Shear viscosity in $\phi^4$ theory from an extended ladder resummation

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Abstract

We study shear viscosity in weakly coupled hot $\phi^4$ theory using the closed time path formalism (CTP) of real time finite temperature field theory. We show that the viscosity can be obtained as the integral of a three-point function. Although the three-point function has seven components in the CTP formalism, we show that the viscosity is given by a decoupled integral equation which involves only one retarded three-point function. Non-perturbative corrections to the bare one loop result can be obtained by solving a Schwinger-Dyson type integral equation for this vertex. This integral equation represents the resummation of an infinite series of ladder diagrams which all contribute to the leading order result. It can be shown that this integral equation has exactly the same form as the Boltzmann equation. We show that the integral equation for the viscosity can be reexpressed by writing the vertex as a combination of polarization tensors. An expression for this polarization tensor can be obtained by solving another Schwinger-Dyson type integral equation. This procedure results in an expression for the viscosity which represents a non-perturbative resummation of contributions to the viscosity which includes certain non-ladder graphs, as well as the usual ladders. We discuss the significance of this set of graphs. We show that these resummations can also be obtained by writing the viscosity as an integral equation involving a single four-point function. Finally, we show that when the viscosity is expressed in terms of a four-point function, it is possible to further extend the set of graphs included in the resummation by treating vertex and propagator corrections self-consistently. We discuss the significance of such a self-consistent resummation and show that the integral equations that are involved contain cancellations between vertex and propagator corrections. Lastly, we discuss the prospect of generalizing our technique to gauge theories.

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I. INTRODUCTION

The investigation of the transport properties of hot dense matter is a topic of great interest in the context of relativistic heavy-ion collisions and astrophysics [1,2]. The application of kinetic theories to the systems which are relativistic and quantum in nature involves difficulties which have been only partially overcome [3]. There are two methods that are used to calculate transport coefficients. The first method involves solving the transport equations. Transport coefficients are calculated from classical transport equations using the relaxation time approximation [4]. The second method is to use the Kubo formulae which relate transport coefficients to the low frequency, zero momentum limits of the spectral densities of the appropriate composite operators [5–8]. The Kubo formulae provide a framework for calculating transport coefficients starting from first principles using finite temperature quantum field theory.

The diagrammatic rules for computing spectral densities have been derived in both the real time formalism [9] and the imaginary time formalism [10] of finite temperature field theory. One-loop calculations of transport coefficients using Kubo formulae with effective propagators (including the single particle thermal lifetimes) have appeared previously [10,11]. However, these calculations are incomplete, even in the weak coupling limit, because they do not include an infinite class of planar ladder diagrams, which contribute at the same order as the one-loop diagrams. The resummation of this infinite series of ladder diagrams is highly nontrivial and has been performed by Jeon [10,12] using the imaginary time formalism. In the imaginary formalism, this resummation is extremely complicated. In this paper we perform the ladder resummation in a more efficient way using the closed time path (CTP) framework [13–18]. We give a compact derivation of the resummation of the infinite series of ladder diagrams that contribute to the shear viscosity. We can write the viscosity as an integral equation involving a single three-point function which can be obtained as the solution of a further decoupled linear integral equation. We show that these results are the same as those obtained in [10]. We will also show that the ladder resummation can be obtained by writing the viscosity as an integral equation involving a single four-point function, which is obtained, as before, from a decoupled linear integral equation. The four-point function is introduced because it allows us to obtain a special type of self-consistent resummation which includes additional non-ladder graphs. It is important to note that in the CTP formalism the three-point function has seven components and the four-point function has fifteen components, which means that the fact that these integral equations are decoupled is highly non-trivial.

We also study extended resummations which include non-ladder contributions. We derive expressions for the three- and four-point functions in terms of a difference of self energies. By writing the self energy as the solution of a decoupled linear integral equation we obtain an extended resummation. We discuss the importance of non-ladder contributions to this resummation.

Finally, we show that when the viscosity is expressed in terms of a four-point function, it is possible to perform a self-consistent resummation in which corrections to the vertex and the propagators are put on the same footing. We find there are cancellations between vertex and propagator corrections. For $\phi^4$ theory, these cancellations do not significantly simplify the form of the integral equation, however, for a gauge theory with a three-point interaction, like...
scalar QED, we expect that the resulting integral equation will be considerably simplified. We present the motivation for this argument. In the last section we discuss our conclusions. This paper is organized as follows.

I. Introduction

II. Notation

II-A. The CTP Formalism

II-B. Definition of Viscosity

III. Viscosity From Corrected Three-Point Functions

III-A. Notation

III-B. Integral equation for viscosity

IV. Viscosity From Corrected Four-Point Functions

IV-A. Notation

IV-B. Integral equation for viscosity

V. Ladder Resummations

V-A. Ladders from three-point vertices

V-A-1. The infrared divergence

V-A-2. The Boltzmann Equation

V-B. Ladders from four-point vertices

VI. Another Way to Resum Graphs

VI-A. The corrected three-point function

VI-A-1. The splitting relation for the three-point function

VI-A-2. An integral equation for II

VI-A-3. Contributions to the viscosity

VI-A-4. The infrared divergence

VI-B. The corrected four-point function

VI-B-1. The splitting relation for the four-point function

VI-B-2. An integral equation for II

VI-B-3. Contributions to the viscosity

VI-B-4. The infrared divergence

VII. Self Consistent Resummations

VIII. Conclusions

Appendix A

References

Figures
II. NOTATION

A. The CTP formalism

The CTP formalism of finite temperature field theory was introduced by Keldysh [13] and Schwinger [14]. Good reviews are found in [16,15,18]. The CTP contour has two branches: \( C_1 \) runs from negative infinity to positive infinity just above the real axis, and \( C_2 \) runs back from positive infinity to negative infinity just below the real axis. All fields can take values on either branch of this contour, which results in a doubling in the number of degrees of freedom. We will consider \( \phi^4 \) theory. The Lagrangian is given by,

\[
L = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4
\]

At finite temperature, when hard thermal loop corrections are included, the mass term is usually dropped relative to the real part of the hard thermal loop self energy. The scalar propagator is given by,

\[
D(X - Y) = -i \langle T_c \phi(X)\phi(Y) \rangle
\]

where \( T_c \) is the operator that time orders along the CTP contour. We also use the notation \( X = (t, \vec{x}) \) and \( P = (p_0, \vec{p}) \). The propagator has \( 2^2 = 4 \) components and can be written as a \( 2 \times 2 \) matrix

\[
D = \begin{pmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{pmatrix}
\]

with

\[
\begin{align*}
D_{11}(X - Y) &= -i \langle T(\phi(X)\phi(Y)) \rangle, \\
D_{12}(X - Y) &= -i \langle \phi(Y)\phi(X) \rangle, \\
D_{21}(X - Y) &= -i \langle \phi(X)\phi(Y) \rangle, \\
D_{22}(X - Y) &= -i \langle \tilde{T}(\phi(X)\phi(Y)) \rangle,
\end{align*}
\]

where \( T \) is the usual time ordering operator, and \( \tilde{T} \) is the anti-chronological time ordering operator. These four components satisfy,

\[
D_{11} - D_{12} - D_{21} + D_{22} = 0
\]

as a consequence of the identity \( \theta(x) + \theta(-x) = 1 \).

It is more useful to write the propagator in terms of the three functions

\[
\begin{align*}
D_R &= D_{11} - D_{12}, \\
D_A &= D_{11} - D_{21}, \\
D_F &= D_{11} + D_{22}.
\end{align*}
\]

\( D_R \) and \( D_A \) are the usual retarded and advanced propagators, satisfying

\[
D_R \text{ and } D_A \text{ are the usual retarded and advanced propagators, satisfying}
\]
\[ D_R(X - Y) - D_A(X - Y) = -i \langle [\phi(X), \phi(Y)] \rangle, \]  
\[ (5) \]

and \( D_F \) is the symmetric combination

\[ D_F(X - Y) = -i \langle \{ \phi(X), \phi(Y) \} \rangle, \]  
\[ (6) \]

which satisfies the KMS condition. In momentum space

\[
D_{R/A}(P) = \frac{1}{(p_0 \pm i \epsilon)^2 - \vec{p}^2 - m^2}, \\
D_F(P) = (1 + 2n(p_0))(D_R(P) - D_A(P)), 
\]  
\[ (7) \]

where \( n(p_0) \) is the thermal Bose-Einstein distribution,

\[
n(p_0) = \frac{1}{e^{\beta p_0} - 1}, \quad n(-p_0) = -(1 + n(p_0)), \quad N_p = 1 + 2n(p_0) \]  
\[ (8) \]

The propagator can be rewritten as an outer product of two component column vectors:

\[
2D = D_R \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + D_A \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + D_F \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. 
\]  
\[ (9) \]

Using the KMS condition (7) this expression can be rewritten,

\[
D(p) = D_R(p) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 + n(p_0) \\ n(p_0) \end{pmatrix} - D_A(p) \begin{pmatrix} n(p_0) \\ 1 + n(p_0) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. 
\]  
\[ (10) \]

We can extract the 1PI two-point function, or self energy, by removing external legs in the usual way. We find,

\[
\Pi_R = \Pi_{11} + \Pi_{12} \\
\Pi_A = \Pi_{11} + \Pi_{21} \\
\Pi_F = \Pi_{11} + \Pi_{22} 
\]  
\[ (11) \]

where \( \Pi_R \) and \( \Pi_A \) are the usual retarded and advanced self energies. The symmetric component \( \Pi_F \) satisfies the KMS condition,

\[
\Pi_F = (1 + 2n(p_0))(\Pi_R - \Pi_A) 
\]
and the four CTP components satisfy the constraint,

\[
\Pi_{11} + \Pi_{12} + \Pi_{21} + \Pi_{22} = 0 
\]
B. Definition of Viscosity

If a system is slightly perturbed away from equilibrium, fluctuations will occur. The response to these fluctuations will be characterized by transport coefficients, for instance shear and bulk viscosities in a system with no conserved particle number. The shear viscosity characterizes the diffusive relaxation of transverse momentum density fluctuations and is proportional to the two body elastic scattering mean free path. The bulk viscosity characterizes the departure from equilibrium during a uniform expansion and is proportional to the mean free path for particle number changing processes. Since the bulk viscosity involves particle number changing processes, its calculation is more difficult. However, at high temperature in a weakly coupled system, the shear viscosity is much larger than the bulk viscosity. For simplicity we will consider only the shear viscosity in this paper.

Using standard linear response theory one may express the shear viscosity in terms of the stress tensor-stress tensor correlation function. One obtains the Kubo formula \[ \eta = \frac{\beta}{20} \lim_{q_0 \to 0} \lim_{\vec{q} \to 0} \sigma(Q) \]

where,

\[ \sigma(Q) = \int dX e^{iQ \cdot (X-Y)} \langle \pi_{lm}(X) \pi^{lm}(Y) \rangle \]

and

\[ \pi_{lm}(X) = \nabla_l \phi \nabla_m \phi - \frac{1}{3} \delta_{lm} (\nabla \phi)^2 \]

in $\phi^4$ theory. We substitute (14) into (13) and use Wick’s theorem and (2). We obtain,

\[ \sigma(Q) = 2 \int dK I_{lm}(k, k) D_{12}(K) D_{21}(K + Q) I_{lm}(k, k) \]

This result is shown diagrammatically in Fig. [1]. The factor $I_{lm}$ is associated with the joining of two propagators and has the form,

\[ I_{lm}(k, k) = k_l k_m - \frac{1}{3} \delta_{lm} k^2 \]

where we have dropped the $q$ dependence because we intend to take $q$ to zero at the end of the calculation. In the factors $I_{lm}$ no difficulties will arise from dropping the $q$ dependence at the beginning of the calculation. If we attempt to work perturbatively, the lowest order contribution to (13) is the one loop graph, as shown in Fig. [1]. For calculational purposes, it is easier to work with the combination of one loop diagrams shown in Fig. [2]. The first is $\sigma$ and the second we call $\sigma'$. The difference of these graphs is related to $\sigma$ as follows. Using the KMS condition [4] in the form,

\[ (1 + n(p_0)) D_{12}(P) = n(p_0) D_{21}(P) \]

it is straightforward to show that,
\[ n(q_0)\sigma = (1 + n(q_0))\sigma' \] (17)

which means that we can obtain \( \sigma \) from the difference \( \sigma - \sigma' \):

\[ \sigma = (1 + n(q_0))(\sigma - \sigma') \] (18)

It has been known for some time that there are graphs that are higher order in the loop expansion than the one loop graph which, nevertheless, contribute to the same order in perturbation theory. The purpose of this paper is to identify these graphs, and to develop techniques for resumming them.

