Estimation by Stable Motions and its Applications

Anirban Das, Manfred Denker, Anna Levina and Lucia Tabacu

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Dedicated to the memory of Wojbor A. Woyczynski (1943–2021)

Abstract

We propose a family of confidence intervals for nonparametric moment estimators if the observations have large or infinite variances. The theoretical underpinnings which guarantee the soundness of the method are demonstrated. Extensive numerical simulations show its superiority over bootstrap and normal approximation and its wide applicability. Finally, a confidence interval to estimate the coupling strength in neuronal networks is proposed.

1 Introduction

Resampling by using stable distributions was introduced in [7], however no application and no simulation about its performance was provided at that time. The purpose of this note is to fill this gap, to provide an effective algorithm for this resampling procedure and to apply the method for nonparametric moment estimators.

Most existing methods for establishing asymptotically consistent confidence intervals of parameters rely on exact distributions or on normal approximation (the Central Limit Theorem (CLT)), provided the underlying statistics has finite variance. However, when the variance of the underlying distribution is infinite the CLT does not hold, no numerically efficient method seems to be known, again when the variance is large, the confidence intervals become too wide to permit reliable conclusions. Thus the natural question here is to find new statistics not changing the parameter, but allowing for calculations of asymptotic confidence intervals for moments. A class of such new statistics is called resampling using stable motions in [7], here abbreviated as stable resampling.

The limiting distribution for our resampling method depends on stochastic integrals with respect to stable motions, introduced in [17] by Rosiński and Woyczynski, but the quantiles of the limiting distribution are not directly calculable. It is possible to use almost sure versions for the convergence to the limiting distribution (abbreviated as ASLT in analogy to the normal case) which are obtained in [11].
We begin in Section 2 with a description of the program to be used in the simulations. It splits into two subroutines, the first is the estimation of confidence intervals, once quantiles are estimated. This is a standard formality, but relies on the limit theorem for stochastic integrals. The other novelty lies in the second subroutine which consists of an estimation of the limiting distribution function using the ASLT method. In Section 3 the theoretical background is briefly sketched leading to the specific form used to calculate confidence intervals and to the formula which permits to estimate the unknown distribution from the data directly. This resembles vaguely to a bootstrap method, though it is almost surely and a different resampling method.

Simulations are collected in Section 4. It splits into several subsections. First, using a simple form of the ASLT-algorithm, simulations for the performance of the estimate of the unknown distribution function are presented.

The second and major part of the section builds the core of this note. Its applications are rather wide, due to the fact that it works for $U$-statistics in general. In Section 4 one specific class of $U$-statistic is used, to estimate confidence intervals for the mean of a distribution which has large or infinite variance. Its algorithm uses the full strength of the resampling algorithm. Clearly, other moment estimators can be handled in the same way. The purpose of this subsection is to present simulation results of various types. First of all we use distributions with heavy tails (power like distributions) since we shall make use of such distributions in the last subsection. For these models (varying power law and perturbations thereof) we examine different statistics (parametrized by the order of the stable motion), different sample sizes and different confidence levels. In addition we compare the new method with the bootstrap method (established in [7]) and approximation by normal distribution.

Many processes observed in biology and neural science are subject to heavy tail distributions or those where estimation of the mean is affected by large variances even if sample sizes are large. The last subsection provides such an example. We apply our algorithm, called the Stable Resampling for Moments (SRM) to estimate the connection strength in a complete neuronal network [9] from the expectation $E(X)$ of its avalanche size distribution. It is shown that the method has advantages over the classical confidence interval estimation for asymptotically normal observables. This is due to the fact that the variance of the distribution is about $(E(X))^3$ and $E(X) \to \infty$ as the number of neurons increases.

## 2 Stable Resampling: The Algorithm

This section contains the description of the algorithm on which our estimation procedure for moments is based upon. It will be called Stable Resampling and has a more formal description which easily enables the transformation into a code: It consists of two subroutines, where the first one relies on the second one.
2.1 Stable resampling for moments

This subroutine will be called \textit{stable resampling for moments} and is abbreviated as \textit{SRM}(q, q', p, r_l, r_u, \delta).

Let \( q > p > 1, q' < \frac{q}{p}, r_u, r_l \in \mathbb{N}, 10 < n \in \mathbb{N} \) and \( \delta \in \{0, 1\} \). For each choice of these parameters the SRM-algorithm to estimate the \( q' \)-th moment based on a sample of size \( n \) proceeds as follows\footnote{The requirement \( 10 < n \) can be replaced by a similar requirement which prevents small number of observations having a big influence on the estimation of the limiting distribution.}

**Data:** \( X_1, X_2, \ldots X_n \) iid sample following the distribution of \( X \), with \( E\left[|X|^q\right] < \infty \).

