On the signed Selmer groups of congruent elliptic curves with semistable reduction at all primes above \( p \)

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Abstract

Let \( p \) be an odd prime. We attach appropriate signed Selmer groups to an elliptic curve \( E \), where \( E \) is assumed to have semistable reduction at all primes above \( p \). We then compare the Iwasawa \( \lambda \)-invariants of these signed Selmer groups for two congruent elliptic curves over the cyclotomic \( \mathbb{Z}_p \)-extension in the spirit of Greenberg-Vatsal and B. D. Kim. As an application of our comparison formula, we show that if the \( p \)-parity conjecture is true for one of the congruent elliptic curves, then it is also true for the other elliptic curve. In the midst of proving this latter result, we also generalize an observation of Hatley on the parity of the signed Selmer groups.

Keywords and Phrases: Signed Selmer groups, congruent elliptic curves, \( p \)-parity conjecture.

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1 Introduction

Let \( \mathbb{Q}^{\text{cyc}} \) be the cyclotomic \( \mathbb{Z}_p \)-extension of \( \mathbb{Q} \), where \( p \) is a fixed odd prime. In [10], Greenberg and Vatsal showed that if two congruent elliptic curves have good ordinary reduction at \( p \), then the Iwasawa \( \lambda \)-invariants of their Selmer groups are related in an explicit manner. This was later extended by B. D. Kim [19] to the situation of an elliptic curve with good supersingular reduction at \( p \).

The goal of this paper is to extend the results of Greenberg-Vatsal and B. D. Kim to the situation, where the elliptic curves have no additive reduction at all primes above \( p \). In other words, our elliptic curves, which are now defined over a finite extension of \( \mathbb{Q} \), can have either good ordinary/supersingular reduction or split/non-split multiplicative reduction at each prime above \( p \). We shall briefly describe our results here.

Let \( L' \) be a number field and \( L \) a finite extension of \( L' \). Suppose that \( E \) is an elliptic curve defined over \( L' \) which satisfies the following hypotheses.

\( \text{(NA)} \) The elliptic curve \( E \) has no additive reduction at all primes of \( L' \) above \( p \).

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(SS) For each prime $u$ of $L'$ above $p$ which has good supersingular reduction, one assume that that $u$ is unramified in $L/L'$, $L'_u = \mathbb{Q}_p$ and $a_u = 1 + p - |\tilde{E}_u(\mathbb{F}_p)| = 0$, where $\tilde{E}_u$ is the reduction of $E$ at $u$.

The (possible) presence of primes of good supersingular reduction requires us to work with an appropriate signed Selmer group $\text{Sel}^{\tilde{\mu}}(E/F^{\text{cyc}})$ following [18 19 20 21 22 23]. In the event that the elliptic curve in question has no supersingular reduction at any prime above $p$, our signed Selmer group coincides with the usual $p$-primary Selmer group. Once, such appropriate signed Selmer groups are defined, it is then natural to ask for an analogous formula in the vein of Greenberg-Vatsal and B. D. Kim which relates the Iwasawa $\lambda$-invariants of the signed Selmer groups for congruent elliptic curves.

In other words, we now have two elliptic curves $E_1$ and $E_2$ which are defined over the number field $L'$ such that they both satisfy (NA) and (SS), and that $E_1[p] \cong E_2[p]$ as $\text{Gal}(\overline{L}/L)$-modules. As the definition of the signed Selmer groups involves local conditions, we need to be able to compare the local cohomology groups for primes above $p$ which leads us to impose the following two hypotheses.

(RED) The elliptic curves $E_1$ and $E_2$ have the same reduction type at each prime of $L'$ above $p$. Furthermore, in the event that $E_1$ and $E_2$ have non-split multiplicative reduction at a prime $u$ of $L'$ above $p$, suppose further $E_1$ and $E_2$ have the same reduction type at all primes of $F$ above $u$, where $F$ denotes either $L$ or $L(\mu_p)$.

(RU) For each prime $u$ of $L'$ above $p$ which is either a prime of good ordinary reduction or non-split multiplicative reduction, assume that $L'_u$ does not contain $\mu_p$.

We say a little bit on these two hypotheses. In [1 10 32], their elliptic curves are defined over $\mathbb{Q}$ and so (RU) is automatically satisfied. Hypothesis (RED) is necessary for us to make meaningful comparison for the coefficients in the local cohomology groups. This hypothesis is also innate in [1 10 14 19 32], as their congruent elliptic curves are always assumed to have only good ordinary reduction or only good supersingular reduction above $p$.

We now state our first main theorem.

**Theorem** (Theorem 4.1.1). Let $E_1$ and $E_2$ be two elliptic curves defined over $L'$ with $E_1[p] \cong E_2[p]$ as $\text{Gal}(\overline{L}/L)$-modules. Let $L$ be a finite extension of $L'$. Write $F$ for either $L$ or $L(\mu_p)$. Suppose that (NA), (SS), (RED) and (RU) hold for $E_1$ and $E_2$.

Then $\text{Sel}^{\tilde{\mu}}(E_1/F^{\text{cyc}})$ is cofinitely generated over $\mathbb{Z}_p$ if and only if $\text{Sel}^{\tilde{\mu}}(E_2/F^{\text{cyc}})$ is cofinitely generated over $\mathbb{Z}_p$. Moreover, if this is so, we have the following equality

$$\text{corank}_{\mathbb{Z}_p} \left( \text{Sel}^{\tilde{\mu}}(E_1/F^{\text{cyc}}) \right) + \sum_{w \in S^t(F^{\text{cyc}})} \text{corank}_{\mathbb{Z}_p} \left( H^1(F_w^{\text{cyc}}, E_1(p)) \right)$$

$$= \text{corank}_{\mathbb{Z}_p} \left( \text{Sel}^{\tilde{\mu}}(E_2/F^{\text{cyc}}) \right) + \sum_{w \in S^t(F^{\text{cyc}})} \text{corank}_{\mathbb{Z}_p} \left( H^1(F_w^{\text{cyc}}, E_2(p)) \right).$$

We note that even in the case where the elliptic curves have ordinary reduction at all primes above $p$, our result is an improvement of the results of Greenberg-Vatsal [10] and Ahmed-Aribam-Shekhar [1 32].
for we do not require the assumption that $E(F)[p] = 0$. We should mention that the results of Greenberg-Vatsal and Kim have also been established for modular forms of higher weight and even more general Galois representations (for instances, see [6, 12, 15, 30]). However in these prior works, they always work with coherent reduction types above $p$. In other words, their Galois representations are assumed to have either ordinary reduction at all primes above $p$ or non-ordinary reduction at all primes above $p$.

Finally, we shall apply our result to investigate the $p$-parity conjecture following [1, 14, 32]. For an elliptic curve $E$ over $F$, the $p$-parity conjecture predicts that $w(E/F) = (-1)^{s_p(E)}$ (see [5] and references loc. cit), where we write $w(E/F)$ for its global root number and $s_p(E)$ for the $Z_p$-corank of Sel$(E/F)$. Our next main result is then as follows.

**Theorem** (Theorem 4.2.3). Retain the settings as above. Suppose further that at least one of the following statements is also valid.

(a) Sel$\rightarrow$ (E$_1$/Fcyc) is cofinitely generated over $Z_p$.

(b) Sel$\rightarrow$ (E$_1$/Fcyc) is cofinitely generated over $Z_p$ for some $\rightarrow s \neq \rightarrow t$ and (S+) is valid.

Then one has

$$w(E_1/F) w(E_2/F) = (-1)^{s_p(E_1) - s_p(E_2)}.$$  

In particular, the $p$-parity conjecture holds for $E_1$ over $F$ if and only if it holds for $E_2$ over $F$.

When the elliptic curves either have good ordinary reduction at all primes above $p$ or good supersingular reduction at all primes above $p$, the above result was proved under the assumption of statement (a) by the first named author of this paper with Aribam and Shekhar in [1, Corollary 5.6]. Our result is an improvement of this, where we allow mixed reduction types above $p$ and mixed signs in the definition of our signed Selmer groups. We should also mention that the proof of [1, Corollary 5.6] contains a slight gap which we have addressed in this paper (see Remark after Proposition 4.2.1).

We now give an outline of our paper. In Section 2, we collect various results concerning the arithmetic of an elliptic curve over a local field which will be needed for the subsequent discussion of the paper. Section 3 is where we introduce the signed Selmer groups and give various equivalent descriptions of them. The point of having these different descriptions is that in proving Theorem 4.2.3 it is more natural to work with the description of the signed Selmer groups as given in Proposition 3.1.2. On the other hand, while proving Theorem 4.2.3 one needs to work with the so-called strict signed Selmer groups which coincides with the signed Selmer groups on the level of $F$ and $F^{\text{cyc}}$. We do remark that the strict signed Selmer groups and the signed Selmer groups need not coincide on the intermediate subextensions of $F^{\text{cyc}}/F$ (see the proof of Proposition 3.1.3). Finally, Section 4 is where all our main results will be established. In the midst of proving our second main theorem, we also generalize an observation of Hatley [14, Corollary 4.2] on the parity of the signed Selmer groups (see Corollary 4.2.1). Although this latter result is not required for the proof of our second main theorem, we have thought that it is interesting enough to be noted down (also see Remark after Corollary 4.2.1).
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2 Local consideration

In this section, we collect certain results on elliptic curves defined over a local field.

