Motivated by applications in quantum information and quantum control, a new type of $C$-numerical range, the relative $C$-numerical range denoted $W_K(C, A)$, is introduced. It arises upon replacing the unitary group $U(N)$ in the definition of the classical $C$-numerical range by any of its compact and connected subgroups $K \subset U(N)$.

The geometric properties of the relative $C$-numerical range are analysed in detail. Counterexamples prove its geometry is more intricate than in the classical case: e.g. $W_K(C, A)$ is neither star-shaped nor simply-connected. Yet, a well-known result on the rotational symmetry of the classical $C$-numerical range extends to $W_K(C, A)$, as shown by a new approach based on Lie theory.

Furthermore, we concentrate on the subgroup $SU_{\text{loc}}(2^n) := SU(2) \otimes \cdots \otimes SU(2)$, i.e. the $n$-fold tensor product of $SU(2)$, which is of particular interest in applications. In this case, sufficient conditions are derived for $W_K(C, A)$ being a circular disc centered at origin of the complex plane. Finally, the previous results are illustrated in detail for $SU(2) \otimes SU(2)$.

1 Introduction

$C$-numerical ranges and $C$-numerical radii [1, 10, 14] naturally relate to optimization problems on the unitary orbit of a given matrix. A particularly interesting class of applications concerns quantum systems as shown in more detail in the accompanying paper [21]. This is because quantum dynamics and quantum control of closed Hamiltonian systems are governed by the unitary group of the underlying Hilbert space and its subgroups.

For instance, in quantum information and quantum computation, where spin-$\frac{1}{2}$ two-level systems are exploited as elementary units—called quantum bits or qubits for short—the full dynamics of $n$ qubits take place on the unitary
group $SU(2^n)$. This includes two types of coherent time evolutions, (i) within each individual qubit independently as well as (ii) between coupled qubits. The former is brought about by so-called local actions. Hence, we refer to the corresponding subgroup as the subgroup of local time evolution $SU_{\text{loc}}(2^n)$. It takes the form of an $n$-fold tensor product of $SU(2)$ with itself. Clearly, matrices (states) that are themselves $n$-fold tensor-products remain in this product form under conjugation with elements of $SU_{\text{loc}}(2^n)$. Therefore, quantum optimization problems which can be expressed via local actions motivate in a most natural way the study of a subset of the classical $C$-numerical range, namely the local $C$-numerical range. It is but a special instance of the more general structure which we introduce here as the relative $C$-numerical range, where the full unitary group is restricted to some arbitrary compact connected subgroup.

To be mathematically precise, let $U(N)$ denote the compact and connected Lie group of all unitary matrices of size $N$, i.e. $U \in U(N)$ if and only if $U \in \mathbb{C}^{N \times N}$ with $UU^\dagger = U^\dagger U = I_N$, where $U^\dagger$ stands for the conjugate transpose. Moreover, let $SU(N)$ be the subgroup of all special unitary matrices, i.e. $U \in SU(N)$ if and only if $U \in U(N)$ with $\det U = 1$. Finally, let

$$SU_{\text{loc}}(2^n) := SU(2) \otimes \cdots \otimes SU(2)$$

be the $n$-fold tensor product of $SU(2)$, which consists of all $n$-fold Kronecker products of the from

$$U_1 \otimes \cdots \otimes U_n, \quad U_k \in SU(2)$$

for all $k = 1, \ldots, n$. Following terminology from quantum information—as mentioned above—we refer to $SU_{\text{loc}}(2^n)$ as the subgroup of local unitary transformations. Now, for arbitrary complex matrices $C, A \in \mathbb{C}^{2^n \times 2^n}$ the local $C$-numerical range of $A$ is defined as

$$W_{\text{loc}}(C, A) := \{ \text{tr} (C^\dagger UAU^\dagger) \mid U \in SU_{\text{loc}}(2^n) \} \subset \mathbb{C}. \quad (1)$$

Obviously, $W_{\text{loc}}(C, A)$ is a subset of the classical $C$-numerical range of $A$, denoted $W(C, A)$. However, its geometry is significantly more intricate than in the classical case and has yet not been studied in any systematic way, cf. [1]. Analysing its basic properties leads naturally to the following slightly more general object

$$W_K(C, A) := \{ \text{tr} (C^\dagger UAU^\dagger) \mid U \in K \}, \quad (2)$$
which we call the relative $C$-numerical range of $A$. Here, $K$ may be any compact and connected subgroup of $U(N)$.

Clearly, $W_K(C, A)$ is invariant under unitary similarities of $A$ and $C$ by elements of $K$. Moreover, with $W_K(C, A)$ being a continuous image of $K$, it is compact and connected. However, in contrast to the classical $C$-numerical range, cf. [4], counterexamples show that $W_K(C, A)$ is neither star-shaped nor simply connected. In general, the actual size and shape of $W_K(C, A)$ are mostly unknown.

The purpose of the present paper is to initiate the study of the relative $C$-numerical range and in particular its geometry. Starting points for our investigations are well-known results on the classical $C$-numerical range. Therefore, we have summarized the fundamental geometric properties of $W(C, A)$ at the beginning of Section 2. Subsequently, the above mentioned counterexamples and straightforward consequences of (2) are collected in Subsection 2.1. The main result, a generalisation on rotationally symmetric $C$-numerical ranges, is presented in Subsection 2.2. Particularly, the original work by Li and Tsing [15] is recovered as an immediate corollary. In Section 3 we return to local $C$-numerical ranges. Here, the central conclusion is that rotationally symmetric local $C$-numerical ranges are essentially circular discs. Finally, we illustrate the previous results by determining all complex $(4\times 4)$-matrices, the local $C$-numerical range of which is a circular disc centered at the origin of the complex plane. These computations lead to a conjecture on unitary similarity to block-shift form of such matrices. Some concluding remarks on numerical issue and prospects for future work can be found in Section 4.

### 2 The Relative $C$-Numerical Range

In Section 1 we introduced the $C$-numerical range of $A \in \mathbb{C}^{N\times N}$ relative to a compact and connected subgroup $K$ of $U(N)$ as follows

$$W_K(C, A) := \{ \text{tr} (C^\dagger UAU^\dagger) \mid U \in K \}. \quad (3)$$

For simplicity, we call $W_K(C, A)$ the relative $C$-numerical range whenever any misinterpretation is excluded.

Below, we aim at a better understanding of the geometry of the relative $C$-numerical range. In particular, we focus on geometric properties of the classical $C$-numerical range which are preserved by passing to the relative one. For this reason, we first recall some basic facts about the classical $C$-numerical range.
The Classical $C$-Numerical Range: A Short Survey

Let $C \in \mathbb{C}^{N \times N}$ be any complex matrix. The classical numerical and $C$-numerical range of $A \in \mathbb{C}^{N \times N}$ are defined by

$$W(A) := \{ \text{tr} (x^\dagger Ax) \mid x \in \mathbb{C}^N, \|x\| = 1 \}$$

and

$$W(C, A) := \{ \text{tr} (C^\dagger UAU^\dagger) \mid U \in U(N) \},$$

respectively. Since any unitary transformation $U$ can be factored in the form $U = e^{i\varphi}U'$ with $\det U' = 1$, the set $W(C, A)$ does not change, if we restrict $U$ to $SU(N)$. Moreover, if $C := xx^\dagger$ for some $x \in \mathbb{C}^N$ with $\|x\|_2 = 1$ we have

$$W(C, A) = \{ \text{tr} ((Ux)^\dagger AU^\dagger x) \mid U \in U(n) \}$$

$$= \{ \text{tr} (y^\dagger Ay) \mid y \in \mathbb{C}^n, \|y\| = 1 \} = W(A).$$

Hence $W(C, A)$ coincides with $W(A)$, if $C$ is of the above form. Here, we used the fact, that $U(N)$ acts transitively on the unit sphere of $\mathbb{C}^N$. This will be of interest later on, if one wants to define a numerical range of $A$ relative to $K$, see also Section 3, Remark 7.

Probably the most basic geometric properties of the numerical and $C$-numerical range are compactness and connectedness, both following from the fact that $W(A)$ and $W(C, A)$ are continuous images of compact and connected sets. The first deep result on the geometry of the numerical range, its convexity, was obtained independently by Hausdorff [11] and Toeplitz [22]. For a proof in a modern textbook we refer to [9]. Later, the result was extended by Westwick [23] and Poon [19] to $C$-numerical ranges for one of the operators $A$ or $C$ being Hermitian. Yet, in general $C$-numerical ranges are non-convex except for $N = 2$, and even for normal matrices convexity may fail for $N > 2$, cf. [23]. However, by a result of Cheung and Tsing [4], they are always star-shaped with respect to the star-center $(\text{tr} C^\dagger)(\text{tr} A)/N$.

The above mentioned theorems describe features which are in common to all $C$-numerical ranges. Another type of results aims at characterising $C$-numerical ranges $W(C, A)$ of particularly simple form such as an interval or a circular disc. Here, we mention only two results in this direction.

**Theorem 2.1** [2, 18] If $C, A \in \mathbb{C}^{N \times N}$ are Hermitian, then $W(C, A)$ degen-
erates to a compact interval \( I = [a, b] \) on the real line with

\[
b = \sum_{j=1}^{N} \alpha_j \gamma_j \quad \text{and} \quad a = \sum_{j=1}^{N} \alpha_j \gamma_{n-j},
\]

(7)

where \( \alpha_1 \geq \cdots \geq \alpha_n \) and \( \gamma_1 \geq \cdots \geq \gamma_n \) are the eigenvalues of \( A \) and \( C \), respectively.

**Theorem 2.2** [15] Let \( A \in \mathbb{C}^{N \times N} \) and let \( \mathcal{O}_u(A) := \{UAU^\dagger \mid U \in U(N)\} \) denote the unitary orbit of \( A \). The following statements are equivalent:

(a) The unitary orbit of \( A \) satisfies \( e^{i\varphi} \mathcal{O}_u(A) = \mathcal{O}_u(A) \) for all \( \varphi \in \mathbb{R} \).

