ON THE COHEN-MACaulayNESS AND DEFINING IDEAL OF REES ALGEBRAS OF MODULES

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Abstract. In the first part of this paper, we consider a finite, torsion-free, orientable module $E$ over a Gorenstein local ring. We provide a sufficient condition for the Rees algebra $R(E)$ of $E$ to be Cohen-Macaulay. In the second part, we consider a finite module $E$ of projective dimension one over $k[X_1, \ldots, X_n]$. We describe the defining ideal of $R(E)$, under the assumption that the presentation matrix $\varphi$ of $E$ is almost linear, i.e. the entries of all but one column of $\varphi$ are linear.

1. Introduction

Let $R$ be a Noetherian ring, and $E$ a finitely generated $R$-module with a rank. The Rees algebra $R(E)$ of $E$ is defined as the symmetric algebra $S(E)$ modulo its $R$-torsion submodule. It can be presented in terms of generators and relations as a quotient of a polynomial ring $R[T_1, \ldots, T_n]$ modulo an ideal $J$, called the defining ideal of $R(E)$. In this work, we aim to provide a sufficient condition for $R(E)$ to be Cohen-Macaulay, and to describe the defining ideal of certain modules whose Rees algebra is usually not Cohen-Macaulay.

Rees algebras arise in Algebraic Geometry as homogeneous coordinate rings of blow-ups of schemes along a subscheme or as graphs of rational maps between varieties. In many geometric situations, one is interested in Rees algebras of modules which are not ideals. For instance, the homogeneous coordinate ring of a sequence of successive blow-ups of a scheme along two or more subschemes is the Rees algebra of a direct sum of two or more ideals. Moreover, given a subvariety $X$ of an affine space, the conormal variety of $X$ and the graph of the Gauss map from $X$ are projective spectra of the Rees algebras of modules which are not ideals. Geometrically, the Cohen-Macaulay property of Rees algebras encodes information on the vanishing of their cohomology modules.

When $E = I$ is an $R$-ideal, the Rees algebra $R(I)$ coincides with the subalgebra $R[I] = \oplus_{j \geq 0} I^{j+1}$ of the polynomial ring $R[t]$. When $R$ is Cohen-Macaulay, the first step to study the Cohen-Macaulayness of $R(I)$ is to investigate the Cohen-Macaulayness of the associated graded ring $G(I) = \oplus_{j \geq 0} I^j/I^{j+1}$. In fact, in this case, $G(I)$ is Cohen-Macaulay whenever $R(I)$ is ([10]), and although the converse is not true in general, it holds if furthermore some numerical conditions are satisfied ([11, 13, 21, 26, 19]). Now, by analogy with the case of ideals, one defines the powers of an $R$-module $E$ as $E^j = [R(E)]_j$, where $[R(E)]_j$ denotes the degree-$j$ component of $R(E)$. However, since $E^n+1$ is not contained in $E^n$, there is no module analogue for the associated graded ring. Hence, the study of the Cohen-Macaulayness of Rees algebras of modules requires a completely different approach than in the case of ideals, and little is known about it (see [8, 22, 23, 16]).
In the first part of this work, our main goal is to provide a sufficient condition for the Cohen-Macaulayness of $\mathcal{R}(E)$, under suitable assumptions on the depth of finitely many powers of $E$. More specifically, we aim to generalize the following result, due to Johnson and Ulrich ([12, 3.1 and 3.4]) and Goto, Nakamura and Nishida ([6, 1.1 and 6.3]). We refer the reader to Section 2 for the definitions of the $G_s$ condition and of the Artin-Nagata property $AN_s$.

**Theorem 1.1** ([6, 12]). Let $R$ be a local Cohen-Macaulay ring of dimension $d$ with infinite residue field. Let $I$ be an $R$-ideal with analytic spread $\ell$, height $g$ and reduction number $r$, and let $k$ be an integer. Assume that $g \geq 1$, $1 \leq k \leq \ell - 1$, $r \leq k$, and $I$ satisfies $G_1$ and $AN_{\ell-k-1}$. If depth $I^j \geq d - \ell + k - j + 1$ for $1 \leq j \leq k$, then $\mathcal{R}(I)$ is Cohen-Macaulay.

We will approach the problem using the generic Bourbaki ideal $I$ of $E$. Generic Bourbaki ideals (see Section 2) were introduced by Simis, Ulrich and Vasconcelos in [23], to reduce the study of the Cohen-Macaulayness of Rees algebras of modules to the case of ideals. In fact, $\mathcal{R}(E)$ is Cohen-Macaulay if and only if the Rees algebra of its generic Bourbaki ideal $I$ is. Using this technique, the problem is then reduced to imposing suitable assumptions on $E$ so that $I$ satisfies Theorem 1.1. This is not straightforward, most notably because there is no module analogue for the Artin-Nagata property. In fact, it is usually difficult to identify which property of $E$ will guarantee that $I$ satisfies the necessary Artin-Nagata condition. Our main result, Theorem 3.6, generalizes previous work of Lin (see [16, 3.4]).

In the second part of this paper, we will focus on the problem of determining the defining ideal $J$ of $\mathcal{R}(E)$. This is usually a difficult task, and the problem in its full generality is wide open. However, given a presentation $R^s \xrightarrow{\varphi} R^n \to E$ of $E$, one can attempt to deduce information about $J$ by exploiting the connection between $\mathcal{R}(E)$ and the symmetric algebra $S(E)$. In fact, $S(E)$ is isomorphic to a quotient of $R[T_1, \ldots, T_n]$ modulo the ideal generated by the entries of the row vector $[T_1, \ldots, T_n] \cdot \varphi$. This strategy is usually successful when $\varphi$ has a particularly rich structure (see, for instance, [22, 24, 20, 15]). Moreover, in certain situations the Cohen-Macaulayness of the Rees algebra is helpful in order to describe its defining ideal, often providing bounds on the degrees of its generators or even an explicit generating set (see [19, 23, 17, 23]). Less is known on the defining ideal of Rees algebras which are not Cohen-Macaulay ([14, 15, 3]).

