Dirichlet-to-Neumann and elliptic operators on $C^{1+\kappa}$-domains: Poisson and Gaussian bounds

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Abstract

We prove Poisson upper bounds for the heat kernel of the Dirichlet-to-Neumann operator with variable Hölder coefficients when the underlying domain is bounded and has a $C^{1+\kappa}$-boundary for some $\kappa > 0$. We also prove a number of other results such as gradient estimates for heat kernels and Green functions $G$ of elliptic operators with possibly complex-valued coefficients. We establish Hölder continuity of $\nabla_x \nabla_y G$ up to the boundary. These results are used to prove $L_p$-estimates for commutators of Dirichlet-to-Neumann operators on the boundary of $C^{1+\kappa}$-domains. Such estimates are the keystone in our approach for the Poisson bounds.

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1 Introduction

Let Ω ⊂ R^d be a bounded connected open set with Lipschitz boundary and d ≥ 2. Denote by Γ = ∂Ω the boundary of Ω, endowed with the (d − 1)-dimensional Hausdorff measure. Note that Γ is not connected in general. Let C := (c_{kl})_{1≤k,l≤d} be real-valued matrix satisfying the usual ellipticity condition and c_{kl} = c_{lk} ∈ L_∞(Ω) for all k, l ∈ {1, ..., d}. Let V ∈ L_∞(Ω, R). The Dirichlet-to-Neumann operator N_V is an unbounded operator on L_2(Γ) defined as follows. Given φ ∈ L_2(Γ), we solve (if possible) the Dirichlet problem

\[- \sum_{k,l=1}^{d} \partial_l (c_{kl} \partial_k u) + V u = 0 \text{ weakly on } Ω,\]

\[u|_Γ = φ\]

with u ∈ W_0^{1,2}(Ω). We define the weak conormal derivative \(\partial_C u\), which is formally equal to

\[\sum_{k,l=1}^{d} n_l c_{kl} \partial_l u,\]

where \((n_1, \ldots, n_d)\) is the outer normal vector to Ω. If u has a weak conormal derivative in L_2(Γ), then we say that φ ∈ D(N_V) and N_Vφ = \(\partial_C u\). If V = 0 we write N instead of N_0. See the beginning of Section 2 for more details on this definition. In particular, we shall always assume that 0 ∉ σ(A_D + V), where A_D := -\sum_{k,l=1}^{d} \partial_l (c_{kl} \partial_k) and subject to the Dirichlet boundary condition.

The Dirichlet-to-Neumann operator, also known as voltage-to-current map, arises in the problem of electrical impedance tomography and in various inverse problems (e.g., Calderón’s problem). It is also used in the theory of homogenization and analysis of elliptic systems with rapidly oscillating coefficients (see Kenig, Lin and Shen [KLS] and
the references there). Our aim in the present paper is to address another problem, namely upper bounds for the heat kernel associated with the Dirichlet-to-Neumann operator. Heat kernel bounds (mainly Gaussian bounds) for various differential operators on domains of \( \mathbb{R}^d \) as well as on Riemannian manifolds have attracted a lot of attention in recent years. It turns out that they are a powerful tool to attack problems in harmonic analysis, such as Calderón-Zygmund operators, Riesz transforms, spectral multipliers as well as other problems in spectral theory and evolution equations. See for example the monograph [Ouh] and the references therein.

It is well known that \( N_V \) is a lower-bounded and self-adjoint operator on \( L^2(\Gamma) \) with compact resolvent. Therefore, \( -N_V \) generates a \( C_0 \)-semigroup \( S^V \) on \( L^2(\Gamma) \). If \( \Omega \) has \( C^\infty \)-boundary, \( c_{kl} = \delta_{kl} \) and \( V \geq 0 \), then it was shown by ter Elst and Ouhabaz [EO2] that \( S^V \) is given by a kernel which satisfies Poisson upper bounds. In the present paper we extend considerably this result to deal with variable Hölder-continuous coefficients \( c_{kl} \) and less regular domains. Our main result in this direction reads as follows.

**Theorem 1.1.** Suppose \( \Omega \subset \mathbb{R}^d \) is bounded connected with a \( C^{1+\kappa} \)-boundary \( \Gamma \) for some \( \kappa \in (0,1) \). Suppose also each \( c_{kl} = c_{lk} \) is real valued and Hölder continuous on \( \Omega \). Let \( V \in L^\infty(\Omega, \mathbb{R}) \) and suppose that \( 0 \notin \sigma(A_D+V) \). Denote by \( N_V \) the corresponding Dirichlet-to-Neumann operator. Then the semigroup generated by \( -N_V \) has a kernel \( K^V \) and there exists a \( c > 0 \) such that

\[
|K^V_t(z,w)| \leq \frac{c(t \wedge 1)^{-(d-1)} e^{-\lambda_1 t}}{(1 + \frac{|z-w|}{t})^d}
\]

for all \( z,w \in \Gamma \) and \( t > 0 \), where \( \lambda_1 \) is the first eigenvalue of the operator \( N_V \).

One immediate consequence of this result is that the semigroup \( S^V \) acts as a holomorphic semigroup on \( L^1(\Gamma) \). Even the existence of such semigroup on \( L^1(\Gamma) \) as a \( C_0 \)-semigroup is new in this generality. The holomorphy of the semigroup follows as in Theorem 7.1 in [EO2]. We can also draw further information, for example \( N_V \) has a holomorphic functional calculus on \( L^p(\Gamma) \) for all \( p \in (1, \infty) \) with angle \( \mu \in (\frac{\pi(d-1)}{2d}, \pi) \), see Theorem 7.2 in [EO2]. The previous theorem has another consequence. It allows to establish existence results for evolution equations on \( C(\Gamma) \) (the space of continuous functions on \( \Gamma \)). This subject will be addressed in a forthcoming paper.

The strategy of proof is similar to [EO2] in the sense that we prove appropriate \( L_p^p - L_q^q \) estimates for iterated commutators of the semigroup \( S^V = (e^{-tN_V})_{t>0} \) with \( M_g \), a multiplication operator by a Lipschitz continuous function \( g \) on \( \Gamma \). In [EO2] these estimates are essentially deduced from \( L_p^p - L_q^q \) estimates of \( S \) together with commutator estimates of Coifman-Meyer for pseudo-differential operators and this is the reason why we assumed there that the boundary is \( C^\infty \).

One cannot use these commutator results of Coifman-Meyer on \( C^{1+\kappa} \)-domains and this is the major difficulty here. We shall instead rely solely on a recent \( L^2_2 - L_2^2 \) estimate for the commutator \([N^\ast, M_g]\) proved by Shen [She]. The result of [She] is valid even for \( \Omega \) with Lipschitz boundary. We extend this commutator estimate to \( L_p^p(\Gamma) \) for all \( p \in (1, \infty) \) under the assumption that \( \Omega \) has a \( C^{1+\kappa} \)-boundary by appealing to Calderón-Zygmund theory.
In order to do so we need appropriate bounds for the Schwartz kernel $K_{N_V}$ of $N_V$, namely

$$|K_{N_V}(z, w)| \leq \frac{c}{|z - w|^d}$$

and

$$|K_{N_V}(z, w) - K_{N_V}(z', w')| \leq c \frac{(|z - z'| + |w - w'|)^\alpha}{|z - w|^{d + \alpha}}$$

for all $z, z', w, w' \in \Gamma$ with $z \neq w$ and $|z - z'| + |w - w'| \leq \frac{1}{2}|z - w|$. It turns out that one can express the Schwartz kernel $K_{N_V}$ in terms of the trace on the boundary of second order derivatives $\partial_k^{(1)} \partial_l^{(2)} G_V$ of the Green function $G_V$ of $A_D + V$. Therefore, we need appropriate bounds and Hölder continuity for $\partial_k^{(1)} \partial_l^{(2)} G_V$. We take the opportunity to prove these bounds in the general setting of elliptic operators with complex-valued coefficients. We prove that the heat kernel $H_t$ of $A_D + V$ satisfies bounds

$$|((\partial_x^\alpha \partial_y^\beta H_t)(x, y)| \leq a t^{-d/2} t^{-(|\alpha|+|\beta|)/2} e^{-b \frac{|x-y|^2}{t}} e^{\omega t}$$

and

$$|((\partial_x^\alpha \partial_y^\beta H_t)(x + h, y + k) - (\partial_x^\alpha \partial_y^\beta H_t)(x, y)| \leq a t^{-d/2} t^{-(|\alpha|+|\beta|)/2} \left( \frac{|h| + |k|}{\sqrt{t} + |x - y|} \right)^\alpha e^{-b \frac{|x-y|^2}{t}} e^{\omega t}$$

for all $x, y \in \Omega$ and $h, k \in \mathbb{R}^d$ with $x + h, y + k \in \Omega$, $|h| + |k| \leq \tau \sqrt{t} + \tau' |x - y|$ and all $|\alpha|, |\beta| \leq 1$. These bounds are proved using Morrey and Campanato spaces. The idea of using these spaces in order to obtain Gaussian upper bounds together with Hölder regularity for heat kernels of elliptic operators on $\mathbb{R}^d$ originates in a work of Auscher [Aus], see also ter Elst and Robinson [ERo1] and for derivatives of the kernel on Lie groups see [ERo2]. Here the new difficulty is that we have boundary conditions and the approach needs to be adjusted to this setting. In addition, not only Gaussian upper bounds for the heat kernel are proved here but also Gaussian upper bounds and Hölder continuity for the derivatives $\partial_x \partial_y H_t$ with $|\alpha|, |\beta| \leq 1$. In order to obtain the necessary De Giorgi or energy estimates for derivatives of weak solutions, we use estimates of Campanato [Can].

The previous bounds on the heat kernel readily imply for the Green function $G_V$ the bounds

$$|((\partial_x^\alpha \partial_y^\beta G_V)(x, y)| \leq c |x - y|^{-(d-2+|\alpha|+|\beta|)}$$

and

$$|((\partial_x^\alpha \partial_y^\beta G_V)(x', y') - (\partial_x^\alpha \partial_y^\beta G_V)(x, y)| \leq c \frac{|x' - x| + |y' - y|}{|x - y|^{d-2+|\alpha|+|\beta|}}\right|^\alpha$$

for all $x, x', y, y' \in \Omega$ with $x \neq y$ and $|x - x'| + |y - y'| \leq \frac{1}{2}|x - y|$ if $d \geq 3$. If $\Re V \geq 0$, we have uniform constants $c$ (with respect to the coefficients $c_{kl}$ and $V$), a very useful fact when using approximation by smooth coefficients as we shall do in our proofs. If $d = 2$, then the estimates are the same when $|\alpha| + |\beta| \neq 0$. Otherwise a logarithmic term appears.

These estimates on the Green function are used to prove the previous estimates on the Schwartz kernel $K_{N_V}$ of the Dirichlet-to-Neumann operator.
We emphasize that if $c_{kl} = c_{lk}$ are real-valued then upper bounds for $\nabla_x \nabla_y G$ are known (see for example Avellaneda and Lin [AL] and Kenig, Lin and Shen [KLS]). Note however that Hölder continuity of $\nabla_x \nabla_y G$ as stated above seems to be missing in the literature.

We return now to final step used in the proof of the Poisson bounds. Once $L_p - L_q$ estimates for the commutator $[N_V, M_g]$ are proved we obtain $L_p - L_q$ bounds for iterated commutators $\delta^j g(N_V) := [M_g, [...], M_g, N_V]...$ for all $j \in \{1, ..., d\}$. This together with $L_p - L_q$ estimates for the semigroup $S^t$ is used to estimate the $L_1 - L_\infty$ norm of the iterated commutator $\delta^d g(S^t) := [M_g, [...], [M_g, S^t], ...]$. We then optimize over $g$ and obtain the Poisson bounds.

**Notation**

Throughout this paper we use the following notation. For a function $R$ of two variables we denote by $\partial_j^k R$ the $k$th-partial derivatives with respect to the $j$th variable with $j = 1, 2$.

We identify a uniformly continuous function on $\Omega$ with a uniformly continuous function on $\overline{\Omega}$. We emphasize that a function in $C^1(\Omega)$ is not bounded in general, nor it is an element of $L_1(\Omega)$ in general, even if $\Omega$ is bounded. We define $C^1(\Omega) = \{u \in C^1(\Omega) : \partial_k u$ is uniformly continuous for all $k \in \{1, ..., d\}\}$.

For a bounded domain $\Omega$ with Lipschitz boundary $\Gamma$, let $C^{0,1}(\Gamma)$ denote the space of Lipschitz continuous functions on $\Gamma$. It is endowed with the norm

$$\|g\|_{C^{0,1}(\Gamma)} = \|g\|_{L_\infty(\Gamma)} + \sup_{z, w \in \Gamma, z \neq w} \frac{|g(z) - g(w)|}{|z - w|}.$$

For all $g \in C^{0,1}(\Gamma)$ we use the notation $\text{Lip}_\Gamma(g) = \sup_{z, w \in \Gamma, z \neq w} \frac{|g(z) - g(w)|}{|z - w|}$. If $f \in L_\infty(\Omega)$ and $p \in [1, \infty]$, then we denote by $M_f$ the multiplication operator by the function $f$ on $L_p(\Omega)$. Finally, the $L_p - L_q$ norm of an operator $T$ will be denotes by $\|T\|_{p \to q}$.

**2 Preliminaries and the first auxiliary results**

We assume throughout this section that $\Omega$ is a bounded Lipschitz domain of $\mathbb{R}^d$ with $d \geq 2$. We assume that $c_{kl} = c_{lk} \in L_\infty(\Omega, \mathbb{R})$ such that

$$\sum_{k,l=1}^d c_{kl}(x) \xi_k \xi_l \geq \mu |\xi|^2$$

for all $\xi \in \mathbb{C}^d$ and a.e. $x \in \Omega$, where $\mu > 0$ is a positive constant. Let $V \in L_\infty(\Omega, \mathbb{R})$ be a real-valued potential. We define the space $H_V$ of harmonic functions for the operator $-\sum_{k,l=1}^d \partial_l (c_{kl} \partial_k) + V$ by

$$H_V = \{u \in W^{1,2}(\Omega) : -\sum_{k,l=1}^d \partial_l (c_{kl} \partial_k u) + Vu = 0 \text{ weakly on } \Omega\}.$$
Here and in what follows \(- \sum_{k,l=1}^{d} \partial_{l} (c_{kl} \partial_{k} u) + Vu = 0\) weakly on \(\Omega\) means that \(u \in W^{1,2}(\Omega)\) and
\[
\int_{\Omega} \sum_{k,l=1}^{d} c_{kl} (\partial_{k} u) \overline{\partial_{l}} \chi + \int_{\Omega} V u \overline{\chi} = 0
\]
for all \(\chi \in C_{0}^{\infty}(\Omega)\).

Define the continuous sesquilinear form \(a_{V}: W^{1,2}(\Omega) \times W^{1,2}(\Omega) \rightarrow \mathbb{C}\) by
\[
a_{V}(u, v) = \int_{\Omega} \sum_{k,l=1}^{d} c_{kl} (\partial_{k} u) \overline{\partial_{l} v} + \int_{\Omega} V u \overline{v}.
\]
(1)

It is clear that \(H_{V}\) is a closed subspace of \(W^{1,2}(\Omega)\) and
\[
H_{V} = \{ u \in W^{1,2}(\Omega) : a_{V}(u, v) = 0 \text{ for all } v \in \ker \text{Tr} \},
\]
where \(\text{Tr}: W^{1,2}(\Omega) \rightarrow L^{2}(\Gamma)\) is the trace operator.

Define the form \(a_{V}^{D}: W_{0}^{1,2}(\Omega) \times W_{0}^{1,2}(\Omega) \rightarrow \mathbb{C}\) by \(a_{V}^{D} = a_{V}|_{W_{0}^{1,2}(\Omega) \times W_{0}^{1,2}(\Omega)}\). Then the associated operator is \(A_{D} + V\), where \(A_{D}\) is the operator associated to the form \(a_{0}^{D}\). Formally, \(A_{D} = - \sum_{k,l=1}^{d} \partial_{l} (c_{kl} \partial_{k})\), subject to the Dirichlet boundary condition.

As in Section 2 in [EO2], one proves easily that if \(0 \notin \sigma(A_{D} + V)\), then the space \(W^{1,2}(\Omega)\) has the decomposition
\[
W^{1,2}(\Omega) = W_{0}^{1,2}(\Omega) \oplus H_{V}.
\]
(2)

In particular
\[
\text{Tr}(H_{V}) = \text{Tr}(W^{1,2}(\Omega)).
\]

A direct corollary is that \(\text{Tr}\) is injective as an operator from \(H_{V}\) into \(L^{2}(\Gamma)\). Thus, we may define the form \(b_{V}: \text{Tr}(W^{1,2}(\Omega)) \times \text{Tr}(W^{1,2}(\Omega)) \rightarrow \mathbb{C}\) by
\[
b_{V}(\varphi, \psi) = a_{V}(u, v),
\]
where \(u, v \in H_{V}\) are such that \(\text{Tr} u = \varphi\) and \(\text{Tr} v = \psi\). One obtains as in [EO2] that \(b_{V}\) is bounded from below and is a closed symmetric form. Hence there exists an associated self-adjoint operator \(N_{V}\) associated with \(b_{V}\). This is the Dirichlet-to-Neumann operator.

Let \(u \in W^{1,2}(\Omega)\) and \(f \in L^{2}(\Omega)\). We say that \(Au = f\) if \(a_{0}(u, v) = (f, v)_{L^{2}(\Omega)}\) for all \(v \in W_{0}^{1,2}(\Omega)\). In particular, \(u \in H_{V}\) if and only if \(Au = -V u\). If \(u \in W^{1,2}(\Omega)\), then we say that \(Au \in L^{2}(\Omega)\) if there exists an \(f \in L^{2}(\Omega)\) such that \(Au = f\). Let \(u \in W^{1,2}(\Omega)\) with \(Au \in L^{2}(\Omega)\). Then we say that \(u\) has a weak conormal derivative if there exists a \(\psi \in L^{2}(\Gamma)\) such that
\[
\int_{\Omega} \sum_{k,l=1}^{d} c_{kl} (\partial_{k} u) \overline{\partial_{l} v} - \int_{\Omega} (Au) \overline{v} = \int_{\Gamma} \psi \overline{\text{Tr} v}
\]
(3)
for all \(v \in W^{1,2}(\Omega)\). In that case \(\psi\) is unique and we write \(\partial_{\nu} u = \psi\).

We next present a couple of equivalent descriptions for the Dirichlet-to-Neumann operator \(N_{V}\).
Lemma 2.1. Let $\varphi, \psi \in L_2(\Gamma)$. Then the following are equivalent.

(i) $\varphi \in D(N_V)$ and $N_V \varphi = \psi$.

(ii) There exists a $u \in H_V$ such that $\text{Tr } u = \varphi$ and $\partial^C_\nu u = \psi$.

(iii) There exists a $u \in W^{1,2}(\Omega)$ such that $\text{Tr } u = \varphi$ and

$$a_V(u, v) = (\psi, \text{Tr } v)_{L_2(\Gamma)}$$

for all $v \in W^{1,2}(\Omega)$.

Proof. ‘(i)$\Rightarrow$(ii)’. By definition there exists a $u \in H_V$ such that $\text{Tr } u = \varphi$ and

$$\int_\Omega \sum_{k,l=1}^d c_{kl}(\partial_k u) \overline{\partial_l v} + \int_\Omega V u \nu = a_V(u, v) = \int_{\Gamma} \psi \text{Tr } v$$

for all $v \in H_V$. Since $u \in H_V$ obviously (5) is valid for all $v \in W^{1,2}_0(\Omega)$. Then by (2) one deduces that (5) is valid for all $v \in W^{1,2}(\Omega)$. Moreover, since $Au + Vu = 0$ it follows from (3) that $\partial^C_\nu u = \psi$.

‘(ii)$\Rightarrow$(iii)’. Since $u \in H_V$ it follows that $Au + Vu = 0$. Then (3) implies that (5) is valid for all $v \in W^{1,2}(\Omega)$. But this is just (4).

‘(iii)$\Rightarrow$(i)’. Now (5) with $v \in W^{1,2}_0(\Omega)$ gives $Au + Vu = 0$, that is $u \in H_V$. By definition of $b_V$ one deduces that $b_V(\varphi, \tau) = (\psi, \tau)_{L_2(\Gamma)}$ for all $\tau \in \text{Tr } (H_V)$. Hence Condition (i) is valid.

For additional information regarding Condition (iii) we refer to [AE1].

The self-adjoint operator $-N_V$ generates a quasi-contraction holomorphic semigroup $S^V$ on $L_2(\Gamma)$. When $V = 0$ we write for simplicity $N = N_0$ and $S = S^0$. We also denote by $\lambda_1$ the first eigenvalue of the Dirichlet-to-Neumann operator $N_V$ without specifying the dependence on $V$.

We summarize in the following two theorems some important properties of the semigroups $S^V$ and $S$. The proofs are the same as in [EO2] Section 2, where these results are proved in the case $c_{kl} = \delta_{kl}$.

Theorem 2.2. Suppose that $\Omega$ is bounded Lipschitz, $c_{kl} = c_{lk} \in L_\infty(\Omega, \mathbb{R})$ satisfying the ellipticity condition and $V \in L_\infty(\Omega, \mathbb{R})$ with $0 \notin \sigma(A_D + V)$.

(a) If $A_D + V \geq 0$ then the semigroup $S^V$ is positive (it maps positive functions on $\Gamma$ into positive functions).

(b) If $V \geq 0$ then $S^V$ is sub-Markovian. Therefore $S^V$ acts as a contraction $C_0$-semigroup on $L_p(\Gamma)$ for all $p \in [1, \infty)$.

(c) If $V \geq 0$ then $S_t^V \varphi \leq S_t \varphi$ for all $t \geq 0$ and all positive $\varphi \in L_2(\Gamma)$.

We note that in the first assertion, if the assumption $A_D + V \geq 0$ is not satisfied then the semigroup $S^V$ may not be positive for all $t > 0$ (see [Dan]). This is the reason why our Poisson bound in the main theorem is formulated for $|K_t^V(x, y)|$ and not for $K_t^V(x, y)$.

Now we state $L_p - L_q$ estimates for the semigroup $S^V$. Note that $\lambda_1 \geq 0$ in the next theorem.
Theorem 2.3. Let $0 \leq V \in L_{\infty}(\Omega)$ and let $\lambda_1 \in \sigma(N_V)$ be the first eigenvalue of $N_V$. Then for all $1 \leq p \leq q \leq \infty$ and $t > 0$ the operator $S^V_t$ is bounded from $L_p(\Gamma)$ into $L_q(\Gamma)$. Moreover, there exists a $C > 0$ such that

$$\|S^V_t\|_{p \rightarrow q} \leq C (t \wedge 1)^{-\left(\frac{d-1}{2} - \frac{1}{q}\right)} e^{-\lambda_1 t}$$

for all $t > 0$ and $p, q \in [1, \infty]$ with $p \leq q$.

Actually, it will follow from Theorem 1.1 that this theorem is also valid for general $V \in L_{\infty}(\Omega)$, possibly with $\lambda_1 < 0$.

