Accelerating SGD for Highly Ill-Conditioned Huge-Scale Online Matrix Completion

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Abstract

The matrix completion problem seeks to recover a $d \times d$ ground truth matrix of low rank $r \ll d$ from observations of its individual elements. Real-world matrix completion is often a huge-scale optimization problem, with $d$ so large that even the simplest full-dimension vector operations with $O(d)$ time complexity become prohibitively expensive. Stochastic gradient descent (SGD) is one of the few algorithms capable of solving matrix completion on a huge scale, and can also naturally handle streaming data over an evolving ground truth. Unfortunately, SGD experiences a dramatic slow-down when the underlying ground truth is ill-conditioned; it requires at least $O(\kappa \log(1/\epsilon))$ iterations to get $\epsilon$-close to ground truth matrix with condition number $\kappa$. In this paper, we propose a preconditioned version of SGD that preserves all the favorable practical qualities of SGD for huge-scale online optimization while also making it agnostic to $\kappa$. For a symmetric ground truth and the Root Mean Square Error (RMSE) loss, we prove that the preconditioned SGD converges to $\epsilon$-accuracy in $O(\log(1/\epsilon))$ iterations, with a rapid linear convergence rate as if the ground truth were perfectly conditioned with $\kappa = 1$. In our experiments, we observe a similar acceleration for item-item collaborative filtering on the MovieLens25M dataset via a pair-wise ranking loss, with 100 million training pairs and 10 million testing pairs. [See supporting code at https://github.com/Hong-Ming/ScaledSGD.]

1 Introduction

The matrix completion problem seeks to recover an underlying $d \times d$ ground truth matrix $M$ of low rank $r \ll d$ from observations of its individual matrix elements $M_{i,j}$. The problem appears most prominently in the context of collaborative filtering and recommendation system, but also numerous other applications. In this paper, we focus on the symmetric and positive semidefinite variant of the problem, in which the underlying matrix $M$ can be factored as $M = ZZ^T$ where the factor matrix $Z$ is $d \times r$, though that our methods have natural extensions to the nonsymmetric case. We note that the symmetric positive semidefinite variant is actually far more common in collaborative filtering, due to the prevalence of item-item models, which enjoy better data (most platforms contain several orders of magnitude more users than items) and more stable recommendations (the similarity between items tends to change slowly over time) than user-user and user-item models.

For the full-scale, online instances of matrix completion that arise in real-world collaborative filtering, stochastic gradient descent or SGD is the only viable algorithm for learning the underlying matrix $M$. The basic idea is to formulate a candidate matrix of the form $XX^T$ with respect to a
learned factor matrix $X \in \mathbb{R}^{d \times r}$, and to minimize a cost function of the form $\phi(XX^T - M)$. Earlier work used the root mean square error (RMSE) loss $\|XX^T - M\|_F^2 = \sum_{i,j} (XX^T - M)_{ij}^2$, though later work have focused on pairwise losses like the BPR [1] that optimize for ordering and therefore give better recommendations. For the RMSE loss, the corresponding SGD iterations with (rescaled) learning rate $\alpha > 0$ reads

$$x_{i,+} = x_i - \alpha \cdot (x_i^T x_j - M_{ij}) x_j, \quad x_{j,+} = x_j - \alpha \cdot (x_j^T x_i - M_{ij}) x_i,$$  \hspace{1cm} (1)

where $M_{ij}$ is the sampled $(i,j)$-th element of the ground truth matrix $M$, and $x_i, x_j$ and $x_{i,+}, x_{j,+}$ denote the $i$-th and $j$-th rows of the current iterate $X_t$ and new iterate $X_{t+1}$. Pairwise losses like the BPR can be shown to have a similar update equation over three rows of $X$ [1]. Given that only two or three rows of $X$ are accessed and updated at any time, SGD is readily accessible to massive parallelization and distributed computing. For very large values of $d$, the update equation (1) can be run by multiple workers in parallel without locks, with vanishing probability of collision [2]. The blocks of $X$ that are more frequently accessed together can be stored on the same node in a distributed memory system.

Unfortunately, the convergence rate of SGD can sometimes be extremely slow. One possible explanation, as many recent authors have pointed out [3–6], is that matrix factorization models are very sensitive to ill-conditioning of the ground truth matrix $M$. The number of SGD iterations grows at least linearly the condition number $\kappa$, which here is defined as the ratio between the largest and the $r$-th largest singular values of $M$. Ill-conditioning causes particular concern because most real-world data are ill-conditioned. In one widely cited study [7], it was found that the dominant singular value accounts for only ~80% prediction accuracy, with diversity of individual preferences making up the remainder ill-conditioned singular values. Cloninger et al. [8] notes that there are certain applications of matrix completion that have condition numbers as high as $\kappa = 10^{15}$.

This paper is inspired by a recent full-batch gradient method called ScaledGD [4, 9] and a closely related algorithm PrecGD [5] in which gradient descent is made immune to ill-conditioning in the ground truth by right-rescaling the full-batch gradient by the matrix $(X^T X)^{-1}$. Applying this same strategy to the SGD update equation (1) yields the row-wise updates

$$x_{i,+} = x_i - \alpha \cdot (x_i^T x_j - M_{ij}) P x_j, \quad x_{j,+} = x_j - \alpha \cdot (x_j^T x_i - M_{ij}) P x_i,$$  \hspace{1cm} (2a)

in which we precompute and cache the preconditioner $P = (X^T X)^{-1}$ ahead of time\footnote{For an initialization, if the $d$ rows of $X_0$ are selected from the unit Gaussian as in $x_1, \ldots, x_d \sim \mathcal{N}(0, \sigma^2 I_d)$, then we can simply set $P_0 = \sigma^2 I$ without incurring the $O(d)$ time needed in explicitly computing $P_0 = (X_0^T X_0)^{-1}$.}, and update it after the iteration as

$$P_+ = (P^{-1} + x_{i,+} x_{i,+}^T + x_{j,+} x_{j,+}^T - x_i x_i^T - x_j x_j^T)^{-1}$$  \hspace{1cm} (2b)

by making four calls to the Sherman–Morrison rank-1 update formula

$$(P^{-1} + uu^T)^{-1} = P - \frac{Puu^TP}{1 + uu^TPu}.$$  \hspace{1cm} (3)

This way, the rescaled update equations use just $O(r^2)$ arithmetic operations, which for modest values of $r$ is only marginally more than the $O(r)$ cost of the unscaled update equations (1). Indeed, the nearest-neighbor algorithms inside most collaborative filters have exponential complexity with respect to the latent dimensionality $r$, and so are often implemented with $r$ small enough for (1) and (2) to have essentially the same runtime. Here, we observe that the rescaled update equations (2) preserve essentially all of the practical advantages of SGD for huge-scale, online optimization: it can also be run by multiple workers in parallel without locks, and it can also be easily implemented over distributed memory. The only minor difference is that separate copies of $P$ should be maintained by each worker, and resynchronized once differences grow large.

Contributions In this paper, we provide a rigorous proof that the rescaled update equations (1), which we name ScaledSGD, become immune to the effects of ill-conditioning in the underlying ground truth matrix. For symmetric matrix completion under the root mean squared error (RMSE) loss function, regular SGD is known to have an iteration count of $O(\kappa^2 \cdot d^2 \log(d/\epsilon))$ within a local neighborhood of the ground truth [10]. This figure is optimal in the dimension $d$, the rank
r, and the final accuracy $\epsilon$, but suboptimal by four exponents with respect to condition number $\kappa$. In contrast, we prove for the same setting that ScaledSGD attains an optimal convergence rate, converging to $\epsilon$-accuracy in $O(\kappa^{3}\log d/\epsilon)$ iterations for all values of the condition number $\kappa$. In fact, our theoretical result predicts that ScaledSGD converges as if the ground truth matrix is perfectly conditioned, with a condition number of $\kappa = 1$.

At first sight, it appears quite natural that applying the ScaledGD preconditioner to SGD should result in accelerated convergence. However, the core challenge of stochastic algorithms like SGD is that each iteration can have substantial variance that “drown out” the expected progress made in the iteration. In the case of ScaledSGD, a rough analysis would suggest that the highly ill-conditioned preconditioner should improve convergence in expectation, but at the cost of dramatically worsening the variance.

Surprisingly, we find in this paper that the specific scaling $(X^T X)^{-1}$ used in ScaledSGD not only does not worsen the variance, but in fact improves it. Our key insight and main theoretical contribution is Lemma 4, which shows that the same mechanism that allows ScaledGD to converge faster (compared to regular GD) also allows ScaledSGD to enjoy reduced variance (compared to regular SGD). In fact, it is this effect of variance reduction that is responsible for most ($\kappa^3$ out of $\kappa^4$) of our improvement over the previous state-of-the-art. It turns out that a careful choice of preconditioner can be used as a mechanism for variance reduction, while at the same time also fulfilling its usual, classical purpose, which is to accelerate convergence in expectation.

Related work Earlier work on matrix completion analyzed a convex relaxation of the original problem, showing that nuclear norm minimization can recover the ground truth from a few incoherent measurements [11–15]. This approach enjoys a near optimal sample complexity but incurs an $O(d^3)$ per-iteration computational cost, which is prohibitive for a even moderately large $d$. More recent work has focused more on a nonconvex formulation based on Burer and Monteiro [16], which factors the optimization variable as $M = XX^T$ where $X \in \mathbb{R}^{d \times r}$ and applies a local search method such as alternating-minimization [17–20], projected gradient descent [21, 22] and regular gradient descent [23–26]. A separate line of work [27–35] focused on global properties of nonconvex matrix recovery problems, showing that the problem has no spurious local minima if sampling operator satisfies certain regularity conditions such as incoherence or restricted isometry.

The convergence rate of SGD has been well-studied for general classes of functions [36–39]. For matrix completion in particular, Jin et al. [10] proved that SGD converges towards an $\epsilon$-accurate solution in $O(\kappa d^4 \log(1/\epsilon))$ iterations where $\kappa$ is the condition number of $M$. Unfortunately, this quartic dependence on $\kappa$ makes SGD extremely slow and impractical for huge-scale applications.

This dramatic slow down of gradient descent and its variants caused by ill-conditioning has become well-known in recent years. Several recent papers have proposed full-batch algorithms to overcome this issue [9, 40, 41], but these methods cannot be used in the huge-scale optimization setting where $d$ is so large that even full-vector operations with $O(d)$ time complexity are too expensive. As a deterministic full-batch method, ScaledGD [9] requires a projection onto the set of incoherent matrices at every iteration in order to maintain rapid convergence. Instead our key finding here is that the stochasticity of SGD alone is enough to keep the iterates as incoherent as the ground truth, which allows for rapid progress to be made. The second-order method proposed in [41] costs at least $O(d^3)$ per-iteration and has no straightforward stochastic analog. PreeGD [5] only applies to matrices that satisfies matrices satisfying the restricted isometry property, which does not hold for matrix completion.

2 Background: Linear convergence of SGD

In our theoretical analysis, we restrict our attention to symmetric matrix completion under the root mean squared error (RMSE) loss function. Our goal is to solve the following nonconvex optimization

$$
\min_{X \in \mathbb{R}^{d \times r}} \frac{1}{2} \|XX^T - ZZ^T\|_F^2 \quad \text{where } Z = [z_1, z_2, \ldots, z_n]^T \in \mathbb{R}^{d \times r}
$$

in which we assume that the $d \times d$ ground truth $ZZ^T \succeq 0$ matrix is exactly rank-$r$, with a finite condition number

$$
\kappa \overset{\text{def}}{=} \frac{\lambda_{\max}(ZZ^T)}{\lambda_r(ZZ^T)} = \frac{\lambda_{\max}(Z^T Z)}{\lambda_{\min}(Z^T Z)} < \infty.
$$
In order to be able to reconstruct $ZZ^T$ from a small number of measurements, we will also need to assume that the ground truth has small *coherence* \cite{42}

$$\mu \overset{\text{def}}{=} \frac{d}{r} \cdot \max_{1 \leq i \leq d} \|e_i^T Z (Z^T Z)^{-1/2}\|^2. \quad (5)$$

Recall that $\mu$ takes on a value from 1 to $d/r$, with the smallest achieved by dense, orthonormal choices of $Z$ whose rows all have magnitudes of $1/\sqrt{d}$, and the largest achieved by a ground truth $ZZ^T$ containing a single nonzero element. Assuming incoherence $\mu = O(1)$ with respect to $d$, it is a well-known result that all $d^2$ matrix elements of $ZZ^T$ can be perfectly reconstructed from just $O(dr \log d)$ random samples of its matrix elements \cite{12, 43}.

This paper considers solving (3) in the huge-scale, online optimization setting, in which individual matrix elements of the ground truth $(ZZ^T)_{i,j} = z_i^T z_j$ are revealed one-at-a-time, uniformly at random with replacement, and that a current iterate $X$ is continuously updated to streaming data. We note that this is a reasonably accurate model for how recommendation engines are tuned to user preferences in practice, although the uniformity of random sampling is admittedly an assumption made to ease theoretical analysis. Define the stochastic gradient operator as

$$SG(X) = 2d^2 \cdot (x_i^T x_j - z_i^T z_j)(e_i x_j^T + e_j x_i^T) \quad \text{where } (i, j) \sim \text{Unif}([d] \times [d]),$$

where $x_i, x_j \in \mathbb{R}^r$ are the $i$-th and $j$-th rows of $X$, and the scaling $d^2$ is chosen that, over the randomness of the sampled index $(i, j)$, we have exactly $\mathbb{E}[SG(X)] = \nabla f(X)$. Then, the classical online SGD algorithm can be written as

$$X_{t+1} = X_t - \alpha SG(X_t) \quad \text{where } \alpha > 0. \quad \text{(SGD)}$$

Here, we observe that a single iteration of SGD coincides with full-batch gradient descent in expectation, as in $\mathbb{E}[X_{t+1} | X_t] = X_t - \alpha \nabla f(X_t)$. Therefore, assuming that bounded deviations and bounded variances, it follows from standard arguments that the behavior of many iterations of SGD should concentrate about that of full-batch gradient descent $X_{t+1} = X_t - \alpha \nabla f(X_t)$.

Within a region sufficiently close to the ground truth, full-batch gradient descent is well-known to converge at a linear rate to the ground truth \cite{23, 44}. Within this same regime, Jin et al. \cite{10} proved that SGD also converges linearly. For an incoherent ground truth with $\mu = O(1)$, they proved that SGD with an aggressive choice of step-size is able to recover the ground truth to $\epsilon$-accuracy $O(\kappa^4 dr \log (d/\epsilon))$ iterations, with each iteration costing $O(r)$ arithmetic operations and selecting 1 random sample. This iteration count is optimal with respect to $d$, $r$, and $1/\epsilon$, although its dependence on $\kappa$ is a cubic factor (i.e. a factor of $\kappa^3$) worse than full-batch gradient descent’s figure of $O(\kappa \log(1/\epsilon))$, which is itself already quite bad, given that $\kappa$ in practice can readily take on values of $10^3$ to $10^6$.

