JANUS: A bit-wise reversible integrator for N-body dynamics

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ABSTRACT
Hamiltonian systems such as the gravitational N-body problem have time-reversal symmetry. However, all numerical N-body integration schemes, including symplectic ones, respect this property only approximately. In this paper, we present the new N-body integrator JANUS, for which we achieve exact time-reversal symmetry by combining integer and floating point arithmetic. JANUS is explicit, formally symplectic and satisfies Liouville’s theorem exactly. Its order is even and can be adjusted between two and ten. We discuss the implementation of JANUS and present tests of its accuracy and speed by performing and analyzing long-term integrations of the Solar System. We show that JANUS is fast and accurate enough to tackle a broad class of dynamical problems. We also discuss the practical and philosophical implications of running exactly time-reversible simulations.

Key words: methods: numerical — gravitation — planets and satellites: dynamical evolution and stability

1 INTRODUCTION
Many astrophysical systems including the gravitational N-body problem can be modeled as Hamiltonian systems. We approximate all N bodies as point particles with positions $r_i$, velocities $v_i$, and constant masses $m_i$. Ignoring any external perturbations, the Hamiltonian of the system is then given by

$$H = \sum_i \frac{1}{2} m_i v_i^2 - \sum_i \sum_{j \neq i} \frac{G m_i m_j}{|r_i - r_j|},$$

(1)

The resulting equations of motion exhibit time-reversal symmetry or time symmetry, i.e. they are invariant under the reversal of the direction of time. An equivalent statement is that if we evolve a system for an arbitrary time $t$, reverse the sign of the velocity of each particle, and evolve the system once again for a time $t$, then the system will return to its initial conditions.

In the context of the gravitational dynamics of planetary systems, the equations of motion are too complex to solve analytically except in the simplest cases. One therefore has to rely on numerical methods to solve for the system’s evolution. Finite floating point representation on a computer introduces roundoff errors that cause deviations from the true trajectory. Furthermore, integrations forward and then backward in time will not return to the initial point in phase space, since floating point operations are in general irreversible and thus the errors committed in each direction are different. This is true even for schemes that are formally time-reversible (i.e., if computers had infinite precision), like leap-frog and many, though not all, implicit and symplectic methods.

It is desirable to have the same symmetries in numerical schemes as in the original dynamical system since they lead to conserved quantities (Noether’s theorem). Furthermore time-reversibility is also intimately related to Liouville’s Theorem. Any integrator that is not bijective (and therefore also not time symmetric) does not keep the phase-space distribution function constant along trajectories, and will introduce numerical dissipation.

In this paper, we present a new formally symplectic integrator JANUS that is exactly time-reversible. We achieve this by replacing critical floating point operations with integer arithmetic. The force calculation is still subject to round-off errors due to floating point arithmetic. However, since we make exactly the same errors going forward and backwards in time, the integrator is time-reversible. JANUS is explicit, based on a generalized leap-frog method and its order can be chosen between 2, 4, 6, 8 and 10. We show that JANUS is fast enough to run long-term integration of complex systems like the Solar System over dynamically interesting and astronomically relevant timescales.

The rest of this paper is structured as follows. In Sec. 2 we describe bit-wise reversible integrators including the new JANUS algorithm. Sec. 3 discusses the implementation of JANUS. Then in Sec. 4 we present tests of the algorithm, including a long-term integration of the Solar System. Finally, in Sec. 5 we discuss our results and speculate about implications, both practical and philosophical.
2 BIT-WISE REVERSIBLE NUMERICAL INTEGRATORS

We consider an $N$-particle system and assume that the equations of motions can be written in the form
\[ \dot{r}_i(t) = F_i(r_0, \ldots, r_{n-1}, t) \quad i = 0, \ldots, N - 1. \]  
Specifically, we assume that the force is not velocity dependent. This is the case for the equations of motion derived from the Hamiltonian in Eq. [1] and in general from any Hamiltonian that can be split into a kinetic and potential term.

