On the real projection of the zeros of \(1 + 2^s + \ldots + n^s\)

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ABSTRACT

This paper proves that the real projections of the simple zeros of each partial sum \(G_n(s) \equiv 1 + 2s + \ldots + ns, \ n \geq 2\), of the Riemann zeta function on the half-plane \(\text{Re} s < -1\), are not isolated points of that set.

AMS Subject Classification: 30Axx, 30D05.

Key Words: Zeros of Entire Functions, Exponential Polynomials, Almost-Periodic Functions, Partial Sums of the Riemann Zeta Function.

1 Introduction

On zeros of partial sums of the Riemann zeta function we find several works. Among them might be quoted the classical papers of Turan [16] and Montgomery [10], and more recently Borwein et al. [4], and Gonek and Ledoan [6].

In this paper we focus on the existence of accumulation points of the subset defined by the real projection of the zeros of the functions

\[ U_n(s) := \sum_{k=1}^{n} \frac{1}{k^s}, \quad s = \sigma + it. \]

That would imply the existence of an infinite amount of zeros of \(U_n(s)\) arbitrarily close to a line parallel to the imaginary axis passing through every accumulation point. In order to study this problem, we begin by considering the set defined by the real projection of the zeros of the functions

\[ G_n(s) := 1 + 2^s + \ldots + n^s \]

and then, by virtue of the equality

\[ U_n(s) = G_n(-s) \quad \text{for all } s \in \mathbb{C}, \]

we have that the set of the zeros of \(U_n(s)\), called \(Z_{U_n(s)}\), is the same as \(-Z_{G_n(s)}\) where \(Z_{G_n(s)}\) denotes the set of the zeros of \(G_n(s)\).

Each \(G_n(s), \ n \geq 2\), is an entire function of order 1, of exponential type \(\sigma = \ln n\), and it has an infinite amount of zeros, not all of them located on the imaginary axis, except for \(G_2(s)\) [11]. Furthermore, since for any \(t\)

\[ \lim_{\sigma \to -\infty} G_n(\sigma + it) = 1 \]

and

\[ \lim_{\sigma \to +\infty} \frac{G_n(\sigma + it)}{n^{\sigma + it}} = 1, \]
there exist two values of $\sigma$, $\sigma_{n,1} < 0 < \sigma_{n,2}$, such that
\[ |G_n(s) - 1| < 1 \text{ for all } s \text{ with } \text{Re } s \leq \sigma_{n,1} \]
and
\[ \left| \frac{G_n(\sigma + it)}{n^{\sigma + it}} - 1 \right| < 1 \text{ for all } s \text{ with } \text{Re } s \geq \sigma_{n,2}. \]

Thus, every function $G_n(s)$ has its zeros in a critical strip $S_n$ defined by
\[ S_n := \{ s = \sigma + it : a_n \leq \Re s \leq b_n \}, \]
where the bounds
\[ a_n := \inf \{ \text{Re } s : G_n(s) = 0 \} \]
and
\[ b_n := \sup \{ \text{Re } s : G_n(s) = 0 \} \]
are given by
\[ a_n = -1 - \left( \frac{4}{\pi} - 1 + o(1) \right) \frac{\log \log n}{\log n} \]
and
\[ b_n = n \log 2 + o(1) \]
as it was proved by Montgomery [10] and Balazard and Velásquez-Castañón [3], respectively.

Since all zeros of the function $G_2(s) = 1 + 2^s$ are on the imaginary axis, the set defined by their real projections is reduced to $\{0\}$, so $a_2 = b_2 = 0$ and therefore we consider $n = 2$ as the trivial case.

In the case of $n > 2$, we can use the bounds $a_n$, $b_n$ to define the critical interval
\[ I_n := [a_n, b_n] \]
associated with each function $G_n(s)$ and that contains the set of the real projections of the zeros of $G_n(s)$. The study on the existence of accumulation points within that set is the main goal of the present paper. In that sense, we will answer as well to several questions about this subject that were pointed out in other publications like [8], [9] and [14].

Finally, we would like to note that, according to the author in [13], the real projections of the zeros of exponential polynomials with the form
\[ \varphi(s) = \sum_{k=1}^{m} A_k e^{\alpha_k s} \]
where $\alpha_k$ are real exponents called frequencies, and under rather restrictive conditions (refer to [13]) satisfies the following so-called Main Theorem that we would like to quote here:
MAIN THEOREM (Quoted from [13]). Assume that $1, \alpha_1, \ldots, \alpha_m$ are real numbers linearly independent over the rationals. Consider the exponential polynomial
\[
\varphi(s) = m \sum_{k=1}^{m} A_k e^{\alpha_k s}, \quad s = \sigma + it,
\]
where the $A_k$ are complex numbers. Then a necessary and sufficient condition for $\varphi(s)$ to have zeros arbitrarily close to any line parallel to the imaginary axis inside the strip
\[
I = \{ \sigma + it : \sigma_0 < \sigma < \sigma_1, \quad -\infty < t < \infty \}
\]
is that
\[
|A_j e^{\sigma \alpha_j}| \leq \sum_{k=1, k \neq j}^{m} |A_k e^{\sigma \alpha_k}|, \quad (j = 1, 2, \ldots, m) \quad (1.1)
\]
for any $\sigma$ with $\sigma + it \in I$.

From the above result it follows that the real projections of the zeros of an exponential polynomial of the form $\varphi(s)$, satisfying the hypothesis of this theorem, define a dense subset of the real interval $(\sigma_0, \sigma_1)$ if and only if the geometric principle (1.1) holds. In the case of the $G_n(s)$ functions, since only $G_2(s)$ and $G_3(s)$ are of the type $\varphi(s)$, in the sense that both satisfy the condition of the linear independence over the rationals of their frequencies, the Main Theorem can be exclusively applied for $n = 2$ and $n = 3$. Nevertheless, it does not show that the projection on the real axis of the zeros of $G_n(s)$, for $n > 3$, might also have some density properties in their critical intervals $[a_n, b_n]$, as it happens, for instance, for the cases $n = 4, 5$ and other $n$’s, as we will see in this paper.

2 A characterization of the density of the real projections of the zeros of $G_n(s)$

As we said in the foregoing section it is crucial to study the existence of accumulation points of the set of the real projection of the zeros of $G_n(s)$ within its critical interval $I_n$. Therefore, firstly we are going to formulate a general theorem of characterization of the set
\[
R_n := \{ \text{Res} : G_n(s) = 0 \}
\]
which can be considered as an ad hoc version of [2, Theorem 3.1] and that can be directly applied to our functions $G_n(s)$.

Theorem 1 For each integer $n > 2$, let $\{p_1, p_2, \ldots, p_{k_n}\}$ be the set of all prime numbers less than or equal to $n$. Let us define $p = (\log p_1, \log p_2, \ldots, \log p_{k_n})$ and let $c_m$ be the unique vector of $\mathbb{R}^{k_n}$ with non-negative integer components
such that \( \log m = \langle c_m, p \rangle \), \( 1 \leq m \leq n \), where \( \langle \cdot, \cdot \rangle \) is the standard inner product in \( \mathbb{R}^{k_n} \). Let us define the function \( F_n : \mathbb{R} \times \mathbb{R}^{k_n} \rightarrow \mathbb{C} \) as

\[
F_n(\sigma, x) := \sum_{m=1}^{n} m^\sigma e^{\langle c_m, x \rangle i}
\]  

(2.1)

for \( \sigma \) real and \( x = (x_1, x_2, ..., x_{k_n}) \) a vector of \( \mathbb{R}^{k_n} \). Then, \( \sigma \in R_n := \{ \text{Res : } G_n(s) = 0 \} \) if and only if there exists some vector \( x \in \mathbb{R}^{k_n} \) such that \( F_n(\sigma, x) = 0 \).