**Theorem 1** (Jin, Kakade, and Netrapalli \cite{10}). For $Z \in \mathbb{R}^{d \times r}$ with $\sigma_{\max}(Z) = 1$ and $f(X) = \|XX^T - ZZ^T\|_F^2$ and $h_i(X) = \|e_i^T X\|^2$, define the following

$$f_{\max} \overset{\text{def}}{=} \left(\frac{1}{10\kappa}\right)^2, \quad h_{\max} \overset{\text{def}}{=} 20 \cdot \kappa^2 \cdot \frac{\mu r}{d}.$$  

For an initial point $X_0 \in \mathbb{R}^{d \times r}$ that satisfies $f(X_0) \leq \frac{1}{2} f_{\max}$ and $\max_i h_i(X_0) \leq \frac{1}{2} h_{\max}$, there exists some constant $c$ such that for any learning rate $\alpha < c \cdot (\kappa \cdot h_{\max} \cdot d^2 \log d)^{-1}$, with probability at least $1 - T/d^{10}$, we will have for all $t \leq T$ iterations of SGD that

$$f(X_t) \leq \left(1 - \frac{\alpha}{2\kappa}\right)^t \cdot f_{\max}, \quad \max_i h_i(X_t) \leq h_{\max}.$$  

The reason for Theorem 1’s additional $\kappa^3$ dependence beyond full-batch gradient descent is due to its need to maintain *incoherence* in its iterates. Using standard techniques on martingale concentration, one can readily show that SGD replicates a single iteration of full-batch gradient descent over an epoch of $d^2$ iterations. This results in an iteration count $O(\kappa \cdot d^2 \log (1/\epsilon))$ with an optimal dependence on $\kappa$, but the entire matrix is already fully observed after collecting $d^2$ samples. Instead, Jin et al. \cite{10} noted that the variance of SGD iterations is controlled by the step-size $\alpha$ times the maximum coherence $\mu_X = \frac{d}{r} \cdot \max_{i,t} \|e_i^T X_t\|^2$ over the iterates $X_1, X_{t-1}, \ldots, X_0$. If the iterates can be kept incoherent with $\mu_X = O(1)$, then SGD with a more aggressive step-size will reproduce an iteration of full-batch gradient descent after an epoch of just $O(dr \log d)$ iterations.
The main finding in Jin et al. [10]’s proof of Theorem 1 is that the stochasticity of SGD is enough to keep the iterates incoherent. This contrasts with full-batch methods at the time, which required an added regularizer [20, 30, 45] or an explicit projection step [9]. (As pointed out by a reviewer, it was later shown by Ma et al. [46] that full-batch gradient descent is able to maintain incoherence without a regularizer nor a projection.) Unfortunately, maintaining incoherence requires shrinking the step-size by a factor of $\kappa$, and the actual value of $\mu_X$ that results is also a factor of $\kappa^2$ worse than the original coherence $\mu$ of the ground truth $Z$. The resulting iteration count $O(\kappa^4 \cdot d \cdot \log(d/\epsilon))$ is made optimal with respect to $d$, $r$, and $1/\epsilon$, but only at the cost of worsening its the dependence on the condition number $\kappa$ by another three exponents.

Finally, the quality of the initial point $X_0$ also has a dependence on the condition number $\kappa$. In order to guarantee linear convergence, Theorem 1 requires $X_0$ to lie in the neighborhood $\|X_0 X_0^T - Z Z^T\|_F < \lambda_{\min}(Z Z^T) = O(\kappa^{-1})$. This dependence on $\kappa$ is optimal, because full-batch gradient descent must lose its ability to converge linearly in the limit $\kappa \to \infty$ [5, 6]. However, the leading constant can be very pessimistic, because the theorem must formally exclude spurious critical points $X_{\text{spur}}$ that have $\nabla f(X_{\text{spur}}) = 0$ but $f(X_{\text{spur}}) > 0$ in order to be provably correct. In practice, it is commonly observed that SGD converges globally, starting from an arbitrary, possibly random initialization [30], at a linear rate that is consistent with local convergence theorems like Theorem 1. It is now commonly argued that gradient methods can escape saddle points with high probability [47], and so their performance is primarily dictated by local convergence behavior [48, 49].

3  Proposed algorithm and main result

Inspired by a recent full-batch gradient method called ScaledGD [4, 9] and a closely related algorithm PrecGD [5], we proposed the following algorithm

$$X_{t+1} = X_t - \alpha S G(X_t) (X_t^T X_t)^{-1} \quad \text{where } \alpha > 0. \quad \text{(ScaledSGD)}$$

As we mentioned in the introduction, the preconditioner $P = (X^T X)^{-1}$ can be precomputed and cached in a practical implementation, and afterwards efficiently updated using the Sherman–Morrison formula. The per-iteration cost of ScaledSGD is $O(r^2)$ arithmetic operations and 1 random sample, which for modest values of $r$ is only marginally more than the cost of SGD.

Our main result in this paper is that, with a region sufficiently close to the ground truth, this simple rescaling allows ScaledSGD to converge globally to $\epsilon$-accuracy $O(\kappa^4 \cdot d \cdot \log(d/\epsilon))$ iterations, with no further dependence on the condition number $\kappa$. This iteration count is optimal with respect to $d$, $r$, $1/\epsilon$, and $\kappa$, and in fact matches SGD with a perfectly conditioned ground truth $\kappa = 1$. In our numerical experiments, we observe that ScaledSGD converges globally from a random initialization at the same rate as SGD as if $\kappa = 1$.

**Theorem 2** (Main). For $Z \in \mathbb{R}^{d \times r}$ with $\sigma_{\max}(Z) = 1$ and $f(X) = \|X X^T - Z Z^T\|_F^2$ and $g_i(X) = c_i^T X (X^T X)^{-1} X^T c_i$, select a radius $\rho < 1/2$ and set

$$f_{\text{max}} \overset{\text{def}}{=} \left(\frac{\rho}{\kappa}\right)^2, \quad g_{\text{max}} \overset{\text{def}}{=} \frac{2^4}{(1 - 2\rho)^2} \cdot \frac{\mu r}{d}.$$ 

For an initial point $X_0 \in \mathbb{R}^{d \times r}$ that satisfies $f(X_0) \leq \frac{1}{2} f_{\text{max}}$ and $\max_i g_i(X_0) \leq \frac{1}{2} g_{\text{max}}$, there exists some constant $c$ such that for any learning rate $\alpha < c \cdot (g_{\text{max}} + \rho) \cdot d^2 \log(d/\epsilon)^{-1}$, with probability at least $1 - T/d^\alpha$, we will have for all $t \leq T$ iterations of ScaledSGD that:

$$f(X_t) \leq \left(1 - \frac{\alpha}{2}\right)^t \cdot f_{\text{max}}, \quad \max_i g_i(X_t) \leq g_{\text{max}}.$$

Theorem 2 eliminates all dependencies on the condition number $\kappa$ in Theorem 1 except for the quality of the initial point, which we had already noted earlier as being optimal. Our main finding is that it is possible to maintain incoherence while making aggressive step-sizes towards a highly ill-conditioned ground truth $ZZ^T$. In fact, Theorem 2 says that, with high probability, the maximum coherence $\mu_X$ of any iterate $X_t$ will only be a mild constant factor of $\approx 16$ times worse than the coherence $\mu$ of the ground truth $ZZ^T$. This is particularly surprising in view of the fact that every iteration of ScaledSGD involves inverting a potentially highly ill-conditioned matrix $(X^T X)^{-1}$.
contrast, even without inverting matrices, Theorem 1 says that SGD is only able to keep \( \mu_X \) within a factor of \( \kappa^2 \) of \( \mu \), and only by shrinking the step-size \( \alpha \) by another factor of \( \kappa \).

However, the price we pay for maintaining incoherence is that the quality of the initial point \( X_0 \) now gains a dependence on dimension \( d \), in addition to the condition number \( \kappa \). In order to guarantee fast linear convergence independent of \( \kappa \), Theorem 2 requires \( X_0 \) to lie in the neighborhood \( \|X_0X_0^T - ZZ^T\|_F < \mu_0 \lambda_{\min}(ZZ^T)/d = (nd)^{-1} \), so that \( \rho \) can be set to be the same order of magnitude as \( \eta_{\max} \). In essence, the “effective” condition number of the ground truth has been worsened by another factor of \( d \). This shrinks the size of our local neighborhood by a factor of \( d \), but has no impact on the convergence rate of the resulting iterations.

In the limit that \( \kappa \to \infty \) and the search rank \( r \) becomes overparameterized with respect to the true rank \( r^* < r \) of \( ZZ^T \), both full-batch gradient descent and SGD slows down to a sublinear convergence rate, in theory and in practice [5, 6]. While Theorem 2 is no longer applicable, we observe in our numerical experiments that ScaledSGD nevertheless maintains its fast linear convergence rate as \( \kappa = 1 \). Following PrecGD [5], we believe that introducing a small identity perturbation to the scaling matrix of ScaledSGD, as in \((X^TX + \eta I)^{-1}\) for some \( \eta \approx \sqrt{f(X)} \), should be enough to rigorously extend Theorem 2 to the overparameterized regime. We leave this extension as future work.

## 4 Key ideas for the proof

We begin by explaining the mechanism by which SGD slows down when converging towards an ill-conditioned ground truth. Recall that

\[
E[SG(X)] = E[2d^2 \cdot (XX^T - ZZ^T) \langle \cdot \rangle \cdot (e_ie_i^T + e_je_j^T)] = 4(XX^T - ZZ^T)X = \nabla f(X).
\]

As \( XX^T \) converges towards an ill-conditioned ground truth \( ZZ^T \), the factor matrix \( X \) must become progressively ill-conditioned, with

\[
\lambda_{\min}(XX^T) = \lambda_{\min}(ZZ^T) \leq \lambda_{\min}(ZZ^T) + \|XX^T - ZZ^T\|_F \leq \frac{1 + \rho}{\kappa}.
\]

Therefore, it is possible for components of the error vector \( XX^T - ZZ^T \) to become “invisible” by aligning within the ill-conditioned subspaces of \( X \). As SGD progresses towards the solution, these ill-conditioned subspaces of \( X \) become the slowest components of the error vector to converge to zero. On the other hand, the maximum step-size that can be taken is controlled by the most well-conditioned subspaces of \( X \). A simple idea, therefore, is to rescale the ill-conditioned components of the gradient \( \nabla f(X) \) in order to make the ill-conditioned subspaces of \( X \) more “visible”.

More concretely, define the local norm of the gradient as \( \|\nabla f(X)\|_X = \|\nabla f(X)(XX^T)\|_F \) and its corresponding dual norm as \( \|\nabla f(X)\|_X^* = \|\nabla f(X)(XX^T)^{-1}\|_F \). It has long been known (see e.g. [23, 44]) that rescaling the gradient yields

\[
\|\nabla f(X)\|_X \overset{\text{def}}{=} 4(XX^T - ZZ^T)X(XX^T)^{-1/2}\|_F = 4 \cos \theta \cdot \|XX^T - ZZ^T\|_F,
\]

where \( \theta \) is the angle between the error vector \( XX^T - ZZ^T \) and the linear subspace \( \{YY^T + YXX^T : Y \in \mathbb{R}^{d \times r}\} \). This insight immediately suggests an iteration like \( X_+ = X - \alpha \nabla f(X)(XX^T)^{-1} \).

In fact, the gradients of \( f \) have some Lipschitz constant \( L \), so

\[
f(X_+) \leq f(X) - \alpha \langle \nabla f(X), \nabla f(X)(XX^T)^{-1} \rangle + \frac{L}{2} \alpha^2 \|\nabla f(X)(XX^T)^{-1}\|_F^2,
\]

\[
\leq f(X) - \alpha (\|\nabla f(X)\|_X^*)^2 + \frac{L}{2} \alpha^2 (\|\nabla f(X)^*\|_X^*)^2,
\]

\[
\leq \left[ 1 - \alpha \cdot 8 \cos^2 \theta \right] f(X) \quad \text{for } \alpha \leq 1/L_X.
\]

However, a naive analysis finds that \( L_X = L/\lambda_{\min}(XX^T) \approx L \cdot \kappa \), and this causes the step-size to shrink by a factor of \( \kappa \). The main motivating insight behind ScaledGD [4, 9] and later PrecGD [5] is that, with a finer analysis, it is possible to prove Lipschitz continuity under a local change of norm.

**Lemma 3** (Function descent). Let \( X, Z \in \mathbb{R}^{n \times r} \) satisfy \( \|XX^T - ZZ^T\|_F \leq \rho \lambda_{\min}(ZZ^T) \) where \( \rho < 1/2 \). Then, the function \( f(X) = \|XX^T - ZZ^T\|_F^2 \) satisfies

\[
f(X + V) \leq f(X) + \langle \nabla f(X), V \rangle + \frac{L_X}{2} \|V\|_X^2, \quad (\|\nabla f(X)\|_X^*)^2 \geq 13 \cdot f(X)
\]

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for all $\|V\|_X \leq C \cdot \sqrt{f(X)}$ with $L_X = 6 + 8C + 2C^2 = O(1 + C^2)$.

This same idea can be “stochastified” in a straightforward manner. Conditioning on the current iterate $X$, then the new iterate $X_+ = X - \alpha SG(X)(X^TX)^{-1}$ has expectation

$$
\mathbb{E}[f(X_+)] \leq f(X) - \alpha \langle \nabla f(X), \mathbb{E}[SG(X)(X^TX)^{-1}] \rangle + \alpha \frac{L_X}{2} \mathbb{E}[\|\|SG(X)\|_X^2] \tag{1}.
$$

The linear term evaluates as $\mathbb{E}[SG(X)(X^TX)^{-1}] = \nabla f(X)(X^TX)^{-1}$, while the quadratic term is

$$
\mathbb{E}[\|\|SG(X)\|_X^2] \leq \sum_{i,j} 4 d^2 \cdot (X X^T - Z Z^T)_{i,j}^2 \cdot 4 \max_i (\|\|e_i^T X\|_X^2) = 16 \cdot f(X) \cdot \max g_i(X),
$$

where $g_i(X) = e_i^T X (X^TX)^{-1} X^T e_i = (\|\|e_i^T X\|_X^2)^2$. Combined, we obtain geometric convergence

$$
\mathbb{E}[f(X_+)] \leq (1 - \alpha \cdot 8 \cos^2 \theta) f(X) \quad \text{for } \alpha = O(g_\text{max}^{-1} \cdot d^{-2}). \tag{6}
$$

We see that the step-size depends crucially on the incoherence $g_i(X) \leq g_\text{max}$ of the current iterate. If the current iterate $X$ is incoherent with $g_\text{max} = O(1/d)$, then a step-size of $\alpha = O(1/d)$ is possible, resulting in convergence in $O(dr \log(d/\epsilon))$ iterations, which can be shown using standard martingale techniques [10]. But if the current iterate is $g_\text{max} = O(1)$, then only a step-size of $\alpha = O(1/d^2)$ is possible, which forces us to compute $d^2$ iterations, thereby obviating the need to complete the matrix in the first place.

Therefore, in order for proof rapid linear convergence, we need to additionally show that with high probability, the coherency $g_k(X) = (\|\|e_k^T X\|_X^2)^2$ remains $O(1)$ throughout ScaledGD iterations. This is the most challenging part of our proof. Previous methods that applied a similar scaling to full-batch GD [9] required an explicit projection onto the set of incoherent matrices at each iteration. Applying a similar projection to ScaledSGD will take $O(d)$ time, which destroys the scalability of our method. On the other hand, Jin et al. [10] showed that the randomness in SGD is enough to keep the coherency of the iterates within a factor of $\kappa^2$ times worse than the coherence of the ground truth, and only by a step-size of at most $\alpha = O(\kappa^{-1})$.

Surprisingly, here we show that the randomness in ScaledSGD is enough to keep the coherence of the iterates with a constant factor of the coherence the ground truth, using a step-size with no dependence on $\kappa$. The following key lemma is the crucial insight of our proof. First, it says that function $g_k(X)$ satisfies a “descent lemma” with respect to the local norm $\|\cdot\|_X$. Second, and much more importantly, it says that descending $g_k(X)$ along the scaled gradient direction $\nabla f(X)(X^TX)^{-1}$ incurs a linear decrement $\frac{1-2\rho}{\rho} g_k(X)$ with no dependence of the condition number $\kappa$. This is in direct analogy to the function value decrement in (6), which has no dependence on $\kappa$, and in direct contrast to the proof of Jin et al. [10], which is only able to achieve a decrement of $(8/\kappa)g_k(X)$ due to the lack of rescaling by $(X^TX)^{-1}$.

