Universal Lower-Bounds on Classification Error under Adversarial Attacks and Random Corruption

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Abstract

We theoretically analyse the limits of robustness to test-time adversarial and noisy examples in classification. Our work focuses on deriving bounds which uniformly apply to all classifiers (i.e. all measurable functions from features to labels) for a given problem. Our contributions are three-fold. (1) In the classical framework of adversarial attacks, we use optimal transport theory to derive variational formulae for the Bayes-optimal error a classifier can make on a given classification problem, subject to adversarial attacks. The optimal adversarial attack is then an optimal transport plan for a certain binary cost-function induced by the specific attack model, and can be computed via a simple algorithm based on maximal matching on bipartite graphs. (2) We derive explicit lower-bounds on the Bayes-optimal error in the case of the popular distance-based attacks. These bounds are universal in the sense that they depend on the geometry of the class-conditional distributions of the data, but not on a particular classifier. Our results are in sharp contrast with the existing literature, wherein adversarial vulnerability of classifiers is derived as a consequence of nonzero ordinary test error. (3) For our third contribution, we study robustness to random noise corruption, wherein the attacker (or nature) is allowed to inject random noise into examples at test time. We establish nonlinear data-processing inequalities induced by such corruptions, and use them to obtain lower-bounds on the Bayes-optimal error for noisy problem.

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1 Introduction

1.1 Context

Despite their popularization, machine-learning powered systems (assisted-driving, natural language processing, facial recognition, etc.) are not likely to be deployed for critical tasks which require a stringent error margin, in a closed-loop regime any time soon. One of the main blockers which has been identified by practitioners and ML researchers alike is the phenomenon of adversarial examples [Szegedy et al., 2013]. There is now an arms race [Athalye et al., 2018] between adversarial attack developers and defenders, and there is some speculation that adversarial examples in machine-learning might simply be inevitable.

In a nutshell an (evasion) attack operates as follows. A classifier is trained and deployed (e.g the road traffic sign recognition sub-system on an AI-assisted car). At test / inference time, an attacker (aka adversary) may submit queries to the classifier by sampling a data point $x$ with true label $y$, and modifying it $x \rightarrow x_{adv}$ according to a prescribed threat model. For example, modifying a few pixels on a road traffic sign [Su et al., 2017], modifying intensity of pixels by a limited amount determined by a prescribed tolerance level variance per pixel, etc. The goal of the attacker is to fool the classifier into classifying $x_{adv}$ with a label different from $y$. A robust classifier tries to limit this failure mode, for a prescribed attack model.

In this manuscript, we establish universal lower-bounds on the test error any classifier can attain under test-time adversarial attacks and corruption by random noise, on a given classification problem.

1.2 Overview of related works

Questions around adversarial examples and fundamental limits of defense mechanisms, are an active area of research in machine-learning, with a large body of scientific literature. Let us overview a representative sample of this literature which is relevant to our own contributions.

Classifier-dependent bounds. There is a rich array of works which study adversarial examples as a consequence of nonzero ordinary / non-adversarial test error [Szegedy et al., 2013, Tsipras et al., 2018, Schmidt et al., 2018, Shafahi et al., 2018, Gilmer et al., 2018, Dohmatob, 2019, Fawzi et al., 2018, Mahloujifar et al., 2018]. These all use a form of the Gaussian isoperimetric inequality [Boucheron et al., 2013]: in these theories, adversarial examples exist as a consequence of ordinary test-error in high-dimensional problems with concentrated class-conditional distributions. On such problems, for a classifier which does not attain 100% on clean test examples (which is likely to be the case in practice), every test example will be close to a misclassified example, i.e can be misclassified by adding a small perturbation. Still using Gaussian isoperimetry, [Gilmer et al., 2019] has studied the relationship between robustness to adversaries and robustness to random noise. The authors argued that adversarial examples are a natural consequence of errors made by a classifier on noisy images. In section 5 we will extend this line of thought and obtain universal bounds in this scenario.

One should also mention some works which exploit curvature of the decision boundary of neural networks to exhibit the existence of vectors in low-dimensional subspaces, which when added to every example in a target class, can fool a classifier on a fraction of the samples [Moosavi-Dezfooli et al., 2017, Moosavi-Dezfooli et al., 2017].

Universal / classifier-independent bounds. To our knowledge, [Bhagoji et al., 2019] is the first work to derive universal / classifier-independent lower-bounds for adversarial robustness. The authors considered general adversarial attacks (i.e beyond distance-based models of attack), and show that Bayes-optimal error for the resulting classification problem under such adversaries is linked to a certain transport distance between the class-conditional distributions (see our Theorem 3.1 for a
generalization of the result). This result is singularly different from the previous literature as it applies even to classifiers which have zero test-error in the normal / non-adversarial sense. Thus, there adversarial examples that exist solely as a consequence of the geometry of the problem. The results in section 3 of our paper are strict extension of the bounds in [Bhagoji et al., 2019]. The main idea in [Bhagoji et al., 2019] is to construct an optimal-transport metric (w.r.t a certain binary cost-function induced by the attack), and then use Kantorovich-Rubenstein duality to relate this metric to the infimal adversarial error a classifier can attain under the adversarial attack. After the first version of our manuscript was submitted to ArXiv, Muni Pydi and Varun Jog told us (in private communication) that they had recently done work [Pydi and Jog, 2019] that further pursued the optimal-transport characterization proposed in [Bhagoji et al., 2019]. Some of their findings are special cases of our independently obtained general results in section 3.

Finally, one should mention [Cranko et al., 2019] which studies vulnerability of hypothesis classes in connection to loss functions used.

1.3 Summary of our main contributions

Our main contributions can be summarized as follows.

• Adversarial attacks. In section 3 after developing some background material in section 2, we use optimal transport theory to derive variational formulae for the Bayes-optimal error (aka smallest possible test error) of a classifier under adversarial attack, as a function of the "budget" of the attacker. These formulae suggest that instead of doing adversarial training, practitioners should rather do normal training on adversarially augmented data. Incidentally, this is a well-known trick to boost up the adversarial robustness of classifiers to known attacks, and is usually used in practice under the umbrella name of "adversarial data-augmentation". See [Yang et al., 2019], for example. In our manuscript, this principle appears as a natural consequence of our variational formulae. In section 3.5 we also provide a realistic algorithm for computing the optimal universal attack plan via maximal matching of bipartite graphs, inspired by [Harel and Mannor, 2015]. For the special case of distance-based attacks, we proceed in 4 to (1) Establish universal lower-bounds on the adversarial Bayes-optimal error. These bounds are a consequence of concentration properties of light-tailed class-conditional distributions of the features (e.g sub-Gaussianity, etc.). (2) Establish universal bounds under more general moment constraints conditions on the class-conditional distributions (e.g existence of covariance matrices for the class-conditional distributions of the features).

• Random corruption. Section 5 considers robustness to the corruption by random noise, in which the adversary (or nature!) corrupts each example at test-time by adding an instance of noise drawn from the same underlying distribution (e.g Gaussian noise). This is a new paradigm for studying the robustness of classifiers, and captures noise which might be inherent in measurement instruments (e.g imperfect sensors) or natural noise which might occur in the wild (e.g dirty on road traffic signs), say. For our analysis, we borrow and extend certain tools from information theory (more precisely, strong nonlinear data-processing inequalities), to give exact bounds on the Bayes-optimal error in such noisy environments. In the case of classification problems in $\mathbb{R}^m$ we derive explicit bounds which exploit the covariance structure of the features.

2 Preliminaries

2.1 General notations and classification framework

All through this manuscript, the feature space will be denoted $X$. The label (aka classification target) is a random variable $Y$ with values in $\mathcal{Y} = \{1, 2\}$, and random variable $X$ called the features, with values in $X$. We only consider binary classification problems in this work. The goal is to predict $Y$ given $X$. This corresponds to prescribing a measurable function $h : X \to \{1, 2\}$, called a classifier. The joint distribution $P_{X,Y}$ of $(X, Y)$ is unkown. The goal of learning is to find a classifier $h$ (e.g a deep neural net) such that $h(X) = Y$ as often as possible, possibly under additional constraints.

In this work, as in [Bhagoji et al., 2019], we will only consider balanced binary classification problems, where the labels are equiprobable, i.e $P(Y = 1) = P(Y = 2) = 1/2$. Multiclass problems can be considered in one-versus-all fashion. For each label $k \in \{1, 2\}$, we define the
(unnormalized) probability measure \( P^k \) on the feature space \( \mathcal{X} \) by
\[
P^k(A) := P(X \in A, Y = k) = \frac{1}{2}P(X \in A|Y = k), \forall \text{measurable } A \subseteq X. \tag{1}
\]
Thus, \( P^k \) is an unnormalized probability distribution on the feature space \( \mathcal{X} \) which integrates to \( 1/2 \), and the classification problem is therefore entirely captured by the pair \( P = (P^1, P^2) \), also called a binary experiment [Reid and Williamson, 2011].

The notions of metric and pseudo-metric spaces will come of often in the manuscript.

Definition 2.1 (Metric and pseudo-metric spaces). A mapping \( d : \mathcal{X}^2 \to \mathcal{X} \) is called a pseudo-metric on \( \mathcal{X} \) iff for all \( x, x', z \in \mathcal{X} \), the following hold:
- Reflexivity: \( d(x, x) = 0 \).
- Symmetry: \( d(x, x') = d(x', x) \).
- Triangle inequality: \( d(x, x') \leq d(x, z) + d(z, x') \).

The pair \((\mathcal{X}, d)\) is then called a pseudo-metric space. If in addition, \( d(x, x') = 0 \implies x = x' \), then we say \( d \) is a metric (or distance) on \( \mathcal{X} \), and the pair \((\mathcal{X}, d)\) is called a metric space.

An example of a metric space is \( \mathbb{R}^m \) equipped with the euclidean distance. An example of a pseudo-metric space is the manahobis "distance" \( d_{\Sigma}(x, x') := \|x - x'\|_{\Sigma} := \sqrt{(x - x')^T \Sigma (x - x')} \) on \( \mathbb{R}^m \) induced by a positive semi-definite matrix \( \Sigma \) with pseudo-inverse \( \Sigma^\dagger \). \( d_{\Sigma} \) fails to be a metric if \( \Sigma \) is not strictly positive-definite, which happens if \( \Sigma \) is rank-deficient.

2.2 Models of adversarial attack

In full generality, an adversarial attack model on the feature space \( \mathcal{X} \) (a topological space) is any closed subset \( \Omega \subseteq \mathcal{X}^2 \). Given \( x', x \in \mathcal{X} \), we call \( x' \) an adversarial example of \( x \) if \( (x, x') \in \Omega \). The subset \( \text{diag}(\mathcal{X}^2) := \{(x, x) \mid x \in \mathcal{X} \} \) corresponds to classical / standard classification theory where there is no adversary. A nontrivial example is the case of so-called distance-based attacks, where \( d \) is a metric on \( \mathcal{X} \) and the attack model \( \Omega = D_{\varepsilon} \), where
\[
D_{\varepsilon} = \{(x, x') \in \mathcal{X}^2 \mid d(x, x') \leq \varepsilon \}, \tag{2}
\]
with \( \varepsilon \geq 0 \) being the budget of the attacker. These include the well-known \( \ell_p \)-norm attacks in finite-dimensional euclidean spaces usually studied in the literature (e.g. [Szegedy et al., 2013] [Tsipras et al., 2018] [Schmidt et al., 2018] [Shafahi et al., 2018] [Gilmer et al., 2018]).

