Efficient fiber coupling of down-conversion photon pairs

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Abstract

We develop and apply an effective analytic theory of a non-collinear, broadband type-I parametric down-conversion to study a coupling efficiency of the generated photon pairs into single mode optical fibers. We derive conditions necessary for highly efficient coupling for single and double type-I crystal producing polarization entangled states of light. We compare the obtained approximate analytic expressions with the exact numerical solutions and discuss the results for a case of BBO crystals.
I. INTRODUCTION

Sources of polarization entangled photon pairs are important in a wide range of modern applications. Efficient generation of photon pairs led to an opportunity of testing several fundamental problems of quantum mechanics and to a practical realization of various quantum-information or cryptographic schemes. Among all accessible entanglement sources, down-conversion crystals turned out to be the most effective and easy to implement so far. Therefore the development of down-conversion techniques started to be an important issue for many branches of quantum optics. In a search for enhancement of counting rates, it has been noticed that focusing of the pump beam can essentially increase the efficiency of the process. Another important observation was that coupling of the generated photons into single mode fibers can optimize the collection efficiency and is a very practical method of preparing photons in well defined spatial modes. This idea has already inspired some authors to study various aspects of fiber coupling of the down-converted photon pairs. Efficient fiber coupling of the entangled photon pairs can also help in enhancing the fiber communication efficiency by encoding additional information in the polarization degree of freedom.

The paper delivers a detailed analysis of the fiber coupling efficiency and properties of the coupled type-I down-conversion photon pairs in a general case of a pumping with ultrashort pulses. To our best knowledge this issue has not been analyzed so far. One of the advantages of using the pulsed pump sources is the fact, that generated photon pairs arrive at the detectors at a well-defined temporal window which may be very helpful in a number of experimental schemes.

We develop an approximate analytic theory of type-I down-conversion to study the dependence of the coincidence spectrum and coupling efficiency on several relevant parameters characterizing the system such as properties of the nonlinear medium, pumping beam, spectral filters, geometry of the setup, etc. We point out what are the optimum (for the coupling efficiency and a spectrum separability, which is necessary for a high-purity polarization entanglement generation) settings of the setup and compare the result with the numerical calculations.

The paper is organized as follows: in Sec. II we derive explicitly a wave function of the type-I down-conversion generated photon pairs and discuss the applied approximations. In
Sec. III we discuss the coupling process, properties of the spectra of the coupled photons, derive the spectrum separability condition and derive expressions for the coupling probability. In Sec. IV we apply the obtained results to discuss the case of a double type-I crystal generating polarization entangled states and finally Sec. V concludes the paper and in the Appendix we derive a relation between geometrical parameters of the crystal and a dispersion relation.

II. WAVE FUNCTION

Consider a type-I down-conversion process taking place in an anisotropic medium with a second-order nonlinearity. Interaction between an extraordinarily polarized laser pump beam \( E_p(r, t) \) (which will be assumed to have a finite time) and the nonlinear crystal generates pairs of daughter photons whose properties are fully determined by the properties of the pump beam, indices of refraction and geometry of the process - see Figure I. In the first order perturbation, only a single photon pair can be created in the down-conversion and the output quantum state has the approximate form:

\[
|\Psi\rangle \approx |0\rangle - \frac{i\chi}{\hbar} \int_{-\infty}^{\infty} dt \int_V d^3r \, E_p(r, t) \left( \hat{E}^{(-)}(r, t) \right)^2 |0\rangle, \tag{1}
\]