**Proof.** Firstly observe that if \( s = \sigma + it \) is an arbitrary zero of \( G_n(s) \), since \( \log m = \langle c_m, p \rangle \), we can write

\[
0 = G_n(s) = \sum_{m=1}^{n} e^{\langle c_m, p \rangle s} = \sum_{m=1}^{n} e^{\langle c_m, p \rangle \sigma} e^{\langle c_m, p \rangle ti} = \sum_{m=1}^{n} m^\sigma e^{\langle c_m, t p \rangle i} = F_n(\sigma, t p).
\]  

(2.2)

Now assume \( \sigma \in R_n \). Then there exists a sequence \( (s_j = \sigma_j + it_j)_{j=1,2,...} \) of zeros of \( G_n(s) \) such that \( \sigma = \lim_{j \to \infty} \sigma_j \) and, because of (2.2), we have

\[
F_n(\sigma_j, t_j p) = 0, \text{ for all } j = 1, 2, ...
\]

Hence, from (2.1), we obtain

\[
0 = \sum_{m=1}^{n} m^\sigma_j e^{\langle c_m, t_j p \rangle i}, \text{ for all } j = 1, 2, ...
\]  

(3.3)

Now consider the sequence \( (e^{\langle c_2, t_j p \rangle i})_{j=1,2,...} \) of points of the unit circle, thus bounded, and let

\[
(e^{\langle c_2, t_{h,2} p \rangle i})_{h=1,2,...}
\]

be a convergent subsequence to \( e^{\theta_2 i} \) for some \( \theta_2 \in [0, 2\pi) \). Since the sequence \( (e^{\langle c_3, t_{h,2} p \rangle i})_{h=1,2,...} \) is also bounded, there exists a convergent subsequence

\[
(e^{\langle c_3, t_{h,3} p \rangle i})_{h=1,2,...}
\]

to \( e^{\theta_3 i} \) for some \( \theta_3 \in [0, 2\pi) \) and so on. Then, by means of this process, for every prime number \( m \) of the set \( \{1, 2, 3, ..., n\} \), we determine a subsequence

\[
(e^{\langle c_m, t_{h,p_k n} p \rangle i})_{h=1,2,...}
\]
which converges to $e^{\theta_m}$ for some $\theta_m \in [0, 2\pi)$. For the other numbers of the set \{1, 2, 3, ..., n\} one has

$$c_1 = (0, ..., 0), \quad c_4 = 2c_2, \quad c_6 = c_2 + c_3, ...$$

so, by considering (2.3) for the $j$’s of the subsequence $j_{h,p_{kn}}$, $h = 1, 2, ...$, and taking the limit as $h \to \infty$ we get

$$0 = \lim_{h \to \infty} \sum_{m=1}^{n} M^{\sigma_j,r_{kn}} e^{(\mathbf{c}_m \cdot \mathbf{t}_{j_{h,p_{kn}}})i} =$$

$$= 1 + 2^\sigma e^{\theta_2i} + 3^\sigma e^{\theta_3i} + 4^\sigma e^{2\theta_2i} + 5^\sigma e^{3\theta_3i} + 6^\sigma e^{\theta_2+\theta_3i} + ... =$$

$$= 1 + 2^\sigma e^{(x_2,\theta_2)i} + 3^\sigma e^{(x_3,\theta_3)i} + 4^\sigma e^{(x_4,\theta_4)i} + ... + n^\sigma e^{(x_n,\theta_n)i} = F_n(\sigma, \theta),$$

where $\theta = (\theta_2, \theta_3, \theta_4, ..., \theta_{p_{kn}})$.

Conversely, suppose

$$F_n(\sigma, \mathbf{x}) = 0$$

for some real number $\sigma$ and a vector $\mathbf{x} = (x_1, x_2, ..., x_{k_n}) \in \mathbb{R}^{k_n}$. As the components of $\frac{1}{2\pi} \mathbf{p} = \left(\frac{1}{2\pi} \log p_1, ..., \frac{1}{2\pi} \log p_{k_n}\right)$ are linearly independent over the rationals, given the numbers $\frac{1}{2\pi} \langle \mathbf{c}_{p_1}, \mathbf{x} \rangle$, ..., $\frac{1}{2\pi} \langle \mathbf{c}_{p_{k_n}}, \mathbf{x} \rangle$, $T = 1$ and $\frac{e^\sigma}{2\pi}$, for each $j = 1, 2, ...$, by applying the Kronecker theorem [7, p. 382] there exist a sequence $(T_j)_{j=1,2,...}$, $T_j > 1$, and integers $(N_j,l)_{l=1,2,...,k_n}$ such that for each $j = 1, 2, ...$ it holds that

$$|T_j \frac{1}{2\pi} \log p_l - \frac{1}{2\pi} \langle \mathbf{c}_{p_l}, \mathbf{x} \rangle - N_j,l| < \frac{e}{2\pi}$$

for all $l = 1, 2, ..., k_n$. (2.4)

Multiplying by $2\pi$ and substituting $\log p_l$ by $\langle \mathbf{c}_{p_l}, \mathbf{p} \rangle$, the inequality (2.4) becomes

$$|\langle \mathbf{c}_{p_l}, T_j \mathbf{p} - \mathbf{x} \rangle - 2\pi N_j,l| < \frac{e}{2\pi}$$

for all $l = 1, 2, ..., k_n$, which means that

$$\lim_{j \to \infty} e^{\langle \mathbf{c}_{p_l}, T_j \mathbf{p} - \mathbf{x} \rangle i} = 1$$

and then

$$\lim_{j \to \infty} e^{\langle \mathbf{c}_{p_l}, \mathbf{x} - T_j \mathbf{p} \rangle i} = 1.$$ (2.5)

Now, for each $m \in \{1, 2, 3, ..., n\}$ and noticing that the vector $\mathbf{c}_m$ is a linear combination with non-negative integer coefficients of the vectors $\mathbf{c}_{p_l}$, $l = 1, 2, ..., k_n$, it can be deduced from (2.5) that

$$\lim_{j \to \infty} e^{\langle \mathbf{c}_m, \mathbf{x} - T_j \mathbf{p} \rangle i} = 1.$$ (2.6)

Thus, by using (2.6), we have

$$\lim_{j \to \infty} G_n(\sigma + iT_j) = \lim_{j \to \infty} \sum_{m=1}^{n} e^{\langle \mathbf{c}_m, \mathbf{p} \rangle \sigma + \langle \mathbf{c}_m, \mathbf{p} \rangle iT_j} e^{\langle \mathbf{c}_m, \mathbf{x} - T_j \mathbf{p} \rangle i} =$$
which means that \( \sigma \in R_n \) and then the result follows. \( \blacksquare \)

By applying the preceding theorem to our function \( G_4(s) \) we are going to deduce the existence of a subinterval of its critical interval contained in \( R_4 \). This means that there is an infinite amount of zeros of \( G_4(s) \) arbitrarily close to any line parallel to the imaginary axis passing through any point of this subinterval.

**Corollary 2** The interval \([-0.55, 1]\) is contained in \( R_4 := \{ \text{Res} : G_4(s) = 0 \} \), where \( G_4(s) = 1 + 2^s + 3^s + 4^s \).

**Proof.** Because of (2.1), the function associated with \( G_4(s) \) is given by

\[
F_4(\sigma, x_1, x_2) = 1 + 2^\sigma e^{x_1i} + 3^\sigma e^{x_2i} + 4^\sigma e^{2x_1i}.
\]

By defining the function

\[
f_4(\sigma, x_1) := 1 + 2^\sigma e^{x_1i} + 4^\sigma e^{2x_1i},
\]

we claim that for every \( \sigma \in [-0.55, 1] \) there exists a value of \( x_1 \) depending on \( \sigma \), say \( x_{1,\sigma} \), such that

\[
|f_4(\sigma, x_{1,\sigma})| = 3^\sigma.
\]

Indeed, the function \( f_4(\sigma, x_1) \) can be rewritten as

\[
f_4(\sigma, x_1) = (2^\sigma e^{x_1i} + e^{\pi i}) (2^\sigma e^{x_1i} + e^{-\pi i}),
\]

so its modulus is the product of the distances from the point \( s = 2^\sigma e^{x_1i} \) to the fixed points \( s_0 = -e^{\pi i} \) and \( s_1 = -e^{-\pi i} \) of the unit circle. According to \( \frac{1}{2} < 2^\sigma \) for all \( \sigma \in [-0.55, 1] \), the vertical line of equation \( x = -\frac{1}{2} \) intersects the circles of equation \( |s| = 2^\sigma \) at points \( P_1, P_2 \) and \( P_3 \) of the upper half-plane for each possible value of \( \sigma \) in the interval \([-0.55, 1]\), namely, \( \sigma < 0 \), \( \sigma = 0 \) and \( \sigma > 0 \), respectively. Let us denote \( Q_1, Q_2, Q_3 \) as the points where the positive real half-axis intersects the circles \( |s| = 2^\sigma \) for \( \sigma < 0 \), \( \sigma = 0 \) and \( \sigma > 0 \) respectively, then we may have the following situations:

1) Case \(-0.55 \leq \sigma < 0\). The modulus \( |f_4(P_1)| = 1 - 4^\sigma \) satisfies the inequality \( 1 - 4^\sigma \leq 3^\sigma \). On the other hand, \( |f_4(Q_1)| = 1 + 2^\sigma + 4^\sigma \geq 3^\sigma \) for any \(-0.55 \leq \sigma < 0\).