We finish this section with a known formula. Again let $V \in L_{\infty}(\Omega)$ with $0 \notin \sigma(A_D + V)$. Define the harmonic lifting $\gamma_V: \text{Tr}(W^{1,2}(\Omega)) \rightarrow H^1(\Omega)$ as follows. Given $\varphi \in H^{1/2}(\Gamma) := \text{Tr}(W^{1,2}(\Omega))$ it follows from (2) that there exists a unique $u \in H_V$ with $\text{Tr} u = \varphi$. We define

$$\gamma_V \varphi := u.$$

There is a simple relation between $\gamma_V$ and $\gamma_0$, where the latter is the harmonic lifting in case $V = 0$. Let $\varphi \in \text{Tr}(W^{1,2}(\Omega))$. Write $u_0 = \gamma_0 \varphi$ and $u = \gamma_V \varphi$. Then $u - u_0 \in W_0^{1,2}(\Omega)$.

Moreover, $(A + V)u = 0$ and $A u_0 = 0$. So $(A_D + V)(u - u_0) = (A + V)(u - u_0) = -V u_0$. Therefore $u - u_0 = -(A_D + V)^{-1} M_V u_0$ and

$$\gamma_V = \gamma_0 - (A_D + V)^{-1} M_V \gamma_0. \quad (6)$$

We shall use this relation in Sections 5.

3 Heat kernel bounds for elliptic operators on $C^{1+\kappa}$-domains

Let $\Omega \subset \mathbb{R}^d$ be an open set. Let $\mu, M > 0$. We define $E(\Omega, \mu, M)$ to be the set of all measurable $C: \Omega \rightarrow \mathbb{C}^{d \times d}$ such that

$$\text{Re}\langle C(x) \xi, \xi \rangle \geq \mu |\xi|^2 \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{C}^d, \text{ and,}$$

$$\|C(x)\| \leq M \quad \text{for all } x \in \Omega,$$

where $\|C(x)\|$ is the $\ell_2$-norm of $C(x)$ in $\mathbb{C}^d$ and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{C}^d$. Here and in the sequel $c_{kl}(x)$ is the appropriate matrix coefficient of $C(x)$. We define $E(\Omega) = \bigcup_{\mu, M > 0} E(\Omega, \mu, M)$. For all $C \in E(\Omega)$ define the closed sectorial form

$$a: W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \rightarrow \mathbb{C}$$

by

$$a(u, v) = \int_{\Omega} \sum_{k,l=1}^d c_{kl} (\partial_k u) (\partial_l v)$$

and let $A_D^C$ be the associated operator. Note that $A_D^C$ has Dirichlet boundary conditions. If no confusion is possible then we drop the $C$ and write $A_D = A_D^C$. 

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Let $\kappa \in (0, 1)$. The space $C^\kappa(\Omega)$ is the space of all Hölder continuous functions of order $\kappa$ on $\Omega$ with semi-norm
\[ |||u|||_{C^\kappa(\Omega)} = \sup\{|u(x) - u(y)| : x, y \in \Omega, \ 0 < |x - y| \leq 1\}.\]

Let $\mu, M > 0$. We define $\mathcal{E}^\kappa(\Omega, \mu, M)$ to be the set of all continuous $C \in \mathcal{E}(\Omega, \mu, M)$ such that $c_{kl} \in C^\kappa(\Omega)$ for all $k, l \in \{1, \ldots, d\}$, and
\[ |||c_{kl}|||_{C^\kappa(\Omega)} \leq M \quad \text{for all } k, l \in \{1, \ldots, d\}.\]

Define $\mathcal{E}^\kappa(\Omega) = \bigcup_{\mu, M > 0} \mathcal{E}^\kappa(\Omega, \mu, M)$.

The main theorem of this section is the following.

**Theorem 3.1.** Let $\kappa, \tau' \in (0, 1)$ and $\mu, M, \tau > 0$. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with a $C^{1+\kappa}$-boundary. Then there exist $a, b > 0$ and $\omega \in \mathbb{R}$ such that for every $C \in \mathcal{E}^\kappa(\Omega, \mu, M)$ and $V \in L_\infty(\Omega, \mathbb{R})$ with $||V||_\infty \leq M$ there exists a function $(t, x, y) \mapsto H_t(x, y)$ from $(0, \infty) \times \Omega \times \Omega$ into $C$ such that the following is valid.

(a) The function $(t, x, y) \mapsto H_t(x, y)$ is continuous from $(0, \infty) \times \Omega \times \Omega$ into $C$.

(b) For all $t \in (0, \infty)$ the function $H_t$ is the kernel of the operator $e^{-t(A_D + V)}$.

(c) For all $t \in (0, \infty)$ the function $H_t$ is once differentiable in each variable and the derivative with respect to one variable is differentiable in the other variable. Moreover, for every multi-index $\alpha, \beta$ with $0 \leq |\alpha|, |\beta| \leq 1$ one has
\[ ||(\partial_\alpha^\beta H_t)(x, y)|| \leq a \ t^{-d/2} \ t^{-|\alpha|+|\beta|/2} \ e^{-b \frac{|x-y|^2}{t}} \ e^{\omega t} \]
and
\[ ||(\partial_\alpha^\beta H_t)(x+h, y+k) - (\partial_\alpha^\beta H_t)(x, y)|| \]
\[ \leq a \ t^{-d/2} \ t^{-|\alpha|+|\beta|/2} \ \left( \frac{|h|+|k|}{\sqrt{t}+|x-y|} \right)^\kappa \ e^{-b \frac{|x-y|^2}{t}} \ e^{\omega t} \]
for all $x, y \in \Omega$ and $h, k \in \mathbb{R}^d$ with $x + h, y + k \in \Omega$ and $|h| + |k| \leq \tau \sqrt{t} + \tau' |x-y|$.

(d) If $\Re V \geq 0$, then $\omega < 0$.

The proof requires a lot of preparation. First we introduce the pointwise Morrey and Campanato semi-norms as in [ERE].

Let $\Omega \subset \mathbb{R}^d$ be open. For all $x \in \mathbb{R}^d$ and $r > 0$ define $\Omega(x, r) = \Omega \cap B(x, r)$. For all $\gamma \in [0, d]$, $R_e \in (0, 1]$ and $x \in \Omega$ define $|| \cdot ||_{M, \gamma, x, \Omega, R_e} : L_2(\Omega) \to [0, \infty]$ by
\[ ||u||_{M, \gamma, x, \Omega, R_e} = \sup_{r \in (0, R_e]} \left( r^{-\gamma} \int_{\Omega(x, r)} |u|^2 \right)^{1/2}. \]

Next, for all $\gamma \in [0, d+2]$, $R_e \in (0, 1]$ and $x \in \Omega$ define $||| \cdot |||_{M, \gamma, x, \Omega, R_e} : L_2(\Omega) \to [0, \infty]$ by
\[ |||u|||_{M, \gamma, x, \Omega, R_e} = \sup_{r \in (0, R_e]} \left( r^{-\gamma} \int_{\Omega(x, r)} |u - \langle u \rangle_{\Omega(x, r)}|^2 \right)^{1/2}, \]
where for an $L_2$ function $v$ we denote by $\langle v \rangle_\Omega = \frac{1}{|D|} \int_D v$ the average of $v$ over a bounded measurable subset $D$ of the domain of $v$ with $|D| > 0$. If no confusion is possible, then we drop the dependence of $\Omega$.

As for Morrey and Campanato spaces, one has the following connections.

**Lemma 3.2.**

(a) For all $\gamma \in [0, d)$, $\bar{c} > 0$ and $R_0 \in (0, 1]$ there exist $c_1, c_2 > 0$ such that

$$|||u|||_{M, \gamma, x, R_0}^2 \leq |||u|||_{M, \gamma, x, R_0}^2 \leq c_1 |||u|||_{M, \gamma, x, R_0}^2 + c_2 \int_{\Omega(x, R_0)} |u|^2$$

for all open $\Omega \subset \mathbb{R}^d$, $x \in \Omega$ and $u \in L_2(\Omega)$ such that $|\Omega(x, r)| \geq \bar{c}r^d$ for all $r \in (0, R_0]$.

(b) Let $\Omega \subset \mathbb{R}^d$ be open, $\gamma \in (d, d+2)$, $\bar{c} > 0$, $x \in \Omega$, $u \in L_2(\Omega)$ and $R_0 \in (0, 1]$. Assume that $|||u|||_{M, \gamma, x, R_0} < \infty$ and $|\Omega(x, r)| \geq \bar{c}r^d$ for all $r \in (0, R_0]$. Then $\lim_{R \downarrow 0} \langle u \rangle_{\Omega(x, R)}$ exists. Write $\hat{u}(x) = \lim_{R \downarrow 0} \langle u \rangle_{\Omega(x, R)}$. Then

$$|||\langle u \rangle_{\Omega(x, R)} - \hat{u}(x)||| \leq \frac{2^{1+d/2}}{\sqrt{\bar{c}(1 - 2^{-(\gamma-d)/2})}} R^{(\gamma-d)/2} |||u|||_{M, \gamma, x, R_0}$$

for all $R \in (0, R_0]$.

(c) Let $\gamma \in (d, d+2)$ and $\bar{c} > 0$. Then there exists a $c > 0$ such that

$$|\hat{u}(x) - \hat{u}(y)| \leq c (|||u|||_{M, \gamma, x, R_0} + |||u|||_{M, \gamma, y, R_0}) |x - y|^{(\gamma-d)/2}$$

for all open $\Omega \subset \mathbb{R}^d$, $x, y \in \Omega$, $R_0 \in (0, 1]$ and $u \in L_2(\Omega)$ such that $|||u|||_{M, \gamma, x, R_0} < \infty$, $|||u|||_{M, \gamma, y, R_0} < \infty$, $|x - y| \leq \frac{R_0}{2}$ and, in addition, $|\Omega(x, r)| \geq \bar{c}r^d$ and $|\Omega(y, r)| \geq \bar{c}r^d$ for all $r \in (0, R_0]$, where $\hat{u}(x)$ and $\hat{u}(y)$ are as in (b).

**Proof.** See the appendix in [ERe].

Let $\Omega \subset \mathbb{R}^d$ be open and $C \in \mathcal{E}(\Omega)$. Let $u \in W^{1,2}(\Omega)$. Then we say that $A^C u = 0$ weakly on $\Omega$ if

$$\int_\Omega \sum_{k,l=1}^d c_{kl} (\partial_k u) (\partial_l v) = 0 \quad (7)$$

for all $v \in C_0^\infty(\Omega)$. Then by density (7) is valid for all $v \in W^{1,2}_0(\Omega)$.

We need various De Giorgi estimates. First we need interior De Giorgi estimates.

**Lemma 3.3.** Let $\Omega \subset \mathbb{R}^d$ be open and $\mu, M > 0$. Then there exists a $c_{DG} > 0$ such that

$$\int_{B(x, r)} |\nabla u|^2 \leq c_{DG} \left( \frac{r}{R} \right)^d \int_{B(x, R)} |\nabla u|^2 \quad \text{and}$$

$$\int_{B(x, r)} |\partial_k u - \langle \partial_k u \rangle_{B(x, r)}|^2 \leq c_{DG} \left( \frac{r}{R} \right)^{d+2} \int_{B(x, R)} |\partial_k u - \langle \partial_k u \rangle_{B(x, R)}|^2 \quad (9)$$

for all $k \in \{1, \ldots, d\}$, $x \in \Omega$, $R \in (0, 1]$, $r \in (0, R]$, $u \in W^{1,2}(B(x, R))$ and constant coefficient $C \in \mathcal{E}(\Omega, \mu, M)$ satisfying $B(x, R) \subset \Omega$ and $A^C u = 0$ weakly on $B(x, R)$. 

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Proof. The estimate (8) was first proved by De Giorgi. For a proof, see Corollario [7.I] in Campanato [Cam]. The estimate (9) is in Corollario [7.II] of the same paper. The uniformity of the constants follows from the proof. Note that the coefficients can be complex and non-symmetric. The proofs in [Cam] also work for complex and non-symmetric coefficients with obvious modifications.

We also need De Giorgi estimates on the boundary. Define

$$E = (-4, 4)^d \quad \text{and} \quad E^- = (-4, 4)^{d-1} \times (-4, 0)$$

the open cube in $\mathbb{R}^d$ and its lower half $E^-$. The midplate is $P = E \cap \{x \in \mathbb{R}^d : x_d = 0\}$ We also need the cubes, lower halves and midplates with half and a quarter sizes, denoted by $\frac{1}{2} E$, $\frac{1}{2} E^-$, $\frac{1}{2} P$, etc. Recall that $E^-(x, r) = E^- \cap B(x, r)$ for all $x \in \mathbb{R}^d$ and $r > 0$.

Lemma 3.4. There exists a $c_{DG} > 0$ such that

$$\int_{E^-(x, r)} |\partial_i u|^2 \leq c_{DG} \left(\frac{r}{R}\right)^{d+2} \int_{E^-(x, R)} |\partial_i u|^2, \quad (10)$$

$$\int_{E^-(x, r)} |\partial_d u|^2 \leq c_{DG} \left(\frac{r}{R}\right)^d \int_{E^-(x, R)} |\partial_d u|^2 \quad \text{and} \quad (11)$$

$$\int_{E^-(x, r)} |\partial_d u - (\partial_d u)_{E^-(x, r)}|^2 \leq c_{DG} \left(\frac{r}{R}\right)^{d+2} \int_{E^-(x, R)} |\partial_d u - (\partial_d u)_{E^-(x, R)}|^2 \quad (12)$$

for all $i \in \{1, \ldots, d-1\}$, $x \in \frac{1}{2} P$, $R \in (0, 1]$, $r \in (0, R]$, $u \in W^{1,2}(E^-(x, R))$ and constant coefficient $C \in \mathcal{E}(E^-, \mu, M)$ satisfying $(\text{Tr} u)|_{P \cap E(x, R)} = 0$ and $A^C u = 0$ weakly on $E^-(x, R)$.

Proof. Estimate (10) is Corollario [11.I] and the other two are in Lemma [11.II] in [Cam]. Again the uniformity of the constants follows from the proof and the coefficients can be complex.

We now turn to regularity. Close to the boundary we have to take a coordinate transformation. For good bounds we have to combine the coordinate transformation together with the regularity improvement theorem. In the next lemma we first collect some easy estimates.

Lemma 3.5. Let $\kappa \in (0, 1)$ and $K \geq 1$. Let $\Omega, U \subset \mathbb{R}^d$ open. Let $\Phi$ be a $C^{1+\kappa}$-diffeomorphism from $U$ onto $E$ such that $\Phi(U \cap \Omega) = E^-$ and $\Phi(U \cap \partial \Omega) = P$. Suppose that $K$ is larger than the Lipschitz constant for $\Phi$ and $\Phi^{-1}$. Moreover, suppose that $|||(D\Phi)_{ij}|||_{C^\kappa} \leq K$ and $|||(D(\Phi^{-1}))_{ij}|||_{C^\kappa} \leq K$ for all $i, j \in \{1, \ldots, d\}$, where $D\Phi$ denotes the derivative of $\Phi$. Then one has the following.

(a) Let $\mu, M > 0$. Let $C \in \mathcal{E}(\Omega, \mu, M)$. Define $C^\Phi : E^- \to \mathbb{C}^{d \times d}$ by

$$C^\Phi(y) = \frac{1}{|\det(D\Phi)(\Phi^{-1}(y))|} (D\Phi)(\Phi^{-1}(y)) A(\Phi^{-1}(y)) (D\Phi)^T(\Phi^{-1}(y)). \quad (13)$$
Then $C^Φ ∈ E^κ(E^−, (d!K^{d+2})−1 μ, d!d^2 K^{d+2} M)$. Moreover, if $u, v ∈ W^{1,2}(U ∩ Ω)$, then
\begin{align*}
\sum_{k,l=1}^d \int_{U ∩ Ω} c_{kl}(∂_k u) (∂_l v) = \sum_{k,l=1}^d \int_{E^{-}} (C^Φ)_{kl}(∂_k (u ∘ Φ^{-1})) (∂_l (v ∘ Φ^{-1})).
\end{align*}

(b) If $x, x′ ∈ U ∩ Ω$, then $|Φ(x) − Φ(x′)| ≤ K |x − x′|$. Conversely, if $y, y′ ∈ E^−$, then $|Φ^{-1}(y) − Φ^{-1}(y′)| ≤ K |y − y′|$.

(c) If $u ∈ W^{1,2}(Ω)$, then
\begin{align*}
(d K)^{-1} ∥∇(u ∘ Φ^{-1})∥_{L^∞(\frac{1}{d}E^−)} ≤ ∥∇u∥_{L^∞(Ω)} \leq d K ∥∇(u ∘ Φ^{-1})∥_{L^∞(\frac{1}{d}E^−)},
\end{align*}
possibly both norms are infinite.

(d) If $u ∈ W^{1,2}_0(Ω)$, then $(Tr(u ∘ Φ^{-1}))|_P = 0$.

**Proof.** Statements (a)–(c) are elementary. For the proof of Statement (d), first note that the map $v → v ∘ Φ^{-1}$ is continuous from $W^{1,2}(Ω)$ into $W^{1,2}(E^−)$ and the map $v → 1_P · Tr v$ is continuous from $W^{1,2}(E^−)$ into $L_2(∂E^−)$. So the map $v → 1_P · Tr v ∘ Φ^{-1}$ is continuous from $W^{1,2}(Ω)$ into $L_2(∂E^−)$. Obviously $1_P · Tr v ∘ Φ^{-1} = 0$ for all $v ∈ C^∞_c(Ω)$. Then Statement (d) follows from the density of $C^∞_c(Ω)$ in $W^{1,2}_0(Ω)$.

The first regularity lemma is with half-balls and points on the boundary. It is a variation of Teorema [13.1] in [Cam], with an additional term $(f_0, v)_{L_2(Ω)}$. The most interesting case occurs for $δ = 0$ in the next lemma, but we also need the lemma with $δ > 0$ to avoid a technical complication in the proof of Proposition 3.13.

**Lemma 3.6.** Let $κ ∈ (0, 1)$, $K ≥ 1$, $δ ∈ [0, κ]$ and $μ, M > 0$. Then there exists a $c ≥ 1$ such that the following is valid.

Let $Ω, U ⊂ R^d$ open. Let $Φ$ be a $C^{1+κ}$-diffeomorphism from $U$ onto $E$ such that $Φ(U ∩ Ω) = E^−$ and $Φ(U ∩ ∂Ω) = P$. Suppose that $K$ is larger than the Lipschitz constant for $Φ$ and $Φ^{-1}$. Moreover, suppose that $∥(DΦ)_{ij}∥_{C^κ} ≤ K$ and $∥(D(Φ^{-1}))_{ij}∥_{C^κ} ≤ K$ for all $i, j ∈ \{1, \ldots, d\}$, where $DΦ$ denotes the derivative of $Φ$. Let $C ∈ E^κ(Ω, μ, M)$, $u ∈ W^{1,2}_0(Ω)$ and $f_0, f_1, \ldots, f_d ∈ L_2(Ω)$ and suppose that
\begin{align}
\int_Ω \sum_{k,l=1}^d c_{kl}(∂_k u)(∂_l v) = (f_0, v)_{L_2(Ω)} − \sum_{k=1}^d (f_k, ∂_k v)_{L_2(Ω)}
\end{align}

for all $v ∈ W^{1,2}_0(Ω)$. Define $\tilde{u}_0: E^− → C$ by $\tilde{u} = u ∘ Φ^{-1}$. Let $x ∈ \frac{1}{2} P$. For all $ρ ∈ (0, 1]$ define

\begin{align*}
Ψ(ρ) &= \int_{E^−(x, ρ)} |∂_d \tilde{u} − \langle ∂_d \tilde{u}, E^−(x, ρ) \rangle|^2 + \sum_{i=1}^{d-1} \int_{E^−(x, ρ)} |∂_i \tilde{u}|^2,
\end{align*}

set $γ = d + 2κ − δ$ and

\begin{align*}
c_0 &= ∥f_0 ∘ Φ^{-1}∥_{M, γ−2, x, E^−, 1} + \sum_{k=1}^d ∥f_k ∘ Φ^{-1}∥_{M, γ, x, E^−, 1} + ∥∇\tilde{u}∥_{M, d−κ, x, E^−, 1}.
\end{align*}
Then
\[ \Psi(r) \leq c \left( \frac{r}{R} \right)^\gamma \Psi(R) + c c_0^2 r^\gamma \]
for all \( r, R \in (0, 1] \) with \( 0 < r \leq R \).

