SUPERALEGBRAIC INTERPRETATION OF THE
QUANTIZATION
MAPS OF WEIL ALGEBRAS

LI YU

Abstract. In [2], A. Alekseev and E. Meinrenken construct an explicit $G$-differential space homomorphism $Q$, called the quantization map, between the Weil algebra $\mathcal{W}_g = S(\mathfrak{g}^*) \otimes \wedge (\mathfrak{g}^*)$ and $\mathcal{W}_e = U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{g})$ (which they called the noncommutative Weil algebra) for any quadratic Lie algebra $\mathfrak{g}$. They showed that $Q$ induces an algebra isomorphism between the basic cohomology rings $H^{\text{bas}}_*(\mathcal{W}_g)$ and $H^{\text{bas}}_*(\mathcal{W}_e)$. In this paper, I will interpret the quantization map $Q$ as the super Duflo map between the symmetric algebra $S(\tilde{T}_\mathfrak{g}[1])$ and the universal enveloping algebra $U(\tilde{T}_\mathfrak{g}[1])$ of a super Lie algebra $\tilde{T}_\mathfrak{g}[1]$ which is canonically related to the quadratic Lie algebra $\mathfrak{g}$. The basic cohomology rings $H^{\text{bas}}_*(\mathcal{W}_g)$ and $H^{\text{bas}}_*(\mathcal{W}_e)$ correspond exactly to $S(\tilde{T}_\mathfrak{g}[1])^{\text{inv}}$ and $U(\tilde{T}_\mathfrak{g}[1])$ respectively. So what they proved is equivalent to the fact that the Duflo map commutes with the adjoint action of the Lie algebra, and that the Duflo map is an algebra homomorphism when restricted to the space of invariants. In addition, I will explain how the diagrammatic analogue of the Duflo map introduced in [6] can be also made for the quantization map $Q$.

Acknowledgements: First of all, I would like to thank my advisor J. Roberts and professor N. Wallach for their support and insightful advices. I would also like to thank professor E. Meinrenken for pointing out some mistakes I wrote in the draft. In addition, the argument in section 5.5 is due to Dylan Thurston. I also understand that several people including E. Meinrenken and P. Severa have already conjectured the main results of this paper and probably have their way of proving them. However, no formal proof had yet been published. So I think it is worth writing it out.

Date: May 1, 2005.
Department of Mathematics, University of California, San Diego, La Jolla, California 92093-0112, USA. Email: lyu@math.ucsd.edu.
1. Introduction

The classical Weil algebra $W(g)$ of a Lie algebra $g$ is introduced in the algebraic framework of equivariant geometry where people investigate the geometry and topology of group actions on smooth manifolds. As a vector space, $W(g)$ is just $S(g^*) \otimes \Lambda(g^*)$. It has a well-known structure as a $G$-differential algebra and is very useful in equivariant De Rham theory.

In 2000, A. Alekseev and E. Meinrenken introduced, for a Lie algebra with an ad-invariant metric (called a quadratic Lie algebra), the non-commutative version $W_g = U(g) \otimes Cl(g)$, which is also a $G$-differential algebra. They called it the noncommutative Weil algebra of $g$. They then constructed a map $Q : W_g \rightarrow W_g$, called the quantization map, between these two algebras which has three main properties:

- it is an isomorphism of vector spaces
- it is an isomorphism of $G$-differential spaces
- it is not a map of algebras, but it does induce an algebra isomorphism between the basic cohomologies of the two algebras.

However, their definition of $Q$ and their proof are both quite complicated, and it is difficult to understand the geometric meaning of their formulae.

Their theorem resembles another theorem, the Duflo isomorphism theorem in Lie theory. M. Duflo in [10] constructed, for any Lie algebra $g$, a map $\Upsilon : S(g) \rightarrow U(g)$ which has the properties:

- it is an isomorphism of vector spaces
- it is a map of $g$-modules
- it is not a map of algebras, but induces an algebra isomorphism between the spaces of invariants.

This theorem is highly non-trivial although for semisimple Lie algebras, it follows from the Weyl character formula.

In 2003 Dror Bar-Natan, Le Thang and Dylan Thurston proved in [6] yet another similar result, the "wheeling theorem". They constructed a map $\Phi : B \rightarrow A$ between certain spaces of diagrams appearing in Vassiliev theory, and proved that $\Phi$ is an algebra isomorphism with respect to some natural algebraic structures on these diagrams.

Finally, there is Kontsevich’s famous work [17] on deformation quantization of Poisson Manifolds. In his viewpoint, the Duflo map of a Lie algebra $g$ comes from the deformation of the Lie-Poisson structure of $g^*$. In addition, his proof can be generalized to the case of super Lie algebras.
In this paper I will study the relationships between these theorems. The main result is: the Alekseev-Meinrenken quantization map $Q$ can be identified with the Duflo map for a certain Lie superalgebra $\tilde{T}g[1]$, which is the Lie algebra of a central extension of the odd tangent bundle of $G$. Many properties of $Q$ can therefore be proved by using Kontsevich’s proof of the super-Duflo theorem, or by a diagrammatic proof using wheeling.

Here are some logical relations between these theorems:

- Kontsevich’s deformation quantization $\Rightarrow$ Duflo theorem (for any Lie superalgebras)
- Wheeling theorem $\Rightarrow$ Duflo theorem for quadratic Lie (super)algebras
- SuperDuflo $\Rightarrow$ Alekseev-Meinrenken theorem (this is the main theorem in this thesis)

The supergeometric interpretation of the non-commutative Weil algebra ought to be useful for future applications in geometry.

**Remark 1.1.** The noncommutative Weil algebra can so far only be defined for quadratic Lie algebras. It is unclear whether any of the results of Alekseev and Meinrenken can be extended to the non-quadratic case.

**1.1. Plan of Paper.** I first present some standard introductory materials to make this thesis as self-contained as possible in chapter 2. I will review some basic facts about $G$-differential algebras and the Weil algebra $W_g$ of a Lie algebra $g$ which are discussed extensively in [15]. A canonical $G$-differential structure and its slightly varied form are defined on $W_g$. In addition I will define the $G$-differential structure on Clifford algebra $Cl(g)$ and the noncommutative Weil algebra, which is introduced in [11] by A.Alekseev and E.Meinrenken. Then I introduce the Duflo map for a Lie (super)algebra and the quantization map of the Weil algebra $W_g$.

In chapter 3, I will build a critical connection between the $G$-differential structure on the Weil algebra $W_g$ and the Lie super algebra structure on $\tilde{T}g[1]$, which can help us to understand the quantization map by the theory of Lie algebras. Then I state the main theorem of this thesis.

In chapter 4, I will present a proof of the main theorem. To do that, I have to first discuss spin representations for Clifford algebras and its generalization to a super algebra. Then I discuss the factorization of the spin representation constructed by A.Alekseev and E.Meinrenken in [3]. The proof of the main theorem is put at the end of the chapter 4.
In Chapter 5, I will introduce Jacobi diagrams and diagrammatic representation of tensors in the category of Lie (super) algebras. People can find the standard exposition of these in [5] and [6]. In Chapter 6, I will explain how the method of using Jacobi diagrams to prove the Duflo isomorphism in [6] could be naturally extended to interpret the quantization map.

2. Definitions and Preliminary facts

2.1. G-differential algebras. Suppose $G$ is a Lie group with Lie algebra $\mathfrak{g}$. Choose a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$ and let $e^1, \ldots, e^n$ be the dual basis in $\mathfrak{g}^*$. Let $\{f_{ab}^c\}$ be the structure constants defined by

$$[e_a, e_b] = f_{ab}^c e_c$$

define $\hat{\mathfrak{g}}$ to be the Lie super algebra

$$\hat{\mathfrak{g}} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

where $\mathfrak{g}_{-1}$ is an $n$-dimensional vector space with basis $\tau_1, \ldots, \tau_n$, $\mathfrak{g}_0$ is an $n$-dimensional vector space with basis $L_1, \ldots, L_n$ and $\mathfrak{g}_1$ is a one-dimensional vector space with basis $d$. The super Lie bracket is defined in terms of this basis by

$$[\tau_a, \tau_b] = \tau_a \tau_b + \tau_b \tau_a = 0,$$
$$[L_a, \tau_b] = L_a \tau_b - \tau_b L_a = f_{ab}^c \tau_c,$$
$$[L_a, L_b] = L_a L_b - L_b L_a = f_{ab}^c L_c,$$
$$[d, \tau_a] = dt_a + \tau_a d = L_a,$$
$$[d, L_a] = dL_a - L_a d = 0,$$
$$[d, d] = 2d^2 = 0.$$

Recall that a Lie super algebra is just a $\mathbb{Z}$-graded vector space

$$V = \bigoplus_{i \in \mathbb{Z}} V_i$$

equipped with a bracket operation

$$[\ , \ ] : V_i \times V_j \longrightarrow V_{i+j}$$

which is super anti-commutative in the sense that

$$[x, y] + (-1)^{ij}[y, x] = 0, \ \forall x \in V_i, y \in V_j$$

and satisfies the super Jacobi identity

$$[x, [y, z]] = [[x, y], z] + (-1)^{ij}[y, [x, z]], \ \forall x \in V_i, y \in V_j.$$
It is easy to see that the bracket relations defined above do give a super Lie algebra structure on \( \hat{\mathfrak{g}} \). See [9] for more details on super vector spaces and super Lie algebras.

**Definition 2.1.** A **differential space** is a super vector space \( E \) with a differential, i.e. an odd endomorphism \( d \in \text{End}(E) \) satisfying \( d \circ d = 0 \). Endomorphisms of \( E \) that commute with the differential \( d \) will be called chain maps or differential space homomorphisms. We use \( H^*(E, d) \) to denote the cohomology of \( E \) with respect to \( d \). \( (E, d) \) is called **acyclic** if

\[
H^k(E, d) = \begin{cases} 
\mathbb{R} & k = 0 \\
0 & k \neq 0 
\end{cases}
\]

**Definition 2.2.** A **homotopy operator** between two chain maps \( \varphi_1, \varphi_2 : E \to E' \) is an odd linear map \( h \) (if \( E \) and \( E' \) are \( \mathbb{Z} \)-graded, we require \( h \) to be of degree -1) such that \( dh + hd = \varphi_1 - \varphi_2 \). If \( \varphi_1, \varphi_2 \) are chain homotopic, they induce the same map in cohomology.

If we have a Lie group \( G \) acting on a differential space \( V \), we introduce a notion of **\( G \)-differential space**.

**Definition 2.3.** A **\( G \)-differential space** is a super vector space \( V \), together with a super Lie algebra homomorphism \( \rho : \hat{\mathfrak{g}} \to \text{End}(V) \). The horizontal subspace \( V_{\text{hor}} \) is the space fixed by the action of \( \mathfrak{g}_{-1} \) under \( \rho \), the invariant subspace \( V^G \) is the space fixed by \( \mathfrak{g}_0 \), and the space \( V_{\text{basic}} \) of basic elements is \( V_{\text{hor}} \cap V^G \). It is easy to see that \( d : V_{\text{basic}} \to V_{\text{basic}} \), i.e. \( V_{\text{basic}} \) is a differential subspace of \( V \), we call the the cohomology of \( (V_{\text{basic}}, d) \) the **basic cohomology** of the \( G \)-differential space \( V \), denoted by \( H^*_\text{bas}(V) \).

A **\( G \)-differential algebra** is a super algebra \( B \) with a structure of a \( G \)-differential space such that \( \rho \) takes values in the derivation space \( \text{Der}(B) \) of the algebra.

**Example 2.4.** \( \hat{\mathfrak{g}} \) is a \( G \)-differential space with respect to the adjoint action of itself.

**Example 2.5.** Suppose a Lie group \( G \) acts on a smooth manifold \( M \), i.e. we have a group homomorphism \( \rho : G \to \text{Diff}(M) \). Then the infinitesimal action \( d\rho : \mathfrak{g} \to \text{Vect}(M) \) give a representation of the Lie algebra \( \mathfrak{g} \) of \( G \). For \( \forall \xi \in \mathfrak{g} \), let \( X_\xi \) be the vector field on \( M \) corresponds \( \xi \) under \( d\rho \). Let \( L_\xi \) and \( \iota_\xi \) be the Lie derivative and interior product of \( X_\xi \) in the algebra \( \Omega^*(M) \) of smooth differential forms. Then \( \Omega^*(M) \) is a \( G \)-differential algebra with the action of \( \hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) by \( \iota_\xi \), \( d \) and \( L_\xi \). This example is actually the geometric origin of the concept of \( G \)-differential algebras.
Example 2.6. Choosing a basis $\theta^1, \ldots, \theta^n$ of $g^*$, we can make the exterior algebra $\wedge(g^*)$ a $G$-differential algebra by defining:

$$\iota_a \theta^b = \delta^b_a,$$

(9)

$$L_a \theta^b = -f^b_{ab} \theta^c;$$

(10)

$$d \theta^a = -\frac{1}{2} f^a_{bc} \theta^b \theta^c$$

(11)

Please note here, I omit the $\wedge$ symbol between two odd elements. In my thesis, I will always omit it as long as there is no confusion. For a compact Lie group $G$, its De Rham cohomology $H^*_{DR}(G)$ coincides with $H^*(\wedge(g^*), d)$ defined above. So in general $(\wedge(g^*), d)$ is not acyclic.

