Quantum Entanglement under Non-Markovian Dynamics of Two Qubits Interacting with a common Electromagnetic Field

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Abstract

We study the non-equilibrium dynamics of a pair of qubits made of two-level atoms separated in space with distance \( r \) and interacting with one common electromagnetic field but not directly with each other. Our calculation makes a weak coupling assumption but no Born or Markov approximation. We derived a non-Markovian master equation for the evolution of the reduced density matrix of the two-qubit system after integrating out the electromagnetic field modes. It contains a Markovian part with a Lindblad type operator and a non-Markovian contribution, the physics of which is the main focus of this study. We use the concurrence function as a measure of quantum entanglement between the two qubits. Two classes of states are studied in detail: Class A is a one parameter family of states which are the superposition of the highest energy \( |I\rangle \equiv |11\rangle \) and lowest energy \( |O\rangle \equiv |00\rangle \) states, viz, \( |A\rangle \equiv \sqrt{p}|I\rangle + \sqrt{1-p}|O\rangle \), with \( 0 \leq p \leq 1 \); and Class B states \( |B\rangle \) are linear combinations of the symmetric \( |+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \) and the antisymmetric \( |\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \) Bell states. We obtain similar behavior for the Bell states as in earlier results derived by using the Born-Markov approximation [40] on the same model. However, in the Class \( |A\rangle \) states the behavior is qualitatively different: under the non-Markovian evolution we do not see sudden death of quantum entanglement and subsequent revivals, except when the qubits are sufficiently far apart. (The existence of sudden death was first reported for two qubits in two disjoint cavity electromagnetic fields [38], and the dark period and revival were found from calculations using the Born-Markov approximation [40]). For an initial Bell state, our findings based on non-Markovian dynamics agree with those obtained under the Born-Markov approximation. We provide explanations for such differences of behavior both between these two classes of states and between the predictions from the Markov and non-Markovian dynamics. We also study the decoherence of this two-qubit system and find that the decoherence rate in the case of one qubit initially in an excited state does not change significantly with the qubits separation whereas it does for the case when one qubit is initially in the ground state. Furthermore, when the two qubits are close together, the coherence of the whole system is preserved longer than it does in the single qubit case or when the two qubits are far apart.
I. INTRODUCTION

Investigation of quantum entanglement is both of practical and theoretical significance: It is viewed as a basic resource for quantum information processing (QIP) [1] and it is a basic issue in understanding the nature of nonlocality in quantum mechanics [2, 3, 4]. However, even its very definition and accurate characterization are by no means easy, especially for multi-partite states (see, e.g., [5, 6, 7, 8, 9, 10, 11]). Nonetheless there are useful criteria proposed for the separability of a bipartite state, pure and mixed (e.g., [12, 13, 14, 15, 16, 17, 18]), theorems proven (e.g., [19, 20]), and new mathematical tools introduced (e.g., [21, 22]), which add to advances in the last decade of this new endeavor [23].

Realistic quantum systems designed for QIP cannot avoid interactions with their environments, which can alter their quantum coherence and entanglement properties. Thus quantum decoherence and disentanglement are two essential obstacles to overcome for the design of quantum computers and the implementation of QIP. Environment-induced decoherence in the context of QIP has been under study for over a decade [24] and studies of environment-induced disentanglement has seen a rapid increase in recent years. There are now experimental proposals to measure finite time disentanglement induced by a reservoir [25]. The relation between decoherence and disentanglement is an interesting one because both are attributable to the decay of quantum interference in the system upon the interaction with an environment. (See, e.g., [26, 27, 28, 29, 30])

In addition to the mathematical investigations mentioned above which could provide rather general characterizations of quantum entanglement, detailed studies of physical models targeting actual designs of quantum computer components can add precious insight into its behavior in concrete settings. Two classes of models relevant to condensed matter and atomic-optical QIP schemes are of particular interest to us. The first class consists of the quantum Brownian motion model (QBM) and the spin-boson model (SBM). Quantum decoherence has been studied in detail in both models, and results on quantum disentanglement are also appearing (See [26, 27, 28, 29] for QBM under high temperature, negligible dissipation, and [31] for an attempt towards the full non-Markovian regimes.) The second class of models describes atoms or ions in a cavity with a quantum electromagnetic field at zero or finite temperature. The model consists of two two level atoms (2LA) in an electromagnetic field (EMF). For a primary source on this topic, read, e.g., [32]. For a more recent
description of its dynamics under the Born-Markov approximation, see the review of \cite{33}.

A. Two-atom entanglement via an electromagnetic field

Quantum decoherence and entanglement between one 2LA and an EMF has been treated by us and many other authors earlier \cite{34,35} and by Cummings and Hu recently towards the strong coupling regime \cite{36}, which provide insight in how the atom-field interaction affects their entanglement. There is recent report of exact solutions found for a 2LA in an EMF using the underlying pseudo-supersymmetry of the system \cite{37}. In the 2 atom-EMF model, the two atoms can be assumed to interact only with its own cavity EMF, or with a common EMF, and they can also interact with each other. The noninteracting case of separate fields was first studied by Yu and Eberly \cite{38,39} where ‘sudden death’ of quantum entanglement was sighted. The noninteracting case of a common field was studied recently by Ficek and Tanas \cite{40}, An \textit{et al} \cite{41}. Quantum decoherence of N-qubits in an electromagnetic field was studied by Palma, Suominen and Ekert \cite{42}. For entanglement of ions in cavities, see, e.g., \cite{43}.

For the purpose of quantum information processing, we have emphasized earlier in our studies of quantum decoherence that it is absolutely essential to keep track fully of the mutual influence of, or the interplay between, the system and the environment. If one chooses to focus only on the system dynamics, one needs to take into consideration the back-action of the environment, and vice versa.

In our prior work \cite{34,35}, we used the influence functional formalism with a Grassmannian algebra for the qubits (system) and a coherent state path integral representation for the EMF (environment). Here, we employ a more standard operator method through perturbation theory, because the assumption of an initial vacuum state for the EMF allows a full resummation of the perturbative series, thus leading to an exact and closed expression for the evolution of the reduced density matrix of the two qubits. This approach incorporates the back-action of the environment on the system self-consistently and in so doing generates non-Markovian dynamics. We shall see that these features make a fundamental difference in the depiction of evolution of quantum entanglement in the qubit dynamics.
B. The importance of including back-action self-consistently

Since quantum entanglement is a more delicate quantity to track down than decoherence, an accurate description is even more crucial. For this, one needs to pay extra attention to back-actions. For example, in the case of two 2LA (system) in a cavity EMF (environment), the two parties are equally important. This means that we should include both the back-action of the field on the atoms, and the atoms on the field. In a more complete treatment as attempted here, we obtain results qualitatively different from earlier treatments where the back-action is not fully included or properly treated \[33\]. Some special effects like ‘sudden death’ \[38\] can in this broader context be seen as consequences only of rather special arrangements: Each atom interacting with its own EMF precludes the fields from talking to each other and in turn cuts off the atoms’ natural inclination (by the dictum of quantum mechanics) to be entangled. In effect, this is only a limiting case of the full dynamics we explored here for the two-qubit entanglement via the EMF. This limit corresponds to the qubits being separated by distances much larger than the correlation length characterizing the total system. For a wide range of spatial separations within the correlation length, entanglement is robust: Our results for the full atom-field dynamics reveal that there is no sudden death.

C. Non-Markovian dynamics from back-action

It is common knowledge in nonequilibrium statistical mechanics \[44\] that for two interacting subsystems the two ordinary differential equations governing each subsystem can be written as an integro-differential equation governing one such subsystem, thus rendering its dynamics non-Markovian, with the memory of the other subsystem’s dynamics registered in the nonlocal kernels (which are responsible for the appearance of dissipation and noise should the other subsystem possess a much greater number of degrees of freedom and are coarse-grained in some way). Thus inclusion of back-action self-consistently in general engenders non-Markovian dynamics. Invoking the Markov approximation as is usually done may lead to the loss of valuable information, especially for quantum systems. These assumptions need to be scrutinized carefully with due consideration of the different time scales in the system and the specific physical quantities that are of interest in the study.
For monitoring the evolution of quantum entanglement which is usually a more delicate process than coherence, if one lacks detailed knowledge of how the different important processes interplay, our experience is that it is safer not to put in any ad hoc assumption at the beginning (e.g., Markovian approximation, high temperature, white noise) but to start with a first principles treatment of the dynamics (which is likely non-Markovian) involving all subsystems concerned and respecting full self-consistency. This is because entanglement can be artificially and unknowingly curtailed or removed in these ad hoc assumptions. What is described here is not a procedural, but a substantive issue, if one seeks to coherently follow or manipulate any quantum system, as in QIP, because doing it otherwise can generate quantitatively inaccurate or even qualitatively wrong results.

Thus the inclusion of backreaction (which depends on the type of coupling and the features of the environment) usually leads to non-Markovian dynamics [51]. Also, under extreme conditions such as imposing infinite cutoff frequency and at high temperatures, the dynamics of, say, a quantum harmonic oscillator bilinearly coupled to an Ohmic bath can become Markovian [43]. Other factors leading to or effecting nonMarkovian behavior include the choice of special initial conditions. For example, the factorizable initial condition introduces a fiducial or special choice of time into the dynamics which destroys time-homogeneity.

A word about terminology might be helpful here: One usually refers to Markovian dynamics as that governed by a master equation with constant-in-time coefficients, i.e., described by a Linblad operator, and non-Markovian for all other types of dynamics. A more restricted condition limits the definition of non-Markovian dynamics to cases with non-trivial (nonlocal in time) integral kernels appearing in the master equation. This more stringent definition would refer to dynamics (depicted by master equations containing coefficients which are) both time-homogeneous and non-time-homogeneous as Markovian. We use the first and more common convention of terminology, in which the master equation (85) which is local in time but non-time-homogeneous would be nonMarkovian. The Markovian regime emerges in the limit when the two qubits are far separated. This feature is similar to the HPZ master equation [46] for quantum Brownian motion, where Markovian (time-homogeneous) dynamics appears only in specific limiting conditions (high temperature and ohmic distribution of environmental modes, as alluded to above).

Our present study of the two qubit (2qb)- EMF system is also aimed at addressing a common folklore, namely, that in quantum optics one does not need to worry about non-
Markovian effects. We will see that there is memory effect residing in the off diagonal components of the reduced density matrix for the 2 qubit system which comes from virtual photon exchange processes mediated by the field and which depends on the qubit separation. Perhaps the simplest yet strongest reason for the need to take non-Markovian effects seriously is that results from the Markovian approximation are incomplete and lead to qualitatively wrong predictions.

