Isomorphism rigidity of algebraic $\mathbb{Z}^d$-actions

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Abstract

An algebraic $\mathbb{Z}^d$-action is an action of $\mathbb{Z}^d$ on a compact abelian group $X$ by continuous automorphisms of $X$. We prove that that for $d \geq 8$, there exist mixing zero entropy algebraic $\mathbb{Z}^d$-actions which do not exhibit isomorphism rigidity property.

1 Introduction

An algebraic $\mathbb{Z}^d$-action is an action $\alpha : \mathbb{n} \to \alpha(\mathbb{n})$ of $\mathbb{Z}^d$ on a compact abelian group $X$ by continuous automorphisms of $X$. It is easy to see that any such action preserves $\lambda_X$, the Haar measure on $X$. If $\alpha$ is a homomorphism from $\mathbb{Z}^d$ to $GL(n, \mathbb{Z})$ for some $n \geq 1$, then the natural action of $\alpha(\mathbb{Z}^d)$ on $\mathbb{R}^n$ induces an algebraic $\mathbb{Z}^d$-action on $\mathbb{T}^n \cong \mathbb{R}^n/\mathbb{Z}^n$. Another class of examples is given by group shifts: let $F$ be a finite abelian group and let $S$ be the shift action of $\mathbb{Z}^d$ on $F^{\mathbb{Z}^d}$. A group shift is a closed shift invariant subgroup $X \subset F^{\mathbb{Z}^d}$, together with the shift action of $\mathbb{Z}^d$ restricted to $X$.

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An algebraic $\mathbb{Z}^d$-action $(X, \alpha)$ is said to be irreducible if $X$ does not admit proper closed $\alpha$-invariant infinite subgroups. If $(X, \alpha)$ and $(Y, \beta)$ are two algebraic $\mathbb{Z}^d$-actions and $f : X \to Y$ is a measurable map then $f$ is said to be a measurable conjugacy if $f$ is a measure space isomorphism from $(X, \lambda_X)$ to $(Y, \lambda_Y)$ and for all $\mathbb{n} \in \mathbb{Z}^d$, $f \circ \alpha(\mathbb{n}) = \beta(\mathbb{n}) \circ f$ a.e. $\lambda_X$. An algebraic conjugacy from $(X, \alpha)$ to $(Y, \beta)$ is a continuous isomorphism $\theta$ from $X$ to $Y$,
which satisfies $\theta \circ \alpha(n) = \beta(n) \circ \theta$ for all $n$ in $\mathbb{Z}^d$. Two algebraic $\mathbb{Z}^d$-actions $(X, \alpha)$ and $(Y, \beta)$ are said to be measurably conjugate (resp. algebraically conjugate) if there is a measurable conjugacy (resp. algebraic conjugacy) from $(X, \alpha)$ to $(Y, \beta)$. If $X$ and $Y$ are compact abelian groups and $f : X \to Y$ is a measurable map, then $f$ is said to be affine if there exists an element $c \in Y$ and a continuous surjective group homomorphism $\theta : X \to Y$ such that $f(x) = c + \theta(x)$ a.e. $\lambda_X$.

Recently, in [1] and [3] it has been shown that the measurable orbit structure of a certain class of mixing zero entropy algebraic $\mathbb{Z}^d$-actions exhibit strong rigidity properties. More specifically, it has been proved that if $(X, \alpha)$ and $(Y, \beta)$ are two algebraic $\mathbb{Z}^d$-actions such that the actions $\alpha$ and $\beta$ are irreducible, expansive and mixing then every measurable conjugacy from $(X, \alpha)$ to $(Y, \beta)$ is an affine map (cf. [3], Corollary 1.2). The question whether this form of rigidity occurs for all mixing zero entropy algebraic $\mathbb{Z}^d$-actions has been raised by several authors. In various degrees of generality, several questions and conjectures about this aspect of mixing zero entropy algebraic $\mathbb{Z}^d$-actions can be found in [2], [3] and [4]. All these questions can be viewed as special cases of the following more general conjecture due to K. Schmidt (cf. [6], Conjecture 3.5).