Similarly, we can define the notion of homomorphism and homotopy in the category of $G$-differential spaces.

Definition 2.7. A $G$-homomorphism between $G$-differential spaces $(V_1, \rho_1)$ and $(V_2, \rho_2)$ is a homomorphism of super vector spaces $\phi : V_1 \to V_2$ that commutes with the actions of $\hat{g}$ on $V_1$ and $V_2$, i.e. for $\forall x \in \hat{g}, v \in V_1$, $\phi(\rho_1(x) \cdot v) = \rho_2(x) \cdot \phi(v)$.

Definition 2.8. Two $G$-homomorphism $\phi_1, \phi_2$ between two $G$-differential spaces $V_1$ and $V_2$ is called $G$-chain homotopic if there is an odd linear map $h : V_1 \to V_2$ such that $\phi_1 - \phi_2 = dh + hd$, $\iota_a h + h \iota_a = 0$, and $L_a h - h L_a = 0$.

2.2. Koszul complex. Let $V$ be an $n$-dimensional vector space, and let $\wedge(V)$ be the exterior algebra of $V$. Koszul algebra $K_V$ is the tensor product $\wedge(V) \otimes S(V)$. The elements $x \otimes 1 \in \wedge^1(V) \otimes S^0(V)$ and $1 \otimes x \in \wedge^0(V) \otimes S^1(V)$ generate $E_V$. The Koszul operator $d_K$ is defined as the derivation extending the operator on the generators given by

$$d_K(x \otimes 1) = 1 \otimes x$$

$$d_K(1 \otimes x) = 0$$

Clearly $d_K^2 = 0$ on generators, and hence everywhere, since $d_K^2$ is a derivation. In addition, $K_V$ can be naturally graded by the sum of the natural gradings of $\wedge(V)$ and $S(V)$. It is easy to see that $(K_V, d_K)$ is an acyclic space. Let $x_1, \ldots, x_n$ be a basis of $V$ and define

$$\theta_i := x_i \otimes 1$$

$$z_i := 1 \otimes x_i$$

Then $d = d_K$ is expressed in terms of these generators as

$$d \theta_i = z_i$$

$$dz_i = 0$$
2.3. Weil algebra and its G-differential structures. For any Lie algebra \( \mathfrak{g} \), we define the Weil algebra \( W_\mathfrak{g} = S(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*) \). It has a natural \( G \)-differential algebra structure. Let \( \nu^a = e^a \otimes 1 \) and \( \theta^a = 1 \otimes e^a \). They generate the \( S(\mathfrak{g}^*) \) and \( \wedge(\mathfrak{g}^*) \) in \( W_\mathfrak{g} \), respectively. We define the action of \( \hat{\mathfrak{g}} \) on \( W_\mathfrak{g} \) by

\[
L_a \nu^b = -f_{ak}^b \nu^k, \\
L_a \theta^b = -f_{ak}^b \theta^k, \\
\iota_a \nu^b = 0, \\
\iota_a \theta^b = \delta_a^b, \\
d\nu^a = -f_{bc}^a \theta^b \nu^c, \\
d\theta^a = \nu^a - \frac{1}{2}f_{bc}^a \theta^b \theta^c.
\]

(12) (13) (14) (15) (16) (17)

Equation (12) and (13) mean that \( W_\mathfrak{g} \) is a Koszul complex generated by \( \nu^a \) and \( \theta^a \). So it is acyclic.

Proposition 2.9. \((W_\mathfrak{g}, d)\) is an acyclic differential space.

Proof. The easiest way to see this is considering a variable change in \( W_\mathfrak{g} \). Let

\[
\tilde{\nu}^a = \nu^a - \frac{1}{2}f_{jk}^a \theta^j \theta^k.
\]

(18)

Extend this naturally to all elements in \( W_\mathfrak{g} \). Observe that \( \tilde{\nu}^1, \ldots, \tilde{\nu}^n, \theta^1, \ldots, \theta^n \) also generates \( W_\mathfrak{g} \). The induced \( G \)-differential structure under this set of generators is

\[
L_a \tilde{\nu}^b = -f_{ak}^b \tilde{\nu}^k, \\
L_a \theta^b = -f_{ak}^b \theta^k, \\
\iota_a \tilde{\nu}^b = -f_{ak}^b \theta^k, \\
\iota_a \theta^b = \delta_a^b, \\
d\tilde{\nu}^a = 0, \\
d\theta^a = \tilde{\theta}^a.
\]

(19) (20) (21) (22) (23) (24)

Equation (23) and (24) mean that \( W_\mathfrak{g} \) is a Koszul complex generated by \( \tilde{\nu}^a \) and \( \theta^a \). So it is acyclic.

□
We will find that this different presentation of the $G$-differential structure on $W(\mathfrak{g})$ in (19)—(24) is useful in next chapter. So we have the following definition.

**Definition 2.10.** Let $W^K_\mathfrak{g} = S(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*)$ with generators $\hat{v}^a$ and $\theta^a$, and define its $G$-differential structure by (19)—(24). We call it Koszul $G$-differential structure of $W_\mathfrak{g}$.

By definition, $W_\mathfrak{g}$ and $W^K_\mathfrak{g}$ are isomorphic as $G$-differential algebras. The isomorphism is just the super variable change.

$$\tau_0 : W^K_\mathfrak{g} \rightarrow W_\mathfrak{g} \quad \hat{v}^a \mapsto v^a - \frac{1}{2} f^a_{jk} \theta^j \theta^k$$

(25)

### 2.4. $G$-differential structure of Clifford algebra

Suppose a Lie algebra $\mathfrak{g}$ is equipped with an $ad(\mathfrak{g})$-invariant inner product $B(,)$, we call this type of Lie algebra *quadratic Lie algebra*. Semisimple Lie algebras and compact Lie algebras are all quadratic. However, not every Lie algebra allows an invariant inner product. From now on, we always assume $\mathfrak{g}$ is quadratic. Let $Cl(\mathfrak{g})$ be the Clifford algebra of $\mathfrak{g}$, i.e. the quotient of the tensor algebra $T(\mathfrak{g})$ by the ideal generated by all $2x \otimes x - B(x,x)$ with $x \in \mathfrak{g}$. It inherits a natural $\mathbb{Z}_2$-grading from the tensor algebra and a filtration,

$$\mathbb{R} = Cl^{(0)}(\mathfrak{g}) \subset Cl^{(1)}(\mathfrak{g}) \subset \ldots$$

So $Cl(\mathfrak{g})$ is a filtered super algebra. The associated graded algebra $Gr^*(Cl(\mathfrak{g}))$ is isomorphic to $\wedge(\mathfrak{g})$. Let elements in $\mathfrak{g}$ have odd grading and define a bracket operation in $\mathfrak{g}^{odd} := \mathfrak{g} \oplus \mathbb{R}\mathfrak{c}$ by

$$[x, y]_{odd} = B(x,y) \mathfrak{c}, \quad \forall x, y \in \mathfrak{g}$$

$$[x, \mathfrak{c}]_{odd} = 0, \quad \forall x \in \mathfrak{g}$$

Here $\mathfrak{c}$ is an even element. We use a subscript $odd$ to distinguish this bracket from the original Lie bracket of $\mathfrak{g}$. Then we can look on $Cl(\mathfrak{g})$ as the universal enveloping algebra of $\mathfrak{g} \oplus \mathbb{R}\mathfrak{c}$ modulo $\mathfrak{c} = 1$. The odd bracket of $\mathfrak{g}^{odd}$ can be extended naturally to $Cl(\mathfrak{g})$. The general formula is :

$$[x_{a_1}, \ldots, x_{a_k}, x_{b_1}, \ldots, x_{b_l}]_{odd}$$

$$= \sum_{\substack{1 \leq i \leq k \leq l \leq l}} (-1)^{k-i-j-1} [x_{a_i}, x_{b_j}]_{odd} x_{a_1} \ldots \hat{x}_{a_i} \ldots x_{a_k} x_{b_1} \ldots \hat{x}_{b_j} \ldots x_{b_l}$$

(26)
The Poincaré-Birkhoff-Witt map in this case is just the anti-symmetrization, which is also called Chevalley quantization map.

\[ q : \wedge(\mathfrak{g}) \rightarrow Cl(\mathfrak{g}) \]

\[ q(x_1 \wedge \ldots \wedge x_k) = \sum_{\sigma \in S_k} (-1)^{\text{sgn}(\sigma)} x_{\sigma(1)} \ldots x_{\sigma(k)} \]

Next, we make \( Cl(\mathfrak{g}) \) a \( G \)-differential algebra by first defining the action of \( \hat{g} \) on a basis of \( \mathfrak{g} \) and then extend it by derivations to the whole \( Cl(\mathfrak{g}) \) as follows. Suppose \( e_1, \ldots, e_n \) is an orthonormal basis of \( \mathfrak{g} \) with respect to the invariant metric \( B(\cdot,\cdot) \). If \( [e_a,e_b] = f_{ab}^c e_c \), then from the fact \( B([x_a, x_b], x_c) + B(x_b, [x_a, x_c]) = 0 \), we can easily see that \( f_{ab}^c = -f_{bc}^a \).

So under this basis, the structure constant \( f_{abc} \) is totally anti-symmetric with respect to its indices. For convenience, we use \( f_{abc} \) to denote \( f_{abc} \) under an orthonormal basis. In addition, we can naturally identify \( \mathfrak{g}^* \) and \( \mathfrak{g} \) using \( B(\cdot,\cdot) \).

**Proposition 2.11.** The above definition gives a \( G \)-differential algebra structure on \( Cl(\mathfrak{g}) \) and \( (Cl(\mathfrak{g}), d^{Cl}) \) has trivial cohomology in all filtration degrees for any quadratic Lie algebra \( \mathfrak{g} \).

**Proof.** See [2] chapter3. \( \square \)

So generally speaking, the exterior algebra \( \wedge(\mathfrak{g}^*) \) and the Clifford algebra \( Cl(\mathfrak{g}) \) are not isomorphic as differential spaces (unless \( \mathfrak{g} \) is abelian).

### 2.5. Noncommutative Weil algebra

In [2], A. Alekseev and E. Meinrenken defines a noncommutative version of Weil algebra, they call it non-commutative Weil algebra.

**Definition 2.12.** For a quadratic Lie algebra \( \mathfrak{g} \) with invariant metric \( B(\cdot,\cdot) \), if \( \hat{u}_1, \ldots, \hat{u}_n \) is a basis of \( \mathfrak{g} \) and \( \xi_1, \ldots, \xi_n \) be the corresponding basis of another copy \( \mathfrak{g}_1 \) of \( \mathfrak{g} \), the noncommutative Weil algebra \( \mathcal{W}_\mathfrak{g} \) of \( \mathfrak{g} \) is defined to the the quotient of the tensor algebra \( T(\mathfrak{g} \oplus \mathfrak{g}_1) \) by the following relations:

\[ \hat{u}_a \otimes \hat{u}_b - \hat{u}_b \otimes \hat{u}_a = [\hat{u}_a, \hat{u}_b]_{\mathfrak{g}}, \]

\[ \xi_a \otimes \xi_b - \xi_b \otimes \xi_a = B(\xi_a, \xi_b), \]

\[ \hat{u}_a \otimes \xi_b - \xi_b \otimes \hat{u}_a = [\xi_a, \xi_b]_{\mathfrak{g}_1} \]
We can define the $G$-differential algebra structure on the noncommutative Weil algebra of $\mathfrak{g}$ as follows: Suppose $\hat{u}_a$ and $\xi_a$ have degree 2 and 1, respectively. Then define:

\begin{align}
L_a \hat{u}_b &= f_{abc} \hat{u}_c, \\
L_a \xi_b &= f_{abc} \xi_c, \\
\iota_a \hat{u}_b &= f_{abc} \xi_c, \\
\iota_a \xi_b &= \delta_{ab}, \\
d\hat{u}_a &= 0, \\
d\xi_a &= \hat{u}_a
\end{align}

(30) (31) (32) (33) (34) (35)

If we identify $\mathfrak{g}^*$ and $\mathfrak{g}$ using the invariant metric on $\mathfrak{g}$, it is easy to see that the $G$-differential structure of $W^g$ defined above corresponds exactly to the $G$-differential structure of $W^K$. So we use $W^K$ to denote $W^g$ with this $G$-differential structure.

**Remark 2.13.** The naive extension of the map $v^a \mapsto u_a$, $\theta_a \mapsto \xi_a$ by symmetrization is not a $G$-differential algebra homomorphism from $W^K$ to $W^K$. It is only a linear map.

