Time decay for solutions to the Stokes equations with drift

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Abstract

In this note, we study the behaviour of Lebesgue norms $\|v(\cdot,t)\|_p$ of solutions $v$ to the Cauchy problem for the Stokes system with drift $u$, which is supposed to be a divergence free smooth vector valued function satisfying a scale invariant condition.

1 Introduction

The main aim of the paper is the following Stokes system with a drift $u$

$$\partial_t v - u \cdot \nabla v - \Delta v - \nabla q = -\text{div } F, \quad \text{div } v = 0 \quad (1.1)$$

in $Q_+ = \mathbb{R}^3 \times ]0, \infty[$ and

$$v(x,0) = 0 \quad (1.2)$$

for $x \in \mathbb{R}^3$.

It is supposed that a tensor-valued field $F$ is smooth and compactly supported in $Q_+$. In addition, let us assume that $F$ is skew symmetric and therefore

$$\text{div } \text{div } F = 0. \quad (1.3)$$

As to the drift $u$, one may assume that $u$ is a bounded divergence free field in $Q_+$, say $|u| \leq 1$ there, whose derivatives of any order exist and are bounded in $Q_+$.

It is not so difficult to prove, see Appendix I, the following statement.

Proposition 1.1. There exists a unique solution $v$ to (1.1) and (1.2) with properties:

$$\nabla^l \partial_t^k v \in L_2(Q_+)$$
for $k, l = 0, 1, \ldots$ except $k + l = 0$,
\[ \nabla^{l+1} \partial_{t}^{k} q \in L_{2}(Q_{+}) \]
for $k, l = 0, 1, \ldots$,
\[ v \in L_{2, \infty}(Q_{+}), \quad q \in L_{2, \infty}(Q_{+}) \]
for any $k = 0, 1, \ldots$.

The goal of the paper is to study how $L_{p}$-norms of the velocity field $v$
\[ \left( \| v(\cdot, t) \|_{p} := \left( \int_{\mathbb{R}^{3}} |v(x, t)|^{p} \, dx \right)^{\frac{1}{p}} \right) \]
behave as $t \to \infty$. In particular, two cases are of great interest: $p = 1$ and $p = 2$.

Let us impose a decay assumption on the drift
\[ |u(x, t)| \leq \frac{c_{d}}{|x| + \sqrt{t}} \quad (1.4) \]
for all $(x, t) \in Q_{+}$.

Two results will be proven in the paper.

**Theorem 1.2.** Let $v$ be a solution $v$ to (1.1) and (1.2) and let $u$ satisfy
(1.4). Then for any $m = 0, 1, \ldots$, two decay estimates are valid:
\[ \| v(\cdot, t) \|_{1} \leq c(m, c_{d}) \sqrt{t} \frac{1}{\ln^{m}(t + e)} \quad (1.5) \]
and
\[ \| v(\cdot, t) \|_{2} \leq \frac{c(m, c_{d})}{\ln^{m}(t + e)} \quad (1.6) \]

To motivate the aforesaid problem and the assumptions made, consider the
Navier-Stokes system
\[ \partial_{t} w + w \cdot \nabla w - \Delta w = -\nabla r, \quad \div w = 0 \]
in the unit parabolic ball $Q = B \times [-1, 0[$ for functions $w \in L_{\infty}(-1, 0; L_{2}(B)) \cap L_{2}(-1, 0; W_{2}^{1}(B))$ and $r \in L_{2}^{3}(Q)$ satisfying the additional restriction
\[ |w(x, t)| \leq \frac{c_{d}}{|x| + \sqrt{-t}} \quad (1.7) \]
for all \((x,t) \in Q\). Our aim is to understand whether or not the origin \(z = (x,t) = (0,0)\) is a regular point of \(w\), i.e., there exists \(\delta > 0\) such that \(v\) is essentially bounded in the parabolic ball \(Q(\delta) = B(\delta) \times ] - \delta^2, 0[^\ast\). Here, as usual, \(B(r)\) stands for the ball of radius \(r\) centered at the origin. The answer is certainly positive if \(c_d\) is sufficiently small. However, we would not like to make such an assumption at this point. In [8], it has been shown that if \(z = 0\) is a singular point of \(w\) then a so-called a mild bounded ancient solution \(\tilde u\) to the Navier-Stokes equations in \(Q_- = \mathbb{R}^3 \times ] - \infty, 0[^\ast\) exists and it is non-trivial. The latter means the following: \(\tilde u \in L^\infty(Q_-) (|\tilde u| \leq 1\text{ a.e. in } Q_-\text{ and } |u(0)| = 1)\) and there exists a scalar function \(\tilde \rho \in L^\infty(-\infty, 0; BMO(\mathbb{R}^3))\) such that the pair \(\tilde u\) and \(\tilde \rho\) satisfy the classical Navier-Stokes system

\[
\partial_t \tilde u + \tilde u \cdot \nabla \tilde u - \Delta \tilde u = -\nabla \tilde \rho, \quad \text{div } \tilde u = 0 \quad (1.8)
\]

in \(Q_-\) in the sense of distributions. It is known, see [4], that \(\tilde u\) is infinitely smooth and all its derivatives are bounded. Moreover, it can be shown, see Appendix II, that, for \(u(x,t) = \tilde u(x, -t)\),

\[
\int_{Q_+} u \cdot \text{div} F dx dt = - \lim_{T \to \infty} \int_{\mathbb{R}^3} u(x, T) \cdot v(x, T) dx. \quad (1.9)
\]

If time decay of \(v\) is such that, for any tensor-valued field \(F \in C_0^\infty(\mathbb{R}^3)\), obeying condition (1.3), the limit on the right hand side of (1.9) vanishes, then one can easily show that \(u\) must be a function of time only. Indeed, we then have

\[
\int_{Q_+} \nabla u : F dx dt = 0.
\]

The latter means that the skew symmetric part of \(\nabla u\) vanishes in \(Q_+\). Since \(u\) is a divergence free field, \(u\) is a bounded harmonic function and so does \(\tilde u\) in \(Q_-\). But \(\tilde u\) is a bounded mild ancient solution to the Navier-Stokes equation and thus must be a constant in \(Q_-\) as well as \(u\) in \(Q_+\). But condition (1.4) means that \(\tilde u\) is identically zero. This finally would prove that \(z = 0\) is not a singular point of \(w\) and condition (1.7) is in fact a regularity condition.

