WEIGHTED SUMS OF THE SQUARES OF THE DISTANCES OF A POINT TO THE SIDELINES OF A TRIANGLE

GEORGI GANCHEV AND NIKOLAI NIKOLOV

Abstract. We study a function, which is a weighted sum of the squares of the distances of an arbitrary point to the sidelines of a triangle. The given weights, considered as barycentric coordinates, determine a point \( M \). We prove that the function reaches its minimum (maximum) at a point, which is isogonal conjugate to \( M \).

1. Introduction

As usual, we denote by \( a, b, c \) the sides of a given \( \triangle ABC \) and by \( S \) its area. The positive orientation of the plane is determined by \( \triangle ABC \).

Let \( x, y, z \) be trilinear coordinates of a point with respect to \( \triangle ABC \).

It is well known that the Lemoine point \( K \) minimizes the sum \( x^2 + y^2 + z^2 \), i.e. \[ x^2 + y^2 + z^2 \geq \frac{4S^2}{a^2 + b^2 + c^2}. \]

Recently Kimberling \(^{[1]}\) obtained several inequalities for the power sums \( x^q + y^q + z^q \).

In this note for an arbitrary point \( X \) in the plane of \( \triangle ABC \) we study a weighted sum \( F(X) \) of the squares of the distances of \( X \) to the sidelines of the triangle. We give a geometric interpretation of the minimum (maximum) of the function \( F(X) \).

2. Preliminaries

Let us recall some properties of isogonal conjugate points with respect to a given \( \triangle ABC \).

Given the basic \( \triangle ABC \) and its circumcircle \( k(ABC) \). Denote by \( t \) the isogonal conjugation with respect to the triangle. The action of \( t \) in the domains (with respect to the vertex \( A \)) (Fig. 1) is as follows:

\[ \sigma, \sigma_1, \sigma_{12}, \sigma_{13}, \sigma_{13}' \]

\[ k \]

\[ \triangle ABC \]

\[ E, F, G \]

\[ A, B, C \]

\[ D \]

\[ \text{2000 Mathematics Subject Classification.} \quad \text{Primary 51M04, Secondary 51M16.} \]

\[ \text{Key words and phrases.} \quad \text{Weighted sum of the squares of the distances, isogonal conjugate points, barycentric coordinates.} \]
1) \( \iota(\sigma) = \sigma \).

If \( M \) is a point on the side \( BC \), then \( \iota(M) = A \).

2) \( \iota(\sigma_{12}) = \sigma_{12} \).

For any point \( M \) on the ray \( BD^+ \) we have \( \iota(M) = C \); for any point \( M \in CF^- \) \( \iota(M) = B \).

3) \( \iota(\sigma_{13}) = \sigma_{13}' \), \( \iota(\sigma_{13}') = \sigma_{13} \).

For any point \( M \in AE^- \) \( \iota(M) = C \); for any point \( M \in AG^- \) \( \iota(M) = B \).

The transformation \( \iota \) can also be defined for points on the circumcircle \( k \), different from the vertices of the triangle. If \( M \) is a point on the arc \( BC \), then \( \iota(M) \) is the point at infinity of the line, which is symmetric to the line \( AM \) with respect to the bisector of \( \angle BAC \).

For an arbitrary point \( X \) in the plane of \( \triangle ABC \) we denote by \( x, y, z \), the directed distances of the point \( X \) to the lines \( BC, CA, AB \), respectively. Then \( (x, y, z) \) is the triple of trilinear coordinates of \( X \) with respect to the basic triangle. The trilinear coordinates satisfy the equality \( ax + by + cz = 2S \). Further we denote by \( S_1, S_2, S_3 \) the oriented areas of the triangles \( BCX, CAX, ABX \), respectively. Then \( \lambda = \frac{S_1}{S}, \mu = \frac{S_2}{S}, \nu = \frac{S_3}{S}; \) \( \lambda + \mu + \nu = 1 \) are the barycentric coordinates of \( X \) with respect to the \( \triangle ABC \). The relation between the trilinear coordinates and the barycentric coordinates of the point \( X \) is given by

\[
\lambda = \frac{ax}{2S}, \quad \mu = \frac{by}{2S}, \quad \nu = \frac{cz}{2S}.
\]

We also consider the function \( J = \frac{a^2}{\lambda} + \frac{b^2}{\mu} + \frac{c^2}{\nu} = \left( \frac{a}{x} + \frac{b}{y} + \frac{c}{z} \right) 2S \), which is defined for all points that do not lie on the sidelines of \( \triangle ABC \).

The following characterization of the points in the \( \angle BAC \) out of \( \triangle ABC \) is useful.

