Fidelity freeze for a random matrix model with off–diagonal perturbation

H.-J. Stöckmann† and H. Kohler‡
†Fachbereich Physik der Philipps-Universität Marburg, Renthof 5, D-35032 Marburg, Germany
‡Institut für theoretische Physik, Universität Heidelberg, Philosophenweg 19, D-69120 Heidelberg, Germany

(Dated: March 30, 2022)

The concept of fidelity has been introduced to characterize the stability of a quantum-mechanical system against perturbations. The fidelity amplitude is defined as the overlap integral of a wave packet with itself after the development forth and back under the influence of two slightly different Hamiltonians. It was shown by Prosen and Žnidarič in the linear-response approximation that the decay of the fidelity is frozen if the Hamiltonian of the perturbation contains off-diagonal elements only. In the present work the results of Prosen and Žnidarič are extended by a supersymmetry calculation to arbitrary strengths of the perturbation for the case of an unperturbed Hamiltonian taken from the Gaussian orthogonal ensemble and a purely unitary antisymmetric perturbation. It is found that for the exact calculation the freeze of fidelity is only slightly reduced as compared to the linear-response approximation. This may have important consequences for the design of quantum computers.

PACS numbers: 05.45.Mt, 03.65.Yz, 03.67.Lx

Keywords: Fidelity, Random matrix theory, Quantum Computation

I. INTRODUCTION

The concept of fidelity was originally introduced by Peres [1] to characterize the quantum-mechanical stability of a system against perturbations. Recently it enjoys a renewed popularity because of its obvious relevance for quantum computing. In the present context the works which focus on random matrix aspects are of particular relevance. First the paper by Gorin et al. [2] has to be mentioned where the Gaussian average of the decay of the fidelity amplitude was calculated in linear response approximation. For small perturbations the authors found a predominantly Gaussian decay, with a cross-over to exponential decay for strong perturbations, in accordance with literature [3, 4]. The results of the paper could be experimentally verified in an ultrasound experiment [5] and in a microwave billiard [6]. Using supersymmetry techniques, the limitations of the linear response approximation could be overcome, yielding analytic expressions for the decay of the fidelity amplitude for the Gaussian orthogonal (GOE) and Gaussian unitary (GUE) ensemble. Quite surprisingly a recovery of the fidelity was found at the Heisenberg time [7] which was interpreted as a spectral analogue of an Debye-Waller factor [8]. Reference [7] is the basis for the present work.

The Gaussian decay observed for small perturbation is caused by the diagonal part of the perturbation in the eigenbasis of the unperturbed Hamiltonian. This was the motivation for Prosen and Žnidarič to look for perturbations with zero diagonal, first in classically integrable systems [9]. Later on they extended those studies to classically chaotic systems [10]. In linear response approximation they found an plateau in the decay of the fidelity. Only after extraordinarily long times the decay started again, exponentially below, and Gaussian beyond the Heisenberg time. It remained, however, an open question whether this freeze of the fidelity is reality or whether it is just an artifact of the approximation. It was the motivation for the present work to answer this question by extending the previous supersymmetry calculation to the freeze situation. It will be shown that the freeze is present also in the exact calculation. This may have important consequences for quantum computing. If one succeeds in imbedding the atoms representing the qubits into an environment coupled only via an off-diagonal perturbation to the atoms, an enhancement of the system’s stability by orders of magnitude is expected.

The present results are not restricted to random matrices. In reference [11] it is shown that, e.g., kicked tops with a corresponding dynamics follow exactly the random matrix predictions of the present paper.

II. THE LINEAR RESPONSE APPROXIMATION

The fidelity amplitude is defined as the overlap integral of an initial wave function $|\psi\rangle$ with itself after the time evolution due to two slightly different Hamiltonians $H_0$ and $H_\lambda = H_0 + \lambda V$,

$$f_\lambda(\tau) = \langle \psi | e^{2\pi i H_\lambda \tau} e^{-2\pi i H_0 \tau} | \psi \rangle ,$$

where the time $\tau$ is given in units of the Heisenberg time. For chaotic systems $f_\lambda(\tau)$ is independent on the initial condition, and we may replace expression (1) by its average over $|\psi\rangle$, i.e.,

$$f_\lambda(\tau) = \frac{1}{N} \langle \text{tr} \left[ e^{2\pi i H_\lambda \tau} e^{-2\pi i H_0 \tau} \right] \rangle ,$$

(2)
where $N$ is the rank of the Hamiltonians, and the brackets denote an ensemble average. Under the assumptions that (i) $H_0$ is taken either from the Gaussian orthogonal (GOE) or the Gaussian unitary ensemble (GUE) with a mean level spacing of one in the band centre and (ii) that the variances of the matrix elements of $V$ are given by

$$\langle V_{ij}V_{kl} \rangle = \begin{cases} 
\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}, & \text{(GOE)} \\
\delta_{il}\delta_{jk}, & \text{(GUE)} 
\end{cases}$$ (3)

