The infinite Dirac operator

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Abstract. In this article, we define the infinite Dirac operator and explore some key properties, particularly its conformal invariance. En route, we also establish the conformal invariance of the $p$-Dirac equation. We also introduce the infinite Dirac operator on the sphere $S^n$ and establish the relationship between the two infinite Dirac operators via the Cayley transformation. Also we introduce an infinite Laplace operator on $S^n$.

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1. Introduction

Associated to most Laplace operators is a first order Dirac-type operator. For the Laplacian in Euclidean space, there is the Dirac operator arising in Clifford analysis in euclidean space. For the Laplace-Beltrami operator on a Riemannian manifold, there is the Hodge-Dirac operator $d + *d^*$. Furthermore, on $S^n$, for the conformal Laplacian, or Yamabe operator, there is the spherical Dirac operator, and for a spin manifold associated to the Atiyah-Singer-Dirac operator, there is the spinorial Laplacian.

In [6] it is shown that associated to the second-order non-linear $p$-Laplace operator ($1 < p < \infty$) there is also a first order non-linear $p$-Dirac operator, and the conformal structure associated with the $p$-Laplace operator carries over naturally to the $p$-Dirac operator. In this paper we extend that work by introducing a first order non-linear infinite Dirac operator associated to the non-linear infinite Laplacian in $\mathbb{R}^n$. We introduce its “fundamental solution”. We establish its conformal invariance and the conformal covariance of its solution space. En route we need to establish the conformal invariance of the $p$-Dirac operator and the conformal covariance of the solution set of the $p$-Dirac equation. This differs from the $p$-Laplace case where conformal invariance of the operator only exists for $p = n$. For all other choices of $p$ the $p$-Laplace operator transforms to an $(A,p)$-Laplace operator where $A(x)$ is a weight function dependant on the choice of Möbius transformation. We conclude by using the Cayley transformation to extend some of our results to the sphere, and introducing a $p$-Laplace operator, and infinite Laplace operator on $S^n$ together with their fundamental solutions.

2. Background

One way to derive the Laplace equation in $\mathbb{R}^n$ is from the Dirichlet integral $\int_U \|\nabla f\|^2 dx^n$, where $f : U \to \mathbb{R}$ and $U$ is a domain in $\mathbb{R}^n$. Assuming $U$ is bounded and also assuming appropriate
conditions on the boundary of $U$, minimizing this integral leads to the vanishing of the first moment integral. So we have

$$\int_U \langle \nabla f, \nabla \psi \rangle dx^n = 0,$$

where $\psi \in C_0^\infty(U)$. Applying Stokes’ Theorem to this integral leads to Laplace’s equation.

To get to the $p$-Laplace equation we replace the 2 in the Dirichlet integral by $p$ to get

$$\int_U \|\nabla f\|^p dx^n \text{ where } 1 < p < \infty. \text{ Again minimizing leads to the integral equation}$$

$$\int_U \langle \|\nabla f\|^{p-2}\nabla f, \nabla \psi \rangle dx^n = 0.$$ 

Applying Stokes’ Theorem gives rise to the equation

$$- \text{div}(\|\nabla f\|^{p-2}\nabla f) = 0.$$

This is the $p$-Laplace equation, and the term $- \text{div}(\|\nabla f\|^{p-2}\nabla f)$ is denoted by $\Delta_p f$. For $p \neq 2$ the $p$-Laplace equation is nonlinear. Letting $g : \partial U \to \mathbb{R}$ be a continuous function $g : \partial U \to \mathbb{R}$, there exist unique solutions to the Dirichlet problem

$$\left\{ \begin{array}{ll}
\Delta_p f = 0 & \text{in } U \\
f = g & \text{on } \partial U.
\end{array} \right.$$

In addition, the fundamental solution to the $p$-Laplace equation is given by

$$h_p(x) = \left\{ \begin{array}{ll}
\|x\|^\frac{2-p}{p} & \text{for } p \neq n \\
\log \|x\| & \text{for } p = n.
\end{array} \right.$$ 

That is, for appropriate constants $C_{p,n} = C(p,n)$ we have

$$\Delta_p h_p(x) = C_{p,n} \delta_0$$

in the sense of distributions. For further discussion on the $p$-Laplace equation in the Euclidean environment, the interested reader is directed to [3, 4] and the references therein.

