ON DUAL EQUATION IN THEORY OF THE SECOND ORDER ODE’s

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Abstract

We study the relations between the second order nonlinear differential equations

\[ y'' + a_1(x, y)y'^3 + 3a_2(x, y)y'^2 + 3a_3(x, y)y' + a_4(x, y) = 0 \]

with arbitrary coefficients \( a_i(x, y) \) and dual the second order nonlinear differential equations

\[ b'' = g(a, b, b') \]

with the function \( g(a, b, b' = c) \) satisfying the nonlinear partial differential equation

\[
g_{aacc} + 2cg_{abcc} + 2gg_{accc} + c^2g_{bccc} + 2cgg_{bccc} + 3gg_{bccc} - g_cg_{accc} + 4gcg_{bc} - 3g_bg_{cc} + 6g_b = 0.
\]

1 Introduction

The relation between the equations in form

\[ y'' + a_1(x, y)y'^3 + 3a_2(x, y)y'^2 + 3a_3(x, y)y' + a_4(x, y) = 0 \] (1)

and

\[ b'' = g(a, b, b') \] (2)

with function \( g(a, b, b') \) satisfying the p.d.e

\[
g_{aacc} + 2cg_{abcc} + 2gg_{accc} + c^2g_{bccc} + 2cgg_{bccc} + \]

\[
g^2 g_{cccc} + (g_a + cg_b)g_{cc} - 4g_{abc} - 4cg_{bc} - cg_cg_{bc} - \]

\[
3gg_{bccc} - g_cg_{accc} + 4gcg_{bc} - 3g_bg_{cc} + 6g_b = 0.
\]

from geometrical point of view was studied by E.Cartan [1].
In fact, according to the expressions on curvature of the space of linear elements \((x,y,y')\) connected with equation (1)

\[
\Omega^1_2 = a[\omega^2 \wedge \omega^1], \quad \Omega^0_1 = b[\omega^1 \wedge \omega^2], \quad \Omega^0_2 = h[\omega^1 \wedge \omega^2] + k[\omega^2 \wedge \omega^1].
\]

where:

\[
a = -\frac{1}{6} \frac{\partial^4 f}{\partial y'^4}, \quad h = \frac{\partial b}{\partial y'}, \quad k = -\frac{\partial \mu}{\partial y'} - \frac{1}{6} \frac{\partial^2 f}{\partial y'^2} \frac{\partial^3 f}{\partial y'^3},
\]

and

\[
6b = f_{xxy'y'} + 2y'f_{xyy'y'} + 2f f_{xxy'y'} + y'^2 f_{yy'yy'} + 2y'f f_{yy'y'y'}
+ f^2 f_{y'y'yy'} + (f_x + y'f_y)f_{y'y'y'} - 4f_{xyy'} - 4y'f_{yy'} - y'f_y f_{yy'}
- 3f f_{yy'y'} - f_y f_{xyy'} + 4f_y f_{yy'} - 3f_y f_{yy'} + 6f_{yy}.
\]

two types of equations by a natural way are evolved: the first type from the condition \(a = 0\) and second type from the condition \(b = 0\).

The first condition \(a = 0\) lead to the the equation in form (1) and the second condition give us the equations (2) where the function \(g(a, b, b')\) satisfies the above p.d.e. (3).

From the elementary point of view the relation between both equations (1) and (2) is a result of the special properties of their General Integral

\[
F(x, y, a, b) = 0
\]

which can be considered as the equation of some 3-dim orbifold.

## 2 Method of solution and reductions

The equation (2) forming dual pair with some equation (1) can be find from the solutions of the p.d.e.(3).

For solutions of this type of equation we use the method of solution of the p.d.e.’s described first in [3].

To integrate the partial nonlinear first order differential equation

\[
F(x, y, z, f_x, f_y, f_z, f_{xx}, f_{xy}, f_{x}, f_{yy}, f_{yz}, f_{xxx}, f_{xyy}, f_{xxy}, \ldots) = 0
\]

(4)
can be applied a following approach.

We use the following parametric presentation of the functions and variables

\[
f(x, y, z) \rightarrow u(x, t, z), \quad y \rightarrow v(x, t, z), \quad f_x \rightarrow u_x - \frac{u_t}{v_t} v_x,
\]

\[
f_z \rightarrow u_z - \frac{u_t}{v_t} v_z, \quad f_y \rightarrow \frac{u_t}{v_t}, \quad f_{yy} \rightarrow \frac{(u_v v_t)_t}{v_t}, \quad f_{xy} \rightarrow \frac{(u_x - u_v v_x)_t}{v_t}, \ldots
\]

(5)

where variable \(t\) is considered as parameter.

Remark that conditions of the type

\[
f_{xy} = f_{yx}, \quad f_{xz} = f_{zx}, \ldots
\]

are fulfilled at the such type of presentation.
In result instead of equation (4) one get the relation between the new variables \( u(x, t, z) \) and \( v(x, t, z) \) and their partial derivatives

\[
\Phi(u, v, u_x, u_z, u_t, v_x, v_z, v_t, \ldots) = 0. \tag{6}
\]

In some cases the solution of such type of indefinite equation is more simple problem than solution of the equation (4).

The equation (3) has many types of reductions and the simplest of them are

\[
g = c^\alpha \omega [ac^{\alpha - 1}], \quad g = c^\alpha \omega [bc^{\alpha - 2}], \quad g = a^{-\alpha} \omega [ca^{\alpha - 1}],
\]

\[
g = b^{1-2\alpha} \omega [cb^{\alpha - 1}], \quad g = a^{-1} \omega (c - b/a), \quad g = a^{-3} \omega [b/a, b - ac], \quad g = a^{3/\alpha - 2} \omega [b^\alpha/a^\beta, c^\alpha/a^{\beta - \alpha}].
\]

For every type of reduction we can write corresponding equation (3) and then integrate it. Remark that the first examples of solutions of equation (3) were obtained in [2-8].

Proposition 1 Equation (3) can be represent in form

\[
g_{ac} + gg_{cc} - g_c^2/2 + cg_{bc} - 2gb = h(a, b, c),
\]

\[
h_{ac} + gh_{cc} - gc_h + ch_{bc} - 3hb = 0.
\]

So in standard name of variables we get

\[
f_{xy} + ff_{yy} - f_y^2/2 + yf_{yz} - 2f_z = h(x, z, y),
\]

\[
h_{xy} + fh_{yy} - f_yh_y + yh_{xz} - 3h_z = 0.
\] \tag{8}

3 Two-dimensional short-cut \((x, y)\) equation

At the condition \(h(x, z, y) = 0\) we get the equation

\[
f_{xy} + ff_{yy} - f_y^2/2 + yf_{yz} - 2f_z = 0. \tag{9}
\]

In particular case \(f(x, z, y) = f(x, y)\) the equation takes the form

\[
f_{xy} + ff_{yy} - f_y^2/2 = 0. \tag{10}
\]

It was integrated in the [4] by the Legendre transformation and another methods in more latest publications.

