Three-Neutrino Mixing and Combined Vacuum Oscillations and MSW Transitions of Solar Neutrinos

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Abstract

Assuming three flavour neutrino mixing takes place in vacuum, we investigate the possibility that the solar $\nu_e$ take part in MSW transitions in the Sun due to $\Delta m_{31}^2 \sim (10^{-7} - 10^{-4})$ eV$^2$, followed by long wave length vacuum oscillations on the way to the Earth, triggered by $\Delta m_{21}^2$ (or $\Delta m_{32}^2$) $\sim (10^{-12} - 10^{-10})$ eV$^2$, $\Delta m_{31}^2$ and $\Delta m_{21}^2$ ($\Delta m_{32}^2$) being the corresponding neutrino mass squared differences. The solar $\nu_e$ survival probability is shown to be described in this case by a simple analytic expression. Depending on whether the vacuum oscillations are due to $\Delta m_{21}^2$ or $\Delta m_{32}^2$ there are two very different types of interplay between the MSW transitions and the vacuum oscillations of the solar $\nu_e$. Performing an analysis of the most recently published solar neutrino data we have found several qualitatively new solutions of the solar neutrino problem of the hybrid MSW transitions + vacuum oscillations type. The solutions differ in the way the $pp$, $^7$Be and $^8$B neutrino fluxes are affected by the transitions in the Sun and the oscillations in vacuum. The specific features of the new solutions are discussed.

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1. Introduction

It is well known that the solar neutrino problem \[1–7\] admits, in particular, two quite different neutrino physics solutions: one based on the old idea of Pontecorvo \[8\] that the solar $\nu_e$ take part in vacuum oscillations when they travel from the Sun to the Earth, and a second based on the more recent hypothesis \[9,10\] of the solar $\nu_e$ undergoing matter-enhanced (MSW) transitions into neutrinos of a different type when they propagate from the central part to the surface of the Sun. Both these solutions and the possibilities to test them and to distinguish between them in the future solar neutrino experiments, have been extensively studied in the simplest case of two-neutrino mixing (see, e.g., refs. \[11–16\] and the articles quoted therein). The vacuum oscillations and the MSW transitions of the solar $\nu_e$ are characterized by the same two parameters in this case \[8–10,17\]: $\sin^2 2\theta$, $\theta$ being the neutrino (lepton) mixing angle in vacuum, and $\Delta m^2 = m_2^2 - m_1^2 > 0$, where $m_{1,2}$ are the masses of two neutrinos $\nu_{1,2}$ with definite mass in vacuum. The vacuum oscillation interpretation of the solar neutrino problem requires that \[16\]

\[5.0 \times 10^{-11} eV^2 \lesssim \Delta m^2 \lesssim 1.1 \times 10^{-10} eV^2, \quad (1a)\]

\[0.67 \lesssim \sin^2 2\theta \leq 1.0, \quad (1b)\]

while the MSW solution is possible for \[16\]

\[3.6 \times 10^{-6} eV^2 \lesssim \Delta m^2 \lesssim 9.8 \times 10^{-6} eV^2, \quad (2a)\]

\[4.5 \times 10^{-3} \lesssim \sin^2 2\theta \lesssim 1.3 \times 10^{-2}, \quad (2b)\]

and for

\[5.7 \times 10^{-6} eV^2 \lesssim \Delta m^2 \lesssim 9.5 \times 10^{-5} eV^2, \quad (3a)\]

\[0.51 \lesssim \sin^2 2\theta \lesssim 0.92, \quad (3b)\]

if the MSW transitions are into an active neutrino, $\nu_{\mu(\tau)}$ \[1\]. The $\Delta m^2$ solution interval (1a) \[1\]

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\[1\] For a detailed study of the MSW solution of the solar neutrino problem with solar $\nu_e$ transitions into a sterile neutrino, $\nu_s$, see ref. \[18\].
corresponds to values of the solar $\nu_e$ oscillation length in vacuum comparable to the Sun-Earth distance.

Detailed studies of the vacuum oscillation or MSW transition solution of the solar neutrino problem under the more natural assumption of three flavour neutrino mixing are still lacking, partly because of the relatively large number of parameters involved. The three flavour neutrino mixing hypothesis implies the existence of three neutrinos $\nu_k$ with definite vacuum masses, $m_k$, $k = 1, 2, 3$, and the relevant vacuum oscillation and/or MSW transition probabilities depend now on two different neutrino mass squared differences $\Delta m^2_{21} > 0$ and $\Delta m^2_{31} > 0$, and on at least two mixing angles, say, $\theta_{12}$ and $\theta_{13}$. The general features of the three-neutrino MSW transitions of solar neutrinos have been analyzed by many authors (see, e.g., refs. [19,20]). Certain aspects of the possible three-neutrino MSW solution of the solar neutrino problem have been discussed as well [21,22]. However, comprehensive results based on the more recent and more precise data from the four solar neutrino experiments have been obtained only in specific cases when, for instance, $\Delta m^2_{21}$ has a value in one of the solution intervals $(2a), (3a)$, while $\Delta m^2_{31} >> 10^{-4} eV^2$ [23,24]. Under these conditions the solar $\nu_e$ MSW transitions are essentially of the two-neutrino mixing type [19,20], although the regions of solutions can differ somewhat from those given in eqs. $(2a) - (3b)$ [23,24].

In the present article we study the qualitatively new possibility of a hybrid vacuum oscillation and MSW transition solution of the solar neutrino problem. This possibility is rather natural if three-neutrino mixing takes place in vacuum.

One would guess a priori that such a solution would require $\Delta m^2_{21}$ (or $\Delta m^2_{32}$) and $\Delta m^2_{31}$

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2 One can always choose $m_1 < m_2 < m_3$ without loss of generality and we will work with this convention in what follows.

3 An earlier rather brief and qualitative discussion of combined solar $\nu_e$ MSW transitions and long wave length vacuum oscillations in connection with the solar neutrino problem can be found in ref. [22].
to have values in the regions specified by eq. (1a) and by eqs. (2a) or (3a), respectively. This corresponds to the case

$$\Delta m_{21}^2 \ll \Delta m_{31}^2 \approx \Delta m_{32}^2,$$

(4)

or to

$$\Delta m_{32}^2 \ll \Delta m_{31}^2 \approx \Delta m_{21}^2.$$

(5)

Several patterns of neutrino masses can lead to the inequality (4) or (5) and to the requisite values of $\Delta m_{21}^2$ (or $\Delta m_{32}^2$) and $\Delta m_{31}^2$. Eqs. (1a), (2a) or (3a) and (4) (or (5)) are compatible, for instance, i) with the existence of three quasi-degenerate neutrinos, $m_1 \cong m_2 \cong m_3$, with a very different splitting between their masses, $(m_2 - m_1) \ll (m_3 - m_1)$, ii) with a neutrino mass spectrum consisting of two close masses which are very different from the third one, $m_1 \cong m_2 \ll m_3$, and iii) with a hierarchical neutrino mass spectrum, $m_1 \ll m_2 \ll m_3$. In the first case neutrinos can have masses in the cosmologically relevant region, $m_i \sim (1 - 2) \text{ eV}$, with $\frac{1}{2} (m_3 - m_1) \sim (10^{-6} - 10^{-5}) \text{ eV}$ and $(m_2 - m_1) \sim (10^{-12} - 10^{-10}) \text{ eV}$, while in the other two cases the neutrino masses are too small to be of significance for the solution of the dark matter problem: $m_3 \sim 10^{-3} \text{ eV}$ and $m_2 \sim m_1 \sim (m_2 - m_1) \sim (10^{-6} - 10^{-5}) \text{ eV}$ (case ii) or $m_2 \sim (10^{-5} - 10^{-6}) \text{ eV}$ (case iii). In case ii) the pattern $m_1 \ll m_2 \cong m_3$ with $m_{2,3} \sim 10^{-3} \text{ eV}$ and $(m_3 - m_2) \sim (10^{-8} - 10^{-7}) \text{ eV}$ is also possible. It corresponds to eq. (5) and arises in the Zee model of neutrino mass generation \cite{Zee1980, Zee1982} and its various GUT generalizations. Obviously, if $\Delta m_{21}^2$ ($\Delta m_{32}^2$) and $\Delta m_{31}^2$ lie in the intervals (1a) and (2a) or (3a) respectively, there will be no observable neutrino oscillation effects due to $\Delta m_{21}^2$ ($\Delta m_{32}^2$) and $\Delta m_{31}^2$ in the oscillation experiments performed with terrestrial (reactor, accelerator, etc.) or atmospheric neutrinos.

\footnote{If the massive neutrinos $\nu_i$ are Majorana particles, the neutrinoless double beta decay would be allowed and, for $m_i \sim (1 - 2) \text{ eV}$, can proceed with a rate which is in the range of sensitivity of the new generation of experiments searching for this decay (see, e.g., \cite{Zyla2018}).}
All the different patterns of neutrino masses mentioned above can arise in gauge theories of electroweak interactions with massive neutrinos, and in particular, in GUT theories and in the superstring inspired models (see, e.g., [29]).

This paper is organized as follows. Section 2 is devoted to the discussion of the average solar $\nu_e$ survival probability, $\bar{P}(\nu_e \rightarrow \nu_e)$, when three-neutrino mixing takes place in vacuum and the solar neutrinos undergo MSW transitions in the Sun, which are followed by long wave length neutrino oscillations in vacuum. It is shown, in particular, that the probability $\bar{P}(\nu_e \rightarrow \nu_e)$ is described by a simple analytic expression and that inequalities (4) and (5) imply two very different types of interplay between the MSW transitions and the vacuum oscillations of the solar $\nu_e$. In Section 3 the hypothesis that the solar $\nu_e$ take part in MSW transitions and long wave length vacuum oscillations is confronted with the solar neutrino data. Several new hybrid MSW transition + vacuum oscillation (MSW + VO) solutions of the solar neutrino problem are found and their specific features are discussed. Section 4 contains our conclusions. A derivation of the analytic expression for the probability $\bar{P}(\nu_e \rightarrow \nu_e)$ used in the present study is given in the Appendix.

2. The Solar $\nu_e$ Survival Probability

It is possible to obtain a very simple analytic expression for the solar $\nu_e$ survival probability in the case of interest, namely, when the solar $\nu_e$ undergo matter-enhanced (MSW) transitions in the Sun and vacuum oscillations on their way from the surface of the Sun to the Earth. The vacuum oscillations are supposed to proceed for neutrinos having energy $E \sim 1 \text{ MeV}$ with an oscillation length comparable to the Sun–Earth distance, $R_0 \cong 1.4966 \times 10^8 \text{ km}$.

We assume three-flavour neutrino mixing takes place in vacuum,

$$ |\nu_l > = \sum_{k=1}^{3} U_{lk}^* |\nu_k >, \quad l = e, \mu, \tau, $$

where $|\nu_l >$ is the state vector of the (left-handed) flavour neutrino $\nu_l$ having momentum $\vec{p}$, $|\nu_k >$ is the state vector of a neutrino $\nu_k$ possessing a definite mass $m_k$ and momentum $\vec{p}$, $m_k \neq m_j$, $k \neq j = 1, 2, 3$, $m_1 < m_2 < m_3$, and $U$ is a $3 \times 3$ unitary matrix – the
lepton mixing matrix. The neutrinos $\nu_k$ are assumed to be stable and relativistic. We shall suppose also that the matter-enhanced transitions of the solar $\nu_e$ into $\nu_\mu$ and/or $\nu_\tau$ in the Sun are associated with $\Delta m^2_{31}$, where, as usual, $\Delta m^2_{ij} = m_i^2 - m_j^2$, while $\Delta m^2_{21}$ (or $\Delta m^2_{32}$) is responsible for the long wave length ($\sim 1.5 \times 10^8$ km) $\nu_e \leftrightarrow \nu_\mu, \tau$ oscillations taking place between the Sun and the Earth. This implies that either relation (4) or (5) is valid. We shall consider in the present analysis a somewhat wider range of values of $\Delta m^2_{21}$ ($\Delta m^2_{32}$) and $\Delta m^2_{31}$ than is given by the two-neutrino mixing solution intervals eqs. (1a) and (2a) or (3a), respectively:

$$10^{-12} \text{ eV}^2 \leq \Delta m^2_{21} \leq 5.0 \times 10^{-10} \text{ eV}^2,$$

$$10^{-7} \text{ eV}^2 \leq \Delta m^2_{31} \leq 10^{-4} \text{ eV}^2.$$

Under the above conditions and assuming the inequality (4), $\Delta m^2_{21} \ll \Delta m^2_{31}$, holds, the analytic expression for the solar $\nu_e$ survival probability of interest, $\bar{P}(\nu_e \rightarrow \nu_e)$, can be easily deduced from the general expression for $\bar{P}(\nu_e \rightarrow \nu_e)$ derived in ref. [20] (see eq. (29) in [20]) for the relevant case of three (flavour) neutrino mixing. It has the following simple form:

$$\bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0) = \bar{P}^{(31)}_{2\text{MSW}}(\nu_e \rightarrow \nu_e; t_\odot, t_0)$$

$$+ \frac{1 - |U_{e3}|^2}{1 - 2|U_{e3}|^2} \left[ 1 - P^{(21)}_{2\text{V}}(\nu_e \rightarrow \nu_e; t_0, t_\odot) \right] \left[ |U_{e3}|^2 - P^{(31)}_{2\text{MSW}}(\nu_e \rightarrow \nu_e; t_\odot, t_0) \right].$$