In the case where \( \mathcal{X} \) is a normed vector-space, there is an subset \( L_{\varepsilon} \) of \( D_{\varepsilon} \) which corresponds to the attack model in so-called universal adversarial perturbations (UAPs) ([Moosavi-Dezfooli et al., 2017] [Moosavi-Dezfooli et al., 2017]), namely \( L_{\varepsilon} := \bigcup_{\|v\| \leq \varepsilon} \{(x, x + v) \mid x \in \mathcal{X} \} \), a union of lines in \( \mathcal{X}^2 \).

Another instance of our general formulation is when \( \Omega = \mathcal{A}^\times \), where \( \mathcal{A}^\times := \{(x, x') \in \mathcal{X}^2 \mid \mathcal{A}_x \cap \mathcal{A}_{x'} \neq \emptyset \} \), for a system \( \langle \mathcal{A}_x \rangle_{x \in \mathcal{X}} \) of subsets of \( \mathcal{X} \). This framework is already much more general than the distance-based framework (which is the default setting in the literature), and corresponds to the setting considered in [Bhagoji et al., 2019]. Working at this level of generality allows the possibility to study general attacks like pixel-erasure attacks [Su et al., 2017], for example, which cannot be metrically expressed.

A type-\( \Omega \) adversarial attacker on the feature space \( \mathcal{X} \) is then a measurable mapping \( a : \mathcal{X} \to \mathcal{X} \), such that \((x, a(x)) \in \Omega \) for all \( x \in \mathcal{X} \). For example, in the distance-based attacks, it corresponds to a measurable selection for every \( x \in \mathcal{X} \), of some \( x' = a(x) \in \mathcal{X} \) with \( d(x, x') \leq \varepsilon \).

For \( x \in \mathcal{X} \), denote \( \Omega(x) := \{x' \in \mathcal{X} \mid (x, x') \in \Omega\} \). Given a classifier \( h : \mathcal{X} \to \{1, 2\} \), and a label \( k \in \{1, 2\} \), define \( \Omega_{h,k} := \{x \in \mathcal{X} \mid h(x') \neq k \text{ for some } x' \in \Omega(x)\} \), \( \tag{3} \)

![Figure 1: Showing a generic distance-based attack model. The green region corresponds to the set \( D_{\varepsilon} \subseteq \mathcal{X}^2 \) defined in (2). An attacker is allowed to swap any point \( x \in \mathcal{X} \) with another point \( x' \in \mathcal{X} \), as long as the pair \((x, x')\) lies in the green region.](image-url)
the set of examples with a "neighbor" whose predicted label is
different from \( k \). Conditioned on the event \( Y = k \), the "size" of the set \( \Omega^{h,k} \) is the adversarial error /
risk of the classifier \( h \), on the class \( k \). This will be made precise in the passage.

**Adversarial Bayes-optimal error.** The adversarial error / risk of a classifier \( h \) under type-\( \Omega \)
adversarial attacks, is defined by

\[
R_\Omega(h; P^1, P^2) := P_{X,Y}[h(x') \neq Y \text{ for some } x' \in \Omega(X)] = \sum_{k=1}^{2} P^k(\Omega^{h,k}).
\]  

(4)

Thus, \( R_\Omega(h; P^1, P^2) \) is the least possible classification error suffered by \( h \) under type-\( \Omega \) attacks. The
adversarial Bayes-optimal error \( R^*_{\Omega}(P^1, P^2) \) for type-\( \Omega \) attacks is defined by

\[
R^*_{\Omega}(P^1, P^2) := \inf_h R_\Omega(h; P^1, P^2),
\]  

(5)

where the infimum is taken over all measurable functions \( h : \mathcal{X} \to \{1, 2\} \), i.e over all classifiers.

Econometrically, adversarial Bayes-optimal error \( R^*_{\Omega}(P^1, P^2) \) corresponds to the maximal payoff of
a type-\( \Omega \) adversarial attacker who tries to uniformly “blunt” all classifiers at the task of solving the
classification problem \( (P^1, P^2) \).

For the special case of distance-based attacks with budget \( \varepsilon \), where the attack model is \( \Omega = \Delta \varepsilon \)
(defined in Eq. [2]), we will simply write \( R^*_{\varepsilon}(P^1, P^2) \) in lieu of \( R_{\varepsilon}(P^1, P^2) \), that is

\[
R^*_{\varepsilon}(P^1, P^2) = \sum_{k=1}^{2} P^k(D^h_{\varepsilon,k}) = \sum_{k=1}^{2} P^k(\{x \in \mathcal{X} \mid \exists x' \in \mathcal{X}, \ d(x, x') \leq \varepsilon, \ h(x') \neq k\}).
\]  

(6)

3 Optimal transport characterization of adversarial vulnerability

3.1 Adversarial attacks as optimal transport plans

Consider the binary cost-function \( c_\Omega : \mathcal{X}^2 \to \{0, 1\} \), defined by

\[
c_\Omega(x, x') := \begin{cases} 
0, & \text{if } (x, x') \in \Omega, \\
1, & \text{else.}
\end{cases}
\]  

(7)

This cost-function is special in that, for every \( (x, x') \in \Omega \), one can transport \( x \) to \( x' \) without incurring
any cost at all. If \( x \) and \( x' \) happen to belong to different classes, then an adversarial attack which
replaces \( x \) with \( x' \) will be perfectly undetectable. As in [Bhagoji et al., 2019], we start with a
variational formula for measuring the cost of a type-\( \Omega \) for the task of “blunting” the Bayes-optimal
classifier for the classification problem \( (P^1, P^2) \).

**Definition 3.1** (Adversarial total-variation). Let \( OT_{\Omega}(P^1, P^2) \) be the optimal transport distance
between \( P^1 \) and \( P^2 \) w.r.t to the ground cost \( c_\Omega \) defined in Eq. (7), i.e

\[
OT_{\Omega}(P^1, P^2) := \inf_{\gamma \in \Pi(P^1, P^2)} \int_{\mathcal{X}^2} c_\Omega(x_1, x_2) \, d\gamma(x_1, x_2) = \inf_{\gamma \in \Pi(P^1, P^2)} \mathbb{E}_\gamma[c_\Omega(X_1, X_2)],
\]  

(8)

where \( \Pi(P^1, P^2) \) is the set of all couplings of \( P^1 \) and \( P^2 \), i.e the set of all measures on \( \mathcal{X}^2 \) with
marginals \( P^1 \) and \( P^2 \), and \( (X_1, X_2) \) is a pair of r.v.s on \( \mathcal{X} \) with joint distribution \( \gamma \).

If \( \gamma \) is a coupling of \( P^1 \) and \( P^2 \) and \( (X_1, X_2) \sim \gamma \), with abuse of language we shall also refer to
\((X_1, X_2)\) as a coupling of \( P^1 \) and \( P^2 \).

Note that, since \( \Omega \subseteq \mathcal{X}^2 \) is closed, \( c_\Omega \) is lower-semicontinuous (l.s.c) and is therefore an admissible
ground cost-function for transportation between distributions on \( \mathcal{X} \). Thus, \( OT_{\Omega}(\cdot, \cdot) \) defines a distance
over measures on the feature space \( \mathcal{X} \).

In the particular case of distance-based attacks, we have \( \Omega = \Delta \varepsilon \) as defined in Eq. [2], and formula
(8) can be equivalently written as

\[
OT_{\varepsilon}(P^1, P^2) := \inf_{(X_1, X_2)} \mathbb{P}(d(X_1, X_2) > \varepsilon),
\]  

(9)
where the infimum is taken over all couplings \((X_1, X_2)\) of \(P^1\) and \(P^2\). The joint distribution \(\gamma_\varepsilon\) of \((X_1, X_2)\) is then an optimal adversarial attack plan for the classification problem \((P^1, P^2)\). Note that the case \(\varepsilon = 0\) conveniently corresponds to the usual definition of total-variation, namely
\[
TV(P^1, P^2) := \sup_{A \subseteq \mathcal{X}} \mathbb{P}(\mathbb{P}(A) - P^2(A) = \inf_{(X_1, X_2)} \mathbb{P}(X_1 \neq X_2), \quad (10)
\]
The RHS of the above formula is usually referred to as Strassen’s formula for total-variation.

Coincidentally, the metric \(TV_\varepsilon\) in (9) has been studied in context of statistical testing, under the name “perturbed variation” [Harel and Mannor, 2015] as robust replace for usual total-variation. Moreover, the authors proposed an efficient algorithm for computing both the optimal plan \(\gamma_\varepsilon\) as maximal graph matching in a bipartite graph. In has also been studied in [Yang et al., 2019] in the context of adversarial attacks.

**Link to classical theory of classification.** It is well-known [Reid and Williamson, 2011] in standard classification theory that the Bayes-optimal error is exactly equal to
\[
R^*(P^1, P^2) := \frac{1}{2}(1 - TV(P^1, P^2)). \quad (11)
\]
Thus, one might expect that the adversarial total-variation metric \(TV_\Omega(\cdot\cdot)\) defined in Eq. (8) would play a role in control of the adversarial Bayes-optimal error \(R^*_\Omega(\cdot\cdot)\) (defined in Eq. (5)) which is similar to the role played by ordinary total-variation \(TV(\cdot\cdot)\) (defined in Eq. (10)) plays in formula (11) for the classical / standard Bayes-optimal error. This is indeed the case.

**Remark 3.1.** The adversarial Bayes-optimal error \(R^*_\Omega(P^1, P^2)\) under type-\(\Omega\) adversarial attacks should not be confused with the adversarial error of the standard / classical Bayes-optimal classifier \(h^* := \arg\min_h R(P^1, P^2)\) for the unattacked classification problem. In fact \(R^*_\Omega(P^1, P^2) \leq R^*_\Omega(h^*; P^1, P^2)\), and we can construct explicit scenarios in which the inequality is strict (e.g one-dimensional classification problem whose class-conditional distributions are gaussians with different means and same variance, under the distance-based attack model \(\Omega := \{(x, x') \in \mathbb{R} | |x - x'| \leq \varepsilon\}\).

**Proposition 3.1** (Extension of Theorem 1 of [Bhagoji et al., 2019]). For any attack model \(\Omega\) on the feature space \(\mathcal{X}\), the adversarial Bayes-optimal error under type-\(\Omega\) attacks
\[
R^*_\Omega(P^1, P^2) \geq \frac{1}{2}(1 - OT_\Omega(P^1, P^2)). \quad (12)
\]
Note that the reverse inequality does not hold in general. A remarkable exception is the case of distance-based attacks with a distance \(d\) that turns the feature space \(\mathcal{X}\) into a complete separable metric space with the midpoint property. Examples of such spaces include complete riemannian manifolds and any closed convex subset of a separable Banach space. We shall return to such spaces in section 3.3.

### 3.2 Characterizing the adversarial Bayes-optimal error via optimal transport

Henceforth, assume the feature space \(\mathcal{X}\) is Polish (i.e \(\mathcal{X}\) is metrizable, complete, and separable). Finally, given a subset \(U \subseteq \mathcal{X}\), define its \(\Omega\)-closure \(\overline{U}_\Omega\) by
\[
\overline{U}_\Omega := \{x \in \mathcal{X} \mid (x, x') \in \Omega \text{ for some } x' \in U\}. \quad (13)
\]
In the case of metric attacks where \(\Omega = D_\varepsilon := \{(x, x') \in \mathcal{X}^2 \mid d(x, x') \leq \varepsilon\}\), we have \(\overline{U}_\Omega = U^\varepsilon\), where \(U^\varepsilon\) is the \(\varepsilon\)-neighborhood of \(U\) defined by
\[
U^\varepsilon := \{x \in \mathcal{X} \mid d(x, x') \leq \varepsilon \text{ for some } x' \in U\}. \quad (14)
\]
The following theorem is a direct application of Strassen’s Marriage Theorem (see [Villami, 2003, Theorem 1.27]), and is as a first simplification of the complicated distance \(OT_\Omega\) that appears in Proposition 3.1. For distance-based attacks has been, a special case of our result has been independently obtained in [Pydi and Jog, 2019].