(b) The \( C \)-numerical range of \( A \) satisfies \( e^{i\varphi} W(C, A) = W(C, A) \) for all \( C \in \mathbb{C}^{N \times N} \) and all \( \varphi \in \mathbb{R} \), i.e. \( W(C, A) \) is rotationally symmetric for all \( C \in \mathbb{C}^{N \times N} \).

(c) The \( A \)-numerical range of \( A \) satisfies \( e^{i\varphi} W(A, A) = W(A, A) \) for all \( \varphi \in \mathbb{R} \), i.e. \( W(A, A) \) is rotationally symmetric.

(d) The matrix \( A \) is unitarily similar to a block matrix \( M = (M_{kl})_{1 \leq k,l \leq m} \) such that all \( M_{kk} \) are square blocks and \( M_{kl} = 0 \) if \( l + 1 \neq k \).

(e) \( \mathbb{C}^N \) can be factored into a direct sum of mutually orthogonal subspaces \( \mathbb{C}^N = S_1 \oplus \cdots \oplus S_m \) such that \( A(S_k) \subset S_{k+1} \) for \( k = 1, \ldots, m-1 \) and \( A(S_m) = 0 \).

(f) The \( C \)-numerical range \( W(C, A) \) is a circular disc in the complex plane centered at the origin for all \( C \in \mathbb{C}^{N \times N} \).

(g) \( W(A, A) \) is a circular disc in the complex plane centered at the origin.

Theorem 2.1 can be traced back to von Neumann. For a proof we refer to [2, 12]. Theorem 2.2 is a resume of Theorem 2.1 and Corollary 2.2 in [15]. More on \( C \)-numerical ranges and the \( C \)-numerical radii can be found in a special issue of *Linear and Multilinear Algebra* by Ando and Li [1].

**Remark 1** It is essential in Theorem 2.2 to require the symmetry condition in statement (b) and (f) for all \( C \in \mathbb{C}^{N \times N} \). In fact, \( W(C, A) \) can be a circular disc, although neither \( A \) nor \( C \) satisfy any of the above conditions, for instance, if \( A \) is Hermitian and \( C \) skew-Hermitian.

### 2.1 Basic Properties

Our starting points for analysing the geometry of the relative \( C \)-numerical range are the above mentioned facts on \( W(C, A) \). Obviously, \( W_K(C, A) \) is compact and connected by the same argument as before. However, the previous results on convexity and star-shapedness fail as the following three counterexamples show. In particular, the first and third one prove that \( W_K(C, A) \)
is in general not even simply connected.

Here, the first example exploits the fact that the subgroup $K$ itself need not be simply connected, while the second and third one are based on the fact that the product of convex or star-shaped subsets of $\mathbb{C}$ is in general neither convex nor star-shaped.

**Example 2.3** Let

$$K := \left\{ \left[ \begin{array}{cc} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{array} \right] \, | \, \varphi \in \mathbb{R} \right\} \cong U(1) \quad \text{and} \quad A := C := \left[ \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right].$$

(8)

Then we have

$$W_K(C, A) = \{ e^{i\varphi} | \, \varphi \in \mathbb{R} \} \cong S^1.$$  

(9)

Obviously, $W_K(C, A)$ is not simply connected and therefore neither star-shaped nor convex. However, this is not really surprising, since $K \cong U(1)$ is a closed one-parameter subgroup and hence itself not simply connected.

**Example 2.4** Let $K := U(2) \otimes U(2)$ and let

$$A := A_1 \otimes A_2 \quad \text{with} \quad A_1 := \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right], \quad A_2 := \left[ \begin{array}{cc} 1 + i & 0 \\ 0 & 1 - i \end{array} \right],$$

$$C := C_1 \otimes C_2 \quad \text{with} \quad C_1 := C_2 := \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right].$$

By the trace identity $\text{tr} (A \otimes B) = \text{tr} A \cdot \text{tr} B$ for all $A \in \mathbb{C}^{N \times N}$ and $B \in \mathbb{C}^{N' \times N'}$ it is easy to see that $W_K(C, A) = W(C_1, A_1) \cdot W(C_2, A_2)$. Moreover, as $A_j$ is normal and $C_j$ is Hermitian for $j = 1, 2$, a straightforward calculation yields

$$W(C_1, A_1) = [-1, 1] \quad \text{and} \quad W(C_2, A_2) = \{ 1 + ib \mid -1 \leq b \leq 1 \}.$$ 

Hence we obtain

$$W_K(C, A) = \{ a + ib \mid -1 \leq a \leq 1, \, |b| \leq |a| \}$$

which is obviously not convex.
**Example 2.5** Let $K := U(2) \otimes U(2) \otimes U(2)$ and let

$$A := A_1 \otimes A_2 \otimes A_3 \quad \text{with} \quad A_1 := A_2 := A_3 := \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{bmatrix},$$

$$C := C_1 \otimes C_2 \otimes C_3 \quad \text{with} \quad C_1 := C_2 := C_3 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$ 

As in Example 2.4, we have $W_K(C, A) = W(C_1, A_1) \cdot W(C_2, A_2) \cdot W(C_3, A_3)$ and

$$W(C_j, A_j) = \{(1 - t) + te^{\frac{2\pi i}{3}} \mid t \in [0, 1]\} \quad \text{for} \quad j = 1, 2, 3.$$ 

Thus, it is easy to see that the equilateral triangle

$$\Delta := W(C_1, A_1) \cup e^{\frac{2\pi i}{3}} W(C_1, A_1) \cup e^{\frac{4\pi i}{3}} W(C_1, A_1)$$

forms the outer boundary of $W_K(C, A)$, while the origin and a large part of the interior of $\Delta$ does not belong to $W_K(C, A)$. Therefore $W_K(C, A)$ is not simply connected and hence not star-shaped either.

In both figures, the outer boundary of $W_K(C, A)$ has been computed analytically and drawn in for a better visualization. In some sense, Example 2.5 is more significant than 2.4. Nevertheless, we presented Example 2.4 because it seems to be impossible to imitate 2.5 in dimension $N = 4$. This raises the question which is the lowest dimension $N$ for which a compact, simply
connected subgroup $K \subset U(N)$ and matrices $A, C$ exist such that $W_K(C, A)$ is not simply connected. Moreover, the reader should note that both can be interpreted as relative numerical ranges, cf. Remark 1 of Section 3.

Having seen that most of the geometric properties of the classical $C$-numerical range are not preserved, we present a series of results which guarantee that some geometric features of $W(C, A)$ are passed to $W_K(C, A)$ under additional assumptions on $A, C,$ and $K$. The first one is a trivial fact based on the properties of the Hermitian form $(A, C) \mapsto \text{tr} C^\dagger A$.

**Lemma 2.6** Let $K$ be any compact and connected subgroup of $U(N)$ and let $A, C$ be Hermitian. Then the $C$-numerical range of $A$ is a compact subinterval of $\mathbb{R}$.

**Proof** The identity $\text{tr} C^\dagger A = \text{tr} A^\dagger C = \text{tr} CA^\dagger = \text{tr} C^\dagger A$ for all Hermitian matrices $A, C$ implies immediately the above assertion. □

The next two propositions focus on special types of subgroups—direct sums and tensor products—which are of particular interest for applications, cf. [21]. Both results show that the properties of the classical $C$-numerical range are maintained, if the matrices $A, C$ are compatible with the corresponding subgroup structure.

**Proposition 2.7** Let $K_1 \subset U(N_1)$ and $K_2 \subset U(N_2)$ be compact and connected subgroups and let $K$ be the direct sum of $K_1$ and $K_2$, i.e. $K := K_1 \oplus K_1 := \{ U_1 \oplus U_2 \mid U_1 \in K_1, U_2 \in K_2 \}$, where $U_1 \oplus U_2 \in U(N_1 + N_2)$ is defined by

$$U_1 \oplus U_2 := \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}.$$  

Moreover, let $A := A_1 \oplus A_2$ and $C := C_1 \oplus C_2$ with $A_i, C_i \in C^{N_i \times N_i}$ for $i = 1, 2$. Then the $C$-numerical range of $A$ is given by

$$W_K(C, A) = W_{K_1}(C_1, A_1) + W_{K_2}(C_2, A_2).$$  (10)

In particular, $W_K(C, A)$ is star-shaped (convex), if $W_{K_1}(C_1, A_1)$ and $W_{K_2}(C_2, A_2)$ are star-shaped (convex).

**Proof** The result follows from $\text{tr} (C_1 \oplus C_2)^\dagger (A_1 \oplus A_2) = \text{tr} C_1^\dagger A_1 + \text{tr} C_2^\dagger A_2$ and the fact that the sum of two star-shaped (convex) sets is again star-shaped (convex). □

**Proposition 2.8** Let $K_1 \subset U(N_1)$ and $K_2 \subset U(N_2)$ be compact and connected subgroups and let $K$ be the tensor product of $K_1$ and $K_2$, i.e.
$K := K_1 \otimes K_2 := \{ U_1 \otimes U_2 \mid U_1 \in K_1, U_2 \in K_2 \}$, where $U_1 \otimes U_2 \in U(N_1 N_2)$ denotes the Kronecker product of $U_1$ and $U_2$. Moreover, let $A := A_1 \otimes A_2$ and $C := C_1 \otimes C_2$ with $A_i, C_i \in C^{N_i \times N_i}$ for $i = 1, 2$. Then the $C$-numerical range of $A$ is given by

$$W_K(C, A) = W_{K_1}(C_1, A_1) \cdot W_{K_2}(C_2, A_2). \tag{11}$$

In particular, $W_K(C, A)$ is star-shaped (convex), if either $W_{K_1}(C_1, A_1)$ is star-shaped (convex) and $W_{K_2}(C_1, A_2)$ is contained in a ray $\{ re^{i\varphi} \mid r \geq 0 \}$ of the complex plane or vice versa.

**Proof** Eq. (11) is obtained by the trace identity $\text{tr} (A \otimes B) = \text{tr} A \cdot \text{tr} B$. Moreover, $W_K(C, A)$ is star-shaped (convex), if $W_{K_1}(C_1, A_1)$ is as the following two operations preserve star-shapedness and convexity: (a) rotation by a fixed complex number, (b) multiplication by a subinterval of $\mathbb{R}_0^+$, cf. Appendix, Lemma A.1. □

The following corollary is an immediate consequence of the previous results.