D. Relation to prior work and organization of this paper

In this paper we study the non-Markovian dynamics of a pair of qubits (2LA) separated in space by distance $r$ interacting with one common electromagnetic field (EMF) through a Jaynes-Cummings type interaction Hamiltonian. We use the concurrence function as a measure of quantum entanglement between the two qubits. The same model was studied before in detail by Ficek and Tanas [33] using the Born-Markov approximation. In a more recent paper [40] they show the existence of dark periods and revival of quantum entanglement in contrast to the findings of Yu and Eberly which assumes two qubits in disjoint EMFs.

Our calculation makes a weak coupling assumption and ignores the two- and higher-photon- exchanges between the qubits, but it makes no Born or Markov approximation. We derive a non-Markovian master equation, which differs from the usual one of the Lindblad type: it contains extra terms that correspond to off-diagonal elements of the density matrix propagator. We concentrate on two classes of states, superpositions of highest and lowest energy states and the usual antisymmetric $|\psi^-\rangle$ and symmetric $|\psi^+\rangle$ Bell states [3] and observe very different behavior. These are described in detail in the Discussions section. The difference between our results and that of Ref. [40] highlights the effect of non-Markovian (with memory) evolution of quantum entanglement. In short, we find similar behavior in the Class B (Bell) states but qualitative different behavior in the evolution of Class A states. Ref [40] found that their evolution leads generically to sudden death of entanglement and a subsequent revival. In our more complete treatment of the atom-field dynamics we indeed see the former effect present for large values of the inter-qubit distances. However, sudden death is absent for short distances, while there is no regime in which a revival of entanglement can take place. This calls for caution.

Another set of papers close to our work reported here is that of [47] who considered two
2LA in an infinite temperature field bath. When the atoms are separated at large distance the authors assume that they are located inside two independent baths. (The severance of the field is subject to the same criticism above: A small but finite quantum entanglement cannot be equated to zero because the small amount can later grow.) For these conditions and under the Markovian approximation, the time evolution of the two-atom system is given by the ergodic dynamical semi-group. They ignore without justification the effect of distance on the interaction between the qubits. A paper of interest not directly related to the present model but which does show the dependence of the disentanglement rate on distance, like ours reported here, is that by Roszak and Machnikowski [48]. They consider a system of excitons with different coupling, and with a very different infrared behavior of the bath modes. The latter seems not to be relevant to the two atoms’ case here.

This paper is organized as follows: Section 2 contains the main derivation. We write down the Hamiltonian for two 2-level atoms (2LA) interacting with a common electromagnetic field (EMF) at zero temperature, and we compute the relevant matrix elements for the propagator of the total system by resummation of the perturbative series (Appendix A). We then determine the evolution of the reduced density matrix of the atoms, which is expressed in terms of seven functions of time. We compute these functions using an approximation that amounts to keeping the contribution of the lowest loop order for the exchange of photons between the qubits. In Section 3 we examine the evolution of the reduced density matrix for two classes of initial states. We then describe the time evolution of quantum entanglement with spatial separation dependence in these states via the concurrence plotted for some representative cases. In Section 4, we study the decoherence of this system when the two qubits are initially disentangled. We consider two cases that correspond to one of the qubits being initially in a vacuum state and in an excited state. We compare these results with the single qubit cases and highlight the lessening of decoherence due to the presence of other qubit(s). In Section 5 we discuss and compare our results on disentanglement with the work of Yu and Eberly for two 2LA in separate EMF baths, and with the work of Fizek and Tanas on two 2LA in a common EMF bath under the Born-Markov approximation. We identify the point of departure of quantum dynamics under the Markovian approximation from the full non-Markovian dynamics and thereby demonstrate the limitations of the Born-Markov approximation. Finally, we discuss the domain of validity of the rotating wave approximation in describing these systems. In Appendix C we sketch an alternative derivation through the
Feynman-Vernon influence functional technique, in which Grassmann variables are employed for the study of the atomic degrees of freedom.

II. TWO-ATOMS INTERACTING VIA A COMMON ELECTROMAGNETIC FIELD

A. The Hamiltonian

We consider two 2-level atoms (2LA), acting as two qubits, labeled 1 and 2, and an electromagnetic field described by the free Hamiltonian

\[ 
\hat{H}_0 = \hbar \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k + \hbar \omega_o \hat{S}_+^{(1)} \hat{S}_-^{(1)} + \hbar \omega_o \hat{S}_+^{(2)} \hat{S}_-^{(2)} 
\]

where \( \omega_k \) is the frequency of the \( k \)th electromagnetic field mode and \( \omega_o \) the atomic frequency between the two levels of the atom, assumed to be the same for the two atoms. The electromagnetic field creation (annihilation) operator is \( \hat{b}_k^\dagger (\hat{b}_k) \), while \( S_+^{(n)} (S_-^{(n)}) \) are the spin raising (lowering) operators for the \( n \)th atom. We will define the pointing vector from 1 to 2 as \( \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \) and we will assume without loss of generality that \( \mathbf{r}_1 + \mathbf{r}_2 = 0 \).

The two 2LAs do not interact with each other directly but only through the common electromagnetic field via the interaction Hamiltonian

\[ 
\hat{H}_I = \hbar \sum_k g_k \left( \hat{b}_k^\dagger \left( e^{-ik\mathbf{r}/2} \hat{S}_-^{(1)} + e^{ik\mathbf{r}/2} \hat{S}_-^{(2)} \right) + b_k \left( e^{ik\mathbf{r}/2} \hat{S}_+^{(1)} + e^{-ik\mathbf{r}/2} \hat{S}_+^{(2)} \right) \right), 
\]

where \( g_k = \lambda / \sqrt{\omega_k} \), \( \lambda \) being the coupling constant. We have assumed that the dipole moments of the atoms are parallel. The total Hamiltonian of the atom-field system is

\[ 
\hat{H} = \hat{H}_0 + \hat{H}_I. 
\]

B. Perturbative expansion and resummation

We assume that at \( t = 0 \) the state of the combined system of atoms+field is factorized and that the initial state of the EMF is the vacuum \( |O\rangle \). For this reason we need to identify the action of the evolution operator \( e^{-i\hat{H}t} \) on vectors of the form \( |O\rangle \otimes |\psi\rangle \), where \( |\psi\rangle \) is a vector on the Hilbert space of the two 2LA’s.
For this purpose, we use the resolvent expansion of the Hamiltonian
\[ e^{-i\hat{H}t} = \int \frac{dE e^{-iEt}}{E - \hat{H} + i\eta} \]
and we expand
\[ (E - \hat{H})^{-1} = (E - \hat{H}_0)^{-1} + (E - \hat{H}_0)^{-1}\hat{H}_1(E - \hat{H}_0)^{-1} \]
\[ + (E - \hat{H}_0)^{-1}\hat{H}_1(E - \hat{H}_0)^{-1}\hat{H}_1(E - \hat{H}_0)^{-1} + \ldots \]  
(5)

Of relevance for the computation of the reduced density matrix of the two qubits are matrix elements of the form \( \langle z; i', j'|(E - \hat{H})^{-1}|O; i, j \rangle \), where \( z \) refers to a coherent state of the EM field and \( i, j = 0, 1 \), the value \( i = 0 \) corresponding to the ground state of a single qubit and \( i = 1 \) to the excited state. We compute the matrix elements above through the perturbation expansion (5). It turns out that we can effect a resummation of the perturbative series and thus obtain an exact expression for the matrix elements–see Appendix A for details of the resummation.

The non-vanishing matrix elements are the following
\[
\langle z; 0, 0|(E - \hat{H})^{-1}|O; 0, 1 \rangle = \sum_k \frac{g_k z_k^* e^{i\frac{\omega_k}{2}}}{(E - \omega_o - \alpha(E) - \beta(E, r)e^{i\omega_k})(E - \omega_k)}
\]
(6)
\[
\langle z; 0, 1|(E - \hat{H})^{-1}|O; 0, 1 \rangle = \frac{1}{2} \left[ \frac{1}{E - \omega_o - \alpha(E) - \beta(E, r)} \right. \\
+ \left. \frac{1}{E - \Omega - \alpha(E) + \beta(E, r)} \right]
\]
(7)
\[
\langle z; 1, 0|(E - \hat{H})^{-1}|O; 0, 1 \rangle = \frac{1}{2} \left[ \frac{1}{E - \omega_o - \alpha(E) - \beta(E, r)} \right. \\
- \left. \frac{1}{E - \omega_o - \alpha(E) + \beta(E, r)} \right]
\]
(8)
\[
\langle z; 0, 0|(E - \hat{H})^{-1}|O; 0, 0 \rangle = E^{-1}
\]
(9)
\[
\langle z; 1, 1|(E - \hat{H})^{-1}|O; 1, 1 \rangle = \frac{1}{E - 2\omega_o - 2\alpha(E - \omega_o) - f(E, r)}
\]
(10)
\[
\langle z; 0, 0|(E - \hat{H})^{-1}|O; 1, 1 \rangle = \sum_{kk'} \frac{\hat{H}_{kk'} z_k^* z_{k'}^*}{E - 2\omega_o - 2\alpha(E - \omega_o) - f(E, r)}
\]
(11)
\[
\begin{pmatrix}
\langle z; 0, 1|(E - \hat{H})^{-1}|O; 1, 1 \rangle \\
\langle z; 1, 0|(E - \hat{H})^{-1}|O; 1, 1 \rangle
\end{pmatrix} = \sum_{kk'} \frac{g_{kk'} z_k^*}{(E - 2\omega_o)(E - \omega_o - \omega_{k'})} \left[ \frac{e^{-i\frac{\omega_k}{2}}}{e^{i\frac{\omega_k}{2}}} (1 - L)_{kk'} \right]
\]
(12)
In the equations above the functions \( \alpha(E), \beta(E, r) \) are
\[
\alpha(E) := \sum_k \frac{g_k^2}{E - \omega_k}
\]
(13)
\[
\beta(E, r) : = \sum_k \frac{g_k^2}{E - \omega_k} e^{i k \cdot r}.
\]  

The definitions of the kernel \( H_{kk'} \) and of the function \( f \) involve complicated expressions. However, the term involving \( H_{kk'} \) does not contribute to the evolution of the reduced density matrix, while the function \( f(E, r) \) is of order \( \lambda^4 \) and it can be ignored in the approximation we effect in Sec. II.E. Thus for the purpose of this investigation, the explicit definitions of \( H \) and \( f \) are not needed, and hence not given here.

Finally, the matrix \( L \) is defined as
\[
L := \begin{pmatrix}
\Xi & \Theta \\
\Theta & \Xi
\end{pmatrix},
\]
where \( \Xi \) and \( \Theta \) are matrices on the space of momenta
\[
\Xi_{kk'} = \frac{1}{E - \Omega - \omega_k} \left( \alpha(E - \omega_k) \delta_{kk'} + g_k g_{k'} \left( \frac{1}{E - 2\Omega} + \frac{e^{i(k-k') \cdot r}}{E - \omega_k - \omega_{k'}} \right) \right),
\]
\[
\Theta_{kk'} = \frac{1}{E - \Omega - \omega_k} \left( \beta(E - \omega_k, r) \delta_{kk'} + g_k g_{k'} \left( \frac{1}{E - 2\Omega} + \frac{1}{E - \omega_k - \omega_{k'}} \right) \right),
\]
and the overbar denotes complex conjugation.

