More 1-cocycles for classical knots

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Abstract

Let \( n \) be a natural number and let \( M \) be the moduli space of all knots in the solid torus \( V^3 \) which represent the homology class \( n \in \mathbb{Z} \cong H_1(V^3) \). Let \( K \) be a framed oriented long knot and let \( nK \) be its \( n \)-cable twisted by a fixed string link \( T \) to a knot in the solid torus \( V^3 \). Let \( M_n^{reg} \) be the topological moduli space of all such knots \( nK \) up to regular isotopy with respect to a fixed projection into the annulus. We construct two new sorts of 1-cocycles: two integer 1-cocycles \( R_{[a,b,c]} \) and \( R_{(a,b,c)} \) for \( M \), which use linear weights and which depend on three integer parameters \( a, b, c \in \mathbb{Z} \), and two integer 1-cocycles \( R_{a,\pm}^{(2)} \) for \( M_n^{reg} \), which use quadratic weights and which depend on a natural number \( 0 < a < n \). The Lagrange interpolation polynomials of \( R_{a,\pm}^{(2)}(\gamma) \) are candidates for new polynomial knot invariants for classical knots \( K \), which can be calculated with polynomial complexity for each homology class \( [\gamma] \in H_1(M_n^{reg}) \). The 1-cocycles \( R_{[a,b,c]} \) and \( R_{(a,b,c)} \) are candidates to distinguish the orientations of classical knots.

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1 Introduction

In the monograph "Polynomial One-cocycles for Knots and Closed Braids" [6] we have laid the foundation of the theory of combinatorial 1-cocycles which depend on integer parameters for knots in the solid torus. In particular, these 1-cocycles give new invariants for knots in $\mathbb{R}^3$, often called classical knots, when they are evaluated on certain canonical loops in the topological moduli spaces of knots in the solid torus, i.e. loops which are universally defined in all connected components of the moduli space. (We will often use [6] as a reference for definitions, notations and conventions.)

The present paper is the sequel of the monograph and we construct two new sorts of combinatorial 1-cocycles.

There is a natural projection $pr : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ of the 3-space into the plan and knots in 3-space can be given by knot diagrams, i.e. a smoothly embedded oriented circle in 3-space together with its projection into the plan. It is well known that oriented knot types are in one-one correspondence with knot types of oriented long knots. We close now the long knot with the 1-braid to a knot in the standard embedded solid torus $V^3 \subset \mathbb{R}^3$. Moreover, a framed long knot can be replaced by its parallel n-cable (with the same orientation of all strands) which we close to a knot in $V^3$ by a cyclic permutation braid or more generally by a fixed n-component string link $T$, which induces a cyclic permutation of its end points. We denote the resulting knot in $V^3$ simply by $nK$. The projection into the plan becomes now a projection into the annulus $pr : \mathbb{C}^* \times \mathbb{R} \to \mathbb{C}^*$. We chose as generator of $H_1(V^3)$ the class which is represented by the closure of the oriented 1-braid. Hence the knot in $V^3$, which is obtained from the n-cable, represents the homology class $n \in H_1(V^3)$. We consider the infinite dimensional space $M_n$ of all diagrams of knots $K$ in $V^3$, which represent the homology class $n$, and such that there is a compressing disc of $V^3$ which intersects $K$ transversely and in exactly $n$ points. $M_n$ is called the moduli space of knot diagrams without negative loops in the solid torus. The knot types in 3-space correspond to the connected components of
Given a generic knot diagram $K \subset V^3$ we consider the oriented curve $pr(K)$ in the annulus. A loop in $pr(K)$ is a piecewise smoothly oriented immersed circle in $pr(K)$ which respects the orientation of $pr(K)$. In other words, we go along $pr(K)$ following its orientation and at a double point we are allowed to switch perhaps to the other branch, but still following the orientation of $pr(K)$. Naturally, a loop in $pr(K)$ is called negative (respectively positive) if it represents a negative (respectively positive) homology class in $H_1(V^3)$. One easily sees that $pr(K)$ contains only positive loops if and only if $K \subset V^3$ is isotopic to a closed braid with respect to the disc fibration of $V^3$, and that knots which arise as cables of long knots by the above construction, contain never negative loops. The space $M_n^{reg}$ is the subspace of $M_n$ which consists of all those knots on which $pr$ induces an immersion. The space $M_n$ is a natural subspace of the space $M$ of all knots in the solid torus which represent the homology class $n$ (without loss of generality we can assume that $n \geq 0$), i.e. negative loops in diagrams are now allowed as well. The natural inclusions $M_n^{reg} \subset M_n \subset M$ are all strict.

Victor Vassiliev has started the combinatorial study of knot spaces (in fact the space $M_1$) in "Combinatorial formulas of cohomology of knot spaces" [15] by studying the simplicial resolution of the discriminant of long singular knots in $\mathbb{R}^3$. Our approach is very different, because it is based on the study of another discriminant, namely the discriminant of non-generic diagrams of knots in the solid torus [6].

Why do we need to construct 1-cocycles? Classical knots can be transformed into knots in the solid torus. The moduli space of knots in the solid torus is essentially the only moduli space of knots in 3-manifolds with infinite $H_1$, compare [6]. We have to use this, because so far the study of $H_0$ (which has given plenty of invariants) was not enough to distinguish all classical knots!

But why do we need to construct 1-cocycles in a combinatorial way? Of course it would be of great importance for better understanding to construct differential 1-forms on the moduli space, which depend on natural numbers as parameter and which represent the same cohomology classes as our combinatorial 1-cocycles. But they do not exist yet and we have to put up with the difficult combinatorial approach.

Combinatorial integer 0-cocycles are usually called Gauss diagram formulas, compare [14] and also [5]. They correspond to finite type invariants and are solutions of the 4T- and 1T-relations. Dror Bar-Natan has shown in
"On the Vassiliev knot invariants" [1] that such solutions can be constructed systematically by using the representation theory of Lie algebras. Arnaud Mortier has constructed for 1-cocycles of finite type for long knots the analog of the Kontsevich integral in "A Kontsevich integral of order 1" [13]. The 4T-relations are now replaced by three 16T- and three 28T-relations and 4x4T-relations! There is actually no representation theory which could help to construct such solutions. The reason for this is simple. The well known representation theory related to the tetrahedron equation, see e.g. [11], is as usual of a local nature. But $H_1$ of the moduli space of (closed) non-satellite knots in $\mathbb{R}^3$ is only torsion (in contrast to $H_0$), see [9], [2] and [3], and hence all integer 1-cocycles from local solutions of the tetrahedron equation are trivial (in contrast to the local solutions from the Yang-Baxter equation, see e.g. [12])! We construct therefore in a combinatorial way solutions of the global tetrahedron equation, i.e. the contribution of a R III move depends on the whole knot in the solid torus and not only on the local picture of the move. This is an equation which is much more complicated as the well known Yang-Baxter equation.

So far, all our 1-cocycles constructed in [6] have used linear weights for the contributions of R III moves to the 1-cocycles, i.e. besides the triangle (which corresponds to the move in the Gauss diagram) we consider just the position of individual arrows with respect to the triangle. They depend at most on two integer parameters.

First, we construct new 1-cocycles in $M$ with linear weights too, but which depend now on three integer parameters and an interesting cyclical order of the parameters appears. Then we go one step further and we construct two 1-cocycles, called $R^{(2)}_a$ and $R^{(2)}_{a+}$, which use quadratic weights for the contributions of R III moves, i.e. they depend on the position of couples of arrows with respect to the triangle. Their construction is very complex but the outcome are rather beautiful formulas. Surprisingly, it becomes now really essential that knots are framed and that their projection into the annulus contains no loops which represent a negative homology class. Hence, the 1-cocycles live only in $M^{reg}_n$ and at least some of them can not be extended to $M_n$ nor to $M^{reg}$. 

In the case of quadratic weights there are only three types of equations to solve, and in the order given below, because we consider only regular isotopy and only R III moves contribute to the 1-cocycle (compare [6]):
(1) a R III move with simultaneously another Reidemeister move
(2) the positive global tetrahedron equation
(3) the cube equations.

We want to construct a weight $W_2(p)$ for each R III move $p$ which is
defined by using couples of crossings (i.e. arrows in the Gauss diagram which
always go from the under-cross to the overcross, compare [6]) outside of the
R III move $p$ (i.e. arrows not of the triangle which corresponds to the R
III move). But these couples have to be related to the R III move $p$. The
equations (1) force the weight $W_2(p)$ to be a knot invariant if we consider all
couples, and not only those which are related to the move $p$. R I moves force
the weight not to contain isolated arrows with homological markings 0 or
$n$. R II moves force that each crossing of the weight contributes with its sign
(writhe). Consequently, it is then sufficient to show that the weight $W_2(p)$ is
invariant under a simultaneous R III move with the move $p$.

An invariant of long knots which is given by a Gauss diagram formula,
which uses only couples of arrows, is an invariant of degree 2. Hence, it can
be defined by the beautiful formulas of Michael Polyak and Oleg Viro for
$v_2(K)$ of long knots [14], see Fig. 2. The point on the circle corresponds
to the point at infinity on the knot. For a crossing $c$ we call $K_+(c)$ the
knot which is obtained by smoothing the crossing $c$ from the under-cross
to the over-cross (and the remaining knot is called $K_-(c)$), see Fig. 1. The
homological marking of $c$ is the homology class in $\mathbb{Z} \cong H_1(V^3)$ represented
by $K_+(c)$ (compare [6]). Consequently, the point at infinity is in $K_+(c)$ if
and only if the homological marking $[c] = 1$.

All arrows in a (generic) Gauss diagram are generic straight segments. A
Gauss diagram is called connected if the set of arrows without the circle is a
connected set in the plan. One easily sees that in a connected Gauss diagram
the homological markings determine the point at infinity. This is not always
true if the Gauss diagram is not connected, see Fig. 3.

We say that a Gauss diagram (sometimes also called a configuration)
Figure 2: Polyak-Viro formulas for $v_2(K)$ in $M_1$

\[ \begin{array}{ccc}
\includegraphics[width=0.2\textwidth]{figure2a} & = & \includegraphics[width=0.2\textwidth]{figure2b} \\
\includegraphics[width=0.2\textwidth]{figure2c} & = & \includegraphics[width=0.2\textwidth]{figure2d}
\end{array} \]

Figure 3: The markings do not always determine the point at infinity

is homological determined, if the homological markings determine the point at infinity in a unique way (this corresponds just to some symmetry of the Gauss diagram).

Here is an example of a non connected Gauss diagram which is homological determined, see Fig. 4.

**Proposition 1** Let $G$ be a Gauss diagram formula for long knots, which defines a knot invariant. We assume that each Gauss diagram in the formula is homological determined and that the proof of the invariance does not use Gauss diagram identities. Let $K \subset V^3$ be a knot which belongs to $M_n$ (i.e. no negative loops). We keep the markings 0 in $G$ but we replace each marking 1 by the marking $n$. Then the resulting Gauss diagram formula defines a knot invariant for $K$ in $M_n$.

*Proof.*

\[ \begin{array}{ccc}
\includegraphics[width=0.2\textwidth]{figure4a} & = & \includegraphics[width=0.2\textwidth]{figure4b} \\
\includegraphics[width=0.2\textwidth]{figure4c} & = & \includegraphics[width=0.2\textwidth]{figure4d}
\end{array} \]

Figure 4: A homological determined configuration
Figure 5: The Polyak-Viro formula for $v_2(K)$ is not true in $M$

$V_3 = \begin{array}{cccc}
\quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad \\
\end{array}$

Figure 6: Chmutov-Polyak formula for $v_3(K)$ in $M_1$

There are exactly six global types of R III moves for long knots (compare [6]). If we replace 1 by $n$ then there are exactly six corresponding global types of R III moves with only markings 0 and $n$ in the triangle. Consequently, exactly the same proof of the invariance of the Gauss diagram formula $G$ still works, because the markings determine the point at infinity. □

Notice that the result is no longer true in $M$ already for the invariant of degree two, see Fig. 5.

Sergei Chmutov and Michael Polyak in "Elementary combinatorics of the HOMFLYPT polynomial" [4] have given a method to obtain Gauss diagram formulas for all those finite type invariants which can be extracted from the HOMFLYPT polynomial for long knots. We repeat here their formula for $v_3$ (up to some normalization), see Fig. 6.

Notice, that up to degree 3 all Gauss diagrams in the formulas are connected. Chmutov and Polyak’s proof of the formula for $v_3$ uses a Gauss diagram identity (they explore F. Jaeger’s state model for the HOMFLYPT polynomial [10] and they use the symmetry of linking numbers). However, it is easy to see that once the formula is known, then it can be checked directly for the six global types of R III moves without using a Gauss diagram.
identity (which is perhaps no longer true for $K$ by replacing 1 by $n$, because there are now also crossings with markings between 1 and $n$).

