ON GROWTH IN TOTALLY ACYCLIC MINIMAL COMPLEXES

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ABSTRACT. Given a commutative Noetherian local ring, we provide a criterion under which a totally acyclic minimal complex of free modules has symmetric growth.

1. INTRODUCTION

Given a commutative Noetherian local ring, when does a totally acyclic minimal complex of free modules have symmetric growth? In other words, given such a complex, does the left growth of the ranks of its free modules equal the right growth? Avramov and Buchweitz show in [AvB] that this is always the case for totally acyclic minimal complexes of free modules over local complete intersections. However, Jorgensen and Šega showed in [JoS] that it does not hold for a local ring in general, even when the ring is Gorenstein. In fact, they constructed such a ring and a totally acyclic minimal complex whose left growth is exponential and right growth is constant. (The characteristics of growth in the dual complex are thus reversed.)

Totally acyclic complexes can be used to compute Tate, or stable (co)homology. Since characteristics of the underlying complex are often reflected in characteristics of the derived (co)homology, it is of interest to study general properties of the underlying complexes.

In this paper, we give a criterion under which symmetric growth of totally acyclic minimal complexes holds. This criterion is given in terms of the cohomology of the image of a given differential in the complex. Namely, we show that if the cohomology is finitely generated with respect to a ring acting centrally on the derived category, and the ring action commutes with dualization, then the complex has symmetric polynomial growth. As a corollary of our main theorem we prove that whenever an image in the complex and its dual have complete intersection dimension zero, then the complex has symmetric polynomial growth. Since all such images and their duals have complete intersection dimension zero when the ring is a local complete intersection, we recover the result of Avramov and Buchweitz cited above.

2. NOTATION AND TERMINOLOGY

Throughout this section, we fix a local (meaning commutative Noetherian local) ring \((A, \mathfrak{m}, k)\) and a finitely generated \(A\)-module \(M\). All modules we encounter are assumed to be finitely generated. We denote by \(M^*\) the \(A\)-dual of \(M\), that

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is, the $A$-module $\text{Hom}_A(M, A)$. If the canonical homomorphism $M \rightarrow M^{**}$ is an isomorphism, then $M$ is said to be reflexive. Furthermore, if $M$ is reflexive, and
\[ \text{Ext}^n_A(M, A) = 0 = \text{Ext}^3_A(M^*, A) \]
for $n \geq 1$, then $M$ is a module of Gorenstein dimension zero (alternatively, $M$ is said to be totally reflexive). We shall write “G-dimension” instead of “Gorenstein dimension”.

Suppose now that $M$ has G-dimension zero, and fix free resolutions
\[ \cdots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \xrightarrow{d_0} M \rightarrow 0 \]
\[ \cdots \rightarrow C_{-3} \rightarrow C_{-2} \rightarrow C_{-1} \rightarrow M^* \rightarrow 0 \]
of $M$ and $M^*$, respectively. Dualizing the latter resolution, we obtain a complex
\[ 0 \rightarrow M \rightarrow C_{-1} \rightarrow C_{-2} \rightarrow C_{-3} \rightarrow \cdots \]
which is exact since $\text{Ext}^n_A(M^*, A) = 0$ for $n \geq 1$. Splicing this sequence with the free resolution of $M$, we obtain a doubly infinite exact sequence
\[ C : \cdots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \xrightarrow{d_0} \cdots \]
of free modules, in which $M \cong \text{Im} d_0$. The “left part” of the dualized complex $\text{Hom}_A(C, A)$ is exact, since it is just the free resolution of $M^*$. Moreover, the “right part” is also exact, since $\text{Ext}^n_A(M, A) = 0$ for $n \geq 1$. Consequently, the complex $\text{Hom}_A(C, A)$ is exact; thus $C$ is totally acyclic. Conversely, if $C$ is a totally acyclic complex of free modules, then the image of any of its differentials has G-dimension zero. Given such a complex, we denote by $M_C$ the image of the zeroth differential.

When $M$ has G-dimension zero, then the totally acyclic complex constructed above is a complete resolution of $M$. By [Buc, CoK], it is unique up to homotopy equivalence. Consequently, for every $n \in \mathbb{Z}$ and every $A$-module $N$, the Tate cohomology module (or stable cohomology module)
\[ \widehat{\text{Ext}}^n_A(M, N) \overset{\text{def}}{=} \text{H}._{-n}(\text{Hom}_A(C, N)) \]
is independent of the choice of complete resolution of $M$. By construction, there is an isomorphism $\widehat{\text{Ext}}^n_A(M, N) \cong \text{Ext}_A^n(M, N)$ whenever $n > 0$. We refer to [AvM] for general properties of Tate cohomology modules.

