Growth of Sobolev norms in quasi integrable quantum systems

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Abstract

We prove an abstract result giving a $\langle t \rangle^\epsilon$ upper bound on the growth of the Sobolev norms of a time dependent Schrödinger equation of the form $i\dot{\psi} = H_0\psi + V(t)\psi$. $H_0$ is assumed to be the Hamiltonian of a steep quantum integrable system and to be a pseudodifferential operator of order $d > 1$; $V(t)$ is a time dependent family of pseudodifferential operators, unbounded, but of order $b < d$. The abstract theorem is then applied to perturbations of the quantum anharmonic oscillators in dimension 2 and to perturbations of the Laplacian on a manifold with integrable geodesic flow, and in particular Zoll manifolds, rotation invariant surfaces and Lie groups. We also cover the case of several particles on a Zoll manifold or on a Lie group, possibly obeying some restrictions due to the Fermionic or Bosonic nature of the particles. The proof is based on quantum version of the proof of the classical Nekhoroshev theorem.

Keywords: Schrödinger operator, normal form, Nekhoroshev theorem, pseudo differential operators

MSC 2010: 37K10, 35Q55

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1 Introduction

In this paper we prove an abstract theorem giving a \((t)^\epsilon\) upper bound for the Sobolev norms of an abstract Schrödinger equation of the form

\[
i\dot{\psi} = (H_0 + V(t))\psi, \quad \psi \in \mathcal{H}
\]

where \(\mathcal{H}\) is a Hilbert space, and \(H_0\) is the Hamiltonian of a quantum system which is globally integrable in a sense defined below; \(H_0\) is also assumed to be a pseudodifferential operator of order \(d > 1\), and \(V(t)\) is a smooth time dependent family of self-adjoint pseudodifferential operators of order \(b < d\). The abstract result is applied to several cases in which \(H_0\) also fulfills a quantum analogue of the steepness assumption of the Nekhoroshev theorem of classical mechanics (see e.g. [Nek77, Nek79, Nie06, GCB16]).

The novelty of this result is twofold: it extends the class of unperturbed systems for which upper bounds of the Sobolev norms can be obtained, and it allows to treat the case of unbounded perturbations with order just smaller than the order of \(H_0\). We emphasize that the class of unperturbed systems treated here (globally integrable quantum systems, see Definition 2.10 below) strictly contains all the systems of order \(d > 1\) for which estimates on the growth of Sobolev norms have been obtained; it also contains all the systems of order \(d > 1\) on which KAM or Birkhoff normal form results exist and also new systems, like the quantum 2-d anharmonic oscillator. For this reason we think that our result opens the way to new extensions of KAM and Birkhoff normal form theory to systems in more than 1 space dimension (see [FM21] for a first result in this direction).

We now describe more in detail our result. First, following [BGMR21] (see also [BFKG12, Fis13]) we introduce some abstract algebras of linear operators in \(\mathcal{H}\) that enjoy the properties typical of pseudodifferential operators. We actually call the elements of these algebras pseudodifferential operators.

Then we give the definition of globally integrable quantum system. The idea is to introduce some operators \(A_j, j = 1, ..., d\) that are the quantum analogue of the classical action variables and to consider quantum systems with
Hamiltonian $H_0 = h_0(A_1, \ldots, A_d)$, namely Hamiltonians which are functions of the quantum action variables. In turn, the quantum action variables are defined to be $d$ commuting self-adjoint pseudodifferential operators of order 1, whose joint spectrum is contained in $\mathbb{Z}^d + \kappa$, with some $\kappa \in \mathbb{R}^d$. The motivation of this definition rests in the classical results by Duistermaat–Guillemin [DG75], Colin de Verdiere [CdV80], and Helffer–Robert [HR82], which ensure that, under suitable assumptions, the quantization of a classical action variable is a perturbation of a pseudodifferential operator with spectrum contained in $\mathbb{Z} + \kappa$ with $\kappa \in \mathbb{R}$. We point out that our definition, which does not involve directly the action variables of the classical system, is quite flexible, since it applies also to systems whose classical action variables are poorly known and to quantization of some superintegrable systems [Nek72, Fas05] in which the Hamiltonian is independent of some of the action variables. Finally, we assume that the function $h_0$ is homogeneous of degree $d > 1$ and fulfills the steepness assumption of the classical Nekhoroshev’s theorem (see Definition 2.11 below).

We come now to the applications of the abstract result.

The most original application is to perturbations of a 2 dimensional quantum anharmonic oscillator with Hamiltonian

$$H_0 = -\frac{\Delta}{2} + \frac{\|x\|^{2\ell}}{2\ell}, \quad x \in \mathbb{R}^2, \quad \ell \in \mathbb{N}, \quad \ell \geq 2,$$

for which we prove the $\langle t \rangle^\epsilon$ upper bound on the growth of Sobolev norms. We recall that for anharmonic oscillators the situation was clear only for the 1-d case [BGMR21], while for the higher dimensional case only the trivial $\langle t \rangle^s$ estimate for the $\mathcal{H}^s$ norm was known, for the case of bounded perturbations (see [MR17]). A $\langle t \rangle^\epsilon$ estimate was completely out of reach for the 2d case with previous methods. In our application to the anharmonic oscillator the quantum actions are constructed by exploiting the theory of [CdV80, Cha83] in order to quantize the classical action variables. The steepness assumption is verified using the results of [Nie06, BF17, BFS18]. We remark that at present we are unable to deal with the 3 dimensional case, since the topology of the foliation of the classical integrable Hamiltonian system is very different in the 2 and in the 3 dimensional cases [BF16] and the results by Colin de Verdière [CdV80] do not apply to the 3-d case.

The second application is to Schrödinger equations on manifolds with integrable geodesic flow. In this framework, in order to be determined, we present some specific examples. First (i) we recover the known results for tori

\footnote{The case of unbounded perturbation was also treated in [MR17], but in that case they got an upper bound of the form $\langle t \rangle^{\alpha_s}$, with an exponent $\alpha_s > 1$ which diverges as $b \to d$. See also [HM20] for an interesting semiclassical lower bound on the Sobolev norms.}
and Zoll Manifolds \cite{BGMR21}, then (ii) we consider rotation invariant surfaces (following \cite{CdV80, Del10}) and construct the operators $A_j$ by quantizing the classical action variables. The steepness of the classical Hamiltonian is then verified using tools from classical Hamiltonian theory (\cite{Nie06, BFS18}). The novelty of the result we get for rotation invariant surfaces is that we can deal with unbounded perturbations of the Laplacian. Another example (iii) is that of the Schrödinger equation on a Lie group (following \cite{BP11, BCP15}): it is know that the geodesic flow on a compact Lie group is integrable (\cite{MF78, Mis82, Bol04}), but very little is known on the action angle variables, so we use the intrinsic pseudodifferential calculus on Lie groups (\cite{Fis15, RT09}) to construct directly the quantum actions in the case of compact, simply connected Lie groups. Here the lattice $\mathbb{Z}^d$ is essentially the lattice of the dominant weights of the irreducible representations of the Lie group. This result is new.

The last result that we present deals with several quantum particles on a Zoll manifold or on a Lie group, including the torus $\mathbb{T}^d$; in particular we consider systems of particles which are either Fermions or Bosons, so that the wave function is either symmetric or antisymmetric in the exchange of the particle variables. Actually this application is essentially a remark, which is made possible by the fact of working in an abstract framework.

We come to a description of the proof of the abstract result. The present paper is a direct continuation of the works \cite{BLM20b, BLM22, BLM20a, BLR21}, and is based on the quantization of the proof of the classical Nekhoroshev theorem. Here in particular we develop an abstract version of the proof, which is based just on the use of the lattice of the joint spectrum of the actions.

We recall that the idea of \cite{BLM20b, BLM22, BLM20a} is to proceed in two steps: first one looks for a family of unitary time dependent operators conjugating the original Hamiltonian (1.1) to a Hamiltonian of the form

$$H_0 + Z(t) + R(t)$$

with $Z(t)$ which is a “normal form” operator and $R(t)$ a smoothing operator, which plays the role of a remainder. This was done in \cite{BLM22, BLM20a} for the Schrödinger operator on $\mathbb{T}^d$ by quantizing the classical normal form procedure. In \cite{BLM22, BLM20a}, the operator $Z(t)$ was the quantization of a symbol in normal form according to the standard definition of classical Hamiltonian perturbation theory. The second step of the proof in \cite{BLM22, BLM20a} consists in developing a quantum version of the geometric construction of Nekhoroshev theorem (see e.g. \cite{Nek77, Nek79, Gio03, GCB16, BL20} for the classical construction) to show that $Z(t)$ has a block diagonal structure and
furthermore the blocks are dyadic, so that for the dynamics of $H_0 + Z(t)$ the Sobolev norms remain bounded forever. The addition of the remainder $R(t)$ is the responsible for the $\langle t \rangle^c$ estimate on the growth.

To develop an abstract version of the above construction one has to develop several new tools. A nontrivial point consists in developing a Fourier expansion and a notion of normal form based just on the lattice of the joint spectrum. These are key steps of the proof, and some ideas are inspired by the methods developed in [BLR21], while some others are new. Moreover, the geometric part of the proof is considerably more complicated here than in [BLM22, BLM20a] because in the case of $T^d$ the operator $H_0$ is the quantization of the function $\sum_j \xi_j^2$ which is explicitly known and furthermore is convex, while we deal here with an abstract steep Hamiltonian.

We conclude this introduction by comparing the present result with previous ones. Our main references is the paper [Bou99] (see also [Del10, BM19]) in which Bourgain deals with the Schrödinger equation on the torus. Bourgain’s approach is based on a dyadic decomposition of $Z^d$ which is almost invariant for $H$ and only allows to deal with bounded perturbations of $-\Delta$ on tori. Bourgain’s result were extended to the case of unbounded perturbations in [BLM20a] which in turn is the starting point of the present work. The main new point of [BLM20a] is that it relates Bourgain’s decomposition to the resonance properties of the classical frequencies, a property which extends quite naturally to more general systems. This generalization is what is developed in the present paper. We emphasize that this approach also allows to keep track of the pseudodifferential structure of the initial operators and thus to deal with unbounded operators.

The other direct reference for our work is [BGMR21] (which in turn is closely related to [Bam18, Bam17, BM18, MR17, BGMR18]). In [BGMR21] essentially two cases were considered: systems with periodic classical flows (1-d systems and Zoll manifolds) and systems of degree $d = 1$ (half wave equation in dimension 1 and harmonic oscillators). The main novelty of the present approach is that we are able to deal with the case of quantization of “general” classical integrable systems, in which the flow is either periodic or quasiperiodic depending on the initial datum. However we restrict here to the case $d > 1$, since our approach does not apply to the case $d = 1$ in the present form. We leave this case for future work.

When $V$ is time independent, the present approach can also be used to deduce spectral properties of quasi integrable systems, following [BLM20b, BLM21, Par08, PS10, PS12]. We leave this for future work.

Acknowledgments During the preliminary work which led to this result we had
many discussions on integrable systems and Fourier integral operators, which were essential in order to understand the theory of [CdV80] and to arrive to our definition of quantum integrable system: we warmly thank Francesco Fassò, Didier Robert and San Vũ Ngọc. We also thank Bert Van Geemen and Michela Procesi for some discussions and suggestions on Lie groups, Santiago Barbieri and Laurent Niederman for discussions about steepness, and Massimiliano Berti, Benoît Grébert, Alberto Maspero and Riccardo Montalto for comments on a first version of this paper.

We acknowledge the support of GNFM.

Part I

Statements

2 Main results

2.1 Abstract pseudodifferential operators

Let $\mathcal{H}$ be a Hilbert space and $K_0$ a self-adjoint positive operator with compact inverse.

We define a scale of Hilbert spaces by $\mathcal{H}^r := D(K_0^r)$ (the domain of the operator $K_0^r$) if $r \geq 0$, and $\mathcal{H}^r := (\mathcal{H}^{-r})'$ (the dual space) if $r < 0$. We denote by $\mathcal{H}^{-\infty} = \bigcup_{r \in \mathbb{R}} \mathcal{H}^r$ and $\mathcal{H}^{+\infty} = \bigcap_{r \in \mathbb{R}} \mathcal{H}^r$. We endow $\mathcal{H}^r$ with the natural norm $\|\psi\|_r := \|(K_0)^r\psi\|_0$, where $\|\cdot\|_0$ is the norm of $\mathcal{H}^0 \equiv \mathcal{H}$. From the spectral decomposition of $K_0$ it follows that for any $m \in \mathbb{R}$, $\mathcal{H}^{+\infty}$ is a dense linear subspace of $\mathcal{H}^m$. In the following, when we say that an operator is self-adjoint or unitary, we always mean that it is self-adjoint or unitary with respect to the scalar product in $\mathcal{H}$.

We denote by $\mathcal{B}(\mathcal{H}^{s_1}; \mathcal{H}^{s_2})$ the space of bounded linear operators from $\mathcal{H}^{s_1}$ to $\mathcal{H}^{s_2}$. We will in particular consider the space $\bigcap_{s \in \mathbb{R}} \mathcal{B}(\mathcal{H}^s, \mathcal{H}^{s-m})$ which is a Fréchet space when endowed by the semi-norms $\|A\|_{\mathcal{B}(\mathcal{H}^s, \mathcal{H}^{s-m})}$.

Definition 2.1. We will say that $F$ is of order $m$ if $F \in \bigcap_{s \in \mathbb{R}} \mathcal{B}(\mathcal{H}^s, \mathcal{H}^{s-m})$. In the case of negative $m = -N$, we will also say that $F$ is $N$-smoothing.

Definition 2.2. If $\mathcal{F}$ is a Fréchet (or a Banach) space, we denote by $C^k_0(\mathbb{R}^d; \mathcal{F})$ the space of the functions $F \in C^k_0(\mathbb{R}^d; \mathcal{F})$, such that all the seminorms of
For all $j \leq k$ we write $F \in C^\infty_b(\mathbb{R}^d,F)$. Given $0 \leq \rho_0 < 1$ we introduce a family of graded algebras $A_\rho$ with $\rho \in (\rho_0,1]$ of operators encoding the fundamental properties of pseudodifferential operators. We recall that a similar construction was done in [BGMR21], but there only one algebra, which corresponds to our $A_1$, was needed.

For $m \in \mathbb{R}$ and $\rho \in (\rho_0,1)$ let $A^m_\rho$ be a linear subspace of $\bigcap_{s \in \mathbb{R}} \mathcal{B}(\mathcal{H}^s,\mathcal{H}^{s-m})$ and define $A_\rho := \bigcup_{m \in \mathbb{R}} A^m_\rho$.

**Assumption I:**

i. For each $m \in \mathbb{R}$, one has $K^m_0 \subseteq A^m_1$.

ii. For each $m \in \mathbb{R}$, and $\rho \in (\rho_0,1]$, $A^m_\rho$ is a Fréchet space for a family of semi-norms $\{p^m_{\rho,j}\}_{j \geq 1}$ such that the embedding $A^m_\rho \hookrightarrow \bigcap_{s \in \mathbb{R}} \mathcal{B}(\mathcal{H}^s,\mathcal{H}^{s-m})$ is continuous.

If $m' < m$ then $A^m_\rho \subseteq A^{m'}_\rho$ with a continuous embedding.

If $\rho_1 < \rho_2$, then $A^m_{\rho_2} \hookrightarrow A^m_{\rho_1}$ for all $m$, with continuous embedding.

iii. $A_\rho$ is a graded algebra, i.e $\forall m, n \in \mathbb{R}$: if $F \in A^m_\rho$ and $G \in A^n_\rho$ then $FG \in A^{m+n}_\rho$ and the map $(F,G) \mapsto FG$ is continuous from $A^m_\rho \times A^n_\rho$ into $A^{m+n}_\rho$.

iv. $A_\rho$ is a graded Lie-algebra: if $F \in A^m_\rho$ and $G \in A^n_\rho$ then the commutator $[F,G] \in A^{m+n-\rho}_\rho$ and the map $(F,G) \mapsto [F,G]$ is continuous from $A^m_\rho \times A^n_\rho$ into $A^{m+n-\rho}_\rho$.

v. $A_\rho$ is closed under perturbations by smoothing operators in the following sense: let $F : \mathcal{H}^{+\infty} \to \mathcal{H}^{-\infty}$ be a linear map. If there exists $m \in \mathbb{R}$ such that for every $N > 0$ we have a decomposition $F = F^{(N)} + S^{(N)}$, with $F^{(N)} \in A^m_\rho$ and $S^{(N)}$ is $N$-smoothing, then $F \in A^m_\rho$.

vi. If $F \in A^m_\rho$, then also the adjoint operator $F^* \in A^m_\rho$.

vii. For any $F \in A^m_1$, any $G \in A^1_1$, the map $\mathbb{R} \ni t \mapsto e^{itG} Fe^{-itG} \in C^\infty_b(\mathbb{R},A^m_1)$.

**Remark 2.3.** Property I.iv is the one which makes the algebras $A_\rho$ different for different values of $\rho$.

**Remark 2.4.** Property I.vii is an abstract version of Egorov theorem.
2.2 Globally integrable quantum systems

The idea is to define a quantum integrable system as a system whose Hamiltonian operator can be written as a function of some Action Operators. The Action Operators are \( d \) self-adjoint pairwise commuting operators \( A_1, \ldots, A_d \), of order 1 fulfilling the the following Assumption A. Since in particular we will give an assumption allowing to use functional calculus, we first give a definition:

**Definition 2.5.** Let \( m \in \mathbb{R} \) and let \( 0 < \varsigma \leq 1 \) be a parameter; a function \( f \in C^\infty(\mathbb{R}^d) \) is said to be a symbol of class \( S_m^\varsigma \) if

\[
|\partial^\alpha f(a)| \leq C_\alpha \langle a \rangle^{m-\varsigma |\alpha|} \quad \forall \alpha \in \mathbb{N}^d, \forall a \in \mathbb{R}^d,
\]

where \( \langle a \rangle := \sqrt{1 + \sum_j a_j^2} \).

The best constants s.t. (2.1) holds are seminorms for \( S_m^\varsigma \).

**Assumption A:**

i. For \( j = 1, \ldots, d \), the operators \( A_j \) fulfill \( A_j \in A_1^1 \).

ii. \( \exists c_1 > 0 \) s.t. \( c_1 K_0^2 < 1 + \sum_{j=0}^d A_j^2 \).

iii. There exist a convex closed cone \( C \subseteq \mathbb{R}^d \) and a vector

\[
\kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{R}^d,
\]

such that the joint spectrum \( \Lambda \) of the \( A_j \)'s fulfills

\[
\Lambda \subset (\mathbb{Z}^d + \kappa) \cap C.
\]

iv. There exist \( \varsigma_0 \in [0,1] \) and an increasing continuous function \( (\varsigma_0,1) \ni \varsigma \mapsto \rho(\varsigma) \in (\rho_0,1] \), with \( \rho(1) = 1 \), s.t., if \( f \in S_\varsigma^m \), then

\[
f(A_1, \ldots, A_d) \in A_\rho_{\rho(\varsigma)}^m.
\]

Furthermore its seminorms depend only on the seminorms of \( f \) and on the seminorms of the \( A_j \)'s.

**Remark 2.6.** Assumption A.iv allows to use functional calculus in our abstract context.
Remark 2.7. By A.ii, the operator $1 + \sum_{j=0}^{d} A_j^2$ has compact inverse, therefore the spectrum $\sigma(A_j)$ of each one of the $A_j$’s is pure point and formed by a sequence of eigenvalues.

Remark 2.8. We recall that the joint spectrum $\Lambda$ of the operators $A_j$ is defined as the set of the $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$ s.t. there exists $\psi_a \in \mathcal{H}$ with $\psi_a \neq 0$ and

$$A_j \psi_a = a_j \psi_a \ , \ \forall j = 1, \ldots, d \ . \quad (2.3)$$

Remark 2.9. The assumption A is invariant if one substitutes the operators $A_j$ with new operators $A_j'$ defined by

$$A_j' := \sum_{i=1}^{d} U_{ji} A_i \ , \quad (2.4)$$

with $U_{ji}$ a unimodular matrix with integer coefficients. This is the same invariance property of the action variables of classical integrable systems.

We are now ready to give our definition of a globally integrable quantum system:

**Definition 2.10.** [Globally integrable quantum system] We say that $H_0$ is a globally integrable quantum system if there exists $h_0 \in C^\infty(\mathbb{R}^d; \mathbb{R})$ such that

$$H_0 = h_0(A_1, \ldots, A_d) \ , \quad (2.5)$$

where $A_1, \ldots, A_d$ satisfy Assumption A and the function (2.5) is spectrally defined.

2.3 The Statement

To state our assumptions on the function $h_0$ defining $H_0$ we still need a couple of definitions:

**Definition 2.11.** Given $m \in \mathbb{R}$, a function $f \in C^\infty(\mathbb{R}^d \setminus \{0\})$ will be said to be homogeneous of degree $m$ if

$$f(\lambda a) = \lambda^m f(a) \ , \ \forall \lambda > 0 \ . \quad (2.6)$$

Homogeneous functions are typically singular at the origin, but, in the context of pseudodifferential operators, the behavior of functions in a neighborhood of the origin is not important. This is captured by the next definition.
Definition 2.12. A function \( f \in C^\infty(\mathbb{R}^d) \) will be said to be homogeneous of degree \( m \) at infinity if it fulfills (2.6) \( \forall a \in \mathbb{R}^d \setminus B_{1/4}, \ B_r \) being the ball of radius \( r \) centered at the origin.

Remark 2.13. Of course the value 1/4 has been inserted just to be determined.

We recall, from [GCB16], the definition of steepness:

Definition 2.14. [Steepness] Let \( U \subset \mathbb{R}^d \) be a bounded connected open set with nonempty interior. A function \( h_0 \in C^1(U) \), is said to be steep in \( U \) with steepness radius \( r \), steepness indices \( \alpha_1, \ldots, \alpha_{d-1} \) and (strictly positive) steepness coefficients \( B_1, \ldots, B_{d-1} \), if its gradient \( \omega(a) := \frac{\partial h_0}{\partial a}(a) \) satisfies the following estimates: 
\[
\inf_{a \in U} \| \omega(a) \| > 0 \quad \text{and for any } a \in U \quad \text{and for any } s \text{ dimensional linear subspace } M \subset \mathbb{R}^d \text{ orthogonal to } \omega(a), \quad \text{with } 1 \leq s \leq d-1, \text{ one has}
\]
\[
\max_{0 \leq \eta \leq \xi} \min_{u \in M : \| u \| = 1} \left\| \Pi_M \omega(a + \eta u) \right\| \geq B_s \xi^{\alpha_s} \forall \xi \in (0, r], \tag{2.7}
\]
where \( \Pi_M \) is the orthogonal projector on \( M \); the quantities \( u \) and \( \eta \) are also subject to the limitation \( a + \eta u \in U \).

