Uniform function constants of motion
and their first-order perturbation

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Abstract
The main purpose of this work is to present some uniform function constants of motion rather than the well-known quantities arising from spacetime symmetries. These constants are usually associated with the intrinsic characteristics of the trajectories of a particle in a central potential field. We treat two cases. The first is the Lenz vector which sometimes is found in the literature [1, 2]; the other is associated with the isotropic harmonic oscillator, of relative importance in some simple models of the classical molecular interaction. The first example is applied to describe the perturbation of the trajectories in the Rutherford scattering and the precession of the Keplerian orbit of a planet. In the other case the conserved quantity is a symmetric tensor. We find the eigenvectors and eigenvalues of that tensor while at the same time we obtain the solution to the problem of calculating the rotation rate of the orbits in first order of a perturbation parameter in the potential energy, by performing a simple coordinate transformation in the Cartesian plane. We think that the present work addresses many aspects of mechanics with a didactical interest in other physics or mathematics courses.

1. Introduction

It is well known that an isolated mechanical system has a set of conserved quantities associated with its invariance property under the Galilean group. In the Newtonian formulation of the equations of motion this property is characterized by the absence of a total external force and an external net torque acting on the system; it is also required that the internal force between a pair of particles depends only on the distance between the particles. These conserved

1 For the conservation of the angular momentum it is enough that the internal force between a pair of particles is along the same line of action.
quantities are the well-known linear momentum and angular momentum, both as conserved vectors, and the total energy\(^2\) considered a scalar quantity.

Curiously, these are not all the conserved quantities for a given particular system. It may well happen that we have other conserved quantities (which may be scalars, vectors or, in general, tensors), even for systems which are not isolated. In all cases the conserved quantities are uniform functions of the state of the system (positions and velocities) [1]. Some known examples are the Lenz vector for a particle in the Coulomb or Kepler central potential and the ‘position–velocity’ tensor for the spatial isotropic oscillator [2–4]. The existence of these conserved quantities is useful to find the equation of the trajectory in a simple and quite straightforward manner and to study the changes in those trajectories due to a small perturbation in the original potential energy of the system.

In this work we shall consider the cases of central potentials of the forms \(1/r\) and \(r^2\) and their perturbations of the type \(r^n\) for some particular values of \(n\).

2. The Lenz vector

Let us consider a particle of mass \(m\) in a central field with the potential energy

\[ U(r) = \alpha/r, \]

where \(\alpha\) is a positive or negative constant. The vector expression (the Lenz vector)

\[ \vec{A} = \vec{v} \wedge \vec{J} + \alpha \vec{r}/r, \]

is a uniform function of the constant of motion \((\vec{r}, \vec{v})\) [1]. In equation (2) \(\vec{J}\) is the constant angular momentum of the particle, with respect to the centre of force, and \(\vec{v}\) is the velocity of the particle.

Since \(\vec{A}\) is a constant vector it may be computed at any point on the trajectory of the particle. The trajectory may be an ellipse, a parabola or a hyperbola, depending on the value of \(\alpha\) and the energy of the particle. The orbits are symmetric with respect to the pericentre where \(\vec{r}\) and \(\vec{v}\) are mutually orthogonal. A simple calculation shows that \(\vec{A}\) is along the line joining the centre of symmetry with the pericentre. A first application of the Lenz vector it is to obtain the equation of the trajectory by taking the scalar product of equation (2) with \(\vec{r}\).

For instance, if we are interested in the scattering of positive particles by an atomic nucleus (the Rutherford scattering) we take \(\alpha > 0\). In this case the energy can only be positive or null and the motion is infinite. The trajectory is a hyperbola \((E > 0)\), or a parabola \((E = 0)\). Let us consider the case of a hyperbola. The initial data may be chosen as the initial velocity at infinity, \(\vec{v}_0\), and the impact parameter \(\rho\) (see figure 1). Let \((\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)\) be a basis of Cartesian orthonormal vectors; then

\[ \vec{J} = mv_0 \rho \hat{\mathbf{e}}_z, \quad \vec{v}_0 = -v_0 \hat{\mathbf{e}}_x, \]  
\[ \vec{A} = mv_0^2 \rho \hat{\mathbf{e}}_y + \alpha \hat{\mathbf{e}}_z. \]

The particle is scattered in the direction \(\hat{\mathbf{e}}\) at an angle \(\psi\) with respect to the incoming direction. The vector \(\vec{A}\) can also be expressed in terms of the outgoing parameters

\[ \vec{A} = mv_0^2 \rho \hat{\mathbf{e}} \wedge \hat{\mathbf{e}}_z + \alpha \hat{\mathbf{e}}. \]

