TQFT at work for IR-renormalons, resurgence and Lefschetz decomposition

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We investigate the implications of coupling a topological quantum field theory (TQFT) to Yang-Mills theory with $SU(N)$ gauge group in the context of the IR-renormalon problem. Coupling a TQFT to QFT does not change the local dynamics and perturbation theory, but it does change the bundle topology. Crucially, the configurations with integer topological charge but fractional action seem unrelated. We claim that in a rather general manifold $M$, the TQFT coupling changes only the sum over bundle topologies, while changing nothing locally in an arbitrarily large ball $B_d$ in a $d$-manifold $M_d$. The two notions seem unrelated. We claim that in a rather general class of theories, the TQFT coupling carries crucial data to solve the IR-renormalon problem. This may sound strange since the TQFT coupling does nothing locally, but it is exactly this curious fact that will be our guide.

We consider $SU(N)$ pure Yang-Mills theory on a 4-manifold $M_4 = T^4, \mathbb{R}^4$ (and remark briefly on the $\mathbb{CP}^{N-1}$ model in 2d), and their supersymmetric completions, and an IR-conformal theory. YM theory has a $\mathbb{Z}_N^1$ 1-form center-symmetry, for which we can turn on a background field or gauge. The gauged version is called the center-symmetry, for which we can turn on a background field or gauge. The gauged version is called the YM theory, where the IR-renormalon problem is much closer to the origin than the TQFT coupling does nothing locally, but it is exactly this curious fact that will be our guide.

On the other hand, their sums over bundle topologies are different. The topological sectors in the $SU(N)$ theory are classified according to instanton charge density (which integrates to an integer) \cite{11, 12}, while the bundles in the $PSU(N)_p$ theory are much finer, being classified by two topological quantities: instanton charge density and 2nd Stiefel-Whitney class $w_2 \in H^2(M_4, \mathbb{Z}_N)$. The topological charge in the $PSU(N)_p$ theory takes fractional values, $\frac{1}{N}\mathbb{Z}$, and in the BPS bound, the action of such configurations are also a fraction of an instanton action $S_I/N$. What should we make of this different classical data?

Standard lore about perturbation theory on $\mathbb{R}^4$:

Since perturbation theory is independent of the topological $\theta$ angle, so too is the ambiguity in the Borel resummation of the perturbation theory. As a consequence, no topological configuration that carries $\theta$ angle dependence can be associated with a singularity in the Borel plane. The so-called instanton singularity in the Borel plane of $SU(N)$ theory is always an instanton-antiinstanton singularity \cite{13, 14}. The $[I\bar{I}]$ singularity in the Borel plane is located at:

$$t_{[I\bar{I}]}^R = 2S_I g^2 = 16\pi^2$$

and the corresponding ambiguity is of order $\pm ie^{-2S_I}$. This is sourced by the factorial growth of the number of Feynman diagrams.

However, there are other more important singularities in the Borel plane, much closer to the origin \cite{1, 2}. These arise from the integration over low and high momentum domains combined with the singularity in the running of the coupling constant. For example, integration over low energy domain in the Adler function produces a factorial growth \cite{2}.

$$D(Q^2) \sim \frac{\alpha_s}{2} \sum_{n=0}^{\infty} \frac{n!}{(4\pi^2/\beta_0)^n}$$

Borel resummation of this series produces an ambiguity of order $\pm ie^{-4S_I/\beta_0} \sim \pm i\Lambda^4$, located at

$$t_{\text{ren.}} = \frac{4}{\beta_0} S_I g^2 = \frac{2}{\beta_0} t_{[I\bar{I}]}^R = \frac{6}{11N} t_{[I\bar{I}]}$$

which is much closer to the origin than $t_{[I\bar{I}]}$ is. The renormalon on $\mathbb{R}^4$ is believed to be fixed by an ambiguity in the gluon condensate, $\text{Im} \left[ \frac{1}{N} \text{tr} F_{\mu\nu} F^{\mu\nu} / s \right] \sim \pm \Lambda^4$, as argued in \cite{15} based on OPE analysis of \cite{16}. There is also recent lattice evidence \cite{17, 18}. However, a microscopic mechanism through which this take place in the strong coupling regime is not yet known. For recent works on renormalons, see \cite{19–33}.
Using \( \mathbb{Z}_N \) PSUs