Remark 2.15. It is well known that steepness is generic. Examples of steep functions are given by functions which are convex or quasiconvex. In the applications we will verify steepness by verifying an equivalent condition due to Niederman [Nie06] (see Theorem C.1 below).

On \( H_0 \) we assume:

Assumption H

i. \( H_0 \) is the Hamiltonian of a globally integrable quantum system, and the function \( h_0 \) of (2.5) is homogeneous of degree \( d > 1 \) at infinity.

ii. There exists a convex open set \( U \subset \mathbb{R}^d \), s.t. \( U \supset \overline{(B_2 \setminus B_{1/2}) \cap C} \), with the property that \( h_0 \) is steep on \( U \).

Theorem 2.16. Let \( H = H(t) \) be of the form
\[
H(t) := H_0 + V(t) \tag{2.8}
\]
with \( H_0 \) the Hamiltonian of a globally integrable quantum system. Assume that Assumption H holds and that \( V(\cdot) \in C^\infty_b(\mathbb{R}; \mathcal{A}_b^1) \) is a family of self-adjoint operators. Assume \( b < d \); then for any \( s \geq 0 \) and for any initial
datum $\psi \in \mathcal{H}^s$ there exists a unique global solution $\psi(t) := U(t, \tau)\psi \in \mathcal{H}^s$ of the initial value problem

$$i\partial_t \psi(t) = H(t)\psi(t), \quad \psi(\tau) = \psi,$$

furthermore, for any $s > 0$ and $\varepsilon > 0$ there exists a positive constant $K_{s, \varepsilon}$ such that for any $\psi \in \mathcal{H}^s$

$$\|U_H(t, \tau)\psi\|_s \leq K_{s, \varepsilon} (t - \tau)^{\varepsilon} \|\psi\|_s, \quad \forall t, \tau \in \mathbb{R}. \quad (2.10)$$

**Remark 2.17.** Nekhoroshev gave a counterexample showing that non steep Hamiltonian systems can exhibit unbounded growth of the actions. A quantum variant of Nekhoroshev’s counterexample is given by the Schrödinger equation

$$i\dot{\psi} = \left(-\partial_{x_1}^2 + \partial_{x_2}^2 + \varepsilon \sin(x_1 - x_2 - 2nt)\right)\psi, \quad n \in \mathbb{Z} \setminus \{0\} \quad (2.11)$$
on $\mathbb{T}^2$. Indeed, for any $n \neq 0$ it is easy to see that a particular solution of (2.11) is

$$\psi(x, t) = e^{-\varepsilon t \sin(x_1 - x_2 - 2nt)}e^{i(n^2 t + nx_2)}, \quad (2.12)$$

whose $s$ Sobolev norm grows like $(\varepsilon t)^s$.

### 3 Application 1: The anharmonic oscillator in dimension 2

We define $\mathcal{H} := L^2(\mathbb{R}^2)$ and, for $\ell \in \mathbb{N}$, $\ell \geq 2$, consider the Hamiltonian of the quantum anharmonic oscillator

$$H_0 := -\frac{\Delta}{2} + \frac{\|x\|^{2\ell}}{2\ell}, \quad x \in \mathbb{R}^2. \quad (3.1)$$

In order to define the scale of Hilbert spaces $\mathcal{H}^s$, we define

$$K_0 := (1 + H_0)^{\frac{\ell + 1}{2\ell}}.$$

To introduce the class of symbols we need, consider

$$k_0(x, \xi) := (1 + \|x\|^{2\ell} + \|\xi\|^2)^{\frac{\ell + 1}{2\ell}}.$$

For $\rho \in (\frac{\ell - 1}{\ell + 1}, 1]$, define

$$\delta_1 := \frac{1}{2} \left(\rho - \frac{\ell - 1}{\ell + 1}\right), \quad \delta_2 := \frac{1}{2} \left(\rho + \frac{\ell - 1}{\ell + 1}\right). \quad (3.2)$$
Definition 3.1. Given \( f \in C^\infty(\mathbb{R}^4) \), we will write \( f \in S_{AN, \rho}^m \) if \( \forall \alpha, \beta \in \mathbb{N}^2 \), there exists \( C_{\alpha, \beta} > 0 \) s.t.
\[
|\partial_x^\alpha \partial_\xi^\beta f(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-\rho|\alpha|-(1-\rho)|\beta|} \quad \forall (x, \xi) \in \mathbb{R}^4,
\]
with \( \delta_1, \delta_2 \) given by (3.2). We will say that an operator \( F \) is a pseudodifferential operator of class \( A_m^\rho \) if there exists a symbol \( f \in S_{AN, \rho}^m \) s.t. \( F \) is the Weyl quantization of \( f \).

Remark 3.2. When \( \rho = 1 \) the class of symbol \( A_1^m \) reduces to the standard classes used to study the anharmonic oscillator (see e.g. [HR82]). The case with \( \rho < 1 \) was studied in [BLR21].

The properties I are an immediate consequence of standard pseudodifferential calculus in \( \mathbb{R}^4 \). Following [CdV80, Cha83, BLR21], the operators \( A_1, A_2 \) will be constructed in Subsection 9.1 by quantizing the classical actions. The assumption A will be verified in Subsection 9.1 where we will prove the following:

Theorem 3.3. Consider the Schrödinger equation (2.9) with \( H_0 \) given by (3.1) and \( V(.) \in C^\infty(\mathbb{R}; A_{b1}^1) \), with \( b < \frac{2\ell}{\ell+1} \), then the corresponding evolution operator fulfills (2.10).

Remark 3.4. An example of a perturbation fulfilling the assumption of the Theorem 3.3 is the Weyl quantization of
\[
v(x, \xi, t) = \sum_{|k|+|\ell|<2\ell} c_{k, \ell}(t)x^k \xi^\ell;
\]
with \( k \in \mathbb{N}^2, |k| := k_1 + k_2 \), similarly for \( j \) and \( c_{k, \ell}(.) \in C^\infty(\mathbb{R}) \).

4 Application 2: Manifolds with integrable geodesic flow

Let \( (M, g) \) be a compact \( n \)-dimensional Riemannian manifold without boundary; to fit our scheme we define \( H := L^2(M), K_0 := \sqrt{1-\Delta_g} \), with \( \Delta_g \) the negative Laplace Beltrami operator relative to the metric \( g \), so that \( H^s \) coincides with the classical Sobolev space \( H^s \).

Definition 4.1. A function \( f \in C^\infty(T^*M) \) is said to be a symbol of class \( S_{H, \varphi}^m \) if, when written in any canonical coordinate system (in the sense of \( T^*M \) ) it fulfills
\[
|\partial_x^\alpha \partial_\xi^\beta f(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-\varphi|\beta|+(1-\varphi)|\alpha|}, \quad \forall \alpha, \beta \in \mathbb{N}^n, \quad \forall (x, \xi) \in T^*M.
\]

(4.1)
Definition 4.2. We say that $F \in A^m_{\rho}$, if it is a pseudodifferential operator (in the sense of Hörmander [Hör85]) with Weyl symbol of class $S^m_{\rho, \rho}$, with $\rho = \frac{\rho + 1}{2}$.

Then assumption I holds. Furthermore, Assumption A.iv with $\rho(\varsigma) = 2\varsigma - 1$ follows from functional calculus for any pairwise commuting operators $A_1, \ldots, A_d \in A^1_{\rho}$ such that $A := \sum_{j=1}^{d} A_j^2$ has elliptic symbol, namely $A$ is the Weyl quantization of a symbol $a$ and

$$a(x, \xi) \geq c\|\xi\|^2, \quad \forall\|\xi\| \geq R$$

and for some $c, R > 0$ (see for instance the argument in [Str72], in particular equation (4) and Lemma 1 therein).

We are now going study different specific manifolds $M$, for which we will construct the quantum actions $A_1, \ldots, A_d$ and verify the remaining assumptions. The applications are dealt with in different subsections, since the construction is different in each specific case. The result will always be that the solution of the Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = (-\Delta_g + V(t))\psi, \quad \psi \in H^s(M)$$

with $V(.) \in C^\infty_b(\mathbb{R}; A^b_1)$, $b < 2$, fulfills the estimate (2.10).

4.1 Flat tori

Let $\Gamma$ be a lattice of dimension $n$ in $\mathbb{R}^n$, with basis $e_1, e_2, \ldots, e_n$, namely

$$\Gamma := \left\{ \sum_{i=1}^{n} k_i e_i : k_1, \ldots, k_n \in \mathbb{Z} \right\},$$

and define

$$M \equiv \mathbb{T}_\Gamma := \mathbb{R}^n / \Gamma.$$  

By introducing in $\mathbb{T}_\Gamma$ the basis of the vectors $e_i$, the Laplacian is transformed in the operator

$$H_0 := \sum_{k,l} g^{kl} (-i\partial_k)(-i\partial_l),$$

with $g^{kl}$ the inverse matrix of $g_{jk} := e_j \cdot e_k$, namely $g^{kl}$ is defined by $\sum_k g^{jk} g^{kl} = \delta_j^l$. In this case one has $A_j := -i\partial_j, j = 1, \ldots, n \equiv d$ and $h_0(\xi) := \sum_{ijkl} g^{kl} \xi_i \xi_k$, which is convex and thus steep. So Theorem (2.10) applies and we get the estimate (2.10). This result was already obtained in [BLM20a], which improved the results by Bourgain [Bou99] [Del10] and [BM19].
4.2 Zoll Manifolds

We recall that a Zoll manifold is a compact manifold s.t. all its geodesics are closed; the typical example of a Zoll manifold is a sphere. By Theorem 1 of [CdV79], there exists a pseudodifferential operator $Q$ of order $-1$, commuting with $-\Delta_g$, s.t. $\text{spec}(\sqrt{-\Delta_g} + Q) \subset N + \kappa$, with $\kappa \geq 0$. We put

$$A := \sqrt{-\Delta_g} + Q,$$

(4.7)

so that

$$-\Delta_g = A^2 - 2AQ + Q^2.$$

Then the assumptions on $A$ and $H_0$ are easily verified. We remark that in this case one has $d = 1$ and the cone $C$ coincides with $\mathbb{R}$. Furthermore the Schrödinger operator at r.h.s. of (4.3) takes the form $H_0 + \tilde{V}$ with $\tilde{V} := V - 2AQ + Q^2$.

We thus get that the solutions of (4.3) on a Zoll manifold fulfill (2.10). We recall that this result was already obtained in [BGMR21].

4.3 Rotation invariant surfaces

Consider a real analytic function $f : \mathbb{R}^3 \to \mathbb{R}$ invariant by rotations around the $z$ axis, and assume it is a submersion at $f(x, y, z) = 1$. Denote by $M$ the level surface $f(x, y, z) = 1$ and endow it by the natural metric $g$ induced by the euclidean metric of $\mathbb{R}^3$, then $M$ has integrable geodesic flow.

Following [CdV80], we introduce suitable coordinates in $M$ as follows: let $N$ and $S$ be the north and the south poles (intersection of $M$ with the $z$ axis) and denote by $\theta \in [0, L]$ the curvilinear ascissa along the geodesic given by the intersection of $M$ with the $xz$ plane; we orient it as going from $N$ to $S$ and consider also the cylindrical coordinates $(r, \phi, z)$ of $\mathbb{R}^3$: we will use the coordinates

$$(\theta, \phi) \in (0, L) \times (0, 2\pi)$$

as coordinates in $M$. Using such coordinates, one can write the equation of $M$ by expressing the cylindrical coordinates of a point in $\mathbb{R}^3$ as a function of $(\theta, \phi)$ getting

$$M = \{(r(\theta), \phi, z(\theta)) \, , \, (\theta, \phi) \in (0, L) \times T^1\}.$$

Since $\theta$ is a geodesic parameter, the metric takes the form

$$g = r^2(\theta)d\phi^2 + d\theta^2.$$
We assume that the function \( r(\theta) \) has only one critical point \( \theta_0 \in (0, L) \). Furthermore, in order to state the nondegeneracy condition we need to ensure steepness. To this aim, consider the following Taylor expansion at \( \theta = \theta_0 \):

\[
\frac{1}{2r^2(\theta)} = \beta_0 + \frac{1}{2} \beta_2 (\theta_0^2 + 1) \beta_3 (\theta_0^3 + 1) \beta_4 (\theta_0^4 + 1) O(|\theta - \theta_0|^5) .
\] (4.8)

**Theorem 4.3.** Consider the Schrödinger equation (1.3) with \( V(.) \in C^\infty_b(\mathbb{R}; A_1) \), with \( b < 2 \). Assume also that \( \beta_2 \neq 0 \), and that

\[
\beta_0 \left( \frac{-5\beta^2_2 + 3\beta_2 \beta_4}{24\beta^2_2} \right) - \beta_2 \neq 0
\] (4.9)

then (2.10) holds.

This theorem will be proved in Subsect. 9.2. In this example, the actions were actually constructed in [CdV80] by quantizing the classical action variables. Such a construction will be recalled in Subsection 9.2.

**Remark 4.4.** For the case \( V(.) \in C^\infty_b(\mathbb{R}; A_1) \) this theorem was proved in [Del10] where the condition (4.9) was not required.

**Remark 4.5.** In principle our theory applies to all the manifolds considered in Sect. 2 of [CdV80], namely manifolds with integrable geodesic flow in which the singularities of the invariant foliation of the classical actions are only elliptic. In order to apply our result one should also verify that the Hamiltonian of the geodesic flow is a steep function. Here we decided to avoid a more detailed statement, since we do not know any nontrivial example of manifold fulfilling the assumptions of [CdV80].

### 4.4 Compact, simply connected Lie groups

Let \( M \equiv G \) be a simply connected compact Lie group endowed by the bi-invariant metric \( g \). To apply Theorem 2.16 to equation (1.3) on \( G \) we use the intrinsic formulation of pseudodifferential calculus on Lie groups, developed in [RT09, Fis15]. In particular this will be needed to construct the quantum actions \( A_j \) and to verify their properties. We remark that some examples of Lie groups which are simply connected and compact are given by the special unitary groups \( SU(n) \) with \( n \geq 2 \), the compact symplectic groups \( Sp(n) \) with \( n \geq 3 \) and the spin groups \( Spin(n) \) with \( n \geq 7 \).

The starting point of the construction is the fact that in Lie groups the Fourier coefficients of a smooth function are labeled by the irreducible unitary representations of the group and each Fourier coefficient is a unitary operator.
in the representation space. Furthermore, if \( d \) is the rank of the group, then
the unitary representations are in 1-1 correspondence with the points of a
subset of a \( d \) dimensional lattice, more precisely the cone of the dominant
weights. Then the Laplacian is a “Fourier multiplier” and the quantum
actions too will be defined as Fourier multipliers.

More precisely, denote by \( \hat{G} \) the set of unitary irreducible representations
of \( G \), modulo unitary equivalence and by \( \text{Rep}(G) \) the set of unitary repre-
sentations of \( G \) (still modulo equivalence). Given \( \xi \in \text{Rep}(G) \), denote by
\( \mathcal{H}_\xi \) the corresponding representation space; then the Fourier coefficients of a
function \( \psi : G \to \mathbb{R} \) are a sequence \( \{ \hat{\psi}_\xi \mid \xi \in \hat{G} \} \) with \( \hat{\psi}_\xi \in \mathcal{B}(\mathcal{H}_\xi) \).

There is a way of defining symbols of pseudodifferential operators as maps
\( \sigma(x, \xi) \)
\[ G \times \text{Rep}(G) \ni (x, \xi) \mapsto \sigma(x, \xi) \in \mathcal{B}(\mathcal{H}_\xi) , \tag{4.10} \]
with suitable properties (see Definition 9.7 below for the precise definition
taken from [Fis15]). Actually a symbol is usually defined by its action on \( \hat{G} \)
and extended to \( \text{Rep}(G) \) by direct sum.

The remarkable fact is that, as proved in [Fis15], the pseudodifferential
calculus constructed in this way is equivalent to the pseudodifferential calcu-
lus constructed considering \( G \) as a manifold and defining symbols according
to Definition 4.1.

To define the actions we need a further step in the theory of Lie groups:
to a representation \( \xi \in \hat{G} \), one associates its highest weight \( \mathfrak{w}_\xi \), and it turns
out (see e.g. [FH13]) that there is a 1-1 correspondence between the elements
of \( \hat{G} \) and the elements of the cone \( \Lambda^+(G) \) of dominant weights, defined by
\[ \Lambda^+(G) = \{ \mathfrak{w} \in \mathbb{R}^d \mid \mathfrak{w} = \mathfrak{w}_j \mathbf{f}_1 + \cdots + \mathfrak{w}_d \mathbf{f}_d , \quad \mathfrak{w}_j \in \mathbb{N} , \quad \forall j = 1, \ldots, d \} \] \tag{4.11}
where \( \mathbf{f}_1, \ldots, \mathbf{f}_d \in \mathbb{R}^d \) are the fundamental weights of \( G \). In the following we
also denote \( \mathbf{f} := \sum_{j=1}^d \mathbf{f}_j \in \mathbb{R}^d \) and, given a dominant weight \( \mathfrak{w} \) we denote by
\( \mathfrak{w}_j \) its components on the basis \( \mathbf{f}_j \), namely the numbers such that
\[ \mathfrak{w} = \sum_{j=1}^d \mathfrak{w}_j \mathbf{f}_j . \]

With these notations, the Laplacian \( -\Delta_g \) acts in Fourier space as follows:
\[ (-\Delta_g \phi)_\xi = (\|\mathfrak{w}_\xi + \mathbf{f}\|^2 - \|\mathbf{f}\|^2) \hat{\phi}_\xi , \] \tag{4.12}
and its symbol is given by
\[ \sigma_{-\Delta_g} (\xi) = (\|\mathfrak{w}_\xi + \mathbf{f}\|^2 - \|\mathbf{f}\|^2) 1_{\mathcal{H}_\xi} . \]
We are now ready to define the quantum actions \( A_1, \ldots, A_d \) as the operators acting in Fourier space as follows:

\[
\widehat{(A_j \phi)}(\xi) = (w_j^d + 1) \widehat{\phi}(\xi),
\]
whose symbol is given by

\[
\sigma_{A_j}(\xi) = (w_j^d + 1) \mathbf{1}_{\mathbb{R}^d}. \tag{4.14}
\]

By direct computation one can see that the operators \( A_j \) commute, that their joint spectrum is

\[
\Lambda = \mathbb{N}^d + \kappa, \quad \kappa = (1, \ldots, 1),
\]
and that

\[
-\Delta_g = \sum_{i,j=1}^d A_i A_j f_i \cdot f_j - \|f\|^2,
\]
so that we will define

\[
h_0(A) := \sum_{i,j=1}^n A_i A_j f_i \cdot f_j. \tag{4.16}
\]

Note that \( h_0 \) is homogeneous of degree 2, convex and thus steep. In Section 9.3 we will prove that the \( A_j \)'s are pseudodifferential operators, so that we can apply Theorem 2.16 and deduce that the estimate (2.10) holds for the solutions of the equation (4.3) on a compact, simply connected Lie group.

We also point out that it would be interesting to study growth of Sobolev norms on homogeneous spaces; we leave this for future work.

5 Application 3: Several quantum particles on a Zoll manifold or a Lie Group

Let \((M, g)\) be a Zoll manifold or a compact, simply connected Lie group\(^2\), and consider a system composed by \(n\) quantum particles on \(M\) which interact through a potential possibly depending on the velocity and also depending on time (external forcing). Then the Hamiltonian of the system takes the form

\[
H = \sum_{\ell=1}^n (-\Delta_g)\ell + V(t) \tag{5.1}
\]

\(^2\)the case of \(M = \mathbb{T}_\Gamma\) can also be dealt with trivially
with \((-\Delta_g)_\ell\) the Laplacian with respect to the variables of the \(\ell\)-th particle. To describe the application, we consider only the case of a Lie group, the case of a Zoll manifold being simpler. Following Subsection 4.4 one introduces the action variable \(A_{\ell,j}, j = 1, \ldots, d\) for \((-\Delta_g)_\ell\) as in (4.13), (4.15) and gets a Hamiltonian of the form (2.8) with

\[ H_0 := \sum_{\ell=1}^{n} h_0(A_{\ell,1}, \ldots, A_{\ell,d}), \]  

(5.2)

\(h_0\) being the function (4.16), and a redefined potential. Now, it is clear that the Hamiltonian (5.2) is convex, thus also steep. The cone \(\mathcal{C}\) coincides with \((\mathbb{R}^+)^{nd}\). So Theorem 2.16 applies. As far as we know, this result was known [Bou99, BM19, BLM20a] only for the case of \(T^n\).

We emphasize that in particular, equation (2.10) gives a bound on the rate at which the energy of every single particle can grow.

We also remark that one can easily introduce the statistics by restricting to the subspace of the symmetric or antisymmetric wave functions and the result holds unchanged.

The above treatment easily extends to the case of a manifold \(M\) which is the product of an arbitrary number of Zoll manifolds, compact simply connected groups and copies of \(T^d\).

Finally, we remark that we are not able to deal with the the case of several particles on a rotation invariant surface since, in general the sum of \(n\) copies of a steep Hamiltonian is not steep.

Part II
Proofs

6 Analytic Part

We start by fixing some notations and definitions that will be used in the rest of the paper. Given two real valued functions \(f\) and \(g\), sometimes we will use the notation \(f \lesssim g\) to mean that there exists a constant \(C > 0\), independent on all the relevant quantities, such that \(f \leq Cg\). If it is important to remember that the constant \(C\) depends on some parameters \(\alpha_1, \ldots, \alpha_n\), we will write \(f \lesssim_{\alpha_1, \ldots, \alpha_n} g\). If \(f \lesssim g\) and \(g \lesssim f\), we will write \(f \simeq g\).

We recall that we denote

\[ \omega(a) := \frac{\partial h_0}{\partial a}(a), \]
which is homogeneous at infinity of degree
\[ M := d - 1. \]  
(6.1)

Furthermore, given \( \delta \) such that \( \max\{0, M - 1\} < \delta < M \), we set
\[ \varsigma := 1 - (M - \delta) \]  
(6.2)

and
\[ \rho := \rho(\varsigma), \]  
(6.3)

where \( \rho(\cdot) \) is the function defined in Assumption A.iv.

