Remarks on the interaction between Born-Infeld solitons

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(October 31, 2018)

Abstract

We consider the Abelian Higgs model as well as the SU(2) Georgi-Glashow model in which the gauge field action is replaced by a nonlinear Born-Infeld action. We study soliton solutions arising in these models, namely the vortex and monopole solutions, respectively. We construct formulas which provide good approximations for the mass of the Born-Infeld deformed solitons using only the data of the undeformed solutions. The results obtained indicate that in the self-dual limit, the Born-Infeld interaction leads to bound vortices, while for monopoles it gives rise to repulsion.

PACS numbers: 11.10.Lm, 11.27.+d, 14.80.Hv
I. INTRODUCTION

In recent years it became apparent that when studying low energy effective actions of string theory, the part of the Lagrangian containing the abelian Maxwell field strength tensor and its non-abelian counterpart in Yang-Mills field theories, respectively, must be replaced by a corresponding (resp. abelian and non-abelian) Born-Infeld term. This idea is not new and was first suggested by Born and Infeld in the 1930s [1] to get rid of singularities associated with point-like charges in electrodynamics. A subtle point in the generalisation to non-abelian Born-Infeld (NBI) theory is how to specify the trace over the gauge group generators. From the viewpoint of string theory the symmetrized trace [2] seems more favourable than the ordinary trace. In general, the Lagrangian involving a symmetrized trace is only known in a perturbative expansion [3]. Recently, however, an explicit expression of the SU(2) NBI action for static, spherically symmetric, magnetic configurations involving a symmetrized trace was constructed [4]. It was found that the qualitative results are in good agreement with the ones obtained previously [5] using the ordinary trace.

A number of classical field theory model have been studied with respect to the effects of the Born-Infeld (BI) interaction. It was found that in SU(2) Yang-Mills theory the presence of the Born-Infeld term leads to the existence of particle-like solutions, so called ”glueballs” [6], which are absent in the standard case (i.e. without BI interaction). The reason for this is that similar to gravity in the Einstein-Yang-Mills model [6], the Born-Infeld term breaks the scale invariance and thus admits finite energy solutions. For models in which soliton-type solutions already exist in the standard case, it is of interest to study the BI deformation of these configurations. This has been done for a number of models including the Abelian Higgs model [7] and the SU(2) Georgi-Glashow model [8].

Both, the Abelian Higgs model [7] and the Georgi-Glashow model [8] admit soliton solutions characterized by an integer of topological origin and possess a so-called ”self-dual” (BPS) limit for a specific value of the Higgs self-coupling constant. In this limit, the second order Euler-Lagrange equations can be replaced by a set of first order (”self-dual”) equations and the mass of the $n$-soliton solution is given by $M(n) = nM(n = 1)$ indicating that no interaction between the solitons exists [11]. The repulsion due to the long range gauge field is exactly compensated by the attraction of the Higgs field which, because of its masslessness, is also long range.

A natural question arising in the study of the Born-Infeld interaction is whether it can lead to attraction between vortices and monopoles, respectively. In the case of vortices, it was found that in the self-dual limit the Born-Infeld interaction leads to attraction [7]. For monopoles the question has not be answered yet and one of the aims of this paper is to give an indication of whether the Born-Infeld interaction similar to gravity [12] can lead to attraction.

Here we present a formula which provides a good approximation for the energy of the BI-deformed solitons using only the data of the standard case. In Section II, we study this approximation for the Abelian Higgs model, in section III for the Georgi-Glashow model. We give our conclusion in section IV.
II. THE ABELIAN HIGGS MODEL

The Lagrangian of the Abelian Higgs model reads [9]:

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \Phi D^\mu \Phi^* - \frac{\lambda}{4} (\Phi^* \Phi - v^2)^2 \]  

(1)

where \( D_\mu = \partial_\mu - ie A_\mu \), \( A_\mu \) is the U(1) gauge potential with coupling \( e \) and \( \Phi \) is a complex Higgs field with vacuum expectation value \( v \) and self-coupling constant \( \lambda \).

