Thermodynamic pressure for massless QCD and the trace anomaly

H. Arthur Weldon

Department of Physics and Astronomy, West Virginia University, Morgantown, West Virginia, 26506-6315

(Dated: November 11, 2022)

From statistical mechanics the trace of the thermal average of any energy-momentum tensor is 
\[ \langle T^\mu_\mu \rangle = T^0 T / \partial T - 4 P. \]

The renormalization group formula 
\[ \langle T^\mu_\mu \rangle = \beta(g_M) \partial P / \partial \mu M \]

for QCD with massless fermions requires the pressure to have the structure

\[ P = T^4 \sum_{n=0}^{\infty} \phi_n(g_M) \left( \frac{g_M}{4\pi T} \right)^n, \]

where the factor 4\pi is for later convenience. The functions \( \phi_n(g_M) \) for \( n \geq 1 \) may be calculated from \( \phi_0(g_M) \) using the recursion relation

\[ n \phi_n(g_M) = -\beta(g_M) \phi_{n-1} / \partial g_M. \]

This is checked against known perturbation theory results by using the terms of order \( (g_M)^1, (g_M)^3, (g_M)^4 \) in \( \phi_0(g_M) \) to obtain the known terms of order \( (g_M)^1, (g_M)^3, (g_M)^6 \) in \( \phi_2(g_M) \). The above series may be summed and gives the same result as choosing \( M = 4\pi T \), viz. \( T^4 \phi_0(g_M T^3) \).

I. INTRODUCTION

For a symmetric energy-momentum tensor \( T^\mu_\nu \) the dilatation current \( S^\mu = T^{\mu\lambda\beta\alpha} x_\lambda x_\beta \) and the four conformal currents \( K^\alpha \) are conserved if the energy-momentum tensor is traceless:

\[ \partial_\mu S^\mu = T^\mu_\mu, \]

\[ \partial_\mu K^{\mu\lambda} = -2T^{\mu\lambda}. \]

The classical energy-momentum tensor for QCD with massless fermions is traceless but quantum corrections introduce a renormalization scale that spoils the conservation of scale and conformal currents and renders the trace nonzero.

The trace of the thermally averaged energy-momentum tensor is \( \langle T^\mu_\mu \rangle = u - 3P \) where \( u = \langle T^0_0 \rangle \) is the energy density and \( P = -\sum_{j=1}^3 \langle T^j_j \rangle / 3 \) is the pressure. The relation

\[ \exp(\beta P V) = Z = \text{Tr}\{e^{-\beta H}\} \]

between the pressure and the partition function implies that

\[ \frac{\partial}{\partial \beta} (\beta P) = -\frac{\langle H \rangle}{V} = -u, \]

or equivalently

\[ T \frac{\partial P}{\partial T} = u + P. \]

The trace of the energy-momentum tensor becomes

\[ \langle T^\mu_\mu \rangle = u - 3P = T \frac{\partial P}{\partial T} - 4P. \]

For non-Abelian gauge fields with massless fermions the pressure has the form

\[ P = T^4 \Phi(g_M, M/T), \]

where \( M \) is the renormalization scale. From (1) the trace of the energy-momentum tensor is

\[ \langle T^\mu_\mu \rangle = T^5 \frac{\partial \Phi}{\partial T}. \]

One would expect the calculation of \( \Phi \) to be primary and the trace anomaly only an afterthought. However with the theorem of Drummond, Horgan, Landshoff, and Rebhan that

\[ \langle T^\mu_\mu \rangle = \beta(g_M) \frac{\partial P}{\partial g_M}, \]

the anomaly becomes predictive in that the combination of (3) and (4) gives

\[ T \frac{\partial \Phi}{\partial T} = \beta(g_M) \frac{\partial \Phi}{\partial g_M}, \]

which is Eq. (3.11) of Drummond et al.

