Determination of the Boltzmann constant by the equipartition theorem for capacitors

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Received 11 October 2018, revised 5 January 2019
Accepted for publication 18 February 2019
Published 8 April 2019

Abstract

A new experimental set-up for measurement of the Boltzmann constant is described. The statistically averaged square of voltage $\langle U^2 \rangle$ is measured for different capacitances $C$. The Boltzmann constant is determined by the equipartition theorem $C \langle U^2 \rangle = k_B T$. For fixed capacitance, voltages could be measured for different temperatures. The set-up consists of low-noise, high-frequency operational amplifiers ADA4898-2. An instrumental amplifier is followed by an inverting amplifier, the square of the voltage is created by an analog multiplier AD633, and finally, the averaged signal is measured by a multimeter. More than ten high-school students were able to measure the Boltzmann constant with the experimental set-up in the 5th Experimental Physics Olympiad with excellent accuracy compared to the price, conditions and available time for the experiment. A new derivation of the important statistical
physics theorems by Nyquist and Callen–Welton is given in an appendix at the level of introductory courses in physics studied by future teachers. To understand the work of the experimental set-up, it is only necessary to know the equipartition theorem.

Keywords: Boltzmann constant, equipartition theorem, voltage thermal fluctuations of a capacitor

(Some figures may appear in colour only in the online journal)

1. Introduction

The equipartition theorem [1, 2] describing the relation between the mean thermal energy of a quadratic degree of freedom
\[ \frac{1}{2} \langle m v_x^2 \rangle = \frac{1}{2} \langle k x^2 \rangle = \frac{1}{2} \langle \hat{I} \omega^2 \rangle = \frac{1}{2} \langle C U^2 \rangle = \cdots = \frac{1}{2} k_B T \]

is one of the first quantitative results of statistical physics. The coefficient \( k_B \) in front of the temperature is the Boltzmann constant. The physical nature of the variables is irrelevant: \( m \) can be the mass of a molecule and \( v_x \) is the \( x \)-component of the velocity, \( k \) can be the elastic constant of a spring and \( x \) can be the deformation, for a torsion magnetometer \( x \) could be the angle, \( \hat{I} \) can be the moment of inertia of a molecule and \( \omega \) is the angular velocity. It is not a thoughtcrime to denote mass by \( C \) and velocity by \( U \). Now the equipartition theorem is included in all high-school textbooks. Perhaps the most famous example is the mean kinetic energy of a single atom:
\[ \frac{1}{2} \langle m v^2 \rangle = \frac{3}{2} k_B T, \quad v^2 = v_x^2 + v_y^2 + v_z^2. \] (1)

Here, it is implicitly assumed that \( \langle v \rangle = 0 \); no wind in the room. Analogously, the mean voltage of the connected resistor and capacitor shown in figure 1 is zero \( \langle U \rangle = 0 \).

At the beginning of the development of statistical physics Albert Einstein suggested [3] that the Boltzmann constant \( k_B \) could be determined using the equipartition theorem for the thermally averaged square of the voltage \( \langle U^2 \rangle \) of a capacitor \( C \):
\[ C \langle U^2 \rangle = k_B T. \] (2)

The first attempt at the realization of this idea was made by the Habicht brothers [4] in 1910, but unfortunately their electrostatic amplifier with mechanically rotating parts suffered from floating off the zero. It is strange that for more than a century this simple experiment has not yet been performed. Up to now, only Johnson and Schottky spectral noise experiments are conducted in student laboratories [5–9].

Imagine that we have a set of capacitors with capacities given in table 1 with \( V_0 \propto \langle U^2 \rangle \).

According to the equipartition theorem at a temperature of 24 °C for 6.89 nF and 132 nF we have, for the voltage fluctuations \( \delta U = \sqrt{\langle U^2 \rangle} = \sqrt{k_B T/C} \), 771 nV and 176 nV, correspondingly. If these voltages are amplified 1 million times, 771 mV and 176 mV can be measured, even using cheap multimeters. After several time constants \( \tau_{nc} = RC \), the random voltage \( U(t) \) is already independent, and to measure the mean square
\[ \langle U^2 \rangle = \frac{1}{T} \int_0^T U^2(t) \, dt, \] (3)
it is necessary for the averaging time $T$ to significantly exceed $\tau_{nc}$; the accuracy is $\sqrt{\tau_{nc}/T} \ll 1$.

All those theorems are actually simple illustrations of the kinetic principle of the detailed balance applied to the most simple physical system—the harmonic oscillator—and can be considered as a comment to the Planck work [10] from the 19th century. To explain the spectral density of the black body radiation, Planck postulated the equidistant spectrum $E_n = h\omega n$; it was the simplest realization of the Boltzmann idea for the atomic structure of energy. In some sense, the discrete energy spectrum is associated with quantum mechanics, and it is interesting to trace the origin of quantum mechanics. At the DPG conference in Halle in 1891, Boltzmann presented his famous formula for the statistical interpretation of the entropy $S$ and the number of quantum states (in contemporary terminology) with one and the same energy. Asked provocatively ‘Do you consider that ...’, unexpectedly for the auditorium, Boltzmann replied ‘I do not exclude that energy can have atomic structure’ [11]. In such a way, the birthyear of quantum mechanics is 1891.

The manuscript by Waterston [2] was rejected from publication by the polite referee report by Sir John William Lubbock: ‘the paper is nothing but nonsense ...’ [12]. Even nowadays, mathematical physicists feel obliged to convince physicists that they did not lead Boltzmann to suicide. In such a way, the Einstein idea to determine the Boltzmann constant by the equipartition theorem is in a good company. The idea was not realized in the 20th century and had to wait for the appearance of low-noise, high-frequency operational amplifiers to be realized in student education. The building of an experimental set-up for determination of the Boltzmann constant from scratch was a modus for students at the University of Sofia to pass the exam in statistical physics.
The purpose of the present work is to present a simple self-made set-up for determination of the Boltzmann constant in high-school and university laboratories. This is a good methodological illustration of the equipartition theorem. Firstly, the signal should be amplified a million times $Y \approx 10^6$ using low-noise, high-frequency operational amplifiers. The problem of floating off the zero is solved with large fast capacitors sequentially connected to the gain resistors of the circuit, depicted in figure 2. Later on the signal has to be squared by an analog multiplier, and this squared signal has to be averaged by a low-pass filter. Finally, the time-averaged signal $\langle V \rangle$ is measured by an ordinary multimeter. The simplicity of the experiment and set-up, as well as the availability of the electronic elements nowadays, mean that this experimental set-up is suitable for high-school physics labs. In such a way, our ultimate goal is to overturn the deteriorating trend in physics and engineering education worldwide, and this experimental set-up is a valuable tool in this endeavor. In short, the set-up can be used by students to measure a fundamental constant $k_B$, teachers can give explanations not only for the physical principles, but also to explain how the set-up works. An enthusiastic teacher, together with his or her students, can make the set-up in one day.

