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Grubb, Gerd

Published in:
Mathematica Scandinavica

DOI:
10.7146/math.scand.a-138002

Publication date:
2023

Document version
Peer reviewed version

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Citation for published version (APA):
Grubb, G. (2023). Weyl asymptotics for fractional-order Dirichlet realizations in nonsmooth cases. Mathematica Scandinavica, 129(3), 593-612. https://doi.org/10.7146/math.scand.a-138002
WEYL ASYMPTOTICS FOR FRACTIONAL-ORDER DIRICHLET REALIZATIONS IN NONSMOOTH CASES

GERD GRUBB

Dedicated to Vsevolod A. Solonnikov, on the occasion of his ninetieth birthday

Abstract
Let $P$ be a symmetric $2a$-order classical strongly elliptic pseudodifferential operator with even symbol $p(x,\xi)$ on $\mathbb{R}^n$ ($0 < a < 1$), for example a perturbation of $(-\Delta)^a$. Let $\Omega \subset \mathbb{R}^n$ be bounded, and let $P_D$ be the Dirichlet realization in $L_2(\Omega)$ defined under the exterior condition $u = 0$ in $\mathbb{R}^n \setminus \Omega$. When $p(x,\xi)$ and $\Omega$ are $C^\infty$, it is known that the eigenvalues $\lambda_j$ (ordered in a nondecreasing sequence for $j \to \infty$) satisfy a Weyl asymptotic formula
\[
\lambda_j(P_D) = C(P,\Omega)j^{2a/n} + o(j^{2a/n}) \text{ for } j \to \infty,
\]
with $C(P,\Omega)$ determined from the principal symbol of $P$. We now show that this result is valid for more general operators with a possibly nonsmooth $x$-dependence, over Lipschitz domains, and that it extends to $\tilde{P} = P + P' + P''$ where $P'$ is an operator of order $< \min\{2a, a + \frac{1}{2}\}$ with certain mapping properties, and $P''$ is bounded in $L_2(\Omega)$ (e.g. $P'' = V(x) \in L_\infty(\Omega)$). Also the regularity of eigenfunctions of $P_D$ is discussed.

1. Introduction
Consider a $2a$-order classical strongly elliptic pseudodifferential operator $P$ on $\mathbb{R}^n$ ($0 < a < 1$) with a symbol $p(x,\xi)$ with even parity in $\xi$ (cf. [8]), having $C^\tau$-dependence on $x$ for some $\tau > 2a$. A standard example is the fractional Laplacian $(-\Delta)^a$, which is $x$-independent, but our methods also allow $x$-dependent symbols. Let $\Omega \subset \mathbb{R}^n$ be bounded open with $C^{1+\tau}$-boundary. Denote by $P_D$ the $L_2$-Dirichlet realization of $P$ with domain $D(P_D) = \{u \in H^a(\Omega) \mid (Pu)_{|\Omega} \in L_2(\Omega)\}$. Its spectrum $\Sigma$ is discrete and contained in a convex sectorial region in $\mathbb{C}$ opening to the right.

Section 2 contains preliminary material.
In Section 3, we recall the results from [17] on the regularity of the eigenfunctions $u_\lambda$:
\[
u_\lambda \in \mathcal{D}'(\Omega) \text{ for } t = \min\{2a, \tau - a\} - \varepsilon; \text{ small } \varepsilon > 0,
\]

2020 Mathematics Subject Classification. 35S15; 47G30; 35J25; 60G51.
where \( d(x) = \text{dist}(x, \partial \Omega) \). Also the \( L_q \) Dirichlet realizations are considered for \( 1 < q < \infty \); they have the same spectrum and eigenfunctions as for \( q = 2 \). Slightly sharper results than \([4]\) hold in \( C^\infty \)-cases \([14]\).

In Sections 4 and 5, we study the asymptotic behavior of the eigenvalues, restricting the attention to selfadjoint operators. The main purpose is to show that the eigenvalues \( \lambda_j(P_D) \), ordered nondecreasingly and repeated according to multiplicity, satisfy

\[
\lambda_j(P_D) = C(P, \Omega) j^{2a/n} + o(j^{2a/n}) \quad \text{for } j \to \infty,
\]

(2)

where \( C(P, \Omega) = \left( \frac{1}{n(2\pi)^n} \int \int_{|\xi|=1} |p_0(x, \xi)|^{-n/2a} d\omega(\xi) dx \right)^{-2a/n} \); with \( p_0 \) denoting the principal symbol of \( P \).

This was shown for \((-\Delta)^a\) already by Blumenthal and Getoor in \([2]\) when \( \Omega \) has bounded volume. See also e.g. Chen and Song \([3]\), Geisinger \([7]\), Dyda, Kuznetsov and Kwapisz \([5]\). Also Frank and others contributed to the question, see the survey Frank \([6]\) for an account of related eigenvalue estimates (such as the behavior of sums of eigenvalues), including fractional Schrödinger operators \((-\Delta)^a + V(x)\). Two-term asymptotics were put forward by Ivrii \([19]\). In \([14]\), we established (2) for \( x \)-dependent operators \( P \) with smooth symbols over smooth regions \( \Omega \), also giving estimates of \( s \)-numbers (singular values) in nonsymmetric cases.

Our present aim is to allow nonsmooth regions \( \Omega \) as well as more general operators \( P \) with possibly nonsmooth \( x \)-dependence. For the main results we focus on selfadjoint cases, where there are convenient perturbation tools. Nonsmooth symbols \( p(x, \xi) \) are treated in Section 4, and estimates when \( \Omega \) is Lipschitz are obtained in Section 5.

Besides \( P \), we consider suitable lower-order perturbations

\[ \tilde{P} = P + P' + P'', \]

(3)

where \( P' \) is of order \( < \min\{2a, a+\frac{1}{2}\} \) with certain mapping properties in Sobolev spaces, and \( P'' \) is bounded in \( L_2(\Omega) \) (e.g. the multiplication by a potential \( V(x) \in L_\infty(\Omega) \)). Then the Dirichlet realization \( \tilde{P}_D \) is shown to have the same domain \( H^{a(2a)}(\Omega) \) as \( P_D \) (Theorem \([11]\)), and (2) is shown to hold for \( \tilde{P}_D \) with the same constant \( C(P, \Omega) \) when \( \Omega \) is Lipschitz (Theorem \([5,3]\)).

2. Preliminaries

The setting is the same as in \([17]\), so we shall just rapidly repeat the part of the notation that will be needed here.

The space \( C^k(\mathbb{R}^n) \equiv C^k_0(\mathbb{R}^n) \) consists of \( k \)-times differentiable functions with bounded norms \( \|u\|_{C^k} = \sup_{|\alpha| \leq k, x \in \mathbb{R}^n} |D^\alpha u(x)| \) (\( k \in \mathbb{N}_0 = \{0, 1, 2, \ldots \} \)). The
The Bessel-potential spaces \( H^s_\nu(\mathbb{R}^n) \) are defined for \( s \in \mathbb{R} \), \( \nu \in \mathbb{R} \), and are denoted \( C^{k,\sigma}(\mathbb{R}^n) \) for \( k \in \mathbb{N}_0 \) and \( 0 < \sigma \leq 1 \), consist of function with bounded norms \( \|u\|_{C^{k,\sigma}} = \|u\|_{C^k} + \sup_{|\alpha|=k} \sup_{x,y} |D^\alpha u(x) - D^\alpha u(y)|/|x-y|^\sigma \). For \( \sigma = 1 \), they are the Hölder spaces, also denoted \( C^{k+\sigma}(\mathbb{R}^n) \). For \( \sigma = 1 \), they are Lipschitz spaces. Occasionally, we refer to the scale of Hölder-Zygmund spaces \( C^\alpha_s(\mathbb{R}^n) \), \( s \in \mathbb{R} \), which coincide with the Hölder spaces for \( s > 0 \), \( s \notin \mathbb{N} \), and is preserved under interpolation. There are similar spaces over subsets of \( \mathbb{R}^n \).