In the case of the invariant $v_{4,1}$ of degree 4 there is a unique non connected Gauss diagram in the corresponding formula of Chmutov and Polyak (which has already 21 terms) but it is homological determined (it is just the example given above).

Let’s come back to the formula for $v_2$, which will be important in this paper. In this particular case there are only four global types of R III moves to study in order to show its invariance, see Fig. 7.

We will use for $W_2(p)$ the following configuration (where as usual we take the product of the signs of the two crossings), which we denote by $(n, 0)$, see Fig. 8.

The individual contribution of each $n$-crossing (i.e. crossing with homological marking $n$) is already invariant for the above global types II, III and IV. However it changes for the global type I, see Fig. 9.

But we notice that the two $n$-crossings in the case I have their foot in the same arc on the circle! This is the fact which we will use to make a connection with the R III move $p$, because the triangle cuts the circle into three arcs. Let the natural number $a$ with $0 < a < n$ be fixed. Let $p$ be a R III move where the distinguished crossing $d$ (i.e. highest with lowest branch) has the marking $[d] = a$ (we don’t care about the markings of the crossings $ml$ and $hm$ in the move).
Definition 1 The weight of $p$ is defined by

$$W_2(p) = \sum (n, 0),$$

where the sum is taken only over all those $n$-crossings with the foot in $K_-(d)$ and which are not a crossing of the move $p$ (i.e. not a crossing from the triangle).

It follows immediately from our constructions that $W_2(p)$ is invariant by passing a simultaneous Reidemeister move with the move $p$ and hence it satisfies the equations (1). Notice also that each weight of degree 1 satisfies automatically the equations (1) (besides isolated arrows of marking 0 or $n$ of coarse, which could appear or disappear by a Reidemeister I move).

The weight $W_2(p)$ solves the equations (1), but it does not solve the positive global tetrahedron equation (2). The strata of R III moves in the meridian of a positive quadruple crossing come in pairs with different signs: $P_i$ and $\bar{P}_i$ (compare [6]). They can differ by n-crossings for which the foot has slide over the head or the foot of the distinguished crossing $d$ of $P_i$ or $\bar{P}_i$ and hence these n-crossings do no longer contribute to $W_2(p)$. But it turns out that such a n-crossing is always a crossing $d$, $hm$ or $ml$ for another R III move $P_j$ and $\bar{P}_j$ of a very particular global type in the meridian of the quadruple crossing. We use this to define a weight of degree 1 for $P_j$ and $\bar{P}_j$, which we multiply by a certain linking number and which will contribute to the 1-cocycle too. The weight of degree 1 is essentially the same for $P_j$ and $\bar{P}_j$ but the linking numbers are different. This is really extremely complex but leads finally to a solution of the positive global tetrahedron equation (2). The solution of the cube equations (3) is then relatively easy and leads just to some simple linear correction terms in the 1-cocycle. Let us mention that the construction of $R_a^{(2)}$ completely breaks down for the parameters $a = 0$ and $a = n$. We will give the final result in Section 3.

Notice that we have made several choices. The two Polyak-Viro formulas for $v_2$ lead to four different 1-cocycles: head instead of foot for n-crossings,
foot or head in $K_+(d)$ instead of $K_-(d)$. Also the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acts naturally on the moduli spaces, generated by orientation reversing of $K$ together with the hyper elliptic involution of the solid torus, and by taking the mirror image. Some of the corresponding 1-cocycles $R^{(2)}_a$ correspond to different choices in the construction of $R^{(2)}_a$.

We observe then that the 1-cocycle $R^{(2)}_a$ can be refined, because we can fix at the same time the arcs for the foots of the 0-crossings (i.e. the crossings of homological marking 0) in the weight! Indeed, the 0-crossing in the couple $(n, 0)$ with a n-crossing can change only in the case IV in the figure. But the two 0-crossings have the foot in the same arc on the circle. This allows us to define now a weight $W^+_2(p)$ by all couples of n-crossings with 0-crossings, such that the 0-crossing cuts the n-crossing from the left to the right, the foot of the 0-crossing is in $K_+(d)$ and the foot of the n-crossing is in $K_-(d)$ for the move $p$. We have now to study also in the tetrahedron equation the foots of the 0-crossings which slide over the foot or the head of the distinguished crossing $d$ of the move $p$. As a consequence, new global types of $R$ III moves will contribute too. We give the resulting 1-cocycle $R^{(2)}_{a+}$ in Section 4. Of course, we could also construct a 1-cocycle $R^{(2)}_{a-}$, by fixing the foots of the 0-crossings in $K_-(d)$ instead of $K_+(d)$ for $p$. But this is not necessary, because $R^{(2)}_a - R^{(2)}_{a+}$ is of course a 1-cocycle which represents the same cohomology class as the 1-cocycle $R^{(2)}_{a-}$.

In this paper we concentrate on the construction of the new 1-cocycles. Calculations by hand of interesting examples are too complicated because for $R^{(2)}_a$ and $R^{(2)}_{a+}$ we need that $n \geq 2$. We calculate just the easiest example in order to show that $R^{(2)}_a$ and $R^{(2)}_{a+}$ are not always trivial already for $n = 2$.

The 1-cocycle $R_{[a,b,c]}$ is most interesting for classical knots if $a, b$ and $c$ are all different, because the cyclical orders are then different. This implies that we need $n \geq 3$.

In common work with Roland van der Veen and Jorge Becerra we hope to create a computer program and to calculate lots of examples and to answer many open questions in knot theory in 3-space: how to distinguish efficiently mutants, orientations of knots, all the knots up to 16 crossings from the table of Hoste-Thistlewait-Weeks, diagrams of the unknot, diagrams of torus knots from diagrams of hyperbolic knots and so on.

Let us finish the Introduction just by indicating to the interested reader
how to go perhaps further and to define 1-cocycles with weights of even higher
degrees.

Let’s consider the Chmutov-Polyak formula for \( v_3 \). It follows from Proposition 1 that we can replace the point at infinity by the markings 0 and \( n \). For each of the seven configurations we have to study now which extremities of the \( n \)-crossings can be fixed in arcs on the circle for each of the six global types of R III moves with only markings 0 and \( n \). If at the end for a (or some) configuration in the invariant there would be a non trivial intersection of the fixed arcs for the heads or foots, then we could chose a distribution of them to \( K_+(d) \) and \( K_-(d) \) for the move \( p \), and hope to extend the corresponding weight of degree 3 (which satisfies then automatically the equations (1)) to a solution of the tetrahedron equation, by adding this time weights of degree 2, multiplied by linking numbers, for very particular global types of R III moves in the meridian of the quadruple crossing. But this will be extremely complicated!

To illustrate the method for the reader, we consider just one case of a 0-crossing for the R III move where all three crossings have marking \( n \), see Fig. 10. We see clearly that \( v_3 \) (with markings 0 and \( n \)) is invariant (because those configurations which contain only exactly one crossing of the triangle will evidently not change), and we see which extremities of \( n \)-crossings in which configurations can be fixed in arcs \( i,j,k \) on the circle in this case, see Fig. 11.

## 2 The integer 1-cocycles \( R_{[a,b,c]} \) and \( R_{(a,b,c)} \)

Let \( K \) be an arbitrary oriented knot in the solid torus with \([K] = n\).

A Reidemeister III move corresponds to a triangle in the Gauss diagram. The **global type of a Reidemeister III move** is now shown in Fig. 12, where \( m+h, m+h-[K] \), \( m \) and \( h \) are the homological markings of the corresponding arrows. Here, \([K]\) is the homology class represented by the knot \( K \). Moreover, we indicate whether the arrow \( ml \) in the triangle goes to the left, denoted by \( l \), or it goes to the right, denoted by \( r \). We will consider Gauss diagram formulas for R III moves. The arrows in the configurations which are not arrows of the triangle are called the **weights** of the formulas.

**Definition 2** The coorientation for a Reidemeister III move is the direction from two intersection points of the corresponding three arrows to one
Figure 10: A R III move with only n-crossings and a 0-crossing

Figure 11: Arcs which can be fixed in the configurations for a certain R IIII move

Figure 12: The global types of R III moves for knots in the solid torus
intersection point and of no intersection point of the three arrows to three inter-
section points, compare Fig. 13. (In [6] we have studied the discriminant
\( \Sigma \) of non generic diagrams in \( M \). In particular, we have shown in the cube
equations for \( \Sigma_{\text{trans-self}}^{(2)} \), that the two coorientations for triple crossings fit
together for the strata of \( \Sigma_{\text{tri}}^{(1)} \) which come together in \( \Sigma_{\text{trans-self}}^{(2)} \).
Evidently, our coorientation is completely determined by the corresponding planar curves
and therefore we can draw just chords instead of arrows in Fig. 13. We call
the side of the complement of the codimension 1 strata of R III moves \( \Sigma_{\text{tri}}^{(1)} \)
in \( M \) into which points the coorientation, the positive side of \( \Sigma_{\text{tri}}^{(1)} \).

Each transverse intersection point \( p \) of an oriented generic arc in \( M \) with
\( \Sigma_{\text{tri}}^{(1)} \) has now an intersection index +1 or −1, called \( \text{sign}(p) \), by comparing
the orientation of the arc with the coorientation of \( \Sigma_{\text{tri}}^{(1)} \).

We define a 1-cocycle, which depends on three integer parameters: \( a, b, c \in \mathbb{Z} \).
Let \([a, b, c]\) denote the equivalence class of \((a, b, c)\) under cyclic permutations.

**Definition 3** The 1-cocycle \( R_{[a,b,c]} \) is defined with the usual notation con-
ventions (i.e. each R III move of the given global type in an oriented generic
loop in \( M \) contributes by its sign multiplied with the sum of the signs of all
crossings not in the triangle and which realize the given configuration of the
weight in the Gauss diagram, compare [6]) by the formula given in Fig. 14.

**Theorem 1** \( R_{[a,b,c]} \) is a 1-cocycle in \( M^{\text{reg}} \) for all \( a, b, c \). If \( a, b \) and \( c \) are
neither 0 nor \( n \), then \( R_{[a,b,c]} \) is a 1-cocycle in \( M \).
Moreover, its restriction for
\( 0 < a, b, c < n \)
\( n \leq a + b \)
Figure 14: The 1-cocycle $R_{[a,b,c]}$

Figure 15: The 1-cocycle $R_{(a,b,c)}$

$n \leq a + c$
$n \leq b + c$
$2n \leq a + b + c$

could be a non-trivial 1-cocycle in $M_n$. Again, if one of $a$, $b$ or $c$ is 0 or $n$, then it is only a 1-cocycle in $M_n^{reg}$. A priory, the 1-cocycles $R_{[a,b,c]}$ and $R_{[a,c,b]}$ are different.

In order to have the cyclical orderings different we need that $a$, $b$ and $c$ are all three different. Therefore $R_{[a,b,c]}$ could be interesting for classical knots starting from $n = 3$ with $a = 1$, $b = 2$ and $c = 3$.

Let now $(a, b, c)$ be an ordered triple.

**Definition 4** The 1-cocycle $R_{(a,b,c)}$ is defined by the formula given in Fig. 15.

**Theorem 2** $R_{(a,b,c)}$ is a 1-cocycle in $M_n^{reg}$ for all $a, b, c$. If $a$, $b$ and $c$ are neither 0 nor $n$, then $R_{(a,b,c)}$ is a 1-cocycle in $M$.

Moreover, its restriction for

$0 < a, b, c < n$

$a + c \leq n$

$n \leq b + c$

$a \leq b$

$a + b + c \leq 2n$

could be a non-trivial 1-cocycle in $M_n$. Again, if one of $a$, $b$ or $c$ is 0 or $n$, then it is only a 1-cocycle in $M_n^{reg}$.
$R_{(a,b,c)}$ could be interesting for classical knots starting from $n = 2$ with $a = 0$, $b = 1$ and $c = 2$.

For the convenience of the reader we recall here the definitions of some canonical loops in $M_n^{reg}$, on which we could evaluate our 1-cocycles in order to get invariants of classical knots.

**Definition 5** Let $K$ be an oriented framed long knot and let $T$ be a string link which induces a cyclical permutation of its end points. We denote by $nK$ the knot in $M_n$ which is obtained by closing the parallel $n$-cable of $K$ (with respect to the framing and with the induced orientation) by the string link $T$ to a knot in the solid torus. The loop $\text{push}(T,K)$ is defined by pushing once $T$ through the parallel $n$-cable of $K$ in the solid torus in counter-clockwise direction [6].

Adding to $nK$ a full-twist in form of a positive $n$-curl, compare Fig. 22 in the next section, we could push then $nK$ once through the curl. This is a nice representative in $M_n^{reg}$ of Gramain’s loop $\text{rot}(nK)$, see [8], which is induced by the full rotation of $V^3$ around its core.