When we start with minimal free resolutions of $M$ and $M^*$, we end up with an almost minimal totally acyclic complex $C$, meaning that the image of the differential $C_{n+1} \rightarrow C_n$ is contained in $\mathfrak{m} C_n$ for all $n \neq -1$. The complex $C$ will be minimal if in addition the image of $d_0$ lies in $\mathfrak{m} C_{-1}$ (and this happens precisely when $M$ has no nonzero free summands). Since minimal free resolutions are unique up to isomorphism, the integers $\beta_n(M) \overset{\text{def}}{=} \text{rank} C_n$ are well-defined for all $n \in \mathbb{Z}$. The integer $\beta_n(M)$ is called the $n$th Betti number of $M$, and for $n \neq 0, -1$ it is equal to the dimension of the $k$-vector space $\widehat{\text{Ext}}^n_A(M, k)$. We define the positive complexity and negative complexity of $M$ as
\[ \text{cx}^+_A M \overset{\text{def}}{=} \inf \{ t \in \mathbb{N} \cup \{ 0 \} \mid \exists a \in \mathbb{R} \text{ such that } \beta_n(M) \leq an^{t-1} \text{ for } n \gg 0 \}, \]
\[ \text{cx}^-_A M \overset{\text{def}}{=} \inf \{ t \in \mathbb{N} \cup \{ 0 \} \mid \exists a \in \mathbb{R} \text{ such that } \beta_n(M) \leq a(-n)^{t-1} \text{ for } n \ll 0 \}. \]
These measure on a polynomial scale the left and the right, respectively, rate of growth of the minimal complete resolution of $M$. We also define the corresponding Poincaré series

$$P^+_A(M, t) \overset{\text{def}}{=} \sum_{n \geq 0} \beta_n(M) t^n,$$

$$P^-_A(M, t) \overset{\text{def}}{=} \sum_{n \geq 0} \beta_{-n}(M) t^n,$$

that is, the generating functions of the Betti numbers. Note that the positive complexity coincides with the “ordinary” complexity of a module, that is, the polynomial rate of growth in its minimal free resolution.

The aim of this paper is to give sufficient conditions for a totally acyclic minimal complex of free modules to have symmetric growth. In other words, given such a complex $C$, we give a criterion for when $\text{cx}^+_A MC$ and $\text{cx}^-_A MC$ are equal. Namely, we show that $C$ has symmetric growth whenever the cohomology of $M_C$ behaves well with respect to dualization and is “finitely generated.” To explain these notions, consider the bounded derived category $D^b(A)$ of finitely generated $A$-modules. This is a triangulated category, whose suspension functor $\Sigma$ is just the left shift of a complex. Given complexes $X$ and $Y$ and an integer $n$, we denote the graded $A$-module $\text{Hom}_{D^b(A)}(X, \Sigma^n Y)$ by $\text{Ext}^n_A(X, Y)$; for modules, this is just the usual Ext. We denote by $\text{Ext}_A(X, Y)$ the graded module $\bigoplus_{n \in \mathbb{Z}} \text{Ext}^n_A(X, Y)$, and if $n_0$ is an integer then we set $\text{Ext}^\geq_{n_0}_A(X, Y) = \bigoplus_{n \geq n_0} \text{Ext}^n_A(X, Y)$. The graded center of $D^b(D)$, denoted $Z(D^b(A))$, is a graded ring $Z(D^b(A)) = \bigoplus_{n \in \mathbb{Z}} Z^n(D^b(A))$, whose degree $n$ component $Z^n(D^b(A))$ is the set of natural transformations $\text{Id} \rightarrow \Sigma^n$ satisfying $f_{X, Y} = (-1)^n \Sigma^n f_X$ on the level of objects. For details and properties of the graded center, see [BuF].

