An implicit method for the finite time horizon Hamilton-Jacobi-Bellman quasi-variational inequalities

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We propose a new numerical method to solve the parabolic Hamilton-Jacobi-Bellman quasi variational inequality (HJBQVI) associated with the combined impulse and stochastic optimal control problem over the finite time horizon. Our method is regarded as an implicit method in the field of the numerical method of the partial differential equation and thus it is advantageous in the sense that the stability condition is independent of the discretization parameters. We apply our method to the finite time horizon optimal forest harvesting problem considering the going out of business at the finite time. We show that the behaviour of the obtained optimal harvesting strategy of the extended problem coincides with our intuition.

1. Introduction

Solving the Hamilton-Jacobi-bellman quasi-variational inequality (HJBQVI) is one of the most challenging issue in the stochastic optimal control problem. The HJBQVI is associated with the combined impulse and stochastic optimal control which is able to formulate the system which changes drastically by our control. The combined stochastic optimal control is a quite applicable framework. We present some of the literature which deal with the applications for the mathematical finance: Pliska and Suzuki [1], Palczewski and Zabczyk [2] and Kharroubi and Pham [3] treat the portfolio optimization

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with transaction costs; Mundaca and Øksendal [4] and Cadenillas and Zapatero [5] study
the control of the exchange rate by the Central bank; Korn [6] provides the overview
relating to the applications of the impulse control. The applications to other areas, for
instance, the management of electricity management, the problems of maintenance and
quality control and information technology are found in Bensoussan and Lions [7] and
the references therein.

The major approach to apply the impulse control framework has the aspect that prob-
lems or models are formulated to have an analytical solution. For the HJBQVI associated
with the one-dimensional infinite horizon combined stochastic optimal control problem,
the smooth-fit technique is an established method to obtain the solution. However this
technique is not valid for the general HJBQVI and to our best knowledge, there is no
established method for the general HJBQVI. Hence the development of the numerical
method to solve the HJBQVI is significantly required.

The HJBQVI is categorized into the elliptic type associated with the infinite time
horizon combined optimal control problem and the parabolic one associated with the
finite time horizon case. We first mention the numerical approach to the elliptic type.
Bensoussan and Lions [7] approximate the impulse control problem by iterations of
the optimal stopping problem and hence HJBQVI is translated to the HJB variational
inequalities (HJBVIs). The numerical method for the HJBVI is well studied due to
the motivation for the pricing of American options in the mathematical finance. An
alternative method is proposed in Chancelier et. al. [8]. In this paper authors provide
the solving method for a fixed point problem which consists of the contractive operator
and the non-expansive operator and the numerical algorithm for the elliptic HJBQVI is
appeared as an application. They discretize the elliptic HJBQVI by the finite difference
scheme and lead an equivalent fixed point problem which is solvable by their algorithm.

In the case of the parabolic type, we can employ the dynamic programming, the
backward induction in a similar fashion as other optimal control problems which have the
terminal condition. Chen and Forsyth [9] solves the parabolic type HJBQVI associated
with the annuity pricing problem with a guaranteed minimum withdrawal benefit. If
the time grid size is small enough this method gives an approximated solution although
it does not satisfy the original HJBQVI exactly. Since the HJBQVI discretized by the
backward difference, this approach corresponds to the explicit method in the numerical
method of partial differential equations.

In this paper we propose a new numerical algorithm solving the parabolic type HJBQVI.
Our method is regarded as an implicit method for the HJBQVI and hence it is advan-
tageous than the previous explicit ones in the the sense that (i) there is no need for
the time grid to satisfies the condition imposed in the explicit one; (ii) the stability
condition is independent of the discretization parameters. The outline of our approach
is that we discretize the HJBQVI by the forward difference and lead the equivalent fixed
point problem which is solvable by the algorithm proposed by Chancelier et. al [8]. The
secondary aim of this paper is to provide the detailed procedure of the algorithm by
which readers are able to implement our algorithm easily. To accomplish it we describe
the procedure in the matrix form.

This paper is organized as follows. We present the mathematical formulation of the
HJBQVI in Section \[2\]. The goal of this section is to display the discretized HJBQVI in the matrix form. Section \[3\] provides a detail of our algorithm. The matrix form HJBQVI obtained in Section \[2\] is translated into an equivalent fixed point problem. We describe the procedure in detail from the viewpoint of the computational implementation. In Section \[4\] we apply the proposed method to the optimal forest harvesting problem. This problem that determines the optimal harvesting strategy for ongoing forest. We employ the mathematical formulation of the problem proposed by Willassen [10] based on the infinite time horizon impulse control framework having analytical solutions of the value function and optimal strategy are provided. We introduce the terminal time representing the time of going out the business and then the above problem turns into the finite time horizon one. In this case the analytical solution is unavailable and thus we solve it numerically by the proposed algorithm.

2. Mathematical formulation and discretization

We consider the following combined stochastic and impulse control problems over a finite time horizon \([0, T]\) with performance criterion

\[
J^w_t(x) = \mathbb{E} \left[ \int_t^T f(s, X^w_s, u_s) dt + g(X^w_T) + \sum_{t < \tau_j < T} K(\tau_j, X^w_{\tau_j -}, \zeta_j) \right] X^w_T = x. \tag{1}
\]