**Result:** Given \( \alpha \in [0, 1] \), the output is a one- or two-sided \( \alpha \)-level confidence interval for \( E[X^{q'}] \).

**Sequential Steps:**

(a) Calculate \( \hat{\mu} := \frac{1}{n} \sum_{i=1}^{n} X_i^{q'} \) \hspace{1cm} (2.1)

(b) Generate an iid sample \( \left\{ Y_i \right\}_{i=1}^{n} \) (independent of the samples \( X_1, \ldots X_n \)) from a stable distribution with location parameter 1, skewing parameter 0, stability parameter \( p \), scale parameter \( 0.5 \)\footnote{Again, the scale parameter may be chosen differently for variants of the subroutine.}

(c) For \( i = 1, \ldots, n \) calculate

\[ W_i = X_i^{q'} Y_i - \hat{\mu} Y_i \] \hspace{1cm} (2.2)

Pass the sample \( \left\{ W_i \right\}_{i=1}^{n} \) as input to the \textit{ASLT}(p, r_l, r_u) in Section 2.2 to obtain two estimated distribution functions \( \text{Edf}_l \) and \( \text{Edf}_u \). From \( \text{Edf}_l \) calculate the lower \( \frac{\alpha}{2} \) quantile \( L \) and from \( \text{Edf}_u \) calculate the the upper \( \frac{\alpha}{2} \) quantile \( U \).

(d) \( XY \leftarrow \frac{\sum_{i=1}^{n} X_i^{q'} Y_i}{n}, \quad Y \leftarrow \frac{\sum_{i=1}^{n} Y_i}{n} \).

(e) If \( \delta = 1 \) \( C_U \leftarrow \frac{XY}{Y} - \frac{n^{\frac{1}{2}-1} L}{\sqrt{p}} \) \hspace{1cm} \( C_L \leftarrow \frac{XY}{Y} - \frac{n^{\frac{1}{2}-1} U}{\sqrt{p}} \);

If \( \delta = 0 \) \( C_U \leftarrow \frac{\hat{\mu}}{Y} - \frac{n^{\frac{1}{2}-1} L}{\sqrt{p}} \) \hspace{1cm} \( C_L \leftarrow \frac{\hat{\mu}}{Y} - \frac{n^{\frac{1}{2}-1} U}{\sqrt{p}} \)

(f) Output the \( \alpha \)-level confidence interval \( C(q, p, r_l, r_u, \delta) = (C_L, C_U) \) and the \( \alpha/2 \)-level confidence intervals \(( -\infty, C_U) \) and \(( C_L, \infty) \)
2.2 Estimating the p-re-sampled distribution function

The two distribution functions needed for the SRN subroutine are called p-
resampled distribution functions and are calculated according to the following
subroutine ASLT(p, rl, ru).

Data: \( W_1, W_2, \ldots, W_n \) iid sample following the distribution of \( W \), \( n > 10 \).

Result: A pair of distribution functions \( \text{Edf}_\text{l} \) and \( \text{Edf}_\text{u} \)

Initialization \( \text{CL}_{\text{vec}} \leftarrow \text{Vector(size: } r_l \text{)} \), \( \text{CU}_{\text{vec}} \leftarrow \text{Vector(size: } r_u \text{)} \)

Sequential Steps \( r_m \leftarrow \max(r_l, r_u) \)

For \( \text{perm} \leftarrow 1 \) to \( r_m \) by 1 do

(a) randomly permute \( W_1, W_2, \ldots, W_n \) to obtain \( W'_1, W'_2, \ldots, W'_n \).

(b) Define \( T_1, T_2, \ldots, T_{n-9} \) as

\[
T_i = (i + 9)^{-\frac{1}{p}} \sum_{i' = 1}^{i+9} W'_{i'} 
\]  

(2.3)

(c) For any \( t \), define

\[
C(t) = \left( \sum_{i' = 10}^{n} \frac{1}{i'^{\frac{1}{p}}} \right)^{-1} \left( \sum_{i' = 10}^{n} \frac{1}{i'^{\frac{1}{p}}} 1_{(-\infty,t]}(T'_{i' - 9}) \right). 
\]  

(2.4)

If \( \text{perm} \leq r_l \) \( \text{CL}_{\text{vec}} \left[ \text{perm} \right] \leftarrow C \)

If \( \text{perm} \leq r_u \) \( \text{CU}_{\text{vec}} \left[ \text{perm} \right] \leftarrow C \)

\( \text{Edf}_\text{l}(t) = \frac{1}{r_l} \sum_{c \in \text{CL}_{\text{vec}}} c(t), \quad \text{Edf}_\text{u}(t) = \frac{1}{r_u} \sum_{c \in \text{CU}_{\text{vec}}} c(t). \)