**III. VISCOSITY FROM CORRECTED THREE-POINT FUNCTIONS**

**A. Notation**

We define the bare three-point vertex as,

\[ \Gamma_{cb}^{(0)lm} = 1_{c}i\gamma_{b}^{3}\tau_{3}b_{a}\delta_{ab}\delta_{cb}I^{lm}(k,k) \] (19)

where \( 1 \) is the two by two identity matrix and \( \tau_3 \) is the third Pauli matrix. Corrections to this vertex are given by the 1PI part of the three-point function,

\[ \Gamma_{cba}^{Clm}(Z,X,Y) = \langle T_{c}\phi_{c}(Z)\tau_{3}^{b}\pi^{lm}(X)\phi_{a}(Y) \rangle \] (20)

In the real time formalism, the three-point function has \( 2^3 = 8 \) components, since each of the fields can take values on either branch of the contour. Only seven of these components are independent because of the identity

\[ \sum_{a=1}^{2} \sum_{b=1}^{2} \sum_{c=1}^{2} (-1)^{a+b+c-3} \Gamma_{abc}^{Clm} = 0. \] (21)

The corrected three-point vertex is obtained by truncating external legs to obtain the 1PI part of the three-point function. These 1PI vertices obey the constraint,

\[ \sum_{a=1}^{2} \sum_{b=1}^{2} \sum_{c=1}^{2} \Gamma_{abc}^{lm} = 0 \]

We define the combinations,

\[ \begin{align*}
\Gamma_{R}^{lm} &= \Gamma_{111}^{lm} + \Gamma_{112}^{lm} + \Gamma_{211}^{lm} + \Gamma_{212}^{lm} \\
\Gamma_{R}^{lm} &= \Gamma_{111}^{lm} + \Gamma_{112}^{lm} + \Gamma_{211}^{lm} + \Gamma_{212}^{lm} \\
\Gamma_{R}^{lm} &= \Gamma_{111}^{lm} + \Gamma_{112}^{lm} + \Gamma_{211}^{lm} + \Gamma_{212}^{lm} \\
\Gamma_{F}^{lm} &= \Gamma_{111}^{lm} + \Gamma_{112}^{lm} + \Gamma_{211}^{lm} + \Gamma_{212}^{lm} \\
\Gamma_{F}^{lm} &= \Gamma_{111}^{lm} + \Gamma_{112}^{lm} + \Gamma_{211}^{lm} + \Gamma_{212}^{lm} \\
\Gamma_{F}^{lm} &= \Gamma_{111}^{lm} + \Gamma_{112}^{lm} + \Gamma_{211}^{lm} + \Gamma_{212}^{lm} \\
\Gamma_{E}^{lm} &= \Gamma_{111}^{lm} + \Gamma_{112}^{lm} + \Gamma_{211}^{lm} + \Gamma_{212}^{lm}. \end{align*} \] (22)
The first three are the retarded three-point functions; $\Gamma^lm_{Ri}$ is retarded with respect to the first leg, $\Gamma^lm_{R}$ is retarded with respect to the second leg, and $\Gamma^lm_{Ro}$ is retarded with respect to the third leg. The other four vertices are related to the retarded ones through the KMS conditions \[21,22\],

\[
\begin{align*}
\Gamma^lm_F &= N_1(\Gamma^*lm_{Ro} - \Gamma^lm_{Ri}) + N_2(\Gamma^*lm_{Ro} - \Gamma^lm_{R}) + N_3(\Gamma^*lm_{Ro} - \Gamma^lm_{R}), \\
\Gamma^lm_{Fi} &= N_2(\Gamma^*lm_{Ro} - \Gamma^lm_{Ri}) + N_3(\Gamma^*lm_{Ro} - \Gamma^lm_{R}) + N_1N_3(\Gamma^*lm_{Ro} + \Gamma^*lm_{R}) \\
&+ N_1N_3(\Gamma^*lm_{Ro} + \Gamma^*lm_{R}) + N_2N_3(\Gamma^*lm_{Ro} + \Gamma^*lm_{R}), \\
\Gamma^lm_{Fo} &= N_2(\Gamma^*lm_{Ro} - \Gamma^lm_{Ri}) + N_3(\Gamma^*lm_{Ro} - \Gamma^lm_{R}) + N_1N_3(\Gamma^*lm_{Ro} + \Gamma^*lm_{R}) \\
&+ N_1N_3(\Gamma^*lm_{Ro} + \Gamma^*lm_{R}) + N_2N_3(\Gamma^*lm_{Ro} + \Gamma^*lm_{R}), \\
\Gamma^lm_E &= \Gamma^*lm_{Ri} + \Gamma^*lm_{Ro} + N_2N_3(\Gamma^*lm_{Ro} + \Gamma^*lm_{R}) \\
&+ N_1N_3(\Gamma^*lm_{Ro} + \Gamma^*lm_{R}) + N_2N_3(\Gamma^*lm_{Ro} + \Gamma^*lm_{R}),
\end{align*}
\]

The vertex can be written as the outer product of three column vectors:

\[
4 \Gamma^lm = \Gamma^lm_{Ri} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \Gamma^lm_{Ro} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \Gamma^lm_{Ri} F \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \Gamma^lm_{Ro} F \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \Gamma^lm_E \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}.
\]

B. Integral equation for viscosity

In order to perform a resummation we look at an integral equation of the form represented in Fig. [3]. In these diagrams, the vertex with the solid box is a full vertex whose form will be specified later. As we will see, it is the form of this vertex that determines which diagrams are being resummed. The four diagrams with corrected vertices in Fig. [3] are necessary in order to obtain a result for $\sigma$ that is pure real. Using corrected vertices on the right and left sides alternately is necessary to correctly take cuts of the vertex itself. Notice that diagrams with corrected vertices include an extra factor of $\tau_3$ which carries the index that corresponds to the momentum $Q$. The reason is as follows. The places where lines join in the bare one loop diagrams are not vertices and do not carry any factors, numerical or otherwise (except for the kinematic factor $I_{lm}$). In the diagrams with corrected vertices, the corrected vertex is given by \[21\]. The factor of $\tau_3$ which is included in the definition of the vertex must be compensated for by an extra $\tau_3$ which carries the index that corresponds to the momentum $Q$. This extra factor of $\tau_3$ ensures that the diagram with the full vertex would reduce to the bare one loop diagram in the tree limit, which means that the combinatorics are correct in the integral equation that derives from Fig. [3]. The integral equation corresponding to the diagrams in Fig. [3a] is of the form,

\[
\lambda^{(1)}_{ab} = 2 \int dK I_{lm}(k, k)D_{ab}(K + Q)D_{ba}(K)I_{lm}(k, k)
\]

Using the decomposition \[9\] we obtain,
\[(\sigma - \sigma')(^{(1)}1) = (\lambda_{21} - \lambda_{12})^{(1)}\]
\[= \int dK I_{lm}(k,k)[f_{k+q}(a_k - r_k) - f_k(a_{k+q} - r_{k+q})]I_{lm}(k,k)\]

where we define \(a_k = D_A(K), r_k = D_R(K), f_k = D_F(K)\) etc. Using the KMS condition (7) we obtain,
\[(\sigma - \sigma')(^{(1)}1) = -\int dK I_{lm}(k,k)(N_k - N_{k+q})(a_k r_{k+q} + h.c.)I_{lm}(k,k)\]

In obtaining this result we have used the fact that we will eventually take the limit \(Q \to 0\) to obtain the viscosity (12). This limit gives rise to what is known as the pinch effect: terms with a product of factors \(a_k r_{k+q}\) can be dropped. (The large terms occur when the contour is “pinched” between the poles with products of propagators \(a_k a_{k+q}\) or \(r_k r_{k+q}\). Thus terms proportional to \(a_k a_{k+q}\) or \(r_k r_{k+q}\) can be dropped. (The large terms occur when the contour is “pinched” between the poles of the two propagators, which gives rise to a factor in the denominator that is proportional to the imaginary part of these propagators).

The diagrams in Fig. [3b] correspond to an expression of the form,

\[2\lambda_{ab}^{(2)} = \int dK \sigma_3^a \Gamma_{cd}^{lm}(K, Q, -K - Q) D_{db}(K + Q) D_{bc}(K) I_{lm}^k(k,k)\]
\[+ \int dK I_{lm}(k,k)D_{ac}(K + Q) D_{da}(K)\Gamma_{db}^{lm}(-K, -Q, K + Q) r_3^b\]

Using the decompositions (9) and (24) and the KMS condition (7) we obtain,
\[(\sigma - \sigma')^{(2)} = (\lambda_{21} - \lambda_{12})^{(2)}\]
\[= \frac{1}{4} \int dK \left\{ a_k r_{k+q} [(N_{k+q} - N_k)\Gamma_{R}^{lm} + \bar{N}_{k+q} \Gamma_{R_i}^{lm} - N_k \Gamma_{R_0}^{lm} - \Gamma_{F}^{lm}] I_{lm}(k,k)\right\}
\[\hspace{1cm} - r_k a_{k+q} \Gamma_{lm}^{k}(k,k) [(N_k - N_{k+q})\Gamma_{R}^{lm(-)} + \bar{N}_k \Gamma_{R_0}^{lm(-)} - \Gamma_{k+q} \Gamma_{R_i}^{lm(-)} + \Gamma_{F}^{lm(-)}]\}

where \(\Gamma_x := \Gamma_x(K, Q, -K - Q)\) and \(\Gamma_{x}^{(-)} := \Gamma_x(-K, -Q, K + Q)\) and we have again used the pinch limit. We can further simplify this result by using the KMS condition (23) with the relations \(\Gamma_{R_x}^{lm(-)} = [\Gamma_{R_x}^{lm}]^*\) and \(\Gamma_{F_x}^{lm(-)} = -[\Gamma_{F_x}^{lm}]^*\). We obtain,
\[(\sigma - \sigma')^{(2)} = \frac{1}{2} \int dK \left\{ \Gamma_{R}^{lm}(K, Q, -K - Q) a_k r_{k+q} + h.c.\right\} (N_{k+q} - N_k) I_{lm}(k,k)\]

Combining (27) and (30) we obtain,
\[\sigma - \sigma' = \int dK \left\{ (I_{lm}(k,k) + \frac{1}{2} \Gamma_{R}^{lm}(K, Q, -K - Q)) a_k r_{k+q} + h.c.\right\} (N_{k+q} - N_k) I_{lm}(k,k)\]

Notice that this expression depends only on the vertex \(\Gamma_{lm}^{R}\).