Unfortunately, decay bounds in Theorem 1.2 do not provide the above scenario. Let us give a couple of bounds on \(c_d\) that give a required time decay.

To describe the first case, we are going to use a solution formula for the Stokes system with non-divergence free right hand side.
Let
\[ F = -v \otimes u + F. \]
The solution to problem (1.1), (1.2) has the form, see for instance [4],
\[ v(x, t) = \int_0^t \int_{\mathbb{R}^3} K(x - y, t - s) F(y, s) dy ds, \] (1.10)
where the potential \( K = (K_{ij}) \) defined with the help of the standard heat kernel in the following way
\[ \Delta \Phi(x, t) = \Gamma(x, t) \]
and
\[ K_{ij} = \Phi_{ij} - \delta_{ii} \Phi_{kk}. \]
It is easy to check that the following bound is valid:
\[ |K(x, t)| \leq \frac{c_1}{(t + |x|^2)^2} \] (1.11)
and therefore
\[ \int_{\mathbb{R}^3} |K(x, t)| dx \leq \frac{c_*}{\sqrt{t}} \] (1.12)
with \( c_* = cc_1 \), where \( c \) is an absolute constant.

**Theorem 1.3.** Assume that
\[ 4c_*c_d < 1. \] (1.13)
Then
\[ \int_{\mathbb{R}^3} v(x, T) \cdot u(x, T) dx \to 0 \] (1.14)
as \( T \to \infty \).

To describe the second case, let us introduce the operator \( K : \mathcal{L}_2 \to J_2 \), where \( \mathcal{L}_2 \) consists of all tensor-valued functions, belonging to \( L_2(\mathbb{R}^3) \) and satisfying condition (1.3), and \( J_2 \) is a space of square integrable divergence.
free fields in $\mathbb{R}^3$. The action of this operator is defined as $A_F = K F$, where $A_F$ is the unique solution to the following problem

$$\text{rot} \ A_F = - \text{div} \ F.$$  

The elliptic theory reads that operator $K$ is bounded.

In addition, one may introduce the second operator $M : L_2(\mathbb{R}^3; M^{3 \times 3}) \to L_2(\mathbb{R}^3)$ so that

$$\Delta q_F = - \text{div} \ \text{div} \ F,$$

where $q_F = M F$.

Actually, we have fixed the pressure $q = q_{\text{v}} \otimes u$ in Proposition 1.1. This will be done everywhere in what follows. Our result is the following.

**Theorem 1.4.** Let

$$c_d \leq \frac{\sqrt{3}}{2 \|K\|(1 + \sqrt{3}\|M\|)}.$$  

Then (1.14) is true.

## 2 Time Decay of $L_1$-Norm

Now, from (1.10), it follows

$$\|v(\cdot, t)\|_p \leq \int_0^t \int_{\mathbb{R}^3} |K(\cdot - y, t - s)| \mathcal{F}(y, s) dy \|_p ds$$

Applying Hölder inequality and taking into account (1.12), we find

$$\|v(\cdot, t)\|_p \leq \int_0^t \left( \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} K(x - y', t - s) dy' \right)^\frac{p}{2} \right. \times$$

$$\left. \times \int_{\mathbb{R}^3} |K(x - y, t - s)| |\mathcal{F}(y, s)|^p dy dx \right)^{\frac{1}{p}} ds \leq c \int_0^t \frac{1}{\sqrt{t - s}} \|\mathcal{F}(\cdot, s)\| ds$$

for any $p \geq 1$.

Now, for $p$, satisfying the condition

$$p \in ]6/5, 2[, \quad (2.1)$$
Hölder inequality gives the following estimate
\[
\|v(\cdot, t)\|_p \leq c \int_0^t \frac{ds}{\sqrt{t-s}} \left( \int_{\mathbb{R}^3} |\mathcal{F}(y, s)|^2 (\sqrt{s} + |y|)^2 dy \right)^{\frac{1}{2}} \times \\
\times \left( \int_{\mathbb{R}^3} \left( \frac{1}{\sqrt{s} + |y|} \right)^{\frac{2p}{2-p}} dy \right)^{\frac{2-p}{2p}}.
\]
By changing variables \( y = z\sqrt{s} \),
\[
\left( \int_{\mathbb{R}^3} \left( \frac{1}{\sqrt{s} + |y|} \right)^{\frac{2p}{2-p}} dy \right)^{\frac{2-p}{2p}} \leq \sqrt{s}^{\frac{5p-6}{2p}} \left( \int_{\mathbb{R}^3} \left( \frac{1}{1 + |z|} \right)^{\frac{2p}{2-p}} dz \right)^{\frac{2-p}{2p}} = \\
= C(p) \sqrt{s}^{\frac{5p-6}{2p}}
\]
with
\[
C(p) := \left( \int_{\mathbb{R}^3} \left( \frac{1}{1 + |z|} \right)^{\frac{2p}{2-p}} dz \right)^{\frac{2-p}{2p}} \to \infty
\]
as \( p \to 6/5 + 0 \). So,
\[
\|v(\cdot, t)\|_p \leq C(p) \int_0^t \frac{ds}{\sqrt{t-s}} \sqrt{s}^{\frac{5p-6}{2p}} \left( \int_{\mathbb{R}^3} |\mathcal{F}(y, s)|^2 (\sqrt{s} + |y|)^2 dy \right)^{\frac{1}{2}}.
\]
Now, by our assumptions on \( F \) and by (1.4),
\[
\int_{\mathbb{R}^3} |\mathcal{F}(y, s)|^2 (\sqrt{s} + |y|)^2 dy \leq c(c_d\|v(\cdot, s)\|_2 + \|G(\cdot, s)\|_2)^2,
\]
where \( G(y, s) = F(y, s)(\sqrt{s} + |y|) \), and thus
\[
\|v(\cdot, t)\|_p \leq C(p) \int_0^t \frac{ds}{\sqrt{t-s}} \sqrt{s}^{\frac{5p-6}{2p}} (c_d\|v(\cdot, s)\|_2 + \|G(\cdot, s)\|_2)
\]
\[
\leq C(p) A_p(s)
\]
with
\[ A_p(s) := \int_0^t \frac{ds}{\sqrt{t-s}} \sqrt{s^{5p-6}} (c_d \| v(\cdot, s) \|_2 + \| G(\cdot, s) \|_2). \]  