The halfspaces \( \mathbb{R}^n_{\pm} \) are defined by \( \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n \mid x_n \geq 0 \} \), with points denoted \( x = (x',x_n) \), \( x' = (x_1,\ldots,x_{n-1}) \). For a given real function \( \zeta \in C^t(\mathbb{R}^{n-1}) \), \( t \geq 0 \), we define the curved halfspace \( \mathbb{R}^n_\zeta \) by \( \mathbb{R}^n_\zeta = \{ x \in \mathbb{R}^n \mid x_n > \zeta(x') \} \); it is a \( C^t \)-domain. By a bounded \( C^t \)-domain \( \Omega \) we mean the following: \( \Omega \subset \mathbb{R}^n \) is open and bounded, and every boundary point \( x_0 \) has an open neighborhood \( U \) such that, after a translation of \( x_0 \) to \( 0 \) and a suitable rotation, \( U \cap \Omega \) equals \( U \cap \mathbb{R}^n_\zeta \) for a function \( \zeta \in C^t(\mathbb{R}^{n-1}) \) with \( \zeta(0) = 0 \). There are similar definitions for \( C^{k,\sigma}_t \)-spaces.

Restriction from \( \mathbb{R}^n_\zeta \) to \( \mathbb{R}^n_\pm \) (or from \( \mathbb{R}^n_\zeta \) to \( \Omega \)) is denoted \( r^\pm \), extension by zero from \( \mathbb{R}^n_\pm \) to \( \mathbb{R}^n_\zeta \) (or from \( \Omega \) resp. \( \mathbb{R}^n_\zeta \) to \( \mathbb{R}^n_\pm \)) is denoted \( e^\pm \).

When \( \Omega \) is a \( C^{1+\tau} \)-domain with \( \tau > 0 \), we denote by \( d(x) \) a function that is \( C^{1+\tau} \) on \( \overline{\Omega} \), positive on \( \Omega \) and vanishes only to the first order on \( \partial \Omega \) (i.e., \( d(x) = 0 \) and \( \nabla d(x) \neq 0 \) for \( x \in \partial \Omega \)). It is equivalent to the distance \( d_{\Omega}(x) = \text{dist}(x,\partial \Omega) \) near \( \partial \Omega \). (More details in [11, I.7].)

The Bessel-potential spaces \( H^s_\nu(\mathbb{R}^n) \) are defined for \( s \in \mathbb{R} \), \( 1 < q < \infty \), by

\[
H^s_\nu(\mathbb{R}^n) = \{ u \in S'(\mathbb{R}^n) \mid F^{-1}(\xi^s \hat{u}) \in L_q(\mathbb{R}^n) \},
\]

where \( (\xi) = (|\xi|^2 + 1)^{\frac{\nu}{2}} \), and \( F \) is the Fourier transform \( \hat{u}(\xi) = (Fu)(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) \, dx \), with inverse \( F^{-1}v(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} v(\xi) \, d\xi \).

Along with the spaces \( H^s_\nu(\mathbb{R}^n) \) defined in [11], there are the two scales of spaces associated with \( \Omega \) for \( s \in \mathbb{R} \):

\[
\mathcal{D} \nu_q(\Omega) = \{ u \in \mathcal{D}'(\Omega) \mid u = r^+ U \text{ for some } U \in H^s_\nu(\mathbb{R}^n) \},
\]

the restricted space,

\[
\hat{H}^s_\nu(\Omega) = \{ u \in H^s_\nu(\mathbb{R}^n) \mid \text{supp } u \subset \Omega \},
\]

the supported space;

here \( \text{supp } u \) denotes the support of \( u \) (the complement of the largest open set where \( u = 0 \)).
A pseudodifferential operator (ψdo) \( P \) on \( \mathbb{R}^n \) is defined from a function \( p(x, \xi) \) on \( \mathbb{R}^n \times \mathbb{R}^n \), called the symbol, by

\[
P u = \text{Op}(p(x, \xi)) u = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) \, d\xi = \mathcal{F}^{-1}_{\xi \rightarrow x}(p(x, \xi) \mathcal{F} u(\xi)),
\]

using the Fourier transform \( \mathcal{F} \). An introduction to ψdo’s is given e.g. in [13], Ch. 7–8, and a description with further references and an inclusion of results for operators with nonsmooth symbols can be found in [1].

The subspace of classical symbols is denoted \( S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \). A classical symbol \( p(x, \xi) \) is said to be strongly elliptic when \( \text{Re} \, p_0(x, \xi) \geq c|\xi|^m \) for all \( \xi \in \mathbb{R}^n \), with global estimates in \( x \in \mathbb{R}^n \). For \( x \) then, the terms in the symbol expansion \( p(x, \xi) \sim \sum_{j \in \mathbb{N}_0} p_j(x, \xi) \) with \( p_j \) homogeneous in \( \xi \) of degree \( m - j \) for all \( |\xi| \geq 1 \) and \( j \in \mathbb{N}_0 \), such that

\[
\partial^\alpha_p \partial^\beta_p (p(x, \xi) - \sum_{j < J \in \mathbb{N}_0} p_j(x, \xi)) = O(|\xi|^{m-|\alpha|-J}) \quad \text{for all } \alpha, \beta \in \mathbb{N}_0^n, J \in \mathbb{N}_0.
\]

The subspace of classical symbols is denoted \( S^m(\mathbb{R}^n \times \mathbb{R}^n) \).

A classical symbol \( p(x, \xi) \) of order \( m \in \mathbb{R} \) (and the associated operator \( P \)) is said to be strongly elliptic when \( \text{Re} \, p_0(x, \xi) \geq c|\xi|^m \) for all \( \xi \geq 1 \), with \( c > 0 \). Moreover, a classical ψdo \( P = \text{Op}(p(x, \xi)) \) of order \( m \) is said to be even, when the terms in the symbol expansion \( p \sim \sum_{j \in \mathbb{N}_0} p_j \) satisfy

\[
p_j(x, -\xi) = (-1)^j p_j(x, \xi) \quad \text{for all } x \in \mathbb{R}^n, |\xi| \geq 1, j \in \mathbb{N}_0.
\]

There are also defined spaces of symbols with finite smoothness in \( x \). In the case of \( C^\tau \)-smoothness (\( \tau > 0 \)), the symbol spaces are denoted \( C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \), \( C^\tau S^m(\mathbb{R}^n \times \mathbb{R}^n) \), see details e.g. in [13]. Also here, strongly elliptic symbols and even symbols are well-defined subclasses. When \( p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \), it defines a continuous operator

\[
\text{Op}(p) : H^s_{q+m}(\mathbb{R}^n) \rightarrow H^s_{q}(\mathbb{R}^n) \quad \text{for } |s| < \tau.
\]

There are two families of pseudodifferential operators that play a special role, the “order-reducing operators” \( \Lambda_\pm^t \) defined for bounded \( C^\infty \)-domains \( \Omega \) in [13]. They are of order \( t \), and have the homeomorphism properties:

\[
\Lambda_+^t : H^s_q(\Omega) \xrightarrow{\sim} H^{-t-s}_q(\Omega), \quad r^+\Lambda_+^t e^+ : \overline{H^s_q(\Omega)} \xrightarrow{\sim} \overline{H^{-t-s}_q(\Omega)}, \quad \text{all } s \in \mathbb{R}.
\]

Here the operator \( r^+\Lambda_+^t e^+ \), also denoted \( \Lambda_{\pm}^{t_{\pm}} \), is the adjoint of \( \Lambda_\pm^t \). The operators are modeled after the operators \( \Xi^\pm = \text{Op}((\xi') \pm i\xi \eta') \) that act in this way relative to \( \mathbb{R}^n_+ \). (For \( C^{1+\tau} \)-domains we have not made the effort to
introduce global versions of $\Lambda(t)$, but obtain the relevant results by using versions of $\Xi_t$ in local coordinates at the boundary.) They play a role in the study of our special $2\alpha$-order operators $P$, because they enter as factors in the analysis of the homogeneous Dirichlet problem (13) for $P$. In fact, the solution space for the problem with right-hand side $f$ in $H^s_q(\Omega)$ is the so-called $\alpha$-transmission space:

$$H^{a(s+2\alpha)}_q(\Omega) = \Lambda^{-(a)}_+ e^\tau H^{a(s+2\alpha)}_q(\Omega)$$

defined via local coordinates when $\Omega$ is nonsmooth ($\frac{1}{q} + \frac{1}{q'} = 1$). This regularity result is formulated below as Theorem 3.3. A pedestrian introduction to $\alpha$-transmission spaces can be found in [16].

