FOR THE SPLITTING METHOD OF THE NONLINEAR HEAT EQUATION
WITH INITIAL DATUM IN \( W^{1,q} \)

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ABSTRACT. In this paper, we analyze an operator splitting scheme of the nonlinear heat equation in \( \Omega \subset \mathbb{R}^d \) \((d \geq 1)\):

\[
\begin{aligned}
\partial_t u &= \Delta u + \lambda |u|^{p-1} u \quad \text{in } \Omega \times (0, \infty), \\
u &= 0 \quad \text{in } \partial \Omega \times (0, \infty), \\
u(x, 0) &= \phi(x) \quad \text{in } \Omega,
\end{aligned}
\]

where \( \lambda \in \{-1, 1\} \) and \( \phi \in W^{1,q}(\Omega) \cap L^\infty(\Omega) \) with \( 2 \leq p < \infty \) and \( d(p - 1)/2 < q < \infty \). We establish the well-posedness of the approximation of \( u \) in \( L^r \)-space \((r \geq q)\), and furthermore, we derive its convergence rate of order \( \mathcal{O}(\tau) \) for a time step \( \tau > 0 \). Finally, we give some numerical examples to confirm the reliability of the analyzed result.

1. INTRODUCTION

We are interested in the effective operator splitting method of the nonlinear heat equation which is regarded as a fundamental problem consisting of the diffusion part and the nonlinear reaction part. Let \( \Omega \subset \mathbb{R}^d, d \geq 1 \). Our concerned problem is

\[
\begin{aligned}
\partial_t u &= \Delta u + \lambda |u|^{p-1} u \quad \text{in } \Omega \times (0, \infty), \\
u &= 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
u(x, 0) &= \phi(x) \quad \text{for } x \in \Omega,
\end{aligned}
\]

where \( 1 < p < \infty \), \( \lambda \in \{-1, 1\} \) and \( \phi(x) \) is an initial temperature function. Regarding the equation (1.1), various mathematical issues have been investigated by several researchers. The existence theory for initial data in \( H^1_0(\Omega) \) was studied in [6, 12]. The local well-posedness was established in [4, 27] for initial data in \( L^q(\Omega) \) with \( q > 1 \). We also refer to several literatures [8, 13, 21, 22] which are for the global well-posedness and blow-up solutions.

We state the well-posedness of the solution \( u \) to the problem (1.1), which is shown in [4, 27] (also refer to [24, Theorem 15.2]).

**Theorem A.** Let \( p > 1, q \geq 1 \), \( \frac{d(p-1)}{2} < q < \infty \) and \( r \in [q, \infty] \). If we assume that \( \phi \in L^q(\Omega) \), then there exists a time \( T_0 > 0 \) such that the problem (1.1) has a unique classical \( L^r \)-solution in \([0, T_0)\) and the following estimate holds:

\[
\sup_{t \in [0, T_0)} t^{\frac{1}{2}(\frac{1}{q} - \frac{1}{r})} \| u(t) \|_{L^r(\Omega)} \leq C_{d,p,q} \| \phi \|_{L^q(\Omega)},
\]

where \( C_{d,p,q} > 0 \) is a constant independent of the domain \( \Omega \). Furthermore, the time \( T_0 = T_0(d, p, q, \| \phi \|_{L^q(\Omega)}) \) in [12, 22] can be precisely determined by

\[
T_0 = c_{d,p,q} \left( 1/ \| \phi \|_{L^q(\Omega)} \right)^{q - \frac{d(p-1)}{2d} \frac{1}{q}}.
\]
for some positive constant \( c_{d,p,q} > 0 \).

In this paper, we are concerned with the operator splitting scheme of the problem (1.1). Such numerical method is useful in the numerical computation of the semilinear-type problem, and it can be proposed by splitting (1.1) into a linear flow and a nonlinear one as follows:

**Linear Part** Let \( v_0(x) \in \mathbb{R} \) be a given function. For \( t > 0 \), the operator \( S(t) \) is defined by
\[
S(t)v_0 = e^{t\Delta}v_0 \text{ which denotes the solution } v \text{ satisfying the following linear heat propagation:}
\[
\begin{align*}
\partial_t v &= \Delta v \quad \text{in } \Omega \times (0, \infty), \\
v &= 0 \quad \text{on } \partial\Omega \times (0, \infty), \\
v(x, 0) &= v_0(x) \quad \text{for } x \in \Omega.
\end{align*}
\]

**Nonlinear Part** For a bounded function \( w_0(x) \in \mathbb{R} \), there exists a time \( T_1 > 0 \) and a unique solution \( w \) satisfying
\[
\begin{align*}
\partial_t w &= \lambda |w|^{p-1}w \quad \text{in } \Omega \times (0, T_1), \\
w &= 0 \quad \text{on } \partial\Omega \times (0, T_1), \\
w(x, 0) &= w_0(x) \quad \text{for } x \in \Omega.
\end{align*}
\]

The operator \( N(t) \) is defined by \( N(t)w_0 = w \), where \( w \) is the solution of (1.4). Indeed, the explicit form of \( w \) in (1.4) can be expressed by
\[
N(t)w_0 = w_0(x) \left( \frac{1}{1 - (p - 1)\lambda t|w_0(x)|^{p-1}} \right)^{\frac{1}{p-1}} \quad \text{for } t \in (0, T_1). \tag{1.5}
\]

Here, the time \( T_1 = T_1(p, \lambda, \|w_0\|_{L_\infty(\Omega)}) > 0 \) in (1.4) can be determined by
\[
T_1 = \left( (p - 1) |\lambda| \|w_0\|_{L_\infty(\Omega)} \right)^{-1}. \tag{1.6}
\]

Using the operators \( N(t) \) and \( S(t) \), the solution \( u \) of (1.1) is separated with a small switching time \( \tau \ll T_1 \) as follows: On a fixed time interval \([0, T_1]\), we define the Lie approximation \( Z(n\tau) \) for \( n \in \mathbb{N} \) and \( 0 < n\tau < T_1 \) by
\[
Z(n\tau)\phi = (S(\tau)N(\tau))^n\phi. \tag{1.7}
\]

We note that \( Z(n\tau)\phi \) is well-defined for \( n \in \mathbb{N} \) satisfying \( 0 < n\tau < T_1 \) (see Proposition 2.3 later). Furthermore, the Duhamel-type formula for \( Z \) is given by
\[
Z(n\tau) = S(n\tau) + \frac{n\tau}{\tau} \sum_{k=0}^{n-1} S(n\tau - k\tau) \left( \frac{N(\tau) - I}{\tau} \right) Z(k\tau), \tag{1.8}
\]
where \( I \) denotes the identity operator. On the other hand, the Duhamel formula of the solution \( u \) to (1.1) is obtained by
\[
u(t) = S(t)\phi + \int_0^t S(t - s) (\lambda |u(s)|^{p-1}u(s)) \, ds, \quad t \geq 0. \tag{1.9}
\]

The aim of this paper is to establish a convergence result of the proposed approximation (1.7) under the weak regularity on the initial datum \( \phi \). So far, the splitting methods of various problems such as the reaction diffusion equation and the nonlinear Schrödinger equation have been investigated in numerous literatures [3–5, 20] and other references therein. Recently, several literatures [2, 7, 15, 16, 17, 18] studied some convergence results of the Schrödinger equation with the initial datum in low-regularity spaces. On the other hand, regarding the nonlinear parabolic problem,
various convergence results of the approximation based on its splitting scheme can be found in [3, 5, 9, 10, 11, 25], but we do not have rich knowledge of its convergence under the initial datum of low-regularity up to $W^{1,d}(\Omega)$, which is the object of this paper.