2) Case \( \sigma = 0 \). We have \( |f_4(P_2)| = 0 \) and \( |f_4(Q_2)| = 3 \).

3) Case \( 0 < \sigma \leq 1 \). In this case \( |f_4(P_3)| = 4^\sigma - 1 \leq 3^\sigma \) for any \( 0 < \sigma \leq 1 \). On the other hand, \( |f_4(Q_3)| = 1 + 2^\sigma + 4^\sigma \geq 3^\sigma \) for any \( 0 < \sigma \leq 1 \).

Then, taking into account the above three cases and from the continuity of the function \( |f_4(\sigma, x_1)| \), the mean value theorem implies that for each \( \sigma \) in \([-0.55, 1]\) there is a value \( x_{1,\sigma} \) such that

\[
|f_4(\sigma, x_{1,\sigma})| = 3^\sigma,
\]
as we claimed. Thus $f_4(\sigma, x_1, \sigma)$ is a point of the circle $|s| = 3\sigma$, so there exists a real number, say $x_2$, such that

$$f_4(\sigma, x_1, \sigma) = -3\sigma e^{x_2 i}.$$  

Now, taking into account that for any arbitrary real numbers $\sigma, x_1, x_2$ it is true that

$$F_4(\sigma, x_1, x_2) = f_4(\sigma, x_1) + 3\sigma e^{x_2 i},$$

it follows that $F_4(\sigma, x_1, \sigma, x_2) = 0$. Therefore the proof is completed.

In the next result we will prove the existence of a monotone relationship between the sets $R_n$ and $R_{n+1}$ when $n+1$ is a prime number. But before that, we will introduce a new real interval associated with each function $G_n(s)$.

**Definition 3** For each integer $n \geq 2$ we define the numbers

$$x_{n,0} := \inf \{ \sigma \in \mathbb{R} : 1 \leq 2^\sigma + ... + n^\sigma \}$$

and

$$x_{n,1} := \sup \{ \sigma \in \mathbb{R} : 1 + 2^\sigma + ... + (n-1)^\sigma \geq n^\sigma \}.$$  

It is immediate to check that $x_{2,0} = x_{2,1} = 0$, and for $n > 2$ it holds that $[0, 1] \subset [x_{n,0}, x_{n,1}]$.

The relationship between the new interval $[x_{n,0}, x_{n,1}]$ and the critical interval $[a_n, b_n]$ is shown in the following lemma.

**Lemma 4** Let $I_n = [a_n, b_n]$ be the critical interval associated with the function $G_n(s)$. Then $[a_n, b_n] \subset [x_{n,0}, x_{n,1}]$ for all $n \geq 2$.

**Proof.** Since $a_2 = b_2 = x_{2,0} = x_{2,1} = 0$, the lemma trivially follows for $n = 2$. Now, let us assume $n > 2$. In that case, it can be seen that $b_n \leq x_{n,1}$. Indeed, since

$$b_n := \sup \{ \text{Re} : G_n(s) = 0 \},$$

by assuming that $b_n > x_{n,1}$, there exists $w = a + ib$ a zero of $G_n(s)$ such that $a > x_{n,1}$. Thus, because of the definition of $x_{n,1}$, it follows that

$$1 + 2^a + ... + (n-1)^a < n^a. \tag{2.7}$$

On the other hand, as $G_n(w) = 0$, we write

$$1 + 2^w + ... + (n-1)^w = -n^w,$$

and taking the modulus we obtain

$$n^a \leq 1 + 2^a + ... + (n-1)^a,$$

which contradicts (2.7). Hence $b_n \leq x_{n,1}$, as claimed before.
Likewise we are going to prove that $a_n \geq x_{n,0}$ by proof by contradiction. Hence, let us assume that $a_n < x_{n,0}$. By noticing that

$$a_n := \inf \{ \text{Res} : G_n(s) = 0 \},$$

there exists $u = c + id$ a zero of $G_n(s)$ with $c < x_{n,0}$. Therefore, from the definition of $x_{n,0}$, it follows that

$$1 > 2^c + \ldots + n^c. \quad (2.8)$$

Now, as $G_n(u) = 0$, we get

$$2^u + \ldots + n^u = -1$$

and taking the modulus we are led to

$$1 = |2^u + \ldots + n^u| \leq 2^c + \ldots + n^c,$$

which contradicts (2.8). Therefore $a_n \geq x_{n,0}$, as claimed, and so we have

$$[a_n, b_n] \subset [x_{n,0}, x_{n,1}],$$

which proves the lemma. ~ \(\blacksquare\)

**Proposition 5** Let $n + 1$ be a prime number greater than 2, $I_n = [a_n, b_n]$ the critical interval of $G_n(s)$ and $R_n := \{ \text{Res} : G_n(s) = 0 \}$. Then

$$R_n \cap \left[ a_n, \frac{\log 2}{\log(1 + \frac{1}{n})} \right] \subset R_{n+1}.$$

**Proof.** Since $R_2 = \{0\}$, $a_2 = 0$, and the fact that $0 \in R_3$, see [13], the proposition follows for $n = 2$. Therefore let us assume that $n > 2$. Let $\sigma$ be a point of $R_n \cap \left[ a_n, \frac{\log 2}{\log(1 + \frac{1}{n})} \right]$. Because of Theorem 1, there exists a vector $x \in \mathbb{R}^k$ such that the function $F_n(\sigma, x) = 0$, so

$$|F_n(\sigma, x)| = 0. \quad (2.9)$$

On the other hand, from Lemma 4,

$$\sigma \in R_n \subset [a_n, b_n] \subset [x_{n,0}, x_{n,1}],$$

and subsequently

$$1 + 2^\sigma + \ldots + (n - 1)^\sigma \geq n^\sigma.$$

Hence

$$1 + 2^\sigma + \ldots + (n - 1)^\sigma + n^\sigma \geq 2n^\sigma. \quad (2.10)$$

Taking into account that

$$\sigma \leq \frac{\log 2}{\log(1 + \frac{1}{n})},$$
we can write
\[ (1 + \frac{1}{n})^\sigma \leq 2 \]
and multiplying by \( n^\sigma \) it can be deduced that
\[ (n + 1)^\sigma \leq 2n^\sigma. \]
Now, from (2.10), we get
\[ 1 + 2^\sigma + ... + n^\sigma \geq (n + 1)^\sigma \]
and then
\[ |F_n(\sigma,0)| = 1 + 2^\sigma + ... + n^\sigma \geq (n + 1)^\sigma, \tag{2.11} \]
where 0 denotes the vector zero of \( \mathbb{R}^{k_n} \). Furthermore, because of the continuity of the modulus of \( F_n(\sigma,x) \) and using (2.9) and (2.11), there exists a vector \( a = (a_1, ..., a_{k_n}) \in \mathbb{R}^{k_n} \) such that
\[ |F_n(\sigma,a)| = (n + 1)^\sigma, \]
and then, for some \( \alpha \in [0,2\pi) \), we write
\[ F_n(\sigma,a) = (n + 1)^\sigma e^{\alpha i}. \tag{2.12} \]
Since \( n + 1 \) is a prime number, the number of prime numbers, \( k_{n+1} \), of the sequence \( \{1,2, ..., n+1\} \) is so that \( k_{n+1} = k_n + 1 \) and then, noticing (2.1), we put
\[ F_{n+1}(\sigma,y) = F_n(\sigma,xy) + (n + 1)^\sigma e^{y_{k_{n+1}} i}, \tag{2.13} \]
where \( y \) is an arbitrary vector of \( \mathbb{R}^{k_{n+1}} \), \( xy \) is the vector of \( \mathbb{R}^{k_n} \) defined by the first \( k_n \) components of \( y \) and \( y_{k_{n+1}} \) is the last component of \( y \). Thus, by substituting in (2.13) the vector \( y \) by the vector \( b : = (a_1, ..., a_{k_n}, \alpha + \pi) \) and, according to (2.12), it follows
\[ F_{n+1}(\sigma,b) = 0, \]
which means, from Theorem 1, that \( \sigma \in R_{n+1} \). Now the proof is completed and so the proposition follows. ■

**Corollary 6** The interval \([-0.55,1]\) is contained in \( R_5 := \{\text{Res} : G_5(s) = 0\} \), where \( G_5(s) = 1 + 2^s + 3^s + 4^s + 5^s \).