In [23], the defining ideal of $\mathcal{R}(E)$ was determined for linearly presented modules of projective dimension one. In this case, the generic Bourbaki ideal $I$ of $E$ is a linearly presented perfect ideal of height two, hence the defining ideal of $\mathcal{R}(I)$ is well-understood ([17]). Moreover, $\mathcal{R}(I)$ is Cohen-Macaulay, and this makes it possible to ‘lift’ the shape of the defining ideal of $\mathcal{R}(I)$ back to $\mathcal{R}(E)$. Similarly, in Theorem 4.3, we describe the defining ideal in the case when the presentation matrix of $E$ is only almost linear, namely, has linear entries, except possibly for those in the last column, which are homogeneous of degree $m \geq 1$. In this case, $I$ is an almost linearly presented perfect ideal of height two, and the defining ideal of $\mathcal{R}(I)$ was described by Boswell and Mukundan in [3]. However, now $\mathcal{R}(I)$ is only almost Cohen-Macaulay, and so the argument used in the linearly presented case requires a substantial modification.

This paper is organized as follows. In Section 2, we review the main properties of Rees algebras of modules and of generic Bourbaki ideals. We also recall basic definitions and results on residual intersections of ideals that are the foundation of
Theorem 1.1 and will be needed later. Section 3 discusses the Cohen-Macaulayness of \( \mathcal{R}(E) \) when \( E \) is a torsion-free, orientable \( R \)-module (Theorem 3.6). As a byproduct, we also provide a sufficient condition for a module to be of linear type (Theorem 3.5). Finally, in Section 4 we describe the defining ideal of the Rees algebra of almost linearly presented modules of projective dimension one.

2. Preliminaries

In this section we review properties of Rees algebras of modules and generic Bourbaki ideals. We also recall some facts about residual intersections of ideals, as well as the notions of Jacobian dual and iterated Jacobian duals, which will be needed in Sections 3 and 4 respectively.

Let \( R \) be a Noetherian ring. Recall that a finite \( R \)-module \( E \) has a rank, \( \text{rank } E = e \), if \( E \otimes_R \text{Quot}(R) \cong (\text{Quot}(R))^e \), or, equivalently, if \( E_p \cong R_p^e \) for all \( p \in \text{Ass}(R) \). Moreover, \( E \) is a torsion-free module of rank 1 if and only if it is isomorphic to an \( R \)-ideal \( I \) of positive grade. Very often the modules we will consider in this paper will be orientable. This means that \( E \) has a rank \( e > 0 \) and \( (\bigwedge^e E)^* \cong R \), where \((-)^* \) denotes the functor \( \text{Hom}_R(-, R) \).

Given any presentation \( R^e \xrightarrow{\varphi} R^n \to E \) of \( E \), the symmetric algebra of \( E \) is \( S(E) = R[T_1, \ldots, T_n]/(l_1, \ldots, l_s) \), where \( l_1, \ldots, l_s \) are linear forms in \( R[T_1, \ldots, T_n] \) satisfying \( [T_1, \ldots, T_n] \cdot \varphi = [l_1, \ldots, l_s] \). If \( E \) has a rank, the Rees algebra \( \mathcal{R}(E) \) of \( E \) is defined as the quotient of \( S(E) \) modulo its \( R \)-torsion submodule. In particular, \( \mathcal{R}(E) = R[T_1, \ldots, T_n]/J \) for some ideal \( J \), called the defining ideal of \( \mathcal{R}(E) \). The module \( \mathcal{R}(E) \) is said to be of linear type if \( \mathcal{R}(E) \) is naturally isomorphic to \( S(E) \), since in this case \( J = (l_1, \ldots, l_s) \) is an ideal of linear forms. When \( E = I \) is an \( R \)-ideal, \( \mathcal{R}(I) \) coincides with the subalgebra \( \mathcal{R}[t] = \bigoplus_{j \geq 0} t^j \mathcal{R}^{[j]} \) of the polynomial ring \( R[t] \). By analogy with the case of ideals, the powers of an \( R \)-module \( E \) are then defined as \( E^j = [\mathcal{R}(E)]_j \). We refer the reader to [4] for a more general definition of Rees algebra, which also applies to modules not having a rank.

We recall that a module \( E \) of rank \( e \) satisfies \( G_s \) if \( \mu(E_p) \leq \dim R_p - e + 1 \) for every \( p \in \text{Spec}(R) \) with \( 1 \leq \dim R_p \leq s-1 \), or, equivalently, if \( \text{ht Fitt}_i(E) \geq i - e + 2 \) for \( e \leq i \leq e + s - 2 \), where \( \text{Fitt}_i(E) \) is the \( i \)-th Fitting ideal of \( E \). If the same condition holds for all \( s \), then \( E \) is said to satisfy \( G_s \). Notice that for an ideal \( I \), the \( G_s \) condition can be restated as \( \mu(I_p) \leq \dim R_p \) for every \( p \in V(I) \) with \( \dim R_p \leq s - 1 \). Moreover, if \( R \) has dimension \( d \), then \( I \) is \( G_d \) if and only if it is \( G_{d+1} \).

Notation 2.1. ([23, 3.3]). Let \( R \) be a Noetherian ring, \( E = Ra_1 + \cdots + Ra_n \) a finite \( R \)-module with rank \( E = e > 0 \). Let \( Z = \{ Z_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq e - 1 \} \) be a set of indeterminates, and denote \( R' = R[Z], \ E' = E \otimes_R R', \ x_j = \sum_{i=1}^{n} Z_{ij} a_i \in E' \), and \( F' = \sum_{j=1}^{e-1} R' x_j \). If \( R \) is local with maximal ideal \( \mathfrak{m} \), let \( R'' = R[Z] = R[Z]_{\mathfrak{m} R[Z]} \) and similarly denote \( E'' = E \otimes_R R'', \ F'' = F' \otimes_R R'' \).

Theorem 2.2. ([23, 3.1, 3.2 and 3.4]). Let \( R \) be a Noetherian ring, \( E \) a finite \( R \)-module with rank \( E = e > 0 \). Assume that \( E \) is torsion-free and that \( E_p \) is free for all \( p \in \text{Spec}(R) \) with depth \( R_p \leq 1 \).
Theorem and Definition 2.3. ([23, 3.2 and 3.4]). Let $R$ be a Noetherian local ring, and $E$ a finite $R$-module with rank $E = e > 0$. With the assumptions of Theorem 2.2, $E''/F''$ is isomorphic to an $R''$-ideal $I$, called a \textit{generic Bourbaki ideal} of $E$. Moreover, if $K$ is another ideal constructed this way using variables $Y$, then the images of $I$ and $K$ in $S = R(Z,Y)$ coincide up to multiplication by a unit in $\text{Quot}(S)$, and are equal whenever $I$ and $K$ have grade at least 2.

