Rayleigh quotient and left eigenvalues of quaternionic matrices

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\textbf{ABSTRACT}

We study the Rayleigh quotient of a Hermitian matrix with quaternionic coefficients and prove its main properties. As an application, we give some relationships between left and right eigenvalues of Hermitian and symplectic matrices.

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\section{1. Introduction}

The Rayleigh quotient of a matrix, introduced by the British physicist Lord Rayleigh in 1904 in his book “The theory of sound”, is a well known tool which is widely used to obtain estimates of the eigenvalues of real and complex matrices [1, 2].

For quaternionic matrices, however, only a few references about the Rayleigh quotient can be found in the literature, and there is a lack of a general exposition of its properties and main results. Since quaternions have many applications, among them in quantum mechanics, solid body rotations, and signal theory [3], it seems useful to fill that gap. Notice that most of the results and proofs are analogous to the complex case, but they refer to right eigenvalues (see Subsection 2.2).

On the other hand, very little is known about left eigenvalues of quaternionic matrices (see Section 4). Wood [4] proved that every quaternionic matrix has at least one left eigenvalue. Huang and So [5] completely solved the case of $2 \times 2$ matrices. The authors [6, 7] applied Huang and So’s results to $2 \times 2$ symplectic matrices. The case $n = 3$ was studied by So [8] and the authors [9]. Finally, Zhang [10] and Farid, Wang and Zhang [11] gave several Geršgorin type theorems for quaternionic matrices.

Consequently, many problems still remain open, in particular those about the relationship between left and right eigenvalues. In this paper we give some partial answers to this
question, for Hermitian and symplectic matrices, as an application of the previously proved properties of the Rayleigh quotient.

The contents of the paper are as follows. In Section 2 we present some preliminaries about the right eigenvalues of a quaternionic matrix.

In Section 3, we consider a Hermitian quaternionic \( n \times n \) matrix \( S \), and we define its Rayleigh quotient \( h_S \) as a real function defined on the sphere \( S^{4n-1} \). We compute the gradient, the Hessian and the mean value of \( h_S \), and we prove its main properties, among them the min-max principle for right eigenvalues (Section 3.2).

In Section 4, we introduce left eigenvalues and we study the case \( n = 2 \) with some detail, as a testing bench for later results. For an arbitrary Hermitian matrix \( S \), our main result (Theorem 4.8) is that the real part of any left eigenvalue \( \lambda \) is bounded by the right eigenvalues, in a way that depends on the dimension of the \( \lambda \)-eigenspace. As we shall see, this implies that the existence of left eigenvalues with a high-dimensional space of eigenvectors depends on the multiplicity of the right eigenvalues.

Finally, in Section 5 we state similar results for symplectic matrices.

Our results suggest that there are still many more hidden relations between left and right eigenvalues.

\[ 2. \textbf{Preliminaries} \]

As a general reference for quaternionic linear algebra we take Rodman’s book [12]. For a brief survey on quaternions and matrices of quaternions, see Zhang’s paper [13].

\[ 2.1. \textbf{Basic notions} \]

We denote by \( \mathbb{H} \) the non-commutative algebra of quaternions. Each quaternion \( q \in \mathbb{H} \) can be written as \( q = t + xi + yj + zk \), with \( t, x, y, z \in \mathbb{R} \), where \( i, j, k \) verify the Hamilton conditions \( i^2 = j^2 = k^2 = ijk = -1 \). For the quaternion \( q \in \mathbb{H} \) we denote its conjugate by \( \overline{q} = t - xi - yj - zk \), its norm by \( |q| = (t^2 + x^2 + y^2 + z^2)^{1/2} \) and its real part by \( \Re(q) = t \).

Let \( \mathbb{H}^{n \times n} \) be the space of \( n \times n \) matrices with quaternionic coefficients. If \( M \in \mathbb{H}^{n \times n} \), we denote by \( M^* \) its conjugate transpose \( (M^T)^\dagger \). The quaternionic space of \( n \)-tuples \( u = (u_1, \ldots, u_n)^T \), with \( u_i \in \mathbb{H} \), will be denoted by \( \mathbb{H}^n \). We shall always consider it as a right vector space over \( \mathbb{H} \), endowed with the Hermitian product \( \langle u, v \rangle = u^* v \). Notice that \( |u|^2 = \langle u, u \rangle \) is the Euclidean norm in \( \mathbb{R}^{4n} \), so the scalar product is \( u \cdot v = \Re(\langle u, v \rangle) \).

The matrix \( S \in \mathbb{H}^{n \times n} \) is Hermitian if it is self-adjoint for the Hermitian product, that is, \( \langle Su, v \rangle = \langle u, Sv \rangle \), or equivalently, \( S^* = S \). The matrix \( A \in \mathbb{H}^{n \times n} \) is quaternionic unitary or symplectic if the associated linear map preserves the Hermitian product, that is, \( \langle Au, Av \rangle = \langle u, v \rangle \), or equivalently, \( A^* A = A A^* = I_n \).

Two quaternions \( q \) and \( q' \) are similar if there exists some \( r \in \mathbb{H} \), \( r \neq 0 \), such that \( q' = rqr^{-1} \). Equivalently, they have the same norm and the same real part, that is, \( |q| = |q'| \) and \( \Re(q) = \Re(q') \) [13, Theorem 2.2]. As a consequence, any quaternion \( q \) is similar to a complex number, namely \( z = t + s i \), where \( t = \Re(q) \) and \( t^2 + s^2 = |q|^2 \). Notice that \( z \in \mathbb{C} \) and its conjugate \( \overline{z} \) are similar quaternions.

Finally, we shall need the following result, which can be proved by a direct computation.
Lemma 2.1: Let \( \omega, \omega' \) be two quaternions such that \( |\omega| = |\omega'| = 1 \) and \( \Re(\omega) = \Re(\omega') = 0 \). If \( \omega \omega' = \omega' \omega \) then \( \omega = \pm \omega' \).

