We study a weakly coupled, frustrated two-leg spin-1/2 Heisenberg ladder. For vanishing coupling between the chains, elementary excitations are deconfined, gapless spin-1/2 objects called spinons. We investigate the fate of spinons for the case of a weak interchain interaction. We show that despite a drastic change in ground state, which becomes spontaneously dimerised, spinons survive as elementary excitations but acquire a spectral gap. We furthermore determine the exact dynamical structure factor for several values of momentum transfer.

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I. INTRODUCTION

The role of frustration in quasi one-dimensional magnetic materials has attracted much experimental and theoretical attention in recent years. On the theoretical side, the simplest example of a frustrated quantum magnet is the spin-1/2 Heisenberg antiferromagnetic chain with nearest neighbour exchange $J$ and next-nearest neighbour exchange $\delta J$. This model is equivalent to a two-leg ladder (see Fig.1), where the coupling along (between) the legs of the ladder is equal to $J$ ($\delta J$).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Heisenberg zigzag ladder.}
\end{figure}

The zigzag ladder model is believed to describe the quantum magnet $\text{SrCu}_2\text{O}_2$ \cite{1} above the magnetic ordering transition, which takes place at about $T \approx 2K$. The exchange constants are estimated to be $J \approx 1800K$, $|\delta J/J| \approx 0.1 - 0.2$ \cite{2}. A second material with zigzag structure, that has recently attracted much interest, is $\text{Cs}_2\text{CuCl}_4$ \cite{3}. However, in $\text{Cs}_2\text{CuCl}_4$ all neighbouring chains are coupled by a zigzag interaction and no pronounced ladder structure exists.

In Refs. \cite{4,5,6} it was argued that a weak antiferromagnetic zigzag coupling between the chains drives the model to a massive phase, characterised by spontaneous dimerisation (see also \cite{7}). Let us briefly review some important parts of the derivations of \cite{5,6}. The lattice Hamiltonian of the zigzag ladder is

$$H = J \sum_{j=1,2} \sum_n S_{j,n} \cdot S_{j,n+1} + \delta J \sum_n (S_{1,n} + S_{1,n+1}) \cdot S_{2,n},$$

where we assume that $\delta \ll 1$. The low-energy effective action for \cite{5} is now obtained as follows. For $\delta \to 0$ one is dealing with two decoupled Heisenberg chains, which can be bosonised in terms of Wess-Zumino-Novikov-Witten (WZNW) models by using the standard relation between the spin density on chain $j$ and the fields of the WZNW model (see e.g. \cite{8})

$$\frac{S_j^a(x)}{a_0} = [J_j^a(x) + J_j^a(x)] + (-1)^{a_0} \delta n_j^a(x).$$

Here $a_0$ is the lattice spacing, and the fields $J_j^a$ and $\bar{J}_j^a$ are the right and left currents of the WZNW model corresponding to chain $j$. They parametrise the smooth component of the magnetisation. Finally, $\bar{n}_j$ is the staggered component of the magnetisation on chain $j$. Using \cite{8}, the zigzag interchain interaction can be expressed in terms of the WZNW fields. In this way one straightforwardly obtains the current-current interaction \cite{9}

$$\mathcal{H}_c = \lambda_1 (J_1^a + J_2^a)(J_2^a + J_2^a) - \lambda_0 (J_1^a J_1^a + J_2^a J_2^a),$$

where $\lambda_1 \propto \delta J$. A standard renormalisation group (RG) analysis then shows that the antiferromagnetic interchain interaction $\lambda_1$ leads to a spontaneously dimerised ground state \cite{9}. In \cite{9} it was shown that, in addition to the current-current interaction \cite{9}, a “twist” term arises

$$\mathcal{H}_t = \rho (n_1^a \partial_x n_2^a - n_2^a \partial_x n_1^a).$$

In the presence of exchange anisotropies the twist term induces incommensurabilities in the spin correlations \cite{10}. We expect this to be hold true even in the $\text{SU}(2)$ symmetric case (no exchange anisotropies) we are interested in here. In the latter case it can be shown that the twist
term and current-current interaction are equally important in the RG sense: they diverge (i.e. reach strong coupling) simultaneously, with a fixed ratio \([10]\). As far as the SU(2) symmetric zigzag ladder is concerned, it is therefore not possible to separate the effects of the twist and current-current interactions in a simple way.

