A Novel Non-perturbative Self-consistent and General Approximation Method in Quantum Theory*

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Abstract

A new method of approximation scheme with potential application to a general interacting quantum system is presented. The method is non-perturbative, self-consistent, systematically improvable and uniformly applicable for arbitrary strength of interaction. It thus overcomes the various limitations of the existing methods such as the perturbation theory, the variational method, the WKBJ method and other approximation schemes. The current method has been successfully applied to a variety of interacting systems including the anharmonic/ double-well oscillators (with quartic-, sextic- and octic couplings) and the scalar field theory with quartic-coupling in the symmetric phase. The method yields important insight into the structure and stability of the interacting-vacuum of the theory. The results are in good agreement with the exact predictions of supersymmetry wherever applicable. Possible further applications in the areas of quantum statistics, finite temperature field theory, condensed matter physics etc are also discussed.

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1. Introduction

It is now generally believed that quantum theory (QT) describes the observed features of the physical universe at the fundamental level, both at the microscopic- and the macro- scales, which span a vast range in distance, typically from $\sim 10^{-16}$ cm to light years and time scales, from $10^{-15}$ second to the age of the universe $\sim 10^{10}$ years. It is truly amazing that phenomena as diverse as the colour of the sky, the emission of light by heated bodies, stability of matter, the evolution of stars, super conductivity and other properties of matter at extreme conditions of temperatures, density and pressure etc, all yield to description by the QT.

However, inspite of the vastness of the range of applicability of QT, the exact, analytic predictions of the theory based upon the solution of the underlying fundamental dynamical equations are rather sparse. It is not difficult to count and classify the cases where exact analytic solutions are possible. In fact, it is currently believed that the analytic solvability of the dynamical equations of quantum theory is mainly based upon the factorisation property of the corresponding quantum-Hamiltonians. In that context, recent applications of super-symmetric quantum mechanics (SUSYQM) have resulted in considerable extension of the range and class of exactly solvable potentials in non-relativistic QT.

In view of the rather limited range of physical phenomena amenable to exact analytic solutions in QT, it has become inevitable to develop various approximation methods of solution. This aspect was realized soon after the discovery of QT. The history of approximation methods is, therefore, as old as the theory itself. Consequently, this has generated a vast amount of literature devoted to the various approximation schemes (AS).

In discussing the merits and limitations of the various AS in QT certain basic criteria are generally applied. These criteria may, for example, include the following:

(i) general applicability (ii) simplicity of implementation (iii) achieved accuracy (iv) systematic improvability (v) rapidity of convergence and adaptability for numerical computation (when the obtained results are not in closed form) etc. In spite of the vast amount of accumulated literature and the broad range of the existing AS, there is considerable current effort
in searching for further improved AS which would satisfy most of the criteria listed above.

The present thesis records original research aimed at achieving a novel method of approximation in QT, which is non-perturbative, self-consistent, systematically improvable and simple to implement. More importantly, it has the potential to be applicable, in principle, to any general and arbitrary interacting system including quantum field theory and quantum statistics.

In order to further motivate and understand the scope of the present scheme, it is necessary to review and survey the various existing approximation schemes (AS) in QT and discuss their relative merits and limitations as against the set of desirable criteria listed above. However, such a task is understandably rather arduous due to the vast amount of accumulated literature. Moreover, several popular schemes of approximation already find mention in texts on QT. We shall, therefore, confine ourselves to the main AS’s and shall be rather brief. In the following sections a survey along the stated lines is presented for the following main categories of the AS in QT:

1. *Methods based upon Perturbation theory*
2. *The Variation Method of Approximation*
3. *The Hill-determinant and related methods*
4. *Combination of variation method and perturbation techniques*
5. *The Algebraic Method of Operator Expansion*
6. *The JWKB approximation method*
7. *The Method of Approximation employing the Canonical Transformation in QT*
8. *The Approximation methods based upon super symmetric quantum mechanics*
9. *The methods based upon the re-summation techniques*
(10) The Approximation schemes based upon the path-integral formulation of QT

(11) The self-consistent schemes of approximation

(12) Other methods of approximation

In the next Chapter, we make a brief survey in dealing with each scheme of approximation and highlight the main features of each with special emphasis on its merit and limitation. Each method is also applied to the case of the quantum an-harmonic oscillator (QAHO) taken as the reference standard. This quantum system is chosen for the above purpose since it is also the traditional choice as the theoretical laboratory for testing various AS in QT. Consequently, the QAHO is perhaps among the most widely studied quantum system, see, e.g., refs. [8,9].

2. Survey of Approximation Schemes in Quantum Theory

In the following sub-sections we present a brief survey of the different main schemes of approximation in QT as listed in the previous Chapter.

(1) Methods based upon Perturbation-Theory

This scheme is among the earliest approximation methods recorded in the literature. In the context of classical physics, the method was applied by Lord-Rayleigh as early as 1873. In QT the application of the method is as old as the theory itself, see, e.g. ref.[7]. Since the salient features of the method are well discussed in texts (e.g.[3]) only brief discussion follows. For simplicity, we restrict to the case of non-degenerate and discrete spectrum.

The generic central feature of the perturbation method consists in addressing to those problems where the system Hamiltonian can be split in the
following fashion:

\[ H = H_s + \lambda H_I, \]  

(1)

where \( H_s \) corresponds to an exactly solvable Hamiltonian whereas the other part, \( \lambda H_I \) can be regarded as a perturbation-interaction term added to the dominant Hamiltonian represented by \( H_s \), the implicit assumption being that the perturbation term is “small” in some-sense (to be specified below). In the above equation, the parameter ‘\( \lambda \)’ generically corresponds to an expansion-parameter. This may be either naturally present in the interaction term or explicitly introduced by hand to keep a count on the order of successive approximation involving powers of the perturbation-term \( H_I \). In the latter case, it is set to unity, \( \lambda \rightarrow 1 \), at the end of the computation to the desired order of approximation.

In order that the above scheme works, at least the following obvious requirements must be met:

(a) the system Hamiltonian permits a split in the above manner as in eqn.(1).
(b) the perturbation-interaction term \( \lambda H_I \) remains sub-dominant to the solvable unperturbed part, \( H_s \). In view that the Hamiltonian is an operator, the above statement requires precise quantification. A commonly accepted criteria, which ensures the condition of sub-dominance of the perturbation term is the following\(^3\):

\[ | < n_s | H_s | n_s > | \geq | < n_s | \lambda H_I | n_s > |, \]  

(2)

where \( | n_s > \) is defined by the eigen value equation,

\[ H_s | n_s > = E_n^s | n_s > \]  

(3)

(c) The method yields meaningful results if the actual physical spectrum can be computed as a power-series, i.e.

\[ E_n = E_n^s + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + ... , \]  

(4)

and

\[ | n > = | n_s > + \lambda | n^{(1)} > + \lambda^2 | n^{(2)} > + ... , \]  

(5)

(d) It is further required that the above sequences represent either convergent or asymptotic series in order that the eqs.(4) and (5) make sense.
as an approximation if truncation of the same after a \textit{finite} number of terms would yield results representative of the actual physical situation.

(e) An important corollary of the above requirement is, therefore, that the \textit{rate of convergence} of the series (or, the rate of decrease of the sub-asymptotic sequence, as the case may be) are important considerations bearing upon the \textit{practical} application of the method.

The series-development in eqs.(4) and (5) can be generated by iteration starting from the Schrödinger equation for the states $|n\rangle$ and $|n_s\rangle$. Following standard method, one obtains the following general expression valid for arbitrary order:

$$|n\rangle = |n_s\rangle + Q_n (H_s - z)^{-1}(E_n - \lambda H_I - z)|n\rangle,$$

After iteration, eqn.(2.6) yields the following result:

$$|n\rangle = \sum_m [Q_n (H_s - z)^{-1}(E_n - \lambda H_I - z)]^m |n_s\rangle. \quad (7)$$

Similarly, for the energy shift one obtains:

$$E_n = E_s^n + \sum_m <n_s|\lambda H_I [Q_n (H_s - z)^{-1}(E_n - \lambda H_I - z)]^m |n_s\rangle,$$

where, $z$ = arbitrary complex parameter; $Q_n \equiv 1 - |n_s\rangle <n_s|$, is a projection operator with the property of projection to states orthogonal to $|n_s\rangle$, i.e. $<n_s|Q_n |\psi> = 0$, for any arbitrary representative state $|\psi>$. The role of the arbitrary parameter ‘z’ is to generate by iteration, the two popular versions of perturbation expansion, namely the Rayleigh-Schrödinger(RS) and the Brillouin-Wigner (BW) versions for different choice of z. Thus, for $z = E_s^n$ the RS-series is obtained while for $z = E_n$, the BW-series follows. More explicitly, the separate expressions are given below for the two schemes:

\textbf{RS-Scheme}

$$|n\rangle = \sum_m [Q_n (H_s - E_s^n)^{-1}(E_n - \lambda H_I - E_s^n)]^m |n_s\rangle,$$

Similarly, for the energy-eigen value one obtains

$$E_n = E_s^n + \sum_m <n_s|\lambda H_I [Q_n (H_s - E_s^n)^{-1}(E_n - \lambda H_I - E_s^n)]^m |n_s\rangle,$$

(10)
Similar expressions in the BW scheme are given below.

**BW-Scheme**

\[ | n > = \sum_m \left[ Q_n (H_s - E_n)^{-1}(-\lambda H_I) \right]^m | n_s > \]  \hspace{1cm} (11)

\[ E_n = E_n^s + \sum_m < n_s | \lambda H_I [ Q_n (H_s - E_n)^{-1}(-\lambda H_I) ]^m | n_s > . \]  \hspace{1cm} (12)

It may be remarked here that the above (formal) expressions, which are valid to arbitrary order, become useful only when approximated to any given finite order. A comparison of the two schemes reveals that although the BW-expressions look simpler, yet it involves the unknown, exact energy of the system in the denominator, which thus requires further expansion up to the desired order in \( \lambda \). This accounts for the popularity of the RS-scheme. On the other hand, however, there are systems for which, the BW-scheme is preferred as it converges more rapidly than the RS-scheme.

**Merits and limitations of perturbation theory**

Decidedly, the main merit of the method is the built-in provision for systematic, order-by-order improvement over the zeroth-order (unperturbed) result. However, in its naïve formulation as elaborated above (which, we denote as naïve perturbation theory (NPT) in the following) the applicability gets severely restricted due to several reasons. We mention some of these in the following.

The main limitation arises when eqn.(2) fails to hold for the system. This may happen due to several reasons. Firstly, this may mean that the range of values of \( \lambda \) gets restricted (often, the condition: \( \lambda < 1 \) ) provides a necessary but not sufficient condition. Secondly, the nature of the problem may be such that the inequality (eqn.(2)) fails (even if \( \lambda << 1 \)). A third reason leading to the inapplicability of the NPT may arise when the desired condition, eqn.(2) does not hold for arbitrary value of the excitation label ‘n’.

Even when the basic requirement stated in eqn.(2) is satisfied, the applicability of the method may be restricted on the grounds of convergence of the
NPT. This may happen, for example, when the physical quantity sought to be calculated by perturbation series, is independently found to be non-analytic at $\lambda = 0$. There is a large variety of physical phenomena, which fall under this category, e.g. tunneling, decay, phase-transitions, critical-phenomena, collective and cooperative phenomena such as super-conductivity, super-fluidity etc. It may be pointed-out in this context, that in most of the cases listed above, the perturbative ground state (i.e., the ground state of the unperturbed Hamiltonian, $H_s$) is shown to be unstable. In addition to the above limitations of the NPT on theoretical grounds, there may be practical difficulties arising, for example, from computing corrections beyond the first-order, which generally involve the (infinite) sum over the intermediate states. This problem has been addressed by several authors and it has been found in specific examples that the difficulty can be surmounted by appealing to special techniques, such as the Dalgrano-Lewis method$^{11}$.

Some of the above aspects are exemplified when one considers the application of the NPT to the problem of the QAHO as discussed below.

The QAHO problem and the NPT

The Hamiltonian for the QAHO (in one space-dimension) is given by the following expression:

$$H = \frac{1}{2} p^2 + \frac{1}{2} g x^2 + \lambda x^{2k} ; k = 2, 3, 4, .......$$  \hspace{1cm} (13)

In the above equation, $p = -i \frac{\partial}{\partial x}$ and $g, \lambda$ are real, positive parameters. (The units are chosen such that $m = 1$ and $\hbar = 1$).

The NPT can be developed by recognizing that for $\lambda = 0$, the system Hamiltonian reduces to that of the simple harmonic oscillator and is exactly solvable. Hence, it is natural to choose the anharmonic term as the perturbation-correction and express the Hamiltonian in the form given by equation (1), where $\lambda H_I \equiv \lambda x^{2k}$ and $H_s \equiv \frac{1}{2} p^2 + \frac{1}{2} g x^2$.

In order to apply the NPT in the above formulation, it becomes apparent that the coupling strength be restricted to small values: $\lambda < 1$. However, it was discovered that the NPT-series fails to converge, even if the above
restriction is imposed. Specifically, it was demonstrated by the pioneering work by Bender and Wu $^{12}$ who studied the large-order behaviour of the NPT-series for the quartic-AHO and came to the conclusion that the co-efficient of $\lambda^n$ grows as $n!$ for any value of $\lambda$ even when $\lambda \to 0$. It was subsequently established $^{13}$ that the factorial-growth of the series worsened further for the cases of higher-anharmonicity. Even prior to the work under refs.$^{[12,13]}$, it was shown by Lam $^{14}$ that the development of the perturbation theory as a power-series in $\lambda$ has vanishing radius of convergence due to the occurrence of an essential singularity at $\lambda = 0$.

Since then, a lot of investigation into the large-order behaviour of the NPT for the QAHO have been carried out and the results have found mention in texts$^{15}$. From these investigations it has become clear that the divergence of the NPT could be ultimately traced to the failure of the essential condition expressed in eqn.(2) due to the eventual dominance of the perturbation-term $\lambda x^{2k}$ over the unperturbed part for large amplitude of oscillation.

In this context, it has now come to be recognized that the problem of divergence of the NPT may not be restricted to the case of the QAHO alone. For example, in the case of quantum electrodynamics (QED), the possibility of failure of the NPT was conjectured by Dyson $^{16}$ much earlier. This has led to the conjecture that the problem of divergence of the NPT could be generic and the NPT-series may at most be an asymptotic series. A crude estimate of the optimal number, $n_0$ of terms in the NPT-series beyond which, the subsequent terms start diverging could be obtained from the criteria:

$$\lambda^{n_0} R_{n_0} \approx O(1),$$

where, it is assumed that the NPT-series (for the observable, say, energy $E$) is represented by

$$E = \sum_n \lambda^n R_n,$$

where, $\lambda$ is the coupling strength. Then, assuming factorial growth, $R_n \approx n!$ for large $n$, and using the Stirling-approximation, one obtains from eqn.(14), the desired estimate:

$$n_0 \approx O\left(\frac{1}{\lambda}\right).$$

It is clear, therefore, that the relevant coupling-strength must be sufficiently small in order that the NPT-series in eqn.(15) could make sense even as an
asymptotic, divergent series! We elaborate on these aspects further in the following while discussing *summability* methods. (It may be noted, in the context, that eqn.(16) may provide the rationale why the QED-perturbation series can still be trusted as an asymptotic series since the former starts diverging only after $\sim 137$ terms since $\lambda_{QED} \approx 1/137$).

It may be relevant to note also the following aspects while discussing application of the NPT to the QAHO-problem:

(i) For $g = 0$ (see, eqn.(13)), the an-harmonic term: $\lambda x^{2k}$ can not be treated as a perturbation to the remaining part of the Hamiltonian, which is the kinetic-energy term (KET). This is because of the totally different nature of the resultant spectrum- the KET generates a *continuous* spectrum whereas the addition of the an-harmonic term changes this to a discrete one.

(ii) Arguments based upon scaling and dimensional analysis\(^1\) can be used for the QAHO-problem to show that the latter is essentially a single parametric problem even though there are two couplings : $g$ and $\lambda$ occurring in the Hamiltonian. Thus, for the *quartic*-AHO, the following scaling law holds\(^1\):

$$E (g, \lambda) = \lambda^{1/2} E (g \lambda^{-2/3}, 1) \, ,$$

$$\psi (x; g, \lambda) = \psi (x \lambda^{1/6}; g \lambda^{-2/3}, 1) \, ,$$

where $E =$ energy and $\psi =$ wave function of the system.

(iii) The above scaling-property together with re-parameterization techniques can be effectively used\(^2\) to develop perturbation theory for *strong coupling* regime, i.e., for $\lambda >> 1$ since the effective single-coupling for the problem becomes $g \lambda^{-2/3}$ (see, eqns.,17-18). This is useful since the strong-coupling perturbation theory may be used as a complement to the usual small-coupling RS-series in studying smooth transition across $\lambda = 1$.

(iv) It may also be relevant to note that recently a novel-perturbation theory has been proposed by Turbiner\(^1\) where the Schrödinger-equation is first converted to the Riccati-equation in the variable $y(x) \equiv \frac{-d \ln \psi(x)}{dx}$, (where $\psi(x) =$ wave-function) and then developing a perturbative expansion for $y(x)$ through a recursive and iterative procedure. The resultant series for the energy and the wave function are characteristically different from those
obtained in the usual NPT and are claimed to be rapidly convergent.

We close this sub-section noting that the difficulties encountered with the NPT as applied to the QAHO-problem have led to several alternative approaches, as listed earlier which are discussed in subsequent sections. We next turn our attention to the method based upon variational-approximation.

(2) The Variational-Method of Approximation (VMA)

The VMA is a powerful method in obtaining the approximate eigen-values and eigen-states of observables in QT. Like perturbation theory (PT), the application of the method has a long history- initial application of the method being ascribed to Lord Rayleigh\(^{19}\) and to W.Ritz\(^{20}\). The VMA becomes especially suitable when perturbation theory (PT) fails or becomes inapplicable to the considered problem. VMA is also used to test results based upon PT and for analysis of stability of the system considered.

The essential ingredient of the method is based upon the variational theorem (Theorem-I), which provides an upper bound for the ground state energy of the system:

\[
\frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} \equiv E[\psi] \geq E_0 ,
\]

where, \( |\psi\rangle \) is any arbitrary, normalisable state; \( H \) is the Hamiltonian, and \( E_0 \) is the ground-state energy of the system. (In the above equation, we have used the standard Dirac-notation for expectation values and also denoted by \( E[\psi] \), the energy-functional). The proof of the theorem is based upon straightforward application of eigen-function expansion method in QT and can be found in any standard texts\(^3\).

A second theorem (Theorem-II) found useful applications, states that the state \( |\psi\rangle \), which renders the energy-functional \( E[\psi] \) stationary i.e. \( \delta E[\psi] = 0 \), satisfies the stationary Schrödinger equation:

\[
H |\psi\rangle = E |\psi\rangle.
\]

Again the proof can be found in standard texts\(^3\) and is therefore, omitted.

For practical application to obtain a close approximation, one follows the Ritz-method\(^{20}\), which consists in specifying the \textit{trial} -state \( |\psi\rangle \) as a
function of one or more free-parameters \( \{ \alpha_i \} \) and then minimising the energy functional \( E[\psi] \) with respect to these parameters:

\[
\partial E[\psi]/\partial \alpha_i = 0 ; \quad \partial^2 E[\psi]/\partial \alpha_i^2 > 0.
\]  

(20)

Thus, the Ritz-method enables one to obtain a least-upper bound (LUB), for a given choice of the trial-state, which is then the closest approximation to the ground-state energy for that choice. Some other aspects of the VMA are discussed below:

(a) The equality sign in eqn.(19) holds if (and only if) the trial-state, \( | \psi > \) coincides with the true ground state, \( | \psi_0 > \) of the system. (b) An error of 1st order, i.e., \( \sim O(\epsilon) \) in the choice of the trial-state gives rise to an error of 2nd order, i.e., \( \sim O(\epsilon^2) \) in the computed value of the energy, which means that the energy is more accurately determined by the VMA than the wave function and consequently, the over-all accuracy of the method crucially depends on the ingenuity and insight in the choice of the trial-state, consistent with the physical boundary conditions for the system. (c) The generalization of the Rayleigh-Ritz method of the VMA to obtain estimates/approximation for the higher excited states can be achieved by several methods. Some of these methods are outlined below:

(i) A tower of trial states, \( | \psi_\alpha > \); \( \alpha = 1, 2, 3, \ldots \), approximating the higher excited states of the system may be constructed with the requirement that these are mutually orthogonal to each other, as well as, orthogonal to the variational-ground-state. The construction can be achieved by any suitable method of orthogonalisation, such as the Schmidt-method\(^{21}\). Denote by \( E_\alpha \), the energy-functional for the \( \alpha \)-th state minimized with respect to its free-parameters and then arranged as an decreasing sequence, i.e.,

\[
E_\alpha \equiv \left[ \frac{< \psi_\alpha | H | \psi_\alpha >}{< \psi_\alpha | \psi_\alpha >} \right]_{\text{min}} ; \quad \alpha = 1, 2, 3, \ldots
\]  

(21)

The resulting sequence, \( E < E_1 < E_2 < E_3 \ldots \), then represents the variational approximation to the energy of the higher excited states in the order shown.

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The other standard method to deal with the excited states within VMA is known as the “method of linear variational approximation (LVA)”. This consists of the choice of the trial-state as a linear superposition of a suitably chosen set of eigen-functions consistent with the boundary conditions and otherwise appropriate for the system, as follows:

\[ | \psi \rangle = \sum_n c_n | u_n \rangle ; \quad \sum_n |u_n|^2 = 1 ; \quad < u_n | n_m \rangle = \delta_{nm}. \quad (22) \]

The energy functional as given in eqn.(19) computed with this trial-state leads to the following set of equations after minimization with respect to the coefficients \( c_n \) (see, eqn.(22)):

\[ \sum_n c_n^* (H_{nm} - E\delta_{nm}) = 0 ; \quad m = 1, 2, 3, \ldots. \quad (23) \]

Condition for existence of non-trivial solutions of the above set of equation, is given by the vanishing of the secular determinant:

\[ \det (H_{nm} - E\delta_{nm}) = 0 \quad (24) \]

For practical purpose, the determinant has to be truncated at some finite order ‘\( k \)’. The solution of eqn.(24) then provides, in general, \( k \)-roots for the energy \( E \), which then correspond to the approximate energies of the first \( k - \text{excited states} \). It is obvious that the above method becomes efficient in practice when the truncation-error (due to the chosen finite value of \( k \)) becomes small and further that the spectrum stabilizes (with increasing value of \( k \)). Clearly, therefore, intelligent choice of the basis-states, \( | u_n \rangle \) becomes crucial in the above method.

For certain problems, a variant of the above formalism is often called for when it becomes more convenient to work with a non-orthogonal basis set. In that case, eqn.(24) is replaced by the following equation:

\[ \det (H_{nm} - E S_{nm}) = 0, \quad (25) \]

where, now \( S_{nm} \equiv < u_n | u_m > \), is the overlap function.

The above basic formulation of VMA can be improved in several ways some
of which, are described in the following.

(a) Improved choice of the trial-states

This obvious generalization that has to be tried first for a given problem requires the choice of more realistic trial-states, which often involve large number of variation-parameters. These lead in general, to results that are more accurate. The main limitation of this approach is related to the fact that there is no systematic and standard method to follow. In the context of the QAHO, several trial-states have been tried with varying degree of success. A sample list is given under ref.[23].

(b) Obtaining lower bounds on the spectrum

The idea in this approach is to obtain lower bounds to the energy eigenvalues in addition to the upper bound provided by the standard VMA. Carrying out the optimization of both the lower- and upper- bounds then leads more accurately to the actual eigen-value. Lower bounds are obtained by employing standard inequalities and other techniques. In the context of the QAHO some important work, in this direction, are listed in ref.[24].

(c) The Gaussian approximation

This method employs the choice of a Gaussian as the trial state for a general problem. This approach relies upon the general applicability of the theory of small-oscillations to any arbitrary potential/interaction possessing a stable-minimum. The variation parameters are usually chosen to be the ‘width’ and the location of the peak of Gaussian trial-state. For example, in one space-dimension, the trial wave-function for the ground state could be of the generic form, \( \psi(x) = A \exp\left[\frac{-(x-x_0)^2}{\sigma^2}\right] \), where ‘\( x_0 \)’ and ‘\( \sigma \)’ are the free-parameters and ‘\( A \)’ is a normalization constant. This method has a long-history mainly because it has been ‘rediscovered’ and ‘refined’ time-and-again. It has been applied to a variety of problems in QT including quantum field theory. We refer to the work of Stevenson and collaborators\(^{25} \), which also provides guide to the earlier literature.

(d) The method of minimum energy variance (MEV)
In stead of the Hamiltonian, $H$ itself, upper bounds can be derived, in principle, for any arbitrary function $f(H)$ by the generalization of the standard Rayleigh-Ritz method. The generalization when the chosen function is considered as the variance of the Hamiltonian, has been termed\textsuperscript{26} accordingly as “method of minimum energy variance (MEV)”. The MEV has been shown\textsuperscript{26} to result in improved estimate of the trial-state as well as, the energy eigen-value. In particular, it has been demonstrated that the estimates go beyond the simple Gaussian approximation\textsuperscript{26}. 

In addition to the above, there are other methods, which combine the variation method with other approximation methods, e.g., perturbation theory, super symmetry, the Hill-determinant method etc. in order to improve the results from the VMA alone. Some of these latter-schemes are separately discussed in the following subsections.