**Corollary 2.9** Let $K$, $A$ and $C$ either be defined as in Proposition 2.7 or as in Proposition 2.8 and furthermore, let $A_i, C_i$ be Hermitian for $i = 1, 2$. Then

$$W_K(C, A) = [a_1 + a_2, b_1 + b_2]$$

or

$$W_K(C, A) = [\min\{a_1 a_2, b_1 b_2\}, \max\{a_1 a_2, b_1 b_2\}],$$

respectively, where $W_{K_1}(C_1, A_1) = [a_1, b_1]$ and $W_{K_2}(C_2, A_2) = [a_2, b_2]$.

### 2.2 A Extension of the Circular-Disc Theorem

This subsection contains the main result of the paper—a generalisation of the Circular-Disc Theorem by Li and Tsing [15], cf. Theorem 2.2. More precisely, we characterize all matrices $A \in C^{N \times N}$, the relative $C$-numerical range of which is rotationally symmetric for all $C$. But first, we illustrate by a counterexample that the relative $C$-numerical range of a block-shift matrix—as Li and Tsing’s result might suggest—is in general not rotationally symmetric.

**Example 2.10** Let $K := U(2) \otimes U(2)$ and let

$$A := C := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. $$
We claim that $W_K(C, A)$ is not circular, although its classical $C$-numerical range is a circular for all $C \in \mathbb{C}^{4 \times 4}$. A rigorous proof of this fact will be given after Corollary 33. Moreover, we note that easier examples can be constructed via the subgroup of Example 23. However, we favored the above one as it will be of interest also in Section 3.

Yet, for the main result and its proof, we need some further notation. We call the $K$-orbit $O_K(A) := \{UAU^\dagger \mid U \in K\}$ of $A \in \mathbb{C}^{N \times N}$ weakly rotationally symmetric, if

$$e^{i\varphi}O_K(A) = O_K(A)$$

holds for all $\varphi \in \mathbb{R}$. Moreover, let $\mathfrak{k}$ denote the Lie algebra of $K$ and let $ad_\Omega$ for $\Omega \in \mathfrak{k}$ be the operator defined by

$$ad_\Omega : \mathbb{C}^{N \times N} \to \mathbb{C}^{N \times N}, \quad A \mapsto ad_\Omega(A) := [\Omega, A] := \Omega A - A\Omega.$$

Finally, a maximal Abelian subalgebra of $\mathfrak{k}$ is called a torus algebra of $K$.

**Proposition 2.11** Let $A \in \mathbb{C}^{N \times N}$ and let $\varphi_0 \in \mathbb{R}$. Then the following statements are equivalent:

(a) The orbit $O_K(A)$ satisfies the relation $e^{i\varphi_0}O_K(A) = O_K(A)$.
(b) The relative $C$-numerical range of $A$ satisfies $e^{i\varphi_0}W_K(C, A) = W_K(C, A)$ for all $C \in \mathbb{C}^{N \times N}$.
(c) The relative $A$-numerical range of $A$ satisfies $e^{i\varphi_0}W_K(A, A) = W_K(A, A)$.

If $\varphi_0$ is an irrational multiple of $2\pi$ and if one of the above statements (a), (b), or (c) holds for $\varphi_0$, then all of them are satisfied for all $\varphi \in \mathbb{R}$.

*Proof* The implications (a) $\implies$ (b) and (b) $\implies$ (c) are obvious. For proving (c) $\implies$ (a) we assume without loss of generality $A \neq 0$. Now, there is a $U_0 \in K$ such that

$$e^{i\varphi_0}\text{tr} (A^\dagger A) = \text{tr} (A^\dagger U_0 A U_0^\dagger). \quad (12)$$

Moreover, using the Cauchy-Schwarz inequality and the unitary invariance of the Frobenius norm we obtain

$$\|A\|^2 = |e^{i\varphi_0}\text{tr} (A^\dagger A)| = |\text{tr} (A^\dagger U_0 A U_0^\dagger)| \leq \|A\| \cdot \|U_0 A U_0^\dagger\| = \|A\|^2. \quad (13)$$

Thus, in fact equality holds in Eq. (13) and therefore we have $U_0 A U_0^\dagger = \lambda A$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Substituting $U_0 A U_0^\dagger = \lambda A$ in Eq. (12) yields
Relative $C$-numerical ranges

\[ U_0 A U_0^\dagger = e^{i\varphi_0} A \] and hence \( \mathcal{O}_K(A) = \mathcal{O}_K(U_0 A U_0^\dagger)^{\dagger} = e^{i\varphi_0} \mathcal{O}_K(A) \). This finally implies (a).

If one of the above statements (a), (b) or (c) holds for some \( \varphi_0 \in \mathbb{R} \setminus 2\pi \mathbb{Q} \), we know that all of them are satisfied for \( \varphi_0 \). In particular, by (a) we have

\[ e^{i\varphi_0} A = U_0 A U_0^\dagger \] (14)

for some \( U_0 \in K \). Thus, by induction we obtain

\[ e^{ik\varphi_0} A = U_0^k A (U_0^k)^\dagger \] (15)

for all \( k \in \mathbb{Z} \). Now, compactness of \( K \) and Eq. (15) imply that for all \( \varphi \in \mathbb{R} \) there exists a \( U \in K \) such that \( e^{i\varphi} A = U A U^\dagger \) and thus

\[ e^{i\varphi} \mathcal{O}_K(A) = \mathcal{O}_K(A) \] (16)

for all \( \varphi \in \mathbb{R} \). Hence (a) and therefore also (b) and (c) are satisfied for all \( \varphi \in \mathbb{R} \). \( \square \)

**Remark 2** Proposition 2.11 is a slight generalization of the first part of Theorem 2.1 in [15]. The above proof, which is almost the same in [15], is included only for completeness.

As an immediate consequence of the above proposition we obtain.

**Corollary 2.12** Let \( K \subset K' \) be compact and connected subgroups of \( U(N) \).

(a) The relative $C$-numerical range of \( A \) is rotationally symmetric for all \( C \in \mathbb{C}^{N \times N} \) if and only if the orbit \( \mathcal{O}_K(A) \) is weakly rotationally symmetric.

(b) If the relative $C$-numerical range \( W_K(C, A) \) is rotationally symmetric for all \( C \in \mathbb{C}^{N \times N} \), then the relative $C$-numerical range \( W_K'(C, A) \) is as well.

**Proof**

(a) \( \checkmark \)

(b) This follows from (a) and the fact that weak rotational symmetry of \( \mathcal{O}_K(A) \) implies weak rotational symmetry of \( \mathcal{O}_K'(A) \). \( \square \)

Now, by the previous definitions our main theorem reads as follows.

**Theorem 2.13** Let \( K \) be a compact and connected subgroup of \( U(N) \) with Lie algebra \( \mathfrak{k} \) and let \( \mathfrak{t} \) be a torus algebra of \( \mathfrak{k} \). Then the orbit \( \mathcal{O}_K(A) \) of \( A \in \mathbb{C}^{N \times N} \), \( A \neq 0 \) is weakly rotationally symmetric if and only if

(a) there exists \( \Omega \in \mathfrak{k} \) such that \( A \) is an eigenvector of the operator \( \text{ad}_\Omega \) to a non-zero eigenvalue,

or equivalently,
(b) there exist $U \in K$ and $\Delta \in \mathfrak{k}$ such that $UAU^\dagger$ is an eigenvector of the operator $\text{ad}_\Delta$ to a non-zero eigenvalue.

Before proving Theorem 2.13, the subsequent lemma will clarify the relation between the operator $\text{ad}_\Omega$ and the weak rotational symmetry of the orbit $O_K(A)$.

**Lemma 2.14** For $\Omega \in u(N)$ and $\varphi \in [0, 2\pi)$ let

$$E_\varphi(\Omega) := \{ A \in \mathbb{C}^{N \times N} | e^{i\varphi t} A = e^{\Omega t} A e^{-\Omega t} \text{ for all } t \in \mathbb{R} \}.$$  

Then $E_\varphi(\Omega)$ is equal to the eigenspace of the operator $\text{ad}_\Omega$ to the eigenvalue $i\varphi$, i.e.

$$E_\varphi(\Omega) = \{ A \in \mathbb{C}^{N \times N} | \text{ad}_\Omega(A) = i\varphi A \}.$$

**Proof** "$\subset$": Let $A \in E_\varphi(\Omega)$. Then

$$e^{i\varphi t} A = e^{\Omega t} A e^{-\Omega t}$$

holds for all $t \in \mathbb{R}$. Hence differentiating Eq. (17) on both sides, yields

$$i\varphi A = \Omega A - A \Omega = \text{ad}_\Omega(A)$$

for $t = 0$. Therefore, $A$ is an eigenvector of $\text{ad}_\Omega$ to the eigenvalue $i\varphi$.

"$\supset$": Now, let $A$ be an eigenvector of $\text{ad}_\Omega$ to the eigenvalue $i\varphi$. Then we consider the curve $\omega(t) := e^{\Omega t} A e^{-\Omega t}$. An easy calculation shows that $\omega(t)$ satisfies the linear differential equation

$$\dot{\omega}(t) = e^{\Omega t} (\Omega A - A \Omega) e^{-\Omega t} = e^{\Omega t} \text{ad}_\Omega(A) e^{-\Omega t} = i\varphi \omega(t).$$

(18)

By uniqueness of solutions of Eq. (18), we have

$$\omega(t) = e^{i\varphi t} A \text{ for all } t \in \mathbb{R}$$

and thus $A \in E_\varphi(\Omega)$. □

**Proof** [of Theorem 2.13] (a) "$\iff$": Let $\Omega \in \mathfrak{k}$ and assume that $A$ is an eigenvector of the operator $\text{ad}_\Omega$ to some eigenvalue $\lambda \neq 0$. Then $\lambda$ is purely imaginary, i.e. $\lambda = i\varphi$ for some $\varphi \in \mathbb{R} \setminus \{0\}$, since $\text{ad}_\Omega$ is skew-Hermitian with respect to the scalar product $(A, C) \mapsto \text{tr}(C^\dagger A)$. Therefore, Lemma 2.14
implies
\[ e^{\Omega t} A e^{-\Omega t} = e^{i\varphi t} A \quad \text{for all } t \in \mathbb{R} \]
and this shows that \( \mathcal{O}_K(A) \) is weakly rotationally symmetric.