This model has soliton solutions, namely the Nielsen-Olesen vortices [9] and \( n \)-soliton solutions can be constructed with an appropriate axially symmetric ansatz:

\[ \Phi = v f(\rho)e^{i\varphi}, \quad A_\mu dx^\mu = \frac{1}{e}(n - P(\rho))d\varphi \]  

(2)

The self-dual case is given for \( \alpha = \frac{e^2}{\lambda} = 2 \). In the Born-Infeld version of this theory [7], the part of the Lagrangian containing the Maxwell field strength tensor \( F_{\mu\nu} \) is replaced by the corresponding Born-Infeld (BI) expression with BI coupling \( \beta \):

\[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \rightarrow \beta^2 \left( 1 - \sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{16\beta^4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2} \right) \]  

(3)

For \( \beta^2 \gg 1 \), the square root can be evaluated as follows:

\[ \sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{16\beta^4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2} \approx \left( 1 + \frac{1}{4\beta^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{32\beta^4} [(F_{\mu\nu} F^{\mu\nu})^2 + (F_{\mu\nu} \tilde{F}^{\mu\nu})^2] \right) \pm O(\beta^{-6}) \]  

(4)

For \( \beta^2 \to \infty \), the standard expression of the U(1) theory on the lhs of (3) is recovered. In the limit of static and purely magnetic solutions, the part containing the dual field strength tensor \( \tilde{F}_{\mu\nu} \) vanishes.

In this paper, we are interested in static, finite energy solutions of the full BI equations with classical mass denoted by \( M_{bi}(\beta^2) \). Using (4), we obtain the following approximation for this mass:

\[ M_{bi}(\beta^2) \approx M_0 - \frac{1}{32\beta^2} \int d^3 x (F_{(0)jk} F^{(0)jk})^2 \equiv M_0 - \frac{1}{\beta^2} \Delta_{bi} \]  

(5)

Both, the mass of the standard soliton \( M_0 \) and the field strength tensors \( F_{(0)jk} \) are computed from the standard equations. In other words, we approximate the mass of the BI-soliton by the \( \beta^{-2} \)- correction of the BI Lagrangian density using only the solution data of the standard U(1) theory.

For the self-dual case, the first order equations read:

\[ f' = \frac{Pf}{x}, \quad P' = x^2(f^2 - 1) \]  

(6)

In this limit, the mass \( M_0 \) of the Nielsen-Olesen vortex and the correction \( \Delta_{bi} \) are given as follows:
\[ M_0 = \int dx \{ \frac{P'}{x^2} + (f')^2 + \frac{P^2 f^2}{x^2} + \frac{1}{4} (1 - f^2)^2 \} \] (7)

and

\[ \Delta_{b_i} = \frac{1}{2} \int dx \left[ \frac{P'}{x} \right]^4 \] (8)

where \( x \) is a dimensionless coordinate \( x = ev \rho \) and the prime denotes the derivative with respect to \( x \). With the choice of coordinates, the mass per winding number of the \( n \)-soliton solution in the self-dual case is given by \( \frac{M_0}{n} = 1 \) (in units of \( \frac{2 \pi a^2}{\alpha} \)).

Evaluating the correction integral numerically, we obtain \( \Delta_{b_i}(n = 1) \approx 0.0118 \) and \( \Delta_{b_i}(n = 2)/n \approx 0.02037 \). In the table below, we compare the masses per winding number obtained with this correction and the masses per winding number \( M_{b_i}(\beta^2)(n)/n \) computed in [7] with the full equations.

| \( \beta^2 \) | \( M_{b_i}(\beta^2)(n = 1) \) | \( M_0 - \frac{\Delta_{b_i}}{\beta^2}(n = 1) \) | \( M_{b_i}(\beta^2)(n = 2)/2 \) | \( (M_0 - \frac{\Delta_{b_i}}{\beta^2}(n = 2))/2 \) |
|--------------|------------------|------------------|------------------|------------------|
| \( \infty \) | 1.0              | 1.0              | 1.0              | 1.0              |
| 100          | 0.99988          | 0.99988          | 0.99797          | 0.99797          |
| 10           | 0.99880          | 0.99882          | 0.99790          | 0.99796          |
| 5.           | 0.99760          | 0.99764          | 0.99585          | 0.99592          |
| 1.           | 0.9870           | 0.9882           | 0.97780          | 0.97963          |
| 0.5          | 0.97038          | 0.9764           | 0.94980          | 0.95926          |