( 3 ) The Hill-determinant and related methods

In this method, the expectation value of the Hamiltonian of the problem is computed in the trial state function, which is expanded in terms of of a suitably chosen set of basis functions (which need not be orthogonal) with unknown coefficients. The condition for the determination of these coefficients translates into the condition of vanishing of the resulting secular determinant (Hill-determinant). This latter equation is then solved numerically for the approximants of the Hill-determinant obtained by truncation. More specifically, the trial-state proposed by the authors\textsuperscript{27}, who pioneered this approach, is given as :

$$
\psi(x) = \exp\left(-\frac{x^2}{2}\right) \sum_n c_n x^{2n}.
$$

Substitution of this trial-state in the Schrödinger equation for the quartic-AHO then leads to the following three-term difference equation for the coefficients, $c_n$’s :

$$
2 \left( n + 1 \right) \left( 2n + 1 \right) c_{n-1} + (E - 1 - 4n) c_n - \lambda c_{n-2} = 0,
$$

(27)
where, $\lambda$ is the an-harmonic coupling strength and $E = \text{energy of the system.}$ (Alternatively, due to theorem-II mentioned earlier, the same result is obtained after minimisation of the energy functional, $E[\psi]$ with respect to the $c_n^*$ treating the latter as the variation parameters.) The condition of existence of solutions of the eqn.(27) then requires that the corresponding (infinite-dimensional) Hill-determinant $D$ of the problem vanishes, $D = 0.$ (The explicit form of $D$ is given in ref.[27]). The authors of ref.[27] numerically solve the truncated approximant, $D_n = 0$, by noting that these satisfy a three-term recursion relation:

$$D_n - (E - 1 - 4n) D_{n-1} + 16\lambda n(n-1/2)(n-1)(n-3/2)D_{n-3} = 0. \quad (28)$$

The solution, $D_n = 0$, then leads, after ensuring convergence and stability, to $n$-roots for the energy $E$, which corresponds to the energy of the first $n$-levels of the even-parity states, when rearranged as an increasing sequence. (The energy levels of states with odd-parity can similarly be obtained by substituting, $n \to n + 1/2$).

Obviously, therefore, stability, convergence, the computational time and effort needed, are the main criteria for the practical implementation of this method. The authors of ref.[27] obtained, by this method the eigen-values of the $\lambda x^{2k}$-anharmonic oscillators separately for $k = 2, 3$ and 4. The results compare well with the accurate numerical results obtained by other methods.

However, several authors$^{28,29}$ have later pointed out the limitation of the approach in ref.[27]. One main problem concerns the normalizability of the resultant eigen-functions. Another problem is the failure of the method to reproduce the so called “terminated (polynomial)solutions” (later known as the “quasi-exact solutions”$^{30}$). Still another criticism$^{28}$ of the original method$^{27}$ is the failure to reproduce correct spectral-properties when applied to a general an-harmonic oscillator having arbitrary couplings for the quadratic-, cubic- and quartic an-harmonic terms. Several modifications, involving improved choice of the variational trial-state, have been suggested$^{28,29}$. For example, the following substitution of the exponential convergence factor (see, eqn.(26)), has been proposed:

$$\exp \left( - \frac{x^2}{2} \right) \to \exp \left( - \alpha x^4 + \beta x^2 \right), \quad (29)$$

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where, the constants $\alpha, \beta$ are either treated as variation parameters or determined otherwise. Even though the above ‘modified’ Hill-determinant method removes certain limitations of the original approach\cite{27}, there still remain certain controversy\cite{31} regarding correct reproduction of the eigen-values and ‘moments’.

A related approach is the so-called “variational sturmain approximation”. The concept of the sturmain basis functions was introduced\cite{32,33} as a non-perturbative approximation scheme for solving the stationary Schrödinger equation in the context of several potentials considered in atomic- and molecular physics problems. This method involves, akin to the Hill-determinant method, the simultaneous solution of an infinite number of algebraic equation and, therefore, to the vanishing of the secular determinant as the necessary condition. For numerical solution it is necessary to solve the finite order approximant to the secular determinant and thereafter ensure convergence and stability in respect of chosen size of the latter. Recently, it has been shown\cite{34} that the basic sturmain approximation can be variationally improved to yield better results. The authors in ref.\cite{34} have obtained accurate results for the quartic-AHO and the exponential potential. For details, ref.\cite{34} can be consulted, which also provides guide to the earlier literature.

(4) Combination of Variational and Perturbation techniques

In order to retain the merits but to overcome the limitations separately of perturbation theory and variation methods, several approaches have been proposed, where both the methods have been used in a single formulation. Various authors have worked using this combined approach, including the following: Halliday and Suranyi\cite{35}, W.Caswell\cite{36}, J.Killingbeck\cite{37}, Hsue and Chern\cite{38}, Feynman and Kleinert\cite{39} and collaborators\cite{40}, Patnaik\cite{41} and Rath\cite{42}. The review of each individual work as mentioned above falls beyond the scope of the present thesis. However, it may be relevant here to note certain common features of the techniques used by these authors and the consequent achievements attained.

In most of the approaches listed above the basic Hamiltonian of the system is altered by the addition and subtraction of terms involving certain parameters. These parameters are apriori arbitrary but fixed later by the variational
minimisation of energy or by imposing other constraints, which simplify computation. Perturbation techniques can then be applied with redefinition of the unperturbed Hamiltonian and the perturbation correction. The resulting perturbation series is often shown\(^{35,36,41,42}\) to be convergent. Thus, the problem of convergence of the naive perturbation theory as well as the absence of a built-in mechanism of systematic improvement in the VMA, are overcome to a large extent in these hybrid method, which may be called variation-perturbation method (VPM). Apart from the above common underlying feature, each approach differs in detail, which may be found in the individual reference cited above.

(5) The operator Method of Approximation (OMA)

The OMA has been pioneered by Feranchuk and Komarov\(^43\). The application of the OMA to the QAHO-problem and to several other problems has been described in ref.[44], which also provides a guide to earlier works of the authors. The basic idea of this method is to formulate the problem in the Fock-space of operators, instead of working with the co-ordinates and momenta.

The relevant transformation is provided by the following relations:

\[
x = (a + a^\dagger)/\sqrt{2\omega}; \quad p = i (a^\dagger - a)/\sqrt{(2/\omega)}, \quad [a, a^\dagger] = 1 \quad (30)
\]

where, \(a, a^\dagger\) are the annihilation- and creation-operators respectively and \('\omega'\) is a positive real number having the significance of the frequency of the associated simple-harmonic oscillator. The operators satisfy the standard commutation relation as shown in eqn.(30) in order to be compatible with the equal-time commutation relation, \([x, p] = i\).

The development of the OMA then consists of the following steps:

(i) The Hamiltonian of the system is first re-expressed in terms of the Fock-space operators (by the help of eqn.(30)),

(ii) The unperturbed Hamiltonian, \(H_0\) is chosen by clubbing all the diagonal terms in \(H\) (i.e., those involving polynomials of the number operator \(a^\dagger a\)),

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(iii) The remaining off-diagonal terms are together chosen as the perturbation, \( V \),

(iv) A modified perturbation theory is then developed using the above choice of the unperturbed Hamiltonian and the perturbation term,

(v) Finally, the arbitrary parameter ‘\( \omega \)’ is fixed by variational minimization of the energy. The procedure is indicated as shown below:

\[
H(x, p) \rightarrow H(a, a^\dagger) = H_0(a^\dagger a) + V(a, a^\dagger) \quad (31)
\]

To illustrate the method consider the QAHO problem with the Hamiltonian given by

\[
H(x, p) = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \lambda x^4, \quad \lambda > 0. \quad (32)
\]

Then the OMA-unperturbed Hamiltonian \( H_0 \) and the perturbation term \( V \), as defined in eqn.(31) are given by the following expression:

\[
H_0 = (1 + 2N)(\omega^2 + 1)/4\omega + (3\lambda/4\omega^2)(1 + 2N + 2N^2), \quad (33)
\]

\[
V = -[(\omega^2-1)/4\omega](a\dagger a)^2 + (\lambda/4\omega^2)[2(a\dagger)^2(2N+3) + (2N+3)2(a^2) + a\dagger a^\dagger] \quad (34)
\]

In the above equations, \( N \) stands for the number operator, \( N = a\dagger a \). The energy eigen-value corresponding to \( H_0 \) is easily obtained:

\[
E_n^{(0)} = (n + 1/2)[(\omega^2 + 1)/2\omega] + (3\lambda/8\omega^2)(n^2 + n + 1/2) \quad (35)
\]

Minimization of the energy as given in eqn.(35) above then leads\(^4\) to the following equation, for the frequency ‘\( \omega \)’:

\[
\omega^3 - \omega - 6\lambda(1 + 2n + 2n^2)/(1 + 2n) = 0, \quad (36)
\]

which determines this parameter as a function of the excitation level ‘\( n \)’ and the strength of the anharmonic coupling \( \lambda \). Substitution of the solution of eqn.(36) in eqn.(35) then leads to the leading-order approximation of
the energy levels, which are found to be accurate within a few percent over the entire range of ‘n’ and λ. Furthermore, this leading order result can be systematically improved order-by-order by developing the RS-perturbation theory with the perturbation term chosen to be V as given in eqn.(34). Besides, this perturbation series has been argued to be convergent unlike the naïve perturbation theory. Similar results have been obtained for the cases of higher an-harmonicity, double-well oscillators and a variety of other systems.

The main limitation of the model appears to be the implementation of the basic ansatz defining the split given in eqn.(31), particularly for non-polynomial interactions, interaction representing analytic functions expressible in infinite series, problems in higher dimensions, quantum field theory etc. In such cases, additional assumptions/methods/skills have to be employed.

(6) The JWKB approximation method

This method is quite old and popular due to its model-independent nature-it can be applied to any arbitrary, smooth potential. The formalism has been dealt at length in texts. Therefore, it will not be repeated here. We note, however, the main features of the approximation scheme as applied to the discrete bound state problems in QT. The relevant central formula for the above purpose is the so-called JWKB quantization rule:

\[ \int_a^b k(x) \, dx = (n + \frac{1}{2}) \pi, \quad n = 0, 1, 2, 3, \ldots \]  

(37)

where, \( k(x) = \frac{2m}{\hbar^2} (E - V(x)) \); a,b are a set of adjacent “turning points” obtained by solving the equation, \( k(x) = 0 \) and other terms have standard meaning. For those cases where the above integral can be evaluated, eqn.(37) can be expressed as:

\[ f(E, \lambda, g...) = (n + \frac{1}{2}) \pi, \]  

(38)

where, the l.h.s. of eqn.(38) represents the value of the integral as a function of the energy (E) and coupling-strengths (\( \lambda, g,... \)) occurring in the potential V(x).
The determination of energy $E$ as a function of the other parameters $(n, \lambda, g...)$ therefore, requires the inversion of eqn.(38). This task is far from simple for most cases of interest since the inversion cannot be achieved in closed, analytic form except in a few known cases (which, also happen to be exactly solvable by standard methods). However, eqn.(38) can be inverted by numerical-methods to the desired order of accuracy. The complexity of the procedure increases for the cases, for which the WKBJ-integral cannot be obtained in closed form. Similarly, the cases involving multiple (closely spaced) turning points and cases requiring higher order of approximation in the WKBJ-series, (e.g. to deal with rapidly varying potentials) become increasingly difficult to implement. Nevertheless, several useful information about the energy levels can be obtained in limiting cases of the small-coupling regime, the strong-coupling and/or the large-$n$ limits. Application to the case of the QAHO for studying the large-order behaviour of the naïve perturbation theory was carried out in ref.[47]. The case of the DWO has been dealt in ref.[48]. The connection between multi-instanton solutions in the path-integral formalism, the JWKB method and large-order behaviour of perturbation theory has been discussed by several authors. We refer to Garg who discusses the issue and lists earlier references.

(7) Approximation methods based on quantum-canonical transformation (QCT)

This is a powerful non-perturbative method, which has been successfully applied in QT, particularly in many-body systems exhibiting collective- and co-operative phenomena, e.g. superconductivity and super-fluidity. The method had been primarily expounded by Bogoliubov and hence more familiarly known as the “Bogoliubov-Transformation”. These transformations connect the Hillbert-space of the interacting system to that of the interaction-free case, while preserving the canonical structure of the basic (equal-time) commutation rules. The method is particularly useful in variational studies where the ansatz for the interacting vacuum state (IVS) generated through QCT can be tested to dynamically establish the stable ground state of the system. The basic formalism of the method can be illustrated, for a Bosonic-system with single-degree of freedom, through the following example. Consider such a system, which is described, in absence of interaction by the free-field annihilation- and creation operators, $a, a^\dagger$. The dynamic ‘field’ $\phi$ and the canonical momentum $p$ are parameterized in terms of these oper-
ators as: \( \phi = \frac{a + a^\dagger}{\sqrt{2}} \) and \( p = i \frac{a^\dagger - a}{\sqrt{2}} \) and satisfy the standard equal-time commutation relation: \([\phi, p] = i\). In presence of interaction, it is natural to assume that the interaction-free operators, \( a, a^\dagger \) change to the interacting ones, \( b, b^\dagger \). Similarly, let the vacuum states for the corresponding cases be denoted as: \( |0\rangle \) and \( |\text{vac}\rangle \) respectively. By definition, these satisfy: \( a |0\rangle = 0 \), \( b |\text{vac}\rangle = 0 \). The canonical-commutation rules satisfied by both the set of operators are identical by physical requirement, i.e.,

\[
[ a, a^\dagger ] = [ b, b^\dagger ] = 1 \tag{39}
\]

The general Bogoliubov-transformation relating the two sets of operators is given by

\[
b = a \cosh(\alpha) - a^\dagger \sinh(\alpha),
\]
\[
b^\dagger = a^\dagger \cosh(\alpha) - a \sinh(\alpha), \tag{40}
\]

where, the parameter \( \alpha \) is real and a priori arbitrary but the same can be determined dynamically through variational minimization of energy and/or by other physical requirements. The corresponding unitary transformation, which relates the two vacua is given by,

\[
|\text{vac}\rangle = \exp \left[ \frac{1}{2} \tanh(\alpha) (a^\dagger a^\dagger - a a) \right] |0\rangle, \tag{41}
\]

such that the parameter, \( \alpha \) embodies the full effects of interaction: \( |\text{vac}\rangle \rightarrow |0\rangle \) as \( \alpha \rightarrow 0 \). This vacuum state representing the situation in presence of interaction can be used as a trial state for a given Hamiltonian, such as given by eqn.(31) in order to study the stability and to compute the ground state energy. The energy for the excited-states can then be determined by computing the expectation value: \( < n |H(a, a^\dagger)| n > \), where the state \( |n\rangle \) is given by the standard formula: \( |n\rangle = \left[(b^\dagger)^n/\sqrt{n!}\right]|\text{vac}\rangle \).

The formalism has been employed in refs.[53,54] for the QAHO and the DWO problems. In the context of \( \lambda \phi^4 \) quantum-field theory, QCT has been employed in refs.[55,56,57]. QCT also provides important in-sight into the
structure of the interacting vacuum state\textsuperscript{55,56,58,59}.

The limitations\textsuperscript{60} of the method appear to be those of variational methods as discussed earlier.

(8) \textit{Approximation methods based on super symmetric quantum mechanics}

The method of super symmetric quantum mechanics (SUSYQM) has been dealt in texts and several review articles. As a representative text\textsuperscript{6}, may be consulted, which provides guidance to earlier literature. As has been noted earlier, the property of \textit{shape-invariance} in SUSYQM\textsuperscript{6} has been successfully employed to considerably extend the class of exactly solvable potentials in QT. The method of SUSYQM can be profitably applied to improve and extend the scope of known approximation methods, e.g. perturbation theory, variational method and JWKB-approximation scheme. In the context of non-relativistic QT, several exact results follow in SUSYQM valid for \textit{partner potentials}\textsuperscript{6} such as the property of \textit{iso-spectrality level degeneracy and positivity}. These exact results provide a testing ground for approximation schemes applied to such potentials.

The SUSY-improved perturbation theory starts from an initial guess of the ground state wave function, which can be based, for example, on a realistic variational \textit{ansatz}. The \textit{super-potential} corresponding to this trial-ground state wave function can then be constructed by computing the logarithmic derivative of the latter. The unperturbed Hamiltonian is then chosen to be the one obtained from the super potential and the perturbation correction is taken to be the difference of the original Hamiltonian and the unperturbed one. The development of the RS-perturbation theory then becomes straightforward. For illustration of this method, ref.[6] can be consulted.

The SUSY-based JWKB approximation provides the following modification of the quantization-rule:

\[
\int_a^b \sqrt{2m \left[ E_n^{(1)} - W^2(x) \right]} \, dx = n\pi\hbar \ ; \ n = 0, 1, 2, 3, \ldots \quad (42)
\]

where, \( W \) is the \textit{super-potential} and \( E_n^{(1)} \) is the \( n \)-th energy-level of the \( H_1 \), which is one of the partner-Hamiltonians. Similar expression for the energy
levels of the other partner-Hamiltonian, $H_2$ is given by,

$$\int_a^b \sqrt{2m \left[ E_n^{(2)} - W^2(x) \right]} \, dx = (n + 1)\pi\hbar \, ; \, n = 0, 1, 2, 3, \ldots \quad (43)$$

It may be seen from the above equations that the exact result on the iso-spectrality / level-degeneracy of the partner potentials, is respected in the SUSY-JWKB quantization condition. Moreover, the formulae are valid for all values of ‘$n$’ rather than for large-$n$ as in the case of the conventional formulation. The SUSY-JWKB formula is demonstrated\textsuperscript{6} to yield the exact results for shape-invariant potentials and results with improved accuracy for other known cases.

In the context of the QAHO/DWO, SUSYQM- methods have been investigated by several authors. A sample list is provided in refs.[61-71].

(9) The methods based upon re-summation techniques

The generic problem of divergence of the naïve perturbation series (NPS) has led to the various re-summation techniques in order to extract some meaningful result out of the former. We will mention briefly the two popular re-summation techniques described in the literature, namely (i) the Borel-re-summation and (ii) the Padé-approximation in the following.

(i) The Borel re-summation method

The method of Borel re-summation\textsuperscript{72} is a well-studied technique\textsuperscript{73,74}, which permits, under suitable conditions, re-summation up to the optimal number of terms of an asymptotic power series exhibiting the growth at large-order ‘$n$’ as $n!$. Consider such a series :

$$F(g) = \sum_n f_n \, g^n \, , \quad (44)$$

where, $g$ is a real parameter ( ‘coupling-strength’ ) and $f_n \sim n!$, for large-$n$. Then, the associated Borel re-summed series is defined as

$$B(z) = \sum_n f_n \, z^n / n! \, , \quad (45)$$
This latter series is expected to have at least a finite radius of convergence by construction. The original series, (44) can be recovered, at least formally, from eqn.(44) as

\[ F(g) = \frac{1}{g} \int_0^\infty \exp\left(-\frac{z}{g}\right) B(z) \, dz , \quad (46) \]

by noting that \( \int_0^\infty \exp\left(-\frac{z}{g}\right) z^n \, dz = n! g^{n+1} \), provided that the integral on the r.h.s. of eqn.(46) can be uniquely evaluated. The integral representation, eqn.(46) is known as the Borel-transform and this represents, within the radius of convergence of the series for B(z), the sum of the asymptotic perturbation series, eqn.(44). In order to be useful, the detailed knowledge of the analytic structure of B(z) in the complex- \( z \) (Borel-)plane is necessary. If the singularities of B(z) lie off the positive real axis then the integral, eqn.(45) can be evaluated by counter-integral after appropriate distortion of the contour around those singularities.

However, if one (or more) singularity in the Borel-plane lies on the real-axis, then the evaluation by contour integral becomes ambiguous; the integral representation, eqn.(46) no longer exists and, hence the case is not Borel-summable. The other requirement is that the radius of convergence of B(z) must be larger than (or equal to) the magnitude of the coupling-strength, \( g \) in order that the re-summation is relevant.

There is another situation, which invalidates the re-summation. This occurs when the terms in the original series all have equal phase. In that case the Borel-transform, B(z) itself grows too fast at infinity. Consequently, the Borel-integral, eqn.(46) can not be evaluated. There may be other situations in addition to those already stated, when the singularity structure of B(z) prevents Borel re-summation. For the details, ref.[75] may be consulted.

Finally, it may be stressed that the method fails when the coupling becomes large, \( |g| \geq 1 \) because then the original perturbation series becomes invalid due to the divergence of the individual terms. Thus, the re-summation techniques are necessarily restricted to the small coupling regime.

Borel re-summability for the quartic-AHO has been proved in ref.[76].
The relation of the singularity in the Borel-plane to the corresponding Euclidean classical action was first established by Lipatov \(^7\). An early study of Borel re-summation method was made by Jaffe \(^8\). Use of conformal mapping to enhance the rate of convergence of the Borel-transform has been discussed in ref.[79]. Bounds on the optical sum of divergent series, in the context of QT, have been obtained in ref.[80] using a novel variational method for the Borel-transform. Borel-summability of interacting quantum field theory has been reviewed, for example, in ref.[81]. Some modern applications of re-summation techniques have been discussed in ref.[82,83]. Singularities ('Renormalons') in the coupling-plane due to the effects of renormalization of field theories and their significance have been discussed in ref.[84].

(ii) Padé-approximation

Re-summation of the original perturbation series, eqn.(44) via Padé approximation can be expressed as the ratio of two polynomials, \( P_l(z) / Q_m(z) \) of approximately same degree, as follows:

\[
[l/m]_F \equiv P_l(z)/Q_m(z) = (p_0+p_1z+p_2z^2+...+p_lz^l)/(1+q_1z+q_2z^2+...+q_mz^m)
\]  

The \( l + m + 1 \) coefficients, \( p_0, p_1, p_2, .......p_l; q_1, q_2....q_m \) are chosen in such a way that the Taylor-series expansion of the r.h.s. of eqn.(46) agrees with the original perturbation expansion up to the desired order for small \( |z| \), i.e.,

\[
F(z) - [ P_l(z)/Q_m(z) ] = O ( z^{l+m+1} ) , \quad z \to 0
\]  

The condition, eqn.(48) requires the solution of \( l+m+1 \) linear equations for the coefficients of the polynomials \( P_l(z) \) and \( Q_m(z) \). If this system of equations has a solution, the Padé-approximant can be expressed as a ratio of two determinants which depend upon the coefficients, \( f_0, f_1, f_2, ......f_{l+m+1} \) of the original perturbation series, eqn.(44). The generalization to multi-point Padé-approximation is straightforward. If the original asymptotic expansion can be made about a set of given points, \( z = z_\alpha ; \alpha = 0, 1, 2, .... \), then following the above procedure, the approximant to the original series can be developed for expansion about each point. Details may be found in
ref.\[85,86\] and references contained therein.

Earlier works on Padé approximation for the AHO are contained in ref.\[87,88\]. It has been found\[87,88\] that single-point Padé approximation works for the quartic- and sextic-AHO cases although the convergence is poor for the latter case. However, the same method fails\[87–89\] for the case of the octic-AHO. More recently, it has been shown in ref.\[90\], that clever use of scaling and transformation of the expansion variable results in accelerated convergence over a larger domain of \( z \), by the use of two-point Padé approximants for the case of AHO’s with quartic-, sextic- and octic an-harmonicity. It may be noted, however, that sophisticated numerical analysis using symbolic manipulation are required for obtaining results by the above method.

\textbf{(10) Approximation Schemes Based upon the Path-Integral Formulation of QT}

The Path-Integral (PI) formulation of Quantum Theory\[91–93\] has traditionally served as an important basis to formulate and improve upon approximation methods- it may be recalled that the Feynman-diagram technique, which is perhaps the most popular method of computation in perturbation theory, originated from the PI-approach\[94\]. In the context of quantum mechanics, a very comprehensive account of the PI-formulation is given in ref.\[9\], which also provides extensive reference to earlier work. The approximation methods can be formulated either in the real-time formulation\[91,92\] or as the imaginary time, Euclidean-formulation\[9\]. The latter is found to be ideally suited for extending the application to quantum-statistics.

Various approximation schemes including the Semi-classical approximation methods\[9,95\], Stationary-phase approximation\[9,96\], loop-wise expansion\[9\], large-order estimate of perturbation theory\[9,95\], variational estimation using inequalities\[9,96\], variational perturbation theory\[9\] and many more methods\[9\] have been profitably based upon the PI-formulation. Application to quantum field theory and statistical physics /critical phenomena are extensively treated in the text\[95\].

In the context of the QAHO-problem, the stationary-phase approximation was used\[97\] in the PI-formulation to infer the in-applicability of the naïve perturbation theory due to the existence of an essential singularity at the origin of the coupling strength plane. A variational method was proposed in
the Euclidean formalism of the PI in ref.[96] to obtain an approximation to
the partition function of the QAHO and thereby, to obtain an estimate of
the ground state energy. The method has been improved upon and extened
in application in ref.[9] under the name of ‘variational-perturbation theory
(VPT)’. Wide applications and convergence tests of VPT are described in de-
tail in ref.[9]. In ref.[98], variational lower-bound as well as, upper bound on
energy of the QAHO have been obtained by the use of inequilities for the par-
tition function. Semi-classical approximation of partition functions for one-
dimensional potentials using the PI-formalism has been described in ref.[99].
The same in higher dimensions is dealt in ref.[100]. The multi-instanton
effects for potentials with degenerate minima is described in ref.[101]. In
the context of scalar field theory, ref.[102] describes non-standard expansion
techniques for the generating functional in the PI formulation. Several other
applications in field theory and statistical physics can be found in ref.[95].

The major limitation of the PI-formulation of approximation method appears
to be the mathematical complexity involved in evaluating multi-dimensional
integrals - even the simple, exactly solvable potentials such as the hydrogen
atom problem, require rather complex techniques\(^9\) for implementation. In
many cases, it is therefore, preferable to use the standard operator based
formalism for implementing the approximation methods.

(11) The self-consistent schemes of approximation

These schemes are popular in the context of many-body systems and other-
wise known under the name of “mean-field approximation” and the “Hartree-
approximation” including various generalizations of the latter, (for example,
the “Hartree-Fock” method, “Hartree-Fock-Bogoliubov” method etc). The basic
concept of the scheme consists in approximating the given potential
by an exactly solvable one-body potential. If the original Hamiltonian is \(H\)
and the approximating Hamiltonian is \(H_0\), then perhaps the simplest way
to ensure self-consistency in the approximation could be imposition of the
following constraint:

\[
< n | H | n > = < n | H_0 | n > , \quad (49)
\]
where, the states \( |n\rangle \) are the eigen-states of \( H_0 \). In practice, the approximating Hamiltonian may be chosen with \textit{a-priori} unknown, adjustable parameters, which then get determined through the constraint, eqn.(49) and other requirements such as the variational principle and / or further simplifying conditions. The non-linear feedback that usually characterizes the self-consistency condition can then be easily ensured. Thus, the simple-looking equation, eqn.(49) possesses the potential to include interaction effects non-perturbatively and simultaneously preserve the non-linearity of the original Hamiltonian. The above method of implementation often yields results which are reasonably accurate even in the leading order of approximation, which can be further improved through standard recursive procedure as in the case of the Hartree-Scheme.