Theorem 2.4. ([23, 3.5 and 3.8]). In the setting of Notation 2.1 and with the assumptions of Theorems 2.2 and 2.3 then:

(a) $\mathcal{R}(E)$ is Cohen-Macaulay if and only if $\mathcal{R}(I)$ is Cohen-Macaulay.
(b) $E$ is of linear type with grade $\mathcal{R}(E)_+ \geq e$ if and only if $I$ is of linear type.
(c) If grade $\mathcal{R}(E)_+ \geq e$, then $\mathcal{R}(J) \cong \mathcal{R}(E')/(F')$. Moreover, if in addition $E$ is of linear type, then so is $J$.

The following result will be crucial in Section 3 in order to transfer assumptions from $E'$ to $I'$ or $I'$.  

Theorem 2.5. ([23, 3.11]). Let $R$ be a Noetherian ring, $E$ a finite $R$-module with rank $E = e > 0$. Let $0 \rightarrow F \rightarrow E \rightarrow I \rightarrow 0$ be an exact sequence where $F$ is a free $R$-module with free basis $x_1, \ldots, x_{e-1}$ and $I$ is an $R$-ideal. The following are equivalent.

(a) $\mathcal{R}(E)/(F)$ is $R$-torsion free.
(b) $\mathcal{R}(E)/(F) \cong \mathcal{R}(I)$.
(c) $\mathcal{R}(E)/(F) \cong \mathcal{R}(I)$ and $x_1, \ldots, x_{e-1}$ of $F$ form a regular sequence on $\mathcal{R}(E)$. Moreover, if $I$ is of linear type, then so is $E$ and the equivalent conditions above hold.

Let $R$ be a Noetherian local ring with residue field $k$. The next proposition examines the connection between the analytic spread and reduction number of $E$ and those of its generic Bourbaki ideal $I$. Recall that a \textit{minimal reduction} of $E$ is a minimal submodule $U \subseteq E$ so that $E^{r+1} = UE^r$ for some $r \geq 0$. The least such $r$ is denoted by $r_U(E)$, and the \textit{reduction number} of $E$ is $r(E) = \min \{r_U(E) \mid U \text{ a minimal reduction of } E \}$. The \textit{analytic spread} $\ell(E)$ of $E$ is the Krull dimension of the \textit{special fiber ring} $\mathcal{F}(E) = \mathcal{R}(E) \otimes_R k$. If $k$ is infinite, any minimal reductions of $E$ have the same minimal number of generators, equal to $\ell(E)$. Moreover, any submodule $U$ generated by $\ell(E)$ general elements in $E$ is a minimal reduction of $E$, with $r_U(E) = r(E)$.

Proposition 2.6. ([23, 2.3 and 3.10]). Let $R$ be a Noetherian local ring with $\dim R = d$, $E$ a finite $R$-module with rank $E = e$.

(a) If $d > 0$, then $e \leq \ell(E) \leq d + e - 1$.
(b) If $E$ admits a generic Bourbaki ideal $I$, then $\ell(I) = \ell(E) - e + 1$. Moreover, \[ r(I) \leq r(E) \] if the residue field of $R$ is infinite.
When $R$ is Cohen-Macaulay, minimal reductions of ideals can be used to construct residual intersections (see [20, 3.1] or [12, 2.7]). Let $I$ be an $R$-ideal with $\text{ht}(I) = g$, and $s \geq g$ an integer. Recall that a proper ideal $K$ is an $s$-residual intersection of $I$ if there exists an ideal $J \subseteq I$ such that $K = J : I$, $\mu(J) \leq s$ and $J_p = I_p$ for every prime ideal $p$ with $\dim R_p \leq s - 1$. $K$ is called a geometric $s$-residual intersection of $I$ if in addition $J_p = I_p$ for every $p \in V(I)$ with $\dim R_p = s$.

**Definition 2.7 ([2], [24]).** Let $R$ be a Cohen-Macaulay local ring, $I$ an $R$-ideal with $\text{ht}(I) = g$, and $s \geq g$ an integer. $I$ is said to satisfy the Artin-Nagata property $\text{AN}_s$ if for all $g \leq i \leq s$ and every $i$-residual intersection $K$ of $I$, $R/K$ is Cohen-Macaulay. If the same property holds for every geometric $i$-residual intersection, then $I$ is said to satisfy $\text{AN}_s$.

The Artin-Nagata condition $\text{AN}_s$ is satisfied (at least when $R$ is Gorenstein) by a large class of ideals, including perfect ideals of height 2, perfect Gorenstein ideals of height 3, complete intersections, and in general any licci ideal. Notice also that $\text{AN}_s$ is satisfied whenever $s \leq g - 1$, or in case $s = g$, $R$ is Gorenstein and $R/I$ is Cohen-Macaulay (see [18]). Another sufficient condition is given by the following result.

**Theorem 2.8 ([24]).** Let $R$ be a local Gorenstein ring with $\dim(R) = d$, $I$ an $R$-ideal with $\text{ht}(I) = g$. Assume that $I$ satisfies $G_s$ for some $s \geq g$. If $\text{depth}(I^j) \geq d - g - j + 2$ for $1 \leq j \leq s - g + 1$, then $I$ satisfies $\text{AN}_s$.

We next give a sufficient condition for an ideal of a local Gorenstein ring to be of linear type. This follows from work of Chardin, Eisenbud and Ulrich (see [3]), but we include its proof here for lack of a specific reference.

**Proposition 2.9.** Let $R$ be a local Gorenstein ring, $d = \dim R$, $I$ an $R$-ideal with $\text{ht}(I) = g \geq 1$ and $\ell(I) = \ell$. Assume that $I$ is $G_{\ell + 1}$ and that $\text{Ext}^{g+j-1}(I, R) = 0$ for $1 \leq j \leq \min \{\ell - g, d - g - 1\}$. Then, $I$ is of linear type.

**Proof.** Without loss of generality, we may assume that the residue field of $R$ is infinite. Let $J$ be a minimal reduction of $I$ generated by $\ell$ general elements. Since $I$ satisfies $G_{\ell + 1}$, if $J \subsetneq I$, then $J : I$ is a geometric $\ell$-residual intersection of $I$ (see [24, 1.6]). Therefore, since $I$ and $J$ have the same radical, $\text{ht}(J : I) \geq \ell + 1$.

