The minimal number of generators for simple Lie superalgebras

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Abstract: Using the classification theorem due to Kac we prove that any finite dimensional simple Lie superalgebra over an algebraically closed field of characteristic 0 is generated by one element.

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0. Introduction

Throughout we work over an algebraically closed field of characteristic zero, \( F \), and all the vector spaces and algebras are assumed to be finite dimensional. Our principal aim is to determine the minimal number of generators for simple Lie superalgebras. We prove that any simple Lie superalgebra is generated by one element in the super sense, that is, any simple Lie superalgebra coincides with the smallest sub-Lie superalgebra containing some fixed element. We know that for finite simple groups only the groups of prime orders can be generated by one element and that a simple Lie algebra is never generated by one element. Our results are not surprising since in a Lie superalgebra the square of an odd element is even and not necessarily zero. As in finite simple group or simple Lie algebra cases, our proof is dependent on the classification theorem of simple Lie superalgebras [6], but not a one-by-one checking.

This study is mainly motivated by two papers of Bois mentioned as follows. In 2009, Bois [1] proved that any simple Lie algebra in arbitrary characteristic \( p \neq 2, 3 \) is generated by 2 elements and moreover, the classical Lie algebras and the graded Cartan type simple Lie algebras \( W(1, n) \) (Zassenhaus algebras) can be generated by 1.5 elements, that is, any given nonzero element can be paired a suitable element such that these two elements generate the whole algebra. Later, as a continuation of this work, Bois [2] showed that the simple graded Lie algebras of Cartan type \( W(m, n) \) with \( m \neq 1 \) and the ones of the remaining Cartan types \( S, H \) and \( K \) are never generated by 1.5 elements. Papers [1, 2] contain a considerable amount of

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information in characteristic $p$ and cover the earlier results in characteristic 0: In 1976, Ionescu [5] proved that a simple Lie algebra over the field of complex numbers is generated by 1.5 elements; In 1951, Kuranashi [7] proved that a semi-simple Lie algebra in characteristic 0 is generated by 2 elements.

By the classification theorem [6], a simple Lie superalgebra (excluding simple Lie algebras) is either a classical Lie superalgebra or a Cartan Lie superalgebra (see also [8]). The Lie algebra (even part) of a classical Lie superalgebra is reductive and meanwhile there exists a similarity in the structure side between the Cartan Lie superalgebras in characteristic 0 and the simple graded Lie algebras of Cartan type in characteristic $p$. In view of the observation above, we began this study in 2009.

Since the Lie algebra of a classical Lie superalgebra is reductive and the odd part decomposes into at most two irreducible components as adjoint modules (see [6] or [8]), we first proved that the Lie algebra of a classical Lie superalgebra is generated by two elements and then obtained that any classical Lie superalgebra is generated by two (non-homogeneous) elements in the non-super sense; As expected, we also proved that any Cartan Lie superalgebra is generated by two (non-homogeneous) elements (see arXiv 1103.4242v1 math. ph). Thus we obtained that any simple Lie superalgebra is generated by two (non-homogeneous) elements in the non-super sense and then we submitted the manuscript to a journal for publication. Later, a conversation with Professor Yucai Su during a workshop on Lie theory (organized by Professor Bin Shu, spring of 2011) led us to consider the question: Determine all the simple Lie superalgebras which are generated by one element?? (equivalently, by two homogeneous elements in the non-super sense).

Let us briefly explain the outline and ideas in this improved version. A simple fact is that $[L_1, L_1] = L_0$ for a simple Lie superalgebra $L$. So, for a classical Lie superalgebra $L$, our discussion is mainly based on the weight decomposition of $L_1$ as $L_0$-module relative to the standard Cartan subalgebra???: We find the desired generators by starting from the sum of all the odd weight vectors and use the fact that $L_1$ as $L_0$-module is irreducible or a direct sum of two irreducible submodules. For a Cartan Lie superalgebra $L$, we know that $L$ is generated by its local part $L_{-1} + L_0 + L_1$ with respect to the standard grading and moreover, the null $L_0$ is a Lie algebra and $L_i$ as $L_0$-module, $i = \pm 1$, is irreducible or a direct sum of two irreducible submodules. Then, considering the weight space decomposition relative to the standard Cartan subalgebra of $L_0$???, we find the desired generators by choosing a weight vector in each irreducible submodule of $L_i$. The proofs?? of main conclusions are constructive and provide an explicit description of the generator candidates. The process involves certain computational techniques and we use certain information about classical Lie superalgebras from [9].

In this paper we write $\langle X \rangle$ for the sub-Lie superalgebra generated by a subset $X$ in a Lie superalgebra.

