PRINCIPAL EIGENVECTORS OF GENERAL HYPERGRAPHS

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Abstract. In this paper we obtain bounds for the extreme entries of the principal eigenvector of hypergraphs; these bounds are computed using the spectral radius and some classical parameters such as maximum and minimum degrees. We also study inequalities involving the ratio and difference between the two extreme entries of this vector.

Keywords. Hypergraph; Adjacency tensor; Spectral radius; Principal eigenvector.

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1. Introduction

One of the main goals of spectral graph theory is to understand the structure of graphs through its associated matrices and their spectra. For many years, researchers around the world have tried to develop a similar theory for hypergraphs. Several proposals were made between the late 1990s and the early 2000s, and we believe that the main representative attempts presented in this period were given in [4, 5, 13]. None of these proposals seemed to be widely accepted or carried over by the community of researchers in this area. This scenario changed in 2012, with the work [3] presented by Cooper and Dutle. In that paper, the concept of an adjacency tensor of a uniform hypergraph was introduced, based on the developments of the spectral tensor theory, which begun in 2005 by Qi in [11] (for definitions read section 2). Cooper and Dutle generalized many important results of the spectral graph theory, initiating the spectral hypergraph theory via tensors.

Most of the current literature in this area is devoted to uniform hypergraphs. While the spectral theory of uniform hypergraphs already has some recognition, studies of general hypergraphs are still taking their first steps. Our main concern in the present paper is to the study spectral properties of general hypergraphs. To achieve this, we will use the definition of adjacency tensor presented by Benerjee, Char and Mondal in [1]. It is worth mentioning other published papers on the spectra of general hypergraphs such as [18, 7]. Finally, the work [17] proposes a definition of an adjacency tensor different from that proposed in [1], that we use here.

Our paper is anchored in a version of Perron-Frobenius Theorem for hypergraphs, presented below.

Theorem 1 (Theorem 3.1, [18]). Let \( \mathcal{H} \) be a hypergraph and \( \rho \) its spectral radius, then

(a) \( \rho \) is an eigenvalue of \( \mathcal{H} \), with a non-negative real eigenvector.

(b) If \( \mathcal{H} \) is connected then \( \rho \) is the unique eigenvalue of \( \mathcal{H} \) with a strictly positive eigenvector, and this eigenvector is unique (up to a positive multiplier).

We will call principal eigenvector of a connected hypergraph, the vector obtained in part (b) of the Perron-Frobenius Theorem, normalized to norm \( \ell_k \). The main object
of study of this note is this vector, more specifically, we are interested in studying the extreme entries - the largest and the smallest entries - of the principal eigenvector of a connected hypergraph.

The study of the principal eigenvector for hypergraphs is interesting because the value of each of its entry may be seen as a spectral measure of the centrality of the vertex associated with this entry. Another interesting property of this vector is that the quotient and the subtraction of its two extreme entries can be understood as measurements of the irregularity of the hypergraph. In addition, an important optimization problem for graphs is to determine which vertex or edge that, when removed from a graph, causes the greatest decrease in the spectral radius. The answer for this problem, given in [16], is to remove a vertex with maximum entry in the main eigenvector $x = (x_v)$, or remove the edge $e = uv$ such that $x_e = x_u x_v$ has maximum value. We see that many important information from a graph can be obtained through the study of the principal eigenvector. In fact, bounding the entries of the main eigenvector of a graph is a topic of many research works. In particular, Stevanovic has a chapter in his book [16] dedicated to the study of these parameters. The reader is referred to this chapter and references therein.

For tensors and hypergraphs these entries have been studied in the papers [10, 8, 9, 15, 7]. In this paper we obtain bounds for the largest and smallest entries of the principal eigenvector of a general hypergraph. Many of our results are inspired by results proven for graphs by Cioabă and Gregory in [2]. The bounds presented throughout this work are always best possible for regular hypergraphs. We emphasize the result presented in Theorem 2 since it is sharp for a larger class of hypergraphs.

**Theorem 2.** Let $H$ be a connected hypergraph on $n$ vertices and rank $k$. If $(\rho, x)$ is its principal eigenpair, then

$$x_{\min} \leq k \sqrt{\frac{\delta}{\rho + \delta(n-1)}}.$$

Equality holds if and only if there is a vertex $v$ such that, for all $w \in V \setminus \{v\}$, we have $x_w = x_{\min}$ and $x_v = k \sqrt{\frac{\rho}{\delta}} x_{\min}$.

We call the attention for the last section of this note, because we have defined and studied some parameters that have not yet been properly explored. When $H$ is a uniform hypergraph, we define, for each edge $e = \{v_1, \ldots, v_k\}$, the number $x_e = x_{v_1} \cdots x_{v_k}$, where $x = (x_v)$ is the principal eigenvector of this hypergraph. We will say that $x_e$ is the value of the edge $e$ in the principal eigenvector. Thus we define the parameters $x_{\max}$ and $x_{\min}$ as the largest and the smallest value reached by $x_e$ and the parameter $\Gamma(H)$ as the quotient between $x_{\max}$ and $x_{\min}$. Theorem 3 is the main result of this part of the work.