We consider a new approach to integration of this equation.

The change from the equation (10) to the (6) lead to the relation between the functions \( u(x, t) \) and \( v(x, t) \) and their derivatives

\[
2 \left( \frac{\partial^2}{\partial t \partial x} u(x, t) \right) \left( \frac{\partial}{\partial t} v(x, t) \right)^2 - 2 \left( \frac{\partial^2}{\partial t^2} u(x, t) \right) \left( \frac{\partial}{\partial x} v(x, t) \right) \frac{\partial}{\partial t} v(x, t) +
\]

\[
+ 2 \left( \frac{\partial}{\partial t} u(x, t) \right) \left( \frac{\partial}{\partial x} v(x, t) \right) \frac{\partial^2}{\partial t^2} v(x, t) - 2 \left( \frac{\partial}{\partial t} u(x, t) \right) \left( \frac{\partial^2}{\partial t \partial x} v(x, t) \right) \frac{\partial}{\partial t} v(x, t) +
\]

\[
+ 2 u(x, t) \left( \frac{\partial^2}{\partial t^2} u(x, t) \right) \frac{\partial}{\partial t} v(x, t) - 2 u(x, t) \left( \frac{\partial}{\partial t} u(x, t) \right) \frac{\partial^2}{\partial t^2} v(x, t) - \left( \frac{\partial}{\partial t} u(x, t) \right)^2 \frac{\partial}{\partial t} v(x, t) = 0
\]
The substitutions here of the form
\[ u(x, t) = t \frac{\partial}{\partial t} \omega(x, t) - \omega(x, t), \]
\[ v(x, t) = \frac{\partial}{\partial t} \omega(x, t) \]
give us the linear p.d.e. equation
\[ -2 \frac{\partial^2}{\partial t \partial x} \omega(x, t) + 2t \frac{\partial}{\partial x} \omega(x, t) - 2 \omega(x, t) - t^2 \frac{\partial^2}{\partial t^2} \omega(x, t) = 0. \] \hfill (11)

The equation (11) is transformed into the form
\[ \frac{\partial^2}{\partial \eta \partial \xi} \omega(\xi, \eta) + 4 \frac{\partial}{\partial \xi} \omega(\xi, \eta) - \xi + \eta - 2 \omega(\xi, \eta) \left( -\xi + \eta \right) \right)^2 = 0 \]
with the change of variables
\[ \xi = x + 2/t, \quad \eta = x \]
and can be integrated by the Laplace-method.

In particular case the equation (11) admits the solution
\[ \omega(x, t) = -C1 + 4C2 \ln(t) + 4C2 t + C3 t^2 + t \left( C1 t + C2 t^2 \right) \]
with arbitrary parameters \(Ci\) and elimination of the parameter \(t\) from the relations
\[ f(x, y) - t \frac{\partial}{\partial t} \omega(x, t) + \omega(x, t) = 0 \]
and
\[ y - \frac{\partial}{\partial t} \omega(x, t) = 0 \]
give us the solution of the equation (10) taking of of the form at the condition \(C4 = 0\)
\[ f(x, y) = 4C2 \left( LambertW(1/2 x e^{-1/4 \frac{\psi(k, a)+ka+C2+C3+C1 \alpha}{x^2}}) \right)^2 x^{-1} + \]
\[ + 8C2 LambertW(1/2 x e^{-1/4 \frac{\psi(k, a)+ka+C2+C3+C1 \alpha}{x^2}}) x^{-1} + C1. \]

So the equation
\[ \frac{d^2 b}{da^2} = 4C2 \left( LambertW(1/2 ae^{-1/4 \frac{\psi(k, a)+ka+C2+C3+C1 \alpha}{x^2}}) \right)^2 a^{-1} + \]
\[ + 8C2 LambertW(1/2 ae^{-1/4 \frac{\psi(k, a)+ka+C2+C3+C1 \alpha}{x^2}}) a^{-1} + C1 \] \hfill (12)
is dual for the some equation in form (1).

In fact General solution of the equation (12) is defined by the relation
\[ b - 8C2 a + 8x C2 - C3 a + C3 x - 1/2 C1 a^2 + 1/2 C1 x^2 - \]
\[ -4C2 a \ln(2) + 4C2 x \ln(2) - 4C2 a \ln((-a + x)^{-1}) - y(x) = 0, \]
and elimination of the variables $a$, $b$ from here give us the equation in form (1)

$$ \frac{d^2}{dx^2}y(x) + $$

$$ + \frac{1}{4} \left( \frac{d}{dx} y(x) \right)^2 + \frac{1}{4} \left( -8 \cdot C_2 \ln(2) - 2 \cdot C_3 - 2 \cdot C_1 x - 12 \cdot C_2 \right) \frac{d}{dx} y(x) + $$

$$ + \frac{1}{4} \cdot C_1^2 x + 2 \cdot C_1 \cdot C_2 \ln(2) + 1/2 \cdot C_3 \cdot C_1 + 2 \cdot C_2 \cdot C_1 + $$

$$ + \frac{1}{4} \left( 32 \cdot C_2^2 + 12 \cdot C_2 \cdot C_3 + C_3^2 + 8 \cdot C_3 \cdot C_2 \ln(2) + 48 \cdot C_2^2 \ln(2) + 16 \cdot C_2^2 \ln(2)^2 \right) x = 0. $$

(13)

Remark that the equation (13) is the Rikkati equation with respect of variable $z(x) = \frac{dy}{dx}$.

