A STRONG MAXIMUM PRINCIPLE FOR THE FRACTIONAL LAPLACE EQUATION WITH MIXED BOUNDARY CONDITION

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Abstract

In this work we prove a strong maximum principle for fractional elliptic problems with mixed Dirichlet–Neumann boundary data which extends the one proved by J. Dávila (cf. [11]) to the fractional setting. In particular, we present a comparison result for two solutions of the fractional Laplace equation involving the spectral fractional Laplacian endowed with homogeneous mixed boundary condition. This result represents a non–local counterpart to a Hopf’s Lemma for fractional elliptic problems with mixed boundary data.

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1. Introduction

The aim of this paper is to prove a strong maximum principle for elliptic problems involving a fractional Laplacian operator and homogeneous mixed boundary data. In particular, we consider the problem

\[
\begin{cases}
(-\Delta)^s u = f & \text{in } \Omega, \\
B(u) = 0 & \text{on } \partial\Omega,
\end{cases} \quad (P_s)
\]

where \( \frac{1}{2} < s < 1 \), \( f \in C_0^\infty(\Omega) \), \( f \geq 0 \) and \( \Omega \) is a smooth bounded domain of \( \mathbb{R}^N \) with \( N \geq 2 \). Here \( (-\Delta)^s \) denotes the spectral fractional Laplacian.
defined through the spectral decomposition of the classical Laplacian with mixed Dirichlet–Neumann boundary condition $B(u)$ (see Section 2 for further details) given by
\[ B(u) = u\chi_{\Sigma_D} + \frac{\partial u}{\partial \nu}\chi_{\Sigma_N}, \]
where $\nu$ is the outward normal to $\partial \Omega$ and $\chi_A$ stands for the characteristic function of the set $A \subset \partial \Omega$. The sets $\Sigma_D$ and $\Sigma_N$ satisfy the following:
- $\Sigma_D$ and $\Sigma_N$ are $(N-1)$-dimensional smooth submanifolds of $\partial \Omega$,
- $\Sigma_D$ is a closed (with respect to the relative topology) manifold of positive $(N-1)$-dimensional Lebesgue measure,
- $|\Sigma_D| = \alpha \in (0, |\partial \Omega|)$,
- $\Sigma_D \cap \Sigma_N = \emptyset$, $\Sigma_D \cup \Sigma_N = \partial \Omega$ and $\Sigma_D \cap \Sigma_N = \Gamma$,
- $\Gamma$ a smooth $(N-2)$-dimensional submanifold of $\partial \Omega$.

As in the local case, i.e. $s = 1$, by comparison one can easily prove that, for $C_1 = \max_{x \in \Omega} f(x)$,
\[ u(x) \leq C_1 \tilde{v}(x) \quad \text{for} \quad x \in \Omega, \]
with $\tilde{v}$ being the solution to
\[
\begin{cases}
(-\Delta)^s \tilde{v} = 1 & \text{in } \Omega, \\
B(\tilde{v}) = 0 & \text{on } \partial \Omega.
\end{cases}
\] (1.1)
So, a natural question is whether the opposite inequality, namely,
\[ \tilde{v}(x) \leq C_2 u(x) \quad \text{for} \quad x \in \Omega, \] (1.2)
holds true for some constant $C_2 > 0$.

For the local case, Dávila (cf. [11]) proved that inequality (1.2) holds for a positive constant $C_2$ depending on $\Omega$, $\Sigma_D$, $\Sigma_N$ and $\|f\tilde{v}\|_{L^1(\Omega)}$. We will obtain a similar result for the mixed boundary data problem $(P_s)$ by adapting the approach of Dávila to our fractional setting. To that end we will also use the regularity results proved in [8]. Let us remark that in [1] the authors proved a fractional strong maximum principle, but dealing with a different fractional operator which is defined by means of a singular integral.

Our main aim is then to prove the following.

**Theorem 1.1.** Assume that $f \in C^\infty_0(\Omega)$, $f \geq 0$ and let $u$ be the solution to $(P_s)$. Then there exists a constant $C = C(N, s, \Omega, \Sigma_D, \Sigma_N) > 0$ such that
\[ u(x) \geq C \left( \int_\Omega f \tilde{v} dz \right) \tilde{v}(x) \quad \text{for} \quad x \in \Omega, \]
being $\tilde{v}$ the solution to problem (1.1).
The key result we need to prove to obtain Theorem 1.1 is an $L^\infty$ bound on the ratio between the solution to $(P_s)$ with a nonnegative $f \in L^\infty(\Omega)$ and the solution to a suitable auxiliary problem. In particular, let $v \in H_{\Sigma_D}^s(\Omega)$ be the solution to
\[
\begin{cases}
(-\Delta)^s v = g & \text{in } \Omega, \\
B(v) = 0 & \text{on } \partial \Omega,
\end{cases}
\]
with $g \in L^p(\Omega)$, $p > N/s$ and $g \geq 0$. Then, the next result holds.

**Theorem 1.2.** Let $u$ be the solution to $(P_s)$ with $f \in L^\infty(\Omega)$, $f \geq 0$ and $g \in L^p(\Omega)$ for some $p > N/s$ and let $v$ be the solution to (1.3). Then, there exists a constant $C > 0$ such that
\[
\|u - v\|_{L^\infty(\Omega)} \leq C\|g\|_{L^p(\Omega)}
\]
with the constant $C$ depending on $N$, $p$, $s$, $\Omega$, $\Sigma_D$, $\|u\|_{L^\infty(\Omega)}$, $\|f\|_{L^\infty(\Omega)}$ and
\[
1/(\int_{\Omega} f(z) \text{d}(z) \text{d}z)
\]
where $d(x) = \text{dist}(x, \partial \Omega)$.