Gorin et al. [2] obtained for the fidelity amplitude in the linear response approximation

$$f_r(\tau) \sim 1 - \epsilon C(\tau),$$ (4)

where $\epsilon = 4\pi^2 \lambda^2$, and $C(\tau)$ is given by

$$C(\tau) = \frac{\tau^2}{\beta} + \frac{\tau}{2} - \int_0^\tau \int_0^\tau b_{2,\beta}(t')dt'dt.$$ (5)

$b_{2,\beta}(\tau)$ is the two-point form factor, and $\beta$ is the universality index, i.e. $\beta = 1$ for the GOE, and $\beta = 2$ for the GUE. For the Gaussian ensembles $b_{2,\beta}(\tau)$ is known, and $C(\tau)$ can be explicitly calculated [2]. The range of validity of the linear response approximation can be somewhat extended, by exponentiating Eq. (4),

$$f_r(\tau) = e^{-\epsilon C(\tau)}.$$ (6)

The authors argued that the errors of the approximation should be fairly small for $\lambda \sim 0.1$ and negligible for $\lambda \sim 0.01$ (corresponding to $\epsilon = 0.4$ and 0.004, respectively), which was fully confirmed by the exact calculations [7].

The Gaussian decay for small perturbations is caused by the diagonal part of the perturbation. This is immediately evident from Eq. (2): For small perturbations $V$ can be truncated to $V_{\text{diag}}$, its diagonal part in the basis of eigenfunctions of $H_0$. In this regime Eq. (2) reduces to

$$f_r(\tau) = \frac{1}{N} \left\langle \text{tr} e^{2\pi i (H_0 + \lambda V_{\text{diag}}) \tau} e^{-2\pi i H_0 \tau} \right\rangle$$

$$= \frac{1}{N} \left\langle \sum_n e^{2\pi i (E_n + \lambda V_{n\pi}) \tau} e^{-2\pi i E_n \tau} \right\rangle$$

$$= \frac{1}{N} \left\langle \sum_n e^{2\pi i \lambda V_{n\pi} \tau} \right\rangle = \frac{1}{N} \left\langle \text{tr} e^{2\pi i \lambda V_{\text{diag}} \tau} \right\rangle$$

$$= e^{-\frac{\tau^2}{\beta} (V_{\text{diag}}^2)},$$ (7)

where the $E_n$ are the eigenenergies of $H_0$ [4]. This suggests to consider perturbations with vanishing diagonal matrix elements in the eigenbasis of $H_0$ [2]. In the linear response approximation one then obtains

$$f_r(\tau) = e^{-\epsilon C_{\text{freeze}}(\tau)}.$$ (8)

where $C_{\text{freeze}}$ differs from the expression [4] derived previously only by the fact that the term $\tau^2/\beta$ is missing on the right hand side of the equation [2]. The resulting decay of the fidelity amplitude is extremely slow. It will be discussed below and compared with the exact result as obtained from the supersymmetry calculation.

### III. THE PURELY IMAGINARY ANTSYMMETRIC PERTURBATION

To apply the supersymmetric technique of Reference 7 the Hamiltonian needs to be invariant under the action of the orthogonal/unitary group. This is not the case for a GOE perturbation with a deleted diagonal. However a purely imaginary antisymmetric matrix meets with both requirements, zero diagonal elements and orthogonal symmetry. Therefore it is an ideal candidate. We consider the Hamiltonian

$$H_\lambda = H_0 + i\lambda V,$$ (9)

where $H_0$ is taken from the GOE, i.e.

$$\langle (H_0)_{ij}(H_0)_{kl} \rangle = \frac{N}{\pi^2} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$ (10)

and $V$ is real antisymmetric, i.e.

$$\langle V_{ij}V_{kl} \rangle = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$ (11)

The variance of the matrix elements has been chosen to have a mean level spacing of one for $H_0$ and of order $1/\sqrt{N}$ for $V$. Note that in contrast to Reference 7 the mean density of states does not remain constant with increasing perturbation, but decreases with increasing $\lambda$. In fact it is irrelevant in the present context, whether a defolding to a constant mean density of states is performed or not. Such a defolding would imply an additional factor of $1/\sqrt{1+(\pi\lambda)^2/N}$ on the right hand side of Equation 9, which in the final limit $N \to \infty$ reduces to one. These definitions are consistent with the normalization used by Gorin et al. [2]. Expressing $f_r(\tau)$, see Eq. (2), in terms of its Fourier transform

$$f_r(\tau) = \int dE_1 dE_2 e^{2\pi i (E_1 - E_2)\tau} R_e (E_1, E_2),$$ (12)

we have

$$R_e (E_1, E_2) \propto \frac{1}{N} \left\langle \text{tr} \left( \frac{1}{E_1 - H_0 - i\lambda V} \frac{1}{E_2+ - H_0} \right) \right\rangle,$$ (13)

with $E_\pm = E \pm \eta$. We rewrite $R_e (E_1, E_2)$ using the formula

$$\text{tr} \frac{1}{AB} = \sum_{n,m} \frac{\partial}{\partial J_{nm}} \frac{\partial}{\partial K_{mn}} \frac{\text{det}(A+J)\text{det}(B+K)}{\text{det}(A-J)\text{det}(B-K)} \bigg|_{J=K=0},$$ (14)

