Existence and Minimax Theorems for Adversarial Surrogate Risks in Binary Classification

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Editor: Gabor Lugosi

Abstract

We prove existence, minimax, and complementary slackness theorems for adversarial surrogate risks in binary classification. These results extend recent work of Pydi and Jog (2021), who established analogous minimax and existence theorems for the adversarial classification risk. We show that their conclusions continue to hold for a very general class of surrogate losses; moreover, we remove some of the technical restrictions present in prior work. Our results provide an explanation for the phenomenon of transfer attacks and inform new directions in algorithm development.

Keywords: Adversarial Learning, Minimax Theorems, Optimal Transport, Adversarial Bayes risk, Convex Relaxation

1. Introduction

Neural networks are state-of-the-art methods for a variety of machine learning tasks including image classification and speech recognition. However, a concerning problem with these models is their susceptibility to adversarial attacks: small perturbations to inputs can cause incorrect classification by the network (Szegedy et al., 2013; Biggio et al., 2013). This issue has security implications; for instance, Gu et al. (2017) show that a yellow sticker can cause a neural net to misclassify a stop sign. Furthermore, one can find adversarial examples that generalize to other neural nets; these sort of attacks are called transfer attacks. In other words, an adversarial example generated for one neural net will sometimes be an adversarial example for a different neural net trained for the same classification problem (Tramèr et al., 2017; Demontis et al., 2018; Kurakin et al., 2017; Rozsa et al., 2016; Papernot et al., 2016). This phenomenon shows that access to a specific neural net is not necessary for generating adversarial examples. One method for defending against such adversarial perturbations is adversarial training, in which a neural net is trained on adversarially perturbed data points (Kurakin et al., 2017; Madry et al., 2019; Wang et al., 2021). However, adversarial training is not well understood from a theoretical perspective.
From a theoretical standpoint, the most fundamental question is whether it is possible to design models which are robust to such attacks, and what the properties of such robust models might be. In contrast to the classical, non-adversarial setting, much is still unknown about the basic properties of optimal robust models. In the context of binary classification, several prior works study properties of the adversarial classification risk—the expected number of classification errors under adversarial perturbations. Recently, Awasthi et al. (2023), Bungert et al. (2021), and Pydi and Jog (2021) all showed existence of a minimizer to the adversarial classification risk under suitable assumptions, and characterized some of its properties. A crucial observation, emphasized by Pydi and Jog (2021), is that minimizing the adversarial classification risk is equivalent to a dual robust classification problem involving uncertainty sets with respect to the $\infty$-Wasserstein metric. This observation gives rise to a game-theoretic interpretation of robustness, which takes the form of a zero-sum game between a classifier and an adversary who is allowed to perturb the data by a certain amount. As noted by Pydi and Jog (2021), this interpretation has implications for algorithm design by suggesting that robust classifiers can be constructed by jointly optimizing over classification rules and adversarial perturbations.

This recent progress on adversarial binary classification lays the groundwork for a theoretical understanding of adversarial robustness, but it is limited insofar as it focuses only on minimizers of the adversarial classification risk. In practice, minimizing the empirical adversarial classification risk is computationally intractable; as a consequence, the adversarial training procedure typically minimizes an objective called a surrogate risk, which is chosen to be easier to optimize. In the non-adversarial setting, the properties of surrogate risks are well known (see, e.g. Bartlett et al., 2006), but in the adversarial scenario, existing results for the adversarial classification risk fail to carry over to surrogate risks. In particular, the existence and minimax results described above are not known to hold. We close this gap by developing an analogous theory for adversarial surrogate risks. Our main theorems (Theorems 7–9) establish that strong duality holds for the adversarial surrogate risk minimization problem, that solutions to the primal and dual problems exist, and that these optimizers satisfy a complementary slackness condition.

These results suggest explanations for empirical observations, such as the existence of transfer attacks. Specifically, our analysis suggests that adversarial examples are a property of the data distribution rather than a specific model. In fact, the complimentary slackness theorem presented in this paper states that certain attacks are the strongest possible adversary against any minimizer of the adversarial surrogate risk, which might explain why adversarial examples tend to transfer between trained neural nets. Furthermore, our theorems suggest that a training algorithm should optimize over neural nets and adversarial perturbations simultaneously. Adversarial training, the current state of the art method for finding adversarially robust networks, does not follow this procedure. The adversarial training algorithm tracks an estimate of the optimal function $\hat{f}$. To update $\hat{f}$, the algorithm first finds optimal adversarial examples at the current estimate $\hat{f}$, and then performs a gradient descent step. See the papers (Kurakin et al., 2017; Madry et al. 2019; Goodfellow et al. 2014) for more details on adversarial training. Finding these adversarial examples is a computationally intensive procedure. On the other hand, algorithms for optimizing minimax problems in the finite dimensional setting alternate between primal and dual steps (Mokhtari et al., 2019). This observation suggests that designing an algorithm that opti-
mizes over model parameters and adversarial perturbations simultaneously is a promising research direction. Trillos and Trillos (2023); Wang and Chizat (2023); Domingo-Enrich et al. (2021) adopt this approach, and one can view the minimax results of this paper as a mathematical justification for the use of surrogate losses in such algorithms.

Lastly, our theorems are an important first step in understating statistical properties of surrogate losses. Recall that one minimizes a surrogate risk because minimizing the original risk is computationally intractable. If a sequence of functions which minimizes the surrogate risk also minimizes the classification risk, then that surrogate is referred to as a consistent risk. Similarly, if a sequence of functions which minimizes the adversarial surrogate risk also minimizes the adversarial classification risk, then that surrogate is referred to as an adversarially consistent risk. Much prior work studies the consistency of surrogate risks (Bartlett et al., 2006; Lin, 2004; Steinwart, 2007; Philip M. Long, 2013; Mingyuan Zhang, 2020; Zhang, 2004). Alarmingly, Meunier et al. (2022) show that a family of surrogates used in applications is not adversarially consistent. In follow-up work, we show that our results can be used to characterize adversarially consistent supremum-based risks for binary classification (Frank and Niles-Weed, 2023), strengthening results on calibration in the adversarial setting Bao et al. (2021); Meunier et al. (2022); Awasthi et al. (2021).

2. Related Works

This paper extends prior work on the adversarial Bayes classifier. Pydi and Jog (2021) first proved multiple minimax theorems for the adversarial classification risk using optimal transport and Choquet capacities, showing an intimate connection between adversarial learning and optimal transport. Later, follow-up work used optimal transport minimax reformulations of the adversarial learning problem to derive new algorithms for adversarial learning. Trillos et al. (2022) reformulate adversarial learning in terms of a multi-marginal optimal transport problem and then apply existing techniques from optimal transport to find a new algorithm. Trillos and Trillos (2023); Wang and Chizat (2023); Domingo-Enrich et al. (2021) propose ascent-descent algorithms based on optimal transport and use mean-field dynamics to analyze convergence. These approaches leverage the minimax view of the adversarial training problem to optimize over model parameters and optimal attacks simultaneously. Gao et al. (2022) use an optimal transport reformulation to find regularizers that encourage robustness. Wong et al. (2019); Wu et al. (2020) use Wasserstein metrics to formulate adversarial attacks on neural networks.

Further work analyzes properties of the adversarial Bayes classifier. Awasthi et al. (2023), Bhagoji et al. (2019), and Bungert et al. (2021) all prove the existence of the adversarial Bayes classifier, using different techniques. Yang et al. (2020) studied the adversarial Bayes classifier in the context of non-parametric methods. Pydi and Jog (2019) and Bhagoji et al. (2019) further introduced methods from optimal transport to study adversarial learning. Lastly, (Trillos and Murray, 2020) give necessary and sufficient conditions describing the boundary of the adversarial Bayes classifier. Simultaneous work (Li and Telgarsky, 2023) also proves the existence of minimizers to adversarial surrogate risks using prior results on the adversarial Bayes classifier.

The adversarial training algorithm is also well studied from an empirical perspective. Demontis et al. (2018) discussed an explanation of transfer attacks on neural nets trained
using standard methods, but did not extend their analysis to the adversarial training setting. (Wang et al., 2021; Kurakin et al., 2017; Madry et al., 2019) study the adversarial training algorithm and improving the runtime. Two particularly popular attacks used in adversarial training are the FGSM attack (Goodfellow et al., 2014) and the PGD attack (Madry et al., 2019). More recent popular variants of this algorithm include (Shafahi et al., 2019; Xie et al., 2018; Kannan et al., 2018; Wong et al., 2020).

3. Background and Notation

3.1 Adversarial Classification

This paper studies binary classification on \( \mathbb{R}^d \) with two classes encoded as \(-1\) and \(+1\). Data is distributed according to a distribution \( D \) on \( \mathbb{R}^d \times \{-1, +1\} \). We denote the marginals according to the class labels as \( P_0(S) = D(S \times \{-1\}) \) and \( P_1(S) = D(S \times \{+1\}) \). Throughout the paper, we assume \( P_0(\mathbb{R}^d) \) and \( P_1(\mathbb{R}^d) \) are finite but not necessarily that \( P_0(\mathbb{R}^d) + P_1(\mathbb{R}^d) = 1 \).

To classify points in \( \mathbb{R}^d \), algorithms typically learn a real-valued function \( f \) and then classify points \( x \) according to the sign of \( f \) (arbitrarily assigning the sign of 0 to be \(-1\)).

The classification error, also known as the classification risk, of a function \( f \) is

\[
R(f) = \int 1_{f(x) \leq 0} dP_1 + \int 1_{f(x) > 0} dP_0. \tag{1}
\]

Notice that finding minimizers to \( R \) is straightforward: define the measure \( P = P_0 + P_1 \) and let \( \eta = dP_1/dP \). Then the risk \( R \) can be re-written as

\[
R(f) = \int \eta(x) 1_{f(x) \leq 0} + (1 - \eta(x)) 1_{f(x) > 0} dP.
\]

Hence a minimizer of \( R \) must minimize the function \( C(\eta(x), \alpha) = \eta(x) 1_{\alpha \leq 0} + (1 - \eta(x)) 1_{\alpha > 0} \)

at each \( x \) \( P \)-a.e. The optimal Bayes risk is then

\[
\inf_f R(f) = \int C^*(\eta) dP
\]

where \( C^*(\eta) = \inf_\alpha C(\eta, \alpha) = \min(\eta, 1 - \eta) \).

This paper analyzes the evasion attack, in which an adversary knows both the function \( f \) and the true label of the data point, and can perturb each input before it is evaluated by the function \( f \). To constrain the adversary, we assume that perturbations are bounded by \( \epsilon \) in a norm \( \| \cdot \| \). Thus a point \( x \) with label \(+1\) is misclassified if there is a perturbation \( h \) with \( \|h\| \leq \epsilon \) for which \( f(x + h) \leq 0 \) and a point \( x \) with label \(-1\) is misclassified if there is a perturbation \( h \) with \( \|h\| \leq \epsilon \) for which \( f(x + h) > 0 \). Therefore, the adversarial classification risk is

\[
R^*(f) = \int \sup_{\|h\| \leq \epsilon} 1_{f(x + h) \leq 0} dP_1 + \int \sup_{\|h\| \leq \epsilon} 1_{f(x + h) > 0} dP_0 \tag{2}
\]

which is the expected proportion of errors subject to the adversarial evasion attack. The expression \( \sup_{\|h\| \leq \epsilon} 1_{f(x + h) \leq 0} \) evaluates to 1 at a point \( x \) if \( x \) can be moved into the set.
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\[ f \leq 0 \] by a perturbation of size at most \( \epsilon \). Equivalently, this set is the Minkowski sum \( \oplus \) of \( [f \leq 0] \) and \( B_\epsilon(0) \). For any set \( A \), let \( A^\epsilon \) denote

\[ A^\epsilon = \{ x : \exists h \text{ with } ||h|| \leq \epsilon \text{ and } x + h \in A \} = A \oplus B_\epsilon(0) = \bigcup_{a \in A} B_\epsilon(a). \tag{3} \]

This operation ‘thickens’ the boundary of a set by \( \epsilon \). With this notation, (2) can be expressed as

\[ R^\epsilon(f) = \int \mathbb{1}_{\{f \leq 0\}} \cdot dP_1 + \int \mathbb{1}_{\{f > 0\}} \cdot dP_0. \]

Unlike the classification risk \( R \), finding minimizers to \( R^\epsilon \) is nontrivial. One can re-write \( R^\epsilon \) in terms of \( P \) and \( \eta \) but the resulting integrand cannot be minimized in a pointwise fashion. Nevertheless, it can be shown that minimizers of \( R^\epsilon \) exist (Awasthi et al., 2023; Bungert et al., 2021; Pydi and Jog, 2021; Frank and Niles-Weed, 2023).

### 3.2 Surrogate Risks

As minimizing the empirical version of risk in (1) is computationally intractable, typical machine learning algorithms minimize a proxy to the classification risk called a surrogate risk. In fact, Ben-David et al. (2003) show that minimizing the empirical classification risk is NP-hard in general. A popular surrogate is

\[ R_\phi(f) = \int \phi(f) dP_1 + \int \phi(-f) dP_0 \tag{4} \]

where \( \phi \) is a decreasing function.\(^1\) To define a classifier, one then thresholds \( f \) at zero. There are many reasonable choices for \( \phi \)—one typically chooses an upper bound on the zero-one loss which is easy to optimize. We make the following assumption on \( \phi \):

**Assumption 1** The loss \( \phi \) is non-increasing, non-negative, lower semi-continuous, and \( \lim_{\alpha \to \infty} \phi(\alpha) = 0 \).

A particularly important example, which plays a large role in our proofs, is the exponential loss \( \psi(\alpha) = e^{-\alpha} \), which will be denoted by \( \psi \) in the remainder of this paper. Assumption 1 includes many but not all surrogate risks used in practice. Notably, some multiclass surrogate risks with two classes are of a somewhat different form, see for instance (Tewari and Bartlett, 2007) for more details.

In order to find minimizers of \( R_\phi \), we rewrite the risk in terms of \( P \) and \( \eta \) as

\[ R_\phi(f) = \int \eta(x)\phi(f(x)) + (1 - \eta(x))\phi(-f(x)) dP \tag{5} \]

Hence the minimizer of \( R_\phi \) must minimize \( C_\phi(\eta, \cdot) \) pointwise \( P \)-a.e., where

\[ C_\phi(\eta, \alpha) = \eta\phi(\alpha) + (1 - \eta)\phi(-\alpha). \]

In other words, if one defines \( C_\phi^*(\eta) = \inf_\alpha C_\phi(\eta, \alpha) \), then a function \( f^* \) is optimal if and only if

\[ \eta(x)\phi(f^*(x)) + (1 - \eta(x))\phi(-f^*(x)) = C_\phi^*(\eta(x)) \quad P \text{-a.e.} \tag{6} \]

\(^1\) Notice that due to the asymmetry of the sign function at 0 in (1), \( R_\phi \) is not quite a generalization of \( R \).
Thus one can write the minimum value of $R_{\phi}$ as
\[ \inf_f R_{\phi}(f) = \int C_{\phi}^* (\eta) d\mathbb{P}. \tag{7} \]

To guarantee the existence of a function satisfying (6), we must allow our functions to take values in the extended real numbers $\mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\}$. Allowing the value $\alpha = +\infty$ is necessary, for instance, for the exponential loss $\psi(\alpha) = e^{-\alpha}$: when $\eta = 1$, the minimum of $C_{\psi}(1, \alpha) = e^{-\alpha}$ is achieved at $\alpha = +\infty$. In fact, one can express a minimizer as a function of the conditional probability $\eta(x)$ using (6). For a loss $\phi$, define $\alpha_{\phi}: [0, 1] \to \mathbb{R}$ by
\[ \alpha_{\phi}(\eta) = \inf\{\alpha : \alpha \text{ is a minimizer of } C_{\phi}(\eta, \cdot)\}. \tag{8} \]

Lemma 25 in Appendix C shows that the function $\alpha_{\phi}$ is monotonic and $\alpha_{\phi}(\eta)$ is in fact a minimizer of $C_{\phi}(\eta, \cdot)$. Thus
\[ f^*(x) = \alpha_{\phi}(\eta(x)) \tag{9} \]

is measurable and satisfies (6). Therefore, the function $f^*$ must be a minimizer of the risk $R_{\phi}$.

Similarly, rather directly minimizing the adversarial classification risk, practical algorithms minimize an adversarial surrogate. The adversarial counterpart to (4) is
\[ R_{\phi}(f) = \int \sup_{\|h\| \leq \epsilon} \phi(f(x+h)) d\mathbb{P}_1 + \int \sup_{\|h\| \leq \epsilon} \phi(-f(x+h)) d\mathbb{P}_0. \tag{10} \]

Due to the definitions of the adversarial risks (2) and (10), the operation of finding the supremum of a function over $\epsilon$-balls is central to our subsequent analysis. For a function $g$, we define
\[ S_{\epsilon}(g)(x) = \sup_{\|h\| \leq \epsilon} g(x+h) \tag{11} \]

Using this notation, one can re-write the risk $R_{\phi}$ as
\[ R_{\phi}(f) = \int S_{\epsilon}(\phi \circ f) d\mathbb{P}_1 + \int S_{\epsilon}(\phi \circ -f) d\mathbb{P}_0 \]

By analogy to (5), we equivalently write the risk $R_{\phi}$ in terms of $\mathbb{P}$ and $\eta$:
\[ R_{\phi}(f) = \int \eta(x) S_{\epsilon}(\phi \circ f)(x) + (1 - \eta(x)) S_{\epsilon}(\phi \circ -f)(x) d\mathbb{P}. \tag{12} \]

However, unlike (5), because the integrand of $R_{\phi}$ cannot be minimized in a pointwise manner, proving the existence of minimizers to $R_{\phi}$ is non-trivial. In fact, unlike the adversarial classification risk $R_\epsilon$, there is little theoretical understanding of the properties of $R_{\phi}$.