4 Two-dimensional short-cut $(z, y)$ equation

The next example is the equation (9) at the conditions $h(x, z, y) = 0$ and $f(x, z, y) = f(z, y)$

$$ ff_{yy} - f_y^2/2 + yf_{yz} - 2f_z = 0. $$

(14)

For a such equation we get the relation

$$ 2u(z, t) \left( \frac{\partial^2}{\partial t^2} u(z, t) \right) \frac{\partial}{\partial t} v(z, t) - 2u(z, t) \left( \frac{\partial}{\partial t} u(z, t) \right) \frac{\partial^2}{\partial t^2} v(z, t) - \left( \frac{\partial}{\partial t} u(z, t) \right)^2 \frac{\partial}{\partial t} v(z, t) + $$

$$ + 2v(z, t) \left( \frac{\partial^2}{\partial t \partial z} u(z, t) \right) \left( \frac{\partial}{\partial t} v(z, t) \right)^2 - 2v(z, t) \left( \frac{\partial^2}{\partial t^2} u(z, t) \right) \left( \frac{\partial}{\partial z} v(z, t) \right) \frac{\partial}{\partial t} v(z, t) + $$

$$ + 2v(z, t) \left( \frac{\partial}{\partial t} u(z, t) \right) \left( \frac{\partial}{\partial z} v(z, t) \right) \frac{\partial^2}{\partial t^2} v(z, t) - 2v(z, t) \left( \frac{\partial^2}{\partial t \partial z} u(z, t) \right) \frac{\partial}{\partial t} v(z, t) - $$

$$ - 4 \left( \frac{\partial}{\partial z} u(z, t) \right) \left( \frac{\partial}{\partial t} v(z, t) \right)^3 + 4 \left( \frac{\partial}{\partial z} u(z, t) \right) \left( \frac{\partial}{\partial z} v(z, t) \right) \left( \frac{\partial}{\partial t} v(z, t) \right)^2 = 0 $$

which is equivalent the p.d.e

$$ -2\omega(z, t) + 2t \frac{\partial}{\partial t} \omega(z, t) - t^2 \frac{\partial^2}{\partial t^2} \omega(z, t) - 2 \frac{\partial}{\partial t} \omega(z, t) \frac{\partial^2}{\partial t \partial z} \omega(z, t) + $$

$$ + 4 \left( \frac{\partial^2}{\partial t^2} \omega(z, t) \right) \frac{\partial}{\partial z} \omega(z, t) = 0 $$

(15)

after the substitution

$$ u(z, t) = t \partial_t \omega(z, t) - \omega(z, t), $$

$$ v(z, t) = \partial_t \omega(z, t). $$

A simplest solution of the equation (15) can be find in form

$$ \omega(z, t) = A(t) + zt^2 $$
where the function \( A(t) \) satisfies the linear equation

\[-2 A(t) - 2 t \frac{d}{dt} A(t) + 3 t^2 \frac{d^2}{dt^2} A(t) = 0\]

having the solution

\[A(t) = \frac{C_1}{\sqrt{t}} + C_2 t^2.\]

After inverse transformation in the case \( C_2 = 0 \) we find corresponding solution of the equation (14) in implicit form

\[-351918 y^3 z^2 \triangleleft 1^3 \left( f(z, y) \right)^2 + 84672 y^5 z \triangleleft 1^3 f(z, y) - 34992 y^2 \left( f(z, y) \right)^6 z^2 +
+8748 y^4 \left( f(z, y) \right)^5 z + 46656 \left( f(z, y) \right)^7 z^3 - 729 y^6 \left( f(z, y) \right)^4 + 518616 y^3 \triangleleft 1^3 \left( f(z, y) \right)^3 +
+823543 z^4 \triangleleft 1^6 - 6912 y^7 \triangleleft 1^3 = 0.\]

So the second order ODE

\[-351918 \left( \frac{d}{da} b(a) \right)^3 \left( b(a) \right)^2 \triangleleft 1^3 \left( \frac{d^2}{da^2} b(a) \right)^2 + 84672 \left( \frac{d}{da} b(a) \right)^5 b(a) \triangleleft 1^3 \frac{d^2}{da^2} b(a) -
-34992 \left( \frac{d}{da} b(a) \right)^2 \left( \frac{d^2}{da^2} b(a) \right)^6 \left( b(a) \right)^2 + 8748 \left( \frac{d}{da} b(a) \right)^4 \left( \frac{d^2}{da^2} b(a) \right)^5 b(a) +
+46656 \left( \frac{d^2}{da^2} b(a) \right)^7 \left( b(a) \right)^3 - 729 \left( \frac{d}{da} b(a) \right)^6 \left( \frac{d^2}{da^2} b(a) \right)^4 +
+518616 \left( \frac{d}{da} b(a) \right) \left( b(a) \right)^3 \triangleleft 1^3 \left( \frac{d^2}{da^2} b(a) \right)^3 + 823543 \left( b(a) \right)^4 \triangleleft 1^6 -
-6912 \left( \frac{d}{da} b(a) \right)^7 \triangleleft 1^3 = 0\]

is dual equation for the some equation of the form (1).

It can be reduced to the first order ODE

\[-351918 \left( h(b) \right)^5 b^2 \triangleleft 1^3 \left( \frac{d}{db} h(b) \right)^2 + 84672 \left( h(b) \right)^6 b \triangleleft 1^3 \frac{d}{db} h(b) -
-34992 \left( h(b) \right)^8 \left( \frac{d}{db} h(b) \right)^6 b^2 + 8748 \left( h(b) \right)^9 \left( \frac{d}{db} h(b) \right)^5 b +
+46656 \left( \frac{d}{db} h(b) \right)^7 \left( h(b) \right)^7 b^3 - 729 \left( h(b) \right)^{10} \left( \frac{d}{db} h(b) \right)^4 +
+518616 \left( h(b) \right)^4 b^3 \triangleleft 1^3 \left( \frac{d}{db} h(b) \right)^3 + 823543 b^4 \triangleleft 1^6 -
-6912 \left( h(b) \right)^7 \triangleleft 1^3 = 0\]

having singular solution

\[h(b) = \frac{7}{108} 108^{5/7} \sqrt{-b^4 \triangleleft 1^3}.\]
In general case $C_1 \neq 0$, $C_2 \neq 0$ we also get the first order ODE having singular solution in the form
\[
\left( \frac{d}{da} b(a) \right)^7 + \frac{823543}{11664} C_1^3 C_2^4 + \frac{823543}{2916} C_1^3 C_2^3 b + \frac{823543}{1944} C_1^3 C_2^2 b^2 + \frac{823543}{2916} C_1^3 C_2 b^3 + \frac{823543}{11664} C_1^3 b^4 = 0.
\]
which corresponds the function determined from the equation
\[
a + 4 \left( -\frac{1}{12} b(a) - \frac{1}{12} C_2 \right)^7 \sqrt{-\frac{1}{11664} \frac{1}{C_1^3 (C_2 + b(a))^4}} + C_3 = 0.
\]
In more general case the solution of the equation (15) has the form
\[
\omega(z, t) = \left( t^{\frac{k}{1+4k}} \right)^4 - C_1 \left( t^{(-1+4k)^{-1}} \right)^2 t^{-1} + C_2 t^2 + kzt^2
\]
where $k$ is essential parameter.

