ACTIONS OF SYMBOLIC DYNAMICAL SYSTEMS ON $C^*$-ALGEBRAS II.
SIMPLICITY OF $C^*$-SYMBOLIC CROSSED PRODUCTS AND SOME EXAMPLES

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Abstract. We have introduced a notion of $C^*$-symbolic dynamical system in [K. Matsumoto: Actions of symbolic dynamical systems on $C^*$-algebras, to appear in J. Reine Angew. Math.], that is a finite family of endomorphisms of a $C^*$-algebra with some conditions. The endomorphisms are indexed by symbols and yield both a subshift and a $C^*$-algebra of a Hilbert $C^*$-bimodule. The associated $C^*$-algebra with the $C^*$-symbolic dynamical system is regarded as a crossed product by the subshift. We will study a simplicity condition of the $C^*$-algebras of the $C^*$-symbolic dynamical systems. Some examples such as irrational rotation Cuntz-Krieger algebras will be studied.

1. Introduction

In [CK], J. Cuntz and W. Krieger have founded a close relationship between symbolic dynamics and $C^*$-algebras (cf.[C], [C2]). They constructed purely infinite simple $C^*$-algebras from irreducible topological Markov shifts. The $C^*$-algebras are called Cuntz-Krieger algebras.

In [Ma], the author introduced a notion of $\lambda$-graph system, whose matrix version is called symbolic matrix system. A $\lambda$-graph system is a generalization of finite labeled graph and presents a subshift. He constructed $C^*$-algebras from $\lambda$-graph systems [Ma2] as a generalization of the above Cuntz-Krieger algebras. A $\lambda$-graph system gives rise to a finite family $\{\rho_\alpha\}_{\alpha \in \Sigma}$ of endomorphisms of a unital commutative AF-$C^*$-algebra $A_\Delta$ with some conditions stated below. A $C^*$-symbolic dynamical system, introduced in [Ma6], is a generalization of $\lambda$-graph system. It is a finite family $\{\rho_\alpha\}_{\alpha \in \Sigma}$ of endomorphisms of a unital $C^*$-algebra $A$ such that the closed ideal generated by $\rho_\alpha(1), \alpha \in \Sigma$ coincides with $A$. A finite labeled graph gives rise to a $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$ such that $A = C^N$ for some $N \in \mathbb{N}$. Conversely, if $A = C^N$, the $C^*$-symbolic dynamical system comes from a finite labeled graph. A $\lambda$-graph system $\Sigma$ gives rise to a $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$ such that $A$ is $C(\Omega_\Sigma)$ for some compact Hausdorff space $\Omega_\Sigma$.
with \( \dim \Omega_C = 0 \). Conversely, if \( \mathcal{A} \) is \( C(X) \) for a compact Hausdorff space \( X \) with \( \dim X = 0 \), the \( C^* \)-symbolic dynamical system comes from a \( \lambda \)-graph system.

A \( C^* \)-symbolic dynamical system \( (\mathcal{A}, \rho, \Sigma) \) yields a nontrivial subshift \( \Lambda_{(\mathcal{A}, \rho, \Sigma)} \), that we will denote by \( \Lambda_\rho \), over \( \Sigma \) and a Hilbert \( C^* \)-right \( \mathcal{A} \)-module \( (\phi_\rho : \mathcal{A} \to L(\mathcal{H}_\mathcal{A}), \{u_\alpha\}_{\alpha \in \Sigma}) \) that has an orthogonal finite basis \( \{u_\alpha\}_{\alpha \in \Sigma} \) and a unital faithful diagonal left action \( \phi_\rho : \mathcal{A} \to L(\mathcal{H}_\mathcal{A}) \). It is called a Hilbert \( C^* \)-symbolic bimodule over \( \mathcal{A} \), and written as \( (\phi_\rho, \mathcal{H}_\mathcal{A}, \{u_\alpha\}_{\alpha \in \Sigma}) \). By using general construction of \( C^* \)-algebras from Hilbert \( C^* \)-bimodules established by M. Pimsner [Pim] (cf. [Kal]), the author has introduced a \( C^* \)-algebra denoted by \( \mathcal{A} \rtimes_\rho \Lambda \) from the Hilbert \( C^* \)-symbolic bimodule \( (\phi_\rho, \mathcal{H}_\mathcal{A}, \{u_\alpha\}_{\alpha \in \Sigma}) \), where \( \Lambda \) is the subshift \( \Lambda_\rho \) associated with \( (\mathcal{A}, \rho, \Sigma) \). We call the algebra \( \mathcal{A} \rtimes_\rho \Lambda \) the \( C^* \)-symbolic crossed product of \( \mathcal{A} \) by the subshift \( \Lambda \). If \( \mathcal{A} = \mathbb{C} \), the subshift \( \Lambda \) is the full shift \( \Sigma^2 \), and the \( C^* \)-algebra \( \mathcal{A} \rtimes_\rho \Lambda \) is the Cuntz algebra \( \mathcal{O}_{|\Sigma|} \) of order \( |\Sigma| \). If \( \mathcal{A} = C(X) \) with \( \dim X = 0 \), there uniquely exists a \( \lambda \)-graph system \( \mathcal{L} \) up to equivalence such that the subshift \( \Lambda \) is presented by \( \mathcal{L} \) and the \( C^* \)-algebra \( \mathcal{A} \rtimes_\rho \Lambda \) is the \( C^* \)-algebra \( \mathcal{O}_\mathcal{L} \) associated with the \( \lambda \)-graph system \( \mathcal{L} \) Conversely, for any subshift, that is presented by a \( \lambda \)-graph system \( \mathcal{L} \), there exists a \( C^* \)-symbolic dynamical system \( (\mathcal{A}, \rho, \Sigma) \) such that \( \Lambda_\rho \) is the subshift presented by \( \mathcal{L} \), the algebra \( \mathcal{A} \) is \( C(\Omega_\mathcal{L}) \) with \( \dim \Omega_\mathcal{L} = 0 \), and the algebra \( \mathcal{A} \rtimes_\rho \Lambda \) is the \( C^* \)-algebra \( \mathcal{O}_\mathcal{L} \) associated with \( \mathcal{L} \) ([Ma6]). If in particular, \( \mathcal{A} = \mathbb{C}^n \), the subshift \( \Lambda \) is a sofic shift and \( \mathcal{A} \rtimes_\rho \Lambda \) is a Cuntz-Krieger algebra.

In this paper, a condition called (I) on \( (\mathcal{A}, \rho, \Sigma) \) is introduced as a generalization of condition (I) on the finite matrices of Cuntz-Krieger [CK] and on the \( \lambda \)-graph systems [Ma2]. Under the assumption that \( (\mathcal{A}, \rho, \Sigma) \) satisfies condition (I), the simplicity conditions of the algebra \( \mathcal{A} \rtimes_\rho \Lambda \) is discussed in Section 3. We further study ideal structure of \( \mathcal{A} \rtimes_\rho \Lambda \) from the view point of quotients of the \( C^* \)-symbolic dynamical systems in Section 4. Related discussions have been studied in Kajiwara-Pinzari-Watatani’s paper [KPW] for the \( C^* \)-algebras of Hilbert \( C^* \)-bimodules (cf. [Kat], [MS], [Tom], etc.). They have studied simplicity condition and ideal structure of the \( C^* \)-algebras of Hilbert \( C^* \)-bimodules in terms of the language of the Hilbert \( C^* \)-bimodules. Our approach to study the algebras \( \mathcal{A} \rtimes_\rho \lambda \) is from the view point of \( C^* \)-symbolic dynamical systems, that is different from theirs. In Section 5, we will study pure infiniteness of the algebras \( \mathcal{A} \rtimes_\rho \Lambda \). To obtain rich examples of the algebras \( \mathcal{A} \rtimes_\rho \Lambda \), we will in Section 6 construct \( C^* \)-symbolic dynamical systems from a finite family of automorphisms \( \alpha_i \in \text{Aut}(\mathcal{B}), i = 1, \ldots, N \) on a unital \( C^* \)-algebra \( \mathcal{B} \) and a \( C^* \)-symbolic dynamical systems \( (\mathcal{A}, \rho, \Sigma) \) with \( \Sigma = \{\alpha_1, \ldots, \alpha_N\} \). The \( C^* \)-symbolic dynamical system is denoted by \( (\mathcal{B} \otimes \mathcal{A}, \rho^{\otimes \Sigma}, \Sigma) \) that is the tensor product between two \( C^* \)-symbolic dynamical systems \( (\mathcal{B}, \alpha, \Sigma) \) and \( (\mathcal{A}, \rho, \Sigma) \). As examples of \( C^* \)-symbolic crossed products, continuous analogue of Cuntz-Krieger algebras called irrational rotation Cuntz-Krieger algebras denoted by \( \mathcal{O}_{\mathcal{G}, \theta_1, \ldots, \theta_N} \) and irrational rotation Cuntz algebras denoted by \( \mathcal{O}_{\theta_1, \ldots, \theta_N} \) are studied in Sections 8 and 9. They belongs to the class of the \( C^* \)-algebras of continuous graphs by V. Deaconu ([De],[De2]). The fixed point algebras \( \mathcal{F}_{\mathcal{G}, \theta_1, \ldots, \theta_N} \) of \( \mathcal{O}_{\theta_1, \ldots, \theta_N} \) under gauge actions are no longer AF-algebras. They are AT-algebras. In particular, the fixed point algebras \( \mathcal{F}_{\theta_1, \ldots, \theta_N} \) of \( \mathcal{O}_{\theta_1, \ldots, \theta_N} \) under gauge actions are simple AT-algebras of real rank zero with unique tracial state if and only if difference of rotation angles \( \theta_i - \theta_j \) is irrational for some \( i, j = 1, \ldots, N \) (Theorem 9.4).

Throughout this paper, we denote by \( \mathbb{Z}_+ \) and by \( \mathbb{N} \) the set of nonnegative integers and the set of positive integers respectively. A homomorphism and an isomorphism between \( C^* \)-algebras mean a \( * \)-homomorphism and a \( * \)-isomorphism respectively.
An ideal of a $C^*$-algebra means a closed two sided $*$-ideal.

2. $C^*$-symbolic dynamical systems and their crossed products

Let $\mathcal{A}$ be a unital $C^*$-algebra. In what follows, an endomorphism of $\mathcal{A}$ means a $*$-endomorphism of $\mathcal{A}$ that does not necessarily preserve the unit $1_\mathcal{A}$ of $\mathcal{A}$. The unit $1_\mathcal{A}$ is denoted by $1$ unless we specify. We denote by $\text{End}(\mathcal{A})$ the set of all endomorphisms of $\mathcal{A}$. Let $\Sigma$ be a finite set. A finite family of endomorphisms $\rho_\alpha \in \text{End}(\mathcal{A}), \alpha \in \Sigma$ is said to be essential if $\rho_\alpha(1) \neq 0$ for all $\alpha \in \Sigma$ and the closed ideal generated by $\rho_\alpha(1), \alpha \in \Sigma$ coincides with $\mathcal{A}$. It is said to be faithful if for any nonzero $x \in \mathcal{A}$ there exists a symbol $\alpha \in \Sigma$ such that $\rho_\alpha(x) \neq 0$. We note that $\{\rho_\alpha\}_{\alpha \in \Sigma}$ is faithful if and only if the homomorphism $\xi_\rho : a \in \mathcal{A} \rightarrow [\rho_\alpha(a)]_{\alpha \in \Sigma} \in \oplus_{\alpha \in \Sigma}\mathcal{A}$ is injective.

Definition ([Ma6]). A $C^*$-symbolic dynamical system is a triplet $(\mathcal{A}, \rho, \Sigma)$ consisting of a unital $C^*$-algebra $\mathcal{A}$ and an essential and faithful finite family of endomorphisms $\rho_\alpha$ of $\mathcal{A}$ indexed by $\alpha \in \Sigma$.

Two $C^*$-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A'}, \rho', \Sigma')$ are said to be isomorphic if there exist an isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{A'}$ and a bijection $\pi : \Sigma \rightarrow \Sigma'$ such that $\Phi \circ \rho_\alpha = \rho'_{\pi(\alpha)} \circ \Phi$ for all $\alpha \in \Sigma$. A $C^*$-symbolical dynamical system $(\mathcal{A}, \rho, \Sigma)$ yields a subshift $\Lambda_{(\mathcal{A}, \rho, \Sigma)}$ over $\Sigma$ such that a word $\alpha_1 \cdots \alpha_k$ of $\Sigma$ is admissible for $\Lambda_{(\mathcal{A}, \rho, \Sigma)}$ if and only if $(\rho_{\alpha_1} \circ \cdots \circ \rho_{\alpha_k})(1) \neq 0$ ([Ma6; Proposition 2.1]). The subshift $\Lambda_{(\mathcal{A}, \rho, \Sigma)}$ will be denoted by $\Lambda_\rho$ or simply by $\Lambda$ in this paper.

