Statistical properties of zeta functions’ zeros

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Abstract: The paper reviews existing results about the statistical distribution of zeros for three main types of zeta functions: number-theoretical, geometrical, and dynamical. The paper provides necessary background and some details about the proofs of the main results.

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1. Introduction

The distribution of zeros of Riemann’s zeta function is one of the central problems in modern mathematics. The most famous conjecture is that all of the zeros are on the critical line $\text{Re} s = 1/2$. A more precise version says that the zeros behave like eigenvalues of large Hermitian matrices. While these statements are conjectural, a great deal is known about the statistical properties of these zeros and about zeros of closely related functions. In this report we aim to summarize findings in this research area.

We give necessary background information, and we cover the three main types of zeta functions: number-theoretical, Selberg-type, and dynamical zeta functions.

Some interesting and important topics are left outside of the scope of this report. For example, we do not discuss quantum arithmetic chaos or characteristic polynomials of random matrices.

The paper is divided in three main sections according to the type of the zeta function we discuss. Inside each section we tried to separate the discussion of the properties of zeros at the global and local scales.

Let us briefly describe these types of zeta functions and their relationships. First, the number-theoretical zeta functions come from integers in number fields and their generalizations. Due to the additive and multiplicative structures of the integers, and in particular due to the unique decomposition in prime factors, the zeta functions have a functional equation, $\zeta (1 - s) = c(s) \zeta (s)$ and the Euler product formula $\zeta (s) = \prod_p (1 - (Np)^s)^{-1}$.

One can also look at number-theoretical zeta functions from a different prospective if one starts with modular forms, that is, functions on two dimensional lattices that are invariant relative to transformations from $SL_2 (Z)$.

Some modular forms (the cusp forms) can be written as $\sum_{n>0} c_n \exp (2\pi i nz)$, where $z$ is the ratio of the periods of the lattice, and one can associate a zeta function $\sum c_n n^{-s}$ to this modular form. Then the modularity ensures that the zeta function satisfies a functional equation. In addition, the zeta function will have the Euler product property if the original modular form is an eigenvector for certain operators (the Hecke operators).
This class of modular zeta functions overlaps with number-theoretical functions to a large extent.

In fact, a recent significant development occurred when it was proved that all zeta functions associated with elliptic curves come from modular zeta functions. Among other applications, this discovery was a key to the proof of Fermat’s last theorem.

An important representative of the second class of zetas is Selberg’s zeta function. Let $\mathbb{H}$ is the upper half-plane with the hyperbolic metric and $\Gamma$ is a discrete subgroup of $SL_2(\mathbb{Z})$. Selberg showed that sums over eigenvalues of the Laplace operator on Riemann surface $\Gamma \backslash \mathbb{H}$ are related to sums over non-conjugate elements of $\Gamma$. This relation is called Selberg’s trace formula. Selberg’s zeta function is constructed in such a way that it has the same relation to the trace formula as Riemann’s zeta function has to the so-called “explicit formula” for sums over zeros of the zeta function.

While Selberg’s zeta function resembles Riemann’s zeta in some features, there are significant differences. In particular, the statistical behavior of its zeros depends on the group $\Gamma$ and often it is significantly different from the behavior of Riemann’s zeros.

The third class of zetas, the dynamical zeta functions, appeared first as a generalization of Weil’s zeta functions.

Let $k_q$ is a finite field with $q$ elements and $f(X,Y)$ is an absolutely irreducible polynomial over this field. The simplest Weil’s zeta function is the usual number-theoretical zeta functions for integer ideals of the field $k_q[X,Y]/f(X,Y)$. It turns out (non-trivially) that this zeta function can be written as a generating function for fixed points of powers of the Frobenius map: $x \rightarrow x^q$. (The Frobenius map acts on the points of the curve $f(X,Y)$ in the algebraic closure of $k_q$.)

The theory of dynamical zeta functions starts from this setup and defines a zeta function for arbitrary map $F$ on a set $M$ as the generating function for fixed points of powers of $F$. This definition can also be generalized to flows on a set, that is, to maps $F : M \times \mathbb{R}^+ \rightarrow M$. In particular, Selberg’s zeta function can be understood as the dynamical zeta function for the geodesic flow on the surface $\Gamma \backslash \mathbb{H}$.

With this overview in mind we now come to a more detailed description of available results about the statistics of zeta function zeros.

2. Number-Theoretical Zetas

2.1. Riemann’s zeta

There are several excellent sources on Riemann’s zeta and Dirichlet L-functions, for example the books by Davenport [8] and Titchmarsh [47]. In addition, a very good reference for all topics in this report is provided by Iwaniec and Kowalski’s book [21].

By definition Riemann’s zeta function is given by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

for $\text{Re} s > 1$. It can be analytically continued to a meromorphic function in the entire complex
plane and it satisfies the functional equation
\[
\zeta(1 - s) = \pi^{s - \frac{1}{2}} \frac{\Gamma\left(\frac{1}{2} (1 - s)\right)}{\Gamma\left(\frac{1}{2} s\right)} \zeta(s).
\] (2)

Indeed, we can relate \(\zeta(s)\) to the series \(\theta(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x}\),
\[
\frac{\Gamma\left(\frac{1}{2} s\right) \zeta(s)}{\pi^{s/2}} = \sum_{n=1}^{\infty} \int_{0}^{\infty} x^{s/2 - 1} e^{-n^2 \pi x} dx = \int_{0}^{\infty} x^{s/2 - 1} \theta(x) dx.
\]

From the identities for the Jacobi theta-functions, implied by the Poisson summation formula, it follows that
\[
2\theta(x) + 1 = 1 + \sqrt{x} (2\theta\left(\frac{1}{\sqrt{x}}\right) + 1).
\]

Writing
\[
\int_{0}^{\infty} x^{s/2 - 1} \theta(x) dx = \int_{0}^{1} x^{s/2 - 1} \theta(x) dx + \int_{1}^{\infty} x^{s/2 - 1} \theta(x) dx,
\]
and applying the identity to the integral from 0 to 1, we find that
\[
\frac{\Gamma\left(\frac{1}{2} s\right) \zeta(s)}{\pi^{s/2}} = \frac{1}{s(s - 1)} + \int_{1}^{\infty} \left(x^{-s/2 - 1/2} + x^{s/2 - 1}\right) \theta(x) dx,
\]
which is symmetric relative to the change \(s \to 1 - s\).

The second fundamental property of Riemann’s zeta is the Euler product formula:
\[
\zeta(s) = \prod_{\rho} \left(1 - \frac{1}{\rho^s}\right)^{-1}
\] (3)
valid for \(\text{Re} s > 1\).

Let the non-trivial zeros of Riemann’s zeta function be denoted by \(\rho_k = \beta_k + i\gamma_k\). The Riemann’s hypothesis asserts that \(\beta_k = 1/2\) and it is known that \(0 < \beta_k < 1\). The zeros are symmetric relative to the real axis, so it is enough to consider zeros with positive imaginary part, \(\gamma_k > 0\). We order them so that the imaginary part is non-decreasing, \(\gamma_1 \leq \gamma_2 \leq \ldots\).

One of the most important ideas in the field is to relate the sums over prime numbers and sums over zeta function zeros. For example, the Riemann-Mangoldt formula says that
\[
\sum_{n \leq x} \Lambda(n) = x - \sum_{\rho} \frac{x^\rho}{\rho} + \sum_{n \geq 2} \frac{x^{-2n}}{2n} - \frac{\zeta'}{\zeta}(0),
\] (4)
where \(\Lambda(n) = \log p\), if \(n\) is a prime \(p\) or a power of \(p\), and otherwise \(\Lambda(n) = 0\). The idea of the proof (due to Riemann) is to consider the logarithmic derivative
\[
\frac{\zeta'}{\zeta}(s) = -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s},
\] (5)
and then to use the following formula:
\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \frac{ds}{s} = \begin{cases} 0 & \text{if } 0 < y < 1, \\ \frac{1}{2} & \text{if } y = 1, \\ 1 & \text{if } y > 1, \end{cases}
\]
where $c > 0$, in order to pick out the terms in a Dirichlet series with $n \leq x$ by taking $y = x/n$. 

From (5), one gets 

$$
\sum_{n \leq x} \Lambda (n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[ -\frac{\zeta'}{\zeta} (s) \right] x^s \frac{ds}{s}.
$$

Moving the line of integration away to infinity on the left and collecting the residues at the poles one finds formula (4). (See Chapter 17 in Davenport [8] for a detailed proof.)

More generally, the Landau’s formula holds:

$$
\sum_{n \leq x} \frac{\Lambda (n)}{n^s} = \frac{x^{1-s}}{1-s} - \sum_{\rho} \frac{x^{\rho-s}}{\rho - s} + \sum_{n} \frac{x^{-2n-s}}{2n + s} - \frac{\zeta'}{\zeta} (s). \tag{6}
$$

Formula (4) is useful if one want to study the distribution of primes and something is known about the distribution of zeros, and the formula (6) can be used to study the behavior of $\frac{\zeta'}{\zeta} (s)$ if some information about the primes is known. 

One difficulty is that the sums over zeros in these formulas are only conditionally convergent. $(\rho_k \approx 1/2 + 2\pi i k/\log k)$. In particular, in these sums it is assumed that the summation is in the order of increasing $|\text{Im}\rho|$. 

Another variant of this formula was discovered by Selberg and it avoids the problem of conditional convergence. Define

$$
\Lambda_x (n) = \begin{cases} 
\Lambda (n), & \text{for } 1 \leq n \leq x, \\
\Lambda (n) \left( \frac{\log^2 \frac{x^2 - 2 \log^2 x}{2 \log x}}{2 \log x} \right), & \text{for } x \leq n \leq x^2, \\
\Lambda (n) \frac{\log^2 x}{2 \log^3 x}, & \text{for } x^2 \leq n \leq x^3.
\end{cases}
$$

Then,

$$
\frac{\zeta'}{\zeta} (s) = - \sum_{n \leq x^3} \frac{\Lambda_x (n)}{n^s} + \frac{1}{\log^2 x} \sum_{\rho} \frac{x^{\rho-s} (1 - x^{\rho-s})^2}{(s - \rho)^3} + \frac{x^{1-s} (1 - x^{1-s})^2}{\log^2 x \cdot (1-s)^3} + \frac{1}{\log^2 x} \sum_{q=1}^{\infty} \frac{x^{-2q-s} (1 - x^{-2q-s})^2}{(2q+s)^3}. \tag{7}
$$

Finally, there is one more variant of the explicit formula (sometimes called Delsarte’s explicit formula). Suppose that $H (s)$ is an analytic function in the strip $-c \leq \text{Im} \ s \leq 1 + c$ (for $c > 0$) and that $|H (\sigma + it)| \leq A (1 + |t|)^{-(1+c)}$ uniformly in $\sigma$ in the strip. Let $\widehat{h} (t) = H \left( \frac{1}{2} + it \right)$ and define 

$$
\widehat{h} (x) = \int_{\mathbb{R}} h (t) e^{-2\pi i tx} dt.
$$

(Note that analyticity of $H (s)$ implies that $\widehat{h} (x)$ has finite support.) Then,

$$
\sum_{\rho} H (\rho) = H (0) + H (1) - \frac{1}{2\pi} \sum_{n=1}^{\infty} \Lambda (n) \sqrt{n} \left[ \widehat{h} \left( \frac{\log n}{2\pi} \right) + \widehat{h} \left( -\frac{\log n}{2\pi} \right) \right] - \frac{1}{2\pi} \int_{-\infty}^{\infty} h (t) \Psi (t) dt, \tag{8}
$$

where

$$
\Psi (t) = \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} + it \right) + \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} - it \right).
$$
The idea of the proof is similar to the proof of the Riemann-Mangoldt formula. One starts with the formula
\[ \frac{1}{2\pi i} \int_{2-\infty}^{2+i\infty} \left[ -\frac{\zeta'(s)}{\zeta(s)} \right] H(s) \, ds = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1/2}} \int_{-\infty}^{\infty} h(t) e^{-i(\log n)t} \, dt = -\frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1/2}} \tilde{h} \left( \log \frac{n}{2\pi} \right). \]

Next, one can move the line of integration to \((-1-i\infty, -1+i\infty)\) and use the calculus of residues and the functional equation to obtain formula (8).