2.1 Ordinary reduction at $p$

Let $K$ be a finite extension of $\mathbb{Q}_p$ and $E$ an elliptic curve defined over $K$. In this subsection, our elliptic curve $E$ is always assumed to have either good ordinary reduction or multiplicative reduction. Then from [4, P. 150], we have the following short exact sequence

$$0 \rightarrow C \rightarrow E(p) \rightarrow D \rightarrow 0$$

of $\text{Gal}(\bar{K}/K)$-modules, where $C$ and $D$ are cofree $\mathbb{Z}_p$-modules of corank one, and are characterized by the facts that $C$ is divisible and that $D$ is the maximal quotient of $E(p)$ by a divisible subgroup such that $\text{Gal}(\bar{K}/K^{ur})$ acts on $D$ via a finite quotient. Here $K^{ur}$ is the maximal unramified extension of $K$.

In fact, $D$ can be explicitly described as follows (see [4] or [9, Section 3])

$$D = \begin{cases} \bar{E}, & \text{if } E \text{ has good ordinary reduction,} \\ \mathbb{Q}_p/\mathbb{Z}_p, & \text{if } E \text{ has split multiplicative reduction,} \\ \mathbb{Q}_p/\mathbb{Z}_p \otimes \phi, & \text{if } E \text{ has nonsplit multiplicative reduction,} \end{cases}$$

where $\bar{E}$ is the reduction of $E$ and $\phi$ is a unramified character of $\text{Gal}(\bar{K}/K)$. Note that in all these three cases, $D$ is a trivial $\text{Gal}(\bar{K}/K^{ur})$-module which in turn enables us to view $D$ as a $\text{Gal}(K^{cyc,ur}/K^{cyc})$-module. We now break our discussion into three cases.

Case 1: Suppose that $E$ has good ordinary reduction. Since $D(K^{cyc}) = \bar{E}(K^{cyc})$ is finite, the group $\text{Gal}(K^{cyc,ur}/K^{cyc})$ does not act trivially on $D$. Consequently, $D_{\text{Gal}(K^{cyc,ur}/K^{cyc})}$ is a proper quotient of $D$. But since $D$ is divisible of $\mathbb{Z}_p$-corank one, this forces $D_{\text{Gal}(K^{cyc,ur}/K^{cyc})} = 0$, or equivalently, $H^1(\text{Gal}(K^{cyc,ur}/K^{cyc}), D) = 0$. On the other hand, we clearly have $H^2(\text{Gal}(K^{cyc,ur}/K^{cyc}), D) = 0$ as $\text{Gal}(K^{cyc,ur}/K^{cyc}) \cong \hat{\mathbb{Z}}$ has $p$-cohomological dimension one. It then follows from the inflation-restriction sequence that $H^1(K^{cyc}, D) \cong H^1(K^{cyc,ur}, D)$.

Case 2: Suppose that $E$ has non-split multiplicative reduction. Then $D = \mathbb{Q}_p/\mathbb{Z}_p \otimes \phi$ for a unramified character of $\text{Gal}(\bar{K}/K)$ which factors through a quadratic extension of $K$. Since $p$ is odd, the multiplicative reduction remains non-split. Hence $D(K^{cyc})$ is finite. A similar argument as above then yields an isomorphism $H^1(K^{cyc}, D) \cong H^1(K^{cyc,ur}, D)$. 

4
Case 3: Suppose that $E$ has split multiplicative reduction. Then we have $D \cong \mathbb{Q}_p/\mathbb{Z}_p$ as $\text{Gal}(\bar{K}/K^{\text{cyc}})$-modules.

We record the above analysis into the following lemma which will be required in the subsequent of the paper.

**Lemma 2.1.1.** Let $E$ be an elliptic curve defined over a finite extension $K$ of $\mathbb{Q}_p$. Then the following statements hold.

(a) If $E$ has good ordinary reduction or non-split multiplicative reduction, then $D(K^{\text{cyc}})$ is finite and there is an isomorphism $H^1(K^{\text{cyc}}, D) \cong H^1(K^{\text{cyc,ur}}, D)$.

(b) If $E$ has split multiplicative reduction, then $D \cong \mathbb{Q}_p/\mathbb{Z}_p$ as $\text{Gal}(\bar{K}/K^{\text{cyc}})$-modules.

We end the subsection with the following lemma concerning a uniqueness property of $C[p]$ which will be required for the proof of Theorem 4.1.1.

**Lemma 2.1.2.** Suppose that $K$ does not contain $\mu_p$. Then $C[p]$ is the unique $\text{Gal}(\bar{K}/K)$-submodule of $E[p]$ which is isomorphic to $\mu_p$ as $\text{Gal}(\bar{K}/K^{\text{ur}})$-modules.

**Proof.** Let $U$ be a $\text{Gal}(\bar{K}/K)$-submodule of $E[p]$ which is isomorphic to $\mu_p$ as a $\text{Gal}(\bar{K}/K^{\text{ur}})$-module. Now since $U/(U \cap C[p]) \cong (U + C[p])/C[p]$ is contained in $E[p]/C[p] \cong D[p]$, it has trivial $\text{Gal}(\bar{K}/K^{\text{ur}})$-action. On the other hand, as $K$ does not contain $\mu_p$, $\text{Gal}(\bar{K}/K^{\text{ur}})$ does not act trivially on $U$. Therefore, $U/(U \cap C[p])$ must be a proper quotient of $U$. But as $U$ is an one dimensional $\mathbb{F}_p$-vector space, we must have $U/(U \cap C[p]) = 0$, or equivalently, $U \subseteq C[p]$. Now since both $U$ and $C[p]$ are one dimensional $\mathbb{F}_p$-vector spaces, the inclusion is an equality as required. 

### 2.2 Good supersingular reduction at $p$

In this subsection, $E$ will denote an elliptic curve defined over $\mathbb{Q}_p$ with $a_p = 0$. Denote by $\hat{E}$ the formal group of $E$. Let $k$ be an unramified extension of $\mathbb{Q}_p$ of degree $d$. For $n \geq 1$, set $K_n = k(\mu_{p^n})$. In particular, $K_{-1} = k$. Note that the extension $K_{\infty}/k$ is totally ramified and $\text{Gal}(K_0/k) \cong \mathbb{Z}/(p-1)\mathbb{Z}$.

We shall write $\hat{E}(K_n) = \hat{E}(\mathfrak{m}_n)$, where $\mathfrak{m}_n$ is the maximal ideal of the ring of integers of $K_n$. Finally, we write $\hat{E}(K_\infty) = \bigcup_n \hat{E}(K_n)$.

**Lemma 2.2.1.** Retain settings as above. Then $\hat{E}(K_n)$ and $E(K_n)$ have no $p$-torsion for every $n \geq -1$.

**Proof.** The first assertion follows from [22 Proposition 3.1]. The second follows from the first, the following short exact sequence

$$0 \longrightarrow \hat{E}(K_n) \longrightarrow E(K_n) \longrightarrow \hat{E}(\mathbb{F}_{p^n}) \longrightarrow 0$$

and that $\hat{E}(\mathbb{F}_{p^n})$ has no $p$-torsion by the supersingular assumption. 

5
Following \[18, 19, 20, 21, 22, 23\], we define the following groups

\[
\hat{E}^+(K_n) = \{ P \in \hat{E}(K_n) : \text{tr}_{n/m+1}(P) \in \hat{E}(K_m), 2 \mid m, -1 \leq m \leq n - 1 \},
\]

\[
\hat{E}^-(K_n) = \{ P \in \hat{E}(K_n) : \text{tr}_{n/m+1}(P) \in \hat{E}(K_m), 2 \mid m, -1 \leq m \leq n - 1 \},
\]

where \( \text{tr}_{n/m+1} : E(K_n) \to E(K_{m+1}) \) denotes the trace map. We shall then write \( \hat{E}^\pm(K_n) = \cup_n \hat{E}^\pm(K_n) \).

By fixing a topological generator \( \gamma \) of \( \text{Gal}(K_\infty/K_0) \), we identify \( \mathbb{Z}_p[\text{Gal}(K_\infty/k)] \) with the formal power series ring \( (\mathbb{Z}_p[\text{Gal}(K_0/k)])[X] \). For a character \( \eta \) of \( \text{Gal}(K_0/k) \) and a \( (\mathbb{Z}_p[\text{Gal}(K_0/k)])[X] \)-module \( M \), let \( M^\eta \) denotes its \( \eta \)-eigenspace, which is regarded as a \( \mathbb{Z}_p[X] \)-module. We also write

\[
\delta = \begin{cases} 
0, & \text{if } d \neq 0 \text{ (mod } 4) \text{ or } \eta \neq 1, \\
2, & \text{otherwise}.
\end{cases}
\]

With this notation, we can now state the following.