### 3.3 Measurability

In order to define the adversarial risks $R_\epsilon$ and $R_{\phi}$, one must show that $S_{\epsilon}(1_A), S_{\epsilon}(\phi \circ f)$ are measurable. To illustrate this concern, Pydi and Jog (2021) show that for every $\epsilon > 0$ and $d > 1$, there is a Borel set $C$ for which the function $S_{\epsilon}(1_C)(x)$ is not Borel measurable. However, if $g$ is Borel, then $S_{\epsilon}(g)$ is always measurable with respect to a larger $\sigma$-algebra called the universal $\sigma$-algebra $\mathcal{U}(\mathbb{R}^d)$. Such a function is called universally measurable. We prove the following theorem and formally define the universal $\sigma$-algebra in Appendix A.
**Theorem 1** If \( f \) is universally measurable, then \( S_\epsilon(f) \) is also universally measurable.

In fact, in Appendix A, we show that a function defined by a supremum of a universally measurable function over a compact set is universally measurable—a result of independent interest. The universal \( \sigma \)-algebra is smaller than the completion of \( \mathcal{B}(\mathbb{R}^d) \) with respect to any Borel measure. Thus, in the remainder of the paper, unless otherwise noted, all measures will be Borel measures and the expression \( \int S_\epsilon(f) dQ \) will be interpreted as the integral of \( S_\epsilon(f) \) with respect to the completion of \( Q \).

### 3.4 The \( W_\infty \) Metric

In this section, we explain how the integral of a supremum \( \int S_\epsilon(f) dQ \) can be expressed in terms of a supremum of integrals. We start by defining the Wasserstein-\( \infty \) metric.

**Definition 2** Let \( P, Q \) be two finite measures with \( P(\mathbb{R}^d) = Q(\mathbb{R}^d) \). A coupling is a positive measure on the product space \( \mathbb{R}^d \times \mathbb{R}^d \) with marginals \( P, Q \). We denote the set of all couplings with marginals \( P, Q \) by \( \Pi(P, Q) \). The \( \infty \)-Wasserstein distance with respect to a norm \( \| \cdot \| \) is defined as

\[
W_\infty(P, Q) = \inf_{\gamma \in \Pi(P, Q)} \text{ess sup}_{(x, x') \sim \gamma} \| x - x' \|
\]

Jylhä (2014, Theorem 2.6) proves that the infimum is always attained. Therefore, \( P, Q \) are within a Wasserstein-\( \infty \) distance of \( \epsilon \) if there is a coupling \( \gamma \) for \( P \) and \( Q \) for which \( \text{supp} \gamma \) is contained in the set \( \Delta_\epsilon = \{(x, x') : \| x - x' \| \leq \epsilon \} \). This optimal coupling will be a useful tool in proving theorems throughout this paper.

The \( \infty \)-Wasserstein metric is closely related to the to the operation \( S_\epsilon \). First, we show that \( S_\epsilon \) can be expressed as a supremum of integrals over a Wasserstein-\( \infty \) ball. For a measure \( Q \), we write \( B_\infty^\epsilon(Q) = \{ Q' : W_\infty(Q, Q') \leq \epsilon \} \).

**Lemma 3** Let \( Q \) be a finite positive Borel measure and let \( f : \mathbb{R}^d \to \mathbb{R} \cup \{ \infty \} \) be a Borel measurable function. Then

\[
\int S_\epsilon(f) dQ = \sup_{Q' \in B_\infty^\epsilon(Q)} \int f dQ'
\]

(13)

Lemma 5.1 of Pydi and Jog (2021) proves an analogous statement for sets, namely that \( Q(A') = \sup_{Q' \in B_\infty^\epsilon(Q)} Q(A) \), under suitable assumptions on \( Q \) and \( Q' \).

Conversely, the \( W_\infty \) distance between two probability measures can be expressed in terms of the integrals of \( f \) and \( S_\epsilon(f) \). Let \( C_b(X) \) be the set of continuous bounded functions on the topological space \( X \).

**Lemma 4** Let \( P, Q \) be two finite positive Borel measures with \( P(\mathbb{R}^d) = Q(\mathbb{R}^d) \). Then

\[
W_\infty(P, Q) = \inf_{\epsilon \geq 0} \int \epsilon dQ \leq \int S_\epsilon(h) dP \quad \forall h \in C_b(\mathbb{R}^d)
\]

This observation will be central to proving a duality result. See Appendix B for proofs of Lemmas 3 and 4.
4. Main Results and Outline of the Paper

4.1 Summary of Main Results

Our goal in this paper is to understand properties of the surrogate risk minimization problem $\inf_f R^\epsilon_\phi$. The starting point for our results is Lemma 3, which implies that $\inf_f R^\epsilon_\phi$ actually involves a $\inf$ followed by a $\sup$: 

$$ \inf_{f \text{ Borel}} R^\epsilon_\phi(f) = \inf_{f \text{ Borel}} \sup_{\mathcal{P}_0^\epsilon \in \mathcal{B}^\infty_{\mathcal{P}_0}(\mathbb{P}_0)} \sup_{\mathcal{P}_1^\epsilon \in \mathcal{B}^\infty_{\mathcal{P}_1}(\mathbb{P}_1)} \int \phi \circ f d\mathbb{P}_1 + \int \phi \circ \bar{f} d\mathbb{P}_0. $$

We therefore obtain a lower bound on $\inf_f R^\epsilon_\phi$ by swapping the $\sup$ and $\inf$ and recalling the definition of $C^*_\phi(\eta) = \inf_{\alpha} C_{\phi}(\eta, \alpha)$:

$$ \inf_{f \text{ Borel}} R^\epsilon_\phi(f) \geq \sup_{\mathcal{P}_0^\epsilon \in \mathcal{B}^\infty_{\mathcal{P}_0}(\mathbb{P}_0)} \inf_{f \text{ Borel}} \int \phi \circ f d\mathbb{P}_1 + \int \phi \circ \bar{f} d\mathbb{P}_0 $$

$$ = \sup_{\mathcal{P}_0^\epsilon \in \mathcal{B}^\infty_{\mathcal{P}_0}(\mathbb{P}_0)} \inf_{f \text{ Borel}} \int \frac{d\mathbb{P}'_1}{d(\mathbb{P}_0 + \mathbb{P}_1)} \phi(f) + \left(1 - \frac{d\mathbb{P}'_1}{d(\mathbb{P}_0 + \mathbb{P}_1)}\right) \phi(\bar{f}) d(\mathbb{P}_0 + \mathbb{P}_1) $$

$$ \geq \sup_{\mathcal{P}_0^\epsilon \in \mathcal{B}^\infty_{\mathcal{P}_0}(\mathbb{P}_0)} \inf_{f \text{ Borel}} \left( C^*_\phi \left( \frac{d\mathbb{P}'_1}{d(\mathbb{P}_0 + \mathbb{P}_1)} \right) \right) d(\mathbb{P}_0 + \mathbb{P}_1). \quad (14) $$

If we define

$$ \tilde{R}_\phi(\mathbb{P}_0^\epsilon, \mathbb{P}_1^\epsilon) = \int C^*_\phi \left( \frac{d\mathbb{P}'_1}{d(\mathbb{P}_0 + \mathbb{P}_1)} \right) d(\mathbb{P}_0 + \mathbb{P}_1), \quad (15) $$

then we have shown

$$ \inf_{f \text{ Borel}} R^\epsilon_\phi(f) \geq \sup_{\mathbb{P}_0^\epsilon \in \mathcal{B}^\infty_{\mathbb{P}_0}(\mathbb{P}_0)} \tilde{R}_\phi(\mathbb{P}_0^\epsilon, \mathbb{P}_1^\epsilon). \quad (16) $$

This statement is a form of weak duality.

When the surrogate adversarial risk is replaced by the standard adversarial classification risk, Pydi and Jog (2021) proved that the analogue of (16) is actually an equality, so that strong duality holds for the adversarial classification problem. Concretely, by analogy to (15), consider

$$ \tilde{R}(\mathbb{P}_0^\epsilon, \mathbb{P}_1^\epsilon) = \int C^*_\phi \left( \frac{d\mathbb{P}'_1}{d(\mathbb{P}_0 + \mathbb{P}_1)} \right) d(\mathbb{P}_0 + \mathbb{P}_1). $$

Let $\mu$ be the Lebesgue measure and let $\mathcal{L}_\mu(\mathbb{R}^d)$ be the Lebesgue $\sigma$-algebra. Then define

$$ \mathcal{B}^\infty(\mathbb{Q}) = \{Q' : W_\infty(\mathbb{Q}, Q') \leq \epsilon, Q' \text{ a measure on } (\mathbb{R}^d, \mathcal{L}_\mu(\mathbb{R}^d))\}. \quad (17) $$

Pydi and Jog (2021) show the following.

**Theorem 5 (Pydi and Jog, 2021, Theorem 7.1)** Assume that $\mathbb{P}_0, \mathbb{P}_1$ are absolutely continuous with respect to the Lebesgue measure $\mu$. Then

$$ \inf_{f \text{ Lebesgue}} R^\epsilon_\phi(f) = \sup_{\mathbb{P}_0^\epsilon \in \mathcal{B}^\infty_{\mathbb{P}_0}(\mathbb{P}_0)} \tilde{R}(\mathbb{P}_0^\epsilon, \mathbb{P}_1^\epsilon) \quad (18) $$
and furthermore equality is attained at some Lebesgue measurable \( \hat{f} \) and \( \hat{P}_1, \hat{P}_0 \).

Additionally, \( \hat{P}_1 = P_1 \circ \varphi_i^{-1} \) for some universally measurable \( \varphi_i \) with \( \| \varphi_i(x) - x \| \leq \epsilon \), \( \sup_{\|y-x\|\leq\epsilon} 1_{f(y)\leq0} = 1_{f(\varphi_i(x))\leq0} P_1 \text{-a.e.} \), and \( \sup_{\|y-x\|\leq\epsilon} 1_{f(y)>0} = 1_{f(\varphi_i(x))>0} P_0 \text{-a.e.} \).

This is a foundational result in the theory of adversarial classification, but it leaves an open question crucial in applications: Does the strong duality relation extend to surrogate risks and to general measures? In this work, we answer this question in the affirmative.

We start by proving the following:

**Theorem 6 (Strong Duality)** Let \( P_0, P_1 \) be finite Borel measures. Then

\[
\inf_{f \text{ Borel}} R_\phi^*(f) = \sup_{P_0' \in B_\infty^\epsilon(P_0)} \inf_{P_1' \in B_\infty^\epsilon(P_1)} \hat{R}_\phi(P_0', P_1'). \tag{19}
\]

When \( \epsilon = 0 \), we recover the fundamental characterization of the minimum risk for standard (non-adversarial) classification in (7). Theorem 6 can be rephrased as

\[
\inf_{f \text{ Borel}} \sup_{P_0' \in B_\infty^\epsilon(P_0)} \inf_{P_1' \in B_\infty^\epsilon(P_1)} \hat{R}_\phi(f, P_0', P_1') = \sup_{P_0' \in B_\infty^\epsilon(P_0)} \inf_{P_1' \in B_\infty^\epsilon(P_1)} \hat{R}_\phi(f, P_0, P_1') \tag{20}
\]

where

\[
\hat{R}_\phi(f, P_0, P_1') = \int \phi(f) dP_1' + \int \phi(-f) dP_0'
\]

As discussed in Pydi and Jog (2021), this result has an appealing game theoretic interpretation: adversarial learning with a surrogate risk can be though of as a zero-sum game between the learner who selects a function \( f \) and the adversary who chooses perturbations through \( P_0' \) and \( P_1' \). Furthermore, the player to pick first does not have an advantage.

Additionally, (20) suggest that training adversarially robust classifiers could be accomplished by optimizing over primal functions \( f \) and dual distributions \( P_0', P_1' \) simultaneously.

A consequence of Theorem 6 is the following complementary slackness conditions for optimizers \( f^*, P_0^*, P_1^* \):

**Theorem 7 (Complimentary Slackness)** The function \( f^* \) is a minimizer of \( R_\phi^* \) and \((P_0^*, P_1^*) \) is a maximizer of \( \hat{R}_\phi \) over the \( W_\infty \) balls around \( P_0 \) and \( P_1 \) iff the following hold:

1) \[
\int \phi \circ f^* dP_1^* = \int S_\epsilon(\phi(f^*)) dP_1 \quad \text{and} \quad \int \phi \circ -f^* dP_0^* = \int S_\epsilon(\phi(f^*)) dP_0 \tag{21}
\]

2) If we define \( P_* = P_0^* + P_1^* \) and \( \eta^* = dP_1^*/dP_* \), then

\[
\eta^*(x) \phi(f^*(x)) + (1 - \eta^*(x)) \phi(-f^*(x)) = C_\phi^*(\eta^*(x)) \quad P_* \text{-a.e.} \tag{22}
\]

This theorem implies that every minimizer \( f^* \) of \( R_\phi^* \) and every maximizer \((P_0^*, P_1^*) \) of \( \hat{R}_\phi \) forms a primal-dual pair. The condition (21) states that every maximizer of \( \hat{R}_\phi \) is an optimal adversarial-dual attack on \( f^* \) while the condition (22) states that every minimizer...
$f^*$ of $R_\phi^*$ also minimizes the conditional risk $C_\phi(\eta^*, \cdot)$ under the distribution of optimal adversarial attacks. Explicitly: (22) implies that every minimizer $f^*$ minimizes the loss $R_\phi(f, P_0^*, P_1^*) = \int C(\eta^*(x), f(x)) dP^*$ in a pointwise manner $P^*$-a.e., or n other words, the function $f^*$ minimizes the standard surrogate risk with respect to the optimal adversarially perturbed distributions. This fact is the relation (6) with respect to the measures $P^*_0, P^*_1$ that maximize the dual $\hat{R}_\phi$.

This interpretation of Theorems 6 and 7 shed light on the phenomenon of transfer attacks. These theorems suggests that adversarial examples are a property of the data distribution rather than a specific model. Later results in the paper even show that there are maximizers of $R_\phi$ that are independent of the choice of loss function $\phi$ (see Lemma 26). Theorem 7 specifically states that every maximizer of $R_\phi$ is actually an optimal adversarial attack on every minimizer of $R_\phi^*$. Notably, this statement is independent of the choice of minimizer of $R_\phi^*$. Because neural networks are highly expressive model classes, one would hope that some neural net could achieve adversarial error close to $\inf_f R_\phi^*(f)$. If $f^*$ is a minimizer of $R_\phi^*$ and $g$ is a neural net with $R_\phi^*(g) \approx R_\phi^*(f^*)$, one would expect that an optimal adversarial attack against $f^*$ would be a successful attack on $g$ as well. Notice that in this discussion, the adversarial attack is independent of the neural net $g$. A method for calculating these optimal adversarial attacks is an open problem.

Finally, to demonstrate that Theorem 7 and the preceding discussion is non-vacuous, we prove the existence of primal and dual optimizers along with results that elaborate on their structure.

**Theorem 8** Let $\phi$ be a lower-semicontinuous loss function. Then there exists a maximizer $(P_0^*, P_1^*)$ to $R_\phi$ over the set $\mathcal{B}_c^\infty(P_0) \times \mathcal{B}_c^\infty(P_1)$.

Theorem 3.5 of (Jylhä, 2014) implies that when the norm $\| \cdot \|$ is strictly convex and $P_0, P_1$ are absolutely continuous with respect to Lebesgue measure, the optimal $P_0^*, P_1^*$ of Theorem 8 are induced by a transport map. Corollary 3.11 of (Jylhä, 2014) further implies that these transport maps are continuous a.e. with respect to the Lebesgue measure $\mu$. As the $\ell_\infty$ metric is commonly used in practice, whether there exist maximizers of the dual of this type for non-strictly convex norms remains an attractive open problem.

In analogy with (6) and (9) one would hope that due to the complimentary slackness condition (22), one could define a minimizer in terms of the conditional $\eta^*(x)$. Notice, however, that as this quantity is only defined $P^*$-a.e., verifying the other complimentary slackness condition (21) would be a challenge. To circumvent this issue, we construct a function $\hat{\eta} : \mathbb{R}^d \rightarrow [0, 1]$, defined on all of $\mathbb{R}^d$, to which we can apply (9). Concretely, we show that $\alpha_\phi(\hat{\eta}(x))$ is always a minimizer of $R_\phi^*$, with $\alpha_\phi$ as defined in (8).

**Theorem 9** There exists a Borel function $\hat{\eta} : (\text{supp } P)^c \rightarrow [0, 1]$ for which $f^*(x) = \alpha_\phi(\hat{\eta}(x))$ is a minimizer of $R_\phi^*$ for any $\phi$ with $\alpha_\phi$ as defined in (8). In particular, there exists a Borel minimizer of $R_\phi^*$.

In fact, we show that $\hat{\eta}$ is a version of the conditional derivative $dP^*/dP^*$, where $P_0^*, P_1^*$ are the measures which maximize $\hat{R}_\phi$ independently of the choice of $\phi$ (see Lemma 24), as described in the discussion preceding Theorem 8. The fact that the function $\hat{\eta}$ is independent of the choice of loss $\phi$ suggests that the minimizer of $R_\phi^*$ encodes some fundamental quality of the distribution $P_0, P_1$. 


Simultaneous work (Li and Telgarsky, 2023) also proves the existence of a minimizer to the primal $R_\phi^\epsilon$ along with a statement on the structure of this minimizer. Their approach leverages prior results on the adversarial Bayes classifier to construct a minimizer to the adversarial surrogate risk.

4.2 Outline of Main Argument

The central proof strategy of this paper is to apply the Fenchel-Rockafellar duality theorem. This classical result relates the infimum of a convex functional with the supremum of a concave functional. One can argue that $\bar{R}_\phi$ is concave (Lemma 12 below); however, the primal $R_\phi^\epsilon$ is not convex for non-convex $\phi$. Thus the Fenchel-Rockafellar theorem is applied to a convex relaxation $\Theta$ of the primal $R_\phi^\epsilon$.

The remainder of the paper then argues that minimizing $\Theta$ is equivalent to minimizing $R_\phi^\epsilon$. Notice that the Fenchel-Rockafellar theorem actually implies the existence of dual maximizers. We show that that dual maximizers of $\bar{R}_\psi$ for $\psi(\alpha) = e^{-\alpha}$ satisfy certain nice properties that are independent of the loss $\psi$. These properties then allow us to construct the function $\hat{\eta}$ present in Theorem 9 in addition to minimizers of $\Theta$ from the dual maximizers of $\bar{R}_\psi$, for any loss $\phi$. The construction of these minimizers guarantee that they minimize $R_\phi^\epsilon$ in addition to $\Theta$.