Summing over all backgrounds gives the path integral of a theory with a 1-form gauge invariance. To couple the YM action to \( \mathbb{Z}_N \) PSUs, we consider the following theory:

\[
\sum_{W \in \mathbb{Z}} \int_W \text{da} \quad \text{vs.} \quad \sum_{w_2 \in H^2(M_4, \mathbb{Z}_N)} \sum_{W \in \mathbb{Z}} \int_W \text{da} \tag{4}
\]

In the \( SU(N) \) theory, the integral is over connections in a fixed bundle labelled by \( W \) together with a sum over \( W \in \mathbb{Z} \). In the \( PSU(N)_p \) theory, the integral is over connections in a fixed bundle labelled by \( W \) and \( w_2 \) together with a sum over \( W \in \mathbb{Z} \) and \( w_2 \in H^2(M_4, \mathbb{Z}_N) \). A fixed \( w_2 \in H^2(M_4, \mathbb{Z}_N) \) evaluated on various 2-cycles gives the \( \mathbb{Z}_N \) 't Hooft fluxes of the corresponding bundle.

Start with \( SU(N) \) Yang-Mills theory with action:

\[
S = \frac{1}{2g^2} \int \text{tr}[F \wedge *F] + \frac{i \theta}{8 \pi^2} \int \text{tr}[F \wedge F] \tag{5}
\]

where the second term is the properly quantized topological term, \( \partial Q \in \partial \mathbb{Z} \). This theory has a 1-form \( \mathbb{Z}_N \) symmetry.

The \( \mathbb{Z}_N \) TQFT can be defined as a path integral over a pair of properly quantized fields \((B^{(2)}, B^{(1)})\) with the insertion of the phase \( \exp \left[ \frac{i \theta}{2g^2} \int B^{(2)} \wedge B^{(2)} \right] \). The TQFT has a 1-form gauge invariance. To couple the \( SU(N) \) YM theory to the background \( B^{(2)} \) field, we first define a \( U(N) \) field \( \tilde{a} = a + \frac{1}{N} B^{(1)} \). The 1-form gauge invariant combination is \( \tilde{F} = \tilde{B}^{(2)} \) where \( \tilde{F} \) is the \( U(N) \) field strength. The action of the \( SU(N) \) theory in the TQFT background is [9]:

\[
S[B^{(2)}, \tilde{a}] = \frac{1}{2g^2} \int \text{tr}[(\tilde{F} - B^{(2)}) \wedge *(\tilde{F} - B^{(2)})] + \frac{i \theta}{8 \pi^2} \int \text{tr}[\tilde{F} \wedge (\tilde{F} - B^{(2)})] \tag{6}
\]

Summing over all backgrounds gives the \( PSU(N)_p \) theory:

\[
Z_{PSU(N)_p} = \int DB^{(2)} DB^{(1)} D\tilde{a} \delta(NB^{(2)} - dB^{(1)}) e^{i\theta \tilde{W}} \int B^{(2)} \wedge B^{(2)} e^{-S[B^{(2)}, \tilde{a}]} \tag{7}
\]

