Nonequilibrium Renormalization Theory III

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Abstract

In the present paper we develop the general theory of renormalization of some nonequilibrium diagram technique. This technique roughly is the Keldysh diagram technique. To develop our theory we have used the Bogoliubov — Parasiuk $R$-operation method. Our theory can be used for the studying of the divergences in kinetic equations.

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1 Introduction

The main goal of this paper is to prove that for the general class of two-particle interaction there exist (in the sense of formal power series on coupling constant) the stationary translation invariant states on the algebra of canonical commutative relations which satisfy the weak cluster property and can not be described by the Gibbs formula. To solve this problem we use the $R$-operation method [1, 2, 3]. The methods of $R$-operation for studying the large time behavior of quantum systems have been used in [4]. The philosophical consequences of our statement have been discussed in our first paper on this topic. This statement follows from the theorem — construction of the last section of this paper.

2 The Algebra of Canonical Commutative Relations

Let $S(\mathbb{R}^3)$ be a Schwartz space of test functions (infinitely-differentiable functions decaying at infinity faster than any inverse polynomial with all its derivatives). The algebra of canonical commutative relations $\mathcal{C}$ is a unital algebra generated by symbols $a^+(f)$ and $a(f)$ $f \in S(\mathbb{R}^3)$ satisfying the following canonical commutative relations.

a) $a^+(f)$ is a linear functional of $f$,

b) $a(f)$ is an antilinear functional of $f$,

\begin{align}
[a(f), a(g)] &= [a^+(f), a^+(g)] = 0, \\
[a(f), a^+(g)] &= \langle f, g \rangle, \tag{1}
\end{align}

where $\langle f, g \rangle$ is a standard scalar product in $L^2(\mathbb{R}^3)$,

\begin{align}
\langle f, g \rangle &:= \int d^3 x f^*(x) g(x). \tag{2}
\end{align}

Let $\rho_0$ be a Gauss state on $\mathcal{C}$ defined by the following correlator

\begin{align}
\rho_0(a^+(k)a^+(k')) &= \rho_0(a(k)a(k')) = 0, \\
\rho_0(a^+(k)a(k')) &= n(k)\delta(k - k'), \tag{3}
\end{align}

where $\delta$ is the Dirac delta function.
where \( n(k) \) is a real-valued function from the Schwartz space. In the case then

\[
n(k) = \frac{e^{-\beta(\omega(k) - \mu)}}{1 - e^{-\beta(\omega(k) - \mu)}},
\]

where \( \mu \in \mathbb{R} \), \( \mu < 0 \), \( \rho_0 \) is called the Plank state. Here \( \omega(k) = \frac{k^2}{2} \).

Let \( \mathcal{C}' \) be a space of linear functionals on \( \mathcal{C} \), and \( \mathcal{C}_{+1}' \) be a set of all states on \( \mathcal{C} \). Let us make the GNS construction corresponding to the algebra \( \mathcal{C} \) and the Gauss state \( \rho_0 \). We obtain the set \( (\mathcal{H}, D, \hat{\cdot}, \langle \cdot \rangle) \) consisting of the Hilbert space \( \mathcal{H} \), the dense linear subspace \( D \) in \( \mathcal{H} \), the representation \( \hat{\cdot} \) of \( \mathcal{C} \) by linear operators from \( D \) to \( D \), and cyclic vector \( \langle \cdot \rangle \in D \), i.e. the vector such that \( \hat{\cdot} \langle \cdot \rangle = \rho_0(\cdot) \).

Below we will omit the symbol \( \hat{\cdot} \), i.e. we will write \( a \) instead of \( \hat{a} \).

Let us introduce the field operators:

\[
\Psi(x) = \frac{1}{(2\pi)^{3/2}} \int e^{ikx} a(k) dk,
\]

\[
\Psi^+(x) = \frac{1}{(2\pi)^{3/2}} \int e^{-ikx} a^+(k) dk.
\]

We say that the state \( \rho \) on \( \mathcal{C} \) satisfy to the weak cluster property if

\[
\lim_{a \to \infty} \int \langle \Psi^+(t, x_1 + \delta_1 e_1 a)\ldots \Psi^+(t, x_n + \delta_n e_1 a) \rangle f(x_1, \ldots, x_n) d^3x_1 \ldots d^3x_n
\]

\[
= \int \langle \Psi^+(t, x_{i_1})\ldots \Psi^+(t, x_{i_k}) \rangle \langle \Psi^+(t, x_{i_k})\ldots \Psi^+(t, x_{i_n}) \rangle
\]

\[
\times f(x_1, \ldots, x_n) d^3x_1 \ldots d^3x_n,
\]

where \( \delta_i \in \{1, 0\}, i = 1, 2 \ldots n \) and

\[
i_1 < i_2 < \ldots < i_k,
\]

\[
i_{k+1} < i_{k+2} < \ldots < i_n,
\]

\[
\{i_1, i_2, \ldots, i_k\} = \{i = 1, 2 \ldots n | \delta_i = 0\} \neq \emptyset,
\]

\[
\{i_{k+1}, i_{k+2}, \ldots, i_n\} = \{i = 1, 2 \ldots n | \delta_i = 1\} \neq \emptyset.
\]

\[
f(x_1, \ldots, x_n) \text{ is a test function. } e_1 \text{ is an unite vector parallel to the } x\text{-axis.}
\]
**Definition** The vector of the form

\[ \int v(p_1, \ldots, p_n) a^\pm(p_1) \ldots a^\pm(p_n)) d^3p_1 \ldots d^3p_n, \]

\[ v(p_1, \ldots, p_n) \in S(\mathbb{R}^{3n}). \]  

(8)

is called a finite vector. The finite linear combination of the vectors of the form (8) is also called a finite vector.

Let \( f(x_1, \ldots, x_k | y_1, \ldots, y_l | v_1, \ldots, v_m | w_1, \ldots, w_n) \) be a function of the form

\[ f(x_1, \ldots, x_k | y_1, \ldots, y_l | v_1, \ldots, v_m | w_1, \ldots, w_n) = g(x_1, \ldots, x_k | y_1, \ldots, y_l | v_1, \ldots, v_m | w_1, \ldots, w_n) \]

\[ \times \delta(\sum_{i=1}^k x_i + \sum_{j=1}^m w_j - \sum_{f=1}^l y_f - \sum_{g=1}^n v_g), \]  

(9)

where \( g \) is a function from Schwartz space.

Consider the following functional on \( C \)

\[ \rho_f(A) := \int \prod_{i=1}^k dx_i \prod_{j=1}^l dx_j \prod_{f=1}^m dv_f \prod_{g=1}^n dw_g \]

\[ \times f(x_1, \ldots, x_n | y_1, \ldots, y_l | v_1, \ldots, v_m | w_1, \ldots, w_n) \]

\[ \times \rho_0(\ldots a(x_1) \ldots a(x_n) a^+(y_1) \ldots a^+(y_l) : A : a(v_1) \ldots a(v_m) a^+(w_1) \ldots a^+(w_n) :). \]  

(10)

Here the symbol

\[ : (\ldots) : A : (\ldots) : \]  

(11)

means that when one transform the previous expression to the normal form according to the Gauss property of \( \rho_0 \) one must neglects by all correlators \( \rho(a^\pm(x_1) a^\pm(x_n)) \) such that \( a^\pm(x_1) \) and \( a^\pm(x_n) \) both do not came from \( A \).

Let \( \tilde{C}' \) be a subspace in \( C' \) spanned on the states just defined.

Now let us introduce an useful method for the representation of the states just defined.

Let \( C_2 = C_+ \otimes C_- \), where \( C_+ \) and \( C_- \) are the algebras of canonical commutative relations. The algebras \( C_\pm \) are generated by the generators \( a_\pm(k), a^\pm_\pm(k) \).
respectively satisfying to the following relations:

\[
\begin{align*}
[a_+^+(k), a_+^+(k')] &= [a_+(k), a_+(k')] = 0, \\
[a_-(k), a_+(k')] &= [a_-(k), a_+(k')] = 0, \\
[a_+(k), a_+^+(k')] &= \delta(k - k'), \\
[a_-(k), a_+^+(k')] &= \delta(k - k'), \\
[a_+^+(k), a_+^-(k)] &= 0.
\end{align*}
\] (12)

Here we put by definition \(a_\pm := a_\mp\). Let us consider the following Gauss state \(\rho_0\) on \(C_2\) defined by its two-point correlator

\[
\begin{align*}
\rho_0(a_\pm(k)a_\pm(k')) &= \rho_0(a_\pm(k)a_\pm(k')) = \rho_0(a_\pm(k)a_\pm(k')), \\
\rho_0(a_\pm(k)a_\pm(k')) &= \rho_0(a_\pm(k)a_\pm(k')) = \rho_0(a_\pm(k)a_\pm(k')) = 0, \\
\rho_0(a_\pm(k)a_\pm(k')) &= \rho_0(a_\pm(k)a_\pm(k')) = n(k)\delta(k - k'), \\
\rho_0(a_\pm(k)a_\pm(k')) &= (1 + n(k))\delta(k - k').
\end{align*}
\] (13)

Let us make the GNS construction corresponding to the state \(\rho_0\) and the algebra \(C_2\). We obtain the set \((\mathcal{H}', \mathcal{D}', \langle \cdot, \cdot \rangle)\) consisting of the Hilbert space \(\mathcal{H}'\), the dense linear subspace \(\mathcal{D}'\) in \(\mathcal{H}'\), the representation \(\mathcal{C}_2\) by means linear operators from \(\mathcal{D}'\) to \(\mathcal{D}'\), and the cyclic vector \(\langle \cdot \rangle \in \mathcal{D}'\), i.e. the vector such that \(\mathcal{C} \langle \cdot \rangle = \mathcal{D}'\). This set satisfy the following condition: \(\forall a \in \mathcal{C} \langle a \rangle = \rho_0(a)\). Below we will omit the symbol \(\mathcal{C}\), i.e. we will write \(a\) instead of \(\mathcal{C}\).