**Proposition 2.2.2.** Retain settings as above. Write \( \Gamma_n = \text{Gal}(K_\infty/K_n) \). The following statements are then valid.

(a) \( (\hat{E}^\pm(K_\infty)^\eta \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_n} \) are cofree \( \mathbb{Z}_p \)-modules for all \( n \), and

\[
\text{corank}_{\mathbb{Z}_p} \left( (\hat{E}^\ast(K_\infty)^\eta \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_n} \right) = \begin{cases} 
 dp^n + \delta, & \text{if } s = +, \\
 dp^n, & \text{if } s = -.
\end{cases}
\]

(b) One has

\[
\left( \hat{E}^\ast(K_\infty)^\eta \otimes \mathbb{Q}_p/\mathbb{Z}_p \right)^\vee \cong \begin{cases} 
 \mathbb{Z}_p[X]^\oplus d \oplus \mathbb{Z}_p^{\oplus \delta}, & \text{if } s = +, \\
 \mathbb{Z}_p[X]^\oplus d, & \text{if } s = -.
\end{cases}
\]

**Proof.** Statement (a) is \[22\] Corollary 3.25, and statement (b) follows from \[22\] Theorem 3.34 (also see \[21\] Theorems 2.7 and 2.8). \( \square \)

Write \( \mathbb{H}_n^\pm = \hat{E}^\pm(K_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p \) and \( \mathbb{H}_n^\pm = \left( \hat{E}^\pm(K_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p \right)^{\Gamma_n} \). By the Hochshild-Serre spectral sequence and Lemma [2.2.1] we have an isomorphism

\[
H^1(K_n, E(p)) \cong H^1(K_\infty, E(p))^{\Gamma_n}.
\]

Via this isomorphism, we may view \( \mathbb{H}_n^\pm \) as a subgroup of \( H^1(K_n, E(p)) \). Let \( M_n^\pm \) be the exact annihilator of \( \mathbb{H}_n^\pm \) with respect to the local Tate pairing

\[
H^1(K_n, E(p)) \times H^1(K_n, T_pE) \to \mathbb{Q}_p/\mathbb{Z}_p.
\]

In other words, we have short exact sequences

\[
0 \to \mathbb{H}_n^\pm \to H^1(K_n, E(p)) \to (M_n^\pm)^\vee \to 0,
\]

6


\[ 0 \rightarrow M_n^\pm \rightarrow H^1(K_n, T_pE) \rightarrow (\mathbb{H}_n^\pm)^\vee \rightarrow 0. \]

Since \( \mathbb{H}_n^\pm \) is divisible by Proposition 2.2.2(a), it follows that there are two short exact sequences

\[ 0 \rightarrow \mathbb{H}_n^\pm[p^j] \rightarrow H^1(K_n, E(p))[p^j] \rightarrow (M_n^\pm/p^j)^\vee \rightarrow 0, \]
\[ 0 \rightarrow M_n^\pm/p^j \rightarrow H^1(K_n, T_pE)/p^j \rightarrow (\mathbb{H}_n^\pm[p^j])^\vee \rightarrow 0. \]

By appealing to Lemma 2.2.1 again, one has \( H^1(K_n, E[p^j]) \cong H^1(K_n, E[p^j]) \) and \( H^1(K_n, T_pE)/p^j \cong H^1(K_n, E[p^j]) \). Combining this with the above short exact sequences, we see that \( M_n^\pm/p^j \) is the exact annihilator of \( \mathbb{H}_n^\pm[p^j] \) with respect to the local Tate pairing

\[ H^1(K_n, E[p^j]) \times H^1(K_n, E[p^j]) \rightarrow \mathbb{Z}/p^j. \]

In fact, one even has the following.

**Proposition 2.2.3** (Kim). One always has \( \mathbb{H}_n^\pm[p^j] = M_n^\pm/p^j \). If \( d \) is not divisible by 4, then we also have \( \mathbb{H}_n^\pm[p^j] = M_n^\pm/p^j \).

**Proof.** The first equality is established by Kim in [18, Proposition 3.15]. One can check that the same proof carries over for the second equality under the assumption that \( d \) is not divisible by 4 (also see [21, Theorem 2.9]). \( \square \)

### 2.3 Local parity comparison outside \( p \)

In this subsection, \( K \) will denote a finite extension of \( \mathbb{Q}_l \), where \( l \neq p \). Let \( E \) be an elliptic curve defined over \( K \). Writing \( K^{\text{cyc}} \) for the cyclotomic \( \mathbb{Z}_p \)-extension of \( K \), it is well-known that \( H^1(K^{\text{cyc}}, E(p)) \) is cofinitely generated over \( \mathbb{Z}_p \) (cf. [3, Proposition 2]), whose \( \mathbb{Z}_p \)-corank is in turn denoted by \( \sigma_E \). We also write \( w(E/K) \) for the local root number of \( E \) over \( K \) (see [3, Section 3.4] for its definition). The following result of Ahmed-Aribam-Shekhar will be required in the subsequent section.

**Proposition 2.3.1** (Ahmed-Aribam-Shekhar). Suppose that \( E_1 \) and \( E_2 \) are two elliptic curves defined over \( K \) with \( E_1[p] \cong E_2[p] \) as \( \text{Gal}(\bar{K}/K) \)-modules. Then we have

\[ \frac{w(E_1/K)}{w(E_2/K)} = (-1)^{\sigma_{E_1} - \sigma_{E_2}}. \]

**Proof.** See [1, Theorem 5.7]. \( \square \)

### 3 Signed Selmer groups over cyclotomic \( \mathbb{Z}_p \)-extension

We now turn to the global situation, where we begin by fixing some notation and standing assumptions that will be adhered throughout the section. Let \( L' \) be a number field and \( E \) an elliptic curve defined over \( L' \). Fix a finite extension \( L \) of \( L' \). Let \( S \) be a finite set of primes of \( L' \) which contains all the primes
above $p$, all the ramified primes of $L/L'$, the bad reduction primes of $E$ and the archimedean primes. Denote by $S_p$ the set of primes of $L'$ above $p$. Write $S_p^{ss}$ for the set of primes of $L'$ above $p$ at which $E$ has good supersingular reduction. We also write $S' = S - S_p$ and $S_p^{o} = S_p - S_p^{ss}$. The following assumptions will be in full force throughout this section.

(NA) The elliptic curve $E$ has no additive reduction at all primes of $L'$ above $p$.

(SS) For each $u \in S_p^{ss}$, one has that $u$ is unramified in $L/L'$, $L'_u = \mathbb{Q}_p$ and $a_u = 1 + p - |\tilde{E}_u(\mathbb{F}_p)| = 0$, where $\tilde{E}_u$ is the reduction of $E$ at $u$.

For any subset $R$ of $S$ and any extension $F$ of $L'$, we shall write $R(F)$ for the set of primes of $F$ above $R$.

Throughout, $F$ will always denote either $L$ or $L(\mu_p)$, where $\mu_p$ is the group of $p$th roots of unity. We then write $F^{\text{cyc}}$ for the cyclotomic $\mathbb{Z}_p$-extension of $F$ and $F_n$ the intermediate subfield of $F^{\text{cyc}}$ with $|F_n : F| = p^n$.

### 3.1 Signed Selmer groups

In this subsection, we give the definition of the signed Selmer groups. The notion of the signed Selmer groups was first conceived by Kobayashi [23] to handle elliptic curves with good supersingular reduction at all primes above $p$. Later, Kim [21] introduced the signed Selmer groups which allowed mixed signs. Recently, Kitajima and Otsuki [22] considered the signed Selmer groups attached to elliptic curves with mixed good reduction types at primes above $p$. Mimicking these prior works, the authors of this paper introduced the signed Selmer groups which allow mixed signs and mixed good reduction types (see [2]). In this paper, we will slightly refine this even further by allowing the presence of multiplicative reduction primes above $p$.

We begin by taking care of the local conditions at the supersingular primes. By (SS), every prime in $S_p^{ss}(F)$ is totally ramified in $F^{\text{cyc}}/F$. In particular, for each such prime $v$, there is a unique prime of $F_n$ lying above the said prime which, by abuse of notation, is still denoted by $v$. We then write $\tilde{E}_v$ for the formal group $E$ over $L'_u$, where $u$ is a prime of $L'$ below $v$.

