JACOBI FIELDS FOR SECOND-ORDER DIFFERENTIAL
EQUATIONS ON LIE ALGEBROIDS

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ABSTRACT. We generalize the concept of Jacobi field for general second-order differential
equations on a manifold and on a Lie algebroid. The Jacobi equation is expressed in
terms of the dynamical covariant derivative and the generalized Jacobi endomorphism
associated to the given differential equation.

1. Introduction. In Riemannian geometry a Jacobi field $W$ along a geodesic $\gamma$ is defined
as a solution of the Jacobi equation $D_t D_t W + \text{Ric}(W, \dot{\gamma}) \dot{\gamma} = 0$, where $D$ is the Levi-
Civita connection associated to a given metric and $\text{Ric}$ is the curvature tensor. They are
interpreted as infinitesimal variations of the geodesic $\gamma$ by geodesics, or in other words,
as the infinitesimal variation vector field associated to a 1-parameter family of geodesics.
The geodesic equation is a special kind of second-order differential equation known as a
geodesic spray. The purpose of this paper is to generalize these results for general second
order differential equations on a manifold, showing its geometrical origin and finding the
differential equation satisfied by such vector fields in a form which resembles as much as
possible the original Jacobi equation.

For that, we will study the properties satisfied by the infinitesimal variation vector field
of a 1-parameter family of solutions of a general first-order differential equation, which
 corresponds to the linear variational differential equation. These results will be applied to
the case of a second-order differential equation (sode), clarifying the geometrical meaning
of the concept of Jacobi field in terms of the geometry of the tangent bundle. Finally, by
using the non-linear connection associated to the sode, we will find the equation satisfied by
the variation vector fields, which generalizes the Jacobi equation. The form of this equation,
$\nabla\nabla W + \Phi(W) = 0$, is similar to the original Jacobi equation. Our results hold also in the
case of the sode defined by a Finsler metric, by removing the closed set of points where it
is singular.

Finally, we will generalize our results to the framework of Lie algebroids. In this way we
will obtain an equation which is valid for second order systems with holonomic constraints,
systems defined on Lie algebras and systems with symmetry, in addition to the standard
case. For applications of the theory of Lie algebroids in Classical Mechanics, Control Theory
and Field Theory we refer to [8, 4, 9, 10].

2010 Mathematics Subject Classification. 34A26, 58Z05, 58Cxx.
Key words and phrases. Jacobi fields, second-order differential equations, Lie algebroids, Jacobi equation.
Partial financial support from MINECO (Spain) grant number MTM2012-33575, and from Gobierno de
Aragón (Spain) grant DGA-E24/1 is acknowledged.
2. The standard case of sodes on tangent bundles. In this section we will review the basic results about the variational equation and we reformulate the Jacobi equation in a way suitable for the generalization to SODE on Lie algebroids that will be given in Section 3.

2.1. The variational differential equation and its geometric interpretation. We consider a vector field $X \in \mathfrak{X}(M)$ on a manifold $M$ and we denote by $\{\varphi_t\}$ its local flow. We fix an integral curve $\zeta_0: I \subset \mathbb{R} \to M$ of $X$ defined on a compact interval $I = [0, T]$. We set $m = \zeta_0(0)$ the initial point, so that $\zeta_0(t) = \varphi_t(m)$.

A vector field along the curve $\zeta_0$ is a map $Z: I \to TM$ such that $Z(t) \in T_{\zeta(t)} M$ for all $t \in I$. The following definitions are from [3].

**Definition 2.1.** A vector field $Z$ along $\zeta_0$ is said to be Lie transported along the flow of $X$ if there exists $\xi \in T_m M$ such that $Z(t) = T_{\varphi_t}(\xi)$ for every $t \in I$.

**Definition 2.2.** The Lie derivative of a vector field $Z$ along $\zeta_0$ with respect to $X$ is the vector field $\mathcal{L}_X Z$ along $\zeta_0$ defined by

$$\mathcal{L}_X Z(t) = \frac{d}{ds} T_{\varphi_s} Z(t + s) \bigg|_{s=0} = \lim_{h \to 0} \frac{1}{h} [T_{\varphi_h} Z(t + h) - Z(t)].$$

It follows that if $Z \in \mathfrak{X}(M)$ is an extension of $Z$ then $\mathcal{L}_X Z(t) = [X, Z](\zeta_0(t))$.

We recall that given a vector field $W$ along a curve $\zeta_0$ the complete lift of $W$ is the vector field $W^c$ along $\zeta_0$ given by $W^c(t) = \chi_{TM} (W(t))$, where $\chi_{TM}: TT M \to TM$ is the canonical involution, given locally by $\chi_{TM}(q, v, w, a) = (q, w, v, a)$. Similarly, if $Y$ is a vector field on $M$, the complete lift of $Y$ is the vector field on $TM$ defined by $Y^c = \chi_{TM} \circ TY$.

Both definitions are consistent in the sense that the complete lift of the restriction of $Y$ to the curve $\zeta_0$ is the restriction of the complete lift to the curve $\zeta_0$, that is, $Y^c \circ \zeta_0 = (Y \circ \zeta_0)^c$.