**Lemma 1.** Let \( M(x, y, z) \) be with trilinear coordinates satisfying the conditions \( x < 0, y > 0, z > 0 \), i.e. \( M \) is in the \( \angle BAC \) out of \( \triangle ABC \). Then

1) \( M \in \sigma_{12} \) iff \( J > 0 \);
2) \( M \) lies on the arc \( \overline{BC} \) iff \( J = 0 \);
3) \( M \in \sigma_{13} \) iff \( J < 0 \).

We also need the following statement.

**Lemma 2.** Let \( M(x, y, z) \) be with trilinear coordinates satisfying the conditions \( x > 0, y < 0, z < 0 \), i.e. \( M \in \sigma'_1 \). Then \( J < 0 \).

**The isogonal conjugation with respect to barycentric coordinates**

Let \( (\lambda, \mu, \nu), \lambda + \mu + \nu = 1 \) be the barycentric coordinates of a point \( M \), which does not lie on the lines \( AB, BC, CA \) or on the circumcircle \( k(ABC) \). If \( (\lambda', \mu', \nu') \) are the barycentric coordinates of the point \( N \), isogonal conjugate to \( M \), then

\[
(2.1) \quad \lambda' = \frac{a^2}{\lambda J}, \quad \mu' = \frac{b^2}{\mu J}, \quad \nu' = \frac{c^2}{\nu J}.
\]

If \( (\lambda, \mu, \nu) \) are the barycentric coordinates of a point \( M \), it is useful to consider the *homogeneous* barycentric coordinates of \( M \):

\[
(\rho \lambda, \rho \mu, \rho \nu), \quad \rho \neq 0.
\]
WEIGHTED SUMS OF THE SQUARES OF THE DISTANCES

Then the formulas
\[(2.2)\]
\[\lambda' = \frac{a^2}{\lambda}, \quad \mu' = \frac{b^2}{\mu}, \quad \nu' = \frac{c^2}{\nu}\]
represent the isogonal conjugation even on the arcs $BC$, $CA$ or $AB$ of $k$. If $M(\lambda, \mu, \nu)$ lies on the arc $BC$, then the point $N(\lambda', \mu', \nu')$ satisfies the condition $\lambda' + \mu' + \nu' = 0$ and lies on the line at infinity.

A general formulation of the problem

Now, let $(\lambda, \mu, \nu) \neq (0, 0, 0)$ be a triple of fixed real numbers. For any point $X$ with trilinear coordinates $(x, y, z)$ consider the function

\[F(X) = \lambda x^2 + \mu y^2 + \nu z^2,\]

which is a weighted sum of the squares of the directed distances $(x, y, z)$.

Our aim is to investigate the minima and maxima of the above function.

Further we consider three essential cases:
1. $\lambda \mu \nu \neq 0, \quad \lambda + \mu + \nu > 0$;
2. $\lambda \mu \nu \neq 0, \quad \lambda + \mu + \nu < 0$;
3. $\lambda \mu \nu \neq 0, \quad \lambda + \mu + \nu = 0$.

3. Weighted sum with $\lambda \mu \nu \neq 0, \quad \lambda + \mu + \nu > 0$

Obviously both functions

\[\lambda x^2 + \mu y^2 + \nu z^2, \quad \frac{\lambda x^2 + \mu y^2 + \nu z^2}{\lambda + \mu + \nu}\]

have minima and maxima at the same points. Without loss of generality we can assume that $\lambda + \mu + \nu = 1$.

Thus the problem in this section is to find the minimum (maximum) of the function
\[(3.1)\]

\[F(X) = \lambda x^2 + \mu y^2 + \nu z^2, \quad \lambda + \mu + \nu = 1.\]

First we consider the case

1.1. $\lambda > 0, \quad \mu > 0, \quad \nu > 0$.

Problem 1. (i) Find the point $N$ that minimizes the function $F(X)$.

(ii) If $M$ is the point with barycentric coordinates $(\lambda, \mu, \nu)$, prove that $M$ and $N$ are isogonal conjugate.

Solution. To solve (i), consider the system

\[(3.2)\]

\[F(X) = \lambda x^2 + \mu y^2 + \nu z^2, \quad ax + by + cz = 2S; \quad x, y, z \in \mathbb{R}\]

and interpret $(x, y, z)$ as Cartesian coordinates in the three dimensional Euclidean space. The level surfaces of the function $F(X)$ are the ellipsoids

\[\varepsilon(k) : \quad \lambda x^2 + \mu y^2 + \nu z^2 = k, \quad k = \text{const} \in (0, \infty).\]

Geometrically, to find the point that minimizes the function $F(X)$ in (2.2), means to find $k$ so that the ellipsoid $\varepsilon(k)$ is tangent to the plane $\pi : ax + by + cz = 2S$ and then to determine the touch-point $N$ of $\varepsilon(k)$ to $\pi$. 


