the NP-Complete problem studied by [24], in which each variable appears at most three times once with one literal, and twice with the other literal, then the number of our uncertain segments is exactly $2s + 2m$.

3 ALL-EQUAL SEGMENT COVER

In this section, we strengthen our result to show that the SEGMENT COVER remains NP-hard even when we require that the length of the intervals all be equal. We call this problem ALL-EQUAL SEGMENT COVER. We will later deduce that the problem BCU of [16] is NP-hard, even in the 1-dimensional case.

We now describe the modifications to the reduction necessary to keep all the intervals the same length. First, observe that we can make sure that the intervals $B_j1, B_j2, B_j3, B_j4$ for all $j$ have equal length. It remains to make sure that the intervals in the uncertain segments from the $S_i$ have equal length. For simplicity in this argument, we will start by a special 3-SAT problem, namely, the one considered by [24]. They have proved that PLANAR 3-SAT remains NP-Complete when each variable appears at most three times, once as one literal, twice as the other. When applying our reduction to this type of formulas, we will see that in the final uncertain segments intervals $B_j1, B_j2, B_j3, B_j4$ are divided into at most two smaller intervals. With these preliminaries in mind, we will substitute the intervals in the Figure 3 for the corresponding intervals from our original construction. In this figure $B_j1, B_j3, B_j5$ play the roles of $B_j1, B_j2$ and $B_j3$ of the original reduction. Note that we have assumed in the figure that the worst case happens, i.e., each three of the sub-intervals is divided. The other cases are simpler.

**Theorem 2.** The problem ALL-EQUAL SEGMENT COVER is NP-Complete.

**Proof.** Consider a clause $B_j$. To the set $T_j$ of the original reduction we add $s - 1$ new uncertain segments, each of them consisting of the two copies of the same interval. This insures that certain subsets of the interval $I$ are always covered, see Figure 3. The set $S_i$ of uncertain segments for the variable $x_i$ is defined just as in the original reduction, but with the modification that a vertex interval $B_{jk}$ is not partitioned, rather the sub-intervals for the at most two incident edges are copies of the interval $B_{jk}$, one of them slightly moved to the right, the other slightly moved to the left. We just need to check that the new intervals have the required properties used in the reduction. As before, at most two of the intervals $B_j1, B_j3$ and $B_j5$ can be covered by the uncertain segments from (updated) $T_j$. It is easily checked that that interval, say $B_j1$, which is not covered, can only be covered when both of the intervals of the incident edges are present. Hence, the same correctness argument applies here as well.

We next show that the optimization problem called BCU and studied in [16] is NP-hard on the real line. For the definition of BCU refer to Section 1.3.

**Corollary.** The 1-dimensional BCU is NP-hard.
Proof. Let the set $s_1, \ldots, s_n$ be an instance of ALL-EQUAL SEGMENT COVER. For each $s_i$, construct an uncertain region $u_i$ containing two points, namely, the midpoints of the two intervals in $s_i$. We add two more regions defined as follows. Let $x_l$ be the smallest coordinate and $x_r$ be the largest coordinate of any midpoint. Moreover, let $r$ be half the length of an interval. Define $u_0 = \{x_l - 2r, x_l - 3r\}$ and $u_{n+1} = \{x_r + 2r, x_r + 3r\}$. Add these two sets to the problem instance. Then the $u_i$ define an instance of BCU. An algorithm solving BCU returns a minimum $r'$ such that there are $n + 2$ disks of radius $r'$, with centers at the points of the $u_i$, one center from each $u_i$, such that the area they cover is connected. Because of $u_0, u_{n+1}$ we have always $r' \geq r$. Moreover, $r' = r$ if and only if the answer to the original ALL-EQUAL SEGMENT COVER is YES.

4 Approximation

In this section, we consider the approximation of the SEGMENT COVER problem. We can define two natural approximation problems. The first, called MAX-SEGMENT COVER, or MAX-SC for short, asks for each uncertain segment such that a maximum-length connected interval is obtained.

4.1 Approximation of MAX-SC

We first prove hardness of approximation for MAX-SC. Let MAX-E3SAT be MAX-3SAT restricted to formulas in which each clause contains exactly three literals.

\textbf{Theorem 3.} Let $c'$ be a ratio beyond which it is \textit{NP-hard} to approximate MAX-E3SAT. Then it is \textit{NP-hard} to approximate MAX-SC with ratio larger than $c = \frac{c' + 2}{3}$.