As it is well known, if $\{\phi_t\}$ is the flow of $Y$ then the flow of $Y^c$ is $\{T_{\phi_t}\}$.

**Proposition 2.3.** The following properties are equivalent:

1. $Z$ is Lie transported along the flow of $X$.
2. For every $t \in I$ and every $s$ such that $s + t \in I$ we have $T_{\varphi_s} (Z(t)) = Z(t + s)$.
3. The Lie derivative of $Z$ vanishes identically, $\mathcal{L}_X Z(t) = 0$ for all $t \in I$.
4. The curve $Z: I \to TM$ is an integral curve of the complete lift $X^c \in \mathfrak{X}(TM)$ of $X$.

In local natural coordinates $(x^i, v^i)$ in $TM$, the above properties express that the components $x^i = \zeta_0^i(t)$, $v^i = Z^i(t)$ of $Z$ satisfy the linear variational equation

$$\begin{cases}
\dot{x}^i = X^i(x) \\
\dot{v}^i = \frac{\partial X^i}{\partial x^j}(x) v^j.
\end{cases}$$

The proof will be postponed to Section 3 where it will be given for the more general case of Lie algebroids.

Our first aim is to prove that this kind of vector fields is obtained via variation of integral curves of the vector field.

**Definition 2.4.** A 1-parameter family of integral curves of $X$ is a map $\zeta: (-\epsilon, \epsilon) \times I \subset \mathbb{R}^2 \to M$ such that for every $s \in (-\epsilon, \epsilon)$ the curve $\zeta_s: I \to M$, given by $\zeta_s(t) = \zeta(s, t)$ is an integral curve of $X$. The vector field $Z$ along $\zeta_0$ defined by $Z(t) = \frac{\partial}{\partial x^i}(0, t)$ is said to be the infinitesimal variation vector field defined by the 1-parameter family.

We will also say that $\zeta$ is a finite variation of $\zeta_0$ by integral curves of $X$.

**Proposition 2.5.** A vector field $Z$ along an integral curve $\zeta_0$ of $X$ is the infinitesimal variation defined by a 1-parameter family of integral curves of $X$ if and only if it is Lie transported by the flow of $X$.
Proof. If $\zeta(s, t)$ is a 1-parameter family of integral curves of $X$ then we have that $\zeta(s, t) = \varphi_t(\zeta(s, 0))$. Taking the partial derivative with respect to $s$ at $s = 0$ we get that the variational vector field $Z(t) = \frac{\partial \zeta}{\partial s}(0, t)$ satisfies
\[ Z(t) = \frac{\partial \zeta}{\partial s}(0, t) = T\varphi_t \frac{\partial \zeta}{\partial s}(0, 0) = T\varphi_t(Z(0)), \]
so that $Z$ is Lie transported.

Conversely, let $Z(t)$ be Lie transported along the integral curve $\zeta_0(t) = \varphi_t(m)$, that is $Z(t) = T\varphi_t(\xi)$. Consider any curve $\alpha(s)$ in $M$ such that $\alpha(0) = \zeta_0(0) = m$ and $\dot{\alpha}(0) = \xi$. The 1-parameter family $\zeta(s, t) = \varphi_t(\alpha(s))$ is a family of integral curves of $X$ satisfying the given properties. Indeed, for every fixed $s$ we have that $\zeta_s(t) = \varphi_t(\alpha(s))$ is an integral curve of $X$; for $s = 0$ we have $\zeta(0, t) = \varphi_t(\alpha(0)) = \varphi_t(m) = \zeta_0(t)$; and the variational vector field is
\[ \frac{\partial \zeta}{\partial s}(0, t) = T\varphi_t(\dot{\alpha}(0)) = T\varphi_t(\xi) = Z(t), \]
where in the last equality we have used that $Z$ is Lie transported. \hfill \Box

2.2. The case of second-order differential equations. In geometric terms, a second-order differential equation is a vector field $\Gamma$ defined on the tangent bundle $Z$ of a manifold $Q$. We consider now the special case of the constructions above with $M \equiv TQ$ and $X = \Gamma$ a SODE on $Q$. We have that a variation by integral curves of $\Gamma$ is of the form $\zeta(s, t) = \varphi_t(\zeta(s, 0))$, for $\gamma(s, t)$ a 1-parameter family of curves in $Q$ (each one is an integral curve in the base of $\Gamma$). We will denote by $W(t)$ the variational vector field of the base family,
\[ W(t) = \frac{\partial \gamma}{\partial s}(0, t). \]
Then, the variational vector field for the family $\zeta(s, t)$ is
\[ Z(t) = \frac{\partial \zeta}{\partial s}(0, t) = \frac{\partial}{\partial s} \frac{\partial \gamma}{\partial t}(s, t) \bigg|_{s=0} = \chi_{TQ} \frac{\partial}{\partial t} \frac{\partial \gamma}{\partial s}(0, t) = \chi_{TQ} W(t) = W^C(t). \]

Definition 2.6. Given a second-order vector field $\Gamma \in \mathfrak{X}(TQ)$, a vector field $W(t)$ along an integral curve in the base $\gamma_0$ of $\Gamma$ is a Jacobi field if $Z = W^C$ is a variation vector field along the integral curve $\gamma_0$ by integral curves of $\Gamma$.

