THE RATIONALITY OF STARK–HEEGNER CYCLES ATTACHED TO BIANCHI MODULAR FORMS – THE BASE–CHANGE SCENARIO.

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Abstract. We study Stark–Heegner cycles attached to Bianchi modular forms, that is automorphic forms for GL(2) over an imaginary quadratic field $F$. The Stark–Heegner cycles are local cohomology classes in the $p$-adic Galois representation associated to the Bianchi eigenform. They are conjectured to be the restriction (at a prime $p$) of global cohomology classes in the (semistable) Bloch–Kato Selmer group defined over ring class fields of a relative quadratic extension $K/F$. In this article, we show that these conjectures hold when the Bianchi eigenform is the base-change of a classical elliptic cuspform.

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1. Introduction

The arithmetic theory of automorphic $L$-functions has been the subject of mathematical research for long. The Birch & Swinnerton-Dyer Conjecture (BSD) and its generalization, the Bloch-Kato Conjecture predict a mysterious relationship between the arithmetic of an automorphic form and the special values of its $L$-function. For instance, if $E/\mathbb{Q}$ is an elliptic curve over the field of rational numbers, then BSD predicts that the order of vanishing of the Hasse-Weil $L$-function at $s=1$ (analytic rank) equals the Mordell-Weil rank of the group of rational points (algebraic rank), i.e.

$$\text{ord}_{s=1} L(E, s) = \text{rank}_\mathbb{Z} E(\mathbb{Q}).$$

When the analytic rank is precisely one, the theory of Complex Multiplication via Heegner points plays a key role in the celebrated proof of BSD by Gross-Zagier ([GZ86]) and Kolyvagin ([Kol88]). In scenarios that go beyond the realms of the theory of Complex Multiplication, Darmon ([Dar01]) used $p$-adic methods to construct local points in the Mordell-Weil group of the curve, known as Stark-Heegner points. These points are conjectured to be global points and satisfy a reciprocity law under Galois automorphisms similar to the ones satisfied by Heegner points.

When $E$ is replaced by the Galois representation $\rho/K$ attached to the quadratic base-change of a Bianchi modular form (i.e. a modular form over an imaginary quadratic field $F$), then the Bloch-Kato Conjecture predicts that the order of vanishing of its $L$-function at critical values equals the rank of a Selmer group attached to $\rho/K$. Inspired by the ideas of Darmon and generalizing earlier works of Trifković ([Tri06]) & Rotger-Seveso ([RS12]), we construct local Selmer classes via Stark-Heegner cycles in [VW21]. We then conjecture these local classes to be global Selmer classes that satisfy a reciprocity isomorphism as in [Dar01]. The main aim of this article (Theorem 1.4) is to show that the global rationality conjecture of [VW21] (Conjecture 6.8) holds when the Bianchi eigenform is the base-change to the imaginary quadratic field $F$ of a classical elliptic eigenform. We also record the consequences of Theorem 1.4 towards Trifković’s global rationality conjecture for Stark-Heegner points attached to (modular) elliptic curves defined over imaginary quadratic fields ([Tri06 Conjecture 6]) in §1.3 below. In particular, our results fit within Darmon’s conjectural program initiated in [Dar01].
1.1. Set-up. Let $p$ be a rational prime and fix embeddings $\iota_{\infty}: \mathbb{Q} \to \mathbb{C}$ and $\iota_p: \mathbb{Q} \to \overline{\mathbb{Q}}_p$ once and for all. Let $F$ be an imaginary quadratic field with ring of integers $\mathcal{O}_F$ and discriminant $D_F$ such that $p$ is unramified in $F$, and we denote by $p$ to be the prime above $p$ in $\mathcal{O}_F$ corresponding to $\iota_p$. For ease of notation, we assume that the class number of $F$ is one. Let $\mathcal{F} \in S_{k_0+2}(U_0(N))^{\text{new}}$ be a Bianchi cuspidal newform of (parallel) even weight $k_0 + 2$ and square-free level $N = pM \subseteq \mathcal{O}_F$ (such that $(p, M) = 1$). Let $K/F$ be a quadratic extension of $F$ of relative discriminant (resp. absolute discriminant) $D_{K/F}$ (resp. $D_K$) relatively prime to $N$. Further we assume that $K$ satisfies the following Stark–Heegner hypothesis ($\text{SH-Hyp}$)

- $p$ is inert in $K$
- All primes $l | M$ split in $K$

Under ($\text{SH-Hyp}$), the sign of the functional equation of the base-change $L$-function $L(\mathcal{F}/K, s)$ is $-1$ and in particular forces the vanishing of the central critical value, i.e.

$$L(\mathcal{F}/K, k_0/2 + 1) = 0$$

This allows us to force higher orders of vanishing over ring class extensions of the field $K$. Let $\mathcal{C} \subseteq \mathcal{O}_F$ be any ideal relatively prime to $ND_{K/F}$ and let

$$\mathcal{O}_\mathcal{C} := \mathcal{O}_F + \mathcal{C} \mathcal{O}_K$$

be the $\mathcal{O}_F$-order of conductor $\mathcal{C}$ in $K$. Let $H_\mathcal{C}/K$ be the ring class field of conductor $\mathcal{C}$ and let $G_\mathcal{C} := \text{Gal}(H_\mathcal{C}/K)$ which we know by global class field theory is isomorphic to $\text{Pic}(\mathcal{O}_\mathcal{C})$. For any character $\chi: G_\mathcal{C} \to \mathbb{C}^\times$, the sign of the twisted $L$-series $L(\mathcal{F}/K, \chi, s)$ is again $-1$. Further, the $L$-series admits a factorisation

$$L(\mathcal{F}/H_\mathcal{C}, s) = \prod_{\chi \in G_\mathcal{C}^\vee} L(\mathcal{F}/K, \chi, s)$$

and it follows that

$$\text{ord}_{s = k_0/2 + 1} L(\mathcal{F}/H_\mathcal{C}, s) \geq h(\mathcal{O}_\mathcal{C}) := |G_\mathcal{C}|$$

We denote by $V_p(\mathcal{F})$ to be the two dimensional $G_\mathcal{F} := \text{Gal}(\overline{\mathbb{Q}}/F)$-representation attached to $\mathcal{F}$, taking values in a finite extension $L/\mathbb{Q}_p$. The Bloch–Kato conjecture then predicts the existence of a family of non-trivial cohomology classes

$$\{ s_\mathcal{C} \in \text{Sel}_{nt}(H_\mathcal{C}, V_p(\mathcal{F})(k_0/2 + 1)) \}$$

over towers of class fields $H_\mathcal{C}$ for $\mathcal{C}$ relatively prime to $ND_{K/F}$. Following the ideas of [Dar01] and [RS12], in [VW21] we proposed conjectural candidates for such a family of cohomology classes, viz. Stark–Heegner cycles which can be regarded as local cohomology classes

$$s_\Psi \in H^1_{nt}(L, V_p(\mathcal{F})(k_0/2 + 1))$$

associated to optimal embeddings of $\mathcal{O}_F[1/p]$-orders. See §2.6 below where we briefly recall the construction of Stark–Heegner cycles. The aim of this article (See Theorem 1.4 below) is to give some evidence for Conjecture 6.8 of [VW21] by showing that the Stark–Heegner classes are in fact (the restriction at $p$ of) global Selmer classes in the base-change scenario.

\footnote{In [VW21], we denoted the form to have parallel weight $(k_0, k_0)$.}
Before stating Theorem 1.4 precisely, we introduce some notation. Let \( \mathcal{R} \) be the Eichler \( \mathcal{O}_F[1/p] \)-order in \( M_2(\mathcal{O}_F[1/p]) \) of \( 2 \times 2 \) matrices that are upper triangular modulo \( \mathcal{M} \) and let \( \Gamma := \mathcal{R}^\times \) be the set of invertible matrices of \( \mathcal{R} \) of determinant 1. Let \( \Psi : \mathcal{O}_C \rightarrow \mathcal{R} \) be an optimal embedding of \( \mathcal{O}_C \) – an \( \mathcal{O}_F[1/p] \)-order in \( K \) of conductor \( \mathcal{C} \) prime to \( \mathcal{N} \mathcal{D}_K \). The Stark–Heegner (or Darmon) cycle is then a homology class
\[
D_\Psi \in (\Delta_0 \otimes \text{Div}(\mathcal{H}^\text{ur}_p) \otimes V_{k_0,k_0})_\Gamma
\]
where \( \Delta_0 := \text{Div}^0(\mathbb{P}^1(F)) \). \( \text{Div}(\mathcal{H}^\text{ur}_p) \) denotes the subgroup of divisors supported on \( \mathcal{H}^\text{ur}_p := (\mathbb{P}^1(\mathbb{Q}^\text{ur}_p) \setminus \mathbb{P}^1(F_p)) \text{Gal}(\mathbb{Q}^\text{ur}_p/L^0) \) where \( L^0 := L \cap \mathbb{Q}^\text{ur}_p \), and \( V_{k_0,k_0} := V_{k_0} \otimes V_{k_0} \) where \( V_{k_0} \) is the ring of homogenous polynomials of degree \( k_0 \) in two variables with coefficients in \( L \). This space should be regarded as an explicit substitute for the local Chow group.

In [VW21], we developed a ‘modular symbol theoretic’ \( p \)-adic integration theory following [Sev12] (See §2.3 below). Our \( p \)-adic integration theory can be regarded as a morphism
\[
\Phi^\sigma_p : (\Delta_0 \otimes \text{Div}^0(\mathcal{H}^\text{ur}_p) \otimes V_{k_0,k_0})_\Gamma \rightarrow D_{p,L}^\sigma/\text{Fil}^{k_0+2}(D_{p,L})
\]
for each embedding \( \sigma : F_p \rightarrow L \). Here \( D_{p,L}^\sigma \) is a two-dimensional filtered \( L \)-vector space built from the space of overconvergent Bianchi modular symbols over the Bruhat–Tits tree \( \mathcal{T}_\sigma \) for \( \text{GL}_2/F_p \) associated to \( \mathcal{F} \), which we denote by \( \text{MS}_\Gamma(L)_\sigma \). In [VW21], we showed that \( D_{\mathcal{F},L} := \bigoplus_{\sigma} D_{\mathcal{F},L}^\sigma \in \text{MF}(\varphi,N,F_p,L) \) – the category of filtered Frobenius modules over \( F_p \) with coefficients in \( L \). The \( p \)-adic Abel–Jacobi map that we construct is a lift of \( \Phi^\sigma_p \) (See Theorem 2.20 below)
\[
\Phi^{A\sigma}_p : (\Delta_0 \otimes \text{Div}(\mathcal{H}^\text{ur}_p) \otimes V_{k_0,k_0})_\Gamma \rightarrow D_{\mathcal{F},L}^\sigma/\text{Fil}^{k_0+2}(D_{\mathcal{F},L})
\]
removing the condition on the degree of divisors on \( \mathcal{H}^\text{ur}_p \). In §3 below, we show that the \( (D_{\mathcal{F},L}^\sigma/\text{Fil}^{k_0+2}) \)-valued \( p \)-adic integration theory can be realized as an \( \text{MS}_\Gamma(L)_\sigma^{-1} \)-valued integration theory via
\[
\log \Phi^{A\sigma}_p : (\Delta_0 \otimes \text{Div}(\mathcal{H}^\text{ur}_p) \otimes V_{k_0,k_0})_\Gamma \rightarrow D_{\mathcal{F},L}^\sigma/\text{Fil}^{k_0+2}(D_{\mathcal{F},L}) \cong \text{MS}_\Gamma(L)_\sigma
\]
We also define
\[
\log \Phi^{A\sigma} := \sum \sigma \log \Phi^{A\sigma}_p
\]
and show that there exists a \( (D_{\mathcal{F},L}^\sigma/\text{Fil}^{k_0+2}) \)-valued integration theory, viz. \( \Phi^{A\sigma} \), equivalent to \( \log \Phi^{A\sigma} \) for any choice of \( \sigma : F_p \rightarrow L \) (See Remark 3.3 in particular).

Let \( D_{\mathcal{F}} := D_{\text{st}}(V_p(\mathcal{F})) \in MF(\varphi,N,F_p,L) \) be Fontaine’s semistable Dieudonné module attached to the local Galois representation \( V_p(\mathcal{F}) \mid_{G_{F_p}} \). The trivial zero conjecture (See [VW21], Conjecture 4.2) would then afford a \( (\varphi,N) \)-module (over \( F_p \) with coefficients in \( L \)) isomorphism
\[
D_{\mathcal{F}} \cong D_{\mathcal{F}}
\]
which induces an identification of the tangent spaces
\[
\frac{D_{\mathcal{F},L}^{k_0+2}}{\text{Fil}^{k_0+2}(D_{\mathcal{F},L})} \cong \frac{D_{\mathcal{F},L}^{k_0+2}}{\text{Fil}^{k_0+2}(D_{\mathcal{F},L})} \cong \text{H}_{\text{st}}^1(L,V_p(\mathcal{F})(k_0/2 + 1))
\]
where \( D_{\mathcal{F},L}^{k_0+2} := D_{\mathcal{F}} \otimes F_p \otimes L,\sigma L \) and the last isomorphism is given by the Bloch–Kato exponential. Let \( \chi : \text{Gal}(H_C/K) \rightarrow \mathbb{C}^\times \) be any character. Then, the \( \chi \)-twisted Stark–Heegner cycle is defined
as

\[ D_\chi := \sum_{\sigma \in \text{Gal}(H_C/K)} \chi^{-1}(\sigma)D_{\text{rec}^{-1}(\sigma)\psi} \in (\Delta_0 \otimes \text{Div}(\mathcal{H}^\mu) \otimes V_{k_0,k_0})_F \otimes \chi, \]

where \((-) \otimes \chi\) denotes suitable scalar extension by \(\chi\). Let \(H_\chi\) denote the abelian subextension of \(H_C\) cut out by the character \(\chi\). Then similar to \([VW21, \text{Conjecture 6.8}]\), we may formulate

**Conjecture 1.1.** There exists a global Selmer class \(S_\chi \in \text{Sel}_\text{st}(H_\chi, V_p(\mathcal{F})(k_0/2 + 1))^\chi\) such that

\[ \exp_{BK} \circ \varphi (\Phi^{AJ}(D_\chi)) = \text{res}_p (S_\chi) \]

where \((-)^\chi\) denotes the \(\chi\)-isotypic component.

**Remark 1.2.** The formulation of Conjecture 1.1 above is slightly different from that in \([VW21, \text{Conjecture 6.8}]\) which asserts the global rationality of \(\Phi^{AJ}_\sigma(D_\chi)\) over each embedding \(\sigma : F_p \hookrightarrow L\). As explained in \([3]\) below, \(\Phi^{AJ}_\sigma(D_\chi)\) should be considered as the sum of \(\Phi^{AJ}_\sigma(D_\chi)\) over all possible embeddings \(\sigma : F_p \hookrightarrow L\).

**Remark 1.3.** When \(\chi = \chi_{\text{triv}} : \text{Gal}(H_K/K) \to \mathbb{C}^\times\) – the trivial character thought of as an unramified character associated to the maximal order \(\mathcal{O}_C = \mathcal{O}_K\), then we denote the Stark–Heegner cycle \(D_\chi := \sum_{\sigma \in \text{Gal}(H_K/K)} D_{\text{rec}^{-1}(\sigma)\psi}\) simply by \(D_1\). Here \(H_K\) is the Hilbert class field of \(K\) and \(H_\chi = K\).

### 1.2. Main results

Now suppose that \(\mathcal{F} \in S_{k_0+2}(U_0(N))^\text{new}\) is the base-change of an elliptic cuspidal newform \(f \in S_{k_0+2}(\Gamma_0(N))^\text{new}\), where \(N = NO_F\). Note that this is always the case when \((N, D_F) = 1\) which we assume to hold. Since the base-change \(\mathcal{F}\) is cuspidal, we know that \(f\) doesn’t have CM by the imaginary quadratic field \(F\). By Atkin–Lehner–Li theory, we know that \(a_p(f) = -\omega_p p^{k_0}/2\) where \(-\omega_p\) is the eigenvalue of the Atkin-Lehner involution \(W_p\). We shall assume that \(\omega_p = -1\) throughout (i.e. \(f\) has split multiplicative reduction at \(p\)). Further assume that the level \(N\) admits a factorization of relatively prime integers

\[ N = pM = pN^+N^- \]

such that the following Heegner hypothesis holds (**Heeg–Hyp**):

- \(p\) is inert in \(F\)
- All primes dividing \(N^+\) (resp. \(N^-\)) split (resp. are inert) in \(F\)
- \(N^-\) is the square–free product of an odd number of primes

Let \(V_p(f)\) denote Deligne’s two dimensional \(p\)-adic \(G_F\)-representation attached to the newform \(f\). Note that \(V_p(\mathcal{F}) = V_p(f)|_{G_F}\) as \(p\)-adic \(G_F\)-representations. Let \(\omega_M\) be the eigenvalue of the Atkin–Lehner involution \(W_M\) acting on \(S_{k_0+2}(U_0(N))^\text{new}\). The main result of this article is to shed some evidence towards Conjecture 1.1 formulated above.

**Theorem 1.4.** With notation as above, suppose that \(\omega_M = (-1)^{k_0+2}.\) Then there exists a global Selmer class

\[ S_K \in \text{Sel}_\text{st}(K, V_p(\mathcal{F})(k_0/2 + 1)) \]

such that

\[ \exp_{BK} \circ \varphi (\Phi^{AJ}(D_1)) = \text{res}_p (S_K) \]
Theorem 1.5. described above. More precisely, we show that $p\text{K/F}$ ture 5.10 and Theorem 5.13 below). When $p\text{L}$ construct a base-change $p\text{K/F}$ to the arithmetic subgroup $\Gamma$. Let $\text{L} \subseteq W$ $\text{F}$, $\text{U} \subseteq W$ $\text{F}$, $\text{V} \subseteq W$ $\text{F}$ describe. Let $\text{L} \subseteq W_{\text{F,par}} \subseteq W_{\text{F}}(L)$ be a slope-adapted affinoid centred around the point $\lambda_{k_0}$. In §5.4 we construct a base-change $p\text{-adic}$ $\text{L}$-function

$$L_p(\mathcal{F}/K, \chi, -) : U \to \mathbb{C}_p$$

that we expect to interpolate the central critical $\text{L}$-values $L^{\text{alg}}(\mathcal{F}_K/K, \chi, k/2 + 1)$ (See Conjecture 6.10 and Theorem 6.13 below). When $K/F$ is a relative quadratic extension that satisfies (SH-Hyp) mentioned above, we show a $p\text{-adic}$ Gross–Zagier type formula relating the second derivative of this $p\text{-adic}$ $\text{L}$-function to the $p\text{-adic}$ Abel–Jacobi image of the Stark–Heegner cycles described above. More precisely, we show that

**Theorem 1.5.**

$$\frac{d^2}{d\lambda_k^2} [L_p(\mathcal{F}/K, \lambda_k)]_{\kappa = k_0} = \begin{cases} 2 \left( N_{F/Q}(D_{K/F}) \right) \frac{k_0}{2} (\log \Phi_{\mathcal{F}}(D_1)(\Phi_{\mathcal{F}}^{\text{har}}))^2 & \text{if } \omega_M = (-1)^{k_0+2} \\ 0 & \text{if } \omega_M = (-1)^{k_0} \end{cases}$$

where $\Phi_{\mathcal{F}}^{\text{har}} \in \text{MS}_{\mathcal{F}}(L)(\mathcal{F})$ is the harmonic modular symbol attached to $\mathcal{F}$ in §2.4.

Let $\epsilon_K/F$ be the quadratic idèle class character of $F$ that cuts out the relative quadratic extension $K/F$. In §5.5 we show a $p\text{-adic}$ Artin formalism for the base-change $p\text{-adic}$ $\text{L}$-function described above.

**Theorem 1.6.** There exists a $p\text{-adic}$ analytic function $\eta(\kappa) : U \to \mathbb{C}_p$ such that for all $\lambda_k \in U$,

$$L_p(\mathcal{F}/K, \lambda_k) = \eta(\kappa)L_p(\mathcal{F}, \lambda_k)L_p(\mathcal{F}, \epsilon_K/F, \lambda_k)$$

Recall the factorization $N = pM = pN^+N^-$. Let $\mathcal{B}$ be the indefinite quaternion algebra ramified at the primes dividing $pN^-$. Let $X := X_{N^+, pN^-}$ be the Shimura curve associated to $\mathcal{B}$ and an Eichler order of level $N^+$ in $\mathcal{B}$. Let $M_{k_0}/\mathbb{Q}$ be the Chow motive attached to the space of weight $k_0 + 2$ modular forms on the Shimura curve $X$ and let $CH^{k_0+2}(M_{k_0} \otimes F)$ denote the Chow group of co-dimension $k_0/2 + 1$ cycles of $M_{k_0}$ base-changed to $F$. Let $M_{k_0+2}(\Gamma', L)$ denote the space of rigid analytic modular forms, over $L$, on the Mumford curve $X_{\Gamma'}$ associated to the arithmetic subgroup $\Gamma'$ defined in §5.3. The Mumford curve $X_{\Gamma'}$ can be identified with the rigid analytification – $X^{\text{an}}$ of the Shimura curve $X$ via the Cerednik–Drinfeld Theorem of $p\text{-adic}$ uniformization. Then the $p\text{-adic}$ étale Abel–Jacobi map described in §5.3 can be regarded as

(1.3) \[ \log \text{cl}_{f, L} : CH^{k_0+2}(M_{k_0} \otimes F) \to M_{k_0+2}(X, L)^{\text{new}} \to M_{k_0+2}(\Gamma', L)^{\text{new}}. \]

where $f^{\text{rig}} \in M_{k_0+2}(X, L)$ (resp. $f^{\text{rig}} \in M_{k_0}(\Gamma', L)$) is the modular form on the Shimura curve $X$ (resp. on the Mumford curve $X_{\Gamma'}$) associated to $f \in S_{k_0+2}(\Gamma_0(N))^{\text{new}}$ via the Jacquet–Langlands correspondence (resp. via the Cerednik–Drinfeld $p\text{-adic}$ uniformization Theorem).