2.2. Right eigenvalues

The theory of right eigenvalues is well known, and has many properties in common with the complex case.

**Definition 2.2:** The quaternion \( q \in \mathbb{H} \) is a right eigenvalue of the matrix \( M \in \mathbb{H}^{n \times n} \) if there exists some vector \( u \in \mathbb{H}^n, u \neq 0 \), such that \( M u = u q \).

Notice that the eigenvectors associated to a right eigenvalue do not form a vector subspace. Instead, right eigenvalues are organized in similarity classes.

**Proposition 2.3:** Let \( q \) be a right eigenvalue of \( M \in \mathbb{H}^{n \times n} \), and let \( u \) be a \( q \)-eigenvector. If \( r \in \mathbb{H} \) is a non-zero quaternion, then \( r q r^{-1} \) is also a right eigenvalue of \( M \), and \( u r^{-1} \) is an \( r q r^{-1} \)-eigenvector.

**Proof:**

\[
M(u r^{-1}) = (u q) r^{-1} = u r^{-1} (r q r^{-1}).
\]

So, the computation of the right eigenvalues of the matrix \( M \) is reduced to computing the complex representatives of their similarity classes. This can be done as follows. Each quaternion \( q \) can be written in a unique form as \( q = u + j v \), with \( u, v \in \mathbb{C} \) complex numbers. Then the matrix \( M \in \mathbb{H}^{n \times n} \) decomposes as \( M = U + j V \), with \( U, V \in \mathbb{C}^{n \times n} \) complex matrices. We define the associated complex matrix

\[
c(M) = \begin{bmatrix} U & -V \\ V & U \end{bmatrix}.
\]

It is straightforward to verify that the map \( c: \mathbb{H}^{n \times n} \rightarrow \mathbb{C}^{2n \times 2n} \) is an injective morphism of \( \mathbb{R} \)-algebras. It also satisfies \( c(M^*) = c(M)^* \), as it follows from the formulas \( \bar{q} = \bar{u} - j \bar{v} \) and \( M^* = U^* - j V^T \).

**Proposition 2.4:** The right eigenvalues of \( M \) are grouped in \( n \) similarity classes \( [z_1], \ldots, [z_n] \). The complex representatives \( z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n \) are the eigenvalues of the complex matrix \( c(M) \).

See [12, Theorem 5.5.3] for a discussion of the Jordan form of \( M \). There are other methods for computing the right eigenvalues which use a real counterpart of the matrix, see for instance [14, 15].

**Example 2.5:** Let the matrix \( M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). The eigenvalues of \( c(M) \) are \( z_1 = \frac{1}{\sqrt{2}} (1 + i) \), \( z_2 = \frac{1}{\sqrt{2}} (-1 + i) \) and their conjugates \( \bar{z}_1, \bar{z}_2 \). Then, the right eigenvalues of \( M \) are all the quaternions \( q \) such that \( \Re(q) = \pm 1/\sqrt{2} \) and \( |q| = 1 \).

Notice that, unlike the usual complex case, the matrices \( M - qI \) in the latter example are invertible, for all right eigenvalues \( q \). This can be easily seen by computing their kernel. This leads to the notion of left eigenvalue, as a quaternion \( \lambda \in \mathbb{H} \) such that \( M - \lambda I \) is not invertible (see Section 4).
3. The Rayleigh quotient

The well known Rayleigh quotient for complex matrices can be generalized to matrices of quaternions. We now focus on Hermitian matrices.

For the classical version, over the real or complex numbers, of the next results, see [1, Section 4.2] or [2, Section 3.1].

3.1. Definition and first properties

Recall that the \( n \times n \) quaternion matrix \( S \) is Hermitian if \( S = S^* \). Any right eigenvalue \( q \) of \( S \) is real: in fact, if \( Su = uq \) then

\[
|u|^2 q = u^*uq = u^*Su = (u^*Su)^*
\]

is real, hence \( \overline{q} = q \). Moreover, \( S \) is diagonalizable [12, Theorem 5.3.6]. Let \( t_1 \leq \cdots \leq t_n \) be the eigenvalues of \( S \), and let \( u_1, \ldots, u_n \) be an orthonormal basis of eigenvectors. Then \( S \) diagonalizes as \( S = U \text{diag}[t_1, \ldots, t_n] U^* \), where \( U \) is a symplectic matrix (that is, \( UU^* = I \)) whose columns are the \( u_j \)’s.

**Definition 3.1:** If \( S \) is a Hermitian \( n \times n \) matrix, and \( v \in \mathbb{H}^n \) is a vector, \( v \neq 0 \), the Rayleigh quotient is the real number

\[
R(S, v) = \frac{v^*Sv}{|v|^2}.
\]

**Proposition 3.2:** If \( t \in \mathbb{R} \) is a right eigenvalue of \( S \), and \( u \in \mathbb{H}^n \) is a \( t \)-eigenvector, then \( R(S, u) = t \).

If \( v \in \mathbb{H}^n \) is a vector, \( v \neq 0 \), we can write it in coordinates with respect to the orthonormal basis \( \{u_j\}_{j=1,\ldots,n} \) as

\[
v = \sum_{j=1}^{n} u_j x_j, \quad x_j \in \mathbb{H}.
\]

**Proposition 3.3:** The Rayleigh quotient equals the weighted mean

\[
R(S, v) = \frac{\sum_j t_j |x_j|^2}{\sum_j |x_j|^2},
\]

where the real numbers \( t_j \) are the right eigenvalues of \( S \).