Here $\bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0)$ is the average probability that a solar $\nu_e$ having energy $E \simeq |\mathbf{p}| = p$ will not be converted into $\nu_\mu$ and/or $\nu_\tau$ when it propagates from the central part of the Sun, where it was produced at time $t_0$, to the surface of the Earth reached at time $t_E$, $\bar{P}^{(31)}_{2\text{MSW}}(\nu_e \rightarrow \nu_e; t_\odot, t_0)$ is the average solar $\nu_e$ survival probability when the $\nu_e$ undergoes two-neutrino MSW transitions due to $\Delta m^2_{31}$ and $U_{e3} \neq 0$ on its way to the surface of the Sun, $t_\odot$ being the time at which the $\nu_e$ reaches the surface of the Sun, and $P^{(21)}_{2\text{V}}(\nu_e \rightarrow \nu_e; t_0, t_\odot)$ is the solar $\nu_e$ survival probability for a $\nu_e$ taking part in two-neutrino vacuum oscillations due to $\Delta m^2_{21}$ and $U_{e1} \neq 0$, $U_{e2} \neq 0$, on its way from the surface of the Sun to the surface of the Earth. The two-neutrino MSW transition and vacuum oscillation probabilities $\bar{P}^{(31)}_{2\text{MSW}}(\nu_e \rightarrow \nu_e; t_\odot, t_0)$ and $P^{(21)}_{2\text{V}}(\nu_e \rightarrow \nu_e; t_E, t_\odot)$ are given by the well-known expressions
In eqs. (10) and (11) \( P'_{(31)} \) is the so-called “jump” (or “level-crossing”) probability,
\[
\cos 2\theta_{13} = 1 - 2|U_{e3}|^2 > 0, \quad \sin 2\theta_{13} = 2|U_{e3}|\sqrt{1 - |U_{e3}|^2} > 0,
\]
(12)
\( \theta_{13}^m \) is the two-neutrino mixing angle in matter, which in vacuum coincides with \( \theta_{13} \),
\[
\cos 2\theta_{13}^m(t_0) = \frac{1 - N_e(t_0)/N_{e_{res}}}{\sqrt{(1 - N_e(t_0)/N_{e_{res}})^2 + \tan^2 2\theta_{13}}},
\]
(13)
where \( N_e(t_0) \) is the electron number density at the point of \( \nu_e \) production in the Sun and
\( N_{e_{res}} = \Delta m_{31}^2 \cos 2\theta_{13}/(2p\sqrt{2}G_F) \) is the resonance density,
\[
\sin^2 2\theta_{12} = 4\frac{|U_{e1}|^2 |U_{e2}|^2}{(|U_{e1}|^2 + |U_{e2}|^2)^2} ,
\]
(14)
\( R = (t_E - t_\odot) \) is the distance traveled by the neutrinos in vacuum, \( R \approx R_0 \), and \( L_{21}^v = 4\pi p/\Delta m_{21}^2 \) is the oscillation length in vacuum associated with \( \Delta m_{21}^2 \).

Several comments are in order. The requirement
\[
|U_{e3}|^2 < 0.5
\]
(15)
in eq. (12) ensures the possibility of a resonant enhancement of the solar \( \nu_e \) transitions in the Sun [34].

For the values of \( \Delta m_{31}^2 \) of interest, eq. (8), the oscillating terms in the probability
\( P(\nu_e \rightarrow \nu_e; t_E, t_0) \), associated with the solar \( \nu_e \) MSW transitions in the Sun [35] are rendered negligible [36] by the various averagings one has to perform when calculating the effects of the transitions and oscillations on the signals in the solar neutrino detectors.

Further, the probability \( \bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0) \), eq. (9), depends only on the absolute values of the elements of the lepton mixing matrix \( U \), forming the first row of \( U \), more precisely, on \( |U_{ek}|^2, k = 1, 2, 3 \). Of these only two are independent since the unitarity of \( U \) implies:
\(|U_{e1}|^2 + |U_{e2}|^2 + |U_{e3}|^2 = 1\). It follows from eqs. (12), (14) and the preceding remark that \(\theta_{13}\) and \(\theta_{12}\) are independent parameters.

We note next that at least three different expressions for the “jump” probability \(P'_{(31)}\) entering into formula (10) for the MSW probability have been proposed \([30,32]\). We will use the one derived in ref. \([32]\) in the exponential density approximation:

\[
P'_{(31)} = \frac{e^{-2\pi r_0 \frac{\Delta m^2_{31}}{2p} \sin^2 \theta_{13}} - e^{-2\pi r_0 \frac{\Delta m^2_{31}}{2p}}}{1 - e^{-2\pi r_0 \frac{\Delta m^2_{31}}{2p}}} ,
\]

where \(r_0\) is the scale-height of the variation of the electron number density \(N_e\) along the neutrino path in the Sun. Unlike the expressions proposed in ref. \([30]\), the one given by eq. (15) describes correctly the (strongly) nonadiabatic MSW transitions of solar neutrinos for values of \(\sin^2 2\theta_{13} \gtrsim (0.2 - 0.3)\) \([31,32,37]\).

Let us comment on how eq. (9) with probabilities \(\bar{P}_{2\text{MSW}}^{(31)}\) and \(\bar{P}_{2\text{VO}}^{(21)}\) given by eqs. (10) and (11) can be obtained from eq. (29) in ref. \([20]\). The analysis leading to eq. (29) in \([20]\) is valid, in particular, under the condition \(\Delta m^2_{21} \ll \Delta m^2_{31}\). This condition is fulfilled in the case presently considered, which is specified by eqs. (4), (7) and (8). We observe first that in this case one has, in the notations of ref. \([21]\), \(\tilde{c}_{12} \cong 0\), \(\phi'_{13}(t'_0) \cong 0\) (and therefore \(\tilde{c}_{13} \cong c_{13}, \tilde{s}_{13} \cong s_{13}\)) and \(s'_{23} \cong 0\) in eq. (29); as a consequence, the latter simplifies considerably. We notice next that for \(\Delta m^2_{21}\) having values in the interval (7) the times \(t\) and \(t'_0\) in eq. (29) in \([20]\) can be considered to be respectively the times \(t_E\) and \(t_0\) with the meaning they have in eq. (9) above: the derivation of eq. (29) in \([20]\) for \(\bar{P}(\nu_e \rightarrow \nu_e)\) is valid in this case as well. Finally, i) the angles \(\phi_{12}\) and \(\phi_{13}\) in eq. (29) in \([20]\) coincide with the angles \(\theta_{12}\) and \(\theta_{13}\) defined by eqs. (12) and (14) above, ii) the probabilities \(\bar{P}(\nu_e \rightarrow \nu_e; t, t_0)\), \(\bar{P}_H(\nu_e \rightarrow \nu_e; t'_0, t_0)\) and \(\bar{P}_L(\nu_e \rightarrow \nu_e; t, t'_0)\) in eq. (29) coincide for \(t \equiv t_E\) and \(t'_0 \equiv t_0\) with the probabilities \(\bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0)\), \(\bar{P}_{2\text{MSW}}^{(31)}(\nu_e \rightarrow \nu_e; t_0, t_0)\) and \(\bar{P}_{2\text{VO}}^{(21)}(\nu_e \rightarrow \nu_e; t_E, t_0)\), respectively, and iii) the term \(\text{Re} \left[ R_H(t'_0, t_0) \right]\) in eq. (29) in the case of interest is given, as it is not difficult to show using the results in refs. \([32,35,38]\), by the following expression:

\[
\text{Re} \left[ R_H(t'_0, t_0) \right] = - \left( \frac{1}{2} - P'_{(31)} \right) \sin 2\theta_{13} \cos 2\theta^m_{13}(t_0).
\]
Given the above observations it is easy to obtain expression (9) for $P(\nu_e \to \nu_e; t_E, t_0)$ from eq. (29) in [20]. A proof of eq. (9) which does not rely on eq. (29) in [20] and is based only on some of the intermediate results of the analysis performed in [20] is given in the Appendix.

Using eqs. (10), (11) and (12) we can rewrite the second term in eq. (9) in a form which proves convenient for our later discussion:

$$P(\nu_e \to \nu_e; t_E, t_0) = P_{2MSW}(\nu_e \to \nu_e; t_E, t_0) - \frac{1}{2} \sin^2 2\theta_{12} (1 - \cos 2\pi \frac{R}{L_{21}^5}) \cos^2 \theta_{13} \left[ \frac{1}{2} + \left( \frac{1}{2} - P'_{(31)} \right) \cos 2\theta_{m_13}(t_0) \right]$$

(17)

Before proceeding further let us point out that since $\bar{P}(31)_{2MSW} \geq |U_{e3}|^2 = \sin^2 \theta_{13}$ and $|U_{e3}|^2 < 0.5$, for fixed values of the MSW transition parameters one always has

$$\bar{P}(\nu_e \to \nu_e; t_E, t_0) \geq \frac{|U_{e3}|^2}{1 - 2|U_{e3}|^2} \left[ 1 - |U_{e3}|^2 - \bar{P}_{2MSW}^{(31)} \right] = \sin^2 \theta_{13} \left[ \frac{1}{2} + \left( \frac{1}{2} - P'_{(31)} \right) \cos 2\theta_{m_13}(t_0) \right]$$

(18)

as it follows from eq. (9). This lower limit constrains effectively $\bar{P}(\nu_e \to \nu_e; t_E, t_0)$ only for sufficiently large values of $\sin^2 \theta_{13}$. Setting $P_{2VO}^{(21)}(\nu_e \to \nu_e; t_E, t_\odot) = 1$ in eq. (9) we get an absolute upper limit on $\bar{P}(\nu_e \to \nu_e; t_E, t_0)$ for given $\bar{P}_{2MSW}^{(31)}$ and $|U_{e3}|^2$:

$$\bar{P}(\nu_e \to \nu_e; t_E, t_0) \leq \bar{P}_{2MSW}^{(31)}(\nu_e \to \nu_e; t_\odot, t_0).$$

(19)

The three-neutrino mixing solar $\nu_e$ survival probability in the case of interest is always smaller than the corresponding two-neutrino MSW probability.

Eqs. (18) and (19) determine the range in which the probability $\bar{P}(\nu_e \to \nu_e; t_E, t_0)$ can oscillate as a function of $R/p$ due to the presence of the vacuum oscillation component $P_{2VO}^{(21)}$ in it. Let us consider several specific cases relevant to our further discussion.

Note that there is a misprint in eq. (29) in [20]: the probability $\bar{P}_L(\nu_e \to \nu_e; t, t_0')$ in the expression in the curly brackets multiplying the term $\mathrm{Re} \left[ R_H(t_0', t_0) \right]$ in the third row should read $\bar{P}_L(\nu_e \to \nu_e; t, t_0') \equiv 1 - \bar{P}_L(\nu_e \to \nu_e; t, t_0')$. 
If the solar $\nu_e$ undergo extreme nonadiabatic MSW transitions in the Sun ($p/\Delta m^2_{31}$ is “large”), one has [32] $P'_{(31)} \cong \cos^2 \theta_{13}$, $\bar{P}^{(31)}_{2MSW} \cong 1 - \frac{1}{2} \sin^2 2\theta_{13}$ and

$$|U_{e3}|^4 = \sin^4 \theta_{13} \leq \bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0) \leq |U_{e3}|^4 + (1 - |U_{e3}|^2)^2 = \sin^4 \theta_{13} + \cos^4 \theta_{13}. \quad (20)$$

We get the same result, eq. (20), if $N_e(t_0)(1 - \tan 2\theta_{13})^{-1} < N_{e}^{res}$, i.e., when $p/\Delta m^2_{31}$ is sufficiently small. In this case [37] the $\nu_e$ MSW transitions are adiabatic ($P'_{(31)} \cong 0$), the $\nu_e$ oscillate in the Sun due to $\Delta m^2_{31}$ as in vacuum: $\cos 2\theta_{13}^m(t_0) \cong \cos 2\theta_{13}$ and $\bar{P}^{(31)}_{2MSW} \cong 1 - \frac{1}{2} \sin^2 2\theta_{13}$. If, however, in the “small” $p/\Delta m^2_{31}$ region the vacuum oscillation term $\cos 2\pi R/L^e_{21}$ averages out so that effectively $P^{(21)}_{2VO} = 1 - 1/2 \sin^2 2\theta_{12}$, one obtains:

$$P(\nu_e \rightarrow \nu_e; t_E, t_0) \cong |U_{e1}|^4 + |U_{e2}|^4 + |U_{e3}|^4 = 1 - \frac{1}{2} \sin^2 2\theta_{12} \cos^4 \theta_{13} - \frac{1}{2} \sin^2 2\theta_{13}, \quad (21)$$

which is the average three-neutrino vacuum oscillation probability.

If the solar $\nu_e$ MSW transitions are adiabatic (i.e., $P'_{(31)} \cong 0$) and $\cos 2\theta_{13}^m(t_0) \cong -1$ (i.e., $N_e(t_0)/N_{e}^{res} \gg 1$, $\tan 2\theta_{13}$, solar neutrinos are born “above” and “far” (in $N_e$) from the resonance region), one has $\bar{P}^{(31)}_{2MSW} \cong |U_{e3}|^2$ and therefore

$$\bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0) \cong \bar{P}^{(31)}_{2MSW} \cong |U_{e3}|^2 = \sin^2 \theta_{13}. \quad (22)$$

Under the above conditions the $\nu_e$ state in matter at the point of $\nu_e$ production (in the Sun) essentially coincides with the heaviest of the three neutrino matter-eigenstates, which continuously evolves (as the neutrino propagates towards the surface of the Sun) into the mass (energy) eigenstate $|\nu_3 >$ at the surface of the Sun. As a consequence, vacuum oscillations do not take place between the Sun and the Earth and $\bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0)$ coincides with the probability to find $\nu_e$ in the state $|\nu_3 >$.