**Proof.** Let \( c_{DG} > 0 \) be as in Lemma 3.4, but with \( \mu \) replaced by \((d!K^{d+2})^{-1}\mu\) and \( M \) replaced by \( d!d^2K^{d+2}M \). Let \( R \in (0, 1] \). Let \( C^\Phi \) be as in (13). Since \( C^\Phi \) is Hölder continuous, it extends uniquely to a continuous function on \( \overline{E}^\gamma \), which we also denote by \( C^\Phi \). We will freeze the coefficients of \( C^\Phi \) at \( x \). There exists a unique \( \tilde{v} \in W_0^{1,2}(E^-(x, R)) \) such that

\[
\sum_{k,l=1}^d \int_{E^-(x,R)} (C^\Phi)_{kl}(x)(\partial_k \tilde{v}) \frac{\partial_l r}{r} = \sum_{k,l=1}^d \int_{E^-(x,R)} (C^\Phi)_{kl}(x)(\partial_k \tilde{u}) \frac{\partial_l r}{r} \quad (15)
\]

for all \( \tau \in W_0^{1,2}(E^-(x, R)) \). Define \( v : \Omega \to C \) by

\[
v(y) = \begin{cases} 
\tilde{v}(\Phi(y)) & \text{if } y \in \Phi^{-1}(E^-(x, R)), \\
0 & \text{if } y \in \Omega \setminus \Phi^{-1}(E^-(x, R)).
\end{cases}
\]

Then \( v \in W_0^{1,2}(\Omega) \). Set \( w = u - v, \tilde{w} = v \circ \Phi^{-1} \) and \( \tilde{\tilde{w}} = w \circ \Phi^{-1} \). Clearly \( w \in W_0^{1,2}(\Omega) \) and \( \text{Tr} \tilde{\tilde{w}}|_\Gamma = 0 \) by Lemma 3.5(d). Moreover, \( A(c^\Phi)(\tilde{\tilde{w}}) = 0 \) weakly on \( E^-(x, R) \) by (15). Let \( r \in (0, R] \). Using (12), one deduces that

\[
\int_{E^-(x,r)} |\partial_d \tilde{u} - \langle \partial_d \tilde{u} \rangle_{E^-(x,r)}|^2 \\
\leq \int_{E^-(x,r)} |\partial_d \tilde{u} - \langle \partial_d \tilde{u} \rangle_{E^-(x,r)}|^2 \\
\leq 2 \int_{E^-(x,r)} |\partial_d \tilde{w} - \langle \partial_d \tilde{w} \rangle_{E^-(x,r)}|^2 + 2 \int_{E^-(x,r)} |\nabla \tilde{u}|^2 \\
\leq 2c_{DG} \left( \frac{r}{R} \right)^{d+2} \int_{E^-(x,R)} |\partial_d \tilde{u} - \langle \partial_d \tilde{u} \rangle_{E^-(x,R)}|^2 + 2 \int_{E^-(x,R)} |\nabla \tilde{u}|^2 \\
\leq 4c_{DG} \left( \frac{r}{R} \right)^{d+2} \int_{E^-(x,R)} |\partial_d \tilde{u} - \langle \partial_d \tilde{u} \rangle_{E^-(x,R)}|^2 + (4c_{DG} + 2) \int_{E^-(x,R)} |\nabla \tilde{u}|^2. \quad (16)
\]

Similarly, with (10) one deduces that

\[
\int_{E^-(x,r)} |\partial_i \tilde{u}|^2 \leq 4c_{DG} \left( \frac{r}{R} \right)^{d+2} \int_{E^-(x,R)} |\partial_i \tilde{u}|^2 + (4c_{DG} + 2) \int_{E^-(x,R)} |\nabla \tilde{u}|^2
\]

for all \( i \in \{1, \ldots, d-1\} \). So

\[
\Psi(r) \leq 4c_{DG} \left( \frac{r}{R} \right)^{d+2} \Psi(R) + (4c_{DG} + 2) d \int_{E^-(x,R)} |\nabla \tilde{u}|^2. \quad (17)
\]
Finally, since $|\nabla \tilde{v}|^2$. Ellipticity, the equality $\tilde{v}|E^-(x,R) = \tilde{v}|E^-(x,R)$, (15), Lemma 3.5(a) and (14) give

$$(d!K^{d+2})^{-1} \mu \int_{E^-(x,R)} |\nabla \tilde{v}|^2$$

$$\leq \text{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}(x) (\partial_k \tilde{v}) \overline{\partial_l \tilde{v}}$$

$$= \text{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}(x) (\partial_k \tilde{u}) \overline{\partial_l \tilde{v}}$$

$$= \text{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}(x) (\partial_k \tilde{u}) \overline{\partial_l \tilde{v}} + \text{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}(x) - (C^\Phi)_{kl} (\partial_k \tilde{u}) \overline{\partial_l \tilde{v}}$$

$$= \text{Re} \int_{\Omega} \sum_{k,l=1}^{d} c_{kl} (\partial_k u) (\overline{\partial_l v}) + \text{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}(x) - (C^\Phi)_{kl} (\partial_k \tilde{u}) \overline{\partial_l \tilde{v}}$$

$$= \text{Re} (f_0, v)_{L_2(\Omega)} - \text{Re} \sum_{k=1}^{d} (f_k, \partial_k v)_{L_2(\Omega)}$$

$$+ \text{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}(x) - (C^\Phi)_{kl} (\partial_k \tilde{u}) \overline{\partial_l \tilde{v}}.$$

We estimate the terms separately. First

$$\text{Re} (f_0, v)_{L_2(\Omega)} \leq d! K^d \left( \int_{E^-(x,R)} |f_0 \circ \Phi^{-1}|^2 \right)^{1/2} \left( \int_{E^-(x,R)} |\tilde{v}|^2 \right)^{1/2}$$

$$\leq d! K^d \|f_0 \circ \Phi^{-1}\|_{M,\gamma-2,x,E^{-1,1}} R^{2-\gamma} c_D R \left( \int_{E^-(x,R)} |\nabla \tilde{v}|^2 \right)^{1/2}$$

$$= d! K^d c_D \|f_0 \circ \Phi^{-1}\|_{M,\gamma-2,x,E^{-1,1}} R^{2} \left( \int_{E^-(x,R)} |\nabla \tilde{v}|^2 \right)^{1/2},$$

where $c_D$ is the constant in the Dirichlet type Poincaré inequality in the unit half-ball. Secondly, since $v \in W^{1,2}_0(\Omega)$ and has compact support in $\mathbb{R}^d$, it follows that $\int_{\Phi^{-1}(E^-(x,R))} \partial_k \tilde{v} = \int_{\Omega} \partial_k v = 0$ and therefore

$$\text{Re} \sum_{k=1}^{d} (f_k, \partial_k v)_{L_2(\Omega)} = \text{Re} \sum_{k=1}^{d} \int_{\Phi^{-1}(E^-(x,R))} (f_k - (f_k \circ \Phi^{-1})_{E^-(x,R)}) \overline{\partial_k v}$$

$$\leq d! K^d \sum_{k=1}^{d} \|f_k \circ \Phi^{-1}\|_{M,\gamma,x,E^{-1,1}} \|f_k \circ \Phi^{-1}\|_{M,\gamma,x,E^{-1,1}} R^{2} \left( \int_{E^-(x,R)} |\nabla \tilde{v}|^2 \right)^{1/2}.$$

Finally, since $|(C^\Phi)_{kl}(x) - (C^\Phi)_{kl}(y)| \leq |||C^\Phi|||_{C^\alpha} |x - y|^\alpha \leq d! d^2 K^{d+2} M R^\alpha$ for all
\(k, l \in \{1, \ldots, d\}\) and \(y \in E^-(x, R)\), one deduces that

\[
\text{Re} \sum_{k, l=1}^{d} \int_{E^-(x, R)} \left( (C^\Phi)_{kl}(x) - (C^\Phi)_{kl} \right) (\partial_k \tilde{u}) \overline{\partial_l \tilde{v}}
\leq d! d^2 K^{d+2} M R^\gamma \left( \int_{E^-(x, R)} |\nabla \tilde{u}|^2 \right)^{1/2} \left( \int_{E^-(x, R)} |\nabla \tilde{v}|^2 \right)^{1/2}
\leq d! d^2 K^{d+2} M R^\gamma \|\nabla \tilde{u}\|_{M, d-\delta, x, E^{-1}} \left( \int_{E^-(x, R)} |\nabla \tilde{v}|^2 \right)^{1/2}.
\]

Therefore

\[
\left( \int_{E^-(x, R)} |\nabla \tilde{v}|^2 \right)^{1/2} \leq c_0 c_1 R^\gamma,
\]
where \(c_1 = \mu^{-1} d! d^2 K^{2d+4} (1 + c_D + M)\).

It follows from (17) and (18) that

\[
\Psi(r) \leq 4c_{DG} \left( \frac{r}{R} \right)^{d+2} \Psi(R) + (4c_{DG} + 2) c_0^2 c_1^2 d R^\gamma
\]
for all \(0 < r \leq R \leq 1\). These bounds can be improved by use of Lemma III.2.1 of [Gia]. It follows that there exists an \(a > 0\), depending only of \(c_{DG}, \gamma\) and \(d\), such that

\[
\Psi(r) \leq a \left( \frac{r}{R} \right)^{\gamma} \Psi(R) + a (4c_{DG} + 2) c_0^2 c_1^2 d r^\gamma
\]
for all \(0 < r \leq R \leq 1\), as required. \(\Box\)

We next turn to interior regularity.

**Proposition 3.7.** Let \(\kappa \in (0, 1)\), \(\delta \in [0, \kappa]\) and \(\mu, M > 0\). Then there exists a \(c \geq 1\) such that the following is valid. Let \(\Omega \subset \mathbb{R}^d\) be an open set. Let \(C \in \mathcal{E}^\kappa(\Omega, \mu, M), u \in W^{1,2}(\Omega)\) and \(f_0, f_1, \ldots, f_d \in L_2(\Omega)\). Suppose that

\[
\int_{\Omega} \sum_{k, l=1}^{d} c_{kl} (\partial_k u) (\partial_l v) = (f_0, v)_{L_2(\Omega)} - \sum_{k=1}^{d} (f_k, \partial_k v)_{L_2(\Omega)}
\]
for all \(v \in W_0^{1,2}(\Omega)\). Let \(r, R, R_e \in (0, 1]\) and \(x \in \Omega\) and suppose that \(0 < r \leq R \leq R_e\) and \(B(x, R_e) \subset \Omega\). Then

\[
\Psi_0(r) \leq c \left( \frac{r}{R} \right)^{\gamma} \Psi_0(R) + c c_0^2 r^\gamma,
\]
where \(\gamma = d + 2\kappa - \delta\), for all \(\rho \in (0, 1]\) we define

\[
\Psi_0(\rho) = \sum_{k=1}^{d} \int_{B(x, \rho)} |\partial_k u - \langle \partial_k u \rangle_{B(x, \rho)}|^2
\]
and where

\[
c_0 = \|f_0\|_{M, \gamma-2, x, \Omega, R_e} + \sum_{k=1}^{d} \|f_k\|_{M, \gamma, x, \Omega, R_e} + \|\nabla u\|_{M, d-\delta, x, \Omega, R_e}.
\]

Moreover,

\[
\|\nabla u\|_{M, \gamma, x, \Omega, R_e} \leq c c_0 + c R_e^{-\kappa} \|\nabla u\|_{M, d-\delta, x, \Omega, R_e}.
\]

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Proof. Let $c_{DG} > 0$ be as in (9). Let $R \in (0, R_e]$. There exists a unique $v \in W^{1,2}_0(B(x, R))$ such that
\[
\sum_{k,l=1}^{d} \int_{B(x,R)} c_{kl}(x) (\partial_k v) \partial_l \tau = \sum_{k,l=1}^{d} \int_{B(x,R)} c_{kl}(x) (\partial_k u) \partial_l \tau
\]
for all $\tau \in W^{1,2}_0(B(x, R))$. Extend $v$ by zero to a function from $\Omega$ into $\mathbb{C}$, still denoted by $v$. Set $w = u - v$. Then $w \in W^{1,2}(\Omega)$. Moreover, $A^C(x)w = 0$ weakly on $B(x, R)$. Let $r \in (0, R]$. If $k \in \{1, \ldots, d\}$, then it follows as in the proof of (16), but now using (9) instead of (12), that
\[
\int_{B(x,r)} |\partial_k u - (\partial_k u)_{B(x,r)}|^2 \leq 4c_{DG} \left(\frac{r}{R}\right)^{d+2} \int_{B(x,R)} |\partial_k u - (\partial_k u)_{B(x,R)}|^2 + (4c_{DG} + 2) \int_{B(x,R)} |\nabla v|^2.
\]
Then the remaining part of the proof is very similar to the proof of Lemma 3.6. This time one has to use the Dirichlet type Poincaré inequality on the full unit ball. \hfill \Box

We combine the last lemma and proposition to obtain estimates close to the boundary.

**Proposition 3.8.** Let $k \in (0, 1)$, $K \geq 1$, $\delta \in [0, k]$ and $\mu, M > 0$. Then there exists a $c \geq 1$ such that the following is valid.

Let $\Omega, U \subset \mathbb{R}^d$ open. Let $\Phi$ be a $C^{1+k}$-diffeomorphism from $U$ onto $E$ such that $\Phi(U \cap \Omega) = E^-$ and $\Phi(U \cap \partial \Omega) = P$. Suppose that $K$ is larger than the Lipschitz constant for $\Phi$ and $\Phi^{-1}$. Moreover, suppose that $|||\|D(\Phi)_{ij}\||C^\kappa \leq K$ and $|||\|D(\Phi^{-1})_{ij}\||C^\kappa \leq K$ for all $i, j \in \{1, \ldots, d\}$, where $D\Phi$ denotes the derivative of $\Phi$. Let $C \in \mathcal{E}^\kappa(\Omega, \mu, M)$, $u \in W^{1,2}_0(\Omega)$ and $f_0, f_1, \ldots, f_d \in L_2(\Omega)$ and suppose that
\[
\int_{\Omega} \sum_{k,l=1}^{d} c_{kl}(\partial_k u)(\partial_l v) = (f_0, v)_{L_2(\Omega)} - \sum_{k=1}^{d} (f_k, \partial_k v)_{L_2(\Omega)}
\]
for all $v \in W^{1,2}_0(\Omega)$. Define $\bar{u}: E^- \to \mathbb{C}$ by $\bar{u} = u \circ \Phi^{-1}$. Then
\[
|||\nabla \bar{u}||| \mathcal{M}, \gamma, x, E^{-, 1} \leq c \left(||f_0 \circ \Phi^{-1}|| \mathcal{M}, \gamma - 2, x, E^{-, 1} + \sum_{k=1}^{d} ||f_k \circ \Phi^{-1}|| \mathcal{M}, \gamma, x, E^{-, 1} + ||\nabla \bar{u}|| \mathcal{M}, d - \delta, x, E^{-, 1}\right)
\]
for all $x \in \frac{1}{2}E^-$, where $\gamma = d + 2k - \delta$.

**Proof.** Let $c \geq 1$ be as in Lemma 3.6. Set $\tilde{u} = u \circ \Phi^{-1}$. For all $x \in \frac{1}{2}E^-$ and $\rho \in (0, 1]$ define
\[
\Psi(x, \rho) = \int_{E^-(x, \rho)} |\partial_d \tilde{u} - (\partial_d \tilde{u})_{E^-(x, \rho)}|^2 + \sum_{i=1}^{d-1} \int_{E^-(x, \rho)} |\partial_i \tilde{u}|^2 	ext{ and}
\]
\[
\Psi_0(x, \rho) = \sum_{k=1}^{d} \int_{E^-(x, \rho)} |\partial_k \tilde{u} - (\partial_k \tilde{u})_{E^-(x, \rho)}|^2.
\]
Clearly $\Psi_0(x, \rho) \leq \Psi(x, \rho) \leq \Psi(x, 1) \leq \|\nabla \tilde{u}\|_{L^2(E^-(x, 1))}^2 \leq \|\nabla \tilde{u}\|_{M, d-\delta, x, E^-, 1}^2$.

Let $x \in \frac{1}{2} E^-$. Set

$$c_0 = \|f_0\|_{M, \gamma-2, x, E^-, 1} + \sum_{k=1}^{d} \|f_k\|_{M, \gamma, x, E^-, 1} + \|\nabla \tilde{u}\|_{M, d-\delta, x, E^-, 1}.$$

If follows as in the proofs of Proposition 3.7 and Lemma 3.6 that there exists a $\tilde{c} \geq 1$, depending only on $\kappa$, $K$, $\delta$, $\mu$ and $M$, such that

$$\Psi_0(x, r) \leq \tilde{c} \left( \frac{r}{R} \right)^{\gamma} \Psi_0(x, R) + \tilde{c} c_0^2 r^{\gamma},$$

(19)

for all $r, R \in (0, 1]$ with $r \leq R \leq |x_d|$.

Define $y = (x_1, \ldots, x_{d-1}, 0)$. Then $y \in \frac{1}{2} P$. Let $r \in (0, 1]$. We distinguish four cases.

Case 1. Suppose that $r \leq |x_d| \leq \frac{1}{4}$.

Then (19), the inclusion $E^-(x, |x_d|) = B(x, |x_d|) \subset E^-(y, 2|x_d|)$, Lemma 3.6 and the inclusion $E^-(y, \frac{1}{2}) \subset E^-(x, 1)$ give

$$\Psi_0(x, r) \leq \tilde{c} \left( \frac{r}{|x_d|} \right)^{\gamma} \Psi_0(x, |x_d|) + \tilde{c} c_0^2 r^{\gamma}$$

$$\leq \tilde{c} \left( \frac{r}{|x_d|} \right)^{\gamma} \Psi(x, |x_d|) + \tilde{c} c_0^2 r^{\gamma}$$

$$\leq \tilde{c} \left( \frac{r}{|x_d|} \right)^{\gamma} \Psi(y, 2|x_d|) + \tilde{c} c_0^2 r^{\gamma}$$

$$\leq \tilde{c} \left( \frac{r}{|x_d|} \right)^{\gamma} \left( c (4|x_d|)^{\gamma} \Psi(y, \frac{1}{2}) + c c_0^2 (2|x_d|)^{\gamma} \right) + \tilde{c} c_0^2 r^{\gamma}$$

$$\leq 4^{\gamma} c \tilde{c} r^{\gamma} \Psi(x, 1) + 2^{\gamma} c \tilde{c} c_0^2 r^{\gamma} + \tilde{c} c_0^2 r^{\gamma}$$

$$\leq 4^{\gamma + 1} c \tilde{c} c_0^2 r^{\gamma}.$$

Case 2. Suppose that $|x_d| \leq r \leq \frac{1}{4}$.

Then the inclusion $E^-(x, |x_d|) \subset E^-(y, 2r)$ and Lemma 3.6 give

$$\Psi_0(x, r) \leq \Psi(x, r) \leq \Psi(y, 2r) \leq c (4r)^{\gamma} \Psi(y, \frac{1}{2}) + c c_0^2 (2r)^{\gamma}$$

$$\leq c (4r)^{\gamma} \Psi(x, 1) + c c_0^2 (2r)^{\gamma} \leq 4^{\gamma + 1} c c_0^2 r^{\gamma}.$$

Case 3. Suppose that $r \geq \frac{1}{4}$.

Then $\Psi_0(x, r) \leq \|\nabla \tilde{u}\|_{L^2(E^-(x, 1))}^2 \leq 4^{d+2} c_0^2 r^{\gamma}$.

Case 4. Suppose that $r \leq \frac{1}{4} \leq |x_d|$.

Then (19) gives $\Psi_0(x, r) \leq c (4r)^{\gamma} \Psi_0(x, \frac{1}{4}) + \tilde{c} c_0^2 r^{\gamma} \leq 4^{\gamma + 1} c \tilde{c} c_0^2 r^{\gamma}$.

The four cases together complete the proof of the proposition.

Using the De Giorgi estimates (8) one also has interior regularity for $A^C$ in the Morrey-region. The proposition is a modification of a proposition which appears at many places in the literature ([Mor], [GM] Theorem 5.13, [Aus] Theorem 3.6, [AT] Lemma 1.12, [ERo2] Proposition 4.2, [DER] Proposition A.3.1, [ERE] Proposition 3.2.)
Proposition 3.9. Let $\kappa \in (0, 1)$, $\mu, M > 0$, $\gamma \in [0, d)$ and $\delta \in (0, 2]$ with $\gamma + \delta < d$. Then there exists an $c > 0$, such that the following is valid. Let $\Omega \subset \mathbb{R}^d$ be an open set. Let $C \in \mathcal{E}^\kappa(\Omega, \mu, M)$, $u \in W^{1,2}(\Omega)$ and $f_0, f_1, \ldots, f_d \in L_2(\Omega)$. Suppose that

$$
\int_{\Omega} \sum_{k,l=1}^{d} c_{kl} \left( \partial_k u \right) \left( \partial_l v \right) = (f_0, v)_{L_2(\Omega)} - \sum_{k=1}^{d} (f_k, \partial_k v)_{L_2(\Omega)}
$$

for all $v \in W_0^{1,2}(\Omega)$. Let $x \in \Omega$, $R_\varepsilon \in (0, 1]$ and suppose that $B(x, R_\varepsilon) \subset \Omega$. Then

$$
\|\nabla u\|_{M, \gamma + \delta, x, \Omega, R_\varepsilon} \leq c \left( \varepsilon^{2-\delta} \|f_0\|_{M, \gamma, x, \Omega, R_\varepsilon} + \sum_{k=1}^{d} \|f_k\|_{M, \gamma + \delta, x, \Omega, R_\varepsilon} + \varepsilon^{-(\gamma + \delta)} \|\nabla u\|_{L_2(\Omega)} \right)
$$

for all $\varepsilon \in (0, 1]$.

Similarly, using the De Giorgi estimates (10) and (11) one also has boundary regularity in the Morrey region.

Proposition 3.10. Let $\kappa \in (0, 1)$, $K \geq 1$, $\mu, M > 0$, $\gamma \in [0, d)$ and $\delta \in (0, 2]$ with $\gamma + \delta < d$. Then there exists an $c > 0$, such that the following is valid.

Let $\Omega, U \subset \mathbb{R}^d$ open. Let $\Phi$ be a $C^{1+\kappa}$-diffeomorphism from $U$ onto $E$ such that $\Phi(U \cap \Omega) = E^-$ and $\Phi(U \cap \partial \Omega) = \partial \Omega$. Suppose that $K$ is larger than the Lipschitz constant for $\Phi$ and $\Phi^{-1}$. Moreover, suppose that $||| (D\Phi)_{ij} |||_{C^\kappa} \leq K$ and $||| (D(\Phi^{-1}))_{ij} |||_{C^\kappa} \leq K$ for all $i, j \in \{1, \ldots, d\}$, where $D\Phi$ denotes the derivative of $\Phi$. Let $C \in \mathcal{E}^\kappa(\Omega, \mu, M)$, $u \in W_0^{1,2}(\Omega)$ and $f_0, f_1, \ldots, f_d \in L_2(\Omega)$ and suppose that

$$
\int_{\Omega} \sum_{k,l=1}^{d} c_{kl} \left( \partial_k u \right) \left( \partial_l v \right) = (f_0, v)_{L_2(\Omega)} - \sum_{k=1}^{d} (f_k, \partial_k v)_{L_2(\Omega)}
$$

for all $v \in W_0^{1,2}(\Omega)$. Define $\tilde{u}: E^- \to C$ by $\tilde{u} = u \circ \Phi^{-1}$. Then

$$
\|\nabla \tilde{u}\|_{M, \gamma + \delta, x, E^-, -1} \leq c \left( \varepsilon^{2-\delta} \|f_0 \circ \Phi^{-1}\|_{M, \gamma, x, E^-, -1} + \sum_{k=1}^{d} \|f_k \circ \Phi^{-1}\|_{M, \gamma + \delta, x, E^-, -1} + \varepsilon^{-(\gamma + \delta)} \|\nabla \tilde{u}\|_{L_2(\Omega)} \right)
$$

for all $x \in \frac{1}{2} E^-$ and $\varepsilon \in (0, 1]$.

Let $\Omega \subset \mathbb{R}^d$ be an open set, let $C \in \mathcal{E}(\Omega)$ and $V \in L_\infty(\Omega)$. Let $T$ be the semigroup generated by $-(A_D + V)$. We omit the dependence of $T$ on $C$ and $V$ in our notation, since that will be clear from the context. We also need the Davies perturbation. Let

$$
\mathcal{D} = \{ \psi \in C^\infty_c(\mathbb{R}^d, \mathbb{R}) : \|\nabla \psi\|_\infty \leq 1 \}.
$$

For all $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}$ define the multiplication operator $U_R^\rho$ by $U_R^\rho u = e^{-\rho \psi} u$. Note that $U_R^\rho \in W_0^{1,2}(\Omega)$ for all $u \in W_0^{1,2}(\Omega)$. Let $T_t^\rho = U_t^\rho$ be the Davies perturbation for all $t > 0$. Let $-A^\rho$ be the generator of $(T_t^\rho)_{t>0}$. Then $A^\rho$ is the operator associated with the form $f^\rho$ with form domain $D(f^\rho) = W_0^{1,2}(\Omega)$ and

$$
f^\rho(u, v) = a(u, v) + \int_{\Omega} \sum_{i=1}^{d} \left( a_i^\rho \partial_i u \right) \left( \partial_i \psi \right) + b_i^\rho u \left( \partial_i \psi \right) + \int_{\Omega} a_0^\rho u \left( \partial_i \psi \right) \tag{20}
$$
with
\[ a_k^{(p)} = -\rho \sum_{l=1}^{d} c_{kl} \partial_l \psi, \quad b_k^{(p)} = \rho \sum_{l=1}^{d} a_{lk} \partial_l \psi \]
and
\[ a_0^{(p)} = V - \rho^2 \sum_{k,l=1}^{d} c_{kl} (\partial_k \psi) \partial_l \psi. \]

We start with \( L_2 \)-estimates for the perturbed semigroup.

