Lambda: A Mathematica-package for operator product expansions in vertex algebras

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Abstract

We give an introduction to the Mathematica package Lambda, designed for calculating \(\lambda\)-brackets in both vertex algebras, and in SUSY vertex algebras. This is equivalent to calculating operator product expansions in two-dimensional conformal field theory. The syntax of \(\lambda\)-brackets is reviewed, and some simple examples are shown, both in component notation, and in \(N=1\) superfield notation.

1 Introduction

Conformal field theory is one of the cornerstones of theoretical physics. Vertex algebras, originally introduced in [1], provide a rigorous mathematical formulation of the chiral part of two-dimensional conformal field theory. The axioms of a vertex algebra are obtained from an abstract treatment of the properties of quantum field theories, and of operator product expansions (OPEs).

The key idea of OPEs is that a product of local operators defined at nearby locations can be expanded in a series of local operators. The formalism developed within vertex algebras provides a very compact and computational convenient way for calculating OPEs, including quantum effects. The reference to the position of the operators is abstracted away, and the OPEs are encoded in so-called \(\lambda\)-brackets [2]. There is also a normal order product, capturing the regular part of the product of two operators. The relations between these two products, in other words, formulas of OPEs between composite fields, are algebraic. The calculations are not necessarily hard to perform by hand, but they rapidly grow in number, and thus become error-prone. This makes them very appropriate for being implemented in some computational software, like Mathematica. This is the purpose of the Mathematica package at hand: Lambda.

The package grew out of a need to calculate \(\lambda\)-brackets between composed operators, and the need to do so in \(N=1\) superfield formalism. This was motivated by the construction of the Chiral de Rham complex (CDR), introduced in [3]. The CDR is a sheaf of vertex
algebras, and investigation of symmetries of this sheaf relies on the formalism of λ-brackets, see, for example, [4] and [5]. In [6], the CDR was given the interpretation of a formal Hamiltonian quantization of the supersymmetric non-linear sigma model. Thus, insights from the sigma model could be used to further investigate symmetries of the CDR. In [7], the symmetries of the CDR on special holonomy manifolds were investigated. Here, the package Lambda was heavily used, and the OPEs between operators composed of up to five fields were calculated.

There exists other implementations in Mathematica, and similar computational software, for calculating OPEs. The most used seems to be OPEdefs [8]. Also see [9]. The package OPEdefs has been developed further to handle \(N = 2\) superfield in [10]. One main advantage of the package Lambda is that it implements the algebraic approach to OPEs given by vertex algebras. It is the first package to implement the efficient λ-brackets. It also handles indices, and operators can be constructed out of tensors. Both input and output are handled graphically, thus increasing the readability in the usage of the module. It is also tested in Mathematica 7, and therefore offers a newer environment than the previous packages.

The paper is organized as follows. In section 2, the basic properties of vertex algebras are reviewed, along with the relations between OPEs and λ-brackets. The aim is to give an understanding of how to interpret the results of the package Lambda in the, perhaps more familiar, language of OPEs. In section 3, this is generalized to \(N = 1\) superfields. Section 4 lists the basic relations between the different operators and brackets. In section 5, we give some examples of how to use the package Lambda. Section 6 lists the commands provided by Lambda.

Acquiring the package

The package is available from the Computer Physics Communications library, see http://cpc.cs.qub.ac.uk/. It can also be downloaded from http://www.fejston.se/lambda.

2 Operator product expansions and vertex algebras

In operator product expansions, the product of two operators at nearby points is expressed as a sum of operators: \(A_i(z)A_j(w) = \sum_k f_{ij}^k(z - w)A_k(w)\), where each operator \(A_k\) is well behaved at \(w\), and \(f_{ij}^k(z - w)\) is a function, possible divergent when \(z \to w\). In two dimensions, the positions are coordinates on the complex plane, and the OPE is a Laurent series. Writing out the poles, and only keeping divergent terms, we can write the OPE between the operators \(A\) and \(B\) as

\[
A(z)B(w) \sim \frac{(A(0)B)(w)}{(z - w)} + \frac{(A(1)B)(w)}{(z - w)^2} + \frac{(A(2)B)(w)}{(z - w)^3} + \ldots ,
\]

(2.1)

where \((A_{ij})B\) are operators. Two operators are said to be mutually local if there is a highest pole, i.e., if the series (2.1) terminates.
A vertex algebra captures the structure of such OPEs. We here give a brief overview of the concepts of a vertex algebra, emphasizing the similarities with OPEs. For a proper introduction to vertex algebras, see [11] and [12].

A vertex algebra is a vector space $V$ (the \textit{space of states}), with a vector $|0\rangle \in V$ (the \textit{vacuum}), a map $Y$ from a given state $A \in V$ to a field $Y(A, z)$ (called the \textit{state-field correspondence}), and an endomorphism $\partial : V \to V$ (the \textit{translation operator}).