Using \( B^{(2)} = \frac{2 \pi}{N} \ell \phi dx_\mu \wedge dx_\mu \), \( \ell = \ell_i \), \( \ell_{ij} = \epsilon_{ijk} m_k \) to denote 't Hooft fluxes, the partition functions for \( SU(N) \), \( SU(N) \) in \( B^{(2)} \) background, and \( PSU(N)_p \) theories can be expressed as:

\[
Z_{SU(N)} = \sum_{W \in \mathbb{Z}} e^{i\theta W} Z_W \tag{8}
\]

\[
Z_{SU(N)}(\ell, m) = \sum_{W \in \mathbb{Z}} e^{i\theta (W + \ell m/n)} Z_W(\ell, m) \tag{9}
\]

\[
Z_{PSU(N)_p} = \sum_{W, \ell, m \in (\mathbb{Z}_N)^3} e^{i\frac{2 \pi}{N} \ell m} e^{i\theta (W + \ell m/n)} Z_W(\ell, m) \tag{10}
\]

The gauging procedure admits the freedom to add a topological phase, a discrete topological theta angle, \( \theta_p = \frac{2 \pi p}{N} \) to each network configuration of topological defects, hence the subscript in \( PSU(N)_p \). Here are three implications of this discussion for the resurgence program [5]:

1) The self-dual saddles in \( PSU(N)_p \) theory are solutions of the self-duality equation:

\[
(\tilde{F} - B^{(2)}) = \mp * (\tilde{F} - B^{(2)}) \tag{11}
\]

The configurations that saturate the BPS bound have action: [6, 7, 34–38]

\[
S = \frac{8 \pi^2}{g^2} \left( W + \frac{\ell \cdot m}{N} \right) \in \frac{S^1}{N} \mathbb{Z}^{\mathbb{Z}_0} \tag{12}
\]

2) If we take a minimal BPS and anti-BPS configuration in \( PSU(N)_p \), we can construct critical points at infinity with \( W = 0 \) and action \( 2S_I/N \) that can be lifted to \( SU(N) \) bundle [38]. This is the first point where our construction is in sharp disagreement with the widely accepted perspective [1, 2, 14, 39] which as- sumes that the leading non-BPS semi-classic configuration in the \( SU(N) \) theory must have an action \( 2S_I \) and as such, semi-classical saddles must be irrelevant to \( IR \)-renormalon problem.

If the theory is in a weakly coupled domain [19, 40], \( 2S_I/N \) configurations produce a singularity in the Borel plane at

\[
t^* = \frac{1}{N} 2S_I = \frac{2}{N} l_{(1)} \tag{13}
\]

One may be tempted to think that despite the proximity to the origin of the Borel plane by a factor of \( N \) relative to \( l_{(1)} \), the mismatch between (11) and (3) is a clear failure, and the singularity (11) is unrelated to the gluon condensate which is believed to be a macroscopic resolution of renormalon ambiguity in YM theory on \( \mathbb{R}^4 \). In fact, there is some criticism of the semi-classical ideas in this regard, which we also address in this paper [41–48].

3) An important step toward the resolution of this issue is from somewhere unexpected: the non-renormalization theorem for the theta angle, which is the coefficient of the topological term. In \( PSU(N)_p \) theory, \( \theta = \theta + 2 \pi N \) and in \( SU(N) \), \( \theta = \theta + 2 \pi \). However, in both cases, the local observables are \( 2 \pi \) periodic, \( multi-branched \) functions, and each branch is \( 2 \pi N \) periodic [49, 50], both in weak and strong coupling. The form of theta angle dependence will provide severe constraints.

From QFT to QM on \((N-1)\)-simplex with TQFT coupling: Consider Yang-Mills theory in the following setting. First compactify the theory on \( \mathbb{R}^4 \) down to small \( \mathbb{R}^3 \times S^1 \) and insert a double-trace deformation on \( S^1 \) to stabilize the 0-form part of center symmetry. The resulting theory satisfies adiabatic continuity: it is continuously connected to pure YM on \( \mathbb{R}^4 \) in the sense of
all gauge invariant order parameters [51, 52]. This proposal is tested on the lattice, and it works [53–55]. Then, further compactify the theory on $\mathbb{R} \times T^2 \times S^1_L$, where the $T^2$ size is of order $LN$ for reasons to be explained. We map this system to quantum mechanics within the Born-Oppenheimer (BO) approximation. Finally, when we study the partition function in QM, we compactify the Euclidean time direction, and study the theory on the 4-manifold

$$M_4 = S^2_{\beta} \times T^2_{LN} \times S^1_L, \quad \Lambda \ll 1, \quad \beta \Lambda \gg 1,$$

with all TQFT backgrounds turned on.