The most general explicit formula was derived by Weil (see [51]). We will not present it here since it involves adelic language and this would take us too far apart.

2.1.1. Statistics of zeros on global scale

Let \( N(T) \) denote the number of zeros with the imaginary part strictly between 0 and \( T \). If there is a zero with imaginary part equal to \( T \), then we count this zero as \( 1/2 \). Define
\[ S(T) := \frac{1}{\pi} \text{Im} \log \zeta \left( \frac{1}{2} + iT \right), \]
where the logarithm is calculated by continuous variation along the contour \( \sigma + iT \), with \( \sigma \) changing from \(+\infty\) to \( 1/2 \).

By applying the argument principle to \( \zeta \) and utilizing the functional equation, it is possible to show (see Chapter 15 in [8]) that
\[ N(T) = \frac{T}{2\pi} \log T - \frac{T}{8} + S(T) + O \left( \frac{1}{1+T} \right). \]

Let
\[ X(t) := \sqrt{\frac{2\pi S(t)}{\log \log t}}. \]

Then, we have the following theorem by Selberg. (See [42] and [41].)

**Theorem 2.1 (Selberg)** Assume RH, and let \( T^a \leq H \leq T^2 \), where \( a > 0 \). Then for every \( k \geq 1 \)
\[ \frac{1}{H} \int_{T}^{T+H} |X(t)|^{2k} \, dt = \frac{2k!}{k!2^k} + O(1/\log \log T), \]
with the constant in the remainder term that depends only on \( k \) and \( a \).

In other words, \( X(t) \) behaves like a Gaussian random variable. Note that the Riemann Hypothesis is not assumed in this result.

Recently, this was generalized by Bourgade in [7], who found the correlation structure of \( X(t) \).

**Theorem 2.2 (Bourgade)** Let \( \omega \) be uniform on \((0, 1)\), \( \epsilon_t \to 0 \), \( \epsilon_t \gg 1/\log t \), and functions \( 0 \leq f_t^{(1)} < \ldots < f_t^{(i)} < c < \infty \). Suppose that for all \( i \neq j \),
\[ \frac{\log |f_t^{(j)} - f_t^{(i)}|}{\log \epsilon_t} \to c_{i,j} \in [0, \infty]. \]
Then the vector
\[
\frac{1}{\sqrt{-\log \epsilon_t}} \left( \log \zeta \left( \frac{1}{2} + \epsilon_t + i f_t^{(1)} + i \omega t, \ldots, \frac{1}{2} + \epsilon_t + i f_t^{(l)} + i \omega t \right) \right)
\]
converges in law to a complex Gaussian vector \((Y_1, \ldots, Y_l)\) with the zero mean and covariance function
\[
\text{Cov}(Y_i, Y_j) = \begin{cases} 
1 & \text{if } i = j, \\
1 \wedge c_{i,j} & \text{if } i \neq j.
\end{cases}
\]
Moreover, the above result remains true if \(\epsilon_t \ll 1/\log \log t\), replacing the normalization \(-\log \epsilon_t\) with \(\log \log t\).

This result implies the following Corollary for the zeros of the Riemann’s zeta. Let
\[
\Delta(t_1, t_2) = \left( \mathcal{N}(t_2) - \frac{t_2}{2 \pi} \log \frac{t_2}{2 \pi e} \right) - \left( \mathcal{N}(t_1) - \frac{t_1}{2 \pi} \log \frac{t_1}{2 \pi e} \right),
\]
which represents the number of zeros with imaginary part between \(t_1\) and \(t_2\) minus its deterministic prediction.

**Corollary 2.3 (Bourgade)** Let \(K_t\) be such that, for some \(\epsilon > 0\) and all \(t\), \(K_t > \epsilon\). Suppose \(\log K_t/\log \log t \to \delta \in (0, 1)\), as \(t \to \infty\). Then the finite-dimensional distributions of the process
\[
\frac{\Delta(\omega t + \alpha/K_t, \omega t + \beta/K_t)}{\frac{1}{2} \sqrt{(1 - \delta) \log \log t}}, 0 \leq \alpha < \beta < \infty,
\]
converge to those of a centered Gaussian process \((\bar{\Delta}(\alpha, \beta), 0 \leq \alpha < \beta < \infty)\) with the covariance structure
\[
\mathbb{E}(\bar{\Delta}(\alpha, \beta) \bar{\Delta}(\alpha', \beta')) = \begin{cases} 
1 & \text{if } \alpha = \alpha', \text{ and } \beta = \beta', \\
1/2 & \text{if } \alpha = \alpha', \text{ and } \beta \neq \beta', \\
1/2 & \text{if } \alpha \neq \alpha', \text{ and } \beta = \beta', \\
-1/2 & \text{if } \beta = \alpha', \\
0 & \text{elsewhere}.
\end{cases}
\]

Note that \(K_t\) is of order \((\log t)^{-\delta}\) and the average spacing between zeros is \(1/\log t\), hence the number of zeros in the interval \((\omega t + \alpha/K_t, \omega t + \beta/K_t)\) is of order \((\log t)^{1-\delta}\).

This result perfectly corresponds to the results of Diaconis and Evans in [9] about fluctuations of eigenvalues of random unitary matrices.

Selberg and Bourgade’s theorems are based on an approximation the functions \(S(t)\), discovered by Selberg.

**Proposition 2.4 (Selberg)** Suppose \(k \in \mathbb{Z}^+, 0 < a < 1\). Then there exists \(c_{a,k} > 0\) such that for any \(1/2 \leq \sigma \leq 1\) and \(t^{a/k} \leq x \leq t^{1/k}\), it is true that
\[
\frac{1}{t} \int_1^t \left| \log \zeta(\sigma + is) - \sum_{p \leq x} \frac{1}{p^{\sigma+is}} \right|^{2k} \, ds \leq c_{a,k}.
\]

The proof of this statement, in turn, depends on Selberg’s formula (7).
2.1.2. Statistics of zeros on local scale

At the local scale, we have a result due to Montgomery, \[32\]. Assume the Riemann Hypothesis, and define

\[ F(\alpha) = F(\alpha, T) = \frac{2\pi}{T \log T} \sum_{0 < \gamma, \gamma' < T} T^{\alpha(\gamma - \gamma')} w(\gamma - \gamma'), \]

where \(\alpha\) is real, \(T \geq 2\), and \(w(u)\) is a weighting function,

\[ w(u) = \frac{4}{4 + u^2}. \]

**Theorem 2.5 (Montgomery)** The function \(F(\alpha)\) is real and \(F(\alpha) = F(-\alpha)\). If \(T > T_0(\varepsilon)\), then \(F(\alpha) \geq -\varepsilon\) for all \(\alpha\). For fixed \(\alpha\) satisfying \(0 \leq \alpha < 1\) we have

\[ F(\alpha) = (1 + o(1))T^{-2\alpha} \log T + \alpha + o(1), \]

as \(T\) tends to infinity; this holds uniformly for \(0 \leq \alpha \leq 1 - \varepsilon\).

In fact, this result holds uniformly throughout \(0 \leq \alpha \leq 1\), as later proved by Goldstone.

This result can be used through the identity

\[ \sum_{0 < \gamma, \gamma' \leq T} r \left( (\gamma - \gamma') \frac{\log T}{2\pi} \right) w(\gamma - \gamma') = \frac{T}{2\pi} \log T \int_{-\infty}^{\infty} F(\alpha) \hat{r}(\alpha) \, d\alpha, \]

where

\[ \hat{r}(\alpha) = \int_{-\infty}^{\infty} r(u) e^{-2\pi i \alpha u} \, du. \]

Montgomery conjectures that

\[ F(\alpha) = 1 + o(1) \]

for \(\alpha \geq 1\), uniformly in bounded intervals. It can also be formulated as follows.

**Conjecture 2.6 (Montgomery)** For fixed \(\alpha < \beta\),

\[ \sum_{2\pi \alpha / \log T < |\gamma - \gamma'| < 2\pi \beta / \log T} 1 \sim \left( \int_{\alpha}^{\beta} \left[ 1 - \left( \frac{\sin \pi u}{\pi u} \right)^2 \right] du + \delta(\alpha, \beta) \right) \frac{T}{2\pi} \log T, \]

as \(T\) goes to infinity. Here \(\delta(\alpha, \beta) = 1\) if \(0 \in [\alpha, \beta]\), \(\delta(\alpha, \beta) = 0\) otherwise.

The proof of Montgomery’s theorem is based on the following variant of the explicit formula.

**Lemma 2.7 (Montgomery)** If \(1 < \sigma < 2\) and \(x \geq 1\) then

\[ (2\sigma - 1) \sum_{\gamma} \frac{x^{i\gamma}}{(\sigma - \frac{1}{2}) + (t - \gamma)^2} = -x^{-1/2} \left( \sum_{n \leq x} \Lambda(n) \left( \frac{x}{n} \right)^{1-\sigma + it} + \sum_{n > x} \Lambda(n) \left( \frac{x}{n} \right)^{\sigma + it} \right) \]

\[ + x^{1/2-\sigma + it} \left( \log \tau + O_\sigma(1) \right) + O_\sigma \left( x^{1/2-\sigma} \right), \]

where \(\tau = |t| + 2\). The implicit constants depend only on \(\sigma\).
In a further development Hughes and Rudnick in [17] study the sums $f(x_1) + \ldots f(x_N)$ where $x_N$ denote the scaled zeros and calculate their moments. This is possible to do if $f$ is sufficiently smooth. They proved the following result. Write the zeros as $\frac{1}{2} + i\gamma_j$ where $\gamma_j$ are not assumed real. Let $f$ be a real-valued even function with smooth Fourier transform that has compact support. Define

$$N_f(t, T) := \sum_{\gamma_j} f\left(\log T \frac{\gamma_j - t}{2\pi}\right).$$

(If $f$ were the indicator function of an interval $[-1,1]$ and if all $\gamma_j$ are real, then this function would count number of zeros in the interval $[t-2\pi/\log T, t+2\pi/\log T]$. However, the Fourier transform of this indicator function does not have compact support.)