The $\alpha$-transmission spaces are defined similarly for the scale of Hölder-Zygmund spaces:

$$H^{a(s+2\alpha)}_q(\Omega) = \dot{H}^{a(s+2\alpha)}_q(\Omega)$$

if $s < -a + 1/q$, (11)

3. On eigenfunctions and their regularity

Our basic hypothesis is:

HYPOTHESIS 3.1. There are given constants $a, \tau, q$ with $0 < a < 1$, $\tau > 2a$, and $1 < q < \infty$. $P$ is a classical, strongly elliptic $\psi$do of order $2a$, with even symbol in $C^\tau S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)$.

In much of the paper, we moreover assume that $P$ is symmetric:

HYPOTHESIS 3.2. The conditions in Hypothesis 3.1 hold, and in addition, $P$ equals its formal adjoint $P^*$ (in short: is symmetric).

The last hypothesis will be satisfied e.g. if $P$ has real symbol independent of $x$, if $P$ is a power $L^a$ of a symmetric strongly elliptic second-order differential operator $L$, or if $P$ is obtained as the “real part” $P = \frac{1}{2}(P_1 + P_1^*)$ of a suitable nonsymmetric operator $P_1$.

Recall the regularity theorem for the homogeneous Dirichlet problem for $P$:

THEOREM 3.3. Assume Hypothesis 3.2 and let $\Omega$ be a bounded $C^{1+\tau}$-domain in $\mathbb{R}^n$; let $s$ satisfy $-a \leq s < \tau - 2a$.

1. Forward mapping property: $r^+ P$ maps continuously

$$r^+ P : H^{a(s+2\alpha)}_q(\Omega) \rightarrow H^{a(s+2\alpha)}_q(\Omega).$$

2. Regularity (backward mapping property): When $u \in \dot{H}^a_q(\Omega)$ solves the homogeneous Dirichlet problem

$$Pu = f \text{ in } \Omega, \quad \text{supp } u \subset \Omega,$$

(13)
for some \( f \in \overline{\mathcal{D}}_q(\Omega) \), then \( u \in H^{s+2a}_q(\Omega) \). Here if \( q \neq 2 \) and \( \tau < \infty \), we assume that \( s \geq 0 \).

**Proof.** The statements were shown in the smooth case \((\tau = \infty)\) for all \( 1 < q < \infty \) in [13] Th. 4.2, 4.4 (in fact allowing \( s \) to take values down to \(-a - \frac{1}{q}\), and \( u \in \dot{H}^q(\Omega) \) with \( \sigma > a - \frac{1}{q} \) in \( 2^\circ \)). The statements were extended to the nonsmooth cases in [1] Cor. 6.2, Th. 6.9, with \( s \) assumed \( \geq 0 \) for \( 2^\circ \). When \( q = 2 \), the interval \( 0 > s > -a \) can be included in \( 2^\circ \) by use of the variational definition of the Dirichlet realization, recalled below and studied in more detail in [17] Sect. 4; here it is found that \( r^+ P + b : \dot{H}^a(\Omega) \to \overline{\mathcal{D}}^{-a} \) is a homeomorphism for suitable constants \( b \). By interpolation between this and the homeomorphism \( r^+ P + b : H^{a+2s}_0(\Omega) \to H^{-a} \) when \( 0 > s > -a \). Then for \(-a \leq s \leq 0 \), \( u \in \dot{H}^a(\Omega) \) and \( r^+ Pu \in \overline{\mathcal{D}}^{-a} \) (hence \( (r^+ P + b)u \in \overline{\mathcal{D}}^{-a} \)) imply \( u \in H^{a+2s}_0(\Omega) \).

Recall from [17] Sect. 4, that under Hypothesis 3.1 \( P \) satisfies a Gårding inequality (with \( c > 0 \), \( \beta \in \mathbb{R} \))

\[
\text{Re}(Pu, u) \geq c_0 \| u \|^2_{\dot{H}^a(\mathbb{R}^n)} - \beta \| u \|^2_{L_2(\mathbb{R}^n)}, \quad \text{for } u \in C^\infty_0(\mathbb{R}^n),
\]

and the sesquilinear form \( s(u, v) \) obtained by closure in \( \dot{H}^a(\Omega) \) of

\[
s(u, v) = \int_\Omega Pu \overline{v} \, dx \text{ on } C^\infty_0(\Omega),
\]

defines by variational theory (as in e.g. [11] Sect. 12.4, after Lions-Magenes [20]) the \( L_2 \)-Dirichlet realization \( P_{D, 2} \), i.e., the operator in \( L_2(\Omega) \) acting like \( r^+ P \) with domain

\[
D(P_{D, 2}) = \{ u \in \dot{H}^a(\Omega) \mid r^+ Pu \in L_2(\Omega) \}.
\]

Its spectrum and numerical range is contained in a sectorial region with \( c_1 \geq 0 \):

\[
M = \{ \lambda \in \mathbb{C} \mid \text{Re}\lambda \geq c_0 - \beta, |\text{Im}\lambda| \leq c_1(\text{Re}\lambda + \beta) \};
\]

the spectrum is discrete since \( \dot{H}^a(\Omega) \) is compactly injected in \( L_2(\Omega) \). In fact, there holds:

**Proposition 3.4.** [17] Assume Hypothesis 3.1 and let \( \Omega \) be bounded and \( C^{1+\tau} \). The domain \( D(P_{D, 2}) \) equals \( H^{a+2s}_0(\Omega) \), and the spectrum \( \Sigma \) of \( P_{D, 2} \) is discrete.

For \( \lambda \notin \Sigma \), \( P_{D, 2} - \lambda I \) is a homeomorphism of \( H^{a+2s}_0(\Omega) \) onto \( L_2(\Omega) \).
For $\lambda \in \Sigma$, $P_{D,2} - \lambda I$ defines a Fredholm operator with index zero from $H^{(2a)}(\Omega)$ to $L_2(\Omega)$. The kernel is denoted $N_\lambda$, and there is a cokernel $N'_\lambda$, where $N'_\lambda$ denotes the kernel of $P^*_{D,2} - \lambda$. Here

$$\dim N_\lambda = \dim N'_\lambda.$$  \hspace{1cm} (18)

From (11) with $q = 2$, $s = 0$, we have that $H^{2a}(\Omega) = \Lambda_{+}^{-}(\lambda) \mathcal{H}_{\lambda}^{(a)}(\Omega)$ (in a local sense if $\Omega$ is nonsmooth). Here $H^{2a}(\Omega) = H^2(\Omega)$ if $a < \frac{1}{2}$, and $H^{2a}(\Omega) \subset H^{2+\frac{1}{2} - \varepsilon}(\Omega)$ if $a \geq \frac{1}{2}$ (more detailed information in [15]).

We can also define the $L_q$-realization $P_{D,q}$ for general $q$; it acts as $r^+P$ in $L_q(\Omega)$ with domain

$$D(P_{D,q}) = \{ u \in H^a_q(\Omega) | r^+ Pu \in L_q(\Omega) \} = H^a_{q}(2a)(\Omega).$$ \hspace{1cm} (19)

For this we showed in [17] Sect. 4, that the adjoint of $P_{D,q}$ is the analogous operator $(P^*)_{D,q}$ defined from $P^*$ in $L_q(\Omega)$. The spectra and eigenfunctions satisfy:

**Theorem 3.5.** [17] Assume Hypothesis [3,7] and let $\Omega$ be bounded and $C^{1+\tau}$.  
1° For any $q \in [1, \infty[$, the spectrum of $P_{D,q}$ equals $\Sigma$, and when $\lambda \in \Sigma$, the eigenspace equals $N_\lambda$, and $N'_\lambda$ is a cokernel of $P_{D,q} - \lambda$.

2° If $0$ is an eigenvalue, an associated eigenfunction $u_0$ satisfies

$$u_0 \in C^a(\Omega) \subset d^a C^{\tau-a-\varepsilon}(\Omega),$$ \hspace{1cm} (20)

for small $\varepsilon > 0$ (with $\tau - a - \varepsilon > 0$).