Let $\mathcal{Y} \in CH^{k_0+2}(M_{k_0} \otimes F)$ be the Heegner cycle constructed in [LS03, §8] using the theory of Complex Multiplication. We then have the following result of M. Seveso (Sev14) on the $p\text{-adic}$ Abel–Jacobi image of the Heegner cycle :-

**Theorem 1.7.**

$$\frac{d^2}{d\lambda_k^2} [L_p(\mathcal{F}, \lambda_k)]_{\kappa = k_0} = \frac{d^2}{d\lambda_k^2} [L_p(f/F, \lambda_k)]_{\kappa = k_0} = 2\log \text{cl}_{f, L}(\mathcal{Y})(f^{\text{rig}})^2$$
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Under the assumption that $L(F, ε_{K/F}, k_0/2 + 1) \neq 0$, it can be shown that the $p$-adic $L$-value $L_p(\mathcal{F}, ε_{K/F}, λ_{k_0}) \neq 0$. On comparing the $p$-adic Abel–Jacobi image of Stark–Heegner cycles with that of classical Heegner cycles via the $p$-adic Artin formalism described above, we show in \[3\] that (see Theorem 6.1):

**Theorem 1.8.** Suppose that $ω_{\mathcal{M}} = (-1)^{k_0/2}$. Then there exists $\mathcal{Y} \in \text{CH}^{k_0/2 + 1}(\mathcal{M}_{k_0} \otimes F) \subset \text{CH}^{k_0/2 + 1}(\mathcal{M}_{k_0} \otimes K)$ and $s_\mathcal{F} \in \mathbb{Q}(\mathcal{F})^\times$ such that

$$\log \Phi^1(D_1)(\phi_{\mathcal{F}}) = s_\mathcal{F} \cdot \log \text{cl}_{F,L}(\mathcal{Y})(f_{\text{rig}})$$

from which Theorem 1.4 above follows.

1.3. **Comparison to relevant literature.** We have already mentioned that the construction of Stark–Heegner cycles for Bianchi modular forms in \[1\] should be viewed as a higher weight generalization of Trifković’s construction of Stark–Heegner points for (modular) elliptic curves defined over imaginary quadratic fields \[2\]. In particular, Conjecture 1.1 formulated in \[3\] above is a direct generalization of \[4\].

Let $f := f_E \in S_2(Γ(N))^\text{new}$ be the weight two newform associated to an elliptic curve $E/\mathbb{Q}$ of conductor $N$ by modularity and let $\mathcal{F}_E \in S_2(U_0(N))^\text{new}$ (i.e. $k_0 = 0$) be the Bianchi modular form that corresponds to the quadratic base–change $E/K$.

Let $V_p(E) := T_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, denote the $p$-adic Galois representation attached to $E$. Then, the Kummer map gives us the following exact sequence

\[0 \to E(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \twoheadrightarrow \text{Sel}_{\text{st}}(K, V_p(E)) \to \text{III}(E/K)[p^\infty] \to 0\]

where $\text{III}(E/K)[p^\infty]$ denotes the $p$-primary part of the Tate–Shafarevich group. If we assume that $\text{III}(E/K)[p^\infty]$ is finite, which is conjectured to be always true by the Birch and Swinnerton–Dyer conjecture, then $κ$ is an isomorphism and $\dim(\text{Sel}_{\text{st}}(K, V_p(E))) = \text{rank}(E(K))$.

Let $P \in E(K_p)$ be the Stark–Heegner point constructed by Trifkovic in \[3\]. Then Theorem 1.4 above implies

**Theorem 1.9.** The Stark–Heegner point $P \in E(K_p)$ is a global $K$–rational point in $E(K)$.

**Proof.** Firstly, note that when $\mathcal{F}_E$ corresponds to the quadratic base–change of $f = f_E$, then we may take our coefficient field $L/\mathbb{Q}_p$ to be just $K_p$. In particular, the local cohomology classes constructed in \[1\] may be regarded as classes $s_\chi \in H^1_{\text{rig}}(K_p(\chi), V_p(E))$. In particular, we have a commutative diagram for the trivial character $\chi = χ_{\text{triv}}$

\[
\begin{array}{ccc}
E(K) \otimes \mathbb{Q}_p & \xrightarrow{κ} & \text{Sel}_{\text{st}}(K, V_p(E)) \\
\downarrow & & \downarrow \text{resp} \\
E(K_p) \otimes \mathbb{Q}_p & \xrightarrow{κ_p} & H^1_{\text{rig}}(K_p, V_p(E))
\end{array}
\]

It is well known that the local Kummer map $κ_p$ is an isomorphism (see \[5\] 3.10.1]) and maps the Stark–Heegner point $J \in E(K_p)$ to the local Selmer class $s_1 \in H^1_{\text{rig}}(K_p, V_p(E))$ of \[1\]. Theorem 1.4 above shows the existence of a global Selmer class $S_K \in \text{Sel}_{\text{st}}(K, V_p(E))$ such that

$$\text{resp}(S_K) = s_3$$

Further, if we assume that $\text{III}(E/K)[p^\infty]$ is a finite group then $κ$ is an isomorphism and the commutativity of \[1\] above shows that there exists a global $K$–rational point $P \in E(K)$ that is mapped to the Stark–Heegner point $P \in E(K_p)$ under the natural inclusion $E(K) \hookrightarrow E(K_p)$. 

$\square$
Remark 1.10. Theorem 1.9 in particular confirms Trifković’s Conjecture ([10] Conjecture 6]) on the rationality of Stark–Heegner points for modular elliptic curves $E/F$ that are base-change from $\mathbb{Q}$.

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2. Review of Stark–Heegner cycles attached to Bianchi modular forms

In this section, we briefly review the construction of Stark–Heegner cycles attached to Bianchi modular forms from [WW21].

2.1. Bianchi modular forms. Bianchi modular forms are adelic automorphic forms for $\text{GL}_2$ over the imaginary quadratic field $F$. We recall here some basic properties of Bianchi modular forms. Let $U$ be any open compact subgroup of $\text{GL}_2(\mathbb{A}_F)$, and for any $k \geq 0$, there exists a finite-dimensional $\mathbb{C}$-vector space $S_{k+2}(U)$ of Bianchi cusp forms of (parallel) weight $k+2$ and level $U$, which are functions

$$\tilde{F} : \text{GL}_2(F) \backslash \text{GL}_2(\mathbb{A}_F)/U \rightarrow V_{2k+2}(\mathbb{C})$$

that transform appropriately under the subgroup $\mathbb{C}^\times \cdot \text{SU}_2(\mathbb{C})$, and also satisfy suitable harmonicity and growth conditions. We will be chiefly interested in the case where

$$U = U_0(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{O}_F) : c \equiv 0 \pmod{N} \},$$

where $N = pM \subset \mathbb{O}_F$ is square-free and $p \nmid M$ as before.

Bianchi modular forms admit an analogue of $q$-expansions (cf. [Wil17 §1.2]), giving rise to a system of Fourier–Whittaker coefficients $c(I, \tilde{F})$ indexed by the ideals $I \subset \mathcal{D}^{-1}$ (where $\mathcal{D}$ is the different of $F/\mathbb{Q}$). These can be described as the eigenvalues of Hecke operators. In fact, one can define a family of (commuting) Hecke operators indexed by ideals $m \subset \mathcal{O}_F$, defined via double coset operators. When $\tilde{F}$ is a normalised Hecke eigenform (i.e. $c(1, \tilde{F}) = 1$), then the eigenvalue $\lambda_m$ of the $m$-th Hecke operator on $\tilde{F}$ is equal to $c(m\mathcal{D}^{-1}, \tilde{F})$ (see [Hid94 Cor. 6.2]).

For $M$ any module equipped with an action of the Hecke operators, and $\tilde{F}$ a cuspidal Bianchi eigenform, we denote by $M(\tilde{F})$ for the $\tilde{F}$-isotypic part of $M$. This is the generalised eigenspace where the Hecke operators act with the same eigenvalues as on $\tilde{F}$.

2.2. Bianchi modular symbols. For an integer $k \geq 0$, and a ring $R$, we define $V_k(R)$ to be the space of homogeneous polynomials in two variables of degree $k$ over $R$. We define a left $\text{GL}_2(R)$ action via

$$((a \ b) \ c \ d) \cdot P(x, y) = \frac{(a+cy)^k}{(ad-bc)^k} P \left( \frac{by+dx}{ay+cx} \right).$$

Note that the centre acts trivially. In particular, the action descends to $\text{PGL}_2(R)$. We let $V_{k,k}(R) := V_k(R) \otimes_R V_k(R)$ which carries a left action of $\text{PGL}_2(R)^2$, acting on each component via (2.1). In particular, if $L$ is a large enough field containing both the embeddings $\sigma : F \hookrightarrow \mathbb{C}$, then we get a left action of $\text{PGL}_2(F)$ on $V_{k,k}(L)$ acting on the first component via one embedding and on the other via its conjugate. Let $\Delta_0 := \mathbb{P}^1(F)$ denote the space of degree zero divisors supported on the cusps $\mathbb{P}^1(F)$ of the hyperbolic $3$-space $\mathcal{H}_3$. Note that $\text{PGL}_2(F)$ acts on $\Delta_0$ via Möbius transformations $\delta \mapsto (a\delta + b)/(c\delta + d)$. 
For \( \Gamma \subset \text{PGL}_2(F) \) any subgroup, and \( V \) a right \( \Gamma \)-module, we set \( \Delta(V) := \text{Hom}(\Delta_0, V) \). Further, we equip this space with a \( \Gamma \) action by

\[
(\gamma, \phi)(D) := \phi(\gamma D) |_{\gamma}
\]

The space of \( V \)-valued modular symbols for \( \Gamma \) is then defined as the \( \Gamma \)-invariants

\[
\text{Sym}_\Gamma(V) := H^0(\Gamma, \Delta(V))
\]

**Definition 2.1.** The space of Bianchi modular symbols of level \( \Gamma_0(\mathcal{N}) := \mathcal{U}_0(\mathcal{N}) \cap \text{SL}_2(\mathcal{O}_F) \) and parallel weight \( k_0 + 2 \) is defined to be the space \( \text{Sym}_\Gamma(\mathcal{N})(V_{k_0,k_0}(\mathbb{C}_p)^\vee) \) where \( V_{k_0,k_0}(\mathbb{C}_p)^\vee \) is the \( \mathbb{C}_p \)-dual of \( V_{k_0,k_0}(\mathbb{C}_p) \), equipped with the right dual action of \( \Gamma_0(\mathcal{N}) \).

**Remark 2.2.** The space \( \text{Sym}_\Gamma(\mathcal{N})(V_{k_0,k_0}(\mathbb{C}_p)^\vee) \) admits an action of the Hecke operators. In particular, there is a Hecke-equivariant injection (under our assumption that \( F \) has class number one)

\[
S_{k_0+2}(U_0(\mathcal{N})) \hookrightarrow \text{Sym}_\Gamma(\mathcal{N})(V_{k_0,k_0}(\mathbb{C}_p)^\vee)
\]

with the co-kernel consisting of Eisenstein packets. In particular, to each cuspidal eigenform \( \mathcal{F} \in S_{k_0+2}(U_0(\mathcal{N})) \), we can attach an eigensymbol \( \phi_{\mathcal{F}} \in \text{Sym}_\Gamma(\mathcal{N})(V_{k_0,k_0}(\mathbb{C}_p)^\vee) \).

For the rest of the paper, we denote by \( \phi_{k_0} = \phi_{\mathcal{F}} \) to be the Bianchi modular symbol associated to the form \( \mathcal{F} = \mathcal{F}_{k_0} \in S_{k_0+2}(U_0(\mathcal{N}))^{\text{new}} \).

### 2.3. Overconvergent modular symbols

We recall the theory of overconvergent modular symbols of [Wil17]. When \( p \) is inert in \( F \), we resort to the entire module of overconvergent modular symbols. When \( p \) splits as \( \mathbb{p} \mathbb{p} \) in \( F \), we work with the partially overconvergent coefficients, as defined in §6 op. cit. Hence we impose no conditions at \( \mathbb{p} \) here. We refer the reader to [Wil17] §3,§7 for more details.

**Definition 2.3.** Let \( F_{p} \) denote the completion of \( F \) at \( p \) and \( \mathcal{O}_{F_{p}} \) its ring of integers. For an extension \( L/\mathbb{Q}_{p} \), let \( \mathcal{A}_{k_0}(\mathcal{O}_{F_p}, L) \) denote the ring of locally analytic functions \( \mathcal{O}_{F_p} \to L \). Then:

(i) If \( p \) is inert in \( F \), let \( \mathcal{A}^p_{k_0,k_0}(\mathcal{O}_p,L) := \mathcal{A}_{k_0}(\mathcal{O}_{F_p}, L) \);

(ii) If \( p\mathcal{O}_F = \mathbb{p} \mathbb{p} \) splits, let

\[
\mathcal{A}^p_{k_0,k_0}(\mathcal{O}_p,L) = \mathcal{A}_{k_0}(\mathcal{O}_{F_p}, L) \otimes V_{k_0}(\mathcal{O}_{F_p}, L),
\]

where \( V_{k_0}(\mathcal{O}_{F_p}, L) \) is the ring of \( L \)-valued homogeneous polynomials in two variables on \( \mathcal{O}_{F_p} \to L \) of degree \( k_0 \).

We equip these spaces with a natural ‘weight \( k_0 \)’ left action of the semigroup

\[
\Sigma_0(\mathbb{p}) := \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in M_2(\mathcal{O}_F \otimes \mathbb{Z}_p) : v_p(c) > 0, v_p(a) = 0, ad - bc \neq 0 \right\}
\]

given by

\[
\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \cdot g(z) = (a + cz)^{k_0} f \left( \frac{b + cz}{a + cz} \right).
\]

We denote by \( \mathcal{D}^p_{k_0,k_0}(\mathcal{O}_p,L) \) - the space of locally analytic distributions on \( \mathcal{O}_p \), to be the continuous \( L \)-dual of \( \mathcal{A}^p_{k_0,k_0}(\mathcal{O}_p,L) \) equipped with the corresponding dual right (weight \( k_0 \)) action. Note that the inclusion \( V_{k_0,k_0} \subset \mathcal{A}^p_{k_0,k_0} \) induces, on taking the dual, a surjection \( \mathcal{D}^p_{k_0,k_0} \to V_{k_0,k_0}^\vee \). In particular, we have a Hecke-equivariant map

\[
\rho : \text{Sym}_{\Gamma_0(\mathcal{N})}(\mathcal{D}^p_{k_0,k_0}(\mathcal{O}_p,L)) \to \text{Sym}_{\Gamma_0(\mathcal{N})}(V_{k_0,k_0}^\vee(L)).
\]

We recall the following Control Theorem of Williams.
Theorem 2.4 (Williams). We have
\[ \rho|_{(\mathcal{F})} : \text{Symb}_{\Gamma_0(N)}(\mathcal{D}_{k_0}^p, \mathcal{O}_p, L)) \cong \text{Symb}_{\Gamma_0(N)}(V^\vee_{k_0}^p)(\mathcal{F}), \]
that is, the restriction of \( \rho \) to the \( \mathcal{F} \)-isotypic subspaces of the Hecke operators is an isomorphism. In particular, there is a unique overconvergent lift \( \Psi_{\mathcal{F}} \) of \( \phi_{\mathcal{F}} = \phi_{k_0} \) under the map \( \rho \).

2.4. Harmonic modular symbols. In order to define a suitable \( p \)-adic Integration theory which links the Stark–Heegner cycles to arithmetic data, we need to spread out the overconvergent modular symbol \( \Psi_{\mathcal{F}} \) which is invariant under \( \Gamma_0 \) to the larger Ihara group \( \Gamma \), thereby enabling us to define a family of distributions over the projective line \( \mathbb{P}^1(F_p) \). For more details see [VW21, Section 3] and [BSW19].

**Definition 2.5.** Let \( \mathcal{T}_p \) be the Bruhat–Tits tree for \( \text{GL}_2(F_p) \) which is a connected tree with vertices given by homothety classes of \( \mathcal{O}_{F_p} \)-lattices \( \mathcal{L} \subset (F_p)^2 \). Two vertices are joined by an edge \( e \) if one can find representatives of lattices \( \mathcal{L} \) and \( \mathcal{L}' \) such that
\[ (p)\mathcal{L}' \subset \mathcal{L} \subset \mathcal{L}'\]
where by abuse of notation, we denote a uniformizer also by \( (p) \).

Each edge comes with an orientation (given by the source and target vertices) and we denote the set of oriented edges of the Bruhat–Tits tree by \( \mathcal{E}(\mathcal{T}_p) \) and the set of vertices by \( \mathcal{V}(\mathcal{T}_p) \). Further we denote by \( v_* := [\mathcal{L}_*] \) to be the standard vertex corresponding to the homothety class of lattices \( \mathcal{L}_* := \mathcal{O}_{F_p} \oplus \mathcal{O}_{F_p} \). Similarly denote by \( v_\infty := [\mathcal{L}_\infty] \) for \( \mathcal{L}_\infty := \mathcal{O}_{F_p} \oplus p\mathcal{O}_{F_p} \). We also set \( e_\infty \) to be the standard edge connecting \( v_* \) and its neighbour \( v_\infty \). We say a vertex is even (resp. odd) if it is connected to \( v_* \) by an even (resp. odd) number of edges. An edge \( e \in \mathcal{E}(\mathcal{T}_p) \) is called even (resp. odd) if its source vertex \( v_*(e) \). We denote the set of even (resp. odd) vertices and edges by \( \mathcal{V}^+(\mathcal{T}_p) \) and \( \mathcal{E}^+(\mathcal{T}_p) \) (resp. \( \mathcal{V}^-(\mathcal{T}_p) \) and \( \mathcal{E}^-(\mathcal{T}_p) \)). There is a natural transitive action of \( \text{PGL}_2(F_p) \) on the tree \( \mathcal{T}_p \) via Möbius transformation, which we can extended to a larger group. We recall

**Definition 2.6.** Recall that \( \mathcal{N} = p\mathcal{M} \) with \( p \nmid \mathcal{M} \).

(i) For \( v \) a finite place of \( F \), define
\[ R_0(\mathcal{M}_v) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_v) : c \equiv 0 \pmod{\mathcal{M}_v} \right\}. \]
(ii) Let \( R = R_0(\mathcal{M}) := \left\{ \gamma \in M_2(\mathcal{A}_F) : \gamma_v \in R_0(\mathcal{M}_v) \text{ for } v \neq p, \gamma_p \in M_2(F_p) \right\} \).
(iii) Let \( \tilde{\Omega} \) denote the image of \( R^\times \) in \( \text{PGL}_2(\mathcal{A}_F) \).
(iv) Let \( \Omega := \text{PGL}_2^+ (\mathcal{A}_F) \cap \tilde{\Upsilon} \), where
\[ \text{PGL}_2^+ (\mathcal{A}_F) := \left\{ \gamma \in \text{PGL}_2(\mathcal{A}_F) : v_p(\det(\gamma_p)) \equiv 0 \pmod{2} \right\}. \]
(v) Finally, let
\[ \tilde{\Gamma} = \tilde{\Omega} \cap \text{PGL}_2(F), \quad \Gamma = \Omega \cap \text{PGL}_2(F). \]

The groups \( \tilde{\Omega} \) and \( \Omega \) act on \( \mathcal{T}_p \) via projection onto \( \text{PGL}_2(F_p) \). By [Ser80, Theorem 2, Chapter II.1.4], we know that \( \Omega \) acts transitively on the sets \( \mathcal{E}(\mathcal{T}_p) \) and \( \mathcal{V}(\mathcal{T}_p) \) whilst \( \Omega \) acts transitively on \( \mathcal{E}^\pm(\mathcal{T}_p) \) and \( \mathcal{V}^\pm(\mathcal{T}_p) \). Let \( \text{PGL}_2(F_p) \) act on the projective line \( \mathbb{P}^1(F_p) \) via
\[ (a \ b \ c \ d) \cdot x := \frac{b + dx}{a + cx}. \]
For \( e \in \mathcal{E}(T_p) \), let \( \gamma_e \in \bar{\Omega} \) be such that \( e = \gamma_e e_+ \). We associate to the edge \( e \), the open set
\[
U_e := \gamma_e^{-1}(O_{F_p}) := \{ x \in \mathbb{P}^1(F_p) : \gamma_e x \in O_{F_p} \} \subset \mathbb{P}^1(F_p)
\]

**Remark 2.7.** The sets \( U_e \), as \( e \) ranges over \( \mathcal{E}(T_p) \), form a basis of compact open subsets of \( \mathbb{P}^1(F_p) \).

We can define Bianchi modular forms on the Bruhat–Tits tree \( T_p \) which will allow us to extend distributions from \( O_p \) to those that are projective in \( p \). See [BSW19 Section 2] and [VV21 Section 3.2] for details.

