The Classification of Diffeomorphism Classes of Real Bott Manifolds

Admi Nazra

Abstract—A real Bott manifold (RBM) is obtained as the orbit space of the $n$-torus $T^n$ by a free action of an elementary abelian $2$-group $(\mathbb{Z}_2)^n$. This paper deals with the classification of some particular types of RBMs of dimension $n$, so that we know the number of diffeomorphism classes in such RBMs.

Index Terms—Real Bott manifolds, orbit space, diffeomorphism classes, Seifert fiber space.

I. INTRODUCTION

KAMISHIMA et al. [1], [2] defined a real Bott manifold of dimension $n$ (RBM$_n$) as the total space $B_n$ of the sequence of $\mathbb{R}P^1$-bundles

$$B_n \to B_{n-1} \to \cdots \to B_2 \to B_1 \to \{\text{a point}\}$$  \hspace{1cm} (1)

starting with a point, where each $\mathbb{R}P^1$-bundle $B_i \to B_{i-1}$ is the projectivization of the Whitney sum of a real line bundle $L_i$ and the trivial line bundle over $B_{i-1}$, then, from the viewpoint of group actions, it was explained that a RBM$_n$ is the quotient of the torus of dimension $n$, $T^n = S^1 \times \cdots \times S^1$ by the product $(\mathbb{Z}_2)^n$ of cyclic group of order 2. Such RBM$_n$ can be expressed by an upper triangular matrix $A$ of size $n$ (called a Bott matrix of size $n$, BM$_n$) whose entries are either 1 or 0 except the diagonal entries which are 0. Each row of the BM$_n$ $A$ express the free action of $(\mathbb{Z}_2)^n$ on $T^n$ and the orbit space $M_n(A) = T^n/(\mathbb{Z}_2)^n$ is the RBM$_n$. In fact, $M_n(A)$ is a Riemannian flat manifold (compact Euclidean space form). To classify RBM$_n$s, we can apply the Bieberbach Theorem [3] and by this theorem, it was obtained in [1], [4] the classification of RBM$_n$s up to dimension 4.

Kamishima and Nazra proved in [2] that every RBM$_n$ $M_n(A)$ admits an injective Seifert fibred structure which has the form $M_n(A) = T^k \times (\mathbb{Z}_2)^s \times M(B)$, that is there is a $k$-torus action on $M_n(A)$ whose quotient space is an $(n-k)$-dimensional real Bott orbifold $M_{n-k}(B)/((\mathbb{Z}_2)^s)$ by some $(\mathbb{Z}_2)^s$-action ($1 \leq s \leq k$). Moreover, they have proved the smooth rigidity that two RBM$_n$s $M_n(A_1)$ and $M_n(A_2)$ are diffeomorphic if and only if the corresponding actions $((\mathbb{Z}_2)^s_1,M_{n-k_1}(B_1))$ and $((\mathbb{Z}_2)^s_2,M_{n-k_2}(B_2))$ are equivariantly diffeomorphic. By the above rigidity we can determine the diffeomorphism classes of higher dimensional RBM$_n$s when the low dimensional ones with $(\mathbb{Z}_2)^s$-actions are classified. RBM$_n$s up to dimension 5 have been classified (see [5], [6]).

This paper aims to study the number of diffeomorphism classes in some particular types of RBM$_n$s.

II. PRELIMINARIES

In this section, we shall review some concepts from [2] related to the RBM.

A. Seifert fiber space

In a BM$_n$ $A$, each $i$-th row defines a $\mathbb{Z}_2$-action on $T^n$ by

$$g_i(z_1, z_2, \ldots, z_n) = (z_1, \ldots, z_{i-1}, -z_i, \tilde{z}_{i+1}, \ldots, \tilde{z}_n)$$

where $\tilde{z}_m$ is either $z_m$ or $\bar{z}_m$ depending on whether $(i, m)$-entry ($i < m$) is 0 or 1 respectively while $(i, i)$-(diagonal) entry 0 acts as $z_i \to -z_i$. Note that $\tilde{z}$ is the conjugate of the complex number $z \in S^1$. It is always trivial: $z_m \to z_m$ whenever $m < i$. Here $(z_1, \ldots, z_n)$ are the standard coordinates of the $n$-dimensional torus $T^n = S^1 \times \cdots \times S^1$ whose universal covering is the $n$-dimensional Euclidean space $\mathbb{R}^n$. The projection $p: \mathbb{R}^n \to T^n$ is denoted by

$$p(x_1, \ldots, x_n) = (e^{2\pi i x_1}, \ldots, e^{2\pi i x_n}) = (z_1, \ldots, z_n).$$

Those $g_1, \ldots, g_n$ constitute the generators of $(\mathbb{Z}_2)^n$.$^n$. In fact, $(\mathbb{Z}_2)^n$ acts freely on $T^n$ such that the orbit space $M_n(A) = T^n/(\mathbb{Z}_2)^n$ is a smooth compact $n$-dimensional manifold. In this way, given a BM$_n$ $A$, we obtain a free action of $(\mathbb{Z}_2)^n$ on $T^n$.