C. The matrix elements of the propagator

The next step is to Fourier transform the matrix elements of the resolvent in order to obtain the matrix elements of the evolution operator. Explicitly,
\[
\langle z; 0, 0 | e^{-i \hat{H} t} | O; 0, 1 \rangle = \sum_k e^{i k \cdot r / 2} x_k^* s_k(t)
\]
\[
\langle z; 0, 1 | e^{-i \hat{H} t} | O; 0, 1 \rangle = \int dE e^{-i E t} \left[ \frac{1}{E - \omega_o - \alpha(E) - \beta(E, r)} + \frac{1}{E - \omega_o - \alpha(E) + \beta(E, r)} \right] =: v_+(t)
\]
\[
\langle z; 1, 1 | e^{-i \hat{H} t} | O; 0, 1 \rangle = \int dE e^{-i E t} \left[ \frac{1}{E - \omega_o - \alpha(E) - \beta(E, r)} - \frac{1}{E - \omega_o - \alpha(E) + \beta(E, r)} \right] =: v_-(t)
\]
\[
\langle z; 0, 0 | e^{-i \hat{H} t} | O; 0, 0 \rangle = 1
\]
\[
\langle z; 1, 1 | e^{-i \hat{H} t} | O; 1, 1 \rangle = \int dE e^{-i E t} \frac{\delta(E)}{E - 2\omega_o - 2\alpha(E - \omega_o) - f(E, r)} =: u(t)
\]
where we defined the functions $s_k(t), \nu_k(t), \nu'_k(t)$ as

$$s_k(t) = \int \frac{dE e^{-iEt}}{(E - \omega_o - \alpha(E) - \beta(E, r)e^{ikr})(E - \omega_k)}$$

$$\begin{pmatrix} \nu_k(t) \\ \nu'_k(t) \end{pmatrix} = \int \frac{dE e^{-iEt}}{E - 2\omega_o} \sum_{k'} (1 - L)_{kk'} \begin{pmatrix} \frac{g_{kk'}}{E - \omega_o - \omega_{k'}} \\ \frac{g_{kk'}}{E - \omega_o - \omega_{k'}} \end{pmatrix}$$

D. The reduced density matrix

We next compute the elements of the reduced density matrix for the qubit system by integrating out the EM field degrees of freedom

$$\rho_{ij}^{(t)}(t) = \sum_{i_o,j_o,i_{j_o}} \rho_{i_o,j_o}^{(0)}(0) \int [dz][dz^*] \langle O; i_{j_o} | e^{i\hat{H}t} | z; i', j' \rangle \langle z, i, j | e^{-i\hat{H}t} | O, i_o, j_o \rangle,$$

where $[dz]$ is the standard Gaussian integration measure for the coherent states of the EM field.

Substituting Eqs. (18-24) into (27) we obtain through a tedious but straightforward calculation the elements of the reduced density matrix

$$\rho_{11}^{(t)}(t) = \rho_{11}^{(0)} |u|^2(t)$$

$$\rho_{01}^{(t)}(t) = \rho_{01}^{(11)}(0)u(t)v^*_+(t) + \rho_{10}^{(11)}(0)u(t)v^*_-(t)$$

$$\rho_{10}^{(t)}(t) = \rho_{10}^{(11)}(0)u(t)v^*_+(t) + \rho_{01}^{(11)}(0)u(t)v^*_-(t)$$

$$\rho_{00}^{(t)}(t) = \rho_{00}^{(11)}(0)u(t)$$

$$\rho_{01}^{(t)}(t) = \rho_{01}^{(01)}(0)v_+(t) + \rho_{00}^{(01)}(0)v_-(t) + \rho_{00}^{(11)}(0)\mu_1(t) + \rho_{01}^{(11)}(0)\mu_2(t)$$

$$\rho_{10}^{(t)}(t) = \rho_{10}^{(01)}(0)v_+(t) + \rho_{10}^{(00)}(0)v_-(t) + \rho_{01}^{(11)}(0)\mu_2^*(t) + \rho_{01}^{(11)}(0)\mu_1^*(t)$$

$$\rho_{01}^{(t)}(t) = \rho_{01}^{(01)}(0)|v_+|^2(t) + \rho_{10}^{(01)}(0)v_+(t)v^*_-(t) + \rho_{10}^{(10)}(0)|v_-|^2(t)$$

$$+ \rho_{01}^{(10)}(0)v_-(t)v^*_+(t) + \rho_{11}^{(11)}(0)\kappa_1(t)$$

$$\rho_{10}^{(t)}(t) = \rho_{10}^{(01)}(0)|v_+|^2(t) + \rho_{10}^{(10)}(0)|v_-|^2(t) + \rho_{10}^{(01)}(0)v_+(t)v^*_-(t)$$
\[ + \rho_{10}^{10}(0)v_{-}(t)v_{+}^{*}(t) + \rho_{11}^{11}(0)\kappa_{2}(t) \]
\[ \rho_{00}^{00}(t) = 1 - \rho_{11}^{11}(t) - \rho_{01}^{01}(t) - \rho_{10}^{10}(t) \]

where

\[ \mu_1(t) = \sum_{k} g_k \nu_k(t)s_{k}(t) \]
\[ \mu_2(t) = \sum_{k} g_k \nu_k(t)s_{k}^{*}(t)e^{-ik\cdot r} \]
\[ \kappa_1(t) = \sum_{k} |\nu_k|^2(t) \]
\[ \kappa_2(t) = \sum_{k} \nu_k(t)\nu_k^{*}(t)e^{-ik\cdot r}, \]

and the functions \( u(t), v_{\pm}(t) \) were defined in Eqs. (22), (19) and (20).

E. Explicit forms for the evolution functions

Eqs. (28-36) provide an exact expression for the evolution of the reduced density matrix for the system of two qubits interacting with the EM field in the vacuum state. The evolution is determined by seven functions of time \( u, v_{\pm}, \kappa_{1,2}, \mu_{1,2} \), for which we have provided the full definitions. To study the details of the qubits’ evolution we must obtain explicit forms for the functions above. For analytic expressions, we recourse to an approximation: Assuming weak coupling \( (\lambda^2 << 1) \), we ignore the contribution of all processes that involve the exchange of two or more photons between the two qubits.

1. The functions \( u, v_{\pm} \)

With the approximation above, the contribution of the function \( f \) drops out from the definition of \( u \). Thus we obtain

\[ u(t) = \int dE e^{-iEt} \frac{dE e^{-iEt}}{E - 2\omega_o - 2\alpha(E - \omega_o)} \]
\[ v_{\pm} = \int dE e^{-iEt} \frac{1}{2} \left[ \frac{1}{E - \omega_o - \alpha(E) - \beta(E, r)} \pm \frac{1}{E - \omega_o - \alpha(E) + \beta(E, r)} \right]. \]

We evaluate these expressions using an additional approximation. In performing the Fourier transform, we only keep the contribution of the poles in the integral and ignore
that coming from a branch-cut that appears due to the presence of a logarithm in the exact expression of $\alpha(E)$—see Ref. [34] for details. We then obtain

$$u(t) = e^{-2i\omega_o t - 2\Gamma_0 t}$$

$$v_{\pm}(t) = \frac{e^{-i\omega_o t - \Gamma_0 t}}{2} (e^{-i\sigma t - \Gamma_r t} \pm e^{i\sigma t + \Gamma_r t}).$$

In the equations above, we effected a renormalization of the frequency $\omega_o$ by a constant divergent term—see [34]. The parameters $\Gamma_0, \Gamma_r$ and $\sigma(r)$ are defined as

$$\Gamma_0 := -\text{Im} \alpha(\omega_o)$$

$$-\sigma(r) + i\Gamma_r := \beta(\omega_o, r),$$

and they read explicitly

$$\Gamma_0 = \frac{\lambda^2 \omega_o}{2\pi}$$

$$\Gamma_r = \frac{\lambda^2 \sin \omega_or}{2\pi r}$$

$$\sigma(r) = \frac{\lambda^2}{2\pi^2 r} \left[-\cos \omega_or \left[\frac{\pi}{2} - Si(\omega_or)\right]\right]$$

$$+ \sin \omega_or \left[\log(e^{\gamma}\omega_or) + \int_0^{\omega_or} \frac{dz}{z} \frac{1 - \cos z}{z}\right],$$

where $Si$ is the sine-integral function.

The term $\sigma(r)$ is a frequency shift caused by the vacuum fluctuations. It breaks the degeneracy of the two-qubit system and generates an effective dipole coupling between the qubits. At the limit $r \to 0$, this term becomes infinite. One should recall however that the physical range of $r$ is always larger than $a_B$, the Bohr radius of the atoms. As $r \to \infty$, $\sigma(r) \to 0$.

The constant $\Gamma_0$ corresponds to the rate of emission from individual qubits. It coincides with the rate of emission obtained from the consideration of a single qubit interacting with the electromagnetic field. The function $\Gamma_r$ is specific to the two-qubit system. It arises from Feynman diagrams that involve an exchange of photons between the qubits. Heuristically, it expresses the number of virtual photons per unit time exchanged between the qubits. As such, $\Gamma_r^{-1}$ is the characteristic time-scale for the exchange of information between the qubits. As $r \to 0$, $\Gamma_r \to \Gamma_0$ and as $r \to \infty$, $\Gamma_r \to 0$. Note that the ratio $\Gamma/\Gamma_0 = \frac{\sin \omega_or}{\omega_or}$, while smaller than unity, is of the order of unity as long as $r$ is not much larger than $\omega_o^{-1}$.
It is interesting to note that $\Gamma_r = 0$ for $r = n\pi\omega_0^{-1}$, where $n$ an integer. This is a resonant behaviour, similar to that of a classical oscillating dipole when $r = n\lambda/2$, where $\lambda$ is the oscillation wavelength.

2. The functions $\kappa_{1,2}(t)$

We first compute the functions $\nu_k, \nu'_k$ of Eq. (26) keeping terms up to second loop order

$$
\begin{align*}
\left( \begin{array}{c}
\nu_k(t) \\
\nu'_k(t)
\end{array} \right) &= \int \frac{dE e^{-iEt}}{E - 2\omega_o} \sum_{k'} \frac{g_{k'}}{E - \omega_o - \omega_{k'}} \left( \frac{1}{E - \omega_o - \omega_k} \right) \left( \frac{1}{E - \omega_o - \omega_k} \right)
\end{align*}
$$

where $\Xi$ and $\Theta$ are given by Eqs. (16) and (17).