**Lemma 4** (Coherence descent). Let $g_k(X) = e_k^T X (X^TX)^{-1} X^T e_k$. Under the same conditions as Lemma 3, we have

$$
g_k(X + V) \leq g_k(X) + \langle V, \nabla g_k(X) \rangle + \frac{5(\|\|V\|_X^2)^2}{1 - 2\|\|V\|_X^2},
$$

$$
\langle \nabla g_k(X), \nabla f(X)(X^TX)^{-1} \rangle \geq \left[ 1 - \frac{2\rho}{\rho^2} g_k(X) - \frac{1}{1 - \rho} \sqrt{g_k(X)g_k(Z)} \right].
$$

Conditioning on $X$, we have for the search direction $V = SG(X)(X^TX)^{-1}$ and $X_+ = X + V$

$$
\mathbb{E}[g_k(X_+)] \leq g_k(X) - \alpha \langle \nabla g_k(X), \mathbb{E}[V] \rangle + \alpha^2 \cdot \mathbb{E} \left[ \frac{(\|\|V\|_X^2)^2}{1 - 2\|\|V\|_X^2} \right] \tag{7}
$$

$$
\leq \left( 1 - \frac{1 - 2\rho}{\rho^2} \alpha \right) g_k(X) + \alpha \cdot \frac{1}{1 - \rho} \cdot \sqrt{g_k(X)g_k(Z)} + \alpha^2 \cdot \mathbb{E} \left[ \frac{(\|\|V\|_X^2)^2}{1 - 2\|\|V\|_X^2} \right]
$$

$$
\leq \left( 1 - \frac{1 - 2\rho}{\rho^2} \alpha \right) g_k(X) + \alpha \cdot \frac{\sqrt{\mu}}{g_\text{max}} \cdot g_\text{max} + \alpha^2 \cdot \frac{O(d^2 \cdot g_\text{max}^2 \cdot \rho)}{1 - O(g_\text{max}^2 \cdot \rho)}
$$

$$
\leq (1 - \xi) g_k(X) + \alpha \cdot \frac{\zeta}{2 g_\text{max}} \quad \text{for } \alpha = O(\rho^{-1} d^{-2}).
$$
It then follows that $g_k(X_+)$ converges geometrically towards $\frac{1}{2}g_{\max}$ in expectation, with a convergence rate $(1 - \zeta \alpha)$ that is independent of the condition number $\kappa$:

$$E[g_k(X_+) - \frac{1}{2}g_{\max}] \leq \left[(1 - \zeta \alpha) g_k(X) + \alpha \cdot \frac{\zeta}{2}g_{\max}\right] - \frac{1}{2}g_{\max} \leq (1 - \zeta \alpha) \left[g_k(X) - \frac{1}{2}g_{\max}\right].$$

The proof of Theorem 2 then follows from standard techniques, by making the two decrement conditions (6) and (7) into supermartingales and applying a standard concentration inequality. We defer the rigorous proof to appendix E.

5 Experimental validation

In this section we compare the practical performance of ScaledSGD and SGD for the RMSE loss function in Theorem 2 and two real-world loss functions: the pairwise RMSE loss used to complete Euclidean Distance Matrices (EDM) in wireless communication networks; and the Bayesian Personalized Ranking (BRP) loss used to generate personalized item recommendation in collaborative filtering. In each case, ScaledSGD remains highly efficient since it only updates two or three rows at a time, and the preconditioner $P$ can be computed through low-rank updates, for a per-iteration cost of $O(r^2)$. All of our experiments use random Gaussian initializations and an initial $P = \sigma^2 I$. To be able to accurately measure and report the effects of ill-conditioning on ScaledSGD and SGD, we focus on small-scale synthetic datasets in the first two experiments, for which the ground truth is explicitly known, and where the condition numbers can be finely controlled. In addition, to gauge the scalability of ScaledSGD on huge-scale real-world datasets, in the third experiment, we apply ScaledSGD to generate personalized item recommendation using MovieLens25M dataset [50], for which the underlying item-item matrix has more than 62,000 items and 100 million pairwise samples are used during training. (Due to space constraints, we defer the details on the experimental setup, mathematical formulations, and the actual update equations to Appendix A.) The code for all experiments are available at https://github.com/Hong-Ming/ScaledSGD.

Matrix completion with RMSE loss. The problem formulation is discussed in Section 3. Figure 1 plots the error $f(X) = \|XX^T - M\|^2_F$ as the number of epochs increases. As expected, in the well-conditioned case, both ScaledSGD and SGD converges quickly to the ground truth. However, in the ill-conditioned case, SGD slows down significantly while ScaledSGD converges at almost exactly the same rate as in the well-conditioned case.

![Figure 1: Matrix Completion with RMSE loss.](image)

Euclidean distance matrix (EDM) completion. The Euclidean distance matrix (EDM) is a matrix of pairwise distance between $d$ points in Euclidean space [51]. In applications such as wireless sensor networks, estimation of unknown distances, i.e., completing the EDM is often required. We
emphasize that this loss function is a pairwise loss, meaning that each measurement indexes multiple elements of the ground truth matrix.

To demonstrate the efficacy of ScaledSGD, we conduct two experiments where $D$ is well-conditioned and ill-conditioned respectively: **Experiment 1.** We uniformly sample 30 points in a cube center at origin with side length 2, and use them to compute the ground truth EDM $D$. In this case, each row $x_i \in \mathbb{R}^3$ corresponds to the coordinates of the $i$-th sample. The corresponding matrix $X \in \mathbb{R}^{30 \times 3}$ is well-conditioned because of the uniform sampling. **Experiment 2.** The ground truth EDM is generated with 25 samples lie in the same cube in experiment 1, and 5 samples lie far away from the the cube. These five outliers make the corresponding $X$ become ill-conditioned.

![Sample Positions](image1)

**Figure 2:** Euclidean distance matrix (EDM) completion. We compare the convergence rate of ScaledSGD and SGD for EDM completion for two set of samples. **(Upper right)** 30 samples are uniformly distributed in the pink cube center at origin. **(Upper left)** 25 samples (in blue) are uniformly distributed in the cube, 5 outlier samples (in orange) are outside of the cube. **(Lower left)** Sample uniformly in cube. **(Lower right)** Sample with outliers.

**Item-item collaborative filtering (CF).** In the task of item-item collaborative filtering (CF), the ground truth $M$ is a $d \times d$ matrix where $d$ is the number of items we wish to rank and the $i, j$-th of $M$ is a similarity measure between the items. Our goal is to learn a low-rank matrix that preserves the ranking of similarity between the items. For instance, given a pairwise sample $(i, j, k)$, if item $i$ is more similar to item $j$ than item $k$, then $M_{ij} > M_{ik}$. We want to learn a low-rank matrix that also has this property, i.e., the $i, j$-th entry is greater than the $i, k$-th entry.

To gauge the scalability of ScaledSGD on a huge-scale real-world dataset, we perform simulation on item-item collaborative filtering using a $62,000 \times 62,000$ item-item matrix $M$ obtained from MovieLens25M dataset. The CF model is trained using Bayesian Personalized Ranking (BPR) loss [1] on a training set, which consists of 100 million pairwise samples in $M$. The performance of CF model is evaluated using Area Under the ROC Curve (AUC) score [1] on a test set, which consists of 10 million pairwise samples in $M$. The BPR loss is a widely used loss function in the context of collaborative filtering for the task of personalized recommendation, and the AUC score is
a popular evaluation metric to measure the accuracy of the recommendation system. We defer the
detail definition of BPR loss and AUC score to Appendix A.4.

Figure 3 plots the training BPR loss and testing AUC score within the first epoch (filled with red) and
the second epoch (filled with blue). In order to measure the efficacy of ScaledSGD, we compare its
testing AUC score against a standard baseline called the NP-Maximum [1], which is the best possible
AUC score by non-personalized ranking methods. For a rigorous definition, see Appendix A.4.

We emphasize two important points in the Figure 3. First, the percentage of training samples
needed for ScaledSGD to achieve the same testing AUC scores as NP-Maximum is roughly 4 times
smaller than SGD. Though both ScaledSGD and SGD are able to achieve higher AUC score than
NP-Maximum before finishing the first epoch, ScaledSGD achieve the same AUC score as NP-
Maximum after training on 11% of training samples while SGD requires 46% of them. We note that
in this experiment, the size of the training set is 100 million, this means that SGD would require 35
million more iterations than ScaledSGD before it can reach NP-Maximum.

Second, the percentage of training samples needed for ScaledSGD to converge after the first epoch
is roughly 5 times smaller than SGD. Given that both ScaledSGD and SGD converge to AUC score
at around 0.9 within the second epoch (area filled with blue), we indicate the percentage of training
samples when both algorithms reach 0.9 AUC score in Figure 3. As expected, ScaledSGD is able
to converge using fewer samples than SGD, with only 16% of training samples. SGD, on the other
hand, requires 81% training samples.

Figure 3: Huge-scale item-item collaborative filtering. (MovieLens25M dataset with \(|\Omega_{\text{train}}| = 100\) million and \(|\Omega_{\text{test}}| = 10\) million pairwise measurements). We compare the training BPR loss
and testing AUC score of ScaledSGD and SGD. (Left) Training BPR loss on the training set \(\Omega_{\text{train}}\).
(Right) Testing AUC score on the test set \(\Omega_{\text{test}}\).

6 Conclusions

We propose an algorithm called ScaledSGD for huge scale online matrix completion. For the non-
convex approach to solving matrix completion, ill-conditioning in the ground truth causes SGD to
slow down significantly. ScaledSGD preserves all the favorable qualities of SGD while making it
immune to ill-conditioning. For the RMSE loss, we prove that with an initial point close to the
ground truth, ScaledSGD converges to an \(\epsilon\)-accurate solution in \(O(\log(1/\epsilon))\) iterations, independent
of the condition number \(\kappa\). We also run numerical experiments on a wide range of other loss func-
tions commonly used in applications such as collaborative filtering, distance matrix recovery, etc.
We find that ScaledSGD achieves similar acceleration on these losses, which means that it is widely
applicable to many real problems. It remains future work to provide rigorous justification for these
observations.
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A Supplemental Details on Experiments in Main Paper

A.1 Experimental setup and datasets used

Simulation environment. We implement both ScaledSGD and SGD in MATLAB (version R2021a). All simulations in this paper are performed on a computer with Apple silicon M1 pro chip with 10-core CPU, 16-core GPU, and 32GB of RAM.

Datasets. The datasets we use for the experiments in the main paper are described below.

- **Matrix completion with RMSE loss**: In the simulation result shown in Figure 1, we synthetically generate both the well-conditioned and ill-conditioned ground truth matrix $M$. We fix both $M$ to be a rank-3 matrix of size $30 \times 30$. To generate $M$, we sample a random orthonormal matrix $U \in \mathbb{R}^{30 \times 3}$ and set $M = USU^T$. For well-conditioned case, we set $S = \text{diag}(2^{1}, 2^{1}, 2^{1})$, thus $M$ is perfectly conditioned with $\kappa = 1$. For ill-conditioned case, we let $S = \text{diag}(10^{3}, 10^{1}, 10^{-3})$, so that $M$ is ill-conditioned with $\kappa = 10^{4}$.

- **Euclidean distance matrix completion**: In this simulation shown in Figure 2, the ground truth Euclidean distance matrix $D$ for experiments 1 and 2 are generated with respect to their sample matrix $X$ as $D_{ij} = \|x_i - x_j\|_2^2$. For the sample points in **Experiment 1**, we randomly sample (without replacement) 30 points in 3-dimensional cube centered at origin with side length 2, and the corresponding sample matrix $X$ has conditioned number $\kappa = 1$. For the sample points in **Experiment 2**, we take the first 5 sample points in experiment 1 and perturb its x-coordinate by 10, and keep the rest of the 25 samples intact. The corresponding sample matrix $X$ has conditioned number $\kappa = 8.0828$.

- **Item-item collaborative filtering**: In this simulation shown in Figure 3, we use the Movielens25M dataset [50], which is a standard benchmark for algorithms for recommendation systems.\footnote{The MovieLens25M dataset is accessible at https://grouplens.org/datasets/movielens/25m/} This dataset consists of 25 million ratings over 62,000 movies by 162,000 users, the ratings are stored in an user-item matrix $G$ whose $(i,j)$-th entry is the rating that the $i$-th user gives the $j$-th movie. The rating is from 1 to 5, where a higher score indicates a stronger preference. If the $i,j$-th entry is 0, then no rating is given. For the simulation of item-item collaborative filtering, the $(i,j)$-th entry of the ground truth item-item matrix $M$ is the similarity score between the item $i$ and $j$, which can be computed by measuring cosine similarity between the $i$-th and $j$-th column of $G$.

Hyperparameter and initialization. We start ScaledSGD and SGD at the same initial point in each simulation. The initial points for each simulation are drawn from the standard Gaussian distribution.

- **Matrix completion with RMSE loss**: The step-size for both ScaledSGD and SGD are set to be $\alpha = 0.3$. The search rank for both ScaledSGD and SGD are set to be $r = 3$.

- **Euclidean distance matrix completion**: Since SGD is only stable for small step-size in EDM completion problem, while ScaledSGD can tolerance larger step-sizes, we pick the
A centralized dispatcher can later merge the updates to \( P \).

**Euclidean distance matrix (EDM) completion**

In practice, this low-rank update can be "pushed" onto a centralized storage of the preconditioner in practical scenario. Entries in machine learning, require Euclidean distance matrix to provide necessary services. However, loss function. Specifically, let \( X \in \mathbb{R}^{d \times r} \) be a matrix containing \( x_1, \ldots, x_d \) in its row and let \( M = XX^T \) be the Grammian of \( X \). Each entry of \( D \) can be written in terms of three entries in \( M \)

\[
D_{ij} = \|x_i - x_j\|^2 = x_i^T x_i - 2x_i^T x_j + x_j^T x_j = M_{ii} - 2M_{ij} + M_{jj}.
\]

**Item-item collaborative filtering:** The step-sizes for this experiment are set as follows: we first pick a small step-size and train the CF model over a sufficient number of epochs, this allows us to estimate the best achievable AUC score; we then set the step-sizes for both ScaledSGD and SGD to the largest possible step-size for which ScaledSGD and SGD is able to converge to the best achievable AUC score, respectively. The step-size for ScaledSGD \( \alpha = 1.000 \), step-size for SGD \( \alpha = 0.01 \). The search rank for both ScaledSGD and SGD are set to be \( r = 3 \).

### A.2 Matrix completion with RMSE loss

We now turn to the practical aspects of implementing ScaledSGD for RMSE loss function. In practical setting, suppose we are given a set \( \Omega = \{(i, j)\} \) that contains indices for which we know the value of \( M_{ij} \), our goal is to recover the missing elements in \( M \) by solving the following nonconvex optimization

\[
\min_{X \in \mathbb{R}^{d \times r}} f(X) = \frac{1}{2|\Omega|} \sum_{(i,j) \in \Omega} (x_i^T x_j - M_{ij})^2.
\]

The gradient of \( f(X) \) is

\[
\nabla f(X) = \frac{1}{|\Omega|} \sum_{(i,j) \in \Omega} (x_i^T x_j - M_{ij}) (e_i e_j^T + e_j e_i^T) X.
\]

**ScaledSGD update equations for RMSE loss.** Each iteration of ScaledSGD samples one element \((i, j) \in \Omega \) uniformly. The resulting iteration updates only two rows of \( X \)

\[
x_{i,+} = x_i - \alpha \cdot (x_i^T x_j - M_{ij}) P_{ij}, \quad x_{j,+} = x_j - \alpha \cdot (x_i^T x_j - M_{ij}) P_{ij}.
\]

The update on \( P \) is low-rank

\[
P_+ = (P^{-1} + x_{i,+} x_{i,+}^T + x_{j,+} x_{j,+}^T - x_i x_i^T - x_j x_j^T)^{-1},
\]

and can be computed by calling four times of rank-1 Sherman–Morrison–Woodbury (SMW) update formula in \( O(r^3) \) time

\[
(P^{-1} + uu^T)^{-1} = P - \frac{P u u^T P}{1 + u^T P u}, \quad (P^{-1} - uu^T)^{-1} = P + \frac{P u u^T P}{1 - u^T P u}.
\]

In practice, this low-rank update can be "pushed" onto a centralized storage of the preconditioner \( P \). Heuristically, independent copies of \( P \) can be maintained by separate, distributed workers, and a centralized dispatcher can later merge the updates to \( P \) by simply adding the cumulative low-rank updates onto the existing centralized copy.