**Definition 6.1.** Given a joint eigenvalue \( a = (a_1, \ldots, a_d) \in \Lambda \) of the \( A_j \)'s, we consider the corresponding joint eigenspaces, namely the spaces \( \Sigma_a \subset D(K_0) \) with the property that
\[ \psi \in \Sigma_a \iff A_j \psi = a_j \psi, \quad \forall j = 1, \ldots, d. \]  
(6.4)

The orthogonal projector on \( \Sigma_a \) will be denoted by \( \Pi_a \).

**Remark 6.2.** Given \( \psi \in H \), one can consider its spectral decomposition, namely
\[ \psi = \sum_{a \in \Lambda} \Pi_a \psi, \]  
(6.5)

then by Assumption A.ii there exist \( c_1, c_2 > 0 \) such that one has that
\[ c_1^s \| \psi \|^2 \leq \sum_{a \in \Lambda} \langle a \rangle^{2s} \| \Pi_a \psi \|^2 \leq c_2^s \| \psi \|^2, \quad \forall s > 0. \]  
(6.6)

Given \( \mu > 0 \) (typically \( \mu \ll 1 \)) and \( R > 0 \), we give the following definitions:

**Definition 6.3.** We say that a point \( a \in \Lambda \) is resonant with \( k \in \mathbb{Z}^d \setminus \{0\} \) if
\[ \| a \| \geq R \text{ and } |\omega(a) \cdot k| \leq \| a \|^\delta \| k \| \quad \text{and} \quad \| k \| \leq \| a \|^\mu. \]  
(6.7)

**Definition 6.4.** [Normal form] We say that an operator \( Z \in A^m_\rho \) is in normal form if
\[ \langle \Pi_a \psi; Z \Pi_b \psi \rangle \neq 0 \]  
(6.8)

for some \( \psi \in H \) implies that either \( a \) is resonant with \( b - a \), or \( b \) is resonant with \( b - a \).

**Definition 6.5.** We say that a family of unitary operators \( U(t) \), conjugates \( H \) to \( H^+ \), if, when \( \psi(t) = U(t)\phi(t) \), one has
\[ i\dot{\psi}(t) = H(t)\psi(t) \iff i\dot{\phi}(t) = H^+(t)\phi(t). \]  
(6.9)
We are going to prove the following normal form theorem

**Theorem 6.6.** [Normal Form Lemma] Let $H$ be as in equation (2.8), with $V \in C_b^\infty (\mathbb{R}; \mathcal{A}_{b}^\ast)$, $b < d$, and assume that $V(t)$ is a family of self-adjoint operators. There exists $0 < \delta_* < \mathcal{M}$ such that, if $\delta_* < \delta < \mathcal{M}$, then

$$a := \min \{ 2 \rho + \delta - d; \rho + \delta - b; \delta \} > 0 ;$$  

(6.10)

Furthermore for any $N \in \mathbb{N}$, for any $\mu, \mathbb{R}$ there exists a time dependent family of unitary maps $U_N(t)$ which conjugates $H$ to

$$H^{(N)} := H_0 + Z_N(t) + R^{(N)}(t) ,$$  

(6.11)

and the following properties hold

1. $Z_N \in C_b^\infty (\mathbb{R}; \mathcal{A}_{\rho}^\ast)$ is a family of self-adjoint operators in normal form;
2. $R^{(N)} \in C_b^\infty (\mathbb{R}; \mathcal{A}_{\rho-aN}^\ast)$, is a family of self-adjoint operators;
3. For any $s \geq 0$, $U_N, U_N^{-1} \in L^\infty (\mathbb{R}; \mathcal{B}(\mathcal{H}^s; \mathcal{H}^s))$.

The rest of this section is devoted to the proof of this theorem.

The conjugating maps $U_N(t)$ that one looks for are compositions of maps of the form $e^{-iG(t)}$, with $G(\cdot) \in C_b^\infty (\mathbb{R}; \mathcal{A}_{\rho}^\ast)$ a family of self-adjoint pseudodifferential operators with $\eta < \rho$. The study of the properties of $e^{-iG(t)}$ was done in detail in [BGMR21] in a context very similar to the present one. In this section we just state the result we need for the proof of Theorem 6.6; for the sake of completeness, we will report the corresponding proofs in Appendix A.

**Lemma 6.7.** Let $G \in C_b^\infty (\mathbb{R}; \mathcal{A}_{\rho}^\ast)$, $\eta < \rho$ and $H \in C_b^\infty (\mathbb{R}; \mathcal{A}_{\rho}^m)$, be families of self-adjoint operators; then $e^{-iG(t)}$ conjugates $H$ to $H^+$ given by

$$H^+ = H - i[H, G] + C_b^\infty (\mathbb{R}; \mathcal{A}_{\rho}^{m+2(\eta-\rho)}) + C_b^\infty (\mathbb{R}; \mathcal{A}_{\rho}^\ast) .$$  

(6.12)

We use this formula to compute the structure of the transformed Hamiltonian. To this end remark that, since $H_0 \in \mathcal{A}_{d}^\ast$ (by A.iv and H.i) and $V \in C_b^\infty (\mathbb{R}; \mathcal{A}_{b}^\ast)$, taking $G \in C_b^\infty (\mathbb{R}; \mathcal{A}_{\rho}^\ast)$, $\eta < \rho$, one gets that $H^+$ has the structure

$$H^+ = H_0 - i[H_0, G] + V + C_b^\infty (\mathbb{R}; \mathcal{A}_{\rho}^{d+2(\eta-\rho)}) + C_b^\infty (\mathbb{R}; \mathcal{A}_{\rho}^\ast) .$$  

(6.13)

Then the idea is to determine $G$ which solves the so called cohomological equation, namely

$$- i[H_0, G] + V = Z + \text{lower order terms}$$  

(6.14)
with \( Z \) in normal form and then to iterate the construction. The solution of (6.14) is the main issue of this section and will be done by developing an abstract version of the normal form theory of \([\text{BLM20b, BLM22, BLR21}]\). This requires some work that will be done in the next subsections.

6.1 The Fourier expansion

Following \([\text{BGMR21}]\) (see also \([\text{Bam96, BL20}]\)) we consider a suitable Fourier expansion of pseudodifferential operators

**Definition 6.8.** Let \( \mathcal{F} \in \mathcal{A}^m_\rho \) with \( \rho \in (\rho_0, 1] \), then, for \( k \in \mathbb{Z}^d \), we define its \( k \)-th Fourier coefficient to be:

\[
\hat{F}_k := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{i\varphi \cdot A} e^{-i\varphi \cdot k} d\varphi .
\] (6.15)

In the following we will use the notation

\[
F(\varphi) := e^{i\varphi \cdot A} e^{-i\varphi \cdot A}.
\] (6.16)

**Remark 6.9.** By the formula

\[
\frac{d}{d\varphi_j} (e^{i\varphi \cdot A} e^{-i\varphi \cdot A}) = e^{i\varphi \cdot A} (-i[F; A_j]) e^{-i\varphi \cdot A}
\]

Assumption I.vii implies that, for \( F \in \mathcal{A}^m_1 \), the map \( \varphi \mapsto F(\varphi) = e^{i\varphi \cdot A} e^{-i\varphi \cdot A} \in C^\infty_b (\mathbb{R}, \mathcal{A}^m_1) \).

**Remark 6.10.** By assumptions I on the algebra the following holds. For all \( F \in \mathcal{A}^m_\rho \) and all \( G \in \mathcal{A}^n_\rho \) one has

\[
\forall m, n, j, \rho \quad \exists J \text{ s.t. } \varphi_{p,j}^{m+n}(FG) \leq C_1 \varphi_{p,j}^m(F) \varphi_{p,j}^n(G) ,
\] (6.17)

\[
\forall m, n, j, \rho \quad \exists J \text{ s.t. } \varphi_{p,j}^{m+n-\rho}([F, G]) \leq C_2 \varphi_{p,j}^m(F) \varphi_{p,j}^n(G) ,
\] (6.18)

for some positive constants \( C_1(m, n, j, \rho) \), \( C_2(m, n, j, \rho) \).

**Lemma 6.11.** Let \( F \in \mathcal{A}^m_\rho \), then for any \( j \) and \( N \in \mathbb{N} \) there exist \( C > 0 \) and \( J \), independent of \( F \), such that

\[
\varphi_{1,j}^m \left( \hat{F}_k \right) \leq C \frac{\varphi_{1,j}^m(F)}{\langle k \rangle_N} \quad \forall k \in \mathbb{Z}^d .
\] (6.19)

It follows that the series

\[
F(\varphi) = \sum_{k \in \mathbb{Z}^d} \hat{F}_k e^{ik \cdot \varphi} ,
\] (6.20)
is convergent, so that, in particular
\[ F = \sum_{k \in \mathbb{Z}^d} \hat{F}_k. \quad (6.21) \]

The proof is an immediate consequence of Remark 6.9.

**Remark 6.12.** If \( F \in \mathcal{A}_\rho^m, \rho < 1 \), then, due to the fact that
\[ [F; A_j] \in \mathcal{A}_\rho^{m+1-\rho} \not\subset \mathcal{A}_\rho^m, \]
eq (6.19)
does not hold in the algebra \( \mathcal{A}_\rho \), with \( \rho < 1 \).

It turns out that the algebra \( \mathcal{A}_1 \) is not stable under solution of the cohomological equation. For this reason we have to introduce a further class of pseudodifferential operators.

**Definition 6.13.** For \( m \in \mathbb{R}, \rho \in (\rho_0, 1] \), the set of the operators \( F \in \mathcal{A}_\rho^m \), s.t. \( \forall j, \forall N \in \mathbb{N} \)
\[ \varphi_{\rho,j,N}^m (F) := \sum_{k \in \mathbb{Z}^d} \langle k \rangle^N \varphi_{\rho,j}^m \left( \hat{F}_k \right) < \infty, \quad (6.22) \]
will be denoted by \( \mathcal{S}_\rho^m \). This is a Fréchet space with the family of seminorms (6.22).

**Remark 6.14.** By Lemma 6.11 one has \( \mathcal{A}_1^m \hookrightarrow \mathcal{S}_1^m \) continuously.

The algebra properties of \( \mathcal{S}_\rho \) are ensured by the Lemma below.

**Lemma 6.15.** Let \( F \in \mathcal{S}_\rho^m \) and \( G \in \mathcal{S}_\rho^{m'} \). Then \( FG \in \mathcal{S}_\rho^{m+m'} \), \( [F; G] \in \mathcal{S}_\rho^{m+m'-\rho} \) and \( \forall N \in \mathbb{N}, \forall j, \exists J, C \) s.t. one has
\[ \varphi_{\rho,j,N}^{m+m'} (FG) \leq C \varphi_{\rho,j,N}^m (F) \varphi_{\rho,j,N}^{m'} (G), \quad (6.23) \]
\[ \varphi_{\rho,j,N}^{m+m'-\rho} ([F; G]) \leq C \varphi_{\rho,j,N}^m (F) \varphi_{\rho,j,N}^{m'} (G). \quad (6.24) \]

**Proof.** Observe that, \((FG)(\varphi) = F(\varphi)G(\varphi)\), thus we have
\[ FG = \sum_{k,k'} \hat{F}_k \hat{G}_{k'} \]
and therefore
\[ \varphi_{\rho,j,N}^{m+m'} (FG) \leq C \sum_{k,k'} \langle k \rangle^N \varphi_{\rho,j}^m \left( \hat{F}_k \right) \varphi_{\rho,j}^{m'} \left( \hat{G}_{k'} \right) \]
\[ \leq C \sum_{k,k'} \langle k \rangle^N \langle k' \rangle^N \varphi_{\rho,j}^m \left( \hat{F}_k \right) \varphi_{\rho,j}^{m'} \left( \hat{G}_{k'} \right) \leq C \varphi_{\rho,j,N}^m (F) \varphi_{\rho,j,N}^{m'} (G). \]

Equation (6.24) is proved in the same way. \( \square \)
Remark 6.16. From the above Lemma it is immediate to realize that Equations (6.20) and (6.21) hold also for the classes $S_m^\rho$.

Finally, the reason why this Fourier expansion is useful for the solution of the cohomological equation is summarized in the following remark.

Remark 6.17. By deriving $F(\varphi)$ with respect to $\varphi_j$ one gets

$$\frac{\partial F}{\partial \varphi_j}(\varphi) = \sum_k i k_j \hat{F}_k e^{i k \cdot \varphi} = -i[F(\varphi); A_j],$$

which implies

$$\sum_k i k_j \hat{F}_k = -i[F; A_j].$$

(6.25)

The main property relating the lattice $\Lambda$ and the Fourier expansion is given by the following Lemma

Lemma 6.18. Let $F \in A_m^\rho$ for some $m$. For any $a, b \in \Lambda$ and $k \in \mathbb{Z}^d$, if $\psi_a$ is an eigenfunction corresponding to $a$ and $\psi_b$ is an eigenfunction corresponding to $b$, one has

$$\langle \psi_b; \hat{F}_k \psi_a \rangle = \delta_k^{a-b} \langle \psi_b; F \psi_a \rangle.$$  

(6.26)

Proof. Just compute

$$\langle \psi_b; \hat{F}_k \psi_a \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \langle \psi_b; e^{i \varphi \cdot A} F e^{-i \varphi \cdot A} \psi_a \rangle e^{-i \varphi \cdot k} d\varphi$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \langle e^{-i \varphi \cdot A} \psi_b; F e^{-i \varphi \cdot A} \psi_a \rangle e^{-i \varphi \cdot k} d\varphi$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \langle e^{-i \varphi \cdot b} \psi_b; F e^{-i \varphi \cdot a} \psi_a \rangle e^{-i \varphi \cdot k} d\varphi$$

$$= \langle \psi_b; F \psi_a \rangle \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{-i \varphi \cdot (k+a-b)} d\varphi.$$  

\[\Box\]

6.2 Solution of the Cohomological equation

In this subsection we are going to prove the following lemma

Lemma 6.19. There exists $\delta_* < \mathbb{M}$ s.t. for all $\delta_* < \delta < \mathbb{M}$, the following holds true: define $\varsigma := \varsigma(\delta)$ and $\rho = \rho(\varsigma(\delta))$ according to (6.2) and (6.3), then $\forall F \in S^m_\rho$ self-adjoint, there exist self-adjoint operators $G \in S^{m-\delta}_\rho$, $Z \in S^m_\rho$, with $Z$ in normal form, s.t.

$$-i[H_0; G] + F - Z \in S^{-2\rho + \delta - d}_\rho + A^\infty_\rho,$$  

(6.27)

furthermore, for $\delta_* < \delta < \mathbb{M}$ one has $2\rho + \delta - d > 0$.  

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First, following \[BLM20b\], we split the perturbation $F$ in a resonant, a nonresonant and a smoothing part. This will be done with the help of suitable pseudodifferential cutoffs.

Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be a symmetric cutoff function which is equal to 1 in $[-\frac{1}{2}, \frac{1}{2}]$ and has support in $[-1, 1]$, and given $R > 0$, define

$$\chi^{(R)}(t) := \chi(R^{-1}||t||), \quad t \in \mathbb{R}^d.$$  

With their help we define, for $k \in \mathbb{Z}^d \setminus \{0\}$,

$$\tilde{\chi}_k(a) := \chi\left(\frac{||k||}{||a||^\mu}\right) \quad \text{(ultraviolet cutoff)}, \quad (6.28)$$

$$\chi_k(a) := \chi\left(\frac{\omega(a) \cdot k}{||a||^\delta ||k||}\right) \quad \text{(resonant cutoff)}, \quad (6.29)$$

$$d_k(a) := \frac{1}{\omega(a) \cdot k} \left(1 - \chi\left(\frac{\omega(a) \cdot k}{||a||^\delta ||k||}\right)\right) \quad \text{(cutoffed denominators)} \quad (6.30)$$

We also put

$$\chi_0(a) := 1, \quad \tilde{\chi}_0(a) := 0, \quad (6.31)$$

and

$$\chi_k^T(a) := (1 - \chi^{(R)}(a))\chi_k(a), \quad d_k^T(a) := (1 - \chi^{(R)}(a))d_k(a). \quad (6.32)$$

The above functions satisfy the following:

**Lemma 6.20.** \(\forall k \in \mathbb{Z}^d \setminus \{0\}\) one has

$$\tilde{\chi}_k(a), (1 - \chi^{(R)})(1 - \chi_k), \tilde{\chi}_k \in S_0^0, \quad d_k^T \in S_0^{-\delta},$$

with seminorms uniformly bounded in $k$.

**Proof.** The proof follows closely the proof of Lemmas 6.3, 6.4, 6.6 of \[BLR21\]. We start by proving the result for $\tilde{\chi}_k$. First of all we observe that $\tilde{\chi}_k(a) \neq 0$ only if $||a||^\mu \geq ||k|| \geq 1$, which implies that $\tilde{\chi}_k \in C^\infty(\mathbb{R}^d)$. Furthermore, let

$$\tau_k(a) := \frac{||k||}{||a||^\mu},$$

then one has $\tilde{\chi}_k = \chi \circ \tau_k$ and, by the Faa di Bruno formula,

$$|\partial^\beta_a \tilde{\chi}_k(a)| \simeq \sum_{j=1}^{||\beta||} \sum_{\gamma_1 + \cdots + \gamma_j = \beta} |\chi^{(j)} \circ \tau_k| \prod_{i=1}^j |\partial_{a_i}^{\gamma_i} \tau_k|. \quad (6.32)$$

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Now, by homogeneity of the function $\|a\|^{-\mu}$, one has

$$|\partial_a^{\gamma_i} t_k| \lesssim \|k\| \|a\|^{-(\mu + |\gamma_i|)} \lesssim \|a\|^{-|\gamma_i|}$$

on the support of $\tilde{\chi}_k$. Thus one has $|\partial_a^2 \tilde{\chi}_k(a)| \lesssim \|a\|^{-|\beta|}$, which gives $\tilde{\chi}_k \in S_0^0 \subset S_\zeta^0$.

We now prove that $\chi^T = (1 - \chi^{(R)})\chi_k \in S_\zeta^0$. First of all we observe that since $\chi^{(R)}$ has compact support, one immediately has $1 - \chi^{(R)}(\|a\|) \in S_1^0$. Thus it is sufficient to estimate $\tilde{\chi}_k$ and its derivatives in the support of $1 - \chi^{(R)}(\|a\|)$.

We define

$$t_k(a) := \frac{\omega(a) \cdot k}{\|a\|^2 \|k\|}$$

and we use again Faa di Bruno formula: one has

$$|\partial_a^2 \chi_k(a)| \simeq \sum_{j=1}^{\beta} \sum_{\gamma_1 + \cdots + \gamma_j = \beta} |\chi^{(j)} \circ t_k| \prod_{i=1}^{j} |\partial_{\gamma_i} t_k|,$$

with

$$|\partial_{\gamma_i} t_k| \lesssim \|a\|^{M - \delta - |\gamma_i|} \lesssim \|a\|^{-|\gamma_i|},$$

since $t_k$ is homogeneous of degree $M - \delta$. This implies that, in the support of the function $1 - \chi^{(R)}(\|a\|)$, one has $|\partial_a^2 \chi_k(a)| \lesssim \|a\|^{-|\beta|}$, which gives $\chi^T_k \in S_\zeta^0$.

One proves $(1 - \chi^{(R)})(1 - \chi_k) \in S_\zeta^0$ arguing in analogous way.

In order to prove that $(1 - \chi^{(R)})d_k \in S_{-\delta}^\zeta$, one observes that it is the product of $(1 - \chi^{(R)})(1 - \chi_k) \in S_\zeta^0$ with $\frac{1}{\omega \cdot k}$, and that in the support of $(1 - \chi^{(R)})(1 - \chi_k)$ the function $\frac{1}{\omega \cdot k}$ satisfies

$$\left|\partial_a^2 \left( \frac{1}{\omega(a) \cdot k} \right) \right| \lesssim \sum_{j=1}^{\beta} \sum_{\gamma_1 + \cdots + \gamma_j = \beta} \frac{1}{\omega(a) \cdot k} \sum_{i=1}^{j+1} |\partial_{\gamma_i} (\omega(a) \cdot k)|,$$

with

$$\frac{1}{\|\omega(a) \cdot k\|^{j+1}} \lesssim \|a\|^{-\delta(j+1)} \|k\|^{-j+1}, \quad |\partial_{\gamma_i} (\omega(a) \cdot k)| \lesssim \|k\| \|a\|^{M - |\gamma_i|}.$$

This then implies

$$\left|\partial_a^2 \left( \frac{1}{\omega(a) \cdot k} \right) \right| \lesssim \|a\|^{-\delta(|\beta|+1) \cdot |\beta| - |\beta|} \lesssim \|a\|^{-\delta - |\beta|},$$

which gives the thesis. □
Given $F \in S^m_\rho$ self-adjoint, we use the above functions to decompose $F$:

\[ F_0^{(\text{res})} := \sum_{k \in \mathbb{Z}^d} \chi_k^T(A) \tilde{\chi}_k(A) \hat{F}_k, \quad (6.33) \]

\[ F_0^{(\text{nr})} := \sum_{k \in \mathbb{Z}^d \setminus \{0\}} (1 - \chi^{(R)}(A))(1 - \chi_k(A)) \tilde{\chi}_k(A) \hat{F}_k, \quad (6.34) \]

\[ F_0^{(S)} := \sum_{k \in \mathbb{Z}^d \setminus \{0\}} (1 - \chi^{(R)}(A))(1 - \tilde{\chi}_k(A)) \hat{F}_k + \chi^{(R)}(A) F, \quad (6.35) \]

and

\[ F^{(\text{res})} := \frac{F_0^{(\text{res})} + (F_0^{(\text{res})})^*}{2}, \quad F^{(\text{nr})} := \frac{F_0^{(\text{nr})} + (F_0^{(\text{nr})})^*}{2}, \quad (6.36) \]

\[ F^{(S)} := \frac{F_0^{(S)} + (F_0^{(S)})^*}{2}, \quad (6.37) \]

so that each one of the operators is self-adjoint. Remarking that, since $F$ is self-adjoint, one has

\[ F := F + F^*, \]

it is immediate to realize that

\[ F = F^{(\text{nr})} + F^{(\text{res})} + F^{(S)}. \quad (6.38) \]

Furthermore, we have the following result:

**Lemma 6.21.** Let $\varsigma$ and $\rho$ be respectively defined as in (6.2) and (6.3). If $F \in S^m_\rho$ for some $m \in \mathbb{R}$, then $F^{(\text{res})}, F^{(\text{nr})} \in S^m_\rho$.