From now on, we give the main results, i.e., the well-posedness of $Z(n\tau)\phi$ in (1.7) and its error estimate. To compare with $u(n\tau)$ in Theorem A, we need to consider the common time interval $(0, T_2)$, where $T_2 = T_2(d, p, q, \lambda, \|\phi\|_{L^1(\Omega)}, \|\phi\|_{L^\infty(\Omega)}) > 0$ is defined by

$$T_2 = \min\{T_0, T_1\} \in (0, \infty),$$

(1.10)

where $T_0$ and $T_1$ are given by (1.3) and (1.6), respectively. By showing Appendix later, the well-posedness of the approximation $Z(n\tau)\phi$ on the interval $(0, T_2)$ is stated as follows:

**Theorem 1.1.** Let $p > 1$, $q \geq 1$, $\frac{d(p-1)}{2} < q < \infty$ and $r \in [q, \infty]$. If we assume that $\phi \in L^q(\Omega) \cap L^\infty(\Omega)$, then there exists a constant $C_{d,p,q} > 0$, independent of the domain $\Omega$, such that

$$\sup_{0 < n\tau < T_2} (n\tau)^{\frac{d}{d+p}} \|Z(n\tau)\phi\|_{L^r(\Omega)} \leq C_{d,p,q} \|\phi\|_{L^q(\Omega)},$$

(1.11)

where $\tau \in (0, T_2/2]$ is a given small switching time and $T_2 > 0$ is given by (1.10).

In the use of (1.11), we can show the following result for the convergence of $Z(n\tau)\phi$, which will be proved in Section 3 later.

**Theorem 1.2.** Let $p \geq 2$, $q \geq 1$, $\frac{d(p-1)}{2} < q < \infty$ and $r \in [q, \infty]$. If we assume that $\phi \in W^{1,q}(\Omega) \cap L^\infty(\Omega)$, then there exists a constant $C_{d,p,q} > 0$, independent of the small switching time $\tau < 1$ and the domain $\Omega$, such that

$$\sup_{0 < n\tau < T_2} (n\tau)^{\frac{d}{d+p}} \|u(n\tau) - Z(n\tau)\phi\|_{L^r(\Omega)}$$

$$\leq C_{d,p,q} T_2^{1 - \frac{d(p-1)}{2r}} \tau \|\phi\|_{W^{1,q}(\Omega)} (1 + \|\phi\|_{W^{1,q}(\Omega)}^p),$$

(1.12)

where $\tau \in (0, T_2/2]$ is a given small switching time and $T_2 > 0$ is given by (1.10).

Regarding the assumption of Theorem 1.2, it is actually sufficient that $\phi \in W^{1,q}(\Omega)$ for $q \geq d$ by Sobolev embedding $W^{1,d}(\Omega) \hookrightarrow L^\infty(\Omega)$.

**Remark 1.3.** In Section 3, the following error estimate will be proved:

$$\sup_{0 < n\tau < T_2} (n\tau)^{\frac{d}{d+p} - (1-\mu)} \|u(n\tau) - Z(n\tau)\phi\|_{L^r(\Omega)}$$

$$\leq C_{d,p,q} \tau \left( \|\phi\|_{L^\infty(\Omega)}^p + \|\phi\|_{L^{p+1}(\Omega)}^{p+1} + \tau \|\phi\|_{L^\infty(\Omega)}^p \|\phi\|_{L^{p+1}(\Omega)}^2 \right),$$

(1.13)

where $\mu := \frac{d(p-1)}{2q} < 1$. Since $\tau \leq n\tau < T_2 \leq T_1 \leq 1/\|\phi\|_{L^{p+1}(\Omega)}$ and $\|\phi\|_{L^\infty(\Omega)} \leq \|\phi\|_{W^{1,q}(\Omega)}$, the inequality (1.13) obviously implies the main result (1.12).

Indeed, the derivation of the convergence result under the weak regularity on the initial datum $\phi$ has some difficult issues. By the Duhamel formulas (1.9) and (1.8), we carefully try to measure the difference between the solution $u(n\tau)$ and the approximation $Z(n\tau)\phi$. The proof of this result is inspired by the reference [18] which is for the splitting method of the nonlinear Schrödinger equation (see also [7]).

This paper is organized as follows. In Section 2 we give some properties regarding the operators $S(t)$ and $N(t)$, which are essentially used in the proof of Theorem 1.2. In Section 3 we prove the main result: Theorem 1.2 by the use of induction. In Section 4, we try to confirm the analyzed result (1.12) by some numerical experiments.
2. Preliminaries

In this section, we discuss some properties of the linear heat flow $S(t)$ and nonlinear flow $N(t)$. From Proposition 4.8 in [24], we first give the following basic property of the flow $S(t)$: Let $p \geq 1$ and $r \in [q, \infty]$ with $q \geq 1$. If we assume that $\phi \in L^q(\Omega)$, then we have

$$
\|S(t)\phi\|_{L^r(\Omega)} \leq (4\pi t)^{-\frac{d}{2}} \|\phi\|_{L^q(\Omega)} \quad \text{for } t > 0. \quad (2.1)
$$

By (2.1), we show the useful estimate of $u$ to the problem (1.1), which is similar to (1.2).

**Proposition 2.1.** Let $p \geq 2$, $q \geq 1$, $\frac{d(p-1)}{2} < q < \infty$ and $r \in [q, \infty]$. If we assume that $\phi \in W^{1,q}(\Omega)$, then there exists a time $T_0 > 0$ as in (1.3), satisfying

$$
\sup_{t \in [0,T_0)} t^{\frac{d}{2} \left(\frac{1}{r} - \frac{1}{q}\right)} \|u(t)\|_{W^{1,r}(\Omega)} \leq C_{d,p,q} \|\phi\|_{W^{1,q}(\Omega)}, \quad (2.2)
$$

where $C_{d,p,q} > 0$ is a constant independent of the domain $\Omega$.

**Proof.** We now derive the estimate (2.2). Due to the result (1.2), it is enough to show

$$
\sup_{t \in [0,T_0)} t^{\frac{d}{2} \left(\frac{1}{r} - \frac{1}{q}\right)} \|\nabla u(t)\|_{L^r(\Omega)} \leq C_{d,p,q} \|\phi\|_{W^{1,q}(\Omega)}. \quad (2.3)
$$

By a direct calculation, the equality (1.9) implies

$$
\nabla u(t) = S(t)(\nabla \phi) + \int_0^t S(t-s) \left(\nabla (\lambda|u(s)|^{p-1}u(s))\right) ds.
$$

Using (2.1), we obtain

$$
t^{\frac{d}{2} \left(\frac{1}{r} - \frac{1}{q}\right)} \|\nabla u(t)\|_{L^r(\Omega)}
\leq C_d \|\nabla \phi\|_{L^q(\Omega)} + t^{\frac{d}{2} \left(\frac{1}{r} - \frac{1}{q}\right)} \int_0^t \|S(t-s) \left(\nabla (\lambda|u(s)|^{p-1}u(s))\right)\|_{L^r(\Omega)} ds
\leq C_d \|\phi\|_{W^{1,q}(\Omega)} + C_{d,p,q} t^{\frac{d}{2} \left(\frac{1}{r} - \frac{1}{q}\right)} \int_0^t (t-s)^{-\frac{d}{2} \left(\frac{1}{r} - \frac{1}{q}\right)} \|\nabla (\lambda|u(s)|^{p-1}u(s))\|_{L^q(\Omega)} ds. \quad (2.4)
$$