Taking the limit as $p \to \infty$ we obtain the infinite Laplace operator.

3. Preliminaries

We wish to extend the nonlinear infinite Laplace equation by an infinite Dirac equation. To do this we need some background on Clifford algebras and some results from [6] on $p$-Dirac operators.

We shall consider Euclidean $n$ space, $\mathbb{R}^n$, as embedded in the real $2^n$ dimensional Clifford algebra $Cl_n$ where for each $x \in \mathbb{R}^n$ we have $x^2 = -\|x\|^2$. In terms of an orthonormal basis $e_1, \ldots, e_n$ this gives the anticommutation relationship

$$e_i e_j + e_j e_i = -2 \delta_{ij}$$

and a basis

$$1, e_1, \ldots, e_n$$

for $Cl_n$, where $j_1, \ldots, j_r$ and $1 \leq r \leq n$.

We shall need the antiautomorphism $\sim : Cl_n \to Cl_n : \sim (e_{j_1} \ldots e_{j_r} e_{j_{r+1}} \ldots e_{j_{r+s}}) = e_{j_{r+s}} \ldots e_{j_1}$. For $A \in Cl_n$ we write $A$ for $\sim A$.

We also need the antiautomorphism $- : Cl_n \to Cl_n : -(e_{j_1} \ldots e_{j_r}) = (-1)^r e_{j_r} \ldots e_{j_1}$. We write $A$ for $-(A)$. The scalar part of $AA$ gives the square of the norm of $A$. More precisely, if $A = a_0 + \ldots + a_{m,n} e_1 \ldots e_n$ then the scalar part of $AA$ is $a_0^2 + \ldots + a_{m,n}^2$.

In [1] and elsewhere it is shown that if $y = M(x)$ is a Möbius transformation over $\mathbb{R}^n \cup \{\infty\}$ then $y = (ax + b)(cx + d)^{-1}$ where $a, b, c$ and $d \in Cl_n$ with the following properties:
This last equation is to be interpreted in the distributional sense. Furthermore, $f$ to a constant, $R$ the Dirac operator in $\mathbb{R}^n$ is the first order differential operator $D := \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}$. Note that $D^2 = -\Delta$, the Laplacian in $\mathbb{R}^n$. The fundamental solution to the Dirac equation, $Df = 0$ is, up to a constant, $\frac{e^x}{2\pi}$. Here $f$ is defined on a domain $U$ in $\mathbb{R}^n$ and $f$ takes its values in $C_\text{fin}(\mathbb{R}^n)$.

In [2] it is shown that
$$D_y = J_{-1}(M, x)^{-1} D_x J_1(M, x)$$
where $D_y$ is the Dirac operator with respect to $y = M(x)$, $D_x$ is the Dirac operator with respect to $x$ with $J_{-1}(M, x) = \frac{e^{x^t d}}{\|x\|_2}$ and $J_1(M, x) = \frac{e^{x^t d}}{\|x\|_2}$. In [6] the $p$-Dirac equation is defined to be $D_p f = D(|f|^{p-2} f) = 0$. When $p \neq 2$, this equation is nonlinear. When $p = n$ a solution is given by
$$u_n(x) = x^{-1} = -\frac{x}{\|x\|^2}.$$ 
For other choices of $p$ a solution is
$$u_p(x) = \frac{x}{\|x\|^{\frac{n+p-2}{p-1}}}.$$ 
If $f = Dg$ then the $p$-Dirac equation becomes $D(|Dg|^{p-2} Dg) = 0$, and when $g$ is a scalar valued function then the scalar part of this equation becomes the $p$-Laplace equation.