4.3 Paper Outline

Section 5 proves strong duality and complimentary slackness theorems for $\bar{R}_\phi$ and $\Theta$, the convex relaxation of $R_\phi^\epsilon$. Next, in Section 6, a version of the complimentary slackness result is used to prove the existence of minimizers to $\Theta$. Subsequently, Section 7 shows the equivalence between $\Theta$ and $R_\phi^\epsilon$.

Appendix A proves Theorem 1 and further discusses universal measurability. Next, Appendix B proves all the results about the $W_\infty$-norm used in this paper. Appendix C then defines the function $\alpha_\phi$ which is later used in the proof of several results. Appendices D, E, F, and G.3 contain technical deferred proofs.

5. A Duality Result for $\Theta$ and $\bar{R}_\phi$

5.1 Strong Duality

The fundamental duality relation of this paper follows from employing the Fenchel-Rockafellar theorem in conjunction with the Riesz representation theorem, stated below for reference. See e.g. (Villani, 2003) for more on this result.

**Theorem 10 (Fenchel-Rockafellar Duality Theorem)** Let $E$ be a normed vector space $E^*$ its topological dual and $\Theta, \Xi$ two convex functionals on $E$ with values in $\mathbb{R} \cup \{\infty\}$. Let $\Theta^*, \Xi^*$ be the Legendre-Fenchel transforms of $\Theta, \Xi$ respectively. Assume that there exists $z_0 \in E$ such that

$$\Theta(z_0) < \infty, \Xi(z_0) < \infty$$

and that $\Theta$ is continuous at $z_0$. Then

$$\inf_{z \in E} [\Theta(z) + \Xi(z)] = \sup_{z^* \in E^*} [-\Theta^*(z^*) - \Xi^*(-z^*)]$$  \hspace{1cm} (23)
and furthermore, the supremum on the right hand side is attained.

Let $\mathcal{M}(X)$ be the set of finite signed Borel measures on a space $X$ and recall that $C_b(X)$ is the set of bounded continuous functions on the space $X$.

**Theorem 11 (Riesz Representation Theorem)** Let $K$ be any compact subset of $\mathbb{R}^d$. Then the dual of $C_b(K)$ is $\mathcal{M}(K)$.

See Theorem 1.9 of (Villani, 2003) and result 7.17 of (Folland, 1999) for more details.

Notice that in the Fenchel-Rockafellar theorem, the left-hand side of (23) is convex while the right-hand side is concave. However, when $\phi$ is non-convex, $R^*_\phi$ is not convex. In order to apply the Fenchel-Rockafellar theorem, we will relax the primal $R^*_\phi$ will to a convex functional $\Theta$.

We define $\Theta$ as

$$\Theta(h_0, h_1) = \int S_\epsilon(h_1) d\mathbb{P}_1 + \int S_\epsilon(h_0) d\mathbb{P}_0$$

which is convex in $h_0, h_1$ due to the sub-additivity of the supremum operation. Notice that one obtains $\Theta$ from $R^*_\phi$ by replacing $\phi \circ f$ with $h_1$ and $\phi \circ -f$ with $h_0$.

The functional $\Xi$ will be chosen to restrict $h_0, h_1$ in the hope that at the optimal value, $h_1 = \phi(f)$ and $h_0 = \phi(-f)$ for some $f$. Notice that if $h_1 = \phi(f)$, $h_0 = \phi(-f)$ then for all $\eta \in [0, 1]$, $\eta h_1(x) + (1 - \eta) h_0 = \eta \phi(f) + (1 - \eta) \phi(-f) \geq C^*_\phi(\eta)$.

Thus we will optimize $\Theta$ over the set of functions $S_\phi$ defined by

$$S_\phi = \left\{ (h_0, h_1) : h_0, h_1 : K^c \rightarrow \mathbb{R} \text{ Borel, } 0 \leq h_0, h_1 \text{ and for all } x \in \mathbb{R}^d, \eta \in [0, 1], \eta h_0(x) + (1 - \eta) h_1(x) \geq C^*_\phi(\eta) \right\}$$

where $K = \text{supp}(\mathbb{P}_0 \cup \mathbb{P}_1)$. (Notice that the definition of $S_\epsilon(g)$ in (11) assumes that the domain of $g$ must include $\overline{B}_r(x)$. Thus in order to define the integral $\int S_\epsilon(h) d\mathbb{Q}$, the domain of $h$ must include $(\text{supp} \mathbb{Q})^c$.) Thus we define $\Xi$ as

$$\Xi(h_0, h_1) = \begin{cases} 0 & \text{if } (h_0, h_1) \in S_\phi \\ +\infty & \text{otherwise} \end{cases}$$

The following result expresses $\bar{R}_\phi$ as an infimum of linear functionals continuous with respect to the weak topology on probability measures. This lemma will assist in the computation of $\Xi^*$. In the remainder of this section, $\mathcal{M}_+(S)$ will denote the set of positive finite Borel measures on a set $S$.

**Lemma 12** Let $K \subset \mathbb{R}^d$ be compact, $E = C_b(K^c) \times C_b(K^c)$, and $\mathbb{P}_0, \mathbb{P}_1 \in \mathcal{M}_+(K^c)$. Then

$$\inf_{(h_0, h_1) \in S_\phi \cap E} \int h_1 d\mathbb{P}_1 + \int h_0 d\mathbb{P}_0 = \bar{R}_\phi(\mathbb{P}_0, \mathbb{P}_1)$$

Therefore, $\bar{R}_\phi$ is concave and upper semi-continuous on $\mathcal{M}_+(K^c) \times \mathcal{M}_+(K^c)$ with respect to the weak topology on probability measures.
We sketch the proof and formally fill in the details in Appendix D. Let \( \mathbb{P}' = \mathbb{P}'_0 + \mathbb{P}'_1 \), \( \eta' = d\mathbb{P}'_1/d\mathbb{P}' \). Then
\[
\int h_1 d\mathbb{P}'_1 + \int h_0 d\mathbb{P}'_0 = \int \eta' h_1 + (1 - \eta') h_0 d\mathbb{P}'
\]
Clearly, the inequality \( \geq \) holds because \( \eta' h_1 + (1 - \eta') h_0 \geq C_\phi'(\eta') \) for all \( (h_0, h_1) \in S_\phi \). Equality is achieved at \( h_1 = \phi(\alpha_\phi(\eta')) \), \( h_0 = \phi(-\alpha_\phi(\eta')) \), with \( \alpha_\phi \) as in (8). However, these functions may not be continuous. In Appendix D, we show that \( h_0, h_1 \) can be approximated arbitrarily well by elements of \( S_\phi \cap E \).

**Lemma 13** Let \( \phi \) be a non-increasing, lower semi-continuous loss function and let \( \mathbb{P}_0, \mathbb{P}_1 \) be compactly supported finite Borel measures. Let \( S_\phi \) be as in (25).

Then
\[
\inf_{(h_0, h_1) \in S_\phi} \Theta(h_0, h_1) = \sup_{\mathbb{P}'_0 \in \mathcal{B}_c^\infty(\mathbb{P}_0)} \sup_{\mathbb{P}'_1 \in \mathcal{B}_c^\infty(\mathbb{P}_1)} R_\phi(\mathbb{P}'_0, \mathbb{P}'_1) \tag{28}
\]
Furthermore, there exist \( \mathbb{P}'_0, \mathbb{P}'_1 \) which attain the supremum.

**Proof** We will show a version of (28) with the infimum taken over \( S_\phi \cap E \), and then argue that the same claim holds when the infimum is taken over \( S_\phi \).

Notice that if \( h_0, h_1 \) are continuous, then \( S_\epsilon(h_0), S_\epsilon(h_1) \) are also continuous and \( \int S_\epsilon(h_0) dQ, \int S_\epsilon(h_1) dQ \) are well-defined for every Borel \( Q \). Hence we assume that \( \mathbb{P}_0, \mathbb{P}_1 \) are Borel measures rather than their completion.

Let \( K = \text{supp}(\mathbb{P}_0 + \mathbb{P}_1) \). We will apply the Fenchel-Rockafellar Duality Theorem to the functionals given by (24) and (26) on the vector space \( E = C_b(K^c) \times C_b(K^c) \) equipped with the sup norm. By the Riesz representation theorem, dual of the space \( E \) is \( E^* = \mathcal{M}(K^c) \times \mathcal{M}(K^c) \).

To start, we argue that the Fenchel-Rockafellar duality theorem applies to these functionals. First, notice that if \( (h_0, h_1) \in E \), then both \( h_0, h_1 \) are bounded so \( \Theta(h_0, h_1) < \infty \). Furthermore, \( \Theta \) is convex due to the subadditivity of supremum and \( \Xi \) is convex because the constraint \( h_0(x) + (1 - \eta) h_1(x) \geq C_\phi'(\eta) \) is linear in \( h_0 \) and \( h_1 \). Furthermore, \( \Theta \) is continuous on \( E \) because \( \Theta \) is convex and bounded and \( E \) is open, see Theorem 2.14 of (Barbu and Precupanu, 2012).

Because the constant function \( (C_\phi'(1/2), C_\phi'(1/2)) \) is in \( S_\phi \), \( \Xi \) is not identically \( \infty \) and therefore the Fenchel-Rockafellar theorem applies.

Clearly \( \inf_E \Theta(h_0, h_1) + \Xi(h_0, h_1) \) reduces to the left-hand side of (28).

We now compute the dual of \( \Xi \). Lemma 12 implies that
\[
-\Xi^*(-\mathbb{P}'_0, -\mathbb{P}'_1) = -\sup_{(h_0, h_1) \in S_\phi \cap E} -\int h_0 d\mathbb{P}'_0 - \int h_1 d\mathbb{P}'_1
\]
\[
= \begin{cases} 
R_\phi(\mathbb{P}'_0, \mathbb{P}'_1) & \text{if } \mathbb{P}'_1 \geq 0 \\
-\infty & \text{otherwise}
\end{cases}
\tag{29}
\]
This computation implies that the term \( -\Xi^*(-\mathbb{P}'_0, -\mathbb{P}'_1) \) present in the Fenchel-Rockafellar Theorem is not \( -\infty \) if \( \mathbb{P}'_0, \mathbb{P}'_1 \) are positive measures. Next, notice that because \( \Theta(h_0, h_1) < \infty \)
Lemma 14 which is proved formally in Appendix E. Lemma 4 implies that for positive measures $P'_0, P'_1$, 

$$
\Theta^*(P'_0, P'_1) = \sup_{h_0, h_1 \in C_0(K')} \int h_1 dP'_1 + \int h_0 dP'_0 - \left( \int S_\epsilon(h_0) dP'_0 + \int S_\epsilon(h_1) dP'_1 \right)
$$

$$
= \sup_{h_1 \in C_0(K')} \left( \int h_1 dP'_1 - \int S_\epsilon(h_1) dP'_1 \right) + \sup_{h_0 \in C_0(K')} \left( \int h_0 dP'_0 - \int S_\epsilon(h_0) dP'_0 \right)
$$

$$
= \left\{ \begin{array}{ll}
0 & \text{if } P'_0, P'_1 \text{ positive measures, with } W_{\infty}(P'_0, P_0) \leq \epsilon \text{ and } W_{\infty}(P'_1, P_1) \leq \epsilon \\
+\infty & \text{if } P'_0, P'_1 \text{ positive measures, with either } W_{\infty}(P'_0, P_0) > \epsilon \text{ or } W_{\infty}(P'_1, P_1) > \epsilon 
\end{array} \right.
$$

Therefore

$$
\sup_{P'_0, P'_1 \in \mathcal{M}(K')} \Theta(P'_0, P'_1) = \sup_{P'_0 \in \mathcal{B}_c^\infty(P_0)} \Theta(P'_0, P'_1)
$$

and furthermore there exist measures $P'_0, P'_1$ maximizing the dual problem. Therefore the Fenchel-Rockafellar Theorem implies that

$$
\inf_{(h_0, h_1) \in S_\phi} \Theta(h_0, h_1) \leq \inf_{(h_0, h_1) \in S_\phi \cap E} \Theta(h_0, h_1) = \sup_{P'_0 \in \mathcal{B}_c^\infty(P_0)} \sup_{P'_1 \in \mathcal{B}_c^\infty(P_1)} \bar{R}_\phi(P'_0, P'_1)
$$

The opposite inequality follows from the weak duality argument presented in (16) in Section 4.1. See Lemma 45 of Appendix E for a full proof.

Note that this proof does not easily extend to an unbounded domain $X$: for a non-compact space, the dual of $C_b(X)$ is much larger than $\mathcal{M}(X)$, and thus a naive application of the Fenchel-Rockafellar Theorem would result in a different right-hand side than (28). On the other hand, the Reisz representation theorem for an unbounded domain $X$ states that the dual of $C_0(X)$ is $\mathcal{M}(X)$, where $C_0(X)$ is the set of continuous bounded functions vanishing at $\infty$. At the same time, if $h_0, h_1 \in C_0(X)$, then $\eta h_1(x) + (1 - \eta)h_0(x)$ becomes arbitrarily small for large $x$ so the constraint $\eta h_1(x) + (1 - \eta)h_0(x) \geq C_\phi^*(\eta)$ cannot hold for all $\eta$. Thus if $K$ is unbounded, $S_\phi \cap C_0(X) = \emptyset$ and the functional $\Xi$ would be $+\infty$ everywhere on $C_0(X)$, precluding and application of the Fenchel-Rockafellar Theorem.

However, Lemma 13 can be extended to distributions with arbitrary support via a simple approximation argument. By Lemma 13, the strong duality result holds for the distributions defined by $P'_0 = P_0|_{B_n(0)}$, $P'_1 = P_1|_{B_n(0)}$. One then shows strong duality by computing the limit of the primal and dual problems as $n \to \infty$. We therefore obtain the following Lemma, which is proved formally in Appendix E.

**Lemma 14** Let $\phi$ be a non-increasing, lower semi-continuous loss function and let $P_0, P_1$ be finite Borel measures supported on $\mathbb{R}^d$. Let $S_\phi$ be as in (25). Then

$$
\inf_{(h_0, h_1) \in S_\phi} \Theta(h_0, h_1) = \sup_{P'_0 \in \mathcal{B}_c^\infty(P_0)} \sup_{P'_1 \in \mathcal{B}_c^\infty(P_1)} \bar{R}_\phi(P'_0, P'_1)
$$

Furthermore, there exist $P'_0, P'_1$ which attain the supremum.
5.2 Complimentary Slackness

Using a standard argument, strong duality (Lemma 14) allows us to prove a version of the complimentary slackness theorem.

**Lemma 15** Assume that $\mathbb{P}_0, \mathbb{P}_1$ are compactly supported. The functions $h_0^n, h_1^n$ minimize $\Theta$ over $S_\phi$ and $(\mathbb{P}_0^n, \mathbb{P}_1^n)$ maximize $\bar{R}_\phi$ over $\mathcal{B}_c^\infty(\mathbb{P}_0) \times \mathcal{B}_c^\infty(\mathbb{P}_1)$ iff the following hold:

1) $$\int h_1^n d\mathbb{P}_1 = \int S_\epsilon(h_1^n) d\mathbb{P}_1 \quad \text{and} \quad \int h_0^n d\mathbb{P}_0 = \int S_\epsilon(h_0^n) d\mathbb{P}_0$$

2) If we define $\mathbb{P}^* = \mathbb{P}_0^* + \mathbb{P}_1^*$ and $\eta^* = d\mathbb{P}_1^*/d\mathbb{P}^*$, then

$$\eta^*(x) h_1^n(x) + (1 - \eta^*(x)) h_0^n(x) = C_\phi^*(\eta^*) \quad \mathbb{P}^*-a.e. \quad (31)$$

This lemma is proved in Appendix F. Theorem 7 will later follow from this result.

To show that Lemma 15 is non-vacuous, one must prove that there exist minimizers to $\Theta$ over $S_\phi$, which we delay to Sections 6 and 7. Notice that the application of the Fenchel-Rockafellar Theorem in Lemma 13 actually implies the existence of dual maximizers in the case of compactly supported $\mathbb{P}_0, \mathbb{P}_1$.

In fact, the complimentary slackness conditions hold approximately for any maximizer of $\bar{R}_\phi$ and any minimizing sequence of $\Theta$. This result is essential for proving the existence of minimizers to $\Theta$.

**Lemma 16** Let $(h_0^n, h_1^n)$ be a minimizing sequence for $\Theta$ over $S_\phi$. $\lim_{n \to \infty} \Theta(h_0^n, h_1^n) = \inf_{(h_0, h_1) \in S_\phi} \Theta(h_0, h_1)$. Then for any maximizer of the dual problem $(\mathbb{P}_0^*, \mathbb{P}_1^*)$, the following hold:

1) $$\lim_{n \to \infty} \int S_\epsilon(h_0^n) d\mathbb{P}_0 - \int h_0^n d\mathbb{P}_0 = 0 \quad \text{and} \quad \lim_{n \to \infty} \int S_\epsilon(h_1^n) d\mathbb{P}_1 - \int h_1^n d\mathbb{P}_1 = 0 \quad (32)$$

2) If we define $\mathbb{P}^* = \mathbb{P}_0^* + \mathbb{P}_1^*$ and $\eta^* = d\mathbb{P}_1^*/d\mathbb{P}^*$

$$\lim_{n \to \infty} \int \eta^* h_1^n + (1 - \eta^*) h_0^n - C_\phi^*(\eta^*) d\mathbb{P}^* = 0 \quad (33)$$

**Proof** Let $m = \inf_{(h_0, h_1) \in S_\phi} \Theta(h_0, h_1)$.

Then the fact that $(h_0^n, h_1^n) \in S_\phi$ and the duality result (Lemma 14) implies

$$\int h_1^n d\mathbb{P}_1 = \int h_0^n d\mathbb{P}_0 = \int \eta^* h_1^n + (1 - \eta^*) h_0^n d\mathbb{P}^* \geq \int C_\phi^*(\eta^*) d\mathbb{P}^* = m \quad (34)$$

Now pick $\delta > 0$ and an $N$ for which $n \geq N$ implies that $\Theta(h_0^n, h_1^n) \leq m + \delta$. Then

$$m + \delta \geq \int S_\epsilon(h_1^n) d\mathbb{P}_1 + \int S_\epsilon(h_0^n) d\mathbb{P}_0 \geq \int \eta^* h_1^n + (1 - \eta^*) h_0^n d\mathbb{P}^* \geq m$$
Subtracting $m = \int C_\omega^*(\eta^*)dP^*$ from this inequality results in

$$
\delta \geq \int \eta^*h_1^n + (1 - \eta^*)h_0^n dP^* - \int C_\omega^*(\eta^*)dP^* \geq 0
$$

(35)

which implies (33). Next, (34) further implies

$$
m - \int h_1^n dP_1^* + \int h_0^n dP_0^* \leq 0
$$

(36)

Now this inequality implies

$$
\delta \geq \delta + m - \left( \int h_1^n dP_1^* + \int h_0^n dP_0^* \right) \geq \Theta(h_1^n, h_0^n) - \left( \int h_1^n dP_1^* + \int h_0^n dP_0^* \right)
$$

$$
\geq \left( \int S_\epsilon(h_1^n)dP_1 + \int S_\epsilon(h_0^n)dP_0 \right) - \left( \int h_1^n dP_1^* + \int h_0^n dP_0^* \right)
$$

However, Lemma 3 implies that both $\int S_\epsilon(h_1^n)dP_1 - \int h_1^n dP_1^*$, $\int S_\epsilon(h_0^n)dP_0 - \int h_0^n dP_0^*$ are positive quantities. Therefore, we have shown that for any $\delta > 0$, there is an $N$ for which $n \geq N$ implies that

$$
\delta > \int S_\epsilon(h_1^n)dP_1 - \int h_1^n dP_1^* \geq 0 \quad \text{and} \quad \delta > \int S_\epsilon(h_0^n)dP_0 - \int h_0^n dP_0^* \geq 0
$$

which implies (32).

