THE ⋆-VERTEX-REINFORCED JUMP PROCESS

BY CHRISTOPHE SABOT\textsuperscript{1,a}, AND PIERRE TARRÈS\textsuperscript{2,b}

\textsuperscript{1}Université Claude Bernard Lyon 1, Institut Camille Jordan, CNRS UMR 5208, 43, Boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France, and Institut Universitaire de France, \textsuperscript{a}sabot@math.univ-lyon1.fr

\textsuperscript{2}NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai, China; Courant Institute of Mathematical Sciences, New York, USA; CNRS and Université Paris-Sorbonne, Laboratoire de Probabilités, Statistique et Modélisation, 75005 Paris, France., \textsuperscript{b}tarrès@nyu.edu

We investigate the non-reversible generalization of the Vertex-Reinforced Jump Process (VRJP), called the ⋆-Vertex-Reinforced Jump Process (⋆-VRJP) and introduced in [4]. It can be seen as the continuous-time counterpart to the ⋆-Edge-Reinforced Random Walk (⋆-ERRW) (see [3, 4]), which is itself a non-reversible, and in fact Yaglom reversible, generalization of the original ERRW introduced by Coppersmith and Diaconis in 1986 [8]. In contrast to the classical VRJP, the ⋆-VRJP is not exchangeable after time-change, which leads to several difficulties and new phenomena.

Firstly, we show that with some appropriate randomization of the initial local time, it becomes partially exchangeable after time-change. We provide a representation of the "randomized" ⋆-VRJP as a mixture of Yaglom reversible Markov jump processes with an explicit mixing measure, and we prove that the non-randomized ⋆-VRJP can be written as a mixture of conditioned Markov processes.

Secondly, we give a representation of the ⋆-VRJP in terms of a random Schrödinger operator. The corresponding representation for the classical VRJP has proved to be very useful in the understanding of its asymptotic behavior. The construction is based on several new and rather remarkable identities between integrals on the space of ⋆-symmetric and ⋆-antisymmetric functions on vertices. We give a description of the randomized ⋆-VRJP in terms of that random Schrödinger operator, which allows us to prove the representation of the randomized ⋆-VRJP as a mixture of Markov jump processes in a different and more analytic manner. Similarly as for the VRJP, we think that the representation by a random Schrödinger operator and the associated identities are key-features of the ⋆-VRJP.

The purpose of this paper is to investigate a natural non-reversible generalization of the Vertex Reinforced Jump Process (VRJP). The VRJP is a time-continuous self-interacting jump process defined on an undirected conductance network. This process was proposed by Werner and initially investigated by Davis and Volkov ([10, 11], see also [5]). In [27], it was shown to be intimately related on the one hand to the Edge Reinforced Random Walk (ERRW), and on the other hand to a supersymmetric hyperbolic sigma-field introduced by Zimbauer [34] and investigated by Disertori, Spencer and Zimbauer [15, 14]. These relations, together with the crucial estimates of [15, 14] and several other works [1, 13, 28, 30, 24], allowed to clarify the picture about the recurrent/transient phases of the VRJP and the ERRW.

The aim of this paper is to start a similar program for a family of processes interpolating between the non-directed case (ERRW and VRJP) and the fully directed case (directed ERRW, or equivalently the random walk in random Dirichlet environment, see below).

MSC2020 subject classifications: Primary 60K37, 60K35, 82B44; secondary 81T25, 81T60.

Keywords and phrases: Vertex-reinforced jump process, self-interacting random walks, random Schrödinger operator, supersymmetric hyperbolic nonlinear sigma model.
The processes considered here are defined on a directed graph \( \mathcal{G} = (V, E) \) endowed with an involution on the vertices: the involution is denoted by \( \star : V \mapsto V \), and the graph satisfies the property that

\[
(i, j) \in E \iff (j^*, i^*) \in E.
\]

(0.1)

Hence, the involution \( \star \) also acts on the edges and all the weights defined on the edges will be \( \star \)-symmetric. We call \( \mathcal{G} \) a \( \star \)-directed graph.

The natural generalization of the ERRW on \( \star \)-directed graphs was introduced in [4] and generalizes the \( k \)-dependent ERRW defined by Baccalado in [3]. Starting from some positive \( \star \)-symmetric weights \((\alpha_e)_{e \in E}\) on the edges, the \( \star \)-ERRW behaves as follows: each time an edge \((i, j)\) is crossed, the weights of the edge \((i, j)\) and of the \( \star \)-symmetric edge \((j^*, i^*)\) are increased by one, and the process jumps with a probability proportional to the current weights. With that definition, the weights remain \( \star \)-symmetric at all times (see precise definition below).

Under a divergence condition on the weights, we proved in [4] that the \( \star \)-ERRW is partially exchangeable and we compute its mixing measure using a new discrete Feynman-Kac type formula. This yields the counterpart on \( \star \)-directed graphs of the Diaconis-Coppersmith "magic formula" [8, 20].

This model naturally interpolates between the undirected ERRW and the directed Edge reinforced random walk: indeed, when \( \star \) is the identity, then \((j^*, i^*) = (j, i)\) is dual to the edge \((i, j)\) and this yields the standard ERRW. On the contrary, when the graph is composed of two disconnected pieces \( V_1 \) and \( V_1^\star \) which are mapped to one another by the involution \( \star \) then, starting from \( V_1 \), the \( \star \)-ERRW remains on \( V_1 \) and follows the directed ERRW. A more interesting version arises when one glues \( V_1 \) and \( V_1^\star \) by the starting point of the process, see Section 1.2 for more details.

The undirected and directed ERRW have very different behavior: e.g. on \( \mathbb{Z}^d, \ d \geq 3 \), the first exhibits a phase transition between recurrence and transience [27, 1, 13] and the second is always transient [25]. The \( \star \)-ERRW lives on a directed graph but keeps some reversibility in the reinforcement scheme, in that respect it shows some features of both processes.

In [4], following [27], we introduced the \( \star \)-VRJP and proved that the \( \star \)-ERRW is a mixture of \( \star \)-VRJP. Let \((W_e)_{e \in E}\) be a family of positive \( \star \)-symmetric real weights on the edges of the graph. By abuse of terminology, and by analogy with the VRJP, we sometimes call \((W_e)\) the "conductances", even though those are not symmetric in general. In the exponential time-scale, the \( \star \)-VRJP is the continuous-time process which, at time \( t \), jumps from \( i \) to \( j \) at a rate given by

\[
W_{i,j} e^{T_i(t) + T_j^*(t)},
\]

(0.2)

where \( T_i(t) \) is the occupation time of the vertex \( i \in V \) at time \( t \). At fixed time \( t \), the transition rates are obviously not reversible. However the \( \star \)-symmetry on the transition rates, see (1.6), leads to a type of Yaglom reversibility [32, 31, 16] in the limit, see below. A Markov jump process on the graph \( \mathcal{G} \), with jump rates \((K_{i,j})_{(i,j) \in E}\), will be called Yaglom reversible if it admits an invariant measure \((\pi_i)_{i \in V}\) such that \( \pi_i = \pi_{i^*} \) for all \( i \in V \), and \( \pi_i K_{i,j} = \pi_j K_{j^*,i^*} \) for all edge \((i,j) \in E\).

Note that after a change of time similar to the VRJP case, see [27], the jumps rates (0.2) can also be written as \( W_{i,j} (1 + L_j^*(s)) \) where \( L \) is the local of the new process. That corresponds to the original time-scale of the VRJP.

Contrary to the standard VRJP, the \( \star \)-VRJP is not partially exchangeable, even though the \( \star \)-ERRW is partially exchangeable under a condition on the weights. This adds a major difficulty to the model. However, we obtain in Part I a counterpart to the results of [27]:
• Under a suitable randomization of the initial local times, we prove that the $\star$-VRJP becomes partially exchangeable after a proper time change. We identify the limiting measure and represent this randomized $\star$-VRJP as a mixture of Yaglom reversible Markov jump processes (with the definition above).

• Coming back to the non randomized process, we prove that in general the $\star$-VRJP can be written, after a proper time change, as a mixture of some self-interacting jump processes which can be seen as conditioned Markov jump processes.

In a second part, we give a random Schrödinger representation of the randomized $\star$-VRJP, which generalizes the representation obtained in [28] for the VRJP. Several new phenomena and difficulties appear compared to the VRJP, due to the necessary extra randomization of the initial local times. The representation is based on new and rather remarkable identities between integrals on the space of $\star$-symmetric and $\star$-antisymmetric functions. The corresponding representation of the classical VRJP by a random Schrödinger operator has given rise to several important developments, such as recurrence for all reinforcement parameters in dimension two, diffusive behavior in the transient phase in dimension at least 3, a monotonicity property and the uniqueness of phase transition [30, 24, 26, 21]. We present more detailed motivations for that random Schrödinger representation later in Section 5, since it is requires a prior understanding of partial exchangeability and the mixing measure.

A natural question which emerges from these works is that of the existence of a counterpart to the supersymmetric hyperbolic sigma-field $\mathbb{H}^{1/2}$, which is deeply linked to the VRJP (see e.g. [15, 27, 6]). This question will be the object of a further work with Andrew Swan.

CONTENTS

I Exchangeability and mixing measure
1 A non-reversible counterpart to the Edge-Reinforced Random Walk and the Vertex Reinforced Jump Process .................................................... 4
2 The limiting manifold ............................................................. 6
3 Randomization of the initial local time and exchangeable time scale .................... 7
4 The limiting distribution and representation as mixtures .................................... 9

II Random Schrödinger representation
5 Motivation ........................................................................... 12
6 The $\beta$-potentials and fundamental properties .............................................. 14
7 Relation with the $\star$-VRJP ....................................................... 16

III Proofs of the Results
8 Preliminary results and proof of the statements in Sections 2 and 3 ............ 20
9 Proof of Theorems 4.2 and 4.6 ...................................................... 25
10 Proof of the results of Section 6: Lemma 6.5, Theorems 6.2 and 6.6 and Proposition 6.7 ................................................................. 37
11 Proof of Theorem 7.8 .................................................................. 45
12 Proof of Lemma 7.9, Corollary 7.5 and Corollary 7.10 ............................. 49
Appendix: Proof of Lemma 2.1 .......................................................... 54
Appendix: Results on $\mathcal{M}$-matrices .................................................. 56
Funding ..................................................................................... 56
References .................................................................................. 56
Part I
Exchangeability and mixing measure

1. A non-reversible counterpart to the Edge-Reinforced Random Walk and the Vertex Reinforced Jump Process.

1.1. $\star$-directed graphs. In this paper, we consider a directed graph $G = (V, E)$ endowed with an involution on the vertices denoted by $\star$, and such that

\[(i, j) \in E \Rightarrow (j^\star, i^\star) \in E.\]

We write $i \to j$ to mean that $(i, j)$ is a directed edge of the graph, i.e. that $(i, j) \in E$.

Denote by $V_0$ the set of fixed points of $\star$

$V_0 = \{i \in V \text{ s.t. } i = i^\star\},$

and by $V_1$ a subset of $V$ such that $V$ is a disjoint union

$V = V_0 \sqcup V_1 \sqcup V_1^\star.$

Denote by $\tilde{E}$ the set of edges quotiented by the relation $(i, j) \sim (j^\star, i^\star)$. In the sequel we also consider $\tilde{E}$ as a subset of $E$ obtained by choosing a representative among two equivalent edges $(i, j)$ and $(j^\star, i^\star)$.

If $v = (v_i)_{i \in V}$ is a function on the vertices we simply write $v^\star$ for the function given by $v^\star_i = v_i^\star$. The spaces $S$ and $A$ of $\star$-symmetric and $\star$-antisymmetric functions on the vertices, defined below, play a central role throughout the paper,

\[(1.2) \quad A = \{a \in \mathbb{R}^V, a^\star = -a\} \simeq \mathbb{R}^{V_1},\]

\[(1.3) \quad S = \{s \in \mathbb{R}^V, s^\star = s\}.\]

The orthogonal projections on the subspaces $A$ (resp. $S$), are given by

\[(1.4) \quad \mathcal{P}_A(u) = \frac{1}{2}(u - u^\star), \quad \mathcal{P}_S(u) = \frac{1}{2}(u + u^\star).\]

We denote by $\langle \cdot, \cdot \rangle$ the bilinear form on $\mathbb{R}^V \times \mathbb{R}^V$ given by

\[(1.5) \quad \langle x, y \rangle = \sum_{i \in V} x_i \cdot y_i,\]

Note that, in general, $\langle \cdot, \cdot \rangle$ is not a scalar product. We also use sometimes the Euclidian scalar product on $\mathbb{R}^V$ that we denote by $\langle \cdot, \cdot \rangle$, so that $\langle x, y \rangle = (x^\star, y)$.

A function $(\alpha_{i,j})_{(i,j) \in E}$ on the edges is called $\star$-symmetric if it satisfies

\[(1.6) \quad \alpha_{i,j} = \alpha_{j^\star, i^\star} \text{ for all } (i, j) \in E.\]

In Part I we always assume that the graph $G$ is strongly connected, i.e. that for any vertices $i$ and $j$, there exists a directed path in the graph between $i$ and $j$. 
1.2. The $\ast$-Edge Reinforced Random Walk ($\ast$-ERRW). Suppose we are given some $\ast$-symmetric positive weights $(\alpha_{i,j})_{(i,j)\in E}$ on the edges. Let $(X_n)_{n\in \mathbb{N}}$ be a nearest-neighbor discrete-time process taking values in $V$. For all $(i,j) \in E$, denote by $N_{i,j}(n)$ the number of crossings at time $n$ of the directed edge $(i,j)$,

$$N_{i,j}(n) = \sum_{k=0}^{n-1} \mathbb{1}(X_k,X_{k+1})=(i,j),$$

and let

$$\alpha_{i,j}(n) = \alpha_{i,j} + N_{i,j}(n) + N_{j,i}(n).$$

Note that $(\alpha_e(n))_{e\in E}$ remains $\ast$-symmetric at all time $n$.

The process $(X_n)_{n\in \mathbb{N}}$ is called the $\ast$-Edge Reinforced Random Walk ($\ast$-ERRW), with initial weights $(\alpha_e)_{e\in E}$ and starting from $i_0$, if $X_0=i_0$ and, for all $n \in \mathbb{N}$ and $j \in V$,

$$\mathbb{P}(X_{n+1} = j \mid X_0,X_1,\ldots,X_n) = \frac{\alpha_{X_{n,j}}(n)}{\sum_{l,X_{n-1} \neq l} \alpha_{X_{n,l}}(n)}.$$

As mentioned above, the $\ast$-ERRW is a generalisation of the Edge-Reinforced Random Walk (ERRW) introduced in the seminal work of Diaconis and Coppersmith [8], corresponding to the case where $\ast$ is the identity map.

An example of $\ast$-ERRW is the $k$-dependent Reinforced Random Walk, already considered by Bacallado in [2], which takes values in the de Bruijn graph $G = (V,E)$ of a finite set $S$, defined as follows: $V = S^k$, $E = \{(i_1,\ldots,i_k), (i_1,\ldots,i_{k+1}) \mid i_1,\ldots,i_{k+1} \in S \}$. Let $\ast$ be the involution mapping $(i_1,\ldots,i_k) \in V$ to the reversed sequence $(i_k,\ldots,i_1) \in V$, which obviously satisfies (1.1). Note that, in that case, $V_0$ is the set of palindromes. Then the $k$-dependent Reinforced Random Walk is defined as the $\ast$-ERRW on that de Bruijn graph, see [4].

As we pointed out in the introduction, the Random Walk in Random Dirichlet Environment (RWDE), considered in [17, 25] (see also [29] for a review), can also be seen as a particular case of $\ast$-ERRW. Indeed, given a directed graph $G = (V,E)$ of a finite set $S$, consider its "reversed" graph $G_1 = (V_1^*,E_1^*)$, where $V_1^*$ is a copy of $V_1$ and $E_1^*$ is obtained by reversing each edge in $E_1$. Then the full graph $G$ is a disjoint union $(V_1 \sqcup V_1^*, E_1 \sqcup E_1^*)$. If $i_0 \in V_1$, then the $\ast$-ERRW is a Random Walk in Dirichlet environment on $G_1$ under the annealed law. An interesting modification of the previous construction, which makes the $\ast$-directed graph strongly connected, is obtained by gluing the two graphs $G_1$ and $G_1^*$ at the starting point $i_0$. In that case, the condition for partial exchangeability of the $\ast$-ERRW, see condition (1.7) below, coincides with the condition for the key property of statistical invariance by time-reversal of RWDE, see [25] lemma 1 or [29], Lemma 3 and Remark 1 of [4].

The divergence operator $\text{div} : \mathbb{R}^E \to \mathbb{R}^V$ is defined on functions on the edges $(x_e)_{e \in E}$ by

$$\text{div}(x)(i) = \sum_{j,j\to i} x_{i,j} - \sum_{j,j\to i} x_{j,i}.$$
1.3. *-Vertex Reinforced Jump Process (⋆-VRJP). Let us now describe the representation of the ⋆-ERRW in terms of a ⋆-Vertex-Reinforced Jump Process (⋆-VRJP) with independent gamma conductances, similarly as for the ERRW.

Suppose we are given ⋆-symmetric positive weights \( (W_{i,j})_{(i,j) \in E} \), i.e. satisfying (1.6), that we call "conductances" by analogy with VRJP, even though they are not symmetric (see introduction). We often extend \( W \) to a \( V \times V \) matrix by setting \( W_{i,j} = 0 \) if \( (i,j) \notin E \). Clearly, the matrix \( W \) is \( \langle \cdot, \cdot \rangle \)-symmetric, since \( W_{i,j} = W_{j,i}^\star \), where the bilinear form \( \langle \cdot, \cdot \rangle \) is defined in (1.5).

If \( (u_i)_{i \in V} \in \mathbb{R}^V \), define \( W^u \in \mathbb{R}^E \) by

\[
W^u_{i,j} = W_{i,j} e^{u_i + u_j^\star}.
\]

Remark that \( (W^u_e)_{e \in E} \) is ⋆-symmetric, i.e. \( W^u_{i,j} = W^u_{j,i} \), for \( (i,j) \in E \).

Consider the process \( (X_t)_{t \geq 0} \) with state space \( V \), which, conditioned on the past at time \( t \), with \( X_t = i \), jumps from \( i \) to \( j \), such that \( i \rightarrow j \), at a rate

\[
W^u_{i,j} = W_{i,j} e^{T_i(t) + T_j^\star(t)},
\]

where \( T_i(t) \) is the local time of \( X \) at time \( t \), i.e.

\[
T_i(t) = \int_0^t 1_{X_u = i} \, du, \quad \forall i \in V.
\]

We call this process the ⋆-VRJP. When ⋆ is the identity, the jump rates in (1.9) correspond to the ones for the standard VRJP in the exponential time scale considered in [27]. We denote by \( \mathbb{P}^W_{i_0} \) the law of the ⋆-VRJP starting from the point \( i_0 \). We always define the ⋆-VRJP on the canonical space \( \mathcal{D}([0, \infty), V) \) of càdlàg functions from \([0, \infty)\) to \( V \), and \( (X_t)_{t \geq 0} \) will denote the canonical process on \( \mathcal{D}([0, \infty), V) \).

The following relation between the ⋆-VRJP and ⋆-ERRW was given in [4].

**Lemma 1.2.** Let \( (\alpha_e)_{e \in E} \) be ⋆-symmetric positive weights on the edges. Let \( (W_e)_{e \in E} \) be independent Gamma random variables with weights \( (\alpha_e)_{e \in E} \), and let \( (W_e)_{e \in E} \) be the conductances on \( E \) constructed to be ⋆-symmetric. Let \( (X_t)_{t \geq 0} \) be the ⋆-VRJP with conductances \( (W_e) \) starting from \( i_0 \), i.e. the process with law \( \mathbb{P}^W_{i_0} \), and let \( (\tilde{X}_n)_{n \in \mathbb{N}} \) be the discrete time process describing the successive jumps of \( (X_t)_{t \geq 0} \). Then, after taking the expectation with respect to the r.v. \( W \), under \( \mathbb{P}^W_{i_0} \), \( (\tilde{X}_n) \) has the law of the ⋆-ERRW with initial weights \( (\alpha_e)_{e \in E} \) and starting at \( i_0 \).

2. The limiting manifold. We will use the following notation:

\[
\mathcal{H}_0 = \{(u_i)_{i \in V}, \sum_{i \in V} u_i = 0 \}, \quad \mathcal{S}_0 = \mathcal{S} \cap \mathcal{H}_0.
\]

and we remark that \( \mathcal{A} \subseteq \mathcal{H}_0 \). For the ⋆-symmetric conductances \( (W_e)_{e \in E} \) we let \( \mathcal{U}_0^W \) be the following manifold (which, as we will see later, is smooth)

\[
\mathcal{U}_0^W = \{ u \in \mathcal{H}_0, \text{div}(W^u)(i) = 0 \text{ for all } i \in V \},
\]

where \( W^u \) is the element of \((0, \infty)^E\) defined in (1.8). The following lemma, proved in Appendix 12.2, asserts that \( \mathcal{U}_0^W \) is the manifold of limiting values of the local time.

**Lemma 2.1.** For all \( a \in \mathcal{A} \), under \( \mathbb{P}^W_{i_0} \), for all \( i \in V \),

\[
U_i := \lim_{t \to \infty} (a_i + T_i(t) - t/N) \text{ exists a.s.}
\]

and \( U := (U_i)_{i \in V} \in \mathcal{U}_0^W \) a.s.
Remark 2.2. Note that for $a = 0$, $U$ is the limit of the centered local time of the $\ast$-VRJP under $\mathbb{P}^W_{i_0}$. In general, in the sequel, $U$ is defined as in Lemma 2.1, as this will turn out to be convenient for notational purposes.

The following result shows that $A$ is transversal to the manifold $U^W_0$.

**Lemma 2.3 (Projection on $U^W_0$ parallel to $A$).** Let $u$ be in $H_0$. There exists a unique element $a \in A$ such that

$$u - a \in U^W_0.$$

Moreover, $a^\ast$ is the unique minimizer of the strictly convex function $J_u : H_0 \mapsto \mathbb{R}$

$$J_u(v) = \sum_{i \in V, j, i \rightarrow j} W_{i,j}^u (e^{v_j - v_i} - 1).$$

We denote by $p_{U^W_0}(u) := u - a$ this projection on $U^W_0$ parallel to $A$.

Lemma 2.3 gives a natural parametrization of $U^W_0$, by orthogonal projection on the subspace $S_0$:

$$U^W_0 \rightarrow S_0 \quad \text{with} \quad h \mapsto s = \mathcal{P}_S(h) = \frac{1}{2} (h + h^\ast).$$

This parametrization is inverted by the application $h = p_{U^W_0}(s)$, since $s$ is in the fiber $h + A$. Hence it gives a natural positive measure on $U^W_0$, denoted by $d\sigma_{U^W_0}$, and defined by:

$$d\sigma_{U^W_0} = (p_{U^W_0})_\ast (d\lambda_{S_0})$$

where $(p_{U^W_0})_\ast (d\lambda_{S_0})$ is the push-forward of the euclidean volume measure $\lambda_{S_0}$ on $S_0$ by the projection on $U^W_0$ restricted to $S_0$. More precisely we have, for any measurable subset $B$ of $U^W_0$,

$$\sigma_{U^W_0}(B) = \lambda_{S_0}((p_{U^W_0})^{-1}(B)) = \lambda_{S_0}(\mathcal{P}_S(B)).$$

3. Randomization of the initial local time and exchangeable time scale. Contrary to the VRJP, the $\ast$-VRJP is in general not a mixture of time changed Markov Jump Processes. Indeed, otherwise this would imply that, taking expectation with respect to random conductances $W$ as in Lemma 1.2, the $\ast$-ERRW were a mixture of random walks, hence partially exchangeable, but it is not the case when the weights do not have the property of null divergence, see Proposition 1.1. Nevertheless, the $\ast$-VRJP has some exchangeability property after randomization of the initial local times. The results of this Section are proved in Section 8.2.

For $i_0 \in V$, let $\nu_{\ast,i_0}$ be the measure on $A$ defined by

$$\nu_{\ast,i_0}^W(da) = \frac{1}{\sqrt{2\pi|V_1|}} e^{a^2} e^{-\frac{1}{2} \sum_{(i,j) \in E} W_{i,j} (e^{a_i + a_j^\ast} - 1)} \left( \prod_{i \in V_1} da_i \right),$$

and let

$$F^W_{i_0} := \int_A \nu_{\ast,i_0}^W(da)$$

be its normalization constant.

**Lemma 3.1.** Under the assumption that $G$ is strongly connected, we have $F^W_{i_0} < \infty$. 
We will be interested in the \( W \)-VRJP starting from an antisymmetric random initial local time \( A \) distributed according to \( \frac{1}{P_{t_0}} \nu^{W}_{A,t_0} \), i.e. the \( W \)-VRJP with conductances \( W^A \). To that purpose, we define the probability measure \( \mathbb{P}^W_{t_0} (\cdot) \) on the canonical space \( \mathcal{D}([0, \infty), V) \times A \) by

\[
\int \phi(\omega, a) \mathbb{P}^W_{t_0} (d\omega, da) = \int_{\mathcal{D}([0, \infty), V) \times A} \phi(\omega, a) \mathbb{P}^W_{t_0} (d\omega, da) \frac{1}{P_{t_0}} \nu^{W}_{A,t_0}(da),
\]

for any measurable bounded test function \( \phi \). On the canonical space \( \mathcal{D}([0, \infty), V) \times A \), we define \( (X_t)_{t \geq 0} \) as the canonical random variables on path space \( \mathcal{D}([0, \infty), V) \), and \( A \) as the random variable on \( A \) equal to the second coordinate. Hence, under \( \mathbb{P}^W_{t_0} (\cdot) \), \( A \) is distributed according to \( \frac{1}{P_{t_0}} \nu^{W}_{A,t_0} \), and conditionally on \( A \), \( (X_t)_{t \geq 0} \) is distributed according to \( \mathbb{P}^W_{t_0} A^W \). We call the associated process \( (X_t) \) under \( \mathbb{P}^W_{t_0} \), the randomized \( W \)-VRJP.

By a slight abuse of notation, the marginal of \( \mathbb{P}^W_{t_0} (\cdot) \) on the canonical space \( \mathcal{D}([0, \infty), V) \) will also be denoted by \( \mathbb{P}^W (\cdot) \). It will be clear from the context whether \( \mathbb{P}^W (\cdot) \) represents the probability on \( \mathcal{D}([0, \infty), V) \times A \) or its marginal on \( \mathcal{D}([0, \infty), V) \).

The following Lemma 3.2 shows the stability of the family of laws \( \frac{1}{P_{t_0}} \nu^{W}_{A,t_0} \) under the posterior distribution of \( \mathbb{P}^W_{t_0} \), which enables to deduce in Corollary 3.3 the jump rate of the annealed \( W \)-VRJP.

**Lemma 3.2.** Under \( \mathbb{P}^W_{t_0} \), conditionally on \( \sigma \{ X_s, s \leq t \} \), \( (A_i) \) is distributed according to \( \frac{1}{P_{t_0}} \nu^{W}_{A,X_t} \).

**Corollary 3.3.** Under \( \mathbb{P}^W_{t_0} \), conditionally on the past at time \( t \), if the process \( X_t \) is at position \( i \), then it jumps to \( j \) such that \( i \rightarrow j \) at a rate

\[
W_{i,j}^{T(t)} \frac{P^{W^{T(t)}}_{j}}{P^{W^{T(t)}}_{i}}.
\]

The following Proposition 3.4 shows that, with that randomization of initial local time, the \( W \)-VRJP becomes partially exchangeable after a time-change, which is a counterpart to the time change introduced in [27], Theorem 2.

**Proposition 3.4.** Let \( (C(t))_{t \geq 0} \) be the increasing functional of the process \( (X_t) \) defined by

\[
C(t) = \frac{1}{2} \sum_{i \in V} (e^{T_i(t) + T_i^*(t)} - 1).
\]

Let \( (Z_s)_{s \geq 0} \) be the time changed process defined by

\[
Z_s = X_{C^{-1}(s)}.
\]

Then, under the randomized distribution \( \mathbb{P}^W_{t_0} \), \( (Z_s) \) is partially exchangeable in the sense of [18], i.e. for any real \( h > 0 \), the discrete time process \( (Z_{kh})_{k \in \mathbb{N}} \) is partially exchangeable in the sense of Diaconis and Freedman [12].

**Remark 3.5.** This partial exchangeability implies by [12] that the process \( (Z_s)_{s \geq 0} \) under the randomized distribution \( \mathbb{P}^W_{t_0} \) is a mixture of Markov jump processes. This is the object of Theorem 4.2, where we make explicit the law of the mixture of Markov jump processes.
Remark 3.6. Note that, compared to the original time-change for the VRJP in [27], Theorem 2, there is an extra $\frac{1}{2}$ in the definition of $C(t)$, which will lead to a multiplicative factor 2 in the jump rate in Theorem 4.2 (ii), which applied to the VRJP is $W_{i,j}e^{U_{j}-U_{i}}$, instead of $\frac{1}{2}W_{i,j}e^{U_{j}-U_{i}}$ in [27]. This choice removes several annoying factors $\frac{1}{2}$ in the statements.

The following lemma states that the $*$-ERRW is a mixture of these randomized $*$-VRJP, under the condition that the weights of the $*$-ERRW makes it partially exchangeable.

Lemma 3.7. Let $(\alpha e)e\in E$ be $*$-symmetric positive weights which satisfy divergence-free condition (1.7). Assume that $W = (W e)e\in E$ are independent Gamma random variables with weights $(\alpha e)e\in E$ and that $(W e)e\in E$ are the associated $*$-symmetric conductances. If $A$ has the law $\frac{1}{F_{0}}\nu_{A,0}^{W}$, then $W^{A}$ has the same law as $W$. It implies by Lemma 1.2 that after expectation with respect to $(W e)$, the discrete time process associated with the randomized $*$-VRJP is a $*$-ERRW.