Clearly, the approximation is very good for large values of \( \beta^2 \) and as expected gets worse for decreasing \( \beta^2 \), since the approximation used in (4) is not valid anylonger and higher order terms have to be taken into account.

The quality of this approximation for large \( \beta^2 \) also provides a good estimation for the difference between the energy per winding number of the \( n = 1 \) and the \( n = 2 \) vortex:

\[ \delta M = (M_0 - \frac{\Delta_{b_i}}{\beta^2}(n = 1)) - (M_0 - \frac{\Delta_{b_i}}{\beta^2}(n = 2))/2 \approx 0.00875 \frac{1}{\beta^2} \] (9)

This clearly confirms the results in [7], especially that the Born-Infeld interaction stabilizes the axially symmetric 2-vortex solution for \( \alpha = 2 \).

III. THE GEORGI-GLASHOW MODEL

The Lagrangian of the SU(2) Yang-Mills-Higgs theory with the Higgs field in the adjoint representation (the so-called Georgi-Glashow model) reads:

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a} - \frac{1}{2} D_\mu \Phi^a D^\mu \Phi^a - \frac{\lambda}{4} (\Phi^a \Phi^a - v^2)^2, \quad a = 1, 2, 3 \] (10)

where the field strength tensor \( F_{\mu\nu}^a \) and the covariant derivative of the Higgs field \( D_\mu \Phi^a \) are given as follows:

\[ F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e \varepsilon_{abc} A_\mu^b A_\nu^c, \quad D_\mu \Phi^a = \partial_\mu \Phi^a + e \varepsilon_{abc} A_\mu^b \Phi^c \] (11)
The soliton solutions of this model are magnetic monopoles [10].

For a similar approximation as in (3), we expand the ordinary trace version of the non-abelian Born-Infeld (NBI) model. This is a straightforward generalisation of (4) replacing \( F_{\mu\nu}F^{\mu\nu} \) by \( F_{a\mu\nu}F^{a\mu\nu} \). We also give the first term in the approximation of the symmetrized trace version extending the results of [4].

A. Spherically symmetric monopoles

For the gauge and Higgs fields, we use the purely magnetic hedgehog ansatz (a = 1, 2, 3) [10]:

\[
A_r^a = A_t^a = 0, \quad A_\theta^a = \frac{1 - K(r)}{e} e_e^a, \quad A_\phi^a = -\frac{1 - K(r)}{e} \sin \theta e_\theta^a,
\]

\[
\Phi^a = v H(r) e^a.
\]

The spherically symmetric \( n = 1 \) Born-Infeld monopoles were constructed in [8] both by using the ordinary and the symmetrized trace. It was found that for large values of the Born-Infeld coupling \( \beta^2 \), the profiles of the functions don’t differ significantly from those of the ’t Hooft-Polyakov monopole. Equally, the mass only depends slightly on \( \beta^2 \). It was found, however, that the falloff of the Higgs field in the limit of vanishing Higgs coupling now is given by \( c(\beta^2)/r \), where \( c \) is a constant depending on \( \beta^2 \) with \( c = 1 \) in the limit \( \beta^2 \to \infty \). For the ordinary trace model, a critical value of \( \beta^2 \) was found, \( \beta^2_{cr} \), such that for \( \beta^2 < \beta^2_{cr} \) no solutions exist. For vanishing Higgs self-coupling this was determined to be \( \beta^2_{cr} = 0.168 \).