Let $\mathcal{G} = (G, \lambda)$ be a left-resolving finite labeled graph with underlying finite directed graph $G = (V, E)$ and labeling map $\lambda : E \rightarrow \Sigma$ (see [LM; p.76]). Denote by $v_1, \ldots, v_N$ the vertex set $V$. Assume that every vertex has both an incoming edge and an outgoing edge. Consider the $N$-dimensional commutative $C^*$-algebra $\mathcal{A}_\mathcal{G} = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_N$ where each minimal projection $E_i$ corresponds to the vertex $v_i$ for $i = 1, \ldots, N$. Define an $N \times N$-matrix for $\alpha \in \Sigma$ by

\begin{equation}
A^\mathcal{G}(i, \alpha, j) = \begin{cases} 
1 & \text{if there exists an edge } e \text{ from } v_i \text{ to } v_j \text{ with } \lambda(e) = \alpha, \\
0 & \text{otherwise}
\end{cases}
\end{equation}

for $i, j = 1, \ldots, N$. We set $\rho^\mathcal{G}_\alpha(E_i) = \sum_{j=1}^N A^\mathcal{G}(i, \alpha, j)E_j$ for $i = 1, \ldots, N, \alpha \in \Sigma$. Then $\rho^\mathcal{G}_\alpha, \alpha \in \Sigma$ define endomorphisms of $\mathcal{A}_\mathcal{G}$ such that $(\mathcal{A}_\mathcal{G}, \rho^\mathcal{G}, \Sigma)$ is a $C^*$-symbolical dynamical system such that the algebra $\mathcal{A}_\mathcal{G}$ is $\mathbb{C}^N$, and the subshift $\Lambda_{\rho^\mathcal{G}}$ is the sofic shift presented by $\mathcal{G}$. Conversely, for a $C^*$-symbolical dynamical system $(\mathcal{A}, \rho, \Sigma)$, if $\mathcal{A}$ is $\mathbb{C}^N$, there exists a left-resolving labeled graph $\mathcal{G}$ such that $\mathcal{A} = \mathcal{A}_\mathcal{G}$ and $\Lambda_\rho = \Lambda_{\rho^\mathcal{G}}$ the sofic shift presented by $\mathcal{G}$ ([Ma6; Proposition 2.2]).

More generally let $\mathcal{G}$ be a $\lambda$-graph system $(V, E, \lambda, \iota)$ over $\Sigma$ (see [Ma]). Its vertex set $V$ is $\bigcup_{l=0}^{\infty} V_l$. We equip $V_l$ with discrete topology. We denote by $\Omega_\mathcal{G}$ the compact Hausdorff space with dim $\Omega_\mathcal{G} = 0$ of the projective limit $V_0 \leftarrow V_1 \leftarrow V_2 \leftarrow \cdots$, as in [Ma2;Section 2]. The algebra $C(V_\iota)$ of all continuous functions on $V_\iota$, denoted by $\mathcal{A}_{\lambda, \iota}$, is the direct sum $\mathcal{A}_{\lambda, \iota} = \mathbb{C}E_1^\iota \oplus \cdots \oplus \mathbb{C}E_m(\iota)$ where each minimal projection $E_i^\iota$ corresponds to the vertex $v_i^\iota$ for $i = 1, \ldots, m(\iota)$. Let $\mathcal{A}_\mathcal{G}$ be the commutative $C^*$-algebra $C(\Omega_\mathcal{G}) = \lim_{l \rightarrow \infty} \{\iota_* : \mathcal{A}_{\lambda, l} \rightarrow \mathcal{A}_{\lambda, l+1}\}$. Let $A_{l, l+1}, l \in \mathbb{Z}_+$ be the matrices defined in [Ma2; Theorem A]. For a symbol $\alpha \in \Sigma$ we set

\begin{equation}
\rho^\mathcal{G}_\alpha(E_i^\iota) = \sum_{j=1}^{m(l)+1} A_{l, l+1}(i, \alpha, j)E_j^{l+1} \quad \text{for } i = 1, 2, \ldots, m(l),
\end{equation}

for $l \geq 1$. Let $A_{l, l+1}$ be the matrices defined in [Ma2; Theorem A]. For a symbol $\alpha \in \Sigma$ we set

\begin{equation}
\rho^\mathcal{G}_\alpha(E_i^\iota) = \sum_{j=1}^{m(l)+1} A_{l, l+1}(i, \alpha, j)E_j^{l+1} \quad \text{for } i = 1, 2, \ldots, m(l),
\end{equation}

for $l \geq 1$. Let $A_{l, l+1}$ be the matrices defined in [Ma2; Theorem A]. For a symbol $\alpha \in \Sigma$ we set
so that $\rho^e_\alpha$ defines an endomorphism of $A_\Sigma$. We have a $C^*$-symbolic dynamical system $(A_\Sigma, \rho^e, \Sigma)$ such that the $C^*$-algebra $A_\Sigma$ is $C(\Omega_\Sigma)$ with $\dim \Omega_\Sigma = 0$, and the subshift $\Lambda^e_\rho$ coincides with the subshift $\Lambda_\Sigma$ presented by $\mathcal{L}$. Conversely, for a $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$, if the algebra $A$ is $C(X)$ with $\dim X = 0$, there exists a $\lambda$-graph system $\mathcal{L}$ over $\Sigma$ such that the associated $C^*$-symbolic dynamical system $(A_\Sigma, \rho^e, \Sigma)$ is isomorphic to $(A, \rho, \Sigma)$ ([Ma6; Theorem 2.4]).

Let $\mathcal{L}$ and $\mathcal{L}'$ be predecessor-separated $\lambda$-graph systems over $\Sigma$ and $\Sigma'$ respectively. Then $(A_\Sigma, \rho^e, \Sigma)$ is isomorphic to $(A_{\Sigma'}, \rho^{e'}, \Sigma')$ if and only if $\mathcal{L}$ and $\mathcal{L}'$ are equivalent. In this case, the presented subshifts $\Lambda_\Sigma$ and $\Lambda_{\Sigma'}$ are identified through a symbolic conjugacy. Hence the equivalence classes of the $\lambda$-graph systems are identified with the isomorphism classes of the $C^*$-symbolic dynamical systems of the commutative AF-algebras.

We say that a subshift $\Lambda$ acts on a $C^*$-algebra $A$ if there exists a $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$ such that the associated subshift $\Lambda_\rho$ is $\Lambda$. For a $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$, we have a Hilbert $C^*$-bimodule $(\phi, \mathcal{H}_A, \{u_\alpha\}_{\alpha \in \Sigma})$ called a Hilbert $C^*$-symbolic bimodule ([Ma6]). We then have a $C^*$-algebra by using the Pimsner’s general construction of $C^*$-algebras from Hilbert $C^*$-bimodules [Pin] (cf. [Ka], see also [KPW], [KW], [Kat], [MS], [PWY], [Sch] etc.). We denote the $C^*$-algebra by $A \rtimes_\rho \Lambda$, where $\Lambda$ is the subshift $\Lambda_\rho$ associated with $(A, \rho, \Sigma)$. We call the algebra $A \rtimes_\rho \Lambda$ the $C^*$-symbolic crossed product of $A$ by the subshift $\Lambda$.

**Proposition 2.1 ([Ma6; Proposition 4.1]).** The $C^*$-symbolic crossed product $A \rtimes_\rho \Lambda$ is the universal $C^*$-algebra $C^*(x, S_\alpha; x \in A, \alpha \in \Sigma)$ generated by $x \in A$ and partial isometries $S_\alpha, \alpha \in \Sigma$ subject to the following relations called $(\rho)$:

$$
\sum_{\beta \in \Sigma} S_\beta S^*_\beta = 1, \quad S^*_\alpha x S_\alpha = \rho_\alpha(x), \quad x S_\alpha S^*_\alpha x = S_\alpha S^*_\alpha x
$$

for all $x \in A$ and $\alpha \in \Sigma$. Furthermore for $\alpha_1, \ldots, \alpha_k \in \Sigma$, a word $(\alpha_1, \ldots, \alpha_k)$ is admissible for the subshift $\Lambda$ if and only if $S_{\alpha_1} \cdots S_{\alpha_k} \neq 0$.

Assume that $A$ is commutative. Then we know ([Ma6; Theorem 4.2])

(i) If $A = \mathbb{C}$, the subshift $\Lambda$ is the full shift $\Sigma^\mathbb{Z}$, and the $C^*$-algebra $A \rtimes_\rho \Lambda$ is the Cuntz algebra $O_{|\Sigma|}$ of order $|\Sigma|$.

(ii) If $A = \mathbb{C}^N$ for some $N \in \mathbb{N}$, the subshift $\Lambda$ is a sofic shift $\Lambda_\mathcal{G}$ presented by a left-resolving labeled graph $\mathcal{G}$, and the $C^*$-algebra $A \rtimes_\rho \Lambda$ is a Cuntz-Krieger algebra $O_\mathcal{G}$ associated with the labeled graph. Conversely, for any sofic shift $\Lambda_\mathcal{G}$, that is presented by a left-resolving labeled graph $\mathcal{G}$, there exists a $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$ such that the associated subshift is the sofic shift $\Lambda_\mathcal{G}$, the algebra $A$ is $\mathbb{C}^N$ for some $N \in \mathbb{N}$, and the $C^*$-algebra $A \rtimes_\rho \Lambda$ is the Cuntz-Krieger algebra $O_\mathcal{G}$ associated with the labeled graph $\mathcal{G}$.

(iii) If $A = C(X)$ with $\dim X = 0$, there uniquely exists a $\lambda$-graph system $\mathcal{L}$ up to equivalence such that the subshift $\Lambda$ is presented by $\mathcal{L}$ and the $C^*$-algebra $A \rtimes_\rho \Lambda$ is the $C^*$-algebra $O_\mathcal{L}$ associated with the $\lambda$-graph system $\mathcal{L}$. Conversely, for any subshift $\Lambda_\mathcal{L}$, that is presented by a left-resolving $\lambda$-graph system $\mathcal{L}$, there exists a $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$ such that the associated subshift is the subshift $\Lambda_\mathcal{L}$, the algebra $A$ is $C(\Omega_\mathcal{L})$ with $\dim \Omega_\mathcal{L} = 0$, and the $C^*$-algebra $A \rtimes_\rho \Lambda$ is the $C^*$-algebra $O_\mathcal{L}$ associated with the $\lambda$-graph system $\mathcal{L}$.
3. Condition (I) for $C^*$-symbolic dynamical systems

The notion of condition (I) for finite square matrices with entries in $\{0,1\}$ has been introduced in [CK]. The condition gives rise to the uniqueness of the associated Cuntz-Krieger algebras under the canonical relations of the generating partial isometries. The condition has been generalized by many authors to corresponding conditions for generalizations of the Cuntz-Krieger algebras, for instance, infinite directed graphs ([KPRR]), infinite matrices with entries in $\{0,1\}$ ([EL]), Hilbert $C^*$-bimodules ([KPW]), etc. (see also [Re], [Ka2],[Tom2], etc.). The condition (I) for $\lambda$-graph systems has been also defined in [Ma2] to prove the uniqueness of the $C^*$-algebra $O_\Sigma$ under the canonical relations of generators. In this section, we will introduce the notion of condition (I) for $C^*$-symbolic dynamical systems to prove the uniqueness of the $C^*$-algebras $A \rtimes_\rho \Lambda$ under the relation $(\rho)$. In [KPW], a condition called (I)-free has been introduced. The condition is similar to our condition (I). The discussions given in [KPW] is also similar ones to ours in this section. We will give complete descriptions in our discussions for the sake of completeness. Throughout this paper, for a subset $F$ of a $C^*$-algebra $B$, we denote by $C^*(F)$ the $C^*$-subalgebra of $B$ generated by $F$.

In what follows, $(A, \rho, \Sigma)$ denotes a $C^*$-symbolic dynamical system and $\Lambda$ the associated subshift $\Lambda_\rho$. We denote by $\Lambda^k$ the set of admissible words $\mu$ of $\Lambda$ with length $|\mu|=k$. Put $\Lambda^* = \bigcup_{k=0}^\infty \Lambda^k$, where $\Lambda^0$ denotes the empty word. Let $S_\alpha, \alpha \in \Sigma$ be the partial isometries in $A \rtimes_\rho \Lambda$ satisfying the relation $(\rho)$ in Proposition 2.1. For $\mu = (\mu_1, \ldots, \mu_k) \in \Lambda^k$, we put $S_\mu = S_{\mu_1} \cdots S_{\mu_k}$ and $\rho_\mu = \rho_{\mu_k} \circ \cdots \circ \rho_{\mu_1}$. In the algebra $A \rtimes_\rho \Lambda$, we set

$$F_\rho = C^*(S_\mu x S^*_\nu : \mu, \nu \in \Lambda^*, |\mu| = |\nu|, x \in A),$$
$$F^k_\rho = C^*(S_\mu x S^*_\nu : \mu, \nu \in \Lambda^k, x \in A), \text{ for } k \in \mathbb{Z}_+ \quad \text{and}$$
$$D_\rho = C^*(S_\mu x S^*_\mu : \mu, \nu \in \Lambda^*, x \in A).$$

The identity $S_\mu x S^*_\nu = \sum_{\alpha \in \Sigma} S_{\mu \alpha} \rho_\alpha(x) S^*_\alpha$ for $x \in A$ and $\mu, \nu \in \Lambda^k$ holds so that the algebra $F^k_\rho$ is embedded into the algebra $F^{k+1}_\rho$ such that $\bigcup_{k \in \mathbb{Z}_+} F^k_\rho$ is dense in $F_\rho$. The gauge action $\hat{\rho}$ of the circle group $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ on $A \rtimes_\rho \Lambda$ is defined by $\hat{\rho}_z(x) = x$ for $x \in A$ and $\hat{\rho}_z(S_\alpha) = zS_\alpha$ for $\alpha \in \Sigma$. The fixed point algebra of $A \rtimes_\rho \Lambda$ under $\hat{\rho}$ is denoted by $(A \rtimes_\rho \Lambda)^{\hat{\rho}}$. Let $\mathcal{E}_\rho : A \rtimes_\rho \Lambda \longrightarrow (A \rtimes_\rho \Lambda)^{\hat{\rho}}$ be the conditional expectatation defined by

$$\mathcal{E}_\rho(X) = \int_{z \in \mathbb{T}} \hat{\rho}_z(X)dz, \quad X \in A \rtimes_\rho \Lambda.$$

It is routine to check that $(A \rtimes_\rho \Lambda)^{\hat{\rho}} = F_\rho$.