2. Theory of the experimental set-up

As already stated in the previous section, the voltage fluctuations of nF capacitors at room temperature are of the order of a hundred nV. These nV should be amplified $10^6$ times and squared after this to be able to measure them with relatively cheap and commercially available multimeters. The circuit of the experimental set-up performing these actions is shown in figure 2. The capacitor $C$, whose voltage fluctuations are to be measured, is connected parallel
to a resistor $R$. This parallel circuit is connected to the two inputs of a buffer with amplification

$$y_1 = 1 + 2 \frac{R_F}{r_G},$$

which is the first step of the amplifier. Next an MKS2 WIMA-type capacitor $C_G$ is connected to each of the two outputs of the buffer to stop the voltage offset of the low-noise ADA4898-2 operational amplifiers. The second step of the amplifier, which is a difference amplifier with amplification

$$y_2 = - \frac{R'_F}{R_G},$$

is connected to both $C_G$ capacitors, forming an instrumental amplifier together with the buffer. The output of the difference amplifier is connected again to an MKS2 WIMA $C_G$ capacitor for the same reason—to stop the voltage offset—and after $C_G$, the third and last step, which is an inverting amplifier with amplification

$$y_3 = - \frac{R'_F}{R_G},$$

is connected. Finally, the total amplification of the amplifier

$$Y = y_1y_2y_3 = \left(1 + 2 \frac{R_F}{r_G}\right)\left(\frac{R'_F}{R_G}\right)^2.$$

The $Y$ times amplified signal is squared by an AD633 analog multiplier by connecting the amplifier output to both inputs of the multiplier. A voltage divider is connected after the amplifier according to the manufacturer’s instructions \[14,15\], and, finally, there is an averaging low-pass filter consisting of a resistor $R_{av}$ and a WIMA MKS2-type capacitor $C_{av}$. The values of all passive elements and voltage supplies for the ADA4898-2 operational amplifiers and AD633 analog multiplier are given in table 2.

### Table 2. A table of the numerical values of the circuit elements from figure 2.

| Circuit element | Value      |
|-----------------|------------|
| $R$             | 510 $\Omega$ |
| $r_G$           | 20 $\Omega$ |
| $R_F$           | 1 k$\Omega$ |
| $C_F$           | 10 pF      |
| $C_G$           | 10 $\mu$F  |
| $R_G$           | 100 $\Omega$ |
| $R'_F$          | 10 k$\Omega$ |
| $C'_F$          | 10 pF      |
| $R_1$           | 2 k$\Omega$ |
| $R_2$           | 18 k$\Omega$ |
| $R_{av}$        | 1.5 M$\Omega$ |
| $C_{av}$        | 10 $\mu$F  |
| $R_V$           | 1 M$\Omega$ \[17\] |
| $V_{CC}$        | +9 V \[14,15\] |
| $V_{EE}$        | −9 V \[14,15\] |
Lastly, the resistor $R_{av}$ forms another voltage divider with the internal resistance of the used DT-830B multimeter $R_V$ [17]. For small values of $R_{av}$ this voltage division is negligible, but in our case $R_{av}$ and $R_V$ are in the same order of magnitude, and we have to take it into account in the final equation for the amplification.

The time-dependent thermal fluctuations of the voltage $U(t)$ of the parallelly connected $C$ and $R$ are amplified $Y$ times:

$$U_{\text{amp}} = YU.$$  
(8)

Then, the amplified voltage $U_{\text{amp}}$ is squared by an analog multiplier. We use the circuit depicted in [14], figure 17, where $X2 = Y2 = 0$, $X1 = Y1 = U_{\text{amp}}(t)$ and $W \equiv U_{\text{m}}$, i.e. we have

$$U_2 = \frac{U_{\text{amp}}^2 R_1 + R_2}{U_{\text{m}} R_1},$$  
(9)

where the constant factor for this multiplies AD633 [14] $U_{\text{m}} = 10V$. Actually we have a voltage divider for which $Z = WR_2/(R_1 + R_2)$ and $S = 0$. The squared voltage is averaged by an averaging low-pass filter with a large time constant $\tau_{av} = R_{av} C_{av}$ of the order of a quarter of a minute. The large $R_{av}$ resistance is comparable with the internal resistance of the voltmeter $R_V$, and for the time-averaged DC voltage shown by the voltmeter we have to take into account a second voltage divider

$$V_2 = \langle U_2 \rangle \frac{R_V}{R_V + R_{av}},$$  
(10)

while $\langle V_i(t) \rangle = 0$ the fluctuations of the electric noise create a non-zero square. Combining together the amplifying equation (8), squaring equation (9) and averaging equation (10) we arrive at the transfer function of the amplifier

$$V_2 = \frac{Y^2 R_1 + R_2}{U_m R_1} \frac{R_V}{R_V + R_{av}} \langle U^2(t) \rangle,$$  
(11)

where the brackets denote time averaging at steady spectral density of the noise. This can also be written in the form

$$\langle U^2 \rangle = U^* V_2,$$  
(12)

where the time-averaged square of the investigated voltage $\langle U^2 \rangle$ and the constant voltage measured by a voltmeter $V$ are related by a constant with the voltage dimension

$$\frac{1}{U^*} \equiv \frac{Y^2 R_1 + R_2}{U_m R_1} \frac{R_V}{R_V + R_{av}}.$$  
(13)

If we substitute here the expressions for $Y$ from equations (7), (4), (5) and (6) we obtain the final expression for the voltage constant describing our set-up

$$\frac{1}{U^*} = \frac{1}{U_m} \left(1 + 2 \frac{R_E}{R_c} \right)^2 \left( \frac{R_E}{R_G} \right)^2 \frac{R_1 + R_2}{R_1} \frac{R_V}{R_V + R_{av}}.$$  
(14)

As a whole, the circuit can be characterized as a low-noise pre-amplifier (an instrumental amplifier followed by an inverting amplifier, all of them based on ADA4898-2 [15]) and a sequential true root mean square (RMS) meter. If necessary, the cheap AD633 multiplier [14] (maximum total error 2% of full scale, small signal bandwidth 1 MHz) can be substituted with the more precise AD835 [18] (total error 0.1% of full scale, small signal bandwidth 250 MHz). But even in this case, the combination of a commercial low-noise pre-amplifier
and a true RMS-meter costs one or even two orders of magnitude more than our experimental set-up that can be easily reproduced in any school physics classroom.

