Markowitz's Mean-Variance Optimization with Investment and Constrained Reinsurance

Nan Zhang, Ping Chen*, Zhuo Jin and Shuanming Li
Centre for Actuarial Studies, Department of Economics
The University of Melbourne
VIC, 3010, Australia
(Communicated by Christian-Oliver Ewald)

Abstract. This paper deals with the optimal investment-reinsurance strategy for an insurer under the criterion of mean-variance. The risk process is the diffusion approximation of a compound Poisson process and the insurer can invest its wealth into a financial market consisting of one risk-free asset and one risky asset, while short-selling of the risky asset is prohibited. On the side of reinsurance, we require that the proportion of insurer's retained risk belong to $[0,1]$, is adopted. According to the dynamic programming in stochastic optimal control, the resulting Hamilton-Jacobi-Bellman (HJB) equation may not admit a classical solution. In this paper, we construct a viscosity solution for the HJB equation, and based on this solution we find closed form expressions of efficient strategy and efficient frontier when the expected terminal wealth is greater than a certain level. For other possible expected returns, we apply numerical methods to analyse the efficient frontier. Several numerical examples and comparisons between models with constrained and unconstrained proportional reinsurance are provided to illustrate our results.

1. Introduction. Reinsurance is a process whereby one entity (the “reinsurer”) takes on all or part of the risk covered under a policy issued by an insurance company (the “cedent” or “insurer”). It is an important mechanism of risk management for a cedent to spread its underlying risk by paying some premiums to the reinsurer. The research on optimal reinsurance design dated back to the 1960s. As a sound and prudent risk management tool that permits insurance companies to be protected against adverse fluctuations, optimal reinsurance design has remained an active subject. Some typical reinsurance strategies include stop loss, proportional, excess-of-loss, loss-occurring coverage and risk-attaching reinsurance. The proportional (or quota-share) and the excess-of-loss reinsurance have received great attention from the academia and practitioners. Some literatures on the proportional reinsurance include Choulli et al. [7], Højgaard and Taksar [10,11,12], Schmidli [17,18] and Taksar [22]. Some recent works on the excess-of-loss reinsurance are Asmussen et al. [1], Choulli et al. [6], Irgens and Paulsen [13] and Zhang et al. [27]. Specifically, Asmussen et al. [1] study the excess of loss reinsurance through re-parameterizing the controlled risk process by taking the drift as a basic control parameter, then

2010 Mathematics Subject Classification. Primary: 91G10, 91G80; Secondary: 70H20.
Key words and phrases. Mean-variance, HJB equation, viscosity solution, Lagrange multiplier, efficient strategy, efficient frontier.
* Corresponding author: Ping Chen.
the resulting process has a similar form with the dynamics under proportional reinsurance setting. Thus after the re-parametrization, one can solve the stochastic control problem under the excess-of-loss policy by using techniques in proportional settings.

Besides transferring part of risks from insurance claims by purchasing reinsurance, insurance companies may invest their surpluses in financial markets. Hence the investigation of optimal investment-reinsurance problems of insurance companies by applying stochastic optimal control theory has been one of the central research topics in actuarial science and a great attention has been drawn into this area. Browne [4] firstly introduces Brownian motion with drift to describe the surplus of the insurance company and finds the optimal investment strategy to maximize the expected exponential utility of terminal wealth. After this, Irgens and Paulsen [13] incorporate a proportional reinsurance to an optimal investment problem and derive the optimal reinsurance-investment strategy in a diffusion model from an insurer’s perspective. Instead of using diffusion models for insurer’s surpluses, Hipp and Plum [9] employ the Cramér Lundberg model to describe the risk process of an insurance company and suppose that the surplus of insurance company is invested in a risky asset whose price is described by a Geometric Brownian motion. Yang and Zhang [24] study the optimal investment policies of an insurer with jump-diffusion process under three criteria, i.e., maximizing exponential utility at a given terminal time; maximizing the survival probability and a general objective function. Bai and Guo [2] consider a financial market with multiple risky assets and obtain the optimal strategy under the criterion of maximizing expected exponential utility of terminal wealth. Zhang and Siu [26] model the optimal investment-reinsurance problem with uncertainty as a two-player, zero-sum, stochastic differential game between the insurance company and the market.

In this paper, we apply the mean-variance criterion to proportional reinsurance and investment problem of an insurer whose risk process is driven by the diffusion approximation of a controlled compound Poisson process. The mean-variance criterion is firstly proposed in portfolio selection by Markowitz [16] considering the expected return as well as the variance of the investment in a single period. The continuous-time version is solved in Zhou and Li [28] under the framework of linear-quadratic stochastic control theory. Considering that short-selling of risky assets is always restricted by regulatory, we introduce the no-shorting constraint in our model, then the portfolio is constrained to take nonnegative values hence the corresponding HJB equation has no smooth solution. Li et al. [14] overcome this difficulty by constructing a continuous function via two Riccati equations and show that this function is a viscosity solution to the HJB equation. Recently, Wang et al. [23] point out that the mean-variance problem is also of interest in insurance applications. Then there are increasing interests in adopting the mean-variance criterion in insurance modelling. Bai and Zhang [3] derive the optimal proportional reinsurance and investment strategy in both classical model and its diffusion approximation under the mean-variance criterion.

However, in their model the constraint of proportional reinsurance is not considered, which leads to the situation that the proportion of claim risks that insurance company might take is greater than one. In fact, the amount of claims that the company would have to take can even be more than ten times of the original risk. Such kind of optimal policies can hardly be realistic. Thus in this paper, the proportion of reinsurance is constrained to be in $[0, 1]$, which makes our mean-variance...
problem challenging because the value function (i.e. the solution of HJB equation) is no longer a quadratic function. As yet, analytical research on the problem is literally nil according to our best knowledge.

The main contribution of this paper is that we construct a viscosity solution, which is not a quadratic function of the company’s surplus, of the corresponding HJB equation. Based on this, the explicit expressions of efficient strategy and efficient frontier are derived when the objective expected terminal wealth is greater than a certain level. When the expected return is below that level, it is hard to determine the efficient frontier explicitly since the expression of value function, which coincides with the viscosity solution, is too complicated and involves too many parameters. However, by applying numerical methods and with the assistance of mathematical softwares, we find that the expressions of efficient strategy and efficient frontier vary with the alteration of parameters. When comparing the efficient frontiers of models with constrained and unconstrained proportional reinsurance, we find that if the proportion of retained risk $q$ is only required to be nonnegative, the company will take much lower risks than the constrained reinsurance case. However, the optimal reinsurance proportion might be too high to be realistic, which indicates the importance of reinsurance restriction $q \leq 1$.

The rest of this paper is organized as following. In Section 2, we formulate the mean-variance problem under no short-selling constraint and proportional reinsurance setting. We study an auxiliary stochastic control problem in Section 3. A viscosity solution is constructed to the corresponding HJB equation together with the optimal feedback control, where we overcome the difficulties caused by the constrained proportional reinsurance. In Section 4 the efficient strategy and efficient frontier are explicitly derived when the expected return exceeds a certain level and for the remaining objective terminal wealth, numerical analysis is applied. To illustrate the results, two numerical examples and the comparison of efficient frontiers in constrained and unconstrained reinsurance models are presented in Section 5. Finally, some additional remarks are provided in Section 6.

2. Formulation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\{\mathcal{F}_t\}$. Consider the classical Cramér-Lundberg model

$$X_t = x_0 + ct - \sum_{i=1}^{N_t} Y_i, \quad (2.1)$$

where $x_0$ is the initial surplus, the arrival process $N_t$ is a Poisson process with constant intensity $\lambda > 0$ and the random variables $Y_i, i = 1, 2, \ldots$, are i.i.d claim sizes independent of $N_t$. We let $\{T_i, i = 1, 2, \ldots\}$ denote the claim times and $G(x)$ denotes the claim size distribution with finite first and second moments $m_1, m_2$. The premium rate $c$ is assumed to be calculated via the expected value principle, i.e.,

$$c = (1 + \eta)\lambda m_1,$$

where $\eta > 0$ is the relative safety loading factor.

Suppose that the insurer has the choice of both risk management and investment within a finite time horizon $[0, T]$. Risk management takes the form of proportional reinsurance, i.e., insurer could transfer a fraction $1 - q(t)$ of the contingent claims to a reinsurer, where $q(t)$ is $\mathcal{F}_t$-measurable and satisfies $0 \leq q(t) \leq 1$ for all $t$. For this business, the premium rate payable to reinsurer is $(1 + \theta)(1 - q(t))\lambda m_1$, where $\theta (\theta \geq \eta)$ represents the loading factor for the reinsurer. The insurer could invest
its wealth at hand into a financial market consisting of a risk-free asset (bond) and a risky asset (stock). Particularly, the price process of the risk-free asset follows an ordinary differential equation (ODE)

\[ dS_0(t) = rS_0(t)dt, \quad r > 0, \]

and the price process of the risky asset is driven by a geometric Brownian motion

\[ dS(t) = S(t) \left\{ \mu dt + \sigma dW(t) \right\}, \quad \mu > r, \quad \sigma > 0, \]

where \( \{W(t)\} \) is a standard Brownian motion independent of claim process.

A strategy \( \alpha \) is described by a pair of stochastic processes \((\pi(t), q(t))\), where \( \pi(t) \) represents the amount of wealth invested in the risky asset at time \( t \) and \( q(t) \) represents the retention proportion of claims at time \( t \). A restriction considered here is the prohibition of short selling the risky asset, i.e., \( \pi(t) \geq 0 \). While borrowing from the money market (at interest rate \( r \)) is still allowed. A strategy \( \alpha \) is said to be admissible if \((\pi(t), q(t)) \in \mathcal{F}_t\)-progressively measurable, and satisfies \( 0 \leq q(t) \leq 1 \), \( \pi(t) \geq 0 \) and \( \mathbb{E}[\int_0^T \pi^2(s)ds] < \infty \). We denote the set of all admissible strategies by \( \mathcal{A}_S \).