Next, choose a weight function $w(x)$, such that $w \geq 0$, $\int w(x) \, dx = 1$, and $\hat{w}(x)$ is compactly supported, and define an averaging operator

$$\langle W \rangle_{T,H} := \int_{\mathbb{R}} W(t) w\left(\frac{t-T}{H}\right) \, dt.$$

**Theorem 2.8 (Hughes-Rudnick)** Assume that the averaging window $H = T^a$ for $0 < a \leq 1$, and that the support of $\hat{f}(u) = \int f(x) e^{-2\pi i xu} \, dx$ is in $(-2a/m, 2a/m)$ with integer $m \geq 1$. Then, as $T \to \infty$, the first $m$ moments of $N_f$, $\langle N_f^m \rangle_{T,H}$ converge to those of a Gaussian random variable with expectation $\int f(x) \, dx$ and variance

$$\sigma_f^2 = \int \min(|u|, 1) \hat{f}(u)^2 \, du.$$

They note that a similar result holds in the random matrix theory for eigenvalues of a unitary random matrix, and that the higher moments converge to non-Gaussian values. This is again in agreement with the situation in the random matrix theory.

We will describe the ideas of their proof below in the case when they are applied to zeros of Dirichlet’s L-functions. The key idea is to use a variant of the explicit formula (8).

A generalization of Montgomery results about pair correlations of zeros is due to Rudnick and Sarnak in [37] who studied statistical properties of $k$-tuples of zeros. They proved these results for a quite large class of zeta functions, and we will postpone the discussion of their results to a later section.

Rudnick and Sarnak proved their results under some restrictive conditions, similar to conditions imposed by Montgomery. Roughly speaking these conditions require that the test functions used for evaluation of correlation functions are very smooth. It is an outstanding problem to prove that all results about correlations of zeros hold without these restrictive hypotheses. This is supported by numerical evidence and, if this conjecture true, we can conclude that the behavior of the zeta zeros is very similar to the behavior of eigenvalues of random unitary matrices.

### 2.2. Dirichlet’s L-functions

Let $\chi(n)$ denote a multiplicative character modulo a positive integer $q$. That is, the function $\chi$ maps integers to the unit circle; it is multiplicative, $\chi(nm) = \chi(n)\chi(m)$, and $\chi(n) = 0$ if
$n$ and $q$ are not relatively prime. The character which sends every integer relatively prime to $q$ to 1 is called the principal character modulo $q$. The conductor of the character is the minimal integer $N$ such that the character is periodic modulo $N$. For simplicity, let $q$ be a prime in the following. In this case the conductor equals $q$. A character is odd if $\chi(-1) = -1$, and even if $\chi(-1) = 1$.

Dirichlet’s $L$-function corresponding to the character $\chi$ is defined by the series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$  

This series can be continued to a function which is meromorphic in the whole complex plane and satisfies a functional equation. Namely, let $\mu = 0$ if $\chi$ is even and $\mu = 1$ if $\chi$ is odd. Define

$$\Phi(s, \chi) = q^{\frac{s}{2}} \pi^{-\frac{s}{2}} \Gamma\left(\frac{1}{2} (s + \mu)\right) L(s, \chi),$$

then the functional equation has the form

$$\Phi(1-s, \chi) = i^{\mu} \sqrt{q} \tau(\chi) \Phi(s, \chi),$$

where $\tau(\chi)$ is the Gauss sum:

$$\tau(\chi) = \sum_{m=1}^{q} \chi(m) e^{2\pi i m/q}.$$

Note that $|\tau(\chi)| = \sqrt{q}$. For a proof see Chapter 9 in Davenport [8].

Finally, note that if $\chi$ is not principal, then $L(s, \chi)$ is entire.

2.2.1. Global scale

For $T > 0$, let $N(T, \chi)$ denote the number of zeros of $L(s, \chi)$ with $0 < \sigma < 1$ and $0 \leq t \leq T$, possible zeros with $t = 0$ or $t = T$ counting one half only. Let

$$S(t, \chi) = \frac{1}{\pi} \text{Im} \log L\left(\frac{1}{2} + it, \chi\right).$$

Then it can be shown that

$$N(T, \chi) = \frac{T}{2\pi} \log Tq - \frac{\chi(-1)}{8} + S(T, \chi) - S(0, \chi) + O\left(\frac{1}{1 + T}\right).$$

See formula (1.3) in Selberg’s paper [40] and for example, Chapter 16 in Davenport [8].

If the character is fixed and the imaginary part of the argument is large, then the results for Dirichlet’s $L$-functions are similar to those for Riemann’s zeta function.

A different situation arises when the interval $[0, T]$ is fixed and the character $\chi$ is random. This situation was studied by Selberg, who proved the following results. (See [40].)

First, for $|t| \leq q^{1/4-\varepsilon}$, the difference

$$S(t, \chi) - \frac{1}{\pi} \text{Im} \sum_{p \leq q} \frac{\chi(p)}{p^{1/2} + it}$$

is uniformly bounded in the mean as $q \to \infty$, and $|S(t, \chi)|$ is of the order of magnitude $\sqrt{\log \log q}$ in the mean. Selberg also established the following theorem.
Theorem 2.9 (Selberg) For \( |t| \leq q^{1/4 - \varepsilon} \), we have

\[
\frac{1}{q - 2} \sum_{\chi} |S(t, \chi)|^{2r} = \frac{(2r)!}{r! (2\pi)^{2r}} (\log \log q)^r + O((\log \log q)^{r-1}),
\]

where the summation is over all non-principal characters over the base \( q \).

In other words the distribution of \( S(t, \chi) \) approaches the distribution of a Gaussian random variable with variance \( \frac{1}{2\pi^2} \log \log q \). In these results \( t \) is assumed to be fixed.

2.2.2. Local Scale

Hughes and Rudnick in [18] study the linear statistics of low-lying zeros of \( L \)-functions on the local scale. Hughes and Rudnick order the zeros \( \rho_{i,\chi} = \frac{1}{2} + i\gamma_{i,\chi} \) as follows:

\[
\ldots \leq \text{Re} \gamma_{-2,\chi} \leq \text{Re} \gamma_{-1,\chi} < 0 \leq \text{Re} \gamma_{1,\chi} \leq \text{Re} \gamma_{2,\chi} \leq \ldots
\]

and define

\[
x_{i,\chi} = \frac{\log q}{2\pi} \gamma_{i,\chi}.
\]

Then they define

\[
W_f(\chi) = \sum_{i=-\infty}^{\infty} f(x_{i,\chi})
\]

where \( f \) is a rapidly decaying test function.

The question is to understand the behavior of the averages

\[
\langle W_f^m \rangle = \frac{1}{q - 2} \sum_{\chi \neq \chi_0} W_f(\chi).
\]

The basis for their analysis is a variant of the explicit formula (8) that relates a sum over zeros of \( L(s, \chi) \) to a sum over prime powers. This formula is a particular version of the formula from Rudnick and Sarnak [37], which is valid for zeta functions of all cuspidal automorphic representations (which we will not define in this paper). Let

\[
\mu(\chi) = \begin{cases} 
0, & \text{if } \chi \text{ is even}, \\
1, & \text{if } \chi \text{ is odd}, 
\end{cases}
\]

and let \( h(r) \) be any even analytic function in the strip \(-c \leq \text{Im} r \leq 1 + c \) (for \( c > 0 \)) such that \( |h(r)| \leq A (1 + |r|)^{-(1 + \delta)} \) (for \( r \in \mathbb{R}, A > 0, \delta > 0 \)). Set \( \hat{h}(u) = \frac{1}{2\pi} \int h(r) e^{-iru} dr \), so that \( h(r) = \int \hat{h}(u) e^{iru} du \). Then

\[
\sum_j h(\gamma_{j,x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) (\log q + G_\chi(r)) dr
\]

\[
- \sum_n \frac{\Lambda(n)}{\sqrt{n}} \hat{h}(\log n) (\chi(n) + \overline{\chi}(n)),
\]

where

\[
G_\chi(r) = \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} + \mu(\chi) + ir \right) + \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} + \mu(\chi) - ir \right) - \frac{1}{2} \log \pi.
\]
By using this formula, Hughes and Rudnick separate \( W_f(\chi) \) into two components. Take \( h(r) = f \left( \frac{\log q}{2\pi} r \right) \), so that \( \hat{h}(u) = \frac{1}{\log q} \hat{f} \left( \frac{u}{\log q} \right) \). We have to impose some conditions on \( f \) so that the conditions on \( h(r) \) are satisfied. We say that \( f \) is *admissible*, if it is a real, even function whose Fourier transform \( \hat{f}(u) := \int f(r) e^{-2\pi i u r} dr \) is compactly supported, and such that \( |f(r)| \leq A (1 + |r|)^{-(1+\delta)} \). Then, we get the following decomposition:

\[
W_f(\chi) = W_f^r(\chi) + W_f^{osc}(\chi),
\]

where

\[
W_f^r(\chi) := \int_{-\infty}^{\infty} f \left( \frac{\log q}{2\pi} r \right) (\log q + G_\chi(r)) dr,
\]

and

\[
W_f^{osc}(\chi) := -\frac{1}{\log q} \sum_n \Lambda(n) \sqrt{n} \hat{f} \left( \frac{\log n}{\log q} \right) (\chi(n) + \overline{\chi}(n)).
\]

For large \( q \),

\[
W_f(\chi) := \int_{-\infty}^{\infty} f(x) dx + O \left( \frac{1}{\log q} \right),
\]

which is asymptotically independent of \( \chi \).

First, Hughes and Rudnick show that if \( \text{Supp} \left( \hat{f} \right) \subseteq [-2, 2] \), then

\[
\langle W_f^{osc}(\chi) \rangle_q = O \left( \frac{1}{\log q} \right).
\]

Next, they prove the following result.

**Theorem 2.10 (Hughes and Rudnick)** Let \( f \) be an admissible function and assume that

\[
\text{Supp} \left( \hat{f} \right) \subseteq [-\alpha, \alpha], \alpha > 0.
\]

If \( m < 2/\alpha \), then the \( m \)-th moment of \( W_f^{osc} \) is

\[
\lim_{q \to \infty} \langle (W_f^{osc})^m \rangle_q = \begin{cases} 
\frac{m!}{2^{m/2} (m/2)!} \sigma(f)^m, & \text{if } m \text{ is even}, \\
0, & \text{if } m \text{ is odd},
\end{cases}
\]

where

\[
\sigma(f)^2 = \int_{-1}^{1} |u| \hat{f}(u)^2 du.
\]

They also made a conjecture about the limit when the support of \( \hat{f} \) is not bounded to \([-2/m, 2/m]\) by performing an analogous calculation for unitary random matrices (where all moments of the eigenvalue distribution are known from the work of Diaconis and Shahshahani [10]). A consequence of this conjecture is that the limit of the linear statistic \( W_f(\chi) \) is not Gaussian for high moments.