In our case $A = E_{2+} - H_0$ is real symmetric and $B = E_{1-} - H_0 - i\lambda V$ is Hermitean. According to the universality classes of $A$ and $B$ we write the determinants as Gaussian integrals over real and complex wave functions, respectively. We obtain

$$\sum_{n,m} \frac{\partial}{\partial J_{nm}} \frac{\text{det}(A+J)}{\text{det}(A-J)} \bigg|_{J=0} = -i \int d[x] d[y] d[\xi] d[\xi^*]$$

$$\sum_{n,m} (x_n x_m + y_n y_m - \xi_n^* \xi_m - \xi_m^* \xi_n)$$

$$e^{-ix^T A x - iy^T A y - i\xi^T A \xi - i\xi^T A \xi^*},$$ (15)
where the commuting integration variables are real. We adopt the usual convention and use Latin letters for commuting, and greek ones for anticommuting variables, respectively. For $B$ we obtain instead

$$
\sum_{m,n} \frac{\partial}{\partial K_{mn}} \det(B + K) \bigg|_{K=0} = \int d[a] d[b] d[\eta] d[\eta^*] \sum_{m,n} (a_m a_n + b_m b_n - \eta_m^* \eta_n - \eta_m \eta_n^*)
$$

The commuting integration variables are all real. Now the average is taken over real symmetric $H_0$ using Eq. (10)

$$
\langle \ldots \rangle_{H_0} = \exp \left( -\frac{N}{\pi^4} \text{Str}(LZ)^2 \right) \quad (18)
$$

where $Z$ is a supermatrix given by $Z = \sum_n z_n z_n^\dagger$ with $z_n^\dagger = (x_n, y_n, \xi_n, \xi_n^*, a_n, b_n, \eta_n, \eta_n^*)$, and $L = \text{diag}(1_4, -1_4)$ in the advanced–retarded block notation. $Z$ is exactly the matrix given in Table 4.1 of Reference 13, denoted by VWZ in the following. The matrix $B$ has an orthosymplectic symmetry, i.e. in Boson–Fermion block notation the Boson–Boson block is real symmetric and the Fermion–Fermion block is Hermitian selfdual. For the $V$ average we obtain with Eq. (11)

$$
\langle \ldots \rangle_V = \exp \left( -\lambda^2 \text{Str}(KT^2) \right) \quad (19)
$$

Here $T = \sum_n a_n a_n^\dagger$ with $a_n^\dagger = (a_n, b_n, \eta_n, \eta_n^*)$ the supermatrix $K$ is given by

$$
K = \begin{pmatrix}
-\sigma_y & 0 \\
0 & \sigma_z
\end{pmatrix},
$$

where we used the Pauli matrices

$$
\sigma_x = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
$$

In $K$ the off–diagonal nature of the perturbation is encoded.

The subsequent steps are the same as described in [5]. After transforming equations (18) and (19) by means of two Hubbard–Stratonovich transformations, the integrations over the the $a, b, x, y$ variables, and over the auxiliary variables of one Hubbard–Stratonovich transformation can be performed resulting in

$$
R_e (E_1, E_2) \propto \frac{\pi^4}{4N^3} \int d[\sigma] \text{Str} \left( \mathbf{P}_{\sigma RA} \mathbf{P}_{\sigma RA}^\dagger \right) e^{-\frac{\pi^2}{N^2} \text{Str} \left( \mathbf{K}_{\sigma RR} \right)^2} \left[ \text{Sdet} \left( \frac{\sigma_{AA} - E_1}{\sigma_{RA}} \frac{\sigma_{RA}}{\sigma_{RR} - E_2} \right) \right]^{-N/2} \quad (22)
$$

Here $\mathbf{P} = \text{diag}(1, 1, -1, -1)$. The matrix $\sigma$ has the same orthosymplectic symmetry as $B$ and reads in advanced–retarded block notation

$$
\sigma = \begin{pmatrix}
\sigma_{AA} & \sigma_{RA} \\
\sigma_{RA} & \sigma_{RR}
\end{pmatrix}.
$$

Introducing the notation $E_{1/2} = \bar{E} \pm E/2$, and substituting $\sigma_{AA}$ and $\sigma_{RR}$ by $\sigma_{AA} + E/2$ and $\sigma_{RR} - E/2$, respectively, we obtain

$$
R_e (E_1, E_2) \propto \frac{\pi^4}{4N^3} \int d[\sigma] \text{Str} \left( \mathbf{P}_{\sigma RA} \mathbf{P}_{\sigma RA}^\dagger \right) e^{-\frac{\pi^2}{N^2} \text{Str} \left( \mathbf{K}_{\sigma RR} \right)^2} \left[ \text{Sdet} \left( \frac{\sigma - \bar{E}}{\overline{E}} \right) \right]^{-N/2} \quad (24)
$$

This expression can be evaluated in the limit $N \to \infty$ by a saddle point approximation.