### A.3 Euclidean distance matrix (EDM) completion

Suppose that we have \( d \) points \( x_1, \ldots, x_d \in \mathbb{R}^r \) in \( r \) dimensional space, the Euclidean distance matrix \( D \in \mathbb{R}^{d \times d} \) is a matrix of pairwise squared distance between \( d \) points in Euclidean space, namely, \( D_{ij} = \|x_i - x_j\|^2 \). Many applications, such as wireless sensor networks, communication and machine learning, require Euclidean distance matrix to provide necessary services. However, in practical scenario, entries in \( D \) that correspond to points far apart are often missing due to high uncertainty or equipment limitations in distance measurement. The task of Euclidean distance matrix completion is to recover the missing entries in \( D \) from a set of available measurement, and this problem can be formulated as a rank \( r \) matrix completion problem with respect to pairwise square loss function. Specifically, let \( X \in \mathbb{R}^{d \times r} \) be a matrix containing \( x_1, \ldots, x_d \) in its row and let \( M = XX^T \) be the Grammian of \( X \). Each entry of \( D \) can be written in terms of three entries in \( M \)

\[
D_{ij} = \|x_i - x_j\|^2 = x_i^T x_i - 2x_i^T x_j + x_j^T x_j = M_{ii} - 2M_{ij} + M_{jj}.
\]
Hence, given a set of sample $\Omega = \{(i, j)\}$ in $D$, the pairwise square loss function for EDM completion reads

$$\min_{X \in \mathbb{R}^{d \times r}} f(X) = \frac{1}{4|\Omega|} \sum_{(i,j) \in \Omega} \left( x_i^T x_i - 2x_i^T x_j + x_j^T x_j - D_{ij} \right)^2.$$ 

The gradient of $f(X)$ is

$$\nabla f(X) = \frac{1}{|\Omega|} \sum_{(i,j) \in \Omega} \left( x_i^T x_i - 2x_i^T x_j + x_j^T x_j - D_{ij} \right) \left[ (e_i e_i^T + e_j e_j^T) - (e_i e_j^T + e_j e_i^T) \right] X.$$ 

**ScaledSGD update equations for EDM completion.** Each iteration of ScaledSGD samples one element $(i, j) \in \Omega$. The resulting iteration updates only two rows of $X$

$$x_{i,+} = x_i - \alpha \cdot (x_i^T x_i - 2x_i^T x_j + x_j^T x_j - D_{ij}) P(x_i - x_j),$$

$$x_{j,+} = x_j - \alpha \cdot (x_j^T x_i - 2x_i^T x_j + x_j^T x_j - D_{ij}) P(x_j - x_i).$$

Similarly, the update on $P$ is low-rank and can be computed by calling four times of equation (8).

### A.4 Item-item collaborative filtering (CF)

In the task of item-item collaborative filtering, the ground truth $M$ is an $d \times d$ matrix where $d$ is the number of items we wish to rank and the $i, j$-th of $M$ is a similarity measure between the items. Our goal is to learn a low-rank matrix that preserves the ranking of similarity between the items. For instance, suppose that item $i$ is more similar to item $j$ than item $k$, then $M_{ij} > M_{ik}$, we want to learn a low-rank matrix $XX^T$ that also has this property, i.e., $x_i^T x_j > x_i^T x_k$ where $x_i$ is the $i$-th row of $X$.

**Similarity score.** An important building block of item-item recommendation systems is the so-called item-item similarity matrix [52–54], which we denote by $M$. The $i, j$-th entry of this matrix is the pairwise similarity scores of items $i$ and $j$. There are various measures of similarity. In our experiments we adopt a common similarity measure known as cosine similarity [53]. As a result, the item-item matrix can be computed from the user-item matrix. In particular, let $g_i, g_j$ denote the $i$-th and $j$-th columns of the user-item matrix $G$, corresponding to the ratings given by all users to the $i$-th and $j$-th items. Then the $(i, j)$-th element of the item-item matrix $M$ is set to

$$M_{ij} = g_i^T g_j/\left\|g_i\right\|\left\|g_j\right\|.$$ 

In general, the item-item matrix computed this way will be very sparse and not capable of generating good recommendations. Our goal is to complete the missing entries of this matrix, assuming that $M$ is low-rank. As we will see, we can formulate this completion problem as an optimization problem over the set of rank-$r$ matrices.

**Pairwise entropy loss (BPR loss).** The Bayesian Personalized Ranking (BRP) loss [1] is a widely used loss function in the context of collaborative filtering. For the task of predicting a personalized ranking of a set of items (videos, products, etc.), BPR loss often outperforms RMSE loss because it is directly optimized for ranking; most collaborative filtering models that use RMSE loss are essentially scoring each individual item based on user implicit feedbacks, in applications that only positive feedbacks are available, the models will not be able to learn to distinguish between negative feedbacks and missing entries.

The BPR loss in this context can be defined as follows. Let $\Omega = \{(i, j, k)\}$ denote a set of indices for which we observe the ranking of similarity between items $i, j, k$. Our observations are of the form $Y_{ijk} = 1$ if $M_{ij} > M_{ik}$ and $Y_{ijk} = 0$ otherwise. In other words, $Y_{ijk} = 1$ if item $i$ is more similar to item $j$ than to item $k$. We form a candidate matrix of the form $XX^T$, where $X \in \mathbb{R}^{d \times r}$. Our hope is that $XX^T$ preserves the ranking between the items. The BPR loss function is designed to enforce this property.

Let $x_i$ denote the $i$-th row of $X$ and set $z_{ijk} = (XX^T)_{ij} - (XX^T)_{ik} = x_i^T (x_j - x_k)$. The BPR loss attempts to preserve the ranking of samples in each row of $M$ by minimizing the logistic loss with respect to $(Y_{ijk}, \sigma(z_{ijk}))$, where $\sigma(\cdot)$ is the sigmoid function:

$$\min_{X \in \mathbb{R}^{n \times r}} f(X) = \frac{1}{|\Omega|} \sum_{(i,j,k) \in \Omega} -Y_{ijk} \log (\sigma(z_{ijk})) - (1 - Y_{ijk}) \log (1 - \sigma(z_{ijk})).$$
Thus, a higher AUC score indicates that the candidate matrix

\[ \text{AUC} = \sum_{(i,j,k) \in \Omega} \delta_{ijk}. \]

**Upper bounds on the non-personalized ranking AUC score (NP-Maximum).** As opposed to personalized ranking methods, non-personalized ranking methods generate the same ranking for every pair of item \( j \) and \( k \), independent of item \( i \). In the context of item-item collaborative filtering, the non-personalized ranking method can be defined as follows. Given a set of pairwise comparisons \( \Omega = \{(i,j,k)\} \) and observations \( Y_{ijk} \), we optimized the ranking between item \( j \) and \( k \) on a candidate vector \( x \), where \( x \in \mathbb{R}^d \).

Let \( x_i \) denote the \( i \)-th entry of \( x \), the non-personalized ranking method attempts to preserve the ranking between the \( x_j \) and \( x_k \) by minimizing the logistic loss with respect to \( \{Y_{ijk}, \sigma(x_j - x_k)\} \) where \( \sigma(\cdot) \) is the sigmoid function:

\[
\min_{x \in \mathbb{R}^d} f(x) = \frac{1}{|\Omega|} \sum_{(i,j,k) \in \Omega} -Y_{ijk} \log (\sigma(x_j - x_k)) - (1 - Y_{ijk}) \log (1 - \sigma(x_j - x_k)).
\]
The gradient of $f(x)$ is
\[
\nabla f(x) = \frac{1}{|\Omega|} \sum_{(i,j,k) \in \Omega} (\sigma(x_j - x_k) - Y_{ijk}) \left[ (e_j^T - e_k^T) \right] x,
\]
and the SGD update equations for $x_j$ and $x_k$ are
\[
x_{j,+} = x_j - \alpha \cdot (\sigma(x_j - x_k) - Y_{ijk}) x_j, \quad x_{k,+} = x_k + \alpha \cdot (\sigma(x_j - x_k) - Y_{ijk}) x_k.
\]

Notice that non-personalized ranking method is not a matrix completion problem, the regular SGD is used to minimized $f(x)$. To find the upper bound on the non-personalized ranking AUC score, we directly optimize the non-personalized ranking on the test set $\Omega_{test}$, and evaluated the corresponding AUC score on $\Omega_{test}$. Since we perform both training and evaluation on $\Omega_{test}$, this corresponding AUC score is the upper bound on the best achievable AUC score on $\Omega_{test}$.

**B Additional Experiments on pointwise cross-entropy loss**

This problem is also known as 1-bit matrix completion [55]. Here our goal is to recover a rank-$r$ matrix $M$ through binary measurements. Specifically, we are allowed to take independent measurements on every entry $M_{ij}$, which we denote by $Y_{ij}$. Let $\sigma(\cdot)$ denote the sigmoid function, then $Y_{ij} = 1$ with probability $\sigma(M_{ij})$ and $Y_{ij} = 0$ otherwise. After a number of measurements are taken on each entry in the set $\Omega$, let $y_{ij}$ denote the percentage of measurements on the $i, j$-th entry that is equal to 1. The plan is to find the maximum likelihood estimator for $M$ by minimizing a cross-entropy loss defined as follow
\[
\min_{X \in \mathbb{R}^{d \times r}} f(X) = \frac{1}{|\Omega|} \sum_{(i,j) \in \Omega} -y_{ij} \log (\sigma(x_i^T x_j)) - (1 - y_{ij}) \log (1 - \sigma(x_i^T x_j)).
\]

We assume an ideal case where the number of measurements is large enough so that $y_{ij} = \sigma(M_{ij})$ and the entries are fully observed. The gradient of $f(X)$ is
\[
\nabla f(X) = \frac{1}{|\Omega|} \sum_{(i,j) \in \Omega} (\sigma(x_i^T x_j) - y_{ij}) (e_i e_j^T + e_j e_i^T) X.
\]

**ScaledSGD update equations for pointwise cross-entropy loss.** Each iteration of ScaledSGD samples one element $(i,j) \in \Omega$ uniformly. The resulting iteration updates only two rows of $X$
\[
x_{i,+} = x_i - \alpha \cdot (\sigma(x_i^T x_j) - y_{ij}) P x_j, \quad x_{j,+} = x_j - \alpha \cdot (\sigma(x_i^T x_j) - y_{ij}) P x_i.
\]
The preconditioner $P$ can be updated by calling four times of equation (8) as in RMSE loss.

**Matrix completion with pointwise cross-entropy loss.** We apply ScaledSGD to perform matrix completion through minimizing pointwise cross-entropy loss. In this experiment, the well-conditioned and ill-conditioned ground truth matrix $M$ is the same as those in Figure 1, and the process of data generation are described in A.1. The learning rate for both ScaledSGD and SGD are set to be $\alpha = 1$. The search rank for both ScaledSGD and SGD are set to be $r = 3$.

Figure 4 plots the error $f(X) = \|XX^T - M\|_F^2$ against the number of epochs. Observe that the results shown in Figure 4 are almost identical to that of the RMSE loss shown in Figure 1. Ill-conditioning causes SGD to slow down significantly while ScaledSGD is unaffected.
C Additional Experiments with Noise

To mimic the real-world datasets, we corrupt each entry of ground truth matrix $M$ by white Gaussian noise. We first generate a noiseless well-conditioned and ill-conditioned matrix $\tilde{M}$ following the same procedure as the one described in Section 4.1. For the well-conditioned case, we set the singular value as $S = \text{diag}(10, 10, 10)$. For the ill-conditioned case, we set $S = \text{diag}(10, 10^{-1}, 10^{-3})$. To obtain a noisy ground truth, we generate a matrix of white Gaussian noise $W$ corresponding to a fixed signal to noise ratio (SNR), which is defined as $\text{SNR} = 20 \log_{10}(\|M\|_F/\|W\|_F)$. Finally, we set $M = \tilde{M} + W$. For the experiments in this section, we set $\text{SNR} = 15$ dB. For the case of well-conditioned $\tilde{M}$, the resulting $M = \tilde{M} + W$ is full-rank with condition number $\kappa = 310.72$. For the case of ill-conditioned $\tilde{M}$, the resulting $M$ is full-rank with condition number $\kappa = 423.5022$.

Matrix completion with RMSE loss on noisy datasets. We plot the convergence rate of ScaledSGD and SGD under the noisy setting in Figure 5. In this experiment, we pick a larger search rank $r = 5$ to accommodate the noisy ground truth. Observe that SGD slows down in both the well-conditioned and ill-conditioned case due to the addition of white Gaussian noise and the larger search rank $r$, while ScaledSGD converge linearly toward the noise floor.

We also plot the noise floor, which can be computed as follows. First we take the eigendecomposition of $M = QAQ^T$, where $Q$ is an orthonormal matrix and $A$ is a diagonal matrix containing the eigenvalues of $M$ sorted in descending order in its diagonal entries. Let $A'$ be a diagonal matrix such that $A'_{ii} = A_{ii}$ if $i \leq r$, and $A'_{ii} = 0$ otherwise, then the noise floor is defined as the RMSE between $M$ and its best rank-$r$ approximation $M' = QA'Q^T$, which is equal to $\frac{1}{\left|M\right|} \sum_{(i,j) \in \Omega} (M'_{ij} - M_{ij})^2$.

The step-sizes in the simulation are set to be the largest possible step-sizes for which ScaledSGD and SGD can converge to the noise floor. For ScaleSGD, the step-size is set to be $\alpha = 0.15$. For SGD, the step-size is set to be $\alpha = 0.01$. SNR = $20 \log_{10}(\|M\|_F/\|W\|_F) = 15$ dB.

Matrix completion with pointwise cross-entropy loss on noisy datasets. We plot the convergence rate of ScaledSGD and SGD under the noisy setting in Figure 6. Similar to RMSE loss in noisy setting, SGD show down in both well-conditioned and ill-conditioned case, while ScaledSGD converge linearly toward the noise floor. In this simulation, the search rank is set to be $r = 5$. The step-size are set to be the largest possible step-sizes for which ScaledSGD and SGD can converge to the noise floor. For ScaleSGD, the step-size is set to be $\alpha = 0.15$. For SGD, the step-size is set to be $\alpha = 0.01$. SNR = $15$ dB.
Figure 5: **Matrix Completion with RMSE loss in noisy setting.** We compare the convergence rate of ScaledSGD and SGD for noisy ground truth matrix $M = \tilde{M} + W$ computed with respect to a well-conditioned and ill-conditioned $\tilde{M}$ and white Gaussian noise $W$. (Left) Well-conditioned $\tilde{M}$. (Right) Ill-conditioned $\tilde{M}$.

Figure 6: **Matrix Completion with pointwise cross-entropy loss in the noisy setting.** We compare the convergence rate of ScaledSGD and SGD for noisy ground truth matrix $M = \tilde{M} + W$ computed with respect to a well-conditioned and ill-conditioned $\tilde{M}$ and white Gaussian noise $W$. (Left) Well-conditioned $\tilde{M}$. (Right) Ill-conditioned $\tilde{M}$.

### D Additional simulation on item-item collaborative filtering

Finally, we perform three additional experiments on item-item collaborative filtering in order to compare the ability of ScaledSGD and SGD to generate good recommendations using matrix factorization.

**Dataset.** For additional simulations on item-time collaborative filtering, we use the MovieLens-Latest-Small and MovieLens-Latest-Full datasets [50] in order to gauge the performance of our algorithm on different scales. First, we run a small-scale experiment on the MovieLens-Latest-Small dataset that has 100,000 ratings over 9,000 movies by 600 users. Second, we run a medium-scale and a large-scale experiment on the MovieLens-Latest-Full dataset with 27 million total ratings over 58,000 movies by 280,000 users.

**Experimental Setup.** The process of training a collaborative filtering model is described in A.4. The hyperparameters for the three experiments in this section are described below.

---

3 Both datasets are accessible at [https://grouplens.org/datasets/movielens/latest/](https://grouplens.org/datasets/movielens/latest/)
• **MovieLens-Latest-Small dataset:** In the small-scale experiment, we sample $|\Omega_{\text{train}}| = 1$ million and $|\Omega_{\text{test}}| = 100,000$ pairwise observations for training and testing, respectively. We set our search rank to be $r = 3$, so the optimization variable $X$ is of size $9000 \times 3$. Both ScaledSGD and SGD are initialized using a random Gaussian initial point. For ScaledSGD the step-size is $10^{-3}$ and for SGD the step-size is $5 \times 10^{-2}$.

• **MovieLens-Latest-Full dataset:** In the medium-scale experiment, we sample $|\Omega_{\text{train}}| = 10$ million and $|\Omega_{\text{test}}| = 1$ million pairwise observations for training and testing, respectively. In the large-scale experiment, we sample $|\Omega_{\text{train}}| = 30$ million and $|\Omega_{\text{test}}| = 3$ million pairwise observations for training and testing, respectively. In both cases, we set our search rank to be $r = 3$, so the optimization variable $X$ is of size $58000 \times 3$. For ScaledSGD the step-size is $5 \times 10^{-3}$ and for SGD the step-size is $5 \times 10^{-2}$.