2. Functional setting and preliminaries

As far as the fractional Laplace operator is concerned, we recall its definition given through the spectral decomposition. Let $(\varphi_i, \lambda_i)$ be the eigenfunctions (normalized with respect to the $L^2(\Omega)$-norm) and the eigenvalues of $(-\Delta)$ equipped with homogeneous mixed Dirichlet–Neumann boundary data, respectively. Then, $(\varphi_i, \lambda_i^s)$ are the eigenfunctions and eigenvalues of $(-\Delta)^s$, where, given $u_i(x) = \sum_{j \geq 1} \langle u_i, \varphi_j \rangle \varphi_j$, $i = 1, 2$, it holds
\[
\langle (-\Delta)^s u_1, u_2 \rangle = \sum_{j \geq 1} \lambda_j^s \langle u_1, \varphi_j \rangle \langle u_2, \varphi_j \rangle,
\]
i.e., the action of the fractional operator on a smooth function $u_1$ is
\[
(-\Delta)^s u_1 = \sum_{j \geq 1} \lambda_j^s \langle u_1, \varphi_j \rangle \varphi_j.
\]
As a consequence, the fractional Laplace operator $(-\Delta)^s$ is well defined through its spectral decomposition in the following space of functions that vanish on $\Sigma_D$,

\[
H_{\Sigma_D}^s(\Omega) = \left\{ u = \sum_{j \geq 1} a_j \varphi_j \in L^2(\Omega) : \|u\|_{H_{\Sigma_D}^s}^2 = \sum_{j \geq 1} a_j^2 \lambda_j^s < \infty \right\}.
\]
Observe that since $u \in H_{\Sigma_D}^s(\Omega)$, it follows that
\[
\|u\|_{H_{\Sigma_D}^s(\Omega)} = \left\|(-\Delta)^{\frac{s}{2}} u\right\|_{L^2(\Omega)}.
\]
As it is proved in [14, Theorem 11.1], if $0 < s \leq \frac{1}{2}$, then $H^s_0(\Omega) = H^s(\Omega)$ and, therefore, also $H^s_{\Sigma_D}(\Omega) = H^s(\Omega)$, while for $\frac{1}{2} < s < 1$, $H^s_0(\Omega) \subsetneq H^s(\Omega)$. Hence, the range $\frac{1}{2} < s < 1$ guarantees that $H^s_{\Sigma_D}(\Omega) \subsetneq H^s(\Omega)$ and it provides us the correct functional space to study the mixed boundary problem $(P_s)$.

This definition of the fractional powers of the Laplace operator allows us to integrate by parts in the appropriate spaces, so that a natural definition of weak solution to problem $(P_s)$ is the following.

**Definition 2.1.** We say that $u \in H^s_{\Sigma_D}(\Omega)$ is a solution to $(P_s)$ if

$$\int_{\Omega} (-\Delta)^{s/2} u (-\Delta)^{s/2} \psi dx = \int_{\Omega} f \psi dx$$

for any $\psi \in H^s_{\Sigma_D}(\Omega)$.

Due to the nonlocal nature of the fractional operator $(-\Delta)^s$ some difficulties arise when one tries to obtain an explicit expression of the action of the fractional Laplacian on a given function. In order to overcome these difficulties, we use the ideas by Caffarelli and Silvestre (see [6]) together with those of [2, 5, 7] to give an equivalent definition of the operator $(-\Delta)^s$ by means of an auxiliary problem that we introduce next.

Given a domain $\Omega \subset \mathbb{R}^N$, we set the cylinder $\mathcal{C}_\Omega = \Omega \times (0, \infty) \subset \mathbb{R}_+^{N+1}$. We denote by $(x, y)$ those points that belong to $\mathcal{C}_\Omega$ and by $\partial_L \mathcal{C}_\Omega = \partial \mathcal{C}_\Omega \times [0, \infty)$ the lateral boundary of the cylinder. Let us also denote by $\Sigma^*_D = \Sigma_D \times [0, \infty)$ and $\Sigma^*_N = \Sigma_N \times [0, \infty)$ as well as $\Gamma^* = \Gamma \times [0, \infty)$. It is clear that, by construction,

$$\Sigma^*_D \cap \Sigma^*_N = \emptyset, \quad \Sigma^*_D \cup \Sigma^*_N = \partial_L \mathcal{C}_\Omega \quad \text{and} \quad \Sigma^*_D \cap \Sigma^*_N = \Gamma^*.$$  

Given a function $u \in H^s_{\Sigma_D}(\Omega)$ we define its $s$-harmonic extension, denoted by $U(x, y) = E_s[u(x)]$, as the solution to the problem

$$\begin{cases}
-\text{div}(y^{1-2s} \nabla U(x, y)) = 0 & \text{in } \mathcal{C}_\Omega, \\
B(U(x, y)) = 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\
U(x, 0) = u(x) & \text{on } \Omega \times \{y = 0\},
\end{cases}$$

where $B(U) = U\chi_{\Sigma_D^*} + \frac{\partial U}{\partial \nu}\chi_{\Sigma_N^*}$, being $\nu$, with an abuse of notation, the outward normal to $\partial_L \mathcal{C}_\Omega$. Note that, if $\nu(x, y)$ denotes the outward normal to $\mathcal{C}_\Omega$ then, by construction, $\nu(x, y) = (\nu, 0)$, $y > 0$ with $\nu$ the outward normal to $\partial \Omega$. The extension function belongs to the space

$$X^s_{\Sigma_D}(\mathcal{C}_\Omega) := C^\infty((\Omega \cup \Sigma_N) \times [0, \infty)) \big/ \|\cdot\|_{X^s_{\Sigma_D}(\mathcal{C}_\Omega)},$$

where we define
\[ \| \cdot \|_{X_{\Sigma_D}^s(\mathcal{C}_\Omega)} := \kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla(\cdot)|^2 dx dy, \]

with \( \kappa_s = 2^{2s-1} \frac{\Gamma(s)}{\Gamma(1-s)} \) being \( \Gamma(s) \) the Gamma function.

Note that the space \( X_{\Sigma_D}^s(\mathcal{C}_\Omega) \) is a Hilbert space equipped with the norm \( \| \cdot \|_{X_{\Sigma_D}^s(\mathcal{C}_\Omega)} \) which is induced by the scalar product

\[ \langle U, V \rangle_{X_{\Sigma_D}^s(\mathcal{C}_\Omega)} = \kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s}(\nabla U, \nabla V) dx dy. \]

Moreover, the following inclusions are satisfied,

\[ X_0^s(\mathcal{C}_\Omega) \subset X_{\Sigma_D}^s(\mathcal{C}_\Omega) \subset X_s(\mathcal{C}_\Omega), \tag{2.1} \]

with \( X_0^s(\mathcal{C}_\Omega) \) the space of functions in \( X_s(\mathcal{C}_\Omega) \equiv H^1(\mathcal{C}_\Omega, y^{1-2s} dx dy) \) and vanishing on the lateral boundary of \( \mathcal{C}_\Omega \), denoted by \( \partial_L \mathcal{C}_\Omega \).