However, from a purely theoretical point of view it clearly is desirable to develop a thorough understanding of the physics due to isolated current-current and twist interactions. Their effects can be disentangled by introducing an exchange anisotropy \([9]\), which makes the twist more and the current-current interaction less relevant in the RG sense. Using this trick, a pure twist interaction was studied in \([9]\).

The role of an isolated current-current interaction has been previously investigated in connection with the zigzag ladder in \([7-9]\). In particular, the spectrum of the states and elementary excitations. In section VI we determine the exact dynamical structure factor for several values of momentum transfer and show that there are no coherent contributions to the structure factor. We conclude with a summary and discussion of our results.

II. A FRUSTRATED LADDER WITHOUT TWIST

The model we consider is a generalisation \([13]\) of the standard two-leg spin ladder which, apart from the on-run coupling \(J_\perp\), also includes an interaction \(J_\times\) across both diagonals of the plaquettes. The Hamiltonian reads

\[
H = J \sum_{j=1,2} \sum_n S_{j,n} \cdot S_{j,n+1} + J_\perp \sum_n S_{1,n} \cdot S_{2,n} + J_\times \sum_n [S_{1,n} \cdot S_{2,n+1} + S_{1,n+1} \cdot S_{2,n}] \tag{5}
\]

We assume that

\[
J, J_\perp, J_\times > 0, \quad J \gg J_\perp, J_\times . \tag{6}
\]

The low-energy effective action can be derived by non-abelian bosonisation in the usual way. The Hamiltonian density is found to be of the form

\[
\mathcal{H}(x) = \mathcal{H}_1(x) + \mathcal{H}_2(x) + \mathcal{H}_{\text{int}}(x) , \tag{7}
\]

where \(\mathcal{H}_{1,2}\) are critical SU\((1)\) WZNW models with a marginally irrelevant current-current perturbation \((\lambda_0)\):

\[
\mathcal{H}_j = \frac{2\pi v_s}{3} (\bar{\mathbf{J}}_j \cdot \mathbf{J}_j : + : \mathbf{J}_j \cdot \mathbf{J}_j :) - \lambda_0 \bar{\mathbf{J}}_j \cdot \mathbf{J}_j , \quad j = 1, 2. \tag{8}
\]

The interaction part is given by

\[
\mathcal{H}_{\text{int}} = \lambda_1 (\mathbf{J}_1 + \bar{\mathbf{J}}_1) \cdot (\mathbf{J}_2 + \bar{\mathbf{J}}_2) + \lambda_2 \mathbf{n}_1 \cdot \mathbf{n}_2 , \tag{9}
\]

with the coupling constants

\[
\lambda_1 = (J_\perp + 2J_\times) a_0, \quad \lambda_2 = (J_\perp - 2J_\times) a_0 . \tag{10}
\]

No marginal perturbation with the twist-term structure arises because the staggered magnetisation operators add rather than subtract due to the geometry of the problem. The absence of such term can also be deduced from the existence of discrete (reflection) symmetries of the lattice Hamiltonian \((\bar{\mathbf{J}})\). If \(J_\perp = 2J_\times\) only the marginal (current-current) interaction survives. This is the case we study in the remainder of this paper.

We note that for generic values of \(J_\perp\) and \(J_\times\) the interaction of staggered magnetisations dominates and the resulting physics is essentially the same as for the standard ladder \((J_\times = 0)\) \([14]\) (see also chapter 21 of \([13]\)).

III. DUALITY TRANSFORMATION

The low-energy effective action \([\ref{e:2}]\)-\([\ref{e:4}]\) can be recast as a theory of four massive, interacting, real (Majorana)
fermions, or equivalently, four weakly coupled Ising models [8]

\[ \mathcal{H} = -i \sum_{a=0}^{3} v_0 (\psi_0 \partial_x \psi_0 - \bar{\psi}_0 \partial_x \bar{\psi}_0) \]

\[ + \frac{\lambda_1 - \lambda_0}{2} \sum_{j > i=1}^{3} \psi_i \bar{\psi}_j \psi_j \bar{\psi}_i + \frac{\lambda_1 + \lambda_0}{2} \psi_0 \bar{\psi}_0 \sum_{i=1}^{3} \psi_i \bar{\psi}_i. \]  

(11)