Taking into account that

\[ \hat{\mathbf{e}} = -\cos \psi \hat{\mathbf{e}}_x + \sin \psi \hat{\mathbf{e}}_y, \]

\(^2\) The sum of the internal potential energy plus the kinetic energy corresponding to the motion of the system as a whole.
we obtain
\[ \alpha = m v_0^2 \rho \sin \psi - \alpha \cos \psi, \]  
(7)
\[ m v_0^2 \rho = m v_0^2 \rho \cos \psi + \alpha \sin \psi. \]  
(8)

From equations (7), (8) we have
\[ \tan \frac{\psi}{2} = \frac{\alpha}{m v_0^2 \rho}, \]  
(9)
which is the usual expression for the scattering angle \( \psi \) in terms of the initial data \( v_0 \) and \( \rho \).

The equation of the orbit in polar coordinates \((r, \theta)\) is obtained by taking the scalar product of equation (2) with \( \vec{r} = r(\cos \theta \hat{e}_x + \sin \theta \hat{e}_y) \):
\[ \vec{A} \cdot \vec{r} = \left( J^2/m \right) + \alpha r \Rightarrow \]  
\[ \frac{1}{r} = \frac{m}{J^2} \left[ \alpha (\cos \theta - 1) + m v_0^2 \rho \sin \theta \right]. \]  
(11)

2.1. The perturbed Rutherford scattering

Let the potential energy be of the form
\[ U(r) = \frac{\alpha}{r} + \frac{\beta}{n r^n} \Rightarrow \vec{f} = \frac{\alpha \vec{r}}{r^3} + \frac{\beta \vec{r}}{r^{n+2}}, \]  
(12)
while at the same time we keep expression (2) for the Lenz vector; \( \vec{f} \) is the central force acting on the particle. Then
\[ \frac{d\vec{A}}{dr} = \frac{\beta}{m r^{n+2}} \vec{r} \wedge \vec{f} \Rightarrow \frac{d\vec{A}}{d\theta} = \frac{\beta}{J r^n} \vec{r} \wedge \vec{J}. \]  
(13)

In writing equation (13) we have used that \((d\theta/dr) = J/m r^2\). We wish to compute the total change of \( A \), in first order in the parameter \( \beta \). To this end, we may integrate equation (13) along the unperturbed trajectory for \( \theta \) varying from 0 to \( \Theta \), where \( \Theta = \pi - \psi \). Let us recall that the Lenz vector corresponding to the unperturbed trajectory is a constant vector pointing to the pericentre in the direction \( \gamma = \Theta/2 \) (see figure 1). According to equation (13), the elementary change in \( \vec{A} \) due to the perturbation is in the plane of the trajectory and it is perpendicular to \( \vec{r} \) at each point. Because of the symmetry of the trajectory with respect to
the point of minimal distance to the centre, the total change in \( \vec{A} \) is other than zero only in the direction perpendicular to the line joining the centre of force and the pericentre. Then, to simplify the calculation it is convenient to choose a new system of orthogonal coordinates \((x', y')\) in the plane of the orbit with \(x'\) in the direction of the pericentre (see figure 2). Then,

\[
\vec{r} = r \cos \delta \hat{e}_x' + r \sin \delta \hat{e}_y'.
\]

The only component of \( \vec{r} \), in equation (14), that contributes to the integration of equation (13) is the \( x' \) component. Thus,

\[
\vec{A}(\Theta_1) - \vec{A}(0) = -\hat{e}_y' \int_0^{\Theta_1} \frac{\cos \delta}{r^{n-1}} d\theta,
\]

where \(\delta = \theta - \Theta_1/2\). The integration is along the unperturbed trajectory given by equation (11). To show a definite result let us consider \(n = 2\). The \(\cos \delta\) is obtained taking the scalar product \(\vec{A} \cdot \vec{r}\) in the \((x', y')\) coordinate system and using that \(A = (\alpha^2 + m^2 v_0^2 \rho^2)^{1/2}\). Finally,

\[
\Delta \vec{A} := \vec{A}(\Theta_1) - \vec{A}(0) = -\hat{e}_y' \int_0^{\gamma} d\theta \frac{2m\beta (\alpha \cos \theta + m v_0^2 \rho \sin \theta) \left[ \alpha (\cos \theta - 1) + m v_0^2 \rho \sin \theta \right]}{J^2 (\alpha^2 + m^2 v_0^2 \rho^2)^{1/2}}.
\]