**IV. VISCOSITY FROM CORRECTED FOUR POINT FUNCTIONS**

It is also possible to obtain a ladder resummation by writing the viscosity as an integral involving a corrected four-point function. This approach has advantages over the three-point function because it can be used to generate self consistent resumptions. This point will be explained in detail below.
A. Notation

The connected four-point function is given by the contour ordered expectation value,
\[ M_{abcd}(X, Y, Z, W) = \langle T_c \phi_a(X) \phi_b(Y) \phi_c(Z) \phi_d(W) \rangle \]

The 1PI four-point function is obtained by truncating external legs and forms a 16 component tensor which we can write as the outer product of four two component vectors,
\[ M = \begin{pmatrix} x \\ y \\ u \\ v \\ w \\ z \\ s \\ t \end{pmatrix} \]

The retarded 1PI four-point functions are given by
\[ M_{R1} = M_{1111} + M_{1112} + M_{1121} + M_{1211} + M_{1212} + M_{1221} + M_{1222} 
= \frac{1}{2}(x - y)(u + v)(w + z)(s + t) \]
\[ M_{R2} = M_{1111} + M_{1112} + M_{1121} + M_{2111} + M_{1211} + M_{2121} + M_{2122} 
= \frac{1}{2}(x + y)(u - v)(w + z)(s + t) \]
\[ M_{R3} = M_{1111} + M_{1112} + M_{2111} + M_{1211} + M_{1212} + M_{2211} + M_{2212} 
= \frac{1}{2}(x + y)(u + v)(w - z)(s + t) \]
\[ M_{R4} = M_{1111} + M_{2111} + M_{1121} + M_{1211} + M_{2121} + M_{2211} + M_{2221} 
= \frac{1}{2}(x + y)(u + v)(w + z)(s - t) \]

where we have used the relation
\[ \sum_{a,b,c,d=1}^2 M_{abcd} = 0. \] (33)

The other combinations are,
\[ M_A = \frac{1}{2}(x + y)(u + v)(w - z)(s - t) \]
\[ M_B = \frac{1}{2}(x - y)(u + v)(w - z)(s + t) \]
\[ M_C = \frac{1}{2}(x + y)(u - v)(w - z)(s + t) \]
\[ M_D = \frac{1}{2}(x + y)(u - v)(w + z)(s - t) \]
\[ M_E = \frac{1}{2}(x - y)(u - v)(w + z)(s + t) \]
\[ M_F = \frac{1}{2}(x - y)(u + v)(w + z)(s - t) \]
\[ M_\alpha = \frac{1}{2}(x + y)(u - v)(w - z)(s - t) \] (34)
$$M_{\beta} = \frac{1}{2}(x - y)(u + v)(w - z)(s - t)$$
$$M_{\gamma} = \frac{1}{2}(x - y)(u - v)(w + z)(s - t)$$
$$M_{\delta} = \frac{1}{2}(x - y)(u - v)(w - z)(s + t)$$
$$M_{T} = \frac{1}{2}(x - y)(u - v)(w - z)(s - t)$$

We use the decomposition of the four-point vertex:

$$8M = M_{R1} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + M_{R2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + M_{R3} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + M_{R4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + M_{A} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + M_{B} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} + M_{C} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + M_{D} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + M_{E} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + M_{F} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + M_{G} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + M_{H} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} + M_{I} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$+ M_{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + M_{3} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + M_{4} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$+ M_{4} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

B. Integral equation for viscosity

The integral equation for the viscosity is obtained from Fig [4]. The one loop diagrams give, as before (27). The second diagrams are of the form,

$$\lambda_{ab} = \int dP dK I_{lm}(p, p) I_{lm}(k, k)$$
$$D_{ac}(P + Q) D_{ab}(K + Q) D_{bf}(K) D_{ef}(P) M_{cd ef}(P + Q, -K - Q, -P, K)$$

(36)

We want to extract the combination \(\lambda_{21} - \lambda_{12} = (\sigma - \sigma')(2)\). The result is,

$$\begin{align*}
(\sigma - \sigma')(2) &= \frac{1}{4} \int dP dK I_{lm}(p, p) I_{lm}(k, k) \left[ M_{R1} (r_{p+q} f_{p} (a_{k} f_{k+q} f_{k} r_{k+q} + f_{k} f_{k+q} r_{p+q} r_{p}) \\
- M_{R2} (f_{k} a_{k+q} (p_{f} f_{p+q} + f_{p+q} r_{p})) + M_{R3} (f_{p+q} a_{p} (a_{k} f_{k+q} + f_{k} r_{k+q})) \\
+ M_{R4} (f_{p+q} r_{k} (f_{p} r_{k+q} - f_{k} r_{q+q} r_{p}) - r_{k} f_{k+q} a_{p+q} f_{p}) \\
+ M_{B} (r_{p+q} a_{p} (a_{k} f_{k+q} + f_{k} r_{k+q})) - M_{D} (r_{k} a_{k+q} (r_{p} f_{p+q} + f_{p} a_{p+q})) \right]
\end{align*}$$

(37)

where we have suppressed the momentum arguments of the four-point functions.

To simplify this result, we take the pinch limit and use the KMS conditions. These conditions for the four-point functions are analogous to the KMS conditions for the three-point functions (23) and are derived in [22]. The only one that we need is,
\[(N_k - N_{k+q})[M_B - N_p M_{R1} + N_{p+q} M_{R3}] = (N_p - N_{p+q})[M_D^* + N_k M_{R2} - N_{k+q} M_{R4}^*] \quad (38)\]

Substituting (38) into (37) we get,
\[
(\sigma - \sigma')^{(2)} = -\frac{1}{4} \int dP\ dK\ I_{lm}(p, p) I_{lm}(k, k) (N_k - N_{k+q}) \\
\left( (M_B - N_p M_{R1} + N_{p+q} M_{R3}) r_{k+q} a_k r_{p+q} a_p + h.c. \right) \quad (39)
\]

Making the definition
\[
\tilde{M}(P + Q, -K - Q, -P, K) = M_B(P + Q, -K - Q, -P, K) \\
- N_p M_{R1}(P + Q, -K - Q, -P, K) + N_{p+q} M_{R3}(P + Q, -K - Q, -P, K) \quad (40)
\]
we can rewrite to obtain,
\[
(\sigma - \sigma')^{(2)} = -\frac{1}{4} \int dP\ dK\ I_{lm}(p, p) I_{lm}(k, k) (N_k - N_{k+q}) \left[ \tilde{M} r_{k+q} a_k r_{p+q} a_p + h.c. \right] \quad (41)
\]

Combining (27) and (41) we obtain from Fig. [4],
\[
\sigma - \sigma' = -\int dK (N_k - N_{k+q}) I_{lm}(k, k) \{ a_k r_{k+q} \\
I_{lm}(k, k) + \frac{1}{2} \int dP I_{lm}(p, p) \tilde{M} a_p r_{p+q} \} + h.c. \quad (42)
\]

V. LADDER RESUMMATIONS

An argument similar to the pinch argument described in the section III-B gives rise to the conclusion that ladder graphs of the form shown in Fig. [5] contribute to the viscosity to the same order in perturbation theory as the bare one loop graph, and thus need to be resummed [10,12]. This effect occurs for the following reason. It appears that the ladder graphs are suppressed relative to the one loop graph by extra powers of the coupling, which come from the extra vertex factors that one obtains when one adds rungs (vertical lines). However, these extra factors of the coupling are compensated for by a kinematical factor \(\sim 1/Q\) from the pinch effect, which becomes large when \(Q\) becomes small. One can also argue that the ladder graphs are larger than other graphs of the same order in the loop expansion. This conclusion is based on the following argument. We compare the graphs shown in Fig. [6a] and [6b]. The ladder graph shown in Fig. [6a] contains three pairs of propagators of the form \(D_x D_{x+q}\) which will give pinch contributions in the limit that \(Q \rightarrow 0\). The graph in Fig. [6b] contains only two such pairs; the third pair is replaced by a pair of the form \(D_{x+y} D_{x+q}\). The momentum variable \(y\) is an internal momentum which must be integrated over and thus there is a region of phase space for which it will be small. However, one expects that the measure will also be small in this region of phase space, and therefore that the contribution of all non-ladder graphs will be suppressed. This argument will be discussed in more detail in the next section. In this section, we will show that we can resum the ladder graphs by solving an integral equation for the corrected vertex.
A. Ladders from three-point vertices

The ladder graphs are resummed by solving an integral equation for the three-point vertex. We can obtain $\Gamma_{lm}^R$ self-consistently from the Schwinger-Dyson equation that corresponds to the diagrams in Fig. [7]. This expression for $\Gamma_{lm}^R$ can then be substituted into the integral equation (31) from which we obtain the viscosity. The diagrams in Fig. [7] give,

\[
\Gamma_{lm}^R(K, Q, -K - Q) = \int dP dR \tau_3^b \tau_3^c I_{lm}(p, p) D_{ca}(R) D_{ac}(K + R - P) D_{bc}(P + Q) D_{ab}(P) + \frac{1}{2} \int dP dR \Gamma_{lm}^{\text{abc}}(P, Q, -P - Q) D_{ad}(P) D_{ec}(P + Q) D_{ac}(R + K - P) \tau_3^c D_{ca}(R) \tau_3^a
\]

We expand this expression by using the decompositions (9) and (24) and performing the contractions. In the limit $Q \to 0$ we make use of the pinch limit to obtain,

\[
\Gamma_{lm}^R(K, Q, -K - Q) = -\lambda^2 \int dP dR (I_{lm}(p, p) + \frac{1}{2} \Gamma_{lm}^{\text{abc}}(P, Q, -P - Q)) \left[ f_p r^a(p') + a_p r^a + a_p r^a + f_r r^a \right]
\]

where we have defined $r^a = D_R(P')$; $P' = R + K - P$, etc. The key point is that this integral equation is decoupled: there are no vertex components other than $\Gamma_{lm}^R$ on the right hand side. We introduce the notation,

\[
\delta(r, k) = a_r r_k + r_r a_k + f_r f_k \\
\phi_1(r, k) = f_r a_k + r_r f_k \\
\phi_2(r, k) = f_r r_k + a_r f_k
\]

which allows us to write,

\[
\Gamma_{lm}^R(K, Q, -K - Q) = -\lambda^2 \int dP dR (I_{lm}(p, p) + \frac{1}{2} \Gamma_{lm}^{\text{abc}}(P, Q, -P - Q)) \left[ f_p r^a \phi_1(r, r + k - p) + a_p r^a \phi_1(r, r + k - p) + a_p f_p^q \phi_2(r, r + k - p) \right]
\]

In conclusion then, we can obtain a resummation of ladder graphs by solving (46) for $\Gamma_{lm}^R$ and substituting into the expression (31) from which we obtain the viscosity.

1. The Infrared Divergence

It is easy to see that when the limit $Q \to 0$ is taken, the resummation of ladder graphs is divergent. In fact, this result can be seen immediately from (31): the difference $\sigma - \sigma'$ is of order $Q^0$ as $Q \to 0$ and therefore the viscosity, as obtained from (12) and (18), is divergent. This problem is customarily remedied by replacing the bare propagators on the rails (the horizontal lines) with hard thermal loop (HTL) effective propagators [19,23,24]. We write the HTL propagators as

\[
D^\ast(P) = \frac{1}{P^2 - \Sigma(P)}
\]
where $\Sigma(P)$ is the HTL contribution to the self energy. Note that the superscript $^*$ indicates a corrected propagator (and not complex conjugation). It is easy to see that this replacement regulates the divergence, but does not invalidate the conclusion that all of the ladder graphs need to be resummed. When resummed propagators are used, products of propagators of the form $r_{k+q}^*a_k^*$ and $r_k^*a_{k+q}^*$ contribute factors on the order of one over the imaginary part of the HTL self energy (instead of the factors $\sim 1/Q$ that occur for bare propagators). The imaginary part of the HTL self energy is of order $\lambda^2$, and such a factor arises for each pair of rails. These factors will compensate the extra factors of $\lambda^2$ in the numerator that accompany the addition of each rung. Since each additional rung is accompanied by an additional pair of rails, the result is that all of the ladder graphs have a piece that contributes to leading order.

To resum ladder diagrams with HTL effective propagators, we rewrite (31) with HTL propagators,

$$\sigma - \sigma' \equiv - \int dK (N_k - N_{k+q}) [\langle I_{lm}(k, k) + \frac{1}{2} \Gamma_{lm}^{\text{in}}(K, Q, -K - Q) a_k^* r_{k+q}^* + h.c.] I_{lm}(k, k)$$

and replace the vertex with the solution to the integral equation (compare (46)),

$$\Gamma_{lm}^{\text{in}}(K, Q, -K - Q) = -\frac{\lambda^2}{4} \int dP dR [I_{lm}(p, p) + \frac{1}{2} \Gamma_{lm}^{\text{in}}(P, Q, -P - Q)]$$

$$[f_p^* r_{p+q}^* f_1^* (r, r + k - p) + a_p^* r_{p+q}^* \delta^* (r, r + k - p) + a_p^* f_{p+q}^* \phi_2^* (r, r + k - p)]$$

where $\phi_1^*$, $\phi_2^*$, and $\delta^*$ are obtained from (45) with bare propagators replaced by HTL propagators. When the solution to (49) is substituted into (48) we find that $\sigma - \sigma'$ is of order $Q$, and the viscosity, as obtained from (12) and (18) is finite.