Let us discuss next the properties of the probability $\bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0)$ if inequality (5), i.e., a Zee model type relation between $\Delta m^2_{32}$ and $\Delta m^2_{31}$ ($\Delta m^2_{21}$), is valid and $\Delta m^2_{32}$ is assumed to give rise to solar neutrino long wave length vacuum oscillations on the path between the Sun and the Earth. The probability $\bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0)$ in this case has the form:

$$\bar{P}^{Z}(\nu_e \rightarrow \nu_e; t_E, t_0) = \bar{P}^{(13)}_{2MSW}(\nu_e \rightarrow \nu_e; t_0)$$
\[ + \frac{1 - |U_{e1}|^2}{1 - 2|U_{e1}|^2} \left[ 1 - P^{(23)}_{2V\Omega}(\nu_e \to \nu_e; t_E, t_\odot) \right] \left[ |U_{e1}|^2 - \bar{P}^{(13)}_{2M\text{SW}}(\nu_e \to \nu_e; t_\odot, t_0) \right] \quad (23a) \]
\[ = \bar{P}^{(13)}_{2M\text{SW}} - \frac{1}{2} \sin^2 2\theta_{23} \left( 1 - \cos 2\pi \frac{R}{L^v_{32}} \right) \sin^2 \theta'_{13} \left[ \frac{1}{2} - \left( \frac{1}{2} - P'_{(13)} \right) \cos 2\theta'_{13}(t_0) \right]. \quad (23b) \]

The angles \( \theta_{23} \) and \( \theta'_{13} \) in eq. (23b) are determined by
\[
\sin^2 2\theta_{23} = 4 \frac{|U_{e3}|^2 |U_{e2}|^2}{(|U_{e3}|^2 + |U_{e2}|^2)^2}, \quad (24) \]
and
\[
\cos 2\theta'_{13} = 2|U_{e1}|^2 - 1 > 0, \quad \sin 2\theta'_{13} = -2|U_{e1}| \sqrt{1 - |U_{e1}|^2} < 0, \quad (25) \]
and \( L^v_{32} = 4\pi p/\Delta m^2_{32} \); the requirement
\[
|U_{e1}|^2 > 0.5 \quad (26) \]
guarantees the possibility of resonant enhancement of the MSW transitions in the Sun.

The two-neutrino mixing MSW probability \( \bar{P}^{(13)}_{2M\text{SW}}(\nu_e \to \nu_e; t_\odot, t_0) \equiv \bar{P}^{(13)}_{2M\text{SW}} \) is given by eq. (10) in which the angles \( \theta_{13}, \theta^m_{13}(t_0) \) and the “jump” probability \( P'_{(31)} \) are replaced respectively by \( \theta'_{13}, \theta'^m_{13}(t_0) \) and \( P'_{(13)} \), where \( \theta^m_{13}(t_0) \) and \( P'_{(13)} \) are defined by eq. (13) and by eq. (15) in which \( \theta_{13} \) is substituted by \( \theta'_{13} \). The two-neutrino vacuum oscillation probability \( P^{(23)}_{2V\Omega}(\nu_e \to \nu_e; t_E, t_\odot) \equiv P^{(23)}_{2V\Omega} \) can be obtained by replacing in eq. (11) \( \theta_{12} \) and \( L^v_{21} \) with \( \theta_{23} \) and \( L^v_{32} \).

Expressions (23) follow from eqs. (9) and (17) if we interchange the indices 1 and 3 in the quantities \( |U_{ej}|^2 \) and \( \Delta m^2_{ik} \), or equivalently, if we replace \( |U_{e3(1)}|^2 \) with \( |U_{e1(3)}|^2 \) and \( \Delta m^2_{31} (\Delta m^2_{21}) \) with \( \Delta m^2_{13} (\Delta m^2_{23}) \) in all the terms entering into eqs. (9) and (17), except in the “jump” probability \( P'_{(31)} \), and take into account the definitions of \( \theta_{13} \) (eq. (12)) and \( \theta'_{13} \) (eq. (24)). It can be shown that \( P'_{(13)} \) coincides with the expression for \( P'_{(31)} \), eq. (15), in which \( \theta_{13} \) is changed to \( \theta'_{13} \).

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6In the particular case of the Zee model lepton mixing matrix \( U \) \( [26,27] \) in which two of the three mixing angles have specific values and all elements of \( U \) are real, expression (23a) for \( \bar{P}^Z(\nu_e \to \nu_e; t_E, t_0) \) reduces to the one derived in ref. \( [28] \) in the indicated case.
Although expressions (9) ((17)) and (23a) ((23b)) formally look quite similar, the two probabilities \( \bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0) \) and \( \bar{P}^Z(\nu_e \rightarrow \nu_e; t_E, t_0) \) actually have very different properties.

Expression (17) implies that if \( \Delta m_{21}^2 \ll \Delta m_{31}^2 \), a genuinely hybrid small mixing angle, \( \sin^2 \theta_{13} \ll 1 \), MSW transition + vacuum oscillation solution(s) of the solar neutrino problem is (are) in principle possible. In contrast, if \( \sin^2 \theta'_{13} \ll 1 \) the contribution of the vacuum oscillation probability \( P_{2V0}^{(23)} \) in \( \bar{P}^Z(\nu_e \rightarrow \nu_e; t_E, t_0) \) is strongly suppressed and \( \bar{P}^Z(\nu_e \rightarrow \nu_e; t_E, t_0) \approx \bar{P}_{2MSW}^{(13)}(\nu_e \rightarrow \nu_e; t_0) \). Thus, if inequality (5) holds, hybrid MSW + VO solutions can exist only for large values of the MSW mixing angle \( \theta'_{13} \).

Similarly to \( \bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0) \), the probability \( \bar{P}^Z(\nu_e \rightarrow \nu_e; t_E, t_0) \) is limited from above by \( \bar{P}^{(13)}_{2MSW} \): \( \bar{P}^Z(\nu_e \rightarrow \nu_e; t_E, t_0) \leq \bar{P}^{(13)}_{2MSW}(\nu_e \rightarrow \nu_e; t_0) \). This follows from the fact that \( 0.5 < |U_{e1}|^2 \leq 1 \) and \( |U_{e1}|^2 - \bar{P}^{(13)}_{2MSW} > 0 \). However, the lower limit on \( \bar{P}^Z(\nu_e \rightarrow \nu_e; t_E, t_0) \) is physically very different from the lower limit on \( \bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0) \), eq. (18):

\[
\bar{P}^Z(\nu_e \rightarrow \nu_e; t_E, t_0) \geq \frac{|U_{e1}|^2}{2|U_{e1}|^2 - 1} \left[ \bar{P}^{(13)}_{2MSW} - (1 - |U_{e1}|^2) \right] = \cos^2 \theta'_{13} \left[ \frac{1}{2} + \left( 1 - P^{(13)} \right) \cos 2\theta'_{13} (t_0) \right]
\]

(27)

In the two “asymptotic” regions of “small” (adiabatic MSW transitions) and “large” (extreme nonadiabatic MSW transitions) values of \( p/\Delta m_{31}^2 \) the probability \( \bar{P}^Z(\nu_e \rightarrow \nu_e; t_E, t_0) \) has the form:

\[
\bar{P}^Z(\nu_e \rightarrow \nu_e; t_E, t_0) \approx |U_{e1}|^4 + (1 - |U_{e1}|^2)^2 P_{2V0}^{(23)}(\nu_e \rightarrow \nu_e; t_0).
\]

(28)

From eq. (28) we get the following band of allowed values of \( \bar{P}^Z(\nu_e \rightarrow \nu_e; t_E, t_0) \), which is

---

7For the specific case of the Zee model mixing matrix \( U \) \[26,27\] this upper bound was obtained in ref. \[28\].

8We have used the upper bound \( \bar{P}^{(13)}_{2MSW} \leq 1 - 2|U_{e1}|^2(1 - |U_{e1}|^2) = 1 - 1/2 \sin^2 2\theta'_{13} \) to get the last inequality.
the same for the two regions:

$$|U_{e1}|^4 = \cos^4 \theta'_{13} \leq P^Z(\nu_e \to \nu_e; t_E, t_0) \leq |U_{e1}|^4 + (1 - |U_{e1}|^2)^2 = \cos^4 \theta'_{13} + \sin^4 \theta'_{13}. \quad (29)$$

The corresponding band for the probability $P(\nu_e \to \nu_e; t_E, t_0)$ is given in (20).

Finally, if the MSW transitions in the Sun are adiabatic, $P'_{(13)} = 0$, and $\cos 2\theta_{13}^m(t_0) = -1$ ($\bar{P}_{2MSW}^{(13)} = \sin^2 \theta'_{13}$), one has:

$$0 \leq \bar{P}^Z(\nu_e \to \nu_e; t_E, t_0) = (1 - |U_{e1}|^2)P_{2VO}^{(23)}(\nu_e \to \nu_e; t_E, t_0) \leq \sin^2 \theta'_{13}, \quad (30)$$

to be compared with eq. (22).

It follows from the above discussion that if $\Delta m^2_{21} \ll \Delta m^2_{31}$, the amplitude of the oscillations of $\bar{P}(\nu_e \to \nu_e; t_E, t_0)$ (due to the $P_{2VO}^{(23)}$ term) as $R/p$ varies is equal in the regions of “large” and “small” $p/\Delta m^2_{31}$ to $\cos^4 \theta_{13} \sin^2 2\theta_{12}$ and can be maximal; for $\sin^2 2\theta_{12} \sim 1$ it is always greater than 1/4. At the same time, this amplitude is strongly suppressed and vacuum oscillations practically do not take place (even if $\sin^2 2\theta_{12} \sim 1$) in the region of the adiabatic MSW transitions where $\cos 2\theta_{13}^m(t_0) \cong -1$ and $\bar{P}_{2MSW}^{(31)}$ has its minimal value, $\bar{P}_{2MSW}^{(31)} = \sin^2 \theta_{13}$. In contrast, if the Zee model type relation $\Delta m^2_{32} \ll \Delta m^2_{31}$ holds, the vacuum oscillations are suppressed in the two “asymptotic” regions: the amplitude of the oscillations is equal to $\sin^4 \theta'_{13} \sin^2 2\theta_{23}$ and is always constrained to be smaller than 1/4. In the region of adiabatic MSW transitions, where $\cos 2\theta'_{13}^m(t_0) \cong -1$ and $\bar{P}_{2MSW}^{(13)} = \sin^2 \theta'_{13}$, the probability $\bar{P}^Z(\nu_e \to \nu_e; t_E, t_0)$ oscillates (due to $P_{2VO}^{(23)}$) as a function of $R/p$ with an amplitude $\sin^2 \theta_{13}^m \sin^2 2\theta_{23}$, which does not exceed $0.5 \sin^2 2\theta_{23}$. Obviously, the vacuum oscillations can be important in this case only for sufficiently large values of $\sin^2 \theta'_{13}$.

Because of the aforementioned differences in the properties of the probabilities $\bar{P}(\nu_e \to \nu_e; t_E, t_0)$ and $\bar{P}^Z(\nu_e \to \nu_e; t_E, t_0)$, the case (4) of relations between the neutrino mass squared differences provides a much richer spectrum of possibilities of genuinely new solutions of the

\footnote{Note that the band of allowed values of $\bar{P}^Z(\nu_e \to \nu_e; t_E, t_0)$ in the “large” $p/\Delta m^2_{31}$ region shown in Fig. 2 in ref. [28] does not correspond to eqs. (28) and (29).}
3. Hybrid MSW Transition + Vacuum Oscillation Solutions of the Solar Neutrino Problem

We have analyzed the most recently published data from the four solar neutrino experiments [1–4] searching for solutions of the solar neutrino problem of the hybrid MSW transitions + vacuum oscillations type. Only the case of inequality (4) with $\Delta m^2_{31}$ and $\Delta m^2_{21}$ having values in the intervals (7) and (8) was studied.