Let \( X_t^\alpha \) be the resulting surplus process after incorporating strategy \( \alpha \) into (2.1), the dynamics of \( X_t^\alpha \) can be preserved as follows

\[ dX_t^\alpha = \left[ q(t)(1 + \theta) - (\theta - \eta) \right] \lambda m_1 dt - d \sum_{i=1}^{N_t} q(T_i)Y_i \]

\[ + \pi(t)\left[ \mu dt + \sigma dW(t) \right] + r[X_t^\alpha - \pi(t)]dt. \]

Similar to (1.5) in Højgaard and Taksar [12], we approximate this controlled surplus process by a diffusion process with identical mean and variance, i.e.

\[ dX_t^\alpha = \left\{ rX_t^\alpha + (\mu - r)\pi(t) + [q(t)\theta - (\theta - \eta)\lambda m_1] \right\} dt \]

\[ + \sigma \pi(t)dW(t) + q(t)\sqrt{\lambda m_2}dW_0(t), \]

where \( \{W_0(t)\} \) is a standard \( \mathcal{F}_t \)-adapted Brownian motion independent of \( \{W(t)\} \).

Accordingly, the jump term \( d \sum_{i=1}^{N_t} q(T_i)Y_i \) is approximated by \( q(t)[\lambda m_1 dt + q(t)\sqrt{\lambda m_2}dW_0(t)] \). The independence between \( \{W(t)\} \) and \( \{W_0(t)\} \) comes from the fact that claim process and investment return are uncorrelated.

In this paper, we apply the mean-variance principle, under which our aim is to find an admissible strategy such that the expected terminal wealth satisfies \( \mathbb{E}[X_T^\alpha] = d \), while the risk measured by the variance of the terminal wealth, i.e.,

\[ \text{Var}X_T^\alpha = \mathbb{E}[X_T^\alpha - \mathbb{E}X_T^\alpha]^2 = \mathbb{E}[X_T^\alpha - d]^2, \]

is minimized.

**Remark 1.** It is reasonable to impose \( d \geq d_0 \), where \( d_0 := X_0e^{rT} + \frac{\lambda m_1(\theta - \eta)}{r}(1 - e^{rT}) \) is the terminal wealth at time \( T \) if insurance company invests all of its wealth at hand into the risk-free asset and transfers all forthcoming risks to the reinsurer.

**Remark 2.** The mean-variance principle considers the tradeoff between the expected return and variance of insurer’s surplus at the end of time horizon \( T \), regardless of the ruin issue during the period \((0, T)\). As formulated in many papers, for example, Chen et al. [5], we assume that the insurance company can continue its operation even if its surplus is negative and then the ruin problem is ignored in our model setting.
The above problem can be formulated as the following optimization problem parameterized by $d$:

$$\min Var_X = E[(X^\alpha_T - d)^2],$$

subject to

$$\begin{align*}
\mathbb{E}X^\alpha_T &= d, \\
(\pi(t), q(t)) &\in \alpha_S, \\
(X^\alpha_t, \pi(t), q(t)) &\text{ satisfy (2.3)}. 
\end{align*}$$

(2.4)

The optimal strategy of (2.4) is called an efficient strategy, and $(Var_X^\alpha, d)$, where $Var_X^\alpha$ is the optimal value of (2.4) corresponding to $d$, is called an efficient point.

The set of all efficient points, when the parameter $d$ runs over $[d_0, +\infty)$, is called the efficient frontier.

Since (2.4) is a convex optimization problem, the equality constraint $\mathbb{E}X^\alpha_T = d$ can be dealt with by introducing a Lagrange multiplier $\beta \in \mathbb{R}$. In this way the problem (2.4) can be solved via its dual problem (for every fixed $\beta$):

$$\min E\left\{ (X^\alpha_T - d)^2 + 2\beta[\mathbb{E}X^\alpha_T - d] \right\},$$

subject to

$$\begin{align*}
(\pi(t), q(t)) &\in \alpha_S, \\
(X^\alpha_t, \pi(t), q(t)) &\text{ satisfy (2.3)}. 
\end{align*}$$

(2.5)

Let $b = d - \beta$. The above problem is equivalent to

$$\min E[(X^\alpha_T - b)^2],$$

subject to

$$\begin{align*}
(\pi(t), q(t)) &\in \alpha_S, \\
(X^\alpha_t, \pi(t), q(t)) &\text{ satisfy (2.3)}. 
\end{align*}$$

(2.6)

3. Value function for the auxiliary problem. For the auxiliary problem (2.6), we define the associated optimal value function by

$$V(t, x) = \min_{\alpha \in \alpha_S} E[(X^\alpha_T - b)^2 | X_t = x].$$

(3.7)

Then we can use dynamic programming approach to find the optimal control for the auxiliary problem (2.6).

From standard arguments, we see that if the optimal value function $V(t, x)$ is twice continuously differentiable (i.e., $V \in C^{1,2}$), then it satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{align*}
\min_{0 \leq q \leq 1} \left\{ v_t + [rx + (\mu - r)\pi + [\theta q - (\theta - \eta)]\lambda m_1]v_x + \frac{1}{2}(\sigma^2\pi^2 + \lambda m_2 q^2)v_{xx} \right\} &= 0, \\
\pi &\geq 0
\end{align*}$$

(3.8)

with the boundary condition

$$v(T, x) = (x - b)^2.$$

(3.9)

For calculation convenience, we rewrite the HJB equation (3.8) as

$$\begin{align*}
v_t + rx v_x + \min_{\pi \geq 0} \left\{ (\mu - r)v_x + \frac{1}{2}\sigma^2 \pi^2 v_{xx} \right\} \\
+ \min_{0 \leq q \leq 1} \left\{ [\theta q - (\theta - \eta)]\lambda m_1 v_x + \frac{1}{2}\lambda m_2 q^2 v_{xx} \right\} &= 0.
\end{align*}$$

(3.10)

Since $V \in C^{1,2}$ is not be satisfied in most of the cases, we study the viscosity solution of this HJB equation. The notion of viscosity solution was introduced by...
Crandell and Lions [8] for first-order Hamilton-Jacobi equations and by Lions [15] for second-order partial differential equations. Nowadays, it is a standard tool for studying HJB equations. Adopting the notion of viscosity solution introduced by Crandell and Lions [8] and by Soner [20, 21], we define

**Definition 3.1.** A continuous function $v : [0, T] \times \mathbb{R} \to \mathbb{R}^+$ is said to be a viscosity subsolution of (3.9)-(3.10) at $(t, x) \in [0, T] \times \mathbb{R}$, if any twice continuously differentiable function $\varphi : [0, T] \times \mathbb{R} \to \mathbb{R}^+$ with $v(t, x) = \varphi(t, x)$ such that $v - \varphi$ reaches its maximum at $(t, x)$ satisfies

$$
\varphi_t + rx \varphi_x + \min_{\pi \geq 0} \left\{ (\mu - r)\pi \varphi_x + \frac{1}{2} \sigma^2 \pi^2 \varphi_{xx} \right\} \\
+ \min_{0 \leq q \leq 1} \left\{ [\theta q - (\theta - \eta)] \lambda m_1 \varphi_x + \frac{1}{2} \lambda m_2 q^2 \varphi_{xx} \right\} \geq 0.
$$

A continuous function $v : [0, T] \times \mathbb{R} \to \mathbb{R}^+$ is said to be a viscosity supersolution of (3.9)-(3.10) at $(t, x) \in [0, T] \times \mathbb{R}$, if any twice continuously differentiable function $\varphi : [0, T] \times \mathbb{R} \to \mathbb{R}^+$ with $v(t, x) = \varphi(t, x)$ such that $v - \varphi$ reaches its minimum at $(t, x)$ satisfies

$$
\varphi_t + rx \varphi_x + \min_{\pi \geq 0} \left\{ (\mu - r)\pi \varphi_x + \frac{1}{2} \sigma^2 \pi^2 \varphi_{xx} \right\} \\
+ \min_{0 \leq q \leq 1} \left\{ [\theta q - (\theta - \eta)] \lambda m_1 \varphi_x + \frac{1}{2} \lambda m_2 q^2 \varphi_{xx} \right\} \leq 0.
$$

Finally, a continuous function $v : [0, T] \times \mathbb{R} \to \mathbb{R}^+$ is said to be a viscosity solution of (3.9)-(3.10) if it is both a viscosity subsolution and a viscosity supersolution at any $(t, x) \in [0, T] \times \mathbb{R}$.

The following theorem shows that (3.9)-(3.10) has a continuous viscosity solution.