The summation over $k'$ yields within an order of $\lambda^5$

$$
\begin{align*}
\left( \begin{array}{c}
\nu_k(t) \\
\nu'_k(t)
\end{array} \right) &= \int \frac{dE e^{-iEt}}{E - 2\omega_o} \sum_{k'} \frac{g_{k'}}{E - \omega_o - \omega_{k'}} \left( 1 + \frac{\alpha(E - \omega_k) + \beta(E - \omega_k)}{E - \omega_o - \omega_k} \right) \left[ 1 + 2\frac{\alpha(E - \omega_o)}{E - 2\omega_o} \right] \\
&\quad \times \left( 1 + \frac{1}{1 + \frac{1}{(E - \omega_{k'} - \omega_{k})(E - \omega_o - \omega_{k'})}} \frac{g_{k'}^2(1 + e^{-i(k - k')r})}{g_{k'}^2(1 + e^{-i(k - k')r})} \left( 1 + \frac{1}{1 + \frac{1}{(E - \omega_{k'} - \omega_{k})(E - \omega_o - \omega_{k'})}} \right) \right)
\end{align*}
$$

The terms in brackets in the first line of the equation above can be absorbed in the leading-order denominators—see Eq. (52). The term in the second line, however, only gives rise to a (time-independent) multiplicative term of the form $1 + O(\lambda^2)$. Hence, if we keep the leading order terms in the expression of $\nu_k$, we may ignore this term. Then $\nu_k = \nu'_k$, and within an error of order $\lambda^5$

$$
\nu_k(t) = g_k \int \frac{dE e^{-iEt}}{E - 2\omega_o - 2\alpha(E - \omega_o)} \left[ E - \omega_o - \omega_k - \alpha(E - \omega_k) - \beta(E - \omega_k, r) \right].
$$

Using the same approximation as in Sec. II.1 for the Fourier transform, we obtain

$$
\nu_k(t) = g_k \frac{e^{-i\omega_o t - \Gamma_o t}}{\omega - \omega_k - \sigma - i\Gamma_0 + i\Gamma_r} \left( e^{-i\omega_k t - \Gamma_r t} - e^{-i\omega_k t - \sigma t - \Gamma_r t} \right)
$$

We then substitute the expression above for $\nu_k$ into Eqs. (39) and (40) to get

$$
\kappa_1(t) = \frac{\lambda^2}{2\pi^2} e^{-2\Gamma_o t} \int_0^\infty dk \frac{e^{-2\Gamma_o t} + e^{-2\Gamma_r t} - 2e^{-(\Gamma_0 + \Gamma_r)t} \cos[(\omega_o - k - \sigma)t]}{(k - \omega_o + \sigma)^2 + (\Gamma_0 - \Gamma_r)^2} \\
\kappa_2(t) = \frac{\lambda^2}{2\pi^2} e^{-2\Gamma_o t} \int_0^\infty dk \frac{e^{-2\Gamma_o t} + e^{-2\Gamma_r t} - 2e^{-(\Gamma_0 + \Gamma_r)t} \cos[(\omega_o - k - \sigma)t]}{(k - \omega_o + \sigma)^2 + (\Gamma_0 - \Gamma_r)^2} \sin kr
$$
For $\omega_0 t >> 1$, it is a reasonable approximation to substitute the Lorentzian with a delta function. Hence,

$$\kappa_1(t) = \Gamma_0 \kappa(t)$$
(56)

$$\kappa_2(t) = \Gamma_r \kappa(t)$$
(57)

where

$$\kappa(t) \simeq \frac{1}{\Gamma_0 - \Gamma_r} e^{-2\Gamma_0 t} (e^{-\Gamma_0 t} - e^{-\Gamma_r t})^2.$$  
(58)

3. The functions $\mu_{1,2}$

We first compute the functions $s_k(t)$ of Eq. (25)

$$s_k(t) = e^{-i\omega_0 t - \Gamma_0 t - (\Gamma_r + i\sigma) e^{i k \cdot r} t} - e^{-i\omega_k t}$$
(59)

Substituting into Eqs. (37) and (38) we obtain

$$\mu_1(t) = \frac{\lambda^2}{4\pi^2} \int_1^0 d\xi \int k dk e^{-i\omega_0 t - \Gamma_0 t} e^{-i\omega_k t - \Gamma_0 t} - e^{-i\omega_k t - i\sigma t - \Gamma_r t} \Omega - \omega_k - \sigma - i\Gamma_0 + i\Gamma_r
\times e^{i\omega_0 t - \Gamma_0 t - (\Gamma_r - i\sigma) e^{-i k \cdot r} t} - e^{i\omega_k t}$$
(60)

$$\mu_2(t) = \frac{\lambda^2}{4\pi^2} \int_1^0 d\xi \int k dk e^{-i k \cdot r} e^{-i\omega_0 t - \Gamma_0 t} e^{-i\omega_k t - \Gamma_0 t} - e^{-i\omega_k t - i\sigma t - \Gamma_r t} \Omega - \omega_k - \sigma - i\Gamma_0 + i\Gamma_r
\times e^{i\omega_0 t - \Gamma_0 t - (\Gamma_r - i\sigma) e^{-i k \cdot r} t} - e^{i\omega_k t}$$
(61)

An approximate evaluation of the $\xi$-integral followed by the further approximation $\frac{1}{x+i\epsilon} \simeq i\pi \delta(x)$, gives an estimation of the leading order contribution

$$\mu_1(t) = \Gamma_0 [\mu(t) + i\nu(t)]$$
(62)

$$\mu_2(t) = \Gamma_r [\mu(t) + i\nu(t)]$$
(63)

where $\mu + i\nu$ is the complex-valued function

$$\mu(t) + i\nu(t) \simeq \frac{1}{\Gamma_0 + \frac{2\sin \omega_0 r}{\omega_0 r} \Gamma_r - i\sigma(1 + \frac{2\sin \omega_0 r}{\omega_0 r})} e^{-i\omega_0 t - \Gamma_0 t}$$
\times (e^{-\Gamma_0 t} - e^{-\Gamma_r t})(e^{-\Gamma_0 t - 2\sin \omega_0 r [\Gamma_r - i\sigma] t} - e^{i\sigma t}).$$  
(64)
F. The master equation

Given the explicit form of the functions computed in Sec. II.E, we write the evolution equations in the following form

\[
\rho_{I}^{I}(t) = e^{-4\Gamma_0 t}\rho_{I}^{I}(0) \tag{65}
\]

\[
\rho_{O}^{I}(t) = e^{-2\omega_0 t+2\Gamma_0 t}\rho_{O}^{I}(0) \tag{66}
\]

\[
\rho_{-}^{I}(t) = e^{-\omega_0 t-3\Gamma_0 t-i\sigma t+\Gamma_r t}\rho_{-}^{I}(0) \tag{67}
\]

\[
\rho_{+}^{I}(t) = e^{-\omega_0 t-3\Gamma_0 t-i\sigma t-\Gamma_r t}\rho_{+}^{I}(0) \tag{68}
\]

\[
\rho_{O}^{-}(t) = e^{-\omega_0 t-\Gamma_0 t+i\sigma t+\Gamma_r t}\rho_{O}^{-}(0) + i(\Gamma_0 + \Gamma_r)\nu(t)\rho_{+}^{I}(0) + (\Gamma_0 - \Gamma_r)\mu(t)\rho_{-}^{I}(0) \tag{69}
\]

\[
\rho_{O}^{+}(t) = e^{-\omega_0 t-\Gamma_0 t-\sigma t-\Gamma_r t}\rho_{O}^{+}(0) + (\Gamma_0 + \Gamma_r)\mu(t)\rho_{+}^{I}(0) + i(\Gamma_0 - \Gamma_r)\nu(t)\rho_{-}^{I}(0) \tag{70}
\]

\[
\rho_{+}^{-}(t) = e^{-2\Gamma_0 t-2\Gamma_r t}\rho_{+}^{-}(0) + (\Gamma_0 + \Gamma_r)\kappa(t)\rho_{I}^{I}(0) \tag{71}
\]

\[
\rho_{-}^{-}(t) = e^{-2\Gamma_0 t-2\Gamma_r t}\rho_{-}^{-}(0) + (\Gamma_0 - \Gamma_r)\kappa(t)\rho_{I}^{I}(0) \tag{72}
\]

\[
\rho_{+}^{+}(t) = e^{2\sigma t-2\Gamma_0 t}\rho_{+}^{+}(0). \tag{73}
\]

In the equations above we wrote the density matrix in a basis defined by \( |I\rangle, |O\rangle, |+\rangle, |-\rangle \), where

\[
|+\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \tag{74}
\]

\[
|--\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle). \tag{75}
\]

We note that in this approximation, the diagonal elements of the density matrix propagator (i.e. the ones that map \( \rho_{a}^{b}(0) \) to \( \rho_{a}^{b}(t) \) where \( a, b \in \{O, +, -, I\} \)) decay exponentially (except for the \( \rho_{O}^{O} \) which is functionally dependent on the other matrix elements due to normalization). We shall see in Sec. V that this behavior is in accordance with the Born-Markov approximation. The situation is different for the off-diagonal elements of the density matrix propagator, i.e. the ones that map \( \rho_{a}^{b}(0) \) to \( \rho_{a}^{b'}(t) \) for \( a \neq a' \) and \( b \neq b' \). They are given by more complex functions of time and they differ from the ones predicted by the Born-Markov approximation.

As we have the solution \( \rho(t) = M_t[\rho(0)] \), where \( M_t \) is the density matrix propagator defined by Eqs. (65-73), we can identify the master equation through the relation \( \dot{\rho}(t) = M_t^{-1}[M_t^{-1}[\rho(t)]] \). We obtain

\[
\dot{\rho}_{I}^{I} = -4\Gamma_0 \rho_{I}^{I} \tag{76}
\]
\[ \dot{\rho}_I = (-i\omega - 3\Gamma_0 + i\Gamma_r)\rho_I \] 
\[ \dot{\rho}_+ = (-i\omega - 3\Gamma_0 - i\Gamma_r)\rho_+ \] 
\[ \dot{\rho}_O = (-2i\omega - 2\Gamma_0)\rho_O \] 
\[ \dot{\rho}_O = = (-i\omega - \Gamma_0 + i\Gamma_r)\rho_O + (\Gamma_0 + \Gamma_r)\alpha_1(t)\rho_+ + (\Gamma_0 - \Gamma_r)\alpha_2(t)\rho_- \] 
\[ \dot{\rho}_O = = (-i\omega - \Gamma_0 + i\Gamma_r)\rho_O + (\Gamma_0 + \Gamma_r)\alpha_3(t)\rho_+ + (\Gamma_0 - \Gamma_r)\alpha_4(t)\rho_- \] 
\[ \dot{\rho}_+ = -2(\Gamma_0 + \Gamma_r)\rho_+ + (\Gamma_0 + \Gamma_r)\alpha_5(t)\rho_I \] 
\[ \dot{\rho}_- = -2(\Gamma_0 - \Gamma_r)\rho_+ + (\Gamma_0 - \Gamma_r)\alpha_6(t)\rho_I \] 
\[ \dot{\rho}_+ = 2(i\sigma - \Gamma_0)\rho_+ \] 

Explicit expressions for the functions of time \( \alpha_i(t), i = 1, \ldots, 6 \) appearing in Eqs. \(76-84\) are given in the Appendix B.