**IV. SADDLE POINT APPROXIMATION**

The next steps are a direct repetition of the corresponding ones in Reference 5. We shall adopt the nota-
tion from VWZ, which is the main source for the following calculations. First we diagonalize $\sigma$,
\[
\sigma = T_0^{-1}R^{-1}\sigma_DRT_0,
\]
where $R$ is block diagonal and
\[
T_0 = \begin{pmatrix}
\sqrt{1+t_{12}t_{21}} & t_{12} \\
-t_{21} & \sqrt{1+t_{21}t_{12}}
\end{pmatrix}
\]  
(26)
(see VWZ, Eqs. (5.28+29)). The integration of the diagonal variables of $\sigma$ can be performed by means of the saddle point approximation. $\sigma_D$ at the saddle point reads
\[
\sigma_D = \begin{pmatrix}
s_A & 0 \\
0 & s_R
\end{pmatrix},
\]
where the advanced and retarded saddle points are given by
\[
s_{A/R} = \frac{1}{2} \left( E \pm i\Delta \right),
\]
\[
\Delta = \frac{2N}{\pi} \sqrt{1 - \left( \frac{\pi E}{2N} \right)^2} = \frac{2N}{\pi} \rho,
\]
(28)
and $\rho$ is the density of states. In the following we shall restrict ourselves to the band centre, $E = 0$, where Eq. (28) reduces to $s_{A/R} = \pm i\pi/N$. We then have for the matrix $\sigma$ at the saddle point
\[
\sigma = \frac{N}{\pi} \begin{pmatrix}
\left(1 + 2t_{12}t_{21}\right) & 2t_{12}\sqrt{1+t_{12}t_{21}} \\
2t_{21}\sqrt{1+t_{21}t_{12}} & \left(1 - 2t_{21}t_{12}\right)
\end{pmatrix}.
\]
(29)
The matrix $R$ (see Eq. [28]) does not enter, since it commutes with $\sigma_D$ at the saddle point. We obtain
\[
R_e(E_1, E_2) \propto \frac{\pi^4}{4N^2} \int F(t_{12})d[t_{12}] \text{Str}(P\sigma_{AR}P\sigma_{RA})
\]
\[
e^{-\frac{\pi^4}{4N^2} \left[ \text{Str}(\sigma_{AA} - \sigma_{BB}) - \frac{\pi^4}{2N^2} \text{Str}(\sigma_{AA})^2 \right]},
\]
(30)
where the integral is over the elements of the matrix $t_{12}$, parametrizing the saddle-point manifold. The function $F(t_{12}) = \text{Sdet}^{-1/2}(1 + t_{12}t_{21})$ is the Berezinian of the coordinate transformation Eq. [28]. Using Eq. [29] the various terms entering Eq. [30] may be written as
\[
\text{Str}(\sigma_{AA} - \sigma_{RR}) = i\frac{4N}{\pi} \text{Str}(t_{12}t_{21}t_{12}),
\]
(31)
\[
\text{Str}(K\sigma_{RR})^2 = \left(\frac{1N}{\pi}\right)^2 \text{Str}[K(1 + 2t_{12}t_{21})]^2(32)
\]
\[
\text{Str}(\sigma_{AR}P\sigma_{RA}P) = \left(\frac{2N}{\pi}\right)^2 \text{Str} \left( t_{21}\sqrt{1 + t_{12}t_{21}}P \times t_{12}\sqrt{1 + t_{21}t_{12}}P \right). \quad (33)
\]
We proceed further by diagonalizing the matrices $t_{12}$ and $t_{21}$. This is achieved by the radial decomposition
\[
t_{12} = U_1^{-1}MU_2, \quad t_{21} = U_2^{-1}U_1MU_1,
\]
(34)
with diagonal $M = \text{diag}(\mu_1, \mu_2, \mu, \mu)$. The matrix $U = U_1 \oplus U_2$ is a $4 \times 4$ block diagonal matrix, with $U \in \text{SU}(2)$ (see VWZ, Eq. (I.18)). The $U_i (i = 1, 2)$ may be parameterised as $U_i = \tilde{V}_iO_i$, where the $O_i \in O(12)$ are $4 \times 4$ block diagonal, with $\tilde{O}_i \in \text{SO}(2)$. The parameterisation of the $V_i$ in terms of anti-commuting variables is postponed to App. A If we moreover introduce
\[
X = M^2 = \text{diag}(x, y, -z, -z),
\]
\[
x = \mu_1^2, \quad y = \mu_2^2, \quad z = \mu^2,
\]
(35)
we can write Eqs. (31) to (33), using Eq. (34) and Eq. (35)
\[
\text{Str}(\sigma_{AA} - \sigma_{RR}) = i\frac{4N}{\pi} \text{Str} X,
\]
(36)
\[
\text{Str}(K\sigma_{RR})^2 = \left(\frac{1N}{\pi}\right)^2 \text{Str}[K(1 + 2X)]^2(37)
\]
\[
\text{Str}(\sigma_{AR}P\sigma_{RA}P) = \left(\frac{2N}{\pi}\right)^2 \text{Str} \left( \sqrt{X}\sqrt{1 + XP} \right) \times \sqrt{X}\sqrt{1 + XP}.
\]
(38)
where
\[
K_1 = U_1KU_1^{-1}, \quad P_1 = U_1PU_1^{-1}, \quad P_2 = U_2PU_2^{-1}U^{-1}.
\]
(39)
Under the transformations Eq. (34) and Eq. (35) the measure transforms as
\[
d[t_{12}] = \mathcal{G}(X)d\mu(U_1)d\mu(U_2)d\mu(U)d[X].
\]
(40)
The function $\mathcal{G}$ has been calculated in VWZ (Eq. K.17). The average in Eq. (39) is over the elements of the matrices $X, U_1, U_2$. Only $P_2$ depends on the matrix elements of $U_2$. It will be shown in App. A that $U_2PU_2^{-1}$ averaged over the matrix elements of $U_2$ is nothing but a multiple of the four-dimensional unit matrix. Thus the $U$ dependence cancels. We are then left with an average over $x, y, z$, and the matrix elements of $U_1$. Inserting these results into Eq. (40), we get
\[
R_e(E_1, E_2) \propto \frac{1}{N} \int F(X)\mathcal{G}(X)d[X]d\mu(U_1)
\]
\[
\text{Str}[X + 1]P_1e^{-\frac{\pi^2}{2N} \text{Str}[1 + 2X]K_1^2},
\]
(41)
where we employed the definition $\varepsilon = 4\pi^2\lambda^2$, see above. The Berezinians $F(X)$ and $\mathcal{G}(X)$ can be comprised in one measure function $\mu(X)$ which was given in VWZ:
\[
\mu(X) = \frac{|x - y|}{\sqrt{xy(x + 1)(y + 1)(z - x)^2(z - y)^2}}.
\]
(42)
Substituting expression (41) for $R_e(E_1, E_2)$ into Eq. (12), and introducing $E = (E_1 - E_2)/2$ and $\tilde{E} = (E_1 + E_2)/2$ as new integration variables, the $E$ integration generates a delta function, whereas the $\tilde{E}$ integration corresponds to an energy average. The result is (see Reference [2] for details)
\[ f_\epsilon(\tau) \propto \frac{1}{N} \int_0^\infty du \int_0^\infty dv \int_0^1 dz \mu(u,v,z) d\mu(U_1) \]
\[
\delta(\tau - u - z) \text{Str}(X(X + 1)P_1) \exp \left( -\frac{\epsilon}{16} \text{Str} [(1 + 2X)K_1]^2 \right), \quad (43)
\]
with \( u = (x+y)/2 \). In addition we shall use \( v = (x-y)/2 \) as another new variable, and replace \( z \) by \( \tau - u \) everywhere, which is admissible because of the presence of the delta function. The integration domains of the radial variables \( u, v \) and \( z \) are dictated by the hyperbolic symmetry of the saddle point manifold, i.e., we have non-compact integration domains \( 0 < x, y < \infty \) for the bosonic coordinates \( x, y \) and a compact integration domain \( 0 < z < 1 \) for the fermionic coordinate \( z \). For more detail on this point, see VWZ. We still have to integrate over the matrix elements of \( U_1 \).