**Theorem 3.2.** The function

$$
v(t, x) =
\begin{cases}
e^{-2r(t-T)}[x - g_1(t)e^{r(t-T)}]^2, & \text{if } (t, x) \in A_1, \\
e^{(A(1)-2r)(t-T)}[x - g_1(t)e^{r(t-T)}]^2, & \text{if } (t, x) \in A_2, \\
\frac{m_2}{m_2}e^{(A(k)-2r)(t-T)} + \frac{\lambda m_2 (1-k)^2}{A(k)-2r}(1 - e^{(A(k)-2r)(t-T)}), & \text{if } (t, x) \in A_4, \\
e^{(A(0)-2r)(t-T)}[x - g_0(t)e^{r(t-T)}]^2 + \frac{\lambda m_2}{A(0)-2r}(1 - e^{(A(0)-2r)(t-T)}), & \text{if } (t, x) \in A_3,
\end{cases}
$$

is a continuous viscosity solution to the HJB equation (3.10) with the boundary condition (3.9), where

$$
A(k) = \frac{(\mu - r)^2}{\sigma^2} + \frac{\lambda m_1^2 \theta^2 k^2}{m_2}, \quad k \in [0, 1],
$$

$$
g_0(t) = b + \frac{\lambda m_1 \eta}{r}(1 - e^{-r(t-T)}),
$$

$$
g_1(t) = b - \frac{\lambda m_1 (\theta - \eta)}{r}(1 - e^{-r(t-T)}),
$$

$$
k = k(t, x) = \frac{\eta}{\lambda m_1 \theta} + \frac{r}{\lambda m_1 \theta} \frac{b - e^{-r(t-T)}(x + \frac{m_2}{m_1 \theta})}{1 - e^{-r(t-T)}}.
$$
\[ (\pi_1^*(t), q_1^*(x)) = \begin{cases} (0,0), & \text{if} \ (t,x) \in A_1, \\ \left( \frac{-\theta - r}{2}x + g_1(t)e^{r(T-t)} \right), & \text{if} \ (t,x) \in A_2, \\ \left( \frac{m_2}{m_1}(\theta - r), 1 \right), & \text{if} \ (t,x) \in A_3, \\ \left( \frac{-\theta - r}{2}x - g_0(t)e^{r(T-t)} \right), & \text{if} \ (t,x) \in A_4. \end{cases} \]

**Proof.** To illustrate the method, we solve (3.9)-(3.10) analytically. We now construct a viscosity solution to (3.10) with boundary condition (3.9). Considering the assumption (3.16), the minimum of the left hand side of HJB equation (3.10) is attained at \( \alpha = (\pi, q) = (0,0) \). Assuming that \( v(t,x) \) has the following trivial form: 

\[ v(t,x) = P(t)x^2 + Q(t)x + R(t), \quad (3.14) \]

where \( P(\cdot), Q(\cdot), R(\cdot) \) are three suitable functions of \( t \) to be determined. Inserting \( (3.14) \) and \( \alpha^* = (\pi^*, q^*) = (0,0) \) into (3.10), the HJB equation becomes 

\[ [P'(t) + 2rP(t)]x^2 + [Q'(t) + rQ(t) - 2(\theta - \eta)\lambda m_1 P(t)]x + R'(t) - (\theta - \eta)\lambda m_1 Q(t) = 0. \]

Therefore, \( P(\cdot), Q(\cdot), \) and \( R(\cdot) \) should satisfy the following differential equations (the first being a special Riccati equation)

\[ \begin{align*}
\begin{cases} P'(t) + 2rP(t) &= 0, \\
Q'(t) + rQ(t) - 2(\theta - \eta)\lambda m_1 P(t) &= 0, \\
R'(t) - (\theta - \eta)\lambda m_1 Q(t) &= 0,
\end{cases} \quad (3.15)
\end{align*} \]

with the boundary conditions

\[ P(T) = 1, Q(T) = -2b, R(T) = b^2. \quad (3.16) \]

Solving (3.15) and (3.16), we obtain 

\[ v(t,x) = P_1(t)x^2 + Q_1(t)x + R_1(t), \]

where

\[ \begin{align*}
\begin{cases} P_1(t) &= e^{-2r(T-t)}, \\
Q_1(t) &= -2g_1(t)e^{r(T-t)}, \\
R_1(t) &= R_2(t),
\end{cases} \quad (3.13)
\end{align*} \]

and \( g_1(t) \) is given by (3.13).

Considering the assumption \( v_x \geq 0 \), we have 

\[ v(t,x) = e^{-2r(T-t)}[x - g_1(t)e^{r(T-t)}]^2, \]
in the region:
\[ A_1 = \left\{ (t, x) \in [0, T] \times \mathbb{R} : x - g_1(t)e^{r(t-T)} \geq 0 \right\}, \]
and the minimum is attained at \((\pi^*, q^*) = (0, 0)\).

For \((t, x) \in \left\{ (t, x) \in [0, T] \times \mathbb{R} : x - g_1(t)e^{r(t-T)} < 0 \right\}\), we have \(v_x < 0\). Assume that the minimum of (3.10) is attained in the interior of the control region, i.e., the optimal \(\pi^*(t, x)\) is non-negative and \(q^*(t, x)\) lies in \([0, 1]\) for all \((t, x) \in [0, T] \times \mathbb{R}\). Then,
\[
\pi^0(t, x) = -\frac{\mu - r}{\sigma^2} v_x, \\
q^0(t, x) = -\frac{m_1 \theta}{m_2} v_x.
\]

Note that \(\pi^0(t, x) > 0\), thus \(\pi^*(t, x) : \pi^0(t, x) = -\frac{\mu - r}{\sigma^2} v_x\). Considering that \(q^0(t, x) > 0\), then if \(q^0(t, x) \leq 1\), \(q^*(t, x)\) will coincide with \(q^0(t, x)\). Otherwise, we simply set \(q^*(t, x) = 1\).

For \(q^0(t, x) \leq 1\), we have \(q^*(t, x) = q^0(t, x) = -\frac{m_1 \theta}{m_2} v_x\). Inserting \(\alpha^* = (\pi^*, q^*)\) into (3.10), the HJB equation becomes
\[
v_t + r x v_x - (\theta - \eta) \lambda m_1 v_x - \frac{(\mu - r)^2}{2 \sigma^2} v_x^2 - \frac{\lambda m_1^2 \theta^2}{2 m_2} v_x^2 = 0. \tag{3.17}
\]
Inserting the trivial solution (3.14) and \(\alpha^* = (\pi^*, q^*)\) into (3.17), we obtain
\[
\begin{cases}
P'(t) + [2r - A(1)]P(t) = 0, \\
Q'(t) + [r - A(1)]Q(t) - 2 \lambda m_1 (\theta - \eta) P(t) = 0, \\
R'(t) - \frac{A(1)Q(t)}{2P(t)} - \lambda m_1 (\theta - \eta) Q(t) = 0,
\end{cases} \tag{3.18}
\]
with the boundary conditions (3.16), where \(A(1) = \frac{(\mu - r)^2}{\sigma^2} + \frac{\lambda m_1^2 \theta^2}{m_2}\). Solving (3.18) and (3.16), we have
\[ v(t, x) = P_2(t)x^2 + Q_2(t)x + R_2(t), \]
where
\[
\begin{align*}
P_2(t) &= e^{(A(1)-2r)(t-T)}, \\
Q_2(t) &= -2g_1(t)e^{(A(1)-r)(t-T)}, \\
R_2(t) &= g_1^2(t)e^{A(1)(t-T)},
\end{align*}
\]
and \(g_1(t)\) is given by (3.13).

Since \(v_x < 0\) and \(q^*(t, x) \leq 1\), we have
\[ v(t, x) = e^{(A(1)-2r)(t-T)} [x - g_1(t)e^{r(t-T)}]^2, \]
in the region
\[ A_2 = \left\{ (t, x) \in [0, T] \times \mathbb{R} : -\frac{m_2}{m_1 \theta} \leq x - g_1(t)e^{r(t-T)} < 0 \right\}, \]
and the minimum is attained at
\[ \alpha^* = (\pi^*, q^*) = \left( -\frac{\mu - r}{\sigma^2} \left[ x - g_1(t)e^{r(t-T)} \right], -\frac{m_1 \theta}{m_2} \left[ x - g_1(t)e^{r(t-T)} \right] \right). \]
For \( q^0(t, x) > 1 \), it is reasonable to let \( q^*(t, x) = 1 \), so the HJB equation (3.10) becomes
\[
v_t + rxv_x + \lambda m_1 \eta v_x - \frac{(\mu - r)^2}{2\sigma^2} v_x^2 + \frac{1}{2} \lambda m_2 v_{xx} = 0.
\] (3.19)
Inserting the trivial solution (3.14) and \( \alpha^* = (\pi^*, 1) \) into (3.19), we obtain
\[
\begin{align*}
P'(t) + [2r - A(0)]P(t) &= 0, \\
Q'(t) + [r - A(0)]Q(t) + 2\lambda m_1 \eta P(t) &= 0, \\
R'(t) - \frac{A(0)Q^2(t)}{4P(t)} + \lambda m_1 \eta Q(t) + \lambda m_2 P(t) &= 0,
\end{align*}
\] (3.20)
with the boundary conditions (3.16), where \( A(0) = \frac{(\mu - r)^2}{\sigma^2} \). Solving (3.20) and (3.16), we have
\[v(t, x) = P_3(t)x^2 + Q_3(t)x + R_3(t),\]
where
\[
\begin{align*}
P_3(t) &= e^{(A(0)-2r)(t-T)}, \\
Q_3(t) &= -2g_0(t)e^{(A(0)-r)(t-T)}, \\
R_3(t) &= g_0(t)e^{A(0)(t-T)} + \frac{\lambda m_2}{A(0)-2r}(1 - e^{(A(0)-2r)(t-T)}),
\end{align*}
\] and \( g_0(t) \) is given by (3.12).