An analogous approximate complimentary slackness result typically holds in other applications of the Fenchel-Rockafellar theorem. Consider a convex optimization problem which can be written as $\inf_x \Theta(x) + \Xi(x)$ in such a way that the Fenchel-Rockafellar theorem applies and for which $\Xi$ and $\Theta^*$ are indicator functions of the convex sets $C_P, C_D$ respectively. Then the Fenchel-Rockafellar Theorem states that

$$
\inf_{x \in C_P} \Theta(x) = \inf_{x \in C_P} \sup_{y \in C_D} \langle y, x \rangle = \sup_{y \in C_D} \inf_{x \in C_P} \langle y, x \rangle = \sup_{y \in C_D} \Xi^*(y)
$$

(37)

Let $y^*$ be a maximizer of the dual problem and let $m$ be the minimal value of $\Theta$ over $C_P$. If $x_k$ is a minimizing sequence of $\Theta$, then for $\delta > 0$ and sufficiently large $k$, $\delta + m > \Theta(x_k)$ and thus by (37),

$$
m + \delta > \Theta(x_k) = \sup_{y \in C_P} \langle y, x_k \rangle \geq \langle y^*, x_k \rangle \geq \inf_{x \in C_D} \langle y^*, x \rangle = \inf_{x \in C_D} \Xi^*(x) = m
$$

(38)

and therefore $\lim_{k \to \infty} \langle y^*, x_k \rangle = m$. Condition (31) is this statement adapted to the adversarial learning problem. Furthermore, subtracting $\Theta(x_k)$ from (38) and taking the limit $k \to \infty$ results in $\lim_{k \to \infty} \Theta(x_k) - \langle y^*, x_k \rangle = 0$. In our adversarial learning scenario, this statement is equivalent to the conditions in (32) due to Lemma 3. Furthermore, just like the standard complimentary slackness theorems, the relations $\lim_{k \to \infty} \langle y^*, x_k \rangle = m$, $\lim_{k \to \infty} \Theta(x_k) - \langle y^*, x_k \rangle = 0$ imply that $x_k$ is a minimizing sequence for $\Theta$.

In the classical proof of the Kantorovich duality, one can choose $\Theta, \Xi$ of a form similar to the discussion above, see for instance Theorem 1.3 of Villani (2003). Using an argument similar to (38), one can prove approximate complimentary slackness for the Kantorovich problem called the quantitative Knott-Smith criteria, see Theorems 2.15, 2.16 of Villani (2003) for further discussion.
6. Existence of Minimizers of $\Theta$ over $S_\psi$

After proving the existence of maximizers to the dual problem, we can now use the approximate complementary slackness conditions to prove the existence of minimizers to the primal. The exponential loss $\psi$ has certain properties that make it particularly easy to study:

**Lemma 18** Let $\psi(\alpha) = e^{-\alpha}$. Then $C^*_\psi(\eta) = 2\sqrt{\eta(1-\eta)}$ and $\alpha_\psi(\eta) = 1/2\log(\eta/1-\eta)$ is the unique minimizer of $C_\psi(\eta,cdot)$, with $\alpha_\psi(0),\alpha_\psi(1)$ interpreted as $-\infty, +\infty$ respectively. Furthermore, $\partial C^*_\psi(\eta)$ is the singleton $\partial C^*_\psi(\eta) = \{\psi(\alpha_\psi(\eta)) - \psi(-\alpha_\psi(\eta))\}$.

See Appendix G.1 for a proof. The existence of minimizers of $\Theta$ for the exponential loss then follows from properties of $C_\psi$. Let $(h_0^n, h_1^n)$ be a minimizing sequence of $\Theta$. Because the function $C_\psi$ is strictly concave, one can use the condition (33) to show that there is a subsequence $\{\eta_k\}$ along which $h_0^{n_k}(x')$, $h_1^{n_k}(x')$ converge $P_0^\star, P_1^\star$-a.e. respectively. Due to (32), $S_\epsilon(h_0^{n_k})(x)$, $S_\epsilon(h_1^{n_k})$ also converge $P_0, P_1$-a.e. respectively along this subsequence. This observation suffices to show the existence of functions that minimize $\Theta$ over $S_\psi$.

The first step of this proof is to formalize this argument for sequences in $\mathbb{R}$.

**Lemma 18** Let $(a_n, b_n)$ be a sequence for which $a_n, b_n \geq 0$ and

$$\eta a_n + (1-\eta) b_n \geq C^*_\psi(\eta)$$

for all $\eta \in [0,1]$ (39)

and

$$\lim_{n \to \infty} \eta a_n + (1-\eta) b_n = C^*_\psi(\eta_0)$$

for some $\eta_0$. Then $\lim_{n \to \infty} a_n = \psi(\alpha_\psi(\eta_0))$ and $\lim_{n \to \infty} b_n = \psi(-\alpha_\psi(\eta_0))$.

Notice that if $\eta a + (1-\eta) b \geq C^*_\psi(\eta)$ and $\eta a + (1-\eta) b = C^*_\psi(\eta_0)$, then this lemma implies that $a = \psi(\alpha_\psi(\eta_0))$ and $b = \psi(-\alpha_\psi(\eta_0))$.

To prove Lemma 18, we show that all convergent subsequences of $\{a_n\}$ and $\{b_n\}$ must converge to $a$ and $b$ that satisfy $\eta a + (1-\eta) b = C^*_\psi(\eta_0)$ and $a - b \in \partial C^*_\psi(\eta_0)$. As the set $\partial C^*_\psi(\eta_0)$ is a singleton, the values $a = \psi(\alpha_\psi(\eta_0))$ and $b = \psi(-\alpha_\psi(\eta_0))$ uniquely solve these equations for $a$ and $b$. Therefore the sequences $\{a_n\}$ and $\{b_n\}$ must converge to $a$ and $b$ as well. See Appendix G.2 for a formal proof. This result applied to a minimizing sequence of $\Theta$ allows one to find a subsequence with certain convergence properties.

**Lemma 19** Let $(h_0^n, h_1^n)$ be a minimizing sequence of $\Theta$ over $S_\psi$. Then there exists a subsequence $n_k$ for which $S_\epsilon(h_0^{n_k})$, $S_\epsilon(h_1^{n_k})$ converge $P_0, P_1$-a.e. respectively.

**Proof** Let $P_0^\star, P_1^\star$ be maximizers of the dual problem. Let $\gamma_i$ be the coupling between $P_i, P_i^\star$ with supp $\gamma_i \subset \Delta_i$.

Then (33) of Lemma 16 implies that

$$\lim_{n \to \infty} \int \eta^+(x') h_1^n(x') + (1-\eta^+(x')) h_0^n(x') - C_\psi(\eta^+(x')) d(\gamma_1 + \gamma_0)(x, x') = 0$$

and (32) implies that

$$\lim_{n \to \infty} \int S_\epsilon(h_0^n)(x) - h_0^n(x') d\gamma_1(x, x') = 0, \quad \lim_{n \to \infty} \int S_\epsilon(h_0^n)(x) - h_0^n(x') d\gamma_0(x, x') = 0$$
Recall that on a bounded measure space, $L^1$ convergence implies a.e. convergence along a subsequence (see Corollary 2.32 of (Folland, 1999)). Thus one can pick a subsequence $n_k$ along which

$$\lim_{k \to \infty} \eta^*(x')h_1^{n_k}(x') + (1 - \eta^*(x'))h_0^{n_k}(x') - C_\psi(\eta^*(x')) = 0 \quad (41)$$

$\gamma_1 + \gamma_0$-a.e. and

$$\lim_{k \to \infty} S_\epsilon(h_1^{n_k})(x) - h_1^{n_k}(x') = 0, \quad \lim_{k \to \infty} S_\epsilon(h_0^{n_k})(x) - h_0^{n_k}(x') = 0 \quad (42)$$

$\gamma_1, \gamma_0$-a.e. respectively.

Furthermore, $\eta h_1^{n_k} + (1 - \eta)h_0^{n_k} \geq C_\psi^*(\eta)$ for all $\eta \in [0, 1]$. Thus (41) and Lemma 18 imply that $h_1^{n_k}$ converges to $\psi(\alpha_\psi(\eta^*))$ and $h_0^{n_k}$ converges to $\psi(-\alpha_\psi(\eta^*))$ $\gamma_0 + \gamma_1$-a.e. Equation 42 then implies that $S_\epsilon(h_1^{n_k})(x), S_\epsilon(h_0^{n_k})(x)$ converge $\gamma_1, \gamma_0$-a.e. respectively. Because $P_1, P_0$ are marginals of $\gamma_1, \gamma_0$, this statement implies the result.

The existence of a minimizer then follows from the fact that $S_\epsilon(h_1^{n_k})$ converges. The next lemma describes how the $S_\epsilon$ operation interacts with lim infs and lim supers.

**Lemma 20** Let $h_n$ be any sequence of functions. Then the sequence $h_n$ satisfies

$$\liminf_{n \to \infty} S_\epsilon(h_n) \geq S_\epsilon(\liminf_{n \to \infty} h_n) \quad (43)$$

and

$$\limsup_{n \to \infty} S_\epsilon(h_n) \geq S_\epsilon(\limsup_{n \to \infty} h_n) \quad (44)$$

See Appendix G.3 for a proof.

Finally, we prove that there exists a minimizer to $\Theta$ over $S_\psi$.

**Lemma 21** There exists a minimizer $(h_0^*, h_1^*)$ to $\Theta$ over the set $S_\psi$.

**Proof** Let $(h_0^n, h_1^n)$ be a sequence minimizing $\Theta$ over $S_\psi$. Lemma 19 implies that there is a subsequence $\{n_k\}$ for which $\lim_{k \to \infty} S_\epsilon(h_0^{n_k})$ exists $P_0$-a.e.

Thus

$$\limsup_{k \to \infty} S_\epsilon(h_0^{n_k}) = \liminf_{k \to \infty} S_\epsilon(h_0^{n_k}) \quad P_0\text{-a.e.} \quad (45)$$

Next, we will argue that the pair $(\limsup_k h_0^{n_k}, \liminf_k h_1^{n_k})$ is in $S_\psi$. Because

$$C_\psi^*(\eta) \leq \eta h_1^{n_k} + (1 - \eta)h_0^{n_k},$$

one can conclude that

$$C_\psi(\eta) \leq \eta \liminf_{k \to \infty} (h_1^{n_k} + (1 - \eta)h_0^{n_k}) \leq \eta \liminf_{k \to \infty} h_1^{n_k} + (1 - \eta)\limsup_{k \to \infty} h_0^{n_k},$$

Define

$$h_1^* = \liminf_k h_1^{n_k}, \quad h_0^* = \limsup_k h_0^{n_k}$$
7. Reducing $\Theta$ to $R^*_\phi$

Using the properties of $C^*_\psi(\eta)$, we showed in the previous section that there exist minimizers to $\Theta$ over the set $S_\psi$. The inequality $\eta h^*_1 + (1 - \eta^*)h^*_0 \geq C^*_\phi(\eta)$ together with (31) imply that $h^*_1(x) - h^*_0(x)$ is a supergradient of $C^*_\psi(\eta^*(x))$ and thus $h^*_1 - h^*_0 = (C^*_\psi)^\prime(\eta)$. This relation together with (31) provides two equations in two variables that can be uniquely solved for $h^*_0, h^*_1$, resulting in $h^*_0 = \psi \circ -\alpha_\phi(\eta^*), h^*_1 = \psi \circ \alpha_\phi(\eta^*)$.

Next, primal minimizers of $\Theta$ over $S_\phi$ for any $\phi$ will be constructed from the dual maximizers $P^*_0, P^*_1$ of $R_\phi$. Because $\alpha_\psi(\eta) = 1/2\log(\eta/1 - \eta)$ is a strictly increasing function, the compositions $\psi \circ \alpha_\phi, \psi \circ -\alpha_\phi$ are strictly monotonic. Thus the complimentary slackness condition (30) applied to $h^*_1 = \psi(\alpha_\phi(\eta^*)), h^*_0 = \psi(-\alpha_\phi(\eta^*))$ implies that $\sup P^*_1$ is contained in the set of points $x'$ for which $\eta^*$ assumes its supremum over some $\epsilon$-ball at $x'$ and $\sup P^*_0$ is contained in the set of points $x'$ where $\eta^*$ assumes its supremum over some $\epsilon$-ball at $x'$. Thus, the functions $\phi \circ \alpha_\phi(\eta^*), \phi \circ -\alpha_\phi(\eta^*)$ satisfy (30) for the loss $\phi$. The definition of $\alpha_\phi$ further implies they satisfy (31). Therefore, Lemma 15 implies that for any $\phi, h^*_1 = \phi \circ \alpha_\phi(\eta^*), h^*_0 = \phi \circ -\alpha_\phi(\eta^*)$ are primal optimal and $P^*_0, P^*_1$ are dual optimal!

This reasoning about $\eta^*$ is technically wrong but correct in spirit. Because $\eta^*$ is a Raydon-Nikodym derivative, it is only defined $P^*$-a.e. As a result, the supremum over an $\epsilon$-ball of the function $\phi(\alpha_\phi(\eta^*))$ is not well-defined. The solution is to replace $\eta^*$ in the discussion above by a function $\tilde{\eta}$ that is defined everywhere. The function $\tilde{\eta}$ is actually a version of the Raydon-Nikodym derivative $dP^*_1/dP^*$. The next two lemmas describe how one constructs this function $\tilde{\eta}$.

The next two lemmas discuss the analog of the $c^*$ transform for the Kantorovich problem in optimal transport (see for instance Chapter 1 of (Santambrogio, 2015) or Section 2.5 of (Villani, 2003)).

**Lemma 22** Assume that $h_0, h_1 \geq 0$ and $(h_0(x), h_1(x))$ satisfy $\eta h_1 + (1 - \eta)h_0 \geq C^*_\phi(\eta)$ for all $\eta$. Then if we define $h^*_{0c^*}$ via

$$h^*_{0c^*} = \sup_{\eta \in [0, 1]} \frac{C^*_\phi(\eta) - \eta h_1}{1 - \eta}$$

Now Fatou’s lemma, Lemma 20, and Equation 45 imply that

$$\lim_{k \to \infty} \Theta(h^*_0, h^*_1) \geq \int \liminf_{k \to \infty} S_\epsilon(h^*_0) dP^*_1 + \int \liminf_{k \to \infty} S_\epsilon(h^*_1) dP^*_0 \quad \text{(Fatou’s Lemma)}$$

$$= \int \liminf_{k \to \infty} S_\epsilon(h^*_1) dP^*_1 + \int \limsup_{k \to \infty} S_\epsilon(h^*_0) dP^*_0 \quad \text{(Equation 45)}$$

$$\geq \int S_\epsilon(\liminf_{k \to \infty} h^*_1) dP^*_1 + \int S_\epsilon(\limsup_{k \to \infty} h^*_0) dP^*_0 \quad \text{(Lemma 20)}$$

$$= \int S_\epsilon(h^*_1) dP^*_1 + \int S_\epsilon(h^*_0) dP^*_0$$

Therefore, $(h^*_0, h^*_1)$ must be a minimizer.
then \( h^C_0 \leq h_0 \) while and \( h_1 + (1 - \eta)h_0^C \geq C^*_\phi(\eta) \) for all \( \eta \), and \( h^C_0 \) is the smallest function \( h_0 \) for which \((h_0, h_1) \in S_\phi \). Furthermore, the function \( h^C_0 \) is Borel and there exists a function \( \tilde{h} \) on \( \mathbb{R}^d \rightarrow [0, 1] \) for which \( \tilde{h}(x)h_1(x) + (1 - \tilde{h}(x))h_0^C(x) = C^*_\phi(\tilde{h}(x)) \).

**Proof** For convenience, set \( \tilde{h}_0 = h^C_1 \). Notice that \( \tilde{h}_0 \geq 0 \) because the right-hand side of (46) evaluates to 0 at \( \eta = 0 \). We will show that \( \tilde{h}_0 \) is Borel and that \((\tilde{h}_0, h_1) \) is a feasible pair.