**Theorem 3.** Let $H$ a $k$-graph. $\Gamma(H) = 1$, if and only if, for each edge the product of the degrees of its vertices is constant.

We notice that if a hypergraph $H$ has the parameter $\Gamma(H)$ greater than 1, then the product of the vertices in each edge is not constant, so we can say that $\Gamma(H)$ is a measure of the distribution of degrees of vertices along the edges of the hypergraph.

The remaining of the paper is organized as follows. In Section 2 we present some basic definitions about hypergraphs and tensors. In Section 3 we obtain bounds for the difference and for the ratio between the two extreme entries of the principal eigenvector. In Section 4 we will prove some relations between the extreme entries of the principal
eigenvector with other important parameters. In Section 5 we will prove Theorem 3, and we will construct some inequations involving the parameters $x_{\text{max}}$, $x_{\text{min}}$ and $\Gamma$.

2. Preliminaries

In this section, we shall present some basic definitions about hypergraphs and tensors, as well as terminology, notation and concepts that will be useful in our proofs. More details can be found in [1, 3, 9, 14].

**Definition 1.** A tensor (or hypermatrix) $A$ of dimension $n$ and order $r$ is a collection of $n^r$ elements $a_{i_1 \ldots i_r} \in \mathbb{C}$ where $i_1, \ldots, i_r \in [n] = \{1, 2, \ldots, n\}$.

Let $A$ be a tensor of dimension $n$ and order $r$ and $B$ be a tensor of dimension $n$ and order $(r-1)(s-1) + 1$, where

$$c_{j\alpha_1 \ldots \alpha_{r-1}} = \sum_{i_2, \ldots, i_r = 1}^{n} a_{ji_2 \ldots i_r}b_{i_2\alpha_1} \cdots b_{i_r\alpha_{r-1}} \text{ with } j \in [n] \text{ and } \alpha_1, \ldots, \alpha_{r-1} \in [n]^{s-1}.$$

In particular, if $x \in \mathbb{C}^n$ is a vector, then the $i$th component of the product $Ax$ is given by

$$(Ax)_i = \sum_{i_2, \ldots, i_r = 1}^{n} a_{ii_2 \ldots i_r}x_{i_2} \cdots x_{i_r} \quad \forall i \in [n].$$

If $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ is a vector and $r$ is a positive integer, we let $x^{[r]}$ denote the vector in $\mathbb{C}^n$ whose $i$th component is given by $x_i^r$.

**Definition 2.** A number $\lambda \in \mathbb{C}$ is an eigenvalue of a tensor $A$ of dimension $n$ and order $r$ if there is a nonzero vector $x \in \mathbb{C}^n$ such that

$$Ax = \lambda x^{[r-1]}.$$

We say that $x$ is an eigenvector of $A$ associated with the eigenvalue $\lambda$ and that $(\lambda, x)$ is an eigenpair of $A$.

An eigenpair $(\lambda, x)$ is strictly positive if the eigenvalue $\lambda > 0$ and all the entries of the eigenvector $x$ are positive.

**Definition 3.** The spectral radius of a tensor $A$ of order $k$, is the largest modulus of an eigenvalue of $A$, that is

$$\rho(A) = \max\{|\lambda| : Ax = \lambda x^{k-1}, \text{ for } x \neq 0\}.$$

**Definition 4.** A hypergraph $\mathcal{H} = (V, E)$ is a pair formed by a set of vertices $V(\mathcal{H})$ and a set of edges $E(\mathcal{H}) \subseteq 2^V$, where $2^V$ is the power set of $V$.

$\mathcal{H}$ is said to be a $k$-uniform (or a $k$-graph) for an integer $k \geq 2$, if all the edges of $\mathcal{H}$ have cardinality $k$.

**Definition 5.** Let $\mathcal{H}$ be a hypergraph. The neighborhood of a vertex $v$, denoted by $N(v)$, is the set formed by all distinct vertices of $v$, that have some common edge with $v$.

**Definition 6.** Let $\mathcal{H}$ be a hypergraph. We define the following sets

$$E_{(i)} = \{e - i : i \in e \in E(\mathcal{H})\}, \quad E_{[i]} = \{e : i \in e \in E(\mathcal{H})\}$$
Definition 7. Let $\mathcal{H} = (V, E)$ be a hypergraph. A path is a sequence of vertices and edges $v_0e_1v_1e_2\ldots e_kv_l$ where $v_{i-1}$ and $v_i$ are contained in $e_i$ for each $i \in [l]$ and the vertices $v_0, v_1, \ldots, v_l$ are all distinct, as well as the edges $e_1, \ldots, e_l$ are also all distinct. In these conditions we say that the hypergraph $\mathcal{H}$ is connected, if for each pair of vertices $u, w \in V$ there is a path $v_0e_1v_1e_2\ldots e_kv_l$ where $u = v_0$ and $w = v_l$. Otherwise $\mathcal{H}$ is said to be disconnected.