Considering the condition \( v_x < 0 \) and \( q^0(t, x) > 1 \), we have
\[v(t, x) = e^{(A(0)-2r)(t-T)} \left[ x - g_0(t)e^{r(t-T)} \right]^2 + \frac{\lambda m_2}{A(0)-2r} \left( 1 - e^{(A(0)-2r)(t-T)} \right),\]
in the region
\[\mathcal{A}_3 = \left\{(t, x) \in [0, T] \times \mathbb{R} : x - g_0(t)e^{r(t-T)} \leq -\frac{m_2}{m_1 \eta} \right\},\]
and the minimum is attained at
\[\alpha^* = (\pi^*, q^*) = \left( -\frac{\mu - r}{\sigma^2} \left[ x - g_0(t)e^{r(t-T)} \right], 1 \right).\]

If we draw a graph of regions where we have already obtained the solution (see Figure 1), it is interesting to notice that there is still one region where we have no solution yet. Considering that \( q^0(t, x) \leq 1 \) in \( \mathcal{A}_2 \) and \( q^0(t, x) > 1 \) in \( \mathcal{A}_3 \), in the meanwhile, on the lower boundary of \( \mathcal{A}_2 \) and upper boundary of \( \mathcal{A}_3 \), we have \( q^0(t, x) = 1 \). So it is not surprising to expect that \( q^0(t, x) \equiv 1 \) in the remaining region \( \mathcal{A}_4 = \{(t, x) \in [0, T] \times \mathbb{R} : x - g_0(t)e^{r(t-T)} \leq -\frac{m_2}{m_1 \eta} \leq x - g_0(t)e^{r(t-T)} \}. \) For this remaining region, we cannot get the solution by the previous standard method. This is because the solution of the HJB equation corresponding to the mean-variance problem we considered is no longer a quadratic function, which makes it difficult to solve the HJB equation analytically. Next we will focus on the construction of solutions in \( \mathcal{A}_4 \). By comparing the expression of \( v(t, x) \) on upper and lower boundaries of \( \mathcal{A}_4 \), we guess \( v \) has the form
\[
v(t, x) = e^{(A(k)-2r)(t-T)} \left[ x - g(k, t)e^{r(t-T)} \right]^2 + H(t)
\] (3.21)
on the curve \( \mathcal{C}_k = \{(t, x) \in [0, T] \times \mathbb{R} : x - g(k, t)e^{r(t-T)} = -\frac{m_2}{m_1 \eta} \} \), where
\[g(k, t) = b - \frac{\lambda m_1 (k \theta - \eta)}{r} \left( 1 - e^{-r(t-T)} \right), \quad 0 < k < 1,
\] (3.22)
Figure 1. the region of $A_i$, $i = 1, 2, 3, 4$. For the region $A_4$, we can deem it as a family of curves $\{C_k\}_{0 \leq k \leq 1}$ (i.e., the red dot curve) and construct a solution to the HJB equation on each curve and $A(k)$ is given by (3.11).

Plugging (3.21) into $q^0(t, x)$, we obtain $q^0(t, x) \equiv 1$ in $A_4$, which is exactly what we expect, thus $q^* = 1$. Inserting (3.21) into the HJB equation and rearranging the terms, we obtain

$$H(t) = \frac{\lambda m_2(1 - k)^2}{A(k) - 2r} \left( 1 - e^{(A(k) - 2r)(t - T)} \right).$$

Thus for $(t, x) \in A_4$, we have

$$v(t, x) = e^{(A(k) - 2r)(t - T)} \left[ x - g(k, t) e^{r(t - T)} \right]^2 + \frac{\lambda m_2(1 - k)^2}{A(k) - 2r} \left( 1 - e^{(A(k) - 2r)(t - T)} \right)$$

$$= \left( \frac{m_2}{m_1 \theta} \right)^2 e^{(A(k) - 2r)(t - T)} + \frac{\lambda m_2(1 - k)^2}{A(k) - 2r} \left( 1 - e^{(A(k) - 2r)(t - T)} \right),$$

with

$$k = k(t, x) = \eta + \frac{r}{\lambda m_1 \theta} \frac{b - e^{-r(t-T)}(x + \frac{m_2}{m_1 \theta})}{1 - e^{-r(t-T)}},$$

and the minimum is attained at

$$\alpha^* = (\pi^*, q^*) = \left( \frac{m_2(\mu - r)}{m_1 \theta \sigma^2}, 1 \right).$$

Now we have constructed a solution $v(t, x)$ to HJB equation (3.10) with boundary condition (3.9), which is given in Theorem 3.2.

**Remark 3.** In the inner region of $A_i$ ($i = 1, 2, 3, 4$), $v(t, x) \in C^{1,2}$ thus it is a classical solution inside these regions. From the method used in Bai and Zhang [3], we can prove $v(t, x)$ is a viscosity solution on the switching curve $\{(t, x) \in [0, T] \times \mathbb{R} : x - g_1(t) e^{r(t - T)} = 0\}$ by Definition 3.1. On switching curves $C_0$ and $C_1$,
the second order derivative is too complicated to compare so the proof of viscosity needs further study.

**Remark 4.** From the verification theorem in Yong and Zhou [25], we know that the value function is a viscosity solution of HJB equation and the HJB equation admits at most one viscosity solution. Here, \( v(t,x) \) we constructed is a viscosity solution, then we can draw the conclusion: The value function \( V(t,x) \) for the auxiliary problem (2.6) coincides with the solution \( v(t,x) \) constructed in Theorem 3.2. And the optimal feedback control is given by the \((\pi_t^*, q_t^*)\) in Theorem 3.2 where \( x = X_t^\alpha \) is the corresponding dynamic process.

4. **Efficient strategy and efficient frontier.** For notational convenience, in this section we define:

\[
\begin{align*}
\kappa_0 & := X_0 - g_0(0)e^{-rT}, & \kappa_1 & := X_0 - g_1(0)e^{-rT}, \\
\kappa_0^* & := X_t^\alpha - g_0(t)e^{r(t-T)}, & \kappa_1^* & := X_t^\alpha - g_1(t)e^{r(t-T)},
\end{align*}
\]

where

\[
\begin{align*}
g_0(0) &= d - \beta + \frac{\lambda m_1 \eta}{r}(1 - e^{rT}), \\
g_1(0) &= d - \beta - \frac{\lambda m_1 (\theta - \eta)}{r}(1 - e^{rT}), \\
g_0(t) &= d - \beta^* + \frac{\lambda m_1 \eta}{r}(1 - e^{-r(t-T)}), \\
g_1(t) &= d - \beta^* - \frac{\lambda m_1 (\theta - \eta)}{r}(1 - e^{-r(t-T)}), \\
\beta^* &= \frac{d - X_0e^{rT} + \frac{\lambda m_1 \eta}{r}(1 - e^{rT})}{1 - e^{A(0)T}}.
\end{align*}
\]

According to Theorem 2.5.2 (Fritz John Conditions) in Shi [19], we know that the Lagrange multiplier \( \beta \) in our model exists. Then we could apply the results in Section 3 to the mean-variance problem to find the efficient strategy. Let \( t = 0 \) and \( x = X_0 \) in \( V(t,x) \), we obtain

\[
V_\beta(0,X_0) = V(0,X_0) - \beta^2 = \begin{cases} 
    e^{2rT} k_1^2 - \beta^2, & \text{if } \kappa_1 \geq 0, \\
    e^{-(A(1)-2r)T} k_1^2 - \beta^2, & \text{if } -\frac{m_2}{m_1 \theta} \leq \kappa_1 < 0, \\
    \left(\frac{m_2}{m_1 \theta}\right)^2 e^{-(A(k)-2r)T} + \frac{\lambda m_2 (1-k)^2}{A(k)^{-2r}} (1 - e^{-(A(k)-2r)T}) - \beta^2, & \text{if } \kappa_1 < -\frac{m_2}{m_1 \theta} < \kappa_0, \\
    e^{-(A(0)-2r)T} k_0^2 + \frac{\lambda m_2}{A(0)^{-2r}} (1 - e^{-(A(0)-2r)T}) - \beta^2, & \text{if } \kappa_0 \leq -\frac{m_2}{m_1 \theta}.
\end{cases}
\]

The value of \( V_\beta(0,X_0) \) depends on the Lagrange multiplier \( \beta \). To obtain the optimal value (i.e. the minimum variance of \( X_T \)) and optimal strategy for original problem, we only need to maximize the value of \( V_\beta(0,X_0) \) where \( \beta \) runs over \( \mathbb{R} \).

4.1. **The case of** \( d \geq X_0e^{rT} + \frac{\lambda m_1 \eta}{r}(e^{rT} - 1) + \frac{m_2}{m_1 \theta} e^{rT} \).
Theorem 4.1. For $d \geq X_0e^{rT} + \frac{\lambda m_1\eta}{r}(e^{rT} - 1) + \frac{m_2}{m_1}\theta e^{rT}$, the efficient strategy of the problem \cite{2,4} corresponding to the expected terminal wealth $EX_T = d$ is

$$(\pi^*_1, q^*_1) = \begin{cases} (0,0), & \text{if } \kappa_1^* \geq 0, \\ \left(-\frac{\theta - \kappa_1^*}{\sigma^2}, -\frac{m_1\theta}{m_2}\kappa_1^*\right), & \text{if } -\frac{m_2}{m_1}\theta \leq \kappa_1^* < 0, \\ \left(\frac{m_2(\theta - r)}{m_1\theta}, 1\right), & \text{if } \kappa_1^* < -\frac{m_2}{m_1}\theta < \kappa_0^*, \\ \left(-\frac{\theta - \kappa_0^*}{\sigma^2}, \kappa_0^*\right), & \text{if } \kappa_0^* \leq -\frac{m_2}{m_1}\theta. \end{cases}$$

The efficient frontier is

$$VarX_T = \frac{\left[d - X_0e^{rT} + \frac{\lambda m_1\eta}{r}(1 - e^{rT})\right]^2}{e^{A(0)T} - 1} + \frac{\lambda m_2 \left(1 - e^{-\left(A(0) - 2r\right)T}\right)}{A(0) - 2r}, \quad (4.23)$$

where $A(0) = \frac{(\mu - r)^2}{\sigma^2}$.

Proof. We can find the piecewise maximum first and then compare them to obtain the maximum of $V_{\beta}(0, X_0)$ for $\beta$ running over $\mathbb{R}$.

1. For $\kappa_1 \geq 0$, i.e., $\beta \geq \beta_2$, where $\beta_2 := d - X_0e^{rT} - \frac{\lambda m_1\eta}{r}(1 - e^{rT})$, $V_{\beta}(0, X_0)$ is linear and is decreasing in $\beta$ (since $d \geq X_0e^{rT} + \frac{\lambda m_1\eta}{r}(1 - e^{rT})$), then we have:

$$\max_{\beta \geq \beta_2} V_{\beta}(0, X_0) = V_{\beta_2}(0, X_0) = -\left[d - X_0e^{rT} - \frac{\lambda m_1\eta}{r}(1 - e^{rT})\right]^2.$$

2. For $-\frac{m_2}{m_1}\theta \leq \kappa_1 \leq 0$, i.e., $\beta_1 \leq \beta \leq \beta_2$, where $\beta_1 := d - X_0e^{rT} - \frac{\lambda m_1\eta}{r}(1 - e^{rT}) - \frac{m_2}{m_1}\theta e^{rT}$, $V_{\beta}(0, X_0)$ is quadratic and concave in $\beta$. A tedious calculation shows that:

(a) If $d \leq d_1$, where $d_1 := X_0e^{rT} + \frac{\lambda m_1\eta}{r}(1 - e^{rT}) + \frac{m_2}{m_1}\theta e^{rT}(1 - e^{-A(1)T})$, we have:

$$\max_{\beta_1 \leq \beta \leq \beta_2} V_{\beta}(0, X_0) = V_{\beta_2}(0, X_0) = \frac{\left[d - X_0e^{rT} - \frac{\lambda m_1\eta}{r}(1 - e^{rT})\right]^2}{e^{A(1)T} - 1},$$

where $\beta^* = \frac{d - X_0e^{rT} - \frac{\lambda m_1\eta}{r}(1 - e^{rT})}{1 - e^{A(1)T}}$.