In the case of spherically symmetric solutions in the BPS limit, the first order (self-dual) differential equations read:

\[
K' = HK, \quad x^2 H' = (K^2 - 1)
\]

In the BPS limit, the non-abelian counterparts to the integrals (7) and (8) read:

\[
M_0(n = 1) = \int \! dx x^2 \frac{(K')^2}{x^2} + \frac{(K^2 - 1)^2}{2x^4} + \frac{1}{2}(H')^2 + \frac{K^2 H^2}{x^2}
\]

\[
\Delta_{bi}^{tr} = \frac{1}{2} \int \! dx x^2 \left[ \frac{(K')^2}{x^2} + \frac{(K^2 - 1)^2}{2x^4} \right]^{1/2}
\]

for the ordinary trace denoted by \( tr \), and

\[
\Delta_{bi}^{Str} = \frac{1}{2} \int \! dx x^2 \left[ \frac{2}{3} \left( \frac{(K')^2}{x^2} + \frac{(K^2 - 1)^2}{2x^4} \right)^2 + \frac{2}{3} \frac{(K^2 - 1)^2 (K')^2}{x^2} \right]
\]

for the symmetrized trace denoted by \( Str \). The mass \( M_0 \) is given in unit of \( 4\pi g^2 e \), \( x = e vr \) and the prime denotes the derivative with respect to \( x \). The numerical evaluation of the integrals ([10] and [17]) for the \( n = 1 \)-monopole gives \( \Delta_{bi}^{tr} \approx 0.00919 \) and \( \Delta_{bi}^{Str} \approx 0.00515 \). Similar as in the case of vortices, this provides a very good approximation of the BI-monopole mass for large \( \beta^2 \) as the table below demonstrates:
$M_{bi}^n(\beta^2)$ was computed using the full Born-Infeld equations. The integration of the full equations in the symmetrized trace version becomes very involved and thus the mass $M_{Str}^n(\beta^2)$ denotes the mass of the BI monopole computed from the Lagrangian involving terms up to order $\beta^{-2}$. The reason why we don’t give the numbers for $\beta^2 = 0.5$ in the symmetrized trace version is that we couldn’t integrate the equations for $\beta^2 < 0.6$. This leaves us with the assumption that for the equations derived from the symmetrized trace Lagrangian involving terms up to order $\beta^{-2}$ a $\beta^2_{cr}$ also exists and that in the case studied here $\beta^2_{cr} \approx 0.6$. However, further investigation of this is out of the aim of this paper.

### B. Axially symmetric monopoles

The axially symmetric Ansatz for the gauge fields is given by [13]:

$$A_\mu dx^\mu = \frac{1}{2} A_\mu^a r^a dx^\mu = \frac{1}{2\epsilon_{cr}} [\tau_n^a (H_1 dr + (1 - H_2) r d\theta)$$

$$- n(\tau_n^a H_3 + \tau_n^\theta(1 - H_4)) r \sin \theta d\varphi],$$

while for the Higgs field it reads

$$\Phi = \Phi^a r^a = (\Phi_1 \tau_n^a + \Phi_2 \tau_n^\theta)$$

(19)

where the matter field functions $H_1, H_2, H_3, H_4, \Phi_1$ and $\Phi_2$ depend only on $r$ and $\theta$. The symbols $\tau_n^r, \tau_n^\theta$ and $\tau_n^\varphi$ denote the dot products of the cartesian vector of Pauli matrices, $\vec{\tau} = (\tau^1, \tau^2, \tau^3)$, with the spatial unit vectors

$$\vec{e}_r^n = (\sin \theta \cos n\varphi, \sin \theta \sin n\varphi, \cos \theta),$$

$$\vec{e}_\theta^n = (\cos \theta \cos n\varphi, \cos \theta \sin n\varphi, - \sin \theta),$$

$$\vec{e}_\varphi^n = (- \sin n\varphi, \cos n\varphi, 0),$$

(20)