Remark 3.8. The stability property in Lemma 3.7 cannot hold when the divergence condition (1.7) is not satisfied, since otherwise it would imply by Proposition 3.4 above that the $*$-ERRW is always exchangeable, irrespective of that divergence condition.

Remark 3.9. In the classical reversible model, the VRJP was instrumental in the analysis of the asymptotic behavior of the ERRW. In this respect, the previous Lemma is important, since it means that the exchangeable $*$-ERRW can be described by randomized $*$-VRJP with independent gamma random conductances.

4. The limiting distribution and representation as mixtures.

4.1. The case of the randomized $*$-VRJP. Let us first introduce some notation. For $u \in U_{0}^{W}$, we denote by $K_{i,j}^{u} = (K_{i,j}^{u})_{i,j\in E}$ the infinitesimal generator of the Markov jump process at rate $W_{i,j}^{u}$ from $i$ to $j$, i.e. we let $K_{i,j}^{u} = (K_{i,j}^{u})_{i,j\in E}$ be the matrix defined by

\[
K_{i,j}^{u} = \begin{cases} W_{i,j}^{u}, & \text{if } i \neq j, \\ -\sum_{l \in V} W_{i,l}^{u}, & \text{if } i = j. \end{cases}
\]

We set

\[
D(W) := \sum_{T \in T_{j_{0}}} \prod_{(i,j) \in T} W_{i,j},
\]

where the sum is on the set $T_{j_{0}}$ of directed spanning trees $T$ of the graph rooted at some vertex $j_{0} \in V$. By the matrix-tree theorem, we also have

\[
D(W^{u}) = \det(-K_{V\setminus\{j_{0}\},V\setminus\{j_{0}\}}^{u}),
\]

where $\det(-K_{V\setminus\{j_{0}\},V\setminus\{j_{0}\}}^{u})$ is the diagonal minor obtained after removing the line and the column $j_{0}$ of the matrix $-K^{u}$. The value of this determinant does not depend on the choice of $j_{0}$ since the sums on any line or column of $K^{u}$ is null, using $u \in U_{0}^{W}$.

We denote by

\[
\det_{A}(K^{u})
\]

the determinant of the linear operator $P_{A}K^{u}P_{A}$ restricted to the subspace $A$, where $P_{A}$ is the projection on $A$, see (1.4).
DEFINITION 4.1. We introduce the following positive measure on the manifold $U_0^W$:

\begin{equation}
\mu_{i_0}^W(du) := C_G e^{u_{i_0} - \sum_{i \in V_0} u_i} e^{\frac{1}{2} \sum_{i,j} W_{i,j} (e^{s_{i,j} - s_{i,j}} - 1)} \frac{\sqrt{D(W_u)}}{\det_A(-K_u)} \sigma_{U_0}^W(du),
\end{equation}

where

\[ C_G := \frac{\sqrt{|W| \sqrt{2} |V_1|}}{\sqrt{2\pi} |V_1^0| |V_1|}, \]

and $\sigma_{U_0}^W$ is the volume measure on the manifold $U_0^W$ defined in Section 2.

THEOREM 4.2. (i) Under $P_{i_0}^W$, the following limit exists a.s.

$$U_i := \lim_{t \to \infty} (A_i + T_i(t) - t/N),$$

and $U \in U_0^W$ is distributed according to

$$\frac{1}{F_{i_0}^W} \mu_{i_0}^W(du).$$

(ii) Under $P_{i_0}^W$, conditionally on $U$, the $\ast$-VRJP in exchangeable time scale, $(Z_t)_{t \geq 0}$, is a Yaglom reversible Markov jump process with jump rate from $i$ to $j$ equal to

$$W_{i,j} e^{U_{i,j}} - U_{i,j}.$$ 

REMARK 4.3. We deduce from (i) that $\frac{1}{F_{i_0}^W} \mu_{i_0}^W$ is a probability measure, which implies the following identity of integrals:

$$\int_{U_0^W} \mu_{i_0}^W(du) = F_{i_0}^W = \int_{A} \nu_{A,i_0}^W(du).$$

This is a "non trivial" identity involving a measure on the limiting manifold $U_0^W$ and a measure on the transversal space $A$. We provide two different proofs of this equality: a probabilistic one using Feynman-Kac formula in this Part I, and another one is by direct computation, in Part II, see Corollary 7.10. A supersymmetric interpretation of that identity will also be given in a forthcoming work with Swan.

REMARK 4.4. When all points are self-dual, i.e. $V_0 = V$ and $V_1 = \emptyset$, one can check that the limiting distribution (4.2) coincides with the one of Theorem 1 of [27]. The difference in the factor $\sqrt{|V|}$ instead of $|V|$ comes from the choice of the volume measure $\sigma_0^W(du)$; indeed, in the case of a self-dual graph, $\sigma_0^W$ is the Euclidean measure on $H_0 = \{ (u_i), \sum_i u_i = 0 \}$, whereas in [27] we chose the measure $\prod_{i \neq i_0} du_i$ (there is a factor $\sqrt{|V|}$ between the two).

4.2. The case of the (non-randomized) $\ast$-VRJP. Recall that $(X_t)_{t \geq 0}$ is the canonical process on the space $D([0, \infty), V)$, and that on that space, $P_{i_0}^W$ is the law of the $\ast$-VRJP starting at $i_0$, and $P_{i_0}^W$ the law of the randomized $\ast$-VRJP. Besides, $(Z_s)_{s \geq 0} := (X_{C^{-1}(s)})_{s \geq 0}$ is the time-changed of the canonical process $X$, as defined in Proposition 3.4. We will denote by $(\ell_i(s))_{i \in V, s \geq 0}$ the local time process of $(Z_s)$:

\begin{equation}
\ell_i(s) = \int_0^s 1_{Z_u = i} \, du, \quad i \in V, \ s \geq 0.
\end{equation}
Given \( u \in \mathcal{U}_0^W \), we define \( P_{i_0}^{W,u} \) as the law on the path space \( \mathcal{D}([0, \infty), V) \) such that the time-changed process \((Z_t)_{t \geq 0}\) on the graph \( \mathcal{G} \), is a Markov jump process starting from \( i_0 \) and with jump rate from \( i \) to \( j \) given by

\[
W_{i,j}e^{u_i - u_j}.
\]

It is equivalent to say that, under \( P_{i_0}^{W,u} \), the canonical process \((X_t)_{t \geq 0}\) is the jump process which, conditioned on the past at time \( t \), jumps from \( i \) to \( j \), \( i \to j \), at rate

\[
W_{i,j}e^{u_i - u_j} e^{T_i(t) + T_j(t)}.
\]

see forthcoming Lemma \( 9.1 \). With this notation, Theorem \( 4.2 \) (ii) means that on \( \mathcal{D}([0, \infty), V) \), we have the following equality of probabilities

\[
\mathbb{P}_i^W(\cdot) = \int_{\mathcal{U}_0^W} P_{i_0}^{W,u}(\cdot) \frac{1}{\mu_{i_0}^W} \mu_i^W(du).
\]

We introduce the following functional of the process \((Z_s)\), which plays a crucial role in the analysis of the non-randomized \( \ast \)-VRJP: for all \((\theta_i)_{i \in V} \in \mathcal{S} \), let

\[
(4.4) \quad B^\theta_i(s) = \frac{1}{2} \int_0^s \frac{\mathbf{1}_{Z_u = i} - \mathbf{1}_{Z_u = i^*}}{\ell_i(u) + \ell_{i^*}(u)} \, du, \quad \forall i \in V, \ s \geq 0.
\]

We denote by \( B^\theta(s) = (B^\theta_i(s))_{i \in V} \) the associated random vector, taking values in \( \mathcal{A} \). We simply write \( B(s) := B^1(s) \) when \( \theta_i = 1 \) for all \( i \in V \).

Obviously, under \( P_{i_0}^{W,u} \), \((Z_s)\) is a Markov jump process with invariant measure \( e^{u_i + u_j} \), since \( u \in \mathcal{U}_0^W \). This implies the following simple result.

**Proposition 4.5.** Under \( P_{i_0}^{W,u} \), a.s., there exists a random vector \( B^\theta(\infty) \in \mathcal{A} \) such that

\[
\lim_{s \to \infty} B^\theta_i(s) = B^\theta_i(\infty).
\]

Moreover, \( B^\theta(\infty) \) has a density on \( \mathcal{A} \) that we denote by \( f_{i_0}^{W,u,\theta} \). We simply write \( f_{i_0}^{W,u} := f_{i_0}^{W,u,1} \) when \( \theta_i = 1 \) for all \( i \in V \).

Besides, \( B^\theta(\infty) \) has the following additivity property:

\[
(4.5) \quad B^\theta(\infty) = B^\theta(s) + B^\theta(\ell(s) + \ell^*(s))(\infty) \circ \Theta^Z_s,
\]

where \( \Theta^Z_s \) is the time-shift of the trajectories of \( Z \) (i.e. \( \Theta^Z_s = \Theta_{C^{-1}(s)} \) if \( \Theta_t \) is the time shift on the canonical space).

We will be interested in the conditioned law, defined for a.e. \( b \in \mathcal{A} \),

\[
P_{i_0}^{W,u}(\cdot | B^1(\infty) = b),
\]

under which \((Z_s)\) is a conditioned Markov process. The previous conditioned law should be understood as an \( h \)-process, which has jump rate at time \( t \) from \( i \) to \( j \) given by

\[
W_{ij}^{u} f_j^{W,u,1 + \ell(t) + \ell^*(t)}(b - B^1(t))
\]

We are now ready to state the main theorem concerning the non-randomized \( \ast \)-VRJP.
THEOREM 4.6 (Limit theorem for the non-randomized *-VRJP). Given $i_0 \in V$, for Lebesgue-a.e. conductances $W$, under the law of the non-randomized *-VRJP $\mathbb{P}^W_{i_0}$, $U$ has law

$$j_{i_0}^W((u - u^*)/2) \cdot \mu_{i_0}^W(du).$$

Conditionally on $U$, the time-changed *-VRJP $Z$ has the law of a Markov jump process $P_{i_0}^{W,u}$ conditioned on $B^1(\infty) = \frac{1}{2}(U - U^*)$, and thus it has a jump rate at time $t$ from $i$ to $j$ given by

$$W_{U,j}^{U_i,-U_{j}} = J_{ij}^{W,U,1+\ell(t)+\ell'(t)} \left( \frac{1}{2}(U - U^*) - B^1(t) \right) / J_{ij}^{W,U,1+\ell(t)+\ell'(t)} \left( \frac{1}{2}(U - U^*) - B^1(t) \right).$$

REMARK 4.7. Theorem 4.6 is an easy consequence of Theorem 4.2 by Bayes formula, and is proved in Section 9. The statement is a.s. in $W$, even though it should be true for all $W$, but the latter would require some regularity of the density $f_i^W(b)$, both in its variable and its parameters. This does not seem completely obvious but should be doable with some more work.

Part II
Random Schrödinger representation

5. Motivation. As explained in Remark 4.3 above, Theorem 4.2 implies the following identity we rewrite for convenience:

$$\int_A \nu_{i_0}^W(da) = \int_{\mathcal{E}} \mu_{i_0}^W(du) := F_{i_0}^W.$$

In the special case of the VRJP, i.e. when $*$ is the identity, the left-hand side term is trivially equal to 1 and $\mu_{i_0}^W(du)$ is the first marginal of the supersymmetric $\mathbb{H}^{2|2}$ model considered by Disertori, Spencer and Zirnbauer [15].

Thus the equality above generalizes the property that the mixing measure of the VRJP is normalized by a constant which does not depend on the weights $W$. That property plays a fundamental role in the analysis of the VRJP and has been proved in three very different ways. The first proof is due to Disertori Spencer and Zirnbauer ([15], Theorem (1.4) page 437): it uses that the measure is the marginal of the $\mathbb{H}^{2|2}$ model, and that the $\mathbb{H}^{2|2}$ model is invariant by a supergroup of transformations. The second proof, due to the authors of this paper, is probabilistic ([27], Theorem 2), and is a consequence of the representation of the VRJP as a mixture of Markov jump processes: Theorem 4.2 above generalizes that approach to the *-VRJP. The third proof, due to the Sabot, Tarres and Zeng [28] is based on a direct computation going through a representation of the measure by a random Schrödinger operator. That representation has proved to be very useful in the investigation of the properties of the VRJP, see below. In Part II, we generalize that representation to the *-VRJP, several new phenomena appear due to the extra randomization by the measure $\nu_{A, i_0}^W$.

Before we state the main results of this part, we briefly recall the corresponding results in the standard VRJP case. As explained above, the VRJP corresponds to the case where $*$ is the identity, in which case we can consider $G = (V, E)$ as a non-directed graph with some conductances on the non-directed edges $(W_{ij})_{i,j \in V}$. Assume that $V$ is finite. If $\beta = (\beta_i)_{i \in V}$ is a function of the vertices we define the Schrödinger operator $H_\beta = -W + \beta$ where $W$ is the (symmetric) matrix of conductances (with value 0 at $(i,j)$ when $\{i,j\}$ is not an edge of the
graph) and $\beta$ is the operator of multiplication by $\beta$ (considered as a potential on the graph). We write $H_\beta > 0$ when $H_\beta$ is positive definite, in which case we set $G_\beta = (H_\beta)^{-1}$, which has positive coefficients if the graph is connected, as the inverse of an $M$-matrix. In [28], definition 1, we introduced a probability distribution on the set of potentials $\beta = (\beta_i)_{i \in V}$, which in its simplest form has the following expression

$$\nu^W_V(d\beta) = \frac{1_{H_\beta > 0}}{\sqrt{2\pi} |V|} \frac{1_{H_\beta > 0}}{\sqrt{\det(H_\beta)}} d\beta,$$

where the 1 above denotes the vector in $\mathbb{R}^V$ equal to 1 in each coordinate. $^1$ The property

$$\int \nu^W_V(d\beta) = 1$$

is non-trivial and was proved by direct computation in [28]. It is also related to the identity (5.1) above in the case of the VRJP, where the left-hand side is 1. Indeed, [28] Theorem 3 states that, for any $i_0 \in V$, the random field $t = (t_i)_{i \in V}$ defined by

$$e^{t_i} = \frac{G_\beta(i_0, i)}{G_\beta(i_0, i_0)}, \quad \forall i \in V,$$

where $\beta$ is a random potential distributed according to $\nu^W_V$, is the mixing field of the VRJP starting at $i_0$, rooted at $t_{i_0} = 0$: as a consequence, $(u_i)_{i \in V} = \left( t_i - |V|^{-1} \sum_{j \in V} t_j \right)_{i \in V}$ has the law $\mu^W_{i_0}$ of definition (4.2).

Letac and Wesołowski noticed that the distribution (5.2) belongs to a larger and more natural family of measures, which has a remarkable property of stability by restriction and conditioning (proved independently in [30, 23]). The representation of the VRJP by a random Schrödinger operator was instrumental in several directions, in particular it enabled the representation by Sabot and Zeng of the VRJP as a mixture of Markov processes on infinite graphs and the characterization of the recurrence/transience of the VRJP in terms of the existence of a certain delocalized eigenvector of the Schrödinger operator at the ground state [30], and the proof of a monotonicity property [24]. The random Schrödinger representation and its consequences were instrumental in proving recurrence in 2D of the ERRW and the VRJP for any constant reinforcement parameter [30, 26, 21], diffusive behaviour in dimensions larger or equal to 3 at weak reinforcement [30], and the uniqueness of the phase transition in dimension $d \geq 3$ [24].

We generalize below the construction of the random potential $\beta$ of (5.2) to the case of the $\star$-VRJP. The potential $\beta = (\beta_i)_{i \in V}$ lives on the space $S$ of $\star$-symmetric functions on $V$, defined in (1.3), and we denote below by $\nu^W_S(d\beta)$ the corresponding measure, which is introduced later. The main results are the following:

- We generalize the key property (5.3) to an identity of the type $\int_S \nu^W_S(d\beta) = \int_A \nu^W_A(da)$, see (6.3) below, where $\nu^W_A(da)$ is a positive measure living on the space $A$ of $\star$-antisymmetric functions and closely related to the law of the initial local time defined in (3.1). The proof works by induction and goes through a family of measures involving both $\beta$ and $a$ variables in complementary $\star$-symmetric subsets. Surprisingly, the initialization of this induction is difficult and uses the Lagrange resolvent method for solving fourth degree polynomial equations.

$^1$Note that our $\beta$ stands for $2\beta$ in [28].
• We relate the distribution $\nu^W_S$ to the randomized $\ast$-VRJP, see Section 7 below: roughly speaking, the random potential $\beta$ describes the law of the asymptotic jump rates of the time-changed $\ast$-VRJP. For any $\ast$-symmetric subset $I \subseteq V$, we identify the law of the randomized $\ast$-VRJP conditioned on $\beta_I$ and $A_I$: this representation is new, even in the standard VRJP case. This enables us to express the mixing law of the $\ast$-VRJP $\nu^W_\nu$ defined in (4.2) in terms of the law of the random potential $\nu^W_S$, and in particular it gives a new proof of Theorem 4.2 and a purely computational proof of the identity (5.1).

6. The $\beta$-potentials and fundamental properties.

6.1. Definition. Recall the notations and definitions of Section 1.1. However, in this Part II, it is more convenient not to suppose that the $\ast$-directed graph $G$ is strongly connected. We remind from the beginning of Section 1.3 that $(W_{i,j})_{(i,j) \in E}$ are $\ast$-symmetric positive weights, called conductances by abuse of terminology, on the graph $G$ and that $W = (W_{i,j})_{i,j \in V}$ also denotes the matrix of conductances with $W_{i,j} = 0$ when $(i, j) \notin E$.

For all $\beta \in S$, let $H_\beta$ be the Schrödinger operator defined by

$$H_\beta = \beta - W,$$

where $\beta$ is the operator of coordinate multiplication by $(\beta_i)_{i \in V}$. We write $H_\beta > 0$ when $H_\beta$ is positive stable, i.e. all its eigenvalues have positive real parts. When $H_\beta > 0$, $H_\beta$ is a non-singular $M$-matrix (see e.g. [7] or Appendix A.5 in Appendix 12.2) and the inverse matrix

$$G_\beta = H_\beta^{-1}$$

is well-defined and has positive coefficients between any two vertices $i$ and $j$ such that there is a directed path in $G$ from $i$ to $j$, and coefficient 0 otherwise, see [7], Theorem 2.3 ($N_{38}$) or Proposition A.5 in Appendix 12.2.

Clearly, the matrices $W$, $H_\beta$, for $\beta \in S$, are $\langle \cdot, \cdot \rangle$-symmetric since $W_{i,j} = W_{j,i}$ and $\beta_i = \beta_j$, where $\langle \cdot, \cdot \rangle$ is the bilinear form defined in (1.5).

The random Schrödinger representation of the $\ast$-VRJP is based on key integral identities on the spaces $S$ and $A$. The following results give an extension of Theorem 1 in [28] for the $\ast$-VRJP, and of its generalized form given in Theorem 2.2 of [23].

**Definition 6.1.** For all $\theta \in (0, \infty)^V$, $\eta \in [0, \infty)^V$, we define $\nu^W_S^{\theta, \eta}(d\beta)$ as the measure on $S$ defined by

$$(6.1)$$

$$\nu^W_S^{\theta, \eta}(d\beta) = \left( \prod_{i \in V_0} \theta_i \right) \frac{1}{\sqrt{2\pi \dim(S)}} \exp \left( -\frac{1}{2} \langle \theta, H_\beta \theta \rangle - \frac{1}{2} \langle \eta, G_\beta \eta \rangle + \langle \theta, \eta \rangle \right) \frac{d\beta_{\beta > 0}}{\sqrt{|H_\beta|}},$$

where $d\beta_{\beta > 0} = \prod_{i \in V_0} d\beta_i$, and $\nu^W_A^{\theta, \eta}(da)$ as the measure on $A$ defined by

$$(6.2)$$

$$\nu^W_A^{\theta, \eta}(da) = \frac{1}{\sqrt{2\pi \dim(A)}} \exp \left( -\frac{1}{2} \langle e^a \theta, W e^a \theta \rangle + \frac{1}{2} \langle \theta, W \theta \rangle - \langle \eta, e^a \theta - \theta \rangle \right) da,$$

where $da = \prod_{i \in V_0} da_i$ and where $e^a \theta = (e_{a_i i} \theta_i)_{i \in V}$. If $V_1 = \emptyset$ then $A = \{0\}$ and by convention we set $\nu^W_A^{\theta, \eta} = 1$. When $\theta_i = 1$ and $\eta_i = 0$ for all $i \in V$ we simply write $\nu^W_S(d\beta)$ and $\nu^W_A(da)$.
Theorem 6.2. For all \( \theta \in (0, \infty)^V, \eta \in (\mathbb{R}_+)^V \), we have

\[
\int_S \nu^W_{S, \theta, \eta}(d\beta) = \int_A \nu^W_{A, \theta, \eta}(da).
\]

We denote by \( F^W_{\theta, \eta} = \int_A \nu^W_{A, \theta, \eta}(da) = \int_S \nu^W_{S, \theta, \eta}(d\beta) \) the integrals corresponding to (6.3). We simply write \( F^W \) when \( \theta = 1 \) and \( \eta = 0 \). Finally, \( F^W_{\theta, \eta} < \infty \) as soon as for any vertex \( i \in V \), there exists a directed path in \( \mathcal{G} \) from \( i \) to \( i^* \) or a directed path from \( i \) to a vertex \( j \) such that \( \eta_j > 0 \).

Remark 6.3. Remark that, when \( \theta = 1 \) and \( \eta = 0 \), the measures \( \nu^W_A \) above and \( \nu^W_{A,0} \) introduced in (3.1) are related by the simple formula: \( \nu^W_{A,0}(da) = e^a \nu^W_A(da) \). In particular, they are equal when \( i_0 = i^*_0 \) and \( F^W = F^W_{i_0} \) in this case, where \( F^W_{i_0} \) is defined in (3.2).

Remark 6.4. Note that, for the study of the *-VRJP, it is natural to assume that the graph is strongly connected, in which case the condition ensuring that \( F^W_{\theta, \eta} < \infty \) is satisfied. However, the induction in Theorem (6.6) below involves a restriction of the graph which may no longer be strongly connected, even when the graph \( \mathcal{G} \) is connected, which explains that we do not restrict the definition to that case.

6.2. Restriction and conditioning properties. Theorem 6.2 stated above is a special case of a more general theorem, that we state below, which involves also a measure in both the \( \beta \) and the \( \alpha \) variables. This measure appears naturally in the induction step of the proof of Theorem 6.2. It is also related to the *-VRJP, see Theorems 7.1 and 7.8 below.

Fix \( I \) self-dual subset of \( V \) (i.e. \( I^* = I \)), and let \( S_I \) and \( A_I \) be the corresponding *-symmetric and *-antisymmetric subspaces of \( \mathbb{R}^I \):

\[
S_I = \{ x \in \mathbb{R}^I, \ x_i = x_{i^*} \}, \ A_I = \{ x \in \mathbb{R}^I, \ x_i = -x_{i^*} \}.
\]

We also adopt the following notations: given \( y \in \mathbb{R}^V \), denote by \( y_I = (y_i)_{i \in I} \) the restriction of \( y \) to the indices of \( I \), and for a \( V \times V \) matrix \( M \), let

\[
M_{I,I^*}, \ M_{I^*,I}, \ M_{I,I^*}, \ M_{I^*,I^*},
\]

be the block matrices obtained by the restriction of \( M \) to the corresponding subsets.

Given \( a_I \in A_I, \ \beta_I \in S_I, \ \theta \in (0, \infty)^V \) and \( \eta \in \mathbb{R}^V_+ \), define

\[
\begin{align*}
\hat{H}_I &= (H_\beta)_I = \beta_I - W_{I,I}, \ \hat{G}_\beta = (\hat{H}_\beta)^{-1} \\
\theta^a_I &= e^{a_i} \theta_I = (e^{a_i} \theta_i)_{i \in I} \\
\hat{\eta}_I &= \eta_I + W_{I,I^*} \theta_{I^*}, \ \hat{\eta}^a_I = \eta_I + W_{I,I} \theta^a_I \\
\hat{W}_{I^*,I} &= W_{I^*,I} + W_{I,I^*} \hat{G}_\beta W_{I,I^*}, \\
\hat{\eta}_{I^*} &= \eta_{I^*} + W_{I,I^*} \hat{G}_\beta \eta_I
\end{align*}
\]

Note that \( \hat{\eta}_I \) does not depend on \( \beta \) but on \( \eta_I \) and \( \theta_{I^*} \), while \( \hat{W}_{I^*,I}, \hat{\eta}_{I^*} \), depends on \( \beta_I \) but not on \( \beta_{I^*} \). Note also that \( \hat{\eta}_{I^*} = 0 \) if \( \eta = 0 \). The following lemma plays a key role in the proof of Theorems 6.2 and 6.6 below.

Lemma 6.5. For all non-empty subset \( I \subsetneq V, \ I^* = I \), for \( \theta \in (0, \infty)^V, \ \eta \in \mathbb{R}_+^V \), and with the notations above, we have the following equality of measures:

\[
\nu^W_{S, \theta, \eta}(d\beta) = \nu^W_{S_I, \theta_I, \eta_I}(d\beta_I) \nu^W_{S_{I^*}, \theta_{I^*}, \eta_{I^*}}(d\beta_{I^*});
\]
We denote by $Q^W_{I}(d\beta_1, da_{I^c})$ the measure on $S_I \times A_I$ given by the equivalent expressions above.

Theorem below generalizes Theorem 6.2 above.

**Theorem 6.6.** For all $I \subseteq V$ such that $I^* = I$, $\theta \in (0, \infty)^V$ and $\eta \in \mathbb{R}_+^V$,

$$\int_S \nu^W_{S \times A_{I^c}}(d\beta) = \int_{S_I \times A_{I^c}} Q^W_{I}(d\beta_1, da_{I^c}) = \int_A \nu^W_{A_{I^c}}(da)$$

If $F^W,\eta < \infty$, we denote by $\nu^W_S, \nu^W_A$ and $Q^W_I$ the probability measures $\frac{1}{F^W,\eta} \nu^W_S, \frac{1}{F^W,\eta} \nu^W_A$ and $\frac{1}{F^W,\eta} Q^W_I$.

Finally, we prove that the laws introduced in Theorems 6.2 and 6.6 are stable by conditioning by the values on a subset. This gives a counterpart to the result proved independently in Lemma 5 of [30] and [23] Section 4.

**Proposition 6.7.** Let $I \subseteq V$ be such that $I^* = I$, $\theta \in (0, \infty)^V$ and $\eta \in \mathbb{R}_+^V$. Assume that $F^W,\eta < \infty$.

i) (Conditioning) We have the equalities of conditioned probabilities:

- under $\nu^W_S(d\beta)$, conditioned on $\beta_1$, $\beta_{I^c}$ has law $\nu^W_{S_1}(da_{I^c})$;
- under $\nu^W_A(d\beta)$, conditioned on $\beta_1$, $\beta_{I^c}$ has law $\nu^W_{A_{I^c}}(d\beta_{I^c})$;
- under $Q^W_I(d\beta_1, da_{I^c})$: conditioned on $\beta_1$, $\beta_{I^c}$ has law $\nu^W_{S_1}(da_{I^c})$ and conditioned on $\beta_1$, $\beta_{I^c}$ has law $\nu^W_{A_{I^c}}(d\beta_{I^c})$.

ii) (Restriction) $\beta_1 = (\beta_i)_{i \in I}$ has the same law under $\nu^W_S(d\beta)$ and $Q^W_I(d\beta_1, da_{I^c})$; while $a_{I^c} = (a_i)_{i \in I^c}$ has the same law under $\nu^W_A(d\beta)$ and $Q^W_I(d\beta_1, da_{I^c})$.

**Remark 6.8.** Implicitly, Theorem 6.7 implies that if $F^W,\eta < \infty$ for the full graph, then it is also the case for all the integrals that appear on the restrictions to $I$ and $I^c$ with the corresponding parameters $W, \theta, \eta$, for instance in the first statement it implies that $F^W_{I\cap I^c, \theta_{I\cap I^c}, \eta_{I\cap I^c}} < \infty$.

**Remark 6.9.** Let us point out that Theorem 6.7 yields a generalization in the case of the $\ast$-VRJP of the restriction and conditioning identities stated in Lemma 5 of [30] and [23] Section 4, corresponding to the case where $\ast$ is the identity: (i) coincides with Lemma 5 (ii) in [30], whereas (ii) coincides with Lemma 5 (i) in [30]: indeed, in this case $Q^W_I(d\beta_1, da_{I^c})$ is just $\nu^W_{S_1}(da_{I^c})$ since $A_{I^c} = \{0\}$ and $\nu^W_{A_{I^c}}(da_{I^c}) = 1$.

7. Relation with the $\ast$-VRJP.
7.1. Results in the simpler case where \( i_0 = i^*_0 \). In the case where the starting point of the \( \star \)-VRJP is self-dual, i.e. \( i_0 = i^*_0 \), the relation between the \( \beta \)-potential and the \( \star \)-VRJP, and its mixing measure \( \mu^W_{i_0} \), is easier to state. In the general case where \( i_0 \neq i^*_0 \), the formulas are slightly more involved and one needs to tilt the measures to take into account the fact the \( \star \)-VRJP starts from \( i_0 \) or \( i^*_0 \). Hence, for clarity, we start by stating the results in the case of a self-dual starting point and state the general case in the next section.

For \( \theta = 1 \) and \( \eta = 0 \), the measure \( Q^W_t \) defined in Lemma 6.5 appears as the joint law of the asymptotic jump rates of the randomized \( \star \)-VRJP on \( I \) and the initial random local time \( A \) on \( I^c \).