**Conjecture.** Let $d > 1$, and let $\alpha$ and $\beta$ be mixing algebraic $\mathbb{Z}^d$-actions on compact abelian groups $X$ and $Y$, respectively. If $h(\alpha) = 0$, and if $\phi : X \to Y$ is a measurable conjugacy of $\alpha$ and $\beta$, then $\phi$ is $\lambda_X$ a.e. equal to an affine map. In particular, measurable conjugacy implies algebraic conjugacy.

In this note we give a counter-example to the above conjecture. More specifically we prove the following result.

**Theorem 1.1** For every $d \geq 8$, there exists a mixing zero entropy algebraic $\mathbb{Z}^d$-action $\alpha$ on a compact zero dimensional abelian group $X$, and a non-affine homeomorphism $f : X \to X$, such that $f$ preserves the Haar measure on $X$ and commutes with the action $\alpha$.

## 2 Markov subgroups

For any $d \geq 1$, by $X_d$ we denote the group $(\mathbb{Z}/2\mathbb{Z})^{zd}$, equipped with pointwise addition and the topology of pointwise convergence. It is easy to see that $X_d$
is a compact zero dimensional abelian group. By $S$ we denote shift action of $\mathbb{Z}^d$ on $X_d$ defined by

$$S(j)(x)(i) = x(i + j) \quad \forall i, j \in \mathbb{Z}^d.$$ 

A Markov subgroup of $X_d$ is a closed subgroup which is invariant under the shift action. In [2] it was shown that the dynamics of the shift action on Markov subgroups can be studied using algebraic methods. We briefly recall the results that are needed in our construction. For proofs, the reader is referred to [5], Theorem 6.5 and Proposition 19.4.

Let $\mathbb{F}_2$ denote the field with two elements and for $d \geq 1$, let $\mathcal{R}^{(2)}_d$ denote the group-ring of $\mathbb{Z}^d$ with coefficients in $\mathbb{F}_2$. The ring $\mathcal{R}^{(2)}_d$ can be identified with $\mathbb{F}_2[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$, the ring of Laurent polynomials in $d$ commuting variables with coefficients in $\mathbb{F}_2$. Every element $p \in \mathcal{R}^{(2)}_d$ is written as

$$p = \sum_{m \in \mathbb{Z}^d} c_p(m) u^m,$$

with $u^m = u_1^{m_1} \cdots u_d^{m_d}$ and $c_p(m) \in \mathbb{F}_2$, where $c_p(m) = 0$ for all but finitely many $m$. For any $d \geq 1$, the group $X_d$ can be viewed as a $\mathcal{R}^{(2)}_d$-module via the operation $p \cdot x = \sum c_p(m)S(m)(x)$. For any ideal $I \subset \mathcal{R}^{(2)}_d$ we define $X(I) \subset X_d$ by

$$X(I) = \{ x \in X_d \mid p \cdot x = 0 \ \forall p \in I \}. $$

It is easy to see that $X(I)$ is a Markov subgroup of $X_d$. Conversely, given any Markov subgroup $H$ of $X_d$, we define an ideal $I(H) \subset \mathcal{R}^{(2)}_d$ by

$$I(H) = \{ p \in \mathcal{R}^{(2)}_d \mid p \cdot x = 0 \ \forall x \in H \}. $$

Using duality theory of compact abelian groups it can be shown that for any ideal $J \subset \mathcal{R}^{(2)}_d$ and for any Markov subgroup $H \subset X_d$, $I(X(J)) = J$ and $X(I(H)) = H$. Hence the correspondence $H \mapsto I(H)$ is an order reversing bijection from the set of all Markov subgroups of $X_d$ to the set of all ideals in $\mathcal{R}^{(2)}_d$.

**Proposition 2.1** Let $d \geq 1$ and let $H \subset X_d$ be a Markov subgroup.

1. The action $(H, S)$ has zero entropy if and only if $H$ is a proper subgroup.

2. If $I(H)$ is a prime ideal then the action $(H, S)$ is mixing if and only if for every non-zero $m$, $u^m - 1$ does not lie in $I(H)$.
3 Binary linear codes