With help of the function $\omega(z, t)$ a large class of solutions of the equation (14) radically depending from the choice of parameter $k$ can be produced.

5 Two-dimensional full $(x, y)$- equation

In the case $f(x, z, y) = f(x, y)$, $h(x, z, y) = h(x, y) \neq 0$ from the system (8) we find full $(x, y)$- equation
\[
\frac{\partial^4}{\partial y \partial x \partial y} f(x, y) + \left( \frac{\partial}{\partial x} f(x, y) \right) \frac{\partial^3}{\partial y^3} f(x, y) + 2 f(x, y) \frac{\partial^4}{\partial y^2 \partial x \partial y} f(x, y) + (f(x, y))^2 \frac{\partial^4}{\partial y^4} f(x, y) -
\]
\[
- \left( \frac{\partial}{\partial y} f(x, y) \right) \frac{\partial^3}{\partial y \partial x \partial y} f(x, y) = 0.
\]
This equation can be transformed into the form ([5])
\[
\frac{\Omega_x}{\Omega_y} \Omega_{(fff)} + \frac{1}{\Omega_y} \Omega_{(xff)} - 1 = 0
\]
(17)
with the help of presentation
\[
y - \Omega(f(x, y), x) = 0.
\]
From the equation (17) we find that the function $\Omega_x(f, x)$ defined by the relation
\[
\Omega(f, x) = \frac{\partial \Lambda(x, f)}{\partial x}
\]
satisfies the equation
\[
\frac{\partial^2}{\partial f^2} \Lambda(x, f) = \frac{1}{6} \left( \frac{\partial}{\partial f} \Lambda(x, f) \right)^3 + \alpha(f) \left( \frac{\partial}{\partial f} \Lambda(x, f) \right)^2 + \beta(f) \frac{\partial}{\partial f} \Lambda(x, f) + \mu(f),
\]
with arbitrary coefficients.
From solutions of this equation we can find the function \( g(a,c) \) from the relation
\[
c - \Omega(g(a,c),a) = 0
\]
and the equations forming dual pair with the equations
\[
y'' + a_1(x)y'^3 + 3a_2(x)y'^2 + 3a_3(x)y' + a_4(x) = 0 \tag{18}
\]
can be obtained by such a way.

Remark that the equation (18) has the form of Abel’s equation with respect the variable \( z(x) = y' \).

In the case of its solvability (Bernoulli and others) there are a lot possibilities to get an examples of of dual equation in explicit form.

Let us consider the solutions of the equation (16) in form
\[
Ay'' + (By^3 + Cy^2 + Ey')y'' + Hy^6 + Ky^5 + Ly^4 + My^3 + Ny^2 + Py' + Q = 0
\]
where the coefficients depend from the variable \( x A = A(x), \ B = B(x) \,... \)  
Joint consideration of the relation
\[
Af(x,y)^2 + (By^3 + Cy^2 + Ey + F)f(x,y) + Hy^6 + Ky^5 + Ly^4 + My^3 + Ny^2 + Py + Q = 0
\]
and (16) lead to the conditions for determination of the coefficients.

As example we find
\[
-26244 A^2 (f(x,y))^2 + (18x^3y^3 + 2916 Axy^2) f(x,y) + 27x^2y^4 + 3888 y^3A = 0
\]
or
\[
-2916 A^2 \left( \frac{d^2}{dx^2}y(x) \right)^2 - 324 A \left( \frac{d^2}{dx^2}y(x) \right) \left( \frac{d}{dx}y(x) \right) y(x) +
+3 (y(x))^2 \left( \frac{d}{dx}y(x) \right)^2 - 2 (y(x))^3 \frac{d^2}{dx^2}y(x) + 432 A \left( \frac{d}{dx}y(x) \right)^3 = 0.
\]

From General Integral
\[
y_1 x^2 + 2 y_1 x_1 x_2 + y_1 x_1 x_1 x_2^2 + 108 A - 108 A_1 x - 108 A_1 x_1 x_2 = 0,
\]
we find dual equation
\[
3 \frac{d}{da}b(a) + \left( \frac{d}{da}b(a) \right)^3 a^4 + 2 \left( \frac{d^2}{da^2}b(a) \right) a = 0.
\]
The equation
\[
b'' = A/b^3
\]
has General Integral
\[
b^2 - x (a - y)^2 - \frac{A}{x} = 0
\]

Corresponding dual equation looks as
\[
-Ax^6 \left( \frac{d^2}{dx^2}y(x) \right)^2 + \left( -6 \left( \frac{d}{dx}y(x) \right) x^5 A - 2 \left( \frac{d}{dx}y(x) \right)^3 x^9 \right) \frac{d^2}{dx^2}y(x) + A^2 -
-3 x^8 \left( \frac{d}{dx}y(x) \right)^4 - 6 x^4 \left( \frac{d}{dx}y(x) \right)^2 A = 0
\]
6 Short-cut \((x, y, z)\) equation

The equation has the form

\[
\begin{align*}
\frac{\partial^2}{\partial x \partial y} f(x, y, z) &+ \frac{\partial^2}{\partial y^2} f(x, y, z) - \frac{1}{2} \left( \frac{\partial}{\partial y} f(x, y, z) \right)^2 + y \frac{\partial^2}{\partial y \partial z} f(x, y, z) - 2 \frac{\partial}{\partial z} f(x, y, z) = 0
\end{align*}
\]