**Definition 2.8.** Let \( \mathcal{A}_{k_0}(\mathbb{P}^1(F_p), L) \) denote the space of \( L \)-valued functions on \( \mathbb{P}^1(F_p) \) that are locally analytic except for a pole of order at most \( k_0 \) at \( \infty. \) Then:

(i) If \( p \) is inert in \( F \), define \( \mathbb{P}^1(F_p) = \mathbb{P}^1(F_p) \), and \( \mathcal{A}_{k_0}(\mathbb{P}^1(F_p), L) = \mathcal{A}_{k_0}(\mathbb{P}^1(F_p), L) \).

(ii) If \( pO_F = p \mathfrak{p} \) in \( F \), define \( \mathbb{P}^1(F_p) = \mathbb{P}^1(F_p) \times \mathbb{C}_p/\mathfrak{p} \), and let
\[
\mathcal{A}_{k_0}(\mathbb{P}^1(F_p), L) = \mathcal{A}_{k_0}(\mathbb{P}^1(F_p), L) \odot V_{k_0}(O_{\mathfrak{p}}).
\]

Let \( \mathcal{D}^p_{k_0}(\mathbb{P}^1(F_p), L) := \text{Hom}_{cts}(\mathcal{A}_{k_0}(\mathbb{P}^1(F_p), L), L) \).

This space of distributions is a right \( \Gamma \)-module, and there is a natural restriction map
\[
\mathcal{D}^p_{k_0}(\mathbb{P}^1(F_p), L) \rightarrow \mathcal{D}^p_{k_0,k_0}(O_p, L),
\]

inducing
\[
\rho_{\mathcal{F}} : \text{Sym}^p_{\mathcal{F}}(\mathcal{D}^p_{k_0}(\mathbb{P}^1(F_p), L)) \rightarrow \text{Sym}^p_{\mathcal{F}_0(N)}(\mathcal{D}^p_{k_0,k_0}(O_p, L)).
\]

We call the domain \( \text{Sym}^p_{\mathcal{F}}(\mathcal{D}^p_{k_0}(\mathbb{P}^1(F_p), L)) \) as \( \text{‘harmonic modular symbols’ on } T_p \). There is a natural action of the Hecke operators making the map \( \rho_{\mathcal{F}} \) Hecke equivariant. Recall that \( \mathcal{F} \in S_{k_0+2}(U_0(N)) \) is a cuspidal Bianchi eigenform that is new at \( p \). Then Theorem 3.8 of [VV21] shows that

**Theorem 2.9.** We have an isomorphism on the \( \mathcal{F} \)-isotypic Hecke-eigenspace
\[
\rho_{\mathcal{F}}|_{\mathcal{F}} : \text{Sym}^p_{\mathcal{F}}(\mathcal{D}^p_{k_0}(\mathbb{P}^1(F_p), L)),_{\mathcal{F}} \cong \text{Sym}^p_{\mathcal{F}_0(N)}(\mathcal{D}^p_{k_0,k_0}(O_p, L)),_{\mathcal{F}}.
\]

In particular, after combining with Theorem 2.3 above, we obtain a canonical element \( \Phi_{k_0}^{\text{har}} \in \text{Sym}^p_{\mathcal{F}}(\mathcal{D}^p_{k_0}(\mathbb{P}^1(F_p), L)) \) attached to the newform \( \mathcal{F} \). For brevity, we set \( \text{MS}_{\mathcal{F}}(L) := \text{Sym}^p_{\mathcal{F}}(\mathcal{D}^p_{k_0}(\mathbb{P}^1(F_p), L)) \). In particular \( \text{MS}_{\mathcal{F}}(L),_{\mathcal{F}} \) is a one-dimensional \( L \)-vector space (See [VV21 Remark 3.9]). \( \Phi_{k_0}^{\text{har}} \) is called the ‘harmonic modular symbol’ attached to the Bianchi eigenform \( \mathcal{F} \).

### 2.5. Double integrals and \( p \)-adic Integration

We now recall the theory of double integrals developed in [BSW19 Section 6]. We fix forever a choice of the \( p \)-adic logarithm \( \log_p : \mathbb{C}_p^\times \rightarrow \mathbb{C}_p \) such that \( \log_p(p) = 0 \). Recall that \( \mathcal{H}^w_p := (\mathbb{P}^1(\mathbb{Q}_p^w) \setminus \mathbb{P}^1(F_p))^{\text{Gal}(\mathbb{Q}_p^w/L^0)} \) where \( L^0 := L \cap \mathbb{Q}_p^w \).

**Definition 2.10.** Let \( \tau_1, \tau_2 \in \mathcal{H}^w_p, \ P \in V_{k_0,k_0}(\mathbb{C}_p) \) and \( r, s \in \mathbb{P}^1(F) \).

(i) For each \( \sigma : F_p \rightarrow L \) an embedding, define the ‘log’ double integral at \( \sigma \) by
\[
\int_{\tau_2}^{\tau_1} \int_r^s (P) \omega_{\mu} \log_p, \sigma := \int_{\mathbb{P}^1(F_p)} \log_p \left( \frac{t - \tau_1}{t - \tau_2} \right)^\sigma P(t) d\mu(r - s)(t),
\]

where \( t_p \) is the projection of \( t \in \mathbb{P}^1(F_p) \) to \( \mathbb{P}^1(F_p) \).
We record the following results from [VW21] by \( \Phi \).

Here we normalise so that \( \text{ord}_p(p) = 1 \), noting that \( p \) is a uniformiser in \( F_p \).

For \( \delta \) either \( (\log, \sigma) \) or \( (\text{ord}, \sigma) \), we think of the double integrals defined above as maps

\[
\Phi^\delta : [\Delta_0 \otimes \text{Div}^0(\mathcal{H}^\text{ur}_p) \otimes V_{k_0,k_0}] \otimes \text{Hom}(\Delta_0, \mathcal{D}^p_{k_0}(\mathbb{P}^1, L)) \to L
\]

\[
[(r-s) \otimes (\tau_1 - \tau_2) \otimes P] \otimes \mu \mapsto \int_{\tau_1}^{\tau_2} \int_r^s (P) \omega^\sigma_{\mu}.
\]

Also let

\[
\Phi_{\log_p} : \Phi_{\log_p} \circ N_{F_p/Q_p} := \sum_\sigma \Phi_{\log_p, \sigma}.
\]

Since the pairings \( \Phi_{\log_p, \sigma} \) and \( \Phi_{\text{ord}_p} \) are \( \Gamma \)-invariant, we may interpret them as morphisms

\[
\Phi^\delta, \Phi_{\log_p, \sigma}, \Phi_{\text{ord}_p} : (\Delta_0 \otimes \text{Div}^0(\mathcal{H}^\text{ur}_p) \otimes V_{k_0,k_0})_\Gamma \to \text{MS}_\Gamma(L)^\vee.
\]

We will denote the projection onto the \( (\mathcal{F}) \)-isotypic component of \( \text{MS}_\Gamma(L)^\vee \), \( \text{pr}_\mathcal{F} \circ \Phi^\delta \) simply by \( \Phi^\delta_\mathcal{F} \). Note that we have an exact sequence

\[
0 \to \Delta_0 \otimes \text{Div}^0(\mathcal{H}^\text{ur}_p) \otimes V_{k_0,k_0} \to \Delta_0 \otimes \text{Div}(\mathcal{H}^\text{ur}_p) \otimes V_{k_0,k_0} \to \Delta_0 \otimes V_{k_0,k_0} \to 0,
\]

obtained by tensoring \( 0 \to \text{Div}^0 \to \text{Div} \to \mathbb{Z} \to 0 \) with the flat \( \mathbb{Z} \)-module \( \Delta_0 \otimes V_{k_0,k_0} \). On taking \( \Gamma \)-homology, we have

\[
\cdots \to H_i(\Delta_0 \otimes \text{Div}^0(\mathcal{H}^\text{ur}_p) \otimes V_{k_0,k_0}) \to H_i(\Delta_0 \otimes \text{Div}(\mathcal{H}^\text{ur}_p) \otimes V_{k_0,k_0}) \to H_i(\Delta_0 \otimes V_{k_0,k_0}) \to \cdots
\]

In particular, we have the connecting morphism

\[
H_1(\Delta_0 \otimes V_{k_0,k_0}) \xrightarrow{\partial} (\Delta_0 \otimes \text{Div}^0(\mathcal{H}^\text{ur}_p) \otimes V_{k_0,k_0})_\Gamma.
\]

We record the following results from [VW21]

**Theorem 2.11.**

(i) The morphism between the \( L \)-vector spaces

\[
\Phi^\delta_{\mathcal{F}} \circ \delta : H_1(\Delta_0 \otimes V_{k_0,k_0}) \to \text{MS}_\Gamma(L)^\vee_{(\mathcal{F})}
\]

is surjective.

(ii) For each embedding \( \sigma : F_p \to L \), there exists a unique \( \mathcal{L}^\sigma_p \in \mathbb{C}_p \) such that

\[
\Phi_{\log_p, \sigma} \circ \delta = \mathcal{L}^\sigma_p \circ \Phi^\delta_{\mathcal{F}} \circ \delta : H_1(\Delta_0 \otimes V_{k_0,k_0}) \to \text{MS}_\Gamma(L)^\vee_{(\mathcal{F})}.
\]
Definition 2.13. A
We recall the definition of a two-dimensional filtration given by
Remarks 2.14. For each $\sigma$
where $\sigma$
where each $\sigma$

Proof. Part (i) follows from [VW21, Theorem 3.15] while Part (ii) is [VW21, Corollary 3.17]. □

Remark 2.12. For each prime $p$, we have an equality

$$L_p^{\text{BW}} := \sum_{\sigma} L_p^\sigma,$$

where $\sigma$ ranges over all embeddings and $L_p^{\text{BW}}$ is the $L$-invariant of [BSW19]. In particular, we also have

$$\Phi_p^{\log} \circ \delta = L_p^{\text{BW}} \circ \Phi_p^{\text{ord}} \circ \delta : H_1(\Gamma, \Delta_0 \otimes V_{k_0,k_0}) \rightarrow \text{MS}_\Gamma(L)\!(\mathcal{J}),$$

where we set

$$(\Delta_0 \otimes \text{Div}(\mathcal{H}_p^\text{ur}) \otimes V_{k_0,k_0})_\Gamma \rightarrow D_\mathcal{J},$$

where we set

$$D_{\mathcal{J},L} := \text{MS}_\Gamma(L)\!(\mathcal{J}) \oplus \text{MS}_\Gamma(L)\!(\mathcal{J}).$$

Further, let

$$(2.10)$$

$$D_{\mathcal{J}} := \bigoplus_{\sigma:F_p \hookrightarrow L} D_{\mathcal{J},L}$$

where each $D_{\mathcal{J},L}$ is a two-dimensional $L$-vector space but with scalar action of $F_p$ (viewed as a subfield of $L$) given by $\sigma$. In [VW21, Section 4], we had given $D_{\mathcal{J}}$ the structure of a rank two filtered $(\varphi, N)$-module over $F_p$ with coefficients in $L$. $D_{\mathcal{J},L}$ is then a filtered $L$-vector space of dimension two with the filtration given by

$$D_{\mathcal{J},L} = \text{Fil}^0 \supseteq \text{Fil}^1 = \ldots = \text{Fil}^{k_0+1} \supseteq \text{Fil}^{k_0+2} = 0$$

where

$$\text{Fil}^{k_0+2}_{\mathcal{J},L} := \{(-L_p^\sigma x, x) : x \in \text{MS}_\Gamma(L)\!(\mathcal{J})\}$$

for each $\sigma : F_p \hookrightarrow L$. From the $\Gamma$-homology exact sequence (2.7), we have the connecting morphisms

$$(2.11)$$

$$\frac{(\Delta_0 \otimes \text{Div}(\mathcal{H}_p^\text{ur}) \otimes V_{k_0,k_0})_\Gamma}{\delta(\text{H}_1(\Gamma, \Delta_0 \otimes V_{k_0,k_0}))} \otimes \partial_1 (\Delta_0 \otimes \text{Div}(\mathcal{H}_p^\text{ur}) \otimes V_{k_0,k_0})_\Gamma \otimes \partial_2 (\Delta_0 \otimes V_{k_0,k_0})_\Gamma.$$

We recall the definition of a $p$-adic Abel–Jacobi map from [VW21, Definition 5.1]

Definition 2.13. A $p$-adic Abel–Jacobi map is a morphism

$$\Phi_{\mathcal{J}} : (\Delta_0 \otimes \text{Div}(H_p^\text{ur}) \otimes V_{k_0,k_0})_\Gamma \rightarrow D_{\mathcal{J},L}/\text{Fil}^{k_0+2}_{\mathcal{J}}(D_{\mathcal{J},L})$$

such that the following diagram commutes:

$$(2.12)$$

$$\frac{(\Delta_0 \otimes \text{Div}(\mathcal{H}_p^\text{ur}) \otimes V_{k_0,k_0})_\Gamma}{\delta(\text{H}_1(\Gamma, \Delta_0 \otimes V_{k_0,k_0}))} \otimes \Phi_{\mathcal{J}} D_{\mathcal{J},L}/\text{Fil}^{k_0+2}_{\mathcal{J}}(D_{\mathcal{J},L})$$

for $\sigma : F_p \hookrightarrow L$. In other words, a $p$-adic Abel–Jacobi map $\Phi_{\mathcal{J}}$ is a lift of the morphism $\Phi_{\mathcal{J}}$.

Remark 2.14. Note that while there is no unique choice of a lift of $\Phi_{\mathcal{J}}$ to a $p$-adic Abel–Jacobi map, we have shown in [VW21, Theorem 6.5] that the image of the $p$-adic Abel–Jacobi image of the Stark–Heegner cycle is independent of such a choice (See also [VW21, Remark 5.2]).
2.6. Stark–Heegner cycles. Recall from the Introduction (§1) that \( K/F \) is a quadratic extension of relative discriminant \( D_{K/F} \) prime to the level \( \mathcal{N} = \mathfrak{p}\mathcal{M} \) and satisfies the Stark–Heegner hypothesis (SH-Hyp). In particular, the completion \( K_\mathfrak{p} \) of \( K \) at the prime \( \mathfrak{p} \) is the quadratic unramified extension of \( F_\mathfrak{p} \). We fix \( \delta_\mathfrak{p} \in \mathcal{O}_K/\mathcal{O}_F \) such that \( \delta_\mathfrak{p}_F \in \mathcal{O}_F \) is a generator of the discriminant ideal \( \mathcal{D}_{K/F} \) (recall that we have assumed \( F \) to have class number one). We will regard \( \delta_\mathfrak{p} \) as an element of \( K_\mathfrak{p} \) via \( \iota_\mathfrak{p} \). Let \( \mathcal{O} \) be an \( \mathcal{O}_F[1/\mathfrak{p}] \)-order of conductor \( \mathcal{C} \) prime to \( \mathcal{D}_{K/F}\mathcal{N} \) and let \( \mathcal{R} \) be the Eichler \( \mathcal{O}_F[1/p] \)-order in \( M_2(\mathcal{O}_F[1/p]) \) that are upper triangular modulo \( \mathcal{M} \).

Definition 2.15. We say an embedding \( \Psi : K \hookrightarrow M_2(\mathcal{F}) \) is optimal if \( \Psi(K) \cap \mathcal{R} = \Psi(\mathcal{O}) \).

Denote the set of \( \mathcal{O}_F[1/p] \)-optimal embeddings by \( \text{Emb}(\mathcal{O}, \mathcal{R}) \). To an embedding \( \Psi \in \text{Emb}(\mathcal{O}, \mathcal{R}) \), we associate the following data.

- The two points \( \tau_\Psi \) and \( \tau^0_\Psi \in \mathcal{H}^w_\mathfrak{p}(K) := \mathcal{H}^w_\mathfrak{p} \cap K \) that are fixed by the action of \( \Psi(K^\times) \).
- Here \( \tau^0_\Psi := \theta(\tau_\Psi) \) for \( \theta \in \text{Gal}(K/F) \), \( \theta \neq \text{id} \).
- The fixed vertex \( v_\Psi \in \mathcal{V} \) in the Bruhat–Tits tree for the action of \( \Psi(K^\times) \) on \( \mathcal{V} \).
- The polynomial \( P_\Psi(x, y) := (cx^2 + (a - d)xy - by^2)(c\tau_\Psi^2 + (a - d)\tau_\Psi y - by^2) \in V_{2,2} \), where \( \Psi(\delta_\mathfrak{p}) = \left( \frac{a}{c} \frac{b}{d} \right) \).
- Let \( u \) be a fixed generator of \( \mathcal{O}_1^\times / \{ \text{torsion} \} \cong \mathbb{Z} \) (by Dirichlet’s Unit theorem), where \( \mathcal{O}_1 := \{ x \in \mathcal{O} \mid N_{K/F}(x) = 1 \} \). Let \( \gamma_\Psi := \Psi(u) \) and \( \Gamma_\Psi \) be the cyclic subgroup of \( \Gamma \) generated by \( \gamma_\Psi \). In particular \( \Gamma_\Psi = \text{Stab}(\Psi) \subseteq \Gamma \) and \( \mathcal{P}_\Psi \in (V_{2,2})^{\Gamma_\Psi} \).

\( \Psi \) is said to have positive (resp. negative) orientation (at \( \mathfrak{p} \)) if \( v_\Psi \in \mathcal{V}^+(\mathcal{V}_\mathfrak{p}) \) (resp. \( \mathcal{V}^-(\mathcal{V}_\mathfrak{p}) \)). Then

\[ \text{Emb}(\mathcal{O}, \mathcal{R}) = \text{Emb}^+(\mathcal{O}, \mathcal{R}) \sqcup \text{Emb}^-(\mathcal{O}, \mathcal{R}) \]

where \( \text{Emb}^\pm(\mathcal{O}, \mathcal{R}) \) denotes the set of embeddings with positive/negative orientation. \( \Gamma \) acts naturally on the set \( \text{Emb}(\mathcal{O}, \mathcal{R}) \) by conjugation and it preserves the subsets \( \text{Emb}^\pm(\mathcal{O}, \mathcal{R}) \). Further, we know that the association

\[ \Psi \mapsto (\tau_\Psi, P_\Psi, \gamma_\Psi) \]

under conjugation action by any \( \gamma \in \Gamma \) satisfies

\[ (\tau_{\gamma\gamma_\Psi^{-1}}, P_{\gamma_\Psi^{-1}}, \gamma_{\gamma_\Psi^{-1}}) = (\gamma \cdot \tau_\Psi, \gamma^{-1} \cdot P_\Psi, \gamma \gamma_\Psi \gamma^{-1}) \]  

(2.13)

Remark 2.16. Let \( \theta \in \text{Gal}(K/F) \) be the non-trivial element. We let \( \Psi^0 \in \text{Emb}(\mathcal{O}, \mathcal{R}) \) be the embedding defined as \( \Psi^0(-) := \Psi(\theta(-)) \). Then, a simple calculation shows that

\[ (\tau_{\Psi^0}, P_{\Psi^0}, \gamma_{\Psi^0}) = (\tau_\Psi^{-}, -P_\Psi, \gamma^{-1}_\Psi). \]

Once we fix a cusp \( x \in \mathbb{P}^1(F) \), we define

\[ D : \text{Emb}(\mathcal{O}, \mathcal{R}) \rightarrow \Delta_0 \otimes \text{Div}(\mathcal{H}^w_\mathfrak{p}) \otimes V_{k_0,k_0}, \]

\[ D_\Psi := D(\Psi) := (\gamma_\Psi \cdot x - x) \otimes \tau_\Psi \otimes \left( \frac{1}{\sqrt{N_{F/Q}(\mathcal{D}_{K/F})}} \right)^{k_0/2} P_{\Psi}^{k_0/2}. \]

Remark 2.17. Note that there is a subtle error in [VW21, §6.1] in defining the classes \( D_\Psi \) that has been corrected above.

Lemma 2.18 (Lemma 6.3, [VW21]). The image of \( D_\Psi \) in \( (\Delta_0 \otimes \text{Div}(\mathcal{H}^w_\mathfrak{p}) \otimes V_{k_0,k_0})_\Gamma \), which is denoted by \( [D_\Psi] \), does not depend on the choice of the class \( y \in \Gamma x \). Further, \( [D_\Psi] \) is invariant under the conjugation action of \( \Gamma \) on \( \text{Emb}(\mathcal{O}, \mathcal{R}) \).
In particular, there is a well defined map
\[
D : \Gamma / \text{Emb}(\mathcal{O}, \mathcal{R}) \rightarrow (\Delta_0 \otimes \text{Div}(\mathcal{H}^\ur_p) \otimes V_{k_0,k_0})_\Gamma.
\]

**Definition 2.21.** We call \([D_\Psi]\) the *Stark–Heegner cycle* attached to the conjugacy class of optimal embeddings \([\Psi]\).