Definition 8. Let $\mathcal{H} = (V, E)$ be a hypergraph. The degree of a vertex $v \in V$, denoted by $d(v)$, is the number of edges that contain $v$.

A hypergraph is $r$-regular if $d(v) = r$ for all $v \in V$. We will also define the maximum, minimum and average degrees as follows

$$\Delta(\mathcal{H}) = \max_{v \in V} \{d(v)\}, \quad \delta(\mathcal{H}) = \min_{v \in V} \{d(v)\}, \quad d(\mathcal{H}) = \frac{1}{n} \sum_{v \in V} d(v)$$

Definition 9. Let $\mathcal{H}$ be a hypergraph. The rank of $\mathcal{H}$ is the maximum cardinality of the edges in the hypergraph, let us denote this parameter by $r(\mathcal{H})$.

Observe that, if a hypergraph is $k$-uniform then its rank is $k$, because all its edges have size $k$. From this point on, $k$ represents the rank of the hypergraph being treated.

Definition 10. Let $\mathcal{H} = (V, E)$ be a hypergraph.

1. For each edge $e \in E$, we say that an ordered sequence $\alpha = (v_1, v_2, \ldots, v_k)$ is a $k$-expanded edge from $e$, if its set of the distinct elements is equal to edge $e$, and denote this by $e \prec \alpha$.
2. For each edge $e \in E$, we define $S(e) = \{\alpha \mid e \prec \alpha\}$, the set of all $k$-expanded edges from $e$.
3. For each edge $e \in E$ and vertex $v \in e$, we define $S(e)_v = \{\alpha \in S(e) \mid \alpha = (v, v_2, \ldots, v_k)\}$, the set of all $k$-expanded edges in $S(e)$, starting with $v$.
4. We define $S(\mathcal{H}) = \bigcup_{e \in E} S(e)$, the set of all $k$-expanded edges.
5. For each vertex $v \in V$, we define $S(\mathcal{H})_v = \bigcup_{e \in E: e \ni v} S(e)_v$, the set of all $k$-expanded edges in $S(\mathcal{H})$, starting with $v$.

Remark 1. Let $\mathcal{H}$ be a hypergraph. For each edge $e \in E$ it holds that

1. If $|e| = r$ then $|S(e)| = \sum_{s_1+\cdots+s_r=k} \frac{k!}{s_1!\cdots s_r!}$, with $s_i \geq 1$ for $i \in [r]$.
2. If $v \in e$ then $|S(\alpha)| = |e||S(e)_v|$

Definition 11. Let $\mathcal{H}$ be a hypergraph on $n$ vertices, we define the adjacency tensor $A_{\mathcal{H}}$, as a tensor of dimension $n$ and order $k$ where

$$a_{i_1\ldots i_k} = \begin{cases} \frac{|e|}{|S(e)|} & \text{if } e \prec (i_1, \ldots, i_k) \text{ for some } e \in E(\mathcal{H}) \\ 0 & \text{otherwise} \end{cases}$$

For each edge $e \in E(\mathcal{H})$ denote $a(e) = \frac{|e|}{|S(e)|}$. For each $k$-expanded edge $\alpha = (i_1, \ldots, i_k) \in S(\mathcal{H})$ denote $a_\alpha = a_{i_1i_2\ldots i_k}$, $x_\alpha = x_{i_1} \cdots x_{i_k}$ and $x^{\alpha-i} = x_{i_1} \cdots x_{i_{l-1}}x_{i_{l+1}} \cdots x_{i_k}$. With these considerations we have
\((A_Hx)_i = \sum_{i_2, \ldots, i_k=1}^n a_{i_2 \cdots i_k} x_{i_2} \cdots x_{i_k} = \sum_{\alpha \in S(H)_i} a_{\alpha} x^{\alpha - i} = \sum_{e \in E[i]} a(e) \sum_{\alpha \in S(e)_i} x^{\alpha - i}\)

Therefore to determine the eigenvalues of \(H\) we need to solve the following system

\[
\sum_{e \in E[i]} a(e) \sum_{\alpha \in S(e)_i} x^{\alpha - i} = \lambda x^{k-1}_i \quad \forall i \in V(H)
\]

(1)

Another interesting formula is

\[
x^T A_H x = \sum_{i \in V} \left( x_i \sum_{e \in E[i]} a(e) \sum_{\alpha \in S(e)_i} x^{\alpha - i} \right) = \sum_{e \in E} a(e) \sum_{\alpha \in S(e)} x^{\alpha} \quad \text{(2)}
\]

For uniform hypergraphs the formulas 1 and 2 are reduced to

\[
(A_Hx)_i = \sum_{e \in E[i]} x^{e-i}, \quad x^T A_H x = k \sum_{e \in E} x^e
\]

**Definition 12.** Let \(H\) be a connected hypergraph on \(n\) vertices, and \(x\) the positive eigenvector associated to \(\rho(H)\) obtained in Theorem 1 with \(||x||_k = 1\). We will call \((\rho, x)\) of principal eigenpair and \(x\) of principal eigenvector.