**Theorem 7.1.** Assume that \( \mathcal{G} \) is finite and strongly connected. Let \( I \subseteq V \), \( I^* = I \). Assume that \( i_0 = i^*_0 \) and \( i_0 \in I^c \).

i) Under the law of the randomized \( \star \)-VRJP starting at \( i_0 \), i.e. under \( \mathbb{P}^W_{i_0} \), denote by \( (U_i)_{i\in V} \) the limiting local time defined in Theorem 4.2 i) and by

\[
    B_i = \sum_{j, i \to j} W_{i,j} e^{U_j - U_i}, \quad i \in I.
\]

Then, \((B_I, A_{I^*})\) has the law \( \mathbb{P}^W_{i_0} Q^W_t \).

ii) Under \( \mathbb{P}^W_{i_0} \), conditionally on \((B_I, A_{I^*})\), the \( \star \)-VRJP \( X(t) \) has the law of the jump process with jump rates at time \( t \), from \( i \) to \( j \) given by

\[
    W_{i,j} e^{T_i(t) + T_j(t) + V_j(t) - V_i(t)},
\]

where \( V(t) = (V_i(t))_{i\in V} \) is defined by

\[
    \begin{cases}
    V_i(t) = T_i(t) + A_i, & \text{if } i \in I^c, \\
    (H_B(e^{V_i(t)}))_I = 0.
    \end{cases}
\]

**N.B.** Note that the equation \((H_B(e^{V_i(t)}))_I = 0\) only involves \( B_I \) and not \( B \) on \( I^c \).

**Remark 7.2.** Let us clarify the meaning of the statement ii):

- When the process \( X(t) \) is in \( I \), then \( V(t) \) is constant, as \( \beta \) is fixed and the boundary value of \( V(t) \) on \( I^c \) is constant. Hence, while \( X(t) \) belongs to \( I \), the process jumps at rate \( W_{i,j} e^{T_i(t) + T_j(t) + v_j - v_i} \) for a constant function \( v \) and, after the change of time given in Theorem 4.2 ii), the process jumps with constant rate \( W_{i,j} e^{v_j - v_i} \). In other words, the process behaves as a time-changed Markov jump process during its excursions on the subset \( I \), but the transition probabilities may change between two different excursions.

- On the contrary, when \( i \in I^c \), and \( j \in I^c \), then

\[
    W_{i,j} e^{T_i(t) + T_j(t) + v_j - v_i} = W_{i,j} e^{T_i(t) + A_i + T_j(t) + A_j},
\]

hence the jump rate is the jump rate of the \( \star \)-VRJP with initial local time \( A \).

In summary, Theorem 7.1 ii) gives a representation of the randomized \( \star \)-VRJP by a self-interacting process on \( I^c \), and a mixture of Markov jump processes during the excursions on \( I \).

**Remark 7.3.** Note that Theorem 7.1 is also new in the case of the standard VRJP: then \( A = 0 \), and the statement gives the law of the VRJP conditioned on the asymptotic jump rate \( \beta \) restricted to a subset \( I \subseteq V \).
Remark 7.4. The law of the asymptotic jump rates $\mathcal{B}_i = \sum_{j, i \rightarrow j} W_{i,j} e^{U_{i,j}} - U_{i,j}$ on the full set of vertices $V$ does not belong to the family of laws $\mathcal{P}_{W}^{\theta, \eta}$, since it is biased by the starting point of the process. In fact, we always have $\det(H_\beta) = 0$ while $H_\beta > 0$ a.s. under any of the laws $\mathcal{P}_{W}^{\theta, \eta}(d\beta)$. Note that the same happens for the standard VRJP. However, the expression of the law of $\mathcal{B}$ on $V$ appears naturally in the course of the proof of Corollary 7.10, see Remark 12.3 below.

Theorem 7.1 enables us to retrieve the mixing measure of the VRJP, more precisely it gives a different proof of Theorem 4.2. Indeed, consider the case $I^c = \{i_0\}$, and recall that we have fixed $i_0 = i^*_0$ in this section. Then $A_{i_0} = 0$ and

$$V_j(t) - V_i(t) = u_{j^*} - u_{i^*}, \forall t \geq 0,$$

where $u$ is the solution of $u_{i_0} = 0$ and $H_\beta(e^u) t = 0$, i.e.

$$e^{u_j} = \frac{G_B(i_0, j)}{G_B(i_0, i_0)}, \forall j \in V.$$  

Remark that this only involves the values of $\mathcal{B}$ on $I$, and in particular not $\mathcal{B}_{i_0}$. It leads to the following corollary, which is a generalization to the case of the $\star$-VRJP of Theorem 3 i) of [28], and which also gives a different proof of Theorem 4.2.

Corollary 7.5. Assume that $i_0 = i^*_0$ and that the graph is strongly connected. Let $\theta = 1$ and $\eta = 0$. Then $\int_A \nu_W^W(d\alpha) = F^W = F_{i_0}^W$ and, under the distribution $\frac{1}{\nu_W^W}(d\beta)$,

$$\left( \frac{G_B(i_0, j)}{G_B(i_0, i_0)} \right)_{j \in V}$$

is distributed as $(e^{u_j - u_{i_0}})_{j \in V}$ under the mixing law $\frac{1}{\nu_W^W}(d\alpha)$. In particular, under $\nu_W^W$, the randomized $\star$-VRJP in exchangeable time scale $(Z_t)_{t \geq 0}$ (defined in Theorem 4.2 ii) is a mixture of Markov jump processes with jump rates from $i$ to $j$

$$W_{i,j} \frac{G_B(i_0, j^*)}{G_B(i_0, i^*)},$$

where $\beta$ is distributed according to $\frac{1}{\nu_W^W}(d\beta)$.

7.2. Results in the the general case $i_0 \neq i^*_0$. As explained above, we need to introduce a tilted version of the measures $Q_{W}^{\theta, \eta}(d\beta)$ in order to take into account the fact that the $\star$-VRJP starts from the vertex $i_0$ rather than from $i^*_0$.

Definition-Proposition 7.6. i) For $i_0 \in V$, with the notations of Section 6, we define

$$\nu_{\alpha, i_0, \beta}(da) = \theta_{i_0}^{\alpha} \nu_{\alpha, i_0}^{\theta, \eta}(da) = \theta_{i_0}^{\alpha} e^{\alpha \beta_0} \nu_{\alpha, i_0}^{\theta, \eta}(da).$$

Remark that when $\theta_1 = 1$ for all $i \in V$ and $\eta = 0$, it coincides with Definition (3.1).

ii) Let $I \subsetneq V$, $I = I^*$, and assume that $i_0 \in I^c$. Define

$$Q_{W}^{\theta, \eta}(d\beta_1, da_{I^*}) = \theta_{i_0}^{\alpha} Q_{I^*}^{\theta, \eta}(d\beta_1, da_{I^*}).$$

Then

$$\int_{S_{I^*} \times A_{I^*}} Q_{I, i_0}^{\theta, \eta}(d\beta_1, da_{I^*}) = \int_{A_{I^*}} \nu_{\alpha, i_0}^{\theta, \eta}(da).$$

Denote by $F_{i_0}^{W, \theta, \eta}$ the corresponding value. When $\theta = 1$, $\eta = 0$, we simply write $F^W_{i_0}$ and it corresponds to the integral defined in (3.2).
Remark 7.7. We could also define a tilted version of the measure on $S$ when $\eta \neq 0$, with the same integral as (7.1), by $\nu_{\mathcal{S},i_0,\theta,\eta}^{W}(d\beta) = (G_{\beta}\eta)_{i_0}\nu_{\mathcal{S},\theta,\eta}^{W}(d\beta)$. However, we do not introduce it formally since we do not really need it here.

Theorem 7.8. Let $G$ be finite and strongly connected. Let $I \subseteq V$, $I^* = I$ and $i_0 \in I^c$. Then the statement of Theorem 7.1 i) and ii) holds, with a single modification: $\frac{1}{\mu_{i_0}^W}Q_{I,i_0}^W$ replaces $\frac{1}{\mu_{i_0}^W}Q_{I,i_0}^W$ in i) as the law of $(B_I, A_I)$.

We can deduce from Theorem 7.8 a second proof of the representation of the $\ast$-VRJP, i.e. of Theorem 4.2. It is the content of Corollary 7.10 below. Let us fix some notations: in this subsection we take a starting point $i_0 \in V$ and set

$$I = V \setminus \{i_0, i_0^*\}.$$ 

We use notation (6.4) above for this subset $I$, so that for $\beta_I = (\beta_i)_{i \in I} \in \mathcal{S}_I$, we have

$$\tilde{H}_\beta = \beta_I - W_{I,I}, \quad \tilde{G}_\beta = (\tilde{H}_\beta)^{-1}.$$ 

Theorem 7.8 i) gives us a relation between the law $\frac{1}{\mu_{i_0}^W}\mu_{i_0}^W$ of Theorem 4.2 and the distribution $\frac{1}{\mu_{i_0}^W}Q_{I,i_0}^W$. The first step is to prove that in the case where $I = V \setminus \{i_0, i_0^*\}$, there is a bijection between the random variables $(U_i)_{i \in V}$ and the variables $(B_i)_{i \in I}$ which appear in Theorem 7.8, on good domains. With this goal, we introduce some notation. Denote by

$$(\mathcal{D}_{i_0}) = \left\{ \beta_I = (\beta_i)_{i \in I} \in \mathcal{S}_I : \tilde{H}_\beta > 0 \right\}.$$ 

Note that the probability measure $Q_{I,i_0}^W(d\beta_I, da_{I'})$ is supported on $\mathcal{D}_{i_0} \times A_{I'}$. Finally, let $\Xi_{i_0} : \mathcal{U}_0^W \mapsto \mathcal{S}_I$ be the map given by

$$\Xi_{i_0}(u) = \sum_{j,i \to j} W_{ij}e^{u_j - u_i}, \quad \forall i \in I.$$ 

Note that $\Xi_{i_0}(u) \in \mathcal{S}_I$ since $u \in \mathcal{U}_0^W$. We have the following lemma.

Lemma 7.9. For all $i_0 \in V$, $\Xi_{i_0}$ is a $C^1$-diffeomorphism from $\mathcal{U}_0^W$ onto $\mathcal{D}_{i_0}$.

Applying Theorem 7.8 to the case $I = V \setminus \{i_0, i_0^*\}$ and the previous lemma, we see that under the law $\frac{1}{\mu_{i_0}^W}$ of the randomized $\ast$-VRJP starting at $i_0$, the joint law of $((U_i)_{i \in V}, (A_{i_0}, A_{i_0^*}))$ is the same as the law of $(\Xi_{i_0}^{-1}(B_I), A_{I'})$, where $(B_I, A_{I'})$ is distributed according to $\frac{1}{\mu_{i_0}^W}Q_{I,i_0}^W$. In order to retrieve Theorem 4.2, it is thus enough to prove that $(\Xi_{i_0})^{-1}(B_I)$ has law $\frac{1}{\mu_{i_0}^W}\mu_{i_0}^W$, which is the purpose of the corollary below.

Corollary 7.10. Assume that the graph $G$ is strongly connected and let $I = V \setminus \{i_0, i_0^*\}$. If $(B_I, A_{I'})$ is distributed according to $\frac{1}{\mu_{i_0}^W}Q_{I,i_0}^W$, then $\Xi_{i_0}^{-1}(B_I)$ is distributed according to $\frac{1}{\mu_{i_0}^W}\mu_{i_0}^W$.

Remark 7.11. Together with Theorem 7.8, Corollary 7.10 gives a different proof of Theorem 4.2, and in particular of the identity (5.1). Remark that it is also the general form of Corollary 7.5 since in the case $i_0 = i_0^*$, we easily see that if we define $U$ by $U = (\Xi_{i_0})^{-1}(B_I)$, then $e^{U_j - U_{i_0}} = \frac{G_B(i_0,j)}{G_B(i_0,i_{i_0})}$, by definition of $G_{\beta}$. 
Part III
Proofs of the Results

8. Preliminary results and proof of the statements in Sections 2 and 3.

8.1. Proofs of the results of Section 2 and preliminary results concerning the limiting manifold.

Proof of Lemma 2.3. We denote by

$$DJ_u(v) = \left( \frac{\partial}{\partial v_i} J_u(v) \right)_{i \in V}, \quad D^2 J_u(v) = \left( \frac{\partial^2}{\partial v_i \partial v_j} J_u(v) \right)_{i,j \in V},$$

the gradient and Hessian of the function $J_u$. A direct computation gives

$$\frac{\partial}{\partial v_i} J_u(v) = -\sum_{j,i \to j} W_{i,j}^u v_j - v_i + \sum_{j,j \to i} W_{j,i}^u v_i - v_j,$$

$$\frac{\partial^2}{\partial v_i \partial v_j} J_u(v) = \begin{cases} -W_{i,j}^u v_i - v_j, & i \neq j, \\ \sum_{l} (W_{i,l}^u v_l + W_{l,i}^u v_i), & i = j. \end{cases}$$

From this last formula, we easily deduce that $J_u$ is strictly convex, since for all $b \in \mathcal{H}_0$,

$$(b, D^2 J_u(v)b) = \sum_{i,j} (W_{i,j}^u v_i - v_j + W_{j,i}^u v_i - v_j)(b_i - b_j)^2.$$

If $a \in \mathcal{A}$, denoting as above $h = u - a$ and using that $W_{i,j}^u e^{a_j - a_i} = W_{i,j}^h$, we deduce

$$(8.1) \quad (D J_u)(a^*)_i = -\text{div}(W^h)(i), \quad \forall i \in V.$$  

Now consider $J_u$ restricted to $\mathcal{A}$ : by the asymptotic behavior of $J_u$ and its convexity, we know that $J_u$ admits a unique minimizer on $\mathcal{A}$. It implies that there is a unique $a \in \mathcal{A}$ such that

$$(D J_u)(a^*)_i = 0$$

for all $b \in \mathcal{A}$, which is equivalent to $\text{div}(W^{u-a}) = 0$, since $\text{div}(W^{u-a}) \in \mathcal{A}$. Hence there is a unique $a \in \mathcal{A}$ such that $h = u - a \in \mathcal{U}^W_\theta$. Note finally that by (8.1) $a^*$ is also the minimizer of $J_u$ on $\mathcal{H}_0$.

We will also need a result on the tangent space to the limiting manifold $\mathcal{U}^W_\theta$. Given $h \in \mathbb{R}^V$, $K^h$ defined in (4.1) can be understood as the operator $K^h : \mathbb{R}^V \to \mathbb{R}^V$ defined by

$$(8.2) \quad K^h(g)(i) = \sum_{j,i \to j} W_{i,j}^h (g(j) - g(i)).$$

In other words, $K^h$ is the infinitesimal generator of the Markov Jump Process with jump rates $W_{i,j}^h$. We write $\mathcal{T} K^h$ for the transpose of $K^h$. When $h \in \mathcal{U}^W_\theta$, $\text{Im}(\mathcal{T} K^h) \subseteq \mathcal{H}_0$ : indeed, using $\text{div}(W^h) = 0$,

$$\sum_{i \in V} \mathcal{T} K^h(g)(i) = \sum_{i \in V} g(i)\text{div}(W^h)(i) = 0.$$  

Hence, when $h \in \mathcal{U}^W_\theta$, $\mathcal{T} K^h$ is bijective on $\mathcal{H}_0$ : in the sequel we always understand the inverse of $\mathcal{T} K^h$ as the operator $(\mathcal{T} K^h)^{-1} : \mathcal{H}_0 \to \mathcal{H}_0$, with a slight abuse of notation, obtained as the inverse of $\mathcal{T} K^h|_{\mathcal{H}_0}$. 
LEMMA 8.1. The tangent space of \( \mathcal{U}_0^W \) at point \( h \), denoted by \( T_h^W \), is equal to
\[
T_h^W = \{ x \in \mathcal{H}_0, \ tK^h x \in \mathcal{S}_0 \} = (tK^h)^{-1}(\mathcal{S}_0),
\]
where, as explained above, \((tK^h)^{-1}\) is understood as the inverse of the restriction of \( tK^h \) to \( \mathcal{H}_0 \).

PROOF. By definition, \( h \in \mathcal{U}_0^W \) if and only if \( h \in \mathcal{H}_0 \) and \( \text{div}(W^h) = 0 \). Differentiating the last identity, we see that an element \( dh \) is in \( T_h(\mathcal{U}_0^W) \) if and only if \( dh \in \mathcal{H}_0 \) and for all \( i \in V \), using \( \text{div} \),
\[
\sum_{j:i\to j} W_{i,j}^h (dh_i + dh_j) - \sum_{j:j\to i} W_{j,i}^h (dh_i + dh_j) = 0
\]

\[
\iff \sum_{j:i\to j} W_{j,i}^h (dh_j - dh_i) - \sum_{j:j\to i} W_{j,i}^h (dh_j - dh_i) = 0
\]

\[
\iff tK^h(dh)(i) = tK^h(dh)(i^*).
\]

\[\square\]

Finally, the following simple property will be used several times throughout the paper.

PROPOSITION 8.2. For \( i \in V \), define \( \tilde{F}_i^W \) by
\[
(8.3) \quad \tilde{F}_i^W = e^{-\frac{1}{2}\sum_{(i,j)\in E} W_{i,j}^W} F_i^W = \frac{1}{\sqrt{2\pi|V_i|}} \int_{\mathcal{A}} e^{a_{i^*}} e^{-\frac{1}{2}\sum_{(i,j)\in E} W_{i,j}^W} e^{a_{i^*}+a_j} \left( \prod_{i\in V_i} da_i \right)
\]

If \( \tilde{a} \in \mathcal{A} \) then \( \tilde{F}_{i_{0}^W}^W = e^{\tilde{a}_{i_0}} \tilde{F}_{i_0}^W \).

PROOF. We have
\[
\tilde{F}_{i_0}^W = \frac{1}{\sqrt{2\pi|V_i|}} \int_{\mathcal{A}} e^{a_{i^*}} e^{-\sum_{(i,j)\in E} W_{i,j}^W} e^{a_{i^*}+a_j} \left( \prod_{i\in V_i} da_i \right)
\]

The change of variables \( a' = a + \tilde{a} \) yields the result. \[\square\]

8.2. Proof of the results of Section 3.

8.2.1. Proof of Lemma 3.1. In fact, \( F_{i_0}^W < \infty \) is true under the weaker assumption that, for any \( i \in V \) there is a directed path in \( G \) from \( i \) to \( i^* \), which is obviously satisfied when the graph is strongly connected. Set \( w := \inf_{(i,j)\in E} W_{i,j} > 0 \). For \( i \in V \), let \( \mathcal{A}_i = \{ a \in \mathcal{A}, a_i \geq \max_{j\in V} |a_j| \} \), we have
\[
\int_{\mathcal{A}} \nu_{i_0}^W(da) \leq \sum_{i\in V} \int_{\mathcal{A}_i} \nu_{i_0}^W(da).
\]

Fix \( i \in V \). Since there exists a directed path from \( i \) to \( i^* \), we denote by \( \sigma \) a shortest one. Since \( a_{i^*} = -a_i \), for \( a \in \mathcal{A}_i \) there exists \( k \) such that \( a_{\sigma_k} - a_{\sigma_{k+1}} \geq \frac{2w}{|\sigma|} \), where \( |\sigma| \) is the length of the path \( \sigma \). Besides, we have
\[
\nu_{i_0}^W(da) \leq Ce^{-a_{i_0}} \exp \left( -\frac{1}{2} \sum_{i\to j} W_{i,j} e^{a_i-a_j} \right) da \leq Ce^{-a_{i_0}} \exp \left( -\frac{1}{2} w \frac{2w}{|\sigma|} \right) da,
\]

\[\square\]
where $C$ is a constant depending only on $W$. Since $|a_j| \leq a_i$ on $A_i$, we have
\[
\int_{A_i} \nu^{W,1,\eta}_A(da) \leq C \int_0^\infty |2a_i| |\nu^l_i| e^{a_i} \exp \left(-\frac{1}{2}e^{\frac{2a_i}{|\nu^l_i|}}\right) da_i,
\]
since for all $j \notin \{i, i^*\}$, we integrate $a_i = -a_{i^*}$ on the interval $[-a_i, a_i]$ if $j \neq j^*$ or $a_j = 0$ if $j = j^*$. The last expression is finite. This concludes the proof.

8.2.2. Proof of Lemma 3.2. Under $\mathbb{P}^W_{i_0}$, conditionally on $A$, the process $X$ is a $*-$VRJP with conductances $W^A_{i,j} = W_{i,j} e^{A_i + A_j}$. Hence, conditionally on $A$, the probability that the process $X$ at time $t$ has performed $n$ jumps in infinitesimal time intervals $[t_i, t_i + dt_i]$, $0 < t_1 < \cdots < t_n < t$, following the trajectory $\sigma_0 = i_0, \ldots, \sigma_n = j_0$ is equal to

\[
(8.4) \exp \left(-\int_0^t \sum_{j: X_s \rightarrow j} W^A_{X_s,j} e^{T_{j,s}(s) + T_{j^*,s}(s)} ds \right) \left(\prod_{l=1}^n W^A_{\sigma_{l-1},\sigma_l} e^{T_{\sigma_{l-1},l}(t_l) + T_{\sigma_l}(t_l)} dt_l\right)
\]

Indeed, the exponential term accounts for the holding probability since, at time $s$, the process $X$ leaves the position $X_s$ at a rate $\sum_{j: X_s \rightarrow j} W^A_{X_s,j} e^{T_{j,s}(s) + T_{j^*,s}(s)}$, and the product terms accounts for the probabilities to jump from $\sigma_{l-1}$ to $\sigma_l$ in time intervals $[t_l, t_l + dt_l]$. Remark now that

\[
d \left( \sum_{(i,j) \in E} W^A_{i,j} (e^{T_i(t)+T_j(t)} - 1) \right)
= \sum_{j: X_t \rightarrow j} W^A_{X_t,j} e^{T_{j,t}(t)} dt + \sum_{j^*: X_t \rightarrow j^*} W^A_{j^*,X_t} e^{T_{j^*,t}(t) + T_{X_t}(t)} dt
= 2 \sum_{j: X_t \rightarrow j} W^A_{X_t,j} e^{T_{X_t}(t) + T_j(t)} dt.
\]

This implies that,
\[
\int_0^t \sum_{j: X_s \rightarrow j} W^A_{X_s,j} e^{T_{j,s}(s) + T_{j^*,s}(s)} ds = \frac{1}{2} \sum_{(i,j) \in E} W^A_{i,j} (e^{T_i(t)+T_j(t)} - 1).
\]

Since $A \in A$, we have $W^A_{i,j} = W_{i,j} e^{A_i + A_j} = W_{i,j} e^{A_{i^*} - A_{j^*}}$, and it leads to the following expression for the probability (8.4)

\[
(8.5) \left(\prod_{l=1}^n W_{\sigma_{l-1},\sigma_l} e^{T_{\sigma_{l-1},l}(t_l) + T_{\sigma_l}(t_l)} dt_l\right) e^{A_{i^*} - A_{j^*} e^{-\frac{1}{2} \sum_{(i,j) \in E} W_{i,j} e^{A_{i^*} - A_{j^*} (e^{T_i(t)+T_j(t)} - 1)}}}.
\]

Hence, if $\phi$ is a test function,
\[
\mathbb{P}^W_{i_0}(\phi(A) | \sigma(X_u, u \leq t))
= \int_A \phi(a) e^{ax_i - a_{i^*}} \exp \left(-\frac{1}{2} \sum_{(i,j) \in E} W_{i,j} e^{a_{i^*} - a_{j^*}} (e^{T_i(t)+T_j(t)} - 1)\right) \nu^{W}_{A,i_0}(da)
= \int_A e^{a x_i - a_{i^*}} \exp \left(-\frac{1}{2} \sum_{(i,j) \in E} W_{i,j} e^{a_{i^*} - a_{j^*}} (e^{T_i(t)+T_j(t)} - 1)\right) \nu^{W}_{A,i_0}(da)
= \int_A \phi(a) e^{ax_i} \exp \left(-\frac{1}{2} \sum_{(i,j) \in E} W_{i,j}^{T(t)} e^{a_{i^*} - a_{j^*}}\right) \nu^{W}_{A,i_0}(da)
= \phi(a) \int_A \frac{1}{F^W_{X_i}} \nu^{W,T(t)}_{A,X_i}(da).
\]
8.2.3. Proof of Corollary 3.3. Under $\mathbb{P}^W_{t_0}$, conditionally on $A$ and on $\mathcal{F}_t^X$, if $X_t = i$, then $X$ jumps from $i$ to $j$, $i \rightarrow j$, at a rate $W_{i,j}^{A \cup T(t)} = W_{i,j}^{T(t)} e^{A_j - A_i}$. By Lemma 3.2, conditionally on the past at time $t$, $A$ has law $\frac{F_{i,j}^{W^T(t)}}{F_{i,j}^{W^T(t)}}$. Hence, conditionally on the past at time $t$, $X$ jumps from $i$ to $j$ at rate

$$W_{i,j}^{T(t)} = \int e^{a_j - a_i} \frac{1}{F_{i,j}^{W^T(t)}} e^{W_{i,j}^{T(t)}} (da) = W_{i,j}^{T(t)} \frac{F_{i,j}^{W^T(t)}}{F_{i,j}^{W^T(t)}}.$$

8.2.4. Proof of Lemma 3.7. We will prove that, if $\alpha$ satisfies the divergence condition (1.7), if $(W_e)_{e \in E}$ are independent gamma random variables with parameters $(\alpha_e)_{e \in E}$, and, if, conditionally on $W$, $A$ is distributed according to $\nu_{A,e}^W$, then $W_A \stackrel{law}{\equiv} W$. This will conclude the proof, using Lemma 1.2.

Let $\phi$ be any positive measurable test function, and let $C_\alpha := \prod_{(i,j) \in E} \Gamma(\alpha_{i,j})$. Then

$$\mathbb{E}(\phi(W^A)) = C_{\alpha}^{-1} \int_{\mathbb{R}_+^E} \left( \int_A \phi(W^a) \frac{1}{F_{i_0}^W} e^{W_{i_0}^a} (da) \right) \prod_{(i,j) \in E} W_{i,j}^{\alpha_{i,j} - 1} e^{-W_{i,j}} dW_{i,j}$$

$$= C_{\alpha}^{-1} \int_{\mathbb{R}_+^E \times A} \phi(W^a) \left( \prod_{(i,j) \in E} W_{i,j}^{\alpha_{i,j} - 1} e^{-W_{i,j}} dW_{i,j} \right) \frac{e^{a_i \gamma} - \frac{1}{2} \sum_{(i,j) \in E} W_{i,j} e^{s_i + \sigma_j}}{\prod_{i \in V_i} da_i},$$

where we recall that $\tilde{F}_i^W$ is defined in proposition 8.2. Let us perform the change of variables

$$(W,a) \rightarrow (\tilde{W} = W^a, a);$$

then

$$\prod_{(i,j) \in E} W_{i,j}^{\alpha_{i,j} - 1} e^{-W_{i,j}} dW_{i,j}$$

$$= \left( \prod_{(i,j) \in E} \tilde{W}_{i,j}^{\alpha_{i,j} - 1} d\tilde{W}_{i,j} \right) e^{-\frac{1}{2} \sum_{(i,j) \in E} \alpha_{i,j} (a_i - a_j)} e^{-\sum_{(i,j) \in E} \tilde{W}_{i,j} e^{-s_i - s_j}}$$

$$= \left( \prod_{(i,j) \in E} \tilde{W}_{i,j}^{\alpha_{i,j} - 1} d\tilde{W}_{i,j} \right) e^{-\frac{1}{2} \sum_{i \in V} a_i \text{div}(\alpha)(i)} e^{-\frac{1}{2} \sum_{(i,j) \in E} \tilde{W}_{i,j} e^{-s_i - s_j}}$$

$$= e^{-a_i \gamma} \left( \prod_{(i,j) \in E} \tilde{W}_{i,j}^{\alpha_{i,j} - 1} d\tilde{W}_{i,j} \right) e^{-\frac{1}{2} \sum_{(i,j) \in E} \tilde{W}_{i,j} e^{-s_i - s_j}}$$

where we use the divergence condition (1.7) in the last equality.

Using also that, by proposition 8.2, $e^{a_i \gamma} \tilde{F}_i^W = \tilde{F}_i^{W^a}$, we deduce that

$$\mathbb{E}(\phi(W^A)) = C_{\alpha}^{-1} \int_{\mathbb{R}_+^E \times A} \phi(\tilde{W}) \left( \prod_{(i,j) \in E} \tilde{W}_{i,j}^{\alpha_{i,j} - 1} e^{-\tilde{W}_{i,j}} d\tilde{W}_{i,j} \right) \frac{e^{-a_i \gamma}}{\tilde{F}_i^{W^a}} e^{-\sum_{(i,j) \in E} \tilde{W}_{i,j} e^{-s_i - s_j}} \left( \prod_{i \in V_i} da_i \right).$$
Changing to variables $a' = -a$ and integrating on $a'$, we deduce
\[
E\left( \phi(W^A) \right) = C_\alpha^{-1} \int_{\mathbb{R}_+^E} \phi(W) \left( \prod_{(i,j) \in E} \tilde{W}_{i,j}^{(a')_{i,j}} e^{-\tilde{W}_{i,j}} d\tilde{W}_{i,j} \right).
\]

8.2.5. Proof of Proposition 3.4. From (8.5), integrating on the initial random time $A \sim \nu_{io}^{W,A}$, the probability that, under $\nu_0^W$, the randomized $\nu$-VRJP $X$ at time $t$ has performed $n$ jumps in infinitesimal time intervals $[t_l, t_l + dt_l)$, $l = 1, \cdots, n$ with $0 < t_1 < \cdots < t_n < t = t_{n+1}$, following the trajectory $\sigma_0 = i_0, \sigma_1, \cdots, \sigma_n = j_0$ is equal to
\[
\prod_{l=1}^{n} W_{\sigma_{l-1}, \sigma_l} e^{T_{\sigma_{l-1}}(t_l) + T_{\sigma_l}(t_l)} \frac{\tilde{F}_i^W}{\tilde{F}_0^W}
\]
where we remind that $\tilde{F}_i^W$ is defined in (8.3).