Now we can rewrite the functional, defined in (10) \(\rho_f\) as follows

\[
\rho_f(A) = \langle A'S_f \rangle,
\] (14)

where \(A'\) is an element of \(C_2\) such that it contains only the operators \(a_-\), \(a_+^\dagger\) and can be represented from \(a_-\), \(a_+^\dagger\) as the same way as \(A\) can be represented from \(a, a^\dagger\). \(S_f\) is an element of \(C_2\) of the form

\[
\begin{align*}
S_f &= \int \prod_{i=1}^k dx_i \prod_{j=1}^l dx_j \prod_{f=1}^m dv_f \prod_{g=1}^n dw_g \\
&\times f(x_1, ..., x_n|y_1, ..., y_l|v_1, ..., v_m|w_1, ..., w_n) \\
&\times :a_+^\dagger(x_1) ... a_+^\dagger(x_n)a_+(y_1) ... a_+(y_l)a_-(v_1) ... a_-(v_m)a_+^\dagger(w_1) ... a_+^\dagger(w_n):.
\end{align*}
\] (15)

Here the symbol \(\cdot : \cdot \cdot\) is a normal ordering with respect to the state \(\rho_0\).
Denote by $\tilde{D}'$ the space dual to $\tilde{D}$. We just construct the injection from $C'$ into $\tilde{D}'$. Denote its image by $\mathcal{H}'$.

By definition the space $C''$ is a space of all functionals on $C$ which can be represented as finite linear combinations of the following states

$$\rho(A) = \langle A' : S_{f_1}...S_{f_n} \rangle.$$  \hfill (16)

Here $A'$ is an element of $C_3$ such that it contains only the operators $a_-, a_+$ and can be represented from $a_-, a_+$ as the same way as $A$ can be represented from $a, a^+$ and $S_{f_i}$ are the elements of the form $[15]$. Denote by $\mathcal{H}''$ the subspace in $\tilde{D}'$ spanned on the vectors : $S_{f_1}...S_{f_n}$ ::

There exists an involution $*$ on $\mathcal{H}'$ defined by the following formula:

$$\left\{ \int \prod_{i=1}^{k} dx_i \prod_{j=1}^{l} dx_j \prod_{f=1}^{m} dv_f \prod_{g=1}^{n} dw_g \right. $$

$$\times f(x_1, ..., x_n|y_1, ..., y_l|v_1, ..., v_m|w_1, ..., w_n)$$

$$\times : a_+^+(x_1)...a_+^+(x_n)a_+(y_1)...a_+(y_l)a_-(v_1)...a_-(v_m)a_+^+(w_1)...a_+^+(w_n) : \right\}^*$$

$$= \int \prod_{i=1}^{k} dx_i \prod_{j=1}^{l} dx_j \prod_{f=1}^{m} dv_f \prod_{g=1}^{n} dw_g$$

$$\times f^*(x_1, ..., x_n|y_1, ..., y_l|v_1, ..., v_m|w_1, ..., w_n)$$

$$\times : a_+^+(x_1)...a_+^+(x_n)a_-(y_1)...a_-(y_l)a_+(v_1)...a_+(v_m)a_+^+(w_1)...a_+^+(w_n) :. \hfill (17)$$

We define the involution $*$ on $\text{Hom}(\tilde{H}', \tilde{H}')$ by the following equation:

$$(a|f)^* = a^*(|f)^*, \hfill (18)$$

where $a \in \text{Hom}(\tilde{H}', \tilde{H}')$ and $|f \in \tilde{H}'$.

We define also the involution $*$ on $C^2$ by the following equation:

$$\left\{ \int \prod_{i=1}^{k} dx_i \prod_{j=1}^{l} dx_j \prod_{f=1}^{m} dv_f \prod_{g=1}^{n} dw_g \right. $$

$$\times f(x_1, ..., x_n|y_1, ..., y_l|v_1, ..., v_m|w_1, ..., w_n)$$

$$\times : a_+^+(x_1)...a_+^+(x_n)a_+(y_1)...a_+(y_l)a_-(v_1)...a_-(v_m)a_+^+(w_1)...a_+^+(w_n) : \right\}^*$$

$$= \int \prod_{i=1}^{k} dx_i \prod_{j=1}^{l} dx_j \prod_{f=1}^{m} dv_f \prod_{g=1}^{n} dw_g$$

$$\times f^*(x_1, ..., x_n|y_1, ..., y_l|v_1, ..., v_m|w_1, ..., w_n)$$

$$\times : a_+^+(x_1)...a_+^+(x_n)a_-(y_1)...a_-(y_l)a_+(v_1)...a_+(v_m)a_+^+(w_1)...a_+^+(w_n) :. \hfill (19)$$
where \( f(x_1, \ldots, x_k | y_1, \ldots, y_l | v_1, \ldots, v_m | w_1, \ldots, w_n) \) be a test function of its arguments. Note that the involution on \( \text{Hom}(\tilde{H}', \tilde{H}') \) extends the involution on \( C^2 \). We say that the element \( a \in C_2 \) is real if \( a^* = a \). The involution on \( \tilde{H}'' \) can be defined by the similar way.

### 3 The von Neumann Dynamics

Suppose that our system described by the following Hamiltonian

\[
H = H_0 + \lambda V,
\]

(20)

where

\[
H_0 = \int d^3k (\omega(k) - \mu) a^+(k) a(k) \quad \text{and} \quad V = \int d^3p_1 d^3p_2 d^3q_1 d^3q_2 v(p_1, p_2 | q_1, q_2) \times \delta(p_1 + p_2 - q_1 - q_2) a^+(p_1) a^+(p_2) a(q_1) a(q_2).
\]

(21)

Here the kernel \( v(p_1, p_2 | q_1, q_2) \) belongs to the Schwartz space of test functions. To point out the fact that \( H \) is represented through the operators \( a^+, a^- \) we will write \( H(a^+, a^-) \).

The von Neumann dynamics takes place in the space \( \tilde{H}'' \) and defined by the following differential equation:

\[
\frac{d}{dt} |f\rangle = \mathcal{L} |f\rangle,
\]

(22)

where the von Neumann operator has the form

\[
\mathcal{L} = -iH(a^+, a^-) + iH^+(a^+, a^-),
\]

(23)

where we put by definition:

\[
\left( \int \prod_{i=1}^n dp_i \prod_{j=1}^m dq_j v(p_1, \ldots, p_n | q_1, \ldots, q_m) : a^+(p_1) \ldots a^+(p_n) a(q_1) \ldots a(q_m) : \right)^{\dagger}
\]

\[
= \int \prod_{i=1}^n dp_i \prod_{j=1}^m dq_j v(p_1, \ldots, p_n | q_1, \ldots, q_m)^* : a^+(p_1) \ldots a^+(p_n) a(q_1) \ldots a(q_m) : .
\]

(24)
Let us divide the von Neumann operator into the free operator $\mathcal{L}$ and the interaction $\mathcal{L}_{\text{int}}$, $\mathcal{L} = \mathcal{L}_0 + \lambda \mathcal{L}_{\text{int}}$, where

$$\begin{align*}
\mathcal{L}_0 &= -iH_0(a_+^+a_-^-) + iH_0^\dagger(a_+^+,a_-^-), \\
\mathcal{L}_{\text{int}} &= -iH_{\text{int}}(a_+^+a_-^-) + iH_{\text{int}}^\dagger(a_+^+,a_-^-).
\end{align*}$$

(25)

Note that the operators $\mathcal{L}_0$ and $\mathcal{L}_1$ are real.

Let us introduce kinetic evolution operator (in the interaction representation)

$$U(t'',t') = e^{-\mathcal{L}_0 t''} e^{\mathcal{L}(t''-t')} e^{\mathcal{L}_0 t'}.$$  

(26)

After differentiating on $t$ we find the differential equation for $U(t,t')$.

$$\frac{d}{dt} U(t,t') = \mathcal{L}_{\text{int}}(t) U(t,t'),$$

(27)

where

$$\mathcal{L}_{\text{int}}(t) = e^{-\mathcal{L}_0 t} \mathcal{L}_{\text{int}} e^{\mathcal{L}_0 t}.$$  

(28)

So the state $\langle \rho$ under consideration in the interaction representation in the space $\mathcal{H}''$ has the form

$$\langle \rho = T \exp\left( \int_{-\infty}^0 \mathcal{L}_{\text{int}}(t) dt \right),$$

(29)

where $T$ is the time-ordering operator.