In addition, similar to Weil algebra, we can define a super variable change $\tau_1$:

\begin{align}
\hat{u}_a &= u_a - \frac{1}{2} f_{abc} \xi_b \xi_c \\
\xi_a &= \xi_a
\end{align}

(36)

Notice $u_a$ and $\xi_a$ also generate $W^g$, and they commute. In fact

\begin{align}
[u_a, \xi_b] = [\hat{u}_a + \frac{1}{2} f_{aij} \xi_i \xi_j, \xi_b] &= f_{abi} \xi_i + \frac{1}{2} f_{aij} \xi_i \delta_{jb} - \frac{1}{2} f_{aij} \xi_j \delta_{ib} \\
&= f_{abi} \xi_i + \frac{1}{2} f_{aij} \xi_i - \frac{1}{2} f_{aij} \xi_j = 0
\end{align}

(37)

Under this variable change, the induced $G$-differential structure on $W^g$ defined on generators $u_a$ and $\xi_a$ is:

\begin{align}
L_a u_b &= f_{abc} u_c, \\
L_a \xi_b &= f_{abc} \xi_c, \\
\iota_a u_b &= 0, \\
\iota_a \xi_b &= \delta_{ab}, \\
du_a &= - f_{abc} \xi_b u_c, \\
d\xi_a &= u_a - \frac{1}{2} f_{abc} \xi_b \xi_c
\end{align}

(38) (39) (40) (41) (42)
Obviously, this form of $G$-differential structure on $\mathcal{W}_g$ corresponds to the $G$-differential structure of $W_g$ defined in \[12\]– \[17\]. When we write $W_g$ without the capital $K$, we mean its generators are $u_a$ and $\xi_a$.

**Remark 2.14.** It is not hard to see that the bracket $[\hat{u}_a, \hat{u}_b]$ corresponds exactly to $[u_a, u_b]$ under the super variable change. Thus we can think of $u_a$ being a basis of a Lie algebra $g_0$ which isomorphic to $g$. Then as vector spaces, $W_g = W_g^K = U(g_0) \otimes Cl(g_1)$.

2.6. **Notation summary.** To avoid confusions, I summarize different notations of basis elements of $g$ or $g^*$ we use in this paper so far.

\[
\begin{array}{cccc}
g & g^* & S(g^*) & \wedge (g^*) \\
e_a & e^a & v^a, \tilde{v}^a & \theta^a \\
u_a, \hat{u}_a & \xi_a & & 
\end{array}
\]  

(43)

2.7. **Duflo isomorphism and Quantization map.** Every Lie algebra $g$ has two associated algebras: the symmetric algebra $S(g)$, generated by $g$ with relations $xy - yx = 0$, and the universal enveloping algebra $U(g)$, generated by $g$ with relations $xy - yx = [x, y]$. There is a natural map between the two algebras,

\[\chi : S(g) \to U(g)\]

given by taking a monomial $x_1x_2\ldots x_n$ in $S(g)$ and averaging over the product in $U(g)$ of all the $x_i$ in all possible orders. By the Poincaré-Birkhoff-Witt(PBW) theorem, $\chi$ is an isomorphism of vector spaces and $g$-modules. $\chi$ is not an algebra isomorphism, even restricted to the invariant subspaces of both sides. in \[10\]. M. Duflo modifies $\chi$ a little and gives an algebra isomorphism

\[\Upsilon : S(g)^g \to U(g)^g\]

where

\[\Upsilon = \chi \circ \partial_{j^{\frac{1}{2}}}(x) = det^{\frac{1}{2}} \left( \frac{\sinh(\frac{1}{2}ad_x)}{\frac{1}{2}ad_x} \right) = det^{\frac{1}{2}} \left( \frac{e^{ad(x)/2} - e^{-ad(x)/2}}{ad(x)} \right)\]

(45)

The notation $ad_x$ is the adjoint action of an element $x$ on $g$. The $\partial_{j^{\frac{1}{2}}}$ means to consider $j^{\frac{1}{2}}(x)$ as a power series on $g$ and so we can think of it as an infinite-order differential operator on $g^*$, which we can then apply to polynomials on $g^*$ (i.e., elements in $S(g)$). $j^{\frac{1}{2}}$ is an important function in the theory of Lie algebras. Its square, $j(x)$, is the Jacobian of the exponential mapping from $g$ to its Lie group $G$ when $g$ is unimodular. The map $\Upsilon$ here is called Duflo isomorphism. When Lie algebra $g$ is semisimple, Duflo isomorphism coincides with the Harish-Chandra isomorphism. The general proof in \[10\] is highly nontrivial, it used...
certain facts about finite-dimensional Lie algebras which follow only from the classification theory. In addition, Duflo isomorphism is also true in the case of Lie super algebras, which is proved by M. Kontsevich in [17].

In addition, we have another way of understanding the Duflo map which is more convenient to use in many cases. We can identify $\mathcal{S}(\mathfrak{g})$ with the space of distributions which are supported at the origin of $\mathfrak{g}$. In fact, on any vector space $V$ with a basis $\{e_a\}$, let $x_a$ be the coordinate function of $e_a$, the algebra $\mathcal{E}_0(V)$ of distributions with support at the origin on $V$ is canonically isomorphic to its symmetric algebra $\mathcal{S}(V)$, by identifying each basis element $e_a$ with $-\frac{\partial}{\partial x_a}\delta_0$, where $\delta_0$ is the Dirac delta function at the origin of $V$.

$$e_a = -\frac{\partial}{\partial x_a}\delta_0,$$

The algebra structure on $\mathcal{S}(V)$ corresponds to the convolution $\ast_V$ of distributions (see [13] for details). In addition, any smooth function $f(x)$ can act on $\mathcal{E}_0(V)$ by

$$\langle f(x) \cdot D, \phi(x) \rangle = \langle D, f(x)\phi(x) \rangle, \quad \forall \ D \in \mathcal{E}_0(V), \phi(x) \in C_c^\infty(V).$$

Similarly for $U(\mathfrak{g})$, we can identify the generator $u_a$ with a distribution on $G$ which is supported at the identity element $1$ of $G$. We have

$$u_a = \frac{d}{dt} \bigg|_{t=0} \delta_{\exp(te_a)}$$

In this way, $U(\mathfrak{g})$ can be identified algebraically with the space of distributions $\mathcal{E}_1(G)$ on $G$ supported at the identity $1$ of $G$. The product in $U(\mathfrak{g})$ corresponds to the convolution product $\ast_G$ of distributions in $\mathcal{E}_1(G)$.

If we view $S(\mathfrak{g})$, $U(\mathfrak{g})$ and the action of $j^{\pm}(x)$ as above described way, the PBW map $\chi$ is interpreted as pushing forward distributions on $\mathfrak{g}$ to distributions on $G$ by the exponential map of $\mathfrak{g}$. In fact, this is the point of view adopted by M. Duflo in his original work [10]. So we can write Duflo’s theorem as follows:

$$\Upsilon(\eta) = \exp_s(j^{\pm}(x) \cdot \eta), \quad \forall \ \eta \in \mathcal{E}_0(\mathfrak{g}) = S(\mathfrak{g})$$

(46)

$$\Upsilon(\eta_1 \ast_\mathfrak{g} \eta_2) = \Upsilon(\eta_1) \ast_G \Upsilon(\eta_2), \quad \forall \ \eta_1, \eta_2 \in S(\mathfrak{g})^\mathfrak{g}$$

(47)

In fact, these two ways of understanding Duflo map are Fourier transform of each other in a canonical way.

More recently, A. Alekseev and E. Meinrenken establishes in [2] an interesting isomorphism $\mathcal{Q}$ between $\Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*)$ and $Cl(\mathfrak{g}) \otimes U(\mathfrak{g})$
for any quadratic Lie algebra \( \mathfrak{g} \). They called \( Q \) the quantization map. If we identify \( \mathfrak{g}^* \) and \( \mathfrak{g} \) using the metric, when restricted to the symmetric algebra \( \Lambda(\mathfrak{g}^*) \otimes 1 \), \( Q \) becomes the usual Duflo isomorphism from \( S(\mathfrak{g}) \) to \( U(\mathfrak{g}) \). When restricted to the exterior algebra \( \wedge(\mathfrak{g}^*) \otimes 1 \), \( Q \) becomes the antisymmetrization map \( q \) from \( \wedge(\mathfrak{g}) \) to \( \text{Cl}(\mathfrak{g}) \). However, \( Q \) is not just the direct product \( Duf \otimes q \). It has the more complicated form

\[
Q = (Duf \otimes q) \circ \exp \left( \frac{1}{2} T_{ab}(x) t_{a} t_{b} \right) = (\chi \otimes q) \circ \partial \frac{j}{2}(x) \circ \exp \left( \frac{1}{2} T_{ab}(x) t_{a} t_{b} \right)
\]

where \( T_{ab} \) is a certain anti-symmetric tensor field on \( \mathfrak{g} \), i.e. an \( \wedge^2(\mathfrak{g}) \) value function on \( \mathfrak{g} \). It was obtained by Etingof-Varchenko [11] as a solution of the classical dynamical Yang-Baxter equation.

\[
T_{ab}(x) := (\ln(j')(ad_x))_{ab}
\]

When acting on the elements in Weil algebra \( W(\mathfrak{g}) \), \( T_{ab}(x) \) is treated as a differential operator like \( j^\frac{1}{2}(x) \), \( t_{a} t_{b} \) is understood as contractions of odd variables.

We will see in Chapter 4 that, the quantization map \( Q \) can be understood as the super Duflo map of a super Lie algebra.

### 3. Supergeometric Interpretation of the Quantization Map

Suppose \( G \) is a Lie group with Lie algebra \( \mathfrak{g} \). Set \( TG[1] = G \times \mathfrak{g} \) which we look upon as a Lie group with multiplication given as follows

\[
(g, X)(h, Y) = (gh, Ad_h^{-1}(X) + Y), \quad \forall g, h \in G, X, Y \in \mathfrak{g}
\]

\( TG[1] \) is just the odd tangent bundle of \( G \) with the natural group structure induced from the group structure of \( G \). The Lie algebra, \( T\mathfrak{g}[1] \), of \( TG[1] \) is \( \mathfrak{g} \times \mathfrak{g} \) with bracket given by

\[
[(X, Y), (X', Y')] = ([X, X'], [Y, X'] + [X, Y']), \quad \forall X, X', Y, Y' \in \mathfrak{g}
\]

In \( T\mathfrak{g}[1] \), \( \mathfrak{g} \times 0 \) is a Lie subalgebra isomorphic to \( \mathfrak{g} \) and \( 0 \times \mathfrak{g} \) is an abelian Lie subalgebra. When \( \mathfrak{g} \) is a quadratic Lie algebra, we can make a central extension of the Lie algebra \( T\mathfrak{g}[1] \) using its invariant metric \( B( , , ) \),

\[
0 \longrightarrow \mathbb{R} \longrightarrow \widehat{T\mathfrak{g}[1]} \longrightarrow T\mathfrak{g}[1] \longrightarrow 0 \quad (48)
\]

And the Lie bracket of \( 0 \times \mathfrak{g} \) in \( T\mathfrak{g}[1] \) becomes

\[
[(0, X), (0, Y)] = B(X, Y)c
\]
where $c$ is a central element in $\tilde{Tg[1]}$. It generates the $\mathbb{R}$ in (48).

3.1. Relations between $S(\tilde{Tg[1]})$ and $W_g$ and $W_K^g$. If we identify $g^*$ and $g$ via the invariant metric $B(\cdot, \cdot)$, $W_K^g$ is then identified with $S(\tilde{Tg[1]})/ <c = 1 >$. In fact, we can think of $W_K^g$ as the space of differential forms on $g$ with polynomial coefficients. Let $\sigma_0 : S(\tilde{Tg[1]}) \rightarrow W_g$ be a map which make the following diagram commute:

$$
\begin{array}{c}
W_K^g \xrightarrow{\varepsilon = 1_{B(\cdot, \cdot)}} S(\tilde{Tg[1]}) \\
\downarrow \tau_0 \quad \downarrow \sigma_0 \\
W_g
\end{array}
$$

Since $\tau_0(v^a) = v^a - \frac{1}{2} f_{abc} \theta^b \theta^c$, if $e_a$ and $\tilde{e}_a$ are basis of $\tilde{Tg[1]}$ and $\tilde{Tg}[1]$ respectively, $\sigma_0$ must be:

$$
\sigma_0(e_a) = v^a - \frac{1}{2} f_{abc} \theta^b \theta^c \\
\sigma_0(\tilde{e}_a) = \theta^a
$$

The following lemma tells us how the super Lie algebra structure of $\tilde{Tg[1]}$ is related to the $G$-differential structure of $W_g$ under the map $\sigma_0$.

**Lemma 3.1.** The map $\sigma_0 : S(\tilde{Tg[1]}) \rightarrow W_g$ satisfies:

$$
\begin{align*}
\sigma_0([e_a, e_b]) &= L_a(\sigma_0(e_b)) 
\sigma_0([e_a, \tilde{e}_b]) &= L_a(\sigma_0(\tilde{e}_b)) \\
\sigma_0([v_b, e_a]) &= \iota_b(\sigma_0(e_a)) \\
\sigma_0([\tilde{e}_a, \tilde{e}_b]) &= \iota_a(\sigma_0(\tilde{e}_b))
\end{align*}
$$

**Proof.** Here I only give the proof of (49), the proof of the other two equations are similar. By the definition of $\sigma_0$, we have

$$
\sigma_0([e_a, e_b]) = \sigma_0(f_{abc} e_c) = f_{abc}(v^c - \frac{1}{2} f_{cqp} \theta^p \theta^q)
$$
On the other hand,

\[
L_a(\sigma_0(e_b)) = L_a(v^b - \frac{1}{2} f_{bij} \theta^i \theta^j) = f_{abc} v^c - L_a(\frac{1}{2} f_{bij} \theta^i \theta^j) \\
= f_{abc} v^c - \frac{1}{2} f_{bij} f_{aijk} \theta^k \theta^j - \frac{1}{2} f_{bij} f_{ajk} \theta^i \theta^k \\
= f_{abc} v^c - \frac{1}{2} f_{abk} f_{kij} \theta^i \theta^j
\]

The last equality is because of the Jacobi identity. So the (49) holds. □

By the above lemma, we have the following correspondence:

\[
\sigma_0(e_a) \leftrightarrow e_a, \quad L_a \leftrightarrow [e_a, \cdot], \quad \theta_a \leftrightarrow \overline{e}_a, \quad \iota_a \leftrightarrow \overline{e}_a.
\]

Since \( \sigma_0 \) is an algebra isomorphism, we can easily see the following.