Now, if $\ell \leq d - 1$, then by assumption $\text{Ext}^{g+j-1}(I, R) = 0$ for $1 \leq j \leq \ell - g$. Thus, $\text{ht}(J : I) = \ell$ by [3] 4.1 and 3.4], a contradiction. Hence, it must be $\ell = d$, so that $I$ is $G_d$ and $I = J$. But then, by assumption $\text{Ext}^{g+j-1}(I, R) = 0$ for $1 \leq j \leq d - g - 1$. Hence, by [3] 4.1 and 3.6(b)] $I = J$ is generated by a $d$-sequence, thus it is of linear type by [9].

**Remark 2.10.** Let $R = k[Y_1, \ldots, Y_d](Y_1, \ldots, Y_d)$, where $k$ is a field. A minimal presentation $R^s \xrightarrow{\varphi} R^n \rightarrow E \rightarrow 0$ of $E$ induces a minimal presentation of a generic Bourbaki ideal $I$ of $E$ as follows. With $Z$ and $x_j$ as in Notation [24], by possibly multiplying $\varphi$ from the left by an invertible matrix with coefficients in $k(Z)$, we may assume that $\varphi$ presents $E^n$ with respect to a minimal generating set of the form $x_1, \ldots, x_{e-1}, a_e, \ldots, a_n$. Then, $\varphi = \begin{bmatrix} A \\ \psi \end{bmatrix}$, where $A$ and $\psi$ are submatrices of size $(e - 1) \times s$ and $(n - e + 1) \times s$ respectively. By construction,
ψ is a presentation of \( I \), and is minimal since \( \mu(I) = \mu(E) - e + 1 = n - e + 1 \). Also, if the entries of \( \varphi \) are homogeneous polynomials of constant degrees \( \delta_1, \ldots, \delta_s \) along each column, then the entries of \( \psi \) are homogeneous polynomials of constant \( Y \)-degrees \( \delta_1, \ldots, \delta_s \) along each column.

In Section 4 we will be interested in transferring information on iterated Jacobian duals of \( \varphi \) to those of \( \psi \) (see Lemma 4.4). Iterated Jacobian duals of a matrix were introduced in [3], in order to describe the defining ideal of the Rees algebra of a certain class of ideals. We recall the construction here.

Let \( R = k[Y_1, \ldots, Y_d] \) be a standard graded polynomial ring over a field \( k \), let \( S = R[T_1, \ldots, T_n] \) be bigraded, and set \( Y = Y_1, \ldots, Y_d \), \( T = T_1, \ldots, T_n \). Let \( R^d \xrightarrow{\varphi} R^n \to E \to 0 \) be a presentation of a finite \( R \)-module \( E \), such that \( I_1(\varphi) \subseteq (Y) \) and the entries of \( \varphi \) are homogeneous of constant \( Y \)-degrees \( \delta_1, \ldots, \delta_s \) along each column. Let \( l_1, \ldots, l_s \) be linear forms in the \( T_i \) variables, satisfying \([l_1, \ldots, l_s] = [T] \cdot \varphi \) (i.e. such that \( S(E) \cong S/(l_1, \ldots, l_s) \)). Since the entries of \( \varphi \) are contained in \( (Y) \), the equation can be rewritten as \([l_1, \ldots, l_s] = [Y] \cdot B(\varphi) \), where \( B(\varphi) \) is a \( d \times s \) matrix whose entries are linear in the \( T_i \) variables, and homogeneous of constant \( Y \)-degrees \( \delta_1 - 1, \ldots, \delta_s - 1 \) along each column. \( B(\varphi) \) is called a Jacobian dual of \( \varphi \) (see [22]). Notice that \( B(\varphi) \) is not necessarily unique.

**Theorem and Definition 2.11** ([3]). With \( R, S \) and \( \varphi \) as above, let \( B_1(\varphi) = B(\varphi) \) for some Jacobian dual \( B(\varphi) \). Assume that matrices \( B_j(\varphi) \) with \( d \) rows have been inductively constructed for \( 1 \leq j \leq i \), such that each \( B_j(\varphi) \) has homogeneous entries of constant \( Y \)-degrees and \( T \)-degrees along each column. There exists a matrix \( C_i \), with entries in \( S \) which are homogeneous of constant \( Y \)-degrees and \( T \)-degrees in each column, such that:

1. \( B_{i+1}(\varphi) = [B_i(\varphi) \mid C_i] \)
2. \([Y \cdot B_i(\varphi)] + (I_d(B_i(\varphi)) \cap (Y)) = (Y \cdot B_i(\varphi)) + (Y \cdot C_i) \), where \((Y \cdot B_i(\varphi)) \)
   denotes the ideal generated by the entries of the row vector \([Y \cdot B_i(\varphi)]\).

\( B_i(\varphi) \) is called the \( i \)-th iterated Jacobian dual of \( \varphi \). Moreover, for all \( i \geq 1 \):

1. The ideal \( (Y \cdot B(\varphi)) + I_d(B_i(\varphi)) \) only depends on \( \varphi \).
2. \((Y \cdot B_i(\varphi)) + I_d(B_i(\varphi)) = (Y \cdot B(\varphi)) + I_d(B_i(\varphi)) \subseteq (Y \cdot B(\varphi)) + I_d(B_{i+1}(\varphi)) \).
   In particular, the procedure stops after finitely many iterations.
3. \((Y \cdot B(\varphi)) + I_d(B_i(\varphi)) \subseteq ((Y \cdot B(\varphi)) : (Y)^r) \).

### 3. Modules with Cohen-Macaulay Rees algebra

Throughout this section, we will assume the following.

**Setting 3.1.** Let \( R \) be a local Cohen-Macaulay ring with \( \dim R = d \). Let \( E = Ra_1 + \ldots + Ra_n \) be a finite \( R \)-module with \( \operatorname{rank}(E) = e > 0 \) and analytic spread \( \ell(E) = \ell \). With \( E', E'', x_1, \ldots, x_{\ell-1}, F' \) and \( F'' \) as in Notation [21], assume that \( E'/F' \) is isomorphic to an \( R' \)-ideal \( J \), let \( I = JR'' \) be a generic Bourbaki ideal of \( E \), and let \( g = \operatorname{ht} I \).