Here \( v_1 = v_2 = v_3 = v_4 \neq v_0 \) are the velocities of the four Majorana fermions. The lattice spin operators are expressed in terms of the Majorana fields and order and disorder operators of the four Ising models as

\[ S_\pm (x) = -i (\psi_1 \psi_2 + \bar{\psi}_1 \bar{\psi}_2) - \mathcal{A}(-1)^{x/\alpha} \mu_1 \mu_2 \sigma_3 \sigma_4, \]

\[ S^z (x) = i (\psi_3 \psi_0 + \bar{\psi}_3 \bar{\psi}_0) + \mathcal{A}(-1)^{x/\alpha} \sigma_1 \sigma_2 \mu_3 \mu_0, \]  

(12)

where \( S_\pm (x) = S^+_1 (x) \pm S^+_2 (x) \) and \( \mathcal{A} \) is a nonuniversal constant. Analogous expressions are available for the other components of the spin operators [8]. A standard one-loop RG analysis shows that the coupling \( \lambda_0 \) flows to zero, so we will ignore it in what follows. In order to further simplify the problem, we also neglect the small difference between the velocities \( v_4 \) and \( v_0 \), and finally perform a duality transformation on the Majorana fields

\[ \psi_0 \rightarrow \psi_4, \quad \bar{\psi}_0 \rightarrow -\bar{\psi}_4, \quad \sigma_0 \rightarrow \mu_4, \quad \mu_0 \rightarrow \sigma_4. \]  

(13)

This yields the Hamiltonian of the O(4) Gross-Neveu model [12]

\[ \mathcal{H} = -i \sum_{i=1}^{4} \psi_i \partial_x \psi_i - \bar{\psi}_i \partial_x \bar{\psi}_i + \frac{\lambda_1}{2} \sum_{j > i=1}^{4} \psi_i \bar{\psi}_j \psi_j \bar{\psi}_i. \]  

(14)

Under [12] the spin-densities transform to

\[ S_\pm (x) \propto -i (\psi_1 \psi_2 + \bar{\psi}_1 \bar{\psi}_2) - \mathcal{A}(-1)^{x/\alpha} \mu_1 \mu_2 \sigma_3 \sigma_4, \]

\[ S^z (x) \propto i (\psi_3 \psi_0 + \bar{\psi}_3 \bar{\psi}_0) + \mathcal{A}(-1)^{x/\alpha} \sigma_1 \sigma_2 \mu_3 \mu_0. \]  

(15)

IV. GROUND STATE

In order to proceed, it is convenient to use the representation of (14) in terms of two sine-Gordon models [18]. Ignoring terms that only renormalise the velocity we find that (14) is equivalent to

\[ \mathcal{H} = \sum_{i=\pm} \frac{v_i}{2} \left( (\partial_x \varphi_i)^2 + (\partial_x \theta_i)^2 \right) \]

\[ + 2\lambda_1 \left[ \frac{1}{8\pi} (\partial_x \varphi_i)^2 - (\partial_x \theta_i)^2 \right] - \frac{1}{(2\pi\alpha_0)^2} \cos \sqrt{8\pi \varphi_i}, \]  

(16)

where \( \theta_i \) are the dual fields. The two sine-Gordon models [13] occur on the SU(2) invariant strong-coupling separatrix of the Kosterlitz-Thouless phase diagram and are thus in the massive regime.

A. Twistless Ladder

The low-energy effective model [13] exhibits a local \( Z_2 \) symmetry related to independent translations by one lattice spacing on each chain \( (\varphi_\pm \rightarrow \varphi_\pm + \sqrt{\pi/2}) \). This symmetry is spontaneously broken in the ground state and leads to a nonvanishing dimerization. Notice that the \( Z_2 \) symmetry appears to be a feature of the low-energy sector only and follows from the fact that spin currents \( J_{1,2} \) are translationally invariant objects. The transformation \( S_1 (n) \rightarrow 1/2 [S_{1,n+1} + S_{1,n-1}] \), or a similar one with \( S_{1,k} \rightarrow S_{2,k} \), changes the lattice Hamiltonian but leaves the low-energy effective field theory invariant and maps the two ground states onto one another.