On rearrangement we find the relation

\[
\begin{align*}
2 \left( \frac{\partial^2}{\partial t \partial x} u(x, t, z) \right) \left( \frac{\partial}{\partial t} v(x, t, z) \right) &- 2 \left( \frac{\partial}{\partial t} u(x, t, z) \right) \left( \frac{\partial^2}{\partial t \partial x} v(x, t, z) \right) \frac{\partial}{\partial t} v(x, t, z) - \\
-2 \left( \frac{\partial}{\partial x} v(x, t, z) \right) \left( \frac{\partial^2}{\partial t^2} u(x, t, z) \right) \frac{\partial}{\partial t} v(x, t, z) &+ 2 \left( \frac{\partial}{\partial x} v(x, t, z) \right) \left( \frac{\partial}{\partial t} u(x, t, z) \right) \frac{\partial^2}{\partial t^2} v(x, t, z) + \\
+2 u(x, t, z) \left( \frac{\partial^2}{\partial t^2} u(x, t, z) \right) \frac{\partial}{\partial t} v(x, t, z) &- 2 u(x, t, z) \left( \frac{\partial}{\partial t} u(x, t, z) \right) \frac{\partial^2}{\partial t^2} v(x, t, z) - \\
- \left( \frac{\partial}{\partial t} u(x, t, z) \right)^2 &\frac{\partial}{\partial t} v(x, t, z) + 2 v(x, t, z) \left( \frac{\partial^2}{\partial t \partial z} u(x, t, z) \right) \left( \frac{\partial}{\partial t} v(x, t, z) \right)^2 - \\
-2 v(x, t, z) &\left( \frac{\partial}{\partial t} v(x, t, z) \right) \left( \frac{\partial^2}{\partial t^2} u(x, t, z) \right) \frac{\partial}{\partial z} v(x, t, z) - \\
-2 v(x, t, z) &\left( \frac{\partial}{\partial t} v(x, t, z) \right) \left( \frac{\partial}{\partial t} u(x, t, z) \right) \frac{\partial^2}{\partial t \partial z} v(x, t, z) + \\
+2 v(x, t, z) &\left( \frac{\partial^2}{\partial t^2} v(x, t, z) \right) \left( \frac{\partial}{\partial t} u(x, t, z) \right) \frac{\partial}{\partial z} v(x, t, z) - \\
-4 \left( \frac{\partial}{\partial t} v(x, t, z) \right)^3 &\frac{\partial}{\partial z} u(x, t, z) + 4 \left( \frac{\partial}{\partial t} v(x, t, z) \right)^2 \left( \frac{\partial}{\partial t} u(x, t, z) \right) \frac{\partial}{\partial z} v(x, t, z) = 0.
\end{align*}
\]

From here with the help of substitution

\[
\begin{align*}
\omega(x, t, z) &= t \frac{\partial}{\partial t} \omega(z, t) - \omega(z, t), \\
v(x, t, z) &= \frac{\partial}{\partial t} \omega(z, t)
\end{align*}
\]

we get the equation

\[
\begin{align*}
-2 \frac{\partial^2}{\partial t \partial x} \omega(x, t, z) + 2 t \frac{\partial}{\partial t} \omega(x, t, z) - 2 \omega(x, t, z) - t^2 \frac{\partial^2}{\partial t^2} \omega(x, t, z) - \\
-2 \left( \frac{\partial}{\partial t} \omega(x, t, z) \right) \frac{\partial^2}{\partial t \partial z} \omega(x, t, z) &+ 4 \left( \frac{\partial}{\partial t} \omega(x, t, z) \right)^2 \left( \frac{\partial}{\partial t} \omega(x, t, z) \right) \frac{\partial}{\partial z} \omega(x, t, z) = 0. \tag{19}
\end{align*}
\]

We consider the solutions of the equation (19) in form

\[
\omega(x, t, z) = A(x, t) + kzt^2
\]

where \(k\) is parameter.
In result we get linear equation for the function $A(x,t)$

$$(2t - 4kt) \frac{\partial}{\partial t} A(x,t) + \left(-t^2 + 4kt^2\right) \frac{\partial^2}{\partial t^2} A(x,t) - 2 \frac{\partial^2}{\partial t \partial x} A(x,t) - 2 A(x,t) = 0.$$ 

It can be transformed into the form

$$4 \ (3k - 1) \ (-\xi + \eta) \frac{\partial}{\partial \eta} A(\xi, \eta) - \left( \frac{\partial^2}{\partial \eta \partial \xi} A(\xi, \eta) \right) (-\xi + \eta)^2 \ (-1 + 4k) - 2 A(\xi, \eta) = 0 \quad (20)$$

with help of the substitutions

$$\left\{ \begin{array}{l} \xi = x, \eta = -\frac{x}{-1 + 4k} + 4 \frac{xk}{-1 + 4k} - 2 \frac{1}{(-1 + 4k)t} \end{array} \right\}.$$ 

In result the Laplace-equation

$$\frac{\partial^2}{\partial \eta \partial \xi} A(\xi, \eta) - \left( \frac{(12k - 4) \frac{\partial}{\partial \eta} A(\xi, \eta)}{(-1 + 4k) (-\xi + \eta)} + 2 \frac{A(\xi, \eta)}{(-\xi + \eta)^2 (-1 + 4k)} \right) = 0 \quad (21)$$

with the invariants

$$H = -2 \frac{1}{(-\xi + \eta)^2 (-1 + 4k)}$$

and

$$K = 6 \frac{2k - 1}{(-\xi + \eta)^2 (-1 + 4k)}$$

has been obtained.

For a given case we have

$$p = \frac{K}{H} = -6k + 3$$

and

$$q = \frac{\partial \xi \partial \eta (\ln H)}{H} = -1 + 4k$$

i.e. all invariants of the Laplace-sequence of the equation (19) are in a fixed ratio ([9]).

In particular the condition

$$H^N / H = 1 + (1 - p) N - 1/2 qN (N + 1) =$$

$$= 1 + (6k - 2) N - 1/2 (-1 + 4k) N (N + 1)$$

is fulfilled.

From here we have

$$N = 2, \quad N = -(-1 + 4k)^{-1}.$$
7 Full \((x, y, z)\)- equation

The construction of non trivial solutions of a full \((x, y, z)\)-equation

\[
\frac{\partial^4}{\partial y \partial x^2 \partial y} f(x, y, z) + \left( \frac{\partial}{\partial x} f(x, y, z) \right) \frac{\partial^3}{\partial y^3} f(x, y, z) + 2 f(x, y, z) \frac{\partial^4}{\partial y^2 \partial x \partial y} f(x, y, z) -
\]

\[
-4 \frac{\partial^3}{\partial y \partial x \partial z} f(x, y, z) + 2 y \frac{\partial^4}{\partial y^2 \partial x \partial z} f(x, y, z) + (f(x, y, z))^2 \frac{\partial^4}{\partial y^4} f(x, y, z) +
\]

\[
+ 2 f(x, y, z) y \frac{\partial^4}{\partial y^3 \partial z} f(x, y, z) - \left( \frac{\partial}{\partial y} f(x, y, z) \right) \frac{\partial^3}{\partial y \partial x \partial y} f(x, y, z) +
\]

\[
+ 4 \left( \frac{\partial}{\partial y} f(x, y, z) \right) \frac{\partial^2}{\partial y \partial z} f(x, y, z) - \left( \frac{\partial}{\partial y} f(x, y, z) \right) y \frac{\partial^3}{\partial y^2 \partial z} f(x, y, z) +
\]

\[
y \left( \frac{\partial}{\partial z} f(x, y, z) \right) \frac{\partial^3}{\partial y^3} f(x, y, z) - 4 y \frac{\partial^3}{\partial z \partial y \partial z} f(x, y, z) + y^2 \frac{\partial^4}{\partial z \partial y^2 \partial z} f(x, y, z) -
\]

\[
- 3 \left( \frac{\partial}{\partial z} f(x, y, z) \right) \frac{\partial^2}{\partial y^2} f(x, y, z) - 3 f(x, y, z) \frac{\partial^3}{\partial y^2 \partial z} f(x, y, z) +
\]

\[
+ 6 \frac{\partial^2}{\partial z^2} f(x, y, z) = 0 \quad (22)
\]

may be interested to understanding of the properties of 3-dim orbifolds

\[F(x, y, a, b) = 0\]

defined by the second order ODE's \((1)\).