**Proposition 3.2.** The basic complex of \( W^K_g \) corresponds exactly to the invariant symmetric tensor of \( T[g][1] \) under \( \sigma_0 \). Therefore, \( S(T[g][1])^{inv} \cong S(g)^g \).

\[
\sigma_0 : S(T[g][1])^{inv} \leftrightarrow (W_g)_{basic}
\]

In addition, it is easy to see that \( W_g = U(T[g][1])/\langle \epsilon = 1 \rangle \) by definition. So we get a similar diagram for the noncommutative Weil algebra \( W_g \) and \( W^K_g \).

\[
\begin{array}{c}
U(T[g][1]) \xrightarrow{\epsilon = 1} W^K_g \\
\sigma_1 \\
\tau_1 \\
W_g
\end{array}
\]

Where \( \tau_1 \) is the super variable change. The same argument as the preceding lemma shows that \( \sigma_1 \) maps \( U(T[g][1])^{inv} \) isomorphically onto \( (W_g)_{basic} \).

**3.2. Duflo isomorphism of \( T[g][1] \).** In chapter two, we have introduced the Duflo isomorphism of a Lie algebra and it actually makes sense for any super Lie algebras. Now, let us examine the Duflo isomorphism of \( T[g][1] \).
Let \( \{e_1, \ldots, e_n, \overline{e}_1, \ldots, \overline{e}_n, \epsilon\} \) be a basis \( \hat{T}\mathfrak{g}[1] \). By definition, \( \text{ad}(e_a) \) and \( \text{ad}(\overline{e}_a) \) are represented by the following matrices:

\[
\begin{align*}
\text{even} & \quad \text{odd} & \quad \epsilon \\
\text{ad}(e_a) &= \begin{pmatrix}
M_a & 0 & 0 \\
0 & M_a & 0 \\
0 & 0 & 0
\end{pmatrix} & \text{even} \\
\text{ad}(\overline{e}_a) &= \begin{pmatrix}
0 & 0 & 0 \\
M_a & 0 & 0 \\
\epsilon_a & 0 & 0
\end{pmatrix} & \text{odd}
\end{align*}
\]

Where \((M_a)_{ij} = f_{aji}\) and \(\epsilon_a = (0, \ldots, 0, 1, 0, \ldots, 0)\).

Then it is easy to see that for any element \( x \in \hat{T}\mathfrak{g}[1] \), \( \text{Tr}(\text{ad}^k(x)) \equiv 0 \) for \( \forall k \in \mathbb{N} \) (here \( \text{Tr} \) is the super trace, see \([9]\) for definition).

\[
\left(\frac{\sinh(\frac{1}{2} ad_x)}{\frac{1}{2} ad_x}\right) = \exp \left( \sum_{k=1}^{\infty} b_{2k} \text{Tr}(\text{ad}^k(x)) \right) = 1
\]

Where \( b_{2k} \)'s are modified Bernoulli numbers defined by the power series expansion

\[
\sum_{k=0}^{\infty} b_{2k} x^{2k} = \frac{1}{2} \ln \left( \frac{\sinh \frac{x}{2}}{\frac{x}{2}} \right).
\]  \(53\)

So we have proved the following.

**Proposition 3.3.** the Duflo isomorphism for \( \hat{T}\mathfrak{g}[1] \) is just the (super) symmetrization map.

3.3. **The Main Theorem.** I want to show that the quantization map between Weil algebra and noncommutative Weil algebra is essentially equivalent to the Duflo map for the super Lie algebra \( \hat{T}\mathfrak{g}[1] \). I put this in the main theorem as follows.
Main Theorem. The quantization map \( Q \) is the (super) Duflo isomorphism of the super Lie algebra \( \widetilde{T}_g[1] \), i.e. the following diagram commutes.

\[
\begin{array}{ccc}
W^K_g & \xrightarrow{\varepsilon = 1} & S(\widetilde{T}_g[1]) \\
\downarrow \tau_0 & & \downarrow \sigma_0 \\
W_g & \xrightarrow{Q} & W_g
\end{array}
\quad \text{vector space homo.}
\]

\[
\begin{array}{ccc}
U(\widetilde{T}_g[1]) & \xrightarrow{\varepsilon = 1} & \mathcal{W}^K_g \\
\downarrow \sigma_1 & & \downarrow \tau_1 \\
\mathcal{W}_g & \xrightarrow{\text{vector space homo.}} & \mathcal{W}_g
\end{array}
\quad \text{Duflо vector space homo.}
\]

(54)

Remark 3.4. When we apply \( Q \) to elements in \( W^K_g \), we use the metric \( B(\, , \, ) \) to identify \( W_g \) with \( S(g) \otimes \wedge(g) \).

Corollary 3.5. The basic cohomology of \( W_g \) is algebraically isomorphic to the basic cohomology of \( \mathcal{W}_g \).

Proof. Since Duflo map is an algebra isomorphism when restricted to the invariants and \( \sigma_0, \sigma_1 \) in the diagram are also algebra homomorphisms, so the quantization map \( Q|_{(W_g)_{\text{basic}}} \) is an algebra homomorphism by the commutativity of the diagram above. So we have the following diagram.

\[
\begin{array}{ccc}
S(\widetilde{T}_g[1])^{\text{inv}} & \xrightarrow{\text{Duflо algebra homo.}} & U(\widetilde{T}_g[1])^{\text{inv}} \\
\downarrow \sigma_0 & & \downarrow \sigma_1 \\
(W_g)_{\text{basic}} & \xrightarrow{Q} & (W_g)_{\text{basic}}
\end{array}
\]

(55)

Notice the differential \( d \) is actually trivial on \( (W_g)_{\text{bas}} \), so we have \( H^*_{\text{bas}}(W_g) = (W_g)_{\text{basic}} = S(g^*)^g \). Therefore the quantization map \( Q \) induces an algebra homomorphism in the basic cohomology. \( \square \)

Remark 3.6.

(1) The action of the quantization map \( Q \) on \( H^*_{\text{bas}}(W_g) \) is just the usual Duflo map for the Lie algebra \( g \). Although this is a little disappointing, we will see in the last chapter that the quantization map \( Q \) itself has an interesting diagrammatic interpretation.
The above corollary can also be derived from a very strong theorem proved in [1] which asserts that any $G$-differential space homomorphism from the Weil algebra of $\mathfrak{g}$ to a locally free $G$-differential algebra (may not be commutative) always induces an algebra homomorphism in basic cohomology.

4. Proof of the Main Theorem

In this chapter, I will present an algebraic proof of the main theorem in this thesis. To do that, we need to investigate the structure of the spin representation of the spin group. The techniques in the proof are described by A. Alekseev and E. Meinrenken in [1], [2], [3]. So I will quote some theorems directly from their papers without giving the proof since they are quite complicated themselves. First, let’s look at the spin representation.

4.1. Spin group of $\mathfrak{g}$ and spin representation. In Chapter two, I have introduced the Clifford algebra of a quadratic Lie algebra $\mathfrak{g}$ and defined a $G$-differential structure on it. In this section, I will discuss an important object $\text{Spin}(\mathfrak{g})$ in $\text{Cl}(\mathfrak{g})$ which will be used in the proof of the main theorem. People can find more details of Clifford algebra and $\text{Spin}(\mathfrak{g})$ and its representations in [19], [14], [7] and [18]. Most of the definitions and properties of $\text{Cl}(\mathfrak{g})$ and $\text{Spin}(\mathfrak{g})$ introduced here can be easily generalized to the Clifford algebra of any vector space with a non-degenerate symmetric bilinear form.

4.2. The action of $\text{Cl}(\mathfrak{g})$ on $\wedge(\mathfrak{g})$. Using the ad-invariant inner product $B(\cdot, \cdot)$ on the Lie algebra $\mathfrak{g}$, we define an $\text{Cl}(\mathfrak{g})$-module structure on $\wedge(\mathfrak{g})$ as follows. For $\forall x \in \mathfrak{g}$, define

$$\varrho(x) \cdot \alpha = x \wedge \alpha + \frac{1}{2} l_x \alpha \quad \forall \alpha \in \wedge(\mathfrak{g})$$

where $l_x(y_1 \wedge \ldots \wedge y_k) = \sum_{i=1}^{k} (-1)^{i-1} B(x, y_i) y_1 \wedge \ldots \hat{y}_i \ldots \wedge y_k$. It is easy to see that $\varrho(x_1) \varrho(x_2) + \varrho(x_2) \varrho(x_1) = B(x_1, x_2)$. So by the universal property of $\text{Cl}(\mathfrak{g})$, $\varrho$ could be extended to the whole $\text{Cl}(\mathfrak{g})$ by:

$$\varrho(x_1 x_2 \ldots x_s) (\alpha) = \varrho(x_1) \circ \varrho(x_2) \ldots \circ \varrho(x_s) (\alpha).$$

Lemma 4.1. The inverse of quantization map $q : \wedge(\mathfrak{g}) \to \text{Cl}(\mathfrak{g})$ can be expressed in terms of $\varrho$ as $q^{-1}(x) = \varrho(x) \cdot 1$ for $\forall x \in \text{Cl}(\mathfrak{g})$. 
Proof. Under an orthonormal basis $e_1, \ldots, e_n$ of $\mathfrak{g}$, it is easy to see that, for any $1 \leq i_1 < \ldots < i_k \leq n$,

\[
\vartheta(e_{i_1} \ldots e_{i_k}) \cdot 1 = \vartheta(e_{i_1} \ldots e_{i_{k-1}}) \cdot e_{i_k} \\
= \vartheta(e_{i_1} \ldots e_{i_{k-2}}) \cdot (e_{i_{k-1}} \wedge e_{i_k}) \\
= \ldots = e_{i_1} \wedge \ldots \wedge e_{i_k}.
\]

and conversely

\[
\varrho(e_{i_1} \wedge \ldots \wedge e_{i_k}) = e_{i_1} \ldots e_{i_k}
\]

So the lemma follows from the fact that any element in $Cl(\mathfrak{g})$ is a linear combination of $e_{i_1} \ldots e_{i_k}$ as above. □

People often call $q^{-1}$ symbol map. It gives the isomorphism from the associated graded algebra $Gr(Cl(\mathfrak{g}))$ to $\wedge(\mathfrak{g})$.

4.3. Definition of Spin group and spin representation. We can think of $Cl(\mathfrak{g})$ itself as a finite dimensional Lie algebra with the odd bracket defined by (26). Notice the subspace $Cl^{(2)}(\mathfrak{g})$ is actually closed under the odd bracket, so $Cl^{(2)}(\mathfrak{g})$ is a Lie subalgebra of $Cl(\mathfrak{g})$. In addition, let $SO(\mathfrak{g})$ be the special orthogonal group with respect to the inner product of $\mathfrak{g}$. Its Lie algebra is denoted by $so(\mathfrak{g})$.

Lemma 4.2. $Cl^{(2)}(\mathfrak{g})$ is isomorphic to the Lie algebra $so(\mathfrak{g})$.

Proof. We can identify $\mathfrak{g}$ as $Cl^{(1)}(\mathfrak{g})$ and let $\tau_{\mathfrak{g}} : Cl^{(2)}(\mathfrak{g}) \rightarrow so(\mathfrak{g})$ be defined by:

\[
\tau_{\mathfrak{g}}(a) \cdot v = [a, x]_{odd}, \quad \forall \ a \in Cl^{(2)}(\mathfrak{g}), \ x \in \mathfrak{g}
\]

It is easy to check $\tau(a)$ does preserve $Cl^{(1)}(\mathfrak{g})$ and so defines a Lie algebra homomorphism from $Cl^{(2)}(\mathfrak{g})$ to $\mathfrak{gl}(\mathfrak{g})$. To see $\tau_{\mathfrak{g}}(a)$ is in $so(\mathfrak{g})$, observe that

\[
B(\tau_{\mathfrak{g}}(a) \cdot x, y) + B(x, \tau_{\mathfrak{g}}(a) \cdot y) = [ [a, x]_{odd}, y]_{odd} + [ x, [a, y]_{odd}]_{odd}
\]

\[
= -[a, [x, y]_{odd}]_{odd} = 0
\]

The second equality is a consequence of the Jacobi identity in $(Cl(\mathfrak{g}), [\cdot, \cdot]_{odd})$.