In the context of \( \lambda \phi^4 \) quantum field theory and the QAHO problem, such a scheme was considered in ref.[103] where, however, the expectation value (see,eqn.(49)) was restricted to the ground state only, in implementing the self-consistency requirement. The limitation has been overcome in proposing a “generalised Hartree-method” in ref.[58]. The ‘mean-field’ approach has also been tried in ref.[104].

The main limitations of the method appear to be the lack of \textit{uniqueness of the initial choice of the approximating Hamiltonian and the rate of convergence of the recursive procedure for subsequent improvement}.

(12) \textit{Other Methods of Approximation}

In addition to the methods listed above there are several other methods of approximation in QT. Since it is not possible to discuss all such methods within the limited scope of the present thesis, we list below only a small sample which, we believe, are worthy of mention- the choice is entirely personal.

(i) \textit{The \( \delta \)-expansion method}

This method\textsuperscript{105} is similar in many respects to the approaches of the Gaussian approximation\textsuperscript{25}, the method of Caswell\textsuperscript{36}, Killingbeck\textsuperscript{37}, Halliday and Suranyi\textsuperscript{35} and other methods discussed above under the category : \textit{Variation-perturbation methods’}. The basic idea is to split the Hamiltonian into a ‘free’ part and
an ‘interaction-part’, both involving one (or more) additional parameter, say Ω, which is not present in the original Hamiltonian. In addition, an expansion parameter, ‘δ’ is multiplied to the interaction part for the purpose of development of perturbation theory. After calculation to the desired order, this parameter is set to unity, δ → 1 at the end. Hence, the method is intrinsically non-perturbative although calculation can be done order-by-order in the artificial parameter, δ. The result of approximation to any finite order depends upon Ω whereas the sum to all orders must be independent of the same. The sensitivity of the result to this parameter at any given order is minimized by application of the “principle of minimum sensitivity (PMS)”\textsuperscript{106}. The convergence of the ‘perturbation theory’ through the δ-expansion method, has been rigorously established for the QAHO/QDWO problem\textsuperscript{107}. Application to quantum field theories have been described in ref.[108]. However, the method, in its original formulation, fails to reproduce the wave-function at any order due to the variational nature of the approximation implied by the PMS. This defect has been remedied in ref.[109] by generalization of the scheme to allow coordinate dependence of the Ω parameter and using the PMS criteria also for the wave-function.

(ii) The non-perturbative renormalization group (NPRG) method

The NPRG formulation uses the Wilson-effective action (WEA)\textsuperscript{110} as developed to study the critical phenomena, non-perturbative aspects in statistical mechanics and quantum field theory. The method has been applied to quantum mechanics by Aoki and collaborators\textsuperscript{111}. The basic equation in the formalism was derived by Wegner and Houghton (WH)\textsuperscript{112}. However, the WH-equation, which represents the exact cut-off dependence of the WEA, can not be analytically solved exactly. Hence some approximation has to be made. In the so-called “local potential approximation (LPA)”, quantum mechanical systems, including the AHO and the DWO, have been numerically investigated\textsuperscript{113} and shown to yield accurate results. Details can be found in ref.[113] and references to earlier related work contained therein.

(iii) The method of multiple scale perturbation theory (MSPT)

The above method has been propounded in ref.[114]. It is especially suitable to the cases where the ordinary perturbation theory diverges due to secular terms (terms that grow rapidly with the co-ordinate or the time variable)

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in the chosen perturbation. The AHO/DWO problem is an obvious example where the perturbation $\lambda x^4$ soon predominates over the harmonic term, $gx^2$ no matter how small $\lambda$ is. MSPT recognizes different characteristic properties of the system, such as the frequency of oscillation, different asymptotic behaviour of the wave function etc, at different time/length-scales when secular terms are present. In ref.[114], the MSPT techniques have been applied to solve the Heisenberg-equations of motion for the QAHO problem, obtain the energy spectrum and the eigen-states. It also provides references to related earlier work.

(iv) The coupled cluster expansion method (CCEM)

The CCEM is a known technique in many-body theory. In that area, the CCEM has been widely applied\textsuperscript{115} and extensively reviewed\textsuperscript{116}. The basic idea is due to Hubbard\textsuperscript{117} who proposed that the true interacting vacuum state of the system can be generated by an operator, $\exp(S)$ acting on the free, non-interacting vacuum state. In that respect, the above relation can be generated through appropriate canonical transformation, as discussed above. The method has been extended to quantum mechanics and quantum field theory in ref.[118]. In ref.[119], the technique has been applied to a general AHO with quadratic, cubic and quartic self-interaction such that various cases of QHO/DWO are covered. The starting point is a parameterization of the interacting vacuum, given by the relation:

$$|\text{vac}'\rangle = \exp ( sa^\dagger + ta^{\dagger 2} ) |0\rangle,$$  \hspace{1cm} (50)

where, $s$ and $t$ are real parameters and other terms have standard meaning as explained earlier. The creation/annihilation operators $b$, $b^\dagger$ for the interacting vacuum state $|\text{vac}'\rangle$ can be constructed by standard method\textsuperscript{119}. Using the above trial vacuum state and the tower of higher excited states generated there from, the usual variational calculation can be done\textsuperscript{119} for the AHO and it yields fairly accurate result as discussed earlier.

For the situation of degenerate ground state as happens in case of the DWO, the trial vacuum-state can be constructed as a linear superposition of the degenerate vacua and variational calculation can be made to calculate the energy split as well as the average energy of a given level. The CCEM method consists in systematically improving the above initial trial state through a
A series of linked cluster operators parameterized by the relation:

$$|\text{vac}\rangle = \exp\left(\sum_{k=l}^{\infty} S_k\right)|\text{vac}'\rangle,$$  \hspace{1cm} (51)

where, $$S_k = u_k(b^\dagger)^k.$$ Computation of the energy eigenvalues of quantum systems can be based upon the above trial state and higher excited states based upon it after suitable approximation to restrict the infinite sum in the argument of the exponential in eqn. (51). For details, ref.[119] can be consulted.

This completes our survey of the various approximation methods in QT.

In summary, it is revealed from the above brief yet hopefully representative survey that considerable progress has been achieved in the development of approximation schemes to tackle various problems in quantum theory. At the same time it is also brought to focus by the above survey that the ‘ideal’ method of approximation, based upon the earlier stated, desirable set of requirements of general applicability, simplicity, systematic improvability and efficiency of computation etc, remains largely elusive till date. In view of the above situation, there still remains a lot of scope to incorporate, if possible, the above set of desirable criteria into a single-scheme.

The present dissertation is devoted to an attempt at achieving the above ambitious objective, namely, to develop a self-consistent, non-perturbative, simple yet efficient scheme of approximation, which is systematically improv-able and generally applicable, in principle to an arbitrary interacting system in QT including the quantum field theory. The scope of the present dissertation is, however, limited in application to the self-interacting Bosonic systems in quantum theory in which, we include the quartic an-harmonic oscillator, quartic double-well potential, sextic an-harmonic and double-well oscillator, the octic-an-harmonic oscillator and the $\lambda \phi^4$ quantum field theory without spontaneous symmetry breaking. In the next Chapter, we describe the formulation$^{120}$.

3. The New General Approximation Scheme (NGAS)
3.1 Formulation of NGAS for self-interacting Systems

Consider a generic Hamiltonian $H_\lambda(\phi, p)$ describing a self-interacting quantum system involving the field $\phi$ and conjugate momentum $p$, given by

$$H_\lambda(\phi, p) = H_s(\phi, p) + \lambda H_I(\phi), \quad (52)$$

where, $H_s$ is the unperturbed Hamiltonian and $\lambda H_I$ is the self-interaction with $\lambda$ as the coupling strength. We use the language of field theory identifying quantum mechanics as field theory in $(0+1)$-dimensions.

Many physically important systems are described by the above Hamiltonian including the anharmonic oscillators (AHO); the double-well oscillators (DWO); the $\lambda \phi^4$ quantum field theory in the symmetric-phase as well as, in the spontaneously symmetry broken (SSB) phase; pure Yang-Mills fields with the quartic ‘gluon’-self interaction etc.

As has been discussed earlier, perturbation theory treating $H_s(\phi, p)$ as the unperturbed Hamiltonian and the entire interaction $\lambda H_I(\phi)$ as perturbation, fails to converge even for infinitesimal $\lambda$. Other non-perturbative methods often suffer from limitations as reviewed in Chapter-2. With a view to overcome these limitations and difficulties the formulation of NGAS consists of the following steps:

3.1.1 Choice of the approximating potential : $V(\phi)$

The aim is to replace the original interaction $\lambda H_I(\phi)$ by a “suitably chosen” approximating potential $\lambda V(\phi)$ such that it preserves the symmetries of the original interaction but leads to an “exactly solvable” Hamiltonian, i.e. the “Effective Hamiltonian (EH)” generated by $V(\phi)$ and defined by

$$H_0(\phi, p) = H_s(\phi, p) + \lambda V(\phi), \quad (53)$$

is exactly solvable, i.e.

$$H_0|n> = E_n^{(0)}|n>, \quad <m|n> = \delta_{mn}, \quad (54)$$

where, the spectrum $|n>$ and the eigen-values $E_n^{(0)}$, are known. (We consider, for simplicity, that the spectrum is discrete and non-degenerate.)
We refer to this requirement, eqn.(54), as the “condition of exact solvability” (CES). The next requirement on $V(\phi)$ is the condition of equal quantum average, as explained below.

### 3.1.2 The Principle of Equal Quantum Average

The effective Hamiltonian is constrained to yield the same quantum average (QA) as the original, i.e.

$$<n|H(\phi)|n> = <n|H_0(\phi)|n>,$$  \hspace{1cm} (55)

which implies,

$$<n|H_I(\phi)|n> = <n|V(\phi)|n>$$  \hspace{1cm} (56)

We refer to this requirement, eqs.(55-56) as the “condition of equal quantum average (CEQA)”. The next step is to optimize the approximation as described below:

### 3.1.3 Variational Optimisation of $V(\phi)$

Let $V(\phi)$ involve a set of free-parameters: $\{\alpha_i\}$. Then the requirement of optimisation consists in the variational minimisation of $H_0$ with respect to the free-parameters ($\alpha_i$) which characterize $V(\phi)$:

$$\frac{\partial}{\partial \alpha_i} <H_0>= 0$$  \hspace{1cm} (57)

where, the notation is

$$<\hat{A}> \equiv <n|\hat{A}|n>$$  \hspace{1cm} (58)

We refer to this condition, eqn.(57) as the “condition of optimality (CO)”.

### 3.1.4 The Self-Consistency and other features of the Method

The steps which are outlined above are the essential ingredients of the proposed approximation scheme in the leading order (LO). The following several observations regarding the approximation scheme are in order:
(i) It is to be noted that in a restricted form, i.e. when the quantum average in eqn.(57) is restricted to the ground state only, the CEQA, as expressed in eqn.(55), corresponds to the Hartree-approximation/ mean field approximation in quantum field theory\(^{103}\). In view of this, the NGAS can be regarded as a “generalised” Hartree-approximation method\(^{58}\).

(ii) The self-consistency of the procedure is implicit in eqs.(53-55): the states \(|n\rangle\) which are obtained as the solution of \(H_0\) (see, eqn.(54)), are used as input (in eqn.(56)) to determine \(V(\phi)\) which, in turn, defines \(H_0\) (eqn.53), thus making the “feed-back loop” complete. This can be schematically represented as:

\[
|n\rangle \implies V(\phi) \implies H_0 \implies |n\rangle.
\]

(iii) The leading order (LO) approximation in NGAS consists of finding the spectrum \(|n\rangle\) and the energy eigen-values \(E_0^n\). This is easily achieved because of the CES (eqn.(54)). It may be emphasized, however, that even the LO results capture the dominant contribution of the full (nonlinear) interaction through the requirement, eqn.(55), even though one always deals with an exactly solvable Hamiltonian, \(H_0\). We consider this as a key feature of the approximation method.

3.1.5 The Vanishing Quantum Average of the Effective Interaction and its Significance

The other important result which follows trivially from eqn.(55) is that the modified interaction \(\lambda H'\) defined by the relation:

\[
\lambda H' \equiv \lambda (H_1 - V),
\]

has vanishing QA for arbitrary ‘\(\lambda\)’ and ‘\(n\)’, i.e.

\[
<n|\lambda H'|n> = 0.
\]

In otherwords, only the non-diagonal matrix elements of \(H'\) are non- vanishing in the basis which renders \(H_0\) diagonal.

This result is of considerable significance and it naturally suggests a scheme of improved perturbation theory (IPT) in which, \(H_0\) is chosen as
the unperturbed Hamiltonian and $\lambda H'$ is considered to be the perturbation. The IPT thus developed, can be shown to be convergent, owing mainly to eqn.(60) and thus can be used to systematically improve the LO result order-by-order (Chapter 10). The various steps outlined in equations (53-60) define the scheme in the LO.

In the following Chapters, we implement and illustrate the general approach described above to specific quantum systems with self-interaction.

4. Application of NGAS to the case of the Quartic-Anharmonic Oscillator: formulation for the leading order (LO)

4.1 Application of the NGAS to the case of the Quartic Anharmonic Oscillator in Quantum Theory

In the following subsections we illustrate the NGAS applied to the case of the quartic anharmonic oscillator (QAHO) in quantum theory. However, it may be worth while to first describe the importance of the QAHO problem due to its applications in diverse areas of quantum physics and its significance from a theoretical stand-point as well.

4.1.1 The Quartic Anharmonic Oscillator (QAHO) in Quantum Theory: its importance and applications

The quantisation of the simple harmonic oscillator is perhaps among the most celebrated examples$^{8,9}$ due to its immense significance and application. In particular, small-oscillations about the equilibrium position are described by (normal-mode) simple-harmonic oscillators. Thus, quantum theory of the simple harmonic oscillator describes small oscillations of complex dynamical systems about the equilibrium. This property leads, for example, to quantisation of free-field systems, which are shown equivalent to an assembly of independent harmonic oscillators. Similarly, most lattice-dynamic
studies in the area of condensed matter physics employ the harmonic approximation as the leading description of the underlying physics. However, realistic approximation of actual physical systems need the incorporation of the anharmonic-interaction when deviation from small-oscillations or inclusion of self-interaction among oscillators become necessary on physical ground. This is the primary reason which endows the study of anharmonic oscillators with considerable importance and significance and explains its application in diverse areas of physics. Anharmonic oscillators are, therefore, among the most widely studied models in quantum theory in which the quartic-anharmonic oscillator (QAHO) is the simplest system exhibiting self-interaction. This system has been extensively investigated leading to a vast amount of literature. Its importance arises due to physical applications in areas including field theory, condensed matter physics, statistical mechanics, non-linear systems, classical and quantum chaos, inflationary cosmology, lattice dynamics, plasma oscillations etc, to cite only a few cases. Besides, the QAHO has also served as a theoretical laboratory to study convergence of perturbation theory, development of non-perturbative approximation methods, renormalisation, vacuum structure and stability analysis etc.

The Hamiltonian of the system is given by

\[ H = \frac{1}{2}p^2 + \frac{1}{2}g\phi^2 + \lambda\phi^4, \]  

where \(\lambda, g\) are real and positive. Note that the canonical momentum conjugate to the ‘field’ \(\phi(t)\) is given by \(p(t) = d\phi(t)/dt\). In the notation of eqn.(52), the free field Hamiltonian corresponds to

\[ H_s(\phi) = \frac{1}{2}p^2 + \frac{1}{2}g\phi^2 \]  

and the interaction is

\[ \lambda H_I(\phi) = \lambda\phi^4. \]  

In order to develop the NGAS for the QAHO, we follow the steps outlined in the Chapter-3, as described below.

4.2 Formulation of the NGAS for the QAHO
4.2.1 Choice of $V(\phi)$

The first step is to find a suitable approximating potential $V(\phi)$. The following ansatz is naturally suggested on the grounds of simplicity and exact solvability:

$$V(\phi) = A \phi^2 - B \phi + C.$$  \hspace{1cm} (64)

where A,B,C are parameters to be determined self consistently. It would appear from eqn.(64) that the global symmetry of the original Hamiltonian, eqn.(61), under $\phi \to -\phi$, is not respected by the ansatz, eqn.(64). However, this is illusory since the coefficient ‘B’ in eqn.(64) is dynamically determined to be proportional to $<\phi>$ (see eqn. (87) below), which ensures the preservation of the above global symmetry.

4.2.2 Solution of the eigen value equation of the Resultant Effective Hamiltonian(EH)

To obtain the exact analytic solution of the spectrum of $V(\phi)$ we proceed as follows.

We define the “Effective Hamiltonian (EH)” by the following equation

$$H_0(\phi, p) = \frac{1}{2} p^2 + \frac{1}{2} g \phi^2 + \lambda V(\phi)$$  \hspace{1cm} (65)

The EH defined in eqn.(65) can now be recast in the form of a diagonalizable structure as outlined below. Substitution of eqn.(64) in eqn.(65) leads to the following equation

$$H_0(\phi, p) = \frac{1}{2} p^2 + \frac{1}{2} g \phi^2 + \lambda (A \phi^2 - B \phi + C)$$  \hspace{1cm} (66)

The simplification of the eqn.(66) leads into the following Hamiltonian:

$$H_0 = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 (\phi - \sigma)^2 + h_0,$$  \hspace{1cm} (67)

where,

$$\omega^2 = g + 2\lambda A,$$  \hspace{1cm} (68)
\[ \sigma = \frac{\lambda B}{\omega^2}, \quad (69) \]

and
\[ h_0 = \lambda C - \frac{1}{2} \omega^2 \sigma^2. \quad (70) \]

It may be atonce recognized that the EH given by eqn.(67) corresponds to a “shifted”, effective, harmonic oscillator where both the field, as well as, the energy are respectively shifted by ‘\( \sigma \)’ and ‘\( h_0 \)’. Note further that the c-number parameters ‘\( \omega \)’ and ‘\( \sigma \)’ are restricted by physical requirement, to satisfy \( \omega > 0; \sigma = \text{real} \) (since, \( \phi = \phi^\dagger \)). Diagonalisation of \( H_0 \) can be achieved by the standard method of invoking the creation-and annihilation operators, defined by

\[ \phi(t) = \sigma + (b + b^\dagger)/\sqrt{2\omega}, \quad (71) \]
\[ p(t) = i\sqrt{\omega/2} (b^\dagger - b), \quad (72) \]

along with the standard equal-time (canonical) commutation relation (ETCR) given by

\[ [b, b^\dagger] = 1. \quad (73) \]

We first introduce the number operator: \( N_b \equiv b^\dagger b \) and its eigen-states by the equation
\[ N_b |n> = n |n>. \quad (74) \]

Because of the hermiticity and positivity of the operator \( N_b \); the eigen values are positive-definite and the eigen states are orthonormal:
\[ <m|n> = \delta_{mn}, \quad (75) \]
\[ \sum_n |n><n| = 1. \quad (76) \]

Further, as proven in standard text books\textsuperscript{130}, by the use of the ETCR (eqn. 73), the spectrum of eigen values is restricted to:
\[ n = 0, 1, 2, ........ \quad (77) \]

Then, using the defining parametrisation in eqns.(71-72) one immedi-
ately obtains the Hamiltonian in diagonal form:

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The energy spectrum is then trivially obtained and given by

\[ E_n^{(0)} = \omega \xi + h_0, \]

where \( \xi = (n + 1/2) \) with 'n' taking values according to eqn. (77).

At this stage, the following remarks/observations and results may be noted:

(i) By using the standard properties of the creation-/annihilation operators, the QA of polynomials of field \( \phi \) and momentum \( p \) can be easily evaluated. In particular, we note the following results for subsequent use (in the following, the vacuum-state is denoted by \( |\text{vac}> \) and is defined by the property, \( b|\text{vac}> = 0 \)):

\[
\langle \text{vac}|\phi|\text{vac} > = \langle n|\phi|n > \equiv \langle \phi > = \sigma, \quad (80)
\]

\[
\langle \phi^2 > = \sigma^2 + (\xi/\omega); \quad \langle p^2 > = \omega \xi, \quad (81)
\]

\[
\langle \phi^3 > = \sigma^3 + 3 \sigma (\xi/\omega), \quad (82)
\]

\[
\langle \phi^4 > = \sigma^4 + 6 \sigma^2 (\xi/\omega) + 3(1 + 4 \xi^2)/8\omega^2, \quad (83)
\]

\[
\langle \phi^6 > = \sigma^6 + 15 \sigma^4 (\xi/\omega) + 45 \sigma^2 (1 + 4 \xi^2)/8\omega^2 \\
+ (5/8)(\xi/\omega^3)(5 + 4 \xi^2). \quad (84)
\]

The QA of the original Hamiltonian \( H \) defined in eqn.(61) can now be evaluated with the help of eqs.(80-84) and given by

\[
\langle H > = \frac{1}{2} \langle p^2 > + \frac{1}{2} g \langle \phi^2 > + \lambda \langle \phi^4 >. \quad (85)
\]

Substituting the QA values from eqs.(80, 81, 83) and noting that the eqn.(55) holds, eqn.(85) is transformed to:

\[
\langle n|H|n > \equiv \langle n|H_0|n > = \omega \xi /2 + (g + 12 \lambda \sigma^2) (\xi /2\omega) \\
+ (3\lambda/8\omega^2)(1 + 4 \xi^2) + g \sigma^2 /2 + \lambda \sigma^4. \quad (86)
\]
(ii) Eqn.(80) shows that ‘$\sigma$’ corresponds to the vacuum expectation value (VEV) of ‘$\phi$’. In view of this result and eqn.(69), the coefficient ‘B’ of the linear term in $V(\phi)$ (see eqn.(64)) can be re-expressed as:

$$B = (\omega^2/\lambda) <\phi>.$$  

Eqn.(87) demonstrates that the global symmetry of the original Hamiltonian under the transformation, $\phi \rightarrow -\phi$, is preserved by the potential $V(\phi)$, which is not otherwise transparent in eqn.(64).

(iii) If one denotes by $a$ and $a^\dagger$, the corresponding creation- and annihilation operators of the ‘free’ theory (i.e. defined by $H_s(\phi, p)$), then ‘$\phi$’ and ‘$p$’ can also be expressed in terms of these operators analogous to eqs.(71-72) as:

$$\phi(t) = \sigma + (a + a^\dagger)/\sqrt{2}\omega_0,$$  

$$p(t) = i\sqrt{\omega_0}/2 (a^\dagger - a),$$

where $\omega_0 \equiv \sqrt{g}$. It is important to note here that both the sets of creation- and annihilation operators satisfy identical (equal-time) commutation relation:

$$[a, a^\dagger] = 1 = [b, b^\dagger].$$  

It may also be mentioned that the VEV of $\phi$ ($<\phi> = \sigma$) remains invariant in the two descriptions. Eqn.(90) implies that the two sets of operators must be related by a quantum canonical transformation (“Bogoliubov transformation")\textsuperscript{52}. This result has crucial implications for the vacuum structure and stability of the approximate theory. This is discussed in detail in Chapter 9.

Returning to the implementation of the NGAS, the next task is to determine the free parameters, ‘$\sigma$’ and ‘$\omega$’ (or equivalently, A, B and C occurring in eqn.(64)). This is achieved as follows:

4.2.3 Determination of the free parameters

The conditions expressed under CEQA and CO given in eqs( 56-57) are sufficient to fully determine the free parameters involved in the approximation. These requirements translate in this case, to the following equations:
\[<\phi^4> = A <\phi^2> - B <\phi> + C; \quad (91)\]

\[\partial <H_0>/\partial \omega = 0 \quad \text{and} \quad \partial <H_0>/\partial \sigma = 0 \quad \text{where,} \quad <H_0>\]

is given by eqn.(86). Carrying out the explicit variational minimisation of
\[<H_0>\] with respect to ‘\(\omega\)’ i.e., \(\partial <H_0>/\partial \omega = 0\) one obtains the following
equation:

\[
\omega^3 - \omega(12\lambda \sigma^2 + g) - 6\lambda f(\xi) = 0, \quad (92)\]

Again carrying out the explicit variational minimisation of \(<H_0>\) with
respect to ‘\(\sigma\)’ i.e., \(\partial <H_0>/\partial \sigma = 0\), the following equation is obtained:

\[
\sigma(4\lambda \sigma^2 + g + 12\lambda \xi/\omega) = 0, \quad (93)\]

where, in eqn.(92), \(f(\xi) \equiv \xi + 1/4\xi\). Eqs.(92-93) are to be solved simultaneously to determine ‘\(\omega\)’ and ‘\(\sigma\)’ as functions of \(\lambda\), \(g\) and \(\xi\). In the following, we refer to these eqs.(92-93) as the “gap equation (GE)” and “the equation for the ground state (EGS)” respectively. The constants \(A\), \(B\) and \(C\) appearing in eqn.(64) can now be determined as follows:

To determine the parameter ‘\(A\)’ we use the eqs.(68) and (92). Rearranging
the “gap-equation” given in eqn(92) we obtain the relation

\[
\omega^2 = (g + 12\lambda \sigma^2) + \frac{6\lambda f(\xi)}{\omega} \quad (94)\]

Eqn.(94), considered together with eqn.(68) lead to a determination of
the constant ‘\(A\)’, given by

\[A = 6\sigma^2 + 3f(\xi)/\omega \quad (95)\]

To determine the parameter ‘\(B\)’ we use the eqs.(69) and (93). Rearranging
the eqn.(69) leads to the following relation:

\[\sigma(-1 + \frac{\lambda B}{\sigma \omega^2}) = 0 \quad (96)\]

Comparing eqs.(93) and (96) we obtain the parameter ‘\(B\)’ as

\[B = (1 + g)(\sigma \omega^2/\lambda) + 4\omega^2 \sigma^3 + 12\omega \sigma \xi \quad (97)\]
To obtain the parameter ‘C’ we use equation (91):

\[ C = <\phi^4 > - A <\phi^2 > + B <\phi >, \]  

(98)

On substitution of the values \( <\phi >, <\phi^2 >, \) and \( <\phi^4 > \) from eqs.(80, 81, 83), along with the values of ‘A’ and ‘B’ determined above, the parameter ‘C’ is determined. Using the gap-equation (92) further the parameter C is given by

\[ C = \frac{3}{8\omega^2}(1 + 4\xi^2) - \frac{3\xi f(\xi)}{\omega^2} + \frac{\sigma^2\omega}{\lambda}[\omega^3 + \omega \xi^2f(\xi)] - \sigma^4(5 + 8\omega^2) \]  

(99)

On substitution of eqn.(99) in eqn.(70) we obtain the relation

\[ h_0 = \frac{3\lambda}{8\omega^2}(1 + 4\xi^2) - \frac{3\lambda f(\xi)}{\omega^2} + \frac{\sigma^2\omega}{2}[\omega^3 + \omega^2 \xi^2f(\xi)] - \lambda\sigma^4(5 + 8\omega^2) \]  

(100)

Thus, the parameters defining the approximating potential \( V(\phi) \) are completely determined.