A binary linear code of length \(d\) is a subspace \(C\) of \(\mathbb{F}_2^d\). For any \(v \in \mathbb{F}_2^d\), by \(|v|\) we denote the number of non-zero coordinates of \(v\). A set \(A \subset \mathbb{F}_2^d\) is said to be even (resp. doubly even) if for every element \(v\) of \(A\), \(|v|\) is divisible by 2 (resp. divisible by 4). If \(v\) and \(w\) are two elements of \(\mathbb{F}_2^d\) then their dot product \(v \cdot w\) is defined by \(v \cdot w = \sum v_iw_i\). For any set \(A \subset \mathbb{F}_2^d\), by \(A^\perp\) we denote the binary linear code defined by

\[
A^\perp = \{v \in \mathbb{F}_2^d \mid v \cdot w = 0 \ \forall w \in A\}.
\]

A set \(A \subset \mathbb{F}_2^d\) is said to be self orthogonal if \(A \subset A^\perp\).

**Example 1:** For any \(d \geq 2\), let \(E_d\) be the subspace consisting of all \(v\) such that \(|v|\) is even. If \(1\) denotes the vector \((1, 1, \ldots, 1)\) in \(\mathbb{F}_2^d\) then \(E^\perp_4\) is an one dimensional subspace of \(\mathbb{F}_2^d\), consisting of \(1\) and \(0\).

**Example 2:** We define a \(4 \times 8\) matrix \(M\) with entries in \(\mathbb{F}_2\) by

\[
M = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}.
\]

Let \(C \subset \mathbb{F}_2^8\) be the row space of \(M\). It can be easily verified that the row vectors of \(M\) are linearly independent. Hence \(\dim(C) = 4\). Throughout the paper we will denote this code by \(C_8\).

If \(v, w\) are two elements of \(\mathbb{F}_2^d\), then we define their product \(v \times w\) by \(v \times w = (v_1w_1, \ldots, v_dw_d)\). It is easy to see that \(v \cdot w = 0\) if and only if \(|v \times w|\) is an even integer.

**Proposition 3.1** Let \(C \subset \mathbb{F}_2^d\) be a binary linear code.

1. If \(C\) admits a self-orthogonal doubly even basis \(A\), then \(C\) itself is self orthogonal and doubly even.

2. \(\dim(C) + \dim(C^\perp) = d\).

**Proof.** Let \(A\) be the collection of all self orthogonal doubly even subsets of \(\mathbb{F}_2^d\) which contains \(A\), and let \(W\) be a maximal element of \(A\). Let \(w, w'\)
be any two elements of $W$. Since the dot product is a bilinear form on $\mathbb{F}_2^d$, the set $W \cup \{w + w'\}$ is self orthogonal. We note that for any two elements $v, w \in \mathbb{F}_2^d$,

$$|v + w| = |v| + |w| - 2|v \times w|.$$  

Hence $W \cup \{w + w'\}$ is doubly even. By the maximality of $W, w + w' \in W$. Therefore $W$ is a subspace of $\mathbb{F}_2^d$. Since $A$ is a basis of $C$, this proves the first part. The second assertion is a consequence of the non-degeneracy of the dot product. \hfill $\Box$

Now we introduce a notion of non-degeneracy on binary linear codes. For any $d \geq 1$ we define a map $B : \mathbb{Z}^d \times \mathbb{F}_2^d \to \mathbb{Z}$ by

$$B(n, v) = \sum_{i, n_i = 1} n_i.$$  

**Definition:** Let $C \subset \mathbb{F}_2^d$ be a binary linear code. Then $C$ is said to be *integratedly non-degenerate* if for all non-zero $n \in \mathbb{Z}^d$, there exists a $v \in C$ such that $B(n, v)$ is non-zero.

**Proposition 3.2** For every $d \geq 8$ there exist proper subspaces $C, C' \subset \mathbb{F}_2^d$ such that $C$ is an integratedly non-degenerate code containing $1$ and for any two $x, y$ in $C$, their product $x \times y$ lies in $C'$.

**Proof:** First we will consider the case when $d = 8$. We claim that the pair $(C_8, E_8)$ has the required properties. It is easy to see that $1 \in C_8$. Let $A \subset \mathbb{F}_2^8$ denote the set of row vectors of the matrix $M$, as defined in Example 2. It is easy to check that $A$ is a doubly even self orthogonal basis of $C_8$. By the previous proposition, $C_8$ is doubly even and self orthogonal. In particular for any two $x, y \in C_8$, $x \cdot y = 0$ i.e. $x \times y \in E_8$.