It follows that a vector field $W(t)$ is a Jacobi field if and only if it satisfies the equation $\mathcal{L}_\Gamma W^C = 0$, or equivalently $W^C$ is Lie transported along the flow $\phi_t$ of $\Gamma$ or equivalently $T\phi_s(W^C(t)) = W^C(t + s)$.

2.3. The Jacobi equation. The above equation $\mathcal{L}_\Gamma W^C = 0$ is a second-order linear differential equation, as it can be easily seen in local coordinates. In the case of a geodesic spray one can rewrite such equation in terms of the associated covariant derivative. A general SODE does not define such a covariant derivative, but has an associated canonical non-linear connection. We will use such nonlinear connection to find an explicit expression of that equation. See [12] for a detailed construction of the objects in this section.

If $S$ is the vertical endomorphism on $TQ$, it is well known that the tensor $\mathcal{L}_\Gamma S$ satisfies $(\mathcal{L}_\Gamma S)^2 = I$. The eigenvectors of $\mathcal{L}_\Gamma S$ of eigenvalue 1 are precisely the vertical vectors. Therefore, at every point, the eigenspace of eigenvalue $-1$ is complementary to the vertical subspace, and hence it defines a splitting $TTQ = \text{Hor}(TQ) \oplus \text{Ver}(TQ)$. Both subbundles $\text{Hor}(TQ)$ and $\text{Ver}(TQ)$ are isomorphic to $\tau^Q(TQ) = TQ \times_{\phi} TQ$ via horizontal and vertical lift, respectively. It follows that every vector field $Z \in \mathfrak{X}(TQ)$ can be written in a unique way as $Z = X^u + Y^v$, with $X, Y$ vector fields along the map $\tau_Q$. 


Taking into account this decomposition, the Lie derivative of a horizontal lift defines two geometrical objects: the generalized Jacobi endomorphism $\Phi$ and the covariant derivative $\nabla$, by means of
\[ \mathcal{L}_\Gamma X^i = (\nabla X)^i + \Phi(X)^i. \]
Applying this relation to the vector field $fX$ for a function $f \in C^\infty(TQ)$ we find that $\nabla$ is a derivation along $\Gamma$, that is, it satisfies $\nabla(fX) = (\mathcal{L}_\Gamma f)X + f\nabla X$, while $\Phi$ is $(1,1)$-tensorial, $\Phi(fX) = f\Phi(X)$. The Lie derivative of a vertical lift does not define any new object, but can be expressed in terms of $\nabla$,
\[ \mathcal{L}_\Gamma X^\gamma = -X^\gamma + (\nabla X)^\gamma. \]

Obviously, the same relations hold for a vector field $W$ along an integral curve in the base $\gamma$ of $\Gamma$
\[ \mathcal{L}_\Gamma W^i = (\nabla W)^i + \Phi(W)^i \quad \text{and} \quad \mathcal{L}_\Gamma W^\gamma = -W^\gamma + (\nabla W)^\gamma. \]
In these expressions we should understand $\Phi(X)$ as $\Phi(X)(v) = \Phi_v(X(v))$, and similarly $\Phi(W)$ as $\Phi(W)(t) = \Phi_{\gamma_0(t)}(W(t))$.

In local coordinates $(x^i, v^j)$ on $TQ$, if the sode vector field is $\Gamma = v^i \partial/\partial x^i + f^j \partial/\partial v^j$, the horizontal lift of a coordinate vector field $\partial/\partial x^i$ is $\partial/\partial x^i - \Gamma^i_j \partial/\partial v^j$ where $\Gamma^i_j$ are the connection coefficients given by
\[ \Gamma^i_j = -\frac{1}{2} \frac{\partial f^i}{\partial v^j}. \]
If $W$ is of the form $W(t) = W^i(t) \partial/\partial x^i$ then the dynamical covariant derivative of $W$ takes the expression
\[ \nabla W(t) = \left( \dot{W}^i(t) + \Gamma^i_j (\gamma_0(t)) W^j(t) \right) \frac{\partial}{\partial x^i}, \]
expression which is also valid for the derivative of a vector field along $\Gamma_0$ if we interpret $X^i = \Gamma X^i$.

The components of the Jacobi endomorphism are
\[ \Phi^i_j = -\frac{\partial f^i}{\partial x^j} - \Gamma^i_k \Gamma^k_j - \Gamma^i_j, \]
and therefore $\Phi(W)(t) = \Phi^j_i (\gamma_0(t)) W^i(t) \frac{\partial}{\partial v^j}$.