The tangent plane to $\varepsilon(k)$ at a point $(x_0, y_0, z_0)$ is given by the equality

$$\tau : \lambda x_0 + \mu y_0 + \nu z_0 = k.$$ 

Then the condition $\tau \equiv \pi$ implies that

$$\frac{\lambda x_0}{a} = \frac{\mu y_0}{b} = \frac{\nu z_0}{c} = t,$$

(3.3) 

$$t(\lambda x_0 + \mu y_0 + \nu z_0) = k,$$

$$\lambda x_0 + \mu y_0 + \nu z_0 = 2S.$$ 

Solving (3.3), we find:

$$t = \frac{2S}{J}, \quad k = \frac{4S^2}{J}, \quad \left( J = \frac{a^2}{\lambda} + \frac{b^2}{\mu} + \frac{c^2}{\nu} \right)$$

and

(3.4) 

$$x_0 = \frac{2S a}{J \lambda'}, \quad y_0 = \frac{2S b}{J \mu'}, \quad z_0 = \frac{2S c}{J \nu'}.$$ 

Now, taking into account (3.4), we conclude that the point $N$ minimizing the function $F(X)$ has barycentric coordinates $(\lambda', \mu', \nu')$ with respect to $\triangle ABC$ given by

(3.5) 

$$\lambda' = \frac{ax_0}{2S} = \frac{a^2}{J \lambda'}, \quad \mu' = \frac{by_0}{2S} = \frac{b^2}{J \mu'}, \quad \nu' = \frac{cz_0}{2S} = \frac{c^2}{J \nu'}$$

and $F_{\text{min}} = k = \frac{4S^2}{J}$, which solves (i).

To prove (ii), let us denote by $M$ the point with homogeneous barycentric coordinates $(\lambda, \mu, \nu)$. Comparing with (1.1) we conclude that formulas (3.5) are a representation of the isogonal conjugation in barycentric coordinates. Hence, the point $N(\lambda', \mu', \nu')$ is the isogonal conjugate one to the point $M(\lambda, \mu, \nu)$.

In this case $M$ and $N$ are in $\sigma$.

Next we consider the case

1.2. $\lambda < 0, \mu > 0, \nu > 0$.

In this case the problem states as follows:

**Problem 2.** Prove that $F(X)$ has a minimum if and only if

$$J = \frac{a^2}{\lambda} + \frac{b^2}{\mu} + \frac{c^2}{\nu} < 0.$$ 

(i) Find the point $N$ that minimizes the function $F(X)$.

(ii) If $M$ is the point with homogeneous barycentric coordinates $(\lambda, \mu, \nu)$, prove that $M$ and $N$ are isogonal conjugate.

**Solution.** Consider the system (3.2). In this case any level surface of the function $F(X)$

$$\varepsilon(k) : \lambda x^2 + \mu y^2 + \nu z^2 = k, \quad k = \text{const} \in \mathbb{R}$$

is one of the following: one sheet hyperboloid if $k > 0$; cone if $k = 0$; two sheet hyperboloid if $k < 0$.

Geometrically, to find the point that minimizes the function $F(X)$ in (3.2), means to find $k < 0$ so that the two sheet hyperboloid $\varepsilon(k)$ is tangent to the plane $\pi : ax + by + cz = 2S$ and then to determine the touch-point $N$ of $\varepsilon(k)$ to $\pi$. 

The plane $\pi$ can be tangent to $\varepsilon(k)$ only if $k < 0$.
Similarly to the solution of Problem 1 we obtain the system (3.4), which implies that
$$
t J = 2S, \quad 2St = k.$$
Therefore $\varepsilon(k)$ is tangent to the plane $\pi$ only in the case $J < 0$.
Further, we find the coordinates of the touch-point

$$
x_0 = \frac{2S a}{J}, \quad y_0 = \frac{2S b}{J}, \quad z_0 = \frac{2S c}{J}.
$$

Thus, the point $N$ minimizing the function $F(X)$ lies in the domain $\sigma'_{13}$.