Now let $H = \bigoplus_{n \geq 0} H_n$ be a positively graded ring which is graded-commutative, that is, $\eta \theta = (-1)^{|\eta||\theta|} \theta \eta$ for all homogeneous elements $\eta, \theta \in H$. We say that $H$ acts centrally on $D^b(A)$ if there exists a homomorphism $H \rightarrow Z(D^b(A))$ of graded rings. In this case, for every complex $X \in D^b(A)$ there is a graded ring homomorphism

$$H \xrightarrow{\varphi_X} \text{Ext}_A(X, X),$$

and for every complex $Y$ and all homogeneous elements $\eta \in H, \theta \in \text{Ext}_A(X, Y)$ the equality $\varphi_Y(\eta) \circ \theta = (-1)^{|\eta||\theta|} \theta \circ \varphi_X(\eta)$ holds. In other words, the left and right $H$-module structures on $\text{Ext}_A(X, Y)$ coincide up to a sign.

**Definition.** We say that $\text{Ext}_A(X, Y)$ is an *eventually Noetherian $H$-module of finite length*, and write $\text{Ext}_A(X, Y) \in \text{Noeth}_H$, if the following holds: there is a number $n_0$ such that the $H$-module $\text{Ext}^\geq_{n_0}_A(X, Y)$ is Noetherian and the length of $\text{Ext}^n_A(X, Y)$ as an $H_{n_0}$-module, denoted by $\ell_{H_{n_0}}$, is finite for each $n \geq n_0$.

Let $M$ be an $A$-module of $G$-dimension zero, $C$ be a totally acyclic complex of free modules with $M = M_C$, and $\theta$ be an element of $\text{Ext}^n_A(M, M)$ for some $n$. This element $\theta$ corresponds to a chain map $C \rightarrow \Sigma^n C$ (and equivalent elements in $\text{Ext}^n_A(M, M)$ correspond to homotopic chain maps). Dualizing, we obtain a chain map $\Sigma^{-n}(C^*) = (\Sigma^n C)^* \rightarrow C^*$. Applying the shift functor $\Sigma^n$ we now get a chain map $C^* \rightarrow \Sigma^n(C^*)$, and this corresponds to an element in $\text{Ext}^n_A(M^*, M^*)$. One checks easily that this defines an anti-isomorphism

$$\mathcal{D} : \text{Ext}_A(M, M) \rightarrow \text{Ext}_A(M^*, M^*)$$
of graded rings.

**Definition.** Let $M$ be an $A$-module of $G$-dimension zero. We say that the central ring action from $H$ on $D^b(A)$ commutes with dualization of $M$, provided that the diagram

$$
\begin{array}{ccc}
\text{Ext}_A(M, M) & \xrightarrow{D} & \text{Ext}_A(M^*, M^*) \\
\varphi_M & \downarrow & \varphi_{M^*} \\
\end{array}
$$

commutes, that is, $D(\varphi_M(\eta)) = \varphi_{M^*}(\eta)$ for every homogeneous element $\eta \in H$.

3. Symmetric growth

In this section, we prove the main result on symmetric growth in totally acyclic minimal complexes of finitely generated free modules. We start with the following proposition, which provides a criterion for an extension to induce an eventually surjective chain map on a minimal free resolution.

**Proposition 3.1.** Let $(A, m, k)$ be a local ring, and let $M$ be a finitely generated $A$-module with minimal free resolution $F$. Let $\eta \in \text{Ext}_A(M, M)$ be a homogeneous element of positive degree, and suppose that it induces injective maps $\text{Ext}_A^n(M, k) \cdot \eta \rightarrow \text{Ext}_A^{n+|\eta|}(M, k)$ for $n \gg 0$. Then any chain map on $F$ induced by $\eta$ is eventually surjective.

**Proof.** Let $\Omega^n_A(M) \xrightarrow{f_n} M$ be a map representing the element $\eta$. Lifting this map along $F$ gives a chain map

$$
\begin{array}{ccccccc}
\cdots & \xrightarrow{f_2} & F_{|\eta|+1} & \xrightarrow{f_1} & F_{|\eta|} & \xrightarrow{f_0} & F_0 & \xrightarrow{f_n} & M & \rightarrow & 0 \\
\end{array}
$$

of degree $-|\eta|$ induced by $\eta$. Since the complex $F$ is minimal, that is, the image of the differential $F_n \rightarrow F_{n-1}$ is contained in $m F_{n-1}$ for all $n$, the differentials in the complex $\text{Hom}_A(F, k)$ are all trivial. We may therefore identify $\text{Ext}_A^n(M, k)$ with $\text{Hom}_A(F_n, k)$ for all $n$. Moreover, under this identification, for all $n$ the map