In order to characterise the dimerisation patterns of the two ground states, we determine the expectation values

\[ \langle \tilde{S}_{1,n} \cdot \tilde{S}_{2,n} \rangle \propto \langle \tilde{J}_1 (x) \cdot \tilde{J}_2 (x) \rangle + \langle \tilde{n}_1 (x) \cdot \tilde{n}_2 (x) \rangle, \]

\[ \langle \tilde{S}_{1,n} \cdot \tilde{S}_{2,n+1} \rangle \propto \langle \tilde{J}_1 (x) \cdot \tilde{J}_2 (x) \rangle - \langle \tilde{n}_1 (x) \cdot \tilde{n}_2 (x) \rangle, \]

\[ \langle \tilde{S}_{2,n} \cdot \tilde{S}_{1,n+1} \rangle \propto \langle \tilde{J}_1 (x) \cdot \tilde{J}_2 (x) \rangle - \langle \tilde{n}_1 (x) \cdot \tilde{n}_2 (x) \rangle. \]  

(17)

After performing the duality transformation to the O(4) Gross-Neveu model and bosonising, we obtain

\[ \langle \tilde{n}_1 (x) \cdot \tilde{n}_2 (x) \rangle \propto \langle \cos \sqrt{2\pi \varphi_+} \cos \sqrt{2\pi \varphi_-} \rangle = \pm \text{const } m, \]

\[ \langle \tilde{J}_1 (x) \cdot \tilde{J}_2 (x) \rangle \propto \left( \langle \cos \sqrt{2\pi \varphi_+} \cos \sqrt{2\pi \varphi_-} \rangle \right)^2 = \text{const } m^2, \]  

(18)

where \( m \propto \exp(-\text{const} J / J_\perp) \) is the (exponentially small) soliton mass in the Sine-Gordon model. Due to the smallness of \( m \), the \( \langle \tilde{n}_1 (x) \cdot \tilde{n}_2 (x) \rangle \) expectation value dominates in (17), so that within the exponential accuracy the dimerisation is proportional to the quantum soliton mass.

![FIG. 3. Qualitative picture of the two degenerate dimerised ground states: spins connected by the solid (dotted) lines have a tendency to form singlets (triplets).](image-url)
The $\mathbb{Z}_2$ symmetry of the low-energy effective Hamiltonian (16), that manifests itself in the degeneracy of the two ground states corresponding to different signs in (18), is spontaneously broken, implying the existence of massive $\mathbb{Z}_2$ kinks. It turns out (see below) that these kinks are elementary excitations of the model.

B. Zig-Zag Ladder

Let us discuss the implications of the emergence of spontaneous dimerisation for the case of the zig-zag ladder if we ignore the twist term. For the zig-zag ladder the appropriate definition for the dimerisation is

$$d = \langle \vec{S}_1(x) \cdot \left( \vec{S}_2(x + a_0/2) - \vec{S}_2(x - a_0/2) \right) \rangle. \quad (19)$$

In the continuum limit we find

$$d \propto \langle \cos \sqrt{2} \pi \phi_+ \cos \sqrt{2} \pi \phi_- \rangle = \pm \text{const } m. \quad (20)$$

The resulting dimerisation patterns are shown in Fig.4.

![FIG. 4. Qualitative picture of the spin configuration in the dimerised ground states: spins along the thick diagonal bonds have a tendency to form singlets.](image)

We believe that taking into account the twist term will not qualitatively change this picture.

V. EXCITATIONS

From the exact solution of the Sine-Gordon models (16) we infer that there are only four elementary excitations corresponding to solitons and antisolitons in the $\pm$ sectors. We denote these by $s_\pm$ and $\bar{s}_\pm$. The elementary excitation have a simple interpretation in terms of dimerisation kinks, i.e. domain walls separating regions of dimerisation with opposite sign. It can be shown along the lines of (16) that these particles carry spin $\pm 1/2$. In terms of the low-energy effective theory of two Sine-Gordon models (16), the total spin density is given by

$$S_1^z(x) + S_2^z(x) = \frac{1}{\sqrt{2}} \left[ \partial_x \varphi_+(x) + \partial_x \varphi_-(x) \right]. \quad (21)$$

Kinks interpolate between asymptotic values of the fields $\varphi_i$ differing by $\pm \sqrt{\pi/2}$ as is most easily deduced from the fact that the classical vacua of (16) are located at

$$\langle \varphi_i \rangle_{\text{class}} = \sqrt{\frac{\pi}{2}} n_i, \quad i = \pm, \quad (22)$$

where $n_i$ are arbitrary integers. Integration of (21) then yields that a single kink carries spin

$$S^z = \pm \frac{1}{\sqrt{2}} \sqrt{\frac{\pi}{2}} = \pm \frac{1}{2}. \quad (23)$$

The results presented below for the dynamical structure factor are consistent with the interpretation of these kinks as gapped spinons. Altogether there are two spin-1/2 multiplets, corresponding to one multiplet for each leg of the ladder. The emerging physical picture is quite simple and pretty: the two-spinon states observed in the structure factor simply correspond to the kinks related to the spontaneous breakdown of the discrete $\mathbb{Z}_2$ symmetry.