1. Classical Lie superalgebras

A classical Lie superalgebra by definition is a simple Lie superalgebra $L = L_0 \oplus L_1$ for which $L_1$ as $L_0$-module is completely reducible [6, 8]. The information of
classical Lie superalgebras is as follows [1]:

\[
\begin{array}{|c|c|c|}
\hline
L & L_0 & L_1 \text{ as } L_0\text{-module} \\
\hline
A(m, n), m, n \geq 0, n \neq m & A_m \oplus A_n \oplus F & \mathfrak{sl}_{m+1} \oplus \mathfrak{sl}_{n+1} \oplus F \oplus (\text{its dual}) \\
A(n, n), n > 0 & A_n \oplus A_n & \mathfrak{sl}_{n+1} \oplus \mathfrak{sl}_{n+1} \oplus (\text{its dual}) \\
B(m, n), m \geq 0, n > 0 & B_m \oplus C_n & \mathfrak{so}_{2m+1} \oplus \mathfrak{sp}_{2n} \\
D(m, n), m \geq 2, n > 0 & D_m \oplus C_n & \mathfrak{so}_{2m} \oplus \mathfrak{sp}_{2n} \\
C(n), n \geq 2 & C_{n-1} \oplus F & \mathfrak{csp}_{2n-2} \oplus (\text{its dual}) \\
P(n), n \geq 2 & \mathfrak{a}_n & \mathfrak{a}^* \mathfrak{sl}_{n+1} \oplus \mathfrak{S}^* \mathfrak{sl}_{n+1} \\
Q(n), n \geq 2 & \mathfrak{a}_n & \text{ad}\mathfrak{sl}_{n+1} \\
D(2, 1; \alpha), \alpha \in \mathbb{F} \setminus \{-1, 0\} & \mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1 & \mathfrak{sl}_3 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \\
G(3) & \mathfrak{g}_2 \oplus \mathfrak{a}_1 & \mathfrak{g}_2 \oplus \mathfrak{g}_2 \\
F(4) & \mathfrak{b}_3 \oplus \mathfrak{a}_1 & \mathfrak{sp}_7 \oplus \mathfrak{sl}_2 \\
\hline
\end{array}
\]

From Table 1.1, one sees that \( L_0 \) is reductive and \( L_1 \) as \( L_0 \)-module is irreducible or a direct sum of two irreducible submodules.

Throughout this section \( L \) denotes a classical Lie superalgebra with the standard Cartan subalgebra \( H \). The corresponding weight (root) decompositions are

\[
L_0 = H \oplus \bigoplus_{\alpha \in \Delta_0} L_0^\alpha, \quad L_1 = \bigoplus_{\beta \in \Delta_1} L_1^\beta; \\
L = H \oplus \bigoplus_{\alpha \in \Delta_0} L_0^\alpha \oplus \bigoplus_{\beta \in \Delta_1} L_1^\beta.
\]

Write

\[
\Delta := \Delta_0 \cup \Delta_1 \quad \text{and} \quad L^\gamma := L_0^\gamma \oplus L_1^\gamma \quad \text{for } \gamma \in \Delta.
\]

Note that the standard Cartan subalgebra of a classical Lie superalgebra is diagonal:

\[
ad h(x) = \gamma(h)x \quad \text{for } h \in H, \ x \in L^\gamma, \ \gamma \in \Delta.
\]

Let \( V \) be a vector space and \( \mathfrak{f} := \{f_1, \ldots, f_n\} \) a finite set of non-zero linear functions on \( V \). Write

\[
\Omega_\mathfrak{f} := \{v \in V \mid \Pi_{i \neq j \leq n} (f_i - f_j)(v) \neq 0\}.
\]

It is a standard fact that \( \Omega_\mathfrak{f} \neq \emptyset \) (see also [1, Lemma 2.2.1]). The following technical lemma is a basic fact in Linear Algebra. For convenience, we write down a proof:

**Lemma 1.1.** Let \( \mathfrak{a} \) be an algebra. For \( a \in \mathfrak{a} \) write \( L_a \) for the left-multiplication operator given by \( a \). Suppose \( x = x_1 + x_2 + \cdots + x_n \) is a sum of eigenvectors of \( L_a \) associated with mutually distinct eigenvalues. Then all \( x_i \)’s lie in the subalgebra generated by \( a \) and \( x \).

**Proof.** Let \( \lambda_i \) be the eigenvalues of \( L_a \) corresponding to \( x_i \). Suppose for a moment that all the \( \lambda_i \)’s are nonzero. Then

\[
(L_a)^k(x) = \lambda_1^k x_1 + \lambda_2^k x_2 + \cdots + \lambda_n^k x_n \quad \text{for } k \geq 1.
\]

Our conclusion in this case follows from the fact that the Van der Monde determinant given by \( \lambda_1, \lambda_2, \ldots, \lambda_n \) is nonzero and thereby the general situation is clear. \( \square \)
Lemmas. \[ \text{Proposition 1, p.137]}

(1) If \( L \neq A(1,1), P(3) \) or \( Q(n) \) then \( \dim L_{\gamma} = 1 \) for every \( \gamma \in \Delta \).

(2) If \( L \neq Q(n) \) then \( 0 \notin \Delta_1 \).

Notice that, for \( L = A(1,1), P(3) \) or \( Q(n) \), from Table 1.1 \( L_0 \) is a semi-simple Lie algebra. Then by a standard result in Lie algebras (see \[8\]) we have

\[ \dim L_{\alpha}^\gamma = 1 \quad \text{for} \quad \alpha \in \Delta_0, \quad H = \sum_{\alpha \in \Delta_0} [L_{\alpha}^\gamma, L_0]. \tag{1.1} \]

**Theorem 1.3.** Any classical Lie superalgebra is generated by 1 element.

**Proof.** By Lemma 1.2, we treat two cases separately:

**Case 1.** If \( L \neq A(1,1), Q(n) \) or \( P(3) \), then all the weight spaces are 1-dimensional. Choose any \( h \in \Omega_{\Delta_1} \subset H \) and an element \( x = \sum_{\gamma \in \Delta_1} x_\gamma \), where \( x_\gamma \) is a weight vector of \( \gamma \). By Lemmas 1.2 and 1.3 all components \( x_\gamma \) belong to \( \langle x + h \rangle \). Since \( \dim L_{\gamma} = 1 \), we conclude that \( L_\gamma \subset \langle x + h \rangle \) for \( \gamma \in \Delta_1 \) and then \( L_1 \subset L \). By \[8\] Proposition 1.2.7(1), p.20, \( L_0 = [L_1, L_1] \) and then \( \langle x + h \rangle = L \).

