ON THE NON-DEGENERACY OF RADIAL VORTEX SOLUTIONS FOR A COUPLED GINZBURG-LANDAU SYSTEM

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ABSTRACT. For the coupled Ginzburg-Landau system in $\mathbb{R}^2$
\begin{align*}
-\Delta w^+ + [A_+ (|w^+|^2 - t^+^2) + B(|w^-|^2 - t^-^2)] w^+ &= 0, \\
-\Delta w^- + [A_- (|w^-|^2 - t^-^2) + B(|w^+|^2 - t^+^2)] w^- &= 0,
\end{align*}
with following constraints for the constant coefficients
\[ A_+, A_- > 0, \quad B^2 < A_+ A_-, \quad t^+, t^- > 0, \]
the radially symmetric solution $w(x) = (w^+, w^-) : \mathbb{R}^2 \to \mathbb{C}^2$ of degree pair $(1, 1)$ was given by A. Alama and Q. Gao in J. Differential Equations 255 (2013), 3564-3591. We will concern its linearized operator $L$ around $w$ and prove the non-degeneracy result under one more assumption $B < 0$: the kernel of $L$ is spanned by the functions $\frac{\partial w}{\partial x^1}$ and $\frac{\partial w}{\partial x^2}$ in a natural Hilbert space. As an application of the non-degeneracy result, a solvability theory for the linearized operator $L$ will be given.

1. Introduction.

1.1. Existence of symmetric vortex solutions. We consider the Ginzburg-Landau system in $\mathbb{R}^2$
\begin{align*}
-\Delta w^+ + [A_+ (|w^+|^2 - t^+^2) + B(|w^-|^2 - t^-^2)] w^+ &= 0, \\
-\Delta w^- + [A_- (|w^-|^2 - t^-^2) + B(|w^+|^2 - t^+^2)] w^- &= 0,
\end{align*}
(1)
where $w = (w^+, w^-) : \mathbb{R}^2 \to \mathbb{C}^2$ is a complex vector-valued function. The Ginzburg-Landau system of this type has been introduced in physical modes of Bose-Einstein condensates and for the superconductors. The reader can refer to the papers [3], [5].

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In particular, \( w = (w^+, w^-) : \mathbb{R}^2 \to \mathbb{C}^2 \).

For the balanced case, \( A_+ = A_- \) and \( t^+ = t^- = 1/\sqrt{2} \), S. Alama, L. Bronsard and P. Mironescu proved the existence, uniqueness and monotonicity of radial solutions in [1]. By making the following assumptions

\[ A_+, A_- > 0, \quad B^2 < A_+ A_-, \quad t^+, t^- > 0, \]

(H1)

S. Alama and Q. Gao in [3] addressed the general case and showed that the energy functional of the system (1) is continuous, coercive, and convex, and then they proved that \( E(w) \) has a radially symmetric minimizer \( w_n = (w_n^+, w_n^-) \) in the forms

\[
\begin{align*}
  w_n^+(r, \theta) &= U^+_n(r)e^{in+\theta}, \\
  w_n^-(r, \theta) &= U^-_n(r)e^{in-\theta},
\end{align*}
\]

(2)

with given degree pair \( n = (n_+, n_-) \in \mathbb{Z}^2 \). Here and in the sequel, we will use the convention \( x_1 = r \cos \theta, \quad x_2 = r \sin \theta. \)

By substituting (2) in (1), we can obtain an ODE system for the vortex profiles \( U_n^\pm \) for \( r \in (0, \infty) \),

\[
\begin{aligned}
  -U''_n^\pm - \frac{1}{r}U'_n^\pm + \frac{n^2}{r^2}U_n^\pm + \left[ A_+(U_n^2 - t^2) + B(U_n^{-2} - t^{-2}) \right] U_n^\pm &= 0, \\
  -U''_n^\mp - \frac{1}{r}U'_n^\mp + \frac{n^2}{r^2}U_n^\mp + \left[ A_-(U_n^{-2} - t^{-2}) + B(U_n^{-2} + t^{-2}) \right] U_n^\mp &= 0,
\end{aligned}
\]

(3)

with the conditions

\[
U_n^\pm \geq 0 \quad \text{for all } r \geq 0, \quad U_n^\pm \to t^\pm \quad \text{as } r \to \infty,
\]

(4)

\[
U_n^\pm(0) = 0 \quad \text{if } n_\pm \neq 0, \quad U_n^\pm(0) = 0 \quad \text{if } n_\pm = 0.
\]

(5)

For the convenience of the readers, we give the following existence result and fundamental properties of solutions to (3)-(5). The readers can refer to Proposition 1.1 in [3] for more details.

**Proposition 1.** [3] Let (H1) hold. Then there exists a unique solution \((U_n^+, U_n^-)\) to (3)-(5). Moreover, we have

\[
U_n^\pm \in C^\infty(0, \infty), \quad U_n^\pm > 0 \text{ for all } r > 0,
\]

\[
U_n^\pm \sim r^{\lfloor n\rfloor} \text{ as } r \to 0, \quad \int_0^\infty |U_n^\pm|^2 r \, dr < \infty.
\]

In particular, \( w_n = (U_n^+ e^{in+\theta}, U_n^- e^{in-\theta}) \) is an entire solution of (1) in \( \mathbb{R}^2 \).

The asymptotic behavior of vortex profiles \( U_n^\pm \) as \( r \to \infty \) are derived from the following proposition, see Theorem 3.2 in [3].

**Proposition 2.** [3] Let (H1) hold. Suppose that \((U_n^+, U_n^-)\) is the solution of (3)-(5). Then we have

\[
U_n^\pm = t^\pm + \frac{\tilde{c}_\pm}{2r^2} + O(r^{-4}) \quad \text{and} \quad U_n^\pm' = -\frac{\tilde{c}_\pm}{r^3} + O(r^{-5}), \quad \text{as } r \to \infty,
\]
with
\[ \tilde{c}_\pm = \frac{Bn_\pm^2 - A_\pm n_\pm^2}{(A_+ A_- - B^2) t^\pm}. \]

The monotonicity properties of the vortex profiles were also given in Theorem 1.2 of [3]. The validity of the properties is strongly dependent on the interaction coefficient \( B \).

**Proposition 3.** [3] Let (H1) hold. Assume that \( w_n = (U_n e^{i n_+ \theta}, U_n e^{i n_- \theta}) \) is the entire equivariant solution of (1), which is described in Proposition 1.

(1). If \( B < 0 \), then \( U_n^\pm(r) \geq 0 \) for all \( r > 0 \) and any degree pair \( n = (n_+, n_-) \).

(2). If \( B > 0 \), \( n_+ \geq 1 \) and \( n_- = 0 \), then \( U_n^+(r) \geq 0 \) and \( U_n^-(r) \leq 0 \) for all \( r > 0 \).

(3). For any pair \( (n_+, n_-) \) with \( n_+ \neq 0 \neq n_- \), there exists \( B_0 > 0 \) such that \( U_n^+(r) \geq 0 \) for all \( r > 0 \) and all \( B \) with \( 0 \leq B \leq B_0 \). \( \square \)

1.2. Stability and non-degeneracy of symmetric vortex solutions. The stability or non-degeneracy of the minimizer is of fundamental importance in analysis. There are many works involving non-degeneracy and stability, see, e.g., [4], [6], [8], [18], [20], [22] and so on.

For the classical (single complex component) Ginzburg-Landau equation on the disc \( B_R(0) \subset \mathbb{R}^2 \),
\[ -\Delta u = u(1 - |u|^2), \] (6)

P. Mironescu [20] considered the radial solution of degree one and gave the stability result in the sense that corresponding quadratic form is positive definite. By using the Fourier decomposition method, T. C. Lin [18] studied the (single complex component) Ginzburg-Landau equation in \( B_1(0) \) with boundary condition
\[ -\Delta u = \frac{1}{\epsilon^2} u(1 - |u|^2) \quad \text{in} \quad B_1(0), \quad u = g \quad \text{on} \quad \partial B_1(0), \]
and proved the stability result for the radial symmetric solutions of degree one. M. del Pino, P. Felmer and M. Kowalczyk [10] studied the classical (single complex component) Ginzburg-Landau equation (6) in \( \mathbb{R}^2 \), and they proved the non-degeneracy result for the radial symmetric vortex solution of degree one in a given Hilbert space. For the solutions of degree \( d \geq 2 \) of Ginzburg-Landau mode, there are stability or instability results (see, e.g., [8], [20], [22]).

In [5] and [6], Y. Almog, L. Berlyand, D. Golovaty and I. Shafrir considered the minimization of a \( p \)-Ginzburg-Landau energy functional
\[ E_p(u) = \int_{\mathbb{R}^2} |\nabla u|^p + \frac{1}{2} (1 - |u|^2)^2, \]
with \( p > 2 \), over the class of radially symmetric functions of degree one. The radially symmetric minimizer, i.e., \( u_p = f_p(r)e^{i \theta} \), is the radially symmetric solution of degree one to
\[ \frac{\partial}{\partial r} \cdot (|\nabla u|^p - 2 \nabla u) + u(1 - |u|^2) = 0. \]
Moreover they studied the existence, uniqueness and the asymptotic behavior results for the radially symmetric minimizer. In particular, when \( 2 < p \leq 4 \), the radially symmetric solution is locally stable in the sense that the second variation of the functional \( E_p(u) \) at the minimizer \( u_p = f_p(r)e^{i \theta} \) is positive and the kernel
of linearized operator $\mathcal{L}_p$ around the minimizer is spanned by $\{\partial u_p/\partial x^1, \partial u_p/\partial x^2, \partial u_p/\partial \theta\}$ in a natural Hilbert space.

Let us go back to the system (1). For the balanced case, $A_+ = A_-$ and $t^+ = t^- = 1/\sqrt{2}$, S. Alama, L. Bronsard and P. Minorencu concerned the stability and bifurcation of radial solutions on a disc [2]. Recently, S. Alama and Q. Gao [4] considered the Ginzburg-Landau system (1) on the disk in $\mathbb{R}^2$ with a symmetric, degree-one boundary Dirichlet condition, and studied its stability for $B < 0$ case and instability for case of positive $B$ close to zero in sense of the spectrum of the second variation of the energy. By suitable rescaling, they formally proposed that the translation invariance of the entire radial solution will lead to the whole null space for the second variation of energy if $B < 0$.