\textbf{Proof.} Suppose we are given an instance $\phi$ of MAX-E3SAT with $n$ variables. We shall apply the reduction of Section 2 to $\phi$ and obtain an instance of MAX-SC, however, we need some modifications. Consider a graph $G_i$ constructed in the reduction. If $|P_i| = |N_i|$ we leave the graph as it is, otherwise, let $|P_i| < |N_i|$. We add $|N_i| - |P_i|$ dummy vertices to $|P_i|$ to make the two sets equal. We do analogously in the other case. Let $G_i$ denote the modified graphs, $i = 1, \ldots, n$. We build the uncertain segments $S_i$ from $G_i$ as follows. Let $v$ be a vertex of $G_i$ and $m = |P_i| = |N_i|$. If $v$ is not a dummy vertex, it has associated with it a sub-interval of a clause-interval. We make sure all these sub-intervals have length 1, and a clause interval has length 3. If $v$ is a dummy vertex, associate to it the fixed interval $J'$ of very small length $\epsilon > 0$, anywhere outside all of the clause intervals.

Next, we build the uncertain segments $S_i$ as before from the graphs $\tilde{G}_i$ and associated intervals. Let $W$ be the total length of the union of the intervals of uncertain segments $S_i$, then by construction

$$W = 3s + \epsilon.$$ 

Note that any two intervals of (possibly different) uncertain segments defined here are disjoint other than when both intervals are sub-intervals of $J'$.

We run the approximation algorithm for MAX-SC on our instance. The algorithm makes a choice from each uncertain segment. We modify this choice slightly. If any uncertain segment has chosen a sub-interval of $J'$ we reverse this choice. It is clear that at the end we have at worst decreased the total approximated length by $\epsilon$. And we have not decreased the approximated length over the original clause intervals.

Observe that the total length that the uncertain segments chosen from $T_i$ contribute is at most $2W/3 = s$. If from any clause-interval the choice from $T_i$ covers only 1/3 of the interval, then the middle interval $B_{j2}$ is covered. We change the choices so that only 1/3 of the interval is not covered, by covering either of $B_{j1}$ or $B_{j3}$. This insures that from any clause interval exactly one sub-interval is not covered by the $T_i$.

Now from the modified choice of uncertain segments and the graphs $G_i$ define the graphs $\tilde{G}_i$ as follows. For each $i$, from the graph $\tilde{G}_i$, remove any vertex whose interval is covered in the approximation by intervals.
from $T_i$. Denote the new graph by $G'_i$. The total length of the intervals corresponding to the non-dummy vertices of $G'_i$ is $W/3 = s$. Next, define an assignment as follows. We distinguish five cases from each other.

Case 1: The graph has original vertices in positive part only, and, dummy vertices are in positive part. For any edge $e \in \tilde{G}_i$ that is not incident with a dummy vertex, we redirect the choice to the positive side. Note that since any interval we uncover is covered by $T_i$ this does not decrease the length of the approximation. After these re-directions, any non-dummy vertex in the positive side of $\tilde{G}_i$ has their intervals chosen.

Case 2: The graph has original vertices in positive part only, and, dummy vertices are in negative part. For any edge $e \in \tilde{G}_i$ that is not incident with a dummy vertex, we redirect the choice to the positive side. Recall that all the other edges have also chosen the positive side. Then again after this re-direction of choice all the vertices in positive part of $\tilde{G}_i$, have their intervals covered. Again this operation does not decrease the total approximated length.

Cases 3,4: These are analogous to the previous cases, where non-dummy vertices appear in the negative part only. We perform analogously as in those cases.

Case 5: The graph has non-dummy vertices in both parts, or it has only dummy vertices. In this case, we can assign an arbitrary value to $x_i$. We choose the side which does not have dummy vertices and redirect all the edges of $\tilde{G}_i$ towards that side. Re-direction of the choice for an edge not incident on a dummy vertex does not change the approximated weight. Also we had set the choice for edges incident on dummy vertices away from them. It follows that all the intervals associated to the vertices of the chosen side are covered.

Thus we have defined an assignment. Now we compute the number of clauses satisfied by our assignment. The length not covered by the $T_i$ and covered by the $S_i$ in the approximation is at least $c(W + \epsilon) - \frac{2}{3}W$. After the above redirection of the choices, an interval corresponding to a clause is either all covered or covered in exactly $2/3$ of its length. Therefore, $c(W + \epsilon) - \frac{2}{3}W$ is (lower bound for) the total number of the intervals satisfied by our assignment. For any algorithm that runs in polynomial times we must have $c(W + \epsilon) - \frac{2}{3}W = 3cs + c\epsilon - 2s < c's$. This implies

$$c < \frac{c' + 2}{3 + \frac{2}{s}}.$$  

The claim follows.