**Remark 1** The new 1-cocycles $R_{[a,b,c]}$ and $R_{(a,b,c)}$ have a chance to detect knot orientation. Indeed, let $-K$ be the inverse knot in the solid torus, i.e. the image of $K$ with reversed orientation after applying the hyperelliptic involution to the solid torus. It is not difficult to see that all signs, markings and relative positions of two generic disjoint chords in the Gauss diagram stay invariant, but the crossings $ml$ and $hm$ in the triangle change places. Consequently, for each oriented loop $\gamma \subset M$ we have $R_{[a,b,c]}(\gamma(K)) = R_{[a,c,b]}(\gamma(-K))$. This implies that if $R_{[a,b,c]}$ is not always equal to $R_{[a,c,b]}$ (in particular if it is not always trivial), then it would detect the knot orientation.

$R_{(a,b,c)}$ changes also by replacing $K$ by $-K$, but in a more complicated way.

### 3 The 1-cocycle $R^{(2)}_a$ in $M_n^{reg}$

The local types of Reidemeister moves for unoriented knots are shown in Fig. [16].

For oriented knots there are exactly eight local types of $R$ III moves, see Fig. [17]. The sign corresponds to the side of the complement of the
Figure 16: The Reidemeister moves for unoriented knots
Figure 17: Local types of a triple crossing
discriminant. It coincides with our coorientation for the global type $r$ and it is the opposite for the global type $l$.

To each Reidemeister move of type III corresponds a diagram with a *triple crossing* $p$: three branches of the knot (the highest, middle and lowest with respect to the projection $pr : \mathbb{C}^* \times \mathbb{R} \to \mathbb{C}^*$) have a common point in the projection into the plane. A small perturbation of the triple crossing leads to an ordinary diagram with three crossings near $pr(p)$.

**Definition 6** We call the crossing between the highest and the lowest branch of the triple crossing $p$ the distinguished crossing of $p$ and we denote it by $d$ ($d$ stands for distinguished). The crossing between the highest branch and the middle branch is denoted by $hm$ and that of the middle branch with the lowest is denoted by $ml$, compare Fig. 18. For better visualization we draw the crossing $d$ always with a thicker arrow.

**Definition 7** Let $c$ be a crossing of marking $n$ in a $R$ III move $p$. Then $W_1(c) = \sum_{0 \in (c,0)} w(0)$.

In other words, we take the algebraic sum of all 0-crossings which cut the n-crossing $c$ from the left to the right (in the Gauss diagram). The difference with $(n,0)$ is the fact, that we do not multiply by the sign $w(c)$ of the crossing $c$. If $[c] \neq n$ then of course $W_1(c) = 0$.

Let the solid torus $V^3$ be embedded in $\mathbb{R}^3$ in the standard way.

**Definition 8** Let $c = ml$ for a $R$ III move $p$. Then the linking number $l(c) \in \mathbb{Z}$ is defined as the linking number of $K_+(c)$ with $K_-(c)$ in $V^3 \subset \mathbb{R}^3$. Let $c = d$ for a $R$ III move $p$. Then the linking number $l(c) \in \mathbb{Z}$ is defined as the linking number of $K_+(c)$ with $K_-(c)$ in $V^3 \subset \mathbb{R}^3$ on the positive side of the $R$ III move $p$. 

Figure 18: The names of the crossings in a $R$ III-move
\[ [d] = a \quad \text{and} \quad l(n, n-a, a) = r(a, a, n) \]

Figure 19: The R III moves in \( R_a^{(2)} \)

The linking number \( l(ml) \) does not depend on the side of the move but \( l(d) \) changes by 1. We have therefore to make a choice of the side in this case. It will follow from the tetrahedron equation that we have to chose the positive side. Notice that the linking numbers do not use the homological markings of the crossings of \( K_+(c) \) with \( K_-(c) \) in the diagram in \( V^3 \).

We use the following notation for the global types of R III moves: \( r \) if the crossing \( ml \) goes from the left to the right in the Gauss diagram and \( l \) if it goes from the right to the left. The homological markings are always given in the following order: \( (d, hm, ml) \). If the marking of a crossing is arbitrary, then we write a point at the place.

We illustrate our notations in Fig. 19.

We are now ready to define our first 1-cocycle with quadratic weights.

**Definition 9** Let \( \gamma \) be an oriented generic loop in \( M^r_{n} \) and let \( p \) be the R III moves in \( \gamma \). Let \( 0 < a < n \) be fixed. Then
Figure 20: The only possible 0-crossings in the weights $W_1$ for $ml$, $hm$ and $d$

Figure 21: The 0-crossing would imply the existence of a loop $s$ with $[s] = a - n < 0$

$$R^{(2)}_a(\gamma) =$$
$$\sum_{r(a,a,a)} \text{sign}(p)(W_2(p) + (w(hm) - 1)W_1(hm) + 1/2(w(ml) - 1)W_1(ml))$$
$$+ \sum_{l(a,a,n)} \text{sign}(p)W_2(p)$$
$$- \sum_{r(a,a,n)} \text{sign}(p)(l(d) + 1/2(w(d) - 1))W_1(ml)$$
$$- \sum_{r(a,n,a)} \text{sign}(p)l(ml)W_1(hm)$$
$$- \sum_{l(n,n-a,a)} \text{sign}(p)l(ml)W_1(d).$$

Actually, the position of the 0-crossings in the weights $W_1$ is determined by the fact that there are no negative loops in diagrams for knots in $M_n$. We show this in Fig. 20 and Fig. 21. Notice that the linear terms in the formula disappear for positive triple crossings.

One canonical loop in $M^{reg}_n$ is the pushing of $nK$ through the n-cable of a positive curl (compare Definition 5). This corresponds to Gramain’s loop
which is induced by a full rotation of the solid torus around its core. We call the first half of the loop push the scan-arc and denote it by scan(nK).

A picture of scan(nK) is given in Fig. 22. Notice, that it is only an arc in $M_{reg}^n$ and not a loop.

Instead of classical knots our invariants are of course also defined for arbitrary coherently oriented n-component string links $T$ in $\mathbb{R}^3$, by evaluating the 1-cocycles e.g. on scan$(T)$ or on rot$(T)$.

**Theorem 3** Let $0 < a < n$. Then $R_a^{(2)}$ is an integer valued 1-cocycle in $M_{reg}^n$, which is not always trivial. Already $R_a^{(2)}$(scan$(nK)$) is an invariant of $nK$ up to regular isotopy (and hence a knot invariant of $K$). The corresponding Lagrange interpolation polynomials $LR^{(2)}$(push$(nK)$) and $LR^{(2)}$(rot$(nK)$) are knot polynomials of degree at most $n - 2$.

Evidently, R I moves do not change the value of $R_a^{(2)}$. However, $R_a^{(2)}$ does not vanish in general on the meridian of a cusp with a transverse branch in the projection $pr$ (because it could contain just one R III move $p$ with $[d] = a$ and no control over $W_2(p)$ at all), i.e. it does not satisfy the wandering cusps equations (compare [6]). Consequently, it is not a 1-cocycle in $M_n$ but only in $M_{reg}^n$.

We consider only the most simple example of the loop push$(T,K)$: let $K$ be the standard positive trefoil, $n = 2$ and let $T$ be the 2-braid $\sigma_1$, see

![Diagram of scan of nK]

Figure 22: The scan of $nK$
Figure 23: We push $\sigma_1$ through the 2-cable of the standard positive trefoil.

Fig. 23. We indicate in Fig. 23 the 0-crossings and the n-crossings. All remaining crossings have the marking 1.

We have calculated that for this example $R_1^{(2)}(\text{push}(T, K)) = -1$. There are exactly twelve R III moves, which we consider in their natural order. We give just the contribution from $W_2(d)$ and from $W_1$ for each of the eight moves which could a priori contribute (because they have the right global type). An interested reader could check this easily.

move 1: 0, move 3: $1 - 3$, move 4: 3, move 5: 1, move 8: 0, move 9: $-3$, move 10: $-1 + 2$, move 12: $-1$.

There are even more canonical loops, e.g. the Fox-Hatcher loops, in $M_n$ and in $M$, compare [6] and references therein. However, the Fox-Hatcher loop is not a regular isotopy. But of course, by using Whitney tricks, it can be approximated by a regular isotopy, see e.g. [5]. This approximation is not unique, but we show that for each component of $M_n^{\text{reg}}$ the kernel of the natural inclusion $\text{in}: H_1(M_n^{\text{reg}}) \rightarrow H_1(M_n)$ is generated by sliding small curls of marking 0 and of marking n along the whole knot. It turns out that $R_n^{(2)} = 0$ on these loops.

**Proposition 2** Let $\text{in}: M_n^{\text{reg}} \rightarrow M_n$ be the inclusion. Then the kernel of $\text{in}_*: H_1(M_n^{\text{reg}}) \rightarrow H_1(M_n)$ for each component of $M_n^{\text{reg}}$ is generated by sliding a small curl with marking 0 and sliding a small curl with marking n along the whole knot.

**Proof.**
Let $\gamma(s) \subset M^r_n$, $s \in [0,1]$, be a generic loop with base point $*$ which contracts in $M_n$. We try to contract it in $M^r_n$ to $*$. Let $\gamma_t$, $t \in [0,1]$, be a generic homotopy of $\gamma_0 = \gamma$ to $\gamma_1 = *$ in $M_n$.

We have only to study the following events in the 2-parameter family $\gamma(s)_t$, $s,t \in [0,1]$, (compare [8]):

• (1) $\gamma_t$ becomes tangential to $\Sigma^{(1)}_{cusp} \subset M_n$. In case (a) a cusp is born and dies immediately after and in case (b) a cusp dies and is immediately reborn.

• (2) $\gamma_t$ passes through the transverse intersection of a stratum $\Sigma^{(1)}_{cusp}$ with another stratum of $\Sigma^{(1)}$

• (3) $\gamma_t$ passes through $\Sigma^{(2)}_{trans-cusp}$

• (4) $\gamma_t$ passes through $\Sigma^{(2)}_{cusp-deg}$

In (1a) we replace the birth and the following death of a cusp in $\gamma_{t_0}$ by a Whitney trick and its inverse and we continue the homotopy $\gamma_t \subset M^r_n$, $t > t_0$. The effect are just two additional small curls on the diagrams for a small arc $s$ in the loops $\gamma_t$, $t > t_0$. We keep the small curls in the rest of the homotopy.

If $\gamma_t$ touches $\Sigma^{(1)}_{cusp}$ from the other side, i.e. in (1b), then we simply keep the corresponding small curl on the arc $s$.

In (2) there is no problem at all because we can make Whitney tricks simultaneously at different places of a diagram.

In (3) a branch moves over or under a cusp. But we can perform a Whitney trick which creates or eliminates the cusp simultaneously with moving the brunch over or under the Whitney trick.

In (4) a Reidemeister II move is replaced by the birth or the death of two cusps with different writhe and different Whitney index. We can replace this by a single Whitney trick, which is followed by sliding one of the two curls once over the other in order to obtain the diagram which allows the Reidemeister II move.

We end up with a diagram $*'$ which is regularly isotopic to $*$, but which differs from $*$ by lots of small curls on it. But because $*'$ and $*$ are regularly isotopic the sum of the writhe's of the small curls as well as the sum of their Whitney indices vanishes. This implies that we can eliminate them two by two with Whitney tricks. This approximation of the homotopy $\gamma_t$ by
a homotopy in $M_n^{reg}$ is unique up to adding loops in $M_n^{reg}$ which consist of sliding small curls through small curls and sliding small curls all along the knot. These loops are evidently contractible in $M_n$, but our method does not show that they are contractible in $M_n^{reg}$, because we are not allowed to contract small curls (as in (1b)). □

(Let $U$ denote a diagram of an unknot which consists of small curls on a small arc of the knot $K$. In fact, one can prove that e.g. the loop rot($U$), which is obtained by exchanging two positive curls with the same negative Whitney index, is not contractible in $M_n^{reg}$ by using the techniques of trace graphs developed in [7], compare also [6]. The trace graph of a loop in $M_n$ is an oriented singular link in the thickened torus. All its singularities are ordinary triple points. The parity of the number of non-contractible components of any resolution of the trace graph in the thickened torus is an invariant of the homotopy class of the loop in $M_n^{reg}$, compare [7]. The trace graph of the constant loop is just the standard closure of the trivial n-braid, where $n$ is the number of crossings of the knot, hence $n = 2$ in our case of $U$. On the other hand, one easily sees that any resolution of the trace graph of rot($U$) has only one component, because the two crossings are interchanged by the monodromy. This implies that the kernel of $in_* : H_1(M_n^{reg}) \rightarrow H_1(M_n)$ is never trivial.)