A field is defined as an $\text{End}(V)$-valued distribution in a formal parameter $z$:

$$A(z) = \sum_{j \in \mathbb{Z}} \frac{1}{z^{j+1}} A(j), \quad \text{where } A(j) \in \text{End}(V), \quad (2.2)$$

and for all $B \in V$, $A(z)B$ contains only finitely many negative powers of $z$. The field $Y(A, z)$ will also be denoted by $A(z)$.

These structures are subject to certain axioms. For instance, the vacuum should be invariant under translations: $\partial |0\rangle = 0$. Acting with $\partial$ on the endomorphisms of a field, should be the same as differentiation of the field with respect to the formal parameter $z$:

$$[\partial, Y(A, z)] = \partial_z Y(A, z). \quad (2.3)$$

We will use $\partial$ to denote both the endomorphism and $\partial_z$. The state-field correspondence creates a given state $A$ from the vacuum in the limit $z \to 0$:

$$Y(A, z)|0\rangle|_{z=0} = A^{(-1)}|0\rangle = A. \quad (2.4)$$

From the endomorphisms $A(j)$ of $Y(A, z)$ (called the \textit{Fourier modes}), we can define the $\lambda$-\textit{bracket}:

$$[A \lambda B] = \sum_{j \geq 0} \frac{\lambda^j}{j!} (A(j)B). \quad (2.5)$$

The $\lambda$-bracket can be viewed as a formal Fourier transformation of $Y(A, z)B$:

$$[A \lambda B] = \text{Res}_z e^{\lambda z} Y(A, z)B, \quad (2.6)$$

where $\text{Res}_z$ picks the $z^{-1}$-part of the expression. The locality axiom of the vertex algebra says that the series (2.5) terminates for all $A$ and $B$, in other words, all fields in a vertex algebra are mutually local.

The $\lambda$-bracket (2.5) encodes the same information as the OPE (2.1). For example, the operator expansion of a Virasoro generator $L$ is

$$L(z)L(w) \sim \frac{1}{(z-w)} \partial L(w) + \frac{2}{(z-w)^2} L(w) + \frac{c}{2(z-w)^4}, \quad (2.7)$$

where $c$ is the central charge. In $\lambda$-bracket notation, this is written

$$[L \lambda L] = (\partial + 2\lambda)L + \frac{c}{12} \lambda^3. \quad (2.8)$$
One way to define a normal order product of operators in quantum field theory is by point splitting. The singular terms captured by the OPE between the operators, are subtracted from the product:

\[(AB)(z) = \lim_{w \to z} (A(w)B(z) - \text{singular terms}) , \quad (2.9)\]

and the result is a well-defined operator at \(z\). In vertex algebras, the normal ordering product is defined in a similar way; by acting with the non-singular part of a field \(A(z)\), in the limit \(z \to 0\):

\[ : AB : = A(\cdot^{-1})B . \quad (2.10)\]

It is important to stress that this product is neither commutative, nor associative, see (4.7) and (4.8). This is a quantum effect. In all further considerations, we drop the notation : : , and the normal ordering is always assumed in products.

The above construction easily extends to the case when \(V\) is a super vector space. The state-field correspondence \(Y\) should respect this grading, \(\partial\) should be an even endomorphism, and the vacuum should be even.

For instance, in a Neveu-Schwarz superconformal algebra, there is, in addition to \(L\) with the OPE (2.7), an odd operator \(G\), with the OPEs

\[G(z)G(w) \sim \frac{2}{(z-w)}L(w) + \frac{2c}{3(z-w)^3} , \quad (2.11a)\]
\[L(z)G(w) \sim \frac{3}{2(z-w)^2}G(w) + \frac{1}{(z-w)}\partial G(w) . \quad (2.11b)\]

In the language of \(\lambda\)-brackets, this is equivalent to

\[ [G \lambda G] = 2L + \frac{c}{2}\lambda^2 , \quad (2.12a)\]
\[ [L \lambda G] = (\partial + \frac{c}{2}\lambda) G . \quad (2.12b)\]

The relations and properties of the introduced operators \(\partial, [\cdot \cdot \cdot]\) and the normal order product will be listed in section 4. Before that, we introduce a superfield formulation of OPEs and of vertex algebras.

### 3 OPEs of superfields and SUSY vertex algebras

Combining conformal field theory with supersymmetry, one gets superconformal field theory. This can be formulated in terms of \(N = 1\) superfields. Adding a fermionic (odd) partner \(\theta\) to the bosonic (even) coordinate \(z\), allows us to combine two fields of opposite grading, \(A(z)\) and \(B(z)\), into a superfield: \(C(z, \theta) = A(z) + \theta B(z)\). To describe the operator product expansions of superfields, we let \(Z_i = (z_i, \theta_i)\), and define the displacements

\[ Z_{12} = z_1 - z_2 - \theta_1 \theta_2 , \quad \theta_{12} = \theta_1 - \theta_2 . \quad (3.1)\]
and the derivative $D_i = \partial_{\theta_i} + \theta_i \partial_{z_i}$. An OPE between superfields can now be written as