3 Theoretical Justifications for the Stable Re-
sampling for Moments

We first demonstrate the theoretical underpinnings which guarantee the consis-
tency of the method \( \text{SRM}(q, q', p, r_l, r_u, \delta) \) described in Subsection 2.1, where \( q' \)
is the order of the moment being estimated. From \([7]\) (Theorem 3.3) we have:

**Theorem 3.1.** Given an iid sample \( X_1, X_2, \ldots, X_n, \ldots \) following the distri-
bution of \( X \), any function \( h \) satisfying \( E \left[ |h(X)|^r \right] < \infty \), and an iid sam-
ple \( Y_1, Y_2, \ldots, Y_n, \ldots \) following a centered \( p \)-stable distribution with \( p \) satisfying \( r > p > 1 \), and also \( Y_i \) being independent of \( X_i, \forall i, i' \), then

\[
\frac{1}{n^r} \sum_{i \leq n} \left( h(X_i) - E_{X \sim X} [h(x)] \right) Y_i \xrightarrow{\text{weakly}} G_{X,h,p}, \quad (3.1)
\]

for some random variable \( G_{X,h,p} \) whose distribution depends on \( X, h, \) and \( p \).
The setting considered in Algorithm SRM\((q, q', p, r_l, r_u, \delta)\) is recovered if we apply the above theorem with \(h(x) = x^{q'}\), such that \(E\left[|X|^{q'}\right] < \infty\) and \(p < \frac{q}{q'}\).

Next we demonstrate that in Step (iii) of Algorithm 2.1, the call to Algorithm 2.2 yields two distribution functions approximating \(G_{X,h,p}\). This in conjunction with (3.1) will establish the veracity of Algorithm 2.1.

**Definition 3.2. p-resampled distribution:** For the setting considered in Algorithm SRM\((q, q', p, r_l, r_u, \delta)\), the p-resampled distribution \(G_{X,q',p}\) is defined as the unique limit (in the sense of converge in law) of \(\frac{1}{n} \sum_{i=1}^{n} W_i\) as \(n\) tends to infinity, with \(W_i\) defined in (2.2).

From Theorem 4.1 of [11] we get:

**Theorem 3.3.** Given \(X_1, X_2, \ldots, X_n\) iid samples following the distribution of \(X\), \(h\) any real-valued function, \(Y_1, Y_2, \ldots, Y_n\) iid samples following a p-stable distribution with mean 0, and

\[
T_n(h) := \frac{1}{n^p} \sum_{i \leq n} (h(X_i) - E_{x \sim X}[h(x)]) Y_i.
\]

If \(Y_i\) is independent of \(X_i\), \(\forall i, i'\), and \(T_n(h) \xrightarrow{\text{weakly}} G\), for some random variable \(G\), then for any \(-\infty < a < b < \infty\) such that \(a, b\) are continuity points of \(G\), we have

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} 1_{(a,b)}(T_k(h)) \to \mathcal{P}\left(G \in (a,b)\right) \quad \text{as.} \quad (3.2)
\]

If Theorem 3.3 holds true under the relaxed condition that \(E[Y_i] = 1\) (see proceeding paragraph), it implies that when SRM\((q, q', p, r_l, r_u, \delta)\) calls the subroutine ASLT\((p, r_l, r_u)\) in Step (c), the two distributions which are returned both converge in distribution to the p-resampled distribution \(G_{X,q',p}\).

Theorem 3.3 is only applicable when \(E[Y_i] = 0\), we show it’s veracity for non zero values of \(E[Y_1]\) when \(E_{x \sim X}[|h(x)|^r] < \infty\) for some \(r > p > 1\). Without loss of generality let \(E_{x \sim X}[h(x)] = 0\), then

\[
T_n(h) = \frac{1}{n^p} \sum_{i \leq n} h(X_i) Y_i = \frac{1}{n^p} \sum_{i \leq n} h(X_i)(Y_i - E(Y_i)) + \frac{E(Y_i)}{n^p} \sum_{i \leq n} h(X_i)
\]

Theorem 3.3 is applicable for the first part of the sum above, the second part goes to zero because \(E_{x \sim X}[|h(x)|^r] < \infty\), and \(r > p\) (see [3]).

**Remark 3.4.** The sequence \(\{T_i\}_{i=1}^{n-9}\) defined in (2.3) of Algorithm 2.2 are different for different permutations \(\{W_i\}_{i=1}^n\). This enables different estimates of
the quantiles to be derived by permuting the data. One can use this to reduce
variance of the final estimate by averaging across the estimates from different
permutations. This is not possible for statistics which do not depend on the
order of the samples.

Remark 3.5. The results of [7] and [11] are also applicable to $U$-statistics in
general. Some additional assumption has to be imposed. Since we are only
simulating moment estimators we are not formulating this here.