“\( \Rightarrow \)”: Now, let \( \mathcal{O}_K(A) \) be weakly rotationally symmetric and choose \( \varphi_0 \not\in 2\pi\mathbb{Q} \). Thus there is a \( U_0 \in K \) such that
\[ e^{i\varphi_0} A = U_0 A U_0^\dagger \quad (19) \]
and hence
\[ e^{i k \varphi_0} A = (U_0^k)^\dagger A U_0^k \quad \text{for all } k \in \mathbb{Z}. \quad (20) \]

Now, we consider the closed Abelian subgroup \( K_0 \) generated by \( \{ U_0^k \mid k \in \mathbb{Z} \} \). Since \( K \) is compact, \( K_0 \) is the direct product of a torus \( T_0 \) and a finite subgroup, c.f. [3]. Continuity and Eq. (20) imply, that for all \( U \in T_0 \) there exists a unique \( \varphi \) depending on \( U \) such that
\[ e^{i\varphi} A = U A U^\dagger. \quad (21) \]

Furthermore, let \( \{ \Omega_1, \ldots, \Omega_m \} \) be a basis of the Lie algebra of \( T_0 \) such that the following conditions hold:
\[ e^{\Omega_j t} = I_N \text{ for } t = 2\pi \quad \text{and} \quad e^{\Omega_j t} \neq I_N \text{ for } 0 < t < 2\pi \quad (22) \]
for all \( j = 1, \ldots, m \). Since \( U_0^k \in T_0 \) for some \( k \in \mathbb{Z} \), we can assume \( U_0 \in T_0 \). Hence, there are \( \alpha_j \in \mathbb{R} \) such that \( U_0 = e^{\alpha_1 \Omega_1 + \cdots + \alpha_m \Omega_m} \). Moreover, by Eq. (21) there exist \( \varphi_j \in \mathbb{R} \) for \( j = 1, \ldots, m \) such that
\[ e^{i \varphi_j} A = e^{\alpha_j \Omega_j} A e^{-\alpha_j \Omega_j}. \quad (23) \]

Now, from Eq. (19) it follows that at least one \( \varphi_j \) is an irrational multiple of \( 2\pi \). Without loss of generality let \( \varphi_1 \not\in 2\pi\mathbb{Q} \). Hence, we obtain
\[ e^{i \varphi_1} A = e^{\alpha_1 \Omega_1} A e^{-\alpha_1 \Omega_1} \quad (24) \]
and thus
\[ e^{i k \varphi_1} A = e^{\alpha_1 \Omega_1 k} A e^{-\alpha_1 \Omega_1 k} \quad (25) \]
for all \( k \in \mathbb{Z} \). Finally, let \( t \in \mathbb{R} \) and \((k_l)_{l \in \mathbb{N}}\) be a sequence such that \( \lim_{l \to \infty} e^{\alpha_l k_l \Omega} = e^{\Omega t} \). Since the map \( e^{it} \mapsto e^{\Omega t} \) is a diffeomorphism of \( S^1 \) onto the one-parameter subgroup \( \{ e^{\Omega t} \mid t \in \mathbb{R} \} \) by Eq. (22), we conclude
\[
\lim_{l \to \infty} e^{i\alpha_l k_l \Omega} = e^{i t} \quad \text{and so}
\]
\[
e^{-\Omega t} A e^{\Omega t} = \lim_{l \to \infty} e^{-\alpha_l k_l \Omega} A e^{\alpha_l k_l \Omega} = \lim_{l \to \infty} e^{i k_l \varphi_l} A = e^{i \frac{\pi}{2} t} A. \tag{26}
\]
Therefore, Lemma 2.14 yields the desired result.

(b) \( \Rightarrow \) : If \( UAU^\dagger \) is an eigenvector of \( \text{ad}_\Delta \) to a non-zero eigenvalue part (a) implies that the \( K \)-orbit of \( UAU^\dagger \) is weakly rotationally symmetric. However, the \( K \)-orbit of \( A \) and \( UAU^\dagger \) are equal. This proves the first part of (b).

\( \Leftarrow \) : Let \( \mathcal{O}_K(A) \) be weakly rotationally symmetric. Then by part (a) there exist \( \Omega \in \mathfrak{k} \) and \( \varphi \in \mathbb{R} \setminus \{0\} \) such that
\[
\text{ad}_\Omega(A) = i \varphi A. \tag{27}
\]
Moreover, by a well-known fact from Lie theory, \( \Omega \) is \( K \)-conjugate to some element \( \Delta \in \mathfrak{t} \), i.e. \( \Omega = U^\dagger \Delta U \), cf. [3] and thus
\[
\text{ad}_\Delta(UAU^\dagger) = U \text{ad}_U^\dagger \Delta U(A) U^\dagger. \tag{28}
\]
Now, Eqs. (27) and (28) imply \( \text{ad}_\Delta(UAU^\dagger) = i \varphi UAU^\dagger \). Hence, \( UAU^\dagger \) is an eigenvector of \( \text{ad}_\Delta \) and thus the proof of (b) is complete.

\( \square \)

Remark 3 Equation (20) in the above proof—which can be rewritten as
\[
e^{i k \varphi_0} A = e^{\Omega_0 k} A e^{-\Omega_0 k} \tag{29}
\]
for all \( k \in \mathbb{Z} \) and some \( \Omega_0 \in \mathfrak{k} \)—does in general not imply
\[
e^{i \varphi_0 t} A = e^{\Omega_0 t} A e^{-\Omega_0 t} \tag{30}
\]
for all \( t \in \mathbb{R} \) as the following example shows.

**Example 2.15** For \( a, b \in \mathbb{R} \) let
\[
A := \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \quad \text{and} \quad U_0 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = e^{\Omega_0} \quad \text{with} \quad \Omega_0 := \frac{\pi}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

Then a straightforward computation yields \( U_0 A U_0^\dagger = -A \) and thus
\[
U_0^k A(U_0^k)^\dagger = e^{\Omega k} A e^{-\Omega k} = e^{k \pi i} A
\]
for all \( k \in \mathbb{Z} \). However, \( A \) and \( \Omega_0 \) do not satisfy Eq. (30), as the necessary condition of Lemma 2.14 is not met, i.e., \( A \) is not an eigenvector of \( \text{ad}_{\Omega_0} \) to the eigenvalue \(-i\pi\).

Problem 2.16 Here, we have \( \varphi_0 = \pi \), which is obvious a rational multiple of \( 2\pi \) and thus does not completely fit the previous requirements. Actually, we have the following conjecture.

Conjecture 2.17 It is impossible to find a counterexample also satisfying \( \varphi_0 \notin 2\pi\mathbb{Q} \). To see this, one probably has to use deeper results on the structure of the torus algebra and its root space decomposition of \( K \).

Theorem 2.13 suggests that the set

\[
E(t) := \bigcup_{\Delta \in t, \varphi \neq 0} E_{\varphi}(\Delta),
\]

where \( t \) is an arbitrary torus algebra of \( \mathfrak{k} \), plays a crucial role in the further analysis of matrices with rotationally symmetric relative \( C \)-numerical range. More precisely, we have the following corollary.

**Corollary 2.18** Let \( t \) be a torus algebra of \( \mathfrak{k} \). The relative \( C \)-numerical range of \( A \in \mathbb{C}^{N \times N}, A \neq 0 \) is rotationally symmetric for all \( C \in \mathbb{C}^{N \times N} \) if and only if there exists \( U \in K \) such that \( UAU^\dagger \) is in \( E(t) \).

**Proof** This follows immediately from Theorem 2.13 and Proposition 2.11. \( \square \)

Next, we show that part (d) and (e) of Theorem 2.2 can easily be derived from Theorem 2.13 by choosing \( K = U(N) \).

**Corollary 2.19** The classical \( C \)-numerical range of \( A \) is rotationally symmetric for all \( C \in \mathbb{C}^{N \times N} \) if and only if \( A \) is unitarily similar to a block-shift matrix \( M \), i.e. \( M = (M_{kl})_{1 \leq k,l \leq m} \) is of block form such that all the \( M_{kk} \) are square blocks and \( M_{kl} = 0 \) if \( l + 1 \neq k \).

**Proof** We may assume without loss of generality that \( A \neq 0 \). Applying Theorem 2.13(b) to \( K := U(N) \) and the torus algebra \( t := \{ \Delta \in u(N) \mid \Delta \text{ diagonal} \} \) yields that the unitary orbit of \( A \) is weakly rotationally symmetric if and only if \( A \) is unitarily similar to a matrix \( M \) such that

\[
\text{ad}_\Delta(M) = i\varphi M
\]

(31)
for some $\Delta \in t$ and $\varphi \in \mathbb{R} \setminus \{0\}$. Now, $\Delta$ can be arranged such that

$$\Delta = i \cdot \text{diag}(\lambda_1, \ldots, \lambda_{n_1}, \ldots, \lambda_{n_m}, \ldots, \lambda_m), \quad \sum_{j=1}^{m} n_j = n$$  \hspace{1cm} (32)

and

$$\lambda_k - \lambda_l = \varphi \quad \implies \quad k = l + 1 \hspace{1cm} (33)$$

for all $1 \leq k, l \leq m$. Choosing a block partition of $M$ corresponding to the one of $\Delta$, Eqs. (31) and (32) yield

$$(\lambda_k - \lambda_l - \varphi)M_{kl} = 0 \hspace{1cm} (34)$$

for all $1 \leq k, l \leq m$. Thus condition (33) implies $M_{kl} = 0$ if $k \neq l + 1$ and hence $M$ has the required form. \hfill \Box

Remark 4 In contrast to part (f) of Theorem 2.2 on the classical $C$-numerical range of $A$, the relative one need not be a circular disc in order to be rotationally symmetric, a counterexample being provided by Example 2.3.

Finally, we want to obtain some information on the Lie-algebraic structure of the set of all matrices with rotationally symmetric relative $C$-numerical range.