**Definition 13.** Let \(H\) be a hypergraph and \(x = (x_v)\) its principal eigenvector.

1. \(x_{\text{min}} = \min_{v \in V(H)} \{x_v\}\)
2. \(x_{\text{max}} = \max_{v \in V(H)} \{x_v\}\)
3. \(\sigma(H) = x_{\text{max}} - x_{\text{min}}\)
4. \(\gamma(H) = \frac{x_{\text{max}}}{x_{\text{min}}}\)

3. **Irregularity Measurements of Hypergraphs**

In this section we will study the parameters \(\sigma(H)\) and \(\gamma(H)\), they can be used as measurements of the irregularity of the hypergraph \(H\), this occurs because \(H\) is regular only when its principal eigenvector has all coordinates equals. More precisely, we will prove some results that help to estimate the value of these parameters.

**Theorem 4.** A hypergraph \(H\) is \(r\)-regular, if and only if, \((r, x)\) is its principal eigenpair, with \(x = \left( \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right)\).

**Proof.** If \(H\) is \(r\)-regular then

\[
(A_Hx)_i = \sum_{e \in E[i]} a(e) \sum_{\alpha \in S(e)_i} \left( \frac{1}{\sqrt{n}} \right)^{k-1} = \sum_{e \in E[i]} a(e|S(e)_i| \left( \frac{1}{\sqrt{n}} \right)^{k-1} = d(i) \left( \frac{1}{\sqrt{n}} \right)^{k-1} = rx^{k-1}_i
\]

Therefore \((r, x)\) is an eigenpair of \(H\) and by Theorem 1 we conclude that this is the principal eigenpair.

Conversely, if \((r, x)\) is the principal eigenpair of \(H\), then for each \(u \in V\) we have

\[
r \left( \frac{1}{\sqrt{n}} \right)^{k-1} = \sum_{e \in E[u]} a(e) \sum_{\alpha \in S(e)_u} \left( \frac{1}{\sqrt{n}} \right)^{k-1} = \sum_{e \in E[u]} a(e|S(e)_u| \left( \frac{1}{\sqrt{n}} \right)^{k-1} = d(u) \left( \frac{1}{\sqrt{n}} \right)^{k-1}.
\]
Hence, \( r = d(u), \forall u \in V \). That is \( \mathcal{H} \), is \( r \)-regular.

The sufficient condition of Theorem 4 has already been demonstrated in [1].

Theorem 5 below is a generalization for hypergraphs of Theorem 2.8 for graphs due to Cioabă and Gregory in [2].

**Theorem 5.** Let \( \mathcal{H} \) be a connected hypergraph. If \( (\rho, x) \) is its principal eigenpair, then

\[
\gamma(\mathcal{H}) \geq \max \left\{ \left( \frac{\Delta}{\rho} \right)^{1-\frac{1}{r}}, \left( \frac{\rho}{\delta} \right)^{1-\frac{1}{r}} \right\}
\]

Moreover

1. If all the vertices of maximum and minimum degrees in \( \mathcal{H} \) are contained only in edges of maximum cardinality, then equality occurs, if and only if, both statements are true
   (a) For each \( u \) of maximum degree we have \( x_u = x_{\max} \) and \( x_p = x_{\min} \) whenever \( u \) and \( p \) are adjacent.
   (b) For each \( v \) of minimum degree we have \( x_v = x_{\min} \) and \( x_q = x_{\max} \) whenever \( v \) and \( q \) are adjacent.

2. If any vertex of maximum or minimum degree in \( \mathcal{H} \) is contained in an edge which does not have maximum cardinality, then equality occurs, if and only if, \( \mathcal{H} \) is regular.

**Proof.** Let \( u, v \in V \) be vertices, such that \( d(u) = \Delta \) and \( d(v) = \delta \), so we have

\[
\rho x_{\max}^{k-1} = \rho x_u^{k-1} \geq \sum_{e \in E [u]} a(e) \sum_{\alpha \in S(e)_u} x^{\alpha - u} \geq \sum_{e \in E [u]} x_{\min}^{k-1} = \Delta x_{\min}^{k-1} \Rightarrow \left( \frac{x_{\max}}{x_{\min}} \right)^{k-1} \geq \frac{\Delta}{\rho}
\]

\[
\rho x_{\min}^{k-1} \leq \rho x_v^{k-1} = \sum_{e \in E [v]} a(e) \sum_{\alpha \in S(e)_v} x^{\alpha - v} \leq \sum_{e \in E [v]} x_{\max}^{k-1} = \delta x_{\max}^{k-1} \Rightarrow \left( \frac{x_{\max}}{x_{\min}} \right)^{k-1} \geq \frac{\rho}{\delta}
\]

(1) Notice that

\[
\rho x_{\max}^{k-1} = \rho x_u^{k-1} \Leftrightarrow x_u = x_{\max}, \quad \sum_{e \in E [u]} x^{e - u} = \Delta x_{\min}^{k-1} \Leftrightarrow x_p = x_{\min} \quad \forall p \in N(u)
\]

\[
\rho x_{\min}^{k-1} = \rho x_v^{k-1} \Leftrightarrow x_v = x_{\min}, \quad \sum_{e \in E [v]} x^{e - u} = \delta x_{\max}^{k-1} \Leftrightarrow x_q = x_{\max} \quad \forall q \in N(v)
\]