(b) If $d \geq d_1$, we have:

$$\max_{\beta_1 \leq \beta \leq \beta_2} V_{\beta}(0, X_0) = V_{\beta_0}(0, X_0) = \left(\frac{m_2}{m_1\theta}\right)^2 e^{-\left(A(1) - 2r\right)T} - \beta_1^2.$$

In both cases, we can prove that $\max_{\beta_1 \leq \beta \leq \beta_2} V_{\beta}(0, X_0) > V_{\beta_2}(0, X_0) = \max_{\beta \geq \beta_2} V_{\beta}(0, X_0)$ by some simple calculations.

3. For $\kappa_0 \leq -\frac{m_2}{m_1}\theta$, i.e., $\beta \leq \beta_0$, where $\beta_0 := d - X_0e^{rT} + \frac{\lambda m_1\eta}{r}(1 - e^{rT}) - \frac{m_2}{m_1}\theta e^{rT}$, $V_{\beta}(0, X_0)$ is quadratic and concave in $\beta$. We can obtain:

(a) If $d \leq d_2$, where $d_2 := X_0e^{rT} - \frac{\lambda m_1\eta}{r}(1 - e^{rT}) + \frac{m_2}{m_1}\theta e^{rT}(1 - e^{-A(0)T})$, we have:

$$\max_{\beta \leq \beta_0} V_{\beta}(0, X_0) = V_{\beta_0}(0, X_0)$$

$$= \left(\frac{m_2}{m_1\theta}\right)^2 e^{-\left(A(0) - 2r\right)T} + \frac{\lambda m_2\eta}{A(0) - 2r} - \beta_0^2.$$
Thus, $V_{\beta}(0, X_0) = V_{\beta}(0, X_0)$

$$= \frac{d - X_0 e^{rT} + \frac{\lambda m_1}{r}(1 - e^{rT})^2}{e^{A(0)T} - 1} + \frac{\lambda m_2(1 - e^{-(A(0) - 2r)T})}{A(0) - 2r},$$

where $\beta^* = \frac{d - X_0 e^{rT} + \frac{\lambda m_1}{r}(1 - e^{rT})}{1 - e^{A(0)T}}$.

4. For $\kappa_1 \leq -\frac{m_2}{m_1 \theta} \leq \kappa_0$, i.e., $\beta_0 \leq \beta \leq \beta_1$, we have:

$$V_{\beta}(0, X_0) = \left(\frac{m_2}{m_1 \theta}\right)^2 e^{-(A(k) - 2r)T} + \lambda m_2(1 - k)^2 \frac{1 - e^{-(A(k) - 2r)T}}{A(k) - 2r} - \beta^2,$$

where

$$k = k(\beta) = \frac{\eta}{\theta} + \frac{r - d - e^{-T}(0 + \frac{m_2}{m_1 \theta} e^{T})}{1 - e^{T}}.$$

If we deem $V_{\beta}(0, X_0)$ as a function of $k$ instead of $\beta$, we obtain:

$$V_{\beta}(0, X_0) = \left(\frac{m_2}{m_1 \theta}\right)^2 e^{-(A(k) - 2r)T} + \lambda m_2(1 - k)^2 \frac{1 - e^{-(A(k) - 2r)T}}{A(k) - 2r} - \beta(k)^2,$$

where

$$\beta(k) \equiv d - X_0 e^{rT} - \frac{\lambda m_1(k \theta - \eta)}{r}(1 - e^{rT}) - \frac{m_2}{m_1 \theta} e^{rT}, \quad 0 \leq k \leq 1.$$

As $V_{\beta}(0, X_0)$ is continuous over $[\beta_0, \beta_1]$, its maximum in this region must exist. However, the explicit expressions for the maximum point and maximum value are hard to determine since the function is too complicated. But for $d \geq d$, where $d := x_0 e^{rT} + \frac{\lambda m_1}{r}(e^{rT} - 1) + \frac{m_2}{m_1 \theta} e^{rT}$, we can compare it with $V_{\beta_0}(0, X_0)$. Then we will show that for $d \geq d$, we always have $V_{\beta}(0, X_0) \leq V_{\beta_0}(0, X_0)$, for any $\beta \in [\beta_0, \beta_1]$.

It is easy to prove that $f(\alpha) = \frac{1 - \alpha e^T}{\alpha}$ is a decreasing function for $\alpha > 0$ and is always positive. $A(0) \leq A(k) \leq A(1)$ and for $d \geq d$, we have $0 \leq \beta_0 \leq \beta(k) \leq \beta_1$. Thus for $\beta_0 \leq \beta \leq \beta_1$,

$$V_{\beta}(0, X_0) \leq \left(\frac{m_2}{m_1 \theta}\right)^2 e^{-(A(0) - 2r)T} + \lambda m_2(1 - k)^2 \frac{1 - e^{-(A(0) - 2r)T}}{A(0) - 2r} - \beta^2_0$$

$$\leq \left(\frac{m_2}{m_1 \theta}\right)^2 e^{-(A(0) - 2r)T} - \beta^2_0 < V_{\beta_0}(0, X_0).$$

To obtain the maximum of $V_{\beta}(0, X_0)$ for $\beta \in \mathbb{R}$, we only need to compare $V_{\beta_1}(0, X_0)$ and $V_{\beta_2}(0, X_0)$ (since $d \geq d$ and $d > d_i$, $i = 1, 2$):

$$V_{\beta_1}(0, X_0) = \frac{(d - X_0 e^{rT} + \frac{\lambda m_1}{r}(1 - e^{rT}))^2}{e^{A(0)T} - 1} + \frac{\lambda m_2(1 - e^{-(A(0) - 2r)T})}{A(0) - 2r}$$

$$> \frac{(d - X_0 e^{rT} + \frac{\lambda m_1}{r}(1 - e^{rT}))^2}{e^{A(0)T} - 1} \geq \frac{m_2}{m_1 \theta} e^{rT} \frac{[m_2}{m_1 \theta} e^{rT}]^2{e^{A(0)T} - 1}$$

$$\geq \frac{m_2}{m_1 \theta} e^{2rT} \frac{m_2}{e^{A(1)T}} \geq V_{\beta_1}(0, X_0).$$

Thus,

$$\max_{\beta \in \mathbb{R}} V_{\beta}(0, X_0) = V_{\beta_1}(0, X_0), \quad \text{for } d \geq d,$$

and the efficient frontier is given by $\{4.23\}$. \hfill \square
4.2. The case of \( d < X_0 e^{rT} + \frac{\lambda m_1 n}{r} (e^{rT} - 1) + \frac{m_2}{m_1} e^{rT}. \) For \( d_0 \leq d \leq \bar{d}, \) it is hard to obtain the explicit expression of efficient frontier since the expression of \( V_\beta(0, X_0) \) is too complicated to ascertain its maximum. However, from the numerical results obtained by MatLab, we draw the following conclusion: If \( d_1 \leq d_2, \) we have:

- For \( d_0 \leq d \leq d_1, \) \( \max_{\beta \in \mathbb{R}} V_\beta(0, X_0) = V_{\overline{\beta}}(0, X_0), \) thus the efficient frontier (corresponding to \( EX_T^* = d \)) is:

\[
Var X_T^* = \frac{[d - X_0 e^{rT} - \frac{\lambda m_1 n}{r} (1 - e^{rT})]^2}{e^{A(1)r} - 1}.
\] (4.24)

- For \( d_1 < d < d_2, \) the maximum of \( V_\beta(0, X_0) \) when \( \beta \) runs over \( \mathbb{R} \) is attained at \( \hat{\beta}^* \in (\beta_0, \beta_1) \) and the efficient frontier is given by the corresponding \( V_{\hat{\beta}^*}(0, X_0). \) In this case, explicit expressions for the efficient strategy and efficient frontier are hard to determine;

- For \( d \geq d_2, \) \( \max_{\beta \in \mathbb{R}} V_\beta(0, X_0) = V_{\hat{\beta}^*}(0, X_0) \) and the efficient frontier is given by (4.23).

If \( d_1 > d_2, \) as \( \beta \) runs over the interval \([\beta_0, \beta_1],\) \( V_{\beta}(0, X_0) \) will gradually increase to its local maximum, then go downwards to the local minimum and thenceforth start to bound back. Moreover, there exists a unique \( \hat{d} \in (\frac{d_0 + d_2}{2}, d_2) \) that satisfies \( \max_{\beta_0 \leq \beta \leq \beta_1} V_\beta(0, X_0) = V_{\hat{\beta}}(0, X_0) \) and exists a unique \( \hat{d} \in (d_2, \frac{d_1 + d_2}{2}) \) such that \( \max_{\beta_0 \leq \beta \leq \beta_1} V_{\hat{\beta}^*}(0, X_0) = V_{\hat{\beta}^*}(0, X_0). \) Numerical results show that:

- For \( d_0 \leq d \leq \hat{d}, \) \( \max_{\beta \in \mathbb{R}} V_{\hat{\beta}}(0, X_0) = V_{\hat{\beta}}(0, X_0) \) and the efficient frontier is given by (4.24);

- For \( \hat{d} < d < d_1, \) \( \max_{\beta \in \mathbb{R}} V_\beta(0, X_0) = V_{\hat{\beta}}(0, X_0), \) where \( \hat{\beta}^* \in (\beta_0, \beta_1). \) Like in the case of \( d_1 \leq d_2, \) explicit expressions for the efficient strategy and efficient frontier are hard to find out;

- For \( d \geq \hat{d}, \) \( \max_{\beta \in \mathbb{R}} V_{\hat{\beta}^*}(0, X_0) = V_{\hat{\beta}^*}(0, X_0), \) thereof the efficient frontier is given by (4.23).