**Results.** The results of our experiments for ScaledSGD and SGD are plotted in Figures 7, 8, and 9. In all three cases, ScaledSGD reaches the AUC scores that are greater than NP-Maximum’s within the first epoch, while SGD requires more than one epoch to achieve the same AUC score as NP-Maximum’s in the small-scale (Figure 7) and medium-scale (Figure 8) setting. In addition, of all three cases, ScaledSGD is able to converge to the asymptote of AUC score within the second epoch, while SGD needs more than 2 epochs to converge to the asymptote in the small-scale (Figure 7) and medium-scale (Figure 8) setting. These results demonstrate that ScaledSGD remain highly efficient across small-scale (Figure 7), medium-scale (Figure 8), large-scale (Figure 9) and huge-scale (Figure 3) settings.

![Figure 7](image.png)

**Figure 7:** Small-scale item-item collaborative filtering. (MovieLens-Latest-Small dataset with $|\Omega_{\text{train}}| = 1$ million and $|\Omega_{\text{test}}| = 100,000$ pairwise measurements). We compare the training BPR loss and testing AUC score of ScaledSGD and SGD. (Left) Training BPR loss on the training set $\Omega_{\text{train}}$. (Right) Testing AUC score on the test set $\Omega_{\text{test}}$.

**E Proof of the theoretical results**

In this section, we show that, in expectation, the search direction $V = SG(X)(X^TX)^{-1}$ makes a geometric decrement to both the function value $f$ and the incoherence $g$. A key idea is to show that the size of the decrement in $f$ is controlled by the coherence $g_{\text{max}} \geq g_k(X)$ of the current iterate, and this motivates the need to decrement $g_k$ in order to keep the iterates incoherent. Our key result is that both decrements are independent of the condition number $\kappa$.

**E.1 Preliminaries**

We define the inner product between two matrices as $\langle X, Y \rangle \overset{\text{def}}{=} \text{tr}(X^TY)$, which induces the Frobenius norm as $\|X\|_F = \sqrt{\langle X, X \rangle}$. The vectorization $\text{vec}(X)$ is the column-stacking operation that turns an $m \times n$ matrix into a length-$mn$ vector; it preserves the matrix inner product $\langle X, Y \rangle = \text{vec}(X)^T\text{vec}(Y)$ and the Frobenius norm $\|\text{vec}(X)\| = \|X\|_F$.

We denote $\lambda_i(M)$ and $\sigma_i(M)$ as the $i$-th eigenvalue and singular value of a symmetric matrix $M = M^T$, ordered from the most positive to the most negative. We will often write $\lambda_{\text{max}}(M)$ and
Figure 8: **Medium-scale item-item collaborative filtering.** (MovieLens-Latest-Full dataset with $|\Omega_{\text{train}}| = 10$ million and $|\Omega_{\text{test}}| = 1$ million pairwise measurements). We compare the BPR loss and AUC score of ScaledSGD and SGD (**Left**) Training BPR loss on the training set $\Omega_{\text{train}}$. (**Right**) Testing AUC score on the test set $\Omega_{\text{test}}$.

Figure 9: **Large-scale item-item collaborative filtering.** (MovieLens-Latest-Full dataset with $|\Omega_{\text{train}}| = 30$ million and $|\Omega_{\text{test}}| = 3$ million pairwise measurements). We compare the BPR loss and AUC score of ScaledSGD and SGD (**Left**) Training BPR loss on the training set $\Omega_{\text{train}}$. (**Right**) Testing AUC score on the test set $\Omega_{\text{test}}$.

Recall for any matrix $V \in \mathbb{R}^{d \times r}$, we define its local norm with respect to $X \in \mathbb{R}^{d \times r}$ as

$$
\|V\|_X = \|V(X^TX)^{1/2}\|_F, \quad \|V\|_X^* = \|V(X^TX)^{-1/2}\|_F.
$$

Also recall that we have defined the stochastic gradient operator

$$
SG(X) = 2d^2 \cdot (XX^T - ZZ^T)_{i,j} \cdot (e_i e_j^T + e_j e_i^T) X
$$

where $(i,j) \sim [d]^2$ is selected uniformly at random. This way SGD is written $X_+ = X - \alpha SG(X)$ and ScaledSGD is written $X_+ = X - \alpha SG(X)(X^TX)^{-1}$ for step-size $\alpha > 0$.

### E.2 Function value convergence

Recall that Theorem 1, due to Jin et al. [10], says that SGD converges to $\epsilon$ accuracy in $O(\kappa^4 \log(1/\epsilon))$ iterations, with a four-orders-of-magnitude dependence on the condition number $\kappa$. By comparison, our main result Theorem 2 says that ScaledSGD converges to $\epsilon$ accuracy in $O(\log(1/\epsilon))$ iterations, completely independence on the condition number $\kappa$. In this section, we explain how the first two factors of $\kappa$ are eliminated, by considering the full-batch counterparts of these two algorithms.
First, consider full-batch gradient descent on the function \( f(X) = \|XX^T - ZZ^T\|_F^2 \). It follows from the local Lipschitz continuity of \( f \) that

\[
f(X_{t+1}) \leq f(X_t) - \alpha \langle \nabla f(X_t), \nabla f(X_t) \rangle + \alpha \cdot (L/2) \cdot \|\nabla f(X_t)\|_F^2
\]

\[
= f(X_t) - \alpha \|\nabla f(X_t)\|_F^2 + \alpha^2 \cdot (L/2) \cdot \|\nabla f(X_t)\|_F^2
\]

\[\text{(10)}\]

where \( X_{t+1} = X_t - \alpha \nabla f(X_t) \). Here, the linear progress term determines the amount of progress that can proportionally be made with a sufficiently small step-size \( \alpha \), whereas the inverse step-size term basically controls how large the step-size can be. In the case of full-batch gradient descent, it is long known that an \( X_t \) that is sufficiently close to \( Z \) will satisfy the following

\[
8\lambda_{\min}(Z^T Z) \cdot f(X_t) \leq \|\nabla f(X_t)\|_F^2 \leq 16\lambda_{\max}(Z^T Z) \cdot f(X_t)
\]

and therefore, taking \( \lambda_{\max}(Z^T Z) = 1 \) and \( \lambda_{\min}(Z^T Z) = \kappa^{-1} \) where \( \kappa \) is the condition number, we have linear convergence

\[
f(X_{t+1}) \leq (1 - \alpha \cdot 8\kappa^{-1} + \alpha^2 \cdot 8L) \cdot f(X_t) \leq (1 - \alpha \cdot 4\kappa^{-1}) \cdot f(X_t)
\]

for step-sizes of \( \alpha \leq 2/(\kappa L) \). Therefore, it follows from this analysis that full-batch gradient descent takes \( T = O(\kappa^2 \log(1/\epsilon)) \) iterations to converge to \( \epsilon \)-accuracy. In this iteration count, one factor of \( \kappa \) arises from the linear progress term, which shrinks as \( O(\kappa^{-1}) \) as \( \kappa \) grows large. The second factor of \( \kappa \) arises because the inverse step-size term is a factor of \( \kappa \) larger than the linear progress term, which restricts the maximum step-size to be no more than \( O(\kappa^{-1}) \).

The following lemma, restated from the main text, shows that an analogous analysis for full-batch ScaledGD proves an iteration count of \( T = O(\log(1/\epsilon)) \) with no dependence on the condition number \( \kappa \). In fact, it proves that full-batch ScaledGD converges like full-batch gradient descent with a perfect condition number \( \kappa = 1 \).

**Lemma 5 (Function descent, Lemma 3 restated).** Let \( X, Z \in \mathbb{R}^{n \times r} \) satisfy \( \|XX^T - ZZ^T\|_F \leq \rho \lambda_{\min}(Z^T Z) \) where \( \rho < 1/2 \). Then, the function \( f(X) = \|XX^T - ZZ^T\|_F^2 \) satisfies

\[
|f(X + V) - f(X) - \langle \nabla f(X), V \rangle| \leq \frac{L_X}{2} \cdot \|V\|_X^2,
\]

\[
13 \cdot f(X) \leq \|\nabla f(X)\|_X^2 \leq 16 \cdot f(X),
\]

\[\text{(11)}\]

\[\text{(12)}\]

for all \( \|V\|_X \leq C \cdot \sqrt{f(X)} \) with \( L_X = 6 + 8C + 2C^2 = O(1 + C^2) \).

It follows that the iteration \( X_{t+1} = X_t - \alpha \nabla f(X_t)(X_t^T X_t)^{-1} \) yields

\[
f(X_{t+1}) \leq f(X_t) - \alpha \langle \nabla f(X_t), \nabla f(X_t)(X_t^T X_t)^{-1} \rangle + \alpha \cdot (L_X/2) \cdot \|\nabla f(X_t)(X_t^T X_t)^{-1}\|_X^2
\]

\[
= f(X_t) - \alpha \langle \nabla f(X_t), (X_t^T X_t)^{-1} \rangle^2 + \alpha^2 \cdot (L_X/2) \cdot \|\nabla f(X_t)(X_t^T X_t)^{-1}\|_X^2
\]

\[\text{(13)}\]

\[
\leq (1 - \alpha \cdot 8 + \alpha^2 \cdot 8L_X) \cdot f(X_t) \leq (1 - \alpha \cdot 4) \cdot f(X_t)
\]

\[\text{(14)}\]

for step-sizes of \( \alpha \leq 2/L_X \), where \( L_X = 6 + 8(4) + 2(4)^2 \). Therefore, we conclude that full-batch ScaledGD takes \( T = O(\log(1/\epsilon)) \) iterations to converge to \( \epsilon \)-accuracy, as if the condition number were perfectly \( \kappa = 1 \).

Note that Lemma 5 has been proved in both Tong et al. [9] and Zhang et al. [5]. For completeness, we give a proof inspired by Zhang et al. [8].

**Proof of Lemma 5.** We prove (11) via a direct expansion of the quadratic

\[
\frac{f(X + V)}{2}(\nabla^2 f(X)[V,V]) = \frac{f(X)}{2}(\nabla^2 f(X)[V,V]) + 2\langle XX^T - ZZ^T, VX^T + VX \rangle + 2\langle XX^T - ZZ^T, VV \rangle + 2\langle VX^T, VX \rangle
\]

\[
+ \langle VV^T \rangle + \frac{1}{2}(\nabla^3 f(X)[V,V,V]) + \frac{1}{2}(\nabla^4 f(X)[V,V,V,V])
\]

\[23\]
and it then follows by simple counting that
\[ |f(X + V) - f(X) - \langle \nabla f(X), V \rangle| \leq \frac{L_2}{2} \|V\|^2_X + \frac{L_3}{6} \|V\|^3_X + \frac{L_4}{24} \|V\|^4_X, \]
\[ L_2 = 4 + 2 \frac{\|X^T - ZZ^T\|_F}{\lambda_{\min}(X^TX)}, \quad L_3 = \frac{24}{\lambda_{\min}(X^TX)^2}, \quad L_4 = \frac{24}{\lambda_{\min}(X^TX)^3}. \]
Now, from Weyl’s inequality that
\[ \lambda_{\min}(X^TX) = \lambda_1(X^TI) \geq \lambda_2(Z^TZ) - \|XX^T - ZZ^T\|_F \geq (1 - \rho) \cdot \lambda_{\min}(Z^TZ) \]
and therefore \( \|XX^T - ZZ^T\|_F / \lambda_{\min}(X^TX) \leq \rho / (1 - \rho) \leq 1 \) because \( \rho < 1/2 \). It follows that \( L_2 \leq 6 \). If \( \|V\|_X \leq C \cdot \sqrt{f(X)} \), then \( \frac{L_2}{2} \|V\|_X \leq 8C \) and \( \frac{L_4}{24} \|V\|^3_X \leq 2C^2. \)
For the upper-bound in (12), we have simply
\[ \|\nabla f(X)\|_X = 4 \|XX^T - ZZ^T\|_F \leq 4 \|XX^T - ZZ^T\|_F. \]
For the lower-bound in (12), we evoke Zhang et al. [5, Lemma 12] with RIP constant \( \eta = 0 \) to yield
\[ \|\nabla f(X)\|_X = \max_{Y \in [d \times r]} 2(YY^T + YY^T, XX^T - ZZ^T) = 4 \|XX^T - ZZ^T\|_F \cdot \cos \theta \]
in which \( \cos \theta \) is defined between \( XX^T - ZZ^T \) and the set \( \{ XX^T + YY^T : Y \in [d \times r] \} \), as in
\[ \cos \theta = \max_{Y \in [d \times r]} \|XX^T + YY^T\|_F \cdot \|XX^T - ZZ^T\|_F. \]
It follows from Zhang et al. [5, Lemma 13] that
\[ \sin \theta = \frac{\|Z\|_F \cdot \|X - XX^T\|_F}{\|XX^T - ZZ^T\|_F} \leq \frac{1}{\sqrt{2}} \frac{\rho}{\sqrt{1 - \rho^2}}. \]
Hence, for \( \rho < 1/2 \), we have
\[ \frac{(\|\nabla f(X)\|_X)^2}{\|XX^T - ZZ^T\|_F^2} = 16 \cos^2 \theta \geq 16 \left( 1 - \frac{\rho^2}{2(1 - \rho^2)} \right) \geq 16 \left( 1 - \frac{1}{6} \right) = \frac{40}{3} > 13. \]
\[ \square \]
E.3 Coherence convergence
We now explain that ScaledSGD eliminates the last two factors of \( \kappa \) from SGD because it is able to keep its iterations a factor of \( \kappa^2 \) more coherent. First, consider regular SGD on the function \( f(X) = \|XX^T - ZZ^T\|_F^2 \). Conditioning on the current iterate, we have via the local Lipschitz continuity of \( f \):
\[ \mathbb{E}[f(X_+)] \leq f(X) - \alpha \langle \nabla f(X), \mathbb{E}[SG(X)] \rangle + \alpha \cdot (L/2) \cdot \mathbb{E}[\|SG(X)\|_F^2] \]
\[ = f(X) - \alpha \langle \nabla f(X), \|SG(X)\|_F^2 \rangle + \alpha^2 \cdot (L/2) \cdot \mathbb{E}[\|SG(X)\|_F^2] \]
\[ \text{linear progress} \quad \text{inverse step-size} \quad (15) \]
where \( X_+ = X - \alpha SG(X) \). In expectation, the linear progress term of SGD coincides with that of full-batch gradient descent in (10). The inverse step-size term, however, is up to a factor of \( d^2 \) times larger. To see this, observe that
\[ \mathbb{E}[\|SG(X)\|_F^2] = \frac{1}{d^2} \sum_{i=1}^d \sum_{j=1}^d 2d^2 \cdot (XX^T - ZZ^T)_{i,j} \cdot (e_i e_j^T + e_j e_i^T) X_{i,j} \]
\[ = 4d^2 \cdot \sum_{i=1}^d \sum_{j=1}^d (XX^T - ZZ^T)_{i,j} ^2 \cdot (e_i e_j^T + e_j e_i^T) X_{i,j} \]
\[ = 4d^2 \cdot f(X) \cdot \|e_i e_j^T + e_j e_i^T\|_F^2 \cdot \max_i \|e_i e_i^T\|_F^2. \]
\[
\text{Here, we correct for a factor-of-two error in Zhang et al. [5, Lemma 12].}
\]
In a coarse analysis, we can simply bound \( \max_i \|e_i^T X\|^2_F \leq \lambda_{\max}(X^T X) = O(1) \) to yield
\[
f(X_+) \leq (1 - \alpha \cdot 8\kappa^{-1} + 16d^2 \cdot \alpha^2) f(X) \leq (1 - \alpha \cdot 4\kappa^{-1}) f(X)
\]
for step-sizes of \( \alpha \leq 4/(d\kappa^2) \). Hence, we conclude that it takes \( T = O(\kappa^2d^2\log(1/\epsilon)) \) iterations to converge to \( \epsilon \)-accuracy, with an epoch of \( d^2 \) iterations of SGD essentially recreating a single iteration of full-batch gradient descent. Unfortunately, the matrix is already fully observed after \( d^2 \) iterations, and so this result is essentially vacuous.