For a nonzero $\lambda \in \Sigma$, the eigenfunctions $u_\lambda$ satisfy

$$u_\lambda \in C^a(t+a)(\Omega) \subset d^a C^t(\Omega) \text{ for } t = \min\{2a, \tau - a\} - \varepsilon; \text{ small } \varepsilon > 0.$$ \hspace{1cm} (21)

3° If $\tau = \infty$ or $P$ is symmetric, the eigenfunctions of $(P^*)_{D,q}$ have the same behavior as in 2°. In nonsymmetric cases, when $\tau$ is finite, they satisfy at least

$$u_\lambda \in C^a(t+a)(\Omega) \subset d^a C^t(\Omega) \text{ for } t = \min\{2a, 1, \tau - a\} - \varepsilon; \text{ small } \varepsilon > 0.$$ \hspace{1cm} (22)

4° In particular, the cokernels $N'_\lambda$ of $P_{D,2} - \lambda$, $\lambda \in \Sigma$, have the regularity stated in 3°, and both kernels and cokernels are in all cases contained in $\tilde{C}^a(\Omega)$.

It is understood here that $\varepsilon$ is chosen such that $\tau - a$, $t$ and $t - a$ are not integer.

It is perhaps interesting to recall from [17] that the identification of eigenspaces for different $q$ was easy to show for $q \geq 2$, but demanded some effort for $q < 2$, and vice versa for the analysis of cokernels. Moreover, extra efforts were necessary for results on the adjoint in the nonsymmetric case (when $\tau < \infty$) because of remainder terms in the pseudodifferential calculus. More details in [17].
When $\tau \geq 3a$, the result in (21) is only an $\varepsilon$ weaker than the result obtained for the $C^\infty$-case in [14] Th. 2.3.

4. Weyl asymptotics for nonsmooth operators over smooth domains

It was shown in [14] in the smooth case ($\tau = \infty$) that the eigenvalues of $P_{D,2}$ (when selfadjoint), and more generally the $s$-numbers, satisfy a Weyl asymptotic estimate where the coefficient is determined by an integral formed of the principal symbol over the domain. This will now be generalized to nonsmooth cases. In the present paper we restrict the attention to eigenvalues in symmetric cases (in the Hilbert space setting), where there are some convenient auxiliary tools.

When $P$ is symmetric, the realization $P_{D,2}$ is selfadjoint in $L^2(\Omega)$, and the sectorial set $M$ in (17) is simply $M = [c_0 - \beta, \infty[$. Henceforth we drop the subscript 2, writing $P_{D,2} = P_D$. Here $P_D + \beta$ is a positive, bijective operator from $D(P_D)$ to $L^2(\Omega)$.

As already noted, the spectrum $\Sigma$ of $P_D$ is discrete. The spectrum of $P_D + \beta$ is a sequence of positive eigenvalues (repeated according to multiplicities)

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty.$$ (23)

The inverses of the $\lambda_j$ are the nonzero eigenvalues $\mu_j$ of the compact inverse of $P_D + \beta$. In the Appendix, Lemma 6.1, we recall the equivalent formulations of asymptotic formulas for a compact selfadjoint injective nonnegative operator and its inverse.

For the extension of the Weyl formula from the smooth case, we shall use some well-known properties of $s$-numbers (also called singular values):

When $B$ is a compact linear operator from a Hilbert space $H$ to another $H_1$, the $s$-numbers $s_j(B)$ (singular values) are defined as the numbers $s_j(B) = \mu_j(B^*B)^{1/2}$, where $\mu_j(B^*B)$ denotes the $j$-th positive eigenvalue of $B^*B$, arranged nonincreasingly and repeated according to multiplicities. A standard reference is the book of Gohberg and Krein [8]. For $p > 0$, the so-called weak Schatten class $\mathfrak{S}_p(\Omega, H_1)$ (with notation as in [12], consists of the compact operators $B$ such that

$$s_j(B) \leq Cj^{-1/p} \text{ for all } j.$$ (24)

(The indication $(H, H_1)$ is replaced by $(H)$ if $H = H_1$; it can be omitted when it is clear from the context.) The convention for $p$ stems from the fact that $\mathfrak{S}_p(\Omega, H_1)$ is close to the standard Schatten class $\mathfrak{S}_p$ consisting of the operators $B$ with $(s_j(B))_{j\in\mathbb{N}} \in \ell_p(\mathbb{N})$. In fact, $\mathfrak{S}_p(\Omega, H_1) \subset \mathfrak{S}_{p+\varepsilon}$ for any $\varepsilon > 0$. They are linear spaces. We recall some useful properties:

$$\mathfrak{S}_p \cdot \mathfrak{S}_q \subset \mathfrak{S}_{r}, \text{ where } r^{-1} = p^{-1} + q^{-1},$$

$$s_j(B^*) = s_j(B), \quad s_j(EBF) \leq \|E\|s_j(B)\|F\|,$$ (25)
when $E: H_1 \to H_3$ and $F: H_2 \to H$ are bounded linear maps between Hilbert spaces. Moreover, there are perturbation rules:

**Lemma 4.1.**

1° If $s_j(B)j^{1/p} \to C_0$ and $s_j(B')j^{1/p} \to 0$ for $j \to \infty$, then $s_j(B + B')j^{1/p} \to C_0$ for $j \to \infty$.

2° If $B = B_M + B'_M$ for each $M \in \mathbb{N}$, where $s_j(B_M)j^{1/p} \to C_M$ for $j \to \infty$ and $s_j(B'_M)j^{1/p} \leq c_M$ for $j \in \mathbb{N}$, with $C_M \to C_0$ and $c_M \to 0$ for $M \to \infty$, then $s_j(B)j^{1/p} \to C_0$ for $j \to \infty$.

The statement in 1° is the Weyl-Ky Fan theorem (cf. e.g. [1] Th. II 2.3), and 2° is a refinement shown in [9] Lemma 6.2.2. More details in [10] Sect. A.6. One also defines singular values for unbounded operators $A$ with discrete spectrum (including inverses of operators $B$ as above); we shall use the notation $r_j(A) = \lambda_j(A^*A)^{1/2}$ as in [10]. By an abuse of notation, they were called $s_j$ again in [14].

In the case where the symbol of $P$ depends smoothly on $x$, we have from [14] Th. 2.7:

**Theorem 4.2.** Assume Hypothesis [7] with $\tau = \infty$, and let $\Omega$ be bounded and $C^\infty$. The singular values $r_j(P_D)$ of $P_D$ (the eigenvalues of $(P_D^*P_D)^{1/2}$), as well as the singular values of $P_D + b$ for any $b \in \mathbb{C}$, have the asymptotic behavior

$$r_j(P_D + b) = C(P,\Omega)j^{2a/n} + o(j^{2a/n}) \text{ for } j \to \infty,$$

(26)

where $C(P,\Omega)$ is defined from the principal symbol $p_0(x,\xi)$ by:

$$C(P,\Omega) = C'(P,\Omega)^{-2a/n}, \quad C'(P,\Omega) = \frac{1}{\pi^{n(2\tau)}} \int_{\Omega} \int_{|\xi|=1} |p_0(x,\xi)|^{-n/2a} d\omega(\xi)dx.$$

(27)

We remark here that the proof of Theorem 2.7 in [14] first shows that the asymptotic behavior holds for $P_D + b$ when $P_D + b$ is invertible, and then concludes it for $P_D$ by a perturbation argument. (One can here take $b = \beta$ in [14].) An analogous perturbation argument allows to include arbitrary $b$.

Now consider nonsmooth symbols. When $\tau < \infty$, we approximate the symbol $p(x,\xi)$ in $C^{\tau'} S^{2a}$ ($\tau' < \tau$ close to $\tau$) by smooth symbols $p_k(x,\xi)$, $k \to \infty$, obtained by convolutions in $x$ with a resolution of the identity as in [12]; the $p_k$ are likewise strongly elliptic and even. They give rise to operators $P_k$ on $\mathbb{R}^n$ and Dirichlet realizations $P_{k,D}$ on $\Omega$. By [12] (2.18), the operator norm of $P - P_k$ in $\mathcal{L}(H^a(\mathbb{R}^n), H^{-a}(\mathbb{R}^n))$ goes to 0, so we get from [14] that

$$\text{Re}(P_k u, u) \geq c/2\|u\|^2_{H^a(\mathbb{R}^n)} - (\beta + 1)\|u\|^2_{L^2(\mathbb{R}^n)}$$

(28)
for sufficiently large $k$; discarding the first terms, we can assume that (28) holds for all $k$.