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1That is, for every \(x, x' \in \mathcal{X}\), there exists \(z \in \mathcal{X}\) such that \(d(x, z) = d(x', z) = d(x, x')/2\).

2This paper was kindly brought to our attention after the first version of our manuscript appeared on ArXiv.
We now turn to distance-based attacks and refine representation presented in the previous lemma.

We now present a lemma which allows us to rewrite the optimal transport distance in adversarial attacks, satisfies the measurable midpoint property. Indeed, generic examples of such spaces include normed vector-spaces and Riemannian manifolds. For our next result, it will be useful to have the measurable midpoint (MM) property if for every Borel measure \( \eta \), there is a point \( z \) in a measurable manner almost-everywhere.

\section{Adversarial robustness / vulnerability through the lens of adversarially augmented data}

We now turn to distance-based attacks and refine representation presented in the previous lemma. Recall that a metric space is said to have the midpoint property if for every pair of points \( z \) and \( z' \), there is a point \( \eta(z, z') \) in the space which sits exactly halfway between them. Examples of such spaces include normed vector-spaces and Riemannian manifolds. For our next result, it will be important to be able to select the midpoint \( \eta(z, z') \) in a measurable manner almost-everywhere.

\section{Condition 3.1 (Measurable Midpoint (MM) property). A metric space \( Z = (Z, d) \) is said to satify the measurable midpoint (MM) property if for every Borel measure \( Q \) on \( Z \) there exists a \( Q \)-measurable map \( \eta : Z \rightarrow \mathcal{Z} \) such that \( d(z, \eta(z, z')) = d(z', \eta(z, z')) = d(z, z')/2 \) for all \( z, z' \in Z \).

The feature space for most problems in machine learning together with the distances usually used in adversarial attacks, satisfies the measurable midpoint property. Indeed, generic examples of metric spaces which satisfy this condition include:

- Hilbert spaces.
- Closed convex subsets of Banach spaces.
- Complete Riemannian manifolds (equipped with the geodesic distance).
- Complete separable metric spaces with the midpoint property.

In fact, in the first two examples, the midpoint mapping \( \eta \) can be chosen to be continuous everywhere. The last example, which can be proved via the classical Kuratowski-Ryll-Nardzewski measurable selection theorem, is the most general and most remarkable, and deserves an explicit restatement.

\section{Lemma 3.2. Every complete separable metric space which has the midpoint property also has the measurable midpoint property.}

The following theorem, which is proved in the appendix (as are all the other theorems in this manuscript), is one of our main results.

\section{Theorem 3.2 (Adversarially augmented data, a proxy for adversarial robustness). Consider a classification problem \( (P^1, P^2) \). Suppose \( d \) is a distance on the feature space \( X \) with the MM property
and consider the distance-based attack model \( D_\varepsilon := \{(x, x') \in \mathcal{X}^2 \mid d(x, x') \leq \varepsilon \} \). Recall the definition of \( \text{TV}_\varepsilon(P^1, P^2) \) from Eq. (9). Define

\[
\tilde{\text{TV}}_\varepsilon(P^1, P^2) := \inf_{\gamma_1, \gamma_2} \text{TV}(\text{proj}^2_\gamma \# \gamma_1, \text{proj}^1_\gamma \# \gamma_2),
\]

\[
\tilde{\text{TV}}_\varepsilon(P^1, P^2) := \inf_{a_1, a_2 \text{ type-}D_{\varepsilon/2}} \text{TV}(a_1 \# P^2, a_2 \# P^1),
\]

where "#" denotes pushforward of measures and the \( 1 \text{st inf.} \) is taken over all pairs of distributions \( (\gamma_1, \gamma_2) \) on \( \mathcal{X}^2 \) concentrated on \( D_{\varepsilon/2} \) such that \( \text{proj}^1_\gamma \# \gamma_1 = P^2 \) and \( \text{proj}^2_\gamma \# \gamma_2 = P^1 \). It holds that

\[
\text{TV}_\varepsilon(P^1, P^2) = \tilde{\text{TV}}_\varepsilon(P^1, P^2) \leq \tilde{\text{TV}}_\varepsilon(P^1, P^2),
\]

and there is equality if \( P^1 \) and \( P^2 \) have densities w.r.t the Borel measure on \( \mathcal{X} \).

Consequently, we have the following lower-bound for the adversarial Bayes-optimal error:

\[
R^*_{\varepsilon}(P^1, P^2) \geq \frac{1}{2} (1 - \text{TV}_\varepsilon(P^1, P^2)) = \frac{1}{2} (1 - \tilde{\text{TV}}_\varepsilon(P^1, P^2)) \geq \frac{1}{2} (1 - \tilde{\text{TV}}_\varepsilon(P^1, P^2)).
\]

**Practical recommendation:** Standard machine learning on adversarially-augmented data.

The variational formulae \([19] \) in Thm. 3.2 suggest that instead of doing adversarial training, practitioners should rather do normal training on adversarially augmented data. Incidentally, this is a well-known trick to boost up the adversarial robustness of classifiers to known attacks, and is usually used in practice under the umbrella name of "adversarial data-augmentation". This rule of thumb falls out of Thm. 3.2 as a natural consequence.

### 3.4 Case study: (separable) Banach spaces

Thm. 3.2 has several important consequences, which will be heavily explored in the sequel. A particularly simple consequence is the Consider the special case where \( \mathcal{X} = (\mathcal{X}, \| \cdot \|) \), a separable Banach space. Given a point \( z \in \mathcal{X} \), let \( P^1 + z \) be the translation of \( P^1 \) by \( z \). For \( z, z' \in \text{Ball}_{\mathcal{X}}(0, \varepsilon/2) \), consider the type-\( D_{\varepsilon/2} \) distance-based attacks \( a_1^z, a_2^z : \mathcal{X} \to \mathcal{X} \) defined by \( a_1^z(x) = x - z \) and \( a_2^z(x) = x + z' \). One computes

\[
\text{TV}_\varepsilon(P^1, P^2) := \inf_{a_1, a_2 \text{ type-}D_{\varepsilon/2}} \text{TV}(a_1 \# P^1, a_2 \# P^2) \leq \inf_{\|z\| \leq \varepsilon/2} \|z\| \leq \varepsilon/2 \inf_{\|z\| \leq \varepsilon/2} \text{TV}(a_1^{z} \# P^1, a_2^{z'} \# P^1)
\]

\[
= \inf_{\|z\| \leq \varepsilon/2} \|z'\| \leq \varepsilon/2 \text{TV}(P^1 - z, P^2 + z') \leq \inf_{\|z\| \leq \varepsilon} \text{TV}(P^1, P^2 + z),
\]

where \( P^2 + z \) is the translation of distribution \( P^2 \) by the vector \( z \). Note that in the above upper bound, the LHS can be made very concrete in case the distributions are prototypical (e.g multivariate Gaussians with same covariance matrix; etc.). Thus we have the following result

**Corollary 3.1.** Let the feature space \( \mathcal{X} \) be a normed vector space and consider a distance-based attack model \( D_\varepsilon = \{(x, x') \in \mathcal{X}^2 \mid \|x' - x\| \leq \varepsilon \} \). Then, it holds that

\[
R^*_{\varepsilon}(P^1, P^2) \geq \frac{1}{2} \left(1 - \sup_{\|z\| \leq \varepsilon} \text{TV}(P^1, P^2 + z)\right).
\]

A solution in \( z^* \) to optimization problem in the RHS of \((20)\) would be a (doubly) universal adversarial perturbation: a single fixed small vector which fools all classifiers on proportion of test samples. Such a phenomenon has been reported in [Moosavi-Dezfooli et al., 2017].

As a concrete application of Corollary 3.1, consider the following problem inspired by [Tsipras et al., 2018]: the classification target is uniformly distributed on \( \{1, 2\} \) and the class-conditional distribution of the features are multi-variate Gaussians with the same covariance matrix for both classes. More formally, the joint distribution of \( X \) and \( Y \) is given by \( Y \sim U\{1, 2\}, \quad X|Y = k \sim \mathcal{N}(\mu_k, \Sigma) \) where \( \mu_1, \mu_2 \in \mathbb{R}^m \) and \( \Sigma \) is p.s.d matrix of size \( d \), with singular values \( \sigma_1 \geq \ldots \geq \sigma_d \geq 0 \). We have the following corollary.
We will exploit geometric properties of the class-conditional distributions $P^k = (1/2)\mathcal{N}(\mu_k, \Sigma)$. For $\ell_\infty$-norm attacks with budge $\varepsilon$, we have

$$R^\ast_k(P^1, P^2) \geq 1 - \Phi(\Delta(\varepsilon)/2) \geq 1 - \Phi(\|s(\varepsilon)\|_\infty/2),$$

where $\Delta(\varepsilon) := \sqrt{\Delta^2 + \Sigma - 1} \Delta \mu / \varepsilon$, $\Delta \mu := \mu_1 - \mu_2$, and $s(\varepsilon) \in \mathbb{R}^m$ is defined by $s_j(\varepsilon) := |\sigma_j|^{-1}(d_j|\mu_j - \varepsilon|)$, and $\sigma_1^2, \ldots, \sigma_m^2$ are the eigenvalues of $\Sigma$.

### 3.5 Computing the optimal attack plan

It turns out that the optimal transport plan $\gamma_\Omega$ which realizes the distance $\text{TV}_\Omega(P^1, P^2)$ in Thm. 3.2 can be efficiently computed via matching (graph theory), by using iid samples from both distributions. The recent work [14] has studied a metric on probability measures which coincidentally corresponds to the metric $\text{TV}_\varepsilon(\cdot, \cdot)$ we defined in Eq. (9). This metric even goes back to the authors of [5], who proposed it under the name of “perturbed variation”, for the purposes of robust statistical hypothesis testing.

The following proposition is an adaptation of [5], and the proof is similar and therefore omitted.

**Proposition 3.2** (Optimal universal attacks via maximal matching). Suppose $P^1 = \sum_{i=1}^{n_1} \mu_1^i \delta_{x_1^i}$ and $P^2 = \sum_{j=1}^{n_2} \mu_2^j \delta_{x_2^j}$ are distributions with finite supports $V^1 := \{x_1^1, \ldots, x_1^{n_1}\} \subseteq \mathcal{X}$, $V^2 := \{x_2^1, \ldots, x_2^{n_2}\} \subseteq \mathcal{X}$, with weights $(\mu_1^1, \ldots, \mu_1^{n_1}) \in \Delta_n$, $(\mu_2^1, \ldots, \mu_2^{n_2}) \in \Delta_n$. Let $G$ be the bipartite graph with vertices $V^1 \cup V^2$ and edges $(V^1 \times V^2) \cap \Omega$. Then we can compute $\text{OT}_\Omega(P^1, P^2)$ via the following linear program:

$$\text{OT}_\Omega(P^1, P^2) = \min_{w^1, w^2, \gamma} \sum_{i=1}^{n_1} w^1_i + \sum_{j=1}^{n_2} w^2_j$$

subject to:

$$w^1_i \in \mathbb{R}^n_+, \quad w^2_j \in \mathbb{R}^n_+ \quad \gamma_{i,j} = 0 \forall (i, j) \notin \Omega$$

$$\sum_{j \sim x_1^i} \gamma_{i,j} + w^1_i = \mu_1^i \forall i \in \{n_1\}, \quad \sum_{i \sim x_2^j} \gamma_{i,j} + w^2_j = \mu_2^j \forall j \in \{n_2\}. \quad (22)$$

Moreover, the optimal transport plan $\gamma_\Omega$ to the above LP, and the can be computed in $O(k \min(n_1, n_2)/\sqrt{\max(n_1, n_2)}$ time, where $k$ is the average number of pairs edges in the graph $G$.