2.1 Levesque-Verlet integrator

Levesque & Verlet [1993] presented the first integrator for molecular dynamics that is time-reversible bit by bit. The Levesque-Verlet integrator uses the position vectors at times $t_n$ and $t_0$ to generate new position vectors at time $t_{n+1}$:
\[ r^{n+1}_i = 2 \cdot r^n_i - r^{n-1}_i + h^2 F^n_i \]  
where $F^n_i$ is the force felt by particle $i$ at time $t_n$ and $h$ is the time step $h = t_{n+1} - t_n = t_n - t_{n-1}$. With this algorithm, the state of the system is fully specified by the particles’ positions at the current and previous timestep, so the particle velocities are never explicitly used or calculated. It therefore requires a warmup step at some time $t_0$ to generate the second set of particle positions from the initial conditions. Aside from this complication, the above algorithm is formally time-reversible. To reverse the integration, one only has to swap the two position vectors in Eq. [3]
\[ r^{n+1}_i \leftrightarrow r^{n-1}_i \]  
However, a naïve implementation would not be reversible bit by bit on a computer. This is because in floating point arithmetic, the finite representation of numbers leads in general to an irreversible loss of digits under even basic operations like additions and subtractions. This roadblock can be circumvented by instead storing the positions as integers, whose arithmetic operations are bijective and therefore invertible. To achieve this, we put a sufficient fine grid on the computational domain. Assuming a typical dynamic range in the coordinates in problems we are trying to solve, a grid of $2^{128}$ on the computational domain. Assuming a typical dynamic range in the coordinates in problems we are trying to solve, a grid of $2^{128}$ on the computational domain.

The force $F$, however, must still be calculated using standard floating point arithmetic because of square root and division operators. These operations cannot be bijectively implemented with integer arithmetic. After multiplying the force with the timestep squared, we convert (round) from floating point numbers to the nearest integer on our grid. We denote this rounding operation with square brackets, $[\cdot]$, in Eq. [3].

Note that even though we perform part of the calculation with floating point operations, the algorithm is nevertheless exactly bitwise time-reversible. The crucial feature is that all the irreversible operations are performed in the middle of the step using the $r^n$ coordinates only, so that we re-calculate exactly the same floating point force value (and rounded integer representation) whether stepping forward or backward. Performing the final operations in integer arithmetic guarantees that the step is exactly time-reversible.

We note that [Rannou (1974)] and [Earn & Tremaine (1992)] looked at similar methods to the one described here, but focused on the low dimensional standard map.

2.2 Leap-frog

There are two disadvantages that make the Levesque-Verlet integrator uninteresting for the gravitational N-body problem. First, a warmup step needs to be performed at the beginning and end of the simulation, as well as whenever velocity information is needed. Second, the integrator is only second order. Thus, a relative precision of $\sim 10^{-10}$ in a simulation of the Solar System would require a timestep of $\sim 0.0008$ days. This is not competitive with standard mixed-variable symplectic integrators, which achieve similar precision with timesteps that are 10000 times longer.

Inspired by the Levesque-Verlet scheme, we came up with a bit-wise reversible version of the standard leap-frog algorithm that is symplectic and can be easily generalized to higher order methods.

As in the Levesque-Verlet integrator, we use an integer grid to represent the positions of the particles; however, we now also discretize the particle velocities. We can then write an integer version of the standard leap-frog integrator:
\[ r_{n+0.5} = r_n + \left[ \frac{h}{2} v_n \right] \]  
\[ v_{n+1} = v_n + [h F_{n+0.5}] \]  
\[ r_{n+1} = r_{n+0.5} + \left[ \frac{h}{2} v_{n+1} \right]. \]

As before, the force is evaluated using floating point arithmetic, as are the multiplications with the timestep. We also reuse the square bracket, $[\cdot]$, to denote the rounding operation from floating point numbers to the integer grid.

The nontrivial force calculation that modifies the particle velocities occurs in the middle of the timestep, yielding identical results running forward or backward. Furthermore, the positions are updated with the velocities evaluated at their respective ends of the timestep. These features, combined with the integer grid, yield a formally and bitwise time-reversible algorithm.