Since

$$
\nabla (|u(s)|^{p-1}u(s)) = ((p-1)|u(s)|^{p-2}u(s) + |u(s)|^{p-1}) \nabla u(s),
$$

and by Hölder’s inequality and (1.2) with $p \geq 2 - \frac{d}{r}$ and $q \leq r$, one yields

$$
\|\nabla (|u(s)|^{p-1}u(s))\|_{L^q(\Omega)}
\leq p \|u(s)|^{p-1}\|L^r(\Omega)\| L^q(\Omega)
\leq p \left(\|u(s)\|_{L^q(\Omega)}^{\frac{p-1}{p-1}}\right)^{p-1} \|\nabla u(s)\|_{L^r(\Omega)}
\leq C_{d,p,q} \left(s^{-\frac{d}{2} \left(\frac{1}{r} - \frac{1}{q}\right)} \|\phi\|_{L^q(\Omega)}\right)^{p-1} \|\nabla u(s)\|_{L^r(\Omega)}
\leq C_{d,p,q} \left(s^{-\frac{d}{2} \left(\frac{1}{r} - \frac{1}{q}\right)} \|\phi\|_{L^q(\Omega)}\right)^{p-1} \left(s^{\frac{d}{2} \left(\frac{1}{r} - \frac{1}{q}\right)} \|\nabla u(s)\|_{L^r(\Omega)}\right)^{p-1} \left(\sup_{s \in [0,T_0)} s^{\frac{d}{2} \left(\frac{1}{r} - \frac{1}{q}\right)} \|\nabla u(s)\|_{L^r(\Omega)}\right). \quad (2.5)
$$

On the other hand, we note that for $\alpha, \beta \in [0,1)$ and $t > 0$,

$$
\int_0^t (t-s)^{-\alpha} s^{-\beta} ds \leq \int_0^{t/2} \left(\frac{t}{2}\right)^{-\alpha} s^{-\beta} ds + \int_{t/2}^t (t-s)^{-\alpha} \left(\frac{t}{2}\right)^{-\beta} ds
= C_{\alpha,\beta} t^{-\alpha-\beta+1}, \quad (2.6)
$$
where \( C_{\alpha, \beta} := 2^{\alpha + \beta - 1} \left( \frac{1}{1 - \alpha} + \frac{1}{1 - \beta} \right) > 0 \). By (2.5) and (2.6), and since \( \frac{d}{q} \left( \frac{1}{q} - \frac{1}{p} \right) < 1 \) and \( \frac{d(p-1)}{2q} < 1 \) for \( p \geq 2 \) and \( \frac{d(p-1)}{2} < q \), the inequality (2.4) becomes

\[
\| e^{\frac{d(p-1)}{2q}} \|_{L^r(\Omega)} \left\| \nabla u(t) \right\|_{L^r(\Omega)} \\
\leq C_d \| \phi \|_{W^{1,q}(\Omega)} + C_{d,p,q} \| \phi \|_{L^t(\Omega)}^{p-1} \left( \sup_{s \in [0,T_0]} s^{\frac{d(p-1)}{2q}} \| \nabla u(s) \|_{L^r(\Omega)} \right) \\
\times t^{\frac{d(p-1)}{2q}} \int_0^t (t-s)^{-\frac{d(p-1)}{2q}} s^{-\frac{d(p-1)}{2q}} ds \\
\leq C_d \| \phi \|_{W^{1,q}(\Omega)} + C_{d,p,q} \| \phi \|_{L^t(\Omega)}^{p-1} \left( \sup_{s \in [0,T_0]} s^{\frac{d(p-1)}{2q}} \| \nabla u(s) \|_{L^r(\Omega)} \right) t^{-\frac{d(p-1)}{2q}}. \tag{2.7}
\]

Using the definition (1.3) of \( T_0 \), one sees that for \( t < T_0 \),

\[
C_{d,p,q} \| \phi \|_{L^t(\Omega)}^{p-1} t^{-\frac{d(p-1)}{2q}} \leq C_{d,p,q} \| \phi \|_{L^t(\Omega)}^{p-1} T_0^{1-\frac{d(p-1)}{2q}} \\
\leq C_{d,p,q} \| \phi \|_{L^t(\Omega)}^{p-1} C_{d,p,q} \left( 1/\| \phi \|_{L^t(\Omega)}^{(p-1)q} \right)^{\frac{1}{q}} \\
\leq \frac{1}{2}. \tag{2.8}
\]

where \( C_{d,p,q} > 0 \) is a sufficiently small constant given in Theorem A. From (2.8), the inequality (2.7) gives that for \( t \in [0, T_0) \),

\[
C_d \| \phi \|_{W^{1,q}(\Omega)} + \frac{1}{2} \left( \sup_{s \in [0,T_0]} s^{\frac{d(p-1)}{2q}} \| \nabla u(s) \|_{L^r(\Omega)} \right), \tag{2.9}
\]

and then taking the supremum of (2.9) on \([0, T_0)\), the desired estimate (2.3) follows. \( \square \)

**Corollary 2.2.** Let \( p \geq 2 \), \( q \geq 1 \) and \( \frac{d(p-1)}{2} < q < \infty \). Suppose that \( \phi \in W^{1,q}(\Omega) \). Then for \( t \in [0, T_0) \), we have the following estimates:

\[
(i) \quad \| u(t) \|^2 \| \nabla u(t) \|^2 \|_{L^t(\Omega)} \leq C_{d,p,q} t^{-\mu} \| \phi \|_{L^t(\Omega)} \| \phi \|^2_{W^{1,q}(\Omega)}, \\
(ii) \quad \| u(t) \|^2 \| \nabla u(t) \|^2 \|_{L^t(\Omega)} \leq C_{d,p,q} t^{-\mu} \| \phi \|_{L^t(\Omega)} \| \phi \|^2_{W^{1,q}(\Omega)},
\]

where \( \mu := \frac{d(p-1)}{2} < 1 \) and \( C_{d,p,q} > 0 \) is a constant that does not depend on the domain \( \Omega \). Furthermore, if we assume that \( \phi \in W^{1,q}(\Omega) \cap L^\infty(\Omega) \) and \( 0 < t < T_0 \leq 1/\| \phi \|_{L^\infty(\Omega)} \), then we have

\[
(iii) \quad \| u(t) \|^2 \| \nabla u(t) \|^2 \|_{L^t(\Omega)} \leq C_{d,p,q} t^{-\mu} \| \phi \|_{L^t(\Omega)} \| \phi \|^2_{W^{1,q}(\Omega)}. \tag{2.10}
\]

**Proof.** First, we use Hölder’s inequality and apply the estimates (1.2) and (2.2) to deduce

\[
\| u(t) \|^2 \| \nabla u(t) \|^2 \|_{L^t(\Omega)} \leq \| u(t) \|^2 \| \nabla u(t) \|_{L^t(\Omega)} \| \nabla u(t) \|_{L^t(\Omega)} \\
\leq C_{d,p,q} t^{-\frac{d(p-1)}{2q}} \| \phi \|_{L^t(\Omega)} \| \phi \|^2_{W^{1,q}(\Omega)}, \tag{2.11}
\]

which is the estimate (i). Also, using Hölder’s inequality, (1.2) and Sobolev embedding [1], the second estimate (ii) is obtained as follows:

\[
\| u(t) \|^2 \| \nabla u(t) \|^2 \|_{L^t(\Omega)} \leq C_{d,p,q} t^{-\frac{d(p-1)}{2q}} \| \phi \|_{L^t(\Omega)} \| \phi \|^2_{W^{1,q}(\Omega)} \\
\leq C_{d,p,q} t^{-\frac{d(p-1)}{2q}} \| \phi \|_{L^t(\Omega)} \| \phi \|^2_{W^{1,q}(\Omega)}.
\]

and

\[
\leq C_{d,p,q} t^{-\frac{d(p-1)}{2q}} \| \phi \|_{L^t(\Omega)} \| \phi \|^2_{W^{1,q}(\Omega)}.
\]
where the last inequality holds true since \( d\left(\frac{1}{q} - \frac{2}{(p+1)q}\right) = \frac{d(p-1)}{(p+1)q} < 1 \) since \( q > \frac{d(p-1)}{2} \) and \( p \geq 2 \).