Let \(\sigma : F_p \hookrightarrow L\) be any embedding as before. Recall from [VW21, Theorem 6.5] that

**Theorem 2.20.** The \(p\)-adic Abel–Jacobi image of the Stark–Heegner cycle \(D_\Psi\) is independent of the choice of a \(p\)-adic Abel–Jacobi map. In other words, if
\[
\Phi^{A1}_{\sigma,i} : (\Delta_0 \otimes \text{Div}(\mathcal{H}^\ur_p) \otimes V_{k_0,k_0})_\Gamma \rightarrow D^{\sigma}_{F,L}/\text{Fil}^{k_0/2}(D^{\sigma}_{F,L}), \quad i = 1, 2
\]
are any two \(p\)-adic Abel–Jacobi maps lifting \(\Phi^{\sigma}_p\), then
\[
\Phi^A_{\sigma,1}\left(\left[(\gamma_\Psi \cdot x - x) \otimes \tau_\Psi \otimes \left(\sqrt{N_{F/Q}(D_{K/F})}^{-k_0/2} P^{k_0/2}_\Psi\right)\right]\right) = \Phi^A_{\sigma,2}\left(\left[(\gamma_\Psi \cdot x - x) \otimes \tau_\Psi \otimes \left(\sqrt{N_{F/Q}(D_{K/F})}^{-k_0/2} P^{k_0/2}_\Psi\right)\right]\right)
\]

**Definition 2.21.** The *Stark–Heegner cohomology class* associated to a conjugacy class of embeddings \([\Psi]\) \(\in \Gamma / \text{Emb}(\mathcal{O}, \mathcal{R})\) is defined as
\[
\mathfrak{a}_\Psi : \Phi^A_{\sigma,1}(D_\Psi) \in D^{\sigma}_{F,L}/\text{Fil}^{k_0/2}(D^{\sigma}_{F,L}).
\]

There is a natural action of \(\text{Pic}(\mathcal{O})\) (by conjugation) on the set \(\Gamma / \text{Emb}(\mathcal{O}, \mathcal{R})\). By the reciprocity isomorphism of class field theory,
\[
\text{rec} : \text{Pic}(\mathcal{O}) \cong \text{Gal}(H_C/K)
\]
we get an action of \(\text{Gal}(H_C/K)\) on \(\Gamma / \text{Emb}(\mathcal{O}, \mathcal{R})\). Here \(H_C\) is the ring class field of conductor \(\mathcal{C}\).

2.6.1. *Picard group torsors.* We will now fix orientations (at \(\mathcal{M}\)) for the set of optimal embeddings. Note that by the Stark–Heegner hypothesis \((\text{SH-Hyp})\), there exists \(\mathcal{M}' \subseteq \mathcal{O}_K\) such that \(N_{K/F} \mathcal{M}' = \mathcal{M}\). Now the ideal \(\mathcal{M} = \mathcal{M}' \mathcal{O}_F [1/p]\) is the kernel of a unique surjective \(\mathcal{O}_F\)-algebra homomorphism (since \((\mathcal{C}_p, \mathcal{M}) = 1)\)
\[
(2.16) \quad \mathfrak{o} : \mathcal{O} \rightarrow \mathcal{O}_F / \mathcal{M}
\]
The homomorphism \(\mathfrak{o}\) is called an *orientation* of the order \(\mathcal{O}\) (at \(\mathcal{M}\)) which we fix along with the ideal \(\mathcal{M}\). To an optimal embedding \(\Psi \in \text{Emb}(\mathcal{O}, \mathcal{R})\), we can associate an orientation
\[
\mathfrak{o}_\Psi : \mathcal{O} \rightarrow \mathcal{O}_F / \mathcal{M}
\]
which maps \(a \in \mathcal{O}\) to the upper-left hand entry of the matrix \(\Psi(a)\) which can easily be verified to be an \(\mathcal{O}_F\)-algebra morphism since \(\Psi(a)\) is upper triangular modulo \(\mathcal{M}\).

**Definition 2.22.** An optimal embedding \(\Psi\) is said to be oriented if \(\mathfrak{o}_\Psi = \mathfrak{o}\). We denote the set of oriented optimal embeddings by \(\text{Emb}^o(\mathcal{O}, \mathcal{R}) \subseteq \text{Emb}(\mathcal{O}, \mathcal{R})\).

The conjugation action of \(\Gamma\) on \(\text{Emb}(\mathcal{O}, \mathcal{R})\) in fact preserves \(\text{Emb}^o(\mathcal{O}, \mathcal{R})\). Furthermore, we have

**Proposition 2.23.** There exists a bijection
\[
\text{Pic}(\mathcal{O}) \cong \text{Gal}(H_C/K) \cong \Gamma / \text{Emb}^o(\mathcal{O}, \mathcal{R})
\]
Proof. See [Tuy09, Proposition 2]. □

In particular, Proposition 2.23 shows that the set $\Gamma/\text{Emb}^o(\mathcal{O}, \mathcal{R})$ is a $\text{Gal}(H_C/K)$-torsor with the group action as described above.

Remark 2.24. For primes $l \mid N$, let $\alpha_l$ be the Atkin-Lehner matrices that define the Atkin-Lehner operators on the space of Bianchi eigenforms of level $N$ (See [Cre13, §3.3] and [Lin05, §5.3] for a precise definition), i.e.

$$F | W_l := F | \alpha_l = \omega_l F$$

where $\omega_l \in \{\pm 1\}$. For an oriented optimal embedding $\Psi \in \text{Emb}^o(\mathcal{O}, \mathcal{R})$, the optimal embedding $\alpha_M \Psi^0 \alpha_M^{-1}$ has the same orientation as $\Psi$, i.e.

$$\alpha_M \Psi^0 \alpha_M^{-1} \in \text{Emb}^o(\mathcal{O}, \mathcal{R})$$

Definition 2.25. Let $\chi : \text{Gal}(H_C/K) \to \mathbb{C}^\times$ be any character. The $\chi$-twisted Stark–Heegner cycle is then defined as

$$D_\chi := \sum_{\sigma \in \text{Gal}(H_C/K)} \chi^{-1}(\sigma)D_{[\text{rec}^{-1}(\sigma)\Psi]} \in (\Delta_0 \otimes \text{Div}(H_p^{ur}) \otimes V_{k_0,k_0})_{\Gamma} \otimes \chi,$$

where $(-) \otimes \chi$ denotes suitable scalar extension by $\chi$. We may also similarly define

$$D^\theta_\chi := \sum_{\sigma \in \text{Gal}(H_C/K)} \chi^{-1}(\sigma)D_{[\text{rec}^{-1}(\sigma)\Psi]^{\theta}} \in (\Delta_0 \otimes \text{Div}(H_p^{ur}) \otimes V_{k_0,k_0})_{\Gamma} \otimes \chi.$$ 

We set $s^\chi_\theta := \Phi^1_{\sigma}(D_\chi)$ (resp. $s^\chi_\theta := \Phi^1_{\sigma}(D^\theta_\chi)$).

When $\chi = \chi_{\text{triv}}$ is the trivial character, we will denote the $\chi_{\text{triv}}$-twisted Stark–Heegner cycles by $D_1$ and $D^\theta_1$ respectively.

Remark 2.26. In [VW21, Section 5 and 6], we had dropped the dependence of the embedding $\sigma : F_p \hookrightarrow L$ from the notation in the $p$-adic Abel–Jacobi map as well as the image of the Stark–Heegner cycle under it. However, in the sequel we will need to consider the $p$-adic Abel–Jacobi image of the Stark–Heegner cycles over all possible embeddings when we relate them to $p$-adic $L$-functions, thus making the notation used here slightly different from that in [VW21]. We explain this in detail in §3 below.

3. Review of the $p$-adic Abel–Jacobi Map

For $\sigma : F_p \hookrightarrow L$, we consider the following commutative diagram:

\[
\begin{array}{ccc}
(\Delta_0 \otimes \text{Div}(H_p^{ur}) \otimes V_{k_0,k_0})_{\Gamma} & \xrightarrow{\Phi^p_{\sigma}} & D^\theta_{F_p,L}/\text{Fil}^{k_0+2} (D^\theta_{F_p,L}) \\
\downarrow{\text{Pr}^{\sigma}} & & \downarrow{\text{Pr}^{\sigma}} \\
(\Delta_0 \otimes \text{Div}(H_p^{ur}) \otimes V_{k_0,k_0})_{\Gamma} & \xrightarrow{\log \Phi^p_{\sigma}} & \text{MS}_\Gamma(L)^{\vee}_{(F)}
\end{array}
\]

where $\text{Pr}^{\sigma}(x,y) := -x - L_p^\sigma y$ is well-defined & an isomorphism and

\[
\log \Phi^p_{\sigma} := \Phi^p_{\log \sigma} - L_p^\sigma \Phi^p_{\delta_{\sigma}}.
\]

By Theorem 2.11 we know that $\log \Phi^p_{\sigma} \circ \delta = 0$. Since $\text{Pr}^{\sigma}$ is an isomorphism, we may identify $\log \Phi^p_{\sigma}$ and $\Phi^p_{\delta_{\sigma}}$. We also define

\[
(\Delta_0 \otimes \text{Div}(H_p^{ur}) \otimes V_{k_0,k_0})_{\Gamma} \xrightarrow{\log \Phi_{\sigma}} \text{MS}_\Gamma(L)^{\vee}_{(F)}
\]
as $\log \Phi_x := \sum_{\sigma} \text{Pr}^\sigma \circ \Phi_x^\sigma$. In particular, we have

\[
(3.4) \quad \log \Phi_x = \sum_{\sigma} \log \Phi_x^\sigma = \Phi_x^\text{log} - L_p^{\text{BW}} \Phi_x^\text{ordp}
\]

. Recall that we have fixed a branch of the $p$-adic logarithm such that $\log_p(p) = 0$. For every $\ell \in L$, we define

\[
(3.5) \quad \log_{\ell}: \log_p - \text{ord}_{\ell} : L^\times \to L
\]

to be the branch of the $p$-adic logarithm such that $\log_{\ell}(p) = -\ell$. (Note that in our convention $\log_p(p) = 0$ and not $-p$.) Corresponding to this branch of the $p$-adic logarithm, we may define $\Phi_x^\text{log}_{\ell,\sigma}, \Phi_x^\sigma, L_p^\sigma, D_{x,L}^\sigma, \text{and Pr}^\sigma,\ell$ accordingly. Note that the dependence of $\ell$ in $D_{x,L}^\sigma$ is from the definition of the filtration Fil_{\ell}. We then have,

**Proposition 3.1.** For each $\sigma : F_p \hookrightarrow L$ and for every $\ell \in L$,

\[
\Phi_x^{\text{log}_{\ell,\sigma}} = \Phi_x^{\log_{\text{p},\sigma}} - \ell \Phi_x^{\text{ordp}} \in \text{Hom}\left((\Delta_0 \otimes \text{Div}^0(H^{ur}_p) \otimes V_{k_0,k_0})_\Gamma, MS_\Gamma(L)_{(\mathcal{F})}\right)
\]

and

\[
L_p^\sigma - \ell = L_p^\sigma,\ell
\]

**Proof.** See Proposition 3.1 and Lemma 3.2 of [Sev12]. \hfill $\square$

Now, let us choose the branch of the $p$-adic logarithm, $\log_{L_p^\sigma}$, corresponding to $\ell = L_p^\sigma \in L$. Then, by Proposition 3.1, the commutative diagram (3.1) above simplifies as

- $\text{Pr}^{\sigma} \circ L_p^\sigma (x,y) = -x$.
- $\log \Phi_x^{L_p^\sigma} = \log \Phi_x = \Phi_x^{\log_{L_p^\sigma}}$.

In particular, $\log \Phi_x^{L_p^\sigma}$ is independent of the choice of a branch of the $p$-adic logarithm and

\[
(3.6) \quad \log \Phi_x = \sum_{\sigma} \Phi_x^{\log_{L_p^\sigma}}
\]

3.0.1. **Galois Action.** Since we consider only the case when $p$ inert in $F$, we have $F_p \cong Q_{p,a}$ -- the unique quadratic unramified extension of $Q_p$. In this setting, we have two (Galois) embeddings, viz. the identity and the (lift of the) Frobenius, $\sigma_{id}, \sigma_{Fr} : F_p \hookrightarrow L$. We fix an identification

\[
\sigma^* : \frac{D_{x,L}^\text{id}}{\text{Fil}_{k_0+2}(D_{x,L}^\text{id})} \xrightarrow{\cong} \frac{D_{x,L}^{\text{Fr}}}{\text{Fil}_{k_0+2}(D_{x,L}^{\text{Fr}})}
\]

such that the following diagram commutes

\[
\begin{array}{ccc}
\frac{D_{x,L}^\text{id}}{\text{Fil}_{k_0+2}(D_{x,L}^\text{id})} & \xrightarrow{\sigma^*} & \frac{D_{x,L}^{\text{Fr}}}{\text{Fil}_{k_0+2}(D_{x,L}^{\text{Fr}})} \\
\downarrow & & \downarrow \\
\text{MS}_\Gamma(L)_{(\mathcal{F})} & \xrightarrow{(\text{Pr}^{\sigma_{Fr}})^{-1}} & \text{MS}_\Gamma(L)_{(\mathcal{F})}
\end{array}
\]

Via the identification $\sigma^*$, we have

\[
\text{Pr}^{\sigma_{Fr}} \circ \sigma^* \circ \Phi_x^{id} = \text{Pr}^{\sigma_{id}} \circ \Phi_x^{id} \in \text{Hom}\left((\Delta_0 \otimes \text{Div}^0(H^{ur}_p) \otimes V_{k_0,k_0})_\Gamma, MS_\Gamma(L)_{(\mathcal{F})}\right)
\]
Then,

(3.8) \[ \log \Phi_F = \Pr^{\sigma_F} \circ (\Phi^{\sigma_F} + \sigma^* \Phi^{\sigma_{lid}}) \in \text{Hom} \left( (\Delta_0 \otimes \text{Div}^0(H^\text{ur}_p) \otimes V_{k_0,k_0})_\Gamma, \text{MS}_\Gamma(L)^\vee \right). \]

This motivates us to set

(3.9) \[ \Phi_F := \Phi^{\sigma_F} + \sigma^* \Phi^{\sigma_{lid}} \in \text{Hom} \left( (\Delta_0 \otimes \text{Div}^0(H^\text{ur}_p) \otimes V_{k_0,k_0})_\Gamma, \frac{D^{\sigma_F}_{F,L}}{F} \right) \]

and

(3.10) \[ \Phi^\lambda := \Phi^{\lambda_F} + \sigma^* \Phi^{\lambda_{lid}} \in \text{Hom} \left( (\Delta_0 \otimes \text{Div}(H^\text{ur}_p) \otimes V_{k_0,k_0})_\Gamma, \frac{D^{\lambda_F}_{F,L}}{F} \right) \]

so that we have the following commutative diagrams :

(3.11) \[
\begin{array}{c}
\text{Fil}^{k_0+2} (\Phi^{\sigma_F}) \\
\Phi^{\sigma_F} \\
\Phi_F \\
\Phi_F \\
\Phi_F^\lambda \\
\text{Fil}^{k_0+2} (\Phi^{\lambda_F}) 
\end{array}
\]

(3.12) \[
\begin{array}{c}
\text{Fil}^{k_0+2} (\Phi^{\sigma_F}) \\
\Phi^{\sigma_F} \\
\Phi_F \\
\Phi_F \\
\Phi_F^\lambda \\
\text{Fil}^{k_0+2} (\Phi^{\lambda_F}) 
\end{array}
\]

\( \Phi_F \) (resp. \( \Phi^\lambda \)) should be considered as the sum of \( \Phi^{\sigma_F} \) (resp. \( \Phi^{\lambda_F} \)) over the possible embeddings \( \sigma : F_p \hookrightarrow L \).

**Remark 3.2.** Since \( F \) is the base–change to \( F \) of \( f \in S_{k_0+2}(\Gamma_0(N))^{\text{new}} \), we know by [VW21] Lemma 4.4] that

\[
\mathcal{L}^{\sigma_{lid}}_p = \mathcal{L}^{\sigma_F}_p = \mathcal{L}_p(f)
\]

where \( \mathcal{L}_p(f) \) is the Darmon–Orton \( \mathcal{L} \)-invariant attached to the newform \( f \). Then, by (3.6),

(3.13) \[ \log \Phi_F = \Phi^{\log_p \epsilon_p(f) \cdot \sigma_{lid}}_F + \Phi^{\log_p \epsilon_p(f) \cdot \sigma_F}_F = \Phi^{\log_p \epsilon_p(f)}_F \]

i.e. the \( D^{\sigma_F}_{F,L} / \text{Fil}^{k_0+2} \)-valued integration theory \( \Phi_F \) is equivalent, via \( \Pr^{\sigma_F} \), to the \( \text{MS}_\Gamma(L)^\vee \)-valued integration theory

\[
\log \Phi_F = \Phi^{\log_p}_F - \Phi^{\text{BW}}_F \Phi^{\text{ord}_p}_F = \Phi^{\log_p \epsilon_p(f)}_F
\]

**Remark 3.3.** Note that, without loss of generality, we could also have set

(3.9) \[ \Phi_F := \Phi^{\sigma_F} + (\sigma^*)^{-1} \Phi^{\sigma_{lid}}_F \in \text{Hom} \left( (\Delta_0 \otimes \text{Div}^0(H^\text{ur}_p) \otimes V_{k_0,k_0})_\Gamma, \frac{D^{\sigma_{lid}}_{F,L}}{F} \right) \]

and

(3.10) \[ \Phi^\lambda := \Phi^{\lambda_F} + (\sigma^*)^{-1} \Phi^{\lambda_{lid}}_F \in \text{Hom} \left( (\Delta_0 \otimes \text{Div}(H^\text{ur}_p) \otimes V_{k_0,k_0})_\Gamma, \frac{D^{\lambda_{lid}}_{F,L}}{F} \right) \]
which would then fit in the following commutative diagrams:

\[
\begin{align*}
&\Delta_0 \otimes \text{Div}^0(\mathcal{H}_p^{ur}) \otimes V_{k_0, k_0})_\Gamma \xrightarrow{\Phi_F} D_{\text{Fil}, L/F, \mathbb{Z}, r_{\text{ld}}}^{k_0+k_2}(D_{\text{Fil}, L/F, \mathbb{Z}, r_{\text{ld}}}) \\
&\Delta_0 \otimes \text{Div}^0(\mathcal{H}_p^{ur}) \otimes V_{k_0, k_0})_\Gamma \xrightarrow{\log \Phi_F} \text{MS}_\Gamma(L)(\mathbb{C})
\end{align*}
\]

(3.14)

and

\[
\begin{align*}
&\Delta_0 \otimes \text{Div}(\mathcal{H}_p^{ur}) \otimes V_{k_0, k_0})_\Gamma \xrightarrow{\Phi^A_{\mathcal{J}}} D_{\text{Fil}, L/F, \mathbb{Z}, r_{\text{ld}}}^{k_0+k_2}(D_{\text{Fil}, L/F, \mathbb{Z}, r_{\text{ld}}}) \\
&\Delta_0 \otimes \text{Div}(\mathcal{H}_p^{ur}) \otimes V_{k_0, k_0})_\Gamma \xrightarrow{\log \Phi^A_{\mathcal{J}}} \text{MS}_\Gamma(L)(\mathbb{C})
\end{align*}
\]

(3.15)

In particular, we may think of \(\Phi_F\) (resp. \(\Phi^A_{\mathcal{J}}\)) as being \((D_{\text{Fil}, L/F, \mathbb{Z}, r_{\text{ld}}}^{k_0+k_2})(D_{\text{Fil}, L/F, \mathbb{Z}, r_{\text{ld}}})\)-valued for either choice of an embedding \(\sigma : F_p \hookrightarrow L\).

4. Families of Bianchi modular forms and families of Bianchi modular symbols

The goal of this section is to recall some of the requisite results on \(p\)-adic families of Bianchi modular forms, mainly following the exposition in [BSW21 §3] (See also [Han17]). Let \(L/\mathbb{Q}_p\) be a sufficiently large finite extension of \(\mathbb{Q}_p\) as before.

**Definition 4.1.** The Bianchi weight space of level \(U_0(N)\) is defined as the rigid analytic space whose \(L\)-points are given by

\[
\mathcal{W}_{F,N}(L) := \text{Hom}_{cts}((\mathcal{O}_F \otimes \mathbb{Z})^x, E(N), L^x)
\]

where \(E(N) := F^x \cap U_0(N) = \mathcal{O}_F^x \simeq \mu(\mathbb{O}_F)\) - the roots of unity in \(\mathcal{O}_F\).

A weight \(\lambda \in \mathcal{W}_{F,N}(L)\) is said to be classical if it is of the form \(c \lambda_{\text{alg}}\) for \(c\) a finite order character and \(\lambda_{\text{alg}}(z) = z^k := (z^{k_1})(z^{k_2})\) for \(k = (k_1, k_2)\) with \(k_1, k_2 \in \mathbb{Z}\). Here the over–line indicates complex conjugation (i.e. action by the non-trivial element of \(\text{Gal}(F/\mathbb{Q})\)).

**Remark 4.2.** When the level of the \(p\)-adic weight space is clear, we shall drop it from the notation and denote the weight space simply by \(\mathcal{W}_F\).

**Remark 4.3.** When \(c\) is the trivial character and \(k = (k, k)\) for \(k \in \mathbb{Z}\), we shall call \(\lambda = \lambda_k\) a parallel weight. Note that in this case

\[
\lambda_k(z) = (z^k)(z^k) = N_{F/\mathbb{Q}}(z)^k
\]

We shall see later that the parallel weights \(\lambda_k\) parametrize classical Bianchi modular forms of (parallel) weight \(k + 2\).