Simple visualizations of this picture are shown in Fig.5 for the twistless ladder and in Fig.6 for the zig-zag ladder.

![FIG. 5. A two-spinon state in the twistless ladder. Spinons correspond to kinks between domains with different sign of the dimerisation. Solid lines depict bonds along which there is a tendency to form singlets.](image)

For the twistless ladder the kinks correspond to vertical domain walls between regions with different signs of dimerisation. There is a spin-1/2 associated with each domain wall, although this is not immediately obvious from Fig.5. In order to get a feeling why a spin-1/2 might be associated with each kink, let us think of the translationally invariant, “double-zigzag” ground state shown in Fig.3 as a symmetric superposition of two dimerised states. Each such state represents a sequence of plaquettes with ideal singlet bonds across the plaquette diagonals (with each spin involved in one bond only), has a period $2a_0$ and is shifted with respect to the other state by one lattice spacing. If the “double-zigzag” phase occupies a finite domain of the ladder, for the two $2a_0$-periodic dimerised states to resonate, the number of rungs within such a domain should be odd. Then the two-kink configuration in Fig.5 can equivalently be viewed as the superposition of states shown in Fig.6.
VI. DYNAMICAL STRUCTURE FACTOR

The long-distance asymptotics of the spin-spin correlation functions are dominated by the soft modes at $q = 0, \pi, q_\perp = 0, \pi$, where $q$ and $q_\perp$ denote the wave-numbers along and perpendicular to the two chains, respectively. In what follows we will determine the dynamical structure factor for wave numbers in the vicinity of the above four points in $\vec{q}$-space. Due to the spin-rotational symmetry the dynamical structure factor is given by

$$S(\omega, q, q_\perp) \propto \text{Im} \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dt e^{-i\omega t + iqx} \times \langle \{ S^z_1(t, x) \pm S^z_2(t, x) \}, \{ S^z_1(0, 0) \pm S^z_2(0, 0) \} \rangle .$$

(24)

where the positive (negative) sign corresponds to $q_\perp = 0$ ($q_\perp = \pi$).

A. Summary of large-N results

The dynamical structure factor has been previously calculated in the framework of a large-$N$ approach. The limit $N \to \infty$ of (14) is equivalent to a theory of free massive Majorana fermions

$$\mathcal{H} = \frac{i\psi}{2} \sum_{i=1}^{N} \psi_i \partial_x \bar{\psi}_i - \bar{\psi}_i \partial_x \psi_i + im\psi_i \bar{\psi}_i .$$

(25)

The presence of the mass term reflects the spontaneous breakdown of parity, which in turn implies the existence of two degenerate ground states. The sign of the mass terms, as well as the expectation values $\langle \sigma_3 \rangle$ and $\langle \mu_3 \rangle$, depend on the choice of ground state. The case where $\langle \sigma_3 \rangle \neq 0$ (the mass of the triplet is positive), the structure factor for $q_\perp = 0$ and $q$ around $0, \pi$ was shown to be

$$S(\omega, q \approx \pi, 0) \propto \frac{m}{|\omega|} \delta \left( \omega - \sqrt{v_s^2(q - \pi)^2 + m^2} \right) ,$$

$$S(\omega, q \approx 0, 0) \propto \frac{m^2 q^2}{s^2 \sqrt{s^2 - 4m^2}} ,$$

(26)

where $s^2 = \omega^2 - v_s^2 q^2$. The explicit expressions for the structure factor around $(q, q_\perp) = (0, \pi), (\pi, \pi)$ are complicated, but reveal the presence of incoherent two and three particle continua, respectively. We will now show that the results obtained in the large-$N$ limit are qualitatively incorrect. The reason for this failure of the large-$N$ approach is that it entirely neglects the existence of topological kinks interpolating between the two degenerate ordered ground states. Extrapolation of the large-$N$ results to lower values of $N$ should be done with caution because the spectrum of the $O(N)$ Gross-Neveu model is very sensitive to the value of $N$.