**Proof.** From Corollary 2 we have \([-0.55,1] \subset R_4 \subset [a_4,b_4] \). On the other hand, as 5 is a prime number, because of the above proposition, we get
\[ R_4 \cap \left[ a_4, \frac{\log 2}{\log(1 + \frac{1}{4})} \right] \subset R_5. \]
Now, noticing
\[ [-0.55,1] \subset \left[ a_4, \frac{\log 2}{\log(1 + \frac{1}{4})} \right], \]
the result follows. ■
3 Density properties of $G_n(s)$

In this section we are going to study some properties of the functions $G_n(s)$ about the existence of zeros arbitrarily close to any right-line parallel to the imaginary axis contained in its critical strip. Firstly, taking into account the definition of $x_{n,0}$ and $x_{n,1}$, we observe that in the strip

$$\{s = \sigma + it : x_{n,0} < \sigma < x_{n,1}\},$$

the functions $G_n(s)$ satisfy the following geometric principle.

**Lemma 7** Let $n$ be an integer greater than 2. Then, for arbitrary $s = \sigma + it$ inside the strip $\{s \in \mathbb{C} : x_{n,0} \leq Re(s) \leq x_{n,1}\}$, it follows that

$$|j^s| \leq \sum_{k=1; k \neq j}^{n} |k^s|, \text{ for every } j = 1, 2, ..., n. \quad (3.1)$$

**Proof.** For $j = 1$ and $j = n$, inequality (3.1) is immediate by virtue of the definitions of $x_{n,0}$ and $x_{n,1}$, respectively. For any $j \neq 1, n$, inequality (3.1) is true for arbitrary $s \in \mathbb{C}$. Then the lemma follows. $\blacksquare$

**Corollary 8** For any $\sigma \in [a_n, b_n]$ there exists at least one $n$-sided polygon whose sides have lengths $1^\sigma, 2^\sigma, ..., n^\sigma$.

**Proof.** Because of Lemma 4, we have $[a_n, b_n] \subset [x_{n,0}, x_{n,1}]$. Now, taking into account the preceding lemma, $\sigma$ is a real number such that the lengths $1^\sigma, 2^\sigma, ..., n^\sigma$ satisfy all the inequalities (3.1). Hence, from [13, p. 71], the conclusion of the corollary is valid. $\blacksquare$

**Lemma 9** Every function $G_n(s) := 1 + 2^s + ... + n^s$, $n \geq 2$, satisfies the following properties:

(a) $G_n(s)$ is bounded on the strip $S_n := \{s = \sigma + it : a_n \leq \sigma \leq b_n\}$.

(b) There exists some $\sigma_0 \in [a_n, b_n]$ such that the vertical line of equation $x = \sigma_0$ contains a sequence of points $(\sigma_0 + iT_j)_{j=1,2,...}$ such that

$$\lim_{j \to \infty} G_n(\sigma_0 + iT_j) = 0.$$

(c) There exist positive numbers $\delta$ and $l$ such that on any segment of length $l$ of the line $x = \sigma_0$ there is a point $\sigma_0 + iT$ such that $|G_n(\sigma_0 + iT)| \geq \delta$.

**Proof.** The case $n = 2$ is trivial. Indeed, $S_2$ coincides with the imaginary axis so $|G_2(s)| \leq 2$ for all $s \in S_2$ and then (a) holds. The line of equation $x = 0$ contains all the zeros of $G_2(s)$, namely, the imaginary numbers $s = \frac{i \pi (2k+1)}{\ln 2}$, $k \in \mathbb{Z}$, so (b) is true. Moreover, by taking $\delta = 2$ and an arbitrary $l > \frac{2 \pi}{\ln 2}$, any segment of length $l$ of the right-line $x = 0$ contains a point $iT$, with $T = \frac{i \pi 2k}{\ln 2}$ for some integer $k$, such that $|G_n(iT)| \geq 2$. Hence, for $n = 2$, the result follows. Now let us assume that $n > 2$. 

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It is clear that $G_n(s)$ is bounded on any finite vertical strip, so, in particular, $G_n(s)$ is bounded on $S_n$. Therefore part (a) is proved.

Consider an arbitrary zero $s_0 = \sigma_0 + i\tau_0$ of $G_n(s)$. By applying Theorem 1, the vector 

$$t_0p = t_0(\log p_1, \log p_2, ..., \log p_{k_n})$$

is such that $F_n(\sigma_0, t_0p) = 0$, where $F_n$ is the function defined by (2.1). Now, by following the proof of the sufficiency of the mentioned Theorem 1, there exists a sequence of points $(\sigma_0 + iT_j)_{j=1,2,...}$ on the line $x = \sigma_0$ such that 

$$\lim_{j \to \infty} G_n(\sigma_0 + iT_j) = 0.$$ 

Consequently, part (b) has been shown.

Finally, since $G_n(\sigma_0) > 0$, and taking into account that $G_n(s)$ is an almost-periodic function, given $\delta = G_n(\sigma_0)2$, there exists a real number $l = l(\delta)$ such that every interval of length $l$ on the imaginary axis contains at least one translation number $T$, associated with $\delta$, satisfying the inequality 

$$|G_n(s + iT) - G_n(s)| \leq \delta,$$

for all $s \in \mathbb{C}$.

In particular, for $s = \sigma_0$, one has $|G_n(\sigma_0 + iT) - G_n(\sigma_0)| \leq \delta$ and, according to the choice of $\delta$, it implies that 

$$|G_n(\sigma_0 + iT)| \geq \delta.$$ 

Now, the proof of the lemma is completed.

The density properties on the real projections of the zeros of $G_n(s)$ are given by means of the following result.

**Theorem 10** Let $\sigma_0$ be a point of the critical interval $[a_n, b_n]$ of $G_n(s)$, $n \geq 2$, verifying properties (b) and (c) of the above lemma. Then $G_n(s)$ has zeros in the strip

$$\{s \in \mathbb{C} : \sigma_0 - \delta \leq \text{Re}s \leq \sigma_0 + \delta\},$$

for an arbitrary $\delta$ satisfying $0 \leq \delta \leq \min \{\sigma_0 - a_n, b_n - \sigma_0\}$.

**Proof.** The function $G_n(s)$ and $\sigma_0$ satisfy the properties (a), (b) and (c) of the previous lemma. Then, it suffices to apply the result attributed in [13, p. 74] to H. Bohr for demonstrating the validity of the theorem.

A more detailed result about the existence of zeros of the $G_n(s)$ functions in a strip is provided in Section 4 of this paper.

4 A characterization of the density of the real projections of the zeros of $G_n(s)$ by means of level curves

In this section a new characterization of the sets $R_n := \{\text{Re} : G_n(s) = 0\}$ in terms of the old concept of level curve [15] is shown. Firstly we examine an important property of the $G_n(s)$ functions.
Lemma 11  Given $\sigma \in \mathbb{R}$, for each $n \geq 2$ the function $G_n(s) := 1 + 2^s + ... + n^s$ satisfies
\[
\operatorname{Max} \{|G_n(s)| : \Re s \leq \sigma\} = G_n(\sigma).
\]
Furthermore, for $n > 2$, $s = \sigma$ is the unique point where the maximum is attained.

Proof. Given $\sigma$, let $s = x + iy$ be any complex number with $x \leq \sigma$. Since
\[
|G_n(s)| \leq G_n(x) \leq G_n(\sigma),
\]

it can be deduced that
\[
\operatorname{Max} \{|G_n(s)| : \Re s \leq \sigma\} = G_n(\sigma),
\]

so the first part of the lemma follows. Now let us assume that $n > 2$, to prove that $s = \sigma$ is the unique point where the maximum is attained it suffices to show that
\[
|1 + 2^{\sigma+it} + 3^{\sigma+it}| < 1 + 2^\sigma + 3^\sigma \quad \text{for all real } t \neq 0. \tag{4.1}
\]

Indeed,
\[
|1 + 2^{\sigma+it} + 3^{\sigma+it}| = \left| \frac{1}{2} + 2^{\sigma+it} + \frac{1}{2} + 3^{\sigma+it} \right| \leq \left| \frac{1}{2} + 2^{\sigma+it} \right| + \left| \frac{1}{2} + 3^{\sigma+it} \right|.
\]

Now we claim that at least one of the two inequalities
\[
\left| \frac{1}{2} + 2^{\sigma+it} \right| \leq \frac{1}{2} + 2^\sigma, \quad \left| \frac{1}{2} + 3^{\sigma+it} \right| \leq \frac{1}{2} + 3^\sigma
\]

must be strict. Otherwise, there will be two positive $\lambda$ and $\mu$ such that
\[
2^{\sigma+it} = \frac{\lambda}{2}, \quad 3^{\sigma+it} = \frac{\mu}{2}.
\]

However there are integers $k$, $l \neq 0$ such that $\frac{\log 2}{\log 3} = \frac{k}{l}$ which contradicts the Fundamental Theorem of Arithmetic. Hence (4.1) is true and the proof of the lemma is completed. \[\square\]

Following [15, p.121] we recall the concept of level curves.