The analysis we performed is based on the analytic expression eq. (9) for the average solar $\nu_e$ survival probability, $\bar{P}(\nu_e \to \nu_e; t_E, t_0)$, with $\bar{P}_{2M\text{SW}}^{(31)}(\nu_e \to \nu_e; t_\odot, t_0)$ given by eqs. (10), (13) and (15). As is well known, the solar neutrino flux consists of several components, six of which are relevant for the interpretation of the results of the solar neutrino experiments [3] (see also, e.g., [3,4]): the $pp$, $pep$, $^7\text{Be}$, $^8\text{B}$ and the $\text{CNO}$ (two components). We utilized the predictions of the solar model of Bahcall and Pinsonneault from 1995 with heavy element diffusion [39] for the $pp$, $pep$, etc. neutrino fluxes in this study. The estimated uncertainties in the theoretical predictions for the indicated fluxes [39] were not taken into account. For each given neutrino flux the probability $\bar{P}(\nu_e \to \nu_e; t_E, t_0)$ was averaged over the corresponding region of $\nu_e$ production in the Sun. Since the data analyzed was accumulated over a period of several years, $\bar{P}(\nu_e \to \nu_e; t_E, t_0)$ was also averaged over an interval of time equal to one year [11]. The probability $\bar{P}(\nu_e \to \nu_e; t_E, t_0)$ depends on the time of the year $t_y$ through the dependence of $P_{2\text{VO}}^{(31)}(\nu_e \to \nu_e; t_E, t_\odot)$ on the distance between the Sun and the Earth, $t_E - t_\odot \approx R$, which is a function of $t_y$:

$$ R = R(t_y) = R_0 \left[1 - \epsilon \cos 2\pi \frac{t_y}{T}\right], \quad (31) $$

where $\epsilon = 0.0167$ is the ellipticity of the Earth orbit around the Sun and $T = 365$ days. The solar neutrino fluxes are functions of $R^{-2}(t_y)$ and their dependence on $t_y$ was effectively accounted for when we performed the time averaging of the probability $\bar{P}(\nu_e \to \nu_e; t_E, t_0)$. 

solar neutrino problem of the hybrid MSW transition + vacuum oscillation type, than the case of relations (5).
The results we shall report have been obtained utilizing the $\chi^2$—method. The regions of the relevant MSW transition and vacuum oscillation parameters allowed by the data were determined by requiring that $\chi^2 < 3.841$. This choice is motivated by the following arguments. In the $\chi^2$—analysis in the case under study there are zero degrees of freedom: we have four experimental results and the theory tested contains four parameters ($\theta_{12}, \Delta m^2_{21}, \theta_{13}$ and $\Delta m^2_{31}$). Nevertheless, the existence of regions of values of the parameters for which the $\chi^2$—function has a distinct minimum and a sufficiently low value at the minimum is a strong and unambiguous indication that the mechanism of solar $\nu_e$ flux depletion considered gives a good quality of the fit of the data and therefore provides a solution of the solar neutrino problem. At the same time, in the case of zero degrees of freedom it is impossible to assign a precise confidence level to the $\chi^2 = 3.841$ contours. It should be noted, however, that in grand unified theories with massive neutrinos the quantities $\Delta m^2_{21}$ and $\Delta m^2_{31}$ are typically not independent: their ratio is a function of the ratio of quark or charged lepton masses. If such a relation holds there would be three independent parameters in the theory, and correspondingly one degree of freedom in the $\chi^2$—analysis. Under these conditions $\chi^2 = 3.841$ corresponds to 95% C.L.

We have searched for MSW + VO solutions of the solar neutrino problem for values of $\sin^2 2\theta_{12}$ and $\sin^2 2\theta_{13}$ from the intervals:

\begin{align}
0.20 & \lesssim \sin^2 2\theta_{12} \leq 1.0, \\
10^{-4} & \leq \sin^2 2\theta_{13} < 1.0.
\end{align}

The intervals (32) and (33) are considerably wider than the corresponding ones in the case of purely two-neutrino mixing ($2\nu$) vacuum oscillation and MSW transition solutions (see, e.g., [13–16]). We did not perform a complete scan of the 4-parameter space defined by eqs. (7), (8), (32) and (33). We were primarily interested in solutions which do not converge

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10In certain classes of theories some of the lepton mixing angles are also functions of the charged lepton or quark masses.
continuously (via regions where $\chi^2 \leq 3.841$) to the $2\nu$ vacuum oscillation or MSW transition solution in the limit of $\sin^2 2\theta_{13} \to 0$ or, correspondingly, of $\sin^2 2\theta_{12} \to 0$ or $\Delta m_{12}^2 \to 0$.

Nevertheless, we have found five different regions of values of the four parameters, for which $\chi^2 \leq 3.841$. These regions (solutions) are denoted by $A$, $B$, $C$, $D$ and $E$ below. Sections of the indicated regions in the planes of the MSW transition and of the vacuum oscillation parameters $\Delta m_{31}^2 - \sin^2 2\theta_{13}$ and $\Delta m_{21}^2 - \sin^2 2\theta_{12}$ are shown respectively in Figs. 1a, 1b and 2a, 2b. Shown in Figs. 1a – 2b are also the points in which the $\chi^2$ function has a minimal value.

We shall discuss next the physical features of the solutions found. A detailed analysis of the implications of the MSW transition + vacuum oscillation solutions for the solar neutrino experiments Super-Kamiokande, SNO, ICARUS, BOREXINO, HELLAZ and HERON will be given elsewhere. Here we will limit ourselves only to qualitative remarks.

**Solution A.** For this solution (see Figs. 1a and 2a) the vacuum oscillation parameters lie in the intervals

$$4 \times 10^{-12} \text{ eV}^2 \lesssim \Delta m_{21}^2 \lesssim 8 \times 10^{-12} \text{ eV}^2, \quad (34a)$$

$$0.65 \lesssim \sin^2 2\theta_{12} \lesssim 1.0, \quad (34b)$$

while the region formed by the values of the MSW transition parameters extends from the domain of the $2\nu$ nonadiabatic solution at $\Delta m_{31}^2 \approx 5.0 \times 10^{-6} \text{ eV}^2$, $\sin^2 2\theta_{13} \approx 8 \times 10^{-3}$, first to larger (smaller) values of $\Delta m_{31}^2$ ($\sin^2 2\theta_{13}$) reaching $\Delta m_{31}^2 \approx 10^{-4} \text{ eV}^2$ and $\sin^2 2\theta_{13} \approx 3.1 \times 10^{-4}$, and then to larger values of $\sin^2 2\theta_{13}$ up to 0.5, spanning approximately 3 orders of magnitude in $\sin^2 2\theta_{13}$ at practically constant $\Delta m_{31}^2 \approx (1.1 - 1.3) \times 10^{-4} \text{ eV}^2$. The minimum $\chi^2$ value for this solution is $\chi^2_{\text{min}} \approx 0.071$ and corresponds to $(\Delta m_{21}^2, \sin^2 2\theta_{12}, \Delta m_{31}^2, \sin^2 2\theta_{13}) \approx (5.6 \times 10^{-12} \text{ eV}^2, 0.98, 4.2 \times 10^{-5} \text{ eV}^2, 10^{-3})$ (the black triangle-down in Fig. 1a). In the “horizontal” region where $\sin^2 2\theta_{13} \gtrsim 10^{-3}$ and $\Delta m_{31}^2 \approx (1.1 - 1.3) \times 10^{-4} \text{ eV}^2$ there exists a local minimum with a somewhat larger value of $\chi^2_{\text{min}}$: one finds $\chi^2_{\text{min}} \approx 1.0$. This minimum is reached for $(\Delta m_{21}^2, \sin^2 2\theta_{12}, \Delta m_{31}^2, \sin^2 2\theta_{13}) \approx (5.4 \times 10^{-12} \text{ eV}^2, 1.0, 1.2 \times 10^{-4} \text{ eV}^2, 0.14)$ (the black dot in Fig. 1a).
For $p \cong E \geq 5$ MeV and $\Delta m_{21}^2 \leq 8.0 \times 10^{-12}$ eV$^2$ we have $L_{21}^v \geq 1.6 \times 10^9$ km, $\cos(2\pi R/L_{21}^v) \geq 0.82$ and, as follows from eq. (11), $P_{2VO}^{(21)}(\nu_e \rightarrow \nu_e; t_E, t_0) \approx P_{2MSW}^{(31)}(\nu_e \rightarrow \nu_e; t_\odot, t_0)$, i.e., most of the $^8$B neutrinos undergo only MSW transitions. The MSW transitions of the $^8$B neutrinos having energy $E \geq 5$ MeV are adiabatic for values of $\Delta m_{31}^2 \cong (1.1 - 1.3) \times 10^{-4}$ eV$^2$ and $\sin^2 2\theta_{13} \cong (3.0 - 4.0) \times 10^{-3}$ from the “horizontal” region of the solution (see Fig. 1a). They are nonadiabatic for values of $\Delta m_{31}^2$ and $\sin^2 2\theta_{13}$ from the remaining part of the allowed region.

For the $pp$ and the major part of the $^7$Be neutrinos we have $E \leq 0.41$ MeV and $E = 0.862$ MeV, respectively, and for $\Delta m_{31}^2 \cong 1.3 \times 10^{-5}$ eV$^2$, their energies fall in the “small” $p/\Delta m_{31}^2$ domain (see Section 2) where

$$P(\nu_e \rightarrow \nu_e; t_E, t_0) \cong \sin^4 \theta_{13} + \cos^4 \theta_{13} P_{2VO}^{(21)}(\nu_e \rightarrow \nu_e; t_E, t_\odot),$$

(35)

with $\sin^2 \theta_{13} \leq 0.15$ for the solution under discussion. Thus, in contrast to the main fraction of $^8$B neutrinos, the $pp$ and $^7$Be $\nu_e$ do not undergo resonant MSW transitions but take part in vacuum oscillations between the Sun and the Earth. Actually, the $^7$Be neutrino energy of 0.862 MeV is in the region of the first minimum of $P_{2VO}^{(21)}$ as $E$ decreases from the “asymptotic” values at which $P_{2VO}^{(21)} \cong 1$, while the interval of energies of the $pp$ neutrinos, relevant for the current Ga–Ge and the presently discussed future solar neutrino experiments (HELLAZ, HERON), 0.22 MeV $\lesssim E \lesssim 0.41$ MeV, is in the region of the first maximum of $P_{2VO}^{(21)}$ as $E$ decreases further. The dependence of the probability $P(\nu_e \rightarrow \nu_e; t_E, t_0)$ on the neutrino energy $E$ in the case of solution A is illustrated in Figs. 3a - 3c.

For $5.0 \times 10^{-6}$ eV$^2 \leq \Delta m_{31}^2 \lesssim 1.3 \times 10^{-5}$ eV$^2$, i.e., in the minor sub-region of solution A, which overlaps with the region of the $2\nu$ nonadiabatic solution (see Fig. 1a), the 0.862 MeV $^7$Be neutrinos take part either in MSW transitions and vacuum oscillations or in MSW

\[ ^{11} \text{Obviously, if, for instance, } \Delta m_{21}^2 = 4.0 \times 10^{-12} \text{ eV}^2, \text{ the same result will be valid for the}\]

The neutrinos with $E \geq 2.5$ MeV.
transitions only. This small sub-region converges continuously to the region of the 2ν nonadiabatic solution when \( \sin^2 \theta_{12} \to 0 \) and we are not going to discuss it further. Note that the domain of solution A

in the \( \Delta m_{21}^2 - \sin^2 \theta_{12} \) plane shown in Fig. 2a corresponds to values of \( \Delta m_{31}^2 \) and \( \sin^2 \theta_{13} \) which lie outside the sub-region in question.

For solution A, the Kamiokande signal and the signal due to the pp (\(^8\)B) neutrinos in the Ga–Ge (Cl–Ar) experiments are smaller approximately by factors of \((0.33 - 0.42)\) and \((0.70 - 0.80)\) \((\sim 0.30)\), respectively, than the corresponding signals predicted in ref. \[39\]. The \(^7\)Be \(\nu_e\) flux is suppressed rather strongly – by a factor of \((0.13 - 0.14)\), with respect to the flux in the solar model of ref. \[39\].

Let us note that the solution A region in the \( \Delta m_{21}^2 - \sin^2 \theta_{12} \) plane of the vacuum oscillation parameters is quite similar to the region of the “low” \(^8\)B \(\nu_e\) flux 2ν vacuum oscillation solution found in ref. \[11\]. The latter is possible for values of the \(^8\)B neutrino flux which are lower by a factor of 0.35 to 0.43 (of 0.30 to 0.37) than the flux predicted in \[11\] (in \[39\] and used in the present study). Note, however, that the \(\chi^2_{\text{min}}\) for the indicated purely vacuum oscillation solution, \(\chi^2_{\text{min}} = 4.4\) \((2 \text{ d.f.})\) \[11\], is considerably larger than the value of \(\chi^2_{\text{min}}\) for solution A. The two solutions differ drastically in the way the \(^8\)B neutrino flux is affected by the transitions and/or the oscillations.

The implications of solution A for the future solar neutrino experiments aimed at detection and studies of the \(^7\)Be and/or \(pp\) neutrinos like BOREXINO, HELLAZ or HERON, are practically the same as those discussed in detail in ref. \[11\] for the case of the corresponding purely vacuum oscillation “low” \(^8\)B neutrino flux solution. However, the implications for the Super-Kamiokande, SNO and ICARUS experiments which are sensitive only to \(^8\)B neutrinos are completely different. They will be discussed elsewhere. Let us mention only here that in the case of solution A: i) the spectrum of \(^8\)B neutrinos will be strongly deformed (see Figs. 3a - 3c), ii) the magnitude of the day-night asymmetry in the signals in the indicated detectors can be very different from that predicted in the case of the 2ν MSW solution (see, e.g., \[40\]), and iii) the seasonal variation of the \(^8\)B \(\nu_e\) flux \[17,13\] practically coincides with the standard (geometrical) one of 6.68%. 
**Solution B.** For solution B (see Figs. 1a, 2a and 2b) the allowed region in the $\Delta m^2_{31} - \sin^2 2\theta_{13}$ plane of the MSW transition parameters,

$$5.1 \times 10^{-6} \ eV^2 \lesssim \Delta m^2_{31} \lesssim 1.2 \times 10^{-5} \ eV^2,$$

$$3.2 \times 10^{-3} \lesssim \sin^2 2\theta_{13} \lesssim 6.6 \times 10^{-3},$$

looks approximately like the region of the $2\nu$ MSW nonadiabatic solution shifted as a whole to somewhat larger values of $\Delta m^2_{31}$ (by a factor $\sim 1.3$) and to smaller values of $\sin^2 2\theta_{13}$ (on average by a factor $\sim 1.6$). The two regions have a small common sub-region (see Fig. 1a) and our further discussion does not extend to this sub-region of overlapping. Solution B can be regarded as an “improved” MSW transitions + vacuum oscillations version of the purely $2\nu$ MSW nonadiabatic solution.