A simple algorithm for computing optimal matching $\gamma$ is given in Alg. 1 below. This algorithm is an adaptation of the algorithm in [5], for computing their “perturbed variation”, a robust version of total-variation which corresponds to our $\text{TV}_\varepsilon(P^1, P^2)$ defined in (9). Also see [14] and [15] for related work.

**Algorithm 1** Empirical approx. of $\text{TV}_\Omega(P^1, P^2)$ and optimal attack plan $\gamma_\Omega$, for an attack model $\Omega$

Input: $V^k = \{x_1^k, \ldots, x_{n_k}^k\}$, where $x_1^k, \ldots, x_{n_k}^k \sim P^k$ is a sample of size $n_k$ for $k \in \{1, 2\}$.

Construct $G$ a bipartite graph $G$ with vertex set $V^1 \cup V^2$ and edges $(V^1 \times V^2) \cap \Omega$.

Compute maximal matching on $\gamma_\Omega$ on $G$.

Return $\gamma_\Omega$ and $1/2 \left( \frac{n_1}{n_1} + \frac{n_2}{n_2} \right)$, where $u_k$ is the number of unmatched vertices in $V^k$.

**Remark 3.2** (No Free lunch for the attacker). Unfortunately for the attacker, convergence of the above algorithm (or any other algorithm) for computing $\text{TV}_\Omega(P^1, P^2)$ from samples will typically suffer from the curse of dimensionality. For example, in the case of distance-based attacks on $\mathbb{R}^m$, this remark is a direct consequence of [5] Theorems 3 and 4 where it is shown that the sample complexity is exponential in the dimensionality $m$.

### 4 Universal bounds for general distance-based attacks

We now turn to the special case distance-based attacks on a metricized feature space $\mathcal{X} = (\mathcal{X}, d)$. We will exploit geometric properties of the class-conditional distributions $P^k$ to obtain upper-bounds on $\text{TV}_\varepsilon(P^1, P^2)$, which will in turn imply lower lower bounds on optimal error (thanks to Thm. 3.2).
4.1 Bounds for light-tailed class-conditional distributions

We now establish a series of upper-bounds on $\text{TV}_\epsilon(P^1, P^2)$, which in turn provide hard lower bounds for the adversarial robustness error on any classifier for the binary classification experiment $(P^1, P^2)$, namely $R_\epsilon^*(P^1, P^2)$. These bounds are a consequence of light-tailed class conditional distributions.

**Name of the game.** We always have the upper-bound $R_\epsilon^*(P^1, P^2) \leq 1/2$ (attained by random guessing). Thus, the real challenge is to show that $R_\epsilon^*(P^1, P^2) = 1/2 + o_\epsilon(1)$, where $o_\epsilon(1)$ goes to zero as the attack “budget” $\epsilon$ is increased.

**Definition 4.1 (Bounded tails).** Let $\alpha : [0, \infty) \to [0, 1]$ be a function. We say the a distribution $Q$ on $(\mathcal{X}, d)$ has $\alpha$-light tail about the point $x_0 \in \mathcal{X}$ if $\mathbb{P}_{x \sim Q}(d(x, x_0) > t) \leq \alpha(t)$ $\forall t \geq 0$.

**Theorem 4.1 (The curse of light-tailed class-conditional distributions).** Suppose $P^1$ and $P^2$ have $\alpha$-light tails about a points $\mu_1 \in \mathcal{X}$ and $\mu_2 \in \mathcal{X}$ resp. Then for every $\epsilon \geq d(\mu_1, \mu_2)$, we have

$$R_\epsilon^*(P^1, P^2) \geq 1/2 - \alpha(\epsilon - d(\mu_1, \mu_2)/2).$$

**Proof of Theorem 4.1** Define $\bar{\epsilon} := (\epsilon - d(m_1, m_2))/2$ and let $(X_1, X_2)$ be a any coupling of $P^1$ and $P^2$. By definition of $\text{TV}_\epsilon(P^1, P^2)$, we have

$$\text{TV}_\epsilon(P^1, P^2) \leq \mathbb{P}(d(X_1, X_2) > \epsilon) \leq \mathbb{P}(d(X_1, m_1) + d(X_2, m_2) > \epsilon - d(m_1, m_2))$$

$$\leq \mathbb{P}(d(X_1, m_1) > \bar{\epsilon}) + \mathbb{P}(d(X_2, m_2) > \bar{\epsilon})$$

$$\leq \alpha(\bar{\epsilon}) + \alpha(\bar{\epsilon}) = 2\alpha(\bar{\epsilon}),$$

where the 1st inequality is the triangle inequality and the 2nd is a union bound. \square

As an example, take $\mathcal{X} = (\mathbb{R}^m, \|\cdot\|_\infty)$ and $P^k = \mathcal{N}(\mu_k, \sigma^2 I_m)$. Then for every $t \geq 0$, one computes

$$\mathbb{P}(d(x, \mu_1) > t) = \mathbb{P}(\|N(0, \sigma^2 I_m)\|_\infty > t) \leq m\mathbb{P}(\|N(0, \sigma^2)\| > t) \leq 2me^{-t^2/(2\sigma^2)}.$$ 

Thus, we can take $\alpha(t) = 2me^{-\frac{t^2}{2\sigma^2}} \forall t \geq 0$ and obtain that the test error of any classifier under $\ell_\infty$-norm adversarial attacks of size $\epsilon$ is at least $R_\epsilon^*(P^1, P^2) \geq (1 - \text{TV}_\epsilon(P^1, P^2))/2 \geq 1/2 - m\epsilon(|\mu_1 - \mu_2|/\sigma^2)^2/(4\sigma^2) = 1/2 + o(1)$, which increases to $1/2$, i.e the performance of random guessing, exponentially fast as $\epsilon$ is increased. This is just another manifestation of the concentration of measure in high-dimensions (large $m$), for distributions which are sufficiently “curved”.

4.2 Bounds under general moment and tail constraints on the class-conditional distributions

The following condition will be central for the rest of the manuscript.

**Condition 4.1 (Moment constraints).** There exists $\alpha > 0$ and $M : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing convex function (fixed, once and for all) such that $M(0) = 0$. We will occasionally assume that the classification problem satisfies the moment condition

$$\exists x_0 \in \mathcal{X}, \mathbb{E}_{x_1 \sim P^1}[M(d(x_1, x_0))] + \mathbb{E}_{x_2 \sim P^2}[M(d(x_2, x_0))] \leq 2\alpha.$$ 

(24)

For example, if each $P^k$ is $\sigma$-subGaussian about $\mu_k \in \mathbb{R}^m$, then we may take $M(r) := e^{r^2/\sigma^2} - 1$, $x_0 = (\mu_1 + \mu_2)/2$ to satisfy the condition. More generally, recall that the Orlicz $M$-norm of a random variable $X_k \sim P^k$ (relative to the reference point $x_0$) is defined by

$$\|X_k\|_M := \inf\{C > 0 \mid \mathbb{E}[M(d(X_k, x_0)/C)] \leq 1\}.$$ 

(25)

Thus, Condition 4.1 is more general than demanding that both $P^1$ and $P^2$ have Orlicz $M$-norm at most $\alpha$ about a point $x_0$.

**Theorem 4.2 (The curse of bounded moments).** Suppose $(P^1, P^2)$ satisfies Condition 4.1 Then

$$R_\epsilon^*(P^1, P^2) \geq 1/2(1 - \alpha/M(\epsilon)), \forall \epsilon \geq 0.$$ 

(26)
We take an information-theoretic point of view and think of the scenario as that of transmitting information via a noisy communication channel as in Fig. 2. For the analysis, we develop strong Data Processing Inequalities (DPIs) that bound the transmission rate contractively. Already the basic / weak DPI, we know that a random noise corruption mechanism would only reduce the total-variation distance between class-conditional probabilities (making the classification problem harder), and so increase Bayes-optimal error for the problem. We will provide quantitative problem-specific lower-bounds which reflect the increase Bayes-optimal error due to the noise corruption.

A variety of corollaries to Thm. 4.2 can be obtained by considering different choices for the moment function $M$ and the parameter $\alpha$. More are presented in the supplementary materials. For example if $P^1$ and $P^2$ have $d(\cdot, x_0) \in L^p(P^1) \cap L^p(P^2)$ for some (and therefore all) $x_0 \in \mathcal{X}$, we may take $M(r) := r^p$ and $\alpha = W_{d,p}(P^1, P^2)^p$, where $W_{d,p}(P^1, P^2)$ is the order-$p$ Wasserstein distance between $P^1$ and $P^2$, and obtain the following corollary, which was also obtained independently in the recent paper [Pydi and Jog, 2019]. We have

**Corollary 4.1 (Lower-bound from Wasserstein distance).** Under the conditions in the previous paragraph, we have

$$R^*_{\epsilon}(P^1, P^2) \geq \frac{1}{2} \left( 1 - \left( \frac{W_{d,p}(P^1, P^2)}{\epsilon} \right)^p \right).$$

\[\text{(27)}\]

**Proof.** Follows from Theorem 4.2 with $M(r) \equiv r^p$ and $\alpha = W_{d,p}(P^1, P^2)^p$. □

### 5 Universal bounds for random corruption

We now analyse what happens when the attacker is replaced with a random noise corruption mechanism. For example, think of measurement errors in collected data, or natural dirt on road traffic signs (this can be problematic for the artificial vision system on board a self-driving car). Robustness of classifiers to random perturbations of inputs has been studied as a proxy of their robustness to adversarial attacks (e.g [Gilmer et al., 2019]). We do not claim that deep learning is vulnerable to random noise. Of course, certain deep-learning architectures are known to be robust / invariant w.r.t to adversarial attacks (e.g [Ollivier, 2009] for a synthetic theory of ricci curvature, built using Markov kernels. Markov kernels are a generalization of Markov chains. Intuitively, given a point $x \in \mathcal{X}$, the distribution $\omega_x$, prescribes how to "jump" from a point $x$. If $Q$ is a measure on $\mathcal{X}$, then the probability of starting at a point $x \sim Q$, and then landing in a prescribed Borel $A \subseteq \mathcal{X}$ is given by the convolution

$$\omega_x \ast Q(A) := \int_{x \in \mathcal{X}} \omega_x(A) \, dQ(x) = \int_{x \in \mathcal{X}} \int_{\tilde{x} \in A} d\omega_x(\tilde{x}) \, dQ(x).$$

\[\text{(28)}\]

Here, $L^p(P^k)$ is the space of measurable functions $g : \mathcal{X} \to \mathbb{R}$ such that $|g|^p$ is $P^k$-integrable.
More formally, a Markov kernel on $X$ is a map $\omega : X \to \mathcal{P}(X)$, $x \mapsto \omega_x$ such that $\omega_x$ depends measurably on $x$, i.e. for every Borel $A \subseteq X$ and every Borel $I \subseteq \mathbb{R}$, the set $\{ x \in X \mid \omega_x(A) \in I \}$ is a Borel subset of $X$. $\omega$ can be seen as a linear operator acting on distributions according to \cite{20}. In particular, note that $\delta_x \ast \omega = \omega_x \; \forall x \in X$. The trivial Markov kernel given by setting $\omega_x = \delta_x \; \forall x \in X$ of course corresponds to classical machine learning with no noise corruption of the input data. A less trivial example is the isotropic Gaussian Markov kernel (in a possibly infinite-dimensional Hilbert space), for which $\omega_x \equiv \mathcal{N}(x, \varepsilon^2 I) = x + \mathcal{N}(0, \varepsilon^2 I)$. Refer to Fig. 2.