To illustrate how the pre-given expected terminal wealth level \( d \) and the sizing of \( d_1 \) and \( d_2 \) affect the efficient strategies and efficient frontiers, we list some numerical results in Table 1 and Table 2. In both tables, the potential piecewise maximum value of \( V_{\beta}(0, X_0) \) are listed under several commonly used claim size distributions and different expected returns. The global maximum (i.e., \( Var X_T^* \)) in each case is represented in boldface.

In Table 1 with parameters specified in the table, we have \( d_1 < d_2 \) under all distributions. Corresponding to all the three categories of \( d \) (i.e., \( d_0 \leq d < d_1, \) \( d_1 \leq d < d_2 \) and \( d \geq d_2 \)), we pick one specific value in each case (i.e., \( d = \frac{d_0 + d_1}{2}, \) \( d = \frac{d_1 + d_2}{2} \) and \( d = \frac{d_2 + d_3}{2} \)). By comparing the potential piecewise maximum values, we find that the global maximum are attained at \( \hat{\beta}^* = \hat{\beta}^* \) and \( \hat{\beta}^* \) respectively. In Table 2 we always have \( d_1 > d_2 \) with the granted parameters. To make the calculation more efficient in mathematical softwares, we choose a particular expected return \( d = d_0 - 100 \) instead of \( \frac{d_1 + d_2}{2} \) for the category \( d \in (d, \hat{d}). \) Results indicate that the global maximum is attained at \( \hat{\beta}^*, \hat{\beta}^* \) and \( \hat{\beta}^* \) when the expected terminal wealth is in the range \((d_0, \hat{d}), (\hat{d}, d)\) and \((\hat{d}, +\infty), \) respectively.
In both tables, we notice that the global maximum value of \( V_\beta \) goes up with the increase of targeted terminal wealth \( d \), which indicates that the variance and expected terminal wealth are positively correlated. This is because lower risk investments, while good for peace of mind, will generally provide a lower expected long term return than a high risk investment. Considering that the expected returns \( d \) in Table 2 are much larger than those in Table 1, the variances in Table 2 are

### Table 1. Piecewise and global maximum values of \( V_\beta(0, X_0) \) under different distributions, if \( \lambda = 10, \theta = 0.3, \eta = 0.2, \mu = 0.06, r = 0.04, \sigma = 1, T = 100 \) and \( X_0 = 50 \), which lead to \( d_1 < d_2 \) in all the following distributions

| Distribution | \( \max_{d_0 \leq d_1} V_\beta \) | \( \max_{d_0 \leq d \leq d_1} V_\beta \) | \( \max_{d_1 \leq d \leq d_2} V_\beta \) | \( \max_{d_2 \leq d} V_\beta \) |
|-------------|----------------|----------------|----------------|----------------|
| \( U(0, 1) \) | \( d = \frac{d_0 + d_1}{2} \) | N/A | -1.533 | -0.0037 | \( 1.7 \times 10^{-20} \) |
| \( \times d \) | \( d = \frac{d_2}{2} \) | N/A | -1.420 | -0.0037 | -0.0037 |
| \( \times \) | \( d = \frac{d_1 + d_2}{2} \) | 0.2173 | 0.1306 | 0.1306 | -3.8089 | -4.2972 |
| \( \times \) | \( d = \frac{d_1 + d_2}{2} \) | 0.1596 | 0.0816 | 0.0816 | -1.4784 | -1.7716 |
| \( \times \) | \( d = \frac{d_1 + d_2}{2} \) | 0.4130 | 0.2376 | 0.2376 | -6.0489 | -6.9279 |
| \( \times \) | \( d = \frac{d_1 + d_2}{2} \) | 0.2413 | 0.1546 | 0.1546 | -5.5396 | -6.1254 |
| \( \times \) | \( d = \frac{d_1 + d_2}{2} \) | 1.2376 | 0.4564 | -3.3675 | -3.3675 | -4.8359 |
| \( \times \) | \( d = \frac{d_1 + d_2}{2} \) | 0.0030 | -0.0207 | -0.0207 | -0.0207 | -0.0207 |
| \( \times \) | \( d = \frac{d_1 + d_2}{2} \) | 2.30 \times 10^5 | -6.010 | 6.30 \times 10^5 | -6.010 |
| \( \times \) | \( d = \frac{d_1 + d_2}{2} \) | 0.030 | -0.0207 | -0.0207 | -0.0207 | -0.0207 |
| \( \times \) | \( d = \frac{d_1 + d_2}{2} \) | 0.4176 | 0.2376 | 0.2376 | -1.8364 | -2.0184 |
| \( \times \) | \( d = \frac{d_1 + d_2}{2} \) | 0.5216 | 0.2323 | 0.2323 | -2.8733 | -3.6740 |
| \( \times \) | \( d = \frac{d_1 + d_2}{2} \) | 0.5664 | 0.2545 | 0.2545 | -3.2263 | -4.1063 |
bigger, which coincides with the mechanism that greater return comes along with more risks.

| Table 2. Piecewise and global maximum values of $V_{\beta}(0, X_0)$ under different distributions, if $\lambda = 1$, $\theta = 0.25$, $\eta = 0.2$, $\mu = 0.12$, $r = 0.1$, $\sigma = 1$, $T = 100$ and $X_0 = 50$, which lead to $d_1 > d_2$ in all the following distributions |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| $U(0, 1)$      | $d = \frac{d_0 + d_2}{2}$ | $N/A$           | $-0.9978$       | $-0.9212$       | $-1.8896$       | $0.0020$        | $-0.2225$       |
| ($\times 10^3$) | $d = d_2 - 100$   | $N/A$           | $0.8971$        | $0.9296$        | $-0.8106$       | $0.0080$        | $-0.8842$       |
|                 | $d = \frac{d_0 + d_2}{2}$ | $7.4450$        | $2.3088$        | $2.3166$        | $-0.1859$       | $0.0173$        | $-1.9383$       |
|                 | $d = \frac{d_1 + d_2}{2}$ | $49.763$        | $3.8965$        | $3.8965$        | $-0.1519$       | $N/A$           | $-5.2191$       |
| $Exp(1)$       | $d = \frac{d_0 + d_2}{2}$ | $N/A$           | $-0.5612$       | $-0.5220$       | $-1.9780$       | $0.0042$        | $-0.0960$       |
| ($\times 10^{10}$) | $d = d_2 - 100$   | $N/A$           | $0.5807$        | $0.6015$        | $-1.1763$       | $0.0169$        | $-0.8382$       |
|                 | $d = \frac{d_0 + d_2}{2}$ | $9.3795$        | $2.1070$        | $2.1098$        | $-0.2391$       | $0.0587$        | $-1.3311$       |
|                 | $d = \frac{d_1 + d_2}{2}$ | $51.957$        | $3.3527$        | $3.3527$        | $0.0744$        | $N/A$           | $-4.0010$       |
| $\Gamma(2, 1)$ | $d = \frac{d_0 + d_2}{2}$ | $N/A$           | $-1.7584$       | $-1.6274$       | $-4.0605$       | $0.0055$        | $-4.0605$       |
| ($\times 10^{10}$) | $d = d_2 - 100$   | $N/A$           | $1.6593$        | $1.7189$        | $-1.9674$       | $0.0219$        | $-1.4494$       |
|                 | $d = \frac{d_0 + d_2}{2}$ | $17.205$        | $4.7277$        | $4.7395$        | $-0.4418$       | $0.0548$        | $-3.6266$       |
|                 | $d = \frac{d_1 + d_2}{2}$ | $106.69$        | $7.7884$        | $7.7884$        | $0.1778$        | $N/A$           | $-10.075$       |
| $Erlang(3, 0.5)$ | $d = \frac{d_0 + d_2}{2}$ | $N/A$           | $-1.2426$       | $-1.2423$       | $-1.6745$       | $0.0011$        | $-0.3123$       |
| ($\times 10^{11}$) | $d = d_2 - 100$   | $N/A$           | $1.0630$        | $1.1002$        | $-0.5359$       | $0.0043$        | $-1.2483$       |
|                 | $d = \frac{d_0 + d_2}{2}$ | $5.6680$        | $2.2385$        | $2.2512$        | $-0.1273$       | $0.0077$        | $-2.2031$       |
|                 | $d = \frac{d_1 + d_2}{2}$ | $44.398$        | $3.9729$        | $3.9727$        | $-0.2576$       | $N/A$           | $-5.6482$       |
| $Pareto(3, 1)$  | $d = \frac{d_0 + d_2}{2}$ | $N/A$           | $-0.2622$       | $-0.2459$       | $-1.9025$       | $0.0075$        | $-0.0297$       |
| ($\times 10^{10}$) | $d = d_2 - 100$   | $N/A$           | $0.3471$        | $0.3584$        | $-1.3874$       | $0.0297$        | $-0.1179$       |
|                 | $d = \frac{d_0 + d_2}{2}$ | $9.0697$        | $1.8675$        | $1.8690$        | $-0.1602$       | $0.1934$        | $-0.7671$       |
|                 | $d = \frac{d_1 + d_2}{2}$ | $51.540$        | $3.1176$        | $3.1176$        | $0.6238$        | $N/A$           | $-2.9664$       |
| $N(1, 2^2)$    | $d = \frac{d_0 + d_2}{2}$ | $N/A$           | $-0.1294$       | $-0.1215$       | $-1.1009$       | $0.0050$        | $-0.0131$       |
| ($\times 10^{11}$) | $d = d_2 - 100$   | $N/A$           | $0.1890$        | $0.1948$        | $-0.8222$       | $0.0198$        | $-0.0522$       |
|                 | $d = \frac{d_0 + d_2}{2}$ | $4.9701$        | $1.0826$        | $1.0836$        | $-0.0647$       | $0.1456$        | $-0.3834$       |
|                 | $d = \frac{d_1 + d_2}{2}$ | $30.540$        | $1.9036$        | $1.9036$        | $0.5233$        | $N/A$           | $-1.6600$       |
| $LN(1, 1)$     | $d = \frac{d_0 + d_2}{2}$ | $N/A$           | $-1.4785$       | $-1.3810$       | $-7.5183$       | $0.0223$        | $-0.2086$       |
| ($\times 10^{11}$) | $d = d_2 - 100$   | $N/A$           | $1.7070$        | $1.7662$        | $-5.0451$       | $0.0890$        | $-0.8339$       |
|                 | $d = \frac{d_0 + d_2}{2}$ | $37.143$        | $7.5727$        | $7.5794$        | $-0.8606$       | $0.4226$        | $-3.9611$       |
|                 | $d = \frac{d_1 + d_2}{2}$ | $198.97$        | $12.037$        | $12.037$        | $1.0593$        | $N/A$           | $-13.110$       |
| $NB(1, 0.6)$   | $d = \frac{d_0 + d_2}{2}$ | $N/A$           | $-1.6279$       | $-1.5202$       | $-8.1096$       | $0.0236$        | $-0.2324$       |
| ($\times 10^{10}$) | $d = d_2 - 100$   | $N/A$           | $1.8624$        | $1.9273$        | $-5.4140$       | $0.0942$        | $-0.9276$       |
|                 | $d = \frac{d_0 + d_2}{2}$ | $40.041$        | $8.1885$        | $8.1958$        | $-0.9357$       | $0.4402$        | $-4.3335$       |
|                 | $d = \frac{d_1 + d_2}{2}$ | $214.47$        | $13.003$        | $13.003$        | $1.0824$        | $N/A$           | $-14.249$       |
5. Numerical examples and comparisons with unconstrained reinsurance model. In this section, we present two numerical examples, which are selected from previous two tables, to illustrate the results we obtained in previous sections. Then we compare the main results in our model with the one with unconstrained proportional reinsurance (i.e., when \( q \geq 0 \)).