**Proof:** Since \( u_j^*u_j = \delta_{ij} \), we have

\[
|v|^2 = v^*v = \left( \sum_i \overline{x}_i u_i^* \right) \left( \sum_j u_j x_j \right) = \sum_{ij} \overline{x}_i u_i^* u_j x_j = \sum_j \overline{x}_j x_j = \sum_j |x_j|^2.
\]

Analogously, since \( Su_j = u_j t_j \),

\[
v^*Sv = \left( \sum_i \overline{x}_i u_i^* \right) \left( \sum_j u_j t_j x_j \right) = \sum_j \overline{x}_j t_j x_j = \sum_j t_j |x_j|^2.
\]
By diagonalization, we have reduced the problem of an arbitrary bilinear form to the corresponding quadratic form. As a consequence we have:

**Proposition 3.4:** The minimum value of the function \( R(S, v) \), defined in \( \mathbb{H}^n \setminus \{ 0 \} \), is the lowest right eigenvalue of \( S \). The maximum value is the highest right eigenvalue of \( S \).

**Proof:** First, notice that \( R(S, v/|v|) = R(S, v) \), so we can assume that \( |v| = 1 \). It is known \([1, Appendix B]\) that a convex function \( \sum_{j=1}^n t_j s_j \) with \( \sum_{j=1}^n s_j = 1, s_j \geq 0 \), attains its maximum value at \( t_n = \max t_j \) and its minimum value at \( t_1 = \min t_j \). By taking into account Proposition 3.3, the result follows. \[\square\]

In fact, we shall compute all the critical values of the function. This is a variational characterization of eigenvalues, which was discovered in connection with problems of physics \([2, 3.7]\).

The classical version of Proposition 10 appears, for instance, in \([16, Fact 1.8]\). For the computation of the critical points, it is possible to use coordinates and to optimize a quadratic form on the sphere by using the method of Lagrange multipliers. However, we have chosen a more synthetic proof, which is similar to that in \([17, Propositions 4.6.1 and 4.6.2]\), computing the gradient and the Hessian on the sphere \( S^{4n-1} \).

**Proposition 3.5:** The critical values of the function \( R(S, v) \) are the right eigenvalues \( t_j \) of \( S \). The index of \( t_j \) equals \( \sum_{i < j} l_i \), where \( l_i \) is the multiplicity of \( t_i \).

**Proof:** By differentiating the function \( R = R(S, v) : \mathbb{H} \setminus \{ 0 \} \rightarrow \mathbb{R} \) we obtain

\[
R_{sv}(w) = \frac{1}{|v|^4} \left( (w^* S v + v^* S w) |v|^2 - (v^* S v)(w^* v + v^* w) \right) \\
= \frac{2}{|v|^4} \left( \Re(v^* S w) |v|^2 - (v^* S v) \Re(v^* w) \right) \\
= \frac{2}{|v|^4} \left( (S v \cdot w) |v|^2 - (v^* S v)(v \cdot w) \right) \\
= \frac{2}{|v|^2} \left( (S v \cdot w) - R(S, v)(v \cdot w) \right) \\
= \frac{2}{|v|^2} (S v - R(S, v)v) \cdot w,
\]

where \( (\cdot) \) represents the scalar product in \( \mathbb{R}^{4n} \).

Hence the gradient of \( R(S, v) \) is

\[
G_v = \frac{2}{|v|^2} (S v - R(S, v)v).
\]

Since \( R(S, v/|v|) = R(S, v) \), we can assume that \( |v| = 1 \). Let us denote by \( h \) the restriction of the Rayleigh function to the sphere \( S^{4n-1} \subset \mathbb{H}^n \) of unitary vectors. We check that \( G_v \perp v \),
because
\[
\frac{1}{2} \langle v, G_v \rangle = v^*(Sv - R(S, v)v) = v^*Sv - R(S, v)|v|^2 = 0.
\]

Hence, \( v \cdot G_v = \Re \langle v, G_v \rangle = 0 \) and \( G_v \) is tangent to the sphere for the scalar product, so it is also the gradient of the restriction \( h \).

It follows that the point \( v \) is critical (both for the function and its restriction) if and only if \( Sv = R(S, v)v \), that is, \( v \) is an eigenvector of the eigenvalue \( R(S, v) = t \in \mathbb{R} \).

This will allow us to compute the Hessian
\[
H_v(w) = (G_v)_*(w) = \frac{2}{|v|^4} (Sw - (R_{sv}(w)v + R(S, v)w)|v|^2
- (Sv - R(S, v)v)(w^*v + v^*w))
= \frac{2}{|v|^2} (Sw - (R_{sv}(w)v + R(S, v)w))
= \frac{2}{|v|^2} (Sw - R(S, v)w)
= \frac{2}{|v|^2} (Sw - tw) = \frac{2}{|v|^2} (S - tI)w.
\]

Moreover, if \(|v| = 1\) and \( w \in T_vS^{4n-1} \), that is, \( v \cdot w = \Re(v^*w) = 0 \), then
\[
v \cdot H_v(w) = \Re(v^*H_v(w)) = 2\Re(v^*Sw) = 2\Re(w^*Sv) = 2w \cdot vt = 0.
\]

Hence \( H_v(w) \in T_vS^{4n-1} \), and \( H_v(w) \) is also the Hessian of the restriction \( h \).

Now, we compute the index of \( t = R(S, v) \), which is the number of negative eigenvalues of the Hessian at the critical point \( v \). If \( \mu \) is an eigenvalue of the Hessian, we have \( H_v(w) = \mu w \), for some \( w \neq 0 \), that is,
\[
2(S - tI)w = \mu w,
\]
so we are looking for the eigenvalues \( \mu \) of of the shifted matrix \( S - tI \). Hence, the eigenvalues of the Hessian are \( \mu_k = 2(t_k - t) \), twice the differences with the other right eigenvalues \( t_k \) of \( S \), and the result follows.

For instance, the minimum \( t_1 \) has index 0. The maximum \( t_n \) has index \( n - l_n \).