The controlled process \(\{X^w_t\}_{t \geq 0}\) is governed by

\[
\begin{aligned}
\frac{dX^w_t}{\tau_j + 1} &= \mu(t, X^w_t, u_t) dt + \sigma(t, X^w_t, u_t) dW_t, \quad \tau_j \leq t < \tau_{j+1}, \\
X_{\tau_{j+1}} &= \Gamma(X_{\tau_{j+1}}, \zeta_{j+1}), \quad j = 0, 1, 2, \cdots,
\end{aligned} \tag{2}
\]

where \(W_t\) is a \(d\)-dimensional Brownian Motion on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), \(\mu : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n\), \(\sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times d}\), \(\Gamma : \mathbb{R}^n \times Z \to \mathbb{R}^n\), \(U \subset \mathbb{R}^l\), and \(Z \subset \mathbb{R}^l\). The combined control \(w\) consists of the Markov control strategy \(u = \{u_t\}_{t \geq 0}\), an \(U\)-valued stochastic process which is of the form \(u_t = \alpha(t, X^w_t)\) for some function \(\alpha : [0, T] \times \mathbb{R}^n \to U\) and the impulse control strategy \(v = \{(\tau_j, \zeta_j)\}_{j=1}^\infty\). Here \(\tau_0 = 0\), \(\tau_1 < \tau_2 < \cdots\) are \(\mathcal{F}_t\)-stopping times and \(\zeta_j \in Z\), \(j \geq 1\), are \(\mathcal{F}_{\tau_j}\)-measurable random variables. The performance is measured by the following three functions: a profit rate function \(f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}\), a bequest function \(g : \mathbb{R}^n \to \mathbb{R}\), and an intervention profit function \(K : [0, T], \mathbb{R}^n \times Z \to \mathbb{R}\).

We denote by \(\mathcal{W}\) the set of admissible combined controls, i.e., \(w \in \mathcal{W}\) satisfies: (i) a unique strong solution of the SDE \(2\) with control \(w\) exists; (ii) \(\lim_{j \to \infty} \tau_j = T\) a.s. We also assume that for \(w \in \mathcal{W}\),

\[
\mathbb{E} \left[ \int_0^T |f(t, X^w_t, u_t)| dt \right] < \infty, \quad \mathbb{E} [g(X^w_T)] < \infty, \quad \mathbb{E} \left[ \sum_{\tau_j < T} K(\tau_j, X^w_{\tau_j -}, \zeta_j) \right] < \infty.
\]
The value function corresponding to our problem \([\mathbf{1}]\) is defined by 
\[ V_t(x) = \sup_{w \in W} J_t^w(x), \quad (t, x) \in [0, T] \times \mathbb{R}^n. \]
Hence the corresponding HJBQVI is given by
\[
\max \left( \sup_{\alpha \in \mathcal{U}} \{ \partial_t V_t(x) + \mathcal{L}_t^\alpha V_t(x) + f(t, x, \alpha) \} , \sup_{\zeta \in \mathbb{Z}} \{ V_t(\Gamma(x, \zeta)) + K(t, x, \zeta) - V_t(x) \} \right) = 0, 
\]
with terminal condition 
\[ V_T(x) = g(x), \quad x \in \mathbb{R}^n, \]
where \( \partial_t \) is partial differential operator with respect to \( t \), \( \mathcal{L}_t^\alpha \) is the infinitesimal generator of the process \( X^w \) at time \( t \):
\[
\mathcal{L}_t^\alpha \Psi(x) = \mu(t, x, \alpha)\partial_x \Psi(x) + \frac{1}{2} \text{Tr} \left( (\sigma \sigma^*) (t, x, \alpha) \partial^2_x \Psi(x) \right)
\]
for \( \Psi \in C^2(\mathbb{R}^n) \). Here \( \partial^j_x \) is the \( j \)-th order partial differential operator with respect to \( x \) and the asterisk means transposition.

The continuation set \( D_t \) at time \( t \) is defined by
\[
D_t = \left\{ x \in \mathbb{R}^n \left| \sup_{\zeta \in \mathbb{Z}} \{ V_t(\Gamma(x, \zeta)) + K(t, x, \zeta) - V_t(x) \} < 0 \right. \right\}.
\]
We impose the condition \( D_t \neq \emptyset, t \in [0, T] \) on the control set \( \mathcal{U} \), i.e., the control which intervenes at all region is not admissible. As similar to other numerical problems, we face the problem that the computer does not deal with the unbounded domain. Since our state process \( X^w \) leaves any bounded region until the termination time \( T \) with non-zero probability, this problem is inevitable. To cope with it, we restrict ourselves to the problem with the bounded domain. Let \( S \subset \mathbb{R}^n \) be the domain and \( \partial S \) be the boundary. We assume that (i) \( D_t \cap S \neq \emptyset, t \in [0, T] \); (ii) the intervention function \( \Gamma \) satisfies \( \Gamma : S \rightarrow S \). The boundary condition is giving by the function \( \psi : [0, T] \times \partial S \rightarrow \mathbb{R} \).