We see that the evolution equation for the reduced density matrix of the two qubits, while it is local-in-time, it does not have constant-in-time coefficients. Hence, it does not correspond to a Markov master equation of the Lindblad type. Again, we note that the non-Markovian behavior is solely found in the off-diagonal terms of the evolution law and that the diagonal ones involve constant coefficients.

To facilitate comparison with the expressions obtained from the Born-Markov approximations we cast equations \(76-84\) into an operator form.

\[ \dot{\hat{\rho}} = -i[\hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_i, \hat{\rho}] + \sum_{i,j=1}^{2} \Gamma_{ij}(\hat{S}^+(i)\hat{S}^-(j)\hat{\rho} + \hat{\rho}\hat{S}^+(i)\hat{S}^-(j) - 2\hat{S}^-(j)\hat{\rho}\hat{S}^+(i)) 
+ (\Gamma_0 + \Gamma_r)\mathbf{F}_t[\hat{\rho}] + (\Gamma_0 - \Gamma_r)\mathbf{G}_t[\hat{\rho}]. \] 

The first term on the right-hand-side of Eq. \(85\) corresponds to the usual Hamiltonian evolution: the total Hamiltonian is a sum of the free Hamiltonian \(\hat{\mathcal{H}}_0\) and of a dipole interaction Hamiltonian

\[ \hat{\mathcal{H}}_i = -\sigma(\hat{S}^- \otimes \hat{\mathcal{S}}_+ + \hat{\mathcal{S}}^+ \otimes \hat{S}^-). \] 

The second term in the right-hand-side of Eq. \(85\) is the usual Lindblad term (see e.g., \(33\)), where we defined \(\Gamma_{11} = \Gamma_{22} = \Gamma_0\) and \(\Gamma_{12} = \Gamma_{21} = \Gamma_r\).

The last two terms contain effects that pertain to the off-diagonal terms of the reduced density matrix propagator and they are non-Markovian: \(\mathbf{F}_t\) and \(\mathbf{G}_t\) are trace-preserving linear operators on the space of density matrices and their explicit form is given in Appendix
B. Assuming that the Markovian regime corresponds to the vanishing of $F_t$ and $G_t$, we find from Eqs. (B12–B13) that in this regime the functions $\alpha_i(t)$ in Eqs. (76–84) should reduce to the following constants

$$\alpha_1 = \alpha_4 = 0; \quad \alpha_3 = \alpha_5 = \alpha_6 = 2; \quad \alpha_2 = -2.$$  

(87)

In Sec. V, we shall discuss in more detail the physical origin of non-Markovian behavior in this two-qubit system.

III. DISENTANGLEMENT OF TWO QUBITS

In this section, we employ the results obtained above to study the evolution of the two qubits initially in an entangled state. We shall focus on the process of disentanglement induced by their interaction with the field.

A. Class A states: Initial superposition of $|00\rangle$ and $|11\rangle$

We first examine the class of initial states we call Class A of the following type

$$|\psi_0\rangle = \sqrt{1-p}|00\rangle + \sqrt{p}|11\rangle,$$  

(88)

where $0 \leq p \leq 1$. Recall our definition $|I\rangle = |11\rangle$ and $|O\rangle = |00\rangle$. From Eqs. (28 - 36) we obtain

$$\dot{\rho}(t) = p^2 e^{-4\Gamma_0 t}|I\rangle\langle I| + e^{-2\Gamma_0 t}\sqrt{p(1-p)}(e^{2i\omega_0 t}|I\rangle\langle O| + e^{-2i\omega_0 t}|O\rangle\langle I|)$$

$$+ [\kappa_1(t) - \kappa_2(t)]|+\rangle\langle -| + [\kappa_1(t) + \kappa_2(t)]|\rangle\langle +| + [1 - p^2 e^{-4\Gamma_0 t} - 2\kappa_1(t)]|O\rangle\langle O|,$$  

(89)

where the functions $\kappa_1(t)$ and $\kappa_2(t)$ are given by Eqs. (56 - 57).

These results are quite different from those reported in Ref. [40], which were obtained under the Born-Markov approximation. While the $|I\rangle\langle I|$ and $|I\rangle\langle O|$ terms are essentially the same, the $|+\rangle\langle -|$ and $|+\rangle\langle +|$ ones are not, as they involve non-diagonal elements of the density matrix propagator. For comparison, we reproduce here the explicit form of these matrix elements in our calculation

$$\rho^+ = p\frac{\Gamma_0 + \Gamma_r}{\Gamma_0 - \Gamma_r} e^{-\Gamma_0 t} (e^{-\Gamma_0 t} - e^{-\Gamma_r t})^2$$

$$\rho^- = p e^{-2\Gamma_0 t} (e^{-\Gamma_0 t} - e^{-\Gamma_r t})^2,$$  

(90) (91)
FIG. 1: A plot of the concurrence as a function of time for initial Class A state $\rho_{\text{SS}}$ with $p = 0.5$ for different values of $\omega_0 r$.

and in that of Ref. [40] (translated into our notation):

$$
\rho_+(t) = p \frac{\Gamma_0 + \Gamma_r}{\Gamma_0 - \Gamma_r} e^{-2\Gamma_0 t}(e^{-2\Gamma_r t} - e^{-2\Gamma_0 t})
$$

(92)

$$
\rho_-(t) = p \frac{\Gamma_0 - \Gamma_r}{\Gamma_0 + \Gamma_r} e^{-2\Gamma_0 t}(e^{2\Gamma_r t} - e^{-2\Gamma_0 t}).
$$

(93)

For large values of $r$, $\Gamma_r \ll \Gamma_0$ and the expressions above coincide. However, for $\omega_0 r \ll 1$ their difference is substantial. In this regime, $\Gamma_0 \approx \Gamma_r$ and at times $\Gamma_0 t \sim 1$ we obtain $(\Gamma_0 - \Gamma_r)t \ll 1$. According to the Markovian results of [40], in this regime the $|+\rangle\langle+|$ term is of order $O(\lambda^0)$ and hence comparable in size to the other terms appearing in the evolution of the density matrix. However, according to our results, which are based on the full non-Markovian dynamics, the $|+\rangle\langle+|$ term is of order $\frac{\Gamma_0 - \Gamma_r}{\Gamma_0}$ and hence much smaller.

In general, for $\omega_0 r \ll 1$, we find that the $|--\rangle\langle-|$ and $|+\rangle\langle+|$ terms contribute little to the evolution of the reduced density matrix and they can be ignored. Since these terms are responsible for the sudden death and subsequent revival of entanglement studied in [40], we conclude that these effects are absent in this regime. Indeed, this can be verified by the study of concurrence as a function of time as appearing in Figs. 1 and 2. For large inter-qubit separations and for specific initial states ($p > 0.5$) there is sudden death of entanglement—but no subsequent revivals.
FIG. 2: A plot of the concurrence as a function of time for initial Class A state \( |\psi\rangle \) for different values of \( p \) and fixed inter-qubit distance \( (\omega_0 r = 1) \).

B. Initial antisymmetric \(|-\rangle\) Bell state

We next consider the case of an initial antisymmetric \(|-\rangle\) state for the two qubits. We find

\[
\hat{\rho}(t) = e^{-2[\Gamma_0 - \Gamma_r]t} |-\rangle \langle -| + \left( 1 - e^{-2[\Gamma_0 - \Gamma_r]t} \right) |O\rangle \langle O| \quad (94)
\]

We see that the decay rate is \( \Gamma_0 - \Gamma_r \). The effect of photon exchange between the qubits essentially slows down the overall emission of photons from the two-qubit system (subradiant behavior). As the qubits are brought closer, the decay is slower. At the limit \( r \to 0 \), there is no decay.

The results agree qualitatively with those obtained in Ref. \[33\] through the Born-Markov approximation. Fig. 3 shows the time evolution of concurrence for an initial antisymmetric \(|-\rangle\) state and for different inter-qubit distances.

C. Initial symmetric \(|+\rangle\) Bell state

For an initial symmetric \(|+\rangle\) Bell state the reduced density matrix of the two qubits is

\[
\hat{\rho}(t) = e^{-2[\Gamma_0 + \Gamma_r]t} |+\rangle \langle +| + \left( 1 - e^{-2[\Gamma_0 + \Gamma_r]t} \right) |O\rangle \langle O|. \quad (95)
\]
FIG. 3: A plot of the concurrence as a function of time for the initial antisymmetric $|\!-\!\rangle$ Bell state and for different values of the inter-qubit distance $r$ (in units of $\omega_0^{-1}$). The decay of concurrence proceeds more slowly when the qubits are closer together.

FIG. 4: Plot of the concurrence as a function of time for initial symmetric $|+\rangle$ Bell state and for different values of the inter-qubit distance $r$ (in units of $\omega_0^{-1}$).

Here we have a super-radiant decay with rate given by $\Gamma_0 + \Gamma_r$. Note that for both the antisymmetric $|\!-\!\rangle$ and symmetric $|+\rangle$ states, a resonance appears when $r = n\pi\omega_0^{-1}$, and the decay becomes radiant. Fig. 4 shows the behavior of concurrence, in qualitative agreement with the corresponding results of Ref. [33].
IV. DECOHERENCE OF QUBITS

In this section we study the evolution of factorized initial states, in order to understand how the decoherence of one qubit is affected by the presence of another.

A. One qubit in vacuum state

We assume that the first qubit is prepared initially in a superposition of the \(|0\rangle\) and \(|1\rangle\) states, and that the second qubit lies on the ground state. Therefore, the initial state is of the form

\[
\left( \sqrt{p}|1\rangle + \sqrt{1-p}|0\rangle \right) \otimes |0\rangle = \sqrt{p}|10\rangle + \sqrt{1-p}|00\rangle.
\] (96)

From Eqs. (28–36), we obtain the density matrix of the combined qubit system

\[
\hat{\rho}(t) = p \left( |v_+|^2 |10\rangle \langle 10| + |v_-|^2 |01\rangle \langle 01| + v_-^* v_+ |01\rangle \langle 10| + v_+^* v_- |10\rangle \langle 01| \right)
+ \sqrt{p(1-p)} \left( v_+^* |10\rangle \langle 00| + v_-^* |01\rangle \langle 00| + v_+ |00\rangle \langle 10| + v_- |00\rangle \langle 01| \right)
+ \left( 1 - p(|v_+|^2 + |v_-|^2) \right) |00\rangle \langle 00|,
\] (97)

where the functions \(v_\pm(t)\) are given by Eq. (44).