(2) Suppose that there is a vertex \( u \) of maximum degree contained in one edge that does not have maximum cardinality, the other case is analogous.

\[
\rho x_{\max}^{k-1} = \rho x_u^{k-1} \Leftrightarrow x_u = x_{\max}, \quad \sum_{e \in E [u]} a(e) \sum_{\alpha \in S(e)_u} x^{\alpha - u} = \Delta x_{\min}^{k-1} \Leftrightarrow x_p = x_{\min} \quad \forall p \in N(u) \cup \{u\}
\]

That is, we would have \( x_{\max} = x_{\min} \) and therefore equality is true, if and only if, \( \mathcal{H} \) is regular.

\( \square \)

We observe that in Theorem 5 the equality is true, for example, for the star \( S_n \).
Corollary 1. Let $\mathcal{H}$ be a hypergraph. If $(\rho, \mathbf{x})$ is its principal eigenpair, then
\[
\gamma(\mathcal{H}) \geq \left( \frac{\Delta}{\delta} \right)^{1/(k-1)}
\]
Equality holds under the same conditions of Theorem 5.

Proof. We notice that $\left( \frac{\Delta}{\delta} \right)^{1/(k-1)}$ is the geometric mean between $\left( \frac{\Delta}{\rho} \right)^{1/(k-1)}$ and $\left( \frac{\rho}{\delta} \right)^{1/(k-1)}$, so
\[
\gamma(\mathcal{H}) \geq \max \left\{ \left( \frac{\Delta}{\rho} \right)^{1/(k-1)}, \left( \frac{\rho}{\delta} \right)^{1/(k-1)} \right\} \geq \left( \frac{\Delta}{\delta} \right)^{1/(k-1)}
\]
\[\square\]

Theorem 6. Let $\mathcal{H}$ be a connected hypergraph on $n$ vertices. If $(\rho, \mathbf{x})$ is its principal eigenpair, then
\[
\sigma(\mathcal{H}) \geq \frac{\Delta^{1/(k-1)} - \delta^{1/(k-1)}}{\Delta^{1/(k-1)} n^{1/k}}
\]
Equality holds if and only if $\mathcal{H}$ is regular.

Proof. Firstly, we observe that
\[
\frac{x_{\max}}{x_{\min}} \geq \left( \frac{\Delta}{\delta} \right)^{1/(k-1)} \Rightarrow x_{\min} \leq \left( \frac{\delta}{\Delta} \right)^{1/(k-1)} x_{\max}
\]
Multiplying the inequality by $-1$ and adding $x_{\max}$ to both sides, we arrive at the following inequality
\[
x_{\max} - x_{\min} \geq \left( 1 - \left( \frac{\delta}{\Delta} \right)^{1/(k-1)} \right) x_{\max} = \frac{\Delta^{1/(k-1)} - \delta^{1/(k-1)}}{\Delta^{1/(k-1)} n^{1/k}} x_{\max}
\]
Note that $x_{\max} \geq \frac{1}{\sqrt[k]{n}}$, so we conclude that
\[
x_{\max} - x_{\min} \geq \frac{\Delta^{1/(k-1)} - \delta^{1/(k-1)}}{\Delta^{1/(k-1)} n^{1/k}}
\]
Equality occurs, if and only if, $x_{\max} = \frac{1}{\sqrt[k]{n}}$, that is whenever $\mathcal{H}$ is regular. \[\square\]

4. bounds for the principal eigenvector entries

In this section we present some results on the extreme entries of the principal eigenvector of a general hypergraph, relating it with important classical parameters.

The Theorem 7 is a generalization for hypergraphs of Lemma 3.3 for graphs given by Cioabă and Gregory in [2].

Theorem 7. Let $\mathcal{H}$ be a hypergraph on $n$ vertices. If $(\rho, \mathbf{x})$ is its principal eigenpair, then
\[
\begin{align*}
(a) \quad x_{\max} & \geq \frac{1}{k \sqrt[k]{\frac{\Delta}{\delta}^{1/(k-1)}} + n^{-1}}. \text{ For } k \geq 3, \text{ the equality holds if and only if } \mathcal{H} \text{ is regular.} \\
(b) \quad x_{\min} & \leq \frac{1}{k \sqrt[k]{\frac{\Delta}{\delta}^{1/(k-1)}} + n^{-1}}. \text{ For } k \geq 3, \text{ the equality holds if and only if } \mathcal{H} \text{ is regular.}
\end{align*}
\]
Proof. To prove part (a), we observe that

\[ 1 = \sum_{u \in V} x_v^k \leq x_{\min}^k + (n-1)x_{\max}^k = (\gamma^{-k} + n-1)x_{\max}^k \leq \left( \frac{\delta}{\Delta} \right)^{\frac{k}{2(k-1)}} + n-1 \]

Now note that the equality occurs if, and only if, \( n-1 \) entries of \( x \) are equal \( x_{\max} \) and one entry is equals \( x_{\min} \). Observe yet that equality occurs only if the equality on Theorem 5 occurs as well. Under these conditions we can assume that all vertices of maximum or minimum degree are contained only in edges of maximum cardinality – otherwise, the hypergraph would be regular. I.e., for each \( u \in V \) such that \( d(u) = \Delta \) must be true \( x_q = x_{\min} \) for every neighbor \( q \) of \( u \). Since \( k \geq 3 \), then \( u \) must have at least two neighbors, but only one can take on a value other than \( x_{\max} \), therefore \( x_{\min} = x_{\max} \), that is \( H \) is regular.