We discretize the QVI \([\mathbf{3}]\) using the standard finite difference scheme with the central difference. Let \( \delta_t, \delta = (\delta_1, \cdots, \delta_n)^* \) be the finite difference steps respect to \( t \) and \( x \). We denote by \( S_\delta \) the spacial grid and then \( S_\delta = S \cap \prod_{i=1}^n (\delta_i \mathbb{Z}) \). The discretized boundary \( \partial S_\delta \) is also represented by \( \partial S_\delta = \partial S \cap \prod_{i=1}^n (\delta_i \mathbb{Z}) \). For the convenience we introduce symbols of the time grid points \( \{ t_i \}_{0 \leq i \leq N^t}, t_i \in [0, T] \) and the spacial grid points including boundary \( \{ x_i \}_{1 \leq i \leq N_x + \tilde{N}_x}, x_i \in S_\delta \cup \partial S_\delta \), where \( N^t \) is the number of the time grid points, \( N_x \) is the number of the spacial grid points and \( \tilde{N}_x \) are the number of the discretized boundary points. We note that the subsequences \( \{ x_i \}_{1 \leq i \leq N_x} \) and \( \{ x_i \}_{N_x < i \leq N_x + \tilde{N}_x} \) represent the spacial (internal) grid points and discretized boundary points respectively. Furthermore the time grid points are defined sequentially: \( t_0 = 0, t_1 = \delta_t, \cdots, t_{N^t} = N^t \delta_t = T \). Here we implicitly assume that \( \delta_t \) is the number supporting the existence of \( N_x \in \mathbb{N} \) s.t. \( N^t \delta_t = T \). We impose the following conditions on the intervention function \( \Gamma \): (i) \( \Gamma : S_\delta \times \mathbb{Z} \rightarrow S_\delta \), (ii) there exist an integer function \( \eta : \{ 1, \cdots, N^x \} \times \mathbb{Z} \rightarrow \{ 1, \cdots, N^x \} \) s.t.
\[
\Gamma(x_i, \zeta) = x_{\eta(i, \zeta)}, \quad i \in \{ 1, \cdots, N^x \}, \quad \zeta \in \mathbb{Z}.
\]
The discretized QVI of the QVI (3) is defined as follow:

\[
\max \left\{ \frac{\Phi^{k+1}(x) - \Phi^k(x)}{\delta_t} + \mathcal{L}^{\alpha,k}\Phi^k(x) + f(t_k, x, \alpha) \right\},
\]

\[
\sup_{\zeta \in \mathcal{Z}} \left\{ \Phi^k(\Gamma(x, \zeta)) + K(x, \zeta) - \Phi^k(x) \right\} = 0, \quad x \in S_\delta,
\]

\[
\Phi^k(x) = \psi(t_k, x), \quad x \in \partial S_\delta,
\]

for \( k \in \{0, \ldots, N_t - 1\} \), with terminal condition \( \Phi^{N_t}(x) = g(x) \), \( x \in S_\delta \cup \partial S_\delta \), where \( \Phi^k : S_\delta \cup \partial S_\delta \to \mathbb{R} \) and \( \mathcal{L}^{\alpha,k} \) is the operator such that

\[
\mathcal{L}^{\alpha,k}\Psi(x) = \Psi(x) \left\{ \sum_{i=1}^{n} \frac{-(\sigma\sigma^*)_{ii}}{\delta_t^2} \right\} + \sum_{j \in \mathcal{J}(i)} \frac{|(\sigma\sigma^*)_{ij}|}{2\delta_t\delta_j} \Psi(x + \kappa\delta_i e_i) \left\{ \frac{-(\sigma\sigma^*)_{ii}}{\delta_t^2} - \sum_{j \in \mathcal{J}(i)} \frac{|(\sigma\sigma^*)_{ij}|}{\delta_t\delta_j} + \frac{\kappa\mu_i}{\delta_t} \right\} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j \in \mathcal{J}(i)} \sum_{\kappa,\lambda = \pm 1} \Psi(x + \kappa\delta_i e_i + \lambda\delta_j e_j) \frac{(\sigma\sigma^*)_{ij}^{[\kappa\lambda]}}{\delta_t^2}, \quad x \in S_\delta,
\]

for \( \Psi : S_\delta \cup \partial S_\delta \to \mathbb{R} \). Here we have used the notations

\[
(\sigma\sigma^*)_{ij}^{[\kappa\lambda]} = \begin{cases} \max (0, (\sigma\sigma^*)_{ij}), & \kappa\lambda = 1, \\ -\min (0, (\sigma\sigma^*)_{ij}), & \kappa\lambda = -1, \end{cases}
\]

\[
\mathcal{J}(i) = \{1, \ldots, n\} \setminus \{i\},
\]

and have omitted the arguments of \((\sigma\sigma^*)\) and \(\mu\), i.e, \((\sigma\sigma^*) = (\sigma\sigma^*)(t_k, x, \alpha)\) and \(\mu = \mu(t_k, x, \alpha)\).

We have employed the central difference to obtain the discretized QVI (4). Therefore to assure that \( \Phi^k(x) \) converges to the viscosity solution of (3), we assume that the functions \( \mu \) and \( \sigma \) satisfy the condition

\[
|\mu_i(t, x, \alpha)| \leq \frac{(\sigma\sigma^*)_{ii}(t, x, \alpha)}{\delta_t} - \sum_{j \in \mathcal{J}(i)} |(\sigma\sigma^*)_{ij}(t, x, \alpha)| \frac{1}{\delta_j}, \quad (t, x, \alpha) \in [0, T) \times S_\delta \times \mathcal{U}.
\]

(5)

Here we remark that if we have employed the one-sided difference the condition (5) becomes the milder one:

\[
0 \leq \frac{(\sigma\sigma^*)_{ii}(t, x, \alpha)}{\delta_t} - \sum_{j \in \mathcal{J}(i)} |(\sigma\sigma^*)_{ij}(t, x, \alpha)| \frac{1}{\delta_j}, \quad (t, x, \alpha) \in [0, T) \times S_\delta \times \mathcal{U},
\]

even though the convergence speed becomes slow.
Finally we represent the discretized QVI \((6)\) as the matrix QVI form:

\[
\begin{align*}
    \max \left( \sup_{a \in \mathbb{U}^{N_x}} \left\{ \frac{\phi_i^{k+1} - \phi_i^k}{\delta_t} + \left( L^{a,k} \phi_i^k \right)_i + f_i^{a,k} \right\} \right), \\
    \sup_{z \in \mathbb{Z}^{N_x}} \left\{ \left( M^z \phi_i^k \right)_i + K_i^{z,k} - \phi_i^k \right\} = 0, \quad 1 \leq i \leq N_x, \\
    \phi_i^k = \psi(t_i,x_i; \phi), \quad N_x < i \leq N_x + \bar{N},
\end{align*}
\]