B. Exact results

We now determine the dynamical structure factor using exact results on formfactors in the Sine-Gordon model. We start with the case $q_\perp = 0, q \approx 0$. The smooth component of the sum of the two spin densities is expressed in terms of the Sine-Gordon models as follows

$$S^z_1(x) + S^z_2(x) \bigg|_{\text{smooth}} \propto \partial_x \varphi_+ + \partial_x \varphi_- .$$

(27)
This is nothing but the sum of the temporal components of the current operators in the two Sine-Gordon models \( (j^0_1 + j^0_2) \). We are interested in the structure factor, i.e.,

\[
S(\omega, q \approx 0, 0) \propto \text{Im} \sum_{\sigma = \pm} \int_0^\infty dt \int_0^\infty \text{d}x \ e^{i(\omega \pm \epsilon) t - i v_q x} \\
\times \langle [j^0_\sigma(x, t), j^0_\sigma(0, 0)] \rangle ,
\]

(28)

where \( v_q \) is the velocity of the excitations. We express [28] in the spectral representation using our knowledge of a complete set of states in terms of (anti) soliton scattering states. Energy and momentum are parametrised in terms of the rapidity variable \( \theta \) as

\[
p = m \sinh \theta \quad \epsilon = m \cosh \theta ,
\]

(29)

where \( m \) is the mass of the four elementary excitations. The resolution of the identity is given by

\[
1 = \sum_{n=0}^{\infty} \sum_{\alpha_1, \cdots, \alpha_n} \int \frac{d\theta_1 \cdots d\theta_n}{(2\pi)^n n!} \langle \theta_n \cdots \theta_1 \rangle_{\alpha_n \cdots \alpha_1} \langle \theta_1 \cdots \theta_n \rangle_{\alpha_1 \cdots \alpha_n} ,
\]

(30)

where \( n \) is the number of particles and \( \alpha_i \in \{ s_\pm, s_\pm \} \) specifies their respective “flavour” (soliton or antisoliton in + or − sector). Inserting (30) in (28) and using Poincaré invariance yields

\[
S(\omega, q \approx 0, 0) \propto \sum_{n=0}^{\infty} \sum_{\alpha_1, \cdots, \alpha_n} \int \frac{d\theta_1 \cdots d\theta_n}{(2\pi)^n n!} |F_{\overset{\circ}{\rho}}(\theta_1 \cdots \theta_n)_{\alpha_1 \cdots \alpha_n}|^2 \\
\times \left[ \delta (v_q - m \sum_j \sinh \theta_j) - \delta (v_q + m \sum_j \sinh \theta_j) \right] - \left[ \omega + m \sum_j \cosh \theta_j + i \epsilon \right].
\]

(31)

where \( F_{\overset{\circ}{\rho}}(\theta_1 \cdots \theta_n)_{\alpha_1 \cdots \alpha_n} \) is the Sine-Gordon current form factor

\[
F_{\overset{\circ}{\rho}}(\theta_1 \cdots \theta_n)_{\alpha_1 \cdots \alpha_n} = \langle 0 | j^0(0, 0) | \theta_n \cdots \theta_1 \rangle_{\alpha_n \cdots \alpha_1} .
\]

(32)

We note that an \( n \)-particle state only contributes to (31) above the \( n \)-particle threshold, i.e., \( s^2 = \omega^2 - v_q^2 q^2 \geq n^2 m^2 \). Thus, at low energies \( s^2 \leq 16 m^2 \) only two-particle states contribute. The corresponding formfactor is [31]

\[
F_{\overset{\circ}{\rho}}(\theta_1, \theta_2)_{s_k} = -2 m \sinh \left( \frac{\theta_1 + \theta_2}{2} \right) f(\theta_1 - \theta_2) ,
\]

\[
f(\theta) = \frac{i \sinh \theta/2}{2\pi} \\
\times \exp \left( \int_0^\infty \frac{\text{d}k}{\kappa} \frac{\sin^2 \left( \frac{\pi}{2} (\theta - \pi i) \right)}{\kappa \ sinh(\pi \kappa)} \left[ \text{th} \left( \frac{\pi \kappa}{2} \right) - 1 \right] \right).
\]

(33)

After performing the \( \theta \)-integrations we obtain

\[
S(\omega, q \approx 0, 0) \propto \frac{m^2 v_q^2 |f(2\theta(s))|^2}{s^2 \sqrt{s^2 - 4m^2}} ,
\]

(34)

where \( \theta(s) = \text{arccosh} \left( \frac{s}{2m} \right) \) and \( 4m^2 < s^2 < 16m^2 \). As we already mentioned, the result (34) is exact as long as \( s^2 < 16m^2 \). For larger energy transfers there are (small) corrections due to four, six, eight etc particle states. These can be calculated in the same way as the two-particle contribution. Approaching the threshold \( s = 2m \) from above, (34) goes to zero like \( \sqrt{s - 2m} \).