Since we are interested in the equation $\mathcal{L}_\Gamma W^c = 0$, we need to decompose the complete lift $W^c$ in its horizontal and vertical components. We have that
\[ W^c = W^h + (\nabla W)^\gamma. \]
Indeed, the difference between $X^c$ and $X^h$ is a vertical vector field $X^c - X^h = Y^\gamma$. Applying $\mathcal{L}_\Gamma S$ to this expression we find
\[ Y^\gamma = (\mathcal{L}_\Gamma S) Y^\gamma = (\mathcal{L}_\Gamma S) X^c - (\mathcal{T}_\Gamma S) X^h = \mathcal{L}_\Gamma (S X^c) - S (\mathcal{L}_\Gamma X^c) + X^h = \mathcal{L}_\Gamma X^\gamma + X^h = (\nabla X)^\gamma \]
where we have used that $\mathcal{L}_\Gamma X^c$ is vertical (as it can be proved easily in coordinates).

From all this facts it is now easy to prove the following result.

**Theorem 2.7.** A vector field $W$ along an integral curve in the base $\gamma_0$ of $\Gamma$ is a Jacobi field if and only if it satisfies the second-order differential equation
\[ \nabla\nabla W + \Phi(W) = 0. \]
This equation will be called the (generalized) Jacobi equation.
Proof. Taking the Lie derivative of \( W^c = W^n + (\nabla W)^y \) with respect to \( \Gamma \) we have
\[
\mathcal{L}_\Gamma W^c = \mathcal{L}_\Gamma W^n + \mathcal{L}_\Gamma (\nabla W)^y
= \left( (\nabla W)^n + \Phi(W)^y \right) + \left( - (\nabla W)^n + (\nabla W)^y \right)
= [\nabla \nabla W + \Phi(W)]^y.
\]
From where the result follows immediately. \( \square \)

Jacobi fields are related to infinitesimal symmetries of the SODE \( \Gamma \). If \( Y \) is an infinitesimal symmetry of \( \Gamma \), that is \( \mathcal{L}_\Gamma Y = 0 \), then it is of the form \( Y = X^n + (\nabla X)^y \) for some vector field \( X \) along the projection \( \tau_Q \). Therefore, if \( \gamma \) is an integral curve in the base of \( \Gamma \), the vector field \( W(t) = X \circ \gamma \) satisfies \( Y \circ \gamma = W^c \) and \( \mathcal{L}_\Gamma W^c = 0 \), and hence \( W \) is a Jacobi field. This result already appeared in this form in [1].

We remark that our results apply to any second order differential equation on a manifold. In particular, after removing the points where the SODE is singular, everything applies to Finsler geometry. Notice however that in that case the relation between \( \nabla \) and any one of the linear connections associated to the Finsler spray is not obvious.

2.4. The variational SODE. The theory developed here for SODE in terms of complete lifts fits nicely into the general first-order theory that we saw in the previous paragraphs and allowed us to find easily the Jacobi equation, generalizing also the results given in [5, 13] for sprays and homogeneous SODEs. However, it is clear from its local expression that the Jacobi equation is a second order differential equation for the components of the Jacobi fields. Therefore, it is natural to look for a SODE on \( TQ \) whose solutions are the Jacobi fields together with the solutions of the original SODE.

As we have seen, \( W \) is a Jacobi field if \( W^c \) is an integral curve of \( \Gamma^c \). Taking into account that \( W^c = \chi_{TQ} \circ \dot{W} \) we have that the derivative of \( \dot{W} \) satisfies
\[
\frac{d}{dt} W = \frac{d}{dt} (\chi_{TQ} \circ W^c) = T \chi_{TQ} \circ \frac{d}{dt} W^c = T \chi_{TQ} \circ \Gamma^c \circ \chi_{TQ} \circ \dot{W}.
\]
Therefore \( \dot{W} \) is an integral curve of the vector field \( T \chi_{TQ} \circ \Gamma^c \circ \chi_{TQ} \).

**Theorem 2.8.** The map \( \Gamma^{\text{var}} = T \chi_{TQ} \circ \Gamma^c \circ \chi_{TQ} \) is a vector field on \( TTQ \) and satisfies the following properties:

1. \( \Gamma^{\text{var}} \) is \( \chi_{TQ} \)-related to \( \Gamma^c \).
2. \( \Gamma^{\text{var}} \) is a SODE vector field in \( TQ \).
3. \( \Gamma^{\text{var}} \) is \( T_{\tau_Q} \)-related to \( \Gamma \).
4. If \( \dot{\phi}_t \) is the local flow of \( \Gamma \), then the local flow of \( \Gamma^{\text{var}} \) is \( \dot{\phi}_t^{\text{var}} = \chi_{TQ} \circ T \phi_t \circ \chi_{TQ} \).

If \( W : \mathbb{R} \to TQ \) is an integral curve in the base of \( \Gamma^{\text{var}} \), then \( \gamma = \tau_Q \circ W \) is an integral curve on the base of \( \Gamma \) and \( W \) is a Jacobi field along \( \gamma \). Conversely, if \( \gamma \) is an integral curve in the base of \( \Gamma \) and \( W \) is a Jacobi field along \( \gamma \) then \( W \) is an integral curve in the base of \( \Gamma^{\text{var}} \).