**Proof.** By I.vi, it is sufficient to prove that $F_0^{(\text{res})}$ and $F_0^{(\text{nr})}$ are in $S^m_\rho$. We prove that $F_0^{(\text{res})} \in S^m_\rho$. The result for $F_0^{(\text{nr})}$ is proved analogously. By Eq. (6.33), the Fourier coefficients of $F_0^{(\text{res})}$ are given by

\[ \hat{F}_{0,k} = \chi_k^T(A) \tilde{\chi}_k(A) \hat{F}_k \quad \forall k \in \mathbb{Z}^d. \]

Lemma 6.20 ensures that $\tilde{\chi}_k$ and $\chi_k^T$ are in $S^0_\varsigma$, with seminorms uniformly bounded in $k$. Then by Assumption I.v we obtain $\tilde{\chi}_k(A), \chi_k^T(A) \in A^0_\rho$, with seminorms $\{\psi_{\rho,j}^{0}(\tilde{\chi}_k(A))\}_j, \{\psi_{\rho,j}^{0}(\chi_k^T(A))\}_j$ uniformly bounded in $k$. Then Lemma 6.15 implies that $\forall j$ there exists $J$ such that

\[ \psi^m_{\rho,j}(\hat{F}_{0,k}) \lesssim \psi^m_{\rho,j}(\tilde{\chi}_k(A)) \psi^m_{\rho,j}(\chi_k^T(A)) \psi^m_{\rho,j}(\hat{F}_k) \lesssim \psi^m_{\rho,j}(\hat{F}_k) \quad \forall k \in \mathbb{Z}^d, \]

which gives $\psi^m_{\rho,j,N}(F_0^{(\text{res})}) \lesssim \psi^m_{\rho,j,N}(F)$ for any $N$ and $j$, and thus the thesis.

\[ \square \]
Remark 6.22. By Lemma 6.15, one has that $F^{(nr)} = F_0^{(nr)} + S_{\rho}^{m - \rho}$.

Concerning $F^{(S)}$, we have the following lemma.

Lemma 6.23. Assume $F \in S_{\rho}^{m}$, then $F^{(S)} \in A_{\rho}^{-\infty}$.

Proof. First remark that the statement is obviously true for the term coming from second term of (6.35). Consider now the first term of (6.35). We are going to prove that $\forall s$ it is smoothing of order $s - m$, then, by Assumption I.v the thesis follows for $F_0^{(S)}$, while it follows for $F^{(S)}$ thanks to Assumption I.vi.

To start with consider the case $m = 0$; we prove that $F_0^{(S)}$ maps $H^0$ to $\mathcal{H}^s$ for all $s$. Using the spectral decomposition of the operators $A$, we have $F_0^{(S)} \psi = \sum_{a,k} \Pi_a \chi_k^T(A) \hat{F}_k \psi = \sum_{a,k} \chi_k^T(a) \Pi_a \hat{F}_k \psi$, so that, using that $\chi_k^T(a)$ is different from zero only if $\|a\|^\mu < \|k\|$, one has

$$\|F_0^{(S)} \psi\|^2_s = \sum_{a} \langle a \rangle^{2s} \left( \sum_{\|k\| > \|a\|^\mu} \|\Pi_a \hat{F}_k \psi\|_0 \right)^2 \leq \sum_{a} \langle a \rangle^{2s} \left( \sum_{\|k\| > \|a\|^\mu} \|\hat{F}_k\|_{B(H^0, H^0)} \|\psi\|_0 \right)^2 \leq C \sum_{a} \langle a \rangle^{2s} \left( \sum_{\|k\| > \|a\|^\mu} \frac{\langle k \rangle^N \psi_{\rho,j}^0(\hat{F}_k)}{\langle k \rangle^N} \right)^2 \|\psi\|_0^2 \leq C \sum_{a} \langle a \rangle^{2s} \frac{1}{\|a\|^{2s\mu N}} \left( \sum_{k} \langle k \rangle^N \psi_{\rho,j}^0(\hat{F}_k) \right)^2 \|\psi\|_0^2$$

Taking $N > \frac{s + d}{\mu}$, one gets that the above quantity is smaller than

$$C \sum_{a} \langle a \rangle^{2s-\mu N} \left( \psi_{\rho,j,N}^0(F) \right)^2 \|\psi\|_0^2 \leq C' \left( \psi_{\rho,j,N}^0(F) \right)^2 \|\psi\|_0^2,$$

which proves the statement in the considered case. The general case $m \neq 0$ and $\psi \in H^s$ is easily reduced to the previous one by considering the operator $\langle A \rangle^{-m+s} F \langle A \rangle^{-s}$, where

$$\langle A \rangle := \left( 1 + \sum_{j} A_j^2 \right)^{1/2}.$$
Lemma 6.24. \( F^{\text{res}} \) is in resonant normal form.

Proof. One has

\[
F^{\text{res}} := \frac{1}{2} \sum_{k \in \mathbb{Z}^d} \left[ (1 - \chi^{(R)}(A))\chi_k(A)\tilde{x}_k(A)\hat{F}_k + \tilde{F}_{-k}(1 - \chi^{(R)}(A))\chi_k(A)\tilde{x}_k(A) \right].
\]

Consider first the first term in the sum. By Lemma 6.18 and by the definition of the cutoffs one has

\[
\langle \psi_b; (1 - \chi^{(R)}(A))\chi_k(A)\tilde{x}_k(A)\hat{F}_k\psi_a \rangle = \langle (1 - \chi^{(R)}(b))\chi_k(b)\tilde{x}_k(b)\hat{F}_k\psi_a; \psi_b \rangle = (1 - \chi^{(R)}(b))\chi_k(b)\delta_k^{\omega+k}\langle \psi_b; F\psi_a \rangle,
\]

which means that this term does not vanish only when \( b \) is resonant with \( k = b - a \), so this term is in normal form. An equal computation shows that the second term is different from zero only if \( a \) is resonant with \( k = b - a \), so that the lemma is proven.

Proof of Lemma 6.19. First, we remark that for \( \delta_* \) sufficiently close to \( \mathcal{M} = d - 1 \) one has that \( \varsigma := 1 - (\mathcal{M} - \delta) \) is close to 1. Consequently, since the function \( \rho(\cdot) \) defined in Assumption A.iv is continuous, also \( \rho(\varsigma) \) is close to 1, and one gets

\[
2\rho + \delta - d = 2\rho + \varsigma - 2 > 0.
\]

Then we put \( Z := F^{\text{res}} \) and we include \( F^{(S)} \) in the remainder term \( \mathcal{A}_\rho^{-\infty} \).

Then we define

\[
G_0 := \sum_{k \neq 0} d_k^R(A)\hat{F}_k,
\]

Define furthermore

\[
G := \frac{G_0 + G_0^*}{2} = G_0 + \mathcal{S}_\rho^{m-\delta-\rho},
\]

which is self-adjoint.

Concerning the properties of this operator, we remark that, by Lemma 6.20 and Assumption A.iv, one obtains \( d_k^R(A) \in \mathcal{A}_\rho^{-\delta} \). Thus, arguing as in the proof of Lemma 6.21 one has \( G_0 \in \mathcal{S}_\rho^{m-\delta} \) with seminorms bounded by the seminorms of \( F \). By Assumption I.vi, the same holds for \( G \).

We verify now that with such a \( G \) the cohomological equation is solved.

By the generalized Commutator Lemma 3.1 one has

\[
-i[H_0; G] = -i[H_0; G_0] + \mathcal{S}_\rho^{m-\delta+d-2\rho} = \sum_{j=1}^d -i\frac{\partial h_0}{\partial a_j}(A)[A_j; G_0] + \mathcal{S}_\rho^{m-\delta+d-2\rho},
\]

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but, by Remark 6.17 $i \frac{\partial h_{0}}{\partial a_{j}} (A) [A_{j}; G_{0}]$ is equal to

$$\sum_{j=1}^{d} i \omega_{j} (A) k_{j} G_{0,k} = \sum_{j=1}^{d} \chi_{k}^{0} (A) \hat{F}_{k} = F_{0}^{(nr)}.$$  

So we have

$$-i[H_{0}; G] = F_{0}^{(nr)} + S_{\rho}^{m-\delta+d-2\rho} = F^{(nr)} + S_{\rho}^{m-\delta+d-2\rho},$$

since $F_{0}^{(nr)} - F^{(nr)} \in S_{\rho}^{m-\rho}$ and $\rho > \rho + (\rho - 1) + (\delta - M) = 2\rho + \delta - d$.  

6.3 End of the proof of Theorem 6.6

We are now going to prove the following iterative lemma, which immediately yields Theorem 6.6

**Lemma 6.25.** Let $H$ be as in equation (2.8), with $V \in C_{b}^{\infty}(\mathbb{R}; S_{\rho}^{b})$, $b < d$, a family of self-adjoint operators. There exists $0 < \delta_{*} < M$ such that, if $\delta_{*} < \delta < M$, then $a$ defined as in (6.10) satisfies $a > 0$ and the following holds. For any $\mu, \mathcal{R}$ and $\forall n \geq 0$ there exists a time dependent family $U_{n}(t)$ of unitary maps conjugating the operator $H$ of (2.8) to

$$H_{n}(t) = H_{0} + Z_{n}(t) + R_{n}(t) + \tilde{R}_{n}(t),$$

where:

1. $Z_{n} \in C_{b}^{\infty}(\mathbb{R}; S_{\rho}^{b})$ is a family of self-adjoint operators in normal form
2. $R_{n} \in C_{b}^{\infty}(\mathbb{R}; S_{\rho}^{b-na})$ and $R_{n}(t)$ is a family of self-adjoint operators
3. $\tilde{R}_{n} \in C^{\infty}(\mathbb{R}; A_{\rho}^{-\infty})$ and $\tilde{R}_{n}(t)$ is a family of self-adjoint operators
4. For any $s \geq 0$, $U_{N}, U_{N}^{-1} \in L^{\infty}(\mathbb{R}; B(\mathcal{H}^{s}; \mathcal{H}^{s})).$

**Proof.** First of all we observe that there exists $\delta_{*} = \delta_{*}(b) > 0$ such that, if $\delta_{*} < \delta < M$, then $a > 0$. We prove the theorem by induction. In the case $n = 0$, the claim is trivially true taking $U_{0}(t) = 1$, $Z_{0} = 0$, $R_{0} = V$ and $\tilde{R}_{0} = 0$.

We consider now the case $n > 0$. Warning: here time only plays the role of a parameter, so in this proof we write $F \in S_{\rho}^{m}$ to mean $F \in C_{b}^{\infty}(\mathbb{R}; S_{\rho}^{m})$ and so on.
Denote \( m := b - na \); we determine \( G_{n+1} \in S^\eta, \eta = b - na - \delta \), according to Lemma 6.19 with \( F \) replaced by \( R_n \). Up to increasing again the value of \( \delta_m \), one has that
\[
\eta = b - na - \delta \leq b - \delta = -(d - b) + (M - \delta) + 1 < 1.
\]
Then one uses \( e^{iG_{n+1}} \) to conjugate \( H_n \) to \( H^+ \) given by (see Lemma 6.7)
\[
H^+ = H - i [H_n; G_{n+1}] + S^{n+2(\eta - \rho)} + S^\eta + A^{-\infty}_\rho
\]
\[
= H_0 + Z_n + R_n - i [H_0; G_{n+1}] + S^{b+\eta - \rho} + S^{m+2(\eta - \rho)} + S^\eta + A^{-\infty}_\rho
\]
\[
= H_0 + Z_n + R^{(res)}_n + S^{m-(2\rho + \delta - d)} + S^{b+\eta - \rho} + S^{m+2(\eta - \rho)} + S^\eta + A^{-\infty}_\rho,
\]
where we go from the first to the second line by inserting the expression of \( H_n \) and from the second to the third line using Lemma 6.19.

Writing explicitly the different exponents of the classes of the remainder terms in the last line of (6.41), we get that they are given by
\[
e_1 := b - na - (2\rho + \delta - d) = b - na - a_1, \quad a_1 := 2\rho + \delta - d
\]
\[
e_2 := b + b - na - \delta - \rho = b - na - a_2, \quad a_2 := \delta + \rho - b
\]
\[
e_3 := b - na + 2(b - na - \delta - \rho) = b - na - a_3, \quad a_3 := 2(na + \delta + \rho - b)
\]
\[
e_4 := b - na - \delta = b - na - a_4, \quad a_4 := \delta.
\]

Remarking that \( a_3 \geq a_2 \) and taking the smallest \( a \) one immediately gets the thesis. \( \square \)

7 Geometric part

Definition 7.1. Given a subset \( E \subset \Lambda \) we define \( \Pi_E := \sum_{a \in E} \Pi_a \).

In the present section we prove the following result:

Theorem 7.2. Suppose that \( H_0 \) satisfies Assumption \( H \), then there exists a partition \( \{W_\ell\}_{\ell \in \mathbb{N}} \) of \( \Lambda \) with the following properties.

(1) The sets \( W_\ell \) are dyadic, namely
\[
\max\{\|a\| : a \in W_\ell\} \leq 2 \min\{\|a\| : a \in W_\ell\} \quad \forall \ell.
\]

(2) The sets \( W_\ell \) are invariant for any operator in normal form operator \( Z \), namely
\[
[\Pi_{W_\ell}, Z] = 0, \quad \forall \ell \in \mathbb{N}.
\]
The construction of the sets $W_{\ell}$ is a generalization to the abstract steep case of that done in [BLM22, BLM20a, PS12] for quadratic Hamiltonians. It represents a quantum counterpart of the geometric decomposition at the basis of the proof of Nekhoroshev theorem (see [Nek77, Nek79, GCB16, Gio03, BL20]).

The key point to have in mind is that an operator in normal form only connects points $a$ and $b$ of the lattice $\Lambda$ s.t. $b - a$ is either resonant with $a$ or resonant with $b$ (see Lemma 6.24 and its proof).

First, we recall the following:

**Definition 7.3.** We say that an additive subgroup $M$ of $\mathbb{Z}^d$ is a module if $\text{span}_R\{M\} \cap \mathbb{Z}^d = M$. We will often refer to a module $M$ as a resonant module. If $M \neq \{0\}$ and $M \neq \mathbb{Z}^d$, we say $M$ is a proper or nontrivial module.

### 7.1 The invariant partition: definitions and statement

#### 7.1.1 Resonant zones

Following [Nek77, Nek79, BLM22, GCB16, Gio03, BL20], the first step of the construction consists in classifying all points $a \in \Lambda$ according to the resonant relations they fulfill. The key points of this subsection are Definitions 7.9, 7.11, 7.18 and 7.19, which actually classify the points. The rest of the subsection gives some properties of the defined objects.

In order to perform our construction we take $\delta, \mu$ fulfilling

$$\alpha d(d - 1)\mu < M - \delta, \quad \alpha := \alpha_1 \cdots \alpha_{d - 1},$$

where the steepness indexes $\alpha_1, \ldots, \alpha_{d - 1}$ have been introduced in Definition 2.14. We also take positive parameters $\gamma_1, \ldots, \gamma_d, C_1, \ldots, C_d, D_1, \ldots, D_d$ fulfilling

$$\gamma_1 = M - \delta, \quad \gamma_{s+1} = \frac{\gamma_s - s\mu}{\alpha_s}, \quad \forall s = 1, \ldots, d - 1,$$

$$1 = C_1 < C_2 \cdots < C_d,$$

$$1 = D_1 < D_2 \cdots < D_d.$$  

**Remark 7.4.** Equation (7.3) ensures that the parameters $\gamma_1, \ldots, \gamma_d$ are positive, with $\gamma_1 > \cdots > \gamma_d$.

In the following, neighborhoods of the origin are not relevant, so, we will only consider the set

$$\|a\| \geq \frac{1}{2}.$$  

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We fix now an open convex cone $C_e$ s.t. $C_e \supset \overline{C}\setminus \{0\}$ and $(B_2 \setminus B_{\frac{1}{2}}) \cap C_e \subset U$, with $U$ as in Assumption H.ii.

**Remark 7.5.** By the steepness assumption one has in particular $\min_{\|\hat{a}\|=1} \|\omega(\hat{a})\| > 0$. Therefore, by homogeneity of $\omega$ there exists $C > 0$ such that

$$\|a\|^M C^{-1} \leq \|\omega(a)\| \leq C\|a\|^M \quad \forall a \in \mathbb{R}^d \cap C_e .$$

(7.6)

**Remark 7.6.** By homogeneity, the steepness condition (2.7) implies that $\forall r \geq 1$ and $\forall a \in (B_{2r} \setminus B_{r/2}) \cap C_e$, one has

$$\max_{0 \leq n \leq \xi} \min_{u \in M: \|u\|=1} \|\Pi_M \omega(a + \eta u)\| \geq B_s r^{\xi - \alpha_s} \xi^{\alpha_s} \quad \forall \xi \in (0, r) ,$$

(7.7)

This is the condition we will use in the proof.

We recall that the definition of normal form depends on the two parameters $\mu$ and $R$.

We start with the following definition:

**Definition 7.7.** Let $j = 1, \ldots, d$. Given $a \in \Lambda$ and $k \in \mathbb{Z}^d$, we say that $a$ is resonant with $k$ at order $j$ if

$$\|a\| \geq R \quad \wedge \quad \|k\| \leq D_j \|a\|^\mu$$

(7.8)

and

$$|\omega(a) \cdot k| \leq C_j \|k\| \|a\|^{M - \gamma_j} .$$

(7.9)

If $j = 1$ and $a$ is not resonant with $k$ at order 1, we say that $a$ is nonresonant with $k$.

**Remark 7.8.** Notice that, since $C_1 = D_1 = 1$ and $M - \gamma_1 = \delta$, $a$ is resonant with $k$ at order 1 if and only if $a$ is resonant with $k$ according to Definition 6.3.

In all the construction a parameter $\sigma \in \{0, 1\}$ will appear. The case $\sigma = 0$ corresponds to the case $b - a$ resonant with $a$, while $\sigma = 1$ corresponds to $b - a$ resonant with $b$.

First we identify the nonresonant points.

**Definition 7.9 (Non resonant region).** For $\sigma = 0, 1$, consider the sets $\Omega_\sigma \subset \Lambda$ defined by the following property: $a \in \Omega_\sigma$ if and only if $\forall k \in \Lambda$ such that $a + k \in \Lambda$, the vector $a + \sigma k$ is non resonant with $k$. Then the non resonant region is defined by

$$\Omega := \Omega_0 \cap \Omega_1 .$$

(7.10)
Remark 7.10. For \( a, b \in \Lambda \) denote \( k := b - a \), then one easily sees that:

\[
\begin{align*}
\Omega_0 &= \{ a \in \Lambda \mid a \text{ is non resonant with } b - a \quad \forall b \in \Lambda \}, \\
\Omega_1 &= \{ a \in \Lambda \mid b \text{ is non resonant with } b - a \quad \forall b \in \Lambda \}.
\end{align*}
\]

We then give the following definitions:

Definition 7.11 (Resonant zones). Let \( M \) be a module of \( \mathbb{Z}^d \) of dimension \( s \).

(i) If \( s = 0 \), namely \( M = \{0\} \), we set

\[
Z_M^{(0)} = \Omega := \Omega_{\{0\}}.
\]  

(ii) If \( s \geq 1 \), for any set of linearly independent vectors \( \{k_1, \ldots, k_s\} \) in \( M \), we define, for \( \sigma = 0, 1 \)

\[
Z_{k_1,\ldots,k_s}^{\sigma} = \{ a \in \Lambda \mid (a + k_1 \in \Lambda) \land (a + \sigma k_1 \text{ is resonant with } k_j \text{ at order } j \quad \forall j = 1, \ldots, s) \},
\]

and

\[
Z_M^{(s)} := \bigcup_{\text{lin. ind. in } M} (Z_{k_1,\ldots,k_s}^{0} \cup Z_{k_1,\ldots,k_s}^{1}).
\]

The sets \( Z_M^{(s)} \) are called resonant zones.

The sets \( Z_M^{(s)} \) contain lattice points \( a \) which are in resonance with at least \( s \) linearly independent vectors in \( M \).

Remark 7.12. Fix \( r, s \in \{1, \ldots, d\} \) with \( 1 \leq r < s \), then for any \( M \) with \( \dim M = s \), one has

\[
Z_M^{(s)} \subseteq \bigcup_{\substack{M' \subseteq M \\ \dim M' = r}} Z_{M'}^{(r)}.
\]

Remark 7.13. By the very definition of \( Z_k^{\sigma} \) and \( \Omega_\sigma \) one has

\[
(\cup_{k_1} Z_{k_1}^{\sigma}) \cap \Omega_\sigma = \emptyset , \quad (\cup_{k_1} Z_{k_1}^{\sigma}) \cup \Omega_\sigma = \Lambda ,
\]

and therefore

\[
(\cup_{\sigma \cup k_1} Z_{k_1}^{\sigma}) = \Lambda \setminus \Omega .
\]

In particular,

\[
\left( \bigcup_{M \text{ of dim.1} \cap M = \{0\}} Z_M^{(1)} \right) \cap \Omega = (\cup_{\sigma \cup k_1} Z_{k_1}^{\sigma}) \cap \Omega = \emptyset .
\]
Remark 7.14. By (7.14), one also has that the resonant zones $\bigcup_{0 \leq s \leq d} \bigcup_{M} Z^{(s)}_{M}$ are a covering of $\Lambda$.