**Definition 4.4.** Let \(A(\mathcal{O}_F \otimes \mathbb{Z}, L)\) denote the ring of \(L\)-valued locally analytic functions on \(\mathcal{O}_F \otimes \mathbb{Z}\). For \(\lambda \in \mathcal{W}_F(L)\), we equip this space with a natural weight \(\lambda\) left action of the semi-group

\[
\Sigma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_F \otimes \mathbb{Z}) : \nu_p(c) > 0, \nu_p(a) = 0, ad - bc \neq 0 \right\}
\]

given by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \lambda f(z) := \lambda(a + cz)f \left( \frac{b + dz}{a + cz} \right)
\]
This transcends to a dual weight $\lambda$ right action on $\mathcal{D}(L) := \text{Hom}_{\text{cts}}(\mathcal{A}(\mathcal{O}_F \otimes_{\mathcal{O}_F} \mathbb{Z}_p), L)$ - the space of $L$-valued locally analytic distributions on $\mathcal{O}_F \otimes_{\mathcal{O}_F} \mathbb{Z}_p$. We will denote this space by $\mathcal{D}_\lambda(L)$ to make the weight $\lambda$ action implicit.

Let $U \subset \mathcal{W}_F$ be an affinoid with associated universal character $\lambda_{\text{un}}^U : (\mathcal{O}_F \otimes_{\mathbb{Z}_p})^\times \to \mathcal{O}(U)^\times$ i.e. any weight $\lambda : (\mathcal{O}_F \otimes_{\mathbb{Z}_p})^\times \to L^\times$ in $U(L)$ factors via

$$(\mathcal{O}_F \otimes_{\mathbb{Z}_p})^\times \xrightarrow{\lambda_{\text{un}}^U} \mathcal{O}(U)^\times \xrightarrow{\text{ev}_\lambda} L^\times$$

where the last map is evaluation at $\lambda$. The universal character enables us to equip $\mathcal{A}_U := \mathcal{A}(\mathcal{O}(U))$ with a \textquote{weight $U$} action of the semi-group $\Sigma_0(\mathfrak{p})$ as follows

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot U f(z) = \lambda_{\text{un}}^U(a + cz)f\left(\frac{b + dz}{a + cz}\right)$$

and correspondingly a dual \textquote{weight $U$} right action on $\mathcal{D}_U := \mathcal{D}(\mathcal{O}(U))$. For $W \subset U$ a closed subset, we have an isomorphism

$$\mathcal{D}_U \otimes_{\mathcal{O}(U)} \mathcal{O}(W) \cong \mathcal{D}_W$$

Particularly, if $\lambda \in U(L)$ corresponds to a maximal ideal $\mathfrak{m}_\lambda \subset \mathcal{O}(U)$, then we have $\mathcal{D}_U \otimes_{\mathcal{O}(U)} \mathcal{O}(U)/\mathfrak{m}_\lambda \cong \mathcal{D}_\lambda$.

. For the rest of this paper, we shall fix an affinoid $U \subset \mathcal{W}_F$ that contains the classical (parallel) weight $\lambda_{k_0}$. Also, since $p$ is inert in $F$ under (Heeg-Hyp), we have

$$\mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \cong \mathcal{O}_{F_p}$$

where $\mathfrak{p}$ is the unique prime in $\mathcal{O}_F$ above $p$. We will also denote $E(N)$ simply by $\mu(\mathcal{O}_F)$ henceforth. By Remark 4.3 we have

$$\lambda_{k_0}(z) = N_{F/F_p}^1(z)^{k_0} = N_{F_p/\mathbb{Q}_p}(z)^{k_0}$$

Further since $(\mathcal{O}_F \otimes_{\mathbb{Z}_p})^\times \cong \mathcal{O}_{F_p}^\times \cong (F_p)^\times \times (1 + \mathfrak{p}\mathcal{O}_{F_p})$, we may write any $z \in \mathcal{O}_{F_p}$ in the form $z := [z](z)$, where $[z] \in (F_p)^\times$ and $(z) \in (1 + \mathfrak{p}\mathcal{O}_{F_p})$ is the projection to the group of principal units. Upto shrinking the affinoid $U \subset \mathcal{W}_F$, we may assume that any $\lambda \in U(L)$ is of the form $\lambda(z) = [z]^{k_0}(z)^s := [z]^{k_0}\exp(s \log_{\mathfrak{p}}(z))$ for $s \in \mathcal{O}_{F_p}$. In particular, any classical (parallel) weight $\lambda_k \in U(L)$ is of the form

$$\lambda_k(z) = [N_{F_p/\mathbb{Q}_p}(z)]^{k_0} N_{F_p/\mathbb{Q}_p}(z)^k$$

Note that $\forall \lambda_k \in U$ we have $k \equiv k_0 \text{ mod } (p^2 - 1/|\mu(\mathcal{O}_F)|)$.\n
**Remark 4.5.** Let $\mathcal{W}_{F,\text{par}} \subset \mathcal{W}_F$ be the parallel weight line defined as the image of $\mathcal{W}_Q := \text{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, L^\times)$ in $\mathcal{W}_F$. Our choice of the affinoid $U$ will be such that $U \subset \mathcal{W}_{F,\text{par}}$. In particular $(U, k_0)$ will be a slope-adapted affinoid as defined in [BSW21 Section 4.1]. By abuse of notation, we shall also denote the pre-image of $U$ in $\mathcal{W}_Q$ by $U$.

. Let $\chi_Q^{\text{cycl}} : G_Q \to \mathbb{Z}_p^\times$ be the $p$-adic cyclotomic character. We denote by $\chi_F^{\text{cycl}}$ its restriction to $G_F$ which corresponds, via global class field theory, to a character

$$\chi_F^{\text{cycl}} : F^\times/\mathbb{A}_F^\times \to \mathbb{Z}_p^\times$$

normalized in such a way that the restriction $\chi_F^{\text{cycl}}|_{\mathcal{O}_{F_p}^\times}$ equals $N_{F_p/\mathbb{Q}_p}$.
Set \( \mathcal{W} := (F_F)^2 - \{0, 0\} \) and denote by \( \mathcal{W} \) the space of orbits \( \mathcal{W} := \mu(O_F)\backslash \mathcal{W} \), where \( \mu(O_F) \) acts diagonally. Consider the projection given by
\[
\pi : \mathcal{W} \longrightarrow \mathbb{P}^1(F_F)
\]
\[
\pi((x, y)) := [x : y]
\]
Note that \( \pi \) factors via \( \mathcal{W} \). For \( \mathcal{L} \subset (F_F)^2 \) any \( O_{F_F} \)-lattice, let \( \mathcal{L}' \) denote the set of primitive vectors of \( \mathcal{L} \). For \( g \) any \( O_{F_F} \)-basis of \( \mathcal{L} \), we set \( |\mathcal{L}| := (N_{F/F(p)})^{\text{ord}_p(\det(g))} \). As before, let \( \mathcal{L}_\star \) denote the standard lattice \( O_{F_F} \oplus pO_{F_F} \) and \( \mathcal{L}_\infty := O_{F_F} \oplus pO_{F_F} \) which correspond to the vertices \( v_\star \) and \( v_\infty \) in the Bruhat–Tits tree \( \mathcal{T}_p \). Let \( e_\infty \in \mathcal{E}(\mathcal{T}_p) \) denote the (oriented) edge joining \( v_\star \) and the \( v_\infty \). In fact, for any \( e \in \mathcal{E}(\mathcal{T}_p) \), we fix lattices \( \mathcal{L}_s(e) \) and \( \mathcal{L}_t(e) \) such that the homothety classes \( [\mathcal{L}_s(e)] \) and \( [\mathcal{L}_t(e)] \) represent the source and target vertices in the Bruhat–Tits tree respectively. For every \( e \in \mathcal{E}(\mathcal{T}_p) \), denote by \( W_e := \mathcal{L}_s(e) \cap \mathcal{L}_t(e) \) and its image in \( \mathcal{W} \) by \( Y_e \). Let \( U_e \subset \mathbb{P}^1(F_F) \) be the open compact subset as in [BSW19 Proposition 2.4]. Note that we have \( p^\infty W_e := \bigcup p^nW_e = \pi^{-1}(U_e) \). In particular \( U_\infty = \mathcal{O}_{F_F} \). For brevity, we denote \( W_\infty = O_{F_F} \oplus pO_{F_F} \) and \( Y_\infty \) simply by \( W_\infty \) and \( Y_\infty \) respectively.

For \( \mathcal{X} \) any open compact subset of \( \mathcal{W} \), \( \mathcal{Y} \) or \( \mathbb{P}^1(F_F) \), we denote by \( \mathcal{A}(\mathcal{X}) \) the space of \( L \)-valued locally analytic functions on \( \mathcal{X} \) and accordingly by \( \mathcal{D}(\mathcal{X}) \) the space of locally analytic distributions. For any \( \mu \in \mathcal{D}(\mathcal{X}) \) and any \( F \in \mathcal{A}(\mathcal{X}) \), we use the measure theoretic definition \( \int_{\mathcal{X}}Fd\mu \) to denote \( \mu(F) \). Further, if \( \mathcal{X}' \subset \mathcal{X} \) is any subset, then by \( \int_{\mathcal{X}} \), we mean \( \mu(F, \mathcal{X}') \) where \( \chi_{\mathcal{X}'} \) is the characteristic function on \( \mathcal{X}' \).

By viewing elements of \( (F_F)^2 \) as column vectors, we have a natural left action of \( \text{GL}_2(F_F) \) on \( (F_F)^2 \) which induces a left action on the spaces \( \mathcal{W}, \mathcal{Y} \) and the Bruhat–Tits tree \( \mathcal{T}_p \). For \( \mathcal{L} \) any lattice, we have an induced left action of \( \text{GL}_2(O_{F_F}) \) on \( \mathcal{L}' \) and on \( \mathcal{L}' := \mu(O_F)\backslash \mathcal{L}' \subset \mathcal{W} \). The diagonal action of \( O_{F_F}^\times \) on \( \mathcal{L}' \), given by \( t.(x, y) := (tx, ty) \), descends to an action of \( O_{F_F}^\times / \mu(O_F) \) on \( \mathcal{L}' \). Note that we have a natural \( \text{GL}_2(O_{F_F}) \) action on \( \mathcal{A}(\mathcal{X}) \) and an induced left \( \text{GL}_2(O_{F_F}) \) action on \( \mathcal{D}(\mathcal{X}) \). We set \( \mathcal{D} := \mathcal{D}(\mathcal{L}_\infty) \) where \( \mathcal{L}_\infty := \mu(O_F)\backslash \mathcal{L}' \). We give \( \mathcal{D} \) a \( \mathcal{D}(O_{F_F}^\times / \mu(O_F)) \)-module structure as follows
\[
\mathcal{D}(O_{F_F}^\times / \mu(O_F)) \times \mathcal{D} \longrightarrow \mathcal{D}
\]
\[
(\nu, \mu) \mapsto \nu\mu
\]
where
\[
\int_{\mathcal{L}_\infty} F(x, y)d\mu(x, y) := \int_{O_{F_F}^\times / \mu(O_F)} \left( \int_{\mathcal{L}_\infty} F(zx, zy)d\mu(x, y) \right) d\nu(z)
\]
for \( z \in O_{F_F}^\times / \mu(O_F) \). Further, we also give \( \mathcal{O}(U) \) a \( \mathcal{D}(O_{F_F}^\times / \mu(O_F)) \)-module structure via the Fourier transform given by
\[
\nu \mapsto \left[ \lambda \mapsto \int_{O_{F_F}^\times / \mu(O_F)} \lambda(z)d\nu(z) \right]
\]
Let \( \mathcal{D}_U := \mathcal{O}(U)\hat{\otimes}_{\mathcal{D}(O_{F_F}^\times / \mu(O_F))} \mathcal{D} \) denote the completed tensor product over \( \mathcal{D}(O_{F_F}^\times / \mu(O_F)) \).

**Definition 4.6.** Let \( \lambda \in U(L) \). A function \( f \in \mathcal{A}(\mathcal{X}) \) is said to be homogeneous of weight \( \lambda \) if
\[
f(zx, zy) = \lambda(z)f(x, y) = |z|^{k_0(N_{F_F/F}(z))} f(x, y)
\]
for all \( z \in O_{F_F}^\times / \mu(O_F) \). We denote the subspace of homogeneous of weight \( \lambda \) functions by \( \mathcal{A}^\lambda(\mathcal{X}) \subset \mathcal{A}(\mathcal{X}) \)
Remark 4.7. Note that for any integer $k \in \mathbb{Z} \geq 2$, the space $V_{k,k}(X)$ is homogeneous of weight $\lambda_k$. In particular $V_{k,k}(X) \subset A^{\lambda_k}(X)$.

For $\lambda \in U(L)$, we define

$$B_\lambda : \mathcal{O}(U) \times \mathbb{D} \rightarrow \text{Hom}_{\text{cont}}(A^{\lambda}(\tilde{L}_*), L)$$

$$B_\lambda(\alpha, \mu)(F) := \alpha(\lambda) \int_{\tilde{L}_*} F(x, y) d\mu(x, y)$$

In particular, we have a bilinear pairing

$$B : \mathbb{D}_U \times A^{\lambda}(\tilde{L}_*) \rightarrow L$$

We once again use the measure-theoretic notation to denote $B(\mu_U, F)$ as $\int_{\tilde{L}_*} F(x, y) d\mu_{U}(x, y)$. Fix $\tau \in H^\text{ur}$. For $P \in A(\tilde{L}_*)$ and $\mu_U \in \mathbb{D}_U$, we will make sense of the following quantity

$$\int_{\tilde{L}_*} P(x, y) \log_p ((x - \tau y)(\overline{x} - \tau \overline{y})) \mu_U$$

below in Definition 4.10. Here $\log_p$ is the Iwasawa $p$-adic logarithm as before (i.e. the branch such that $\log_p(p) = 0$). Define

$$F : U(L) \times \tilde{L}_* \rightarrow \mathbb{C}_p$$

$$F(\lambda, (x, y)) := P(x, y) ((x - \tau y)(\overline{x} - \tau \overline{y}))^{\lambda - \lambda_{k_0}}$$

Here $(x - \tau y)^{\lambda - \lambda_{k_0}}$ stands for $\langle x - \tau y \rangle^{\lambda - \lambda_{k_0}} = \exp((s - k_0) \log(x - \tau y))$, where $s \in \mathcal{O}_{F_p}$ is such that $\lambda(z) = [z]^{k_0}(z)^s$.

Remark 4.8. The above integral can also be written as $\int_{\tilde{L}_*} P(x, y) \log_p \circ \text{N}_{F_p/Q_p}(x - \tau y) \mu_U$.

For the rest of this section, assume that the affinoid $U$ is smooth at $\lambda = \lambda_{k_0}$ (See Remark 4.12) below. Then

Lemma 4.9. For $\mu_U \in \mathbb{D}_U$, the function $U(L) \rightarrow \mathbb{C}_p$ given by

$$\lambda \mapsto \int_{\tilde{L}_*} F(\lambda, (x, y)) \mu_U(x, y)$$

is analytic around the point $\lambda_{k_0} \in U(L)$.

Proof. The same proof as in [BDI10, Lemma 4.5] goes through. \qed

With notations in place as above, Lemma 4.9 motivates us to make the following definition.

Definition 4.10. We define the derivative

$$\left( \frac{d}{d\lambda} \int_{\tilde{L}_*} F(\lambda, (x, y)) \mu_U(x, y) \right)_{\lambda = \lambda_{k_0}}$$

to be $\int_{\tilde{L}_*} P(x, y) \log_p ((x - \tau y)(\overline{x} - \tau \overline{y})) \mu_U(x, y) = \int_{\tilde{L}_*} P(x, y) \log_p \circ \text{N}_{F_p/Q_p}(x - \tau y) \mu_U(x, y)$.

The following result will be useful to compute derivatives of $p$-adic $L$-functions later on in the sequel.
Proposition 4.11. Let $P(x, y) \in V_{k_0, k_0}(L'_s)$ and $\tau_1, \tau_2 \in \mathcal{H}_p^{ur}$. Then for $\mu \in \mathbb{D}_U$, we have

$$
\frac{d}{d\lambda} \left( \int_{L'_s} P(x, y) \langle (x - \tau_1 y)(x - \tau_2 y) \rangle^{\frac{\lambda - \lambda_0}{2}} d\mu_U(x, y) \right)_{\lambda = \lambda_0} \\
= \frac{1}{2} \frac{d}{d\lambda} \left( \int_{L'_s} P(x, y) \langle (x - \tau_1 y)(x - \tau_2 y) \rangle^{\lambda - \lambda_0} d\mu_U(x, y) \right)_{\lambda = \lambda_0} \\
+ \frac{1}{2} \frac{d}{d\lambda} \left( \int_{L'_s} P(x, y) \langle (x - \tau_2 y) \rangle^{\lambda - \lambda_0} d\mu_U(x, y) \right)_{\lambda = \lambda_0}
$$

Proof. This follows from Definition 4.10. See also [BDI10] Remark 4.7 and [Sev12] Proposition 4.2.

Remark 4.12. There is, in particular, a paucity of smooth points in the Bianchi eigenvariety. However, striking results of Barrera Salazar–Williams (c.f. [BSW21] Sections 5.2 & 5.3) show that base-change points are smooth in $\mathcal{E}_{par}$—the parallel weight eigenvariety, which is the case that we are interested in.

4.1. Families of Bianchi modular symbols. Recall that $\mathcal{F} \in S_{k_0+2}(U_0(N))^{new}$ is a (parallel) weight $k_0 + 2$ newform, where $N = pM$ as ideals in $\mathcal{O}_p$ and $(p, M) = 1$. By [BSW19] Corollary 4.8, we know that

$$\mathcal{F} | U_p = \omega_p N_{F/Q}(p)^{k_0/2} \mathcal{F} = \pm p^{k_0} \mathcal{F}$$

where $U_p$ is the Hecke operator at $p$ and $-\omega_p \in \{\pm 1\}$ is the Atkin–Lehner eigenvalue at $p$. In particular $\mathcal{F}$ is of non-critical slope in the sense of [BSW21]. Let $x_{k_0} \in E(L)$ denote the classical cuspidal point in the Bianchi eigenvariety $\mathcal{E}$ defined over $L/Q_p$. Upto shrinking $U$, we may assume that $x_{k_0}$ varies in a family over the affinoid $U$ (See [BSW21] Definition 4.1 and Theorem 3.8). We will denote the connected component containing $x_{k_0}$ in $\mathcal{E}$ by $V = \text{Sp}(T)$. It follows from [BSW21] that $U \subset \mathcal{W}_{F, par}$ and $V \subset \mathcal{E}_{BC} \subset \mathcal{E}_{par}$ where $\mathcal{E}_{par}$ (resp. $\mathcal{E}_{BC}$) is the parallel weight eigenvariety (resp. base-change eigenvariety) of [BSW21] Section 5.

Remark 4.13. Note that since $\mathcal{F}$ is, by assumption, the base-change to $F$ of an elliptic cuspidal newform $f \in S_{k_0+2}(\Gamma_0(N))$, we have that $\mathcal{F} | U_p = (a_p(f))^2 = p^{k_0} \mathcal{F}$. In particular $\omega_p = 1$. This will be crucial when we consider the trivial zero phenomenon in §6 later.

We recall the following result of Barrera Salazar and Williams.

Proposition 4.14 (Barrera Salazar–Williams). Every irreducible component of $\mathcal{E}_{par}$ is one-dimensional and contains a Zariski-dense set of classical points.

Proof. See [BSW21] Proposition 5.1. □

By assumption, $(U, k_0)$ is a slope-adapted affinoid in $\mathcal{W}_{F, par}$ containing the classical weight $\lambda_{k_0}$. In particular, for each classical (parallel) weight $\lambda_k \in U$, the point $x_k \in V \subset \mathcal{E}_{BC}$ lying above $\lambda_k$ (i.e. $\lambda_k = w(x_k)$ for the weight map $w$) corresponds to a classical cuspidal (base-change) Bianchi eigenform of slope $h = k_0$ by the Control Theorem of [Wil17]. We will denote the forms corresponding to $x_k \in V$ by $\mathcal{F}_k \in S_{k_0+2}(U_0(N))$ and correspondingly the elliptic cuspidal eigenforms by $f_k \in S_k(\Gamma_0(N))$. For all $k \neq k_0$, the eigenforms $\mathcal{F}_k$ (resp. $f_k$) are old
at \( p \) (resp. at \( p \)) and there exists a unique normalized newform \( F_k^\# \in S_{k+2}(U_0(M))_{\text{new}} \) (resp. \( f_k^\# \in S_{k+2}(\Gamma_0(M))_{\text{new}} \), such that for all \( g \in \text{GL}_2(\mathbb{A}_F) \):

\[
F_k(g) = F_k^\#(g) - \frac{N_{F/Q}(p)^{k+1}}{a_p(F_k)} F_k^\# \left( \begin{pmatrix} \pi_p & 0 \\ 0 & 1 \end{pmatrix} g \right)
\]

and

\[
f_k(q) = f_k^\#(q) - \frac{p^{k+1}}{a_p(k)} f_k^\#(q^p)
\]

where \( a_p(F_k) \) (resp. \( a_p(f_k) \)) is the \( U_p \)-eigenvalue of \( F_k \) (resp. \( U_p \)-eigenvalue of \( f_k \)) and \( \pi_p \) is the idelic representative of \( p \). We will also set \( F_k^\#_{k_0} := F_{k_0} = F \) (resp. \( f_k^\#_{k_0} := f_{k_0} = f \)). To ease our notation, we will denote by \( F(g) \) (resp. \( f(q) \)) to denote the Coleman family, over the affinoid \( U \), of cupidal Bianchi eigenforms (resp. cuspidal elliptic eigenforms) passing through the forms \( F_k \) (resp. \( f_k \)) for classical weights \( \lambda \in U \). We will also occasionally denote by \( f/F \) to denote the (base-change) family \( F \).