Here we describe some approach to solution of this problem.

With this aim we introduce the function \(K(\tau, x, z)\) by definition

\[y'' = f(x, y, z) = \sqrt{1 + y'^2} K(\tau, x, z) = \frac{K(\tau, x, z)}{\cos(\tau)^3},\]

where

\[\tau = \arctan(y')\]

Function \(K(\tau, x, z)\) is the curvature along the curve.

From the equation \((22)\) we find the equation for the function \(K(\tau, x, z)\)

\[-4 \left( \frac{\partial^3}{\partial z \partial \tau \partial z} K(\tau, x, z) \right) \sin(2 \tau) + 4 \left( \frac{\partial^3}{\partial x \partial \tau \partial x} K(\tau, x, z) \right) \sin(2 \tau) - 6 \left( \frac{\partial^2}{\partial x \partial z} K(\tau, x, z) \right) \sin(2 \tau) +
\]

\[+ 2 \left( \frac{\partial^4}{\partial x^2 \partial \tau^2 \partial z} K(\tau, x, z) \right) \sin(2 \tau) + 9 \frac{\partial^2}{\partial x^2} K(\tau, x, z) + \frac{\partial^4}{\partial x^2 \partial \tau^2 \partial x} K(\tau, x, z) + 9 \frac{\partial^2}{\partial \tau^2} K(\tau, x, z) +
\]

\[+ \frac{\partial^4}{\partial z \partial \tau^2 \partial z} K(\tau, x, z) - 8 \left( \frac{\partial^3}{\partial x \partial \tau \partial z} K(\tau, x, z) \right) \cos(2 \tau) + 20 \left( K(\tau, x, z) \right)^2 \frac{\partial^2}{\partial \tau^2} K(\tau, x, z) +
\]

\[+ 2 \left( K(\tau, x, z) \right)^2 \frac{\partial^4}{\partial \tau^4} K(\tau, x, z) + 18 \left( K(\tau, x, z) \right)^3 - \left( \frac{\partial^4}{\partial z \partial \tau^2 \partial z} K(\tau, x, z) \right) \cos(2 \tau) +
\]
where a pair of the second order ODE's.

and condition (24) corresponds the equation

\( +3 \left( \frac{\partial^2}{\partial z^2} K(\tau, x, z) \right) \cos(2\tau) - 3 \left( \frac{\partial^2}{\partial x^2} K(\tau, x, z) \right) \cos(2\tau) + \left( \frac{\partial^4}{\partial x \partial \tau^2 \partial x} K(\tau, x, z) \right) \cos(2\tau) - \\
-8 \left( \frac{\partial}{\partial \tau} K(\tau, x, z) \right) \left( \frac{\partial^2}{\partial \tau \partial x} K(\tau, x, z) \right) \sin(\tau) + 8 \left( \frac{\partial}{\partial \tau} K(\tau, x, z) \right) \cos(\tau) \frac{\partial^2}{\partial \tau \partial z} K(\tau, x, z) - \\
-2 \left( \frac{\partial}{\partial \tau} K(\tau, x, z) \right) \left( \frac{\partial^3}{\partial \tau^2 \partial z} K(\tau, x, z) \right) \sin(\tau) + 8 \left( \frac{\partial}{\partial \tau} K(\tau, x, z) \right) \left( \frac{\partial}{\partial z} K(\tau, x, z) \right) \sin(\tau) + \\
+6 \left( \frac{\partial}{\partial x} K(\tau, x, z) \right) \left( \frac{\partial^2}{\partial \tau^2 \partial x} K(\tau, x, z) \right) \sin(\tau) + 4 K(\tau, x, z) \sin(\tau) \frac{\partial^4}{\partial \tau^4 \partial x} K(\tau, x, z) - \\
-2 \left( \frac{\partial}{\partial \tau} K(\tau, x, z) \right) \cos(\tau) \frac{\partial^3}{\partial \tau^3} K(\tau, x, z) + \\
+6 K(\tau, x, z) \frac{\partial^3}{\partial \tau^3} K(\tau, x, z) \cos(\tau) + 2 \left( \frac{\partial}{\partial z} K(\tau, x, z) \right) \sin(\tau) \frac{\partial^3}{\partial z \partial \tau} K(\tau, x, z) + \\
+2 \left( \frac{\partial}{\partial x} K(\tau, x, z) \right) \cos(\tau) \frac{\partial^3}{\partial \tau^3} K(\tau, x, z) - 6 K(\tau, x, z) \left( \frac{\partial^3}{\partial \tau^2 \partial z} K(\tau, x, z) \right) \cos(\tau) + \\
+8 \left( \frac{\partial}{\partial \tau} K(\tau, x, z) \right) \cos(\tau) \frac{\partial}{\partial x} K(\tau, x, z) - 6 \left( \frac{\partial}{\partial z} K(\tau, x, z) \right) \left( \frac{\partial^2}{\partial \tau^2} K(\tau, x, z) \right) \cos(\tau) - \\
-36 K(\tau, x, z) \left( \frac{\partial}{\partial z} K(\tau, x, z) \right) \cos(\tau) + 4 K(\tau, x, z) \cos(\tau) \frac{\partial^4}{\partial \tau^4 \partial x} K(\tau, x, z) = 0. \tag{23} \)