For $H_1 = H_3 = \Phi_2 = 0, H_2 = H_4 = K(r), \Phi_1 = H(r)$ and $n = 1$, the Ansatz (12), (13) for the ‘t Hooft-Polyakov monopole [10] is recovered. The self-dual equations read:

$$F_{ij}^a = \pm \varepsilon_{ijk} \sqrt{-g} D_k^{a\Phi} \Phi = \pm \varepsilon_{ijk} r^2 \sin \theta D_k^a \Phi g^{kk}$$

(21)

It would be interesting to construct the Born-Infeld $n = 2$-monopole and compare its mass with the one of the $n = 1$ BI-monopole [8] as well as with the standard $n = 2$-monopole [13,14]. This can be done only by using the full axially symmetric ansatz [13], [19]. The occurrence of non-polynomial terms in the Born-Infeld action leads to the fact that the corresponding partial differential equations are not manifestly elliptic (-mixed derivatives
of the fields are unavoidable). The numerical integration seems to be very involved and is left for future work. However, encouraged by the results described above, we propose an estimation for the energy of the $n = 2$-axially symmetric Born-Infeld monopole. For the $n = 2$ monopole, the relevant integrals in the BPS limit read:

$$M_0(n) = \frac{1}{2} \int \int \sin \theta d\theta dx \left[ \frac{1}{x^2} \left( (x \partial_x \Phi_1 + H_1 \Phi_1)^2 + (x \partial_x \Phi_2 - H_1 \Phi_2)^2 + (\partial_\Phi \Phi_1 - H_2 \Phi_2)^2 + (\partial_\Phi \Phi_2 - H_2 \Phi_1)^2 \right) + \frac{1}{x^4} \left\{ (x \partial_x H_2 + \partial_\theta H_1)^2 + n^2 (H_4 \Phi_1 + (H_3 + \cot \theta) \Phi_2)^2 \right\} \right]$$

and

$$\Delta_{bi}^{tr} = \frac{1}{8} \int \int \sin \theta d\theta dx \frac{1}{x^6} \left[ (x \partial_x H_2 + \partial_\theta H_1)^2 + n^2 (x \partial_x H_3 - H_1 H_4)^2 + (x \partial_x H_4 + H_1 (H_3 + \cot \theta))^2 + (\partial_\theta H_3 - 1 + \cot \theta H_3 + H_2 H_4)^2 + (\partial_\theta H_4 + \cot \theta (H_4 - H_2) - H_2 H_3)^2 \right]{^2}$$

We derive the expression for $\Delta_{bi}^{Str}$ in the case of axially symmetric monopoles in the Appendix. We obtain:

$$\Delta_{bi}^{Str} = \frac{1}{16} \int \int \sin \theta d\theta dx \frac{1}{x^6} \left[ (x \partial_x H_2 + \partial_\theta H_1)^4 + n^4 [(x \partial_x H_3 - H_1 H_4)^2 + (\partial_\theta H_3 - 1 + \cot \theta H_3 + H_2 H_4)^2] + 4n^4 \left\{ (x \partial_x H_4 - H_1 H_3 + \cot \theta H_1)^2 + (\partial_\theta H_4 - H_2 H_3 - \cot \theta (H_2 - H_4))^2 \right\} + 4n^4 \left\{ (x \partial_x H_4 + H_1 H_3 + \cot \theta H_1)^2 + (\partial_\theta H_4 - H_2 H_3 - \cot \theta (H_2 - H_4))^2 \right\} + 4n^4 \left\{ (x \partial_x H_4 - H_1 H_3 + \cot \theta H_1)^2 + (\partial_\theta H_4 - H_2 H_3 - \cot \theta (H_2 - H_4))^2 \right\} \right]$$

We find $(1/n)\Delta_{bi}^{tr} = (1/2)\Delta_{bi}^{tr} \approx 0.00303$ and $(1/n)\Delta_{bi}^{Str} = (1/2)\Delta_{bi}^{Str} \approx 0.00187$. With this approximation, the difference between the mass per winding number of the $n = 1$ and the $n = 2$ monopole in the BPS limit is given by:

$$\delta M^{tr} = (M_0 - \frac{\Delta_{bi}^{tr}}{\beta^2}(n = 1)) - (M_0 - \frac{\Delta_{bi}^{tr}}{\beta^2}(n = 2))/2 \approx -0.006 \frac{1}{\beta^2}$$

and

$$\delta M^{Str} = (M_0 - \frac{\Delta_{bi}^{Str}}{\beta^2}(n = 1)) - (M_0 - \frac{\Delta_{bi}^{Str}}{\beta^2}(n = 2))/2 \approx -0.003 \frac{1}{\beta^2}$$
Note that with our choice of coordinates, the mass per winding number of the BPS monopoles is given by $M = 1$.