(2.3)

So,
\[ \| v(\cdot, t) \|_p \leq C(p) A_p(t). \]  

(2.4)

Now, one can repeat the above arguments for \( p = 1 \) and find
\[
\| v(\cdot, t) \|_1 \leq \int_0^t \frac{c}{\sqrt{t-s}} \int_{\mathbb{R}^3} |\mathcal{F}(y, s)| dy ds.
\]

Since
\[
|\mathcal{F}(y, s)| \leq c \frac{c_d |v(y, s)| + |G(y, s)|}{\sqrt{s + |y|}},
\]

the latter estimate can be transform as follows:
\[
\| v(\cdot, t) \|_1 \leq c \int_0^t \frac{ds}{\sqrt{t-s}} \int_{\mathbb{R}^3} c_d |v(y, s)| + |G(y, s)| \frac{1}{\sqrt{s + |y|}} dy \leq
\]
\[
\leq c \int_0^t \frac{ds}{\sqrt{t-s}} \left( \int_{\mathbb{R}^3} \left( \frac{1}{\sqrt{s + |y|}} \right)^{1+5\varepsilon} dy \right)^{\frac{1}{1+5\varepsilon}} \left( \int_{\mathbb{R}^3} (c_d |v(y, s)| + |G(y, s)|)^{\frac{1+5\varepsilon}{5}} dy \right)^{\frac{5}{1+5\varepsilon}}
\]

for some positive \( 0 < \varepsilon < 3/10 \). Hence,
\[
\| v(\cdot, t) \|_1 \leq C_1(\varepsilon) \int_0^t \frac{ds}{\sqrt{t-s}} \sqrt{s^{1+5\varepsilon} - 1} \left( \int_{\mathbb{R}^3} (c_d |v(y, s)| + |G(y, s)|)^{\frac{1+5\varepsilon}{5}} dy \right)^{\frac{5}{1+5\varepsilon}}
\]

with
\[
C_1(\varepsilon) := \left( \int_{\mathbb{R}^3} \left( \frac{1}{1+|z|} \right)^{\frac{6+5\varepsilon}{1+5\varepsilon}} dz \right)^{\frac{1+5\varepsilon}{6+5\varepsilon}}.
\]
Simplifying slightly the previous bound, we have
\[
\|v(\cdot, t)\|_1 \leq C_1(\varepsilon) \int_0^t \frac{ds}{\sqrt{t-s}} \sqrt{s}^{-\frac{3+10\varepsilon}{8+5\varepsilon}} (\|c_d v(\cdot, s)\|_{\frac{6+5\varepsilon}{5}} + \|G(\cdot, s)\|_{\frac{6+5\varepsilon}{5}}) dy.
\]

By (2.4),
\[
\|v(\cdot, s)\|_{\frac{6+5\varepsilon}{5}} \leq C(6/5 + \varepsilon) A_{\frac{6}{5} + \varepsilon}(t).
\]
So, the final estimate of L1-norm is:
\[
\|v(\cdot, t)\|_1 \leq C_3(\varepsilon) \int_0^t \frac{ds}{\sqrt{t-s}} \sqrt{s}^{-\frac{3+10\varepsilon}{8+5\varepsilon}} (c_d A_{\frac{6}{5} + \varepsilon}(s) + \|G(\cdot, s)\|_{\frac{6+5\varepsilon}{5}}) dy
\]
with \(C_3(\varepsilon) \to \infty\) as \(\varepsilon \to 0\).

Since the energy of \(v\) is bounded, one can derive from (2.3) the following:
\[
A_p(t) \leq c(s)(c_d)p \|v\|_{2,\infty} + \|G\|_{2,\infty} \sqrt{t}^{\frac{6p}{2p}}
\]
and thus
\[
A_{\frac{6}{5} + \varepsilon}(t) \leq c(\varepsilon)(c_d)2 \|v\|_{2,\infty} + \|G\|_{2,\infty} \sqrt{t}^{\frac{12 \cdot 15}{2(6+5\varepsilon)}}.
\]
Now, (2.5) is giving to us:
\[
\|v(\cdot, t)\|_1 \leq c \sqrt{t}^{\frac{3}{2}}
\]
where \(c\) depends on the data of the problem.

3 Improvement for \(L_2\)-norm

Here, we are going to use methods developed in [5] and [11], see also [1] and [3].

We have the energy inequality
\[
\partial_t y(t) + \|\nabla v(\cdot, t)\|_2^2 \leq \|F(\cdot, t)\|_2^2
\]
with \(y(t) = \|v(\cdot, t)\|_2^2\).

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The Fourier transform and Plancherel identity give us
\[
\partial_t y(t) \leq - \int_{\mathbb{R}^3} |\xi|^2 |\hat{v}(\xi, t)|^2 d\xi + \|F(\cdot, t)\|_2^2 \\
= - \int_{|\xi| > g(t)} |\xi|^2 |\hat{v}(\xi, t)|^2 d\xi - \int_{|\xi| \leq g(t)} |\xi|^2 |\hat{v}(\xi, t)|^2 d\xi + \|F(\cdot, t)\|_2^2,
\]
where \(g(t)\) is a given function which will be specified later on. The latter implies
\[
y'(t) + g^2(t) y(t) \leq \int_{|\xi| \leq g(t)} (g^2(t) - |\xi|^2) |\hat{v}(\xi, t)|^2 d\xi + \|F(\cdot, t)\|_2^2.
\]
Taking the Fourier transform of the Navier-Stokes equation, we find
\[
\partial_t \hat{v} + |\xi|^2 \hat{v} = -\hat{H},
\]
where
\[
H = -\text{div} (v \otimes u + \|q - F\|_2).
\]
Clearly,
\[
\hat{v}(\xi, t) = - \int_0^t \exp\{-|\xi|^2(t - s)\} \hat{H}(\xi, s) ds
\]
and
\[
|\hat{H}(\xi, s)| \leq |\xi| \left(\|v(\cdot, s)\| u(\cdot, s)\|_1 + \|F(\cdot, s)\|_1\right).
\]
Denoting
\[
k(t) = \|v(\cdot, t)\|_1,
\]
we notice
\[
\|v(\cdot, s)\| u(\cdot, s)\|_1 \leq \sqrt{s}^{-1} c_d k(s).
\]
So,
\[
|\hat{v}(\xi, t)| \leq c \int_0^t \exp\{-|\xi|^2(t - s)\} |\xi| (\sqrt{s}^{-1} k(s) + \|F(\cdot, s)\|_1) ds.
\]
Applying the Hölder inequality, we show
\[
y'(t) + g^2(t) y(t) \leq
\]
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\begin{align*}
\leq c \int_{|\xi| \leq g(t)} (g^2(t) - |\xi|^2) \left( \int_0^t \exp\{-|\xi|^2(t-s)\} |\xi| \left( \sqrt{s^{-1}k(s)} + \|F(\cdot, s\|_1) \right) ds \right)^2 \leq \\
\leq c \int_0^t (s^{-1}k^2(s) + \|F(\cdot, s\|_1^2) ds \times \\
\times \int_0^t \int_{|\xi| \leq g(t)} (g^2(t) - |\xi|^2) |\xi|^2 \exp\{-|\xi|^2(t-s_1)\} d\xi ds_1 + \|F(\cdot, t\|_2^2.
\end{align*}