**Case 2.** Let \( L = A(1,1), Q(n) \) or \( P(3) \). In this case, there exists a weight space which is not 1-dimensional.

Let \( L = A(1,1) \). For simplicity, write \( e_{ij} \) for \( e_{ij} + F_1 \). By Table 1.1, let \( L_1 = L_1^1 \oplus L_1^2 \) be a direct sum of two irreducible \( L_0 \)-modules. The standard basis of \( A(1,1) \) is listed below:

| \( L_0 \) | \( H \) | \( L_1 \) | \( L_1^1 \) | \( L_1^2 \) |
| --- | --- | --- | --- | --- |
| | \( e_{11} + e_{33} \), \( e_{11} + e_{14} \) | \( e_{12} \), \( e_{21} \), \( e_{33} \), \( e_{43} \) | \( e_{13} \), \( e_{14} \), \( e_{23} \), \( e_{24} \) | \( e_{31} \), \( e_{32} \), \( e_{41} \), \( e_{42} \) |

Let \( x \) be the sum of all the standard odd basis elements (weight vectors) in Table 1.2, that is,

\[ x = e_{13} + e_{14} + e_{23} + e_{24} + e_{31} + e_{32} + e_{41} + e_{42} \]

Choose an element \( h \in \Omega_{\Delta_1} \). Assert \( \langle x + h \rangle = L \). To that aim, define \( e_{ij}^\alpha \) to be the linear function on \( H \) given by \( e_{ij}^\alpha (e_{kl} + e_{k+j+2l}) = \delta_{ij} \) for \( 1 \leq i,j \leq 2 \). All the odd weights and the corresponding odd weight vectors are listed below:

| weights | \( e_{ij}^\alpha \) | \( e_{ij}^\beta \) | \( -e_{ij}^\alpha \) | \( -e_{ij}^\beta \) |
| --- | --- | --- | --- | --- |
| vectors | \( e_{13} \), \( e_{42} \) | \( e_{14} \), \( e_{32} \) | \( e_{23} \), \( e_{41} \) | \( e_{31} \), \( e_{24} \) |

Then, by Lemma 1.1 and Table 1.3, the elements

\[ e_{13} + e_{42}, \quad e_{14} + e_{32}, \quad e_{23} + e_{41}, \quad e_{31} + e_{24} \]

lie in \( \langle x + h \rangle \). A direct computation shows that

\[ e_{34} = \frac{1}{2}[e_{14} + e_{32}, e_{24} + e_{31}], \quad e_{12} = \frac{1}{2}[e_{14} + e_{32}, e_{13} + e_{42}], \]

\[ e_{21} = \frac{1}{2}[e_{24} + e_{31}, e_{23} + e_{41}], \quad e_{43} = \frac{1}{2}[e_{13} + e_{42}, e_{23} + e_{41}]. \]
Then by Table 1.2, all the standard even basis elements $e_{12}, e_{21}, e_{43}, e_{34}$ lie in $\langle x + h \rangle$. According to (1.1), we have $L_0 \subset \langle x + h \rangle$. As

$$[e_{14} + e_{32}, e_{21}] = -e_{24} + e_{31} \in \langle x + h \rangle,$$

we have $e_{24}, e_{31} \in \langle x + h \rangle$. Since $e_{24} \in L_1^1$, $e_{31} \in L_1^2$ (see Table 1.2) and $L_1^i$ is irreducible as $L_0$-module, where $i = 1, 2$, we have $L_1 \not\subset \langle x + h \rangle$. Therefore $(x + h) = L$.

Let $L = P(3)$. Note that $P(3)$ is a subalgebra of $A(3,3)$ consisting of the matrices of the form: $\begin{pmatrix} a & b \\ -c & -a \end{pmatrix}$, where tr$(a) = 0$, $b$ is symmetric and $c$ is skew-symmetric (see [3]). By Table 1.1, let $L_1 = L_1^1 \oplus L_1^2$ be a direct sum of two irreducible $L_0$-modules. The standard basis of $P(3)$ is as follows:

$$\begin{array}{|c|c|}
\hline
L_0 & H \\
\hline
\epsilon_1 - \epsilon_2 - \epsilon_55 + \epsilon_66, & \epsilon_1 - \epsilon_33 - \epsilon_55 + \epsilon_77, \\
\epsilon_11 - \epsilon_44 - \epsilon_55 + \epsilon_88 & \\
e_2 - \epsilon_65, e_3 - \epsilon_75, e_4 - \epsilon_85, & e_23 - \epsilon_76, e_24 - \epsilon_86, e_34 - \epsilon_78, \\
e_21 - \epsilon_56, e_31 - \epsilon_55, & e_41 - \epsilon_58, e_42 - \epsilon_68, e_43 - \epsilon_78, e_32 - \epsilon_67 \\
\hline
L_1 & L_1^1 \\
\epsilon_15, \epsilon_26, & \epsilon_28, e_36, e_38 + e_47, \\
\epsilon_37, \epsilon_48, e_16 + e_25, & e_17 + e_35, e_27 + e_36 \\
\hline
L_1^2 & e_52 - e_61, e_53 - e_71, & e_54 - e_61, e_63 - e_72, e_64 - e_82, e_74 - e_83 \\
\hline
\end{array}$$

