Abstract

The formulation of the Gerasimov-Drell-Hearn sum rule is revisited, showing its connection with other sum rules occurring in electron scattering and discussing the problem of its saturation, both from the theoretical and experimental point of view. The generalisation to other nuclear targets is also reported.

1 Introduction

This Conference is the second of the series, following the Mainz edition of two years ago [1], testifying the growing interest of the hadronic physics community. Actually, the GDH sum rule is considered an important test of our knowledge of the electromagnetic excitation of the nucleon, since it is based on very general (and accepted) principles. On the other hand, the interest in the GDH has also been the starting point for a large variety of activity, both experimental and theoretical, concerning, among others, the electromagnetic production of meson, the excitation of baryon resonances, the spin structure of the nucleon and of various nuclear targets. In the last Conference, we have seen many interesting results, some of them, specially the experimental ones, still at a preliminary level. In particular a big effort is devoted to the experimental verification of the saturation of the GDH, work which is now feasible thanks to the modern facilities. We all expect that on these (and many other) topics, the Conference will produce exciting results.

In this introductory talk, I will briefly review the derivation of the GDH sum rule and point out some of the results presented two years ago for which some improvements and developments are expected.
2 An introduction to GDH

Here the main assumptions leading to the GDH sum rule are briefly reported [2].

The starting point is the consideration of the nucleon Compton scattering amplitude \( T(\nu, \theta) \), where \( \nu \) is the photon energy and \( \theta \) is the scattering angle. In the forward direction, the amplitude \( T(\nu, 0) = T(\nu) \) can be written as

\[
T(\nu) = \vec{\epsilon}_f^* \cdot \vec{\epsilon}_i f(\nu) + i (\vec{\epsilon}_f^* \times \vec{\epsilon}_i) \cdot \vec{\sigma} g(\nu),
\]

where \( \vec{\epsilon}_f \) and \( \vec{\epsilon}_i \) are respectively the final and initial photon polarisation vectors, \( \vec{\sigma} \) is the nucleon spin and \( f(\nu) \) and \( g(\nu) \) the non-spin-flip and the spin-flip amplitudes, respectively. According to crossing symmetry

\[
T(-\nu, -\gamma_f, -\gamma_i) = T(\nu, \gamma_i, \gamma_f).
\]

This means that \( f(\nu) \) (\( g(\nu) \)) is even (odd) in \( \nu \).

For small values of the photon energy, the amplitudes can be expanded in a power series. The first terms are fixed by the Low Energy Theorem (LET), which is a consequence of Lorentz and gauge invariance [3]. For the even amplitude \( f(\nu) \) we have:

\[
4\pi f(\nu) = -\frac{e^2}{m} + 4\pi (\alpha + \beta) \nu^2 + O(\nu^4),
\]

where \( e \), \( m \) being respectively the proton charge and mass, is the Thomson scattering amplitude; \( \alpha \) is the electric and \( \beta \) the magnetic polarisability of the nucleon. The odd amplitude \( g(\nu) \) is written as

\[
4\pi g(\nu) = -\frac{e^2}{2 m^2} \kappa^2 \nu + 4\pi \gamma \nu^3 + O(\nu^5),
\]

where \( \kappa \) is the anomalous magnetic moment of the nucleon and \( \gamma \) the forward polarisability.

The next step is made by invoking causality, which, in a time dependent description of scattering, states that any amplitude \( a(t) \) is equal to zero for \( t < 0 \), that is the scattering wave vanishes before that the incoming particle collides with the target. This obvious statement has important consequences on the properties of the scattering amplitudes. In fact, according to the Titchmarsh theorem [4], the following three statements are equivalent:

1. \( a(t) = 0 \) for \( t < 0 \),
2. the Fourier transform \( a(\nu) \) is an analytic function,
3. \( \text{Re} \ a(\nu) \) and \( \text{Im} \ a(\nu) \) satisfy the dispersion relations ("Hilbert transforms"):

\[
\text{Re} \ a(\nu) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\text{Im} \ a(\nu)}{(\nu' - \nu)} \, d\nu',
\]
\[ Im a(\nu) = - \frac{1}{\pi} P \int^{\infty}_{-\infty} \frac{Re a(\nu)}{\nu' - \nu} d\nu', \]  

(6)

Before applying Eq. (5), we have to remind the Optical theorem, a direct consequence of unitarity, which, in terms of the two amplitudes \( f(\nu) \) and \( g(\nu) \) is written as

\[ 4\pi Im f(\nu) = \frac{\nu}{2} (\sigma_{1/2} + \sigma_{3/2}) = \nu \sigma_{tot}, \]  

(7)

\[ 4\pi Im g(\nu) = \frac{\nu}{2} (\sigma_{1/2} - \sigma_{3/2}) = \nu \Delta \sigma_{tot}, \]  

(8)

where \( \sigma_h \), with \( h = 1/2, 3/2 \), are the helicity dependent cross sections: \( h = 1/2 \) (\( h = 3/2 \)) means that the incident photon and the target nucleon have antiparallel (parallel) spins.