**Proof.** We have that \( \Gamma^{\text{var}} = T \chi_{TQ} \circ \Gamma^c \circ \chi_{TQ} = T \chi_{TQ} \circ \Gamma^c \circ \chi_{TQ} \), which proves the first. As a consequence, the flow of \( \Gamma^{\text{var}} \) is \( \chi_{TQ} \circ \phi_t \circ \chi_{TQ} \) where \( \phi_t \) is the flow of \( \Gamma \), which proves (4). Taking into account that \( T \tau_Q \circ \chi_{TQ} = T \tau_Q \), we have
\[
T (T \tau_Q \circ \Gamma^{\text{var}}) \circ \Gamma^{\text{var}} = T (T \tau_Q \circ \Gamma^{\text{var}}) \circ \Gamma^c \circ \chi_{TQ} = T \tau_Q \circ \Gamma^c \circ \chi_{TQ} = \Gamma \circ T \tau_Q \circ \chi_{TQ} = \Gamma \circ T \tau_Q,
\]
which proves (3). Finally (2) follows easily from the coordinate expression of \( \Gamma^c \). \( \square \)

3. The case of SODEs on Lie algebroids. We consider here the generalization of the results in the previous section to the case of SODEs on Lie algebroids. For the general theory of Lie algebroids we refer to the reader to [7]. For the notation and the definition and properties of the main objects used here we refer to [4, 2, 11].
Let $\tau: E \to M$ be a Lie algebroid with anchor $\rho$ and bracket $[\cdot, \cdot]$. A curve $a: \mathbb{R} \to E$ is said to be admissible if $\rho a = \frac{d}{dt}(\tau a)$. By an integral curve of a section $\sigma$ of $E$ we just mean an integral curve of the vector field $\rho(\sigma) \in \mathfrak{X}(M)$.

A local coordinate system $(x^i)$ in $M$ and a local basis $\{e_\alpha\}$ of sections of $E$ determine coordinates $(x^i, y^\alpha)$ on $E$, where $y^\alpha(a)$ are the components of $a \in E_m$ in the basis $e_\alpha(m)$. The Lie algebroid structure is locally given by the structure functions $\rho^i_a, C^i_{\alpha\beta} \in C^\infty(M)$ defined by $\rho(e_\alpha) = \rho^i_a \partial/\partial x^i$ and $[e_\alpha, e_\beta] = C^i_{\alpha\beta} e_i$.

The $E$-tangent to $E$ is the vector bundle $\mathcal{T}_g E: \mathcal{T}E \to E$ whose fiber at the point $a \in E_m$ is $\mathcal{T}_a E = \{(b, v) \in E_{\tau(a)} \times T_{\tau(a)} E \mid T \tau(v) = \rho(b)\}$, and can be endowed with a natural Lie algebroid structure. The anchor and the bracket will also be denoted $\rho$ and $[\cdot, \cdot]$. An element $(b, v) \in \mathcal{T}_g E$ will be denoted $(a, b, v)$.

On $\mathcal{T}_g E$ there is a canonical involution $\chi_E: \mathcal{T}E \to \mathcal{T}E$ similar to the canonical involution on a manifold. In terms of it, the complete lift of a section $\sigma \in \text{Sec}(E)$ is the section of $\mathcal{T}_g E$ given by $\sigma^c(a) = \chi_E(\sigma(\tau(a), a, T\sigma(\rho(a))))$ for every $a \in E$. Similarly, given an admissible curve $a(t)$ and a section $W$ of $E$ along $\gamma = \tau a$ we define the complete lift of $W$ with respect to $a$ as the section of $\mathcal{T}E$ along $a$ given by $W^c(t) = \chi_E(W(t), a(t), W(t))$. It follows that $\sigma^c a = (\sigma^c a)^c$ for every admissible curve $a$.

By the flow of the section $\sigma$ we mean the pair $(\phi_t, \varphi_t)$, where $\varphi_t$ is the flow of the vector field $\rho(\sigma)$ and $\phi_t$ is the flow of the vector field $\rho(\sigma^c)$. For every fixed $t$, the map $\phi_t$ is a morphism of Lie algebroids over $\varphi_t$.

We will first generalize the results about Lie transport and the Lie derivative. We set a point $m \in M$ and consider the integral curve $\zeta_0(t) = \varphi_t(m)$ of $\rho(\sigma)$ starting at $m$ and defined on an interval $I$.

**Definition 3.1.** A section $Z$ of $E$ along $\zeta_0$ is said to be obtained by Lie transport along the flow of $\sigma$ if there exists $\xi \in E_m$ such that $Z(t) = \phi_t(\xi)$ for all $t \in I$.

**Definition 3.2.** The Lie derivative of a section $Z$ of $E$ along $\zeta_0$ with respect to $\sigma$ is the section along $\zeta_0$ given by

$$
(d_\sigma Z)(t) = \frac{d}{ds} \phi_{-s} Z(t + s) \bigg|_{s=0} = \lim_{h \to 0} \frac{1}{h} [\phi_{-h} Z(t + h) - Z(t)].
$$

It follows from the definition that if $\eta \in \text{Sec}(E)$ then $[\sigma, \eta] \circ \zeta_0 = d_\sigma(\eta \circ \zeta_0)$.