**Lemma 2.20** Let $\Omega$ be skew-Hermitian and let $A \neq 0$ be an eigenvector of $\text{ad}_\Omega$ to the non-trivial eigenvalue $i \varphi$, $\varphi \in \mathbb{R}$. Then we have

(a) $A^\dagger$ is an eigenvector of $\text{ad}_\Omega$ to a non-trivial eigenvalue $-i \varphi$.

(b) $A$ is nilpotent.

(c) $[A, A^\dagger] \neq 0$ and $[\Omega, [A, A^\dagger]] = 0$.

**Proof** (a) Let $\Omega$ be skew-Hermitian and $\varphi \neq 0$ such that $\text{ad}_\Omega(A) = i \varphi A$. Then

$$(\text{ad}_\Omega(A^\dagger)) = (\text{ad}_\Omega(A))^\dagger = -i \varphi A^\dagger.$$  \hspace{1cm} (36)

(b) From the identity $[\Omega, A] = i \varphi A$ we obtain $\Omega A^n - A^n \Omega = n i \varphi A^n$ for all $n \in \mathbb{N}$ by induction. Therefore, we have

$$n |\varphi| ||A||^n \leq 2 \|\Omega\| \cdot ||A||^n \hspace{1cm} (35)$$

for all $n \in \mathbb{N}$, where $|| \cdot ||$ denotes the Frobenius norm. This implies $A^n = 0$ for some $n \in \mathbb{N}$, otherwise Eq. (35) would contradict the fact $||\Omega|| < \infty$. 
(c) Again, let $Ω$ be skew-Hermitian and $ϕ ≠ 0$ such that $\text{ad}_Ω(A) = iϕA$. Then by the Jacobi-identity for the double commutator we obtain

$$[Ω, [A, A^\dagger]] = − ([A, [A^\dagger, Ω]] + [A^\dagger, [Ω, A]])$$

$$= [A, \text{ad}_Ω(A^\dagger)] − [A^\dagger, \text{ad}_Ω(A)]$$

$$= −iϕ[A, A^\dagger] − iϕ[A^\dagger, A] = 0,$$

i.e., $Ω$ and $[A, A^\dagger]$ commute. To prove that $[A, A^\dagger]$ does not vanish, we assume the converse. Hence, $A$ is normal and thus part (b) implies $A = 0$. This, however, contradicts our assumptions on $A$ and therefore $[A, A^\dagger] ≠ 0$. □

**Corollary 2.21**

(a) The $K$-orbit of $A$ is weakly rotationally symmetric if and only if the $K$-orbit of $A^\dagger$ is as well.

(b) If the $K$-orbit of $A$ is weakly rotationally symmetric then also the $K$-orbit of $[[[A, A^\dagger], A]$.

**Proof** (a) This follows immediately by Lemma 2.20(a) and Theorem 2.13(a).

(b) Lemma 2.20 and the identity $[\text{ad}_X, \text{ad}_Y] = \text{ad}_{[X,Y]}$ imply that $\text{ad}_Ω$ and $\text{ad}_{[A,A^\dagger]}$ commute and thus they admit a simultaneous eigenspace decomposition. Hence $E_ϕ(Ω)$ is invariant under $\text{ad}_{[A,A^\dagger]}$. In particular, $\text{ad}_{[A,A^\dagger]}(A) = [[A, A^\dagger], A]$ is again an eigenvector of $\text{ad}_Ω$ to the eigenvalue $iϕ$. Therefore, Theorem 2.13(a) yields the desired result. □

So far we have seen that $A$ and $[A, A^\dagger]$ are contained in $E_ϕ(Ω)$. This, however, does not imply that $[A, A^\dagger] = λA$ for some $λ ∈ \mathbb{C}$. Therefore, we introduce the following notion to analyse the situation in more detail. For $A ∈ \mathbb{C}^{N×N}$ let the *separation index* $I_s(A)$ of $A$ be defined by

$$I_s(A) := \min\{\dim E_ϕ(Ω) \mid A ∈ E_ϕ(Ω), Ω ∈ \mathfrak{u}(N), ϕ ∈ \mathbb{R}, ϕ ≠ 0\}. \quad (36)$$

If $A$ is not contained in any eigenspace $E_ϕ(Ω)$, then we set $I_s(A) := −∞$.

**Proposition 2.22** If the separation index of $A$ is equal to 1 then the Lie algebra generated by $A − A^\dagger$, $iA + iA^\dagger$ and $i[A, A^\dagger]$ is isomorphic to $\mathfrak{su}(2)$.

**Proof** By assumption there exist $Ω ∈ \mathfrak{u}(N)$ and $ϕ ∈ \mathbb{R}$, $ϕ ≠ 0$ such that $A ∈ E_ϕ(Ω)$ with $\dim E_ϕ(Ω) = 1$. As in Corollary 2.21(b), we obtain the invariance of $E_ϕ(Ω)$ under $\text{ad}_{[A,A^\dagger]}$ and thus

$$\text{ad}_{[A,A^\dagger]}(A) = λA$$
for some $\lambda \in \mathbb{R}$. Here, $\lambda$ has to be real, since the operator $\text{ad}_{[A,A^\dagger]}$ is Hermitian with respect to the scalar product $(A,C) \mapsto \text{tr}(C^\dagger A)$. Moreover, as in Lemma 2.20 we obtain

$$\text{ad}_{[A,A^\dagger]}(A^\dagger) = -\lambda A^\dagger.$$ 

Hence, we have the following commutator relations:

$$[(A - A^\dagger), i(A + A^\dagger)] = 2i[A, A^\dagger],$$

$$[i(A + A^\dagger), [A, A^\dagger]] = \lambda(A - A^\dagger),$$

$$[i[A, A^\dagger], (A - A^\dagger)] = \lambda i(A + A^\dagger).$$

Therefore, $X := i[A, A^\dagger]$, $Y := A - A^\dagger$ and $Z := iA + iA^\dagger$ generate a 3-dimensional Lie subalgebra of $\mathfrak{u}(N)$. If $\lambda \neq 0$ then a straightforward rescaling of $X, Y$ and $Z$ shows that the generated Lie subalgebra is isomorphic to $\mathfrak{su}(2)$. If $\lambda = 0$ then $X$ and $Y$ as well as $X$ and $Z$ commute. Hence we can assume that $X$ and $Y$ are diagonal. This, however, contradicts $[Y, Z] = X$ and thus we are done. □

An alternative approach to Proposition 2.22 is given by the following lemma, which explicitly determines all $A$ with $I_s(A) = 1$. It is a straightforward consequence of Corollary 2.19 and therefore stated without proof.

**Lemma 2.23** The separation index of $A$ is 1 if and only if $A$ is unitarily similar to $\lambda E_{ij}$ for some $\lambda \in \mathbb{C}, \lambda \neq 0$, where all entries of $E_{ij}$ are zero except the one in the $i$-th row and $j$-th column with $1 \leq i, j \leq N, i \neq j$.

**Remark 5**

(a) Proposition 2.22 is independent of the subgroup $K \subset U(N)$. In particular, $A$ can be the eigenvector of some $\Omega \in \mathfrak{k}$ with $I_s(A) = 1$, while $[A, A^\dagger]$ is not contained in the Lie algebra $\mathfrak{k}$ of $K$. For instance, let

$$K := \left\{ \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \mid U \in SU(2) \right\},$$

$$\Omega := \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}$$

and

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

but

$$[A, A^\dagger] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
Thus the subgroup corresponding to the subalgebra generated by $A - A^\dagger$, $iA + iA^\dagger$ and $i[A, A^\dagger]$ is in general not contained in $K$, even if the $K$-orbit of $A$ is weakly rotationally symmetric. On the other hand, characterizing all subgroups $K$ having the inclusion property

$$A - A^\dagger, iA + iA^\dagger, i[A, A^\dagger] \in \mathfrak{k}$$

for all $A$ with weakly rotationally symmetric $K$-orbit seems to be an open problem.

(b) If the separation index of $A$ is greater than 1, then Proposition 2.22 is in general not true, as the following example shows.

$$A := \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0
\end{bmatrix} \quad \text{and} \quad [A, A^\dagger] := \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -4 & 0 \\
0 & 0 & 0 & 4
\end{bmatrix}.$$ 

By Corollary 2.19 we have $I_S(A) = 2$. Moreover, the Lie algebra generated by $A - A^\dagger$, $iA + iA^\dagger$ and $i[A, A^\dagger]$ is not isomorphic to $\mathfrak{su}(2)$, as claimed above.

3 The Local $C$-Numerical Range

In this subsection we specify the previous results to the $n$-fold tensor product of $SU(2)$, i.e.

$$K := SU_{\text{loc}}(2^n) := SU(2) \otimes \cdots \otimes SU(2) \subset SU(2^N).$$

In quantum mechanics and, in particular, in quantum information, $SU_{\text{loc}}(2^n)$ is called the subgroup of local action. Therefore, we call the corresponding relative $C$-numerical range the local $C$-numerical range of $A$ and introduce the short-hand notation

$$W_{\text{loc}}(C, A) := W_{SU_{\text{loc}}(2^n)}(A, C). \quad (37)$$

Note that replacing $SU_{\text{loc}}(2^n)$ by $U(2) \otimes \cdots \otimes U(2)$ in Definition 37 would yield the same local $C$-numerical range, which can easily be seen by the identity

$$(e^{i\varphi_1}U_1) \otimes \cdots \otimes (e^{i\varphi_n}U_n) = e^{i\varphi_1 + \cdots + \varphi_n}(U_1 \otimes \cdots \otimes U_n).$$
Remark 1 Following Eq. (6) one might naively assign a relative numerical range \( W_K(A) \) to an operator \( A \in \mathbb{C}^{N \times N} \) by a definition like \( W_K(A) := W_K(xx^\dagger, A) \) with \( x \in \mathbb{C}^N \), \( \|x\|_2 = 1 \). However, such a concept is inappropriate as it would depend on the particular choice of \( x \in \mathbb{C}^N \). Yet, for the local case or more general, if \( K = SU(N_1) \otimes \cdots \otimes SU(N_n) \) is a tensor product of special unitary groups, there is a canonical subset of the unit sphere, to wit the set of all \( x = x_1 \otimes \cdots \otimes x_n \) with \( x_k \in \mathbb{C}^{N_k} \), \( \|x_k\| = 1 \), on which \( K \) acts transitively. This allows for properly defining the local numerical range of \( A \) as the set

\[
W_{\text{loc}}(A) := \{ \text{tr} (x^\dagger Ax) \mid x = x_1 \otimes \cdots \otimes x_n, x_k \in \mathbb{C}^{2}, \|x_k\| = 1 \},
\]

which in turn immediately yields the local analogue of Eq. (6). Moreover, note that in physical terms, the local numerical range is nothing else than the classical numerical range restricted to the set of all pure product states. Some of its implications are analysed in the accompanying paper [21].