5.1. Numerical calculations.

**Example 1.** Suppose that the claim sizes are exponentially distributed with parameter 1, then we have \( m_1 = 1 \) and \( m_2 = 2 \). Assume that \( \lambda = 10, \theta = 0.3, \eta = 0.2, \mu = 0.06, r = 0.04, \sigma = 1, T = 100 \) and \( X_0 = 50 \).

A simple calculation shows that \( d_1 < d_2 \). For a given expected return \( d \), we calculate \( V_\beta(0, X_0) \), then to determine the global maximum point and global maximum value of it, in this way we can obtain \( \text{Var}X^*_T \) thereof the efficient strategy and efficient frontier, the graphs of \( V_\beta(0, X_0) \) are presented in Figure 2(a) - Figure 2(d), corresponding to \( d \) belongs to four distinct categories (i.e., \( d_0 \leq d < d_1 \), \( d_1 \leq d < d_2 \), \( d_2 \leq d < \hat{d} \) and \( d \geq \hat{d} \)). Figure 2(e) exhibits the three-dimensional graph of \( V_\beta(0, X_0) \) if we consider it as a function of two variables \( d \) and \( \beta \).

From Figure 2 we notice that \( V_\beta(0, X_0) \) is a concave function of \( \beta \) in this specific setting. It monotonically increases to the maximum point, which is \( \beta^* \) when \( d_0 \leq d < d_1 \), is \( \beta^* \) when \( d_1 \leq d < d_2 \) and is \( \beta^* \) when \( d \geq d_2 \), henceforth start to shrink. When \( d \) is close to \( d_0 \), which is the company’s risk free return at time \( T \), the variance under efficient strategy is close to 0. The risk represented by variance gradually goes up as \( d \) increases, which manifests that the company should face more risks if it seeks after higher earnings.

**Example 2.** Suppose that the claim sizes follow i.i.d Erlang(3, 0.5) distributions, then we have \( m_1 = 6 \) and \( m_2 = 40 \). Assume that \( \lambda = 1, \theta = 0.25, \eta = 0.2, \mu = 0.12, r = 0.1, \sigma = 1, T = 100 \) and \( X_0 = 50 \). Then we have \( d_1 > d_2 \).

Figure 3(a) - Figure 3(d) present the graphs of \( V_\beta(0, X_0) \) with different values of pre-given expected terminal wealth \( d \). If we deem \( V_\beta(0, X_0) \) as a function of two variables, \( d \) and \( \beta \), the three-dimensional graph is provided by Figure 3(e).

In this example, for \( \beta \in (\beta_0, \beta_1) \), \( V_\beta(0, X_0) \) will firstly increase to the local maximum then monotonically decreases to its local minimum point and then start to bounce back. Therefore, comparing with the previous example, the curve of \( V_\beta(0, X_0) \) has at least one more inflection point. When \( d_0 \leq d < d_2 \), \( \beta^* \) does not exist. \( V_\beta(0, X_0) \) keeps going upwards for \( \beta \in (-\infty, \beta_0] \) and expanding until reaching the locally maximum point \( \beta^* \), then gradually shrinks to its local minimum and bounce back after that point. Once hitting the local maximum at \( \beta^* \), it will start to diminish. Thus the global maximum can only be \( V_{\hat{\beta}_1}(0, X_0) \) or \( V_{\hat{\beta}_2}(0, X_0) \), depending on \( d \leq d_0 \) or \( d > d_0 \) (corresponding to Figure 3(a) and Figure 3(b)). When \( d \geq d_2 \), we always have \( V_{\hat{\beta}}(0, X_0) > \max_{\beta \geq \beta_1} V_\beta(0, X_0) \). So the global maximum will be \( V_{\hat{\beta}}(0, X_0) \) or \( V_{\hat{\beta}_2}(0, X_0) \), depending on \( d_2 \leq d < \hat{d} \) or \( d > \hat{d} \). From Figure 3(a) - Figure 3(d) we notice that the variance and expected terminal wealth are positively correlated, which coincides with the conclusion in previous example.

5.2. Main results from unconstrained reinsurance model. When the proportional reinsurance in our model is not constrained, i.e., we assume that \( q \geq 0 \), the problem coincides with the diffusion model with \( \rho_S = 0 \) in Bai and Zhang [3]. To
(a) If $d_0 \leq d \leq d_1$, $V_{\beta}(0,X_0)$ is maximized at $\beta^*$

(b) If $d_1 \leq d \leq d_2$, $V_{\beta}(0,X_0)$ is maximized at $\hat{\beta}^* \in (\beta_0, \beta_1)$

(c) If $d_2 \leq d \leq \overline{d}$, $V_{\beta}(0,X_0)$ is maximized at $\beta^*$

(d) If $d \geq \overline{d}$, $V_{\beta}(0,X_0)$ is maximized at $\beta^*$

(e) $V_{\beta}(0,X_0)$ as a function of $d$ and $\beta$

Figure 2. The value of $V_{\beta}(0,X_0)$ in Example 1
(a) If $d_0 \leq d \leq \tilde{d}$, $V_{\beta}(0,X_0)$ is maximized at $\beta^*$

(b) If $\tilde{d} \leq d \leq \hat{d}$, $V_{\beta}(0,X_0)$ is maximized at $\hat{\beta}^* \in (\beta_0, \beta_1)$

(c) If $\hat{d} \leq d \leq d_1$, $V_{\beta}(0,X_0)$ is maximized at $\beta^*$

(d) If $d \geq d_1$, $V_{\beta}(0,X_0)$ is maximized at $\beta^*$

(e) $V_{\beta}(0,X_0)$ as a function of $d$ and $\beta$

**Figure 3.** The value of $V_{\beta}(0,X_0)$ in Example 2
demonstrate the impact of the constraint $q \leq 1$, we first list some results obtained in Bai and Zhang [3], which we formalize in following propositions. The procedure of proof is omitted here.

**Proposition 1.** If reinsurance proportion $q(t)$ belongs to $[0, +\infty)$, the value function and the optimal feedback strategy are as follows:

$$v(t, x) = \begin{cases} e^{-2r(t-T)}[x - g_1(t)e^{r(t-T)}]^2, & \text{if } (t, x) \in A_1, \\ e^{(A(1)-2r)(t-T)}[x - g_1(t)e^{r(t-T)}]^2, & \text{otherwise}. \end{cases}$$

$$(\pi^*_t(x), q^*_t(x)) = \begin{cases} (0, 0), & \text{if } (t, x) \in A_1, \\ (-\frac{\mu - r}{\sigma^2}[x - g_1(t)e^{r(t-T)}], -\frac{m_1}{m^2}[x - g_1(t)e^{r(t-T)}]), & \text{otherwise}. \end{cases}$$

**Proposition 2.** When $q(t) \in [0, +\infty)$, for any expected terminal wealth $d \geq X_0e^{rT} + \frac{\lambda m_1(\theta - \eta)}{r}(1 - e^{rT})$, corresponding to the expected terminal wealth $EX_t^* = d$, the efficient frontier is given by (4.24) and the efficient strategy is:

$$(\pi^*_t(x), q^*_t(x)) = \begin{cases} (0, 0), & \text{if } \pi_1^* \geq 0, \\ \left(-\frac{\mu - r}{\sigma^2}\pi_1^*, -\frac{m_1}{m^2}\pi_1^*\right), & \text{if } \pi_1^* < 0, \end{cases}$$

where

$$\pi_1^* = X_t^* - g_1(t)e^{r(t-T)},$$

$$g_1(t) = d - \bar{\beta}^* - \frac{\lambda m_1(\theta - \eta)}{r}(1 - e^{r(t-T)}),$$

$$\bar{\beta}^* = \frac{d - X_0e^{rT} - \frac{\lambda m_1(\theta - \eta)}{r}(1 - e^{rT})}{1 - e^{A(1)}T}.$$ 

**Remark 5.** The expressions of value function and optimal strategy in the constrained reinsurance model look similar to, but more complicated than the results in these two propositions. However, the reinsurance proportion constraint $q \leq 1$ can have significant impact on the efficient frontier, which will be shown in Figure 4.