Now, recalling that \( Re f(\nu) \) is even in \( \nu \), while \( Im f(\nu) \) is odd and vanishes below the pion threshold \( \nu_0 \), we can write, for \( \nu < \nu_0 \)

\[ 4\pi f(\nu) = \frac{2}{\pi} \int^{\infty}_{\nu_0} d\nu' \frac{\nu'^2 \sigma_{tot}(\nu')}{(\nu'^2 - \nu^2)}. \]  

(9)

The high energy behaviour of the total cross section poses a problem of convergence of the integral in Eq. (9), therefore it is convenient to adopt a subtraction technique

\[ 4\pi [f(\nu) - f(0)] = \frac{2}{\pi} \nu^2 \int^{\infty}_{\nu_0} d\nu' \frac{\sigma_{tot}(\nu')}{(\nu'^2 - \nu^2)}. \]  

(10)

A similar procedure for the amplitude \( g(\nu) \) leads to the unsubtracted dispersion relation

\[ 4\pi g(\nu) = \frac{2}{\pi} \nu \int^{\infty}_{\nu_0} d\nu' \frac{\Delta \sigma(\nu')}{(\nu'^2 - \nu^2)}. \]  

(11)

Finally, using the LET and the Taylor expansion of the dispersion integrals, comparing terms of the same order in \( \nu \), we arrive at the Baldin sum rule [5]

\[ \alpha + \beta = \frac{1}{2\pi^2} \int^{\infty}_{\nu_0} d\nu' \frac{\sigma_{tot}(\nu')}{\nu'^2}, \]  

(12)

the Gerasimov-Drell-Hearn sum rule [6]

\[ -\frac{e^2 \kappa^2}{2m^2} = \frac{1}{\pi} \int^{\infty}_{\nu_0} d\nu' \frac{\sigma_{1/2}(\nu') - \sigma_{3/2}(\nu')}{\nu'}, \]  

(13)

and the Gell-Mann-Goldberger-Thirring [7] formula for the spin polarisability

\[ \gamma = \frac{1}{4\pi^2} \int^{\infty}_{\nu_0} d\nu' \frac{\sigma_{1/2}(\nu') - \sigma_{3/2}(\nu')}{\nu'^3}. \]  

(14)
3 The $Q^2$-behaviour

The GDH integral can be generalised to virtual photon absorption, that is to electron scattering, by introducing an explicit $Q^2$ dependence. In this way, it can be put in relation with the spin structure function of the nucleon. Let us introduce the definition

$$I_1(Q^2) = \frac{2m^2}{Q^2} \int_0^1 g_1(x, Q^2) \, dx,$$

where $g_1(x, Q^2)$ is the spin structure function of the nucleon, $Q^2$ the momentum transfer and $x = \frac{Q^2}{2m \nu}$ is the Bjorken variable. Then Eq. (13) can be rewritten

$$-\frac{2\pi^2 \alpha}{m^2} \kappa^2 = \int_{\nu_0}^{\infty} d\nu' \frac{\sigma_{1/2}(\nu') - \sigma_{3/2}(\nu')}{\nu'} = \frac{8\pi^2 \alpha}{m^2} I_1(0),$$

in this way at the photon point we have

$$I_1(0) = -\frac{\kappa^2}{4}. \quad (17)$$

The extension to non zero $Q^2$ allows in particular a connection of the GDH field to the sum rules involving the spin structure functions of the nucleon. As examples one can quote the Ellis-Jaffe sum rule for the proton spin structure [9]

$$\Gamma_p^1(Q^2) = \int_0^1 g_p^1(x, Q^2) \, dx = 0.185, \quad (18)$$

whose value turns out to be different from the experimental results. For instance [10],

$$\Gamma_p^1_{\text{exp}} = 0.141 \pm 0.011 \quad \text{at} \quad Q^2 = 5 (\text{GeV/c})^2, \quad (19)$$

as it is well known, this kind of discrepancy has triggered the so call "spin crisis".