for \(k \in \{0, \ldots, N^t - 1\}\) with terminal condition \(\phi_i^{N^t} = g(x_i), i \in \{1, \ldots, N_x\}\), where \(\phi_i^k\) is a \((N_x + \bar{N})\)-dimensional vector s.t. \(\phi_i^k = \Phi(x_i), a = \{a_i\}_{1 \leq i \leq N_x}, a_i \in \mathbb{U}, z = \{z_i\}_{1 \leq i \leq N_x}, z_i \in \mathbb{Z}, L^{a,k}\) is a \(N_x \times (N_x + \bar{N})\) matrix such that for \(i \in \{1, \ldots, N_x\}\),

\[
L^{a,k}_{ij} = \begin{cases} 
    \frac{-(\sigma^*)_{ij}}{\delta_t^2} + \sum_{j' \in J(i)} \frac{|(\sigma^*)_{ij'|j}|}{2 \delta_t \delta_j}, & \text{if } i = j, \\
    \frac{1}{2} \left\{ \frac{-(\sigma^*)_{ij}}{\delta_t^2} - \sum_{j' \in J(i)} \frac{|(\sigma^*)_{ij'|j}|}{\delta_t \delta_j} + \frac{\kappa \mu_i}{\delta_t} \right\}, & \text{if } x_j = x_i + \kappa \delta_t e_i, \\
    \frac{1}{2} \frac{|(\sigma^*)_{ij'|j}|}{\delta_t^2}, & \text{if } x_j = x_i + \kappa \delta_t e_i + \lambda \delta_j e_j, \\
    0, & \text{otherwise},
\end{cases}
\]

where \(f_i^{a,k}\) is a \(N_x\)-dimensional vector s.t. \(f_i^{a,k} = f(t_i,x_i,a_i), M^z_i\) is a \(N_x \times (N_x + \bar{N})\) matrix s.t. \(M^z_{ij} = \mathbb{1}_{j = (i,z_i)}\) and \(K_i^{z,k}\) is a \(N_x\)-dimensional vector such that \(K_i^{z,k} = K(t_i,x_i,z_i)\). Here we have omitted the arguments of \((\sigma^*)\) and \(\mu\) again, i.e., \((\sigma^*) = (\sigma^*)(t_k,x_i,a_i)\) and \(\mu = \mu(t_k,x_i,a_i)\).

3. Algorithm

We convert the matrix QVI \((6)\) to the following equivalent fixed point problem:

\[
\begin{align*}
    \phi_i = \max \left( \sup_{a \in \mathbb{U}^{N_x}} \left\{ \left( \bar{L}^{a,k} \phi_i^k \right)_i + \bar{f}^{a,k}_i \right\}, \sup_{z \in \mathbb{Z}^{N_x}} \left\{ \left( M^z \phi_i^k \right)_i + K_i^{z,k} \right\} \right), \quad 1 \leq i \leq N_x, \\
    \phi_i^k = \psi(t_i,x_i), \quad N_x < i \leq N_x + \bar{N},
\end{align*}
\]

where \(\bar{f}^{a,k}\) is a \(N_x\)-dimensional vector, \(\bar{L}^{a,k}\) is a \(N_x \times (N_x + \bar{N})\) matrix and they are defined as follows:

\[
\begin{align*}
    \bar{f}^{a,k}_i &= \frac{h^{a,k}}{h^{a,k} + \delta_t} f_i^{a,k} + \frac{h^k}{h^k + \delta_t} \phi_i^{k+1}, \\
    \bar{L}^{a,k}_i &= \frac{h^{a,k}}{h^{a,k} + \delta_t} (I + h^k L^{a,k}).
\end{align*}
\]
Here \( h^k \) is a positive number such that
\[
h^k \leq \inf_{a \in U^{N_x}} \min_{1 \leq i \leq N_x} \frac{1}{I_{ii}^a,k}
\]
and \( I \) is a \( N_x \times (N_x + \bar{N}_x) \) matrix such that \( I_{ij} = \delta_{ij} \) where \( \delta_{ij} \) is the Kronecker delta. Then the fixed point problem (7) satisfies the conditions to apply the method proposed by Chancelier et. al. [8]. The detail is discussed in Appendix A. We are able to solve the problem (7) by the backward method. In the following steps we assume that we have already found the \( \phi^{k+1} \).

**step1** set \( \phi^k = \phi^{k+1} \).

**step2** search the controls \( \hat{a}^k \) and \( \hat{z}^k \) such that
\[
\hat{a}^k = \arg\max_{a \in U^{N_x}} \left\{ \bar{L}_{i,j}^{\hat{a},k} \phi^k + \bar{f}_{i,j}^{\hat{a},k} \right\}, \quad \hat{z}^k = \arg\max_{z \in Z^{N_x}} \left\{ M^z \phi^k + K^z \right\}
\]
and define an index set \( \mathcal{I}^k \) such that
\[
\mathcal{I}^k = \left\{ i \in \{1, \ldots, N_x\} \mid \left( \bar{L}_{i,j} \hat{a}_{i}^{k,k} \phi^k + \bar{f}_{i,j}^{\hat{a},k} \geq (M^z \phi^k)_i + K^z_{k,k} \right) \right\}
\]