Furthermore, the scheme does not require a warmup step, and both velocity and position information is available at every timestep. To reverse the integration one can either change the sign of the timestep, $h \rightarrow -h$, or apply the involution $(r_n, v_n) \rightarrow (r_{n+1} - v_n)$. Note that this works only because the IEEE754 rounding conventions are symmetric about zero.

The first and third step in Eq. [5] include a rounding operation because we express the timestep as a floating point number. Note that one can also express the timestep as an integer (Syer & Tremaine [1995]). If the timestep is an inverse power of two, then the multiplication can be implemented as a simple bit-shift on the integer representation of the velocity.

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1 Although this depends somewhat on the range of scales in the problem. For example, a system with a tight binary and an additional wide companion might require a finer grid.

2 The Levesque-Verlet integrator can be thought of as a kick-drift-kick leap-frog with a specific warmup-step.
2.3 Higher order leap-frog: JANUS

The bit-wise time-reversible leap frog integrator described in Sect. 2.2 can be generalized to higher order with the Baker-Campbell-Hausdorff (BCH) formula (see e.g. Haar et al. [2006]). Let us define \( \varphi_{\gamma} \) as an operator that evolves the system for a time \( \gamma \cdot h \) with the bit-wise reversible leap-frog integrator. Higher order integrators can then be constructed using compositions of \( \varphi \). For example, consider the operator

\[
\Phi_{h}^{(6)} = \varphi_{\gamma_{1}} \circ \varphi_{\gamma_{2}} \circ \varphi_{\gamma_{3}} \circ \varphi_{\gamma_{4}} \circ \varphi_{\gamma_{5}} \circ \varphi_{\gamma_{6}} \circ \varphi_{\gamma_{7}} \circ \varphi_{\gamma_{8}}. \tag{6}
\]

For a suitable choice of constants \( \gamma_{1}, \ldots, \gamma_{8} \), the operator \( \Phi_{h}^{(6)} \) corresponds to a 6th order integrator in \( h \) (Kahan & Li [1997], Pihajoki [2015]). Integrators with other orders can be constructed in a similar way. We have implemented orders 2 (i.e. just the operator \( \varphi \)), 4, 6, 8, and 10. We will refer to these integrators as JANUS, and more specifically as \( \Phi^{(2)} \), \( \Phi^{(4)} \), \( \Phi^{(6)} \), \( \Phi^{(8)} \), and \( \Phi^{(10)} \), depending on the order used. Although it is straightforward to construct even higher order integrators this way, we show in Sect. 3 that it seems unlikely that those would be useful for the gravitational N-body problem.

We note that for higher order methods, some of the \( \gamma \) coefficients are negative. Whereas this gives a formally high order integrator, these methods are in general not very popular. In practical terms, the issue is that negative sub-steps lead to longer sub-steps (in absolute terms) and some of them can become comparable to the original timestep. So despite more function evaluations, the integrator does not sample smaller timescales. Often a lower order integrator with more timesteps and therefore better sampling of small timescales is more reliable than a high order integrator. This is the case with any integrator that can be described by Eq. 6 and is not specific to bit-wise reversibility.

2.4 Symplecticity

The question of whether to call JANUS symplectic or not is more complicated than it might seem.

We first note that any numerical N-body scheme, including JANUS, can only be symplectic to the level of the floating point precision. An integrator’s deviation from perfect symplecticity can be measured directly and it is indeed non-zero for all numerical schemes (Hernández [2016]). Thus, statements in the literature on whether a given integrator is symplectic refer only to the formal scheme, ignoring the floating point representation of coordinates. We refer to these integrators as being formally symplectic.

If we ignore the floating point errors for a moment, then an important property of such formally symplectic schemes is that they solve the nearby surrogate Hamiltonian exactly, explaining their good long-term numerical behaviour (e.g. Saha & Tremaine [1992]).