Another example of related sum rule is the Bjorken one [11]

$$\Gamma_1^p - \Gamma_1^n = \frac{1}{6} \frac{g_A}{g_V} [1 - \frac{\alpha_S(Q^2)}{\pi}] = 0.191 \pm 0.002, \quad (20)$$

where $\Gamma_1^n$ denotes the integral of the neutron spin structure, $g_A$ and $g_V$ are the axial-vector and vector constants, respectively, and $\alpha_S(Q^2)$ is the strong coupling constant. The Bjorken sum rule is in good agreement with the experimental value of about 0.20 [12].

There are many ways of generalising the GDH sum rule [13], that is of obtaining a $Q^2$-dependent equation which, at the photon point, reproduces the original Eq. (13). One of them uses the integral defined in Eq. (15)

$$I_1(Q^2) = \frac{2m^2}{Q^2} \int_0^1 g_1(x, Q^2) \, dx \quad (21)$$
Another one introduces the second spin structure function of the nucleon

\[ I_2(Q^2) = \frac{2m^2}{Q^2} \int_0^1 g_2(x, Q^2) \, dx \]

(23)

\[ = \frac{m^2}{8\pi^2\alpha} \int_{\nu_0}^{i} \frac{1 - x}{1 + y^2} \left( \sigma_{1/2} - \sigma_{3/2} - 2y \sigma_{LT'} \right) \frac{d\nu}{\nu}. \]

(24)

In both Eqs. (22) and (24) \( y = Q/\nu \) and \( \sigma_{LT'} \) is the longitudinal-transverse cross section.

As already mentioned, the integral \( I_1(Q^2) \) enters into the Ellis-Jaffe sum rule, while \( I_2(Q^2) \) is involved in the Burkhardt-Cottingham sum rule

\[ I_2(Q^2) = \frac{1}{4} \frac{G_M(Q^2) - G_E(Q^2)}{1 + \frac{Q^2}{4m^2}} \]

(25)

There are further generalisations, which make use of integrals involving different combinations of the cross sections, namely

\[ \sigma_{1/2} - \sigma_{3/2}, \quad \sigma_{1/2} - \sigma_{3/2} \frac{1 - x}{\sqrt{1 + y^2}}, \quad \sigma_{1/2} - \sigma_{3/2} (1 - x). \]

(26)

4 The saturation of the GDH sum rule for the nucleon

The l.h. side of Eq. (13) is known, since it is determined by the experimental value of the anomalous magnetic moment of the nucleon; in order to verify that the r.h. side has the same value one needs a complete knowledge of the total photoabsorption cross section as a function of the photon energy. Here arises the problem of its saturation, since such knowledge is limited by the availability of experimental data.

Many analyses have been performed in the past, based on the older data. The results are reported in Table 1, where the various evaluations of the GDH-integral (r.h. side of Eq. (13)) are given separately for the proton and the neutron, together with their differences, and compared with the experimental value given by the anomalous magnetic moment.

All the analyses, except the one by Bianchi and Thomas, overestimate the proton integral and underestimate the neutron one, leading to wrong (positive) values of the differences.

The recent measurements of the total photoabsorption cross section of the proton at MAMI allow the evaluation of the integral directly from the data. The resulting value, integrating the experimental data in the interval 200 – 800 MeV is 216 ± 6 ± 13 and seems to be somewhat higher than the expected sum rule. Further information on this point will be provided by the new data at higher energy from ELSA (see [23]).
Table 1: Different phenomenological analyses of the GDH sum rule.

| Name                  | $I_{GDH}^p$ (µb) | $I_{GDH}^n$ (µb) | $I_{p-n}^{GDH}$ (µb) |
|-----------------------|------------------|------------------|----------------------|
| Karliner [16]         | 204              | 232              | −28                  |
| Workman-Arndt [17]    | 257              | 189              | 68                   |
| Burkert-Li [18]       | 203              | 125              | 78                   |
| Sandorfi et al. [19]  | 289              | 160              | 130                  |
| L.N. Chang et al. [20]| 294              | 185              | 109                  |
| Drechsel-Krein [21]   | 261              | 180              | 81                   |
| Bianchi-Thomas [22]   | 207 ± 23         | 226 ± 22         | −19 ± 37             |

The energy range spanned by the present data covers for the major part the proton resonance region. It is therefore meaningful to try to analyze the GDH sum rule by means of the Constituent Quark Model, which gives a fairly consistent description of all resonances. In a simplified, un retarded approach to the photo-excitation of baryon resonances [24, 25], the saturation is substantially provided by the excitation of the $\Delta$-resonance, as it can be seen in Table 2 [24], where the multipole contributions relevant for the considered energy interval are reported. In fact, summing up all the multipoles, one is left with the operator $e^2 \sigma_0(3)$ coming from the $M_1$-multipole, which is mainly due to the $\Delta$-excitation.