### 3.2. The min-max principle for right eigenvalues

As in the complex case, it is possible to refine Proposition 3.5. Now, we constrain \( v \) to a \( k \)-dimensional subspace, in order to obtain a quaternionic version of the so-called min-max Courant-Fischer-Weyl theorem [1, Theorem 4.2.6].

Fix some \( k \in \{1, \ldots, n\} \) and let \( \mathbb{E} \subset \mathbb{H}^n \) be any \( \mathbb{H} \)-subspace of dimension \( k \). We shall denote \( \mathbb{E}^* = \mathbb{E} \setminus \{0\} \).
Let \( \{u_1, \ldots, u_n\} \) be again an orthonormal basis of eigenvectors. We have that
\[
\mathbb{E} \cap \langle u_k, \ldots, u_n \rangle \neq 0,
\]
due to dimension reasons. Then there exists \( v = \sum_{j=k}^{n} u_j x_j \in \mathbb{E}^* \), and its Rayleigh quotient is
\[
R(S, v) = \frac{\sum_{j=k}^{n} t_j |x_j|^2}{\sum_{j=k}^{n} |x_j|^2} \geq t_k,
\]
because \( t_j \geq t_k \) for all \( j \geq k \).

This implies that
\[
M_{\mathbb{E}} := \max\{R(S, v) : v \in \mathbb{E}^*\} \geq t_k.
\]
Since this is true for all \( \mathbb{E} \) we conclude that \( \min\{M_{\mathbb{E}} : \dim \mathbb{E} = k\} \geq t_k \). This is in fact an equality:

**Theorem 3.6:** \( t_k = \min\{M_{\mathbb{E}} : \dim \mathbb{E} = k\} \).

**Proof:** It only remains to prove that \( M_{\mathbb{E}} \leq t_k \) for some \( \mathbb{E} \) with \( \dim \mathbb{E} = k \). Take \( \mathbb{E} = \langle u_1, \ldots, u_k \rangle \). Then, for all \( v \in \mathbb{E}^* \), we have
\[
R(S, v) = \frac{\sum_{j=1}^{k} t_j |x_j|^2}{\sum_{j=1}^{k} |x_j|^2} \leq t_k,
\]
because \( t_j \leq t_k \) if \( j \in \{1, \ldots, k\} \).

Analogously, if we denote
\[
m_{\mathbb{E}} := \min\{R(S, v) : v \in \mathbb{E}^*\},
\]
we have the following result.

**Corollary 3.7:** \( t_{n-k+1} = \max\{m_{\mathbb{E}} : \dim \mathbb{E} = k\} \).

The proof is immediate if we take into account that \( t_{n-k+1}(S) = -t_k(-S) \).
As particular cases, for \( k = 1, n \) we have Proposition 3.4.

### 3.3. Mean value

We want to compute the mean value of the Rayleigh function over the sphere \( S^{N-1} \subset \mathbb{H}^n \), where \( N = 4n \).

Let \( h : S^{N-1} \to \mathbb{R} \) be the restriction given by \( h(v) = v^* Sv, |v| = 1 \). In order to compute the mean value of \( h \),
\[
M(h) = \frac{1}{\text{Vol}(S^{N-1})} \int_{S^{N-1}} v^* Sv \, dv,
\]
one can consider hyper-spherical coordinates and to undertake a long direct computation. Another proof follows by using Pizzetti’s formula [18, Formula (11.3)]. However, in order
to have a similar result for the variance, we shall use moments, as explained in Gray’s book [18, Appendix A.2].

For any integrable function $F = F(u_1, \ldots, u_N): \mathbb{R}^N \to \mathbb{R}$ we denote by $M(F)$ the average of $F$ over the unit sphere $S^{N-1} \subset \mathbb{R}^N$.

**Lemma 3.8 ([18, Theorem 1.5].):**

$$M(u_i^2) = \frac{1}{N}, \quad M(u_i^4) = \frac{3}{N(N+2)}, \quad M(u_i^2 u_j^2) = \frac{1}{N(N+2)}$$ \text{ if } i \neq j.

**Theorem 3.9:** The expected value of the Rayleigh quotient $R(S, v)$ over the sphere $S^{4n-1} \subset \mathbb{H}^n$ equals

$$M(h) = \frac{1}{n} \text{Trace } S.$$

**Proof:** According to Proposition 3.3, $h(v) = \sum_{j=1}^n t_j |x_j|^2$, where $x_j \in \mathbb{H}$. Since each $x_j$ has four real coordinates, our function can be written as

$$h(u_1, \ldots, u_N) = t_1(u_1^2 + \cdots + u_4^2) + \cdots + t_n(u_{N-3}^2 + \cdots + u_N^2), \quad N = 4n,$$

where $u_1^2 + \cdots + u_N^2 = 1$.

Then, by Lemma 3.8,

$$M(h) = 4 \frac{1}{N} (t_1 \cdots + t_n) = \frac{1}{n} (t_1 + \cdots + t_n),$$

is the arithmetic mean of the eigenvalues and the result follows.

A similar computation gives us the relationship between the second central moment of $h$ and the variance of the eigenvalues. We denote by

$$\mu = \frac{1}{n} \sum_{i=1}^n t_i$$

the mean of the eigenvalues and by

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^n (t_i - \mu)^2$$

its variance.

**Theorem 3.10:** The second central moment of the Rayleigh quotient over the sphere is proportional to the variance of the right eigenvalues,

$$M((h - \mu)^2) = \frac{1}{2n + 1} \sigma^2.$$

**Proof:** The proof follows from the well-known identity $M((h - \mu)^2) = M(h^2) - \mu^2$, and Lemma 3.8.
Remark 1: To the best of our knowledge, Theorems 3.9 and 3.10 are new, in its present form. They can be deduced from von Neumann’s paper [19, Section 4], where there is a computation of ‘the moments of the distribution law of a quantity \( \gamma = \sum_{\mu=1}^{m} B_\mu x_\mu^2 \) where the point \( x_1, \ldots, x_m \) is equidistributed over the spherical surface \( \sum x_\mu^2 = 1 \.)’