Now, for applying Theorem 2.13, we have to choose a torus algebra in the Lie algebra of \( SU_{\text{loc}}(2^N) \). A straightforward way of doing so is presented in the following. Let \( K_1 \subset \mathbb{C}^{N_1 \times N_1} \) and \( K_2 \subset \mathbb{C}^{N_2 \times N_2} \) be Lie subgroups with Lie algebras \( \mathfrak{k}_1 \) and \( \mathfrak{k}_2 \), respectively. Then the Lie algebra of the tensor product \( K_1 \otimes K_2 \) is denoted by \( \mathfrak{k}_1 \hat{\otimes} \mathfrak{k}_2 \). It is given by

\[
\mathfrak{k}_1 \hat{\otimes} \mathfrak{k}_2 := \{ \Omega_1 \otimes I_{N_2} + I_{N_1} \otimes \Omega_2 \mid \Omega_1 \in \mathfrak{k}_1, \Omega_2 \in \mathfrak{k}_2 \} \subset \mathbb{C}^{N_1N_2 \times N_1N_2}. \tag{38}
\]

Moreover, let \( \text{sl}_\mathbb{C}(N) \) denote the set of all \( A \in \mathbb{C}^{N \times N} \) with \( \text{tr} A = 0 \).

Lemma 3.1 Let \( t_1 \) and \( t_2 \) be torus algebras of the subalgebras \( \mathfrak{k}_1 \subset \mathbb{C}^{N_1 \times N_1} \) and \( \mathfrak{k}_2 \subset \mathbb{C}^{N_2 \times N_2} \), respectively. Then \( t_1 \hat{\otimes} t_2 \) is a torus algebra of \( \mathfrak{k}_1 \hat{\otimes} \mathfrak{k}_2 \). If, moreover, \( \mathfrak{k}_1 \subset \text{sl}_\mathbb{C}(N_1) \) and \( \mathfrak{k}_2 \subset \text{sl}_\mathbb{C}(N_2) \), then the converse is also true.

Proof Let \( \Omega = \Omega_1 \otimes I_{N_2} + I_{N_1} \otimes \Omega_2 \in \mathfrak{k}_1 \hat{\otimes} \mathfrak{k}_2 \) such that \( [\Omega, \Omega'] = 0 \) for all \( \Omega' \in t_1 \hat{\otimes} t_2 \), i.e.

\[
[\Omega_1 \otimes I_{N_2} + I_{N_1} \otimes \Omega_2, \Omega'_1 \otimes I_{N_2} + I_{N_1} \otimes \Omega'_2] =
\]

\[
= [\Omega_1, \Omega'_1] \otimes I_{N_2} + I_{N_1} \otimes [\Omega_2, \Omega'_2] = 0
\]

for all \( \Omega'_1 \in t_1 \) and \( \Omega'_2 \in t_2 \). By the fact that \( [\Omega_1, \Omega'_1] \otimes I_{N_2} \) and \( I_{N_1} \otimes [\Omega_2, \Omega'_2] \) are orthogonal with respect to the scalar product \( (A, C) \mapsto \text{tr} (C^\dagger A) \), we obtain
the equivalence
\[
[\Omega, \Omega'] = 0 \quad \text{for all } \Omega' \in t_1 \oplus t_2
\]
\[
\iff \quad [\Omega_1, \Omega'_1] = 0 \quad \text{and} \quad [\Omega_2, \Omega'_2] = 0 \quad \text{for all } \Omega'_i \in t_i, \Omega_i \in t_i
\]

Now, if \( t_1 \) and \( t_2 \) are maximal Abelian, then \( \Omega_1 \) and \( \Omega_2 \) are contained in \( t_1 \) and \( t_2 \), respectively, and thus \( t_1 \oplus t_2 \) is maximal Abelian, too. On the other hand, let \( t_1, t_2 \) be Abelian and suppose maximality of \( t_1 \oplus t_2 \), the above equivalence shows that \( \Omega_1 \otimes I_{N_2} + I_{N_1} \otimes \Omega_2 \) belongs to \( t_1 \oplus t_2 \), if \([\Omega_i, \Omega'_i] = 0 \) for all \( \Omega'_i \in t_i \) and \( i = 1, 2 \). Hence, it follows \( \Omega_1 \in t_1 \) and \( \Omega_2 \in t_2 \), if the map
\[
(\Omega_1, \Omega_2) \mapsto \Omega_1 \otimes I_{N_2} + I_{N_1} \otimes \Omega_2
\]
is one-to-one. This, however, is guaranteed under the additional assumption \( k_1 \subset sl_C(N_1) \) and \( k_2 \subset sl_C(N_2) \). Therefore, \( t_1 \) and \( t_2 \) are maximal Abelian, too.

Note that the additional assumption for the converse in Lemma 3.1 is necessary as the following example shows.

**Example 3.2** Let \( k_1 := k_2 := u(2) \) and define
\[
t_1 := \left\{ \begin{bmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{bmatrix} \mid \lambda \in \mathbb{R} \right\} \quad \text{and} \quad t_2 := \left\{ \begin{bmatrix} i\lambda & 0 \\ 0 & i\mu \end{bmatrix} \mid \lambda, \mu \in \mathbb{R} \right\}.
\]
Then \( t_1 \oplus t_2 \) is a torus algebra in \( u(2) \oplus u(2) \). However, \( t_1 \) is not maximal Abelian in \( k_1 = u(2) \).

Now, let \( su_{\text{loc}}(2^n) \) be the Lie algebra of \( SU_{\text{loc}}(2^n) \) and let \( t_{\text{loc}} \subset su_{\text{loc}}(2^n) \) be the subset of all diagonal matrices. Obviously, \( t_{\text{loc}} \) is a torus algebra of \( su_{\text{loc}}(2^n) \) by Lemma 3.1 which yields the following corollary.

**Corollary 3.3** The local \( C \)-numerical range \( W_{\text{loc}}(C, A) \) of \( A \in \mathbb{C}^{2^n \times 2^n} \) is rotationally symmetric for all \( C \in \mathbb{C}^{2^n \times 2^n} \) if and only if there exists \( U \in SU_{\text{loc}}(2^n) \) such that \( UAU^\dagger \in E(t_{\text{loc}}) \), i.e.
\[
[\Delta, UAU^\dagger] = i\varphi UAU^\dagger.
\]

for some \( \Delta \in t_{\text{loc}} \) and \( \varphi \in \mathbb{R}, \varphi \neq 0 \).

**Proof** This follows immediately from Corollary 2.18 and Lemma 3.1. \( \square \)
Finally, we are prepared to present the main result of this section, which roughly speaking excludes the possibility of an annulus for rotationally symmetric local $C$-numerical ranges.

**THEOREM 3.4** The local $C$-numerical range $W_{\text{loc}}(C, A)$ of $A \in \mathbb{C}^{2^n \times 2^n}$ is rotationally symmetric for all $C \in \mathbb{C}^{2^n \times 2^n}$ if and only if it is a circular disc in the complex plane centered at the origin for all $C \in \mathbb{C}^{2^n \times 2^n}$.

Before approaching Theorem 3.4 we provide the following technical lemma.

**LEMMA 3.5** Let $\Delta \in t_{\text{loc}}$ and $A = (a_{ij}) \in \mathbb{C}^{2^n \times 2^n}$ satisfy the relation

$$[\Delta, A] = i\varphi A \quad \text{for some } \varphi \in \mathbb{Q}. \quad (41)$$

Then there exists a rational $\Delta' \in t_{\text{loc}}$ such that Eq. $(41)$ holds.

**Proof** Let $\Delta \in t_{\text{loc}}$, i.e.

$$\Delta = \sum_{j=1}^{n} I_2 \otimes \cdots \otimes I_2 \otimes \begin{bmatrix} i\lambda_j & 0 \\ 0 & -i\lambda_j \end{bmatrix} \otimes I_2 \otimes \cdots \otimes I_2$$

and let $\mu := (\mu_1, \ldots, \mu_{2^n})^\top$ denote the diagonal entries of $\Delta$, i.e. $\Delta = i \cdot \text{diag}(\mu_1, \ldots, \mu_{2^n})$. Then one can find a matrix $X_{\text{loc}} \in \mathbb{Q}^{(2^n-n) \times 2^n}$ such that

$$\Delta \in t_{\text{loc}} \iff X_{\text{loc}} \mu = 0, \quad (42)$$

cf. Lemma 3.1. Moreover, a straightforward calculation shows that

$$[\Delta, A] = i\varphi A \iff \begin{cases} a_{ii} = 0 & \text{for all } i = 1, \ldots, 2^n \\ X_{\text{ad}} \mu = (\varphi, \ldots, \varphi)^\top, \end{cases} \quad (43)$$

where $X_{\text{ad}}$ is a matrix of appropriate size depending on $A$ with entries equal to $\pm 1$ or $0$. In particular, $X_{\text{ad}} \in \mathbb{Q}^{m \times 2^n}$. Hence, we have

$$[\Delta, A] = i\varphi A, \quad \Delta \in t_{\text{loc}} \iff \begin{cases} a_{ii} = 0 & \text{for all } i = 1, \ldots, 2^n \\ X_{\text{loc}} \mu = 0 \\ X_{\text{ad}} \mu = (\varphi, \ldots, \varphi)^\top \end{cases} \quad (44)$$

with $[X_{\text{loc}}^\top \ X_{\text{ad}}^\top] \in \mathbb{Q}^{2^n \times (2^n-n)+m}$. Now, by assumption there exists a $\mu$ such that Eq. $(44)$ is satisfied for some $\varphi \in \mathbb{Q}$. This, however, implies that Eq. $(44)$ has in particular rational solutions, i.e. solutions in $\mathbb{Q}^{2^n}$. \hfill \square