A truly symplectic integrator, or in other words a symplectic map, is also a diffeomorphism, i.e. a one-to-one map on the phase-space that is smooth and has a smooth inverse. No numerical integrator implemented on a computer can satisfy this condition for two reasons. First, coordinates always have to be discretized, either on a floating point or integer grid. Thus the standard notion of smoothness or differentiability breaks down. Second, all integrators, with the exception of JANUS, are also not bijective, a requirement for being a diffeomorphism. We could therefore conclude that JANUS is more symplectic than all other integrators in the sense that it at least is a bijection on the phase-space, even though it is still not a diffeomorphism.

One reason as to why this might matter is that even though JANUS is not exactly symplectic in the above sense, it does satisfy Liouville’s theorem exactly. Liouville’s theorem states that the phase-space distribution function along trajectories is constant. Satisfying this condition is not equivalent to symplecticity because it does not state anything about the topology. To our knowledge, JANUS is the only N-body integrator that does satisfy Liouville’s theorem exactly. This is a direct consequence of JANUS being a bijective operator on the discretized phase-space.

We can show that time reversibility implies bijectivity through a simple proof by contradiction. Suppose that two phase space points at time \( t \) were mapped by the time-reversible JANUS scheme onto the same phase space point at time \( t_{n+1} \). Then integrating backwards from the new phase space point by one timestep, one would again arrive at a single point in phase space. This means that at least one of the trajectories was not time reversible, contradicting our assumption. Thus time reversibility and bijectivity are equivalent properties of Hamiltonian systems. Intuitively, they prevent the contraction or expansion of bundles of trajectories in phase space, as required by Liouville’s theorem.

We speculate about the possible implications in the discussion section (Sect. 5). For now let us summarize that the family of JANUS integrators, \( \Phi^{(n)} \), is explicit, both formally and bit-wise time-reversible, \( n \)-th order accurate, exactly satisfying Liouville’s theorem, and formally symplectic.

3 IMPLEMENTATION

We implemented the five bit-wise reversible integrators, \( \Phi^{(2)} \), \( \Phi^{(4)} \), \( \Phi^{(6)} \), \( \Phi^{(8)} \), and \( \Phi^{(10)} \), which we collectively refer to as JANUS, in the open source REBOUND framework (Rein & Liu [2012]). We also modified the Simulation Archive (Rein & Tamayo [2017]) to work seamlessly with JANUS.

We found that a 64-bit integer grid for both the positions and velocities is well suited for the Solar System case. Using 32-bit integers would impose a floor on the relative position and velocity accuracy of \( 2^{-32} \approx 10^{-10} \). Since the semi-major axes of Mercury and Neptune differ by almost two orders of magnitude, this would translate to relative position and velocity accuracies of at best 10\(^{-8}\). In order to compare results to standard Wisdom-Holman schemes (Wisdom & Holman [1991], Kinoshita et al. [1991]), which typically have relative energy errors of less than 10\(^{-9}\), we need at least 64-bit integers. Using 128-bit integers yields no advantages since the forces are evaluated in floating point numbers, which have a relative precision limit of approximately 10\(^{-16}\). We thus set 64-bit integers as the default data-type in JANUS. However, our implementation allows us to easily change the integer datatype should another problem require a different precision.

Signed 64-bit integers range from approximately \(-2^{19}\) to \(2^{19}\). To convert from a floating point number \( x_f \) to an integer \( x_i \), we need to define a grid scale \( s \). We can then calculate the integer representation of \( x_f \) on our grid \( x_i = \lfloor x_f / s \rfloor \), where the square bracket denotes the rounding operation. The inverse operation is

\[ x_f = s \lfloor x_f / s \rfloor. \]

\[ \text{Exact in the sense that there is no discretization error from the finite timestep.} \]

\[ \text{Since we are working on an integer grid, JANUS satisfies a discrete version of Liouville's theorem.} \]
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(a) leap-frog, \( t = 0 \) (b) leap-frog, \( t = 35 \) (c) leap-frog, \( t = 500 \) (d) leap-frog, \( t = 965 \) (e) leap-frog, \( t = 1000 \)