Let $M$ be the point with barycentric coordinates $(\lambda, \mu, \nu)$. Then the formulas (3.6) show
that the point $N(\lambda', \mu', \nu')$ has barycentric coordinates

$$
\lambda' = \frac{a^2}{\lambda J}, \quad \mu' = \frac{b^2}{\mu J}, \quad \nu' = \frac{c^2}{\nu J}
$$

and it is isogonal conjugate to the point $M$.
Hence, the triple $(\lambda, \mu, \nu)$ determines a point $M$ in the domain $\sigma_{13}$ and

$$
F_{min} = F(N) = \frac{4S^2}{J} < 0,
$$

where $N \in \sigma'_{13}$ is the isogonal conjugate to the point $M$.

4. Weighted sum with $\lambda \mu \nu \neq 0$, $\lambda + \mu + \nu < 0$

If $\lambda + \mu + \nu < 0$, then we put $\bar{\lambda} = -\lambda$, $\bar{\mu} = -\mu$, $\bar{\nu} = -\nu$ and consider the function

$$
\bar{F}(X) = \bar{\lambda} x^2 + \bar{\mu} y^2 + \bar{\nu} z^2 = -F(X), \quad \bar{\lambda} + \bar{\mu} + \bar{\nu} > 0.
$$

We suppose again that $\bar{\lambda} + \bar{\mu} + \bar{\nu} = 1$ and consider the point $M$ with barycentric coordinates $(\bar{\lambda}, \bar{\mu}, \bar{\nu})$.
Comparing with Section 2 we have the following.

2.1. $\lambda < 0$, $\mu < 0$, $\nu < 0$

Under these conditions $J = \frac{a^2}{\lambda} + \frac{b^2}{\mu} + \frac{c^2}{\nu} < 0$.
Then

$$
M(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \in \sigma, \quad N = \iota(M) \in \sigma, \quad \bar{J} > 0, \quad F_{max} = F(N) = \frac{4S^2}{\bar{J}} < 0.
$$

2.2. $\lambda > 0$, $\mu < 0$, $\nu < 0$
2.2.1. \( J < 0 \)

The function \( \bar{F} = -F \) has no minimum or maximum.

2.2.2. \( J > 0 \)

\[ M(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \in \sigma_{13}, \quad N = \iota(M) \in \sigma'_{13}, \quad \bar{J} < 0, \quad F_{\max} = F(N) = \frac{4S^2}{J} > 0. \]

2.3. \( \lambda < 0, \mu > 0, \nu > 0 \)

The function \( \bar{F} = -F \) has no minimum or maximum.

5. Weighted sum with \( \lambda\mu\nu \neq 0, \lambda + \mu + \nu = 0 \)

Let us consider the system of mass points \( \{A(\lambda), B(\mu), C(\nu)\} \) and denote by \( P(\mu + \nu) \) the center of mass of the system \( \{B(\mu), C(\nu)\} \) (Fig. 2). If \( O \) is an arbitrary point, we have

\[ \lambda \overrightarrow{OA} + \mu \overrightarrow{OB} + \nu \overrightarrow{OC} = -\lambda \overrightarrow{AP} = \mu \overrightarrow{AB} + \nu \overrightarrow{AC} = \overrightarrow{v} = \text{const}. \]

Calculating

\[ \overrightarrow{v}^2 = -\lambda\mu\nu \left( \frac{a^2}{\lambda} + \frac{b^2}{\mu} + \frac{c^2}{\nu} \right) = -\lambda\mu\nu J, \]

we obtain the geometric meaning of \( J \) in the case \( \lambda + \mu + \nu = 0 \).

The condition \( \overrightarrow{v}^2 > 0 \) implies that

\[ (5.1) \quad \lambda\mu\nu J < 0. \]

Fig. 2

In this section we consider two cases: \( \lambda < 0, \mu > 0, \nu > 0 \) and \( \lambda > 0, \mu < 0, \nu < 0 \).

3.1. \( \lambda\mu\nu < 0 \).

In this case \( M(\lambda, \mu, \nu) \) can be interpreted as a point at infinity, i.e. the point at infinity of the line \( AP \) (Fig. 2). The inequality (5.1) implies that \( J > 0 \). Using similar arguments as in Section 2 we conclude that the function \( F(X) \) has neither minimum nor maximum.

3.2. \( \lambda\mu\nu > 0 \).

Under these conditions the inequality (5.1) implies that \( J < 0 \). We consider the function \( \tilde{F} = -F \). Since \( \tilde{\lambda}\tilde{\mu}\tilde{\nu} < 0 \), comparing with the previous case we conclude that \( \tilde{F} \) i.e. \( F \) has no minimum or maximum.
6. Weighted Sum with One or Two Zero Weights

4.1. $\lambda = 0$, $\mu > 0$, $\nu > 0$.