At the limit of large interqubit distances, \(\Gamma_r = 0 = \sigma(r)\), whence \(v_+ \simeq e^{-i\omega_0 t - \Gamma_0 t}\), the off-diagonal elements decay within a characteristic time-scale of order \(\Gamma_0^{-1}\). These results coincide with those for the single qubit case–see Refs. [34, 35]. However, in the regime \(\omega_{\rho r} << 1\), the results are substantially different. The entanglement with the second qubit leads to a departure from pure exponential decay. In this regime, \(\Gamma_r \simeq \Gamma_0\). This implies for times longer than \(\Gamma_0^{-1}\) a substantial fraction of the off-diagonal elements persists. This decays eventually to zero within a time-scale of order \([\Gamma_0 - \Gamma_r]^{-1} \gg \Gamma_0^{-1}\). Hence, the qubit preserves its coherence longer. (At the limit \(r \to 0\) there is no decoherence.)

The reduced density matrix of the second qubit is

\[
\hat{\rho}_2(t) = p|v_-|^2 |1\rangle \langle 1| + \sqrt{p(1-p)} \left( v_- |0\rangle \langle 1| + v_+^* |1\rangle \langle 0| \right) + \left( 1 - p|v_-|^2 \right) |0\rangle \langle 0|.
\] (99)
Note that at the limit of small inter-qubit distances the asymptotic behavior (for $\Gamma_0 t \gg 1$) of $\hat{\rho}_1(t)$ is identical to that of $\hat{\rho}_2(t)$. The second qubit (even though initially on its ground state) develops a persistent quantum coherence as a result of the interaction with the first one.

B. One qubit in excited state

We also consider the case of a factorized initial state, in which the second qubit is excited

$$\left(\sqrt{p}|1\rangle + \sqrt{1-p}|0\rangle\right) \otimes |1\rangle = \sqrt{p}|I\rangle + \sqrt{1-p}|01\rangle.$$  (100)

This system behaves differently from that of Sec. IV.A. The matrix elements of the reduced density matrix read

$$\rho_{11}^{11} = pe^{-4\Gamma_0 t}$$  (101)
$$\rho_{01}^{11} = \sqrt{p(1-p)}e^{-2i\omega_0 t-2\Gamma_0 t}v_-^*$$  (102)
$$\rho_{00}^{00} = \sqrt{p(1-p)}\mu_1(t)$$  (103)
$$\rho_{00}^{10} = \sqrt{p(1-p)}\mu_2^*(t)$$  (104)
$$\rho_{01}^{01} = (1-p)|v_+|^2 + p\kappa_1$$  (105)
$$\rho_{10}^{01} = (1-p)v_+v_+^* + p\kappa_2,$$  (106)

where the functions $\mu_{1,2}$ are given by Eqs. (62–64), the functions $\kappa_{1,2}$ by Eqs. (56) and (57) and the functions $v_{\pm}$ by Eq. (44).

The reduced density matrix of the first qubit is

$$\hat{\rho}_1(t) = \left( (1-p)|v_+|^2 + p\kappa_1 + pe^{-4\Gamma_0 t} \right) |1\rangle\langle 1| + \left( 1 - (1-p)|v_+|^2 - p\kappa_1 - pe^{-4\Gamma_0 t} \right) |0\rangle\langle 0|$$
$$+ \sqrt{p(1-p)} \left( (\mu_2^* e^{-2i\omega_0 t-2\Gamma_0 t} v_-^*) |0\rangle\langle 1| + (\mu_2 + e^{2i\omega_0 t-2\Gamma_0 t} v_-^*) |1\rangle\langle 0| \right).$$  (107)

At times $t \gg \Gamma_0^{-1}$, the decay of the off-diagonal elements at is dominated by the function $\mu_2$, which then reads

$$\mu_2(t) \simeq \frac{\Gamma e^{-i(\omega_0-\sigma)t-\Gamma_0 t}}{\Gamma_0 + 2\sin \omega_0 r} \Gamma_r - i\sigma(1 + 2\sin \omega_0 r) (e^{-\Gamma_r t} - e^{-\Gamma_0 t}).$$  (108)

In the limit of large inter-qubit distance $r$, the off-diagonal element falls like $r^{-2\Gamma_0 t}$, while in the small $r$ limit it falls like $e^{-\Gamma_0 t}$. Comparing with the case of Sec. IV.A, we see that the
initial excitation of the second qubit leads to a lesser degree of entanglement of the total system, as it cannot absorb any virtual photons emitted from the first one. For this reason, the asymptotic decoherence rate does not vary significantly with the interqubit distance.

V. DISCUSSION

In this section, we first summarize our results for the evolution of various initial states, we then discuss the origin of the non-Markovian behavior and finally, a possible limitation of our results that is due to the restricted domain of validity of the Rotating-Wave approximation.

A. Description of the results

For an initial Class A state $|88\rangle$, the $|I\rangle$ component decays to the vacuum, but it also evolves into a linear combination of antisymmetric $|-\rangle$ and symmetric $|+\rangle$ Bell states. However, if the qubits are close together the evolution to Bell states is suppressed. This behavior is qualitatively different from that of Ref. [40], which was obtained through the Born-Markov approximation. The corresponding terms differ by many orders of magnitude at the physically relevant time-scales. As a consequence, we find that there is neither sudden death nor revival of entanglement in this regime.

In retrospect, this difference should not be considered surprising. The Born-Markov method involves two approximations: i) that the back-action of the field on the atoms is negligible and ii) that all memory effects in the system are insignificant. When the qubits are found within a distance much smaller than the one corresponding to their characteristic wavelength, it is not possible to ignore back-action. The virtual photons exchanged by the qubits (at a rate given by $\Gamma_r$) substantially alter the state of the electromagnetic field.

On the other hand, the effect of virtual photons exchange between qubits drops off quickly at large separations $r$ – the two qubits decay almost independently one of the other. Hence, the Born-Markov approximation – reliable for the case of two separate qubits each interacting with an individual field – also gives reasonable results for the two qubits interacting with a common field. In this regime sudden death is possible, but not revival of entanglement. In this sense, our results effectively reduces to those of Ref. [38]: when the distance between the qubits is much larger than any characteristic correlation length scale of the system it
looks as though the two qubits are found in different reservoirs.

Therefore, our results for initial states of Bell type are qualitatively similar to those obtained in Ref. [33] through the Born-Markov approximation. The symmetric $|+\rangle$ state decays super-radiantly and the antisymmetric $|\rangle$ one sub-radiantly. For small values of $\omega_o r$, the antisymmetric $|\rangle$ state decays very slowly and entanglement is preserved.

Concerning decoherence, when the inter-qubit separation is large and the second qubit lies on its ground state, our two qubit calculation reproduces previous results on the single qubit case [34, 35]. However, if the qubits are close together the coherence is preserved longer. These results seem to suggest that in a many-qubit system, the inter-qubit quantum coherence can be sustained for times larger than the decoherence time of the single qubit case. This may suggest some physical mechanisms to resist decoherence in multi-qubit realistic systems.

B. The origin and significance of the non-Markovian behavior

With the exact solutions we have obtained (under weak coupling but no Born or Markov approximation) for the two qubit - EMF system we want to elaborate on the origin and significance of the non-Markovian behavior started in the Introduction. In the evolution equations (65–73) we note that the diagonal terms of the reduced density matrix propagator all decay exponentially. Their decay rate is therefore constant. It is well known that this feature is a sign of Markovian behavior. In fact, it characterizes the domain of validity of Fermi’s golden rule: one could obtain the decay rates by direct application of this rule. Hence, as far as this part of the evolution is concerned, our results are fully compatible with the Markovian predictions.

However, the behavior of the non-diagonal terms in the reduced density matrix propagator is different. A look at Eqs. (65, 73) will convince the reader that the only non-zero such terms are ones that describe the effect of successive decays, for example the $|11\rangle$ state first decaying into $|\rangle$ and then $|\rangle$ decaying into the ground state $|00\rangle$. Hence, the $\rho_{-\rangle}$ term consists of one component that contains the remaining of the $|\rangle\langle-|\rangle$ part of the initial state and another component that incorporates the decay of the $|11\rangle\langle11|\rangle$ part of the initial state towards the state $|\rangle$. In our calculation, the latter term is encoded into the functions $\kappa_{1,2}(t)$, which are obtained by squaring the amplitudes $\nu_k(t)$ as in Eqs. (39–40). At second loop
order, these amplitudes are obtained from the summation of two Feynman diagrams—see Eq. (51). The structure of the poles in Eq. (51) reveals that the first Feynman diagram describes the decays of the \(|11\rangle\) state, while the second one corresponds to processes involving the \(|01\rangle\) and \(|10\rangle\) states. When we construct the evolution functions \(\kappa_{1,2}(t)\), we obtain terms that are both diagonal and off-diagonal with respect to the two types of process. It is the presence of the off-diagonal terms that is primarily (but not solely) responsible for the deviation of our results from the Markovian prediction. In the Markov approximation, the corresponding term involves summation (subtraction) of probabilities rather than amplitudes.

To justify the last statement, we note from Eqs. (28–36) that the evolution decouples the diagonal from the off-diagonal elements of the density matrix (the Markovian equations also have this property). Hence, the probabilities \(p_a(t) = \rho^a_a(t), a \in \{I, +, -, O\}\) evolve autonomously. Time–homogeneity (i.e. Lindblad time evolution) implies that their evolution can be given by a transfer matrix

\[
\dot{p}_a(t) = \sum_b T^b_a p_b(t).
\]  

Here \(T^a_b\) is the constant decay rate for the process \(b \rightarrow a\). Noting that Eqs. (28–36) contain no transitions between + and − and no transitions from − to I, Eq. (109) yields

\[
\dot{p}_-(t) = -2(\Gamma_0 - \Gamma_r) p_-(t) + wp_I(t),
\]  

where \(w = T^I_-\). Since \(p_I(t)\) is determined by Eq. (76) as \(p_I(0)e^{-4\Gamma_0 t}\), we obtain for the initial condition \(p_-(0) = 0\),

\[
p_-(t) \sim p_I(0)(e^{-2(\Gamma_0 - \Gamma_r)t} - e^{-4\Gamma_0 t}),
\]  
in full agreement with Eq. (93) obtained from the Lindblad master equation.