Let $x$ be the sum of all standard odd basis elements (weight vectors) in Table 1.4. Choose an element $h \in \Omega_{\Delta_i}$. Assert $\langle x + h \rangle = L$. To that aim, define $e_i^\mu$ to be the linear function on $H$ given by

$$e_i^\mu(\epsilon_{11} - \epsilon_{1+j,1+j} - \epsilon_{55} + \epsilon_{5+j,5+j}) = \delta_{ij}$$

for $1 \leq i, j \leq 3$. All the odd weights and the corresponding odd weight vectors are listed below:

$$\begin{array}{|c|c|c|c|}
\hline
\text{Weight} & 2e_1^\mu + 2e_2^\mu + 2e_3^\mu & -2e_1^\mu & -2e_2^\mu & -2e_3^\mu \\
\hline
\text{Vectors} & e_{15} & e_{26} & e_{37} & e_{48} \\
\hline
\text{Weight} & e_{1}^\mu + e_{2}^\mu & -e_{1}^\mu - e_{2}^\mu & -e_{1}^\mu - e_{2}^\mu & -e_{1}^\mu - e_{2}^\mu \\
\hline
\text{Vectors} & e_{17} + e_{35}, e_{64} - e_{82}, & e_{27} + e_{36}, e_{64} - e_{81}, & e_{28} + e_{46}, e_{53} - e_{71}, & \\
\hline
\text{Weight} & e_{1}^\mu + e_{2}^\mu & e_{1}^\mu + e_{2}^\mu & -e_{1}^\mu - e_{2}^\mu & -e_{1}^\mu - e_{2}^\mu \\
\hline
\text{Vectors} & e_{16} + e_{25}, e_{74} - e_{83}, & e_{18} + e_{45}, e_{63} - e_{72}, & e_{38} + e_{47}, e_{52} - e_{61}, & \\
\hline
\end{array}$$

By Lemma 1.1 and Table 1.5, one sees that $\langle x + h \rangle$ contains the following elements

$$e_{37}, e_{48}, e_{38} + e_{47} + e_{52} - e_{61}, e_{16} + e_{25} + e_{74} - e_{83}, e_{17} + e_{35} + e_{64} - e_{82},$$

$$e_{15}, e_{26}, e_{27} + e_{36} + e_{54} - e_{61}, e_{18} + e_{45} + e_{63} - e_{72}, e_{28} + e_{46} + e_{53} - e_{71}.$$  

Lie superbrackets of the above odd elements yield

$$e_{57} - e_{31} = [e_{37}, e_{28} + e_{46} + e_{53} - e_{71}], e_{58} - e_{41} = [e_{48}, e_{27} + e_{36} + e_{54} - e_{81}],$$

$$e_{56} - e_{21} = [e_{26}, e_{38} + e_{47} + e_{52} - e_{61}], e_{12} - e_{65} = [e_{15}, e_{38} + e_{47} + e_{52} - e_{61}],$$

$$e_{13} - e_{75} = [e_{15}, e_{28} + e_{46} + e_{53} - e_{71}], e_{23} - e_{76} = [e_{26}, e_{18} + e_{45} + e_{63} - e_{72}],$$

$$e_{24} - e_{86} = [e_{26}, e_{17} + e_{35} + e_{64} - e_{82}], e_{14} - e_{85} = [e_{15}, e_{27} + e_{36} + e_{54} - e_{81}],$$

$$e_{34} - e_{87} = [e_{37}, e_{16} + e_{25} + e_{74} - e_{83}], e_{67} - e_{32} = [e_{37}, e_{18} + e_{45} + e_{63} - e_{72}],$$

$$e_{68} - e_{42} = [e_{48}, e_{17} + e_{35} + e_{64} - e_{82}], e_{78} - e_{43} = [e_{48}, e_{16} + e_{25} + e_{74} - e_{83}].$$


Then, according to Table 1.4 and (1.1) one sees $L_0 \subset \langle x + h \rangle$. Since

$$e_{18} + e_{45} - e_{63} + e_{72} = [e_{17} + e_{35} + e_{64} - e_{82}, e_{78} - e_{43}] \in \langle x + h \rangle,$$

and $e_{18} + e_{45} + e_{63} - e_{72} \in \langle x + h \rangle$, we have

$$e_{18} + e_{45}, e_{63} - e_{72} \in \langle x + h \rangle.$$

By Table 1.4 we have

$$e_{18} + e_{45} \in L^1_1, e_{63} - e_{72} \in L^2_1.$$

Then the irreducibility of $L^1_1$ as $L_0$-module ensures that $L^1_1 \subset \langle x + h \rangle$, where $i = 1, 2$. Therefore, $\langle x + h \rangle = L$.