Remark that
\[
\prod_{l=1}^{n} W_{\sigma_{l-1}, \sigma_l} e^{T_{\sigma_{l-1}}(t_l) + T_{\sigma_l}(t_l)} = \left( \prod_{l=1}^{n} W_{\sigma_{l-1}, \sigma_l} e^{T_{\sigma_{l-1}}(t_l) + T_{\sigma_l}(t_l)} \right) \exp \left( \sum_{l=1}^{n} \left( T_{\sigma_l}(t_l) - T_{\sigma_{l-1}}(t_l) \right) \right),
\]
and that
\[
\sum_{l=1}^{n} \left( T_{\sigma_l}(t_l) - T_{\sigma_{l-1}}(t_l) \right) = -\sum_{l=0}^{n-1} \left( T_{\sigma_{l+1}}(t_{l+1}) - T_{\sigma_l}(t_l) \right) + T_{j_0}(t).
\]

Next, we observe that:
- If $\sigma_l \notin V_0$, then $T_{\sigma_l}(t_l) = T_{\sigma_l}(t_{l+1})$, since the process is at site $\sigma_l$ between times $t_l$ and $t_{l+1}$.
- If $\sigma_l = \sigma_l^* = i \in V_0$, then the sum of $T_{\sigma_l}(t_{l+1}) - T_{\sigma_l}(t_l)$ restricted on the indices $i$ such that $\sigma_l = i$ is equal to $T_i(t)$ (indeed, the local time $T_i$ is constant when the walk is not at vertex $i$),

which imply that (8.7) equals $T_{j_0}(t) - \sum_{i \in V_0} T_i(t)$.

This implies that the expression (8.6) is equal to
\[
\exp \left( T_{j_0}(t) - \sum_{i \in V_0} T_i(t) \right) \left( \prod_{l=1}^{n} W_{\sigma_{l-1}, \sigma_l} e^{T_{\sigma_{l-1}}(t_l) + T_{\sigma_l}(t_l)} dt_l \right) \frac{\tilde{F}_i^W}{\tilde{F}_0^W}.
\]

Setting $M_i(t) = \frac{1}{2} (T_i(t) + T_{i^*}(t))$, and using Proposition 8.2, (8.8) is equal to
\[
\frac{e^{M_i(t)}}{\prod_{i \in V_0} e^{M_i(t)}} \left( \prod_{l=1}^{n} W_{\sigma_{l-1}, \sigma_l} e^{T_{\sigma_{l-1}}(t_l) + T_{\sigma_l}(t_l)} dt_l \right) \frac{\tilde{F}_i^W}{\tilde{F}_0^W}.
\]

Changing to time $s = C(t)$, we have
\[
ds = e^{T_{X_i}(t) + T_{X_{i^*}}(t)} dt.
\]

We deduce that
\[
e^{T_{X_i}(t) + T_{X_{i^*}}(t)} - 1 = \int_0^t e^{T_{X_i}(u) + T_{X_{i^*}}(u)} (1_{X_{i^*} = i} + 1_{X_{i^*} = i^*}) du
\]
\[
= \int_0^s (1_{Z_{i^*} = i} + 1_{Z_{i^*} = i^*}) dv = \ell_{i^*}(s) + \ell_i^*(s).
\]
where $\ell^Z$ is the local time of the process $Z$. Hence,

$$(8.12) \quad e^{M_i(t)} = \sqrt{1 + \ell^Z_i(s) + \ell^Z_{\overline{i}}(s)},$$

Changing to time $s$ in the expression (8.9), the probability that, under $\mathbb{P}_i^W$, the time-changed randomized $\star$-VRJP $Z$ at time $s$ has performed $n$ jumps in infinitesimal time intervals $[s_l, s_l + ds_l)$, $l = 1, \ldots, n$ with $0 < s_1 < \cdots < s_n < s$, following the trajectory $\sigma_0 = i_0, \sigma_1, \ldots, \sigma_n = j_0$ is equal to

$$\sqrt{1 + \ell^Z_{j_0}(s) + \ell^Z_{\overline{j_0}}(s)} \frac{\tilde{W}_i^W(\ell^Z(s))}{\prod_{j \in V_0} \sqrt{1 + 2\ell^Z_i(s)}} \left( \prod_{l=1}^n W_{\sigma_{l-1}, \sigma_l} ds_l \right),$$

where $\tilde{W}_{i,j}(\ell^Z(s)) = W_{i,j} \sqrt{1 + \ell^Z_i(s) + \ell^Z_{\overline{i}}(s)} \sqrt{1 + \ell^Z_j(s) + \ell^Z_{\overline{j}}(s)}$.

Hence, the law of the process $Z$ on a time interval $[0, s]$ depends only on the number of crossings of edges and on the local time at final time $s$. This implies that $Z$ is partially exchangeable in the sense of [18], see [33] Proposition 1.

**Remark 8.3.** If one performs a similar computation under the non-randomized law $\mathbb{P}_i^W$, then the probability of a path depends on the whole local time $(T_i(t))$, not only on its projection $(M_i(t))$ on $S$, contrary to the outcome for the randomized VRJP, under law $\mathbb{P}_i^W$. Indeed, that local time $(T_i(t))$ cannot be expressed in terms of $\ell^Z_i(s)$: in fact we can prove that

$$(8.13) \quad e^{T_i(t)} = \sqrt{1 + \ell^Z_i(s) + \ell^Z_{\overline{i}}(s)}e^{B^1_i(s)},$$

see Remark 9.7 below. Here $B^1_i(s)$ is the functional defined in (4.4), which depends on the trajectory of $Z$ up to time $s$, not just on its final local time.

**9. Proof of Theorems 4.2 and 4.6.** In this section, we first prove Theorem 4.2 (the randomized case), from which we deduce Theorem 4.6.

9.1. **Notation.** We remind that $(X(t))_{t \geq 0}$ represents the canonical process on $\mathcal{D}([0, \infty), V)$, and that $\mathbb{P}_i^W$ is the law of the randomized $\star$-VRJP defined in Section 3. Besides, $Z = X \circ C^{-1}$ is the time changed process defined in Section 3. Under $\mathbb{P}_i^W$, $Z$ is the randomized $\star$-VRJP in exchangeable time-scale.

For $u \in \mathcal{U}_0^W$, let $P_{i_0}^{W,u}$ be the law on $\mathcal{D}([0, \infty), V)$, such that under $P_{i_0}^{W,u}$, $(X(t))$ is the process starting from $i_0$ which, conditioned on the past at time $t$, jumps from $i$ to $j$ at rate

$$W_{i,j} e^{u_j - u_i} e^{T_i(t) + T_j(t)}.\)$$

The following simple lemma shows that the definition of $P_{i_0}^{W,u}$ is consistent with the definition given in Section 4.2.

**Lemma 9.1.** Under $P_{i_0}^{W,u}$, $(Z_s)_{s \geq 0} = (X_{C^{-1}(s)})_{s \geq 0}$ is the Markov jump process starting at $i_0$ with jump rates $W_{i,j} e^{u_j - u_i}$.

**Proof.** As in the proof of Proposition 3.4, changing to time $s = C(t)$ we deduce that

$$ds = e^{T_{i_0}(t) + T_{i_0}(t)} dt,$$

where we note that, in the definition of $C(t)$ in Proposition 3.4, $e^{T_i(t) + T_i(t)}$ appears twice if $i \in V_1$. This implies $W_{i,j} e^{u_j - u_i} e^{T_i(t) + T_j(t)} dt = W_{i,j} e^{u_j - u_i} ds$. \qed
Next, define the process $\tilde{X}(t) = (X(t), T(t))$, which is the joint process of position and local time. By Corollary 3.3, under $P_{i_0}^W$, $(\tilde{X}(t))$ is a Markov process with generator

\begin{align}
L^W f(i, t) = \frac{\partial}{\partial t} f(i, t) + \sum_{j, i \to j} W_{i,j}^t \frac{F_{i}^{W^t}}{F_{i}^{W}} (f(j, t) - f(i, t)).
\end{align}

Denote by $P_{i_0, t^0}^W$ the law of the process $(\tilde{X}(t))$ with generator $L^W$, starting from initial value $(i_0, t^0)$.

Let $L^{W,u}$ be the generator of $(\tilde{X}(t))$ under the law $P_{i_0}^{W,u}$. Then

\begin{align}
L^{W,u} f(i, t) = \frac{\partial}{\partial t} f(i, t) + \sum_{j, i \to j} W_{i,j} e^{t_i \tau_i - t_j \tau_j} (f(j, t) - f(i, t)).
\end{align}

Also denote by $P_{i_0, t^0}^{W,u}$ the law of the Markov process with generator $L^{W,u}$, starting from initial value $(i_0, t^0)$.

9.2. Proof of Theorem 4.2 (i).

Step 1: Feynman-Kac identity

From now on, we fix $\varphi : U^W_0 \mapsto [0, \infty)$ a positive bounded measurable function with compact support in $U^W_0$. We denote by $C_0 \subseteq U^W_0$ its support. The main result of the step 1 of the proof is the following key Feynman-Kac identity.

**Lemma 9.2.** For all $(j, \tau) \in V \times \mathbb{R}^+_0$, define the function

\begin{align}
\Psi(j, \tau) = \int_{U^W_0} \varphi(u + \tau) \frac{1}{F_{j}^{W^\tau}} \mu_j^{W^\tau}(du),
\end{align}

where $\tau = (\tau_i)_{i \in V}$, $\tau_i = \tau_i - \frac{1}{|V|} \sum_{j \in V} \tau_j$. Then, for any starting point $i_0 \in V$,

\begin{align}
\Psi(i_0, 0) = E_{i_0}^W (\Psi(X_t, T(t))).
\end{align}

N.B.: By an elementary computation we have that $u + \tau \in U^W_0$ if $u \in U^W_0$ so that $\varphi(u + \tau)$ is well defined in (9.3).

We could prove that lemma by direct computation, and directly verify that $L^W(\Psi) = 0$. We use a more constructive approach in Lemma 9.3, as this also gives an insight on how the mixing measure has to be related to the law of the *-VRJP, in order to satisfy the Feynman-Kac identity. Our approach is also useful in the second part of the proof of Theorem 4.2.

Define, for all $u \in U^W_0$, $R^{W,u} : V \times \mathbb{R}^+_0 \rightarrow \mathbb{R}_+$ by

\begin{align}
R^{W,u}(i_0, \tau) = F_{i_0}^{W^\tau} e^{-\frac{1}{2} \sum_{i, j \to i} W_{i,j} (e^{\tau_i + \tau_j} - e^{-\tau_i + \tau_j}) \sum_{j \in V} e^{-\tau_j}} \prod_{i \in V} e^{\tau_i}.
\end{align}

**Lemma 9.3.** (i) For all $(i_0, \tau) \in V \times \mathbb{R}^+_0$, the Radon-Nykodym derivative of the randomized *-VRJP $\tilde{X}$ under the law $P_{i_0, \tau}^W$, with respect to the law of the Markov Jump Process $P_{i_0, \tau}^{W,u}$ on time interval $[0, t]$, is given by

\begin{align}
\left( \frac{dP_{i_0, \tau}^W}{dP_{i_0, \tau}^{W,u}} \right)_{[0,t]} = \frac{R^{W,u}(\tilde{X}(t))}{R^{W,u}(i_0, \tau)}.
\end{align}
(ii) Let $\tau \in \mathbb{R}_+^Y$, $i_0, j_0 \in V$. For any positive measurable test function $\phi$, we have

$$\int_{U_{i_0}^W} \phi(u) \frac{R_{W,u}(i_0, 0)}{R_{W,u}(j_0, \tau)} \frac{1}{F_{i_0}^W} \mu_{i_0}^W (du)$$

$$= \int_{U_{j_0}^W} \phi \left( \tilde{u}_i + \tau_i - \frac{1}{|V|} \sum_{j \in V} T_j \right) \frac{1}{F_{j_0}^W} \mu_{j_0}^W (d\tilde{u}).$$

**Proof of Lemma 9.3.** i) By a direct adaptation of (8.8) to the case where the initial local time is $T(0) = \tau$, the probability under $P_{i_0}^{W,\tau}$ that the probability under $P_{i_0}^{W,\tau}$ that the randomized $\ast$-VRJP $X$ at time $t$ has performed $n$ jumps at times in $[t_l, t_l + dt_l]$, $l = 1, \cdots, n$ with $0 < t_1 < \cdots < t_n < t$, following the trajectory $\sigma_0 = i_0, \sigma_1, \cdots, \sigma_n = j_0$ is equal to

$$\exp \left( T_{X(t),*}(t) - \tau_0 - \sum_{i \in V_0} (T_i(t) - \tau_i) \right) \left( \prod_{l=1}^n W_{\sigma_{l-1}, \sigma_l} e^{T_{\sigma_{l-1}, \sigma_l}(t_l) + T_{\sigma_{l-1}, \sigma_l}(t_l)} dt_l \right) \frac{F_{j_0}^{W,T(t)}}{F_{i_0}^W}.$$ 

which can be written, using definition (8.3), as

$$\left( \prod_{l=1}^n W_{\sigma_{l-1}, \sigma_l} e^{T_{\sigma_{l-1}, \sigma_l}(t_l) + T_{\sigma_{l-1}, \sigma_l}(t_l)} dt_l \right) \frac{e^{T_{X(t),*}(t) - \tau_0}}{\prod_{l \in V} e^{T(t) - \tau_l}} e^{\frac{1}{2} \sum_{i,j,i \rightarrow j} W_{i,j} (e^{T_{i,j}(t) + T_{i,j}(t) - e^{T_{i,j}}}) \frac{F_{j_0}^{W,T(t)}}{F_{i_0}^W}}.$$

On the other hand, the probability that, under $P_{i_0}^{W,u}$, $X$ follows the same path is equal to

$$\exp \left( - \int_0^t \sum_{j,X(s) \rightarrow j} W_{X(s),j} e^{u_{j,s} - u_{X(s),s}} e^{T_{X(s),s}(s) + T_{X(s),s}(s)} ds \right) \times \left( \prod_{l=1}^n W_{\sigma_{l-1}, \sigma_l} e^{T_{\sigma_{l-1}, \sigma_l}(t_l) + T_{\sigma_{l-1}, \sigma_l}(t_l)} e^{u_{\sigma_l} - u_{\sigma_{l-1}}} dt_l \right).$$

Indeed, the first expression comes from the fact that under $P_{i_0}^{W,u}$, $X$ jumps from $i$ to $j$, $i \rightarrow j$, at a rate $W_{i,j} e^{T_{i,j}(t) + T_{i,j}(t)} e^{u_{j} - u_i}$, conditionally on the past at time $t$, so that the jump rate at time $t$ is

$$\sum_{j,X(t) \rightarrow j} W_{X(t),j} e^{u_{j,t} - u_{X(t),t}} e^{T_{X(t),t}(t) + T_{X(t),t}(t)}.$$ 

This explains the integral in the first exponential term. The product comes from the probability to jump in time interval $[t_l, t_l + dt_l]$. Now note that

$$\frac{d}{dt} \left( \sum_{i,j:i \rightarrow j} W_{i,j} e^{u_{j} - u_{i}} e^{T_{i,j}(t) + T_{i,j}(t)} \right) =$$

$$\sum_{j,X(t) \rightarrow j} W_{X(t),j} e^{u_{j,t} - u_{X(t),t}} e^{T_{X(t),t}(t) + T_{X(t),t}(t)} + \sum_{j \in V} W_{X(t),j} e^{u_{j,t} - u_{X(t),t}} e^{T_{X(t),t}(t) + T_{X(t),t}(t)}.$$ 

But, since $u \in U_{i_0}^W$, we have, for all $i \in V$

$$\sum_{j,i \rightarrow j} W_{i,j} e^{u_{j} - u_{i}} = \sum_{j,j \rightarrow i} W_{j,i} e^{u_{j} - u_{i}} = \sum_{j,i \rightarrow j} W_{i,j} e^{u_{j} - u_{i}}.$$
This implies that
\[
\int_0^t \sum_{j : X(s) \to j} W_{X(s)j} e^{u_j - u_{X(s)} - T_{X(s)}^s} \, ds = \frac{1}{2} \sum_{i,j : i \to j} W_{i,j} e^{u_j - u_i - T^s} (e^{T^s} - e^{-T^s}).
\]
Hence, (9.6) is equal to (9.7).

(ii) From the definition of \( \mu_{i_0}^W \) and \( R_{i, u}^W \), for \( i_0, j_0 \in V, \tau \in \mathbb{R}^+_V \),
\[
\frac{R_{i, u}^W(i_0, 0)}{R_{j, u}^W(j_0, \tau)} \frac{1}{F_{i_0}^W} \mu_{i_0}^W(du) = e^{-\frac{1}{2} \sum_{i,j : i \to j} W_{i,j} (\tau_i - \tau_j)} e^{\sum_{i \in V_0} u_i - \sum_{i \in V_0} u_i - \tau} \frac{\sqrt{D(W_{i,j})}}{\sqrt{2\pi |V_0|}} \frac{1}{\det_A(-K^u)} \sigma_{i_0}^W(du)
\]

Changing from variables \((u)_{i \in V}\) to \((\tilde{u})_{i \in V}\), given by
\[
\tilde{u}_i := u_i - \tau_i + \frac{1}{|V|} \sum_{j \in V} \tau_j, \quad \forall i \in V,
\]
we deduce:
\[
e^{-\frac{1}{2} \sum_{i,j : i \to j} W_{i,j} (\tau_i - \tau_j)} e^{\sum_{i \in V_0} u_i - \sum_{i \in V_0} u_i - \tau} \frac{\sqrt{D(W_{i,j})}}{\sqrt{2\pi |V_0|}} \frac{1}{\det_A(-K^u)} \sigma_{i_0}^W(du)
\]

and
\[
\sqrt{D(W_{i,j})} = e^{-\frac{1}{2} \sum_{i \in V} \tau_i} \sqrt{D(W_{i,j} + \tilde{u})}, \quad \det_A(K^u) = e^{\frac{-2 \text{dim}(A)}{|V|} \sum_{i \in V} \tau_i} \det_A(-K^{\tilde{u}}).
\]

Moreover, \( u \in U_0^W \) iff \( \tilde{u} \in U_0^{W_T} \) and \( \sigma_{i_0}^W(du) = \sigma_{i_0}^{W_T}(d\tilde{u}) \), see (2.2). Since \(|V_0| = |V| - 2 \text{dim}(A)\), this concludes the proof. \( \square \)

**PROOF OF LEMMA 9.2.** By Lemma 9.3 (ii) applied to \( \phi = \varphi \), we have
\[
\mathbb{E}_{i_0}^W \left( \int_{U_0^W} \varphi(u + T(t)) \frac{1}{F_{X(t)}^W} \mu_{X(t)}^W(du) \right) = \mathbb{E}_{i_0}^W \left( \int_{U_0^W} \varphi(u + T(t)) \frac{R_{i, u}^W(i_0, 0)}{R_{i, u}^W(X(t), T(t))} \frac{1}{F_{i_0}^W} \mu_{i_0}^W(du) \right)
\]

\[
= \int_{U_0^W} \varphi(u) \mathbb{E}_{i_0}^W \left( \frac{R_{i, u}^W(i_0, 0)}{R_{i, u}^W(X(t), T(t))} \right) \frac{1}{F_{i_0}^W} \mu_{i_0}^W(du)
\]

\[
= \int_{U_0^W} \varphi(u) E_{i_0}^{W_{i_0}}(1) \frac{1}{F_{i_0}^W} \mu_{i_0}^W(du) = \Psi(i_0, 0),
\]
Step 2: Asymptotic Gaussian estimates.

The strategy is now to prove that $\psi(X_t, T(t))$ converges a.s. to $\varphi(U)$ where $U$ is the limit defined in Theorem 4.2 i), and to obtain a good bound on $\psi(X_t, T(t))$ to apply dominated convergence. Remind that, at this stage of the proof, we do not know that $\frac{1}{F_{t_0}} \mu_W^t$ is a probability measure, so that we do not have an obvious bound on $\psi(X_t, T(t))$, even though $\varphi$ is bounded.

Following the notation in Lemma 9.2, we set

$$\mathcal{T}(t) = T(t) - t/|V| \in \mathcal{H}_0 \quad \text{and} \quad \mathcal{P}(t) = p_{tU}^W(\mathcal{T}(t)).$$

the projection of $\mathcal{T}(t)$ on the limiting manifold $\mathcal{U}_0 W$. Besides we set $H(t) = \mathcal{P}(t) + t/|V|$. Let $A(t) := \mathcal{H}(t) - \mathcal{T}(t)$ which is in $\mathcal{A}$ by Lemma 2.3. We also have $A(t) = H(t) - T(t)$. Lemma 2.1 yields, with the notation of Theorem 4.2 i), that

$$\lim_{t \to \infty} A + \mathcal{T}(t) = U.$$ 

This implies subsequently, using $U \in \mathcal{U}_0 W$, that

$$\lim_{t \to \infty} p_{tU}^W(A + \mathcal{T}(t)) = \lim_{t \to \infty} \mathcal{P}(t) = U,$$

and therefore that $\lim_{t \to \infty} A(t) = A$.

If $u \in \mathcal{U}_0 W$, set

$$\eta(W^u) = \sqrt{\frac{|V|}{2\pi}} \frac{V^{|V|/2}}{\det(K^W)}.$$

With this notation, for all $(i_0, \tau) \in V \times \mathbb{R}_+$, we have, using $\mathcal{U}_0 W^\tau = \mathcal{U}_0 W_\tau$,

$$\psi(i_0, \tau) = \frac{1}{F_{i_0}^\tau} \int_{\mathcal{U}_0 W_\tau} \varphi(u + \mathcal{T}) e^{u_\tau \mathcal{T}} e^{-\sum_{i \in \mathcal{V}_0} u_i e^{-\frac{1}{2} \sum_{(i,j) \in E} W_{i,j} (e^{u_j} - u_j - 1)}} \eta(W^u + \mathcal{T}) d\sigma_{\mathcal{U}_0 W_\tau}(u)$$

$$\psi(i_0, \tau) = \frac{1}{F_{i_0}^\tau} \int_{\mathcal{U}_0 W_\tau} \varphi(u + \mathcal{T}) e^{u_\tau \mathcal{T}} e^{-\sum_{i \in \mathcal{V}_0} u_i e^{-\frac{1}{2} \sum_{(i,j) \in E} W_{i,j} e^{u_j} - u_j}} \eta(W^u + \mathcal{T}) d\sigma_{\mathcal{U}_0 W_\tau}(u).$$

Using that

$$F_{i_0}^{H\tau(t)} = e^{A\tau(t)} F_{i_0}^{H\tau(t)}$$

(see Proposition 8.2), and changing to variable $\tilde{u}_i = u_i - A_i(t)$, we have $\tilde{u} \in \mathcal{U}_0 W^{\tilde{\tau}(t)}$ and

$$\psi(i_0, T(t)) = \frac{1}{F_{i_0}^{H\tau(t)}} \int_{\mathcal{U}_0 W^{\tilde{\tau}(t)}} \varphi(\tilde{u} + \mathcal{T}(t)) e^{\tilde{u}_\tau \mathcal{T}} e^{-\sum_{i \in \mathcal{V}_0} \tilde{u}_i e^{-\frac{1}{2} \sum_{(i,j) \in E} W_{i,j} (e^{\tilde{u}_j} - \tilde{u}_j - 1)}} \eta(W^{H\tau(t) + \tilde{u}}) d\sigma_{\mathcal{U}_0 W^{\tilde{\tau}(t)}}(\tilde{u})$$

$$\psi(i_0, T(t)) = \frac{1}{F_{i_0}^{H\tau(t)}} \int_{\mathcal{U}_0 W^{\tilde{\tau}(t)}} \varphi(\tilde{u} + \mathcal{T}(t)) e^{\tilde{u}_\tau \mathcal{T}} e^{-\sum_{i \in \mathcal{V}_0} \tilde{u}_i e^{-\frac{1}{2} \sum_{(i,j) \in E} W_{i,j} (e^{\tilde{u}_j} - \tilde{u}_j - 1)}} \eta(W^{H\tau(t) + \tilde{u}}) d\sigma_{\mathcal{U}_0 W^{\tilde{\tau}(t)}}(\tilde{u})$$

:= \frac{\psi(i_0, H(t))}{F_{i_0}^{H\tau(t)}}.
where, by definition, $\tilde{\Psi}(i_0, H(t))$ is the integral term in the penultimate expression.

Set $w = \inf_{(i,j) \in E} W_{i,j}$, and

$$c_0 = \max_{u \in C_0} \max_{(i,j) \in E} |\nabla u_{i^*, j^*}|,$$

where $\nabla u_{i^*, j^*} = u_{j^*} - u_{i^*}$, which is finite since $C_0$ is compact.

We now introduce the compact set $C_1$ of all $u \in U_0^W$ such that

$$\max_{(i,j) \in E} |\nabla u_{i^*, j^*}| \leq c_0 + 1.$$

Note that the interior of $C_1$ contains $C_0$, and that, for all $u' \in C_1^r$,

$$\max_{(i,j) \in E} |\nabla u'_{i^*, j^*}| > c_0 + 1.$$

The strategy is to treat separately the case $\overline{P}(t) \in C_1$, where we use the compactness of $C_1$ to have uniform estimates, and the case $\overline{P}(t) \in C_1^c$, where we use that $\supp(\varphi) = C_0$ to prove a uniform convergence to 0 of $\tilde{\psi}(X(t), T(t))$.

Recall that $P_S$ and $P_A$ are the orthogonal projections on subspaces $S$ and $A$, and that $K^u$ was introduced in (4.1). For $u \in U_0^W, y \in \mathbb{R}^V$,

$$t(K^u)(y^*)(i^*) = \sum_{j:j \to i^*} W_{j,i^*}(y_{j^*} - y_i) = \sum_{j:j \to i^*} W_{i,j}(y_{i} - y_{j}) = K^u(y)(i),$$

hence

$$P_S K^u P_S = P_S \left( \frac{1}{2} (K^u + t(K^u)) \right) P_S,$$

$$P_A K^u P_A = P_A \left( \frac{1}{2} (K^u + t(K^u)) \right) P_A,$$

which implies that $P_S K^u P_S$ and $P_A K^u P_A$ are symmetric. For simplicity, we denote as before by $\det_{S_0}((-K^u)^{-1})$ the determinant of the operator $P_{S_0}(-K^u)^{-1}P_{S_0}$ restricted to $S_0$, and by $\det_A(-K^u)$ the determinant of the operator $P_A(-K^u)P_A$ restricted to $A$.

**Lemma 9.4.** We have, when $t \to \infty$,

$$\mathbbm{1}_{\overline{P}(t) \in C_1} \tilde{\psi}(X(t), H(t)) \sim \mathbbm{1}_{\overline{P}(t) \in C_1} e^{-\frac{1}{2} |V_1| \sqrt{2\pi} |\supp(\varphi)|^{-1} \chi(\overline{P}(t)) \eta(W(\overline{P}(t))) \sqrt{\det_{S_0}(-K(\overline{P}(t))^{-1})}},$$

with a uniform control of the same order for each term.

**Lemma 9.5.** There exist positive constants $c$ and $c'$ such that

$$\mathbbm{1}_{\overline{P}(t) \in C_1^c} \tilde{\psi}(X(t), H(t)) \leq c \exp \left( -c' e^{t/|V_1|} \right).$$

**Proof of Lemma 9.4.** When $t$ tends to $\infty$, $W^{H(t)} = e^{2t/N} W^{\overline{P}(t)}$ is equivalent to $e^{2t/N} W^U$. The main exponential term in the integrand of $\tilde{\Psi}(i, H(t))$ is

$$-e^{2t/N} \frac{1}{2} \sum_{(i,j) \in E} W_{i,j}(t)(e^{u_{j^*}} - u_{i^*} - 1).$$

The maximum of the last expression is obtained for $u = 0$ since $H(t) \in U_0^W$, by Lemma 2.3. This implies that the first order Taylor expansion cancels out. A second order expansion yields

$$\sum_{(i,j) \in E} W_{i,j}(t)(e^{u_{j^*}} - u_{i^*} - 1) = \frac{1}{2} \sum_{(i,j) \in E} W_{i,j}(t)(u_{j^*} - u_{i^*})^2 + o(\|u\|^2).$$
Besides, the remainder term $o(\|u\|^2)$ is uniform for $\overline{H}(t)$ in $C_1$ and $u$ such that $\overline{H}(t) + u \in C_0$.

We recall from Lemma 8.1 that $T_0^{W_0(\pi)} = \left(tK\overline{\pi}(t)\right)^{-1}(S_0)$ is the tangent space at 0 of $U_0^{W_0(\pi)}$. We denote by $\sigma_{T_0^{W_0(\pi)}}$ the volume measure induced by $\sigma_{tW_0(\pi)}$ so that, by (2.1), $\sigma_{T_0^{W_0(\pi)}}(B) = \lambda S_0(P_S(B))$. For large $t$, the integral in $\Psi(i, H(t))$ concentrates around $u = 0$.

Since we have a uniform quadratic estimate of the exponential term, we can localize the integral in a ball of size $B(0, e^{(1+\epsilon)t/N})$, change to variables $x_i = e^{t/N} \bar{u}_i$, and integrate on the tangent plane $T_0^{W_0(\pi)}$ (details are easy and left to the reader). Using that $U_0^{W_0(\pi)}$ has dimension $|V_0| + |V_1| - 1$ and that $\eta$ is $C^1$ homogeneous, $1_{\overline{\pi}(t) \in C_1} (\Psi(i, H(t)))$ is equivalent to

$$
e^{-\frac{i}{2}\|V_1\|} 1_{\overline{\pi}(t) \in C_1} \varphi(\overline{\pi}(t)) \eta(W\overline{\pi}(t)) \int_{T_0^{W_0(\pi)}} e^{-\frac{1}{2} \sum_i \sum_{j,j-i} W_{i,j}^{(0)} (x_i - x_j)^2} d\sigma_{T_0^{W_0(\pi)}}(x)$$

$$= e^{-\frac{i}{2}\|V_1\|} 1_{\overline{\pi}(t) \in C_1} \varphi(\overline{\pi}(t)) \eta(W\overline{\pi}(t)) \int_{T_0^{W_0(\pi)}} e^{-\frac{1}{2} \sum_i \sum_{j,j-i} W_{i,j}^{(0)} (x_i - x_j)^2} d\sigma_{T_0^{W_0(\pi)}}(x)$$

$$= e^{-\frac{i}{2}\|V_1\|} 1_{\overline{\pi}(t) \in C_1} \varphi(\overline{\pi}(t)) \eta(W\overline{\pi}(t)) \int_{T_0^{W_0(\pi)}} e^{\frac{i}{2}(tK\overline{\pi}(t)x,x)} d\sigma_{T_0^{W_0(\pi)}}(x).$$

By Lemma 8.1, we use the change of variables $y = t(K\overline{\pi}(t))^2 \in S_0$, which yields that the previous expression is equal to

$$e^{-\frac{i}{2}\|V_1\|} 1_{\overline{\pi}(t) \in C_1} \varphi(\overline{\pi}(t)) \eta(W\overline{\pi}(t)) \det S_0 \left(-(K\overline{\pi}(t))^{-1}\right) \int_{S_0} e^{\frac{i}{2}((K\overline{\pi}(t))^{-1}y,y)} d\lambda S_0(dy).$$

(9.13)

$$= e^{-\frac{i}{2}\|V_1\|} 1_{\overline{\pi}(t) \in C_1} \sqrt{2\pi}^{V_0|V_1|^{-1}} \varphi(\overline{\pi}(t)) \eta(W\overline{\pi}(t)) \det S_0 \left(-(K\overline{\pi}(t))^{-1}\right)^{\frac{i}{2}}$$

since $\overline{H}(t) \to U$ and since $\varphi$ is supported on $C_0 \subseteq C_1$.