The derivation of Eq. (111) provides an example of a more general fact: the Markovian assumption forces the off-diagonal terms of the density matrix propagator (in this case the one mapping \(\rho^I_I(0)\) to \(\rho_-(t)\)) to be subsumed by the diagonal ones. The off-diagonal elements can have no independent dynamics of their own, unlike what would be the case if they were derived from a full calculation. One should also note that the diagonal terms of the propagator are obtained from the first order perturbation theory. Since they determine the off-diagonal terms within the Markov approximation, the latter only contains the information obtained from first-order perturbation theory. However, in our calculation the
off-diagonal terms involve second–order effects and hence, reveals dynamical correlations that are inaccessible within the context of the Born-Markov approximation.

For the reasons above, the matrix elements \((92–93)\) given by \([40]\) obtained under the Markovian approximation–carry the characteristic superradiant and subradiant behavior of the decay of the \(|+\rangle\) and \(|−\rangle\) states respectively, a property that does not arise from our calculation.

Moreover, we note that Eq. \((110)\) obtained by making the Markov approximation is related to arguments from classical probability, i.e. it involves addition of the probabilities associated to different processes \([52]\). On the other hand, a proper quantum mechanical calculation involves the addition of amplitudes–e.g. the ones appearing in the definition of the functions \(κ_{1,2}(t)\)– and as such it must also contain ‘interference’ terms. In effect, the Markovian approximation introduces by hand a degree of partial ‘decoherence’ (or classicality) and we think that this explains why in general it predicts a faster classicalization of the system than the full analysis does.

We believe that the feature discussed above is generic. In the single qubit system, it was not present because its structure was too simple. There could be no intermediate decays. However, this effect should in principle be present in any system that contains intermediate states. The Markovian approximation would then be valid only if specific conditions hold that render the ‘interference’ terms negligible–for example if there exists a sharp separation of the relevant timescales.

To summarize, the Markov approximation essentially ignores interference terms that are relevant to processes that involve successive decays. These processes appear through off-diagonal terms in the reduced density matrix propagator. The Markov approximation misrepresents the intrinsic dynamics for these terms and ties them –by forcing additivity of probabilities–to the evolution of the diagonal ones. As a results, the off-diagonal terms are subsumed by the diagonal ones.

C. The use of the rotating wave approximation

Finally, we add a few words on the accuracy of our model for the interaction between the 2LA’s and the EMF. The Hamiltonian \([2]\) is obtained from the study of the interaction of the atomic degrees of freedom with the electromagnetic field. Its derivation involves
the dipole and the rotating wave (RW) approximations—see Appendix A in Ref. [34]. The RW approximation consists in ignoring rapidly oscillating terms in the interaction-picture Hamiltonian and keeping only the ones that correspond to resonant coupling. (One ignores processes during which a photon is emitted and the atom becomes excited.) The terms that are dropped out in the RW approximation oscillate with a frequency of order $\omega_0$.

For a single qubit system the RW approximation is self-consistent. However, in the two-qubit system we keep terms in the Hamiltonian that vary in space as $e^{ik \cdot r}$. For the RW approximation to be consistent, one has to assume that $k \cdot r << \omega_0 t$. Since $k$ is peaked around the resonance frequency, this condition is equivalent to $r << t$. Since the physically interesting time-scale for the study of disentanglement and decoherence corresponds to $\Gamma_0^{-1} \sim [\lambda^2 \omega_0]^{-1}$, we expect the RW approximation to be reliable in this context, as long as $\lambda^2 \omega_0 r << 1$. This is sufficient for realistic situations, in which $r$ is bounded by the size of the cavity and $\lambda^2 << 1$. However, the condition above does not hold at the formal limit $r \to \infty$. In this regime the RW approximation may break down. Indeed, in section IV.B the reduced density matrix for the single qubit in the limit $r \to \infty$ does not coincide with the corresponding expression obtained in the study of the single qubit. The presence of an excited second qubit, even if it is situated far away, affects the time evolution significantly. This effect is also present in the Born-Markov approximation. This is arguably an unphysical behavior, and we believe that it arises as an artefact of the RW approximation.

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APPENDIX A: PERTURBATIVE RESUMMATION

We notice that the action of the operator \([|E - H_0\rangle^{-1}H_1|^2\) on any linear combination of the vectors \(|O; 0, 1\rangle\) and \(|O; 1, 0\rangle\) yields another linear combination of these vectors. If we denote by \(p_n\) and \(q_n\) the coefficients of \(|O; 0, 1\rangle\) and \(|O; 1, 0\rangle\) respectively after the \(n\)-th action of \([|E - H_0\rangle^{-1}H_1|^2\) (2\(n\)-th order of perturbation theory) we obtain

\[
\begin{pmatrix}
p_n \\
q_n
\end{pmatrix} = \frac{1}{E - \Omega} \begin{pmatrix}
|\alpha(E) \beta^*(E, r)\
\beta(E, r) \alpha(E)
\end{pmatrix} \begin{pmatrix}
p_{n-1} \\
q_{n-1}
\end{pmatrix} =: A \begin{pmatrix}
p_{n-1} \\
q_{n-1}
\end{pmatrix}
\]

(A1)

Consequently,

\[
\begin{pmatrix}
p_n \\
q_n
\end{pmatrix} = A^n \begin{pmatrix}
p_0 \\
q_0
\end{pmatrix},
\]

(A2)

and summing over all \(n\) we obtain

\[
\sum_{n=0}^{\infty} \begin{pmatrix}
p_n \\
q_n
\end{pmatrix} = \left(\sum_{n=0}^{\infty} A^n\right) \begin{pmatrix}
p_0 \\
q_0
\end{pmatrix} = (1 - A)^{-1} \begin{pmatrix}
p_0 \\
q_0
\end{pmatrix}
\]

(A3)

To compute \(\langle z; 0, 1|(|E - H|^{-1}|O; 01\rangle\) and \(\langle z; 1, 0|(|E - H|^{-1}|O; 01\rangle\) according to the perturbation series we have \(p_0 = (E - \omega_o)^{-1}, q_0 = 0\). The results in the main text then follow.

The resummation for \(\langle z; 0, 0|(|E - H|^{-1}|O; 0, 1\rangle\) proceeds from the fact that the terms in the \(2n + 1\) order of the perturbative expansion are of the form

\[
\sum_k g_k \zeta_k^n e^{\frac{i k r}{2}} b_k^0 |O, 0, 0\rangle
\]

(A4)

We then note that

\[
\zeta_k^n = [\frac{\alpha(E) + \beta(E, r)e^{ikr}}{E - \omega_o}]\zeta_k^{n-1},
\]

(A5)

where \(\zeta_k^0 = (E - \omega_o)^{-1}\). This geometric series can be summed to give the result quoted in the main text.

A similar summation is achieved for the terms \(\langle z; 0, 1|(|E - H|^{-1}|O; 1, 1\rangle\) and \(\langle z; 1, 0|(|E - H|^{-1}|O; 1, 1\rangle\). In the \(2n\)-th order of perturbation theory the action of the series on \(O; 1, 1\) yields

\[
\sum_k \frac{g_k}{(E - 2\Omega)(E - \Omega - \omega_k)} \left( e^{-ikr/2} \gamma_k^n |O; 0, 1\rangle + e^{ikr/2} \delta_k^n |O; 1, 0\rangle \right)
\]

(A6)
APPENDIX B: THE TIME-DEPENDENT TERMS OF THE MASTER EQUATION

The time-dependent coefficients \( \alpha_i, i = 1, \ldots, 6 \) appearing in the master equation (76–84) are defined as

\[
\begin{align*}
\alpha_1(t) &= i \frac{d}{dt} \left( \nu(t) e^{i \omega_0 t + \Gamma_0 t - i \sigma t - \Gamma_r t} \right) e^{2 \Gamma_0 t + 2 i \sigma t + 2 \Gamma_r t} \\
\alpha_2(t) &= \frac{d}{dt} \left( \mu(t) e^{i \omega_0 t + \Gamma_0 t - i \sigma t - \Gamma_r t} \right) e^{2 \Gamma_0 t} \\
\alpha_3(t) &= \frac{d}{dt} \left( \mu(t) e^{i \omega_0 t + \Gamma_0 t + i \sigma t + \Gamma_r t} \right) e^{2 \Gamma_0 t} \\
\alpha_4(t) &= i \frac{d}{dt} \left( \nu(t) e^{i \omega_0 t + \Gamma_0 t + i \sigma t + \Gamma_r t} \right) e^{2 \Gamma_0 t - 2 i \sigma t - 2 \Gamma_r t} \\
\alpha_5(t) &= \frac{d}{dt} \left( \kappa(t) e^{2 \Gamma_0 t + 2 \Gamma_r t} \right) e^{2 \Gamma_0 t - 2 \Gamma_r t} \\
\alpha_6(t) &= \frac{d}{dt} \left( \kappa(t) e^{2 \Gamma_0 t - 2 \Gamma_r t} \right) e^{2 \Gamma_0 t + 2 \Gamma_r t}.
\end{align*}
\]

Using the notation

\[
\begin{align*}
\hat{A}_{+I} &= |+\rangle \langle I|, \\
\hat{A}_{OI} &= |O\rangle \langle I| \\
\hat{A}_{++} &= |+\rangle \langle +| \\
\hat{A}_{+-} &= |+\rangle \langle -| \\
\hat{A}_{--} &= |-\rangle \langle -|,
\end{align*}
\]

the linear functional \( F_t \) and \( G_t \) of Eq. (85) are defined as

\[
\begin{align*}
F_t[\hat{\rho}] &= -[2 - \alpha_5(t)](\hat{A}_{+I} \hat{\rho} \hat{A}_{+I}^\dagger - \hat{A}_{OI} \hat{\rho} \hat{A}_{OI}^\dagger) - [2 - \alpha_3(t)] \hat{A}_{OI} \hat{\rho} \hat{A}_{++} - [2 - \alpha_3^*(t)] \hat{A}_{++} \hat{\rho} \hat{A}_{OI}^\dagger \\
&\quad + \alpha_1(t) \hat{A}_{OI} \hat{\rho} \hat{A}_{+-} + \alpha_1^*(t) \hat{A}_{+-} \hat{\rho} \hat{A}_{OI}^\dagger \\
G_t[\hat{\rho}] &= -[2 - \alpha_6(t)](\hat{A}_{-I} \hat{\rho} \hat{A}_{-I}^\dagger - \hat{A}_{OI} \hat{\rho} \hat{A}_{OI}^\dagger) + [2 + \alpha_2(t)] \hat{A}_{OI} \hat{\rho} \hat{A}_{--} + [2 + \alpha_2^*(t)] \hat{A}_{--} \hat{\rho} \hat{A}_{OI}^\dagger \\
&\quad + \alpha_4(t) \hat{A}_{OI} \hat{\rho} \hat{A}_{+-} + \alpha_4^*(t) \hat{A}_{+-} \hat{\rho} \hat{A}_{OI}^\dagger.
\end{align*}
\]

APPENDIX C: A GRASSMANN PATH-INTEGRAL DERIVATION

In Refs. [34, 35] the evolution of the reduced density matrix for a 2LA interacting with the EM field was determined with a version of the Feynman-Vernon path integral.
method, using Grassmann variables for the atomic degrees of freedom. The same method can be applied to the present problem, even though the perturbative approach turned out to be more convenient because the vacuum initial state allowed for a summation of the perturbative series. However, the path-integral method may allow for a simpler treatment of other systems, such as an N-qubit system or the EM field at a finite temperature. For this reason, we include a sketch of the path-integral treatment in this Appendix, noting that it leads to the same results as the ones obtained in Sec. II.