For each $\vec{s} = (s_v)_{v \in S_p^{ss}(F)} \in \{\pm\}^{S_p^{ss}(F)}$, we write

$$H_n^{\vec{s}} = \bigoplus_{v \in S_p^{ss}(F)} H^1(F_n, E_p, E(p)) / E^{\text{cyc}}_v(F_n,v) \otimes \mathbb{Q}_p / \mathbb{Z}_p.$$  

The signed Selmer groups are then defined by

$$\text{Sel}^{\vec{s}}(E/F_n) = \ker \left( H^1(G_S(F_n), E(p)) \xrightarrow{\psi} H_n^{\vec{s}} \times \bigoplus_{w \in S_p^{\text{op}}(F_n)} H^1(F_n, E(p)) / E^{\text{cyc}}_v(F_n,v) \otimes \mathbb{Q}_p / \mathbb{Z}_p \times \bigoplus_{w \in S_p^{\text{op}}(F_n)} H^1(F_n, E(p)) \right).$$

Note that in the event that $S_p^{ss} = \emptyset$, the above definition coincides with the usual $p$-primary Selmer group $\text{Sel}(E/F_n)$. In the event that all the signs occurring in the definition are $-$ signs, the signed Selmer group is then denoted by $\text{Sel}^{\vec{s}}(E/F_n)$. We also note that one always has $\text{Sel}^{\vec{s}}(E/F) = \text{Sel}(E/F)$.
regardless of the presence of supersingular primes. For a general $n$, the signed Selmer group and the classical Selmer group fit into the following commutative diagram

$$
\begin{array}{c}
0 \rightarrow \text{Sel}^{-\infty}(E/F_n) \rightarrow H^1(G_S(F_n), E(p)) \xrightarrow{\psi^{-\infty}} \mathcal{H}_{n}^{-\infty} \times \bigoplus_{w \in S_n(F_n)} H^1(F_{n,w}, E(p)) \times \bigoplus_{w \in S'(F_n)} H^1(F_{n,w}, E(p)) \\
0 \rightarrow \text{Sel}(E/F_n) \rightarrow H^1(G_S(F_n), E(p)) \xrightarrow{\phi} \bigoplus_{w \in S_p(F_n)} H^1(F_{n,w}, E(p)) \times \bigoplus_{w \in S'(F_n)} H^1(F_{n,w}, E(p))
\end{array}
$$

with exact rows. Denote by $\psi^{-\infty}_{ss}$ the map from $\text{Sel}(E/F_n)$ to $\mathcal{H}_{n}^{-\infty}$ that is induced by $\psi^{-\infty}$. Then one has the following equivalent description of the signed Selmer groups.

**Lemma 3.1.1.** We have the following identification

$$\text{Sel}^{-\infty}(E/F_n) = \ker \left( \text{Sel}(E/F_n) \xrightarrow{\psi^{-\infty}} \mathcal{H}_{n}^{-\infty} \right).$$

**Proof.** This follows from a straightforward analysis of the above commutative diagram with the definition of the signed Selmer groups. \qed

We then define $\text{Sel}^{-\infty}(E/F^\text{cyc}) = \lim_n \text{Sel}^{-\infty}(E/F_n)$ and $\mathcal{H}_{n}^{-\infty} = \lim_n \mathcal{H}_{n}^{-\infty}$. It is not difficult to verify that $\text{Sel}^{-\infty}(E/F^\text{cyc})$ is cofinitely generated over $\mathbb{Z}_p[\Gamma]$, where $\Gamma = \text{Gal}(F^\text{cyc}/F)$. In fact, one expects the following conjecture which is a natural extension of Mazur [27] and Kobayashi [23].

**Conjecture.** $\text{Sel}^{-\infty}(E/F^\text{cyc})^\wedge$ is a torsion $\mathbb{Z}_p[\Gamma]$-module.

When $E$ has good ordinary reduction at all primes above $p$, the above conjecture is precisely Mazur’s conjecture [27] which is known in the case when $E$ is defined over $\mathbb{Q}$ and $F$ an abelian extension of $\mathbb{Q}$ (see [17]). For an elliptic curve over $\mathbb{Q}$ with good supersingular reduction at $p$, this conjecture was established by Kobayashi (cf. [23]; also see [3] for some recent progress on this conjecture).

For our subsequent discussion, we require another description of $\text{Sel}^{-\infty}(E/F^\text{cyc})$. Let $v$ be a prime of $F$ above $p$ which is not a good supersingular reduction prime of $E$. Then from Subsection 2.1 we have the following short exact sequence

$$0 \rightarrow C_v \rightarrow E(p) \rightarrow D_v \rightarrow 0$$

of $\text{Gal}(\bar{F}_v/F_v)$-modules, where both $C_v$ and $D_v$ are cofree with $\mathbb{Z}_p$-corank 1. If $w$ is a prime of $F^\text{cyc}$ above $v$, we sometimes write $D_w$ for $D_v$. Now the results of Coates and Greenberg [4, Propositions 4.3 and 4.8] tell us that that $H^1(F^\text{cyc}_w, E)(p) \cong H^1(F^\text{cyc}_w, D_v)$ which is the key to obtaining the following alternative useful description of the signed Selmer groups.
Proposition 3.1.2. We have the following two equivalent descriptions of the signed Selmer group, namely

\[
\text{Sel}^\pm(E/F^{\text{cyc}}) \cong \ker \left( H^1(G_S(F^{\text{cyc}}), E(p)) \xrightarrow{\psi^\pm} \mathcal{H}^\pm_{\infty} \times \bigoplus_{w \in S_p(F^{\text{cyc}})} H^1(F_{w^cyc}^w, D_w) \times \bigoplus_{w \in S'(F^{\text{cyc}})} H^1(F_{w^cyc}^w, E(p)) \right)
\]

\[
\cong \ker \left( H^1(G_S(F^{\text{cyc}}), E(p)) \xrightarrow{\psi^\pm} \mathcal{H}^\pm_{\infty} \times \bigoplus_{w \in S_p(F^{\text{cyc}})} H^1(E/F^{\text{cyc}}) \times \bigoplus_{w \in S'(F^{\text{cyc}})} H^1(F_{w^cyc}^w, E(p)) \right),
\]

where

\[
H^1_w(E/F_{w^cyc}) = \begin{cases} 
H^1(F_{w^cyc,ur}^w, D_w), & \text{if } w \text{ is a prime of good ordinary reduction} \\
H^1(F_{w^cyc}^w, Q_p/Z_p), & \text{if } w \text{ is a prime of split multiplicative reduction.}
\end{cases}
\]

Proof. The first isomorphism follows immediately from the discussion before the proposition. If \(D_v(F_{w^cyc}^w)\) is infinite, then it is \(Q_p/Z_p\) as \((F_{w^cyc}^w/F_{w^cyc}^w)\)-module by Lemma 2.1.1. The same lemma says that if \(w\) is a prime of good ordinary reduction or non-split multiplicative reduction, then \(H^1(F_{w^cyc}^w, D_v) \cong H^1(F_{w^cyc,ur}^w, D_v)\). Hence we may replace \(H^1(F_{w^cyc}^w, D_v)\) by \(H^1_w(E/F_{w^cyc})\) for \(w \in S_p(F^{\text{cyc}})\) in the first isomorphism and still recover \(\text{Sel}^\pm(E/F^{\text{cyc}})\). (Note: in view of this identification, the localisation maps in both isomorphisms are then identified with maps \(\psi^\pm\) as given in the definition of the signed Selmer groups, and by abuse of notation, we also denote both localisation maps by \(\psi^\pm\) as stated in the proposition.) \(\square\)

We shall require another equivalent description of the signed Selmer groups on the level of \(F\) and \(F^{\text{cyc}}\). For each \(n\), we define the strict signed Selmer group

\[
\text{Sel}^\pm,\text{str}(E/F_n) = \ker \left( H^1(G_S(F_n), E(p)) \rightarrow \bigoplus_{w \in S(F_n)} H^1(F_{n,w}, E(p)) \right),
\]

where

\[
L_w(E/F_n) = \begin{cases} 
\left( E_{w\text{cyc}}^w \otimes Q_p/Z_p \right)^{\Gamma_n}, & \text{if } w \in S_p^s(F_n), \\
\text{im} \left( H^1(F_{n,w}, C_w) \rightarrow H^1(F_{n,w}, E(p)) \right), & \text{if } w \in S_p^a(F_n), \\
0, & \text{if } w \in S'(F_n),
\end{cases}
\]

where \(\Gamma_n\) denotes \(\text{Gal}(F^{\text{cyc}}/F_n)\) and \((M)_{\text{div}}\) is the maximal \(p\)-divisible subgroup of \(M\).

Remark. Note that in the case when \(S_p^s = \emptyset\), this is the strict Selmer group as defined in the sense of Greenberg [8] (also see [11]). Our choice of naming the above as the strict signed Selmer group is inspired by this observation.

We now give a result which compares the strict signed Selmer groups with the usual \(p\)-primary Selmer group at the level of \(F\), and compares the strict signed Selmer groups with the signed Selmer groups at
the level of $F^{\astyc}$. We should mention that the strict Selmer groups $\text{Sel}^\astyc(E/F_n)$ need not coincide with $\text{Sel}^\ast(E/F_n)$ in general. Before stating and proving our result, we introduce another assumption that will play a role in subsequent discussion especially in the presence of a “+” sign in the signed Selmer groups.