Hughes and Rudnick used their results to derive some results for the smallest zero of \( L(s, \chi) \). In particular they show that for infinitely many \( q \) there are characters \( \chi \) such that the imaginary part of the zero is smaller than \( 1/4 \), and that for all sufficiently large \( q \), a positive proportion of \( L(s, \chi) \) has a zero smaller than a certain constant.
2.3. L-functions for modular forms (The Hecke L-functions)

A good introductory source for the material in this section is Chapter V of Milne’s book [30] about elliptic curves.

2.3.1. L-functions from modular forms

Let \( \Gamma \) be a subgroup of finite index in \( SL_2(\mathbb{Z}) \) and let \( \mathbb{H} \) denote the upper half-plane \( \{ z | \text{Im} z > 0 \} \). The set \( \mathbb{H}^* = \mathbb{H} \cup \{ \infty \} \cup \mathbb{Q} \) can be made into a Hausdorff topological space and one can define a continuous action of \( SL_2(\mathbb{Z}) \) on \( \mathbb{H}^* \) as an extension of the action of \( SL_2(\mathbb{Z}) \) on \( \mathbb{H} \):

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = f \left( \frac{az + b}{cz + d} \right).
\]

The cusps are points in \( \mathbb{H}^* \setminus \mathbb{H} \). One can also show that \( \Gamma \setminus \mathbb{H}^* \) is a compact Hausdorff space (that is, a compact space in which every two points have disjoint open neighbourhoods), which is a Riemann surface (that is, it admits a complex structure). Recall the following definition.

**Definition 2.11** A modular form for \( \Gamma \) of weight \( 2k \) is a function \( \mathbb{H} \to \mathbb{C} \) such that

(a) \( f \) is holomorphic on \( \mathbb{H} \);

(b) for any \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \), \( f(\gamma z) = (cz + d)^{2k} f(z) \);

(c) \( f \) is holomorphic at the cusps.

In particular, analyticity at infinity implies that a modular form can be written as

\[
f(z) = \sum_{n \geq 0} c(n) q^n, \quad \text{where} \quad q = e^{2\pi iz/h},
\]

where \( h \) is the minimal integer \( h > 0 \), such that \( \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma \).

A cusp form is a modular form which is zero at all the cusps. It can be written

\[
f(z) = \sum_{n \geq 1} c(n) q^n.
\]

It is usual to write \( M_{2k}(\Gamma) \) and \( S_{2k}(\Gamma) \) to denote the vector spaces of all modular and cusp forms.

Recall that \( \Gamma_0(N) \) is a subgroup of \( SL_2(\mathbb{Z}) \) defined as follows:

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \, (\text{mod } N) \right\}.
\]

**Definition 2.12** Let \( f \) be a cusp form of weight \( 2k \) for \( \Gamma_0(N) \),

\[
f(z) = \sum_{n \geq 1} c(n) q^n.
\]

The \( L \)-series of the cusp form \( f \) is the Dirichlet series

\[
L(f, s) = \sum_{n \geq 1} c(n) n^{-s}.
\]
It is possible to estimate that $|c(n)| \leq Cn^k$, and therefore this series is convergent for $\text{Re}s > k + 1$.

Let $\alpha_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. Then

$$\alpha_N \begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha_N^{-1} = \begin{pmatrix} d & -c/N \\ -Nb & a \end{pmatrix},$$

and so the conjugation by $\alpha_N$ preserves $\Gamma_0(N)$. This implies the following fact about cusp forms. Define operator $w_N$ by the following formula:

$$(w_N f)(z) := \left(\sqrt{Nz}\right)^{2k} f\left(-1/z\right).$$

Then $w_N$ preserves $S_{2k}(\Gamma_0(N))$ and $w_N^2 = 1$. Hence the only eigenvalues of $w_N$ are $\pm 1$, and $S_{2k}(\Gamma_0(N))$ is a direct sum of the corresponding eigenspaces, $S_{2k} = S_{2k}^+ + S_{2k}^-$. This implies the following fact about cusp forms. Define operator $w_N$ by the following formula:

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Then $w_N$ preserves $S_{2k}(\Gamma_0(N))$ and $w_N^2 = 1$. Hence the only eigenvalues of $w_N$ are $\pm 1$, and $S_{2k}(\Gamma_0(N))$ is a direct sum of the corresponding eigenspaces, $S_{2k} = S_{2k}^+ + S_{2k}^-$. It turns out that the zeta functions corresponding to these eigenvector cusp forms satisfy functional equations analogous to the functional equation of the Riemann’s zeta.

**Theorem 2.13 (Hecke)** Let $f \in S_{2k}(\Gamma_0(N))$ be a cusp form in the $\varepsilon$-eigenspace, $\varepsilon = 1$ or $-1$. Then $f$ extends analytically to a holomorphic function on the whole complex plane, and satisfies the functional equation

$$\Lambda(s, f) = \varepsilon (-1)^k \Lambda(k - s, f),$$

where

$$\Lambda(s, f) = N^{s/2} (2\pi)^{-s} \Gamma(s) L(f, s).$$

Moreover, under certain conditions, which we explain below, these zeta functions can be written as a Euler product over primes.

Namely, the Hecke operators $T(n)$ ($n \geq 1$) are linear operators on $S_{2k}(\Gamma_0(N))$. We skip their definition here, which can be found in Section V.4 of Milne [30] or Section VII.5 of Serre [43], and only list their properties.

**Theorem 2.14 (Hecke)** The maps $T(n)$ have the following properties:

(a) $T(mn) = T(m) T(n)$ if $\gcd(m, n) = 1$;
(b) $T(p^r) = T(p^{r+1}) + p^{2k-1} T(p^{r-1})$ if $p$ does not divide $N$;
(c) $T(p^r) = T(p) T(r) \quad r \geq 1, \text{ if } p | N$;
(d) all $T(n)$ commute.

Since the Hecke operators commute we can look for the functions which are eigenfunctions for all of them. Then, it turns out that these eigenfunctions have Euler products.

**Theorem 2.15 (Hecke)** Let $f$ be a cusp form of weight $2k$ for $\Gamma_0(N)$ that is simultaneously an eigenvector for all $T(n)$, say $T(n) f = \lambda(n) f$, and let

$$f = \sum_{n \geq 1} c(n) q^n, \quad q = e^{2\pi i z}.$$

Let $c(1) = 1$. Then, (i) coefficients of the series are eigenvalues of the Hecke operators,

$$c(n) = \lambda(n)$$
and (ii)
\[
L(s, f) = \prod_{p \mid N} \frac{1}{1 - c(p) p^{-s}} \prod_{p \not\mid N} \frac{1}{1 - c(p) p^{-s} + p^{2k - 1 - s}}.
\]

In particular, \(S_{12}(\Gamma_0(1))\) has dimension 1, and therefore it is generated by a single function, which is called the \(\Delta\)-function:
\[
\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum \tau(n) q^n,
\]
where \(\tau(n)\) is called the Ramanujan \(\tau\)-function. It follows that the \(L\)-function associated to the \(\Delta\)-function has both a functional equation and the Euler product property.

In a more general situation, if we wish to find forms that have both a functional equation and a Euler product, then we must overcome the obstacle that operators \(w_N\) and \(T(n)\) do not always commute. However, this obstacle can be circumvented and it can be proved that such modular forms do exist. They are called newforms.

In summary, the \(L\)-functions of modular newforms have both a functional equation and the Euler product property.

2.3.2. \(L\)-functions from Maass forms

A nice source for the material in this section is the lecture notes by Liu [25]. A lot of additional information about Maass forms can be found in the book by Iwaniec [20].

Modular forms are holomorphic and so are comparably rare animals. It is not immediately clear how to compute them. We can try to relax the assumption of complex analyticity, and in this way we come to the concept of a Maass form.

**Definition 2.16** A smooth function \(f \neq 0\) is called a Maass form for group \(\Gamma\), if
(i) for all \(g \in \Gamma\) and all \(z \in \mathbb{H}\), \(f(gz) = f(z)\);
(ii) \(f\) is an eigenfunction of the non-Euclidean Laplace operator:
\[
-y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f = \lambda f,
\]
and
(iii) there exists a positive integer \(N\), such that
\[
f(z) \ll y^N, \; y \to \infty.
\]

Note that it is relatively easy to generate Maass forms as eigenfunctions of the Laplace operator on a fundamental domain of the group \(\Gamma\).

**Definition 2.17** A Maass form \(f\) is said to be a cusp form if the equality
\[
\int_0^1 f(z + b) \, db = 0
\]
holds for all \(z \in \mathbb{H}\).
A Maass form $f$ is call odd if $f(-x + iy) = -f(x + iy)$, and even if $f(-x + iy) = f(x + iy)$.

One example of a Maass form is provided by the Eisenstein series:

$$E(z, s) = \frac{\Gamma(s)}{\pi^s} \sum_{(m,n) \neq (0,0)} \frac{y^s}{|mz + n|^{2s}}$$

$$= \pi^{-s} \Gamma(s) \zeta(2s) \left(2y^s + \sum_{(c,d) = 1 \atop c \neq 0} \frac{y^s}{|cz + d|^{2s}}\right)$$

$$= \pi^{-s} \Gamma(s) \zeta(2s) \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} (\text{Im} (\gamma z))^s,$$

where $\Gamma_\infty = \{ \gamma \in \Gamma | \gamma \infty = \infty \}$. \hfill (9)

(The factor $\pi^{-s} \Gamma(s)$ is introduced to get a simpler functional equation.) It can be checked that this series is an eigenfunction of the Laplace operator with eigenvalue $s(1-s)$, and it turns out that it possess a nice functional equation. Eisenstein series can also be defined for different subgroups $\Gamma$ and different cusps.

From the spectral point of view, Eisenstein series correspond to the continuous part of the spectrum of the Laplacian. There are also other forms that correspond to the discrete part of the spectrum.

By expanding a Maass form in Fourier series and taking the Fourier coefficients as the coefficients of a Dirichlet series, one can construct new L-functions that satisfy a functional equation. The construction is similar to the case of modular forms.

**Theorem 2.18** Let $f$ be a Maass form with eigenvalue $1/4 + \nu^2$. Let $\varepsilon = 0$ or $1$ depending on whether $f$ is even or odd. Let

$$\Lambda(s, f) = \pi^{-s} \Gamma \left(\frac{s + \varepsilon + i\nu}{2}\right) \Gamma \left(\frac{s + \varepsilon - i\nu}{2}\right) L(s, f).$$

Then $\Lambda(s, f)$ is an entire function that satisfies

$$\Lambda(s, f) = (-1)^\varepsilon \Lambda(1-s, f).$$

Since the Laplace operator commutes with Hecke operators, therefore they can be simultaneously diagonalized.