For each \( k \neq k_0 \) such that \( \lambda \in U(L) \), we will denote by

\[
\phi_k^\# := \frac{\phi_k^\#}{\Omega_k^\#} \in \text{Sym}^{1}_{\text{sym}}(V_{k,k}(\mathbb{E}_k)^\vee)
\]

for the Bianchi modular symbol attached to \( F_k^\# \) in [Wil17], where \( \Omega_k^\# \in \mathbb{C}^\times \) is some complex period and \( \mathbb{E}_k/Q \) is some number field. Similarly, let \( \phi_k \in \text{Sym}^{1}_{\text{sym}}(V_{k,k}(\mathbb{E}_k)^\vee) \) denote the Bianchi modular symbol attached to \( F_k \). Note that, similar to (4.3), we have a relation between the modular symbols \( \phi_k \) and \( \phi_k^\# \).

Recall the space \( D_U := \mathcal{O}(U) \otimes_{\mathcal{O}(\mathbb{F}_p)/\mu(\mathcal{O}_F)} \mathcal{D}(\overline{\mathbb{F}_p}) \). For each classical weight \( \lambda \in U(L) \), we can define a weight \( \lambda \)-specialization map

\[
\rho_{\lambda} : \text{Sym}^{1}_{\text{sym}}(\mathbb{D}_U) \rightarrow \text{Sym}^{1}_{\text{sym}}(\mathbb{D}_{k,k}(L)) \rightarrow \text{Sym}^{1}_{\text{sym}}(V_{k,k}(\mathbb{E}_k)^\vee)
\]

given by

\[
\rho_{\lambda}(\Phi([r-s]))(P) := \int_{Y_\infty} P(x,y)d\Phi([r-s])(x,y)
\]

for \( P(x,y) \in V_{k,k}(L) \) and \( r,s \in \mathbb{F}_k^1(F) \). Recall that \( Y_\infty = \mu(\mathcal{O}_F) \setminus (\mathcal{O}_F^\times \oplus p\mathcal{O}_F) \). We now state the following Theorem which follows from a combination of results of [Wil17] and [BSW21].

**Theorem 4.15** (Williams, Barrera Salazar–Williams). There exists \( \Phi_\infty \in \text{Sym}^{1}_{\text{sym}}(\mathbb{D}_U) \) such that

- For every classical weight \( \lambda \in U(L) \), we have \( \rho_{\lambda}(\Phi_\infty) = c(k)\phi_k \) for some \( p \)-adic periods \( c(k) \in L^\times \).
- \( \rho_{\lambda_0}(\Phi_\infty) = \phi_k \).

**Proof.** This follows from a combination of [BSW21] Proposition 6.7 and the Control Theorem [Wil17] Corollary 4.3. \( \square \)

In particular for all \( \lambda \in U(L) \), we can explicitly write down the weight \( \lambda \)-specialization as

\[
\rho_{\lambda}(\Phi_\infty)([r-s])(P,\chi_{Y_\infty}) = c(k)\phi_k([r-s])(P)
\]
Lemma 4.18. For all $P(x, y) \in V_{k,k}(L)$. Recall that for a lattice $\mathcal{L} \subset (F_p)^2$, we denote by $\bar{L} = \mu(\mathcal{O}_F) \backslash \mathcal{L}$ its image in $\mathcal{Y}$. We have the following result that defines a family of of distributions $\{\Phi_{\bar{L}}\}_{\bar{L}} \in \text{Symbr}_{\mathcal{Y}}(\mathcal{D}_U)$ indexed by the sets $\bar{L}$ in $\mathcal{Y}$.

**Proposition 4.16.** There exists a family of distributions $\{\Phi_{\bar{L}}\}_{\bar{L}} \in \text{Symbr}_{\mathcal{Y}}(\mathcal{D}_U)$ that satisfy

- $\Phi_{\bar{L}} = \Phi_\infty$
- For all $\gamma \in \bar{\Gamma}$ and for all compact open sets $Y \subset \mathcal{Y}$,

$$\int_{\gamma Y} (F|\gamma^{-1})d\Phi_{\gamma Y}\{\gamma r - \gamma s\} = \int_Y (F)d\Phi_{\bar{L}}\{r - s\}$$

**Proof.** The proof follows [BD07] Proposition 1.8. Since $\bar{\Gamma}$ acts transitively on the set of lattices $\mathcal{L} \subset (F_p)^2$, the induced action on the sets $\bar{L}$ is also transitive. The stabilizer of $\bar{L}_s$ under the action is $\Gamma_0(\mathcal{N})$. Since $\Phi_\infty$ is invariant under the action of $\Gamma_0(\mathcal{N})$, the family of distributions is well-defined and determines $\Phi_{\bar{L}}$ uniquely once we set $\Phi_{\bar{L}_s} := \Phi_\infty$. \qed

We will now recall some useful Lemmas of similar nature to those recorded in [Sev12], [BD07], [BD10] and [BD09] and skip the proofs.

**Lemma 4.17.** Let $\mathcal{L}_1, \mathcal{L}_2$ be two lattices in $(F_p)^2$ such that $\mathcal{L}_2 \subset \mathcal{L}_1$ is a sub-lattice of index $N_{F/Q}(p)$. Let $e \in \mathcal{E}(T)$ denote the (ordered) edge joining the vertices corresponding to the homothety classes $[\mathcal{L}_1]$ and $[\mathcal{L}_2]$. Then for all $\lambda_k \in U(L)$ and for all $F \in \mathcal{A}_<(Y_e)$, we have

$$\int_{Y_e} F d\Phi_{\mathcal{L}_2}\{r - s\} = a_p(F_k) \int_{Y_e} F d\Phi_{\mathcal{L}_1}\{r - s\}$$

where $Y_e = \bar{L}_1 \cap \bar{L}_2$.

**Proof.** The proof is similar to that of [BD07] Lemma 1.10. \qed

**Lemma 4.18.** For all $\lambda_k \in U(L)$ and all $P(x, y) \in V_{k,k}(L)$, we have

$$\rho_{\lambda_k}(\Phi_\infty)\{(r - s)\}(P) = c(k) \left(1 - \frac{N_{F/Q}(p)^k}{a_p(F_k)^2}\right) (\phi_{k_0}^\#)\{(r - s)\}(P)$$

**Proof.** The proof goes through as in [BD09] Proposition 2.4. \qed

For every set $\bar{L} = \mu(\mathcal{O}_F) \backslash \mathcal{L}$, we can define a pushforward modular symbol $\pi_*(\Phi_{\bar{L}}) \in \text{Symbr}_1(D_{k_0}^E(\mathbb{P}_1(F_p),L))$ given by

$$\pi_*(\Phi_{\bar{L}})\{(r - s)\}(F(t)) := |\mathcal{L}|^{-k_0/2}\Phi_{\bar{L}}\{(r - s)\}(F(x,y))$$

where $F(t) \in \mathcal{A}_{k_0}(\mathbb{P}_1(F_p),L)$ is a locally analytic function on $\mathbb{P}_1(F_p)$ except for a pole of order at most $k_0$ at $\infty$. Here we use the convention that $F(t) := F(t,1)$ and $F(x,y) := (N_{F/Q}(y))^{k_0}F(x/y)$ accordingly. Recall that there exists a unique harmonic modular symbol $\Phi_{\bar{L}}^{\text{har}} \in \text{MS}_F(L)$ that lifts $\phi_{k_0} = \phi_{k_0}^\#$. The following Corollary relates the pushforward modular symbols with the harmonic modular symbol $\Phi_{k_0}^{\text{har}}$.

**Corollary 4.19.** For all lattices $\mathcal{L}$, $\pi_*(\Phi_{\bar{L}}) = \Phi_{k_0}^{\text{har}} \in \text{MS}_F(L)$
Proof. The Lemma follows by a repeated application of Lemma 4.17 combined with the weight $\lambda_k$-specialization of (4.6). Compare with [Sev12, Corollary 4.7] and [BD07, Proposition 2.12]. Since our form $F_{k_0}$ is split–multiplicative (See Remark 4.13), we don’t need to restrict to even vertices as in [Sev12]. □

Following [Sev12, Definition 4.8], we define

Definition 4.20. The semidefinite integral attached to $r, s \in \mathbb{P}^1(F)$; $\tau \in H_p^{ur}$ and $P(x, y) \in V_{k_0,k_0}$ is defined as

\[(4.9) \quad \int_r^s \int_r^\tau P(x, y) \omega_F := |\mathcal{L}_\tau|^{-k_0/2} \]
\[
\frac{d}{d\lambda} \left( \int_{\mathcal{L}_\tau} P(x, y)(N_{F/Q}(x - \tau y))^{\lambda-k_0} d\Phi_{\mathcal{L}_\tau}(r-s)(x, y) \right)_{\lambda=\lambda_k_0}
\]

where $[\mathcal{L}_\tau] = \text{red}(\tau)$.

Proposition 4.21. For all $\gamma \in \Gamma$ and $\tau \in H_p^{ur}$, we have

\[
\int_{r' \gamma} \int_{r' \gamma} P(x, y) \omega_F = \int_r^s \int_r^\tau (\gamma P(x, y)) \omega_F
\]

Proof. See [BDI10, Proposition 6.6]. □

Proposition 4.22. For $r, s \in \mathbb{P}^1(F)$ and $\tau_1, \tau_2 \in H_p^{ur}$, we have

\[
\int_r^s \int_r^\tau P(x, y) \omega_F - \int_r^s \int_r^\tau P(x, y) \omega_F = \int_r^s \int_{\tau_1}^{\tau_2} P(x, y) \omega_F^{\text{log}_p} + 2 \frac{a_p(F)}{a_p(F)} \int_r^s \int_{\tau_1}^{\tau_2} P(x, y) \omega_F^{\text{ord}_p}
\]

Proof. The method of proof is similar to that in [BDI10, Proposition 6.7] where a similar result is proved for elliptic cusp forms with split–multiplicative reduction at a prime $p$. See also [Sev12, Proposition 4.10]. Note that here we have the ‘normed log’ and ‘ord’ double integrals of Definition 2.10. □

Proposition 4.23. The $L$-invariant attached to the Bianchi cusp form $F \in S_{k_0+2}(U_0(N))^{\text{new}}$ is given by

\[
\mathcal{L}_F^{BW} = -2 \frac{a_p'(F)}{a_p(F)}
\]

Proof. Recall that the Bianchi cuspform $F$ is the base–change to $F$ of the elliptic cuspidal newform $f \in S_{k_0+2}(\Gamma_0(N))$. By [VW21, Lemma 4.4], we know that $\mathcal{L}_F^{BW} = 2\mathcal{L}_p(f)$. Moreover, since $(p) = (p)$ is inert in $F$, we know that $a_p(F) = \alpha_p(f)^2$ where $\alpha_p$ is the $U_p$-eigenvalue of $f$. The result follows since it is known that (See for eg. [Sev12, Theorem 4.11])

\[
\mathcal{L}_p(f) = -2 \frac{a_p'(f)}{a_p(f)}
\]

□
Corollary 4.24. Choose the branch of the $p$-adic logarithm corresponding to $\ell = L_p(f)$. Then the semidefinite integral $\int_r^s \int \tau_2 P(x,y)\omega_F$ satisfies
\[
\int_r^s \int \tau_2 P(x,y)\omega_F - \int_r^s \int \tau_1 P(x,y)\omega_F = \int_r^s \int \tau_1 P(x,y)\omega_F^{\log L_p(f)}
\]

Proof. By Propositions 4.22 and 4.23 above, we have
\[
\int_r^s \int \tau_2 P(x,y)\omega_F - \int_r^s \int \tau_1 P(x,y)\omega_F = \int_r^s \int \tau_1 P(x,y)\omega_F^{\log L_p} - L_{p}^{BW} \int_r^s \int \tau_1 P(x,y)\omega_F^{\text{ord}_p}
\]

The Corollary now follows from Remark 3.2 \(\square\)

Theorem 4.25. Let $\Psi \in \text{Emb}(O, R)$ and let $r \in \PP^1(F)$ be an arbitrary base point. Then
\[
\int_r^s \int \tau_2 P_{\Psi}^{k_0/2} \omega_F = \left(\sqrt{N_{F/Q}(D_{K/F})}\right)^{k_0/2} \log \Phi_{\text{AJ}}(D|\Psi|)(\Phi_{\text{bar}})
\]

Proof. By Corollary 4.24, the $\text{MS}_F(L)^{\vee}(F)$-valued semidefinite integration theory
\[
\int_s^r \int \tau_2 P(x,y)\omega_F
\]

lifts
\[
\int_s^r \int \tau_2 P(x,y)\omega_F^{\log L_p(f)}
\]

The result now follows from Remark 3.2 on combining with Theorem 2.20 (See [VW21, Theorem 6.5] for more details) since the $p$-adic Abel–Jacobi image of Stark–Heegner cycles are independent of the choice of the $p$-adic Abel–Jacobi map. \(\square\)

5. $p$-adic $L$-functions

In this section, we briefly recall the construction and interpolation properties of the various $p$-adic $L$-functions that are germane to this article.

5.1. Stevens–Mazur–Kitagawa $p$-adic $L$-functions. Let $\chi$ be a quadratic Dirichlet character of conductor $c(\chi)$ relatively prime to $N$. For $k \in U$ a classical weight, which we view as a classical weight on the weight space $W_{Q} := \text{Hom}_{cts}(\mathbb{Z}_{p}^{\times}, L^{\times})$ when the context is clear, let $L(f_k^#, \chi, s)$ be the analytic continuation of the $\chi$-twisted Dirichlet $L$-series $\sum_{n \geq 1} \frac{\chi(n)\eta(n)}{n^s}$ for $s \in \mathbb{C}$. We denote by $\Omega_{\infty}(f_k^#)^{\pm} \in \mathbb{C}^{\times}$ to be the Shimura periods that satisfy
\[
\Omega_{\infty}(f_k^#)^{+} \cdot \Omega_{\infty}(f_k^#)^{-} = \langle f_k^#, f_k^# \rangle_{M_{p}^{\ast}(\kappa)}
\]

where $r(k_0) = 1$ and zero otherwise, and $\langle , \rangle$ is the usual Petersson norm. Then the algebraic part of the special $L$-values, for $1 \leq j \leq k + 1$, are given by
\[
L_{\text{alg}}(f_k^#, \chi, j) = \frac{(j - 1)! \cdot \tau(\chi) \cdot c(\chi)^{j} \cdot L(f_k^#, \chi, j)}{(-2\pi i)^{j - 1} \cdot \Omega_{\infty}(f_k^#)^{\pm}} \in \mathbb{Q}(f_k^#, \chi)
\]
where \( \tau(\chi) = \sum_{a \in (\mathbb{Z}/c(\chi)\mathbb{Z})^*} \chi(a) \cdot c_c^a(\chi) \) is the Gauss sum of \( \chi \). The sign of the Shimura period is chosen according to the sign of

\[
\chi(-1) = \pm (-1)^{j-1}
\]

**Theorem 5.1** (Stevens, Mazur, Kitagawa). There exists \( p \)-adic periods \( \lambda^\pm(k) \in L \) for \( k \in U^{\text{cl}} \) (the set of classical weights in the affinoid \( U \)) such that for any quadratic Dirichlet character \( \chi \), there is a locally analytic \( p \)-adic function \( L_p(f, \chi) \) on \( U \times \mathbb{Z}_p \) that satisfies, for \( 1 \leq j \leq k+1 \),

\[
L_{f, \chi}(k, j) := L_p(f, \chi)(k, j) = \begin{cases} 
\lambda^\pm(k) \left( 1 - \frac{\chi(p)p^{-1}}{\omega_p(k)} \right) L^{\text{alg}}(f_k, \chi, j), & \text{if } k = k_0 \\
\lambda^\pm(k) \left( 1 - \frac{\chi(p)p^{-1}}{\omega_p(k)} \right) \left( 1 - \frac{\chi(p)p^{k-j+1}}{\omega_p(k)} \right) L^{\text{alg}}(f_k^\#, \chi, j), & \text{if } k \neq k_0
\end{cases}
\]

**Remark 5.2.** We can normalize \( L_{f, \chi} \) such that \( \lambda^\pm(k_0) = 1 \).

\( L_{f, \chi}(k, s) \) is the two-variable \( p \)-adic \( L \)-function attached to the Coleman family \( f(q) \) and the character \( \chi \) constructed by Mazur (unpublished) and Kitagawa ([Kit94]) in the slope zero case (Hida families) and by Stevens in the finite slope scenario ([PS11] & [PS13]). We will be primarily interested in the slice of the two variable \( p \)-adic \( L \)-function along the central critical line \( s = k/2 + 1 \). Namely, set

\[
\mathcal{L}_p(f, \chi, k) := L_p(f, \chi)(k, k/2 + 1)
\]

for all classical weights \( k \in U^{\text{cl}} \). Theorem 5.1 above implies that

\[
\mathcal{L}_p(f, \chi, k) = \begin{cases} 
\left( 1 - \frac{\chi(p)p^{k/2}}{\omega_p(k)} \right) L^{\text{alg}}(f_k, \chi, k/2 + 1), & \text{if } k = k_0 \\
\lambda^\pm(k) \left( 1 - \frac{\chi(p)p^{k/2}}{\omega_p(k)} \right)^2 L^{\text{alg}}(f_k^\#, \chi, k/2 + 1), & \text{if } k \neq k_0
\end{cases}
\]

### 5.2. \( p \)-adic \( L \)-functions attached to Bianchi modular forms.

We briefly summarize the construction of \( p \)-adic \( L \)-functions attached to Bianchi modular forms in [Wil17] following the exposition of [BSW19, §3.4] where the \( p \)-adic \( L \)-function is described in terms of analytic functions on \( \mathcal{O}_F \otimes \mathbb{Z}_p \cong \mathcal{O}_F \) (since \( p \) is inert in \( F \)) rather than as locally analytic distributions on \( \text{Cl}_F(p^{\infty}) \) - the ray class group of \( F \) of conductor \( p^{\infty} \). Let \( \mathcal{G} \subseteq \mathcal{O}_F \) be any ideal relatively prime to \( p \). It can be shown that

\[
\text{Cl}_F(\mathcal{G}p^{\infty}) \cong [(\mathcal{O}_F/\mathcal{G})^\times \times (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times]/\mathcal{O}_F^\times
\]

Let \( \mathcal{G} \in S_{k_0+2}(U_0(\mathcal{N})) \) be a small slope cuspidal Bianchi eigenform and let \( \Phi_\mathcal{G} \) be the overconvergent modular symbol of [Wil17] attached to \( \mathcal{G} \). Let \( \mu'_{\mathcal{G}(\text{mod} \mathcal{G})} \) be a distribution on \( \{[a] \times (\mathcal{O}_F \otimes \mathbb{Z}_p) \subset (\mathcal{O}_F/\mathcal{G})^\times \times (\mathcal{O}_F \otimes \mathbb{Z}_p) \} \) (which can be seen as a copy of \( (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \)) defined as

\[
\mu'_{\mathcal{G}(\text{mod} \mathcal{G})} := (g\overline{g})^{k_0/2} \left[ \Phi_\mathcal{G} \left( \begin{array}{cc} 1 & b \\ 0 & g \end{array} \right) \right] \{0 - \infty\}
\]

where \( b \) is some lift of \( a \text{mod} \mathcal{G} \) and \( \mathcal{G}\mathcal{O}_F = (g) \). Combining the distributions for different \( a \in (\mathcal{O}_F/\mathcal{G})^\times \), we get a distribution \( \mu'_{\mathcal{G}} \) on \( (\mathcal{O}_F/\mathcal{G})^\times \times (\mathcal{O}_F \otimes \mathbb{Z}_p) \). On restricting to units in the second variable and then to restricting to functions invariant under \( \mathcal{O}_F^\times \), we obtain a distribution on \( \text{Cl}_F(\mathcal{G}p^{\infty}) \) afforded by the identification above.

**Definition 5.3.** Let \( \chi \) be a finite order Hecke character of conductor \( \mathcal{G} \) where \( \mathcal{G} \) is coprime to \( p \) and \( f \mid p^{\infty} \) (which can be seen as a finite order character of \( \text{Cl}_F(\mathcal{G}p^{\infty}) \)). The \( p \)-adic \( L \)-function
associated to $G \in S_{k+2}(U_0(N))$ is defined to be the analytic function on $(O_F \otimes \mathbb{Z})$ given by

$$L_p(G, \chi, s) := \int_{Cl(p)\infty} (z_p)^s \chi(z) d\mu_p(z),$$

where $s \in O_F \otimes \mathbb{Z}$ and $z_p$ is the projection of $z \in Cl(p)\infty$ to $Cl(p)\infty$.