The latter integral can be estimated in the following way:

\begin{align*}
\int_0^t \int_{|\xi| \leq g(t)} (g^2(t) - |\xi|^2) |\xi|^2 \exp\{-|\xi|^2(t-s_1)\} d\xi ds_1 &= \\
= c \int_0^t \int_0^{g(t)} (g^2(t) - r^2) r^4 \exp\{-2r^2(t-s_1)\} dr ds_1 \leq \\
\leq c g^6(t) \int_0^t \int_0^{g(t)} \exp\{-2r^2(t-s_1)\} d(r \sqrt{t-s_1}) \frac{ds_1}{\sqrt{t-s_1}} \leq \\
\leq c g^6(t) \int_0^t \frac{ds_1}{\sqrt{t-s_1}} \int_0^\infty \exp\{-2z^2\} dz \leq c g^6(t) \sqrt{t}.
\end{align*}

Coming back to our energy inequality, we find

\begin{align*}
y'(t) + g^2(t)y(t) &\leq \\
\leq K(t) := c g^6(t) \sqrt{t} \int_0^t \left( s^{-1}k^2(s) + \|F(\cdot, s\|_1^2 \right) ds + \|F(\cdot, t\|_2^2.
\end{align*}

Then Gronwall inequality implies

\begin{align*}
y(t) \leq c \int_0^t \exp \left\{ - \int_0^t g^2(\tau) d\tau \right\} K(s) ds.
\end{align*}
4 Proof of Theorem 1.2

The proof is on induction in $m$. The basis of induction has been already established in Section II. Let us assume that our statement is true for $m$ and show that it is true for $m + 1$.

We can estimate $K(t)$ using the fact that $F$ has a compact support

$$K(t) \leq cg^6(t)\sqrt{t} \int_0^t (\sqrt{s} \ln^{-2m}(s + e) + \|F(\cdot, s)\|_2^2) ds + \|F(\cdot, t)\|_2^2 \leq$$

$$\leq C(\|F\|_{1, \infty}, m)g^6(t)\sqrt{t} \int_0^t \sqrt{s} \ln^{-2m}(s + e) ds + \|F(\cdot, t)\|_2^2 \leq$$

$$\leq C(\|F\|_{1, \infty}, m)g^6(t)t^2 \ln^{-2m}(t + e) + \|F(\cdot, t)\|_2^2.$$

Let

$$g^2(t) = \frac{h'(t)}{h(t)}. \tag{4.1}$$

Then

$$\int_0^t \exp \left\{ - \int_0^t g^2(\tau)d\tau \right\} (g^6(s)s^2 \ln^{-2m}(s + e) + \|F(\cdot, s)\|_2^2) ds =$$

$$= \frac{1}{h(t)} \int_0^t \left( \frac{s^2 \ln^{-2m}(s + e)}{h^2(s)} (h'(s))^3 + h(s)\|F(\cdot, s)\|_2^2 \right) ds.$$

Now, one specify function $g$ by a particular choice of fuction $h$, setting

$$h(t) = \ln^k(t + e) \tag{4.2}$$

for some $k > 2m + 2$. Then

$$\frac{1}{h(t)} \int_0^t \frac{s^2 \ln^{-2m}(s + e)}{h^2(s)} (h'(s))^3 ds =$$

$$= \frac{1}{\ln^k(t + e)} \int_0^t \frac{s^2 \ln^{-2m}(s + e)}{(s + e)^3} k^3 \ln^{-3}(s + e) ds \leq$$
\begin{align*}
&= \frac{1}{\ln^k(t + e)} \int_0^t \frac{s^2}{(s + e)^3} k^3 \ln^{k-2m-3}(s + e) ds \\
&\leq \frac{1}{\ln^k(t + e)} \frac{k^3}{k - 2m - 2} (\ln^{k-2m-2}(t + e) - 1) \\
&\leq c(k, m) \frac{1}{\ln^{2m+2}(t + e)}.
\end{align*}

Since \( s \mapsto \|F(\cdot, s)\|_2^2 \) has a compact support in \([0, \infty[\), we find
\[
\|v(\cdot, t)\|_2 \leq \frac{c}{\ln^{m+1}(t + e)}.
\]

Then, as it follows from (2.3),
\[
A_p(t) \leq C(\|G\|_{2, \infty}, p, m) \sqrt{t} \frac{6-3p}{2p} \frac{1}{\ln^{m+1}(t + e)}
\]
and thus
\[
A_{6-\varepsilon}(t) \leq C(\|G\|_{2, \infty}, \varepsilon, m) \sqrt{t} \frac{12-15\varepsilon}{2(6+5\varepsilon)} \frac{1}{\ln^{m+1}(t + e)}.
\]

And again from (2.5), it follows finally that
\[
\|v(\cdot, t)\|_1 \leq c \sqrt{t} \frac{3}{\ln^{m+1}(t + e)}.
\]