Here, Jin et al. [10] pointed out that the term \( h_{\max} = \max_i \|e_i^T X\|^2_F \) measures the coherence of the \( d \times r \) iterate \( X \), and can be as small as \( O(1/d) \) for small values of rank \( r = O(1) \). Conditioned on the current iterate \( X \), they observed that the function \( h_i(X) = \|e_i^T X\|^2_F \) converges towards a finite value in expectation
\[
E[h_i(X_+)] \leq (1 - \alpha \cdot 8\kappa^{-1}) h_i(X) + \alpha \cdot 8\sqrt{h_i(X)h_i(Z)} + 4\alpha^2/2 \cdot E[\|e_i^T S G(X)\|^2_F]
\]
\[
\leq (1 - \alpha \cdot 8\kappa^{-1}) h_i(X) + \alpha \cdot 8\sqrt{h_i(X)h_i(Z)} + 8\alpha^2 \cdot O(d^2\kappa^{-1}).
\]

Let us define \( \gamma \) as the ratio between the coherences of the ground truth \( Z \) and the iterate \( X \):
\[
\gamma = \max_j \|e_j^T Z\|^2_F / \max_j \|e_j^T Z\|^2_F \quad \implies \quad \max_j \|e_j^T Z\|^2_F \leq \gamma^{-1} \cdot h_{\max}.
\]
Crucially, we require \( \gamma = \kappa^2 \) in order for \( h_i(X) \) to converge towards \( 1/2 h_{\max} \) in expectation:
\[
E[h_i(X_+) - \frac{1}{2} h_{\max}] \leq (1 - \alpha \cdot 8\kappa^{-1}) h_i(X) + \alpha \cdot 4\gamma^{-1/2} h_{\max} - \frac{1}{2} h_{\max}
\]
\[
\leq (1 - \alpha \cdot 8\kappa^{-1}) \left[ h_i(X) - \frac{1}{1 - \alpha \cdot 8\kappa^{-1}} \right] \frac{1}{2} h_{\max}.
\]
As a consequence, we conclude that, while SGD is able to keep its iterates \( X \) incoherent, their actual coherence \( h_{\max} = \max_i \|e_i^T X\|^2_F \) is up to a factor of \( \kappa^2 \) worse than the coherence \( \max_j \|e_j^T Z\|^2_F \) of the ground truth \( Z \).

Using a standard supermartingale argument, Jin et al. [10] extended the analysis above to prove that if the ground truth \( Z \) has coherence \( \max_j \|e_j^T Z\|^2_F = O(1/d) \), then the SGD generates iterates \( X \) that have coherence \( \max_i \|e_i^T X\|^2_F \leq h_{\max} = O((\kappa^2/d) \log d) \), which is two factors worse in \( \kappa \) as expected. Combined, this proves that SGD converges to \( \epsilon \) accuracy in \( T = O(\kappa^4d^2 \log(d/\epsilon)) \) iterations with the step-size of \( \alpha = O(\kappa^{-1}d^{-1} h_{\max}^{-1}) \) and iterate coherence \( h_{\max} = O(\kappa^{-1}d^{-1} \log d) \), which is another two factors of \( \kappa \) worse than full-batch gradient descent.

The following lemma, restated from the main text, shows that an analogous analysis for ScaledSGD proves that the algorithm maintains iterates \( X \) whose coherences have no dependence on \( \kappa \). Here, we need to define a different incoherence function \( g_k(X) = \|e_i X(X^T X)^{-1}\|^2_F \equiv (\|e_i X\|^2_F) \) in order to “stochastic” our previous analysis for full-batch ScaledGD. Surprisingly, the factors of (\( X^T X^{-1} \) in both the new definition of \( g_k(X) \) and the search direction \( S G(X)(X^T X)^{-1} \)) do not hurt incoherence, but in fact improves it.

**Lemma 6** (Coherence descent, Lemma 4 restated). Let \( X, Z \in \mathbb{R}^{n \times r} \) satisfy \( \|XX^T - ZZ^T\|_F \leq \rho \lambda_{\min}(Z^TZ) \) where \( \rho < 1/2 \). Then, the functions \( f(X) = \|XX^T - ZZ^T\|_F^2 \) and \( g_k(X) = e_k^T X(X^T X)^{-1}X^T e_k \) satisfy
\[
|g_k(X + V) - g_k(X) - \langle V, \nabla g_k(X) \rangle| \leq \frac{5 \|V\|^2_F}{1 - 2\|V\|^2_F} \rho \|e_k\|^2_F \cdot \|X\|_F^2, .
\]
\[
\langle \nabla g_k(X), \nabla f(X)(X^T X)^{-1} \rangle \geq \left[ \frac{1 - 2\rho}{1 - \rho} g_k(X) - \frac{1}{1 - \rho} \sqrt{g_k(X)g_k(Z)} \right].
\]
Conditioning on \( X \), we have for the search direction \( V = SG(X)(X^T X)^{-1} \) and \( X_+ = X + V \)
\[
E[g_k(X_+)] \leq g_k(X) - \alpha \langle \nabla g_k(X), \epsilon[V] \rangle + \alpha^2 \cdot E \left[ \frac{(\|V\|^2_F)^2}{1 - 2\|V\|^2_F} \right]
\]
\[
\leq (1 - \zeta \alpha) g_k(X) + \alpha \cdot \frac{\zeta}{2} g_{\max} \quad \text{for } \alpha = O(\rho^{-1}d^{-2}) \tag{16}
\]
where $\zeta = \frac{1-2\rho}{1-\rho}$. It then follows that $g_k(X_+)$ converges geometrically towards $\frac{1}{2}g_{\max}$ in expectation, with a convergence rate $(1-\zeta\alpha)$ that is independent of the condition number $\kappa$:

$$\mathbb{E}\left[g_k(X_+) - \frac{1}{2}g_{\max}\right] \leq \left[(1-\zeta\alpha)g_k(X) + \alpha \cdot \frac{\zeta}{2}g_{\max}\right] - \frac{1}{2}g_{\max} \leq (1-\zeta\alpha)\left[g_k(X) - \frac{1}{2}g_{\max}\right].$$

Before we prove Lemma 6, we first prove a simple claim.

**Lemma 7** (Change of norm). The local norm $\|V\|_X^* = \|V(X^TX)^{-1/2}\|$ satisfies

$$\frac{(\|V\|_X)^2}{1+2\|Y-X\|_X + (\|Y-X\|_X)^2} \leq (\|V\|_Y)^2 \leq \frac{(\|V\|_X)^2}{1-2\|Y-X\|_X^*}.$$

**Proof.** The upper-bound follows because

$$\text{tr}(VP_Y V^T) = \text{tr}(VP_X^{1/2}[P_X^{-1/2} P_Y P_X^{-1/2}] P_X^{1/2} V^T) \leq \frac{\text{tr}(VP_X V^T)}{\lambda_{\min}[P_X^{1/2} P_Y^{-1} P_X^{1/2}]}$$

where $P_Y = (Y^TY)^{-1}$ and $P_X = (X^TX)^{-1}$ and therefore

$$P_X^{1/2} P_Y^{-1} P_X^{1/2} \succeq I + P_X^{1/2}[X^T(Y-X) + (Y-X)^T X]P_X^{1/2}$$

and $\sigma_{\max}[P_X^{1/2}[X^T(Y-X)P_X^{1/2}] \leq \|Y-X\|_X$ because $XP_X^{1/2}$ is orthonormal. The lower-bound follows similarly.

We are ready to prove Lemma 6.

**Proof of Lemma 6.** It follows from the intermediate value version of Taylor’s theorem that there exists some $\tilde{X} = X + tV$ with $t \in [0,1]$ that

$$g_i(X + V) - g_i(X) - \langle \nabla g_i(X), V \rangle = \frac{1}{2} \langle \nabla^2 g_i(\tilde{X}) [V], V \rangle.$$

Let $P = (X^TX)^{-1}$ and $U = e_i e_i^T (I - XPX^T)V$ and $G = VPX^T e_i e_i^T$. By direct computation, we have

$$\frac{1}{2} \langle \nabla g_i(X), V \rangle = \langle (I - XPX^T) e_i e_i^T XP, V \rangle = \langle U, XP \rangle = \langle I - XPX^T, G \rangle,$n

$$\frac{1}{2} \langle \nabla^2 g_i(X)[V], V \rangle = \langle UP - XP(U^TX + X^TU)P, V \rangle - \langle (I - XPX^T)(G + G^T)XP, V \rangle,$$

by differentiating $XP$ and $XPX^T$ respectively. A coarse count yields $\frac{1}{2} \langle \nabla^2 g_i(X)[V], V \rangle \leq 5(\|V\|_X^2)^2$ and therefore

$$\|g_i(X + V) - g_i(X) - \langle \nabla g_i(X), V \rangle \| \leq 5\|V\|_X^2 \leq \frac{5(\|V\|_X^2)^2}{1-2\|V\|_X^2} \leq \frac{5(\|V\|_X^2)^2}{1-2\|V\|_X^*},$$

which is the first claim. Now, observe that the two functions have gradient

$$\nabla g_i(X) = 2[I - X(X^TX)^{-1}] e_i e_i^T X(X^TX)^{-1}, \quad \nabla f(X) = 4(X^TX - ZZ^T)X.$$

Directly substituting yields

$$\frac{1}{8} \langle \nabla g_i(X), \nabla f(X)(X^TX)^{-1} \rangle = \langle (I - X(X^TX)^{-1}) X e_i e_i^T X(X^TX)^{-1}, (X^TX - ZZ^T)X(X^TX)^{-1} \rangle$$

$$= e_i^T X(X^TX)^{-2} X^T Z Z^T X(X^TX)^{-1} X^T e_i - e_i^T Z Z^T X(X^TX)^{-2} X^T e_i,$$

where the second line follows from the fact that

$$\langle (I - X(X^TX)^{-1}) X e_i e_i^T X(X^TX)^{-1}, X^T Z Z^T X(X^TX)^{-1} \rangle = 0.$$

The second claim follows from the following three identities

$$\lambda_{\min}(X^TX) \geq (1-\rho)\lambda_{\min}(Z^TZ) \quad (17)$$

$$e_i^T Z Z^T X(X^TX)^{-2} X^T e_i \leq \frac{1}{1-\rho} \|e_i^T X\|_X \cdot \|e_i^T Z\|_Z^2 \quad (18)$$

$$e_i^T X(X^TX)^{-2} X^T Z Z^T X(X^TX)^{-1} X^T e_i \geq \frac{1-2\rho}{1-\rho} \cdot (\|e_i^T X\|_X^*)^2 \quad (19)$$

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We have (17) via Weyl’s inequality:
\[ \lambda_{\min}(X^TX) = \lambda_r(XX^T) = \lambda_r(ZZ^T + XX^T - ZZ^T) \geq \lambda_r(ZZ^T) - \|XX^T - ZZ^T\|_F. \]
We have (18) by rewriting
\[ e_i^TZZ^T(X^TX)^{-2}XX^Te_i = (e_i^TP)(P^TZZ^T(X^TX)^{-2}XX^TP^TQ)Q^Te_i \]
\[ \leq \|e_i^TP\||ZZ^T(X^TX)^{-2}XX^T||Q^TQ\| \]
and rewriting \( ZZ^T = XX^T - E \) where \( E = XX^T - ZZ^T \) and evoking (17) as in
\[ \|ZZ^T(X^TX)^{-2}XX^T\| \leq \|XX^T(X^TX)^{-2}XX^T\| + \|E\| \|XX^T(X^TX)^{-2}XX^T\| \]
and noting that \( 1 + \frac{\rho}{1 - \rho} = \frac{1}{1 - \rho}. \) We have (19) again by substituting \( ZZ^T = XX^T - E \)
\[ e_i^T(X^TX)^{-2}XX^TZZ^T(X^TX)^{-1}XX^Te_i = e_i^T(X^TX)^{-1}XX^Te_i - e_i^T(X^TX)^{-2}XX^TXX^T(ZZ^T)(X^TX)^{-1}XX^Te_i \]
\[ \geq e_i^T(X^TX)^{-1}XX^Te_i \cdot (1 - \|E\| \|XX^T(X^TX)^{-2}XX^T\|) \leq \rho/(1 - \rho). \]

and then noting that \( 1 - \frac{\rho}{1 - \rho} = \frac{1 - 2\rho}{1 - \rho}. \)

\[ \square \]

E.4 Proof of the main result

In the previous two subsections, we showed that when conditioned on the current iterate \( X_t \), a single step of ScaledSGD \( X_{t+1} = X_t - \alpha SG(X_t)(X^TX_t)^{-1} \) is expected to geometrically converge both the loss function \( f \) and each of the incoherence functions \( g_i \), as in
\[ E[f(X_{t+1})] \leq (1 - \alpha)f(X_t), \quad E[g_i(X_{t+1}) - \frac{1}{2}g_{\max}] \leq \left( 1 - \frac{1 - 2\rho}{1 - \rho} \right) \left[ g_i(X_t) - \frac{1}{2}g_{\max} \right]. \]

In this section, we will extend this geometric convergence to \( T \) iterations of ScaledSGD. Our key challenge is to verify that the variances and maximum deviations of the sequences \( f(X_0), f(X_1), \ldots, f(X_T) \) and \( g_i(X_0), g_i(X_1), \ldots, g_i(X_T) \) have the right dependence on the dimension \( d \), the radius \( \rho \), the condition number \( \kappa \), the maximum coherence \( g_{\max} \), and the iteration count \( t \), so that \( T \) iterations of ScaledSGD with a step-size of \( \alpha \leq c/(\|g_{\max} + \rho\|^2 \log d) \) results in no more than a multiplicative factor of 2 deviation from expectation. Crucially, we must check that the cumulated deviation over \( T \) iterations does not grow with the iteration count \( T \), and that the convergence rate is independent of the condition number \( \kappa \). We emphasize that the actual approach of our proof via the Azuma–Bernstein inequality is textbook; to facilitate a direct comparison with SGD, we organize this section to closely mirror Jin et al. [10]’s proof of Theorem 1.

Let \( f_{\max} = \rho^2 \cdot \chi_{\min}(Z^TZ) \) and \( g_{\max} = \frac{16}{(1 - 2\rho)^2} \|g_{\max}\|_{\infty} \). Our goal is to show that the following event happens with probability \( 1 - T/d^{10} \):
\[ \mathcal{E}_t = \left\{ f(X) \leq \left( 1 - \frac{\alpha}{2} \right)^t f_{\max}, \quad \max_i g_i(X) \leq g_{\max} \right\} \quad \text{for all } t \leq T. \] (20)

Equivalently, conditioned on event \( \mathcal{E}_t \), we want to prove that the probability of failure at time \( t + 1 \) is \( \delta = 1/d^{10} \). We split this failure event into a probability of \( \delta \) that the function value clause fails to hold, as in \( f(X_{t+1}) > (1 - \alpha/2)^t f_{\max} \), and a probability of \( \delta \) that any one of the \( d \) incoherence clauses fails to hold, as in \( g_i(X_{t+1}) > g_{\max} \). Then, cumulated over \( T \) steps, the total probability of failure would be \( T \cdot \delta = T/d^{10} \) as desired.

We begin by setting up a supermartingale on the loss function \( f \). Our goal is to show that the variance and the maximum deviation of this supermartingale have the right dependence on \( \alpha, d, \rho, \kappa, g_{\max} \), so that a step-size of \( \alpha \leq c/(g_{\max}d^2 \log d) \) with a sufficiently small \( c > 0 \) will keep the cumulative deviations over \( T \) iterations within a factor of 2. Note that, by our careful choice of the coherence function \( g_i \), the following statement for ScaledSGD match the equivalent statements for SGD with a perfect condition number \( \kappa = 1 \); see Jin et al. [10, Section B.2].
Lemma 8 (Function value supermartingale). Let \( f(X) = \|XX^T - ZZ^T\|^2 \). Define \( f_{\max} = \rho^2 \cdot \lambda_{\text{max}}^2(Z^TZ) \) and \( g_{\max} = \frac{16}{(1-2\rho)^2} \lambda_{\text{max}} g_i(Z) \). For a sufficiently small \( c > 0 \), the following with learning rate \( \alpha \leq c/(g_{\max} \log d) \) is a supermartingale

\[
F_t = (1 - \alpha)^{-t} f(X_t) \cdot 1_{e_t},
\]

meaning that \( \mathbb{E}[F_{t+1}|X_t, \ldots, X_0] \leq F_t \) holds for all \( t \in \{0, 1, 2, \ldots, \} \). Moreover, there exist sufficiently large constants \( C_{\text{dev}}, C_{\text{var}} > 0 \) such that the following holds with probability one:

\[
\begin{align*}
\mathbb{E}[F_t|X_{t-1}, \ldots, X_0] - F_t &\leq C_{\text{dev}} \cdot \alpha \cdot d^2 \cdot g_{\max} \cdot (1 - \alpha)^{-t} \left( 1 - \frac{\alpha}{2} \right)^t f_{\max}, \\
\text{Var}[F_t|X_{t-1}, \ldots, X_0] &\leq C_{\text{var}} \cdot \alpha^2 \cdot d^2 \cdot g_{\max} \cdot (1 - \alpha)^{-2t} \left( 1 - \frac{\alpha}{2} \right)^{2t} f_{\max}^2.
\end{align*}
\]

Proof. The proof is technical but straightforward; it is deferred to Section E.5.