**Lemma 3.11.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded open set. For all \( \mu, M > 0 \) there exist \( c_0, \omega_0, \omega_1 > 0 \) such that
\[
\| T_t^\rho u \|_{L_2(\Omega)} \leq e^{-\omega_1 t} e^{\| (Re \nabla V) \|_\infty t} e^{\omega_0 \rho^2 t} u \|_{L_2(\Omega)} , \quad \| \nabla T_t^\rho u \|_{L_2(\Omega)} \leq c_0 t^{-1/2} e^{\omega_0 (1+\rho^2) t} \| u \|_{L_2(\Omega)}
\]
and
\[
\| A^{(p)} T_t^\rho u \|_{L_2(\Omega)} \leq c_0 t^{-1} e^{\omega_0 (1+\rho^2) t} \| u \|_{L_2(\Omega)}
\]
for all \( \kappa \in (0,1), C \in \mathcal{E}^\kappa(\Omega, \mu, M), V \in L_{\infty}(\Omega), u \in L_2(\Omega), t > 0, \rho \in \mathbb{R} \) and \( \psi \in \mathcal{D} \).

**Proof.** By the Dirichlet type Poincaré inequality there exists a \( \lambda > 0 \) such that \( \lambda \int_\Omega |u|^2 \leq \int_\Omega |\nabla u|^2 \) for all \( u \in W_0^{1,2}(\Omega) \). Without loss of generality we may assume that \( \mu \leq 1 \leq M \).

Let \( u \in L_2(\Omega) \). It follows from (20) that
\[
\mu \| \nabla T_t^\rho u \|_{L_2(\Omega)}^2 \leq \text{Re} a(T_t^\rho u)
\]
\[
\leq \text{Re} a(T_t^\rho u) + ((Re V)^+ T_t^\rho u, T_t^\rho u)_{L_2(\Omega)}
\]
\[
\leq \text{Re} t^{(p)}(T_t^\rho u) + 2d M |\rho| \| \nabla T_t^\rho u \|_{L_2(\Omega)} \| T_t^\rho u \|_{L_2(\Omega)}
\]
\[
+ \|(Re V)^-\|_\infty \| T_t^\rho u \|_{L_2(\Omega)}^2 + d M \rho^2 \| T_t^\rho u \|_{L_2(\Omega)}^2
\]
\[
\leq \text{Re} t^{(p)}(T_t^\rho u) + \frac{1}{2} \mu \| \nabla T_t^\rho u \|_{L_2(\Omega)}^2 + \frac{2d^2 M^2 \rho^2}{\mu} \| T_t^\rho u \|_{L_2(\Omega)}^2
\]
\[
+ \|(Re V)^-\|_\infty \| T_t^\rho u \|_{L_2(\Omega)}^2 + d M \rho^2 \| T_t^\rho u \|_{L_2(\Omega)}^2
\]
for all \( t > 0 \). So
\[
\frac{1}{2} \lambda \mu \| T_t^\rho u \|_{L_2(\Omega)}^2 \leq \frac{1}{2} \mu \| \nabla T_t^\rho u \|_{L_2(\Omega)}^2
\]
\[
\leq \text{Re} t^{(p)}(T_t^\rho u) + ((Re V)^-\|_\infty + \omega_1 \rho^2) \| T_t^\rho u \|_{L_2(\Omega)}^2,
\]
where \( \omega_1 = 3d^2 M^2 \mu^{-1} \). Hence
\[
\frac{d}{dt} \| T_t^\rho u \|_{L_2(\Omega)}^2 = -2 \text{Re} (A^{(p)} T_t^\rho u, T_t^\rho u)_{L_2(\Omega)}
\]
\[
= -2 \text{Re} t^{(p)}(T_t^\rho u) \leq 2(-\frac{1}{2} \lambda \mu + \|(Re V)^-\|_\infty + \omega_1 \rho^2) \| T_t^\rho u \|_{L_2(\Omega)}^2
\]
for all \( t > 0 \). This implies that
\[
\| T_t^\rho u \|_{L_2(\Omega)} \leq e^{-\frac{1}{2} \lambda \mu t} e^{\| (Re V)^-\|_\infty t} e^{\omega_1 \rho^2 t} \| u \|_{L_2(\Omega)}
\]
for all \( t > 0 \).

The other estimates of the lemma follow as in the proof of Lemma 2.1 in [EO3]. \( \square \)
By a Neumann type Poincaré inequality there is a relation between the Campanato norm and the Morrey norm of the gradient of a function.

**Lemma 3.12.** There exists a $c_N > 0$ such that

$$\|u\|_{M,\gamma+2,x,E^{-},1} \leq c_N \|\nabla u\|_{M,\gamma,x,E^{-},1}$$

and

$$\|v\|_{M,\gamma+2,y,\Omega, R_e} \leq c_N \|\nabla v\|_{M,\gamma,y,\Omega, R_e}$$

for all $\gamma \in [0,d)$, $u \in W^{1,2}(E^{-})$, $x \in \frac{1}{2}E^{-}$, open $\Omega \subset \mathbb{R}^{d}$, $v \in W^{1,2}(\Omega)$, $y \in \Omega$ and $R_e \in (0,1]$ with $B(y, R_e) \subset \Omega$.

Next we consider $L_2$–$W^{1+\kappa,\infty}$ estimates for the perturbed semigroup. We start with bounds close to the boundary.

**Proposition 3.13.** Let $\kappa \in (0,1)$, $K \geq 1$ and $\mu, M > 0$. Then there exist $c, \omega > 0$ such that the following is valid.

Let $\Omega, U \subset \mathbb{R}^{d}$ open. Let $\Phi$ be a $C^{1+\kappa}$–diffeomorphism from $U$ onto $E$ such that $\Phi(U \cap \Omega) = E^{-}$ and $\Phi(U \cap \partial \Omega) = P$. Suppose that $K$ is larger than the Lipschitz constant for $\Phi$ and $\Phi^{-1}$. Moreover, suppose that $\|((D\Phi)_{ij})\|_{C^\kappa} \leq K$ and $\|((D\Phi^{-1})_{ij})\|_{C^\kappa} \leq K$ for all $i, j \in \{1, \ldots, d\}$, where $D\Phi$ denotes the derivative of $\Phi$. Let $C \in \mathcal{E}^\kappa(\Omega, \mu, M)$, $V \in L_\infty(\Omega)$, $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}$, with $\|V\|_{\infty} \leq M$. Then $\nabla((T_t^\rho u) \circ \Phi^{-1})$ is continuous on $\frac{1}{2}E^{-}$. Moreover,

$$\|T_t^\rho u\|_{L_\infty(\Phi^{-1}(\frac{1}{2}E^{-}))} \leq c t^{-d/4} e^{\omega(1+\rho^2)t} \|u\|_{L_2(\Omega)},$$

$$\|\nabla T_t^\rho u\|_{L_\infty(\Phi^{-1}(\frac{1}{2}E^{-}))} \leq c t^{-d/4} t^{-1/2} e^{\omega(1+\rho^2)t} \|u\|_{L_2(\Omega)},$$

and

$$|\nabla T_t^\rho u(x) - \nabla T_t^\rho u(y)| \leq c t^{-d/4} t^{-1/2} \kappa/2 e^{\omega(1+\rho^2)t} \|u\|_{L_2(\Omega)} |x - y|^\kappa$$

for all $x, y \in \Phi^{-1}(\frac{1}{2}E^{-})$ with $|x - y| \leq \frac{1}{4K}$.

**Proof.** For all $\gamma \in [0,d-2)$ let $P(\gamma)$ be the hypothesis

There exist $c, \omega > 0$, depending only on $\Omega$, $\kappa$, $\mu$ and $M$, such that

$$\|(T_t^\rho u) \circ \Phi^{-1}\|_{M,\gamma,x,E^{-},1} \leq c t^{-\gamma/4} e^{\omega(1+\rho^2)t} \|u\|_{L_2(\Omega)}$$

and

$$\|\nabla((T_t^\rho u) \circ \Phi^{-1})\|_{M,\gamma,x,E^{-},1} \leq c t^{-\gamma/4} t^{-1/2} e^{\omega(1+\rho^2)t} \|u\|_{L_2(\Omega)}$$

for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x \in \frac{1}{2}E^{-}$.

Clearly $P(0)$ is valid by Lemma 3.11. Arguing as in the proof of Proposition 4.3 in [ERo2], Lemma 3.3 in [EO1] or Lemma 7.1 in [ERe], it follows from Lemma 3.11 and Proposition 3.10 that $P(\gamma)$ is valid for all $\gamma \in [0,d)$.

For all $\gamma \in [0,d+2\kappa]$ let $P'(\gamma)$ be the hypothesis
There exist $c, \omega > 0$, depending only on $\Omega, \kappa, \mu$ and $M$, such that
\[
\| (T_t^\rho u) \circ \Phi^{-1} \|_{M, \gamma, x, E^{-1}, 1} \leq c t^{-\gamma/4} e^{\omega(1+\rho^2) t} \| u \|_{L_2(\Omega)}
\] (25)
and
\[
\| \nabla ((T_t^\rho u) \circ \Phi^{-1}) \|_{M, \gamma, x, E^{-1}, 1} \leq c t^{-\gamma/4} t^{-1/2} e^{\omega(1+\rho^2) t} \| u \|_{L_2(\Omega)}
\]
for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in D$ and $x \in \frac{1}{2} E^-$. If $\gamma \in [0, d)$, then $P(\gamma)$ and Lemma 3.2(a) imply that $P'(\gamma)$ is valid. Then the Poincaré inequality of Lemma 3.12 and (24) give that (25) is valid for all $\gamma \in [0, d+2\kappa]$ (even for all $\gamma \in [0, d+\kappa]$). Arguing similarly, using the regularity estimates of Proposition 3.8, it follows that for all $\delta \in [0, \kappa]$ there exist $c, \omega > 0$, depending only on $\Omega, \kappa, \delta, \mu$ and $M$, such that
\[
\| \nabla ((T_t^\rho u) \circ \Phi^{-1}) \|_{M, \gamma, x, E^{-1}, 1} \leq c t^{-\gamma/4} t^{-1/2} e^{\omega(1+\rho^2) t} \| u \|_{L_2(\Omega)} + c \| \nabla ((T_t^\rho u) \circ \Phi^{-1}) \|_{M, d-\delta, x, E^{-1}, 1}
\] (26)
for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in D$ and $x \in \frac{1}{2} E^-$, where $\gamma = d+2\kappa - \delta$. Choose $\delta = \kappa$. Then (24) gives
\[
\| \nabla ((T_t^\rho u) \circ \Phi^{-1}) \|_{M, \gamma, x, E^{-1}, 1} \leq c t^{-(d+\kappa)/4} t^{-1/2} e^{\omega(1+\rho^2) t} \| u \|_{L_2(\Omega)}
\]
for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in D$ and $x \in \frac{1}{2} E^-$, for suitable $c', \omega' > 0$. So $\lim_{R \to 0} \| \nabla ((T_t^\rho u) \circ \Phi^{-1}) \|_{E^{-}(x,R)}$ exists for all $x \in \frac{1}{2} E^-$ by Lemma 3.2(b). Therefore the function $\nabla ((T_t^\rho u) \circ \Phi^{-1})$ is continuous on $\frac{1}{2} E^-$. Choose $R = t^{1/2} e^{-t}$. Then $R \leq 1$ and Lemma 3.2(b) gives that there exists a $c'' > 0$, depending only on $d$ and $\kappa$, such that
\[
\| \nabla ((T_t^\rho u) \circ \Phi^{-1}(x)) \| \leq c'' R^{\kappa/2} \| \nabla ((T_t^\rho u) \circ \Phi^{-1}) \|_{M, d+\gamma, x, E^{-1}, 1} + \langle \nabla ((T_t^\rho u) \circ \Phi^{-1}) \rangle_{E^{-}(x,R)}
\]
\[
\leq c'' t^{-d/4} t^{-1/2} e^{\omega'(1+\rho^2) t} \| u \|_{L_2(\Omega)} + \omega_d^{-1/2} R^{-d/2} \| \nabla ((T_t^\rho u) \circ \Phi^{-1}) \|_{L_2(E^-)}
\]
for all $x \in \frac{1}{2} E^-$. Then (22) follows from (21).

Finally, use (26) with $\delta = 0$ and $x \in \frac{1}{4} E^-$, the quarter lower half of $E$. It follows that there are suitable $c'''$, $\omega'' > 0$ such that
\[
\| \nabla ((T_t^\rho u) \circ \Phi^{-1}) \|_{M, d+2\kappa, x, E^{-1}, 1} \leq c t^{-(d+2\kappa)/4} t^{-1/2} e^{\omega(1+\rho^2) t} \| u \|_{L_2(\Omega)}
\]
for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in D$ and $x \in \frac{1}{4} E^-$. Then (23) follows from Lemma 3.2(c). \hfill \Box

Similar estimates are valid far away from the boundary.
Proposition 3.14. Let \( \kappa \in (0, 1) \), \( R_e \in (0, 1] \) and \( \mu, M > 0 \). Then there exist \( c, \omega > 0 \) such that the following is valid.

Let \( \Omega \subset \mathbb{R}^d \) be open, \( x_0 \in \Omega \) and suppose that \( B(x_0, R_e) \subset \Omega \). Let \( C \in \mathcal{E}^\kappa(\Omega, \mu, M) \) and \( V \in L_\infty(\Omega) \) with \( \|V\|_\infty \leq M \). Then

\[
\|T_t^0 u\|_{L_\infty(B(x_0, \frac{1}{2} R_e))} \leq c t^{-d/4} e^{\omega(1+\rho^2)t} \|u\|_{L_2(\Omega)},
\]

\[
\|\nabla T_t^0 u\|_{L_\infty(B(x_0, \frac{1}{2} R_e))} \leq c t^{-d/4} t^{-1/2} e^{\omega(1+\rho^2)t} \|u\|_{L_2(\Omega)}, \text{ and}
\]

\[
\|\nabla T_t^0 u\|_2(x) - \|\nabla T_t^0 u\|_2(y) \leq c t^{-d/4} t^{-1/2} t^{-\kappa/2} e^{\omega(1+\rho^2)t} \|u\|_{L_2(\Omega)} |x - y|^\kappa
\]

for all \( t > 0 \), \( u \in L_2(\Omega) \), \( \rho \in \mathbb{R} \), \( \psi \in D \) and \( x, y \in B(x_0, \frac{1}{4} R_e) \) with \( |x - y| \leq \frac{1}{4} R_e \).

**Proof.** This follows similarly to the proof of Proposition 3.13, using Propositions 3.7 and 3.9 instead of Propositions 3.8 and 3.10. We leave the proof to the reader. \(\square\)

Proposition 3.15. Let \( \kappa \in (0, 1) \), and \( \mu, M > 0 \). Let \( \Omega \subset \mathbb{R}^d \) be a bounded open set with \( C^{1+\kappa}\)-boundary. Then there exist \( c, \omega > 0 \) such that such that the following is valid.

Let \( C \in \mathcal{E}^\kappa(\Omega, \mu, M) \) and \( V \in L_\infty(\Omega) \) with \( \|V\|_\infty \leq M \). Then

\[
\|T_t^0 u\|_{L_\infty(\Omega)} \leq c t^{-d/4} e^{\omega(1+\rho^2)t} \|u\|_{L_2(\Omega)},
\]

\[
\|\nabla T_t^0 u\|_{L_\infty(\Omega)} \leq c t^{-d/4} t^{-1/2} e^{\omega(1+\rho^2)t} \|u\|_{L_2(\Omega)}, \text{ and}
\]

\[
\|\nabla T_t^0 u\|_{C^\kappa} \leq c t^{-d/4} t^{-1/2} t^{-\kappa/2} e^{\omega(1+\rho^2)t} \|u\|_{L_2(\Omega)}
\]

for all \( t > 0 \), \( u \in L_2(\Omega) \), \( \rho \in \mathbb{R} \) and \( \psi \in D \).

**Proof.** This follows from a compactness argument from Propositions 3.13 and 3.14. \(\square\)

We can now prove the Gaussian Hölder kernel bounds of Theorem 3.1.

**Proof of Theorem 3.1.** Let \( c_0, \omega_0, \omega_1 > 0 \) be as in Lemma 3.11. Then

\[
\|T_t^0 u\|_{L_2(\Omega)} \leq e^{-\omega_1 t} e^{(\Re V)^- t} e^{\omega_0 \rho^2 t} \|u\|_{L_2(\Omega)}
\]

for all \( u \in L_2(\Omega) \), \( t > 0 \), \( \rho \in \mathbb{R} \) and \( \psi \in D \). Then the semigroup \( (e^{(\omega_1 - \|\Re V\|_\infty) t} T_t)^{t>0} \) satisfies the bounds of Proposition 3.15. Therefore the Gaussian Hölder kernel bounds of Theorem 3.1 follows as in the proof of Lemma A.1 in [EO3]. \(\square\)

Since all our estimates are locally uniform, we also obtain the following theorem which is valid for unbounded domains.

**Theorem 3.16.** Let \( \kappa, \kappa' \in (0, 1) \) and \( \mu, M, \tau, K > 0 \). Then there exist \( a, b > 0 \) and \( \omega \in \mathbb{R} \) such that the following is valid.

Let \( \Omega \subset \mathbb{R}^d \) be an open set. Suppose for all \( x \in \partial \Omega \) there exists an open neighbourhood \( U \) of \( x \) and a \( C^{1+\kappa'} \)-diffeomorphism \( \Phi \) from \( U \) onto \( E \) such that

- \( \Phi(x) = 0 \),
• \( \Phi(U \cap \Omega) = E^- \)

• \( \Phi(U \cap \partial \Omega) = P \),

• \( K \) is larger than the Lipschitz constant for \( \Phi \) and \( \Phi^{-1} \), and

• \( \| |(D \Phi)_{ij}| |_{C^\infty} \leq K \) and \( \| |(D(\Phi^{-1}))_{ij}| |_{C^\infty} \leq K \) for all \( i, j \in \{1, \ldots, d\} \) where \( D \Phi \) denotes the derivative of \( \Phi \).

Let \( C \in \mathcal{E}^\kappa(\Omega, \mu, M) \) and \( V \in L_\infty(\Omega) \) with \( \| V \|_\infty \leq M \). Then there exists a function \( (t, x, y) \mapsto H_t(x, y) \) from \( (0, \infty) \times \Omega \times \Omega \) into \( C \) such that the following is valid.

(a) The function \( (t, x, y) \mapsto H_t(x, y) \) is continuous from \( (0, \infty) \times \Omega \times \Omega \) into \( C \).

(b) For all \( t \in (0, \infty) \) the function \( H_t \) is the kernel of the operator \( e^{-t(A_D + V)} \).

(c) For all \( t \in (0, \infty) \) the function \( H_t \) is once differentiable in each variable and the derivative with respect to one variable is differentiable in the other variable. Moreover, for every multi-index \( \alpha, \beta \) with \( 0 \leq |\alpha|, |\beta| \leq 1 \) one has

\[
| (\partial_x^\alpha \partial_y^\beta H_t)(x, y) | \leq a t^{-d/2} t^{-|\alpha|+|\beta|/2} e^{-b \frac{\sqrt{x-y}^2}{t}} e^{\omega t}
\]

and

\[
| (\partial_x^\alpha \partial_y^\beta H_t)(x + h, y + k) - (\partial_x^\alpha \partial_y^\beta H_t)(x, y) | \\
\leq a t^{-d/2} t^{-|\alpha|+|\beta|/2} \left( \frac{|h| + |k|}{\sqrt{t} + |x-y|} \right) ^\kappa e^{-b \frac{\sqrt{x-y}^2}{t}} e^{\omega t}
\]

for all \( x, y \in \Omega \) and \( h, k \in \mathbb{R}^d \) with \( x + h, y + k \in \Omega \) and \( |h| + |k| \leq \tau \sqrt{t} + \tau' |x-y| \).

By a small additional argument one can also add first-order terms to the operator with \( C^\kappa \)-coefficients. We do not need first-order terms in this paper.

4 Green function bounds and regularity properties

This section is devoted to estimates and regularity properties of the resolvent operators \((A_D + V)^{-1}\). We prove estimates for the Green function and its derivatives. We emphasise that in the first theorem the constants are uniform with respect to the complex coefficients \( C \) if \( \text{Re} V \) is positive.

**Theorem 4.1.** Let \( \kappa \in (0, 1) \) and \( \mu, M > 0 \). Let \( \Omega \subset \mathbb{R}^d \) be an open bounded set with a \( C^{1+\kappa} \)-boundary. Then there exists a \( c > 0 \) such that for all \( C \in \mathcal{E}^\kappa(\Omega, \mu, M) \) and \( V \in L_\infty(\Omega) \) with \( \text{Re} V \geq 0 \) and \( \| V \|_\infty \leq M \) the operator \((A_D + V)^{-1}\) has a kernel \( G_V : \{ (x, y) \in \Omega \times \Omega : x \neq y \} \to \mathbb{C} \), which is differentiable in each variable and the derivative is differentiable in the other variable. Moreover, for every multi-index \( \alpha, \beta \) with
0 \leq |\alpha|, |\beta| \leq 1 the function $\partial^\alpha_x \partial^\beta_y G_V$ extends to a locally $\kappa$-Hölder continuous function on $\{(x, y) \in \Omega \times \Omega : x \neq y\}$ with estimates

$$|\langle \partial^\alpha_x \partial^\beta_y G_V \rangle(x, y)| \leq \begin{cases} c |x - y|^{-(d-2+|\alpha|+|\beta|)} & \text{if } d - 2 + |\alpha| + |\beta| \neq 0 \\ c \log(1 + \frac{1}{|x - y|}) & \text{if } d - 2 + |\alpha| + |\beta| = 0 \end{cases}$$ (27)

and

$$|\langle \partial^\alpha_x \partial^\beta_y G_V \rangle(x', y') - \langle \partial^\alpha_x \partial^\beta_y G_V \rangle(x, y)| \leq c \frac{(|x' - x| + |y' - y|)^\kappa}{|x - y|^{d-2+|\alpha|+|\beta|+\kappa}}$$ (28)

for all $x, x', y, y' \in \Omega$ with $x \neq y$ and $|x - x'| + |y - y'| \leq \frac{1}{2} |x - y|$. 