In the $\Delta m^2_{21} - \sin^2 2\theta_{12}$ plane the region of solution B extends in $\Delta m^2_{21}$ from $1.8 \times 10^{-11} eV^2$ at least up to $10^{-9} eV^2$ (we did not perform a search for allowed values of $\Delta m^2_{21}$ beyond $10^{-9} eV^2$) and from 0.15 to practically 1.0 in $\sin^2 2\theta_{12}$:

$$0.15 \lesssim \sin^2 2\theta_{12} \lesssim 0.98.$$  

(37)

For $\Delta m^2_{21}$ from the domain $\sim (0.5 - 1.0) \times 10^{-10} eV^2$ of the $2\nu$ vacuum oscillation solution, however, solution B takes place for values of $\sin^2 2\theta_{12}$ which are systematically smaller than the values of the same parameter in the $2\nu$ vacuum oscillation solution. As is seen in Fig. 2b, in the $\Delta m^2_{21} - \sin^2 2\theta_{12}$ plane there are two marginally disconnected regions, which we will call “down” and “up”: the separation “line” is approximately at $\Delta m^2_{21} \approx 1.5 \times 10^{-10} eV^2$. Except in the region where $\Delta m^2_{21} \approx (1.0 - 2.0) \times 10^{-10} eV^2$, the vacuum oscillation mixing parameter is relatively small for solution B: one has $\sin^2 2\theta_{12} \approx 0.8$, and for most of the allowed values of $\Delta m^2_{21}$ actually $\sin^2 2\theta_{12} \approx 0.7$.

The minimum $\chi^2$ value for the solution under discussion is even smaller than for solution A: we have $\chi^2_{\text{min}} \approx 0.025$ at $(\Delta m^2_{21}, \sin^2 2\theta_{12}, \Delta m^2_{31}, \sin^2 2\theta_{13}) \approx (1.4 \times 10^{-10} eV^2, 0.43,$
9.9 × 10⁻⁶ eV², 4.0 × 10⁻³) (the black triangle-up in Figs. 1a and 2a) [13].

In the “up” region in the $\Delta m_{21}^{2} - \sin^{2} 2\theta_{12}$ plane one has $\chi^{2}_{\text{min}} \cong 0.03$ and this minimum is located at $(\Delta m_{21}^{2}, \sin^{2} 2\theta_{12}, \Delta m_{31}^{2}, \sin^{2} 2\theta_{13}) \cong (5.8 \times 10^{-10} \text{ eV}^{2}, 0.43, 9.9 \times 10^{-6} \text{ eV}^{2}, 3.8 \times 10^{-3})$.

For $E \leq 0.41 \text{ MeV}$ and values of $\Delta m_{31}^{2}$ and $\sin^{2} 2\theta_{13}$ from the solution intervals (36a) and (36b), one has $\tilde{P}_{2\text{MSW}}^{(31)} \cong 1 - 1/2 \sin^{2} 2\theta_{13} \cong 1$, and consequently we obtain from eq. (9): $\tilde{P}(\nu_{e} \rightarrow \nu_{e}; t_{E}, t_{0}) \cong P^{(21)}_{2\text{V/O}}(\nu_{e} \rightarrow \nu_{e}; t_{E}, t_{\odot})$. This implies that the effect of the MSW transitions for the $pp$ neutrinos is negligible and they effectively take part in vacuum oscillations only. Moreover, for $E \leq 0.41 \text{ MeV}$ and $\Delta m_{21}^{2} \cong 3 \times 10^{-11} \text{ eV}^{2}$ we find $L_{21}^{v} \cong 3.4 \times 10^{7} \text{ km}$ and $2\pi R/L_{21}^{v} \cong 27$. Correspondingly, for $E \leq 0.41 \text{ MeV}$ the term $\cos(2\pi R/L_{21}^{v})$ in $P^{(21)}_{2\text{V/O}}$ is a fastly oscillating function of $R/p$, which is averaged practically to zero, e.g., by the integration over the $pp$ neutrino energy, or by averaging over an uncertainty $\Delta E$ in the measured value of the energy $E$, $\Delta E/E \gg 4 \times 10^{-2}$. Therefore for $\Delta m_{21}^{2} \cong 3 \times 10^{-11} \text{ eV}^{2}$ the $pp$ neutrino contribution to the event rate of the Ga–Ge detectors, for instance, is suppressed by the energy-independent factor $\tilde{P}(\nu_{e} \rightarrow \nu_{e}; t_{E}, t_{0}) \cong \tilde{P}_{2\text{V/O}}^{(21)} \cong 1 - 1/2 \sin^{2} 2\theta_{12}$.

The 0.862 MeV $^7\text{Be}$ neutrinos take part in adiabatic MSW transitions in the Sun, while the $^8\text{B}$ neutrinos with $E \gtrsim 4 \text{ MeV}$ undergo nonadiabatic transitions. Both the $^7\text{Be}$ and $^8\text{B}$ neutrinos, as well as the $\nu_{\mu}$ and/or $\nu_{\tau}$ into which a fraction of the $\nu_{e}$ has been converted by the MSW effect in the Sun, participate in vacuum oscillations after leaving the Sun. These oscillations are modulated by the MSW probability $\tilde{P}_{2\text{MSW}}^{(31)}$ (see Figs. 4a - 4b). Let us note that for $\Delta m_{21}^{2} \cong 3.0 \times 10^{-10} \text{ eV}^{2}$ the effects of the averaging of $\tilde{P}(\nu_{e} \rightarrow \nu_{e}; t_{E}, t_{0})$ over the time period of 1 year is negligible for the 0.862 MeV $^7\text{Be}$ neutrinos; this effect becomes important for the $^7\text{Be} \nu_{e}$ for $\Delta m_{21}^{2} \cong 5.0 \times 10^{-10} \text{ eV}^{2}$, as Fig. 4b illustrates. For the range of $\Delta m_{21}^{2}$ of interest, $\Delta m_{21}^{2} \cong (5.0 - 6.0) \times 10^{-10} \text{ eV}^{2}$, the time averaging does not change the

\footnotesize
\begin{itemize}
  \item It is very close to the lowest $\chi^2$ value possible, $\chi^2_{\text{min}} = 0.024$, with the set of the data \cite{1-4} we use in the present analysis.
\end{itemize}

\normalsize
probability $P(\nu_e \rightarrow \nu_e; t_E, t_0)$ for solar neutrino energies $E \gtrsim 3.0$ MeV, i.e., for the dominant fraction of the $^8$B neutrinos.

With respect to the predictions in ref. [39], the signal in the Kamiokande detector and the contribution of the $^8$B neutrinos to the signals in the Cl–Ar detector are smaller typically by factors of $\sim (0.43 - 0.47)$ and $\sim (0.32 - 0.36)$. The $pp$ and the 0.862 MeV $^7$Be $\nu_e$ fluxes are suppressed by factors of $\sim (0.65 - 0.90)$ and $\sim (0.11 - 0.27)$ for most of the values of the parameters from the allowed region. However, for $\sin^2 2\theta_{12} \sim 0.9$, for instance, one has $P(\nu_e \rightarrow \nu_e; t_E, t_0) \approx 0.55$ for the $pp$ neutrinos. Even in this case the 0.862 MeV $^7$Be $\nu_e$ flux is reduced by a factor of $\sim 0.3$, but the indicated possibility is rather marginal.

There are rather large and distinctive distortions of the $^8$B $\nu_e$ spectrum (see Figs. 4a and 4b) in the case of solution B. The seasonal variations due to the vacuum oscillations of the signals in the Super-Kamiokande, SNO and ICARUS detectors are estimated to be smaller than the variations in the case of the $2\nu$ vacuum oscillation solution [3], except possibly in the small region of the $\Delta m_{21}^2 - \sin^2 2\theta_{12}$ plane where $\Delta m_{21}^2 \gtrsim 10^{-10}$ eV$^2$ and $\sin^2 2\theta_{12} \gtrsim 0.7$. The range of the predicted values of the day-night asymmetry in these detectors is different from the one expected for the $2\nu$ MSW solution. The seasonal variation of the 0.862 MeV $^7$Be $\nu_e$ flux caused by the vacuum oscillations is expected to be considerably smaller than in the $2\nu$ case, while the day-night asymmetry is estimated to be somewhat smaller than the one predicted for the $2\nu$ MSW nonadiabatic solution. The seasonal variation, nevertheless, may be observable. Obviously, the experimental detection both of a deviation from the standard (geometrical) 6.68% seasonal variation of the solar neutrino flux and of a nonzero day-night effect will be a proof that solar neutrinos take part in MSW transitions and vacuum oscillations.

**Solution C.** The values of the parameters corresponding to solution C (see Figs. 1a and

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These variations were shown [13] to be not larger than 15% for the $2\nu$ vacuum oscillation solution.
2a) form a rather large region in the $\Delta m^2_{31} - \sin^2 2\theta_{13}$ plane,

$$1.1 \times 10^{-5} \text{eV}^2 \lesssim \Delta m^2_{13} \lesssim 1.2 \times 10^{-4} \text{eV}^2,$$  
(38a)

$$1.7 \times 10^{-4} \lesssim \sin^2 2\theta_{13} \lesssim 2.0 \times 10^{-3},$$  
(38b)

and a relatively small one in the $\Delta m^2_{21} - \sin^2 2\theta_{12}$ plane,

$$1.1 \times 10^{-10} \text{eV}^2 \lesssim \Delta m^2_{21} \lesssim 1.3 \times 10^{-10} \text{eV}^2,$$  
(39a)

$$0.7 \lesssim \sin^2 2\theta_{12} \leq 1.0.$$  
(39b)

The $\chi^2_{min}$ for this solution is larger than for solutions A and B: $\chi^2_{min} \approx 1.49$ at $(\Delta m^2_{21}, \sin^2 2\theta_{12}, \Delta m^2_{31}, \sin^2 2\theta_{13}) \approx (1.2 \times 10^{-10} \text{eV}^2, 0.78, 4.6 \times 10^{-5} \text{eV}^2, 5.9 \times 10^{-4})$ (the black square in Figs. 1a and 2a).

For the values (38a) and (38b) of $\Delta m^2_{31}$ and $\sin^2 2\theta_{13}$ and the energies of the $pp$ neutrinos of interest, we have $\bar{P}^{(31)}_{2V} \approx 1 - 1/2 \sin^2 2\theta_{13} \approx 1$ and the $pp$ neutrino flux is effectively suppressed by the energy-independent factor $\bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0) \approx \bar{P}^{(21)}_{2V} \approx 1 - 1/2 \sin^2 2\theta_{12}$ (Figs. 5a and 5b). The energy of the dominant fraction of $^7$Be neutrinos, 0.862 MeV, is for this solution in the region of energies for which the effect of the time-averaging of the probability $\bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0)$ is important. This effect is negligible for $E \gtrsim 2$ MeV, i.e., for most of the $^8$B neutrinos. The 0.862 MeV $^7$Be $\nu_e$ flux is suppressed either due to the vacuum oscillations only (Fig. 5b) or due to the combined effect of the vacuum oscillations and MSW transitions (Fig. 5a). The suppression of the $^8$B $\nu_e$ flux is a result of the interplay of the vacuum oscillations and the MSW transitions (Figs. 5a and 5b). As a consequence, i) the $^8$B $\nu_e$ spectrum is strongly deformed and ii) one expects observably large seasonal variations due to the vacuum oscillations of the $^8$B and/or, for $\Delta m^2_{31} \approx 3 \times 10^{-5} \text{eV}^2$, of the $^7$Be $\nu_e$ fluxes. The day-night asymmetry in the signals of the Super-Kamiokande, SNO and ICARUS experiments is not expected to be larger than a few percent. In the case of the $^7$Be $\nu_e$ flux the asymmetry is either absent (if $\Delta m^2_{31} \approx 3 \times 10^{-5} \text{eV}^2$) or does not exceed a few percent.
In comparison with the predictions of ref. [39], the Kamiokande signal and the signals produced by the $^8$B neutrinos in the Cl–Ar detector are reduced respectively by factors of $\sim (0.35 - 0.41)$ and $\sim (0.24 - 0.32)$, while the ranges of suppression of the $pp$ and of the 0.862 MeV $^7$Be $\nu_e$ fluxes read respectively 0.50 - 0.65 and 0.17 - 0.30.

**Solution D.** In the $\Delta m^2_{31} - \sin^2 2\theta_{13}$ plane the region of solution D (see Figs. 1b and 2a) can formally be obtained from the one of solution A by moving the latter as a whole to larger values of $\Delta m^2_{31}$ and smaller values of $\sin^2 2\theta_{13}$ and by truncating two relatively small sub-regions located at $\sin^2 2\theta_{13} \geq 0.2$ and at $\Delta m^2_{31} \leq 1.1 \times 10^{-5} eV^2$. In the $\Delta m^2_{21} - \sin^2 2\theta_{12}$ plane the solution D region is determined by

$$3.2 \times 10^{-11} eV^2 \lesssim \Delta m^2_{21} \lesssim 4.0 \times 10^{-11} eV^2, \quad (40a)$$

$$0.62 \lesssim \sin^2 2\theta_{12} \leq 1.0. \quad (40b)$$

The minimum $\chi^2$ value for this solution is similar to that for solution C, $\chi^2_{\min} \approx 1.35$; it is located at $(\Delta m^2_{21}, \sin^2 2\theta_{12}, \Delta m^2_{31}, \sin^2 2\theta_{13}) \approx (3.4 \times 10^{-11} eV^2, 0.80, 1.5 \times 10^{-5} eV^2, 2.5 \times 10^{-3})$ (the black diamond in Figs. 1b and 2a).