Let $\pi(A) = \langle \tilde{g}_1, \ldots, \tilde{g}_n \rangle$ be the lift of $(\mathbb{Z}_2)^n = \langle g_1, \ldots, g_n \rangle$ to $\mathbb{R}^n$. Then, we get

$$\tilde{g}_i(x_1, x_2, \ldots, x_n) = (x_1, \ldots, x_{i-1}, \frac{1}{2} + x_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_n)$$

where $\tilde{x}_m$ is either $x_m$ or $-x_m$. One can see that $\pi(A)$ acts properly discontinuously and freely on $\mathbb{R}^n$ as Euclidean motions. Note that $\pi(A)$ is a Bieberbach group which is a discrete uniform subgroup of the Euclidean group $E(n) = \mathbb{R}^n \rtimes O(n)$ (cf. [3]). It follows that

$$\mathbb{R}^n/\pi(A) = T^n/(\mathbb{Z}_2)^n = M_n(A).$$

Now, we consider the following moves (I, II, III) to $A$ under which the diffeomorphism class of RBM$_n$ $M_n(A)$ does not change.

I. If the $j$-th column has all 0-entries for some $j > 1$, then interchange the $j$-th column and the $(j - 1)$-th column. Next, interchange the $j$-th row and the $(j - 1)$-th row.

We perform move I iteratively to get a BM$_n$ $A'$.

$$A = \begin{pmatrix} 0 & * & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \\ \end{pmatrix}, A' = \begin{pmatrix} O_k & C \\ 0 & B \\ \end{pmatrix},$$

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\[
B = \begin{pmatrix}
0 & * \\
. & . \\
0 & 0
\end{pmatrix}.
\]

\(O_k\) is a \(k \times k\) zero matrix \((1 \leq k \leq n)\) and we call it a block zero matrix of size \(k\).

Note the following.

1. \(O_k\) is a maximal block of zero matrix.
2. As \(B\) is an \((n-k)\)-dimensional Bott matrix, we obtain a real Bott manifold \(M_{n-k}(B) = T^{n-k}/(\mathbb{Z}_2)^{n-k}\).

\[
M_n(A) = \frac{T^k \times T^{n-k}}{(\mathbb{Z}_2)^{k} \times (\mathbb{Z}_2)^{n-k}} = T^k \times M_{n-k}(B) = M_n(A').
\]

4. The matrix \(C\) corresponds to \((\mathbb{Z}_2)^k\)-action on \(T^{n-k}\).

**II** For an \(m\)-th row \((1 \leq m \leq k)\) whose entries in \(C\) are all zero, divide \(T^k \times M_{n-k}(B)\) by the corresponding \(\mathbb{Z}_2\)-action.

**III** If there are two rows, \(p\)-th row and \(ℓ\)-th row \((1 \leq p < \ell \leq k)\), having the common entries in \(C\), then compose the \(\mathbb{Z}_2\)-action of \(p\)-th row and \(\ell\)-th row and divide \(T^k \times M_{n-k}(B)\) by \(\mathbb{Z}_2\)-action.

By using **II**, **III**, the quotient is again diffeomorphic to \(T^k \times (\mathbb{Z}_2)^k M_{n-k}(B)\) but consequently the \((\mathbb{Z}_2)^k\)-action is reduced to the effective \((\mathbb{Z}_2)^*\)-action on \(T^k \times M_{n-k}(B)\).

Therefore \(A'\) reduces to

\[
A'' = \begin{pmatrix}
0_{k-s} & 0 & 0 \\
0 & 0_s & * \\
0 & 0 & B
\end{pmatrix} \tag{2}
\]

in which \(M_{n}(A') = T^k \times (\mathbb{Z}_2)^k M_{n-k}(B) = \frac{T^{k-s} \times T^s \times M_{n-k}(B)}{(\mathbb{Z}_2)^k \times (\mathbb{Z}_2)^s} = M_{n}(A'').\) Since \((\mathbb{Z}_2)^k\)-acts trivially on \(T^s \times M_{n-k}(B)\), we have \(M_{n}(A'') \cong T^k \times (\mathbb{Z}_2)^{n-k}(B)\).

Hereinafter, we write \(M_{n}(A)\) in place of \(M_{n}(A'')\).