Let $L = Q(n)$. Note that $Q(n) = \tilde{Q}(n)/\mathbb{F}I_{2n+2}$ and $\tilde{Q}(n)$ is the subalgebra of $\mathfrak{sl}(n + 1, n + 1)$ consisting of the matrices of the form: $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where $\text{tr}(b) = 0$ (see [3]). For simplicity, write $e_{ij}$ for $e_{ij} + \mathbb{F}I_{2n+2}$. The standard basis of $Q(n)$ is listed below:

| $L_0$ | $H$ | $e_{ij} + e_{n+1+i,n+1+j}, 1 \leq i \leq n$ |
|-------|-----|---------------------------------------------|
|       |     | $e_{ij} + e_{n+1+i,n+1+j}, 1 \leq i \leq j \leq n + 1$ |

| $L_1$ | $e_{1,n+2} + e_{n+2,1} - e_{n+1+i} - e_{n+1+i,n+2}, 2 \leq i \leq n + 1, n+1$ |
|-------|-------------------------------|
|       | $e_{1,n+1+k} + e_{n+1+j,k}, 1 \leq k \neq j \leq n + 1$ |

Let $x$ be the sum of all standard odd basis elements (weight vectors) in Table 1.6. Choose an element $h \in \Omega_{\Delta_i}$. Assert $\langle x + h \rangle = L$. To that aim, define $\varepsilon_i^Q$ to be the linear function on $H$ by $\varepsilon_i^Q(e_{jj} + e_{n+1+j,n+1+j}) = \delta_{ij}$ for $1 \leq i, j \leq n$. All the odd weights and the corresponding odd weight vectors are as follows:

| weights | $\varepsilon_i^Q - \varepsilon_j^Q, 1 \leq k \neq j \leq n$ |
|---------|------------------------------------------------|
| vectors | $e_{j,n+1+k} + e_{n+1+j,k}, 1 \leq k \neq j \leq n$ |
| weights | $\varepsilon_j^Q, 1 \leq j \leq n$ |
| vectors | $e_{j,2n+2} + e_{n+1+j,n+1}, 1 \leq j \leq n$ |
| weights | $-\varepsilon_j^Q, 1 \leq k \leq n$ |
| vectors | $e_{n+1,n+1+k} + e_{2n+2,k}, 1 \leq k \leq n$ |
| weights | $0$ |
| vectors | $e_{1,n+2} + e_{n+2,1} - e_{i,n+1+i} - e_{n+1+i,i}, 2 \leq i \leq n + 1$ |

By Lemma[10] and Table 1.7, one sees that $\langle x + h \rangle$ contains the following elements

$$e_{j,n+1+k} + e_{n+1+j,k}, 1 \leq k \neq j \leq n,$$

$$e_{j,2n+2} + e_{n+1+j,n+1}, 1 \leq j \leq n,$$

$$e_{n+1,n+1+k} + e_{2n+2,k}, 1 \leq k \leq n,$$

$$\sum_{j=2}^{n+1}(e_{1,n+2} + e_{n+2,1} - e_{j,n+1+j} - e_{n+1+j,j}).$$
Write $Z$ for $\sum_{j=2}^{n+1}(e_{1,n+2} + e_{n+2,1} - e_{j,n+1+j} - e_{n+1+j,j})$. Then

$$[e_{1,n+1+k} + e_{n+1+i,k}, Z] = \delta_{ij}(e_{jk} + e_{n+1+j,n+1+k}) - \delta_{ik}(e_{1k} + e_{n+2,n+1+k}) + \delta_{k1}(e_{ii} + e_{n+1,i,n+2}) - \delta_{kj}(e_{ij} + e_{n+1+i,n+1+j}), \; 1 \leq i \neq k \leq n + 1.$$  

Hence

$$e_{ik} + e_{n+1+i,n+1+k} \in (x + h), \; 1 \leq i \neq k \leq n + 1.$$  

So, by Table 1.6 and (6.1) we have $L_0 \subset (x + h)$. Since $x$ lies in $(x + h)$ and $L_1$ is irreducible as $L_0$-module (see Table 1.1), we have $L_1 \subset (x + h)$. Furthermore, $(x + h) = L$.  

\[\square\]

2. Cartan Lie superalgebras

All the Cartan Lie superalgebras are listed below \cite{6, 8}:

- $W(n)$ with $n \geq 3$, $S(n)$ with $n \geq 4$, $\tilde{S}(2m)$ with $m \geq 2$ and $H(n)$ with $n \geq 5$.

Let $\Lambda(n)$ be the Grassmann superalgebra with generators $\xi_1, \ldots, \xi_n$. For a $k$-shuffle $u := (i_1, i_2, \ldots, i_k)$, that is, a strictly increasing sequence between 1 and $n$, we write $|u| := k$ and $x^u := \xi_{i_1}\xi_{i_2}\cdots\xi_{i_k}$. Letting $\deg \xi_i = 1$, $i = 1, \ldots, n$, we obtain the standard consistent $\mathbb{Z}$-grading of $\Lambda(n)$. Let us briefly describe the Cartan Lie superalgebras.