**Proposition 3** The 1-cocycle $R_a^{(2)}$ vanishes on the kernel of $in_* : H_1(M_n^{reg}) \rightarrow H_1(M_n)$.

**Proof.**

Sliding a small curl of marking 0 does not lead to any R III moves which contribute to $R_a^{(2)}$. We have only to consider the sliding of a curl of marking $n$. If a crossing is not of marking $a$ then sliding the curl over or under the crossing does not contribute neither. Let us consider a crossing of marking $a$. By using Whitney tricks and Reidemeister II moves we can easily show that it is sufficient to consider a positive crossing and a positive curl. But the curl slides exactly one time over and one time under the crossing, as shown in Fig. 24. The first R III move here is of type $-r(a,n,a)$ and the second is of type $+r(a,a,n)$. Evidently, in the first move $W_1([hm] = n) = 0$ and in the second move $W_1([ml] = n) = 0$. But the crossing $d$ is essentially the same crossing for the two moves, and consequently $W_2(p)$ is the same for the two moves too (remember that by definition the n-crossing $ml$ or $hm$
Figure 24: Sliding a small n-curl twice through a crossing of marking $a$

does never contribute to $W_2(p)$. It follows that the contributions of the two moves cancel out together. □

Let $fh(nK)$ be the Fox-Hatcher loop, see [6], and let $LR^{(2)} \in \mathbb{Q}[x]$ be the Lagrange interpolation polynomial of $R^{(2)}_a$ with respect to $0 < a < n$. Proposition 3 implies that $LR^{(2)}(fh(nK))$ is well defined. Using a fundamental result of Allen Hatcher [9] it was shown in [6], that if $K$ is a non-trivial torus knot, then the non-trivial classes $[\text{rot}(nK)]$ and $[fh(nK)]$ are linearly dependent in $H_1(M_n; \mathbb{Q})$ for each $n > 0$ and each closure to a knot in the solid torus with a string link $T$.

It follows that if $K$ is a non-trivial torus knot, then the rational function $LR^{(2)}(fh(nK))/LR^{(2)}(\text{rot}(nK)) \in \mathbb{Q}(x)$ is in fact only a rational number!

Is this number a complete invariant for non-trivial torus knots if $n$ is big enough?

Remark 2 Let $c$ denote the number of crossings of a classical knot and let $n$ be the number of strands in the cable of $K$. It is well known that finite type invariants behave functorial under cabling, see e.g. [10], but quantum knot invariants do not (higher dimensional representations correspond to linear combinations of cables). It is expected that invariants from 1-cocycles do not behave functorial under cabling neither, because already the possible values of the parameters $a, b, c$ are different and the $n$-component string links $T$, in order to close the parallel $n$-cable to a knot, are necessarily different too.

It is therefore natural to measure the complexity of the calculation of the invariants by the two parameters $c$ and $n$.

The knot $nK$ has about $n^2c$ crossings. The loops push and rot contain therefore about $n$ times $n^2c$ RIII moves. But only about $1/n$ of these moves have $[d] = a$ (the number of moves with markings $a$ and $n$ or $a$ and $0$ is
not of the same order). Only crossings of marking \( n \) or \( 0 \) contribute to the weights and there are about \( nc \) such crossings. Therefore there are \( O((nc)^2) \) operations necessary to calculate the weight in \( R_a^{(2)} \) for a \( R III \) move. Consequently, the calculation of \( R_a^{(2)}(\text{push}(nK)) \) is of complexity \( O(n^4c^3) \) for each given homological parameter \( a \), and the calculations of the corresponding Lagrange interpolation polynomials \( LR^{(2)}(\text{push}(nK)) \) or \( LR^{(2)}(\text{rot}(nK)) \) in \( \mathbb{Q}[x] \) are of complexity \( O(n^5c^3) \). One can show that the calculation of \( LR^{(2)}(fh(nK)) \) is of complexity \( O(n^6c^4) \).

4 The 1-cocycle \( R_{a+}^{(2)} \) in \( M_{\text{reg}}^n \)

In this section we refine the 1-cocycle \( R_a^{(2)} \).

**Definition 10** Let \( p \) be a \( R III \) move with \([d] = a\). Then the refined weight of \( p \) is defined by \( W_2^+(p) = \sum (n,0) \), where the sum is taken only over all those \( n \)-crossings and \( 0 \)-crossings which are not crossings of the move \( p \), and such that the \( 0 \)-crossing cuts the \( n \)-crossing from the left to the right, the foot of the \( n \)-crossing is in \( K^-((d) \) and the foot of the \( 0 \)-crossing is in \( K^+(d) \).

The only difference with \( W_2^-(p) \) is that the foot of the \( 0 \)-crossing has to be now in \( K^+(d) \).

**Definition 11** Let \( c \) be a crossing \( \text{hm} \) in a \( R III \) move \( p \). If \([c] = n\) then \( W_1^+(c) = \sum_{0 \in (c,0)} w(0) \) and the foot of the \( 0 \)-crossing has to be in \( K^+(d) \). If \([c] = 0\) then \( W_1^+(c) = \sum_{n \in (n,0)} w(n) \) and the foot of the \( n \)-crossing has to be in \( K^-(d) \). For \( l(a,0,a) \) the foot of the \( n \)-crossing has to be in \( K^-(d) \). For \( r(a,n,a) \) the foot of the \( 0 \)-crossing has to be in \( K^+(d) \). If \([c]\) is neither \( n \) nor \( 0 \) then \( W_1^+(c) = 0 \).

Notice that a move \( p \), which contributes to the 1-cocycle, can never contain a \( n \)-crossing and a \( 0 \)-crossing at the same time, because it contains always a crossing of marking \( a \) and \( 0 < a < n \).

The linking numbers \( l(d) \) and \( l(ml) \) are still the same as in Definition 8.

We are now ready to define \( R_{a+}^{(2)} \). The formula is more symmetric than that for \( R_a^{(2)} \). It contains only four quadratic terms, which give already a solution of the positive global tetrahedron equation, and the cube equations contribute just two linear terms in this case! The following is our most beautiful formula, because it is relatively short.

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Figure 25: The only possible 0-crossings and n-crossings in the weights $W_i^+$ for $hm$

**Definition 12** Let $\gamma$ be an oriented generic loop in $M_{n\text{reg}}$ and let $p$ be the R III moves in $\gamma$. Let $0 < a < n$ be fixed. Then

$$R_{a+}^{(2)}(\gamma) =$$

$$\sum_{r(a,\ldots)} \text{sign}(p)(W_2^+(p) + (w(hm) - 1)W_1^+(hm))$$

$$+ \sum_{l(a,\ldots)} \text{sign}(p)(W_2^+(p) + (w(hm) - 1)W_1^+(hm))$$

$$- \sum_{r(a,n,a)} \text{sign}(p)l(ml)W_1^+(hm)$$

$$- \sum_{l(a,0,a)} \text{sign}(p)l(ml)W_1^+(hm)$$

**Theorem 4** Let $0 < a < n$. Then $R_{a+}^{(2)}$ is an integer valued 1-cocycle in $M_{n\text{reg}}$, which is not always trivial. Already $R_{a+}^{(2)}(\text{scan}(nK))$ is an invariant of $nK$ up to regular isotopy (and hence a knot invariant of $K$).

The positions of the 0-crossings and the n-crossings in the weights $W_i^+$ are also determined by the fact that there are no negative loops in diagrams for knots in $M_n$. We show this in Fig. 25. This excludes automatically the global types $r(a,a,n)$, $l(a,a,0)$ and $l(n,n-a,a)$, because the possible foots of the 0-crossings or of the n-crossings are not in the right arcs on the circle in order to contribute to $R_{a+}^{(2)}$.

We consider again only the example of the loop $\text{push}(T,K)$, where $K$ is the standard positive trefoil, $n = 2$ and $T$ is the 2-braid $\sigma_1$ (compare the previous section). We have calculated that for this example $R_{1+}^{(2)}(\text{push}(T,K)) = -1$ too. We give just the contribution from $W_2^+(d)$ and from $W_1^+$ for each
of the six R III moves which could a priory contribute. An interested reader could check this easily too.

move 1: 0, move 3: $0 - 3$, move 5: $+1$, move 8: $-0 + 0$, move 10: $-1$, move 12: $-1 + 3$.

The example shows that $R_{a}^{(2)}$ and $R_{a+}^{(2)}$ are different as 1-cocycles. However, our example does not exclude yet that $[R_{a}^{(2)}] = [R_{a+}^{(2)}] \in H^{1}(M_{n}^{reg}; \mathbb{Z})$. But this would imply that $R_{a}^{(2)} = 0$ for all loops, which is very unlikely.

Notice that Proposition 3 can be easily extended to $R_{a+}^{(2)}$ too.

5 Proofs

5.1 Generalities and preparations

We use the technology which was developed in [6]. For the convenience of the reader we recall here the main lines of our approach.

We study the discriminant $\Sigma$ of non-generic diagrams in $M$ or $M_{n}^{reg}$, together with its natural stratification. Our strategy is the following: for an oriented generic loop in $M$ or $M_{n}^{reg}$ we associate an integer to the intersection with each stratum in $\Sigma^{(1)}_{\text{tri}}$, i.e. to each Reidemeister III move, and we sum up over all moves in the loop.

In order to show that our 1-cochains are 1-cocycles, we have to prove that the sum is 0 for each meridian of strata of codimension 2, i.e. in $\Sigma^{(2)}$. This is very complex but we have used strata from $\Sigma^{(3)}$ in order to reduce the proof to a few strata in $\Sigma^{(2)}$. It follows that our sum is invariant under generic homotopies of loops in $M$ or $M_{n}^{reg}$. But it takes its values in an abelian ring and hence it is a 1-cocycle. Showing that the 1-cocycle is 0 on the meridians of the quadruple crossings $\Sigma^{(2)}_{\text{quad}}$ is by far the hardest part. This corresponds to finding a new solution of the tetrahedron equation.

Consider four oriented straight lines which form a braid and such that the intersection of their projection into $\mathbb{C}$ consists of a single point. We call this an ordinary quadruple crossing. After a generic perturbation of the four lines we will see now exactly six ordinary crossings. We assume that all six crossings are positive and we call the corresponding quadruple crossing a positive quadruple crossing. Quadruple crossings form smooth strata of codimension 2 in the topological moduli space of lines in 3-space which is equipped with a fixed projection $pr$. Each generic point in such a stratum...
is adjacent to exactly eight smooth strata of codimension 1. Each of them
 corresponds to configurations of lines which have exactly one ordinary triple
 crossing besides the remaining ordinary crossings. We number the lines from
 1 to 4 from the lowest to the highest (with respect to the projection \( pr \)).

The eight strata of triple crossings glue pairwise together to form four
 smooth strata which intersect pairwise transversely in the stratum of the
 quadruple crossing, see Fig. 27, compare [6] and [7]. The strata of triple
 crossings are determined by the names of the three lines which give the
 triple crossing. For shorter writing we give them names from \( P_1 \) to \( P_4 \) and
 \( \bar{P}_1 \) to \( \bar{P}_4 \) for the corresponding stratum on the other side of the quadruple
 crossing. We show the intersection of a normal 2-disc of the stratum of
 codimension 2 of a positive quadruple crossing with the strata of codimension
 1 in Fig. 26. The strata of codimension 1 have a natural coorientation,
 compare Definition 2. We could interpret the six ordinary crossings as the
 edges of a tetrahedron and the four triple crossings likewise as the vertices
 or the 2-faces of the tetrahedron. For the classical tetrahedron equation one
 associates to each stratum \( P_i \), i.e. to each vertex or equivalently to each 2-face
 of the tetrahedron, some operator (or some R-matrix) which depends only
 on the names of the three lines and to each stratum \( \bar{P}_i \) the inverse operator.
 The tetrahedron equation says now that if we go along the meridian then
 the composition of these operators is equal to the identity. Notice, that
 in the literature, see e.g. [11], one considers planar configurations of lines.
 But this is of course equivalent to our situation because all crossings are
 positive and hence the lift of the lines into 3-space is determined by the
 planar picture. Moreover, each move of the lines in the plane which preserves
 the transversality lifts to an isotopy of the lines in 3-space. The tetrahedron
 equation has many solutions, the first one was found by Zamolodchikov, see
 e.g. [11].

However, the solutions of the classical tetrahedron equation are not well
 adapted in order to construct 1-cocycles for moduli spaces of knots. A a
 local solution of the tetrahedron equation is of no use for us, because as
 already pointed out there are no integer valued 1-cocycles for all knots in
 the 3-sphere. We have to replace them by long knots or more generally
 by points in \( M_n \). For knots in the solid torus \( V^3 \) we can now associate
 to each crossing in the diagram a winding number (i.e. a homology class
 in \( H_1(V^3) \)) in a canonical way. Therefore we have to consider six different
 positive tetrahedron equations, corresponding to the six different abstract
 closures of the four lines to a circle and in each of the six cases we have to
Figure 26: The intersection of a normal 2-disc of a positive quadruple crossing with the strata of triple crossings

consider all possible winding numbers of the six crossings. We call this the positive global tetrahedron equations. There are exactly six global types of positive quadruple crossings without the homological markings. We show them in Fig. 28.