Next, using eqn.(86) and noting that \( <H_0> = <H> \equiv E_n^{(0)} \) one obtains the energy spectrum:

\[ E_n^{(0)} = \omega\xi/2 + (g + 12\lambda\sigma^2)(\xi/2\omega) + (3\lambda/8\omega^2)(1 + 4\xi^2) + g\sigma^2/2 + \lambda\sigma^4, \]  

(101)

where ‘\( \omega \)’ and ‘\( \sigma \)’ are solutions of eqs.(92-93). The different solutions thus obtained are discussed in the following:

4.2.4 The Leading-Order (LO) Results - Determination of the Spectrum of \( H_0 \)

Solution of the gap-equation (GE) eqn.(92) and the equation for ground state (EGS) eqn.(93) constitute the key ingredients in the calculation of the energy spectrum. It is convenient to first obtain the solution of the EGS, eqn.(93). For the case of the QAHO \( (g, \lambda > 0) \) , eqn.(93) has two solutions:

\( (i) \quad \sigma = 0 \)  

(102)
and

\[(ii) \quad g + 4\lambda \sigma^2 + 12\lambda \xi / \omega = 0. \quad (103)\]

We now analyse the eqn.(103) which gives

\[\sigma^2 = -\frac{1}{4\lambda} (g + 12\lambda \xi / \omega) < 0 \quad (104)\]

Since \( g, \lambda > 0 \), eqn.(104) does not lead to acceptable solution, which requires \( \sigma^2 > 0 \). Therefore, the only physically accepted solution for QAHO is the solution (i) given in eqn.(102), i.e., \( \sigma = 0 \). (This is also intuitively obvious since the single-well shape of the “classical” potential can not get altered to double-well shape by quantum fluctuations). Substitution of eqn.(102) in eqn.(92), then leads to the following simplified GE for the QAHO:

\[\omega^3 - g\omega - 6\lambda f(\xi) = 0. \quad (105)\]

It may be emphasized at this point that this GE (with \( g=1 \)) has been derived by several authors\(^{131}\), but starting from widely different considerations. On substitution of \( \sigma = 0 \) in eqn.(101) we obtain the equation

\[E_n^{(0)} = \frac{1}{2} \omega \xi + \frac{1}{2} \left( \frac{g \xi}{\omega} \right) + \frac{3\lambda}{8\omega^2} (1 + 4\xi^2). \quad (106)\]

Using the GE, eqn.(105) and noting that

\[f(\xi) \equiv \frac{(1 + 4\xi^2)}{4\xi}, \quad (107)\]

the last term in eqn.(106) can be simplified to be

\[\frac{3\lambda}{8\omega^2} (1 + 4\xi^2) = \left( \frac{\xi}{4} \right) (\omega - \frac{g}{\omega}) \quad (108)\]

Substitution of eqn.(108) in eqn.(106) then leads to the following simple expression for the energy spectrum of the QAHO in the LO:

\[E_n^0 = \frac{\xi}{4} (3\omega + \frac{g}{\omega}) \quad (109)\]

where ‘\( \omega \)’ is obtained as a solution of the GE for the QAHO given by eqn.(105). It is to be noted that the “gap-equation” given by eqn.(105) is in the form of a cubic equation of the type.
\[ x^3 - 3Px - 2Q = 0; \quad P, Q > 0 \quad (110) \]

The real solution of this eqn.(110) is given by,

\[ x = Q^{1/3}[(1 + \sqrt{(1 - \frac{P^3}{Q^2})})^{1/3} + (1 - \sqrt{(1 - \frac{P^3}{Q^2})})^{1/3}] \quad (111) \]

Then comparing the coefficients of eqs.(110) and (105) the real solution is obtained explicitly as

\[ \omega = (3\lambda f(\xi))^{1/3}[(1 + \sqrt{(1 - \rho)})^{1/3} + (1 - \sqrt{(1 - \rho)})^{1/3}], \quad (112) \]

where, \( \rho^{-1} = 243\lambda^2 f^2(\xi)/g^3 \). It may be noted that, for the case when \( g = 1 \), the solution for \( \omega \) as given above has the correct limiting behaviour, \( \omega \rightarrow 1 \) for \( \lambda \rightarrow 0 \) and further that it exhibits the non-analytic dependence on the coupling \( \lambda \) at the origin characteristic of the non-perturbative nature of the NGAS.

As has been noted by several authors \(^{131}\), the formula, eqn.(109) is accurate to within a few percent of the ‘exact’ result for the full allowed range of \( g, \lambda > 0 \) (both in the ‘weak coupling’- and the ‘strong coupling’ regimes) and for all values of the excitation level \( n \geq 0 \). In particular, the accuracy in the strong coupling regime can be judged by the following result for the computation of the ground state energy. The ‘exact’ asymptotic result (for \( g = 1 \)) is given by\(^4\), for \( \lambda \rightarrow \infty \), \( E_0|_{\text{exact}} = 0.668\lambda^{1/3} \) which is to be compared to the LO-result in NGAS: \( E_0|_{\text{NGAS}} = 0.681\lambda^{1/3} \) in the same limit. In Table-1, the LO results of the present approximation scheme for \( g = 1 \) are presented for sample values of ‘\( \lambda \)’ and ‘\( n \)’ along with ‘exact’ results (obtained by numerical methods (Hsue and Chern, ref.[23])) and results from earlier computation\(^4\) (obtained by different analytical methods) for comparison. As can be seen from the comparison, the LO results, lie within 0.2 – 2 % of the results from the ‘exact’ numerical calculations\(^{23}\). In the same Table-1, we also dispaly the improvement of results obtained by inclusion of the first non-trivial correction in the improved perturbation theory (IPT), which is discussed in Chapter-10. We next apply the method to the case of the quartic -double well oscillator in the following Chapter.
Table- 1:  *Leading order(LO) results and perturbation correction in the 2nd order of IPT for the energy levels of the QAHO computed for $g = 1$ and sample values of ‘$\lambda$’ and ‘$n$’ shown along with analogous results of ref.[44], compared with the ‘exact’(numerical) results of (Hsue and Chern ref.[23]). The relative percentage errors in the two schemes are also shown.*

| $\lambda$ | n  | $E_n^{(0)}$ | Exact ref.[23] | $E_n^{(2)}$ | Error(%) | $E_n^{(2)}$ ref.[44] | Error(%) ref.[44] |
|-----------|----|-------------|-----------------|-------------|-----------|----------------------|------------------|
| 0.1       | 0  | 0.5603      | 0.5591          | 0.5591      | 0.007     | 0.5591               | 0.007            |
|           | 1  | 1.7734      | 1.7695          | 1.7694      | 0.005     | 1.7694               | 0.005            |
|           | 2  | 3.1382      | 3.1386          | 3.1391      | 0.016     | 2.9006               | 7.580            |
|           | 4  | 6.2052      | 6.2203          | 6.2239      | 0.058     | 5.4795               | 11.96            |
|           | 10 | 17.266      | 17.352          | 17.374      | 0.127     | 14.539               | 16.32            |
|           | 40 | 94.843      | 90.562          | 95.766      | 0.75      | 76.152               | 15.91            |
| 1.0       | 0  | 0.8125      | 0.8038          | 0.8032      | 0.070     | 0.8032               | 0.07             |
|           | 1  | 2.7599      | 2.7379          | 2.7367      | 0.043     | 2.7367               | 0.043            |
|           | 2  | 5.1724      | 5.1792          | 5.1824      | 0.061     | 4.4440               | 14.19            |
|           | 4  | 10.900      | 10.964          | 10.982      | 0.17      | 8.8890               | 18.93            |
|           | 10 | 32.663      | 32.933          | 33.013      | 0.243     | 25.833               | 21.56            |
|           | 40 | 192.79      | 194.60          | 195.15      | 0.282     | 149.87               | 22.99            |
| 10.0      | 0  | 1.5313      | 1.5050          | 1.5030      | 0.131     | 1.5030               | 0.131            |
|           | 1  | 5.3821      | 5.3216          | 5.3177      | 0.070     | 5.3177               | 0.070            |
|           | 2  | 10.324      | 10.347          | 10.356      | 0.090     | 8.6131               | 16.76            |
|           | 4  | 22.248      | 22.409          | 22.457      | 0.210     | 17.651               | 21.23            |
|           | 10 | 68.171      | 68.804          | 68.996      | 0.270     | 52.943               | 23.05            |
|           | 40 | 409.89      | 413.94          | 415.18      | 0.300     | 316.13               | 23.62            |
| 100.0     | 0  | 3.1924      | 3.1314          | 3.1266      | 0.150     | 3.1266               | 0.150            |
|           | 1  | 11.325      | 11.187          | 11.178      | 0.080     | 11.178               | 0.080            |
|           | 2  | 21.853      | 21.907          | 21.927      | 0.090     | 18.095               | 17.40            |
|           | 4  | 47.349      | 47.707          | 47.817      | 0.230     | 37.314               | 21.80            |
|           | 10 | 145.84      | 147.23          | 147.65      | 0.285     | 112.79               | 23.40            |
|           | 40 | 880.55      | 889.32          | 892.03      | 0.300     | 677.91               | 23.77            |
5. Application to the case of the Quartic Double Well Oscillator (QDWO): Leading Order (LO) Results

5.1 Importance of the QDWO and its Applications

The QDWO is also an extensively studied system because of its theoretical importance and practical application. The Hamiltonian of the system is given by:

\[ H = \frac{1}{2}p^2 - \frac{1}{2}g \phi^2 + \lambda \phi^4; \quad g, \lambda > 0 \]  

The crucial sign of the \( \phi^2 \) term generates a quite different physical situation than the case of the QAHO, even in the classical limit. The 'classical' potential, \( V_c \equiv -\frac{1}{2}g\phi^2 + \lambda \phi^4 \) exhibits the familiar double-well shape with symmetric minima. For \( g = 1 \), these minima are located at positions \( \pm \frac{1}{2\sqrt{\lambda}} \) and each with depth \( \frac{1}{16\lambda} \). As \( \lambda \) becomes smaller and smaller the depth of the two- wells become deeper and deeper. The actual low lying energy eigen-states of the problem become radically different from the trial wave function offered by the harmonic basis. This fact severely handicaps the convergence of the resultant perturbation theory. In principle, the natural solution would be the simultaneous use of two harmonic basis centers around the positions of the minima at \( \pm \frac{1}{2\sqrt{\lambda}} \). However, the implementation of that idea, although possible, implies the use of nonorthogonal states which is rather cumbersome.

The other difficulty is that the theory is not defined for \( \lambda \to 0 \), because the ground state does not exist in that limit due to the non-existence of a lower limit to \( V_c \). In that sense, the SHO is not the free-field limit of the QDWO. Therefore, the naïve perturbation theory (NPT) is not applicable as such, to this case. The perturbation expansion of the eigen values \( E_n(g, \lambda) \) in powers of \( \lambda \) is divergent for all \( \lambda > 0 \). This fact may be understood qualitatively by noting that the addition of the term \( \lambda x^4 \) turns a completely continuous eigen- value spectrum of \( p^2 - g x^2 \) into a com-
pletely discrete spectrum bounded from below. A nonperturbative treatment is therefore, necessary. The WKB method is one such method which is well suited especially for lower energy eigen values. However, as shown below, the NGAS can be successfully applied to the case of the QDWO in completely analogous procedure as in the case of the QAHO, for a considerable larger range of values of \( n, g \) and \( \lambda \).

5.2 Application of NGAS to the QDWO

5.2.1 The Gap Equation for the QDWO

To develop the NGAS for QDWO we use the identical ansatz for \( V(\phi) \) as given in eqn.(64). The parameters A, B, C can also be determined self-consistently as before. The “Effective Hamiltonian (EH)” for the system is analogously defined as:

\[
H_0(\phi, p) = \frac{1}{2} p^2 - \frac{1}{2} g \phi^2 + \lambda V(\phi); \quad \lambda, \; g > 0 \quad (114)
\]

Using the ansatz for \( V(\phi) \) given in eqn.(64), the EH for QDWO is then recast into the diagonal structure analogous to the case of QAHO and is given by

\[
H_0 = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 (\phi - \sigma)^2 + h_0, \quad (115)
\]

where, the parameters are defined as before:

\[
\omega^2 = g + 2\lambda A, \quad (116)
\]

\[
\sigma = \lambda B / \omega^2, \quad (117)
\]

\[
h_0 = \lambda C - \frac{1}{2} \omega^2 \sigma^2. \quad (118)
\]

In this case also, the EH defined in eqn.(115) is to be interpreted as a “shifted” harmonic oscillator. Also in this case, the frequency \( \omega > 0 \); and the vacuum configuration-parameter \( \sigma \) = real, are the physical requirement.
Using eqs.(71,72,73) and proceeding as in the case of the QAHO, one can express $H_0$ into the desired diagonal form given by the eqn.(78) from which one obtains the energy spectrum

$$E_n^{(0)} = \omega \xi + h_0,$$

where $\xi = (n + 1/2) ; n = 0, 1, 2, ...$

In order to implement the NGAS, the next task is the determination of the free parameters ‘$\sigma$’ and ‘$\omega$’ (or, equivalently, A, B and C occuring in eqn.(64) for the QDWO.

For this purpose, taking the quantum average of original Hamiltonian defined in eqn.(113) and using eqs.(80)(81)(83) leads the following equation

$$<n|H|n> \equiv <n|H_0|n> = \omega \xi / 2 - \frac{1}{2} g \left(\sigma^2 + (\xi/\omega)\right) + \lambda \left(\sigma^4 + 6 \sigma^2 (\xi/\omega) + 3 \left(1 + 4 \xi^2 / 8 \omega^2\right)\right),$$

(120)

Carrying out the variational minimisation of EH given in eqn.(120) with respect to free parameter ‘$\omega$’ i.e., $\partial <H_0>/\partial \omega = 0$ leads to the following “gap-equation (GE)” for the QDWO:

$$\omega^3 - \omega (12\lambda \sigma^2 - g) - 6\lambda f(\xi) = 0,$$

(121)

Similar procedure with respect to the free parameter ‘$\sigma$’ i.e., $\partial <H_0>/\partial \sigma = 0$, leads to the following equation for the ground state(EGS) configuration:

$$\sigma (4\lambda \sigma^2 - g + 12\lambda \xi / \omega) = 0.$$

(122)

It is here to be noted that these equations differ from the analogous equations, (92-93) for the QAHO by the substitution, $g \rightarrow -g$, as expected.

It is again convenient to solve the EGS first. It should be noted however that, in contrast to the case of the QAHO, there are now two realizable quantum phases of the system corresponding to the solution of eqn.(122) for the ground states. These are characterised by the solutions:

$$4\lambda \sigma^2 = g - 12\lambda (\xi / \omega),$$

(123)
respectively. The two solutions lead to different “quantum phases” for the QDWO, which is discussed below.

### 5.2.2 Different Quantum-Phases of the QDWO and the Critical Coupling

Following standard terminology used in the literature, the solution given by eqn.(123) leads to the “Spontaneously Symmetry Broken (SSB)” phase whereas, the other solution, eqn.(124), corresponds to the “Symmetry-Restored (SR)” phase. It is shown below that the dynamic realization of the two “phases” is controlled by the coupling ‘$\lambda$’ such that the SSB phase is energetically favoured when $\lambda \leq \lambda_c$, whereas the SR phase is preferred for $\lambda > \lambda_c$ where, $\lambda_c$ is a ’critical’ coupling. To demonstrate this we consider the GE in the respective phases:

### 5.2.3 Solution of the Gap Equation and Determination of the Spectrum in the Different Phases

(i) **The SSB-phase of the QDWO**

The GE for SSB-phase is obtained by substitution of eqn.(123) in eqn.(121) and is given by

\[
\omega_{a}^3 - 2g\omega_{a} + 6\lambda p(\xi) = 0,
\]  

(125)

where, $p(\xi) \equiv 5\xi - 1/(4\xi)$ and we have denoted by $\omega_a$, the frequency in the SSB phase. To get physically accepted solution of eqn.(125), it is convenient to adopt the ‘trial’ solution given by:

\[
\omega_{a} = \rho \left[ e^{i\theta/3} + e^{-i\theta/3} \right] \equiv 2 \rho \cos(\theta/3).
\]  

(126)

where ‘$\rho$’ and ‘$\theta$’ are to be determined. Substitution of eqn.(126) in eqn.(125) leads to a determination of ‘$\rho$’ and $\cos \theta$:

\[
\rho = \sqrt{(2g/3)},
\]  

(127)

and
\[
\cos \theta = - (\lambda / \lambda_c), \quad (128)
\]

with
\[
\lambda_c \equiv (2g/3)^{3/2} / 3p(\xi) \equiv \lambda_c(g, \xi) \quad (129)
\]

From eqn.(128), one gets either:
\[
\theta = \pi + \cos^{-1}(\lambda / \lambda_c), \quad (130)
\]
or
\[
\theta = \frac{\pi}{2} + \sin^{-1}(\lambda / \lambda_c). \quad (131)
\]

However, on stability grounds it is found that eqn.(131) leads to the acceptable solution as the corresponding ground state energy lies lower than that obtained using the other solution, eqn.(130). Thus the physical solution of eqn.(125) is given by
\[
\omega_a = 2 \sqrt{\frac{2g}{3}} \cos \left[ \frac{\pi}{6} + \frac{1}{3} \sin^{-1}(\lambda / \lambda_c) \right] \quad (132)
\]

An estimate of \(\lambda_c\) for the ground state and for \(g = 1\) is \(\lambda_c(g = 1, \xi = 1/2) = 0.0362886\). Clearly, the solution given in eqn.(131) is valid only when, \(\lambda \leq \lambda_c\).

The energy-levels in the LO of the SSB-Phase are computed in analogous manner to the QAHO as follows. One notes that
\[
E_n^{(0)|Q^\text{DWO}} = \langle n|H^{Q^\text{DWO}}|n \rangle \equiv \langle n|H_0^{Q^\text{DWO}}|n \rangle
\]
\[
= \frac{1}{2} \omega \xi - \frac{1}{2} g \left( \frac{\xi}{\omega} \right) + \frac{3 \lambda}{8 \omega^2} (1 + 4 \xi^2)
\]
\[
+ \sigma^2 \left( -\frac{1}{2} g + \frac{6 \lambda \xi}{\omega} \right) + \lambda \sigma^4, \quad (133)
\]

where \(H_0^{Q^\text{DWO}}\) is given by eqn.(114) and the QA values, as given in eqs.(80-83) have been substituted. From the defining eqn. of the SSB-Phase i.e., eqn.(123) one can substitute for \(\sigma^2\) and obtain:
\[
E_n^{(0)|Q^\text{DWO}} = \frac{1}{2} \omega_a \xi + \frac{g \xi}{\omega_a} + \frac{3 \lambda}{8 \omega_a^2} (1 - 20 \xi^2) - \frac{g^2}{16 \lambda} \quad (134)
\]
The third-term in eqn.(134) can be further simplified by the use of the GE in eqn.(125) and given by

\[ \frac{3\lambda}{8\omega_a^2}(1 - 20\xi^2) = \left( \frac{\xi}{4} \right) \left( \omega_a - \frac{2g}{\omega_a} \right). \]  (135)

Substitution of eqn.(135) in eqn.(134) then leads to the final expression for the energy levels given by

\[ E_n^{(0)}|_{SSB}^{QDWO} = \left( \frac{\xi}{4} \right)(3\omega_a + \frac{2g}{\omega_a}) - \frac{g^2}{16\lambda}. \]  (136)

(ii) The SR-Phase of the QDWO

For \( \lambda > \lambda_c \), the SR-Phase is dynamically favoured. The stability analysis of the phase-structure confirms this. The GE, in this case, is obtained by substituing eqn.(124) in eqn.(121):

\[ \omega_s^3 + g\omega_s - 6\lambda f(\xi) = 0, \]  (137)

where, we have denoted by \( \omega_s \), the frequency in the SR-Phase. Note that the above equation simply follows from the GE of the QAHO, eqn. (105), by the substitution: \( g \rightarrow -g \), as expected, due to the underlying single well shape. The energy levels in this phase can be calculated rather easily. On substitution of eqn.(124) in eqn.(133) one obtains the equation:

\[ E_n^{(0)}|_{SR}^{QDWO} \equiv <n|H_0^{QDWO}|n> |_{\sigma=0} = 
\frac{1}{2}\omega_s\xi - \frac{1}{2}(\frac{g}{\omega_s}) + \frac{3\lambda}{8\omega_s^2}(1 + 4\xi^2) \]  (138)

Rearranging the GE for SR-Phase given in eqn.(137) leads to the following equation for the last term in eqn.(138):

\[ \frac{3\lambda}{8\omega_s^2}(1 + 4\xi^2) = \left( \frac{\xi}{4} \right) \left( \omega_s + \frac{g}{\omega_s} \right). \]  (139)

Using the result obtained in eqn.(139) in the simplified eqn.(138), the energy levels for SR-Phase are given by the following simple expression:

\[ E_n^{(0)}|_{SR}^{QDWO} = \left( \frac{\xi}{4} \right)(3\omega_s - \frac{g}{\omega_s}). \]  (140)
which, again follows from the corresponding formula for the QAHO, eqn.(109), by the substitution, \( g \rightarrow -g \). In eqn.(140), \( \omega_s \) is the solution of eqn.(137) which can be obtained in analogous manner and is given by

\[
\omega_s = (3\lambda f(\xi))^{1/3}[(\sqrt{(1+\rho)} + 1)^{1/3} - (\sqrt{(1+\rho)} - 1)^{1/3}] \tag{141}
\]

where, \( \rho^{-1} = 243\lambda^2 f^2(\xi)/g^2 \).

In Table-2, we present the energy-levels of the QDWO in the LO, over a wide range of ‘\( \lambda \)’ and ‘\( n \)’ for \( g = 1 \). The results are compared with an earlier computation\(^{36}\), which employs a modified perturbation theory and includes correction up to twenty orders of perturbation. In the same Table-2, we also display the improvement of results obtained by inclusion of the first, non-trivial correction in the improved perturbation theory (IPT), which is discussed in Chapter 10. From the comparison with earlier calculation\(^{36}\), it is seen that the LO results are already quite accurate.

In the next Chapter we consider the case of the sextic- anharmonic oscillator and the sextic- double-well oscillator in the NGAS.
Table- 2: The computed energy levels of the quartic - DWO in the lowest order of NGAS for sample values of ‘λ’ and ‘n’ compared with the results of ref.[36] which includes perturbation correction up to twenty orders in a “modified” perturbation theory. Also shown are the results obtained after inclusion of the perturbation correction $E^{(2)}_n$, at the next order in IPT.

| $\lambda$ | $n$ | $E^{(0)}_n$ | $E^{(2)}_n$ | ref.[36] |
|-----------|-----|-------------|-------------|---------|
| 0.1       | 0   | 0.5496      | 0.4606      | 0.4702  |
|           | 1   | 0.8430      | 0.7553      | 0.7703  |
|           | 2   | 1.5636      | 1.6547      | 1.6300  |
|           | 4   | 3.5805      | 3.7232      | 3.6802  |
|           | 10  | 12.192      | 12.517      | 12.400  |
| 1.0       | 0   | 0.5989      | 0.5752      | 0.5800  |
|           | 1   | 2.1250      | 2.0800      | 2.1800  |
|           | 2   | 4.2324      | 4.2600      | 4.2500  |
|           | 4   | 9.4680      | 9.5950      | 9.5600  |
|           | 10  | 30.530      | 30.650      | 30.420  |
| 10.0      | 0   | 1.4098      | 1.3752      | 1.3800  |
|           | 1   | 5.0650      | 4.9910      | 5.0900  |
|           | 2   | 9.8660      | 9.9050      | 9.8900  |
|           | 4   | 21.561      | 21.791      | 21.700  |
|           | 10  | 66.950      | 67.820      | 67.620  |
| 100.0     | 0   | 3.1340      | 3.0650      | 3.0700  |
|           | 1   | 11.175      | 11.024      | 11.002  |
|           | 2   | 21.638      | 21.715      | 21.700  |
|           | 4   | 47.023      | 47.505      | 47.200  |
|           | 10  | 145.27      | 147.10      | 146.70  |
6. Application of NGAS to the Sextic-Anharmonic and Double Well Oscillator

6.1 Importance of the Sextic-Anharmonic Oscillator and its Applications

The sextic AHO-system is an example of higher anharmonicity, which is also widely investigated\textsuperscript{12,13,133}. This system is interesting and important in its own right as it finds application in diverse areas of physics. As a theoretical laboratory, this system together with its double-well counter-part provide perhaps the simplest examples, for which supersymmetric quantum mechanics (SUSYQM) has definite predictions\textsuperscript{61-71} for energy-levels for specific values of the Hamiltonian-parameters. Hence, various models and approximation methods can be tested against the exact analytic results of SUSYQM. Besides, owing to the higher anharmonicity, the divergence of the naive-perturbation theory at higher-orders becomes even more severe\textsuperscript{133}, \[ E_n \sim \Gamma[n(m-1)] \] for \( \lambda \phi^{2m} \) type of AHO. This result endows the sextic-AHO with added importance for testing convergent-approximation methods. It may also be noted that the “Wick-ordering” method of Caswell\textsuperscript{36} is not directly applicable to this case since the method generates \( \phi^4 \) counter-terms which are not present in the original (non-ordered) Hamiltonian. For the above stated reasons of practical applicability and theoretical importance, the sextic-AHO provides a unique testing ground for the NGAS, which is described in the following.