Even more detailed derivation of all those formulae requires only Ohm’s law applied to a voltage divider, and a sequential chain of simple problems for high-school students is described in great detail in [19]: and it is remarkable that even a high-school student was able to solve the university electronic problem.

The substitution of \( U^2 \) from the equipartition theorem equation (2) into the property of the circuit we derived in equation (12) gives a linear dependence between the DC voltage \( V_2 \) and the reciprocal capacitance \( 1/C \):

\[
V_2 = q_0 \frac{1}{C} + v_0, \quad q_0 = \frac{2kT}{U^*}, \quad Z \approx 1.
\] (15)

The irrelevant for the experiment constant \( v_0 \) describes the internal noise of the circuit determined mainly by the voltage noise of the first dual ADA4898-2 amplifier in the buffer (double non-inverting amplifier with a virtual common point). For \( v_0 \) to be small, we need to use low-noise operational amplifiers.

The slope \( q_0 \) of the linear regression \( V_2 \) versus \( 1/C \)

\[
q_0 = \frac{dV_2}{dC} \bigg|_{\text{regr}}
\] (16)
determines the Boltzmann constant

\[
k_B = \frac{q_0 U^*}{2T}.
\] (17)

For high-school students the correction multiplier \( Z = 1 - \varepsilon \) is indistinguishable from one, but for university students at undergraduate level the several percent correction \( \varepsilon \) can be calculated by analyzing the frequency dependence of the amplifier, as described in great detail in appendix A.

### 3. Frequency-dependent considerations

Up to now, frequency dependence has been neglected. For high school, and even undergraduate level, this is perfectly acceptable, but for a detailed analysis of the amplifier the frequency dependence has to be included. This analysis is far more complicated and is performed in appendix A.

In this section we present a short analysis of time constants and frequencies of the presented experimental set-up.

The ADA4898-2 operational amplifier time constant \( \tau \) is calculated from the crossover frequency \( f_0 \) with \( \tau = 1/2\pi f_0 \) [15]. For amplification maximally close to the frequency-independent one (the considered scenario in the last section), \( \tau \ll \tau_c \), where \( \tau_c = R_C C_c \) is the time constant of the buffer feedback. In other words, the operational amplifier should not ‘feel’ that there is a capacitance in its feedback. The signal amplified by the buffer now has a higher time constant \( \tau_b = \tau b \) or \( f_b = f_0/\gamma_1 \) or lower frequency due to the lower amplification of higher frequencies by the operational amplifier and \( \tau_b \gg \tau_c \), because of the requirement for the feedback capacitance. After the buffer, this signal passes through a high-pass filter consisting of the resistor \( R_G \) and the capacitor \( C_G \) with a large time constant \( \tau_0 = R_G C_G \), whose function is to filter out the voltage offsets of the ADA4898-2 operational amplifiers of the buffer. For proper operation of the amplifier up to now, the time constants should be ordered:
There is another capacitor $C'$ in the feedback of the inverting amplifier (IA), and analogously to the buffer, its presence should be barely ‘felt’; therefore the time constant of the IA feedback $\tau = \tau_0 C_G$, $\tau_0 = R_C C_G$, $\tau_0 \ll \tau_{ac} \ll \tau_0 \ll \tau_{av}$.

Now let us return to the input signal, which is the time-dependent thermal fluctuations of the voltage of the parallelly connected capacitor $C$ and resistor $R$, whose time constant is $\tau_\lambda = RC$. This time constant should be much larger than $\tau_\lambda$ for a maximum amplification but much smaller than $\tau_\tau$ for a maximum transfer between the three steps of the whole amplifier. Therefore, the requirement for the optimal amplification of our experimental set-up is

$$\tau \ll \tau_\tau \ll \tau_\lambda \ll \tau_\tau.$$

There is another capacitor $C'$ in the feedback of the inverting amplifier (IA), and analogously to the buffer, its presence should be barely ‘felt’; therefore the time constant of the IA feedback $\tau' = R' C'_e < \tau_\lambda$. And, finally, the averaging low-pass filter should have the largest time constant $\tau_{av} = R_m C_m$, to reliably average the squared voltage.

Now let us return to the input signal, which is the time-dependent thermal fluctuations of the voltage of the parallelly connected capacitor $C$ and resistor $R$, whose time constant is $\tau_\lambda = RC$. This time constant should be much larger than $\tau_\lambda$ for a maximum amplification but much smaller than $\tau_\tau$ for a maximum transfer between the three steps of the whole amplifier. Therefore, the requirement for the optimal amplification of our experimental set-up is

$$\tau \ll \tau_\tau \ll \tau_\lambda \ll \tau_\tau.$$

A set of the calculated time constants, frequencies and additional calculated parameters of the experimental set-up is given in Table 3.

Why do we need capacitors? (1) Even during the first attempt at measuring the Boltzmann constant, the Habicht brothers noted that their amplifier suffered from floating off the zero [3, 4]. Even nowadays, there are no low-noise auto-zero operational amplifiers commercially available. To remove this unpleasant floating off the zero and the low-noise 1/f noise in frequencies over 100 Hz, large $C_G = 10 \mu F$ metallized polyester capacitors, whose price is comparable to those of the batteries and operational amplifiers, are put in the path of the signal. (2) The small ceramic $C_F$ and $C'_F$ connected parallel to the feedback resistors reduce the amplification in high frequencies ($> 1/R_C C_F$, $> 1/R'_C C'_F$), and in this way stabilise the amplification in terms of ringing.

In short, the removal of the offset and the ringing of the amplifier causes the addition of capacitors. If we additionally assume the operational amplifiers to be ideal $\tau = 0$, in the approximation $C_F = C'_F = 0$ and $C_G = \infty$, for the amplification of the amplifier we have the frequency-independent approximation $Y(\omega) \approx y_1 y_3 y_3$, for which we have put in a great deal of effort for it to be a very good approximation for engineering the amplifier. The infinite
capacitance $C_G = \infty$ indicates a short circuit and that the signal is going directly to the gain resistor, while $C_F = C'_F = 0$ indicates absence of the parallel connected capacitors.