Let us turn on a background magnetic ’t Hooft and GNO flux [6, 56] through the the 12-plane:

$$\frac{N}{2\pi} \int_{\Sigma_{12}} B^{(2)} = k \equiv \ell_{12}, \quad \Phi = \int_{\Sigma_{12}} B = \frac{2\pi}{g} \mu_k$$

where $\mu_k \in \Gamma_{\nu}^\nu$ is an element of the co-weight lattice. The energy of the background flux configuration is [57]:

$$E = \frac{1}{2} \int_{\Sigma_{12}} B^2 = \frac{\Phi^2}{2 \text{Area}(T^2)}$$

In both $SU(N)$ and $PSU(N)_{p}$ theory, the dynamical monopoles are elements of the co-root lattice, $\alpha \in \Gamma_{\nu}^\nu$.

Take $k = 1$. First, we note that there are two classes of tunneling events [38]. Since the energy depends on the area of $T^2$ inversely as in (14), one type of tunneling is among the states which become degenerate only in $\text{Area}(T^2) \to \infty$ limit (see [57], page 226). However, in the flux background (13), there exist $N$ exactly degenerate minima even at finite $\text{Area}(T^2)$, given by:

$$\{\nu_a\} \equiv \{\nu_1 - \alpha_1, \ldots - \alpha_{a-1}\} \quad a = 1, \ldots, N$$

These states can be visualized as the $N$ vertices of an $(N-1)$-simplex. Despite the fact that these points form an $(N-1)$-simplex, the system does not have the $S_N$ permutation symmetry. Only the tunneling between adjacent pairs are minimal and all others are non-minimal action. The $Z^0$ center-symmetry acting on the Polyakov loop along $S^1_L$ cyclically permutes these tunneling events. The minimal tunneling events form a closed loop in field space:

$$|\nu_1\rangle \xrightarrow{+} |\nu_2\rangle \xrightarrow{+} \cdots \xrightarrow{+} |\nu_N\rangle \xrightarrow{+} |\nu_1\rangle$$

Note that the vacuum of YM theory on $M_4$ with flux (13) is identical to the $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1_L$ with twisted boundary condition [21, 22, 58, 59]. If we apply the BO approximation a second time, see Fig. 1, Yang-Mills theory maps to a particle on a circle with $N$ minima. In the latter, Ref. [60], generalizing [61], proved all-orders resurgent cancellations by using exact WKB. What is the implication of this all-orders resurgent cancellation for IR-renormalon problem?

**Gluon condensate from small to large-$L$:** We can first calculate the gluon condensate on $T^3_{\text{large}} \times S^1_L$ at small-$L$ by weak coupling methods. Then, by using dimensional analysis and multi-branch structure of theta angle dependence, we show compatibility with large $T^3_{\text{large}}$ limit. The hero of this story is non-renormalization of theta, and multi-branching structure of the condensate.

In the first two orders in semi-classics, we can consider the Euclidean description of the vacuum of the quantum mechanical system as a dilute gas of fractional instantons and bions. There are $N$ types of fractional-instantons $M_i$ with topological charge $Q = 1/N$, and complex fugacity $e^{-S_i/N + i\theta}/N$. The fractional topological charge follows from $\frac{N}{2\pi} \int B^{(2)} \wedge B^{(2)} \in \frac{1}{N} \mathbb{Z}$ which is valid both at small and large $M_4$ where semi-classics is no longer valid [38]. There are also $N$ types of neutral bions $B_{i \pm} = [M_i, \overline{M}_i]_{\pm}$ with zero topological charge, but action $2S_i/N$. These configurations in quantum mechanics are attached to critical points at infinity [62, 63]. Integration over the quasi-zero mode Lefschetz thimble produces a two-fold ambiguous result, whose imaginary part is $\pm i e^{-2S_i/N}$. What do we make of this at strong coupling?