Lemma 9 (Function value concentration). Let the initial point satisfy \( f(X_0) \leq \frac{1}{2} f_{\max} \). Then, there exists a sufficiently small constant \( c > 0 \) such that for all learning rates \( \alpha < c/(g_{\max} \log d) \), we have

\[
\Pr \left( f_t(X_{t+1}) \leq (1 - \alpha)^t f(X_0) \right) = \Pr \left( \mathcal{E}_t \cap \left\{ f_t(X_{t+1}) > (1 - \alpha)^t f_{\max} \right\} \right) \leq \frac{1}{2} d^{10}.
\]

Proof. Let \( \sigma^2 = \sum_{\tau=1}^t \text{Var}[F_\tau|X_{\tau-1}, \ldots, X_0] \) and let \( R \) satisfy \( \mathbb{E}[F_\tau|X_{\tau-1}, \ldots, X_0] \leq X_\tau + R \) almost surely for all \( \tau \in \{1, 2, \ldots, t\} \). Recall via the standard Azuma–Bernstein concentration inequality for supermartingales that \( \Pr \left( F_t \geq F_0 + s \right) \leq \exp \left( -\frac{s^2}{2 \mathbb{E}^2 \mathbb{E}^2} \right) \). Equivalently, there exists a large enough constant \( C > 0 \) in \( s = C \cdot (1 - \alpha)^t \left[ \sqrt{\sigma^2} \log d + R \log d \right] \) such that the following is true

\[
\Pr \left( f(X_{t+1}) \leq (1 - \alpha)^t f(X_0) + s \right) \leq \frac{1}{2} d^{10}.
\]

Given that \( f(X_0) \leq \frac{1}{2} f_{\max} \) and therefore \( (1 - \alpha)^t f(X_0) \leq \frac{1}{2} \left( 1 - \frac{\alpha}{2} \right)^t \cdot f_{\max} \), holds by hypothesis, the desired claim is true if we can show that \( s \leq \frac{1}{2} \left( 1 - \frac{\alpha}{2} \right)^t \cdot f_{\max} \). Crucially, we observe that the variance term in \( s \) does not blow-up with time \( t \)

\[
(1 - \alpha)^{2t} \cdot \sigma^2 \leq f_{\max}^2 \cdot C_{\text{var}} \cdot d^2 \cdot g_{\max} \cdot \alpha^2 \cdot \sum_{\tau=1}^t (1 - \alpha)^{2t-2\tau} \left( 1 - \frac{\alpha}{2} \right)^{2\tau} \leq f_{\max}^2 \cdot C_{\text{var}} \cdot d^2 \cdot g_{\max} \cdot \alpha^2 \cdot \sum_{\tau=1}^t \left( 1 - \frac{\alpha}{1 - \alpha/2} \right)^{2t-2\tau} \leq f_{\max}^2 \cdot C_{\text{var}} \cdot d^2 \cdot g_{\max} \cdot \alpha
\]

due to the geometric series expansion \( \sum_{\tau=0}^t \beta^{t-\tau} = (1 - \beta^{t+1})/(1 - \beta) \). Substituting the deviations term, choosing a step-size \( \alpha \leq c/(\rho \rho^2 \log d) \) for sufficiently small \( c \) yields

\[
s = \left( 1 - \frac{\alpha}{2} \right)^t \left[ \sqrt{C_{\text{var}} \cdot d^2 \cdot g_{\max} \cdot \alpha \cdot \log d + C_{\text{dev}} \cdot d^2 \cdot g_{\max} \cdot \alpha \cdot \log d} \right] \cdot f_{\max} \leq \frac{1}{2} \left( 1 - \frac{\alpha}{2} \right)^t \cdot f_{\max}.
\]

We now set up a supermartingale on each of the incoherence functions \( g_i \). Again, our goal is to show that the variance and the maximum deviation of this supermartingale have the right dependence on \( \alpha, d, \rho, \kappa, g_{\max} \), so that a step-size of \( \alpha \leq c/(\rho \rho^2 \log d) \) with a sufficiently small \( c > 0 \) will keep the cumulative deviations over \( T \) iterations within a factor of 2. Note that Jin et al. [10, Section B.2]’s proof tracks a different function \( h_t(X) = e^T X^T \cdot e_t \) that is substantially simpler, but pays a penalty of two to three factors of the condition number \( \kappa \).
Lemma 10 (Incoherence supermartingale). Let \( g_i(X) = \epsilon_i^T X (X^T X)^{-1} X^T \epsilon_i \). Define \( g_{\max} = \max_{i \leq n} g_i(Z) \). For a fixed \( i \in [n] \) with sufficiently small \( c > 0 \), the following with learning rate \( \alpha < c/(\rho d^2 \log d) \) is a supermartingale

\[
G_{it} = (1 - \zeta \cdot \alpha)^t \left( g(X_t) - \frac{1}{2} g_{\max} \right) \text{ where } \zeta = 1 - \frac{2 \rho}{1 - \rho} < 1,
\]

meaning that \( \mathbb{E}[G_{it+1} | X_t, \ldots, X_0] \leq G_{it} \) holds for all \( t \in \{0, 1, 2, \ldots \} \). Moreover, there exist sufficiently large constants \( C_{\text{dev}}, C_{\text{var}} > 0 \) with no dependence on \( g_{\max}, n, t \) such that

\[
\mathbb{E}[G_{it} | X_t, \ldots, X_0] - G_{it} \leq C_{\text{dev}} \cdot \alpha \cdot d^2 \cdot \rho \cdot (1 - \zeta \cdot \alpha)^t g_{\max},
\]

\[
\mathbb{Var}[G_{it} | X_t, \ldots, X_0] \leq C_{\text{var}} \cdot \alpha^2 \cdot d^2 \cdot \rho^2 \cdot (1 - \zeta \cdot \alpha)^{-2t} g_{\max}^2.
\]

Proof. The proof is long but straightforward; it is deferred to Section E.5.

Lemma 11 (Incoherence concentration). Let the initial point satisfy \( \max_i g_i(X_0) \leq \frac{1}{2} g_{\max} \). Then, there exists a sufficiently small constant \( c > 0 \) such that for all learning rates \( \alpha < c/(\rho d^2 \log d) \), we have

\[
\Pr(g_i(X_{t+1}) \epsilon_t > g_{\max}) = \Pr(\epsilon_t \cap \{ g_i(X_{t+1}) > g_{\max} \}) \leq \frac{1}{2d^{1/4}}.
\]

Proof. Let \( \sigma^2 = \sum_{t=1}^T \mathbb{Var}[g_i | X_{t-1}, \ldots, X_0] \) and let \( R \) satisfy \( \mathbb{E}[G_{it} | X_{t-1}, \ldots, X_0] \leq X_t + R \) almost surely for all \( \tau \in \{1, 2, \ldots, t\} \). Recall via the standard Azuma–Bernstein concentration inequality for supermartingales that

\[
\Pr(G_{it} \geq G_{i0} + s) \leq \exp \left( -\frac{s^2 / 2}{\sigma^2 + Rs / 3} \right).
\]

Equivalently, there exists a large enough constant \( C > 0 \) such that the following is true

\[
\Pr \left( g_i(X_{t+1}) \epsilon_t \geq \frac{1}{2} g_{\max} + (1 - \zeta \cdot \alpha)^t \left( g(X_0) - \frac{1}{2} g_{\max} \right) + s' \right) \leq \frac{1}{2d^{1/4}}
\]

where \( s' = C \cdot (1 - \zeta \cdot \alpha)^t \left[ \sqrt{\sigma^2 \log d + R \log d} \right] \).

Given that \( g(X_0) \leq \frac{1}{2} g_{\max} \) holds by hypothesis, the desired claim is true if we can show that \( s' \leq \frac{1}{2} g_{\max} \). Crucially, we observe that the variance term in \( s' \) does not blow-up with time \( t \)

\[
(1 - \zeta \cdot \alpha)^{2t} \rho^2 \leq g_{\max}^2 \cdot C_{\var} \cdot d^2 \cdot \rho^2 \cdot \alpha^2 \sum_{t=1}^T (1 - \zeta \cdot \alpha)^{2t - 2t} \leq g_{\max}^2 \cdot C_{\var} \cdot d^2 \cdot \rho^2 \cdot \alpha
\]

due to the geometric series expansion \( \sum_{\tau=0}^T \beta^{T-\tau} = (1 - \beta^{T+1})/(1 - \beta) \). Substituting the deviations term, choosing a step-size \( \alpha \leq c/(\rho d^2 \log d) \) for sufficiently small \( c \) yields

\[
s' = O \left( \sqrt{\alpha \cdot d^2 \cdot \rho^2 \cdot g_{\max}^2 \cdot \log d} \right) + O \left( \alpha \cdot d^2 \cdot \rho \cdot g_{\max} \cdot \log d \right) = \frac{g_{\max}}{2}.
\]

In summary, Lemma 9 requires a step-size of \( \alpha \leq c/(g_{\max} d^2 \log d) \) to keep deviations on \( f \) small, while Lemma 11 requires a step-size of \( \alpha \leq c/(\rho d^2 \log d) \) to keep deviations on \( g_i \) small. Therefore, it follows that a step-size \( \alpha \leq c/(g_{\max} + \rho d^2 \log d) \) will keep both deviations small.

Proof of Theorem 2. For a step-size \( \alpha \leq c/(g_{\max} + \rho d^2 \log d) \) with sufficiently small \( c > 0 \), both concentration bounds Lemma 9 and Lemma 11 are valid. Combined, we take the trivial union bound to determine the probability of failure at the \((t+1)\text{-th}\) step, after succeeding after \( t \) steps:

\[
\Pr(\epsilon_t \cap \epsilon_{t+1}) = \sum_{i=1}^d \Pr(\epsilon_t \cap \{ g_i(X_{t+1}) \geq g_{\max} \}) + \Pr(\epsilon_t \cap \{ f(X_{t+1}) \geq (1 - \frac{\alpha}{2}) f_{\max} \}) \leq \frac{d}{2d^{1/4}}.
\]
Here, $\overline{E}_{t+1}$ denotes the complement of $E_{t+1}$. The probability of failure of the $T$-th step is then the cumulative probability of failing at the $(t+1)$-th step, after succeeding after $t$ steps, over all $t \leq T$:

$$\Pr(\overline{E}_T) \leq \sum_{t=1}^{T} \Pr(\overline{E}_{t-1} \cap \overline{E}_t) \leq \frac{T}{d^{10}}$$

and this proves that $E_T$ happens with probability $1 - T/d^{10}$ as desired. \hfill \blackslug

### E.5 Proofs of supermartingale deviations and variances

We will now verify the supermartingales and their deviations and variances in detail. We first begin by proving the following bounds on the size of the stochastic gradient.

**Lemma 12.** Let $X, Z \in \mathbb{R}^{n \times r}$ satisfy $\|XX^T - ZZ^T\|_F \leq \rho \cdot \lambda_{\min}(Z^T Z)$ with $\rho < 1/2$ and $\max_i e_i^T X (X^T X)^{-1}X e_i \leq g_{\max}$ and $\max_i e_i^T Z (Z^T Z)^{-1}Z e_i \leq g_{\max}$. Then, with respect to the randomness of the following

$$SG(X) = 2d^2 \cdot (XX^T - ZZ^T)_{i,j} \cdot (e_i^T e_j + e_j^T e_i) X$$

where $(i, j) \sim [d]^2$ is selected uniformly at random, we have:

1. $\|SG(X)\|_{\chi}^\prime \leq 8d^2 \cdot \frac{g_{\max}}{\lambda_{\min}(X^T X)} \cdot \|XX^T - ZZ^T\|_F$.

2. $\|SG(X)(X^T X)^{-1}\|_{\chi}^\prime \leq 16d^2 \cdot \frac{1}{\lambda_{\min}(X^T X)} \cdot \rho$.

3. $\mathbb{E}((\|SG(X)\|_{\chi}^\prime)^p \leq 2^{2p} \cdot d^{2(p-1)} \cdot \frac{g_{\max}^p}{\lambda_{\min}(X^T X)}$.

4. $\mathbb{E}((\|SG(X)(X^T X)^{-1}\|_{\chi}^\prime)^p \leq 2^{3p} \cdot d^{2(p-1)} \cdot \rho^p$.

**Proof.** Let us write $E = XX^T - ZZ^T$. To prove (i) we have

$$\|SG(X)\|_{\chi}^\prime = 2d^2 \cdot E_{i,j} \cdot (\|e_i^T e_j + e_j^T e_i\| X) \leq 4d^2 \cdot E_{i,j} \cdot \max_i \|e_i^T X\|_{\chi}^\prime$$

and if we write $Q_X = X(X^T X)^{-1/2}$ and $Q_Z = Z(Z^T Z)^{-1/2}$ we have

$$E_{i,j} = e_i^T Q_X e_j = e_i^T Q_X Q_X^T (XX^T - ZZ^T) e_j - e_i^T (I - Q_X Q_X^T) ZZ^T e_j$$

$$\leq \|e_i^T Q_X\| \|Q_X^T (XX^T - ZZ^T) e_j\| + \|e_i^T (I - Q_X Q_X^T) ZZ^T Q_Z\| \|Q_Z e_j\|$$

$$\leq g_{\max} \cdot \|XX^T - ZZ^T\|_F + \|XX^T - ZZ^T\|_F \cdot g_{\max}^2 \cdot \rho$$

we use the fact that $(I - Q_X Q_X^T)(XX^T - ZZ^T) = -(I - Q_X Q_X^T) ZZ^T$ in the first and last lines. To prove (ii) we have

$$\|SG(X)(X^T X)^{-1}\|_{\chi}^\prime = \frac{\|SG(X)(X^T X)^{-1}\|_{\chi}}{\lambda_{\min}(X^T X)} \leq \frac{8d^2 \cdot \frac{g_{\max}}{\lambda_{\min}(X^T X)} \cdot \rho \cdot \lambda_{\min}(Z^T Z)}{(1 - \rho) \cdot \lambda_{\min}(Z^T Z)} \leq 16d^2 \cdot \frac{g_{\max}^2}{\lambda_{\min}(X^T X)} \cdot \rho$$

where we used Weyl's inequality $\lambda_r(XX^T) \geq \lambda_r(ZZ^T) - \|XX^T - ZZ^T\|_F$. To prove (iii) we have

$$\mathbb{E}((\|SG(X)\|_{\chi}^\prime)^p = \frac{1}{d^2} \sum_{i,j} d^{2p} \cdot (2E_{i,j})^p \cdot (\|e_i^T e_j + e_j^T e_i\| X) \leq 2^{2p} \cdot d^{2(p-1)} \cdot \|E\|_F \cdot \max_i \|e_i^T X\|_{\chi}^\prime$$

where we used $(\sum_i x_i^p)^{1/2} \geq (\sum_i x_i^p)^{1/p}$ for any $p \geq 2$. The proof of (iv) follows identically by applying the proof of (ii) to the proof of (iii). \hfill \blackslug

We now prove the properties of the function value supermartingale $F_t$.