Following the well known result by Caffarelli and Silvestre (cf. \cite{6}), \( U \) is related to the fractional Laplacian of the original function through the formula

\[ \frac{\partial U}{\partial \nu^s} := -\kappa_s \lim_{y \to 0^+} y^{1-2s} \frac{\partial U}{\partial y} = (-\Delta)^s u(x). \]

Using the above arguments we can reformulate the problem \((P_s)\) in terms of the extension problem as follows:

\[ \begin{cases} 
-\text{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathcal{C}_\Omega, \\
B(U) = 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\
\frac{\partial U}{\partial \nu^s} = f & \text{on } \Omega \times \{y = 0\},
\end{cases} \tag{P_s^*} \]

and we have that \( u(x) = U(x, 0) \).

Next, we specify the meaning of solution to problem \((P_s^*)\) and its relationship with the solutions to problem \((P_s)\).

**Definition 2.2.** An energy solution to problem \((P_s^*)\) is a function \( U \in X_{\Sigma_D}^s(\mathcal{C}_\Omega) \) such that, for all \( \varphi \in X_{\Sigma_D}^s(\mathcal{C}_\Omega) \),

\[ \kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s}\nabla U \nabla \varphi dx dy = \int_{\Omega} f(x)\varphi(x, 0) dx. \tag{2.2} \]

If \( U \in X_{\Sigma_D}^s(\mathcal{C}_\Omega) \) is the solution to problem \((P_s^*)\), we can associate the function \( u(x) = Tr[U(x, y)] = U(x, 0) \), that belongs to \( H_{\Sigma_D}^s(\Omega) \), and solves problem \((P_s)\). Moreover, also the vice versa is true: given the solution \( u \in H_{\Sigma_D}^s(\Omega) \) to \((P_s)\) its \( s \)-harmonic extension \( U = E_s[u(x)] \in X_{\Sigma_D}^s(\mathcal{C}_\Omega) \)
is the solution to $(P^*_s)$. Thus, both formulations are equivalent and the Extension operator

$$E_s : H^s_{\Sigma_D}(\Omega) \to \mathcal{X}_s(\mathcal{E}_\Omega),$$

allows us to switch between each other. Moreover, according to [2, 6], due to the choice of the constant $\kappa_s$, the extension operator $E_s$ is an isometry, i.e.,

$$\|E_s[\varphi](x, y)\|_{\mathcal{X}_s(\mathcal{E}_\Omega)} = \|\varphi(x)\|_{H^s_{\Sigma_D}(\Omega)} \quad \text{for all } \varphi \in H^s_{\Sigma_D}(\Omega). \quad (2.3)$$

Let us also recall the trace inequality (cf. [2]) that is a useful tool to be exploited along this paper: there exists $C = C(N, s, r, |\Omega|)$ such that for all $z \in \mathcal{X}_0^s(\mathcal{E}_\Omega)$ we have

$$C \left( \int_{\Omega} |z(x, 0)|^r dx \right)^\frac{2}{r} \leq \int_{\mathcal{E}_\Omega} y^{1-2s} |\nabla z(x, y)|^2 dx dy,$$

with $1 \leq r \leq 2^*_s$, $N > 2s$, with $2^*_s = \frac{2N}{N-2s}$. Observe that, because of (2.3), the trace inequality turns out to be, in fact, equivalent to the fractional Sobolev inequality:

$$C \left( \int_{\Omega} |v|^r dx \right)^\frac{2}{r} \leq \int_{\Omega} |(-\Delta)^{\frac{s}{2}}v|^2 dx,$$  \quad (2.4)

for all $v \in H^s_0(\Omega)$, with $1 \leq r \leq 2^*_s$ and $N > 2s$. For the critical exponent $r = 2^*_s$ the best constant in (2.4), namely the fractional Sobolev constant, denoted by $S(N, s)$, is independent of the domain $\Omega$ and its exact value is given by $S(N, s) = 2^{2s-N} \pi^s \frac{\Gamma(N+2s)}{\Gamma(N)} \left( \frac{\Gamma(N)}{\Gamma(\frac{N}{2})} \right)^{\frac{2s}{N}}$.

When mixed boundary conditions are considered, the situation is quite similar since the Dirichlet condition is imposed on a set $\Sigma_D \subset \partial \Omega$ such that $|\Sigma_D| = \alpha > 0$. Hence, thanks to (2.1), there exists a positive constant $C_D = C_D(N, s, |\Sigma_D|)$ such that

$$0 < C_D := \inf_{u \in H^s_{\Sigma_D}(\Omega) \setminus \{0\}} \frac{\|u\|_{H^s_{\Sigma_D}(\Omega)}}{\|u\|_{L^{2^*_s}(\Omega)}} < \inf_{u \in H^s_0(\Omega) \setminus \{0\}} \frac{\|u\|_{H^s_0(\Omega)}}{\|u\|_{L^{2^*_s}(\Omega)}}. \quad (2.5)$$

Moreover, $C_D(N, s, |\Sigma_D|) \leq 2^{-\frac{2s}{N}} S(N, s)$, (cf. [9, Proposition 3.6]).

**Remark 2.1.** Due to the spectral definition of the fractional operator, using Hölder’s inequality, we have $C_D \leq |\Omega|^{\frac{2s}{N}} \lambda^s_1(\alpha)$, where $\lambda^s_1(\alpha)$ denotes the first eigenvalue of the Laplace operator endowed with mixed boundary conditions on $\Sigma_D = \Sigma_D(\alpha)$ and $\Sigma_N = \Sigma_N(\alpha)$. Since $\lambda^s_1(\alpha) \to 0$ as $\alpha \to 0^+$, (cf. [10, Lemma 4.3]), we have $C_D \to 0$ as $\alpha \to 0^+$. 