**step3** determine a \( (N_x + \bar{N}_x) \times (N_x + \bar{N}_x) \) matrix \( A \) and \( (N_x + \bar{N}_x) \)-dimensional vector \( b \) as follows:
\[
A_{ij} = \\
\begin{cases}
\bar{L}_{i,j} \hat{a}_{i}^{k,k} & \text{if } i \in \mathcal{I}, \\
M_{i,j} & \text{if } i \in \{1, \ldots, N_x\} \setminus \mathcal{I}, \\
0 & \text{if } i \in \{N_x + 1, \ldots, N_x + \bar{N}_x\},
\end{cases}
\]
\[
b_i = \\
\begin{cases}
\bar{f}_{i,j}^{\hat{a},k} & \text{if } i \in \mathcal{I}, \\
K_{i}^{\hat{z},k} & \text{if } i \in \{1, \ldots, N_x\} \setminus \mathcal{I}, \\
\psi(t_k, x_i) & \text{if } i \in \{N_x + 1, \ldots, N_x + \bar{N}_x\},
\end{cases}
\]

**step4** solve the linear equation \((I-A)\phi' = b\) where \( \phi' \) is a \( (N_x + \bar{N}_x) \)-dimensional vector and \( I \) is a \( (N_x + \bar{N}_x) \times (N_x + \bar{N}_x) \) identity matrix,

**step5** if \( \max|\phi' - \hat{\phi}^k| \) exceeds the admissible error replace \( \phi^k \) by \( \phi' \) and back to step2, else also replace \( \phi^k \) by \( \phi' \) and go to step6,

**step6** if \( k \neq 1 \) replace \( k \) by \( k - 1 \) and go back to step1, else determine the Markov control \( \hat{\alpha} \), the set \( \hat{D}_{t_k} \) and the impulse control \( \hat{v} = (\hat{\tau}_i, \hat{\zeta}_i)_{i \geq 1} \) as follows:
\[
\hat{\alpha}(t_k, x_i) = \hat{a}^k_i, \quad \hat{D}_{t_k} = \left\{ x_i \mid i \in \mathcal{I}^k \right\},
\]
\[
\hat{\tau}_i = \min \left\{ t_k \in \{t_1, \ldots, t_{N^\mu} \} \mid t_k > \hat{\tau}_{i-1}; X^w_{t_k} \notin \hat{D}_{t_k} \right\},
\]
\[
\hat{\zeta}_i(x_j) = \hat{z}^k_{j,k'}, \quad k' \in \{1, \ldots, N^t\} \text{ s.t. } t_{k'} = \hat{\tau}_i,
\]
where \( \hat{\tau}_0 = 0 \) and \( \hat{\phi} = (\hat{\alpha}, \hat{v}) \).

The control \( \hat{\phi} \) obtained by the above procedure satisfies \( J_{t_k}^w(x_i) \geq J_{t_k}^w(x_i), \) \( t_k \in \{t_0, \ldots, t_{N^\mu-1}\}, \)
\( x_i \in \mathcal{S}_8, \) \( w \in \mathcal{W} \) and hence \( \hat{\phi} \) is optimal control.

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4. Numerical results

We apply our method to the finite horizon optimal forest harvesting problem based on the Willassen’s formulation. The mathematical formulations of the original and extended problems are described in Section 4.1. The original work considers the infinite time horizon impulse control problem and gives the analytical solution in the unbounded domain. We first restrict the domain to the bounded one and add the boundary condition which gives the equivalent solution to the unbounded domain. In Section 4.2 we discuss the validity of the our boundary condition using the numerical experiment. Our main target problem, the finite time horizon optimal forest harvesting problem which solution is not obtained analytically is discussed in Section 4.3.

4.1. Optimal forest harvesting problem

Let $X_t$ be the biomass of a forest at time $t$ and $\tau_1 < \tau_2 < \cdots$ be the tree harvesting times. We cut all trees in the forest at time $\tau_i$ and replant the biomass $\tilde{x} \in \mathbb{R}$. Suppose that the growth of the biomass follows the Geometric Brownian motion and then $X_t$ is governed by

\[
\begin{cases}
  dX_t^v = \mu X_t^v dt + \sigma X_t^v dW_t, & \tau_j < t < \tau_{j+1}, \\
  X_{\tau_j}^v = \tilde{x}, & j = 1, 2, \cdots,
\end{cases}
\]

where $\mu$ and $\sigma$ are positive constants. Furthermore we suppose that $\tau_i$ satisfies the conditions to be the intervention time, i.e., $\tau_i$ is an $\mathcal{F}_t$-stopping time and $\tau_i < \infty$ a.s. Then $X_t^v$ has a unique strong solution and $v := (\tau_1, \tau_2, \cdots)$ is the admissible impulse control strategy.

The original work formulated by Willassen [10] is defined as the infinite time horizon optimal impulse control problem. Let $\beta \in (0, 1)$ be the proportional harvesting cost and $Q > 0$ be the replanting cost of the biomass $\tilde{x}$. Then the performance criterion is

\[
J^v(x) = \mathbb{E} \left[ \sum_{i=1}^{\infty} e^{-\lambda \tau_i} \left( (1 - \beta) X_{\tau_i} - Q \right) \left| X_0^v = x \right. \right],
\]

where $\lambda > 0$ is the discounting factor. We impose $\tilde{x}$, $\beta$ and $Q$ on the condition $(1 - \beta) \tilde{x} < Q$: if this is not the case then the optimal strategy is that we harvest trees immediately after the replant which is a quite vacuity situation.