6.2 Application of NGAS to the Sextic-AHO

6.2.1 The Gap Equation for the Sextic-AHO

The Hamiltonian for the system is given by

\[ H = \frac{1}{2}p^2 + g \phi^2 + \lambda \phi^6, \] (142)
where, $\lambda, g > 0$. As in other cases,

$$H_s(\phi) = \frac{1}{2} p^2 + \frac{1}{2} g \phi^2,$$  \hspace{1cm} (143)

corresponds to the free-field Hamiltonian. The interaction-term is, therefore,

$$\lambda H_I(\phi) = \lambda \phi^6.$$  \hspace{1cm} (144)

To apply the NGAS, we follow identical steps as in previous cases. An identical ansatz is assumed for the potential $V(\phi)$, which is given in eqn.(64). Hence the Effective Hamiltonian (EH) for this case is analogously defined as:

$$H_0 = \frac{1}{2} p^2 + \frac{1}{2} g \phi^2 + \lambda V(\phi)$$  \hspace{1cm} (145)

Substituting the ansatz for $V(\phi)$ in eqn.(64) in eqn.(145), the EH can be recast into the diagonizable structure, as in earlier cases:

$$H_0 = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 (\phi - \sigma)^2 + h_0,$$  \hspace{1cm} (146)

where, analogously

$$\omega^2 = g + 2\lambda A,$$  \hspace{1cm} (147)

$$\sigma = \lambda B / \omega^2;$$  \hspace{1cm} (148)

$$h_0 = \lambda C - \frac{1}{2} \omega^2 \sigma^2.$$  \hspace{1cm} (149)

The interpretation of $H_0$ in eqn.(146) is also identical- it is the Hamiltonian for a “shifted” effective harmonic oscillator. Following steps identical to those in the case of quartic-oscillators, the diagonal form for $H_0$ is obtained as:

$$H_0 = \omega (N_b + 1/2) + h_0,$$  \hspace{1cm} (150)

where again $N_b \equiv b^\dagger b$ with $b, b^\dagger$ defined as before (see eqs.(71-72)). The eigen value equation as given in eqn.(54) then leads to the energy spectrum as given below:

$$E_n^{(0)} = \omega \xi + h_0,$$  \hspace{1cm} (151)
where \( \xi = (n + 1/2); \ n = 0, 1, 2, \ldots \) and \( h_0 \) is given by eqn.(149).

For the determination of the parameters \( \omega \) and \( \sigma \), we use, as before, the variational minimisation conditions: \( \partial < H_0 > /\partial \omega = 0 \) and \( \partial < H_0 > /\partial \sigma = 0 \). For this purpose the quantum average of eqn.(142) evaluated as:

\[
< H > =< H_0 > = \frac{1}{2} < p^2 > + \frac{1}{2} g < \phi^2 > + \lambda < \phi^6 > \tag{152}
\]

On substitution of the results given in eqs.(80,81,84) in eqn.(152), the following equation results:

\[
<n|H|n> \equiv <n|H_0|n> = \frac{\omega \xi}{2} + \frac{1}{2} g (\sigma^2 + (\xi/\omega))
+ \lambda (\sigma^6 + 15 \sigma^4 (\xi/\omega) + \frac{45 \sigma^2 (1 + 4 \xi^2)}{8 \omega^2} + \frac{5 \xi (4 \xi^2 + 5)}{8 \omega^3}), \ g > 0 \tag{153}
\]

Carrying out variational minimisation of eqn.(153) with respect to the frequency parameter \( \omega \) we obtain the relation

\[
\omega^4 - \omega^2 (g + 30 \lambda \sigma^4) - 45 \lambda (\sigma^2 \omega/2 \xi)(1 + 4 \xi^2) - (15 \lambda/4)(5 + 4 \xi^2) = 0, \tag{154}
\]

Further, carrying out the variational minimisation with respect to \( \sigma \) leads the following equation:

\[
\sigma [g + 6 \lambda (\sigma^4 + \frac{10 \xi \sigma^2}{\omega} + \frac{15 (4 \xi^2 + 1)}{8 \omega^2})] = 0. \tag{155}
\]

Eqn.(155) which defines \( \sigma \) provides the corresponding ground state configuration (EGS) of this system.

As in earlier cases, it is convenient to first analyse the EGS which leads to the following relations:

\[
\sigma = 0 \tag{156}
\]

and

\[
g + 6 \lambda (\sigma^4 + 10 \xi \sigma^2/\omega + 15 (4 \xi^2 + 1)/8 \omega^2) = 0. \tag{157}
\]
On inspection of eqn.(157), it is easily seen that no physical solution for $\sigma^2$ (i.e. $\sigma^2 > 0$) exists for $g > 0, \lambda > 0$. Hence, eqn.(157) is not physically realisable for the sextic AHO. Therefore, the physical ground state of the sextic AHO is uniquely determined by the $'\sigma = 0'$ solution as given by eqn.(156).

6.2.2 Solution of the Gap Equation and Determination of the Energy Spectrum

The GE given in eqn.(154) simplifies to the following form on substitution of the correct EGS expression, i.e. $\sigma = 0$:

$$\omega^4 - g\omega^2 - (15\lambda/4)(5 + 4\xi^2) = 0$$

(158)

The parameters A, B, C defining $V(\phi)$ can be determined in analogous manner as given below. Rearranging the gap eqn.(154) and using the eqn.(147) ‘A’ is determined as

$$A = 15\sigma^4 + 45\sigma^2(1 + 4\xi^2)/4\xi\omega + (15/8\omega^2)(5 + 4\xi^2);$$

(159)

Similarly, following the same procedure, ‘B’ is calculated using eqn(148) and the EGS given in eqn.(155) and given by

$$B = \sigma[(1 + g)(\omega^2/\lambda) + 6\omega^2\sigma^4 + 60\sigma^2\omega\xi + (45/4)(1 + 4\xi^2)];$$

(160)

To determine ‘C’ we use the equation as given below:

$$C = <\phi^6> - A <\phi^2> + B <\phi>,$$

(161)

together with expressions for $<\phi^6>$, $<\phi^2>$ and $<\phi>$ as given in eqs.(80-84). On substitution of the physical solution, eqn.(156) for the EGS in eqn.(153) and using the GE as given in eqn.(158) one obtains the following simple expression for the energy-levels of the sextic AHO in the LO:

$$E_n^{(0)}|_{\text{sextic-AHO}} = \frac{\xi}{3}(2\omega + \frac{g}{\omega}); g > 0.$$

(162)

where, of course, ‘$\omega$’ is the solution of the GE given in eqn.(158). In Table-3, sample results for the energy levels of the sextic AHO computed in LO are presented for $g = 1$, over a wide range of ‘$\lambda$’ and ‘$n$’ and compared to
the results\textsuperscript{133} (shown within parenthesis) with percentage deviation from the latter (shown within square bracket). It can be seen from this tabulation that the LO- results obtained in the NGAS are quite accurate, compared to those obtained by sophisticated numerical calculations\textsuperscript{133}. Further improvement in accuracy is achieved by application of the IPT as discussed in Chapter-10.
Table 3: Sample results in the lowest order (LO) of (NGAS) for the sextic AHO over a wide range of $\lambda$ and $n$ compared for $g = 1$ with the results of ref. [133] (shown in the parentheses). The relative percentage error is shown in square brackets.

| n | $\lambda = 0.2$ | 2.0 | 10.0 | 100.0 | 400.0 | 2000.0 |
|---|----------------|-----|------|-------|-------|--------|
| 0 | 1.193          | 1.676 | 2.323 | 3.947 | 5.521 | 8.206 |
|   | (1.174)        | (1.610) | (2.206) | (3.717) | (5.188) | (7.702) |
|   | [1.611]        | [4.079] | [5.313] | [6.188] | [6.415] | [6.544] |
| 1 | 3.966          | 5.931 | 8.420 | 14.52 | 20.39 | 30.37 |
|   | (3.901)        | (5.749) | (8.115) | (13.95) | (19.56) | (29.12) |
|   | [1.681]        | [3.165] | [3.762] | [4.148] | [4.244] | [4.298] |
| 2 | 7.420          | 11.61 | 16.74 | 29.16 | 41.03 | 61.18 |
|   | (7.382)        | (11.54) | (16.64) | (28.98) | (40.78) | (60.81) |
|   | [0.523]        | [0.612] | [0.6179] | [0.6157] | [0.6145] | [0.6138] |
| 4 | 16.15          | 26.48 | 38.73 | 68.01 | 95.90 | 143.2 |
|   | (16.30)        | (26.83) | (39.29) | (69.05) | (97.38) | (145.4) |
|   | [0.9170]       | [1.302] | [1.426] | [1.499] | [1.517] | [1.527] |
| 6 | 26.88          | 45.08 | 66.36 | 117.0 | 165.1 | 246.5 |
|   | (27.29)        | (45.94) | (67.70) | (119.4) | (168.5) | (251.7) |
|   | [1.50]         | [1.870] | [1.98] | [2.043] | [2.058] | [2.067] |
| 10| 53.24          | 91.17 | 135.0 | 238.7 | 337.1 | 503.8 |
|   | (54.31)        | (93.26) | (138.2) | (244.5) | (345.3) | (516.1) |
|   | [1.967]        | [2.245] | [2.323] | [2.367] | [2.377] | [2.383] |
| 14| 85.01          | 147.0 | 218.3 | 386.6 | 546.2 | 816.3 |
|   | (86.78)        | (150.4) | (223.4) | (395.7) | (559.1) | (835.6) |
|   | [2.047]        | [2.230] | [2.279] | [2.306] | [2.313] | [2.316] |
| 17| 111.9          | 194.4 | 289.0 | 512.1 | 723.7 | 1082.0 |
|   | (114.0)        | (198.3) | (294.9) | (522.7) | (738.6) | (1104.0) |
|   | [1.868]        | [1.974] | [2.001] | [2.016] | [2.020] | [2.022] |
6.3 The Sextic-Double Well Oscillator (Sextic-DWO)

6.3.1 Importance of Sextic-Double Well Oscillator

The sextic double well oscillator has been the subject of much investigation and discussion for a pretty long time\textsuperscript{134}, both from the analytical and the numerical points of view. This is because it has important applications in quantum field theory and molecular physics\textsuperscript{135} and in other branches also. By using Hill-determinant method, Biswas et al\textsuperscript{27} have calculated the eigenvalues of the oscillators of the type $\lambda x^{2n}$. Banerjee and Bhattacharjee have obtained energy eigenvalues for the potential of the form $x^2 + \lambda x^4$ using scaled Hill determinant method\textsuperscript{136}. However, it has been pointed out that\textsuperscript{137} this method has limited domain of applicability for the sextic oscillator potential of the type $V(x) = \mu x^2 - \lambda x^4 + \eta x^6; \eta > 0$. There have been some improvements\textsuperscript{138} of the procedure for removal\textsuperscript{139,140} of difficulties in the Hill-determinant approach\textsuperscript{141}. The potential cited above is particularly of great interest in scalar field theory\textsuperscript{142} and also in the calculations of the vibrational spectra of molecules\textsuperscript{143}. It has been discussed by authors of ref.[144] that the potential admits exact analytic solutions for the ground state under certain conditions. It has also been shown that\textsuperscript{145} the potential is “quasi exactly solvable (QES)” i.e., exact solutions can be obtained if the coupling constants satisfy some constraints. Moreover, supersymmetric quantum mechanics (SUSYQM) can be applied\textsuperscript{71} to compute the eigenvalues of the double-well potential $x^6 - 3x^2$ and the spectrum can be related to that of the “partner” potential: $x^6 + 3x^2$, which is the anharmonic counterpart. We apply our scheme of approximation in the next section, to calculate the energy eigen values for the sextic-DWO as a method of confirmation of our scheme, as well as, to test the various exact results following from QES and SUSYQM.

6.3.2 Application of NGAS to the Sextic-DWO

The Hamiltonian for such a system is given by the following expression:

$$H = \frac{1}{2} p^2 - \frac{1}{2} g \phi^2 + \lambda \phi^6,$$  \hspace{1cm} (163)

where, $\lambda > 0$ and $g > 0$. To apply NGAS, we follow identical steps
as in previous case of the QDWO. An identical ansatz is suggested for the potential \( V(\phi) \) which is given in eqn.(64). Hence the (Effective Hamiltonian (EH)) for this case can be written as:

\[
H_0 = \frac{1}{2} p^2 - \frac{1}{2} g \phi^2 + \lambda V(\phi)
\]  

(164)

On Substitution the value for \( V(\phi) \) given in eqn.(64) in eqn.(164) the EH is now expressed into the diagonalizable structure:

\[
H_0 = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 (\phi - \sigma)^2 + h_0,
\]  

(165)

where,

\[
\omega^2 = -g + 2\lambda A,
\]  

(166)

\[
\sigma = \frac{\lambda B}{\omega^2},
\]  

(167)

\[
h_0 = \frac{\lambda C}{2} - \frac{1}{2}\omega^2\sigma^2.
\]  

(168)

The parameters A,B,C can be determined through identical procedure as in earlier cases. In particular, we get back relations given by eqs.(159-161) with the substitution: \( g \leftarrow -g \). Again the eqn.(165) can be regarded as the hamiltonian for the “shifted” effective harmonic oscillator for sextic-DWO, which is identical with the equations developed earlier for the case of QAHO eqn.(67), QDWO(eqn.(115) and sextic-AHO (eqn.(146)). This, therefore, confirms the general applicability of the method formulated in **Chapter 3**. As usual ‘ \( \omega \) ’ is identified as the frequency of the “shifted” harmonic oscillator which has the restriction of satisfying the physical requirement \( \omega > 0 \). The parameter ‘ \( \sigma \) ’ accounts for the field-shift of the “shifted” harmonic sextic-DWO which must be real, since the field is hermitian, \( \phi = \phi^\dagger \). Parameters ‘ \( \omega \) ’ and ‘ \( \sigma \) ’ are determined by following the analogous procedure (see below). Finally the EH defined in eqn.(165) is transformed by standard method into diagonalised form as:

\[
H_0 = \omega (N_b + 1/2) + h_0,
\]  

(169)

where the creation- and annihilation operators and ETCR given in eqs.(71-73) have been used. The energy spectrum for sextic-DWO is then trivially obtained as:
\[ E_n^{(0)} = \omega \xi + h_0 , \]  

where \( \xi = (n+1/2) ; \) \( n = 0, 1, 2, \ldots \) and \( h_0 \) is given by eqn.(168).

(i) The Gap-Equation and the different Quantum- Phases of the Sextic- QDWO

For the determination of the parameter ‘ \( \omega \) ’ and ‘ \( \sigma \) ’ we use the standard variational-minimisation conditions \( \partial < H_0 > / \partial \omega = 0 \) and \( \partial < H_0 > / \partial \sigma = 0 \). On taking the quantum average of eqn.(163) and taking the advantage of eqn.(55) we obtain \( < H_0 > \) as given below:

\[
<n|H|n> \equiv <n|H_0|n> = \omega \xi / 2 - \frac{1}{2} g \left( \sigma^2 + (\xi/\omega) \right) + \lambda \left( \sigma^6 + 15 \sigma^4 (\xi/\omega) + 45 \sigma^2 (1 + 4 \xi^2)/8\omega^2 + \frac{5\xi(4\xi^2 + 5)}{8\omega^3} \right) ,
\]

(171)

From the variational minimisation of eqn.(171) with respect to ‘ \( \omega \) ’, we obtain the relation

\[
\omega^4 - \omega^2 (-g + 30 \lambda \sigma^4) - 45 \lambda \sigma^2 / 2 \xi (1 + 4 \xi^2) - (15 \lambda/4)(5 + 4 \xi^2) = 0 ,
\]

(172)

Eqn.(172) corresponds to “gap-equation” GE for the case of sextic-DWO. Similarly, from the variational minimisation of eqn.(171) with respect to ‘ \( \sigma \) ’ leads to the following equation:

\[
\sigma \left[ -g + 6\lambda (\sigma^4 + 10\xi \sigma^2/\omega + 15(4\xi^2 + 1)/8\omega^2) \right] = 0 .
\]

(173)

Eqn.(173) has the significance of defining the ground state (EGS). This equation leads, as in the case of the QDWO, to the SSB-phase and the SR-phase, corresponding to the ‘ \( \sigma^2 \neq 0 \)’ and the ‘ \( \sigma = 0 \)’ solutions respectively. It is found on computation, however, that the SR-phase is energetically favoured for all values of \( \lambda \) since the energy levels for this phase, always lie below the corresponding ones for the SSB-phase. This result has important consequence in the context of supersymmetry, as discussed subsequently.
(ii) Solution of the Gap-Equation and determination of the energy spectrum of the SR-phase

The GE, in the case of sextic-QDWO for the SR-phase, is simply obtained by substituting \( \sigma = 0 \) in the “gap-equation” eqn.(172) which corresponds to the GE for the AHO (eqn.(158)) by the replacement, \( g \rightarrow -g \):

\[
\omega_a^4 + g\omega_a^2 - \left( \frac{15\lambda}{4} \right)(5 + 4\xi^2) = 0,
\]

where again, we have distinguished the frequency of the DWO, by the subscript ‘a’. To determine the energy levels one substitutes \( \sigma = 0 \) in the eqn.(171) for \( <H_0> \) one obtains:

\[
E_n^{(0)}|_{\text{sextic-DWO}} \equiv <H_0> |_{\sigma=0} = \frac{\omega_a^2}{2} - \frac{1}{2} \left( \frac{g\xi}{\omega_a} \right) + \frac{5\lambda\xi}{8\omega_a^3}(5 + 4\xi^2). \quad (175)
\]

Next, using the GE, eqn.(174), the last term in eqn.(175) can be simplified as:

\[
\frac{5\lambda\xi}{8\omega_a^3}(5 + 4\xi^2) = \left( \frac{\xi}{6} \right)(\omega_a + \frac{g}{\omega_a}). \quad (176)
\]

Substitution of the above equation in eqn.(175) then leads to energy spectrum given below:

\[
E_n^{(0)}|_{\text{sextic-DWO}} = \frac{\xi}{3}(2\omega_a - \frac{g}{\omega_a}), \quad g > 0.
\]

In Table-4, sample results for the energy levels of the sextic DWO computed in the LO are presented and are compared to the results of ref.[71]. It can be seen from this tabulation that the LO- results obtained in NGAS are quite accurate, compared to those obtained by sophisticated numerical calculations of ref.[71]. On comparison it is seen that there is good agreement for all values of ‘n’, expect for the ground state. This is further discussed in the next Chapter. Further improvement in accuracy is achieved by application of IPT as discussed in Chapter-10.

In the Chapter-7, we consider the implication of supper symmetric quantum mechanics (SUSYQM) for the case of the sextic oscillators and comparison with the results of NGAS.
Table 4: Sample results for the energy levels of the sextic- AHO and DWO in the LO of NGAS (for $\beta = 1$) displaying the approximate validity of ISPP relation (see eqn. (191)). Also shown for comparison, are the corresponding results of ref.[71] obtained by numerical methods based upon SUSY. (Note that the ISPP relations are preserved for arbitrary value of ‘$\beta$’ due to the ‘scaling’ property given by eqn. (191) of the text).

| $n$ | $E_n^{(AHO)}$ | $E_{n+1}^{(DWO)}$ | $E_n^{(AHO)}$ | $E_{n+1}^{(DWO)}$ |
|-----|--------------|-------------------|--------------|-------------------|
|     |              |                    | ref.[71]     | ref.[71]          |
| 0   | 1.95608      | 2.38721           | 1.93548      | 1.93548           |
| 1   | 6.37732      | 6.24897           | 6.29849      | 6.29849           |
| 2   | 11.7352      | 11.3668           | 11.6810      | 11.6810           |
| 3   | 17.9931      | 17.4785           | 18.0426      | 18.0426           |
| 4   | 25.0597      | 24.4375           | 25.2546      | 25.2546           |
| 5   | 32.8581      | 32.1484           | 33.2261      | 33.2261           |
| 6   | 41.3276      | 40.5427           | 41.8910      | 41.8910           |
| 7   | 50.4197      | 49.5679           | 51.1979      | 51.1979           |
| 8   | 60.0950      | 59.1822           | 61.1053      | 61.1053           |
| 9   | 70.3204      | 69.3513           | 71.5790      | 71.5790           |
| 10  | 81.0680      | 80.0462           | 82.5899      | 82.5899           |
| 11  | 92.3136      | 91.2421           | 94.1129      | 94.1129           |
| 12  | 104.036      | 102.917           | 106.126      | 106.126           |
| 13  | 116.217      | 115.053           | 118.611      | 118.611           |
| 14  | 128.839      | 127.632           | 131.549      | 131.549           |
| 15  | 141.889      | 140.640           | 144.927      | 144.927           |
| 16  | 155.351      | 154.062           | 158.728      | 158.728           |
| 17  | 169.214      | 167.887           | 172.942      | 172.942           |
| 18  | 183.467      | 182.102           | 187.557      | 187.557           |
| 19  | 198.099      | 196.698           | 202.561      | 202.561           |
7. Comparison of the Results of NGAS with the Exact Predictions of Super Symmetry for the Sextic Oscillator

7.1 Super Symmetric Quantum Mechanics (SUSYQM) in Brief

Supersymmetry (SUSY) is a symmetry between Fermions and Bosons which was invoked by physicists to obtain a unified description of all basic interaction of nature although there has been no experimental evidence of SUSY being realised in nature. Nevertheless, in the last fifteen years, the ideas of supersymmetry have stimulated new approaches and applications to the different branches of physics including atomic, molecular, statistical and condensed matter physics as well as non-relativistic quantum mechanics.

In this section, we will be concerned with the application of SUSY in the area of non-relativistic quantum mechanics - a field now known as super symmetric quantum mechanics (SUSYQM). The investigations in this area have rapidly proliferated in the past two-decades which have established SUSYQM as an important and interesting area of research. The application of SUSY to quantum mechanics has yielded several new and significant results. In particular, the class of potentials yielding exact analytic solution of the non-relativistic Schrödinger’s equation have been classified and understood readily on the basis of SUSY. Moreover, the results and concept of “partner-potentials” yielding iso-spectral Hamiltonians are among the significant predictions of SUSYQM. For reviews of the subject, the ref.[6] may be consulted.

Our immediate interest in the subject of SUSYQM arises in this thesis from the fact that the latter has exact-predictions of iso-spectrality of the sextic-AHO and the sextic-DWO which form partner Hamiltonians for specific choices of the parameters defining the potentials. To this topic, we turn to in the next section.

7.2 Super Symmetry and Iso-spectrality of Partner Potentials

66
One of the simplest non-trivial applications of SUSYQM\textsuperscript{12,13} is made for the case of the sextic-oscillators (AHO and DWO). Consider the “super potential” ref.[6]:

\[ W(\phi) = \beta \phi^3, \quad (178) \]

This is the simplest anharmonic, supersymmetric and “parity even” potential with no known analytic solutions. This is perhaps the simplest choice beyond the linear solvable case. According to SUSY quantum mechanics\textsuperscript{6} the “partner-potentials” are given by:

\[ V(-) \equiv \frac{1}{2}(W^2 - W') \quad (179) \]
\[ V(+) \equiv \frac{1}{2}(W^2 + W') \quad (180) \]

By using eqn.(178) in eqs.(179) and (180) the “partner-potentials” are generated as given below:

\[ V(-) = \frac{1}{2}(\beta^2 \phi^6 - 3\beta \phi^2) \quad (181) \]
\[ V(+) = \frac{1}{2}(\beta^2 \phi^6 + 3\beta \phi^2) \quad (182) \]

Both the “partner-potentials” correspond to parity- even, confining potentials. One of these, \( V(-) \) corresponds to the double-well. The ground state energy of the “partner-potential” \( V(-) \) occurs at \( E = 0 \). In this case there is no trace of the double well structure as the zero-point energy is just equal to the well depth.

The potentials for the sextic DWO and the AHO which are used in the Hamiltonian given by the eqn.(163) and eqn.(142) respectively can be written as:

\[ V_{1,2} = \lambda \phi^6 \mp \frac{1}{2}g\phi^2 \quad (183) \]

where \( \lambda > 0 \) and \( g > 0 \). On the other hand, the Hamiltonians corresponding to the “partner-potentials” are:
\[ H^{(-)} = \frac{1}{2}p^2 + V^{(-)} \] 

\[ H^{(+)} = \frac{1}{2}p^2 + V^{(+)} \] 

In standard notation, the ‘exact’ results of SUSY for the above Hamiltonians as given in eqs.(184-185) can be summarized as follows:\(^6\):

(i) \[ E_{n+1}^{(-)} = E_{n}^{(+)} \] 

(ii) \[ E_{0}^{(-)} = 0 \] 

(iii) \[ \psi_0^{(-)}(\phi) = A \exp(-\int \phi W(y)dy) \] 

where, n=0,1,2,...; the ground state wave function for \( H^{(-)} \) is denoted by \( \psi_0^{(-)}(\phi) \) and ‘A’ denotes its normalisation. The property given by eqn.(186) is referred to [6] as “Iso-spectrality” of Partner Potentials (ISPP). Eqn.(187) is a rigorous result of exact (unbroken) super symmetry, while eqn.(188) is the prediction for the ground state wave function of \( H^{(-)} \).