Let $n$ be a non-zero vector in $\mathbb{Z}^d$. Clearly, we can choose $i, j$ in $\{1, \ldots, 8\}$ such that $n_i + n_j \neq 0$. Let $\phi$ be the vector space homomorphism from $C_8$ to $\mathbb{F}_2^2$ defined by $\phi(v) = (v_i, v_j)$ and let $E \subset C_8$ be the set defined by

$$E = \{v \in C_8 \mid \phi(v) = (1, 1)\}.$$  

It is easy to see that for every $k \in \{1, \ldots, 8\}$ there exists a vector $v'$ in $A$ such that $v'_k = 1$. Hence we can choose vectors $v, w$ in $A$ such that $v_i = 1$ and $w_j = 1$. It is easy to see that there exists $x$ in the set $\{v, w, v + w\}$ which lies in $E$. Since $C_8$ has 16 elements and $\mathbb{F}_2^2$ has 4 elements, the set $E = x + \ker(\phi)$
contains at least 4 elements. Since $C_8$ is doubly even this implies that we can find two distinct vectors $v, w$ in $E$ such that $|v| = |w| = 4$. As $v \cdot w = 0$, this shows that $(v \times w)_k = 1$ if $k = i, j$ and zero otherwise. In particular, $B(n, v \times w) = n_i + n_j \neq 0$. We also note that the map $B$ satisfies the identity

$$2B(m, x \times y) = B(m, x) + B(m, y) - B(m, x + y).$$

Hence we can find a vector $v_0$ in the set $\{v, w, v + w\} \subset C_8$ such that $B(n, v_0)$ is non-zero. This proves the claim. For $d > 8$, we define $C, C' \subset F_2^d = F_2^8 \oplus F_2^{d-8}$ by $C = C_8 \oplus F_2^{d-8}$ and $C' = E_8 \oplus F_2^{d-8}$. It is easy to verify that the pair $(C, C')$ has the desired properties. \(\square\)

4 Non rigid actions

As in Section 2, for any $d \geq 1$ by $X_d$ we denote the group $(\mathbb{Z}/2\mathbb{Z})^\mathbb{Z}$ and by $S$ we denote shift action of $\mathbb{Z}^d$ on $X_d$. For any $x, y \in X_d$ we define their product $x \star y \in X_d$ by $x \star y(i) = x(i)y(i)$. It is easy to see that $X_d$ becomes a compact topological ring with respect to this product and $S(i)(x \star y) = S(i)(x) \star S(i)(y)$ for all $i$ in $\mathbb{Z}^d$.

The following proposition is the basis of our construction.

**Proposition 4.1** Let $H, K \subset X_d$ be proper Markov subgroups such that the actions $(H, S)$ and $(K, S)$ are mixing, and for all $x, y$ in $H$, $x \star y \in K$. We define a $\mathbb{Z}^d$-action $(X, \alpha)$ and a map $f : X \to X$ by

$$(X, \alpha) = (H, S) \times (H, S) \times (K, S), \quad f(x, y, z) = (x, y, x \star y + z).$$

Then $(X, \alpha)$ is a mixing zero entropy action of $\mathbb{Z}^d$, and the map $f$ is a non-affine homeomorphism which preserves the Haar measure on $X$ and commutes with the action $\alpha$.

**Proof**: Since $H$ and $K$ are proper subgroups of $X_d$, by Proposition 2.1, both $(H, S)$ and $(K, S)$ have zero entropy. Since both $(H, S)$ and $(K, S)$ are mixing by our assumption, it follows that $(X, \alpha)$ is mixing and has zero entropy. It is easy to see that $f$ is a homeomorphism which commutes with the action $\alpha$. From the standard results on skew products if follows that $f$ preserves the Haar measure on $X$. So it remains to show that $f$ is a non-affine map. Suppose this is not the case. Comparing the last coordinate we
see that there exists a constant $c_0 \in K$ and homomorphisms $\theta_1, \theta_2 : H \to K$ and $\theta_3 : K \to K$ such that

$$x \star y + z = c_0 + \theta_1(x) + \theta_2(y) + \theta_3(z) \ \forall x, y, z.$$ 

Putting $x = y = 0$ we see that $c_0 = 0$ and $\theta_3 = \text{Id}$. Putting $x = 0$ (resp. $y = 0$) we see that $\theta_2 = 0$ (resp. $\theta_1 = 0$). Hence $x \star y = 0$ for all $x$ and $y$. On the other hand, $x \star x \neq 0$ for any non-zero $x$. This contradiction completes the proof. 