**Proposition 3.3.** The following properties are equivalent.

1. $Z$ is Lie transported along the flow of $\sigma$.
2. For every $t \in I$ and every $s$ such that $s + t \in I$ we have $\phi_s(Z(t)) = Z(t + s)$.
3. The Lie derivative of $Z$ vanishes identically, $d_\sigma Z(t) = 0$ for all $t \in I$.
4. The curve $Z: I \to E$ is an integral curve of the complete lift vector field $\rho(\sigma^c) \in \mathfrak{X}(E)$ of the section $\sigma$.

In local coordinates $(x^i, y^\alpha)$ in $E$, the above properties express that the components $x^i = \zeta^i_0(t)$, $y^\alpha = Z^\alpha(t)$ of $Z$ satisfy the variational equation

$$
\begin{align*}
\dot{x}^i &= \rho^i_{\alpha} \cdot \sigma^\alpha \\
\dot{y}^\alpha &= \frac{\partial \sigma^\alpha}{\partial x^i} \rho^i_{\alpha} y^\beta + C^\alpha_{\beta\gamma} y^\beta \cdot \sigma^\gamma.
\end{align*}
$$

**Proof.** We first prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and later on (2) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1). The final statement follows from (4) and the local expression of $\rho(\sigma^c) \in \mathfrak{X}(E)$.

1. $\Rightarrow$ (2) If $Z(t) = \phi_t(\xi)$ then $\phi_t(Z(t)) = \phi_t(\phi_t(\xi)) = \phi_{s+t}(\xi) = Z(t + s)$.
2. $\Rightarrow$ (3) Aplying $\phi_{-s}$ we get $\phi_{-s}(Z(t + s)) = Z(t)$ so that $d_\sigma Z(t) = 0$. 

Therefore it is constant, \( \phi_t(Z(t)) = \xi \) and hence \( Z(t) = \phi_t(\xi) \).

(2) \( \Rightarrow \) (4) The flow of the complete lift vector field \( \rho(\sigma^c) \) is \( \{\phi_t\} \). Taking the derivative of \( \phi_s(Z(t)) = Z(t + s) \) with respect to \( s \) at \( s = 0 \) we get \( \rho(\sigma^c)(Z(t)) = Z(t) \). Therefore \( Z(t) \) is an integral curve of \( \rho(\sigma^c) \).

(4) \( \Rightarrow \) (1) If \( Z(t) \) is integral curve of \( \rho(\sigma^c) \), then \( Z(t) = \phi_t(Z(0)) \) because \( \{\phi_t\} \) is the flow of \( \rho(\sigma^c) \). Taking \( \xi = Z(0) \) we get that \( Z(t) \) is Lie transported. \( \square \)

For the standard Lie algebroid \( E = TM \) we have that \( \phi_t = T\varphi_t \), and we recover the standard definitions and properties, and in particular a proof of Proposition 2.3.

### 3.1. One-parameter families of solutions

In the context of Lie algebroids, finite variations with fixed base endpoints are to be considered as \( E \)-homotopies. More generally, finite variations are to be considered as morphisms \( \theta: \mathbb{T}\mathbb{R}^2 \to E \), defined over a map \( \zeta: \mathbb{R}^2 \to M \) on some open subset of \( \mathbb{R}^2 \). In some sense, the idea is to mimic the situation in the standard case \( E = TM \), where the tangent map to a variation \( \zeta(s,t) \) is the map

\[
\theta(s,t) \equiv T\zeta(s,t) = \frac{\partial\zeta}{\partial t}(s,t)dt + \frac{\partial\zeta}{\partial s}(s,t)ds = X(\zeta(s,t))dt + \beta(s,t)ds.
\]

The map \( \theta \equiv T\zeta: \mathbb{T}\mathbb{R}^2 \to TM = E \) being a tangent map, it is a morphism of Lie algebroids. The variation vector field is \( Z(t) = \beta(0,t) = \frac{\partial\zeta}{\partial s}(0,t) \).

We now consider a section \( \sigma \in \text{Sec}(E) \) and the integral curve \( \zeta_0 \) of the vector field \( \rho(\sigma) \) starting at \( m \in M \), that is \( \zeta_0(t) = \varphi_t(m) \).

**Definition 3.4.** A 1-parameter family of integral curves of \( \sigma \in \text{Sec}(E) \) is a morphism of Lie algebroids \( \theta: \mathbb{T}\mathbb{R}^2 \to E \) over \( \zeta: \mathbb{T}\mathbb{R}^2 \times I \subset \mathbb{R}^2 \to M \) of the form \( \theta = \sigma(\zeta(s,t))dt + \beta(s,t)ds \). The section \( \tilde{Z} \) along \( \zeta_0 \) defined by \( \tilde{Z}(t) = \beta(0,t) \) is said to be the infinitesimal variation defined by the 1-parameter family.

It follows from the definition that the base map of \( \theta \) is \( \zeta \). Moreover, for every fixed \( s \), the curve \( \zeta_s(t) = \zeta(s,t) \) is an integral curve of \( \rho(\sigma) \). Indeed,

\[
\rho(\sigma)(\zeta_s(t)) = \rho(\sigma(\zeta(s,t))) = \frac{\partial\zeta}{\partial t}(s,t) = \dot{\zeta}_s(t),
\]

where we have used that \( t \mapsto \sigma(\zeta(s,t)) \) is admissible, because \( \theta \) is a morphism.