Similarly, we prove part (b) \( \square \)

Theorem 8 below is a generalization for hypergraphs of Theorem 3.4 for graphs due to Cioabă and Gregory [2].

**Theorem 8.** Let \( H \) be a hypergraph. If \( (\rho, x) \) is its principal eigenpair, then

(a) \( x_{\max} \geq \sqrt[k]{\frac{\rho}{nd(H)}} \). Equality holds if and only if \( H \) is regular.

(b) \( x_{\max} \geq \frac{\rho^{k-1}}{\left( \sum_{u \in V} d(u) \right)^{k-1}} \). Equality holds if and only if \( H \) is regular.

Proof. To prove the part (a), observe that, for each \( u \in V \) must be true

\[ \rho x_u^{k-1} = \sum_{e \in E(u)} a(e) \sum_{\alpha \in S(e)u} x_{\alpha-u}^{k-1} \leq \sum_{e \in E(u)} x_{\max}^{k-1} = d(u) x_{\max}^{k-1} \]

So it is worth \( \rho x_u^k \leq d(u)x_{\max}^k \). Summing over the set of vertices, we have

\[ \rho \sum_{u \in V} x_u^k \leq \sum_{u \in V} d(u)x_{\max}^k \Rightarrow \rho \leq nd(H)x_{\max}^k \Rightarrow x_{\max} \geq \sqrt[k]{\frac{\rho}{nd(H)}} \]

Note that the equality occurs, if and only if, \( x_u = x_{\max} \) for all \( u \in V \). That is, equality occurs only when \( H \) is regular.

For the part (b), notice that \( \rho x_u^{k-1} \leq d(u)x_{\max}^{k-1} \), hence \( \rho^{\frac{k}{k-1}}x_u^k \leq d(u)^{\frac{k}{k-1}}x_{\max}^k \). Adding on the set of vertices, we have

\[ \rho^{\frac{k}{k-1}} \sum_{u \in V} x_u^k \leq \sum_{u \in V} d(u)^{\frac{k}{k-1}}x_{\max}^k \Rightarrow \rho^{\frac{k}{k-1}} \leq x_{\max}^k \left( \sum_{u \in V} d(u)^{\frac{k}{k-1}} \right) \]

For the same reason of the part (a), the equality holds only when \( H \) is regular. \( \square \)

Now we will prove Theorem 2, which we state again for easy reference.

**Theorem 2.** Let \( H \) be a connected hypergraph on \( n \) vertices. If \( (\rho, x) \) is its principal eigenpair, then

\[ x_{\min} \leq k \sqrt[\frac{k}{\rho + \delta(n-1)}]{} \]
The equality holds if and only if there is a vertex \( v \) such that, for all \( w \in V - \{v\} \), we have \( x_w = x_{\min} \) and \( x_v = \sqrt[k]{\frac{\rho}{\delta}} x_{\min} \).

**Proof.** We notice that

\[
\rho x_i^k = \sum_{e \in E_i} a(e) \sum_{\alpha \in S(e)} x^\alpha \leq \sum_{e \in E_i} a(e) \sum_{\alpha \in S(e)} x^\alpha = d(i)x_{\max}^k, \forall i \in V(H).
\]

Therefore

\[
\rho x_{\min}^k \leq \delta x_{\max}^k \tag{3}
\]

Let us choose a vertex \( v \) such that \( x_v = x_{\max} \), so the following inequality holds

\[
\delta(n - 1)x_{\min}^k \leq \delta \sum_{i \in V - \{v\}} x_i^k \tag{4}
\]

Adding the inequalities 3 and 4 we conclude that

\[
x_{\min}^k(\rho + \delta(n - 1)) \leq \delta \sum_{i \in V} x_i^k \Rightarrow x_{\min} \leq \sqrt[k]{\frac{\delta}{\rho + \delta(n - 1)}}
\]

To finish the proof, we just notice that

\[
\delta(n - 1)x_{\min}^k = \delta \sum_{i \in V - \{v\}} x_i^k \Leftrightarrow x_w = x_{\min} \forall w \neq v.
\]

Further

\[
\rho x_{\min}^k = \delta x_{\max}^k \Leftrightarrow x_v = \sqrt[k]{\frac{\rho}{\delta}} x_{\min}
\]

□

Note that the equality in Theorem 2 is true, for example, for regular hypergraphs or for the star \( S_n \).