4. Experiment

A photograph of the realized circuit on a PCB is shown in figure 3, 200 copies of which were produced and given to high-school students in the 5th Experimental Physics Olympiad [19] (EPO5). The set of the parameters from the circuit and the integral schemes is listed in table 2. The experimental data measured with an ordinary, cheap All-Sun DT-830B multimeter corresponding to equation (17) is presented in table 1 and graphically shown in figure 4. The obtained value of the slope $q_0 = (1718 \pm 18)$ pC and the achieved accuracy for determination of the Boltzmann constant is rather good:

$$k_n = (1.40 \pm 0.08) \times 10^{-23} \text{ J/K}$$

(20)

for a $50 set-up which can be further elaborated. The random thermal voltage can be observed with an oscilloscope, and the temperature $T$ of the resistor $R$ can be varied from the freezing to boiling point of the water, but we are only presenting the simplest experiment by only varying the capacitor at room temperature, which can be realized in every high school. It is intriguing to measure a fundamental constant using a set-up which can be created from scratch within a week by a novice.

5. Conclusions

Some administrators related to education in third world countries can conclude that very few high schools or introductory college courses would have the background, interest and/or
resources to adopt this experiment. However, the equipartition theorem has already been a part of high-school education for more than a century. For more than a century it has been taught that temperature is related to motion and fluctuations. The vacuum technique will remain expensive, and even at good universities Maxwell velocity distribution is not experimentally demonstrated. However, for the last century electronics have made significant progress and prices have reduced a thousand times. This is why the described set-up can be industrially produced for illustration of thermal fluctuations in high schools, even within the frameworks of existing school programs. But even without commercially available set-ups the experimental set-up contains three integral circuits: two operational amplifiers, a multiplier, three large, fast capacitors and 9 V batteries. Such a set-up can be performed within one day in every high school and complete the teaching of thermal fluctuations and electronics. The problem of measurement of the Boltzmann constant using the described experimental set-up was given in EPO5 [19] to more than 100 high-school students, and more than 10% of them were able to perform the experiment and to process their data to obtain a final answer; this was without any special training and only within the framework of standard education. We can certainly conclude that the whole experiment is appropriate for high-school physics laboratories all over the world, where in addition a hardware or software oscilloscope can be used to visualize the thermal noise amplified one million times. As a by-product of EPO5 the authors of the present article can send one set-up free of charge to the physics laboratory to the first 137 teachers who write to us. When the set-up of the PCB version is outsourced in China, measurement of the Boltzmann constant can reach any student from the 1st, 2nd and perhaps 3rd world. In short, we conclude that it is time to introduce simple experimental set-ups for high-school level physics education, for the determination of fundamental constants: not only for $k_B$, but also for electron charge $q_e$ [21], the speed of light $c$ [22] and Planck’s constant $\hbar$ [23]. Touching on fundamental theorems determines the thinking of the next generation.
Acknowledgments

The authors are grateful to Vasil Yordanov for his contribution at the early stages of the present research [24], to Alexander Petkov for making the first measurements, to Gary White for stimulating comments, to Nikolay Zografov for introducing order in the lab, to Andreana Andreeva for animation of the spirit in the lab, to Riste Popeski-Dimovski, Marina Poposka, Sladjana Nikolic, Slavoljub Mitic and Stojan Manolev for the invaluable help and unforgettable moments during EPO. One of the authors, AMV, is grateful to the ‘Young Scientists and Postdoctoral Students’ Bulgarian national program. Finally, the authors are thankful to Valentin Popov for the indispensable assistance in the founding of the Laboratory for Measurements of Fundamental Constants at the University of Sofia.

Appendix A. Analog electronics in a nutshell

This appendix is aimed at colleagues involved with the construction of similar devices and with modifying the scheme. This recall of the standard electronics notion is not aimed at students.

A.1. Operational amplifier master equation

In the beginning there was the approximate master equation of operational amplifiers [20]:

\[ G^{-1} U_0(t) = \left( \frac{1}{G_0} + \tau \frac{d}{dt} \right) U_0 = U_+ - U_-, \quad (A1) \]

giving the relation between the output voltage \( U_0(t) \) and the difference between the (+) and (−) inputs of the operational amplifier, cf [25]. For harmonic signals, introducing the j-imaginary unit

\[ U \propto e^{j \omega t} = e^{st}, \quad j = -i, \quad s \equiv j \omega, \quad (A2) \]

we have

\[ G^{-1}(\omega) U_0 = \left( \frac{1}{G_0} + j \omega \tau \right) U_0 = U_+ - U_-, \quad (A3) \]

\[ G^{-1} = \left( \frac{1}{G_0} + s \tau \right) \frac{1 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots}{1 + b_1 s + b_2 s^2 + b_3 s^3 + \ldots}, \quad (A4) \]

where we present a Padé approximant for the frequency-dependent open loop gain in the second row. For the operational amplifier ADA4898 amplifier used by us [15] the static open loop gain is approximately 100 dB:

\[ G_0 = 10^5, \quad f_0 \equiv \frac{1}{2 \pi \tau} = 65 \text{ MHz}, \quad (A5) \]

and the time \( \tau \) constant is parameterized by the crossover frequency \( f_0 \).

In the next subsections we will recall how the frequency-dependent open loop gain determines the frequency-dependent transmission function of different amplifiers.
A.2. Buffer

For the buffer, consisting of two non-inverting amplifiers (NIA) depicted in figure 5, the input voltages $U_1$ and $U_2$ are applied directly to the (+) inputs of each operational amplifier (OpAmp) $U_{\text{in}} = U_1$, and $U'_{\text{in}} = U_2$, here the primed notations refer to the lower NIA. Since no currents enter in the OpAmp, the only current $I$ starts from the output $U_0$, passes through the upper feedback $Z_F$, the gain resistor $r_G$, the lower feedback $Z_F$ and terminates in the output $U'_0$.