**Links with adversarial attacks.** It should be noted that our Markov kernel model of corruption is general enough to encompass even adversarial attacks considered in sections \cite{2,3} and \cite{4}. Indeed, an attacker $a : X \to X$ can be encoded as a Markov kernel $\omega^a$ given by $\omega^a_x := \delta_{a(x)}$. Thus, adversarial attacks are special cases of noise corruption in our framework of Markov kernels. However, this link is only anecdotal because, for the bounds we obtain in this section, we require the Markov kernel $\omega$ to have (w.r.t Lebesgue), which is not the case for singular kernels like $\omega^a$. This is not an issue per se, since in section \cite{4} we already obtained explicit bounds in case of distance-based adversarial attacks.

**Definition 5.1** (Misclassification error under random corruption). Consider a classification problem $(P^1, P^2)$. Given a Markov kernel $\omega$ on $X$, the noisy test error of a classifier $h : X \to \{1, 2\}$ under corruption by $\omega$ is defined by $R_\omega(h; P^1, P^2) = \sum_{k=1}^{2} \mathbb{E}_{x \sim P^1 \ast \omega^k \sim \omega} \left[ \mathbb{I} [h(x) \neq k] \right]$. The noisy Bayes-optimal error under corruption by $\omega$ is defined by $R^\omega_\omega(P^1, P^2) := \inf_{h} R_\omega(h; P^1, P^2)$.

**Lemma 5.1.** For any classifier $h : X \to \{1, 2\}$, we have the identity $R_\omega(h; P^1, P^2) = R(h; P^1_\omega, P^2_\omega)$, where $P^k_\omega := P^k \ast \omega$. Furthermore, it holds that $R^\omega_\omega(P^1, P^2) = \frac{1}{2} (1 - \text{TV}(P^1_\omega, P^2_\omega))$.

Henceforth let $d$ be a pseudo-metric on the feature space $X$. Refer to definition \cite{2}. We will need the following notion of regularity of Markov kernels.

**Condition 5.1** (Regularity of Markov kernels). Given a measurable function $\theta : \mathbb{R}_+ \to (0, 1]$, a Markov kernel $\omega$ on $X$ is said to be $\theta$-regular if $\text{TV}(\omega_x, \omega_x') \leq \theta(d(x, x'))$, $\forall x, x' \in X$.

As we shall see, the above condition is a key ingredient for obtaining strong DPs. As an example, consider the Gaussian smoothing Markov kernel given by $\omega_x = \mathcal{N}(x, \varepsilon^2 I_m)$. From \cite{Barsov and Ulyanov, 1987} Theorem 1, we know that $\text{TV}(\omega_x, \omega_x') = \text{TV}(\mathcal{N}(x, \varepsilon^2 I), \mathcal{N}(x', \varepsilon^2 I)) = 2\Phi(\|x - x'\|/(2\varepsilon)) - 1$. So we can take $\theta(r) := 2\Phi(r/(2\varepsilon)) - 1$ for $r \geq 0$, a concave non-decreasing function, and Condition 5.1 will be satisfied.

**Theorem 5.1** (Strong DPs from moment conditions). Let $\omega$ be a Markov kernel on $X$ and let $d : X^2 \to \mathbb{R}_+$ be a measurable function. Finally, let $\theta : \mathbb{R}_+ \to (0, 1]$ be a convex non-decreasing function such that Condition 5.1 is satisfied. Then, for any strictly-increasing convex function $M : \mathbb{R}_+ \to \mathbb{R}_+$, and for all $P^1, P^2 \in \mathcal{P}_{\text{Mod}}(X)$, it holds that $\text{TV}(P^1 \ast \omega, P^2 \ast \omega) \leq \theta(W_{M,d}(P^1, P^2))$, where $W_{M,d}(P^1, P^2) := M^{-1} (\inf \mathbb{E}[M(d(X_1, X_2))]$ and the infimum is taken over all couplings $(X_1, X_2)$ of the distributions $P^1$ and $P^2$.

**Proof.** Take $M(r) := r^p$ for all $r \geq 0$ and invoke Theorem 5.1 and note that $W_{r^p,d} = W_{r^p,d}$, the $p$-Wasserstein distance between $P^1$ and $P^2$, for the ground cost $d$. This is actually a distance since $d$ is lower-semicontinuous.

**Corollary 5.1.** Let $X, \omega, \theta$ be as in Thm. 5.1, with the added condition that $d$ is upper-semicontinuous. For every $P^1, P^2 \in \mathcal{P}_{dp}(X)$, we have $\text{TV}(P^1 \ast \omega, P^2 \ast \omega) \leq \theta(W_{d,p}(P^1, P^2))$, where $W_{d,p}(P^1, P^2)$ is the $p$-Wasserstein distance between $P^1$ and $P^2$, for the ground cost $d$.

The above corollary takes us to the right direction: we link the degree of separation distributions $P^1 \ast \omega$ and $P^2 \ast \omega$ in the smoothed problem, measured in total-variation, to the degree of separation between the distributions $P^1$ and $P^2$ in the original problem, measured in Wasserstein distance. However, we are still missing a link to the ultimate complexity parameter of the original problem.
namely the total-variation between $P^1$ and $P^2$. Such a link is unlikely to be obtainable from the above corollary as there is in general no nonvacuous inequalities between TV and Wasserstein metrics (except for extreme cases where the underlying metric space is discrete and bounded).

### 5.2 Main results: universal bounds via total-variation Dobrushin curves

Information theorists know that in general, the TV-based data-processing inequalities are not contractions. That is, an inequality of the form $TV(P^1 \ast \omega, P^2 \ast \omega) \leq c \cdot TV(P^1, P^2)$ is generally only true for $c = 1$. If we could have $c < 1$, then in view of Theorem 5.2, such a contraction would immediately lead to universal results of the form: “if ordinary learning on a binary classification problem $P = (P^1, P^2)$ is hard, then adversarially-robust learning is much harder!”. Lack of contraction is really not a problem. Afterall, all we need are inequalities of the form $TV(P^1 \ast \omega, P^2 \ast \omega) \leq \mathcal{D}(\mathcal{T}(P^1, P^2))$, for suitable functions $\mathcal{D} : [0, 1] \rightarrow [0, 1]$ such that $\mathcal{D}(t) < t$. Note that if $TV(P^1, P^2) \leq t$, then the Bayes-optimal error is at least $(1 - t)/2$ in the noiseless regime, and as a consequence of Lemma 5.1, $R^\ast_\omega(P^1, P^2) \geq \frac{1}{2}(1 - \mathcal{D}(t))$, in the noisy regime we consider here. Thus, upper bounds on $\mathcal{D}(t)$ immediately induce lower bounds on bayes-optimal error in the corrupted regime. A quantity which can play the role of such a nonlinear operator $\mathcal{D}$ is given by the theory of nonlinear strong DPIs developed by Polyanskiy and co-workers [Polyanskiy and Wu, 2016] will be the inspiration for the work in the rest of this section.

**Definition 5.2.** Given $t \in (0, 1]$, let $\mathcal{G}_{M, \alpha}(t)$ denote the set of all binary classification problems $(P^1, P^2)$ such that $TV(P^1, P^2) \leq t$ and Condition 4.1 holds.

The following concept, first introduced in [Polyanskiy and Wu, 2016], will be instrumental to us.

**Definition 5.3 (Dobrushin curve).** Given a Markov kernel $\omega$ on $\mathcal{X}$, its total-variation Dobrushin curve is the function $\mathcal{D}_{\omega, M, \alpha} : [0, 1] \rightarrow [0, 1]$ defined by

\[
\mathcal{D}_{\omega, M, \alpha}(t) := \sup_{(P^1, P^2) \in \mathcal{G}_{M, \alpha}(t)} TV(P^1 \ast \omega, P^2 \ast \omega).
\]  

(29)

Our interest in Dobrushin curves lies in the fact that if the classification problem $(P^1, P^2)$ satisfies condition 4.1 and $TV(P^1, P^2) \leq t$, then

\[
R^\ast_\omega(P^1, P^2) \geq \frac{1}{2}(1 - \mathcal{D}_{\omega, M, \alpha}(t)).
\]  

(30)

This is a direct consequence of Lemma 5.1. The following theorem is one of our main results.

**Theorem 5.2 (Strong DPIs from moment / tail bounds).** Let $\omega$ be a Markov kernel on $\mathcal{X}$ and let $\theta : [0, \infty) \rightarrow [0, 1]$ be a function such that Condition 5.1 holds. Define the function $\bar{\theta} : [0, \infty) \rightarrow [0, 1]$ by $\bar{\theta}(t) = \sup_{0 \leq s \leq t} \theta(s)$ and let $\bar{\theta}^{cc}$ be the concave envelope (i.e smallest concave majorant) of $\bar{\theta}$. Then for every $t \in (0, 1]$, we have

- **Bounds on Dobrushin curve.** We have

\[
\mathcal{D}_{\omega, M, \alpha}(t) \leq t\bar{\theta}^{cc}(2M^{-1}(\alpha/t)).
\]  

(31)

Moreover, if the inequality in Condition 5.1 is an equality, then the bound in (31) is tight, i.e

\[
\mathcal{D}_{\omega, M, \alpha}(t) \geq t\bar{\theta}(2M^{-1}(\alpha/t)).
\]  

(32)

- **Bounds on noisy Bayes-optimal error.** We have

\[
\inf_{(P^1, P^2) \in \mathcal{G}_{M, \alpha}(t)} R^\ast_\omega(P^1, P^2) \geq \frac{1}{2}(1 - t\bar{\theta}^{cc}(2M^{-1}(\alpha/t))).
\]  

(33)

The above theorem gives precise control on the increase in hardness of statistically testing between $P^1$ and $P^2$, as a function of the noise and moment constraints on the $P^k$’s.
5.3 Optimal adversarial noise model from covariance structure of data

We now show how to exploit the covariance structure of the features to engineer a multivariate Gaussian noise kernel which will do the most damage to Bayes-optimal error.