The result (34) has the same structure as the one obtained in the large-\( N \) approximation. We note that the vanishing of the structure factor for \( q = 0 \) \( (S(\omega, 0, 0) = 0) \) reflects the fact that the \( z \)-component of spin is a conserved quantity.

Next, we consider the structure factor at \( (q \approx 0, \pi) \). The smooth component of the difference of spin densities is

\[
S^\pi_1(x) - S^\pi_2(x) \mid_{\text{smooth}} \propto \partial_x \varphi_+ - \partial_x \varphi_-. \]

(35)

This is precisely the difference of the spatial components of the currents in the two Sine-Gordon models \( (j_1^z - j_2^z) \). Using the exact two-particle formfactor we obtain the leading contribution to the structure factor

\[
S(\omega, q \approx 0, \pi) \propto \frac{m^2 \omega^2 |f(2\theta(s))|^2}{s^3 \sqrt{s^2 - 4m^2}} ,
\]

(36)

where \( f(\theta) \) is given by (33) and again \( 4m^2 < s^2 < 16m^2 \). Note that the structure factor does not vanish for \( q \to 0 \) as the magnetisation difference between chains is not conserved. This is due to the fact that our starting point does not have \( \text{O}(4) \) symmetry: after the duality transformation we obtain an \( \text{O}(4) \) symmetric Lagrangian, but correlation functions transform nontrivially. This result is of course expected, since the interchain interaction must break the \( \text{O}(4) \sim SU(2) \times SU(2) \) down to \( SU(2) \).

Finally, we examine the structure factor at \( (q \approx \pi, 0) \) and \( (q \approx \pi, \pi) \). The bosonised forms for the staggered components of the sum and difference of the spin densities are found to be

\[
S^\pi_1(x) + S^\pi_2(x) \mid_{\text{stagg}} \propto \cos \sqrt{\pi \Phi} \cos \sqrt{\pi \Theta} ,
\]

\[
S^\pi_1(x) - S^\pi_2(x) \mid_{\text{stagg}} \propto \sin \sqrt{\pi \Phi} \sin \sqrt{\pi \Theta} ,
\]

(37)

where \( \Phi = (\varphi_+ + \varphi_-)/\sqrt{2} \) and \( \Theta = (\varphi_+ - \varphi_-)/\sqrt{2} \). At present it is not known how to calculate formfactors for the operators appearing in (37) as they involve both the field and the dual field. However, it is still possible to determine the qualitative behaviour of the structure factor. From (47) it is clear that the structure factor involves the calculation of formfactors of operators

\[
[\cos \text{ or sin}] \left( \sqrt{\frac{\pi}{2}} \varphi_+ \right) [\cos \text{ or sin}] \left( \sqrt{\frac{\pi}{2}} \varphi_- \right).
\]
\[ \times [\cos \text{ or } \sin] \left( \sqrt{\frac{\pi}{2}} \varphi_+ \right) [\cos \text{ or } \sin] \left( \sqrt{\frac{\pi}{2}} \varphi_- \right). \] (38)

These formfactors are obviously products of formfactors in the two Sine-Gordon models. Let us therefore concentrate on the + sector for the time being. It was shown in [22] that the operators \( \cos \sqrt{\frac{\pi}{2}} \theta_+ \) and \( \sin \sqrt{\frac{\pi}{2}} \theta_+ \) in the Sine-Gordon model with coupling constant \( \beta = \sqrt{8\pi} \) have fermionic character and thus have nontrivial formfactors with one-soliton states. On the other hand, we know from [14] that \( \cos \sqrt{\frac{\pi}{2}} \varphi_+ \) and \( \sin \sqrt{\frac{\pi}{2}} \varphi_+ \) are of bosonic character. We therefore conclude that + part of the operator (18) has fermionic character. This implies that it couples only to states with at least one (anti) soliton. An analogous statement holds true for the - sector, so that the leading contribution to the structure factor comes from two-particle states. In other words no coherent one-particle excitation exists.

From the above results for the dynamical structure factor we deduce that the low-lying excitations are described in terms of a gapped two-particle scattering continuum. As we have mentioned above, the elementary excitations carry spin-1/2. This leads us to identify them as massive spinons.