V. INTEGRATION OVER THE GRASSMANN VARIABLES

We recall that \( U_1 = V_1 O_1 \). Since \( O_1 \) commutes with \( P \) and with \( K \), the \( O_1 \) integration is trivial, and we are left with the integration over \( V_1 \). The parametrization of \( V_1 \) in terms of Grassmannian variables and the calculation of the traces in equation (43) is quite involved, and is postponed to the appendices. Here we note only the results:

\[
\text{Str} [X(X + 1)P_1] = 4v(2u + 1)B \\
 + 2[2u(u + 1) - \tau(2u + 1 - \tau) + v^2] \\
+ 4[\tau(2u + 1 - \tau) + v^2](A - 2\bar{a}), \quad (44)
\]

and

\[
\frac{1}{8} \text{Str} [(1 + 2X)K_1]^2 = \tau(2u + 1 - \tau) - v^2 + 2(A - D)(\tau^2 - v^2), \quad (45)
\]

where

\[
A = \alpha^\dagger \beta^\dagger, \quad B = \alpha \beta^\dagger, \quad D = i(\alpha^\dagger \beta - \beta^\dagger \alpha^\dagger). \quad (46)
\]

and \( \bar{a} = \alpha^\dagger \beta^\dagger \). \( \alpha, \alpha^\dagger, \beta, \beta^\dagger \) are anticommuting variables. It follows

\[
e^{-\frac{\epsilon}{16} \text{Str} [(1 + 2X)K_1]^2} = e^{-\frac{\epsilon}{8} [\tau(2u + 1 - \tau) - v^2 + 2(A + D)(\tau^2 - v^2)]} \\
[1 - \epsilon(A - D)(\tau^2 - v^2)] e^{-\frac{\epsilon}{2} \tau(2u + 1 - \tau) - v^2}. \quad (47)
\]