Note that (3) is similar to the zero temperature operator identity \( T^\mu_\mu = \beta(g_M) \partial \mathcal{L} / \partial g_M \).

Sec. II shows how Eq. (3) ensures that \( P \) is independent of the renormalization scale \( M \) and requires \( P \) to have the structure shown in the abstract. Sec. III tests the recursion relation using known results for \( \phi_0(g_M) \) from perturbation theory to calculate the three known terms in \( \phi_2(g_M) \) and the only known term of \( \phi_2(g_M) \) and illustrates how to improve perturbation theory.

II. STRUCTURE OF P

1. Independence of the renormalization scale \( M \)

As indicated in Eq (2) the renormalization scale appears in \( \Phi \) through \( g_M \) and through \( r = M/T \). The full
\( M \) derivative of \( \Phi \) is
\[
M \frac{d\Phi}{dM} = M \frac{d\Phi}{dM} \frac{\partial \Phi}{\partial g_M} \bigg|_r + M \frac{dr}{dM} \frac{\partial \Phi}{\partial T} \bigg|_{g_M}.
\] (6)

In the first term use \( M \frac{d\Phi}{dM} = \beta(g_M) \); in the second, \( M \frac{dr}{dM} = r \) and \( r \frac{\partial \Phi}{\partial r} = -T \frac{\partial \Phi}{\partial T} \) so that
\[
M \frac{d\Phi}{dM} = \beta(g_M) \frac{\partial \Phi}{\partial g_M} \bigg|_r - T \frac{\partial \Phi}{\partial T} \bigg|_{g_M} = 0
\] (7)
after using Eq (5).

Comment: One can reverse the argument and derive the anomaly relation (3) of Drummond et al. by starting with the assertion that \( P \) is a physical quantity and must therefore be independent of the renormalization scale.

2. Origin of \([\ln(M/T)]\)^n

Since \( \Phi(g_M, M/T) \) is independent of \( M \) it must be only a function of \( T/\Lambda_{QCD} \). It is convenient to consider \( \Phi \) as a function of \( u = \ln(M/\Lambda_{QCD}) \), where \( \xi \) is some constant
\[
\Phi(g_M, T/M) = \phi_0(\ln(T/\Lambda_{QCD})),
\] (8)
and to introduce variables
\[
\begin{align*}
 u &= \ln(M/\Lambda_{QCD}) \\
 v &= \ln(M/\xi T).
\end{align*}
\] (9)
The running coupling is a function of \( u \) determined by \( \beta(g_M) = d\phi_M/du; \Phi \) is a function of \( u - v \):
\[
\Phi(g_M, M/T) = \phi_0(u - v) = \sum_{n=0}^{\infty} (-1)^n \frac{d^n\phi_0(u)}{du^n} v^n.
\] (10)
after at Taylor series expansion. The definition
\[
\phi_n(g_M) = \frac{(-1)^n \frac{d^n\phi_0(g_M)}{du^n}}{n!}
\] (11)
allows the series to be written
\[
\Phi(g_M, M/T) = \sum_{n=0}^{\infty} \phi_n(g_M) \left[ \ln \left( \frac{M}{\xi T} \right) \right]^n.
\] (12)
The recursion relation \( n \phi_n(g_M) = -d\phi_{n-1}/du \), which follows from (11), may be expressed as
\[
\phi_n(g_M) = -\frac{1}{n} \beta(g_M) \frac{d\phi_{n-1}}{dg_M} \quad (n \geq 1).
\] (13)
One can confirm directly that the series (12) satisfies \( d\Phi/dM = 0 \).

Comment: If \( \xi \) is changed to \( \xi' \) then
\[
\ln \left( \frac{M}{\xi T} \right) = \ln \left( \frac{M}{\xi' T} \right) + \ln \left( \frac{\xi'}{\xi} \right).
\] (14)
The binomial theorem allow the series (12) to be expressed in terms of powers of \( \ln(M/\xi'T) \) with modified functions \( \phi'_n(g_M) \).