VII. SUMMARY

We have studied the effects of pure current-current interactions in a frustrated two-leg spin ladder. We have shown that spinons, which are gapless topological excitations propagating along decoupled Heisenberg chains, survive as elementary excitations in the frustrated ladder, but acquire a finite mass gap. We have given an interpretation of these massive spinons as quantum dimerisation kinks. The kinks are deconfined and, in all physical states, appear only in pairs. As a result their contribution to the dynamical structure factor is entirely incoherent. Our findings bear a strong resemblance to those of [23].

We believe that our results not only apply to the ladder (3), but with some modifications also to the zigzag ladder (1). As discussed above, in the zigzag case there is a twist term in addition to the current-current interaction. We conjecture, that the effect of the twist term is merely to shift the minimum of the two-spinon continua at \( (q = \pi, 0) \) and \( (q = \pi, \pi) \) to incommensurate wave numbers, i.e. to \( (q = \pi + \delta, 0) \) and \( (q = \pi + \delta, \pi) \), where \( |\delta| \ll 1 \). Such a picture is consistent with what is known from numerical studies [14, 29] and also fits well to what one would expect on the basis of an (uncontrolled) extrapolation of the results for \( \delta = O(1) \) [27, 28] to \( |\delta| \ll 1 \).

Coming back to the twistless chain (3), it should be pointed out that its ground state and excitations have been previously studied for the special case \( J_\perp = J \) [29, 30] ("Bose-Gayen model"). In this case, the Hamiltonian (3) exhibits an enlarged (local) symmetry, related to the interchange \( S_1(n) \leftrightarrow S_2(n) \) at arbitrary rung \( n \), and decouples into two commuting parts describing either an array of entirely decoupled on-rung singlets or an effective S=1 chain [31]. In both cases, the ground state belongs to the universality class of the (undimerised) Haldane spin liquids with the spin-1 massive magnons being coherent elementary excitations [14, 23]. This is in marked contrast with our findings for \( J_\perp = \frac{1}{2} J_\parallel \ll J \) and implies the existence of a crossover between the two regimes at some intermediate coupling.

It should be understood that the region where the marginally perturbed ladder \( \lambda_2 = 0 \) and the Bose-Gayen model start overlapping, i.e. the vicinity of the point \( J_\perp = 2 J_\parallel = 2 J \), is not accessible within our continuum approach, based on the assumption that \( J_\perp, J_\parallel \ll J \). Staying on the line \( J_\perp = 2 J_\parallel \) and increasing \( J_\parallel \) would enforce the amplitude of the current-current perturbation \( \lambda_1 \) to increase, in which case no reliable conclusions are available. On the other hand, one can start approaching the Bose-Gayen regime by keeping \( J_\parallel \) fixed and increasing \( J_\parallel \). In this case one inevitably deviates from the line \( J_\perp = 2 J_\parallel \), and that gives rise to the appearance of the strongly relevant perturbation \( \lambda_2 \mathbf{n}_1 \cdot \mathbf{n}_2 \). The latter introduces an extra potential,

\[ U \sim \lambda_2 [2 \cos \sqrt{2\pi} (\varphi_+ - \varphi_-) - \cos \sqrt{2\pi} (\varphi_+ + \varphi_-) + \cos \sqrt{2\pi} (\theta_+ - \theta_-)], \] (39)

that couples the two Sine-Gordon models (16), removes the \( Z_2 \) degeneracy between the two dimerised ground states and thus leads to soliton confinement. The soliton-antisoliton pairs start forming triplet and singlet massive bound states and transform to coherent single-particle excitations. If the deviation from the line \( J_\perp = 2 J_\parallel \) is large enough, the \( \lambda_2 \) perturbation takes over, and the effective low-energy field theory becomes that of four Majorana fermions, with a mass term

\[ \propto i \lambda_2 \left( 3 \sum_{a=1}^3 \psi_a \bar{\psi}_a - 3 \psi_0 \bar{\psi}_0 \right) \]

as it is the case for the standard (nonfrustrated) ladder (1). It is therefore tempting to speculate that the two massive Haldane phases on the both sides of the line \( J_\perp = 2 J_\parallel \) can be smoothly connected with those of the Bose-Gayen model. This, however, does not exclude the existence of other phases in the 3-parameter space of the model (3).

As discussed in Refs. [27, 28], a similar soliton confinement scenario is realized if one adds an explicit dimerisation to the zigzag Hamiltonian (1).

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