For example, the Eisenstein series $E(z, 1/2 + it)$ is an eigenfunction of all Hecke operators with an eigenvalue

$$\eta_t(n) = \sum_{ad=n} \left(\frac{\alpha}{d}\right)^i.$$

The Maass forms which are eigenfunctions of Hecke operators lead to L-functions with an Euler product property and in many respects the theory is parallel to the theory of L-functions which are obtained from modular newforms.

In particular, for the L-series corresponding to the Eisenstein series $E_r = E(z, 1/2 + ir)$, one can check that

$$L(s, E_r) = \zeta(s+ir) \zeta(s-ir).$$ \hfill (10)
2.3.3. Statistical properties of zeros: Global scale

Let
\[ S(t, \psi) := \frac{1}{\pi} \arg L \left( \frac{1}{2} + it, \psi \right), \]
where \( \psi \) is the Maass form with eigenvalue \( \lambda(\psi) \) and \( L \) is the corresponding \( L \)-function. \( S(t, \psi) \) is related to the number of zeros of \( L \) in the critical strip in the same way as the usual \( S(t) \) function is related to the number of zeros of Riemann’s zeta function.

For this function \( S(t, \psi) \), Hejhal and Luo [16] proved analogues of Selberg’s results for the function \( S(t) \).

For the Eisenstein series \( E_r = E(z, \frac{1}{2} + ir) \), formula (10) implies that
\[ S(t, E_r) = S(t) - S(-t), \]
and the results about the statistical properties of \( S(t, E_r) \) are reduced to the question about the statistical properties of \( S(t) \).

For the case of Maass cusp forms with eigenvalues from the discrete part of the spectrum, Hejhal and Luo proved the following result.

Let \( S_j(t) := S(t, \psi_j) \) where \( \psi_j \) has an eigenvalue \( \lambda_j = \frac{1}{4} + r_j^2 \). Since \( \psi_j(z) \) is periodic along the real axis, it has a Fourier series, and it is known that it can be written as follows:
\[ \psi_j(x + iy) = \sum_{n \neq 0} \rho_j(n) \sqrt{\pi} K_{i\nu_j}(2\pi|n|y) e^{2\pi i n x}, \]
where \( K_{\nu}(z) \) are the standard Bessel functions. Let \( \nu_j(n) = \rho_j(n) / \sqrt{\cosh \pi r_j} \). The numbers \( \nu_j(1) \) will be used as weights in the limiting procedure. One knows that they are \( O(\sqrt{r_j}) \) and
\[ \frac{1}{T^2} \sum_{r_j \leq T} |\nu_j(1)|^2 = \frac{1}{\pi^2} + O \left( \frac{\log T}{T} \right). \]

Since by the Weyl law, the number of \( r_j \) below \( T \) is proportionate to \( T^2 \), these weights can be thought as having bounded magnitude and not too sparse.

Now we formulate one of the results in this paper (Theorem 3 in [16]).

**Theorem 2.19 (Hejhal-Luo)** Let \( h > 0 \) and \( t > 0 \) be fixed. Then we have
\[ \lim_{T \to \infty} \frac{1}{2HT} \sum_{|r_j - T| \leq h} \frac{\pi^2 |\nu_j(1)|^2}{2} \left( S_j(t) \right)^n = C_n, \]
where \( C_n \) are moments of the Gaussian distribution.

The main new ingredient in the proof is the following lemma (Lemma 1 in [16]).

**Lemma 2.20** Let \( T \geq 3, \beta > 0, \frac{1}{T} > \varepsilon > 0, \) and let \( \max(n_1, n_2) \leq T^{1-\varepsilon} \). Then
\[ \sum_{j=1}^{\infty} e^{-\beta(T-r_j)^2} \nu_j(n_1) \nu_j(n_2) = \frac{2\pi^{-3/2}}{\sqrt{\beta}} T \delta_{n_1, n_2} + O_{\beta, \varepsilon} \left( d(n_1) d(n_2) (\log T)^2 \right), \]
where \( d(n) \) denotes the number of divisors of \( n \).

The proof of this Lemma is based on the so-called Kuznetsov trace formula.
2.3.4. Local scale

Rudnick and Sarnak [37] extended the results of Montgomery to zeta functions that arise from modular forms and from Maass forms. In fact, they work in greater generality and study the zeta functions that arise from the general automorphic cuspidal representations of $GL_m$. The Hecke modular L-functions correspond to the case $m = 2$. Their main tool is the following explicit formula, which we formulate for the case of the Hecke L-functions. Let

$$L(s, f) = \prod_{p \nmid N} \frac{1}{1 - c(p)p^{-s}} \prod_{p \mid N} \frac{1}{1 - c(p)p^{-s} + p^{2k-1-s}}$$

where

$$L_p(s, f) = \frac{1}{(1 - \alpha_1(p)p^{-s})(1 - \alpha_2(p)p^{-s})}$$

with the convention that for $p \mid N$ one of $\alpha_i(p)$ is zero. Let $a(p^k) = \alpha_1(p)^k + \alpha_2(p)^k$, and define $b(n) = \Lambda(n) a(n)$. Then

$$\frac{L'}{L} = -\sum_{n=1}^{\infty} \frac{b(n)}{n^s}.$$  

**Theorem 2.21 (Rudnick and Sarnak)** Let $\hat{h} \in C_c^\infty(\mathbb{R})$ be a smooth compactly supported function, and let $h(r) = \int \hat{h}(u) e^{iru} du$. Then

$$\sum h(\gamma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \left( \log Q + \sum_j \left[ \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} + \mu_j + ir \right) + \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} + \mu_j - ir \right) \right] \right) dr$$

$$- \sum_{n=1}^{\infty} \left( \frac{b(n)}{\sqrt{n}} \hat{h}(\log n) + \frac{b(n)}{\sqrt{n}} \hat{h}(-\log n) \right),$$

where $\mu_j$ are some parameters that depend on the form $f$, and $Q$ is the conductor of the form.

By using this result and estimates on the size of coefficients $b(n)$, Rudnick and Sarnak proved a generalization of the Montgomery theorem. Their result is valid not only for the Riemann zeta function, but also for Dirichlet L-functions, for Hecke modular L-functions and for L-functions that correspond to automorphic cuspidal representations of $GL_3$. We formulate it for Hecke modular L-functions.

Consider the class of smooth test functions $F(x_1, \ldots, x_n)$ that satisfy the following conditions:

- **TF 1.** $F(x_1, \ldots, x_n)$ is symmetric.
- **TF 2.** $F(x + t(1, \ldots, 1)) = F(x)$ for all $t \in \mathbb{R}$.
- **TF 3.** $F(x) \to 0$ rapidly as $|x| \to \infty$ in the hyperplane $\sum x_j = 0$.

If $B_N$ is a set of $N$ numbers $x_1, \ldots, x_N$, then the $n$-level correlation sum is defined by

$$R_N(B_N, F) = \frac{n!}{N} \sum_{S \subseteq B_N, |S| = n} F(S).$$
Define the $n$-level correlation density by

$$W_n(x_1, \ldots, x_n) = \det (K(x_i - x_j)), \quad K(x) = \frac{\sin \pi x}{\pi x}.$$ 

**Theorem 2.22** Assume the Riemann hypothesis for the zeros of $L(s, f)$. Let $F(x_1, \ldots, x_N)$ satisfy $TF 1, 2, 3$ and in addition assume that $\hat{F}(\xi)$ is supported in $\sum_j |\xi_j| < 1$. Then,

$$R_n(B_N, F) \to \int F(x) W_n(x) \delta \left(\frac{x_1 + \cdots + x_n}{n}\right) dx_1 \cdots dx_n$$

as $N \to \infty$.

Rudnick and Sarnak mention that the result can probably be proven for functions $F$ with the Fourier transform supported in $\sum_j |\xi_j| < 2$ by an improvement of their method, and conjecture that it holds without any assumption on the support of $\hat{F}(\xi)$.

### 2.4. Elliptic curve zeta functions

The main source for this section is the book [30] by Milne. Consider an elliptic curve

$$E : Y^2Z = X^3 + aXZ^2 + bZ^3,$$

where $a$ and $b$ are integer, and assume that $|\Delta| = |4a^3 + 27b^2|$ cannot be made smaller by a change of variable $X \to X/c^2, Y \to Y/c^3$. This equation is called minimal. The equation

$$\overline{E} : Y^2Z = X^3 + \overline{a}XZ^2 + \overline{b}Z^3,$$

with $\overline{a}$ and $\overline{b}$ the images of $a$ and $b$ in $\mathbb{F}_p$ (the finite field with $p$ elements) is called the reduction of $E$ modulo $p$. (It is assumed here that $p \neq 2, 3$. In the case when $p$ is 2 or 3, a somewhat different notion of the minimal equation is needed.) Let $N_p$ be the number of solutions of this equation in $\mathbb{F}_p$.

There are four possible cases:

(a) **Good reduction.** $E$ is an elliptic curve. (That is, the determinant does not vanish and therefore the curve is smooth.) This happens if $p \neq 2$ and $p$ does not divide $\Delta$. In this case we define the number of points of $E$ by $N_p$.

(b) **Cuspidal, or additive, reduction.** This is the case in which the reduced curve has a cusp. For $p \neq 2, 3$, this case occurs exactly when $p|4a^3 + 27b^2$ and $p| -2ab$.

(c) **Nodal, or multiplicative, reduction.** The reduced curve has a node. For $p \neq 2, 3$, it occurs exactly when $p|4a^3 + 27b^2$, $p \nmid -2ab$.

(c1) **Split case.** The tangents at the node are rational over $\mathbb{F}_p$. This happens when $-2ab$ is a square in $\mathbb{F}_p$.

(c2) **Non-split case.** The tangents at the node are not rational over $\mathbb{F}_p$. This occurs when $-2ab$ is not a square in $\mathbb{F}_p$.

The names additive and multiplicative refer to the group of points on the reduced curve, which in these cases is isomorphic either to $(\mathbb{F}_p, +)$, or $(\mathbb{F}_p^*, \times)$.