From this closed loop, it is straightforward to obtain an expression for the current

$$I = \frac{(U'_0 - U_0)}{2Z_F + r_G} = \frac{-\Delta U_0}{Z_F(\omega)} = \frac{1}{R_F} + j\omega C_F. \quad (A6)$$

The voltage difference between the (−) inputs of both OpAmps of the buffer is

$$U_+ - U'_+ = -ir_g = \Delta U_0 \frac{r_g}{2Z_F + r_g}. \quad (A7)$$

The master equations for both OpAmps are

$$U_0 G^{-1}(\omega) = U_1 - U_+ \quad (A8)$$

$$U'_0 G^{-1}(\omega) = U_2 - U'_+ \quad (A9)$$

and after subtracting them, we obtain a single equation:

$$\Delta U_0 G^{-1} = (U_1 - U_2) - (U_+ - U'_+). \quad (A10)$$
Substituting equation (A7) into the last equation, the frequency-dependent amplification of the buffer

$$\gamma_{\text{NIA}}(\omega) \equiv \frac{\Delta U_0}{U_1 - U_2} = \frac{1}{G^{-1}(\omega) + \gamma^{-1}(\omega)}, \quad (A11)$$

$$\gamma(\omega) \equiv \frac{Z_F(\omega)}{r_w} + 1 \quad (A12)$$

in agreement with [13], equation (4). Using complex numbers in programming we can simply calculate $|\gamma_{\text{NIA}}(\omega)|^2$; however, using only real numbers we have to apply some effort using complex algebra, and after a straightforward calculation from A11 we obtain

$$|\gamma_{\text{NIA}}(\omega)|^2 = \frac{N^2(\omega)}{|G_0^{-1} + (\gamma^{-1}_1 + \omega^2 \tau^{-2}_0)I_0^2 + (\omega \tau_0)^2\left[\frac{1}{I_0^2} N + I_0^2\right]|^2}, \quad (A13)$$

where

$$N(\omega) = 1 + \omega^2 \tau^{-2}_0, \quad I_0 \equiv 1 - \gamma^{-1}_1, \quad \tau_0 \equiv \frac{C_F R_F}{\gamma_1}. \quad (A14)$$

The calculation of the pass bandwidth requires the square of the modulus of the complex amplifications for all steps of the amplifier, which we are presenting in full detail.

A.3. Difference amplifier

The difference amplifier shown in figure 6 has a capacitor $C_G$ at each of its inputs. The current $I_F$ flows from the first input $U_0$ of the difference amplifier to the output $U_\Delta$ and

$$I_F = \frac{U_\Delta - U_0}{Z_G + R_F}, \quad Z_G(\omega) = R_G + \frac{1}{j\omega C_G}, \quad (A15)$$

while the current $I_0$ flows from the second input $U_0'$ to the common point and

$$I_0 = \frac{-U_0'}{Z_G + R_F}. \quad (A16)$$

The voltage drops at the inputs of the OpAmp:

$$U_\Delta = I_F Z_G + U_0 = \frac{U_\Delta - U_0}{Z_G + R_F} Z_G + U_0, \quad (A17)$$
\[ U_+ = I_0 G + U_0' = -\frac{U_0'}{Z_G + R_f'} Z_G + U_0'. \]  
(A18)

Subtracting the last two equations, we obtain an expression for the voltage difference of the OpAmp inputs:
\[ U_+ - U_- = -(U_0 - U_0') \left( 1 - \frac{Z_G}{Z_G + R_f'} \right) \frac{U_\Delta Z_G}{Z_G + R_f'}. \]  
(A19)

This expression is equal to \( G^{-1}(\omega) U_\Delta \) according to equation (A1) (here \( U_\Delta \) is the output voltage), and therefore, after a little rearrangement of the terms:
\[ U_\Delta \left( G^{-1} + \frac{Z_G}{Z_G + R_f'} \right) = -(U_0 - U_0') \frac{R_f'}{Z_G + R_f'}. \]  
(A20)

The frequency-dependent amplification of the difference amplifier is therefore
\[ \Upsilon_\Delta(\omega) \equiv \frac{U_\Delta}{U_0 - U_0'} = -\frac{1}{\Lambda(\omega) + G^{-1}(\omega)[1 + \Lambda(\omega)]}. \]  
(A21)

where \( \Lambda(\omega) \equiv Z_G(\omega)/R_f' \). The calculation of the pass bandwidth requires the square of the modulus of the complex amplification, which after a straightforward calculation from A21:
\[ |\Upsilon_\Delta(\omega)|^2 = \frac{(\omega \tau_G)^2}{[1 + G_0^{-1} - M \omega^2 \tau_G^2 (\lambda_0 + \Lambda_0)]^2}, \]
\[ \Lambda_0 \equiv \frac{R_G}{R_f'}, \quad M \equiv 1 + \Lambda_0, \quad \tau_G \equiv C_G R_f'. \]  
(A22)

This complicated expression is necessary if we wish to use real numbers in the numerical integration, which needs to be preformed for the pass bandwidth. The last step of the amplifier is the inverting step described in the next subsection.

**A.4. Inverting amplifier**

Setting \( U_0' = 0 \) in the difference amplifier from the last subsection, we obtain an IA. In our case there is one more difference, which is a capacitor \( C_F \) connected parallel to the feedback resistor \( R_f' \), and therefore, the reciprocal of the impedance feedback
\[ \frac{1}{Z_f'(\omega)} = \frac{1}{R_f'} + j \omega C_F. \]  
(A23)

Re-denoting \( \Lambda(\omega) \) to \( \Gamma(\omega) \equiv Z_G(\omega)/Z_f' \), the frequency-dependent amplification of the IA
\[ \Upsilon_{\Lambda}(\omega) = \frac{1}{\Gamma(\omega) + G^{-1}(\omega)[1 + \Gamma(\omega)]} \]  
(A24)

in agreement with [13], equation (7). This agreement provides implicit proof of the applicability of the time-dependent equation (A1) and its frequency transformation equation (A3) for circuits with low-noise operational amplifiers ADA4898, ADA4817, AD711, etc. The calculation of the pass bandwidth requires the square of the modulus of the complex amplification, which after a straightforward but slightly more complicated calculation from (A24):
\[ |\hat{N}_A(\omega)|^2 = \frac{(\omega \tau_{\text{F}})^2}{[(1 + G_0^{-1})(1 - \omega^2 \tau_{\text{F}}) - M \omega^2 \tau_{\text{F}} - \omega^2 \tau_{\text{F}}^2 + (\omega \tau_{\text{F}})^2 \left[ \frac{1}{\tau_{\text{F}}} + \Lambda_0 + G_0^{-1} M + \frac{G_0^{-1}}{\tau_{\text{F}}} (G_0^{-1} + 1) - \omega^2 \tau_{\text{F}} \right]^2}], \]

where

\[ \tau_{\text{F}}' \equiv C_R R_F, \quad \tau_{\text{Fg}} \equiv C_F R_G. \]

For \( C_F = 0, \tau_{\text{F}}' = \tau_{\text{Fg}} = 0, \) and the modulus of the amplification of the IA equation (A25) becomes equal to the modulus of the amplification of the difference amplifier equation (A22).