Consider the integral
$$\int_U \|f(x)\|^{p-2} f(x) D\psi(x) dx^n$$
where $f : U \to C_\text{fin}(\mathbb{R}^n)$ and $\psi \in C_\infty^\omega(U, C_{\text{fin}})$, where $C_\infty^\omega(U, C_{\text{fin}})$ is the space of $C_{\text{fin}}$-valued test functions with support in $U$. If this integral vanishes for all $\psi \in C_\infty^\omega(U, C_{\text{fin}})$ then $f$ is called a weak solution to the $p$-Dirac equation. Note that
$$\frac{1}{\omega_n} \int_U \|u_p(x - y)\|^{p-2} u_p(x - y) D\psi(x) dx^n = \psi(y)$$
for all $y \in U$, where $\omega_n$ is the surface area of the unit sphere $S^{n-1}$. This identity follows from the fact that
$$\|u_p(x - y)\|^{p-2} u_p(x - y) = \frac{x - y}{\|x - y\|^n},$$
and this, up to a scalar, is the fundamental solution of the Dirac operator $D$. In this sense $u_p$ can be regarded as the fundamental solution to the $p$-Dirac equation. Alternatively, as $\psi$ is smooth on $\mathbb{R}^n$ and $u_p(x - y)$ is smooth on $\mathbb{R}^n \setminus \{y\}$, applying Stokes’ Theorem to the previous integral we obtain the identity
$$\int_U D_p u_p(x - y) \psi(x) dx^n = \psi(y).$$
This last equation is to be interpreted in the distributional sense. Furthermore
$$D \frac{1}{\omega_n} \int_U \|u_p(x - y)\|^{p-2} u_p(x - y) \psi(x) dx^n = \psi(y).$$
4. The infinite Dirac operator in $\mathbb{R}^n$

We now define the infinite Dirac operator acting on a function $f$ as $D_{\infty}f(x)$ by taking the formal limit of $D_p f(x)$ as $p \to \infty$. Taking the limit of the fundamental solutions $u_p(x)$, as $\lim_{p \to \infty} \frac{n+p-2}{p-1} = 1$ we obtain

$$u_{\infty}(x) = \lim_{p \to \infty} u_p(x) = \frac{x}{\|x\|}$$

which leads to the following theorem.

**Theorem 4.1** Let $u_{\infty}$ be defined as above. Then for all $x \in \mathbb{R}^n \setminus \{0\}$, we have

$$D_{\infty}u_{\infty}(x) = 0.$$

Note that as $\frac{1}{\omega_n} \int_U \|u_p(x-y)\|^{p-2}u_p(x-y)D\psi(x)dx^n = \psi(y)$ for all $y \in U$ and $\psi \in C_0^\infty(U, Cl_n)$ then applying Stokes’ Theorem to this expression and taking limits in $p$ we have

$$\frac{1}{\omega_n} \int_U D_{\infty}u_{\infty}(x-y)\psi(x)dx^n = \psi(y).$$

This equation is valid in the distributional sense.

As previously noted, if $f = Dg$ then the $p$-Dirac operator becomes $D(\|Dg(x)\|^{p-2}Dg(x))$ and the scalar part of this is the usual $p$-Laplacian operator whenever $g$ is scalar valued. If we take the limit of this operator as $p$ tends to infinity then we obtain a Clifford valued operator acting on $g(x)$ which we denote by $\Delta'_{\infty}g(x)$. So $\Delta'_{\infty}g(x) = D_{\infty}Dg(x)$. Note that when $g$ is scalar-valued we retrieve the usual infinite Laplacian acting on $g$, $\Delta_{\infty}g(x)$. Further when $\|g(x)\| = \|x\|$ then $Dg(x) = n\frac{x}{\|x\|^2}$, and in this case $\Delta_{\infty}g(x) = \Delta_{\infty}g(x) = nD_{\infty}\frac{x}{\|x\|}$. These provide some links between $\Delta_{\infty}$ and $D_{\infty}$. In fact if we have a $Cl_n$ valued differentiable function, $g(x)$, defined on $U$ and we place $Dg(x) = f(x)$ we see that $\Delta'_{p}g(x) = D_{p}f(x)$. So if $\|Dg(x)\|^{p-2}Dg(x)$ converges in the $Cl_n$ valued Sobolev space $W_0^{1,1}(U, Cl_n)$ as $p$ tends to infinity then $\Delta'_{\infty}g(x) = D_{\infty}f(x)$ in the weak sense. In greater generality suppose that for each $p \in (1, \infty)$ the function $g_p : U \to Cl_n$ is differentiable almost everywhere and converges pointwise almost everywhere to the function $g_{\infty}(x)$. Suppose further that $f_p$ is defined to be $Dg_p$ and that $f_p$ converges almost everywhere to $f_{\infty}$ and $\|f_p(x)\|^{p-2}f_p(x)$ converges as $p$ tends to infinity in the Sobolev space $W_0^{1,1}(U, Cl_n)$ then $\Delta'_{\infty}g_{\infty} = D_{\infty}f_{\infty}$ in the weak sense.