**Theorem 3.5.** A section \( \tilde{Z} \) along \( \zeta_0 \) is the infinitesimal variation defined by a 1-parameter family of integral curves of \( \sigma \) if and only if it is Lie transported along the flow of \( \sigma \).

**Proof.** Let \( Z(t) = \phi_t(\xi) \) for some \( \xi \in \mathfrak{e}_m \). Consider an admissible curve \( \mu: (-\epsilon, \epsilon) \to E \) such that \( \mu(0) = \xi \), and denote by \( \nu \) the base path, \( \nu = \tau \circ \mu \). The map

\[
\theta(s,t) = \alpha(s,t)dt + \beta(s,t)ds = \sigma(\varphi_t(\nu(s)))dt + \phi_t(\mu(s))ds,
\]

is a 1-parameter family of integral curves of \( \sigma \). The base family is \( \zeta(s,t) = \tau(\alpha(s,t)) = \varphi_t(\nu(s)) \). To prove it, we recall (see [11]) that \( \theta \) is a morphism of Lie algebroids if and only if \( a_s(t) \equiv \alpha(s,t) \) is admissible, \( b_t(s) = \beta(s,t) \) is admissible and \( X_E(\alpha, \beta, \frac{\partial\alpha}{\partial t}) = (\beta, \alpha, \frac{\partial\beta}{\partial t}) \).

Indeed, from \( [\sigma, \sigma] = 0 \) we get that \( \phi_t \sigma \sigma = \sigma \circ \varphi_t \), and therefore we can rewrite \( a_s(t) = \phi_t(\sigma(\nu(s))) \), which is admissible because it is an integral curve of \( \rho(\sigma^c) \). On the other hand, \( b_t(s) = \phi_t(\mu(s)) \) is admissible because \( \phi_t \) is a morphism of Lie algebroids (morphisms
transforms admissible curves into admissible curves). Finally, to prove $\chi_E(\alpha, \beta, \frac{\partial \alpha}{\partial s}) = (\beta, \alpha, \frac{\partial \beta}{\partial t})$ it is enough to show that $\rho(\chi_E(\alpha, \beta, \frac{\partial \alpha}{\partial s})) = \frac{\partial \beta}{\partial t}$. On one hand we have

$$\frac{\partial \beta}{\partial t}(s,t) = \frac{d}{dt}\phi_t(\mu(s)) = \rho(\sigma^c)(\phi_t(\mu(s))) = (\rho(\sigma^c)\circ \beta)(s,t)$$

and on the other

$$\chi_E\left(\alpha, \beta, \frac{\partial \alpha}{\partial s}\right) = \chi_E\left(\sigma \circ \zeta, \beta, T\sigma \circ \frac{\partial \zeta}{\partial s}\right) = \chi_E(\sigma \circ \zeta, \beta, T\sigma \circ \rho \circ \beta) = \sigma^c \circ \beta,$$

and applying $\rho$ to it we get $\rho(\chi_E(\alpha, \beta, \frac{\partial \alpha}{\partial s})) = \rho(\sigma^c \circ \beta) = \rho(\sigma^c) \circ \beta = \frac{\partial \beta}{\partial t}$.

Therefore $\theta$ is a morphism of Lie algebroids. The infinitesimal variation defined by $\theta$ is $\beta(0,t) = \phi_t(\mu(0)) = \phi_t(\xi) = Z(t)$.

Conversely, given a 1-parameter family $\theta(s,t) = \sigma(\zeta(s,t)) dt + \beta(s,t) ds$ of integral curves of $\sigma$ we will prove that $Z(t) = \beta(0,t)$ is Lie transported by showing that it is an integral curve of $\rho(\sigma^c)$. Since $\theta$ is a morphism we have $\frac{\partial \beta}{\partial t} = \rho(\chi_E(\alpha, \beta, \frac{\partial \alpha}{\partial s}))$, where $\alpha(s,t) = \sigma(\zeta(s,t))$, which at $s = 0$ gives

$$Z(t) = \frac{\partial \beta}{\partial t}(0,t) = \rho(\chi_E(\sigma(\zeta(0,t)), \beta(0,t), T\sigma_t(\partial s)(0,t))) =$$

$$= \rho(\chi_E(\sigma(\zeta(0,t)), Z(t), T\sigma(\rho(Z(t))))) = \rho(\sigma^c)(Z(t)),$$

where we have used that $\sigma^c(a) = \chi_E(\sigma(\tau(a)), a, T\sigma(\rho(a)))$, for $a \in E$.

From the proof we deduce that every 1-parameter family of integral curves is necessarily of the form $\theta(s,t) = \sigma(\varphi_t(\nu(s))) dt + \phi_t(\mu(s)) ds$ and we have $\nu(s) = \zeta(s,0)$, $\mu(s) = \beta(s,0)$ and $Z(t) = \phi_t(\mu(0))$.