4 Simulation Results

As explained in the introduction, this section summarises our simulation results
for the p-resampled distribution in Subsection 4.1, the stable re-sampling for
moments in Subsection 2.1 and an application to neuronal avalanches.

4.1 Simulations for estimating p-resampled distributions

The Algorithm 2.2 when called from Step (c) of Algorithm 2.1 returns estimates
for the p-resampled distribution (def 3.2). Here we demonstrate the robustness
of the distribution functions inferred by varying the number of samples, and
comparing to estimates of the p-resampled distribution obtained by bootstrapping.

We took $N$ random samples $Z_i$ from a Pareto distribution with shape pa-
rameters 2 and location parameter 3, and independently we took $N$ samples $Y_i$
from a stable distribution with order $p = 1.2$, shape $\gamma = 1$, skewness $\beta = 0$ and
mean $\delta = 1$. We transformed the data of the first sample using the map

$$x \mapsto f(x) = x \max\{\log|x|, 1\},$$

in order to get a distribution not in the strict domain of attraction of a stable
distribution (here called Pareto-like). Then, for the sample

$$X_i = f(Z_i),$$

define (in likeness of (2.2) with $q' = 1$)

$$W_i = X_i Y_i - \hat{\mu} Y_i, \text{ where } \hat{\mu} := \frac{1}{n} \sum_{i=1}^{n} X_i. \quad (4.1)$$

The data was generated using the seed 1345 in the R-software. We compute
the empirical distributions produced by passing the sequence $W_i$ to ASLT(1.2, 1, 1)
(using using $T_1, T_2, \ldots, T_n$ instead of $T_1, T_2, \ldots, T_{n-9}$ in (2.4)), we also estimate
the p-resampled distribution by applying bootstrapping in two ways: 1. boot-
strapping the quantities $W_i = X_i Y_i - \hat{\mu} Y_i$, where $\hat{\mu} := \frac{1}{10^5} \sum_{i=1}^{10^5} X_i$ and 2.
bootstrapping the quantities $W_i = X_i Y_i - 12.565 \times Y_i$. Note 12.565 is the mean
of the random variable $X_1$ calculated using $10^6$ samples.
The distribution functions from applying Algorithm 2.2 clearly show the heavy tail behavior of distributions: Large simulated values occur rarely, so are seen only in larger sample size simulations. The mean differs considerably from its median, hence the distribution is not symmetric around 0. The distribution does not have second moments but is in the domain of attraction of a normal. The graphics show that the estimation of the distribution function stabilizes quite well as \( n \to \infty \) as expected from Theorem 3.3.

Comparing the approximations in Fig 1a and Fig 1b, it can be said that the ASLT approach is at least as good - if not better - than the bootstrap approximation. It also should be noticed that the bootstrap distribution seems to become symmetric around 0 and does not show the convergence pattern as the ASLT. It is known that the ASLT approach can even be improved by using some permutations of the data and deleting some initial terms in the summation procedure. This has been incorporated in the algorithms in Section 2.

Figure 1c shows the same approximation using bootstrap when the true mean \( \mu \) is replaced by an estimated \( \hat{\mu} \). The graphics shows the same type of approximation, slightly shifted to the left, an effect due to the underestimation of \( \mu \).
4.2 Confidence intervals for the mean of power-like distributions

Here we will investigate the performance of the SRM method for synthetic data-sets. First we introduce some metrics which will be used to evaluate the performance.

Recall that the cover probability is the proportion \( p_c \) of runs where the unknown parameter \( \theta^* \) lies in the \( \alpha \)-confidence interval divined by the estimation. If \( p_c > \alpha \), then the method is called conservative. If \( p_c < \alpha \), then the method is termed permissive.

Also the length of a confidence (CI length) is used to evaluate the accuracy of an estimation method. We take the average of the lengths of confidence intervals across all runs to obtain the CI length.

We will explicitly restrict ourselves to examples for estimating the first moment. Except otherwise stated, the data for the examples studied in this section are generated from a random variable \( X \sim Z \log(Z) \), where \( Z \) follows a Pareto distribution with scale parameter set to 1, and shape parameter \( q_P \) (\( q_P \) is varied for various examples), these distributions are loosely called “nearly power laws”.

Given these settings \( E[|X|^q] \) is finite for all \( q < q_P \), and is \( \infty \) for all \( q \) satisfying \( q \geq q_P \). Whenever we apply \( \text{SRM}(q, 1, p, r_l, r_u, \delta) \), we are careful to choose a value of \( p \), satisfying \( p < q_P \).