When $g$ is scalar-valued, we have

$$\Delta'_{\infty}g(x) = \Delta_{\infty}g(x) + \Delta''_{\infty}g(x)$$

where $\Delta''_{\infty}g(x)$ is the part of $D(\|Dg(x)\|^{2}Dg(x))$ taking its values in the subspace $<e_1e_2, \ldots, e_{n-1}e_n>$ of $Cl_n$ and $\Delta''_{\infty}g(x) = \lim_{p \to \infty} \Delta''_{p}g(x)$. When $n = 3$ applying the Hodge star map reveals that when $g$ is scalar-valued then

$$\Delta''_{\infty}g(x) \equiv \lim_{p \to \infty} \nabla \times \|\nabla g(x)\|^{p-2}\nabla g(x).$$

Now let us return to the integral $\int_U \|f(y)\|^{p-2}f(y)D_y\psi(y)dy^n$. If $f = M(x) = (ax + b)(cx + d)^{-1}$ then using the intertwining operators for $D_x$ and $D_y$ and the fact that the Jacobian for $M(x)$ is $\frac{1}{\|cx+dy\|^n}$, we obtain

$$\int_U \|f(y)\|^{p-2}f(y)D_y\psi(y)dy^n =$$

$$\int_{M^{-1}(U)} \|f(M(x))\|^{p-2} \frac{cx + d}{\|cx+dy\|^n} f(M(x))D_xJ_1(M, x)\psi(M(x))dx^n.$$
Redistributing terms in \(||cx + d||\) this integral becomes
\[
\int_{M^{-1}(U)} \|k(M, p, x)f(M(x))\|^{p-2}k(M, p, x)f(M(x))D_xJ_1(M, x)\psi(M(x))dx^n
\]
where
\[
k(M, p, x) = \frac{cx + d}{||cx + d||^{n+p-2}}.
\]

This establishes the conformal invariance of the operator \(D_p\) and the conformal covariance of its solution set.

As \(\psi\) is arbitrary in \(C_0^\infty(U, Cl_n)\), we then have:

**Theorem 4.2** The function \(f : U \rightarrow Cl_n\) is a weak solution to the \(p\)-Dirac equation if and only if
\[
k(M, p, x)f(M(x))
\]
is a weak solution to the \(p\)-Dirac equation on \(M^{-1}(U)\) where \(y = M(x) = (ax + b)(cx + d)^{-1}\).

In particular the fundamental solution
\[
u_p(y) = \frac{y}{||y||^{n+p-2}}
\]
is transformed to
\[
u_p = \frac{cx + d}{||cx + d||^{n+p-2}} \frac{(ax + b)(cx + d)^{-1}}{||ax + b||^{n+p-2}}
\]
Now, \((ax + b)(cx + d)^{-1}\) is a vector in \(\mathbb{R}^n\) and so \((ax + b)(cx + d)^{-1} = (cx + d)^{-1}(ax + b)\). We then have
\[
u_p = \frac{ax + b}{||ax + b||^{n+p-2}}
\]
and we obtain a solution to the \(p\)-Dirac equation.

Furthermore, for each \(\psi \in C_0^\infty(U, Cl_n)\) and for each Möbius transformation \(y = M(x) = (ax + b)(cx + d)^{-1}\), we have that
\[
\lim_{p \rightarrow \infty} \int_U \|f(y)\|^{p-2}f(y)D_y\psi(y)dy^n
\]
can be expressed as
\[
\lim_{p \rightarrow \infty} \int_{M^{-1}(U)} \|k(M, p, x)f(M(x))\|^{p-2}k(M, p, x)f(M(x))D_xJ_1(M, x)\psi(M(x))dx^n.
\]

Suppose \(\lim_{p \rightarrow \infty} \int_U \|f(y)\|^{p-2}f(y)D_y\psi(y)dy^n = 0\) then we have established:

**Theorem 4.3** A function \(f : U \rightarrow Cl_n\) satisfies \(D_\infty f(y) = 0\) if and only if \(D_\infty \frac{cx + d}{||cx + d||} f(M(x)) = 0\) where \(y = M(x) = (ax + b)(cx + d)^{-1}\).