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Gathering together (2.5) and (2.3) it follows that, for all \( \varphi \in X^s_\Sigma (\mathcal{E}_\Omega) \),

\[
C_D \left( \int_\Omega |\varphi(x,0)|^{2^*_s} \, dx \right)^{\frac{1}{2^*_s}} \leq \|\varphi(x,0)\|_{H^s_{\Sigma_D}(\Omega)}^2 = \|E_s[\varphi(x,0)]\|_{X^s_{\Sigma_D}(\mathcal{E}_\Omega)}^2.
\]

This Sobolev–type inequality provides a trace inequality adapted to the mixed boundary data framework.

**Lemma 2.1.** [9, Lemma 2.4] There exists \( C_D = C_D(N, s, |\Sigma_D|) > 0 \) constant such that, for all \( \varphi \in X^s_\Sigma (\mathcal{E}_\Omega) \), we have

\[
C_D \left( \int_\Omega |\varphi(x,0)|^{2^*_s} \, dx \right)^{\frac{1}{2^*_s}} \leq \int_{\mathcal{E}_\Omega} y^{1-2s}|\nabla \varphi|^2 \, dx \, dy.
\]

Along the proof of the Theorem 1.1 we will make use of the following fractional Hardy inequality (cf. [12, Theorem 3]): given \( \frac{1}{2} \leq s < 1 \), there exist a constant \( C > 0 \) such that

\[
C \int_\Omega \left( \frac{f(x)}{d^s(x)} \right)^2 \, dx \leq \|f\|_{H^s_0(\Omega)}^2 \quad \text{for all } f \in H^s_0(\Omega),
\]

where \( d(x) = \text{dist}(x, \partial \Omega) \).

In order to establish the validity of (2.7) we need to impose some geometrical or smoothness assumptions on the domain \( \Omega \). From the geometrical point of view, if one assumes that \( \Omega \) is such that, in the sense of distributions,

\[-\Delta d(x) \geq 0 \quad \text{for } x \in \Omega,
\]

then, inequality (2.7) holds for the constant \( C = C(s) = \frac{2^{2s}1^{-2s}(\frac{3+2s}{4})}{1^{-2s}+1} \).

The above condition is related to, but weaker than, the assumption of convexity of the domain \( \Omega \). From the regularity point of view, if one considers a smooth domain \( \Omega \), then inequality (2.7) holds for a constant \( C \leq \frac{2^{2s}1^{-2s}(\frac{3+2s}{4})}{1^{-2s}+1} \). Finally, in terms of the \( s \)-harmonic extension the fractional Hardy inequality reads (cf. [12, Theorem 1])

\[
\int_\Omega \left( \frac{F(x,0)}{d^s(x)} \right)^2 \, dx \leq C \int_{\mathcal{E}_\Omega} y^{1-2s}|\nabla F(x,y)|^2 \, dx \, dy,
\]

for all \( F \in X^s_0(\mathcal{E}_\Omega) \) and some constant \( C > 0 \).

### 3. Proof of main results

In this section we prove Theorem 1.2 and, as a consequence, we deduce Theorem 1.1. Following the approach of [11], we start by proving the following weighted Sobolev–type inequality.
Lemma 3.1. Let $u \in H^s_{\Sigma_D}(\Omega)$ be the solution to $(P_s)$ with $f \in L^\infty(\Omega)$, $f \geq 0$ and denote by $U \in \mathcal{X}^s_{\Sigma_D}(\mathcal{C}_\Omega)$ its $s$-harmonic extension. Then, there exists a constant $C > 0$ such that, for every $\varphi \in \mathcal{X}^s_{\Sigma_D}(\mathcal{C}_\Omega) \cap L^\infty(\mathcal{C}_\Omega)$,  
\[
\int_\Omega |u|^r \varphi(x,0)|^q dx \leq C \left( \int_{\mathcal{C}_\Omega} y^{1-2s} U^2 |\nabla \varphi|^2 dx dy + \int_\Omega u^2 \varphi^2(x,0) dx \right)^{\frac{q}{r}}, \quad (3.1)
\]
where $0 \leq r \leq 2_s^*$ and $\frac{q}{r} = 1 + \frac{s}{N}$ and the constant $C > 0$ depends on $N$, $s$, $\Omega$, $\|u\|_{L^\infty(\Omega)}$, $\|f\|_{L^\infty(\Omega)}$ and $1/(\int_\Omega f(z)dz)$.  

Proof. We divide the proof into three steps according to the cases $r = 0$, $r = 2_s^*$ and the interpolation case $r \in (0, 2_s^*)$.  

Step 1: Case $r = 0$.  
We start by proving that, for all $\varphi \in \mathcal{X}^s_{\Sigma_D}(\mathcal{C}_\Omega) \cap L^\infty(\mathcal{C}_\Omega)$,  
\[
\int_\Omega \varphi^2(x,0) dx \leq C \left( \int_{\mathcal{C}_\Omega} y^{1-2s} U^2 |\nabla \varphi|^2 dx dy + \int_\Omega u^2(x) \varphi^2(x,0) dx \right).
\]
Let $\phi_1$ be the first eigenfunction of $(-\Delta)^s$ under homogeneous Dirichlet boundary condition,  
\[
\begin{cases}
(-\Delta)^s \phi_1 = \lambda_1^s \phi_1 & \text{in } \Omega, \\
\phi_1 = 0 & \text{on } \partial \Omega.
\end{cases}
\]
As in (2.2), in terms of the $s$-harmonic extension of the function $\phi_1$, denoted by $\Phi_1 = E_s[\phi_1]$, we have that, for all $\Psi \in \mathcal{X}^s_{\Sigma_D}(\mathcal{C}_\Omega)$,  
\[
\kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s} \nabla \Phi_1 \nabla \Psi dx dy = \lambda_1^s \int_{\Omega} \phi_1(x) \Psi(x,0) dx. \quad (3.2)
\]
Let us remark that $\Phi_1 \in C^\gamma(\overline{\mathcal{C}_\Omega}) \cap L^\infty(\mathcal{C}_\Omega)$ for some $\gamma \in (0, 1)$, (cf. [2, Theorem 4.7] and [2, Corollary 4.8]). Because of the spectral definition of the fractional Laplace operator, the function $\phi_1$ is also the first eigenfunction of the classical Laplace operator $(-\Delta)$ under homogeneous Dirichlet boundary condition, hence, there exists a constant $c_1 > 0$ (depending only on $\Omega$) such that  
\[
\phi_1(x) \geq c_1 d(x) \quad \text{for } x \in \Omega. \quad (3.3)
\]
Using (3.3) and (2.8) with $F = \varphi \Phi_1^s$ (note that $\varphi \Phi_1^s \in \mathcal{X}^s_{\Sigma_D}(\mathcal{C}_\Omega)$), we get  
\[
\int_\Omega \varphi^2(x,0) dx \leq c_1 \int_\Omega \frac{\varphi^2(x,0) \phi_1^{2s}(x)}{d^{2s}(x)} dx 
\leq C \int_{\mathcal{C}_\Omega} y^{1-2s} |\nabla (\varphi \Phi_1^s)|^2 dx dy. \quad (3.4)
\]
Next, we observe that
\[
|\nabla(\varphi\Phi_1^s)|^2 = \Phi_1^{2s}|\nabla\varphi|^2 + 2s\varphi\Phi_1^{2s-1}\nabla\varphi\nabla\Phi_1 + s^2\varphi^2\Phi_1^{2s-2}|
abla\Phi_1|^2
\]
\[
= \Phi_1^{2s}|\nabla\varphi|^2 + s\nabla\Phi_1\nabla(\varphi^2\Phi_1^{2s-1})
\]
\[
+ s(1-s)\varphi^2\Phi_1^{2s-2}|
abla\Phi_1|^2.
\] (3.5)