For the values of $\Delta m^2_{21}$ given in eq. (40a) and $0.22 \ MeV \leq E \leq 0.41 \ MeV$, the effect of the time averaging of the probability $\bar{P}(\nu_e \to \nu_e; t_E, t_0)$ cannot be neglected; for $E \gtrsim 0.7 \ MeV$ it becomes inessential. The $\nu_e$ survival probability for the 0.862 MeV $^7$Be neutrinos is given by expression (35). The same expression is approximately valid for the $pp$ neutrinos (see Figs. 6a - 6c). However, the $pp$ neutrino contribution to the signals in the Ga-Ge detectors is suppressed by the factor given in eq. (21). The $^8$B $\nu_e$ flux is suppressed due to the combined effect of the MSW transitions and vacuum oscillations (Figs. 6a - 6c).

With respect to the predictions in ref. [39], the contributions of the $pp$ and $^7$Be ($^8$B) neutrinos in the signals of the Ga–Ge (Cl–Ar) detectors are smaller by factors of $\sim (0.50 - 0.70)$ and $\sim (0.31 - 0.49)$ ($\sim (0.22 - 0.28)$), while the signal in the Kamiokande detector is reduced by a factor of $\sim (0.32 - 0.40)$.

As in the case of solutions B and C, the $^8$B $\nu_e$ spectrum is strongly deformed by the combined effect of the MSW transitions and the vacuum oscillations. The seasonal variations
of the $pp$ and of the 0.862 MeV $^7$Be $\nu_e$ fluxes generated by the vacuum oscillations are estimated to be greater than $\sim 10\%$; they are expected to be smaller than a few percent for the $^8$B $\nu_e$ having energy $E \geq 5$ MeV. The day-night asymmetry in the signals of the Super-Kamiokande, SNO and ICARUS detectors may be observable only for values of $\sin^2 2\theta_{13} \gtrsim 10^{-3}$; even in this region it is not expected to be bigger than a few percent. The day-night effect for the 0.862 MeV $^7$Be $\nu_e$ flux is negligible.

Solution D has as a $\theta_{13} \to 0$ limit the second “low” $^8$B $\nu_e$ flux $2\nu$ vacuum oscillation solution discussed in ref. [11]: the regions of values of the vacuum oscillation parameters of the two solutions practically coincide. This $2\nu$ solution was found to be possible for values of the initial $^8$B $\nu_e$ flux which are by a factor $\sim (0.45 - 0.65)$ ($\sim (0.39 - 0.56)$) smaller than the flux predicted in ref. [11] (in ref. [38]). The value of $\chi^2_{\text{min}}$ for the indicated purely $2\nu$ vacuum oscillation solution, $\chi^2_{\text{min}} \cong 5.0$ (2 d.f.) [11], is considerably larger than $\chi^2_{\text{min}}$ for the solution D. The MSW + VO effects and correspondingly the purely VO effects on the $^7$Be and/or $^8$B neutrino fluxes in the cases of the two solutions are also very different.

**Solution E.** This solution holds for relatively large values of $\sin^2 2\theta_{12}$ and $\sin^2 2\theta_{13}$,

\[
0.70 \lesssim \sin^2 2\theta_{12} \lesssim 0.80, \quad (41a)
\]

\[
0.14 \lesssim \sin^2 2\theta_{13} \lesssim 0.39, \quad (41b)
\]

and values of $\Delta m^2_{21}$ forming a tiny region around the point $\Delta m^2_{21} \cong 9.4 \times 10^{-10}$ eV$^2$, with the values of $\Delta m^2_{31}$ belonging to the narrow interval $1.2 \times 10^{-4}$ eV$^2 \lesssim \Delta m^2_{31} \lesssim 1.4 \times 10^{-4}$ eV$^2$ (Figs. 1b and 2b). The $\chi^2_{\text{min}}$ value is rather large, $\chi^2_{\text{min}} \cong 3.3$, and we shall not discuss this solution in greater detail. Let us mention only that the $pp$ and $^7$Be $\nu_e$ fluxes are suppressed by one and the same energy-independent factor given in eq. (21) - the average $\nu_e$ survival probability in the case of $3\nu$ vacuum oscillations, while both MSW transitions and vacuum oscillations contribute to the suppression of the $^8$B $\nu_e$ flux. This is illustrated in Fig. 7.

Finally, we would like to point out that, as Fig. 8 indicates, the MSW transitions + vacuum oscillations may provide an alternative to the purely MSW mechanism of suppression.
of the solar $\nu_e$ flux in the case discussed in ref. [12].

4. Conclusions.

Assuming three flavour neutrino mixing takes place in vacuum, we have investigated the possibility that the observed depletion of the solar $\nu_e$ flux is caused by an interplay of MSW transitions of the solar $\nu_e$ in the Sun followed by long wave length neutrino oscillations in vacuum on the way from the Sun to the Earth. Choosing the neutrino mass squared difference $\Delta m^2_{31} > 0$ to be responsible for the MSW transitions in the Sun, the long wave length vacuum oscillations can be due either to $\Delta m^2_{21} > 0$ (inequality (4) is valid) or to $\Delta m^2_{32} > 0$ (5 holds). In both cases the solar $\nu_e$ survival probability is described, as we have shown, by simple analytic expressions, eq. (9) and eq. (23a). However, the two cases were found to be very different in their physics implications. In particular, if (5) is realized, the vacuum oscillations are not suppressed, in principle, only for large values of the MSW mixing angle parameter. The case of (4) appeared to provide a much richer spectrum of possible new solutions of the solar neutrino problem of the hybrid MSW transitions + vacuum oscillations type and we have analyzed only this case. We have found for the values of $\Delta m^2_{21}$ and $\Delta m^2_{31}$ within the ranges given in (7) and (8) several versions of this solution, denoted by A, B, C, D, and E in the text, most of which are distinctly different from the two-neutrino mixing purely vacuum oscillation or MSW solution. In particular, the values of the MSW parameters $\Delta m^2_{31}$ and $\sin^2 2\theta_{13}$ of the different solutions found lie outside the regions of the $2\nu$ MSW solutions (including the regions obtained by varying the $^8$B and $^7$Be neutrino fluxes) [13-16], except in two cases (solutions A and B) in which very small sub-regions are located within the region of the $2\nu$ MSW nonadiabatic solution. Two of the solutions studied, A and D, are similar in what regards the effective mechanism and the magnitude of suppression of the $pp$ flux to the two “low” $^8$B $\nu_e$ flux $2\nu$ vacuum oscillation solutions discussed in [14]. However, the $\chi^2_{\text{min}}$ values for the former are much smaller than for the latter. In addition, the MSW + VO solutions A and D and the two corresponding purely vacuum oscillation solutions [11] imply very different mechanisms of suppression of the $^7$Be and/or the $^8$B neutrino fluxes.
A general feature of the MSW + VO solutions studied by us is that the \( pp \ \nu_e \) flux is suppressed (albeit not strongly - by a factor not smaller than 0.5) primarily due to the vacuum oscillations of the \( \nu_e \), the suppression of the 0.862 MeV \(^7\text{Be} \ \nu_e \) flux is caused either by the vacuum oscillations or by the combined effect of the MSW transitions and the vacuum oscillations, while the \(^8\text{B} \ \nu_e \) flux is suppressed either due to the MSW transitions only or by the interplay of the MSW transitions in the Sun and the oscillations in vacuum on the way to the Earth. The solutions differ in the way the \( pp, \ ^7\text{Be} \) and the \(^8\text{B} \) neutrinos are affected by the \( \nu_e \) MSW transitions and/or the oscillations in vacuum.

Searching for MSW + VO solutions we did not scan the entire region of the parameter space defined by eqs. (7), (8), (32) and (33). In particular, solutions which correspond to a suppression (by a factor \( \gtrsim 0.5 \)) of the \( pp \) neutrino flux due to the MSW transitions, concomitant with the requisite suppression of the \(^7\text{Be} \) and of the \(^8\text{B} \) neutrino fluxes due to the MSW effect and/or the vacuum oscillations and respectively the vacuum oscillations alone, are possible for \( \Delta m^2_{31} \sim (10^{-8} - 10^{-7}) \text{ eV}^2 \) and \( \Delta m^2_{21} \sim (0.3 - 1.0) \times 10^{-10} \text{ eV}^2 \).

For all MSW + VO solutions we have considered, the \(^8\text{B} \ \nu_e \) spectrum is predicted to be rather strongly deformed. The \(^8\text{B} \) neutrinos undergo nonadiabatic transitions for values of the MSW parameters \( \Delta m^2_{31} \approx (0.1 - 1.2) \times 10^{-4} \text{ eV}^2 \) and \( \sin^2 2\theta_{13} \approx (0.2 - 3.0) \times 10^{-3} \) which are respectively larger and smaller than the values corresponding to the \( 2\nu \) MSW nonadiabatic solution. The \(^8\text{B} \ \nu_e \) adiabatic transitions take place for \( \Delta m^2_{31} \approx (1.1 - 1.5) \times 10^{-4} \text{ eV}^2 \) and \( \sin^2 2\theta_{13} \approx (3 \times 10^{-3} - 0.5) \). Clearly, the Kamiokande data on the shape of the \(^8\text{B} \) neutrino spectrum can be used to further constrain the solutions we have found. Such an analysis, however, lies outside the scope of the present study.

For the MSW + VO solutions considered by us the day-night asymmetry in the signals of the detectors sensitive only to \(^8\text{B} \) or \(^7\text{Be} \) neutrinos are estimated to be rather small, not exceeding a few percent. The seasonal variation effect caused by the vacuum oscillations can be observable for \(^7\text{Be} \) neutrinos and, for certain relatively small regions of the allowed values of the parameters, can also be observable for the \(^8\text{B} \) or for the \( pp \) neutrinos if the \( pp \) neutrino flux is measured with detectors like HELLAZ or HERON.
Finally, the MSW transitions + vacuum oscillations can be an alternative to the purely MSW mechanism of suppression of the solar $\nu_e$ flux in the case considered in ref. [42].

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Appendix. Derivation of the Expression for the Solar $\nu_e$ Survival Probability

In this Appendix we present a derivation of expression (9) for the average probability $\bar{P}(\nu_e \to \nu_e; t_E, t_0)$, which is not based on the general analytic result for the average solar $\nu_e$ survival probability in the case of three-neutrino mixing $\bar{P}(\nu_e \to \nu_e; t, t_0)$, derived in ref. [20]. We remind the reader that the two relevant MSW transition and vacuum oscillation parameters $\Delta m_{31}^2$ and $\Delta m_{21}^2$ (or $\Delta m_{32}^2$) are assumed to satisfy inequality (4) (or (5)) and to lie in the intervals (8) and (7), respectively.

The probability amplitude that the solar $\nu_e$ will not be converted into $\nu_\mu$ and/or $\nu_\tau$ when it propagates from the central part of the Sun, where it was produced at time $t_0$, to the surface of the Earth, reached at time $t_E$, $A(\nu_e \to \nu_e; t_E, t_0)$, can be written in the following form:

$$A(\nu_e \to \nu_e; t_E, t_0) = \sum_{l=e,\mu,\tau} A_{MSW}(\nu_e \to \nu_l; t_\odot, t_0)A_{VO}(\nu_l \to \nu_e; t_E, t_\odot),$$  \hspace{1cm} (A1)

Here $t_\odot$ is the time at which the neutrino reaches the surface of the Sun, $A_{MSW}(\nu_e \to \nu_l; t_\odot, t_0)$ is the amplitude of the probability to find neutrino $\nu_l$, $l = e, \mu, \tau$, at the surface of the Sun after the $\nu_e$ underwent MSW transitions into $\nu_\mu$ and/or $\nu_\tau$ while propagating in the Sun, and $A_{VO}(\nu_l \to \nu_e; t_E, t_\odot)$ is the amplitude of the probability that $\nu_l$ will oscillate into $\nu_e$ while traveling from the surface of the Sun to the surface of the Earth. For stable and relativistic neutrinos $\nu_k$ the amplitude $A_{VO}(\nu_l \to \nu_e; t_E, t_\odot)$ is given by the well-known
expression (see, e.g., [17,33]):

\[ A_{VO}(\nu_l \rightarrow \nu_e; t_E, t_\odot) = \sum_{k=1}^{3} U_{ek} e^{-iE_k(t_E-t_\odot)} U_{lk}^* \cong e^{-iE_1 R} \sum_{k=1}^{3} U_{ek} e^{-i\frac{\Delta m^2_{1k}}{2p} R} U_{lk}^*, \quad (A2) \]

where \( R = (t_E - t_\odot) \cong R_0 = 1.4966 \times 10^8 \text{ km} \), \( R_0 \) being the mean Sun-Earth distance, and we have used \( E_k - E_1 \cong \Delta m^2_{1k}/2p \).