Let $B$ be a unital $C^*$-algebra. Suppose that there exist an injective homomorphism $\pi : A \longrightarrow B$ preserving their units and a family $s_\alpha \in B, \alpha \in \Sigma$ of partial isometries satisfying

$$\sum_{\beta \in \Sigma} s_\beta s^*_\beta = 1, \quad s^*_\alpha \pi(x) s_\alpha = \pi(\rho_\alpha(x)), \quad \pi(x) s_\alpha s^*_\alpha = s_\alpha s^*_\alpha \pi(x)$$

for all $x \in A$ and $\alpha \in \Sigma$. Put $\tilde{A} = \pi(A) \subset B$ and $\tilde{\rho}_\alpha(\pi(x)) = \pi(\rho_\alpha(x)), x \in A$. We then have
Lemma 3.1. The triple $(\tilde{A}, \tilde{\rho}, \Sigma)$ is a $C^*$-symbolic dynamical system such that the presented subshift $\Lambda_{\tilde{\rho}}$ is the same as the one $\Lambda(= \Lambda_{\rho})$ presented by $(A, \rho, \Sigma)$.

Let $O_{\pi,s}$ be the $C^*$-subalgebra of $B$ generated by $\pi(x)$ and $s_\alpha$ for $x \in A, \alpha \in \Sigma$. In the algebra $O_{\pi,s}$, we set
\[
\mathcal{F}_{\pi,s} = C^*(s_\mu \pi(x)s_\nu^* : \mu, \nu \in \Lambda^*, |\mu| = |\nu|, x \in A),
\]
\[
\mathcal{F}_{\pi,s}^k = C^*(s_\mu \pi(x)s_\nu^* : \mu, \nu \in \Lambda^k, x \in A) \text{ for } k \in \mathbb{Z}_+ \text{ and}
\]
\[
D_{\pi,s} = C^*(s_\mu \pi(x)s_\mu^* : \mu \in \Lambda^*, x \in A).
\]

By the universality of the algebra $A \rtimes_{\rho} \Lambda$, the correspondence
\[
x \in A \rightarrow \pi(x) \in \tilde{A}, \quad S_\alpha \rightarrow s_\alpha, \quad \alpha \in \Sigma
\]
extends to a surjective homomorphism $\tilde{\pi} : A \rtimes_{\rho} \Lambda \rightarrow O_{\pi,s}$.

Lemma 3.2. The restriction of $\tilde{\pi}$ to the subalgebra $F_\rho$ is an isomorphism from $F_\rho$ to $F_{\pi,s}$.

Proof. It suffices to show that $\tilde{\pi}$ is injective on $F_{\rho}^k$. Suppose that $\sum_{\mu, \nu \in \Lambda^k} s_\mu \pi(x, \nu)s_\nu^* = 0$ for $\sum_{\mu, \nu \in A} S_\mu x_\mu, \nu S_\nu^* \in F_\rho$ with $x_\mu, \nu \in A$. For $\xi, \eta \in \Lambda^k$, it follows that
\[
\pi(\rho(1)x_\xi, \eta, \rho(1)) = s_\xi^*(\sum_{\mu, \nu \in \Lambda^k} s_\mu \pi(x_\mu, \nu)s_\nu^*)s_\eta = 0.
\]
As $\pi : A \rightarrow B$ is injective, one has $\rho(1)x_\xi, \eta, \rho(1) = 0$ so that $S_\xi x_\xi, \eta S_\eta^* = 0$. This implies that $\sum_{\mu, \nu \in \Lambda^k} S_\mu x_\mu, \nu S_\nu^* = 0$. \(\square\)

Definition. A $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$ satisfies condition (I) if there exists a unital increasing sequence
\[
A_0 \subset A_1 \subset \cdots \subset A
\]
of $C^*$-subalgebras of $A$ such that $\rho(\alpha)(A_1) \subset A_{l+1}$ for all $l \in \mathbb{Z}_+, \alpha \in \Sigma$ and the union $\cup_{l \in \mathbb{Z}_+} A_l$ is dense in $A$ and for $k, l \in \mathbb{N}$ with $k \leq l$, there exists a projection $q^l_k \in D_\rho \cap A_l^l(= \{ x \in D_\rho \mid x a = ax \} \text{ for } a \in A_l)$ such that
\begin{enumerate}
  \item $q^l_k a \neq 0$ for all nonzero $a \in A_l$,
  \item $q^l_k \phi^m(y^l_k) = 0$ for all $m = 1, 2, \ldots, k$, where $\phi^m(\lambda) = \sum_{\mu \in \Lambda^m} S_\mu X S_\mu^*$.
\end{enumerate}
As the projection $q^l_k$ belongs to the diagonal subalgebra $D_\rho$ of $F_\rho$, the condition (I) of $(A, \rho, \Sigma)$ is intrinsically determined by $(A, \rho, \Sigma)$ by virtue of Lemma 3.2.

If a $\lambda$-graph system $\Sigma$ over $\Sigma$ satisfies condition (I), then $(A_2, \rho^2, \Sigma)$ satisfies condition (I) (cf. [Ma2; lemma 4.1]).

We now assume that $(A, \rho, \Sigma)$ satisfies condition (I). We set for $k \leq l$
\[
F_{\rho, l}^k = C^*(S_\mu x S_\nu^* : \mu, \nu \in \Lambda^k, x \in A_l).
\]
There exists an inclusion relation $F_{\rho, l}^k \subset F_{\rho, l'}^{k'}$ for $k \leq k'$ and $l \leq l'$. We put a projection $Q^l_k = \phi^l_k(q^l_k)$ in $D_\rho$.\[6\]
Lemma 3.3. The map \( X \in \mathcal{F}_\rho^k \rightarrow Q_k^l X Q_k^l \in Q_k^l \mathcal{F}_\rho^k Q_k^l \) is a surjective isomorphism.

Proof. As \( q_k^l \) commutes with \( A_l \), for \( x \in A_l \) and \( \mu, \nu \in \Lambda^k \), we have

\[
Q_k^l S_\mu x_s S_\nu^* = \sum_{\xi \in \Lambda^k} S_\xi q_k^l S_\xi^* S_\mu x_s S_\nu^* = S_\mu q_k^l S_\mu^* S_\mu x_s S_\nu^* = S_\mu x_q S_\nu^*,
\]

and similarly \( S_\mu x_s^* Q_k^l = S_\mu x_q^* S_\nu^* \) so that \( Q_k^l \) commutes with \( S_\mu x_s^* \). Hence the map \( X \in \mathcal{F}_\rho^k \rightarrow Q_k^l X Q_k^l \in Q_k^l \mathcal{F}_\rho^k Q_k^l \) defines a surjective homomorphism. It remains to show that it is injective. Suppose that \( Q_k^l (\sum_{\mu, \nu \in \Lambda^k} S_\mu x_\mu S_\nu^*) Q_k^l = 0 \) for \( X = \sum_{\mu, \nu \in \Lambda^k} S_\mu x_\mu S_\nu^* \) with \( x_\mu, x_\nu \in A_l \). For \( \xi, \eta \in \Lambda^k \), one has

\[
0 = S_\xi Q_k^l (\sum_{\mu, \nu \in \Lambda^k} S_\mu x_\mu S_\nu^*) Q_k^l S_\eta = Q_k^l S_\xi x_\xi S_\eta^*,
\]

so that \( 0 = S_\xi^* Q_k^l S_\xi x_\xi S_\eta S_\eta^* = S_\xi^* S_\xi^* Q_k^l S_\xi x_\xi S_\eta S_\eta^* = q_k^l \rho_\xi(1) x_\xi S_\eta S_\eta^* = q_k^l \rho_\xi(1) x_\xi \eta S_\eta^* \). Hence \( \rho_\xi(1) x_\xi \eta S_\eta^* = 0 \) by condition (I). Thus \( S_\xi x_\xi S_\eta^* = 0 \), so that \( \sum_{\xi, \eta \in \Lambda^k} S_\xi x_\xi S_\eta^* = 0 \). \( \square \)

Lemma 3.4. \( Q_k^l S_\mu Q_k^l = 0 \) for \( \mu \in \Lambda^* \) with \(|\mu| \leq k \leq l \).

Proof. By condition (I), we have \( Q_k^l \phi_\rho^m(Q_k^l) = 0 \) for \( 1 \leq m \leq k \). For \( \mu \in \Lambda^* \) with \(|\mu| \leq k \), one has \( \phi_\rho^{|\mu|}(Q_k^l) S_\mu = S_\mu Q_k^l S_\mu^* S_\mu = S_\mu Q_k^l \). Hence we have \( 0 = Q_k^l \phi_\rho^{|\mu|}(Q_k^l) S_\mu = Q_k^l S_\mu Q_k^l \). \( \square \)

As a result, we have

Lemma 3.5. The projections \( Q_k^l \) in \( D_\rho \) satisfy the following conditions:

(a) \( Q_k^l F \rightarrow Q_k^l \) converges to 0 as \( k, l \rightarrow \infty \) for \( F \in \mathcal{F}_\rho \).
(b) \( \|Q_k^l F\| \rightarrow \|F\| \) as \( k, l \rightarrow \infty \) for \( F \in \mathcal{F}_\rho \).
(c) \( Q_k^l S_\mu Q_k^l = 0 \) for \( \mu \in \Lambda^* \) with \(|\mu| \leq k \leq l \).

We note that \( Q_k^l S_\mu Q_k^l = 0 \) if and only if \( Q_k^l S_\mu Q_k^l S_\mu^* = 0 \). Since \( Q_k^l S_\mu Q_k^l S_\mu^* \) belongs to the algebra \( \mathcal{F}_\rho \), the condition \( Q_k^l S_\mu Q_k^l = 0 \) is determined in the algebraic structure of \( \mathcal{F}_\rho \). As the restriction of \( \tilde{\pi} : A \rtimes_\rho \Lambda \rightarrow O_{\pi, s} \) to \( \mathcal{F}_\rho \) yields an isomorphism onto \( \mathcal{F}_{\pi, s} \), by putting \( \tilde{Q}_k = \tilde{\pi}(Q_k) \) we have

Lemma 3.6. The projections \( \tilde{Q}_k \) in \( D_{\pi, s} \) satisfy the following conditions:

(a') \( \tilde{Q}_k F \rightarrow \tilde{Q}_k \) converges to 0 as \( k, l \rightarrow \infty \) for \( F \in \mathcal{F}_{\pi, s} \).
(b') \( \|\tilde{Q}_k F\| \rightarrow \|F\| \) as \( k, l \rightarrow \infty \) for \( F \in \mathcal{F}_{\pi, s} \).
(c') \( \tilde{Q}_k S_\mu \tilde{Q}_k = 0 \) for \( \mu \in \Lambda^* \) with \(|\mu| \leq k \leq l \).

Proposition 3.7. There exists a conditional expectation \( E_{\pi, s} : O_{\pi, s} \rightarrow \mathcal{F}_{\pi, s} \) such that \( E_{\pi, s} \circ \tilde{\pi} = \tilde{\pi} \circ E_\rho \).

Proof. Let \( P_{\pi, s} \) be the \( \ast \)-subalgebra of \( O_{\pi, s} \) generated algebraically by \( \pi(x), s_\alpha \) for \( x \in A, \alpha \in \Sigma \). Then any \( X \in P_{\pi, s} \) can be written as a finite sum

\[
X = \sum_{|\nu| \geq 1} X_{-\nu} S_\nu^* + X_0 + \sum_{|\mu| \geq 1} s_\mu X_\mu \quad \text{for some } X_{-\nu}, X_0, X_\mu \in \mathcal{F}_{\pi, s}.
\]
Thanks to the previous lemma and a usual argument of [CK], the element $X_0 \in F_{\pi,s}$ is unique for $X \in P_{\pi,s}$ and the inequality $\|X_0\| \leq \|X\|$ holds. The map $X \in P_{\pi,s} \rightarrow X_0 \in F_{\pi,s}$ can be extended to the desired expectation $\mathcal{E}_{\pi,s} : O_{\pi,s} \rightarrow F_{\pi,s}$. □

Therefore we have

**Theorem 3.8.** Assume that $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I). The homomorphism $	ilde{\pi} : \mathcal{A} \rtimes_{\rho} \Lambda \rightarrow O_{\pi,s}$ defined by

$$
\tilde{\pi}(x) = \pi(x), \quad x \in \mathcal{A}, \quad \tilde{\pi}(S_\alpha) = s_\alpha, \quad \alpha \in \Sigma.
$$

becomes a surjective isomorphism, and hence the $C^*$-algebras $\mathcal{A} \rtimes_{\rho} \Lambda$ and $O_{\pi,s}$ are canonically isomorphic through $\tilde{\pi}$.