Since \( \frac{1}{2} < s < 1 \), using the Cauchy–Schwarz and \( \varepsilon \)-Young inequalities, we get
\[
\nabla\Phi_1\nabla(\varphi^2\Phi_1^{2s-1}) = 2\varphi^2\Phi_1^{2s-1}\nabla\Phi_1 \nabla\varphi + (2s-1)\varphi^2\Phi_1^{2s-2}|\nabla\Phi_1|^2
\]
\[
\geq -\frac{1}{\varepsilon}\Phi_1^{2s}|\nabla\varphi|^2 + (2s-1-\varepsilon)\varphi^2\Phi_1^{2s-2}|\nabla\Phi_1|^2.
\]
for some \( \varepsilon > 0 \) such that \( 2s-1-\varepsilon > 0 \). Then,
\[
\varphi^2\Phi_1^{2s-2}|\nabla\Phi_1|^2 \leq \frac{1}{2s-1-\varepsilon} \left( \frac{1}{\varepsilon}\Phi_1^{2s}|\nabla\varphi|^2 + \Phi_1\nabla(\varphi^2\Phi_1^{2s-1}) \right).
\]

As a consequence, from (3.5), we find
\[
|\nabla(\varphi\Phi_1^s)|^2 \leq \left( 1 + \frac{s(1-s)}{\varepsilon(2s-1-\varepsilon)} \right) \Phi_1^{2s}|\nabla\varphi|^2
\]
\[
+ \left( s + \frac{s(1-s)}{2s-1-\varepsilon} \right) \nabla\Phi_1\nabla(\varphi^2\Phi_1^{2s-1}).
\]

Therefore, because of (3.4) and (3.2),
\[
\int_{\Omega} \varphi^2(x,0)dx \leq C \int_{\mathcal{E}_{\Omega}} y^{1-2s}|\nabla(\varphi\Phi_1^s)|^2dxdy
\]
\[
\leq C_1 \int_{\mathcal{E}_{\Omega}} y^{1-2s}\Phi_1^{2s}|\nabla\varphi|^2dxdy + C_2 \int_{\mathcal{E}_{\Omega}} y^{1-2s}\nabla\Phi_1\nabla(\varphi^2\Phi_1^{2s-1})dxdy
\]
\[
= C_1 \int_{\mathcal{E}_{\Omega}} y^{1-2s}\Phi_1^{2s}|\nabla\varphi|^2dxdy + \frac{C_2\lambda^s}{\kappa s} \int_{\Omega} \phi_1^s(x)\varphi^2(x,0)dx.
\]

Finally, given \( u \) the solution to (\( P_u \)), because of the Hölder regularity of solutions to fractional elliptic problems with mixed boundary data (cf. [8, Theorem 1.1]), we have \( u \in C^\gamma(\overline{\Omega}) \) for some \( \gamma \in (0,\frac{1}{s}) \). Moreover, since \( \Omega \) is a smooth bounded domain, we also have \( \phi_1 \in C^\infty(\overline{\Omega}) \) and, thus, \( \phi_1^s \in C^s(\overline{\Omega}) \).

As consequence, since \( \frac{1}{2} < s < 1 \), there exists a constant \( C > 0 \) such that \( \phi_1^s \leq Cu \) and, hence, \( \Phi_1^s \leq CU \). Then, we conclude
\[
\int_{\Omega} \varphi^2(x,0)dx \leq C \left( \int_{\mathcal{E}_{\Omega}} y^{1-2s}U^2|\nabla\varphi|^2dxdy + \int_{\Omega} u^2(x)\varphi^2(x,0)dx \right),
\]
for some constant \( C > 0 \).

**Step 2:** Case \( r = 2^*_s \).

We continue by proving that, for all \( \varphi \in X_{\Sigma_D}^s(\mathcal{E}_{\Omega}) \cap L^\infty(\mathcal{E}_{\Omega}) \),
\[
\left( \int_{\Omega} (u | \varphi(x,0)|^{2^*_s}) dx \right)^{\frac{2}{2^*_s}} \leq C \left( \int_{\bar{\Omega}} y^{1-2s} U^2 |\nabla \varphi|^2 dx dy + \int_{\Omega} u^2 \varphi^2 (x,0) dx \right)
\]

Since by hypothesis \( f \in L^\infty (\Omega) \), repeating step by step the Moser-type proof done for fractional elliptic problems with Dirichlet boundary data (cf. [2, Theorem 4.7]), we get that \( U = E_s[u] \in L^\infty (\bar{\Omega}) \), being \( u \) the solution to \((P_s)\). Thus, \( U \varphi^2 \in \mathcal{K}_{\Sigma_D}^s (\bar{\Omega}) \) and, because of (2.6) and (2.2), we obtain (the constants may vary line to line)

\[
\left( \int_{\Omega} (u | \varphi(x,0)|^{2^*_s}) dx \right)^{\frac{2}{2^*_s}} \leq C \left( \int_{\bar{\Omega}} y^{1-2s} |\nabla (U \varphi)|^2 dx dy + \int_{\Omega} u^2 \varphi^2 (x,0) dx \right)
\]

where we have used that \( u \in L^\infty (\Omega) \) (cf. [8, Theorem 3.7]) and Step 1 in the last inequality.