$$
\text{Ext}_A^n(M, k) \xrightarrow{\eta} \text{Ext}_A^{n+|\eta|}(M, k)
$$

is the map

$$
\text{Hom}_A(F_n, k) \xrightarrow{f_n^*} \text{Hom}_A(F_{n+|\eta|}, k)
$$

induced by $f_n$. Applying $\text{Hom}_A(-, k)$ to the right exact sequence

$$
F_{n+|\eta|} \xrightarrow{f_n} F_n \rightarrow \text{Coker} f_n \rightarrow 0
$$

induces the left exact sequence

$$
0 \rightarrow \text{Hom}_A(\text{Coker} f_n, k) \rightarrow \text{Hom}_A(F_n, k) \xrightarrow{f_n^*} \text{Hom}_A(F_{n+|\eta|}, k).
$$

By assumption, the map $f_n^*$ is injective for large $n$, and so $\text{Hom}_A(\text{Coker} f_n, k)$ must vanish for $n \gg 0$. However, the group $\text{Hom}_A(X, k)$ is nonzero whenever $X$ is a nonzero finitely generated $A$-module. Consequently, the map $f_n$ must be surjective for $n \gg 0$. \qed
The next result shows that when the cohomology of a module is finitely generated, then its (positive) complexity is finite and its Poincaré series is rational.

**Lemma 3.2.** Let \((A, m, k)\) be a local ring, and let \(M\) be a finitely generated \(A\)-module.

(i) Suppose that \(\text{Ext}_A^*(M, k)\) belongs to \(\text{Noeth}^H H\) for some graded-commutative ring \(H\) acting centrally on \(D^b(A)\). Then \(\text{cx}^+_A M\) is finite and the Poincaré series \(P_A^+(M, t)\) is rational. Moreover, \(\text{cx}^+_A M\) equals the order of the pole of \(P_A^+(M, t)\) at \(t = 1\).

(ii) Suppose that \(M\) has \(G\)-dimension zero and that \(\text{Ext}_A(M^+, k)\) belongs to \(\text{Noeth}^H H\) for some graded-commutative graded ring \(H\) acting centrally on \(D^b(A)\). Then \(\text{cx}^+_A M\) is finite and the Poincaré series \(P_A^+(M, t)\) is rational. Moreover, \(\text{cx}^+_A M\) equals the order of the pole of \(P_A^+(M, t)\) at \(t = 1\).

**Proof.** It suffices to prove (i); claim (ii) follows from (i) and the fact that \(\beta_n(M^+) = \beta_{-n+1}(M)\) for all \(n \in \mathbb{Z}\).

Since the scalar action from \(H_0\) on \(\text{Ext}_A^n(M, k)\) factors through \(\text{Hom}_A(\cdot, k)\), the positive complexity of \(M\) is the complexity of the sequence \(\{\ell_{H_0} \text{Ext}_A^n(M, k)\}_{n=0}^{\infty}\). Therefore, by [BKO] Lemma 2.6], the positive complexity of \(M\) is finite.

Let \(n_0\) be an integer such that the \(H\)-module \(\text{Ext}_A^{2n_0}(M, k)\) is Noetherian and \(\ell_{H_0}(\text{Ext}_A^n(M, k)) < \infty\) for each \(n \geq n_0\), and denote the ideal \(\text{Ann}_H \text{Ext}_A^{2n_0}(M, k)\) in \(H\) by \(I\). By [BKO] Remark 2.1], the quotient ring \(H/I\) is Noetherian, and its degree zero part \((H/I)_0\) is Artin. The rationality of \(P_A^+(M, t)\) now follows from the Hilbert-Serre Theorem (cf. [AtM] Theorem 11.1]). The last part is a standard result on Poincaré series (cf. [Ben] Proposition 5.3.2]).

In order to prove the main result, we also need the following elementary lemma. It shows that the category of modules of \(G\)-dimension zero, and the category of modules whose cohomology is finitely generated, are closed under extensions.

**Lemma 3.3.** Let \((A, m, k)\) be a local ring, and let

\[ 0 \to L \to M \to N \to 0 \]

be an exact sequence of finitely generated \(A\)-modules.

(i) If \(L\) and \(N\) are both of \(G\)-dimension zero, then so is \(M\).