A simplest solutions of the equation (23) has the form

\[ K(\tau, x, z) = - \left( \frac{\partial}{\partial x} U(x, z) \right) \sin(\tau) + \left( \frac{\partial}{\partial z} U(x, z) \right) \cos(\tau) \tag{24} \]

where

\[-6 \frac{\partial^3}{\partial x^3} U(x, z) - 6 \frac{\partial^3}{\partial z \partial x^2} U(x, z) + 12 \left( \frac{\partial^2}{\partial z^2} U(x, z) \right) \frac{\partial}{\partial x} U(x, z) + 12 \left( \frac{\partial^2}{\partial x^2} U(x, z) \right) \frac{\partial}{\partial x} U(x, z) = 0 \]

and

\[-12 \left( \frac{\partial^2}{\partial x^2} U(x, z) \right) \frac{\partial}{\partial z} U(x, z) - 12 \left( \frac{\partial}{\partial z} U(x, z) \right) \frac{\partial^2}{\partial z^2} U(x, z) + 6 \frac{\partial^3}{\partial z^3} U(x, z) + 6 \frac{\partial^3}{\partial x^2 \partial z} U(x, z) = 0. \]

It follows that the function \( U(x, z) \) is solution of the equation

\[ U_{xx} + U_{zz} = \exp(2U) \]

and condition (24) corresponds the equation

\[ z'' = (U_z - z'U_x)(1 + z'^2) \]

This equation is cubic on the first derivative and this case corresponds the projectively flat pair of the second order ODE's.
As an illustration of nontrivial example can be considered the function

\[ K(\tau, x, z) = - \]

\[ = - \frac{(\cos(\tau))^3 \left( 2 \tan(\tau)x^3 (1 + (\tan(\tau))^2 x^6) + 2 (1 + (\tan(\tau))^2 x^6)^{3/2} + 3 \tan(\tau)x^3 \right)}{x^4} \]

which correspond the second order ODE

\[ z'' = - \frac{2 z' x^3 (1 + z' x^6) + 2 (1 + z' x^6)^{3/2} + 3 z' x^3}{x^4}. \] (25)

Equation (25) is dual equation for the some of the second order ODE cubic on the first derivative.

It enters into the composition of the equations

\[ b'' = g(a, b, b') = \frac{A(b^a - 1)}{a^b}, \]

where the function \( A(b^a - 1) = A(\xi) \) satisfies the equation

\[ (A + (\mu - 1)\xi^2)A^{IV} + 3(\mu - 2)(A + (\mu - 1)\xi^3)A^{III} + (2 - \mu)A^I A^{II} + (\mu^2 - 5\mu + 6)A^{III} = 0. \]

This solution was considered here as an example of solution of the full \( f(x, y, z) \) - equation.

In a most general case the solution of the equation (23) can be considered in the form

\[ K(\tau, x, z) = \sum_{n=0}^{\infty} \frac{(A_n(x, z) \sin(\tau n) + B_n(x, z) \cos(\tau n) + C_n(x, z))}{\cos(\tau)^3}. \]

**Remark 1** The system (8) after the change of variables and function according to the rule

\[ g(x, b', b) = b'\phi(x, \ln(b), b/b') \]

takes the form

\[ 2 \frac{\partial}{\partial x} \phi(x, \eta, \xi) + 2 \frac{\partial^2}{\partial \eta \partial x} \phi(x, \eta, \xi) - 2 \left( \frac{\partial^2}{\partial x \partial \xi} \phi(x, \eta, \xi) \right) \xi + 2 \phi(x, \eta, \xi) \frac{\partial^2}{\partial \eta^2} \phi(x, \eta, \xi) - \\
- 4 \phi(x, \eta, \xi) \left( \frac{\partial^2}{\partial \eta \partial \xi} \phi(x, \eta, \xi) \right) \xi + 2 \phi(x, \eta, \xi) \left( \frac{\partial^2}{\partial \xi^2} \phi(x, \eta, \xi) \right) \xi^2 - (\phi(x, \eta, \xi))^2 + \\
+ 2 \phi(x, \eta, \xi) \left( \frac{\partial}{\partial \xi} \phi(x, \eta, \xi) \right) \xi - \left( \frac{\partial}{\partial \eta} \phi(x, \eta, \xi) \right)^2 + 2 \left( \frac{\partial}{\partial \eta} \phi(x, \eta, \xi) \right) \left( \frac{\partial}{\partial \xi} \phi(x, \eta, \xi) \right) \xi - \\
- \left( \frac{\partial}{\partial \xi} \phi(x, \eta, \xi) \right)^2 \xi^2 + 2 \frac{\partial^2}{\partial \eta \partial \xi} \phi(x, \eta, \xi) - 2 \left( \frac{\partial^2}{\partial \xi^2} \phi(x, \eta, \xi) \right) \xi - 4 \frac{\partial}{\partial \xi} \phi(x, \eta, \xi) = 2\kappa(x, \eta, \xi) \]

\[ \frac{\partial^2}{\partial \eta \partial x} \kappa(x, \eta, \xi) - \left( \frac{\partial^2}{\partial x \partial \xi} \kappa(x, \eta, \xi) \right) \xi + \phi(x, \eta, \xi) \frac{\partial^2}{\partial \eta^2} \kappa(x, \eta, \xi) - 2 \phi(x, \eta, \xi) \left( \frac{\partial^2}{\partial \eta \partial \xi} \kappa(x, \eta, \xi) \right) \xi - \\
- 2 \phi(x, \eta, \xi) \frac{\partial}{\partial \eta} \kappa(x, \eta, \xi) + \phi(x, \eta, \xi) \left( \frac{\partial^2}{\partial \xi^2} \kappa(x, \eta, \xi) \right) \xi^2 + 3 \phi(x, \eta, \xi) \left( \frac{\partial}{\partial \xi} \kappa(x, \eta, \xi) \right) \xi - \\
\]
\[-\left( \frac{\partial}{\partial \eta} \phi(x,\eta,\xi) \right) \frac{\partial}{\partial \eta} \kappa(x,\eta,\xi) + \left( \frac{\partial}{\partial \eta} \phi(x,\eta,\xi) \right) \left( \frac{\partial}{\partial \xi} \kappa(x,\eta,\xi) \right) \xi + \left( \frac{\partial}{\partial \xi} \phi(x,\eta,\xi) \right) \frac{\partial}{\partial \eta} \kappa(x,\eta,\xi) - \left( \frac{\partial}{\partial \xi} \phi(x,\eta,\xi) \right) \frac{\partial^2}{\partial \eta^2} \kappa(x,\eta,\xi) \xi^2 \right] \\
- \left( \frac{\partial}{\partial \xi} \phi(x,\eta,\xi) \right) \frac{\partial^2}{\partial \eta \partial \xi} \kappa(x,\eta,\xi) \xi - 4 \frac{\partial}{\partial \eta} \kappa(x,\eta,\xi) = 0,
\]

where
\[\eta = \ln(b'), \quad \xi = b/b'.\]

This form of equation can be used for construction of new examples of dual equations. In more general case exists the reduction in the form
\[f = y^k \omega(xy^{k-1}, \ln(y), xy^{k-2})\]

### 8 Examples

Here we discuss possibility to reception of new examples of dual equations.