The value function corresponding to the criterion (10) is defined by $V(x) = \sup_v J^v(x)$, $x \in \mathbb{R}$ and hence the corresponding HJBQVI is given by

\[
\max \left( \frac{\sigma^2 x^2}{2} \partial^2_x V(x) + \mu x \partial_x V(x) - \lambda V(x), V(\tilde{x}) + (1 - \beta)x - Q - V(x) \right) = 0
\]

Willassen solved the HJBQVI (11) explicitly:

\[
V(x) = \begin{cases}
  \Psi(x) & \text{for } x < y, \\
  (1 - \beta)x - Q + \Psi(\tilde{x}) & \text{for } x \geq y,
\end{cases}
\]
where

\[ \Psi(x) = \frac{(1 - \beta)y}{\gamma} \left( \frac{x}{y} \right)^\gamma, \quad \gamma = \frac{\sigma^2 - 2\mu + \sqrt{(\sigma^2 - 2\mu)^2 + 8\sigma^2\lambda}}{2\sigma^2} \]

and \( y > \tilde{x} \) is a solution of

\[ y = \frac{\gamma Q - (1 - \beta)y(\tilde{x}/y)}{(1 - \beta)(\gamma - 1)}. \]

We call \( y \) the strategy switch point. The key ideas to obtain the solution are as follows:
(i) the condition for the cost suggests that we should wait for the harvesting until \( X_t \) exceeds a certain value \( y \); (ii) since the partial differential equation \( \frac{\sigma^2x^2}{2} \partial^2_x V(x) + \mu x \partial_x V(x) - \lambda V(x) = 0 \) has the analytical general solution, we obtain the value function by connecting this analytical solution and \( V(\tilde{x}) + (1 - \beta)x - Q \) smoothly at \( y \).

We extend the above problem by considering to the case that the former goes out of the forest business at time \( T \): he harvests all the trees and does not replant at time \( T \). Then the problem turns into the finite time horizon problem and the performance criterion is modified as follow:

\[ J_t^v(x) = \mathbb{E} \left[ \sum_{t < \tau < T} e^{-\lambda \tau} (1 - \beta)X_{\tau^-}^v + e^{-\lambda T} ((1 - \beta)X_T^v - Q) \right| X_t = x \]  \hspace{1cm} (13)

The value function corresponding to this problem is defined by \( V_t(x) = \sup_v J_t^v(x) \), \( t \in [0,T), x \in \mathbb{R} \) and thus the corresponding HJBQVI is given by

\[ \max \left( \partial_t V_t(x) + \frac{\sigma^2x^2}{2} \partial^2_x V_t(x) + \mu x \partial_x V_t(x), \right. \]

\[ \left. V_t(\tilde{x}) + e^{-\lambda T} ((1 - \beta)x - Q) - V_t(x) \right) = 0 \]  \hspace{1cm} (14)

with terminal condition \( V_T(x) = e^{-\lambda T} (1 - \beta)x \).

In this case the analytical solution is not available, however we can expect the behaviour of the solution of the HJBQVI \[ \text{(14)} \]. Since the performance criterion \( J_0^v \) is equivalent to that of the infinite time horizon case \[ \text{(10)} \] on the limit \( T \to \infty \), the value function \( V_t \) and the optimal control \( \hat{v} \) coincide with the infinite horizon ones if \( T \) is large enough and \( t \ll T \).

4.2. Determination of the bounded domain and boundary condition

The idea introducing the strategy switch point \( y \) is significant for determination of the candidate finite domain and boundary condition. Let \( x_{\text{max}} \) be positive real value, \( \mathcal{S} = (0, x_{\text{max}}) \) be the candidate bounded domain and boundary \( \partial \mathcal{S} = \{0, x_{\text{max}}\} \) be the candidate boundary. We first define \( \psi \) as the function giving the boundary condition of the infinite time horizon case such that

\[
\begin{align*}
\psi(0) &= 0, \\
\psi(x_{\text{max}}) &= V(\tilde{x}) + (1 - \beta)x_{\text{max}} - Q.
\end{align*}
\]  \hspace{1cm} (15)
Since we cannot expect the forest growth after the biomass reaches to 0, the value function should be 0 at $x = 0$. We are able to expect that this boundary condition gives the same solution in the infinite domain case if $x_{\text{max}}$ is enough larger than $y$.

We examine this candidate domain and boundary condition by solving the HJBQVI (11) with them numerically. The method to solve the infinite horizon HJBQVI is proposed by Øksendal and Sulem [11]. Since our state process is the 1-dimensional geometric Brownian motion, we are able to employ the method without extra assumptions. The HJBQVI is discretized by the standard finite difference scheme with central difference and we denote by $\delta_x$ be the finite finite difference step and $N^x$ be the number of the grid points. We determine the parameters as table 1. In this situation the value of strategy switch point $y$ is 5.495503.

| Parameter | Description                  | Value |
|-----------|------------------------------|-------|
| $x_{\text{max}}$ | right limit of the domain | 10    |
| $\tilde{x}$   | initial                     | 1     |
| $\beta$     | harvesting cost rate        | 10%   |
| $Q$         | replanting cost             | 2     |
| $\mu$      | expected growth rate        | 1     |
| $\sigma$   | volatility                  | 1     |
| $\lambda$ | discount factor             | 2     |

Table 1: Parameters

The numerical results obtained by the algorithm are as follows. Figure 1 shows the maximum error of the value function comparing the analytical solution (12) and the numerical solution. The order of the error is $O(\delta_x^2)$ which coincides with that implied by the finite difference scheme with central difference. Figure 2 displaying the analytical strategy switch point with red line and the numerical one with blue line indicates that the numerical result of switch point is well accorded with the $\delta_x$-order. Therefore we
conclude that our candidate domain and boundary condition are valid.