(ii) If \(\text{Ext}_A(L \oplus N, k)\) belongs to \(\text{Noeth}^H H\) for some graded-commutative ring \(H\) acting centrally on \(D^b(A)\), then so does \(\text{Ext}_A(M, k)\).

**Proof.** For a proof of (i), see for example [Ch] Lemma 1.1.10]. As for (ii), note that the exact sequence induces an exact sequence

\[ \text{Ext}_A(N, k) \to \text{Ext}_A(M, k) \to \text{Ext}_A(L, k) \]

of graded \(H\)-modules. It follows that the middle term must be eventually Noetherian of piecewise finite length.

We now prove the main theorem. It shows that a totally acyclic minimal complex of finitely generated free modules has symmetric growth, provided its cohomology is finitely generated, and the central action from \(H\) commutes with dualization.

**Theorem 3.4.** Let \((A, m, k)\) be a local ring, and \(C\) be a totally acyclic minimal complex of finitely generated free \(A\)-modules. Suppose that \(\text{Ext}_A(M_G \oplus M'_G, k)\) belongs to \(\text{Noeth}^H H\) for some graded-commutative ring \(H\) acting centrally on \(D^b(A)\)
and such that its action commutes with dualization of $M$. Then $C$ has symmetric growth.

**Proof.** Denote the module $M_C$ by $M$. We prove the result by induction on the positive complexity of $M$, which is finite by Lemma [3.2]. If $\text{cx}_A^1 M = 0$, then $M$ has finite projective dimension and is therefore free. Thus, in this case, the complex $C$ is bounded, and the result trivially follows.

Next, suppose that $\text{cx}_A^1 M$ is nonzero. By [HKO] Lemma 2.5, there exists a homogeneous element $\eta \in H$, of positive degree, inducing injective maps

\[
\begin{align*}
\text{Ext}^n_A(M, k) &\xrightarrow{\eta} \text{Ext}^{n+|\eta|}_A(M, k) \\
\text{Ext}^n_A(M^*, k) &\xrightarrow{\eta} \text{Ext}^{n+|\eta|}_A(M^*, k)
\end{align*}
\]

for $n \gg 0$. Choose maps $\Omega^{|\eta|}_A(M) \xrightarrow{f_0} M$ and $\Omega^{|\eta|}_A(M^*) \xrightarrow{g_0} M^*$ representing the elements $\varphi_M(\eta)$ and $\varphi_{M^*}(\eta)$ in $\text{Ext}_A(M, M)$ and $\text{Ext}_A(M^*, M^*)$, and note that the complexes $C_{>0}$ and $(C_{<1})^*$ are minimal free resolutions of $M$ and $M^*$, respectively. By Proposition 3.1, the maps $f_0$ and $g_0$ induce eventually surjective chain maps

\[
\cdots \rightarrow C_{[n]+2} \rightarrow C_{[n]+1} \rightarrow C_{[n]} \rightarrow \Omega^{|\eta|}_A(M) \rightarrow 0 \\
\cdots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow 0
\]

\[
\cdots \rightarrow C^*_{-(|n|+3)} \rightarrow C^*_{-(|n|+2)} \rightarrow C^*_{-(|n|+1)} \rightarrow \Omega^{|\eta|}_A(M^*) \rightarrow 0 \\
\cdots \rightarrow C^*_{-3} \rightarrow C^*_{-2} \rightarrow C^*_{-1} \rightarrow M^* \rightarrow 0
\]

on $C_{>0}$ and $(C_{<1})^*$, respectively. Since $g_n$ is split surjective for $n \gg 0$, when dualizing the lower diagram we obtain a chain map

\[
\cdots \rightarrow 0 \rightarrow M \rightarrow C_{-1} \rightarrow C_{-2} \rightarrow C_{-3} \rightarrow \cdots \\
\cdots \rightarrow 0 \rightarrow \Omega^{-|\eta|}_A(M) \rightarrow C_{-(|n|+1)} \rightarrow C_{-(|n|+2)} \rightarrow C_{-(|n|+3)} \rightarrow \cdots
\]

which is eventually split injective.