We can apply a very similar reasoning to $F_i^{W_0(\pi)}$. Changing variables $a$ to $\bar{a} = e^{t/N}a$, we deduce

$$1_{\overline{\pi}(t) \in C_1} F_i^{W_0(\pi)} \sim 1_{\overline{\pi}(t) \in C_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{i}{2}\|V_1\|} 1_{\overline{\pi}(t) \in C_1} \int_{\mathcal{A}} e^{-\frac{i}{2} \sum_i \sum_{j,j-i} W_{i,j}^{(0)} (\bar{a}_j - \bar{a}_i)^2} d\bar{a}$$

$$= 1_{\overline{\pi}(t) \in C_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{i}{2}\|V_1\|} 1_{\overline{\pi}(t) \in C_1} \int_{\mathcal{A}} e^{\frac{i}{2}(\bar{a}, K\overline{\pi}(\bar{a}))} d\bar{a}$$

where as before $(\cdot, \cdot)$ is the usual scalar product on $\mathbb{R}^V$. Let us denote by $\lambda_\mathcal{A}$ the Euclidean volume measure on $\mathcal{A}$. Then we have $d\lambda_\mathcal{A} = \sqrt{2\|V_1\|} \prod_{i \in V_1} da_i$, hence

$$1_{\overline{\pi}(t) \in C_1} F_i^{W_0(\pi)} \sim 1_{\overline{\pi}(t) \in C_1} \frac{\sqrt{2\|V_1\|}}{\sqrt{2\pi}} e^{-\frac{i}{2}\|V_1\|} 1_{\overline{\pi}(t) \in C_1} \int_{\mathcal{A}} e^{\frac{i}{2}(\bar{a}, K\overline{\pi}(\bar{a}))} d\lambda_\mathcal{A}(a)$$

(9.14)

$$= \sqrt{2\|V_1\|} e^{-\frac{i}{2}\|V_1\|} 1_{\overline{\pi}(t) \in C_1} \det \mathcal{A}(-K\overline{\pi}(t))^{-\frac{i}{2}}.$$

\[ \square \]

**Proof of Lemma 9.5.** In this proof, $c, c', c''$ are positive constants (depending only on the parameters of the model and on $\varphi$), whose values can change from line to line. We first show an exponential upper bound for $1_{\overline{\pi}(t) \in C_1} \psi(i, H(t))$ valid for any $i \in V$, hence in particular for $X_i$. 

We start by observing that, \( \varphi \) having compact support \( C_0 \), we have a uniform bound
\[
1_{u+\overline{W}(t) \in C_0} \varphi(u + \overline{W}(t)) \eta(u + \overline{W}(t)) \leq c1_{u+\overline{W}(t) \in C_0}.
\]

Besides, in \( \tilde{\psi}(i, H(t)) \), we integrate on the compact set \( C_0 - \overline{W}(t) \), which has measure \( \lambda_{\sigma_0}(P\overline{W}(C_0 - \overline{W}(t))) = \lambda_{\sigma_0}(P\overline{W}(C_0)) \), see (2.1). Hence
\[
\tilde{\psi}(i, H(t)) \leq c \max_{u \in \mathcal{U}_0^W, u+\overline{W}(t) \in C_0} \sum_{j \in \mathcal{E}} u_j e^{-\frac{1}{2} e^{2t/|V|} \sum_{i,j \in \mathcal{E}} W_{i,j}(t) (e^{u_j} - e^{u_i} - 1)}
\]
\[
\leq ce^{(|V_0|+1) \max_{u \in \mathcal{V}} |\overline{W}(t)|} \max_{u \in \mathcal{U}_0^W, u+\overline{W}(t) \in C_0} \sum_{j \in \mathcal{E}} u_j e^{-\frac{1}{2} e^{2t/|V|} \sum_{i,j \in \mathcal{E}} W_{i,j}(t) (e^{u_j} - e^{u_i} - 1)}.
\]

Also note that
\[
(9.15) \quad \max_{i \in \mathcal{V}} |\overline{W}(t)| \leq c't + c.
\]

Indeed, \( T_i(t) \leq t \) implies \( |\overline{W}(t)| \leq t \). Also, as \( \overline{W}(t) = \overline{W}(t) + A(t) \) and \( A(t) \in \mathcal{A} \), we have \( |\overline{W}(t) + \overline{W}(t)| = |\overline{W}(t) + \overline{W}(t)| \leq 2t \). Besides, Lemma 2.3 implies
\[
\sum_{i,j \in \mathcal{E}} W_{i,j} e^{\overline{W}(t)} \overline{W}(t) \leq \sum_{i,j \in \mathcal{E}} W_{i,j} e^{\overline{W}(t)} \overline{W}(t),
\]

thus, for \((i,j) \in \mathcal{E}, \overline{W}(t) + \overline{W}(t) \leq 2t + c \) for \( t \) large enough, which implies subsequently that for \((i,j) \in \mathcal{E}, \overline{W}(t) \leq \overline{W}(t) + \overline{W}(t) - \overline{W}(t) - \overline{W}(t) \leq 2t + c \). We deduce (9.15) since \( \sum_{i \in \mathcal{V}} |\overline{W}(t)| = 0 \), and since the graph \( G \) is strongly connected.

Hence
\[
(9.16) \quad \tilde{\psi}(i, H(t)) \leq ce^{c't} \max_{u \in \mathcal{U}_0^W, u+\overline{W}(t) \in C_0} \sum_{j \in \mathcal{E}} u_j e^{-\frac{1}{2} e^{2t/|V|} \sum_{i,j \in \mathcal{E}} W_{i,j}(t) (e^{u_j} - e^{u_i} - 1)}.
\]

The aim is now to prove that
\[
(9.17) \quad \min_{u \in \mathcal{U}_0^W, u+\overline{W} \in C_0} \left( \sum_{(i,j) \in \mathcal{E}} W_{i,j}(t) (e^{u_j} - e^{u_i} - 1) \right) > 0.
\]

We now fix \( \overline{W} \in \mathcal{C}_j \) and \( u \in \mathcal{U}_0^W \) such that \( u + \overline{W} \in C_0 \). Since \( \overline{W} \in \mathcal{U}_0^W \), we have
\[
\sum_{(i,j) \in \mathcal{E}} W_{i,j}(t) (e^{u_j} - e^{u_i} - 1) = \sum_{(i,j) \in \mathcal{E}} W_{i,j}(t) (e^{u_j} - e^{u_i} - 1).
\]

Remark that \( e^s - s - 1 \) is positive convex and minimal at \( s = 0 \), and \( e^s - s - 1 \geq \varepsilon_0 \) for \( |s| \geq 1 \) and that \( e^s - s - 1 \geq \varepsilon \) for \( s \geq 1 \), for some constants \( \varepsilon_0 > 0 \) and \( \varepsilon > 0 \).

Consider the set
\[
E' = \{(i,j) \in \mathcal{E}, \text{ such that } |\nabla \overline{W}_{i,j}| \leq c_0 + 1 \text{ and } |\nabla \overline{W}_{i,j}| \leq c_0 + 1 \}.
\]

Remark that \( E' \) can be considered as a subset of \( \overline{E} \) since \( (i,j) \in E' \) implies \( (j^*, i^*) \in E' \). By (9.11), since \( \overline{W} \in \mathcal{C}_j \), we know that \( E' \) is strictly contained in \( \overline{E} \).

Using \( \sum_{i \in \mathcal{V}} \overline{W}_{i} = 0 \), we can find \( i_0 \in \mathcal{V} \) such that \( \overline{W}_{i_0} + \overline{W}_{i_0} \geq 0 \). Consider now the set \( V' \) of vertices which can be reached from \( i_0 \) by a directed path in \( E' \). If \( j_0 \in V' \) then \( H_{j_0} + H_{j_0} \geq -(c_0 + 1) |\overline{E}| + 1 \).

Since \( E' \) is strictly included in \( \overline{E} \) there exist \( j_0 \in V' \) and \( j_1 \in V \) such that \( (j_0, j_1) \in E \setminus E' \).

If \( |\nabla \overline{W}_{i_0,j_0}| > c_0 + 1 \), then \( |\nabla u_{i_0,j_1}| > c_0 + 1 \), using \( u + \overline{W} \in C_0 \) and (9.10).

We consider two cases: if \( H_{j_0} + H_{j_1} \geq -(c_0 + 1)(2|\overline{E}| + 1) \), then
\[
W_{j_0,j_1}(e^{\nabla u_{i_0,j_1}} - e^{\nabla u_{j_0,j_1}} - 1) \geq \varepsilon_0 e^{-(c_0 + 1)(2|\overline{E}| + 1)},
\]
while if \( H_{j_0} + H_j \leq -(c_0 + 1)(2E) + 1 \) then \( \nabla \overline{H}_{j_0,j} \leq -(c_0 + 1) \) since, which implies subsequently that \( \nabla u_{j_0,j} \leq 1 \) since \( |\nabla (u + \overline{H})|_{j_0,j} | \leq c_0 \), using \( u + \overline{H} \in C_0 \). This implies that

\[
W^{\overline{H}}_{j_0,j} (e^{\nabla u_{j_0,j}^*} - \nabla u_{j_0,j} - 1) \geq \varepsilon_1 W_{j_0,j_1} e^{\overline{H}_{j_0,j} + \overline{H} \partial u} e^{\nabla (u + \overline{H})_{j_0,j}^*} \geq \varepsilon_1 we^{-(c_0 + 1)2E}e^{-(c_0 + 1)2E}e^{-(c_0 + 1)}.
\]

On the other hand, if \( |\nabla \overline{H}_{j_1,j_0} | \leq c_0 + 1 \) and \( |\nabla \overline{H}_{j_1,j_0} | > c_0 + 1 \) we have

\[
e^{H_{j_0} + H_j} = e^{H_{j_0} + H_j \partial u} e^{\nabla \overline{H}_{j_0,j}^*} \geq e^{-(c_0 + 1)} e^{H_{j_0} + H_{j_0}^*},
\]

hence, considering the term associated with the edge \((j_1^*, j_0^*)\),

\[
W^{\overline{H}}_{j_1^*,j_0^*} (e^{\nabla u_{j_1,j_0} - \nabla u_{j_1,j_0} - 1} \geq \varepsilon_0 W_{j_0,j_1} \geq \frac{1}{2} W_{j_0,j_1} e^{H_{j_0} + H_{j_0}^*} e^{-(c_0 + 1)} \geq \varepsilon_0 we^{-(c_0 + 1)2E}e^{-(c_0 + 1)2E}e^{-(c_0 + 1)}.
\]

This concludes the proof of the estimate (9.17).

Combining (9.16) and (9.17), we deduce that

\[
|\cdot|_{\Psi} (i, H(t)) \leq c e^{c t} e^{-c |t|^{V_i}}.
\]

We now need a lower bound for \( F^W(t) \). Since \( \overline{H}(t) \in U^W \), the exponential term of the integrand can be rewritten as follows:

\[
- e^{2|t|V_i} \frac{1}{2} \sum_{(i,j) \in E} W_{i,j} (e^{a_{i,j} - a_{i,j}^*} - 1) = - e^{2|t|V_i} \frac{1}{2} \sum_{(i,j) \in E} W_{i,j} (e^{a_{i,j}^* - a_{i,j}^*} - \nabla a_{i,j}^* - 1).
\]

By (9.15) we know that \( e^{2|t|V_i} e^{a_{i,j}^* - a_{i,j}^*} \leq e^{2e t} \). We set \( A(t) = \{ a \in \mathcal{A}, a_i \leq e^{-ct} \} \) with the same constant \( c \). Using that \( e^a - s - 1 \leq 2s^2 \) for \( s \leq 1 \), we deduce

\[
F^W(t) \geq c' \int_{\mathcal{A}(t)} e^{-\sum_{i,j} W_{i,j}} da = c' (2e^{-ct}) |V_i| e^{-\sum_{i,j} W_{i,j}}.
\]

Combining with (9.18) concludes the proof.

**Step 3: final computations.**

From (9.4) and (9.9), and since \( \overline{H}(t) \rightarrow U \), we can apply the dominated convergence theorem to deduce that

\[
\psi(i_0, 0) = \lim_{t \rightarrow \infty} W_{i_0} \left( \psi(X(t), T(t)) \right)
\]

\[
= \lim_{t \rightarrow \infty} W_{i_0} \left( 1_{\overline{H}(t) \in C_1} \psi(X(t), T(t)) \right)
\]

\[
= \mathbb{E}_{i_0} \left( 2^{V_i} \eta(U) \mathbb{E}_{i_0} \left[ (V_i) \eta(U) \sqrt{\det \mathcal{S}_0 (-KU)^{-1}) \det \mathcal{A}(-KU) \right] \right).
\]

Indeed, we have a uniform control in Lemma 9.4, since the functions involved are continuous in \( \overline{H}(t) \). This implies that \( 1_{\overline{H}(t) \in C_1} \psi(X(t), T(t)) \) is bounded, and Lemma 9.5 yields a control outside \( C_1 \)

Coming back to the definition of \( \eta(U) \), see (9.8), we have

\[
\sqrt{2^{V_i} \eta(U) \sqrt{\det \mathcal{S}_0 (-KU)^{-1}) \det \mathcal{A}(-KU) = \sqrt{|U| \sqrt{D(U) \sqrt{\det \mathcal{S}_0 (-KU)^{-1}) \det \mathcal{A}(-KU)}}}}.
\]

For simplicity, we simply write \( K \) for \( KU \) below. Firstly, we have

\[
\frac{\det \mathcal{A}(-K)}{\det \mathcal{S}_0 (-KU)} = \det H_{\mathcal{R}}(-K).
\]
Indeed, if we write \( K \) by blocks on \( \mathcal{H}_0 = A \oplus S_0 \) as
\[
K = \begin{pmatrix}
P_A K P_A & P_A K P_{S_0} \\
P_{S_0} K P_A & P_{S_0} K P_{S_0}
\end{pmatrix},
\]
we note that
\[
\begin{pmatrix}
P_A K P_A & P_A K P_{S_0} \\
0 & \text{Id}_{S_0}
\end{pmatrix} K^{-1} = \begin{pmatrix}
\text{Id}_A & 0 \\
P_{S_0} K^{-1} P_A & P_{S_0} K^{-1} P_{S_0}
\end{pmatrix}.
\]

Next, we prove that
(9.20) \[ |V| D(W^U) = \det_{\mathcal{H}_0}(-K). \]
In order to compute the determinant of the operator \( K|_{\mathcal{H}_0} \), we choose a basis of \( \mathcal{H}_0 \): a convenient one in this context is the basis \((e_i = \delta_i - \delta_{i_0})_{i \neq i_0}\). In this basis, we have
\[
K(e_i) = \sum_{j \neq i_0} (K_{i,j} - K_{i_0,j}) e_j.
\]

We deduce
\[
det_{\mathcal{H}_0}(-K) = det(-(K_{i,j} - K_{i_0,j})_{i \neq i_0, j \neq i_0}).
\]
Since the sum of columns of \( K \) is null, summing all columns in a column \( i_1 \), now note that
\[
det(-(K_{i,j} - K_{i_0,j})_{i \neq i_0, j \neq i_0}) = det(-(K_{i,j} - K_{i_0,j})_{i \neq i_0, j \neq i_0, i_1}, |-V| K_{i_0,j})_{i \neq i_0, j = i_1})
\]
\[
= |V| det(-(K_{i,j} - K_{i_0,j})_{i \neq i_0, j \neq i_0, i_1}, (-K_{i_1,j})_{i \neq i_0, j = i_1})
\]
\[
= |V| det(-(K_{i,j})_{i \neq i_0, j \neq i_0}) = |V| D(W^U).
\]
We conclude that \( \psi(i_0, 0) = \mathbb{E}^W_{i_0} (\varphi(U)) \), which implies that \( \frac{1}{F^W_{i_0}} \mu^W_{i_0} \) is the law of \( U \), hence in particular that it must be a probability measure.

9.3. Proof of Theorem 4.2 (ii). The statement (ii) is equivalent to the following equality of probabilities,
\[
\int_{\mathcal{U}_t^0} \mathbb{P}^W_{i_0} (\cdot) \frac{1}{F^W_{i_0}} \mu^W_{i_0} (du) = \mathbb{P}^W_{i_0} (\cdot),
\]
since under \( P^W_{i_0} \), \((Z_s) = (X_{C^{-1}(s)}) \) is a Markov jump process with jump rate \( W_{i,j} e^{u_j - u_i} \).

Let \( \phi((X_v)_{v \leq t}) \) be a positive measurable test function of the trajectory of \( X \) up to time \( t \). By Lemma 9.3 (i), we have
\[
\int_{\mathcal{U}_t^0} \mathbb{E}^W_{i_0} (\phi((X_v)_{v \leq t})) \frac{1}{F^W_{i_0}} \mu^W_{i_0} (du) = \int_{\mathcal{U}_t^0} \mathbb{E}^W_{i_0} \left( \frac{R^W_{u}(i_0, 0)}{R^W_{u}(X_t)} \phi((X_v)_{v \leq t}) \right) \frac{1}{F^W_{i_0}} \mu^W_{i_0} (du)
\]
\[
= \mathbb{E}^W_{i_0} \left( \phi((X_v)_{v \leq t}) \int_{\mathcal{U}_t^0} \frac{R^W_{u}(i_0, 0)}{R^W_{u}(X_t)} \frac{1}{F^W_{i_0}} \mu^W_{i_0} (du) \right).
\]

Using Lemma 9.3 (ii), we obtain
\[
\int_{\mathcal{U}_t^0} \frac{R^W_{u}(i_0, 0)}{R^W_{u}(X_t)} \frac{1}{F^W_{i_0}} \mu^W_{i_0} (du) = \int_{\mathcal{U}_t^0} \frac{1}{F^W_{X_t}} \mu^{W_{T(v)}}_{X_t} (d\hat{u}) = 1
\]
since $\frac{1}{E_{\langle \tau \rangle}^{W,T(t)}} \mu_{X,t}^{W,T(t)}$ is a probability measure by Theorem 4.2 (i). Hence

$$\int_{\mathcal{U}_0^W} E_{i_0}^{W,u}(\phi((X_v)_{v \leq t})) \frac{1}{E_{i_0}^{W,T(t)}} \mu_{i_0}^{W}(du) = \mathbb{E}_{i_0}^{W,T(t)}(\phi((X_v)_{v \leq t})).$$

Finally, if $u \in \mathcal{U}_0^W$, then under $P_{i_0}^{W,u}$, $(Z_s)$ is Yaglom reversible since $\pi_\ast := e^{u,+u^\ast}$ obviously satisfies $\pi_\ast W_{i,j} e^{u^\ast,-u^\ast} = \pi_j W_{j,i} e^{u^\ast,-u^\ast}$ and is an invariant measure, using $u \in \mathcal{U}_0^W$.

9.4. Proof of Proposition 4.5 and Theorem 4.6.

PROOF OF PROPOSITION 4.5. Since $u \in \mathcal{U}_0^W$, we know that under the law $P_{i_0}^{W,u}$, $(Z_s)$ is the Markov process with jump rates $W_{i,j} e^{u^\ast,-u^\ast}$, which has invariant measure

$$\pi_\ast(u) = \frac{e^{u,+u^\ast}}{\sum_{j \in V} e^{u^\ast,+u_j^\ast}}, \quad \forall i \in V.$$ 

Hence $\lim_{s \to \infty} \frac{1}{s} (\ell_i(\sigma) - \ell_i^c(\sigma)) = 0$ a.s. and, by the central limit theorem, the speed of convergence is of order $\frac{1}{\sqrt{s}}$. In particular, we deduce that $\frac{1}{s} (\ell_i(\sigma) - \ell_i^c(\sigma)) = o(s^{-\frac{1}{2}})$ a.s.. Using $(1_{Z_s = i} - 1_{Z_s = i^c})ds = d(\ell_i(s) - \ell_i^c(s))$, an integration by parts yields

$$B_i^\theta (s) = \frac{1}{2} \int_0^s \frac{d(\ell_i(u) - \ell_i^c(u))}{\theta_i + \ell_i(u) + \ell_i^c(u)} = \frac{1}{2 \theta_i + \ell_i(s) + \ell_i^c(s)} + \frac{1}{2} \int_0^s \frac{d(\ell_i(u) - \ell_i^c(u))(1_{Z_u = i} + 1_{Z_u = i^c})}{(\theta_i + \ell_i(u) + \ell_i^c(u))^2} du.$$ 

The first term converges to 0, and the second is a convergent integral by previous considerations.

The formula (4.5) is a direct consequence of the Markov property of $(Z_s)_{s \geq 0}$ under $P_{i_0}^{W,u}$, and it is also true for a stopping time instead of fixed time $s$.

Moreover, if $\tau$ is the stopping time defined as the first time the Markov Process $Z_s$ leaves the last point it has visited in $V$, then $B_i^\theta(\tau)$ has a density on $\mathbb{R}^V$, since the jump rates are exponential. By (4.5), $B_i^\theta(\infty)$ is the convolution of $B_i^\theta(\tau)$ under $P_{i_0}^{W,u}$ and $B_i^\theta + U + \ell^c(\tau)(\infty)$ under $P_{Z_\tau}^{W,i}$, hence it has a density.

Theorem 4.6 will be a consequence of the following Proposition 9.6.

PROPOSITION 9.6 (Limit theorem for the $\ast$-VRJP conditioned on A). (i) Under $P_{i_0}^W$, conditionally on $A$, $U$ is distributed according to

$$f_{i_0}^{W,u} \left( \frac{1}{2}(u - u^*) - A \right) e^{-A_\delta - \frac{1}{2} \sum_{i,j \in V} W_{i,j}(e^{A_i + A_j^*} - 1) \mu_{i_0}^{W}(du).$$

(ii) Under $P_{i_0}^W$, conditionally on $A$ and $U$, $(Z_s)$ has the law of a conditioned Markov jump process, more precisely:

$$P_{i_0}^W (\cdot \mid A, U) = P_{i_0}^{W,U} \left( \cdot \mid B_i^\theta(\infty) = \frac{1}{2}(U - U^*) - A \right) \quad a.s..$$

Proposition 9.6 implies Theorem 4.6 by the following argument. Since $\nu_{i_0}^W$ has a density on $A$, (i) means that for almost all $a \in A$ under $P_{i_0}^{W,a}$, the law of the non-randomized $\ast$-VRJP at rates $\nu_{i_0}^W$, the law of $U$ is given by the formula above (with $a$ instead of $A$). Hence, for a.e.
By definition of $\mathbb{P}_W^{10}$, the law of $U$ is

$$f_{i_0}^{W,u}(u - u^*)/2 \cdot \mu_{i_0}^W(du)$$

and, if we condition further on $U$, the $\star$-VRJP has the law $P_{i_0}^{W,U}(\cdot \mid B^1(\infty) = (U - U^*)/2)$.

**Proof of Proposition 9.6.** By definition of $U = (U_i)_{i \in V}$ in Theorem 4.2, we have

$$\frac{1}{2}(U_i - U_i^*) = A_i + \lim_{t \to \infty} \frac{1}{2}(T_i(t) - T_i^*(t)).$$

Recall the time change defined by $s = C(t)$ defined in Proposition 3.4. By (8.10),

$$ds = e^{-T_i(t) + T_i^*(t)} dt.$$

Besides, (8.12) implies

$$e^{T_i(t) + T_i^*(t)} = 1 + \ell_i^Z(s) + \ell_i^Z(s),$$

hence

$$dt = \frac{1}{1 + \ell_i^Z(s) + \ell_i^Z(s)} ds.$$

Changing from time $t$ to time $s$, we deduce

$$\frac{1}{2}(T_i(t) - T_i^*(t)) = \frac{1}{2} \int_0^t (\mathbb{1}_{X_i = i} - \mathbb{1}_{X_i^* = i^*}) dv = \frac{1}{2} \int_0^s \frac{\mathbb{1}_{Z_{\cdot=i} - \mathbb{1}_{Z_{\cdot=i^*}}}}{1 + \ell_i^Z(u) + \ell_i^Z(u)} du = B^1_i(s).$$

**Remark 9.7.** Note that this proves the formula (8.13), since

$$e^{T_i(t)} = e^{\frac{1}{2}(T_i(t) + T_i^*(t))} e^{\frac{1}{2}(T_i(t) - T_i^*(t))} = \sqrt{1 + \ell_i^Z(s) + \ell_i^Z(s)} e^{B_i(s)},$$

with the time change $s = C(t)$.

This yields

$$A = \frac{1}{2}(U - U^*) - B^1(\infty).$$

Under $\mathbb{P}_W^{i_0}$, $A$ is distributed according to $\frac{1}{F_{i_0}} \nu_{X_{i_0}}^W$, and, conditionally on $U$, $(Z_i)$ has law $P_{i_0}^{W,U}$. Now, under $P_{i_0}^{W,U}$, $B^1(\infty)$ has density $f_{i_0}^{W,U}(b)$. Hence, under $\mathbb{P}_W^{i_0}$ and conditionally on $U$, $A$ has a density

$$f_{i_0}^{W,U}(\frac{1}{2}(U - U^*) - a).$$

Next, apply Bayes formula: under $\mathbb{P}_W^{i_0}$, $A$ is distributed according to $\frac{1}{P_{i_0}} \nu_{i_0}^W$, and $U$ is distributed according to $\frac{1}{P_{i_0}} \mu_{i_0}^W$. We deduce that, under $\mathbb{P}_W^{i_0}$ and conditionally on $A$, $U$ has distribution

$$f_{i_0}^{W,u}(\frac{1}{2}(u - u^*) - A) e^{-A_0} e^{\frac{1}{2} \sum_{(i,j) \in E} W_{i,j}(e_{A_i + A_j^*} - 1)} \mu_{i_0}^W(du).$$
10. Proof of the results of Section 6: Lemma 6.5, Theorems 6.2 and 6.6 and Proposition 6.7. Equality (6.3) in Theorem 6.2 is a special case of Theorem 6.6. The proof of Theorem 6.6 works by induction on \( \dim(S) \). The section is organized as follows: we start by proving the initialization step of the induction, then we prove the key identities of Lemma 6.5 and deduce Theorem 6.6 and Proposition 6.7. At the end of the section we prove the statement of Theorem 6.2 about the finiteness of the integrals.

10.1. Initialization of the induction, step 1: reduction of the problem. We start by proving Theorem 6.6 in the case of a self-dual point or a pair of dual points. Surprisingly, the proof is rather difficult and relies on the Lagrange resolvent method to solve the polynomial equations of degree 4 ([22] Section 30 and 31, or [9] Section 12.C and [19], Section 3.1 for modern references).

**Lemma 10.1.** Theorem 6.6 is true for \( V = \{i\}, \ i = i^* \), or for a pair of dual points, \( |V| = 2 \) and \( V = \{i, i^*\} \).

**Step 1 of the proof of Lemma 10.1.** By a simple change of variables \( \tilde{\beta}_i = \theta_i \theta_i^* \beta_i \) we obtain \( \int_S \nu_{S}^W,\eta,\eta(d\beta) = \int_S \nu_{S}^W,\eta,\eta(d\beta) \) with \( \tilde{W}_{i,j} = \theta_i \theta_j W_{i,j}, \ \tilde{\eta}_i = \theta_i \eta_i \). On the other hand, from the definition, we have that \( \int_A \nu_{A}^W,\eta,\eta(da) = \int_A \nu_{A}^W,\eta,\eta(da) \). Hence it is enough to prove the Lemma in the case \( \theta = 1 \), which is what we assume below. Besides, note that we have \( W_{i,i} = W_{i^*,i^*} \). Changing \( \beta_i \) into \( \beta_i - W_{i,i} \) in \( \nu_{S}^W,\eta,\eta \), we can always assume that \( W \) is null on the diagonal, since on \( \nu_{A}^W,\eta,\eta \) the diagonal terms of \( W \) cancel directly.

Assume first that \( V = \{i\} \) with \( i = i^* \) a self-dual point, the proof is simple in this case:

\[
\int_S \nu_{S}^W,\eta,\eta(d\beta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{d\beta}{\sqrt{\beta}} \exp\left(-\frac{1}{2}(\beta + \eta \frac{2}{\beta}) + \eta \right)
\]

\[
= \frac{\sqrt{\pi} e^\eta}{\sqrt{2\pi}} \int_0^\infty \exp(-\eta \cosh(y)) e^{\frac{y}{2}} dy
\]

\[
= \frac{\sqrt{\pi} e^\eta}{\sqrt{2\pi}} 2K_{\frac{1}{2}}(\eta) = 1,
\]

where in the third equality we made the change of variable \( \frac{\beta}{\eta} = e^y \), and where \( K_{\frac{1}{2}}(\eta) = \sqrt{\frac{\pi}{2\eta}} e^{-\eta} \) is the modified Bessel function of the second kind with order \( \frac{1}{2} \). On the other hand \( A = \emptyset \) and \( \nu_{A}^W,\eta,\eta \) is the constant 1.