**Remark 1:** Concerning \(*\) in (2), the group \((\mathbb{Z}_2)^s = \langle g_{k-s+1}, \ldots, g_k \rangle\) acts on \(T^k \times M_{n-k}(B)\) by

\[
g_{i}(z_1, \ldots, z_{k-s+1}, \ldots, z_k, z_{k+1}, \ldots, z_n) = (z_1, \ldots, z_{k-s+1}, -z_i, \ldots, z_k, [z_{k+1}, \ldots, z_n]) \tag{3}
\]

where \(\tilde{z} = \tilde{z}\). So there induces an action of \((\mathbb{Z}_2)^s\) on \(M_{n-k}(B)\) by

\[
g_{i}(\tilde{z}_{k+1}, \ldots, \tilde{z}_n) = [\tilde{z}_{k+1}, \ldots, \tilde{z}_n]. \tag{4}
\]

Moreover in [2], it was obtained the following theorem.

**Theorem 1 (Structure):** For a RBM \(_n M_{n}(A)\), there is a maximal \(T^k\)-action \((k \geq 1)\) such that \(M_{n}(A) = T^k \times (\mathbb{Z}_2)^s M_{n-k}(B)\) is an injective Seifert fiber space over the \((n-k)\)-dimensional real Bott orbifold \(M_{n-k}(B)/(\mathbb{Z}_2)^s\):

\[
T^k \rightarrow M_{n}(A) \rightarrow M_{n-k}(B)/(\mathbb{Z}_2)^s. \tag{5}
\]

There exist a central extension of the fundamental group \(\pi(A)\) of \(M_{n}(A)\):

\[
1 \rightarrow \mathbb{Z}^k \rightarrow \pi(A) \rightarrow Q_B \rightarrow 1 \tag{6}
\]

such that

(i) \(\mathbb{Z}^k\) is the maximal central free abelian subgroup

(ii) The induced group \(Q_B\) is the semidirect product \(\pi(B) \rtimes (\mathbb{Z}_2)^s\) for which \(\mathbb{R}^{n-k}/\pi(B) = M_{n-k}(B)\).

See [2] for the proof.

Using this theorem, a RBM \(_n M_{n}(A)\) which admits a maximal \(T^k\)-action \((k \geq 1)\) can be created from an RBM \(_{n-k} M_{n-k}(B)\) by a \((\mathbb{Z}_2)^s\)-action, and the corresponding \(BM_n\) \(A\) has the form as in (2) above.

**B. Affine maps between real Bott manifolds**

Next, to check whether two RBM’s are diffeomorphic, we can apply the following theorem.

**Theorem 2 (Rigidity):** Suppose that \(M_{n}(A_1)\) and \(M_{n}(A_2)\) are RBM’s and \(1 \rightarrow \mathbb{Z}^{k_1} \rightarrow \pi(A_1) \rightarrow Q_{B_1} \rightarrow 1\) is the associated group extensions \((i = 1, 2)\). Then, the following are equivalent:

(i) \(\pi(A_1)\) is isomorphic to \(\pi(A_2)\).

(ii) There exists an isomorphism of \(Q_{B_1} = \pi(B_1) \times (\mathbb{Z}_2)^{s_1}\) onto \(Q_{B_2} = \pi(B_2) \times (\mathbb{Z}_2)^{s_2}\) preserving \(\pi(B_1)\) and \(\pi(B_2)\).

(iii) The action \((\mathbb{Z}_2)^{s_1}, M_{n-k}(B_1))\) is equivariantly diffeomorphic to the action \((\mathbb{Z}_2)^{s_2}, M_{n-k}(B_2))\).

See [2] for the proof. Here Bott matrices \(A_1\) and \(A_2\) are created from \(B_1\) and \(B_2\) respectively.

Note that two RBM’s \(M_{n}(A_1)\) and \(M_{n}(A_2)\) are diffeomorphic if and only if \(\pi(A_1)\) is isomorphic to \(\pi(A_2)\) by the Bieberbach theorem [3]. Moreover, by Theorem 1 and 2 we have,

**Remark 2:** Let \(RB_{n} M_{n}(A_i) = T^{k_i} \times (\mathbb{Z}_2)^{s_i} M_{n-k}(B_i)\) \((i = 1, 2)\). If \(M_{n}(A_1)\) and \(M_{n}(A_2)\) are diffeomorphic then the following hold.

(i) \(k_1 = k_2\).

(ii) \(M_{n-k}(B_1)\) and \(M_{n-k}(B_2)\) are diffeomorphic.

(iii) \(s_1 = s_2\).

If two RBM’s have the same maximal \(T^k\)-action, then the quotients \((\mathbb{Z}_2)^{s_1}, M_{n-k}(B_1))\) are compared. So, what we have to do next is to distinguish the \((\mathbb{Z}_2)^{s_1}\)-action on \(M_{n-k}(B_1)\) when it is the case that \(s_1 = s_2 = s\) and \(M_{n-k}(B_1)\) is diffeomorphic to \(M_{n-k}(B_2)\).