Our main goal in this section is to provide a sufficient condition for \( \mathcal{R}(E) \) to be Cohen-Macaulay under assumptions on the depth of finitely many powers of \( E \), that recover Theorem [11] in the case when \( E \) is an ideal. With this goal in mind and thinking of the construction of generic Bourbaki ideals described in Section 2.
it makes sense to investigate the relationship between powers of $E$ and powers of $I$, and whether it is possible to transfer an assumption on depth $E^j$ to depth $I^j$.

Notice that the exact sequences $0 \to F \to E' \to J \to 0$ and $0 \to F \to E'' \to I \to 0$ induce homogeneous epimorphisms $\mathcal{R}(E') \to \mathcal{R}(J)$ and $\mathcal{R}(E'') \to \mathcal{R}(I)$. Taking degree $j$ components, we then have epimorphisms $(E')^j \to J^j$ and $(E'')^j \to I^j$. Hence, by augmenting the Koszul complexes $[K.(x_1, \ldots, x_{e-1}; \mathcal{R}(E'))]_j$ and $[K.(x_1, \ldots, x_{e-1}; \mathcal{R}(E''))]_j$, respectively, we can define the complexes

$$C'_j: 0 \to \oplus R' \overset{e'_1}{\to} \oplus E' \to \cdots \overset{e'_j}{\to} (E')^{j-1} \overset{e'_j}{\to} J \to 0$$

and

$$C''_j: 0 \to \oplus R'' \overset{e''_1}{\to} \oplus E'' \to \cdots \overset{e''_j}{\to} (E'')^{j-1} \overset{e''_j}{\to} I \to 0.$$
Lemma 3.4. Under the assumptions of Setting 3.1, let $R' = R[Z_1, \ldots, Z_n]$ be a polynomial ring and let $x = \sum_{i=1}^{n} a_i Z_i$. Let $k$ be an integer such that $\text{Ext}^{j+1}(E^j, R) = 0$ for $1 \leq j \leq k$. Then, $\text{Ext}^{j+1}(E^j / x(E^j)^{-1}, R) = 0$ for $1 \leq j \leq k$.

Proof. By [23, 3.6] $x$ is a regular element on $R(E')$, for $1 \leq j \leq k$ we have that $x(E^j)^{-1} \cong (E')^{-1}$. Now, consider the long exact sequence of $\text{Ext}_{R'}(R', R)$ induced by $0 \rightarrow x(E^j)^{-1} \rightarrow (E')^{-1} \rightarrow (E') / x(E^j)^{-1} \rightarrow 0$. Then, our assumptions imply that $\text{Ext}^{j+1}(E^j / x(E^j)^{-1}, R) = 0$ for $1 \leq j \leq k$.

The following result is a module version of Proposition 2.9.

Theorem 3.5. Let $R$ be a local Gorenstein ring of dimension $d$. Let $E$ be a finite, torsion-free and orientable $R$-module with rank $E = e$ and $\ell(E) = \ell$. Assume that $E$ is $G_{\ell-2}$ and that $\text{Ext}^{j+1}(E^j, R) = 0$ for $1 \leq j \leq \min \{\ell - e - 1, d - 3\}$. Then, $E$ is of linear type and $E'/F'$ is isomorphic to an $R'$-ideal of linear type.

Proof. Without loss of generality, we may assume that $E$ is not free and $e > 0$. Since $E$ is torsion-free, orientable and satisfies $G_{\ell-2}$, by Theorem 2.2 $E'/F' \cong J$ and $E'' / F'' \cong I$, where $I$ and $J$ are ideals of height at least 2, satisfying $G_{\ell-2}$, i.e. $G_{\ell-1}$.

If $e = 1$, then $R' = R$ and $E \cong I$, an $R$-ideal of height $g = 2$. In fact, if $g > 2$, by assumption we would have $\text{Ext}^g(R/I^{g-2}, R) \cong \text{Ext}^g(I^{g-2}, R) = 0$, contradicting the fact that $\text{grade}(I^{g-2}) = \text{grade}I = g$. So, $I$ is of linear type by Proposition 2.9.

Now, assume that $e \geq 2$. We induct on $d \geq 2$. If $d = 2$, then $2 = \text{ht} I = \ell(I)$, so $I$ satisfies $G_{\ell}$. Hence, $I$ is a complete intersection ideal, so it is of linear type by [7]. Then, by Theorem 2.4(b) and (c), $E$ and $J$ are of linear type. If $d \geq 2$, we show that $J_q$ is of linear type for all $q \in \text{Spec}(R')$. Then, $J$ and hence $I$ are of linear type, so also $E$ is of linear type by Theorem 2.4(b). For each $q \in \text{Spec}(R')$, let $p = q \cap R$. Using Notation 2.3 over the ring $R_p$, let $F'_p = \sum R'_j x_j$. By Theorem 2.2, $E'_p / F'_p$ is isomorphic to an $R'_p$-ideal $L$ which localizes to $J_q$. Hence, in order for $J_q$ to be of linear type it suffices to show that $L$ is of linear type.

We claim that, for all $p \in \text{Spec}(R)$, $\text{Ext}^{j+1}((E'_p / F'_p)^j, R'_p)$ is isomorphic to $\text{Ext}^{j+1}((E'_p)^j / F'_p(E'_p)^{-1}, R'_p)$ whenever $1 \leq j \leq \min \{\ell - e - 1, d - 3\}$. Then, by iteration of Lemma 3.4 (with $k = \min \{\ell - e - 1, d - 3\}$), we have $\text{Ext}^{j+1}(E^j, R) \cong \text{Ext}^{j+1}((E'_p)^j / F'_p(E'_p)^{-1}, R'_p) = 0$ for $1 \leq j \leq \min \{\ell - e - 1, d - 3\}$. Also, similarly as in the case $e = 1$, $L$ has height exactly 2, and is then of linear type by Proposition 2.9.