**Remark 2.** The equality in Theorem 2 is sharper than the Theorem 7, when

\[
\rho \geq 2^{(k-1)}\sqrt[k]{\Delta k^3k-1}
\]

For completeness, we will prove this statement. Indeed,

\[
\rho \geq 2^{(k-1)}\sqrt[k]{\Delta k^3k-1} \Rightarrow \frac{\rho}{\delta} \geq \left(\frac{\Delta}{\delta}\right)^{\frac{k}{2(k-1)}} \Rightarrow \frac{1}{\frac{\rho}{\delta} + n - 1} \leq \frac{1}{\left(\frac{\Delta}{\delta}\right)^{\frac{k}{2(k-1)}} + n - 1}
\]

Thus we conclude that

\[
\sqrt[k]{\frac{\rho}{\delta} \delta} \leq \sqrt[k]{\left(\frac{\Delta}{\delta}\right)^{\frac{k}{2(k-1)}} + n - 1}
\]
5. Measures for centering and regularity of edges

In this section we will study the parameters $x_{\text{max}}$ and $x_{\text{min}}$ that are still little explored, even for graphs. Just as $x_{\text{max}}$ and $x_{\text{min}}$ can be used to determine the most central and peripheral vertices, we believe that the values $x_{\text{max}}$ and $x_{\text{min}}$ have a similar role for the edges of a uniform hypergraph.

**Definition 14.** Let $H$ be a $k$-graph and $x = (x_v)$ its principal eigenvector.

$$x_{\text{min}} = \min_{e \in E(H)} \{x^e\}, \quad x_{\text{max}} = \max_{e \in E(H)} \{x^e\}, \quad \Gamma(H) = \frac{x_{\text{max}}}{x_{\text{min}}}$$

**Theorem 9.** Let $H$ be a connected $k$-graph. If $(\rho, x)$ is its principal eigenpair, then

(a) $\frac{\delta}{\rho} x_{\text{min}} \leq x_{\text{min}}^k \leq x_{\text{min}}$
(b) $x_{\text{max}}^k \leq x_{\text{max}} \leq \frac{\Delta}{\rho} x_{\text{max}}$

If $H$ is regular then the equalities hold.

*Proof.* For part (a) notice that for every $v \in V$ we have $x_{\text{min}} \leq x_v \leq x_{\text{max}}$, thus given an edge $e = \{v_1, \ldots, v_k\} \in E$, we have

$$x^e = x_{v_1} \cdots x_{v_k} \geq x_{\text{min}} \cdots x_{\text{min}} = x_{\text{min}}^k \quad \forall e \in E \Rightarrow x_{\text{min}} \geq x_{\text{min}}^k.$$  

Let $u \in V$ such that $x_u = x_{\text{min}}$, so

$$\rho x_{\text{min}}^k = \sum_{e \in E[u]} x^e \geq \sum_{e \in E[u]} x_{\text{min}}^k \geq \delta x_{\text{min}}^k.$$  

Similarly we prove the part (b).

If $H$ is regular then $x_{\text{min}} = x_{\text{max}}$ and $\delta = \rho = \Delta$, therefore the equality holds. $\square$

**Theorem 10.** Let $H$ be a connected $k$-graph on $n$ vertices and $m$ edges, if $(\rho, x)$ is its principal eigenpair, then

$$x_{\text{min}} \leq \frac{\rho(H)}{km} \leq x_{\text{max}}$$

If $H$ is regular then the equality holds.

*Proof.* To prove the first inequality, just note that

$$\rho(H) = k \sum_{e \in E} x^e \geq k \sum_{e \in E} x_{\text{min}} = km x_{\text{min}}$$

The other inequality is analogous.

If $H$ is regular then $x_{\text{min}} = \frac{1}{n} = x_{\text{max}}$ and $\rho = \frac{km}{n}$, therefore the equality holds. $\square$

One consequence of this theorem is that $\sqrt[k]{\frac{\rho}{km}} \leq x_{\text{max}}$, because $x_{\text{max}}^k \leq x_{\text{max}}^k$. It is also possible to obtain the following inequality $x_{\text{min}} \leq \sqrt[k]{\frac{\rho}{km}}$, but it is not very interesting because $x_{\text{min}} \leq \frac{1}{\sqrt{n}} \leq \sqrt[k]{\frac{\rho}{km}}$.

**Theorem 11.** Let $H$ be a connected $k$-graph, if $(\rho, x)$ is its principal eigenpair, then

$$\sqrt[k]{\Gamma(H)} \leq \gamma(H) \leq \sqrt[k]{\frac{\Delta}{\delta} \Gamma(H)}.$$  

If $H$ is regular then both equalities hold.
Proof. These inequalities follow from Theorem 9:

\[ x_{\text{max}}^k \leq (x_{\text{max}})^k, \quad x_{\text{min}}^k \geq (x_{\text{min}})^k \Rightarrow \Gamma(H) \leq \gamma(H)^k. \]

\[ x_{\text{max}}^k \leq \frac{\Delta}{\rho} x_{\text{max}}, \quad x_{\text{min}}^k \geq \frac{\delta}{\rho} x_{\text{min}} \Rightarrow \gamma(H)^k \leq \frac{\Delta}{\delta} \Gamma(H) \]

We observed that if \( H \) is regular then \( \gamma = 1, \Gamma = 1 \) and \( \Delta = \delta \), therefore equality holds.