These results are inserted into equation (43). The measure is given by

\[
d\mu(U_1) = 2\pi d\mu(V_1) \propto d\alpha^\dagger d\beta^\dagger d\beta^\dagger d\alpha^\dagger. \quad (48)
\]

Therefore only the terms proportional to \( \bar{a} \) survive the integration over the antisymmetric variables. We obtain

\[
f_\epsilon(\tau) \propto \frac{1}{N} \int \mu(u,v,z) \delta(\tau - u - z) e^{-\frac{\epsilon}{8}(\tau(2u + 1 - \tau) - v^2)} [1 + \epsilon(\tau^2 - v^2)][\tau(2u + 1 - \tau) + v^2] du dv dz, \quad (49)
\]

which is almost our final result.

VI. RESULT AND DISCUSSION

The final result is obtained by an VWZ-like integral (see VWZ, Eq. (8.10)) and is given in the present case by

\[
f_\epsilon(\tau) = 2 \int_{\text{Max}(0,\tau - 1)}^{\tau} du \int_0^u dv \int_0^1 \frac{v \, dv \, \sqrt{[u^2 - v^2]([u + 1]^2 - v^2)}}{(v^2 - \tau^2)^2} \\
\times [1 + \epsilon(\tau^2 - v^2)][\tau(2u + 1 - \tau) + v^2] e^{-\frac{\epsilon}{8}(\tau(2u + 1 - \tau) - v^2)}. \quad (50)
\]

Figure 4 shows the fidelity decay for different perturbations, as calculated from Eq. (50), together with the result from the exponentiated linear response approximation. For comparison the fidelity decay for the case of a GOE perturbation (9) is shown as well. We see that the linear response approximation is able to describe the fidelity decay for quite a long time very well. For still
larger times the linear response approximation underestimates the decay, as compared to the exact result, but still the decay is by orders of magnitude slower as for a GOE perturbation.

Figure 2 shows the fidelity $f_\varepsilon(\tau)$ for three fixed values 0.5, 1.0, 1.5 of $\tau$ as a function of the perturbation $\varepsilon$. The figure demonstrates that the freezing effect not unexpectedly becomes less and less pronounced with increasing perturbation, though the decay is always by orders of magnitude slower than for the case of a GOE perturbation (not shown). It is further seen that the linear response approximation works very well up to about half the Heisenberg time, but underestimates the decay more and more for increasing $\tau$ values.

We thus can conclude that the fidelity freeze is not an artefact of the linear response approximation but is also present in the exact calculation. Due to the perfect and well established correspondence between random matrices and chaotic quantum systems \[ \text{[11, 12, 10]} \] this result provides an important new mechanism of preserving quantum stability.

Acknowledgments

The motivation to this work came from numerous discussions with T. Gorin, T. Prosen, T. Seligman, and M. Žnidarič, as well as from microwave experiments on the subject of fidelity, which had been performed by R. Schäfer together with one of the authors (H.-J. St.). The experiments had been supported by the Deutsche Forschungsgemeinschaft.

APPENDIX A: CALCULATION OF \[ \text{Str}[X(X+1)P_1] \]

In terms of Pauli matrices $X$ may be expressed as

$$X = -z 1 + \left( \begin{array} { c c } { \hat{X} } & { 0 } \\ { 0 } & { 0 } \end{array} \right),$$

(A1)

where

$$\hat{X} = \left( \begin{array} { c c } { x + z } & { 0 } \\ { 0 } & { y + z } \end{array} \right) = \tau 1 + v \sigma_z.$$  

(A2)

It follows

$$\text{Str}[X(X+1)P_1] = 4z(z-1) + (1 - 2z) \text{tr} \left[ \hat{X} (P_1)_{u.l.} \right] + \text{tr} \left[ \hat{X}^2 (P_1)_{u.l.} \right] \quad \text{(A3)}$$

where it was used that $\text{Str}P_1 = \text{Str}P = 4$, and where $(P_1)_{u.l.}$ denotes the upper left submatrix of $P_1$. As was already mentioned, matrices $U_1$ and $U_2$ entering the calculation of $P_1$ and $P_2$ (see Eq. \[ \text{[93]} \]) are parameterized as

$$U_p = V_p O_p , \quad (p = 1, 2) \quad \text{(A4)}$$

where

$$O_p = \left( \begin{array} { c c } { \hat{O}_p } & { 0 } \\ { 0 } & { 1 } \end{array} \right), \quad \text{(A5)}$$

and $\hat{O}_1$ and $\hat{O}_2$ are $2 \times 2$ orthogonal matrices (see VWZ, Eq. (I.13)). The matrices $V_p$ may be parametrized as (see VWZ, Eq. (K.26))

$$(V_p)^{\pm 1} = 1 \pm \nu^{(p-1)}Y_p + \frac{1}{2} \nu^{2(p-1)}Y_p^2 \pm \frac{1}{2} \nu^{3(p-1)}Y_p^3 + \frac{3}{8} Y_p^4 \quad \text{(A6)}$$

where matrices $Y_1$ and $Y_2$ are given by

$$Y_p = \left( \begin{array} { c c } { 0 } & { -\zeta_p^\dagger } \\ { \zeta_p } & { 0 } \end{array} \right) \quad \text{(A7)}$$

where

$$\zeta_p = \left( \begin{array} { c c } { \alpha_p } & { \beta_p } \\ { \beta_p^* } & { \alpha_p^* } \end{array} \right), \quad \zeta_p^\dagger = \left( \begin{array} { c c } { \alpha_p^* } & { -\alpha_p } \\ { \beta_p } & { -\beta_p } \end{array} \right) \quad \text{(A8)}$$