2. The Boltzmann Equation

If we use the HTL self energy to two loop order and include a thermal mass and a thermal width, these results agree with results obtained previously in the imaginary time formalism [12]. In order to sum all planar ladder diagrams, Jeon has constructed a linear integral equation involving a $4 \times 4$ matrix valued kernel for the effective vertex, which is a four component column vector. The four components represent four different “cuts” in the imaginary time formalism. In the limit $Q \to 0$, the $4 \times 4$ matrix valued kernel reduces to a rank one matrix and the original matrix integral equation reduces to a decoupled single component equation for the reduced vertex, which is a complicated linear combination of the four different cuts of the vertex. We will show below that this result is considerably more straightforward to derive in the real time formalism.

We define a new vertex

$$\tilde{\Gamma}_{lm} = I_{lm} + \frac{1}{2} \Gamma_{lm}^{\text{in}}$$

In terms of this definition (18) becomes,

$$\sigma - \sigma' \simeq - \int dK (N_k - N_{k+q}) [\tilde{\Gamma}_{lm}^{\text{in}}(K, Q, -K - Q) a_k^* r_{k+q}^* + h.c.] I_{lm}(k, k)$$
The integral equation for $\bar{\Gamma}_{lm}$ is obtained by adding $I_{lm}$ to both sides of one half of (49) to obtain,

$$\bar{\Gamma}_{lm}^{R}(K, Q, -K - Q) = I_{lm}(k, k) - \frac{\lambda^2}{8} \int dP dR \bar{\Gamma}_{lm}^{R}(P, Q, -P - Q)$$

$$[f_{p}r_{p+q}^{*}(f_{r}a_{r}^{*} + r_{r}^{*}f_{r}^{*}) + a_{p}^{*}r_{p+q}(a_{p}^{*}r_{r}^{*} + r_{p}^{*}f_{r}^{*}) + a_{p}^{*}f_{p+q}(f_{r}r_{r}^{*} + a_{r}^{*}f_{r}^{*})]$$

where we have used the definitions (45).

We define the spectral density function as,

$$\rho_{p} = i(r_{p}^{*} - a_{p}^{*})$$

In the limit that $Q$ is small the pinching terms dominate and we can rewrite the second term in (51) as,

$$-\frac{\lambda^2}{8} \int dP dR \bar{\Gamma}_{lm}^{R}(P, Q, -P - Q)$$

$$a_{p}^{*}r_{p+q}^{*}\left[(r_{r}^{*}a_{r}^{*} + a_{r}^{*}r_{r}^{*} + f_{r}^{*}f_{r}^{*}) - N_{p}(f_{r}a_{r}^{*} + r_{r}^{*}f_{r}^{*}) + N_{p+q}(f_{r}r_{r}^{*} + a_{r}^{*}f_{r}^{*})\right]$$

Using (47) and (52) we have,

$$r_{p}^{*}a_{p}^{*} = -\frac{\rho_{p}}{2\text{Im} \Sigma_{R}(P)}$$

and $f_{p} = -iN_{p}\rho_{p}$. Taking $Q \to 0$ and using these results we find that (53) becomes,

$$-\frac{\lambda^2}{4} \int dP dR \frac{\bar{\Gamma}_{lm}^{R}(P)}{\text{Im} \Sigma_{R}(P)}\rho_{p}\rho_{p^{'}\rho} [n_{r}n_{p^{'} + n_{r}} + n_{p}(n_{r} - n_{p^{'}})]$$

Using the identity

$$n_{r}n_{p^{'} + n_{r}} + n_{p}(n_{r} - n_{p^{'}}) = (1 + n_{p})(1 + n_{p^{'}})n_{r}/(1 + n_{k})$$

and substituting into (54) we have,

$${\bar{\Gamma}}_{lm}^{R}(K) = k_{m}k_{l} - \frac{1}{3}\delta_{ml}k^{2} - \frac{\lambda^2}{4} \int dP dR dP^{'}$$

$$(2\pi)^{4}\delta^{4}(P + P^{'} - R - K)\rho_{p}\rho_{p^{'}\rho} \frac{\bar{\Gamma}_{lm}^{R}(P)}{\text{Im} \Sigma_{R}(P)} (1 + n_{p})(1 + n_{p^{'}})n_{r}/(1 + n_{k})$$

The appearance of the imaginary part of the HTL self energy in the denominator of this result indicates that a finite width is necessary to regulate an infrared divergence in the integral. Once the integral has been expressed in a regulated form, the imaginary part of the self energy can be dropped in the expressions for the spectral densities, since it is subleading. In (57) we take,

$$\rho_{p} \to \rho_{p}^{0} = 2\pi \text{sgn}(p_{0})\delta(P^{2} - m_{th}^{2})$$

where $m_{th}^{2} := m^{2} + \text{Re} \Sigma_{R}$. We define
\[
\bar{\Gamma}^{lm}_{R} := \hat{\Gamma}^{lm}_{R}(k) \Gamma_{R}^{\text{shear}} \\
\hat{\Gamma}^{lm}_{R}(k, k) := \frac{1}{k^2} I^{lm}_{R}(k, k) = (\hat{k}_l \hat{k}_m - \frac{1}{3} \delta_{lm}) \\
B(K) := \frac{\Gamma_{R}^{\text{shear}}(K)}{\Im \Sigma_{R}(K)},
\]

and symmetrize to obtain from (57),

\[
\bar{\Gamma}^{lm}_{R}(K) = I^{lm}_{R}(k, k) - \frac{\lambda^2}{2} \int dP dR dP' (2\pi)^4 \delta^4(P + P' - R - K) \rho_{\rho'}^0 \rho_{\rho'}^0 \\
\frac{1}{3} (\hat{\Gamma}^{lm}_{R}(p) B_p + \hat{\Gamma}^{lm}_{R}(p') B_{p'} - \hat{\Gamma}^{lm}_{R}(r) B_r)(1 + n_p)(1 + n_{p'})n_r/(1 + n_k)
\]

Rearranging we obtain,

\[
I^{lm}_{R}(k, k) = \Im \Sigma_{R}(K) B(K) \hat{\Gamma}^{lm}_{R}(k, k) + \frac{\lambda^2}{12(1 + n_k)} \int dP dR dP' (2\pi)^4 \delta^4(P + P' - R - K) \\
\rho_{\rho'}^0 \rho_{\rho'}^0 [\hat{\Gamma}^{lm}_{R}(p, p) B(P) + \hat{\Gamma}^{lm}_{R}(p', p') B(P') - \hat{\Gamma}^{lm}_{R}(r, r) B(R)](1 + n_p)(1 + n_{p'})n_r
\]

(60)

In \( \phi^4 \) theory \( \Im \Sigma_{R} \) can be expressed as,

\[
\Im \Sigma_{R}(K) = -\frac{\lambda^2}{12} \left( \frac{1}{1 + n_k} \right) \int dP dR dP' (2\pi)^4 \delta^4(P + P' - R - K) \\
\rho_{\rho'}^0 \rho_{\rho'}^0 (1 + n_p)(1 + n_{p'})n_r
\]

which allows us to rewrite (60) as,

\[
I^{lm}_{R}(k, k) = \frac{\lambda^2}{12(1 + n_k)} \int dP dR dP' (2\pi)^4 \delta^4(P + P' - R - K) \rho_{\rho'}^0 \rho_{\rho'}^0 \\
[\hat{\Gamma}^{lm}_{R}(p, p) B(P) + \hat{\Gamma}^{lm}_{R}(p', p') B(P') - \hat{\Gamma}^{lm}_{R}(r, r) B(R)](1 + n_p)(1 + n_{p'})n_r
\]

Equation (62) is an integral equation whose solution represents a non-perturbative re-summation of contributions to \( B \). The last step, is to obtain an expression for the viscosity in terms of the quantity \( B \). Using (12), (18), (50) and (59) we have,

\[
\eta = \frac{\beta}{30} \lim_{q_0 \to 0} \lim_{q \to 0} (1 + n_q) \int dK (N_k - N_{k+q})k^2 [\bar{\Gamma}_{R}^{\text{shear}}(K, Q, -K - Q) a_{k}^* r_{k+q}^* + h.c.] (63)
\]

Using

\[
\lim_{q_0 \to 0} (1 + n_q) (N_k - N_{k+q}) = 2n_k(1 + n_k)
\]

we obtain,

\[
\eta = \frac{\beta}{15} \int dK k^2 n_k(1 + n_k) [\bar{\Gamma}_{R}^{\text{shear}} a_{k}^* r_{k}^* + h.c.] (65)
\]

Using (54) and (59) and making the substitution \( \rho_{\rho} \to \rho_{\rho}^0 \) we have,
\[ \eta = \frac{\beta}{15} \int dK k^2 n_k (1 + n_k) B_k p_k^0 \]  

(66)

It is straightforward to show that our integral equation for the resummation of ladder contributions to the three point vertex \([\mathcal{I}2]\) has exactly the same structure as the equation obtained from the linearized Boltzmann equation, and that our expression for the shear viscosity \((66)\) is exactly the same as that obtained by defining the shear viscosity as a transport coefficient and using the linearized Boltzmann equation to get an integral equation for viscosity. We begin from the Boltzmann equation for the phase space distribution function \(f(X, P)\) which describes the evolution of the phase space probability density for the fundamental particles comprising a fluid,

\[
\frac{K^\mu}{\omega_k} \frac{\partial}{\partial X^\mu} f(X, K) = \frac{1}{2} \int_{123} d\Gamma_{12\to 3k}[f_1 f_2 (1 + f_3)(1 + f_k) - (1 + f_1)(1 + f_2)f_3 f_k] \tag{67}
\]

where \(f_i := f(X, K_i)\), \(f_k := f(X, K)\) and \(d\Gamma_{12\to 3k}\) is the differential transition rate for particles of momentum \(P_1\) and \(P_2\) to scatter into momenta \(P_3\) and \(K\) and is given by

\[
d\Gamma_{12\to 3k} := \frac{1}{2\omega_k} |\mathcal{T}(K, P_3; P_2, P_1)|^2 \prod_{i=1}^3 \frac{d^3 p_i}{(2\pi)^3 2\omega_{p_i}} (2\pi)^4 \delta(P_1 + P_2 - P_3 - K) \tag{68}
\]

\(\mathcal{T}\) is a multiparticle scattering amplitude describing a process in which particles of momenta \(P_1\) and \(P_2\) scatter into momenta \(P_3\) and \(K\). Underlined momentum variables are on shell, and for the purposes of this section we specialize further to the positive mass shell. The Boltzmann equation is valid for distribution functions that describe the distributions of positive energy particles.