where an electric nonlinear polarizability \( \chi \) is a parameter determining the interaction strength, \( V \) is volume of the crystal and a negative part of the electric field operator is \( \hat{E}^{(-)}(r, t) = \int dk_x dk_y d\omega e^{-i(k \cdot r - \omega t)} \hat{a}^\dagger(k_x, k_y, \omega) \) (\( \hat{a} \) is an annihilation operator of an ordinarily polarized mode, at \( z = 0 \)). The integral extends to all positive frequencies \( \omega \) and all perpendicular wave-vectors \( k_x, k_y \). Such a choice of the plane wave decomposition will turn out to be very handy, since the perpendicular wave-vector components and frequency do not change on dielectric media boundaries [8, 9]. Let us introduce signal and idler mode vectors \( s = (k_{s,x}, k_{s,y}, \omega_s) \) and \( i = (k_{i,x}, k_{i,y}, \omega_i) \) to decompose the square of the electric field operator appearing in the expression (1). Then the \( z \)th components of the wave vectors are given by a dispersion relations inside the anisotropic medium \( k_{s,z} = k_{i,z} = k^0_z(s) \) and \( k_{i,z} = k^0_z(i) \) (index \(^0\) refers to ordinary wave of the considered anisotropic medium - see Appendix) and the output quantum state can be written down using a quantity \( \Psi(s, i) \) interpreted as
a probability amplitude for a photon pair to be created in modes \( s \) and \( i \):

\[
|\Psi\rangle \approx |0\rangle - \frac{i \chi}{\hbar} \int d^3s \ d^3i \ \Psi(s, i) \hat{a}^\dagger(s) \hat{a}^\dagger(i) |0\rangle.
\] (2)

The first order approximation that we will study is justified when \( \chi \ll 1 \). The properties of the generated photon pairs are described by the function \( \Psi(s, i) \) and this is a typical situation in many down-conversion experiments.

Suppose that a type-I down-conversion medium occupies the area \(-\frac{L}{2} \leq z \leq \frac{L}{2}\) and it is pumped by a light beam propagating towards \( z \) direction. Then the probability amplitude \( \Psi(s, i) \) at the end of the crystal has the following form:

\[
\Psi(s, i) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-z_1}^{z_2} dz \ E_P(r, t) \times e^{-i[(k_0 + k_z) \cdot r - (\omega_0 + \omega)t]} e^{i(k_0 + k_z) \frac{L}{2}},
\] (3)

where the last exponent is a propagation factor of the amplitude from \( z = 0 \) to \( z = \frac{L}{2} \).

Let us express the pump beam electric field in terms of its Fourier transform defined at \( z = -\frac{L}{2} \): \( E_P(r, t) = \int dk_{p,x} \ d k_{p,y} \ d \omega_p \ e^{i(k_0 + k_z) \cdot r - (\omega_0 + \omega)t} e^{i k_p \cdot z \frac{L}{2}} E_{P}(k_{p,x}, k_{p,y}, \omega_p) \), where we have introduced vector \( p = (k_{p,x}, k_{p,y}, \omega_p) \) parameterizing the plane wave decomposition of the pump beam. To calculate the probability amplitude \( \Psi(s, i) \) we need the dispersion relation \( k_{p,z} = k_z^e(p) \), which is well known for uniaxial crystals [10] - see Appendix. Inserting this into the equation \( (3) \) and performing all the integrals yields [11]:

\[
\Psi(s, i) = \tilde{E}_P(s + i) L \ \text{sinc} \left( \frac{L}{2} \delta_-(s, i) \right) \ \exp \left( i \frac{L}{2} \delta_+(s, i) \right),
\] (4)

where \( \delta_\pm (\delta_-) \) is a so-called phase mismatch parameter equal:

\[
\delta_\pm(s, i) = k_z^e(s + i) \pm k_z^o(s) \pm k_z^o(i).
\] (5)

From the form of equation \( (4) \) it is clear that the process occurs most effectively in the directions for which the longitudinal wave vector components meet the phase matching condition \( \delta_- = 0 \). To determine this condition explicitly one has to use Sellmeier expressions for the extraordinary and ordinary index of refraction as a function of the vectors \( s \) and \( i \).
Since the expression (4) is not analytically integrable, which will turn out to be crucial in our analysis and the explicit form of the mismatch parameter $\delta_-$ is rather complicated, we will perform certain approximations before starting any further analysis. First of all, let us perform a Gaussian approximation of the $\text{sinc} x$ function appearing in (4) with the expression $e^{-x^2/4}$. This rough approximation is fair as long as we consider only photons generated near the phase-matching region of wave vectors and frequencies where the down-conversion occurs most effectively. In this narrow range of parameters, i.e. when $\frac{L^2}{2} \delta_- \ll 1$ the wave tails of the $\text{sinc} x$ function give insignificant contribution to the overall probability distribution of generating a pair of photons.