Germane to this article is the $p$-adic interpolation of the central critical $L$-values for the Bianchi eigenforms $F_k$ (resp. $F_k^\#$, viewed as the base–change $F_k/F$ (resp. $F_k^\#/F$), along the Coleman family $F = f/K$. Let $u_F := [O_F^\times : \mathbb{Z}^\times] = |\mu(O_F)|/2$ and let $\chi$ be a finite order Hecke character of $F$ of conductor $(\epsilon, \chi)$. Then the algebraic part of the central critical $L$-value for $F \in S_{k+2}(U_0(N))$ twisted by $\chi$ is given by

$$(5.4) \quad L_{alg}(F_k^\#, \chi, k/2 + 1) := u_F^2 D_f^{k/2}(k/2)^2 \tau(\chi^{-1})(N_F/Q(\chi_k))^{k/2}(2\pi i)^k \Omega_{\mathfrak{a}_F} L_F(\chi, k/2 + 1) \in \overline{\mathbb{Q}}$$

We have the following interpolation result for the Bianchi $p$-adic $L$-function defined above :-

**Theorem 5.4.** For all classical weights $\lambda_k \in U$, we have

$$(5.5) \quad L_p(F, \chi, k) = \begin{cases} \left(1 - \chi(p)^{\frac{N_F}{\mathfrak{p}(\mathfrak{a}_F)}}\right) \cdot L_{alg}(F_k^\#, \chi, k/2 + 1) & \text{if } k = k_0 \\ c(k) \left(1 - \chi(p)^{\frac{\mathfrak{p}(\mathfrak{a}_F)}}{\mathfrak{a}_F}\right)^2 \cdot L_{alg}(F_k^\#, \chi, k/2 + 1) & \text{if } k \neq k_0 \end{cases}$$

**Proof.** See [BSW19, Theorem 3.12] and also [BD07, Theorem 3.8].

We recall the following factorisation result from [BD07].

**Theorem 5.5.** Let $\epsilon_{F/Q}$ denote the quadratic Dirichlet character associated to the imaginary quadratic field $F/Q$. Then there exists a $p$-adic analytic function $\eta$ of $\kappa \in U$ such that

$$(5.6) \quad L_p(F, \kappa) = L_p(f, \kappa) = \eta(\kappa)u_F^2 L_p(f, \kappa) L_p(f, \epsilon_{F/Q}, \kappa)$$

5.3. **Heegner cycles.** We briefly recall the connection between Heegner cycles and the $p$-adic $L$-functions considered above, primarily following the exposition in [SI03, Sev12, Sev14] and [GSS16]. Recall the factorization $N = pM = pN^+N^-$. Let $B$ (resp. $B$) be the indefinite (resp. definite) quaternion algebra ramified at the primes dividing $pN^-$ (resp. $N^-\infty$). Let $\mathcal{R}' := \mathcal{R}_{N^+,pN^-}$ (resp. $\mathcal{R}'$) be a fixed Eichler order of level $N^+$ in $\mathcal{O}_B$ (resp. of level $pN^+$ in $\mathcal{O}_B$) where $\mathcal{O}_B$ and $\mathcal{O}_B$ are maximal orders in $B$ and $B$ respectively. We set $\mathcal{B} := B \otimes \mathcal{O}_B$ and $\mathcal{R} := \mathcal{R}' \otimes \mathcal{O}_B$. Let $\Sigma = \prod \Sigma_\ell \Rightarrow \mathcal{R}^-\otimes \mathcal{O}_B$. Fix an identification $t_p : B \otimes Q_p \cong M_2(Q_p)$ and set

$$\Gamma'_\Sigma := t_p \left( \mathcal{O}_B[1/p] \cap \prod_{\ell \neq p} \Sigma_\ell \right) = t_p \left( \mathcal{R}'[1/p] \right)$$

and by $\Gamma'_\Sigma = \Gamma'$ to be the subgroups of elements of reduced norm one of $\Gamma'_\Sigma$. Denote by $X := X_{N^+,pN^-}$ the Shimura curve attached to $B$ and by $f^{\text{hol}} \in M_{k+2}(X)$ the weight $k_0 + 2$ modular form on the Shimura curve $X$ attached to $f$ via the Jacquet–Langlands correspondence. By the Cerednik–Drinfeld Theorem of $p$-adic uniformization, we have a rigid analytic isomorphism

$$X_{\Gamma'} := \Gamma' \backslash \mathcal{H}_p \cong X^{an}$$

This identification between the Mumford curve $X_{\Gamma'}$ and the rigid analytification of the Shimura curve $X^{an}$ is defined over $F_p \cong Q_p$ (See [BC91]). We denote by $f^{\text{hol}} \in M_{k+2}(\Gamma', F_p)$ the rigid analytic modular form associated to $f$ via this identification.
Let $\mathcal{M}_{k_0}$ be the Chow motive over $\mathbb{Q}$, attached to the space of weight $k_0 + 2$ modular forms on the Shimura curve $X$ and let $V(k_0/2 + 1) := H_p(\mathcal{M}_{k_0} \otimes \mathbb{Q}_p(k_0/2 + 1))$ be its $p$-adic realization. See [IS03] Appendix 10.1 for the construction. By [IS03, Lemma 5.8], the $G_\mathbb{Q}$-representation $V(k_0/2 + 1)$ maybe realized as the representation attached to weight $k_0 + 2$ cusp forms that are new at the primes dividing $pN^{-}$. In particular, $V_p(f)(k_0/2 + 1)$ maybe realized as the idempotent component (corresponding to $f$) of the representation $V(k_0/2 + 1)$. For $H/\mathbb{Q}$ any number field, we have the global $p$-adic étale Abel–Jacobi map

$$\text{cl}_H := \text{cl}^{k_0/2+1}_{0,H} : \text{CH}^{k_0/2+1}(\mathcal{M}_{k_0} \otimes H) \to \text{Sel}_\text{et}(H, V(k_0/2 + 1))$$

where $\mathcal{M}_{k_0} \otimes H$ is the base–change of $\mathcal{M}_{k_0}/\mathbb{Q}$ to $H$ and $\text{CH}^{k_0/2+1}$ denotes the Chow group of co-dimension $k_0/2 + 1$ cycles. We may also consider the projection $V(k_0/2 + 1) \to V_p(f)(k_0/2 + 1)$ to obtain

$$\text{cl}_{f,H} : \text{CH}^{k_0/2+1}(\mathcal{M}_{k_0} \otimes H) \to \text{Sel}_\text{et}(H, V_p(f)(k_0/2 + 1))$$

Let $\mathfrak{p}$ be a prime in $H$ above $p$ and let $H_{\mathfrak{p}}$ be its $\mathfrak{p}$-adic completion. Then we have a commutative diagram

$$\begin{array}{ccc}
\text{CH}^{k_0/2+1}(\mathcal{M}_{k_0} \otimes H) & \xrightarrow{\text{cl}_{f,H}} & \text{Sel}_\text{et}(H, V_p(f)(k_0/2 + 1)) \\
\downarrow & & \downarrow \text{resp} \\
\text{CH}^{k_0/2+1}(\mathcal{M}_{k_0} \otimes H_{\mathfrak{p}}) & \xrightarrow{\text{cl}_{f,\mathfrak{p}}} & H^1(\mathfrak{p}, V_p(f)(k_0/2 + 1))
\end{array}$$

(5.7)

Let

$$\mathbb{D}_f := \mathbb{D}_\text{st}(V_p(f)|_{G_{\mathfrak{p}}})$$

denote the rank-two $(\wp, N)$-module attached to $V_p(f)$. By [IS03] (49), we have the following identification

$$\text{IS} : H^1(\mathbb{P}, V_p(f)(k_0/2 + 1)) \xrightarrow{\log_{\text{BK}}} \mathbb{D}_f \otimes H_{\mathfrak{p}} \cong \text{Fil}^{k_0/2+1}(\mathbb{D}_f \otimes H_{\mathfrak{p}}) \cong M_{k_0+2}(X, H_{\mathfrak{p}})^{\vee}_{(f, \wp)} \cong M_{k_0+2}(\Gamma', H_{\mathfrak{p}})^{\vee}_{(f, \wp)}$$

where $\log_{\text{BK}}$ is the Bloch–Kato logarithm and the final identification holds assuming $H_{\mathfrak{p}} \supseteq F_{\wp} \cong \mathbb{Q}_p$. Here $(-)^{\vee}$ stands for the $H_{\mathfrak{p}}$-dual and $(f^{\vee})$ stands for the $f^{\vee}$-isotypic component. It would be useful to consider the composition

(5.8)

$$\log \text{cl}_{f,L} : \text{CH}^{k_0/2+1}(\mathcal{M}_{k_0} \otimes F) \to \text{Sel}_\text{et}(F, V_p(f)(k_0/2 + 1)) \to \text{Sel}_\text{et}(K, V_p(f)(k_0/2 + 1)) \xrightarrow{\text{resp}} H^1(\mathcal{M}_{k_0+2}(X, F_{\wp} \otimes L) \cong M_{k_0+2}(\Gamma', L)^{\vee}_{(f, \wp)}$$

where $L/\mathbb{Q}_p$ is as before (recall that $L \supseteq K_p$) and

$$\text{Sel}_\text{et}(F, V_p(f)(k_0/2 + 1)) \to \text{Sel}_\text{et}(K, V_p(f)(k_0/2 + 1))$$

is the usual restriction map.
The following result is proved in [Sev14] generalising the weight the $k_0 = 0$ setting of [BD07]. See also [Sev12 §5.3.2] for more details.

**Theorem 5.6.** There exists a global cycle

$$\mathcal{Y} \in \text{CH}^{k_0/2+1} (\mathcal{M}_{k_0} \otimes F)$$

such that

$$\frac{d^2}{dk^2} [L_p(\mathcal{F}, \kappa)]_{k = k_0} = \frac{d^2}{dk^2} [L_p(\mathcal{F}/F, \kappa)]_{k = k_0} = 2\log \text{cl}_{f,L}(\mathcal{Y})(f^{1/2g})^2$$

**Remark 5.7.** The global cycle $\mathcal{Y} \in \text{CH}^{k_0/2+1} (\mathcal{M}_{k_0} \otimes F)$ is the Heegner cycle (associated to the trivial character $\chi = 1$) constructed in [IS03 §8] using the theory of Complex Multiplication.

5.4. $p$-adic $L$-functions over $K$. Recall that $K/F$ is a relative quadratic extension that satisfies the Stark–Heegner hypothesis ($\text{SH–Hyp}$)

- $p$ is inert in $K$
- All primes $l \mid \mathcal{M}$ split in $K$

Let $\Psi \in \text{Emb}(O, R)$ be an optimal embedding of conductor $C$ relatively prime to $N^D_{K/F}$. Let $G_C := \text{Gal}(H_C/K)$ be the corresponding Galois group of the ring class field of conductor $C$ and $(\tau_\Psi, \psi_\Psi, \gamma_\Psi)$ be the data attached to the embedding $\Psi$ as in [BD09]. Let $\mathcal{L}_\Psi$ be an $O_F$-lattice corresponding to the vertex $\psi_\Psi$ and let $\tilde{L}_\Psi := \mu(O_F)\backslash \mathcal{L}_\Psi$. Following [BD09] and [Sev12], we define a partial square root $p$-adic $L$-function to such an embedding $\Psi$ as follows

**Definition 5.8.** Let $\tau \in \mathbb{P}^1(F)$ be any base point. The partial square root $p$-adic $L$-function attached to $\mathcal{L}(\mathcal{F}, \Psi)$ is defined as

$$L_p(\mathcal{F}/K, \Psi, \lambda_\kappa) := |\mathcal{L}_\Psi|^{-\kappa_0} \int_{\mathcal{L}_\Psi} \langle P_\Psi(x, y) \rangle^{\lambda_\kappa - \lambda_\kappa_0} P_\Psi^{k_0/2}(x, y) d\Phi_{\Psi, \tau} \{\tau - \gamma_\Psi \tau\}$$

The partial square root $p$-adic $L$-function attached to $(\mathcal{F}/K, \psi_\kappa)$ for $\psi_\kappa : G_C \to \mathbb{C}^\times$ is then defined as

$$L_p(\mathcal{F}/K, \psi_\kappa, \lambda_\kappa) := \sum_{\sigma \in G_C} \psi_\kappa^{-1}(\sigma) L_p(\mathcal{F}/K, \sigma \Psi, \lambda_\kappa)$$

and the $p$-adic $L$-function attached to $(\mathcal{F}/K, \psi_\kappa)$ is defined as

$$L_p(\mathcal{F}/K, \psi_\kappa, \lambda_\kappa) := L_p(\mathcal{F}/K, \psi_\kappa, \lambda_\kappa)^2$$

**Remark 5.9.** A priori the $p$-adic $L$-functions defined above depend on the choice of the big modular symbol $\Phi_\infty$ of Theorem 4.13 which gives rise to a family of distributions $\{\Phi_{\ell,L}\}_{\ell}$. However it turns out that the definition depends only on the class of optimal embeddings $[\Psi] \in \Gamma \backslash \text{Emb}(O, R)$. Following [Sev12 Remark 5.6], we choose a lattice $\mathcal{L}_\Psi$ as follows. Since $\Gamma$ acts transitively on the set of vertices $\mathcal{V}(\mathcal{C})$, let $\gamma \in \Gamma$ be such that $\gamma v_\Psi = v_\ast$. Then $v_\ast = v_\Psi v_{\gamma^{-1}}$ and $\mathcal{L}_\ast = \mathcal{L}_{\Psi v_{\gamma^{-1}}}$ is the lattice associated to the optimal embedding $\gamma \Psi \gamma^{-1} \in [\Psi]$. It will be clear later on that this is the canonical choice of a lattice when we record the interpolation property of the $p$-adic $L$-function.

It is expected that the $p$-adic $L$-function $L_p(\mathcal{F}/K, \psi_\kappa, -)$ defined above interpolates the central critical $L$-values $L^{\text{alg}}(\mathcal{F}_{k}/K, \psi_\kappa, k/2 + 1)$. This entails rewriting the adelic toric periods...
Theorem 5.12. The functions introduced in this section central critical \( L \)-values. Further, we also have Theorem 5.10 that relates the \( p \)-adic functions constructed by Víctor Hernández Barrios and Santiago Molina Blanco in [BB21, §5]. See also [FG21, §3.3] such a result is not yet available in the literature. Of particular interest is the interpolation formulas of anticyclotomic \( L \)-functions constructed by Victor Hernández Barrios and Santiago Molina Blanco in [BBL21 §5]. See also [TG21, Theorem 5.10] that relates the \( p \)-adic Abel–Jacobi image of Plectic Stark–Heegner points to central critical \( L \)-values.

Conjecture 5.10. Let \( \psi_K : \text{Gal}(H_K/K) \to \mathbb{C}^\times \) be an unramified character. Then
\[
\mathcal{L}(F^\#_k, \psi_K) = (D_K)^{k/2} \mathcal{L}(F^\#_k/K, \psi_K, k/2 + 1)
\]
where
\[
(5.9) \quad \mathcal{L}(F^\#_k/K, \psi_K, k/2 + 1) := u_K^2 \frac{(k/2)!}{(2\pi i)^{k/2}} \frac{\sqrt{N_{F/Q}(D_{F/K})}}{L(F^\#_k/K, \psi_K, k/2 + 1)} \in \mathbb{Q}(F^\#_k)
\]
where \( \langle F^\#_k, F^\#_k \rangle \) is the Petersson norm normalized (by a factor of \((8\pi^2)^2\)) as in [Mok11, Definition 3.5] and \( u_K := \mu(K) : \mu(F) \).

Remark 5.11. Whilst the identities recorded in Conjecture 5.10 above should follow from rewriting the toric period integrals in Waldspurger formula in terms of geodesic cycle integrals on \( \mathcal{H}_3 \) (as in [Pop06] §6 and [LMH20 §3.3]), such a result is not yet available in the literature.

Theorem 5.12. The \( p \)-adic \( L \)-functions, for all \( \Psi \) and \( \psi_K \) as above, vanish at \( \lambda_{k_0} \):
\[
\mathcal{L}_p(F/K, \Psi, \lambda_{k_0}) = \mathcal{L}_p(F/K, \psi_K, \lambda_{k_0}) = \mathcal{L}_p(F/K, \psi_K, \lambda_{k_0}) = 0
\]
Further, we also have
\[
\frac{d}{d\lambda}[\mathcal{L}_p(F/K, \psi_K, \lambda)]_{\lambda=\lambda_{k_0}} = 0
\]

Proof. By Definition 5.8 above, we have
\[
\mathcal{L}_p(F/K, \Psi, \lambda_{k_0}) = [L_\Phi]^{-1} \int_{L_\Phi} P^{k_0/2}(x, y) d\Phi \langle \tau - \gamma \tau \rangle
\]
where the second equality follows from (4.8). By Corollary 4.19 we then have that
\[
\mathcal{L}_p(F/K, \Psi, \lambda_{k_0}) = \int_{B^1(F_p)} P^{k_0/2}(x, y) d\Phi_{k_0} \langle \tau - \gamma \tau \rangle
\]
The vanishing now follows from [BSW19 Proposition 5.8(i)]. The defining properties of the other \( p \)-adic \( L \)-functions implies the vanishing simultaneously at \( \lambda_{k_0} \).  

Granting Conjecture 5.10 we have the following interpolation property of the \( p \)-adic \( L \)-functions introduced in this section
Theorem 5.13. For all classical weights $\lambda_k \in U$, $k \neq k_0$, we have

$$L_p(F^\#_k/K, \psi_K, \lambda_k) = c(k)^2 \left(1 - \frac{N_{F/Q}(p)^k}{a_p(F^\#_k)^2}\right)^2 (N_{F/Q}(D_{K/F}))^{k/2} L_{\alg}^{k}(F^\#_k/K, \psi_K, k/2 + 1)$$

Proof. By definition,

$$L_p(F/K, \Psi, \lambda_k) = |\mathcal{L}_\Psi|^{\frac{-1}{2}} \int_{L_p} P_{\Psi}^{k/2}(x,y) d\Phi_{\mathcal{L}_\Psi} \{\tau - \gamma \psi \tau\}$$

since $P_{\Psi}^{\lambda_k - \lambda_{k_0}} = P_{\Psi}^{k_0/2} = P_{\Psi}^{k/2}$ (as $[z]^k = [z]^{k_0}$ for all $\lambda_k \in U$ classical). By Remark 5.9 above, we may choose $\mathcal{L}_\Psi$ to be the lattice $\mathcal{L}_\Psi = \mathcal{O}_{F_p} \oplus \mathcal{O}_{F_p}$ and hence we get

$$L_p(F/K, \Psi, \lambda_k) = \int_{L_p} P_{\Psi}^{k/2}(x,y) d\Phi_{\mathcal{L}_\Psi} \{\tau - \gamma \psi \tau\}$$

which by Lemma 4.18 implies that

$$L_p(F/K, \Psi, \lambda_k) = c(k)^2 \left(1 - \frac{N_{F/Q}(p)^k}{a_p(F^\#_k)^2}\right)^2 (P_{\Psi}(x,y))^{k/2}$$

and hence

$$L_p(F^\#_k/K, \psi_K, \lambda_k) = c(k)^2 \left(1 - \frac{N_{F/Q}(p)^k}{a_p(F^\#_k)^2}\right)^2 L(F^\#_k, \psi_K)$$

The proof follows upon granting Conjecture 5.10. \qed

5.5. Factorization of $p$-adic $L$-functions. Let $\epsilon_{K/F}$ be the quadratic idèle class character of $F$ that cuts out the relative quadratic extension $K/F$ and let $\chi$ be any finite order Hecke character of $F$. Then by Artin formalism, we have the following factorization of classical $L$-functions:

$$L(F^\#_k/K, \chi \circ N_{F/Q}, s) = L(F^\#_k, \chi, s)L(F^\#_k, \chi \epsilon_{K/F}, s)$$

We show that a similar factorization (p-adic Artin formalism) also holds at the level of $p$-adic $L$-functions. In particular

Theorem 5.14. There exists a $p$-adic analytic function $\eta(\kappa) : U \to \mathbb{C}_p$ such that for all $\lambda_k \in U$,

$$L_p(F/K, \lambda_k) = \eta(\kappa)L_p(F, \lambda_k)L_p(F, \epsilon_{K/F}, \lambda_k)$$

Proof. Let $\lambda_k \in U$ be a classical weight. Then by (5.10) above, we have

$$L(F^\#_k/K, k/2 + 1) = L(F^\#_k, k/2 + 1)L(F^\#_k, \epsilon_{K/F}, k/2 + 1)$$

Recall that we have the following identities (See [Mok11, Appendix I] for instance)

$$N_{F/Q}(\epsilon_{K/F}) = N_{F/Q}(D_{K/F})$$

$$\tau(\epsilon_{K/F}^{-1}) = \tau(\epsilon_{K/F}) = \sqrt{N_{F/Q}(D_{K/F})}$$

Further, upon combining (5.4) and (5.9), we get

$$D_{F}^{k}(N_{F/Q}(D_{K/F}))^{k/2} L_{\alg}^{k}(F^\#_k/K, k/2 + 1)$$

$$= u_{F}^{k}(\Omega_{F^\#_k})^{2} L_{\alg}^{k}(F^\#_k, k/2 + 1)$$

$$= u_{F}^{k}(\Omega_{F^\#_k})^{2} L_{\alg}^{k}(F^\#_k, k/2 + 1)$$
By Theorem 5.4 and Theorem 5.13 above, using the identity $D_K = N_{F/Q}(D_{K/F})D_F^2$, we have

$$L_p(F/K, \lambda_k) = \frac{u_k^2(\Omega_{F^p})^2}{u_F^4(F^p, F^p)} L_p(F, \lambda_k) L_p(F, \epsilon_{K/F}, \lambda_k)$$

for all $\lambda_k \in U$ classical. We also set

$$\eta(k) = \frac{u_k^2(\Omega_{F^p})^2}{u_F^4(F^p, F^p)}.$$

Since classical weights are Zariski–dense in $U$ (See [BSW21, Proposition 5.1] for instance), we have the following factorization of $p$-adic $L$-functions over the affinoid $U$

$$L_p(F/K, \lambda_k) = \eta(k) L_p(F, \lambda_k) L_p(F, \epsilon_{K/F}, \lambda_k)$$

□

**Remark 5.15.** Note that Theorem 5.14 above, in particular 5.11 also shows that

$$\frac{(\Omega_{F^p})^2}{(F^p, F^p)} \in \mathbb{Q}(F^p) \subset \mathbb{Q}(F^p)$$

5.6. A $p$-adic Gross–Zagier formula. We will now show a $p$-adic Gross–Zagier formula relating the (second derivative of the) base–change $p$-adic $L$-function $L_p(F/K, \psi_K, \lambda_k)$ and the $p$-adic Abel–Jacobi image of Stark–Heegner cycles introduced in §2.6.