Comment: From \( u - 3P = T^3 \frac{\partial \Phi}{\partial T} \) it follows that the energy density and entropy density are
\[
\begin{align*}
 u &= T^4 [3 \Phi + T \frac{\partial \Phi}{\partial T}] \\
 s &= T^4 [4 \Phi + T \frac{\partial \Phi}{\partial T}].
\end{align*}
\] (15)

III. RESULTS FROM PERTURBATION THEORY

The \( O(g_M^3) \) term in \( P \) was calculated by Shuryak [3]; the \( O(g_M^4) \) term by Kapusta [4]; to this order there was no \( \ln(M/T) \). The \( O(g_M^4) \) term was calculated by Arnold and Zhai [5]; the \( O(g_M^5) \) by Zhai and Kastening [6]; in both cases \( \ln(M/T) \) appeared. The same result was obtained by Braaten and Nieto [7] using hard thermal loop resummation.

At \( O(g_M^5) \) nonperturbative magnetic screening effects arise [8–10]. Kajantie et al. [11] were able to calculate the \( O(g_M^5) \) perturbative terms and found both \( \ln(M/T) \) and \( \ln^2(M/T) \). A convenient reference that discusses all the results is Sec. 8.4 of Kapusta and Gale [12].

A. Checks against known results

For comparison with the published results from perturbation theory it is convenient to insert a prefactor in the the series expression for the pressure and choose \( \xi = 4\pi \):
\[
P = \frac{\pi^2 d_A}{9} T^4 \sum_{n=0}^{\infty} \phi_n(g_M) \left[ \ln \left( \frac{M}{4\pi T} \right) \right]^n,
\] (17)
where \( d_A \) is the dimension of the adjoint representation. With the order \( (g_M)^2, (g_M)^3, \) and \( (g_M)^4 \) terms of \( \phi_0(g_M) \) the recursion relation (13) gives the first three terms of \( \phi_1(g_M) \) and the first term of \( \phi_2(g_M) \). Using the notation \( \phi_n^{(k)}(g_M) \) for the \( O(g_M^k) \) term in \( \phi_n(g_M) \) the necessary inputs are
\[
\begin{align*}
\phi_0^{(2)}(g_M) &= -\left( \frac{g_M}{4\pi} \right)^2 (C_A + \frac{5}{2} S_F) \\
\phi_0^{(3)}(g_M) &= \left( \frac{g_M}{4\pi} \right)^3 (C_A + S_F)^{3/2} / 16 \sqrt{3} \\
\phi_0^{(4)}(g_M) &= \left( \frac{g_M}{4\pi} \right)^4 \left\{ 48 C_A (C_A + S_F) \ln W + R \right\},
\end{align*}
\]
where \( W = (g_M/2\pi)^3\sqrt{(C_A + S_F)/3} \) and
\[
R = C_A^2 R_1 + C_A S_F R_2 + S_F^2 R_3 + S_F R_4,
\] (18)
The coefficients \( R_j \) are given in [3, 12] in terms of Riemann zeta functions and the Euler constant. For later
comparison with \([11]\) it is convenient to employ the approximate numerical values:
\[
R_1 = 79.2626 \quad R_2 = 18.9212 \quad R_3 = -0.6914 \quad R_4 = 9.6145. \tag{19}
\]
The standard notation \([12]\) for SU(N) with \(n_f\) fermions in the fundamental representation is \(d_A = N^2 - 1, C_A = N, d_F = N n_f, S_F = n_f/2, S_{2F} = (N^2 - 1) n_f/4 N\). The first two terms in the beta function are
\[
\beta(g_M) = -\beta_0 g_M^3 - \beta_1 g_M^5 + \ldots
\]
\[
\beta_0 = \left( \frac{11}{3} C_A - \frac{4}{3} S_F \right)/(4\pi)^2
\]
\[
\beta_1 = \left( \frac{34}{3} C_A^2 - \frac{20}{3} C_A S_F - 4 S_{2F} \right)/(4\pi)^2. \tag{20}
\]
The predictions of the recursion relation \([13]\) are
\begin{align*}
A. \quad & \phi_1^{(4)}(g_M) = \beta_0 g_M^3 \frac{d}{dg_M} \phi_0^{(2)}(g_M) \\
B. \quad & \phi_1^{(5)}(g_M) = \beta_0 g_M^3 \frac{d}{dg_M} \phi_0^{(3)}(g_M) \\
C. \quad & \phi_1^{(6)}(g_M) = \beta_0 g_M^3 \frac{d}{dg_M} \phi_0^{(4)}(g_M) + \beta_1 g_M^5 \frac{d}{dg_M} \phi_0^{(2)}(g_M) \\
D. \quad & \phi_2^{(6)}(g_M) = \frac{1}{2} \beta_0 g_M^3 \frac{d}{dg_M} \phi_1^{(4)}(g_M). \tag{21}
\end{align*}