**Remark** By a seminal result of Håstad [34] MAX-E3SAT cannot be approximated by a ratio larger than $7/8$. Using this result the above theorem implies that MAX-SC cannot be approximated beyond the ration $23/24$, unless P=NP.

To approximate MAX-SC, we can use the same existing algorithms for weighted MAX-SAT, which is a well-studied problem in the literature, refer to the sequence of papers [35, 36, 37, 38]. We just form a SAT from our SC instance. Any maximal sub-interval $J \subset I = [0, 1]$ that does not contain an endpoint defines a clause, and in it are literals corresponding to uncertain segments covering the interval $J$. We assign the length of $J$ as the weight of the corresponding clause. Given we have an algorithm for weighted MAX-SAT with approximation ratio $0 < c' < 1$, then clearly we have an algorithm with the same ratio for MAX-SC.

It is interesting to see these upper and/or lower bounds improved.

### 4.2 Approximation of CONTIGUOUS MAX-SC

In this section we investigate the CONTIGUOUS MAX-SC. Note that clearly the same argument for hardness of approximation of MAX-SC applies here. However, this problem seems not much meaningful as an approximation problem. Let $s$ be the number of clauses in an instance of CONTIGUOUS MAX-SC.

**Theorem 4.** Let $\epsilon > 0$. There can be no algorithm with ratio $r(n) = n^{-1+\epsilon}$ for CONTIGUOUS MAX-SC over all the input.

**Proof.** We start with an instance $A$ of the problem SC with the underlying interval of length $n$. Assume we have an algorithm for approximating CONTIGUOUS MAX-SC with ratio $r(x) = x^{-1+\epsilon}$. We build an
instance B of CONTIGUOUS MAX-SC. The idea is that we want to concatenate copies of $A$ such that any $r()$-approximation would contain a full copy. Let $f(n)$ be the number of copies required. Then we need

$$\frac{f(n)n}{n+1} f(n)n > 2n.$$  

(1)

Assume $f = f(n)$ satisfies the above relation. Then we construct the instance $B$ by copying the instance $A$, $f$ times. Then, the approximation algorithm applied to $B$, outputs an interval of length $r(fn)fn > 2n$. Thus, at least one copy has to be entirely covered, hence the given instance $A$ has to be satisfiable. To satisfy equation (1) it is enough to choose

$$f > \left(2n^{1-\epsilon}\right)\frac{1}{\epsilon}.$$  

\[\square\]

References

[1] Stephen A. Cook. The complexity of theorem-proving procedures. In Proc. 3rd Ann. ACM Symp. Theory Computing, pp 151–158, 1971.

[2] B. A. Trakhtenbrot. A survey of russian approaches to perebor (brute-force searches) algorithms. Annals of the History of Computing, 6(4):384–400, 1984.

[3] Richard M. Karp. Reducibility among combinatorial problems. In In Raymond E. Miller; James W. Thatcher. Complexity of Computer Computations, pp 85–103. New York, Plenum, 1972.

[4] Michael R Garey and David S Johnson. Computers and intractability. W.H. Freeman New York, 1979.

[5] Wolfgang Mulzer and Günter Rote. Minimum-weight triangulation is NP-hard. J. ACM, 55(2):11, 2008.

[6] Simon Tippenhauer. On planar 3-sat and its variants. Master’s thesis, Freie Universität Berlin, 2016.

[7] David Avis and Godfried T. Toussaint. An optimal algorithm for determining the visibility of a polygon from an edge. IEEE Trans. Computers, 30(12):910–914, 1981.

[8] Boaz Ben-Moshe, Olaf A. Hall-Holt, Matthew J. Katz, and Joseph S. B. Mitchell. Computing the visibility graph of points within a polygon. In Proc. 20th ACM Symp. Comput. Geom., pp 27–35, 2004.

[9] Leonidas J. Guibas, John Hershberger, Daniel Leven, Micha Sharir, and Robert Endre Tarjan. Linear-time algorithms for visibility and shortest path problems inside triangulated simple polygons. Algorithmica, 2:209–233, 1987.

[10] Bernard Chazelle and Leonidas J. Guibas. Visibility and intersection problems in plane geometry. Discrete. & Comput. Geom., 4:551–581, 1989.

[11] Subir Kumar Ghosh and David M. Mount. An output-sensitive algorithm for computing visibility. SIAM J. Comput., 20(5):888–910, 1991.

[12] Alireza Zarei and Mohammad Ghodsi. Efficient computation of query point visibility in polygons with holes. In Proc. 21st Ann. Symp. Comput. Geom., pp 314–320, 2005.

[13] Subir Kumar Ghosh. Visibility algorithms in the plane. Cambridge university press, 2007.