**Theorem 5.16.**

$$\frac{d}{d\lambda_k}[\mathcal{L}_p(F/K, \psi, \lambda_k)]_{\lambda_k = \lambda_0} =$$

$$\frac{1}{2} \left( N_{F/Q}(D_{K/F}) \right)^{\lambda_0/4} \left( \log \Phi^{\lambda_1}(D_{[\psi]})(\phi_F) + (-1)^{\frac{\lambda_0+2}{2}} \log \Phi^{\lambda_1}(D_{[\psi]})(\phi_F) \right)$$

**Proof.** We write

$$P_{\psi}(x, y) = N_{F/Q}(A)(x - \tau_\psi y)(x - \tau_\psi^0 y)(\overline{\sigma} - \tau_\psi \overline{\sigma})(\overline{\sigma} - \tau_\psi^0 \overline{\sigma})$$

$$= N_{F/Q}(A)(x - \tau_\psi y)(x - \tau_\psi^0 y)$$

where $\theta$ is the non-trivial automorphism of $\text{Gal}(K/F)$ (and the overline is the non-trivial automorphism of $\text{Gal}(F/Q)$). Then

$$\mathcal{L}_p(F/K, \psi, \lambda_k) = |L_p|^{-k_0} \left( N_{F/Q}(A) \right)^{\lambda_0-\lambda_k} \int_{L_\psi} \langle N_{F/Q}(x - \tau_\psi y) \rangle^{\lambda_0-\lambda_k} P_{\psi}^{k_0/2}(x, y) d\Phi_{L_\psi} \{ r - \gamma_\psi r \}$$

By Theorem 5.12 we get

$$\frac{d}{d\lambda_k}[\mathcal{L}_p(F/K, \psi, \lambda_k)]_{\lambda_k = \lambda_0} = |L_p|^{-k_0}$$

$$\frac{d}{d\lambda_k} \int_{L_\psi} \langle N_{F/Q}(x - \tau_\psi y)(x - \tau_\psi^0 y) \rangle^{\lambda_0-\lambda_k} P_{\psi}^{k_0/2}(x, y) d\Phi_{L_\psi} \{ r - \gamma_\psi r \}$$

$$_{\lambda_k = \lambda_0}$$
By Proposition 4.11,

\[
|L_p| = \frac{1}{2} \frac{d}{d\lambda_k} \left[ \int_{L_p} (N_{F/Q}(x - \tau y)(x - \tau^0 y))^{\lambda_n - \lambda_{\kappa_0}} P_{\psi}^{1/2}(x, y) d\Phi \right]_{\kappa = k_0} - \frac{1}{2} \frac{d}{d\lambda_k} \left[ \int_{L_p} (N_{F/Q}(x - \tau y)(x - \tau^0 y))^{\lambda_n - \lambda_{\kappa_0}} P_{\psi}^{1/2}(x, y) d\Phi \right]_{\kappa = k_0} + \frac{1}{2} \frac{d}{d\lambda_k} \left[ \int_{L_p} (N_{F/Q}(x - \tau y)(x - \tau^0 y))^{\lambda_n - \lambda_{\kappa_0}} P_{\psi}^{1/2}(x, y) d\Phi \right]_{\kappa = k_0}
\]

Recall from Remark 2.16 that \((\tau_{\psi}, P_{\psi}, \gamma_{\psi}) = (\tau_{\psi}^0, -P_{\psi}, \gamma_{\psi}^{-1})\). By Definition 4.20 on semi-definite integrals, we have

\[
\frac{d}{d\lambda_k} [L_p(F/K, \Psi, \lambda_k)]_{\kappa = k_0} = \frac{1}{2} \left( \int_{T_{\psi}} \int_{T_{\psi}} P_{\psi}(x, y)^{k_0/2} \omega_F + \int_{T_{\psi}} \int_{T_{\psi}} P_{\psi}(x, y)^{k_0/2} \omega_F \right)
\]

Replacing the arbitrary base point \(r \in \mathbb{P}^1(F)\) by \(\gamma_{\psi} r\) and combining with Lemma 2.18 we get that

\[
\int_{T_{\psi}} \int_{T_{\psi}} P_{\psi}(x, y)^{k_0/2} \omega_F = \int_{T_{\psi}} \int_{T_{\psi}} P_{\psi}(x, y)^{k_0/2} \omega_F = - \int_{T_{\psi}} \int_{T_{\psi}} P_{\psi}(x, y)^{k_0/2} \omega_F
\]

The result now follows from Theorem 4.25 above. \(\square\)

An immediate consequence of Theorem 5.16 above is

**Corollary 5.17.** For all unramified characters \(\psi_K : \text{Gal}(H_K/K) \to \mathbb{C}^\times\),

\[
\frac{d^2}{d\lambda_k^2} [L_p(F/K, \psi_K, \lambda_k)]_{\kappa = k_0} = \frac{1}{2} \left( N_{F/Q}(D_{K/F}) \right)^{k_0/2} \left( \log \Phi^{A_1}(D_{\psi_K})(\phi_F) + (-1)^{k_0^2} \log \Phi^{A_1}(D_{\psi_K})(\phi_F) \right)^2
\]

In particular, for \(\psi_K = \psi_{\text{triv}}\) - the trivial character, we may further simplify the expression of Corollary 5.17 above as

**Corollary 5.18.**

\[
\frac{d^2}{d\lambda_k^2} [L_p(F/K, \psi_{\text{triv}}, \lambda_k)]_{\kappa = k_0} = \frac{1}{2} \left( N_{F/Q}(D_{K/F}) \right)^{k_0/2} \left( 1 + (-1)^{k_0^2} \omega_M \right) \left( \log \Phi^{A_1}(D_1)(\phi_F) \right)^2
\]

**Proof.** For \(\sigma \in \text{Gal}(H_K/K)\), let \(\sigma \Psi \in \Gamma/\text{Emb}^{e_{\psi} \sigma}(\mathcal{O}, \mathcal{R})\) be a \(\Gamma\)-conjugacy class of oriented optimal embeddings. Since \(\phi_F\) is an eigensymbol for the Atkin–Lehner involution \(W_M\), we have

\[
\log \Phi^{A_1}(D_{\psi_1})(\phi_F | W_M) = \log \Phi^{A_1}(D_{\psi_1}(\phi_F)) = \omega_M \log \Phi^{A_1}(D_{\psi_1})(\phi_F)
\]
Whilst $(\sigma\Psi)^{\theta}$ doesn’t have the same orientation (at $M$) as $\sigma\Psi$, we know that $\alpha_M(\sigma\Psi)^{\theta}\alpha_M^{-1} \in \Gamma/\text{Emb}^{\sigma\Psi}(O, R)$ (See Remark 2.24). By Proposition 2.24 which exhibits the set of the $\Gamma$-conjugacy class of oriented optimal embeddings as a $\text{Gal}(H/K)$-torsor, we know that there exists $\delta_{\sigma\Psi} \in \text{Gal}(H_K/K)$ such that

$$\alpha_M(\sigma\Psi)^{\theta}\alpha_M^{-1} = \delta_{\sigma\Psi}\sigma\Psi.$$ 

Thus we have

$$\sum_{\sigma \in \text{Gal}(H_K/K)} \log \Phi_{\text{AJ}}(D_{(\sigma\Psi)^{\theta}})(\phi_F) = \omega_M \sum_{\sigma \in \text{Gal}(H_K/K)} \log \Phi_{\text{AJ}}(D_{\delta_{\sigma\Psi}\sigma\Psi})(\phi_F)$$

Yet again using the fact that the $\text{Gal}(H_K/K)$-action on $\Gamma/\text{Emb}^{\sigma\Psi}(O, R)$ is transitive, we get

$$\log \Phi_{\text{AJ}}(D_{\delta_{\sigma\Psi}\sigma\Psi})(\phi_F) = \omega_M \sum_{\sigma \in \text{Gal}(H_K/K)} \psi_{\text{triv}}^{-1}(\sigma) \log \Phi_{\text{AJ}}(D_{\sigma\Psi})(\phi_F)$$

and the result follows from Corollary 5.17 above. 

**Corollary 5.19.**

$$\frac{d^2}{d\lambda_k^2} [L_p(F/K, \psi_{\text{triv}}, \lambda_k)_{|\kappa=k_0}] = \begin{cases} 2 \left( N_{F/Q}(D_{K/F}) \right)^\frac{1}{2} \left( \log \Phi_{\text{AJ}}(D_{1})(\phi_F) \right)^2 & \text{if } \omega_M = (-1)^{\frac{b_0+2}{2}} \\ 0 & \text{if } \omega_M = (-1)^{\frac{b_0}{2}} \end{cases}$$

6. **Proof of the main result**

We begin by noting the vanishing of the several ($p$-adic $L$-functions) introduced above. For $F \in S_{k_0+2}(U_0(N))^{\text{new}}$, we know that the sign of the functional equation of the base-change $L$-function $L(F/K, s)$ is $-1$ by the (SH–Hyp). In particular the central critical $L$-value $L(F/K, k_0/2 + 1)$ vanishes to odd order. The classical Artin formalism (5.10) above;

$$L(F/K, s) = L(F, s)L(F, \epsilon_K/F, s)$$

along with the Heegner hypothesis (Heeg–Hyp) shows that the sign of the functional equation of $L(f/F, s) = L(F, s)$ is $-1$, which forces an odd order of vanishing of the central critical $L$-value $L(F, k_0/2 + 1)$, whilst that of $L(f/F, \epsilon_K/F, s) = L(F, \epsilon_K/F, s)$ is $+1$ (See [Gas21] Theorem 1.1). We shall assume that $L(F, \epsilon_K/F, k_0/2 + 1) \neq 0$. 

By Remark 1.13 $\omega_p = 1$ and hence by [BSW19] Theorem 9.3], we know that the $p$-adic $L$-function $L_p(F, \epsilon_K/F, s)$ has a trivial zero at $s = k_0/2 + 1$. Similarly, the $p$-adic $L$-function $L_p(F, \epsilon_K/F, s)$ doesn’t have a trivial zero at $s = k_0/2 + 1$ since $\epsilon_K/F(p) = -1$ (recall that $p$ is inert in $K$ by (SH–Hyp)). In particular, we may summarize that

$$\text{ord}_{s=k_0/2+1}L_p(F, s) \geq 2$$

We can now compare the $p$-adic Abel–Jacobi image of Stark–Heegner cycles introduced in §2.6 with that of the Heegner cycles that appear in §3.3.
Theorem 6.1. Suppose that $\omega_M = (-1)^{\frac{b_0+2}{2}}$. Then there exists $\mathcal{Y} \in \text{CH}^{b_0/2+1}(\mathcal{M}_{k_0} \otimes F) \subset \text{CH}^{b_0/2+1}(\mathcal{M}_{k_0} \otimes K)$ and $s_\mathcal{Y} \in \mathbb{Q}((\mathcal{F})^\times$ such that

$$\log \Phi^A (D_1)(\phi_\mathcal{Y}) = s_\mathcal{Y} \cdot \log \text{cl}_{f,L}(\mathcal{Y})(f_{\text{rig}}).$$

Proof. By Theorem 5.14 above, we have

$$L_p(\mathcal{F}/K, \lambda_\kappa) = \eta(k_0) L_p(\mathcal{F}, \epsilon_{K,F}, \lambda_\kappa)$$

Further, by Theorem 5.12 together with the discussion above, we have

$$d^2 \frac{d}{dk^2} [L_p(\mathcal{F}/K, \lambda_\kappa)]_{k=k_0} = \eta(k_0) d^2 \frac{d}{dk^2} [L_p(\mathcal{F}, \lambda_\kappa)]_{k=k_0} L_p(\mathcal{F}, \epsilon_{K,F}, \lambda_{k_0})$$

By (5.3), we know that

$$L_p(\mathcal{F}, \epsilon_{K,F}, \lambda_{k_0}) = \left(1 - \epsilon_{K,F}(p) \frac{N_{F/Q}(p)^k}{\alpha_p(F_k)}\right) \cdot L_{\text{alg}}(\mathcal{F}, \epsilon_{K,F}, k_0/2 + 1)$$

Similarly, we have (recall that $\mathcal{F}_k = \mathcal{F}$).

$$\eta(k_0) = \frac{u_k^2(\Omega_{\mathcal{F}})^2}{u_{p,F}(\mathcal{F}, \mathcal{F})}$$

By Corollary 5.19 and Theorem 5.13 above, we know that

$$(\log \Phi^A (D_1)(\phi_\mathcal{Y}))^2 = \frac{2u_k^2(\Omega_{\mathcal{F}})^2}{(N_{F/Q}(D_{K,F}))^k} L_{\text{alg}}(\mathcal{F}, \epsilon_{K,F}, k_0/2 + 1) \left(\log \text{cl}_{f,L}(\mathcal{Y})(f_{\text{rig}})\right)^2$$

for $\mathcal{Y} \in \text{CH}^{b_0/2+1}(\mathcal{M}_{k_0} \otimes F) \subset \text{CH}^{b_0/2+1}(\mathcal{M}_{k_0} \otimes K)$. Note that

$$\left(\frac{2u_k^2(\Omega_{\mathcal{F}})^2}{(N_{F/Q}(D_{K,F}))^k} L_{\text{alg}}(\mathcal{F}, \epsilon_{K,F}, k_0/2 + 1) \in \mathbb{Q}(F)\right)$$

is expected to be a square in $\mathbb{Q}((\mathcal{F})^\times$ as in [BD07, Theorem 5.4(4)] and [Sev12, Theorem 6.1 & Theorem 6.2] which we shall assume to hold. See also [Mok11, Remark 6.6]. Granting this, the Theorem now follows upon extracting square-roots on both sides of (6.2). □

Recall that since $\mathcal{F}$ is the base–change $f/F$, at the level of Galois representations, we have

$$V_p(\mathcal{F}) = V_p(f)|_{G_F}$$

Let $D_\mathcal{F} := D_{\text{st}}(V_p(\mathcal{F}))$ (resp. $D_f := D_{\text{st}}(V_p(f))$) be Fontaine’s semistable Dieudonné module attached to the Galois representation $V_p(\mathcal{F})$ (resp. $V_p(f)$). By (5.4) above, we have an identification

$$D_\mathcal{F} \cong D_f \otimes_{Q_p} F_p$$

Further by [AVW21, Theorem 4.5], we have an isomorphism of $(\varphi, N)$-modules over $F_p$ with coefficients in $L$

$$\bigoplus_\sigma D_{\mathcal{F}, L}^\sigma = D_\mathcal{F} \cong D_\mathcal{F} = \bigoplus_\sigma D_{f,L}^\sigma$$

where $D_{\mathcal{F}, L}^\sigma := D_\mathcal{F} \otimes_{F_p} \otimes_{L, \sigma} L$, which induces an identification of the tangent spaces

$$\text{Fil}^{b_0/2}(D_{\mathcal{F}, L}) \cong \text{Fil}^{b_0/2}(D_{f,L}) \cong \text{Fil}^{b_0/2}(D_{f,L})$$
for each $\sigma : F_p \hookrightarrow L$. We fix an isomorphism

$$\text{MS}_\Gamma(L)^\vee_{(F')} \cong M_{k_0+2}(\Gamma', L)_{(f^{\text{reg}})}$$

defined as follows

$$\alpha : \text{MS}_\Gamma(L)^\vee_{(F')} \xrightarrow{(\text{Pr}^\sigma)^{-1}} \frac{\text{Fil}^{\text{exp}}_2 (D_{f,L})}{\text{Fil}^{\text{exp}}_2 (D_{f,L})} \cong M_{k_0+2}(\Gamma', L)_{(f^{\text{reg}})}$$

for either choice of an embedding $\sigma : F_p \hookrightarrow L$ (See Remark 3.3). In particular, we have a commutative diagram

$$\begin{array}{ccc}
(\Delta_0 \otimes \text{Div}(H^m_{\text{ur}}(V_{k_0,k_0}))_{\Gamma} & \xrightarrow{\text{Pr}^J} & \frac{D_{f,L}^{\vee}}{\text{Fil}^{\text{exp}}_2 (D_{f,L})} \\
(\Delta_0 \otimes \text{Div}(H^m_{\text{ur}}(V_{k_0,k_0}))_{\Gamma} & \xrightarrow{\log \Phi^{A\ldots J}} & \text{MS}_\Gamma(L)^\vee_{(F')} \\
\end{array}$$

Remark 6.2. Note that the isomorphism of [VW21, Theorem 4.5], $D_f \cong \mathbb{D}_f$ is conditional on [VW21, Conjecture 4.2] in addition to the semistability of the local Galois representation $V_p(\mathcal{F})|_{G_{F_p}}$. However, in the base-change scenario both these conditions are satisfied, making the isomorphism $\varphi$ unconditional. See [VW21, Lemma 4.4] for more details.

We can now prove our main theorem, Theorem 1.4 from the Introduction, that sheds evidence towards the global rationality conjectures formulated in [VW21, §6.2] :-

$$\boxed{\begin{align*}
\alpha (\log \Phi^{A\ldots J}(D_1)) &= \log \text{cl}_{f,L}(s_{\mathcal{F}}Y) = \text{IS} (\text{cl}_{f,L}(s_{\mathcal{F}}Y)) \\
\end{align*}}$$

The commutative diagram (6.6) implies that

$$\alpha (\log \Phi^{A\ldots J}(D_1)) = \alpha (\text{Pr}^\sigma (\Phi^{A\ldots J}(D_1))) = \text{IS} (\text{exp}_{\text{BK}} \circ \varphi (\Phi^{A\ldots J}(D_1)))$$

which shows that

$$\text{IS} (\text{exp}_{\text{BK}} \circ \varphi (\Phi^{A\ldots J}(D_1))) = \text{IS} (\text{cl}_{f,L}(s_{\mathcal{F}}Y))$$

from which we conclude that

$$\text{exp}_{\text{BK}} \circ \varphi (\Phi^{A\ldots J}(D_1)) = \text{cl}_{f,L}(s_{\mathcal{F}}Y) \in H^1_{\text{st}}(L, V_p(f)(k_0/2 + 1)).$$

By Theorem 5.6 we know that there exists $S_K \in \text{Sel}_{\text{st}}(K, V_p(f)(k_0/2 + 1))$ such that

$$\text{cl}_{f,L}(s_{\mathcal{F}}Y) = \text{res}_{\sigma} (S_K) \in H^1_{\text{st}}(L, V_p(f)(k_0/2 + 1))$$

Theorem 5.4 now follows since we have an identification of the Bloch–Kato Selmer groups owing to (6.4) above

$$\text{Sel}_{\text{st}}(K, V_p(\mathcal{F})(k_0/2 + 1)) = \text{Sel}_{\text{st}}(K, V_p(f)(k_0/2 + 1))$$

Remark 6.3. Let $\mathbb{T}_{(\mathcal{F})}$ denote the $\mathcal{F}$-isotypic component of the usual Hecke algebra $\mathbb{T}$ acting on $S_{k_0+2}(U_0(N))_{\text{new}}$. Then, via the isomorphism $\mathbb{T}_{(\mathcal{F})} \cong \mathbb{Q}(\mathcal{F})$, we may regard $s_{\mathcal{F}} \in \mathbb{Q}(\mathcal{F})^\times$ as a Hecke operator in $\mathbb{T}_{(\mathcal{F})}$ acting on the Chow groups.
6.1. Concluding Remarks.

(1) Conjectures of Calegari–Mazur ([CM09, Conjecture 1.3]) and Barrera Salazar–Williams ([BSW21 Conjecture 5.13]) predict that the only cuspidal $p$-adic families of Bianchi eigenforms come from (twisted) base–change families over $\mathbb{Q}$. Since the ideas explored in this article heavily rely on Hida/ Coleman families of Bianchi eigenforms, the crucial assumption that the Bianchi eigenform $F$ is the base–change to $\mathbb{Q}$ of a classical cuspidal eigenform $f$ is indispensable. In fact, the reader will realize that no genuine Stark–Heegner cycles are constructed in this article. Similar to [BD09] and [Sev12], in scenarios where the theory of Heegner cycles overlaps with that of their Stark–Heegner counterparts such as base–change, we show that the Stark–Heegner cycles can be expressed in terms of Heegner cycles.

(2) It would be evident to the reader that we have restricted ourselves to the analytic rank one setting in this article. In a forthcoming work, we will consider Plectic Stark–Heegner cycles (See [FG21] for example) that would account for the higher (odd) orders of vanishing of the $L$-series $L(F/K, s)$ at the central critical point.

(3) It would be interesting to give some computational evidence when the Bianchi eigenform $F$ corresponds to the quadratic base–change of an elliptic curve $E_F$ as in §1.3. We hope to get back to this in the future.

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