$$A(Z_1)B(Z_2) \sim \frac{\theta_{12}(A(0|0)B)(Z_2)}{Z_{12}} + \frac{(A(0|1)B)(Z_2)}{Z_{12}} + \frac{\theta_{12}(A(1|0)B)(Z_2)}{Z_{12}^2} + \frac{(A(1|1)B)(Z_2)}{Z_{12}^3} + \frac{\theta_{12}(A(2|0)B)(Z_2)}{Z_{12}^4} + \frac{(A(2|1)B)(Z_2)}{Z_{12}^5} + \ldots \tag{3.2}$$

where $(A_{(j|j)}B)$ are operators. Note that the poles are expanded as

$$\frac{1}{Z_{12}^n} = \frac{1}{(z_1 - z_2)^n} + \frac{n\theta_1\theta_2}{(z_1 - z_2)^{n+1}}, \quad \frac{\theta_{12}}{Z_{12}^n} = \frac{\theta_{12}}{(z_1 - z_2)^n}, \tag{3.3}$$

so, e.g., $Z_{12}^{-1}$ contains a double pole in $z_1 - z_2$.

We exemplify this by combining the operators $L$ and $G$ of the superconformal algebra (2.11), into one odd superfield operator:

$$P(z, \theta) = G(z) + 2\theta L(z). \tag{3.4}$$

To calculate the OPE of $P$ with itself, we need (2.7) and (2.11). The OPE $G(z)L(w)$ can be calculated from (2.11b) by expanding $G(z)$ around $G(w)$:

$$G(z)L(w) \sim \frac{3}{2(z-w)^2}G(w) + \frac{1}{2(z-w)}\theta G(w). \tag{3.5}$$

Using this, the superconformal algebra is given by

$$P(Z_1)P(Z_2) \sim \frac{\theta_{12}}{Z_{12}} 2\partial_2 P(Z_2) + \frac{1}{Z_{12}} D_2 P(Z_2) + \frac{\theta_{12}}{Z_{12}^2} 3P(Z_2) + \frac{2c}{Z_{12}^3}. \tag{3.6}$$

An efficient treatment of the OPEs of $N = 1$ superfields is given by the $N_K = 1$ SUSY vertex algebra [13], which extends the vertex algebra construction of the last section. For details, see [13]. For a similar superfield treatment of vertex algebras, see [14]. Let $\theta$ be an odd formal parameter, with $\theta^2 = 0$. Define superfields by

$$A(z, \theta) = \sum_{j \in \mathbb{Z}} \frac{1}{z^{j+1}} \left( A_{(j|1)} + \theta A_{(j|0)} \right), \quad A_{(j|j)} \in \text{End}(V). \tag{3.7}$$

In a SUSY vertex algebra, the state-field correspondence maps a state $A$ to a superfield: $Y(A, z, \theta) = A(z, \theta)$. We have an odd translation operator $D$, that acts on a field by

$$Y(DA, z, \theta) = D Y(A, z, \theta) = (\partial_\theta + \theta \partial_z) A(z, \theta), \tag{3.8}$$

where we again use the same symbol for the endomorphism, and the action on a field. The square of this operator is the even translation operator: $D^2 = \partial$.

The action of the Fourier modes of a superfield $A(z, \theta)$ on another field is described by the $\Lambda$-bracket:

$$[A \Lambda B] = \sum_{j \geq 0} \frac{\chi_j}{j!} (A_{(j|0)}B + \chi A_{(j|1)}B), \tag{3.9}$$
where $\lambda$ is an even, and $\chi$ an odd parameter, satisfying $\chi^2 = -\lambda$. We write $[\cdot \Lambda \cdot]$, instead of $[\cdot \lambda \cdot]$, to indicate that we work with superfields. The normal order product is given by

$$AB = A_{(-1)[1]}B,$$

and again, this product is neither commutative, nor associative, see (4.7) and (4.8).

The $\Lambda$-bracket (3.9) contains the same information as the OPE (3.2). For instance, the OPE (3.6) is written

$$[P \Lambda P] = (2\partial + \chi D + 3\lambda)P + \frac{c}{3}\lambda^2\chi.$$  

We now review some relations between the introduced operations.

### 4 $\Lambda$-bracket calculus

In this section we collect some properties of $\Lambda$-bracket calculus. For further explanations and details, the reader may consult [11] and [13]. We give a unified description of the vertex algebras and the SUSY vertex algebras. Let $N$ denote whether we are working with superfields ($N = 1$) or not ($N = 0$). The brackets $[\cdot \Lambda \cdot]$ below are to be interpreted as $\Lambda$-brackets ($N = 1$) or $\lambda$-brackets ($N = 0$), depending on in which context they are used. We use the same symbol for an operator and its grading, the meaning should be clear. The relevant commands provided by Lambda are listed (in monospace typewriter font). For further explanations of the commands, see section 6.