**C. Type of fixed point set**

Note that from (4), the action of \((\mathbb{Z}_2)^s\) on \(M_{n-k}(B)\) is defined by \(\alpha(\tilde{z}_1, \ldots, \tilde{z}_{n-k}) = [\alpha(\tilde{z}_1, \ldots, \tilde{z}_{n-k}) = ([\tilde{z}_1, \ldots, \tilde{z}_{n-k}] = \tilde{z}\) for \(\alpha \in (\mathbb{Z}_2)^s\) and \(\tilde{z} = z\) or \(\tilde{z}\). Since \(M_{n-k}(B) = T^{n-k}/(\mathbb{Z}_2)^{n-k}\), the action \(\langle \alpha \rangle\) lifts to a linear (affine) action on \(T^{n-k}\) naturally: \(\alpha(\tilde{z}_1, \ldots, \tilde{z}_{n-k}) = (\tilde{z}_1, \ldots, \tilde{z}_{n-k})\). Then, the fixed point set is characterized by the equation: \(\tilde{z}_1, \ldots, \tilde{z}_{n-k} = g(z_1, \ldots, z_{n-k}) \) for some \(g \in (\mathbb{Z}_2)^{n-k}\). It is also an affine subspace of \(T^{n-k}\). So the fixed point sets of \((\mathbb{Z}_2)^s\) are affine subspaces in \(M_{n-k}(B)\).

Let \(B\) be the Bott matrix as in above. By a repetition of move \(I\), \(B\) has the form

\[
B = \begin{pmatrix}
0_{b_2} & C_{23} & \cdots & \cdots & C_{2t} \\
0_{b_3} & C_{34} & \cdots & C_{3t} \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0_{b_{t-1}} & C_{(t-1)t} & \cdots & C_{tt} \\
0 & 0_{b_t} & \cdots & \cdots & 0
\end{pmatrix} \tag{7}
\]
where rank $B = b_2 + \cdots + b_{\ell} = n - k$ ($b_j \geq 1$), $C_{j\ell}$ ($j = 2, \ldots, \ell - 1$, $t = 3, \ldots, \ell$) is a $b_j \times b_{\ell}$ matrix.

Note that by the Bieberbach theorem (cf. [3]), if $f$ is an isomorphism of $\pi_1(A_1)$ onto $\pi_1(A_2)$, then there exists an affine element $g = (h, H) \in \mathbb{A}(n) = \mathbb{R}^n \rtimes \text{GL}(n, \mathbb{R})$ such that

$$f(r) = g r g^{-1} \quad (\forall r \in \pi_1(A_1)).$$

Recall that if $M_n(A_1)$ is diffeomorphic to $M_n(A_2)$ then $M_{n-k}(B_1)$ is diffeomorphic to $M_{n-k}(B_2)$. This implies that $B_1$ and $B_2$ have the form as in (7).

Using (8) and according to the form of $B$ in (7) we obtain that

$$g = \left(\begin{array}{ccc}
\mathbf{h}_1 & H_1 & \mathbf{0} \\
\mathbf{h}_2 & H_2 & \mathbf{0} \\
\vdots & \vdots & \vdots \\
\mathbf{h}_\ell & H_\ell & \mathbf{0}
\end{array}\right)$$

(9)

where $\mathbf{h}_i$ is an $b_i \times 1$ ($s_i = \text{rank } I_i$) column matrix ($\mathbf{h}_1$ is a $k \times 1$ column matrix), $H_i \in \text{GL}(b_i, \mathbb{R})$ ($i = 2, \ldots, \ell$), $H_1 \in \text{GL}(k, \mathbb{R})$ (see Remark 3.2 [2]).

Let $\tilde{f} : Q_{B_1} \to Q_{B_2}$ be the induced isomorphism from $f$ (cf. Theorem 2). Now the affine equivalence $\tilde{g} : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ has the form

$$\tilde{g} = \left(\begin{array}{ccc}
\mathbf{h}_2 & H_2 & \mathbf{0} \\
\vdots & \vdots & \vdots \\
\mathbf{h}_\ell & H_\ell & \mathbf{0}
\end{array}\right)$$

(10)

which is equivalent with respect to $\tilde{f}$. The pair $(\tilde{f}, \tilde{g})$ induces an equivariant affine diffeomorphism $(f, g) : (\mathbb{Z}_2)^s, M_{n-k}(B_1)) \to (\mathbb{Z}_2)^s, M_{n-k}(B_2)).$