4. Left eigenvalues

As mentioned in Example 2.5, the matrix \( M - qI \) can be invertible for a right eigenvalue \( q \) of \( M \). This motivates the following definition.

Definition 4.1: The quaternion \( \lambda \in \mathbb{H} \) is a left eigenvalue of the matrix \( M \in \mathbb{H}^{n \times n} \) if the matrix \( M - \lambda I_n \) is not invertible.

The existence of left eigenvalues for any quaternionic matrix was proved by Wood in [4]. Notice that the left eigenvalues of a matrix are not invariant by a change of basis [13, Example 7.1].

If \( \lambda \) is a left eigenvalue of \( M \), the set of vectors \( \mathbf{v} \in \mathbb{H}^n \) such that \( M\mathbf{v} = \lambda \mathbf{v} \) is a right \( \mathbb{H} \)-vector subspace \( V(\lambda) \neq \{0\} \) of \( \mathbb{H}^n \). A non-null element \( \mathbf{v} \neq 0 \) of \( V(\lambda) \) is called a \( \lambda \)-eigenvector. By dividing it by its norm we can always assume that \( |\mathbf{v}| = 1 \).

Clearly, if a right eigenvalue is a real number, then it is also a left eigenvalue. So the problem is to determine the non-real left eigenvalues of \( M \), if any.

4.1. \( n = 2 \)

The case \( n = 2 \) was completely solved by Huang and So in [5]. Let \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). If \( b = 0 \) or \( c = 0 \) then the left eigenvalues are \( a, d \in \mathbb{H} \), as it is straightforward to check by hand. When \( bc \neq 0 \), Huang and So gave explicit formulas for the left eigenvalues. In particular they proved [5, Theorems 2.3, 3.1 and 3.2] the following result:

Theorem 4.2: If \( bc \neq 0 \),

1. the left eigenvalues of \( M \) are given by \( \lambda = a + bx \), where \( x \) is any solution of the quadratic equation \( x^2 + a_1x + a_0 = 0 \), with \( a_1 = b^{-1}(a - d) \) and \( a_0 = -b^{-1}c \);
2. the matrix \( M \) has either one, two or infinitely many left eigenvalues;
3. the infinite case happens if and only if \( a_0 \in \mathbb{R}, a_1 \in \mathbb{R}, \) and \( \Delta = a_1^2 - 4a_0 < 0 \);
4. in the latter case, the left eigenvalues can be written as
   \[ \lambda = \frac{1}{2}(a + d + b\xi), \quad \Re(\xi) = 0, |\xi|^2 = |\Delta|. \]  

4.2. Left eigenvalues of \( 2 \times 2 \) Hermitian matrices

We apply the previous results to the Hermitian case. If \( S \) is a \( 2 \times 2 \) Hermitian matrix, the condition \( S = S^* \) means that \( S = \begin{bmatrix} s & b' \\ b & s' \end{bmatrix} \), where \( s, s' \in \mathbb{R} \). If \( b = 0 \), the matrix \( S \) has only two left eigenvalues, the real numbers \( s, s' \), which are also right eigenvalues.

Proposition 4.3: When \( b \neq 0 \),
(1) the matrix $S$ has two real eigenvalues (which may be different or not). They can be computed as the roots of the real equation

$$ (s - t)(s' - t) - |b|^2 = 0. \tag{2} $$

(2) it has also non-real left eigenvalues if and only if $\Re(b) = 0$ and $s = s'$. In this case, the left eigenvalues are given by the formula

$$ \lambda = s + b\omega, \quad \Re(\omega) = 0, |\omega| = 1. \tag{3} $$

Proof: 1. Notice that the discriminant of Equation (2) is

$$ \text{disc} = (s + s')^2 - 4(ss' - |b|^2) = (s - s')^2 + 4|b|^2 \geq 0. $$

It is easy to check that the two real roots are eigenvalues.

2. Since there are already two real eigenvalues, we only have to consider the infinite case of Theorem 4.2. If $b \neq 0$, Huang-So’s conditions are

$$ a_1 = b^{-1}(s - s') = \frac{b^*}{|b|^2}(s - s') \in \mathbb{R}, \tag{4} $$

$$ a_0 = -b^{-1}b^* = -\frac{(b^*)^2}{|b|^2} \in \mathbb{R}, \tag{5} $$

$$ a_1^2 - 4a_0 = \frac{(b^*)^2}{|b|^2} \left[ \frac{(s - s')^2}{|b|^2} + 4 \right] < 0, \tag{6} $$

which imply $(b^*)^2 \in \mathbb{R}$, by (5), and $(b^*)^2 < 0$, by (6).

Hence $b^2 \in \mathbb{R}$ and $b^2 < 0$. This implies $\Re(b) = 0$, $b^* = -b$ and $b^2 = -|b|^2$. But then

$$ \Re(a_1) = \frac{s - s'}{|b|^2} \Re(b^*) = 0, $$

so $s - s' = 0$ by (4).

Since $a_1 = 0$, $a_0 = 1$ and $\Delta = a_1^2 - 4a_0 = -4$, Formula (1) implies (3).

Remark 2: Notice that in the latter case, among the infinite left eigenvalues there are two real ones. In fact, the two solutions of Equation (2) are $t = s \pm |b|$. They correspond to Formula (3) with $\omega = \pm \frac{b}{|b|^2}$.

Example 4.4: [5, Example 2.5] The matrix $S = \begin{bmatrix} 0 & 1+i \\ 1-i & 0 \end{bmatrix}$ has only two left eigenvalues, $\lambda = \pm \sqrt{2}$, which are also its right eigenvalues.