The integration is straightforward and the result is notably simplified if we take into account that

\[
\cos \gamma = \frac{\alpha}{A}; \quad \sin \gamma = \frac{m v_0^2 \rho}{A}.
\]

The change in the scattering angle due to the perturbation in the potential becomes

\[
\delta(\Theta/2) \simeq \frac{|\Delta \vec{A}|}{A} = \frac{\beta m}{2J^2}(\Theta - \sin \Theta).
\]

Writing \(\beta = m \tilde{\beta}\); i.e., considering the perturbing potential energy per unit mass, and introducing the angular momentum per unit mass \(\tilde{H} := J/m\), equation (17) can be expressed as

\[
\delta(\Theta/2) \simeq \frac{\tilde{\beta}}{2\tilde{H}^2}(\Theta - \sin \Theta).
\]

We see from equation (18) that \(\delta(\Theta/2)\) actually does not depend on the mass of the particle as is general in the case of a force field which is proportional to the gradient of a potential energy. This remark will be useful in some applications below.
2.2. The perturbed Kepler orbits

An interesting and somehow important case is to study the shift of the Keplerian orbit of a planet in the solar system due to a possible widening by a massive bulge around the equator of the Sun. Its external gravitational potential energy per unit mass, $\tilde{U}(r, \nu)$, will depend on the distance to the centre of the Sun and the azimuthal angle $\nu$ measured from the polar axis. Hence, if we developed the potential $\tilde{U}(r, \nu)$ in spherical harmonics, the leading two terms are of the form

$$\tilde{U}(r, \nu) = -\tilde{\alpha} \frac{r}{3} \cos^2 \nu - \frac{1}{r^3} + O\left(\frac{1}{r^4}\right). \quad (19)$$

The factor $D$ depends, among other quantities, on the deformation of the sphere represented by the extra mass along the equator. We are usually interested in the orbit of a planet in the equatorial plane where $\nu = \pi/2$, then the potential $\tilde{U}(r, \nu)$ acts on it as it has the purely radial dependence

$$\tilde{U}(r) = -\tilde{\alpha} \frac{r}{3} + \tilde{\beta} \frac{r^3}{3}; \quad \tilde{\alpha} > 0. \quad (20)$$

The potential energy of a planet of mass $m$ is $U(r) = m\tilde{U}(r)$, which is of the form (12) with $n = 3$.

In what follows we shall study the perihelion shift per revolution considering a small perturbation of the Keplerian potential energy of the form $\beta/r^n$, in the cases $n = 2, 3$.

**Case $n = 2$.** Let

$$U(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2}. \quad (21)$$

Then, using the expression $\tilde{A} = \tilde{v} \wedge \tilde{J} - a\tilde{r}/r$, we have

$$\frac{d\tilde{A}}{dr} = \frac{2\beta}{mr^3} \tilde{e}_r \wedge \tilde{J} \Rightarrow \frac{d\tilde{A}}{d\theta} = \frac{2\beta}{r} \tilde{e}_r \wedge \tilde{e}_z, \quad (22)$$

where we have used that $\tilde{J} = mr^2 \dot{\theta} \tilde{e}_z$ in a Cartesian orthogonal system with $(\tilde{e}_x, \tilde{e}_y)$ in the equatorial plane. The unperturbed orbit can be written as

$$\frac{p}{r} = 1 + e \cos \theta, \quad (23)$$

where $e = A/\alpha$; $p = J^2/ma$. To obtain equation (23) we have considered that $\tilde{A}$ is constant and it points to the perihelion. If the (perturbed) potential energy is given by equation (21) the vector $\tilde{A}$ will again point to the perihelion after one revolution but it will be shifted by an angle $\Delta \theta = |\Delta \tilde{A}|/A$. To calculate $\Delta \tilde{A}$, in the first order of $\beta$, we integrate (22) along the unperturbed orbit from $\theta = 0$ to $\theta = 2\pi$:

$$\Delta \tilde{A} := \tilde{A}(2\pi) - \tilde{A}(0) = -\frac{2\beta}{p} \tilde{e}_z \wedge \int_0^{2\pi} \tilde{e}_r(1 + e \cos \theta) d\theta = -\frac{2\beta m \pi A}{J^2} \tilde{e}_y. \quad (24)$$

Then,

$$\Delta \theta = -\frac{2\beta \pi}{H^2}, \quad (25)$$

where we have introduced again the constants per unit mass $\tilde{\beta}$ and $H$.