**Proof.** By Theorem 3.1 there are $a, b, \omega > 0$ such that the operator $e^{-t(A_D + V)}$ has a kernel $H_t$ for all $t > 0$, which is once differentiable in each entry, satisfying the bounds

$$|\langle \partial^\alpha_x \partial^\beta_y H_t \rangle(x, y)| \leq a t^{-d/2} t^{-(|\alpha|+|\beta|)/2} e^{-b \frac{|x - y|^2}{t}} e^{-\omega t},$$

$$|\langle \partial^\alpha_x \partial^\beta_y H_t \rangle(x', y') - \langle \partial^\alpha_x \partial^\beta_y H_t \rangle(x, y)| \leq a t^{-d/2} t^{-(|\alpha|+|\beta|)/2} \left( \frac{|x' - x| + |y' - y|}{\sqrt{t}} \right)^\kappa e^{-b \frac{|x - y|^2}{t}} e^{-\omega t}$$ (29)

for all $x, x', y, y' \in \Omega$ and multi-index $\alpha, \beta$ with $0 \leq |\alpha|, |\beta| \leq 1$ and $|x - x'| + |y - y'| \leq \frac{1}{2} |x - y|$. Define

$$G_V(x, y) = \int_0^\infty H_t(x, y) \, dt$$

for all $x, y \in \Omega$ with $x \neq y$. Then $G_V$ is the kernel of the operator

$$(A_D + V)^{-1} = \int_0^\infty T_t \, dt.$$ 

The estimates (29) give

$$|\langle \partial^\alpha_x \partial^\beta_y G_V \rangle(x, y)| \leq \int_0^\infty a t^{-d/2} t^{-(|\alpha|+|\beta|)/2} e^{-b \frac{|x - y|^2}{t}} \, dt = a c_1 |x - y|^{-(d-2+|\alpha|+|\beta|)}$$

and

$$|\langle \partial^\alpha_x \partial^\beta_y G_V \rangle(x', y') - \langle \partial^\alpha_x \partial^\beta_y G_V \rangle(x, y)|$$

$$\leq \int_0^\infty a t^{-d/2} t^{-(|\alpha|+|\beta|)/2} \left( \frac{|x' - x| + |y' - y|}{\sqrt{t}} \right)^\kappa e^{-b \frac{|x - y|^2}{t}} \, dt$$

$$= a c_2 \frac{(|x' - x| + |y' - y|)^\kappa}{|x - y|^{d-2+|\alpha|+|\beta|+\kappa}},$$

where $c_1 = \int_0^\infty a t^{-d/2} t^{-(|\alpha|+|\beta|)/2} e^{-b/t} \, dt < \infty$, under the condition that $d - 2 + |\alpha| + |\beta| \neq 0$, and $c_2 = \int_0^\infty a t^{-d/2} t^{-(|\alpha|+|\beta|)/2} t^{-\kappa/2} e^{-b/t} \, dt < \infty$. If $d - 2 + |\alpha| + |\beta| = 0$ one obtains a logarithmic term, as is well known. \qed

In the self-adjoint case and real valued $V$ we next drop the condition that $V$ is positive. In contrast to the previous theorem, in this case the constants are not uniform with respect to the coefficients $C$. 

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Theorem 4.2. Let $\kappa \in (0,1)$ and $\mu, M > 0$. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with a $C^{1+\kappa}$-boundary. Let $\mathcal{C} \in \mathcal{E}^k(\Omega, \mu, M)$ be real symmetric, $k, l \in \{1, \ldots, d\}$ and $V \in L_{\infty}(\Omega, \mathbb{R})$. Suppose that $0 \notin \sigma(A_D + V)$. Then the operator $(A_D + V)^{-1}$ has a kernel $G_V: \{(x,y) \in \Omega \times \Omega : x \neq y\} \rightarrow \mathbb{R}$, which is differentiable in each variable and the derivative is differentiable in the other variable. Moreover, for every multi-index $\alpha, \beta$ with $0 \leq |\alpha|, |\beta| \leq 1$ the function $\partial_x^\alpha \partial_y^\beta G_V$ extends to a continuous function on $\{(x,y) \in \overline{\Omega \times \Omega} : x \neq y\}$ with estimates

$$|(\partial_x^\alpha \partial_y^\beta G_V)(x,y)| \leq \begin{cases} c |x - y|^{-(d-2+|\alpha|+|\beta|)} & \text{if } d - 2 + |\alpha| + |\beta| \neq 0 \\ c \log(1 + \frac{1}{|x - y|}) & \text{if } d - 2 + |\alpha| + |\beta| = 0 \end{cases}$$

and

$$|(\partial_x^\alpha \partial_y^\beta G_V)(x',y') - (\partial_x^\alpha \partial_y^\beta G_V)(x,y)| \leq c \frac{(|x' - x| + |y' - y|)^\kappa}{|x - y|^{d-2+|\alpha|+|\beta|+\kappa}}$$

for all $x, x', y, y' \in \Omega$ with $x \neq y$ and $|x - x'| + |y - y'| \leq \frac{1}{2} |x - y|$.

Proof. There exists a $\lambda > 0$ such that $V + \lambda \geq 0$. Replacing $V$ by $V + \lambda$, it suffices to show that for all $\lambda > 0$ and $V \in L_{\infty}(\Omega, \mathbb{R})$ with $V \geq 0$ and $0 \notin \sigma(A_D + V - \lambda I)$ the operator $(A_D + V - \lambda I)^{-1}$ has a kernel, denoted by $G_V$, which is differentiable in each variable and the derivative is differentiable in the other variable. Moreover, for every multi-index $\alpha, \beta$ with $0 \leq |\alpha|, |\beta| \leq 1$ the function $\partial_x^\alpha \partial_y^\beta G_V$ extends to a continuous function on $\{(x,y) \in \overline{\Omega \times \Omega} : x \neq y\}$ and

$$|(\partial_x^\alpha \partial_y^\beta G_V)(x,y)| \leq c |x - y|^{-(d-2+|\alpha|+|\beta|)}$$

for all $x, y \in \Omega$ with $x \neq y$.

Since $A_D + V$ is a positive self-adjoint operator with compact resolvent, there exist an orthonormal basis $(u_n)_{n \in \mathbb{N}}$ for $L_2(\Omega)$ of eigenfunctions for $A_D + V$, and a sequence $(\lambda_n)_{n \in \mathbb{N}}$ in $[0, \infty)$ such that $(A_D + V)u_n = \lambda_n u_n$ for all $n \in \mathbb{N}$. There exists an $N \in \mathbb{N}$ such that $\lambda_n > \lambda$ for all $n \in \{N + 1, N + 2, \ldots\}$. Let $\omega_1 = \min\{\lambda_n : n \in \{N + 1, N + 2, \ldots\}\}$. Then $\lambda < \omega_1$. Let $P: L_2(\Omega) \rightarrow L_2(\Omega)$ be the orthogonal projection onto $\text{span}\{u_1, \ldots, u_N\}$. Write $T_t = e^{-t(A_D + V)}$ for all $t > 0$. So $T$ is the semigroup generated by $-(A_D + V)$. Then $\|(I - P)T_t (I - P)\|_{2 \rightarrow 2} \leq e^{-\omega_1 t}$ for all $t > 0$. Note that $P$ commutes with $T_t$ and the resolvent $(A_D + V - \lambda I)^{-1}$ for all $\lambda > 0$. Hence on $L_2(\Omega)$ one has the decomposition

$$(A_D + V - \lambda I)^{-1} = P (A_D + V - \lambda I)^{-1} P + \int_0^\infty e^{\lambda t} (I - P) T_t (I - P) dt.$$

As a consequence

$$\partial^\alpha (A_D + V - \lambda I)^{-1} \partial^\beta = \partial^\alpha P (A_D + V - \lambda I)^{-1} P \partial^\beta + \int_0^\infty e^{\lambda t} \partial^\alpha (I - P) T_t (I - P) \partial^\beta dt. \quad (30)$$

We shall show that the terms on the right hand side of (30) has a kernel with the appropriate bounds and which extends continuously to $\{(x,y) \in \overline{\Omega \times \Omega} : x \neq y\}$.
Let \( t > 0 \). Then \( T_t \) maps \( L_2(\Omega) \) into \( C^{1+\kappa}(\Omega) \) by Proposition 3.15. Hence \( e^{-\lambda_n t} u_n = T_t u_n \in C^{1+\kappa}(\Omega) \) for all \( n \in \mathbb{N} \). In particular, \( u_n \in C^{1+\kappa}(\Omega) \) and \( \partial^\alpha u_n \in C^\kappa(\Omega) \subset C(\overline{\Omega}) \). Since

\[
P(A_D + V - \lambda I)^{-1} P u = \sum_{n=1}^N \frac{(u, u_n)_{L_2(\Omega)}}{\lambda_n - \lambda} u_n
\]

for all \( u \in L_2(\Omega) \), it follows that

\[
\partial^\alpha P(A_D + V - \lambda I)^{-1} P \partial^\beta u = (-1)^{|eta|} \sum_{n=1}^N \frac{(u, \partial^\beta u_n)_{L_2(\Omega)}}{\lambda_n - \lambda} \partial^\alpha u_n
\]

for all \( u \in W^{1,2}(\Omega) \), where we used that \( u_n \in W^{1,2}_0(\Omega) \) for all \( n \in \{1, \ldots, N\} \). Therefore the operator \( \partial^\alpha P(A_D + V - \lambda I)^{-1} P \partial^\beta \) has as kernel the function

\[
(x, y) \mapsto (-1)^{|eta|} \sum_{n=1}^N \frac{1}{\lambda_n - \lambda} (\partial^\alpha u_n)(x) (\partial^\beta u_n)(y),
\]

which is \( \kappa \)-Hölder continuous and extends to a continuous and bounded function on \( \overline{\Omega} \times \overline{\Omega} \). In particular, it can be estimated by \( c |x - y|^{-\delta} \) for a suitable \( c > 0 \), since \( \Omega \) is bounded. This covers the first term on the right hand side of (30).

We split the integral in (30) in two parts: over \( (0, 3] \) and \( [3, \infty) \). We start with the integral over \( [3, \infty) \). We shall show that the operator

\[
\int_3^\infty e^{\lambda t} \partial^\alpha (I - P) T_t (I - P) \partial^\beta dt
\]

has as kernel the function

\[
(x, y) \mapsto (-1)^{|eta|} \int_3^\infty e^{\lambda t} \sum_{n=N+1}^\infty e^{-\lambda_n t} (\partial^\alpha u_n)(x) (\partial^\beta u_n)(y) dt.
\]

By Proposition 3.15 there exist \( c, \omega > 0 \) such that

\[
\| \partial^\alpha T_s u \|_{L_\infty(\Omega)} \leq c s^{-d/4} s^{-|\alpha|/2} e^{\omega s} \| u \|_{L_2(\Omega)}
\]

(32) for all \( s > 0 \) and \( u \in L_2(\Omega) \). Therefore \( e^{-s\lambda_n} \| \partial^\alpha u_n \|_{L_\infty(\Omega)} \leq c s^{-M} e^{\omega s} \) for all \( n \in \mathbb{N} \) and \( s > 0 \), where \( M = \frac{d}{4} + |\alpha|/2 \). Choosing \( s = \lambda_n^{-1} \) gives \( \| \partial^\alpha u_n \|_{L_\infty(\Omega)} \leq c e^{\lambda_n^M} e^{\omega \lambda_n^{-1}} \leq c e^{1+\omega \lambda_n^{-1}} \lambda_n^M \) if \( n \geq N + 1 \). Let \( \varepsilon > 0 \) be such that \( \omega_1(1 - 2\varepsilon) > \lambda \). There exists a \( c_2 > 0 \) such that \( h^M \leq c_2 e^{ch} \) for all \( h \in (0, \infty) \). Then

\[
\| \partial^\alpha u_n \|_{L_\infty(\Omega)} \leq c_2 c_2 e^{1+\omega \lambda_n^{-1}} t^{-M} e^{\lambda_n t} \leq c_3 e^{\varepsilon \lambda_n t}
\]

for all \( t \in [3, \infty) \) and \( n \in \mathbb{N} \), where \( c_3 = c c_2 e^{1+\omega \lambda_n^{-1}} \). If \( x, y \in \Omega \), then

\[
\int_3^\infty e^{\lambda t} \sum_{n=N+1}^\infty e^{-\lambda_n t} \| (\partial^\alpha u_n)(x) (\partial^\beta u_n)(y) \| dt \leq c_3^2 \int_3^\infty \sum_{n=N+1}^\infty e^{\lambda t} e^{-\lambda_n t} e^{2\varepsilon \lambda_n t} dt
\]

\[
= c_3^2 \sum_{n=N+1}^\infty e^{-3(\lambda_n(1-2\varepsilon)-\lambda)} \lambda_n(1-2\varepsilon) - \lambda
\]

\[
\leq \frac{c_3^2 e^{3\lambda}}{\omega_1(1-2\varepsilon) - \lambda} \sum_{n=N+1}^\infty e^{-3\lambda_n(1-2\varepsilon)}.
\]
Next
\[\sum_{n=N+1}^{\infty} e^{-3\lambda_n(1-2\varepsilon)} \leq \text{Tr} T_{3(1-2\varepsilon)} = \int_{\Omega} H_{3(1-2\varepsilon)}(x, x) \, dx < \infty.\]

Hence one can define $K: \Omega \times \Omega \to \mathbb{C}$ by
\[K(x, y) = (-1)^{|eta|} \int_{3}^{\infty} e^{\lambda t} \sum_{n=N+1}^{\infty} e^{-\lambda_n t} (\partial^\alpha u_n)(x)(\partial^\beta u_n)(y) \, dt.\]

Then $K$ is the kernel of (31). We already proved that $K$ is bounded on $\Omega \times \Omega$.

Using the $C^\kappa$-estimate in Proposition 3.15 instead of (32), it follows similarly as above that there exists a $c_4 > 0$ such that $|||\partial^\alpha u_n|||_{C^\kappa(\Omega)} \leq c_4 e^{\varepsilon \lambda_n t}$ for all $t \in [3, \infty)$ and $n \in \mathbb{N}$. Then
\[|(\partial^\alpha u_n)(x')(\partial^\beta u_n)(y') - (\partial^\alpha u_n)(x)(\partial^\beta u_n)(y)| \leq 2c_3 c_4 (|x - x'|^\kappa + |y - y'|^\kappa) e^{2\varepsilon \lambda_n t}\]
for all $n \in \mathbb{N}$, $t \in [3, \infty)$ and $x, x', y, y' \in \Omega$ with $|x - x'| \leq 1$ and $|y - y'| \leq 1$. Arguing as before we obtain that there exists a $c_5 > 0$ such that
\[|K(x', y') - K(x, y)| \leq c_5 (|x - x'|^\kappa + |y - y'|^\kappa)\]
for all $x, x', y, y' \in \Omega$ with $|x - x'| \leq 1$ and $|y - y'| \leq 1$. Since $\Omega$ is bounded, there exists a $c_6 > 0$ such that
\[|K(x', y') - K(x, y)| \leq c_6 (|x' - x| + |y' - y|)^\kappa |x - y|^{-2+|\alpha|+|\beta|+\kappa}\]
for all $x, x', y, y' \in \Omega$ with $x \neq y$ and $|x - x'| + |y - y'| \leq 1/2 |x - y|$. This completes the part of the integral in (30) over $[3, \infty)$.

We split the part of the integral in (30) over $(0, 3]$ in two parts
\[\int_{0}^{3} e^{\lambda t} \partial^\alpha (I - P) T_t (I - P) \partial^\beta dt = \int_{0}^{3} e^{\lambda t} \partial^\alpha T_t \partial^\beta dt - \int_{0}^{3} e^{\lambda t} \partial^\alpha T_t P \partial^\beta dt.\]

Since
\[\partial^\alpha T_t P \partial^\beta u = (-1)^{|eta|} \sum_{n=1}^{N} e^{-\lambda_n t} (u, \partial^\beta u_n) \partial^\alpha u_n\]
for all $u \in W^{1,2}(\Omega)$ it follows that the kernel of the second term in (33) is
\[(x, y) \mapsto (-1)^{|eta|+1} \sum_{n=1}^{N} (\partial^\alpha u_n)(x)(\partial^\beta u_n)(y) \int_{0}^{3} e^{-\lambda_n t} \, dt,\]

which again is $\kappa$-Hölder continuous and can be extended once more to a continuous and bounded function on $\overline{\Omega} \times \overline{\Omega}$. Finally, it follows from Theorem 3.1 that there are $a, b, \omega > 0$ such that the operator $T_t$ has a kernel $H_t$ for all $t > 0$, which is once differentiable in each entry, satisfying the bounds
\[|((\partial^\alpha \partial^\beta H_t)(x, y)| \leq a t^{-d/2} t^{-(|\alpha|+|\beta|)/2} e^{-b\frac{|x-y|^2}{t}} e^{-\omega t}\]

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for all $t > 0$ and $x, y \in \Omega$. Moreover, $(x, y) \mapsto (\partial_x^\alpha \partial_y^\beta H_t)(x, y)$ extends to a continuous function on $\overline{\Omega} \times \overline{\Omega}$. Hence the operator $\int_0^3 e^\lambda \, \partial^\alpha \partial^\beta T_t \, dt$ has kernel

$$(x, y) \mapsto (-1)^{|\beta|} \int_0^3 e^\lambda \, (\partial_x^\alpha \partial_y^\beta H_t)(x, y) \, dt$$

on $\{(x, y) \in \Omega \times \Omega : x \neq y\}$. This kernel extends to a continuous function on $\{(x, y) \in \overline{\Omega} \times \overline{\Omega} : x \neq y\}$. If $x \neq y$ and $d - 2 + |\alpha| + |\beta| \neq 0$, then

$$\left| \int_0^3 e^\lambda \, (\partial_x^\alpha \partial_y^\beta H_t)(x, y) \, dt \right| \leq a |x - y|^{-(d-2+|\alpha|+|\beta|)} e^{3\lambda} \int_0^\infty t^{-d/2} t^{-(|\alpha|+|\beta|)/2} e^{-b/t} \, dt.$$ 

The Hölder bounds follows similarly. If $d - 2 + |\alpha| + |\beta| = 0$, then the obvious adjustments are needed to obtain a logarithmic term. Then the resolvent kernel bounds follow by adding the terms.

We next consider the operator $\partial_k (A_D + V)^{-1}$. We obtain uniform bounds if $V = 0$.

**Proposition 4.3.** Let $\kappa \in (0, 1)$ and $\mu, M > 0$. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with a $C^{1+\kappa}$-boundary. Let $p \in (d+2\kappa, \infty)$. Then there exists a $c > 0$ such that the following is valid. Let $C \in \mathcal{E}^\kappa(\Omega, \mu, M)$ and $k \in \{1, \ldots, d\}$. Then the operator $\partial_k A_D^{-1}$ is bounded from $L_p(\Omega)$ into $C^{2\kappa/p}(\Omega)$ with norm at most $c$. Moreover, if $V \in L_\infty(\Omega)$ and $0 \notin \sigma(A_D + V)$, then the operator $\partial_k (A_D + V)^{-1}$ is bounded from $L_p(\Omega)$ into $C^{2\kappa/p}(\Omega)$.

**Proof.** Write $T_t = e^{-tA_D}$ for all $t > 0$. Then it follows from Proposition 3.15 and Lemma 3.11 that there exist $c, \omega > 0$ such that the operator $\partial_k T_t$ is bounded from $L_2(\Omega)$ into $C^\kappa(\Omega)$ with norm bounded by $c t^{-d/4} t^{-1/2} t^{-\kappa/2} e^{-\omega t}$, uniformly for all $t \in (0, \infty)$. The Gaussian kernel bounds with one derivative imply that $\partial_k T_t$ is bounded from $L_\infty(\Omega)$ into $L_\infty(\Omega)$ norm with bounded by $c t^{-1/2} e^{-\omega t}$, possibly by increasing the value of $c$ and decreasing $\omega$. Hence by interpolation the operator $\partial_k T_t$ is bounded from $L_p(\Omega)$ into $C^{2\kappa/p}(\Omega)$ with norm bounded by $c t^{-d/(2p)} t^{-1/2} t^{-\kappa/p} e^{-\omega t}$ for all $t \in (0, \infty)$. Since $p \in (d+2\kappa, \infty)$, the latter bound is integrable over $(0, \infty)$. Hence $\partial_k A_D^{-1}$ is bounded from $L_p(\Omega)$ into $C^{2\kappa/p}(\Omega)$. The norm is uniform for all $C \in \mathcal{E}^\kappa(\Omega, \mu, M)$ by construction.

Finally, since $\partial_k (A_D + V)^{-1} = \partial_k A_D^{-1} A_D (A_D + V)^{-1}$ and the operator $A_D (A_D + V)^{-1} = I - M_V (A_D + V)^{-1}$ is bounded from $L_p(\Omega)$ into $L_p(\Omega)$, the last part follows.

**Lemma 4.4.** Let $\kappa \in (0, 1)$ and $\mu, M > 0$. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with a $C^{1+\kappa}$-boundary. Then there exists a $c > 0$ such that the following is valid. Let $C \in \mathcal{E}^\kappa(\Omega, \mu, M)$, $p \in [1, \infty]$ and $k \in \{1, \ldots, d\}$. Then $\|\partial_k A_D^{-1}\|_{p \to p} \leq c$. Moreover, if $V \in L_\infty(\Omega)$ and $0 \notin \sigma(A_D + V)$, then the operator $\partial_k (A_D + V)^{-1}$ is bounded from $L_p(\Omega)$ into $L_p(\Omega)$.

**Proof.** Let $T$ be the semigroup generated by $-A_D$. By Theorem 3.1 there exist $c, \omega > 0$, depending only of $\kappa$, $\mu$, $M$ and $\Omega$, such that $\|\partial_k T_t\|_{p \to p} \leq c t^{-1/2} e^{-\omega t}$ for all $t > 0$, $p \in [1, \infty]$ and $k \in \{1, \ldots, d\}$. Then

$$\|\partial_k A_D^{-1}\|_{p \to p} \leq \int_0^\infty \|\partial_k T_t\|_{p \to p} \, dt \leq c \int_0^\infty t^{-1/2} e^{-\omega t} \, dt.$$ 

Finally, $\partial_k (A_D + V)^{-1} = \partial_k A_D^{-1} (I - M_V (A_D + V)^{-1})$ is bounded on $L_p(\Omega)$ for all $p \in [1, \infty]$ and $k \in \{1, \ldots, d\}$.
In this section we shall prove that $\gamma$

Let $p \in (1, \infty)$. Then there exists a $c > 0$ such that the following is valid. Let $C \in \mathcal{E}^\kappa(\Omega, \mu, M)$ be real symmetric and $k, l \in \{1, \ldots, d\}$. Then the operator $\partial_k A_D^{-1} \partial_l$ extends to a bounded operator on $L_p(\Omega)$ with norm at most $c$. Moreover, if $V \in L_\infty(\Omega)$ and $0 \not\in \sigma(A_D + V)$, then the operator $\partial_k (A_D + V)^{-1} \partial_l$ extends to a bounded operator on $L_p(\Omega)$.

**Proof.** For $p = 2$ the operator $\partial_k A_D^{-1} \partial_l = (\partial_k A_D^{-1/2})(A_D^{-1/2} \partial_l)$ extends to a bounded operator with norm at most $\mu^{-1}$. By Theorem 4.1 it follows that the kernel of $\partial_k A_D^{-1} \partial_l$ has Calderón–Zygmund estimates uniformly in $C$. Hence $\partial_k A_D^{-1} \partial_l$ extends to a bounded operator on $L_p(\Omega)$.