**Step 3:** Case \( r \in (0, 2^*_s) \).

Finally, we prove (3.1). By Hölder’s inequality, Step 1 and Step 2 we conclude

\[
\int_{\Omega} u^r |\varphi(x,0)|^q dx = \int_{\Omega} u^r |\varphi^r(x,0)| |\varphi(x,0)|^{2^*_s - \frac{r}{q}} dx \leq \left( \int_{\Omega} \varphi^2 (x,0) dx \right)^{1 + \frac{r}{2^*_s} - \frac{r}{q}} \left( \int_{\Omega} u^{2^*_s} |\varphi(x,0)|^{2^*_s} dx \right)^{\frac{r}{2^*_s}} \leq C \left( \int_{\bar{\Omega}} y^{1-2s} U^2 |\nabla \varphi|^2 dx dy + \int_{\Omega} u^2 \varphi^2 (x,0) dx \right)^{\frac{q}{2}},
\]

since \( \frac{q}{2} = 1 + \frac{r}{2^*_s} \).

**Proof of Theorem 1.2.** First, we observe that it is enough to prove the result in the case \( g \geq 0 \). The general case is deduced applying this argument to the positive and negative parts of \( g \) respectively.

Let \( u, v \in H_{\Sigma_D}^s (\Omega) \) be the solutions to \((P_s)\) and (1.3) respectively. Then, \( U = E_s[u], V = E_s[v] \in \mathcal{K}_{\Sigma_D}^s (\bar{\Omega}) \) are the respective solutions to
\[-\text{div}(y^{1-2s}\nabla U) = 0 \quad \text{in } \mathcal{C}_{\Omega},
B(U) = 0 \quad \text{on } \partial_L \mathcal{C}_{\Omega},
U(x,0) = u(x) \quad \text{on } \Omega \times \{y = 0\},
\frac{\partial U}{\partial \nu^s} = f \quad \text{on } \Omega \times \{y = 0\},\]

and

\[-\text{div}(y^{1-2s}\nabla V) = 0 \quad \text{in } \mathcal{C}_{\Omega},
B(V) = 0 \quad \text{on } \partial_L \mathcal{C}_{\Omega},
V(x,0) = v(x) \quad \text{on } \Omega \times \{y = 0\},
\frac{\partial V}{\partial \nu^s} = g \quad \text{on } \Omega \times \{y = 0\}.\]

Taking in mind (2.2), for every test function \( \varphi \in \mathcal{X}_{\Sigma^s}^s(\mathcal{C}_{\Omega}) \), we have

\[\kappa_s \int_{\mathcal{C}_{\Omega}} y^{1-2s} U \nabla \varphi dxdy = \int_{\Omega} \varphi(x,0) f(x) dx, \quad (3.6)\]

and

\[\kappa_s \int_{\mathcal{C}_{\Omega}} y^{1-2s} V \nabla \varphi dxdy = \int_{\Omega} \varphi(x,0) g(x) dx. \quad (3.7)\]

Since \( f \in L^\infty(\Omega) \) and \( g \in L^p(\Omega) \), with \( p > N/s \), repeating step by step the proof of [2, Theorem 4.7] it follows that \( U, V \in L^\infty(\mathcal{C}_{\Omega}) \). Let us also stress that \( u, v \in L^\infty(\Omega) \) by [8, Theorem 3.7]. Moreover, since \( g \geq 0 \), by comparison with the respective Dirichlet problem, we can assume that \( v \geq 0 \) and \( V = \mathcal{E}_s[v] \geq 0 \), (cf. [7, Lemma 2.3]). Then, for \( \varepsilon > 0 \) and \( k \geq 0 \), we define

\[\varphi_\varepsilon = \left( \frac{V}{U + \varepsilon} - k \right)_+ \in \mathcal{X}_{\Sigma^s}^s(\mathcal{C}_{\Omega}) \cap L^\infty(\mathcal{C}_{\Omega}),\]

where \((\cdot)_+ = \max\{0, \cdot\}\). Since \( \varphi_\varepsilon \) is bounded, both \( V\varphi_\varepsilon \) and \( U\varphi_\varepsilon \) belong to \( \mathcal{X}_{\Sigma^s}^s(\mathcal{C}_{\Omega}) \). Then, using \( V\varphi_\varepsilon \) and \( U\varphi_\varepsilon \) as a test function in (3.6) and (3.7) respectively, we get

\[\kappa_s \int_{\mathcal{C}_{\Omega}} y^{1-2s} \varphi_\varepsilon \nabla U \nabla V dxdy + \kappa_s \int_{\mathcal{C}_{\Omega}} y^{1-2s} V \nabla U \nabla \varphi_\varepsilon dxdy = \int_{\Omega} v \varphi_\varepsilon(x,0) dx,\]

and

\[\kappa_s \int_{\mathcal{C}_{\Omega}} y^{1-2s} \varphi_\varepsilon \nabla V \nabla U dxdy + \kappa_s \int_{\mathcal{C}_{\Omega}} y^{1-2s} U \nabla V \nabla \varphi_\varepsilon dxdy = \int_{\Omega} u \varphi_\varepsilon(x,0) dx.\]