This establishes the conformal invariance of the operator \(D_\infty\) and the conformal covariance of its solution set.

In the particular case where \(f(y) = \nu_\infty(y) = \frac{y}{||y||}\) we get that the function \(\frac{ax + b}{||ax + b||}(ax + b)(cx + d)^{-1}||ax + b||^{n+p-2} = (ax + b)(cx + d)^{-1}\) is a solution to the infinite Dirac equation. By the same arguments presented here for the \(p\)-Dirac equation this simplifies to \(\frac{ax + b}{||ax + b||}\). So \(\frac{ax + b}{||ax + b||}\) is a solution to the infinite Dirac equation.
5. The infinite Dirac operator on the sphere

As done in [6], we may define the $p$-Dirac operator on the sphere $S^n$. Recall that the Dirac operator on the sphere is given by

$$D_s = x (\Gamma_x + \frac{n}{2})$$

where $x \in S^n$ and

$$\Gamma_x = \sum_{i<j} e_i e_j \left( x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right).$$

The $p$-Dirac operator acting on a function $f : V \rightarrow Cl_{n+1}$, with $V$ a domain on $S^n$, is defined on the sphere for $1 < p < \infty$ by

$$D_{s,p} f = D_s (\|f\|^{p-2} f).$$

See [6]. Further $f$ is called a weak solution on $V$ to the operator $D_{s,p}$ if for each $\psi \in C_0^\infty(V, Cl_{n+1})$

$$\int_V \|f(x)\|^{p-2} \overline{f(x)} D\psi(x) dV = 0$$

where $dV$ is the volume measure of the domain $V$ on $S^n$.

Given a point $y \in S^n$, the fundamental solution to the equation $D_{s,p} h_p = 0$ with singularity $y$ is given by

$$h_p(x) = \frac{x - y}{\|x - y\|^{n+p-2}}.$$

Similar to the previous section, we can define the infinite Dirac operator on the sphere, $D_{s,\infty}$, as the limit of the operators $D_{s,p}$.

The Cayley transformation from $\mathbb{R}^n$ to $S^n$ is given by $y = C(x) = (e_{n+1} x + 1)(x + e_{n+1})^{-1}$ and the relationship between $D$ and $D_s$ is given by $D_s = (x + e_{n+1})^{-1} \|x + e_{n+1}\|^{n+2} D \frac{x + e_{n+1}}{\|x + e_{n+1}\|^{n+1}}$.

See [5]. Adapting arguments from the previous section we can now see that under the Cayley transformation the $p$-Dirac operator $D_{s,p} f = D(\|f(y)\|^{p-2} f(y))$ on a domain $V$ on $S^n$ transforms to the $p$-Dirac operator

$$D \left\| \frac{x + e_{n+1}}{\|x + e_{n+1}\|^{n+p-2}} \right\|^{p-2} \frac{x + e_{n+1}}{\|x + e_{n+1}\|^{n+p-2}} f(M(x))$$

on the domain $C^{-1}(V)$ in $\mathbb{R}^n$. In particular if $D_{s,p} f(y) = 0$ then

$$D_p \frac{x + e_{n+1}}{\|x + e_{n+1}\|^{n+p-2}} f(M(x)) = 0.$$

By taking limits as $p$ tends to infinity we see that $D_{s,\infty} f(y) = 0$ if and only if

$$D_{\infty} \frac{x + e_{n+1}}{\|x + e_{n+1}\|} f(M(x)) = 0.$$

In [5] it is shown for $x$ and $y \in S^n$ that $\|x - y\|^n = 2(1 - <x, y>)$ and that $\Gamma_x (x - y)^{\alpha} = 2^\alpha \Gamma_x (1 - <x, y>) = 2^\alpha \frac{\alpha}{2} (1 - <x, y>)^{\alpha-1} x \land y$. It follows from [5] that $(x\Gamma_x + \frac{\alpha}{2})^\alpha (x - y)^{\alpha} = 2^\alpha (x\Gamma_x + \frac{\alpha}{2}) (1 - <x, y>)^{\alpha-1} x \land y$. But $x \land y = xy + <x, y>$ and $x^2 = -1$. So this term simplifies to give the equation