Finally, for the whole amplifier we have

\[ \Upsilon(\omega) = \Upsilon_{\text{NIA}}(\omega) \Upsilon_{\Delta}(\omega) \Upsilon_{\text{IA}}(\omega). \]

**A.5. Application of the Nyquist theorem for our circuit**

To calculate the mean square of the amplified signal \( \langle U_{\text{amp}}^2(t) \rangle \) we have to apply the Nyquist theorem for the spectral density of the thermal noise of the parallelly connected capacitor \( C \) and resistor \( R \) at the input of the circuit. According to this theorem the spectral density of the noise is given by the real part of the impedance:

\[ (U^2)_f = 4k_n T R(\omega), \quad R(\omega) = \Re(Z(\omega)), \quad \hbar \omega \ll k_n T. \]

A new pedagogical re-derivation of the Nyquist theorem and its extensions is given in appendix B.

Although quantum physics was created from the Planck explanation of the spectral density of electromagnetic fluctuations and three Nobel prizes were given for the black body radiation, there is no definite notation for the spectral density of the voltage noise in electronics. For example, in [26], Chap. 4, in [27], equation (78.3), where \((E^2)_\omega = (E^2)_f / 2\), or in [28], Chap. 24, where \(G_V = (E^2)_f / 2\pi\), or in [29], equation (89.3), where \((E^2)_\omega = (E^2)_f / 4\pi\), or in [30], equation (1.4.47), where \(S(\omega) = (E^2)_f / 4\pi\).

Applying the Nyquist theorem to a parallelly connected capacitor and resistor with impedance

\[ Z(\omega) = \left( \frac{1}{R} + j \omega C \right)^{-1}, \quad j = -i. \]

for the spectral density of the voltage noise we obtain

\[ (U^2)_f = 4k_n T \frac{R}{1 + (\omega RC)^2}. \]

This spectral density is amplified by the three steps of the amplifier equation (A27):

\[ (U_{\text{amp}}^2)_f = |\Upsilon(\omega)|^2 (U^2)_f \]

and for the mean square of the amplified voltage we have

\[ \langle U_{\text{amp}}^2(t) \rangle = \int_0^\infty |\Upsilon(\omega)|^2 \frac{4k_n T R}{1 + (\omega RC)^2} \frac{d\omega}{2\pi}. \]
In the approximation of frequency-independent amplification \( \Upsilon(\omega) \approx Y = y_1y_2y_3 \) we have

\[
\langle U_{\text{amp}}^2(t) \rangle \approx \int_0^\infty Y^2 \frac{4kT}{1 + (\omega RC)^2} \frac{d\omega}{2\pi} = Y^2 \frac{4kT}{4RC}. \tag{A33}
\]

That is why, for simplicity, we will represent the exact results as the approximation formula

\[
\langle U_{\text{amp}}^2(t) \rangle = Y^2 \frac{kT}{C} Z, \tag{A34}
\]

corrected by the close to unity coefficient

\[
Z \equiv 1 - \varepsilon \equiv \int_0^\infty \frac{(\Upsilon(\omega))^2}{1 + (\omega RC)^2} \frac{d\omega}{2\pi} \int_0^\infty \frac{d\omega}{1 + (\omega RC)^2}. \tag{A35}
\]

The small correction \( \varepsilon(C) \) as a function of the capacitance \( C \) is graphically presented in figure A1.

High-school students are freely able to use this figure, while university students can calculate it numerically. We do not recommend the analytical calculation of the integral equation (A32).

After calculating \( \langle U_{\text{amp}}^2(t) \rangle \) we can substitute it in equation (9) and obtain

\[
\langle U_i^2 \rangle = \frac{\langle U_{\text{amp}}^2 \rangle}{U_i} \frac{R_1 + R_2}{R_1}. \tag{A36}
\]

Using the described above correction factor, equation (11) now reads

\[
V_2 = \frac{Z Y^2}{U_i} \frac{R_1 + R_2}{R_1} \frac{R_V}{R_V + R_{av}} \langle U^2(t) \rangle, \tag{A37}
\]
Keeping the expression in equation (14) for $U^*$ unchanged, we finally arrive at the correct formula equation (17) used for the determination of the Boltzmann constant.

**Appendix B. Johnson–Nyquist thermal noise and Callen–Welton fluctuation-dissipation theorem**

To understand the work of the described set-up, only the equipartition theorem is necessary, and it is given in all introductory courses in physics: not only in universities but also in many high-school textbooks. Unfortunately when speaking about the thermal fluctuation of the voltage people erroneously suppose knowledge of the frequency-dependent spectral density as described by the Nyquist and Callen–Welton fluctuation-dissipation theorems (FDTs). These theorems already belong to the extended courses on statistical physics. These theorems definitely cannot be reproduced by at least 51% of physics teachers. That is why in this appendix we will give a new derivation of all those theorems at a level corresponding to the introductory courses on physics read by future physics teachers. This appendix is oriented for readers willing to understand the work of the set-up, starting from the ideas and notions given in the university courses on statistical physics, and nothing is beyond this frame.

**B.1. Nyquist theorem and its generalizations**

If the Nyquist theorem [31] is derived from statistical physics methods, it is viewed as an interesting application of the Callen–Welton FDT [32, 33]. On the Landau–Lifshitz course in theoretical physics, in both the 5th volume Statistical Physics [33] and the first edition of the 8th volume *Electrodynamics of Continuous Media* [29], a derivation following the original one [32] is given. In this approach a Gibbs averaging of the Dirac time-dependent perturbation theory is performed. The practice in teaching physics, however, shows that this derivation meant for professionals is unrepeatable by students paying tuition fees and waiting to receive an educational service; the degradation of physics education is global, and the authors of the present article are unable to point out any contemporary textbooks where FDT is derived. That is why in this appendix we give a new re-derivation of the FDT approbated by many recruited students.