Definition 12  Given an entire function $f(z)$ and a non-negative constant $k$, the curves defined by the equation
\[
|f(x + iy)| = k \tag{4.2}
\]

are called level curves of order $k$. 

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For example, the level curves of the exponential function $e^x$ of order $k > 0$ are the vertical lines of equations $x = \log k$. The level curves of $G_2(z) := 1 + 2z$, $z = x + iy$, are given by the equation

$$1 + 2^{x+1}\cos(y \log 2) + 2^{2x} = k^2,$$

for every $k > 0$, which does not contain any vertical line. Therefore, because of Lemma 11, the level curves corresponding to the functions $G_n(z)$, $n \geq 2$, of order $k > 0$, do not contain any vertical line.

We are going to prove that if equation (4.2) has at least one solution, say $z_0 = x_0 + iy_0$, then there exists only one level curve in a certain neighborhood of $(x_0, y_0)$ that passes through the point $(x_0, y_0)$, provided that $f'(z_0) \neq 0$.

**Lemma 13** Let $f(z)$ be an entire function, $k > 0$ and $z_0 = x_0 + iy_0$ a point satisfying the equation $|f(z)| = k$. Then, if $f'(z_0) \neq 0$, in certain neighborhood of $(x_0, y_0)$ there exists only one level curve, either defined by a function $\varphi$ or by a function $\psi$, both of class $C^\infty$, passing through the point $(x_0, y_0)$ and such that $(\varphi(y), y)$ or $(x, \psi(x))$ satisfy (4.2) on certain neighborhoods of $y$ and $x$, respectively.

**Proof.** By setting $f = u + iv$, equation (4.2) can be written as $\Phi(x, y) = 0$, where $\Phi(x, y) := u^2 + v^2 - k^2$. Then, if we assume $\frac{\partial \Phi}{\partial x}(x_0, y_0) = \frac{\partial \Phi}{\partial y}(x_0, y_0) = 0$, it follows that

$$\begin{align*}
2u \frac{\partial u}{\partial x} + 2v \frac{\partial u}{\partial y} &= 0, \\
2u \frac{\partial v}{\partial x} + 2v \frac{\partial v}{\partial y} &= 0.
\end{align*}$$

(4.3)

Now, by virtue of Cauchy-Riemann equations, the expression (4.3) becomes

$$\begin{align*}
u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} &= 0, \\
v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial x} &= 0.
\end{align*}$$

(4.4)

Now, since the determinant of the matrix of system (4.4) is $-u^2 - v^2 = -k^2 \neq 0$, the system (4.4) only has the solution $\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial u}{\partial y}(x_0, y_0) = 0$. It means that $f'(z_0) = 0$, which is a contradiction. Hence, either $\frac{\partial u}{\partial x}(x_0, y_0) \neq 0$ or $\frac{\partial v}{\partial y}(x_0, y_0) \neq 0$ and then, from the implicit function theorem, the result follows.

The existence of a level curve passing through a non-critical point is guaranteed by means of the following result.

**Proposition 14** Let $f(z)$ be a non-constant entire function. For every $k > 0$, there exists a point $z_0$ with $f'(z_0) \neq 0$ belonging to the level curve $|f(z)| = k$.

**Proof.** According to Picard’s little theorem [5, p. 432], for every $w$ of the circle $C_k := \{w: |w| = k\}$ the equation $f(z) = w$ has a solution except, at most, for a value, say $w_0$, of $C_k$. Since the set $f^{-1}(C_k \setminus \{w_0\})$ is uncountable and $D_f := \{z: f'(z) = 0\}$ is a countable set, let $z_0$ be a point of $f^{-1}(C_k \setminus \{w_0\}) \setminus D_f$.

Then, the point $z_0$ satisfies the equation $|f(z)| = k$ and it is such that $f'(z_0) \neq 0$. Consequently, the proposition follows.
Proposition 15 Let \( f(z) \) be analytic on an open set \( U \). Let \( V \subset U \) be a proper bounded open subset such that \( f'(z) \neq 0 \) for all \( z \in V \). Assume the level curve \( |f(z)| = k, k > 0 \), passes through a point \( z_0 \in V \). Then, either the level curve is a closed curve in \( \overline{V} \) or it has two disjoint arc-connected closed subsets \( A, B \) of the boundary of \( V \).

Proof. As proven in Lemma 13, there exists a unique curve contained in the level curve \( |f(z)| = k \) that contains \( z_0 \) as an interior point. Hence, let \( L_0 \) be the arc-connected component of the level curve \( |f(z)| = k \) contained in \( \overline{V} \) that passes through \( z_0 \). If \( L_0 \) is neither a closed curve nor crosses the boundary of \( V \), then, as \( L_0 \) is a closed set in the topology of the complex plane, necessarily \( L_0 \) would have two end points, say \( z_1 \) and \( z_2 \), contained in \( V \). Because \( z_1, z_2 \) are non-critical points, by applying Lemma 13, there exists a continuation of \( L_0 \), which is a contradiction because \( L_0 \) is an arc-connected component and therefore is a maximal set. In this case, it is clear that the continuation from the points \( z_1, z_2 \) creates two sets \( A, B \) (each set \( A \) and \( B \) could be reduced to one point) verifying that \( A \cap B = \emptyset \). Thus the result follows. ■

The next result is a basic property of the complex numbers.

Lemma 16 Let \( z_1, z_2, z_3 \) be complex numbers:

a) If \( z_2 = \lambda z_1 \) with \( \lambda > 0 \) and \( |z_2| > |z_3| \), then \( |z_1| < |z_1 + z_2 + z_3| \).

b) If \( z_2 = -\lambda z_1 \) with \( \lambda > 0 \) and \( |z_1| > |z_2| > |z_3| \), then \( |z_1| > |z_1 + z_2 + z_3| \).

As we have already seen, a level curve that passes through a non-critical point of an analytic function is locally a simple curve. Nevertheless, if \( z_0 \) is a critical point, then the level curve that passes through \( z_0 \) has, at least, four branches rising from \( z_0 \).

Proposition 17 Let \( f \) be a non-constant entire function and \( z_0 \) a critical point of \( f \) belonging to the level curve \( L_k := \{ z : |f(z)| = k \}, k > 0 \). Let \( m \) be the order of the zero of \( f'(z) \) at the point \( z_0 \). Then \( L_k \) has at least \( 2(m+1) \) branches which meet at the point \( z_0 \).

Proof. Since \( z_0 \) is a zero of order \( m \) of \( f'(z) \), the Taylor expansion of \( f(z) \) is given by

\[
f(z) = f(z_0) + \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n, \text{ with } f^{(m+1)}(z_0) \neq 0.
\]

(4.5)

Then, there exits \( r > 0 \) so that for every \( z \) satisfying \( 0 < |z - z_0| \leq r \) we can write

\[
f(z) = z_1 + z_2 + z_3,
\]

satisfying

\[
f'(z) \neq 0 \quad \text{and} \quad |z_1| > |z_2| > |z_3|,
\]

(4.7)

where \( z_1, z_2, z_3 \) (depending on \( z \)) are given by

\[
z_1 = f(z_0),
\]

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z_2 = \frac{f^{(m+1)}(z_0)}{(m+1)!} (z - z_0)^{m+1}

and

z_3 = \sum_{n=m+2}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.