**Proof.** The map $\tilde{\pi} : F_\rho \rightarrow F_{\pi,s}$ is isomorphic and satisfies $\mathcal{E}_{\pi,s} \circ \tilde{\pi} = \tilde{\pi} \circ \mathcal{E}_\rho$. Since $\mathcal{E}_\rho : \mathcal{A} \rtimes_{\rho} \Lambda \rightarrow F_\rho$ is faithful, a routine argument shows that the homomorphism $\tilde{\pi} : \mathcal{A} \rtimes_{\rho} \Lambda \rightarrow O_{\pi,s}$ is actually an isomorphism. □

Hence the following uniqueness of the $C^*$-algebra $\mathcal{A} \rtimes_{\rho} \Lambda$ holds.

**Theorem 3.9.** Assume that $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I). The $C^*$-algebra $\mathcal{A} \rtimes_{\rho} \Lambda$ is the unique $C^*$-algebra subject to the relation $(\rho)$. This means that if there exist a unital $C^*$-algebra $\mathcal{B}$ and an injective homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{B}$ and a family $s_\alpha \in \mathcal{B}, \alpha \in \Sigma$ of nonzero partial isometries satisfying the following relations:

$$
\sum_{\beta \in \Sigma} s_\beta s_\beta^* = 1, \quad s_\alpha^* \pi(x) s_\alpha = \pi(\rho_\alpha(x)), \quad \pi(x) s_\alpha s_\alpha^* = s_\alpha s_\alpha^* \pi(x)
$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma$, then the correspondence

$$
x \in \mathcal{A} \rightarrow \pi(x) \in \mathcal{B}, \quad S_\alpha \rightarrow s_\alpha \in \mathcal{B}
$$

extends to an isomorphism $\tilde{\pi}$ from $\mathcal{A} \rtimes_{\rho} \Lambda$ onto the $C^*$-subalgebra $O_{\pi,s}$ of $\mathcal{B}$ generated by $\pi(x), x \in \mathcal{A}$ and $s_\alpha, \alpha \in \Sigma$.

As a corollary we have

**Corollary 3.10.** Assume that $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I). For any nontrivial ideal $\mathcal{I}$ of $\mathcal{A} \rtimes_{\rho} \Lambda$, one has $\mathcal{I} \cap \mathcal{A} \neq \{0\}$.

**Proof.** Suppose that $\mathcal{I} \cap \mathcal{A} = \{0\}$. Hence $S_\alpha \notin \mathcal{I}$ for all $\alpha \in \Sigma$. By Theorem 3.9, the quotient map $q : \mathcal{A} \rtimes_{\rho} \Lambda \rightarrow \mathcal{A} \rtimes_{\rho} \Lambda / \mathcal{I}$ must be injective so that $\mathcal{I}$ is trivial. □

Let $\lambda_\rho : \mathcal{A} \rightarrow \mathcal{A}$ be the completely positive map on $\mathcal{A}$ defined by $\lambda_\rho(x) = \sum_{\alpha \in \Sigma} \rho_\alpha(x)$ for $x \in \mathcal{A}$.

**Definition.** $(\mathcal{A}, \rho, \Sigma)$ is said to be irreducible if there exists no nontrivial ideal of $\mathcal{A}$ invariant under $\lambda_\rho$.

Therefore we have

**Corollary 3.11.** Assume that $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I). If $(\mathcal{A}, \rho, \Sigma)$ is irreducible, the $C^*$-algebra $\mathcal{A} \rtimes_{\rho} \Lambda$ is simple.
4. Quotients of $C^*$-symbolic dynamical systems

In this section, we will study ideal structure of the $C^*$-symbolic crossed products $\mathcal{A} \rtimes_\rho \Lambda$, related to quotients of $C^*$-symbolic dynamical systems. The ideal structure of $C^*$-algebras of Hilbert $C^*$-bimodules has been studied in Kajiwara, Pinzari and Watatani’s paper [KPW] (cf. [Kat3]). Their paper is written in the language of Hilbert $C^*$-bimodules. In this section we will directly study ideal structure of the $C^*$-symbolic crossed products $\mathcal{A} \rtimes_\rho \Lambda$ by using the language of $C^*$-symbolic dynamical systems. We fix a $C^*$-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$.

An ideal $J$ of $\mathcal{A}$ is said to be $\rho$-invariant if $\rho_\alpha(J) \subset J$ for all $\alpha \in \Sigma$. It is said to be saturated if $\rho_\alpha(x) \in J$ for all $\alpha \in \Sigma$ implies $x \in J$.

**Lemma 4.1.** Let $J$ be an ideal of $\mathcal{A}$.

(i) $J$ is $\rho$-invariant if and only if $\lambda_\rho(J) \subset J$.

(ii) $J$ is saturated if and only if $\lambda_\rho(a) \in J$ for $0 \leq a \in \mathcal{A}$ implies $a \in J$.

**Proof.** (i) Suppose that $J$ satisfies $\lambda_\rho(J) \subset J$. For $x \in J$ one has $\lambda_\rho(x^*x) \geq \rho_\alpha(x^*x) = \rho_\alpha(x)^*\rho_\alpha(x)$ so that $\rho_\alpha(x)^*\rho_\alpha(x) \in J$ because ideal is hereditary. Hence $\rho_\alpha(x)$ belongs to $J$. The only if part is clear.

(ii) Suppose that $J$ is saturated and $\lambda_\rho(a) \in J$ for $0 \leq a \in \mathcal{A}$. Since $\lambda_\rho(a) \geq \rho_\alpha(a)$ and $J$ is hereditary, one has $a \in J$. Conversely suppose that $x \in \mathcal{A}$ satisfies $\rho_\alpha(x) \in J$ for all $\alpha \in \Sigma$. As $\lambda_\rho(x^*x) = \sum_{\alpha \in \Sigma} \rho_\alpha(x)^*\rho_\alpha(x)$, $\rho_\alpha(x^*x)$ belongs to $J$. Hence the condition of the if part implies that $x^*x \in J$ so that $x \in J$. \hfill \Box

Let $J$ be a $\rho$-invariant saturated ideal of $\mathcal{A}$. We denote by $\mathcal{I}_J$ the ideal of $\mathcal{A} \rtimes_\rho \Lambda$ generated by $J$.

**Lemma 4.2.** The ideal $\mathcal{I}_J$ is the closure of linear combinations of elements of the form $S_\mu a_{\mu,\nu}S_\nu^*$ for $a_{\mu,\nu} \in J$.

**Proof.** Elements $x$ and $y$ of $\mathcal{A} \rtimes_\rho \Lambda$ are approximated by finite sums of elements of the form $S_\mu a_{\mu,\nu}S_\nu^*$ and $S_\xi b_{\xi,\eta}S_\eta^*$ for $a_{\mu,\nu}, b_{\xi,\eta} \in \mathcal{A}$ respectively. Hence $xcy$ is approximated by elements of the form

$$\sum_{\mu,\nu} S_\mu a_{\mu,\nu}S_\nu^* \cdot c \cdot \sum_{\xi,\eta} S_\xi b_{\xi,\eta}S_\eta^* = \sum_{\mu,\nu,\xi,\eta} S_\mu a_{\mu,\nu}S_\nu^*cS_\xi b_{\xi,\eta}S_\eta^*.$$  

In case of $|\nu| \geq |\xi|$, one has $\nu = \tilde{\nu}\nu'$ with $|\tilde{\nu}| = |\xi|$ so that

$$S_\nu^*cS_\xi = \begin{cases} S_{\nu'}^*\rho_{\tilde{\nu}}(c) & \text{if } \tilde{\nu} = \xi, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $S_\mu a_{\mu,\nu}S_\nu^*cS_\xi b_{\xi,\eta}S_\eta^*$ is $S_\mu a_{\mu,\nu}\rho_{\nu'}(\rho_{\tilde{\nu}}(c)b_{\xi,\eta})S_{\eta'}^*$ or zero. Since $J$ is $\rho$-invariant, it is of the form $S_\mu c_{\mu,\nu}S_{\nu'}^*$ for some $c_{\mu,\nu} \in J$ or zero. The argument in case of $|\nu| \leq |\xi|$ is similar. Since the ideal $\mathcal{I}_J$ is the closure of elements of the form

$$\sum_{i=1}^n x_i c_i y_i$$

for $x_i, y_i \in \mathcal{A} \rtimes_\rho \Lambda$ and $c_i \in J$, the assertion is proved. \hfill \Box

We set

$$\mathcal{D}_J = C^*(S_\mu c_{\mu}S_{\mu}^* : \mu \in \Lambda^*, c_{\mu} \in J),$$

$$\mathcal{D}_J^k = C^*(S_\mu c_{\mu}S_{\mu}^* : \mu \in \Lambda^*, |\mu| \leq k, c_{\mu} \in J) \quad \text{for } k \in \mathbb{Z}_+. $$
Lemma 4.3.

(i) \(D_J = I_J \cap D_\rho\) and hence \(D_J \cap A = I_J \cap A\).
(ii) \(D_k^J \cap A = J\) for \(k \in \mathbb{Z}_+\).

Proof. (i) Since the elements of the finite sum \(\sum_{\mu} S_\mu c_\mu S_\mu^*\) for \(c_\mu \in J\) are contained in \(I_J \cap D_\rho\), the inclusion relation \(D_J \subset I_J \cap D_\rho\) is clear. Let \(\mathcal{T}_J^{alg}\) and \(D_J^{alg}\) be the algebraic linear spans of \(S_\mu c_\mu, v S_\mu^*\) for \(c_\mu, v \in J\) and \(S_\mu c_\mu S_\mu^*\) for \(c_\mu \in J\) respectively. For any \(x \in I_J \cap D_\rho\) take \(x_n \in \mathcal{T}_J^{alg}\) such that \(\|x_n - x\| \to 0\). Let \(\mathcal{E}_\rho : A \times_\rho \Lambda \to F_\rho\) be the conditional expectation defined previously, and \(\mathcal{E}_D : F_\rho \to D_\rho\) the conditional expectation defined by taking diagonal elements. The composition \(\mathcal{E}_D \circ \mathcal{E}_\rho\) is the conditional expectation from \(A \times_\rho \Lambda\) to \(D_\rho\) that satisfies \(\mathcal{E}_D(\mathcal{T}_J^{alg}) = D_J^{alg}\). Since \(\mathcal{E}_D(x) = x\) and the inequality \(\|x - \mathcal{E}_D(x_n)\| \leq \|x - x_n\| \) holds, \(x\) belongs to the closure \(D_J\) of \(D_J^{alg}\). Hence we have \(I_J \cap D_\rho \subset D_J\) so that \(D_J = I_J \cap D_\rho\). As \(A\) is a subalgebra of \(D_\rho\), the equality \(D_J \cap A = I_J \cap A\) holds.

(ii) An element \(x \in D_k^J\) is of the form \(\sum_{|\mu| \leq k} S_\mu c_\mu S_\mu^*\) for \(c_\mu \in J\). As \(S_\mu c_\mu S_\mu^* = \sum_{\alpha \in \Sigma} S_{\nu\alpha} c_\nu S_{\nu\alpha}^*\) and \(J\) is \(\rho\)-invariant, \(x\) can be written as \(x = \sum_{|\nu| = k} S_\nu c_\nu S_\nu^*\) for \(c_\nu \in J\), and the element \(\lambda_\rho(x) = \sum_{|\nu| = k} c_\nu \rho_\nu(1)\) belongs to \(J\). Further suppose that \(x\) is an element of \(A\). Since \(J\) is saturated, by Lemma 4.1, one has \(x \in J\). Hence the inclusion relation \(A \cap D_k^J \subset J\) holds. The converse inclusion relation is clear so that \(A \cap D_k^J = J\). \(\square\)

Lemma 4.4. \(A \cap D_J = J\).

Proof. Since the inclusion relation \(A \cap D_J \supset J\) is clear, there exists a natural surjective homomorphism from \(A/J\) onto \(A/A \cap D_J\). For an element \(a\) of a \(C^*\)-algebra \(B\), we denote by \(\|[a]_{B/I}\|\) the norm of the quotient image \([a]_{B/I}\) of \(a\) in the quotient \(B/I\) of \(B\) by an ideal \(I\). As the inclusion \(A \hookrightarrow D_\rho\) induces the inclusions both \(A/A \cap D_J \hookrightarrow D_\rho/D_\rho\) and \(A/A \cap D_k^J \hookrightarrow D_\rho/D_k^J\), one has for \(a \in A\)

\[
\|[a]_{A/A \cap D_J}\| = \|[a]_{D_\rho/D_J}\|, \quad \|[a]_{A/A \cap D_k^J}\| = \|[a]_{D_\rho/D_k^J}\|.
\]

Note that \(D_J\) is the inductive limit of \(D_k^J, k = 0, 1, \ldots\). It then follows that

\[
\|[a]_{D_\rho/D_J}\| = \text{dist}(a, D_J) = \lim_{k \to \infty} \text{dist}(a, D_k^J) = \lim_{k \to \infty} \|[a]_{D_\rho/D_k^J}\| = \lim_{k \to \infty} \|[a]_{A/A \cap D_k^J}\|
\]

and hence \(\|[a]_{A/A \cap D_J}\| = \|[a]_{A/J}\|\) by Lemma 4.3 (ii). Thus the quotient map \(A/J \to A/A \cap D_J\) is isometric so that \(A \cap D_J = J\). \(\square\)

By Lemma 4.3 and Lemma 4.4, one has

Proposition 4.5. \(I_J \cap A = J\).