We next define the boundary condition in the finite time horizon case as the slightly modified one from the infinite time horizon case:

\[
\begin{align*}
\psi(t,0) &= 0, \\
\psi(t,x_{\text{max}}) &= V_t(\bar{x}) + e^{-\lambda t} ((1 - \alpha)x_{\text{max}} - Q).
\end{align*}
\]

This boundary condition works well under the case that \(x_{\text{max}}\) is large enough: the value of \(x_{\text{max}}\) should be larger than one of the infinite time horizon case. Because of the replanting cost \(Q\) we are able to expect that the optimal strategy close to the terminal time \(T\) is not cutting down the trees. However our boundary condition has enforced the cutting down the trees at the boundary point \(x_{\text{max}}\). We can avoid this contradiction by taking the value of \(x_{\text{max}}\) large enough. the contribution of the cost \(Q\) to the value function \(V_t(x)\) vanishes if the biomass \(x\) is large enough: the harvest profit which become larger with the growth of the biomass \(x\); the cost \(Q\) dose not depend on the biomass \(x\). We discuss this issue again in the following section with the numerical results.

In the end of this subsection we mention about the computational load to solve the infinite time horizon HJBQVI by this algorithm. Figure 3 describes the computational time as a function of the number of the grid points. We see that the computational time grows exponentially when \(\delta_x\) becomes finer. The main load is the growth of size of the linear equations which is inevitable until we employ the finite difference scheme. The search of the optimal regular and impulse controls on the each grid points is the second load: the index \(i\) takes the larger range value when \(N^x\) becomes larger. There is the possibility that we can ease up this factor by the massively parallel computing such as the GPGPU or others. Figure 4 shows the inherent load of this algorithm. Since the finer finite difference step allows us to compute the more accurate value function, we need more iteration to reach the fix point.

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1 The detail of our computational resources are as follows. The computer we use has Intel Core i5 650 @3.20GHz and 4GB RAMs. Our code is parallelized by the OpenMP and we used the PARDISO [12], the linear equation solver.
4.3. Finite time horizon case

We now treat the main issue of this section. In this situation the matrix operators of our algorithm are defined as follows:

\[
L_{i,j}^{a,k} = \begin{cases} 
\frac{\sigma^2 x_i^2}{\delta_x^2} & i = j, \\
\frac{\sigma^2 x_i^2}{2\delta_x^2} \pm \frac{\mu x}{2\delta_x} & i = i \pm 1, \\
0 & \text{otherwise},
\end{cases}
\]

\[f_a^{i,k} = 0, \quad \eta(i, z_i) = j \in \mathbb{N} \text{ s.t. } x_j = \tilde{x}, \]

\[K_{i}^{z,k} = e^{-\lambda t} ((1 - \alpha) x_i - Q).\]

We set \(T = 3.0\) and \(N^t = 3000\) and then \(\delta_t = 0.001\). The other parameters are used the same value in table 1 except \(x_{\text{max}}\).

The first numerical result we discuss in this subsection is the behaviour of \(\tilde{y}_t\), the strategy switch point at time \(t\). As mentioned in Section 4.1, we can expect that \(\tilde{y}_t\) converges to \(y\), the strategy switch point discussed in the previous section if \(t\) goes to 0 and \(T\) is large enough. We also remind that \(\tilde{y}_t\) contains the error if \(x_{\text{max}}\) is not large enough. Hence we examine the various value of \(x_{\text{max}}\) and the result is shown in Figure 5.

The remarkable point we first mention is that \(\tilde{y}_t\) converges to \(y\) regardless of the value of \(x_{\text{max}}\). It is coincide with our expectation discussed in previous subsection. We can understand the behaviour of \(\tilde{y}_t\) as follows. The behaviour of \(\tilde{y}_t\) close to \(T\) which takes the much higher value than \(\tilde{y}\) suggests that we should keep the trees and this is quite consistent with the discussion in previous subsection. If the time \(t\) disengages from \(T\), the merit of the waiting for the harvest caused by the replanting cost \(Q\) vanishes and hence the value of \(\tilde{y}_t\) decreases.

![Figure 5: \(\tilde{y}_t\) with various \(x_{\text{max}}\). The orange, green, red and blue lines indicate the strategy switch point obtained with the boundary \(x_{\text{max}} = 100, 50, 20\) and \(10\) respectively. The purple line indicates that of the infinite time horizon case.](image-url)
We discuss the error caused by the improper boundary condition. In the previous subsection we hypothesize that the error vanishes. The behaviour of $\hat{y}_t$ support our hypothesis. The evidence is the time interval that $\hat{y}_t$ is fixed on $x_{\text{max}}$. This time interval decreases with the increase of $x_{\text{max}}$ and we cannot recognise the interval the case of $x_{\text{max}} = 100$ in Figure 5. Another evidence is that $\hat{y}_t$ with the smaller $x_{\text{max}}$ consists with one with the larger $x_{\text{max}}$ if $t$ disengages from $T$. The time interval that they are consistent becomes longer if $x_{\text{max}}$ become large.

We next discuss the computational load. We set $x_{\text{max}} = 100$ and the results are displayed in Figure 6 and Figure 7. Figure 6 describes the computational time whose transverse is the number of the grid points. The new load factor is obviously the time grid size $N^t$. The growth of size of the linear equation and the search cost for the optimal controls $\hat{a}^k$ and $\hat{z}^k$ are quite the same factors as the previous subsection. However the iteration number for convergence to the fix point is slightly different. Figure 7 which shows that the maximum iteration number of each time step suggests that we only need to the approximately 10% iteration comparing to the same grid size infinite time horizon case. This is due to the determination of the initial value of $\phi^k$. Since $\delta_t$ is small enough the difference between $\phi^k$ and $\phi^{k+1}$ is expected to be small. Hence we are able to reduce the iteration number than the infinite case which is the case that we have no informations about the solution.

5. Summary

We have proposed a new numerical method to solve the parabolic HJBQVI associated with the combined impulse and stochastic optimal control problem over the finite time horizon. Our method is regarded as an implicit method in the field of numerical method of the PDEs and thus it is advantageous in the sense that the stability condition is independent of the discretization parameters. We have provided the detailed procedure of the algorithm displayed in the matrix form for the sake of the easily implementation.