- Relations between the translation operators and $\lambda$ and $\chi$:

  $$D^2 = \partial \quad [D, \partial] = 0 \quad [D, \lambda] = 0 \quad [\partial, \lambda] = 0 \quad (4.1)$$

  $$\chi^2 = -\lambda \quad [D, \chi] = 2\lambda \quad [\partial, \chi] = 0 \quad (4.2)$$

- Sesquilinearity:

  $$[Da_\Lambda b] = \chi[a_\Lambda b] \quad [a_\Lambda Db] = -(-1)^a(D + \chi)[a_\Lambda b] \quad (4.3)$$

  $$[\partial a_\Lambda b] = -\lambda[a_\Lambda b] \quad [a_\Lambda \partial b] = (\partial + \lambda)[a_\Lambda b] \quad (4.4)$$

- Skew-symmetry:

  $$[a_\Lambda b] = -(-1)^{ab+N}[b_{-\Lambda -\nabla}a] \quad (4.5)$$

  The bracket on the right-hand side is in the SUSY case computed by first computing $[b_{\Gamma}a]$, where $\Gamma = (\rho, \eta)$, $\rho$ even and $\eta$ odd. Then $\Gamma$ is replaced by $(-\lambda - \partial, -\chi - D)$. In the $N = 0$ case, compute $[b_\rho a]$, and replace $\rho$ by $-\lambda - \partial$. This rule is implemented by the function \texttt{LambdaBracketChangeOrder[a,b]}.

- Jacobi identity:

  $$[a_\Lambda [b_{\Gamma}c]] = (-1)^{N(a+1)}[[a_\Lambda b]_{\Gamma+\Lambda}c] + (-1)^{(a+N)(b+N)}[b_{\Gamma}[a_\Lambda c]] \quad (4.6)$$

  where the first bracket on the right hand side is computed as in (4.5).
• Quasi-commutativity:
\[
ab - (-1)^{ab} ba = \int_0^\nabla [a_{\Lambda} b] d\Lambda
\] (4.7)

The integral \( \int d\Lambda \) is \( \int d\lambda \) in the \( N = 0 \) case, and \( \partial \chi \int d\lambda \) in the SUSY case. The limits of the integral mean that \( \lambda \) should be replaced by \( \partial \). Implemented by \texttt{NormalOrderChangeOrder}\{a, b\}.

• Quasi-associativity:
\[
(ab)c - a(bc) = \left( \int_0^\nabla d\Lambda a \right) [b_{\Lambda} c] + (-1)^{ab} \left( \int_0^\nabla d\Lambda b \right) [a_{\Lambda} c]
\] (4.8)

This formula is to be understood as follows. First, the \( \Lambda \)-brackets are calculated. The \( \lambda \)'s and \( \chi \)'s are integrated as in (4.7). The resulting operators (i.e., a factor times \( \partial \) to some power) act on \( a \) respectively \( b \), and this is normal ordered with the resulting operators from the brackets. This is implemented in \texttt{NormalOrderChangeParenthesis}[expr].

• Quasi-Leibniz (non-commutative Wick formula):
\[
[a_{\Lambda} bc] = [a_{\Lambda} b] c + (-1)^{(a+N)b} [a_{\Lambda} c] + \int_0^\Lambda [\left[ a_{\Lambda} b \right] \Gamma c] d\Gamma
\] (4.9)

The integration is to be understood as in (4.5) and (4.7). The limits of the integral mean replacing \( \rho \) by \( \lambda \).

5 Example of how to use Lambda

In this section we give some examples of how to use Lambda. We demonstrate a realization of the \( N = 1 \) superconformal algebra. We first give the algebra in components, i.e., without using superfields. After this, the algebra is written in \( N = 1 \) superfield formalism. We then show how to extend this to \( N = 2 \) superconformal algebra, by an operator constructed out of a 2-form. The given examples are quite simple, but hopefully still illuminating. It is, in general, however, possible to define algebras with arbitrary complexity, including non-linear ones, by giving consistent definitions of the function \texttt{LambdaBracket}.

First, we must make sure that the package is in a directory where Mathematica can find it, for example by adding \texttt{AppendTo[\$Path, "pathtolambda"]} in the initialization file \$UserBaseDirectory/Kernel/init.m, or by using the command \texttt{SetDirectory}. After this, we can load the package Lambda:

```
In[1]:= << Lambda.m;
```

\texttt{Lambda}: A Mathematica-package for operator product expansions in vertex algebras.

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5.1 \( N = 1 \) superconformal algebra in components

We start without superfields, and write

```mathematica
In[2]:= SetSUSY[False];
```

Let \( \gamma^\mu \) and \( \beta_\nu \) be even fields. We define this with the following code:

```mathematica
In[3]:= Grading[Field[y, ___]] := 0;
Grading[Field[\[Beta], ___]] := 0;
```

A field is written with the command `Field`. The first argument is the symbol of the field, the second is a list of the lower indices that the field carries, the third is the upper indices, and the fourth can be used as target space derivatives on the field. A field is denoted with a hat over it, and the package recognizes any symbol with a hat as a field. As an example, consider the field \( \gamma^i \). It has one upper index, and can be written as

```mathematica
In[5]:= Field[g, 8<, 8i<, 8>D
Out[5]= g`i
```

Mathematica also recognizes this form as input. We want our fields to have the brackets

\[
[\beta_i \gamma^j] = \hbar \delta^j_i, \quad [\beta_i \beta_j] = 0, \quad [\gamma^i \gamma^j] = 0. \tag{5.1}
\]