Assume now that \( V = \{i, i^*\} \) is a pair of dual points, with \( i \neq i^* \). For simplicity we write \( \nu_{S}^W,\eta,\eta(d\beta) = e^{-\frac{1}{2}(1,W1)-(\eta,1)} \nu_{S}^{W,1,\eta}(d\beta) \) and \( \nu_{A}^W,\eta,\eta(da) = e^{-\frac{1}{2}(1,W1)-(\eta,1)} \nu_{A}^{W,1,\eta}(da) \). We first simplify the integral of \( \nu_{S}^{W,1,\eta}(d\beta) \). Set \( w_1 = W_{i,i^*}, \ w_2 = W_{i^*,i} \) and \( \eta_1 = \eta_i, \ \eta_2 = \eta_{i^*} \) (and remind that we assume \( \theta_i = \theta_{i^*} = 1 \) and \( W_{i,i} = W_{i^*,i^*} = 0 \), and \( \beta_i = \beta_{i^*} \) which we simply denote \( \beta \). With these notations, we have

\[
G_\beta = \frac{1}{\beta^2 - w_1 w_2} \left( \begin{array}{c} \beta \ w_1 \\ w_2 \ \beta \end{array} \right) ; \ \eta, \ G_\beta \eta = \frac{2\eta_1 \eta_2 \beta + w_2 \eta_1^2 + w_1 \eta_2^2}{\beta^2 - w_1 w_2}.
\]

With the notation

\[
(10.1) \ h = \sqrt{w_1 w_2}, \ C = \frac{\eta_1}{\sqrt{w_1}}, \ D = \frac{\eta_2}{\sqrt{w_2}}, \ E = \exp\left(\frac{1}{2}(w_1 + w_2) + \eta_1 + \eta_2\right),
\]
changing to variable \( x = \frac{\beta}{\sqrt{w_1 w_2}} \), we obtain that
\[
\int_S \hat{\nu}_S^{W,1,0} (d\beta) = \frac{E}{\sqrt{2\pi}} \int_1^\infty \frac{dx}{\sqrt{x^2 - 1}} \exp \left( -h \left( x + CD \frac{x}{x^2 - 1} + \frac{1}{2}(C^2 + D^2) \frac{1}{x^2 - 1} \right) \right).
\]

Writing
\[
(10.2) \quad P(x) = x^3 + (CD - 1)x + \frac{1}{2}(C^2 + D^2),
\]
the previous integral is equal to
\[
\frac{1}{\sqrt{2\pi}} \int_1^\infty \frac{dx}{\sqrt{x^2 - 1}} \exp \left( -h \frac{P(x)}{x^2 - 1} \right).
\]

Consider now the integral \( \hat{\nu}_A^{W,1,0}(da) \). With the notation above, we have
\[
\int_A \hat{\nu}_A^{W,1,0}(da) = E \int_\mathbb{R} \frac{da}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( w_1 e^{2a} + w_2 e^{-2a} \right) - (\eta_1 e^a + \eta_2 e^{-a}) \right)
= E \int_\mathbb{R} \frac{da}{\sqrt{2\pi}} \exp \left( -h \left( \frac{1}{2} \left( \sqrt{w_1} e^{2a} + \sqrt{w_2} e^{-2a} \right) - (\eta_1 e^a + \eta_2 e^{-a}) \right) \right).
\]

Changing to variable \( \tilde{a} \) such that \( \sqrt{w_1} e^{2a} = e^{2\tilde{a}} \), we have \( \frac{\eta_1}{h} e^a = Ce^{\tilde{a}} \) and \( \frac{\eta_2}{h} e^{-a} = De^{-\tilde{a}} \), so that the previous integral equals
\[
\int_\mathbb{R} \frac{d\tilde{a}}{\sqrt{2\pi}} \exp \left( -h \left( \frac{1}{2} \left( e^{2\tilde{a}} + e^{-2\tilde{a}} \right) - (Ce^{\tilde{a}} + De^{-\tilde{a}}) \right) \right).
\]

Finally, changing to coordinate \( x = e^{\tilde{a}} \), we obtain that
\[
(10.3) \quad \int_A \hat{\nu}_A^{W,1,0}(da) = \frac{E}{\sqrt{2\pi}} \int_0^\infty \exp \left( -h \frac{Q(x)}{2x^2} \right) \frac{dx}{x},
\]
where
\[
(10.4) \quad Q(x) = x^4 + 2Cx^3 + 2Dx + 1.
\]

Hence the lemma is equivalent to the following equality of integrals
\[
(10.5) \quad \int_1^\infty \frac{dx}{\sqrt{x^2 - 1}} \exp \left( -h \frac{P(x)}{x^2 - 1} \right) = \int_0^\infty \exp \left( -h \frac{Q(x)}{2x^2} \right) \frac{dx}{x}.
\]

We note that the left hand side is the Laplace transform of the image of the measure \( \mathbb{1}_{x > 1} \frac{dx}{\sqrt{x^2 - 1}} \) by the function \( \frac{P(x)}{x^2 - 1} \), and the right hand side is the Laplace transform of the image of the measure \( \mathbb{1}_{x > 0} \frac{dx}{x^2} \) by the function \( \frac{Q(x)}{2x^2} \). The equality is equivalent to the fact that these measures are equal. Hence, the strategy is now to make the change of variable \( u = \frac{P(x)}{x^2 - 1} \) or \( u = \frac{Q(x)}{2x^2} \) in each of these equations. This leads to polynomial equations of degree 3 and 4. A difficulty is that these change of variables are not bijective on their domain but 2 to 1. Remarkably, the equation of degree 3 is the Lagrange resolvent of the equation of degree 4. This leads to relations between the roots of these two equations, which are at the heart of the argument. The details of the computation are given in next Section 10.2. \( \square \)
10.2. Initialization of the induction, step 2: proof of equality (10.5) by Lagrange resolvent method. Let us state the equality as a self-contained lemma. The statements and arguments of this Section 10.2 are completely self-contained, and we freely give a different meaning to symbols already used in the rest of the paper. In particular, \( u, \alpha, \beta \) which appear below have nothing to do with the \( u, \alpha \) and \( \beta \) in the rest of the manuscript.

**Lemma 10.2.** Let \( C \geq 0, D \geq 0 \) be two reals, and \( P(x) \) and \( Q(x) \) be the two polynomials

\[
P(x) = x^3 + (CD - 1)x + \frac{1}{2}(C^2 + D^2), \quad Q(x) = x^4 + 2Cx^3 + 2Dx + 1, \]

Then

\[
(10.6) \quad \int_0^\infty \exp \left(-\frac{h}{2x^2}Q(x)\right) \frac{dx}{x} = \int_1^\infty \frac{dx}{\sqrt{x^2 - 1}} \exp \left(-\frac{h}{x^2 - 1}P(x)\right).
\]

for all \( h > 0 \).

**Proof.** If \( C = D = 0 \) the proof is simple: \( \frac{Q(x)}{2x^2} = \frac{1}{2}(x^2 + \frac{1}{x^2}) \), changing to variable \( u \) such that \( e^u = x^2 \) we obtain that

\[
\int_0^\infty \exp \left(-h\frac{Q(x)}{2x^2}\right) \frac{dx}{x} = \frac{1}{2} \int_\mathbb{R} e^{-h \cosh(u)} du = \int_0^\infty e^{-h \cosh(u)} du.
\]

(which is the modified Bessel function \( K_0(h) \)). On the other hand, \( \frac{P(x)}{x^2 - 1} = x \) and changing to variable \( u \) such that \( x = \cosh(u) \), we obtain

\[
\int_1^\infty \frac{dx}{\sqrt{x^2 - 1}} \exp \left(-\frac{h}{x^2 - 1}P(x)\right) = \int_0^\infty e^{-h \cosh(u)} du.
\]

Let us now suppose that \( C > 0 \) or \( D > 0 \). As explained above, the strategy is to make the change of variables \( u = \frac{Q(x)}{x^2} \) and \( u = \frac{P(x)}{x^2} \) in each of the integrals, and to use that \( P \) and \( Q \) are related by the Lagrange resolvent method.

Let us start by \( Q \). The equation \( u = \frac{Q(x)}{2x^2} \) is equivalent to \( Q(x, u) = 0 \) with

\[
(10.7) \quad Q(x, u) = x^4 + 2Cx^3 - 2ux^2 + 2Dx + 1.
\]

We have \( Q(0, u) = 1 \) and \( Q(+\infty, u) = +\infty \). Moreover, if \( x > 0 \), then \( Q(-x, u) \leq Q(x, u) \) since \( C \geq 0, D \geq 0 \) and \( Q(x, u) \) is decreasing in \( u \). It implies that there exists \( u > 0 \) such that \( Q(x, u) \) has two positive simple roots and two negative simple roots for \( u \geq u \), no positive root for \( u < u \), and a double positive root for \( u = u \). For \( u > u \) we denote by \( a(u), b(u), c(u), d(u) \) (we often simply write \( a, b, c, d \)) the roots of the polynomial \( x \to Q(x, u) \) and we choose the roots such that \( 0 < a < b \), and \( c < d < 0 \). Note that the infimum

\[
u = \inf_{x > 0} \frac{Q(x)}{2x^2}
\]

is reached at a unique value \( \bar{x} > 0 \), and that \( x \to \frac{Q(x)}{2x^2} \) is bijective from the interval \((0, \bar{x})\) (resp. \((\bar{x}, +\infty)\)) onto \((u, +\infty)\), with inverse \( a(u) \) (respectively \( b(u) \)). Performing the change of variable \( u = \frac{Q(x)}{2x^2} \) on each of these intervals, we obtain that

\[
(10.8) \quad \int_0^\infty \exp \left(-\frac{h}{2x^2}Q(x)\right) \frac{dx}{x} = \int_u^\infty e^{-hu} \left(-\frac{a'(u)}{a(u)} + \frac{b'(u)}{b(u)}\right) du.
\]
Let us now consider the polynomial \( P \). The equation \( u = \frac{P(x)}{x^2 - 1} \) is equivalent to \( P(x, u) = 0 \), where
\[
P(x, u) = x^3 - ux^2 + (CD - 1)x + \frac{1}{2}(C^2 + D^2) + u.
\]

We remark that \( P(1, u) = \frac{1}{2}(C + D)^2 > 0 \), hence \( P(x, u) \) has at least one root on \((-\infty, 1)\). Besides, for \( x \in (1, \infty) \), \( P(x, u) \) is decreasing in \( u \), hence there is \( u' > 0 \) such that \( x \to P(x, u) \) has two simple roots (resp. no root) on \((1, \infty)\) for \( u > u' \) (resp. \( u < u' \)) and hence a double root on \((1, \infty)\) for \( u = u' \). (We will show later that \( u' = u \).

The key relation between \( Q \) and \( P \) is that, up to a simple change of variable, \( P(x, u) \) is the Lagrange resolvent of \( Q(x, u) \). Let us briefly recall what we need on Lagrange resolvents (cf \([22]\), Section second, 30 and 31, or \([9]\) Section 12.C and \([19]\) Section 3.1 for modern references): if
\[
x^4 + mx^3 + nx^2 + px + q
\]
is a polynomial of degree 4, with roots \( a, b, c, d \), its Lagrange resolvent is the polynomial of degree 3, given by
\[
y^3 - ny^2 + (mp - 4q)y - (m^2 - 4n)q - p^2,
\]
which has roots
\[
ab + cd, \quad ac + bd, \quad ad + bc.
\]

Coming back to our question, the Lagrange resolvent of \( Q(x, u) \) is
\[
R(y) = y^3 + 2uy^2 + y(4CD - 4) - 8\left(\frac{1}{2}(C^2 + D^2) + u\right) = 8P(-\frac{1}{2}y, u).
\]

In particular, it means that \( P(x, u) \) has roots
\[
\gamma(u) := -\frac{1}{2}(ab + cd), \quad \alpha(u) := -\frac{1}{2}(ac + bd), \quad \beta(u) := -\frac{1}{2}(ad + bc).
\]

We remark that for \( u > u_0 \), \( \gamma < 0 \) and \( \beta > 1, \alpha > 1 \) since \( a > 0, b > 0 \) and \( c < 0, d < 0 \) and \( abcd = 1 \) (using the arithmetico-geometric inequality), so that \( x \to P(x, u) \) has two roots on \((1, \infty)\). For \( u = u_0, a = b \) is a double root of \( Q(x, u) \), hence \( \alpha = \beta \) is double root of \( P(x, u) \) on \((1, \infty)\). In particular it implies that \( u = u' \). Moreover, for \( u > u_0 \), we have \( 1 < \alpha < \beta \): indeed, we have \( d < c < 0 < a < b \), hence \( (a - b)(d - c) = ad + bc - (bd + ac) > 0 \). Proceeding as for \( Q \), for \( u > u_0 \), we consider \( x' \) which is the unique point such that \( \frac{P(x, u)}{x^2 - 1} \) reaches its minimum on \((1, \infty)\) and make the change of variables \( u = \frac{P(x, u)}{x^2 - 1} \) on \((1, x')\) and on \((x', +\infty)\). This leads to the following equality

\[
(10.9) \quad \int_1^\infty \frac{dx}{\sqrt{x^2 - 1}} \exp\left(-h \frac{P(x)}{x^2 - 1}\right) = \int_0^\infty e^{-hu} \left(\frac{\beta'(u)}{\beta(u) - 1} - \frac{\alpha'(u)}{\alpha(u) - 1}\right).
\]

In the last step, we use a key argument in the Lagrange method to solve the 4th degree polynomial equation: since \(-2\beta = ac + bd\) and \(abcd = 1\), \(ac\) and \(bd\) are solutions of the equation of degree 2:
\[
il^2 + 2\beta l + 1 = 0.
\]

Since \( d < c < 0 < a < b \) we have that \( bd < ac < 0 \), hence \( ac \) is the solution \( ac = -\beta + \sqrt{\beta^2 - 1} \), hence differentiating in \( u \), we deduce
\[
ab\left(\frac{\alpha'}{a} + \frac{c'}{c}\right) = -\beta' - \beta + \sqrt{\beta^2 - 1}, \quad \text{hence} \quad \frac{\alpha'}{a} + \frac{c'}{c} = \frac{-\beta'}{\sqrt{\beta^2 - 1}}.
\]
Similarly, \( ad \) and \( bc \) are solutions of the equation \( t^2 + 2\alpha t + 1 = 0 \). Moreover, we have \( ad < bc < 0 \): indeed, since for \( u > u > 0 \) we have \( Q(-x, u) < Q(u, x) \) for \( x > 0 \), we deduce \( Q(-b, u) < 0, Q(-a, u) < 0 \) which gives \( d < -b < -a < c \), hence \( |ad| < |bc| \). Thus \( bc \) is the root \( be = -\alpha + \sqrt{\alpha^2 - 1} \). Finally,

\[
\frac{b'}{b} + \frac{c'}{c} = \frac{-\alpha'}{\sqrt{\alpha^2 - 1}}
\]

Taking the difference of the two identities we have obtained, we can conclude that

\[
\frac{a'}{a} - \frac{b'}{b} = \frac{\alpha'}{\sqrt{\alpha^2 - 1}} - \frac{\beta'}{\sqrt{\beta^2 - 1}}.
\]

This concludes the proof with (10.8) and (10.9).

10.3. Proof of Lemma 6.5. By hypothesis, \( I \) is a non-empty subset such that \( I^* = I \) and \( I \not\subseteq V \), so that we can write

\[
V = I \sqcup I^c,
\]

with \( I^c \) non-empty and \((I^c)^* = I^c \). The first part of the proof generalizes some computations done in the proof of lemma 4 in [30] or in section 4 [23]. We can write \( H_\beta \) as the block matrix

\[
H_\beta = \begin{pmatrix}
(H_\beta)_{I,I} & -W_{I,I^c} \\
-W_{I^c,I} & (H_\beta)_{I^c,I^c}
\end{pmatrix}
\]

and set

\[
\tilde{H}_\beta = (H_\beta)_{I,I}, \quad \hat{G}_\beta = \tilde{H}_\beta^{-1}, \quad \tilde{W} = W_{I^c,I^c} + W_{I^c,I} \hat{G}_\beta W_{I,I^c},
\]

\[
\tilde{H}_\beta = \beta_{I^c} - \tilde{W} = (H_\beta)_{I^c,I^c} - W_{I^c,I} \hat{G}_\beta W_{I,I^c}, \quad \hat{G}_\beta = \tilde{H}_\beta^{-1}.
\]

Note that \( \tilde{H}_\beta \) corresponds to the Schur complement of the matrix \( H_\beta \) on the sub-block \( I^c \times I^c \). For this reason, classically, we have

\[
\hat{G}_\beta = (G_\beta)_{I^c,I^c}.
\]

More precisely, with the previous notations we have

\[
H_\beta = \begin{pmatrix}
\text{Id}_{I,I} & 0 \\
-W_{I^c,I} \hat{G}_\beta \text{Id}_{I^c,I^c}
\end{pmatrix}
\begin{pmatrix}
\tilde{H}_\beta & 0 \\
0 & \tilde{H}_\beta
\end{pmatrix}
\begin{pmatrix}
\text{Id}_{I,I} & -\hat{G}_\beta W_{I,I^c} \\
0 & \text{Id}_{I^c,I^c}
\end{pmatrix},
\]

and, subsequently,

\[
G_\beta = \begin{pmatrix}
\text{Id}_{I,I} & \hat{G}_\beta W_{I,I^c} \\
0 & \text{Id}_{I^c,I^c}
\end{pmatrix}
\begin{pmatrix}
\hat{G}_\beta & 0 \\
0 & \hat{G}_\beta
\end{pmatrix}
\begin{pmatrix}
\text{Id}_{I,I} & 0 \\
W_{I^c,I} \hat{G}_\beta \text{Id}_{I^c,I^c}
\end{pmatrix}
\]

By (10.13), we have

\[
\langle \theta, H_\beta \theta \rangle = \langle \theta_{I^c}, \tilde{H}_\beta \theta_{I^c} \rangle + \langle \theta_{I^c}, H_\beta \theta_{I^c} \rangle + \langle \theta_{I^c}, W_{I^c,I} \hat{G}_\beta W_{I,I^c} \theta_{I^c} \rangle - 2 \langle \theta_{I}, W_{I,I^c} \theta_{I^c} \rangle
\]

On the other hand,

\[
\begin{pmatrix}
\text{Id}_{I,I} & 0 \\
W_{I^c,I} \hat{G}_\beta \text{Id}_{I^c,I^c}
\end{pmatrix}
\begin{pmatrix}
\eta_I \\
\eta_{I^c}
\end{pmatrix} = \begin{pmatrix}
\eta_I \\
\tilde{\eta}_{I^c}
\end{pmatrix}.
\]
We deduce from (10.14) that

\begin{equation}
\langle \eta, G_\beta \eta \rangle = \langle \eta I, \hat{G}_\beta \eta I \rangle + \langle \hat{\eta}, \hat{G}_\beta \hat{\eta} \rangle.
\end{equation}

Set \( \hat{\eta} = \hat{\eta}_I \) and \( \hat{\eta} = \hat{\eta}_{I^c} \) for simplicity. Observe that \( W \) and \( \hat{G}_\beta \) are symmetric with respect to \( \langle \cdot, \cdot \rangle \), so that the adjoint of

\[
\begin{pmatrix}
\Id_{I,I} & -\hat{G}_\beta W_{I,I^c} \\
0 & \Id_{I^c,I^c}
\end{pmatrix}
\]

with respect to that bilinear form is

\[
\begin{pmatrix}
\Id_{I,I} & 0 \\
-W_{I^c,I} \hat{G}_\beta & \Id_{I^c,I^c}
\end{pmatrix}.
\]

Hence, using (10.13), we deduce

\begin{equation}
\langle \theta, H_\beta \theta \rangle = \langle \zeta, D_H \zeta \rangle,
\end{equation}

with

\[
D_H = \begin{pmatrix}
\hat{H}_\beta & 0 \\
0 & \hat{H}_\beta
\end{pmatrix},
\]

\[\zeta = \begin{pmatrix}
\Id_{I,I} & -\hat{G}_\beta W_{I,I^c} \\
0 & \Id_{I^c,I^c}
\end{pmatrix} \theta.
\]

Combining (10.15), (10.17) and (10.16), and noting that \( \langle \eta, \theta \rangle = \langle \hat{\eta}, \theta_I \rangle + \langle \hat{\eta}, \theta_{I^c} \rangle \), we deduce

\begin{equation}
\langle \theta, H_\beta \theta \rangle + \langle \eta, G_\beta \eta \rangle - 2 \langle \eta, \theta \rangle = \langle \theta_I, \hat{H}_\beta \theta_I \rangle + \langle \hat{\eta}, \hat{G}_\beta \hat{\eta} \rangle - 2 \langle \hat{\eta}, \theta_{I^c} \rangle + \langle \theta_I, \hat{H}_\beta \theta_I \rangle + \langle \hat{\eta}, \hat{G}_\beta \hat{\eta} \rangle - 2 \langle \hat{\eta}, \theta_I \rangle
\end{equation}

Note that, by (10.13),

\begin{equation}
\det H_\beta = \det \hat{H}_\beta \det \hat{H}_\beta, \quad 1_{H_\beta > 0} = 1_{\hat{H}_\beta > 0} \hat{1}_{\hat{H}_\beta > 0}.
\end{equation}

Combining (10.18) and (10.19), we deduce

\[
\nu^{W_{I^c,I^c}}_{S_I^c} d\beta_I = \frac{\prod_i \theta_i}{\sqrt{2\pi |S_I|}} e^{-\frac{1}{2} \langle \theta, \hat{H}_\beta \theta \rangle - \frac{1}{2} \langle \eta, G_\beta \eta \rangle + \langle \eta, \theta \rangle} \frac{1_{H_\beta > 0}}{\sqrt{\det \hat{H}_\beta}} d\beta_I
\]

\[
= \frac{\prod_i \theta_i}{\sqrt{2\pi |S_I|}} e^{-\frac{1}{2} \langle \theta, \hat{H}_\beta \theta \rangle - \frac{1}{2} \langle \eta, G_\beta \eta \rangle + \langle \eta, \theta \rangle} \frac{1_{H_\beta > 0}}{\sqrt{\det \hat{H}_\beta}} d\beta_I.
\]

Hence, this proves the first identity of Lemma 6.5.

Next we prove the third identity of Lemma 6.5. Using that \( W_{I^c,I^c} = W_{I^c,I^c} + W_{I^c,I \hat{G}_\beta W_{I,I^c}} \), and that \( \hat{\eta}_{I^c} = \eta_{I^c} + W_{I^c,I \hat{G}_\beta \theta_{I^c}} \), and using the \( \langle \cdot, \cdot \rangle \)-symmetry of the matrix \( W \), we deduce:

\[
-\frac{1}{2} \langle \theta_{I^c}^a, W \theta_{I^c}^a \rangle + \frac{1}{2} \langle \theta_{I^c}, W \theta_{I^c} \rangle - \langle \hat{\eta}, (\theta_{I^c}^a - \theta_{I^c}) \rangle - \frac{1}{2} \langle \hat{\eta}, \hat{G}_\beta \hat{\eta} \rangle.
\]
so that

\[ \text{Combined with the previous equality, it gives} \]

This comes from the fact that

\[ \text{I hence} \]

\[ \hat{\eta}, \hat{G}_{\beta} \hat{\eta} \]

\[ \langle \hat{\eta}, \hat{G}_{\beta} \hat{\eta} \rangle = \langle \hat{\eta}^a, \hat{G}_{\beta} \hat{\eta}^a \rangle + \langle \hat{W}_{I,1} \theta_{I^c}, \hat{G}_{\beta} W_{I,1} \theta_{I^c} \rangle + 2 \langle \eta_{I^c}, \theta_{I^c} - \theta_{I^c}^a \rangle - \langle \hat{W}_{I,1} \theta_{I^c}, \hat{G}_{\beta} W_{I,1} \theta_{I^c} \rangle. \]

Combined with the previous equality, it gives

\[ \text{−} \frac{1}{2} \langle \theta_{I^c}, \hat{W} \theta_{I^c} \rangle + \frac{1}{2} \langle \theta_{I^c}, \hat{W} \theta_{I^c} \rangle - \langle \hat{\eta}, (\theta_{I^c}^a - \theta_{I^c}) \rangle - \frac{1}{2} \langle \hat{\eta}, \hat{G}_{\beta} \hat{\eta} \rangle \]

\[ \text{= −} \frac{1}{2} \langle \theta_{I^c}, \hat{W} \theta_{I^c} \rangle + \frac{1}{2} \langle \theta_{I^c}, \hat{W} \theta_{I^c} \rangle - \langle \eta_{I^c}, (\theta_{I^c}^a - \theta_{I^c}) \rangle - \frac{1}{2} \langle \hat{\eta}_{I^c}, \hat{G}_{\beta} \hat{\eta}_{I^c} \rangle. \]

Coming back to the definitions, this yields

\[ \nu_{S_I}^{W_{I,1}, \theta_{I^c}, \hat{\eta}_{I^c}} (d\beta_I) \nu_{A_{I^c}}^{W_{I,1}, \theta_{I^c}, \hat{\eta}_{I^c}} (d\eta_{I^c}) = \nu_{S_I}^{W_{I,1}, \theta_{I^c}, \eta_{I^c}} (d\beta_I) \nu_{A_{I^c}}^{W_{I,1}, \theta_{I^c}, \eta_{I^c}} (d\eta_{I^c}) = Q_I^{W, \theta, \eta} (d\beta_I, d\eta_{I^c}), \]

where the last equality comes from the definition.

The last step is to prove the second equality of Lemma 6.5, i.e. that

\[ \nu_{A_{I^c}}^{W_{I,1}, \theta_{I^c}, \eta_{I^c}} (d\eta_{I^c}) = \nu_{A_{I^c}}^{W, \theta, \eta} (da). \]

This comes from the fact that

\[ \langle \hat{\eta}_{I^c}, \theta_{I^c}^a - \theta_{I^c} \rangle = - \langle \eta_{I^c}, \theta_{I^c}^a - \theta_{I^c} \rangle - \langle \hat{W}_{I,1} \theta_{I^c}^a, \theta_{I^c}^a - \theta_{I^c} \rangle \]

\[ \langle \hat{\eta}_{I^c}, \theta_{I^c}^a - \theta_{I^c} \rangle = - \langle \eta_{I^c}, \theta_{I^c}^a - \theta_{I^c} \rangle - \langle \hat{W}_{I,1} \theta_{I^c}^a, \theta_{I^c}^a - \theta_{I^c} \rangle, \]

hence

\[ \nu_{A_{I^c}}^{W_{I,1}, \theta_{I^c}, \eta_{I^c}} (d\eta_{I^c}) = \nu_{A_{I^c}}^{W_{I,1}, \theta_{I^c}, \eta_{I^c}} (d\eta_{I^c}) = e^{-\frac{1}{2} \langle \theta_{I^c}, W_{I,1} \theta_{I^c} \rangle + \frac{1}{2} \langle \theta_{I^c}, W_{I,1} \theta_{I^c} \rangle} e^{-\frac{1}{2} \langle \theta_{I^c}, W_{I,1} \theta_{I^c} \rangle + \frac{1}{2} \langle \theta_{I^c}, W_{I,1} \theta_{I^c} \rangle} (d\beta_I) = e^{-\frac{1}{2} \langle \theta_{I^c}, W_{I,1} \theta_{I^c} \rangle + \frac{1}{2} \langle \theta_{I^c}, W_{I,1} \theta_{I^c} \rangle} (d\eta_{I^c}) = \nu_{A_{I^c}}^{W, \theta, \eta} (d\eta_{I^c}), \]

which concludes the proof of Lemma 6.5.

10.4. Proof of Theorem 6.6 and Proposition 6.7. The proof works by induction on \( \dim(S) = |V_0 \sqcup V_1| \). The initialization \( \dim(S) = 1 \) has already been done. Assume that the statement is true for \( \dim(S) \leq n \), and consider a graph \( \mathcal{G} \) such that \( \dim(S) = n + 1 \). Consider \( I \subseteq V \) as in Lemma 6.5 so that \( \dim(S_I) \leq n \) and \( \dim(S_{I^c}) \leq n \). From the first equality in Lemma 6.5, we deduce

\[ \nu_{S}^{W, \theta, \eta} (d\beta) = \nu_{S_I}^{W_{I,1}, \theta_{I^c}, \eta_{I^c}} (d\beta_I) \nu_{S_{I^c}}^{W_{I,1}, \theta_{I^c}, \eta_{I^c}} (d\beta_{I^c}). \]

Note that, by definition, the parameters \( \hat{W}_{I,1} \) and \( \hat{\eta}_{I^c} \) only depend on \( \beta_I \) and not on \( \beta_{I^c} \). We now integrate on \( \beta_{I^c} \), while leaving \( \beta_I \) fixed, hence considering \( \hat{W}_{I,1} \) and \( \hat{\eta}_{I^c} \) as fixed parameters. Applying the induction formula on the subgraph with vertices \( I^c \), we obtain

\[ \int_{S_{I^c}} \nu_{S_{I^c}}^{W_{I,1}, \theta_{I^c}, \eta_{I^c}} (d\beta_{I^c}) = \int_{A_{I^c}} \nu_{A_{I^c}}^{W_{I,1}, \theta_{I^c}, \eta_{I^c}} (d\eta_{I^c}). \]
Integrating now on $d\beta_1$, and by definition of $Q^W,\theta,\eta_1_1(d\beta_1, da_{I^c})$ (see Lemma 6.5), we obtain
\[
\int_S \nu_S^W,\theta,\eta_1_1(d\beta) = \int_{S \times A_{I^c}} Q^W,\theta,\eta_1_1(d\beta_1, da_{I^c}),
\]
which proves the first part of the equality of Theorem 6.6. Remark that, by the same argument, we also prove the first part of Proposition 6.7 ii).