[14] Kevin Buchin, Irina Kostitsyna, Maarten Löffler, and Rodrigo I. Silveira. Region-based approximation algorithms for visibility between imprecise locations. In Proc. 17th Workshop Alg. Eng. Exp., ALENEX, 2015, pp 94–103, 2015.
[15] Mohammad Ali Abam, Sharareh Alipour, Mohammad Ghodsi, and Mohammad Mahdian. Visibility testing and counting for uncertain segments. *CCCG 2017*, p 84, 2017.

[16] Erin W. Chambers, Alejandro Erickson, Sándor P. Fekete, Jonathan Lenchner, Jeff Sember, S. Venkatesh, Ulrike Stege, Svetlana Stolpner, Christophe Weibel, and Sue Whitesides. Connectivity graphs of uncertainty regions. *Algorithmica*, 78(3):990–1019, 2017.

[17] Thomas J Schaefer. The complexity of satisfiability problems. In *Proc. 10th ann. ACM sympos. Theory computing*, pp 216–226, 1978.

[18] David Lichtenstein. Planar formulae and their uses. *SIAM J. Computing*, 11(2):329–343, 1982.

[19] P Laroche. Planar 1-in-3 satisfiability is NP-complete. *Comptes Rendus de l’Académie des Sciences Serie I-Mathematiques*, 316(4):389–392, 1993.

[20] Cristopher Moore and J. M. Robson. Hard tiling problems with simple tiles. *Discrete & Comput. Geom.*, 26(4):573–590, 2001.

[21] Jan Kratochvíl. A special planar satisfiability problem and a consequence of its NP-completeness. *Discrete Appl. Math.*, 52(3):233–252, 1994.

[22] Craig A. Tovey. A simplified NP-complete satisfiability problem. *Discrete Appl. Math.*, 8(1):85–89, 1984.

[23] Christos H. Papadimitriou and Mihalis Yannakakis. Optimization, approximation, and complexity classes. *J. Comput. Syst. Sci.*, 43(3):425–440, 1991.

[24] Elias Dahlhaus, David S. Johnson, Christos H. Papadimitriou, Paul D. Seymour, and Mihalis Yannakakis. The complexity of multiterminal cuts. *SIAM J. Computing*, 23(4):864–894, 1994.

[25] Joseph O’Rourke. *Art Gallery Theorems and Algorithms*. Oxford University Press, Inc., New York, NY, USA, 1987.

[26] Svante Carlsson, Håkan Jonsson, and Bengt J. Nilsson. Finding the shortest watchman route in a simple polygon. *Discrete & Comput. Geom.*, 22(3):377–402, 1999.

[27] Joseph S. B. Mitchell. Approximating Watchman Routes, In *Proc. 24th Ann. Symp. Discrete Alg.*, pp 844–855, 2013.

[28] Joachim Gudmundsson and Pat Morin. Planar visibility: testing and counting. In *Proc. 26th Symp. Comput. Geom.*, pp 77–86, 2010.

[29] Inhyuk Moon, Jun Miura, and Yoshiaki Shirai. On-line viewpoint and motion planning for efficient visual navigation under uncertainty. *Robotics Autonomous Systems*, 28(2-3):237–248, 1999.

[30] Amy J. Briggs. An efficient algorithm for one-step planar compliant motion planning with uncertainty. *Algorithmica*, 8(3):195–208, 1992.

[31] Bruce Randall Donald. The complexity of planar compliant motion planning under uncertainty. *Algorithmica*, 5(3):353–382, 1990.

[32] Rafael Murrieta-Cid, Héctor H. González-Baños, and Benjamín Tovar. A reactive motion planner to maintain visibility of unpredictable targets. In *Proc. 2002 IEEE Int. Conf. Robotics Automation, ICRA*, pp 4242–4248, 2002.

[33] Reuven Cohen and David Peleg. Convergence of autonomous mobile robots with inaccurate sensors and movements. *SIAM J. Computing*, 38(1):276–302, 2008.
[34] Johan Håstad. Some optimal inapproximability results. *J. ACM*, 48(4):798–859, 2001.

[35] Michel X Goemans and David P Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *J. ACM*, 42(6):1115–1145, 1995.

[36] Takao Asano and David P. Williamson. Improved approximation algorithms for MAX SAT. *J. Algorithms*, 42(1):173–202, 2002.

[37] Takao Asano. An improved analysis of Goemans and Williamson’s LP-relaxation for MAX SAT. *Theor. Comput. Sci.*, 354(3):339–353, 2006.

[38] Matthias Poloczek and Georg Schnitger. Randomized variants of Johnson’s algorithm for MAX SAT. In *Proc. 22nd Ann. Sympos. Discrete Alg.*, pp 656–663, 2011.