In order for the package to know that the constant \( \hbar \) is not an expression that potentially contains a field, it needs to be declared to be `NumericQ`:

```mathematica
In[6]:= NumericQ[\[HBar]] := True;
```

There is a predefined even Kronecker delta field with symbol \( \delta \) in the package. The bracket \( [\beta_i \gamma^j] = \hbar \delta^j_i \) is then specified by

```mathematica
In[7]:= LambdaBracket[Field[\[Beta], 8a_<, 8<, 8>D, Field[g, 8<, 8b_<, 8<, 8>D]] :=
\[HBar] \cdot Field[\delta, 8a<, 8b<, 8>D;
```

This defines the bracket with the arguments in this specific order. To define the bracket in the opposite order, we use the function `LambdaBracketChangeOrder`, that implements the skew-symmetry property (4.5):

```mathematica
In[8]:= LambdaBracket[ex1 : Field[y, 8<, 8b_<, 8<>}, ex2 : Field[\[Beta], 8a_<, 8<, 8>]] :=
LambdaBracketChangeOrder[ex1, ex2];
```

Note that it is possible to define the opposite order bracket without using `LambdaBracketChangeOrder`. Sometimes this can be useful, if one wish to have a specific form of the output (other than the one given by `LambdaBracketChangeOrder`). Care has then to be taken so that the defined bracket fulfills skew-symmetry (4.5).

The other two brackets are given by

```mathematica
In[9]:= LambdaBracket[Field[\[Beta], ___], Field[\[Beta], ___]] := 0;
LambdaBracket[Field[y, ___], Field[y, ___]] := 0;
```
Now, these fields are a so-called $\beta\gamma$-system. We can define a Virasoro field $L_{bos} = \beta_i \partial \gamma^i$ by

\[
\text{In}[11] := L_{bos} = \beta_i \partial \gamma^i;
\]

The circle denotes normal ordering, and the square $^\Box$ is the (even) derivative $\partial$. The bracket of this field with itself is calculated by

\[
\text{In}[12] := [L_{bos}, L_{bos}];
\]

\[
\text{Out}[12] = \frac{1}{6} \dim \lambda^3 \hbar^2 + 2 \lambda \hbar \left( \hat{\beta}_a \circ \hat{\gamma}^a \right) + \hbar \left( \hat{\beta}_a \circ \hat{\gamma}^a \right) + \hbar \left( \hat{\delta}_a \circ \hat{\gamma}^a \right)
\]

The derivatives of this expression can be collected using CD (an alias for Collect-Derivatives):

\[
\text{In}[13] := \% / / \text{CD};
\]

\[
\text{Out}[13] = \frac{1}{6} \dim \lambda^3 \hbar^2 + 2 \lambda \hbar \left( \hat{\beta}_a \circ \hat{\gamma}^a \right) + \hbar \left( \hat{\delta}_a \circ \hat{\gamma}^a \right)
\]

This shows that $L_{bos}/\hbar$ satisfies the Virasoro algebra (2.8), with central charge $c = 2 \dim$, i.e.,

\[
[L_{bos} \lambda L_{bos}] = \hbar (2 \lambda + \partial) L_{bos} + \hbar^2 \lambda^3 \frac{\dim}{6}. \quad (5.2)
\]

The variable $\dim$ is equal to the trace of the Kronecker delta: $\dim = \delta_a^a$.

We now define two odd fields, $b_i$ and $c^j$, with the brackets

\[
\begin{align*}
[b_i \lambda c^j] &= \hbar \delta_i^j, & [b_i \lambda b_j] &= 0, & [c^i \lambda c^j] &= 0.
\end{align*} \quad (5.3)
\]

This is a $\beta\gamma bc$-system. This is implemented in a similar way as above. The basic fields, $\beta$, $\gamma$, $b$ and $c$, are predefined, along with their brackets, in the file basicfields.m (included in the distribution of Lambda), which we now load:

\[
\text{In}[14] := \langle basicfields.m;\rangle
\]

Of course, this file could have been loaded directly, and the above definitions are only included in order to show the basic syntax. To make the calculations less cluttered, we set $\hbar = 1$.

\[
\text{In}[15] := \hbar = 1;
\]

We can define another Virasoro field $L_{form} = \frac{1}{2} (\partial b_i c^i + \partial c^i b_i)$ by

\[
\text{In}[16] := L_{form} = \frac{1}{2} \left( \Box \hat{\beta}_1 \circ \hat{c}^1 + \Box \hat{c}^1 \circ \hat{b}_1 \right);
\]

The bracket is

\[1\) The reason to use a square is that it is an operator without a built in meaning in Mathematica, in contrast to the symbol $\partial$.}
In order to read off the algebra, some reordering is needed. Let us change the order in the second term using NormalOrderChangeOrder (with alias NOCO), and collect the derivatives with CD:

```
In[18]:= \langle Lferm, Lferm \rangle
Out[18]= \frac{\text{dim} \lambda^3}{12} + \frac{1}{4} \lambda^2 (\hat{\alpha} \circ \hat{\alpha}) + \frac{1}{4} \lambda^2 (\hat{\alpha} \circ \hat{\alpha}) + \lambda \left( \Box \hat{\alpha} \circ \hat{\alpha} \right) + \frac{1}{2} \left( \Box \hat{\alpha} \circ \hat{\alpha} \right) + \frac{1}{2} \left( \Box \hat{\alpha} \circ \hat{\alpha} \right) + \frac{1}{2} \left( \Box \hat{\alpha} \circ \hat{\alpha} \right)
```