In a weak coupling \(\phi^4\) theory, one can linearize the Boltzmann equation by expanding the non-equilibrium distribution function \(f(X, P)\) around a local equilibrium function \(f_0(X, P)\):

\[
f(X, P) = f_0(X, P) \{1 - \chi(X, P)[1 + f_0(X, P)]\} \tag{69}
\]

where

\[
f_0(X, P) = n(|u^\mu P_\mu|); \quad n(\omega) = \frac{1}{e^\beta \omega - 1} \tag{70}
\]

Expanding in powers of \(\nabla u\) one finds \([\mathcal{I}2]\),

\[
\chi(X, P) = \beta(X) A(X, P) \nabla \cdot u(X) + \beta(X) B(X, P) [\hat{P} \cdot \nabla (u(X) \cdot \hat{P})] - \frac{1}{3} \nabla \cdot u(X) \tag{71}
\]

The coefficient \(A\) multiplying the divergence of the flow is related to the bulk viscosity. We obtain the shear viscosity from the coefficient \(B\) which multiples the shear in the flow. \(B\) satisfies the linear inhomogeneous integral equation \([\mathcal{I}0]\):

\[
k_l k_m - \frac{1}{3} \delta_{lm} k^2 = \frac{1}{4} \int \prod_{i=1}^3 \frac{d^3 p_i}{2E_i (2\pi)^3} (2\pi)^4 \delta(P_1 + P_2 - P_3 - K) |\mathcal{T}(P_1, P_2, P_3, K)|^2 \times \{1 + n(\omega_1)\} [1 + n(\omega_2)]\omega_3 / [1 + n(\omega_3)] \times (\hat{I}_{lm}(k, k) B(K) + \hat{I}_{lm}(p_3, p_3) B(P_3) - \hat{I}_{lm}(p_2, p_2) B(P_2) - \hat{I}_{lm}(p_1, p_1) B(P_1)) \tag{72}
\]
We want to compare this result with (62). The two equations have exactly the same form once the delta functions have been used to do the frequency integrals in (62) and the classical piece, which corresponds to the positive mass shell, is extracted. The shear viscosity can be written in terms of $B$ by looking at first order corrections to the energy momentum tensor, and comparing with the constitutive relation,

$$\langle T_{ij} \rangle \simeq -\frac{\eta}{\langle \epsilon + p \rangle} \left[ \nabla_i \langle T^0_j \rangle + \nabla_j \langle T^0_i \rangle - \frac{2}{3} \delta_{ij} \nabla^l \langle T^0_l \rangle \right] - \frac{\zeta}{\langle \epsilon + p \rangle} \delta_{ij} \nabla^l \langle T^0_l \rangle + \delta_{ij} \langle p \rangle \right]$$

(73)

where $\epsilon := T_{00}$ is the energy density, $P = \frac{1}{3} T^i_i$ is the pressure, and $\eta$ and $\zeta$ are the shear and bulk viscosities respectively. The result is \[\text{(6)}\][\text{12}]

$$\eta = \frac{\beta}{15} \int \frac{d^3k}{(2\pi)^3} k^2 n(\omega_k) (1 + n(\omega_k)) B(K)$$

(74)

We want to compare this expression with our result (60). After doing the integral over the frequency components in (60) and extracting the classical piece, the two expressions agree.

**B. Ladders from four-point vertices**

We can also resum ladder contributions to the viscosity by starting from the integral equation for viscosity in terms of the four-point function (39) and replacing the four-point function by the solution of the integral equation shown in Fig. [8]. The zeroth order terms come from the first graph on the right hand side of the figure. We obtain the term corresponding to the second graph in the figure by using the decomposition (35). We have,

$$M_B = \frac{\lambda^2}{4} \int dR \delta(r, r + k - p)$$

$$- \frac{\lambda^2}{32} \int dR dS \left[ M_B \delta(r, r + s - p) a_s r_{s+q} + M_{R3} (\delta(r, r + s - p) a_s f_{s+q} + \phi_2 (r, r + s - p) a_s a_{s+q}) + M_{R1} (\delta(r, r + s - p) f_s r_{s+q} + \phi_1 (r, r + s - p) r_s r_{s+q}) \right]$$

(75)

$$M_{R1} = \frac{\lambda^2}{4} \int dR \phi_1 (r, r + k - p)$$

$$- \frac{\lambda^2}{32} \int dR dS \left[ M_{R1} (\delta(r, r + s - p) f_s r_{s+q} + \phi_1 (r, r + s - p) f_s r_{s+q}) + M_{R3} \phi_1 (r, r + s - p) r_s r_{s+q} \right]$$

$$M_{R3} = \frac{\lambda^2}{4} \int dR \phi_2 (r, r + k - p)$$

$$- \frac{\lambda^2}{32} \int dR dS \left[ M_{R3} (\delta(r, r + s - p) a_s a_{s+q} + \phi_2 (r, r + s - p) f_{s+q} a_s) + M_{R1} \phi_2 (r, r + s - p) r_{s+q} f_s + M_B \phi_2 (r, r + s - p) r_{s+q} a_s \right]$$

We can rewrite this result by taking the pinch limit and using the definition (40). We obtain,

$$\tilde{M}(P + Q, -K - Q, -P, K) = \tilde{M}(0)(P + Q, -K - Q, -P, K) - \frac{\lambda^2}{32} \int dR dS a_s r_{s+q}$$

(76)

$$\tilde{M}(S + Q, -K - S, -P, K) [\delta(r, r + s - p) - N_p \phi_1 (r, r + s - p) + N_{p+q} \phi_2 (r, r + s - p)]$$

18
Note that as with (16), this integral equation is decoupled. As a consequence, we can obtain a resummation of ladder graphs by solving (76) for $\tilde{M}$ and substituting into the integral equation (42) from which we obtain the viscosity.

VI. ANOTHER WAY TO RESUM GRAPHS

In this section we will show that there is another way to obtain a resummation of contributions to the viscosity. We will see that the method discussed in this section indicates that the ladder graphs are not the only graphs that need to be resummed, in spite of the arguments discussed in the previous section. We will identify the extra graphs that are included in our resummation, which are not part of the conventional ladder resummation. The idea is as follows. In [20] it is shown that in $\phi^3$ theory, the retarded three-point vertex can be related to the self energy by a relation of the form,

\[ \Gamma_R(K, Q, -K - Q) \simeq \frac{1}{Q^2 + 2Q \cdot K} \left( \Pi_A(K) - \Pi_R(Q + K) \right) \]  

where the symbol $\simeq$ indicates that we are looking at the infrared region of the momentum integrals in the various terms in the loop expansion of the three-point vertex and the self energy. This condition will be made explicit below. We will obtain an equation of this form for the corrected three-point vertices involved in (31), and for the corrected four-point vertices in (42). In both cases, a resummation is obtained by replacing the self energy in these expressions by the solution of a separate coupled integral equation that resums an infinite series of contributions to the self energy. This procedure is presented in detail below.

A. The corrected three-point function

In (31) the vertex (20) is similar to the three-point vertex in $\phi^3$ theory except that it contains the composite operator $\pi_{lm}(X)$ (14). We will show that in the limit of infrared loop momenta this three-point function is related to a function that is similar to the sunset self energy from $\phi^4$ theory. The equation that relates these quantities is similar to (77). At one loop, the relevant diagrams are shown in Fig [9]. In the vertex diagram, the place where the two lines join is a bare vertex $\Gamma^{(0)lm}_{abc}$ (19). In the self energy diagram, a line carrying momentum $P$ with an asterix contains a factor $I_{lm}(p, p)$. Note that this notation does not indicate an insertion, but merely a multiplicative factor. We will show explicitly that the diagrams in Fig [9] satisfy the relation,

\[ \Gamma^{lm}_R(K, Q, -K - Q) \simeq \frac{2}{Q^2 + 2Q \cdot K} \left( \Pi^{lm}_A(K) - \Pi^{lm}_R(Q + K) \right) \]  

(78)

Note that in order to obtain this result we have rewritten $I_{lm}(p+k, p+k+q) = I_{lm}(p+k, p+k) + I_{lm}(p+k, q)$ and taken only the first term. We have not included the extra contributions that result from the second term, since these terms will not contribute to the final result when the limit $Q \to 0$ is taken. We verify this relation explicitly at one loop, and we also discuss the relation at higher orders in the loop expansion, working at zero temperature for simplicity, and discuss what sets of graphs are included. Finally, we will show that when
this expression for the vertex is substituted into the integral equation (34) from which we obtain the viscosity, and the self energy $\Pi_{lm}$ is replaced by the solution of a separate integral equation, the result for the viscosity corresponds to an infinite resummation of graphs.

1. The splitting relation for the three-point function

We begin by deriving (78). The three-point function in Fig. [9a] is given by the expression

$$
\Gamma_{ebd}^{lm} = \int dP \int dR (-i\lambda)^2 i D_{de}(R) r_3^e i D_{ed}' r_3^d i D_{eb}(P) i D_{bd}(P + Q) r_3^b I_{lm}(p, p)
$$

(79)

We expand this equation using the vector column representation of the propagators (9) and extract $\Gamma_{ebd}^{lm}$ (22). We make the change of variable $P \rightarrow P + K$. We will need the functions $\delta(r, r - p)$, $\phi_1(r, r - p)$ and $\phi_2(r, r - p)$. These expressions are defined in (45). In the equations that follow we will suppress the momentum arguments for these quantities. We obtain,

$$
\Gamma_{R}^{lm}(K, Q, -K - Q) = -\frac{\lambda^2}{4} \int dP dR I_{lm}(p + k, p + k)
$$

$$
(f_{p+k} r_{p+k+q} \phi_1 + f_{p+k+q} a_{p+k} \phi_2 + r_{p+k+q} a_{p+k} \delta)
$$

(80)

Splitting the propagator pair and looking at the region of phase space where

$$
2P \cdot Q \ll Q^2 + 2K \cdot Q
$$

we have,

$$
D(P + K + Q)D(P + K) \simeq \frac{D(P + K) - D(P + K + Q)}{Q^2 + 2K \cdot Q}
$$

(82)

which gives,

$$
\Gamma_{R}^{lm}(K, Q, -K - Q) = -\frac{\lambda^2}{4(Q^2 + 2Q \cdot K)} \int dP dR I_{lm}(p + k, p + k)
$$

$$
[(a_{p+k} - r_{p+k+q}) \delta + f_{p+k} \phi_1 - f_{p+k+q} \phi_2]
$$

(83)

We compare this result with the sunset self energy as shown in Fig. [9b]. We obtain,

$$
\Pi_0(K)_{ab}^{lm} = \frac{i}{2} \int dR dP (i\lambda)^2 r_3^a r_3^b D_{ba}(R) D_{ab}(P + K) I_{lm}(p + k, p + k) D_{ab}(R - P)
$$

(84)

Contracting indices and using the definitions of retarded and advanced self-energies (11), we obtain,

$$
\Pi_0(K)_{R}^{lm} = -\frac{\lambda^2}{8} \int dR dP I_{lm}(p + k, p + k)(r_{p+k} \delta(r, r - p) + f_{p+k} \phi_2(r, r - p))
$$

(85)

$$
\Pi_0(K)_{A}^{lm} = -\frac{\lambda^2}{8} \int dR dP I_{lm}(p + k, p + k)(a_{p+k} \delta(r, r - p) + f_{p+k} \phi_1(r, r - p))
$$

Comparing (83) with (85) gives the result (78).
When we look at (78) at higher order in the loop expansion, we find that the sets of graphs that satisfy this relation do not correspond to only the ladder graphs. Working at zero temperature for simplicity, it is straightforward to show that the vertex graphs shown in Fig. [10] correspond to the second order sunset graph shown in Fig. [11]. The important point is that there is a cancellation between contributions from the ladder graph (Fig. [10a]) and contributions from the two non-ladder graphs (Fig [10b, 10c]). This cancellation indicates that the leading contributions from the ladder graphs must be of the same order as the leading contributions from the non-ladder graphs, in contradiction to the prediction of the pinch argument, as described in the previous section. As explained previously, the conclusion that non-ladders are suppressed relative to ladders is based on the following argument. There is only a limited region of phase space (the infrared region) in which non-ladders are significant, and since the measure that corresponds to this region is small, the non-ladders should be suppressed. However, our derivation of (78) is based on the realization that in the limit that $Q \to 0$ it is precisely this region of phase space that is important, since both ladders and non-ladders are largest in this region. Since we have singled out the infrared part of the phase space, by imposing the condition $2P \cdot Q \ll Q^2 + 2K \cdot Q$, the argument about the measure being small is no longer relevant and there are contributions from the non-ladder graphs that are as large as contributions from the ladder graphs.