Let $\Sigma$ be an $m$-by-$m$ positive semi-definite matrix and consider the (possibly degenerate) elliptic Gaussian noise kernel $\Sigma^2$ on $\mathbb{R}^m$ given by $\Sigma^2 = \mathcal{N}(x, \Sigma)$ for all $x \in \mathbb{R}^m$. The matrix $\Sigma$ indices a pseudo-metric $\mathbb{R}^m$ (usually referred to as Mahalanobis “distance”), given by $d_{\Sigma}(x, x') := \|x - x'\|_{\Sigma^2} = \sqrt{(x - x')^\top \Sigma^2 (x - x')}$, for all $x, x' \in \mathbb{R}^m$. Our interest in this pseudo-metric lies in the fact that $\text{TV}(\omega_{\Sigma}^2, \omega_{\Sigma'}^2) = \theta(d_{\Sigma}(x, x'))$, where $\theta : [0, \infty) \to [0, 1)$ is the increasing function defined by $\theta(r) = 2\Phi(r) - 1$, and $\Phi$ is CDF of the standard normal distribution. This is a direct consequence of [Barsov and Ulyanov, 1987] Theorem 1. Now, consider the quadratic moment function $M(r) := r^2$. Let $\Sigma_0$ and $\mu_k$ bet the covariance matrix and mean of the features $X$ conditioned on $Y = k$, and set $x_0 := (\mu_1 + \mu_2)/2$, $\Delta \mu := (\mu_1 - \mu_2)/2$, and $\Sigma_0 := (\Sigma_1 + \Sigma_2)/2$. Note that positive semi-definite matrix $\Sigma_0$ is the unconditional covariance matrix of the features, $x_0$ is their unconditional mean, and $\Delta \mu$ is the displacement between the class-conditional means. Also define the positive semi-definite matrix $C := \frac{1}{2}(\Sigma_0 + \Delta \mu \Delta \mu^T)$. Then, one computes

$$
\mathbb{E}_{x \sim P_1} M(d_{\Sigma}(x, x_0)) = \mathbb{E}_{x \sim P_1} \|x - (\mu_1 + \mu_2)/2\|_{\Sigma_0}^2 = \mathbb{E}_{x \sim P_1} \|x - \mu_1 + \Delta \mu\|_{\Sigma_0}^2
$$

$$
= \frac{1}{2} \left( \text{tr}(\Sigma_0 \Sigma_0) + \|\Delta \mu\|_{\Sigma_0}^2 \right) = \frac{1}{2} \text{tr}(\Sigma_0 (\Sigma_0 + \Delta \mu \Delta \mu^T)).
$$

Thus $\mathbb{E}_{x \sim P_1} M(d_{\Sigma}(x, x_0)) + \mathbb{E}_{x \sim P_2} M(d_{\Sigma}(x, x_0)) = \text{tr}(\Sigma_0 (\Sigma_0 + \Delta \mu \Delta \mu^T)) = 2\alpha(\Sigma)$, where

$$\alpha(\Sigma) := \text{tr}(\Sigma_0 C).$$

Therefore, for each potential covariance matrix $\Sigma$ for the adversarial noise distribution, the classification problem $(P_1, P_2)$ satisfies Condition 4.1 for the $d = d_{\Sigma}$ and $\alpha = \alpha(\Sigma)$.

The idea is then to construct, under power constraints, a noise covariance matrix $\Sigma$ which recks the classification problem $(P_1, P_2)$ as much as possible. To this end, let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be the vector eigenvalues of $\Sigma_0^{1/2}$ (i.e. the positive square-roots of the eigenvalues of $\Sigma_0$), and set $s := c_0^2 \|\lambda\|_2^2 := \sum_{j=1}^m \lambda_j^2$, with $c_0 > 0$.

**Lemma 5.2.** Subject to the constraints $\Sigma \succeq 0$ and $\text{tr} \Sigma \leq s$, the minimal value of $\alpha(\Sigma)$ is $\alpha^* = (\text{tr} C^{1/2})^2 / s$, and this value is attained at $\Sigma = \frac{s}{\text{tr} C^{1/2}} C^{1/2}$. Moreover, we have the upper-bound

$$\alpha^* \leq \frac{1}{c_0^2} \left( \frac{\|\lambda\|_1}{\|\lambda\|_2} + \frac{\|\Delta \mu\|_2}{\|\lambda\|_2} \right)^2.$$

This leads us to the following theorem, which is one of our main results.

**Theorem 5.3.** Consider a classification problem $(P_1, P_2)$ on $\mathbb{R}^m$ with finite moments of order 2 and $\text{TV}(P_1, P_2) \leq t$. Let $C \in \mathbb{R}^{m \times m}$, $c_0 > 0$, $s > 0$, and $\lambda := (\lambda_1, \ldots, \lambda_m)$ be as in Lemma 5.2. Then, for $\Sigma = \frac{s}{\text{tr} C^{1/2}} C^{1/2}$, the noisy Bayes-optimal error for the classification problem $(P_1, P_2)$ under corruption from the elliptic Gaussian noise kernel $\Sigma^2$ satisfies

$$R^*_{\Sigma^2}(P_1, P_2) \geq \frac{1}{2} \left( 1 - t \left( 2\Phi \left( \frac{c}{c_0 \sqrt{t}} \right) - 1 \right) \right),$$

where $c := \eta + \theta$, with $\eta := \|\lambda\|_1/\|\lambda\|_2$ and $\theta := \|\Delta \mu\|_2/\|\lambda\|_2$.

**Proof.** Follows from Lemma 5.2 and Thm. 5.2 with the quadratic moment function $M(r) := r^2$. 

The constants $\eta$ and $\theta$ in the theorem has an geometric interpretation. $\eta$ is a measure of the “effective” sparsity of the spectrum of the unconditional covariance matrix $\Sigma_0$ of the features, since one easy verifies that $\eta \leq \text{card}(\{1 \leq j \leq m \mid \lambda_j \neq 0\})$. One the other hand, $\theta$ is a measure of the signal-to-noise ratio in the classification problem.
6 Concluding remarks

Our results extend the current theory on the limitations of adversarial robustness in machine learning. Using techniques from optimal transport theory, we have obtained explicitly variational formulae and lower-bounds on the Bayes-optimal error classifiers can attain under adversarial attack. These formulae suggest that instead of doing adversarial training on normal data, practitioners should strive to do normal training on adversarially augmented data. Going further, in the case of metric attacks, we have obtained explicit bounds which exploit the high-dimensional geometry of the class-conditional distribution of the data. These bounds are universal in that the are classifier-independent; they only depend on the geometric properties of the class-conditional distribution of the data (e.g. finite moments, light-tailness, etc.).

In the case where the perturbations are corruption by random noise, we have used information-theory techniques to obtain analogous bounds. Our main result in this direction is a strong data-processing inequality which induces hard limits on the bayes-optimal error in this scenario.

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In this appendix we provide complete proofs for the theorems, corollaries, etc. which were stated without proof in the manuscript. For clarity, each result from the manuscript (theorems, corollaries, etc.) is restated in this supplemental before proved.

A.1 Proofs for results in section 3

Proposition 3.1 (Extension of Theorem 1 of [Bhagoj et al., 2019]). For any attack model $\Omega$ on the feature space $\mathcal{X}$, the adversarial Bayes-optimal error under type-$\Omega$ attacks

$$R^*_\Omega(P^1, P^2) \geq \frac{1}{2}(1 - OT_\Omega(P^1, P^2)).$$

(12)

First note that the $\Omega$ defined is not automatically a closed subset of $\mathcal{X}^2$. A sufficient condition is that the metric space $(\mathcal{X}, d)$ has the mid-point property.

Let us further suppose that...
• $\Omega$ is symmetric, i.e $(x, x') \in \Omega$ iff $(x', x) \in \Omega$, and that
• $\Omega$ contains the diagonal of $X^2$, i.e $(x, x) \in \Omega$ for all $x \in X$.

**Proof of Proposition 3.1.** For $x \in X$, define $\Omega(x) := \{x' \in X \mid (x, x') \in \Omega\}$. For a classifier $h$, consider the derived classifier $h^\perp : X \to \{1, 2, \perp\}$ defined by

$$h^\perp(x) := \begin{cases} y, & \text{if } h(x') = y \forall x' \in \Omega(x), \\ \perp, & \text{else.} \end{cases}$$

(36)

Here, the special symbol $\perp \notin \{1, 2\}$ should be read as “I don’t know!”. Let $X_1$ (resp. $X_2$) be a random variable that has the same distribution as $X$ conditioned on the event $Y = 1$ (resp. $Y = 2$). One easily computes $1 - R_{\Omega}(h; P^1, P^2) = \frac{1}{2} \mathbb{P}(h(X_1) = 1) + \frac{1}{2} \mathbb{P}(h(X_2) = 2)$, from which

$$2(1 - R_{\Omega}(h; P^1, P^2)) = \mathbb{E}[1[h(X_1) = 1]] + \mathbb{E}[1[h(X_2) = 2]].$$

(37)

Now, define $g_0(x') := 1[h(x') = 1]$ and $f_0(x) = 1[h(x) \neq 2] = 1 - 1[h(x) = 2]$. Then $f_0$ and $g_0$ are bounded, and $P^2$- (resp. $P^1$-) a.s continuous. Moreover, given $x', x \in X$, if $c_{\Omega}(x', x) = 1$, then $x' \notin \Omega(x)$. Since $\{h^{-1}((1))\}_{y=1,2}$ is a partitioning of $X$, at most one of $\Omega(x) \subseteq h^{-1}((1))$ and $\Omega(x) \subseteq h^{-1}((2))$ holds. Thus $1[h(x') = 1] + 1[h(x) = 2] \leq 1$, and so $g_0(x') - f_0(x) = 1[h(x') = 1] + 1[h(x') = 2] = 1 - c_{\Omega}(x', x)$, and so $(f_0, g_0)$ is a pair of Kantorovich potentials for the cost function $c_{\Omega}$. Consequently, from the Kantorovich-Rubinstein duality formula, we have

$$OT_{\Omega}(P^2, P^1) = \sup_{P^2 \succcurlyeq P^1} \mathbb{E}[g(X_1)] - \mathbb{E}[f(X_2)] \geq \mathbb{E}[g_0(X_1)] - \mathbb{E}[f_0(X_2)]$$

$$= \mathbb{E}[1[h(X_1) = 1]] + \mathbb{E}[1[h(X_2) = 2]]$$

(38)

Since $h$ is an arbitrary classifier, we obtain that $2 \inf_{h} R_{\Omega}(h; P^1, P^2) \geq 1 - OT_{\Omega}(P^2, P^1)$ as claimed.

**Theorem 3.1.** Let $\Omega$ be an attack model on $X$. Then we have the identity

$$OT_{\Omega}(P^1, P^2) = \sup_{U \subseteq X \text{ closed}} P^1(U) - P^2(U_{\Omega}).$$

(15)

In particular, for distance-based attacks we have $OT_{\varepsilon}(P^1, P^2) = \sup_{U \subseteq X \text{ closed}} P^2(U) - P^1(U_{\varepsilon})$.

**Proof.** One computes

$$OT_{\Omega}(P^1, P^2) := \inf_{\gamma \in \Pi(P^1, P^2)} \int_{X^2} c_{\Omega}(x, x') \, d\gamma(x, x') = \inf_{\gamma \in \Pi(P^1, P^2)} \int_{X^2} 1[(x, x') \in \Omega] \, d\gamma(x, x')$$

$$= \inf_{\gamma \in \Pi(P^1, P^2)} \int_{\Omega} d\gamma(x, x') = \inf_{\gamma \in \Pi(P^1, P^2)} \gamma(\Omega),$$

(39)

On the other hand, by Strassen’s Marriage Theorem (see [Villani, 2003, Theorem 1.27 of]) and the definition of $U_{\Omega}$ in Eq. (13) one has

$$\inf_{\gamma \in \Pi(P^1, P^2)} \gamma(\Omega) = \sup_{U \subseteq X \text{ closed}} P^2(U) - P^1(U_{\Omega}),$$

and the result follows. The particular case of distance-based attacks corresponds to letting $\Omega := D_{\varepsilon} := \{(x, x') \in X^2 \mid d(x, x') \leq \varepsilon\}$, so that $U_{\Omega} = U_{\varepsilon}$, the $\varepsilon$-neighborhood of $U$.