Whence, in the present paper, we will concern the non-degeneracy of the radial vortex solution with degree pair $(1, 1)$ for (1) and give an affirmative answer to the above problem. In the rest part of this paper, we use the notation

$$w = (w^+, w^-) \text{ with } w^\pm = U^\pm e^{i\theta},$$

(7)

to denote the radially symmetric vortex solution with degree pair $(n_+, n_-) = (1, 1)$. Then we get the corresponding ODE system

$$\begin{cases}
-U^{++} - \frac{1}{r}U^{+} + \frac{1}{r^2}U^+ + \left[A_+(U^{+2} - t^{+2}) + B(U^{-2} - t^{-2})\right]U^+ = 0, \\
-U^{-''} - \frac{1}{r}U^{-'} + \frac{1}{r^2}U^- + \left[A_-(U^{-2} - t^{-2}) + B(U^{+2} - t^{+2})\right]U^- = 0.
\end{cases}$$

(8)

Together with Proposition 1 and Proposition 2, when $(n_+, n_-) = (1, 1)$, the properties of the vortex profiles $U^\pm$ are provided in the following

$$\begin{cases}
U^\pm \in C^\infty(0, \infty), & 0 < U^\pm < t^\pm \text{ for all } r > 0, \\
U^\pm \sim r \text{ as } r \to 0, & U^\pm \sim t^\pm - \frac{c_\pm}{r^2} \text{ as } r \to \infty, \\
U^+ > 0 \text{ when } B < 0, & U^+ \sim \frac{t^+}{r} \text{ as } r \to \infty,
\end{cases}$$

(9)

where $c_\pm = \frac{A_+ - B}{A_+ A_- - B^2}$. The linearized operator $\mathcal{L} = (\mathcal{L}_+, \mathcal{L}_-)$ of (1) around $w$ has the components

$$\mathcal{L}_\pm(\phi) = -\Delta \phi^\pm + \left[A_\pm(U^{\pm2} - t^{\pm2}) + B(U^{\mp2} - t^{\mp2})\right]\phi^\pm + 2A_\pm \Re\left(w^\pm \bar{\phi}^\mp\right)w^\pm + 2B\Re\left(w^+ \bar{\phi}^-\right)w^\pm.$$  

(10)

Direct computation shows that

$$\mathcal{L}\left(\frac{\partial w}{\partial x_1}\right) = \mathcal{L}\left(\frac{\partial w}{\partial x_2}\right) = \mathcal{L}(iw) = 0.$$  

(11)

The quadratic form given by the second variation of the energy functional $E$ around $w$ is

$$B(\phi, \phi) = \int_{\mathbb{R}^2} |\nabla \phi|^2 + \int_{\mathbb{R}^2} \left[A_+(U^{+2} - t^{+2}) + B(U^{-2} - t^{-2})\right]|\phi^+|^2 + \int_{\mathbb{R}^2} \left[A_-(U^{-2} - t^{-2}) + B(U^{+2} - t^{+2})\right]|\phi^-|^2.$$
\[
+ \int_{\mathbb{R}^2} \left[ 2A_+ \left| \text{Re}(\overline{w^+}\phi^+) \right|^2 + 2A_- \left| \text{Re}(\overline{w^-}\phi^-) \right|^2 \right] \\
+ \int_{\mathbb{R}^2} 4B\text{Re}(\phi^+\overline{w^+})\text{Re}(\phi^-\overline{w^-}).
\]  
(12)

By defining the inner product, for any \( u = (u^+, u^-), \ v = (v^+, v^-) : \mathbb{R}^2 \to \mathbb{C}^2, \)

\[
\langle u, v \rangle = \langle u^+ + v^+, u^- + v^- \rangle \mathbb{R}^2 + \langle u^+ - v^+, u^- - v^- \rangle \mathbb{R}^2,
\]

(13)
we can get

\[
B(\phi, \phi) = \langle \mathcal{L}(\phi), \phi \rangle = \langle \mathcal{L}_+(\phi), \phi^+ \rangle + \langle \mathcal{L}_-(\phi), \phi^- \rangle \mathbb{R}^2.
\]

As a result of (11), we have

\[
B(\phi, \phi) = 0,
\]

for the special functions

\[
\phi = \frac{\partial w}{\partial x_1} = \left( \frac{\partial w^+}{\partial x_1}, \frac{\partial w^-}{\partial x_1} \right) \quad \text{or} \quad \phi = \frac{\partial w}{\partial x_2} = \left( \frac{\partial w^+}{\partial x_2}, \frac{\partial w^-}{\partial x_2} \right),
\]

(14)

where

\[
\frac{\partial w^+_\pm}{\partial x_1} = \frac{1}{2} \left( U^+_{\pm'} - \frac{U^\pm}{r} \right) e^{2i\theta} + \frac{1}{2} \left( U^\pm_{\pm'} + \frac{U^\pm}{r} \right),
\]

(15)

\[
\frac{\partial w^+_\pm}{\partial x_2} = -i \left( U^+_{\pm'} - \frac{U^\pm}{r} \right) e^{2i\theta} + i \left( U^\pm_{\pm'} + \frac{U^\pm}{r} \right).
\]

(16)

For the non-degeneracy, we mean the positivity of the quadratic form \( B(\phi, \phi) \) and that the kernel of the linearized operator \( \mathcal{L} \) at \( w \) is a linear combination of \( \left\{ \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2} \right\} \) in the space \( \mathcal{H} \). Because of (12) and analogously to [10], we consider perturbations in a “natural” Hilbert space \( \mathcal{H} \) with the norm as follows

\[
\| \phi \|_{\mathcal{H}}^2 = \int_{\mathbb{R}^2} |\nabla \phi|^2 + \int_{\mathbb{R}^2} \left[ A_+ (t^{\pm^2} - U^{\pm^2}) - B(t^{-\pm^2} - U^{-\pm^2}) \right] |\phi^+|^2 \\
+ \int_{\mathbb{R}^2} \left[ A_- (t^{\pm^2} - U^{\pm^2}) - B(t^{\pm^2} - U^{\pm^2}) \right] |\phi^-|^2,
\]

(17)
for any \( \phi = (\phi^+, \phi^-) : \mathbb{R}^2 \to \mathbb{C}^2 \). In fact, as \( A_{\pm} > 0, B < 0, 0 < U^\pm < t^\pm \), we can easily show that \( \| \cdot \| \) is a norm. Together with the properties (9) of the vortex profiles \( U^{\pm} \) of the minimizer \( w \), we can know that \( iw \) is not contained in the space \( \mathcal{H} \).

In the paper [10], the authors proved the non-degeneracy of the solution of degree one for the classical (single complex component) Ginzburg-Landau equation in \( \mathbb{R}^2 \) by using Fourier decomposition method and some ODE techniques. However, it is more complicated to get the non-degeneracy for the minimizer of the Ginzburg-Landau system (1) due to the existence of the interaction terms by using the method in [10]. In this paper, we will overcome the difficulty by the method from [6] (using Picone’s identities) under one more assumption

\[
B < 0,
\]

(H2)
due to technical reasons, see the Remark 1.

Now we give the main results.
Theorem 1.1. Assume that \( H1 \) and \( H2 \) hold. Suppose \( L(\phi) = 0 \) for some \( \phi \in \mathcal{H} \). Then we have
\[
\phi = c_1 \frac{\partial w}{\partial x_1} + c_2 \frac{\partial w}{\partial x_2},
\]
for certain constants \( c_1 \) and \( c_2 \).

Note that we can use the same method to get the non-degeneracy results under the validity of \( H1 \) and \( H2 \) if \( w \) has degree pair \((-1, -1), (-1, 1) \) or \((1, -1) \). An interesting problem is to consider the validity of the non-degeneracy result for the minimizer of Ginzburg-Landau system (1) in \( \mathbb{R}^2 \) for \( B > 0 \) case. On the other hand, to the best knowledge of the authors, there are no non-degeneracy results for the vortex solutions with higher degrees of the Ginzburg-Landau equation with single complex component or a system like (1).

As an application of Theorem 1.1, we give the following Fredholm alternative theorem for the linearized operator \( L \).

Theorem 1.2. Assume that \( H1 \) and \( H2 \) hold. Consider the equation
\[
L(\psi) = h \quad \text{in} \quad \mathbb{R}^2,
\]
with given \( h = (h^+, h^-) : \mathbb{R}^2 \to \mathbb{C}^2 \) satisfying
\[
\int_{\mathbb{R}^2} |h|^2 (1 + r^{2+\sigma}) < +\infty \quad \text{for some} \ \sigma > 0.
\]
If
\[
\langle h^\pm, iw^\pm \rangle = \langle h^\pm, \frac{\partial w^\pm}{\partial x_1} \rangle = \langle h^\pm, \frac{\partial w^\pm}{\partial x_2} \rangle = 0,
\]
then (18) has a solution \( \psi \in \mathcal{H} \) which satisfies
\[
\|\psi\|_{\mathcal{H}}^2 \leq \int_{\mathbb{R}^2} |h|^2 (1 + r^{2+\sigma}).
\]
Moreover, for any solution \( \hat{\psi} \in \mathcal{H} \) of the equation (18), \( \psi \) has the form
\[
\psi = \hat{\psi} + c_3 \frac{\partial w}{\partial x_1} + c_4 \frac{\partial w}{\partial x_2},
\]
where \( c_3 \) and \( c_4 \) are two constants.

By following the work of [10], in Section 3 we will prove the solvability theory in Theorem 1.2. A solvability theory for the linearized operator is of crucial importance in the use of singular perturbation methods for the construction of various solutions with much more complicated vortex structures of problems where the rescaled vortex \( w \) provide a canonical profile. More details about the subject for the single complex component Ginzburg-Landau equation are shown in the literature [11], [9], [19], [21], [23] and [25]. For more results, the reader can refer to [19], [24], [13] and the references therein.

Plan of the paper
We organize the paper as follows. In section 2, we will give the proof of non-degeneracy results by using the method in the paper [6]. In section 3, similar to the paper [10], based on the non-degeneracy result, we will give a proof of the solvability theory (Theorem 1.2) for the linear equation (18).
Notation

Throughout this paper, we employ $C, c$ or $C_j, c_j, j = 0, 1, 2, \cdots$ to denote certain constants. Furthermore, we may make the abuse of notation by writing $\psi_j, \phi_j, j = 0, 1, 2, \cdots$, from line to line in the present paper.