To show the estimate \((iii)\), we recall that \( \|S(t)v\|_{L^\infty(\Omega)} \leq \|v\|_{L^\infty(\Omega)} \) for any \( v \in L^\infty(\Omega) \) (see e.g. Proposition 48.4 in [24]). By this inequality, the solution \( u(t) \) formulated by (1.9) is estimated as
\[
\|u(t)\|_{L^\infty(\Omega)} \leq \|\phi\|_{L^\infty(\Omega)} + \int_0^t \|u(s)\|_{L^p(\Omega)}^p \, ds, \quad \forall t \geq 0.
\]
So, it follows by the standard argument that
\[
\|u(t)\|_{L^\infty(\Omega)} \leq C\|\phi\|_{L^\infty(\Omega)} \quad \text{for} \quad 0 < t < T_2 \leq 1/\|\phi\|_{L^\infty(\Omega)}^{-1}.
\]
Combining this with Hölder’s inequality and estimate \((i)\), one yields
\[
\left\|u(t)\right\|_{L^\infty(\Omega)}^{2p-3}|\nabla u(t)|_{L^\infty(\Omega)}^2 \leq \left\|u(t)\right\|_{L^\infty(\Omega)}^{p-1}\left\|u(t)\right\|_{L^\infty(\Omega)}^{p-2}|\nabla u(t)|_{L^\infty(\Omega)}^2 \leq C_{d,p,q} \left\|\phi\right\|_{L^\infty(\Omega)}^{p-1}\left\|\phi\right\|_{L^\infty(\Omega)}^{p-2}\left\|\phi\right\|_{W^{1,q}(\Omega)}^2,
\]
which gives the estimate \((iii)\).

Next, we show the well-definedness of the operator \( Z(k\tau) \) for \( k = 1, 2, \cdots, N \).

**Proposition 2.3.** Suppose that \( \phi \in L^\infty(\Omega) \) and there is a number \( N \in \mathbb{N} \) such that \( N\tau < T_1 \) for a given time step \( \tau > 0 \). Then \( Z(k\tau)\phi = (S(\tau)k\tau)^k \phi \) is well-defined for \( k = 1, 2, \cdots, N \).

**Proof.** First, we consider the case of \( \lambda > 0 \). To show this proposition, we shall use an induction argument. Let \( M > 0 \) be a constant given by \( \|\phi\|_{L^\infty(\Omega)}^{p-1} \leq M \).

**Step 1 (Base case).** From \([2.1]\) and \([1.5]\), we have
\[
\|S(\tau)N(\tau)\phi\|_{L^\infty(\Omega)}^{p-1} \leq \|N(\tau)\phi\|_{L^\infty(\Omega)}^{p-1} \leq \frac{\|\phi\|_{L^\infty(\Omega)}^{p-1}}{1 - (p-1)\lambda\tau\|\phi\|_{L^\infty(\Omega)}},
\]
and
\[
M \leq \frac{M}{1 - (p-1)\lambda\tau M}. \tag{2.14}
\]

**Step 2 (Inductive step).** Assume that for any \( k = 1, 2, \cdots, N-1 \),
\[
\left\|(S(\tau)N(\tau))^k \phi\right\|_{L^\infty(\Omega)}^{p-1} \leq \frac{M}{1 - (p-1)\lambda(k+1)\tau M}.
\]
Then we obtain
\[
\left\|(S(\tau)N(\tau))^{k+1} \phi\right\|_{L^\infty(\Omega)}^{p-1} = \left\|S(\tau)N(\tau) (S(\tau)N(\tau))^k \phi\right\|_{L^\infty(\Omega)}^{p-1} \leq \frac{\left\|(S(\tau)N(\tau))^k \phi\right\|_{L^\infty(\Omega)}^{p-1}}{1 - (p-1)\lambda\tau\left\|(S(\tau)N(\tau))^k \phi\right\|_{L^\infty(\Omega)}},
\]
and
\[
\left\|(S(\tau)N(\tau))^k \phi\right\|_{L^\infty(\Omega)} \leq \frac{M}{1 - (p-1)\lambda(k+1)\tau M}.
\]
Since \( N\tau < T_1 = ((p-1)\lambda M)^{-1} \) from \([1.0]\), one gets
\[
\left\|(S(\tau)N(\tau))^N \phi\right\|_{L^\infty(\Omega)}^{p-1} \leq \frac{M}{1 - (p-1)\lambda(N\tau)M} < \infty. \tag{2.15}
\]
Using the induction argument with \([2.13]\) and \([2.15]\), \( Z(k\tau)\phi \) is well-defined in the case of \( \lambda > 0 \). Furthermore, the case of \( \lambda \leq 0 \) is obvious, so this proposition is concluded. \( \square \)
Lemma 2.4. Let $p > 1$ and $\lambda \in \{-1, 1\}$. Assume that
\[
0 < \tau \leq \frac{1}{2(p-1)} \min \left\{ \frac{1}{|u|^{p-1}}, \frac{1}{|v|^{p-1}} \right\}.
\] (2.16)

Then, there exists a constant $c_p > 0$ such that
\[
\left| \left( \frac{N(\tau) - I}{\tau} \right) u - \left( \frac{N(\tau) - I}{\tau} \right) v \right| \leq c_p |u - v| \left( |u|^{p-1} + |v|^{p-1} \right),
\] (2.17)
and
\[
\left| \left( \frac{N(\tau) - I}{\tau} \right) u - \lambda |u|^{p-1} u \right| \leq c_p \tau |u|^{2p-1}.
\] (2.18)

Proof. From \cite{165}, one sees that
\[
\left( \frac{N(\tau) - I}{\tau} \right) u - \left( \frac{N(\tau) - I}{\tau} \right) v = \frac{u - v}{\tau} \left( \frac{1}{1 - (p-1)\lambda \tau |u|^{p-1}} - 1 \right) + \frac{v}{\tau} \left( \frac{1}{1 - (p-1)\lambda \tau |u|^{p-1}} - \frac{1}{1 - (p-1)\lambda \tau |v|^{p-1}} \right).
\]

By (2.16) we notice that $(p-1)|u|^{p-1} < 1/2$ and $(p-1)|v|^{p-1} < 1/2$. Without loss of generality, it is assumed that $0 \leq |v(x)| \leq |u(x)|$. Since
\[
a \frac{1}{p-1} - b \frac{1}{p-1} \leq 2p(a-b) \left( a \frac{1}{p-1} - b \frac{1}{p-1} \right)
\]
for all $0 \leq b \leq a$ and $p > 1$, we have
\[
\left| \left( \frac{N(\tau) - I}{\tau} \right) u - \left( \frac{N(\tau) - I}{\tau} \right) v \right| \leq \frac{|u - v|}{\tau} \left( \frac{1}{1 - (p-1)\lambda \tau |u|^{p-1}} - 1 \right) \left( \frac{1}{1 - (p-1)\lambda \tau |v|^{p-1}} + 1 \right) + \frac{|v|}{\tau} \left( \frac{1}{1 - (p-1)\lambda \tau |u|^{p-1}} - \frac{1}{1 - (p-1)\lambda \tau |v|^{p-1}} \right) \times \left( \frac{1}{1 - (p-1)\lambda \tau |u|^{p-1}} - \frac{1}{1 - (p-1)\lambda \tau |v|^{p-1}} \right),
\]
and then it implies
\[
\left| \left( \frac{N(\tau) - I}{\tau} \right) u - \left( \frac{N(\tau) - I}{\tau} \right) v \right| \leq \frac{2|u - v|}{\tau} A_1 + \frac{2|v|}{\tau} A_2,
\] (2.19)
where
\[
A_1 := \left| \frac{1}{1 - (p-1)\lambda \tau |u|^{p-1}} - 1 \right|,
\]
\[
A_2 := \left| \frac{1}{1 - (p-1)\lambda \tau |u|^{p-1}} - \frac{1}{1 - (p-1)\lambda \tau |v|^{p-1}} \right|.
\]
Indeed, a direct calculation gives that
\[
A_1 = \left| \frac{(p-1)\lambda \tau |u|^{p-1}}{1 - (p-1)\lambda \tau |u|^{p-1}} \right| \leq 2(p-1)\tau |u|^{p-1}
\] (2.20)
and

\[
A_2 = \left| \frac{(p-1)\lambda \tau (|u|^{p-1} - |v|^{p-1})}{(1 - (p-1)\lambda \tau |u|^{p-1})(1 - (p-1)\lambda \tau |v|^{p-1})} \right|
\leq 4(p-1)\tau (|u| - |v|)(|u|^{p-2} + |v|^{p-2})
\leq 4(p-1)\tau |u - v| (|u|^{p-2} + |v|^{p-2}). \tag{2.21}
\]