Obviously if \( H \) is regular then \( \gamma = 1 \Rightarrow x_{\text{max}} = x_{\text{min}} \Rightarrow x_{\text{max}} = x_{\text{min}} \Rightarrow \Gamma = 1. \)

But it is not true that if \( \Gamma = 1 \) then \( H \) is regular. Thus the following questions naturally arise. For which hypergraphs \( H \), do we have \( \Gamma(H) = 1? \) What is the meaning of the parameter \( \Gamma? \) We know that \( \gamma \) measures the regularity of vertices of the hypergraph, would \( \Gamma \) have a similar meaning for the edges?

We believe that these questions are answered by Theorem 3 which will be proved below, which we state here again for easy reference.

**Theorema 3.** Let \( H \) a \( k \)-graph. \( \Gamma(H) = 1 \), if and only if, for each edge the product of the degrees of its vertices is constant.

Proof. If \( \Gamma(H) = 1 \) then \( x_{\text{min}}^k = x_{\text{max}}^k \), so by Theorem 10 we have \( x^e = \frac{\rho}{km} \), for all \( e \in E \). For each \( u \in V \) we have that

\[ \rho x_u^k = \sum_{e \in E[u]} x^e = d(u) \frac{\rho}{km} \Rightarrow x_u = \sqrt[k]{d(u)} \frac{\sqrt[k]{D}}{km}. \]

Therefore

\[ \frac{\rho}{km} = x^e = \sqrt[k]{d(v_1) \cdots d(v_k)} \frac{\sqrt[k]{D}}{km} \Rightarrow \rho^k = d(v_1) \cdots d(v_k) \forall e = \{v_1, \ldots, v_k\} \in E. \]

That is, for any edge, the product of the degrees of its vertices is always \( \rho^k \).

Conversely, if \( d(v_1) \cdots d(v_k) = D \) for all \( e = \{v_1, \ldots, v_k\} \in E \), we define the vector \( x \) by \( x_u = \sqrt[k]{d(u)} \frac{\sqrt[k]{D}}{km} \) for all \( u \in V \), and then

\[ \sum_{e \in E[u]} x^e = \sum_{e \in E[u]} \sqrt[k]{d(v_1) \cdots d(v_k)} \frac{\sqrt[k]{D}}{km} = \sum_{e \in E[u]} \sqrt[k]{D} \frac{\sqrt[k]{D}}{km} = \sqrt[k]{D} x_u^k. \]

Hence \( (A_H x)_u = \sqrt[k]{D} x_u^{k-1} \) for all \( u \in V \). That is, \( (\sqrt[k]{D}, x) \) is an eigenpair of \( H \) and by Theorem 1 we know that \( x \) is the principal eigenvector of \( H \), so \( x_{\text{max}} = \frac{\sqrt[k]{D}}{km} = x_{\text{min}} \) and therefore \( \Gamma(H) = 1. \)

Under these conditions, it is reasonable to say that the parameter \( \Gamma \) measures the balance of the distribution of vertex degrees at the edges in the hypergraph, since if \( \Gamma \) is greater than 1 then the product of the vertices of each edge is not constant.

**Definition 15.** Let \( H = (V, E) \) be an \( k \)-graph, let \( s \geq 1 \) and \( r \geq ks \) two integers. We define a generalized power hypergraph of \( H \) as the \( r \)-graph \( H_r^s \), obtained by replacing each vertex \( v_i \in V(H) \) by a set with \( s \) vertices \( \zeta_i = \{v_1, \ldots, v_s\} \) and adding a new set
with \( r - k \) vertices of degree one \( \varsigma_e = \{v_e^1, \ldots, v_e^{r-k}\} \) on each edge \( e \in E(\mathcal{H}). \) More precisely, the sets of vertices and edges of \( \mathcal{H}_s^r \) are

\[
V(\mathcal{H}_s^r) = \left( \bigcup_{v \in V} \varsigma_v \right) \cup \left( \bigcup_{e \in E} \varsigma_e \right)
\]

and \( E(\mathcal{H}_s^r) = \{\varsigma_e \cup \varsigma_{v_1} \cup \cdots \cup \varsigma_{v_k} : e = \{v_1, \ldots, v_k\} \in E\}. \)

**Example 1.** Two examples of families of uniform hypergraphs that verify the conditions of the Theorem 3 are

(a) The generalized power hypergraph \( \mathcal{H}_s^r \) where \( \mathcal{H} \) is a regular \( k \)-uniform hypergraph and \( r \geq ks \).

(b) The generalized power hypergraph \( (S_n)_s^r \) where \( S_n \) is the graph star with \( n \) vertices and \( r \geq 2s \).

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