**Theorem 3.2** (Adversarially augmented data, a proxy for adversarial robustness). Consider a classification problem $(P^1, P^2)$. Suppose $d$ is a distance on the feature space $X$ with the MM property [3.1] and consider the distance-based attack model $D_{\varepsilon} := \{(x, x') \in X^2 \mid d(x, x') \leq \varepsilon\}$. Recall the definition of $TV_{\varepsilon}(P^1, P^2)$ from Eq. (8). Define

$$\tilde{TV}_{\varepsilon}(P^1, P^2) := \inf_{\gamma_{1,2}} TV_{\varepsilon}(\gamma_{1,2}),$$

(17)

$$\tilde{TV}_{\varepsilon}(P^1, P^2) := \inf_{a_1, a_2 \text{ type } D_{\varepsilon} \neq /} TV(a_1 \neq P^2, a_2 \neq P^1),$$

(17)
where "#" denotes pushfoward of measures and the 1st inf. is taken over all pairs of distributions $(\gamma_1, \gamma_2)$ on $\mathcal{X}^2$ concentrated on $D_{\varepsilon}/2$ such that $\text{proj}_1^1 \gamma_1 = P^1$ and $\text{proj}_2^1 \gamma_2 = P^1$. It holds that
\[
TV_\varepsilon(P^1, P^2) = \widetilde{TV}_\varepsilon(P^1, P^2) \leq \widetilde{TV}_\varepsilon(P^1, P^2),
\]
and there is equality if $P^1$ and $P^2$ have densities w.r.t the Borel measure on $\mathcal{X}$.

Consequently, we have the following lower-bound for the adversarial Bayes-optimal error:
\[
R^c_\varepsilon(P^1, P^2) \geq \frac{1}{2}(1 - \widetilde{TV}_\varepsilon(P^1, P^2)) = \frac{1}{2}(1 - \widetilde{TV}_\varepsilon(P^1, P^2)) \geq \frac{1}{2}(1 - \widetilde{TV}_\varepsilon(P^1, P^2)).
\]

**Proof.** Note that each $P^k$ is a Borel measure on $\mathcal{X}$ which integrates to $1/2$. For the convenience of the proof, we rescale each $P^k$ by 2, so that it integrates to 1.

Let $D'_\varepsilon := \mathcal{X}^2 \setminus D_\varepsilon$. To prove the theorem, we consider the following intermediate quantity
\[
E(P^1, P^2) := \sup_{\gamma \in \Pi_\varepsilon(P^1, P^2), \supp(\gamma) \subseteq D_\varepsilon} 1 - \gamma(\mathcal{X}^2).
\]

Applying Lemma A.1 with $\Omega = D_\varepsilon$, we know that $\text{OT}_\varepsilon = E$. The rest of the proof is divided into separate steps.

**Step 1: proving the equality $E = \widetilde{TV}_\varepsilon$.** Let $\gamma$ be feasible for $E$. Because $(\mathcal{X}, d)$ satisfies the MM property (Condition 3.1), there exists a $\gamma$-measurable map $\eta : \mathcal{X}^2 \rightarrow \mathcal{X}$ such that $\eta(x, x')$ is a midpoint of $x$ and $x'$ for all $x, x' \in \mathcal{X}$. Now, consider the $\gamma$-measurable maps $T_1, T_2 : \mathcal{X}^2 \rightarrow \mathcal{X}^2$, $D : \mathcal{X} \rightarrow \mathcal{X}$ defined by
\[
T_1(x, x') := (x, \eta(x, x')), \quad T_2(x, x') := (\eta(x, x'), x'), \quad T_3(x) := (x, x).
\]

Construct couplings $\gamma_1 = (T_1)_\# \gamma + T_3(\mathcal{X}^2 - \text{proj}_\# \gamma)$ and $\gamma_2 = (T_2)_\# \gamma + T_3(\mathcal{X}^2 - \text{proj}_\# \gamma)$. Then $(\gamma_1, \gamma_2)$ is feasible for $\widetilde{TV}_\varepsilon$ and
\[
TV(\text{proj}_\# \gamma_1, \text{proj}_\# \gamma_2) \leq (P^1 - \text{proj}_\# \gamma)(\mathcal{X}) + (P^2 - \text{proj}_\# \gamma)(\mathcal{X}) = 2(1 - \gamma(\mathcal{X}^2))
\]
because the second marginal of $(T_1)_\# \gamma$ and the first marginal of $(T_2)_\# \gamma$ agree by construction. Thus $TV_\varepsilon \leq E$. Conversely, let $\gamma_1, \gamma_2$ be feasible for $\widetilde{TV}_\varepsilon$, and let $\gamma \leq \gamma_1$ and $\gamma \leq \gamma_2$ be such that $\text{proj}_\# \gamma = \text{proj}_\# \gamma_1 = (\text{proj}_\# \gamma_1) \wedge (\text{proj}_\# \gamma_2)$ where $\wedge$ is the "pointwise" minimum of two measures (they can be built with the disintegration theorem). Now build $\gamma$ feasible for $E$ by gluing together $\gamma_1$ and $\gamma_2$. It holds
\[
TV(\text{proj}_\# \gamma_1, \text{proj}_\# \gamma_2) = 2(1 - \text{proj}_\# \gamma_1 \wedge \text{proj}_\# \gamma_2)(\mathcal{X}) = 2(1 - \gamma(\mathcal{X}^2)).
\]

Thus $E \leq \widetilde{TV}_\varepsilon$ hence $E = \widetilde{TV}_\varepsilon$.

**Step 2: proving the inequality $\widetilde{TV}_\varepsilon \leq \widetilde{TV}_\varepsilon$.** Now, the fact that $\widetilde{TV}_\varepsilon \leq \widetilde{TV}_\varepsilon$ in general is due to the fact to any transport map $a$ satisfying $d(a(x), x) \leq \varepsilon$ corresponds a deterministic transport plan $(id, a)_\# P^1$ supported on $D_\varepsilon$. In general, equality in the theorem will fail to hold. For example, on the real line, $P^1 = \frac{1}{3} \delta_{-\varepsilon} + \frac{1}{3} \delta_0 + \frac{1}{3} \delta_{\varepsilon}$ and $P^2 = \frac{1}{2} \delta_{-\varepsilon} + \frac{1}{2} \delta_{\varepsilon}$ has $\widetilde{TV}_\varepsilon(P^1, P^2) = 2/6$ and $\widetilde{TV}_\varepsilon(P^1, P^2) = 0$.

**Step 3: proving the inequality $\widetilde{TV}_\varepsilon \geq \widetilde{TV}_\varepsilon$ for absolutely continuous $P^k$’s.** Finally, the fact that $\widetilde{TV}_\varepsilon \geq \widetilde{TV}_\varepsilon$ when $P^1$ and $P^2$ are absolutely continuous is a consequence of the existence of an optimal transport map for the $W_\infty$ distance. Indeed, if $(\gamma_1, \gamma_2)$ is feasible for $\widetilde{TV}_\varepsilon$, then $W_\infty(P^1, \text{proj}_\# \gamma_1) \leq \varepsilon$ and there exists a measurable map $a_1 : \mathcal{X} \rightarrow \mathcal{X}$ such that $d(a_1(x), x) \leq \varepsilon$ $P^1$-a.e. and $(a_1)_\# P^1 = \text{proj}_\# \gamma_1$ (one can build $a_2$ similarly). 

\[\square\]
Corollary 3.2. Consider a classification problem \((P^1, P^2)\) on \(\mathbb{R}^m\) with \(P^k = (1/2)N(\mu_k, \Sigma)\). For \(\ell_\infty\)-norm attacks with budget \(\varepsilon\), we have

\[
R_\varepsilon^* (P^1, P^2) \geq 1 - \Phi(\Delta(\varepsilon)/2) \geq 1 - \Phi(\|s(\varepsilon)\|_2/2).
\]  

(21)

where \(\Delta(\varepsilon) := \sqrt{\Delta \mu^T \Sigma^{-1} \Delta \mu/\varepsilon}, \Delta \mu := \mu_1 - \mu_2,\) and \(s(\varepsilon) \in \mathbb{R}^m\) is defined by \(s_j(\varepsilon) := |\sigma_j|^{-1}(\Delta \mu_j - \varepsilon)_+,\) and \(\sigma_1^2, \ldots, \sigma_m^2\) are the eigenvalues of \(\Sigma\).

Proof. The first part of the claim follows from a direct application of \cite{Barsov and Ulyanov, 1987} Theorem 1:

\[
TV(N(\mu_1, \Sigma), N(\mu_2, \Sigma)) = 2\Phi(\|\mu\|_{\|\Sigma\|}/2),
\]

where \(\mu := \mu_1 - \mu_2 \in \mathbb{R}^d\). Thus \(R_\varepsilon^* \geq 1 - \Phi(\Delta(\varepsilon)/2),\) where \(\Delta(\varepsilon) := \min_{\|x\| \leq \varepsilon} \|x - \mu\|_{\|\Sigma\|}^2\).

It now remains to bound \(\Phi(\Delta(\varepsilon))\), and we are led to consider the computation of quantities of the following form.

Bounding the quantity \(\Delta(\varepsilon)\). We are led to consider problems of the form

\[
\alpha := \max_{\|w\|_{\|\Sigma\|} \leq \varepsilon} w^T a - \varepsilon \|w\|_1,
\]

(40)

where \(a \in \mathbb{R}^d\) and \(\Sigma\) be a positive definite matrix of size \(d\). Of course, the solution value might not be analytically expressible in general, but there is some hope, when the matrix \(\Sigma\) is diagonal. That notwithstanding, using the dual representation of the \(\ell_1\)-norm, one has

\[
\alpha = \max_{\|w\|_{\|\Sigma\|} \leq \varepsilon} \min_{\|z\| \leq \varepsilon} w^T a - w^T z = \max_{\|z\| \leq \varepsilon} \min_{\|w\| \leq \varepsilon} w^T (z - a)
\]

\[
= \min_{\|z\| \leq \varepsilon} \max_{\|w\| \leq \varepsilon} w^T (z - a) = \min_{\|z\| \leq \varepsilon} \max_{\|w\| \leq \varepsilon} \Sigma^{-1} w^T (z - a)
\]

(41)

where we have used Sion’s minimax theorem to interchange min and max in the first line, and we have introduced the auxiliary variable \(\tilde{w} := \Sigma^{-1/2}w\) in the fourth line. We note that given a value for the dual variable \(z\), the optimal value of the primal variable \(w\) is

\[
w \propto \Sigma^{-1}(a - z).
\]

(42)

The above expression for the optimal objective value \(\alpha\) is unlikely to be computable analytically in general, due to the non-separability of the objective (even though the constraint is perfectly separable as a product of 1D constraints). In any case, it follows from the above display that \(\alpha \leq 0\), with equality iff \(\|a\|_{\|\Sigma\|} \leq \varepsilon\).

Exact formula for diagonal \(\Sigma\). In the special case where \(\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_d)\), the square of the optimal objective value \(\alpha^2\) can be separated as

\[
\alpha^2 = \sum_{j=1}^d \min_{\|z_j\| \leq \varepsilon} \sigma_j^{-2} (z_j - a_j)^2 = \sum_{j=1}^d \sigma_j^{-2} \left\{ \begin{array}{ll}
(a_j + \varepsilon)^2, & \text{if } a_j \leq -\varepsilon, \\
0, & \text{if } -\varepsilon < a_j \leq \varepsilon, \\
(a_j - \varepsilon)^2, & \text{if } a_j > \varepsilon,
\end{array} \right.
\]

(43)

Thus \(\alpha^2 = \sqrt{\sum_{j=1}^d \sigma_j^{-2} ((|a_j| - \varepsilon)_+)^2}\). By the way, the optimum is attained at

\[
z_j = \left\{ \begin{array}{ll}
-\varepsilon, & \text{if } a_j \leq -\varepsilon, \\
a_j, & \text{if } -\varepsilon < a_j \leq \varepsilon, \\
\varepsilon, & \text{if } a_j > \varepsilon,
\end{array} \right.
\]

(44)

Plugging this into (42) yields the optimal weights

\[
w_j \propto \sigma_j^{-2} \text{sign}(a_j)(|a_j| - \varepsilon)_+.
\]
Upper / lower bounds for general $\Sigma$. Let $\sigma_1, \sigma_2, \ldots, \sigma_d > 0$ be the eigenvalues of $\Sigma$. Then

$$\|z - a\|_2 := (z - a)^T \Sigma^{-1} (w - a) \leq \sum_{j=1}^d (z_j - a_j)^2 / \sigma_j^2 =: \|z - a\|_{\text{diag}(1/\sigma_1, \ldots, 1/\sigma_d)}^2.$$  

Therefore in view of the previous computations for diagonal covariance matrices, one has the bound $\alpha \leq \sqrt{\sum_{j=1}^d \sigma_j^{-2} ((|a_j| - \xi_j)_+)^2}$. \hfill \Box

A.2 Proofs for results in section \S

Lemma 5.1. For any classifier $h : X \rightarrow \{1, 2\}$, we have the identity $R_\omega(h; P^1, P^2) = R(h; P^1_\omega, P^2_\omega)$, where $P^k_\omega := P^k \ast \omega$. Furthermore, it holds that $R^*_\omega(P^1, P^2) = \frac{1}{2}(1 - TV(P^1_\omega, P^2_\omega))$.