Due to the trace anomaly, the gluon condensate is proportional to minus vacuum energy density. Vacuum energy for QM shown in Fig. 1 is easiest to diagonalize using the tight binding Hamiltonian. Incorporating first and second order effects in semi-classics, we find:

$$\left\langle \frac{1}{N} \text{tr} F^2 \right\rangle_{\pm} = \max_q \left[ c_1 \Lambda^{\frac{1}{4}} \mathcal{R}^{-\frac{1}{4}} \cos \left( 2\pi q + \theta \right) \frac{N}{\Lambda} \right]$$

$$+ c_2 \Lambda^{\frac{1}{4}} \mathcal{R}^{\frac{1}{2}} \cos \left( 2\pi q + \theta \right) + \cdots$$

$$+ c_3 \Lambda^{\frac{1}{4}} \mathcal{R}^{\frac{3}{2}} \pm i \pi c_4 \Lambda^{\frac{3}{4}} \mathcal{R}^{\frac{3}{2}} + \cdots$$

where $\mathcal{R} \equiv \mathcal{R}(LN)$ is the the fractional instanton size function, proportional to $LN \sim n_{\text{min}}$ in the semi-classical
regime. On large $T_4$, the condensate is expected to be:

$$\left\langle \frac{1}{N} \text{tr} F^2 \right\rangle \pm \theta = \max_q \Lambda^4 h(\frac{\theta + 2\pi q}{N}) \pm ia\Lambda^4,$$  \hspace{1cm} (18)

where $h(\theta/N)$ is $2\pi N$ periodic function [49, 50]. Here comes the important part of the story.

**Fourier vs. Lefschetz decomposition:** The appearance of $e^{i\frac{\pi}{4} k}$, $k \in \mathbb{Z}$ in (17) arises from calculable semi-classics, both on $M_4$ (12) and $\mathbb{R}^3 \times S^1$. This is the weak coupling regime. The function that appears in (18) $h(\theta/N)$ can be expressed in terms of a complete Fourier basis:

$$\{\text{Span}(e^{i\frac{\pi}{4} k}), k \in \mathbb{Z}\}.$$

The decomposition into $e^{i\frac{\pi}{4} k}$, $k \in \mathbb{Z}$ is inevitable, either on small or large $M_4$ as a consequence of thinking $SU(N)$ theory in TQFT backgrounds. All local non-perturbative observables must be multi-branched where each branch can be decomposed in the basis (19).

$e^{\pm i\frac{\pi}{4}}$ is minimally accompanied with $e^{-S_I/N}$ in weak coupling, and the expansion organizes itself in positive integer powers of $\Lambda^{11/3}$. At second order in semi-classics, an important effect is due to neutral bions. Since $\text{Arg}(g^2) = 0$ is a Stokes line, the neutral bion amplitude is two-fold ambiguous [62]. This is of order $\pm ie^{-2S_I/N} \sim \pm i\Lambda^{22/3}$ and is the configuration that enters to all orders resurgent cancellations [60] in QM limit.

However, the expected ambiguity in the condensate on $\mathbb{R}^4$ is of the form $\pm ie^{-4S_I/\beta_0}$, corresponding to $\pm i\Lambda^4$. Wouldn’t it be nicer if semi-classics produced $\Lambda^4$ factors as in (18)? The answer is, no. The only terms allowed in the semi-classical expansion compatible with $PSU(N)_p$ bundle are of the form (set $p = 0$ for convenience):

$$\left\{ \Lambda^{\frac{n+\pi}{2}} R^{\frac{(n+\pi)}{2}} e^{i\frac{\pi}{4} (n-\pi)} \right\}$$

where $n \geq 0, \pi \geq 0, (n, \pi) \neq (0, 0)$. Based on this, it is clear that if we were to obtain $\Lambda^4$ in semiclassical domain of $PSU(N)_0$ theory, it would accompany $e^{i\frac{\pi}{4} k}$ and be a contradiction with $2\pi N$ periodicity. Therefore, it is not only futile, it is in fact wrong to look for a saddle which would produce $\pm ie^{-4S_I/\beta_0}$, answering a concern in [48].