To prove our claim, we distinguish two cases. If $\dim R_p \leq d - 1$, since all of our assumptions are preserved under localization, by induction hypothesis $E_p$ and $L$ are of linear type. Hence, by Theorem 2.4 the complexes $(C'_p)_j$ are exact for each $j$. In particular, $L^j \cong (E'_p / F'_p)^j \cong (E'_p)^j / F'_p(E'_p)^{-1}$, so by iteration of Lemma 3.4 (with $k = \min \{\ell - e - 1, d - 3\}$), $\text{Ext}^{j+1}((E'_p)^j / F'_p(E'_p)^{-1}, R'_p) = 0$ whenever $1 \leq j \leq \min \{\ell - e - 1, d - 3\}$. Now, assume that $\dim R_p = d$. For $1 \leq j \leq \min \{\ell - e - 1, d - 3\}$, let $K_j$ be the kernel of the epimorphisms $(E')^j / F'(E')^{-1} \rightarrow J^j$. The discussion above shows that $\text{grade}(K_j) \geq d$. Now, since $d \geq \max \{\ell - e + 1, d - 1\}$, the long exact sequence of $\text{Ext}(-, R')$ induced by the exact sequence $0 \rightarrow K_j \rightarrow (E')^j / F'(E')^{-1} \rightarrow J^j \rightarrow 0$ gives that $\text{Ext}^{j+1}(J^j, R') \cong \text{Ext}^{j+1}((E')^j / F'(E')^{-1}, R')$ for all $1 \leq j \leq \min \{\ell - e - 1, d - 3\}$.
This finally proves the claim. ■

**Theorem 3.6.** Let $R$ be a local Gorenstein ring of dimension $d$ with infinite residue field. Let $E$ be a finite, torsion-free, orientable $R$-module, with rank $E = e > 0$ and $\ell(E) = \ell$. Let $g$ be the height of the generic Bourbaki ideal of $E$, and assume that the following conditions hold.

(a) $E$ satisfies $G_{\ell+1}$.
(b) $r(I) \leq k$ for some integer $1 \leq k \leq \ell - e$.
(c) depth $E^j \geq d - g - j + 2$ for $1 \leq j \leq \ell - e - k - g + 1$ and depth $E^j \geq d - \ell + e + k - j$ for $\ell - e - k - g + 2 \leq j \leq k$.
(d) If $g = 2$, then $\operatorname{Ext}^{j+1}_R(E^j, R_p) = 0$ for $1 \leq j \leq \min\{\ell - e - 1, d - 3\}$, for all $p \in \operatorname{Spec}(R)$ with $\dim R_p = \ell - e$ such that $E_p$ is not free.

Then, $\mathcal{R}(E)$ is Cohen-Macaulay.

**Proof.** By Theorem 2.2 $E$ admits a generic Bourbaki ideal $I$ with $ht I = g \geq 2$ and $r(I) \leq r(E) \leq k$, which satisfies $G_{\ell+1}$, i.e. $G_{\ell(1)}$. We will prove that $\mathcal{R}(I)$ is Cohen-Macaulay by Theorem 2.3. If $e = 1$, then $R'' = R$ and $E \cong I$, an $R$-ideal with depth $I \geq d - g - j + 2$ for $1 \leq j \leq \ell - k - 1 - g + 1$. Hence, $I$ satisfies $AN_{\ell-1}$ by Theorem 2.8. Therefore, $\mathcal{R}(I)$ is Cohen-Macaulay by Theorem 1.1. Now, assume $e \geq 2$. It suffices to prove that depth $I \geq d - g - j + 2$ for $1 \leq j \leq \ell - e - k - g + 1$ and depth $I \geq d - \ell + e + k - j$ for $\ell - e - k - g + 2 \leq j \leq k$. In fact, then, again by Theorem 2.8, $I$ satisfies $AN_{\ell-1}$, i.e. $AN_{\ell(1)-k}$, and hence $\mathcal{R}(I)$ is Cohen-Macaulay by Theorem 1.1.

Notice that if $g \geq 3$, then by Theorem 2.2(c) $I$ is a free direct summand of $E''$, so it satisfies the desired depth conditions, since $E$ does. Hence, we may assume that $g = 2$. By Lemma 3.2 and Lemma 3.3 (with $s = \ell - e$), it suffices to show that the complexes $(C'_j)_q$ are exact for all $q \in \operatorname{Spec}(R')$ with $\dim R'_q \leq \ell - e$ and all $1 \leq j \leq k$.

For each such $q$, let $p = q \cap R$. If $E_p$ is free, then $\mathcal{R}(E'_p)$ is Cohen-Macaulay, so the $(C'_j)_q$ are exact for all $q$ by Theorems 2.4(c) and 2.4(b), whence the $(C'_j)_q$ are exact for all $j$ as well. If $E_p$ is not free, then by assumption $\operatorname{Ext}^{j+1}_p(E^j, R_p) = 0$ for all $p \in \operatorname{Spec}(R)$ with $\dim R_p \leq \ell - e$ and all $1 \leq j \leq \min\{\ell - e - 1, d - 3\}$. Hence, $E'_p/F'_p$ is isomorphic to an $R'_p$-ideal of linear type by Theorem 3.5, so its localization $(E'_p/F'_p)_q$ is isomorphic to an $R'_p$-ideal of linear type. Hence, the $(C'_j)_q$ are exact by Theorem 2.5. ■

Notice that assumption (e) implies that $k < \ell - e - g + 2$, and can be simplified to depth $E^j \geq d - \ell + e + k - j$ for $i \leq j \leq k$ whenever $k > \ell - e - g + 1$. In this case, $\ell - e - k < g$, so the generic Bourbaki ideal $I$ of $E$ satisfies $AN_{\ell-e-k}$ automatically. In particular, if $k \geq \ell - e - g + 1$, in the proof of Theorem 3.6 we only need $R$ to be Gorenstein locally in codimension $\ell - e$. Finally, notice that assumption (d) is automatically satisfied when $k = \ell - e - g + 2$. These observations prove the following corollaries (see also [16 6.4 and 6.5]).

**Corollary 3.7.** Let $R$ be a local Cohen-Macaulay ring with infinite residue field, and assume that $R$ is Gorenstein locally in codimension $\ell - e$. Let $E$ be a finite, torsion-free, orientable $R$-module, rank $E = e > 0$, $\ell(E) = \ell$, $\ell - e + 1 \geq 2$. 

Assume that $E$ satisfies $G_{\ell - e + 1}$, $r(E) \leq \ell - e - g + 2$ and $\text{depth}(E^j) \geq d - j - g + 2$ for $1 \leq j \leq \ell - e - g + 2$, where $g$ is the height of the generic Bourbaki ideal of $E$. Then, $\mathcal{R}(E)$ is Cohen-Macaulay.