By (2.21) and (2.22), the inequality (2.19) becomes

\[
\left| \left( \frac{N(\tau) - I}{\tau} \right) u - \left( \frac{N(\tau) - I}{\tau} \right) v \right| \leq 4(p-1)|u - v| (|u|^{p-1} + 2|u|^{p-2}|v| + 2|v|^{p-1}),
\]

so the desired estimate (2.17) follows. Furthermore, one sees

\[
\left( \frac{N(\tau) - I}{\tau} \right) u - \lambda |u|^{p-1}u = \frac{u}{\tau} \left( \frac{1}{1 - (p-1)\lambda \tau |u|^{p-1}} \right)^{\frac{1}{p-1}} - 1 \right) - \lambda |u|^{p-1}u, \tag{2.22}
\]

and Taylor series expansion of \( f(x) := \left( \frac{1}{1 - (p-1)\lambda \tau x} \right)^{\frac{1}{p-1}} \) gives that for some \( x_0 \in (0, x) \),

\[
\left( \frac{1}{1 - (p-1)\lambda \tau x} \right)^{\frac{1}{p-1}} = 1 + \lambda \tau x + \frac{p}{2} \left( \frac{1}{1 - (p-1)\lambda \tau x_0} \right)^{\frac{1}{p-1} + 2} \lambda^2 \tau^2 x^2. \tag{2.23}
\]

Here, the inequality (2.23) can be derived by seeing

\[
f'(x) = \lambda \tau \left( \frac{1}{1 - (p-1)\lambda \tau x} \right)^{\frac{1}{p-1} + 1},
\]

\[
f''(x) = p\lambda^2 \tau^2 \left( \frac{1}{1 - (p-1)\lambda \tau x} \right)^{\frac{1}{p-1} + 2}.
\]

By (2.23), and since \((p-1)\tau x_0 \leq (p-1)\tau |u|^{p-1} \leq 1/2\), the quantity of (2.22) is estimated by

\[
\left| \left( \frac{N(\tau) - I}{\tau} \right) u - \lambda |u|^{p-1}u \right| = \left| \frac{u}{\tau} \left( f(|u|^{p-1}) - 1 \right) - \lambda |u|^{p-1}u \right|
\leq \left| \frac{p}{2} \left( \frac{1}{1 - (p-1)\lambda \tau x_0} \right)^{\frac{1}{p-1} + 2} \lambda^2 \tau |u|^{2(p-1)}u \right|
\leq 4p\tau |u|^{2p-1},
\]

and then the inequality (2.18) is obtained. \( \square \)

**Lemma 2.5.** Let \( p \geq 2 \) and \( \lambda \in \{-1, 1\} \). For any \( \tau \in (0, T_1/2) \), there is a constant \( c_p > 0 \) such that the solution \( u \) of (1.1) satisfies the following inequality:

\[
\left| (\partial_t - \Delta) \left( \left( \frac{N(\tau) - I}{\tau} \right) u(t) \right) \right| \leq c_p \left( |u|^{2p-1} + |u|^{p-2} |\nabla u|^2 + \tau |u|^{2p-3} |\nabla u|^2 \right).
\]
Proof. By a direct calculation, one sees

$$
\partial_t \left( \frac{N(\tau) - I}{\tau} u(t) \right) = \frac{u_t}{\tau} \left( \frac{1}{1 - (p-1)\lambda \tau |u|^{p-1}} \right)^{-1} - 1
$$

and

$$
\nabla \left( \frac{N(\tau) - I}{\tau} u(t) \right) = \frac{\nabla u}{\tau} \left( \frac{1}{1 - (p-1)\lambda \tau |u|^{p-1}} \right)^{-1} - 1
$$

and

$$
\Delta \left( \frac{N(\tau) - I}{\tau} u(t) \right) = \frac{\Delta u}{\tau} \left( \frac{1}{1 - (p-1)\lambda \tau |u|^{p-1}} \right)^{-1} - 1
$$

Since \((\partial_t - \Delta)u = \lambda |u|^{p-1}u\), we have

$$
(\partial_t - \Delta) \left( \frac{N(\tau) - I}{\tau} u(t) \right) = \lambda |u|^{p-1}u \left( \frac{1}{1 - (p-1)\lambda \tau |u|^{p-1}} \right)^{-1} - 1
$$

By (2.20), and since \((p-1)\tau |u|^{p-1} \leq 1/2\), the proof of this lemma is concluded. □

**Lemma 2.6.** Let \(p \geq 2\), \(q \geq 1\), \(\frac{d(p-1)}{2} < q < \infty\) and \(r \in [q, \infty]\). For any \(\eta \in C^1\mathcal{C}^2_x((0, n\tau), \Omega)\), the following estimate holds upon that two integrals of both sides are finite:

$$
\left\| \int_0^{n\tau} S(n\tau - s)\eta(s)ds - \tau \sum_{k=0}^{n-1} S(n\tau - k\tau)\eta(k\tau) \right\|_{L^r(\Omega)}
\leq \tau \int_0^{n\tau} (n\tau - t)^{-\frac{d}{2}(1 - \frac{q}{r})} \| \partial_t \eta(t) - \Delta \eta(t) \|_{L^r(\Omega)} dt.
$$

(2.25)
Proof. We use a similar argument as in Lemma 4.6 of [18]. A direct calculation gives
\[
\int_0^{n\tau} S(n\tau - s)\eta(s)ds - \tau \sum_{k=0}^{n-1} S(n\tau - k\tau)\eta(k\tau)
\]
\[
= \sum_{k=0}^{n-1} \int_{k\tau}^{(k+1)\tau} (S(n\tau - s)\eta(s) - S(n\tau - k\tau)\eta(k\tau)) \, ds
\]
\[
= \sum_{k=0}^{n-1} \int_{k\tau}^{(k+1)\tau} \int_{k\tau}^{s} \partial_{t} (S(n\tau - t)\eta(t)) \, dt \, ds
\]
\[
=: Q.
\]
Since
\[
\partial_{t} (S(n\tau - t)\eta(t))^\wedge = \partial_{t} \left( e^{(n\tau-t)|\xi|^2}\hat{\eta} \right)
\]
\[
= -|\xi|^2 e^{(n\tau-t)|\xi|^2}\hat{\eta} + e^{(n\tau-t)|\xi|^2}\hat{\partial_{t}\eta},
\]
we have the following identity:
\[
\partial_{k} (S(n\tau - t)\eta(t)) = (S(n\tau - t)\partial_{k}\eta(t) - S(n\tau - t)\Delta\eta(t)).
\]  (2.26)
By (2.26), we have
\[
Q = \sum_{k=0}^{n-1} \int_{k\tau}^{(k+1)\tau} \int_{k\tau}^{s} (S(n\tau - t)\partial_{k}\eta(t) - S(n\tau - t)\Delta\eta(t)) \, dt \, ds
\]
\[
= \sum_{k=0}^{n-1} \int_{k\tau}^{(k+1)\tau} ((k+1)\tau - t) S(n\tau - t) (\partial_{k}\eta(t) - \Delta\eta(t)) \, dt
\]
\[
= \int_0^{n\tau} \sum_{k=0}^{n-1} X_{(k\tau,(k+1)\tau)}(t) ((k+1)\tau - t) S(n\tau - t) (\partial_{k}\eta(t) - \Delta\eta(t)) \, dt.
\]
Hence, the quantity $Q$ is estimated by
\[
\|Q\|_{L^r(\Omega)}
\]
\[
\leq \int_0^{n\tau} \tau \|S(n\tau - t) (\partial_{k}\eta(t) - \Delta\eta(t))\|_{L^r(\Omega)} \, dt
\]
\[
\leq \tau \int_0^{n\tau} (n\tau - t)^{-\frac{2}{d}\left(\frac{d}{r} - \frac{1}{q}\right)} \|\partial_{k}\eta(t) - \Delta\eta(t)\|_{L^q(\Omega)} \, dt,
\]
and then the desired inequality (2.25) follows. \qed