Example 4.5: [13, Example 5.3] Let $S = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$. The real eigenvalues are $\pm 1$. The left eigenvalues are $\lambda = i\omega$, where $\Re(\omega) = 0$ and $|\omega| = 1$, that is, $\lambda = t + yj + zk$, with $t, y, z \in \mathbb{R}, t^2 + y^2 + z^2 = 1$. 
4.3. **Relationship with the Rayleigh quotient**

The previous section gives an idea of the difficulty of computing the left eigenvalues of a given (Hermitian) matrix. It has been conjectured that the left eigenvalues of an $n \times n$ quaternion matrix could be found by solving quaternionic polynomials of degree at most $n$ [8]. Actually, finding the left eigenvalues of an $n \times n$ quaternionic matrix is the same as solving a system of real polynomial equations of degree $4n - 3$ in four variables [20].

In this section, we shall give a new relationship between left and right eigenvalues.

Let $S$ be a Hermitian matrix. Let $\lambda \in \mathbb{H}$ be a left eigenvalue of $S$ and let $v$ be a $\lambda$-eigenvector, with $|v| = 1$. Then $R(S, v) = v^* \lambda v$ is a real number.

**Lemma 4.6:** $R(S, v) = \Re(\lambda)$, the real part of $\lambda$.

**Proof:** Since $v^* v = |v|^2 = 1$, and $v^* \lambda v = v^* \bar{\lambda} v$, we have

$$2R(S, v) = 2v^* \lambda v = v^* (\lambda + \bar{\lambda}) v = (\lambda + \bar{\lambda}) v^* v = \lambda + \bar{\lambda} = 2 \Re(\lambda).$$

Then, from Proposition 3.4 it follows that:

**Proposition 4.7:** If $\lambda$ is a left eigenvalue of $S$ and $v$ is a $\lambda$-eigenvector, then

$$t_1 \leq R(S, v) = \Re(\lambda) \leq t_n.$$  \hspace{1cm} (7)

Next Theorem refines the latter formula, as an application of the min-max theorems of Section 3.2. It gives a new relationship between left and right eigenvalues.

**Theorem 4.8:** Let $\lambda$ be a left eigenvalue of the Hermitian matrix $S$, with real eigenvalues $t_1 \leq \cdots \leq t_n$. If the $\lambda$-eigenspace $V(\lambda)$ verifies $\dim V(\lambda) \geq k$ then

$$t_k \leq \Re(\lambda) \leq t_{n-k+1}.$$  \hspace{1cm} (8)

**Proof:** Let $E = V(\lambda)$. By Lemma 4.6, the Rayleigh function is constant on $E$, so $m_E = \Re(\lambda) = M_E$. Let $\dim V(\lambda) = j \geq k$, then, by Theorem 3.6 and Corollary 3.7, we have

$$t_k \leq t_j \leq \Re(\lambda) \leq t_{n-j+1} \leq t_{n-k+1}. \hspace{1cm} \blacksquare$$

Notice that the inequality (7) is a particular case, since $\dim V(\lambda) \geq 1$.

**Remark 3:** Using orthonormal coordinates one can prove that $|\lambda| \leq \max |t_i|$, where the right term is the (right) spectral radius of the Hermitian matrix [10].

**Example 4.9:** For the Hermitian matrices $S = \begin{bmatrix} s & b \\ b^* & s \end{bmatrix}$ with non-real left eigenvalues (that is, with $\Re(b) = 0$, see Section 4.2), we have that $\Re(\lambda) = s$ by (3), and $|\lambda|^2 = s^2 + |b|^2$, while $t_1 = s - |b|$ and $t_2 = s + |b|$.

The next Corollary shows the influence of the left eigenvalues on the right ones.

The notation $\lceil x \rceil$ stands for the ceiling function, which maps $x$ to the least integer greater than or equal to $x$. 
Corollary 4.10: Assume that the \( n \times n \) Hermitian matrix \( S \) has a left eigenvalue \( \lambda \) such that \( k = \dim V(\lambda) > \lceil n/2 \rceil \). Then, the right eigenvalues \( t_{n-k+1} = \cdots = t_k \) have multiplicity \( l_k \geq 2k - n \) and they are equal to \( \Re(\lambda) \).

Proof: By Theorem 4.8, we have \( t_k \leq \Re(\lambda) \leq t_{n-k+1} \). Moreover, \( k > \lceil n/2 \rceil \) implies \( n - k + 1 \leq k \), hence \( t_{n-k+1} \leq t_k \). That means that \( t_{n-k+1} = t_k \), hence \( \Re(\lambda) = t_j = t_k \), for all \( n - k + 1 \leq j \leq k \). This implies that the multiplicity \( l_k \) of \( t_k \) is at least \( 2k - n \). 

Example 4.11: Let \( S \) be a Hermitian matrix of order 5, diagonalizable to diag\([t_1, \ldots, t_5]\). Assume that \( S \) has some left eigenvalue \( \lambda \) with \( \dim V(\lambda) = 4 \). Then \( \Re(\lambda) = t_4 = t_3 = t_2 \) has at least multiplicity 3.

Example 4.12: Let \( S \) be a Hermitian matrix of order 6, diagonalizable to diag\([t_1, \ldots, t_6]\). Assume that \( S \) has some left eigenvalue \( \lambda \) with \( \dim V(\lambda) = 4 \). Then \( \Re(\lambda) = t_4 = t_3 \) has at least multiplicity 2.

5. Symplectic matrices

In this section, we extend our results to symplectic matrices with quaternionic coefficients.