**SMA\((q, 1, 1.2, r, r, \delta)\) versus normal approximation**

We compare the performance of \( \text{SRM}(q, 1, 1.2, r, r, \delta) \), for both \( \delta = 0 \), and \( \delta = 1 \) to a standard CLT based method (which makes normal approximation) in a series of examples where we vary \( q < q_P \). We use the cover probability and length of confidence intervals to make comparisons. 500 runs of both methods were used, in each run 1000 samples were used, a two sided 95 percentile interval was constructed, the exact results for the case \( \delta = 0 \) are noted in Table (1), the case \( \delta = 1 \) showed similar results. For smaller values of \( q_P \) (underlying distribution is more heavily tailed) the cover probability of normal approximations is very poor. At \( q_P = 4 \), the performance of the normal method is excellent (Table 1). The cover probability produced by \( \text{SRM}(q, 1, 1.2, r, r, 0) \) is near the required .95 mark for all values of \( q_P \), similar trends hold for \( \text{SRM}(q, 1, 1.2, r, r, 1) \).
Table 1: Cover probability ($P$) and CI length ($L$) of SRM($q, 1, 1.2, r, r, 0$) for the nearly power laws, $P_{nor}$ and $L_{nor}$ are the cover probability and CI length of confidence interval for methods using normal approximations.

| $q_P$  | $P$  | $L$  | $P_{nor}$ | $L_{nor}$ |
|-------|------|------|-----------|-----------|
| $q_P = 1.5, r=5$ | 0.948 | 72.85 | 0.582 | 11.04 |
| $q_P = 1.6, r=5$ | 0.968 | 38.06 | 0.648 | 4.72 |
| $q_P = 1.7, r=4$ | 0.97 | 27.4 | 0.738 | 2.86 |
| $q_P = 2.1, r=2$ | 0.946 | 8.42 | 0.804 | 0.95 |
| $q_P = 3.1, r=2$ | 0.96 | 2.28 | 0.904 | 0.21 |
| $q_P = 4, r=2$ | 0.97 | 1.33 | 0.942 | 0.11 |

Variation of the nearly power law

We investigate the performance of the SRM($q, 1, 1.2, r_l, r_u, 0$) method for different kinds of “nearly power laws”. In particular we generate a Pareto distribution with scale parameter set to 1, and shape parameter set to 1.5 ($q < 1.5$), and pass it through a function $f$ to generate the ground truth data. The performance of the method is evaluated for various choices of $f$, see Table 2. The method gives acceptable performance for deriving both one sided and two sided confidence intervals for a variety of choices for $f$, which shows a type of robustness of the method with respect to perturbations.

Table 2: $P(0)$ is the cover probability when 95% symmetric two sided confidence interval is built for the mean, $P_l(0)$ is the cover probability when 97.5% one sided upper confidence interval is built for the mean, $P_l(0)$ is the cover probability when 97.5% one sided lower confidence interval is built for the mean, $n$ is the number of samples, $r_l$ and $r_u$ are the parameters used for the method SRM($q, 1, p, r_l, r_u, 0$), $q \sim 1.5$.

| function | $r_l$ | $r_u$ | $n$ | $P(0)$ | $P_l(0)$ | $P_u(0)$ | $L$ |
|--------|------|------|-----|--------|---------|---------|-----|
| $x \log(|x|)$ | 5 | 5 | 1000 | 0.958 | 0.978 | 0.98 | 43.82 |
| $x/(1 + \log(x))^{-2}$ | 2 | 2 | 1000 | 0.93 | 0.992 | 0.938 | 0.30 |
| $x$ | 5 | 5 | 1000 | 0.976 | 0.978 | 0.998 | 10.64 |
| $x$ | 4 | 2 | 1000 | 0.958 | 0.98 | 0.978 | 9.06 |
| $x(1 + \frac{1}{2} \cos(x))$ | 5 | 5 | 1000 | 0.964 | 0.964 | 1 | 10.32 |
| $x(1 + \frac{1}{2} \cos(x))$ | 5 | 2 | 1000 | 0.968 | 0.97 | 0.998 | 12.93 |
| $x \cos(x)$ | 5 | 5 | 1000 | 0.994 | 0.998 | 0.996 | 8.02 |
| $x \cos(x)$ | 2 | 2 | 1000 | 0.954 | 0.976 | 0.978 | 5.78 |

Using a single random permutation
As noted in Remark 3.4 the ASLT part of the algorithm has the ability to be able to reduce variance by taking permutations, here we demonstrate taking only a single permutation in Algorithm 2.1 results in a rather poor final confidence interval (Table 3).