In figure B1 a resonance LC circuit with a resistor $R$ creating random noise is depicted. The random thermal voltage $\mathcal{E}_w$ creates a current amplitude $I_w = \sigma(\omega)\mathcal{E}(\omega)$, where the conductivity $\sigma(\omega) = 1/Z(\omega)$ is determined by the total impedance of the sequentially connected inductance, resistor and capacitor.
As a gedanken experiment, let us analyse a high quality resonance circuit, for which

\[ R \ll \sqrt{\frac{1}{\omega L C}} = \frac{C \sqrt{L}}{R}, \quad Q \equiv \frac{\sqrt{L/C}}{R} \gg 1. \]  

(\text{B2})

In this case the square of the modulus of the conductivity

\[ |\sigma(\omega)|^2 = \frac{1}{R^2 + \left(\frac{\omega L - 1}{C}\right)^2} \approx \frac{\pi}{2} R L \delta(\omega - \omega_0) \]  

(\text{B3})

has a sharp maximum at the resonance frequency \( \omega_0 = 1/\sqrt{LC} \) and is negligible far from the resonance, and we can use the \( \delta \)-function approximation

\[ F(\omega_0) = \int_{-\infty}^{\infty} F(\omega) \delta(\omega - \omega_0) d\omega, \quad \omega_0 > 0. \]  

(\text{B4})

The coefficient in front of the \( \delta \) function is given by the integral

\[ \int_{-\infty}^{\infty} \frac{\omega^2 d\omega}{R^2 \omega^2 + (\omega L - 1/C)^2} = \frac{\pi}{2} \frac{1}{RL}, \]  

(\text{B5})

which does not depend on the capacitance \( C \) and was solved by Schottky \([34]\), equation (2).

Introducing the dimensionless variable \( x \equiv \omega L / R \), and dimensionless parameter \( Q \), the corresponding mathematical problem is

\[ \frac{d}{dx} \left( x^2 \frac{dx}{dx + (x^2 - Q^2)^2} \right) = \frac{\pi}{2}, \]  

(\text{B6})

which can be solved both analytically (given in section B.2) and numerically for different values of \( a \). For \( Q = 0 \) we have a table integral \( \int_{-\infty}^{\infty} dx / (1 + x^2) = \pi \).

Using this \( \delta \)-function approximation, for the spectral density of the current we obtain

\[ (I^2)_f = \frac{\pi}{2} \frac{(E^2)_f}{RL} \delta(\omega - \omega_0), \quad (E^2)_f = 4R\xi. \]  

(\text{B7})

Experimentally such frequency dependence of the spectral density can be investigated using Fourier transformation of digital oscilloscopes and generators. Such equipment is not typical in high schools and that is why, for physics teachers, the present appendix is only additional material completing the theory from general introductory courses. Whence for the thermal-averaged energies of the capacitor and inductance

\[ \langle E_C \rangle = \frac{1}{2} C \langle U_C^2 \rangle, \quad \langle \dot{U}_C^2 \rangle = \int_0^{\infty} |Z_C|^2 (I^2)_f \frac{d\omega}{2\pi}, \]

\[ \langle E_L \rangle = \frac{1}{2} L \langle I^2 \rangle, \quad \langle I \rangle = \int_0^{\infty} (I^2)_f \frac{d\omega}{2\pi}, \]

a trivial integration of the \( \delta \)-functions equation (\text{B4}) gives

\[ \langle E_C \rangle = \langle E_L \rangle = \frac{1}{2} \xi, \quad \xi = \frac{1}{2} \frac{1}{\hbar \omega_0} \coth \left( \frac{\hbar \omega_0}{2k_B T} \right). \]  

(\text{B8})

The Nyquist result for the spectral density of the noise can be presented in a more general form.
B.2. Solution of the Schottky integral

Contemporary students use Mathematica or Maple, but 100 years ago physicists were able to perform elementary calculus of integrals [34]. In this subsection we give the calculation of the integral considered by Schottky when he analyzed the influence of noise of a resonance circuit. Let us consider the integrand from equation (B6):

\[ \frac{x^2}{x^2 + (x^2 - Q^2)^2} = \frac{C}{x^2 + a^2} + \frac{D}{x^2 + b^2}, \]  

(B9)

where ±a and ±b are the x values for which the denominator on the left is equal to 0, and C and D are the coefficients that we are going to find. Expanding the divisors on the left and right sides into a polynomial, we obtain

\[ (x^2 + a^2)(x^2 + b^2) = x^4 + (a^2 + b^2)x^2 + a^2b^2, \]  

(B10)

\[ x^2 + (x^2 - Q^2)^2 = x^4 + (1 - 2Q^2)x^2 + Q^4. \]  

(B11)

Both polynomials have to be identical, meaning that the coefficients of the respective x degree terms should be equal. Clearly, the coefficients of the \(x^4\) terms are both equal to 1, and comparing the quadratic and zero degree terms, we obtain a set of two equations (one per each degree term):

\[ a^2 + b^2 = 1 - 2Q^2 \quad \text{(the terms with } x^2), \]
\[ a^2b^2 = Q^4 \quad \text{(the terms with } x^0). \]  

(B12)

From the first equation it is evident that for real values of \(a\) and \(b\), \(Q < 1/\sqrt{2}\), while for \(Q > 1/\sqrt{2}\) we have \(a^2 + b^2 < 0\), and hence, \(a\) and \(b\) are complex.

B.2.1. Real values of \(a\) and \(b, Q < 1/\sqrt{2}\). First we are going for a solution in the case of real values for \(a\) and \(b\). Let us take a closer look at the first of these two equations. Now, going back to equation (B9), we combine both fractions on the right-hand side into one and the nominator becomes

\[ Cx^2 +Cb^2 +Dx^2 +Da^2 = (C + D)x^2 + (Cb^2 + Da^2). \]

Comparing it with the nominator on the left-hand side \(x^2\), we obtain the following two equations:

\[ C + D = 1 \quad \text{(the terms with } x^2), \]  

(B13)

\[ Cb^2 + Da^2 = 0 \quad \text{(the terms with } x^0). \]  

(B14)

Solving these equations for \(C\) and \(D\), we get

\[ C = \frac{a^2}{a^2 - b^2}, \quad D = -\frac{b^2}{a^2 - b^2}. \]  

(B15)

Substituting these expressions for \(C\) and \(D\) into equation (B6), the integral

\[ \mathcal{I} = \int_0^\infty \frac{x^2 dx}{x^2 + (x^2 - Q^2)^2} = \int_0^\infty \left( \frac{C}{x^2 + a^2} + \frac{D}{x^2 + b^2} \right) dx \]

\[ = \frac{a^2}{a^2 - b^2} \int_0^\infty \frac{dx}{x^2 + a^2} - \frac{b^2}{a^2 - b^2} \int_0^\infty \frac{dx}{x^2 + b^2}. \]
The solution of such a table integral is

\[ \int_0^\infty \frac{dx}{x^2 + a^2} = \frac{1}{a} \int_0^\infty \frac{d(x/a)}{(x/a)^2 + 1} = \frac{\pi}{a}. \]

and hence, for the solution of the integral we obtain

\[ \mathcal{I} = \frac{1}{a^2 - b^2} (a - b) \frac{\pi}{2} = \frac{1}{a + b} \frac{\pi}{2}. \]