3. Proof of Theorem 1.2: Convergence of $Z(n\tau)\phi$

In this section, we will prove Theorem 1.2 in the use of Theorem 1.1. Let $d \geq 1$, $p \in [2, \infty)$, $q \geq 1$, $\frac{d(n-1)}{2} < q < \infty$ and $r \in [q, \infty)$. Assume that $\phi \in W^{1,\phi}(\Omega) \cap L^\infty(\Omega)$. To derive the main result (1.13), we now use an induction as follows: For $n \geq 1$, we assume that
\[
(k\tau)^{\frac{2}{d}\left(\frac{d}{r} - \frac{1}{q}\right)} u(k\tau) - Z(k\tau)\phi \|_{L^r(\Omega)} \leq C_* \tau M_{p,q,\phi} \quad \text{for } k = 0, \ldots, n - 1,
\]  (3.1)
where $C_* > 0$ is a constant chosen later, and
\[
M_{p,q,\phi} := \|\phi\|_{L^q(\Omega)} \|\phi\|_{W^{1,q}(\Omega)}^{p-1} + \|\phi\|_{L^q(\Omega)} \|\phi\|_{W^{1,q}(\Omega)}^{p-1} \|\phi\|_{L^\infty(\Omega)} + \tau \|\phi\|_{L^\infty(\Omega)}^{p-1} \|\phi\|_{L^q(\Omega)}^{p-2} \|\phi\|_{W^{1,q}(\Omega)}^{2}.
\]
From (1.8) and (1.9), we have
\[
(nt)^{\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-(1-\mu)}\|u(nt) - Z(n\tau)\phi\|_{L^q(\Omega)} \\
\leq (nt)^{\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-(1-\mu)} \left(\|Q_1(n\tau)\|_{L^q(\Omega)} + \|Q_2(n\tau)\|_{L^q(\Omega)} + \|Q_3(n\tau)\|_{L^q(\Omega)} \right),
\] (3.2)
where
\[
Q_1(n\tau) := \int_0^{n\tau} S(n\tau - s) \left(\lambda|u|^pu(s) - \left(\frac{N(\tau)-1}{\tau}\right) u(s)\right) ds,
\]
\[
Q_2(n\tau) := \int_0^{n\tau} S(n\tau - s) \left(\frac{N(\tau)-1}{\tau}\right) u(s)ds - \tau \sum_{0 \leq k < n} S(n\tau - k\tau) \left(\frac{N(\tau)-1}{\tau}\right) u(k\tau),
\]
\[
Q_3(n\tau) := \tau \sum_{0 \leq k < n} S(n\tau - k\tau) \left[\left(\frac{N(\tau)-1}{\tau}\right) u(k\tau) - \left(\frac{N(\tau)-1}{\tau}\right) Z(k\tau)\right].
\]
By Theorem A, the quantity $Q_1(n\tau)$ is estimated by
\[
\|Q_1(n\tau)\|_{L^q(\Omega)} \leq \int_0^{n\tau} \left\|S(n\tau - s) \left(\lambda|u|^pu(s) - \left(\frac{N(\tau)-1}{\tau}\right) u(s)\right) \right\|_{L^q(\Omega)} ds
\]
\[
\leq C_{d,p,q} \int_0^{n\tau} (n\tau - s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})} \left\|\lambda|u|^pu(s) - \left(\frac{N(\tau)-1}{\tau}\right) u(s)\right\|_{L^q(\Omega)} ds. \tag{3.3}
\]
Using (2.18) and Corollary 2.2, one yields
\[
\left\|\lambda|u|^pu(s) - \left(\frac{N(\tau)-1}{\tau}\right) u(s)\right\|_{L^q(\Omega)} \leq c_p \tau \|u(s)\|_{L^{q}(\Omega)}^{2p-1} \|\phi\|_{W^{1,q}(\Omega)}^{p+1}.
\] (3.4)
By (3.4), the inequality (3.3) implies
\[
(nt)^{\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-(1-\mu)} \|Q_1(n\tau)\|_{L^q(\Omega)}
\]
\[
\leq C_{d,p,q} \tau \|\phi\|_{L^q(\Omega)}^{p-2} \|\phi\|_{W^{1,q}(\Omega)}^{p+1} \int_0^{n\tau} (n\tau - s)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})} s^{-\mu} ds
\]
\[
\leq C_{d,p,q} \tau \|\phi\|_{L^q(\Omega)}^{p-2} \|\phi\|_{W^{1,q}(\Omega)}^{p+1}. \tag{3.5}
\]
Next, we consider the estimate for $Q_2(n\tau)$ in (3.2). Using Lemma 2.6 with $\eta(t) = \left(\frac{N(\tau)-1}{\tau}\right) u(t)$ and Lemma 2.2, we have
\[
\|Q_2(n\tau)\|_{L^q(\Omega)} \leq \tau \int_0^{n\tau} (n\tau - t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})} \left\|(\partial_t - \Delta) \left(\frac{N(\tau)-1}{\tau}\right) u(t)\right\|_{L^q(\Omega)} dt
\]
\[
\leq C_p \tau \int_0^{n\tau} (n\tau - t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})} A(t) dt, \tag{3.6}
\]
where
\[
A(t) := \|u(t)\|^{2p-1}_{L^q(\Omega)} + \|u(t)|^{p-2}|\nabla u(t)|^2\|_{L^q(\Omega)} + \tau \|u(t)^{2p-3}|\nabla u(t)|^2\|_{L^q(\Omega)}.
\]
By Corollary 2.2, one gets
\[
A(t) \leq C_{d,p,q} t^{-\mu} M_{p,q,\phi}, \tag{3.7}
\]
so the inequality (3.6) implies
\[
(nt)^{\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-(1-\mu)} \|Q_2(n\tau)\|_{L^q(\Omega)} \leq C_{d,p,q} \tau M_{p,q,\phi}. \tag{3.8}
\]
Finally, we derive the estimate for \( Q_n(\tau) \) in (3.2). By (2.1) and Lemma 2.4, one has
\[
\begin{align*}
\|Q_n(\tau)\|_{L^r(\Omega)} & \leq \tau \sum_{0 \leq k < n} \left\| S(n\tau - k\tau) \left[ \left( \frac{N(\tau) - I}{\tau} \right) u(k\tau) - \left( \frac{N(\tau) - I}{\tau} \right) Z(k\tau) \right] \right\|_{L^r(\Omega)} \\
& \leq C_{d,p,q} \tau \sum_{0 \leq k < n} (n\tau - k\tau)^{-\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \left\| \left( \frac{N(\tau) - I}{\tau} \right) u(k\tau) - \left( \frac{N(\tau) - I}{\tau} \right) Z(k\tau) \right\|_{L^r(\Omega)} \\
& \leq C_{d,p,q} \tau \sum_{0 \leq k < n} (n\tau - k\tau)^{-\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \left( \|u(k\tau)\|_{L^\frac{r(p-1)}{r-q}(\Omega)}^{p-1} + \|Z(k\tau)\|_{L^\frac{r(p-1)}{r-q}(\Omega)}^{p-1} \right) \\
& \quad \times \|u(k\tau) - Z(k\tau)\|_{L^r(\Omega)}. \tag{3.9}
\end{align*}
\]
From Theorem 1.1 and Theorem 1.1 one sees
\[
\|u(k\tau)\|_{L^\frac{r(p-1)}{r-q}(\Omega)}^{p-1} + \|Z(k\tau)\|_{L^\frac{r(p-1)}{r-q}(\Omega)}^{p-1} \leq C_{d,p,q}(k\tau)^{-\frac{d}{2}\left(\frac{2}{r} + \frac{1}{q}\right)} \|\phi\|_{L^r(\Omega)}, \tag{3.10}
\]
and on the other hand, the assumption (3.11) is rewritten as
\[
\|u(k\tau) - Z(k\tau)\|_{L^r(\Omega)} \leq C_s \tau (k\tau)^{-\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right) + (1-\mu)M_{p,q,\phi}} \quad \text{for } k = 0, \ldots, n - 1. \tag{3.11}
\]
By (3.10) and (3.11), the inequality (3.9) gives
\[
\begin{align*}
(n\tau)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \mu}\|Q_n(\tau)\|_{L^r(\Omega)} & \leq C_{d,p,q} C_s \tau^2 (n\tau)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - (1-\mu)\|\phi\|_{L^r(\Omega)}^{p-1} \|\phi\|_{L^1(\Omega)}}^{2p-1} \\
& \quad \times \sum_{0 \leq k < n} (n\tau - k\tau)^{-\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)(k\tau)} \|k\tau\|_{L^\frac{r(p-1)}{r-q}(\Omega)}^{p-1} \\
& \leq C_{d,p,q} C_s \tau (n\tau)^{1-\frac{d(p-1)}{2} + \mu}\|\phi\|_{L^r(\Omega)}^{p-1} M_{p,q,\phi}. \tag{3.12}
\end{align*}
\]
As seen in (2.8), we note that since \((n\tau) < T_0\), and by (1.3),
\[
C_{d,p,q}(n\tau)^{1-\frac{d(p-1)}{2} + \mu}\|\phi\|_{L^r(\Omega)}^{p-1} \leq \frac{1}{2}. \tag{3.13}
\]
Applying (3.13) to (3.12), one gets
\[
(n\tau)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - (1-\mu)}\|Q_n(\tau)\|_{L^r(\Omega)} \leq \frac{1}{2} C_s \tau M_{p,q,\phi}. \tag{3.14}
\]
From (3.5), (3.8) and (3.14), the error \( u(n\tau) - Z(n\tau)\phi \) in (3.2) is estimated by
\[
(n\tau)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - (1-\mu)}\|u(n\tau) - Z(n\tau)\phi\|_{L^r(\Omega)} \leq \left( C_{d,p,q} + \frac{1}{2} C_s \right) \tau M_{p,q,\phi}.
\]
If the constant \( C_s > 0 \) is chosen by \( C_s \geq 2C_{d,p,q} \), then the error estimate (3.1) for \( k = n \) holds, so the proof of Theorem 1.2 is concluded.