2. Non-degeneracy results. By following the method in [6], now we give the proof of non-degeneracy results.

2.1. Preliminaries for the proof of Theorem 1.1. The following decomposition by the Fourier series plays an important role in our study. Now for any $\phi \in H$ (see (17)), we can decompose $\phi$ in its Fourier modes and write the quadratic form $B(\phi, \phi)$ as a direct sum. For any $k \in \mathbb{Z}$, as a direct sum.

For any $k \in \mathbb{Z}$, let $\phi_k \in S := \{ \psi \in H_{\text{loc}}(\mathbb{R}^+, \mathbb{C}^2) \cap L^2(\mathbb{R}^+, \mathbb{C}^2) : \int_0^\infty |\psi|^2 + \frac{1}{r^2}|\psi|^2 \, dr < +\infty \}$.

And then we get

$$\frac{1}{2\pi} B(\phi, \phi) = 2\mathbb{B}_0(\phi_1, \phi_1) + \sum_{k=1}^{\infty} B_k(\phi_{1+k}, \phi_{1-k}),$$  

(23)

where

$$\mathbb{B}_0(\phi_1, \phi_1) = \int_0^\infty \left[ |(\phi_1^+)\|^2 + \frac{1}{r^2}|\phi_1^+|^2 \right] \, dr + \int_0^\infty \left[ |(\phi_1^-)\|^2 + \frac{1}{r^2}|\phi_1^-|^2 \right] \, dr$$

$$+ \int_0^\infty 2 \left[ A_U^+ U \text{Re}(\phi_1^+)^2 + A_U^- \text{Re}(\phi_1^-)^2 \right] \, dr$$

$$+ \int_0^\infty 4BU^+ U \text{Re}(\phi_1^+) \text{Re}(\phi_1^-) \, dr$$

$$+ \int_0^\infty \left[ A_+ (U^2 - t^2) + B(U^2 - t^2) \right] |\phi_1^+|^2 \, dr$$

$$+ \int_0^\infty \left[ A_- (U^2 - t^2) + B(U^2 - t^2) \right] |\phi_1^-|^2 \, dr,$$  

and for $k \geq 1$

$$\mathbb{B}_k(\phi_{1+k}, \phi_{1-k})$$

$$= \int_0^\infty \left[ |(\phi_{1+k}^+)\|^2 + |(\phi_{1-k}^+)\|^2 + \frac{(1+k)^2}{r^2}|\phi_{1+k}^+|^2 + \frac{(1-k)^2}{r^2}|\phi_{1-k}^+|^2 \right] \, dr$$

$$+ \int_0^\infty \left[ |(\phi_{1+k}^-)\|^2 + |(\phi_{1-k}^-)\|^2 + \frac{(1+k)^2}{r^2}|\phi_{1+k}^-|^2 + \frac{(1-k)^2}{r^2}|\phi_{1-k}^-|^2 \right] \, dr$$

$$+ \int_0^\infty \left[ A_U^+ U \phi_{1+k}^+ \phi_{1-k}^- + A_U^- \phi_{1+k}^- \phi_{1-k}^+ \right] \, dr$$

$$+ \int_0^\infty 2BU^+ U \text{Re}(\phi_{1+k}^+ \phi_{1-k}^-) \, dr$$
\[ + \int_0^\infty \left[ A_+(U^{+2} - t^{+2}) + B(U^{-2} - t^{-2}) \right] (|\phi_{1+k}^+|^2 + |\phi_{1-k}^+|^2) r\, dr \\
+ \int_0^\infty \left[ A_-(U^{+2} - t^{+2}) + B(U^{-2} - t^{-2}) \right] (|\phi_{1+k}^-|^2 + |\phi_{1-k}^-|^2) r\, dr. \tag{24} \]

For the proof of the above results, the reader can refer to the paper [4], and we omit it here.

The decomposition in (22) is naturally associated to the functions \( \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}, iw \) in the kernel of the linearized operator \( L \). In fact, we can write

\[ iw = i\phi_1 e^{i\theta}, \quad \frac{\partial w}{\partial x_1} = \phi_2 e^{2i\theta} + \phi_0, \quad \frac{\partial w}{\partial x_2} = -i\phi_2 e^{2i\theta} + i\phi_0, \]

where the functions have the special forms

\[ \phi_1 = (\phi_1^+, \phi_1^-) \text{ with } \phi_1^\pm = U^\pm, \tag{25} \]
\[ \phi_2 = (\phi_2^+, \phi_2^-) \text{ with } \phi_2^\pm = \frac{1}{2} \left( U^{\pm} - U^\pm \right), \tag{26} \]
\[ \phi_0 = (\phi_0^+, \phi_0^-) \text{ with } \phi_0^\pm = \frac{1}{2} \left( U^{\pm} + U^\pm \right). \tag{27} \]

As a result of (23), we will discuss the quadratic forms \( \{\mathbb{B}_k\} \) on \( S \times S \) to get the non-degeneracy result. It is easy to show that

\[ \mathbb{B}_0(\phi_1, \phi_1) \geq 0 \text{ and } \mathbb{B}_k(\phi_{1+k}, \phi_{1-k}) > 0, \forall k \geq 2, \]

see Propositions 4 and 5. The key step is to establish that \( \mathbb{B}_1(\phi_2, \phi_0) \geq 0 \), and the equality holds if and only if \( \phi_2, \phi_0 \) have the special forms in (26)-(27), see Proposition 6 together with (30) and (34).

2.2. Proof of Theorem 1.1. Before giving the complete proof of Theorem 1.1, we analyze the quadratic forms \( \{\mathbb{B}_k\} \) separately and give several propositions.

**Proposition 4.** We have that

\[ \mathbb{B}_0(\phi, \phi) \geq 0, \quad \forall \phi \in S, \]

and the equality holds if and only if \( \phi = i(c_5 U^+, c_6 U^-) \) where \( c_5, c_6 \) are real constants.

**Proof of Proposition 4.** We can easily get that

\[ \mathbb{B}_0(i\phi, i\phi) \leq \mathbb{B}_0(\phi, \phi), \]

for any \( \phi \in S \), and the strict inequality holds unless \( \phi \) takes only purely imaginary values. Hence, it is natural to consider the functional on \( S \times S \)

\[ \tilde{\mathbb{B}}_0(\phi, \phi) = \int_0^\infty \left[ (|\phi^+|^2 + \frac{1}{r^2} |\phi^+|^2) r\, dr + \int_0^\infty \left[ (|\phi^-|^2 + \frac{1}{r^2} |\phi^-|^2) r\, dr \\
+ \int_0^\infty \left[ A_+(U^{+2} - t^{+2}) + B(U^{-2} - t^{-2}) \right] |\phi^+|^2 r\, dr \\
+ \int_0^\infty \left[ A_-(U^{-2} - t^{-2}) + B(U^{+2} - t^{+2}) \right] |\phi^-|^2 r\, dr. \]


For any $\phi \in C_c^\infty(0, \infty; \mathbb{R}^2)$, whose support does not contain the origin, we write $\phi = (U^+\xi_1, U^-\xi_2)$ for some real functions $\xi_1, \xi_2$. Integration by part together with (8) yields

$$\hat{\mathbb{B}}_0(\phi, \phi) = \int_0^\infty (U'^2 + U'^2) r \, dr.$$  \hfill (28)

The formula (28) also holds for any smooth function $\phi$ whose support contains the origin. The proof relies on a cutoff function. Using a density argument, (28) is also true for any $\phi \in \mathcal{H}$. Then if $\hat{\mathbb{B}}_0(\phi, \phi) = 0$, it is clearly that $\xi_1, \xi_2$ are constants. This finishes the proof of Proposition 4. \hfill $\square$

**Proposition 5.** For each $k \geq 2$, we have $\mathbb{B}_k(\phi_1, \phi_2) > 0$ for all $(\phi_1, \phi_2) \in \mathcal{S} \times \{0, 0\}$.

**Proof of Proposition 5.** From the definition of the functional $\mathbb{B}_k(\phi_1, \phi_2)$ in (24), we can know easily that

$$\mathbb{B}_k(\phi_1, \phi_2) \geq \hat{\mathbb{B}}_0(|\phi_1|, |\phi_1|) + \hat{\mathbb{B}}_0(|\phi_2|, |\phi_2|) \text{ for all } (\phi_1, \phi_2) \in \mathcal{S} \times \mathcal{S} \setminus \{0, 0\}.$$  

By using Proposition 4, we conclude that $\mathbb{B}_k(\phi_1, \phi_2) > 0$ and equality hold if and only if $\phi_1 = \phi_2 = 0$. This finishes the proof of Proposition 5. \hfill $\square$

Now, we discuss the quadratic form $\mathbb{B}_1(\phi_2, \phi_0)$, which is the most delicate one. Note that

$$\mathbb{B}_1(\phi_2, \phi_0) = \int_0^\infty \left[ |(\phi_2^+)^2| + |(\phi_0^+)^2| + \frac{4}{r^2} |\phi_2^+|^2 \right] r \, dr$$

$$+ \int_0^\infty \left[ |(\phi_2^-)^2| + |(\phi_0^-)^2| + \frac{4}{r^2} |\phi_2^-|^2 \right] r \, dr$$

$$+ \int_0^\infty \left[ A_+U'^2 |\phi_2^+ + \phi_0^+|^2 + A_-U'^2 |\phi_2^- + \phi_0^-|^2 \right] r \, dr$$

$$+ \int_0^\infty 2BU'^2 - \text{Re}((\phi_2^+ + \phi_0^+)|\phi_2^- + \phi_0^-|) + \text{Re}((\phi_2^- + \phi_0^-)|\phi_2^+ + \phi_0^+|) \right] r \, dr$$

$$+ \int_0^\infty \left[ A_+(U'^2 + t^2) + B(U'^2 - t^2) \right] |(\phi_2^+)^2| + |(\phi_0^+)^2|^2 r \, dr$$

$$+ \int_0^\infty \left[ A_-(U'^2 - t^2) + B(U'^2 + t^2) \right] |(\phi_2^-)^2| + |(\phi_0^-)^2|^2 r \, dr.$$  \hfill (29)

For any $\phi \in \mathcal{S}$, we can write it as $\phi = \phi^R + i\phi^I$, and then get

$$\mathbb{B}_1(\phi_2, \phi_0) = \mathbb{B}_1(\phi_2^R, \phi_0^R) + \mathbb{B}_1(\phi_2^I, \phi_0^I) = \mathbb{D}(-\phi_2^R, \phi_0^R) + \mathbb{D}(\phi_2^I, \phi_0^I).$$  \hfill (30)

with the quadratic form $\mathbb{D}$ defined as

$$\mathbb{D}(\varpi, h) = \mathbb{B}_1(i\varpi, ih), \text{ for any } (\varpi, h) \in \tilde{\mathcal{S}} \times \tilde{\mathcal{S}}.$$  

Here we denote by $\tilde{\mathcal{S}}$ the subspace of real-valued functions in $\mathcal{S}$. The relation in (30) implies that we shall concern the quadratic form $\mathbb{D}$ on $\tilde{\mathcal{S}} \times \tilde{\mathcal{S}}$.