Choose a short exact sequence

\[
0 \rightarrow M \rightarrow K \rightarrow \Omega^{|\eta|-1}_A(M) \rightarrow 0
\]

representing the element $\varphi_M(\eta)$ in $\text{Ext}_A(M, M)$. By Lemma [3.3] the module $K$ also has G-dimension zero, and both $\text{Ext}_A(K, k)$ and $\text{Ext}_A(K^*, k)$ belong to $\text{Noeth}^b H$. From the sequence we obtain a long exact sequence

\[
\cdots \rightarrow \text{Ext}^n_A(M, k) \xrightarrow{\partial_n} \text{Ext}^{n+|\eta|}_A(M, k) \rightarrow \text{Ext}^{n+1}_A(K, k) \rightarrow \text{Ext}^{n+1}_A(M, k) \xrightarrow{\partial_{n+1}} \cdots
\]

in Tate cohomology. For $n \geq 0$, the connecting homomorphism $\partial_n$ is induced by $f_n$, and so since $f_n$ is surjective for large $n$, we see that $\partial_n$ is injective for $n \gg 0$. Moreover, the central action from $H$ on $D^b(A)$ commutes with dualizations, that
is, $D(\varphi_M(\eta)) = \varphi_{M^*}(\eta)$, hence $\partial_n$ is induced by $g_{n+|\eta|+1}$ for $n \ll 0$. The map $g_n^\ast$ is split injective for large $n$, and consequently $\partial_n$ is surjective for $n \ll 0$.

Choose a number $n_0$ with the property that $\partial_n$ is injective for $n \geq n_0$ and surjective for $n \leq -n_0$. Then the sequences

$$0 \to \widehat{\text{Ext}}_A^n(M, k) \to \widehat{\text{Ext}}_A^{n+|\eta|}(M, k) \to \widehat{\text{Ext}}_A^{n+1}(K, k) \to 0$$

$$0 \to \widehat{\text{Ext}}_A^{-n}(K, k) \to \widehat{\text{Ext}}_A^{-n}(M, k) \to \widehat{\text{Ext}}_A^{-|\eta|}(M, k) \to 0$$

are exact for $n \geq n_0$, giving equalities

$$\beta_{n+1}(K) = \beta_{n+|\eta|}(M) - \beta_n(M)$$
$$\beta_{-n}(K) = \beta_{|\eta|+n}(M) - \beta_{-n}(M)$$

of Betti numbers. Computing Poincaré series, we obtain

$$P_A^+(K, t) = \frac{1 - t^{|\eta|}P_A^+(M, t)}{1 - t} + f(t)$$
$$P_A^-(K, t) = (t^{|\eta|} - 1)P_A^-(M, t) + g(t)$$

for some polynomials $f(t), g(t) \in \mathbb{Z}[t]$. Consequently, the order of the pole of $P_A^+(K, t)$ at $t = 1$ is one less than that of $P_A^+(M, t)$, whereas the pole of $P_A^-(K, t)$ is one less than that of $P_A^-(M, t)$. Therefore, by Lemma 3.2, the positive complexity of $K$ is $\text{cx}_A^+ M - 1$, the negative complexity of $K$ is $\text{cx}_A^- M - 1$, and by induction we obtain

$$\text{cx}_A^+ M = \text{cx}_A^+ K + 1 = \text{cx}_A^- K + 1 = \text{cx}_A^- M.$$ 

This shows that $C$ has symmetric growth. \hfill \Box

We include the following equivalent version of the theorem as a corollary. It follows from the elementary fact that for a module of G-dimension zero, the negative complexity equals the positive complexity of its dual.

Corollary 3.5. Let $(A, m, k)$ be a local ring, and let $M$ be a finitely generated $A$-module of G-dimension zero. If $\text{Ext}_A(M \oplus M^*, k)$ belongs to $\text{N}o\text{e}_H$ for some graded-commutative ring $H$ acting centrally on $D^b(A)$ and such that its action commutes with dualization of $M$, then $\text{cx}_A^+ M = \text{cx}_A^- M = \text{cx}_A^+ M^* = \text{cx}_A^- M^*$.

4. Complete intersection dimension zero

Assume that $A = B/(x_1, \ldots, x_c)$, where $B$ is a local ring and $x_1, \ldots, x_c$ is a $B$-regular sequence. Then there exists a polynomial ring

$$H = A[\chi_1, \ldots, \chi_c]$$

acting centrally on $D^b(A)$, with each cohomology operator $\chi_i$ of degree two. For purposes below, we recall the definition of the elements $\varphi_M(\chi_i) \in \text{Ext}_A^2(M, M)$, and their action on $\text{Ext}_A(M, N)$ and $\text{Ext}_A(N, M)$. (There are actually several definitions for these elements, but they all agree up to sign; see [AS]. The one we give is from [Eis]).