**Proof** [of Theorem 3.4] “$\Rightarrow$”:

✓
Suppose that $W_{\text{loc}}(C, A)$ is rotationally symmetric for all $C \in \mathbb{C}^{2^n \times 2^n}$. It is sufficient to show that zero is contained in $W_{\text{loc}}(C, A)$. Therefore, we can assume $\text{tr}(C^\dagger A) \neq 0$ without loss of generality. Thus, by Proposition 2.11 and Theorem 2.13(a) there exists $\Omega \in \mathfrak{su}_{\text{loc}}(2^n)$ such that

$$t \mapsto \omega(t) := \text{tr}(C^\dagger e^{it\Omega} A e^{-it\Omega}) = e^{i\varphi t} \text{tr}(C^\dagger A), \quad t \in \mathbb{R}$$

is a circle around the origin in the complex plane. By Theorem 2.13(b) and Lemma 3.1 we can assume that $\Omega$ is of diagonal form

$$\Omega = \sum_{j=1}^{N} I_2 \otimes \cdots \otimes I_2 \otimes \begin{bmatrix} i\lambda_j & 0 \\ 0 & -i\lambda_j \end{bmatrix} \otimes I_2 \otimes \cdots \otimes I_2.$$  \hspace{1cm} (45)

and satisfies the relation $[\Omega, A] = i\varphi A$. By rescaling $\Omega$ such that $\varphi$ is rational and by invoking Lemma 3.5 we further suppose that for $j = 1, \ldots, n$ all $\lambda_j$ are rational. Now, let $m$ be the least common multiple of the denominaters of all $\lambda_j$ for $j = 1, \ldots, N$. Then $m\varphi \in \mathbb{Z}$ and thus $\omega|_{[0,2m\pi]}$ is a circle in the complex plane surrounding the origin $(m\varphi)$-times. Moreover, we have

$$e^{2m\pi\Omega} = I_{2^n}.$$

Therefore, the homotopy $H : [0, 2m\pi] \times [0, \frac{\pi}{2}] \to SU_{\text{loc}}(2^n)$ of the form $H(t, s) = U(s)^\dagger e^{it\Omega} U(s)$ with

$$U(s) := \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix}$$

satisfies

$$H(t, 0) = e^{it\Omega}, \quad H(t, \frac{\pi}{2}) = e^{-it\Omega} \quad \text{and} \quad H(0, s) = H(2m\pi, s) = I_{2^n}.$$ \hspace{1cm} (47)

for all $(t, s) \in [0, 2m\pi] \times [0, \frac{\pi}{2}]$. It follows that $\omega|_{[0,2m\pi]}$ is homotopic to its inverse by the homotopy

$$h(t, s) := \text{tr}(C^\dagger H(s, t)\dagger AH(t, s)).$$

Hence, $h$ has to cross the origin, cf. Appendix A, Lemma A.2, and thus, the origin is contained in $W_{\text{loc}}(C, A)$. Therefore, $W_{\text{loc}}(C, A)$ is a circular disc. \hspace{1cm} $\square$

In the remainder of this section, we exemplify the previous results by determining the set of all matrices $A \in \mathbb{C}^{4 \times 4}$, the local $C$-numerical range of which is
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a circular disc centered at the origin. These investigations will lead to a conjecture about “local” similarity to block-shift form. But first—as promised—we tackle the problem of computing all \( A \in \mathbb{C}^{4 \times 4} \) with circular local \( C \)-numerical range. By Corollary 3.3 we can focus on the set of all \( \hat{A} \in \mathbb{C}^{4 \times 4} \) which satisfy

\[
[\Delta, \hat{A}] = i\varphi \hat{A}.
\]  

(48)

for some \( \Delta \in t_{\text{loc}} \) and \( \varphi \neq 0 \). Let \( \hat{A} := (\hat{a}_{kl}) \) and \( \Delta := \text{diag}(\lambda_1, \ldots, \lambda_4) \in t_{\text{loc}} \). Then Eq. (48) can be rewritten as

\[
(\lambda_k - \lambda_l)\hat{a}_{kl} = \varphi \hat{a}_{kl}
\]  

(49)

for all \( k, l = 1, \ldots, 4 \). Moreover, a straightforward calculation shows

\[
\Delta \in t_{\text{loc}} \iff \Delta = \begin{bmatrix} i\lambda & 0 & 0 & 0 \\ 0 & i\mu & 0 & 0 \\ 0 & 0 & -i\mu & 0 \\ 0 & 0 & 0 & -i\lambda \end{bmatrix} \quad \text{with } \lambda, \mu \in \mathbb{R}.
\]  

(50)

Now, considering all possibilities for \( \lambda_k - \lambda_l \) to be equal to \( \varphi \neq 0 \) and taking into account Eq. (50) leads to 32 different cases. However, by the symmetry of Eq. (49), these can be reduced to the following 16 ones, while the remaining ones can be obtained by transposition.

| Case | Eigenvalue | Case | Eigenvalue |
|------|------------|------|------------|
| 1    | \( \varphi = \mu - \lambda \) | 9    | \( \varphi = \mu - \lambda = -2\mu \) |
| 2    | \( \varphi = -\mu - \lambda \) | 10   | \( \varphi = \mu - \lambda = 2\mu \) |
| 3    | \( \varphi = -2\lambda \)    | 11   | \( \varphi = -\mu - \lambda = -2\lambda \) |
| 4    | \( \varphi = -2\mu \)     | 12   | \( \varphi = -\mu - \lambda = 2\lambda \) |
| 5    | \( \varphi = \mu - \lambda = -\mu - \lambda \) | 13   | \( \varphi = -\mu - \lambda = -2\mu \) |
| 6    | \( \varphi = \mu - \lambda = \mu + \lambda \) | 14   | \( \varphi = -\mu - \lambda = 2\mu \) |
| 7    | \( \varphi = \mu - \lambda = -2\lambda \) | 15   | \( \varphi = -2\lambda = -2\mu \) |
| 8    | \( \varphi = \mu - \lambda = +2\lambda \) | 16   | \( \varphi = -2\lambda = 2\mu \) |

For example, Case 1 in the above table means that \( \lambda_2 - \lambda_1 = \lambda_4 - \lambda_3 = \mu - \lambda = \varphi \neq 0 \) and all other differences \( \lambda_k - \lambda_l \) do not equal \( \varphi \). Hence, Eq. (49) reads...
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$\varphi \hat{a}_{21} = \varphi \hat{a}_{43} = \varphi \hat{a}_{43}$, and $\hat{a}_{kl} = 0$ otherwise. Thus $\hat{A}$ has the form

$$\hat{A} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & * & 0 \\
\end{bmatrix},$$

where the symbol $*$ denotes an arbitrary complex number. In the same way one can compute $\hat{A}$ in all other cases. Here, we only list the solutions for the above table. The remaining ones—as mentioned before—can be obtained by transposition.

Case 1: $\begin{bmatrix}
0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & * & 0 \\
\end{bmatrix}$, Case 2: $\begin{bmatrix}
0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & * & 0 \\
\end{bmatrix}$, Case 3: $\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}$, Case 4: $\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
\end{bmatrix}$, Case 5: $\begin{bmatrix}
0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}$, Case 6: $\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
\end{bmatrix}$, Case 7: $\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
\end{bmatrix}$, Case 8: $\begin{bmatrix}
0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
\end{bmatrix}$, Case 9: $\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}$, Case 10: $\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
\end{bmatrix}$, Case 11: $\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
\end{bmatrix}$, Case 12: $\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
\end{bmatrix}$, Case 13: $\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}$, Case 14: $\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
\end{bmatrix}$, Case 15: $\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
\end{bmatrix}$, Case 16: $\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}$.

Finally, we are prepared to provide a rigorous proof of what we claimed in
Example 2.10  The local $C$-numerical range of the block-shift matrix
\[
A := \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{bmatrix}
\]
is not rotationally symmetric for all $C \in \mathbb{C}^{4 \times 4}$. According to the above classification, we have to show that $A$ is not locally unitarily similar, i.e. similar via an element in $SU(2) \otimes SU(2)$, to one of the above cases. This, however, can be checked by “brute force” and is left to the reader.

The above computations reveal an interesting contrast to Corollary 2.19 on the one hand, we have seen by Example 2.10 that not every matrix which is similar to block-shift from via a local unitary transformation has a circular local $C$-numerical range. On the other hand, there are matrices with circular local $C$-numerical range, which are not locally unitarily similar to block-shift form, cf. Case 16 in the above table. However, combining Corollary 2.12 and 2.19 every matrix with circular local $C$-numerical range has to be “globally” unitarily similar to block-shift form, i.e. via a transformation in $U(2^n)$. This raises the question: what is a smallest subgroup $K'$ of $U(2^n)$ containing $SU_{loc}(2^n)$ such that every matrix with circular local $C$-numerical range is similar to block-shift form via a transformation in $K'$? By Corollary 3.3 we can reduce the problem to studying the smallest subgroup $\Pi'$ of (signed) permutations, such that any element in $E(t_{loc})$ is similar to block-shift form via a permutation in $\Pi'$.

One should note that in the above table all matrices are either in block-shift form or similar to block-shift form via a permutation of the following type:

\[
P_{loc} := P_1 \otimes P_2, \quad P_1, P_2 \in \left\{ \pm I_2, \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\} \quad \text{or} \quad P_{out} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]

This leads to the following conjecture:

Conjecture 3.6 Every element of $E(t_{loc})$ is similar to a block-shift matrix via an element of $\Pi_{loc}^\times := \Pi_{loc} \cdot \Pi_{out}$. Here, $\Pi_{loc}$ denotes the subgroup of all (signed) local permutations, i.e. $\Pi_{loc}$ consists of all matrices of the form $P_1 \otimes \cdots \otimes P_n$ with

\[
P_k \in \left\{ \pm I_2, \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}
\]
for \( k = 1, \ldots, n \), and \( \Pi_{\text{out}} \) stands for the subgroup generated by

\[
I_2 \otimes \cdots \otimes I_2 \otimes P_{\text{out}} \otimes I_2 \otimes \cdots \otimes I_2 \in U(2^n), \quad n \geq 2,
\]

where the factor \( P_{\text{out}} \) appears in all possible positions.