The sum rule can be explicitly verified using the Isgur-Karl model [26]. The anomalous magnetic moment term in the r.h. side of Eq. (13) can be evaluated giving

\[ \frac{e^2 \kappa^2}{2 m^2} = 4 - 8 a_M^2 - 10 a_D^2 - 2 \tau_0 a_D^2, \]  

(27)

where $a_M$ and $a_D$ are the mixing amplitudes of, respectively, the mixed symmetry and D-wave states.

If the r.h. side is calculated by averaging the operator $e^2_3 \sigma_0(3)$ in the nucleon state as given by the Isgur-Karl model one gets a slightly different result. The equality is recovered [25] if relativistic corrections are taken into account, which affect the electric dipole contribution only.

5 GDH sum rule with nuclear targets

The GDH sum rule can be generalized to the case of a nuclear target with mass $M_i$, charge $Q$, spin $I$ and anomalous magnetic moment $\kappa$ in the
Table 2: Multipole contributions to the photo-excitation of the baryon resonance in the CQM. \( e_i \) and \( \sigma_0(i) \) are, respectively, the charge and the third spin component of the i-th quark, \( l_{\lambda 0} \) the third component of the angular momentum of the third quark.

| Multipole      | Operator                                                                 |
|----------------|---------------------------------------------------------------------------|
| M1/M1          | \( e_3^2 \sigma_0(3) + \frac{1}{9}(2e_3^2 + e_1e_2)l_{\lambda 0} \)     |
| M1/E2          | 0                                                                         |
| E2/E2          | \( -\frac{1}{9}(2e_3^2 + e_1e_2)l_{\lambda 0} \)                        |
| E1/E1          | \( \frac{2}{3}e_3^2\sigma_0(3) - \frac{1}{3}e_1e_2[\sigma_0(1) + \sigma_0(2)] + \frac{1}{9}(e_3^2 + e_1e_2)l_{\lambda 0} \) |
| E1/M2          | \( -\frac{2}{3}e_3^2\sigma_0(3) + \frac{1}{3}e_1e_2[\sigma_0(1) + \sigma_0(2)] - \frac{4}{9}(e_3^2 + e_1e_2)l_{\lambda 0} \) |
| M2/M2          | 0                                                                         |

following way \[27\]

\[
4\pi^2\frac{e^2}{M_i^2}\frac{\kappa^2}{I} = \int_{\nu_0}^{\infty} d\nu \frac{\sigma^P(\nu) - \sigma^A(\nu)}{\nu} = I_{GDH}, \quad (28)
\]

where \( \sigma^P \) and \( \sigma^A \) are the photoabsorption cross sections for parallel or antiparallel spins, respectively. The phenomenological value of the r.h. side of Eq. (28) is 0.65 \( \mu b \). The theoretical contributions, up to 550 \( MeV \), of the three channels \( \gamma d \rightarrow n p \), \( \gamma d \rightarrow d \pi^0 \), \( \gamma d \rightarrow N N \pi \) are, respectively, -413 \( \mu b \), 63 \( \mu b \), 167 \( \mu b \), which sum up to \( I_{GDH}(550) = -183 \mu b \). If, instead of the theoretical value 167 \( \mu b \) for the \( \gamma d \rightarrow N N \pi \) channel, one uses the sum of the corresponding proton and neutron quantities evaluated by means of a multipole analysis of experimental data, 331 \( \mu b \), one gets \( I_{GDH}(550) = -19 \mu b \), showing in any case that the higher energy contributions should be positive and of about the same size \[27\].

A particular attention is being devoted to the deuterium and \(^3He\) targets. Besides being interesting by themselves, such studies are performed with the hope of extracting information on the neutron photoabsorption cross section. Actually in a deuterium with \( J_z = +1 \), both proton and neutron have aligned spins if one neglects the presence of the \( D- \)state, which has a probability ranging from 4 to 7 \%. In the case of \(^3He\), the \( S- \)state, in the currently accepted descriptions of the three-nucleon system, has a probability of about 90 \%: in the corresponding configuration the proton pair have antiparallel spins, so that the target polarization is given by the neutron only. Of course, the results have to be corrected in order to take into account the presence of higher waves (at the 10 \% level).
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