For $b \in \mathbb{R}$, denote $P_{D} + b = P_{D,b}$ and $P_{k,D,b} = P_{k,D} + b$; then these operators have positive lower bound and are invertible when $b \geq \beta + 1$.

The asymptotic estimate of the eigenvalues of $P_{D,b}$ will be obtained by approximation from the corresponding result for the $s$-numbers of $P_{k,D,b}$, that holds according to Theorem 4.2. For that purpose, we need to estimate the $s$-numbers of $P_{D,b}^{-1} - P_{k,D,b}^{-1} = P_{k,D,b}^{-1}(P_{k,D,b} - P_{D,b})P_{D,b}^{-1}$. (29)

Since $H^{a(2a)}(\Omega)$ is only for $0 < a < \frac{1}{2}$ known to be contained in a $2a$-order Sobolev space, the embedding of $H^{a(2a)}(\Omega)$ into $L^2(\Omega)$ needs special considerations:

**Lemma 4.3.** When $\Omega \subset \mathbb{R}^n$ is smooth and bounded, and $T$ is a linear operator in $L^2(\Omega)$ that is continuous from $L^2(\Omega)$ to $H^{a(2a)}(\Omega)$, then its $s$-numbers satisfy

$$s_j(T) \leq C_1\|T\|_{L(\mathcal{L}(H^{a(2a)}),L^2))} j^{-2a/n}, \text{ all } j,$$

(30)

where $C_1$ is a constant depending only on $\Omega$ and $a$.

**Proof.** This follows since we know from [14] that for $P_\Delta = (-\Delta)^a$, the inverse of the Dirichlet realization is a homeomorphism $P_\Delta^{-1}$ from $L^2(\Omega)$ to $H^{a(2a)}(\Omega)$ satisfying an estimate (as a consequence of Theorem 4.2):

$$s_j(P_\Delta^{-1}) \leq C_0 j^{-2a/n}, \text{ all } j.$$

Then, since $T$ is the composition of $P_\Delta^{-1}$ and the bounded operator $B = P_\Delta T$ in $L^2(\Omega)$, the property follows for $T$ by the last rule in (25):

$$s_j(T) = s_j(P_\Delta^{-1}B) \leq C_0 j^{-2a/n} \|B\| \leq C_0 \|P_\Delta B\|_{L(\mathcal{L}(H^{2a(n)},L^2))} \|T\|_{L(\mathcal{L}(H^{2a(n)},L^2))} j^{-2a/n} = C_1 \|T\|_{L(\mathcal{L}(H^{2a(n)},L^2))} j^{-2a/n}.$$

**Proposition 4.4.** Let $b \geq \beta + 1$. The operator norm of the difference between $P_{D,b}^{-1}$ and $P_{k,D,b}^{-1}$ satisfies

$$\|P_{D,b}^{-1} - P_{k,D,b}^{-1}\|_{L(\mathcal{L}(H^{a(2a)}),L^2))} \to 0 \text{ for } k \to \infty,$$

(31)

and its $s$-numbers satisfy inequalities

$$s_j(P_{D,b}^{-1} - P_{k,D,b}^{-1}) j^{-2a/n} \leq C_k \text{ for all } j, \text{ where } C_k \to 0 \text{ for } k \to \infty.$$

(32)
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PROOF. We use the representation (29). Since \( D(P_{D,b}) = H^{a(2a)}(\Omega) \),

\[
\|P_{D,b}^{-1}\|_{\mathcal{L}(L_2, H^{a(2a)})} \leq C'.
\]  (33)

To the difference \( P_{k,D,b} - P_{D,b} \) we apply Theorem 5.11 of [1], supplied with norm considerations as in Theorem 5.13 there. Namely, the boundedness of \( P_{k,D,b} - P_{D,b} \) follows from Theorem 5.11, with \( \mu = a, m = 2a, s = 0, q = 2 \), and \( \tau \) replaced by \( \tau' < \tau \) close to \( \tau \); and for the pieces localized as in the proof of Theorem 5.11, the convergence to zero in operator norm follows from the convergence to zero of a sufficiently high-numbered symbol seminorm, as shown in Theorem 5.13 there. Thus

\[
\|P_{D,b} - P_{k,D,b}\|_{\mathcal{L}(H^{a(2a)}, L_2)} \to 0 \text{ for } k \to \infty.
\]  (34)

It follows in particular that

\[
P_{k,D,b}P_{D,b}^{-1} \to I \text{ in } \mathcal{L}(L_2(\Omega)) \text{ for } k \to \infty,
\]

so by a Neumann series argument, the inverse \( P_{D,b}P_{k,D,b}^{-1} \) likewise has operator norm in \( L_2(\Omega) \) bounded with respect to \( k \), and hence

\[
\|P_{k,D,b}^{-1}\|_{\mathcal{L}(L_2, H^{a(2a)})} \leq C'' \text{ for all } k.
\]  (35)

Applying (33), (34) and (35) to the factors in (29), we conclude (31), and then (32) follows by Lemma 4.3.

Theorem 4.5. Assume Hypothesis 3.2, let \( \Omega \subset \mathbb{R}^n \) be bounded and \( C^\infty \), and let \( P_D \) be the Dirichlet realization of \( P \) in \( L_2(\Omega) \). Let \( b \in \mathbb{R} \) be such that \( P_{D,b} = P_D + b \) has positive lower bound. Then the eigenvalues \( \mu_j(P_{D,b}^{-1}) \) of its inverse satisfy:

\[
\mu_j(P_{D,b}^{-1}) = C(P, \Omega)^{-1}j^{-2a/n} + o(j^{-2a/n}) \text{ for } j \to \infty,
\]  (36)

where \( C(P, \Omega) \) is defined in (27). Equivalently, the eigenvalues \( \lambda_j(P_D) \) satisfy:

\[
\lambda_j(P_D) = C(P, \Omega)j^{2a/n} + o(j^{2a/n}) \text{ for } j \to \infty.
\]  (37)

PROOF. Define the approximating operators \( P_{k,D,b} \) as described above. We have from Theorem 4.2 that their \( s \)-numbers satisfy:

\[
s_j(P_{k,D,b}^{-1}) = C(P_k, \Omega)^{-1}j^{-2a/n} + o(j^{-2a/n}) \text{ for } j \to \infty,
\]  (38)

where the constants are defined as in (27). For the differences \( P_{D,b}^{-1} - P_{k,D,b}^{-1} \) we have the estimates of \( s \)-numbers in (32). Clearly, \( C(P_k, \Omega) \to C(P, \Omega) \) for \( k \to \infty \) (by the convergence of the principal symbols). Then the result follows by use of Lemma 4.1 2°, noting that \( \mu_j(P_{D,b}^{-1}) = s_j(P_{D,b}^{-1}) \).
The last statement follows in view of Lemma A.1 applied to $P_{D,b}$. Clearly, the asymptotic estimate (37) is equivalent with the estimate where $P$ is replaced by $P + b$.

The result extends to various perturbations of $P$. One is to add to $P$ an operator $P'$ with lower positive order and suitable mapping properties. Another is to add to $P$ an $L_2$-bounded operator $P''$.

**Theorem 4.6.** Assume Hypothesis 3.2 and let $\Omega \subset \mathbb{R}^n$ be bounded and $C^\infty$. Let $P'$ and $P''$ be symmetric operators that map continuously, for some $0 < \delta \leq a$,

$$P': H^t(\mathbb{R}^n) \to H^{t-2a+\delta}(\mathbb{R}^n) \text{ for } a - \delta \leq t \leq 2a,$$  

$$P': H^t(\mathbb{R}^n) \to H^{t-a-\frac{\delta}{2}}(\mathbb{R}^n) \text{ for } \frac{1}{2} - \delta \leq t \leq a + \frac{1}{2},$$  

$$P'': L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n);$$

$P''$ can for example be the multiplication by a function $V(x) \in L_\infty(\mathbb{R}^n, \mathbb{R})$.