On the other hand, since for each \( \lambda > 0 \) the equation \( z_2 = \lambda z_1 \) has exactly \( m + 1 \) solutions in a certain disk \( D(z_0, r_\lambda) := \{ z : |z - z_0| \leq r_\lambda \} \), there exists some \( L > 0 \) such that the set of solutions of the equation \( z_2 = \lambda z_1 \), when \( \lambda \in [0, L] \), coincides with the set determined by other \( m + 1 \) radii of the disk \( D(z_0, r) \). Therefore, taking into account (4.6), (4.7) and the preceding lemma, \( D(z_0, r) \) is partitioned into \( 2(m+1) \) sectors in such a way that given one of these sectors, we have that

\[ |f(z_0)| < |f(z)| \]

for all \( z \) belonging to one radius, and

\[ |f(z_0)| > |f(z)| \]

for any \( z \) of the adjacent radius. Now, by applying Proposition 15 on the interior of each sector, there exists a branch of the level curve that passes through \( z_0 \) contained in each sector. The proof is completed and then the result follows. 

**Definition 18** For every integer \( n > 2 \) we define

\[ G^*_n(s) := G_n(s) - p_{k_n} \]

where \( p_{k_n} \) is the last prime number such that \( p_{k_n} \leq n \).

The characterization of \( R_n \) in terms of level curves is given in the following result.

**Theorem 19** A real number \( \sigma \in R_n := \{ \text{Res} : G_n(s) = 0 \} \), \( n > 2 \), if and only if the level curve \( |G^*_n(z)| = p_{k_n}^\sigma \) intersects the vertical line \( x = \sigma \).

**Proof.** First of all we will prove the sufficiency. Let \( s = \sigma + it \) be a point of the line \( x = \sigma \) belonging to the level curve \( |G^*_n(z)| = p_{k_n}^\sigma \), then

\[ |G^*_n(\sigma + it)| = p_{k_n}^\sigma \]

and thus there exists some \( \theta \in [0, 2\pi) \) such that

\[ G^*_n(\sigma + it) = p_{k_n}^\sigma e^{i\theta}. \]
That means that
\[ 1 + 2^{\sigma+i\theta} + 3^{\sigma+i\theta} + \ldots + p_k^{\sigma+i\theta} - p_k^{\sigma}e^{i\theta} + \ldots + n^{\sigma+i\theta} = 0. \tag{4.8} \]

Now, noticing (4.8), for
\[ x = t(\log 2, \log 3, \ldots, \log p_{kn-1}, \theta + \pi), \]
the function \( F_n(\sigma, x) \), defined by (2.1), satisfies
\[ F_n(\sigma, x) = 1 + 2^{\sigma+i\theta} + 3^{\sigma+i\theta} + \ldots + p_k^{\sigma+i\theta} - p_k^{\sigma}e^{i\theta} + \ldots + n^{\sigma+i\theta} = 0. \]

Then, from Theorem 1, it follows that \( \sigma \in R_n \).

Reciprocally, suppose that \( \sigma \in R_n \). Then there exists a sequence of zeros \((s_m = \sigma_m + it_m)_{m=1,2,\ldots}\) of \( G_n(s) \) such that
\[ \sigma = \lim_{m \to \infty} \sigma_m. \]

Because of the definition of \( G^*_n(s) \), at each zero \( s_m \) of \( G_n(s) \), one has
\[ G^*_n(s_m) = -p_k^{s_m}. \]

Then, by taking the modulus, we obtain
\[ |1 + 2^{\sigma_m+i\theta m} + \ldots + p_k^{\sigma_m+i\theta m} + \ldots + n^{\sigma_m+i\theta m}| = p_k^{\sigma_m} \quad \text{for all } m = 1, 2, \ldots \tag{4.9} \]

Now, since the sequence \((e^{it_m})_{m=1,2,\ldots}\) is contained in the unit circle, there exists a subsequence \((e^{it_{mj}})_{j=1,2,\ldots}\) such that
\[ \lim_{j \to \infty} e^{it_{mj}} = e^{i\lambda} \quad \text{for some } \lambda \in [0, 2\pi). \]

At this point, we can rewrite the expression (4.9), for \( m_j, j = 1, 2, \ldots \), and taking the limit \( j \to \infty \), into the expression
\[ |1 + 2^{\sigma+i\lambda \log 2} + 3^{\sigma+i\lambda \log 3} + \ldots + p_k^{\sigma}e^{i\lambda \log p_{kn-1}} + \ldots + n^{\sigma}e^{i\lambda \log n}| = p_k^{\sigma}, \]
which is equivalent to say
\[ |G^*_n(\sigma + i\lambda)| = p_k^{\sigma}. \]

That is, the level curve \(|G^*_n(x)| = p_k^{\sigma}\) meets the vertical line \( x = \sigma \). Now the proof is completed. \( \blacksquare \)

As a consequence of the theorem above, we will introduce a real function with the aim of characterizing the sets \( R_n \).
Definition 20 For $n > 2$, we define the real function $A_n(x,y)$ as

$$A_n(x,y) := |G_n^*(x+iy)| - p_n^x; x, y \in \mathbb{R}.$$

The characterization of $R_n$ by means of $A_n(x,y)$ is given in the following result.

Corollary 21 A real number $x$ belongs to $R_n$ if and only if $A_n(x,y) = 0$ for some $y \in \mathbb{R}$. Furthermore

$$A_n(x,0) \geq 0 \text{ for all } x \in [a_n,b_n]. \quad (4.10)$$

Proof. The first part is a direct consequence of Definition 20 and Theorem 19. The second part immediately follows from Lemma 4 and the inequality in (3.1).

In order to improve Proposition 5 we will use this real function $A_n(x,y)$ for the next result.

Proposition 22 Let $n + 1$ be a prime number greater than 2 and

$$R_n := \{ \text{Res} : G_n(s) = 0 \}, \ n = 2, 3, ...$$

Then $R_n \subset R_{n+1}$.

Proof. As we saw in the proof of Proposition 5, $R_3 \subset R_3$, and thus the result is valid for $n = 2$. Hence let us assume $n > 2$. Let $\sigma$ be a point of $R_n$, then there exists a sequence $(s_j = \sigma_j + it_j)_{j=1,2,...}$ of zeros of $G_n(s)$ such that $\sigma = \lim_{j \to \infty} \sigma_j$. Since $n + 1$ is a prime number, $G_{n+1}^*(s) = G_n(s)$. Then, for every zero $s_j$, we have

$$A_{n+1}(\sigma_j, t_j) = -(n+1)^{\sigma_j} < 0$$

and, noticing (4.10),

$$A_{n+1}(\sigma_j, 0) \geq 0.$$ 

Then, according to the continuity of $A_{n+1}(x,y)$, there exists $t'_j$ such that

$$A_{n+1}(\sigma_j, t'_j) = 0.$$ 

Now, because of Corollary 21, $\sigma_j \in R_{n+1}$ for all $j$, and then, since $R_{n+1}$ is closed, we get

$$\lim_{j \to \infty} \sigma_j = \sigma \in R_{n+1}.$$ 

This completes the proof.

Observe that the previous proof, mutatis mutandis, applies to the zeros of $G_n^*(s)$ for arbitrary $n > 2$ as well. Indeed, the zeros of $G_n^*(s)$ supply non-void intervals contained in $R_n$, as we will prove in the next result.

Theorem 23 Let $s = \sigma + it$ be a zero of $G_n^*(s)$ with $a_n \leq \sigma < b_n$, $n > 2$. Then there exists a non-void interval $J$ such that $\sigma \in J \subset R_n$. 

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Proof. Since $G_n^*(s) = 0$, given $\epsilon = p_{k_n}^0 > 0$, by continuity, there exists $r > 0$ such that for any $z = x + iy \in \mathcal{D}(s, r)$ (the closed disk of center $s$ and radius $r$) we have

$$|G_n^*(x + iy)| \leq p_{k_n}^0.$$ 

Hence, in particular, by taking $z_\delta = \sigma + \delta + it$, with $0 \leq \delta \leq r$, we get

$$|G_n^*(\sigma + \delta + it)| \leq p_{k_n}^0 \leq p_{k_n}^{\sigma + \delta}.$$ 

Then, because of Definition 20, it can be deduced that

$$A_n(\sigma + \delta, t) \leq 0, \text{ for any } \delta \in [0, r].$$

Therefore, noticing (4.10) and the continuity of the function $A_n(x, y)$, there exist $t'_\delta$ such that

$$A_n(\sigma + \delta, t'_\delta) = 0, \text{ for every } \delta \in [0, r].$$

Consequently, from Corollary 21, it follows that $J := [\sigma, \sigma + r] \subset R_n$. The proof is then completed. □

Lemma 24 Let $z_0$ be a zero of $G_n(z)$. Then there exists a disk

$$U := \{z : |z - z_0| < r\}$$

such that for all $z \in U$ the formula

$$\frac{\partial \arg G_n^*(z)}{\partial x} = -\frac{1}{\partial G_n^*(z)} \frac{\partial |G_n^*(z)|}{\partial y} \quad (4.11)$$

holds.