The Virasoro algebra (2.8) is now apparent, and the central charge is $c = \text{dim}$. Since the two Virasoro fields commute:

```
In[19]:= \langle Lferm, Lbos \rangle
Out[19]= 0
```

d they can be combined into one Virasoro field $L$:

```
In[20]:= L = Lferm + Lbos;
```

The field $L$ fulfills the Virasoro algebra with central charge $c = 3 \text{dim}$. Let us define an odd field $G$ by

```
In[21]:= G = (\hat{\alpha} \circ \hat{\alpha}) + (\nabla \hat{\alpha} \circ \hat{\alpha});
```

This field has conformal weight $\frac{3}{2}$ with respect to $L$

```
In[22]:= \langle L, G \rangle // CD
Out[22]= \frac{3}{2} \lambda \left( \hat{\alpha} \circ \hat{\alpha} \right) + \frac{3}{2} \lambda \left( \nabla \hat{\alpha} \circ \hat{\alpha} \right) + \frac{1}{2} \left( \nabla \hat{\alpha} \circ \hat{\alpha} \right) + \frac{1}{2} \left( \nabla \hat{\alpha} \circ \hat{\alpha} \right)
```

and $[G, G]$ is

```
In[23]:= \langle G, G \rangle // SortExpression[\#, \{b, c, \beta, \gamma\}] &
Out[23]= \text{dim} \lambda^2 - \left( \hat{\alpha} \circ \hat{\alpha} \right) + 2 \left( \hat{\beta} \circ \hat{\gamma} \right) + \left( \nabla \hat{\alpha} \circ \hat{\alpha} \right)
```

where we have sorted the expression. Let us compare this with $2L + \lambda^2 \text{dim}$. We sort the expression, and rename the dummy indices in a canonical way, using MakeDummyIndices-Canonical (with alias MDIC):

```
In[24]:= 2L + 3 \ast \text{dim} \lambda^2 / 3 // SortExpression[\#, \{\}\] & // MDIC // Expand
Out[24]= \text{dim} \lambda^2 - \left( \hat{\alpha} \circ \hat{\alpha} \right) + 2 \left( \hat{\beta} \circ \hat{\gamma} \right) + \left( \nabla \hat{\alpha} \circ \hat{\alpha} \right)
```

We see that we have the $N = 1$ superconformal algebra, given by (2.8) and (2.12).


5.2 $N = 1$ superconformal algebra in superfields

We now want to write the $N = 1$ superconformal algebra in superfields, and write

\[
\text{In[25]:= SetSUSY[True];}
\]

Let us combine the $\beta\gamma$-system into two superfields. One even, $\Phi^i = \gamma^i + \theta c^i$, and one odd, $S_i = b_i + \theta \beta_i$, with the brackets

\[
\begin{align*}
[S_i, \Phi^j] &= \delta^j_i, \\
[S_i, S_j] &= 0, \\
[\Phi^i, \Phi^j] &= 0.
\end{align*}
\] (5.4)

The brackets and the gradings are defined as for the $\beta\gamma$-system, with the only difference that $S$ is declared to have odd grading. All the definitions are in the file `basicfields.m`. The components of the superfields are named using `DefineComponents`:

\[
\text{In[26]:= DefineComponents[S, b, \beta];}
\text{In[26]:= DefineComponents[F, g, c];}
\]

The Virasoro field and the supercurrent are now given by $P = \partial \Phi^i S_i + D \Phi^i DS_i$:

\[
\text{In[28]:= P = \partial \Phi^i S_i + D \Phi^i DS_i;}
\]

Here, the $\nabla$-symbol represents the odd derivative $D$. The components of $P$ are $P^i = G + \theta 2L$, which can be seen by using `ExpandSuperFields`. This returns a list, with the components, and their grading:

\[
\text{In[29]:= ExpandSuperFields[P][[1]][[1]]}
\]

As expected, the field $P$ fulfills the $N = 1$ superconformal algebra (3.11), with central charge $c = 3 \text{dim}$, e.g.,

\[
[P, P] = (2\partial + \chi D + 3\lambda)P + \text{dim} \lambda^2 \chi.
\] (5.5)