Finally, if $V \in L_\infty(\Omega)$ and $0 \not\in \sigma(A_D + V)$, then

$$
\partial_k (A_D + V)^{-1} \partial_l = \partial_k A_D^{-1} \partial_l - \left(\partial_k (A_D + V)^{-1}\right) M_V \left(A_D^{-1} \partial_l\right)
$$

and use Lemma 4.4.

\[\square\]

5 The harmonic lifting

Let $C \in \mathcal{E}^\kappa(\Omega)$ be real symmetric and $V \in L_\infty(\Omega, \mathbb{R})$, where $\Omega$ is a bounded Lipschitz domain. Recall that the harmonic lifting $\gamma_V : \text{Tr} (W^{1,2}(\Omega)) \to W^{1,2}(\Omega)$ is defined by

$$
\gamma_V \varphi = u
$$

for all $\varphi \in \text{Tr} (W^{1,2}(\Omega))$, where $u \in W^{1,2}(\Omega)$ is such that $(Au + V)u = 0$ and $\text{Tr} u = \varphi$.

In this section we shall prove that $\gamma_V$ has a kernel and we obtain good kernel bounds if $\Omega$ has a $C^{1+\kappa}$-boundary. We also show that the map $\gamma_V$ extends to a continuous map from $L_p(\Gamma)$ into $L_p(\Omega)$ for all $p \in [1, \infty]$.

For the proof of these results we need a delicate version of the divergence theorem.

**Lemma 5.1.** Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with $C^1$-boundary. Let $F : \Omega \to \mathbb{C}^d$ be a function. Suppose $F \in C(\overline{\Omega}, \mathbb{C}^d) \cap C^1(\Omega, \mathbb{C}^d)$ and suppose that $\text{div} F \in L_1(\Omega)$. Then $\int_\Omega \text{div} F = \int_\Gamma n \cdot F$.

**Proof.** See [Alt].

We use this divergence theorem to obtain a classical expression of the normal derivative.

**Lemma 5.2.** Let $\kappa \in (0, 1)$. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with a $C^{1+\kappa}$-boundary. Let $C \in \mathcal{E}(\Omega)$ be real symmetric and suppose that $c_{kl} \in C^{1+\kappa}(\Omega)$ for all $k, l \in \{1, \ldots, d\}$. Let $p \in (d, \infty)$ and $u \in C^1(\overline{\Omega})$. Suppose that $Au \in L_2(\Omega)$. Then $u$ has a weak conormal derivative and

$$
\partial^\kappa_{\nu} u = \sum_{k,l=1}^d n_k \langle c_{kl} \partial_l u \rangle_\Gamma.
$$
Proof. Interior regularity gives \( u \in C^2(\Omega) \). Let \( v \in C^\infty_0(\Omega) \). Define \( F = (F_1, \ldots, F_d): \overline{\Omega} \to C^d \) by \( F_k = \sum_{l=1}^d c_{kl} (\partial_l u) \mathbf{v} \). Then \( F \in C(\overline{\Omega}, C^d) \cap C^1(\Omega, C^d) \). Moreover,

\[
\text{div} \, F = -(Au) \mathbf{v} + \sum_{k,l=1}^d c_{kl} (\partial_l u) \partial_k v \in L_2(\Omega) \subset L_1(\Omega).
\]

Then the divergence theorem, Lemma 5.1 gives

\[
\int_{\Omega} \sum_{k,l=1}^d c_{kl} (\partial_k u) \partial_l v - \int_{\Omega} (Au) \mathbf{v} = \int_{\Omega} \text{div} \, F = \int_{\Gamma} n \cdot F = \int_{\Gamma} \sum_{k,l=1}^d n_k c_{kl} (\partial_l u) \mathbf{v}.
\]

Since \( \sum_{k,l=1}^d n_k (c_{kl} \partial_l u)|_{\Gamma} \in C(\Gamma) \subset L_2(\Gamma) \), this proves the lemma. \( \square \)

Note that we required \( c_{kl} \in C^{1+\kappa}(\Omega) \) in Lemma 5.2, which is much more than the condition \( c_{kl} \in C^{\kappa}(\Omega) \) in Theorem 1.1. This is the reason why we use a regularisation of the coefficients below.

Proposition 5.3. Let \( \kappa \in (0,1) \). Let \( \Omega \subset \mathbb{R}^d \) be an open bounded set with a \( C^{1+\kappa} \)-boundary. Let \( C \in \mathcal{S}^\kappa(\Omega) \) be real symmetric and \( V \in L_\infty(\Omega, \mathbb{R}) \). Suppose that \( 0 \notin \sigma(A_D + V) \). Let \( p \in (d+2\kappa, \infty) \) and \( u \in L_p(\Omega) \). Then \((A_D + V)^{-1} u\) has a weak conormal derivative and

\[
\partial^C \nu (A_D + V)^{-1} u = \sum_{k,l=1}^d n_k \text{Tr} \left( c_{kl} \partial_l (A_D + V)^{-1} u \right).
\]  (34)

Proof. Step 1. Suppose \( V = 0 \) and \( c_{kl} \in C^{1+\kappa}(\Omega) \) for all \( k,l \in \{1, \ldots, d\} \).

Let \( u \in L_p(\Omega) \). Then \( A_D^{-1} u \in C^1(\overline{\Omega}) \) by Proposition 4.3. So by Lemma 5.2 one deduces that \( A_D^{-1} u \) has a conormal derivative and (34) is valid.

Step 2. Suppose \( V = 0 \).

We can extend the function \( c_{kl} \) to a \( C^\kappa \)-function \( \tilde{c}_{kl}: \mathbb{R}^d \to \mathbb{R} \) such that \( \tilde{c}_{kl} = c_{ik} \) for all \( k,l \in \{1, \ldots, d\} \). Let \( (\rho_n)_n \in \mathbb{N} \) be a bounded approximation of the identity. For all \( n \in \mathbb{N} \) and \( k,l \in \{1, \ldots, d\} \) define \( c_{kl}^{(n)} = (\tilde{c}_{kl} * \rho_n)|_{\Omega} \) and set \( C^{(n)} = (c_{kl}^{(n)})_{k,l} \). Then there are \( \mu, M > 0 \) such that \( C^{(n)} \in \mathcal{S}^\kappa(\Omega, \mu, M) \) for all large \( n \in \mathbb{N} \) and without loss of generality for all \( n \in \mathbb{N} \). Define \( A_D^{(n)} = A_D^{C^{(n)}} \) for all \( n \in \mathbb{N} \).

Let \( u \in L_p(\Omega) \). Then it follows from from Step 1 that

\[
\int_{\Omega} \sum_{k,l=1}^d c_{kl}^{(n)} (\partial_k (A_D^{(n)})^{-1} u) \partial_l v - \int_{\Omega} u \mathbf{v} = \int_{\Gamma} \sum_{k,l=1}^d n_k \text{Tr} \left( c_{kl}^{(n)} \partial_l (A_D^{(n)})^{-1} u \right) \mathbf{v} \quad (35)
\]

for all \( n \in \mathbb{N} \) and \( v \in W^{1,2}(\Omega) \). Clearly \( \lim c_{kl}^{(n)} = c_{kl} \) uniformly on \( \overline{\Omega} \) for all \( k,l \in \{1, \ldots, d\} \). Let \( k \in \{1, \ldots, d\} \). If \( w \in D(A_D) \) and \( v \in L_2(\Omega) \), then

\[
(w, v)_{L_2(\Omega)} = (A_D w, A_D^{-1} v)_{L_2(\Omega)} = \sum_{j,l=1}^d (c_{jl} \partial_l w, \partial_j A_D^{-1} v)_{L_2(\Omega)}.
\]

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Hence by density \((w, v)_{L^2(\Omega)} = \sum_{j,l=1}^d (c_{jl} \partial_t w, \partial_j A_D^{-1} v)_{L^2(\Omega)}\) for all \(w \in W_0^{1,2}(\Omega)\) and \(v \in L_2(\Omega)\). Substituting \(w = (A_D^{(n)})^{-1} u\), replacing \(v\) by \(\partial_k v\) and integration by parts gives

\[-(\partial_k (A_D^{(n)})^{-1} u, v)_{L^2(\Omega)} = \sum_{j,l=1}^d (c_{jl} \partial_t (A_D^{(n)})^{-1} u, \partial_j A_D^{-1} \partial_k v)_{L^2(\Omega)}\]

for all \(v \in C_c^\infty(\Omega)\). Similarly and slightly easier one proves

\[-(\partial_k A_D^{-1} u, v)_{L^2(\Omega)} = \sum_{j,l=1}^d (c_{jl} \partial_t (A_D^{(n)})^{-1} u, \partial_j A_D^{-1} \partial_k v)_{L^2(\Omega)}\]

for all \(v \in C_c^\infty(\Omega)\). Therefore

\[(\partial_k (A_D^{(n)})^{-1} u - \partial_k A_D^{-1} u, v)_{L^2(\Omega)} = \sum_{j,l=1}^d ((c_{jl}^{(n)} - c_{jl}) \partial_t (A_D^{(n)})^{-1} u, \partial_j A_D^{-1} \partial_k v)_{L^2(\Omega)}\]

for all \(v \in C_c^\infty(\Omega)\). Let \(q\) be the dual exponent of \(p\). By Proposition 4.5 the operator \(\partial_j A_D^{-1} \partial_k\) extends to a bounded operator \(T_{jk}\) on \(L_q(\Omega)\) for all \(j \in \{1, \ldots, d\}\). Then

\[
\| (\partial_k (A_D^{(n)})^{-1} u - \partial_k A_D^{-1} u, v)_{L^2(\Omega)} \|
\leq \sum_{j,l=1}^d \| c_{jl}^{(n)} - c_{jl} \|_{L_\infty(\Omega)} \| \partial_t (A_D^{(n)})^{-1} \|_{L_p(\Omega)} \| u \|_{L_p(\Omega)} \| T_{jk} \|_{q\to q} \| v \|_{L_q(\Omega)}
\]

for all \(v \in C_c^\infty(\Omega)\). Since the operators \(\partial_t (A_D^{(n)})^{-1}\) are bounded on \(L_p(\Omega)\) uniformly in \(n\) by Lemma 4.4, one deduces that

\[
\lim_{n \to \infty} \partial_k (A_D^{(n)})^{-1} u = \partial_k A_D^{-1} u
\]

in \(L_p(\Omega)\). Therefore the left hand side of (35) converges to \(\int_{\Omega} \sum_{k,l=1}^d c_{kl} (\partial_k A_D^{-1} u) \overline{\partial_t v}\) for all \(v \in C_b^\infty(\Omega)\).

Next we consider the right hand side of (35). Let \(l \in \{1, \ldots, d\}\). Then \((\partial_l (A_D^{(n)})^{-1} u)_{n \in \mathbb{N}}\) is bounded in \(C^{2n/p}(\Omega)\) and in \(C(\overline{\Omega})\) by Proposition 4.3. So by the Arzelà–Ascoli theorem and passing to a subsequence if necessary there exists a \(w \in C(\overline{\Omega})\) such that \(\lim_{n \to \infty} \partial_l (A_D^{(n)})^{-1} u = w\) in \(C(\overline{\Omega})\). Since \(\lim_{n \to \infty} \partial_l (A_D^{(n)})^{-1} u = \partial_l A_D^{-1} u\) in \(L_p(\Omega)\), one deduces that \(w = \partial_l A_D^{-1} u\). So

\[
\lim_{n \to \infty} \int_{\Gamma} \sum_{k,l=1}^d n_k \text{Tr} (c_{kl}^{(n)} \partial_l (A_D^{(n)})^{-1} u) \overline{\partial_t v} = \int_{\Gamma} \sum_{k,l=1}^d n_k \text{Tr} (c_{kl} \partial_l A_D^{-1} u) \overline{\partial_t v}
\]

for all \(v \in C_b^\infty(\Omega)\). Then the equality in (35) implies that \(A_D^{-1} u\) has a weak conormal derivative and (34) is valid.

**Step 3.** Suppose \(V \in L_\infty(\Omega, \mathbb{R})\). Let \(u \in L_p(\Omega)\). Then apply Step 2 to \(A_D (A_D + V)^{-1} u \in L_p(\Omega)\).
Lemma 5.4. Let $\kappa \in (0,1)$. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with a $C^{1+\kappa}$-boundary. Let $C \in \mathcal{E}^\kappa(\Omega)$ be real symmetric and $V \in L_\infty(\Omega, \mathbb{R})$. Suppose that $0 \not\in \sigma(A_D + V)$. Let $p \in (d + 2\kappa, \infty)$ and $v \in L_p(\Omega)$. Then

$$(\gamma_V \varphi, v)_{L_2(\Omega)} = - (\varphi, \partial^C \nu (A_D + V)^{-1}v)_{L_2(\Gamma)}$$

for all $\varphi \in \text{Tr} (W^{1,2}(\Omega))$.

Proof. It follows from Proposition 5.3 that $(A_D + V)^{-1}v$ has a weak conormal derivative. Then the equality follows as in [BE] Corollary 5.4. For more details, see [AE2] Proposition 6.4.

Let $\kappa \in (0,1)$, $\Omega \subset \mathbb{R}^d$ be an open bounded set with a $C^{1+\kappa}$-boundary, $C \in \mathcal{E}^\kappa(\Omega)$ and $V \in L_\infty(\Omega, \mathbb{R})$. Suppose that $0 \not\in \sigma(A_D + V)$. Let $G_V$ be the Green kernel of $(A_D + V)^{-1}$. Then $G_V$ is differentiable on $\{(x,y) \in \Omega \times \Omega : x \neq y\}$ by Theorem 4.2 and the derivative extends to a continuous function on $\{(x,y) \in \overline{\Omega} \times \overline{\Omega} : x \neq y\}$. Define the function $K_{\gamma_V} : \Omega \times \Gamma \rightarrow \mathbb{C}$ by

$$K_{\gamma_V}(x, z) = - \sum_{k,l=1}^d n_k(z) c_{kl}(z)(\partial_l^{(1)} G_V)(z,x).$$

We next show that $K_{\gamma_V}$ is the kernel of $\gamma_V$.

Proposition 5.5. Let $\kappa \in (0,1)$, $\Omega \subset \mathbb{R}^d$ be an open bounded set with a $C^{1+\kappa}$-boundary, $C \in \mathcal{E}^\kappa(\Omega)$ real symmetric and $V \in L_\infty(\Omega, \mathbb{R})$. Suppose that $0 \not\in \sigma(A_D + V)$. Then one has the following.

(a) The map $K_{\gamma_V}$ is continuous.

Define $T : L_1(\Gamma) \rightarrow C(\Omega)$ by

$$(T\varphi)(x) = \int_\Gamma K_{\gamma_V}(x, z) \varphi(z) \, dz.$$

(b) If $\varphi \in \text{Tr} (W^{1,2}(\Omega))$, then $\gamma_V \varphi = T\varphi$ a.e.

(c) There exists a $c > 0$ such that

$$|K_{\gamma_V}(x, z)| \leq \frac{c}{|x - z|^{d-1}}$$

and

$$|K_{\gamma_V}(x', z') - K_{\gamma_V}(x, z)| \leq c \frac{|x' - x| + |z' - z|)^\kappa}{|x - z|^{d-1+\kappa}}$$

for all $x, x' \in \Omega$ and $z, z' \in \Gamma$ with $|x' - x| + |z' - z| \leq \frac{1}{2} |x - z|$.

(d) Let $p \in [1, \infty]$. Then the map $\gamma_V \vert_{L_p(\Gamma) \cap \text{Tr}(W^{1,2}(\Omega))}$ extends to a bounded map from $L_p(\Gamma)$ into $L_p(\Omega)$. Explicitly, the restriction $T \vert_{L_p(\Gamma)}$ is continuous from $L_p(\Gamma)$ into $L_p(\Omega)$.

Proof. ‘(a)’. This follows immediately from Theorem 4.2.
(b)’. Let \( \varphi \in \text{Tr}(W^{1,2}(\Omega)) \) and \( v \in C^\infty_c(\Omega) \). Then Lemma 5.4 and Proposition 5.3 give

\[
(\gamma_V \varphi, v)_{L^2(\Omega)} = -(\varphi, \partial^C \nu (A_D + V)^{-1}v)_{L^2(\Gamma)}
\]

\[
= -\int_\Gamma \varphi(z) \sum_{k,l=1}^d n_k(z) c_{kl}(z) (\partial_l (A_D + V)^{-1}v)(z) \, dz
\]

\[
= -\int_\Gamma \int_\Omega \sum_{k,l=1}^d \varphi(z) n_k(z) c_{kl}(z) (\partial_l (G_V)(z, x) v(x)) \, dx \, dz
\]

\[
= \int_\Omega (T\varphi)(x) \, v(x) \, dx = (T\varphi, v)_{L^2(\Omega)}.
\]

So \( \gamma_V \varphi = T\varphi \) a.e.

(c)’. This is a consequence of Theorem 4.2.

(d)’. We divide the proof in several steps.

Step 1. Suppose \( p = 1 \).

Since \( \sup_{z \in \Gamma} \int_\Omega |K_{\gamma_V}(x, z)| \, dx < \infty \) the operator \( T \) is bounded from \( L^1(\Gamma) \) into \( L^1(\Omega) \).

Step 2. Suppose \( p = \infty \) and \( V = 0 \).

We shall show that \( T|_{L^\infty(\Gamma)} \) is bounded from \( L^\infty(\Gamma) \) into \( L^\infty(\Omega) \). The maximum principle, [GT] Theorem 8.1, gives that \( \|\gamma_0 \varphi\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^\infty(\Gamma)} \) for all \( \varphi \in C(\Gamma) \cap \text{Tr}(W^{1,2}(\Omega)) \). So \( \|T\varphi\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^\infty(\Gamma)} \) for all \( \varphi \in C(\Gamma) \cap \text{Tr}(W^{1,2}(\Omega)) \). Now let \( \varphi \in L^\infty(\Gamma) \). Since \( \Omega \) has a Lipschitz boundary, one can regularise \( \varphi \). On a special Lipschitz domain one can regularise an \( L^\infty \)-function \( \psi \) on the boundary to obtain a sequence of continuous \( W^{1,2}_{\text{loc}} \)-functions on the boundary which converges to \( \psi \) in the weak*-topology on \( L^\infty \) and such that the \( L^\infty \)-norm of the approximants is bounded by \( \|\psi\|_{L^\infty} \). Since \( \Omega \) is bounded and Lipschitz one can use a partition of the unity so that \( \Omega \) is split as a finite number, say \( N \), of parts of special Lipschitz domains. Summing up the corresponding smooth approximants one obtains a sequence \( (\varphi_n)_{n \in \mathbb{N}} \) in \( W^{1,2}(\Gamma) \cap C(\Gamma) \) such that \( \lim_{n \to \infty} \varphi_n = \varphi \) weak* in \( L^\infty(\Gamma) \) and \( \|\varphi_n\|_{L^\infty(\Gamma)} \leq N \|\varphi\|_{L^\infty(\Gamma)} \) for all \( n \in \mathbb{N} \). Now let \( x \in \Omega \). Then \( z \mapsto K_{\gamma_0}(x, z) \) is an element of \( L^1(\Gamma) \). So

\[
|(T\varphi)(x)| = \lim_{n \to \infty} |(T\varphi_n)(x)| \leq \limsup_{n \to \infty} \|T\varphi_n\|_{L^\infty(\Omega)} \leq \limsup_{n \to \infty} \|\varphi_n\|_{L^\infty(\Gamma)} \leq N \|\varphi\|_{L^\infty(\Gamma)}.
\]

So \( T|_{L^\infty(\Gamma)} \) is a bounded extension of \( \gamma_0|_{L^\infty(\Gamma) \cap \text{Tr}(W^{1,2}(\Omega))} \) from \( L^\infty(\Gamma) \) into \( L^\infty(\Omega) \).

Step 3. Suppose \( p \in [1, \infty] \) and \( V = 0 \).

This follows by interpolation from Steps 1 and 2.

Step 4. Suppose \( p \in [1, \infty] \) and \( V \in L^\infty(\Omega, \mathbb{R}) \).

Since \( \gamma_V = \gamma_0 - (A_D + V)^{-1} M_V \gamma_0 \) by (6) the general case follows.

As a consequence we deduce that \( N_V \) is a perturbation of \( N \).

Corollary 5.6. Let \( \kappa \in (0, 1) \), \( \Omega \subset \mathbb{R}^d \) be an open bounded set with a \( C^{1+\kappa} \)-boundary, \( C \in \mathcal{E}^\kappa(\Omega) \) real symmetric and \( V \in L^\infty(\Omega, \mathbb{R}) \). Suppose that \( 0 \not\in \sigma(A_D + V) \). Then \( N_V = N + \gamma_0^* M_V \gamma_V \).
Proof. Let \( \phi \in D(N_V) \) and \( \psi \in D(N) \). Write \( u = \gamma_0 \phi \) and \( v = \gamma_0 \phi \). Then
\[
(N_V \phi, \psi)_{L^2(\Gamma)} - (\phi, N \psi)_{L^2(\Gamma)} = a_0(u, v) - a(V(u, v)) = \int_\Gamma V u \psi = (M_V \gamma_0 \phi, \gamma_0 \phi)_L^{2(\Gamma)} = (\gamma_0^* M_V \gamma_0 \phi, \phi)_L^{2(\Gamma)}.
\]
Since \( \gamma_0^* M_V \gamma_0 \) is bounded on \( L^2(\Gamma) \) by Proposition 5.5(d) it follows that \( \phi \in D(N^*) = D(N) \) and similarly \( \psi \in D(N^* V) = D(N_V) \). Then
\[
((N_V - N) \phi, \psi)_{L^2(\Gamma)} = (N_V \phi, \psi)_{L^2(\Gamma)} - (\phi, N \psi)_{L^2(\Gamma)} = (\gamma_0^* M_V \gamma_0 \phi, \psi)_{L^2(\Gamma)}.
\]
Since \( D(N) \) is dense in \( L^2(\Gamma) \) the corollary follows.

6 The Schwartz kernel of the Dirichlet-to-Neumann operator

Our main aim in this section is to show that the Dirichlet-to-Neumann operators \( N \) and \( N_V \) are given by Schwartz kernels that satisfy Calderón–Zygmund-type bounds. The principle step in the proof is that the Schwartz kernel of \( N \) can be expressed in terms of the coefficients \( c_{kl} \) and partial derivatives of the Green kernel. We start with a definition. Let \( \kappa \in (0, 1), \Omega \subset \mathbb{R}^d \) be an open bounded set with a \( C^{1+\kappa} \)-boundary and \( C \in E^\kappa(\Omega) \) real symmetric. Then by Theorem 4.1 the elliptic operator \( A_D \) has a Green kernel \( G \) which is differentiable in each entry and the partial derivatives extend to a continuous function on \( \{(x, y) \in \overline{\Omega} \times \overline{\Omega} : x \neq y\} \). We define \( K_N : \{(z, w) \in \Gamma \times \Gamma : z \neq w\} \rightarrow \mathbb{R} \) by
\[
K_N(z, w) = -\sum_{k,l,k',l'=1}^d n_{k'}(w) n_k(z) c_{k'l'}(w) c_{kl}(z) (\partial^{(1)}_l \partial^{(2)}_{l'} G)(z, w).
\] (37)

Our first aim is to prove that \( K_N \) is the Schwartz kernel of \( N \). In the literature sometimes \( K_N \) is written as \( \partial^{\nu}_{\nu'} G \), the conormal derivatives with respect to the two variables. It far from clear, however, whether the weak conormal derivatives of \( G \) exist in the sense of (3). Even if these weak conormal derivatives would exist, then it is again unclear whether they coincide with (37).