Let $\text{rank} H_i = b_i$ ($i = 2, \ldots, \ell$). (Note that $b_2 + \cdots + b_{\ell} = n - k$.) Since $M_{n-k}(B_1) = T^{n-k}/(\mathbb{Z}_2)^{n-k}$, $\tilde{g}$ induces an affine map $\tilde{g}$ of $T^{n-k}$. Put $X_{b_2} = \{x_1, \ldots, x_{b_2}\}$, $X_{b_\ell} = \{x_{b_\ell+1}, \ldots, x_{b_\ell+b_\ell-1}\}$, $w_{b_i} = p(X_{b_i}) \in T^{b_i}$ ($i = 2, \ldots, \ell$), $v_{b_{\ell-1}} = b_2 + \cdots + b_{\ell-1}$. Since $\tilde{g}p = p\tilde{g}$, $\tilde{g}(t w_{b_2}, \ldots, t w_{b_\ell}) = (t w'_{b_2}, \ldots, t w'_{b_\ell})$ where $w'_{b_i} = p(h_i + H_i X_{b_i}) \in T^{b_i}$. That is, $\tilde{g}$ preserves each $T^{b_i}$ of $T^{n-k} = T^{b_2} \times \cdots \times T^{b_{\ell}}$, so does $\tilde{g}$ on $M_{n-k}(B_1) = \{(z_1, \ldots, z_{b_2}; z_{b_2+1}, \ldots, z_{b_2+b_2}; \ldots; z_{b_{\ell-1}+1}, \ldots, z_{b_{\ell-1}+b_{\ell-1}})\}$.

We say that $\tilde{g}$ preserves the type $(b_2, \ldots, b_{\ell})$ of $M_{n-k}(B_1)$.

As $\tilde{f}$ is $\tilde{g}$-equivariant, it also preserves the type corresponding to the fixed point sets between $((\mathbb{Z}_2)^s, M_{n-k}(B_1))$ and $((\mathbb{Z}_2)^s, M_{n-k}(B_2))$.

**Proposition 1:** The $(\mathbb{Z}_2)^s$-action on $M_{n-k}(B)$ is distinguished by the number of components and types and each positive dimensional fixed point subsets.

See [2] for the proof.

**Definition 1:** We say that two Bott matrices $A$ and $A'$ are equivalent (denoted by $A \sim A'$) if $M_n(A)$ and $M_n(A')$ are diffeomorphic.

### III. CLASSIFICATION OF PARTICULAR TYPES OF RBM$_n$s

In this part, we will review some results from [6] and prove some new results regarding the classification of certain $n$-dimensional real Bott manifolds in order to obtain how many diffeomorphism classes of some particular types of RBM$_n$s.

**Proposition 2:** [6] There are 4 diffeomorphism classes of RBM$_n$s ($n \geq 4$) which admit the maximal $T^{n-2}$-actions (i.e., $s = 1, 2$):

$$M_n(A) = T^{(n-2)} \times M_2(B).$$

**Proposition 3:** [6] The diffeomorphism class is unique for the RBM of the form $M_n(A) = T^k \times Z_2 \times S^1$ for any $k$ ($1 \leq k \leq n - 1$). In particular, if $k = n$ then $M_n(A) = T^n$.

**Remark 3:** By Proposition 3, for $n \geq 2$ there are $n$ distinct diffeomorphism classes of RBM$_n$s $M_n(A), M_n(B)$ of the form $M_n(A) = T^k \times Z_2 \times S^1$ for $k \leq n - 1$.

**Corollary 1:** [6] If the RBM $M(A) = S^1 \times Z_2 \times S^1$ where $M(B) = T^k \times Z_2 \times S^1$, then for any $k \geq 1$ there is only one diffeomorphism class.

**Remark 4:** By Corollary 1, for $n \geq 3$ there are $n - 2$ distinct diffeomorphism classes of RBM$_n$s $M_n(A), M_n(B)$ of the form $M_n(A) = T^k \times Z_2 \times S^1$, $k \leq n - 2$.

**Corollary 2:** [6] Let $M(A)$ be a real Bott manifold which fibers $S^1$ over the real Bott manifold $M(B)$ for which $M(B)$ is $T^k \times Z_2 \times S^1$, $k \geq 2$. Here $K$ is Klein bottle. Then the number of diffeomorphism classes of such $M(A)$ is $3$.

**Remark 5:** By Corollary 2, for $n \geq 5$ there are $(3(n - 4))$ distinct diffeomorphism classes of RBM$_n$s $M_n(A), M_n(B)$ of the form $M_n(A) = T^k \times Z_2 \times S^1$, $k \leq n - 2, s = 1, 2$.

**Corollary 3:** [6] Let $M(A)$ be a real Bott manifold which fibers $S^1$ over the real Bott manifold $M(B)$ for which $M(B)$ is $T^k \times Z_2 \times S^1$, $k \geq 2$. Then the number of diffeomorphism classes of $M(A)$ is $3$.

**Remark 6:** By Corollary 3, for $n \geq 5$ there are $(3(n - 4))$ distinct diffeomorphism classes of RBM$_n$s $M_n(A), M_n(B)$ of the form $M_n(A) = T^k \times Z_2 \times S^1$, $k \leq n - 2, s = 1, 2$.

**Proposition 4:** [6] Let $M(A)$ be a real Bott manifold which fibers $S^1$ over the real Bott manifold $M(B)$, where $M(B) = S^1 \times Z_2 \times T^k$, $k \geq 2$, then the diffeomorphism classes of such $M(A)$ is $[2]^k$. Here $[x]$ is the integer part of $x$.