Next, Notice that the map

\[
G(\eta, \alpha) = \begin{cases} 
\frac{C^*_\phi(\eta) - \eta \alpha}{1 - \eta} & \text{if } \eta < 1 \\
\lim_{\eta \to 1} \frac{C^*_\phi(\eta) - \eta \alpha}{1 - \eta} & \text{if } \eta = 1
\end{cases}
\]  \hspace{1cm} (47)

is continuous in \( \eta \). Thus, the supremum in (46) can be taken over the countable set \( \mathbb{Q} \cap [0, 1] \) and hence the function \( \tilde{h}_0(x) = \sup_{\eta \in (0, 1) \cap \mathbb{Q}} G(\eta, h_1(x)) \) is Borel measurable. Because \( G(\eta, h_1(x)) \) is continuous in \( \eta \) for each fixed \( x \), \( G(\cdot, h_1(x)) \) assumes its maximum on \( \eta \in [0, 1] \) for each fixed \( x \). Thus there exists a function \( \tilde{h}(x) \) that maps \( x \) to a maximizer of \( G(\cdot, h_1(x)) \).

For this function \( \tilde{h}(x) \), one can conclude that \( \tilde{h}_0(x) = G(\tilde{h}(x), x) \) and hence

\[
\tilde{h}(x)h_1(x) + (1 - \tilde{h}(x))\tilde{h}_0(x) = C^*_\phi(\tilde{h}(x)).
\]  \hspace{1cm} (48)

Equation 48 implies that if \( f(x) < \tilde{h}_0(x) \) at any \( x \), then \( \eta h_1(x) + (1 - \eta) f(x) < C^*_\phi(\eta(x)) \) so \((f, h_1) \) is not in the feasible set \( S_\phi \). Therefore, \( \tilde{h}_0 \) is the smallest function \( f \) for which \((f, h_1) \in S_\phi \). \( \blacksquare \)

Next we use this result to define an extension of \( \eta^* \) to all of \( \mathbb{R}^d \).

**Lemma 23** There exist a Borel minimizer \((h_0^*, h_1^*) \) to \( \Theta \) over \( S_\psi \) for which

\[
\tilde{h}(x)h_1^*(x) + (1 - \tilde{h}(x))h_0^*(x) = C^*_\psi(\tilde{h}(x))
\]  \hspace{1cm} (49)

for all \( x \) and some Borel measurable function \( \tilde{h} : (\text{supp } \mathbb{P})^c \rightarrow [0, 1] \).

**Proof** Let \((h_0, h_1) \), be an arbitrary Borel minimizer to the primal (Lemma 21 implies that such a minimizer exists). Set \( h_1^* = h_1 \) and \( h_0^* = h_1^C_0 \). Then Lemma 22 implies that \( h_0^* \leq h_0 \), so \((h_0^*, h_1^*) \) is also optimal and \( \eta h_1^* + (1 - \eta) h_0^* \geq C^*_\psi(\eta) \) for all \( \eta \). Furthermore, Lemma 22 implies that there is a function \( \tilde{h} \) for which \( \tilde{h}(x)h_1^*(x) + (1 - \tilde{h}(x))h_0^*(x) = C^*_\psi(\tilde{h}(x)) \).

It remains to show that \( \tilde{h} \) is Borel measurable. We will express \( \tilde{h}(x) \) in terms of \( h_1^*(x) \), and because \( h_1^*(x) \) is Borel measurable, it will follow that \( \tilde{h} \) is Borel measurable as well. Because \( \eta h_1^*(x) + (1 - \eta) h_0^*(x) \geq C^*_\psi(\eta) \) with equality at \( \eta = \tilde{h}(x) \), it follows that \( h_1^*(x) - h_0^*(x) \) is a supergradient of \( C^*_\psi \) at \( \eta = \tilde{h}(x) \). Thus Lemma 17 implies that \( h_1^* - h_0^* = (1 - 2\tilde{h})/\sqrt{\tilde{h}(1 - \tilde{h})} \iff h_1^* = h_0^* + (1 - 2\tilde{h})/\sqrt{\tilde{h}(1 - \tilde{h})} \). Plugging this expression and the formula \( C^*_\psi(\eta) = 2\sqrt{\tilde{h}(1 - \eta)} \) into the relation \( \eta h_1^* + (1 - \eta) h_0^* = C^*_\psi(\eta) \) results in the equation \( h_0^* + \tilde{h}(1 - 2\tilde{h})/\sqrt{\tilde{h}(1 - \tilde{h})} = 2\sqrt{\tilde{h}(1 - \tilde{h})} \). Solving for \( \tilde{h} \) then results in \( \tilde{h} = (h_0^*)^2/(1 + (h_0^*)^2) \). Because \( h_0^* \) is Borel measurable, \( \tilde{h} \) is measurable as well. \( \blacksquare \)
Notice that this result together with Lemma 18 immediately implies that \( h^*_i = \psi(\alpha_\psi(\hat{\eta})) \) and \( h^*_0 = \psi(-\alpha_\psi(\hat{\eta})) \), immediately proving that minimizing \( \Theta \) over \( S_\psi \) is equivalent to minimizing \( R_\psi \). Next, this observation is extended to arbitrary losses using properties of \( \hat{\eta} \).

Because both \( \psi \) and \( \alpha_\psi \) are strictly monotonic, \( \hat{\eta} \) interacts in a particularly nice way with maximizers of the dual problem:

**Lemma 24** Let \( \mathbb{P}^*_0, \mathbb{P}^*_1 \) be any maximizer of \( R_\psi \) over \( \mathcal{B}^∞_c(\mathbb{P}_0) \times \mathcal{B}^∞_c(\mathbb{P}_1) \). Set \( \mathbb{P}^* = \mathbb{P}^*_0 + \mathbb{P}^*_1 \), \( \eta^* = d\mathbb{P}^*_1/d\mathbb{P}^* \). Let \( \hat{\eta} \) be defined as in Lemma 23. Then \( \hat{\eta} = \eta^* \mathbb{P}^*-a.e. \)

Furthermore, let \( \gamma_i \) be a coupling between \( \mathbb{P}^*_i, \mathbb{P}^*_i \) with \( \supp \gamma_i \subset \Delta_\epsilon \). Then

\[
\supp \gamma_1 \subset \{ (x, x') : \ \inf_{\|y-x\| \leq \epsilon} \hat{\eta}(y) = \hat{\eta}(x') \} \quad (50)
\]

\[
\supp \gamma_0 \subset \{ (x, x') : \ \sup_{\|y-x\| \leq \epsilon} \hat{\eta}(y) = \hat{\eta}(x') \} \quad (51)
\]

The statement \( \hat{\eta} = \eta^* \mathbb{P}^*-a.e. \) implies that \( \hat{\eta} \) is in fact a version of the Raydon-Nikodym derivative \( d\mathbb{P}^*_1/d\mathbb{P}^* \).

For convenience, in this proof, we introduce the notation

\[
I_\epsilon(f)(x) = \inf_{\|y-x\| \leq \epsilon} f(y).
\]

**Proof** Let \( h^*_i, h^*_0 \) be the minimizer described by Lemma 23. Then Lemma 18 implies that \( h^*_i = \psi(\alpha_\psi(\hat{\eta})) \) and \( h^*_0 = \psi(-\alpha_\psi(\hat{\eta})) \).

The complimentary slackness condition (31) implies that \( \eta^* h^*_i + (1-\eta^*) h^*_0 = C^*_\psi(\eta^*) \mathbb{P}^*-a.e. \), and thus Lemma 18 implies that \( h^*_i = \psi(\alpha_\psi(\eta^*)) \) and \( h^*_0 = \psi(-\alpha_\psi(\eta^*)) \mathbb{P}^*-a.e. \). Therefore, \( \psi(\alpha_\psi(\eta^*)) = \psi(\alpha_\psi(\hat{\eta})) \mathbb{P}^*-a.e. \) Now because the functions \( \psi, \alpha_\psi \) are strictly monotonic, they are invertible. Thus it follows that \( \hat{\eta} = \eta^* \mathbb{P}^*-a.e. \)

The complimentary slackness condition (30) states that

\[
\int S_\epsilon(h_i)(x) - h^*_i(x')d\gamma_i = 0.
\]

Therefore,

\[
S_\epsilon(\psi(\alpha_\psi(\hat{\eta}))(x)) = \psi(\alpha_\psi(\hat{\eta}(x'))) \gamma_1-a.e. \quad \text{and} \quad S_\epsilon(\psi(-\alpha_\psi(\hat{\eta}))(x)) = \psi(-\alpha_\psi(\hat{\eta}(x'))) \gamma_0-a.e.
\]

which implies

\[
\psi(\alpha_\psi(I_\epsilon(\hat{\eta})(x))) = \psi(\alpha_\psi(\hat{\eta}(x')) \gamma_1-a.e. \quad \text{and} \quad \psi(-\alpha_\psi(S_\epsilon(\hat{\eta})(x))) = \psi(-\alpha_\psi(\hat{\eta}(x'))) \gamma_0-a.e.
\]

Now \( \psi, \alpha_\psi \) are both strictly monotonic and thus invertible. Therefore

\[
I_\epsilon(\hat{\eta})(x) = \hat{\eta}(x') \gamma_1-a.e. \quad \text{and} \quad S_\epsilon(\hat{\eta})(x) = \hat{\eta}(x') \gamma_0-a.e.
\]

Next, the relation (49) suggests that \( h^*_i = \phi \circ f^*, \ h^*_0 = \phi \circ -f^* \), where \( f^* \) is a function satisfying \( C^*_\psi(\hat{\eta}(x), f^*(x)) = C^*_\psi(\hat{\eta}(x)) \). In fact, Lemma 24 implies that this relation holds for all loss functions, and not just the exponential loss \( \psi \). To formalize this idea, we prove the following result about minimizers of \( C^*_\psi(\eta, \cdot) \) in Appendix C:
Lemma 25 Fix a loss function $\phi$ and let $\alpha_{\phi}(\eta)$ be as in (8). Then $\alpha_{\phi}$ maps $\eta$ to the smallest minimizer of $C_{\phi}(\eta, \cdot)$. Furthermore, the function $\alpha_{\phi}(\eta)$ non-decreasing in $\eta$.

Finally, we use the complimentary slackness conditions of Lemma 15 to construct a minimizer $(h_0^*, h_1^*)$ to $\Theta$ over $S_\phi$ for which $h_0^* = \phi \circ f^*$, $h_0^* = \phi \circ -f^*$ for some function $f^*$.

Lemma 26 Let $\psi = e^{-\alpha}$ be the exponential loss and let $\phi$ be any arbitrary loss. Let $\mathbb{P}^*_0, \mathbb{P}^*_1$ be any maximizer of $\bar{R}_\phi$ over $B_\epsilon^\infty(\mathbb{P}_0) \times B_\epsilon^\infty(\mathbb{P}_1)$. Define $\mathbb{P}^* = \mathbb{P}_0^* + \mathbb{P}_1^*$ and $\eta^* = d\mathbb{P}^*_1/d\mathbb{P}^*$. Let $\tilde{\eta}$ be defined as in Lemma 23.

Then $h_0^* = \phi(-\alpha_{\phi}(\tilde{\eta})), h_1^* = \phi(\alpha_{\phi}(\tilde{\eta}))$ minimize $\Theta$ over $S_\phi$ and $(\mathbb{P}^*_0, \mathbb{P}^*_1)$ maximize $\bar{R}_\phi$ over $B_\epsilon^\infty(\mathbb{P}_0) \times B_\epsilon^\infty(\mathbb{P}_1)$.

Thus there exists a Borel minimizer to $R_\phi^*$ and $\inf_{(h_0, h_1) \in S_\phi} \Theta(h_0, h_1) = \Theta(h_0^*, h_1^*)$.

Proof We will verify the complimentary slackness conditions of Lemma 15.

Lemma 24 implies that $\tilde{\eta} = \eta^* \mathbb{P}^*-a.e.$ Therefore, $\mathbb{P}^*-a.e.,$

$$C_{\phi}^*(\eta^*) = C_{\phi}^*(\tilde{\eta}) = \tilde{\eta} h_1 + (1 - \tilde{\eta}) h_0 = \eta^* h_1 + (1 - \eta^*) h_0$$

This calculation verifies the complimentary slackness condition (31).

We next verify the other complimentary slackness condition (30). Let $\gamma_i$ be a coupling between $\mathbb{P}_0^*, \mathbb{P}_1^*$ with $\text{supp} \gamma_i \subseteq \Delta_\epsilon$. Next, because $\phi \circ \alpha_{\phi}, \phi \circ -\alpha_{\phi}$ are monotonic, the conditions (50) and (51) imply that

$$\int \phi(\alpha_{\phi}(\tilde{\eta})) d\mathbb{P}_1^* = \int \phi(\alpha_{\phi}(\tilde{\eta}(x'))) d\gamma_1(x, x') = \int S_\epsilon(\phi(\alpha_{\phi}(\tilde{\eta}))(x)) d\gamma_1(x, x') = \int S_\epsilon(\phi(\alpha_{\phi}(\tilde{\eta}))) d\mathbb{P}_1$$

$$\int \phi(-\alpha_{\phi}(\tilde{\eta})) d\mathbb{P}_0^* = \int \phi(-\alpha_{\phi}(\tilde{\eta}(x'))) d\gamma_0(x, x') = \int S_\epsilon(\phi(-\alpha_{\phi}(\tilde{\eta}))(x)) d\gamma_0(x, x') = \int S_\epsilon(\phi(-\alpha_{\phi}(\tilde{\eta}))) d\mathbb{P}_0$$

This calculation verifies the complimentary slackness condition (30).

Theorems 6 and 9 immediately follow from Lemmas 14 and 26.

8. Conclusion

We initiated the study of minimizers and minimax relations for adversarial surrogate risks. Specifically, we proved that there always exists a minimizer to the adversarial surrogate risk when perturbing in a closed $\epsilon$-ball and a maximizer to the dual problem. Just like the results of (Pydi and Jog, 2021), our minimax theorem provides an interpretation of the adversarial learning problem as a game between two players. This theory helps explain the phenomenon of transfer attacks. We hope the insights gained from this research will assist in the development of algorithms for training classifiers robust to adversarial perturbations.

Acknowledgments

Natalie Frank was supported in part by the Research Training Group in Modeling and Simulation funded by the National Science Foundation via grant RTG/DMS – 1646339. Jonathan Niles-Weed was supported in part by a Sloan Research Fellowship.
Appendix A. The Universal σ-Algebra and a Generalization of Theorem 1

A.1 Definition of the Universal σ-Algebra and Main Measurability Result

In this Appendix, we prove results for supremums over an arbitrary compact set, not necessarily a unit ball. For a function \( g: \mathbb{R}^d \to \mathbb{R}^d \), we will abuse notation and denote the supremum of \( g \) over the compact set \( B \) by

\[
S_B(g)(x) = \sup_{h \in B} g(x + h).
\]

Let \( X \) be a separable metric space and let \( \mathcal{B}(X) \) be the Borel σ-algebra on \( X \). Denote the completion of \( \mathcal{B}(X) \) with respect to a Borel measure \( \nu \) by \( \mathcal{L}_\nu(X) \). Let \( \mathcal{M}_+(X) \) be the set of all finite positive Borel measures on \( X \). Then the universal σ-algebra on \( X \), \( U(X) \), is

\[
U(X) = \bigcap_{\nu \in \mathcal{M}_+(X)} \mathcal{L}_\nu(X).
\] (52)

In other words, the universal σ-algebra is the sigma-algebra of sets which are measurable with respect to the completion of every Borel measure. Thus \( U(X) \) is contained in \( \mathcal{L}_\nu(X) \) for every Borel measure \( \nu \). The goal of this appendix is to prove

**Theorem 27** If \( f \) is universally measurable and \( B \) is a compact set, then \( S_B(f) \) is universally measurable.

Thus, if \( \mathbb{P}_0, \mathbb{P}_1, \) and \( g \) are Borel, integrals of the form \( \int S_\epsilon(g)d\mathbb{P}_i \) in (10) can be interpreted as the integral of \( S_\epsilon(g) \) with respect to the completion of \( \mathbb{P}_i \).

A.2 Proof Outline

To prove Theorem 27, we analyze the level sets of \( S_B(g) \). One can compute the level set \([S_B(g)(x) > a]\) using a direct sum.

**Lemma 28** Let \( g: \mathbb{R}^d \to \mathbb{R}^d \) be any function. For a set \( B \), define \( -B = \{-b: b \in B\} \). Then

\[
[S_B(g) > a] = [g > a] \oplus -B
\]

**Proof** To start, notice that \( S_B(g)(x) > a \) iff there is some \( h \in B \) for which \( g(x + h) > a \). Thus

\[
x \in [S_B(g) > a] \iff x + h \in [g > a] \text{ for some } h \in B \iff x \in [g > a] \oplus -B
\]

Therefore, to show that \( S_B(g) \) is measurable for measurable \( g \), it suffices to show that the direct sum of a measurable set and the compact set \( B \) is measurable. Thus, to prove Theorem 27, it suffices to demonstrate the following result:

**Theorem 29** Let \( A \in \mathcal{U}(\mathbb{R}^d) \) and let \( B \) be a compact set. Then \( A \oplus B \in \mathcal{U}(\mathbb{R}^d) \).

---

2. Alternatively, one could compute the intersection in (52) over all σ-finite measures. These two approaches are equivalent because for every σ-finite measure \( \lambda \) and compact set \( K \), the restriction \( \lambda|_K \) is a finite measure with \( \lambda|_K(X) \supseteq \mathcal{L}_\lambda(X) \).
The proof of Theorem 29 follows from fundamental concepts of measure theory. A classical measure theory result states that if \( f : X \to Y \) is a continuous function, \( f^{-1} \) maps Borel sets in \( Y \) to Borel sets in \( X \). Consider now the function \( w : B \times \mathbb{R}^d \to B \times \mathbb{R}^d \) given by \( w(h, x) = (h, x + h) \). Then \( w \) is invertible and the inverse of \( w \) is \( w^{-1}(h, x + h) = (h, x - h) \). Furthermore, \( w^{-1} \) maps the set \( B \times A \) to \( B \times A \oplus B \). Therefore, if \( A \in \mathcal{B}(\mathbb{R}^d) \), then \( B \times A \oplus B \) is Borel in \( \mathcal{B}(B \times \mathbb{R}^d) \). However, from this statement, one cannot conclude that \( A \oplus B \) is Borel in \( \mathbb{R}^d \)!