(f) JANUS, \( t = 0 \) (g) JANUS, \( t = 35 \) (h) JANUS, \( t = 500 \) (i) JANUS, \( t = 965 \) (j) JANUS, \( t = 1000 \)

Figure 1. Snapshots of two integrations with 1000 gravitationally interacting particles each. The particles are initially at rest in panels (a) and (f). They collapse in the first 500 timesteps due to self-gravity. The sign of the velocities is flipped after 500 timesteps. The system is then integrated for another 500 steps. Because the equations of motion are time symmetric, the initial conditions should be recovered. The first row shows a run with the standard leap-frog integrator which does not reproduce the initial conditions because it is not bit-wise time-reversible. The second row shows a simulation with the new JANUS integrator. JANUS reproduces the initial conditions exactly because it is bit-wise time-reversible.

4 TESTS

4.1 Time symmetry

We first test the time-reversal symmetry of JANUS. To do that we consider a simulation of 1000 gravitationally interacting particles that are initially at rest as shown in panels (a) and (f) of Fig. 1. To avoid singularities, the gravitational interactions are softened on small scales. We integrate this system forward in time for 500 timesteps. By then the system has collapsed under its own self-gravity, resembling a star cluster. We then reverse the sign of all velocities and integrate for another 500 timesteps. The top row of Fig. 1 shows snapshots of an integration with a standard leap-frog integrator whereas the bottom row shows snapshots of an integration with our new JANUS integrator (\( \Phi(2) \)).

Both integrators are formally second order, time symmetric, and symplectic. One would therefore naively expect to recover the initial conditions at the end of the above procedure. As one can see in Fig. 1: this is not the case for the standard leap-frog integrator. This is a manifestation of the non-reversibility of floating point operations as described above.

The JANUS integrator, on the other hand, recovers the initial conditions. This can be seen by comparing panels (f) and (j) of Fig. 1. The recovery of the initial conditions is exact, down to the last bit of every coordinate of every particle. We repeated the tests for the higher order version of JANUS and for longer timescales. The results are the identical.

4.2 Order

Next, we verify the order of JANUS. In this section, and for the rest of this paper, we test JANUS on the Solar System, a complex dynamical system that has been studied extensively (for a comprehensive historical review see Laskar 2012). We use initial conditions from the NASA Horizons System. More accurate ephemerides are available (Fienga et al. 2011), but given the other approximations we make (see below), these initial conditions are accurate enough for our purposes.

We integrate all eight planets forward in time for 1000 years...
with different timesteps. We set scale parameter for the positions to $10^{-16}$ AU. Thus, positions given in floating point notation are divided by $10^{-16}$ AU before being rounded to the nearest integer on our grid. The scale for the velocities is $10^{-16}$ $2\pi$ AU/yr.

The final relative energy error at the end of the integration is shown in Fig. 2. For large timesteps (right side of the plot), the error is dominated by the scheme error, i.e. the finite timestep. One can see that in this regime (for timesteps greater than $\approx 10^{-2}$), the slope of the curves for integrators $\Phi^{(n)}$ indeed correspond to the order $n$. For small timesteps (left side of the plot), the error is dominated by the larger number of timesteps required to integrate to a fixed time, and the associated increased accumulation of roundoff errors. These floating point errors are more important for higher order schemes that evaluate the forces more often than low order schemes because the high order schemes use more substeps. We can quantify this somewhat by noting that the 10th order scheme has about twice as many substeps as the 8th order one. The error scales as the square root of the number of steps (i.e. approximately a factor of 1.4). Because the plot is shown in log-scale over 18 orders of magnitude, these small factors are hardly visible in the plot. However, one can just about see that the 10th order curve is higher than the 8th order one. We note that these errors are due to a computer’s inability to calculate the forces exactly, and are unavoidable despite the fact that we have a bit-wise time-reversible integrator.