We define the $L$ function associated with the elliptic curve $E$ as follows.

$$L(s, E) := \prod_p \frac{1}{L_p(p^{-s})}.$$
Here, the local factors $L_p(T)$ are defined as follows:

\[
L_p(T) = \begin{cases} 
1 - a_p T + pT^2, & \text{if } p \text{ is good, with } a_p = p + 1 - N_p, \\
1 - T & \text{if } E \text{ has split multiplicative reduction,} \\
1 + T & \text{if } E \text{ has non-split multiplicative reduction,} \\
1 & \text{if } E \text{ has additive reduction.}
\end{cases}
\]

Let $S$ be the (finite) set of primes with bad reduction. Then we can also write

\[
L(s, E) = \prod_{p \in S} \frac{1}{L_p(p^{-s})} \prod_{p \notin S} \frac{1}{1 + (N_p - p - 1)p^{-s} + p^{1-2s}} 
= \prod_{p \in S} \frac{1}{L_p(p^{-s})} \prod_{p \notin S} \frac{1}{1 - \alpha_p p^{-s}} (1 - \beta_p p^{-s}).
\]

The Hasse-Weil conjecture says that $L(s, E)$ can be analytically continued to a meromorphic function on the whole of $\mathbb{C}$ and satisfies a functional equation. The recent work by Wiles and others confirmed this conjecture by showing that all elliptic curves are “modular”, in particular, their $L$-functions arise from modular forms. To a certain extent, this result reduces the study of the elliptic $L$-functions to the study of the Hecke modular $L$-functions.

It is known that the numbers $a_p$ do not exceed $2\sqrt{p}$ in absolute value. For a fixed elliptic curve and different primes $p$, these numbers are believed to be distributed on the interval $[-2\sqrt{p}, 2\sqrt{p}]$ according to the semicircle distribution but this is not proven. In fact, this conjecture is related to the Birch-Swinnerton conjecture that states that

\[
L(s, E) \sim C (s - 1)^r \text{ as } s \to 1,
\]

where $r$ is the rank of the group of rational points on $E$, and $C$ is a certain predicted constant.

A stronger statement, which can still be true, is that

\[
\prod_{p \notin S} \frac{N_p}{p} = \prod_{p \notin S, p \leq x} \left(1 - \frac{a_p - 1}{p}\right) = C (\log x)^r.
\]

It is possible to construct zeta functions for other nonsingular projective varieties and the conjecture by Hasse and Weil states that these zeta functions satisfy a functional equation (and the Riemann hypothesis). However, apparently not much is known beyond the cases of projective spaces and elliptic curves.

3. Selberg’s zeta functions for compact and non-compact manifolds

The main source for this section is Hejhal’s book [15].

3.1. Selberg’s zeta function and trace formula

Let $M$ be a compact Riemann surface of genus $g \geq 2$. Then, $M$ can be identified with a quotient space $\Gamma \backslash \mathbb{H}$, where $\mathbb{H}$ is the upper half-plane and $\Gamma$ is a discrete subgroup of $SL_2(R)$. 
We assume that \( \mathbb{H} \) has the Poincaré metric \( |dz|/y \) with the area element \( dx dy/y^2 \), and therefore it has the constant negative curvature. This metric is naturally projected on the surface \( M \).

This is not the most general situation of interest since most of the quotient spaces \( \Gamma \backslash \mathbb{H} \) occurring in arithmetic applications have cusps and therefore are non-compact. However, the theory is most clear and transparent for the compact surfaces.

The Laplace operator on \( M \) can be defined by the following formula.

\[
-\Delta : u \rightarrow -y^2 (u_{xx} + u_{yy}).
\]

It can be shown that this operator has a discrete set of non-positive eigenvalues:

\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots,
\]

and the only point of accumulation of these eigenvalues is \( \infty \).

Let us define

\[
r_n = \begin{cases} \sqrt{\lambda_n - \frac{1}{4}}, & \text{if } \lambda_n \geq \frac{1}{4}, \\ i \sqrt{-\lambda_n + \frac{1}{4}}, & \text{if } \lambda_n < 1/4, \end{cases}
\]

so that \( \lambda_n = \frac{1}{4} + r_n^2 \).

Also let \( m = \max \{ k : \lambda_k < 1/4 \} \).

Let \( \mathcal{G}(M) \) be the set of all closed geodesics on \( M \), and let \( \mathcal{P}(M) \) be the subset of all prime closed geodesics (that is, the closed geodesics that cannot be represented as a non-trivial multiple of another closed geodesic). It is known that it is a countable set, which we can order by the lengths of its elements. Closed geodesics correspond to hyperbolic elements of the group \( \Gamma \) (that is, the elements of \( \Gamma \) with the trace outside of \([-2, 2]\)) up to conjugacy of these elements. If \( P \in \Gamma \) corresponds to a geodesic \( \gamma \), then \( \gamma \) is prime if and only if there is no \( P_0 \in \Gamma \) such that \( P = P_0^k \) for an integer \( k > 1 \).

If \( l(\gamma) \) denotes the length of the geodesic \( \gamma \), then we set

\[
|\gamma| := e^{l(\gamma)}.
\]

If \( P \) is the hyperbolic element of \( \Gamma \) that corresponds to \( \gamma \), then

\[
|\gamma|^{1/2} + |\gamma|^{-1/2} = |\text{Tr}P|.
\]

We will also write \( |\gamma| = N[P] \) (meaning norm of \( P \)).

The Selberg trace formula relates sums over eigenvalues \( \lambda_k \) to sums over hyperbolic elements (geodesics) \( [P] \). Let \( h(u) \) be a function which (i) is analytic in the strip \( |\text{Im}u| \leq 1/2 + \delta \), (ii) is even: \( h(u) = h(-u) \), and (iii) declines sufficiently fast in the strip: \( |h(u)| = O \left((1 + |\text{Re}u|)^{-2-\delta}\right) \). This ensures absolute convergence in the following formulas.

Let \( \hat{h}(t) = \frac{1}{2\pi} \int h(u) e^{-itu} du \).

Then

\[
\sum_{n=0}^{\infty} h(r_n) = \frac{\mu(F)}{2\pi} \int_r rh(r) \tanh(\pi r) dr + \sum_{[T]} \frac{\ln N[T_0]}{N[T]^{1/2} - N[T]^{-1/2}} \hat{h}(\ln N[T]),
\]
where the sum is over all distinct conjugacy classes of hyperbolic elements \([T]\) and \([T_0]\) is the primitive element for \(T, T = T_0^k\).

It is instructive to compare this formula with formula (11). Since it resembles the explicit formulas from number theory, it is natural to seek for a corresponding zeta functions.

We define the Selberg’s zeta function as follows:

\[
Z(s) = \prod_{\gamma \in \mathcal{P}(M)} \prod_{k=0}^{\infty} \left(1 - |\gamma|^s - k\right), \quad \text{Res} > 1.
\] (13)

It turns out that the Selberg’s zeta function is closely related to the eigenvalues of the Laplace operator on \(M\).

**Theorem 3.1 (Hejhal-Selberg)**

(a) \(Z(s)\) is actually an entire function;

(b) Let \(\beta\) be a real number \(\geq 2\). For all \(s\), the following identity holds:

\[
\frac{1}{2s-1}Z'(s) = \frac{1}{2\beta} Z\left(\frac{1}{2} + \beta\right) + \sum_{n=0}^{\infty} \left[ \frac{1}{r_n^2 + (s - \frac{1}{2})^2} - \frac{1}{r_n^2 + \beta^2} \right] + \frac{\mu(F)}{2\pi} \sum_{k=0}^{\infty} \left[ \frac{1}{\beta + \frac{1}{2} + k} - \frac{1}{s + k} \right],
\]

where \(\mu(F)\) is the area of the fundamental region of the group \(\Gamma\).

(c) \(Z(s)\) has “trivial” zeros \(s = -k, k \geq 1\), with multiplicity \((2g - 2)(2k + 1)\);

(d) \(s = 0\) is a zero of multiplicity \(2g - 1\);

(e) \(s = 1\) is a zero of multiplicity \(1\);

(f) the nontrivial zeros of \(Z(s)\) are located at \(\frac{1}{2} \pm ir_n\).

Since all but a finite number of eigenvalues are greater than \(1/4\) hence all but a finite number of \(r_n\) is real and therefore the claim (f) implies that all but a finite number of zeros of \(Z(s)\) are located on the line \(\text{Im} z = 1/2\).

The formula in claim (b) of this theorem follows from Selberg’s trace formula and it can be thought as a functional equation for the logarithmic derivative of \(Z(s)\). In particular, it implies the functional equation for the zeta functions itself.

**Theorem 3.2 (Hejhal-Selberg)** Selberg’s zeta function satisfies the following functional equation:

\[
Z(s) = Z(1-s) \exp \left[ \mu(F) \int_0^{s-\frac{1}{2}} v \tan(\pi v) dv \right].
\]

### 3.2. Statistics of zeros

The number of zeta zeros in a long interval has the following asymptotics:

\[
N[k : 0 \leq r_k \leq T] = \frac{\mu(F)}{4\pi} T^2 + S(T) + E(T),
\]

where

\[
S(T) = \frac{1}{\pi} \arg Z\left(\frac{1}{2} + iT\right),
\]

\[
E(T) = \frac{1}{\pi} \arg Z\left(\frac{1}{2} + iT\right),
\]
Statistics of zetas’ zeros

and

$$E(T) = O(1) = 2c \int_0^T t \left[ \tanh(\pi t) - 1 \right] dt - (\pi + 1).$$

In other words, the number of zeros in a unit interval is $\sim cT$. In comparison, for Riemann’s zeta we have $\sim c \log T$ zeros in the unit interval.

It is known that $S(T) = O\left( \frac{T}{\log T} \right)$, and $S(T) = \Omega_\pm \left[ \left( \frac{\log T}{\log \log T} \right)^{1/2} \right]$.

(Recall that the notation $f(x) = \Omega_+ [g(x)]$ means that $\lim \sup \frac{f(x)}{g(x)} > 0$, and $f(x) = \Omega_- [g(x)]$ means that $\lim \inf \frac{f(x)}{g(x)} < 0$.)

It was found that it is difficult to generalize the results concerning the moments of the $S(x)$ function to the Selberg’s zeta. Since these results are essential for the study of statistical properties of zeta zeros, there is a stumbling block here.

Selberg managed to generalize some of these statistical results for a particular choice of the group $\Gamma$.

Let $p \geq 3$ be a prime and $A$ be a quadratic non-residue modulo $p$. Define

$$\Gamma = \Gamma(A, p) = \left\{ \left( \begin{array}{cc} y_0 + y_1\sqrt{A} & y_2\sqrt{p} + y_3\sqrt{Ap} \\ y_2\sqrt{p} - y_3\sqrt{Ap} & y_0 - y_1\sqrt{A} \end{array} \right) : y_0, y_1, y_2, y_3 \text{ are integer} \right\}$$

and call it a quaternion group.

It turns out that $\Gamma(A, p)$ has no parabolic elements, (that is, it has no group elements with trace equal to $\pm 2$, except $\pm I$). Moreover, it corresponds to a compact Riemann surface.

If $p \equiv 1 \pmod{4}$, then $\Gamma(A, p)$ has no elliptic elements.

Let $S(t) = S^+(t) - S^-(t)$, where $S^+(t) = \max \{0, S(t)\}$ and $S^-(t) = \max \{0, -S(t)\}$. Then the following theorem holds.