On the otherhand, one can use regularity of measures to conclude that \( A \oplus B \) is in \( \mathcal{U}(\mathbb{R}^d) \). Therefore, to prove Theorem 29, we prove the following two results:

**Lemma 30** Let \( B \subset \mathbb{R}^d \) be a compact set. Then \( B \times A \in \mathcal{U}(B \times \mathbb{R}^d) \) iff \( A \in \mathcal{U}(\mathbb{R}^d) \).

In this document, we say a function \( f : X \to Y \) is universally measurable if \( f^{-1}(E) \in \mathcal{U}(X) \) whenever \( E \in \mathcal{U}(Y) \).

**Lemma 31** Let \( f : X \to Y \) be a Borel measurable function. Then \( f \) is universally measurable as well.

This result is stated on page 171 of (Bertsekas and Shreve, 1996), but we include a proof below for completeness.

Lemma 31 applied to \( w \) implies that the set \( B \times A \oplus B \) is universally measurable while Lemma 30 implies that \( A \oplus B \) is universally measurable.

**A.3 Proof of Theorem 29**

We begin by proving Lemma 31.

**Proof** [Proof of Lemma 31] Let \( A \) be a Borel set in \( Y \). We will show that for any finite measure \( \nu \) on \( X \), \( f^{-1}(A) \in \mathcal{L}_\nu(X) \). As \( \nu \) is arbitrary, this statement will imply that \( f^{-1}(A) \in \mathcal{U}(X) \).

Consider the pushforward measure \( \mu = f\# \nu \). This measure is a finite measure on \( Y \), so by the definition of \( \mathcal{U}(Y) \), \( A \in \mathcal{L}_\mu(Y) \). Therefore, there are Borel sets \( B_1 \subset A \subset B_2 \) in \( Y \) for which \( \mu(B_1) = \mu(B_2) \). Thus, \( f^{-1}(B_1), f^{-1}(B_2) \) are Borel sets in \( X \) for which \( f^{-1}(B_1) \subset f^{-1}(A) \subset f^{-1}(B_2) \) and \( \nu(f^{-1}(B_1)) = \nu(f^{-1}(B_2)) \). Therefore, \( f^{-1}(A) \in \mathcal{L}_\nu(X) \).

On the other hand, the proof of Lemma 30 relies on the definition of a regular space \( X \):

**Definition 32** A measure \( \nu \) is inner regular if for every Borel set \( A \),

\[
\nu(A) = \sup_{K \text{ compact}} \nu(K).
\]

The topological space \( X \) is regular if every finite Borel measure on \( X \) is inner regular.

The following result implies that most topological spaces encountered in applications are regular.

**Theorem 33** A \( \sigma \)-compact locally compact Hausdorff space is regular.
This theorem is a consequence of Theorem 7.8 of (Folland, 1999).

The notion of regularity extends to complete measures.

**Lemma 34** Let \( \overline{\nu} \) be the completion of a measure \( \nu \) on a regular space \( X \). Then for any \( A \in \mathcal{L}_\nu(X) \),

\[
\nu(A) = \sup_{K \text{ compact} \subseteq A} \nu(K).
\]

The proof of this result is left as an exercise to the reader.

Now using the concept of regularity, we prove Lemma 30.

**Proof** [Proof of Lemma 30] We first prove the forward direction. Consider the projection function \( \Pi_2 : B \times \mathbb{R}^d \to \mathbb{R}^d \) given by \( \Pi_2(x,y) = y \). Then \( \Pi_2 \) is a continuous function and \( \Pi_2^{-1}(A) = B \times A \). Therefore Lemma 31 implies that if \( A \) is universally measurable in \( \mathbb{R}^d \), then \( B \times A \) is universally measurable in \( B \times \mathbb{R}^d \).

To prove the other direction, assume that \( B \times A \) is universally measurable in \( B \times \mathbb{R}^d \). Let \( \nu \) be any finite Borel measure on \( \mathbb{R}^d \). We will find Borel sets \( B_1, B_2 \) with \( B_1 \subset A \subset B_2 \) for which \( \nu(B_1) = \nu(B_2) \), and thus \( A \in \mathcal{L}_\nu(\mathbb{R}^d) \). As \( \nu \) was arbitrary, it follows that \( A \) is universally measurable.

Theorem 33 implies that \( B \times \mathbb{R}^d \) is a regular space. Fix a Borel probability measure \( \lambda \) on \( B \). Then \( \lambda \times \nu \) is a finite Borel measure on \( B \times \mathbb{R}^d \), so it is inner regular. Let \( \lambda \times \nu \) be the completion of \( \lambda \times \nu \). Then by Lemma 34,

\[
\lambda \times \nu(B \times A) = \sup_{K \text{ compact} \subseteq B \times A} \lambda \times \nu(K).
\]

We will now argue that

\[
\sup_{K \text{ compact} \subseteq B \times A} \lambda \times \nu(K) = \sup_{K \text{ compact} \subseteq A} \nu(K).
\]

Let \( K \subset B \times A \) and let \( \Pi_2 \) be projection onto the second coordinate. Because the continuous image of a compact set is compact, \( K = \Pi_2(K) \) is compact and contained in \( A \). Thus \( B \times A \supset B \times K' \supset K \), which implies (53). Now (53) applied to \( A^c \) implies that

\[
\lambda \times \nu(X \times A) = \inf_{U \supset B \times A} \lambda \times \nu(U) = \inf_{U \supset A} \nu(U).
\]

Thus

\[
\sup_{K \text{ compact} \subseteq A} \nu(K) = \inf_{U \supset A} \nu(U) := m.
\]

Let \( K_n \) be a sequence of compact sets contained in \( A \) for which \( \lim_{n \to \infty} \nu(K_n) = m \) and \( U_n \) a sequence of sets containing \( A \) for which \( U_n^c \) is compact and \( \lim_{n \to \infty} \nu(U_n) = m \). Because a finite union of compact sets is compact, one can choose such sequences that satisfy \( K_{n+1} \supset K_n \) and \( U_{n+1} \subset U_n \). Then \( B_1 = \bigcup K_n \), \( B_2 = \bigcap U_n \) are Borel sets that satisfy \( B_1 \subset A \subset B_2 \) and \( \nu(B_1) = \nu(B_2) \) so \( A \in \mathcal{L}_\nu(\mathbb{R}^d) \).
Lastly, we formally prove Theorem 29.

**Proof** Consider the function \( w: B \times \mathbb{R}^d \to B \times \mathbb{R}^d \) given by \( w(h, x) = (h, x - h) \). Then \( w \) is continuous, invertible, and \( w^{-1}(h, x) = (x, x + h) \).

Now let \( A \in \mathcal{B}(\mathbb{R}^d) \). Then Lemma 30 implies that \( B \times \mathbb{R}^d \) is universally measurable in \( B \times A \). Lemma 31 then implies that \( w^{-1}(B \times A) = B \times A \oplus B \) is universally measurable as well. Finally, Lemma 30 implies that \( A \oplus B \in \mathcal{B}(\mathbb{R}^d) \) as well.

### Appendix B. Alternative Characterizations of the \( W_\infty \) Metric

We start with proving Lemma 3 using a measurable selection theorem.

**Theorem 35** Let \( X, Y \) be Borel sets and assume that \( D \subset X \times Y \) is also Borel. Let \( D_x \) denote

\[
D_x = \{ y: (x, y) \in D \}
\]

and

\[
\text{Proj}_X(D) = \{ x: (x, y) \in D \}
\]

Let \( f: D \to \mathbb{R} \) be a Borel function mapping \( D \) to \( \mathbb{R} \) and define

\[
f^*(x) = \inf_{y \in D_x} f(x, y)
\]

Assume that \( f^*(x) > -\infty \) for all \( x \). Then for any \( \delta > 0 \), there is a universally measurable \( \varphi: \text{Proj}_X(D) \to Y \) for which

\[
f(x, \varphi(x)) \leq f(x) + \delta
\]

This statement is a consequence of Proposition 7.50 from (Bertsekas and Shreve, 1996).

We use the following results about universally measurable functions, see Lemma 7.27 of (Bertsekas and Shreve, 1996).

**Lemma 36** Let \( g: \mathbb{R}^d \to \mathbb{R} \) be a universally measurable function and let \( Q \) be a Borel measure. Then there is a Borel measurable function \( \varphi \) for which \( \varphi = g \) \( Q \)-a.e.

This result can be extended to \( \mathbb{R}^d \)-valued functions:

**Lemma 37** Let \( g: \mathbb{R}^d \to \mathbb{R}^d \) be a universally measurable function and let \( Q \) be a Borel measure. Then there is a Borel measurable function \( \varphi \) for which \( \varphi = g \) \( Q \)-a.e.

**Proof** Let \( e_i \) denote the \( i \)th basis vector. Then \( g_i := e_i \cdot g \) is a universally measurable function from \( \mathbb{R}^d \) to \( \mathbb{R} \), so by Lemma 36, there is a Borel function \( \varphi_i \) for which \( \varphi_i = g_i \) \( Q \)-a.e. Then if we define \( \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_d) \), this function is equal to \( g \) \( Q \)-a.e.

Finally, we prove Lemma 3. Due to Lemmas 36 and 37, this lemma heavily relies on the fact that the domain of our functions is \( \mathbb{R}^d \) rather than an arbitrary metric space.
Lemma 3 Let $Q$ be a finite positive Borel measure and let $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ be a Borel measurable function. Then
\[
\int S_\epsilon(f)dQ = \sup_{Q' \in B_\infty^\times(Q)} \int fdQ'.
\]

Recall that this paper defines the left-left hand side of (13) as the integral of $S_\epsilon(f)$ with respect to the completion of $Q$.

Proof To start, let $Q'$ be a Borel measure satisfying $W_\infty(Q', Q) \leq \epsilon$. Let $\gamma$ be a coupling with marginals $Q$ and $Q'$ supported on $\Delta_\epsilon$. Then
\[
\int fdQ' = \int f(x')d\gamma(x, x') = \int f(x')1_{|x' - x| \leq \epsilon}d\gamma(x, x') 
\leq \int S_\epsilon(f)(x)1_{|x' - x| \leq \epsilon}d\gamma(x, x') = \int S_\epsilon(f)(x)d\gamma(x, x') = \int S_\epsilon(f)dQ
\]
Therefore, we can conclude that
\[
\sup_{Q' \in B_\infty^\times(Q)} \int fdQ' \leq \int S_\epsilon(f)dQ.
\]
We will show the opposite inequality by applying the measurable selection theorem. Theorem 35 implies for each $\delta > 0$, one can find a universally measurable function $\varphi : \mathbb{R}^d \to \overline{B_\epsilon(x)}$ for which $f(\varphi(x)) + \delta \geq S_\epsilon(f)(x)$. By Lemma 37, one can find a Borel measurable function $T$ for which $T = \varphi$ $Q$-a.e.

Let $Q' = Q \circ T^{-1}$. Because $T$ is Borel measurable, $Q'$ and $f \circ T$ are Borel. We will now argue that $\int fdQ' + \delta \geq \int S_\epsilon(f)dQ$. Recall that $\varphi$ is always measurable with respect to the completion of $Q$, and by convention $\int gdQ$ means integration with respect to the completion of $Q$. Then if we define $M = Q(\mathbb{R}^d)$,
\[
\int fdQ' = \int f(T(x))dQ = \int f(\varphi(x))dQ \geq \int S_\epsilon(f) - \delta dQ = \int S_\epsilon(f)dQ - \delta M
\]
Because $\delta > 0$ was arbitrary and $Q' \in B_\infty^\times(Q)$,
\[
\int S_\epsilon(f)dQ \leq \sup_{Q' \in B_\infty^\times(Q)} \int fdQ'.
\]
It remains to show that $W_\infty(Q, Q') \leq \epsilon$. Define a function $G : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$, $G(x) = (x, T(x))$ and a coupling $\gamma$ by $\gamma = G^*Q$. Then $\gamma(\Delta_\epsilon) = G^*Q(\Delta_\epsilon) = Q(\Gamma^{-1}(\Delta_\epsilon)) = 1$, so $\text{supp}(\gamma) \subseteq \Delta_\epsilon$.

Next we prove Lemma 4. We begin by presenting Strassen’s theorem, see Corollary 1.28 of (Villani, 2003) for more details.

Theorem 38 (Strassen’s Theorem) Let $\mathbb{P}, \mathbb{Q}$ be positive finite measures with the same mass and let $\epsilon \geq 0$. Let $\Pi(\mathbb{P}, \mathbb{Q})$ denote the set couplings of $\mathbb{P}$ and $\mathbb{Q}$. Then
\[
\inf_{\pi \in \Pi(\mathbb{P}, \mathbb{Q})} \pi(\{|x - y| > \epsilon\}) = \sup_{A \text{ closed}} Q(A) - P(A')
\]
Strassen’s theorem is usually written with $A^\epsilon$ in (54) replaced by $A^{\epsilon\downset} = \{x : \text{dist}(x, A) \leq \epsilon\}$—however, for closed sets $A^{\epsilon\downset} = A^\epsilon$. Strassen’s theorem together with Urysohn’s lemma then immediately proves Lemma 4.

**Lemma 39 (Urysohn’s Lemma)** Let $A$ and $B$ be two closed and disjoint subsets of $\mathbb{R}^d$. Then there exists a function $f : \mathbb{R}^d \to [0, 1]$ for which $f = 0$ on $A$ and $f = 1$ on $B$.

See for instance result 4.15 of (Folland, 1999).

**Lemma 4** Let $\mathbb{P}, \mathbb{Q}$ be two finite positive Borel measures with $\mathbb{P}(\mathbb{R}^d) = \mathbb{Q}(\mathbb{R}^d)$. Then

$$W_\infty(\mathbb{P}, \mathbb{Q}) = \inf_\epsilon \{\epsilon \geq 0 : \int h d\mathbb{Q} \leq \int S_\epsilon(h) d\mathbb{P} \ \forall h \in C_b(\mathbb{R}^d)\}$$

**Proof** First, notice that Lemma 3 implies that if $Q \in B^\infty(\mathbb{P})$, then $\int S_\epsilon(h) d\mathbb{P} \geq \int h d\mathbb{Q}$ for all $h \in C_b(\mathbb{R}^d)$, proving the inequality $\geq$ in the statement of the lemma.

We will now argue the other inequality: specifically, we will show that

$$\sup_{A \text{ closed}} \mathbb{Q}(A) - \mathbb{P}(A^\epsilon) \leq \sup_{h \in C_b(\mathbb{R}^d)} \int h d\mathbb{Q} - \int S_\epsilon(h) d\mathbb{P} \quad (55)$$

Strassen’s theorem will then imply that $W_\infty(\mathbb{P}, \mathbb{Q}) \leq \epsilon$. Let $\delta$ be arbitrary and let $A$ be a closed set that satisfies $\sup_{A \text{ closed}} \mathbb{Q}(A) - \mathbb{P}(A^\epsilon) \leq \mathbb{Q}(A) - \mathbb{P}(A^\epsilon) + \delta$. Now because $A$ is closed, $A_n = A \oplus B_{1/n}(0)$ is a series of open sets decreasing to $A$ and $A_n^\epsilon = A^\epsilon \oplus B_{1/n}(0)$ is a sequence of open sets decreasing to $A^\epsilon$. Thus pick $n$ sufficiently large so that $\mathbb{P}(A_n^\epsilon - \mathbb{P}(A^\epsilon)) \leq \delta$. By Urysohn’s lemma, one can choose a function $h$ which is 1 on $A$, 0 on $A_n^\epsilon$, and between 0 and 1 on $A_n - A^\epsilon$. Then $S_\epsilon(h)$ is 1 on $A^\epsilon$, 0 on $(A_n^\epsilon)^C$ and between 0 and 1 on $A_n - A^\epsilon$. Then $\int h d\mathbb{Q} - \mathbb{Q}(A) \geq 0$ and thus

$$\left(\int h d\mathbb{Q} - \int S_\epsilon(h) d\mathbb{P}\right) - (\mathbb{Q}(A) - \mathbb{P}(A^\epsilon)) \geq \mathbb{P}(A^\epsilon) - \mathbb{P}(A_n^\epsilon) \geq -\delta.$$

Because $\delta$ was arbitrary, (55) follows.

**Appendix C. Minimizers of $C_\phi(\eta, \cdot)$: Proof of Lemma 25**

**Lemma 25** Fix a loss function $\phi$ and let $\alpha_\phi(\eta)$ be as in (8). Then $\alpha_\phi$ maps $\eta$ to the smallest minimizer of $C_\phi(\eta, \cdot)$. Furthermore, the function $\alpha_\phi(\eta)$ non-decreasing in $\eta$.

**Proof** To start, we will show that $\alpha_\phi(\eta)$ as defined in (8) is a minimizer of $C_\phi(\eta, \cdot)$. Let $S$ be the set of minimizers of $C_\phi^*(\eta, \cdot)$, which is non-empty due to the lower semi-continuity of $\phi$. Let $a = \inf S = \alpha_\phi(\eta)$ and let $s_i \in S$ be a sequence converging to $a$. Then because $\phi$ is lower semi-continuous,

$$C_\phi^*(\eta) = \liminf_{i \to \infty} \eta \phi(s_i) + (1 - \eta)\phi(-s_i) \geq \eta \phi(a) + (1 - \eta)\phi(-a)$$

Then $a$ is in fact a minimizer of $C_\phi^*(\eta, \cdot)$, so it is the smallest minimizer of $C_\phi^*(\eta, \cdot)$.
We will now show that the function $\alpha \phi$ is non-decreasing.

One can write

$$C_\phi(\eta_2, \alpha) = \eta_2 \phi(\alpha) + (1 - \eta_2) \phi(-\alpha) = \eta_1 \phi(\alpha) + (1 - \eta_1) \phi(-\alpha) + (\eta_2 - \eta_1) (\phi(\alpha) - \phi(-\alpha)) = C_\phi(\eta_1, \alpha) + (\eta_2 - \eta_1) (\phi(\alpha) - \phi(-\alpha))$$

(56)

Notice that the function $\alpha \mapsto \phi(\alpha) - \phi(-\alpha)$ is non-increasing. Then because $\alpha \phi(\eta_1)$ is the smallest minimizer of $C_\phi(\eta_1, \alpha)$, if $\alpha < \alpha \phi(\eta_1)$, then $C_\phi(\eta_1, \alpha)$ > $C_\phi(\eta_1, \alpha \phi(\eta_1))$. Therefore, (56) implies that $C_\phi(\eta_2, \alpha) > C_\phi(\eta_2, \alpha \phi(\eta_1))$, and thus $\alpha$ cannot be a minimizer of $C_\phi(\eta_2, \cdot)$. Therefore, $\alpha \phi(\eta_2) \geq \alpha \phi(\eta_1)$.