Application of eqs.(181,182,184-188) to the case of the sextic AHO and DWO characterised by potentials in eqn.(183) becomes at once obvious, when the following specific values for ‘\( \lambda \)’ and ‘\( g \)’ are chosen:

\[ \lambda = \frac{1}{2}\beta^2, \; g = 3\beta; \; \beta > 0. \] 

For the above choice of ‘\( \lambda \)’ and ‘\( g \)’, eqn.(186) can then be rewritten as:

\[ E_{n+1}^{(DWO)}(\lambda, g) = E_{n}^{(AHO)}(\lambda, g) \] 

### 7.3 Comparison of the Energy Levels obtained in NGAS (LO) with the Exact Results from SUSY
Surprisingly, it is found that the relation given in eqn.(186) (or eqn.(190)) is obeyed to a very good accuracy by the LO results from NGAS for all allowed values of $\lambda > 0$! In Table-5, we demonstrate the (approximate) validity of the ISPP relation of SUSY in NGAS, by comparing the energy level of the DWO for the excitation label ‘(n+1)’ with that of the AHO for the label ‘n’. The agreement is seen to be impressive, particularly at large values of ‘n’, considering that only the LO-results are used. It may be observed, in this context, that the formulae for energy levels of sextic oscillators in NGAS, given by eqs.(162) and (177), obey the following interesting “scaling” law:

$$E_n^{(0)}(\beta) = \sqrt{\beta} E_n^{(0)}(1)$$

(191)

This scaling property guarantees the validity of the ISPP relation, eqn.(190) for arbitrary values of $\beta$, once the relation is established for any particular given value of the latter.

7.4 Positivity Property of the Energy Eigenvalues of the Sextic-DWO predicted by SUSYQM and NGAS

The other observation is regarding the “positivity” property of the energy-eigen values of the sextic-DWO predicted by SUSY through the eqs. (186-187), which is otherwise not obvious owing to the double-well structure of the potential (at least, this is not the case for the SSB-phase of the QDWO!). Interestingly, the positivity of the energy levels of the sextic-DWO, as predicted by SUSY, is also dynamically realized in NGAS. This is because of the fact that the SSB-phase is ruled out on grounds of stability (see, remarks following eqn.(173)).

As a final confirmation of consistency with the exact results from SUSY on energy levels of above systems, it is necessary to establish not only the ISPP -relation, but also the absolute magnitudes of the former. In Table-4 (page-66), we compare the results of ref.[71], on the energy levels of sextic-oscillators obtained by sophisticated numerical methods, with those based upon the simple formulae, eqs.(162,177) in LO of NGAS for $\lambda = 0.5$ and $n \leq 20$. It can be seen from this comparison that there is good agreement for all values of ‘n’, except for the ground state-energy of the DWO. In the latter case there is some deviation from the exact result of SUSY, given by eqn.(146). It may be plausible that the discrepancy could be due to the
departure from the predicted *exact* ground state wave function as given by eqn.(147) from the wave function of the DWO in LO of NGAS. This aspect is further investigated below:

### 7.5 The DWO-Ground State Wave Function in NGAS and SUSYQM

Having demonstrated the approximate validity in NGAS, of the ISPP relations and positivity property of energy levels predicted by SUSY, it remains to compare the respective ground state wave functions. The exact result from SUSY is given by eqn.(188). For the case of sextic DWO, this result can be made more specific as given below. Using eqn.(178) in eqn.(188) the ground state wave function is written as:

\[ \psi_0^{(-)} = A \exp(-\beta \phi^4/4) \]  

(192)

The normalisation constant ‘A’ can be calculated easily and is given by

\[ A = (8\beta)^{1/8} / \sqrt{\Gamma(1/4)}, \]  

(193)

where \( \Gamma(z) \) is the Euler Gamma-function. Hence, the SUSY prediction for the ground state wave function is given by

\[ \psi_0^{(DWO)}(\phi, \beta)|_{SUSY} = (8\beta)^{1/8} \exp(-\beta \phi^4/4) / \sqrt{\Gamma(1/4)} \]  

(194)

Using the value of \( \Gamma(1/4) \), eqn.(7.17) can be approximately represented as:

\[ \psi_0^{(DWO)}(\phi, \beta)|_{SUSY} = (0.68108)\beta^{1/8} \exp(-\beta \phi^4/4). \]  

(195)

On the other hand, the ground state wave function in LO-NGAS corresponds to that of an *effective simple harmonic oscillator* with variable frequency determined by the corresponding *gap-equation*. For the case of the sextic-DWO, the NGAS result is given by

\[ \psi_0^{(DWO)}(\phi, \beta)|_{NGAS} = \left( \frac{\omega_a}{\pi} \right)^{1/4} e^{-\omega_a \phi^2/2} \]  

(196)

where ‘\( \omega_a \)’ is the solution for the ground state of “gap-equation” given in eqn.(174) i.e.,

\[ \omega_a^4 + g \omega_a^2 - \left( \frac{45}{2} \right) \lambda = 0 \]  

(197)
The physical, acceptable solution of the above “gap-equation” is given by

\[ \omega = \left[ \sqrt{\left( g^2 + 90\lambda \right) - g} \right]^{1/2} \tag{198} \]

On substitution of \( \lambda = \frac{1}{2} \beta^2 \); \( g = 3\beta \), the eqn.(198) leads the following result

\[ \left( \frac{\omega(\beta)}{\pi} \right)^{1/4} = \left( \frac{1.4747}{\pi} \right)^{1/4} \beta^{1/8} = 0.828 \beta^{1/8} \tag{199} \]

Using eqn.(7.22) in eqn.(7.19) we have the following equation

\[ \psi_{0}^{(DWO)}(\phi, \beta) |_{LO-NGAS} = \left( \omega_{a}(\beta) / \pi \right)^{1/4} \exp(-\omega_{a}(\beta)\phi^2/2) \]
\[ = 0.828 \beta^{1/8} \exp(-\omega_{a}\phi^2/2). \tag{200} \]

where ‘\( \omega_{a} \)’ satisfies the gap-equation, eqn.(197); ‘\( \beta \)’ is defined by eqn.(189) and \( \xi = 1/2 \), corresponding to the ground state of the DWO. Comparison of the coefficients of the exponential terms in eqs. (194) and (200) interestingly reveals that not only the “\( \beta^{1/8} \)” factor is common but also that the coefficients: 0.68108 and 0.828 are comparable. We compare the two results in Figure 1 for \( \beta = 100 \). The quality of the approximation can be judged from this figure. It is plausible that the inclusion of higher order corrections to the ground state wave function in IPT of NGAS (Chapter-10) may further improve the agreement.

To summarize the results of this Chapter, it is shown that NGAS respects and preserves the exact results of SUSYQM with good accuracy. It would be interesting to extend the comparison\(^{120}\) to the system of self-interacting oscillators described in SUSYQM by the super potentials:

\( W_{\pm} \equiv \beta\phi^3 \pm \gamma\phi \), which generate a family of sextic-, quartic- and quadratic-AHO/DWO for different values of the parameters ‘\( \beta \)’ and ‘\( \gamma \)’. This is, however, beyond the scope of the present thesis.
Figure 1:
Comparision of the ground state wave function of the sextic-DWO predicted by SUSY (curve with sharper peak) with that obtained in LO of NGAS for $\beta = 100$, see eqs. (195) and (200) of text.
8. The NGAS for the Otic-Anharmonic Oscillator (Leading Order(LO) Results)

8.1 The Otic-Anharmonic Oscillator: Introduction

The octic-anharmonic oscillator, like its quartic- and sextic-counterparts, finds applications in modelling molecular physics, lattice-vibrations in solids and in quantum chemistry. Because of the higher anharmonicity, the system also provides the theoretical laboratory for more stringent-tests for non-perturbative approximation schemes in quantum theory since the divergence of the na"ive (Rayleigh-Schr"odinger) perturbation theory becomes still more severe\cite{12,13,14} in this case.

The system is considered here to test the generality of the application of NGAS to the case of still higher anharmonicity and hence to test the reliability of the said scheme. The detailed application is described below.

To demonstrate further the generality and uniformity of the approximation (NGAS), we apply the method to the case of the next higher anharmonicity, i.e., the octic-anharmonic oscillator, described by the following Hamiltonian:

8.2 Application of NGAS to the Otic-AHO

8.2.1 The derivation of the Gap-equation for the Otic-AHO

As before, we start from the Hamiltonian describing the system:

$$H = \frac{1}{2}p^2 + \frac{1}{2}g\phi^2 + \lambda\phi^8; \ g, \ \lambda > 0.$$ \hspace{1cm} (201)

The free field Hamiltonian corresponds to $H_s = \frac{1}{2}p^2 + \frac{1}{2}g\phi^2$ and the interaction term is given by $\lambda H_I(\phi) = \lambda\phi^8$. To develop the NGAS for the octic AHO we follow the identical ansatz for $V(\phi)$ as given in eqn.(62):

$$V(\phi) = A\phi^2 - B\phi + C$$ \hspace{1cm} (202)
The parameters A, B, C can be determined self-consistently as before. For this purpose, consider the “Effective Hamiltonian (EH)” in this case given by,

\[ H_0(\phi, p) = \frac{1}{2} p^2 + \frac{1}{2} g\phi^2 + \lambda V(\phi) \]  \hspace{1cm} (203)

As in the case of quartic- and sextic- anharmonicity, eqn.(203) is transformed into the following diagonal structure by using the eqn.(202) and given by:

\[ H_0(\phi, p) = \frac{1}{2} p^2 + \frac{1}{2} \omega^2(\phi - \sigma)^2 + h_0 \]  \hspace{1cm} (204)

where again,

\[ \omega^2 = g + 2\lambda A, \]  \hspace{1cm} (205)

\[ \sigma = \frac{\lambda B}{\omega^2}, \]  \hspace{1cm} (206)

\[ h_0 = \lambda C - \frac{1}{2} \omega^2\sigma^2. \]  \hspace{1cm} (207)

Eqn.(204) corresponds to “shifted” effective Harmonic oscillator. The field ‘\( \phi \)’ is shifted by ‘\( \sigma \)’ and the energy is shifted by ‘\( h_0 \)’. As in other cases here also the physical requirement is that \( \omega > 0 ; \sigma = \text{real} \).

Using the creation- and annihilation operators defined by the eqs.(67) and (68) along with the equal-time commutation relation defined by the eqn.(69) the EH, \( H_0 \) given in eqn.(204) is expressed into the desired diagonal form

\[ H_0 = \omega \left( N_b + 1/2 \right) + h_0. \]  \hspace{1cm} (208)

To obtain eqn.(208), we have introduced the number operator \( N_b \equiv b^\dagger b \) and its eigen-states by, \( N_b|n> = n|n> \), \( <m|n> = \delta_{mn} \) as usual. This leads to the energy spectrum as in the earlier cases:

\[ E^{(0)}_n = \omega \xi + h_0, \]  \hspace{1cm} (209)

where \( \xi = (n + 1/2) ; n = 0, 1, 2, ..... \)
It is next required to determine the frequency \( \omega \) and \( h_0 \) defined by the eqn.(207). For this purpose, we note that the quantum average of the eqn.(201) is given by:

\[
< H > = < H_0 > = \frac{1}{2} < p^2 > + \frac{1}{2} g < \phi^2 > + \lambda < \phi^8 >
\]  (210)

where eqn.(55) has been used and \( < \phi > \), \( < \phi^2 > \), \( < p^2 > \) have been calculated using the standard properties of creation-/annihilation operators which are given in eqs.(80) and (81). The QA of the last term in eqn.(210) can be obtained by standard procedure and is given by:

\[
< \phi^8 > = \sigma^8 + 28\sigma^6(\frac{\xi}{\omega}) + \frac{105}{4\omega^2}\sigma^4(4\xi^2 + 1) + \frac{35}{2}\left(\frac{\sigma^2}{\omega^3}\right)(4\xi^2 + 5) + \frac{35}{128}\left(\frac{1}{\omega^4}\right)(16\xi^4 + 56\xi^2 + 9)
\]  (211)

Substituting the QA- values in eqn.(210) we obtain the equation

\[
< H_0 > = \frac{1}{2} \omega \xi + g\left[\sigma^2 + (\xi/\omega)\right] + \lambda\left[\sigma^8 + 28\sigma^6(\frac{\xi}{\omega}) + \frac{105}{4\omega^2}\sigma^4(4\xi^2 + 1) + \frac{35}{2}\left(\frac{\sigma^2}{\omega^3}\right)(4\xi^2 + 5) + \frac{35}{128}\left(\frac{1}{\omega^4}\right)(16\xi^4 + 56\xi^2 + 9)\right]
\]  (212)

Applying the variational minimisation condition \( \partial < H_0 > / \partial \omega = 0 \) leads to the following equation

\[
\omega^5 - \omega^3(g + 56\lambda\sigma^6) - 105\omega^2(\lambda\sigma^4/\xi)(4\xi^2 + 1) - 105\omega\lambda\sigma^2(4\xi^2 + 5) - 35\lambda h(\xi) = 0,
\]  (213)

where \( h(\xi) = \xi^3 + (7\xi/2) + (9/16\xi) \). Eqn.(213) is referred as the “gap-equation” for the case of octic AHO.

Again carrying out minimisation with respect to \( \sigma \) i.e., \( \partial < H_0 > / \partial \sigma = 0 \) we have the following equation

\[
\sigma \left[ g + \lambda(8\sigma^6 + 168\sigma^4(\xi/\omega) + 105\sigma^2(4\xi^2 + 1)/\omega^2 + 35\xi(4\xi^2 + 5)/\omega^3) \right] = 0.
\]  (214)

Eqn.(214) corresponds to equation to ground state (EGS) for this case. The solution of eqn.(214) for \( \sigma \) leads the two following equations:

\[
(i) \quad \sigma = 0.
\]  (215)
\( (ii) \ [g + \lambda (8\sigma^6 + 168\sigma^4(\xi/\omega) + 105\sigma^2(4\xi^2 + 1)/\omega^2 + 35\xi(4\xi^2 + 5)/\omega^3)] = 0. \)  

(216)

Obviously, eqn.(216) has no physically acceptable solution for \( \lambda, g, \omega > 0 \). Hence the ‘physical’ solution of the EGS, eqn.(214), is at \( \sigma = 0 \). Substitution of this value in eqn.(213) leads to the simplified GE, given by:

\[
\omega^5 - g\omega^3 - 35\lambda h(\xi) = 0 \tag{217}
\]

It is then straightforward to evaluate \( A, B, C \) appearing in the approximating potential, eqn.(202) in terms of ‘\( \omega \)’ and ‘\( \sigma \)’. These are given by the following expressions:

\[
A = 28\sigma^6 + \frac{105\sigma^4(1+4\xi^2)}{2\xi\omega} + \left( \frac{105\sigma^2}{2\omega^2} \right)(5+4\xi^2) + \frac{35h(\xi)}{2\omega^3}, \tag{218}
\]

\[
B = \sigma \left[ (1+g)\frac{\omega^2}{\lambda} + 8\omega^2\sigma^6 + 168\sigma^4\xi\omega + 105\sigma^2(1+4\xi^2) + \left( \frac{35\xi}{\omega} \right)(5+4\xi^2) \right], \tag{219}
\]

\[
C = <\phi^8> - A <\phi^2> + B <\phi>. \tag{220}
\]

8.2.2 Solution of the Gap Equation and Determination of the Energy Spectrum

The solution of eqn.(217) determinines the frequency ‘\( \omega \)’ of the “shifted” harmonic oscillator. To obtain the energy levels one substitutes \( \sigma = 0 \) and ‘\( \omega \)’ as the solution of eqn.(217). This leads, after some simplification, to the following simple formula:

\[
E_n^{(0)}|_{octic-AHO} = \left( \frac{\xi}{8} \right)(5\omega + \frac{3g}{\omega}), \ g > 0. \tag{221}
\]

where ‘\( \omega \)’ is obtained by solving eqn.(217) numerically.

In Table-5, we compare this LO- result in NGAS with earlier computations over a wide range of values of ‘\( \lambda \)’ and ‘\( n \)’. It can be seen from this comparison.
that the results obtained in the LO of NGAS are already quite accurate over the full range of the parameters, which demonstrates the generality of the method and uniformity of the approximation with increasing anharmonicity.

We next turn our attention to the physics of the effective vacuum state: $|\text{vac}>$, obtained as an approximation to the true vacuum of the theory.
Table- 5: Sample results for the octic- AHO in the LO NGAS compared with results of earlier calculations from ref.[133] (shown in parentheses), over a wide range of ‘λ’ and ‘n’.

| n | λ = 0.1 | 1.0  | 5.0  | 50.0 | 200.0 |
|---|----------|------|------|------|-------|
| 0 | 1.3005   | 1.7794| 2.3290| 3.5565| 4.6425 |
|   | (1.2410) | (1.6413)| (2.1145)| (3.1886)| (4.1461)|
| 1 | 4.4717   | 6.3946| 8.5167| 13.172| 17.259 |
|   | (4.2754) | (5.9996)| (7.9296)| (12.1950)| (15.9519)|
| 2 | 8.6264   | 12.717| 17.126| 26.698| 35.062 |
|   | (8.4530) | (12.421)| (16.711)| (26.033)| (34.183)|
| 4 | 19.763   | 30.026| 40.863| 64.165| 84.444 |
|   | (19.9930)| (30.4605)| (41.495)| (65.202)| (85.8251)|
| 6 | 34.217   | 52.669| 72.044| 113.48| 149.47 |
|   | (35.0560)| (54.1403)| (74.0830)| (116.7629)| (153.83)|
| 8 | 51.570   | 80.013| 109.65| 172.99| 227.97 |
|   | (53.146)| (82.6496)| (113.3486)| (178.9215)| (235.82)|
| 9 | 61.239   | 95.255| 130.64| 206.23| 271.81 |
|   | (63.225)| (98.5529)| (135.26)| (213.6157)| (281.5864)|
| 10| 71.532   | 111.49| 153.01| 242.64| 318.52 |
|   | (73.954)| (115.49)| (158.5991)| (250.5751)| (330.3433)|
| 11| 82.424   | 128.68| 176.69| 279.14| 368.06 |
|   | (85.308)| (133.42)| (183.3103)| (289.71)| (381.97)|
| 12| 93.893   | 146.79| 201.65| 318.67| 420.14 |
|   | (97.2636)| (152.31)| (209.3443)| (330.9440)| (436.3695)|
| 13| 105.92   | 165.79| 227.84| 360.14| 474.85 |
|   | (109.7967)| (172.11)| (236.6436)| (374.1834)| (493.4143)|
| 14| 118.49   | 185.65| 255.21| 403.50| 532.06 |
|   | (122.89)| (192.81)| (265.1732)| (419.3737)| (553.0335)|
9. The Property, Structure, Stability and the Significance of the “Effective” Vacuum in NGAS

9.1 The Bogoliubov Transformation relating the Free-Field Vacuum to the Effective Vacuum

The study of the properties and the structure of the vacuum of interacting quantum systems are of considerable importance\(^{146}\). In the present scheme, the vacuum state, \(|\text{vac} >\) of the effective Hamiltonian \(H_0\), approximates the vacuum of the true interacting theory in the leading order. To study its properties and structure in comparison to the “free”-field vacuum, \(|0 >\), it is useful to start with eqs.(71-73, 88-90). In view of eqn.(90), the creation- and annihilation operators of the ‘free-theory’ and the approximated theory with self-interaction, are related by quantum-canonical transformation (“Bogoliubov-Transformation”)\(^{52}\), given by:

\[
b = a \cosh(\alpha) - \tilde{a}^\dagger \sinh(\alpha) \tag{222}
\]

\[
b^\dagger = \tilde{a}^\dagger \cosh(\alpha) - a \sinh(\alpha) \tag{223}
\]

The two vacua are then related by the following equations:

\[
|\text{vac} > = \exp[(1/2) \tanh(\alpha) (a^\dagger a^\dagger - aa)]|0 > \equiv U(\alpha; a, a^\dagger)|0 > \tag{224}
\]

The derivation of eqn.(224) follows from eqn.(222) by using the defining property of the vacuum, \(b|\text{vac} > = 0\) and the representation of the annihilation operator given by, \(a = \delta/\delta\tilde{a}^\dagger\). The parameter ‘\(\alpha\’) occurring in the above equations, can be simply related to ‘\(\omega\’) by using the eqs.(71,88,222) as given below:

From eqs.(71) and (88) one can obtain,

\[
b + b^\dagger = \sqrt{\frac{\omega}{\omega_0}} (a + a^\dagger) \tag{225}
\]

Again from eqs.(222) and (223) we have the relation

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By solving eqs.(225) and (226) we get the relation between ‘\(\alpha\)’ and ‘\(\omega\)’ and is given by

\[
e^{\alpha} = \sqrt{\frac{\omega_0}{\omega}}
\]  

(227)

Then,

\[
\alpha = \frac{1}{2} \ln(\omega_0/\omega), \ \omega_0 = \sqrt{g}
\]

(228)

It is useful to have the transformation inverse to eqs.(222-223). This is given by

\[
a = b \cosh(\alpha) + b^{\dagger} \sinh(\alpha)
\]

(229)

\[
a^{\dagger} = b^{\dagger} \cosh(\alpha) + b \sinh(\alpha)
\]

(230)

The following significant physical results follow from the above equations, eqs.(222-225).

### 9.2 Structure of the effective vacuum

(i) A non-trivial structure (“dressing”) of the “effective” vacuum (EV) of the theory emerges from the equations. The situation could be analogous to the case of the ground state of the super-fluid\textsuperscript{147} and the hard sphere Bose-gas\textsuperscript{49}. The structure is characterized by the non-vanishing number density of the free particles in the EV and its critical dependence on the strength of the interaction, which is shown below:

The number density of the free particles in the effective vacuum (EV) is defined by the relation

\[
n_0 \equiv <\text{vac}|a^{\dagger}a|\text{vac}>
\]

(231)

From eqs.(229) and (230) and using the defining property of the vacuum, \(b|\text{vac} > = 0 = <\text{vac}|b^{\dagger}\) one obtains the relation

\[
<\text{vac}|a^{\dagger}a|\text{vac} > = \sinh^2(\alpha)
\]

(232)
Hence by using eqn.(227) we have the following equation

\[ n_0 \equiv \langle \text{vac}|a^\dagger a|\text{vac} \rangle = \sinh^2(\alpha) = \frac{1}{4} \left( \frac{\omega}{\omega_0} + \frac{\omega_0}{\omega} - 2 \right) \quad (233) \]

Considering the case of the quartic- AHO, it can then be shown using the ‘gap- equation’ that \( n_0 \sim \lambda^{1/3} \) for \( \lambda \gg 1 \). In the limit of vanishing interaction, one recovers the expected behaviour, \( n_0 \to 0 \) for \( \lambda \to 0 \).

(ii) Secondly, eqs.(222-224,228) imply an entirely new physical interpretation of the parameter, ‘\( \omega \)’ which determines (through eqs.(228) and (235)) the “vacuum structure function” ‘\( \alpha \)’ in the sense that \( \alpha \neq 0 \) (i.e., \( \omega \neq \omega_0 \)) signifies the non-trivial structure of the EV in presence of interaction.

9.3 Stability of the Effective Vacuum

In the remaining part of this Chapter, we investigate the stability properties of the EV.

Instability of the Perturbative(“free-field”) Vacuum

It is shown below that the perturbative vacuum, \( |0> \) becomes unstable compared to the effective vacuum \( |\text{vac}> \) for all values of the coupling strength ‘\( \lambda \)’. For this demonstration we consider, for reasons of simplicity, the case of the QAHO. The standard method for studying the stability properties is to consider the “effective potential (EP)”. The EP, for any given choice of a vacuum state, is defined to be the expectation value of the Hamiltonian in the chosen vacuum-state and expressed as a function of the VEV of the “field” ‘\( \phi \)’. For the case of the QAHO, this is obtained in LO of NGAS, from eqn.(86) by choosing, \( g = 1, n = 0 \) and ‘\( \omega \)’ constrained to satisfy eqn.(92). The resulting expression is as follows:

\[ V_{NGAS}^{eff}(\sigma) = \frac{\omega}{4} + \frac{(1 + 12\lambda \sigma^2)}{4\omega} + \frac{3\lambda}{4\omega^2} + V_c, \quad (234) \]

where,

\[ V_c = \frac{1}{2} \sigma^2 + \lambda \sigma^4, \quad (235) \]
is the “classical potential” and ‘ω’ satisfies eqn.(92).

An analogous expression for the corresponding EP based upon the perturba-
tive (free-field) vacuum is obtained by the substitution, \( \omega \to 1 \) in eqn.(234) above (this follows by comparing, eqs.(71) and (88)) and is given by

\[
V_{eff}^{Pert}(\sigma) = \frac{1}{2} + 3\lambda(\sigma^2 + \frac{1}{4}) + V_c
\]  

The ground state energy is defined to be the \textit{global} minimum of the effective potential and corresponds to \( \sigma = 0 \) in either case. We thus obtain the respective ground state energies from eqn.(234) given by the following equations:

\[
E_0 = \frac{\omega}{4} + \frac{1}{4\omega} + \frac{3\lambda}{4\omega^2} = \frac{1}{8}(3\omega + 1/\omega)
\]  

The above relation is obtained by using the GE for the ground state: \( \omega^3 - \omega - 6\lambda = 0 \). However, \( E_0^{Pert} \) is obtained from eqn.(236), by using \( \sigma = 0 \) and given by

\[
E_0^{Pert} = 1/2 + 3\lambda/4.
\]  

(Note that eqn.(237) is also contained in eqn.(109) for the special case considered here (i.e. \( g = 1, n = 0 \))). Recalling that the GE for the ground state is given by: \( \omega^3 - \omega - 6\lambda = 0 \), it is straightforward to establish that:

\[
E_0 - E_0^{Pert} < 0, \text{ for all values of } \lambda.
\]  

which is shown below.

By using eqs.(237) and (238) we have the relation

\[
\Delta E(\lambda) = E_0 - E_0^{Pert} = \frac{1}{4}(\omega + \frac{1}{\omega}) + \frac{3\lambda}{4}(\frac{1}{\omega^2} - 1) - 1/2.
\]  

Here it is to be noted that as \( \lambda \to 0 \; ; \; \omega \to 1 \) then \( \Delta E(\lambda) \to 0 \) as it should! Now let us study how \( \omega(\lambda) \) and \( \Delta E(\lambda) \) behave as \( \lambda \simeq 0 \). Recalling the gap equation for the ground state we have the equation:

\[
\omega^3 - \omega - 6\lambda = 0
\]  

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By substituting $\omega = 1 + \xi(\lambda) ; |\xi(\lambda)| << 1$ in eqn.(241) then $\xi \simeq 3\lambda + O(\lambda^2) $ as $\lambda \to 0$. Hence in the $\lim_{\lambda \to 0}$, $\omega(\lambda) \simeq 1 + 3\lambda$. So it is seen that $'\omega'$ is monotonically increasing with $'\lambda'$ as $'\lambda'$ increases!