Let $(C, d)$ be a complex of free $A$-modules. We lift $C$ to a sequence of maps $(\tilde{C}, \tilde{d})$ of free $B$-modules. Since $\tilde{d}^2 \equiv 0$ modulo $(x_1, \ldots, x_c)$ we can decompose $\tilde{d}^2$ as

$$\tilde{d}^2 = \sum_{i=1}^c x_i \tilde{t}_i$$
for some family $(\tilde{t}_i)_{i=1}^4$ of degree $-2$ endomorphisms of the graded $B$-module $\tilde{C}$. Then $t_i = \tilde{t}_i \otimes_B A$ become degree $-2$ chain maps on the complex $C$ which are well-defined and commute up to homotopy (see [Eis]). The chain maps $t_i$ on a free resolution $C$ of $M$ then define the elements $\varphi_M(\chi_i) \in \text{Ext}_A^2(M, M)$. The action of $\varphi_M(\chi_i)$ on $\text{Ext}_A(M, N)$ and $\text{Ext}_A(N, M)$ is thus determined by composition of chain maps. If $M = M_C$ for a totally acyclic complex $C$ of free $A$-modules, the element $\varphi_M(\chi_i) \in \text{Ext}_A^2(M, M)$ is determined by the chain map $t_i : C \to \Sigma^2 C$.

Recall that a quasi-deformation of local rings is a diagram of local homomorphisms $A \to A' \leftarrow B$ such that the map $A \to A'$ is flat and $A' \leftarrow B$ is onto with kernel generated by a $B$-regular sequence contained in the maximal ideal of $A$. A finitely generated module $M$ over a local ring $A$ is said to have finite complete intersection dimension, denoted $\text{CI-dim}_A M < \infty$, if there exists a quasi-deformation of local rings $A \to A' \leftarrow B$ such that $\text{pd}_B M \otimes_A A' < \infty$ [AGP]. In loc. cit. it is shown that if $M$ has finite CI-dimension, then $M$ has finite G-dimension, and when both are finite, that

$$\text{CI-dim}_A M = \text{G-dim}_A M = \text{depth}_A A - \text{depth}_A M$$

**Lemma 4.1.** Let $A = B/(x_1, \ldots, x_c)$ where $(B, n, k)$ is a local ring, and $x_1, \ldots, x_c$ is a $B$-regular sequence contained in $n$. Suppose that $M$ is an $A$-module with $\text{depth}_A M = \text{depth}_A A$, and $\text{pd}_B M < \infty$. Then the central ring action of the polynomial ring of cohomology operators $H = A[\chi_1, \ldots, \chi_c]$ on $D^b(A)$ commutes with dualization of $M$.

**Proof.** The hypotheses show that the CI-dimension of $M$ is zero, and therefore $M$ has G-dimension zero. Let $C$ be a totally acyclic complex of free modules with $M = M_C$. Choose $\chi = \chi_1 \in H$. It suffices to prove that $D(\varphi_M(\chi)) = \varphi_{M^*}(\chi)$. The element $\varphi_M(\chi) \in \text{Ext}_A(M, M)$ is determined by the chain map $t : C \to \Sigma^2 C$, as described above. Therefore $D(\varphi_M(\chi))$ corresponds to the chain map $\Sigma^2 t^* : C^* \to \Sigma^2 C^*$. On the other hand, if one uses the lifting $\text{Hom}_B(\tilde{C}, B)$ of $C^*$, and the factorization of $(\tilde{D})^2$ dual to that of $\tilde{D}^2$ we see that the chain map $\Sigma^2 t^* : C^* \to \Sigma^2 C^*$ defines the element $\varphi_{M^*}(\chi) \in \text{Ext}_A(M^*, M^*)$, and this is what we wanted to show. 

We have the following corollary of Theorem 3.4 in the case of CI-dimension zero.

**Corollary 4.2.** Let $(A, m, k)$ be a local ring, and $C$ be a totally acyclic minimal complex of finitely generated free $A$-modules. If $M_C \oplus (M_C^*)^* has CI-dimension zero, then $C$ has symmetric growth.