![Figure 4](image-url)
5.3. Impacts of reinsurance constraint. Figure 4(a) presents the efficient frontiers in models with constrained and unconstrained reinsurance in Example 1. From this figure we can find that when proportional reinsurance is unconstrained, the frontier (the red line) is linear for \( d \geq d_0 \). In fact, the standard deviation is almost 0 even when the expected terminal wealth is very large \( 7 \times 10^{-7} \) when \( d = \hat{d} \). The corresponding reinsurance strategy is \( q^* = 12.04 \), which means the insurance company should act as a reinsurer and soar a twelvefold business. Hardly can this be true and the optimal variance is not realistic.

If strict proportional reinsurance is considered, i.e., \( q \in [0, 1] \), the efficient frontier (the blue curve) is linear when \( d \in [d_0, d_1] \). When \( d \in (d_1, d_2) \) or \( d \in [d_3, +\infty) \), the efficient frontier is part of a hyperbola while the two hyperbolas are different. As for the efficient strategies, when \( d \) increases from \( d_0 \) to \( d_1 \), the retained proportion of claim risks is moving up from 0 to 1 and the amount invested in risky asset is gradually rising up to \( m_2(\mu - r)/(m_1 \theta \sigma^2) \). In this spectrum, both insurance and investment will lead to more profits to the company, as well as more risks. If \( d \) continues to increase, the company should always keep all the claim risks in order to meet its expectation for high yields. The amount invested in risky asset will remain stationary at \( m_2(\mu - r)/(m_1 \theta \sigma^2) \) until the company decides to pursue a wealth greater than \( d_2 \) at terminal time \( T \). Thus \([d_1, d_2]\) can be deemed as an inaction region, during which the company do not need to adjust its investment and risk management strategies. In this region, the additional claim risks and profits are weighted equally to the company. If the company prefers a even higher profit, it should restructure its investment strategy to place even more money into risky asset and thus put itself into a more vulnerable situation.

The graph of efficient frontiers for Example 2 is given by Figure 4(b). The red line represents the frontier of unconstrained reinsurance model. Though the minimum variances corresponding to expected return \( d \) are not as good as those in Example 1, they are still much smaller than those in constrained reinsurance case. The frontier for constrained reinsurance model (the blue curve) is linear when \( d \in [d_0, \hat{d}] \) and is hyperbola when \( d \in (\hat{d}, d) \) or \( d \in [d, +\infty) \). As for the efficient strategies, as \( d \) increases from \( d_0 \), the company’s retained proportion of risk gradually goes up from 0 to 1 and then always keeps all the risk. The amounts invested in risky asset will keep increasing when \( d < \hat{d} \), then stay at \( m_2(\mu - r)/(m_1 \theta \sigma^2) \) in the inaction region \((\hat{d}, d)\). To hunt for a return higher than \( \hat{d} \), company should put additional money in risky asset, which leads to even greater risks.

6. Concluding remarks. This paper derives the viscosity solution of HJB equation for mean-variance problem under proportional reinsurance and no-shorting investment. Explicit expressions for efficient strategies and efficient frontiers are derived when \( d \geq \bar{d} \). For \( d_0 \leq d < \bar{d} \), considering that the solution of HJB equation is not quadratic and the expression of \( \nu_{\beta}(0, X_0) \) involves too many parameters thus is overly complicated, it is hard to compare the size of piecewise maximum thereof to obtain the global maximum depending on their explicit expressions. In this paper, by the assistance of mathematical software, we apply numerical method to obtain the efficient frontiers. However, rigorous proofs are left as an open problem.

In reality, to reduce the risk and improve the profit, insurance company will invest its wealth into multiple risky assets, which lead to another open problem. Suppose the price processes of risky assets are all driven by geometric Brownian motions, then we can obtain viscosity solutions of the corresponding HJB equation.
by the same procedure applied in this paper. However, the efficient strategies and efficient frontiers are difficult to derive. This is because when determining the optimal strategies, we could only derive the implicit expressions of optimal investment strategies. Without explicit expressions, hardly can we obtain the maximum value of $V_0(0, X_0)$ and efficient frontiers.

**Acknowledgments.** The authors gratefully acknowledge valuable comments and suggestions of the two anonymous referees that helped improve the paper and clarify the presentation.

**REFERENCES**

[1] S. Asmussen, B. Højgaard and M. Taksar, Optimal risk control and dividend distribution policies: Example of excess-of-loss reinsurance for an insurance corporation. *Finance Stochastics*, 4 (2000), 299–324.

[2] L. H. Bai and J. Y. Guo, Optimal proportional reinsurance and investment with multiple risky assets and no-shorting constraint. *Insurance: Mathematics and Economics*, 42 (2008), 968–975.

[3] L. H. Bai and H. Y. Zhang, Dynamic mean-variance problem with constrained risk control for the insurers. *Mathematical Methods of Operations Research*, 68 (2008), 181–205.

[4] S. Browne, Optimal investment policies for a firm with a random risk process: Exponential utility and minimizing the probability of ruin. *Mathematics of Operations Research*, 20 (1995), 937–958.

[5] P. Chen, H. L. Yang and G. Yin, Markowitz’s mean-variance asset-liability management with regime switching: A continuous-time model. *Insurance: Mathematics and Economics*, 43 (2008), 456–465.

[6] T. Choulli, M. Taksar and X. Y. Zhou, Excess-of-loss reinsurance for a company with debt liability and constraints on risk reduction. *Quantitative Finance*, 1 (2001), 573–596.

[7] T. Choulli, M. Taksar and X. Y. Zhou, A diffusion model for optimal dividend distribution for a company with constraints on risk control. *SIAM Journal on Control and Optimization*, 41 (2003), 1946–1979.

[8] M. G. Crandell and P. Lions, Viscosity solution of Hamilton-Jacobi equations. *Transactions of the American Mathematical Society*, 277 (1983), 1–42.

[9] C. Hipp and M. Plum, Optimal investment for insurers. *Insurance: Mathematics and Economics*, 27 (2000), 215–228.

[10] B. Højgaard and M. Taksar, Optimal proportional reinsurance policies for diffusion models. *Scandinavian Actuarial Journal*, 1998 (1998), 166–180.

[11] B. Højgaard and M. Taksar, Optimal proportional reinsurance policies for diffusion models with transaction costs. *Insurance: Mathematics and Economics*, 22 (1998), 41–51.

[12] B. Højgaard and M. Taksar, Optimal dynamic portfolio selection for a corporation with controllable risk and dividend distribution policy. *Quantitative Finance*, 4 (2004), 315–327.

[13] C. Irgens and J. Paulsen, Optimal control of risk exposure, reinsurance and investments for insurance portfolios. *Insurance: Mathematics and Economics*, 35 (2004), 21–51.

[14] X. Li, X. Y. Zhou and A. E. B. Lim, Dynamic mean-variance portfolio selection with no-shorting constraints. *SIAM Journal on Control and Optimization*, 41 (2002), 1540–1555.

[15] P. Lions, Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. II. Viscosity solutions and uniqueness. *Communications in Partial Differential Equations*, 8 (1983), 1229–1276.

[16] H. Markowitz, Portfolio selection. *The Journal of Finance*, 7 (1952), 77–91.

[17] H. Schmidli, Optimal proportional reinsurance policies in a dynamic setting. *Scandinavian Actuarial Journal*, 2001 (2001), 55–68.

[18] H. Schmidli, On minimising the ruin probability by investment and reinsurance. *The Annals of Applied Probability*, 12 (2002), 890–907.

[19] S. Z. Shi, *Convex Analysis*, Shanghai Science and Technology Press, 1990.

[20] H. M. Soner, Optimal control with state-space constrain I. *SIAM Journal on Control and Optimization*, 24 (1986), 552–561.

[21] H. M. Soner, Optimal control with state-space constrain II. *SIAM Journal on Control and Optimization*, 24 (1986), 1110–1122.
[22] M. Taksar, Optimal risk and dividend distribution control models for an insurance company, Mathematical Methods of Operations Research, 51 (2000), 1–42.
[23] Z. W. Wang, J. M. Xia and L. H. Zhang, Optimal investment for an insurer: The martingale approach, Insurance: Mathematics and Economics, 40 (2007), 322–334.
[24] H. L. Yang and L. H. Zhang, Optimal investment for insurer with jump-diffusion risk process, Insurance: Mathematics and Economics, 37 (2005), 615–634.
[25] J. M. Yong and X. Y. Zhou, Stochastic Controls: Hamiltonian Systems and HJB Equations, Springer-Verlag, New York, 1999.
[26] X. Zhang and T. K. Siu, Optimal investment and reinsurance of an insurer with model uncertainty, Insurance: Mathematics and Economics, 45 (2009), 81–88.
[27] X. Zhang, M. Zhou and J. Y. Guo, Optimal combinational quota-share and excess-of-loss reinsurance policies in a dynamic setting, Applied Stochastic Models in Business and Industry, 23 (2007), 63–71.
[28] X. Y. Zhou and D. Li, Continuous-time mean-variance portfolio selection: A stochastic LQ framework, Applied Mathematics and Optimization, 42 (2000), 19–33.

Received January 2015; 1st revision October 2015; 2nd revision December 2015.
E-mail address: nzhan@student.unimelb.edu.au
E-mail address: pche@unimelb.edu.au
E-mail address: zhuo.jin@unimelb.edu.au
E-mail address: shli@unimelb.edu.au