We will now consider quotient \(C^*\)-symbolic dynamical systems. Let \(J\) be a \(\rho\)-invariant saturated ideal of \(A\). We set \(\Sigma_J = \{\alpha \in \Sigma \mid \rho_\alpha(1) \notin J\}\). We denote by \([x]\) the class of \(x \in A\) in the quotient \(A/J\). Put

\[
\rho_\alpha^J([x]) = [\rho_\alpha(x)] \quad \text{for} \ [x] \in A/J, \ \alpha \in \Sigma_J.
\]

As \(J\) is \(\rho\)-invariant and saturated, \(\rho_\alpha^J\) is well-defined and the family \(\{\rho_\alpha^J\}_{\alpha \in \Sigma_J}\) is a faithful and essential endomorphisms of \(A/J\). We call the \(C^*\)-symbolic dynamical
system \((A/J, \rho^J, \Sigma_J)\) the quotient of \((A, \rho, \Sigma)\) by the ideal \(J\). We denote by \(\Lambda_J\) the associated subshift for the quotient \((A/J, \rho^J, \Sigma_J)\).

**Definition.** A \(C^*\)-symbolic dynamical system \((A, \rho, \Sigma)\) is said to satisfy **condition** (II) if for any proper \(\rho\)-invariant saturated ideal \(J\) of \(A\), the quotient \(C^*\)-symbolic dynamical system \((A/J, \rho^J, \Sigma_J)\) satisfies condition (I).

Let \(I\) be a proper ideal of \(A \rtimes_\rho \Lambda\). Put \(J_I := I \cap A\).

**Lemma 4.6.**

(i) If \((A, \rho, \Sigma)\) satisfies condition (I), then \(J_I\) is a proper \(\rho\)-invariant saturated ideal of \(A\). We then have \(J_{I_J} = J\).

(ii) If \((A, \rho, \Sigma)\) satisfies condition (II), then the \(C^*\)-symbolic crossed product \((A/J_I) \rtimes_{\rho^J} \Lambda_{J_I}\) is canonically isomorphic to the quotient algebra \(A \rtimes_\rho \Lambda/I\).

**Proof.** (i) By condition (I), \(J_I\) is a nozero ideal of \(A\), that is \(\rho\)-invariant. If \(\rho_\alpha(x)\) belongs to \(J_I\) for all \(\alpha \in \Sigma\), the identity \(x = \sum_{\alpha \in \Sigma} S_\alpha \rho_\alpha(x) S_\alpha^*\) implies \(x \in I\), so that \(J_I\) is saturated. The equality \(J_{I_J} = J\) follows from Proposition 4.5.

(ii) Let \(\pi_I : A \rtimes_\rho \Lambda \to A \rtimes_\rho \Lambda/I\) be the quotient map. Put \(s_\alpha = \pi_I(S_\alpha)\). Then \(\alpha \in \Sigma_{J_I}\) if and only if \(s_\alpha \neq 0\). The following relations

\[
\sum_{\beta \in \Sigma_{J_I}} s_\beta s_\beta^* = 1, \quad s_\alpha^* \pi_I(x) s_\alpha = \pi_I(\rho_\alpha(x)) \quad \pi_I(x) s_\alpha s_\alpha^* = s_\alpha s_\alpha^* \pi_I(x)
\]

for \(x \in A, \alpha \in \Sigma_{J_I}\) hold. As \((A/J_I, \rho^{J_I}, \Sigma_{J_I})\) satisfies condition (I), the uniqueness of the \(C^*\)-symbolic crossed product \((A/J_I) \rtimes_{\rho^{J_I}} \Lambda_{J_I}\) yields a canonical isomorphism to the quotient algebra \(A \rtimes_\rho \Lambda/I\). □

Let \(I_{J_I}\) be the ideal of \(A \rtimes_\rho \Lambda\) generated by \(J_I\). Since \(J_I \subset I\), the inclusion relation \(I_{J_I} \subset I\) is clear.

**Lemma 4.7.** If \((A, \rho, \Sigma)\) satisfies condition (II), then there exists a canonical isomorphism from \((A/J_I) \rtimes_{\rho^{J_I}} \Lambda_{J_I}\) to the quotient algebra \(A \rtimes_\rho \Lambda/I_{J_I}\).

**Proof.** Take an arbitrary element \(x \in A\). If \(x \in J_I\), then \(x \in I_{J_I}\). Conversely \(x \in I_{J_I}\) implies \(x \in J_I\) by Proposition 4.5. Hence \(x \in J_I\) if and only if \(x \in I_{J_I}\).

For \(\alpha \in \Sigma\), we have \(S_\alpha \in I_{J_I}\) if and only if \(S_\alpha^* S_\alpha \in I_{J_I} \cap A\). By Proposition 4.5, the latter condition is equivalent to the condition \(\rho_\alpha(1) \in J_I\). We know that \(\alpha \notin \Sigma_{J_I}\) if and only if \(S_\alpha \in I_{J_I}\). By the uniqueness of the algebra \((A/J_I) \rtimes_{\rho^{J_I}} \Lambda_{J_I}\), it is canonically isomorphic to the quotient algebra \(A \rtimes_\rho \Lambda/I_{J_I}\). □

**Proposition 4.8.** Suppose that \((A, \rho, \Sigma)\) satisfies condition (II). For a proper ideal \(I\) of \(A \rtimes_\rho \Lambda\), let \(I_{J_I}\) be the ideal of \(A \rtimes_\rho \Lambda\) generated by \(J_I\). Then we have \(I_{J_I} = I\).

**Proof.** Since \(I_{J_I} \subset I\), there exists a quotient map \(q_I : A \rtimes_\rho \Lambda/I_{J_I} \to A \rtimes_\rho \Lambda/I\).

By Lemma 4.6, and Lemma 4.7, there exist canonical isomorphisms

\[
\pi_1 : (A/J_I) \rtimes_{\rho^{J_I}} \Lambda_{J_I} \to A \rtimes_\rho \Lambda/I, \quad \pi_2 : (A/J_I) \rtimes_{\rho^{J_I}} \Lambda_{J_I} \to A \rtimes_\rho \Lambda/I_{J_I}.
\]

Since \(q_I = \pi_1 \circ \pi_2^{-1}\), it is isomorphism so that we have \(I_{J_I} = I\). □

Therefore we have

**Theorem 4.9.** Suppose that \((A, \rho, \Sigma)\) satisfies condition (II). There exists an inclusion preserving bijective correspondence between \(\rho\)-invariant saturated ideals of \(A\) and ideals of \(A \rtimes_\rho \Lambda\), through the correspondences: \(J \to I_J\) and \(J_I \leftarrow I\).
5. Pure infiniteness

In this section we will show that the $C^*$-symbolic crossed product $\mathcal{A} \rtimes_{\rho} \Lambda$ is purely infinite if $(\mathcal{A}, \rho, \Sigma)$ satisfies some conditions.

**Definition.** A $C^*$-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to be *central* if the projections $\{\rho_\mu(1) | \mu \in \Lambda^*\}$ contained in the center $Z_\mathcal{A}$ of $\mathcal{A}$. It is said to be *commutative* if $\mathcal{A}$ is commutative. Hence if $(\mathcal{A}, \rho, \Sigma)$ is central, the inequality $\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$ holds. Let $\mathcal{A}_\rho$ be the $C^*$-subalgebra of $\mathcal{A}$ generated by the projections $\rho_\mu(1), \mu \in \Lambda^*$.

**Lemma 5.1.** Assume that $(\mathcal{A}, \rho, \Sigma)$ is central. Then there exists a $\lambda$-graph system $\mathcal{L}_\rho$ over $\Sigma$ such that the presented subshift $\Lambda_{\mathcal{L}_\rho}$ coincides with the subshift $\Lambda$ presented by $(\mathcal{A}, \rho, \Sigma)$, and there exists a unital embedding of $\mathcal{O}_{\mathcal{L}_\rho}$ into $\mathcal{A} \rtimes_{\rho} \Lambda$.

**Proof.** Put $\mathcal{A}_{\rho,0} = \mathbb{C}$. For $l \in \mathbb{Z}_+$, we define the $C^*$-algebra $\mathcal{A}_{\rho,l+1}$ to be the $C^*$-subalgebra of $\mathcal{A}$ generated by the elements $\rho_\alpha(x)$ for $\alpha \in \Sigma, x \in \mathcal{A}_{\rho,l}$. Hence the $C^*$-algebra $\mathcal{A}_\rho$ is generated by $\bigcup_{l=0}^{\infty} \mathcal{A}_{\rho,l}$. Then $(\mathcal{A}_\rho, \rho, \Sigma)$ is a $C^*$-symbolic dynamical system such that $\mathcal{A}_\rho$ is commutative and AF, so that there exists a $\lambda$-graph system $\mathcal{L}_\rho$ over $\Sigma$ such that $\mathcal{A}_\rho = \mathcal{A}_{\mathcal{L}_\rho}$. The presented subshift $\Lambda_{\mathcal{L}_\rho}$ coincides with the subshift $\Lambda$. It is easy to see that there exists a unital embedding of $\mathcal{O}_{\mathcal{L}_\rho}$ into $\mathcal{A} \rtimes_{\rho} \Lambda$ by their universalities. □

In the rest of this section we assume that $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I).

**Definition.** $(\mathcal{A}, \rho, \Sigma)$ is said to be *effective* if for $l \in \mathbb{Z}_+$ and a nonzero positive element $a \in \mathcal{A}_l$, there exist $K \in \mathbb{N}$ and a nonzero positive element $b \in \mathcal{A}_\rho$ such that

\begin{equation}
\sum_{\mu \in \Lambda^K} \rho_\mu(a) \geq b
\end{equation}

where $\mathcal{A}_l$ is a $C^*$-subalgebra of $\mathcal{A}$ appearing in the definition of condition (I).

In what follows, we assume that $(\mathcal{A}, \rho, \Sigma)$ is effective, and central. Let $\mathcal{L} = \mathcal{L}_\rho$ be the $\lambda$-graph system associated to $(\mathcal{A}, \rho, \Sigma)$ as in Lemma 5.1. We further assume that the algebra $\mathcal{O}_{\mathcal{L}}$ is simple, purely infinite. In [Ma2], [Ma3], conditions that the algebra $\mathcal{O}_{\mathcal{L}}$ becomes simple, purely infinite is studied.

**Lemma 5.2.** For $k \leq l \in \mathbb{Z}_+$ and a nonzero positive element $a \in \mathcal{F}_{\rho,l}^k$, there exists an element $V \in \mathcal{A} \rtimes_{\rho} \Lambda$ such that $VaV^* = 1$.

**Proof.** An element $a \in \mathcal{F}_{\rho,l}^k$ is of the form $a = \sum_{\mu,\nu \in \Lambda^K} S_\mu a_{\mu,\nu} S_\nu^*$ for some $a_{\mu,\nu} \in \mathcal{A}_l$ such that $S_\mu^* a S_\nu = a_{\mu,\nu}$. Since $a$ is a nonzero positive element, there exists $\xi \in \Lambda^k$ such that $S_\xi^* a S_\xi (\neq a_{\xi,\xi}) \neq 0$. As we are assuming that $(\mathcal{A}, \rho, \Sigma)$ is effective, there exists $K \in \mathbb{N}$ and a nonzero positive element $b \in \mathcal{A}_\rho$ such that

\begin{equation}
\sum_{\mu \in \Lambda^K} \rho_\mu(S_\xi^* a S_\xi) \geq b.
\end{equation}

Put $T = \sum_{\mu \in \Lambda^K} S_\mu \in \mathcal{A} \rtimes_{\rho} \Lambda$. One has $T^* S_\xi^* a S_\xi T \geq b$. Now $b \in \mathcal{A}_\rho \subset \mathcal{O}_{\mathcal{L}}$ and $\mathcal{O}_{\mathcal{L}}$ is simple, purely infinite. We may find $V_0 \in \mathcal{O}_{\mathcal{L}}$ such that $V_0 b V_0^* = 1$ so that $V_0 T^* S_\xi^* a S_\xi TV_0^* \geq 1$. Hence there exists $V \in \mathcal{A} \rtimes_{\rho} \Lambda$ such that $VaV^* = 1$. □
Lemma 5.3. Keep the above situation. We may take \( V \in A \rtimes_r \Lambda \) in the preceding lemma such as \( VaV^* = 1 \) and \( \|V\| < \|a\|^{-\frac{1}{2}} + \epsilon \) for a given \( \epsilon > 0 \).