Substituting this value of $'\omega'$ in the eqn.(240) and after some simplification we get the relation

$$\lim_{\lambda \to 0} \Delta E(\lambda) = E_0 - E_0^{Pert} = -\left( -\frac{9\lambda^2}{2} \right) < 0 \quad (242)$$

From this relation it is evident that $\Delta E(\lambda)$ is monotonically decreasing with increasing $'\lambda'$ (for $\lambda > 0$)! This proves the instability of the perturbative (free-field) vacuum of the QAHO. It may be noted that, although we have established the above result for the QAHO, the same can be rigorously demonstrated in all other cases of anharmonicity considered here.

**9.4 Significance of the Effective Vacuum state**

In the context of quantum-mechanics, the results presented in the previous sections: 9.1 - 9.3 are significant in many respects which are listed below:

(a) In the first-place, the results contained in eqs.(231-233) demonstrate that the effective vacuum state is endowed with non-trivial particle-content and structure, which is directly attributable to interaction since the particle-content vanishes as $\lambda \to 0$. This result is analogous to the “dressing” of the “physical”-vacuum and is made more transparent when one considers the $\lambda\phi^4$-field theory discussed subsequently in Chapter-11.

(b) The other significant result is the equivalence of the NGAS-vacuum with that obtained through quantum canonical/ Bogoliubov-transformation as depicted in eqs.(222-230). It also demonstrates that the effective-vacuum (EV) state is not obtainable in perturbation-theory due to non-analytic dependence of the “vacuum-structure function (VSF)” $'\alpha'$ on the coupling-strength $'\lambda'$ (see,eqn.(228) and the gap-equation, e.g., eqn.(241)).

(c) The third significant observation is contained in the conclusive demonstration of the instability of the “perturbative”/ free-field vacuum in respect of the EV, as contained in eqn.(242). In this context it may be further significant that the convergence of the perturbation theory about the free-field vacuum discussed in the next Chapter may have a direct bearing with the
stability properties of the theory. This is discussed in greater-detail in the next Chapter.

10. Improved Perturbation Theory (IPT) in NGAS

10.1 Considerations of the Convergence of the ‘naive’ Perturbation Theory and other Variants

One of the main motivations for proposing the NGAS as described earlier, is the possibility of construction of an improved perturbation theory (IPT) which could be convergent for all allowed values of of the coupling strenth, ‘g’ and ‘λ’. This expectation is based upon the result, eqn.(59), which is reproduced below:

\[ <n|\lambda H'|n> = 0 \]  \hspace{1cm} (243)

Since \( H = H_0 + \lambda H' \), eqn.(243) naturally suggests that the IPT be constructed by choosing \( H_0 \) as the unperturbed Hamiltonian and \( \lambda H' \) as the perturbation. The convergence of the resulting IPT is intuitively suggested since the basic condition of convergence is satisfied by ensuring that the magnitude of the perturbation remains always sub-dominant to the unperturbed contribution:

\[ |<n|\lambda H'|n>| \equiv 0 < | <n|H_0|n>| \]  \hspace{1cm} (244)

The important point to note is that eqn.(244) holds for arbitrary values of ‘g’, ‘λ’ and ‘n’. In this context, it may be noted that the analogous requirement, which is the necessary condition for convergence of perturbation expansion, does not hold good in the case of naive perturbation theory (NPT), where the entire self-interaction, \( \lambda H_I(\phi) \) is chosen as the perturbation to the ‘free’ Hamiltonian, \( H_s(p, \phi) \) (see eqn.(52)). Consequently, the divergence of the NPT is anticipated as it could be traced to the eventual dominance of the perturbation-contribution over the unperturbed one for any value of \( \lambda > 0 \), no matter however small. This was explicitly demonstrated in ref.[12,13,87].
In the next section, we formulate the improvement of the LO results through the “(improved) perturbation theory” (IPT) developed using $\lambda H'$ as the perturbation while treating $H_0$ as the unperturbed Hamiltonian.

**10.2 Improvement of the LO Results through the IPT in NGAS**

The unique feature of NGAS summarized in eqn.(244) leads to the systematic further (order-by-order) improvement of the LO results (which are already accurate to within a few percent of the exact result). We demonstrate below, the improvement in accuracy, by inclusion of the higher-order contribution in IPT for the case of the QAHO and the QDWO.

In the Rayleigh-Schrödinger (RS) development of the perturbation series, the perturbative correction to the energy levels is given by the standard expansion:

$$E_n = E_n^{(0)} + \Delta E_n^{(1)} + \Delta E_n^{(2)} + \Delta E_n^{(3)} + \ldots$$

(245)

where, the LO-contribution, $E_n^{(0)}$ has already been defined, (see, eqs.(54-55)). It is important to note that the first order contribution $\Delta E_n^{(1)}$ vanishes due to eqn.(243):

$$\Delta E_n^{(1)} = <n|\lambda H'|n> = 0,$$

(246)

(In the above sense, the IPT can be regarded as optimal and this result, eqn.(246), distinguishes the IPT from many other variants of perturbation theory used earlier, for the problem). Using eqn.(246), the next higher order (HO) contributions are given by the following expressions:

$$\Delta E_n^{(2)} = \sum_{m \neq n} |(\lambda H'_{nm})|^2 / \Delta_{nm},$$

(247)

$$\Delta E_n^{(3)} = \sum_{m \neq n, k \neq n} \frac{(\lambda H'_{nm})(\lambda H'_{mk})(\lambda H'_{kn})}{\Delta_{nm} \Delta_{nk}}$$

(248)

Similar expressions for still higher-order corrections can be obtained by standard methods. In the above equations, we have used the following notations: $(\lambda H')_{mn} \equiv <m|\lambda H'|n>$ and $\Delta_{nm} \equiv (E_n^{(0)} - E_n^{(0)})$. The above formulae can
be applied to the case of the QAHO as discussed below.

10.3 IPT- Applied to Quartic-Oscillator (AHO/DWO)

The Hamiltonian for the systems given in eqn.(61) is

$$H = \frac{1}{2} p^2 + \frac{1}{2} g \phi^2 + \lambda \phi^4,$$  \hspace{1cm} (249)

The free field Hamiltonian is given by

$$H_s(\phi) = \frac{1}{2} p^2 + \frac{1}{2} g \phi^2$$  \hspace{1cm} (250)

and the self-interaction term is given by

$$\lambda H_I(\phi) = \lambda \phi^4.$$  \hspace{1cm} (251)

With the new ansatz $V(\phi)$ the effective Hamiltonian is given by

$$H_0(\phi, p) = \frac{1}{2} p^2 + \frac{1}{2} g \phi^2 + \lambda V(\phi)$$  \hspace{1cm} (252)

where $V(\phi)$ is given by the equation

$$V(\phi) = A \phi^2 - B \phi + C.$$  \hspace{1cm} (253)

The parameters A,B,C are determined and given by the ens.(95), (97), (99) with the choice $\sigma = 0$ (for the case of the QAHO). The modified interaction $\lambda H' = H - H_0 = (\lambda \phi^4 - V(\phi))$ have vanishing quantum average for arbitrary $\lambda$, $g$ and ‘n’ as has been noted. Hence, the first order contribution vanishes as given in eqn.(246). To calculate the higher order perturbative contribution, the matrix elements for the case of the QAHO, are given by

$$<m|m|n> = \frac{1}{2} \omega \delta_{m,n+2} \sqrt{(n+1)(n+2)} + \sqrt{n(n-1)}.$$  \hspace{1cm} (255)

where,

$$<m|\phi^2|n> = (1/2 \omega) \delta_{m,n+2} \sqrt{(n+1)(n+2)} + \delta_{m,n+2} \sqrt{n(n-1)}.$$  \hspace{1cm} (254)
\( < m | \phi^4 | n > = (1/4 \omega^2) (\delta_{m,n+4} \sqrt{(n+1)(n+2)(n+3)(n+4)} + \delta_{m,n-4} \sqrt{n(n-1)(n-2)(n-3)} + 2(\delta_{m,n+2} (2n+3) \sqrt{(n+1)(n+2)}) + 2(\delta_{m,n-2} (2n-1) \sqrt{n(n-1)}) \) (256)

(The above matrix elements are easily calculated by using eqn.(71) and introducing the basis states \(|\text{vac} >\), and \(|n >\); \(n = 1, 2, 3, ..\) with the defining property of the effective vacuum \(b|\text{vac} > = 0 = < \text{vac}|b^\dagger\), and further relations: \(|n > = (b^\dagger)^n |\text{vac} >; \hat{N} = b^\dagger b; \hat{N}|n > = n|n >; b^\dagger|n > = \sqrt{n+1}|n >; b|n > = \sqrt{n}|n-1 >\).

We present in Table-1, (see page -46) the results for the energy levels of the QAHO, with the inclusion of the second-order perturbation correction. In the same Table we also compare our results with available ‘exact’ numerical results and results of calculation in second order perturbation theory of ref.[44], which is based upon the operator methods. It may be seen from this Table that the accuracy is considerably improved by inclusion of the perturbation correction and further that the convergence of the IPT is found superior (order-by-order) to that in ref.[44]. Similar results for the QDWO after inclusion of the second order perturbative correction in IPT, is presented in Table-2 (at page-54). In this Table, we also compare our results with those obtained by inclusion of twenty orders in the ‘modified’ perturbation theory of ref.[36]. Again uniform improvement in accuracy is seen. It is also seen from the Table that the results of the present analysis which includes only the first non-trivial perturbative correction, compares well with the results of ref.[36] obtained by sophisticated numerical methods. In the context of the above results, the following observations may be relevant:

10.4 Remarks on Convergence and other aspects of IPT in NGAS

(a) Corrections up to the fourth-order in IPT have been computed for the QAHO and the QDWO although we have reported only the second order
correction in the Tables-1,2. It is seen that these higher order corrections remain uniformly small compared to the LO results over the full range of ‘λ’ and ‘n’ and decrease fast with the order of correction, which is consistent with the expectations from a rapidly converging sequence.

(b) As has been demonstrated in the previous section, the ‘perturbative’ ground state (i.e., corresponding to the free-field Hamiltonian $H_s(\phi, p)$, see eqn.(52)) becomes unstable compared to the ground state of the ‘effective’ Hamiltonian, $H_0$. Thus the stability of the theory, and the convergence of the IPT- both appear to critically depend on the choice of $H_0$. It may perhaps be plausible, therefore, to conjecture that the convergence of the perturbation theory may be intimately connected with the choice of a stable vacuum resulting from a proper effective Hamiltonian chosen as the unperturbed part.

(c) When compared with the results of some other variants of perturbation theories$^{150}$ applied to the above systems of anharmonic and double well oscillators, the IPT appears to provide better convergence, when compared at each order.

(d) For the case of the QDWO, the results of IPT, as well as those from the other variants of perturbation theories, show poor convergence near the transition point: $\lambda \sim \lambda_c(\xi)$ as intuitively expected$^{151}$. However, since $\lambda_c$ is small ($\lambda_c(\xi) \leq 0.0362886$), this limitation does not affect most applications of practical interest. In particular, the strong coupling regime, $\lambda >> 1$ is excellently described by the IPT.

(e) Although we have provided only plausible argument in support of the convergence of the IPT, a formal proof can be attempted following the methods available in the literature$^{152}$.

In the following Chapter, we extend the method to $\lambda \phi^4$ - quantum field theory in (3 + 1) dimensions.

11. Application of NGAS to $\lambda \phi^4$ - Field Theory in (3 + 1) dimensions
11.1 Formulation of the NGAS for $\lambda\phi^4$ quantum field theory

We consider, in this chapter, the application of the NGAS to $\lambda\phi^4$ theory in (3+1) dimensions. In view of the successful application of the NGAS to the AHO’s and DWO’s as described in earlier chapters, it is but natural to extend the formalism to $\lambda\phi^4$ quantum field-theory and test the consequences of the scheme.

It may be worth while pointing out here that the $\lambda\phi^4$ - field theory in physical dimensions ( and in lower dimensions ) is an important physical system which finds crucial applications in diverse areas of physics, e.g., the standard model of particle-physics\textsuperscript{84}, cosmology\textsuperscript{153}, condensed matter physics\textsuperscript{121}, phase-transitions and critical phenomena\textsuperscript{75} etc. Besides, this theory provides the simplest theoretical laboratory for testing the various approximation schemes in quantum field-theory(QFT). Hence it becomes imperative to test the current approximation scheme NGAS by applying the same to $\lambda\phi^4$ - QFT.

In this thesis, we consider the theory in the massive, symmetric-phase described by the Lagrangian:

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi)(\partial^{\mu} \phi) - \frac{1}{2} m^2 \phi^2 - \lambda \phi^4, \quad (257)$$

where $m^2 > 0$. The Hamiltonian density derived from the above Lagrangian is given by

$$\mathcal{H} = \frac{1}{2} (m^2 \phi^2 + \phi_t^2 + \phi_\alpha^2) + \lambda \phi^4 \quad (258)$$

where we have defined: $\phi_t \equiv \partial \phi(\vec{x},t)/\partial t$ and $\phi_\alpha \equiv \partial \phi(\vec{x},t)/\partial x^\alpha$.

To formulate the NGAS for the above theory we follow analogous steps as in the cases of AHO/DWO considered earlier and choose an approximating potential (AP) denoted by $V(\phi)$ such that ideally the defining constraints as given by eqs.(52-57) are satisfied. However, unlike the case in quantum
mechanics, the PEQA involving multiparticle-states, is hard to implement in QFT. Hence, on grounds of simplicity, we relax the condition, eqn.(56) by restricting the QA to be evaluated in the “few-particle” states only. To be more specific, we proceed as follows:

11.1.1 Choice of $V(\phi)$

Using the most general ansatz quadratic in the fields $\phi$, we parametrise $V(\phi)$ as given below:

$$V(\phi) = A\phi^2 - B\phi + C$$  \hspace{1cm} (259)

As before, the “effective Hamiltonian” (EP) $H_0$ is defined as:

$$H_0 \equiv \frac{1}{2}(m^2\phi^2 + \phi_t^2 + \phi_\alpha^2) + \lambda V(\phi)$$  \hspace{1cm} (260)

Substitution of eqn.(259) in eqn.(260) leads, after some simplification, to the following expression:

$$H_0 = \frac{1}{2}M^2\xi^2 + \frac{1}{2}\xi^2 +$$  \hspace{1cm} (261)

where,

$$\xi(\vec{x},t) \equiv \phi(\vec{x},t) - \sigma ,$$  \hspace{1cm} (262)

$$\sigma \equiv \frac{\lambda B}{M^2} ,$$  \hspace{1cm} (263)

$$M^2 \equiv m^2 + 2\lambda A ,$$  \hspace{1cm} (264)

and

$$h_0 \equiv \lambda C - \frac{1}{2}M^2\sigma^2.$$  \hspace{1cm} (265)

In addition, $\xi_t \equiv \partial \xi / \partial t$; $\xi_\alpha \equiv \partial \xi / \partial x^\alpha$ etc. Eqn.(261) is atonce identified to be the Hamiltonian density of the hermitian scalar-field $\xi(\vec{x},t)$. This is not surprising since the AP was chosen accordingly. However, the important point to emphasize is that the AP is proposed to incorporate the effects of self-interaction even though the EH corresponds to that of an exactly solvable system. To demonstrate this aspect it is first necessary to obtain the spectrum of $H_0$ which is done as follows.
The diagonalisation of the EH given by eqn.(261) is straightforward by using the Fourier expansion in terms of creation- and annihilation operators:

\[ \xi(\vec{x},t) = \phi(\vec{x},t) - \sigma = \int \frac{d^3k}{\Omega_k(M)} [b(\vec{k})e^{-ikx} + b^\dagger(\vec{k})e^{ikx}] , \]  

(266)

where

\[ \Omega_k(M) \equiv 2(2\pi)^3 \sqrt{\frac{1}{k^2|M^2 + \omega_k(M)\omega_k(M)}}, \]  

(267)

and \( kx \equiv k^0t - \vec{k} \cdot \vec{x} \), as usual. The operators \( b(\vec{k}), b^\dagger(\vec{k}) \) satisfy the standard (equal-time) commutation relations (ETCR) :

\[ [b(\vec{k}), b^\dagger(\vec{q})] = \Omega_k(M) \delta^3(\vec{k} - \vec{q}), \]  

(268)

which is a consequence of the ETCR between the ‘field’ \( \phi(\vec{x},t) \) and its canonical conjugate momentum : \( \pi(\vec{x},t) \equiv \partial L/\partial \dot{\phi} \), given by :

\[ [\phi(\vec{x},t), \pi(\vec{y},t)] = i \delta^3(\vec{x} - \vec{y}). \]  

(269)

11.1.2 Diagonalisation of the Effective Hamiltonian

The energy of the system described by \( H_0 \) is obtained by standard methods and given by :

\[ H_0 \equiv \int d^3x \mathcal{H}_0(\vec{x},t) = \frac{1}{2} \int \frac{d^3k}{\Omega_k(M)} [b(\vec{k})b^\dagger(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})] + \int d^3\vec{x} h_0 \]  

(270)

The spectrum of the states are analogously obtained and denoted by : \( |\text{vac} > \), \( |\vec{p} >, |\vec{p}_1, \vec{p}_2 >, .... \) etc where the effective vacuum state \( |\text{vac} > \) is defined by :

\[ b(\vec{k}) |\text{vac} > = 0 , \]  

(271)

and the multi particle-states are generated by multiple application of the creation-operator \( b^\dagger(\vec{p}) \) on \( |\text{vac} > \) :

\[ b^\dagger(\vec{p}) |\text{vac} > = |\vec{p} > \]  

(272)
\[
\frac{b^\dagger(p_1) b^\dagger(p_2)}{\sqrt{2!}} = | p_1, p_2 >, \text{ etc}
\]

Let us note that, \( H_0 | \text{vac} > = E_0 | \text{vac} >; H_0 | \vec{p} > = E_1 | \vec{p} >, \text{ etc} \). where \( E_0, E_1 \) etc correspond to the energy of the corresponding states.

The next step is the implementation of the PEQA.

11.1.3 PEQA for \( \lambda \phi^4 \) QFT in NGAS

This requirement translates to the following constraints:

\[
< \text{vac} | \phi^4 | \text{vac} > = < \text{vac} | V(\phi) | \text{vac} > ,
\]

\[
< \vec{p} | \phi^4 | \vec{p} > = < \vec{p} | V(\phi) | \vec{p} > ,
\]

and similarly for multi-particle states. The implementation of eqs.(274, 275) require the evaluation of the QA of monomials of the field \( \phi(\vec{x},t) \) such as:

\[
< \text{vac} | \phi^n(\vec{x},t) | \text{vac} > = < \phi^n(\vec{x},t) >, \text{ etc.}
\]

We first turn to evaluation of \( < \phi^n(\vec{x},t) > \). This is readily done using \textit{translational – invariance} of the vacuum-state : \( | \text{vac} > \) and the ETCR, as given by eqn.(268). Some useful results thus obtained are given below:

\[
< \phi(\vec{x},t) > = \sigma ,
\]

\[
< \phi^2(\vec{x},t) > = \sigma^2 + \int \frac{d^3k}{\Omega_k(M)} \equiv \sigma^2 + I_0 ,
\]

\[
< \phi^4(\vec{x},t) > = \sigma^4 + 6\sigma^2 I_0 + 3I_0^2 , \text{ etc.}
\]

Similarly,

\[
< \phi^2_\alpha(\vec{x},t) > = \int \frac{d^3k}{\Omega_k(M)} |k|^2 ,
\]

\[
< \phi^2_\beta(\vec{x},t) > = \int \frac{d^3k}{\Omega_k(M)} \omega_k^2(M)
\]

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11.1.4 Determination of the AP

Using the above results in eqn.(274) one determines the AP uniquely, which is given by

\[ V(\phi) = 6(\sigma^2 + I_0)\phi^2 - 2\sigma^3 \phi - 3(\sigma^2 + I_0)^2. \]  

(281)

In other words, the coefficients A, B, C defining the AP (see, eqn.(259)), are uniquely determined:

\[ A = 6(\sigma^2 + I_0) \]  

(282)

\[ B = 2\sigma^3 \]  

(283)

\[ C = -3(\sigma^2 + I_0)^2. \]  

(284)

It is further verified by explicit calculation that the above choice of the AP guarantees not only the equality of the QA over the vacuum state; i.e. \( <V(\phi)> = <\phi^4> \) (eqn. (274)), but also that for the one-particle states as well, i.e. \( <\vec{p}|V(\phi)|\vec{p}> = <\vec{p}|\phi^4|\vec{p}> \), as required by eqn.(275).

It is important to emphasize at this point that eqns.(282-284) when considered together with eqns.(263-265) form a complete set of self-consistency conditions which uniquely specify the physical consequences of the theory in the leading-order (LO), such as the spectrum, renormalisation, stability properties and the structure of the effective vacuum. These physical consequences of the theory are discussed in the following sections.

11.2 The “Effective Potential (EP)” and Renormalisation in LO

Before we discuss the (non-perturbative) renormalisation programme in LO, it is useful to first investigate the consequences of eqs.(262-265) considered together with eqs.(282-284). Substitution of eqn.(282) in eqn.(264) leads to the following equation:

\[ M^2(\lambda, \sigma) \equiv m^2 + 12\lambda\sigma^2 + 12\lambda I_0(M^2), \]  

(285)
i.e.,

\[ M^2(\lambda, \sigma) \equiv m^2 + 12\lambda \sigma^2 + 6\lambda \int \frac{d^3(\vec{k})}{(2\pi)^3 \sqrt{|\vec{k}|^2 + M^2}} . \]  

(286)

Eqn.(286) can be interpreted as the generation of the ‘mass-gap’ (i.e. shift in the bare-mass) due to interaction. In analogy with the terminology used earlier, we refer eqs.(285-286) as the “gap-equation(GE)” of the theory. This equation plays crucial role in the subsequent discussions.

Similarly, consideration of eqn.(283) together with eqn.(263) leads to the “ equation for the ground-state (EGS)”

\[ \sigma \left[ \sigma^2 - \frac{M^2}{2\lambda} \right] = 0 \]  

(287)

As in case of the QAHO and QDWO’s considered in earlier chapters, it is convenient to first obtain the solution of the EGS, eqn.(287). The two solutions of eqn.(287) are given by:

\[ \sigma = 0 , \]  

(288)

and

\[ \sigma^2 = \frac{M^2}{2\lambda} \]  

(289)

We show below that eqn.(289) is not an acceptable solution on physical grounds. To establish this result substitute eqn.(289) in eqn. (285) which leads to:

\[ 5M^2 = -(m^2 + 12\lambda I_0(M^2)) \]  

(290)

Since \( M^2 \geq 0 \) is the physical requirement for definition of the theory (see, e.g. eqn.(267) and eqn.(270)), equation (290) can not lead to acceptable solution unless

\[ \lambda < 0 \text{ i.e. } \lambda \equiv -g , \ g > 0 ; \]  

(291)

and

\[ 12gI_0(M^2) - m^2 \geq 0 \]  

(292)
(Note that $\lambda < 0$ can not apriori be ruled out since bare-parameters in the Lagrangian are unobservable, see later). However, now the defining equation, eqn.(289) can then be rewritten as:

$$\sigma^2 = -\frac{M^2}{2g} < 0,$$  \hfill (293)

which is not acceptable on grounds of the hermiticity i.e. $\phi^\dagger = \phi$! Thus eqn. (289) is ruled out as a solution and the unique physical solution of the ground state corresponds to eqn. (288). Having thus fixed the ground-state configuration, the implementation of the renormalisation programme can be done by defining the effective-potential (EP). The latter is defined as:

$$U_0(\sigma) \equiv <\text{vac}|H_0|\text{vac}>,$$  \hfill (294)

such that

$$\sigma \equiv <\text{vac}|\phi(\vec{x},t)|\text{vac}>).$$  \hfill (295)

It may be noted that the l.h.s. of eqn. (294) is defined to be a function of $\sigma$ alone. This means that any other parameter occurring in $H_0$ is to be variationally fixed by minimisation of $<H_0>$. The procedure is made explicit below by working out the current example, with $H_0$ defined in eqn.(260). To this end, we first calculate $<H_0>$ by using the eqn.(274) which guarantees the following equation:

$$<H_0> \equiv <\mathcal{H}> = \frac{1}{2}m^2 <\phi^2> + \frac{1}{2} <\phi_t^2>
+ \frac{1}{2} <\phi_\alpha^2> + \lambda <\phi^4> .$$  \hfill (296)

This works out to be:

$$<H_0> = \frac{1}{2}m^2(\sigma^2 + I_0) + \frac{1}{2} \int \frac{d^3(\vec{k})}{\Omega_k(M)} \left( \omega_k^2(M^2) + |\tilde{k}|^2 \right)
+ \lambda \left( \sigma^4 + 6\sigma^2I_0 + 3I_0^2 \right).$$  \hfill (297)

This can be rewritten as, (using $\omega_k^2(M^2) \equiv |\tilde{k}|^2 + M^2$)

$$<H_0> = I_1 - \frac{1}{2}I_0 \left( m^2 + 12\lambda\sigma^2 + 12\lambda I_0 \right)
+ \frac{1}{2}m^2 \left( \sigma^2 + I_0 \right) + \lambda \left( \sigma^4 + 6\sigma^2I_0 + 3I_0^2 \right),$$  \hfill (298)

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where, we have defined:

$$I_n(x) \equiv \int \frac{d^3k}{\Omega_k(x)} [\omega_k^2(x)]^n, \ n = 0, \pm 1, \pm 2, \ldots \quad (299)$$

These integrals were first introduced by Stevenson$^{154}$. Eqn.(298) can be simplified further by using the "gap-equation" as given by eqn.(285). One then obtains:

$$\langle H_0 \rangle = I_1(M) - 3\lambda I_0^2(M) + \frac{1}{2} m^2 \sigma^2 + \lambda \sigma^4 \quad (300)$$

We thus derive the LO-effective potential of NGAS as given by

$$U_0(\sigma) = \frac{1}{2} m^2 \sigma^2 + \lambda \sigma^4 + I_1(M) - 3\lambda I_0^2(M), \quad (301)$$

where, it is implicitly understood that the "gap-equation", eqn.(285) is to be first solved to obtain $M^2$ as a function of $\sigma$.