4. Numerical experiments

In this section, we give some numerical results based on the operator splitting method (1.7) and confirm the analyzed convergence rate in Theorem 1.2. For \( 0 < h < 1 \), let \( \mathcal{T}_h \) be a family of regular partitions of \( \Omega \) into disjoint triangular elements. The finite element spaces are defined by
\[
\begin{align*}
V_h &= \{ v \in C(\overline{\Omega}) : \forall K \in P_1(K) \} \quad \forall K \in \mathcal{T}_h, \\
V_h^0 &= \{ v \in V_h : v|_{\partial \Omega} = 0 \} \subset H^1_0(\Omega),
\end{align*}
\]
where \( P_1(K) \) means the space of linear functions defined on \( K \). Now, we propose the numerical scheme related to \( \frac{Z(n\tau)}{\phi} \) as follows: Let \( u_{h,\tau}^n \in V_h^0 \) be the approximation of \( Z(n\tau)\phi \) for \( n \in \mathbb{N} \), and \( \tau \ll 1 \) the given switching time.

**Algorithm A**

1. Set \( n \leftarrow 0 \) and \( u_{h,\tau}^0 = \pi_h \phi \), where \( \pi_h \) denotes the interpolation operator on \( V_h^0 \).
2. Set \( n \leftarrow n + 1 \). Compute
   \[
   u_{h,\tau}^{n-1/2} = u_{h,\tau}^{n-1} \left( \frac{1}{1 - (p-1)\lambda \tau \left| u_{h,\tau}^{n-1} \right|^{\frac{p-1}{p}}} \right)^{\frac{1}{p-1}}.
   \]
3. Find \( u_{h,\tau}^n \in V_h^0 \) such that
   \[
   \left( \frac{u_{h,\tau}^n - u_{h,\tau}^{n-1/2}}{\tau}, v_h \right) + \left( \nabla u_{h,\tau}^n, \nabla v_h \right) = 0 \quad \forall v_h \in V_h^0. \tag{4.1}
   \]
4. Repeat 2-3 until \( 0 < n\tau < T_2 \), where \( T_2 > 0 \) is a time given by \( (1.10) \).

**Figure 1.** The computational domain

**Example 1.** We check the rate of convergence for \( u_{h,\tau}^n \) on the domain \( \Omega = (0,1)^2 \) depicted by Figure 1. For \( p = 5/2 \), we try to simulate the procedure: **Algorithm A** with the initial function \( \phi \) given by

\[
\phi = \sin(\pi x) \sin(\pi y). \tag{4.2}
\]

Set the meshsize \( h_j = 2^{-j} \) on a level number \( j \geq 1 \) and the time step \( \tau_k = 2^{-k} \) on a level number \( k \geq 1 \). Since the exact formula of \( u \) satisfying \( (1.1) \) are not known, we define the sequential error for the time step \( \tau_k \) on the fixed meshsize \( h_j \) as follows:

\[
\mathcal{E}_u = \left\| u_{h_j,\tau_k}^N - u_{h_j,\tau_{k-1}} \right\|_{L^2(\Omega)},
\]

where \( u_{h_j,\tau_k} \) denotes the discrete solution \( u_{h_j,\tau_k}^N \) with \( N_k := 1/\tau_k \), i.e., it means the approximation of \( u(T) \) on a time \( T = 1 \). Also, the convergence rate is defined by \( Rate := \log_2(\epsilon_{k-1}/\epsilon_k) \), provided that \( \epsilon_k \) is the error on the \( k \)-th level.
On Figure 2 we plot the graphs of $E_u$ on the fixed meshsize $h = 2^{-7}$ or $h = 2^{-10}$. If we assume that $\|Z(n\tau)\phi - u_{h,\tau}^n\|_{L^2(\Omega)} \leq C(\tau + h^2)$ which is perhaps expected by the $L^2$-error estimate for parabolic problem (cf. [23] [26]), and by Theorem 1.2, one sees
\[
\|u(n\tau) - u_{h,\tau}^n\|_{L^2(\Omega)} \leq \|u(n\tau) - Z(n\tau)\phi\|_{L^2(\Omega)} + \|Z(n\tau)\phi - u_{h,\tau}^n\|_{L^2(\Omega)} \\
\leq C(\tau + h^2),
\] where $C > 0$ is a constant independent of $h$ and $\tau$. From the estimate (4.3), we expect that $E_u = O(\tau)$ for a sufficiently small $h$. Seeing the graphs of $E_u$ in Figure 2 it is confirmed that the $L^2$-error $E_u$ has the expected convergence rate 1 as the time step $\tau_k \to 0$.

Example 2. In the second experiment, we try to find the approximation $u_{h,\tau}^n$ obtained by Algorithm A, when the initial functions $\phi = \phi_i$ for $i = 1, 2$ are given by
\begin{align}
\phi_1 &= \begin{cases} 
x \sin(\pi y) & \text{for } x \leq 1/2, \\
(1-x) \sin(\pi y) & \text{for } x > 1/2,
\end{cases} \\
\phi_2 &= \begin{cases} 
x \sin(\pi y) & \text{for } x \leq 1/2, \\
2(1-x) \sin(\pi y) & \text{for } x > 1/2.
\end{cases}
\end{align}

Compared with the function $\phi$ of (4.2) in Example 1, we notice that two functions $\phi_1$ and $\phi_2$ lose the smoothness and furthermore, the function $\phi_2$ is even discontinuous at $x = 1/2$. Nevertheless,
Figure 3 describes that each convergence rate of the $L^2$-error $E_u$ corresponding to the both initial functions is identical to the predicted value 1.