Using the well-known binomial theorem \((a + b)^2 = a^2 + 2ab + b^2\), we substitute the values for \(a^2 + b^2\) and \(ab\) from equations (B12) to obtain

\[ (a + b)^2 = 1 - 2Q^2 + 2Q^2 = 1, \]

and hence for the solution we have

\[ \mathcal{I} = \frac{1}{a + b} \frac{\pi}{2} = \frac{\pi}{2}, \]

where we have taken only the positive value.

**B.2.2. Complex values of \(a\) and \(b, Q > 1/\sqrt{2}\).** The solution of the integral in this case is analogous. The parameters \(a\) and \(b\) are complex. However, for the interesting for us case of \(Q \gg 1\) we can recall the analytical continuation. If \(\mathcal{I}(Q) = \pi/2\) for \(0 < Q < 1/2\) this result has a unique analytical continuation and can be extended from a finite segment to the whole axis \(Q > 0\). Now we address a new derivation of the FDT theorem.

**B.3. Callen–Welton FDT**

In the spectral density of the voltage noise

\[ \langle \mathcal{E}^2 \rangle_f = 2R(\omega) \hbar \omega \coth(\hbar \omega / 2k_b T), \]

\[ R(\omega) = \Re(Z(\omega)), \quad Z(\omega) = \frac{1}{\imath \omega C}, \]

the frequency-dependent resistance \(R(\omega)\) is the real part of the complex impedance, which can also be represented by a frequency-dependent capacitance

\[ \omega R(\omega) = \omega \Re \left( \frac{\imath}{\omega C(\omega)} \right) = \frac{\imath}{2} (C^{-1} - C^{-1}) = \frac{C''}{|C|^2}, \]

where \(C' = \Re(C(\omega)), C'' = \Im(C(\omega))\) and \(C = C' + \imath C''\). Here, we suppose an arbitrary frequency of the impedance represented by a generalized capacitance. On the other hand, the spectral density of the charge is also given by the capacitance

\[ \langle Q^2 \rangle_f = |C(\omega)|^2 \langle \mathcal{E}^2 \rangle_f. \]

In such a way, the Nyquist theorem for the thermal noise can be rewritten as

\[ \langle Q^2 \rangle_f = 2\hbar C'' \coth(\hbar \omega / 2k_b T) \]

and for thermal-averaged charge fluctuations we finally arrive at

\[ \langle Q^2 \rangle = \int_0^\infty \langle Q^2 \rangle_f \frac{d\omega}{2\pi} = \frac{\hbar}{\pi} \int_0^\infty C''(\omega) \coth \left( \frac{\hbar \omega}{2k_b T} \right) d\omega. \]

We use the spectral density \(\langle \mathcal{E}^2 \rangle_f = 2\langle \mathcal{E}^2 \rangle_f\), which takes into account the folding of positive and negative frequencies \(\omega\).
Let us again recall the formula for the energy when the considered capacitor $C$ is under the voltage $E = Q_0 / C_0$ created by a large capacitor (charge reservoir) $C_0 \gg C$

$$
E = \frac{Q^2}{2C} + \frac{(Q_0 - Q)^2}{2C_0} = \frac{Q^2}{2C} - E_0 + \text{const.}
$$

One can consider that

$$
V = -E(t)Q
$$

is the interaction energy of the capacitor with the voltage source. The formulae equations (B21) and (B23) are so general that it deserves to change the notations:

$$
\hat{\alpha}(\omega) = C(\omega), \quad \hat{x} = Q, \quad f = E, \quad \hat{V} = V, \quad \langle \hat{x} \rangle_\omega = \alpha_\omega f_\omega.
$$

Analogously in time representation

$$
\tilde{Q}(t) = \int_{-\infty}^{t'} C(t - t')U(t')dt'
$$

becomes

$$
\tilde{x}(t) = \int_{-\infty}^{\infty} \alpha(\tau)f(t - \tau)d\tau.
$$

In this case the Nyquist theorem can be rewritten as

$$
\langle x^2 \rangle = \frac{\hbar}{\pi} \int_{0}^{\infty} \alpha'(\omega) \coth \left( \frac{\hbar \omega}{2k_B T} \right) d\omega, \quad \hat{V} = -f(t)\hat{x}.
$$

cf. equation (V.124.10) and equation (V.123.1) of the Landau–Lifshitz course on theoretical physics [33]. The terminology also has to be changed, and the Nyquist theorem is now well known as the FDT for generalized susceptibility $\alpha(\omega)$ giving the general relations between the fluctuations of some quantum variable $\hat{x}$ and dissipation (absorptive) part of the generalized susceptibility $\alpha'' = \Im(\alpha(\omega))$.

Callen and Welton re-derived the Nyquist FDT applying Gibbs thermal averaging of the quantum mechanical second order perturbation theory, as it is now described in every professionally written textbook on statistical physics. Even the history of physics is rewritten, and in some textbooks one can read that the Nyquist theorem is an interesting application [27] of the FDT. The Sutherland–Einstein relations between the diffusion coefficient and mobility is one of the first examples of the FDT [35–37].

In short we represented all those theorems of statistical physics as simple consequences of the principle of the detailed balance applied to the simplest physical system—the harmonic oscillator. But each creation is a child of its own time. Nowadays, after more than a century of technology development, every teacher is able to illustrate the thermodynamic fluctuations of the voltage and the equipartition theorem after only a single day’s work in the building of the experimental set-up. More than a tenth of the students used the described experimental set-up during the EPO5, and also successfully determined the Boltzmann constant and the elementary formulae describing the electronic circuit operation.

What has changed since the time of the Habicht brothers? That is the invention/appearance of low-noise operational amplifiers, such as the ADA4898 and multipliers of the AD633 type, which make possible the RMS voltage measurements of thermal noise amplified a million times. Now one could be motivated to follow our new derivation of the Nyquist and Callen–Welton theorems, which is actually a detailed balance principle applied to the oscillator written in electric variables.
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