Let
\[\phi(x, y, y') = C\]
be the first integral of the second order ODE
\[y'' = f(x, y, y').\]

Then from the relation
\[\phi_x + y' \phi_y + f(x, y, y') \phi_{y'} = 0\]
we find
\[f(x, y, y') = -\frac{\phi_x + y' \phi_y}{\phi_{y'}}.\] (26)

After substitution of this expression into the relation (8) we get the equation to determination of the function \(\phi(x, y, y').\)

Let us consider an examples.

Substitution of the expression (26) into the (9) lead to the equation
\[2 \left( \frac{d}{d \eta} A(\eta) \right)^2 \left( \frac{d^2}{d \eta^2} A(\eta) \right) A(\eta) - 2 \left( \frac{d}{d \eta} A(\eta) \right) \left( \frac{d^3}{d \eta^3} A(\eta) \right) (A(\eta))^2 - 3 \left( \frac{d}{d \eta} A(\eta) \right)^4 + \]
\[+ 3 \left( \frac{d^2}{d \eta^2} A(\eta) \right) (A(\eta))^2 = 0\] (27)
on the function
\[\phi(x, y, z) = A \left( \frac{y}{z} \right) x^{-1}, \quad \eta = \frac{y}{z}.\]

The solution of the equation (27) is
\[A(\eta) = \frac{LambertW(1/2 e^{1/4 C1 e^{1/4 x_1} e^{-1}})}{LambertW(1/2 e^{1/4 C1 e^{1/4 x_1} e^{-1}})} + C3\]
or
\[A \left( \frac{y}{z} \right) = LambertW(1/2 e^{1/4 \frac{\omega C1}{e^{-1}}} C3 \left( LambertW(1/2 e^{1/4 \frac{\omega C1}{e^{-1}}} + 1 \right)^{-1}.\]
Using this expression it is possible to find the second order ODE

\[
\left( \frac{d^2}{dx^2} y(x) \right) y(x) x C1 - 4 (y(x))^2 \left( \text{LambertW} \left( 1/2 e^{-1/4} \frac{\left( \frac{d}{dx} y(x) \right) C1 + 4y(x)}{y(x)} \right) \right)^2 - 8 (y(x))^2 \text{LambertW} \left( 1/2 e^{-1/4} \frac{\left( \frac{d}{dx} y(x) \right) C1 + 4y(x)}{y(x)} \right) - 4 (y(x))^2 - \left( \frac{d}{dx} y(x) \right)^2 C1 x = 0
\]

with General Integral

\[
y(x) = \left( -\frac{a}{x - a} \right)^{-4a} 16^x \left( -\frac{x}{x - a} \right) 4^x b (-2x + 2a)^{-4a} = 0. \quad (28)
\]

From here is followed that the equation

\[b'' = g(a, b, b')\]

must be cubic on the first derivative \(b'\).

Substitution of the expression (26) into the full equation (22) lead to the equation on the function \(\phi(x, y, z)\) having the solution in form

\[
\phi(x, y, z) = A \left( \frac{y}{z} \right) x^{-1}, \quad \eta = \frac{y}{z},
\]

where the function \(A(\eta)\) satisfies the condition

\[
- (A(\eta))^3 \left( \frac{d}{d\eta} A(\eta) \right)^3 + \frac{d^5}{d\eta^5} A(\eta) + 4 (A(\eta))^2 \left( \frac{d}{d\eta} A(\eta) \right)^2 \left( \frac{d^3}{d\eta^3} A(\eta) \right)^2 - 2 A(\eta) \left( \frac{d}{d\eta} A(\eta) \right)^4 \left( \frac{d^3}{d\eta^3} A(\eta) \right)^2 - 36 (A(\eta))^3 \left( \frac{d}{d\eta} A(\eta) \right) \left( \frac{d^3}{d\eta^3} A(\eta) \right)^2 - 8 (A(\eta))^3 \left( \frac{d^3}{d\eta^3} A(\eta) \right)^2 - 2 A(\eta) \left( \frac{d}{d\eta} A(\eta) \right)^4 \left( \frac{d^3}{d\eta^3} A(\eta) \right)^2 - 3 (A(\eta))^3 \left( \frac{d^3}{d\eta^3} A(\eta) \right)^2 + 3 (A(\eta))^3 \left( \frac{d^3}{d\eta^3} A(\eta) \right)^2 - 3 (A(\eta))^3 \left( \frac{d^3}{d\eta^3} A(\eta) \right)^2 + 3 (A(\eta))^2 \left( \frac{d}{d\eta} A(\eta) \right)^3 \left( \frac{d^3}{d\eta^3} A(\eta) \right) \left( \frac{d^3}{d\eta^3} A(\eta) \right) = 0
\]

which is generalization of the equation (27).

Its solution can be presented in form

\[
\eta = \int_{g}^{h(g)dg + C2} \frac{h(g)}{g^2} dg + C1,
\]

\[
A(\eta) = e^{\int_{g}^{h(g)dg + C3} \frac{h(g)}{g^2} dg},
\]

where the function \(h(g)\) satisfies the equation

\[
-2 \frac{(g^2 - 2g + 1) (h(g))^3}{g^2} - \frac{(4g - 3) (h(g))^2}{g^2} + \frac{3g^2 \frac{d}{dg} h(g)}{g^2} - 3 \left( \frac{d}{dg} h(g) \right) g - 1 \right) h(g) +
\]
\[ + \left( \frac{\frac{d^2}{dg^2}h(g)}{g} \right) g + 4 \frac{\frac{dh}{dg}}{h(g)} - 3 \left( \frac{\frac{dh}{dg}}{h(g)} \right)^2 = 0. \]

Its solution in turn is expressed through the solution of the Abel equation

\[ \frac{d}{da} b(a) = -a \left( 2a^2 + 7a + 6 \right) (b(a))^3 + (-3a - 7) (b(a))^2 - 3 \frac{b(a)}{a} \]

having elementary particular solutions.

In explicit form we get

\[ h(g) = -ae^{\int b(a)da + C_1} + a \left( e^{\int b(a)da + C_1} \right)^2 \]

and

\[ g = \frac{-1 + e^{\int b(a)da + C_1}}{e^{\int b(a)da + C_1}}. \]

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