- $W(n) = \text{der} \Lambda(n)$ is $\mathbb{Z}$-graded, $W(n) = \oplus_{k=1}^{n-1} W(n)_k$, where
  
  $$W(n)_k = \text{span}_\mathbb{F}\{x^u \partial/\partial \xi_i \mid |u| = k + 1, \; 1 \leq i \leq n\}.$$  

- $S(n) = \oplus_{k=1}^{n-2} S(n)_k$ is a $\mathbb{Z}$-graded subalgebra of $W(n)$, where
  
  $$S(n)_k = \text{span}_\mathbb{F}\{D_{ij}(x^u) \mid |u| = k + 2, \; 1 \leq i, j \leq n\}.$$  

Hereafter, $D_{ij}(f) := \partial(f)/\partial \xi_i \partial/\partial \xi_j + \partial(f)/\partial \xi_j \partial/\partial \xi_i$ for $f \in \Lambda(n)$.

- $\tilde{S}(2m)$ is a subalgebra of $W(2m)$ and as a $\mathbb{Z}$-graded subspace,
  
  $$\tilde{S}(2m) = \oplus_{k=1}^{2m-2} \tilde{S}(2m)_k \quad \text{with} \quad m \geq 2,$$  

where

$$\tilde{S}(2m)_1 = \text{span}_\mathbb{F}\{(1 + \xi_1\cdots\xi_{2m}) \partial/\partial \xi_j \mid 1 \leq j \leq 2m\},$$  

$$\tilde{S}(2m)_k = S(2m)_k, \; 0 \leq k \leq 2m - 2.$$  

Notice that $\tilde{S}(2m)$ is not a $\mathbb{Z}$-graded subalgebra of $W(2m)$.

- $H(n) = \oplus_{k=1}^{n-3} H(n)_k$ is a $\mathbb{Z}$-graded subalgebra of $W(n)$, where
  
  $$H(n)_k = \text{span}_\mathbb{F}\{D_H(x^u) \mid |u| = k + 2\}.$$  

Hereafter, $D_H$ is a linear mapping of $\Lambda(n)$ to $W(n)$ such that $D_H(f) := (-1)^{|f|} \sum_{i=1}^n \partial(f)/\partial \xi_{i'} \partial/\partial \xi_{i'}$ for $f \in \Lambda(n)$, where $i'$ is the involution of the index set $\{1, \ldots, n\}$ satisfying that $i' = i + \lceil \frac{n}{2} \rceil$ for $i \leq \lceil \frac{n}{2} \rceil$ and $n' = n$ if $n$ is odd. Here, $\lceil \frac{n}{2} \rceil$ is the biggest integer less than $\frac{n}{2}$ ($n \geq 5$).
In the sequel, we write $W, S, \tilde{S}, H$ instead of $W(n), S(n), \tilde{S}(2m), H(n)$, respectively. Throughout this section $L$ denotes one of the Cartan Lie superalgebras $W, S, \tilde{S}$, or $H$. Consider its decomposition of subspaces:

$$L = L_{-1} \oplus \cdots \oplus L_s. \quad (2.1)$$

For $W, S, \tilde{S}$ or $H$, the height $s = n - 1, n - 2, 2m - 2$ or $n - 3$, respectively. Note that $S$ and $H$ are $\mathbb{Z}$-graded subalgebras of $W$ with respect to 1.1, but $\tilde{S}$ is not. The null $L_0$ is isomorphic to $\mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{sl}(2m), \mathfrak{so}(n)$ for $L = W, S, \tilde{S}, H$, respectively.

From [6, 8], we can write down the following facts:

**Lemma 2.1.** Keep notations as above.

1. **The subspace** $L_{-1}$ **is an irreducible** $L_0$-**module.**
2. **A Cartan Lie superalgebra** $L$ **is generated by** the local part $L_{-1} \oplus L_0 \oplus L_1$.
3. **The subspace** $L_1$ **is an irreducible** $L_0$-**module** for $L = S, \tilde{S}$ **or** $H$, **except for** $H(6)$. **For** $L = H(6)$ **or** $W$, **the subspace** $L_1$ **is a direct sum of** two irreducible $L_0$-**submodules.**

The following is a list of bases of the standard Cartan subalgebras $\mathfrak{h}_{L_0}$ of $L_0$.

| $L$ | $\mathfrak{h}_{L_0}$ |
|-----|---------------------|
| $W(n)$ | $\xi_i \partial / \partial \xi_i, 1 \leq i \leq n$ |
| $S(n)$ | $\xi_i \partial / \partial \xi_1 - \xi_j \partial / \partial \xi_1, 2 \leq j \leq n$ |
| $S(2m)$ | $\xi_i \partial / \partial \xi_1 - \xi_j \partial / \partial \xi_2, 2 \leq j \leq 2m$ |
| $H(2m)$ | $\xi_i \partial / \partial \xi_1 - \xi_j \partial / \partial \xi_k, 1 \leq i \leq m$ |
| $H(2m + 1)$ | $\xi_i \partial / \partial \xi_1 - \xi_j \partial / \partial \xi_k, 1 \leq i \leq m$ |

The weight space decomposition of the subspace $L_k$ relative to $\mathfrak{h}_{L_0}$ is:

$$L_k = \delta_{k,0} \mathfrak{h}_{L_0} \oplus_{\alpha \in \Delta_k} L_k^\alpha, \text{ where } -1 \leq k \leq s.$$  