1° The Dirichlet realization $\tilde{P}_D$ of $\tilde{P} = P + P' + P''$ defined from the sesquilinear form $\tilde{s}$ on $\dot{H}^a(\Omega)$ extending

$$\tilde{s}(u,v) = \int_\Omega (P + P' + P'')u \overline{v} \, dx \text{ on } C_0^\infty(\Omega),$$

is selfadjoint lower bounded and has $D(\tilde{P}_D) = H^{a(2a)}(\Omega)$.

2° The eigenvalues of $\tilde{P}_D$ satisfy

$$\lambda_j(\tilde{P}_D) = C(P, \Omega)a^{2/n} + o(j^{2a/n}) \text{ for } j \to \infty,$$

with the constant $C(P, \Omega)$ defined from $p_0$ as in (27).

**Proof.** 1°. We first show how to handle the addition of $P'$. Let $s'(u,v)$ be the form $s'(u,v) = (P'u, v)$ for $u,v \in C_0^\infty(\Omega)$.

The case $a < \frac{1}{2}$. By use of the order-reducing operators $\Lambda^{(t)}_+$ and their adjoints $\Lambda^{(-t)}_+ \equiv r^+ \Lambda^{(t)}_- \equiv r^+ \Lambda^{(-t)}_+$ recalled in (10), we have that for $u,v \in C_0^\infty(\Omega)$,

$$|\langle P'u, v \rangle_{L_2(\Omega)}| = |\langle (r^+ P', \Lambda_+^{(-a)} \Lambda_+^{(a)} v \rangle_{L_2(\Omega)}| = |\langle (\Lambda_+^{(-a)} v, \Lambda_+^{(a)} v \rangle_{L_2(\Omega)}|$$

$$\leq c \|\Lambda_+^{(-a)} v\|_{L_2(\Omega)} \|\Lambda_+^{(a)} v\|_{L_2(\Omega)} \leq c' \|\Lambda_+^{(a)} v\|_{L_2(\Omega)} \|\Lambda_+^{(-a)} v\|_{L_2(\Omega)};$$

where we used (39) for $t = a - \delta$ (here $\Lambda_+^{(-a)}$ lifts $\overline{H}^{-a}(\Omega)$ to $L_2(\Omega)$). So the form $s'(u,v)$ extends to a form $s'(u,v)$ continuous on $\dot{H}^a(\Omega)$, and moreover, by interpolating $\dot{H}^{a-\delta}(\Omega)$ between $\dot{H}^a(\Omega)$ and $L_2(\Omega)$ we can deduce that for any $\varepsilon > 0$,

$$|s'(u,u)| \leq \varepsilon \|u\|^2_{\dot{H}^a(\Omega)} + C(\varepsilon) \|u\|^2_{L_2(\Omega)}.$$  

(45)
Taking $\varepsilon \leq c_0/2$, we conclude that $s(u, v) + s'(u, v)$ satisfies a coerciveness estimate like (13) over $\Omega$:

$$s(u, u) + s'(u, u) \geq c_0/2\|u\|^2_{H^2(\Omega)} - \beta \|u\|^2_{L^2(\Omega)}, \text{ for } u \in \dot{H}^a(\Omega),$$

with a larger $\beta'$. Then the variational construction carries through to define the Dirichlet realization $(P + P')D$, selfadjoint and with lower bound $\geq c_0/2 - \beta'$.

We shall show that its domain (defined by (10) for $P + P'$) is again $H^a(\Omega)$, in the present case equal to $\dot{H}^{2a}(\Omega)$: When $u \in \dot{H}^{2a}(\Omega)$, then we know that $r^+ Pu \in L^2(\Omega)$, and we have $r^+ P'u \in L^2(\Omega)$ by (39) for $t = 2a$. Conversely, let $u \in \dot{H}^a(\Omega)$ satisfy

$$r^+ Pu + r^+ P'u = f, \quad f \in L^2(\Omega).$$

By (39) with $t = a$, $f - r^+ P'u \in H^{\min\{0,-a+\delta\}}(\Omega)$, so by the regularity of the Dirichlet problem for $P$, $u \in H^{\min\{2a,a+\delta\}}(\Omega)$, cf. Theorem 3.3. If $\delta = a$, we are through. If not, we use (39) again, now with $t = a + \delta$, to see that $f - r^+ P'u \in H^{\min\{0,-a+2\delta\}}(\Omega)$, hence $u \in H^{\min\{2a,a+2\delta\}}(\Omega)$. In finitely many steps of this kind, the conclusion $u \in \dot{H}^{2a}(\Omega)$ is reached.

The case $a \geq \frac{1}{2}$. Here we find (14) by use of (10) with $t = \frac{1}{2} - \delta$. Then (15) can be concluded, leading to a selfadjoint lower bounded Dirichlet realization $(P + P')D$.

To show that the domain is $H^a(\Omega)$, we proceed as follows: Since $H^a(\Omega) \subset \dot{H}^{a+\frac{1}{2}-\frac{1}{2}}(\Omega)$, $r^+(P + P')$ maps $H^a(\Omega)$ into $L^2(\Omega)$ by (10) for $t = a + \frac{1}{2}$. Conversely, consider (10). By (10) with $t = a$, $f - r^+ P'u \in H^{\min\{0,-\frac{1}{2}+\delta\}}(\Omega)$, so by Theorem 3.3 $u \in H^{\min\{2a,2a-\frac{1}{2}+\delta\}}(\Omega)$. If $\delta \geq \frac{1}{2}$, we are through. If not, we observe that $H^{\min\{2a,2a-\frac{1}{2}+\delta\}}(\Omega) \subset H^{\alpha(a+\delta)}(\Omega)$ since $a \geq \frac{1}{2}$; here

$$H^{\alpha(a+\delta)}(\Omega) = \Lambda_+^{(-a)}H^{\delta}(\Omega) = \Lambda_+^{(-a)}\dot{H}^{\delta}(\Omega) = H^{a+\delta}(\Omega),$$

since $\delta < \frac{1}{2}$. By (10) with $t = a + \delta$, $r^+ P'u \in H^{-\frac{1}{2}+2\delta}(\Omega)$, hence $f - r^+ P'u \in H^{\min\{0,-\frac{1}{2}+2\delta\}}(\Omega)$. It follows that $u \in H^{\min\{2a,2a-\frac{1}{2}+2\delta\}}(\Omega)$. In finitely many steps of this kind, we reach the conclusion $u \in H^{\alpha(a+\delta)}(\Omega)$.

Adding $P''$ to $P + P'$ is much easier, since $r^+ P''e^+\dot{P}u$ defines a bounded map in $L^2(\Omega)$. When we add $(P''u, v)$ to $s(u, v) + s'(u, v)$, we can easily verify the $H^a(\Omega)$-continuity and coerciveness for the resulting form $\tilde{s}(u, v)$. Since $H^{\alpha(2a)}(\Omega) \subset L^2(\Omega)$, $\tilde{P}$ is continuous from $H^{\alpha(2a)}(\Omega)$ to $L^2(\Omega)$. In the equation

$$r^+(P + P' + P'')u = f, \quad f \in L^2(\Omega), \quad u \in H^a(\Omega),$$

the right-hand side remains in $L^2(\Omega)$, when $r^+ P''u \in L^2(\Omega)$ is moved there. So we can conclude that $D(\tilde{P}D) = H^{\alpha(2a)}(\Omega)$. 