Hence, subtracting the above equalities,

\[\kappa_s \int_{\mathcal{C}_{\Omega}} y^{1-2s}(U \nabla V - V \nabla U) \nabla \varphi_\varepsilon dxdy = \int_{\Omega} \varphi_\varepsilon(x,0)(ug - vf) dx. \quad (3.8)\]
We observe now that $(U + \varepsilon)^2|\nabla \varphi_\varepsilon|^2 = (U \nabla V - V \nabla U) \nabla \varphi_\varepsilon + \varepsilon \nabla V \nabla \varphi_\varepsilon$, and consequently, by (3.8),

$$
\kappa_s \int_{\Omega} y^{1-2s}(U + \varepsilon)^2|\nabla \varphi_\varepsilon|^2 dxdy = \varepsilon \kappa_s \int_{\Omega} y^{1-2s} \nabla V \nabla \varphi_\varepsilon dxdy + \int_{\Omega} \varphi_\varepsilon(x, 0)(ug - vf)dx
$$

$$
= \int_{\Omega} \varphi_\varepsilon(x, 0)((u + \varepsilon)g - vf)dx
$$

$$
\leq \int_{\Omega} \varphi_\varepsilon(x, 0)(u + \varepsilon)g(x)dx.
$$

(3.9)

Finally, because of Lemma 3.1, inequality (3.9) reads as

$$
\left( \int_{\Omega} u^r |\varphi_\varepsilon(x, 0)|^q dx \right)^{\frac{2}{q}} \leq C \left( \int_{\Omega} \varphi_\varepsilon(x, 0)(u + \varepsilon)gdx + \int_{\Omega} u^2 \varphi_\varepsilon^2(x, 0)dx \right),
$$

with $\frac{q}{2} = 1 + \frac{s}{p}$. On the other hand, we observe that, for $\varepsilon \to 0$,

$$
(u + \varepsilon)\varphi_\varepsilon(x, 0) = (v - k(u + \varepsilon))_+ \nearrow (v - ku)_+
$$

and $\varphi_\varepsilon(x, 0) \nearrow \left( \frac{v}{u} - k \right)_+$. Thus, denoting $w = \frac{v}{u}$, by the monotone convergence theorem we obtain

$$
\left( \int_{\Omega} u^r (w - k)_+^q dx \right)^{\frac{2}{q}} \leq C \left( \int_{\Omega} u(w - k)_+ dx + \int_{\Omega} u^2 (w - k)_+^2 dx \right). \quad (3.10)
$$

Once we get inequality (3.10), the rest of the proof follows by means of an iterative Stampacchia–type method. We include the argument for the reader’s convenience.

Let us set $r = \frac{p}{p - 1} \in (1, 2^*_s)$. Thus, using Hölder’s inequality, from (3.10) we obtain that

$$
\left( \int_{\Omega} u^r (w - k)_+^q dx \right)^{\frac{2}{q}} \leq C \left( \int_{\Omega} u^{1 - \frac{q}{p}} u^\frac{q}{p} (w - k)_+ dx + \int_{\Omega} u^{2 - \frac{q}{p}} u^\frac{q}{p} (w - k)_+^2 dx \right)
$$

$$
\leq C\|g\|_{L^p(\Omega)} \left( \int_{\Omega} u^r (w - k)_+^q dx \right)^{\frac{2}{q}} \left( \int_{\{w > k\}} u^r dx \right)^{1 - \frac{1}{p} - \frac{1}{q}}
$$

$$
+ C \left( \int_{\Omega} u^r (w - k)_+^q dx \right)^{\frac{2}{q}} \left( \int_{\{w > k\}} u^{\frac{2(q - r)}{q - 2}} dx \right)^{1 - \frac{2}{q}}.
$$

(3.11)
Let us define

Using Hölder’s inequality once more, we deduce

Therefore, given

which satisfies

where \( 2 = \frac{2}{q} + \frac{2s}{N} > 1 \). Moreover, from (3.13), we have

so that, for

Then, denoting by

Observe that

\[
\left( \int_{\{w>k\}} u^r((w-k)_+)^q dx \right)^\frac{1}{q} \leq C \|g\|_{L^p} \left( \int_{\{w>k\}} u^r dx \right)^{1 - \frac{1}{p} - \frac{1}{q}}.
\]

Using Hölder’s inequality once more, we deduce

where \( 2 - \frac{1}{p} - \frac{2}{q} = 1 + \frac{1}{r} - \frac{2}{q} > 1 \) since \( p > \frac{N}{s} \). Next, we set the function

which satisfies

Then, denoting by \( \gamma = 2 - \frac{1}{p} - \frac{2}{q} > 1 \), from (3.13) we find

Therefore, given \( k > k_0 \) and integrating in the interval \([k_0, k]\), we get

Since \( a(k) \) is nonnegative and nondecreasing and \( \gamma > 1 \) the above inequality implies that \( a(k) = 0 \) for some \( k \leq C \|g\|_{L^p(\Omega)}^{-\frac{1}{2}} (k_0) + k_0 \) and, hence,

\[
w \leq C \|g\|_{L^p(\Omega)}^{-\frac{1}{2}} (k_0) + k_0. \tag{3.14}
\]

In addition, from (3.13), we have
\[ a(k_0) := \int_{\Omega} u^r (w - k_0)_+ \, dx \leq C\|g\|_{L^p(\Omega)} \left( \int_{\{w > k_0\}} u^r \, dx \right)^{2 - \frac{1}{p} \frac{s}{q}} \]

\[ \leq C\|g\|_{L^p(\Omega)}. \]

Finally, by [8, Theorem 3.7], we have

\[ \|v\|_{L^\infty(\Omega)} \leq C(N,s,|\Sigma_D|,|\Omega|^{\frac{2s}{N}}) \left( \int_{\{w > k_0\}} u^r \, dx \right)^{\frac{2}{p} - \frac{1}{q}} \leq C\|g\|_{L^p(\Omega)}, \]

and, hence, using (3.14) and (3.12), we conclude

\[ \frac{v}{u} = w \leq C\|g\|_{L^p(\Omega)} + \|v\|_{L^\infty(\Omega)} \leq C\|g\|_{L^p(\Omega)}. \]

Using Theorem 1.2 we can now prove Theorem 1.1.