**Remark 2**

(a) Note that the two subgroups \( \Pi_{\text{loc}} \) and \( \Pi_{\text{out}} \) operate via similarity on the set of tensor products \( A_1 \otimes \cdots \otimes A_n \) with \( A_1, \ldots, A_n \in \mathbb{C}^{2 \times 2} \) in a completely different way. While \( \Pi_{\text{loc}} \) acts on each factor \( A_k \) separately, \( \Pi_{\text{out}} \) does not effect the factors themselves but interchanges their order. Moreover, \( \Pi_{\text{loc}} \) and \( \Pi_{\text{out}} \) commute and hence \( \Pi_{\text{loc}}^{\text{ex}} := \Pi_{\text{loc}} \cdot \Pi_{\text{out}} = \Pi_{\text{out}} \cdot \Pi_{\text{loc}} \) is isomorphic to the direct product of \( \Pi_{\text{loc}} \) and \( \Pi_{\text{out}} \). We call it the extended local permutation group. It is easy to check that \( E(t_{\text{loc}}) \) is invariant under conjugation with elements of \( \Pi_{\text{loc}}^{\text{ex}} \). Therefore, we expect \( \Pi_{\text{loc}}^{\text{ex}} \) to be the smallest subgroup of permutations satisfying the above conjecture.

(b) The group \( \Pi_{\text{loc}}^{\text{ex}} \) is closely related to the Weyl group of \( SU_{\text{loc}}(2^n) \). More precisely, the action of \( \Pi_{\text{loc}} \) on \( t_{\text{loc}} \) coincides with the Weyl group action of \( SU_{\text{loc}}(2^n) \) on the torus algebra \( t_{\text{loc}} \). However, conjugation by elements of \( \Pi_{\text{out}} \) cannot be achieved by elements of the Weyl group. These ideas suggest how to generalise the above result to the settings of arbitrary compact Lie groups.

**4 Conclusions**

We introduced a new mathematical object, the relative \( C \)-numerical range \( W_K(C, A) \) of an operator \( A \). In particular, we studied its geometry by comparing its properties with the classical \( C \)-numerical range. We showed that although the relative \( C \)-numerical range is compact and connected as in the classical case, it is neither star-shaped nor simply connected. Moreover, necessary and sufficient conditions for circular symmetry of \( W_K(C, A) \) have been derived. These results generalise a former theorem by Li and Tsing [15] and lead also to a deeper understanding of the classical case in Lie theoretical terms. Moreover, in view of applications in quantum information, we introduced the local \( C \)-numerical range \( W_{\text{loc}}(C, A) \) as a special case of the relative \( C \)-numerical range thus inheriting its mathematical structure.

In particular detail, we analysed circular symmetry of local \( C \)-numerical ranges, which are of special interest in quantum control and quantum information [21]. Here, we proved that local \( C \)-numerical ranges with circular symmetry have to be circular discs centered at the origin of the complex plane. This is not evident as relative \( C \)-numerical ranges are in general not simply
connected. Finally, we applied our results to characterise all $(4 \times 4)$-matrices with circular local $C$-numerical range.

However, explicit formulas for the radius of a circular relative $C$-numerical range or, more general, for the relative $C$-numerical radius of $A$

$$r_K(C, A) := \max \{ |\text{tr} (C^1 U A U^1))| \mid U \in K\}.$$ (52)

lead to open research problems. Therefore, general numerical algorithms for finding sharp bounds on the size of $W_K(C, A)$ are highly desirable. Geometric optimisation methods for the classical as well as the local $C$-numerical radius can be found in [2, 6–8, 12, 13, 21]. Yet another interesting open problem is to find out conditions which guarantee convergence of these methods, such as intrinsic gradient flows, to the relative $C$-numerical radius or at least to boundary points of the respective relative $C$-numerical range. Problems of this kind are anticipated to be illuminating both for mathematical structure and for quantum applications.

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References

[1] T. Ando and C.-K. Li, editors. Special Issue: The Numerical Range and Numerical Radius, volume 37, 1–3 of Linear and Multilinear Algebra, pages 1–238. Gordon and Breach, 1994.
[2] R.W. Brockett. Dynamical systems that sort lists, diagonalize matrices, and solve linear programming problems. In Proc. IEEE of the 27th Conference on Decision and Control, 799–803, Austin, Texas, 1988. See also Lin. Algebra & Applic., 146:79–91, 1991.
[3] T. Bröcker and T. tom Dieck. Representations of Compact Lie Groups. Graduate Texts in Mathematics. Springer, New York, 1985.
[4] W.-S. Cheung and N.-K. Tsing. The $C$-numerical range of matrices is star-shaped. Linear and Multilinear Algebra, 41:245–250, 1996.
[5] J. Dazord. On the $C$-numerical range of a matrix. Lin. Algebra & Applic., 212/213:21–29, 1994.
[6] G. Dirr and U. Helmke and M. Kleinsteuber and S. Glaser and Th. Schulte-Herbrüggen. The local numerical range: Examples, conjectures and numerical algorithms Proc. MTNS, Kyoto, 2006.
Appendix A: Two Technical Lemmas

The following appendix contains two technical lemmas which we referred to in the previous sections. Although most readers will be familiar with these results, we include them for completeness.

**Lemma A.1** Let $W$ be a convex (or star-shaped) subset of a complex or real vector space and let $[r, s]$ be a non-negative, real interval, i.e. $0 \leq r \leq s$. Then the set $[r, s] \cdot W := \{ \lambda \cdot w \mid \lambda \in [r, s], w \in W \}$ is also convex (star-shaped).

**Proof** For $s = 0$ there is nothing to prove thus we may assume without loss of generality $s = 1$, as for $s > 0$ we can use the identity $[r, s] \cdot W = [r/s, 1] \cdot sW$ and the fact that $sW$ is convex or star-shaped if $W$ is convex or star-shaped.

Firstly, we consider the case that $W$ is convex. Let $v_1, v_2 \in [r, 1] \cdot W$, i.e. there are $w_1, w_2 \in W$ and $\lambda_1, \lambda_2 \in [r, 1]$ such that $v_1 = \lambda_1 w_1$ and $v_2 = \lambda_2 w_2$. $\square$
We have to show
\[ v_1 + t(v_2 - v_1) \in [r, 1] \cdot W \]
for all \( t \in [0, 1] \). Without loss of generality let \( \lambda_1 \leq \lambda_2 \) and define \( \lambda^* \) and \( t^* \) by
\[
\lambda^* := \lambda_1 + t(\lambda_2 - \lambda_1)
\]
\[
t^* := \begin{cases} 
\frac{t\lambda_2}{\lambda_1 + t(\lambda_2 - \lambda_1)} & \text{for } \lambda_1 \neq 0, \\
1 & \text{for } \lambda_1 = 0.
\end{cases}
\]
As \( t \in [0, 1] \) it holds that \( \lambda_1 \leq \lambda^* \leq \lambda_2 \) and \( 0 \leq t^* \leq 1 \). Furthermore, we have
\[
\lambda^*(w_1 + t^*(w_2 - w_1)) = (\lambda_1 + t(\lambda_2 - \lambda_1))w_1 + t\lambda_2(w_2 - w_1) \\
= \lambda_1 w_1 + t(\lambda_2 w_2 - \lambda_1 w_1) \\
= v_1 + t(v_2 - v_1). \tag{A1}
\]
Hence \( v_1 + t(v_2 - v_1) \) is in \( [r, 1] \cdot W \), as convexity of \( W \) implies that the left side of Eq. (A1) is obviously in \( [r, 1] \cdot W \).

Now, we assume that \( W \) is star-shaped with star center \( w_0 \). Let \( v \in [r, 1] \cdot W \), i.e. there is \( w \in W \) and \( \lambda \in [r, 1] \) such that \( v = \lambda w \). We have to show that there exists a star center \( v_0 \in [r, 1] \cdot W \) such that
\[
v + t(v_0 - v) \in [r, 1] \cdot W
\]
for all \( t \in [0, 1] \). Let \( v_0 := w_0 \) and define \( \lambda^* \) and \( t^* \) in the same way as in Eq. (A1) with \( \lambda_1 = \lambda \) and \( \lambda_2 = 1 \). Then we have
\[
\lambda^*(w + t^*(w_0 - w)) = \lambda w + t(w_0 - \lambda w) = v + t(v_0 - v). \tag{A2}
\]
As \( v_0 \) does not depend on \( v \), it follows that \( [r, 1] \cdot W \) is star-shaped with star center \( v_0 \).

**Lemma A.2** Let \( \gamma : [a, b] \rightarrow \mathbb{C} \) be a closed curve and let \( \gamma^{-1} \) be its inverse, i.e. \( \gamma^{-1}(t) := \gamma(b + a - t) \). Moreover, let \( z_0 \in \mathbb{C} \) be any point in the interior of \( \gamma \), i.e. the winding number of \( \gamma \) with respect to \( z_0 \) is not equal to zero. Then any homotopy from \( \gamma \) to its inverse \( \gamma^{-1} \) has to cross \( z_0 \).
Proof Let \( w(\cdot, z_0) \) denote the winding number of a closed curve with respect to \( z_0 \) and assume that \( h : [a, b] \times [c, d] \to \mathbb{C} \) is a homotopy from \( \gamma \) to its inverse \( \gamma^{-1} \) such that \( h(t, s) \neq z_0 \) for all \((t, s) \in [a, b] \times [c, d] \). As well-known, the winding number assumes only integer values and satisfies the equality \( w(\gamma, z_0) = -w(\gamma^{-1}, z_0) \). Therefore, by continuity of \( h \), the winding number of \( \gamma \) with respect to \( z_0 \) has to be zero. This, in turn, contradicts our hypothesis and thus \( h \) has to cross \( z_0 \), i.e. \( h(t, s) = z_0 \) for some \((t, s) \in [a, b] \times [c, d] \). \( \square \)