### 4 Dynamics of Correlations

Let us construct some new representation of dynamics useful for the renormalization program. This representation is called the dynamics of correlations. The ideas of the dynamics of correlations belongs to I.R. Prigogin [5]. The dynamics of correlations take place in the space

$$\mathcal{H}_c := \bigoplus_{0}^{\infty} \text{sym} \otimes^n \mathcal{H}'.$$  

(30)
Now let us describe how the operators $L_0^c$ and $C_{int}$ acts in the space $\mathcal{H}_c$.

Let us define the action of operators $L_0^c$ and $L_{int}^c$ which are corresponds to the operators $L_0$ and $L_{int}$.

By definition all the space $\otimes^n \mathcal{H}'$ are invariant under the action of operators $L_0^c$. Note that the space $\tilde{\mathcal{H}}'$ is invariant under the action of operator $L_0$. Let us denote the restriction of $L_0^c$ to the space $\tilde{\mathcal{H}}'$ by the symbol $L_0''$.

By definition the restriction of $L_0^c$ to the each subspace $\otimes^n \tilde{\mathcal{H}}'$ of $\mathcal{H}_c$ has the form

$$L_0^c \otimes 1 \otimes ... \otimes 1 + 1 \otimes L_0^c \otimes ... \otimes 1 + ... + 1 \otimes 1 \otimes ... \otimes L_0''.$$ (31)

Now let us define $L_{int}^c$. Let $|f\rangle \in \mathcal{H}_c$, belongs to the subspace $\otimes^n \tilde{\mathcal{H}}'$ and has the form:

$$|f\rangle = \sum_{i=0}^{m} f_i^1 \otimes ... \otimes f_i^n,$$ (32)

where $f_i^j$ has the form

$$f_i^j = \int \prod_{i=1}^{k} dx_i \prod_{j=1}^{l} dx_j \prod_{f=1}^{m} dv_f \prod_{g=1}^{n} dw_g$$

$$\times f(x_1,...,x_n|y_1,...,y_l|v_1,...,v_m|w_1,...,w_n)$$

$$\times :a^+_1(x_1)...a^+_1(x_n)a_+(y_1)...a_+(y_l)a_-(v_1)...a_-(v_m)a^+_1(w_1)...a^+_1(w_n):.$$ (33)

By definition,

$$C_{int}^c |f\rangle = 0$$ (34)

if $l > n$. Let us consider the following vector in $\tilde{\mathcal{H}}''$

$$\sum_{i=1}^{m} : \prod_{j=1}^{n} f_i^j :.$$ (35)

Let us transform the expression $L_{int} \sum_{i=1}^{m} : \prod_{j=1}^{n} f_i^j :$ to the normal form. Let us denote by $h_l$ the sum of all the terms in the previous expression such that precisely $l$ operators $f_i^j$ couples with $L_{int}$. We find that $h_l \rangle$ has the following form

$$h_l = \sum_{i=1}^{k} : g_1^i...g_{n-l+1}^i :$$ (36)
for some $k$. Here $g^i_k$ has the form of right hand side of (33). Now let us consider the following vector

$$|f\rangle^c_i = \text{sym} \sum_{i=1}^k :g^i_1:\otimes \ldots \otimes :g^i_{n-l+1}:,$$  \hspace{1cm} (37)

where we define symmetrization operator as follows

$$\text{sym}(f_1 \otimes \ldots \otimes f_n) = \frac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)}.$$ \hspace{1cm} (38)

($S_n$ — the group of permutation of $n$ elements.) Put by definition

$$\mathcal{L}_{\text{int}}^{c,l}|f\rangle = |f\rangle^c_i.$$ \hspace{1cm} (39)

Analogously, in the following expression

$$\mathcal{L}_{\text{int}} \sum_{i=1}^m : \prod_{j=1}^n f^i_j :$$ \hspace{1cm} (40)

let us keep only the terms such that $\mathcal{L}_{\text{int}}$ do not couples with any of $f^i_j$. Let us write the sum of such terms as follows

$$\sum_{i=1}^f : \prod_{j=1}^{n+1} h^i_j :.$$ \hspace{1cm} (41)

Here $h^i_j$ has the form of right hand side of (33). Let $|h\rangle$ be a vector in $\text{sym} \otimes^{n+1} \mathcal{H}'$ defined as follows

$$|h\rangle = \text{sym} \sum_{i=1}^f \otimes_{j=1}^{n+1} : h^i_j :.$$ \hspace{1cm} (42)

Put by definition

$$\mathcal{L}_{\text{int}}^{c,0}|f\rangle = |h\rangle.$$ \hspace{1cm} (43)

We have the evident linear map $F : \mathcal{H}_c \to \mathcal{H}''$ which assigns to each vector $\text{sym} : f_1 : \otimes \ldots \otimes : f_n :$ the vector $: f_1 \ldots f_n :$. Denote by $U^c$ the evolution operator in interaction representation in the dynamics of correlation. The following relation holds:

$$F \circ U^c(t', t'') = U(t', t'') \circ F.$$ \hspace{1cm} (44)
5 The correlations tree

The useful representation of dynamics in $\mathcal{H}_c$ is a decomposition by so called the correlation’s trees.

**Definition.** A graph is a triple $T = (V, R, f)$, where $V$, $R$ are finite sets called the set of vertices and lines respectively and $f$ is a map:

$$h : R \to V^{(2)} \cup V \times \{+\} \cup V \times \{-\},$$

(45)

where $V^{(2)}$ is a set of all disordered pairs $(v_1, v_2)$, $v_1, v_2 \in V$, $v_1 \neq v_2$.

If $(v_1, v_2) = f(r)$ for some $r \in R$ we say that the vertices $v_1$ and $v_2$ are connected by a line $r$. If $f(r) = (v_1, v_2)$, $v_1, v_2 \in V$ we say that the line $r$ is internal.

**Remark.** We use this non-usual definition of graphs only in purpose of this section to simplify our notations.

**Definition.** The graph $\Gamma$ is called connected if for two any vertices $v, v'$ there exists a sequence of vertices $v = v_0, v_1, ..., v_{n-1}$ such that $\forall i = 0, ..., n - 1$ the vertices $v_i$ and $v_{i+1}$ are connected by some line.

By definition we say that the line $r$ is an internal line if $f(r) = (v_1, v_2)$ for some vertices $v_1$ and $v_2$.

For each graph $\Gamma$ we define its connected components by the obvious way.

**Definition.** We say that the graph $\Gamma$ is a tree or an acyclic graph if the number of its connected components increases after removing an arbitrary line.

**Definition.** The elements of the set $f^{-1}(V \times \{-\})$ we call the shoots. Put by definition $R_{sh} = f^{-1}(V \times \{-\})$. The elements of the set $f^{-1}(V \times \{+\})$ we call the roots. Put by definition $R_{root} = f^{-1}(V \times \{+\})$.

**Definition.** Directed tree is triple $(T, \Phi_v, \Phi_{sh})$, where $T$ is a tree and $\Phi_v$ and $\Phi_{sh}$ are the following maps:

$$\Phi_v : V \to \{1, 2, ..., \#V\},$$

$$\Phi_{sh} : R_{sh} \to \{1, 2, ..., \#R_{sh}\}.$$  

(46)

**Definition.** We will consider the following two directed trees $(T, \Phi_v, \Phi_{sh})$ and $(T', \Phi'_v, \Phi'_{sh})$ as identical if we can identify the sets of lines $R$ and $R'$ of $T$ and $T'$ respectively and identify the sets of vertices $V$ and $V'$ of $T$ and $T'$ respectively such that after these identification the trees $T$ and $T'$ become the same, the functions $\Phi_v$ and $\Phi'_v$ become the same and the functions $\Phi_{sh}$ and $\Phi'_{sh}$ become the same.
Denote by \( r(T) \) the number of roots of \( T \) and by \( s(T) \) the number of shoots of \( T \). Below, we will denote each directed tree \( (T, \Phi_v, \Phi_{sh}) \) by the same symbol \( T \) as a tree omitting the referring to \( \Phi_v, \Phi_{sh} \) and write simply tree instead of the directed tree.

We say that the connected directed tree \( T \) is right if there exists precisely one line from \( f^{-1}(V \times \{+\}) \).

We say that the tree \( T \) is right if each its connected component is right.

The vertex \( v \) of the tree \( T \) is called a root vertex if \((v, +) \in f^{-1}(R)\).

To point out the fact that some object \( A \) corresponds to a tree \( T \) we will often write \( A_T \). For example we will write \( T = (V_T, R_T, f_T) \) instead of \( T = (V, R, f) \).

