FRACTIONAL-QUANTUM-HALL EDGES AT FILLING FACTOR $\nu = 1 - 1/m$

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We consider the edge of a two-dimensional electron system that is in the quantum-Hall-effect regime at filling factor $\nu = 1 - 1/m$ with $m$ being an odd integer, where microscopic theory explaining the occurrence of the quantum Hall effect in the bulk predicts the existence of two counterpropagating edge-excitation modes. These two modes are the classical edge-magnetoplasmon mode and a slow-moving neutral mode. Assuming the electrons to be confined by a coplanar neutralizing background of positive charges, and taking careful account of long-range Coulomb interactions, we determine microscopically the velocity $v_n$ of the neutral mode and the edge width $d$. Our results are intended to guide experimental efforts aimed at verifying the existence of the neutral mode, which would provide a powerful confirmation of the current microscopic understanding of quantum-Hall physics at the simplest hierarchical filling factors $\nu = 1 - 1/m$.

1 Introduction

The quantum Hall (QH) effect occurs in two-dimensional (2D) electron systems whenever an incompressibility develops in the bulk at a magnetic-field($B$)-dependent value of the electronic sheet density $n_e$. Experimentally, the QH effect is observed when the filling factor $\nu = 2\pi\ell^2 n_e$ is equal to an integer or certain fractions. (Here we defined the magnetic length $\ell = \sqrt{\hbar c/|eB|}$.) The physical origin of the incompressibility, i.e., a gap for excitation of unbound particle-hole pairs, is quite different for the integer and fractional QH effects. In the integer case, the incompressibility arises from Landau quantization of the kinetic energy of a charged 2D particle moving in a perpendicular magnetic field. At fractional filling factors where partially filled Landau levels exist, incompressibility is a consequence of electron-electron interactions. Exactly how interactions give rise to bulk incompressibilities is most well-understood microscopically for filling factors that are the inverse of an odd number, i.e., $\nu = 1/m$ with $m = 3, 5, \ldots$. To explain incompressibility at other fractional values of the filling factor where the QH effect is observed, hierarchical models have been proposed. We focus here on QH systems at $\nu = 1 - 1/m$, which can be regarded as the simplest hierarchical filling factors.

In both the integer and fractional cases, the only low-lying excitations present in a QH sample are localized at the boundary. In a magnetic field, collective modes known as edge-magnetoplasmons (EMP) occur at the edge of a 2D electron system even when the bulk is compressible. Outside of the QH regime, however, these modes have a finite life time due to decay into incoherent particle-hole excitations and are most aptly described using a hydrodynamic picture. In the QH regime, provided that the edge of the 2D electron system is sufficiently sharp, the microscopic physics simplifies and there is no particle-hole continuum into which the modes can decay. Generalizations of models familiar from the study of one-dimensional (1D) electron systems can then be used to provide a fully microscopic description of integer and fractional QH edges. In particular, the sharp edge of a QH system at $\nu = 1/m$ supports a single branch of chiral edge excitations. These are EMP modes which, in this case, have an especially simple microscopic description. For the hierarchical filling factors, however, both microscopic theory and phenomenological considerations suggest that even a sharp edge
supports additional branches of chiral excitations, some of which can be propagating in the direction opposite to the EMP mode. For \( \nu = 1 - 1/m \), a single counterpropagating edge mode is expected to exist in addition to the EMP mode. The prediction of this additional mode is entirely due to our microscopic understanding of why the QH effect is observed at \( \nu = 1 - 1/m \); such a prediction would not arise within a purely classical, hydrodynamic theory for the interacting 2D electron system. Hence, experimental confirmation of the additional counterpropagating edge mode would provide a strong confirmation for the predictive power of microscopic theory describing fractional-QH physics. However, time-domain studies of edge modes at \( \nu = 2/3 \) have turned up no evidence for this mode.

Here we review the microscopic model for a finite QH sample that is at a filling factor \( \nu = 1 - 1/m \). We determine the two parameters characterizing the edge, which are its stiffness to long-wave-length neutral excitations, and the effective edge width. Performing a careful separation of long-range and short-range pieces of interactions between edge excitations, we are able to calculate the velocity of the counterpropagating mode.