We use the definition of \( N \) by the symmetric form \( a_0 \), see (4) and (1) with \( V = 0 \).

Lemma 6.1. Let \( \kappa \in (0, 1), \Omega \subset \mathbb{R}^d \) be an open bounded set with a \( C^{1+\kappa} \)-boundary and \( C \in E(\Omega) \) real symmetric. Suppose that \( c_{kl} \in C^\infty(\Omega) \) for all \( k, l \in \{1, \ldots, d\} \). Let \( \varphi \in \text{Tr} (W^{1,2}(\Omega)) \). Then
\[
a_0(\gamma_0 \varphi, v) = \int_\Gamma \int_\Gamma K_N(z, w) \varphi(w) (\overline{\text{Tr} v}(z) dw dz
\]
for all \( v \in W^{1,2}(\Omega) \) with \( \text{supp} \varphi \cap \text{supp} v = \emptyset \).
Proof. Let $\tau \in C^\infty_c(\mathbb{R}^d)$ be such that $\text{supp} \varphi \cap \text{supp} \tau = \emptyset$. Set $v = \tau|_\Omega$. By definition

$$a_0(\gamma_0 \varphi, v) = \sum_{k,l=1}^d \int_\Omega c_{kl} (\partial_k \gamma_0 \varphi) \overline{\partial_l v}. \tag{38}$$

Let $k \in \{1, \ldots, d\}$. Define $F_k : \Omega \to \mathbb{C}$ by $F_k = \sum_{l=1}^d \overline{\tau} c_{kl} (\partial_l \gamma_0 \varphi)$. Then $F_k \in C^1(\Omega)$ by classical elliptic regularity and the fact that $A(\gamma_0 \varphi) = 0$ weakly in $\Omega$. We next show that $F_k \in C(\overline{\Omega})$. Indeed, by Proposition 5.5(b) we have

$$(\partial_l \gamma_0 \varphi)(x) = -\sum_{k',l' = 1}^d \int_\Gamma n_{k'} (z) c_{k'l'} (z) (\partial_1^{(2)} \partial_{l'}^{(1)} G)(z, x) \varphi(z) \, dz \tag{39}$$

for all $x \in \text{supp} v$. This integral is actually taken over $z \in \text{supp} \varphi$. Since $\text{supp} \varphi \cap \text{supp} v = \emptyset$ we can apply Theorem 4.1, which shows immediately that $F_k \in C(\overline{\Omega})$. Let $F = (F_1, \ldots, F_d)$. Then $\text{div} F = \sum_{k,l=1}^d c_{kl} (\partial_k \gamma_0 \varphi) \overline{\partial_l v} \in L_2(\Omega)$. Hence we can apply Lemma 5.1 to write the RHS of (38) by

$$\sum_{k,l=1}^d \int_\Omega c_{kl} (\partial_k \gamma_0 \varphi) \overline{\partial_l v} = \int_\Omega \text{div} F = \int_\Gamma n \cdot F = \sum_{k,l=1}^d \int_\Gamma n_k (w) (\text{Tr} \overline{c_{kl} (\partial_k \gamma_0 \varphi)}))(w) \, dw. \tag{40}$$

Hence

$$a_0(\gamma_0 \varphi, v) = -\sum_{k,l,k',l' = 1}^d \int_\Gamma \int_\Gamma n_k (w) n_{k'} (z) (\text{Tr} \overline{v})(w) c_{kl}(w) c_{k'l'}(z) (\partial_{l'}^{(1)} \partial_l^{(1)} G)(z, w) \varphi(z) \, dz \, dw$$

$$= \int_\Gamma \int_\Gamma K_N(z, w) \varphi(w) (\text{Tr} \overline{v})(z) \, dz \, dw,$$

which proves the lemma if $v = \tau|_\Omega$ with $\tau \in C^\infty_c(\mathbb{R}^d)$. This extends to all $v \in W^{1,2}(\Omega)$ with $\text{supp} \varphi \cap \text{supp} v = \emptyset$ by a standard approximation argument.

Next we extend the previous lemma to the case of Hölder continuous coefficients.

Lemma 6.2. Let $\kappa \in (0, 1)$, $\Omega \subset \mathbb{R}^d$ be an open bounded set with a $C^{1+\kappa}$-boundary and $C \in E(\Omega)$ real symmetric. Let $\varphi \in \text{Tr}(W^{1,2}(\Omega))$. Then

$$a_0(\gamma_0 \varphi, v) = \int_\Gamma \int_\Gamma K_N(z, w) \varphi(w) (\text{Tr} \overline{v})(z) \, dz \, dw$$

for all $v \in W^{1,2}(\Omega)$ with $\text{supp} \varphi \cap \text{supp} v = \emptyset$.

Proof. As one expects, we proceed by a regularization argument. For all $n \in \mathbb{N}$ let $C_n$ be as in Step 2 in the proof of Proposition 5.3. There exist $\mu, M > 0$ such that $C_n \in E^{\kappa}(\Omega, \mu, M)$ for all $n \in \mathbb{N}$. We denote by $A^{(n)}$, $a_0^{(n)}$, $\gamma_0^{(n)}$, $K_N^{(n)}$, $G^{(n)}$ the same quantities as before with $c_{kl}$ replaced by the new coefficients $c_{kl}^{(n)}$. We apply Lemma 6.1 to obtain

$$a_0^{(n)}(\gamma_0^{(n)} \varphi, v) = \int_\Gamma \int_\Gamma K_N^{(n)}(z, w) \varphi(w) (\text{Tr} \overline{v})(z) \, dz \, dw \tag{40}$$

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for all $\varphi \in \text{Tr}(W^{1,2}(\Omega))$ and $v \in W^{1,2}(\Omega)$ such that $\text{supp} \varphi \cap \text{supp} v = \emptyset$. Since

$$K_{N'}^{(n)}(z, w) = - \sum_{k,l,k',l'=1}^d n_{k'}(w) n_k(z) c_{k'l'}^{(n)}(w) c_{kl}^{(n)}(z) (\partial^{(1)}_l \partial^{(2)}_{l'} G^{(n)})(z, w)$$

it follows from Proposition 6.6 below that $\lim_{n \to \infty} K_{N'}^{(n)}(z, w) = K_N(z, w)$ uniformly in $z \in \Gamma \cap \text{supp} v$ and $w \in \text{supp} \varphi$. On the other hand, by (39) and again Proposition 6.6 we see that $\lim_{n \to \infty} (\partial_k \gamma_0^{(n)})(x) = (\partial_k \gamma_0 \varphi)(x)$ uniformly for all $x \in \text{supp} v$. Since

$$a_0^{(n)}(\gamma_0^{(n)} \varphi, v) = \sum_{k,l=1}^d \int_\Omega c_{kl}^{(n)} (\partial_k \gamma_0^{(n)} \varphi) \overline{\partial_l v}$$

for all $n \in \mathbb{N}$ one deduces that $\lim a_0^{(n)}(\gamma_0^{(n)} \varphi, v) = a_0(\gamma_0 \varphi, v)$. Hence passing to the limit in (40) gives the lemma.

\[ \square \]

**Corollary 6.3.** Let $\kappa \in (0, 1)$, $\Omega \subset \mathbb{R}^d$ be an open bounded set with a $C^{1+\kappa}$-boundary and $C \in \mathcal{E}(\Omega)$ real symmetric. Then $K_N$ is the Schwartz kernel of $\mathcal{N}$.

**Proof.** Let $\varphi \in D(\mathcal{N})$ and $v \in W^{1,2}(\Omega)$ with $\text{supp} \varphi \cap \text{supp} v = \emptyset$. Then by definition of $\mathcal{N}$ and Lemma 6.2 one deduces that

$$(\mathcal{N} \varphi, \text{Tr} v)_{L^2(\Gamma)} = a_0(\gamma_0 \varphi, v) = \int_\Gamma \int_\Gamma K_N(z, w) \varphi(w) \overline{\text{Tr}(v)(z)} \, dw \, dz.$$

This gives the corollary. \[ \square \]

**Proposition 6.4.** Let $\kappa \in (0, 1)$, $\Omega \subset \mathbb{R}^d$ be an open bounded set with a $C^{1+\kappa}$-boundary and $C \in \mathcal{E}(\Omega)$ real symmetric. Then there exists a $c > 0$ such that the Schwartz kernel $K_N$ of $\mathcal{N}$ satisfies

$$|K_N(z, w)| \leq \frac{c}{|z-w|^d}$$

and

$$|K_N(z, w) - K_N(z', w')| \leq c \frac{(|z-z'| + |w-w'|)^\kappa}{|z-w|^{d+\kappa}}$$

for all $z, z', w, w' \in \Gamma$ with $z \neq w$ and $|z-z'| + |w-w'| \leq \frac{1}{2} |z-w|$.\[ \square \]

**Proof.** We have seen in Corollary 6.3 that $\mathcal{N}$ has a Schwartz kernel $K_N$ given by

$$K_N(z, w) = - \sum_{k,l,k',l'=1}^d n_{k'}(w) n_k(z) c_{k'l'}(w) c_{kl}(z) (\partial^{(1)}_l \partial^{(2)}_{l'} G)(z, w)$$

for all $z, w \in \Gamma$ with $z \neq w$. Here $G$ is the Green kernel of the elliptic operator $A_D$. It follows immediately from (27) in Theorem 4.1 and the fact that the coefficients $c_{kl}$ are all bounded on $\overline{\Omega}$ that

$$|K_N(z, w)| \leq a |z-w|^{-d}$$

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for a suitable constant $a > 0$ and all $z, w \in \Gamma$ with $z \neq w$. On the other hand, the bounds (28) of the same theorem show that there exists a $c > 0$ such that

\[
|n_{k'}(w) n_k(z) c_{k'V}(w) c_{kl}(z) \left( \partial_1^{(1)} \partial_{l'}^{(2)} G \right)(z, w) - n_{k'}(w') n_k(z') c_{k'V}(w') c_{kl}(z') \left( \partial_1^{(1)} \partial_{l'}^{(2)} G \right)(z', w')|
\]

\[
= \left| \left( n_{k'}(w) n_k(z) c_{k'V}(w) c_{kl}(z) - n_{k'}(w') n_k(z') c_{k'V}(w') c_{kl}(z') \right) \left( \partial_1^{(1)} \partial_{l'}^{(2)} G \right)(z, w) + n_{k'}(w') n_k(z') c_{k'V}(w') c_{kl}(z') \left( \left( \partial_1^{(1)} \partial_{l'}^{(2)} G \right)(z, w) - \left( \partial_1^{(1)} \partial_{l'}^{(2)} G \right)(z', w') \right) \right|
\]

\[
 \leq c \frac{|z - z'| + |w - w'|}{|z - w|^{d+\kappa}} c \frac{|z - z'| + |w - w'|}{|z - w|^{d+\kappa}}
\]

for all $z, z', w, w' \in \Gamma$ with $z \neq w, z' \neq w'$ and $|z - z'| + |w - w'| \leq \frac{1}{2} |z - w|$. Using the fact that $\Gamma$ is bounded we obtain the bound

\[
|n_{k'}(w) n_k(z) c_{k'V}(w) c_{kl}(z) \left( \partial_1^{(1)} \partial_{l'}^{(2)} G \right)(z, w) - n_{k'}(w') n_k(z') c_{k'V}(w') c_{kl}(z') \left( \partial_1^{(1)} \partial_{l'}^{(2)} G \right)(z', w')|
\]

\[
\leq c_1 \frac{|z - z'| + |w - w'|}{|z - w|^{d+\kappa}}
\]

for all $z, z', w, w' \in \Gamma$ with $z \neq w, z' \neq w'$ and $|z - z'| + |w - w'| \leq \frac{1}{2} |z - w|$. This shows the second bounds of the proposition. \qed

Next we extend the previous estimates to the Schwartz kernel $K_{\mathcal{N}_V}$ of the Dirichlet-to-Neumann operator with a potential $V \in L_\infty(\Omega)$.

**Proposition 6.5.** Let $\kappa \in (0, 1)$, $\Omega \subset \mathbb{R}^d$ be an open bounded set with a $C^{1+\kappa}$-boundary and $C \in \mathcal{E}(\Omega)$ real symmetric. Let $V \in L_\infty(\Omega, \mathbb{R})$ and assume that $0 \notin \sigma(A_D + V)$. Then there exists a constant $c > 0$ such that the Schwartz kernel $K_{\mathcal{N}_V}$ of $\mathcal{N}_V$ satisfies

\[
|K_{\mathcal{N}_V}(z, w)| \leq \frac{c}{|z - w|^d}
\]

and

\[
|K_{\mathcal{N}_V}(z, w) - K_{\mathcal{N}_V}(z', w')| \leq c \frac{|z - z'| + |w - w'|}{|z - w|^{d+\kappa}}
\]

for all $z, z', w, w' \in \Gamma$ with $z \neq w$ and $|z - z'| + |w - w'| \leq \frac{1}{2} |z - w|$. **Proof.** First we have $\mathcal{N}_V = \mathcal{N} + \gamma_V^* M_V \gamma_0$ by Corollary 5.6. We already have the desired estimates for the kernel $K_{\mathcal{N}_V}$. It remains to prove the same estimates for the Schwartz kernel $K_Q$ of $Q = \gamma_V^* M_V \gamma_0$. Recall that $K_{\gamma_V}$ is the kernel of $\gamma_V$. Obviously,

\[
K_Q(z, w) = \int_{\Omega} K_{\gamma_V}(x, z) V(x) K_{\gamma_0}(x, w) \, dx
\]
for all \(z, w \in \Gamma\) with \(z \neq w\). We use Proposition 5.5(c). There exists a constant \(c > 0\) such that
\[
|K_Q(z, w)| \leq c \int_{\Omega} \frac{1}{|x-z|^{d-1}} \frac{1}{|x-w|^{d-1}} dx \leq c \operatorname{diam}(\Omega) \int_{\Omega} \frac{1}{|x-z|^{d-\frac{1}{2}}} \frac{1}{|x-w|^{d-\frac{1}{2}}} dx
\]
for all \(z, w \in \Gamma\) with \(z \neq w\). We apply [Fri] (Lemma 2, Section 4, Chapter 1) to estimate the RHS by \(\frac{c}{|z-w|^{d-1}}\), uniformly for all \(z, w \in \Gamma\) with \(z \neq w\). Next we prove Hölder bounds. Let \(z, z', w, w' \in \Gamma\) with \(z \neq w, z' \neq w'\) and \(|z - z'| \leq \frac{1}{2}|z - w|\). We write
\[
|K_Q(z, w) - K_Q(z', w')| = \int_{\Omega} \left( K_{\gamma V}(x, z) - K_{\gamma V}(x, z') \right) V(x) K_{\gamma 0}(x, w) dx
\]
\[
+ \int_{\Omega} K_{\gamma V}(x, z') V(x) \left( K_{\gamma 0}(x, w) - K_{\gamma 0}(x, w') \right) dx
\]
\[= I + II.\]
The estimates of \(I\) and \(II\) are similar. We split the integral \(I\) into two parts
\[
I = \int_{|z-z'| \leq \frac{1}{2}|z-z'|} \left( K_{\gamma V}(x, z) - K_{\gamma V}(x, z') \right) V(x) K_{\gamma 0}(x, w) dx
\]
\[
+ \int_{|z-z'| > \frac{1}{2}|z-z'|} \left( K_{\gamma V}(x, z) - K_{\gamma V}(x, z') \right) V(x) K_{\gamma 0}(x, w) dx. \quad (41)
\]
For the first term we use Proposition 5.5(c) and it can be estimated by
\[
c |z - z'|^\kappa \int_{\Omega} \frac{1}{|x-z|^{d-1+\kappa}} \frac{1}{|x-w|^{d-1}} dx
\]
for a suitable \(c > 0\). We apply again [Fri] (Lemma 2, Section 4, Chapter 1) to estimate the latter integral by \(c' |z - z'|^\kappa \frac{1}{|z-w|^{d-1+\kappa}}\), for a suitable \(c' > 0\).

The second integral in (41) is more delicate. If \(x \in \Omega\) and \(|z - z'| > \frac{1}{2}|x - z|\), then \(|x-z'| \leq |z-z'| + |x-z| \leq 3|z-z'|\). Moreover, \(|z-w| \leq 2|z'-w|\), since \(|z-z'| \leq \frac{1}{2}|z-w|\) by assumption. Then by Proposition 5.5(c) and [Fri] (Lemma 2, Section 4, Chapter 1) there are suitable \(c_1, c_2 > 0\) such that
\[
\int_{|z-z'| > \frac{1}{2}|z-z'|} \left| K_{\gamma V}(x, z) - K_{\gamma V}(x, z') \right| V(x) K_{\gamma 0}(x, w) dx
\]
\[
\leq c_1 \int_{|z-z'| > \frac{1}{2}|z-z'|} \left( \frac{1}{|x-z|^{d-1}} + \frac{1}{|x-z'|^{d-1}} \right) \frac{1}{|x-w|^{d-1}} dx
\]
\[
\leq 3^\kappa c_1 |z - z'|^\kappa \int_{\Omega} \left( \frac{1}{|x-z|^{d-1+\kappa}} + \frac{1}{|x-z'|^{d-1+\kappa}} \right) \frac{1}{|x-w|^{d-1}} dx
\]
\[
\leq c_2 |z - z'|^\kappa \left( \frac{1}{|z-w|^{d-2+\kappa}} + \frac{1}{|z'-w|^{d-2+\kappa}} \right)
\]
\[
\leq (1 + 2^{d-2+\kappa}) c_2 |z - z'|^\kappa \frac{1}{|z-w|^{d-2+\kappa}}.
\]
The same estimate holds for \(II\).
We finish this section with the proof of the approximation property used in the proof of Lemma 6.2.

**Proposition 6.6.** Let $\kappa \in (0,1)$, $\Omega \subset \mathbb{R}^d$ be an open bounded set with a $C^{1+\kappa}$-boundary and $C \in \mathcal{E}(\Omega)$ real symmetric. Let $c_{kl}^{(n)}$, $A^{(n)}$, $G^{(n)}$, . . . be as in the proof of Lemma 6.2. Let $K_1$ and $K_2$ be compact and disjoint subsets of $\overline{\Omega}$ such that $K_1 \times K_2 = K_1 \times K_2$. Let $k,l \in \{1,\ldots,d\}$. Then

$$\lim_{n \to \infty} (\partial_k^{(1)} \partial_l^{(2)} G^{(n)})(x,y) = (\partial_k^{(1)} \partial_l^{(2)} G)(x,y)$$

uniformly for all $x \in K_1$ and $y \in K_2$.

**Proof.** Let $u, v \in W^{1,2}(\Omega)$ be such that $\text{supp } u \subset K_1$ and $\text{supp } v \subset K_2$. Then

$$\int_{K_1 \times K_2} (\partial_k^{(1)} \partial_l^{(2)} G)(x,y) u(x) \overline{v(y)} \, dx \, dy = \int_{\Omega} \int_{\Omega} (\partial_k^{(1)} \partial_l^{(2)} G)(x,y) u(x) \overline{v(y)} \, dx \, dy$$

$$= \int_{\Omega} \int_{\Omega} G(x,y) (\partial_k u)(x) (\overline{\partial_l v})(y) \, dx \, dy$$

$$= (A_D^{-1} \partial_k u, \partial_l v)_{L_2(\Omega)}$$

$$= \lim_{n \to \infty} ((A_D^{(n)})^{-1} \partial_k u, \partial_l v)_{L_2(\Omega)}$$

$$= \lim_{n \to \infty} \int_{K_1 \times K_2} (\partial_k^{(1)} \partial_l^{(2)} G^{(n)})(x,y) u(x) \overline{v(y)} \, dx \, dy,$$

where the forth equality follows as in (36). Hence

$$\int_{K_1 \times K_2} (\partial_k^{(1)} \partial_l^{(2)} G)(x,y) u(x) \overline{v(y)} \, dx \, dy = \lim_{n \to \infty} \int_{K_1 \times K_2} (\partial_k^{(1)} \partial_l^{(2)} G^{(n)})(x,y) u(x) \overline{v(y)} \, dx \, dy. \tag{42}$$

By Theorem 4.1 there exists a $c > 0$ such that

$$| (\partial_k^{(1)} \partial_l^{(2)} G^{(n)})(x,y) | \leq \frac{c}{|x-y|^d} \leq \frac{c}{\text{dist}(K_1,K_2)^d}$$

uniformly for all $n \in \mathbb{N}$ and $(x,y) \in K_1 \times K_2$. Note that we also have Hölder bounds for $(\partial_k^{(1)} \partial_l^{(2)} G^{(n)})(x,y)$ which are uniform in $n$ and $(x,y) \in K_1 \times K_2$ by the same theorem. Therefore $(\partial_k^{(1)} \partial_l^{(2)} G^{(n)})_{n \in \mathbb{N}}$ is equicontinuous on $K_1 \times K_2$. By the Ascoli–Arzelà theorem there exists a $\Phi \in C(K_1 \times K_2)$ such that after passing to a subsequence if necessary $(\partial_k^{(1)} \partial_l^{(2)} G^{(n)})_{n \in \mathbb{N}}$ converges to $\Phi$ uniformly on $K_1 \times K_2$. Then (42) gives

$$\int_{K_1 \times K_2} \Phi(x,y) u(x) \overline{v(y)} \, dx \, dy = \lim_{n \to \infty} \int_{K_1 \times K_2} (\partial_k^{(1)} \partial_l^{(2)} G^{(n)})(x,y) u(x) \overline{v(y)} \, dx \, dy$$

$$= \int_{K_1 \times K_2} (\partial_k^{(1)} \partial_l^{(2)} G)(x,y) u(x) \overline{v(y)} \, dx \, dy.$$ 

Note that by $\partial_k^{(1)} \partial_l^{(2)} G$ is continuous on $K_1 \times K_2$ by Theorem 4.1. Hence $\Phi(x,y) = (\partial_k^{(1)} \partial_l^{(2)} G)(x,y)$ for all $(x,y) \in (K_1 \times K_2)^c$. By continuity this equality extends to all $(x,y) \in K_1 \times K_2$. \qed
7 $L_p$-commutator estimates

In this section we aim to derive good bounds on $L_p(\Gamma)$ for the commutator of the Dirichlet-to-Neumann operator $\mathcal{N}_V$ and a multiplication operator $M_g$, where $g$ is a Lipschitz continuous function on $\Gamma$. A key ingredient is a commutator estimate by Shen [She].