The result for A,
\[
\phi_1^{(4)}(g_M) = \left( \frac{g_M}{4\pi} \right)^4 \left\{ -C_A \frac{22}{3} - C_A S_F \frac{47}{3} + S_F^2 \frac{20}{3} \right\}, \tag{22}
\]
agrees with \([5, 6, 11]\).

The result for B,
\[
\phi_1^{(5)}(g_M) = \left( \frac{g_M}{4\pi} \right)^5 \left( \frac{C_A + S_F}{3} \right)^{1/2} \times \left( C_A^2 176 + C_A S_F 112 - S_F^2 64 \right). \tag{23}
\]
agrees with \([6, 7, 11]\).

The result for C is
\begin{align*}
\phi_1^{(6)}(g_M) &= 4 \left( \frac{g_M}{4\pi} \right)^6 \left\{ \left( \frac{11}{3} C_A - \frac{4}{3} S_F \right) R \\
&+ \left( C_A + \frac{5}{2} S_F \right) \left( -\frac{17}{3} C_A + \frac{10}{3} C_A S_F + S_{2F} \right) \\
&+ \left( \frac{11}{3} C_A - \frac{4}{3} S_F \right) C_A (C_A + S_F) (12 + 48 \ln W) \right\} \tag{24}
\end{align*}

To compare this with \([11]\) it is necessary to evaluate \([24]\) for SU(3):
\begin{align*}
\phi_1^{(6)}(g_M) &= 4 \left( \frac{g_M}{4\pi} \right)^6 \left\{ 432 (11 - \frac{2}{3} n_f) (1 + \frac{1}{6} n_f) \ln W \\
&+ 1035 + \frac{325}{4} n_f - \frac{49}{12} n_f^2 + \left( 11 - \frac{2}{3} n_f \right) R \right\}. \tag{25}
\end{align*}

Substituting the numerical values of \(R\) gives the final result
\[
\phi_1^{(6)}(g_M) = 4 \left( \frac{g_M}{4\pi} \right)^6 \left\{ 432 (11 - \frac{2}{3} n_f) (1 + \frac{1}{6} n_f) \ln W \\
&+ 8882 - 11.6186 n_f - 29.1767 n_f^2 + 0.1152 n_f^3 \right\}. \tag{26}
\]

In \([11]\) the \(O(g_M^6)\) results are expressed in terms of \((\alpha M/\pi)^3\) and \(\ln(M/2\pi T)\). When \([11]\) is reexpressed in terms of \((g_M/4\pi)^6\) and \(\ln(M/4\pi T)\) it agrees completely with Eq. \([26]\).