**Theorem 3.3 (Hejhal-Selberg)** Let $\Gamma = \Gamma(A, p)$ with $p \equiv 1 \pmod{4}$. Then (for large $T$):

$$\frac{1}{T} \int_T^{qT} S^+(t)^2 dt \geq c_1 \frac{T}{(\log T)^2}$$

where $c_1$ is a positive constant that depends only on $\Gamma$. A similar inequality holds for $S^-(t)$.

In order to appreciate this result note that it implies that the average deviation of $S(T)$ from its mean (if it exists) is of the order larger than $\sqrt{T}/(\log T)$ which should be compared with the number of zeros in the interval $[0, T]$, that is, $cT^2$. To put it in prospective note that the average deviation of the zeros of $S(T)$ for Riemann’s zeta function is of the order $(\log \log T)^{1/2}$ which should be compared with $cT \log T$, the number of Riemann’s zeros in $[0, T]$. These situations appear to be quite different.

Moreover, recently there was some numeric and theoretical work on the eigenvalues of the Laplace operator on manifolds $\Gamma \setminus \mathbb{H}$ for arithmetic groups $\Gamma$. First, numeric and heuristic analysis showed that the spacings between eigenvalues resemble spacings between points from a Poisson point process rather than spacings between eigenvalues of a random matrix ensemble (see Bogomolny et al [5] and references therein). Next some rigorous explanations of this
finding have been given. They are based on exceptionally large multiplicities of closed geodesics with the same length. See Luo and Sarnak ([27] and [28]) and Bogomolny et al. [6].

There is also some work on correlations of closed geodesics - see Pollicott and Sharp [34].

**3.3. Comparison with the circle problem**

If we consider the compact Riemann surface of genus 1, then we are led to similar questions. Such a surface can be represented as quotient space \( \Lambda \setminus \mathbb{C} \), where \( \Lambda \) is a lattice. Consider, for concreteness, \( \Lambda = [1,i] \). Then the eigenvalues of the Laplace operator are \( 4\pi^2 (m^2 + n^2) \), where \( m \) and \( n \) are integer, and the number of the eigenvalues below \( t \) equals the number of integer points in the circle \( t/\pi \).

Let \( r(n) = N \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : a^2 + b^2 = n \} \), and

\[
A(x) = \sum_{0 \leq n \leq x} r(n) = \pi x + R(x).
\]

The function \( A(x) \) can be thought as the counting function both for eigenvalues of Laplace operator and for closed geodesics of bounded length.

Then by using the Poisson summation formula it is possible to derive the following result:

\[
\sum r(n) f(n) = \pi \sum r(n) \int_0^\infty f(x) J_0(2\pi \sqrt{n}x) \, dx.
\]

Given that it is possible to apply this to the step function \( f(x) \), one can obtain the explicit formula:

\[
R(x) = \sqrt{x} \sum_{n=1}^{\infty} \frac{r(n)}{\sqrt{n}} J_1(2\pi \sqrt{n}x).
\]

This leads to various estimates on \( R(x) \), in particular it is known that

\[
R = O \left( x^{1/3} \right) \quad \text{and} \quad R = \Omega_x \left( x^{1/4} \right),
\]

and that

\[
\frac{1}{x} \int_0^x R(t)^2 \, dt = cx^{1/2} + O \left( (\log x)^3 \right).
\]

This suggest that the "standard deviation" of \( R(t) \) is \( x^{1/4} \). Again this does not resemble the situation with Riemann’s zeta zeros.

Some more details about this problem and references to the early papers can be found in [24]. More recent research include [14] and [4].

**4. Zeta functions of dynamical systems**

The main sources for this section are reviews by Ruelle ([38] and [39]) and Pollicott ([36] and [35]).
4.1. Zetas for maps

Let $f$ be a map of a set $M$ to itself, let the periodic orbits of $f$ be denoted by $P$, and let $|P|$ denote the period of orbit $P$. Then, we can define the zeta of $f$ by the following formula:

$$\zeta(z) = \prod_P \left(1 - z^{|P|}\right)^{-1}$$

$$= \exp \sum_{m=1}^{\infty} \frac{z^m}{m} |\text{Fix } f^m|,$$

where $|\text{Fix } f^m|$ denote the number of fixed points of $f^m$.

4.1.1. Permutations

Let $M$ be a finite set, and let $f$ be given by a permutation matrix $A$. Then the number of fixed points of $f^m$ is given by $\text{Tr} A^m$. Hence, we have

$$\zeta(z) = \exp \text{Tr} \sum_{m=1}^{\infty} \frac{(zA)^m}{m}$$

$$= \exp (-\text{Tr} \log (1 - zA))$$

$$= 1/\det (1 - zA),$$

which is closely related to the characteristic polynomial of matrix $A$. In particular, all poles of the zeta are on the unit circle.

4.1.2. Diffeomorphisms

Let $M$ be a torus $\mathbb{R}^2/\mathbb{Z}^2$ and let $f$ be induced by a linear transformation $A \in SL_2(\mathbb{Z})$. Assume that the eigenvalues of $A$ are positive and not on the unit circle: $\lambda_1 > 1 > \lambda_2 > 0$. Then the number of the fixed points of $f^m$ is the number of solutions of the equation $I - A^m = 0 \pmod{\mathbb{Z}^2}$, which can be computed as $|\det (I - A^m)| = (\lambda_1^m - 1)(1 - \lambda_2^m)$. Hence, we have

$$\zeta(z) = \exp \text{Tr} \sum_{m=1}^{\infty} \frac{1}{m} [(z\lambda_1)^m + (z\lambda_2)^m - z^m - (z\lambda_1\lambda_2)^m]$$

$$= \frac{(1 - z\lambda_1)(1 - z\lambda_2)}{(1 - z)(1 - z\lambda_1\lambda_2)}$$

$$= \frac{(1 - z\lambda_1)(1 - z\lambda_2)}{(1 - z)^2}.$$ 

This example can be generalized to linear transformations $A \in GL_d(\mathbb{Z})$ acting on a $d$-dimensional torus or more generally to discrete groups acting on compact quotients of nilpotent Lie groups (nil-manifolds).

If we wish to generalize this example to diffeomorphisms of smooth orientable manifolds, it is natural to count the fixed points by taking into account the degree of map $f$ at the fixed points. Recall that the degree equals to 1 if the map is non-singular and preserves orientation at the fixed point, and to $-1$ if it reverses the orientation. It can also be generalized to the
situation when the map is singular at the fixed point. If \( d(x,f) \) denotes the degree of \( f \) at a fixed point \( x \), then we define the Lefschetz zeta function as

\[
\zeta_L(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \text{Fix}(f^m)} d(x, f^m).
\]

In this case one can use the Lefschetz fixed point formula that says:

\[
\sum_{x \in \text{Fix}(f^m)} d(x, f^m) = \dim M \sum_{i=0}^{\dim M} (-1)^i \text{Tr} ((f^m)^* : H_i \to H_i),
\]

where \( H_i \) is the \( i \)-th homology group of the compact manifold \( M \) with real coefficients, and \((f^m)^*\) is the map induced by \( f^m \) on \( H_i \).

In particular, if \( \lambda_{ij} \) are eigenvalues of \( f^* \), then we get

\[
\zeta_L(z) = \exp \sum_{m=1}^{\infty} \frac{1}{m} \sum_{i=0}^{\dim M} (-1)^i \sum_{j=1}^{\dim H_i} (z \lambda_{ij})^{(-1)^i},
\]

which is a rational function.

If \( f \) is a diffeomorphism of a compact manifold \( M \), then the original dynamical zeta (counted without taking into account the degree of \( f^m \)) is often called the Artin-Mazur zeta function (see Artin-Mazur [2]). If the diffeomorphism satisfies some additional conditions (hyperbolicity or Axiom A), then it is known that this function is rational. (This was conjectured by Smale [45], and proved by Guckenheimer [11] and Manning [29].)

### 4.1.3. Subshifts

Suppose next that \( A \) is an \( N \)-by-\( N \) matrix of zeros and ones, and that the set \( M \) consists of doubly infinite sequences \( \{x_i\} \) of symbols \( 1, \ldots, N \), which satisfy the following criterion. A sequence \( \{x_i\} \) belongs to \( M \) if and only if \( A_{x_ix_{i+1}} = 1 \) for every \( i \). In other words, the symbol \( x_i \) determines which of the other symbols are possible candidates for \( x_{i+1} \). The map \( f \) is simply a shift on this set \( M: \{x_i\} \to \{x_{i+1}\} \). In this case, the number of fixed points of \( f^m \) is simply \( \text{Tr} \left( A^m \right) \), and we have

\[
\xi \left( z \right) = 1 / \det \left( 1 - zA \right),
\]

similar to Example 1. Note that if \( A \) is symmetric then the dynamics of the system is in a certain sense reversible. On the other hand the poles of the zeta in this case are all real. So this example illustrate a connection between the distribution of zeros and the reversibility of a system.

### 4.1.4. Ihara-Selberg zeta function

A basic reference for this section is a book by Terras [46].
Let $G$ be a finite graph. Orient its edges arbitrarily. Let the $2|E|$ oriented edges be denoted $e_1, e_2, \ldots, e_n, e_{n+1} = e_1^{-1}, \ldots, e_{2n} = e_n^{-1}$. A path is a sequence of oriented edges such that the end of one edge equals to the beginning of the next edge. A path is closed if the end of the last edge corresponds to the beginning of the first edge. A closed path $(e_1, e_2, \ldots, e_s)$ is equivalent to the path $(e_2, \ldots, e_s, e_1)$. A closed path $P$ is primitive if $P \neq D^m$ for $m \geq 2$ and any other path $D$. A path is non-backtracking if $e_{i+1} \neq e_i^{-1}$ for any $i$. A closed path is tailless if $a_s \neq a_1^{-1}$.

If $P = (e_1, e_2, \ldots, e_s)$, then $l(P) = s$ is the length of the path.

A prime $[P]$ is an equivalence class of primitive closed non-backtracking tailless paths. (It is an analogue of a geodesic on a manifold.)

The Ihara zeta function is defined as follows:

$$
\zeta_G (u) = \prod_{[P]} (1 - u^{l(P)})^{-1}.
$$

It has a representation as a determinant which was first discovered by Ihara for regular graphs ([19]) and then by Bass [3] and Hashimoto [12] for arbitrary finite graphs:

$$
\zeta_G (u)^{-1} = (1 - u^2)^{|E| - |V|} \det (I - A_G u + Q_G u^2),
$$

where $A_G$ is the adjacency matrix of $G$, and $Q_G$ is the diagonal matrix whose $j$-th diagonal entry is $(-1 + \text{degree of } j\text{-th vertex})$.

The Ihara zeta function can be though as a particular case of a dynamical zeta function for a subshift associated with graph $G$. The alphabet consists of $2|E|$ oriented edges, and the subshift matrix equals to the edge adjacency matrix $W_G$ which is defined as follows.

**Definition 4.1** The edge adjacency matrix $W_G$ is the $2|E| \times 2|E|$ such that $i,j$ entry equals $1$ if the terminal vertex of edge $i$ equals the initial vertex of edge $j$ and edge $j$ is not the inverse of edge $i$.