5 Liouville type theorems

**Proof of Theorem 1.3** From (1.10) and from (1.4), one can derive
\[
f(t) \leq c_s \int_0^t \frac{1}{\sqrt{t-s}} \left( \frac{c_d}{\sqrt{s}} f(s) + \|F(\cdot, s)\|_1 \right) ds,
\]
where \( f(t) := \|v(\cdot, t)\|_1 \). Since \( F \) is compactly supported, (5.1) can be reduced to the following form:
\[
f(t) \leq A + c_s c_d \int_0^t \frac{1}{\sqrt{t-s}} f(s) ds.
\]
Now, fix an arbitrary $T > 0$. Then, for any $t \in [0, T]$, we have

$$f(t) \leq A + 4c_s c_d M(T),$$

where $M(T) = \sup_{0 \leq t \leq T} f(t)$. Hence,

$$M(T) \leq A + 4c_s c_d M(T)$$

for any $T > 0$. Finally, we see that

$$\|v(\cdot, t)\|_1 \leq c = \frac{A}{1 - 4c_s c_d}$$

for all $t > 0$. Therefore,

$$\left| \int_{\mathbb{R}^3} u(x, t) \cdot v(x, t) \, dx \right| \leq \frac{c}{\sqrt{t}} \to 0$$

as $t \to \infty$. *

**Proof of Theorem 1.4** Assume that $F$ is skew symmetric and therefore satisfies condition (1.3).

Equation (1.1) can be written as follows:

$$\partial_t v - \Delta v = \text{div } F_0, \quad (5.2)$$

where

$$F_0 = v \otimes u + \nabla qI - F.$$

We know from previous results that

$$F_0 \in L_{2, \infty}(Q_+), \quad \text{div } F_0 \in L_2(Q_+). \quad (5.3)$$

Since div $\text{div } F_0 = 0$, we can apply the elliptic theory and conclude that there exists a divergence free field $A(\cdot, t)$ such that

$$\text{rot } A(\cdot, t) = \text{div } F_0(\cdot, t) \quad (5.4)$$

in $\mathbb{R}^3$ and the following estimate holds

$$\|A(\cdot, t)\|_2 \leq \|K\| \|F_0(\cdot, t)\|_2 \quad (5.5)$$

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for all $t \in [0, \infty[$. Taking into account the definition of the operator $M$, one can go further and derive from (5.5)

$$
\|A(\cdot, t)\|_2 \leq \|K\|\|v(\cdot, t) \otimes u(\cdot, t)\|_2 + \|q_{0} \otimes u(\cdot, t)\|_2 + \|F(\cdot, t)\|_2 \leq
$$

$$
\leq \|K\|(1 + \sqrt{3}\|M\|)\|v(\cdot, t) \otimes u(\cdot, t)\|_2 + h(t),
$$

where $h(t) = \|K\|\|F(\cdot, t)\|_2$ and thus

$$
\|A(\cdot, t)\|_2 \leq \|K\|(1 + \sqrt{3}\|M\|) \frac{c_d}{\sqrt{t}}\|v(\cdot, t)\|_2 + h(t) \tag{5.6}
$$

With the above $A$, let us consider the Cauchy problem

$$
\partial_t B - \Delta B = A \tag{5.7}
$$

$$
B(\cdot, 0) = 0. \tag{5.8}
$$

Problem (5.7), (5.8) has a unique solution defined for all positive $t$ and $B \in W^{2,1}_2(Q_T)$ for all $T > 0$. Since $A(\cdot, t)$ is divergence free, so is $B(\cdot, t)$. Now, let $w = \text{rot } B$. Then we can see that $w$ is a solution to equation (5.2) and since it vanishes at $t = 0$, we can state that $w = v$.

Now, let us analyse the Cauchy problem for $B$. It is easy to see that $B$ satisfies the energy identity

$$
\frac{1}{2} \partial_t \|B(\cdot, t)\|_2^2 + \|\nabla B(\cdot, t)\|_2^2 = \int_{\mathbb{R}^3} A(x, t) \cdot B(x, t) dx. \tag{5.9}
$$

Taking into account the simple identity

$$
\|v(\cdot, t)\|_2 = \|\nabla B(\cdot, t)\|_2,
$$

one can derive from (5.6) the following estimate

$$
\frac{1}{2} \partial_t \|B(\cdot, t)\|_2^2 + \|v(\cdot, t)\|_2^2 \leq \|K\|(1 + \sqrt{3}\|M\|) \frac{c_d}{\sqrt{t}}\|v(\cdot, t)\|_2\|B(\cdot, t)\|_2 + h(t)\|B(\cdot, t)\|_2.
$$

Applying the Young inequality, we find

$$
\frac{1}{2} \partial_t \|B(\cdot, t)\|_2^2 \leq \|K\|^2(1 + \sqrt{3}\|M\|)^2 \frac{c_d^2}{4t}\|B(\cdot, t)\|_2^2 + \frac{1}{2} h(t)(\|B(\cdot, t)\|_2^2 + 1)
$$
Let us introduce the important constant
\[ l = \|K\|^2 (1 + \sqrt{3}\|M\|)^2 \frac{c_F^2}{2}. \]

Then the previous inequality leads to
\[ \|B(\cdot,t)\|_2^2 \leq t^l \int_0^t \frac{h(\tau)}{\tau^l} \exp \left( - \int_0^\tau h(s) ds \right) d\tau. \]

Taking into account that \( F \) is compactly supported in \( Q_+ \), we have
\[ \|B(\cdot,t)\|_2^2 \leq c_F t^l. \]

From here, it is easy to derive the following:
\[ \int_0^t \|v(\cdot,s)\|_2^2 ds \leq c_F t^l. \quad (5.10) \]

We denote all the constant depending of \( F \) and its support by \( c_F \).