Proof. We may assume that \( \|a\| = 1 \) and there exists \( p \in Sp(a) \) such that \( 0 < p < 1 \). Take \( 0 < \epsilon < \frac{1}{2} \) such that \( \epsilon < 1 - p \). Define a function \( f \in C([0,1]) \) by setting

\[
f(t) = \begin{cases} 
0 & (0 \leq t \leq 1 - \epsilon) \\
1 - \epsilon^{-1}(1-t) & 1 - \epsilon < t \leq 1 
\end{cases}
\]

Put \( b = f(a) \), that is not invertible. By Lemma 5.2, there exists \( V \in A \rtimes_r \Lambda \) such that \( VbV^* = 1 \). We set \( S = b^\frac{1}{2}V^* \) and \( P = SS^* \). Then \( S \) is a proper isometry such that \( P \leq \|V\|b \). As \( P \leq E_a([1 - \epsilon, 1]) \), \( E_a([1 - \epsilon, 1]) \) is the spectral measure of \( a \) for the interval \([1 - \epsilon, 1]\), one has \( PaP \geq (1 - \epsilon)P \). Put \( D = S^*aS \) so that \( D \geq S^*(1 - \epsilon)PS = (1 - \epsilon)1 \). Hence \( D \) is invertible. Set \( V_1 = D^{-\frac{1}{2}}S^* \). Then one sees that \( V_1aV_1^* = 1 \) and \( \|V_1\| < (1 - \epsilon)^{-\frac{1}{2}} < 1 + \epsilon \). \( \square \)

Let \( \mathcal{E}_\rho : A \rtimes_r \Lambda \rightarrow \mathcal{F}_\rho \) be the conditional expectation defined in Section 3.

Lemma 5.4. For a nonzero \( X \in A \rtimes_r \Lambda \) and \( \epsilon > 0 \), there exists a projection \( Q \in \mathcal{D}_\rho \) and a nonzero positive element \( Z \in \mathcal{F}_{\rho,k} \) for some \( k \leq l \) such that

\[
\|QX^*XQ - Z\| < \epsilon, \quad \|\mathcal{E}_\rho(X^*X)\| - \epsilon < \|Z\| < \|\mathcal{E}_\rho(X^*X)\| + \epsilon.
\]

Proof. We may assume that \( \|\mathcal{E}_\rho(X^*X)\| = 1 \). Let \( \mathcal{P}_\rho \) be the \(*\)-algebra generated algebraically by \( S_\alpha, \alpha \in \Sigma \) and \( x \in A \). For any \( 0 < \epsilon < \frac{1}{2} \), find \( 0 \leq Y \in \mathcal{P}_\rho \) such that \( \|X^*X - Y\| < \frac{\epsilon}{2} \) so that \( \|\mathcal{E}_\rho(Y)\| > 1 - \frac{\epsilon}{2} \). As in the discussion in [Ma3;Section 3], the element \( Y \) is expressed as

\[
Y = \sum_{|\nu| \geq 1} Y_{-\nu}S_{\nu}^* + Y_0 + \sum_{|\mu| \geq 1} S_{\mu}Y_{\mu} \quad \text{for some} \quad Y_{-\nu}, Y_0, Y_{\mu} \in \mathcal{F}_\rho \cap \mathcal{P}_\rho.
\]

Take \( k \leq l \) large enough such that \( Y_{-\nu}, Y_0, Y_{\mu} \in \mathcal{F}_{\rho,k} \) for all \( \mu, \nu \) in the above expression. Now \((A, \rho, \Sigma)\) satisfies condition (I). Take a sequence \( Q_k^l \in \mathcal{D}_\rho \) of projections as in Section 3. As \( \mathcal{E}_\rho(Y) = Y_0 \) and \( Q_k^l \) commutes with \( \mathcal{F}_{\rho,k} \), it follows that by Lemma 3.5 (c), \( Q_k^l YQ_k^l = Q_k^l \mathcal{E}_\rho(Y) Q_k^l \). Since \( Q_k^l \mathcal{E}_\rho(Y) Q_k^l \in \mathcal{F}_\rho \), there exists \( 0 \leq Z \in \mathcal{F}_{\rho,k} \), for some \( k' \leq l' \) such that \( \|Q_k^l \mathcal{E}_\rho(Y) Q_k^l - Z\| < \frac{\epsilon}{2} \). By Lemma 3.3, we note \( \|Q_k^l \mathcal{E}_\rho(Y) Q_k^l\| = \|\mathcal{E}_\rho(Y)\| \) so that

\[
\|Z\| \geq \|\mathcal{E}_\rho(Y)\| - \frac{\epsilon}{2} > 1 - \epsilon
\]

and

\[
\|Z\| < \|Q_k^l \mathcal{E}_\rho(Y) Q_k^l\| + \frac{\epsilon}{2} \leq \|\mathcal{E}_\rho(X^*X)\| + \frac{\epsilon}{2} + \frac{\epsilon}{2} < 1 + \epsilon.
\]

\( \square \)

Therefore we have
Theorem 5.5. Assume that \((A, \rho, \Sigma)\) is central, irreducible and satisfies condition (I). Let \(L\) be the associated \(\lambda\)-graph system to \((A, \rho, \Sigma)\). If \((A, \rho, \Sigma)\) is effective and \(O_L\) is simple, purely infinite, then \(A \rtimes_\rho \Lambda\) is simple, purely infinite.

Proof. It suffices to show that for any nonzero \(X \in A \rtimes_\rho \Lambda\), there exist \(A, B \in A \rtimes_\rho \Lambda\) such that \(AXB = 1\). By the previous lemma there exists a projection \(Q \in D_\rho\) and a nonzero positive element \(Z \in \mathcal{F}_{\rho}^l\) for some \(k \leq l\) such that \(\|QX^*XQ - Z\| < \epsilon\). We may assume that \(\|\mathcal{E}_\rho(X^*X)\| = 1\) so that \(1 - \epsilon < \|Z\| < 1 + \epsilon\). By Lemma 5.3, take an element \(V \in A \rtimes_\rho \Lambda\) such that

\[
VZV^* = 1, \quad \|V\| < \frac{1}{\sqrt{\|Z\|}} + \epsilon < \frac{1}{\sqrt{1 - \epsilon}} + \epsilon.
\]

It follows that

\[
\|VQX^*XQV^* - 1\| < \|V\|^2\|QX^*XQ - Z\| < (\frac{1}{\sqrt{1 - \epsilon}} + \epsilon)^2 \cdot \epsilon.
\]

We may take \(\epsilon > 0\) small enough so that \(\|VQX^*XQV^* - 1\| < 1\) and hence \(VQX^*XQV^*\) is invertible in \(A \rtimes_\rho \Lambda\). Thus we complete the proof. \(\Box\)

6. Tensor Products of \(C^*\)-Symbolic Dynamical Systems

In this section, we will consider tensor products between \(C^*\)-symbolic dynamical systems and finite families of automorphisms of unital \(C^*\)-algebras. This construction yields interesting examples of \(C^*\)-symbolic dynamical systems beyond \(\lambda\)-graph systems, that will be studied in the following sections. Throughout this section, we fix a unital \(C^*\)-algebra \(B\) and a finite family of automorphisms \(\alpha_i \in \text{Aut}(B), i = 1, \ldots, N\) of \(B\). Tensor products \(\otimes\) between \(C^*\)-algebras always mean the minimal \(C^*\)-tensor products \(\otimes_{\text{min}}\). We set \(\Sigma = \{\alpha_1, \ldots, \alpha_N\}\). Consider a \(C^*\)-symbolic dynamical system \((A, \rho, \Sigma)\).

Proposition 6.1. For \(\alpha_i \in \Sigma, i = 1, \ldots, N\), define \(\rho_{\alpha_i}^{\Sigma_\otimes} \in \text{End}(B \otimes A)\) by setting

\[
\rho_{\alpha_i}^{\Sigma_\otimes}(b \otimes a) = \alpha_i(b) \otimes \rho_{\alpha_i}(a) \quad \text{for } b \in B, a \in A.
\]

Then \((B \otimes A, \rho_{\Sigma_\otimes}^{\otimes}, \Sigma)\) becomes a \(C^*\)-symbolic dynamical system over \(\Sigma\) such that the presented subshift \(\Lambda_{\rho_{\Sigma_\otimes}^{\otimes}}\) is the same as the subshift \(\Lambda_{\rho}\) presented by \((A, \rho, \Sigma)\).

Proof. We will first prove that \((B \otimes A, \rho_{\Sigma_\otimes}^{\otimes}, \Sigma)\) is a \(C^*\)-symbolic dynamical system. Since \(\{\rho_{\alpha_i}\}_{i=1}^N\) is essential, for \(\epsilon > 0\), there exist \(x_{i,j}, y_{i,j} \in A, j = 1, \ldots, n(i), i = 1, \ldots, N\) such that

\[
\left\| \sum_{i=1}^N \sum_{j=1}^{n(i)} x_{i,j} \rho_{\alpha_i}(1)y_{i,j} - 1 \right\| < \epsilon
\]

so that we have

\[
\left\| \sum_{i=1}^N \sum_{j=1}^{n(i)} (1 \otimes x_{i,j})(\rho_{\alpha_i}^{\Sigma_\otimes}(1))(1 \otimes y_{i,j}) - 1 \right\| < \epsilon.
\]

Hence the closed ideal generated by \(\{\rho_{\alpha_i}^{\Sigma_\otimes}(1) : i = 1, \ldots, N\}\) is all of \(B \otimes A\), so that \(\{\rho_{\alpha_i}^{\Sigma_\otimes}\}_{i=1}^N\) is essential.
Since \( \{\rho_{\alpha_i}\}_{i=1}^{N} \) is faithful on \( \mathcal{A} \), the homomorphism \( \xi_\rho : \mathcal{A} \to \bigoplus_{i=1}^{N} \mathcal{A}_i \), where \( \mathcal{A}_i = \mathcal{A}, i = 1, \ldots, N \) defined by \( \xi_\rho(a) = \bigoplus_{i=1}^{N} \rho_{\alpha_i}(a) \) is injective. Consider the homomorphisms:

\[
\begin{align*}
id_{\mathcal{B}} \otimes \xi_\rho : b \otimes a \in \mathcal{B} \otimes \mathcal{A} \to b \otimes \xi_\rho(a) \in \mathcal{B} \otimes \xi_\rho(\mathcal{A}), \\
\bigoplus_{i=1}^{N} (\alpha_i \otimes \text{id}) : (b_i \otimes a_i) \bigoplus_{i=1}^{N} \in \bigoplus_{i=1}^{N} (\mathcal{B} \otimes \mathcal{A}_i) \to (\alpha_i(b_i) \otimes a_i) \bigoplus_{i=1}^{N} \in \bigoplus_{i=1}^{N} (\mathcal{B} \otimes \mathcal{A}_i).
\end{align*}
\]

Since \( \mathcal{B} \otimes \xi_\rho(\mathcal{A}) \) is a subalgebra of \( \mathcal{B} \otimes \bigoplus_{i=1}^{N} \mathcal{A}_i \) and both \( \text{id}_B \otimes \xi_\rho \\) and \( \bigoplus_{i=1}^{N} (\alpha_i \otimes \text{id}) \) are isomorphisms, the composition \( \bigoplus_{i=1}^{N} (\alpha_i \otimes \text{id}) \circ (\text{id} \otimes \xi_\rho) \) is isomorphic. Hence

\[
\bigoplus_{i=1}^{N} \rho_{\alpha_i \otimes \alpha_i} = \bigoplus_{i=1}^{N} (\alpha_i \otimes \rho_{\alpha_i}) : \mathcal{B} \otimes \mathcal{A} \to \bigoplus_{i=1}^{N} (\mathcal{B} \otimes \mathcal{A}_i)
\]

is injective. This implies that \( \{\rho_{\alpha_i}^{\alpha \otimes \alpha}\}_{i=1}^{N} \) is faithful.

By the equality

\[
\rho_{\alpha_i}^{\alpha \otimes \alpha} \circ \cdots \circ \rho_{\alpha_1}^{\alpha \otimes \alpha}(1) = \rho_{\alpha_i} \circ \cdots \circ \rho_{\alpha_1}(1)
\]

for \( \alpha_i, \ldots, \alpha_i \in \Sigma \), the presented subshifts \( \Lambda_{\rho, \Sigma} \) and \( \Lambda_{\rho} \) coincide. □

We denote by \( \Lambda \) the presented subshift \( \Lambda_{\rho, \Sigma} \). Let \( S_{\alpha_i} \) be the generating partial isometries of \( \mathcal{A} \rtimes_{\rho} \Lambda \) satisfying \( S_{\alpha_i}^* S_{\alpha_i} = \rho_{\alpha_i}(\mathcal{A}) \) for \( x \in \mathcal{A}, i = 1, \ldots, N \), and \( \tilde{S}_{\alpha_i} \) those of \( (\mathcal{B} \otimes \mathcal{A}) \rtimes_{\rho, \Sigma} \Lambda \) satisfying \( \tilde{S}_{\alpha_i}^* y \tilde{S}_{\alpha_i} = \rho_{\alpha_i}(\mathcal{A}) \) for \( y \in \mathcal{B} \otimes \mathcal{A}, i = 1, \ldots, N \).

**Proposition 6.2.** There exists a unital embedding \( i \) of \( \mathcal{A} \rtimes_{\rho} \Lambda \) into \( (\mathcal{B} \otimes \mathcal{A}) \rtimes_{\rho, \Sigma} \Lambda \) in a canonical way.