The map $\tau_{\mathfrak{g}}$ must be an isomorphism, since it is injective by the non-degeneracy of the metric and since the dimensions of $Cl^{(2)}(\mathfrak{g})$ and $so(\mathfrak{g})$ are the same, namely $n(n - 1)/2$. □
Using the map $\tau_g$, any skew-symmetric matrix $A = (a_{ij}) \in \mathfrak{so}(\mathfrak{g})$ under an orthonormal basis $e_1, \cdots, e_n$ corresponds to the Clifford element

$$\tau^{-1}_g(A) = q\left(\frac{1}{2} \sum_{i,j} (Ae_i) \wedge e_j\right) = q\left(\frac{1}{2} \sum_{i,j} a_{ji} e_i \wedge e_j\right)$$

$$= q\left(-\frac{1}{2} \sum_{i,j} a_{ij} e_i \wedge e_j\right) = -\sum_{i<j} a_{ij} \xi_i \xi_j \tag{56}$$

**Definition 4.3.** The group $\text{Spin}(\mathfrak{g})$ is the group obtained by exponentiating the Lie algebra $Cl^{(2)}(\mathfrak{g})$ inside the Clifford algebra $Cl(\mathfrak{g})$. The restriction of $\varrho$ on $\text{Spin}(\mathfrak{g})$ is called the Spin representation of $\text{Spin}(\mathfrak{g})$.

The action $\tau_g$ of $Cl^{(2)}(\mathfrak{g})$ on $\mathfrak{g}$ exponentiates to an orthogonal action still denoted by $\tau_g$. So we have the following diagram.

$$\begin{array}{ccc}
Cl^{(2)}(\mathfrak{g}) & \overset{\exp}{\longrightarrow} & \text{Spin}(\mathfrak{g}) \subset Cl(\mathfrak{g}) \\
\tau_g \downarrow & & \tau_g \\
\mathfrak{so}(\mathfrak{g}) & \overset{\exp}{\longrightarrow} & \text{SO}(\mathfrak{g})
\end{array} \tag{57}$$

It is well known that $\tau_g : \text{Spin}(\mathfrak{g}) \longrightarrow \text{SO}(\mathfrak{g})$ is a double covering map. But $\text{Spin}(\mathfrak{g})$ is much harder to handle than $\text{SO}(\mathfrak{g})$. For $\text{SO}(\mathfrak{g})$, we have a very nice representation in terms of matrices. We can investigate the structures of $\text{SO}(\mathfrak{g})$ using all kinds of decompositions of matrices. However, it is not easy to see how to factor the spin representation of an arbitrary element in $\text{Spin}(\mathfrak{g})$. In \cite{3}, A.Alekseev and E.Meinkenren constructed a very special factorization of $\text{Spin}(\mathfrak{g})$ which is interesting in many senses. We will see that the proof of the main theorem has to use this factorization.

**Remark 4.4.** The discussions above can be applied to any vector space $V$ with a non-degenerate symmetric bilinear form.

4.4. **Factorization of spin representation of $\text{Spin}(\mathfrak{g})$.** For any $x \in \text{Spin}(\mathfrak{g})$, the spin representation $\varrho(x)$ of $x$ is a linear transformation on the vector space $\wedge(\mathfrak{g})$. In this section, we will write $\varrho(x)$ as the product of two special types of linear transformations on $\wedge(\mathfrak{g})$. We call this factorization of spin representation. The two special transformations are:

1. $\wedge(\mathfrak{g}) \longrightarrow \wedge(\mathfrak{g})$, $\alpha \mapsto \beta \wedge \alpha$ for a fixed $\beta \in \wedge(\mathfrak{g})$
2. $\wedge(\mathfrak{g}) \longrightarrow \wedge(\mathfrak{g})$, $\alpha \mapsto \iota_\gamma(\alpha)$ for a fixed $\gamma \in Cl(\mathfrak{g})$

where $\iota_{x_1 \cdots x_s}(\alpha) = \iota_{x_1} \circ \cdots \circ \iota_{x_s}(\alpha)$. 
It is not clear how to do the factorization directly. So let us first introduce an auxiliary space. Since $\mathfrak{g}$ is a finite-dimensional real vector space, the direct sum $W = \mathfrak{g} \oplus \mathfrak{g}^*$ carries a natural non-degenerate bilinear form

$$B_W(x, y) = 0, \forall x, y \in \mathfrak{g},$$
$$B_W(\alpha, \beta) = 0, \forall \alpha, \beta \in \mathfrak{g}^*,$$
$$B_W(x, \alpha) = 2\alpha(x), \forall x \in \mathfrak{g}, \alpha \in \mathfrak{g}^* \quad (58)$$

Let $Cl(W)$ be the Clifford algebra of $(W, B_W)$. The above discussion of $Cl(\mathfrak{g})$ can be applied to $Cl(W)$ without any change. So we have the Lie algebra isomorphism $\tau_W : Cl^{(2)}(W) \rightarrow \mathfrak{so}(W)$ and the double cover of $SO(W)$ by $Spin(W)$.

We can define an algebra representation $\pi$ of $Cl(W)$ on $\wedge(\mathfrak{g})$ by:

$$\pi : Cl(W) \rightarrow \mathfrak{gl}(\wedge(\mathfrak{g}))$$

Where generators $x \in \mathfrak{g}$ act by wedge product and generators $\alpha \in \mathfrak{g}^*$ act by contraction (denoted again by $\iota_\alpha$).

Next, Let $\bar{\mathfrak{g}}$ be the same vector space as $\mathfrak{g}$ with metric $\bar{B}(\cdot, \cdot) = -B(\cdot, \cdot)$. Then we can construct an linear map $\kappa$ between $\mathfrak{g} \oplus \bar{\mathfrak{g}}$ and $W$ as follows.

$$\kappa : \mathfrak{g} \oplus \bar{\mathfrak{g}} \rightarrow W, (x, y) \mapsto \left( x + y, B\left(\frac{1}{2}(x - y), -\right) \right)$$

Assume the metric on $\mathfrak{g} \oplus \bar{\mathfrak{g}}$ is just the direct sum of their metrics. We have the following lemma.

**Lemma 4.5.** $\kappa$ is an isometry between $\mathfrak{g} \oplus \bar{\mathfrak{g}}$ and $W$ with respect to their metrics, its inverse is $\kappa^{-1}(x, B(y, -)) = \left( \frac{1}{2}x + y, \frac{1}{2}x - y \right)$.

Using $\kappa$, we can identify $\mathfrak{g}$ with a subspace of $W$. So $Cl(\mathfrak{g})$ can be thought of as a subalgebra of $Cl(W)$. Correspondingly, $SO(\mathfrak{g})$ and $Spin(\mathfrak{g})$ are subgroups of $SO(W)$ and $Spin(W)$ respectively. Notice the restriction of $\kappa$ to $\mathfrak{g}$ is:

$$\kappa|_{\mathfrak{g}} : \mathfrak{g} \rightarrow W, x \mapsto \left( x, B\left(\frac{1}{2}x, -\right) \right)$$

So the representation $\rho : Cl(\mathfrak{g}) \rightarrow \mathfrak{gl}(\wedge(\mathfrak{g}))$ is simply the restriction of $\pi : Cl(W) \rightarrow \mathfrak{gl}(\wedge(\mathfrak{g}))$ to $Cl(\mathfrak{g})$.

The inclusion $h : SO(\mathfrak{g}) \rightarrow SO(W)$ is given by:

$$h : SO(\mathfrak{g}) \rightarrow SO(W), C \mapsto \kappa \circ \left( \begin{array}{cc} C & 0 \\ 0 & I \end{array} \right) \circ \kappa^{-1} = \left( \begin{array}{cc} \frac{1}{4}(C + I) & \frac{1}{4}(C - I) \\ \frac{1}{4}(C - I) & \frac{1}{4}(C + I) \end{array} \right).$$

Let $C = \exp(tA)$ where $A \in \mathfrak{so}(\mathfrak{g})$ and differential at $t = 0$, we get the inclusion $dh : \mathfrak{so}(\mathfrak{g}) \rightarrow \mathfrak{so}(W)$. 
\[ dh : \mathfrak{so}(g) \to \mathfrak{so}(W), \quad A \mapsto \kappa \circ \left( \begin{array}{cc} A & 0 \\ 0 & 0 \end{array} \right) \circ \kappa^{-1} = \left( \begin{array}{cc} 1 & A \\ \frac{1}{4}A & \frac{1}{2}A \end{array} \right). \]

Then we can write down how \( Cl^{(2)}(g) \) and \( Spin(g) \) are included in \( Cl^{(2)}(W) \) and \( Spin(W) \). However, this won’t directly give any decomposition of the spin representation of \( Spin(g) \).

In [3], A. Alekseev and E. Meinrenken constructed a very interesting factorization of the spin representation of \( Spin(g) \). The idea is to first factor \( SO(g) \) inside \( GL(W) \) and then lift the factorization to its double cover \( Spin(g) \) via the Lie algebra isomorphism \( \tau_W : \mathfrak{so}(W) \to Cl^{(2)}(W) \). Please see the original paper for more details. Here, I just list the results they proved in [3] for the future use. In fact, the statements in [3] make sense for any vector space with an inner product.

**Theorem 4.6.** (Proposition 5.1 in [3]) Let \( C \in SO(g) \) with \( \det(C - I) \neq 0 \), and suppose that \( D \in \mathfrak{so}(g) \) is invertible and commutes with \( C \). Then there is a unique factorization

\[
h(C) = \left( \begin{array}{cc} I & 0 \\ E_1 & I \end{array} \right) \left( \begin{array}{cc} I & D \\ 0 & I \end{array} \right) \left( \begin{array}{cc} I & 0 \\ E_2 & I \end{array} \right) \left( \begin{array}{cc} R & 0 \\ 0 & (R^{-1})^t \end{array} \right)
\]

such that \( E_1, E_2 \in \mathfrak{so}(g) \) and \( R \in GL(g) \) commute with \( C \) and \( D \). One finds

\[
E_1 = \frac{1}{2} C + I - \frac{1}{D}, \quad E_2 = \frac{1}{D^2} \left( \frac{C - C^{-1}}{2} - D \right), \quad R = \frac{D}{I - C^{-1}}.
\]

**Remark 4.7.** Notice \( D \in \mathfrak{so}(g) \) is invertible forces the dimension of \( g \) to be even. When the dimension of \( g \) is odd, we can define \( W \) to be \( g \oplus \mathbb{R} \oplus g^* \) instead and let the copy of \( \mathbb{R} \) act on \( \wedge(g) \) by scalar (see [3] and [14] for details). This won’t effect our formula in (59) anyway.

We know how to lift each component in the factorization (59) to \( Spin(W) \). In fact, under a basis \( e_1, \ldots, e_n, e^1, \ldots, e^n \) of \( W \), where \( \{e_a\} \) is an orthonormal basis with respect to the metric \( B( , ) \) on \( g \) and \( \{e^a\} \) is its dual basis in \( g^* \), the lifting rules are:

1. For a matrix \( D = (D_{ij}) \) which represents a skew-adjoint linear map from \( g^* \) to \( g \), we have

\[
\left( \begin{array}{cc} I & D \\ 0 & I \end{array} \right) \quad \text{lift} \quad \exp\left( -\frac{1}{2} \sum_{i,j} D_{ij} e_i e_j \right) \in Spin(W).
\]
(2) For a matrix $E = (E_{ij})$ which represents a skew-adjoint linear map from $\mathfrak{g}$ to $\mathfrak{g}^*$, we have

$$\begin{pmatrix} I & 0 \\ E & I \end{pmatrix} \xrightarrow{\text{lift}} \exp\left(-\frac{1}{2} \sum_{i,j} E_{ij} e^i e^j\right) \in \text{Spin}(W).$$

(3) For any matrix $R \in \text{GL}(\mathfrak{g})$, there is a natural inclusion

$$\text{GL}(\mathfrak{g}) \to \text{SO}(W), \quad R \mapsto \begin{pmatrix} R & 0 \\ 0 & (R^{-1})^* \end{pmatrix}.$$ 

The lifting of this matrix to $\text{Spin}(W)$ is the same as lifting it to the metaplectic group $\text{Mp}(2n, \mathbb{R})$ (See [12] for the definition). It is not easy to write down the lifting explicitly in this case. But we do know how its lifting acts on $\wedge(\mathfrak{g})$ defined by $\pi$, which is enough to do our job of factoring spin representation of $\text{Spin}(\mathfrak{g})$ here. Let $\hat{R}$ be an element in $\text{Mp}(n, \mathfrak{g})$ (considered as a subgroup of $\text{Spin}(W)$) covering $R \in \text{GL}(\mathfrak{g})$. Then its action on $\wedge(\mathfrak{g})$ is given by:

$$\pi(\hat{R}) \cdot \alpha = \frac{R \cdot \alpha}{|\det|^1/2(\hat{R})}. \tag{60}$$

where $|\det|^{1/2} : \text{Mp}(n, \mathbb{R}) \to \mathbb{R}^\times$ is a suitable choice of square root of $|\det| : \text{GL}(n, \mathfrak{g}) \to \mathbb{R}^+$, and $R \cdot \alpha$ is defined by the unique extension of $R \in \text{GL}(\mathfrak{g})$ to an algebra automorphism of $\wedge(\mathfrak{g})$.

By the definition of $\pi$, for $\forall \alpha \in \wedge(\mathfrak{g})$,

$$\pi(\exp\left(-\frac{1}{2} \sum_{i,j} D_{ij} e^i e^j\right)) \cdot \alpha = \exp\left(-\frac{1}{2} \sum_{i,j} D_{ij} e^i \wedge e^j\right) \land \alpha$$

$$\pi(\exp\left(-\frac{1}{2} \sum_{i,j} E_{ij} e^i e^j\right)) \cdot \alpha = \exp\left(-\frac{1}{2} \sum_{i,j} \iota_{E_{ij} e^i e^j}\right) \cdot \alpha \tag{61}$$

So let $\lambda(D) = -\frac{1}{2} \sum_{i,j} D_{ij} e_i \wedge e_j$ and $\gamma(E) = -\frac{1}{2} \sum_{i,j} E_{ij} e_i e_j$, we get the following theorem.