By the third equality of Lemma 6.5, using the other expression of $Q^W,\theta,\eta_1_1$, we also have
\[
Q^W,\theta,\eta_1_1(d\beta_1, da_{I^c}) = \nu_{S}^{W,\theta,\eta_1_1}(d\beta_1)\nu_{A_{I^c}}^{W,\theta,\eta_1_1}(da_{I^c}).
\]
Remark that, on the right-hand side in $\nu_{A_{I^c}}^{W,\theta,\eta_1_1}(da_{I^c})$, the parameters do not depend on $\beta_1$, while in $\tilde{\eta}_{1_1}^j$ only depends on $a_{I^c}$. Hence, applying the recurrence hypothesis on the subgraph with vertices $I$, conditioned on $a_{I^c}$, we deduce
\[
\int_{S \times A_{I^c}} Q^W,\theta,\eta_1_1(d\beta_1, da_{I^c}) = \int_{A_{I^c}} \nu_{A_{I^c}}^{W,\theta,\eta_1_1}(da_{I^c}).
\]
Integrating now on $a_{I^c}$, and using the second equality of Lemma 6.5, this implies
\[
\int_{S \times A_{I^c}} Q^W,\theta,\eta_1_1(d\beta_1, da_{I^c}) = \int_A \nu_{A_{I^c}}^{W,\theta,\eta_1_1}(da_{I^c}) = \int_A \nu_{A}^{W,\theta,\eta_1_1}(da).
\]
This proves the second part of the equality of Theorem 6.6 and, by a similar argument, the second part of Proposition 6.7 ii).

The conditioning properties, Proposition 6.7 i), are direct consequences of each of the identities of Lemma 6.5.

10.5. Proof of the integrability condition of Theorem 6.2. Under the assumption of Theorem 6.2, we will prove that $\int_A \nu_{A}^{W,\theta,\eta_1_1}(da) < \infty$, which is equivalent to $F^{W,\theta,\eta_1} < \infty$ by (6.3) and definition. Changing $W_{i,j}$ into $\tilde{W}_{i,j} = \theta_i^j \theta_j^i W_{i,j}$ and $\eta_i$ into $\tilde{\eta}_i = \theta_i^j \eta_j$, we can always assume that $\theta_i = 1$ for all $i \in V$, which is what we do in the sequel. Set $w := \inf_{(i,j) \in E} W_{i,j} > 0$. For $i \in V$, let $A_i = \{ a \in A, a_i \geq \max_{j \in V} |a_j| \}$, we have
\[
\int_A \nu_{A}^{W,\theta,\eta_1_1}(da) \leq \sum_{i \in V} \int_{A_i} \nu_{A_i}^{W,\theta,\eta_1_1}(da).
\]
Fix $i \in V$. If there exists a directed path from $i$ to $i^*$, then the proof is essentially the same as the proof of Part I, Lemma 3.1.

Suppose now that the other assumption is satisfied for $i$, i.e. that there exists $j \in V$ such that there is a directed path from $i$ to $j$ and $\eta_j > 0$. Let $\sigma$ be a shortest path from $i$ to $j$, then there exists $k$ such that $\sigma_k - \sigma_{k+1} \geq \frac{a_i - a_j}{|\sigma|}$. Hence
\[
\nu_{A}^{W,\theta,\eta_1_1}(da) \leq C_{W,\eta_1} \exp \left( -\frac{1}{2} \sum_{i \rightarrow j} W_{i,j} e^{a_i - a_j} - \eta_j e^{a_j} \right) \leq \exp \left( -\frac{1}{2} w e^{\frac{a_i - a_j}{|\sigma|}} - \eta_j e^{a_j} \right) da.
\]
We easily obtain $\max(\frac{a_i - a_j}{|\sigma|}, a_j) \geq \frac{a_i}{|\sigma| + 1}$, hence, by the same argument as in the proof of Part I, Lemma 3.1.
\[
\int_{A_i} \nu_{A_i}^{W,\theta,\eta_1_1}(da) \leq C_{W,\eta_1} \int_0^{\infty} |V| \exp \left( -\min(\frac{1}{2} w, \eta_j) e^{\frac{a_i}{|\sigma| + 1}} \right) da_i,
\]
which is finite. This concludes the proof.
11. Proof of Theorem 7.8. The strategy of the proof of Theorem 7.8 goes as follows: we pick \((B_I, A_{I'})\) according to the law \(\frac{1}{P_{I_0}}Q_{I,I_0}^W\) and run the process defined in Theorem 7.8 ii).

Then we prove that the law of this process is that of the randomized \(*\)-VRJP \((X_t)\), i.e. \(\mathbb{P}_I^W\).

To complete the proof, we show that the random variables \((B_I, A_{I'})\) appear asymptotically as functions of the path of the process and coincide with the corresponding values defined in Theorem 7.8 i).

Let us fix some notation: we denote by \(\mathbb{P}_{I,i_0}^W\) the joint law \(((X_t), (B_I, A_{I'})\) where \((B_I, A_{I'})\) is distributed according to the law \(\frac{1}{P_{I_0}}Q_{I,i_0}^W\) and conditionally on \((B_I, A_{I'})\), the process \((X_t)\) is defined in Theorem 7.8 ii). By abuse of notation, we sometimes consider \(\mathbb{P}_{I,i_0}^W\) as the law of its marginal \((X_t)\).

By definition, under \(\mathbb{P}_{I,i_0}^W\) and conditioned on \((B_I, A_{I'})\), the jump rate of \(X(t)\) at time \(t\) is

\[
\sum_{j, X_{n-1} \rightarrow j} W_{X_{n-1}, j} e^{T_{X_{n-1}}(u) + T_{X_{n}}(u) - V_{X_{n}}(u)}
\]

By a classical computation, under \(\mathbb{P}_{I,i_0}^W\) and conditioned on \((B_I, A_{I'})\), the probability that the canonical process \(X\) at time \(t\) has performed \(n\) jumps in infinitesimal time intervals \([t_i, t_i + dt_i], 0 < t_1 < \cdots < t_n < t\), following the trajectory \(\sigma_0 = i_0, \sigma_1, \ldots, \sigma_n = j_0\) is equal to

\[
\exp \left(- \int_0^t \sum_{j, X_{n-1} \rightarrow j} W_{X_{n-1}, j} e^{T_{X_{n-1}}(u) + T_{X_{n}}(u) - V_{X_{n}}(u)} du \right)
\cdot \left( \prod_{l=1}^n W_{\sigma_{l-1}, \sigma_l} e^{T_{\sigma_{l-1}}(t_l) + T_{\sigma_l}(t_l) - V_{\sigma_l}(t_l) - V_{\sigma_{l-1}}(t_l)} dt_l \right).
\]

Due to several simplifications detailed just below we have

\[
\prod_{l=1}^n e^{V_{\sigma_l}(t_l) - V_{\sigma_{l-1}}(t_l)} = e^{V_{\sigma_1}(0) - V_{\sigma_1}(t)} - \sum_{t \in \gamma_i \cap I} T_i(t).
\]

Indeed:

- We know that \(V(t)\) is constant when \(X(t)\) is in \(I\). Hence when \(\sigma_l\) and \(\sigma_{l-1}\) are in \(I\), the term \(e^{V_{\sigma_l}(t_l)}\) cancels with the next term \(e^{-V_{\sigma_l}(t_{l+1})}\) since \(V_{\sigma_l}\) is constant on the interval \([t_l, t_{l+1}]\).
- When \(\sigma_l\) and \(\sigma_{l-1}\) are in \(I^c\), we have \(V_{\sigma_l}(t) = T_{\sigma_l}(t) + A_l\). and we have a similar simplification as in the proof of Part I Proposition 3.4 equation (8.8). More precisely,
  - If \(\sigma_l \neq \sigma_{l-1}\), the local time \(T_{\sigma_l}\) does not change between the time \(s_{l+1}\) at which the process jumps to \(\sigma_l\) and the time \(s_{l+1}\) at which it leaves \(\sigma_l\). Hence, \(T_{\sigma_l}(s_l)\) cancels with \(T_{\sigma_{l+1}}(s_{l+1})\).
  - If \(\sigma_l = \sigma_{l-1}\), the local time \(T_{\sigma_l}\) does not change between the time \(s_{l+1}\) at which the process leaves \(\sigma_l\) and the first time after \(s_{l+1}\) at which it comes back to \(\sigma_l\). Hence, there is a cancellation at each return time to a self-dual point. Since the initial local time is 0, it leaves the contribution \(- \sum_{t \in \gamma_i \cap I} T_i(t)\) in the formula above.

Hence, integrating on the distribution \(\frac{1}{P_{I_0}}Q_{I,i_0}^W\) of the random variables \((B_I, A_{I'})\), we see that under \(\mathbb{P}_{I,i_0}^W\), the probability that the process \(X\) at time \(t\) has performed \(n\) jumps in infinitesimal time intervals \([t_i, t_i + dt_i], 0 < t_1 < \cdots < t_n < t\), following the trajectory
\[ \sigma_0 = i_0, \sigma_1, \ldots, \sigma_n = j_0 \text{ is equal to} \]

\[(11.2) \]

\[
\frac{1}{F_{i_0}} \left( \prod_{i=1}^{n} W_{\sigma_{i-1}, \sigma_i} e^{T_{\sigma_{i-1}}(t_i) + T_{\sigma_i}(t_i)} dt_t \right) e^{-\sum_{i \in V_0 \cap I} T_i(t)} \int_{S_{I \times A_{I^c}}} e^{V_0(t) - V_0(0)} - \int_0^t \sum_{j, X_{i \rightarrow j}} W_{X_{i \rightarrow j}} e^{T_X(u)} e^{V_j(u) - V_X(u)} du \quad Q^W_{I, i_0}(d\beta_I, d\beta_{I^c}).
\]

where in the previous formula, \((V_i(u))\) is constructed from the variables \( (\beta_I, a_{I^c}) \) instead of \( (B_I, A_{I^c}) \), i.e. we have

\[
\begin{align*}
V_i(t) &= T_i(t) + a_i, \quad \text{if } i \in I^c, \\
(H_\beta(e^{V^*(t)}))_I &= 0.
\end{align*}
\]

Let \( \theta(t) = e^{T^*(t)} \).

Then (6.4) and the definition of \( V(t) \) imply that

\[(11.3) \quad \theta^a_{I^c}(t) = (e^{T^*(t) + a^*})_{I^c} = (e^{V^*(t)})_{I^c}. \]

Also,

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \langle \theta_I(t), \beta_I \theta_I(t) \rangle + \frac{1}{2} \langle \theta^a_{I^c}(t), W_{I^c, I} \hat{G}_\beta W_{I^c, I} \theta^a_{I^c}(t) \rangle + \frac{1}{2} \langle \theta^a_{I^c}(t), W_{I^c, I} \theta^a_{I^c}(t) \rangle \right)
\]

\[
= \sum_{j, X_{i \rightarrow j}} W_{X_{i \rightarrow j}} e^{T_X(t_i) + T_X(t)} e^{V_j(t_i) - V_X(t)}.
\]

Indeed, we have

\[
\frac{\partial}{\partial t} \frac{1}{2} \langle \theta_I(t), \beta_I \theta_I(t) \rangle = \mathbb{1}_{X_i \in I} e^{T_X(t_i) + T_X(t)} \beta_{X_i} = \mathbb{1}_{X_i \in I} e^{T_X(t_i) + T_X(t)} \sum_{j, X_{i \rightarrow j}} W_{X_{i \rightarrow j}} e^{V_j(t_i) - V_X(t)}
\]

since, for \( i \in I \), \( \beta_i = \sum_{j,i \rightarrow j} W_{i,j} e^{V_j(t_i) - V_X(t)} \), for all time \( t \geq 0 \). Besides,

\[
\frac{\partial}{\partial t} \frac{1}{2} \langle \theta^a_{I^c}(t), W_{I^c, I} \hat{G}_\beta W_{I^c, I} \theta^a_{I^c}(t) \rangle = \mathbb{1}_{X_i \in I} \theta^a_{X_i}(t) \sum_{j, X_{i \rightarrow j}} W_{X_{i \rightarrow j}} \left( \hat{G}_\beta W_{I^c, I} \theta^a_{I^c}(t) \right)_j
\]

\[
= \mathbb{1}_{X_i \in I} e^{T_X(t_i) + T_X(t)} \sum_{j, X_{i \rightarrow j}} W_{X_{i \rightarrow j}} e^{V_j(t_i) - V_X(t)}
\]

since by the definition of \( V(t) \) in Theorem 7.8 ii) we have, for \( i \in I^c \) and \( j \in I \),

\[
\theta^a_{I^c}(t) = e^{V(t)} = e^{T(t)} e^{a(t)} = e^{T(t) + T(t)} e^{-V(t)}
\]

and, using \( H_\beta(e^{V^*(t)})_I = 0 \), or equivalently \( W_{I^c, I^c} (e^{V^*(t)})_I = \tilde{H}_\beta(e^{V^*(t)})_I \),

\[(11.4) \quad \left( \hat{G}_\beta W_{I^c, I^c} \theta^a_{I^c}(t) \right)_j = \left( \hat{G}_\beta W_{I^c, I^c} (e^{V^*(t)})_I \right)_j = e^{V_j(t)}.
\]

Finally, for the last term we have

\[
\frac{\partial}{\partial t} \frac{1}{2} \langle \theta^a_{I^c}(t), W_{I^c, I} \theta^a_{I^c}(t) \rangle = \mathbb{1}_{X_i \in I} \theta^a_{X_i}(t) \sum_{j, X_{i \rightarrow j}} W_{X_{i \rightarrow j}} \theta^a_{j}(t)
\]

\[
= \mathbb{1}_{X_i \in I} e^{T_X(t_i) + T_X(t)} \sum_{j, X_{i \rightarrow j}} W_{X_{i \rightarrow j}} e^{V_j(t_i) - V_X(t)}.
\]
using (11.3).

Hence, recalling notation (6.7), \( \tilde{W}_{I^c} = W_{I^c} + W_{I^c} \tilde{G}_\beta W_{I^c} \), we have

\[
\int_0^t \sum_{j, X_u \to j} W_{X_u, j} e^{T_{X_u(t)} + T_{X_u^*(t)}} e^{V_{j^*}(u) - V_{X_u^*(u)}} du
\]

\[
= \frac{1}{2} \left( \langle \theta_1(t), \beta_1 \theta_1(t) \rangle + \langle \theta_1^a(t), W_{I^c, I^c} \beta_1 \rangle - \langle 1_I, \beta_1 \theta_1 - \langle 1_I, \beta_1 \theta_1 \rangle \right) \).
\]

From the definition of \( Q_t^{W, \theta, 0}(d\beta_1, da_{I^c}) \), for \( \theta \in (0, \infty) \), \( \eta = 0 \), we see that

\[
e^{-\frac{1}{2}(W, \theta)} Q_t^{W, \theta, 0}(d\beta_1, da_{I^c})
\]

\[
= \frac{1}{\sqrt{2\pi} |\det \tilde{H}_\beta|} \prod_{i \in V_{I^c}} t_{i(t)}^\beta \left( e^{-\frac{1}{2}(\theta(t), W \theta(t))} Q_t^{W, \theta(t)}(d\beta_1, da_{I^c}) \right).
\]

Besides, we have the following simple proposition.

**Proposition 11.1.** For all \( t \geq 0 \),

\[
Q_t^{W, \theta(t)}(d\beta_1, da_{I^c}) = e^{V_{\beta_0}(t)} Q_t^{W, \theta(t)}(d\beta_1, da_{I^c})
\]

**Proof.** Using (11.3) and (11.4), we have \( (e^{V_{\theta^0}})_{I^c} = (\theta^0(t))_{I^c} \) and \( e^{V_{\theta^1}} = \tilde{G}_\beta \tilde{V}_{\beta} \) with \( \tilde{V}_{\beta} = W_{I^c, I^c} (\theta^0(t))_{I^c} \) corresponding to (6.6) with \( \theta(t) \) and \( \eta = 0 \). By Definition-Proposition 7.6, this yields the identity. \( \square \)

Set

\[
\tilde{F}_{i_0}^{W, \theta} = e^{-\frac{1}{2}(\theta, W, \theta)} F_{i_0}^{W, \theta}.
\]

Proposition 11.1 applied at times 0 and \( t > 0 \), combined with (11.5), implies that

\[
e^{-\int_0^t \sum_{j, X_u \to j} W_{X_u, j} e^{T_{X_u(t)} + T_{X_u^*(t)}} da_{V_0(t)} - V_0^*(0) - \sum_{i \in V_{I^c}} t_i(t) \frac{1}{F_{i_0}^{W}} Q_{I, i_0}^{W}(d\beta_1, da_{I^c})}
\]

\[
= e^{-\int_0^t \sum_{j, X_u \to j} W_{X_u, j} e^{T_{X_u(t)} + T_{X_u^*(t)}} da_{V_0(t)} - \sum_{i \in V_{I^c}} t_i(t) e^{-\frac{1}{2}(W, \theta)} Q_{I, i_0}^{W}(d\beta_1, da_{I^c})}
\]

\[
= \prod_{i \in V_{I^c}} t_i(t) F_{i_0}^{W, \theta(t)} e^{V_{\beta_0}(t)} Q_t^{W, \theta(t)}(d\beta_1, da_{I^c})
\]

\[
= e^{-\sum_{i \in V_{I^c}} t_i(t) \frac{\tilde{F}_{i_0}^{W, \theta(t)}}{F_{i_0}^{W, \theta(t)}} Q_{I, i_0}^{W, \theta(t)}(d\beta_1, da_{I^c})}.
\]
Hence, using \((11.2)\) and integrating the last expression on \(S_I \times A_I\), we obtain that under \(\tilde{\mathbb{P}}^W_{t_0} \) of the process in Theorem 7.1, the probability that the process \(X\) at time \(t\) has performed \(n\) jumps in infinitesimal time intervals \([t_i, t_i + dt_i]\), \(0 < t_1 < \cdots < t_n < t\), following the trajectory \(\sigma_0 = \sigma_1, \ldots, \sigma_n = j_0\), is equal to

\[
(11.6) \quad \left( \prod_{l=1}^{n} W_{\sigma_{l-1}, \sigma_l} e^{T_{\sigma_{l-1}}(t_l) + T_{\sigma_l}(t_l)} dt_l \right) e^{-\sum_{l \in V_0} T_{i_l}(t_l) \frac{\tilde{F}_t^W(\theta(t))}{F_{t_0}^W}}.
\]

From the definitions of \(\theta(t)\) and of \(\tilde{F}_t^W(\theta(t))\), we deduce that

\[
\tilde{F}_t^W(\theta(t)) = \theta_j \left( t \right) \tilde{F}_t^W(\theta(t)) = e^{T_{j_0}(t)} \tilde{F}_t^W(\theta(t)).
\]

Hence the expression \((11.6)\) is equal to the expression \((8.8)\) in Part I, which means that the probability of trajectories of \((X_t)\) are the same under \(\tilde{\mathbb{P}}^W_{t_0}\) and \(\tilde{\mathbb{P}}^W_{I,i_0}\), and thus \((X_t)\) has the same law under \(\tilde{\mathbb{P}}^W_{t_0}\) and \(\tilde{\mathbb{P}}^W_{I,i_0}\).

Let us now deduce i) of Theorem 7.8. Under \(\tilde{\mathbb{P}}^W_{t_0}\), the initial local time \((A_i)_{i \in V}\) is distributed according to \(\nu^W_{A_i, t_0}\) and \((U_i)_{i \in V}\) is defined in Theorem 4.2 i) by

\[
(11.7) \quad U_i := \lim_{t \to \infty} A_i + T_i(t) - t/N.
\]

Note that the almost sure convergence of the previous limit is not a consequence of Theorem 4.2 but of a more elementary Lemma 2.1 of Part I. Besides, \(U \in U^W_{t_0}\), which implies

\[
(11.8) \quad U_i := \lim_{t \to \infty} p_{t_0}^W(A + T(t) - t/N) = \lim_{t \to \infty} p_{t_0}^W(T(t) - t/N)
\]

since \(A \in A\) (see Lemma 2.3 of Part I). Hence, a.s. \(U\) can be retrieved from the infinite trajectory of the process \((X(t))\) (i.e. it is a.s. equal to a measurable function of the path). Moreover, consider the time changed process \(Z_t = X_{C^{a_1}(s)}\) defined in Theorem 4.2 ii). By an easy computation, conditionally on \(A\), the jump rate of \(Z\) at time \(t\) from \(i\) to \(j\) is

\[
W_{i,j} e^{T_{i}^j (C^{a_1}(s) - T_i^j (C^{a_1}(s)) + A_j^i - A_i^j)}.
\]

By \((11.8)\) it means that the asymptotic jump rate of \(Z\) from \(i\) to \(j\) is a.s.

\[
(11.9) \quad W_{i,j} e^U_{i,j} - U_{i,j}.
\]

Note that, since the graph is strongly recurrent and the *-VRJP visits infinitely often each vertex, the asymptotic jump rate of \(Z\) is a measurable function of the path of \((X_t)_{t \geq 0}\). This property will be useful to identify the limit of \((V_i(t))\) in terms of \((U_i)\). From \((11.7)\), we also obtain

\[
(11.10) \quad A = \frac{1}{2} (U - U^*) - \lim_{t \to \infty} \frac{1}{2} (T(t) - T^*(t)).
\]

It means that \((A_i)\) can also a.s. be retrieved from the trajectory of \((X_t)\). Besides, in Theorem 7.8 i), we have for all \(i \in I\)

\[
(11.11) \quad B_i = \sum_{j, i \to j} W_{i,j} e^{U_{i,j}^*} - U_{i,j}^*.
\]

Consider now \((X_t)\) under the law \(\tilde{\mathbb{P}}^W_{I,i_0}\) as defined at the beginning of the section. The process is defined as a mixture of processes with \((B_I, A_{I'})\) distributed according to \(Q^W_{I,i_0}\). The aim is to prove that \((B_I, A_{I'})\) can be retrieved a.s. by the same measurable functionals of the path of \((X_t)\) as in \((11.10)\) and \((11.11)\). Let \(U\) be defined from \((X_t)_{t \geq 0}\) by the last term
of formula (11.8): as we have proved that \((X_t)\) has the same law under \(\mathbb{P}^{W}_{I,i_0} \) and \(\mathbb{P}^{W}_{i_0}\), clearly the limit exists and is finite a.s. since it is the case under the law \(\mathbb{P}^{W}_{i_0}\). By definition, for all \(i,j \in V\) such that \(i \rightarrow j\), the jump rate of the process \((X_t)\) from \(i\) to \(j\) is equal to 

\[ e^{T_i(t)+T_j(t)}e^{V_j(t)-V_i(t)}, \]

hence for the time changed process \(Z\), at time \(s\) the jump rate is equal to 

\[ e^{V_j(s)}(C^{-1}(s))-V_i(s)(C^{-1}(s)). \]

Since the asymptotic jump rate is a measurable function of the path, \((X_t)\) has the same law under \(\mathbb{P}^{W}_{I,i_0} \) and \(\mathbb{P}^{W}_{i_0}\), and by (11.9), this implies that 

\[ \lim_{t \to \infty} V_j(t) - V_i(t) = U_{j*} - U_{i*}. \]

Since the graph is strongly connected, we deduce that the previous equality is true for all \(i,j \in V\). Since \(V_i(t) = T_i(t) + A_i\) for all \(i \in I^c\), we have 

\[ A_i = \lim_{t \to \infty} \left( \frac{1}{2}(V_i(t) - V_i(t)) - \frac{1}{2}(T_i(t) - T_i(t)) \right) = \frac{1}{2}(U_i - U_i) - \lim_{t \to \infty} \frac{1}{2}(T_i(t) - T_i(t)), \]

which matches with (11.10) on \(I^c\). Besides, by definition, we have \(B_i = \sum_{j,i \rightarrow j} W_{i,j}e^{V_j(t)-V_i(t)}\), for all \(i \in I\) and for all time \(t \geq 0\). Letting \(t\) go to infinity, we obtain \(B_i = \sum_{j,i \rightarrow j} W_{i,j}e^{U_j} - U_i\), for all \(i \in I\) which matches (11.11). This concludes the proof.

12. Proof of Lemma 7.9, Corollary 7.5 and Corollary 7.10.

12.1. Proof of Lemma 7.9. Fix \(i_0 \in V\), and let \(I = V \setminus \{i_0, i_0^*\}\). We first prove that \(\Xi_{i_0}(U^W_0) \subseteq D_{i_0}\). Given \(u \in U^W_0\), we denote by \(\beta_i = (\beta_i)_{i \in V}\) the vector defined by 

\[ \beta_i = \sum_{j,i \rightarrow j} W_{i,j}e^{u_j} - e^{u_i}, \quad \forall i \in V, \]

so that \(\beta_I = \Xi_{i_0}(u)\), see (7.3). Then \(u \in U^W_0\) implies \(\beta_i = \beta_{i^*}\) for all \(i \in V\), so that \(\beta_I \in S_I\). Furthermore, \(H_{\beta}e^{u^*} = 0\), where \(e^{u^*} = (e^{u_i^*})_{i \in V}\). Let 

\[ \hat{H}_\beta = \beta_I - W_{I,I}, \]

be defined as in (10.11): using the block matrix decomposition (10.10), we have 

\[ \left( \hat{H}_\beta e^{u^*} \right)_I = W_{I,I^c}(e^{u^*})_{I^c}. \]

Assumption (2) of Proposition A.5 in Appendix 12.2 is satisfied with \(A = \hat{H}_\beta\), \(x = (e^{u^*})_I\) and \(y = W_{I,I^c}(e^{u^*})_{I^c}\), using that \(G\) is strongly connected: therefore, \(\hat{H}_\beta\) is a non-singular \(M\)-matrix in the sense of Definition A.4, i.e. \(\hat{H}_\beta > 0\). Hence \(\beta_I \in D_{i_0}\).

Next, our goal is to show that, given \((\beta_i)_{i \in I} \in D_{i_0}\), there exists a unique \(u \in U^W_0\) such that \((\beta_i)_{i \in I} = \Xi_{i_0}(u)\), and that \(u\) is a differentiable function of \((\beta_i)_{i \in I}\). First assume that such an \(u \in U^W_0\) exists, and define \(\beta\) by (12.1). As before we have \(G_{\beta}, W\) and \(\hat{H}_\beta\) as in (10.11) and (10.12). Now, using (10.13), \(H_{\beta}e^{u^*} = 0\) is equivalent to

\[ \begin{pmatrix} \hat{H}_\beta - W_{I,I^c} & 0 \\ 0 & \hat{H}_\beta \end{pmatrix} \begin{pmatrix} (e^{u^*})_I \\ (e^{u^*})_{I^c} \end{pmatrix} = 0. \]
Therefore
\begin{equation}
(12.4) \quad \tilde{H}_\beta \left( e^{u_\ast} \right)_{I^*} = 0,
\end{equation}
which implies
\begin{equation}
(12.5) \quad \det \tilde{H}_\beta = 0.
\end{equation}

Let us first assume that \( i_0 \) is self-dual: then (12.5) is equivalent to
\begin{equation}
(12.6) \quad \beta_{i_0} = \tilde{W}_{i_0,i_0}
\end{equation}
and, using (12.2), \( e^{u_\ast} \) is given by
\begin{equation}
(12.7) \quad \begin{cases}
    e^{u_{i_0}\ast} = C \\
    (e^{u_\ast})_{I} = \hat{G}_\beta W_{I,i_0} e^{u_{i_0}}
\end{cases},
\end{equation}
where the constant \( C > 0 \) is uniquely determined by \( \sum_{i \in V} u_i = 0 \).

Let us now assume that \( i_0 \) is not self-dual, i.e. \( i_0 \neq i_0^* \). Then (12.5) implies
\begin{equation}
(12.8) \quad (\beta_{i_0} - \tilde{W}_{i_0,i_0})^2 = \tilde{W}_{i_0,i_0} \tilde{W}_{i_0,i_0}
\end{equation}
on the other hand, using \( e^{u_{i_0}}, e^{u_{i_0}^*}, W_{i_0,i_0} > 0 \) and (12.4), it holds that \( \beta_{i_0} > \tilde{W}_{i_0,i_0} \) and, subsequently
\begin{equation}
(12.9) \quad \beta_{i_0} = \tilde{W}_{i_0,i_0} + \sqrt{\tilde{W}_{i_0,i_0} \tilde{W}_{i_0,i_0}}.
\end{equation}
Therefore \( \beta_{i_0} = \beta_{i_0}^* \) is uniquely determined from \( (\beta_i)_{i \in I} \) by (12.9). On the other hand, (12.4) is equivalent to the combination of (12.9) and
\begin{equation}
(12.10) \quad e^{u_{i_0}^\ast} - u_{i_0} = \sqrt{\tilde{W}_{i_0,i_0} \tilde{W}_{i_0,i_0}}.
\end{equation}
Therefore, using (12.2), \( e^{u_\ast} \) is given by
\begin{equation}
(12.11) \quad \begin{cases}
    e^{u_{i_0}^\ast} = C \sqrt{\tilde{W}_{i_0,i_0}^\ast} \\
    e^{u_{i_0}^\ast} = C \sqrt{\tilde{W}_{i_0,i_0}^\ast} \\
    (e^{u_\ast})_{I} = \hat{G}_\beta W_{I,i_0} e^{u_{i_0}}
\end{cases},
\end{equation}
where again the constant \( C > 0 \) is uniquely determined by \( \sum_{i \in V} u_i = 0 \). Remark also that \( \hat{G}_\beta \) and \( \tilde{W}_{i_0,i_0}, W_{i_0,i_0} \), are continuous functions of \( \beta_I \) on the domain \( D_{i_0} \).

In summary, given \( (\beta_i)_{i \in I} \in D_{i_0} \), if \( i_0 \) self-dual (respectively if \( i_0 \neq i_0^* \)), then \( e^{u_\ast} \) is uniquely determined by (12.7) (resp. by (12.11)) which is \( C^1 \) for \( \beta_I \in D_{i_0} \). Besides, \( \beta_{i_0} \) is equal to (12.6) (resp. by (12.9)). Conversely, those definitions of \( e^{u_\ast} \) ensure that \( H_\beta e^{u_\ast} = 0 \), which enables us to conclude the proof of Lemma 7.9.