**Corollary 3.8.** Let $R$ be a local Cohen-Macaulay ring with infinite residue field, and assume that $R$ is Gorenstein locally in codimension $\ell - e$. Let $E$ be a finite, torsion-free, orientable $R$-module, rank $E = e > 0$, $\ell(E) = \ell$, $\ell - e + 1 \geq 2$. Assume that $E$ satisfies $G_{\ell - e + 1}$, and let $g$ be the height of the generic Bourbaki ideal of $E$. Then, $\mathcal{R}(E)$ is Cohen-Macaulay if the following conditions hold:

(a) $r(E) \leq \ell - e - g + 1$.

(b) $\text{depth}(E^j) \geq d - g - j + 1$ for $1 \leq j \leq \ell - e - g + 1$.

(c) If $g = 2$, $\text{Ext}^j_{R_0}(E_p, R_p) = 0$ for $1 \leq j \leq \ell - e - 1$, for all $p \in \text{Spec}(R)$ with $\dim R_p = \ell - e$ such that $E_p$ is not free.

4. **Almost linearly presented modules**

In this section, we will focus on the problem of describing the defining ideal of Rees algebras of modules.

As noticed in Remark 2.10 if $E$ admits a generic Bourbaki ideal $I$, then a minimal presentation matrix of $I$ can be constructed from a minimal presentation matrix of $E$. Using this approach, in [23, 4.11] the defining ideal of $\mathcal{R}(E)$ was described in the case when $E$ is a linearly presented module of projective dimension one, and shown to have the same shape as the defining ideal of $\mathcal{R}(I)$. The key ingredients in the proof were that in that situation $\mathcal{R}(I)$ is Cohen-Macaulay, and Cohen-Macaulayness can be transferred to $\mathcal{R}(E)$.

Using a similar strategy, we will approach the problem under the assumption that $E$ is only almost linearly presented, in which case, however, $\mathcal{R}(I)$ is usually not Cohen-Macaulay (see [3, 5.6]).

**Setting 4.1.** Let $R = k[Y_1, \ldots, Y_d]$ be a polynomial ring over a field $k$, where $d \geq 2$. Let $E$ be a finite $R$-module with projective dimension one, satisfying $G_d$. Then:

(i) $E$ is torsion-free, has positive rank $e$, and admits a minimal free resolution of the form $0 \rightarrow R^{n-e} \rightarrow R^n \rightarrow E \rightarrow 0$, where $n = \mu(E)$. We assume that $\varphi$ is almost linear, i.e. has linear entries, except possibly for those in the last column, which are homogeneous of degree $m \geq 1$.

(ii) After localizing at the unique homogeneous maximal ideal, by Theorem 2.2, $E$ admits a generic Bourbaki ideal $I$, which is perfect of grade 2. Let $\psi$ be a minimal presentation of $I$ obtained from $\varphi$ as in Remark 2.10. By construction, $\psi$ is also almost linear.

(iii) By [11] Propositions 3 and 4, $E_p$ is of linear type for all $p \in \text{Spec}(R) \setminus V(R_+)$. As a consequence, if $\mathcal{L}$ is the defining ideal of $\mathcal{S}(E)$, then the defining ideal $\mathcal{J}$ of $\mathcal{R}(E)$ satisfies $\mathcal{J} \supseteq \mathcal{L} : (Y_1, \ldots, Y_d)^i$ for all $i$.

The defining ideal of the Rees algebra of almost linearly presented perfect ideals of grade 2 was described in [3, 5.3], using the notion of iterated Jacobian duals, which we recalled in Definition 2.11. Moreover, in the situation of [3, 5.3], the Rees algebra is almost Cohen-Macaulay, i.e. $\text{depth}(R(I)) \geq \dim R(I) - 1$. In Theorem 4.3, we will prove that a similar description is possible for the defining ideal of the Rees.
algebra of modules as in Setting 4.1. The main ingredient in the proof will be the following technical result, which will allow to transfer almost Cohen-Macaulayness from $\mathcal{R}(I)$ to $\mathcal{R}(E)$.

**Proposition 4.2.** Let $R$ be a Noetherian local ring, $E$ a finite $R$-module with rank $E = e \geq 2$. Let $U = Ra_1 + \ldots + Ra_n$ be a reduction of $E$, $Z_1, \ldots, Z_n$ indeterminates. Let $R'' = R(Z_1, \ldots, Z_n)$, $E'' = E \otimes_R R''$, $x = \sum_{i=1}^n Z_i a_i$, $E''_x = E''/(x)$. Assume that depth $\mathcal{R}(E''_x) \geq 2$ for all $x \in \text{Spec}(R'')$ such that $E''_x$ is not of linear type. Then, the natural epimorphism $\pi: \mathcal{R}(E''/(x)) \rightarrow \mathcal{R}(E''_x)$ is an isomorphism, and $x$ is regular on $\mathcal{R}(E'')$.

**Proof.** We modify the proof of [23, 3.7]. Since $x$ is regular on $\mathcal{R}(E'')$ by [23, 3.6], we only need to show that $K = \ker(\pi)$ is zero. In fact, we only need to prove this locally at primes $q \in \text{Spec}(R'')$ such that $E''_q$ is not of linear type. Indeed, if $E''_q$ is of linear type, then $\mathcal{R}(E''_q) \cong \mathcal{S}(E''_q)$ is isomorphic to $\mathcal{S}(E''_q)/(x)$ by construction, whence $K_q = 0$.

Let $\overline{E''}_q$ denote $\mathcal{R}(E''/(x))$ and let $M = (m, \mathcal{R}(E''))_+$ be the unique homogeneous maximal ideal of $\mathcal{R}(E'')$. Notice that $K \subseteq H^0_M(\overline{E''}_q)$. In fact, after localizing $R''$ if needed, we may assume that $K$ vanishes locally on the punctured spectrum of $R''$. Hence, $K$ is annihilated by a power of $m$. Also, by [23, 3.6], $K$ is annihilated by a power of $U \overline{R''}$, and hence by a power of $E U \overline{R''} = (\overline{R''})_+$, since $E$ is integral over $U$.