\textbf{Proof of Theorem 1.1.} First, we observe the following: Since 
\( f \in C^\infty_0(\Omega) \) we have that \((-\Delta)^{1-s} f\) is bounded and we can choose \( c_f > 0 \) such that

\[ (-\Delta)^{1-s} f + c_f \geq 0 \quad \text{in} \quad \Omega, \]

and \( w \) such that

\[ \begin{cases} 
(-\Delta)w = c_f & \text{in} \quad \Omega, \\
w = 0 & \text{on} \quad \partial\Omega, 
\end{cases} \] \hspace{1cm} (3.15)

At one hand, fix \( x_0 \in \Omega \) and let \( \rho < \frac{1}{4} \text{dist}(x_0, \partial\Omega) \). Let us recall that, for \( w \) satisfying (3.15), we have (cf. [13, p.71 Problem 4.5])

\[ w(x) = \frac{1}{|B_{2\rho}(x)|} \int_{B_{2\rho}(x)} w(z) \, dz + \frac{2c_f}{N(N+2)} \rho^2 \quad \text{for} \quad x \in B_{\rho}(x_0). \] \hspace{1cm} (3.16)

Moreover, since \( f \in C_0^\infty(\Omega) \), by interior regularity (cf. [7, Lemma 2.9], [4, Lemma 4.4]) we have \( u \in C^\infty(B_{2\rho}(x)) \) for any \( x \in B_{\rho}(x_0) \). Thus, \((-\Delta)u\) is well defined in \( B_{2\rho}(x) \) and

\[ (-\Delta)(u + w) = (-\Delta)^{1-s} f + (-\Delta)w \geq 0 \quad \text{in} \quad B_{2\rho}(x), \]

for any \( x \in B_{\rho}(x_0) \). Then, as \( u + w \) is superharmonic,

\[ u(x) + w(x) \geq \frac{1}{|B_{2\rho}(x)|} \int_{B_{2\rho}(x)} (u(z) + w(z)) \, dz \quad \text{for} \quad x \in B_{\rho}(x_0). \]

and, hence, by (3.16),

\[ u(x) + \frac{2c_f}{N(N+2)} \rho^2 \geq \frac{1}{|B_{2\rho}(x)|} \int_{B_{2\rho}(x)} u(z) \, dz \quad \text{for} \quad x \in B_{\rho}(x_0). \] \hspace{1cm} (3.17)

On the other hand, since \( u \in C^\gamma(\overline{\Omega}) \) for some \( \gamma \in (0, \frac{1}{2}) \) (cf. [8, Theorem 1.2]) and \( \frac{1}{2} < s < 1 \), there exists a constant \( c_1 > 0 \) such that \( u(x) \geq c_1 d^s(x) \)
in the whole $\Omega$. Then, we can choose $\rho$ small enough in order to have $c_1 \rho^s \geq \frac{4c_f}{N(N+2)} \rho^2$ and hence
\[
\frac{1}{|B_{2\rho}(x)|} \int_{B_{2\rho}(x)} u(z) dz \geq \frac{1}{|B_{2\rho}(x)|} \int_{B_{2\rho}(x)} c_1 d^s(z) dz \geq c_1 \rho^s \geq \frac{4c_f}{N(N+2)} \rho^2,
\]

since $\text{dist}(\partial B_{2\rho}(x), \partial \Omega) \geq \rho$ for any $x \in B_{\rho}(x_0)$. Then, by (3.17), we get
\[
u(x) \geq \frac{1}{|B_{2\rho}(x)|} \int_{B_{2\rho}(x)} u(z) dz - \frac{2c_f}{N(N+2)} \rho^2 \geq \frac{1}{2|B_{2\rho}(x)|} \int_{B_{2\rho}(x)} u(z) dz.
\]

(3.18)

Now we proceed as in [11, Theorem 1] which in turn is based on [3, Lemma 3.2].

Fixed $x_0 \in \Omega$ and $\rho < \min \left\{ \frac{1}{2}d(x_0), \left( \frac{c_1N+(2c_f)}{4c_f} \right)^{\frac{1}{4N}} \right\}$, we consider a function $f_0 \in C_0^\infty(B_{\rho}(x_0))$ with $0 \leq f_0 \leq 1$, $f \not\equiv 0$. Let $u_0$ be the solution to
\[
\left\{ \begin{array}{l}
(-\Delta)^s u_0 = f_0 \quad \text{in } \Omega,
B(u_0) = 0 \quad \text{on } \partial \Omega.
\end{array} \right.
\]

Using Theorem 1.2, there exists a constant $C > 0$, depending only on $\Omega$, $\|u_0\|_{L^\infty(\Omega)}$ and $\|\tilde{v}\|_{L^\infty(\Omega)}$ such that $u_0 \geq C\tilde{v}$ in $\Omega$. Next, for $x \in B_{2\rho}(x_0)$ we have $\text{supp}(f_0) \subset B_{2\rho}(y) \subset \Omega$. Then, by (3.18), we have for every $x \in B_{2\rho}(x_0)$
\[
u(x) \geq \frac{1}{2|B_{2\rho}(x)|} \int_{B_{2\rho}(x)} u(z) dz \geq \frac{1}{2|\Omega|} \int_{\Omega} u f_0 dz \geq c' \int_{\Omega} u f_0 dz
\]
\[
= c' \int_{\Omega} f u_0 dz \geq c'' \int_{\Omega} f u_0 dz \geq \lambda u_0(x),
\]

where $\lambda := \frac{c''}{\|u_0\|_{L^\infty(\Omega)}} \left( \int_{\Omega} f u_0 dz \right)$. Then $u - \lambda u_0 \geq 0$ in $\overline{B_{\rho}(x_0)}$ and, in particular, $u - \lambda u_0 \geq 0$ in $\partial B_{\rho}(x_0)$. Therefore,
\[
\left\{ \begin{array}{l}
(-\Delta)^s(u - \lambda u_0) = f \geq 0 \quad \text{in } \Omega \setminus \overline{B_{\rho}(x_0)},
u(u - \lambda u_0) \geq 0 \quad \text{on } \partial B_{\rho}(x_0),
B(u - \lambda u_0) = 0 \quad \text{on } \partial \Omega.
\end{array} \right.
\]

Thus, by comparison, $u - \lambda u_0 \geq 0$ in $\Omega \setminus \overline{B_{\rho}(x_0)}$ and, hence,
\[
u(x) \geq c'' \left( \int_{\Omega} f u_0 dz \right) u_0(x) \geq c \left( \int_{\Omega} f u_0 dz \right) \tilde{v}(x) \quad \text{for all } \forall x \in \Omega,
\]

which gives us the desired conclusion. \qed
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