We start the study of the properties of the resonant and non resonant zones by a simple lemma:

Lemma 7.15. Assume $R$ large enough, then

$$B_{R/2} \subset \Omega.$$  

Proof. Given $a \in \Lambda$, suppose that there exist $j = 1, \ldots, d$, $\sigma \in \{0, 1\}$ and $k \in \mathbb{Z}^{d}$ such that $a + k \in \Lambda$ and $a + \sigma k$ is resonant with $k$ at order $j$. When $\sigma = 0$, (7.8) gives $\|a\| \geq R$, therefore $\Omega_0 \supset B_{R}$. On the other hand, for $\sigma = 1$, by (7.8) with $a$ replaced by $a + k$ one has $\|a + k\| \geq R$, and

$$\|a\| \geq \|a + k\| - \|k\| \geq \|a + k\| - \|a + k\|^\mu,$$

but the r.h.s. turns out to be larger than $R/2$, provided $\|a + \sigma k\|$ is large enough. This gives $\Omega_1 \supset B_{R/2}$, and therefore $\Omega = \Omega_0 \cap \Omega_1 \supset B_{R/2}$. \qed

Remark 7.16. Let $M$ be a module with $\dim M \geq 1$. Since by Lemma 7.15 one has $B_{R/2} \subset \Omega$, and by Remarks 7.12, 7.13

$$Z^{(s)}_{M} \cap \Omega \subseteq \left( \bigcup_{M \text{ of dim.1}} Z^{(1)}_{M} \right) \cap \Omega = \emptyset,$$

it follows that

$$B_{R/2} \cap Z^{(s)}_{M} = \emptyset.$$  

In the next Lemma, whose proof is deferred to Subsect. 7.2.1 we claim that the “completely resonant zone” is empty:

Lemma 7.17. Provided $R$ is large enough, the resonant zone $Z^{(d)}_{2d}$ is empty.

We proceed now in partitioning the resonant zones in sets which are not coupled by operators in normal form.

Definition 7.18. On $Z^{(s)}_{M}$ we define the following pre-equivalence relation: $a \sim b$ if $a - b \in M$ and

$$\|a - b\| \leq \max\{\|a\|^\mu, \|b\|^\mu\}. \quad (7.16)$$

We then complete such a pre-equivalence relation to an equivalence relation setting

$$a \sim b$$

if there exists a finite sequence $\{a_j\}_{j=1}^{N}$ such that $a_1 = a$, $a_N = b$ and $a_j \sim a_{j+1}$ for any $j = 1, \ldots, N - 1$.  

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Definition 7.19. For any module $M$ of dimension $s$, we define the sets $\{A^{(s)}_{M,j}\}_{j \in \mathcal{J}_M}$ as the equivalence classes with respect to the equivalence relation $\sim$.

Remark 7.20. By definition, if $Z$ is a normal form operator, then one has $\Pi_a Z \Pi_b \neq 0$ only if either $a$, or $b$, are resonant with $b - a$. Thus in particular $b$ and $a$ must satisfy (7.16), and if $a \in \mathcal{Z}^{(s)}_M$ one must also have $b - a \in M$. The blocks $A^{(s)}_{M,j}$ are designed to group together all such couples of points $a, b$.

Remark 7.21. In the case $M = \{0\}$, the condition $a - b \in M$ implies that each equivalence class $A^{(0)}_{\{0\},j}$ contains only one element of $\mathcal{Z}^{(0)}_{\{0\}}$, and $\mathcal{J}_M = \sharp \mathcal{Z}^{(0)}_{\{0\}}$.

Lemma 7.22. Provided $R$ is large enough, there exists a positive constant $C$, depending only on $\gamma_s, C_s$ and $D_s$, such that the following holds: for the partition $\{A^{(s)}_{M,j}\}_{j \in \mathcal{J}_M}$, the index set $\mathcal{J}_M$ is at most countable, and the following properties hold:

1. $a, b \in A^{(s)}_{M,j} \Rightarrow a - b \in M$

2. If $a \in A^{(s)}_{M,j_1}$ and $b \in A^{(s)}_{M,j_2}$ with $j_1 \neq j_2$, then either $\|a - b\| > \max\{\|a\|^{\mu}, \|b\|^{\mu}\}$, or $a - b \notin M$

3. $a, b \in A^{(s)}_{M,j} \Rightarrow \|a - b\| \leq C\|a\|^{1 - \gamma_{s+1}}$.

We postpone the proof of this lemma to Subsection 7.2.2.

7.1.2 The invariant partition

Clearly the regions $\mathcal{Z}^{(s)}_M$ are not reciprocally disjoint. Following the construction of [BLM22], we identify now sets of points $a \in \Lambda$ which admit exactly $s$ linearly independent resonance relations.

Definition 7.23 (Resonant blocks).

1. (Nonresonant Blocks $B^{(s)}_{M,j}$): If $M = \{0\}$,

   $$B^{(0)}_{\{0\},j} := A^{(0)}_{\{0\},j} \quad \forall j \in \mathcal{J}_{\{0\}}$$

2. ($s$-resonant Blocks): Given a resonance module $M \subset \mathbb{Z}^d$ of dimension $s \in \{1, \ldots, d - 1\}$, we define

   $$B^{(s)}_{M,j} := A^{(s)}_{M,j} \setminus \left( \bigcup_{M' \text{s.t.} \dim M' = s+1} \mathcal{Z}^{(s+1)}_{M'} \right), \quad j \in \mathcal{J}_M,$$
where \( J_M \) and \( \{ A_{M,j}^{(s)} \}_{j \in J_M} \) are the sets whose existence is ensured in Lemma 7.22.

**Remark 7.24.** The resonant blocks form a covering of \( \Lambda \).

As we will prove in the following section, \( \forall s \) there exists a suitable choice of the constants \( C_s, D_s \) such that the blocks \( B_{M,j}^{(s)} \) are reciprocally disjoint. However, still they do not provide an invariant partition of \( \Lambda \). The invariant partition will be given by the extended blocks, that we are now going to define.

Recall first that, given two sets \( A \) and \( B \), their Minkowski sum \( A + B \) is defined by:

\[
A + B := \{ a + b \mid a \in A, \ b \in B \}.
\]

**Definition 7.25 (Extended blocks \( E_{M,j}^{(s)} \)).**

1. \( E_{\{0\},j}^{(0)} := E_{\{0\},j}^{(0)} = A_{\{0\},j}^{(0)} \) \( j \in J_{\{0\}} \)

2. Given a resonance module \( M \) of dimension 1, \( \forall j \in J_M \) we define

\[
E_{M,j}^{(1)} := \left( B_{M,j}^{(1)} + M \right) \cap A_{M,j}^{(1)}, \quad E^{(1)} := \bigcup_{M \text{ of dim.1}, j \in J_M} E_{M,j}^{(1)}.
\]

3. Given a resonance module \( M \) of dimension \( s \), with \( 2 \leq s < d \), for any \( j \in J_M \) we define

\[
E_{M,j}^{(s)} := \left( B_{M,j}^{(s)} + M \right) \cap A_{M,j}^{(s)} \cap \bigcap_{k=1}^{s-1} \left( E^{(s-k)} \right)^c, \quad E^{(s)} := \bigcup_{M \text{ of dim.s}, j \in J_M} E_{M,j}^{(s)},
\]

where, given \( E \subseteq \Lambda \), \( E^c := \Lambda \setminus E \).

**Remark 7.26.** The extended blocks \( \{ E_{M,j}^{(s)} \} \) are still a covering of \( \Lambda \). Indeed, by Remark 7.24 the blocks \( \{ B_{M,j}^{(s)} \} \) form a covering and, by \( B_{M,j}^{(s)} \subseteq \left( B_{M,j}^{(s)} + M \right) \cap A_{M,j}^{(s)} \), it follows that \( \bigcup E_{M,j}^{(s)} = \bigcup \left( B_{M,j}^{(s)} + M \right) \cap A_{M,j}^{(s)} = \Lambda \).

The following theorem is the main result of the present section. Its proof is postponed to Subsect. 7.3.

**Theorem 7.27.** There exists a choice of the parameters \( C_1, \ldots, C_d, D_1, \ldots, D_d \) and \( R \) such that the blocks \( \{ E_{M,j}^{(s)} \}_{M \in \mathbb{Z}^d, j \in J_M} \) are a partition of \( \Lambda \), which is left invariant by operators \( Z \) which are in normal form, namely one has

\[
[\Pi_{E_{M,j}^{(s)}}, Z] = 0 \quad \forall M \in \mathbb{Z}^d, \forall j \in J_M.
\]

(7.17)
Furthermore, one has
\[ \max\{\|a\| : a \in E_{M,j}^{(s)}\} \leq 2 \min\{\|a\| : a \in E_{M,j}^{(s)}\} \quad \forall M \subset \mathbb{Z}^d, \forall j \in J_M. \quad (7.18) \]

### 7.2 Proof of the properties of the partition

#### 7.2.1 Proof of Lemma 7.17

As in the proof of the classical Nekhoroshev Theorem (see also [BLM22]), the following Lemma plays a fundamental role.

**Lemma 7.28.** [Lemma 5.7 of [Gio03]] Let \( s \in \{1, \ldots, d\} \) and let \( \{u_1, \ldots, u_s\} \) be linearly independent vectors in \( \mathbb{R}^d \). Let \( w \in \text{span}\{u_1, \ldots, u_s\} \) be any vector. If \( \alpha, N \) are such that
\[
\|u_j\| \leq N \quad \forall j = 1, \ldots, s,
\]
\[
|w \cdot u_j| \leq \alpha \quad \forall j = 1, \ldots, s,
\]
then
\[
\|w\| \leq \frac{sNs^{-1}\alpha}{\text{Vol}\{u_1| \cdots |u_s\}}.
\]

**Proof of Lemma 7.17.** Assume that \( Z_d^{(d)} \) is not empty and take \( a \in Z_d^{(d)} \). First, by Remark 7.15, one has
\[ \|a\| \geq R/2. \quad (7.19) \]

Furthermore there exists \( \sigma \in \{0, 1\} \) and \( \{k_1, \ldots, k_d\} \subset \mathbb{Z}^d \) linear independent vectors such that \( \forall j \)
\[ \|k_j\| \leq D_j\|a + \sigma k_1\|^\mu \leq D_j\|a + \sigma k_1\|^\mu, \]
\[ |\omega(a + \sigma k_1) \cdot k_j| \leq C_j\|k_j\||a + \sigma k_1|^{M-\gamma} \leq C_dD_d\|a + \sigma k_1|^{M-\gamma d+\mu}. \quad (7.20) \]

Consider first the case \( \sigma = 0 \). By the second of (7.20), using Lemma 7.28 we deduce
\[ \|\omega(a)\| \leq d(D_d)^dC_d\|a|^{M-\gamma d+\mu}. \]

Then by Equation (7.6), one has
\[ K^{-1}\|a\|^M \leq \|\omega(a)\| \leq d(D_d)^dC_d\|a|^{M-\gamma d+\mu}. \quad (7.21) \]

Recalling that \( M - \gamma + d\mu < M \), (7.21) implies \( \|a\| \leq R_0 \) for some positive \( R_0 \), but if \( R/2 > R_0 \) this is in contradiction with (7.19).

If instead \( \sigma = 1 \), in order to eliminate the presence of the vectors \( k_1 \) in
estimates (7.20), we apply Lemma D.3 of the Appendix, with \( k = k_1, h = k_j, a = b, l = 0 \), obtaining the existence of positive constants \( D' = D'(\mu, \gamma_d, D_d, C_d), C' = C'(\mu, \gamma_d, D_d, C_d) \) such that
\[
\| k_j \| \leq D'|a|^{\mu} \quad \forall j, \quad |\omega(a) \cdot k_j| \leq C'|a|^{\|k_j\|} \leq C'D'|a|^{\|k_j\|} \quad \forall j.
\]
(7.22)

Then one applies Lemma 7.28 and argues as in the case \( \sigma = 0 \) in order to obtain the thesis.

\[ \square \]

### 7.2.2 Proof of Lemma 7.22

We first point out that by their very definition, the sets \( \{ A(s_{M,j}) \}_{j \in J_M} \) automatically satisfy Items 1 and 2 of Lemma 7.22. Due to Remark 7.21, Item 3 also immediately follows in the case \( M = \{0\} \). We now tackle the case \( M \neq \{0\} \); this is the heart of the proof of Theorem 7.2, in particular it is where steepness comes into play.

In the remaining part of the present subsection, we will restrict to the case where \( M \) is a proper modulus, namely \( 1 \leq \text{dim} M \leq d - 1 \). The first result we state is the following:

**Lemma 7.29.** If \( a \in \mathcal{Z}_{M}^{(s)} \), then there exists a positive constant \( K \) depending only on \( d, \mu, \gamma_s, C_s, D_s \), such that
\[
\| \Pi_M \omega(a) \| \leq K\|a\|^{\|k_j\|} \quad \forall j = 1, \ldots, d.
\]
(7.23)

**Proof.** By definition of resonant zones, for \( a \in \mathcal{Z}_{M}^{(s)} \), there exist \( k_1, \ldots, k_s \) and \( \sigma \in \{0, 1\} \) s.t. \( |\omega(a + \sigma k_1) \cdot k_j| \leq C_s \|k_j\| \|a + \sigma k_1\|^{\|k_j\|} \), \( \forall j = 1, \ldots, d \). The proof is then completed arguing as in the proof of Lemma 7.17, namely by combining Lemma 7.28 and Lemma D.3. \[ \square \]

**Lemma 7.30.** [Interpolation] For any \( s = 1, \ldots, d - 1 \), there exists a positive constant \( C \) depending only on \( d, \mu, \gamma_s, C_s, D_s \), such that the following holds. For any \( M, j \) and for any \( a, b \in A_{M,j}^{(s)} \), there exists a curve \( \gamma : [0, 1] \to (\{a\} + \text{span}_\mathbb{R} M) \cap C \) such that
\[
\gamma(0) = a, \quad \gamma(1) = b,
\]
(7.24)

and
\[
\| \Pi_M \omega(\gamma(t)) \| \leq C\|\gamma(t)\|^{\|k_j\|} \quad \forall t \in [0, 1] \quad \forall j = 1, \ldots, d.
\]
(7.25)

**Proof.** Suppose first that \( a \) and \( b \) are such that \( a \sim b \), so that \( \|a - b\| \leq \max\{\|a\|^{\mu}, \|b\|^{\mu}\} \). Then we can define \( \gamma \) by \( \gamma(t) = a + t(b - a) \), and since
a and b are contained in the cone C, (see assumption (2.2)), one also has \( \gamma(t) \in C, \forall t \). Furthermore, by Remark D.2 there exists a positive constant \( C \) such that

\[
C^{-1} ||a|| < ||a + t(b - a)|| \leq C ||a|| \quad \forall t \in [0, 1],
\]

so that (by the \( (M - 1) \)-homogeneity of \( \partial \omega/\partial a \)):

\[
\|\Pi_M \omega(a + t(b - a))\| \leq \|\Pi_M \omega(a)\| + \left|\int_0^1 \frac{\partial \omega}{\partial a} (a + t_1 t(b - a))t(b - a) dt_1\right|
\]

\[
\lesssim \|\Pi_M \omega(a)\| + ||a||^{s-1} \|b - a\|
\]

\[
\lesssim ||a||^{\gamma - \gamma_s + s\mu} + ||a||^{s-1+\mu} \lesssim ||a||^{\gamma - \gamma_s + s\mu},
\]

where in the last line we used Lemma 7.29. Then, using again (7.26), one also obtains

\[
\|\Pi_M \omega(a + t(b - a))\| \lesssim ||a + t(b - a)||^{\gamma - \gamma_s + s\mu} \quad \forall t.
\]

In the general case where \( a \sim b \), let \( \{a_l\}_{l=1}^N \) be a finite sequence such that \( a_1 = a, a_N = b \) and \( a_l \sim a_{l+1} \) for any \( l = 1, \ldots, N - 1 \). Then for any \( l \) we can exhibit a curve \( \gamma_l \) connecting \( a_l \) and \( a_{l+1} \) and satisfying the required properties. To obtain the thesis, it is sufficient to consider a curve \( \gamma \) whose support is the union of the supports of \( \gamma_l \) for any \( l \). □

Roughly speaking, the idea in order to prove Item 3 of Lemma 7.22 is that one would like to consider the curve joining \( a \) and \( b \) and to exploit steepness in order to deduce that, if by contradiction Item 3 of Lemma 7.22 does not hold true, then, by steepness (7.25) is violated. This is obtained essentially as in the classical case. The general idea is to exploit homogeneity in order to reduce the study to a compact region.

**Definition 7.31.** For any \( a \in \mathcal{Z}^{(s)}_M \), define \( \omega^\perp(a) \) as the \( d - 1 \) dimensional subspace of \( \mathbb{R}^d \) orthogonal to \( \omega(a) \), and

\[
M_a := \Pi_{\omega^\perp(a)} \text{span}_{\mathbb{R}} M.
\]

The following lemma ensures that in an appropriate sense \( M_a \) is close to \( M \).

**Lemma 7.32.** Let \( M \) be a modulus of dimension \( s \), \( 1 \leq s < d \). There exists positive constants \( C \) and \( R_0 \), depending only on \( \gamma_s, \mu, C_s, D_s \), such that, if \( R > R_0 \), then for any \( a \in \mathcal{Z}^{(s)}_M \),

\[
\|\Pi_{M_a} - \Pi_M\| \leq C ||a||^{-\gamma_s + s\mu}.
\]
Proof. By Lemma 7.29 there exists a positive constant $C_1$, depending only on $C_s, D_s, \gamma_s, \mu$, such that for any $a \in \mathcal{Z}_M^{(s)}$ we have

$$\|\Pi_M \omega(a)\| \leq C_1 \|a\|^{|\gamma_s + s\mu|}. $$

Then Lemma D.4 of the Appendix and Equation (7.6) give

$$\|\Pi_M - \Pi_M\| \leq 9C_1 \|a\|^{|\gamma_s + s\mu|} \|\omega(a)\|^{-1} \leq 9CC_1 \|a\|^{|\gamma_s + s\mu|}$$

(with $C$ the constant in eq. (7.6)), provided

$$\varepsilon := C_1 \|a\|^{|\gamma_s + s\mu|} \leq \frac{1}{2} \|\omega(a)\|;$$

but the last inequality holds provided $R$ is large enough.

Lemma 7.33. Given a modulus $M$ of dimension $s$, let $a \in \mathcal{Z}_M^{(s)}$ and suppose $u \in \{a\} + \text{span}_M \cap C$ is such that

$$\|\Pi_M \omega(a + u)\| \leq C \|a + u\|^{|\gamma_s + s\mu|},$$

$$\|u\| \leq C \|a\|^{1-\tau}. $$

Assume that

$$a + \Pi_Mu \in C_e,$$

then there exists positive constants $C^+$ and $R_0$, depending only on $\gamma_s, \mu, C_s, D_s, C$ and $\tau$, such that if $R > R_0$ one has

$$\|\Pi_M \omega(a + \Pi_Mu)\| \leq C^+ \|a\|^{|\gamma_s + s\mu|}. $$

Proof. One has

$$\Pi_M \omega(a + \Pi_Mu) = \Pi_M \omega(a + u) + (\Pi_M - \Pi_M) \omega(a + \Pi_Mu) + \Pi_M (\omega(a + \Pi_Mu) - \omega(a + u)),$$

where all the quantities are well defined by (7.33). By (7.31), (7.32), and homogeneity of $\omega$, one has

$$\|\Pi_M \omega(a + u)\| \lesssim \|a + u\|^{|\gamma_s + s\mu|}$$

$$\lesssim \|a\|^{|\gamma_s + s\mu|} + \|u\|^{|\gamma_s + s\mu|} \lesssim \|a\|^{|\gamma_s + s\mu|},$$

since $1 - \tau < 1$.

Eq. (7.36) is easily estimated using again homogeneity and Eq. (7.28).
We come to (7.37). Recalling that \( u \in \text{span}_{\mathbb{R}} M \), one has

\[
\| \Pi_M (\omega(a + \Pi_M a u) - \omega(a + u)) \| = \| \Pi_M (\omega(a + \Pi_M a u) - \omega(a + \Pi_M u)) \| \\
\leq \left\| \int_0^1 \frac{\partial \omega}{\partial a} (a + u + t(\Pi_M a - \Pi_M u)) \, dt \right\| \| (\Pi_M a - \Pi_M u) \| . \tag{7.40}
\]

To estimate the integral term in (7.40), remark that, by the convexity of \( C_e \), one has \( a + u + t(\Pi_M a - \Pi_M u) \in C_e \forall t \in [0, 1] \), and therefore

\[
\left\| \int_0^1 \frac{\partial \omega}{\partial a} (a + u + t(\Pi_M a - \Pi_M u)) \right\| \lesssim \| a \|^{R-1}. \tag{7.41}
\]

Using again Lemma 7.32, one easily concludes the proof. \( \square \)

We use now steepness in order to prove the following lemma.

**Lemma 7.34.** There exist positive constants \( \overline{C} \) and \( R_0 \), depending on \( \gamma_s, \mu, C_s, D_s \) only, such that if \( R \geq R_0, \forall a \in A^{(s)}_{M,j} \) one has

\[
\| a - b \| \leq \overline{C}\| a \|^{1-\gamma_s+1} \quad \forall b \in A^{(s)}_{M,j}. \tag{7.42}
\]

**Proof.** We prove that there exist constants \( \overline{R} \) and \( \overline{C} > 0 \), depending on \( \gamma_s, \mu, D_s, C_s \) only, such that if there exists a couple of points \( a, b \in A^{(s)}_{M,j} \) satisfying \( \| a \| \geq \overline{R} \) and

\[
\| a - b \| \geq \overline{C}\| a \|^{1-\frac{(\gamma_s-\mu)}{\alpha s}}, \tag{7.43}
\]

one gets a contradiction. Then the result will follow taking \( R \geq 2\overline{R} \) and using Remark 7.16.

Let us fix \( a \) and \( b \) in some \( A^{(s)}_{M,j} \) and suppose that (7.43) holds. Consider the curve \( \gamma_{a,b} \) joining \( a \) and \( b \) constructed in Lemma 7.30; then there exists \( t^* > 0 \) such that

\[
\| \gamma_{a,b}(t^*) - a \| = \overline{C}\| a \|^{1-\frac{(\gamma_s-\mu)}{\alpha s}}, \tag{7.44}
\]

\[
\| \gamma_{a,b}(t) - a \| < \overline{C}\| a \|^{1-\frac{(\gamma_s-\mu)}{2\alpha s}} \quad \forall t \in [0, t^*). \tag{7.45}
\]

Remark that, if one can prove that

\[
a + \Pi_{M_a}(\gamma_{a,b}(t) - a) \in C_e, \tag{7.46}
\]

then by (7.45) one gets the existence of a \( \overline{R} > 0 \) such that, if \( \| a \| \geq \overline{R} \),

\[
\| \gamma_{a,b}(t) - a \| < \| a \|^{1-\frac{(\gamma_s-\mu)}{2\alpha s}} \quad \forall t \in [0, t^*). \tag{7.47}
\]
Then by (7.25) and (7.47), one can apply Lemma 7.33 getting the existence of a positive constant $C^+$, depending only on $\gamma_s, \mu, C_s, D_s$, such that
\[ \|\Pi_{Ma}\omega(a + \Pi_{Ma}(\gamma_{ab}(t) - a))\| \leq C^+ \|a\|^{|\alpha_s - \gamma_s + s\mu|}. \] (7.48)

We will show in a while that (7.46) holds, and that (7.48) is in contradiction with the consequences of steepness.