Consider first the case specified by inequality (4), with \( \Delta m^2_{31} \) (and \( \Delta m^2_{32} \)) having values in the interval (7). For \( p \leq 14.4 \text{ MeV} \), which is the maximal energy of the solar neutrinos detected on Earth, one has \( 2p/\Delta m^2_{31(32)} \lesssim 5.6 \times 10^4 \text{ km} \), and therefore \( (\Delta m^2_{31(32)}/2p)R_0 \lesssim 2.7 \times 10^3 \). As a consequence the phases \( (\Delta m^2_{31}/2p)R \) and \( (\Delta m^2_{32}/2p)R \) in eq. (A2) give rise to fastly oscillating (with the change of \( p \) and/or \( R \)) terms in the probability of solar \( \nu_e \) survival:

\[ P(\nu_e \rightarrow \nu_e; t_E, t_0) = |A(\nu_e \rightarrow \nu_e; t_E, t_0)|^2. \quad (A3) \]

When \( P(\nu_e \rightarrow \nu_e; t_E, t_0) \) is averaged over (see, e.g., [17,33]) the region of neutrino production in the Sun, and/or the uncertainty of the energy of the neutrino detected on Earth, and/or over the uncertainty in the position of the solar neutrino detector \([4]\) and/or is integrated over a relevant solar neutrino energy interval, the fastly oscillating terms become strongly suppressed and their contribution in the averaged probability, \( \bar{P}(\nu_e \rightarrow \nu_e) \), can be neglected. Taking this effectively into account by omitting the indicated fastly oscillating terms and using eqs. (A1) - (A3) one obtains an expression for the average probability, \( \bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0) \), which is convenient to represent as a sum of two terms:

\[ \bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0) = \bar{S}_1(t_\odot, t_0) + \bar{S}_2(t_E, t_\odot, t_0) \equiv \bar{S}_1 + \bar{S}_2, \quad (A4) \]

where

\[ \bar{S}_1 = \sum_{k=1}^{3} \sum_{l',l=e,\mu,\tau} |U_{lk}|^2 (U_{lk}^{*} A_{MSW}(\nu_l \rightarrow \nu_l; t_\odot, t_0))(U_{l'k}^{*} A_{MSW}(\nu_{l'} \rightarrow \nu_{l'}; t_\odot, t_0))^*, \quad (A5) \]
\[ S_2 = 2\text{Re} \left[ e^{-\frac{\Delta m^2_{21}}{2p}R} \sum_{l,l'=e,\mu,\tau} (U_{e2}A_{\text{MSW}}(\nu_e \rightarrow \nu_l; t_\odot, t_0)U_{l2}^*)(U_{e1}A_{\text{MSW}}(\nu_e \rightarrow \nu_l'; t_\odot, t_0)U_{l'1}^*) \right] \] (A6)

and the bar on the quantities \( S_{1,2} \) in eq. (A4) means that all fastly oscillating terms in \( S_{1,2} \) arising from the amplitudes \( A_{\text{MSW}}(\nu_e \rightarrow \nu_l; t_\odot, t_0) \), which are rendered negligible by the averaging over the region of neutrino production in the Sun, etc. should be dropped. Obviously, \( \bar{S}_1(t_\odot, t_0) \) is determined only by the solar \( \nu_e \) transitions in the Sun, while \( \bar{S}_2(t_E, t_\odot, t_0) \) contains all the dependence of \( \bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0) \) on the long wave length vacuum oscillations between the Sun and the Earth. Note, however, that \( \bar{S}_2(t_E, t_\odot, t_0) \) depends on the \( \nu_e \) transitions in the Sun as well.

The MSW transition amplitudes \( A_{\text{MSW}}(\nu_e \rightarrow \nu_l; t, t_0) \equiv A_l(t), t \leq t_\odot, l = e, \mu, \tau, \) satisfy the system of evolution equations:

\[ i \frac{d}{dt} A_l(t) = \sum_{l'=e,\mu,\tau} M_{ll'}(t) A_{l'}(t). \] (A7)

Here \( M(t) \) is the evolution matrix which can be chosen in the form (see, e.g., [19,20]):

\[ M_{ll'}(t) = \frac{1}{2p} \left[ \sum_{j=2}^3 U_{ij}\Delta m^2_{ji}U_{jl'}^* + A(t)\delta_{il}\delta_{l'l} - \delta_{ll'}\left( \sum_{k=2}^3 |U_{ek}|^2 \Delta m^2_{kl} + A(t) \right) \right], \] (A8)

where

\[ A(t) = 2p\sqrt{2}G_FN_e(t), \] (A9)

and \( N_e(t) \) is the value of the electron number density at the point of the neutrino trajectory in the Sun, reached at time \( t \). The initial conditions for the system of equations (A8) read:

\[ A_e(t_0) = 1, A_\mu(t_0) = A_\tau(t_0) = 0. \] (A10)

Since \( \Delta m^2_{21} \leq 5 \times 10^{-10} \text{ eV}^2 \) and \( \Delta m^2_{31} \geq 10^{-7} \text{ eV}^2 \), we can expect that the evolution of the neutrino system in the Sun practically does not depend on \( \Delta m^2_{21} \). At the same time the lower energy pp neutrinos, for instance, can take part in vacuum oscillations in the Sun induced by \( \Delta m^2_{21} \) [32,33,34]. Inspecting eq. (A8) for the evolution matrix \( M(t) \) one could conclude that the terms containing \( \Delta m^2_{21} \) as a factor can be neglected provided

\[ |U_{i3}\Delta m^2_{31}U_{i3}^*| \gg |U_{i2}\Delta m^2_{21}U_{i2}^*|, \quad l \neq l' = e, \mu, \tau, \text{ and } ||U_{i3}|^2 - |U_{i3}|^2\Delta m^2_{31}|| \gg ||U_{i2}|^2 - \]
eliminated from $M$. The evolution matrix $\bar{e}, \bar{\mu}, \bar{\tau}$ we find that the amplitudes ($A$ having the same form as the system of equations (A7) with an evolution matrix $M$ performed in ref. [20] one can convince oneself that for $\Delta m_{21}^2$ and $\Delta m_{31}^2$ having values in the intervals (7) and (8) the effects of $\Delta m_{21}^2$ on the solar neutrino transitions/oscillations in the Sun are either effectively accounted for by the term $\bar{S}_2(t_E, t_\odot, t_0)$ or are negligible (even for the pp neutrinos). This implies that one can formally set $\Delta m_{21}^2$ to zero in the expression for $M(t)$, eq. (A8). In this case the evolution matrix $M(t)$ depends only on the elements $U_{i3}$, $l = e, \mu, \tau$, forming the third column of the matrix $U$.

The phases of the elements $U_{i3}$, $U_{i3} = e^{i\alpha_{i3}}|U_{i3}|$, where $\alpha_{i3}$ are real constants, can be eliminated from $M(t)$ by a redefinition of the amplitudes $A_l(t)$: $A_l(t) \rightarrow e^{-i\alpha_{i3}}A_l(t)$, $l = e, \mu, \tau$. The evolution matrix $\bar{M}(t)$ of the system of evolution equations for the amplitudes $e^{-i\alpha_{i3}}A_l(t)$ can formally be obtained from $M(t)$ by replacing $U_{i3}$ by $|U_{i3}|$ in the expression (A8) for $M(t)$. As we shall show, the average solar $\nu_e$ survival probability $\bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0)$ does not depend in the case of interest on the phases of the elements of the lepton mixing matrix $U$, and, in particular, on $\alpha_{i3}$.

Performing the transformation

$$
\begin{pmatrix}
    e^{-i\alpha_{e3}}A_e(t) \\
    e^{-i\alpha_{\mu3}}A_\mu(t) \\
    e^{-i\alpha_{\tau3}}A_\tau(t)
\end{pmatrix} = \begin{pmatrix}
    1 & 0 & 0 \\
    0 & \frac{|U_{e3}|}{\sqrt{|U_{e3}|^2 + |U_{\mu3}|^2}} & \frac{|U_{e3}|}{\sqrt{|U_{e3}|^2 + |U_{\mu3}|^2}} \\
    0 & \frac{|U_{e3}|}{\sqrt{|U_{e3}|^2 + |U_{\mu3}|^2}} & \frac{|U_{e3}|}{\sqrt{|U_{e3}|^2 + |U_{\mu3}|^2}}
\end{pmatrix} \begin{pmatrix}
    e^{-i\alpha_{e3}}A_e(t) \\
    A'_\mu(t) \\
    e^{-i\alpha_{e3}}A'_\tau(t)
\end{pmatrix},
$$

(A11)

we find that the amplitudes $(A_e(t), A'_\mu(t), A'_\tau(t))$ satisfy a system of evolution equations having the same form as the system of equations (A7) with an evolution matrix $M'(t)$ given by the expression:

$$
M'(t) = \frac{1}{2p} \begin{pmatrix}
    0 & 0 & |U_{e3}|\sqrt{1 - |U_{e3}|^2} \Delta m_{31}^2 \\
    0 & -|U_{e3}|^2 \Delta m_{31}^2 - A(t) & 0 \\
    |U_{e3}|\sqrt{1 - |U_{e3}|^2} \Delta m_{31}^2 & 0 & (1 - 2|U_{e3}|^2) \Delta m_{31}^2 - A(t)
\end{pmatrix}
$$

(A12)

In obtaining eq. (A12) we have used the unitarity of the lepton mixing matrix $U$: $|U_{e3}|^2 + |U_{\mu3}|^2 + |U_{\tau3}|^2 = 1$. The initial conditions for the new system of equations follow from eqs.
(A10) and (A11):
\[ A_e(t_0) = 1, A'_\mu(t_0) = A'_\nu(t_0) = 0. \]  

(A13)

It follows from eq. (A12) that the evolution of the amplitude \( A'_\mu(t) \) is decoupled from the evolution of \( A_e(t) \) and \( A'_\nu(t) \), while \( A_e(t) \) and \( A'_\nu(t) \) satisfy a two-neutrino mixing system of evolution equations describing two-neutrino MSW transitions of the solar \( \nu_e \) in the Sun. The transitions can be enhanced by the solar matter effects, as it follows from (A12), provided \( |U_{e3}|^2 < 0.5 \). Explicit expressions for the amplitudes \( A_e(t) \) and \( A'_\nu(t) \) have been obtained, in particular, assuming \( N_e(t) \) decreases exponentially along the neutrino path in the Sun [32,35,38]. In this case the system of evolution equations for \( A_e(t) \) and \( A'_\nu(t) \) can be solved exactly [32,43]. Let us note that the exponential approximation describes with a good precision the change of \( N_e(t) \) along the neutrino path, predicted by the solar models [5,39].

As it is easy to see, eqs. (A11) - (A13) (see also (A8)) imply:
\[ A'_\mu(t) = 0, \]
\[ A_\mu(t) = \frac{U_{\mu3}}{\sqrt{|U_{\mu3}|^2 + |U_{\tau3}|^2}} e^{-i\alpha_{e3}} A'_\nu(t), \]
\[ A_\tau(t) = \frac{U_{\tau3}}{\sqrt{|U_{\tau3}|^2 + |U_{\mu3}|^2}} e^{-i\alpha_{e3}} A'_\nu(t). \]

(A14a)

(A14b)

(A14c)

Evidently, one has:
\[ |A_e(t)|^2 + |A_\mu(t)|^2 + |A_\tau(t)|^2 = |A_e(t)|^2 + |A'_\nu(t)|^2 = 1. \]

(A15)

We shall express next the probability \( \bar{P}(\nu_e \rightarrow \nu_e; t_E, t_0) \) in terms of the two amplitudes \( A_e(t) \) and \( A'_\nu(t) \) which contain the whole information about the MSW transitions of neutrinos in the Sun. Inserting expressions (A14b) and (A14c) for \( A_\mu(t) \) and \( A_\tau(t) \) in equations (A5) and (A6) and using the unitarity of the lepton mixing matrix \( U \) we obtain very simple expressions for the quantities \( S_1 \) and \( S_2 \):
\[ S_1(t_\odot, t_0) = [(1 - |U_{e3}|^2)^2 - 2|U_{e1}|^2|U_{e2}|^2] |A_e(t_\odot) - \frac{|U_{e3}|}{\sqrt{1 - |U_{e3}|^2}} A'_\nu(t_\odot)|^2 \]
\begin{align}
&+ |U_{e3}|^2 |A_e(t)U_{e3}| + \sqrt{1 - |U_{e3}|^2} \left| A'_{\nu}(t) \right|^2, \quad (A16) \\
S_2(t_E, t_\odot, t_0) = 2|U_{e1}|^2|U_{e2}|^2 |A_e(t)_{\odot} - \frac{|U_{e3}|}{\sqrt{1 - |U_{e3}|^2}} A'_{\nu}(t)_{\odot}|^2 \cos 2\pi \frac{R}{L_{\odot}}, \quad (A17)
\end{align}

Let us denote
\[ |A_e(t)|^2 = 1 - |A'_{\nu}(t)|^2 \equiv P_{2MSW}^{(31)}(\nu_e \rightarrow \nu_e; t_\odot, t_0). \quad (A18) \]

Explicit expression for the probability \( P_{2MSW}^{(31)}(\nu_e \rightarrow \nu_e; t_\odot, t_0) \) can be found in ref. [33]. However, as we have already emphasized, the oscillating terms in \( P_{2MSW}^{(31)}(\nu_e \rightarrow \nu_e; t_\odot, t_0) \) give a negligible contribution in the averaged solar \( \nu_e \) survival probability \[30\]. Therefore only the average term \( \bar{P}_{2MSW}^{(31)}(\nu_e \rightarrow \nu_e; t_\odot, t_0) \) in \( P_{2MSW}^{(31)}(\nu_e \rightarrow \nu_e; t_\odot, t_0) \) should be taken into account when one obtains \( \bar{S}_1(t_\odot, t_0) \) and \( \bar{S}_2(t_E, t_\odot, t_0) \) from eqs. (A16) - (A18).