In this case the level surfaces of the function $F(X)$ are the elliptic cylinders

$$\mu y^2 + \nu z^2 = k, \quad k = \text{const} \in (0, \infty),$$

and the axis $Ox$, when $k = 0$. The level surfaces intersect the plane $ax + by + cz = 2S$ into ellipses by $k > 0$ and in the point $(2S, 0, 0)$ by $k = 0$.

The minimum of the function $F(X)$ is $F_{\text{min}} = 0$ and it occurs when $y = z = 0$. Thus $M(0, \mu, \nu)$ is a point on the sideline $BC$ and the minimum $F_{\text{min}} = 0$ occurs when $N \equiv A$, i.e. $M$ and $N$ are again isogonal conjugate.

4.2. $\lambda = 0$, $\mu < 0$, $\nu < 0$.

$M(0, \bar{\mu}, \bar{\nu}) \in BC$, $N = \tau(M) = A$, $F_{\text{max}} = F(N) = 0$.

4.3. $\lambda = 0$, $\mu \nu < 0$.

In this case the level surfaces of the function $F(X)$

$$\mu y^2 + \nu z^2 = k, \quad k \in (-\infty, +\infty)$$

are hyperbolic cylinders if $k \neq 0$ and two planes if $k = 0$.

Hence the function $F(X)$ has no minimum or maximum.

5.1. $\lambda = 1$, $\mu = \nu = 0$.

It is clear that $F_{\text{min}} = 0$ and it occurs when $x = 0$.

Thus $M(1, 0, 0) \equiv A$ and $N$ is any point on $BC$, i.e. $M$ and $N$ are again isogonal conjugate.

5.2. $\lambda = -1$, $\mu = 0$, $\nu = 0$.

$M(1, 0, 0) \equiv A$, $N \in BC$, $F_{\text{max}} = F(N) = 0$.

Summarizing we get the following:

1. $\lambda + \mu + \nu > 0$.

If $M(\lambda, \mu, \nu)$ is an inner point for the circumcircle $k$, or coincides with a vertex of $\triangle ABC$, then $F$ has a minimum and $F_{\text{min}} = F(N)$, $N = \tau(M)$.

2. $\lambda + \mu + \nu < 0$.

If $M(-\lambda, -\mu, -\nu)$ is an inner point for the circumcircle $k$, or coincides with a vertex of $\triangle ABC$, then $F$ has a maximum and $F_{\text{max}} = F(N)$, $N = \tau(M)$.

3. $\lambda + \mu + \nu = 0$.

In this case the function $F$ has neither a minimum nor a maximum.

7. Examples

We choose as a typical example the following pair of conjugate points: the circumcenter $O$ and the orthocenter $H$.

1. Let $M \equiv O$. Then $\lambda = \sin 2\alpha$, $\mu = \sin 2\beta$, $\nu = \sin 2\gamma$ and the point $O$ generates the function

$$F(X) = \sin 2\alpha x^2 + \sin 2\beta y^2 + \sin 2\gamma z^2.$$
1.1. $\triangle ABC$ is acute-angled and $O \in \sigma$. Simple calculations show that

$$J = \frac{4S}{\cos \alpha \cos \beta \cos \gamma}, \quad F_{\text{min}} = F(H) = 4S \cos \alpha \cos \beta \cos \gamma.$$

1.2. $\alpha = 90^0$ and $O$ is the midpoint of $BC$. Then

$$F_{\text{min}} = F(A) = 0.$$

1.3. $\alpha > 90^0$ and $O \in \sigma_{13}$. Then

$$J = \frac{4S}{\cos \alpha \cos \beta \cos \gamma} < 0, \quad F_{\text{min}} = F(H) = 4S \cos \alpha \cos \beta \cos \gamma < 0.$$

**Remark.** The function $F(X)$ generates geometric inequalities.

Let $M(\lambda, \mu, \nu) \in \sigma$ generate the function (3.1). If we choose a concrete triangle center $X$ in $\sigma$ with trilinear coordinates $(x, y, z)$ and replace in (3.1), then we obtain the geometric inequality

$$F(X) \geq F_{\text{min}} = \frac{4S^2}{J}.$$

The equality occurs if and only if $X \equiv \iota(M)$.

In a similar way the case $M \in \sigma_{13}$ generates even more interesting geometric inequalities.

**REFERENCES**

[1] C. Kimberling, *Trilinear Distance Inequalities for the Symmedian Point, the Centroid, and Other Triangle Centers*, Forum Geometricorum, 10 (2010), 135-139.

[2] E. Lemoine, *Sur quelques propriétés d’un point remarquable d’un triangle*, Association francaise pour l’avancement des sciences, Congrès (002; 1873; Lyon), (1874), 90-95.