**Theorem 4.8.** (Proposition 5.2 in [3]) Suppose $\hat{C} \in \text{Spin}(\mathfrak{g})$ maps to $C \in \text{SO}(\mathfrak{g})$ with $\det(C - I) \neq 0$. Assume $h(C)$ has the factorization written in [50]. Then the operator $\varrho(\hat{C})$ on $\wedge(\mathfrak{g})$ has the following factorization:

$$\pi(\hat{C}) \cdot \alpha = \varrho(\hat{C}) \cdot \alpha = \frac{\exp(\lambda(D)) \exp(\lambda(E_1)) \exp(\gamma(D)) \exp(\gamma(E_2)) R \cdot \alpha}{|\det|^{1/2}(\hat{R})}. \tag{62}$$
In particular, when $\alpha = 1$, we get

$$\varrho(\hat{C}) \cdot 1 = q^{-1}(\hat{C}) = \frac{\exp\left(\lambda(E_1)\right)}{|\det|^{1/2}(\hat{R})} \cdot \exp(\lambda(D)).$$

(63)

**Remark 4.9.**

1. This factorization actually makes sense for any vector space with an inner product.
2. There are two ways to think of this decomposition. First, we can identify $\wedge(g)$ with (odd) functions over $g^*$ and understand the theorem as giving a decomposition of some differential operators on the odd vector space $g^*$. The second viewpoint is to think of $\wedge g$ as the space of (odd) distributions supported at the origin of $g$ (see chapter 2). Correspondingly, elements in the Clifford algebra $Cl(g)$ are thought of as the distributions with support at the identity element on a (super) Lie group of the (super) Lie algebra $(g, [\ , \ ]_{odd})$. So (63) just tells us what the action of $\frac{\exp(\lambda(E_1))}{|\det|^{1/2}(\hat{R})}$ on the distribution $\exp(\lambda(D))$ is under the quantization map (which can be thought of as pushing forward of the distribution by the exponential map).
3. We will see that if we set $C = \exp(ad\mu)$ and $D = ad\mu$, the theorem gives the expression of our quantization map $Q$.

4.5. **Proof of the Main theorem.** The basic idea is to apply the factorization of spin representation we established in the last chapter to some special elements in $Spin(g)$ and get a highly nontrivial relation in the Clifford algebra. Now let us begin.

**Proof.** Recall in chapter 2, we used $v^a$ and $\theta^a$ to denote the generators of $S(g^*)$ and $\wedge g^*$ in $W_{g_0}$. And we use $\hat{v}^a$ and $\hat{\theta}^a$ to denote the generators of $S(\hat{g}^*)$ and $\wedge \hat{g}^*$ in $W_{\hat{g}}^{even}$. Let their dual generators in $S(g)$ and $\wedge (g)$ be $e_a$, $\hat{e}_a$ and $\bar{e}_a$. By definition of $v^a$ and $\hat{v}^a$, we have $\hat{e}_a = e_a - \frac{1}{2}f_{abc}\bar{e}_b\bar{e}_c$. So $e_a$ commutes with $\bar{e}_b$ while $\hat{e}_a$ doesn’t. Let $g_0$ and $g_1$ be the Lie algebras $\widetilde{Tg}[1]^{even}$ and $\widetilde{Tg}[1]^{odd}$ respectively.

In formula (62), for $\mu^a \in g_0$, set $C = \exp_{se(g)}(ad\mu), \hat{C} = \exp_{CI(g)}\left(\frac{1}{2} \sum_{i,j} (ad\mu)_{ij}\bar{e}_i\bar{e}_j\right)$, and $D = ad\mu$. Then we get:

$$\frac{1}{|\det|^{1/2}(\hat{R})} = j(\hat{C})^{1/2}(ad\mu), \quad E_1 = f(ad\mu),$$
\[ f(s) = \frac{1}{2} e^s + 1 - \frac{1}{s} = \frac{d ln j(s)}{ds} \]

Then \( \exp(v_\gamma(E_1)) \) is exactly the \( \exp\left(\frac{1}{2} T_{ab} \epsilon_a \epsilon_b\right) \) in the quantization map \( \mathcal{Q} \). If we work in the Fourier transformed picture of the usual interpretation of Duflo map, elements in \( g_0 \) and \( g_1 \) are considered as even and odd distributions supported at the origin respectively.

Let \( \iota_{C(x)} \) denote multiplying the function \( j^{\frac{1}{2}} \cdot \exp\left(\frac{1}{2} T_{ab}(x) \epsilon_a \epsilon_b\right) \) to a distribution, then the factorization reads:

\[ q^{-1} \left( \exp_{\mathcal{C}(\mathfrak{g})} \left( -\frac{1}{2} \sum_{i,j} (ad \mu)_{ij} \epsilon_i \epsilon_j \right) \right) = \iota_{C(\mu)} \cdot \exp_{\wedge(\mathfrak{g})} \left( -\frac{1}{2} (ad \mu)_{ij} \epsilon_i \epsilon_j \right) \]

(The \( \mu \) part serve as even functions). Furthermore, the main theorem in [3] says the following is also true.

\[ q^{-1} \left( \exp_{\mathcal{C}(\mathfrak{g})} \left( -\frac{1}{2} \sum_{i,j} ((ad \mu)_{ij} \epsilon_i \epsilon_j) + \nu^a \bar{e}_a \right) \right) \]

\[ = \iota_{C(\mu)} \circ \exp_{\wedge(\mathfrak{g})} \left( -\frac{1}{2} \sum_{i,j} ((ad \mu)_{ij} \epsilon_i \epsilon_j) + \nu^a \bar{e}_a \right) \]

The proof of this is essentially an application of theorem (4.8) to a slightly larger space. (See [3] for detail). In addition, since

\[ \sum_{i,j} (ad \mu)_{ij} \epsilon_i \epsilon_j = \sum_{a,b,c} \mu_a f_{abc} \bar{e}_b \bar{e}_c. \]

We get

\[ q^{-1} \left( \exp_{\mathcal{C}(\mathfrak{g})} \left( -\frac{1}{2} \sum_{a,b,c} (\mu^a f_{abc} \bar{e}_b \bar{e}_c) + \nu^a \bar{e}_a \right) \right) \]

\[ = \iota_{C(\mu)} \circ \exp_{\wedge(\mathfrak{g})} \left( -\frac{1}{2} \sum_{a,b,c} (\mu^a f_{abc} \bar{e}_b \bar{e}_c) + \nu^a \bar{e}_a \right) \]

(65)

Since we will deal with several different algebras in our proof here, it is convenient to just use some subscripts under \( \exp \) to indicate the multiplications in different algebras and omit the \( \wedge \) symbols between odd elements.

First, Let us show that the theorem is true for the special element \( \exp_{\mathcal{S}(\mathfrak{g}[I])}(\sum_a (\mu^a \bar{e}_a + \nu^a \bar{e}_a)) \) where \( \mu_a \) and \( \nu_a \) are parameters, i.e. we
have to show:

\[ Q \circ \sigma_0 \left( \exp_{S(\mathcal{T}_g[1])} \left( \sum_a (\mu^a \hat{e}_a + \nu^a \bar{e}_a) \right) \right) \]

\[ = \sigma_1 \circ Duflo \left( \exp_{S(\mathcal{T}_g[1])} \left( \sum_a (\mu^a \hat{e}_a + \nu^a \bar{e}_a) \right) \right) \quad (66) \]

Since Duflo map here is just the super-symmetrization, so the right side of (66) is:

\[ \sigma_1 \circ Duflo \left( \exp_{S(\mathcal{T}_g[1])} \left( \sum_a (\mu^a \hat{e}_a + \nu^a \bar{e}_a) \right) \right) \]

\[ = \exp_{U(\mathcal{T}_g[1])} \left( \sum_a \left( \mu^a e_a - \frac{1}{2} f_{abc} \bar{e}_b \bar{e}_c \right) + \nu^a \bar{e}_a \right) \]

\[ = \exp_{U(\mathcal{T}_g[1])} \left( \sum_a \left( \mu^a e_a + \left( -\frac{1}{2} \mu^a f_{abc} \bar{e}_b \bar{e}_c + \nu^a \bar{e}_a \right) \right) \right) \]

\[ = \exp_{U(\mathcal{T}_g[1])} \left( \sum_a \mu^a e_a \right) \otimes \exp_{\text{Cl}(g)} \left( \sum_a \left( -\frac{1}{2} \mu^a f_{abc} \bar{e}_b \bar{e}_c + \nu^a \bar{e}_a \right) \right) . \]

The last equality is because \( e_a \) commutes with \( \bar{e}_a \).

Notice the quantization map \( Q = (\chi \otimes q) \circ \iota_{\mathcal{C}(x)} \), the left side of (66) is:

\[ Q \circ \sigma_0 \left( \exp_{S(\mathcal{T}_g[1])} \left( \sum_a (\mu^a \hat{e}_a + \nu^a \bar{e}_a) \right) \right) \]

\[ = Q \cdot \exp_{S(\mathcal{T}_g[1])} \left( \sum_a \left( \mu^a (e_a - \frac{1}{2} f_{abc} \bar{e}_b \bar{e}_c) + \nu^a \bar{e}_a \right) \right) \]

\[ = (\chi \otimes q) \circ \iota_{\mathcal{C}(x)} \left( \exp_{\text{Cl}(g)} \left( \sum_a \mu^a e_a \right) \otimes \exp_{\wedge(g)} \left( \sum_a \left( -\frac{1}{2} \mu^a f_{abc} \bar{e}_b \bar{e}_c + \nu^a \bar{e}_a \right) \right) \right) \]

\[ = \chi \left( \exp_{\text{Cl}(g)} \left( \sum_a \mu^a e_a \right) \right) \otimes q \left( \iota_{\mathcal{C}(\mu)} \cdot \exp_{\wedge(g)} \left( \sum_a \left( -\frac{1}{2} \mu^a f_{abc} \bar{e}_b \bar{e}_c + \nu^a \bar{e}_a \right) \right) \right) \]

\[ = \exp_{U(\mathcal{T}_g[1])} \left( \sum_a \mu^a e_a \right) \otimes \exp_{\text{Cl}(g)} \left( \sum_a \left( -\frac{1}{2} \mu^a f_{abc} \bar{e}_b \bar{e}_c + \nu^a \bar{e}_a \right) \right) . \]
Here, we think of elements in $S(\widehat{Tg}[1])$ as distributions supported at origin (see chapter 2). Hence we have proved the equation (66).

By comparing the coefficients of the parameters $\mu_a$ and $\nu_a$ of both sides in (66), we prove the theorem for any elements in $S(\widehat{Tg}[1])$.

\[\square\]

5. Jacobi diagrams and diagrammatic proof of Duflo isomorphism

This chapter is an explanation of the work [6] of Bar-Natan, Le and Dylan Thurston.

5.1. Tensors and Jacobi diagrams. In general, a tensor with $n$ free indices will be represented by a graph with $n$ legs. The indices of a tensor can belong to different vector spaces or their dual spaces. Correspondingly, the legs of the graph should be labeled to indicate the vector spaces and distinguish a vector space and its dual. For example, a matrix $A \in \text{Hom}(V, V) = V^* \otimes V$ can be represented by the graph in figure (1). By convention, the data flow in the direction of the arrows, so the incoming arrow is the $V^*$ factor and the outgoing arrow is the $V$ factor. In this paper, we mainly deal with the tensors over a Lie algebra $\mathfrak{g}$ with an invariant metric $B( , )$.

Suppose $e_1, \ldots, e_n$ is a basis of $\mathfrak{g}$ and $e^1, \ldots, e^n$ is the dual basis in $\mathfrak{g}^*$. Suppose $[e_a, e_b] = f_{ab}^c e_c$ and $B(e_a, e_b) = t_{ab}$. The matrix $(t_{ab})$ is invertible. We use $(t^{ab})$ to denote its inverse matrix, which defines the dual metric of $B( , )$ on $\mathfrak{g}^*$. Then the invariant metric $B( , )$ and its dual metric could be represented by diagrams in figure (3). It is easy to show that $f_{abc} = f_{abc}^{\prime} = t_{ac}^\prime t_{bc}$ is totally anti-symmetric in its indices. We use a fork (see figure (2)) to represent it.

The antisymmetry and Jacobi identity for the bracket of the Lie algebra $\mathfrak{g}$

\[
[x, y] + [x, y] = 0, \quad ([x, y], z) + ([z, x], y) + ([y, z], x) = 0 \quad \forall x, y, z \in \mathfrak{g}
\]

can be expressed graphically as in figure (4).

The two relations of diagrams in figure (4) are called antisymmetry and IHX relations.
Figure 2. Pictorial representation of Lie bracket

\[ g^* \leftrightarrow f_{abc} e^a e^b e^c \]

Figure 3. Pictorial representation of the invariant metric \( B( , ) \) and its dual

\[ t^{ab} e_a \otimes e_b \]

Figure 4. Pictorial representation of antisymmetry and Jacobi identity of Lie bracket

Notice that the trivalent vertices in a diagram have to be oriented, otherwise it is not clear how to read the corresponding tensor from the diagram.
Definition 5.1. An open Jacobi diagram (also called uni-trivalent graph) is a vertex-oriented uni-trivalent graph, i.e., a graph with only univalent and trivalent vertices where each trivalent vertex is oriented. The univalent vertices are called legs. In planar pictures, the orientation on the edges incident on a vertex is always anticlockwise.