This is calculated by

\[
\text{In[32]:= \{P, P\} // CD // Collect[\#, \{\lambda, \chi\}, Simplify] &}
\]

\[
\text{Out[32]= \text{dim} \lambda^2 \chi + 3 \lambda \left(\left(\nabla \hat{\Phi}^a \circ \nabla \hat{S}_a\right) + \left(\nabla \hat{\Phi}^a \circ \nabla \hat{S}_a\right)\right) +}
\chi \left(\nabla \left(\nabla \hat{\Phi}^a \circ \nabla \hat{S}_a\right) + \nabla \left(\nabla \hat{\Phi}^a \circ \nabla \hat{S}_a\right)\right) + 2 \left(\nabla \left(\nabla \hat{\Phi}^a \circ \nabla \hat{S}_a\right) + \nabla \left(\nabla \hat{\Phi}^a \circ \nabla \hat{S}_a\right)\right)
\]

5.3 \( N = 2 \) superconformal algebra

Now, we follow [5], and extend this algebra to an \( N = 2 \) superconformal algebra, by introducing a constant 2-form \( \omega \), and its inverse. We define the fields by

```math
In[33]:= Grading[Field[\(\omega\), ___]] := 0;
FieldAntiSymmetricLowerIndices[\(\omega\)] := True;
FieldAntiSymmetricUpperIndices[\(\omega\)] := True;
CoordinateFunctionQ[\(\omega\)] := True;
```

We use the same symbol for its inverse, with \( \omega_{ij}\omega^{jk} = \delta_i^k \). This can be implemented by

```math
In[37]:= NormalOrder[Field[\(\omega\), \(\{a__, \text{ind1}_-, \text{b__}\}\), \{\}, \{\}]],
Field[\(\omega\), \{\}, \{\text{c__}, \text{ind1}_-, \text{d__}\}], \{\}]] :=
(-1) \times (1 + \text{Length}\{\{a\}\} + \text{Length}\{\{c\}\}) \times \text{Field}[\delta, \{a, b\}, \{c, d\}], \{\}];
NormalOrder[Field[\(\omega\), \{\}, \{\text{c__}, \text{ind1}_-, \text{d__}\}], \{\}]],
Field[\(\omega\), \{\text{a__}, \text{ind1}_-, \text{b__}\}], \{\}, \{\}]] :=
(-1) \times (1 + \text{Length}\{\{a\}\} + \text{Length}\{\{c\}\}) \times \text{Field}[\delta, \{a, b\}, \{c, d\}], \{\}];
```

For simplicity, we choose \( \omega \) to be constant, and write

```math
In[39]:= Field[\(\omega\), __, __, \{\}]] := 0;
SD[Field[\(\omega\), __, __]] := 0;
TD[Field[\(\omega\), __, __]] := 0;
```

We now construct a composite even field \( J \) by

```math
In[41]:= J = \frac{1}{2} \left( \hat{\omega}^1 \circ \hat{S}_k \circ \hat{S}_j - \hat{\omega}^1 \circ \hat{\omega}^2 \circ \hat{S}_j \right);
```

The operator \( J \) is a primary field with conformal dimension 2 with respect to \( P \): \([P \Lambda J] = (2\partial + 2\lambda + \chi D)J\), which is calculated by

```math
In[42]:= \langle P, J \rangle - (2 \boxdot \circ + 2 \lambda \boxdot + \chi \nabla \gamma) \& @ J // \text{ED} // \text{MDIC} // \text{Expand}
Out[42]= 0
```

The algebra of \( J \) with itself is calculated to \([J \Lambda J] = -(P + \dim \lambda \chi)\):

```math
In[43]:= FixedPoint[
    (\# // \text{SortExpression[\#, \{\(\omega\), \$\}, \$]) \& // \text{MNOC} // \text{ED} // \text{DoDeltaContractions} //
    \text{MDIC}) \&, \langle J, J \rangle]
Out[43]= -\dim \lambda \chi - \left( \nabla \hat{S}_a ^\gamma \circ \nabla \hat{S}_a \right) - \left( \square \hat{S}_a ^\gamma \circ \hat{S}_a \right)
```

So, to conclude, \( P \) and \( J \) generates an \( N = 2 \) superconformal algebra, with central charge \( c = 3 \) dim.
6 List of commands

In this section we list the commands provided by Lambda.

CollectDerivatives[expr] tries to use Leibniz rule to write expressions involving derivatives more compact. Only works on the top-level of an expression. Short name is CD[expr].

DefineComponents[s_Symbol,c_Symbol,t_Symbol] defines the expansion of the superfield with symbol s as Field[c, ...] + \( \theta \) Field[t, ...].

DoDeltaContractions[expr] performs the possible contractions with the special field \( \delta^\beta_\alpha \).

ExpandDerivatives[expr] uses Leibniz rule to expand derivatives. E.g. \( \Box (\hat{a} \circ \hat{b}) \) is expanded to \( (\Box \hat{a} \circ \hat{b}) + (\hat{a} \circ \Box \hat{b}) \). Short name is ED[expr].

ExpandDummyIndices[expr, indRange_List] replaces all dummy indices with values from indRange, e.g., ExpandDummyIndices[\( \hat{a} \circ \hat{b} \), {1, 2}] gives \( \hat{a}_1 \circ \hat{b}^1 + \hat{a}_2 \circ \hat{b}^2 \). This is useful when given concrete realizations of tensors.