## 2 Microscopic Model for the Quantum-Hall Sample at \( \nu = 1 - 1/m \)

Laughlin proposed a set of many-electron states that are incompressible at \( \nu = 1/m \) to explain the occurrence of the QH effect at these filling factors. Hierarchical constructions relate the incompressibility of the 2D electron system at \( \nu = 1/m \) to incompressibility at some other filling factor. For example, due to the fact that the lowest Landau level has perfect particle-hole symmetry, in the absence of Landau-level mixing a system of electrons at a fractional filling factor \( \nu_e = \nu \) is equivalent to a hole system at filling factor \( \nu_h = 1 - \nu \). Based on Laughlin’s theory, the hole system is incompressible at hole filling factors \( \nu_h = 1/m \), which translates into incompressibility of the corresponding electron system at \( \nu_e = 1 - 1/m \). This microscopic explanation of the occurrence of the QH effect at \( \nu = 1 - 1/m \) implies that the ground state of the electron system at that filling factor is the particle-hole conjugate of a Laughlin state for holes at \( \nu_h = 1/m \). In a finite electron system, there are then two distinct ways to add particles at \( \nu = 1 - 1/m \); either before or after performing the particle-hole-conjugation transformation. This leads us to expect two distinct edges of the \( \nu = 1 - 1/m \) QH sample, an inner one and an outer one, which are separated by a distance \( d \). (See Fig. 1.) The inner edge is that of a Laughlin state of holes at \( \nu_h = 1/m \), and the outer one is the edge of a filled lowest Landau level of electrons. Both edges support low-lying excitations corresponding to long-wave-length density fluctuations. However, for now we only consider the ground state configuration. As
a realistic model for the external potential confining the electrons to the finite QH sample, we assume a background of positive charges to be present that would exactly neutralize the electron density if each lowest-Landau-level orbital were occupied with probability \( \nu \). As the local filling factor for the ground-state configuration depicted in Fig. [1] deviates from its bulk value close to the edge, a dipolar strip forms at the edge. While the system is not charge-neutral \textit{locally}, the 2D charge density integrated perpendicular to the edge at any fixed location along the edge yields zero. We call this weaker version of charge neutrality \textit{1D-local} charge neutrality. Note that, in general, 1D-local neutrality will be violated when edge-density fluctuations are present at the inner and/or outer edges.

Conceptionally, it is convenient to work consistently in the Hilbert space for \textit{holes} which has been truncated such that there are no states available with guiding centers beyond the physical boundary of the sample. In hole language, the ground-state configuration depicted in Fig. [1] is that of a hole system which has undergone phase separation into two QH strips: the inner one which is in the Laughlin state for filling factor \( 1 - \nu = 1/m \), and an outer one extending from the physical sample boundary to the outer edge which has filling factor one. In equilibrium, the energy \( \mu_o \) to add a hole at the outer edge must equal the energy \( \mu_i \) to add it at the inner edge:

\[
\mu_o = \mu_i = \mu \ ,
\]

where \( \mu \) is the chemical potential. (Holes cannot be added at the physical boundary of the sample due to the truncation of the Hilbert space.) To add a hole at the inner edge, it must be provided with the energy per particle \( \zeta(1 - \nu) \) of a locally charge-neutral hole system that is in the Laughlin state for filling factor \( 1/m \), plus the electrostatic energy arising from the added hole’s interaction with the above-mentioned dipolar strip of non-neutralized charges. A similar reasoning applies when adding a hole to the outer edge. For \( d > \ell \), we find

\[
\begin{align*}
\mu_i &= \zeta(1 - \nu) + \frac{e^2}{\epsilon \ell \pi} \frac{d}{\ell} (1 - \nu) \ln \left( \frac{1 - \nu}{\nu} \right) , \\
\mu_o &= \zeta(1) + \frac{e^2}{\epsilon \ell \pi} \frac{d}{\ell} \ln \left( \frac{1}{\nu} \right) ,
\end{align*}
\]

where \( \zeta(1) = -\sqrt{\pi/8} e^2/\epsilon \ell \) is the energy per particle for a filled lowest Landau level. The equilibrium condition Eq. ([1]) then determines the ground-state edge separation \( d \); it can be written as

\[
\frac{\hbar v_J}{\ell} \frac{d}{\ell} = \zeta(1 - \nu) - \zeta(1) ,
\]

with a velocity

\[
v_J = -\frac{e^2}{\epsilon \hbar \pi} \left[ \nu \ln(\nu) + (1 - \nu) \ln(1 - \nu) \right] .
\]

Note that \( \zeta(1 - \nu) > \zeta(1) \), which means that we find a positive value for \( d \), as is required by self-consistency.

### 3 Energy of Long-Wave-Length Edge Excitations

The inner edge of a QH sample that is at \( \nu = 1 - 1/m \) supports low-lying excitations that correspond to fluctuations in the 1D edge density which is obtained when integrating the 2D density profile of the inner hole strip over the coordinate perpendicular to the edge. Similarly, the outer hole strip has low-lying excitations that are density fluctuations located at the outer edge. (Note that the edge of the outer hole strip that coincides with the sample boundary originates from the truncation of the Hilbert space in which we perform the particle-hole conjugation and does \textit{not} support physical excitations.) The local electric field, which is related to the slope
expressed in the normal modes reads density, and (ii) the neutral mode $\rho$ the order of the edge perimeter $L$. The first term of the r.h.s. shows a fictitious system where the positive background charge has been adjusted such that the charge density integrated perpendicular to the edge yields zero at any fixed location along the edge. Per definition, the fictitious system is always 1D-locally neutral and long-range electrostatic forces are cut off at distances larger than the edge width $d/\nu \ll L$. The long-range Coulomb forces are accounted for in the second term on the r.h.s. where no charge imbalance occurs in the direction transverse to the edge.