Secondly, we will expand $\delta_\pm$ to the first order in the Taylor series around a point for which the phase-matching condition $\delta_- = 0$ is met in the horizontal plane. If we fix the $x$ axis to be horizontal and the $y$ axis to be vertical, our expansion takes place around the point $\mathbf{s}_0 = \left( \frac{\theta_0}{c}, 0, \omega_0 \right)$ and $\mathbf{i}_0 = \left( -\frac{\theta_0}{c}, 0, \omega_0 \right)$, where $\omega_0$ is the phase-matching degenerate frequency and $\theta_0$ is the angle of propagation of the degenerate daughter photons relative to the $z$ axis outside the crystal. These two parameters can be determined from the properties of the down-conversion medium (given the Sellmeier formulas for extraordinary indices of refraction and direction of the crystal axis, one can calculate $\omega_0$ and $\theta_0$ directly from the condition $\delta_- (\mathbf{s}_0, \mathbf{i}_0) = 0$ and the definition (5)). We obtain an approximate form of the probability amplitude (4):

$$
\Psi(\mathbf{s}, \mathbf{i}) \approx \tilde{E}_p (\mathbf{s} + \mathbf{i}) L \exp \left[ -\frac{L^2}{16} (\mathcal{D}\delta_-)^2 + i\frac{L}{2} \mathcal{D}\delta_+ \right],
$$

where we have introduced the following notation: $\mathcal{D} f(x_1, \ldots, x_N) = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} \Delta x_i$. It is apparent that in our approximation all the relevant information about the structure of the nonlinear medium is contained in the derivatives of $\delta_\pm$. Particular components of these vectors depend on the properties of both, extraordinary and ordinary indices of refraction, as well as on the spatial orientation of the optical axis.

For the reasons that will be made clear in the further parts of the analysis we shall choose the optical axis of the crystal in a plane oriented at the angle 45° in respect to the horizontal plane. In this case all the derivatives considered can be approximately expressed with only 4 real parameters (moreover, it will turn out that one of them is irrelevant, so we end up with only 3 parameters) - see Appendix:
\[
\frac{\partial \delta_\pm}{\partial k_{s,x}} = \frac{\gamma}{\sqrt{2}} \pm \theta_0', \\
\frac{\partial \delta_\pm}{\partial k_{i,x}} = \frac{\gamma}{\sqrt{2}} \pm \theta_0', \\
\frac{\partial \delta_\pm}{\partial k_{s,y}} = \frac{\partial \delta_\pm}{\partial k_{i,y}} = \frac{\gamma}{\sqrt{2}}, \\
\frac{\partial \delta_\pm}{\partial \omega_s} \approx \Delta \beta_{\pm,z},
\]

where \(\theta_0'\) is the angle of propagation of the degenerate daughter photons relative to the \(z\) axis inside the crystal proportional to the \(\theta_0\) (proportionality constant is given by the ordinary index of refraction), \(\gamma\) (equal to the derivative of \(k_e^z\) in the direction of projection of the optical axis onto the plane transversal respect to the \(z\) axis) is a walk-off angle of the extraordinary ray of frequency \(2\omega_0\) incident in the \(z\) direction and \(\Delta \beta_{\pm,z} = \beta_e^z(2\omega_0) \pm \beta_o^z(\omega_0)\) are the combinations of the extraordinary and ordinary inverse group velocities towards \(z\) axis for the frequencies \(2\omega_0\) and \(\omega_0\), respectively. We have also assumed that the \(z\)-th component of the ordinary inverse group velocity \(\beta_o^z(\omega)\) weakly depends on the direction of propagation.

### III. COUPLING PHOTON PAIRS INTO FIBERS

As we have seen, the full information about the modal structure of the generated photon pairs is contained in the function \(\Psi(s, i)\) - the probability amplitude of emission of a photon pair of certain frequencies towards given two directions. In the next step of our analysis, we will study how these photon pairs are coupled into single-mode optical fibers. We may think of such fibers as a pair of spatial filters letting through the photons occupying only selected transverse modes \(u(x, y, \omega)\) propagating through the fiber.