If the boundary $\Gamma$ is $C^\infty$ and $c_{kl} = \delta_{kl}$ it is well known that $\mathcal{N}$ is a pseudo-differential operator. In this case a well known result of Calderón shows that $[\mathcal{N}, M_g]$ acts boundedly on $L_p(\Gamma)$ with norm bounded by $C \| \nabla g \|_{L_\infty(\Gamma)}$ for some constant $C > 0$. See also Coifman and Meyer [CM] for more results on commutators of pseudo-differential operators.

It is our aim here to obtain similar results for less smooth domains and variable coefficients $c_{kl}$. We start with the following recent result of Z. Shen who treated the case of $L^2$-estimates for bounded Lipschitz domains. An additional problem is that it is unclear whether the domain of $\mathcal{N}$ is invariant under the multiplication operator. Also that is a consequence of the same theorem. We formulate the commutator estimate of Shen, [She] Theorem 1.1, in the quadratic form sense.

**Theorem 7.1.** Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with Lipschitz boundary. Let $C \in \mathcal{E}(\Omega)$ be real symmetric and suppose that each $c_{kl}$ is Hölder continuous on $\Omega$. Then there exists a $c > 0$ such that
\[
|b_V(g \varphi, \psi) - b_V(\varphi, g \psi)| \leq c \|g\|_{C^{0,1}(\Gamma)} \|\varphi\|_{L_2(\Gamma)} \|\psi\|_{L_2(\Gamma)}
\]  
for all $g, \varphi \in C^{0,1}(\Gamma)$ and $\psi \in H^{1/2}(\Gamma)$.

The theorem gives invariance of the domain of the Dirichlet-to-Neumann operator under multiplication with a Lipschitz function and commutator estimates. Recall that
\[
\text{Lip}_F(g) = \sup_{z, w \in \Gamma, z \neq w} \frac{|g(z) - g(w)|}{|z - w|}
\]
for every $g \in C^{0,1}(\Gamma)$.

**Theorem 7.2.** Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with Lipschitz boundary. Let $C \in \mathcal{E}(\Omega)$ be real symmetric and suppose that each $c_{kl}$ is Hölder continuous on $\Omega$. Then $g \varphi \in D(\mathcal{N})$ for all $g \in C^{0,1}(\Gamma)$ and $\varphi \in D(\mathcal{N})$. Moreover, there exists a $c > 0$ such that
\[
\|[\mathcal{N}, M_g]\|_{2 \to 2} \leq c \text{Lip}_F(g)
\]
for all $g \in C^{0,1}(\Gamma)$.

**Proof.** Let $c > 0$ be as in Theorem 7.1. Let $g \in C^{0,1}(\Gamma)$. Then $M_g$ is bounded from $L_2(\Gamma)$ into $L_2(\Gamma)$ and from $H^1(\Gamma)$ into $H^1(\Gamma)$. Hence by interpolation the operator $M_g$ is bounded from $H^{1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$. Since $C^{0,1}(\Gamma)$ is dense in $H^{1/2}(\Gamma)$ it follows that (43) extends to all $\varphi, \psi \in H^{1/2}(\Gamma)$.

If $\varphi \in D(\mathcal{N})$, then
\[
|b_V(g \varphi, \psi)| \leq |b_V(\varphi, g \psi)| + c \|g\|_{C^{0,1}(\Gamma)} \|\varphi\|_{L_2(\Gamma)} \|\psi\|_{L_2(\Gamma)}
\]  
\[
\leq \|\mathcal{N} \varphi\|_{L_2(\Gamma)} \|g \psi\|_{L_2(\Gamma)} + c \|g\|_{C^{0,1}(\Gamma)} \|\varphi\|_{L_2(\Gamma)} \|\psi\|_{L_2(\Gamma)} \leq c' \|\psi\|_{L_2(\Gamma)}
\]
for all $\psi \in H^{1/2}(\Gamma) = D(\mathcal{B}_V)$, where $c' = \|N\varphi\|_{L_2(\Gamma)}\|g\|_{L_\infty(\Gamma)} + c\|g\|_{C^{0,1}(\Gamma)}\|\varphi\|_{L_2(\Gamma)}$. Hence $g\varphi \in D(\mathcal{N})$. Then the extended version of (43) gives

$$\|[\mathcal{N}, M_g]\varphi\|_{L_2(\Gamma)} \leq c\|g\|_{C^{0,1}(\Gamma)}\|\varphi\|_{L_2(\Gamma)}$$

for all $\varphi \in D(\mathcal{N})$. So

$$\|[\mathcal{N}, M_g]\|_{2 \to 2} \leq c\|g\|_{C^{0,1}(\Gamma)}. \quad (44)$$

We next observe that one can replace $\|g\|_{C^{0,1}(\Gamma)}$ by Lip$_\Gamma(g)$.

Fix $z_0 \in \Gamma$ and apply (44) to $g - g(z_0)$ to obtain

$$\|[\mathcal{N}, M_g]\|_{2 \to 2} = \|[\mathcal{N}, M_{g-g(z_0)}]\|_{2 \to 2} \leq c\|g - g(z_0)\|_{C^{0,1}(\Gamma)} = c\left(\|g - g(z_0)\|_{L_\infty(\Gamma)} + \text{Lip}_\Gamma(g)\right).$$

On the other hand, since $\Gamma$ is bounded

$$|g(z) - g(z_0)| \leq |z - z_0|\text{Lip}_\Gamma(g) \leq \text{diam}(\Omega)\text{Lip}_\Gamma(g)$$

for all $z \in \Gamma$. Thus the desired estimate holds. \hfill \Box

Next we extend this result to $L_p(\Gamma)$ for all $p \in (1, \infty)$ when the underlying domain is more regular. We even have the result for $\mathcal{N}_V$ with $V \in L_\infty(\Omega)$.

**Theorem 7.3.** Let $\kappa \in (0, 1)$, $\Omega \subset \mathbb{R}^d$ be an open bounded set with a $C^{1+\kappa}$-boundary and $C \in \mathcal{E}(\Omega)$ real symmetric. Let $V \in L_\infty(\Omega, \mathbb{R})$ and assume that $0 \notin \sigma(A_D + V)$. Then for all $p \in (1, \infty)$ there exists a $c > 0$ such that

$$\|[\mathcal{N}_V, M_g]\|_{p \to p} \leq c\text{Lip}_\Gamma(g)$$

for all $g \in C^{0,1}(\Gamma)$.

In addition the operator $[\mathcal{N}_V, M_g]$ is of weak type $(1, 1)$ with an estimate $c\text{Lip}_\Gamma(g)$ as before.

**Proof.** First, recall from Corollary 5.6 that

$$\mathcal{N}_V = \mathcal{N} + Q,$$

where $Q = \gamma_0^*M_V = \gamma_0^*M_V\gamma_0$. Then

$$[\mathcal{N}_V, M_g] = [\mathcal{N}, M_g] + [Q, M_g].$$

On the other hand, by Proposition 5.5(d) the operators $\gamma_V$ and $\gamma_0$ have bounded extensions from $L_p(\Gamma)$ to $L_p(\Omega)$ for all $p \in [1, \infty]$. Therefore $[Q, M_g]$ is a bounded operator on $L_p(\Gamma)$ with norm $\|[Q, M_g]\|_{p \to p} \leq 2\|Q\|_{p \to p}\|g\|_{L_\infty(\Gamma)}$. Then as in the proof of Theorem 7.2, we fix $z_0 \in \Gamma$ and apply the above estimate with the function $g - g(z_0)$ and use the trivial bound $\|g - g(z_0)\|_{L_\infty(\Omega)} \leq \text{diam}(\Omega)\text{Lip}_\Gamma(g)$ to obtain

$$\|[Q, M_g]\|_{p \to p} \leq c'\text{Lip}_\Gamma(g),$$

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where \( c' = 2\|Q\|_{p \to p} \text{diam}(\Omega) \). Hence it remains to prove the correspond estimates for \([\mathcal{N}, M_g]\) if \( p \in (1, \infty) \).

The Schwartz kernel \( K \) of the commutator \([\mathcal{N}, M_g]\) is given by \( K(z, w) := (g(w) - g(z)) K_N(z, w) \), where \( K_N \) denotes the Schwartz kernel of \( \mathcal{N} \). It follows from Proposition 6.4 that there is a suitable \( c > 0 \) such that

\[
|K(z, w)| \leq c \text{Lip}_\Gamma(g) |z - w|^{-(d-1)}
\]

for all \( z, w \in \Gamma \) with \( z \neq w \). On the other hand, using the second bounds in Proposition 6.4 we see that there exists a suitable \( c' > 0 \) such that

\[
|K(z, w) - K(z', w)| = |(g(w) - g(z)) K_N(z, w) - (g(w) - g(z')) K_N(z, w)(z', w)|
\]

\[
\leq (|g(w)| + |g(z')|) |K_N(z, w)(z, w) - K_N(z, w)(z', w)|
\]

\[
+ |(g(z') - g(z)) K_N(z, w)(z, w)|
\]

\[
\leq 2c' \|g\|_{L_\infty(\Gamma)} |z - z'|^\kappa |z - w|^{-(d+\kappa)} + c' |z - z'| \text{Lip}_\Gamma(g) |z - w|^{-d}
\]

for all \( z, z', w \in \Gamma \) with \( z \neq w \) and \( |z - z'| \leq \frac{1}{2} |z - w| \). Again as in the proof of Theorem 7.2, we fix \( z_0 \in \Gamma \) and apply the above estimate with the function \( g - g(z_0) \) and use the trivial bound \( \|g - g(z_0)\|_{L_\infty(\Gamma)} \leq \text{diam}(\Omega) \text{Lip}_\Gamma(g) \), to deduce that there exists a suitable \( M > 0 \) such that

\[
|K(z, w) - K(z', w)| \leq M |z - z'|^\kappa |z - w|^{-(d-1+\kappa)} \text{Lip}_\Gamma(g)
\]

for all \( z, z', w \in \Gamma \) with \( z \neq w \) and \( |z - z'| \leq \frac{1}{2} |z' - w| \). Similarly, one proves that

\[
|K(z, w) - K(z, w')| \leq M_1 |w - w'|^\epsilon |z - w|^{-(d-1+\epsilon)} \text{Lip}_\Gamma(g)
\]

for all \( z, w, w' \in \Gamma \) with \( z \neq w \) and \( |w - w'| \leq \frac{1}{2} |z - w| \). It follows form (45), (46) and (47) that \( K(\cdot, \cdot) \) is a Calderón–Zygmund kernel. We obtain from this and Theorem 7.2 that \([\mathcal{N}, M_g]\) acts as a bounded operator on \( L_p(\Gamma) \) for all \( p \in (1, \infty) \) with norm estimated by \( c'' \text{Lip}_\Gamma(g) \). This shows Theorem 7.3.

\[\square\]

8 Proof of Poisson bounds

Throughout this section we adopt the notation and assumptions of Theorem 1.1.

We start with a lemma.

**Lemma 8.1.** For all \( p \in [1, \infty) \) the semigroup \( S^V \) extends consistently to a \( C_0 \)-semigroup on \( L_p(\Gamma) \).

**Proof.** By Theorem 2.2(b), the semigroup \( S \) generated by \(-\mathcal{N}\) is sub-Markovian and hence it acts as a contraction semigroup on \( L_p(\Gamma) \) for all \( p \in [1, \infty) \) and it is strongly continuous if \( p \in [1, \infty) \). We know from Proposition 5.5(d) that the operator \( Q = \gamma_V M_\gamma \gamma_0 \) is bounded on \( L_p(\Gamma) \) for all \( p \in [1, \infty) \). Since \( \mathcal{N}V = \mathcal{N} + Q \) by Corollary 5.6 it follows from standard perturbation theory that the semigroup \( S^V \) generated by \(-\mathcal{N}V\) acts as a \( C_0 \)-semigroup on \( L_p(\Gamma) \) for all \( p \in [1, \infty) \).

\[\square\]
Let $S^V$ be the semigroup generated by $-\mathcal{N}_V$ on $L_2(\Gamma)$.

**Lemma 8.2.** For all $t > 0$ the operator $S^V_t$ has a kernel $K^V_t$. In addition, there exist $\omega \in \mathbb{R}$ and $c > 0$ such that
\[ |K^V_t(z, w)| \leq c t^{-(d-1)} e^{\omega t} \]
for all $t > 0$ and a.e. $z, w \in \Gamma$.

**Proof.** If $d \geq 3$ the Sobolev embedding of $\text{Tr} (W^{1,2}(\Omega))$ into $L_{\frac{2(d-1)}{d-2}}(\Gamma)$ of [Neč] Theorem 2.4.2 implies easily that there exist $c, \omega > 0$ such that
\[ b_V(\varphi, \varphi) + \omega \int_\Gamma |\varphi|^2 \geq c \|\varphi\|_{L_{\frac{2(d-1)}{d-2}}(\Gamma)} \]
for all $\varphi \in \text{Tr} (W^{1,2}(\Omega))$. Since the semigroup $S^V$ acts on $L_p(\Gamma)$ for all $p \in [1, \infty]$ it is well known that the later inequality implies ultracontractivity estimates. More precisely, there exist $c, \omega > 0$ such that
\[ \|S^V_t\|_{1 \to \infty} \leq c t^{-(d-1)} e^{\omega t} \] (48)
for all $t > 0$. Note that (48) implies that $S^V_t$ is given by a kernel $K^V_t$ such that
\[ |K^V_t(z, w)| \leq c t^{-(d-1)} e^{\omega t} \]
for all $t > 0$ and a.e. $z, w \in \Gamma$.

If $d = 2$, then the proof is precisely the same as in the proof of Theorem 2.6 in [EO2]. \qed

Using the previous two lemmas, duality and interpolation one deduces easily the next lemma.

**Lemma 8.3.** There exists a $c > 0$ such that
\[ \|S^V_t\|_{p \to q} \leq c t^{-(d-1)\left(\frac{1}{p} - \frac{1}{q}\right)} \]
for all $t \in (0, 1]$ and $p, q \in [1, \infty]$ with $p \leq q$.

Let $g \in C^{0,1}(\Gamma)$. Define $\delta^j_g(\mathcal{N}_V) = [M_g, \mathcal{N}_V]$ and for all $j \in \mathbb{N}$ define inductively $\delta^{j+1}_g(\mathcal{N}_V) = [M_g, \delta^j(\mathcal{N}_V)]$. Define similarly $\delta^j_g(S^V_t)$.

**Lemma 8.4.** Suppose either $p, q \in (1, \infty)$ with $p \leq q$ and $(d-1)(\frac{1}{p} - \frac{1}{q}) \in \{0, 1, \ldots, d-1\}$, or $p = 1$ and $q = \infty$. Then there exists a $c_{p,q} > 0$ such that
\[ \|\delta^j_g(\mathcal{N}_V)\|_{p \to q} \leq c_{p,q} (\text{Lip}_\Gamma(g))^j \]
for all $g \in C^{0,1}(\Gamma)$, where $j = 1 + (d-1)(\frac{1}{p} - \frac{1}{q})$.

**Proof.** The kernel $\tilde{K}$ of $\delta^j_g(\mathcal{N})$ is given by $\tilde{K}(z, w) = (g(w) - g(z))^j K_{\mathcal{N}_V}(z, w)$, where we use again $K_{\mathcal{N}_V}$ to denote the Schwartz kernel of $\mathcal{N}_V$. It follows immediately from Proposition 6.5 that there is a suitable $c > 0$ such that
\[ |\tilde{K}(z, w)| \leq c |g(z) - g(w)|^j |z - w|^{-d} \leq c |z - w|^{-(d-j)} (\text{Lip}_\Gamma(g))^j \] (49)
for all \(z,w \in \Gamma\) with \(z \neq w\).

If \(1 < p < q < \infty\), then (49) implies that \(\tilde{K}\) is a Riesz potential. Then the boundedness of \(\delta_g^j(N_V)\) from \(L_p(\Gamma)\) to \(L_q(\Gamma)\) follows from [Ste], Theorem V.1.

If \(1 < p = q < \infty\), then the statement of the lemma is given by Theorem 7.3.

Finally, if \(p = 1\) and \(q = \infty\) then \(j = d\). In this case

\[
|\tilde{K}(z,w)| \leq c (\text{Lip}_\Gamma(g))^d
\]

and hence \(\delta_g^d(N_V)\) is bounded from \(L_1(\Gamma)\) into \(L_\infty(\Gamma)\) with norm estimates by \(c (\text{Lip}_\Gamma(g))^d\).

\[\square\]

In order to prove the Poisson bound for the kernel \(K^V(x,y)\) we proceed as in Section 4 of [EO2]. For the reader’s convenience, we repeat the arguments. Let \(c > 0\) be as in Lemma 8.3. Further let \(c_{p,q}\) be as in Lemma 8.4.

Let \(t \in (0,1]\). Then

\[
\delta_g^d(S^V_t) = \sum_{k=1}^d (-t)^k \sum_{j_1,\ldots,j_k \in \mathbb{N}} \int_{H_k} S^V_{t_{k+1}} \delta^j(N_V) S^V_{t_k} \circ \ldots \circ \\
\circ S^V_{t_2} \delta^j(N_V) S^V_{t_1} d\lambda_k(t_1,\ldots,t_{k+1}),
\]

where

\[
H_k = \{(t_1,\ldots,t_{k+1}) \in (0,\infty)^{k+1} : t_1 + \ldots + t_{k+1} = 1\}
\]

and \(d\lambda_k\) denotes Lebesgue measure of the \(k\)-dimensional surface \(H_k\). We estimate each term in the sum. Let \(k \in \{1,\ldots,d\}\), \((t_1,\ldots,t_{k+1}) \in H_k\), \(g \in C^{0,1}(\Gamma)\), \(t \in (0,1]\) and \(j_1,\ldots,j_k \in \mathbb{N}\) with \(j_1 + \ldots + j_k = d\) and \(\text{Lip}_\Gamma(g) \leq 1\).

If \(k = 1\) then \(j_1 = d\) and we have for \(t \in (0,1]\) and each \(g\) with \(\text{Lip}_\Gamma(g) \leq 1\)

\[
t \|S^V_{t_{k+1}} \delta^j(N_V) S^V_{t_k} \|_{1 \to \infty} \leq t \|S^V_{t_{k+1}}\|_{\infty \to \infty} \|\delta^d(N_V)\|_{1 \to \infty} \|S^V_{t_k}\|_{1 \to 1} \leq c^2 c_{1,\infty} t.
\]

Suppose that \(k \in \{2,\ldots,d\}\). There exists an \(M \in \{1,\ldots,k+1\}\) such that \(t_M \geq \frac{1}{k+1}\). Note that \(\sum_{\ell=1}^k (j_\ell - 1) = d - k < d - 1\). First suppose \(M \notin \{1,k+1\}\). Fix \(1 = q_0 < p_1 \leq q_1 = p_2 \leq q_2 = p_3 \leq \ldots \leq q_{M-2} = p_{M-1} \leq q_{M-1} \leq p_M = q_M = p_{M+1} \leq q_{M+1} \leq \ldots \leq q_k = p_k \leq q_{k+1} = \infty\) such that

\[
1 - \frac{1}{p_1} = 1 - \frac{1}{2(d-1)} = \frac{1}{q_k} , \quad \frac{1}{p_\ell} - \frac{1}{q_\ell} = \frac{j_\ell - 1}{d - 1} \quad \text{and} \quad \frac{1}{q_{M-1}} - \frac{1}{p_M} = \frac{k - 2}{d - 1}
\]

for all \(\ell \in \{1,\ldots,k\}\). Then

\[
t^k \|S^V_{t_{k+1}} \delta^j(N_V) \ldots \delta^j(N_V) S^V_{t_1}\|_{1 \to \infty} \leq t^k \|S^V_{t_{k+1}}\|_{q_0 \to p_1} \prod_{\ell=1}^k \|S^V_{t_{k+1}}\|_{q_\ell \to p_{\ell+1}} \|\delta^j(N_V)\|_{p_\ell \to q_\ell}
\]

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\[ \leq t^k c(t_1 t)^{-(d-1) \left( \frac{1}{m_0} - \frac{1}{m_1} \right)} \prod_{\ell=1}^{k} c_{p_{\ell},q_{\ell}} c(t_{\ell+1} t)^{-(d-1) \left( \frac{1}{q_{\ell}} - \frac{1}{p_{\ell+1}} \right)} \]

\[ = c' t^k t^{-(k-1)} t_1^{-1/2} t_k^{-(k-2)} \leq t_{k+1}^{-1/2} \]

\[ \leq c' (k+1)^k t t_1^{-1/2} t_{k+1}^{-1/2}, \]

where \( c' = c^{k+1} \prod_{\ell=1}^{k} c_{p_{\ell},q_{\ell}} \). If \( M \in \{1,k+1\} \) then a similar estimate is valid with possibly a different constant for \( c' \). Integration and taking the sum gives

\[ \| \delta^d_g(S^V_t) \|_{1 \to \infty} \leq c'' t \]

for a suitable \( c'' > 0 \), uniformly for all \( t \in (0,1] \) and \( g \in C^0(\Gamma) \) such that Lip\( \Gamma(g) \leq 1 \). Therefore

\[ |g(w) - g(z)|^d |K^V_t(z,w)| \leq c'' t \]

(50)

for all \( w,z \in \Gamma \).

Note that the metric \( d_\Gamma:\Gamma \times \Gamma \to [0,\infty) \) given by

\[ d_\Gamma(z,w) := \sup \{ g(z) - g(w) : g \in C^0(\Gamma), \text{ Lip}_\Gamma(g) \leq 1 \} \]

is equivalent to the Euclidean one. Indeed, by the definition of Lip\( \Gamma(g) \leq 1 \) one has

\[ |g(z) - g(w)| \leq |z - w| \]

for all \( z,w \in \Gamma \). Hence \( d_\Gamma(z,w) \leq |z - w| \). To obtain the reverse inequality, let \( k \in \{1,\ldots,d\} \) and choose \( g(z) = z_k \), where \( z = (z_1,\ldots,z_d) \). Then

\[ |z_k - w_k| \leq d_\Gamma(z,w) \]

and hence \( |z - w| \leq d^{d/2} d_\Gamma(z,w) \).

Using this fact and optimizing over \( g \) in (50) we obtain

\[ |w - z|^d |K^V_t(z,w)| \leq c'' d^{d/2} t \]

for all \( t \in (0,1] \) and \( z,w \in \Gamma \). We combine this with Lemma 8.2 and obtain that there is a \( c > 0 \) such that

\[ |K^V_t(z,w)| \leq \frac{c(t \wedge 1)^{-(d-1)} e^{\omega t}}{\left(1 + \frac{|z - w|}{t}\right)^d} \]

for all \( t \in (0,1] \) and \( z,w \in \Gamma \). By [Ouh] Lemma 6.5 and the fact that \( \Gamma \) is bounded we improve this bound and there is a \( c > 0 \)

\[ |K^V_t(z,w)| \leq \frac{c(t \wedge 1)^{-(d-1)} e^{-\lambda_1 t}}{\left(1 + \frac{|z - w|}{t}\right)^d} \]

for all \( t > 0 \) and \( z,w \in \Gamma \). This completes the proof of Theorem 1.1.

\[ \Box \]

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