**Appendix: The well-posedness of the approximation $Z(n\tau)\phi$**

We give the proof of Theorem 15.1 regarding the well-posedness of $Z(n\tau)\phi$ given in (1.7). This proof is essentially similar to the one of Theorem A which is shown in Theorem 15.2 of [24].

Let $d \geq 1$, $p > 1$ and $q \geq 1$ with $\frac{d(p-1)}{2} < q < \infty$. From Proposition 2.3, $Z(n\tau)\phi$ is well-defined for $n\tau \in (0, T_2)$. So we consider the following set:

$$
\Lambda(d, p, q) := \left\{ N \in \mathbb{N} : \max_{1 \leq n \leq N} (n\tau)^{\frac{d}{q} - \frac{1}{q}}\|Z(n\tau)\phi\|_{L^r(\Omega)} \leq C_*\|\phi\|_{L^r(\Omega)} \text{ for all } r \in [q, \infty]\right\},
$$

where $C_* := 4\max\{C_{d,p,q}, 1\}$ for a constant $C_{d,p,q} > 0$ used in (1.2). To derive Theorem 15.1 it is sufficient to show that

$$
\Lambda(d, p, q) \supset \{ n \in \mathbb{N} : n\tau < T_2 \}.
$$

**Step 1 (Base case).** We first consider the estimate of $Z(\tau)\phi$. By (2.1) and the explicit form of $N(t)$, and since $\tau \leq T_3/2 \leq T_1/2$, we have

$$
\tau^{\frac{d}{q} - \frac{1}{q}}\|Z(\tau)\phi\|_{L^r(\Omega)} = \tau^{\frac{d}{q} - \frac{1}{q}}\|S(\tau)N(\tau)\phi\|_{L^r(\Omega)}
\leq (4\pi)^{-\frac{d}{2}\left(\frac{1}{q} - \frac{1}{r}\right)}\|N(\tau)\phi\|_{L^r(\Omega)}
\leq 2\tau^{\frac{d}{2}\left(\frac{1}{q} - \frac{1}{r}\right)}\|\phi\|_{L^r(\Omega)},
$$

and then this inequality implies that $1 \in \Lambda(d, p, q)$.

**Step 2 (Inductive step).** Let $n \in \mathbb{N}$ be given with $n\tau < T_2$. It is assumed that $n-1 \in \Lambda(d, p, q)$.

By the expression (1.8) and (2.1), one yields

$$
(n\tau)^{\frac{d}{q} - \frac{1}{q}}\|Z(n\tau)\phi\|_{L^r(\Omega)} \leq (n\tau)^{\frac{d}{q} - \frac{1}{q}}\|S(n\tau)\phi\|_{L^r(\Omega)} + (n\tau)^{\frac{d}{q} - \frac{1}{q}}B(n, \tau)
\leq C_{d,q}\|\phi\|_{L^r(\Omega)} + (n\tau)^{\frac{d}{q} - \frac{1}{q}}B(n, \tau),
$$

where $C_{d,q} := (4\pi)^{-\frac{d}{2}\left(\frac{1}{q} - \frac{1}{r}\right)}$ and

$$
B(n, \tau) := \tau \sum_{0 \leq k < n} \left\| S(n\tau - k\tau) \left( \frac{N(\tau) - I}{\tau} \right) Z(k\tau)\phi \right\|_{L^r(\Omega)}.
$$

Again, using (2.1), we have

$$
B(n, \tau) \leq \tau \sum_{0 \leq k < n} (n\tau - k\tau)^{-\frac{d}{q} - \frac{1}{q}}\left\| \left( \frac{N(\tau) - I}{\tau} \right) Z(k\tau)\phi \right\|_{L^r(\Omega)}
\leq \tau \sum_{0 \leq k < n} (n\tau - k\tau)^{-\frac{d}{q} - \frac{1}{q}}\|Z(k\tau)\phi\|_{L^{qr}(\Omega)}
= \tau \sum_{0 \leq k < n} (n\tau - k\tau)^{-\frac{d}{q} - \frac{1}{q}}(k\tau)^{-\frac{d}{2}\left(\frac{1}{q} - \frac{1}{r}\right)}\left( \left( k\tau \right)^{\frac{d}{2}\left(\frac{1}{q} - \frac{1}{r}\right)}\|Z(k\tau)\phi\|_{L^{qr}(\Omega)} \right)^p
\leq \tau \sum_{0 \leq k < n} (n\tau - k\tau)^{-\frac{d}{q} - \frac{1}{q}}(k\tau)^{-\frac{d}{2}\left(\frac{1}{q} - \frac{1}{r}\right)}\left( \max_{0 \leq k < n} (k\tau)^{\frac{d}{2}\left(\frac{1}{q} - \frac{1}{r}\right)}\|Z(k\tau)\phi\|_{L^{qr}(\Omega)} \right)^p
\leq C_{d,p,q}(n\tau)^{-\frac{d}{q} - \frac{1}{q} - \frac{d}{2}\left(\frac{1}{q} - \frac{1}{r}\right) + 1} \left( \max_{0 \leq k < n} (k\tau)^{\frac{d}{2}\left(\frac{1}{q} - \frac{1}{r}\right)}\|Z(k\tau)\phi\|_{L^{qr}(\Omega)} \right)^p.
$$
From the inductive hypothesis, one sees
\[
\max_{0 \leq k < n} (k\tau)^{\frac{d}{2}} \left( \frac{1}{p} - \frac{1}{q} \right) \| Z(k\tau) \phi \|_{L^p(\Omega)} \leq C_* \| \phi \|_{L^q(\Omega)},
\]
and then we have
\[
B(n, \tau) \leq C_{d,p,q}(n\tau)^{-\frac{d}{2}} \left( \frac{1}{p} - \frac{1}{q} \right) - \frac{d p}{2} \left( \frac{1}{p} - \frac{1}{q} \right) + 1 \| \phi \|_{L^q(\Omega)}.
\]  
(4.7)

By (4.7), the inequality (4.6) becomes
\[
(n\tau)^{\frac{d}{2}} \left( \frac{1}{p} - \frac{1}{q} \right) \| Z(n\tau) \phi \|_{L^q(\Omega)} \leq C_{d,q} \| \phi \|_{L^q(\Omega)} + C_{d,p,q}(n\tau)^{\frac{1}{2}} \left( \frac{1}{p} - \frac{1}{q} \right) C_{*} \| \phi \|_{L^q(\Omega)}.
\]  
(4.8)

Since
\[
n\tau < T_2 \leq T_0 = c_{d,p,q} \left( \frac{1}{\| \phi \|_{L^q(\Omega)}} \right) \frac{1}{q - \frac{d(p-1)}{2}},
\]
and if \( c_{d,p,q} > 0 \) is a sufficiently small constant satisfying \( C_{d,p,q} c_{d,p,q} \frac{1}{q - \frac{d(p-1)}{2}} C_{*}^{-1} \leq 1/2 \), then it is noted that
\[
C_{d,p,q}(n\tau)^{\frac{1}{2}} \left( \frac{1}{p} - \frac{1}{q} \right) C_{*} \| \phi \|_{L^q(\Omega)} \leq C_{d,p,q} c_{d,p,q} \frac{1}{q - \frac{d(p-1)}{2}} C_{*} \| \phi \|_{L^q(\Omega)} \leq \frac{1}{2} C_* \| \phi \|_{L^q(\Omega)}.
\]  
(4.9)

Using (4.9), the inequality (4.8) becomes
\[
(n\tau)^{\frac{d}{2}} \left( \frac{1}{p} - \frac{1}{q} \right) \| Z(n\tau) \phi \|_{L^q(\Omega)} \leq \left( C_{d,q} + \frac{1}{2} C_* \right) \| \phi \|_{L^q(\Omega)} \leq C_* \| \phi \|_{L^q(\Omega)},
\]
which implies that \( n \in \Lambda(d, p, q) \). As shown in Step 1 and 2, the proof of Theorem \( \ref{thm} \) is concluded.

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