4.3 Long term energy conservation

Let us now test the long-term energy conservation of JANUS. We once again consider the Solar System as a test case. For each planet, we include a post-Newtonian correction for general relativity in the form of a simple potential (Nobili & Roxburgh 1986),

$$\Phi_{\text{GR}} = -\frac{3GmM_{E}}{c^{2}r^{2}},$$

(7)

where $m$ is the mass of the planet, $M_{E}$ is the mass of the Sun, $r$ is the heliocentric distance of the planet, and $G$ and $c$ are the gravitational constant and the speed of light, respectively. This ensures that we reproduce the correct apsidal precession frequencies for the planets, in particular that of Mercury. We ignore all other non-gravitational effects and all planet-moon systems are treated as single particles.

The Solar System is chaotic with a Lyapunov timescale of approximately 5 Myr (Laskar 1989). We integrate the system for 60 times longer, i.e. 300 Myr. We run 24 different realizations where we perturb the initial position of Mercury by 1 m. The simulations use a timestep of 0.6 days and the 6th-order version of JANUS, $\Phi^{(6)}$.

The wall-time for a single simulation in our ensemble is 41 days on an Intel Xeon CPU, E5-2697 v2, 2.70 GHz processor. For comparison, this is about the same performance per timestep as a fast Wisdom-Holman integrator (Wisdom & Holman 1991; Rein & Tamayo 2013). However, the timestep in a simulation of the Solar System run with a Wisdom-Holman integrator is typically an order of magnitude larger than what we use here.

The relative energy error as a function of time is shown in Fig. 3. The points show the energy error in 24 individual simulations whereas the solid black line shows the mean. One can see that during the entire 300 Myr interval, we maintain a relative energy error of $\approx 10^{-11}$. The dashed line shows an error growth proportional to $\sqrt{t}$. As one can see, the energy error for JANUS follows this sub-linear trend. This shows that JANUS is unbiased and follows Brouwer’s law (Brouwer 1937).

4.4 Divergence of nearby trajectories and chaos

The simulations of the Solar System that we presented in the last section will diverge with time because their initial conditions were slightly perturbed. There are two reasons for the divergence. First, because the planets’ action variables were perturbed, nearby trajectories will get out of phase. This effect is typically polynomial with time. Second, dynamical chaos will also move initially nearby trajectories further away from each other. This effect grows exponentially with time.

We can use our simulations to estimate the Lyapunov time, i.e., the timescale for the latter exponential divergence of nearby orbits. We do this by monitoring slowly varying secular quantities, specifically the eccentricity of Mercury. Focusing on variations in these secular quantities (rather than distances in all phase space coordinates) allows us to avoid issues with the quickly growing phase.
The grey dots show the difference between members of our ensemble of simulations is shown in Fig. 4. The Lyapunov timescale of the inner Solar System. Two pairs in the ensemble, the solid black line shows the mean. The dashed red line shows an exponential with a timescale of 6.5 Myr, our best fit for the Lyapunov timescale of the inner Solar System.

differences. However, because the eccentricities evolve secularly only approximately\(^6\), we average them over 500000 yrs.

The difference in the averaged eccentricity of Mercury between members of our ensemble of simulations is shown in Fig. 4. The grey dots show the difference for individual pairs. The black solid curve shows the mean. The red dashed line corresponds to an exponential growth with a timescale of 6.5 Myr. One can see that any two simulations diverge exponentially fast. After 120 Myr the differences reach order unity (the eccentricity of Mercury is \(\approx 10^{-1}\)) and the divergence saturates. Our measurement of the Lyapunov times in the inner Solar System is consistent with that of previous studies (Laskar 1989).

This test confirms that JANUS is able to accurately integrate complex dynamical systems such as the Solar System over extremely long timescales (more than \(10^9\) orbital periods of Mercury) and recover chaotic trajectories with high fidelity.

5 DISCUSSION

In this paper, we presented JANUS, the first bit-wise reversible integrator for \(N\)-body dynamics. JANUS achieves this time-reversal symmetry with a generalized leap-frog scheme that mixes floating point and integer arithmetic. We implemented different orders of JANUS (2, 4, 6, 8 10) which can be chosen by the user at runtime.