(see VWZ, Eqs. (K.23 + 25)). Note the convention $\langle \alpha \rangle^* = -\alpha$ for antisymmetric variables. In VWZ, Eq. (I.13) the sequence of the matrices on the right hand side of Eq. \[ \text{[A3]} \] is reversed. Both parameterizations are equivalent and can be transformed into each other by a straightforward transformation of the $\alpha_p, \beta_p$ variables.

We are now going to calculate $P_1 = U_1P_1U_1^{-1} = V_1O_1P_0^{-1}V_1^{-1}$. To simplify notations, we shall omit the lower index ‘1’ in the following. The calculation for $P_2$ proceeds in the very same way. Since $P = \text{diag}(1, 1, -1, -1)$ commutes with $O$, we are left with

$$P_1 = VPV^{-1} = V \left( \begin{array} { c c } { 1 } & { 0 } \\ { 0 } & { -1 } \end{array} \right) V^{-1}. \quad \text{(A9)}$$

For the further calculation it is suitable to introduce the quantities

$$A = \alpha \alpha^* + \beta \beta^*, \quad B = \alpha \alpha^* - \beta \beta^*, \quad C = \alpha \beta^* + \beta \alpha^*, \quad D = i(\alpha \beta^* - \beta \alpha^*). \quad \text{(A10)}$$

$A, B, C$ obey the relations

$$A^2 = 2\bar{a}, \quad B^2 = -2\bar{a}, \quad C^2 = -2\bar{a}, \quad D^2 = -2\bar{a}, \quad \text{(A11)}$$

where $\bar{a} = \alpha \alpha^* \beta \beta^*$, and

$$AB = AC = AD = BC = BD = CD = 0. \quad \text{(A12)}$$

It follows

$$\zeta^\dagger \zeta = -A1 - B\sigma_z - C\sigma_x, \quad \zeta \zeta^\dagger = A1 \quad \text{(A13)}$$

As a direct consequence we have

$$Y^2 = \left( \begin{array} { c c } { -\zeta^\dagger \zeta } & { 0 } \\ { 0 } & { -\zeta^\dagger \zeta } \end{array} \right) = \left( \begin{array} { c c } { A1 + B\sigma_z + C\sigma_x } & { 0 } \\ { 0 } & { -A1 } \end{array} \right), \quad Y^3 = -AY. \quad \text{(A14)}$$
\[ V^{\pm 1} = 1 + \left( \frac{1}{2} - \frac{3}{8} A \right) Y^2 \pm \left( 1 - \frac{A}{2} \right) Y \]

\[ = \left( \begin{array}{c} w + \omega' \\ -w \end{array} \right), \quad \text{(A15)} \]

where

\[ w = \left( 1 + \frac{A}{2} - \frac{3}{4} \bar{a} \right) + \frac{B}{2} \sigma_z + \frac{C}{2} \sigma_x \]

Inserting the results into Eq. (A9) we have

\[ P_1 = \left( \frac{(1 - 4\bar{a} + 2A)\mathbf{1} + 2(B\sigma_z + C\sigma_x)}{2(1 - A)\zeta} \right) \frac{2(1 - A)\zeta}{(-1 - 4\bar{a} + 2A)\mathbf{1}}. \quad \text{(A17)} \]

A corresponding expression is obtained for \( P_2 \). In the average over the antisymmetric variables only the \( \bar{a} \) terms survive, i.e. \( \langle P_1 \rangle = (P_2) \propto \mathbf{1} \), as was stated above.

Inserting finally the upper left corner element of \( P_1 \) into Eq. (A3), we end up with Eq. (A4).