The last quantity to be computed is the interference term \( \text{Re}[A_e^*(t_\odot)A'_{\nu}(t_\odot)] \) in eqs. (A16) and (A17). This term coincides with the quantity \( \text{Re}[R_H(t'_0, t_0)] \) in eq. (29) in ref. [20] when \( t'_0 = t_\odot \). An explicit analytic expression for \( \text{Re}(A_e^*(t_\odot)A'_{\nu}(t_\odot)) \) can be found exploiting, e.g., the exact analytic results for \( A_e(t_\odot) \) and \( A'_{\nu}(t_\odot) \) derived in ref. [32] (see eqs. (28) and (29)) and cast in a form which is more convenient to use in calculations like the present one in ref. [35,38]. It follows from eqs. (11), (12) and (18) - (20) in ref. [38], for instance, that
\[ \text{Re}[A_e^*(t_\odot)A'_{\nu}(t_\odot)] = - \left( \frac{1}{2} - P'(31) \right) \sin 2\theta_{13} \cos 2\theta^m_{13}(t_0) + Oscillating \ terms, \quad (A19) \]
where the angles \( \theta_{13}, \theta^m_{13}(t_0) \) and the “jump” probability \( P'(31) \) are defined in eqs. (12), (13) and (15). The oscillating terms appearing in eq. (A19) do not contribute in the averaged solar \( \nu_e \) survival probability \[15\]. Let us note that, as can be shown [14], the result eq. (A19) for \( \text{Re}[A_e^*(t_\odot)A'_{\nu}(t_\odot)] \) is general: the fact that we have used the specific (but exact) exponential density solutions for \( A_e(t_\odot) \) and \( A'_{\nu}(t_\odot) \) to derive it is reflected only in the specific form of the “jump” probability \( P'(31) \) (eq. (15)) one gets in eq. (A19).

---

\[15\] Explicit analytic expression for the oscillating terms in eq. (A19) can be obtained from the formulae given in refs. [32,35,38].
Taking into account the remarks we have made concerning the oscillating terms in eqs. (A18) and (A19) and utilizing eqs. (A16) - (A19) and (10) - (14) it is not difficult to get eq. (9) from eq. (A4).

If the Zee model \textsuperscript{20} type relation (5) between $\Delta m_{21}^2$, $\Delta m_{31}^2$ and $\Delta m_{32}^2$ is valid, we have to repeat the above analysis interchanging the indices 1 and 3 in the quantities $U_{ij}$ and in $\Delta m_{ki}^2$ entering into eqs. (A6), (A8), (A11), (A12), (A14b), (A14c), (A16) - (A18). In particular, the evolution matrix $M'(t)$ in the system of evolution equations for the amplitudes $(A_e(t), A'_\mu(t), A'_\tau(t))$ now has the form:

$$M'(t) = \frac{1}{2p} \begin{bmatrix}
0 & 0 & |U_{e1}|\sqrt{1-|U_{e1}|^2} \Delta m_{13}^2 \\
0 & -|U_{e1}|^2 \Delta m_{13}^2 - A(t) & 0 \\
|U_{e1}|\sqrt{1-|U_{e1}|^2} \Delta m_{13}^2 & 0 & (1-2|U_{e1}|^2) \Delta m_{13}^2 - A(t)
\end{bmatrix},$$

(A20)

where $\Delta m_{13}^2 < 0$. The new form of $M'(t)$ and eqs. (A16) and (A17) in which $|U_{e3(1)}|^2$ and $L_{21}^e$ are now replaced respectively by $|U_{e3(1)}|^2$ and $L_{23}^e$, entail the introduction of the new angles $\theta_{23}$, $\theta'_{13}$ and $\theta''_{13}(t_0)$ defined by eqs. (24) - (26), as well as of the “jump” probability $P'_{(13)}$. It also leads to the requirement $|U_{e1}|^2 > 0.5$ which is a necessary condition for the MSW resonance. Comparing the evolution matrices (A12) and (A20) it is easy to deduce the form of $P'_{(13)}$ given the expression for $P'_{(31)}$, eq. (15). The interference term Re$[A_e^*(t_\odot)A'_{\nu_e}(t_\odot)]$ is given by eq. (A19) in which $\theta_{13}$, $\theta''_{13}(t_0)$ and $P'_{(31)}$ are replaced by $\theta'_{13}$, $\theta''_{13}(t_0)$ and $P'_{(13)}$. Using the above remarks it is easy to get expression (24a) for the average probability $\bar{P}^Z(\nu_e \rightarrow \nu_{e';t_E,t_0})$. 

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Figure Captions

Figs. 1a – 1b. Regions of values of the MSW transition parameters $\Delta m_{31}^2$ and $\sin^2 2\theta_{13}$ for which the $\chi^2$—analysis of the solar neutrino data gives $\chi^2 \leq 3.841$. The regions shown are obtained for fixed values of the vacuum oscillation parameters $\Delta m_{21}^2$ and $\sin^2 2\theta_{12}$ which are indicated in the figures. Shown are also the 95% C.L. $2\nu$ MSW nonadiabatic and adiabatic solution regions.

Figs. 2a – 2b. Regions of values of the vacuum oscillation parameters $\Delta m_{21}^2$ and $\sin^2 2\theta_{12}$ allowed by the solar neutrino data ($\chi^2 \leq 3.841$). The regions correspond to fixed values of the MSW transition parameters $\Delta m_{31}^2$ and $\sin^2 2\theta_{13}$ indicated in the figures.

Figs. 3a – 3c. The averaged probability $\bar{P}(\nu_e \rightarrow \nu_e)$ (solid line) as a function of the solar neutrino energy $E$ for four sets of values of the parameters indicated in the figures and belonging to region A. The dependence on $E$ of the MSW probability $\bar{P}^{(31)}_{2\nu MSW}(\nu_e \rightarrow \nu_e)$ averaged over the region of the $^8$B neutrino production is also shown (dashed line).

Figs. 4a – 4b. The probability $\bar{P}(\nu_e \rightarrow \nu_e)$ (solid line) not averaged (a) and averaged (b) over a period of 1 year [11] (see the text), as a function of the neutrino energy $E$ for two different sets of values of the four parameters from region B. The averaged MSW probability $\bar{P}^{(31)}_{2\nu MSW}(\nu_e \rightarrow \nu_e)$ as a function of $E$ is also shown (dashed line).

Figs. 5a – 5b, 6a - 6c, 7. Examples of the dependence of the averaged probabilities $\bar{P}(\nu_e \rightarrow \nu_e)$ (solid line) and $\bar{P}^{(31)}_{2\nu MSW}(\nu_e \rightarrow \nu_e)$ (dashed line) on the neutrino energy $E$ in the cases of solutions C, D and E, respectively, considered in the text.

Fig. 8. The averaged probability $\bar{P}(\nu_e \rightarrow \nu_e)$ (solid line) as a function of $E$ for values of the parameters (indicated in the figure), for which the MSW transitions + vacuum oscillations can provide an alternative to the purely MSW mechanism of suppression of the solar $\nu_e$ flux in the case discussed in ref. [12].
\begin{tabular}{|c|c|c|c|}
\hline
\$\sin^2 2\theta_{12}$ & $\Delta m^2_{21}$ (eV$^2$) & \\
\hline
A: 0.98 & 5.6x10$^{-12}$ & \\
A: 1.0 & 5.4x10$^{-12}$ & \\
B: 0.47 & 5.6x10$^{-10}$ & \\
B: 0.48 & 1.4x10$^{-10}$ & \\
\hline
\end{tabular}

\begin{equation*}
2\nu \text{ MSW (}\chi^2 < 5.991\text{)}
\end{equation*}

Fig. 1a
\[ \sin^2 2\theta_{12} \quad \Delta m^2_{21} (\text{eV}^2) \]

- C: 0.80 \quad 1.2 \times 10^{-10}
- D: 0.80 \quad 3.4 \times 10^{-11}
- E: 0.75 \quad 8.9 \times 10^{-10}

\( \chi^2 < 3.841 \)

- 2ν MSW \( \chi^2 < 5.991 \)

---

**Fig. 1b**
\[ \sin^2 2\theta_3 \quad \Delta m^2_{31} \text{ (eV}^2) \]

- B: 3.8x10^3 \quad 9.6x10^{-6}
- C: 5.1x10^4 \quad 4.5x10^{-5}
- D: 3.0x10^4 \quad 1.0x10^{-4}
- A: 0.3 \quad 1.2x10^{-4}
- A: 2.0x10^3 \quad 1.2x10^{-4}
- A: 6.0x10^2 \quad 1.2x10^{-4}
- A: 8.0x10^3 \quad 1.2x10^{-4}
- A: 1.0x10^3 \quad 4.2x10^{-5}
- A: 2.0x10^3 \quad 2.3x10^{-5}

\( \chi^2 < 3.841 \)
\begin{align*}
\sin^2 2\theta_3 & \quad \Delta m^2_{31} (\text{eV}^2) \\
\hline
\text{B} : 3.8 \times 10^3 & \quad 9.6 \times 10^6 \\
\text{E} : 0.27 & \quad 1.3 \times 10^4 \\
\end{align*}

(\chi^2 < 3.841)

Fig. 2b
\[ \Delta m_{31}^2 = 1.0 \times 10^{-4} \quad \sin^2 \theta_{13} = 3.8 \times 10^{-4} \]
\[ \Delta m_{21}^2 = 5.6 \times 10^{-12} \quad \sin^2 \theta_{12} = 0.98 \]

Fig. 3b
\[ \Delta m^2_{31} = 1.2 \times 10^{-4} \quad \sin^2 \theta_{13} = 0.22 \]
\[ \Delta m^2_{21} = 5.6 \times 10^{-12} \quad \sin^2 \theta_{12} = 0.98 \]

Fig. 3c

\[ P(\nu_e \rightarrow \nu_e) \]
\[ P^{(31)}_{2MSW} \]
\[ \Delta m_{31}^2 = 4.2 \times 10^{-5} \quad \sin^2 \theta_{13} = 1.0 \times 10^{-3} \]
\[ \Delta m_{21}^2 = 5.6 \times 10^{-12} \quad \sin^2 \theta_{12} = 0.98 \]

\[ E \text{ (MeV)} \]

\[ P_{2\text{MSW}} \]

\[ P(\nu_e \rightarrow \nu_e) \]

\[ P^{(31)}_{2\text{MSW}} \]

Fig. 3a
\[ \Delta m^2_{31} = 9.8 \times 10^{-6} \quad \sin^2 \theta_{13} = 4.0 \times 10^{-3} \]
\[ \Delta m^2_{21} = 1.4 \times 10^{-10} \quad \sin^2 \theta_{12} = 0.48 \]
\( \Delta m_{31}^2 = 1.0 \times 10^{-5} \quad \sin^2 2\theta_{13} = 3.8 \times 10^{-3} \)

\( \Delta m_{21}^2 = 5.5 \times 10^{-10} \quad \sin^2 2\theta_{12} = 0.44 \)

\[ E = \left( \frac{\Delta m^2}{2E} \right)^{1/2} \]

\[ \bar{P}(\nu_e \rightarrow \nu_e) \]

\[ \bar{P}_{2\text{MSW}}^{(31)} \]

**Fig. 4b**

**Solution B**

![Graph showing probability as a function of energy](image-url)
\[ \Delta m_{31}^2 = 1.3 \times 10^{-5} \quad \sin^2 \theta_{13} = 1.6 \times 10^{-3} \]
\[ \Delta m_{21}^2 = 1.2 \times 10^{-10} \quad \sin^2 \theta_{12} = 0.80 \]
\[ \Delta m^2_{31} = 4.6 \times 10^{-5} \]
\[ \Delta m^2_{21} = 1.2 \times 10^{-10} \]
\[ \sin^2 \theta_{13} = 5.9 \times 10^{-4} \]
\[ \sin^2 \theta_{12} = 0.80 \]

\[ \overline{P}(v_e \rightarrow v_e) \]
\[ \overline{P}^{(31)}_{2\text{MSW}} \]

Fig. 5b
\[ \Delta m_{31}^2 = 1.5 \times 10^{-5} \quad \sin^2 \theta_{13} = 2.5 \times 10^{-3} \]
\[ \Delta m_{21}^2 = 3.4 \times 10^{-11} \quad \sin^2 \theta_{12} = 0.80 \]
\[ \Delta m^{2}_{31} = 8.2 \times 10^{-5} \quad \sin^{2}\theta_{13} = 3.5 \times 10^{-4} \]
\[ \Delta m^{2}_{21} = 3.4 \times 10^{-11} \quad \sin^{2}\theta_{12} = 0.80 \]

![Graph showing probability as a function of energy with two curves: one solid and one dashed.](image)
\[ \Delta m_{31}^2 = 1.3 \times 10^{-4} \quad \sin^2 2\theta_{13} = 0.1 \]
\[ \Delta m_{21}^2 = 3.4 \times 10^{-11} \quad \sin^2 2\theta_{12} = 0.80 \]
\[ \Delta m^2_{31} = 1.3 \times 10^{-4} \quad \sin^2 \theta_{13} = 0.31 \]
\[ \Delta m^2_{21} = 8.9 \times 10^{-10} \quad \sin^2 \theta_{12} = 0.83 \]
\[ \Delta m_{31}^2 = 1.1 \times 10^{-5} \quad \sin^2 2\theta_{13} = 0.020 \]
\[ \Delta m_{21}^2 = 2.8 \times 10^{-12} \quad \sin^2 2\theta_{12} = 1.0 \]

**Fig. 8**