We presented several tests, showcasing that JANUS is exactly bit-wise reversible and of the desired order. We also demonstrated that JANUS is capable of accurately integrating complex dynamical systems. A total of 24 integrations of the Solar System were integrated for 300 Myr. The relative energy error remained below \(10^{-10}\), and we were able to recover the chaotic motion of the inner Solar System and measure a Lyapunov time of 6.5 Myr.

As an illustration of what bit-wise time-reversibility allows us to do, imagine a simulation of the Solar System where Mercury’s eccentricity chaotically diffuses to near unity, crossing the orbit of Venus (Laskar & Gastineau 2009). With JANUS, we can integrate this realization of the Solar System backwards in time, gradually circularizing the orbit of Mercury, and finally recover the present-day Solar System. Similarly, ignoring dissipative effects, we can integrate backwards from the end of a cosmological simulation with bound galaxies at \(z = 0\) and recapture the adopted primordial density fluctuations\(^6\).

It is important to note that the wide variety of robust results using previously available integrators suggests that bit-wise reversibility is not a limiting feature for accurately representing formally time symmetric dynamical systems. Nevertheless, there are several astrophysical as well as philosophical points and implications that would be fruitful to explore further.

First, and most obviously: if the Hamiltonian system of interest is time symmetric, one would naturally assume that an ideal numerical integrator respects this fundamental property.

Second, to our knowledge JANUS is the first \(N\)-body integrator that satisfies Liouville’s theorem. The scheme’s bijectivity (i.e. reversibility) guarantees that integrated trajectories are unique and thus that phase space distribution functions are constant along trajectories. By contrast, the rounding operations in non-reversible schemes cause multiple trajectories to map to the same point in phase space, violating Liouville’s theorem. In principle, this can lead to unphysical trajectories in phase space. Consider a chaotic trajectory that brings the system close to a previously visited point in phase space. A rounding operation could now shift the system onto the previous point, and incorrectly turn a chaotic trajectory into a periodic one. From then on, there would be two equally valid histories for the point in phase space, which clearly breaks causality (see Figs. 4 (c) and (d) in Earn & Tremaine 1992). JANUS also suffers from rounding errors, but because the scheme guarantees that mappings in phase space across a timestep are one-to-one, cases like the above are impossible.

Third, imagine a non-periodic realization of the Solar System where a planet is ejected after some time (Laskar & Gastineau 2009). By using a bit-wise time symmetric and bijective integrator such as JANUS, it is evident that an integration backwards in time must also lead to an ejection of a planet. Although this might in principle happen only after a very long time in the past, we can make this statement without even running the backwards integration using the same argument as above: Since the orbit is non-periodic, at some point the system will exhaust the volume of phase space that corresponds to bound orbits and a planet will be ejected.

This can be related to Poincare’s recurrence theorem (Caratheodory 1919). Fourth, a bit-wise reversible integrator might be useful to construct other integrators. For example, it might simplify the development of formally symplectic integrators with adaptive timesteps.

Fifth, the reversibility properties of JANUS could be particularly useful for Hamiltonian Markov Chain Monte Carlo methods. There, the time symmetry of the integrator is important to maintain detailed balance (Kendall 1994).

JANUS is available within the REBOUND package which can be downloaded at https://github.com/hanno/rebound. We also provide all notebooks necessary to reproduce the plots.

\(^6\) Small perturbations from nearby planets also change eccentricities on orbital timescales. Often, this is not important. However, it matters for us because we are interested in eccentricity differences of the order of \(10^{-9}\).

\(^7\) This should not be confused with an attempt to recover the primordial density fluctuations from the present day matter distribution, which is not possible.
in this paper at [https://github.com/hannorein/JanusPaper](https://github.com/hannorein/JanusPaper).
The SimulationArchives (Rein & Tamayo 2017b) for the long-term integrations of the Solar System are hosted on zenodo (Rein & Tamayo 2017a).

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