**Proof.** Define the injective homomorphism \( \iota : \mathcal{A} \to \mathcal{B} \otimes \mathcal{A} \) by setting \( \iota(a) = 1 \otimes a \) for \( a \in \mathcal{A} \). Since the equality \( \tilde{S}_{\alpha_i} \iota(a) \tilde{S}_{\alpha_i} = \iota(\rho_{\alpha_i}(a)) \) for \( a \in \mathcal{A}, i = 1, \ldots, N \) holds, there exists a homomorphism \( i \) from \( \mathcal{A} \rtimes_{\rho} \Lambda \) to \( (\mathcal{B} \otimes \mathcal{A}) \rtimes_{\rho, \Sigma} \Lambda \) satisfying \( i(a) = 1 \otimes a, i(S_{\alpha_i}) = \tilde{S}_{\alpha_i} \) for \( a \in \mathcal{A}, i \in \Lambda, \) and \( \tilde{S}_{\alpha_i} \) for \( y \in \mathcal{B} \otimes \mathcal{A}, i = 1, \ldots, N \) by the universality of \( \mathcal{A} \rtimes_{\rho} \Lambda \).

Let \( \varepsilon_{\rho} : \mathcal{A} \rtimes_{\rho} \Lambda \to \mathcal{F}_{\rho} \) and \( \varepsilon_{\rho, \Sigma} : (\mathcal{B} \otimes \mathcal{A}) \rtimes_{\rho, \Sigma} \Lambda \to \mathcal{F}_{\rho, \Sigma} \) be the canonical conditional expectations respectively. Define the \( C^* \)-subalgebras \( \mathcal{F}_{(\mathcal{C} \otimes \mathcal{A}, \rho, \Sigma)^\otimes} \subset (\mathcal{C} \otimes \mathcal{A}) \rtimes_{\rho, \Sigma} \Lambda \) of \( (\mathcal{B} \otimes \mathcal{A}) \rtimes_{\rho, \Sigma} \Lambda \) by setting

\[
(\mathcal{C} \otimes \mathcal{A}) \rtimes_{\rho, \Sigma} \Lambda = C^*(1 \otimes a, \tilde{S}_{\alpha_i} : a \in \mathcal{A}, i = 1, \ldots, N), \\
\mathcal{F}_{(\mathcal{C} \otimes \mathcal{A}, \rho, \Sigma)^\otimes} = C^*(\tilde{S}_{\mu}(1 \otimes a) \tilde{S}_{\nu}^* : a \in \mathcal{A}, \mu, \nu \in \Lambda^*, |\mu| = |\nu|).
\]

The diagrams

\[
\begin{align*}
\mathcal{A} \rtimes_{\rho} \Lambda & \xrightarrow{i} (\mathcal{C} \otimes \mathcal{A}) \rtimes_{\rho, \Sigma} \Lambda \xrightarrow{\varepsilon_{\rho} |_{\mathcal{C} \otimes \mathcal{A}}} (\mathcal{B} \otimes \mathcal{A}) \rtimes_{\rho, \Sigma} \Lambda \\
\varepsilon_{\rho} \downarrow & \quad \downarrow \varepsilon_{\rho, \Sigma} \downarrow \\
\mathcal{F}_{\rho} & \xrightarrow{\iota|_{\mathcal{F}_{\rho}}} \mathcal{F}_{(\mathcal{C} \otimes \mathcal{A}, \rho, \Sigma)^\otimes} \xrightarrow{\varepsilon_{\rho, \Sigma}} \mathcal{F}_{\rho, \Sigma}
\end{align*}
\]

are commutative. Since \( \iota : \mathcal{A} \to \mathcal{C} \otimes \mathcal{A} \) is isomorphic, so is the restriction \( \iota|_{\mathcal{F}_{\rho}} : \mathcal{F}_{\rho} \to \mathcal{F}_{(\mathcal{C} \otimes \mathcal{A}, \rho, \Sigma)^\otimes} \) of \( \mathcal{F}_{\rho} \). One indeed sees that the condition \( S_{\mu}aS_{\nu}^* \neq 0 \) for some \( a \in \mathcal{A}, |\mu| = |\nu| \) implies \( \tilde{S}_{\mu}(1 \otimes a) \tilde{S}_{\nu}^* \neq 0 \) because of the equality \( \iota(\rho_{\mu}(1)a\rho_{\nu}(1)) = 15 \).
\[ \tilde{S}_\mu \tilde{S}_\nu (1 \otimes a) \tilde{S}_\nu \tilde{S}_\nu. \]

For \( \sum_{\mu, \nu \in \Lambda^k} S_{\mu} a_{\mu, \nu} S_{\nu}^* \in \mathcal{F}_\rho \), suppose that \( \iota(\sum_{\mu, \nu \in \Lambda^k} S_{\mu} a_{\mu, \nu} S_{\nu}^*) = 0 \). It follows that for any \( \xi, \eta \in \Lambda^k \),

\[ 0 = \tilde{S}_\xi (\sum_{\mu, \nu \in \Lambda^k} \tilde{S}_\mu (1 \otimes a_{\mu, \nu}) \tilde{S}_\nu)^* \tilde{S}_\eta = \tilde{S}_\xi (1 \otimes a_{\xi, \eta}) \tilde{S}_\eta \tilde{S}_\eta \]

so that \( 0 = \tilde{S}_\xi (1 \otimes a_{\xi, \eta}) \tilde{S}_\eta^* \), and hence \( S_{\xi} a_{\xi, \eta} S_{\eta}^* = 0 \). This implies that \( \iota|_{\mathcal{F}_\rho^k} : \mathcal{F}_\rho^k \to \mathcal{F}_{(\mathbb{C} \otimes \mathbb{A}, \rho^{\otimes \otimes})} \) is injective and so is \( \iota|_{\mathcal{F}_\rho} : \mathcal{F}_\rho \to \mathcal{F}_{(\mathbb{C} \otimes \mathbb{A}, \rho^{\otimes \otimes})} \). Therefore by using a routine argument, one concludes that \( \iota : \mathbb{A} \rtimes_\rho \Lambda \to (\mathbb{C} \otimes \mathbb{A}) \rtimes_\rho^{\otimes \otimes} \Lambda \) is injective and hence isomorphic. \( \square \)

Let us prove that \( (\mathbb{B} \otimes \mathbb{A}, \rho^{\otimes \otimes}, \Sigma) \) satisfies condition (I) if \( (\mathbb{A}, \rho, \Sigma) \) satisfies condition (I). The result will be used in the following sections. We set the \( C^* \)-subalgebras \( \mathcal{D}_{(\mathbb{C} \otimes \mathbb{A}, \rho^{\otimes \otimes})} \subset \mathcal{D}_{\rho^{\otimes \otimes}} \) of \( \mathcal{F}_{\rho^{\otimes \otimes}} \) by setting

\[ \mathcal{D}_{\rho^{\otimes \otimes}} = \mathcal{C}^* (\tilde{S}_\mu x \tilde{S}_\mu^* : \mu \in \Lambda^k, x \in \mathbb{B} \otimes \mathbb{A}), \]

\[ \mathcal{D}_{(\mathbb{C} \otimes \mathbb{A}, \rho^{\otimes \otimes})} = \mathcal{C}^* (\tilde{S}_\mu (1 \otimes a) \tilde{S}_\mu^* : \mu \in \Lambda^k, a \in \mathbb{A}). \]

We may identify the subalgebra \( \mathcal{D}_\rho \) of \( \mathcal{F}_\rho \) with the subalgebra \( \mathcal{D}_{(\mathbb{C} \otimes \mathbb{A}, \rho^{\otimes \otimes})} \) of \( \mathcal{F}_{(\mathbb{C} \otimes \mathbb{A}, \rho^{\otimes \otimes})} \) through the map \( \iota \) as in the preceding proposition.

Let \( \varphi \in \mathbb{B}^* \) be a faithful state on \( \mathbb{B} \). It is well-known that there exists a faithful projection \( \Theta_{\varphi} : \mathbb{B} \otimes \mathbb{A} \to \mathbb{A} \) of norm one satisfying \( \Theta_{\varphi} (b \otimes a) = \varphi(b) a \) for \( b \otimes a \in \mathbb{B} \otimes \mathbb{A} \).

**Lemma 6.3.** Let \( \varphi \in \mathbb{B}^* \) be a faithful state on \( \mathbb{B} \) satisfying \( \varphi \circ \alpha_i = \varphi, i = 1, \ldots, N \). The projection \( \Theta_{\varphi} : \mathbb{B} \otimes \mathbb{A} \to \mathbb{A} \) of norm one can be extended to a projection of norm one \( \Theta_D : \mathcal{D}_{\rho^{\otimes \otimes}} \to \mathcal{D}_\rho \) such that \( \Theta_D (x) = x \) for \( x \in \mathcal{D}_\rho \).

**Proof.** For \( k \in \mathbb{N} \), define the \( C^* \)-subalgebras \( \mathcal{D}_k^\rho \) of \( \mathcal{D}_\rho \) and \( \mathcal{D}_k^{\rho^{\otimes \otimes}} \) of \( \mathcal{D}_{\rho^{\otimes \otimes}} \) by setting

\[ \mathcal{D}_k^\rho = \mathcal{C}^* (S_{\mu} a S_{\mu}^* : \mu \in \Lambda^k, a \in \mathbb{A}), \]

\[ \mathcal{D}_k^{\rho^{\otimes \otimes}} = \mathcal{C}^* (\tilde{S}_\mu x \tilde{S}_\mu^* : \mu \in \Lambda^k, x \in \mathbb{B} \otimes \mathbb{A}). \]

For \( x_{\mu} \in \mathbb{B} \otimes \mathbb{A}, \xi \in \Lambda^k \), the identities

\[ \Theta_{\varphi} (\tilde{S}_\xi^* (\sum_{\mu \in \Lambda^k} \tilde{S}_{\mu} x_{\mu} \tilde{S}_\mu^*) \tilde{S}_\xi) = \Theta_{\varphi} ((1 \otimes \rho_{\xi}(1)) x_{\xi} (1 \otimes \rho_{\xi}(1)) \]

\[ = \rho_{\xi}(1) \Theta_{\varphi} (x_{\xi}) \rho_{\xi}(1) = S_{\xi}^* S_{\xi} \Theta_{\varphi} (x_{\xi}) S_{\xi}^* S_{\xi} \]

hold, so that the map defined by \( \Theta_D^k : \mathcal{D}_{k}^{\rho^{\otimes \otimes}} \to \mathcal{D}_k^\rho \)

\[ \Theta_D^k (\sum_{\mu \in \Lambda^k} \tilde{S}_{\mu} x_{\mu} \tilde{S}_\mu^*) = \sum_{\mu \in \Lambda^k} S_{\mu} \Theta_{\varphi} (x_{\mu}) S_{\mu}^*. \]

is well-defined for each \( k \in \mathbb{Z}_+. \) We will next see the restriction of \( \Theta_D^{k+1} \) to \( \mathcal{D}_k^{\rho^{\otimes \otimes}} \) coincides with \( \Theta_D^k \). Since \( \sum_{\mu \in \Lambda^k} \tilde{S}_{\mu} x_{\mu} \tilde{S}_\mu^* \in \mathcal{D}_{k}^{\rho^{\otimes \otimes}} \) is written as \( \sum_{\mu \in \Lambda^k} \sum_{i=1}^N \tilde{S}_{\mu} \alpha_{\mu i} \tilde{S}_{\alpha_{i} \mu} x_{\mu} \tilde{S}_{\alpha_{i}} \tilde{S}_{\alpha_{i} \mu} \tilde{S}_\mu^* \in \mathcal{D}_{k+1}^{\rho^{\otimes \otimes}} \), it follows that

\[ \Theta_D^{k+1} (\tilde{S}_{\mu} x_{\mu} \tilde{S}_\mu^*) = \sum_{i=1}^N \Theta_D^{k+1} (\tilde{S}_{\alpha_{i} \mu} \rho_{\alpha_{i}} (x_{\mu}) \tilde{S}_{\alpha_{i}}^* \tilde{S}_{\alpha_{i} \mu}^*) = \sum_{i=1}^N S_{\mu} \alpha_{i} \Theta_{\varphi} (\rho_{\alpha_{i}} (x_{\mu})) S_{\mu}^*. \]

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As the state $\varphi$ is $\alpha_i$-invariant for $i = 1, \ldots, N$, one has for $\sum_j b_j \otimes a_j \in \mathcal{B} \otimes \mathcal{A}$,
\[
\Theta_{\varphi}(\rho_{\alpha_i}^{\Sigma}(\sum_j b_j \otimes a_j)) = \sum_j \varphi(\alpha_i(b_j)) \rho_{\alpha_i}(a_j) = \sum_j \varphi(b_j) \rho_{\alpha_i}(a_j) = \rho_{\alpha_i}(\sum_j \varphi(b_j)a_j) = S_{\alpha_i}^* \Theta_{\varphi}(\sum_j b_j \otimes a_j) S_{\alpha_i},
\]
so that $\Theta_{\varphi}(\rho_{\alpha_i}^{\Sigma}(x_{\mu})) = S_{\alpha_i}^* \Theta_{\varphi}(x_{\mu}) S_{\alpha_i}$ for $x_{\mu} \in \mathcal{B} \otimes \mathcal{A}$. It then follows that
\[
\Theta_D^{k+1}(\tilde{S}_{\mu} x_{\mu} \tilde{S}_{\mu}^*) = \sum_{i=1}^{N} S_{\mu \alpha_i} S_{\alpha_i}^* \Theta_{\varphi}(x_{\mu}) S_{\alpha_i} S_{\mu \alpha_i}^* = S_{\mu} \Theta_{\varphi}(x_{\mu}) S_{\mu}^* = \Theta_D^k(\tilde{S}_{\mu} x_{\mu} \tilde{S}_{\mu}^*).
\]
Therefore the sequence $\{\Theta_D^k\}_{k=1}^{\infty}$ defines a projection from $D_{\rho^{\Sigma \otimes}}$ onto $D_{\rho}$, which we denote by $\Theta_D$. \qed

**Lemma 6.4.** Assume that $(\mathcal{A}, \rho, \Sigma)$ is central. Then $\tilde{S}_{\mu}(1 \otimes a)\tilde{S}_{\mu}^*$ commutes with $b \otimes 1$ for $a \in \mathcal{A}$, $\mu \in \Lambda^*$ and $b \in \mathcal{B}$.