Having estimate (5.10) in mind, let us go back to equation (5.2) multiplying it by \( tv \) and integrating result over \( \mathbb{R}^3 \times ]0, t[ \), as a result, we find the following differential inequality
\[
\frac{1}{2} t \|v(\cdot,t)\|_2^2 + \int_0^t \|\nabla v(\cdot,s)\|_2^2 ds = \frac{1}{2} \|v(\cdot,t)\|_2^2 + \int_0^t \int_{\mathbb{R}^3} s F(x,s) \cdot v(x,s) ds \leq \\
\leq c_F (\int_0^t \|v(\cdot,s)\|_2^2 ds + 1).
\]

The latter, together with boundedness of \( \|v(\cdot,t)\|_2 \), implies the bound
\[ \|v(\cdot,t)\|_2^2 \leq c_F (t + 1)^{l-1}, \]

which, in turn, allows to improve the decay of \( \|v(\cdot,t)\|_1 \). To this end, we are going back to (2.4) and (2.5). Indeed, by the assumption of the theorem \( l < 3/4 \),
\[
A_p(t) \leq c \int_0^t \frac{1}{\sqrt{t-s}} s^{-\frac{5p-6}{4p}} (s+1)^{l-1} ds \leq c \int_0^t \frac{1}{\sqrt{t-s}} s^{-\frac{5p-6}{4p}+l-1} ds \leq
\]

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\[ \leq c \epsilon^{6 - 3p + l - 1}. \]

Letting \( p = 6/5 + \epsilon \), for sufficiently small positive \( \epsilon \), we find
\[ \|v(\cdot, t)\|_1 \leq c(\sqrt{t})^{\frac{3}{2} + 2(l-1)}. \]

This shows
\[ \left| \int_{\mathbb{R}^4} v(\cdot, t) \cdot u(\cdot, t) dx \right| \leq c(\sqrt{t})^{\frac{3}{2} + 2(l-1)} \to 0 \]
as \( t \to \infty \) provided \( l < \frac{3}{4} \). 

### 6 Appendix I

**Proof** We recall that all derivatives of \( u \) are bounded.

First of all, there exists a unique energy solution. This follows from the identity
\[ \int_{Q_+} (u \cdot \nabla v) \cdot v dx dt = 0 \]
and from the inequality
\[ \left| - \int_{Q_+} \text{div} F \cdot v dx dt \right| = \left| \int_{Q_+} F : \nabla v dx dt \right| \leq \left( \int_{Q_+} |F|^2 dx dt \right)^{\frac{1}{2}} \left( \int_{Q_+} |\nabla v|^2 dx dt \right)^{\frac{1}{2}} \]

So, we can state that
\[ v \in L_{2,\infty}(Q_+), \quad \nabla v \in L_2(Q_+). \quad (6.1) \]
The latter means that \( u \cdot \nabla v \in L_2(Q_+) \). The pressure can be recovered from the pressure equation
\[ \Delta q = \text{div} \text{div} (F - v \otimes u). \]

One of solutions to the above equation has the form
\[ q_0(x, t) = -\frac{1}{3} v(x, t) \cdot u(x, t) + \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \nabla^2 E(x-y) : v(y, t) \otimes u(y, t) dy, \]
where $E$ is the fundamental solution to the Laplace operator. All others
differs from $q_0$ by a function of time only. Let us fix the pressure by setting
$q = q_0$. The theory of singular integrals implies that
\[ q \in L_{2,\infty}(Q_+), \quad \nabla q \in L_2(Q_+). \]
Then, by properties of solutions to the heat equation, we have
\[ \nabla^2 v \in L_2(Q_+) \quad \partial_t v \in L_2(Q_+). \]
Going back to the pressure equation, let us re-write it in the following way
\[ \triangle q = \text{div} \ \text{div} \ F - u_{+j} v_{+j} \in L_2(Q_+) \]
and thus
\[ \nabla^2 q \in L_2(Q_+). \]
Next, since $u$ is infinitely smooth and all its derivatives are bounded in
space and time, after differentiation with respect to $x_k$, we find
\[ \partial_t v_{+k} - \Delta v_{+k} = \nabla q_{+k} - \text{div} \ F_{+k} + u_{+k} \cdot \nabla v + u \cdot \nabla v_{+k} \in L_2(Q_+) \]
and therefore
\[ \partial_t \nabla v, \ \nabla^3 v \in L_2(Q_+). \]
Arguing in the same way, we find
\[ \partial_t \nabla^k v, \ \nabla^{k+2} v, \ \nabla^{k+1} q \in L_2(Q_+) \]
for each $k = 0, 1, ...$
Now, we differentiate in $t$ the pressure equation
\[ \triangle \partial_t q = \text{div} (\partial_t F - \partial_t u \cdot \nabla v - u \cdot \nabla \partial_t v) \]
and establish
\[ \nabla^k \partial_t q \in L_2(Q_+) \]
for any $k = 1, 2, ...$. Then
\[ \partial_t^2 v - \triangle \partial_t v = -\text{div} \ \partial_t F + \nabla \partial_t q + \partial_t u \cdot \nabla v + u \cdot \nabla \partial_t v \]
and thus
\[ \nabla^k \partial_t^2 v, \ \nabla^{k+2} \partial_t v \in L_2(Q_+) \]
for $k = 0, 1, ...$. And so on. *
7 Appendix II

We recall that \( u(x, t) = \tilde{u}(x, -t) \) and \( p(x, t) = -\tilde{p}(x, -t) \) for \( t > 0 \). Then

\[
- \partial_t u + u \cdot \nabla u - \Delta u = -\nabla p, \quad \text{div} \, u = 0 \quad (7.1)
\]

in \( Q_+ \) in the sense of distributions.

So, let \( v \) be a solution to (1.1) and (1.2). Now, for a compactly supported smooth function \( \psi \) in \( Q_+ \), integration by parts gives

\[
\int_{Q_+} u \cdot \psi \text{div} \, F \, dx \, dt =
\]

\[
= \int_{Q_+} u \cdot \psi \left( -\partial_t v + u \cdot \nabla v + \Delta v + \nabla q \right) \, dx \, dt =
\]

\[
= \int_{Q_+} \left( u \cdot v \partial_t \psi - u \cdot vu \cdot \nabla \psi - u_i v_i \psi_j + u_{i,j} v_i \psi_j - qu \cdot \nabla \psi \right) \, dx \, dt +
\]

\[
\quad + v \psi \left( \partial_t u - u \cdot \nabla u + \Delta u \right) \, dx \, dt =
\]

\[
= \int_{Q_+} \left( u \cdot v \partial_t \psi - u \cdot vu \cdot \nabla \psi - 2u_i v_i \psi_j + (u_{i,j} v_i + u_i v_{i,j}) \psi_j - qu \cdot \nabla \psi \right) \, dx \, dt +
\]

\[
\quad + \int_{Q_+} v \psi \cdot \nabla p \, dx \, dt =
\]

\[
= \int_{Q_+} \left( u \cdot v \partial_t \psi - u \cdot vu \cdot \nabla \psi - 2u_i v_i \psi_j - u \cdot v \Delta \psi - (qu + pv) \cdot \nabla \psi \right) \, dx \, dt.
\]