Consider a scheme in which photons generated in the crystal pass through interference filters and lenses coupling incident light into the single-mode fibers. Let the fibers be oriented in the horizontal plane at the angles \(\theta_s\) and \(\theta_i\) with respect to the \(z\) axis - see Figure III. The coupled modes can be easily characterized if we consider an inverse problem: what is the structure of a light mode emerging from the fiber? The output mode \(u(x, y, \omega)\) can be approximated by a Gaussian beam incident on the down-conversion crystal. Let us consider
a setup in which the coupled beam propagating in the horizontal plane (common with the pump beam) crosses the output face of the crystal of a distance $h$ from the point of intersection of the pump beam and the crystal surface - see Figure 1. In the simplest case, when the output beam is focused on the surface of the crystal, it is parameterized only by the angle of incidence $\theta_s$, the distance $h$ from the pump beam and the waist $w$. The Fourier-transformed mode function at the output surface of the crystal for the signal arm (and analogously, only with $-h$ instead of $h$ for the idler arm) has the form:

\[
\tilde{u}(s) \approx A_F(\omega_s) \frac{w}{\sqrt{2\pi}} \exp \left\{-\frac{w^2}{4} \left[ \left(k_{s,x} - \frac{\omega_s \theta_s}{c} \right)^2 + k_{s,y}^2 \right] - ih \left(k_{s,x} - \frac{\omega_s \theta_s}{c} \right) \right\},
\]

where $A_F(\omega) = \exp \left(-\frac{(\omega-\omega_0)^2}{2\sigma_F^2}\right)$ characterizes transmissivity of the interference filters characterized by their spectral bandwidth $\sigma_F$. The probability amplitude $\psi(\omega_s, \omega_i)$ for the photon pair of frequencies $\omega_s$ and $\omega_i$ to be coupled into the fibers reads:

\[
\psi(\omega_s, \omega_i) = \int dk_{s,x}dk_{s,y}dk_{i,x}dk_{i,y} \Psi(s,i) \tilde{u}^*(s) \tilde{u}^*(i).
\]

Notice, that the above expression involves integrating of the probability amplitudes, not probabilities, because photons with different wave vectors, but the same frequency are indistinguishable after coupling into the fibers.

To write down explicitly the amplitude $\psi(\omega_s, \omega_i)$ it is useful to introduce a set of effective parameters describing the process: a relative pump beam walk-off parameter $\Gamma$, a relative shift of the degenerate photons $\Theta$ and an effective crystal length $L$:

\[
\Gamma = \frac{L\gamma}{\sqrt{w^2 + 2w_P^2}} \\
\Theta = \frac{L\theta_0}{w} \quad \text{and} \quad L = \frac{L}{\sqrt{1 + \Theta^2/2 + \Gamma^2/2}}.
\]

These parameters appear naturally in the below expressions.

To proceed with the analysis, let us assume that the pump field is a Gaussian beam focused on the crystal so that the Rayleigh range is much longer than the crystal length:

\[
\tilde{E}_P(k_x, k_y, \omega) = A_P(\omega)w_P \exp \left[-\frac{w_P^2}{4} \left(k_x^2 + k_y^2 \right) \right],
\]

where the pump beam spectrum is $A_P(\omega) =$
exp \left( -\frac{(\omega - 2\omega_0)^2}{2\sigma_p^2} \right) \) and the fibers are oriented at the angles for which the degenerate photons are collected the most efficiently: \( \theta_s = \theta_0 \) and \( \theta_i = -\theta_0 \).