**Definition.** For each right tree \( T \) there exist an essential partial ordering on the set of its vertices. Let us describe it by induction on the number of its vertices. Suppose that we have defined this relation for all right trees such that the number of their vertices less or equal than \( n-1 \). Let \( T \) be a right tree such that the number of its vertices is equal to \( n \). Let \( v_{\text{max}} \) be a root vertex of \( T \). Put by definition that the vertex \( v_{\text{max}} \) is a maximal vertex. Let \( v_1, \ldots, v_k \) be all its children i.e. the vertices connected with \( v_{\text{max}} \) by lines. By definition each vertex \( v_i < v_{\text{max}}, i = 1, \ldots, k \). We can consider the vertices \( v_1, \ldots, v_k \) as a root vertices of some directed trees \( T_i, i = 1, \ldots, k \). By definition the set of vertices of \( T_i \) consists of all vertices \( v \) which can be connected with \( v_i \) by some path \( v = v'_1, \ldots, v'_l = v_i \) such that \( v_{\text{max}} \neq v'_j \) for all \( j = 1, \ldots, l \). The incident relations on \( T_i \) are induced by incident relations on \( T \). Put by definition that \( \forall (i, j), i, j = 1, \ldots, k, i \neq j \) and for all two vertices \( v'_i \in T_i \) and \( v'_2 \in T_j \) \( v'_i < v'_2 \). If \( v'_i, v'_2 \in T_i \) for some \( T_i \) we put \( v'_i \leq v'_2 \) in \( T \) if and only if \( v'_i \leq v'_2 \) in the sense of ordering on \( T_i \). We put also \( v < v_{\text{max}} \) for all vertex \( v \neq v_{\text{max}} \). These relations are enough to define the partial ordering on \( T \).

Below without loss of generality we suppose that for each the tree of correlation \( T \) and its line \( r \) the pair \((v_1, v_2) = f(r)\) satisfy to the inequality \( v_1 > v_2 \).

**Definition.** The tree of correlations \( C \) is a triple \( C = (T, \varphi, \bar{\tau}) \), where \( T \) is a directed tree, \( \bar{\tau} \) is a map from \( R \setminus R_{sh} \) to \( \mathbb{R}^+ := \{x \in \mathbb{R} | x \geq 0\} \):

\[
\bar{\tau}: R \setminus R_{sh} \to \mathbb{R}^+, \\
\quad r \mapsto \tau(r), \\
(\tau(r))_{r \in \mathbb{R}} = \bar{\tau}(r),
\]  

\( (47) \)
and $\varphi$ is a map which assigns to each vertex $v$ of $T$ an element

$$\varphi(v) \in \text{Hom}(\bigotimes_{r \rightarrow v} \mathcal{H}', \mathcal{H}')$$

(48)

of a space of linear maps from $\bigotimes_{r \rightarrow v} \mathcal{H}'$ to $\mathcal{H}'$.

Here the symbol $r \rightarrow v$ means that $f(r) = (v, v')$ for some vertex $v'$, or $f(r) = (v, +)$.

In $\bigotimes \mathcal{H}'$ the tensor product is over all lines $r$ such that $r \rightarrow v$. Let $v$ be a vertex of the tree $T$. If $f(r) = (v', v)$ for some vertex $v'$ or $f(r) = (v, +)$ we say that the line comes from the vertex $v$. If $f(r) = (v, v')$ for some vertex $v'$ or $f(r) = (v, -)$ we say that the line comes into the vertex $v$.

**Definition.** Let $(T, \varphi, \vec{\tau})$ be a tree of correlations such that for each vertex $v$ $\varphi(v) = L_{\text{int}}$, where $l_v$ is a number of lines coming into $v$. We call this tree the von Neumann tree and denote it by $T_{\vec{\tau}}$. We also say that $\varphi$ is a von Neumann vertex function.

**Definition.** To each tree of correlation $(T, \varphi, \vec{\tau})$ assign an element

$$U^L_{T, \varphi}(\vec{\tau}) \in \text{Hom}(\bigoplus_{R_{\text{sh}}} \mathcal{H}', \bigoplus_{R_{\text{root}}} \mathcal{H}')$$

(49)

by the following way:

If $T$ is disconnected then

$$U^L_{T, \varphi}(\vec{\tau}) f_1 \otimes \ldots \otimes f_n$$

$$= \bigotimes_{CT} \{U^L_{CT, C \varphi}(C \vec{\tau}) \otimes_{i \in R_{\text{sh}}(CT)} f_i \}.$$ 

(50)

Here the number of connected components of $T$ is equal to $n$ and connected components of $T$ are denoted by $CT$. $C \varphi$ and $C \vec{\tau}$ are the restrictions of $\varphi$ and $\vec{\tau}$ to the sets of vertices and lines of $CT$ respectively. $R_{\text{sh}}(CT)$ is a set of shoots of $CT$. Now let $T$ be a connected tree. To define

$$U^L_{T, \varphi}(\vec{\tau}) \bigotimes_{r \in R_{\text{sh}}} f_r$$

(51)

by induction its enough to consider the following two cases.

case 1). The tree $T$ has no shoots.
a) Suppose that the tree $T$ has more than one vertex. Let $v_{\text{min}}$ be some minimal vertex of $T$ and $v_0$ be a vertex such that an unique line $r_0$ comes from $v_{\text{min}}$ into $v_0$. Let $T'$ be a tree obtained from $T$ by removing the vertex $v_{\text{min}}$ of $T$. Let $\bar{\tau}'$ be a restriction of $\bar{\tau}$ to $R \setminus \{r_0\}$. Let $\varphi'$ be a function, defined on $V \setminus \{v_{\text{min}}\}$ as follows: $\varphi'(v) = \varphi(v)$ if $v \neq v_0$ and
\begin{equation}
\varphi'(v_0) \bigotimes_{r \to v_0; r \neq r_0} f_r = \varphi(v_0) \bigotimes_{r \to v_0} h_r,
\end{equation}
where
\begin{align*}
h_r &= f_r \text{ if } r \neq r_0, \text{ and } \\
h_{r_0} &= e^{\mathcal{L}_0 \tau(r_0)} \varphi(v_{\text{min}}).
\end{align*}

Put by definition
\begin{equation}
U^t_{T,\varphi}(\bar{\tau}) = U^t_{T',\varphi'}(\bar{\tau}'),
\end{equation}

b) The tree $T$ has only one vertex $v_{\text{min}}$. Then
\begin{equation}
U^t_{T,\varphi}(\bar{\tau}) = e^{-(t-r)\mathcal{L}_0} \varphi(v_{\text{min}}).
\end{equation}

Case 2.) The tree $T$ has a shoot $r_0$ coming into the vertex $v_0$. In this case instead the tree $(T, \varphi, \bar{\tau})$ consider the tree $(T', \varphi', \bar{\tau}')$, where the tree $T'$ has the same vertices as $T$, the set of lines of $T$ is obtained by removing the line $r_0$ from the set of lines of $T'$, the function $\bar{\tau}'$ is a restriction of the function $\bar{\tau}$ to the set of lines of $T$ and the function $\varphi'$ is defined as follows:
\begin{equation}
\varphi'(v) = \varphi(v), \text{ if } v \neq v_0 \text{ and } \\
\varphi'(v_0) \bigotimes_{r \to v_0; r \neq r_0} h_r = \varphi(v_0) \bigotimes_{r \to v_0} g_r, \text{ where } \\
g_r = h_r, \text{ if } r \neq r_0, \text{ and } \\
g_{r_0} = e^{\mathcal{L}_0(t-r_0)} f_{r_0}.
\end{equation}

Here we put
\begin{equation}
t_r = \sum \tau_{r'},
\end{equation}

where the sum is over all lines $r'$ which forms decreasing way coming from + to $v_0$. Put by definition
\begin{equation}
U^t_{T,\varphi}(\bar{\tau})|f := U^t_{T',\varphi'}(\bar{\tau}')|f',
\end{equation}

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where
\[ |f'_r⟩ = \bigotimes_{r \in (R_{sh})_{T'}} f_r. \] (59)

Let \((T, \varphi, \vec{\tau})\) be some tree of correlations. We can identify the tensor product
\[ \bigotimes_{r \in R_{sh}} \tilde{\mathcal{H}}'_r \] (60)
with
\[ \bigotimes_{i=1}^{sh(T)} \tilde{\mathcal{H}}'_i \] (61)
and the tensor product
\[ \bigotimes_{r \in R_{root}} \tilde{\mathcal{H}}'_r \] (62)
with
\[ \bigotimes_{i=1}^{r(T)} \tilde{\mathcal{H}}'_i \] (63)
Using these identifications consider an operator \(V'_T (\vec{\tau}) : \mathcal{H}^c \to \mathcal{H}^c\) defined by the following formula
\[ V'_T (\vec{\tau}) = \text{sym} \circ U'_T (\varphi) \circ P_{sh(T)}, \] (64)
where \(P_{sh(T)}\) is a projection of \(\mathcal{H}_c\) to \(\text{sym} \bigotimes_{i=1}^{sh(T)} \tilde{\mathcal{H}}'_i\).

Remark. If \((T, \varphi, \vec{\tau})\) is a von Neumann tree of correlations then we will shortly denote the operators \(U'_T (\varphi)\) and \(V'_T (\varphi)\) by \(U'_T\) and \(V'_T\) respectively.