Field[sym_Symbol,lind_List,uind_List,dind_List] is a field with symbol sym, lower indices are given by lind, upper indices by uind and the derivatives by dind. A symbol with a hat, \( \hat{\cdot} \), is interpreted as a field, and the indices are placed as, e.g., \( \hat{a}_{1;12}^{u_1 u_2} \), with a comma separating the lower indices from the derivative indices.

FieldAntiSymmetricLowerIndices[sym_Symbol] declares that the field with the symbol sym is anti-symmetric in its lower indices.

FieldAntiSymmetricUpperIndices[sym_Symbol] declares that the field with the symbol sym is anti-symmetric in its upper indices.

FieldSymmetricLowerIndices[sym_Symbol] declares that the field with the symbol sym is symmetric in its lower indices.

FieldSymmetricUpperIndices[sym_Symbol] declares that the field with the symbol sym is symmetric in its upper indices.

GetSUSY[] returns true, if the \( \Lambda \)-brackets are to be calculated in \( N = 1 \) formalism, and false otherwise.

Grading[a] returns an even value if a is regarded as an even (bosonic) field, and an odd value if a is regarded odd (fermionic).

LambdaBracket[a,b] calculates the \( \Lambda \)-bracket between a and b, i.e., \( [a \Lambda b] \).

LambdaBracketChangeOrder[a,b] calculates \( [a \Lambda b] \) from the bracket \( [b \Lambda a] \) using skew-symmetry.
LambdaBracketHandleIndices[a,b] calculates \([a_\Lambda b]\). First, the dummy indices are replaced with unique names, using MakeDummyIndicesUnique. After the calculation, the indices are made canonical, using MakeDummyIndicesCanonical. Inputed with the AngleBracket, i.e. \((a,b)\).

LambdaBracketJacobiator[a,b,c] calculates the Jacobiator of the \(\Lambda\)-bracket, defined as
\[
[a_\Lambda [b_\Gamma c]] + (-1)^{1+N(a+1)}[[a_\Lambda b]_\Gamma + a_\Lambda c] + (-1)^{1+(a+N)(b+N)}[b_\Gamma [a_\Lambda c]]
\]
, see eq. (4.6).

MakeDummyIndicesCanonical[expr] renames repeated indices by the naming scheme \(\alpha,\beta,\gamma\) etc. Greek letters should therefore be avoided as names for free indices. Short name is MDIC.

MakeDummyIndicesUnique[expr] makes repeated indices unique. Short name is MDIU.

MakeNormalOrderingCanonical[expr] changes the normal ordering to the scheme \((a \circ (b \circ (c \circ \ldots)))\). Short name is MNOC.

MoveTerm[expr,pos,m] moves the element at position pos of the expression expr m steps.

NormalOrder[a,b] represents the normal ordered product of the fields a and b. It is denoted \((a \circ b)\), i.e., using the character SmallCircle. Write the input as \(\text{Esc} \text{sc} \text{Esc} \text{b}\). If more terms are given, the normal ordering binds from the right, so \(a \circ b \circ c\) is interpreted as \((a \circ b) \circ c\).

NormalOrderAsscociator[a,b,c] calculates \(((a \circ b) \circ c) - (a \circ (b \circ c))\).

NormalOrderChangeOrder[a,b] uses quasi-commutativity to express \((a \circ b)\) as \((b \circ a)\) + commutator. Short name is NOC.

NormalOrderChangeParenthesis[expr] uses NormalOrderAssociator to express \(((a \circ b) \circ c)\) or \((a \circ (b \circ c))\) by means of the other expression. The expression \((a \circ b) \circ (c \circ d)\) is transformed to \(((a \circ b) \circ c) \circ d\). Short name NOCP.

NormalOrderChangeParenthesisForward[expr] transforms an expression of the form \((a \circ b) \circ (c \circ d)\) to the form \((a \circ (b \circ (c \circ d)))\).

NormalOrderCommutator[a,b] calculates \((a \circ b) - (-1)^{|a||b|}(b \circ a)\), where \(|a|\) denotes Grading[a].

OddNumericQ indicates that a variable is odd, e.g., like \(\chi\). Note that \(SD[\chi] = 2\lambda\), but this is not implemented for a general OddNumericQ.

SD[expr] is the odd derivative on expr. Denoted \(\nabla \text{expr}\), defining the command Del[expr].

SetSUSY[True] sets the \(\Lambda\)-bracket to be calculated within the \(N=1\) formalisms. With SetSUSY[False], \(\Lambda\)-brackets are calculated without superfields.
SortDerivativesFirst = False|True sets, in case two fields have the same symbol, whether terms with derivatives should be sorted first or not.

SortExpression[expr, order_List] sorts the expression expr, in the order specified by order. The list order has the symbols of the fields as values. E.g., 

SortExpression[\[BB\]^a \[BB\]^b\}, \{b,a\}] yields \[BB\]^b \[BB\]^a + terms from moving the fields. If the symbols of the fields are not included in order, alphabetic order is used. If SortDerivativesFirst is set to true, terms with derivatives are sorted first, in case of equal symbols, otherwise last.

TD[expr] is the even derivative on expr. Denoted □expr, defining the command Square[expr ].

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