We pause here to give an observation of the null space of $\mathbb{D}$. By recalling

$$\frac{\partial u^\pm}{\partial x_1} = \frac{1}{2} \left( \frac{U'^+ - U'^-}{r} \right) e^{2i\theta} + \frac{1}{2} \left( \frac{U'^+ + U'^-}{r} \right),$$  \hfill (31)

$$\frac{\partial u^\pm}{\partial x_2} = \frac{-i}{2} \left( \frac{U'^+ - U'^-}{r} \right) e^{2i\theta} + \frac{i}{2} \left( \frac{U'^+ + U'^-}{r} \right),$$  \hfill (32)
we define
\[ \Phi_2 = (\Phi_2^+, \Phi_2^-), \quad \Phi_0 = (\Phi_0^+, \Phi_0^-), \]
with
\[ \Phi_2^\pm = \frac{1}{2} \left( -U^{\pm} + \frac{U^{\pm}}{r} \right), \quad \Phi_0^\pm = \frac{1}{2} \left( U^{\pm} + \frac{U^{\pm}}{r} \right). \]
Using the result (30) together with (31)-(34), we can obtain
\[ 0 = \frac{1}{\pi} B \left( \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_1} \right) = \mathbb{B}_1(-\Phi_2, \Phi_0) = \mathbb{D}(\Phi_2, \Phi_0), \]
Thus we have
\[ \mathbb{D}(\Phi_2, \Phi_0) = 0. \] (35)
The properties of \( \mathbb{D}(\phi_2, \phi_0) \) on \( \tilde{S} \times \tilde{S} \) will be provided in the following proposition, which confirms the non-negativity of the quadratic form \( \mathbb{B}_1 \) on \( S \times S \) due to the relation (30).

**Proposition 6.** We have
\[ \mathbb{D}(\phi_2, \phi_0) \geq 0, \quad \forall (\phi_2, \phi_0) \in \tilde{S} \times \tilde{S} \]
where the equality holds if and only if \( (\phi_2, \phi_0) = C(\Phi_2, \Phi_0) \) with \( (\Phi_2, \Phi_0) \) in (33)-(34).

**Proof of Proposition 6.** For any \( \phi_2, \phi_0 \in \tilde{S} \times \tilde{S} \), we compute \( \mathbb{D}(\phi_2, \phi_0) \) in the following
\[ \mathbb{D}(\phi_2, \phi_0) = \mathbb{B}_1(i\phi_2, i\phi_0) = \mathbb{B}_1(-\phi_2, \phi_0) \]
\[ = \int_0^\infty \left[ |(\phi_0^+)'|^2 + |(\phi_0^-)'|^2 + \frac{4}{r^2} |\phi_2^+|^2 \right] r \, dr \]
\[ + \int_0^\infty \left[ |(\phi_2^-)'|^2 + |(\phi_2^+)'|^2 + \frac{4}{r^2} |\phi_2^-|^2 \right] r \, dr \]
\[ + \int_0^\infty \left[ A_+U^+|\phi_0^+ - \phi_2^+|^2 + A_-U^-|\phi_0^- - \phi_2^-|^2 \right] r \, dr \]
\[ + \int_0^\infty 2BU^+U^- \left[ \text{Re}(\phi_0^+ - \phi_2^+)(-\phi_2^-) + \text{Re}(\phi_0^- - \phi_2^-)\phi_0^- \right] r \, dr \]
\[ + \int_0^\infty \left[ A_+(U^+ - t^+)^2 + B(U^2 - t^2)^2 \right] |\phi_2^+|^2 + |\phi_0^+|^2 | r \, dr \]
\[ + \int_0^\infty \left[ A_-(U^2 - t^-)^2 + B(U^2 - t^2)^2 \right] |\phi_2^-|^2 + |\phi_0^-|^2 | r \, dr. \] (36)
Furthermore, we denote
\[ C^+ = \phi_0^+ + \phi_2^+, \quad C^- = \phi_0^- + \phi_2^-, \quad D^+ = \phi_0^+ - \phi_2^+, \quad D^- = \phi_0^- - \phi_2^- \] (37)
By substituting (37) in (36), we can obtain
\[ \mathbb{D}(\phi_2, \phi_0) = \int_0^\infty \frac{1}{2} \left[ (C^+)^2 + (D^+)^2 + \frac{2}{r^2} (C^+ - D^+)^2 \right] r \, dr \]
In the above, we also have denoted 

\[ \eta^+ = \Phi_0^+ + \Phi_2^+ = \frac{U^+}{r}, \quad \eta^- = \Phi_0^- + \Phi_2^- = \frac{U^-}{r}, \]

\[ \zeta^+ = \Phi_0^+ - \Phi_2^+ = U^+, \quad \zeta^- = \Phi_0^- - \Phi_2^- = U^- . \]
respectively together with integrating by parts, and using (42), we can get

\[
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\]

This implies that \( \eta \) satisfies the equations

\[
\begin{align*}
( - \alpha_1 \eta^+)' + a_1 \eta + b_1 \zeta^+ &= 0, \\
( - \alpha_2 \eta^-)' + a_2 \eta - b_2 \zeta^- &= 0, \\
( - \beta_1 \zeta^+)' + d_1 \zeta^+ + b_1 \eta^+ + b_3 \zeta^- &= 0, \\
( - \beta_2 \zeta^-)' + d_2 \zeta^- + b_2 \eta^- + b_3 \zeta^+ &= 0.
\end{align*}
\]

(42)

For any smooth functions \( (u^+, v^+, v^- u^-, v^-) \in C_c(0, \infty) \times C_c(0, \infty) \times C_c(0, \infty) \times C_c(0, \infty) \), the Picone’s identities imply that

\[
\begin{align*}
(u^+)'^2 - \left( \frac{u^+}{\eta^+} \right)'(\eta^+)' &= \left( u^+ - \frac{u^+}{\eta^+} \eta^+ \right)^2 \geq 0, \\
(u^-)'^2 - \left( \frac{u^-}{\eta^-} \right)'(\eta^-)' &= \left( u^- - \frac{u^-}{\eta^-} \eta^- \right)^2 \geq 0, \\
(v^+)'^2 - \left( \frac{v^+}{\zeta^+} \right)'(\zeta^+)' &= \left( v^+ - \frac{v^+}{\zeta^+} \zeta^+ \right)^2 \geq 0, \\
(v^-)'^2 - \left( \frac{v^-}{\zeta^-} \right)'(\zeta^-)' &= \left( v^- - \frac{v^-}{\zeta^-} \zeta^- \right)^2 \geq 0.
\end{align*}
\]

(43)-(46)

Now, we consider the properties of \( F \). Multiplying (43)-(46) by \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) respectively together with integrating by parts, and using (42), we can get

\[
\begin{align*}
0 &\leq \int_0^\infty \left[ \alpha_1 (u^+)'^2 + \alpha_2 (u^-)'^2 + \beta_1 (v^+)'^2 + \beta_2 (v^-)'^2 \right] \, dr \\
&\quad + \int_0^\infty \left[ a_1 u^2 + a_2 u^2 - b_1 v^2 + b_2 v^2 \right] \, dr \\
&\quad + \int_0^\infty \left[ b_1 u^2 \frac{\zeta^+}{\eta^+} + b_2 u^2 \frac{\zeta^-}{\eta^-} + b_1 v^2 \frac{\eta^+}{\zeta^+} \\
&\quad + b_2 v^2 \frac{\eta^-}{\zeta^-} + b_3 v^2 \frac{\zeta^+}{\zeta^-} + b_3 v^2 \frac{\zeta^-}{\zeta^+} \right] \, dr \\
= F(u^+, v^+, u^-, v^-) &+ \int_0^\infty b_1 \left[ u^+ \left( \frac{\zeta^+}{\eta^+} \right)^{\frac{1}{2}} - v^+ \left( \frac{\eta^+}{\zeta^+} \right)^{\frac{1}{2}} \right]^2 \, dr \\
&\quad + \int_0^\infty b_2 \left[ u^- \left( \frac{\zeta^-}{\eta^-} \right)^{\frac{1}{2}} - v^- \left( \frac{\eta^-}{\zeta^-} \right)^{\frac{1}{2}} \right]^2 \, dr \\
&\quad + \int_0^\infty b_3 \left[ v^+ \left( \frac{\zeta^-}{\zeta^+} \right)^{\frac{1}{2}} - v^- \left( \frac{\zeta^+}{\zeta^-} \right)^{\frac{1}{2}} \right]^2 \, dr.
\end{align*}
\]

(47)

This implies that

\[
F(u^+, v^+, u^-, v^-) \geq - \int_0^\infty b_1 \left[ u^+ \left( \frac{\zeta^+}{\eta^+} \right)^{\frac{1}{2}} - v^+ \left( \frac{\eta^+}{\zeta^+} \right)^{\frac{1}{2}} \right]^2 \\
- \int_0^\infty b_2 \left[ u^- \left( \frac{\zeta^-}{\eta^-} \right)^{\frac{1}{2}} - v^- \left( \frac{\eta^-}{\zeta^-} \right)^{\frac{1}{2}} \right]^2
\]
\[ - \int_0^\infty b_3 \left[ v^+ \left( \frac{\zeta}{\zeta^+} \right)^{1/2} - v^- \left( \frac{\zeta}{\zeta^-} \right)^{1/2} \right]^2 \geq 0, \quad (48) \]
due to the facts in (39). Inequalities in (48) also hold for \((u^+, v^+, u^-, v^-) \in \tilde{S} \times \tilde{S} \times \tilde{S} \times \tilde{S}\) due to a density argument. The inequalities in (48) imply \(F(u^+, v^+, u^-, v^-) \geq 0\). Moreover, if the equality \(F(u^+, v^+, u^-, v^-) = 0\) holds, then there exists a function \(\vartheta\) such that
\[(u^+, v^+, u^-, v^-) = (\vartheta \eta^+ + \vartheta \zeta^+, \vartheta \eta^- + \vartheta \zeta^-).\]
Applying (43)-(46), we know that the function \(\vartheta\) must be a constant. Therefore, we conclude that \(F(u^+, v^+, u^-, v^-) \geq 0\) and the equality holds if and only if
\[ (u^+, v^+, u^-, v^-) = C(\eta^+, \zeta^+, \eta^-, \zeta^-). \]
Combining with (37), (38), (40), (41) and the results above, we have
\[ D(\phi_2, \phi_0) \geq 0, \]
and if \(D(\phi_2, \phi_0) = 0\), then
\[ (\phi_2, \phi_0) = C(\Phi_2, \Phi_0). \]
This finishes the proof of the Proposition 6.