The explicit form of the probability amplitude modulus \(|\psi(\omega_s, \omega_i)|\) reads:

\[
|\psi(\omega_s, \omega_i)| = \frac{8\pi w_p L}{w^2 + 2w_p^2} \exp \left( -\frac{\Gamma^2}{2 + \Gamma^2} - \frac{1}{2} \frac{\omega^T \Omega \omega}{\sqrt{w^2 + 2w_p^2}} \right) \times \\
\exp \left[ -\frac{2(2 + \Gamma^2)}{2 + \Theta^2 + \Gamma^2} \left( h - \frac{L\theta'_0}{2 \left( 1 + \Gamma^2 / 2 \right)} \right)^2 \right],
\]

(11)

where \( \omega = (\omega_s - \omega_0, \omega_i - \omega_0) \) and particular components of the appearing spectrum matrix \( \Omega \) read:

\[
\Omega = \begin{pmatrix}
\Omega_{ss} & \Omega_{si} \\
\Omega_{is} & \Omega_{ii}
\end{pmatrix}
\]

(12)

\[
\Omega_{ss} = \frac{L^2}{8} \left( \Delta \beta_{z,-} + \frac{\theta_0 \theta'_0}{c} + \frac{\theta_0 \theta'_0}{\sqrt{2c} \sqrt{w^2 + 2w_p^2}} \right)^2 + \frac{w^2 w_p^2 \theta_0^2}{2c^2 (w^2 + 2w_p^2)} + \frac{1}{\sigma_p^2} + \frac{1}{\sigma_F^2}
\]

\[
\Omega_{ii} = \frac{L^2}{8} \left( \Delta \beta_{z,-} + \frac{\theta_0 \theta'_0}{c} - \frac{\theta_0 \theta'_0}{\sqrt{2c} \sqrt{w^2 + 2w_p^2}} \right)^2 + \frac{w^2 w_p^2 \theta_0^2}{2c^2 (w^2 + 2w_p^2)} + \frac{1}{\sigma_p^2} + \frac{1}{\sigma_F^2}
\]

\[
\Omega_{si} = \Omega_{is} = \frac{L^2}{8} \left( \Delta \beta_{z,-} + \frac{\theta_0 \theta'_0}{c} \right)^2 - \frac{\left( \theta_0 \theta'_0 \frac{w \Gamma / \Theta}{\sqrt{2c} \sqrt{w^2 + 2w_p^2}} \right)^2}{2c^2 (w^2 + 2w_p^2)} - \frac{w^2 w_p^2 \theta_0^2}{2c^2 (w^2 + 2w_p^2)} + \frac{1}{\sigma_p^2}.
\]

(13)

Let us notice, that the diagonal elements are not exactly the same. The symmetry of the coincidence spectrum is broken by the non-vertical orientation of the crystal’s optical axis. However, for small angles \( \theta_0 \) the spectrum matrix \( \Omega \) simplifies and the asymmetry of the spectrum disappears:

\[
\Omega_{ss} \approx \Omega_{ii} \approx \frac{L^2 \Delta \beta_{z,-}^2}{8} + \frac{w^2 w_p^2 \theta_0^2}{2c^2 (w^2 + 2w_p^2)} + \frac{1}{\sigma_p^2} + \frac{1}{\sigma_F^2}
\]

\[
\Omega_{si} = \Omega_{is} \approx \frac{L^2 \Delta \beta_{z,-}^2}{8} - \frac{w^2 w_p^2 \theta_0^2}{2c^2 (w^2 + 2w_p^2)} + \frac{1}{\sigma_p^2}.
\]

(14)
Non-separability of the function $\psi(\omega_s, \omega_i)$ is a signature of frequency entanglement, a feature that is undesirable if we are interested in polarization entanglement. To minimize this entanglement one has to adjust the parameters so that the off-diagonal elements of the spectrum matrix $\Omega$ become relatively small, i.e. when $\frac{\Omega_{si}}{\Omega_{ss}}$ reaches its minimum. Inspection of expressions (14) shows that for small angles $\theta_0$ and small mode waists $w$ and $w_P$ the separability can be accomplished only with the use of narrow-band interference filters. However, for large mode waists one can separate the spectrum by fulfilling the following condition:

$$\frac{w^2 w_P^2 \theta_0^2}{2c^2 (w^2 + 2w_P^2)} - \frac{L^2 \Delta \beta^2_{z, -}}{8} = \frac{1}{\sigma_p^2}.$$  \hspace{1cm} (15)

Notice that this condition does not depend on the interference filters’ bandwidth $\sigma_F$.