We write down the following weight sets which will be used in the proof of the following Lemma 2.2. For $W(n)$, define $\varepsilon_i^w$ to be the linear function on $\mathfrak{h}_{W_0}$ by

$$\varepsilon_i^w(\xi_j \partial / \partial \xi_j) = \delta_{ij}, \ 1 \leq i, j \leq n.$$

We have

$$\Delta_{-1} = \{-\varepsilon^w_j | 1 \leq j \leq n\},$$

$$\Delta_1 = \{\varepsilon^w_k + \varepsilon^w_l - \varepsilon^w_j | 1 \leq k \neq l, j \leq n\}. \quad (2.2)$$

For $S(n)$ and $\tilde{S}(n)$, define $\varepsilon_i^s$ to be the linear function on $\mathfrak{h}_{S_0}$ by

$$\varepsilon_i^s(\xi_j \partial / \partial \xi_1 - \xi_j \partial / \partial \xi_1) = \delta_{ij}, \ 2 \leq i, j \leq n$$

and write $\varepsilon_1^s := \sum_{i=2}^n \varepsilon_i^s$. We have

$$\Delta_{-1} = \{\varepsilon^s_j | 1 \leq j \leq n\},$$

$$\Delta_1 = \{\varepsilon^s_k + \varepsilon^s_l - \varepsilon^s_j | 1 \leq k \neq l, j \leq n\}.$$
With the above notations, we have the following properties:

Lemma 2.2. For $H$ for $L$

Proof. All the statements follow directly from the above computations except (4) for $H = H(6)$ or $W$. In this case, from (2.2) and (2.3) one sees that (4) holds. Consequently, (4) holds.

For $L_1 = H(6)_1$ or $W_1$, then by Lemma 2.2(3), $L_1$ is a direct sum of two irreducible $L_0$-modules

$$L_1 = L_1^1 \oplus L_1^2.$$ Let $\Delta_1$ be the weight set of $L_1$, $i = 1, 2$.

Lemma 2.2. With the above notations, we have the following properties:

1. If $L = W$ then $\Delta_1 \cap \Delta_1 = \emptyset$.
2. If $L = S$ or $\tilde{S}$ then $\Delta_1 \cap \Delta_1 = \emptyset$.
3. If $L = H$ then $\Delta_1 \cap \Delta_1 = \emptyset$.
4. If $L = H(6)$ or $W$, there exist nonzero weights $\alpha_1^1 \in \Delta_1$ such that $\alpha_1^1 \neq \alpha_2^1$.

Proof. All the statements follow directly from the above computations except (4) for $L = H(6)$ or $W$. In this case, from (2.2) and (2.3) one sees that $0 \notin \Delta_1$ and $|\Delta_1| > 1$. Consequently, (4) holds.

For $L = W, S, \tilde{S}$ or $H$, fix the corresponding standard Cartan subalgebra, respectively.

Lemma 2.3. For $L$, there exists a weight vector $x_1 \in L_1^n$ for some $\alpha \in \Delta_1$ such that $[x_1, x_1] = 0$.

Proof. It is easy to see that a standard basis vector of $L_1$ is also a weight vector for some weight $\alpha \in \Delta_1$. For $L = W$, we have

$$[x_j x_k \partial/\partial \xi_i, x_j x_k \partial/\partial \xi_i] = 0$$

where $1 \leq i, j, k \leq n$ and $i, j, k$ are pairwise distinct. For $L = S$ or $\tilde{S}$, we have

$$[D_{ij}(x_i x_j x_k), D_{ij}(x_i x_j x_k)'] = 0$$
where $1 \leq i, j, k \leq n$ and $i, j, k$ are pairwise distinct. For $L = H$, we have
\[
[D_H(x_i x_j x_k), D_H(x_i x_j x_k)] = 0
\]
where $1 \leq i, j, k \leq \left\lceil \frac{n}{2} \right\rceil$ and $i, j, k$ are pairwise distinct.

Recall that the null $L_0$ is isomorphic to $\mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{sl}(2m), \mathfrak{so}(n)$ for $L = W, S, \tilde{S}$ or $H$, respectively.

Let $\mathfrak{g}$ be a simple Lie algebra. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, $x \in \mathfrak{g}$ is called $\mathfrak{h}$-balanced provided that $x^n \neq 0$ for $\alpha \in \Phi$, where $\Phi \subset \mathfrak{h}^*$ is the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. The lemma below is abstracted from [1].

**Lemma 2.4.** If $x$ is a non-zero element in $\mathfrak{g}$ then there exists some Cartan subalgebra $\mathfrak{h}$ such that $x$ is $\mathfrak{h}$-balanced.

**Proof.** Suppose $x$ is a non-zero element of $\mathfrak{g}$ and let $\mathfrak{h}'$ be a Cartan subalgebra of $\mathfrak{g}$. By the proof of [1, Theorem 2.2.3], there exists $\varphi \in \text{Aut} \mathfrak{g}$ such that $\varphi(x)$ is $\mathfrak{h}'$-balanced. Letting $\mathfrak{h} = \varphi^{-1}(\mathfrak{h}')$, one sees that $\mathfrak{h}$ is a Cartan subalgebra and $x$ is $\mathfrak{h}$-balanced.