One easily sees that there are exactly forty eight local types of quadruple crossings (analog to the eight local types of triple crossings).

We study the relations of the local types of triple crossings in what we call the cube equations. Triple crossings come together in points of $\Sigma^{(2)}_{\text{trans-self}}$, i.e. an auto-tangency with in addition a transverse branch. But one easily sees that the global type of the triple crossings (i.e. its Gauss diagram with the homological markings but without the writhe) is always preserved. We make now a graph $\Gamma$ for each global type of a triple crossing in the following way: the vertices correspond to the different local types of triple crossings. We connect two vertices by an edge if and only if the corresponding strata of triple crossings are adjacent to a stratum of $\Sigma^{(2)}_{\text{trans-self}}$. We have shown that the resulting graph is the 1-skeleton of the 3-dimensional cube $I^3$, see Fig. 29. In particular, it is connected. The edges of the graph $\Gamma = \text{skl}_1(I^3)$
Figure 27: Unfolding of a positive quadruple crossing
Figure 28: The global types of quadruple crossings

correspond to the types of strata in $\Sigma^{(2)}_{\text{trans-self}}$. The solution of the positive tetrahedron equation tells us what is the contribution to the 1-cocycle of a positive triple crossing (i.e. all three involved crossings are positive). The meridians of the strata from $\Sigma^{(2)}_{\text{trans-self}}$ give equations which allow us to determine the contributions of all other types of triple crossings. However, a global phenomenon occurs: each loop in $\Gamma$ could give an additional equation. Evidently, it suffices to consider the loops which are the boundaries of the 2-faces from $\text{skl}_2(I^3)$. We call all the equations which come from the meridians of $\Sigma^{(2)}_{\text{trans-self}}$ and from the loops in $\Gamma = \text{skl}_1(I^3)$ the cube equations. (Notice that a loop in $\Gamma$ is more general than a loop in $M$ or $M^\text{reg}_n$. For a loop in $\Gamma$ we come back to the same local type of a triple crossing but not necessarily to the same whole diagram of the knot.)

Our strategy is the following: first we find a solution of the positive global tetrahedron equation. We solve then the cube equations by adding correction terms for different local types of triple crossings, but which vanish for positive triple crossings. We have shown in [6] that the resulting 1-cochain is then a 1-cocycle in $M^\text{reg}$ or $M^\text{reg}_n$.

For the convenience of the reader we give here again the figures which are the main tool for our work, see [6]. Let consider the global positive quadruple
crossings. We naturally identify crossings in an isotopy outside Reidemeister moves of type I and II. The Gauss diagrams of the unfoldings of the quadruple crossings are given in Fig. 30 up to Fig. 41. For the convenience of the reader (and for further research) we marque also the different possibilities for the point at infinity in the case of long knots, i.e. $M_1$. The (positive) crossing between the local branch $i$ and the local branch $j$ is always denoted by $ij$. We give then the homological markings, which depend on three parameters $\alpha$, $\beta$ and $\gamma$ on the circle.

Using these figures we show in Fig. 42 up to Fig. 53 the homological markings of the arrows. Here $\alpha$, $\beta$ and $\gamma$ are the homology classes represented by the corresponding arcs in the circle (remember that the circle in the plan is always oriented counter-clockwise). The ordering of the global types here is more adapted to the check for our 1-cocycles.

In $M_n$ we have the unknown parameters $\alpha$, $\beta$ and $\gamma$ in \{0,1,...,n\} and $\alpha + \beta + \gamma \leq n$, because there are no negative loops in the diagrams. In $M$ there are no restrictions at all on the integers $\alpha$, $\beta$ and $\gamma$.

5.2 Proof of Theorems 1 and 2

All triple crossings in the definition of $R_{[a,b,c]}$ are of type $r$. Moreover, the isolated arrow in the configuration is always parallel to $d$ but in the opposite direction. Such configurations are extremely rare in the meridians of the positive quadruple crossings. Inspecting Fig. 30 up to Fig. 41, we see that
Figure 30: first half of the meridian for global type I
Figure 31: second half of the meridian for global type I
Figure 32: first half of the meridian for global type II
Figure 33: second half of the meridian for global type II
Figure 34: first half of the meridian for global type III
Figure 35: second half of the meridian for global type III
Figure 36: first half of the meridian for global type IV
Figure 37: second half of the meridian for global type IV
Figure 38: first half of the meridian for global type V
Figure 39: second half of the meridian for global type V
Figure 40: first half of the meridian for global type VI
Figure 41: second half of the meridian for global type VI
Figure 42: moves of type $r$ for the global type IV
Figure 43: moves of type $l$ for the global type IV
Figure 44: moves of type $r$ for the global type II
Figure 45: moves of type $l$ for the global type II
Figure 46: moves of type $r$ for the global type III
Figure 47: moves of type $l$ for the global type III
Figure 48: moves of type $r$ for the global type $V$
Figure 49: moves of type $l$ for the global type $V$
Figure 50: moves of type $r$ for the global type I
Figure 51: the remaining moves of type r for the global type I
Figure 52: moves of type $l$ for the global type VI
Figure 53: the remaining moves of type $l$ for the global type VI
they appear only for the global type I of the positive quadruple crossing, and only in the strata $P_2$ and $P_3$. Inspecting Fig. 50 and Fig. 51 we see the following contributions.

$-P_2$: \( r(\alpha + \beta, \alpha + \beta + \gamma, n - \gamma) \) and the isolated arrow has marking \( n - \beta \).

$+P_3$: \( r(n - \beta - \gamma, n - \gamma, n - \beta) \) and the isolated arrow \( \alpha + \beta + \gamma \).

Let us define \( a = n - \beta, b = n - \gamma \) and \( c = \alpha + \beta + \gamma \) (remember that \([d] = [hm] + [ml] - n\)). It follows that \( r(b + c - n, c, b) \) with weight \( a \) cancels out with \( r(a + b - n, b, a) \) with weight \( c \).

If $-P_2$ is now of the global type \( r(a + b - n, b, a) \) with weight \( c \) then $+P_3$ is of the global type \( r(a + c - n, a, c) \) with weight \( b \). Consequently, we have to add this configuration in the definition of \( R_{[a,b,c]} \).

If $-P_2$ is now of the global type \( r(a + c - n, a, c) \) with weight \( b \) then $+P_3$ is of the global type \( r(b + c - n, c, b) \) with weight \( a \), but this configuration was already included in the definition of \( R_{[a,b,c]} \). Consequently, \( R_{[a,b,c]} \) is a solution of the positive global tetrahedron equation and the configurations in its definition are just related by cyclic permutations of the homological markings of \( ml, hm \) and the weight.

The solution of the cube equations is trivial in this case. The signs of \( hm, ml \) and \( d \) (which can change by passing \( \Sigma_{trans-self}^{(2)} \)) do not enter into the formula and the arrow of the weight can never become an arrow in the triangle by passing \( \Sigma_{trans-self}^{(2)} \), because it has not the right direction, compare [6] or the next subsection.

\( R_{[a,b,c]} \) can be non-trivial in \( M_n \) only if each loop in the configurations is not negative. This gives the conditions: \( a + b \geq n, a + c \geq n, b + c \geq n, a \geq n - c, b \geq n - c, a + b - n \geq n - c \) and so on. We end up with exactly the conditions from Theorem 1.

Reidemeister I moves create or delete of course only crossings of marking 0 or \( n \), and in the triple crossing in the meridian of a cusp with a transverse branch (compare [6]) there is always a crossing \( hm \) or \( ml \) of marking 0 or \( n \).

This finishes the proof of Theorem 1.

We will see in the next subsection that if the contributions of \( P_2 \) and \( \bar{P_2} \) (i.e. a branch moves under the triple crossing from one side to the other) do not always cancel out together, then the 1-cocycle is not already invariant on the scan-arc. Consequently, \( R_{[a,b,c]} \) is in general not invariant on the scan-arc.

The 1-cocycle \( R_{(a,b,c)} \) exploits the isolated arrows with opposite direction
which are parallel to \( hm \) or \( ml \). We inspect again Fig. 30 up to Fig. 41.

There are no such arrows at all for the global types I and VI. The global type III makes a connection between the first two configurations of \( R_{(a,b,c)} \) and the global type V makes a connection between the last two configurations of \( R_{(a,b,c)} \). The global types II and IV mix then the first two configurations with the last two. We give now the details.

Type III, see Fig. 46 and Fig. 47: + \( \bar{P}_2 \): \( \alpha = a, \ n - \alpha - \gamma = c, \ \alpha + \beta + \gamma = b \) cancels out with a configuration in \( -\bar{P}_3 \).

Type V, see Fig. 48 and Fig. 49: + \( P_2 \): \( n - \gamma = b, \ \alpha + \gamma = c, \ \beta = a \) cancels out with a configuration in \( -P_3 \).

Type II, see Fig. 44 and Fig. 45: - \( \bar{P}_1 \): \( \beta = a, \ n - \gamma = b, \ \alpha + \gamma = c \) cancels out with a configuration in + \( P_2 \).

- \( P_3 \): \( \alpha + \beta + \gamma = b, \ n - \beta - \gamma = c, \ \beta = a \) cancels out with a configuration in + \( P_4 \).

Type IV, see Fig. 42 and Fig. 43: - \( \bar{P}_1 \): \( \alpha + \beta + \gamma = b, \ \alpha = a, \ n - \alpha - \gamma = c \) cancels out with a configuration in + \( P_2 \).

- \( P_3 \): \( \beta = a, \ n - \beta - \gamma = c, \ \alpha + \beta + \gamma = b \) cancels out with a configuration in + \( P_4 \).

The solution of the cube equations is again trivial for exactly the same reasons as for \( R_{(a,b,c)} \).

The condition that there are no negative loops in \( M_n \) leads now to: \( a + c \leq n, \ n \leq b + c, \ a \leq b \) and \( b + c - n \leq n - a \) as stated in the theorem.

Of course, \( R_{(a,b,c)} \) is in general not invariant on the scan-arc neither.

This finishes the proof of Theorem 2.

5.3 Proof of Theorems 3 and 4

We have already shown in Section 1 that the contribution of each of the configurations in \( R^{(2)}_a \) is invariant under: (1) a R III move with simultaneously another Reidemeister move. We have to show now that \( R^{(2)}_a \) without the linear terms (which vanish automatically for positive triple crossings) vanishes on the meridian of each global positive quadruple crossing, i.e. it satisfies the positive global tetrahedron equation.

The eight strata in the meridian come in pairs with different signs: \( P_i \) and \( \bar{P}_i \). The crossings \( d, \ hm \) and \( ml \) are of course the same for \( P_i \) and \( \bar{P}_i \). These two strata differ with respect to the triangle by exactly the three remaining crossings from the quadruple crossing. But only n-crossings with the foot in \( K_-(d) \) contribute to \( W_2(p) \). We have therefore to study when the foot of a
n-crossing has slide over the foot or the head of the crossing \( d \) from \( P_i \) to \( \bar{P}_i \). Inspecting Fig. 30 up to Fig. 41 we see that the crossings, for which the foot slides over \( d \), depend only on the local type of the quadruple crossing. Therefore we have to study only the following crossings:

- \( P_1 \): foot of 12 in \( K_+(d = 14) \), \( \bar{P}_1 \): foot of 12 in \( K_-(d = 14) \)
- \( P_3 \): foot of 14 in \( K_-(d = 13) \), \( \bar{P}_3 \): foot of 14 in \( K_+(d = 13) \)
- \( P_3 \): foot of 34 in \( K_+(d = 13) \), \( \bar{P}_3 \): foot of 34 in \( K_-(d = 13) \)
- \( P_4 \): foot of 13 in \( K_+(d = 14) \), \( \bar{P}_4 \): foot of 13 in \( K_-(d = 14) \)

Notice, that this never happens for \( P_2 \) and \( \bar{P}_2 \). Notice also, that the linking numbers \( l \) never change for \( P_2, \bar{P}_2 \) and \( P_3, \bar{P}_3 \). However, they change always for exactly two crossings \( d, hm \) or \( ml \) for \( P_1, \bar{P}_1 \) as well as for \( P_4, \bar{P}_4 \).

In each of the six global types we have now to consider how \( W_2(p) \) changes and how this changing is compensated by a configuration with \( W_1 \) for another stratum. But we have also to study the possible changing of \( W_1 \) from \( P_i \) to \( \bar{P}_i \) because of different positions of 0-crossings. We do all this by inspecting simultaneously a figure from Fig. 30 up to Fig. 41 and the corresponding figure from Fig. 42 up to Fig. 53. We are conscious that this part of the proof will give a hard time to the reader. We apologize for that.