The following theorem holds:

Theorem. The following representation for the evolution operators holds (in the sense of formal power series on coupling constant \(\lambda\)).
\[ U^c (t', t'') = \sum_T \frac{1}{n_T!} \int_{\forall r \in R_{sh} t_r > t''} V^t_T (\vec{\tau}) d\vec{\tau}. \] (65)
Here \(n_T\) is a number of vertices of the directed tree \(T\).
6 The general theory of renormalization of $U(t, -\infty)\rangle$

In the present section we by using the decomposition of correlation’s dynamics by trees describe the general structure of counterterms of $U(t, -\infty)\rangle$, which subtract the divergences from $U(t, -\infty)\rangle$.

Let $T$ be a tree. Let us give a definition of its right subtree.

**Definition.** Let $v_1, \ldots, v_n$ be vertices of $T$ such that $\forall i, j = 1, \ldots, n, i \neq j$ $v_i \not\leq v_j$. Let us define subtree $T_{v_1, \ldots, v_n}$. The set of vertices $V_{T_{v_1, \ldots, v_n}}$ of $T_{v_1, \ldots, v_n}$ by definition consists of all vertices $v$ such that $v \leq v_i$ for some $i = 1, \ldots, n$.

The set $R_{T_{v_1, \ldots, v_n}}$ of all lines of the tree $T_{v_1, \ldots, v_n}$ consists of all lines $r$ of $R_T$ such that such that $h(r) = (v'', v')$ and $v', v'' \leq v_i$ for some $i = 1, \ldots, n$. The incident relations on $T_{v_1, \ldots, v_n}$ are induced by the incident relations of $T$ except the following point: if the line $r$ comes from the vertex $v$ into $v_i$, $i = 1, \ldots, n$ we put $f_{T_{v_1, \ldots, v_n}}(r) = (v, +)$. In this case the line $r$ is a root and the vertex $v$ is a root vertex of the tree $V_{T_{v_1, \ldots, v_n}}$. The tree $T_{v_1, \ldots, v_n}$ is called a right subtree of $T$.

**The Bogoliubov — Parasiuk renormalization prescription.** Let us define the following operator:

$$W_{r_0}(t) = \bigotimes_{r \in R_{r_0}(T)} Z_{r, r_0}(t),$$

where by definition,

$$Z_{r, r_0}(t) = 1, \text{ if } r \neq r_0, \text{ and } Z_{r, r_0}(t) = e^{-L_0 t}.$$  

(66)

We say that the amplitude $A_{T, \varphi}$ is time — translation invariant amplitude if for each tree $T$ and for each its root line $r_0$

$$W_{r_0}(t)A_{T, \varphi} = A_{T, \varphi}.$$  

(68)

For each set of amplitudes $A_{T, \varphi}$ put by definition:

$$A_{T, \varphi} = F \circ A_{T, \varphi},$$  

(69)

where $T$ is an arbitrary tree without shoots.
The renormalization assign to each tree $T$ without shoots the amplitudes $\Lambda_{T,\varphi}$ satisfying to the following relations:

a) If the tree $T$ is not connected and $\{CT\}$ is a set of its connected components, while $\{C\varphi\}$ is a set of its a restriction of $\varphi$ to $CT$

$$\Lambda_{T,\varphi} = \bigotimes \Lambda_{CT,C\varphi}$$

in obvious notations.

b) The amplitudes $\Lambda_{T,\varphi}$ are real i.e.

$$(\Lambda_{T,\varphi})^* = \Lambda_{T,\varphi^*}$$

(71)

c) The amplitude $\Lambda_{T,\varphi}$ satisfy to the property of time-translation invariance.

It has been proven that

$$U(t, -\infty) = \sum_T \frac{1}{n_T} \int d\vec{\tau} U_T(\vec{\tau}).$$

(72)

In the last formula the summation is over all trees $T$ without shoots.

Let $T$ be a tree without shoots and $T'$ be a right subtree of $T$ in the sense described before. Let us define the amplitude

$$\Lambda_{T,\varphi} \ast U_{T,\varphi}^l(\vec{\tau}).$$

(73)

Let $T\setminus T'$ by definition be a tree obtained by removing from the set $V_T$ all the vertices of $T'$ and from the set $R_T$ all the internal lines of $T'$. In (72) $\vec{\tau}$ is a map from $R_{T\setminus T'}$ into $\mathbb{R}^+$. We can consider the amplitude $U_{T\setminus T'}^l$ as a map

$$\bigotimes_{(R_{T\setminus T'})_{sh}} \hat{H}' \rightarrow \bigotimes_{(R_{T\setminus T'})_{root}} \hat{H}' .$$

(74)

By using this identification we simply put

$$\Lambda_{T',\varphi} \ast U_{T',\varphi}^l(\vec{\tau}) = U_{T\setminus T',\varphi}^l(\vec{\tau}) \Lambda_{T',\varphi}.$$  

(75)
Now let us define the renormalized amplitudes, by means the counterterms $\Lambda_T$ by the following formula:

$$(R_A U)(t, -\infty) = \sum_T \frac{1}{n_T!} \sum_{T' \subset T} \int \Lambda_{T'} \ast U^T_{\tau} (\vec{\tau}) d\vec{\tau}. \quad (76)$$

The renormalized amplitudes satisfy the following properties:

**Property 1.** For each $t \in \mathbb{R}$

$$(R_A U)(t, -\infty) \rangle = e^{-L_A t} (R_A U)(0, -\infty) \rangle. \quad (77)$$

This property simply follows from the definition of $(R_A U)(t, -\infty) \rangle$ and means that the state $(R_A U)(t, -\infty) \rangle$ is a stationary state.

**Property 2.**

$$(R_A U)(t, -\infty) \rangle = U(t, 0)(R_A U)(0, -\infty) \rangle. \quad (78)$$

This property follows from the following representation of $(R_A U)(t, -\infty) \rangle$.

$$(R_A U)(t, -\infty) \rangle = U(t, -\infty) \mathcal{I} \rangle, \quad (79)$$

where

$$\mathcal{I} \rangle = \sum_T \frac{1}{n_T!} \Lambda_T \rangle, \quad (80)$$

and the sum in the last formula is over all von Neumann trees without shoots. Property 2 means that the state $(R_A U)(t, -\infty) \rangle$ satisfy to the von Neumann dynamics.

Below we will prove that we can find the counterterms $\Lambda_T$ such that $(R_A U)(t, -\infty) \rangle$ is finite.

We will prove also that the counterterms $\{\Lambda_T\}$ satisfy to the following property.

d) Let $T$ be an arbitrary connected tree without shoots. Consider the following element of $\mathcal{H}'$:

$$a := \sum_T \sum_{T' \subset T} \int \Lambda_{T'} \ast U^T_{\tau} (\vec{\tau}) d\vec{\tau}. \quad (81)$$
We can represent the element $a$ as follows:

$$a = \sum_{k,l,f,g=0}^{\infty} \int w_m(x_1, ..., x_k|y_1, ..., y_l|v_1, ..., v_f|w_1, ..., w_g)$$

$$\cdot \prod_{i=1}^{k_m} a^+(x_i)dx_i \prod_{i=1}^{l_m} a^+(y_i)dy_i \prod_{i=1}^{f_m} a^-(v_i)dv_i \prod_{i=1}^{g_m} a^+(w_i)dw_i :.$$  \hspace{1cm} (82)

Let $\tilde{w}_{k,l,f,g}(z_1, ..., z_n) (n = k_m + l_m + f_m + g_m)$ be a Fourier transform of $w_{k,l,f,g}(x_1, ..., x_k|y_1, ..., y_l|v_1, ..., v_f|w_1, ..., w_g)$. Then

$$\int dz_1, ..., dz_n \tilde{w}_{k,l,f,g}(z_1 + s(1)e_1a, ..., z_n + s(n)e_1a)f(z_1, ..., z_n),$$  \hspace{1cm} (83)

tends to zero faster than ever inverse polynomial on $a$ as $a \to +\infty$. Here $s(i)$ are the numbers from $\{0, 1\}$ and there exist numbers $i, j, i, j = 1, ..., n$ such that $s(i) = 0, s(j) = 1$ for some $i, j = 1, ..., n$. $f(z_1, ..., z_n)$ is a test function. $e_1$ is a unite vector parallel to the x-axis.

**Remark.** The property d) implies the weak cluster property of the state $(RU)(0, -\infty)$.

(84)

## 7 The Friedrichs diagrams

Now let us start to give a constructive description of the counterterms $\Lambda_T$ such that the amplitude $R(U)(t, -\infty)$ is finite, and the counterterms $\Lambda_T$ satisfy the properties a) — d) from the previous section.

At first we represent $U_T^t(\tau)$, where $T$ is some tree without shoots, as a sum over all so-called Friedrichs graphs $\Phi$ concerned with $T$.

**Definition.** A Friedrichs graph $\Phi_T$ concerted with the directed tree $T$ without shoots is a set $(\hat{V}, R, Or, f^+, f^-, g)$, where $\hat{V}$ is a union of the set of vertices of $T$ and the set $\{\oplus\}$.