For concreteness, we have analyzed the coincidence spectrum (11) on the example of a BBO crystal cut for the degenerate frequency $\omega_0 = 780$nm and the emission angle $\theta_0 = 1.4^\circ$ (using the Sellmeier formulas for the indices of refraction [12] one can calculate that in this case $\Delta \beta_{z, -} = 2.06 \cdot 10^6 \frac{\text{rad}}{\text{m}}$), pump beam spectrum width 5nm to and the interference filter’s width 17nm. The results are shown on Figure 2 for several mode waist diameters. We see that when the beams are strongly focused, the spectrum $|\psi(\omega_s, \omega_i)|^2$ becomes inseparable, and the separability can be achieved only for unfocused beams. In the considered case the optimum waists are approximately $w = w_P \approx 2.5$mm.

Expression (11) can be used to maximize the coincidence probability (9) over all possible choices of the parameter $h$. The optimal shift $h$ of the coupled beam in respect to the pumping beam at $z = z_2$ is:

$$h = \frac{L\theta'_0 \cdot 1 + \Gamma^2}{2 \cdot 1 + \Gamma^2/2}.$$  \hspace{1cm} (16)

One can notice that in the absence of the walk-off effect affecting the propagation of the pumping beam (for $\Gamma = 0$), the optimal choice of $h$ involves crossing of the pumping mode and two fiber coupled modes exactly in the middle of the nonlinear medium. The modification comes from the fact that the pumping beam is transversely shifted due to the walk-off effect during the propagation in the crystal. The effects of the walk-off become more significant when all the modes are strongly focused on the crystal, because only the relative shift of the beam in comparison to its waist and the waist of the coupled modes is relevant.

To obtain the total probability $p$ of coupling any photon pair into the two fibers we need
to integrate the probability $|\psi(\omega_s, \omega_i)|^2$ over all possible frequencies. Notice that this time we integrate the probabilities, not their amplitudes as the photons of different frequencies are perfectly distinguishable:

$$p = \int d\omega_s d\omega_i |\psi(\omega_s, \omega_i)|^2.$$  \hfill (17)

The above integration can be also done analytically if we extend the lower limits to $-\infty$. In this case, for the optimum choice of $h$ we obtain a relatively simple expression for the probability $p$:

$$p = \frac{64\pi^3 w_P^2 L^2}{(w^2 + 2w_P^2)^2 \sqrt{\det \Omega}} \exp \left( -\frac{2\Gamma^2}{2 + \Gamma^2} \right).$$  \hfill (18)

Before we continue with a detailed analysis of the above result, let us stress once more, that it is valid only when the Rayleigh range of all the considered beams is much longer than the thickness of the crystal, i.e. for $\frac{1}{2}k_0 w^2 \gg L$, where $k_0$ is a characteristic wave-vector. This condition guarantees that the diffraction of the pump beam and generated photon pairs inside the crystal can be neglected.

Several conclusions about the efficiency of the coupling process can be drawn from the expression (18). First and the most obvious is that the overall probability $p$ does not depend on $\Delta \beta_{+, z}$. Coupling efficiency is also a decreasing function of the angle $\theta_0$ (and proportional $\theta_0'$), as it appears only in the denominator of the expression (18). This can be easily understood, since increasing $\theta_0$ (depending on the geometry of the crystal and its indices of refraction) decreases the spatial volume of an overlap between the pumping mode and coupled modes. Another clear conclusion is that the walk-off effect represented by $\Gamma$ and responsible for the transverse shift of the pump beam during the propagation through the crystal, also diminishes the coupling probability. The next relevant parameter, inverse group velocity difference of the extraordinary and ordinary beams propagating towards $z$: $\Delta \beta_{-, z}$ suppresses the down-conversion process. This effect is also straightforward, as $\Delta \beta_{-, z}$ is inversely proportional to the down-conversion spectrum width, so the larger $\Delta \beta_{-, z}$ is, the fewer frequencies are efficiently down-converted.