**Global type I**

We proceed as follows: first we go through \( P_1 \) up to \( P_4 \) and we study the compensation of the changing of \( W_2(p) \) by \( W_1 \) of other strata of a special global type. Remember that for \( P_3 \) we have to consider two possible n-crossings. We go then a second time through \( P_1 \) up to \( P_4 \) and study the changing of \( W_1 \) for the remaining special global types.

All global types of triple crossings in the meridian are of type \( r \) in this case. Consequently, the individual contribution of a n-crossing to \( (n,0) \) can not change in the meridian, because it changes only by passing a triple crossing of type \( l(n,0,n) \), compare Section 1.

\( P_1 \): Let \( [d] = \alpha = a \) and \( [12] = n - \beta = n \) implies \( \beta = 0 \). In this case \(-P_1 + \bar{P}_1 \) contribute \(+W_1(12)\). But \( P_4 \) is then of the global type \( r(a,a,n) \) and the crossing \( ml \) is just the crossing 12. We have \( l(d)(P_4) = l(d)(\bar{P}_4) + 1 \) and consequently \(+P_4 - \bar{P}_4 \) contribute \(-((l(d = 14)(\bar{P}_4) + 1)W_1(12) - l(d = 14)(\bar{P}_4)W_1(12)) = -W_1(12) \) and they cancel out together.

\( P_3 \): Let \( [d] = n - \beta - \gamma = a \) and \( [34] = \alpha + \beta + \gamma = n \). Hence \( \beta + \gamma = n - a \) and \( \alpha = a \). In this case \(+P_3 - \bar{P}_3 \) contribute \(-W_1(34)\). But \( P_1 \) is then of the global type \( r(a,n,a) \) and the crossing \( hm \) is just the crossing 34. We have
\( l(ml)(P_1) = l(ml)(\bar{P}_1) + 1 \) and consequently \(-P_1 + \bar{P}_1\) contribute \(+W_1(34)\) and it cancels out.

\( P_3 \): Let \([d] = n - \beta - \gamma = a\) and \([14] = \alpha = n\). This can not happen because negative loops are not allowed.

\( P_4 \): Let \([d] = \alpha = a\) and \([13] = n - \beta - \gamma = n\). Hence \(\beta = \gamma = 0\) (we use all the time that there are no negative loops). In this case \(P_4 - \bar{P}_4\) contribute \(-W_1(13)\). But \(P_1\) is then of the global type \(r(a, a, n)\) and the crossing \(ml\) is just the crossing 13. We have \(l(d)(P_1) = l(d)(\bar{P}_1) + 1\) and hence \(-P_1 + \bar{P}_1\) contribute \(+W_1(13)\) and it cancels out.

\( P_1 \): Both special global types have already appeared.

\( P_3 \): The linking numbers for \(P_3\) and \(\bar{P}_3\) are the same and it cancels out.

\( P_4 \): The linking numbers for \(l(ml)\) in \(P_4\) and \(\bar{P}_4\) are the same and it cancels out.

**Global type II**

We have to be more careful now because of possible changing of the position of 0-crossings for the weights \(W_1\).

\( P_1 \): Let \([d] = \alpha = a\) and \([12] = n - \beta - \gamma = n\) implies \(\beta = \gamma = 0\). Only 23 is a 0-crossing. But 23 does not intersect 12 in \(P_1\). But \(P_1\) is in this case of type \(r(a, a, n)\). We have \(l(d)(P_1) = l(d)(\bar{P}_1) + 1\) and the crossing 23 cuts the crossing \(ml = 13\) in \(P_1\). Consequently, \(-P_1 + \bar{P}_1\) contribute \((l(d = 14)(P_1) + 1)W_1(13) - l(d = 14)(\bar{P}_1)(W_1(13) + 1) + W_1(12) = W_1(13) - l(d = 14)(\bar{P}_1) + W_1(12)\). But \(P_4\) is then also of type \(r(a, a, n)\) and 12 is the crossing \(ml\) and \([13] = n\). We have \(l(d)(P_4) = l(d)(\bar{P}_4) + 1\) and the crossing 23 cuts the crossing \(ml = 12\) in \(\bar{P}_1\) but not the crossing 13. Consequently, \(+P_4 - \bar{P}_4\) contribute \((-l(d = 14)(\bar{P}_4) + 1)W_1(12) + l(d = 14)(P_4)(W_1(12) + 1) - W_1(13) = -W_1(12) + l(d = 14)(\bar{P}_4) - W_1(13)\). It remains to notice that \(l(d = 14)(\bar{P}_1) = l(d = 14)(\bar{P}_4)\) and all the contributions cancel out together.

We have considered this difficult case in all details in order to convince the reader in the correctness of our result. In most of the remaining cases we will adapt shorter explanations.

\( P_2 \): Here we have only to study \(W_1\). Let \([d] = \alpha + \beta + \gamma = n\) and \([ml] = \beta = a\). In \(P_2\) the crossing 14 cuts \(d\) and in \(\bar{P}_2\) it is now the crossing 12 which cuts \(d\). \([14] = \alpha\) and \([12] = n - \beta - \gamma\), but \(\alpha + \beta + \gamma = n\) and hence 14 and 12 can be 0-crossings only simultaneously and they cancel out together.

\( P_3 \): \([d] = n - \gamma\) and \([14] = \alpha\). But \(\gamma = n - a\) and \(\alpha = n\) is not possible in \(M_n\).
\( P_3: \) \([d] = n - \gamma \) and \([34] = \alpha + \gamma \). Let \( \gamma = n - a \) and \( \alpha = a \). Then \([24] = \alpha + \beta + \gamma \neq 0 \) and \(-P_3 + \bar{P}_3 \) contribute \( +W_1(34) \). \( P_1 \) is of type \( r(a,n,a) \) and \( l(ml)(\bar{P}_1) = l(ml)(P_1) + 1 \). \([23] = \beta = 0 \), but it does not cut \( h_{m} = 34 \) from the left to the right. Consequently, \(-P_1 + \bar{P}_1 \) contribute \(-W_1(34)\) and it cancels out.

\( P_1: \) Let \([d] = \alpha = a \) and \([13] = n - \gamma = n \). The case \( \beta = 0 \) was already considered. Therefore we can assume \( \beta \neq 0 \). Then \(+P_1 - \bar{P}_1 \) contribute \(-W_1(13)\). But \( P_1 \) is of type \( r(a,a,n) \) and \( l(d)(P_1) = l(d)(\bar{P}_1) + 1 \). Consequently, it contributes \(+W_1(ml = 13)\) and it cancels out.

\( P_1: \) We have already considered both possibilities for a contribution of \( W_1 \).

\( P_3: \) \([d] = n - \gamma = n \) and \([ml] = n - \beta - \gamma = a \). But there are no crossings at all which cut \( d \) from the left to the right and hence \(-P_3 + \bar{P}_3 \) cancel out.

\( P_4: \) The type \( r(a,a,n) \) was already considered. Let \([d] = \alpha = a \) and \([hm] = \alpha + \beta + \gamma = n \). Only the crossing 34 cuts in \( P_4 \) the crossing \( h_{m} \) from the left to the right, but \([34] = \alpha + \gamma \neq 0 \). It follows that \(+P_4 - \bar{P}_4 \) cancel out because \( l(ml)(P_4) = l(ml)(\bar{P}_4) \).

**Global type III**

\( P_1: \) Let \([d] = \alpha + \beta = a \) and \([12] = \alpha = n \). Not possible.

\( P_2: \) No crossing cuts \( ml \) from the left to the right. Let \([d] = \beta = a \) and \([hm] = \alpha + \beta + \gamma = n \). Then neither 14 or 13 are 0-crossings and hence the contribution of \(-P_2 + \bar{P}_2 \) is 0.

\( P_3: \) Let \([d] = n - \gamma = a \) and \([14] = \alpha + \beta = n \). Not possible.

\( P_3: \) Let \([d] = n - \gamma = a \) and \([34] = \alpha + \beta + \gamma = n \). If \( \beta \neq 0 \) then \([hm] = [23] \neq 0 \) and \(-P_3 + \bar{P}_3 \) contribute \(+W_1(34)\). If \( \beta = 0 \) then \(-P_3 + \bar{P}_3 \) contribute \(+W_1(34) + 1 \). But \( P_1 \) is of type \( r(a,n,a) \) and \( l(ml)(\bar{P}_1) = l(ml)(P_1) + 1 \). The crossings 23 and 24 can be 0-crossings only simultaneously because \( \alpha + \beta + \gamma = n \). If \([24] = \beta \neq 0 \) then the contribution is \(-W_1(34)\) and cancels out. If \( \beta = 0 \) then the contribution is \(-W_1(34) - 1 \) and cancels out too.

\( P_4: \) Let \([d] = \alpha + \beta = a \) and \([13] = n - \gamma = n \). Then \(-P_4 + \bar{P}_4 \) contribute \(+W_1(13)\). But \( P_1 \) is of type \( r(a,a,n) \) and \( l(d)(\bar{P}_1) = l(d)(P_1) + 1 \). We have \([23] = n - a - \gamma \neq 0 \) and hence \(-P_1 + \bar{P}_1 \) contribute \(-W_1(13)\) and cancel out.

\( P_3: \) Both types for \( W_1 \) were already considered.

\( P_3: \) Let \([d] = n - \gamma = n \) and \( \alpha = a \). No crossing cuts \( d \) from the left to the right and hence \(-P_3 + \bar{P}_3 \) cancel out.

\( P_3: \) Let \([d] = \alpha + \beta = n \) and \([ml] = \alpha = a \). Then only 23 could be a
0-crossing, but it cuts $d$ for $P_4$ and also for $\bar{P}_4$. It follows that $+P_4 - \bar{P}_4$
cancel out because $l(ml)(P_4) = l(ml)(\bar{P}_4)$.

**Global type IV**

$P_1$: Let $[d] = n - \gamma = a$ and $[12] = n - \beta - \gamma = n$. Not possible.

$P_2$: No crossing cuts $hm$ from the left to the right. If $[d] = \beta = a$ and $[ml] = \alpha + \beta + \gamma = n$, then 12 and 13 can be 0-crossings only simultaneously and $-P_2 + \bar{P}_2$ is 0.

$P_3$: Let $[d] = \alpha = a$ and $[14] = n - \gamma = n$. No crossing cuts 14 in $P_3$. Hence $P_3 - \bar{P}_3$ contribute $+W_1(14)$. But $P_1$ is of type $l(n, n - a, a)$ and $l(ml)(P_1) = l(ml)(\bar{P}_1) + 1$. If $\beta \neq 0$ then $P_1 - \bar{P}_1$ contribute $-W_1(14)$ and it cancels out.

If $\beta = 0$ then it contributes $-(l(13)(\bar{P}_1) + 1)W_1(14) + l(13)(P_1)(W_1(14) + 1)$. But $P_3$ is of type $r(a, a, n)$ with $d = 13$ and $ml = 12$. The crossing 24 has $[24] = \beta = 0$, but it cuts the crossing $ml$ only in $\bar{P}_3$. Consequently, $+P_3 - \bar{P}_3$ contribute $-l(13)(P_3)(W_1(12) + 1) + l(13)(\bar{P}_3)W_1(12)$. We have to compare now $l(ml = 13)(P_1)$ with $l(d = 13)(P_3) = l(d = 13)(\bar{P}_3)$. For a positive triple crossing $l(ml)$ is just the generic arrows which cut $ml$ plus 1, compare Fig. 9 and Fig. 13 and $l(d)$ is just the generic arrows which cut $d$ (because if we take it on the positive side of the stratum then $ml$ and $hm$ do not cut $d$).

**Inspecting Fig. 37 we see that** $l(ml = 13)(P_1) = l(d = 13)(P_3)$ **if and only if we take $l(d)$ on the positive side of the R III move! In this case all the contributions together cancel out now.**

$P_3$: Let $[d] = \alpha = a$ and $[34] = n - \alpha - \gamma = n$. Not possible.

$P_4$: Let $[d] = n - \gamma = a$ and $[13] = \alpha = n$. Not possible.

$P_1$: The type $l(n, n - a, a)$ was already considered.

$P_4$: The type $r(a, a, n)$ was already considered. For the type $r(a, n, a)$ no crossing cuts $hm$ from the left to the right and it cancels out.

$P_1$: Let $[d] = n - \gamma = n$ and $[ml] = n - \beta - \gamma = a$. The crossing 34 cuts $d$ from the left to the right in $\bar{P}_4$. But $[34] = n - \alpha - \gamma \neq 0$ and $l(ml)(P_4) = l(ml)(\bar{P}_4)$. Hence the contributions cancel out together.