12.2. Proof of Corollary 7.5 and Corollary 7.10. By Proposition 6.7, since \( i_0 \in I^c \), we know that
\begin{equation}
Q_{I,c}^W(d\beta_I, da_I) = \nu_{S_I}^W W_{I,i_0} \nu_{A_{I^c}} W_{I^c,c} (d\beta_I, da_I^c)
\end{equation}
with \( \hat{\eta}_I = W_{I,i_0} \tilde{1}_{I^c} \). Assume first that \( i_0 = i_0^* \), then \( \nu_{A_{I^c}} W_{I^c} = 1 \) is the measure constant equal to 1, and the \( \beta_I \) marginal of \( Q_{I,c}^W(d\beta_I, da_I^c) \) is \( \nu_{S_I}^W W_{I,i_0} \tilde{1}_{I^c} (d\beta_I) \). With this notation we have,
using (12.5),
\[
\nu_W^{W_{I_{r},\hat{I}_{0}},\hat{X}_{I_{r}}}(d\beta_I) = \frac{1}{\sqrt{2\pi |S_1| \sqrt{|H_\beta|}}} e^{-\frac{1}{2} \sum_{i \in I} \beta_i + \frac{1}{2} \langle 1_t, W_{I_{r},I_{r}} \rangle - \frac{1}{2} \langle 1_e, W_{I_{r},I_{r}} G_\beta W_{I_{r},I_{r}} \rangle + \langle 1_t, W_{I_{r},I_{r}} \rangle d\beta_I}
\]
\[
= \frac{1}{\sqrt{2\pi |S_1| \sqrt{|H_\beta|}}} e^{-\frac{1}{2} \sum_{i \in I} \beta_i + \frac{1}{2} \langle 1_t, W_1 \rangle - \frac{1}{2} \hat{W}_{\hat{I}_{0},i_0} d\beta_I}
\]
(12.12) \[
= \frac{1}{\sqrt{2\pi |S_1| \sqrt{|H_\beta|}}} e^{-\frac{1}{2} \sum_{i \in V} \beta_i + \frac{1}{2} \langle 1_t, W_1 \rangle} d\beta_I.
\]
When \( i_0 \neq i_0^* \), \( I^c = \{ i_0, i_0^* \} \), the integral of \( \nu_W^{W_{I_{r},I_{c}}}(da_{I_{r}}) \) can be computed explicitly. Indeed
\[
\int_{A_{I_r}} \nu_W^{W_{I_{r},I_{c}}}(da_{I_{r}}) = \frac{1}{\sqrt{2\pi}} \left( W_{i_0, i_0^*} + W_{i_0^*, i_0} \right) \int_{-\infty}^{+\infty} e^{-a} \exp \left( -\frac{1}{2} W_{i_0, i_0^*} e^{2a} - \frac{1}{2} W_{i_0^*, i_0} e^{-2a} \right) da
\]
\[
= \frac{1}{\sqrt{2\pi}} \left( W_{i_0, i_0^*} + W_{i_0^*, i_0} \right) \int_{-\infty}^{+\infty} e^{-a} \exp \left( -\frac{1}{2} \sqrt{W_{i_0, i_0^*}} W_{i_0^*, i_0} e^{2a} + \sqrt{W_{i_0, i_0^*}} W_{i_0^*, i_0} e^{-2a} \right) da
\]
\[
= \frac{1}{\sqrt{2\pi}} \left( W_{i_0, i_0^*} + W_{i_0^*, i_0} \right) \int_{-\infty}^{+\infty} \cosh(y/2) \exp \left( -\sqrt{W_{i_0, i_0^*}} W_{i_0^*, i_0} \cosh(y) \right) dy
\]
\[
= \frac{1}{\sqrt{2\pi}} \left( W_{i_0, i_0^*} + W_{i_0^*, i_0} \right) \int_{-\infty}^{+\infty} e^{-a} \exp \left( -\frac{1}{2} \sqrt{W_{i_0, i_0^*}} W_{i_0^*, i_0} e^{2a} + \sqrt{W_{i_0, i_0^*}} W_{i_0^*, i_0} e^{-2a} \right) da
\]
where the integral in the third line is the modified Bessel function \( K_{\frac{1}{2}}(\sqrt{W_{i_0, i_0^*}} W_{i_0^*, i_0}) \) (recall that \( K_{\frac{1}{2}} \) has explicit value \( K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \)).

From \( \beta_I \) we can define \( \beta_i = \beta_{i_0} \) by (12.9). With this notation we deduce from Lemma 6.5 that
(12.13)
\[
\int_{A_{I_r}} Q_W^{W_{I_{r},I_{0}}}(d\beta_I, da_{I_{r}})
\]
\[
= \frac{1}{\sqrt{2\pi}} \left( W_{i_0, i_0^*} + W_{i_0^*, i_0} \right) \int_{-\infty}^{+\infty} \cosh(y/2) \exp \left( -\sqrt{W_{i_0, i_0^*}} W_{i_0^*, i_0} \cosh(y) \right) dy
\]
where the integral in the third line is the modified Bessel function \( K_{\frac{1}{2}}(\sqrt{W_{i_0, i_0^*}} W_{i_0^*, i_0}) \) (recall that \( K_{\frac{1}{2}} \) has explicit value \( K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \)).

Corollary 7.5 and Corollary 7.10 are therefore consequences of the following proposition.
PROPOSITION 12.1. The mixing measure $\mu_{i_0}^W$ of $(U_i)_{i \in I}$, starting from $i_0$ with conductances $W$, is equal to the pullback measure by $\Xi_{i_0}$ of the measure (12.12) when $i_0 = i_0^*$ (resp. (12.13) when $i_0 \neq i_0^*$).

PROOF. We start by the computation of the Jacobian.

LEMMA 12.2. By the change of variables $u = \Xi_{i_0}^{-1}(\beta_I)$, the measure $d\beta_I$ on $D_{i_0}$ is transformed to the measure on $U_0^W$ given by

$$d_{i_0} \sqrt{2^{-|V_I|}} \left( \prod_{i \in I} e^{-u_i-u_i^*} \right) \sqrt{|V_I|} \frac{D(W^u)}{\det(-K^u)} \sigma_0^W(du),$$

where $d_{i_0} = 2$ if $i_0 \neq i_0^*$ and $d_{i_0} = 1$ if $i_0 = i_0^*$.

PROOF. We need to remind a few definitions and properties from Part I Section 2. The space $S_0$ is defined by $S_0 = \{ x \in S, \sum_{i \in V} x_i = 0 \}$, and $P_{S_0} : U_0^W \mapsto S_0$ is the projection $P_{S_0}(u) = \frac{1}{2}(u + u^*)$. By Lemma 2.3, for any $u \in S_0$ there is a unique $a \in A$ such that $u + a \in U_0^W$. Hence, $P_{S_0}$ is invertible and we denote by

$$\xi : S_0 \mapsto U_0^W$$

its inverse. Remind that the volume measure on $\sigma_{U_0^W}(du)$ is defined as the pull-back of the euclidean measure $d\lambda_{S_0}$ on $S_0$ by the orthogonal projection $P_{S_0}$, i.e. $\sigma_{U_0^W}(du) = P_{S_0}^*(d\lambda_{S_0})$ (see Part I, (2.1)). The hyperplane tangent to $U_0^W$ at the point $u \in U_0^W$ can be parametrized as $((u^u)^{-1}(S_0))$, see Part I Lemma 8.1.

It will be convenient to define $\Xi : U_0^W \mapsto S$ as the function defined by

$$\Xi(u)_i = \sum_{j, i \in I} W_{i,j} e^{u_j - u_i}, \quad i \in V,$$

which extends the definition (7.3) so that $\Xi(u)_I = \Xi_{i_0}(u)$. Let $\tilde{I} = (V_0 \cup V_1) \cap I$ be the set obtained from $I$ by choosing one representative in each dual pair of points. Let us also denote by $\iota$ be the projection from $\mathbb{R}^V$ to $\mathbb{R}^{\tilde{I}}$. Hence, since on $S_0$ we chose the Euclidian volume measure and on $D_{i_0}$ the measure $\prod_{i \in I} d\beta_i$, we need to compute the determinant of the differential of the application from $S_0$ to $\mathbb{R}^{\tilde{I}}$ given by

$$\iota \circ \Xi \circ \xi,$$

where on $S_0$ we take an orthonormal base and the canonical base on $\mathbb{R}^{\tilde{I}}$.

Remind the definition of the matrix $K^u$ in (4.1). Let us compute the differential of $\Xi$. We have for $i \in V$,

$$\frac{\partial \Xi(u)}{\partial u_j} = W_{ji} e^{u_j - u_i} = e^{-u_i - u^*} K^u_{j,i}, \quad \text{if } j \neq i$$

$$\frac{\partial \Xi(u)}{\partial u_i} = - \sum_{j, i \in I} W_{ji} e^{u_j - u_i} = e^{-u_i - u^*} K^u_{i,i}$$

where in the last equality we used that $u \in U_0^W$, hence that $\sum_{j, j \in I} W_{j,i} = \sum_{j, j \rightarrow i} W_{i,j} = -K^u_{i,i}$. It follows that

$$D\Xi^*(du) = e^{-u - u^*} K^u(du), \quad \forall du \in T_u(U_0^W),$$
where \( e^{-u-u^*} \) is the operator of multiplication by \( e^{-u_i-u_i^*} \) at each point \( i \in V \). From Part I Lemma 8.1, we know that \( K^u \) is bijective from \( T_u(U^W_0) \) to \( S_0 \). Besides, for \( s \in S_0 \), \( (d\xi_s)^{-1} = (\mathcal{P}_{S_0})_{T^W_{\iota^*}}(U^W_0) \). Hence, \( K \circ d\xi_s \) is bijective on \( S_0 \) with determinant

\[
\frac{1}{\det_{S_0}((K^u)^{-1})}.
\]

To compute the Jacobian of the transformation (12.14), we still have to compute the determinant of

\[
dt \circ (e^{-u-u^*}) : S_0 \mapsto \mathbb{R}^\tilde{I},
\]

with the choice of base specified earlier, which is equal to (see below for the proof)

\[
(12.15) \quad \left( \prod_{i \in \tilde{I}} e^{-u_i-u_i^*} \right) \frac{1}{\sqrt{|V|}} \frac{d\nu_0}{\sqrt{2}V_1},
\]

Overall, by the change of variable given by \( \beta_I = \Xi_{\iota^*}(u) \), we obtain that

\[
d\beta_I = \left( \prod_{i \in \tilde{I}} e^{-u_i-u_i^*} \right) \frac{1}{\sqrt{|V|}} \frac{d\nu_0}{\sqrt{2}V_1} \left( \det_{S_0}((K^u)^{-1}) \right)^{-1} \sigma_0^W (du).
\]

We conclude the proof of the lemma using elementary computations on determinants given in formula (9.19) and (9.20).

\[\square\]

**Proof of Formula (12.15).** For \( i \in V \), we set \( d_i = 1 \) if \( i = i^* \) and \( d_i = 2 \) if \( i \neq i^* \). We have that \( dt \circ (e^{-u-u^*}) = (e^{-u-u^*}) \circ dt \) so that we have to compute the Jacobian of the change of variable given by \( \iota \) from \( S_0 \) with the Euclidian volume measure to \( \mathbb{R}^{\tilde{I}} \) with the measure \( \prod_{i \in \tilde{I}} d\beta_i \). On \( S_0 \), we choose the base \( (e_i)_{i \in \tilde{I}} \) given by

\[
e_i = \begin{cases} 
(\delta_i + \delta_{i^*}) - (\delta_{i^*} + \delta_i^*), & \text{if } i \neq i^* \\
\delta_i - \frac{1}{2} (\delta_{i^*} + \delta_i^*), & \text{if } i = i^*
\end{cases}
\]

With the basis \( (e_i)_{i \in \tilde{I}} \) on \( S_0 \) and the canonical base on \( \mathbb{R}^{\tilde{I}} \), \( \iota \) is represented by the identity matrix. Besides, by a simple computation, we have

\[
(e_i, e_j) = \delta_{i=j} d_i + \frac{d_i d_j}{d_0},
\]

so that the determinant of the Gramm matrix of the base \( (e_i)_{i \in \tilde{I}} \) is given by

\[
\det \left( (e_i, e_j) \right)_{i,j \in \tilde{I}} = \left( \prod_{i \in \tilde{I}} d_i \right) \left( 1 + \sum_{i \in \tilde{I}} \frac{d_i}{d_0} \right) = \frac{1}{d_0^2} \left( \prod_{i \in \tilde{I}} d_i \right) \left( d_0 + \sum_{i \in \tilde{I}} d_i \right) = \frac{2|V|}{d_0^2}.
\]

\[\square\]

Let us denote by \( \nu_{i_0}^W(d\beta_I) \) the measures which appear in (12.12) and (12.13), depending on whether \( i_0 = i_0^* \) or \( i_0 \neq i_0^* \). Remind that, if \( u \in U^W_0 \) and \( \beta_I \in D_{i_0} \) are related by \( \beta_I = \Xi_{i_0}(u) \), then \( u \) can be computed with (12.7) and (12.11), and that \( \beta := \Xi(u) \) can be computed on \( i_0, i_0^* \) from \( \beta_I \) by (12.6) and (12.9).

**Remark 12.3.** The law \( \nu_{i_0}^W(d\beta_I) \) is in fact the law of the asymptotic jump rate \( (\mathcal{B}_i)_{i \in V} \) of the \( \ast \)-VRJP on the full set of vertices, mentionned in Remark 7.4. It is an easy consequence of the current proof.
If \( i_0 \neq i_0^* \), using Schur complement, we can write that
\[
\det (H_\beta)_{V \setminus \{i_0\}, V \setminus \{i_0^*\}} = \left( \beta_{i_0} - W_{i_0} \tilde{G}_\beta W_{i_0^*} \right) \det (H_\beta) = \sqrt{W_{i_0^*}^{-1} W_{i_0}^{-1} \det \left( H_\beta \right)},
\]
where we use (12.9) in the last equality. Besides, by factorizing \( e^{-u_i - u_i^*} \) on each line we deduce
\[
\det (H_\beta)_{V \setminus \{i_0\}, V \setminus \{i_0^*\}} = \left( \prod_{i \neq i_0} e^{-u_i - u_i^*} \right) D(W^u).
\]
Combining with (12.13) we obtain that with the notation \( u = \Xi^{-1}(\beta) \) as before,
\[
\nu^W_{i_0^*}(d\beta_1) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{u_{i_0}^*} \left( \prod_{i \neq i_0} e^{u_i + u_i^*} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \sum_{i,j} W_{i,j}(e^{u_i + u_j} - 1)} \frac{1}{\sqrt{D(W^u)}} d\beta_1,
\]
using (12.10) in the last equality. Together with Lemma 12.2, this implies that by the change of variable \( u = \Xi^{-1}(\beta_1) \), we have the relation
\[
\nu^W_{i_0^*}(d\beta_1) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{u_{i_0}^*} \left( \prod_{i \neq i_0} e^{-u_i} \right) e^{-\frac{1}{2} \sum_{i,j} W_{i,j}(e^{u_i + u_j} - 1)} \frac{1}{\sqrt{D(W^u)}} \frac{D(W^u)}{\det A \sigma^0} \sigma^W_{u}(du).
\]
This concludes the proof when \( i_0 \neq i_0^* \). The computation is similar and even simpler in the case \( i_0 = i_0^* \). \( \square \)

**APPENDIX: PROOF OF LEMMA 2.1**

**STEP 1: PROOF FOR THE RANDOMIZED +/- VRJP.** Let us first prove the result when the initial random rates \( W : A \) are random, \( A \sim \nu_{i_0}^W \), as in section 3. We want to show that \( T(t) - t/N \) converges to a random variable \( \alpha \in \mathcal{U}^W_0 \) for \( \nu^W_{A,i_0} \) almost all \( A \), hence for Lebesgue-almost all \( A \in \mathcal{A} \) since \( \nu^W_{A,i_0} \) is absolutely continuous. We use the partial exchangeability of the process proved in Proposition 3.4. We know from Proposition 3.4 (using the notation \( C \) therein) that, under the randomized law \( \mathbb{P}_{i_0}^W \), \( Z_s = X_{C^{-1}(s)} \) is a mixture of Markov jump processes, hence that
\[
\lim_{s \to \infty} \frac{1}{s} \ell^Z_i(s) = V_i
\]
exists a.s. \((V_i \text{ is random})\), where \( \ell^Z_i(s) \) is the local time of \( Z \) at time \( s \), see (4.3).

From (8.11), with \( s = C(t) \),
\[
\ell^Z_i(s) + \ell^Z_i(t - 1) = e^{T_i(t) + T_i(t)} - 1, \quad \forall i \in V,
\]
hence,
\[
\lim_{t \to \infty} \left( T_i(t) + T_i(t) - \log(C(t)) \right) = \log(V_i + V_i^*).
\]
Since
\[
2t = \sum_{i \in V} (T_i(t) + T_i(t)) = \sum_{i \in V} \log(1 + \ell^Z_i(s) + \ell^Z_i(t))
\]
we have,
\[ \lim_{t \to \infty} 2t - N \ln C(t) = \sum_{i \in V} \log(V_i + V_i^*), \]
hence, we deduce
\[ \lim_{t \to \infty} T_i(t) + T_i^*(t) - 2t/N = \log(V_i + V_i^*) - \frac{1}{N} \left( \sum_{i \in V} \log(V_i + V_i^*) \right). \]

Let us now conclude that \( T(t) - t/N \) converges a.s. and that its limit \( U \) is in \( \mathcal{U}_0^{W^A} \). Let
\[ H(t) = \mathcal{P}_{\mathcal{U}_0^{W^A}}(T(t) - t/N). \]
Since \( T_i(t) + T_i^*(t) - 2t/N \) convergences, it implies by lemma 2.3 that \( H(t) \) converges a.s. to a point \( U \) in \( \mathcal{U}_0^{W^A} \). It remains to show that \( \xi_i(t) = T_i(t) - H_i(t) \) converges a.s. to 0.

Let \( N_{i,j}(t) \) be the number of crossings of the directed edges \( (i,j) \) or \( (j^*, i^*) \) at time \( t > 0 \). Since the discrete time process associated with the randomized *-VRJP \( X_t \) is partially exchangeable,
\[ \frac{N_{i,j}(t)}{\sum_{(k,l) \in \tilde{E}} \tilde{N}_{k,l}(t)} \]
converges a.s. to a random variable \( \xi_{i,j} \).

For each edge \( e = (i,j) \sim (j^*, i^*) \) of \( \tilde{E} \), let \( \tau_k^e \) be a point process at rate \( W_{i,j} e^dt \). Then the process \( X_t \) can be represented as follows: when \( X_t \) is at position \( i \), it waits for the first time that, for an adjacent edge \( (i,j) \), \( T_i(t) + T_j^*(t) \) coincides with \( \tau_k^i,j \) for some \( k \), at which time it jumps to \( j \). This implies that, for all \( (i,j) \in E \),
\[ \tau_{N_{i,j}}(t) \leq T_i(t) + T_j^*(t) < \tau_{N_{i,j}+1}(t). \]

Now, direct computation yields that \( e^{\tau_k^e} - 1 \) is a Poisson Point Process at rate \( W_{i,j} e^\tau_k \) for all \( (i,j) \in \tilde{E} \), hence \( W_{i,j} e^\tau_k \sim k \) a.s.. This implies that
\[ W_{i,j} e^{T_i(t) + T_j^*(t)} \sim N_{i,j}(t) \text{ a.s.} \]
Hence the convergence of the ratios \( \frac{N_{i,j}(t)}{\sum_{(k,l) \in \tilde{E}} \tilde{N}_{k,l}(t)} \) implies the a.s. convergence of \( T_i(t) + T_j^*(t) - 2t/N \). Taking into account that
\[ \text{div}(N(t)) = \delta_{i_0} - \delta_{X_t}, \]
it implies that
\[ \lim_{t \to 0} \text{div}(W_{i,j} e^{T_i(t) + T_j^*(t) - 2t/N}) = 0, \]
hence, that \( \lim_{t \to \infty} \xi(t) = 0 \).

**Step 2: General case.** Now we consider the *-VRJP with initial rates \( W_{i,j} \). Let \( i_0 \in V \), we define \( \tau \) as the first return time to \( i_0 \) after the cover time of the graph,
\[ \tau = \inf \{ t > \tilde{\tau} , X_t = i_0 \text{ and } \exists t', \tilde{\tau} < t' < t \text{ such that } X_{t'} \neq i_0 \}, \]
where
\[ \tilde{\tau} = \inf \{ t > 0 , T_i(t) > 0 \forall i \in V \}. \]
From (8.5) (with \( A = 0 \)) we see that, under the law of the non-randomized VRJP, summing on all possible discrete paths, the distribution of the local time \( (T_i(\tau)) \) at time \( \tau \) has a density
on $(0,\infty)^V$, that we denote $g_W^V((t_i)_{i\in V})$, and that this density is continuous in the initial conductances $W$. Besides, conditioned on $F_\tau^X$, $(X_{t+\tau})_{t\geq 0}$ has the law of the $\star$-VRJP starting from the initial rates $W_{i_0}^{T(\tau)}$. Hence, conditioned on $H:=\mathcal{P}_S(T(\tau))$, $\mathcal{P}_A(T(\tau))$ is absolutely continuous with respect to $\nu_{X_\tau}^W$, hence $T(t+\tau) - T(\tau) - t/N$ converges a.s. to a point in $U_0^{W^\tau}$. This implies, that a.s. $T(t) - t/N$ converges a.s. to a point $U \in U_0^W$. 

\section*{APPENDIX: RESULTS ON M-MATRICES}

We will need the following results on $M$-matrices, taken from [7], chapter 6. The following definition is equivalent to the more classical definition (1.2) of \cite{Nonnegative matrices in the mathematical sciences}, using theorem 2.3, property (G20) in [7].

**Definition A.4.** A real $n \times n$ matrix $A$ is called a non-singular $M$-matrix if it has nonpositive off-diagonal coefficients, i.e.

$$a_{i,j} \leq 0, \quad \forall i \neq j,$$

and if the real parts of all of its eigenvalues are positive.

**Proposition A.5** (theorem 2.3, chapter 6, [7]). Assume that $A$ is a real $n \times n$ matrix with nonpositive off-diagonal coefficients. The assertion ”$A$ is a non-singular $M$-matrix ” is equivalent to each of the following assertions

1. (Property (N38) in [7]) $A$ is invertible and $A^{-1}$ has nonnegative coefficients. If moreover $A$ is irreducible, this implies that $A^{-1}$ has positive coefficients by theorem 2.7 [7].
2. (Property (L32) in [7]) There exists a vector $x$ with positive coefficients such that $y := Ax$ has nonnegative coefficients and such that if $y_{j_0} = 0$ for some $j_0$, then there exists a sequence of indices $j_1, \ldots, j_k$ with $y_{j_k} > 0$ such that $a_{j_l,j_{l+1}} \neq 0$ for all $l = 0, \ldots, k-1$. 

**Funding.** This work is supported by National Science Foundation of China (NSFC) grant No. 11771293, by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon and the project ANR LOCAL (ANR-22-CE40-0012-02) operated by the Agence Nationale de la Recherche (ANR)

**REFERENCES**

[1] Angel, O., Crawford, N. and Kozma, G. (2014). Localization for linearly edge reinforced random walks. *Duke Mathematical Journal* **163** 889–921.

[2] Bacallado, S. (2011). Bayesian analysis of variable-order, reversible Markov chains. *Ann. Statist.* **39** 838–864. https://doi.org/10.1214/10-AOS857 MR2816340

[3] Bacallado, S., Chodera, J. D. and Pande, V. (2009). Bayesian comparison of Markov models of molecular dynamics with detailed balance constraint. *J. Chem. Phys.* **131** 045106.

[4] Bacallado, S., Sabot, C. and Tarres, P. (2020). The $\star$-Edge Reinforced Random Walk. *preprint*, arXiv:2102.08984.

[5] Basdevant, A.-L. and Singh, A. (2012). Continuous-time vertex reinforced jump processes on Galton-Watson trees. *Ann. Appl. Probab.* **22** 1728–1743.

[6] Bateurschmidt, R., Helmuth, T. and Swan, A. (2021). The geometry of random walk isomorphism theorems. *Ann. Inst. Henri Poincaré Probab. Stat.* **57** 408–454. https://doi.org/10.1214/20-aap1083 MR4255180

[7] Berman, A. and Plemmons, R. (1979). *Nonnegative matrices in the mathematical sciences*. Academic Press, New York.

[8] Coppersmith, D. and Diaconis, P. (1986). Random walks with reinforcement. *Unpublished manuscript*.

[9] Cox, D. A. (2012). *Galois theory*, second ed. *Pure and Applied Mathematics (Hoboken)*, John Wiley & Sons, Inc., Hoboken, NJ.
[10] DAVIS, B. and VOLKOV, S. (2002). Continuous time vertex-reinforced jump processes. *Probab. Theory Related Fields* **123** 281–300. https://doi.org/10.1007/s004400100189 MR1900324 (2003e:60078)

[11] DAVIS, B. and VOLKOV, S. (2004). Vertex-reinforced jump processes on trees and finite graphs. *Probab. Theory Related Fields* **128** 42–62. https://doi.org/10.1007/s00440-003-0286-y MR2027294 (2004m:60079)

[12] DIACONIS, P. and FREEDMAN, D. (1980). de Finetti’s theorem for Markov chains. *Ann. Probab.* **8** 115–130. MR556418 (81f:60090)

[13] DISERTORI, M., SABOT, C. and TARRES, P. (2015). Transience of edge-reinforced random walk. *Communications in Mathematical Physics* **339** 121–148.

[14] DISERTORI, M. and SPENCER, T. (2010). Anderson localization for a supersymmetric sigma model. *Comm. Math. Phys.* **300** 659–671. https://doi.org/10.1007/s00220-010-1124-6 MR2736958

[15] DISERTORI, M., SPENCER, T. and ZIRNBAUER, M. R. (2010). Quasi-diffusion in a 3D supersymmetric hyperbolic sigma model. *Comm. Math. Phys.* **300** 435–486. https://doi.org/10.1007/s00220-010-1117-5 MR2728731 (2011k:82035)

[16] DOROFFIN, R. L., SUKHOV, Y. M. and FRICTS, U. (1988). A. N. Kolmogorov—founder of the theory of reversible Markov processes. *Uspekhi Mat. Nauk* **43** 167–188.

[17] ENRIQUEZ, N. and SABOT, C. (2006). Random walks in a Dirichlet environment. *Electron. J. Probab.* **11** no. 31, 802–817. https://doi.org/10.1214/EJP11-350 MR2242664

[18] FREEDMAN, D. A. (1965). Bernard Friedman’s urn. *Ann. Math. Statist.* **36** 956–970. MR0177432 (31 #1695)

[19] GALUZZI, M. Equations et substitutions avant Galois : Lagrange et Cauchy. *preprint*.

[20] KEANE, M. S. and ROLLES, S. W. W. (2000). Edge-reinforced random walk on finite graphs. *Infinite dimensional stochastic analysis (Amsterdam, 1999)* R. Neth. Acad. Sci. 217–234.

[21] KOZMA, G. and PELED, R. (2021). Power-law decay of weights and recurrence of the two-dimensional VRJP. *Electron. J. Probab.* **26** Paper No. 82, 19. https://doi.org/10.1214/21-ejp639 MR4278593

[22] LAGRANGE, J. L. (1770). Réflexions sur la résolution algébrique des équations. *Mémoires de l’Académie royale des sciences et Belles-Lettres de Berlin*, pages 205–421. *Publié dans 1772, Oeuvres*, 3, 205–421.

[23] LETAC, G. and WESOLOWSKI, J. (2020). Multivariate reciprocal inverse Gaussian distributions from the Sabot-Tarrès-Zeng integral. *J. Multivariate Anal.* **175** 104559, 18.

[24] POUDDEVIGNE-AUBOIRON, R. (2022). Monotonicity and phase transition for the VRJP and the ERRW. *J. Eur. Math. Soc.*

[25] SABOT, C. (2011, arXiv:0811.4285). Random walks in random Dirichlet environment are transient in dimension $d \geq 3$. *Probab. Theory Related Fields* **151** 297–317. https://doi.org/10.1007/s00440-010-0300-0 MR2834720

[26] SABOT, C. (2021). Polynomial localization of the 2D-vertex reinforced jump process. *Electron. Commun. Probab.* **26** Paper No. 1, 9. https://doi.org/10.1214/20-ecp356 MR4218029

[27] SABOT, C. and TARRÈS, P. (2015). Edge-reinforced random walk, vertex-reinforced jump process and the supersymmetric hyperbolic sigma model. *J. Eur. Math. Soc. (JEMS)* **17** 2353–2378. https://doi.org/10.4171/JEMS/559 MR3420510

[28] SABOT, C., TARRÈS, P. and ZENG, X. (2017). The vertex reinforced jump process and a random Schrödinger operator on finite graphs. *Ann. Probab.* **45** 3967–3986. https://doi.org/10.1214/16-AOP1155 MR3729620

[29] SABOT, C. and TOURNIER, L. (2017). Random walks in Dirichlet environment: an overview. *Ann. Fac. Sci. Toulouse Math.* (6) **26** 463–509. https://doi.org/10.5802/afst.1542 MR3640900

[30] SABOT, C. and ZENG, X. (2019). A random Schrödinger operator associated with the vertex reinforced jump process on infinite graphs. *J. Amer. Math. Soc.* **32** 311–349. https://doi.org/10.1090/jams/906 MR3904155

[31] YAGLOM, A. M. (1947). On the statistical treatment of Brownian motion. *Doklady Akad. Nauk SSSR (N.S.)* **56** 691–694.

[32] YAGLOM, A. M. (1949). On the statistical reversibility of Brownian motion. *Mat. Sbornik N.S.* **24(66)** 457–492.

[33] ZENG, X. (2016). How vertex reinforced jump process arises naturally. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 1061–1075. https://doi.org/10.1214/14-AIHP671 MR3531700

[34] ZIRNBAUER, M. R. (1991). Fourier analysis on a hyperbolic supermanifold with constant curvature. *Communications in mathematical physics* **141** 503–522.