2°. We use Theorem 4.2 to derive the Weyl asymptotics formula for $\tilde{P}_D$ from that of $P_D$ by perturbation:

Take $b$ so large that both $P_D + b$ and $\tilde{P}_D + b$ have positive lower bound. Then

$$(P_D + b)^{-1} - (\tilde{P}_D + b)^{-1} = (\tilde{P}_D + b)^{-1}(r^+ P' + r^+ P'')(P_D + b)^{-1}.$$  

Here, by the assumptions on $P'$, $r^+ P'$ maps $H^{a(2a)}(\Omega)$ into $\mathcal{F}^a(\Omega)$ if $a < \frac{1}{2}$ and into e.g. $\mathcal{F}^a(\Omega)$ if $a \geq \frac{1}{2}$, so $r^+ P'(P_D + b)^{-1}$ maps $L_2(\Omega)$ to $\mathcal{F}^a(\Omega)$ and hence belongs to the weak Schatten class $\mathcal{S}_{n/(\delta/2), \infty}$. As for the contribution from $P''$, we use its continuity in $L_2(\Omega)$ and the last rule in (25) to see that $r^+ P''(P_D + b)^{-1}$ is in $\mathcal{S}_{n/(2a), \infty}$. Then by the product rule in (25),

$$(\tilde{P}_D + b)^{-1}(r^+ P' + r^+ P'')(P_D + b)^{-1} \in \mathcal{S}_{n/(2a), \infty} \mathcal{S}_{n/(\delta/2), \infty} \subset \mathcal{S}_{n/(2a+\delta/2), \infty}.$$  

Finally, the first rule in Lemma 4.1 implies the asserted asymptotics for $\tilde{P}_D$.

1° also holds when $\Omega$ is $C^{1+\tau}$; here the use of the order-reducing operators $\Lambda_t^+$ to show (45) is replaced by the use of $\Xi_t^+ = \text{Op}((\langle \xi \rangle + i\xi_n)^t)$ in local coordinates.

The hypotheses on $P'$ are satisfied if $P'$ is a pseudodifferential operator with symbol in $C^{\tau} S^{a+\min(a, \frac{\delta}{2})-\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ (recall $0 < \delta \leq a$). In particular, $P'$ can be of order $a$ with symbol in $C^\tau S^{\delta}(\mathbb{R}^n \times \mathbb{R}^n)$. And, as mentioned, $P''$ can be a bounded real potential $V(x)$.

Note that only the action of $r^+(P' + P'')$ on functions supported in $\Omega$ really enters, so the requirements (39)–(41) could be replaced by properties referring only to $\Omega$.

5. Nonsmooth operators over nonsmooth sets

The asymptotic formula (43) can be extended to nonsmooth domains by a simple argument in the selfadjoint case. As a generalization of the notion of “contented” in Reed and Simon [21] p. 271, we define:

**Definition 5.1.** We shall say that a bounded open set $\Omega \subset \mathbb{R}^n$ is $C^\infty$-contented, when there exist two sequences of $C^\infty$-domains $\Omega_{in,j} \subset \Omega$, $j \in \mathbb{N}$, and $\Omega_{out,k} \supset \Omega$, $k \in \mathbb{N}$, such that

$$\lim_{j \to \infty} \text{vol}(\Omega_{in,j}) = \lim_{k \to \infty} \text{vol}(\Omega_{out,k}) = \text{vol}(\Omega). \tag{47}$$

This holds for example for a Lipschitz domain (i.e., $C^{0,1}$-domain):

**Lemma 5.2.** When $\Omega$ is bounded with a $C^{0,1}$-boundary, then $\Omega$ is $C^\infty$-contented.
PROOF. This can be shown by an elaboration of the explanation given in Daners [4] Prop. 8.2.1. For the approximation from inside, let

$$V_j = \{x \in \Omega \mid \text{dist}(x, \partial \Omega) > 1/j\},$$

then \(\{V_j\}_{j \in \mathbb{N}}\) is a nested sequence of open sets such that \(\overline{V}_j \subset V_{j+1} \subset \overline{V}_{j+1} \subset \Omega\) for all \(j\). Since \(\Omega\) is \(C^{0,1}\), \(\text{vol}(\Omega \setminus V_j) \to 0\) for \(j \to \infty\). Namely, every boundary point \(x\) has a bounded neighborhood \(W_x\) such that, after a translation and rotation, \(W_x \cap \Omega = W_x \cap \mathbb{R}_x^n = \{x = (x', x_n) \mid x_n > \zeta(x')\}\) for a \(C^{0,1}\)-function \(\zeta(x')\). Here \((W_x \cap (\Omega \setminus V_j)) \cap \mathbb{R}_x^n \subset W_x \cap (\mathbb{R}_x^n \setminus \mathbb{R}_x^n)\) for a constant \(C\), so that the volume goes to 0 for \(j \to \infty\). Since the boundary can be covered by finitely many such sets \(W_x\), \(\text{vol}(\Omega \setminus V_j) \to 0\) for \(j \to \infty\).

Now as shown in [4] Prop. 8.2.1, we can for each \(j\) find a \(C^\infty\)-set \(U_{in,j}\) with \(\overline{V}_j \subset U_{in,j} \subset \overline{U}_{in,j} \subset V_{j+1}\) by choosing a cutoff function \(\psi_j \in C^\infty(V_{j+1}, [0,1])\) that equals 1 on \(V_j\); then \(\psi_j\) has by Sard’s lemma (cf. e.g. Hirsch [18] Th. 3.1.3) a regular value \(t_j \in [0,1[\) such that \(U_{in,j} = \{x \mid \psi(x) > t_j\}\) is \(C^\infty\).

This defines the sets \(U_{in,j}\) approximating \(\Omega\) from inside, and a similar study of \((\mathbb{R}^n \setminus \overline{\Omega}) \cap B_j\), for a large \(R\) such that the ball \(B_j = \{|x| < R\}\) contains \(\overline{\Omega}\), gives an approximating sequence \(U_{out,k}\) from outside.

Let \(\tilde{P}\) be as in Theorem 4.6. Recall that the eigenvalues of \(\tilde{P}_D\) can be described as the Rayleigh quotients, where \(H = L^2(\Omega)\), \(V = \tilde{H}^a(\overline{\Omega})\), \(X\) denotes a linear space:

$$\lambda_j(\tilde{P}_D) = \max_{X \subset H, \dim X < j} \min_{v \in V, v \bot X} \frac{\tilde{s}(v, v)}{\|v\|_H^2},$$

and \(\tilde{s}(u, v)\) is the sesquilinear form on \(\tilde{H}^a(\overline{\Omega})\) defined by (4.12).

There are likewise defined Dirichlet realizations \(\tilde{P}_{D,\Omega_{in,t}}\) of \(\tilde{P}\) on the sets \(\Omega_{in,t}\), with associated forms

$$\tilde{s}_{\Omega_{in,t}}(v, v) = \int_{\Omega_{in,t}} \tilde{P} v \bar{v} \, dx \text{ on } \tilde{H}^a(\overline{\Omega}_{in,t}),$$

coinciding with \(\tilde{s}(v, v)\) for \(v \in \tilde{H}^a(\overline{\Omega}_{in,t})\). Dirichlet realizations \(\tilde{P}_{D,\Omega_{out,t}}\) on \(\Omega_{out,t}\) are similarly defined. Based on the description of the eigenvalues as Rayleigh quotients (14), we can compare the eigenvalues of \(\tilde{P}_{D,\Omega_{in,t}}\) and \(\tilde{P}_D\), showing that the eigenvalues of \(\tilde{P}_{D,\Omega_{in,t}}\) are larger than those of \(\tilde{P}_D\) for each \(j\). This follows by application of a general well-known comparison property Proposition 4.2 that we include in the Appendix (with a proof for the convenience of the reader).

We can then show:
Theorem 5.3. Let \( \tilde{P} = P + P' + P'' \) be as in Theorem 4.4. Let \( \Omega \subset \mathbb{R}^n \) be bounded and \( C^\infty \)-contended; e.g. a Lipschitz domain. Then the eigenvalues of the Dirichlet realization \( \tilde{P}_D \) of \( P \) in \( L_2(\Omega) \) satisfy:

\[
\lambda_j(\tilde{P}_D) = C(P,\Omega)j^{2a/n} + o(j^{2a/n}) \quad \text{for} \quad j \to \infty,
\]

where \( C(P,\Omega) \) is defined by (27).