2. An Integral equation for $\Pi$

The second step in this procedure is to obtain an integral equation that resums an infinite set of contributions to $\Pi$. The integral equation that resums the sunset contributions to the self energy is shown in Fig. [12]. We obtain,

$$\Pi(K)^{lm}_{ab} - \lambda^2 \frac{1}{2} \int dR dP r^3 a^3 b^3 D_{ba}(R) D_{ab}(R - P) D_{ac}(K + P) \Pi^{lm}_{cd}(K + P) D_{db}(K + P)$$

Contracting indices we obtain, for the retarded and advanced combinations,

$$\Pi(K)^{lm}_R = \Pi_0(K)^{lm}_R - \lambda^2 \frac{1}{8} \int dR dP \left( \delta[\Pi(K + P)^{lm}_R f_{k+p} k_{k+p}] + \phi_1[\Pi(K + P)^{lm}_R f_{k+p} k_{k+p} + \Pi(K + P)^{lm}_A a_{k+p} f_{k+p} + \Pi(K + P)^{lm}_F a_{k+p} k_{k+p}] \right)$$

$$\Pi(K)^{lm}_A = \Pi_0(K)^{lm}_A - \lambda^2 \frac{1}{8} \int dR dP \left( \delta[\Pi(K + P)^{lm}_A a_{k+p} k_{k+p}] + \phi_2[\Pi(K + P)^{lm}_R f_{k+p} k_{k+p} + \Pi(K + P)^{lm}_A a_{k+p} f_{k+p} + \Pi(K + P)^{lm}_F a_{k+p} k_{k+p}] \right)$$

where $\delta$, $\phi_1$, $\phi_2$ are functions $f(r, r - p)$ and the momentum dependence has been suppressed.

One of the advantages of the resummation technique outlined in this section is that it can be easily extended to include other types of graphs. By using an integral equation that resums different contributions to the self energy, one can obtain the resummation of a different set of contributions to the viscosity.
3. Contributions to the Viscosity

Finally, we ask what contributions to the viscosity are obtained by substituting the solution of the integral equation (87) into (78) (which is in turn substituted into (31)). The first term in (31) is the one loop graph. We will show that the second term is the first ladder contribution to the viscosity (Fig. [13a]), when we use (78) with the self energy given by the first order sunset graph (83). Substituting (78) and (85) into (31) we obtain for the second term,

\[
(\sigma - \sigma')^{(2)} = \frac{\lambda^2}{8} \int dK \frac{1}{Q^2 + 2Q \cdot K} \int dR dP I_{lm}(p + k, p + k) \{(N_k - N_{k+q})a_k r_{k+q} \}
\]

\[
\quad \quad [(a_{p+k} - r_{p+k+q})\delta + f_{p+k}\phi_1 - f_{p+k+q}\phi_2] + h.c. \} \}
\]

\[
I_{lm}(k, k)
\]

(88)

As before, \( \delta, \phi_1, \phi_2 \) are functions \( f(r, r - p) \) and the momentum dependence has been suppressed. We use (82) to obtain,

\[
(\sigma - \sigma')^{(2)} = -\frac{\lambda^2}{8} \int dK dR dP I_{lm}(p + k, p + k) \{(N_k - N_{k+q})a_k r_{k+q} \}
\]

\[
\quad \quad +\phi_1 f_{p+k} r_{p+k+q} + \phi_2 a_{p+k} f_{p+k+q} + h.c. \} \}
\]

\[
I_{lm}(k, k)
\]

(89)

We want to compare this result with the result obtained from the first ladder graph shown in Fig. [13a]. Defining the quantity in the figure as \( \lambda_{ab} \) we want to look at \( \sigma - \sigma' = \lambda_{21} - \lambda_{12} \). From the figure we have,

\[
\lambda_{ab} = \lambda^2 \int dR dP r_3^c r_3^d D_{cd}(R) D_{ac}(R - P) I_{lm}(p + k, p + k) \]

\[
D_{da}(P + K) D_{ac}(P + K + Q) D_{cb}(K + Q) D_{bd}(K) I_{lm}(k, k)
\]

We substitute in the vector forms for the propagators (9). We keep only the terms that give us non-zero contributions to \((\sigma - \sigma')^{(2)}\). The result is,

\[
(\sigma - \sigma')^{(2)} = -\frac{\lambda^2}{8} \int dK dR dP I_{lm}(p + k, p + k) \{(N_k - N_{k+q})a_k r_{k+q} \}
\]

\[
\quad \quad \delta a_{p+k} r_{p+k+q} + \phi_1 f_{p+k} r_{p+k+q} + \phi_2 a_{p+k} f_{p+k+q} + h.c. \} \}
\]

\[
I_{lm}(k, k)
\]

(90)

where we have made the change of variable \( K \to -K - P - Q \) in the second term in order to obtain a form that is easily recognizable as the hermitian conjugate. Once again, \( \delta, \phi_1, \phi_2 \) are functions \( f(r, r - p) \) and the momentum dependence has been suppressed. Comparison of (89) and (90) verifies our statement that the first order sunset graph corresponds to the first order ladder graph.

At higher orders the calculation is more complicated. In Appendix A, we discuss the two loop contributions, working at zero temperature for simplicity. We show that the second order calculation gives contributions to the viscosity of the form shown in Fig. [13b] and [13c]. The key point is that non-ladder graphs are included. The conclusion is that we can obtain a resummation of ladder and non-ladder graphs by substituting the solution of the integral equation that resums sunset graphs (87) into the expression for the vertex (77), which is in turn substituted into the integral equation (31) from which we obtain the viscosity.
4. The Infrared Divergence

The procedure described above will result in an expression that is divergent in the limit \( Q \to 0 \), in exactly the same way that the previous result, obtained by solving the integral equation for the vertex (14), was divergent. In this case, we expect to encounter difficulties in the limit \( Q \to 0 \) since the condition (81) is not satisfied in this limit. We can remedy the problem by using HTL propagators. The vertex is the same as the vertex shown in Fig. [9a], except that the propagators are now HTL propagators (17). From (80) we obtain,

\[
\tilde{\Gamma}_{lm}^{R}(K, Q, K - Q) = -\frac{\lambda^2}{4} \int dP dR I_{lm}(p + k, p + k)
\]

\[
\left( f^{*}_{p+k} r^{*}_{p+k+q} \phi^{*}_{1} + f^{*}_{p+k+q} a^{*}_{p+k} \phi^{*}_{2} + r^{*}_{p+k+q} a^{*}_{p+k} \delta^{*} \right)
\]  

(91)

We split propagator pairs as before. With HTL propagators, the splitting relation has the form,

\[
D_{R}^{*}(P + K + Q)D_{A}^{*}(P + K) \simeq \frac{D_{A}^{*}(P + K) - D_{R}^{*}(P + K + Q)}{Q^2 + 2K \cdot Q - \Sigma_{R}(K + Q) + \Sigma_{A}(K)}
\]  

(92)

In this case, the notation \( \simeq \) indicates that we are looking at the region of phase space where

\[
2Q \cdot P + P(\Sigma_{A}(K) - \Sigma'_{R}(K + Q)) \ll Q^2 + 2Q \cdot K - \Sigma_{R}(K + Q) + \Sigma_{A}(K)
\]  

(93)

This condition can be compared to the condition for the splitting of bare propagators (81). Since the imaginary part of the HTL sunset self energy is non-zero, the condition (93) is satisfied in the infrared region of the momentum integral, even when \( Q \) is taken to zero. The resulting expression has the form,

\[
\Gamma_{lm}^{im}(K, Q, -K - Q) = -\frac{\lambda^2}{4} \left( \frac{1}{Q^2 + 2Q \cdot K + \Sigma_{A}(K) - \Sigma_{R}(K + Q)} \right)
\]

\[
\int dP dR I_{lm}(p + k, p + k)\left[ (a^{*}_{p+k} - r^{*}_{p+k+q}) \delta^{*} + f^{*}_{p+k} \phi^{*}_{1} - f^{*}_{p+k+q} \phi^{*}_{2} \right]
\]  

(94)

We compare this result with the sunset self energy (compare (85)),

\[
\Pi_{0}(K)^{lm}_{R} = -\frac{\lambda^2}{8} \int dR dP I_{lm}(p + k, p + k)(r^{*}_{p+k} \delta^{*} + f^{*}_{p+k} \phi^{*}_{1})
\]

\[
\Pi_{0}(K)^{lm}_{A} = -\frac{\lambda^2}{8} \int dR dP I_{lm}(p + k, p + k)(a^{*}_{p+k} \delta^{*} + f^{*}_{p+k} \phi^{*}_{2})
\]  

(95)

where \( \delta, \phi_{1}, \phi_{2} \) are functions \( f(r, r - p) \) and the momentum dependence has been suppressed. Comparing (94) and (95) we obtain,

\[
\Gamma_{lm}^{im}(K, Q, -K - Q) = \frac{2}{Q^2 + 2Q \cdot K - \Sigma_{R}(Q + K) + \Sigma_{A}(K)}(\Pi_{A}^{lm}(K) - \Pi_{R}^{lm}(Q + K))
\]  

(96)

We obtain the self energy from the integral equation (compare (87)).
\[ \Pi(K)^{lm}_{R} = \Pi_0(K)^{lm}_{R} - \frac{\lambda^2}{8} \int dR \, dP \left( \delta^* \Pi(K + P)^{lm}_{R} r^*_k r^*_p \right) \]
\[ + \phi_2^* \Pi(K + P)^{lm}_{R} f^*_k r^*_p + \Pi(K + P)^{lm}_{A} a^*_k r^*_p + \Pi(K + P)^{lm}_{F} a^*_k r^*_p \right) \]
\[ \Pi(K)^{lm}_{A} = \Pi_0(K)^{lm}_{A} - \frac{\lambda^2}{2} \int dR \, dP \left( \delta^* \Pi(K + P)^{lm}_{A} a^*_k a^*_p \right) \]
\[ + \phi_1^* \Pi(K + P)^{lm}_{R} f^*_k r^*_p + \Pi(K + P)^{lm}_{A} a^*_k r^*_p + \Pi(K + P)^{lm}_{F} a^*_k r^*_p \right) \]

The corresponding contributions to the viscosity are the same as in Fig. [13], but with all lines taken as HTL propagators. Note that, as usual when calculating with HTL propagators, we must include counter terms to avoid double counting.

B. The corrected four-point function

It is also possible to obtain an expression similar to (74) for the four-point function. We must look at the special case where two of the four fields have the same position co-ordinate, and the same Keldysh index. A vertex of this form is shown in Fig. [14]. Note that this restriction means that the 1PI four-point vertex has essentially the same form as a three-point vertex. We will show that the use of this vertex in the expression for the viscosity (39) leads to a resummation of contributions to the viscosity, all of which contain one chain link. A diagram of this form is shown in Fig. [15]. It has been shown that diagrams containing chain links are suppressed [10]. The argument is as follows. As discussed in section V-A-1 in the case of ladder diagrams, we expect that each additional pair of rails will introduce a factor 1/\lambda^2 from the imaginary part of the HTL self energy. Since each additional chain vertex introduces a factor \lambda, it appears naively that chain diagrams contain a factor (1/\lambda)^n where n is the number of chain links. In fact this conclusion is invalid. A pair of rails in a chain link does not contribute a factor 1/\lambda^2 because of the fact that the discontinuity of a chain link vanishes in the limit of zero external four momentum, and the real part of a bubblechain link does not contain pinching pole contributions. Consequently, chain diagrams are suppressed by a factor \lambda^n where n is the number of links. The four-point vertex is introduced only as a means to illustrate the self-consistent cancellation, as discussed in section [VII].