The final calculation D gives
\[
\phi_2^{(6)}(g_M) = - \left( \frac{g_M}{4\pi} \right)^6 4 \left( C_A + \frac{5}{2} S_F \right) \left( \frac{11}{3} C_A - \frac{4}{3} S_F \right)^2. \tag{27}
\]

For SU(3) with \(n_f\) multiplets of fermions
\[
\phi_2^{(6)}(g_M) = - \left( \frac{g_M}{4\pi} \right)^6 1452 (1 + \frac{5}{12} n_f) (1 - \frac{2}{33} n_f)^2, \tag{28}
\]
which is exactly the same as \([11]\).

### B. Improving perturbation theory

At order \((g_M)^6\) nonperturbative effects appear in \(\phi_0^{(6)}(g_M)\) but not in \(\phi_1^{(6)}(g_M)\) or \(\phi_2^{(6)}(g_M)\) calculated above. The argument of Linde \([8, 9, 12]\) shows that certain diagrams that appear to be of order \((g_M)^k\) with \(k > 6\) are so infrared sensitive that nonperturbative magnetic shielding will render them of order \((g_M)^6\). Thus \(\phi_0^{(6)}(g_M)\) receives contributions from diagrams with infinitely many loops. Nevertheless \(\phi_0(g_M)\) is still a series of the form
\[
\phi_0(g_M) = \sum_{k=0}^{\infty} \phi_0^{(k)}(g_M). \tag{29}
\]

The \(k = 1\) term vanishes; the \(k = 2\) term is the first to depend on \(g_M\). Because the beta function begins with \((g_M)^3\) the recursion relation \([13]\) implies that \(\phi_0^{(k)}(g_M)\) will generate terms of order \((g_M)^{3k}\) for the \(k\) loop contributions. The series \([17]\) for \(P\) may be considered a double series:
\[
P = \frac{\pi^2 d_A}{9} T^4 \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \phi_0^{(2n+k)}(g_M) \ln \left( \frac{M}{4\pi T} \right)^n. \tag{30}
\]

Perturbative calculations through order \((g_M)^5\) determine \(\phi_0^{(2n+k)}(g_M)\) for \(2n + k \leq 5\):
\[
P_{[n]}^{(k \leq 5)} = \frac{\pi^2 d_A}{9} T^4 \sum_{k=0}^{5} \sum_{n=0}^{\infty} \phi_0^{(2n+k)}(g_M) \ln \left( \frac{M}{4\pi T} \right)^n. \tag{31}
\]

The difference between \(P_{[n]}^{(k \leq 5)}\) and \(P_{[n]}^{(k \leq 4)}\) is not small \([6, 7, 13]\).
There is no need to terminate the sum over \( n \); one can easily compute the full sum
\[
P^{(k \leq 5)} = \frac{\pi^2 d_A T^4}{9} \sum_{k=0}^{5} \sum_{n=0}^{\infty} \phi_n^{(2n+k)}(g_M) \left[ \ln \left( \frac{M}{4\pi T} \right) \right]^n.
\]
(32)
The input is of the form
\[
\phi_0^{(k)}(g_M) = \left( \frac{g_M}{4\pi} \right)^k \left\{ A_k + B_k \ln \left\{ \frac{g_M}{2\pi} \sqrt{(C_A + S_F)/3} \right\} \right\},
\]
(33)
where \( A_1 = 0 \) and \( B_4 \) is the only nonzero \( B_k \) for \( k \leq 5 \). As before, define \( u = \ln(M/\Lambda_{QCD}) \). At large \( M \), one can use \((g_M)^2 = [\beta_0 u]^{-1}\) and the parametrization
\[
\phi_0^{(k)}(g_M) = \frac{1}{u^{k/2}} (a_k + b_k \ln u).
\]
(34)
The \( n \)'th order derivatives of \( \phi_0(g_M) \) required by Eq. (11) give
\[
\phi_n^{(2n+k)}(g_M) = \frac{1}{u^{k/2+n}} [a_k S_n - 2 \frac{dS_n}{dk} b_k + S_n b_k \ln u]
\]
(35)
\[S_n = \frac{\Gamma(n+k/2)}{n! \Gamma(k/2)}.
\]
With \( v = \ln(M/4\pi T) \) Eq. (10) requires the sum
\[
\sum_{n=0}^{\infty} S_n \left( \frac{v}{u} \right)^n.
\]
(36)
By the ratio test this sum converges for \( |v/u| < 1 \), which is satisfied provided \( M > \sqrt{4\pi T \Lambda_{QCD}} \) and \( 4\pi T > \Lambda_{QCD} \). The result is
\[
\sum_{n=0}^{\infty} S_n \left( \frac{v}{u} \right)^n = \left[ 1 - \frac{v}{u} \right]^{-k/2}
\]
(37)
Applying \( d/dk \) as required in (35) gives
\[
P^{(k \leq 5)} = \frac{\pi^2 d_A T^4}{9} \sum_{k=0}^{5} \left( \frac{g_4 T}{4\pi} \right)^k \left\{ A_k + B_k \ln \left[ \frac{g_4 T}{2\pi} \sqrt{(C_A + S_F)/3} \right] \right\};
\]
(39)
or more concisely
\[
P^{(k \leq 5)} = \frac{\pi^2 d_A T^4}{9} \sum_{k=0}^{5} \phi_0^{(k)}(g_M) \bigg|_{\mu = 4\pi T}.
\]
(40)
In short, convergence of the infinite sum on \( n \) in (30) is automatic; whether a finite number of \( \phi_0^{(k)}(g_M) \) in the series for \( P^{(k \leq 5)} \) for \( \phi_0(g_M) \) is a good approximation, i.e. whether perturbation theory is reliable, is an open question [13].