**Proof of Lemma 8.** Conditioning on the current iterate $X_t$ and the event $E_t$, the new iterate $X_{t+1} = X_t - \alpha SG(X_t)(X_t^T X_t)^{-1}$ has expectation

$$\mathbb{E}[f(X_{t+1})] \leq f(X_t) - \alpha \langle \nabla f(X_t), \mathbb{E}[SG(X_t)(X_t^T X_t)^{-1}] \rangle + \frac{L_X}{2} \alpha^2 \mathbb{E}[\|SG(X_t)\|_{\chi}^2]$$

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with \( L_X = O(1) \) by evoking Lemma 3 noting that \( \|\alpha SG(X_t)\|_{\chi}^X = O(1) \cdot \sqrt{f(X_t)} \) for the step-size \( \alpha \leq c/(g_{\max}d^2 \log d) \), since

\[
\|\alpha SG(X_t)\|_{\chi}^X = \alpha \cdot 2d^2 \cdot (X_tX_t^T - ZZ^T)_{i,j} \| (e_i e_j^T + e_j e_i^T)X_t \|_{\chi}^X \\
\leq \alpha \cdot 2d^2 \cdot \|X_tX_t^T - ZZ^T\|_{\infty} \cdot 2\sqrt{g_{\max}} \\
\leq \frac{c}{g_{\max}d^2 \log d} \cdot 2d^2 \cdot \sqrt{f(X_t)g_{\max}} \cdot 2\sqrt{g_{\max}} = \frac{4c}{\log d} \sqrt{f(X_t)}.
\]

The linear term evaluates simply as \( \mathbb{E}[SG(X)(X^T X)^{-1}] = \nabla f(X)(X^T X)^{-1} \), while the quadratic term evaluates

\[
\mathbb{E}((\|SG(X)\|_{\chi})^2) = \frac{1}{d^2} \sum_{i,j} 4d^2 \cdot (X X^T - ZZ^T)_{i,j}^2 \| (e_i e_j^T + e_j e_i^T)X \|_{\chi}^X \\
\leq \sum_{i,j} 4d^2 \cdot (X X^T - ZZ^T)_{i,j}^2 \cdot 4g_{\max} = 16 \cdot g_{\max} \cdot d^2 \cdot f(X)
\]

Combined, substituting \( (\|\nabla f(X)\|_{\chi})^2 \geq 13 \cdot f(X) \), it follows that we have geometric convergence

\[
\mathbb{E}[f(X_{t+1})] \leq f(X_t) - \alpha \langle \nabla f(X_t), [SG(X_t)(X_t^T X_t)^{-1}] \rangle + \frac{L_X}{2} \alpha^2 \cdot \mathbb{E}((\|SG(X_t)\|_{\chi})^2) \\
\leq (1 - 2\alpha) f(X_t) + L_X \cdot \alpha^2 \cdot 8 \cdot g_{\max} \cdot d^2 \cdot f(X_t) \leq (1 - \alpha) f(X_t)
\]

where we observe that we can pick a small enough constant \( c \) in the step-size \( \alpha \leq c/(g_{\max}d^2 \log d) \) so that

\[
L_X \cdot \alpha^2 \cdot 8 \cdot g_{\max} \cdot d^2 \cdot f(X_t) = \frac{c \cdot L_X \cdot 8 \cdot g_{\max} \cdot d^2}{g_{\max}d^2 \log d} \alpha f(X_t) \leq \alpha f(X_t).
\]

Now, to confirm that \( F_t \) is a martingale, it remains to see that

\[
\mathbb{E}[F_{t+1} | X_t] = (1 - \alpha)^{-(t+1)} \mathbb{E}[f(X_{t+1}) \cdot 1_{e_i} | X_t] \leq (1 - \alpha)^{-(t+1)} (1 - \alpha) f(X_t) 1_{e_i} \\
\leq (1 - \alpha)^{-t} f(X_t) 1_{e_{i-1}} = F_t,
\]

where the last inequality follows from \( 1_{e_i} \leq 1_{e_{i-1}} \).

We now bound the deviations on \( F_t \). Conditioning on the previous iterates \( X_t, \ldots, X_0 \), we observe that the \( f(X_t) \) terms cancel:

\[
f(X_{t+1}) \cdot 1_{e_i} - \mathbb{E}[f(X_{t+1}) \cdot 1_{e_i}] \leq -\alpha \langle \nabla f(X_t), [SG(X_t) - \mathbb{E} SG(X_t)](X_t^T X_t)^{-1} \rangle \\
+ \frac{\alpha^2 \cdot L_X}{2} (\|SG(X_t)\|_{\chi})^2 + \mathbb{E}((\|SG(X_t)\|_{\chi})^2) \cdot 1_{e_i}.
\]  

Here we have for the linear term

\[
\langle \nabla f(X_t), [SG(X_t)(X_t^T X_t)^{-1}] \rangle \cdot 1_{e_i} \leq \|\nabla f(X_t)\|_{\chi} \cdot \|SG(X_t)\|_{\chi} \cdot 1_{e_i} \\
\leq 4 \sqrt{f(X_t)} \cdot 4d^2 \sqrt{f(X_t)} \cdot g_{\max} \cdot 1_{e_i} = O(d^2g_{\max})f(X_t) \cdot 1_{e_i}
\]

and the quadratic term

\[
(\|SG(X_t)\|_{\chi})^2 \cdot 1_{e_i} \leq d^4 \cdot f(X_t) \cdot g_{\max}^2 \cdot 1_{e_i} = O(d^4g_{\max}^2)f(X_t) \cdot 1_{e_i}.
\]

Therefore, using the maximum value to bound the expectation, we have

\[
F_{t+1} - \mathbb{E}[F_{t+1} | X_t, \ldots, X_0] \leq \alpha (1 - \alpha)^{-t} \cdot \left[ O(d^2g_{\max})f(X_t) + \alpha O(d^4g_{\max}^2)f(X_t) \right] \cdot 1_{e_i} \\
\leq \alpha (1 - \alpha)^{-t} \cdot O(d^2g_{\max})f(X_t) \cdot 1_{e_i} \\
\leq C_{dev} \cdot \alpha (1 - \alpha)^{-t} \left( 1 - \frac{\alpha}{2} \right)^t f_{\max} \cdot d^2 \cdot g_{\max} \cdot 1_{e_i}
\]

where again we observe that a step-size like \( \alpha \leq c/(g_{\max}d^2 \log d) = O(d^{-2}g_{\max}^{-1}) \) yields the cancellation of exponents \( \alpha \cdot O(d^4g_{\max}^2) = O(d^2g_{\max}) \).
Finally, we bound the variance. Conditioned on all previous iterates $X_t, \ldots, X_0$, we have
\[
\text{Var}(\langle \nabla f(X_t), SG(X_t)(X_t^T X_t)^{-1} \cdot 1_{e_t} \rangle) \leq \mathbb{E}[\langle \nabla f(X_t), SG(X_t)(X_t^T X_t)^{-1} \cdot 1_{e_t} \rangle^2 - \mathbb{E}[\langle \nabla f(X_t), SG(X_t)(X_t^T X_t)^{-1} \cdot 1_{e_t} \rangle]^2] \leq O(\ell d g_{\text{max}}^3) \cdot f(X_t)^2 \cdot 1_{e_t},
\]
and also
\[
\text{Var}(\langle SG(X_t) \| X \rangle^2 \cdot 1_{e_t}) \leq \mathbb{E}[\langle SG(X_t) \| X \rangle^4 \cdot 1_{e_t}] = O(\ell^3 d^3 g_{\text{max}}^3) f(X_t)^2 \cdot 1_{e_t}.
\]
By the same expansion in (22) we have
\[
\text{Var}(F_{t+1}|X_t, \ldots, X_0) \leq \alpha^2 (1-\alpha)^{-2t} \cdot \mathbb{E}[\ell d g_{\text{max}}] \cdot f(X_t)^2 + \alpha^2 \mathbb{E}[\ell d g_{\text{max}}] f(X_t)^2 \cdot 1_{e_t} \leq \alpha^2 (1-\alpha)^{-2t} \cdot \mathbb{E}[\ell d g_{\text{max}}] \cdot f(X_t)^2 \cdot 1_{e_t} \leq C_{\text{var}} \cdot \alpha^2 (1-\alpha)^{-2t} \left(1 - \frac{\alpha}{2}\right)^{2t} f_{\text{max}} \cdot d^2 \cdot g_{\text{max}} \cdot 1_{e_t}
\]
where again we observe that a step-size like $\alpha \leq c/(g_{\text{max}} d^2 \log d) = O(d^{-2} g_{\text{max}}^{-1})$ yields the cancellation of exponents $\alpha^2 \cdot (\ell d g_{\text{max}}^3)$.

We now prove properties of the incoherence martingale.

**Proof of Lemma 10.** Conditioning on $X_t$ and the event $e_t$, we have for $V = SG(X_t)(X_t^T X_t)^{-1}$
\[
\mathbb{E}[g_t(X_{t+1})] \leq g_t(X_t) - \alpha \langle \nabla g_t(X_t), \mathbb{E}[V] \rangle + \alpha^2 \cdot \mathbb{E} \left[ \frac{(\| V \|_X^2)}{1 - 2\| V \|_X^2} \right] \leq \left(1 - \frac{1 - 2\rho}{1 - \rho} \alpha\right) g_t(X_t) + \alpha \cdot \frac{1}{1 - \rho} \cdot \mathbb{E} \left[ \frac{(\| V \|_X^2)}{1 - 2\| V \|_X^2} \right] \leq \left(1 - \frac{1 - 2\rho}{1 - \rho} \alpha\right) g_t(X_t) + \alpha \cdot \mathbb{E} \left[ \frac{(\| V \|_X^2)}{1 - 2\| V \|_X^2} \right] + \alpha^2 \cdot \mathbb{E} \left[ \frac{(\| V \|_X^2)}{1 - 2\| V \|_X^2} \right] \leq (1 - \zeta \alpha) g_t(X_t) + \alpha \cdot \frac{\zeta}{2} g_{\text{max}} \quad \text{for } \alpha = O(\rho^{-1} d^{-2}).
\]
Here we note that we have carefully chosen $g_{\text{max}}$ so that the ratio $\mathbb{E}(1 - \rho) \cdot \zeta/4) = 1$. It then follows that the following is a supermartingale
\[
G_{it} = (1 - \zeta \alpha)^{-t} \left( g_t(X_t) \cdot 1_{e_{t-1}} - \frac{\zeta}{2} g_{\text{max}} \right).
\]
Indeed, we have
\[
\mathbb{E}[G_{i(t+1)}|X_t, \ldots, X_0] = (1 - \zeta \alpha)^{-(t+1)} \left( \mathbb{E}[g_t(X_{t+1}) \cdot 1_{e_t}|X_t, \ldots, X_0] - \frac{\zeta}{2} g_{\text{max}} \right) \leq (1 - \zeta \alpha)^{-(t+1)} \left( (1 - \zeta \alpha) g_t(X_t) \cdot 1_{e_t} + \alpha \cdot \frac{\zeta}{2} g_{\text{max}} \cdot 1_{e_t} - \frac{\zeta}{2} g_{\text{max}} \right) \leq (1 - \zeta \alpha)^{-t} \left[ g_t(X_t) \cdot 1_{e_{t-1}} - \frac{1 - \alpha}{1 - \alpha} \cdot \frac{\zeta}{2} g_{\text{max}} \right] = G_{it}
\]
where the final line uses $1_{e_t} \leq \zeta \alpha \cdot 1_{e_{t-1}} \leq 1$.

We now bound the deviations on $G_{it}$. Conditioning on the previous iterates $X_t, \ldots, X_0$, we observe that the $g_t(X_t)$ terms cancel:
\[
g_t(X_{t+1}) \cdot 1_{e_t} - \mathbb{E}[g_t(X_{t+1}) \cdot 1_{e_t}] \leq -\alpha \langle \nabla g_t(X_t), [SG(X_t) - \mathbb{E}SG(X_t)](X_t^T X_t)^{-1} \rangle + 5\alpha^2 \cdot \mathbb{E} \left[ \frac{(\| \nabla g_t(X_t) \|_X^2)}{1 - \alpha \| SG(X_t) \|_X^2} \right] \cdot 1_{e_t}. \tag{23}
\]
Here we have for the linear term
\[
\| \nabla g(X) \| \cdot 1_{e_t} \leq 2\| I - X(X^T X)^{-1} X^T \| e_t e_t^T X(X^T X)^{-1} \| \cdot 1_{e_t} \leq O(\kappa^{1/2} \cdot g_{\text{max}}^{1/2}).
\]

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and \(\|SG(X)\|_X \cdot 1_{\epsilon_1} \leq O(d^2 \cdot \sqrt{g_{\max} f(X)}) = O(d^2 \sqrt{g_{\max} \rho / \kappa})\) noting that \(f_{\max} = \rho^2 / \kappa^2\) and hence
\[
|\langle \nabla g(X_t), SG(X_t) (X_t^T X_t)^{-1} \rangle| \leq \|\nabla g(X_t)\| \cdot \|SG(X_t)\|_X^2 \leq O(\sqrt{\kappa g_{\max}}) \cdot O(d^2 \sqrt{g_{\max} \rho / \kappa}) = O(g_{\max} \cdot d^2 \rho) \cdot 1_{\epsilon_1}.
\]

We have for the quadratic term
\[
5(\|SG(X_t)(X_t^T X_t)^{-1}\|_X^2 \cdot 1_{\epsilon_1}) \leq O(d^4 \cdot g_{\max} \cdot \rho^2) \cdot 1_{\epsilon_1} = O(g_{\max} \cdot d^4 \rho^2) \cdot 1_{\epsilon_1}.
\]

Therefore, using the maximum value to bound the expectation, we have
\[
G_{i(t+1)} - \mathbb{E}[G_{i(t+1)}|X_t, \ldots, X_0] \leq \alpha (1 - \alpha)^{-t} g_{\max} \cdot [O(d^2 \rho) + \alpha O(d^4 \rho^2) f(X_t)] \cdot 1_{\epsilon_1}
\]
\[
\leq \alpha (1 - \alpha)^{-t} g_{\max} \cdot O(d^2 \rho) \cdot 1_{\epsilon_1},
\]
where again we observe that a step-size like \(\alpha \leq c/(\rho d \log d) = O(d^{-2} \rho^{-1})\) yields the cancellation of exponents \(\alpha \cdot O(d^4 \rho^2) = O(d^2 \rho)\).

Finally, we bound the variance. Conditioned on all previous iterates \(X_t, \ldots, X_0\) we have
\[
\text{Var}(\langle \nabla g(X_t), SG(X_t)(X_t^T X_t)^{-1} \rangle \cdot 1_{\epsilon_1}) \leq \mathbb{E}[\langle \nabla g(X_t), SG(X_t)(X_t^T X_t)^{-1} \rangle^2 \cdot 1_{\epsilon_1}]
\]
\[
\leq (\|\nabla g(X_t)\|^2) \cdot \mathbb{E}[(\|SG(X_t)\|_X^2)^2] \cdot 1_{\epsilon_1} \leq O(g_{\max}^2 \cdot d^2 \rho^2) \cdot 1_{\epsilon_1},
\]
and also
\[
\text{Var}\left(\frac{5(\|SG(X_t)(X_t^T X_t)^{-1}\|_X^2 \cdot 1_{\epsilon_1})}{1 - \alpha\|SG(X_t)(X_t^T X_t)^{-1}\|_X^2}\right) \leq \frac{25 \cdot \mathbb{E}[(\|SG(X_t)(X_t^T X_t)^{-1}\|_X^2)^4]}{(1 - \alpha\|SG(X_t)(X_t^T X_t)^{-1}\|_X^2)^2} \cdot 1_{\epsilon_1}
\]
\[
\leq \frac{O(g_{\max}^2 \cdot d^6 \rho^4)}{1 - \alpha \cdot O(d^2 \cdot g_{\max} \rho^2)} \cdot 1_{\epsilon_1} = O(g_{\max}^2 \cdot d^6 \rho^4) \cdot 1_{\epsilon_1}.
\]

By the same expansion in (23) we have
\[
\text{Var}(G_{i(t+1)}|X_t, \ldots, X_0) \leq \alpha^2 (1 - \alpha)^{-2t} \cdot [O(g_{\max}^2 \cdot d^2 \rho^2) \cdot 1_{\epsilon_1} + \alpha^2 O(g_{\max}^2 \cdot d^6 \rho^4) \cdot 1_{\epsilon_1}] \cdot 1_{\epsilon_1}
\]
\[
\leq \alpha^2 (1 - \alpha)^{-2t} g_{\max}^2 \cdot O(d^2 \rho^2) \cdot 1_{\epsilon_1},
\]
where again we observe that a step-size like \(\alpha \leq c/(\rho d \log d) = O(d^{-2} \rho^{-1})\) yields the cancellation of exponents \(\alpha^2 \cdot (d^6 \rho^4) = O(d^2 \rho^2)\). \qed