If we rely on the approximation (3) and extend it to the case of multimonomopoles, the above result indicates that the Born-Infeld interaction leads to repulsion of like-charged monopoles. To have further indication, we computed the integrals (23) and (24) with the axially symmetric Ansatz [14] for winding number up to $n = 7$. The values are given below:

| $n$  | 1    | 2    | 3    | 4    | 5    | 6    | 7    |
|------|------|------|------|------|------|------|------|
| $10^4 \cdot \Delta tr / n$ | 9.198 | 3.035 | 1.600 | 1.008 | 0.699 | 0.517 | 0.397 |
| $10^3 \cdot \Delta sym / n$ | 5.156 | 1.869 | 1.064 | 0.705 | 0.506 | 0.385 | 0.303 |

Clearly, with increasing winding number $n$ the strength of repulsion increases. Plotting the above numbers over $n$ gives smooth curves. This makes us confident that the results obtained are not numerical errors. Of course, the hypothesis of repelling BI monopoles strongly relies on the agreement found in the Abelian Higgs model and in the Georgi-Glashow model for the $n = 1$ monopoles. A direct numerical analysis of the equations is definitely called for.

**IV. CONCLUSIONS**

In this paper, we have studied an approximation for the Born-Infeld term both in the abelian as well as in the non-abelian Higgs model. In the abelian Higgs model, we find that inserting the data of the standard (i.e. undeformed) Nielsen-Olesen vortex in the $\beta^{-2}$-correction we obtain a very good approximation for the mass of BI vortices. For the $n = 1$ monopoles arising in the Georgi-Glashow model this is also true for both the ordinary trace as well as the symmetrized trace version. Using these results, we compute the difference between the mass per winding number of the $n = 1$ and the $n = 2$ monopole, respectively. Our results suggest that the Born-Infeld interaction leads to repulsion between like charged monopoles.
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V. APPENDIX

\[ \Delta_{bi}^{\text{Str}} \] for axially symmetric monopoles

We expand the field strength tensor with respect to the Pauli matrices \( \tau_\lambda^n \), \( \lambda = x, \theta, \varphi \) (see (13)):

\[ F_{\mu\nu} = F_{\mu\nu}^{(\lambda)} \frac{\tau_\lambda^n}{2} \]

The non-vanishing coefficients \( F_{\mu\nu}^{(\lambda)} \) read [15] :

\[
F_{x\theta}^{(\varphi)} = -\frac{1}{x} (\partial_\theta H_1 + x \partial_x H_2) ,
\]

\[
F_{x\varphi}^{(x)} = -n \frac{\sin \theta}{x} (x \partial_x H_3 - H_1 H_4) ,
\]

\[
F_{x\varphi}^{(\theta)} = n \frac{\sin \theta}{x} (x \partial_x H_4 + H_1 H_3 + \cot \theta H_1) ,
\]

\[
F_{\theta\varphi}^{(x)} = -n \sin \theta (\partial_\theta H_3 - 1 + H_2 H_4 + \cot \theta H_3) ,
\]

\[
F_{\theta\varphi}^{(\theta)} = n \sin \theta (\partial_\theta H_4 - H_2 H_3 - \cot \theta (H_2 - H_4)) ,
\]

and \( F_{\mu\nu}^{(\lambda)} = -F_{\nu\mu}^{(\lambda)} \). Thus the first correction of the symmetrized trace version of the theory is given by :

\[ \Delta_{bi}^{\text{Str}} = \int \int \sin \theta d\theta dx^2 \text{Int}^{\text{Str}} \] (28)

with the integrand \( \text{Int}^{\text{Str}} \) :

\[ \text{Int}^{\text{Str}} = \frac{1}{8} \text{Str}(F_{\mu\nu} F_{\mu\nu})^2 = \frac{1}{2} \text{Str}(F_{x\theta}^2 g^{xx} g^{\theta\theta} + F_{x\varphi}^2 g^{xx} g^{\varphi\varphi} + F_{\theta\varphi}^2 g^{\theta\theta} g^{\varphi\varphi})^2 \] (29)