**Remark 7:** By Proposition 4, for $n \geq 4$ there are $\sum_{k=1}^{n-2}(\binom{n-2}{k} + 1)$ distinct diffeomorphism classes of RBM$_n$s $M_n(A), M_n(B)$ of the form $M_n(A) = S^1 \times Z_2 \times T^k$, $k \leq n - 2, s = 1, 2$.

**Proposition 5:** For any $k \geq 1$ and $m \geq 2$ ($n - 3 \geq k + m = t \geq 3$), there are $\left[\frac{n-2}{2}\right] + 1$ diffeomorphism classes in RBM$_n$s $M_n(A), M_n(B)$ of the form $M_n(B) = T^m \times Z_2 \times T^{n-k-m}$.

**Proof:** Similar with the proof of Proposition 4 (see [6]).

**Remark 8:** By Proposition 5, for $n \geq 6$ there are

$$\sum_{k=1}^{n-5} \sum_{m=k+2}^{n-3} \left(\binom{n-2}{k} + 1\right)$$
distinct diffeomorphism classes of $RBM_n$s $M_n(A) = T^k \times Z_2 \ M_{n-k}(B)$ ($k = 1, \ldots, n - 5$) where $M_{n-k}(B) = T^m \times Z_2 \ T^{n-t}$ ($m \geq 2, \ n - 3 \geq t \geq 3$).

Proposition 6: [6] Let $M_n(A) = S^3 \times Z_2 \ M_{n-k}(B)$ be a $RBM_n$. Suppose that $B$ is either one of the list in (11). Then $M_{n-k}(B)$ are diffeomorphic to each other and the number of diffeomorphism classes of such $RBM_n$s $M_n(A)$ above is $(k + 1)2^{n-k-3}$ ($k \geq 2, \ n - k \geq 3$).

\[
B_1 = \begin{pmatrix}
0 & 1 & 0 & 1 & \ldots & 1 \\
n & 0 & 1 & 0 & \ldots & 1 \\
0 & 1 & 0 & 1 & \ldots & 1 \\
0 & 0 & 1 & 0 & \ldots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 1 & \ldots & 1 \\
0 & \ldots & 0 & 1 & 0 & \ldots & 1
\end{pmatrix},
\]

\[
B_2 = \begin{pmatrix}
0 & 1 & 0 & 1 & \ldots & 1 \\
n & 0 & 1 & 0 & \ldots & 1 \\
0 & 1 & 0 & 1 & \ldots & 1 \\
0 & 0 & 1 & 0 & \ldots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 1 & \ldots & 1 \\
0 & \ldots & 0 & 1 & 0 & \ldots & 1
\end{pmatrix},
\]

\[
B_{n-k-1} = \begin{pmatrix}
0 & 1 & 0 & 1 & \ldots & 1 \\
n & 0 & 1 & 0 & \ldots & 1 \\
0 & 1 & 0 & 1 & \ldots & 1 \\
0 & 0 & 1 & 0 & \ldots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 1 & \ldots & 1 \\
0 & \ldots & 0 & 1 & 0 & \ldots & 1
\end{pmatrix},
\]

\[
B_{n-k} = \begin{pmatrix}
0 & 1 & 0 & 1 & \ldots & 1 \\
n & 0 & 1 & 0 & \ldots & 1 \\
0 & 1 & 0 & 1 & \ldots & 1 \\
0 & 0 & 1 & 0 & \ldots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 1 & \ldots & 1 \\
0 & \ldots & 0 & 1 & 0 & \ldots & 1
\end{pmatrix},
\]

\[
B_{n-k+(n-k-4)} = \begin{pmatrix}
0 & 1 & 0 & 1 & \ldots & 1 \\
n & 0 & 1 & 0 & \ldots & 1 \\
0 & 1 & 0 & 1 & \ldots & 1 \\
0 & 0 & 1 & 0 & \ldots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 1 & \ldots & 1 \\
0 & \ldots & 0 & 1 & 0 & \ldots & 1
\end{pmatrix}.
\]  

(11)

Remark 9: By Proposition 6, for $n \geq 5$ there are

\[
\sum_{\ell=3}^{n-2} \sum_{k=2}^{\ell-3} (k+1)2^{\ell-k-3}
\]
distinct diffeomorphism classes of $RBM_n$s $M_n(A) = T^{k'} \times Z_2 \ M_{n-k'}(B)$ ($k' = 1, \ldots, n - 4$) where $B$ is either one of the list in (11).

Now we consider the other type of real Bott manifolds.

Proposition 7: Let $M_n(A) = T^k \times (Z_2)^2 T^{\ell},$ ($n = k + \ell \geq 5, \ell \geq 3$) be a $RBM_n$. Then the number of diffeomorphism classes of such $M_n(A)$ is

\[
\sum_{\ell=3}^{n-2} \sum_{x=1}^{\ell} \left(\frac{\ell}{2} - \frac{x}{2}\right) - (x - 1))
\]

diffeomorphism classes of such $M_n(A)$ corresponding to Bott matrices as in (15).