Appendix D. Continuity Properties of $\bar{C}_\phi$—Proof of Lemma 12

Recall the function $G(\eta, \alpha)$ defined by (47). With this notation, one can write the $C^*_\phi$ transform as $h_{\bar{C}^*_\phi} = \sup_{\eta \in [0,1]} G(\eta, h_1)$.

Lemma 40 Let $c > 0$ and consider $\alpha \geq c$. Let $a(\alpha) = \alpha C^*_\phi$, where the $C^*_\phi$ transform is as in Lemma 22. Then there is a constant $k < 1$ for which

$$a(\alpha) = \sup_{\eta \in [0,k]} \frac{C^*_\phi(\eta) - \eta \alpha}{1 - \eta}$$

(57)

The constants $k$ depends only on $c$.

Proof Recall that the function $G(\eta, \alpha)$ is decreasing in $\alpha$ for fixed $\eta$ and continuous on $[1,0)$. Let $k = \sup \{ \eta: G(\eta, c) > 0 \}$. As $c$ is strictly positive, one can conclude that $\lim_{\eta \to 1} G(\eta, c) = -\infty$ and as a result $k < 1$. Because $G$ is decreasing in $\alpha$, one can conclude that $G(\eta, \alpha) \leq 0$ for all $\eta > k$ and $\alpha \geq c$. However, $\sup_{\eta \in [0,1]} G(\eta, \alpha) \geq 0$ because $G(0, \alpha) = 0$ for all $\alpha$. Thus (57) holds.

Lemma 41 Let $\{f_\alpha\}$ be a set of $L$-Lipschitz functions. Then $\sup_{\alpha} f_\alpha$ is also $L$-Lipschitz.

This statement is proved in Box 1.8 of (Santambrogio, 2015).

Lemma 42 Let $Q$ be any finite measure and assume that $g$ is a non-negative function in $L^1(Q)$. Let $\delta > 0$. Then there is a lower semi-continuous function $\tilde{g}$ for which $\int |g - \tilde{g}| < \delta$ and $g \geq 0$.

See Proposition 7.14 of Folland.
Lemma 43 Let $g$ be a lower semi-continuous function bounded from below. Then there is a sequence of Lipschitz functions that approaches $g$ from below.

This statement appears in Box 1.5 of (Santambrogio, 2015).

Corollary 44 Let $h$ be an $L^1(Q)$ function with $h \geq 0$. Then for any $\delta$, there exists a Lipschitz $\tilde{h}$ for which $\int |h - \tilde{h}|dQ < \delta$.

Proof By Lemma 42, one can pick a lower semi-continuous $\tilde{g}$ for which $\tilde{g} \geq 0$ and $\int |h - \tilde{g}|dQ < \delta/2$. Next, by Lemma 43, one can pick a Lipschitz $\tilde{h}$ for which $\int |\tilde{g} - \tilde{h}|dQ \leq \delta/2$. Thus $\int |h - \tilde{h}|dQ < \delta$.

Lemma 12 Let $K \subset \mathbb{R}^d$ be compact, $E = C_b(K^c) \times C_b(K^c)$, and $\mathbb{P}_0', \mathbb{P}_1' \in \mathcal{M}_+(K^c)$. Then

$$\inf_{(h_0, h_1) \in S_\phi \cap E} \int h_1d\mathbb{P}_1' + \int h_0d\mathbb{P}_0' = \tilde{R}_\phi(\mathbb{P}_0', \mathbb{P}_1')$$

(27)

Therefore, $\tilde{R}_\phi$ is concave and upper semi-continuous on $\mathcal{M}_+(K^c) \times \mathcal{M}_+(K^c)$ with respect to the weak topology on probability measures.

Proof Let $\mathbb{P}' = \mathbb{P}_0' + \mathbb{P}_1'$ and $\eta' = d\mathbb{P}_1'/d\mathbb{P}'$. Then for any $(h_0, h_1) \in S_\phi \cap E$,

$$\int h_1d\mathbb{P}_1' + \int h_0d\mathbb{P}_0' = \int \eta'h_1 + (1 - \eta')h_0d\mathbb{P}' \geq \int C^*_\phi(\eta')d\mathbb{P}' = \tilde{R}_\phi(\mathbb{P}_0', \mathbb{P}_1').$$

We will now focus on showing the other inequality. Define a function $f$ by

$$f(x) = \begin{cases} \alpha_\phi(\eta'(x)) & x \in \text{supp } \mathbb{P}' \\ 0 & x \notin \text{supp } \mathbb{P}' \end{cases}$$

Let $h_1 = \phi \circ f$, $h_0 = \phi \circ -f$. Then $h_1$, $h_0$ satisfy the inequality $\eta h_1 + (1 - \eta)h_0 \geq C^*_\phi(\eta)$ for all $\eta$ while on $\text{supp } \mathbb{P}'$, $\eta'(x)h_1(x) + (1 - \eta'(x))h_0(x) = C^*_\phi(\eta')$ and therefore

$$\int h_1d\mathbb{P}_1' + \int h_0d\mathbb{P}_0' = \int \eta'h_1 + (1 - \eta')h_0d\mathbb{P}' = \int C^*_\phi(\eta')d\mathbb{P}' .$$

However, $(h_0, h_1) \notin E$. We will now approximate $h_0, h_1$ by bounded continuous functions contained in $S_\phi$. Let $\delta > 0$ be arbitrary. Pick a constant $c > 0$ for which $\int cd\mathbb{P}' < \delta$ and set $\hat{h}_1 = \max(h_1, c)$. The pair $(\hat{h}_0, \hat{h}_1)$ are feasible pair for the set $S_\phi$, and thus

$$C^*_\phi(\eta) - \eta \hat{h}_1 - (1 - \eta)\hat{h}_0 \leq 0$$

(58)

Furthermore,

$$\int \hat{h}_1d\mathbb{P}_1' + \int h_0d\mathbb{P}_0' < \tilde{R}_\phi(\mathbb{P}_0', \mathbb{P}_1') + \delta .$$

(59)

Let $k$ be the constant described by Lemma 40 corresponding to $c$. Now by Corollary 44, there is a Lipschitz function $g$ for which $\int |h_1 - g|d\mathbb{P}' < \min((1 - k)/k, 1)\delta$. Let $\hat{h}_1 =$
max(g, c). Then Lemma 41 implies that \( \hat{h}_1 \) has the same Lipschitz constant as \( g \), and the fact that \( \tilde{h}_1 \geq c \) implies that

\[
\int |\tilde{h}_1 - \hat{h}_1| dP' \leq \int |\tilde{h}_1 - g| dP' < \min \left( \frac{1-k}{k}, 1 \right) \delta
\]  

(Equation 60)

Now let \( \hat{h}_0 = \hat{h}_1 \). By Lemma 40, the supremum in the \( C^*_\phi \) transform for computing \( \hat{h}_0 \) can be taken over \([0, k]\). Therefore, if \( L \) is the Lipschitz constant of \( \hat{h}_1 \), Lemma 41 implies that the Lipschitz constant of \( \hat{h}_0 \) is at most \( kL/(1-k) \). Furthermore, \( \hat{h}_0, \hat{h}_1 \) are bounded on \( K^\epsilon \) because Lipschitz functions are bounded over compact sets. Thus \((\hat{h}_0, \hat{h}_1)\) is in \( S_\phi \cap E \).

Next, we will show that \( \int \hat{h}_0 \) is close to \( \int h_0 \).

\[
\int \hat{h}_0 - h_0 dP' = \int \sup_{[0,k]} \frac{C^*_\phi(\eta) - \eta \hat{h}_1}{1-\eta} h_0 dP'_0 = \int \sup_{[0,k]} \frac{C^*_\phi(\eta) - \eta \hat{h}_1 - (1-\eta)h_0}{1-\eta} dP'_0
\]

\[
= \int \sup_{[0,k]} \left( \frac{C^*_\phi(\eta) - \eta \tilde{h}_1 - (1-\eta)h_0}{1-\eta} + \eta \frac{\hat{h}_1 - \tilde{h}_1}{1-\eta} \right) dP'_0 \leq \int \sup_{[0,k]} \frac{\eta}{1-\eta} (\hat{h}_1 - \tilde{h}_1) dP'_0 \leq \delta
\]  

(Equation 58)

\[
\leq \frac{k}{1-k} \int \hat{h}_1 - \tilde{h}_1 dP'_0 \leq \delta
\]  

(Equation 60)

Therefore, by (59), (60), and the computation above,

\[
\int \hat{h}_1 dP'_1 + \int \hat{h}_0 dP'_0 \leq \bar{R}_\phi(P'_0, P'_1) + 3\delta
\]

AS \( \delta > 0 \) is arbitrary, this inequality implies (27). Because \( K^\epsilon \) is compact, the upper semi-continuity and concavity of \( \bar{R}_\phi \) then follows from (27) together with the Reisz representation theorem.

Appendix E. Duality for Distributions with Arbitrary Support—Proof of Lemma 14

We begin with the simple observation that weak duality holds for measures supported on \( \mathbb{R}^d \). This argument is essentially swapping the order of an infimum and a supremum as presented in Section 4.1.

**Lemma 45 (Weak Duality)** Let \( \phi \) be a non-increasing and lower semi-continuous loss function. Let \( S_\phi \) be the set of pairs of functions defined in (25) for \( K = \mathbb{R}^d \).

Then

\[
\inf_{(h_0, h_1) \in S_\phi} \Theta(h_0, h_1) \geq \sup_{P'_0, P'_1 \in \mathbb{R}^\infty(P_0, P_1)} \bar{R}_\phi(P'_0, P'_1)
\]
**Theorem 46 (Portmanteau Theorem)**

The following are equivalent:

1. The sequence \( Q^n \) converges weakly to \( Q \)
2. For all closed sets \( C \), \( \limsup_{n \to \infty} Q^n(C) \leq Q(C) \) and \( \lim_{n \to \infty} Q^n(\mathbb{R}^d) = Q(\mathbb{R}^d) \)
3. For all open sets \( U \), \( \liminf_{n \to \infty} Q^n(U) \geq Q(U) \) and \( \lim_{n \to \infty} Q^n(\mathbb{R}^d) = Q(\mathbb{R}^d) \)

See Theorem 8.2.3 of (Bogachev, 2007). This result allows us to draw conclusions about restrictions of weakly convergent sequences.

**Lemma 47** Let \( Q^n, Q \in \mathcal{M}_+(\mathbb{R}^d) \) and assume that \( Q^n \) converges weakly to \( Q \). Let \( K \) be a compact set with \( Q(\partial K) = 0 \). Then \( Q^n|_K \) converges weakly to \( Q|_K \).

**Proof** We will verify 2) of Theorem 46 for the measures \( Q^n|_K, Q \).

First, because \( Q(K) = Q(\text{int} K) \), Theorem 46 implies that

\[
\limsup_{n \to \infty} Q^n(K) \leq Q(K) = Q(\text{int} K) \leq \liminf_{n \to \infty} Q^n(\text{int} K) \leq \liminf_{n \to \infty} Q^n(K).
\]

Therefore, \( \lim_{n \to \infty} Q^n|_K(\mathbb{R}^d) = \lim_{n \to \infty} Q^n(K) = Q(K) \). Next, for any closed set \( C \), the set \( C \cap K \) is also closed so the fact that \( Q^n \) weakly converges to \( Q \) implies that

\[
\limsup_{n \to \infty} Q^n|_K(C) = \limsup_{n \to \infty} Q^n(K \cap C) \leq Q(K \cap C) = Q|_K(C).
\]

Next, Prokhorov’s theorem allows us to identify weakly convergent subsequences.
Theorem 48 Let $Q^n$ be a sequence of measures for which $\sup_n Q^n(\mathbb{R}^d) < \infty$ and for all $\delta$, there exists a compact $K$ for which $Q^n(K^c) < \delta$ for all $n$. Then $Q^n$ has a weakly convergent subsequence.

See Theorem 8.6.2 of (Bogachev, 2007). These results imply that $\tilde{R}_\phi$ is upper semi-continuous on $\mathcal{M}_+(\mathbb{R}^d) \times \mathcal{M}_+(\mathbb{R}^d)$.

Lemma 49 The functional $\tilde{R}_\phi$ is upper semi-continuous with respect to the weak topology on probability measures (in duality with $C_0(\mathbb{R}^d)$).

Notice that Lemma 12 implies that $\tilde{R}_\phi$ is upper semi-continuous on the space $\mathcal{M}_+(K^c) \times \mathcal{M}_+(K^c)$ for a compact set $K$. However, on $\mathbb{R}^d$, weak convergence of measures is defined with respect to the dual of $C_0(\mathbb{R}^d)$, the set of continuous functions vanishing at $\infty$. This set is strictly smaller than $C_b(\mathbb{R}^d)$, and thus the relation (27) would not immediately imply the upper semi-continuity of $\tilde{R}_\phi^*$.

Proof Let $Q^n_0, Q^n_1$ be sequences of measures converging to $Q_0, Q_1$ respectively. Set $Q = Q_0 + Q_1$.

Define a function $F(R) = Q(B_R(0)^C)$. Then because this function is non-increasing, it has finitely many points of discontinuity.

Let $\delta > 0$ be arbitrary and choose $R$ large enough so that $F(R) < \delta/C^*_\phi(1/2)$ and $F$ is continuous at $R$. Then notice that $\mathbb{P}(\partial B_R(0)) = 0$ and thus one can apply Lemma 47 with the set $B_R(0)$.

Now let $\nu_0, \nu_1$ be arbitrary measures. Consider $\nu^R_i$ defined by $\nu^R_i = \nu_i|_{B_R(0)}$. Set $\nu = \nu_0 + \nu_1, \nu = d\nu_0/d\nu, \nu^R = \nu^R_0 + \nu^R_1, \nu^R = d\nu^R_0/d\nu^R$. Then on $B_R(0)$, $\eta^R = \eta$ a.e. Thus

$$|\tilde{R}_\phi(\nu^R_0, \nu^R_1) - \tilde{R}_\phi(\nu_0, \nu_1)| = \left| \int C^*_\phi(\eta) 1_{B_R(0)} d\nu - \int C^*_\phi(\eta) d\nu \right| \leq C^*_\phi \left( \frac{1}{2} \right) \nu(B_R(0)^C) \tag{61}$$

If we define $Q^n_{i,R}, Q^n_{i,0}$ via $Q^n_{i,R} = Q_i\big|_{B_R(0)}$, $Q^n_{i,0} = Q^n_i\big|_{B_R(0)}$, Lemma 47 implies that $Q^n_{i,R}$ converges weakly to $Q_i$ and $\lim_{n \to \infty} Q^n(B_R(0)^C) = Q(B_R(0)^C) < \delta$. Therefore, for sufficiently large $n$, $Q^n(B_R(0)^C) < \delta/C^*_\phi(1/2)$. By Lemma 12 and (61),

$$\limsup_{n \to \infty} \tilde{R}_\phi(Q^n_0, Q^n_1) \leq \limsup_{n \to \infty} \tilde{R}_\phi(Q^n_{0,R}, Q^n_{1,R}) + 2\delta \leq \tilde{R}_\phi(Q_0, Q_1) + 2\delta \leq \tilde{R}_\phi(Q_0, Q_1) + 3\delta$$

Because $\delta$ was arbitrary, the result follows.

Next we consider an approximation of $\mathbb{P}_0, \mathbb{P}_1$ by compactly supported measures.

Lemma 50 Let $\mathbb{P}_0, \mathbb{P}_1$ be finite measures. Define $\mathbb{P}^n_i = \mathbb{P}_i|_{B^n_i(0)}$ for $n \in \mathbb{N}$. Then $\mathbb{P}^n_i, \mathbb{P}^n_1$ converge weakly to $\mathbb{P}_0, \mathbb{P}_1$ respectively. Furthermore, there are measures $\mathbb{P}^*_0 \in \mathcal{B}_c^\infty(\mathbb{P}_0), \mathbb{P}^*_1 \in \mathcal{B}_c^\infty(\mathbb{P}_1)$ for which

$$\limsup_{n \to \infty} \sup_{\mathbb{P}^*_0 \in \mathcal{B}_c^\infty(\mathbb{P}_0)} \sup_{\mathbb{P}^*_1 \in \mathcal{B}_c^\infty(\mathbb{P}_1)} \tilde{R}_\phi(\mathbb{P}^*_0, \mathbb{P}^*_1) \leq \tilde{R}_\phi(\mathbb{P}^*_0, \mathbb{P}^*_1) \tag{62}$$

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Proof Set $P = P_0 + P_1$, $P^n = P^n_0 + P^n_1$. Notice that 2) of Theorem 46 implies that $P^n_1$ converges weakly to $P_1$. Let $P^n_i, P^n_1$ be maximizers of $R_\phi$ over $\mathcal{B}_c^\infty(P^n_0) \times \mathcal{B}_c^\infty(P^n_1)$. Next, by Strassen’s theorem (Theorem 38), $P^n_i(B_r(0)) \leq P^n_1(B_r(0))$ and thus $P_i(B_r(0)^C) \geq P^n_1(B_r(0)^C)$, therefore, one can apply Prokhorov’s theorem (Theorem 48) to conclude that $P^n_i, P^n_1$ have subsequences $P^n_{i_k}, P^n_{1_k}$ that converge to measures $P^*_i, P^*_1$ respectively. The upper semi-continuity of $R_\phi$ (Lemma 49) then implies that $P^*_i, P^*_1$ satisfy (62).

It remains to show that $P^*_i \in \mathcal{B}_c^\infty(P_i)$. We will apply Lemma 4. Because $P^n_{i_k}, P^n_1$ are finite Borel measures supported on $R$, Lemma 4 implies that for every $f \in C_b(R^d)$, $\int S_i(f) dP^n_{i_k} \geq \int f dP^n_{i_k}$. Because $P^n_{i_k}$ converges weakly to $P_i$ and $P^n_{i_k}$ converges weakly to $P^*_i$, one can take the limit $k \to \infty$ to conclude $\int S_i(f) dP_i \geq \int f dP^*_i$ for all $f \in C_b(R^d)$. Lemma 4 then implies $P^*_i \in \mathcal{B}_c^\infty(P_i)$.