Proof. We prove a more general result for any loss function $\ell : \mathbb{R} \times [K] \rightarrow \mathbb{R}$. Let, $\ell_h(x, y) := \ell(h(x), y)$. Recall the definition of the sub-probability measures $P^\nu$ on $X$ for $y \in [K]$. Now, by direct computation, we have

$$R_\omega(h; \ell) := \sum_y \mathbb{E}_{x \sim P^\nu} [\mathbb{E}_{\tilde{x} \sim P^\nu \ast \omega} [\ell_h(\tilde{x}, y)]] = \sum_y \mathbb{E}_{x \sim (P^\nu \ast \omega)} [\ell_h(x, y)] =: R(\ell_h; P^1 \ast \omega, P^2 \ast \omega),$$

which proves the first identity. The result is then a simple consequence of the classical fact that $R(Q^1, Q^2) = \frac{1}{2}(1 - TV(Q^1, Q^2))$, applied to the classification problem $(Q^1, Q^2) = (P^1 \ast \omega, P^2 \ast \omega)$. \hfill \Box

Theorem 5.2 (Strong DPIs from moment / tail bounds). Let $\omega$ be a Markov kernel on $X$ and let $\theta : [0, \infty) \rightarrow [0, 1]$ be a function such that Condition 5.7 holds. Define the function $\bar{\theta} : [0, \infty) \rightarrow [0, 1]$ by $\bar{\theta}(r) = \sup_{0 \leq s \leq r} \theta(s)$ and let $\theta^{cc}$ be the concave envelope (i.e smallest concave majorant) of $\bar{\theta}$. Then for every $t \in (0, 1]$, we have

- **Bounds on Dobrushin curve.** We have

  $$\mathcal{D}_{\omega, M, \alpha}(t) \leq t \bar{\theta}^{cc}(2M^{-1}(\alpha/t)).$$  

Moreover, if the inequality in Condition 5.7 is an equality, then the bound in \[31\] is tight, i.e

  $$\mathcal{D}_{\omega, M, \alpha}(t) \geq \bar{\theta}(2M^{-1}(\alpha/t)).$$  

- **Bounds on noisy Bayes-optimal error.** We have

  $$\inf_{(P^1, P^2) \in \mathcal{G}_{M, \alpha}(t)} R^*_\omega(P^1, P^2) \geq \frac{1}{2} (1 - t \bar{\theta}^{cc}(2M^{-1}(\alpha/t))).$$  

Proof. The proof is an adaptation of the proof of [Polyanskiy and Wu, 2016]. Let $t \in (0, 1]$ and $(P^1, P^2) \in \mathcal{G}_{M, \alpha}(t)$ with $TV(P^1, P^2) \leq t$, and let $(X_1, X_2)$ be a coupling of $P^1$ and $P^2$. Then

$$TV(P^1 \ast \omega, P^2 \ast \omega) = \mathbb{P}(X_1 \neq X_2) \mathbb{E} TV(P^1 \ast \omega, P^2 \ast \omega \mid X_1 \neq X_2) \leq \mathbb{P}(X_1 \neq X_2) \mathbb{E}[\theta(d(X_1, X_2)) \mid X_1 \neq X_2]$$

$$\leq \mathbb{P}(X_1 \neq X_2) \mathbb{E}[\bar{\theta}^{cc}(d(X_1, X_2)) \mid X_1 \neq X_2]$$

$$\leq \mathbb{P}(X_1 \neq X_2) \bar{\theta}^{cc}(\mathbb{E}[d(X_1, X_2) \mid X_1 \neq X_2]).$$

\[45\]
where the first inequality is due to Condition 5.1 and the last is Jensen’s inequality. On the other hand, one computes

\[ M \left( \mathbb{E} \left[ \frac{d(X_1, X_2)}{2} \right] | X_1 \neq X_2 \right) \leq \mathbb{E} \left[ M \left( \frac{d(X_1, X_2)}{2} \right) \right] \]

\[ \leq \frac{1}{\mathbb{P}(X_1 \neq X_2)} \mathbb{E} \left[ M \left( \frac{d(X_1, X_2)}{2} \right) \right] \]

\[ \leq \frac{1}{\mathbb{P}(X_1 \neq X_2)} \mathbb{E} \left[ M \left( \frac{d(X_1, x_0) + d(X_2, x_0)}{2} \right) \right] \]

\[ \leq \frac{1}{\mathbb{P}(X_1 \neq X_2)} \left( \mathbb{E} \left[ M \left( \frac{d(X_1, x_0)}{2} \right) \right] + \mathbb{E} \left[ M \left( \frac{d(X_2, x_0)}{2} \right) \right] \right) \]

\[ \leq \frac{\alpha/2 + \alpha/2}{\mathbb{P}(X_1 \neq X_2)} = \frac{\alpha}{\mathbb{P}(X_1 \neq X_2)}, \]

where \((a)\) is Jensen’s inequality; \((b)\) is Bayes rule and the fact that \(M(0) = 0\); \((c)\) is by the triangle inequality; \((d)\) is by convexity of \(M\) and \((e)\) is because \((P^1, P^2) \in \mathcal{G}_{M, \alpha}\). Applying \(M^{-1}\) on both sides of the above equation and multiplying by 2 then yields

\[ \mathbb{E} \left[ d(X_1, X_2) \mid X_1 \neq X_2 \right] = 2M^{-1} \left( \frac{\alpha}{\mathbb{P}(X_1 \neq X_2)} \right), \quad (46) \]

Combining with \((45)\) yields

\[ \text{TV}(P^1 \ast \omega, P^2 \ast \omega) \leq \mathbb{P}(X_1 \neq X_2) \hat{\theta}^{cc} \left( 2M^{-1} \left( \frac{\alpha}{\mathbb{P}(X_1 \neq X_2)} \right) \right). \quad (47) \]

But the function \(u \mapsto u \hat{\theta}^{cc}(2M^{-1}(\alpha/u))\) is non-decreasing on \((0, 1)\). Thus, optimizing over the coupling \((X_1, X_2)\), and using the Strassen "marriage" characterization of total-variation, namely

\[ \text{TV}(P^1, P^2) = \inf_{(X_1, X_2)} \mathbb{P}(X_1 \neq X_2), \]

we get the inequalities

\[ \text{TV}(P^1 \ast \omega, P^2 \ast \omega) \leq \text{TV}(P^1, P^2) \hat{\theta}^{cc} \left( 2M^{-1} \left( \frac{\alpha}{\text{TV}(P^1, P^2)} \right) \right) \leq \hat{\theta}^{cc} \left( 2M^{-1}(\alpha/t) \right), \quad (48) \]

where the last inequality is because \(\text{TV}(P^1, P^2) \leq t\) by hypothesis. Since \((P^1, P^2) \in \mathcal{G}_{M, \alpha}(t)\) was chosen arbitrarily, \((31)\) holds as claimed.

For the lower bound \((52)\), indeed, for arbitrary \(x_1, x_2 \in X\) such \(d(x_1, x_2) \leq M^{-1}(\alpha/t)\), consider the pair of distributions \((P^1, P^2)\) defined by \(P^k = (1-t)\delta_{x_0} + t\delta_{x_k}\). It is clear that \((P^1, P^2) \in \mathcal{G}_{M, \alpha}(t)\) (because of the condition \(d(x_1, x_2) \leq M^{-1}(\alpha/t)\) and the triangle inequality). Moreover, it is easy to compute \(\text{TV}(P^1, P^2) = t\) and \(\text{TV}(P^1 \ast \omega, P^2 \ast \omega) = \text{TV}(\omega_{x_1}, \omega_{x_2}) = t\). Optimizing over \((x_1, x_2)\), we get \((32)\).

The second part of the proof follows directly from \((30)\).

\[ \square \]

**Lemma 5.2.** Subject to the constraints \(\Sigma \succeq 0\) and \(\text{tr} \Sigma \leq s\), the minimal value of \(\alpha(\Sigma)\) is \(\alpha^* = (\text{tr} C^{1/2})^2/s\), and this value is attained at \(\Sigma = \frac{s}{\text{tr} C^{1/2}} C^{1/2}\). Moreover, we have the upper-bound

\[ \alpha^* \leq \frac{1}{c_0^2} \left( \frac{\|\lambda\|_1}{\|\lambda\|_2} + \frac{\|\Delta \mu\|_2}{\|\lambda\|_2} \right)^2. \]

**Proof.** If \(C\) is not positive definite, then it \(C\) can be seen as a limit of positive definite matrices. Thus we may restrict \(\Sigma\) to the cone of positive definite matrices. By the method of Lagrange multipliers,

\[ \min_{\text{tr} \Sigma \leq s} \alpha(\Sigma) = \min_{\text{tr} \Sigma \leq s} \text{tr}(\Sigma^{-1} C) = \max_{\mu \geq 0} \min_{\Sigma} \text{tr}(\Sigma^{-1} C) = \text{tr}(\Sigma - s)). \]
Differentiating w.r.t $\Sigma$ and setting to zero gives, $-\Sigma^{-2}C + wI_m = 0$, i.e $\Sigma = C^{1/2}/\sqrt{w}$. Furthermore, the constraint $\text{tr} \Sigma = s$ yields $w = \text{tr} C^{1/2}/s$, and so $\Sigma = (s/\text{tr} C^{1/2})C^{1/2}$. With this choice of $\Sigma$, we have

$$\text{tr}(\Sigma^{-1}C) = \frac{(\text{tr} C^{1/2})^2}{s} = \frac{(\text{tr}(\Sigma_0 + \Delta\mu\Delta\mu^T)^{1/2})^2}{s} \leq \frac{(\text{tr} \Sigma_0^{1/2} + \text{tr}(\Delta\mu\Delta\mu^T)^{1/2})^2}{s}$$

$$= \left(\|\lambda\|_1 + \|\Delta\mu\|_2\right)^2$$

$$= \frac{1}{s^2} \left(\|\lambda\|_1 + \|\Delta\mu\|_2\right)^2,$$

as claimed.

B  Miscellaneous

B.1  Computing optimal adversarial attack plan via bipartite graph matching (section 3.5)

Figure 3: Left: Numerical computation of $\text{TV}_\epsilon(P^1, P^2)$ for two 10-dimensional Gaussians $P^1$ and $P^2$ of same covariance matrix $\sigma I_{10}$ but different means. The maximal graph matching approach is described in Alg. [1] run on empirical samples from 500 iid samples from $P^1$ and $P^2$. 