$(S+)$ For every $v \in S^{ss}_p(L)$, assume further that $[L_v : \mathbb{Q}_p]$ is not divisible by 4.

**Proposition 3.1.3.** The following statements are valid.

(a) $\text{Sel}^\astyc(E/F^{\astyc}) \cong \lim_{\leftarrow n} \text{Sel}^\astyc(E/F_n)$ for every $\vec{s}$.

(b) $\text{Sel}^\astyc(E/F) = \text{Sel}(E/F)$.

(c) If one assumes further that $(S+)$ holds, then we also have $\text{Sel}^\astyc(E/F) = \text{Sel}(E/F)$ for every $\vec{s}$.

**Proof.** To prove assertions (b) and (c), it suffices to show that $L_v(E/F) = E(F_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ for all $v \in S$. This is clear for $v \in S'(F)$. For $v \in S^0_p(F)$, this is a result of Coates-Greenberg (cf. [4, Proposition 4.5]).

Now suppose that $v \in S^{ss}_p(F)$. We have a natural inclusion

$$E(F_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow (E^\pm(F^{\astyc}_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma.$$  

By $(S+)$ and Proposition 2.2.2 (or [22 Corollary 3.25]), we have that $(E^\pm(F^{\astyc}_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma$ is a cofree $\mathbb{Z}_p$-module of $\mathbb{Z}_p$-corank $[F_v : \mathbb{Q}_p]$. On the other hand, it follows from Mattuck’s theorem [26] that $E(F_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is also a cofree $\mathbb{Z}_p$-module of $\mathbb{Z}_p$-corank $[F_v : \mathbb{Q}_p]$. Therefore, the inclusion has to be an isomorphism.

For assertion (a), we note that $L_w(E/F_n) = E(F_{n,w}) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ for all $w \in S^{ss}_p(F_n)$ and so the local condition at $S^{ss}_p(F_n)$ for the strict signed Selmer group and the signed Selmer group at each $F_n$ (and hence $F^{\astyc}$) are the same. Similarly, one has the same conclusion for primes in $S^0_p(F^{\astyc})$. Let $v$ be a prime of $S^{ss}_p$. Recall that we also write $v$ for the prime of $F^{\astyc}$ above $v$. In general, $(E^\pm(F^{\astyc}_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma_n$ and $E^{ss}(F_{n,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ need not agree (one can see this by comparing their $\mathbb{Z}_p$-coranks). But upon taking limit, they are both equal to $E^{ss}(F^{\astyc}_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p$. In conclusion, we have established assertion (a).

### 3.2 Torsionness of signed Selmer groups

As noted in the previous subsection, $\text{Sel}^\astyc(E/F^{\astyc})$ is expected to be cotorsion. In this section, we consider an equivalent characterization of this property which is precisely the content of the next proposition.

**Proposition 3.2.1.** $\text{Sel}^\astyc(E/F^{\astyc})$ is a cotorsion $\mathbb{Z}_p[\Gamma]$-module if and only if $H^2(G_S(F^{\astyc}), E(p)) = 0$ and $\psi^\astyc$ is surjective.

The above result is well-known for the usual $p$-primary Selmer group when the elliptic curve $E$ has either good ordinary reduction or multiplicative reduction at all primes above $p$ (for instances, see [13 Proposition 7.2] or [25 Proposition 3.3]). When the elliptic curve $E$ has good supersingular reduction at all primes above $p$, this was somewhat partially established in [19 Proposition 2.4] (also see [20]...
Proposition 3.10]. That the analogue statement also holds in this mixed reduction situation was made aware to us by Antonio Lei and Ramdorai Sujatha, and we thank them for this and for sharing their work [24] on the subject.

As noted above, if \( S_p^s = \emptyset \), Proposition 3.2.1 is then a consequence of [25, Proposition 3.3]. Hence it remains to establish the equivalence as asserted in Proposition 3.2.1 under the assumption that \( S_p^s \neq \emptyset \). As a start, we record a few preparatory lemmas.

**Lemma 3.2.2.** Suppose that \( S_p^s \neq \emptyset \). Then \( E(F^{\text{cyc}})(p) = 0 \).

**Proof.** For \( w \in S_p^s(F) \), it follows from Lemma 2.2.1 that \( E(E_w^{\text{cyc}})(p) = 0 \). Since \( E(F^{\text{cyc}})(p) \) is contained in \( E(E_w^{\text{cyc}})(p) \), the conclusion of the lemma follows.

For the subsequent discussion, we shall write \( H^1_{1w}(F^{\text{cyc}}/F, T_p E) = \lim_{\rightarrow} H^1(G_S(F_n), T_p E) \), where \( T_p E \) is the Tate module of the elliptic curve \( E \).

**Lemma 3.2.3.** Suppose that \( S_p^s \neq \emptyset \). Then \( H^1_{1w}(E^{\text{cyc}}/F, T_p E) \) is a torsionfree \( \mathbb{Z}_p[\Gamma] \)-module.

**Proof.** By considering the low degree terms of the spectral sequence of Jannsen

\[
\text{Ext}^i_{\mathbb{Z}_p[\Gamma]} \left( H^j(G_S(F^{\text{cyc}})(E(p))^\vee, \mathbb{Z}_p[\Gamma]) \right) \rightarrow H^1_{1w}(E^{\text{cyc}}/F, T_p E)
\]

(cf. [10, Theorem 1]), we obtain the following exact sequence

\[
0 \rightarrow \text{Ext}^1_{\mathbb{Z}_p[\Gamma]} \left( E(F^{\text{cyc}})(p))^\vee, \mathbb{Z}_p[\Gamma]\right) \rightarrow H^1_{1w}(E^{\text{cyc}}/F, T_p E) \rightarrow \text{Ext}^2_{\mathbb{Z}_p[\Gamma]} \left( H^1(G_S(F^{\text{cyc}}), (E(p))^\vee, \mathbb{Z}_p[\Gamma]\right).
\]

In view of Lemma 3.2.2, the leftmost term vanishes. It then follows that \( H^1_{1w}(E^{\text{cyc}}/F, T_p E) \) injects into an \( \text{Ext}^0 \)-term which is a reflexive \( \mathbb{Z}_p[\Gamma] \)-module by [28, Corollary 5.1.3]. Therefore, \( H^1_{1w}(E^{\text{cyc}}/F, T_p E) \) must be torsionfree.

We can now give a proof of Proposition 3.2.1 (compare with [24, Proposition 4.4] and [25, Proposition 3.3]).

**Proof of Proposition 3.2.1.** As noted above, it suffices to prove the proposition under the assumption that \( S_p^s \neq \emptyset \) which we do. For the proof, we shall make use of the first description of \( \text{Sel}^X(E/F^{\text{cyc}}) \) in Proposition 3.1.2. By [29, Proposition A.3.2], there is an exact sequence

\[
0 \rightarrow \text{Sel}^X(E/F^{\text{cyc}}) \rightarrow H^1(G_S(F^{\text{cyc}}), E(p)) \xrightarrow{\psi^\vee} H^1(F_w^{\text{cyc}}, D_w) \times \bigoplus_{w \in S_p^s(F^{\text{cyc}})} H^1(F_w^{\text{cyc}}, E(p)) \xrightarrow{\bigoplus \psi} \text{Sel}^X(E/F^{\text{cyc}}) \rightarrow 0,
\]

where \( \text{Sel}^X(E/F^{\text{cyc}}) \) is a \( \mathbb{Z}_p[\Gamma] \)-submodule of \( H^1_{1w}(E^{\text{cyc}}/F, T_p E) \). (For the precise definition of \( \text{Sel}^X(E/F^{\text{cyc}}) \), we refer readers to loc. cit. For our purposes, the submodule theoretical information suffices.) Standard corank calculations [8, Propositions 1-3] tell us that

\[
\text{corank}_{\mathbb{Z}_p[\Gamma]} \left( H^1(G_S(F^{\text{cyc}}), E(p)) \right) - \text{corank}_{\mathbb{Z}_p[\Gamma]} \left( H^2(G_S(F^{\text{cyc}}), E(p)) \right) = [F : \mathbb{Q}],
\]

12
\[ \text{corank}_{Z_p[\Gamma]} \left( \bigoplus_{w \in S_p(F_{\text{cyc}})} H^1(F_{\text{cyc}}^w, D_w) \right) = \sum_{v \in S_p^c(F)} [F_v : Q_p] \]

and

\[ \text{corank}_{Z_p[\Gamma]} \left( \bigoplus_{w \in S_{p'}(F_{\text{cyc}})} H^1(F_{\text{cyc}}^w, E(p)) \right) = 0. \]

On the other hand, it follows from [22, Proposition 3.32] that

\[ \text{corank}_{Z_p[\Gamma]} (H^2_{\infty}(G_S(F_{\text{cyc}}), E)) = 0. \]

It is now clear from these formulas and the above exact sequence that \( \text{Sel}^\exists(E/F_{\text{cyc}}) \) is a cotorsion \( Z_p[\Gamma] \)-module if and only if \( \mathcal{S}^\exists(E/F_{\text{cyc}}) \) is a torsion \( Z_p[\Gamma] \)-module. Since \( \mathcal{S}^\exists(E/F_{\text{cyc}}) \) is contained in \( H^1_{\text{Iw}}(F_{\text{cyc}}/F, T_pE) \), which is torsionfree by Lemma 3.2.3, the latter statement holds if and only if \( \mathcal{S}^\exists(E/F_{\text{cyc}}) = 0 \). In view of the exact sequence in the beginning of the proof, this is precisely equivalent to saying that \( H^2(G_S(F_{\text{cyc}}), E(p)) = 0 \) and \( \psi^\exists \) is surjective. Hence we have established the proposition.