Proof. Since $G_n(z_0) = 0$, it is true that $G_n^*(z_0) = -p_{k_n}^{z_0} \neq 0$. Then, we can determine a disk $U := \{z : |z - z_0| < r\}$ such that $G_n^*(z) \neq 0$ for all $z \in U$. Now, because $U$ is a simply connected set and noticing [1, p. 52], there exists an analytic logarithm on $U$ such that

$$\log G_n^*(z) = \ln |G_n^*(z)| + i \arg G_n^*(z).$$

Then the real and imaginary part of $\log G_n^*(z)$, i.e. the two functions $\ln |G_n^*(z)|$ and $\arg G_n^*(z)$, belong to class $C^\infty$ on $U$ and are harmonic conjugates, therefore, by the Cauchy-Riemann equations, we obtain

$$\frac{\partial \arg G_n^*(z)}{\partial x} = -\frac{1}{\partial G_n^*(z)} \frac{\partial |G_n^*(z)|}{\partial y} \text{ for all } z \in U,$$

which is the desired formula. □
Lemma 25 Let \( z_0 = x_0 + iy_0 \) be a zero of \( G_n(z) \) with \( a_n < x_0 < b_n \). If for all \( \epsilon > 0 \) there exist two values \( x_1 \in (x_0 - \epsilon, x_0) \) and \( x_2 \in (x_0, x_0 + \epsilon) \) of the critical interval \([a_n, b_n]\) satisfying \( A_n(x_1, y), A_n(x_2, y) \neq 0 \) for all \( y \in \mathbb{R} \), then \( \frac{\partial A_n}{\partial x}(x_0, y_0) = 0 \) and consequently

\[
\frac{\partial |G_n^*(z)|}{\partial x}(z_0) = p_k^{x_0} \log p_k. \tag{4.12}
\]

Proof. Firstly, as \( G_n(z) \) does not have any real zero and \( G_n(\overline{z}) = \overline{G_n(z)} \), we can suppose without loss of generality that \( y_0 > 0 \). On the other hand, since \( G_n(z_0) = 0 \), from Definition 20 we have that

\[ A_n(x_0, y_0) = 0. \]

By assuming \( \frac{\partial A_n}{\partial x}(x_0, y_0) < 0 \), it is deduced the following

\[
\frac{\partial A_n}{\partial x}(x_0, y_0) = \lim_{x \to 0^+} \frac{A_n(x_0 + x, y_0)}{x} < 0.
\]

Therefore, there exists an \( \epsilon > 0 \) such that

\[ A_n(x_0 + x, y_0) < 0, \text{ for all } x \in (0, \epsilon). \]

On the other hand, because of (4.10),

\[ A_n(x_0 + x, 0) \geq 0. \]

Then, by continuity, there exists some value of \( y \), say \( y_x \in (0, y_0) \), such that

\[ A_n(x_0 + x, y_x) = 0, \text{ for every } x \in (0, \epsilon). \]

This contradicts the hypothesis and therefore

\[
\frac{\partial A_n}{\partial x}(x_0, y_0) \geq 0.
\]

Assuming that \( \frac{\partial A_n}{\partial x}(x_0, y_0) > 0 \), we find that

\[
\frac{\partial A_n}{\partial x}(x_0, y_0) = \lim_{x \to 0^+} \frac{A_n(x_0 - x, y_0)}{-x} > 0
\]

and conclude that there exists some \( \epsilon > 0 \) such that

\[ A_n(x_0 - x, y_0) < 0 \text{ for all } x \in (0, \epsilon). \]

Now, by repeating verbatim the above argument, we are led to a contradiction again. Therefore

\[
\frac{\partial A_n}{\partial x}(x_0, y_0) = 0. \tag{4.13}
\]
Finally, looking at (4.13) and according to Definition 20, by taking the partial derivative of \( A_n(x, y) \) with respect to \( x \) at the point \( z_0 = x_0 + iy_0 \), the desired formula is derived:

\[
\frac{\partial |G_n^*(z)|}{\partial x}(z_0) = p_{k_n} \log p_{k_n}.
\]

The main result of the present paper consists of proving that the real projections of the simple zeros of \( G_n(s) := 1 + 2^s + \ldots + n^s \) are not isolated points.

**Theorem 26** Let \( s_0 = \sigma_0 + it_0 \) be a simple zero of \( G_n(s) \), \( n > 2 \). Then, there exist \( \epsilon_1, \epsilon_2 \geq 0 \), with \( \epsilon_1 + \epsilon_2 > 0 \), such that \( (\sigma_0 - \epsilon_1, \sigma_0 + \epsilon_2) \subset R_n \).

**Proof.** As we did in the preceding Lemma we can assume, without loss of generality, that \( t_0 > 0 \). Since \( G_n(s_0) = 0 \), we have \( |G_n^*(s_0)| = p_{k_n}^{s_0} \) and so the level curve \( |G_n^*(s)| = p_{k_n}^{s_0} \) passes through \( s_0 \). Then, let us denote by \( L_0 \) the arc-connected component that passes through the point \( s_0 \). Note that \( L_0 \) is not reduced to \( s_0 \) because either Lemma 13 (if \( s_0 \) is a non-critical point of \( G_n^*(s) \)), or Proposition 17 (if \( s_0 \) is a critical point of \( G_n^*(s) \)) apply. Now let

\[
P_0 := \{ \text{Re} s : s \in L_0 \}
\]

be its projection on the real axis. Firstly, we claim that \( P_0 \) is a real interval not reduced to the point \( \sigma_0 \). Indeed, if we suppose that this is not so, then, necessarily, \( L_0 \) would be contained in the vertical line of equation \( x = \sigma_0 \). On the other hand, from Lemma 11, the point \( \sigma_0 \) does not belong to \( L_0 \). Therefore, as \( L_0 \) is a closed set contained in the vertical \( x = \sigma_0 \), there exists a point \( \omega_0 := \sigma_0 + iT \in L_0 \), where

\[
T = \min \{ t > 0 : \sigma_0 + it \in L_0 \}.
\]

Hence, \( \omega_0 \) is necessarily a non-critical point of \( G_n^*(s) \). Indeed, if \( \omega_0 \) were a critical point, then Proposition 17 implies the existence of at least four branches passing through \( \omega_0 \), which is impossible because we are assuming that \( L_0 \) is contained in \( x = \sigma_0 \). Hence, \( \omega_0 \) is non-critical point and then by applying Lemma 13, there exists a point \( u_0 := \sigma_0 + iT_0 \in L_0 \) with \( 0 < T_0 < T \), which contradicts (4.14). Hence, what we claim is true and therefore there exist \( \delta_1, \delta_2 \geq 0 \), with \( \delta_1 + \delta_2 > 0 \), such that

\[
(\sigma_0 - \delta_1, \sigma_0 + \delta_2) \subset P_0 \cap [a_n, b_n],
\]

where \([a_n, b_n]\) is the critical interval of \( G_n(s) \).

Now, let us consider the two possible cases in virtue of the values of \( \delta_1 \) and \( \delta_2 \).

**Case 1:** \( \delta_2 > 0 \). In this particular case, (4.15) means that

\[
(\sigma_0, \sigma_0 + \delta_2) \subset P_0 \cap [a_n, b_n].
\]
Let $x$ be an arbitrary point of $(\sigma_0, \sigma_0 + \delta_2)$, then $x \in P_0$ and therefore there exists a point $z = x + iy \in L_0$, with $y > 0$. Thus $|G_n^*(z)| = p_n^m < p_n^m$. and, in consequence, we obtain

$$|G_n^*(z)| = p_n^m < p_n^m. \quad (4.16)$$

Now, according to Definition 20, the expression (4.16) is equivalent to having

$$A_n(x, y) < 0.$$

On the other hand, from (4.10) we have

$$A_n(x, 0) \geq 0$$

and hence, because of the continuity of $A_n(x, y)$, there exists $0 \leq y_x \leq y$ such that

$$A_n(x, y_x) = 0.$$

Now, noticing Corollary 21, it follows that $x \in R_n$ and then the Case 1 is proved by just taking $\epsilon_1 = 0$ and $\epsilon_2 = \delta_2$.