The $\theta$-dependence of the semi-classical expansion (20) can also be viewed as a consequence of bundle topology in $PSU(N)_p$ theory. Dimensional analysis tells us that the monomials (20) must be accompanied with powers $R \sim LN$ to match the scaling dimension of the condensate. However, the general form of $R(LN)$-function is unknown.

The completeness of the Fourier basis and its matching with the Lefschetz thimble decomposition of the partition function in terms of critical points (including the ones at infinity) gives us a realistic hope that the Lefschetz decomposition of path integral may well be complete. (See also [21, 64–67] for similar standpoint in different theories.) The Lefschetz decomposition is exact in QM (which is a limit of YM on $T^3 \times \mathbb{R}$) as per translation of exact WKB result to path integral [60], and also produce the correct non-perturbative dynamics at small $S^4 \times \mathbb{R}^3$ as reviewed in [5]. The real difficulty is to understand the implications of this decomposition at strong coupling.

What happens as $\Lambda LN$ gets larger? In the $\Lambda LN \gg 1$ regime, the theory must gradually become a one-scale problem, dependent only on $\Lambda$, and the condensate must saturate to its value on $\mathbb{R}^4$. In the large-$N$ limit, this fact is guaranteed by large-$N$ volume independence, which is proven by using lattice field theory [51, 68, 69]. The semi-classical BO-approximation used to derive (20) is valid provided $\Lambda LN \lesssim 1$.

Although semi-classical effective field theory is not sufficient to address what happens as the theory moves from the semi-classical regime (17) to the strong coupling regime (18), there is a sense in which it can be useful. On $T^3_{large} \times S^1_L$ emulating $\mathbb{R}^3 \times S^1_L$, the Debye length is $\xi \equiv m_{\gamma}^{-1} \sim \Lambda^{-1}(\Lambda R)^{-5/6}$ due to the proliferation of the fractional instantons [51]. The size of the fractional instantons in weak coupling is $R \sim LN$. Polynkov argues that (see [70] page 91), if the finite correlation length cuts off the growth of the fractional instanton size, the EFT pushed to the boundary of its region of validity can provide a *qualitatively* accurate description. We are in search of a qualitative description after all, and take this as our assumption. Given the assumption, the condensate saturates to expected behavior on $\mathbb{R}^4$

$$\left\langle \frac{1}{N} \text{tr} F^2 \right\rangle \pm \theta = \max_q \Lambda^4 \left( \sum_{k \in \mathbb{Z}} c_k e^{i\frac{\theta + 2\pi q}{N}}k \right) \pm ia\Lambda^4$$

At this stage, the ambiguities generated by neutral bions (11) source and transmute to the ambiguities that are associated with condensates (3), and location of leading singularity in Borel plane flows as:

$$r^{\mathbb{R}^4 \times S^1}_{\text{bien}} = \frac{2S_I}{N} \implies r^{\mathbb{R}^4}_{\text{ren.}} = \frac{4S_I}{\beta_0}$$

In fact, neutral bions provide a match to all IR-renormalon singularities located at $t^{\mathbb{R}^4}_{\text{ren.}} = \frac{2S_I}{\beta_0} (4+2a) a = 0, 1, 2, \ldots$. The non-trivial aspect of (21) is that Lefschetz decomposition of saddles extrapolated at the boundary of EFT matches Fourier decomposition of non-trivial multi-branched functions at strong coupling! This perspective provides a microscopic definition of IR-renormalon.