1. Coherent state path integral

The coherent states for the EMF (bosonic) and 2LA (qubit) states are defined respectively by

$$|z_k⟩ = \exp(zi_{k}b_k^\dagger)|0_k⟩ \quad (C1)$$

$$|η^{(n)}⟩ = \exp(ηS_+^{(n)})|0⟩_{(n)}. \quad (C2)$$

The states $|0_k⟩$ and $|0⟩_{(n)}$ are ground states of the electromagnetic field’s $k^{th}$ mode in vacuum and the $n^{th}$ qubit in its lower state, respectively. The bosonic coherent states are labeled by a complex number, $z_k$, and the qubit coherent states are labeled by an anticommuting (i.e. Grassmannian) number, $η^{(n)}$. A non-interacting eigenstate of two qubits and the electromagnetic spectrum can be written as the direct product of coherent states,

$$|\{z_k\}, η^{(1)}, η^{(2)}⟩ = |\{z_k\}⟩ \otimes |η^{(1)}⟩ \otimes |η^{(2)}⟩. \quad (C3)$$

In a path integral approach the transition amplitude between some chosen initial and final states is divided into many infinitesimal time steps. The resulting path integral corresponds to the matrix element of the evolution operator. In the coherent-state path integral representation the Hamiltonian is given by,

$$\langle \tilde{η}^{(1)}, \tilde{η}^{(2)}, \{\tilde{z}_k\} | H | η^{(1)}, η^{(2)}, \{z'_k\}⟩ \exp[\tilde{η}^{(1)}η^{(1)} + \tilde{η}^{(2)}η^{(2)} + \sum_k \tilde{z}_kz'_k] = \hbarω_0\tilde{η}\eta' + \hbar \sum_k [ω_k\tilde{z}_kz'_k + \sum_n (λ_k^{(n)}\tilde{η}^{(n)}z'_k + \bar{λ}_k^{(n)}\tilde{z}_kη^{(n)})] \quad (C4)$$

Spatial dependence is carried in the coupling constants,

$$λ_k^{(n)} = \frac{λ^{(n)}_{ω_k}e^{-ik·r_n}}{ω_k} \quad (C5)$$

$$\bar{λ}_k^{(n)} = \frac{\bar{λ}^{(n)}_{ω_k}e^{+ik·r_n}}{ω_k}. \quad (C6)$$
The transition amplitude for \( N \) atoms after the \( j \)-th infinitesimal time steps is,
\[
K(j, 0) = \exp \left\{ \sum_n \bar{n}^{(n)}_j \left[ \sum_m \psi^{(nm)}_j \eta^{(m)}_0 + \sum_{kl} g^{(n)}_{jkl} z_{0l} \right] + \sum_k \bar{z}_j k \left[ \sum_l f_{jkl} z_{0l} + \sum_m \varphi^{(m)}_j \eta^{(m)}_0 \right] \right\}.
\]
(C7)

The functions in the path integral are determined by finite-difference equations
\[
\psi^{(nm)}_{j+1} = (1 - i\omega_j \epsilon) \psi^{(nm)}_j - i\epsilon \sum_k \lambda^{(n)}_{jk} \phi^{(m)}_{jk} \psi^{(nm)}_j = \delta_{nm} \quad \psi^{(nm)}_0 = 0
\]
(C8)
\[
\phi^{(m)}_{j+1,k} = (1 - i\omega_k \epsilon) \phi^{(m)}_{jk} - i\epsilon \sum_n \lambda^{(n)}_{jk} \psi^{(nm)}_j \phi^{(m)}_{0k} = 0
\]
(C9)

2. Reduced Density Matrix

The reduced density matrix of the two 2LAs is
\[
\rho(t) = \int d\mu(\eta_0^{(2)}) d\mu(\eta^{(1)}) d\mu(\eta^{(1)}) \rho(0) J_R(t, 0).
\]
(C10)

The dynamics of this open system is described by the density matrix propagator
\[
J_R(t, 0)|_{T=0} = \exp \left\{ \sum_{nm} \left[ \eta^{(n)}_t \psi(t)^{nm} \eta^{(m)}_0 + \bar{\eta}^{(n)}_0 \psi(t)^{nm} \bar{\eta}^{(m)}_t + \eta^{(n)}_0 \sum_k \bar{\phi}^{(n)}_k(t) \phi^{(n)}_k(t) \right] \right\}
\]
(C11)

A general initial two-qubit density matrix written in the Grassmann representation is:
\[
\rho(0) = \begin{pmatrix}
\rho_{00} & \rho_{01} & \rho_{10} & \rho_{11} \\
\rho_{01}^* & \rho_{00} & \rho_{11} & \rho_{10}^* \\
\rho_{10}^* & \rho_{11}^* & \rho_{00} & \rho_{01} \\
\rho_{11}^* & \rho_{10}^* & \rho_{01} & \rho_{00}
\end{pmatrix}
\]
(C12)

In addition to Eqs. Eq.(C8)-(C9), we derive finite difference equations for the expanded functions appearing in the reduced density matrix elements
\[
\psi^{(ab)}_{j+1} \psi^{(nm)}_{j+1} = (1 - 2i\omega_j \epsilon) \psi^{(ab)}_j \psi^{(nm)}_j - i\epsilon \sum_k \lambda^{(n)}_{jk} \phi^{(m)}_{jk} \psi^{(nm)}_j - i\epsilon \sum_k \lambda^{(a)}_{jk} \phi^{(b)}_{j1} \psi^{(nm)}_j \quad \psi^{(ab)}_{j+1} \phi^{(m)}_{j+1,k} = (1 - i\omega_j \epsilon - i\omega_k \epsilon) \psi^{(ab)}_j \phi^{(m)}_{jk} - i\epsilon \sum_n \lambda^{(n)}_{jk} \psi^{(nm)}_j \phi^{(ab)}_{j1} \quad \phi^{(b)}_{j+1} \phi^{(m)}_{j+1,k} = (1 - i\omega_j \epsilon - i\omega_k \epsilon) \phi^{(b)}_{j1} \phi^{(m)}_{jk} - i\epsilon \sum_n \lambda^{(n)}_{jk} \phi^{(nm)}_j \phi^{(ab)}_{j1} \quad \phi^{(b)}_{j+1} \phi^{(m)}_{j+1,k} = (1 - i\omega_j \epsilon - i\omega_k \epsilon) \phi^{(b)}_{j1} \phi^{(m)}_{jk} - i\epsilon \sum_n \lambda^{(n)}_{jk} \phi^{(nm)}_j \phi^{(ab)}_{j1}
\]
(C13)-(C15)
For a two-atom system, it is convenient to represent the functions appearing in the propagator by column vectors,

\[
[\psi] = \begin{pmatrix}
\psi(22) \\
\psi(21) \\
\psi(12) \\
\psi(11)
\end{pmatrix}
\quad [\phi_k] = \begin{pmatrix}
\phi_k^{(2)} \\
\phi_k^{(1)}
\end{pmatrix}
\quad [\psi\psi] = \begin{pmatrix}
\psi(11)\psi(22) \\
\psi(21)\psi(12)
\end{pmatrix}
\quad [\psi\phi_k] = \begin{pmatrix}
\psi(22)\phi_k^{(1)} \\
\psi(21)\phi_k^{(2)} \\
\psi(12)\phi_k^{(1)} \\
\psi(11)\phi_k^{(2)}
\end{pmatrix}
\quad [\phi\phi_k] = \begin{pmatrix}
\phi_k^{(1)}\phi_l^{(2)} \\
\phi_k^{(2)}\phi_l^{(1)}
\end{pmatrix}.
\]

(C16)

At the continuous limit the finite difference equations yield a set of differential equations for the column vectors above.

\[
\frac{d}{dt}[\psi] = -i\omega_o[\psi] - i \sum_k G_k^\dagger[\phi_k]
\]

(C18)

\[
\frac{d}{dt}[\phi_k] = -i\omega_k[\phi_k] - i G_k[\psi]
\]

(C19)

\[
\frac{d}{dt}[\psi\psi] = -2i\omega_o[\psi\psi] - i \sum_k A_k[\phi\psi_k]
\]

(C20)

\[
\frac{d}{dt}[\phi\psi_k] = -i(\omega_o + \omega_k)[\phi\psi_k] - iA_k^\dagger[\psi\psi] - i \sum_l L_l[\phi\phi_k]
\]

(C21)

\[
\frac{d}{dt}[\phi\phi_k] = -i(\omega_1 + \omega_k)[\phi\phi_k] - i G_l[\phi\psi_k] - i D_k[\phi\psi_l],
\]

(C22)

in terms of the matrices

\[
G_k = \begin{pmatrix}
\lambda_k^{(2)} & 0 & \bar{\lambda}_k^{(1)} & 0 \\
0 & \lambda_k^{(2)} & 0 & \bar{\lambda}_k^{(1)}
\end{pmatrix}
\quad A_k = \begin{pmatrix}
\lambda_k^{(1)} & 0 & 0 & \lambda_k^{(2)} \\
0 & \lambda_k^{(1)} & \lambda_k^{(2)} & 0
\end{pmatrix}
\]

(C23)

\[
D_k = \begin{pmatrix}
0 & \bar{\lambda}_k^{(2)} & 0 & \bar{\lambda}_k^{(1)} \\
\bar{\lambda}_k^{(2)} & 0 & \bar{\lambda}_k^{(1)} & 0
\end{pmatrix}
\quad L_1 = \begin{pmatrix}
0 & \lambda_l^{(2)} \\
0 & \lambda_l^{(2)}
\end{pmatrix}
\]

(C24)

The solution of Eqs. (C18–C22) allows one to fully reconstruct the propagating amplitude and from this the elements of the reduced density matrix for the qubits. These equations can be solved by implementing an approximation scheme similar to that employed for the operator method in the main text. We do not provide the details, but only note that the results of the two methods coincide.
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[51] This does not preclude the possibility that certain types of backreaction effects when included leads to the same type of open system dynamics as the original (test-field) dynamics, which, if already in the Lindblad form, remains Markovian (e.g., renormalization of the coupling constants, invoking a mean field approximation).
[52] To be precise, in an exponential decay, it is possible to define a probability density for the occupation number of any state as a function of time. This is in general not possible in quantum theory. Eq. [III] is then essentially the Kolmogorov additivity condition for these probabilities.