**APPENDIX B: CALCULATION OF**

\[ \text{Str}[(1 + 2X)K_1]^2 / 8 \]

It is suitable to write

\[ \frac{1}{8} \text{Str}[(1 + 2X)K_1]^2 = \frac{1}{16} \text{Str}[(1 + 2X), K_1]^2 \]

\[ + \frac{1}{8} \text{Str}[(1 + 2X)^2K_1^2] \]

\[ = \frac{1}{4} \text{Str}[X, K_1]^2 \]

\[ + \frac{1}{2} \text{Str}[X(X + 1)] \quad \text{(B1)} \]

where \( K_1^2 = K^2 = \mathbf{1} \) was used. The second term on the right hand side is easily evaluated:

\[ \frac{1}{2} \text{Str}[X(X + 1)] = \frac{1}{2} [x(x + 1) + y(y + 1) + 2z(1 - z)] \]

\[ = u(u + 1) + v^2 + z(1 - z) \]

\[ = -\tau^2 + (2u + 1)\tau + v^2. \quad \text{(B2)} \]

For the first term on the right hand side we need an expression for \( K_1 \). Using \( K_1 = U_1 K U_1^{-1} \) (see Eq. (39)) and \( U = VO \) (see Eq. (A5)) we may write

\[ K_1 = V_1 O_1 K O_1^{-1} V_1^{-1} = V K V^{-1}, \quad \text{(B3)} \]

since \( K \) (see equation (20)) commutes with \( O \). Using Eq. (A15), we obtain

\[ K_1 = \left( \begin{array}{cc} k & k_1 \vspace{1em} \\
\kappa & \bar{k} \end{array} \right), \quad \text{(B4)} \]

where

\[ k = -w\sigma_y w + \omega^\dag \sigma_z \omega, \]

\[ \kappa^\dag = -w\sigma_y \omega^\dag - \omega^\dag \sigma_z w, \]

\[ \bar{k} = -\omega\sigma_y w + \bar{w}\sigma_z w. \quad \text{(B5)} \]

Since \( K_1^2 = 1 \), we have

\[ \frac{1}{4} \text{Str}[X, K_1]^2 = \frac{1}{2} \left[ \text{Str}(XK_1)^2 - \text{Str}X^2 \right] \]

\[ = \frac{1}{2} \left[ \text{Str}(\hat{X}k)^2 - \text{Str}\hat{X}^2 \right], \quad \text{(B6)} \]

where in the second step expression (A1) for \( X \) was used. Using Eqs. (A16) and (B5) we have

\[ k = -(1 + A - D)\sigma_y. \quad \text{(B7)} \]

where \( \zeta^\dag \sigma_z \zeta = D \sigma_y \) was used. Now the calculation of the terms entering the right hand side of Eq. (B6) is straightforward:

\[ \frac{1}{2} \text{Str}(\hat{X}k)^2 = (1 + 2A - 2D)(\tau^2 - v^2), \]

\[ \frac{1}{2} \text{Str}\hat{X}^2 = \tau^2 + v^2. \quad \text{(B8)} \]

Collecting the results of this subsection, we have

\[ \frac{1}{8} \text{Str}[(1 + 2X)K_1]^2 = \]

\[ \tau(2u + 1 - \tau) - v^2 + 2(A - D)(\tau^2 - v^2). \quad \text{(B9)} \]

whence follows Eq. (47).
[1] A. Peres, Phys. Rev. A 30, 1610 (1984).
[2] T. Gorin, T. Prosen, and T. H. Seligman, New J. of Physics 6, 20 (2004).
[3] T. Prosen and M. Žnidarič, J. Phys. A 35, 1455 (2002).
[4] N. R. Cerruti and S. Tomsovic, Phys. Rev. Lett. 88, 054103 (2002).
[5] O. I. Lobkis and R. L. Weaver, Phys. Rev. Lett. 90, 254302 (2003).
[6] R. Schäfer, H.-J. Stöckmann, T. Gorin, and T. H. Seligman, Experimental verification of fidelity decay: From perturbative to Fermi golden rule regime (2004), to be published.
[7] H.-J. Stöckmann and R. Schäfer, New J. of Physics 6, 199 (2004).
[8] H.-J. Stöckmann and R. Schäfer, Phys. Rev. Lett. 94, 244101 (2005).
[9] T. Prosen and M. Žnidarič, New J. of Physics 5, 109 (2003).
[10] T. Prosen and M. Žnidarič, Phys. Rev. Lett. 94, 044101 (2005).
[11] T. Gorin, H. Kohler, T. Prosen, T. Seligman, H.-J. Stöckmann and M. Žnidarič, to be published.
[12] T. Prosen, private communication.
[13] J. J. M. Verbaarschot, H. A. Weidenmüller, and M. R. Zirnbauer, Phys. Rep. 129, 367 (1985).
[14] G. Casati, F. Valz-Gris, and I. Guarneri, Lett. Nuov. Cim. 28, 279 (1980).
[15] O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. 52, 1 (1984).
[16] T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, Phys. Rep. 299, 189 (1998).
FIG. 1: Ensemble average of the fidelity amplitude $f_\epsilon(\tau)$ with $H_0$ taken from the GOE and a purely imaginary antisymmetric perturbation (solid line, calculated from Eq. (50)) for different perturbation strengths $\epsilon$. For comparison the result from the linear response approximation (dashed line), and for a GOE perturbation (dashed-dotted line) are shown as well.
FIG. 2: Ensemble average of the fidelity amplitude $f_\epsilon(\tau)$ for an imaginary antisymmetric perturbation as a function of $\epsilon$ for three fixed values of $\tau = 0.5, 1.0, 1.5$ (from top to bottom, solid lines). Again the results from the linear response approximation are shown for comparison (dashed lines).