### 3.3 Non-primitive signed Selmer group

In comparing Selmer groups of two congruent elliptic curves, it is a standard procedure to do so via an appropriate non-primitive variant of the Selmer group (see [10, 19]). We shall follow this strategy and begin by introducing the non-primitive variant of the signed Selmer groups. This in turn is denoted and defined by

\[ \text{Sel}^\exists_{\text{non}}(E/F_{\text{cyc}}) = \ker \left( H^1(G_S(F_{\text{cyc}}), E(p)) \xrightarrow{\psi^\exists_{\text{non}}} \mathcal{H}_{\infty}^\exists \times \bigoplus_{w \in S_p(F_{\text{cyc}})} H^1_w(E/F_{\text{cyc}}) \right). \]

The non-primitive signed Selmer group and the signed Selmer group are connected via the following commutative diagram

\[ \begin{array}{ccc}
0 & \longrightarrow & \text{Sel}^\exists(E/F_{\text{cyc}}) \\
& \| & \| \\
& \downarrow & \downarrow \\
0 & \longrightarrow & \text{Sel}^\exists_{\text{non}}(E/F_{\text{cyc}}) \\
\end{array} \]

with exact rows. An application of the snake lemma then gives following exact sequence

\[ 0 \longrightarrow \text{Sel}^\exists(E/F_{\text{cyc}}) \longrightarrow \text{Sel}^\exists_{\text{non}}(E/F_{\text{cyc}}) \longrightarrow \bigoplus_{w \in S_p(F_{\text{cyc}})} H^1(F_{\text{cyc}}^w, E(p)). \]

We can now record an important result on the structure of the non-primitive signed Selmer group.
Proposition 3.3.1. Suppose that $\text{Sel}^\wedge(E/F_{\text{cyc}})$ is a cotorsion $\mathbb{Z}_p[\Gamma]$-module. Then the following statements hold.

(a) $\text{Sel}^\wedge_{\text{non}}(E/F_{\text{cyc}})^\vee$ has no nonzero finite $\mathbb{Z}_p[\Gamma]$-submodules.

(b) There is a short exact sequence

$$0 \longrightarrow \text{Sel}^\wedge(E/F_{\text{cyc}}) \longrightarrow \text{Sel}^\wedge_{\text{non}}(E/F_{\text{cyc}}) \longrightarrow \bigoplus_{w \in S'(F_{\text{cyc}})} H^1(F_{w_{\text{cyc}}}, E(p)) \longrightarrow 0.$$ 

Proof. Since $\text{Sel}^\wedge(E/F_{\text{cyc}})$ is assumed to be a cotorsion $\mathbb{Z}_p[\Gamma]$-module, it follows from Proposition 3.2.1 that $H^2(G_S(F_{\text{cyc}}), E(p)) = 0$ and $\psi^\wedge$ is surjective. The former implies that $H^1(G_S(F_{\text{cyc}}), E(p))^\vee$ has no nonzero finite $\mathbb{Z}_p[\Gamma]$-submodules (cf. [8, Proposition 5]), whereas the latter, when combined with a diagram-chasing argument, yields the short exact sequence in (b) and the surjectivity of $\psi^\wedge$. The validity of (a) will now follow from an application of [10, Lemma 2.6].

We record the following corollary of the preceding proposition.

Corollary 3.3.2. The following statements are equivalent.

(a) $\text{Sel}^\wedge(E/F_{\text{cyc}})$ is a cofinitely generated $\mathbb{Z}_p$-module.

(b) $\text{Sel}^\wedge_{\text{non}}(E/F_{\text{cyc}})$ is a cofinitely generated $\mathbb{Z}_p$-module.

(c) $\text{Sel}^\wedge_{\text{non}}(E/F_{\text{cyc}})[p]$ is finite.

Moreover, in the event of that one of (and hence all of) the above statements holds, one has

$$\dim_{\mathbb{F}_p} \left( \text{Sel}^\wedge_{\text{non}}(E/F_{\text{cyc}})[p] \right) = \text{corank}_{\mathbb{Z}_p} \left( \text{Sel}^\wedge_{\text{non}}(E/F_{\text{cyc}}) \right) = \text{corank}_{\mathbb{Z}_p} \left( \text{Sel}^\wedge_{\text{non}}(E/F_{\text{cyc}}) \right) + \sum_{w \in S'(F_{\text{cyc}})} H^1(F_{w_{\text{cyc}}}, E(p)).$$

Proof. The equivalence of (a) and (b) is an immediate consequence of Proposition 3.3.1(b). The equivalence of (b) and (c) is a consequence of Nakayama lemma. Now if $\text{Sel}^\wedge_{\text{non}}(E/F_{\text{cyc}})$ is cofinitely generated over $\mathbb{Z}_p$, then $\text{Sel}^\wedge_{\text{non}}(E/F_{\text{cyc}})[p]$ is a finite quotient of $\text{Sel}^\wedge_{\text{non}}(E/F_{\text{cyc}})$, and by Proposition 3.3.1(a), it has to be trivial. The first equality now follows from this. The final equality follows from Proposition 3.3.1(b) again.

We now consider the non-primitive mod-$p$ signed Selmer group $\text{Sel}^\wedge_{\text{non}}(E[p]/F_{\text{cyc}})$ which is defined to be

$$\ker \left( H^1(G_S(F_{\text{cyc}}), E[p]) \longrightarrow T_{\infty}^s \times \bigoplus_{w \in S'(F_{\text{cyc}})} H^1(w,E[p]/F_{\text{cyc}}) \right),$$

14
The conclusions of the lemma will then follow from combining these observations with Corollary 3.3.2.

is finite. Furthermore, in the event that this finiteness property holds, the argument also shows that

\[ H^1_w(E[p]/F^{cyc}) = \begin{cases} H^1(F_w^{cyc, ur}, D_w[p]), & \text{if } D_v(F_w^{cyc}) \text{ is finite}, \\ H^1(F_w^{cyc}, D_w[p]), & \text{if } D_v(F_w^{cyc}) \text{ is infinite}. \end{cases} \]

This fits into the following commutative diagram

\[
\begin{array}{ccc}
0 \rightarrow \text{Sel}_{\text{non}}^\varpi(E[p]/F^{cyc}) & \rightarrow & H^1(G_S(F^{cyc}), E[p]) \\
\downarrow & & \downarrow \text{h} \\
0 \rightarrow \text{Sel}_{\text{non}}^\varpi(E/F^{cyc})[p] & \rightarrow & H^1(G_S(F^{cyc}), E(p))[p] \\
\end{array}
\]

with exact rows.

**Lemma 3.3.3.** \( \text{Sel}_{\text{non}}^\varpi(E/F^{cyc}) \) is finitely generated over \( \mathbb{Z}_p \) if and only if \( \text{Sel}_{\text{non}}^\varpi(E[p]/F^{cyc}) \) is finite. Furthermore, in the event of such, we have the following equality

\[ \dim_{\mathbb{F}_p}\left( \text{Sel}_{\text{non}}^\varpi(E[p]/F^{cyc}) \right) = \text{corank}_{\mathbb{Z}_p}\left( \text{Sel}_{\text{non}}^\varpi(E/F^{cyc}) \right) + \dim_{\mathbb{F}_p}\left( (E(F^{cyc})[p] \right). \]

**Proof.** The map \( h \) in the above commutative diagram is surjective with \( \ker h = E(F^{cyc})(p)/p \). Since \( E(F^{cyc})(p) \) is finite by [31], we have \( \dim_{\mathbb{F}_p}\left( (E(F^{cyc})[p] \right) = \dim_{\mathbb{F}_p}\left( \ker h \right) \). We now show that each \( c_w \) is an injection. Suppose that \( w \) lies above \( v \in S^*_p(F) \). Then one has

\[ \ker c_w = \begin{cases} D_w(F_w^{cyc, ur})/p, & \text{if } D_v(F_w^{cyc}) \text{ is finite}, \\ D_w(F_w^{cyc})/p, & \text{if } D_v(F_w^{cyc}) \text{ is infinite}. \end{cases} \]

But in either of the two cases, \( \ker c_w \) is a mod-\( p \) quotient of a divisible group and hence must be zero.