Consider the curve $u : [0, 1] \rightarrow \mathbb{R}^d$ defined by
\[ u(t) := \Pi_{Ma}(\gamma_{a,b}(t) - a) \quad \forall t \in [0, 1]; \] (7.49)

We first prove that
\[ \frac{C}{2} \|a\|^{1 - \frac{\gamma_s - \mu}{\alpha_s}} \leq \|u(t)\| \leq 2\bar{C} \|a\|^{1 - \frac{\gamma_s - \mu}{\alpha_s}}, \quad \forall t \in [0, t^*). \] (7.50)

Indeed, since $\gamma_{a,b}(t) - a \in \text{span}_R M$ for all $t$, by Lemma 7.32 we have
\[ \|u(t) - (\gamma_{a,b}(t) - a)\| = \|(\Pi_{Ma} - \Pi_{M})(\gamma_{a,b}(t) - a)\| \leq \frac{1}{2}\|\gamma_{a,b}(t) - a\| \]
provided $\|a\|$ is large enough; this implies
\[ \frac{1}{2}\|\gamma_{a,b}(t) - a\| \leq \|u(t)\| \leq \frac{3}{2}\|\gamma_{a,b}(t) - a\|, \]
from which, using (7.45), Eq. (7.50) follows. This in particular implies that (7.46) holds and therefore implies (7.48).

We are now going to use steepness in the form (7.7). Take $r = \|a\|$ and $\xi := \|u(t^*)\|$ and, for $\eta \in [0, \xi]$ let $t_\eta$ be the smallest time in $[0, t^*)$ such that $\|u(t_\eta)\| = \eta$; by (7.50) one has $\xi < r \|a\|$ (with $r$ the quantity in Eq. (2.7)) and
\[ a + u(t_\eta) \in C_\circ \quad \forall \eta \in [0, \xi]. \] (7.51)

Let $\bar{t}$ be the point realizing the maximum on $[0, \xi]$ of the quantity $\|\Pi_{Ma}\omega(a + u(t_\eta))\|$, then steepness ensures that
\[ \|\Pi_{Ma}\omega(\gamma_{ab}(\bar{t}))\| = \|\Pi_{Ma}\omega(a + u(\bar{t}))\| = \max_{\eta \in [0, \xi]} \|\Pi_{Ma}\omega(a + u(t_\eta))\| \geq B_\star \|a\|^{|\alpha_s - \gamma_s + s\mu|} \geq B_\star (2^{-1}\bar{C})^{\alpha_s} \|a\|^{|\alpha_s - \gamma_s + s\mu|}. \] (7.52)

but, taking $\bar{C}$ large enough, this contradicts Eq. (7.48), and so the thesis follows. \[\square\]
7.3 Proof of Theorem 7.27

This subsection follows very closely the proof given in Subsect. 5.1 of [BLM22] for the convex case. We report it here since the language and the context are slightly different.

7.3.1 Separation properties of the extended blocks

The next two lemmas ensure that, if the parameters $C_j, D_j$ are suitably chosen, an extended block $E_{M,j}^{(s)}$ is separated from every resonant zone associated to a lower dimensional module $M'$ which is not contained in $M$.

**Lemma 7.35.** [Non overlapping of resonances] For all $s = 1, \ldots, d - 1$ there exist positive constants $\bar{R}, \bar{C}_{s+1}$ and $\bar{D}_{s+1}$, depending only on $d, C_s, D_s, \mu, \gamma_s$, such that the following holds: suppose that $M$ and $M'$ are two distinct resonance moduli of respective dimensions $s$ and $s'$ with $s' \leq s$ and $M' \not\subset M$. If

$$C_{s+1} > \bar{C}_{s+1}, \quad D_{s+1} > \bar{D}_{s+1}, \quad R > \bar{R},$$

then

$$E_{M,j}^{(s)} \cap Z_{M'}^{(s')} = \emptyset \quad \forall j \in J_M.$$

**Proof.** Let $M, M'$ be a couple of modules fulfilling the assumptions of the Lemma. Assume by contradiction that there exists $a \in E_{M,j}^{(s)} \cap Z_{M'}^{(s')}$ for some $j \in J_M$

$$a \in E_{M,j}^{(s)} \subseteq \{B_{M,j}^{(s)} + M\} \cap A_{M,j}^{(s)},$$

there exists $b \in B_{M,j}^{(s)}$ such that $a - b \in M$ and $a, b \in A_{M,j}^{(s)}$, so that, by Lemma 7.22 provided $R > \bar{R}$,

$$\|a - b\| \leq C \|a\|^{1 - \gamma_{s+1}}, \quad (7.53)$$

where $C$ and $\bar{R}$ depend only on $\gamma_s, \mu, C_s$ and $D_s$.

On the other hand, since $a \in Z_{M'}^{(s')}$, there exist $\sigma_a \in \{0, 1\}$ and $s'$ integer vectors, $k_1, \ldots, k_{s'} \in M'$ among which at least one does not belong to $M$, s.t.

$$|\omega(a + \sigma_a k_1) \cdot k_j| \leq C_j \||a + \sigma_a k_1||^{\mu - \gamma_j} \|k_j\|, \quad \|k_j\| \leq D_j \|a + \sigma_a k_1\|^{\mu}. \quad (7.54)$$

Let $k_{j_0}$ be the vector which does not belong to $M$; the idea is to show that the resonance relation of $a$ with $k_{j_0}$ implies an analogous relation for $b$, but this will be in contradiction with the fact that $b \in B_{M,j}^{(s)}$ (which contains vectors which are only resonant with $M$).

To start with remark that, since $b \in B_{M,j}^{(s)} \subset Z_{M}^{(s)}$, there exist $\sigma_b \in \{0, 1\}$ and $l_1, \ldots, l_s \in M$, linearly independent, s.t.

$$|\omega(b + \sigma_b l_1) \cdot l_{j_0}| \leq C_j \|b + \sigma_b l_1||^{\mu - \gamma_j} \|l_{j_0}\|, \quad \|l_j\| \leq D_j \|b + \sigma_b l_1\|^{\mu}. \quad (7.55)$$
By Lemma \[\text{Lemma D.3}\] with \( k = \sigma_a k_1, h = \sigma_b l_1 h = k_j \) and \( \delta = \gamma_{s+1} \) there exist constants \( C_{s+1} \) and \( \tilde{D}_{s+1} \), depending only on \( \gamma_s, \mu, C_s, D_s \), such that
\[
|\omega(b + \sigma_b l_1) \cdot j_k| \leq C_{s+1} \|b + \sigma_b l_1\|^{K - \gamma_{s+1}}, \quad \|j_k\| \leq \tilde{D}_{s+1} \|b + \sigma_b l_1\|^\mu.
\]
But, if \( C_{s+1} > \tilde{C}_{s+1} \) and \( \tilde{D}_{s+1} > \tilde{D}_{s+1} \), this means that \( b \) is also resonant with \( k_j \), and thus it belongs to \( Z^{(s+1)}_{M''} \) with \( M'' := \text{span}_{\mathbb{Z}}(M, k_j) \), which contradicts the fact that \( b \in D_{M,j}^{(s)} \).

**Lemma 7.36.** (Separation of resonances) There exist positive constants \( \bar{R}, C_{s+1} \) and \( \tilde{D}_{s+1} \) depending only on \( d, \mu, \gamma_s, C_s, D_s \) such that, if
\[
C_{s+1} > \tilde{C}_{s+1}, \quad D_{s+1} > \tilde{D}_{s+1}, \quad \bar{R} > \bar{R},
\]
then the following holds true. Let \( a \in E^{(s)}_{M,j} \) for some \( M \) of dimension \( s = 1, \ldots, d - 1 \) and some \( j \in J_M \), and let \( k' \in \mathbb{Z}^d \) be such that
\[
\|k'\| \leq \|a + \sigma k'\|^\mu,
\]
for some \( \sigma \in \{0, 1\} \). Then \( \forall M' \subset M \) s. t. \( s' := \dim M' \leq s \) one has
\[
a + k' \notin Z^{(s')}_{M'}.
\]

**Proof.** The proof is very similar to that of Lemma \[\text{Lemma 7.35}\] Assume by contradiction that \( a + k' \in Z^{(s')}_{M'} \) for some \( M' \neq M \). It follows that there exist \( \sigma_a \in \{0, 1\} \) and \( s' \) integer vectors, \( k_1, \ldots, k_{s'} \in M' \) among which at least one does not belong to \( M \), s.t.
\[
|\omega(a + k' + k_1) \cdot j_k| \leq C_j \|a + k' + \sigma_a k_1\|^{K - \gamma_{j_k}} \|j_k\|, \quad \|j_k\| \leq D_j \|a + \sigma_a k_1 + k'\|^\mu.
\]
(7.56)

Let \( k_j \) be the vector which does not belong to \( M \). By Lemma \[\text{Lemma 7.22}\] since \( a \in E^{(s)}_{M,j} \) there exists \( b \in D_{M,j}^{(s)} \) s.t. \( \|a - b\| \leq \|a\|^{1 - \gamma_{s+1}} \). Since in particular \( b \in Z^{(s)}_M \), there exist \( \sigma_b \in \{0, 1\} \) and \( l_1, \ldots, l_s \in M \), linearly independent, s.t.
\[
|\omega(b + \sigma_b l_1) \cdot l_j| \leq C_j \|b + \sigma_b l_1\|^{K - \gamma_{l_j}} \|l_j\|, \quad \|l_j\| \leq D_j \|b + \sigma_b l_1\|^\mu.
\]
(7.57)

We now apply Lemma \[\text{Lemma D.3}\] with \( h := k_j, k = \sigma_a k_1 + k', l = \sigma_b l_1 \) to prove that \( b \) is also resonant with \( k_j \). The only nontrivial assumption of Lemma \[\text{Lemma D.3}\] to verify is the last of \( \text{Lemma D.1} \). To this aim we observe that
\[
\|\sigma_a k_1 + k'\| \leq \|k_1\| + \|k'\| \leq \|a + \sigma_a k_1 + k'\|^{\mu} + \|k'\|.
\]
We now estimate \( \|k'\| : \)
\[
\|k'\| \leq \|a + \sigma k'\|^{\mu} \lesssim_\mu \|a + \sigma_a k_1 + k'\|^\mu + \|\sigma_a k_1\|^\mu + \|(\sigma - 1)k'\|^\mu \lesssim_\mu \|a + \sigma_a k_1 + k'\|^\mu + \|k'\|^\mu,
\]
45
which since \( \mu < 1 \) also implies
\[
\|k'\| \lesssim \|a + \sigma_d k_1 + k'\|^\mu.
\]
Thus Lemma \[D.3\] gives
\[
|\omega(b + \sigma b l_1) \cdot k_j| \leq \tilde{C}_{s+1} \|b + \sigma b l_1\|^{\mu-\gamma_{s+1}} \|k_j\|, \quad \|l_1\| \leq \tilde{D}_{s+1} \|b + \sigma b l_1\|^\mu.
\]

But, if \( C_{s+1} > \tilde{C}_{s+1}, D_{s+1} > \tilde{D}_{s+1} \), this means that \( b \) is also resonant with \( k_j \), and thus \( b \in \mathcal{Z}_{M'\gamma}^{(s+1)} \) with \( M' := \text{span}_\mathbb{Z}(M, k_j) \), and this contradicts the fact that \( b \in B_{M,j}^{(s)} \).

As a consequence of Lemma \[7.35\], we can now prove the following:

**Lemma 7.37.** If the constants \( R, C_1, \ldots, C_d, D_1, \ldots, D_d \) are chosen as in Lemma \[7.35\] then the extended blocks \( \{E_{M,j}^{(s)}\}_{M \in \mathbb{Z}^d, j \in J_M} \) are a partition of \( \Lambda \).

**Proof.** By the definition of the extended blocks, one immediately has that, if \( s \neq s' \), then \( E_{M,j}^{(s)} \cap E_{M',j'}^{(s')} = \emptyset \). Furthermore, again by their very definition, if \( M = M' \) but \( j \neq j' \) the blocks \( E_{M,j}^{(s)} \cap E_{M',j'}^{(s')} \) are disjoint, since \( E_{M,j}^{(s)} \subseteq A_{M,j}^{(s)} \), \( E_{M',j'}^{(s')} \subseteq A_{M',j'}^{(s')} \), and \( A_{M,j}^{(s)} \cap A_{M',j'}^{(s')} = \emptyset \). Thus the only case that it remains to check is \( s = s' \), and \( M \neq M' \). But, due to Lemma \[7.35\] one has that, if the constants \( C_1, \ldots, C_d, D_1, \ldots, D_d, R \) are suitably chosen, then
\[
E_{M,j}^{(s)} \cap E_{M',j'}^{(s)} \subseteq E_{M,j}^{(s)} \cap \mathcal{Z}_{M'}^{(s)} = \emptyset,
\]
which gives the thesis. \( \square \)

### 7.3.2 Invariance of the extended blocks

We are now in position to prove the following:

**Lemma 7.38.** If the constants \( C_1, \ldots, C_d, D_1, \ldots, D_d \) and \( R \) are chosen as in Lemma \[7.36\] and \( Z \) is an operator in normal form, then one has
\[
[\Pi_{E_{M,j}^{(s)}}, Z] = 0 \quad \forall M \subseteq \mathbb{Z}^d, \forall j \in J_M.
\]

**Proof.** Take \( a \in E_{M,j}^{(s)} \) and assume that \( b \) is such that
\[
\Pi_b Z \Pi_a \neq 0.
\] (7.58)

We are going to prove that \( b \in E_{M,j}^{(s)} \), arguing by induction on \( s \).
First of all, we observe that, if \( s = 0 \), then by the definition of normal form
implies $a = b$, otherwise we would have that $a$ is in resonance with $b - a$, or $b$ is in resonance with $b - a$, which is excluded by the definition of $E^{(0)} := Z^{(0)} = \Omega$ (see Remark 7.10).

We then turn to analyze the case $s \geq 1$. To this aim, recall that one has

$$E^{(s)} := \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3,$$

$$\mathcal{E}_1 := \{ B^{(s)}_{M,j} + M \}, \quad \mathcal{E}_2 := A^{(s)}_{M,j}, \quad \mathcal{E}_3 := \bigcap_{k=1}^{s-1} (E^{(s-k)})^c,$$

we are going to prove in three subsequent steps that, if (7.58) holds with $a \in E^{(s)}_{M,j}$, then $b \in E_j \forall j = 1, 2, 3$.

**Step 1:** $b \in \mathcal{E}_1$. By the definition of normal form operator, if (7.58) holds and we denote $k := b - a$, then $a$ is resonant with $k$, or $b = a + k$ is resonant with $k$. Thus in particular $a \in Z^{(1)}_{M'}$, with $M' := \text{span}\{k\}$. Then, by Lemma 7.35 we have that $M' \subseteq M$, namely $k \in M$. Then since $a \in E^{(s)}_{M,j} \subseteq \mathcal{E}_1$, we have

$$b = a + k \in \mathcal{E}_1 + \{k\} = \mathcal{E}_1.$$

**Step 2:** $b \in \mathcal{E}_2$. Assume by contradiction that this is not true. Since the sets $\{ E^{(s)}_{M,j} \}_{M,j}$ form a partition, then there exists $s'$, $M'$ of dimension $s'$ and and $j' \in J_{M'}$ with $(M', j') \neq (M, j)$ such that $b \in E^{(s')}_{M',j'} \subset A^{(s')}_{M',j'}.$

There are three cases:

1) $s' = s$. Since $a$, or $b = a + k$, is resonant with $k$, in particular we have that there exists $\sigma \in \{0, 1\}$ such that

$$\|k\| \leq \|a + \sigma k\|^{\mu}. \quad (7.59)$$

Then Lemma 7.36 implies

$$b \notin Z^{(s)}_{M'}, \quad \text{unless } M = M'.$$

Thus we must have $M = M'$ and $b \in A^{(s)}_{M,j'}$ for some $j' \neq j$. But, by Item 2. of Lemma 7.22 since $b - a = k \in M$, one also has

$$\|b - a\| > \max\{\|a\|^\mu, \|b\|^\mu\},$$

namely

$$\|k\| > \max\{\|a\|^\mu, \|a + k\|^\mu\},$$

which contradicts (7.59). Thus this case is not possible.
2) $s' > s$. By Remark 7.12, this implies $b \in Z_M^{(s)}$ for some $M'$ of dimension $s$, thus in particular $b \in A_{M',j'}^{(s)}$ for some $j' \in J_{M'}$. But by item 1), it must be $M' = M$ and $j = j'$, against the contradiction assumption.

3) $s' < s$. Just remark that, since $Z$ is self-adjoint, (7.58) is equal to

$$
\Pi_a Z \Pi_b = (\Pi_a Z \Pi_b)^* \neq 0,
$$

(7.60)

but the inductive assumption says that $E_m^{(s')}_{M,j'}$ is invariant for $s' < s$, thus (7.60) would imply $a \in E_m^{(s')}_{M,j'}$, which is impossible since the extended blocks form a partition.

Thus we have $a \in E_m^{(s)}$ then $b \in A_m^{(s)} = \mathcal{E}_2$.

**Step 3:** $b \in \mathcal{E}_3$. Again by induction, using (7.60), we already know that $b \in E_m^{(s')}_{M,j'}$, $s' < s$, implies $a \in E_m^{(s')}_{M,j'}$, which is impossible since $a \in E_m^{(s)}_{M,j'}$.

Thus $a \in E_m^{(s)}_{M,j'}$ implies $b \notin E_m^{(s')}_{M,j'}$, $\forall s' < s$, and this concludes the proof of Theorem 7.27.

We finally prove Theorem 7.27.

**Proof of Theorem 7.27.** By Lemma 7.37 and Lemma 7.38 it is possible to choose the constants $C_1, \ldots, C_d, D_1, \ldots, D_d$ and $R$ such that the blocks $\{E_m^{(s)}\}_{s,M,j}$ form a partition, and they are left invariant by any normal form operator $Z$.

It remains to check that the blocks are dyadic, namely that inequality (7.18) holds. To this aim, we observe that if $M = \{0\}$ then the claim is trivial, since $E_m^{(s)} \subseteq A_m^{(s)}$ and $A_m^{(0)}$ is always a singleton by Remark 7.21. Thus it remains to check the case when $M$ is a proper modulus. Let $a$ be a point realizing the minimum of $\| \cdot \|$ in $A_m^{(s)}$ and $b$ a point realizing the maximum of $\| \cdot \|$ in $A_m^{(s)}$. By Item 3 of Lemma 7.22 there exists a constant $C$ such that

$$
\|a - b\| \leq C\|a\|^{1-\gamma_{s+1}},
$$

thus one has

$$
\|b\| \leq \|a\| + \|a - b\| \leq \|a\| + C\|a\|^{1-\gamma_{s+1}}.
$$

(7.61)

Now if $\|a\|$ is large enough, one has that the right-hand side of (7.61) is controlled by $2\|a\|$. Since by Remark 7.16 one has $\|a\| \geq R/2$, up to possibly increasing the value of $R$, one obtains (7.18) and thus the thesis.
8 End of the proof of Theorem 2.16.

The proof is now standard (see e.g. [BLM20a]) and we report it for the sake of completeness.

First we define the sets $W_\ell$ to be the extended blocks $E_{M,j}^{(s)}$. Since they form a numerable set, they can be labeled using integers.

Making reference to Eq. (6.11), denote

$$ \tilde{H}^{(N)} := H_0 + Z_N(t) \quad \text{(8.1)} $$

we start by studying the flow $U_{\tilde{H}^{(N)}}$ of the Schrödinger equation

$$ i\partial_t \psi(t) = \tilde{H}^{(N)}(t)\psi(t). \quad \text{(8.2)} $$

Lemma 8.1. For any $s \geq 0$ there exists a positive constant $K_s$ such that, $\forall \psi \in H^s$

$$ \|U_{\tilde{H}^{(N)}}(t, \tau)\psi\|_s \leq K_s \|\psi\|_s \quad \forall t, \tau \in \mathbb{R}. \quad \text{(8.3)} $$

Proof. We prove the result in each block $W_\ell$ constructed in Theorem 7.2.

Since the blocks form a partition and are invariant under $\tilde{H}^{(N)}$, the result then follows.

To start with remark that, by selfadjointness of $\tilde{H}^{(N)}|_{W_\ell} := \tilde{H}^{(N)}|_{W_\ell}$, the $L^2$ norm is conserved under the flow $U_{\tilde{H}^{(N)}|_{W_\ell}}$. Then, denoting for simplicity $\psi_{W_\ell}(t) := U_{\tilde{H}^{(N)}|_{W_\ell}}(t, \tau)\Pi_{W_\ell}\psi$, since the blocks are dyadic, in each block one has

$$ \|\psi_{W_\ell}(t)\|_s \leq (4 \min_{a \in W_\ell} \|a\|)^s \|\psi_{W_\ell}(t)\|_0 \quad \text{(8.4)} $$

from which the thesis immediately follows.

Proposition 8.2. Consider the Hamiltonian $H^{(N)}$ (cf. (6.11)); there exists $K' > 0$, depending only on $N$, on $K_N$ (cf. Eq. (8.1)) and on the seminorms of $R^{(N)}$, such that for any $\psi \in H^{aN}$ one has

$$ \|U_{H^{(N)}}(t, \tau)\psi\|_{aN} \leq K'(t - \tau)\|\psi\|_{aN} \quad \forall t, \tau \in \mathbb{R}. \quad \text{(8.5)} $$

Proof. Write Duhamel formula

$$ U_{H^{(N)}}(t, \tau)\psi = U_{\tilde{H}^{(N)}}(t, \tau)\psi + \int_{\tau}^{t} U_{\tilde{H}^{(N)}}(t, \tau')(-iR^{(N)}(\tau'))U_{H^{(N)}}(\tau', \tau)\psi \, d\tau', \quad \text{(8.6)} $$
Since $R^{(N)} \in C_b^\infty (\mathbb{R}; B(H^0; H^{aN}))$ and since $H^{(N)}$ is self-adjoint in $L^2$, one gets

$$\|U_{H^{(N)}}(t, \tau)\psi\|_{aN} \leq K_{aN}\|\psi\|_{aN} + \int_\tau^t K_{aN} \|R^{(N)}\|_{B(H^0; H^{aN})} \|U_{H^{(N)}}(\tau', \tau)\psi\|_0 d\tau'$$

$$= K_{aN}\|\psi\|_{aN} + \int_\tau^t K_{aN} \|R^{(N)}\|_{B(H^0; H^{aN})} \|\psi\|_0 d\tau' \leq K_{aN}\|\psi\|_{aN} + K_{aN}C(t - \tau)\|\psi\|_0,$$

for some constant $C$.