Recall that there is a partial order on $V_T$. We define a partial order on the set $\hat{V}$ if we put $\forall v \in V_T \oplus > v$. $f^+$ and $f^-$ are the maps $f^+, f^- : R \to V$ such that $f^+(r) > f^-(r)$. $Or$ is a map $R \to \{+, -\}$ called an orientation. $g$ is a function which to each pair $(v, r), v \in V_T, r \in R$ such that $f^+(r) = v$ or $f^-(r) = v$ assigns $+$ or $-$. The graph $(V, R, Or, f^+, f^-, g)$ must satisfy to the property: if we consider $\oplus$ as a vertex, the obtained graph is connected.
If \( f^+(r) = v \) we will write \( r \to v \), and if \( f^-(r) = v \) we will write \( r \gets v \).

If we want to point out that the object \( B \) concerned with the graph \( \Phi \) we will write \( B_{\Phi} \). For example we will write \( V_{\Phi} \) and \( R_{\Phi} \) for the sets of vertices and lines of \( \Phi \) respectively.

At the picture we will represent the elements of \( V \) by points and the element \( \oplus \) by \( \oplus \). We will represent the elements of \( R \) by lines. The line \( r \) connects the vertices \( f^+(r) \) and \( f^-(r) \) at the picture. We will represent orientation \( Or(r) \) by arrow on \( r \). If \( Or(r) = + \) the arrow oriented from \( f^-(r) \) to \( f^+(r) \). If \( Or(r) = - \) the arrow oriented from \( f^+(r) \) to \( f^-(r) \). To represent the map \( g : (r,v) \to \{+,-\} \) we will draw the symbol \( g((r,v)) \) (+ or −) near each shoot \( (r,v) \). At the picture a shoot \( (r,v) \) is a small segment of the line \( r \) near \( v \).

**Definition.** The Friedrichs diagram \( \Gamma \) is a set \((T, \Phi, \varphi, h)\), where \( T \) is a tree, \( \Phi \) is a Friedrichs graph, \( \varphi \) is a map which assign to each vertex \( v \) of \( T \) a function of momenta \( \{p_r \mid r \in R_\Phi\} \) of the form

\[
\varphi_v(...)p_r \leftrightarrow v... = \psi_v(...)p_r \leftrightarrow v... \prod_{S_i} \delta(\sum_{j=1}^{j_i} \pm p_i^j),
\]

where \( \psi_v \) is a test function of momenta coming into (from) the vertex \( v \). \( \{S_i\}_{i=1}^{n_v} \) is a decomposition of the set of shoots of \( v \) into \( n_v \) disjunctive nonempty sets \( S_i \), \( p_i^1, ..., p_i^{j_i} \) are momenta corresponding to the shoots from \( \{S_i\} \), \( h \) is a function which assign to each pair \( v \in V, r \in R \) such that \( f^+(r) > v > f^-(r) \) a real positive number \( h(v,r) \).

It will be clear that it is enough to consider only the diagrams \( \Gamma \) such that for each its vertex \( v \) and set \( S_i \in \{S_i\}_{i=1}^{n_v} \) there exists a line \( r \) such that \( (r, f^-(r)) \in S_i \).

To each Friedrichs diagram \( \Gamma = (T, \Phi, \varphi) \) assign an element of \( \mathcal{H}_c'' \) of the form

\[
U^t_{(T, \Phi, \varphi)}(\vec{r}) = \int \ldots dp_{ext} \ldots U^t_T(...p_{ext}...) \times \ldots a_{\pm}^+(p_{ext}) :...:
\]

Here \( p_{ext} \) are momenta of external lines, i.e. such lines \( r \) that \( f^+(r) = \oplus \). We chose the lower index of \( a_{\pm}^+(p_{ext}) \) by the following rule. Let \( v \) be a vertex such that \( f^-(r_{ext}) = v \). If \( g((r,v)) = + \) we chose + as a lower index, and
if \( g((r,v)) = - \) we chose \(-\) as a lower index. We chose the upper index of \( a_{\pm}^{\pm}(p_{\text{ext}}) \) by the following rule. If the lower index of \( a_{\pm}^{\pm}(p_{\text{ext}}) \) is \((-\)} then the upper index is equal \(+\) if the corresponding line comes from the vertex \( v \) and this index is equal \(-\) if the corresponding line comes into the vertex \( v \). If the lower index of \( a_{\pm}^{\pm}(p_{\text{ext}}) \) is \((+\)} then the upper index is equal \(-\) if the corresponding line comes from the vertex \( v \) and this index is equal \(+\) if the corresponding line comes into the vertex \( v \).

Now let us describe the amplitude \( U_{\Gamma}^{\pm}(...p_{\text{ext}}...) \). By definition we have

\[
U_{\Gamma}^{\pm}(\vec{\tau})(...p_{\text{ext}}...)
= \int \prod_{r \in R_T} \psi_v(...p_r \leftrightarrow v...)
\times \prod_{r \in R_T} e^{i \text{Or}(r)p_r^2(r) \sum_{r_T \in (R_T)_r} \tau_{r_T} + \sum_{v \in V_T} h(v,r)} dp_r
\prod_{r \in R_T} G(\text{Or}(r), g(f^+(r)), g(f^-(r)))(p).
\]

(88)

Let us describe the elements of this formula. \( R_T \) is a set of all lines of diagram \( \Gamma \). Symbol \( r \leftrightarrow v \) denotes that the line \( r \) comes into (from) the vertex \( v \). In the expression

\[
\psi_v(...p_r \leftrightarrow v...)(\sum_{r \leftrightarrow v} \pm p_r)
\]

(89)

we take the upper sign \(+\) if the line \( r \) comes into the vertex \( v \) and we take lower sign \(-\) in the opposite case. The symbol \( R_T \) denotes the set of lines of the tree \( T \) from the triple \((T, \Phi, \varphi)\) and symbol \( r_T \) means the line from \( R_T \). The symbol \( V_T \) denotes the set of all vertices \( v \) such that \( f^+(r) \geq v \geq f^-(r) \). The symbol \( (R_T)_r \) denotes the set of all lines \( r_T \) of \( R_T \) such that the increasing path coming from \( f^-(r) \) into \( f^+(r) \) contains \( r_T \). \( G(\text{Or}(r), g(f^+(r)), g(f^-(r)))(p) \) is a factor defined as follows

\[
G(\text{Or}(r), g(f^+(r)), g(f^-(r)))(p)\delta(p - p')
= \rho(\sigma\text{sgn}(\text{Or}(r)g((r,f^+(r))))(p), \sigma\text{sgn}(\text{Or}(r)g((r,f^-(r))))(p'))
\]

(90)

Below we will simply write \( G_r(p) \) instead of \( G(\text{Or}(r), g(f^+(r)), g(f^-(r)))(p) \).

It is evident that we can represent \( U_{\Gamma}^{\pm}(\vec{\tau}) \) as a sum over some Friedrichs diagrams \( \Gamma \) corresponding to the tree \( T \) of the quantities \( U_{\Gamma}^{\pm}(\vec{s}) \).
Now let us define the quotient diagrams.

**Definition.** Let $\Gamma = (T, \Phi, \varphi, h)$ be a Friedrichs diagram and $A \subset R_T$ be a subset of the set $R_T$ of lines of $T$ and $\vec{\tau}$ be a map from $R_T$ into $\mathbb{R}^+$. We define the quotient diagram $\Gamma^A_{\vec{\tau}} := (T^A, \Phi^A, \varphi^A_{\vec{\tau}}, h^A)$ by the following way. To obtain the tree $T^A$ we must tighten all lines from $A$ into the point. To obtain $\Phi^A$ we must remove all loops obtained by tightening all lines from $A$ into the point.

Now let us define $\varphi^A_{\vec{\tau}}$. Joint all the vertices of $T$ to $A$. We obtain a tree denoted by $^A_T$. Let $\{C^A_T\}$ be a set of all connected components of $^A_T$. Let $v_0$ be a vertex of $\Phi^A$ corresponding to the connected component $C^A_T$ of $^A_T$. Put by definition:

$$\varphi^A_T(...p_r\mapsto v...)_A^\vec{\tau} = \int \prod_{r \in R_{in}} \prod_{v \in V_r} \varphi(v,... \pm p_r\mapsto v...) \times \prod_{r \in R_{in}} e^{iOr(r)p_r^2(\sum_{r \in (R_T)_r} \tau_{r_T} + \sum_{v \in V_r} h(v,r))}. \quad (91)$$

Let us point out that is that in the previous formula. $R_{in}$ is a set off all lines of $\Phi_A$ such that $f^+(r)$ and $f^-(r)$ are the vertices of $C^A_T$. $(R_T)_r$ denotes the set of all lines $r_T$ of $R_T$ such that the increasing path coming from $f^-(r)$ into $f^+(r)$ contains $r_T$. The symbol $V_r$ denotes the set of all vertices $v$ such that $f^+(r) \geq v \geq f^-(r)$. $h_A(v_0, r) = \sum_{v \in V_{C^A_T}} h(v, r) + \sum_{r_T \in A; r_T \in (R_T)_r} \tau_{r_T}$.

**Definition.** Let $\Gamma$ be a Friedrichs diagram. Let $F_\Gamma$ be a space of all functions of external momenta of the diagram $\Gamma$ of the form:

$$\psi(...p_{ext}...), \quad (92)$$

where $\psi(...p_{ext}...)$ is a test function of external momenta.