**Remark 1.** In the proof of Proposition 6, in order to get the result
\[ F(u^+, v^+, u^-, v^-) \geq 0, \]
we use the facts in (39). That is why we need the technical assumption \(B < 0\) in Theorem 1.1. For the case \(B = 0\), the coupled Ginzburg-Landau system (1) is reduced to the single-component Ginzburg-Landau equation in \(\mathbb{R}^2\), and the non-degeneracy results have been proved in [10]. When \(B > 0\) the non-degeneracy type result for local energy minimizer of the system (1) is not clear.

Now we give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** For any \(\phi \in \mathcal{H}\), we can decompose \(B(\phi, \phi)\) as in (23). By combining Propositions 4-6 and (30), we can know that \(B(\phi, \phi) \geq 0\). In particular, by the equalities in Propositions 4-6, we have that
\[ B(\phi, \phi) = 0 \quad \text{iff} \quad \phi = \phi_0 + \phi_1 e^{i\theta} + \phi_2 e^{2i\theta}, \]
where
\[ (\phi_2^I, \phi_0^I) = c_1(\Phi_2, \Phi_0), \quad (-\phi_2^R, \phi_0^R) = c_2(\Phi_2, \Phi_0), \quad \phi_1 = i(c_3 U^+ + c_4 U^-), \]
for some constants \(c_1, c_2, c_3, c_4 \in \mathbb{R}\). It is easy to verify that
\[ L(\phi) = 0 \quad \text{iff} \quad \phi = c_1 \frac{\partial w}{\partial x_1} + c_2 \frac{\partial w}{\partial x_2} + i(c_3 w^+, c_4 w^-) \text{ for any } \phi \in \mathcal{H}. \]
Since \(i(c_3 w^+, c_4 w^-)\) is not contained in \(\mathcal{H}\), we have
\[ \phi = c_1 \frac{\partial w}{\partial x_1} + c_2 \frac{\partial w}{\partial x_2}. \]
This completes the proof of Theorem 1.1.
3. **Application of non-degeneracy: Fredholm alternative.** In the section, we will find a solution $\psi = (\psi^+, \psi^-): \mathbb{R}^2 \to \mathbb{C}^2$ of the linear equation

$$\mathcal{L}(\psi) = h,$$

for a given $h = (h^+, h^-): \mathbb{R}^2 \to \mathbb{C}^2$. The idea is to find a minimizer of energy functional corresponding to the equation (49)

$$J(\psi) = \frac{1}{2} B(\psi, \psi) - \langle \psi, h \rangle,$$

where $B(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ are defined in (12) and (13).

3.1. **Preliminary work for the proof of Theorem 1.2.** We give some preliminaries for the proof of Theorem 1.2. We decompose $\psi = (\psi^+, \psi^-)$ into the form

$$\psi = \psi_0 + \sum_{j=1}^{\infty} \psi_1^j + \sum_{j=1}^{\infty} \psi_2^j,$$

where

$$\psi_0 = (\psi_{0}^+, \psi_{0}^-), \quad \psi_1^j = (\psi_{1}^+_j, \psi_{1}^-_j), \quad \psi_2^j = (\psi_{2}^+_j, \psi_{2}^-_j),$$

and

$$\psi_0^\pm = e^{i\theta} \left[ \psi_{01}^\pm \pm i \psi_{02}^\pm \right],$$

$$\psi_1^j = e^{i\theta} \left[ \psi_{11}^j \pm \sin j\theta \pm i \psi_{12}^j \pm \cos j\theta \right],$$

$$\psi_2^j = e^{i\theta} \left[ \psi_{21}^j \pm \cos j\theta \pm i \psi_{22}^j \pm \sin j\theta \right].$$

The decomposition in (51)-(53) is naturally associated to the functions $\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}$ in the kernel of the linearized operator $\mathcal{L}$. In fact, we can write

$$\frac{\partial w}{\partial x_1} = e^{i\theta} \left[ U' \cos \theta - \frac{U}{r} \sin \theta \right],$$

$$\frac{\partial w}{\partial x_2} = e^{i\theta} \left[ U' \sin \theta + \frac{U}{r} \cos \theta \right].$$

Similarly, we decompose $h = (h^+, h^-)$

$$h = h_0 + \sum_{j=1}^{\infty} h_1^j + \sum_{j=1}^{\infty} h_2^j,$$

where

$$h_0 = (h_{0}^+, h_{0}^-), \quad h_1^j = (h_{1}^+_j, h_{1}^-_j), \quad h_2^j = (h_{2}^+_j, h_{2}^-_j),$$

and

$$h_0^\pm = e^{i\theta} \left[ h_{01}^\pm \pm i h_{02}^\pm \right],$$

$$h_1^j = e^{i\theta} \left[ h_{11}^j \pm \sin j\theta \pm i h_{11}^j \pm \cos j\theta \right],$$

$$h_2^j = e^{i\theta} \left[ h_{21}^j \pm \cos j\theta \pm i h_{22}^j \pm \sin j\theta \right].$$

Then finding a solution to (49) is equivalent to solving the equations

$$\mathcal{L}(\psi_0) = h_0,$$
\[ \mathcal{L}(\psi^j) = h^j_{\ell}, \quad j \in \mathbb{N}^+, \; \ell = 1, 2. \]  

(58)

We will solve equations (57) and (58) separately and provide the proof for Theorem 1.2 in three cases, see Section 3.2. For that purpose, it is natural to consider the critical point of the functionals

\[ J_0(\psi_0) = \frac{1}{2} B(\psi_0, \psi_0) - \langle \psi_0, h_0 \rangle, \]

(59)

\[ J_0^j(\psi^j) = \frac{1}{2} B(\psi^j, \psi^j) - \langle \psi^j, h_0^j \rangle, \quad j \in \mathbb{N}^+, \; \ell = 1, 2. \]

(60)

3.2. Proof of Theorem 1.2. As we have said in previous section, in order to prove Theorem 1.2, we will solve the equations in (57) and (58).

Case 1: mode \( j = 1 \). We now begin by solving the problem \( \mathcal{L}(\psi^1) = h_1^1 \). Let \( \mathcal{H}_\ast \) be the space of functions

\[ \tilde{\psi} = (\tilde{\psi}^+, \tilde{\psi}^-) = \left( (\tilde{\psi}_1^+, \tilde{\psi}_2^+), (\tilde{\psi}_1^-, \tilde{\psi}_2^-) \right) \]

such that the functions \( \psi = (\psi^+, \psi^-) \) belong to \( \mathcal{H} \) with the forms

\[ \psi^\pm = e^{i\theta} \left[ \tilde{\psi}_1^\pm \sin \theta + i\tilde{\psi}_2^\pm \cos \theta \right]. \]

(61)

Note that for the brevity of notation, we here have used \( \tilde{\psi} \) to denote \( \psi^1 \) in (52), and (61) is just the decomposition for \( \psi^1 \) in (52)-(53). According to the norm for \( \psi \) given in (17), the norm for \( \tilde{\psi} \) in the space \( \mathcal{H}_\ast \) is given by

\[
\| \tilde{\psi} \|^2_{\mathcal{H}_\ast} = \int_0^\infty \left[ |(\tilde{\psi}^+)'|^2 + \frac{1}{r^2} (\tilde{\psi}_1^+ - \tilde{\psi}_2^+)^2 \right] dr \\
+ \int_0^\infty \left[ |(\tilde{\psi}^-)'|^2 + \frac{1}{r^2} (\tilde{\psi}_1^- - \tilde{\psi}_2^-)^2 \right] dr \\
+ \int_0^\infty \left[ A_+ (t^2 - U^+) - B(\ell^2 - U^-) \right] |\tilde{\psi}^+|^2 r dr \\
+ \int_0^\infty \left[ A_- (t^2 - U^-) - B(\ell^2 - U^+) \right] |\tilde{\psi}^-|^2 r dr.
\]

We introduce naturally a new quadratic form \( \mathcal{B}(\tilde{\psi}, \tilde{\psi}) \) for the function \( \tilde{\psi} \in \mathcal{H}_\ast \), which is defined as

\[ \mathcal{B}(\tilde{\psi}, \tilde{\psi}) = \pi B(\psi, \psi), \]

(62)

where \( \psi = (\psi^+, \psi^-) \) is the function defined in (61). By the relations in (62), we can write quadratic form \( \mathcal{B}(\tilde{\psi}, \tilde{\psi}) \) as follows

\[
\mathcal{B}(\tilde{\psi}, \tilde{\psi}) = \int_0^\infty \left( \left[ |(\tilde{\psi}^+)'|^2 + \frac{2}{r^2} (\tilde{\psi}_1^+ - \tilde{\psi}_2^+)^2 \right] + \left[ |(\tilde{\psi}^-)'|^2 + \frac{2}{r^2} (\tilde{\psi}_1^- - \tilde{\psi}_2^-)^2 \right] \right) r dr \\
+ \int_0^\infty \left[ A_+ (U^2 - \ell^2) + B(U^- - \ell^2) \right] |\tilde{\psi}^+|^2 r dr \\
+ \int_0^\infty \left[ A_- (U^2 - \ell^2) + B(U^+ - \ell^2) \right] |\tilde{\psi}^-|^2 r dr \\
+ \int_0^\infty \left[ 2A_+ U |\tilde{\psi}_1^+|^2 + 2A_- U^2 |\tilde{\psi}_1^-|^2 + 4BU^+ U^2 |\tilde{\psi}_1^+ \tilde{\psi}_1^-| \right] r dr.
\]