1. Integral equation for viscosity

We need to express the viscosity in terms of an integral equation of the form shown in Fig. [16]. For the first term we obtain,

\[ \lambda_{ab}^{(1)} = -i \lambda \int dK \, dP \, I_{lm}(k, k) I_{lm}(k, k) D_{ac}(P + Q) D_{ca}(P) r^*_k D_{cb}(K + Q) D_{bc}(K) \]  
(98)

Performing the summations and taking the combination \( \lambda_{21}^{(1)} - \lambda_{12}^{(1)} = (\sigma - \sigma')^{(1)} \) we obtain,

\[ (\sigma - \sigma')^{(1)} = -i \frac{\lambda}{4} \int dK \, dP \, I_{lm}(k, k) I_{lm}(k, k) \left[ (N_p - N_{p+q})(N_k - N_{k+q})(r_p a_{p+q} r_k a_{k+q} + h.c.) \right] \]  
(99)
For the second diagram in Fig. [16] we obtain,

\[
\lambda_{ab}^{(2)} = \int dK dP I_{lm}(p,p)M(P + Q, -K - Q, -P, K)_{eced} \\
D_{ca}(P)D_{ac}(P + Q)D_{bd}(K)D_{cb}(K + Q)I_{lm}(k, k)
\]

Taking the appropriate combination we get,

\[
(\sigma - \sigma')(2) = \frac{1}{8} \int dK dP I_{lm}^{\sigma}(p,p) \\
\{ (N_{k+q} - N_{k})r_{k+q}a_{k}[M_{B}(r_{p}a_{p+q} + a_{p}r_{p+q} + f_{p}f_{p+q}) + (M_{R1} + M_{R3})(a_{p}f_{p+q} + f_{p}r_{p+q})] \\
+ (N_{p+q} - N_{p})r_{p}a_{p+q}(M_{B}(f_{k}f_{k+q} + a_{k}r_{k+q}) + (M_{D} + M_{T})r_{k}a_{k+q} + \\
(M_{R4} + M_{\beta})r_{k}f_{k+q} + (M_{R2} + M_{\delta})f_{k}a_{k+q}) \} I_{lm}(k, k)
\]

This result can be written in terms of a three-point function. Using (23), (35) and (38) we find,

\[
M_{T} = M_{D} := iZ_{F}; \quad M_{B} := 0; \quad M_{R1} = M_{R3} := iZ_{R}; \\
M_{R2} = M_{\delta} := iZ_{R_{hand}}; \quad M_{R4} = M_{\beta} := iZ_{R_{oeo}}
\]

where the momentum arguments for the M’s and the Z’s are \(M_{Z}(P + Q, -K - Q, -P, K)\) and \(Z_{Z}(-K - Q, Q, K)\) and the vertices Z obey the same KMS conditions (23) as the vertices \(\Gamma_{lm}\). Rewriting the result in terms of the three-point function \(Z\) we find that the viscosity can be expressed in terms of one retarded three-point function:

\[
(\sigma - \sigma')(2) = i \int dK dP I_{lm}^{\sigma}(p,p)(N_{k} - N_{k+q})(N_{p} - N_{p+q}) \\
[a_{k}r_{k+q}a_{p}r_{p+q}Z_{R}(-K - Q, Q, K) + h.c.]I_{lm}(k, k)
\]

Combining (39) and (101) we obtain,

\[
\sigma - \sigma' = i \int dK dP I_{lm}^{\sigma}(p,p)(N_{k} - N_{k+q})(N_{p} - N_{p+q}) \\
(a_{k}r_{k+q}a_{p}r_{p+q}[-\lambda + Z_{R}(K, Q, -K - Q)] + h.c.)I_{lm}(p, p)
\]

2. The splitting relation for the four-point function

We consider the one loop diagrams shown in Fig. [17a,b]. The vertex shown in Fig. [17a] is given by,

\[
M_{bcba}(P + Q, -K - Q, -P, K) \\
= i \frac{\lambda^{3}}{2} \int dS dR D_{bc}(R + K + Q)D_{ab}(R + K)D_{ac}(S)D_{ca}(R + S)\tau_{a}\tau_{b}\tau_{c}
\]

Using the form of the propagators given in (9) and extracting the vertex \(M_{R1} = iZ_{R}\) we obtain,
\[ Z_R(K, Q, -K - Q) = \frac{\lambda^3}{4} \int dS dR \left[ r_{r+k+q} a_{r+k} \delta + a_{r+k} f_{r+k+q} \phi_1 + f_{r+k} r_{k+r+q} \phi_2 \right] \]  

where \( \delta, \phi_1, \phi_2 \) are functions of the form \( f(s, r+s) \). Using (82) to split pairs of propagators of the form \( D_{r+k} D_{r+k+q} \) we obtain,

\[ Z_R = \frac{\lambda^3}{4} \frac{1}{Q^2 + 2Q \cdot K} \int dS dR \left[ a_{r+k} \delta + f_{r+k} \phi_2 - r_{k+r+q} \delta - f_{k+r+q} \phi_1 \right] \]

where we are looking at the region of phase space where \( 2Q \cdot R \ll Q^2 + 2Q \cdot K \). We compare this expression with the results for the self energy shown in Fig. [17b]. We find that,

\[ Z_R = -\frac{3\lambda}{Q^2 + 2Q \cdot K} (\Pi_A(K) - \Pi_R(K + Q)) \]

The lowest order contribution to the viscosity is shown in Fig. [17c]. Notice that the one loop graphs shown in Fig. [18] and the two loop graphs shown in Fig. [19] do not contribute to the expression (104) since the tadpole self energies are momentum independent. At higher loop order, the diagrams become increasingly complicated. Working at zero temperature for simplicity, the same relation is satisfied by the diagrams in Figs. [20-24].

3. An Integral equation for \( \Pi \)

In order to perform a resummation of a series of contributions to the viscosity, we need to obtain \( \Pi \) as the solution of an integral equation which resums contributions to the self energy. As in section [VI-A-2], we consider the integral equation which resums sunset diagrams (see Fig. [13]). In this case, none of the lines contain crosses, because we do not need to work with a self energy that contains factors \( I_{lm}(p, p) \). As explained in detail below, this simplification is the motivation for introducing the four-point vertex. We find (compare (87)),

\[ \Pi(K)_R = \Pi_0(K)_R - \frac{\lambda}{8} \int dR dP \left( \delta \Pi(K + P)_{R} r_{k+p} r_{k+p} \right. \]
\[ + \phi_2 [\Pi(K + P)_R f_{k+p} r_{k+p} + \Pi(K + P)_A a_{k+p} f_{k+p} + \Pi(K + P)_F a_{k+p} r_{k+p}] \]
\[ \Pi(K)_A = \Pi_0(K)_A - \frac{\lambda}{8} \int dR dP \left( \delta \Pi(K + P)_A a_{k+p} a_{k+p} \right. \]
\[ + \phi_1 [\Pi(K + P)_R f_{k+p} r_{k+p} + \Pi(K + P)_A a_{k+p} f_{k+p} + \Pi(K + P)_F a_{k+p} r_{k+p}] \]

4. Contributions to the viscosity

As in section [VI-A-3], the last step is to ask what contributions to the viscosity are obtained by substituting the solution of the integral equation (105) into (104) (which is in turn substituted into (102)). It is straightforward to show that the diagrams produced are the same as for the case of the three-point vertex (as shown in Fig. [13]), except for the fact that each diagram contains one bare chain link insertion. One example is shown in Fig. [15]. From power counting arguments, it is known that diagrams containing chain link contributions are suppressed. We reiterate, that the four-point vertex is discussed only as a means to illustrate the self consistent cancellation, as discussed in the next section.
5. The Infrared Divergence

As always, the result is infrared divergent unless the propagators are replaced with HTL propagators. At lowest order, the diagrams that satisfy

\[ Z_R(K, Q, -K - Q) = -\frac{1}{Q^2 + 2Q \cdot K - \sum_R(Q + K) + \sum_A(K)(\Pi_A(K) - \Pi_R(Q + K))} \]  

are the same as in Figs. [17a,b] with the propagators replaced by HTL propagators. Performing the substitution we obtain, at first order, the contribution to the viscosity shown in Fig. [25].

VII. SELF CONSISTENT RESUMMATIONS

It has been shown that the resummation technique introduced in Section [VI] can be used to resum a larger class of diagrams than just the ladders. In the case of the corrected three-point vertex, the ladder graphs are included in the resummation, in addition to another large group of graphs that are shown to contribute at the same order. In the case of the corrected four-point vertex, we include the same large class of diagrams as before, except that all contributions to the viscosity will have a chain link piece to the diagram. Now we want to consider a further increase in the set of diagrams to be resummed. We follow [20]. The basic idea is as follows. Until now, we have considered various forms of vertex corrections, with the propagator lines corrected by HTL self energies to obtain an infrared finite result. Now we will consider diagrams with corrected propagators of the form,

\[ \bar{D}(P) = \frac{1}{P^2 - \Pi(P)} \]  

where the self energies are not HTL self energies, but completely general expressions. Our goal is to find a cancellation between the self energies in expressions for the vertices ((78) and (104)), and the self energies in the corrected propagators. In [20], in a calculation of the self energy in \( \phi^3 \) theory, it has been shown that these cancellations occur and are independent of the form of the self energy. The value of this approach is that the vertex corrections and the propagator corrections are written in the same form and dealt with simultaneously and, in this sense, they are treated self-consistently.

We show below using the four-point function that it is possible to obtain a partial cancellation of the type described above. We remind the reader that in addition to the fact that the cancellation is not complete, there are other problems with the four-point vertex approach. As explained in section [VI-B], the decomposition of the four-point function in terms of self energies cannot be accomplished in general. We must restrict to four-point functions in which two of the fields are at the same space point, and the corresponding two Keldysh indices are equal. The consequence of this restriction is that we only obtain contributions to the viscosity in which there is one chain link piece. It seems likely that these two problems are related. We use the four-point vertex only as a means to illustrate the cancellation in the scalar theory.

We begin with the integral equation shown in Fig. [26]. We have (compare (102)),

27
\[ \sigma - \sigma' = \frac{i}{4} \int dK \, dP \left( N_p - N_{p+q} \right) \left( N_k - N_{k+q} \right) I_{lm} (p, p) \left\{ a_{p+q} r_p \bar{a}_{k+q} \bar{r}_k (-\lambda + Z_R (-K - Q, Q, K)) + h.c. \right\} I_{lm} (k, k) \]  

(108)

The propagators with bars are full propagators \( \bar{\Pi} \), and the two bare propagators will give the bare chain link piece to each viscosity diagram, as discussed previously. Substituting in (106) we obtain,

\[ \sigma - \sigma' = -\frac{i \lambda}{4} \int dK \, dP \left( N_k - N_{k+q} \right) \left( N_p - N_{p+q} \right) I_{lm} (p, p) I_{lm} (k, k) \]

\[ \left[ \left( 1 + \frac{3}{Q^2 + 2Q \cdot K - \Sigma_R (Q + K) + \Sigma_A (K)} \left( \Pi_A (K) - \Pi_R (Q + K) \right) \right) a_p r_{p+q} \bar{a}_k \bar{r}_{k+q} + h.c. \right] \]

\[ \left[ Q^2 + 2Q \cdot K - 3 \Pi_R (Q + K) + 3 \Pi_A (K) - \Sigma_R (Q + K) + \Sigma_A (K) \right] a_p r_{p+q} \bar{a}_k \bar{r}_{k+q} + h.c. \right] \]

(109)

Using \( \bar{D} (P)^{-1} = P^2 - \Pi (P) \) we obtain,

\[ \sigma - \sigma' = -\frac{i \lambda}{4} \int dK \, dP \left( N_k - N_{k+q} \right) \left( N_p - N_{p+q} \right) \frac{a_p r_{p+q} I_{lm} (p, p) I_{lm} (k, k)}{Q^2 + 2Q \cdot K - \Sigma_R (Q + K) + \Sigma_A (K)} \]

\[ \left( \bar{a}_k - \bar{r}_{k+q} + \left[ 2 \Pi_A (K) - 2 \Pi_R (Q + K) + \Sigma_A (K) - \Sigma_R (Q + K) \right] \bar{a}_k \bar{r}_{k+q} + h.c. \right) \]

When \( \Pi \) is obtained as the solution of the integral equation (105), Eqn. (109) represents the resummation of a huge number of contributions to the viscosity. Unfortunately, the cancellation between the vertex and propagator factors is only partial. We expect that the cancellation will be complete in a gauge theory with a three-point interaction like scalar QED. The motivation for this argument is discussed below.