The next issue of practical interest is how the probability $p$ depends on the parameters adjustable for a given nonlinear material, namely the crystal length $L$ and the mode waists $w$ and $w_P$. Again, we have analyzed this issue on the same example of the BBO crystal.
For such setting we have calculated the pair coupling probability $p$ given by Eq. (18) and compared it with the result of a numerical calculation. In the figure we show the probability $p$ as a function of the crystal length $L$ for fixed $w = w_\text{P} = 100\mu\text{m}$ (solid line) and an analogous numerical result (dashed line). The only approximation that the numerical calculus involves is the expansion of the $\delta_{\pm}$ functions to the second order of the Taylor series. In the numerical case we keep the $\text{sinc}\ x$ function unchanged, as well as we do not assume that the Rayleigh range of the considered modes exceeds the crystal length.

It is clear from the Figure 3, that the analytic approximation breaks down when we consider thick crystals. This is when the argument of the $\text{sinc}\ x$ function becomes large, the wave tails of the $\text{sinc}\ x$ cease to be insignificant - and the Gaussian approximation is not legitimate anymore. However for thin crystals all the analytic approximations work well. We will show later that this region is the most interesting when one intends to maximize the coupling efficiency.

In the Figure 4 we have shown the coincidence probability as a function of the pump beam waist $w_\text{P}$. Solid lines again represent the results of the analytical approximation (18), dashed lines - the analogous numerical results. The plots demonstrate several cases: the bottom curves - for $L = 1\text{mm}$ and $w = 50\mu\text{m}$, the middle curves - for $L = 1\text{mm}$ and $w = 150\mu\text{m}$ and the top, bold curves show the probability $p$ as a function of $w_\text{P}$ with optimized values of $L$ and $w$.

The figures reveal a good agreement between the analytical theory and the numerical calculations. The discrepancies are more relevant only in the region of thick crystals or a very intense focusing of the beams.

To find the optimum choice of $L$ and $w$ we have numerically maximized the probability $p$ given by the expression (18) in the region depicted in Figure 4 and performed a linear fit to the obtained dependencies. As a result we have obtained:

$$w_\text{opt} \approx 3.32\mu\text{m} + 0.308w_\text{P},$$

$$L_\text{opt} \approx 167\mu\text{m} + 14.8w_\text{P}.$$  

(19)

The bolder top curves on the Figure 4 (analytical and numerical) involve an application of the above dependencies.
From the Figure 4, one can notice that the coupling process is the most effective for a very small pump beam mode waist. According to the numerical result depicted in this figure we find that the optimum waist $w_{P, \text{opt}} \approx 27 \mu\text{m}$ which, according to the expressions (13), corresponds to $w_{\text{opt}} = 12 \mu\text{m}$ and $L_{\text{opt}} = 0.57\text{mm}$. Rayleigh range for the optimally focused pumping beam equals 98mm. Therefore the condition of the large Rayleigh range in comparison with the crystal length is well preserved.

IV. DOUBLE CRYSTAL

So far we have considered the case of a single nonlinear crystal. However, to generate polarization entanglement with the use of type-I process in the scheme proposed by Kwiat et al. [13] one needs two crystals rotated in respect to each other by the angle of 90° around the $z$ axis. In this scheme a pump beam is linearly polarized so that the chance for a down-conversion in the first and in the second crystal are exactly the same.

If the first crystal is oriented as discussed in the previous sections (the optical axis of the crystal lying in a plane oriented at the angle 45° in respect to the horizontal plane), then we have two options concerning the orientation of the second crystal. It may be either horizontally or vertically flipped in respect to the geometry of the first crystal. Let us notice that in the latter case, when the crystals are symmetric in respect to the horizontal plane, the amplitude $\psi(\omega_s, \omega_i)$ given by equation (9) is exactly the same for both crystals. Therefore the distinguishability of the photon pairs from the both crystals may only be a consequence of either spatial separation of the emission cones or propagation effects: the photons from the first crystal propagate through the second one as the extraordinary rays and undergo a transversal shift due to the walk-off effect. Secondly, the pump beam polarization component down-converting in the second crystal and propagating through the first one is delayed in respect to the other polarization component. Consequently, photons generated in both crystals arrive to the fibers at different times. However, these two dominant propagation effects are related to linear phase shifts easily compensable with additional linear optics. The pump beam delay may be compensated by a birefringent material inserted in front of the double crystal, and the walk-off effect can be compensated by a pair of inverted down-conversion crystals placed in front of the fibers.