$$\frac{x\Gamma_x + \frac{\alpha}{2}}{2} (x - y) = \frac{\alpha}{2} (x - y)^{-\alpha-1}.$$
Now consider the case of \( \| x - y \|^{\frac{p-n}{p-1}} \). In this case

\[
x(\Gamma x + \frac{p-n}{2p-2})\| x - y \|^{\frac{p-n}{p-1}} = \frac{\alpha}{2} \frac{x - y}{\| x - y \|^{\frac{n+p-2}{p-1}}}.
\]

As \( D_s = x(\Gamma x + \frac{2}{p}) \) this previous equation can be rewritten as

\[
(D_s + x \frac{p(n-1)}{p-1})\| x - y \|^{\frac{p-n}{p-1}} = \frac{\alpha}{2} \frac{x - y}{\| x - y \|^{\frac{n+p-2}{p-1}}}.
\]

We shall denote the differential operator \( D_s + x \frac{p(n-1)}{p-1} \) by \( D'_s,p \).

Note that \( \| D'_s,p \|_x - y \|^{\frac{p-n}{p-1}} \| ^{p-2} D'_s,p \|_x - y \|^{\frac{p-n}{p-1}} \) is up to a constant \( \frac{1}{\omega_n \| x - y \|^{n-p}} \) the fundamental solution to the operator \( D_s \). See [5].

It follows that up to a constant, \( \alpha(p) \), depending on \( p \)

\[
\alpha(p) \int_{S^n} D'_s,p \| x - y \|^{\frac{p-n}{p-1}} \| ^{p-2} D'_s,p \|_x - y \|^{\frac{p-n}{p-1}} D_s,p \psi(x) dS^n = \psi(y)
\]

for any \( \psi \in C^\infty(S^n) \). It follows that the p-Laplace operator on \( S^n \) acting on \( g(x) \) is

\[
\Delta_s,p g(x) := D_s \| D'_s,p g(x) \|^{p-2} D'_s,p g(x).
\]

This gives a correction to [6]. We have also shown that \( \| x - y \|^{\frac{p-n}{p-1}} \) is, up to a constant, the fundamental solution for the operator \( \Delta_s,p \).

We can also say that a function \( g(x) \) on a domain \( V \subset S^n \) is a weak solution to the operator \( \Delta_s,p \) if for each \( \psi \in C^\infty_0(V, Cl_{n+1}) \)

\[
\int_V \| D'_s,p g(x) \|^{p-2} D'_s,p g(x) D_s \psi(x) dS^n = 0.
\]

Further we can formally define the infinite Laplace operator on \( S^n \) acting on \( g(x) \) to be \( \Delta_{s,\infty} g(x) \) as \( \lim_{p \to \infty} \Delta_s,p g(x) \). It follows that \( \| x - y \| \) is a solution to the infinite Laplace operator on \( S^n \). We see that \( \lim_{p \to \infty} \frac{2p-2}{p-1} D'_s,p \| x - y \|^{\frac{p-n}{p-1}} = \frac{x - y}{\| x - y \|^{n-p}} \) the fundamental solution of \( \Delta_{s,\infty} \). So we have a link between the operators \( \Delta_{s,\infty} \) and \( \Delta_{s,p} \). In fact if for a differentiable \( Cl_{n+1} \) valued function \( g(x) \) defined on \( V \) we place \( D_s,p g(x) = f_p(x) \) then \( \Delta_{s,p} g(x) = D'_s,p f_p(x) \).

So if \( D'_s,p g(x) \|^{p-2} D'_s,p g(x) \) converges as \( p \) tends to infinity in the \( Cl_{n+1} \) valued Sobolev space \( W^{1,1}(S^n, Cl_{n+1}) \) then \( \Delta_{s,\infty} g(x) = D_{s,\infty} f_p(x) \) in the weak sense.

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