(63)
Define
\[ \tilde{h} = (\tilde{h}^+, \tilde{h}^-) = \left( (h_{11}^+, h_{11}^-), (h_{12}^+, h_{12}^-) \right). \]

Then, for \( \ell = 1, j = 1 \), solving problem (58) in \( \mathcal{H} \) is corresponding exactly to finding a critical point of the functional \( J_1(\tilde{\psi}) \), which is given by
\[ J_1(\tilde{\psi}) = \frac{1}{2} \mathcal{B}(\tilde{\psi}, \tilde{\psi}) - \int_0^\infty \left( h_{11}^+ \tilde{\psi}_1^+ + h_{11}^- \tilde{\psi}_1^- + h_{12}^+ \tilde{\psi}_2^+ + h_{12}^- \tilde{\psi}_2^- \right) r \, dr. \]

Now denote
\[ Z_0 = (Z_0^+, Z_0^-) \quad \text{with} \quad Z_0^+ = \left( U^+, \frac{U^+}{r} \right), \quad Z_0^- = \left( U^-, \frac{U^-}{r} \right). \]

Combining Theorem 1.1 with (55) and (62), we can know easily that \( \mathcal{B}(\tilde{\psi}, \tilde{\psi}) = 0 \) if and only if \( \tilde{\psi} = CZ_0 \). The assumption (19) implies that
\[ \int_0^\infty \left( h_{11}^+ U^{++} + h_{11}^- U^{--} + h_{12}^+ U^{++} + h_{12}^- U^{--} \right) r \, dr = 0. \]

We define a weighted inner product \( \langle \cdot, \cdot \rangle_* \) for the space \( \mathcal{H}_* \)
\[ \langle u, v \rangle_* = \langle u^+, v^+ \rangle_{\mathcal{R}^+} + \langle u^-, v^- \rangle_{\mathcal{R}^-}, \]
for any
\[ u = (u^+, u^-) = \left( (u_1^+, u_2^+), (u_1^-, u_2^-) \right) \in \mathcal{H}_*, \]
\[ v = (v^+, v^-) = \left( (v_1^+, v_2^+), (v_1^-, v_2^-) \right) \in \mathcal{H}_*, \]
where
\[ \langle u^+, v^+ \rangle_{\mathcal{R}^+} = \int_0^\infty \left[ A_+(t^{+2} - U^{+2}) - B(t^{+2} - U^{+2}) \right] (u_1^+ v_1^+ + u_2^+ v_2^+) r \, dr, \]
\[ \langle u^-, v^- \rangle_{\mathcal{R}^-} = \int_0^\infty \left[ A_-(t^{-2} - U^{-2}) - B(t^{-2} - U^{-2}) \right] (u_1^- v_1^- + u_2^- v_2^-) r \, dr. \]

In order to solve the equation (58) for \( j = 1, \ell = 1 \), we need the following lemma.

**Lemma 3.1.** There exists a constant \( C > 0 \) such that for any
\[ \tilde{\psi} = (\tilde{\psi}^+, \tilde{\psi}^-) = \left( (\tilde{\psi}_1^+, \tilde{\psi}_2^+), (\tilde{\psi}_1^-, \tilde{\psi}_2^-) \right) \in \mathcal{H}_*, \]
satisfying
\[ \langle \tilde{\psi}, Z_0 \rangle_* = \langle \tilde{\psi}^+, Z_0^+ \rangle_{\mathcal{R}^+} + \langle \tilde{\psi}^-, Z_0^- \rangle_{\mathcal{R}^-} = 0, \]
we have
\[ C \| \tilde{\psi} \|_{\mathcal{H}_*} \leq \mathcal{B}(\tilde{\psi}, \tilde{\psi}). \]

**Proof of Lemma 3.1.** Recall the expression of the quadratic form \( \mathcal{B}(\tilde{\psi}, \tilde{\psi}) \) in (63). Note that,
\[ t^{\pm 2} - U^{\pm 2} \sim \frac{c_{\pm} t_{\pm}}{r^2}, \quad \text{as} \quad r \to \infty, \quad \text{with} \quad c_{\pm} = \frac{A_{\pm} - B}{(A_+ A_- - B^2) t_{\pm}}. \]

For any given \( \delta > 0 \) small, there exists an \( R > 0 \) large such that, for \( r > R \)
\[ I^+(\tilde{\psi}) = \frac{2 - \delta}{r^2} \left( \tilde{\psi}_1^+ - \tilde{\psi}_2^+ \right)^2 + 2 A_+ U^+ \left| \tilde{\psi}_1^+ \right|^2 + 2 B U^- \tilde{\psi}_1^+ \tilde{\psi}_1^- - \left[ A_+ (t^{+2} - U^{+2}) + B(t^{-2} - U^{-2}) \right] |\tilde{\psi}^+|^2 \]
This implies that, for given \( \delta, R \),

\[
- \frac{\delta}{2\tau^+} \left[ A_+ (t^+ - U^+)^2 - B(t^2 - U^2) \right] |\tilde{\psi}^+|^2
\]

\[
\geq \frac{1 - 2\delta}{r^2} |\tilde{\psi}^+|^2 + 2A_+ U^2 |\tilde{\psi}_1^+|^2 + 2BU^+ U^- \tilde{\psi}_1^+ \tilde{\psi}_1^-
- 2\frac{2 - \delta}{r^2} \tilde{\psi}_1^+ \tilde{\psi}_2^+,
\]

and also

\[
I^- (\tilde{\psi}) = \frac{2 - \delta}{r^2} \left( \tilde{\psi}_1^+ - \tilde{\psi}_2^- \right)^2 + 2A_- U^2 |\tilde{\psi}_1^-|^2 + 2BU^- U^- \tilde{\psi}_1^- \tilde{\psi}_1^-
- \left[ A_- (t^2 - U^2) + B(t^2 - U^2) \right] |\tilde{\psi}^-|^2
\]

\[
- \frac{\delta}{2\tau^-} \left[ A_- (t^2 - U^2) - B(t^2 - U^2) \right] |\tilde{\psi}^-|^2
\]

\[
\geq \frac{1 - 2\delta}{r^2} |\tilde{\psi}^-|^2 + 2A_- U^2 |\tilde{\psi}_1^-|^2 + 2BU^- U^- \tilde{\psi}_1^- \tilde{\psi}_1^-
- 2\frac{2 - \delta}{r^2} \tilde{\psi}_1^- \tilde{\psi}_2^-,
\]

where

\[ \tau^\pm = \max \{ A_\pm c_\pm t^\pm, |B|c_\mp t^\mp \}. \]

Since \( A_+ - B^2 > 0 \), then there exist \( \Lambda > 0 \) such that, for \( r > R \)

\[
2A_+ U^2 |\tilde{\psi}_1^+|^2 + 4BU^+ U^- \tilde{\psi}_1^+ \tilde{\psi}_1^- + 2A_- U^2 |\tilde{\psi}_1^-|^2 \geq \Lambda (|\tilde{\psi}_1^+|^2 + |\tilde{\psi}_1^-|^2).
\]

This implies that, for given \( \delta, R \), when \( r > R \), we have

\[
I^+ (\tilde{\psi}) + I^- (\tilde{\psi}) \geq \frac{1 - 2\delta}{r^2} \left( |\tilde{\psi}^+|^2 + |\tilde{\psi}^-|^2 \right) - \frac{2 - \delta}{r^2} \left( |\tilde{\psi}_1^+|^2 + |\tilde{\psi}_1^-|^2 \right)
+ \Lambda \left( |\tilde{\psi}_1^+|^2 + |\tilde{\psi}_1^-|^2 \right)
\]

\[
\geq \frac{\varsigma_1}{r^2} |\tilde{\psi}^+|^2 + \frac{\varsigma_2}{r^2} |\tilde{\psi}^-|^2 \geq 0,
\]

(67)

with \( \varsigma_1, \varsigma_2 \) small and independent of \( \tilde{\psi} \). From (67), for given \( \delta \) small, we know that there exist \( R > 0 \) such that

\[
\int_R^\infty \left[ \left( |(\tilde{\psi}^+)|^2 + \frac{\delta}{r^2} (\tilde{\psi}_1^+ - \tilde{\psi}_2^-)^2 \right) \right] r \, dr
+ \int_R^\infty \left[ A_+ (U^2 - t^2) + B(U^2 - t^2) \right] |\tilde{\psi}^+|^2 r \, dr
+ \int_R^\infty \left[ A_- (U^2 - t^2) + B(U^2 - t^2) \right] |\tilde{\psi}^-|^2 r \, dr
+ \int_R^\infty \left[ 2A_+ U^2 |\tilde{\psi}_1^+|^2 + 4BU^+ U^- \tilde{\psi}_1^+ \tilde{\psi}_1^- \right] r \, dr
\]

\[
\geq \int_R^\infty \left[ \left( |(\tilde{\psi}^+)|^2 + \frac{\delta}{r^2} (\tilde{\psi}_1^+ - \tilde{\psi}_2^-)^2 \right) \right] r \, dr
+ \int_R^\infty \left[ \left( |(\tilde{\psi}^-)|^2 + \frac{\delta}{r^2} (\tilde{\psi}_1^- - \tilde{\psi}_2^-)^2 \right) \right] r \, dr
+ \frac{\delta}{2\tau^+} \int_R^\infty \left[ A_+ (t^2 - U^2) + B(U^2 - t^2) \right] |\tilde{\psi}^+|^2 r \, dr
+ \frac{\delta}{2\tau^-} \int_R^\infty \left[ A_- (t^2 - U^2) + B(U^2 - t^2) \right] |\tilde{\psi}^-|^2 r \, dr.
\]