Proof. The variational construction of \( \tilde{P}_D \) also works when \( \Omega \) is \( C^{0,1} \) (and even for less smooth domains), giving a selfadjoint lower bounded operator with compact resolvent. We apply the comparison principle recalled in Proposition 6.2 with \( H_1 = L_2(\Omega_{m,l}) \), \( V_1 = \tilde{H}^a(\overline{\Omega}_{m,l}) \), \( s_1(u,v) = \tilde{s}_{\Omega_{m,l}}(u,v) \) (cf. (50)), and \( H_2 = L_2(\Omega) \), \( V_2 = \tilde{H}^a(\overline{\Omega}) \), \( s_2(u,v) = \tilde{s}_{\Omega_{m,l}}(u,v) \) (cf. (52)); this gives that

\[
\lambda_j(\tilde{P}_D) \geq \lambda_j(\tilde{P}_D_{\Omega_{m,l}}), \quad \text{all} \quad j, l.
\]

A similar comparison between \( H_1 = L_2(\Omega) \), \( V_1 = \tilde{H}^a(\overline{\Omega}) \), \( s_1(u,v) = \tilde{s}(u,v) \), and \( H_2 = L_2(\Omega_{out,l}) \), \( V_2 = \tilde{H}^a(\overline{\Omega}_{out,l}) \), \( s_2(u,v) = \tilde{s}_{\Omega_{out,l}}(u,v) \), gives

\[
\lambda_j(\tilde{P}_D) \geq \lambda_j(\tilde{P}_D_{\Omega_{m,l}}), \quad \text{all} \quad j, l.
\]

Thus we have for each \( l \), applying Theorem 4.6 to \( \tilde{P}_D_{\Omega_{m,l}} \),

\[
\limsup_{j \to \infty} \lambda_j(\tilde{P}_D)j^{-2a/n} \leq \limsup_{j \to \infty} \lambda_j(\tilde{P}_D_{\Omega_{m,l}})j^{-2a/n} = C(P,\Omega_{m,l}).
\]

By (27) and (47),

\[
|C(P,\Omega) - C(P,\Omega_{m,l})| \leq C' \text{vol}(\Omega \setminus \Omega_{m,l}) \to 0 \quad \text{for} \quad l \to \infty.
\]

Then letting \( l \to \infty \) in (53), we find that

\[
\limsup_{j \to \infty} \lambda_j(\tilde{P}_D)j^{-2a/n} \leq C(P,\Omega).
\]

It is shown similarly by use of (53) that

\[
\liminf_{j \to \infty} \lambda_j(\tilde{P}_D)j^{-2a/n} \geq C(P,\Omega),
\]

and (51) follows.

Example 5.4. Theorem 5.3 applies e.g. to \( (-\Delta)^{a'} + (-\Delta)^{a''} + V(x) \), with \( 0 < 2a' < \min\{2a, a + 1\} \) and \( V \in L_\infty(\mathbb{R}^n, \mathbb{R}) \); here \( P = (-\Delta)^a \), \( P' = (-\Delta)^{a'} \) and \( P'' = V \). For a more general example, we can replace \( -\Delta \) by a selfadjoint strongly elliptic nonnegative second-order differential operator \( L = \sum_{|\alpha| \leq 2} a_\alpha(x) \partial^\alpha \) with \( C^\infty \)-coefficients, defining the fractional power \( L^{a'} \) by Seeley’s construction [22]. In particular, \( P \) can be the fractional power \( ((i\nabla + A(x))^2 + m^2)^{a'} \) of a magnetic Schrödinger operator \( (i\nabla + A(x))^2 + m^2 \), \( A \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \). For \( P' \) we can take a general symmetric pseudodifferential operator of order \( 2a' \) with \( C^{\text{sym}} \)-symbol.
Besides the examples mentioned above, we note that the theorem applies with $P$ replaced by $P_{\text{H}} = \frac{1}{2}(P + P^*)$ for an operator $P$ satisfying Hypothesis 3.1 with $\tau = \infty$.

Remark 5.5. Departing from Theorem 5.3, one can now perturb $\tilde{P}_{D}$ by suitable lower-order nonsymmetric operators to obtain similar estimates of singular values by use of the principles recalled in the start of Section 4. As for taking $P$ itself nonsymmetric when $\Omega$ is nonsmooth, it seems that the analysis of singular values may demand a more circumstantial effort. One possibility is to use coordinate changes from $\Omega$ to neighboring smooth domains (in the spirit of [12] Th. 6.2), based on the results on nonsmooth coordinate changes for $\psi$do’s in [1].

6. Appendix. Rules for eigenvalues

For the study of eigenvalues we recall the elementary transition between the eigenvalue behavior of a compact nonnegative injective operator $B$, the eigenvalue behavior of its inverse $A$, and the counting function for $A$ (taken up in detail e.g. in [10] Lemma A.5):

Lemma 6.1. Let $B$ be compact selfadjoint $\geq 0$ and injective, with eigenvalues $\mu_j(B)$ going nonincreasingly to 0, let $A = B^{-1}$ with eigenvalues $\lambda_j(A) = \mu_j(B)^{-1}$ going nondecreasingly to $\infty$, and let $N(t; A)$ denote the number of eigenvalues of $A$ in $[0, t]$ (all eigenvalues counted with multiplicity). Let $C_0 > 0$, $p > 0$. The following three statements are equivalent:

\[
\begin{align*}
\mu_j(B) &= C_0^1 j^{-1/p} + o(j^{-1/p}) \text{ for } j \to \infty, \\
\lambda_j(A) &= C_0^{-1} j^{3/p} + o(j^{1/p}) \text{ for } j \to \infty, \\
N(t; A) &= C_0 t^p + o(t^p) \text{ for } t \to \infty.
\end{align*}
\]

(55)

The following is a well-known comparison principle for eigenvalues of operators defined from sesquilinear forms (with notation as in e.g. [11] Sect. 12.4, after Lions-Magenes [20]):

Proposition 6.2. Let $(H_1, V_1, s_1)$ and $(H_2, V_2, s_2)$ be triples giving rise to selfadjoint variational operators $A_1$ resp. $A_2$ by the Lax-Milgram lemma. Assume that $V_1 \subset V_2$ with continuous injection, that $H_1$ is a closed subspace of $H_2$, that the injections of $V_i$ into $H_i$ are compact ($i = 1, 2$), and that $s_1(v, v) \geq s_2(v, v)$ for $v \in V_1$. Then the eigenvalues $\lambda_j(A_1)$ and $\lambda_j(A_2)$ of $A_1$ and $A_2$ (ordered nondecreasingly) satisfy

\[
\lambda_j(A_1) \geq \lambda_j(A_2), \text{ for all } j \in \mathbb{N}.
\]

(56)
Proof. The eigenvalues can be represented as Rayleigh quotients (with \( v \neq 0 \)):

\[
\lambda_j(A_i) = \max_{X \subset H_i, \dim X < j} \min_{v \in V_i, v \perp X} \frac{s_i(v, v)}{\|v\|^2_{H_i}}, \quad i = 1, 2.
\] (57)

Note that when \( X_2 \) is a finite dimensional subspace of \( H_2 \), and \( v \in H_1 \), then \( v \perp X_2 \iff v \perp X_1 \), where \( X_1 = \Pi_{H_1} X_2 \), orthogonal projection. All subspaces of \( H_1 \) of dimension \( \leq j - 1 \) are obtained in the form \( \Pi_{H_1} X \) when \( X \) runs through the subspaces of \( H_2 \) of dimension \( \leq j - 1 \).

For each \( X \subset H_2 \) of dimension \( j - 1 \),

\[
\min\{ s_1(v, v)/\|v\|^2_{H_1} \mid v \in V_1 \setminus \{0\}, v \perp \Pi_{H_1} X \} \\
\geq \min\{ s_2(v, v)/\|v\|^2_{H_2} \mid v \in V_1 \setminus \{0\}, v \perp X \} \\
\geq \min\{ s_2(v, v)/\|v\|^2_{H_2} \mid v \in V_2 \setminus \{0\}, v \perp X \},
\] (58)

since \( s_1(v, v) \geq s_2(v, v) \) on \( V_1 \), and \( V_2 \) contains more elements than \( V_1 \). Taking the maximum over all subspaces \( X \) of \( H_2 \) of dimension \( \leq j - 1 \), we get the \( j \)'th eigenvalues, which then must satisfy the inequality (56).

7. Acknowledgement

The author thanks H. Schlichtkrull for geometrical information, and thanks the referee for valuable criticism.

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Dept. of Mathematical Sciences, Copenhagen University, Universitetsparken 5, DK-2100 Copenhagen, Denmark.
E-mail: grubbmath.ku.dk