**Proof of Theorem 2.16.** In order to get the result one first uses interpolation to get (2.10) for the flow of $H^{(N)}$, then one still has to take into account the effect of the unitary transformation $U_N(t)$, but this is immediately done by using Item 3 of Theorem 6.6.

**9 Proof of the Results on the applications**

**9.1 Proof of the results on the Anharmonic oscillator**

First we introduce the action variables for classical Hamiltonian

$$h_0(x, \xi) = \frac{||\xi||^2}{2} + \frac{||x||^{2\ell}}{2\ell},$$

which corresponds to $H_0$.

The first action (actually we call it $a_2$) is defined to be the angular momentum

$$a_2(x, \xi) := x_1\xi_2 - x_2\xi_1.$$  \hfill (9.2)

Following [BLR21], in order to define the action $a_1$, consider the polar coordinates $(r, \theta)$ in $\mathbb{R}^2$, and the effective potential

$$V^*_L(r) := \frac{L^2}{2r^2} + \frac{r^{2\ell}}{2\ell}.$$  \hfill (9.3)

For $L \neq 0$ we preliminary define

$$a_r = a_r(E, L) := \frac{\sqrt{2}}{\pi} \int_{r_m}^{r_M} \sqrt{E - V^*_L(r)}dr,$$

where $0 < r_m < r_M$ are the two solutions of the equation

$$E - V^*_L(r) = 0.$$  \hfill (9.4)

The cone $C$ is defined by

$$C := \{ a \in \mathbb{R}^2 ; a_1 \geq 0 \text{ if } a_2 \geq 0, \quad a_1 \geq |a_2| \text{ if } a_2 < 0 \}.$$  \hfill (9.5)
Lemma 9.1. [Lemma 4.5 of [BLR21]] The function

\[ a_1(E, L) := \begin{cases} a_r(E, L) & \text{for } L > 0 \\ a_r(E, L) - L & \text{for } L < 0 \end{cases} \]  (9.6)

has the following properties:

1. it extends to a complex analytic function of \( L \) and \( E \) in the region

\[ |L| < \left( \frac{2\ell}{\ell + 1} \right)^{\frac{\ell + 1}{\ell}} E, \quad E > 0 ; \]  (9.7)

2. the map \( E \mapsto a_1(E, a_2) \) admits an inverse \( E = h_0(a_1, a_2) \) which is analytic in the interior of \( C \). Furthermore it is homogeneous of degree \( \frac{2\ell}{\ell + 1} \) as a function of \( (a_1, a_2) \).

3. the function \( a_1(x, \xi) := a_1(h_0(x, \xi), a_2(x, \xi)) \) is quasihomogeneous of degree \( \ell + 1 \), namely

\[ a_1(\lambda x, \lambda^\ell \xi) = \lambda^{\ell + 1} a_1(x, \xi), \quad \forall \lambda > 0 . \]

4. There exist positive constants \( C_1, C_2 \) s.t.

\[ C_1 \langle a \rangle \leq k_0 \leq C_2 \langle a \rangle . \]

Here both \( h_0 \) and \( a_1 \) have a singularity at the origin. To regularize them, consider the decomposition

\[ h_0(x, \xi) = h_0(x, \xi) \chi(Kh_0(x, \xi)) + h_0(x, \xi)(1 - \chi(Kh_0(x, \xi))) , \]  (9.8)

with a large constant \( K \). The first term is smoothing. Denote

\[ h'_0(x, \xi) := h_0(x, \xi)(1 - \chi(Kh_0(x, \xi))) , \]  (9.9)

then \( h'_0 \) and its action variables \( a'_1, a_2 \) coincide with \( h_0 \) and its actions variables outside a neighborhood of the origin. Choosing \( K \) large enough, \( h'_0 \) also fulfills assumption H.i.

Concerning the quantum case, the results of [CdV80] ensure that there exist \( A_1, A_2 \) fulfilling assumptions A.i-A.iii which are perturbations of the Weyl quantizations of \( a_1 \) and \( a_2 \). Precisely, from Theorem 3.1 and Theorem 3.2 of [CdV80] it follows that
Theorem 9.2. There exist two commuting pseudodifferential operators $A_j \in \mathcal{A}_1^1$ satisfying assumption A. In particular there exists $\kappa \in \left( \frac{2}{\ell} \right)$ s.t their joint spectrum is $\Lambda$ and there exists a symbol $h \in S_{AN,1}^{\mathcal{A}_f}$ s.t.

$$H_0 = h(A) . \quad (9.10)$$

Furthermore one has an asymptotic expansion

$$h = h_0' + l.o.t.$$ 

with $h_0'$ the regularized classical Hamiltonian of the anharmonic oscillator written in terms of the action variables.

We are now going to prove that $h_0$ satisfies Assumption H.

Theorem 9.3. Let $h_0'$ be as in (9.9), then $h_0'$ extends to a real analytic function on a cone $C_e$ s.t. $\overline{C} \setminus \{0\} \subset C_e$. Furthermore, its extension is homogeneous at infinity and steep on $C_e$.

Proof. In the following we omit the prime from $h_0$. We start by proving the extension property. The key idea is that, for any fixed value $L \neq 0$ of $a_2$, $\partial C$ corresponds to the circular orbit $a_r = 0$, which in turn is an elliptic equilibrium point of the reduced Hamiltonian system with Hamiltonian

$$h^*_L(p_r, r) := \frac{p_r^2}{2} + \frac{L^2}{2r^2} + \frac{r^{2\ell}}{2\ell} . \quad (9.11)$$

It is well known that the Birkhoff normal form of a 1-d.o.f. system converges in a neighborhood of an elliptic equilibrium, which means that, given a 1 d.o.f. Hamiltonian of the form

$$h(p, q) = \frac{p^2}{2} + V(q) , \quad (9.12)$$

with $V$ an analytic potential having a nondegenerate minimum at zero, there exist analytic canonical variables $(\tilde{p}, \tilde{q})$ (defined in a neighborhood of zero), which give the Hamiltonian (9.12) the form

$$\tilde{h} = \tilde{h} \left( \frac{\tilde{p}^2 + \tilde{q}^2}{2} \right) . \quad (9.13)$$

Furthermore, $\tilde{h}$ extends to a real analytic function of the variable

$$\tilde{a} := \frac{\tilde{p}^2 + \tilde{q}^2}{2} .$$

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defined in a whole neighborhood of zero. Finally, if \( V \) also depends analytically on some parameter \( L \), then all the construction depends analytically on \( L \) (possibly varying in a smaller domain). Birkhoff normal form also allows to compute the Taylor expansion of \( \tilde{h} \) in terms of the Taylor expansion of \( V \).

The explicit computation of the first terms can be found in [FK04], and the result is that if

\[
V(q) = \frac{1}{2} A q^2 + \frac{1}{3!} B q^3 + \frac{1}{4!} C q^4 + O(q^5),
\]

then

\[
\tilde{h}(\tilde{a}) = \sqrt{A} \tilde{a} + \frac{-5B^2 + 3AC}{48A^2} \tilde{a}^2 + O(\tilde{a}^3).
\]

(9.14)

Applying the normal form procedure to the effective Hamiltonian (9.11) with \( L \) in a neighborhood of a fixed \( L_0 \neq 0 \) one gets an action variable \( \tilde{a}_1 = \frac{\tilde{p}^2 + \tilde{\theta}^2}{2} \) such that \( h_L^* \) takes the form (9.14). Now, by construction, the variables

\[
(\tilde{a}_1, a_2) =: \bar{a}
\]

are action variables for \( h_L^* \) (as an Hamiltonian in 2 degrees of freedom). Denote by \( \tilde{h}_0 \) the expression of this Hamiltonian in such action variables. Now one extends \( \tilde{h}_0 \) by homogeneity to a conic neighborhood of \( \bar{a}_* := (0, L_0) \). We remark that the construction is independent of the particular value of \( L_0 \). Indeed, if one denotes \( \bar{a}_* = (0, L'_0) \), and by \( \tilde{h}_0' \) the function constructed using the expansion at \( \bar{a}_* \), one has that \( \tilde{h}_0' \) and \( \tilde{h}_0 (L'_0 \bar{a}) \) are analytic functions which coincide in the domain \( \tilde{a}_1 > 0 \) (by homogeneity of \( h_0 \)) and thus coincide everywhere. Remark now that the variables \( \bar{a} \) are action variables also for the Hamiltonian \( h_0 \), thus there exists a unimodular matrix \( U \) with integer coefficients, s.t. \( \bar{a} = U a \), at least in a neighborhood of the point \( \bar{a}_* = (0, L_0) \). Now, one has that \( a_* := U^{-1} \bar{a}_* \in \partial C \setminus \{0\} \). It follows that

\[
h_0(a) \equiv \tilde{h}_0(Ua)
\]

(9.16)

extends to a conic neighborhood of \( a_* \).

We are finally going to verify steepness, by applying Theorem C.2 of the Appendix. Concerning this point, we remark that the paper [BFS18] contains the proof of the fact that in any central motion problem, but the Harmonic and the Keplerian ones, the condition of Theorem C.2 of the Appendix below is fulfilled (for the case of homogeneous potential this was already verified in [BF17]).

We still have to verify A.iv. This is easily done by using the result of Charbonnel [Cha83] applied to \( h_0' \) extended to \( C_e \) (see also [BLR21]).
9.2 Proof of the results on rotation invariant surfaces

First we recall the Hamiltonian of the classical geodesic flow, which in the coordinates \((\phi, \theta)\) assumes the form

\[
h_0 = \frac{p_\phi^2}{2} + \frac{p_\theta^2}{2r(\theta)^2}. \tag{9.17}\]

The first action (actually we call it \(a_2\)) is defined to be the momentum conjugated to \(\phi\):

\[
a_2 := p_\phi. \tag{9.18}\]

For \(p_\phi \neq 0\) we define

\[
a_1 = a_1(E, p_\phi) := \frac{1}{\pi} \int_{\theta_m}^{\theta_M} \sqrt{E - \frac{p_\phi^2}{r(\theta)^2}} d\theta + |p_\phi|, \tag{9.19}\]

where \(\theta_m\) and \(\theta_M\) are the solutions of \(E = \frac{p_\phi^2}{r(\theta)^2} = 0\).

The cone \(C\) is defined by

\[
C := \{(a_1, a_2) \in \mathbb{R}^2 : a_1 \geq |a_2| \geq 0\}. \tag{9.20}\]

**Lemma 9.4.** [Colin de Verdière] The function \(a_1(E, p_\phi)\) has the following properties:

1. it extends to a complex analytic function of \(L\) and \(E\) in the region \(p_\phi^2 < r(\theta_0)^2 E, \ E > 0\)

2. the map \(E \mapsto a_1(E, a_2)\) admits an inverse \(E = h_0(a_1, a_2)\) which is analytic in the interior of \(C\). Furthermore it is homogeneous of degree 2 as a function of \((a_1, a_2)\).

3. The function \(a_1(p_\phi, p_\theta, \phi, \theta) := a_1(h_0(p_\phi, p_\theta, \phi, \theta), p_\phi)\) extends to a function analytic over \(T^*M \setminus \{0\}\) and is homogeneous of degree 1 in the momenta.

Concerning the quantum case, again the results of [CdV80] ensure that our assumptions \(A\) are satisfied by a perturbation of the Weyl quantization of \(a_1\) and \(a_2\). To precise this point, consider first the unperturbed case of the sphere \(S^2\) (in which \(r(\theta) = \sin \theta\)). In this case the joint spectrum of \(\sqrt{-\Delta + 1/4}\) and of the angular momentum operator \(-i\partial_\phi\) is given by

\[
\Lambda := \left\{ \left( a_1 + \frac{1}{2}, a_2 \right) : |a_2| \leq a_1; \ a_1, a_2 \in \mathbb{Z} \right\}. \tag{9.21}\]
Theorem 9.5. [Colin de Verdière] There exist two commuting pseudodifferential operators $A_j \in A^1_1$ satisfying assumption $A$. Their joint spectrum is $\Lambda$ and there exists a function symbol $h \in S^2_1$ s.t.

$$- \Delta_g = h(A).$$  \hspace{1cm} (9.22)

Furthermore one has an asymptotic expansion

$$h = h_0 + \text{l.o.t.}$$

with $h_0$ the classical Hamiltonian of the geodesic flow written in terms of the action variables.

Actually $h_0$ might have a singularity at $\xi = 0$. To regularize it, we proceed exactly as in the case of the anharmonic oscillator and define

$$h'_0(x,\xi) := h_0(x,\xi)(1 - \chi(Kh_0(x,\xi))) ,$$

for some positive $K$ large enough.

In order to prove Theorem 9.3 it remains to verify that Assumptions H.i and H.ii are satisfied. By Lemma 9.4 point (2), $h'_0$ is analytic on the interior of $C$. It remains to prove that it extends by homogeneity at infinity and that the extension is steep. Precisely, making reference to the expansion (4.8), we prove that

Theorem 9.6. Assume that $\beta_2 \neq 0$, then $h'_0$ extends to a real analytic function on a cone $C_e$ s.t. $\overline{C} \setminus \{0\} \subset C_e$. Assume also (4.9) then $h'_0$ is steep on $C_e$.

Proof. The fact that it can be extended beyond the boundary of $C$ is done exactly as in the case of the anharmonic oscillator. Indeed, one uses the fact that for any fixed $a_2 \neq 0$ $\partial C$ corresponds to the parallel geodesic $\theta = \theta_0$, $\phi \in T^1$, which is an elliptic point of the Hamiltonian

$$h^*_L(p_\theta,\theta) = \frac{p_\theta^2}{2} + \frac{L^2}{r^2(\theta)} ;$$

then it is sufficient to argue as in the proof of Theorem 9.3 to get the result. It remains to show steepness. First, remark that since steepness is independent of the system of action coordinates, it is sufficient to verify it for the function $\tilde{h}_0$ defined as in (9.16). Steepness of $\tilde{h}_0$ is then guaranteed by Corollary C.3 of Theorem C.2 below, since formula (9.14) expresses the condition (C.3) in terms of the Taylor expansion of the effective potential. \hfill \Box
9.3 Proof of results on Lie groups

In order to prove that the quantum actions \( A_1, \ldots, A_d \) defined in (4.13) satisfy Assumptions A, in this section we use the intrinsic formulation of pseudodifferential calculus introduced in [Fis15, RT09], which we briefly recall. We keep the notations of Subsection 4.4, and for any \( \xi \in \hat{G} \), we denote by
\[
\lambda_\xi := \| w_\xi + f \|^2 - \| f \|^2
\]
the corresponding eigenvalue of the Laplace Beltrami operator \(-\Delta_g\).

Following [Fis15], we consider the following class of symbols:

Definition 9.7. Given \( m \in \mathbb{R} \) and \( 0 \leq \delta_1 \leq \delta_2 \leq 1 \), we say that a map \( \sigma : G \times \text{Rep}(G) \ni (x, \xi) \mapsto \sigma(x, \xi) \in \mathcal{B}(\mathcal{H}_\xi) \) is a symbol of order \( m \), and we write \( \sigma \in S^m_{F,\delta_1,\delta_2} \) if for any \( \xi \in \hat{G} \) the map \( x \mapsto \sigma(x, \xi) \) is smooth and for any smooth differential operator \( D^\alpha_x \) of order \( \alpha \) and any \( \tau = (\tau_1, \ldots, \tau_\alpha) \) with \( \tau_i \in \hat{G} \), there exists \( C > 0 \) such that
\[
\| D^\alpha_x \Delta^\beta_\tau \sigma(x, \xi) \|_{\mathcal{B}(\mathcal{H}_\xi^{\otimes \tau})} \leq C(1 + \lambda_\xi)^{m - \delta_2 \beta + \delta_1 \alpha},
\]
for all \( \beta \leq 1 \) and \( \alpha \). Here, \( \Delta^\beta_\tau := \Delta_{\tau_1} \cdots \Delta_{\tau_\beta} \) and \( \Delta_{\tau_\alpha} \sigma(x, \xi) := \sigma(x, \tau_\alpha \otimes \xi) - \sigma(x, \mathbf{1}_{\mathcal{H}_{\tau_\alpha}} \otimes \xi) \) is the operatorial norm on \( \mathcal{H}_{\tau_1} \otimes \cdots \otimes \mathcal{H}_{\tau_\beta} \otimes \mathcal{H}_\xi \).

With the above definition, one defines the quantization \( \text{Op}(\sigma) \) of a symbol \( \sigma \) as
\[
\text{Op}(\sigma)\psi(x) = \sum_{\xi \in \hat{G}} d_\xi \text{Tr} \left( \xi(x) \sigma(x, \xi) \hat{\psi}_\xi \right),
\]
where \( d_\xi = \dim \mathcal{H}_\xi \) and \( \hat{\psi}_\xi \) is the \( \xi \)-th Fourier coefficient of \( \psi \). If there exists \( \sigma \in S^m_{F,\delta_1,\delta_2} \) such that \( \Sigma = \text{Op}(\sigma) \), we write \( \Sigma \in \text{OPS}^m_{F,\delta_1,\delta_2} \).

Remark 9.8. Assume that the symbol \( \sigma \) does not depend on \( x \) and that it has the form \( \sigma(\xi) = m(\xi) \mathbf{1}_{\mathcal{H}_\xi} \) for some function \( m \). Then \( \text{Op}(\sigma) \) defined as in (9.26) acts as a Fourier multiplier, which multiplies each frequency \( \hat{\psi}_\xi \) by the factor \( m(\xi) \). This is for instance the case of the Laplace Beltrami operator \(-\Delta_g\) and of the quantum actions \( A_1, \ldots, A_d \).

The following result was proven in [Fis15]:

Theorem 9.9. Let \( \delta_2 = 1 - \delta_1 =: \varrho \); then the class \( \text{OPS}^m_{F,\delta_1,\delta_2} \) coincides with the class of pseudodifferential operators in the sense of Hörmander with symbol \( S^m_{H,\varrho} \) on \( G \).
By Theorem 9.9, the proof that the operators $A_1, \ldots, A_d$ defined in (1.13) are actually pseudodifferential operators of order 1 reduces to the following:

**Lemma 9.10.** For $j = 1, \ldots, d$ let $\sigma_{A_j}$ be defined as in (1.14), then $\sigma_{A_j} \in S^{1}_{1, 0}$. 

**Proof.** According to Section 3.2 of [Fis15], since the symbols $\sigma_{A_j}$ are independent of $x$, it is sufficient to prove that for any $\beta \in \mathbb{N}$ and $\Delta_{\tau}^\beta = \Delta_{\tau_1} \cdots \Delta_{\tau_\beta}$ there exists $C = C_\tau > 0$ such that

$$\|\Delta_{\tau}^\beta \sigma_{A_j}(\xi)\|_{B(H^{\otimes \tau})} \leq C_\tau (1 + \lambda_\xi)^{\frac{1-\beta}{2}} \quad \forall \xi \in \hat{G}. \quad (9.27)$$

If $\beta = 0$, one has

$$\|\sigma_{A_j}(\xi)\|_{B(H^{\otimes \tau})} = \max_{\xi \in \hat{G}} \{w_\xi^j + 1\} \lesssim (1 + \lambda_\xi)^{\frac{j}{2}}. \quad (9.28)$$

Consider now $\beta = 1$: then one has to estimate $\forall \tau \in \hat{G}$

$$\|\Delta_{\tau} \sigma_{A_j}(\xi)\|_{B(H^{\otimes \tau})} = \|\sigma_{A_j}(\tau \otimes \xi) - \sigma_{A_j}(1_{H_\tau} \otimes \xi)\|_{B(H^{\otimes \tau})}. \quad (9.29)$$

Now (see for instance [FH13], Exercise 25.33), for any $\tau, \xi \in \hat{G}$, if $w_\xi$ is the highest weight of $\xi$, then the representation $\tau \otimes \xi$ is isomorphic to a finite direct sum of representations $\zeta_\mu$ with highest weight $w_\xi + \mu$, for any weight $\mu$ of the representation $\tau$. Thus one has

$$\Delta_{\tau} \sigma_{A_j}(\xi) = \bigoplus_{\mu \in \text{weight of } \tau} ((w_\xi + \mu)^j + 1 - (w_\xi^j + 1)) 1_{H_{\zeta_\mu}} = \bigoplus_{\mu \in \text{weight of } \tau} \mu^j 1_{H_{\zeta_\mu}}. \quad (9.29)$$

This implies

$$\|\Delta_{\tau} \sigma_{A_j}(\xi)\|_{B(H^{\otimes \tau})} = \max_{\mu \in \text{weight of } \tau} |\mu^j| =: C_\tau,$$

which gives (9.27) with $\beta = 1$. Finally, if $\beta \geq 2$, by (9.29) one gets that $\Delta_{\tau}^\beta \sigma_{A_j} = 0$, thus (9.27) is trivially verified. \qed

**A Lie transform**

To study the properties of $e^{-iG(t)}$ we start by the time independent case. First we introduce a notation:
**Definition A.1.** If $F$ and $G$ are linear operators, we define

$$Ad_G F := -i[F, G],$$

(A.1)

and $(Ad_G)^j$ the $j$-th power of this operator.

We have the following Lemma:

**Lemma A.2.** Let $G \in A^\eta_\rho$ with $\eta < \rho$ and suppose that $G$ is self-adjoint. Then for any $\tau, s \in \mathbb{R}$, $e^{i\tau G} \in B(H^s; H^s)$. If $F \in A^m_\rho$, then, for any $N \in \mathbb{N}$ one has

$$e^{i\tau G} F e^{-i\tau G} = \sum_{j=0}^{N} \frac{(i\tau)^j Ad_G^j F}{j!} + R_F,$$

(A.2)

with $R_F \in A^{m+(N+1)(\eta-\rho)}_\rho$.