Table 3: Testing the performance of SRM(1.5, 1, p, r, r, δ) for making 90% confidence intervals. P(δ) is the cover probability when 90% symmetric two-sided confidence interval is built for the mean, Pu(δ) (Pl(δ)) is the cover probability when 95% one-sided upper (lower) confidence interval is built for the mean using SRM(1.5, 1, p, r, r, δ), n is the number of samples. The data is created by generating a Pareto distribution with scale parameter set to 1, and shape parameter set to 1.5, and passing the result through a function \( f(x) = x \log(x) \). 500 runs are made to derive the cover probabilities, for each run n samples are taken.

| \( p \) | n  | P(0)  | Pl(0) | Pu(0) | P(1)  | Pl(1) | Pu(1) | L       |
|------|----|-------|-------|-------|-------|-------|-------|---------|
| 1.1  | 1000 | 0.804 | 0.894 | 0.91  | 0.816 | 0.958 | 0.858 | 38.92   |
| 1.2  | 1000 | 0.74  | 0.872 | 0.868 | 0.76  | 0.956 | 0.804 | 24.7    |
| 1.3  | 1000 | 0.702 | 0.882 | 0.82  | 0.698 | 0.957 | 0.74  | 35.03   |
| 1.4  | 1000 | 0.676 | 0.874 | 0.802 | 0.662 | 0.964 | 0.698 | 14.27   |

Performance under different stable distributions and sample sizes

We demonstrate the performance of SRM(q, 1, p, 5, 5, δ) in two simulations as \( p \) varies and \( q \sim 1.5 \). The data is generated by the same process as in Table 3. 200 runs are made (with \( n \) samples used in every run), for each run \( m \), the output from SRM is a confidence interval \((u(m), l(m))\). To visualize the performance, we observe the mean and 2.5-th percentile of the upper limit \( u(m) \) (calculated across the runs) in various controlled simulations where the value of \( p \) and \( n \) is varied. Similarly we also observe the mean and 97.5-th percentile of the lower limit \( l(m) \) (see Figure 2). It is observed that for \( p = 1.2 \), the 97.5-th percentile of the lower limit and 2.5-th percentile of the upper limit are close to the actual value of the parameter for all values of \( n \). This clearly is not the case when \( p = 1.7 \). The reason for the poor performance at \( p = 1.7 \) is because the condition \( p < q_P \) is violated for this example.

In the second simulation we give the results for SRM under various choices of parameters when the underlying data is generated from the same nearly power laws as the data for Table 3 (with \( q_P = 1.5 \)). The results are summarized in Table 4. It is notable that when \( p, r_l \) and \( r_u \) are fixed, the cover probability increases with the sample size indicating the method gets more conservative.
Figure 2: 200 runs of SRM(1, p, 5, δ) are made, with n samples used in every run. For each value of n, p, δ, the mean (orange •) and 2.5-th percentile (orange △) of the upper limit of confidence intervals are noted. Also the the mean (blue •) and 97.5-th percentile (blue △) of the lower limit of confidence intervals are noted.

Table 4: Performance of SRM(q, 1, p, r_l, r_u, δ) when 500 runs of the method are used, each having n samples. P(δ) denotes the cover probability of the two sided CI for the particular value of δ. Pl(δ) (Pu(δ)) is the cover probability of the one sided CI [z_α, ∞) (CI (−∞, z_α]) where α = 0.025 (α = .975).

| r_l | r_u | n  | P(0)   | Pl(0) | Pu(0) | P(1)   | Pl(1) | Pu(1) | L   |
|-----|-----|----|--------|-------|-------|--------|-------|-------|-----|
| 1.1 | 5   | 5  | 500    | 0.962 | 0.976 | 0.986  | 0.978 | 1     | 0.978| 74.77|
| 1.1 | 5   | 5  | 1000   | 0.986 | 0.988 | 0.998  | 0.99  | 1     | 0.99 | 89.69|
| 1.1 | 5   | 5  | 2000   | 0.976 | 0.98  | 0.996  | 0.996 | 1     | 0.996| 71.7 |
| 1.1 | 2   | 5  | 500    | 0.914 | 0.932 | 0.982  | 0.952 | 0.99  | 0.962| 74.34|
| 1.2 | 5   | 5  | 500    | 0.944 | 0.98  | 0.964  | 0.948 | 1     | 0.948| 48.68|
| 1.2 | 5   | 5  | 1000   | 0.958 | 0.978 | 0.98   | 0.958 | 1     | 0.958| 43.82|
| 1.2 | 5   | 5  | 2000   | 0.974 | 0.984 | 0.99   | 0.966 | 1     | 0.966| 48.67|
| 1.2 | 3   | 6  | 500    | 0.932 | 0.946 | 0.964  | 0.946 | 0.996 | 0.95 | 24.69|
| 1.2 | 6   | 7  | 500    | 0.934 | 0.97  | 0.968  | 0.97  | 1     | 0.97 | 50.45|
| 1.2 | 3   | 6  | 1000   | 0.93  | 0.952 | 0.978  | 0.96  | 0.992 | 0.968| 52.52|
| 1.2 | 6   | 7  | 1000   | 0.968 | 0.982 | 0.986  | 0.97  | 0.998 | 0.972| 168.52|
| 1.2 | 3   | 2  | 2000   | 0.93  | 0.972 | 0.958  | 0.938 | 0.998 | 0.94 | 236.21|
| 1.2 | 2   | 4  | 2000   | 0.946 | 0.956 | 0.99   | 0.946 | 0.986 | 0.96 | 41.4 |
| 1.3 | 5   | 5  | 500    | 0.91  | 0.974 | 0.936  | 0.888 | 1     | 0.888| 44.88|
| 1.3 | 5   | 5  | 1000   | 0.936 | 0.964 | 0.972  | 0.928 | 0.998 | 0.93 | 37.03|
| 1.3 | 5   | 5  | 2000   | 0.954 | 0.972 | 0.982  | 0.942 | 0.998 | 0.944| 35.56|
| 1.3 | 5   | 7  | 500    | 0.89  | 0.964 | 0.934  | 0.91  | 0.998 | 0.912| 37.24|
| 1.4 | 5   | 5  | 500    | 0.858 | 0.952 | 0.906  | 0.864 | 0.998 | 0.866| 36.86|
| 1.4 | 5   | 5  | 1000   | 0.898 | 0.96  | 0.938  | 0.888 | 0.994 | 0.894| 43.51|
Bootstrapping versus ASCLT-algorithm