For any binary linear code $C \subset \mathbb{F}_2^d$ we define a Markov subgroup $X_C \subset X_d$ by

$$X_C = \{ x \in X_d \mid (x(i + e_1), \ldots, x(i + e_d)) \in C \ \forall i \in \mathbb{Z}^d \},$$

where $e_1, \ldots, e_d$ are the standard unit vectors in $\mathbb{Z}^d$. The ideal $I(X_C)$, as defined in Section 2, can be described as follows: For any $v$ in $\mathbb{F}_2^d$ we define a polynomial $p_v$ in $\mathcal{R}_d^{(2)}$ by

$$p_v = \sum_{j=1}^d v_j u_j.$$ 

We note that $p_v \cdot x = 0$ for any $v \in C^\perp$ and $x$ in $X_C$. Since $(C^\perp)^\perp = C$ by Proposition 3.1, it follows that the $I(X_C)$ is the ideal generated by the set \{p_v \mid v \in C^\perp\}. As $p_v + w = p_v + p_w$ for any $v, w \in \mathbb{F}_2^d$, we see that for any basis $A$ of $C^\perp$ the set \{p_v \mid v \in A\} generates the ideal $I(X_C)$.

Examples: If $C = E_d$ then $I(X_C)$ is the principal ideal generated by $u_1 + \cdots + u_d$. Since $C_8^\perp = C_8^\perp$ and the row vectors of the matrix $M$ form a basis of $C_8$, it follows that $I(X_{C_8})$ is the ideal $< p_1, p_2, p_3, p_4 >$, where $p_1, p_2, p_3, p_4$ are given by

$$\begin{align*}
p_1 &= u_1 + u_2 + u_3 + u_4 \\
p_2 &= u_3 + u_4 + u_5 + u_6 \\
p_3 &= u_5 + u_6 + u_7 + u_8 \\
p_4 &= u_1 + u_3 + u_5 + u_7
\end{align*}$$

Lemma 4.2 Let $A \subset \mathbb{F}_2^d$ be a subset and let $I_A \subset \mathcal{R}_d^{(2)}$ be the ideal generated by the set \{p_v \mid v \in A\}. Then $I_A$ is a prime ideal.
Proof. Let $F_2[u_1, \ldots, u_d]$ be the polynomial ring in $d$ variables with coefficients in $F_2$. We can identify $F_2[u_1, \ldots, u_d]$ with the subring of $\mathcal{R}_d^{(2)}$, which consists of all $p$ such that $c_p(n) = 0$ whenever $n_i < 0$ for some $i$. Clearly for each $v$ in $F_2^d$, $p_v$ lies in $F_2[u_1, \ldots, u_d]$. For any set $B \subset F_2^d$ let $I_B' \subset F_2[u_1, \ldots, u_d]$ be the ideal generated by the set $\{p_v \mid v \in B\}$. We claim that $I_B'$ is a prime ideal of $F_2[u_1, \ldots, u_d]$. To prove this we note that $I_B' = I_C$, where $C \subset F_2^d$ is the subspace generated by $B$. If $\dim(C) = k$, we define a subspace $C_1 \subset F_2^d$ by

$$C_1 = \{(v_1, \ldots, v_d) \mid v_i = 0 \forall i > k\}.$$ 

Let $\theta$ be a linear automorphism of $F_2^d$ such that $\theta(C) = C_1$ and let $\overline{\theta}$ be the automorphism of $F_2[u_1, \ldots, u_d]$ satisfying $\overline{\theta}(p_v) = p_{\theta(v)}$ for all $v$ in $F_2^d$. Then $\overline{\theta}(I_C) = I_{C_1}$. It is easy to see that $F_2[u_1, \ldots, u_d]/I_{C_1}$ is isomorphic to $F_2[u_k+1, \ldots, u_d]$. Therefore $I_{C_1}$ is a prime ideal. Since $\overline{\theta}(I_C) = I_{C_1}$ and $\overline{\theta}$ is an automorphism of $F_2[u_1, \ldots, u_d]$, this proves the claim.