When calculating viscosity from the corrected three-point function, the presence of the composite operator (14) leads to difficulties. As shown in section [VI-A-1], this composite operator leads to a three-point vertex which cannot be decomposed in terms of a standard self energy, but rather the modified self energy which we have written as \( \Pi_{lm} \). Even after using translation invariance to write \( \Pi_{lm} = I_{lm} \bar{\Pi} \), there is in general no cancellation between the factor \( \bar{\Pi} \) from the vertex and the factor \( \Pi \) from the full propagator. Note however that the numerical factor seems to be correct. Consider substituting (96) into (compare (31)),

\[ \sigma - \sigma' = \int dK \left\{ (I_{lm} (k, k) + \frac{1}{2} \Gamma^{lm}_R (K, Q, -K - Q)) \bar{a}_k \bar{r}_{k+q} + h.c. \right\} (N_{k+q} - N_k) I_{lm} (k, k) \]

and replacing \( \Pi_{lm} \) by \( I_{lm} \bar{\Pi} \). If we could write \( \Pi = \bar{\Pi} \) then the cancellation between the factors of \( \Pi \) in the propagators and the vertex would be complete: there would be no factors of \( \Pi \) in the numerator of the resulting expression.

We expect that the cancellation that we have described in this section does occur in a gauge theory like scalar QED as a result of the additional constraints imposed by gauge invariance. Note that equations of the form (77) look very much like Ward identities. It has been shown [20] that the corresponding equation for scalar QED is
\[ \Gamma_R^\mu = -ie\frac{(Q + 2K)^\mu}{Q^2 + 2Q \cdot K}\left(\Sigma_A(K) - \Sigma_R(K + Q)\right) \] (111)

and corresponds to the statement that the part of the vertex that is large in the infrared limit must satisfy the usual Ward identity with the polarization tensor. In a future publication, we will look at this type of cancellation in viscosity in scalar QED.

VIII. CONCLUSIONS

We have introduced several different techniques for performing non-perturbative resummations. We used the closed time path to reformulate the traditional resummation of ladder graphs and obtained the same integral equation as that obtained previously in the imaginary time formalism. We have developed a technique which uses a pair of integral equations that gives resummations of both ladder and non-ladder graphs. We have argued that these non-ladder graphs contribute at leading order. Finally, we have shown how this resummation technique can be generalized to treat vertex and propagator corrections self-consistently in a way that gives rise to some cancellation between vertex and propagator corrections. We might expect that, in a gauge theory, the constraints imposed by the Ward identities will facilitate this cancellation.

The various resummation schemes that we introduce in this paper become increasingly difficult to understand in terms of diagrams. It seems likely that a more physical understanding of these resummations can be obtained by interpreting the resulting integral equations in terms of an effective kinetic theory description. The equivalence of the integral equation corresponding to the ladder resummation and the Boltzmann equation was first discussed in [10]. It should be possible to obtain a similar interpretation of the integral equations that correspond to the resummation schemes that we have introduced.

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Appendix A

We start from the integral equation involving the three-point function $\Gamma_{lm}$ (31). We work at zero temperature and consider only the term that corresponds to corrections to the one loop graph. Note that, at zero temperature, it is not possible to regulate the pinch singularity using the imaginary part of the HTL self energy. We use the zero temperature case for illustrative purposes only, to simplify the discussion at higher loop orders. At zero temperature we have,

$$\sigma = 2 \int dK \Gamma_{lm}(K, Q, -K - Q) iD(K) iD(K + Q) I_{lm}(k, k)$$

(112)

We use the zero temperature version of (78),

$$\Gamma_{lm}(K, Q, -K - Q) \simeq \frac{2}{Q^2 + 2Q \cdot K} (\Pi_{lm}(K) - \Pi_{lm}(Q + K))$$

(113)

We make the definitions,

$$a = K - P_1 - P_2$$
$$a' = K + Q - P_1 - P_2$$
$$b = K - P_1 - P_2 + P_3 + P_4$$
$$b' = K + Q - P_1 - P_2 + P_3 + P_4$$

and we write $D(a) = D_a$, etc. We make throughout the approximation

$$I_{lm}(a, a) \simeq I_{lm}(a', a') \simeq I_{lm}(b, b) \simeq I_{lm}(b', b')$$

since there is no difficulty with taking quantities in the numerator to zero.

We look at the self energy shown in Fig. [27]. We obtain,

$$\Pi_{lm}(K) = -\frac{\lambda^4}{4} \int dP_1 dP_2 dP_3 dP_4 D(P_1) D(P_2) D(P_3) D(P_4) D_a^2 D_b I_{lm}(b, b)$$

Substituting into (112) and (113) we have,

$$\sigma = \frac{\lambda^4}{2} \int dK dP_1 dP_2 dP_3 dP_4 \frac{1}{Q^2 + 2Q \cdot K}$$
$$D(P_1) D(P_2) D(P_3) D(P_4) I_{lm}(b, b)(D_a^2 D_b - D_a'^2 D_b') D(K) D(K + Q) I_{lm}(k, k)$$

We can rewrite this equation by using the following identity

$$D_a^2 D_b - D_a'^2 D_b' = D_a D_b (D_a - D'_a) + D'_a D_b' (D_a - D'_a) + D_a D'_a (D_b - D'_b)$$

The next step is to use the splitting relation for the propagators to write,

$$\frac{1}{Q^2 + 2Q \cdot K} (D_a - D'_a) = D_a D'_a$$

etc.
We obtain,

\[
\sigma = \frac{\lambda^4}{2} \int dK dP_1 dP_2 dP_3 dP_4 D(P_1) D(P_2) D(P_3) D(P_4) D(K) D(K + Q) \\
[I_{lm}(a', a) D^2_a D^2_a D_b + I_{lm}(a', a) D_a D^2_a D'_b + I_{lm}(b', b) D_a D'_a D'_b D'_b] I_{lm}(k, k + q)
\]

which corresponds to the diagrams in Fig. [13b]. This procedure can also be extended to higher orders. It can be shown that the polarization tensor in Fig [28] corresponds to the contributions to the viscosity shown in Fig. [29].
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| Figure | Description | Page |
|--------|-------------|------|
| 1      | One-loop skeleton diagram for shear viscosity in $\phi^4$ theory | 34  |
| 2      | Symmetric combination of one-loop diagrams for the shear viscosity in $\phi^4$ theory | 34  |
| 3      | Symmetric diagrams for the shear viscosity in terms of a corrected three-point function. The square blob represents the corrected three-point vertex | 34  |
| 4      | Symmetric diagrams for the shear viscosity in terms of a corrected four-point function. The rectangle represents the corrected four-point vertex | 34  |
| 5      | The infinite series of planar-ladder diagrams in $\lambda\phi^4$ theory | 35  |
| 6      | Five-loop ladder- (a) and non-ladder-graph (b) for the shear viscosity | 35  |
| 7      | Schwinger-Dyson equation for the corrected three-point vertex which includes ladder graphs in $\phi^4$ theory | 35  |
| 8      | Schwinger-Dyson equation for corrected four-point vertex which includes ladder graphs in $\phi^4$ theory | 35  |
| 9      | (a) The three-point vertex including one ladder insertion; (b) the sunset self-energy that corresponds to (a); the asterix represents a factor $I_{lm}$ | 35  |
| 10     | (a) The three-point vertex with two ladders; (b) (c) non-ladder diagrams which contribute at the same order. | 36  |
| 11     | The second order sunset graph that corresponds to Fig. [10] | 36  |
| 12     | The integral equation that resums sunset contributions to the self-energy | 36  |
| 13     | (a) The first order and (b) the second order contributions to the shear viscosity | 36  |
| 14     | Corrected four-point vertex with two joined legs | 36  |
| 15     | Typical ladder diagram for the shear viscosity with one chain | 37  |
| 16     | Contribution one chain to the shear viscosity. The black square represents the corrected four-point vertex | 37  |
| 17     | (a) The four-point vertex and (b) its corresponding sunset diagram; (c) The one-chain and one-ladder diagram for the viscosity | 37  |
| 18     | (a) The one-loop four-point vertex and (b) the corresponding self-energy | 37  |
| 19     | (a) Some two-loop four-point vertices and (b) their corresponding self-energy | 38  |
| 20     | (a) Three-loop four-point vertices with ladders and (b) the corresponding sunset diagram | 38  |
| 21     | (a) Three-loop four-point vertices and (b) the corresponding self-energy | 38  |
| 22     | (a) Four-loop four-point vertex and (b) the corresponding self-energy | 38  |
| 23     | (a) Four-loop four-point vertices and (b) the corresponding self-energy | 39  |
| 24     | Four-loop four-point vertices and (b) the corresponding self-energy | 39  |
| 25     | The shear viscosity with one-chain link and one ladder. The box indicates an effective propagator | 40  |
| 26     | The shear viscosity with corrected 4-point vertex for diagrams with one chain link and effective propagators | 40  |
| 27     | The second order sunset diagram | 40  |
| 28     | The third order sunset diagram | 40  |
| 29     | Some of the ladder and non-ladder diagrams corresponding to the third order sunset diagram | 40  |
FIG. 1. One-loop skeleton diagram for shear viscosity in $\phi^4$ theory.

FIG. 2. Symmetric combination of one-loop diagrams for the shear viscosity in $\phi^4$ theory.

FIG. 3. Symmetric diagrams for the shear viscosity in terms of a corrected three-point function. The square blob represents the corrected three-point vertex.

FIG. 4. Symmetric diagrams for the shear viscosity in terms of a corrected four-point function. The rectangle represents the corrected four-point vertex.
FIG. 5. The infinite series of planar-ladder diagrams in $\lambda \phi^4$ theory.

FIG. 6. Five-loop ladder- (a) and non-ladder-graph (b) for the shear viscosity.

FIG. 7. Schwinger-Dyson equation for the corrected three-point vertex which includes ladder graphs in $\phi^4$ theory.

FIG. 8. Schwinger-Dyson equation for corrected four-point vertex which includes ladder graphs in $\phi^4$ theory.

FIG. 9. (a) The three-point vertex including one ladder insertion; (b) the sunset self-energy that corresponds to (a); the asterix represents a factor $I_{lm}$. 

35
FIG. 10. (a) The three-point vertex with two ladders; (b) (c) non-ladder diagrams which contribute at the same order.

FIG. 11. The second order sunset graph that corresponds to Fig. [10],

\[ \begin{align*}
\text{\bullet} & = \text{\bullet} \ast \text{\bullet} + \text{\bullet} \end{align*} \]

FIG. 12. The integral equation that resums sunset contributions to the self-energy.

FIG. 13. (a) The first order and (b) the second order contributions to the shear viscosity.

FIG. 14. Corrected four-point vertex with two joined legs.
FIG. 15. Typical ladder diagram for the shear viscosity with one chain.

FIG. 16. Contribution one chain to the shear viscosity. The black square represents the corrected four-point vertex.

FIG. 17. (a) The four-point vertex and (b) its corresponding sunset diagram; (c) The one-chain and one-ladder diagram for the viscosity.

FIG. 18. (a) The one-loop four-point vertex and (b) the corresponding self-energy.
FIG. 19. (a) Some two-loop four-point vertices and (b) their corresponding self-energy.

FIG. 20. (a) Three-loop four-point vertices with ladders and (b) the corresponding sunset diagram.

FIG. 21. (a) Three-loop four-point vertices and (b) the corresponding self-energy.

FIG. 22. (a) Four-loop four-point vertex and (b) the corresponding self-energy.
FIG. 23. (a) Four-loop four-point vertices and (b) the corresponding self-energy.

FIG. 24. Four-loop four-point vertices and (b) the corresponding self-energy.
FIG. 25. The shear viscosity with one-chain link and one ladder. The box indicates an effective propagator.

FIG. 26. The shear viscosity with corrected 4-point vertex for diagrams with one chain link and effective propagators.

FIG. 27. The second order sunset diagram.

FIG. 28. The third order sunset diagram.

FIG. 29. Some of the ladder and non-ladder diagrams corresponding to the third order sunset diagram.