of the effective external potential felt by holes, has opposite sign at the location of the inner and outer edges. Hence, density fluctuations at the inner and outer edges propagate in opposite direction. Denoting 1D edge-density fluctuations localized at the inner and outer edges by $\rho_i(x)$ and $\rho_o(x)$, respectively, where $x$ is the coordinate along the edge perimeter, we can write the energy of the system (measured from that of the ground-state configuration depicted in Fig. 1) as a functional $E[\rho_i(x), \rho_o(x)]$. In the long-wave-length limit, this functional can be expanded,

$$E[\rho_i(x), \rho_o(x)] = \sum_{\alpha,\beta \in \{i,o\}} \int \int dx \, dx' \, \rho_{\alpha}(x) \, \rho_{\beta}(x') \frac{\delta^2 E}{\delta \rho_{\alpha} \delta \rho_{\beta}}(x,x') + \ldots ,$$

yielding quadratic terms to lowest order in density fluctuations. Note that the presence of edge excitations leads to non-neutralized charges on a length scale given by the wave length of the edge-density fluctuation. As we consider the case where holes interact via unscreened Coulomb interactions, the expansion coefficients in Eq. (6) diverge as the wave length of the edge-density fluctuations increases, and become approximately equal at the longest wave lengths which are of the order of the edge perimeter $L$. In that limit, the edge-excitation normal modes are (i) the charged mode $\rho_c = \sqrt{1/\nu} (\rho_o + \rho_i)$ which corresponds to fluctuations in the total edge charge density, and (ii) the neutral mode $\rho_n = \sqrt{1(1-\nu)/\nu} (\rho_o + \rho_i/(1-\nu))$. The energy functional expressed in the normal modes reads

$$E[\rho_c(x), \rho_n(x)] \equiv E[\rho_c(x), \rho_n(x)] = \frac{1}{2} \int dx \, dx' \, V_c(x-x') \, \rho_c(x) \, \rho_c(x') + \pi \hbar \nu_n \int dx \, [\rho_n(x)]^2 .$$

The long range of Coulomb interactions enters only the dispersion of the charged mode, which is actually the EMP mode predicted by classical hydrodynamic theory. The neutral mode propagates in the direction opposite to the EMP mode; it has a linear dispersion with a velocity $v_n$ that depends on residual short-range interactions and is much smaller than the characteristic velocity $\sim e^2/\epsilon \hbar \times \ln(L/\ell)$ for EMP propagation. To actually determine $v_n$, a careful separation of long-range and short-range-interaction contributions to the expansion coefficients in Eq. (6) has to be performed. We were able to achieve such a separation by relating the energy of an excitation in the physical QH sample to that of a fictitious system where the background of positive charges has been adjusted to preserve 1D-local charge neutrality, i.e. the 2D charge-density in the fictitious system integrated perpendicular to the edge yields zero at any fixed location along the edge. (See Fig. 3.) Hence, the fictitious system looks neutral on length scales that are larger than the edge width, and effective interactions are short-ranged. The difference
in energy in the real and fictitious systems accounts for the long-range electrostatic interactions and can be calculated straightforwardly. Our approach to separate the long-range and short-range parts of the interaction was inspired by a similar treatment of long-range Coulomb interactions within Landau’s Fermi-liquid theory. As the central result of our calculations, we find the velocity of the counterpropagating neutral mode,

\[ \nu_n = \nu_J , \tag{8} \]

with \(\nu_J\) given by Eq. (5). We see that the velocity of the neutral mode is directly related to the stiffness of the edge against deviations from the ground-state value \(d\) of the edge separation. Stability of the two-branch edge and, therefore, consistency of the description based on particle-hole conjugation requires a finite value of \(\nu_n\). Note that the ratio of the characteristic EMP velocity to \(\nu_n\) is approximately given by \(\ln(L/\ell)\) which is \(\sim 10\) in typical samples.

4 Conclusion

We have studied the two-branch edge realized in a quantum-Hall sample that is at a filling factor \(\nu = 1 - 1/m\). A neutralizing background of positive charges was assumed to confine the electrons to the finite sample, and long-range Coulomb interactions have been taken into account. We find the separation \(d\) between the inner and outer edges in the ground state as well as the stiffness against deviations from \(d\). This stiffness can be parameterized in terms of a velocity \(\nu_J\) which turns out to be equal to the velocity of the neutral counterpropagating edge mode.

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