Inserting the expressions for the field strength tensor gives:

\[
\text{Int}^{\text{Str}} = \frac{1}{2} \text{Str} \left[ \frac{1}{4x^2} (F_{x\theta}^{(\varphi)})^2 (\tau_\varphi^n)^2 + \frac{1}{4 \sin^2 \theta x^2} \left\{ (F_{x\varphi}^{(x)})^2 (\tau_x^n)^2 + (F_{x\varphi}^{(\theta)})^2 (\tau_\theta^n)^2 \\ + F_{x\varphi}^{(x)} F_{x\varphi}^{(\theta)} (\tau_x^n \tau_x^n + \tau_x^n \tau_x^n) \right\} \right] \]

\[
+ \frac{1}{2} \text{Str} \left[ \frac{1}{4 \sin^2 \theta x^4} \left\{ (F_{\theta\varphi}^{(x)})^2 (\tau_x^n)^2 + (F_{\theta\varphi}^{(\theta)})^2 (\tau_\theta^n)^2 \right\} \right] \]

(30)

Now, we use

\[
\text{Str}(\tau_\lambda^n ... \tau_\lambda^n) = \frac{1}{p!} \text{tr}(\tau_\lambda^n ... \tau_\lambda^n + \text{all permutations}) \] (31)

With this, (30) can be evaluated in a straightforward way by noting that

\[
\text{Str} \left( (\tau_\lambda^n)^4 \right) = 2 , \text{Str} \left( (\tau_\lambda^n)^2 (\tau_\lambda^n)^2 \right) = \frac{2}{3} , \text{Str} \left( (\tau_\lambda^n)^3 (\tau_\lambda^n) \right) = 0 , \text{Str} \left( (\tau_\lambda^n)^2 (\tau_\lambda^n)^2 (\tau_\lambda^n) \right) = 0
\]

(32)
where it is understood that \( \lambda_1 \neq \lambda_2 \neq \lambda_3 \). We obtain:

\[
\text{Int}^{Str} = \frac{1}{16x^4} [(F^{(x)}_{x\theta})^4 + \frac{1}{\sin^4 \theta}((F^{(x)}_{x})^2 + \frac{(F^{(x)}_{\theta\varphi})^2}{x^2})^2 + \frac{1}{\sin^4 \theta}((F^{(\theta)}_{x\varphi})^2 + \frac{(F^{(\theta)}_{\theta\varphi})^2}{x^2})^2 \\
+ \frac{4}{3\sin^4 \theta}(F^{(x)}_{x\varphi} F^{(\theta)}_{x\varphi} + \frac{F^{(x)}_{\theta\varphi} F^{(\theta)}_{\theta\varphi}}{x^2})^2 + \frac{2}{3\sin^2 \theta}(F^{(\varphi)}_{x\theta})^2 \{((F^{(x)}_{x\varphi})^2 + \frac{(F^{(x)}_{\theta\varphi})^2}{x^2}) + (F^{(\theta)}_{x\varphi})^2 + \frac{(F^{(\theta)}_{\theta\varphi})^2}{x^2}\} \\
+ \frac{2}{3\sin^4 \theta}\{(F^{(x)}_{x\varphi})^2 + \frac{(F^{(x)}_{\theta\varphi})^2}{x^2}\}\{(F^{(\theta)}_{x\varphi})^2 + \frac{(F^{(\theta)}_{\theta\varphi})^2}{x^2}\} ]
\] (33)