**Case $n = 3$.** Let now

$$U(r) = -\frac{\alpha}{r} + \frac{\gamma}{r^3}. \quad (26)$$

3 This change in the sphericity of the Sun will show as a significant quadrupolar term in its external gravitational potential.
By a similar calculation to the previous case we obtain the result
\[ \Delta \theta = \frac{6 \tilde{\gamma} \pi \tilde{\alpha}}{H^4}, \]  
where \( \gamma = \tilde{\gamma} m, \alpha = \tilde{\alpha} m \) and \( J = H m \). This case corresponds to a flattening of the solar sphere into an ellipsoid (or equivalently, to a surplus mass located at a ring around the equator of the Sun, as we have discussed). Astronomers observe the shift in the perihelion motion after many revolutions of the planet. In particular, we may express the perihelion shift per century by the formula
\[ P := \frac{\Delta \theta}{T} = -\frac{6 \pi \tilde{\gamma} \tilde{\alpha}}{H^4 T}, \]  
where \( T \) is the period of revolution expressed in units of centuries\(^4\). Since \( r^2 \left( \frac{d\theta}{dt} \right) = H \), we have that, approximately, \( 2\pi R^2 = HT \); where \( R \) is the mean of the planet–Sun distance. Then, we can write equation (28) as
\[ P = \frac{6 \pi \tilde{\gamma} \tilde{\alpha} T^3}{(2\pi)^4 R^8}. \]  
Finally, if we use Kepler’s third law: \( (T^2/R^3) = C \), where \( C \) is a constant of the same value for all the planets\(^5\), we have
\[ P = KR^{-7/2}. \]  
where \( K \) is a constant that resumes all the previous constants and it is the same for all the planets. We note that the perihelion shift is greater for the planets closer to the Sun. The measured value [5] of \( P \) for Mercury is \( P_M = (43.11 \pm 0.45) \) arc seconds per century. Since the value of \( \tilde{\gamma} \) for the Sun is quite uncertain, we may obtain the value of \( K \) from the values \( P_M \) and \( R_M \) for Mercury and compute the values of \( P \) from equation (30). The best fit to the observational data [6] is given by a straight line in the form
\[ \ln P = a \ln R + b, \]  
with slope \( a = -2.30 \pm 0.26 \). The theoretical value according to equation (30) is \( a = -3.5 \), which is outside the range of uncertainty of the observational data and shows that the model is not at all satisfactory. To overcome this difficulty was one of the results contained in the three classical tests of general relativity which predicts [6] a slope \( a = -2.5 \) for the residual perihelion advance of the planets observed by the astronomers\(^6\).

3. The isotropic harmonic oscillator

By an isotropic harmonic oscillator we mean a particle of mass \( m \) in a central field of force with a potential energy given by
\[ U(r) = \frac{1}{2} kr^2. \]  
Clearly, the equation of motion is separable in Cartesian coordinates. The motion of the particle is in a plane that we choose as the \((x, y)\) one. Thus the functions of motion, with an adequate choice of the origin of \( t \) and the orientation of the coordinate system, can be written as
\[ x(t) = A \cos \omega t, \quad y(t) = B \sin \omega t, \]
where \( \omega^2 = k/m = 1/\alpha \). The trajectory corresponding to equation (33) is an ellipse with its centre at the origin and axis along the direction of the coordinate axis:

\[
\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1.
\]

(34)

It is straightforward to show that the tensor

\[
F_{ij} = x_i x_j + \alpha v_i v_j,
\]

(35)

where \( x_i = (x, y) \); \( v_i = (v_x, v_y) \); \( \alpha \omega^2 = 1 \) is a constant of motion; i.e.,

\[
\frac{dF_{ij}}{dt} = 0,
\]

(36)

when the equations of motion are satisfied. From equations (33) and (35) the components of \( F \) are

\[
F_{ij} = \begin{pmatrix} A^2 & 0 \\ 0 & B^2 \end{pmatrix}.
\]

(37)

The eigenvalues of \( F_{ij} \) are \( A^2 \) and \( B^2 \); the corresponding eigenvectors are along the direction of the axis of symmetry of the ellipse (34).