Definition 5.2. Let $\mathcal{B}^f$ be the vector space spanned by Jacobi diagrams modulo the IHX relation and the antisymmetry relation. The degree of a diagram in $\mathcal{B}^f$ is half of the number of vertices (trivalent and univalent) of the diagram. Let $\mathcal{B}$ be the completion of $\mathcal{B}^f$ with respect to the degree.

There are some remarkable relations between the space of Jacobi diagrams and the space of Vassiliev invariants. Roughly speaking, any finite type weight system on $\mathcal{B}$ (i.e. a real value function on $\mathcal{B}$ that vanishes on diagrams with degree greater than a certain integer) corresponds to a Vassiliev invariant. The correspondence is established by M. Kontsevich using his famous Kontsevich integral. See Kontsevich’s original paper [16] and Dror Bar-Natan’s paper [5] for the complete exposition on this topic.

If we identify $g^*$ and $g$ using the metric, Any Jacobi diagram $D$ gives a tensor over $g$ in the following way. First we can decompose $D$ into several copies of forks and bars, ignoring the crossings between them. Each of the forks and bars is canonically associated to an invariant tensor as in figure (2) and figure (3). If contracting the legs of these forks and bars, we get a tensor. It is easy to see the tensor we get does not depend on the way we decompose the diagram, hence we denote it by $T_D$. Next, since the symmetric algebra $S(g)$ is a quotient of $T(g)$, let $p : T(g) \rightarrow S(g)$ be the quotient map. It is easy to see that $p(T_D) \in S(g)$ will correspond to the diagram $D$ if we don’t order the legs of $D$. That is to say every Jacobi diagram with unordered legs gives a symmetric tensor over $g$. Unfortunately, not every element in $S(g)$ can be represented in this way because of the following lemma.

Lemma 5.3. The symmetric tensor associated to a diagram in $\mathcal{B}$ is always invariant under the adjoint action of $g$.

Proof. The tensor associated to a diagram in $\mathcal{B}$ is made by contracting copies of structure constants and invariant metric tensors of the Lie algebra, which are all invariant under $ad(g)$.

Remark 5.4. Actually, not every invariant symmetric tensor can be represented by diagrams in $\mathcal{B}$, see [22] for examples.

We can another ingredient into the Jacobi diagrams. If we have a representation $(V, \pi)$ of $g$, it defines an element $R \in V^* \otimes g^* \otimes V$,
\[ g^* \xrightarrow{\leftrightarrow} R^c_{ab} e^a \otimes v^b \otimes v^c \]

**Figure 5.** Jacobi diagram of a representation \( V \) of \( g \)

which is represented by figure (5) where we use a different type of line to denote \( V \). The fact that \( V \) is a representation of \( g \) imposes an IHX type relation in this setting. Now, we can draw diagrams with different type of edges, e.g. figure (6), which allows us to construct more invariant tensors over \( g \).

In addition, \( B \) becomes a commutative algebra if we define the product of two diagram to be the disjoint union \( \sqcup \) of them. This corresponds exactly to the algebra structure on \( S(g) \).

5.2. **Based Jacobi diagrams.** If we order the legs of a Jacobi diagram \( D \), it will represent a tensor with noncommutative indices. To remember the ordering, we can glue the legs of \( D \) to a connected 1-manifold \( X \) (a circle or a oriented line), see figure (7). In this case, we can’t commute any two legs that are attached to \( X \) without changing the diagram. If we impose another diagrammatic relation, called STU relation (see figure (8)), between any two legs attached to \( X \), then the diagram represents an element in the universal enveloping algebra \( U(g) = T(g) / \langle xy - yx = [x, y] \rangle \). We call this new type of diagrams **based Jacobi diagrams** on \( X \).

**Definition 5.5.** For an oriented connected 1-manifold \( X \), let \( A^f(X) \) be the vector space spanned by Jacobi diagrams based on \( X \) modulo the antisymmetry, IHX and STU relations. The **degree** of a diagram in \( A^f(X) \) is half the number of its vertices. Define \( A(X) \) to be the completion of \( A^f(X) \) with respect to the degree.
If we have a representation $V$ of the Lie algebra $\mathfrak{g}$, we can draw diagrams with dashed lines and solid lines with legs attached to $X$.

**Warning:** the solid line denoting $V$ and the based 1-manifold are not the same thing. We don’t put any representation of $\mathfrak{g}$ on the based 1-manifold.

The algebra of Jacobi diagrams based on a oriented circle $A(\bigcirc)$ and Jacobi diagrams based on an oriented line $A(\longrightarrow)$ are actually isomorphic as algebras (see [5]). So we can just use $A$ to denote $A(\bigcirc)$ or $A(\longrightarrow)$.

Similar to $\mathcal{B}$, we can define an algebra structure on $A$: take two based Jacobi diagrams $D_1, D_2$ and place the legs of $D_1$ before the legs of $D_2$ in the total ordering of legs. We denote it by $D_1 \# D_2$. In diagrams, it is just connecting the base lines of $D_1$ and $D_2$ (see figure (8)).

Another useful type of Jacobi diagrams is the *Jacobi diagram with colored legs* in which we give different colors to legs of a Jacobi diagram and don’t distinguish legs with the same color. When we glue the colored legs to an oriented line or a circle, the line or the circle is also colored. See figure (10). We use $A(*_{x_1}, \ldots, *_{x_n})$ to denote the Jacobi diagrams with legs colored by $x_1, \ldots, x_n$.

For more discussion on various kinds of Jacobi diagrams, see [5] and [6].
5.3. Diagrammatic proof of Duflo isomorphism. In [6], Dror Bar-Natan, Thang T. Q. Le and Dylan P. Thurston gave an interesting diagrammatic analogue of Duflo isomorphism in the world of Jacobi diagrams. Their proof essentially uses some special properties of the Kontsevich integral of unknot. For the definition and properties of the Kontsevich integral of knots and tangles, please see [5], [6] and [20].

We can interpret the Duflo isomorphism in terms of Jacobi diagrams as follows. First, the Poincaré-Birkhoff-Witt map can be described diagrammatically as averaging all ways of gluing all the legs of a Jacobi diagram to an oriented line (see figure (11)).

Definition 5.6. For diagrams $C, C' \in B$ so that $C$ has no struts (components like $\cdot \cdot \cdot \cdot$), the inner product of $C$ and $C'$ is defined by

$$\langle C, C' \rangle = \begin{cases} 
\text{the sum of all ways of gluing all legs of } C \text{ to all legs of } D, & \text{if } C \text{ and } C' \text{ have the same number of legs,} \\
0, & \text{otherwise} 
\end{cases}$$

(67)

If $C$ and $C'$ are colored Jacobi diagrams, we require that only when the legs from $C$ and $C'$ have the same color, can they be glued together. So in this case, $C$ and $C'$ must have the same number of colored legs in each color to make their inner product non-zero.

If we write $\iota_C \overset{\text{def}}{=} \langle C', \cdot \rangle$, the PBW map $\chi$ can be pictorially thought of as $\iota_\Gamma$, where $\Gamma$ is a colored Jacobi diagram in $A(\ast_z, \ast_x)$ (see figure (12)).

Next, we interpret the $j^{\frac{1}{2}}$ in Duflo isomorphism in the language of Jacobi diagrams. For any $x \in \mathfrak{g}$, Think of $ad_x$ as a matrix, which is represented by the diagram in figure (13)(a). Then $(ad_x)^n$ can be denoted by the diagram in figure (13)(b). In addition, taking trace of a matrix is just connecting the input and the output, see figure (13)(c).
Figure 11. PBW isomorphism for Jacobi diagrams

\[
\begin{align*}
\Gamma = \exp\left(\frac{x}{z}\right) &= \varnothing + \frac{x}{2} + \frac{1}{n!} z + \ldots \\
&+ \frac{1}{n!} \left(\cdots + \frac{1}{n \text{ legs}} \right)
\end{align*}
\]

Figure 12. The diagram \( \Gamma \) for PBW map \( \chi \)

So we have the diagrammatic representation of \( j^x \) as following.

\[
j^x(x) = \det^\frac{1}{2} \left( \frac{\sinh(\frac{1}{2}ad_x)}{\frac{1}{2}ad_x} \right) = \exp \left( \frac{1}{2} \log \sinh(\frac{1}{2}ad_x) \right) \\
= \exp \left( \frac{1}{2} \log \left( \sum_{n=0}^{\infty} b_{2n} (ad_x)^{2n} \right) \right) \\
= \exp \left( \frac{1}{48} \ldots - \frac{1}{5760} \right) + \frac{1}{362880} \ldots \right) \triangleq \Omega_x
\]

The diagram \( \Omega_x \) is a very important Jacobi diagram. It is proven in [6] to be the Kontsevich integral of the unknot – the only Kontsevich integral of a knot that people can calculate completely so far! We use
Remark 5.7. Notice the diagrammatic interpretation of the Duflo isomorphism in figure (12) and (13) makes sense even without the metric on $g$.

In addition, since $j^\frac{1}{2}(x)$ acts as a differential operator on $S(g)$ in the Duflo isomorphism, we have to define how differential operators are represented in the world of Jacobi diagrams.

Definition 5.8. For a $C \in B$ without struts, the operation of applying $C$ as a differential operator, denoted by $\partial_C : B \rightarrow B$, is defined to be

$$\partial_C = \begin{cases} 
0 & \text{if } C \text{ has more legs than } D, \\
\text{the sum of all ways of gluing all legs of } C \text{ to some(or all) legs of } D & \text{otherwise.}
\end{cases}$$

(69)

If $C$ and $D$ are colored Jacobi diagrams, we can only glue legs of the same color in above definition. In addition, let $\emptyset$ denote the empty diagram, then $\partial_\emptyset(D) = D$.

With this definition, we can interpret the action of $j^\frac{1}{2}$ on symmetric algebra $S(g)$ diagrammatically as $\partial_\Omega$ on $B$.

Before stating the diagrammatic analogue of Duflo isomorphism, we need to introduce two more operations in Jacobi diagrams.

Definition 5.9. For any Jacobi diagram $C$ (no matter if it is colored), $(C)_x$ is just $C$ with all its legs colored by $x$, ignoring the original coloring of $C$. $\Delta_{xy}C$ is defined to be the sum of all possible different colorings on the legs of $C$ by two colors $x$ and $y$, ignoring the original coloring of $C$. 
Lemma 5.10. For any diagrams $C, D_1, D_2 \in \mathcal{B}$,
\[
\langle C, D_1 \sqcup D_2 \rangle = \langle \Delta_{xy} C, (D_1)_x \sqcup (D_2)_y \rangle
\]

Proof. Obvious. \hfill \Box

we have the following theorem which is established in [6].

Theorem 5.11. (Wheeling) The map $\Phi = \partial_\Gamma \circ \partial_\Omega : \mathcal{B} \rightarrow \mathcal{A}$ is an algebra homomorphism with respect to the algebraic structure in $\mathcal{B}$ and $\mathcal{A}$.

Proof. Considering the technical complexity, I only want to give the outline of the proof in this thesis. For the complete proof, please see [6].

First, suppose $H(z; x) \in \mathcal{A}(\star_z, \star_x)$ is the disjoint union of the diagrams $\Omega_x$ and $\Gamma$. We can show that $H$ is in fact the Kontsevich integral of the tangle in figure (14)(a). Next, we compute the Kontsevich integral of the tangles on both side of the diagrammatic equation in figure (14)(b) (so called $2 = 1 + 1$), and we get
\[
\Delta_{x_1x_2} H(z; x) = H(z; x_1) \#_z H(z; x_2) \quad (70)
\]

Then for any Jacobi diagrams $D_1, D_2 \in \mathcal{B}$,
\[
\Phi(D_1 \sqcup D_2) = \langle H(z; x), (D_1 \sqcup D_2)_x \rangle = \langle \Delta_{x_1x_2} H(z; x), (D_1)_{x_1} \sqcup (D_2)_{x_2} \rangle
\]
\[
= \langle H(z; x_1) \#_z H(z; x_2), (D_1)_{x_1} \sqcup (D_2)_{x_2} \rangle
\]
\[
= \Phi(D_1) \#_z \Phi(D_2) \quad (71)
\]

This shows that $\Phi$ is indeed an algebra homomorphism. \hfill \Box
5.4. **Super Version of Wheeling Theorem.** For a super Lie algebra, the super Jacobi identity can be written as:

\[
(-1)^{\text{deg}x\text{deg}z}[[x, y], z] + (-1)^{\text{deg}z\text{deg}y}[[z, x], y] + (-1)^{\text{deg}y\text{deg}x}[[y, z], x] = 0
\]

or

\[
[[x, y], z] + (-1)^{\text{deg}(\text{deg}x+\text{deg}y)}[[z, x], y] + (-1)^{\text{deg}(\text{deg}y+\text{deg}z)}[[y, z], x] = 0
\]

(Pic.72)

Pictorially we can still use figure (4) to represent it, plus letting the quadrivalent crossings in the diagram represent the sign changes caused by the super degree.