**Proof.** The proof is based on the formula of the remainder for the Taylor expansion of

$$F_\tau := e^{i\tau G} F e^{-i\tau G},$$

(A.3)

namely

$$e^{i\tau G} F e^{-i\tau G} = \sum_{j=1}^{N_1} \frac{(i\tau)^j Ad_G^j F}{j!} + \frac{\tau^{N_1+1}}{N_1!} \int_0^1 (1 + s)^{N_1} (Ad_G^{N_1+1} F)^{s\tau} ds,$$

(A.4)

where the lower index of the parentheses is defined according to (A.3). We take $N_1$ larger than $N$ and, according to Assumption I.vi we show that for any $N_1$ one has

$$e^{i\tau G} F e^{-i\tau G} - \sum_{j=1}^{N_1} \frac{(i\tau)^j Ad_G^j F}{j!} = A^{(N_1)} + S^{(N_1)}$$

(A.5)

with $A^{(N_1)} \in A^{\eta-(N+1)(\eta-\rho)}_\rho$ and $S^{(N_1)}$ smoothing of order $(N_1+1)(\eta-\rho) - m \to \infty$.

Of course we define

$$A^{(N_1)} := \sum_{j=N_1+1}^{N_1} \frac{(i\tau)^j Ad_G^j F}{j!}, \quad S^{(N_1)} := \frac{\tau^{N_1+1}}{N_1!} \int_0^1 (1 + s)^{N_1} (Ad_G^{N_1+1} F)^{s\tau} ds$$

and the property of $S^{(N_1)}$ follows from our assumptions on the algebra, in particular from I.v.

The way the Hamiltonian changes in the case of a time dependent $G$ is the content of the forthcoming Lemma whose proof is based on the formula

$$e^{i\tau G(t)} \frac{d e^{-i\tau G(t)}}{d t} = -i \int_0^1 \tau e^{i\tau G(t)} \dot{G}(t) e^{-i\tau G(t)} ds$$

(A.6)

which was proved in detail in Lemma 3.2 of [Bam18].
Lemma A.3. Let $H(t)$ and $G(t)$ be two smooth families of self-adjoint operators. Then $e^{-iG(t)}$ conjugates $H$ to

$$H^+(t) := e^{iG(t)}H(t)e^{-iG(t)} - \int_0^1 e^{irG(t)}\dot{G}(t)e^{-irG(t)} \, d\tau .$$  \hspace{1cm} (A.7)

We now prove a Lemma which allows to show that $H^+(t)$ is still a smooth family of pseudodifferential operators.

Lemma A.4. Let $G \in C_b^\infty(\mathbb{R}; \mathcal{A}_p^n)$ with $\eta < \rho$ and suppose that $G(t)$ is a family of self-adjoint operators; given $m \in \mathbb{R}$ and $F \in C_b^\infty(\mathbb{R}; \mathcal{A}_\rho^m)$, one has

$$F_\tau(\cdot) := e^{irG(\cdot)}F(\cdot)e^{-irG(\cdot)} \in C_b^\infty(\mathbb{R}; \mathcal{A}_\rho^{m})$$  \hspace{1cm} (A.8)

and furthermore, for any $N \in \mathbb{N}$, one has

$$e^{irG(t)}F(t)e^{-irG(t)} = \sum_{j=1}^N \frac{(ir)^j A_d^j \bar{G}(t)}{j!} + C_b^\infty(\mathbb{R}; \mathcal{A}_\rho^{m-(N+1)(\eta-\rho)}) .$$  \hspace{1cm} (A.9)

Proof. First we prove by induction that $F_\tau \in C_b^k(\mathbb{R}; \mathcal{A}_\rho^m)$ for all $k$. For $k = 0$ this is an immediate consequence of Eq. (A.2) with $N = 0$. In order to pass from $k$ to $k + 1$ we deduce a formula for the time derivative of $F_\tau(t)$. To this end we use Eq. (A.6) (and its adjoint) and compute

$$dF_\tau \left( \frac{d}{dt} \right) = \frac{d}{dt} e^{irG} e^{-irG} F e^{-irG} + e^{irG} e^{-irG} \frac{d}{dt} e^{-irG} + e^{irG} F e^{-irG} \frac{de^{-irG}}{dt}$$

$$= i \int_0^1 \tau (\dot{G})_{st} \, ds F_\tau + (\dot{F})_\tau + (-i) F_\tau \int_0^1 \tau (\dot{G})_{st} \, ds ,$$

from which

$$dF_\tau = (\dot{F})_\tau - i \left[ F_\tau; \int_0^1 \tau (\dot{G})_{st} \, ds \right] .$$  \hspace{1cm} (A.10)

Now, the inductive assumption is that, given any operator $A \in C_b^k(\mathbb{R}; \mathcal{A}_\rho^{m+1})$, then $A_\tau \in C_b^k(\mathbb{R}; \mathcal{A}_\rho^{m+1})$. We apply this to both $F_\tau$ and $(\dot{G})_{st}$. It follows that the r.h.s. of (A.10) is of class $C_b^k(\mathbb{R}; \mathcal{A}_\rho^{m+1})$, so that this is true also for the l.h.s, and therefore $F_\tau$ is $C_b^{k+1}(\mathbb{R}; \mathcal{A}_\rho^{m+1})$. \hspace{1cm} $\square$

Remark A.5. If $G \in C_b^\infty(\mathbb{R}; \mathcal{A}_\rho^n)$, then, by Lemma A.4, one has

$$\int_0^1 e^{irG} \dot{G} e^{-irG} \, d\tau = \sum_{j=0}^N \frac{1}{(j+1)!} A_d^j \dot{G} + C_b^\infty(\mathbb{R}; \mathcal{A}_\rho^{n+(N+1)(\eta-\rho)}) .$$  \hspace{1cm} (A.11)
B The generalized commutator Lemma

In this Section we apply the commutator Lemma proved in [Ras12] to our pseudodifferential setting.

Given a multiindex \( j = (j_1, ..., j_d) \), we will denote

\[
Ad_A^j := Ad_{A_1}^{j_1} ... Ad_{A_d}^{j_d}.
\]

Then the following Theorem holds:

**Theorem B.1 (Theorem 3 of [Ras12]).** Let \( B \in \mathcal{B}(\mathcal{H}) \) be such that \( Ad_A^j B \in \mathcal{B}(\mathcal{H}) \) for all multiindexes \( j \). Let \( f \in S^m_1 \), then, for all positive \( t_1, t_2 \) and all integers \( n \) fulfilling

\[
t_1 + t_2 + m < n + 1,
\]

one has

\[
[B; f(A)] = \sum_{|j|=1}^{n} \frac{1}{j!} \partial^j f(A) Ad_A^j B + R_n(A, B).
\] (B.1)

Furthermore there exists a positive constant \( C \) (which depends on \( n \)), independent of \( A \) and \( B \), s.t.

\[
\|R_n(A, B)\|_{\mathcal{B}(\mathcal{H}^{-t_2, \mathcal{H}^{t_1}})} \leq C \sum_{|j| \leq n+1} \|Ad_A^j B\|_{\mathcal{B}(\mathcal{H})}.
\]

**Corollary B.2.** Let \( B \in A^{m'}_\rho \) and let \( f \) be as above; let \( N \) and \( n \) fulfill \( N + m < \rho(n + 1) \) then equation (B.1) holds. Furthermore there exists \( J \) and for any \( s \in \mathbb{R} \) there exists a constant \( C_{s,N} \) s.t. the following estimate holds

\[
\|R_n(A, B)\|_{\mathcal{B}(\mathcal{H}^{s}; \mathcal{H}^{s+N-m'})} \leq C_{s,N} \psi_{s,J}^m(B).
\] (B.2)

To get the corollary, just apply Theorem B.1 to the operator \( \langle A \rangle^{-m'-n(1-\rho)+s} B \langle A \rangle^{-s} \) and take \( t_2 = 0, t_1 = N \).

Remark that the extension to operators \( B \in S^{m'}_\rho \) is trivial.

C Steepness in 2 d.o.f.

We will use here a condition equivalent to steepness introduced in [Nie06].

**Theorem C.1.** [Niederman] Let \( h \) be a real analytic in an open set \( \mathcal{U} \subset \mathbb{R}^d \). Then \( h \) is steep on any compact set \( \Sigma \subset \mathcal{U} \) if and only if \( h \) has no critical points in \( \mathcal{U} \), and its restriction \( h|_M \) to any affine subspace \( M \subset \mathbb{R}^d \) admits only isolated critical points.
If $d = 2$, then, by homogeneity, one gets a condition easy to verify. To state the corresponding result, we recall the definition of the Arnold determinant $D$:

$$D := \det \left[ \frac{\partial^2 h}{\partial a^2} \left( \frac{\partial h}{\partial a} \right) \right].$$ (C.1)

**Theorem C.2.** Let $C_e \subset \mathbb{R}^2$ be an open cone, and let $h : C_e \to \mathbb{R}$ be a real analytic function homogeneous of degree $d$. Assume that there exists $\bar{\alpha} \in C_e$ s.t. $D(\bar{\alpha}) \neq 0$, and that $h(a) > 0 \ \forall a \in C_e$, then $h$ is steep in any compact subset of $C_e$.

**Proof.** First we prove that $h$ has no critical points in $C_e$. Indeed, if $\bar{\alpha}$ is a critical point of $h$ then, $\forall t > 0$, also $t\bar{\alpha}$ is a critical point of $h_0$. It follows that $h$ is constant along the line $t\bar{\alpha}$, $t \in \mathbb{R}^+$, but here one has $h(t\bar{\alpha}) = t^d h(\bar{\alpha})$ which is not constant, since $h$ is positive.

We now recall that the nonvanishing of the Arnold determinant is equivalent to quasiconvexity, namely to the fact that the restriction of $h$ to the line orthogonal to its gradient is convex. This is well known to imply steepness. For our purpose it is useful to see that $D(a_0) \neq 0$ implies that the restriction of $h$ to any line through $a_0$ is a nontrivial analytic function: consider a line through $a_0$, whose equation can be written in the form $a = tn + a_0$ with $n$ a normalized vector and $t$ the parameter along the line. One has

$$h(tn + a_0) = h(a_0) + tdh(a_0)n + \frac{1}{2} d^2h(a_0)(n, n)t^2 + ...$$

but, if $dh(a_0)n \neq 0$, then this is a nontrivial function of $t$, while, if $dh(a_0)n = 0$ then by quasiconvexity one has $d^2h(a_0)(n, n) \neq 0$, so also in this case this is a nontrivial function of $t$. Thus in particular $h$ cannot vanish on any straight line passing through a point $\bar{\alpha}$ with $D(\bar{\alpha}) \neq 0$.

From the assumption of the theorem one has that $D(a)$ can vanish only on an analytic surface $S$, but, since $D$ is homogeneous of degree $(d-2) + 2(d-1)$, if it vanishes at a point it vanishes on the whole line joining the point to the origin. It follows that $S$ is either empty or consists of finite number of radial straight lines. Let $\bar{\alpha} \in S$, and consider a straight line through $\bar{\alpha}$: any nonradial line intersects the region where the Arnold determinant is different from zero, so the restriction of $h$ to such a line is a nontrivial analytic function. Thus the only lines on which $h$ can be trivial can be radial (and then contained in $S$), so of the form $t\bar{\alpha}$. However, one has

$$h(t\bar{\alpha}) = t^d h(\bar{\alpha}) ,$$

which is a nontrivial function of $t$, since $h(\bar{\alpha}) \neq 0$.  

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To be completely explicit, we consider a point \( \bar{a} \equiv (\bar{a}_1, \bar{a}_2) \) with \( \bar{a}_1 = 0 \) and consider the Taylor expansion in \( a_1 \) of \( h \) at \( \bar{a} \): since a homogeneous function of one variable is just a monomial of the corresponding degree, it has the form

\[
h(a_1, a_2) = \alpha_0 a_2^d + \alpha_1 a_2^{d-1} a_1 + \frac{1}{2} \alpha_2 a_2^{d-2} a_1^2 + \ldots,
\]

(C.2)

from which it is straightforward to compute the Arnold determinant. This turns out to be given by

\[
D(\bar{a}) = -\bar{a}_2^{3d-4} d \alpha_0 (2 d \alpha_0 \alpha_2 - \alpha_1^2 (d - 1)) ,
\]

thus we have the following Corollary:

**Corollary C.3.** Under the hypotheses of Theorem C.2, if the Taylor expansion (C.2) of \( h \) at a point \( (0, \bar{a}_2) \in \Lambda \) has coefficients which fulfill

\[
\alpha_0 \neq 0 \text{ and } d \alpha_0 \alpha_2 - (d - 1) \alpha_1^2 \neq 0
\]

(C.3)

then \( h \) is steep in any compact set contained in \( C_e \).

**D Further Technical Lemmas**

In this Appendix we collect a few technical results that we use to prove Theorem 7.27. First of all, we recall the following (see for instance Lemma B.5 of [BLM22]):

**Remark D.1.** Suppose that \( x > 0, y > 1, C > 0 \) and \( 0 < a < 1 \) are such that

\[
x < C (y + x^a).
\]

Then there exists \( C' > 0 \), depending on \( C \) and \( a \) only, such that

\[
x < C' y.
\]

**Remark D.2.** If \( a, b \in \mathbb{R}, C > 0 \) and \( 0 < \mu < 1 \) are such that \( \|a\|, \|b\| \geq 1 \),

\[
\|a - b\| \leq C \|b\|^\mu,
\]

then one has

\[
\|a - b\| \leq C \|b\|^\mu \lesssim_\mu C \|a\|^\mu + C \|a - b\|^\mu,
\]

thus by Remark D.1 one also has

\[
\|a - b\| \lesssim_{C, \mu} \|a\|^\mu.
\]
Lemma D.3. Let \( a, b \in \Lambda \) and \( k, h, l \in \mathbb{Z}^d \) be such that \( a + k, b + l \neq 0 \). Suppose that there exist positive constants \( C, D, R, \gamma, \mu \) and \( \delta \) such that \( \mu \leq 1 - \delta \) and
\[
|\omega(a + k) \cdot h| \leq C\|a + k\|^{R - \gamma} \|h\|,
\]
\[
\|a + k\| \geq R, \quad \|h\| \leq D\|a + k\|^{\mu}, \quad \|k\| \leq D\|a + k\|^{\mu},
\]
and
\[
\|b - a\| \leq D\|a\|^{1 - \delta}, \quad \|l\| \leq D\|b + l\|^{\mu}.
\]
Then there exist positive constants \( C^+, D^+ \), depending only on \( C, D, M, \gamma, \mu, \delta \), such that
\[
|\omega(b + l) \cdot h| \leq C^+\|b + l\|^{R_{\min}(\gamma, \delta)} \|h\|, \quad \|h\| \leq D^+\|b + l\|^{\mu}.
\]

Proof. We start with proving that the following inequality holds:
\[
\|a + k\| \lesssim_{D, \mu, \delta} \|b + l\|.
\]
Indeed, due to hypothesis (D.2) one has
\[
\|a + k\| \leq \|b + l\| + \|l\| + \|a - b\| + \|k\|
\leq \|b + l\| + D\|b + l\|^{\mu} + D\|a\|^{1 - \delta} + D\|a + k\|^{\mu}
\leq (D + 1)\|b + l\| + D\|a\|^{1 - \delta} + D\|a + k\|^{\mu}.
\]
By Remark [D.1] this implies
\[
\|a + k\| \lesssim_{\mu, D} \|b + l\| + \|a\|^{1 - \delta}.
\]
Then one observes that
\[
\|a\| \leq \|b + l\| + \|l\| + \|a - b\| \leq (D + 1)\|b + l\| + D\|b + l\|^{1 - \delta},
\]
which using Remark [D.1] and \( 1 - \delta < 1 \) also implies
\[
\|a\| \lesssim_{D, \delta} \|b + l\|.
\]
Estimate (D.4) then follows combining (D.5) and (D.6).
Then the second estimate in (D.3) follows from hypothesis (D.1) and from estimate (D.4), since one has
\[
\|h\| \leq D\|a + k\|^{\mu} \lesssim_{D, \mu, \delta} \|b + l\|^{\mu}.
\]
In order to prove the first estimate in (D.3), we proceed as follows. Define 
\[ \eta := b + l - (a + k) \]; by assumptions (D.1) and (D.2) one has
\[ \|\eta\| \leq \|a - b\| + \|l\| + \|k\| \]
\[ \leq D\|a\|^{1-\delta} + D\|b + l\|^\mu + \|a + k\|^\mu, \]
and by estimates (D.4), (D.6) and recalling \[ \mu < 1 - \delta \] this implies
\[ \|\eta\| \lesssim_{D,\mu,\delta} \|b + l\|^{1-\delta}. \] (D.7)
Furthermore, using Remark (D.2) and \[ 1 - \delta < 1, \] (D.7) is also equivalent to
\[ \|\eta\| \lesssim_{D,\mu,\delta} \|a + k\|^{1-\delta}. \] (D.8)
Then a first order Taylor development gives
\[ |\omega(b + l) \cdot h| \leq |\omega(a + k) \cdot h| + \int_0^1 \left( \frac{\partial \omega(a + k + t\eta)}{\partial a} \cdot h \right) \, dt, \] (D.9)
which due to homogeneity of the function \( \omega \) also implies
\[ |\omega(b + l) \cdot h| \leq |\omega(a + k) \cdot h| + K \int_0^1 \|a + k + t\eta\|^{\mu-1} \, dt \|\eta\| \|h\|, \] (D.10)
with
\[ K := \sup_{\|\hat{a}\| = 1} \left\{ \left\| \frac{\partial \omega(\hat{a})}{\partial a} \right\| \right\}, \]
and by hypotheses (D.1) and estimates (D.4), (D.7) one has
\[ |\omega(b + l) \cdot h| \leq C\|a + k\|^{\mu-\gamma} \|h\| + K \int_0^1 \|a + k + t\eta\|^{\mu-1} \, dt \|\eta\| \|h\| \]
\[ \lesssim_{C,D,\mu,\delta} \|b + l\|^{\mu-\gamma} \|h\| + K \int_0^1 \|a + k + t\eta\|^{\mu-1} \, dt \|b + l\|^{1-\delta} \|h\|. \] (D.11)
Now estimate (D.8) implies that there exists a constant \( R' \), depending on \( D, \mu, \delta \) only, such that, provided \( \|a + k\| \geq R' \), one has
\[ \frac{1}{2} \|a + k\| \leq \|a + k + t\eta\| \leq 2\|a + k\| \quad \forall t \in [0,1], \]
thus
\[ \|a + k + t\eta\|^{\mu-1} \lesssim_{\mu} \|a + k\|^{\mu-1} \quad \forall t. \] (D.12)
Then plugging in estimate (D.12) into (D.11) and using again estimate (D.4) one obtains

\[ |\omega(b + l) \cdot h| \lesssim C, D, \mu, \delta \|b + l\|^{\gamma-\gamma} \|h\| + \|a + k\|^{\gamma-\gamma} \|b + l\| + \|b + l\|^{\gamma-\gamma} \|h\| \]

(D.13)

Thus if \( \|a + k\| \geq R' \) one obtains the thesis. On the other hand, if \( R < \|a + k\| < R' \), from estimate (D.9) one deduces

\[ |\omega(b + l) \cdot h| \leq |\omega(a + k) \cdot h| + K_2 \|\eta\| \|h\| , \]

with

\[ K_2 := \sup_{\|\tilde{a}\| \in [R, R']} \left\| \frac{\partial \omega(\tilde{a})}{\partial a} \right\| , \]

thus by assumptions (D.7) and by (D.8) one has

\[ |\omega(b + l) \cdot h| \leq C \|a + k\|^{\gamma-\gamma} \|h\| + K_2 \|\eta\| \|h\| \]

\[ \lesssim C, D, \mu, \delta \|b + l\|^{\gamma-\gamma} \|h\| + \|b + l\|^{\gamma-\gamma} \|h\| \]

\[ \lesssim C, D, \mu, \delta, R' \|b + l\|^{\gamma-\gamma} \|h\| . \]

Lemma D.4. Let \( \omega_* \in \mathbb{R}^d \), \( \varepsilon > 0 \) and \( M \subset \mathbb{R}^d \) be such that

\[ \|\Pi_M \omega_*\| \leq \varepsilon . \]  

(D.14)

Define \( M' := \Pi_{\omega_*} M \). If \( \varepsilon \leq \frac{1}{2} \|\omega_*\| \), then one has

\[ \|\Pi_{M'} - \Pi_M\| \leq 9 \varepsilon \|\omega_*\|^{-1} . \]  

(D.15)

Proof. Define \( w = \Pi_M \omega_* \) and \( v = \Pi_{M' \perp} \omega \), then one has

\[ \omega_* = v + w , \quad v \in M^\perp , \quad w \in M , \]

(D.16)

with

\[ \|\omega_* - v\| \leq \varepsilon . \]  

(D.17)

First of all, we prove that

\[ M \oplus v = M' \oplus \omega_* . \]  

(D.18)

where, for simplicity we wrote \( M \oplus v \) instead of \( M \oplus \text{span} v \). We will continue to use this notation.
By the very definition of $M'$ and $v$, one has

$$M' \subset M \oplus v \implies M' \oplus \omega_* \subset M \oplus \omega_* = M \oplus v.$$  

We now prove the opposite inclusion. By the definition of $M'$, for any $m \in M$, one has $\Pi_{M'} m = \Pi_{\omega_*} m$

$$m = \Pi_{\omega_*} m + \Pi_{\omega_*} m = \Pi_{\omega_*} m + \Pi_{M'} m \in M' \oplus \text{span}\{\omega_*\},$$

so we have

$$M \subset M' \oplus \omega_* \implies M' \oplus \omega_* \supset M \oplus \omega_* = M \oplus v.$$  

Let then $E$ be the space defined in \(\text{D.18}\); since $M \perp v$ and $M' \perp \omega_*$, one has that, for any $u \in \mathbb{R}^d$

$$\Pi_M u = \Pi_E u - \frac{v \cdot u}{\|v\|^2} v, \quad \Pi_{M'} u = \Pi_E - \frac{\omega_* \cdot u}{\|\omega_*\|^2} \omega_*.$$  

Thus

$$\|\Pi_M u - \Pi_{M'} u\| = \left\| \frac{v \cdot u}{\|v\|^2} v - \frac{\omega_* \cdot u}{\|\omega_*\|^2} \omega_* \right\|,$$

from which the thesis easily follows. \(\square\)

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