As it has been shown in [9] and [7], one may assume that some scaled invariant energy quantities of \( w \) are bounded. The same quantities remain to be bounded for \( \tilde{u} \) and therefore for \( u \). To be precise, we have

\[
A + E + C + D + C_1 + D_1 + F + H + G = M < \infty, \quad (7.2)
\]

where

\[
A = \sup_{R \geq 0} \sup_{R^2 \geq t > 0} \frac{1}{R} \int_{B(R)} |u(x, t)|^2 \, dx,
\]
\[ E = \sup_{R > 0} \frac{1}{R} \int_{Q_+(R)} |\nabla u(x, t)|^2 dx dt, \]

\[ C = \sup_{R > 0} \frac{1}{R^2} \int_{Q_+(R)} |u|^3 dx dt, \quad D = \sup_{R > 0} \frac{1}{R^2} \int_{Q_+(R)} |p|^3 dx dt, \]

\[ C_1 = \sup_{R > 0} \frac{1}{R^3} \int_{Q_+(R)} |u|^\frac{10}{3} dx dt, \quad D_1 = \sup_{R > 0} \frac{1}{R^3} \int_{Q_+(R)} |p|^\frac{5}{3} dx dt, \]

\[ F = \sup_{R > 0} \frac{1}{R^3} \int_{Q_+(R)} |u|^2 dx dt, \quad H = \sup_{R > 0} \frac{1}{R^2} \int_{Q_+(R)} |u|^\frac{3}{2} dx dt, \]

\[ G = \sup_{R > 0} \frac{1}{R} \int_{Q_+(R)} |u|^4 dx dt \]

and \(Q_+(R) := B(R) \times [0, R^2].\)

We pick \(\psi(x, t) = \chi(t)\phi(x).\) Using simple arguments and smoothness of \(u\) and \(v,\) we can get rid of \(\chi\) and have

\[ J_R(T) = \int_0^T \int_{R^3} u \cdot \phi \text{div} F dx dt = -\int_{R^3} \phi(x)u(x, T) \cdot v(x, T) dx + \]

\[ + \int_0^T \int_{R^3} \left( u \cdot vu \cdot \nabla \phi + 2u v_i v_j \phi_{,j} + u \cdot \phi \Delta \phi + (qu + pv) \cdot \nabla \phi \right) dx dt. \]

Fix a cut-off function \(\phi(x) = \xi(x/R),\) where \(\xi \in C_0^\infty(R^3)\) with the following properties: \(0 \leq \xi \leq 1, \xi(x) = 1\) if \(|x| \leq 1,\) and \(\xi(x) = 0\) if \(|x| \geq 2.\) Our aim is to show that

\[ J_R^2(T) = \int_0^T \int_{R^3} \left( u \cdot vu \cdot \nabla \phi + 2u v_i v_j \phi_{,j} + u \cdot \phi \Delta \phi + (qu + pv) \cdot \nabla \phi \right) dx dt \]

tends to zero if \(R \to \infty.\)

Assuming \(R^2 > T,\) we start with

\[ \left| \int_0^T \int_{R^3} 2u v_i v_j \phi_{,j} dx dt \right| \leq \frac{c}{R} \left( \int_0^T \int_{B(2R)} |u|^2 dx dt \right)^{1/2} \left( \int_0^T \int_{R^3} |\nabla v|^2 dx dt \right)^{1/2} \leq \]

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\[ \leq c\sqrt{A} \sqrt{\frac{T}{R}} \|\nabla v\|_{2,Q+} \to 0 \]

as \( R \to \infty \).

Next, we have

\[ \left| \int_0^T \int_{\mathbb{R}^3} u \cdot v \, \Delta \varphi \, dx \, dt \right| \leq \frac{c}{R^2} \left( \int_0^T \int_{B(2R)} |u|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |v|^2 \, dx \, dt \right)^{\frac{1}{2}} \leq \]

\[ \leq c\sqrt{A} \sqrt{\frac{T^2}{R^3}} \|v\|_{2,\infty, Q+} \to 0 \]

as \( R \to \infty \).

The third term is estimated as follows:

\[ \left| \int_0^T \int_{\mathbb{R}^3} (u \cdot vu \cdot \nabla \varphi) \, dx \, dt \right| \leq \]

\[ \leq \frac{c}{R} \left( \int_0^T \int_{B(2R)} |w|^4 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{B(2R) \setminus B(R)} |v|^2 \, dx \, dt \right)^{\frac{1}{2}} \leq \]

\[ \leq \frac{c}{\sqrt{R}} \left( \frac{1}{2R} \right) \int_{Q+ _{2,R}} |u|^4 \, dx \, dt \left( \int_0^T \int_{\mathbb{R}^3} |v|^2 \, dx \, dt \right)^{\frac{1}{2}} \leq \]

\[ \leq c \sqrt{\frac{GT}{R}} \|v\|_{2,\infty, Q+} \to 0 \]

as \( R \to \infty \).

Now, we are going to estimate terms with pressure

\[ \left| \int_0^T \int_{\mathbb{R}^3} pv \cdot \nabla \varphi \, dx \, dt \right| \leq \frac{C}{R} \left( \int_0^T \int_{B(2R)} |p|^\frac{5}{2} \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{B(2R) \setminus B(R)} |v|^\frac{5}{2} \, dx \, dt \right)^{\frac{1}{2}} \leq \]

\[ \leq cD^{\frac{3}{2}} \left( \int_0^T \int_{B(2R) \setminus B(R)} |v|^\frac{5}{2} \, dx \, dt \right)^{\frac{1}{2}} \to 0 \]
as $R \to \infty$. The latter is true since the integral
\[
\int_0^T \int_{\mathbb{R}^3} |v|^{\frac{3}{2}} dx dt
\]
is finite. Indeed, this follows from the multiplicative inequality
\[
\int_0^T \int_{\mathbb{R}^3} |v|^{\frac{3}{2}} dx dt \leq c T^{\frac{2}{3}} \|v\|_{2, \infty, Q_+}^{\frac{2}{3}} \|\nabla v\|_{2, Q_+}^{\frac{3}{2}}.
\]