---

**Lemma 14** Let $\phi$ be a non-increasing, lower semi-continuous loss function and let $P_0, P_1$ be finite Borel measures supported on $R^d$. Let $S_\phi$ be as in (25). Then

$$\inf_{(h_0, h_1) \in S_\phi} \Theta(h_0, h_1) = \sup_{P'_0 \in \mathcal{B}_c^\infty(P_0)} \inf_{P'_1 \in \mathcal{B}_c^\infty(P_1)} R_\phi(P'_0, P'_1).$$

Furthermore, there exist $P^*_0, P^*_1$ which attain the supremum.

**Proof** Let $P^n_0, P^n_1, P^*_0, P^*_1$ be the the measures described in Lemma 50. Notice that because $P^n_0, P^n_1$ are compactly supported, Lemma 13 applies. Define

$$\Theta^n(h_0, h_1) = \int S_i(h_1) dP^n_1 + \int S_i(h_0) dP^n_0.$$  

Thus Lemmas 13 and Lemma 50 imply that

$$\limsup_{n \to \infty} \inf_{(h_0, h_1) \in S_\phi} \Theta^n(h_0, h_1) = \limsup_{n \to \infty} \sup_{P^n'_0 \in \mathcal{B}_c^\infty(P^n_0)} \inf_{P'_1 \in \mathcal{B}_c^\infty(P^n_1)} R_\phi(P^n'_0, P'_1) \leq \sup_{P'_0 \in \mathcal{B}_c^\infty(P_0)} \inf_{P'_1 \in \mathcal{B}_c^\infty(P_1)} R_\phi(P'_0, P'_1).$$

(63)

We will show

$$\inf_{(h_0, h_1) \in S_\phi} \Theta(h_0, h_1) \leq \limsup_{n \to \infty} \inf_{(h_0, h_1) \in S_\phi} \Theta^n(h_0, h_1).$$

(64)

Equations 63 and 64 imply that

$$\inf_{(h_0, h_1) \in S_\phi} \Theta(h_0, h_1) \leq \sup_{P'_0 \in \mathcal{B}_c^\infty(P_0)} \inf_{P'_1 \in \mathcal{B}_c^\infty(P_1)} R_\phi(P'_0, P'_1).$$

(65)

This relation together with weak duality (Lemma 45) imply that the inequalities in (65) are actually equalities. Therefore strong duality holds and $P^n_0, P^n_1$ maximizes the dual.
Next, we prove the inequality in (64). Let $\delta > 0$ be arbitrary and choose an $n \in \mathbb{N}$ for which $n > 2\epsilon$ and
\[
\mathbb{P}_1(B_{n-2\epsilon}(0)^C) + \mathbb{P}_0(B_{n-2\epsilon}(0)^C) \leq \delta
\]
(66)

Let $(h_0^n, h_1^n) \in S_\phi$ be functions for which
\[
\Theta^n(h_0^n, h_1^n) \leq \inf_{(h_0, h_1) \in S_\phi} \Theta^n(h_0, h_1) + \delta
\]
(67)

Define
\[
\tilde{h}_0^n = \begin{cases} h_0^n & x \in B_{n-\epsilon}(0) \\ C_\phi^* \left( \frac{1}{2} \right) & x \notin B_{n-\epsilon}(0) \end{cases}
\]
\[
\tilde{h}_1^n = \begin{cases} h_1^n & x \in B_{n-\epsilon}(0) \\ C_\phi^* \left( \frac{1}{2} \right) & x \notin B_{n-\epsilon}(0) \end{cases}
\]

Because $\eta h_0^n + (1 - \eta) h_1^n \geq C_\phi^*(\eta) \forall \eta \in [0, 1]$ on $B_{n-\epsilon}(0)$ and $(C_\phi^*(1/2), C_\phi^*(1/2)) \in S_\phi$, one can conclude that $(\tilde{h}_0^n, \tilde{h}_1^n) \in S_\phi$.

Now because $n > 2\epsilon$, the regions $B_{n-\epsilon}(0), B_{n-2\epsilon}(0)$ are non-empty. One can bound $S_\epsilon(h_i)$ in terms of $S_\epsilon(h_i)$ and $C_\phi^*(1/2)$:
\[
S_\epsilon(\tilde{h}_i)(x) = S_\epsilon(h_i)(x) \quad \text{for } x \in B_{n-2\epsilon}(0)
\]
\[
S_\epsilon(\tilde{h}_i)(x) \leq \max(S_\epsilon(h_i)(x), C_\phi^*(1/2)) \leq S_\epsilon(h_i) + C_\phi^*(1/2) \quad \text{for } x \in B_{n}(0)
\]
\[
S_\epsilon(\tilde{h}_i) = C_\phi^*(1/2) \quad \text{for } x \in B_{n}(0)^C
\]

Now for each $i$, these bounds imply that
\[
\int S_\epsilon(\tilde{h}_i^n) d\mathbb{P}_i \leq \int_{B_{n-2\epsilon}(0)} S_\epsilon(\tilde{h}_i^n) d\mathbb{P}_i + \int_{B_{n}(0) - B_{n-2\epsilon}(0)} S_\epsilon(\tilde{h}_i^n) + C_\phi^*(1/2) d\mathbb{P}_i + \int_{B_{n}(0) - B_{n-2\epsilon}(0)} C_\phi^*(1/2) d\mathbb{P}_i
\]
\[
= \int_{B_{n}(0)} S_\epsilon(h_i^n) d\mathbb{P}_i + \int_{B_{n-2\epsilon}(0)} C_\phi^*(1/2) d\mathbb{P}_i
\]

Then, applying this bound for each $i$,
\[
\Theta(\tilde{h}_0^n, \tilde{h}_1^n) = \int_{B_{n}(0)} S_\epsilon(\tilde{h}_0^n) d\mathbb{P}_1 + \int_{B_{n}(0)} S_\epsilon(\tilde{h}_0^n) d\mathbb{P}_0
\]
\[
\leq \left( \int_{B_{n}(0)} S_\epsilon(h_0^n) d\mathbb{P}_1 + \int_{B_{n}(0)} S_\epsilon(h_0^n) d\mathbb{P}_0 \right) + \left( \int_{B_{n-2\epsilon}(0)} C_\phi^*(1/2) d\mathbb{P}_1 + \int_{B_{n-2\epsilon}(0)} C_\phi^*(1/2) d\mathbb{P}_0 \right)
\]
\[
= \Theta^n(h_0^n, h_1^n) + C_\phi^*(1/2) \left( \mathbb{P}_0(B_{n-2\epsilon}(0)^C) + \mathbb{P}_1(B_{n-2\epsilon}(0)^C) \right) \leq \left( \inf_{(h_0, h_1) \in S_\phi} \Theta^n(h_0, h_1) + \delta \right) + \delta C_\phi^*(1/2)
\]

The last inequality follows from Equations 66 and 67. Because $\delta$ arbitrary, (64) holds.



Appendix F. Complimentary Slackness

**Lemma 15** Assume that $\mathbb{P}_0, \mathbb{P}_1$ are compactly supported. The functions $h_0^*, h_1^*$ minimize $\Theta$ over $S_\phi$ and $(\mathbb{P}_0^*, \mathbb{P}_1^*)$ maximize $R_\phi$ over $\mathcal{B}_\epsilon(\mathbb{P}_0) \times \mathcal{B}_\epsilon(\mathbb{P}_1)$ iff the following hold:
1) \[
\int h_1^* d\mathbb{P}_1^* = \int S_\epsilon(h_1^*) d\mathbb{P}_1 \quad \text{and} \quad \int h_0^* d\mathbb{P}_0^* = \int S_\epsilon(h_0^*) d\mathbb{P}_0 \quad (30)
\]

2) If we define \( \mathbb{P}^* = \mathbb{P}_0^* + \mathbb{P}_1^* \) and \( \eta^* = d\mathbb{P}_1^*/d\mathbb{P}^* \), then
\[
\eta^*(x)h_1^*(x) + (1 - \eta^*(x))h_0^*(x) = C_{\phi}^*(\eta^*(x)) \quad \mathbb{P}^*-\text{a.e.} \quad (31)
\]

Notice that the forward direction of this lemma is actually a consequence of the approximate complimentary slackness result in Lemma 16, but we provide a separate self-contained proof below.

**Proof**

First assume that \((\mathbb{P}_0^*, \mathbb{P}_1^*)\) maximizes \( \bar{R}_\phi \) over \( \mathcal{B}_c^\infty(\mathbb{P}_0) \times \mathcal{B}_c^\infty(\mathbb{P}_1) \) and \( (h_0^*, h_1^*) \) minimizes \( \Theta \) over \( S_\phi \). Because \( \mathbb{P}_1^* \in \mathcal{B}_c^\infty(\mathbb{P}_1) \) and \( (h_0^*, h_1^*) \in S_\phi \), by Lemma 3
\[
\Theta(h_0^*, h_1^*) = \int S_\epsilon(h_1^*)d\mathbb{P}_1 + \int S_\epsilon(h_0^*)d\mathbb{P}_0 \geq \int h_1^*d\mathbb{P}_1^* + \int h_0^*d\mathbb{P}_0^* \quad (68)
\]
\[
= \int \eta^*h_1^* + (1 - \eta^*)h_0^*d\mathbb{P}^* \geq \int C_{\phi}^*(\eta^*)d\mathbb{P}^* = \bar{R}_\phi(\mathbb{P}_0^*, \mathbb{P}_1^*) \quad (69)
\]

By Lemma 14, both the first expression of (68) and the last expression of (69) are equal. Thus all the inequalities above must be equalities which implies (31). Next, because (69) implies that
\[
\int S_\epsilon(h_1^*)d\mathbb{P}_1 + \int S_\epsilon(h_0^*)d\mathbb{P}_0 = \int h_1^*d\mathbb{P}_1^* + \int h_0^*d\mathbb{P}_0^*
\]
and Lemma 3 implies that \( \int S_\epsilon(h_0^*)d\mathbb{P}_0 \geq \int h_0^*d\mathbb{P}_0^* \) and \( \int S_\epsilon(h_1^*)d\mathbb{P}_1 \geq \int h_1^*d\mathbb{P}_1^* \) we can conclude (30).

We will now show the opposite implication. Assume that \( h_0^*, h_1^*, \mathbb{P}_0^*, \mathbb{P}_1^* \) satisfy (30) and (31). Then
\[
\Theta(h_0^*, h_1^*) = \int S_\epsilon(h_1^*)d\mathbb{P}_1 + \int S_\epsilon(h_0^*)d\mathbb{P}_0 = \int h_1^*d\mathbb{P}_1^* + \int h_0^*d\mathbb{P}_0^*
\]
\[
= \int \eta^*h_1^* + (1 - \eta^*)h_0^*d\mathbb{P}^* = \int C_{\phi}^*(\eta^*)d\mathbb{P}^* = \bar{R}_\phi(\mathbb{P}_0^*, \mathbb{P}_1^*) \quad (31)
\]

However, Lemma 14 implies that \( \Theta(h_0^*, h_1^*) \geq \bar{R}_\phi(\mathbb{P}_0^*, \mathbb{P}_1^*) \) for any \( h_0, h_1, \mathbb{P}_0^*, \mathbb{P}_1^* \). Therefore, \( h_0^*, h_1^* \) must be optimal for \( \Theta \) and \( \mathbb{P}_0^*, \mathbb{P}_1^* \) must be optimal for \( \bar{R}_\phi \).

Notably, a similar strategy shows that if \( (h_0^*, h_1^*) \in S_\phi \) is a sequence that satisfies 1) and 2) of Lemma 16, then \( (h_0^*, h_1^*) \) must be a minimizing sequence for \( \Theta \).
Appendix G. Technical Lemmas from Section 6

G.1 Proof of Lemma 17

**Lemma 17** Let \( \psi(\alpha) = e^{-\alpha} \). Then \( C^*_\psi(\eta) = 2\sqrt{\eta(1-\eta)} \) and \( \alpha_\psi(\eta) = 1/2 \log(\eta/1-\eta) \) is the unique minimizer of \( C_\psi(\eta) \), with \( \alpha_\psi(0), \alpha_\psi(1) \) interpreted as \(-\infty, +\infty\) respectively. Furthermore, \( \partial C^*_\psi(\eta) \) is the singleton \( \partial C^*_\psi(\eta) = \{ \psi(\alpha_\psi(\eta)) - \psi(-\alpha_\psi(\eta)) \} \).

**Proof** First, one can verify that \(-\infty\) minimizes \( C_\psi(0,\alpha) \) and \( \infty \) minimizes \( C_\psi(1,\alpha) \), and that \( C_\psi^*(0) = C_\psi^*(1) = 0 \). To find minimizers of \( C_\psi(\eta,\alpha) \) for \( \eta \in (0,1) \), we solve \( \partial_\alpha C_\psi(\eta,\alpha) = -\eta e^{-\alpha} + (1-\eta)e^\alpha = 0 \), resulting in \( \alpha_\psi(\eta) = 1/2 \log(\eta/1-\eta) \). This formula allows for computation of \( C_\psi^*(\eta) \) via \( C_\psi^*(\eta) = C_\psi(\eta,\alpha_\psi(\eta)) \).

Next, by definition

\[
\eta \psi(\alpha_\psi(\eta)) + (1-\eta)(-\psi(\alpha_\psi(\eta))) = C^*_\psi(\eta) \quad \text{and} \quad s \psi(\alpha_\psi(\eta)) + (1-s)(-\psi(\alpha_\psi(\eta))) \geq C^*_\psi(s)
\]

for all \( s \in [0,1] \). Therefore, \( \psi(\alpha_\psi(\eta)) - \psi(-\alpha_\psi(\eta)) \) is a supergradient of \( C_\psi^*(\eta) \) at \( \eta \).

The function \( C_\psi^* \) is differentiable on \((0,1)\), and thus the superdifferential is unique on this set. To show that \( \partial C^*_\psi(0) \), \( \partial C^*_\psi(1) \) are singletons, it suffices to observe that

\[
\lim_{\eta \to 0} \frac{d}{d\eta} C^*_\psi(\eta) = +\infty, \quad \lim_{\eta \to 1} \frac{d}{d\eta} C^*_\psi(\eta) = -\infty.
\]

\( \blacksquare \)

G.2 Proof of Lemma 18

**Lemma 18** Let \( (a_n, b_n) \) be a sequence for which \( a_n, b_n \geq 0 \) and

\[
\eta a_n + (1-\eta)b_n \geq C^*_\psi(\eta) \quad \text{for all} \quad \eta \in [0,1]
\]

(39)

and

\[
\lim_{n \to \infty} \eta a_n + (1-\eta_0)b_n = C^*_\psi(\eta_0)
\]

(40)

for some \( \eta_0 \). Then \( \lim_{n \to \infty} a_n = \psi(\alpha_\psi(\eta_0)) \) and \( \lim_{n \to \infty} b_n = \psi(-\alpha_\psi(\eta_0)) \).

**Proof** Recall that on the extended real number line, every subsequence has a convergent subsequence. We will show that \( \lim_{n \to \infty} a_n = \psi(\alpha_\psi(\eta_0)) \) and \( \lim_{n \to \infty} b_n = \psi(-\alpha_\psi(\eta_0)) \) by proving that every convergent subsequence of \( \{a_n\} \) converges to \( \psi(\alpha_\psi(\eta_0)) \) and every convergent subsequence of \( b_n \) converges to \( \psi(\alpha_\psi(\eta_0)) \).

Let \( a_{n_k}, b_{n_k} \) be a convergent subsequences of \( \{a_n\}, \{b_n\} \) respectively. (Again, this convergence is in \( \mathbb{R} \).) Set \( a = \lim_{k \to \infty} a_{n_k}, \ b = \lim_{k \to \infty} b_{n_k} \).

Then (39) (40) imply that

\[
\eta a + (1-\eta)b \geq C^*_\psi(\eta) \quad \text{for all} \quad \eta \in [0,1]
\]

\[
\eta_0 a + (1-\eta_0)b = C^*_\psi(\eta_0)
\]

(70)
These equations imply that $a - b \in \partial C_\psi^*(\eta_0)$ and thus
\[ a - b = \psi(\alpha_\psi(\eta_0)) - \psi(-\alpha_\psi(\eta_0)) \] (71)
while (70) is equivalent to
\[ \eta_0a + (1 - \eta_0)b = \eta_0\psi(\alpha_\psi(\eta_0)) + (1 - \eta_0)\psi(-\alpha_\psi(\eta_0)) \] (72)

The equations (71) and (72) comprise a system of equations in two variables with a unique solution for $a$ and $b$.

G.3 Proof of Lemma 20
Lastly, we prove Lemma 20.

**Lemma 20** Let $h_n$ be any sequence of functions. Then the sequence $h_n$ satisfies
\[ \liminf_{n \to \infty} S_\epsilon(h_n) \geq S_\epsilon(\liminf_{n \to \infty} h_n) \] (43)
and
\[ \limsup_{n \to \infty} S_\epsilon(h_n) \geq S_\epsilon(\limsup_{n \to \infty} h_n) \] (44)

**Proof** We start by showing (43).
\[ \liminf_{n \to \infty} S_\epsilon(h_n)(x) = \liminf_{n \to \infty} \sup_{\|h\| \leq \epsilon} h_n(x + h) = \sup_{N} \inf_{n \geq N} \sup_{\|h\| \leq \epsilon} h_n(x + h) \geq \sup_{N} \inf_{n \geq N} \sup_{\|h\| \leq \epsilon} h_n(x + h) = S_\epsilon(\liminf_{n \to \infty} h_n)(x) \]

Equation 44 can then be proved by the same argument:
\[ \limsup_{n \to \infty} S_\epsilon(h_n)(x) = \limsup_{n \to \infty} \sup_{\|h\| \leq \epsilon} h_n(x + h) = \inf_{N} \sup_{n \geq N} \sup_{\|h\| \leq \epsilon} h_n(x + h) \geq \inf_{N} \sup_{n \geq N} \sup_{\|h\| \leq \epsilon} h_n(x + h) = S_\epsilon(\limsup_{n \to \infty} h_n)(x) \]

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