**Global type V**

$P_1$: Let $[d] = \alpha + \beta = a$ and $[12] = n - \gamma = n$. No arrow cuts 12 from the left to the right. Hence $+P_1 - \bar{P}_1$ contribute $-W_1(12)$. But $P_4$ is of type $r(a, a, n)$. No arrow cuts $ml$ from the left to the right and $l(d)(\bar{P}_4) = l(d)(P_4) + 1$. Hence they cancel out.

$P_2$: Let $[d] = \alpha + \beta + \gamma = n$ and $[ml] = \alpha + \gamma = a$. In $P_2$ the crossing 14 cuts $d$ and in $\bar{P}_2$ it is now the crossing 12 which cuts $d$. $[14] = \alpha + \beta$ and
[12] = n − γ, but \( \alpha + \beta + \gamma = n \) and hence 14 and 12 can be 0-crossings only simultaneously and they cancel out together.

\( P_3: \) Let \([d] = \alpha = a \) and \([14] = \alpha + \beta = n \) and consequently \( \gamma = 0 \). Only the crossing \( ml = 12 \) cuts 14 from the left to the right for \( P_3 \). But \([12] = n − \gamma \neq 0 \). It follows that \( +P_3 - \bar{P}_3 \) contribute \( +W_1(14) \). But \( P_1 \) is of type \( l(n, n - a, a) \). Again, only the crossing 12 cuts \( d = 14 \). We have \( l(ml)(P_1) = l(ml)(P_1) + 1 \) and the contributions cancel out.

\( P_3: \) Let \([d] = \alpha = a \) and \([34] = \beta = n \). Not possible.

\( P_4: \) Let \([d] = \beta + \gamma = n \) and \([ml] = n − \gamma = n \). No arrow cuts \( ml \) from the left to the right and it cancels out.

\( P_3: \) Let \([d] = \alpha = a \) and \([ml] = n − \gamma = a \). If \([34] = \beta \neq 0 \) then no 0-crossing cuts \( hm \) and it cancels out. If \( \beta = 0 \) then it contributes \(-l(ml = 12)(W_1(hm = 23) + 1) + l(12)W_1(23) \). But \( P_4 \) is of type \( r(a, n, a) \) and the crossing 34 cuts \( hm = 24 \) from the left to the right in \( \bar{P}_4 \). We have \( l(ml = 12)(P_4) = l(ml = 12)(P_4) \) and hence it contributes \(-l(12)W_1(24) + l(12)(W_4(24) + 1) \) and it cancels out.

\( P_4: \) Both types \( r(a, a, n) \) and \( r(a, n, a) \) were already considered.

Global type VI

\( P_1: \) Let \([d] = \alpha + \beta + \gamma = a \) and \([12] = \alpha = n \). Not possible.

\( P_2: \) Let \([d] = \beta + \gamma = n \) and \([ml] = \beta = a \). No arrow cuts \( d \) from the left to the right and it cancels out.

\( P_3: \) Let \([d] = \alpha + \beta = a \) and \([14] = \alpha + \beta + \gamma = n \). No arrow cuts \( 14 \) from the left to the right for \( P_3 \). It follows that \(-P_3 + \bar{P}_3 \) contribute \(-W_1(14) \). But \( P_1 \) is of type \( l(n, n - a, a) \). Only 24 cuts \( d = 14 \) from the left to the right for \( \bar{P}_3 \), but \([24] = \beta + \gamma \neq 0 \). We have \( l(ml)(\bar{P}_1) = l(ml)(P_1) + 1 \) and the contributions cancel out.

\( P_3: \) Let \([d] = \alpha + \beta = a \) and \([34] = \gamma = n \). Not possible.

\( P_4: \) Let \([d] = \alpha + \beta + \gamma = a \) and \([13] = \alpha + \beta = n \). Not possible.

\( P_1: \) The type \( l(n, n - a, a) \) was already considered.

\( P_4: \) Let \([d] = \alpha + \beta = n \) and \([ml = 12] = \alpha = a \). Then necessarily \( \gamma = 0 \). It follows that \([34] = \gamma = 0 \) and it cuts \( d = 13 \) in \( P_3 \) from the left to the right. Hence the contribution is \( l(ml = 12)(W_1(13) + 1) - l(12)W(13) \). But \( P_4 \) is of type \( l(n, n - a, a) \) too. The crossing 34 cuts now \( d = 14 \) from the left to the right in \( \bar{P}_4 \). It follows that the contribution is \( l(ml = 12)W_1(14) - l(12)(W_4(14) + 1) \) and they cancel out, because evidently \( l(12) \) does not
depend on the stratum.

$P_4$: The type $l(n, n - a, a)$ was already considered.

We have proven that our 1-cochain $R_2^{(2)}$ satisfies (2): the positive global tetrahedron equations. This was the hardest part!

We have now to study the edges of $\Gamma$, compare Subsection 5.1.

First we observe, that $l(ml)$ is the same for the two vertices of an edge. Indeed, either the crossing $ml$ is the same for the two triple crossings and just one branch of the knot has moved over it. Or the two crossings $ml$ are the crossings of the self-tangency (corresponding to a Reidemeister II move) and one branch of the knot has moved over both. In any case the linking number will not change.

Next we observe, that $W_1(d)$, $W_1(hm)$ and $W_1(ml)$ do not change neither, because they do not use the sign of $d$, $hm$ and $ml$ and there are no 0-crossings in the triangle.

We will study for $R_2^{(2)}$ the cube equations for the global type $r$ of triple crossings. For the convenience of the reader we give here again the corresponding figures from [6]. For the numbers of the local types of triple crossings compare Fig. 17. The positive triple crossing correspond to the type 1. As already mentioned, the global type of two triple crossings of an edge is always the same.

We represent the meridian of $\Sigma^{(2)}_{trans-self}$ in the following way: we create first the self-tangency and then we move the transverse branch from the left to the right and we eliminate again the self-tangency. But we need here only the triple crossings.

The linking number $l(d)$ can only change if the crossings $d$ are the crossings of the self-tangency. This happens exactly for the edges which correspond to Fig. 56, Fig. 58, Fig. 63 and Fig. 65. Inspecting the figures we see that in all cases $l(d)$ on the positive side is greater by 1 as $l(d)$ of the other triple crossing on the positive side, exactly if the sign $w(d) = -1$ (remember that for the local types 2 and 6 on the positive side all three chords pairwise intersect, compare Fig. 13). Consequently, we have to add to $l(d)$ in the formula the correction term $1/2(w(d) - 1)$ and our 1-cochain vanishes on the meridian which correspond to this edge.

It remains to consider how $W_2(p)$ changes for the two triple crossings of an edge. They can evidently only change if the crossings of the self-tangency are n-crossings (there is never a 0-crossing together with a n-crossing in the
Figure 54: $r1 - 7$
Figure 55: \( r: 1−5 \)
Figure 56: \( r: 1−6 \)
$r: 3-6$

Figure 57: $r3 - 6$
$r: 8-2$
Fig. 59: $r: 4 - 6$
Figure 60: $r3 - 8$
Figure 61: $r_{4-8}$
\( r: 5 - 2 \)

Figure 62: \( r5 - 2 \)
$r: 5 - 3$

Figure 63: $r5 - 3$
Figure 64: $r: 7 - 2$
Figure 65: \( r7 - 4 \)
triangle because \( 0 < a < n \). The weight \( W_2(p) \) is only defined if \([d] = a\). Hence we have only to consider the n-crossings \( hm \) and \( ml \).

The crossing \( hm \) is a n-crossing of the self-tangency exactly in Fig. 54, Fig. 59, Fig. 60 and Fig. 62. As already explained, the crossing \( hm \) does not contribute to \( W_2(p) \). But the foot of the two crossings from the self-tangency is always in \( K_-(d) \). Hence the crossing which is not \( hm \) will contribute with its sign. For the local type 1 this adds \( -W_1(hm) \) to \( W_2(p) \) and for the local type 7 this adds \( +W_1(hm) \). Consequently, if we add to \( W_2(p) \) in the formula the correction term \( (w(hm) - 1)W_1(hm) \) then our 1-cochain vanishes on the meridian of this edge. We have used here only the signs of \( hm \), and therefore exactly the same correction term works for the remaining three edges. Evidently, if the two crossings \( hm \) for an edge have the same sign, then the correction terms cancel out together.

The crossing \( ml \) is a n-crossing of the self-tangency exactly in Fig. 55, Fig. 57, Fig. 61 and Fig. 64. The crossing \( ml \) does never contribute to \( W_2(p) \), but the other crossing from the self-tangency will contribute with its sign if and only if its foot is in \( K_-(d) \). Inspecting the four figures we see that surprisingly this happens exactly if the crossing is negative! The crossing \( ml \) is then of course positive. Consequently, if we add to \( W_2(p) \) in the formula the correction term \( \frac{1}{2}(w(ml) - 1)W_1(ml) \) then our 1-cochain vanishes on the meridians of these edges too.

The case of \( W_2(p) \) for the global case \( l \) of triple crossings is completely analogue. The reader can check this easily by using the corresponding figures in [6].

We have proven that our 1-cochain \( R_a(2) \) satisfies also (3): the cube equations. Hence it is a 1-cocycle.

It remains to prove that already \( R_a(2)(scan(nK)) \) is an invariant of \( nK \) up to regular isotopy.

Proof.

More generally, let \( T \) be a diagram of an oriented n-component string link and let \( s \) be a regular isotopy (i.e. without R I moves) which connects \( T \) with a diagram \( T' \). We consider the loop \( -s \circ scan(T') \circ s \circ scan(T) \) in \( M_n^{reg} \), compare Fig. 22 for \( scan \). This loop is contractible in \( M_n^{reg} \) because \( s \) and \( scan \) commute, i.e. we can perform them simultaneously. Consequently, \( R_a(2) \) vanishes on this loop, because \( R_a(2) \) is a 1-cocycle. It suffices to prove now that each contribution of a Reidemeister move \( t \) in \( s \) cancels out with the contribution of the same move \( t \) in \( -s \) (the signs of the contributions are
of course opposite). The difference for the two Reidemeister moves is in a branch which has moved under $t$. It suffices to study the weights and the linking numbers in the R III moves. Evidently, the linking numbers have not changed because the branch has moved under the rest of the diagram. If $t$ is a positive triple crossing now, then the weights are the same just before the branch moves under $t$ and just after it has moved under $t$. Indeed, this follows from the fact that for the positive global tetrahedron equation the contribution from the stratum $-P_2$ cancels always out with that from the stratum $P_2$. If we move the branch further away then the invariance follows from the already proven fact that the values of the 1-cocycles do not change if the loop passes through a stratum of $\Sigma^{(1)} \cap \Sigma^{(1)}$, i.e. two simultaneous Reidemeister moves. We use now again the graph $\Gamma$. The meridian $m$ which corresponds to an arbitrary edge of $\Gamma$ is a contractible loop in $\mathcal{M}_n^{reg}$, no matter what is the position of the branch which moves under everything. Let’s take an edge where one vertex is a triple crossing of local type 1, i.e. a positive triple crossing. Reidemeister II moves do not contribute to $R_a^{(2)}$. Consequently the contribution of the other vertex of the edge doesn’t change neither because the contributions from the two R III moves together sum up to 0. Using the fact that the graph $\Gamma$ is connected we obtain the invariance with respect to the position of the moving branch for all Reidemeister moves $t$ of type III. □

Notice that $R_a^{(2)}$ does not have the scan-property for a branch which moves over everything else because the contributions of the strata $+P_3$ and $-\bar{P}_3$ in the positive tetrahedron equation do not cancel out together at all. But of course one of the ”dual” 1-cocycles will have this property.

The given example shows that $[R_a^{(2)}]$ is not always trivial.
This finishes the proof of Theorem 3.

The proof of Theorem 4 goes along the same lines, but surprisingly it is even a bit easier.

Let us consider e.g. the global type $r(a,a,n)$. The crossing $ml$ can contribute with non-trivial $W_1(ml)$ to $R_a^{(2)}$ but not with non-trivial $W_1^+(ml)$ to $R_{a^+}^{(2)}$, compare Fig. [2]. This happens in fact, because the 0-crossing together with the a-crossing $hm$ form a negative loop. But if the crossing $ml$ would contribute in another stratum in the meridian to $W_2^+(p)$ then the 0-crossing and the a-crossing are almost the same and the negative loop would be still
there. The same is true for the type $l(n, n - a, a)$. This shows us, that whenever in the proof for $R^{(2)}_a$ we had such a cancellation then now it cancels even trivially, because both terms are 0.