Recall that the \( n \times n \) matrix \( A \) is quaternionic unitary or symplectic if \( A^* A = I_n \). Its right eigenvalues have norm 1, because if \( q \) is a right eigenvalue, \( A u = uq \), with \( u \neq 0 \), then
\[
|u|^2 = \langle u, u \rangle = \langle Au, Au \rangle = \langle uq, uq \rangle = \overline{q}u^*uq = |q|^2|u|^2,
\]
so \( |q| = 1 \). Moreover, the matrix is diagonalizable [12, Theorem 5.3.6].

Analogously, the left eigenvalues also have norm 1, because if \( \lambda \) is a left eigenvalue, \( Av = \lambda v \), with \( v \neq 0 \), then
\[
|v|^2 = \langle v, v \rangle = \langle Av, Av \rangle = \langle \lambda v, \lambda v \rangle = v^*\overline{\lambda}v = |\lambda|^2|v|^2,
\]
hence \( |\lambda| = 1 \).

5.1. \( n = 2 \)

For \( n = 2 \), the authors completely characterized in [6] the symplectic matrices which have an infinite number of left eigenvalues.

Theorem 5.1: The only \( 2 \times 2 \) symplectic matrices with an infinite number of left eigenvalues are those of the form
\[
\begin{bmatrix}
    r \cos \theta & -r \sin \theta \\
    r \sin \theta & r \cos \theta 
\end{bmatrix}, \quad r \in \mathbb{H}, |r| = 1, \quad \sin \theta \neq 0.
\]

Proposition 5.2: For the matrix in (8):

(1) The right eigenvalues are the similarity classes of \( q = r(\cos \theta \pm \sin \theta \rho) \), where \( \rho \) is any of the quaternions such that \( \Re(\rho) = 0 \), \( |\rho| = 1 \), and \( r = s + t\rho \), with \( s, t \in \mathbb{R} \).
The left eigenvalues are \( \lambda = r(\cos \theta + \sin \theta \omega) \), where \( \omega \) is an arbitrary quaternion such that \( \Re(\omega) = 0 \) and \( |\omega| = 1 \).

**Proof:** Part (1) follows from definition, by checking the eigenvectors \( (\pm \rho, 1)^T \), and taking into account that \( r \) and \( \rho \) commute, and that \( \rho^2 = -1 \).

Part (2) follows from Proposition 4.3. Notice that there are two left eigenvalues which are right eigenvalues.

### 5.2 Rayleigh quotient of a symplectic matrix

The Rayleigh quotient can be defined for any non-Hermitian matrix. Let us assume that \( A \) is symplectic (that is, quaternionic unitary).

**Definition 5.3:** The Rayleigh quotient of \( A \) is the real function

\[
R(A, v) = \frac{\Re(v^* A v)}{|v|^2}.
\]

As before, we shall consider its restriction \( h_A \) to the sphere \( S^{4n-1} \subset \mathbb{H}^n \).

**Proposition 5.4:** Let \( q \) be a right eigenvalue of the symplectic matrix \( A \). If \( u \) is a \( q \)-eigenvector of \( A \) then \( \bar{q} \) is a right eigenvalue of \( A^* \), and \( u \) is a \( \bar{q} \)-eigenvector of \( A^* \).

**Proof:** We have

\[
Au = uq \Rightarrow u = A^{-1}(uq) = (A^* u)q \Rightarrow uq^{-1} = A^* u.
\]

Moreover \( |q| = 1 \) implies \( q^{-1} = \bar{q} \).

Let \( S = \frac{1}{2}(A + A^*) \) be the Hermitian part of \( A \).

**Corollary 5.5:** If \( q \) is a right eigenvalue of \( A \) and \( u \) is a \( q \)-eigenvector, then \( \Re(q) \) is an eigenvalue of \( S \), and \( u \) is an \( \Re(q) \)-eigenvector.

**Corollary 5.6:**

1. The \( n \) right eigenvalues of \( S \) are the real parts \( t_j = \Re(q_j) \) of the \( n \) similarity classes \( [q_1], \ldots, [q_n] \) of the right eigenvalues of \( A \).
2. The critical values of \( h_A \) are \( \Re(q_1), \ldots, \Re(q_n) \).

**Proposition 5.7:** The Rayleigh functions of \( A \) and \( S \) are equal, \( h_A = h_S \).

**Proof:** We have

\[
h_S(v) = v^* Sv = \frac{1}{2}(v^* A v + v^* A^* v) = \Re(v^* A v) = h_A(v).
\]

Notice that the latter result is still valid even when the matrix \( A \) is not symplectic.
5.3. Left eigenvalues

Now, assume that \( \lambda \) is a left eigenvalue of the symplectic matrix \( A \). We know that there may exist an infinite number of them.

**Proposition 5.8:** If \( \mathbf{v} \) is a \( \lambda \)-eigenvector, that is, \( A \mathbf{v} = \lambda \mathbf{v} \), then \( h_A(\mathbf{v}) = \Re(\lambda) \).

**Proof:** The proof is identical to that of Lemma 4.6. Since \( A \) is normal, it is diagonalizable and we can take an orthonormal basis \( \mathbf{u}_1 \ldots \mathbf{u}_n \) of eigenvectors [12, Theorem 5.3.6]. By taking coordinates \( \mathbf{v} = \sum_j \mathbf{u}_j x_j \), we can assume that \( |\mathbf{v}|^2 = \sum_j |x_j|^2 = 1 \), then

\[
h_A(\mathbf{v}) = \Re(\mathbf{v}^* \lambda \mathbf{v}) = \Re \left( \sum_j \overline{x}_j \lambda x_j \right) = \sum_j \Re(\lambda |x_j|^2) = \Re(\lambda).
\]

**Remark 4:** Since \( |\lambda| = 1 \), notice that \( A \mathbf{v} = \lambda \mathbf{v} \) implies \( A^*(\lambda \mathbf{v}) = \overline{\lambda}(\lambda \mathbf{v}) \), hence \( \lambda \mathbf{v} \) is a \( \overline{\lambda} \)-eigenvector of \( A^* \), but we cannot conclude nothing about the eigenvalues of \( S \), as Example 5.11 shows.