In the final set of simulations we see what happens if we pair the resampling approach with bootstrapping \[8\], instead of the ASLT algorithm. Briefly, the bootstrap method \(BRM(q', p, m)\) follows along the same lines of Algorithm 2.1, with the only difference being that the two estimated distribution functions \(Edf_l\) and \(Edf_u\) are not estimated using Algorithm 2.2 but by using a bootstrap sample of size \(m\) (See \[10\] for details). The results are given in table 5; the underlying data is generated from the same nearly power laws as the data for Table 4 (with \(q_p = 1.5\)). For all the methods studied two-sided symmetric 95% confidence intervals for the mean are derived by making 500 runs, each using 1000 samples. The cover probabilities for the \(SRM\) methods remain relatively closer to the desired 95% for all values of parameters and sample size (see Table 4, section pertaining to \(p = 1.2\) for more details) when compared to the \(BRM\) setting. Table 5 shows that as \(m\) grows \(BRM(1,1.2,m)\) becomes more conservative, similar trends were observed for \(SRM(q,1,1.2,5,5,\delta)\) as the number of samples were increased, however the extent to which the results become conservative is far less for the \(SRM(q,1,1.2,\cdot,\cdot,\cdot)\) methods. (see Table 4, section pertaining to \(p = 1.2\) for more details)

Table 5: Comparison of different estimators for a 95% CI for two-sided symmetric 95% confidence intervals for the mean are derived by making 500 runs, each using 1000 samples

| Estimator                  | Cover Probability | Length of CI |
|----------------------------|-------------------|--------------|
| SRM(1.5,1,1.2,5,5,0)       | 0.96              | 44.27        |
| RBM(1,1.2,50)              | 0.984             | 33.72        |
| RBM(1,1.2,200)             | 0.988             | 35.96        |
| RBM(1,1.2,500)             | 0.994             | 34.87        |
| Normal                     | 0.576             | 6.26         |

4.3 An application to neural avalanches

The Abelian distribution is a distribution that is important in models studying neural avalanches (see \[9\] \[12\] \[13\]), and it belongs to the class of quasi Binomial II distributions (\[4\]). Neural avalanches were observed in field studies, for example by Beggs et al. (\[2\], \[1\]). Cultured slices from the brain were attached to multielectrode ensembles and LFP (Local Field Potential) signals were recorded. The data retrieved showed brief intervals of activity, when electrodes detected LFPs above the threshold. The period in between these short bursts of activity was marked by idleness. A sequence of such sustained activity was called an avalanche. There are models (\[9\]) where the avalanche size (number of neurons firing during an avalanche) statistic follows an Abelian distribution. Recall that this distribution is a probability distribution on \(\{1,2,\cdots,N\}\) defined by the
probability density

\[ P(Z_{N,p} = b) = C_{N,p} \left( \frac{N - 1}{b - 1} \right) p^{b-1} (1 - bp)^{(N-b-1)k}b^{-2}, \]

where \( C_{N,p} \) is the normalization constant defined by

\[ C_{N,p} = \frac{1}{1 - p/1}, \]

where \( N \in \mathbb{N} \) and \( p \in (0, 1) \) (12), see also (13). The \( p \) in the Abelian distribution is often taken as \( \frac{\alpha}{N}, \) where \( 0 < \alpha < 1. \) It is known (12), (13) that:

\[ E(Z_{N,\frac{\alpha}{N}}) = \frac{N}{N - (N - 1)\alpha}, \]

hence \( \lim_{N \to \infty} E(Z_{N,\frac{\alpha}{N}}) = \frac{1}{1 - \alpha}. \) (4.2)

and (6)

\[ \lim_{N \to +\infty} V(Z_{N,\frac{\alpha}{N}}) = \frac{\alpha}{(1 - \alpha)^3}, \] (4.3)