[1] S. Coleman and R. Jackiw, Why Dilation Generators Do Not Generate Dilations, Ann. Phys. 67, 552 (1971).
[2] I.T. Drummond, R.R. Horgan, P.V. Landshoff, and A. Rebhan, QCD Pressure and the trace anomaly, Phys. Lett. B 460, 197 (1999).
[3] E.V. Shuryak, “Quark-gluon plasma and hadronic production of leptons, photons, and psions,” Phys. Lett. 78B, 150 (1978).
[4] J.I. Kapusta, Quantum Chromodynamics at High Temperature, Nucl. Phys. B148, 461 (1979).
[5] P. Arnold and C. Zhai, Three-loop free energy for pure gauge QCD, Phys. Rev. D 50, 7603 (1994); Three-loop free energy for high-temperature QED and QCD with fermions, Phys. Rev. D 51, 1906 (1995).
[6] C. Zhai and B. Kastening, Free energy of hot gauge theories with fermions through \( g^6 \), Phys. Rev. D 52, 7232 (1995).
[7] E. Braaten and A. Nieto, On the Convergence of Perturbative QCD at High Temperature, Phys. Rev. Lett. 76, 1417 (1996); Free energy of QCD at high temperature, Phys. Rev. D 53, 3421 (1996).
[8] A.D. Linde, Infrared problems in the thermodynamics of the Yang-Mills gas, Phys. Lett. B 67, 289 (1980).
[9] D.J. Gross, R.D. Pisarski, and L.G. Yaffe, QCD and instantons at finite temperature, Rev. Mod. Phys. 53, 43 (1981).
[10] I.T. Drummond, R.R. Horgan, P.V. Landshoff, and A. Rebhan, Eliminating infrared divergences in the pressure, Phys. Lett. B 398, 326 (1997).
[11] K. Kajantie, M. Laine, K. Rummukainen, Y. Schröder, Pressure of hot QCD up to \( g^6 \ln(1/g) \), Phys. Rev. D 67, 105008 (2003).
[12] J.I. Kapusta and C. Gale, “Finite Temperature Field Theory Principles and Applications,” 2nd ed, Cambridge Univ. Press, Cambridge, UK, 2006.
[13] E. Braaten, Thermodynamics of Hot QCD, Nucl. Phys. A702, 13 (2002).