One can next carry out the renormalisation programme (in the LO) by noting that$^{154}$:

(i) the vacuum-configuration corresponds to the absolute (global) minimum of $U_0(\sigma)$, i.e. by solving:

$$\left. \frac{dU_0}{d\sigma} \right|_{\sigma_0} = 0; \quad \left. \frac{d^2U_0}{d\sigma^2} \right|_{\sigma_0} > 0 \quad (302)$$

(ii) the renormalised mass in LO is given by:

$$m^2_R \equiv \left. \frac{d^2U_0}{d\sigma^2} \right|_{\sigma = \sigma_0}, \quad (303)$$

(iii) the LO-renormalised coupling strength is likewise defined to be:

$$\lambda_R \equiv \frac{1}{4!} \left. \frac{d^4U_0}{d\sigma^4} \right|_{\sigma = \sigma_0}, \quad (304)$$

where $\sigma_0$ corresponds to the vacuum-configuration as defined by eqn.(302). It is directly verified by minimisation of $U_0(\sigma)$ (see eqn.(302)), that the global minimum of the former occurs at $\sigma_0 = 0$, which is consistent with
eqn.(288) as it should be. Next, evaluating eqn.(303) at \( \sigma_0 = 0 \) one gets the renormalised mass:

\[
m^2_R = m^2 + 12\lambda I_0(m^2_R) \equiv M^2(\lambda, \sigma^2 = 0), \tag{305}
\]

Similarly, after a straightforward calculation, one obtains ref.[154] the renormalised coupling as given by:

\[
\lambda_R = \lambda \left[ \frac{1 - 12\lambda I_1(m_R)}{1 + 6\lambda I_1(m_R)} \right], \tag{306}
\]

where again \( I_1(x) \) is defined as per the general definition given in eqn.(299).

At this stage several remarks / observations are in order:

(a) The results contained in eqs. (285,286) and eqs. (301-306) were first derived by Stevenson\textsuperscript{154} in the context of the “Gaussian effective potential (GEP)” for the symmetric \( \lambda \phi^4 \) theory and obtained by variational calculation using a Gaussian-trial wave-function.

The reproduction of the results of the GEP in ref.[154] in the LO of NGAS demonstrates that the GEP (see, eqn.(301)) is contained in the NGAS as the leading order approximation.

(b) The development of the IPT (see, chapter-10) has therefore, the potential to go beyond the Gaussian approximation in a systematic way, order by order. (The application of IPT to \( \lambda \phi^4 \) theory falls however, beyond the scope of the present dissertation.)

(c) In view of the equivalence with the GEP in the leading order, all the results obtained in the former approximation, are reproduced in the LO of the NGAS. In particular, the demonstration of non-triviality of the symmetric \( \lambda \phi^4 \) theory in the GEP, being entirely based upon the consequences of renormalisation as contained in eqs.(305) and (306), is also reproduced in the LO of NGAS. We discuss in the following sub-section, some of the results concerning the stability and non-triviality of the theory.

(d) It must be emphasized, however, that the current scheme, NGAS is based
upon entirely different starting assumptions and is much more general than the GEP, which is obtained solely due to the choice of the AP as given in eqn.(259) and that too, in the leading order.

We next discuss some of the consequences of the above non-perturbative renormalisation scheme obtained in the LO of NGAS leading to the stability and non-triviality of the theory.

11.3 Stability and non-triviality of $\lambda \phi^4$-theory in the LO of NGAS

For the above purpose, it is convenient to start with eqs.(285) and (286), which involve the divergent integral due to the momentum-integration and which, therefore, need a suitable method of subtraction.

Using the subtraction-procedure devised by Stevenson$^{154}$, eqn.(306) can be inverted to express $\lambda$ in terms of the observable parameters, $\lambda_R$ and $m_R$. This leads to two solutions for $\lambda$ of which, the physical one is given by:

$$\lambda = [-1/6I_{-1}(m_R)][1 + 1/2[\lambda_R I_{-1}(m_R) + .......]$$

( The other solution is $\lambda = -(1/2)\lambda_R + 0(1/I_{-1}(m_R))$. This solution can be shown to lead to instability, since the minimum of the EP corresponding to this solution lies (infinitely) higher than the minimum corresponding to eqn.(307)).

It may be noted at this point that eqn.(307) implies a viable, stable $\lambda \phi^4$ theory when the (unobservable) bare-coupling becomes negative but infinitesimal. This version of the theory has therefore, been designated in ref.[155] as precarious $\lambda \phi^4$ theory.

Substituting for $\lambda$ as given by eqn.(307), one can solve for the bare-mass, by inverting eqn.(305), after carrying out the subtraction as per the Stevenson-prescription. This leads to the following expression for the bare-mass:

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\[ m^2 = m_R^2 + 2I_0(m_R)/I_{-1}(m_R) + (\text{sub-leading terms}) \]  

(308)

With the aid of eqs.(307) and (308), the effective potential, as given by eqn.(301) can be recast in manifestly renormalised form involving the observable parameters: \( \lambda_R \) and \( m_R \) only. The resulting expression is given by:

\[
U_0(\sigma) = U_{\text{min}} + \frac{1}{4} t m_R^2 \sigma^2 - (m_R^4/128\pi^2)(t - 1)^2 - (m_R^4/64\pi^2)(t - 1)\eta,
\]

where

\[
t = M^2(\sigma)/m_R^2; \quad \eta \equiv -4\pi^2/\lambda_R.
\]

(309)

and

\[
U_{\text{min}} = I_1(m_R) - 3\lambda I_0^2(m_R).
\]

(310)

Similarly, the renormalised version of the “gap-equation” is given by:

\[
(1 - \eta)(t - 1) - (16\pi^2/m_R^2)\sigma^2 = t \ln t.
\]

(312)

It must be pointed out that, one has to first solve the gap-equation, eqn.(312) to obtain \( t \equiv t(\sigma) \), which is then to be substituted in eqn.(309) to infer the \( \sigma \)-dependence of \( U_0(\sigma) \).

It may be noted that the gap-equation, eqn.(312) is a transcendental equation and its solution exists only when,

\[
\sigma^2 \leq \sigma^2_{\text{min}} \equiv (m_R^2/16\pi^2)[e^{-\eta} + \eta - 1]
\]

(313)

The domain of validity of the effective potential (EP) is thus restricted by the range of \( \sigma^2 \) determined by eqn.(313) for any value of \( \eta \). In particular, in the regime of large-coupling ( \( \eta \to 0 \) ), the domain of the EP shrinks with \( \eta \) since \( \sigma^2_{\text{min}} \to 0 \) in this limit. This is the situation, therefore, of small oscillations about \( \sigma \approx 0 \) and it corresponds to the pathological situation when \( |\lambda_R| \to \infty \). On the other hand, the small coupling regime ( \( \lambda_R \to 0 \)), which corresponds to \( \eta >> 1 \), the domain of EP increases with
η. One can thus summarise that the LO-EP of the symmetric $\lambda \phi^4$ theory is reasonable and well behaved unless the renormalised coupling is very large. One can thus conclude that a non-trivial and stable theory results in the LO of NGAS provided the physical coupling $\lambda_R$ is not unusually large. We further comment upon the issue of stability of the perturbative vacuum of the theory in the following subsection.

To study the stability issue, it is necessary to compute the effective potential based upon the perturbative vacuum (i.e. the vacuum of the free-field theory). This is easily achieved by letting $M \to m$ in all formulae (see, eqn.(264)). Thus starting from eqn.(297) and letting $M \to m$ one obtains, after simplification the following expression:

$$<H_0>_P = \frac{1}{2}m^2\sigma^2 + \lambda\sigma^4 + \bar{I}_1 + 6\lambda\sigma^2\bar{I}_0 + 3\lambda\bar{I}_0^2$$  \hspace{1cm} (314)$$

In the above, $<H_0>_P$ denotes $<0|H_0|0>$ i.e. VEV of $H_0$ in the perturbative vacuum state; and $\bar{I}_n \equiv I_n(m^2)$. By definition, the effective-potential based upon the perturbative vacuum denoted by $U_P(\sigma)$ is identified with $<H_0>_P$. Hence one obtains:

$$U_P(\sigma) = \frac{1}{2}m^2\sigma^2 + \lambda\sigma^4 + \bar{I}_1 + 6\lambda\sigma^2\bar{I}_0 + 3\lambda\bar{I}_0^2$$  \hspace{1cm} (315)$$

The renormalised-parameters following from the eqn.(315) are like-wise computed and denoted by

$$\bar{m}_R^2 = \frac{d^2U_P}{d\sigma^2}|_{\sigma^2 = 0} ;$$  \hspace{1cm} (316)$$

$$\bar{\lambda}_R = \frac{(1/4!)d^4U_P}{d\sigma^4}|_{\sigma^2 = 0}$$  \hspace{1cm} (317)$$

where $\sigma^2 = 0$ is again the location of the global-minimum of $U_P(\sigma)$, as can be readily verified. In this context, it must be emphasized that the integrals : $\bar{I}_n$ occurring in eqn.(315) are independent of $\sigma$. Next, computing the derivatives of $U_P(\sigma)$ at the minimum, one obtains the following expressions for the renormalised parameters based upon the perturbative vacuum:

$$\bar{m}_R^2 = m^2 + 12\bar{I}_0$$  \hspace{1cm} (318)$$
The requirement of the finiteness of $\bar{m}_R$ and $\bar{\lambda}_R$ then demands that $\lambda$ must be $-ve$ (ref.to eqn.(318)), for otherwise $\bar{m}_R^2$ would be infinitely large since $\bar{I}_0$ is divergent and the bare (unobservable) mass, $m^2 > 0$. However, this would lead to instability since the effective-potential $U^P(\sigma)$ will not have a lower-bound! This is made manifest by explicitly writing the EP in terms of the renormalised parameters:

$$U^P(\sigma) = \frac{1}{2} \bar{m}_R^2 \sigma^2 + \bar{\lambda}_R \sigma^4 + 3 \bar{\lambda}_R \bar{I}_0^2 + \bar{I}_1$$

To prevent instability of the theory when renormalised about the perturbative vacuum it, therefore, becomes inescapable that

$$\bar{\lambda}_R = \lambda = 0,$$

which is nothing but the trivality scenario!

A few remarks/observations are in order, in view of the above results:

(i) We believe that the result in (eqn.(321)) constitutes perhaps, the most direct demonstration of triviality of symmetric $\lambda\phi^4$ theory in physical dimensions.

(ii) At the same time, the result, eqn.(321) also demonstrates that the conclusion of trivality of the theory is an artefact of the naive perturbation theory built and renormalised around the free-field vacuum. As demonstrated earlier, the theory renormalised about the NGAS-vacuum leads to a perfectly acceptable, stable and non-trivial $\lambda\phi^4$-theory (see, eqs.(309-313) and discussions following).

(iii) It may be further pointed out that the ground state of the trivial theory is still unstable as compared to that in the LO of NGAS, i.e.

$$U_{\min} << U_{\min}^P,$$

which is readily established by referring to eqn.(320) (with $\sigma = 0$) and eqn.(311).
This completes our results and discussions regarding the stability and the triviality of $\lambda \phi^4$- theory in the context of NGAS in the LO.

In the next subsection, we discuss the structure of the interacting vacuum in analogy with the results obtained in the case of the AHO/DWO’s considered earlier.

11.4 Properties of the Interacting Vacuum State (IVS) in NGAS

The actual/physical vacuum state in presence of interaction is approximated in the LO of NGAS by the state: $|\text{vac}\rangle$ which is the lowest energy state of $H_0$. In analogy with the results obtained for the AHO/DWO’s in chapter-9, the structure and properties of this state can be inferred from studying the quantum-canonical transformation (Bogoliubov-Valatin transformation, ref.[52]) connecting the interacting vacuum state (IVS) with the free-field vacuum (FFV) state.

For this purpose it is convenient to start from the Fourier-decomposition of the field $\phi(\vec{x},t)$ in terms of the free-field creation- and annihilation operators analogous to eqn.(266):

$$
\phi(\vec{x},t) = \sigma + \int \frac{d^3k}{\Omega_k(m)}[a(\vec{k})e^{-ikx} + a^\dagger(\vec{k})e^{ikx}],
$$

where, now

$$
\Omega_k(m) \equiv 2(2\pi)^3\omega_k(m) ; k^0 = \omega_k(m) \equiv \sqrt{\vec{k}^2 + m^2},
$$

corresponding to the propagation of the free-field quanta satisfying the mass-shell condition: $k^0^2 - |\vec{k}|^2 = m^2$. The free-field operators satisfy the standard commutation relations:

$$
[a(\vec{k}), a^\dagger(\vec{q})] = \Omega_k(m) \delta^3(\vec{k} - \vec{q}).
$$

Comparison of eqs.(268) and (325) implies that the modified operators:

$$
B(\vec{k}) \equiv \frac{b(\vec{k})}{\sqrt{\Omega_k(M)}}
$$
\[ A(\vec{k}) \equiv \frac{a(\vec{k})}{\sqrt{\Omega_k(m)}} \quad (326) \]

satisfy identical commutation relations:

\[ [B(\vec{k}), B^\dagger(\vec{q})] = \delta^3(\vec{k} - \vec{q}) = [A(\vec{k}), A^\dagger(\vec{q})]. \quad (327) \]

It follows, therefore, that the two sets of operators must be connected through Bogoliubov-transformation ref.[52] given by:

\[
\begin{align*}
B(\vec{k}) &= \cosh(\alpha_k)A(\vec{k}) - \sinh(\alpha_k)A^\dagger(-\vec{k}) \\
B^\dagger(\vec{k}) &= \cosh(\alpha_k)A^\dagger(\vec{k}) - \sinh(\alpha_k)A(-\vec{k}),
\end{align*} \quad (328)
\]

whereas the inverse transformation is given by:

\[
\begin{align*}
A(\vec{k}) &= \cosh(\alpha_k)B(\vec{k}) + \sinh(\alpha_k)B^\dagger(-\vec{k}) \\
A^\dagger(\vec{k}) &= \cosh(\alpha_k)B^\dagger(\vec{k}) + \sinh(\alpha_k)B(-\vec{k}).
\end{align*} \quad (329)
\]

In the above, \( \alpha_k = f(|\vec{k}|) \), is apriori an arbitrary real function of \( |\vec{k}| \), i.e.

\[ \alpha_{-\vec{k}} = \alpha_{\vec{k}} = \alpha_{\vec{k}}^*. \quad (330) \]

However, eqs. (268), (323) and (328) considered together further imply that

\[ \exp(2\alpha_k) = \frac{\omega_k(M)}{\omega_k(m)} = \frac{\sqrt{|\vec{k}|^2 + M^2}}{\sqrt{|\vec{k}|^2 + m^2}} \quad (331) \]

To show this, consider eqs.(268) and (323) at \( t = 0 \), which can be written as:

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\[
\phi(\vec{x}, t) = \sigma + \int \frac{d^3\vec{k}}{\sqrt{\Omega_k(m)}} [ A(\vec{k}) + A^\dagger(-\vec{k}) ] e^{i\vec{k} \cdot \vec{x}}
\]

\[
= \sigma + \int \frac{d^3\vec{k}}{\sqrt{\Omega_k(M)}} [ B(\vec{k}) + B^\dagger(-\vec{k}) ] e^{i\vec{k} \cdot \vec{x}},
\]

which implies that:

\[
\{ B(\vec{k}) + B^\dagger(-\vec{k}) \} = \sqrt{\frac{\Omega_k(M)}{\Omega_k(m)}} \{ A(\vec{k}) + A^\dagger(-\vec{k}) \}
\]

(333)

However, from eqn.(328) it follows that

\[
\{ B(\vec{k}) + B^\dagger(-\vec{k}) \} = \exp (\alpha_k) \{ A(\vec{k}) + A^\dagger(-\vec{k}) \}
\]

(334)

thus leading to the desired result, eqn.(331).

To obtain the information regarding the particle-content and other features of the IVS it is instructive to first compute the number-density of the free-field-quanta residing in the IVS. To this end let us note that the free-field-number operator is given by the standard expression:

\[
N \equiv \int \frac{d^3\vec{k}}{\Omega_k(m)} a^\dagger(\vec{k}) a(\vec{k})
\]

\[
= \int d^3\vec{k} A^\dagger(\vec{k}) A(\vec{k})
\]

(335)

Hence the desired number density of the free-field quanta in the IVS is given by

\[
n(\vec{k}) = \langle \text{vac} \mid \frac{A^\dagger(\vec{k}) A(\vec{k})}{v} \mid \text{vac} \rangle,
\]

(336)

where \( v \equiv \text{spatial-volume of quantisation} \equiv \int d^3\vec{x} \). Using eqs.(329), eqn.(336) is easily evaluated:

\[
n(\vec{k}) = \frac{\sinh^2(\alpha_{\text{vac}}(\vec{k}))}{(2\pi)^3}
\]

(337)
where \( \alpha^{\text{vac}}(\mathbf{k}) \) is given by eqn.(331) evaluated for \( M \equiv m_R = \) free-particle-mass renormalised about the IVS, \(|\text{vac}\rangle\). (It may be recalled that \( M(\sigma = 0) = m_R \) and \( \sigma = 0 \) define the IVS.) This leads finally to the expression:

\[
n(\mathbf{k}) = \left( \frac{1}{32\pi^3} \right) \left[ \frac{\omega_k(m)}{\omega_k(m_R)} + \frac{\omega_k(m_R)}{\omega_k(m)} - 2 \right].
\]

(338)

To extract further meaningful content from eqn.(338), we note that the bare-mass is divergent: \( \left( \frac{m}{m_R} \right) \sim 0(\frac{\Lambda}{\sqrt{\ln \Lambda}}) \) where \( \Lambda = \) momentum cut-off (see, eqn.(309)). Since, according to the standard prescription of the renormalisation procedure, the cut-off must be removed (i.e. \( \Lambda \to \infty \)) prior to the calculation of any physical quantity of the theory, one obtains:

\[
\lim_{\Lambda \to \infty} \left( \frac{n(\mathbf{k})}{n(0)} \right) \equiv \rho(\mathbf{k}) = \left( 1 + \frac{|\mathbf{k}|^2}{m_R^2} \right)^{-\frac{3}{2}},
\]

(339)

where \( n(\mathbf{0}) = n(\mathbf{k})|_{\text{max}} = \left( \frac{1}{32\pi^3} \right) \left( \frac{m}{m_R} \right) \), is the maximum value of \( n(\mathbf{k}) \), occurring at \( \mathbf{k} = 0 \).

Equation (339) provides direct physical content for the non-trivial structure of the IVS representing a condensate of off-shell correlated particle-pairs. The situation is analogous to the structure of the physical vacuum state in case of the hard-sphere Bose-gas\(^{49}\) and superfluidity\(^{147}\). It is therefore, plausible that eqn.(339) might lead to interesting consequences for \( T \neq 0 \), as happens in the case of the super-fluid and the hard-sphere Bose-gas.

12. Summary, Conclusions and Outlook

In summary, a new scheme of approximation in quantum theory, is presented which is simple, non-perturbative, self-consistent and systematically improvable. The scheme is, in principle, applicable to arbitrary interacting systems. We have, however, confined the application of the method to the quartic, sextic and octic anharmonic oscillators, to the quartic and sextic double well oscillators and to the \( \lambda \phi^4 \) symmetric QFT in the present work.
The essential method of this scheme of approximation consists of finding a “mapping” which maps the “interacting system” on to an “exactly solvable” model, while preserving the major effects of interaction through the self consistency requirement of equal quantum averages of observables in the two systems.

This approximation method has the advantage over the naive perturbation theory (NPT) and the variational approximation by transcending the limitations of both: unlike the variational method, it is systematically improvable through the development of an improved perturbation theory (IPT) whereas, in contrast to the case of the NPT, the latter satisfies the necessary condition of convergence for all allowed values of the quadratic and the anharmonic coupling strengths ‘$g$’ and ‘$\lambda$’. The method reproduces the results obtained by several earlier methods\textsuperscript{155} but transcends the limitations of these methods in respect of wider applicability, systematic improvement and better convergence.

A remarkable feature of the scheme is that it respects the exact predictions of super symmetric quantum mechanics (SUSYQM) to a good degree of accuracy in case of the sextic anharmonic oscillator and the sextic double well potential, when these form a set of “partner potentials”. In particular, the property of “iso-spectrality”, “positivity” of energy levels and the predictions for the “exact” ground state wave function are reproduced with good accuracy even in the lowest order of approximation.

We have also investigated the stability properties and the structure of the ‘effective’ vacuum (EV) of the exactly solvable Hamiltonian, $H_0$, which models the fully interacting system in the leading order. In particular, it is shown that the free-field (“perturbative”) vacuum is unstable for all values of the couple strength in comparison with the EV. Moreover, the latter is endowed with a rich structure (“dressing”) in terms of the free-field quanta manifested by the increasing number density of particles with the strength of the interaction, in analogy with the case of the super fluid Helium and the hard sphere Bose gas. The application of the method to quantum statistics, non-oscillator systems and field theory appear to be straightforward.

The present approach to $\lambda\phi^4$ theory reproduces the main results of the Gaussian approximation\textsuperscript{154}. This is considered quite significant since the
two approaches are based upon rather different physical assumptions. How-
never, the authors consider the present approach to be more general than
the Gaussian approximation since the former provides a dynamical expla-
nation of the latter through the mechanism of altered vacuum-structure in-
troduced by the interaction. Besides, the authors go beyond the scope of
the Gaussian-approach in establishing new results, e.g., the calculation of
the momentum-distribution of the condensate-structure function ‘n(k)’. It
may be emphasized that by going beyond the LO of NGAS, the results of
the Gaussian approximation can be systematically improved, order-by-order.

As has been demonstrated, application of NGAS in the LO leads to a
nontrivial and stable $\lambda$-$\phi^4$ theory in the symmetric phase. It is well-known,
however, that lattice investigations$^{156}$ of $\phi^4$-field theory indicate the trivial-
ity scenario and miss the non-trivial version arrived at here and in ref.[154].
This result can be succinctly understood as follows: the lattice regularised
version of the theory corresponds to a finite ultraviolet-cut-off. This means
that the bare coupling $\lambda$ is small $\sim 0(1/I_{\text{lattice}})$ and negative for the case
considered in this work. However, the range of the classical field $\sigma$ remains
unrestricted. For this reason, there always exist sufficiently large values of
$\sigma$ (for any given lattice-spacing) such that the term $\lambda \sigma^4$ occurring in the
effective potential (see, Eq.(301) dominates over all other terms (Note that
integrals, $I_n$ are all finite on the lattice). Therefore, $V_{\text{lattice}}(\sigma)$ becomes
unbounded from below hence, unstable since $\lambda$ is negative!

In a continuum theory, however, the ultra-violet cut-off is never actually
present: if a cut-off is introduced to regularise the theory, the same has to
be sent to infinity first prior to considering any other limiting behaviour,
such as $|\sigma| \to \infty$. This crucial difference in the order of taking limits:
(UV-Cut-off) $\Lambda \to \infty$ and $|\sigma| \to \infty$, makes all the difference in the physical
content of the theory in the two approaches and explains why, for any finite
lattice-spacing, it will not be possible to discover the stable and non-trivial
version of the theory presented in this work and in ref.[154]. For a detail
discussion on this important point, see ref.[154].

The resulting momentum distribution of the vacuum condensate struc-
ture function ‘$n(k)$’ deserves special mention as it displays the non-standard
feature of an appreciable spread in $|k|$ about the origin scaled by the renor-
malised mass of the physical quanta. It is reasonable to expect that this
condensate-structure of the physical vacuum persists to finite temperature manifesting in observable consequences in the thermodynamic properties of the associated system. This would, therefore, constitute a test of the basic underlying assumptions of the approximation.

Finally, it may be worthwhile to compare and comment on related work in the recent literature. As has been remarked earlier, analogous ansatz for the ground state and the field-operators derived by Boguliobov transformation, has been used in ref.[38] for the case of the anharmonic oscillator. The important conclusion that emerges from this study is that a convergent and accurate perturbation theory for the energy levels results when the theory is developed about the trial vacuum-state. In contrast, the perturbation theory is badly divergent\textsuperscript{12-14,79} if developed about the non-interacting (perturbative) vacuum.

The relation of the present variational approximation (which is equivalent in the LO to the Gaussian approximation) to the one-loop approximation method\textsuperscript{157} has been discussed in detail in ref.[154]. It has been shown that the one-loop results are contained, as a special case, in the present results (Eqs.(305-306)) in the limit of small bare-coupling, $\lambda \to 0^+$ when $0(\lambda^2)$ terms are neglected. This means that the one-loop results remain essentially perturbative in nature, even though one resums an infinity of ordinary Feynmann diagrams at the one-loop order. Besides, the one-loop effective potential has a rather restricted domain in both $\lambda$ and $\sigma$ beyond which it shows pathological behaviour. In contrast, the effective potential based upon the present approximation, even in the LO, has a considerably larger range of validity and is genuinely non-perturbative in nature.

In a spirit similar to the present approach, massless $\lambda \phi^4$ theory has been variationally investigated in ref.[158]. It has been shown\textsuperscript{158} in this work that the preferred vacuum is also described by a condensate structure albeit with a different momentum distribution function of the condensate particle density. In lower (1+1) dimension, the GEP has been derived\textsuperscript{55} for the $\lambda \phi^4$ theory employing similar ansatz for the trial vacuum state. However, the underlying condensate structure and renormalisation have not been investigated in these works.

As remarked earlier, possible applications of the results derived here are
envisaged in diverse area of current interest including critical phenomena (involving a scalar field as the order-parameter\textsuperscript{122}), inflationary cosmology\textsuperscript{153}, finite temperature field theory\textsuperscript{159}, exploration of the vacuum structure\textsuperscript{160} of pure gluonic-QCD and Higgs sector of the standard model (by extending\textsuperscript{161} the analysis to the spontaneously broken phase, which corresponds to the case of negative bare- mass $m^2 < 0$).

The present work can be extended in different directions to include: finite temperature field theory, quantum-statistics, application to non-oscillator systems, super-symmetric theories and quantum field theories involving fermions and gauge-fields etc.
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