At this point our motivation concerning the diagonal orientation of the crystals becomes
clear. In the proposed configuration only the spatial separation of the photon emission cones can give rise to the photon distinguishability. However for thin crystals separated by a small distance, one can attain the perfect indistinguishability of photon pairs emerging from the both crystals and consequently maximize the polarization entanglement.

V. CONCLUSIONS

We have developed the approximate analytic theory of a type-I down-conversion to analyze the properties of the fiber coupled photon pairs. We have shown that in the regime of the weak focusing one can separate the coincidence spectrum, while for strong focusing one can optimize the coincidence efficiency. We have analytically derived the coincidence spectrum separability condition and the pair fiber-coupling probability and compared the result with the numerical calculation obtaining a good agreement.

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APPENDIX

Let us introduce a following coordinate system: $Z$ axis perpendicular to the crystal and transverse coordinates’ axes $X$ and $Y$, so that the optical axis lies in the plane $XZ$, at the angle $\alpha$ respect to the $Z$ axis. The dispersion relations for the ordinary and extraordinary beams have the form, respectively [10]:
\[ k_Z^0(k_X, k_Y, \omega) = \sqrt{\frac{\omega^2 n^0(\omega)^2}{c^2} - k_X^2 - k_Y^2} \]

\[ k_Z^0(k_X, k_Y, \omega) = \frac{k_X \sin \alpha \cos \alpha \left( 1 - \frac{n^e(\omega)^2}{n^o(\omega)^2} \right)}{\sin^2 \alpha + \frac{n^e(\omega)^2}{n^o(\omega)^2} \cos^2 \alpha} + \]

\[ \sqrt{\left( \frac{\omega^2 n^e(\omega)^2}{c^2} - k_Y^2 \right) \left( \sin^2 \alpha + \frac{n^e(\omega)^2}{n^o(\omega)^2} \cos^2 \alpha \right) - k_X^2 \frac{n^o(\omega)^2}{n^o(\omega)^2}}, \]

where \( n^o(\omega) \) and \( n^e(\omega) \) are the ordinary and extraordinary indices of refraction, respectively, given by the Sellmeier formulas [12].

From the above relations one can calculate the quantities appearing in Equation (7): walk-off angle \( \gamma \), group velocity combinations \( \Delta \beta_{\pm, z} \) and angle of propagation of the degenerate photons inside the crystal \( \theta_0' \) defined as:

\[ \gamma = \frac{\partial k_Z^0(0, 0, 2\omega_0)}{\partial k_X} \]

\[ \Delta \beta_{\pm, z} = \frac{\partial k_Z^e(0, 0, 2\omega_0)}{\partial \omega} \pm \frac{\partial k_Z^0(0, 0, \omega_0)}{\partial \omega} \]

\[ \theta_0' = \frac{\theta_0}{n^e(\omega_0)}. \]

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FIG. 1: Scheme of the setup: a pump beam $E_P(r, t)$ focused on the nonlinear crystal (length $L$) with a waist $w_P$ and the coupled modes with waists $w$ and a shift $h$ at the output face of the crystal in respect to the pump beam. The coupled modes $s$ and $i$ are oriented at the angles $\theta_s$ and $\theta_i$ with respect to the $z$ axis and the coupling lenses are preceded with interference filters.
FIG. 2: Coincidence spectrum $|\psi(\omega_s, \omega_i)|^2$ for several mode waist diameters. For strongly focused beams, the spectrum becomes inseparable, and the separability can be achieved only for unfocused beams.
FIG. 3: Relative coincidence probability as a function of the crystal length $L$ for fixed $w = w_p = 100\mu m$. Solid line - analytical approximation, dashed line - numerical result. The Rayleigh range for the pumping beam is equal to 13cm.
FIG. 4: Relative coincidence probability as a function of the pump beam waist $w_P$: solid line - analytical approximation, dashed line - numerical result for several cases. The bottom curves for $L = 1\text{mm}$ and $w = 50\mu\text{m}$, the middle curves for $L = 1\text{mm}$ and $w = 150\mu\text{m}$ and the top, bold curves for $L$ and $w$ chosen to maximize the result for each $w_P$. 