In particular, from (14) we have another determinantal formula:

$$
\zeta_G (u)^{-1} = \det (I - u W_G).
$$

If the graph is $q+1$ regular, that is, if every vertex has degree $q+1$, then there is a pole of the zeta at $u = 1/q$. Moreover, every non-real pole of the zeta lies on the circle with radius $1/\sqrt{q}$. The Riemann hypothesis for regular graphs says that all non-trivial poles are on this circle. This is not always true, and a useful criterion is as follows. Let the second largest in magnitude eigenvalue of $A_G$ be denoted $\lambda_1$. Then the Riemann hypothesis holds if and only if $|\lambda_1| \leq 2\sqrt{q}$. The graphs that satisfy this condition are often called Ramanujan following a paper by Lubotsky, Phillips, and Sarnak [26], which constructed an infinite family of such graphs.

Note that $2\sqrt{q}$ is critical in the sense that for every $\varepsilon > 0$, there exists a constant $C$ such that for all $(q+1)$-regular graphs with $n$ vertices there is at least $Cn$ eigenvalues in the interval $[2\sqrt{q} - \varepsilon, q+1]$. On the other hand, a random regular graph is approximately Ramanujan with high probability. That is, for arbitrary $\varepsilon > 0$, the probability that $|\lambda_1| \geq 2\sqrt{q} + \varepsilon$ becomes arbitrarily small as the size of the graph grows. This fact is known as the Alon conjecture and
it was recently proved by J. Friedman in Friedman (2008). However, the distribution properties of the largest eigenvalue are still unknown.

In his Ph.D. thesis [33], Derek Newland studied spacings of eigenvalues of random regular graphs and spacings of the Ihara zeta zeros and found numerically that they resemble spacings in the Gaussian Orthogonal Ensemble. See also an earlier paper by Jacobson, Miller, Rivin and Rudnick [22].

For Ihara’s zeta function, there is an analog of Selberg’s trace formula which we formulate for the case of a $q+1$ regular graph.

Let $N_m$ be the number of all non-backtracking tailless closed paths of length $m$. (Note that this is the number of all closed paths, not the number of the equivalence classes.) In particular,

$$u \frac{d}{du} \log \zeta_G(u, X) = \sum_{[P]} i(P) \left( u^{l(P)} + u^{2l(P)} + \ldots \right)$$

$$= \sum_{P} \left( u^{l(P)} + u^{2l(P)} + \ldots \right)$$

$$= \sum N_m u^m.$$

This shows that

$$\zeta_G(u) = \exp \sum_{m \geq 1} N_m \frac{u^m}{m}.$$ 

**Theorem 4.2 (Terras)** Suppose $0 < a < 1/q$. Assume that $h(u)$ is meromorphic in the plane and holomorphic for $|u| > a - \varepsilon$, $\varepsilon > 0$. Assume that $h(u) = O(\left|u\right|^{-1-a})$, $\alpha > 0$. Let $\hat{h}(n) := \frac{1}{2\pi i} \int_{|u|=a} u^nh(u) \, du$ and assume that $\hat{h}(n)$ decays rapidly enough. Then,

$$\sum_{\rho} \rho h(\rho) = \sum_{n \geq 1} N_n \hat{h}(n),$$

where the sum on the left is over the poles of $\zeta_G(u)$.

Some applications of this result can be found in ...

4.1.5. Frobenius maps

Another important example arises if $M$ is the set of solutions of a system of algebraic equations over the algebraic closure of $\mathbb{F}_q$ (i.e., the finite field with $q = p^n$ elements), and $f$ is the Frobenius map $\text{Frob}: x \to x^q$. These functions are often called the Weil zeta functions although it should be noted that the first examples were originally considered by E. Artin.

Weil’s zeta functions directly connected to the number-theoretical zeta-functions. Consider the affine curve given by the equation $f(X, Y) = 0$ over the finite field $\mathbb{F}_q$. Let $p$ denote a prime ideal of the field $\mathbb{F}_q [X, Y] / f (X, Y)$ and let the order of $\mathbb{F}_q [X, Y] / p$ be denoted by $Np$. Then, by analogy with Riemann’s zeta function we can define

$$\zeta(s) = \prod_p \frac{1}{1 - Np^{-s}}.$$ 

It turns out that this definition is in agreement with another definition of the Weil zeta function:

$$\zeta(s) = \exp \sum_{m=1}^{\infty} N_m \frac{p^{-sm}}{m},$$
where \( N_m \) is the number of fixed points of \( Frob^m \), or to put it in other words, the number of solutions of the equation \( f(X, Y) = 0 \) in the finite field \( \mathbb{F}_{q^m} \).

Here is an example. Consider the affine line. Then \( N_m = q^m \), and

\[
\zeta(z) = \exp \sum_{m=1}^{\infty} \frac{N_m z^m}{m} = \exp \left[ -\log (1 - qz) \right] = \frac{1}{1 - qz}.
\]

In another example, the curve is elliptic. Note that the Frobenius map is a linear operator over \( \mathbb{F}_q \) and there is some resemblance to Example 2 about linear maps on a torus. We can guess then that

\[
\zeta(z) = \frac{1 - z \text{Tr} [Frob] + z^2 \text{det} [Frob]}{(1 - z)(1 - z \text{det} [Frob])},
\]

provided \( \text{Tr} [Frob] \) and \( \text{det} [Frob] \) are given some appropriate meaning. If we believe \( \text{det} [Frob] = q \), then it \( N_1 = \text{det} (I - Frob) = \text{det}(Frob) - \text{Tr} (Frob) + 1 \), and so \( \text{Tr} [Frob] = q + 1 - N_1 \). Hence, our guess would be

\[
\zeta(z) = \frac{1 - z (q + 1 - N_1) + z^2 q}{(1 - z)(1 - zq)},
\]

and it turns out that this guess is correct. Of course, \( M \) is in fact infinite-dimensional over \( \mathbb{F}_q \), so we do not know what is the meaning of the determinant and the trace. However, one of the proofs of the rationality of the zeta function uses a two-dimensional module over \( \mathbb{Q}_l \) (the field of \( l \)-adic numbers) and the arguments resemble our heuristic. One can consult the book by Silverman \([44]\) to see more. A more standard proof is based on the Riemann-Roch theorem and counting divisors. It can be found in \([30]\).

The Riemann hypothesis in this example is equivalent to the statement that

\[
|N_1 - q - 1| \leq 2\sqrt{q},
\]

since this implies that the roots of the polynomial in the numerator are on the circle with radius \( q^{-1/2} \).

Artin \([1]\) proved the zetas of the curves can be explicitly computed as rational functions of \( z = p^{-s} \). He checked numerically that in many cases the zeta function satisfies an appropriate variant of the Riemann hypothesis, and conjectured that this should be true for all curves. Later this was proved by Hasse \([13]\) and Weil \((50)\ and [49]\).

Weil also conjectured in \([48]\) that the zeta function of every algebraic variety is rational, that it has a functional equation, and that it satisfies the Riemann hypothesis. The proof of this general conjecture led to an introduction of many new ideas in algebraic geometry by Dwork, Grothendieck, and Deligne.

Katz and Sarnak \([23]\) have studied the distribution of the zeros of the Weil zeta functions when the genus of the corresponding curve grows to infinity. They found that for “most” of the curves the local statistical behavior of these zeros approaches the behavior of the eigenvalues from the random matrix ensembles. It is interesting that there are no explicit constructions of curve families for which it is known to be true. It is only known that it must be true for most of the curves.
4.1.6. Maps of an interval

For yet another example consider the map \( x \mapsto 1 - \mu x^2 \) of the interval \([-1, 1]\) to itself. For a special value of \( \mu \approx 1.401155 \ldots \) (the Feigenbaum value), this map has one periodic orbit of period \( 2^n \) for every integer \( n \geq 0 \). Therefore,

\[
\zeta(z) = \prod_{n=0}^{\infty} \left( 1 - z^{2^n} \right)^{-1} = \prod_{n=0}^{\infty} \left( 1 + z^{2^n} \right)^{n+1}.
\]

This \( \zeta \) satisfies the functional equation \( \zeta \left( z^2 \right) = (1 - z) \zeta(z) \). More generally, the piecewise monotone maps of the interval \([-1, 1]\) to itself correspond to the Milnor-Thurston zeta functions. See [31]. As far as I know there was no systematic study of statistical properties of their poles and zeros.

4.2. Zetas for flows

If \( f \) is a flow on \( M \), that is, a map \( M \times \mathbb{R}^+ \to M \), then we can define the zeta function of this flow as

\[
\zeta(s) = \prod_{\omega} \left( 1 - e^{-sl(\omega)} \right)^{-1},
\]

where \( \omega \) denotes a periodic orbit of \( f \), and \( l(\omega) \) is its length. It is a more general case than the case of maps since there is a construction (“suspension”) that allows us to realize maps as flows but not vice versa. Unsurprisingly, it turns out that zeta functions for flows are more difficult to investigate than zeta functions for maps.

If we imagine that prime numbers correspond to periodic orbits of a flow and that the length of the orbit \( p \) is given by \( \log p \), then the zeta function of the flow will coincide with Riemann’s zeta function.

One particularly important example of a flow is the geodesic flow on a smooth manifold \( M \). In the case when \( M \) has a constant negative curvature, the corresponding dynamical zeta function is closely related to the Selberg zeta function. Namely,

\[
\zeta(s) = \frac{Z(s+1)}{Z(s)} \quad \text{and} \quad Z(s) = \prod_{n=0}^{\infty} \zeta(s+n)^{-1}.
\]

Another important example is the geodesic flow for billiards on polygons.

I am not aware about the functional equation for zeta functions that comes from general geodesic flows (other than flows on manifolds of a constant negative curvature). Also, it appears that not much is known about the statistical properties of the distribution of zeros and poles of these zeta functions.

The main benefit of these functions is that the distribution of closed geodesics depends on the maximal real part of their poles, and this quantity can be studied by methods that do not involve the functional equation or trace formula. Hence, one can study geodesics on spaces of variable curvature. The idea of this approach is that geodesic flows can be written as shifts of sequences of symbols and poles of zeta functions are related to eigenvalues of operators induced by these shifts. (This is the method of symbolic dynamics).
5. Conclusion

We considered the statistical properties of the zeta functions. In summary, the zeros of number-theoretical zeta functions satisfy many properties which are true for eigenvalues of random matrices. The main outstanding problem is the Montgomery conjecture that says that the correlations are the same as for random matrices even if the test functions are not smooth.

The statistical properties of the Selberg zeta zeros are different from the properties of the eigenvalues of random matrices. It is conjectured that the local properties coincide with the properties of the random independent points. However, the exact description of these properties is not known.

The zeros of dynamical zeta functions are often related to eigenvalues of certain matrices, and the study of their properties depend on the study of the matrix properties. Two interesting cases are the case of zeta functions of algebraic curves and the case of Ihara’s zeta functions of graphs.

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