It suffices to show that $H^0_{M_q}(\overline{E''}_q) = 0$ for all $q \in \text{Spec}(R'')$ such that $E''_q$ is not of linear type. Consider the long exact sequence of local cohomology induced by the sequence $0 \rightarrow K_q \rightarrow \overline{E''}_q \rightarrow \mathcal{R}(E''_q) \rightarrow 0$. Since by assumption depth $\mathcal{R}(E''_q) \geq 2$, then $H^i_{M_q}(\overline{E''}_q) \cong H^i_{M_q}(K_q)$ for $i = 0, 1$. In particular, since $K_q \subseteq H^0_{M_q}(\overline{E''}_q)$, $0 = H^1_{M_q}(K_q) \cong H^1_{M_q}(\overline{E''}_q)$. Therefore, the exact sequence $0 \rightarrow \mathcal{R}(E''_q)(-1) \rightarrow \mathcal{R}(E''_q) \rightarrow \overline{E''}_q \rightarrow 0$ induces the exact sequence

$$0 \rightarrow H^0_{M_q}(\overline{E''}_q) \rightarrow H^1_{M_q}(\mathcal{R}(E''_q))(-1) \rightarrow H^1_{M_q}(\mathcal{R}(E''_q)) \rightarrow 0.$$ 

Now, similarly as in [23, 3.7], one can show that $H^1_{M_q}(\mathcal{R}(E''_q))$ is finitely generated, as a consequence of the graded version of the Local Duality Theorem. Thus, by the graded version of Nakayama’s Lemma, $H^1_{M_q}(\mathcal{R}(E''_q)) = 0$, whence also $H^0_{M_q}(\overline{E''}_q) = 0$. ■

**Theorem 4.3.** Under the assumptions of Setting 4.1, set $\Upsilon = [Y_1, \ldots, Y_d]$ and assume that $n = d + e$. Then, $\mathcal{R}(E)$ is almost Cohen-Macaulay, and its defining ideal is $\mathcal{J} = ((Y_i \cdot B_m(\varphi)) : (Y_i^{(m)}) = (Y_i \cdot B_m(\varphi)) + I_d(B_m(\varphi))$, where $B_m(\varphi)$ denotes the $m$-th iterated Jacobian dual as in Definition 2.1.1.

**Proof.** We modify the proof of [23, 4.11]. Let $a_1, \ldots, a_n$ be a minimal generating set for $E$ corresponding to the given presentation matrix $\varphi$, and let $R[T_1, \ldots, T_n] \rightarrow \mathcal{R}(E)$ be the natural epimorphism, mapping $T_i$ to $a_i$ for all $i$. Localization at the unique homogeneous maximal ideal, we may assume that $R$ is local and that $E$ admits a generic Bourbaki ideal $I$, which is perfect of grade 2 and such that $\mu(I) = n - e + 1 = d + 1$. If $e = 1$, then $E \cong I$ and we are done by [3, 5.3].

So, assume that $e \geq 2$. With $x_j$ as in Notation 2.1 for $1 \leq j \leq e - 1$ set $X_j = \sum_{i=1}^n Z_{ij} T_i$, and note that $X_j$ is mapped to $x_j$ under the epimorphism $R''[T_1, \ldots, T_n] \rightarrow \mathcal{R}(E'')$. Set $\Upsilon = [T_1, \ldots, T_n]$. As in Remark 2.10, we can
Also, by [1, Propositions 3 and 4], \( \psi \) is almost Cohen-Macaulay, with defining ideal \((\psi \cdot B(\psi)) + I_d(B_m(\psi)) = (\psi \cdot B(\psi)) : (\psi)^m \), where \( m \) is the degree of the non-linear column of \( \varphi \). In particular, \( \text{depth}(\mathcal{R}(I)) \geq \dim(\mathcal{R}(I)) - 1 = d \geq 2 \).

Since \( \psi \cdot B(\psi) \) and for any Jacobian dual \( \psi \) of \( \varphi \), in \( R^n[T_1, \ldots, T_n] \) we have:

\[
\mathcal{J} = ((\psi \cdot B(\varphi)) + I_d(B_m(\varphi)) + (X_1, \ldots, X_{e-1})) \cap \mathcal{J} = ((\psi \cdot B(\varphi)) + I_d(B_m(\varphi)) + (X_1, \ldots, X_{e-1})). \]

By the graded version of Nakayama’s Lemma, this means that \( \mathcal{J} = (\psi \cdot B(\varphi)) + I_d(B_m(\varphi)) \). Hence, by the inclusions above, also \( \mathcal{J} = ((\psi \cdot B(\varphi)) : (\psi)^m) \).

### Lemma 4.4
Let \( R = k[Y_1, \ldots, Y_d][Y_1, \ldots, Y_d] \), and denote \( \psi = [Y_1, \ldots, Y_d] \). Let \( \varphi, \psi, B(\psi) \) be as in the proof of Theorem 4.3. Then, for all \( i \) and for any Jacobian dual \( B(\varphi) \) of \( \varphi \), in \( R^n[T_1, \ldots, T_n] \) we have \((\psi \cdot B(\varphi)) + I_d(B_i(\varphi)) + (X_1, \ldots, X_{e-1}) = (\psi \cdot B(\varphi)) + I_d(B_i(\varphi)) \).

**Proof.** Choose \( B(\psi) \) such that \([\psi] \cdot B(\psi) = [T] \cdot \begin{bmatrix} 0 \\ \psi \end{bmatrix}\), as in the proof of Theorem 4.3. Then, in \( R^n[T_1, \ldots, T_n] \) we have \([\psi] \cdot B(\varphi) \equiv [\psi] \cdot B(\psi) \) modulo \((X_1, \ldots, X_{e-1})\), so the statement is proved for \( i = 1 \). Now, let \( i + 1 \geq 2 \) and assume that the statement holds for \( B_i(\varphi) \). Let \( C_i \) be a matrix as in Definition 2.11. Since \((\psi \cdot B_i(\varphi)) + (I_d(B_i(\varphi)) \cap (\psi) = (\psi \cdot B_i(\varphi)) + (\psi \cdot C_i)\) and the \( B_i(\varphi) \) are bigraded, going modulo modulo \((X_1, \ldots, X_{e-1})\), in \( R^n[T_1, \ldots, T_n] \) we have:

\[
(\psi \cdot B_i(\varphi)) + (I_d(B_i(\varphi)) \cap (\psi) = (\psi \cdot B_i(\psi)) + (\psi \cdot C_i).
\]

Hence, defining \( B_{i+1}(\psi) = [B_i(\psi) \cap C_i] \) in \( R^n[T_1, \ldots, T_n] \) we have \((\psi \cdot B(\varphi)) + I_d(B_{i+1}(\varphi)) + (X_1, \ldots, X_{e-1}) = (\psi \cdot B(\varphi)) + I_d(B_{i+1}(\psi)) \).

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