We apply our method to the optimal forest harvesting problem. The original problem
formulated by Willassen [10] is defined as the infinite time horizon impulse control problem and has an analytical solution. We introduce the terminal time which represent the time of going out the forest business and then the above problem turns into the finite time horizon one. Since the original problem is defined on the unbounded domain, we have introduced the equivalent bounded domain and boundary condition and we have verified them by the numerical experiment.

The analytical solution of our finite time horizon problem is unavailable and hence we solve it by our algorithm. The behaviour of the obtained optimal strategy is reasonable: the strategy coincides with the infinite time horizon one when the terminal time goes to infinity; the strategy switch point, the threshold of the biomass to harvest the trees is much higher than the infinite horizon case near the terminal time.

A. From the matrix QVI (6) to the fixed point problem (7)

We first verify that the matrix QVI (6) and the fixed point problem (7) are equivalent. The Equations (8) imply that

\[
\begin{align*}
    f^a_{i,k} &= \left(1 + \frac{1}{\delta_t} + \frac{1}{h^k}\right) f^a_{i} - \frac{1}{\delta_t} \phi^{k+1}_i, \\
    L^a_{i,k} &= \left(1 + \frac{1}{\delta_t}\right) L^a_{i} - \frac{1}{h^k} I.
\end{align*}
\]

hence the matrix QVI (6) is rewritten in the following form:

\[
\begin{align*}
    \max \left( \left(1 + \frac{1}{\delta_t} + \frac{1}{h^k}\right) \sup_{a \in \mathbb{U}} \left\{ \left(L^a_{i,k} \phi^k_i\right)_i + f^a_{i,k} - \phi^k_i \right\} \right), \\
    \sup_{z \in \mathbb{Z}^N} \left\{ \left(M^z \phi^k_i\right)_i + K^{z,k}_i - \phi^k_i \right\} = 0, \quad 1 \leq i \leq N^x, \\
    \phi^k_i = \psi(t_k, x_i; \phi), \quad N^x < i \leq N^x + \bar{N}^x.
\end{align*}
\]

The parameters \(\delta_t\) and \(h^k\) are positive numbers thus it is equivalent to \(2\)

\[
\begin{align*}
    \max \left( \sup_{a \in \mathbb{U}^N} \left\{ \left(L^a_{i,k} \phi^k_i\right)_i + f^a_{i,k} - \phi^k_i \right\} \right), \\
    \sup_{z \in \mathbb{Z}^N} \left\{ \left(M^z \phi^k_i\right)_i + K^{z,k}_i - \phi^k_i \right\} = 0, \quad 1 \leq i \leq N^x, \\
    \phi^k_i = \psi(t_k, x_i; \phi), \quad N^x < i \leq N^x + \bar{N}^x.
\end{align*}
\]

Therefore we obtain the fixed point problem (7).

\[\text{max}[cf(x), g(x)] = 0 \text{ is equivalent to } \text{max}[f(x), g(x)] = 0, \text{ for every } c > 0,\]
We next show that $\bar{L}^{a,k}$ is a contraction map displayed in the matrix form. By the definition of $h^k$, the condition (5) and the equations (8), we find that

$$0 \leq \sum_{j=1}^{N^x+N^\bar{x}} \bar{L}^{a,k}_{ij} < 1, \quad i \in 1, \ldots, N^x$$

(16)

Thus we obtain

$$\|\bar{L}^{a,k}\phi' - \bar{L}^{a,k}\phi\| \leq \max_{1 \leq i \leq N^x} \sum_{j=1}^{N^x+N^\bar{x}} \bar{L}^{a,k}_{ij} \phi_j' - \bar{L}^{a,k}_{ij} \phi_j$$

$$\leq \max_{1 \leq i \leq N^x} \sum_{j=1}^{N^x+N^\bar{x}} \bar{L}^{a,k}_{ij} \phi_j' - \bar{L}^{a,k}_{ij} \phi_j$$

$$\leq \max_{1 \leq j \leq N^x} |\phi_j' - \phi_j| \max_{1 \leq i \leq N^x} \sum_{j=1}^{N^x+N^\bar{x}} \bar{L}^{a,k}_{ij}$$

$$< \max_{1 \leq j \leq N^x} |\phi_j' - \phi_j|$$

$$= \|\phi' - \phi\|_{\infty}$$

Hence $\bar{L}^{a,k}$ is a contraction map displayed in the matrix form.

Finally we show that $\bar{L}^{a,k}$ satisfies the discrete maximum principle i.e.,

$$\bar{L}^{a,k}\phi' - \bar{L}^{a,k}\phi \leq \phi' - \phi \Rightarrow \phi' - \phi \geq 0.$$ 

We lead the contra position of the above statement. Assume that $\phi_n' - \phi_n < 0$ and $\phi_i' - \phi_i \geq 0, \ i \neq k$. Then we have

$$(\bar{L}^{a,k}\phi)_n - (\bar{L}^{a,k}\phi)_n = \sum_j \bar{L}^{a,k}_{nj} (\phi_j' - \phi_j)$$

$$= \sum_{j \neq n} \bar{L}^{a,k}_{nj} (\phi_j' - \phi_j) + \bar{L}^{a,k}_{nn} (\phi_n' - \phi_n)$$

$$\geq \bar{L}^{a,k}_{nn} (\phi_n' - \phi_n)$$

$$> (\phi_n' - \phi_n)$$

Hence we obtain

$$\exists n \text{ s.t. } \phi_n' - \phi_n < 0 \Rightarrow \bar{L}^{a,k}\phi' - \bar{L}^{a,k}\phi < \phi' - \phi$$

which establish the statement.
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