**Definition:** The IR-renormalons in 4d gauge theories are composites of (semi-classical) fractional instantons configuration in the $PSU(N)_p$ bundle that lifts to $SU(N)$ bundle, and that are dressed by strong dynamics. The effect of dressing is to saturate the $R$-function around the scale $\Lambda^{-1}$, if the running coupling develops a singularity (pole, branch point) thereof.

As far as we can see, the saturation of $R$-function [70] is a unique way to make 1) theta angle periodicities 2)
multi-branched structure [49, 50], 3) weak coupling analysis on $\mathbb{R}^3 \times S^1$ [20, 21] and 4) strong coupling expectation on $\mathbb{R}^4$ [1, 2] simultaneously consistent.

Examples: We can present three classes of examples that shows that this construction is correct more broadly.

Coupling the 2d $\mathbb{CP}^N$ QFT to a $\mathbb{Z}_N$ TQFT, the gauge field is classified according to both instanton density and $w_2 \in H^2(M_2, \mathbb{Z}_N)$; hence, topological charge is quantized in units of $\frac{1}{N}$. On $\mathbb{R}^2$, the ambiguity in the spin-wave condensate is given by $\text{Im}(\text{d}x_2) \sim \pm i u^2 e^{-2S_1/\beta_0} = \pm i \Lambda^2$ [3]. A pair of minimal BPS and anti-BPS $U(1)/\mathbb{Z}_N$ configurations that can be lifted to $U(1)$ induces an ambiguity $\pm i u^2 e^{-2S_1/N} = \pm i \Delta^2$ due to the fact that $\beta_0 = N$. This is the IR-renormalon.

If we reduce 4d $\mathcal{N} = 1$ $SU(N)$ SYM theory and 2d $\mathcal{N} = (2, 2) \mathbb{CP}^{N-1}$ models down to QM with the $B(2)$ background, both reduce to extended supersymmetric $\mathcal{N} = 2$ QM. In $\mathcal{N} = 2$ QM, there are two types of neutral bions, both of which are ambiguity free, and the vacuum energy vanishes due to a relative hidden topological angle between two types of saddles [71], consistent with the vanishing of condensates (and their ambiguities) [72].

In QCD with $n_W^\text{adj} = 5$ adjoint Weyl and $N_f = (\frac{1}{2} - \epsilon)N$ fundamental Dirac fermions, which can also be coupled to a $\mathbb{Z}_N$ TQFT [73], the ratio $\xi/m_g \sim \epsilon^{2g_s^2/g_s^N} \gg 1$ as $L \rightarrow \infty$, where $g_s^2N \sim \epsilon$ is the value of the coupling in a Banks-Zaks type IR-fixed point [74]. Therefore, semi-classics is valid at any size $S^1_L \times \mathbb{R}^3$. The fractional instantons and bions contribution becomes irrelevant as $L \rightarrow \infty$ as $(LN)^{-4e^{-8\pi^2/\epsilon}} \rightarrow 0$. The condensate vanishes in $\mathbb{R}^4$ limit [72], providing a semi-classical proof for the absence of IR-renormalon in IR-CFTs.

Summary: The fixing of the Borel resummation ambiguities in $\mathbb{R}^4$ via the condensates and the fixing of it in small $M_4$ or $\mathbb{R}^3 \times S^1_L$ via neutral bions are not different mechanisms. Since the gluon and wave-scan condensate are not protected quantities, they have $L$ dependence in the weak coupling domain. But the crucial point is that the non-trivial theta angle dependence of basis functions for Lefshetz decomposition of semi-classics and Fourier decomposition of the expected multi-branched strong coupling result match each other, and does not change with $L$. Our construction shows both the presence of IR-renormalons, along its flow from semi-classical result $\frac{1}{N} t_{[1]}$ to strong coupling result $\frac{1}{N} t_{[1]}$ in Yang-Mills theory. It also proves the absence of IR-renormalon in an IR-CFT in decompactification limit. We believe this reconciles the traditional perspective [1, 2] with the modern semi-classical/TQFT perspective based on neutral bions [19, 21].

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