Suppose that \( w \in S^*_p(F^{cyc}) \). Then we have the following short exact sequence

\[ 0 \rightarrow E^\pm(F_w^{cyc}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow H^1(F_w^{cyc}, E(p)) \rightarrow H^1(F_w^{cyc}, E(p))/E^\pm(F_w^{cyc}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0. \]

Since \( E^\pm(F_w^{cyc}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \) is \( p \)-divisible, the above exact sequence yields the following short exact sequence

\[ 0 \rightarrow E^\pm(F_w^{cyc})/p \rightarrow H^1(F_w^{cyc}, E(p))[p] \rightarrow \left( H^1(F_w^{cyc}, E(p))/E^\pm(F_w^{cyc}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \right)[p] \rightarrow 0. \]

Since \( E(F_w^{cyc})(p) = 0 \), the term \( H^1(F_w^{cyc}, E(p))[p] \) identifies with \( H^1(F_w^{cyc}, E[p]) \). It then follows from this that \( c_w \) is also injective in this case.

In conclusion, the above argument shows that \( \text{Sel}_{\text{non}}^\varpi(E/F^{cyc})[p] \) is finite if and only if \( \text{Sel}_{\text{non}}^\varpi(E[p]/F^{cyc}) \) is finite. Furthermore, in the event that this finiteness property holds, the argument also shows that

\[ \dim_{\mathbb{F}_p}\left( \text{Sel}_{\text{non}}^\varpi(E[p]/F^{cyc}) \right) = \dim_{\mathbb{F}_p}\left( \text{Sel}_{\text{non}}^\varpi(E/F^{cyc})[p] \right) + \dim_{\mathbb{F}_p}\left( (E(F^{cyc})[p] \right). \]

The conclusions of the lemma will then follow from combining these observations with Corollary 3.3.2. \( \square \)
Remark. Note that if $S_p^{ss} \neq \emptyset$, then we even have $\ker h = 0$ and that
\[
\text{Sel}^{\mathfrak{f}}_{\text{non}}(E[p]/F^{\text{cyc}}) \cong \text{Sel}^{\mathfrak{f}}_{\text{non}}(E/F^{\text{cyc}})[p].
\]

**Proposition 3.3.4.** $\text{Sel}^{\mathfrak{f}}(E/F^{\text{cyc}})$ is cofinitely generated over $\mathbb{Z}_p$ if and only if $\text{Sel}^{\mathfrak{f}}_{\text{non}}(E[p]/F^{\text{cyc}})$ is finite. Moreover, in this is so, we have
\[
\text{corank}_{\mathbb{Z}_p} \left( \text{Sel}^{\mathfrak{f}}(E/F^{\text{cyc}}) \right) + \sum_{w \in S'(F^{\text{cyc}})} \text{corank}_{\mathbb{Z}_p} \left( H^1(F_w^{\text{cyc}}, E(p)) \right) + \dim_{\mathbb{F}_p} \left( (E(F^{\text{cyc}})[p] \right)
= \dim_{\mathbb{F}_p} \left( \text{Sel}^{\mathfrak{f}}_{\text{non}}(E[p]/F^{\text{cyc}}) \right).
\]

**Proof.** The first assertion is clear from the exact sequence before Proposition 3.3.1 whereas the second is a consequence of Corollary 3.3.2 and Lemma 3.3.3. \qed

## 4 Main results

In this section, we shall prove our main results. Let $E_1$ and $E_2$ be two elliptic curves defined over $L'$. We shall retain the notation and settings from the previous section. In particular, the two elliptic curves are assumed to satisfy (NA) and (SS). We shall also assume that $E_1[p] \cong E_2[p]$ as $\text{Gal}(L'/L)$-modules. As mentioned in the introduction, we need to be able to transfer this global information into local cohomology groups at primes above $p$ and this leads us to impose the following two hypotheses.

(RED) The elliptic curves $E_1$ and $E_2$ have the same reduction type for each prime of $L'$ above $p$. Furthermore, in the event that $E_1$ and $E_2$ have non-split multiplicative reduction at a prime $u$ of $L'$ above $p$, suppose further $E_1$ and $E_2$ have the same reduction type at all primes of $F$ above $u$.

(RU) For each $u \in S_p^0$ such that $u$ is a prime of good ordinary reduction or non-split multiplicative reduction, suppose that $L'_u$ does not contain $\mu_p$.

### 4.1 Comparing ranks of Selmer groups

We can now state and prove the following result which generalize Greenberg-Vatsal [10] and Kim [19].

**Theorem 4.1.1.** Let $E_1$ and $E_2$ be two elliptic curves defined over $L'$ with $E_1[p] \cong E_2[p]$ as $\text{Gal}(L'/L)$-modules. Let $L$ be a finite extension of $L'$. Write $F$ for either $L$ or $L(\mu_p)$. Suppose that (NA), (SS), (RED) and (RU) hold for $E_1$ and $E_2$.

Then $\text{Sel}^{\mathfrak{f}}(E_1/F^{\text{cyc}})$ is cofinitely generated over $\mathbb{Z}_p$ if and only if $\text{Sel}^{\mathfrak{f}}(E_2/F^{\text{cyc}})$ is cofinitely generated over $\mathbb{Z}_p$. Moreover, if this is so, we have the following equality
\[
\text{corank}_{\mathbb{Z}_p} \left( \text{Sel}^{\mathfrak{f}}(E_1/F^{\text{cyc}}) \right) + \sum_{w \in S'(F^{\text{cyc}})} \text{corank}_{\mathbb{Z}_p} \left( H^1(F_w^{\text{cyc}}, E_1(p)) \right)
= \text{corank}_{\mathbb{Z}_p} \left( \text{Sel}^{\mathfrak{f}}(E_2/F^{\text{cyc}}) \right) + \sum_{w \in S'(F^{\text{cyc}})} \text{corank}_{\mathbb{Z}_p} \left( H^1(F_w^{\text{cyc}}, E_2(p)) \right).
\]
Proof. Clearly, one has \( \dim_F p \left( (E_1(F^{\text{cyc}})[p]) = \dim_F p \left( (E_2(F^{\text{cyc}})[p]) \right) \) by the congruent assumption. Hence, by Proposition 3.3.2 it suffices to show that \( \text{Sel}_{\text{non}}^\sharp (E_1[p]/F^{\text{cyc}}) \cong \text{Sel}_{\text{non}}^\sharp (E_2[p]/F^{\text{cyc}}) \). Plainly, we have \( H^1(G_S(F^{\text{cyc}}), E_1[p]) \cong H^1(G_S(F^{\text{cyc}}), E_2[p]) \). Therefore, it remains to show that the local terms in the definition of the non-primitive mod-\( p \) signed Selmer groups are isomorphic. We note that by (RED), \( E_1 \) and \( E_2 \) have the same reduction type for each prime of \( F \) above \( p \). Let \( \rho \) denote the isomorphism \( E_1[p] \cong E_2[p] \). For \( w \in S^p_0(F^{\text{cyc}}) \), write \( u \) for the prime of \( L' \) below \( w \). Denote by \( D_{i,u} \) the \( \text{Gal}(\overline{w}/L'_u) \)-quotient of \( E_i(p) \) of \( \mathbb{Z}_p \)-corank one as defined in Subsection 2.1.

Now if \( w \) is a prime of split multiplicative reduction for \( E_1 \) and \( E_2 \), then by Lemma 2.1.1 we have \( D_{i,u}(F_w^{\text{cyc}}) \cong \mathbb{Q}_p/\mathbb{Z}_p \) as \( \text{Gal}(F_w^{\text{cyc}}/F_w^{\text{cyc}}) \)-modules for \( i = 1, 2 \). Clearly, in this case, we have \( D_{1,u}[p] \cong D_{2,u}[p] \) which in turn implies that \( H^1(F_w^{\text{cyc}}, D_{1,u}[p]) \cong H^1(F_w^{\text{cyc}}, D_{2,u}[p]) \). Now suppose that \( w \) is a prime of good ordinary reduction or non-split multiplicative reduction for \( E_1 \) and \( E_2 \). Let \( u \) be the prime of \( L' \) below \( w \). Then \( u \) is also prime of good ordinary reduction or non-split multiplicative reduction for \( E_1 \) and \( E_2 \). Since \( \rho \) respects the \( \text{Gal}(F_u/F_u) \)-action, \( \rho(C_{1,u}[p]) \) is a Gal\((F_u/L'_u)\)-submodule of \( E_2[p] \) which is isomorphic to \( \mu_p \) as Gal\((F_u/L'_u)\)-modules. In view of (RU), we may apply Lemma 2.1.2 to conclude that \( \rho(C_{1,u}[p]) = C_{2,u}[p] \). Hence \( \rho \) induces an isomorphism \( D_{1,u}[p] \cong D_{2,u}[p] \) of \( \text{Gal}(F_u/L'_u) \)-modules and hence of Gal\((F_w^{\text{cyc}}/F_w^{cyc, ur})\)-modules. It then follows this that

\[
H^1(F_w^{\text{cyc, ur}}, D_{1,u}[p]) \cong H^1(F_w^{\text{cyc, ur}}, D_{2,u}[p]).
\]