Proof: The proposition follows from Lemmas 1, 2 below.

Lemma 1: Let $M_n(A)$ be an $RBM_n$ ($n \geq 5$) corresponding to the Bott matrix $A = \begin{pmatrix} O_k & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \ldots & 1 \end{pmatrix} (\ell \geq 3).$ (14)

Then the number of diffeomorphism classes of such $M_n(A)$ is $\sum_{\ell=3}^{n-2} \left(\frac{\ell}{2}\right).$

Proof: We associate with the pair $(y, x)$ the Bott matrix (14) where $y = n - x$ and $x$ are the numbers of zero entries in the $(k - 1)$-th row and $\ell$-th row respectively of the right-upper block matrix. Here $1 \leq x \leq \ell - 1.$ Because of move I, we may assume that $x \leq \ell - x.$ So $1 \leq x \leq \left\lfloor \frac{\ell}{2}\right\rfloor.$ For a fixed numbers $\ell$ and $x$, it is easy to check that the fixed point sets of $((Z_2)^2, T^{\ell})$ corresponding to (14) are $2^x$ components $T^{\ell - x}$ and $2^{\ell - x}$ components $T^x$.

For a fixed number $\ell$, suppose that Bott matrices $A_1$ and $A_2$ correspond to the pairs $(y_1, x_1)$ and $(y_2, x_2)$ respectively. If $x_1 \neq x_2$, then by Proposition 1, $M_n(A_1)$ is not diffeomorphic to $M_n(A_2)$.

Therefore for a fixed number $\ell$, there are $\left\lfloor \frac{\ell}{2}\right\rfloor$ diffeomorphism classes of such $RBM_n$s. This implies the lemma.

Lemma 2: Let $M_n(A)$ be a $RBM_n$ ($n \geq 5$) corresponding to the Bott matrix

\[
A = \begin{pmatrix} O_k & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \ldots & 1 \end{pmatrix} (\ell \geq 3).
\]

Then the number of diffeomorphism classes of such $M_n(A)$ is $\sum_{\ell=3}^{n-2} \sum_{x=1}^{\ell} \left(\frac{\ell}{2} - \frac{x}{2}\right) - (x - 1)).$

Proof: We associate with the pair $(t, x)$ the Bott matrix (15) where $x$ is the number of zero entries in the $\ell$-th row of the right-upper block matrix and $t(\neq 0)$ is the number of columns having two non zero entries. Because of move I, we may assume that $1 \leq x \leq \ell - x - t$ and $x \leq \left\lfloor \frac{\ell}{2}\right\rfloor.$ So $1 \leq x \leq \ell - t \leq \left\lfloor \frac{\ell}{2}\right\rfloor.$ For fixed numbers $\ell$, $x$ and $t$, it is easy to check that the fixed point sets of $((Z_2)^2, T^{\ell})$ corresponding to (15) are $2^{t+x}$ components $T^{\ell - x - t}$ and $2^{\ell - x}$ components $T^x$.

For a fixed number $\ell$, suppose that Bott matrices $A_1$ and $A_2$ correspond to the pairs $(t_1, x_1)$ and $(t_2, x_2)$ respectively. If $t_1 \neq t_2$ or(and) $x_1 \neq x_2$, then by Proposition 1, $A_1$ is not equivalent to $A_2$.

Therefore for fixed $\ell$, and $x$, there are $\left(\frac{\ell - x}{2}\right) - (x - 1)$ diffeomorphism classes of such $RBM_n$s. Hence there are $\sum_{\ell=3}^{n-2} \sum_{x=1}^{\ell} \left(\frac{\ell - x}{2}\right) - (x - 1)$
diffeomorphism classes of such $M_n(A)$ corresponding to Bott matrices as in (15).

Since the fixed point sets of $((Z_2)^2, T^{\ell})$ corresponding to Bott matrices (14) and (15) are different, the corresponding real Bott manifolds are not diffeomorphics.
Remark 10: It is hard task algebraically to determine the number of $n$-dimensional $M_n(A) = T^k \times (\mathbb{Z}_2)^r T^t$ for $3 \leq s \leq \min\{n - \ell, \ell\}$. However we shall consider a special type in (12).