The global types $r(a, ..,), l(a, ..,)$ and $r(a, n, a)$ are the only ones which contribute both to $R^{(2)}_a$ and $R^{(2)}_{a+}$. Going again through the proof, we see that for $r(a, ..,)$ and $l(a, ..,)$ the 0-crossings in $P_i$ and in $\bar{P}_i$ are in fact the same crossings. There is just one case (which we study in detail) for the global type $V$, where the individual contributions of n-crossings to $W^+_2(p)$ change. We observe also, that for $r(a, ..,)$ all 0-crossings which cut $hm$ from the left to the right have automatically their foot in $K_+(d)$ and hence they contribute to both $R^{(2)}_a$ and $R^{(2)}_{a+}$. It follows that exactly the same proof as for $R^{(2)}_a$ still works to show that the n-crossings, which change from $P_i$ to $\bar{P}_i$ the region of their foot with respect to $d$, still cancel out for $R^{(2)}_{a+}$ with $W^+_1(hm)$ for the global type $r(a, n, a)$.

It remains to study the 0-crossings which change the region of their foot from $P_i$ to $\bar{P}_i$. Of course, this are exactly the same crossings as in the case of the n-crossings. If their contribution is non-trivial (rather rare) then it will cancel with the contribution of the crossing $hm$ of the new particular global type $l(a, 0, a)$.

We have to be careful now in the global case I too, because the n-crossings can change for the individual 0-crossings by passing a stratum $r(0, n, 0)$. It follows from the formula that for $W^+_1$ it remains only to consider the type $l(a, 0, a)$.

**Global type I**

$P_1$: Let $[d] = \alpha = a$ and $[12] = n - \beta = 0$. Not possible.

$P_2$: n-crossings cut 0-crossings from the right to the left. Only $hm$ and $d$ can contribute with $W^+_1$ in the formula. There are no crossings at all which cut $hm$ or $d$ from the right to the left.

$P_3$: Let $[d] = n - \beta - \gamma = a$ and $[34] = \alpha + \beta + \gamma = 0$. Not possible.

$P_3$: Let $[d] = n - \beta - \gamma = a$ and $[14] = \alpha = 0$. The foot of 14 is in $K_+(d)$ for $-\bar{P}_3$ and no crossing cuts 14. Hence $+P_3 - \bar{P}_3$ contribute $-W^+_1(14)$. But $P_1$ is then of type $r(0, n - a, a)$, which does not contribute because $W^+_1(d = 14) = 0$. It follows that the contribution of $+P_3 - \bar{P}_3$ is also 0 (this is in fact analog to our considerations for the n-crossings in $W^+_2(p)$).

$P_3$: Let $[d] = \alpha = a$ and $[13] = n - \beta - \gamma = 0$. Not possible.

There are no special types $l(a, 0, a)$ at all in the meridian for the global
type I.

Global type II

$P_1$: Let $[d] = \alpha = a$ and $[12] = n - \beta - \gamma = 0$. Not possible.

$P_2$: No crossings cut $hm$ from the right to the left.

$P_3$: Let $[d] = n - \gamma = a$ and $[14] = \alpha = 0$. But then $P_1$ would be of type $r(0, n - a, a)$ and $W_1^+(14) = 0$.

$P_3$: Let $[d] = n - \gamma = a$ and $[34] = \alpha + \gamma = 0$. Not possible.

$P_4$: Let $[d] = \alpha = a$ and $[13] = n - \gamma = 0$. Not possible.

$P_1$: There is no special global type $r$ with a 0.

$P_3$: No crossing cuts $hm$ from the right to the left.

$P_4$: There is no special global type $r$ with a 0.

Global type III

$P_1$: Let $[d] = \alpha + \beta = a$ and $[12] = \alpha = 0$. No crossing cuts 12 from the right to the left. Hence $-P_1 + P_1$ contribute $-W_1^+(14)$. But then $P_4$ is of type $l(a, a, 0)$, which shows that $W_1^+(14) = 0$.

$P_2$: Was already considered for the $n$-crossings.

$P_3$: Let $[d] = n - \gamma = a$ and $[14] = \alpha + \beta = 0$. It follows that $-P_3 + \bar{P}_3$ contribute $+W_1^+(14)$. But $P_1$ is of type $r(0, n - a, a)$ and hence $W_1^+(14) = 0$.

$P_3$: Let $[d] = n - \gamma = a$ and $[34] = \alpha + \beta + \gamma = 0$. Not possible.

$P_4$: Let $[d] = \alpha + \beta = a$ and $[13] = n - \gamma = 0$. Not possible.

$P_1$: Is of type $r$.

$P_3$: If $\gamma = n - a$ and $\alpha = a$ then $[34] = n$ and $P_3$ is of type $l(a, 0, a)$. The crossing 34 cuts $hm$ from the right to the left for $+\bar{P}_3$. It follows that it contributes $-l(12)$. But $P_4$ is also of type $l(a, 0, a)$. The crossing 34 cuts now $hm$ for $-P_4$ and it contributes $+l(12)$, and they cancel out together.

$P_4$: Was already considered.

Global type IV

$P_1$: Let $[d] = n - \gamma = a$ and $[12] = n - \beta - \gamma = 0$. But then $P_4$ is of type $l(a, a, 0)$ and hence $W_1^+(12) = 0$.

$P_2$: Is of type $r$.

$P_3$: Let $[d] = \alpha = a$ and $[14] = n - \gamma = 0$. Not possible.

$P_3$: Let $[d] = \alpha = a$ and $[34] = n - \alpha - \gamma = 0$. Consequently, $\alpha = a$, $\beta = 0$ and $\gamma = n - a$. It follows that $+P_3 - \bar{P}_3$ contribute $+W_1^+(34)$. But $P_1$ is of type $l(a, 0, a)$. Only the crossing 12 cuts $hm = 34$ from the right to
the left, but \([12] \neq n\). We have \(l(ml)(P_1) = l(ml)(\bar{P}_1) + 1\) and consequently 
\(P_1 - \bar{P}_1\) contribute \(-W_{1}^{+}(34)\) and they cancel out together.

\(P_4\): Let \([d] = n - \gamma = a\) and \([13] = \alpha = 0\). Then \(P_1\) is of type \(l(a,a,0)\) and hence \(W_{1}^{+}(13) = 0\).

\(P_1\): \(l(a,0,a)\) was already considered.

\(P_3\) is of type \(r\).

\(P_4\): Let \(\gamma = n - a\) and \(\beta = 0\). But there is no crossing which cuts \(hm\) from the right to the left and which has its foot in \(K_-(d)\). We have \(l(ml)(P_3) = l(ml)(\bar{P}_4)\) and they cancel out.

**Global type V**

\(P_1\): Let \([d] = \alpha + \beta = a\) and \([12] = n - \gamma = 0\). Not possible.

\(P_2\): If \(\alpha + \beta + \gamma = a\) and \(\alpha + \gamma = a\) then \(\beta = 0\). Consequently, \([13] = [14] = \alpha \neq n\) and the contribution of \(P_2 - \bar{P}_2\) cancels out.

\(P_3\): Let \([d] = \alpha = a\) and \([14] = \alpha + \beta = 0\). Not possible.

\(P_3\): Let \([d] = \alpha = a\) and \([34] = \beta = 0\). If \(\gamma \neq n - a\) then \([23] \neq n\) and it contributes \(+W_{1}^{+}(34)\). If \(\gamma = n - a\) then \([hm = 23] = n\) and it still contributes \(+W_{1}^{+}(34)\), because the crossings in the triangle do not contribute to \(W_{2}^{+}(p)\). But \(P_1\) is of type \(l(a,0,a)\). We have \([23] = [24] = \alpha + \gamma\) and hence they are \(n\)-crossings if and only if \(\gamma = n - a\). \(l(ml)(P_1) = l(ml)(\bar{P}_1) + 1\) and hence if \(\gamma \neq n - a\) then it contributes \(-W_{1}^{+}(34)\). If \(\gamma = n - a\) then it contributes \(-W_{2}^{+}(p)\) and \(P_4\) is of type \(r(a,n,a)\) and the n-crossing 23 contributes \(+W_{1}^{+}(23) + 1\) for \(+P_4\) and \(+W_{1}^{+}(23)\) for \(-P_4\). Again the n-crossing 24 is \(hm\) in \(P_1\) and therefore does not contribute to \(W_{2}^{+}(p)\). Consequently the contributions \(-W_{1}^{+}(34) - 1\) from \(P_1\), \(+W_{1}^{+}(34)\) from \(P_3\) and \(+1\) from \(P_4\) cancel out together. We see here, that we have indeed to exclude that the crossing \(hm\) contributes to the weight of a crossing outside of the triangle, compare Definition 10.

\(P_4\): Let \([d] = \alpha + \beta = a\) and \([13] = \alpha = 0\). Then it contributes \(+W_{1}^{+}(13)\). But \(P_1\) is of the type \(l(a,a,0)\) and hence \(W_{1}^{+}(13) = 0\).

\(P_3\) and \(P_4\) are of type \(r\).

**Global type VI**

\(P_1\): Let \([d] = \alpha + \beta + \gamma = a\) and \([12] = \alpha = 0\). It contributes \(+W_{1}^{+}(12)\). But then \(P_4\) is of type \(l(a,a,0)\) and hence \(W_{1}^{+}(12) = 0\).

\(P_2\): Let \(\beta + \gamma = a\) and \(\gamma = 0\). Then 13 and 14 are only simultaneously \(n\)-crossings for \(hm\) and it cancels out.
Let \( d = \alpha + \beta = a \) and \( [14] = \alpha + \beta + \gamma = 0 \). Not possible.

Let \( d = \alpha + \beta = a \) and \( [34] = \gamma = 0 \). Then it contributes \(-W^+_1(34)\).

But \( P_1 \) is of type \( l(a, 0, a) \) and no crossings cut \( hm = 34 \) from the right to the left with its foot in \( K_-(d) \). We have \( l(ml)(P_1) = l(ml)(P_1) + 1 \) and it contributes \(+W^+_1(34)\) and cancels out.

Let \( d = \alpha + \beta + \gamma = a \) and \( [13] = \alpha + \beta = 0 \). Then it contributes \(-W^+_1(13)\).

But \( P_1 \) is of type \( l(a, a, 0) \) and hence \( W^+_1(13) = 0 \).

The type \( l(a, 0, a) \) was already considered.

Let \( \alpha + \beta = a \) and \( \beta = 0 \). No crossing at all cuts \( hm \) from the right to the left and it cancels out.

Let \( \alpha = a \) and \( \beta + \gamma = 0 \). The crossing 13 cuts \( hm \) from the right to the left with its foot in \( K_-(d) \). But \([13] = \alpha + \beta \neq n\). We have \( l(ml)(P_4) = l(ml)(P_4) \) and they cancel out.

We have proven that our 1-cochain \( R^{(2)}_{a+} \) satisfies (2): the positive global tetrahedron equations.

For \( R^{(2)}_{a+} \) only the crossing \( hm \) enters with \( W^+_1 \) and it does not enter in \( W^+_2(p) \). But the foot of the crossing \( hm \) stays for an edge of \( \Gamma \) always in \( K_-(d) \) for the global type \( r \) and in \( K_+(d) \) for the global type \( l \) and the crossings come in couples with different signs from the self-tangency. Consequently, the analogue correction term \((w(hm) - 1)W^+_1(hm)\) gives a solution of (3): the cube equations. Hence \( R^{(2)}_{a+} \) is already a 1-cocycle.

The proof that already \( R^{(2)}_{a+}(\operatorname{scan}(nk)) \) is an invariant of \( nk \) up to regular isotopy is just the same as for \( R^{(2)}_{a} \). The same example shows that \([R^{(2)}_{a+}]\) is not always trivial too.

This finishes the proof of Theorem 4.

Let us consider self-tangencies \( p \) with equal tangent direction and such that the two new crossings have marking \( a \). We define the positive coorientation of the Reidemeister II move \( p \) from no crossings to the two new crossings, which we call both \( d \).

**Definition 13** The weight of \( p \) is defined by

\[
W^+_2(p) = \sum(n, 0), \text{ where the sum is taken only over all those } n\text{-crossings with the foot in } K_-(d) \text{ and all those } 0\text{-crossings with the foot in } K_+(d). \text{ The } 1\text{-cochain } RII_{a+} \text{ is defined by } RII_{a+}(\gamma) = \sum_{p \in \gamma} W^+_2(p), \text{ where the sum is over }
\]
all self-tangencies $p$ with equal tangent direction and homological marking $a$ of $d$.

**Proposition 4** Let $0 < a < n$. Then $RII_{a+}$ is an integer valued 1-cocycle in $M_n^{\text{reg}}$ and $R_{a+}^{(2)} + RII_{a+}$ is an integer valued 1-cocycle in $M_n$.

The proof is left to the reader as an exercise.

We have already shown that $R_{a+}^{(2)}$ is well defined by using approximations in $M_n^{\text{reg}}$ (compare Proposition 3) and we do not know if $R_{a+}^{(2)} + RII_{a+}$ really contains more information than $R_{a+}^{(2)}$.

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