3.1. Orbit perturbation by changing the potential in first order

We wish to study now how the trajectory changes when we perturb the potential energy by adding to it a term of the form, for instance,

\[
\delta U(r) = \frac{\beta}{r^4},
\]

(38)

where \( \beta \ll (A^2 + B^2)^3 k \). It is clear that the tensor (35) will not longer be constant. We have

\[
\frac{dF_{ij}}{dt} = \frac{4\alpha \beta}{m r^6} (x_i v_j + x_j v_i).
\]

(39)

After a lapse of one period the tensor \( F \) will no longer be diagonal as in equation (37); all its components will change by first-order terms keeping its symmetry property. However, it is possible to perform a rotation of the coordinate system by an angle \( \psi \) to have \( F \) in a diagonal form again, with the new axis of symmetry of the ellipse along the axis of the rotated coordinate system (see figure 3). To obtain the angle \( \psi \) it is enough to calculate along the
unperturbed trajectory, in the original coordinate system, the expression for $F$ after a revolution and find its eigenvectors which are, precisely, along the direction of the symmetry axis of the rotated trajectory. Then, the angle $\psi$ is given by the expression

$$\tan \psi = \frac{X_y}{X_x},$$

(40)

where $X_x$ and $X_y$ are the coordinate components of the new eigenvector that form an angle $\psi$ with the $x$-axis. To have the tensor $F$ after a period we integrate equation (39) on the unperturbed orbit for $\omega t$ from 0 to $2\pi$. Thus,

$$F_{ij}(2\pi) - F_{ij}(0) = \frac{4\alpha \beta}{m} \int_0^{2\pi} \frac{(x_i v_j + x_j v_i)}{r^6} \, d(\omega t).$$

(41)

The initial value $F_{ij}(0)$ is the tensor (37). In general $F_{ij}(2\pi)$ will have the form

$$F_{ij}(2\pi) = F_{ij}(0) + O_{ij},$$

(42)

where $O_{ij}$ is of order $\beta$. Similarly, the eigenvectors and eigenvalues of $F_{ij}(2\pi)$ can be calculated up to the first order in $\beta$. We would obtain $X_x = O(\beta)$; $X_y = 1 + O(\beta)$ and the eigenvalues $\lambda_1 = A^2 + O(\beta)$; $\lambda_2 = B^2 + O(\beta)$. By a simple calculation it is possible to show that the components of the eigenvector $X$ can be chosen as

$$X_x = 1; \quad X_y = \frac{F_{12}(2\pi)}{\lambda_1 - F_{22}(2\pi)},$$

(43)

From equations (40), (42) and (43) we can calculate the angle $\psi$ up to order $\beta$ by

$$\psi \simeq \frac{F_{12}(2\pi)}{A^2 - B^2},$$

(44)

Then, from equation (41)

$$F_{12}(2\pi) = \frac{4\alpha \beta A B}{m} \int_0^{2\pi} \frac{(\cos^2 \theta - \sin^2 \theta)}{(A^2 \cos^2 \theta + B^2 \sin^2 \theta)^2} \, d\theta = -\frac{3\pi \alpha \beta (A^4 - B^4)}{mA^4 B^4},$$

(45)

where we have put $\theta = \omega t$. Finally,

$$\psi \simeq \frac{3\pi \alpha \beta (A^2 + B^2)}{mA^4 B^4}.$$  

(46)

The velocity of rotation of the ellipse axis is

$$\Omega := \frac{\psi \omega}{2\pi} = -\frac{3\hat{\beta} (A^2 + B^2)}{2\omega A^4 B^3},$$

(47)

where $\hat{\beta} = \beta/m$ and we have used that $\alpha \omega^2 = 1$. Equation (47) coincides with the result obtained by Kotkin and Serbo [7] through a different method.

4. Final comments

The main purpose of this work was to bring to the attention of students and teachers the existence of another uniform function constant of the motion rather than the well-known quantities arising from spacetime symmetries. We have treated, in some detail, just two cases. The first is the Lenz vector which is sometimes found in the literature [1, 2]; the other is associated with the isotropic harmonic oscillator of relative importance in some simple models of classical molecular interaction. The first example was applied to describe the perturbation in the trajectories by the addition in the potential energy of a small term depending only on the radial variable. As an interesting application of the method, we computed the change in the outgoing direction of a scattered particle as predicted in Rutherford scattering. Another
problem that we considered was the precession of the Keplerian orbits of the planets in the solar system in two cases. One of the cases has especial relevance since it may be used with great simplicity to investigate the magnitude and characteristic of the effect on the trajectories of the planets if the Sun shows a massive protuberance along its equator. We revised this aspect of the problem following the discussion presented by Adler et al [6], that ends up with a comparison with the prediction and validity of general relativity.

On the other hand, we believe that the study of a spherically symmetric perturbation of the potential energy of the isotropic harmonic oscillator is an apparent opportunity to present a case in which the conserved quantity is a tensor. Thus, we have to find the eigenvectors and eigenvalues of that tensor while at the same time we obtain the rate of rotation of the orbits by performing a simple coordinate rotation in the Cartesian plane. We think that the present work addresses many aspects of mechanics with a didactical interest in other physics or mathematics courses.

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