For any $N > 0$ we define $Z_N^d \subset \mathbb{Z}^d$ by

$$Z_N^d = \{(n_1, \ldots, n_d) \in \mathbb{Z}^d \mid n_i \geq N \forall i\}.$$ 

We observe that for any element $p$ in $\mathcal{R}_d^{(2)}$, $p$ lies in $F_2[u_1, \ldots, u_d]$ if and only if there exists $N > 0$ such that $u^n p \in F_2[u_1, \ldots, u_d]$ for all $n \in Z_N^d$. Similarly for any $p$ in $I_A$, $p$ lies in $I_A'$ if and only if there exists $N > 0$ such that $u^n p \in I_A'$ for all $n \in Z_N^d$. Let $p_1, p_2$ be two elements of $\mathcal{R}_d^{(2)}$ such that $p_1 p_2 \in I_A$. Then we can choose $n \in Z_N^d$ such that $u^n p_1, u^n p_2 \in F_2[u_1, \ldots, u_d]$ and $u^{n+1} p_1 p_2 \in I_A'$. By the above claim, $I_A'$ is a prime ideal of $F_2[u_1, \ldots, u_d]$. Hence either $u^n p_1 \in I_A'$ or $u^n p_2 \in I_A'$. This implies that either $p_1 \in I_A$ or $p_2 \in I_A$, which proves the given assertion. \hfill \Box

Lemma 4.3 Let $C \subset F_2^d$ be a binary linear code.

1. If $C$ is integrally non-degenerate and contains 1 then the $\mathbb{Z}^d$-action $(X_C, S)$ is mixing.

2. The action $(X_C, S)$ has zero entropy if and only if $C$ is a proper subspace of $F_2^d$.

Proof : 1) For any $v \in F_2^d$, let $\phi_v$ be the unique homomorphism from $\mathcal{R}_d^{(2)}$ to $\mathcal{R}_1^{(2)} = F_2[z, z^{-1}]$ such that $\phi_v(u_i) = z$ if $v_i = 1$ and $\phi_v(u_i) = 1$ if $v_i = 0$. 

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We choose $v, w \in \mathbb{F}_2 \mathbb{F}^d$ such that $v \in C^\perp$ and $w \in C$. Since $C$ contains $1$ and $v \cdot w = 0$, $|v|$ and $|v \times w|$ are even integers. Hence the sets $\{i \mid v_i = w_i = 1\}$ and $\{i \mid v_i = 1, w_i = 0\}$ contain even number of elements. Therefore

$$\phi_w(p_v) = \sum v_i \phi_w(u_i) = 0.$$ 

This shows that $I(X_C) \subset \ker(\phi_w)$ for all $w$ in $C$. Let $n$ be any non-zero element of $\mathbb{Z}^d$. As $C$ is integrally non-degenerate, there exists a $w$ in $C$ such that $B(n, w)$ is non-zero. Since $\phi_w(u^n) = z^{B(n, w)}$, we conclude that $u^n - 1$ does not lie in $I(X_C)$. By the previous lemma and Proposition 2.1, the action $(X_C, S)$ is mixing. This proves 1). The second assertion is an immediate consequence of Proposition 2.1.

Now we turn to the proof of Theorem 1.1.

Proof of Theorem 1.1: Let $C, C'$ be proper subspaces of $\mathbb{F}_2^d$ satisfying the conditions stated in Proposition 3.2. Since $C$ contains $1$ and $x \times y \in C'$ for all $x, y$ in $C$, it follows that $C \subset C'$. Hence $C$ and $C'$ are integrally non-degenerate codes containing $1$. Since $C'$ is a proper subspace of $\mathbb{F}_2^d$, by Lemma 4.3 the actions $(X_C, S)$ and $(X_{C'}, S)$ are mixing and have zero entropy. It is easy to verify that for any $x, y$ in $X_C$, $x * y$ is an element of $X_{C'}$. Now Theorem 1.1 follows from Proposition 4.1.

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