We note that (4.3) can be proved borrowing results about Quasi Binomial II distributions (5), and asymptotic properties of incomplete gamma integrals. However, an elementary simple proof is given in 6.

The parameter \( N \) represents the number of neurons, in practice it is a large number. Also avalanches have been observed for collections of neurons of various sizes [1, 14, 16, 20, 18, 15, 21], as such it is not presumed to be a phenomenon dependent on \( N. \) For a healthy brain the parameter \( \alpha \) is hypothesized to be close to 1 (9). At \( \alpha = 1 \) it is easy to show that (12):

\[ \lim_{\alpha \to 1} \lim_{k \to \infty} \lim_{N \to \infty} \frac{P(Z_{N,\frac{\alpha}{N}} = k)}{C k^{-1.5}} = 1. \]

All of this has two main consequences:

(a) For neural avalanche data the ratio of the underlying variance and mean will be very large.

(b) The distribution follows a nearly power law with critical exponent 1.5. This is in agreement with experimental observations where avalanche size distributions have been found to follow power-law statistics, possibly with exponential cutoff [19, 1].

(c) The quantity \( \alpha \) is a useful parameter to be estimated from the data, since the extent of it’s closeness to 1 is thought to be a measure of the health of the brain. The quantity \( \alpha \) can be estimated by estimating the mean.

So this motivates us to estimate the mean of neural avalanche (using (4.2), one can estimate confidence interval for \( \alpha \) using confidence intervals for the mean) data using the SRM algorithm.

Outline of simulations.

Data: We use synthetic data. Our data is generated from a 1.5 exponent power law with upper cut-off at \( x_m \) (we will analyze several data sets with different values of \( x_m \)). We will generate \( n = 1000 \) iid instances of the data for
each experiment, denote this by \( X_1, X_2, \ldots X_n \).

Results and discussion:

Figure 3: CLT and SRM method (\( p \) value used is 1.7) for calculating confidence intervals for \( \alpha \) for three different values of \( x_m \): On the \( x \)-axes we indicate the method used to obtain confidence interval for \( \alpha \). On the \( y \)-axes is shown the range of the 4% confidence interval obtained for each method. Red dots indicate the ends of the confidence intervals. The blue \( \triangledown \) symbol indicates a lower bound for the confidence interval cannot be calculated using the method in question. To calculate confidence intervals we use 1000 instances of synthetic data. The points indicated by \( \times \) show the sample mean calculated from 900000 instances of synthetic data. The inset on the leftmost is to show the SRM results for this case more prominently.

Our simulation studies throw up some features worth noting:

(a) To check if our results are accurate we derive the sample mean from a much larger amount of synthetic data than what is used for establishing confidence intervals. This estimate will be called the precise sample mean and is marked by a \( \times \) in Figure 3.

(b) When \( x_m = \infty \), the data is generated from a 1.5 exponent power law over all of the positive integers. This distribution has both infinite first and second moments. In such a setting both the SRM and CLT methods will fail. As \( x_m \) grows larger, the accuracy of both methods deteriorate. However because CLT method depends on higher moment conditions, it’s accuracy deteriorates faster. Note, however for \( x_m = 10^5 \) the precise sample mean is near the center of the confidence interval calculated by the SRM method. But for \( x_m = 8 \times 10^5 \) the lower bound of the confidence is quite far away from the precise sample mean.

(c) It is interesting to note that the methods can sometimes fail to give any lower bound for the confidence interval. The reason for this is as follows: We derive the confidence interval for \( \alpha \) from the confidence interval for the mean \( \mu \) using the understanding \( \mu = \frac{1}{1-\alpha} \). For this one requires that both upper and lower confidence bound estimates for the mean be positive,
absence of such conditions can result in lack of bounds. This happens in the case of the CLT method for $x_m = 8 \times 10^5$ and $x_m = 6 \times 10^5$. Although the underlying data is non-negative valued, the variance is so large that the lower confidence bound obtained for the mean using CLT becomes negative.

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Anirban Das
Yale University

Manfred Denker
The Pennsylvania State University

Anna Levina
University of Tübingen

Lucia Tabacu
Old Dominion University