For any Jacobi diagram \(D\) on the x-y plane, we can always deform it to a general position in the upper half plane such that all the vertices and crossings have different y-level and the tips of legs are placed on the x-axis (see figure (15)). Then from the bottom to the top, we can decompose the diagram into some forks, crossings, cups and caps. If we have a super Lie algebra with an invariant metric, we can associate a canonical invariant tensor to each of the basic components and then contract their legs to get an invariant tensor. It is not hard to show that the result is independent of the general position we use for the diagram.

It is easy to see that the wheeling theorem can be extended to the super case without any change.

5.5. **Wheeling theorem implies the Duflo isomorphism.** Although not every invariant symmetric tensor can be represented by elements in \(B\), we can introduce some labeled blobs with legs to represent arbitrary elements in \(S(g)\) as in figure (16(a)). So a generalized diagram could look like figure (16(b)). The \(g\)-invariance of the elements represented by those blobs can be drawn as diagrammatic relation in figure (17). If we put the legs of a generalized diagram on a solid line, it will represent an element in \(U(g)\).
Figure 16. Generalized Graphs that represent arbitrary invariant symmetric tensors

Figure 17. $\mathfrak{g}$-invariance of the blobs

The wheeling theorem can be easily extended to this larger set of diagrams. So the fact that Duflo isomorphism is an algebra isomorphism follows from the diagram below:

\[
\begin{array}{c}
\mathcal{B}_{\text{blob}} & \xrightarrow{\Phi} & \mathcal{A}_{\text{blob}} \\
\text{surjective} & \text{algebra isom.} & \text{surjective} \\
S(\mathfrak{g}) \to & \Phi & U(\mathfrak{g}) \\
\text{Duflo} & \text{algebra isom.} & \end{array}
\]

(74)

6. Diagrammatic analogue of Alekseev-Meinrenken quantization map

In the last chapter, a diagrammatic analogue of Duflo map in the world of Jacobi diagrams for any quadratic Lie algebra is shown. Although the diagrammatic proof may not be easier than the algebraic proof, it provide us a new way to handle the invariant tensors which could be useful in some other situations.

The most important evidence I found to believe that the quantization map $Q$ is equivalent to a super Duflo map is that the natural diagrammatic representation of the quantization map $Q$ has the same
property as the diagram $H(z,x)$ for the Duflo map in the preceding chapter.

**Remark 6.1.** The super Lie algebra $\widetilde{\mathfrak{g}}[1]$ for a quadratic Lie algebra $\mathfrak{g}$ has a natural invariant non-degenerate (super)symmetric bilinear form $\tilde{B}$ defined by: $\tilde{B}(e_a, e_b) = B(e_a, e_b), \tilde{B}(\overline{c}_a, \overline{c}_b) = 0, \tilde{B}(e_a, \overline{c}_b) = \tilde{B}(\overline{c}_a, e_b), \tilde{B}(e_a, c) = 0, \tilde{B}(\overline{c}_a, c) = 0$, $\tilde{B}(c, c) = 1$ where $\{e_a\}$ and $\{\overline{c}_a\}$ are basis of $\widetilde{\mathfrak{g}}[1]$ even and $\widetilde{\mathfrak{g}}[1]$ odd respectively.

Let’s see what diagrammatic analogue of the quantization map $Q$ should be. It is natural to label legs of a Jacobi diagram by even (e) or odd (o) with respect to the type of variables they represent. We allow partial labelings also. The legs which are not labeled by even or odd can be thought of as super legs.

By the preceding remark, diagrams in $\mathcal{B}$ with legs thus labeled will represent a symmetric tensor on $\widetilde{\mathfrak{g}}[1]$ and diagrams in $\mathcal{A}$ will represent elements in the universal enveloping algebra of $\widetilde{\mathfrak{g}}[1]$. Let $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{A}}$ denote the spaces of corresponding Jacobi diagrams with legs labeled (or partially labeled) by even and odd. The algebraic structures on $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{A}}$ are the same as $\mathcal{B}$ and $\mathcal{A}$ respectively. We can readily call them super labeled Jacobi diagrams.

The definition of the diagrammatic differential operator of a diagram in this situation is the same as (69) in the previous chapter plus that we only glue even legs to even (or super) legs and glue odd legs to odd (or super) legs.

Next, we want to use the Jacobi diagram of $j^\frac{1}{2}$ to define the diagram for the tensor $\exp(\frac{1}{2}T_{ab}(x)\eta_a\eta_b)$. Since $T_{ab}(x) = (\ln(j)'(ad_x))_{ab}$, we need to define the derivative of a Jacobi diagram. First, we call the way of changing a diagram in figure 18 splitting a wheel at a leg. Since the derivative of $x^n$ is $nx^{n-1}$, so it is natural to define the diagrammatic derivative $\mathcal{P}(C)$ of a wheel $C$ to be the diagrammatic sum of all possible ways of splitting $C$ at its legs. For the disjoint union of two wheels $C_1$ and $C_2$, we define $\mathcal{P}(C_1 \sqcup C_2) = \mathcal{P}(C_1) \sqcup \mathcal{P}(C_2)$. With this definition, we have $\mathcal{P}(\exp\sum C_i) = \exp\sum \mathcal{P}(C_i)$ for a collection of wheels $C_i$. In addition, we define the action of $\mathcal{P}$ on a connected diagram other than wheels to be trivial (i.e. kill it).

The diagram for the Duflo map of $\widetilde{\mathfrak{g}}[1]$ has super legs. A super leg consist of an even leg and an odd leg. Then there are four possible cases of splitting. The first case is shown in figure 19 which corresponds to $ad^n(x)\eta_a\eta_b$. The other three cases (see figure 20) are in fact all zero algebraically. The case(a) vanishes because $T_{ab}$ is skew-symmetric,
**Figure 18.** Splitting of a wheel at a leg

![Diagram of wheel splitting](image)

**Figure 19.** $ad^n(x) t_a t_b, \ n = 3$

![Diagram showing cases (a), (b), and (c)](image)

**Figure 20.** Three other possible cases of splitting

![Diagram showing splitting in all possible ways](image)

**Figure 21.** Diagrammatic derivative of a wheel

\[
T_{ab} \frac{\partial}{\partial \mu} \frac{\partial}{\partial \nu} = -T_{ba} \frac{\partial}{\partial \mu} \frac{\partial}{\partial \nu},
\]

and the cases (b) and (c) vanish obviously. The diagrammatic derivative of a 4-leg wheel is shown in figure (21).
Remark 6.2. In figure (19), the even legs in fact correspond to $v^a\left(\frac{1}{2}f_{abe}\bar{e}_b\bar{e}_c\right)$ after the super variable change.

By the Taylor expansion of $\ln(j^s)$ in (53), we get the diagrammatic representation $\Psi_x$ of $\exp\left(\frac{1}{2}T_{ab}(x)\iota_a\iota_b\right)$.

\begin{align*}
\Psi_x &= \exp\left(\frac{1}{2}T_{ab}(x)\iota_a\iota_b\right) \\
&= \exp \left(\frac{1}{24}e \ o - \frac{1}{1440}e^2 \ o + \ldots\right) \\
\end{align*}

(75)

In addition, the map $\chi \otimes q$ in quantization map $Q$ can still be represented by $\Gamma$ (see figure (12)). The legs attached to the z-line are not labeled by even or odd because we allow them to be connected to both even and odd type of legs. Notice we won’t have any legs directly attached to the z-line after the contraction $\partial_\Gamma$ with a Jacobi diagram (see the definition of $\partial_\Gamma$ in (67)).

Let $\tilde{H}(z; x)$ to be the disjoint union of $\Gamma, \Omega_x$ and $\Psi_x$(labeling ignored). by the definition of $\mathcal{P}$, we have the following lemma.

Lemma 6.3. $\tilde{H}(z; x) = (I + \mathcal{P})H(z; x)$, where $I$ is the identity map.

Notice labeling and coloring are different actions on diagrams. More precisely, labeling a leg by even or odd and coloring a leg by x or y are independent to each other. Observe that: the splitting operation $\mathcal{P}$ on wheels commutes with the operation $\Delta_{xy}$ which is just ignore the original coloring of a diagram and coloring all legs of a diagram by two colors x, y in all possible ways (see (5.9) for definition). Then we have the following:

\begin{align*}
\Delta_{x_1x_2}\tilde{H}(z; x) &= \Delta_{x_1x_2}(I + \mathcal{P})H(z; x) = \Delta_{x_1x_2}(I + \mathcal{P})\Delta_{x_1x_2}H(z; x) \\
&= \Delta_{x_1x_2}(I + \mathcal{P})\{H(z; x_1)\#_zH(z; x_2)\} \\
&= \{\Delta_{x_1x_2}(I + \mathcal{P})H(z; x_1)\}\#_z\{\Delta_{x_1x_2}(I + \mathcal{P})H(z; x_2)\} \\
&= \tilde{H}(z; x_1)\#_z\tilde{H}(z; x_2) \\
\end{align*}

(76)

The second equality needs a little explanation. Notice the action of $\mathcal{P}$ will introduce some new odd legs which are not colored by x or y yet, so we need to remove the color we just did and do the coloring again to guarantee each leg is colored.

Next, using the same argument in (71), we can easily prove the following theorem which can be thought of as the diagrammatic analogue of quantization map $Q$ for Weil algebras.
Theorem 6.4. If we label the legs of $\Omega_x$ by even, the map $\tilde{\Phi} = \partial_T \circ \partial_{\ell_y} \circ \partial_{\phi_x} : \tilde{B} \rightarrow \tilde{A}$ is an algebra homomorphism with respect to the algebraic structures of $\tilde{B}$ and $\tilde{A}$.

References

[1] A. Alekseev and E. Meinrenken. Lie theory and the Chern-Weil homomorphism. math.QA/0308135.
[2] A. Alekseev and E. Meinrenken. The non-commutative Weil algebra. Invent. Math., 139(1):135–172, 2000.
[3] A. Alekseev and E. Meinrenken. Clifford algebras and the classical dynamical Yang-Baxter equation. Math. Res. Lett., 10(2-3):253–268, 2003.
[4] M. Andler, A. Dvorsky, and S. Sahi. Kontsevich quantization and invariant distributions on Lie groups. Ann. Sci. École Norm. Sup. (4), 35(3):371–390, 2002.
[5] D. Bar-Natan. On the Vassiliev knot invariants. Topology, 34(2):423–472, 1995.
[6] D. Bar-Natan, T. T. Q. Le, and D. P. Thurston. Two applications of elementary knot theory to Lie algebras and Vassiliev invariants. Geom. Topol., 7:1–31 (electronic), 2003.
[7] N. Berline, E. Getzler, and M. Vergne. Heat kernels and Dirac operators, volume 298 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1992.
[8] L. Corwin, Y. Ne’eman, and S. Sternberg. Graded Lie algebras in mathematics and physics (Bose-Fermi symmetry). Rev. Modern Phys., 47:573–603, 1975.
[9] P. Deligne and J. W. Morgan. Notes on supersymmetry (following Joseph Bernstein). In Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), pages 41–97. Amer. Math. Soc., Providence, RI, 1999.
[10] M. Duflo. Opérateurs différentiels bi-invariants sur un groupe de Lie. Ann. Sci. École Norm. Sup. (4), 10(2):265–288, 1977.
[11] P. Etingof and A. Varchenko. Geometry and classification of solutions of the classical dynamical Yang-Baxter equation. Comm. Math. Phys., 192(1):77–120, 1998.
[12] G. B. Folland. Harmonic analysis in phase space, volume 122 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1989.
[13] F. G. Friedlander. Introduction to the theory of distributions. Cambridge University Press, Cambridge, second edition, 1998. With additional material by M. Joshi.
[14] R. Goodman and N. R. Wallach. Representations and invariants of the classical groups, volume 68 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1998.
[15] V. W. Guillemin and S. Sternberg. Supersymmetry and equivariant de Rham theory. Springer-Verlag, Berlin, 1999.
[16] M. Kontsevich. Vassiliev’s knot invariants. In I. M. Gel’fand Seminar, volume 16 of Adv. Soviet Math., pages 137–150. Amer. Math. Soc., Providence, RI, 1993.
[17] M. Kontsevich. Deformation quantization of Poisson manifolds. *Lett. Math. Phys.*, 66(3):157–216, 2003.

[18] B. Kostant. Clifford algebra analogue of the Hopf-Koszul-Samelson theorem, the $\rho$-decomposition $C(g) = \text{End} V_\rho \otimes C(P)$, and the $g$-module structure of $\bigwedge g$. *Adv. Math.*, 125(2):275–350, 1997.

[19] H. B. Lawson, Jr. and M.-L. Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.

[20] T. Ohtsuki. *Quantum invariants*, volume 29 of *Series on Knots and Everything*. World Scientific Publishing Co. Inc., River Edge, NJ, 2002. A study of knots, 3-manifolds, and their sets.

[21] S. Sternberg. Some recent results on the metaplectic representation. In *Group theoretical methods in physics (Sixth Internat. Colloq., Tübingen, 1977)*, volume 79 of *Lecture Notes in Phys.*, pages 117–143. Springer, Berlin, 1978.

[22] P. Vogel. Algebraic structures on modules of diagrams. *Tech. report, université de Paris*, VII, July 1995.