Case 2: $\delta_2 = 0$. In this case, (4.14) implies that

$$(\sigma_0 - \delta_1, \sigma_0) \subset P_0 \cap [a_n, b_n]$$

with $\delta_1 > 0$. Since $|G_n^*(s_0)| \neq 0$, the function $A_n(x, y)$ is differentiable on a neighborhood of $(\sigma_0, t_0)$. Let us assume the existence of some $\epsilon_1 \in (0, \delta_1]$ such that, for any $x \in (\sigma_0 - \epsilon_1, \sigma_0)$, there exists $y_x \in \mathbb{R}$ such that $A_n(x, y_x) < 0$. At this point, by repeating verbatim the argument of Case 1, the Case 2 follows by taking

$$\epsilon_1 = \delta_1, \; \epsilon_2 = 0.$$

Now, we claim that the Case 2 always leads to the above situation. Indeed, if this were not so we would have the following: for any $\epsilon_1 \in (0, \delta_1]$ there exists some $x_\epsilon \in (\sigma_0 - \epsilon_1, \sigma_0)$ such that

$$A_n(x_\epsilon, y) \geq 0 \text{ for all } y \in \mathbb{R}. \quad (4.17)$$

Then, under this supposition, by taking $\epsilon_1 = \frac{1}{m}$, for sufficiently large $m$, there exists $x_m \in (\sigma_0 - \frac{1}{m}, \sigma_0)$ such that $A_n(x_m, y) \geq 0$ for all $y \in \mathbb{R}$. For every fixed value of $y \in \mathbb{R}$, by taking the limit, it can be found that

$$\lim_{m \to \infty} A_n(x_m, y) = A_n(\sigma_0, y) \geq 0. \quad (4.18)$$

Then, since $A_n(\sigma_0, t_0) = 0$ and taking into account (4.18), we get

$$\frac{\partial A_n}{\partial y}(\sigma_0, t_0) = \lim_{y \to 0} \frac{A_n(\sigma_0, t_0 + y)}{y} \geq 0.$$

On the other hand, (4.18) also implies

$$\frac{\partial A_n}{\partial y}(\sigma_0, t_0) = \lim_{y \to 0} \frac{A_n(\sigma_0, t_0 + y)}{y} \leq 0.$$
and therefore
\[ \frac{\partial A_n}{\partial y}(\sigma_0, t_0) = 0. \]
Consequently, from Definition (20), it follows that
\[ \frac{\partial |G_n^*(x + iy)|}{\partial y}(\sigma_0, t_0) = 0 \]
and, according to formula (4.11), we obtain
\[ \frac{\partial \arg G_n^*(s_0)}{\partial x} = 0. \quad (4.19) \]
Finally, using (4.19), formula (4.12), and taking into account that
\[ \frac{\partial G_n^*}{\partial x}(z) = \frac{\partial G_n}{\partial x}(s_0) \]
\[ \frac{\partial p_k^*}{\partial x} = e^{i \arg G_n^*(s_0)} \frac{\partial |G_n^*(z)|}{\partial x} + i G_n^*(z) \frac{\partial \arg G_n^*(z)}{\partial x} + p_k^* \log p_k \]
at the point \( s_0 \), we get that
\[ G_n'(s_0) = \frac{\partial G_n(s_0)}{\partial x} = e^{i \arg G_n^*(s_0)} p_k^0 \log p_k + p_k \log p_k = \]
\[ \{G_n^*(s_0) + p_k^0\} \log p_k = G_n(s_0) \log p_k = 0, \]
which is a contradiction because \( s_0 \) is a simple zero of \( G_n(s) \). Therefore the claim is true and the proof of the theorem is completed. \( \blacksquare \)

5 Existence of simple zeros of \( G_n(s) \)

In [12, Proposition 1] the authors studied the maximum order of multiplicity of the zeros of the functions \( G_n(s) \). Furthermore, there they also showed that all the zeros of \( G_2(s) \), \( G_3(s) \) and \( G_4(s) \) are simple. Now, in this section, we present a result that improves what was done in that paper by proving the existence of vertical strips where all the zeros of \( G_n(s) \) are simple, provided that \( n \) is a prime number.

**Proposition 27** Let \( n \geq 2 \) be a prime number, \( b_n := \sup \{ \text{Res} : G_n(s) = 0 \} \) and \( b'_n := \sup \{ \text{Res} : G'_n(s) = 0 \} \), where \( G'_n(s) \) denotes the derivative of \( G_n(s) \). Then \( b'_n < b_n \).

**Proof.** Let us define the real functions
\[ f(x) := 1 + 2^x + \ldots + (n-1)^x, \]
\[ g(x) := \frac{\log 2}{\log n} 2^x + \frac{\log 3}{\log n} 3^x + \ldots + \frac{\log(n-1)}{\log n} (n-1)^x \]
and
\[ h(x) := n^x. \]

Firstly let us say that
\[ b_n = b_{n,1}, \]
where \( b_{n,1} \) is defined as
\[ b_{n,1} := \sup \{ x \in \mathbb{R} : f(x) = h(x) \}. \]

Indeed, since \( h(x), f(x) \) are positive real functions satisfying
\[ \lim_{x \to +\infty} \frac{h(x)}{f(x)} = +\infty, \]
there exists some \( x_0 \) such that \( h(x) > f(x) \) for all \( x \geq x_0 \). On the other hand,
\[ \lim_{x \to -\infty} \frac{h(x)}{f(x)} = 0, \]
then, from continuity of \( \frac{h(x)}{f(x)} \), the number \( b_{n,1} \) exists and is not greater than \( x_0 \).

Furthermore, from the definition of \( b_{n,1} \) and the mean value theorem, it follows that
\[ h(x) > f(x), \text{ for all } x > b_{n,1}. \] (5.1)

Then, given \( x > b_{n,1} \), for an arbitrary \( y \), we have
\[
\begin{align*}
|G_n(x + iy)| &= |1 + 2^x 2^iy + \ldots + (n-1)^x(n-1)^iy + n^x + iy| \\
&\geq |n^x + iy| - |1 + 2^x 2^iy + \ldots + (n-1)^x(n-1)^iy| \\
&\geq |n^x - (1 + 2^x + \ldots + (n-1)^x)| > 0,
\end{align*}
\]
which implies that \( \{ z : \text{Re}z > b_{n,1} \} \) is a free-zero region of \( G_n(s) \) and, in consequence,
\[ b_n \leq b_{n,1}. \] (5.2)

On the other hand, as \( n \) is a prime number, \( n = p_{k_n} \), so, \( f(x) = G_n^*(x) \) and \( h(x) = p_{k_n}^x \), for all \( x \in \mathbb{R} \). Now, since \( f(b_{n,1}) = h(b_{n,1}) \), because of Definition 20, we have
\[
A_n(b_{n,1}, 0) = |G_n(b_{n,1})| - p_{k_n}^{b_{n,1}} = f(b_{n,1}) - h(b_{n,1}) = 0,
\]
which means, from Corollary 21, that \( b_{n,1} \in R_n \subset [a_n, b_n] \) and thus
\[ b_{n,1} \leq b_n. \]

Consequently, from (5.2), we get
\[ b_n = b_{n,1}, \] (5.3)
as we claimed.
Now by following a similar procedure to what we did above, we obtain

\[ b'_{n,1} = b'_n \]  

(5.4)

where

\[ b'_{n,1} := \sup \{ x \in \mathbb{R} : g(x) = h(x) \} . \]

Then, according to (5.3) and (5.4), we have

\[ f(b_n) = h(b_n) \text{ and } g(b'_n) = h(b'_n). \]  

(5.5)

Finally, let us say that \( b'_n < b_n \). Indeed, since \( f(x) > g(x) \) for all \( x \in \mathbb{R} \), from (5.5), we deduce that \( b'_n \neq b_n \). If we assume that \( b'_n > b_n \), from (5.3), (5.4) and (5.1), we are led to

\[ h(b'_n) > f(b'_n). \]

Now, noticing (5.5), it follows that

\[ g(b'_n) > f(b'_n), \]

which is a contradiction because \( f(x) > g(x) \) for all \( x \in \mathbb{R} \). Therefore, it is true that

\[ b'_n < b_n \]

and the result follows. ■

From the previous result, the next corollary can be easily derived.

**Corollary 28** Let \( n > 2 \) a prime number, \( b_n := \sup \{ \text{Res} : G_n(s) = 0 \} \) and \( b'_n := \sup \{ \text{Res} : G'_n(s) = 0 \} \). Then, all the zeros of \( G_n(s) \) situated in the vertical strip \( \{ z \in \mathbb{C} : b'_n < \text{Re}z < b_n \} \) are simple.

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