Then we conclude

\[
\mathcal{R}(\tilde{\psi}, \hat{\psi}) \geq \int_R^\infty \left[ \left( |(\tilde{\psi}^+)|^2 + \frac{\delta}{r^2} (\tilde{\psi}_1^+ - \tilde{\psi}_2^-)^2 \right) \right] r \, dr
\]
\[ + \int_{R}^{\infty} \left[ \left| (\tilde{\psi}^-)' \right|^2 + \frac{\delta}{r^2} (\tilde{\psi}_1^- - \tilde{\psi}_2^-)^2 \right] r \, dr \]
\[ + \frac{\delta}{2T} \int_{R}^{\infty} \left[ A_+ (t^+ - U^2) + B(U^{-2} - t^{-2}) \right] |\tilde{\psi}^+|^2 r \, dr \]
\[ + \frac{\delta}{2T} \int_{R}^{\infty} \left[ A_- (t^{-2} - U^{-2}) + B(U^2 + t^2) \right] |\tilde{\psi}^-|^2 r \, dr \]
\[ + \int_{0}^{R} \left[ \left| (\tilde{\psi}^+)' \right|^2 + \frac{2}{r^2} (\tilde{\psi}_1^+ - \tilde{\psi}_2^+)^2 \right] r \, dr \]
\[ + \int_{0}^{R} \left[ \left| (\tilde{\psi}^-)' \right|^2 + \frac{2}{r^2} (\tilde{\psi}_1^- - \tilde{\psi}_2^-)^2 \right] r \, dr \]
\[ + \int_{0}^{R} \left[ A_+ (U^2 + t^2) + B(U^{-2} + t^{-2}) \right] |\tilde{\psi}^+|^2 r \, dr \]
\[ + \int_{0}^{R} \left[ A_- (U^{-2} + t^{-2}) + B(U^2 + t^2) \right] |\tilde{\psi}^-|^2 r \, dr \]
\[ + \int_{0}^{R} \left[ 2A_+ U^+ |\tilde{\psi}^+_1|^2 + 2A_- U^{-2} |\tilde{\psi}^-_1|^2 + 4BU^+ U^{-2} \tilde{\psi}^-_1 \tilde{\psi}^+_1 \right] r \, dr \]
\[ \geq C_1 \|\tilde{\psi}\|^2_{H_*} - C_2 \int_{0}^{R} \left( |\tilde{\psi}^+|^2 + |\tilde{\psi}^-|^2 \right) r \, dr, \quad (68) \]
for some positive constants \( C_1, C_2 \).

Now we prove Lemma 3.1 by contradiction. Suppose that there exists a sequence of functions \( \tilde{\psi}_l = (\tilde{\psi}_1^l, \tilde{\psi}_2^l) \) with \( \|\tilde{\psi}_l\|_{H_*} = 1, l \in \mathbb{N}^+ \) such that
\[ \langle \tilde{\psi}_l, Z_0 \rangle = 0, \quad \forall l \in \mathbb{N}^+, \quad \text{and} \quad \mathcal{B}(\tilde{\psi}_l, \tilde{\psi}_l) \to 0 \quad \text{as} \quad l \to +\infty. \]
Let \( \hat{\psi} \) be the weak limit of \( \tilde{\psi}_l \) in the sense of \( \| \cdot \|_{H_*} \). We claim that \( \hat{\psi} \neq 0 \). Indeed, \( \tilde{\psi}_l \to \hat{\psi} \) locally in \( L^2 \) sense by compactly embedding theorem. Hence, if \( \hat{\psi} \equiv 0 \), we would have
\[ \int_{0}^{R} \left( |\tilde{\psi}_1^l|^2 + |\tilde{\psi}_2^l|^2 \right) r \, dr \to 0. \]
The estimate (68) implies that \( \|\tilde{\psi}_l\|_{H_*} \to 0 \), which is impossible, thus \( \hat{\psi} \equiv 0 \) does not hold. Strong \( L^2 \) convergence over compacts and weak semi-continuity of \( L^2 \)-norm imply that
\[ \mathcal{B}(\hat{\psi}, \hat{\psi}) = 0. \]
Then we have \( \hat{\psi} = CZ_0 \). Since \( \{ \tilde{\psi}_l \}_{l=1}^{\infty} \) are uniformly bounded in \( \| \cdot \|_{H_*} \) norm, then we conclude the weak convergence in \( \| \cdot \|_{H_*} \) norm, which implies that
\[ \langle \tilde{\psi}_l, Z_0 \rangle \to \langle \hat{\psi}, Z_0 \rangle = 0 \quad \text{as} \quad l \to +\infty, \]
so \( C = 0 \). That is a contradiction, so we prove the lemma.

Define the closed subspace \( H_0 \) of \( H_* \) in the form
\[ H_0 = \{ \tilde{\psi} \in H_* : \langle \tilde{\psi}, Z_0 \rangle = 0 \}. \]
Using Lemma 3.1, we can know that the functional \( J_1 \) is continuous, coercive, and convex. Thus there exists a minimizer \( \bar{\psi} \) of the energy functional \( J_1 \) in \( H_0 \). Indeed, by using orthogonal projection onto the closed subspace \( H_0 \), together with the orthogonal condition (64), we can know that \( \bar{\psi} \) is also the minimizer of \( J_1 \) in \( H_* \).
We omit the details here and the readers can refer to the proof of Theorem 1.2 in [10]. It is straightforward to check that the inherited solution \( \psi_1^1 \) of \( L(\psi_1^1) = h_1^1 \) indeed satisfies
\[
\|\psi_1^1\|_H^2 \leq C \int_{\mathbb{R}^2} |h|^2 (1 + r^{2+\sigma}). \tag{69}
\]

In a similar way, we can find a solution \( \psi_2^1 \) to the equation \( L(\psi_2^1) = h_2^1 \) with analogous estimate. This solves the equation (58) for \( j = 1 \).

**Case 2:** modes \( j \geq 2 \). For the case \( j \geq 2 \), we define the closed subspace \( \mathcal{H}^\perp \) of all functions \( \psi \perp \in \mathcal{H} \) that can be written in the form of
\[
\psi \perp = \sum_{j \geq 2} \psi_j^1 + \sum_{j \geq 2} \psi_j^2,
\]
where \( \psi_j^1, \psi_j^2, \forall j \geq 2 \) are given in (51). For a given function
\[
h \perp = \sum_{j \geq 2} h_j^1 + \sum_{j \geq 2} h_j^2,
\]
where \( h_j^1, h_j^2, \forall j \geq 2 \) are given in (56), we now consider the equation
\[
L(\psi \perp) = h \perp. \tag{70}
\]
In order to solve the equation (70), we need to prove the coercivity of the quadratic form
\[
B(\psi \perp, \psi \perp) \geq C\|\psi \perp\|_H^2, \tag{71}
\]
for some constant \( C > 0 \). We will adopt relevant techniques from the work [10] to prove (71).

Recalling the decomposition in (51)-(53), by accurate calculations of all terms in \( B(\psi, \psi) \), we can get a similar result as (1.16) in [10], i.e.
\[
B(\psi, \psi) = B(\psi_0, \psi_0) + \sum_{j=1}^\infty B(\psi_j^1, \psi_j^1) + \sum_{j=1}^\infty B(\psi_j^2, \psi_j^2). \tag{72}
\]
For later use, we pause here to transform the quadratic form \( B(\psi_j^\ell, \psi_j^\ell) \) into an equivalent quadratic form \( B_j^\ell(\varphi_j^\ell, \varphi_j^\ell) \) in (76), which is defined on a space involving radial functions of real vector values. This will be fulfilled in two steps.

**Step 1.** We define \((\varphi^+, \varphi^-)\) by the relation
\[
\psi = (\psi^+, \psi^-) = (iw^+ \varphi^+, iw^- \varphi^-),
\]
where \( w = (w^+, w^-) \) is the radially symmetric vortex solution with degree pair \((n_+, n_-) = (1, 1)\), see (7). Then we introduce the functional for \((\varphi^+, \varphi^-)\)
\[
\mathcal{M}(\varphi, \varphi) = B(iw^+ \varphi^+, iw^- \varphi^-)
\]
\[
= \int_{\mathbb{R}^2} \left[ U^+ |\nabla \varphi^+|^2 + U^- |\nabla \varphi^-|^2 \right]
- 2\text{Re} \int_{\mathbb{R}^2} \frac{1}{r^2} \left[ iU^+ \varphi^+ \frac{\partial \varphi^+}{\partial \theta} + iU^- \varphi^- \frac{\partial \varphi^-}{\partial \theta} \right]
+ \int_{\mathbb{R}^2} \left[ 2A_+ U^+ |\varphi_2^+|^2 + 2A_- U^- |\varphi_2^-|^2 \right]
\]
where we have used the convention
\[\varphi = (\varphi^+, \varphi^-) = (\varphi_1^+ + i\varphi_2^+, \varphi_1^- + i\varphi_2^-).\]

Note that the form for \(M\) is much more simple than \(B\). In fact, the result in (73) is similar to the result in [10] which in the case the quadratic form is defined for complex-valued scalar functions. We here omit the details of the proof for concise.

We now make a decomposition of \(M(\varphi, \varphi)\). By the relations
\[\psi_0 = (iw^+\varphi_0^+, iw^-\varphi_0^-), \quad \psi_j^\ell = (iw^+\varphi_j^+, iw^-\varphi_j^-), \quad \ell = 1, 2, j \in \mathbb{N}^+, \]
where \(\psi_0, \psi_j^\ell, \ell = 1, 2, j \in \mathbb{N}^+\), are the functions in (52)-(53), naturally, we have
\[\varphi = \varphi_0 + \sum_{j=1}^{\infty} \varphi_j^1 + \sum_{j=1}^{\infty} \varphi_j^2,\]
where
\[\varphi_0 = (\varphi_0^+, \varphi_0^-), \quad \varphi_j^\ell = (\varphi_j^+, \varphi_j^-), \quad \ell = 1, 2, j \in \mathbb{N}^+.\]

Using (72), the decomposition is
\[M(\varphi, \varphi) = M(\varphi_0, \varphi_0) + \sum_{j=1}^{\infty} M(\varphi_j^1, \varphi_j^1) + \sum_{j=1}^{\infty} M(\varphi_j^2, \varphi_j^2).\]

**Step 2.** To proceed, let us set
\[\varphi_0^\pm = \varphi_0^1 \pm i\varphi_0^2,\]
\[\varphi_j^\pm = \varphi_j^1 \pm \cos j\theta + i\varphi_j^2 \pm \sin j\theta,\]
\[\varphi_j^2 = \varphi_j^1 \pm \sin j\theta + i\varphi_j^2 \pm \cos j\theta,\]
where
\[\varphi_0 = (\varphi_0^+, \varphi_0^-), \quad \varphi_j^\ell = (\varphi_j^+, \varphi_j^-), \quad \ell = 1, 2, j \in \mathbb{N}^+,\]
are defined in (74). We then define the real vectors \(\varphi_j^\ell\) as follows
\[\varphi_j^\ell = (\varphi_j^+, \varphi_j^-) = \left( (\varphi_j^1 + \varphi_j^2), (\varphi_j^1 + \varphi_j^2) \right). \quad (75)\]