We associate with $(\ell_1, \ell_2, \ldots, \ell_{s-1}, \ell_s)$ the Bott matrix (12) where $\ell_1 = \ell - \sum_{i=2}^{s} \ell_s$, $\ell_2, \ell_3, \ldots, \ell_{s-1}, \ell_s$ are the number of nonzero entries at $k$-row, $(k-1)$-row, $(k-2)$-row, ..., $(k-(s-2))$-row, $(k-(s-1))$-row respectively in the right-upper block matrix. As in the arguments in the proof of Lemmas 1, 2 above, in order to obtain the diffeomorphism classes $RBM M(A)$, we may assume that $\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_{s-1} \geq \ell_s \geq 1$ and $1 \leq \ell_s \leq \lfloor \frac{\ell}{s+1} \rfloor$. For any $\ell_s$ we determine the values of $\ell_{s-1}$, namely $\ell_s \leq \ell_{s-1} \leq \lfloor \frac{\ell-t_s-t_{s-1}}{s-1} \rfloor$. Repeating the previous argument, we obtain that

$$
\ell_{t+1} \leq \ell_t \leq \left[ \frac{\ell - \sum_{i=1}^{s} \ell_s}{t} \right], \quad t = 2, \ldots, s - 2, s - 1.
$$

It is easy to check that for fixed natural numbers $\ell_1, t = 3, 4, \ldots, s - 1, s$ and $\ell$, there are $\frac{[\ell - \sum_{i=2}^{s} \ell_i]}{s} - (\ell_s - 1)$ diffeomorphism classes of $RBM M(A)$.

Hence the number of diffeomorphism classes of $RBM M(A)$ is

$$
N^x_{\ell} = \sum_{\ell_{s-1} = \ell_s}^{\lfloor \frac{\ell}{s+1} \rfloor} \sum_{\ell_1 = \ell_{t+1}}^{[\ell - \sum_{i=2}^{s} \ell_i]} \cdots \sum_{\ell_3 = \ell_4}^{\lfloor \ell - \sum_{i=3}^{s} \ell_i \rfloor} -(\ell_3 - 1)
$$

diffeomorphism classes of $M(A)$ for $3 \leq s \leq \min\{n - \ell, \ell\}$.

The number of diffeomorphism classes of $RBM M(A)$ is

$$
N^x_{\ell} = \sum_{\ell = 3}^{n-3} \sum_{s=3}^{\min\{n-\ell, \ell\}} N^x_{\ell}.
$$

Let $N_n$ be the number of diffeomorphism classes of $RBM_n$.

Choi[7] classified $RBM_n$ corresponding to the following Bott matrices. He considers $\ell \times \ell$ Bott matrices $A_\ell$ of rank $\ell - 1$ all of whose diagonals are 0. Then for such $A_\ell$, $i, i+1$-entry must be 1 for $i = 1, \ldots, \ell - 1$. Masuda[8] proved that for such matrices, there are $2(\ell-2)(\ell-3)/2$ diffeomorphism classes of $\ell$-dimensional real Bott manifolds.

Next Choi considers an $n \times n$ Bott matrix $A$ such that the rank of submatrix consisting of the first $\ell$ columns is $\ell - 1$ and the last $n - \ell$ columns are zero vectors (i.e., $A = \begin{pmatrix} A_\ell & 0 \\ 0 & 0 \end{pmatrix}$).

By move I, the Bott matrix $A$ is equivalent to

$$
A = \begin{pmatrix} 0 & 0 \\ 0 & A_\ell \end{pmatrix}.
$$

By using the result of Masuda above, Choi[7] obtained that the number of diffeomorphism classes of $RBM_n$ corresponding to Bott matrices (16) for $\ell = 2, \ldots, n$ is $\sum_{\ell=2}^{n} 2(\ell-2)(\ell-3)/2$.

Masuda[8] found that

$$
2(\ell-2)(\ell-3)/2 \leq N_n,
$$

by considering the Bott matrices $A_\ell$ above. Then, Choi[7] improved the Masuda’s result where he considers Bott matrices (16).

Theorem 3 ([7]): $2(\ell-2)(\ell-3)/2 < \sum_{\ell=2}^{n} 2(\ell-2)(\ell-3)/2 \leq N_n$.

By using Propositions 7, 2, Theorem 3, Remarks 3, 4, 5, 6, 7, 8, 9, 10, we obtain an improvement of the previous results about $N_n$.

Theorem 4: For $n \geq 4$,

$$
N_n \geq 8n + \sum_{\ell=2}^{n} 2(\ell-2)(\ell-3)/2 + \sum_{\ell=2}^{n} (\ell/2) + 1 + \sum_{k=1}^{n-5} \sum_{m=2}^{n-3} ((\ell-t_k)/2) + 1) \sum_{\ell=3}^{n-3} \sum_{s=3}^{\min\{n-\ell, \ell\}} N^x_{\ell} - 26.
$$

with

$$
N^x_{\ell} = \sum_{\ell_{s-1} = \ell_s}^{\lfloor \frac{\ell}{s+1} \rfloor} \sum_{\ell_1 = \ell_{t+1}}^{[\ell - \sum_{i=2}^{s} \ell_i]} \cdots \sum_{\ell_3 = \ell_4}^{\lfloor \ell - \sum_{i=3}^{s} \ell_i \rfloor} -(\ell_3 - 1)
$$

We assume that if $u < u_0$ in a summation $\sum_{u=u_0}^{n} u$, the value of such summation is equal to zero.
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