From [1], we write down two useful facts:

- If $x \in \mathfrak{g}$ is $\mathfrak{h}$-balanced, then $\mathfrak{g} = \langle x, h \rangle$ for $h \in \Omega_\Phi$, where $\Phi \subset \mathfrak{h}^*$ is the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$.
- If $\mathfrak{sl}(n)$ is generated by an $\mathfrak{h}$-balanced element $x$ and an element $h$ in $\Omega_\Phi$, then $\mathfrak{gl}(n)$ is generated by $x$ and $h + z$, where $z$ is a nonzero central element in $\mathfrak{gl}(n)$.

Note that the $\mathbb{Z}$-grading of a Cartan Lie superalgebra is consistent with the $\mathbb{Z}_2$-grading over $\mathbb{F}$.

**Theorem 2.5.** Any Cartan Lie superalgebra is generated by 1 element.

**Proof.** For $L = S, \tilde{S}$ or $H$, except $H(6)$, fix the standard Cartan subalgebra $\mathfrak{h}$. By Lemmas 2.2 and 2.3, we choose $\alpha_{-1} \neq \alpha_1$ for $\alpha_i \in \Delta_i, i = -1, 1$ and $x_{-1} \in L_{-1}^\alpha$ and $x_1 \in L_1^{\alpha_1}$ such that $[x_{-1}, x_1] \neq 0$ and $[x_1, x_1] = 0$. Let $x_0 := 2[x_{-1}, x_1]$. Choose a suitable Cartan subalgebra $\mathfrak{h}'$ such that $x_0$ is an $\mathfrak{h}'$-balanced element of $L_0$. Let
\[
x := x_{-1} + x_0 + h' + x_1
\]
for $h' \in \Omega_{\Delta_0'}$, where $\Delta_0' \subset \mathfrak{h}'^*$ is the root system of $L$ relative to $\mathfrak{h}'$. Then we have
\[
x_{-1} + x_1, x_0 + h' \in \langle x \rangle
\]
and then
\[
x_0 = [x_{-1} + x_1, x_{-1} + x_1] = 2[x_{-1}, x_1] \in \langle x \rangle.
\]
Furthermore, we have $h' \in \langle x \rangle$. Then $L_0 = \langle x_0, h' \rangle \subset \langle x \rangle$. Choose an element
\[
h \in \Omega_{\langle \alpha_{-1}, \alpha_1 \rangle} \subset \mathfrak{h} \subset L_0.
\]
Then, by Lemma 1.4, we obtain that $x_{-1}$ and $x_1$ lie in $\langle x \rangle$. According to Lemma 2.2.1 (1) and (3), the irreducibility of $L_{-1}$ and $L_1$ as $L_0$-modules ensures that $L_i \subset \langle x \rangle, i = -1, 1$. Furthermore, $L = \langle x \rangle$.

For $L = H(6)$ or $W$, fix the standard Cartan subalgebra $\mathfrak{h}$. From Lemmas 2.2 and 2.3 we choose $\alpha_{-1}, \alpha_1$ and $\alpha_2$ are pairwise distinct for $\alpha_{-1} \in \Delta_{-1}, \alpha_1 \in \Delta_1, i = 1, 2$. 


The minimal number of generators for simple Lie superalgebras

and \( x_{-1} \in L_{-1}^{\alpha_{-1}} \) and \( x_i^1 \in L_{i}^{\alpha_1} \) for \( i = 1, 2 \) such that \([x_{-1}, x_i^1 + x_i^2] \neq 0 \) and \([x_i^1 + x_i^2, x_i^1 + x_i^2] = 0 \). Let \( x_0 := 2[x_{-1}, x_i^1 + x_i^2] \). Choose a suitable Cartan subalgebra \( \mathfrak{h}' \) such that \( x_0 \) is an \( \mathfrak{h}' \)-balanced element of \( L \). Let

\[ x := x_{-1} + x_0 + \delta_{L,W} z + h' + x_1^1 + x_2^2 \]

for \( 0 \neq z \in C(W_0) \) and \( h' \in \Delta_0' \subset \mathfrak{h}'^* \) is the root system of \( L \) relative to \( \mathfrak{h}' \). Then we have

\[ x_{-1} + x_1^1 + x_2^2, \quad x_0 + \delta_{L,W} z + h' \in \langle x \rangle \]

and then

\[ x_0 = [x_{-1} + x_1^1 + x_2^2, x_{-1} + x_1^1 + x_2^2] = 2[x_{-1}, x_1^1 + x_2^2] \in \langle x \rangle. \]

Furthermore, we have \( h' + \delta_{L,W} z \in \langle x \rangle \). Then

\[ L_0 = (x_0, h' + \delta_{L,W} z) \subset \langle x \rangle. \]

Choose an element

\[ h \in \Omega_{(\alpha_{-1}, \alpha_1, \alpha_2)} \subset \mathfrak{h} \subset L_0. \]

Then, by Lemma 1.1 we obtain that \( x_{-1}, x_1^1 \) and \( x_2^2 \) lie in \( \langle x \rangle \). According to Lemma 2.1 (1) and (3), the irreducibility of \( L_{-1}, L_1^1 \) and \( L_2^2 \) as \( L_0 \)-modules ensures that \( L = \langle x \rangle \).

Theorem 2.6. Any simple Lie superalgebra is generated by 1 element.

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