Now suppose that \( w \in S^p_0 \). As before, write \( u \) for a prime of \( L' \) below \( w \). It follows from hypothesis (SS) that \( L'_u = \mathbb{Q}_p \). From the discussion in [14] Proposition 2.8, we see that \( \rho \) induces an isomorphism \( \hat{E}_{1,u}[p] \cong \hat{E}_{2,u}[p] \) of the form \( x \mapsto \lambda(ax) \) for some fixed \( a \in \mathbb{F}_p \) and some fixed isomorphism \( \lambda \) of formal groups \( \hat{E}_{1,u} \cong \hat{E}_{2,u} \) over \( \mathbb{Z}_p \). Such an isomorphism in turn induces an isomorphism \( H^1(F_w^{\text{cyc}}, E_1[p]) \cong H^1(F_w^{\text{cyc}}, E_2[p]) \), which upon restricted to \( E_1^+(F_w^{\text{cyc}})/p \), gives an isomorphism \( E_1^+(F_w^{\text{cyc}})/p \cong E_2^+(F_w^{\text{cyc}})/p \). Putting these together, we obtain

\[
\frac{H^1(F_w^{\text{cyc}}, E_1[p])}{E_1^+(F_w^{\text{cyc}})/p} \cong \frac{H^1(F_w^{\text{cyc}}, E_2[p])}{E_2^+(F_w^{\text{cyc}})/p}.
\]

The proof of the theorem is now completed. \( \square \)

4.2 \( p \)-parity conjecture for congruent elliptic curves

Before proving our next theorem, we first return to the setting of Section 3 where we only work with one elliptic curve \( E \). The following proposition will be required for later discussion.

**Proposition 4.2.1.** Suppose that (NA) and (SS) are valid. Suppose that \( \text{Sel}^\sharp (E/F^{\text{cyc}}) \) is torsion over \( \mathbb{Z}_p[\Gamma] \). In the event that \( \overline{\mathfrak{a}} \neq \overline{\mathfrak{T}} \), assume further that (S+) holds. Then one has

\[
\text{corank}_{\mathbb{Z}_p} (\text{Sel}(E/F)) = \overline{\lambda}^\sharp \quad (\text{mod } 2),
\]

where \( \overline{\lambda}^\sharp \) is the Iwasawa \( \lambda \)-invariant of \( \text{Sel}^\sharp (E/F^{\text{cyc}}) \).

**Proof.** When \( E \) has good ordinary reduction at all primes above \( p \), this was proved in [9] Proposition 3.10. In the case that \( E \) has supersingular reduction, this was proved in [14] Proposition 4.1. The same
argument there essentially carries over which we shall sketch briefly. The key to proving the proposition is to make use of the strict signed Selmer groups. By Proposition 3.1.3 and discussion before the said proposition, the strict signed Selmer group coincides with the usual Selmer group on the level of $F$ and the signed Selmer group on the level of $F^{cyc}$. Now it is easy to see that the kernel of the natural map

$$\text{Sel}^\mathcal{S}_{str}(E/F_n) \xrightarrow{t_n} \text{Sel}^\mathcal{S}_{str}(E/F_n)^\Gamma_n$$

is contained in $H^1(\Gamma_n, E(F^{cyc})(p))$. Since the latter is finite and bounded independently of $n$, so is the $\ker t_n$. The point here is to note that in proving this boundness property, we do not require the full strength of a control theorem as in [27] or [23] which we do have at our disposal in the context we are working in. The remainder of the proof now proceeds as in [9, Proposition 3.10] and [14, Proposition 4.1].

**Remark.** In [1, Theorem 4.6], the authors of that said paper had a result analogous to (but less general than) Proposition 4.2.1. We like to mention that although their result is correct, it would seem that they did not realize that one has to give a proof via the strict signed Selmer groups as what we have done here (also see Hatley [14, Remark 2.3] and Kim [18, Subsection 4.2]). The reason of using such strict signed Selmer groups is because their local conditions at supersingular primes have a self-dual property in the sense of Proposition 2.2.3 which is required in order to be able to apply Flach’s result [7]. We finally note that it is well-known that the local conditions as defined in the strict signed Selmer groups for other primes have the desired self-dual properties (see [11, Theorem 1]).

The following is an immediate corollary of the previous proposition which generalizes a previous observation of Hatley [14, Corollary 4.2].

**Corollary 4.2.2.** Suppose that (NA), (SS) and (S+) hold, and that $\text{Sel}^\mathcal{S}(E/F^{cyc})$ is torsion over $\mathbb{Z}_p[\Gamma]$ for every $\mathcal{S}$. Then the parity of $\lambda^\mathcal{S}$ is independent of $\mathcal{S}$.

**Remark.** We like to take this opportunity to make a remark (which although is not required for subsequent discussion). To the best knowledge of the authors, the signed Selmer groups seem to be treated as independent entities in literature. It is a natural and interesting question to ask whether the signed Selmer groups are related to one another in any way. The above corollary and the result of Hatley [14, Corollary 4.2] seem to suggest (perhaps mildly) that this is so. We also mention another (mild) result in this spirit. In a recent work of the authors [2, Corollary 2.8], we have proved that if the elliptic curve $E$ has good reduction at all primes above $p$ and (S+) holds, then as long as one of the signed Selmer group vanishes, so will the other. It would perhaps be of interest to find out further (possible) relation between the signed Selmer groups.

We come back to the congruent elliptic curves situation. Recall that the global root number of an elliptic curve $E$ is defined by $w(E/F) = \prod_v w(E/F_v)$, where $v$ runs through all primes of $F$ (see [8]). The $p$-parity conjecture then predicts that $w(E/F) = (-1)^{s_p(E)}$, where $s_p(E) = \text{corank}_{\mathbb{Z}_p}(\text{Sel}(E/F))$. We are now in position to prove the following theorem.
Theorem 4.2.3. Let $E_1$ and $E_2$ be two elliptic curves defined over $L'$ with $E_1[p] \cong E_2[p]$ as $\text{Gal}(L'/L')$-modules. Let $L$ be a finite extension of $L'$, and write $F$ for $L$ or $L(\mu_p)$. Suppose that (NA), (SS), (RED) and (RU) hold for $E_1$ and $E_2$. Suppose further that at least one of the following statements is also valid.

(a) $\text{Sel}^\rightarrow (E_1/F_{\text{cyc}})$ is cofinitely generated over $\mathbb{Z}_p$.

(b) $\text{Sel}^\rightarrow (E_1/F_{\text{cyc}})$ is cofinitely generated over $\mathbb{Z}_p$ for some $\rightarrow \neq \rightarrow$ and (S+) is valid.

Then one has

$$\frac{w(E_1/F)}{w(E_2/F)} = (-1)^{s_p(E_1) - s_p(E_2)}.$$

In particular, the $p$-parity conjecture holds for $E_1$ over $F$ if and only if it holds for $E_2$ over $F$.

Proof. Let $v \in S$ be a prime outside $p$. Since $p$ is odd, there are odd number of primes of $S(F_{\text{cyc}})$ above $v$. As $H^1(F_{wv}^{\text{cyc}}, E(p))$ has the common $\mathbb{Z}_p$-corank for every $w|v$, it follows that

$$\text{corank}_{\mathbb{Z}_p} (H^1(F_{wv}^{\text{cyc}}, E(p))) \equiv \sum_{w|v} \text{corank}_{\mathbb{Z}_p} (H^1(F_{wv}^{\text{cyc}}, E(p))) \pmod{2}$$

for some fixed prime $w_v$ of $F_{\text{cyc}}$ above $v$. Combining this with Theorem 4.1.1 and Proposition 4.2.1, we have

$$s_p(E_1) - s_p(E_2) = \sum_{v \in S'(F)} (\sigma_{E_1,w_v} - \sigma_{E_2,w_v}) \pmod{2},$$

where we write $\sigma_{E_i,w_v} = \text{corank}_{\mathbb{Z}_p} (H^1(F_{w_v}^{\text{cyc}}, E_i(p)))$.

Recall that $w(E/F_v) = -1$ when $v$ is archimedean or is a split multiplicative reduction prime of $E$, and $w(E/F_v) = 1$ when $E/F_v$ has good or non-split multiplicative reduction (see [5, Section 3.4]). In view of this and that $E_1$ and $E_2$ have the same reduction type for each prime above $p$, we then have

$$\frac{w(E_1/F)}{w(E_2/F)} = \prod_{v \in S'(F)} \frac{w(E_1/F_v)}{w(E_2/F_v)},$$

and the latter is equal to

$$\prod_{v \in S'(F)} (-1)^{\sigma_{E_1,w_v} - \sigma_{E_2,w_v}}$$

by Proposition 2.3.1. Combining this with the congruence obtained in the first paragraph of the proof, we obtain the required equality of the theorem. The final assertion of the theorem is then immediate from this.

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