### 3.2. Jacobi sections

Our results in Subsection 2.2 for a SODE on a manifold extend easily to the case of a SODE $\Gamma$ on a Lie algebroid, that is, a section of the $E$-tangent $\tau_E^E: T^E_E \rightarrow E$ to $E$, of the form $\Gamma(a) = (a,a, V(a))$ for every $a \in E$. It follows that every integral curve $a$ of $\Gamma$ is admissible, and in local coordinates $(x^i, y^\alpha)$ the differential equations for the integral curves of a SODE section $\Gamma$ are of the form

$$\dot{x}^i = \rho^i_\alpha y^\alpha, \quad \dot{y}^\alpha = f^\alpha(x,y).$$

A 1-parameter family of integral curves of $\Gamma$ is a morphism of the form $\Theta(s,t) = \Gamma(\alpha(s,t)) dt + B(s,t) ds$, where $B$ is of the form $B(s,t) = (\alpha(s,t), \beta(s,t), V(s,t))$. It is easy to see that $V(s,t) = \frac{\partial \alpha}{\partial s}(s,t)$ and that $\theta(s,t) = \alpha(s,t) dt + \beta(s,t) ds$ is a morphism of Lie algebroids. Therefore, at $s = 0$ the infinitesimal variational section is

$$Z(t) = B(0,t) = \left(\alpha_0(t), \beta(0,t), \frac{\partial \alpha}{\partial s}(0,t) \right) =$$

$$= \chi_E(W(t), a_0(t), \frac{\partial \beta}{\partial t}(0,t)) = \chi_E(W(t), 0, \dot{W}(t)) = W^c_{a_0}(t),$$

where $W(t) = \beta(0,t)$ is the infinitesimal variational section associated to $\theta$.

**Definition 3.6.** Given an integral curve $a_0$ of a SODE $\Gamma$ on a Lie algebroid $E$, a section $W$ of $E$ along $\gamma_0 = \tau \circ a_0$ is said to be a Jacobi field along $a_0$ if its complete lift $Z = W^c_{a_0}$ is an infinitesimal variational section.

Equivalently, $W$ is a Jacobi section if it is an integral curve of $\rho(\Gamma^c)$, or equivalently $W^c_{a_0}$ is Lie transported, or equivalently if $d_t W^c_{a_0} = 0$.

The horizontal distribution associated to a SODE is constructed in the same way as in the standard case, as the eigenspace of eigenvalue $-1$ of $d_t S$. Using the same arguments as in the standard case we have that there is a derivation $\nabla$ and a tensor field $\Phi$ satisfying

$$L_\Gamma \eta^\alpha = (\nabla \eta)^\alpha + \Phi(\eta)^\alpha \quad \text{and} \quad L_\Gamma \eta^\nu = -\eta^\nu + (\nabla \eta)^\nu,$$
where $\eta$ is a section of $\pi^*E = E \times_M E$. Similar expressions hold for a section $W$ along the base curve of an integral curve of $\Gamma$.

In local coordinates, the connection coefficients are

$$\Gamma^\alpha_{\beta\gamma} = -\frac{1}{2} \left( \frac{\partial f^\alpha}{\partial y^\beta} - C^\alpha_{\beta\gamma} y^\gamma \right).$$

We also define the functions

$$\gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - C^\alpha_{\beta\gamma} y^\gamma = -\frac{1}{2} \left( \frac{\partial f^\alpha}{\partial y^\beta} + C^\alpha_{\beta\gamma} y^\gamma \right).$$

If the local expression of a section $W$ along $\gamma_0$ is $W(t) = W^\alpha(t)e_\alpha$ then

$$\nabla W(t) = [\dot{W}^\alpha(t) + \gamma^\alpha_{\beta\gamma}(a_0(t))W^\beta(t)]e_\alpha \quad \text{and} \quad \Phi(W) = \Phi^\alpha_{\beta\gamma}(a_0(t))W^\beta(t)e_\alpha,$$

where the components of the Jacobi endomorphism are

$$\Phi^\alpha_{\beta\gamma} = -\rho^\alpha_i \frac{\partial f^\alpha}{\partial x^i} - d\Gamma^\alpha_{\beta\gamma} + \gamma^\alpha_{\beta\gamma} y^\gamma C^\gamma_{\mu\nu} y^\nu.$$

On the other hand, the same arguments as in the standard case show that the complete lift of a section $W$ (with respect to an integral curve $a$ of $\Gamma$, which we omit in the notation) has the expression

$$W^c = W^t + (\nabla W)^t.$$

From these facts, it follows that $L_\Gamma W^c$ is vertical and has the expression

$$L_\Gamma W^c = [\nabla \nabla W + \Phi(W)]^t,$$

and we have proved the following result.

**Theorem 3.7.** A section $W$ along the base curve of an integral curve $a$ of $\Gamma$ is a Jacobi section along $a$ if and only if it satisfies the second-order differential equation

$$\nabla \nabla W + \Phi(W) = 0.$$

This equation is the generalized Jacobi equation.

Many new results and applications can be expected from the theory developed here. For instance, the relation between conjugate points and the failure of the exponential mapping to be a diffeomorphism can also be generalized for the kind of systems considered here. Also the case of a Lagrangian SODE and the relation with the second variation of the action will be developed elsewhere following the ideas in [6].

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Received September 2014; revised January 2015.

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