**Corollary 5.9:** Let \( A \) be a symplectic matrix whose right eigenvalues are organized in \( n \) similarity classes \( [q_1], \ldots, [q_n] \), ordered in such a way that \( \Re(q_1) \leq \cdots \leq \Re(q_n) \). Let \( \lambda \) be a left eigenvalue of \( A \), such that its eigenspace verifies \( \dim V(\lambda) \geq k \). Then

\[
\Re(q_k) \leq \Re(\lambda) \leq \Re(q_{n-k+1}).
\]

**Proof:** Take \( \mathbf{v} \in V_A(\lambda) \), so \( \Re(\lambda) = h_A(\mathbf{v}) = h_S(\mathbf{v}) \). This does not mean that \( \lambda \) is a left eigenvalue of \( S \) (see Example 5.11). But \( \Re(\lambda) = h_S(\mathbf{v}) \) is the constant value of \( h_S \) in the subspace \( E = V_A(\lambda) \), hence \( m_E = \Re(\lambda) = M_E \). If \( \dim E = j \geq k \), then

\[
\Re(q_k) \leq \Re(q_j) \leq \Re(\lambda) \leq \Re(q_{n-j+1}) \leq \Re(q_{n-k+1}),
\]

from Theorem 3.6, Corollaries 3.7 and 5.6. ■

Remember that \( |\lambda| = 1 \).

**Example 5.10:** Let \( A = \begin{pmatrix} \sqrt{2}/2 & 1 \\ 1 & -1 \end{pmatrix} \), as in Proposition 5.2, with \( r = j \) and \( \theta = \pi/4 \). We have \( \rho = j \), and the right eigenvalues are the similarity classes of

\[
q = \cos \theta j \pm \sin \theta j^2 = \frac{\sqrt{2}}{2} (\pm 1 + j)
\]

whose real part is \( \Re(q) = \pm \sqrt{2}/2 \).

On the other hand, the left eigenvalues are

\[
\lambda = \cos \theta j + \sin \theta \omega j = \frac{\sqrt{2}}{2} (1 + \omega j),
\]

where \( \Re(\omega) = 0 \) and \( |\omega| = 1 \), and their real part is \( \Re(\lambda) = \frac{\sqrt{2}}{2} \Re(\omega j) \).
Then, since $|\omega_j| = 1$, it is true that $-1 \leq \Re(\omega_j) \leq 1$, hence $\Re(q_1) \leq \Re(\lambda) \leq \Re(q_2)$.

**Example 5.11:** For the symplectic matrix $A$ in Example 5.10, the Hermitian part is $S = \sqrt{\frac{\pi}{2}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Its right eigenvalues are $q = \pm \sqrt{\frac{\pi}{2}}$, which is the real part of those of $A$, as stated in Corollary 5.5.

Its left eigenvalues are, by (3),

$$\lambda = \frac{\sqrt{2}}{2} j \omega, \quad \Re(\omega) = 1, |\omega| = 1,$$

which are different from those of $A$.

Now, we shall prove a result, analogous to Corollary 4.10, showing that the existence of left eigenvalues with a high-dimensional space of eigenvectors depends on the multiplicity of the right eigenvalues.

**Corollary 5.12:** Let $A$ be an $n \times n$ symplectic matrix. Assume that there is a left eigenvalue $\lambda$ such that $k = \dim V(\lambda) > \lceil n/2 \rceil$. Then, the similarity classes of right eigenvalues $[q_{n-k+1}] = \cdots = [q_k]$ have multiplicity $l_k \geq 2k - n$, and they are equal to the similarity class $[\lambda]$ of $\lambda$.

**Proof:** Remember that the eigenvalue classes $[q_i]$ are ordered according to their real parts, as in Corollary 5.9. From that Corollary, we know that

$$\Re(q_k) \leq \Re(\lambda) \leq \Re(q_{n-k+1}).$$

But $k > \lceil n/2 \rceil$ implies $n - k + 1 \leq k$, hence $\Re(q_{n-k+1}) \leq \Re(q_k)$. Then $\Re(q_{n-k+1}) = \cdots = \Re(q_k) = \Re(\lambda)$. But since $|q_j| = 1$, for all $j$, and $|\lambda| = 1$, it follows that the similarity classes are equal, so we have $[q_{n-k+1}] = \cdots = [q_k] = [\lambda]$. This proves the result. ■

**6. Conclusions**

In this paper, we find a new relationship between left and right eigenvalues, for Hermitian and symplectic quaternionic matrices. The key idea is to consider the Rayleigh quotient, a function that has been widely studied in the real and complex setting. We prove versions for the quaternions of the classical results about that function, filling a gap in the literature. We also prove that the mean value and the variance of the Rayleigh quotient, when seen as a random variable on the sphere, equal the mean and the variance of the right eigenvalues, respectively.

The critical values of the Rayleigh quotient are the right eigenvalues. Our main result states that the real part of a left eigenvalue $\lambda$ lies between two critical levels that depend on the dimension of the eigenspace of $\lambda$. This implies that the existence of left eigenvalues with a high-dimensional eigenspace depends on the multiplicity of the right eigenvalues.

Since the Rayleigh quotient has many applications, mainly in quantum mechanics, this relationship opens the door to find applications to left eigenvalues of quaternion matrices outside pure mathematics. It is expected that some other properties of this function could be used to find new results.
One more advantage of this approach is to benefit of a large literature in mathematical statistics about random quadratic forms [21].

In the background is the idea that, by taking the polar decomposition of a matrix, some information could be extracted about its left eigenvalues from its Hermitian and symplectic parts. For the moment we have not explored this way further.

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