The convolution of the amplitude $A_\Gamma(\vec{\tau})(...p_{ext}...)$ with the function $f \in F_\Gamma$ we denote by $A_\Gamma(\vec{\tau})[f]$.
8 The Bogoliubov — Parasiuk renormalization prescriptions

Let for each Friedrichs diagram $\Gamma = (T, \Phi, \varphi)$ $A_{\Gamma}(\vec{r})(...p_{ext}...)$ be some amplitude. Fix some diagram $\Gamma$ and let $T'$ be some right subtree of the tree $T$ corresponding to $\Gamma$. Let $\Gamma_{T'}$ be a restriction of the diagram $\Gamma$ on $T'$ in obvious sense. Define the amplitude $A_{\Gamma_{T'}} \ast U_{\Gamma}(...p_{ext}...)$ by the following formula:

$$A_{\Gamma_{T'}} \ast U_{\Gamma}(...p_{ext}...).$$

In this formula $V'$ is a set of all vertices $v$ such that $v$ is not a vertex of $V_{T'}$, $R'$ is a set of all lines $r$ of $\Phi_{T}$ such that $f^+(r)$ is not a vertex of $T'$. $(R'_{T})_{r}$ is a set of all lines $r_{T}$ of $T$ such that $r_{T}$ is not a line of $T'$ and there exists an increasing path on $T$ coming from $f^-(r)$ into $f^+(r)$ such that this path contains $r_{T}$. $V'_{v}$ is a set of all vertices $v$ of $T$ such that $v$ is not a vertex of $T'$ and $f^+(r) \geq v \geq f^-(r)$.

Let $A_{\Gamma}(\vec{r})(p)$ be some amplitude. Put by definition:

$$\hat{A}_{\Gamma}(s_1, ..., s_n)(p) := A_{\Gamma}(\frac{1}{s_1}, ..., \frac{1}{s_n})(p) \prod_{i=1}^{n} \frac{1}{s_i^2};$$

where $n$ is a number of lines of $T_{\Gamma}$. Below we will consider the amplitudes $\hat{A}_{\Gamma}(\vec{s})[f]$ as distributions on $(\mathbb{R}^+)^n$ i.e. as an element of the space of tempered distributions $S'((\mathbb{R}^+)^n)$. Let $\psi(\vec{s})$ be a test function from $S((\mathbb{R}^+)^n)$. The convolution of the amplitude $A_{\Gamma}(\vec{s})[f]$ and the function $\psi(\vec{s})$ we denote by:

$$\langle \hat{A}_{\Gamma}(\vec{s})[f], \psi(\vec{s}) \rangle := \int_{(\mathbb{R}^+)^n} d\vec{s} \hat{A}_{\Gamma}(\vec{s})[f] \psi(\vec{s}).$$
r of Γ such that the increasing path on T which connects $f^{-}(r)$ and $f^{+}(r)$ contain $r_T$. Other diagram can be simply subtracted by the counterterms $\Lambda_T$. Below we will consider only such diagrams.

According to the Bogoliubov — Parasiuk prescriptions we must to each diagram Γ assign the counterterm amplitude $\hat{C}_\Gamma(\vec{s})[f]$ $f \in \mathcal{F}_\Gamma$ satisfying the following properties.

a) (Locality.) $\hat{C}_\Gamma(\vec{s})[f]$ is a finite linear combination of $\delta$ functions centered at zero and their derivatives.

b) Let Γ be a Friedrichs diagram and $T$ be a corresponding tree of correlations. Let $A \subseteq R_T$ and $T'$ is some right subtree of $T$ such that:

1) all lines $r_T$ of $T$ such that $r_T$ is not a line of $R_{T'}$ belongs to $A$,
2) All the root lines of $T'$ do not belongs to $A$.

Then

$$\hat{C}_{\Gamma_{A'}}(\vec{s})[f] = (\hat{C}_{\Gamma'_{A'}} \star \hat{U}_\Gamma)(\vec{s})[f], \quad (96)$$

where $A' := A \cap (R_{T'})$ and $\Gamma'$ is a restriction of $\Gamma$ on $T'$.

c) $\hat{C}_\Gamma(\vec{s})[f] = -\mathbb{T}(\sum_{\emptyset \subset A \subset R_T} \hat{C}_{\Gamma_{A'}}(\vec{s})[f] + \hat{U}_\Gamma(\vec{s})[f]), \quad (97)$

where $\vec{\tau} = (\tau_1, ..., \tau_n) = (\frac{1}{s_1}, ..., \frac{1}{s_n})$, the symbol $\subset$ means the strong inclusion and $\mathbb{T}$ is some subtract operator.

d) The amplitudes $\hat{C}_\Gamma(\vec{s})[f]$ satisfy to the property of time-translation invariance, i.e.

$$ie^{r \rho(r) t} \sum_{r \in \{R_{\text{root}}\}_\Gamma} \hat{C}_\Gamma(\tau_1, ..., \tau_n)[f] = \hat{C}_\Gamma(\tau_1 + t, ..., \tau_n)[f]. \quad (98)$$

e) Let Γ be a Friedrichs diagram. Let

$$\hat{R}_\Gamma'(\vec{s})[f] := \hat{U}_\Gamma(\vec{s})[f] + \sum_{\emptyset \subset A \subset R_T} \hat{C}_{\Gamma_{A'}}(\vec{s})[f], \quad \text{and}$$

$$\hat{R}_\Gamma(\vec{s})[f] := \hat{U}_\Gamma(\vec{s})[f] + \sum_{\emptyset \subset A \subset R_T} \hat{C}_{\Gamma_{A'}}(\vec{s})[f] + \hat{C}_\Gamma(\vec{s})[f]. \quad (99)$$

The amplitudes $\hat{R}_\Gamma(\vec{s})$ is well defined distribution on $(\mathbb{R}^+)^n$. 

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f) The amplitudes $\hat{R}_Γ(\vec{s})$ satisfy to the weak cluster property. This property means the following. Let $f(...p_{ext}...) be a test function. Then

$$\int dp \hat{R}_Γ(\vec{s})(...p_{ext}...)f(...p_{ext}...)e^{ia\sum_{r \in A} p^1_r} \to 0,$$

(100)
as $a \to \infty$. Here $p^1_r$ is a projection of $p_r$ to the x-axis. Note that the weak cluster property admit to prove the Gibbs formula if the system has not first integrals except momenta and energy by the usual way.

For each diagram $Γ$ put by definition:

$$\Lambda_Γ(\vec{τ}) = \sum'_{A \subseteq R_Γ} C_{Γ_A}$$

(101)
where ' in the sum means that all the root lines of $T_Γ$ do not belong to $A$. Put

$$\Lambda_T = \sum_{Γ \sim T} \int d\vec{τ} \Lambda_Γ(\vec{τ})(...p_{ext}...)a^±_p(p_{ext}...)$$

(102)
where the symbol $Γ \sim T$ means that the sum is over all diagrams corresponding to $T$ with suitable combinatoric factors. Suppose that the properties a) — f) are satisfied. Then $Λ_T$ are the counterterms needed in the section.

Theorem — Construction. It is possible to find such a subtract operator $T$ such that there exists counterterms $\hat{C}_Γ$ satisfying to the properties a) — f).

Note that we can use not real counterterms. Indeed the evolution operator is real, so after renormalization we can simply take $\text{Re} U(t, -\infty))$.

Before we prove our theorem let us prove the following

Lemma. Let $L_1 = S(\mathbb{R}^k)$, $L_2 = S((\mathbb{R}^+)^n)$, $k, n = 1, 2, ...$. Let $A(p)$ be some quadratic form on $\mathbb{R}^k$. Let $T^{1}_t, t \geq 0$ be a one-parameter semigroup acting in $L_1$ as follows:

$$T^{1}_t : f(...p...) \mapsto e^{iA(p)t} f(...p...)$$

(103)
Let $T^{2}_t, t \geq 0$ be some infinitely differentiable semigroup of continuous operators in $L_2$.

Let $M$ be a subspace of finite codimension in $L_2$. Suppose that $M$ is invariant under the action of $T^{2}_t$, i.e. $\forall t > 0 T^{2}_t M \subset M$.  

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Suppose that there exist the linear independent vectors \( f_1, ..., f_l \) in \( L_2 \) such that

\[
\text{Lin}\{\{f_1, ..., f_i\}, M\} = L_2, \\
M \cap \text{Lin}\{f_1, ..., f_i\} = 0.
\] (104)

and for each \( i = 1, ..., l \)

\[
T_t^2 f_i = f_i + a_{i-1} f_{i-1} + ... + a_1 f_1 + f,
\] (105)

for some coefficients \( a_{i-1}, ..., a_1 \) and the element \( f \in M \).