**Proof.** Set $M = M_C$, and let $A \to A' \leftarrow B$ be a quasi-deformation such that $\text{pd}_B ((M \oplus M^*) \otimes_A A') < \infty$. The complex $C$ has symmetric growth if and only if the complex $C' = C \otimes_A A'$ does, the latter complex being a totally acyclic minimal complex of finitely generated free $A'$-modules. Moreover, $\text{Hom}_A(M, A) \otimes_A A' \cong \text{Hom}_{A'}(M \otimes_A A', A')$, and so $\text{pd}_B ((M \otimes_A A') \oplus \text{Hom}_{A'}(M \otimes_A A', A')) < \infty$. Also $\text{depth}_A A = \text{depth}_{A'} A'$, and $\text{depth}_A M = \text{depth}_{A'} M \otimes_A A'$. Changing notation, we can therefore assume that $A = B/(x_1, \ldots, x_c)$, where $B$ is a local ring and $x_1, \ldots, x_c$ is a $B$-regular sequence contained in the maximal ideal of $B$, that $C$ is a totally acyclic minimal complex of finitely generated free $A$-modules, and that $M = M_C$ and $M^* = \text{Hom}_A(M, A)$ are $A$-modules such that $\text{depth}_A M = \text{depth}_A A$ and $\text{pd}_B M \oplus M^* < \infty$. By Lemma 4.1 we have that the central ring action of the
polynomial ring of cohomology operators $H = A[x_1, \ldots, x_c]$ on $D^b(A)$ commutes with dualization of $M$. Therefore by Theorem 3.4 it suffices to know that $\text{Ext}_A(M \oplus M^*, k)$ belongs to $\text{Noeth}^b H$, but this is implied by the main result of [Gul] (see also [Eis] and [Avr]).

When $A$ is a complete intersection, then every finitely generated $A$-module has finite CI-dimension. Therefore, in this case, every totally acyclic minimal complex of free modules has symmetric growth. This follows already from [AvB, Theorem 5.6], but because of our alternative proof, we include it here as a corollary.

**Corollary 4.3.** Let $(A, m, k)$ be a local complete intersection. Then every totally acyclic minimal complex of finitely generated free $A$-modules has symmetric growth.

The following construction indicates that finitely generated modules $M$ over a non-complete intersection ring such that $M \oplus M^*$ has CI-dimension zero are not at all rare.

**Construction 4.4.** Let $(A_1, m_1, k)$ and $(A_2, m_2, k)$ be local rings which both contain their common residue field $k$. Assume that $A_1 = B_1/(x_1, \ldots, x_c)$ where $(B_1, n_1, k)$ is a regular local ring and $x_1, \ldots, x_c$ is a $B_1$-regular sequence contained in $n_1$ (so that $A_1$ is a complete intersection). Let $m = m_1 \otimes_k A_2 + A_1 \otimes_k m_2$, a maximal ideal of $A_1 \otimes_k A_2$, and set $A = (A_1 \otimes_k A_2)_m$. Then $(A, m, k)$ is a local ring with residue field $k$ also, and which is not a complete intersection if $A_2$ is not a complete intersection.

Now let $M_1$ be any maximal Cohen-Macaulay $A_1$-module, equivalently any $A_1$-module with $\text{CI-dim}_{A_1} M_1 = 0$. Then $\text{Hom}_{A_1}(M_1, A_1)$ is also of CI-dimension zero. It is easy to see that $M = (M_1 \otimes_k A_2)_m$ and

$$M^* = \text{Hom}_A(M, A) \cong (\text{Hom}_{A_1}(M_1, A_1) \otimes_k A_2)_m$$

are finitely generated $A$-modules of CI-dimension zero.

We end with some questions for further study.

**Question 4.5.** Let $M$ be a finitely generated module over a local ring $(A, m, k)$. If $M$ has finite CI-dimension, then does $M^* = \text{Hom}_A(M, A)$ have finite CI-dimension as well? In particular, if $A = B/(x)$ for $(B, n, k)$ a local ring and $x$ a non-zero divisor in $n$, then does $\text{pd}_B M < \infty$ imply that $\text{pd}_B M^* < \infty$ as well?

**Question 4.6.** Do there exist graded commutative rings $H$ acting centrally on $D^b(A)$ such that the action fails to commute with dualization of a finitely generated $A$-module $M$?

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