According to (73), we consider the quadratic forms \(B_j^\ell(\varphi^+, \varphi^-)\) for \(j \in \mathbb{N}^+\) and \(\ell = 1, 2,\)
\[B_j^\ell(\varphi^+, \varphi^-)\]
\[= \int_0^\infty \left[ U^{+2} |\varphi^{+\prime}|^2 + U^{-2} |\varphi^{-\prime}|^2 + U^{+2} M_j^{\ell+} \varphi^{++} + U^{-2} M_j^{\ell-} \varphi^{--} \right] r dr \]
\[+ \int_0^\infty \left[ 2A_+ U^{+4} \varphi_2^{+2} + 2A_- U^{-4} \varphi_2^{-2} + 4BU^{+2} U^{-2} \varphi_2^{+2} \varphi_2^{-2} \right] r dr,\]
where the vector
\[\varphi^+ = (\varphi_1^+, \varphi_2^+), (\varphi_1^-, \varphi_2^-)) : [0, \infty) \to \mathbb{R}^2 \times \mathbb{R}^2,\]
and
\[M_j^{\ell} = \frac{1}{\ell^2} \begin{bmatrix} j^2 & (-1)^{\ell+1} 2j \\ (-1)^{\ell+1} 2j & j^2 \end{bmatrix}.\]
It is easy to check that
\[ B(\psi_j^\ell, \psi_j) = M(\varphi_j^\ell, \varphi_j) = \pi B_j^\ell(\varphi_j^\ell, \varphi_j^\ell), \tag{76} \]

We now turn to the proof for (71). Note that for \( j \geq 2, \ell = 1, 2 \), there hold
\[
(M_j^\ell - M_1^\ell)\varphi_j^\pm \cdot \varphi_j^\pm = \frac{j - 1}{r^2} \left[ j + 1 - (1)^{\ell+1} \right] \left( -1 \right)^{\ell+1} \frac{j + 1}{j - 1} \varphi_j^\pm \cdot \varphi_j^\pm
\]
\[
\geq \frac{(j - 1)^2}{r^2} |\varphi_j^\pm|^2.
\]
Therefore
\[
B_j^\ell(\varphi_j^\ell, \varphi_j) - B_j^\ell(\varphi_1^\ell, \varphi_1^\ell)
\]
\[
= \int_0^\infty \left[ U^{\ell+2} (M_j^\ell - M_1^\ell)\varphi_j^\ell \cdot \varphi_j^\ell + U^{-2} (M_j^\ell - M_1^\ell)\varphi_j^\ell \cdot \varphi_j^\ell \right] \, dr
\]
\[
\geq \int_0^\infty \frac{(j - 1)^2}{r^2} \left( U^{\ell+2} |\varphi_j^\ell|^2 + U^{-2} |\varphi_j^\ell|^2 \right) \, dr.
\]
The positivity of \( B_j^\ell(\varphi_j^\ell, \varphi_j^\ell) \) directly coming from Theorem 1.1 implies that
\[
B_j^\ell(\varphi_j^\ell, \varphi_j) \geq \int_0^\infty \frac{(j - 1)^2}{r^2} \left( U^{\ell+2} |\varphi_j^\ell|^2 + U^{-2} |\varphi_j^\ell|^2 \right) \, dr. \tag{77}
\]
Together with (72), (76), (77), we can get
\[
B(\psi^\pm, \psi^\pm) = \sum_{j \geq 2}^\infty B(\psi_j^1, \psi_j^1) + \sum_{j \geq 2}^\infty B(\psi_j^2, \psi_j^2)
\]
\[
= \sum_{j \geq 2}^\infty M(\varphi_j^1, \varphi_j^1) + \sum_{j \geq 2}^\infty M(\varphi_j^2, \varphi_j^2)
\]
\[
= \pi \sum_{j \geq 2}^\infty B_j^1(\varphi_j^1, \varphi_j^1) + \pi \sum_{j \geq 2}^\infty B_j^2(\varphi_j^2, \varphi_j^2)
\]
\[
\geq \pi \sum_{j \geq 2} \sum_{\ell = 1}^\infty \sum_{j \geq 2} \int_0^\infty \frac{(j - 1)^2}{r^2} \left[ |\varphi_j^\ell |^2 U^{\ell+2} + |\varphi_j^\ell |^2 U^{-2} \right]
\]
\[
\geq c \int_{R^2} \frac{|\psi^\pm|^2}{r^2} \geq c \int_{R^2} \left( A+ (t^2 - U^2) \right) |\psi^\pm|^2 + \left[ A- (t^{-2} - U^{-2}) \right] |\psi^\pm|^2,
\]
where \( \psi_j^\ell, \varphi_j^\ell, \varphi_j^\ell \) are defined in (52), (53), (74), (75) and we used the convention \( \psi^\pm = (\psi^\pm, \psi^\pm) \). From above result, we can get
\[
\int_{R^2} |\nabla \psi|^2 + \int_{R^2} B(U^{-2} - t^{-2}) |\psi^\pm|^2 + \int_{R^2} B(U^{\ell+2} - t^{\ell+2}) |\psi^\pm|^2
\]
\[
+ \int_{R^2} \left( 2A+ |Re(w^\pm |\psi^\pm)|^2 + 2A- |Re(w^- |\psi^-)|^2 \right)
\]
+ \int_{\mathbb{R}^2} 4B \text{Re}(\psi^+ \overline{w^+}) \text{Re}(\psi^- \overline{w^-}) \\
\geq \frac{(1+c)}{1-c} \int_{\mathbb{R}^2} \left( \left| A_+(t^2 - U^2) \right| |\psi^+|^2 + \left| A_-(t^2 - U^2) \right| |\psi^-|^2 \right), \quad (78)

where \( c > 0 \) is small. Then we can easily get the coercivity of the quadratic form \( B(\psi^+, \psi^-) \), see (71).

Since the functional is continuous, coercive, and strictly convex in \( \mathcal{H}^\perp \), it is easy to conclude that for \( h = h^\perp \) in (50) the functional \( J(\psi) \) has a minimizer \( \psi^\perp \) in \( \mathcal{H}^\perp \).

Similar to the estimate (69), we can get
\[
||\psi^\perp||^2_H \leq C \int_{\mathbb{R}^2} |h|^2 (1 + r^{2+\sigma}). \quad (79)
\]

**Case 3: mode \( j = 0 \).** In this part, we will solve the equation (57). Setting \( \chi = (\chi^+, \chi^-) \), \( h = (h^+, h^-) \) by the relations
\[
\psi_0^\pm = iw^\pm \chi_0^\pm = iw^\pm (\chi_1^\pm + i\chi_2^\pm), \quad h_0^\pm = iw^\pm h_0^\pm = iw^\pm (b_1^\pm + ib_2^\pm),
\]
we can get an ODE system from the equation \( L(\phi_0) = h_0 \),
\[
\chi_1^+ + \left( \frac{2U^+}{U^+} + \frac{1}{r} \right) \chi_1^+ = b_1^+, \quad (80)
\]
\[
\chi_1^- + \left( \frac{2U^-}{U^-} + \frac{1}{r} \right) \chi_1^- = b_1^-, \quad (81)
\]
\[
\chi_2^+ + \left( \frac{2U^+}{U^+} + \frac{1}{r} \right) \chi_2^+ - 2A_+ U^+ \chi_2^+ - 2BU^+ \chi_2^- = b_2^+, \quad (82)
\]
\[
\chi_2^- + \left( \frac{2U^-}{U^-} + \frac{1}{r} \right) \chi_2^- - 2A_- U^- \chi_2^- - 2BU^+ \chi_2^+ = b_2^-. \quad (83)
\]

We can get the properties by using the assumptions (19) and (20) made on \( h \)
\[
\int_0^\infty r U^\pm h_1^\pm 0, \quad \int_0^\infty r U^\pm |h_1^\pm|^2 (1 + r^{2+\sigma}) \, dr \leq C \int_{\mathbb{R}^2} |h|^2 (1 + r^{2+\sigma}). \quad (84)
\]

Using the variation of parameters, we get the solutions of (80)-(81)
\[
\chi_1^+ = - \int_r^\infty \frac{1}{sU^\pm(s)^2} ds \int_0^s b_1^+(t)U^\pm(t)^2 \, dt dt,
\]
and
\[
\chi_1^+ = \frac{1}{rU^\pm(r)^2} \int_0^r b_1^+(t)U^\pm(t)^2 \, dt dt.
\]

Then we can deduce that
\[
\chi_1^+ \sim b_1^+(r) r^2, \quad \text{as } r \to +\infty \text{ or } r \to 0,
\]
\[
\chi_1^+ \sim b_1^+(r) r, \quad \text{as } r \to +\infty \text{ or } r \to 0.
\]

Then we can get the estimate
\[
\int_0^\infty \left( |\chi_1^+(r)|^2 + \left[ A_+ (t^\pm - U^\pm) - B(t^\mp - U^\mp) \right] \chi_1^2 \right) U^\pm(r)^2 r \, dr
\]
\[ \leq C \int_0^\infty r U^{p+2} |b_1^+|^2 (1 + r^{2+p}) \, dr \leq C \int_{\mathbb{R}^2} |h|^2 (1 + r^{2+p}). \]

On the other hand, we can solve equations (82) and (83) by finding a minimizer of the functional
\[ J_2(\chi_2^+, \chi_2^-) = \frac{1}{2} \int_0^\infty \left[ |\chi_2^+(r)|^2 U^{-2} + |\chi_2^-(r)|^2 U^{-2} + 2A_+ U^4 \chi_2^+ \chi_2^- + 2A_- U^{-4} \chi_2^- \right] r \, dr \]
\[ + \int_0^\infty \left[ 4BU^{p+2} |\chi_2^+|^2 U^{-2} + b_2^+ \chi_2^+ U^2 + b_2^- \chi_2^- U^{-2} \right] r \, dr. \]

With these works, we obtain the estimate for \( \psi_0 \)
\[ \|\psi_0\|_{H^2}^2 \leq C \int_{\mathbb{R}^2} |h|^2 (1 + r^{2+p}). \] (85)

As a conclusion, together with Steps 1-3 and the estimates (69), (79), (85), we can obtain a solution \( \psi \) of (18) with the required properties. The fact that the solutions of (18) can be written as in (21) is a direct corollary of Theorem 1.1. This completes the proof of Theorem 1.2.

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