Let \( g \) be a functional on \( L_1 \otimes M \) such that \( g \) is continuous with respect to the topology on \( S(\mathbb{R}^k) \times S((\mathbb{R}^+)^n) \). Suppose that \( \forall f \in L_1 \otimes M \) and \( \forall t > 0 \)

\[
\langle g, T_t f \rangle = \langle g, f \rangle.
\]

Then there exists an continuous extension \( \tilde{g} \) of \( g \) on \( L_1 \otimes L_2 \) such that \( \forall f \in L_1 \otimes L_2 \) and \( t > 0 \)

\[
\langle \tilde{g}, T_t f \rangle = \langle \tilde{g}, f \rangle.
\]

By definition we say that the functional \( h \) on \( L_1 \otimes L_2 \) is invariant if

\[
\forall f \in L_1 \otimes L_2 \langle h, T_t f \rangle = \langle h, f \rangle.
\]

**Proof of the lemma.** At first we extend our functional \( g \) to the invariant functional \( \tilde{g} \) on \( L_1 \otimes L_2 \) and then we prove that \( \tilde{g} \) is continuous.

Let \( N \) be a subspace of \( L_1 \) of all functions of the form, \( A(p)f(p) \), where \( f(p) \) is a test function. Let \( M_1 = \text{Lin}\{M \cup \{f_1\}\} \). Let

\[
h := \left( \frac{d}{dt} T_t^2 \right)_{t=0} f_1.
\] (106)

Let \( k \) be a continuous functional on \( N \) defined as follows:

\[
\langle k, \varphi(p)A(p) \rangle = - \langle g, \varphi(p) \otimes h \rangle.
\] (107)

Let \( \tilde{k} \) be an arbitrary continuous extension of \( k \) on whole \( L_1 \). The existence of such continuation follows from Malgrange’s preparation theorem [6]. Now we define the continuous functional \( \tilde{g}_1 \) on \( L_1 \otimes M_1 \) as follows:

\[
\tilde{g}_1 |_{L_1 \otimes M} = g |_{L_1 \otimes M}, \\
\langle \tilde{g}_1, f \otimes f_1 \rangle = \langle \tilde{k}, f \rangle, \forall f \in L_1.
\] (108)

According to (107) we find that \( \tilde{g}_1 \) is an invariant extension of \( g \) on \( L_1 \otimes M_1 \). By the same method step by step we can extend the functional \( g \) to the functionals \( \tilde{g}_2, ..., \tilde{g}_l \) on \( L_1 \otimes M_1, ..., L_l \otimes M_l \) respectively,
where $M_2 = \text{Lin}\{M \cup \{f_1, f_2\}\}, ..., M_l = \text{Lin}\{M \cup \{f_1, f_2, ..., f_l\}\}$ respectively.

Just constructed functional is separately continuous so it is continuous. The lemma is proved.

**Sketch of the proof of the theorem.** We will prove the theorem by induction on the number of lines of the tree of correlation $T_\Gamma$ corresponding to the diagram $\Gamma$.

The base of induction is evident. Suppose that the theorem is proved for all diagrams of order $< n$. (Order is a number of lines of the tree of correlation.)

Let us give some definitions. Let $\xi(t)$ be a smooth function on $[0, +\infty)$ such that $0 \leq \xi(t) \leq 1$, $\xi(t) = 1$ in some small neighborhood of zero and $\xi(t) = 0$ if $t > \frac{1}{3n}$. Let us define a decomposition of unite $\{\eta_A(\vec{s})|A \subset \{1, ..., n\}\}$ by the formula

$$\eta_A(\vec{s}) = \prod_{i \notin A} \xi(s_i) \prod_{i \in A} (1 - \xi(s_i)).$$

(109)

We identify the set of lines of the tree of correlation with $\{1, ..., n\}$ such that the root vertex is a vertex corresponds to 1. Let $\psi(x)$ be some test function on real line such that $\psi(t) \geq 0$, $\int \psi(t) dt = 1$ and $\psi(t) = 0$ if $|t| > \frac{1}{10}$. Put by definition:

$$\delta_\lambda(x) = \frac{x}{\lambda^2} \psi\left(\frac{x}{\lambda}\right).$$

(110)

We have

$$\int_0^{+\infty} d\lambda \delta_\lambda(x - \lambda) = 1.$$  

(111)

Let $S_N((\mathbb{R}^+)^n)$, $N = 1, 2, ..., be a subspace of $S((\mathbb{R}^+)^n)$ of all functions $f$ such that $f$ has a zero at zero of order $\geq N$.

We have:

$$\langle \hat{R}_\Gamma(\vec{s})[f], \Psi(\vec{s}) \rangle$$

$$= \sum_{A \subset \{1, ..., n\}} \int_0^{+\infty} d\lambda \lambda^{n-1} \int_{(\mathbb{R}^+)^n} d\vec{s} \hat{R}_{\Gamma_A}(\vec{s}|_{\{1, ..., n\}\setminus A})$$

$$\delta_1(1 - |\vec{s}|) \Psi(\lambda \vec{s}) \eta_A(\vec{s}).$$

(112)
The inner integral in (112) converges according to the inductive assumption. Therefore if $\Psi(\vec{s}) \in S_N((\mathbb{R}^+)^n)$ and $N$ is enough large the integral at the right hand side of (112) converges. So

$$\langle \hat{R}(\vec{s})[f], \Psi(\vec{s}) \rangle$$

(113)

define a separately continuous functional on $S(\mathbb{R}^3) \otimes S_N((\mathbb{R}^+)^n)$. $f = l - 1$, where $l$ is a number of external lines of $\Gamma$. To define a subtract operator $T$ we must to extend the functional $\langle \hat{R}(\vec{s})[f], \Psi(\vec{s}) \rangle$ to the space $S(\mathbb{R}^3) \otimes S((\mathbb{R})^n)$ such that extended functional will satisfy to time-translation invariant property. To obtain this extension we use the lemma. In our case $L_1 = S(\mathbb{R}^3), L_2 = S((\mathbb{R}^+)^n), A(p) = \sum_{r \in R_{ext}} Or(r)p_r^2$. $T_t^2$ is an operator adjoint to the following operator $T_t^{2*}$ in the $S((\mathbb{R}^+)^n)$.

$$T_t^{2*}f(s_1, ..., s_n) = f(\frac{s_1}{1 - s_1 t}, s_2, ..., s_n) \text{ if } s_i < \frac{1}{t},$$

$$T_t^{2*}f(s_1, ..., s_n) = 0, \text{ if } s_i \geq \frac{1}{t}. \quad (114)$$

The basis $\{f_1, ..., f_l\}$ from the lemma is $\{s_1^{m_1}...s_n^{m_n} \eta_{(1,...,n)}(\vec{s})\}, m_1, ...m_n = 1, 2, 3..., m_1 + m_2 + ... + m_n \leq N$ lexicographically ordered. Now we can directly apply our lemma.

Now let us prove the weak cluster property. Let $p \in \mathbb{R}^3$. Denote by $p^1, p^2, p^3$ the projections of $p$ to the $x, y, z$-axis respectively. To prove the weak cluster property it is enough to prove the following statement: for each connected diagram $\Gamma$ the function $\langle F(\vec{s})(...p_{ext}...), \Psi(\vec{s}) \rangle$ such that

$$\delta(\sum \pm p_{ext})\langle F(\vec{s})(...p_{ext}...), \Psi(\vec{s}) \rangle = \langle \hat{R}(\vec{s})(...p_{ext}...), \Psi(\vec{s}) \rangle$$

(115)

is a distribution on variables $...p^2_{ext}...p^3_{ext}...$ (constrained by momenta conservation law) which depends on $...p^1_{ext}...$ (constrained by momenta conservation law) by the differentiable way. We will prove this statement by induction on the number of lines of the corresponding correlation’s tree. The base of induction is evident. Suppose that the statement is proved for all the correlation’s trees such that the number of its lines $< n$. Let $\Gamma$ be a diagram such that the number of the lines of the corresponding correlation’s tree is equal to $n$. It is evident that if $\Psi(\vec{s})$ has a zero of enough large order at zero then $\langle F(\vec{s})(...p_{ext}...), \Psi(\vec{s}) \rangle$ belongs to the required class (its enough to use our
construction with decomposition of unite). Therefore we need to solve by induction the system of equations of the form:

\[
(i \sum \pm (p_{ext})^2) \langle F_\Gamma(\vec{s})(...p_{ext}...), \Psi(\vec{s}) \rangle = \langle F_\Gamma(\vec{s})(...p_{ext}...), \frac{d}{dt} T^2_t \Psi(\vec{s}) \rangle.
\] (116)

According to Malgrange’s preparation theorem we can choose the solution \( \langle F_\Gamma(\vec{s})(...p_{ext}...) \rangle \) such that it belongs to the required class if \( \langle F_\Gamma(\vec{s})(...p_{ext}...), \frac{d}{dt} T^2_t \Psi(\vec{s}) \rangle \) belongs to the required class. Therefore the statement is proved. So our theorem is proved.

9 Conclusion

In the present theory we have developed the general theory of the renormalization of nonequilibrium diagram technique. To study this problem we have used some ideas of the theory of \( R \)-operation developed by N.N. Bogoliubov and O.S. Parasiuk. This paper is formally independent of the previous paper of this series. But the previous paper can be considered as an illustration of some technical aspects of our theory that have been omitted in the present paper which ends our series.

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