The most difficult term is the last one. To treat it, we split pressure $q$ into two parts $q = P_1 + P_2$ so that
\[
\triangle P_1 = \text{div div } v \otimes u
\]
and
\[
\triangle P_2 = \text{div div } F.
\]
As to the second part $P_2$, we know that it belongs to $L_\infty(0, T; L_2(\mathbb{R}^3))$. This is an immediate consequence of the solution formula
\[
P_2(x, t) = \frac{1}{3} \text{trace } F(x, t) - \int_{\mathbb{R}^3} K(x - y) : F(y, t) dy,
\]
with the kernel $K(x) = \frac{1}{4\pi} \nabla^2 \left( \frac{1}{|x|} \right)$. Then, we have
\[
\left| \int_0^T \int_{\mathbb{R}^3} P_2 u \cdot \nabla \phi dx dt \right| \leq \frac{c}{R} \left( \int_0^T \int_{\mathbb{R}^3} |P_2|^2 dx dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{B(2R)} |u|^2 dx dt \right)^{\frac{1}{2}} \leq
\]
\[
\leq \frac{T}{R} \|P_2\|_{2, \infty, Q_+} \to 0
\]
as $R \to \infty$.

Regarding the second part, we are going to use the following decomposition:
\[
P_1(x, t) = p_{1R}(x, t) + p_{2R}(x, t) + c_R(t),
\]

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where
\[ p_{1R}(x, t) = -\frac{1}{3} u(x, t) \cdot v(x, t) + \int_{B(3R)} K(x - y) : v(y, t) \otimes w(y, t) dy, \]
\[ p_{2R}(x, t) = \int_{\mathbb{R}^3 \setminus B(3R)} \left( K(x - y) - K(-y) \right) : v(y, t) \otimes u(y, t) dy, \]
and
\[ c_R(t) = \int_{\mathbb{R}^3 \setminus B(3R)} K(-y) : v(y, t) \otimes w(y, t) dy. \]

First of all, we observe that
\[ \int_0^T \int_{\mathbb{R}^3} P_1 u \cdot \nabla \varphi dx dt = \int_0^T \int_{\mathbb{R}^3} p_{1R} u \cdot \nabla \varphi dx dt + \int_0^T \int_{\mathbb{R}^3} p_{2R} u \cdot \nabla \varphi dx dt. \]

By the theory of singular integrals,
\[ \int_{B(3R)} |p_{1R}| \frac{4}{3} dx \leq c \int_{B(3R)} |u|^\frac{3}{4} |v|^\frac{1}{2} dx \]
and thus
\[ \int_0^T \int_{B(3R)} |p_{1R}|^\frac{4}{3} dx dt \leq c \left( \int_0^T \int_{B(3R)} |u|^4 dx dt \right)^{\frac{1}{3}} \left( \int_0^T \int_{B(3R)} |v|^2 dx dt \right)^{\frac{2}{3}} \leq \]
\[ \leq c R^\frac{1}{3} G^\frac{1}{4} T^\frac{3}{2} \|v\|_{L^\infty, Q_R}. \]

So,
\[ \left| \int_0^T \int_{\mathbb{R}^3} p_{1R} u \cdot \nabla \varphi dx dt \right| \leq \frac{c}{R} \left( \int_0^T \int_{B(2R)} |p_{1R}|^\frac{4}{3} dx dt \right)^{\frac{1}{3}} \left( \int_0^T \int_{B(3R)} |u|^4 dx dt \right)^{\frac{1}{4}} \leq \]
\[ \leq \frac{c}{R} R^\frac{1}{3} G^\frac{1}{4} T^\frac{1}{2} \|v\|_{L^\infty, Q_R} R^\frac{1}{3} G^\frac{1}{4} \rightarrow 0 \]
as \( R \to \infty. \)
Assuming that $R < |x| < 2R$ and $0 < t < T$, we have for the second counterpart the following estimate

$$|p_{2R}(x, t)| \leq c \int_{R^3 \setminus B(3R)} \frac{|x|}{|y|^4} |u(y, t)||v(y, t)| dy \leq$$

$$\leq cR \sum_{k=0}^{\infty} \frac{1}{(R2^k)^4} \int_{R2^k < |y| < R2^{k+1}} |u(y, t)||v(y, t)| dy \leq$$

$$\leq cR \sum_{k=0}^{\infty} \frac{1}{(R2^k)^4} \left( \int_{B(R2^{k+1})} |u(y, t)|^2 dy \right)^{\frac{1}{2}} \left( \int_{B(R2^{k+1})} |v(y, t)|^2 dy \right)^{\frac{1}{2}} \leq$$

$$\leq cR \left( \int_{R^3} |v(y, t)|^2 dy \right)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{1}{(R2^k)^4} (R2^{k+1})^{\frac{1}{2}} A^{\frac{3}{2}} \leq$$

$$\leq \sqrt{A} \frac{c}{R^2} \|v\|_{2, \infty, Q_+}.$$

Then,

$$\left| \int_0^T \int_{R^3} p_{2R} \mathbf{w} \cdot \nabla \varphi dx dt \right| \leq \frac{c}{R} \int_0^T \int_{B(2R)} \sqrt{A} \frac{1}{R^2} \|v\|_{2, \infty, Q_+} \int_{B(2R)} |u(x, t)| dx dt \leq$$

$$\leq \sqrt{A} \frac{c}{R^2} |B(2R)|^{\frac{1}{2}} \|v\|_{2, \infty, Q_+} \int_0^T \left( \int_{B(2R)} |u(y, t)|^2 dy \right)^{\frac{1}{2}} dt \leq$$

$$\leq (-AT) \frac{c}{R^2} \|v\|_{2, \infty, Q_+} \to 0$$

as $R \to \infty$. So, finally, we have

$$\int_0^T \int_{R^3} u \cdot \text{div} \mathbf{F} dx dt = - \lim_{R \to \infty} \int_{R^3} \varphi(x)u(x, T) \cdot v(x, T) dx.$$

Taking into account $u(\cdot, T) \cdot v(\cdot, T) \in L_1(R^3)$, see (1.5), we conclude that

$$\int_0^T \int_{R^3} u \cdot \text{div} \mathbf{F} dx dt = - \int_{R^3} u(x, T) \cdot v(x, T) dx.$$

for any $T > 0$. 

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