**Proof.** Since $(1 \otimes a)\rho_{\mu}^{\Sigma \otimes}(b \otimes 1) = \rho_{\mu}^{\Sigma \otimes}(b \otimes 1)(1 \otimes a)$, it follows that
\[
\tilde{S}_{\mu}(1 \otimes a)\tilde{S}_{\mu}^*(b \otimes 1) = \tilde{S}_{\mu}(1 \otimes a)\rho_{\mu}^{\Sigma \otimes}(b \otimes 1)\tilde{S}_{\mu}^* = \tilde{S}_{\mu}\tilde{S}_{\mu}^*(b \otimes 1)\tilde{S}_{\mu}(1 \otimes a)\tilde{S}_{\mu}^* = (b \otimes 1)\tilde{S}_{\mu}(1 \otimes a)\tilde{S}_{\mu}^*.
\]
\qed

**Theorem 6.5.** Assume that there exists a faithful state $\varphi$ on $\mathcal{B}$ invariant under $\alpha_i \in \text{Aut}(\mathcal{B})$, $i = 1, \ldots, N$. Suppose that $(\mathcal{A}, \rho, \Sigma)$ is central. If $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I), then $(\mathcal{B} \otimes \mathcal{A}, \rho^{\Sigma \otimes}, \Sigma)$ satisfies condition (I) and is central.

**Proof.** Since $(\mathcal{A}, \rho, \Sigma)$ satisfies condition (I), there exists an increasing sequence $\mathcal{A}_l, l \in \mathbb{Z}_+$ of $C^*$-subalgebras of $\mathcal{A}$ and a projection $q^l_k \in D_{\rho} \cap \mathcal{A}_l$ with $l \geq k$ satisfying the conditions of condition (I). We set $(\mathcal{B} \otimes \mathcal{A})_l = \mathcal{B} \otimes \mathcal{A}_l$ for $l \in \mathbb{Z}_+$. Then the conditions $\bigcup_{l \in \mathbb{N}} ((\mathcal{B} \otimes \mathcal{A})_l) = (\mathcal{B} \otimes \mathcal{A})_l \cap (\mathcal{B} \otimes \mathcal{A})_{l+1}$ are easy to verify. Let $\tilde{i} : \mathcal{A} \rtimes_{\rho} \Lambda \hookrightarrow (\mathcal{B} \otimes \mathcal{A}) \rtimes_{\rho^{\Sigma \otimes}} \Lambda$ be the embedding in Proposition 6.2. Put $\tilde{q}^l_k = \tilde{i}(q^l_k) \in D_{\rho^{\Sigma \otimes}}$ for $l \geq k$. By the preceding lemma, one sees that $\tilde{q}^l_k \in D_{\rho^{\Sigma \otimes}} \cap ((\mathcal{B} \otimes \mathcal{A})_l)'$. We will show that $\tilde{q}^l_k x \neq 0$ for $0 \neq x \in (\mathcal{B} \otimes \mathcal{A})_l$. As $xx^* \in \mathcal{B} \otimes \mathcal{A}_l$, one has $\Theta_D(xx^*) = \Theta_{\varphi}(xx^*) \in \mathcal{A}_l$. Hence $q^l_k \Theta_{\varphi}(xx^*) \neq 0$. By the equality $\Theta_D(\tilde{q}^l_k xx^* \tilde{q}^l_k) = q^l_k \Theta_{\varphi}(xx^*) q^l_k$, one obtains $\tilde{q}^l_k x \neq 0$. Let $\tilde{\varphi}_{\rho^{\Sigma \otimes}}(X) = \sum_{i=1}^{N} \tilde{S}_{\alpha_i} X \tilde{S}_{\alpha_i}^*$ for $X \in D_{\rho^{\Sigma \otimes}}$. One has
\[
\tilde{q}^l_k \tilde{\varphi}_{\rho^{\Sigma \otimes}}(\tilde{q}^l_k) = \tilde{i}(q^l_k \tilde{\varphi}_{\rho}^m(q^l_k)) = 0 \quad \text{for all } m = 1, 2, \ldots, k.
\]
Thus $(\mathcal{B} \otimes \mathcal{A}, \rho^{\Sigma \otimes}, \Sigma)$ satisfies condition (I). If $(\mathcal{A}, \rho, \Sigma)$ is central, the projections $1 \otimes \rho_{\mu}(1)$ for $\mu \in \Lambda^*$ commute with $\mathcal{B} \otimes \mathcal{A}$, so that $(\mathcal{B} \otimes \mathcal{A}, \rho^{\Sigma \otimes}, \Sigma)$ is central. \qed

We will study structure of the fixed point algebra $\mathcal{F}_{\rho^{\Sigma \otimes}}$ of $(\mathcal{B} \otimes \mathcal{A}) \rtimes_{\rho^{\Sigma \otimes}} \Lambda$ under the gauge action $\rho^{\Sigma \otimes}$. Recall that $\mathcal{F}_{\rho}$ denote the fixed point algebra of $\mathcal{A} \rtimes_{\rho} \Lambda$ under the gauge action $\rho$. Recall that for $k \in \mathbb{Z}_+$, the $C^*$-subalgebras $\mathcal{F}_{\rho}^k$ of $\mathcal{F}_{\rho}$ and $\mathcal{F}_{\rho^{\Sigma \otimes}}^k$ of $\mathcal{F}_{\rho^{\Sigma \otimes}}$ are generated by $S_{\mu} a S_{\nu}^*$ for $\mu, \nu \in \Lambda^k, a \in \mathcal{A}$ and $\tilde{S}_{\mu} x \tilde{S}_{\nu}^*$ for $\mu, \nu \in \Lambda^k, x \in \mathcal{B} \otimes \mathcal{A}$ respectively. Then we have
Lemma 6.6. The map $\Phi_k : \tilde{S}_\mu(b \otimes a)\tilde{S}_\nu^* \to b \otimes S_\mu a S_\nu^*$ for $b \otimes a \in B \otimes A$, $\mu, \nu \in \Lambda^k$ extends to an isomorphism from $\mathcal{F}_\rho^{\Sigma \otimes}$ to $B \otimes \mathcal{F}_\rho^k$.

Proof. For $Y = \sum_{\mu, \nu \in \Lambda^k} \tilde{S}_\mu(\sum_{j=1}^n b_j \otimes a_j)\tilde{S}_\nu^* \in \mathcal{F}_\rho^{\Sigma \otimes}$, put

$$\Phi_k(Y) = \sum_{j=1}^n (b_j \otimes \sum_{\mu, \nu \in \Lambda^k} S_\mu a_j S_\nu^*) \in B \otimes \mathcal{F}_\rho^k.$$ 

It follows that for $\xi, \eta \in \Lambda^k$

$$\tilde{S}_\xi^* Y \tilde{S}_\eta = \tilde{S}_\xi^* \tilde{S}_\xi(\sum_{j=1}^n b_j \otimes a_j)\tilde{S}_\eta^* \tilde{S}_\eta = \sum_{j=1}^n b_j \otimes S_\xi^* S_\xi a_j S_\eta^* S_\eta = (1 \otimes S_\xi^*) \Phi_k(Y)(1 \otimes S_\eta)$$

Hence $Y = 0$ if and only if $\Phi_k(Y) = 0$. As $\Phi_k$ is a homomorphism from $\mathcal{F}_\rho^{\Sigma \otimes}$ to $B \otimes \mathcal{F}_\rho^k$, it yields an isomorphism. \square

The following lemma is straightforward.

Lemma 6.7. Let $\alpha \otimes \iota^k : B \otimes \mathcal{F}_\rho^k \to B \otimes \mathcal{F}_\rho^{k+1}$ be the homomorphism defined by

$$(\alpha \otimes \iota^k)(b \otimes S_\mu a S_\nu^*) = \sum_{i=1}^n \alpha_i(b) \otimes S_\mu \alpha_i \rho \alpha_i(a) S_\nu^*$$

for $b \otimes a \in B \otimes A$, $\mu, \nu \in \Lambda^k$.

Then the diagram

$$\begin{array}{ccc}
\mathcal{F}_\rho^{\Sigma \otimes} & \xrightarrow{\iota^k_{\Sigma \otimes}} & \mathcal{F}_\rho^{k+1} \\
\downarrow \Phi_k & & \downarrow \Phi_{k+1} \\
B \otimes \mathcal{F}_\rho^k & \xrightarrow{\alpha \otimes \iota^k} & B \otimes \mathcal{F}_\rho^{k+1}
\end{array}$$

is commutative, where $\iota^k_{\rho \Sigma \otimes} : \mathcal{F}_\rho^{\Sigma \otimes} \to \mathcal{F}_\rho^{k+1}$ denotes the natural inclusion.

Hence we have

Proposition 6.8. The $C^*$-algebra $\mathcal{F}_\rho^{\Sigma \otimes}$ is the inductive limit

$$B \otimes \mathcal{F}_\rho^1 \xrightarrow{\alpha \otimes \iota^1} B \otimes \mathcal{F}_\rho^2 \xrightarrow{\alpha \otimes \iota^2} B \otimes \mathcal{F}_\rho^3 \xrightarrow{\alpha \otimes \iota^3} \cdots.$$ 

Let $B = C(X)$ be the commutative $C^*$-algebra of all continuous functions on a compact Hausdorff space $X$ with a finite family $h_1, \ldots, h_N$ of homeomorphisms on $X$. Define $\alpha_i \in \text{Aut}(C(X))$, $i = 1, \ldots, N$ by $\alpha_i(f)(t) = f(h_i(t))$ for $f \in C(X)$, $t \in X$. Put $\Sigma = \{ \alpha_1, \ldots, \alpha_N \}$ Take $(A_\Sigma, \rho^\Sigma, \Sigma)$ for a $\lambda$-graph system $A$ over $\Sigma$ as $(A, \rho, \Sigma)$. Then the above $C^*$-algebra $\mathcal{F}_\rho^{\Sigma \otimes}$ is an AH-algebra. If in particular $X = \mathbb{T}$, the algebra is an AT-algebra. We will study these examples in the following sections.
7. \(C^*\)-SYMBOLIC DYNAMICAL SYSTEMS FROM HOMEOMORPHISMS AND GRAPHS

Let \(h_1, \ldots, h_N\) be a finite family of homeomorphisms on a compact Hausdorff space \(X\). Put \(\Sigma = \{h_1, \ldots, h_N\}\). Let \(\mathcal{G}\) be a left-resolving finite labeled graph \((G, \lambda)\) over \(\Sigma\) with underlying finite directed graph \(G\) and labeling map \(\lambda : E \to \Sigma\). We denote by \(G = (V, E)\), where \(V = \{v_1, \ldots, v_{N_0}\}\) is the finite set of its vertices and \(E = \{e_1, \ldots, e_{N_1}\}\) is the finite set of its directed edges. As in the beginning of Section 2, we have a \(C^*\)-symbolic dynamical system \((\mathcal{A}_G, (\rho^G)^{\Sigma^\infty}, \Sigma)\). Identify the homeomorphisms \(h_i\) with the induced automorphisms \(\alpha_i\) on \(C(X)\). By Proposition 6.1, the tensor product \((C(X) \otimes \mathcal{A}_G, (\rho^G)^{\Sigma^\infty}, \Sigma)\) of \(C^*\)-symbolic dynamical system is defined. Put \(X_i = X, i = 1, \ldots, N_0\) and

\[
\mathcal{A}_{G,X} = C(X) \otimes \mathcal{A}_G = C(\bigcup_{i=1}^{N_0} X_i), \quad \rho^G_X = (\rho^G)^{\Sigma^\infty}.
\]

We will study the \(C^*\)-symbolic dynamical system \((\mathcal{A}_{G,X}, (\rho^G_X)^{\Sigma^\infty}, \Sigma)\). Note that the presented subshift \(\Lambda_{\rho^G} X\) is the sofic shift \(\Lambda_{\rho^G}\) presented by the labeled graph \(\mathcal{G}\).

For \(u, v \in V\), let \(H_n(u, v)\) be the set \((f_1, \ldots, f_n)\) of \(n\)-edges of the graph \(\mathcal{G}\) satisfying \(s(f_1) = u, t(f_i) = s(f_{i+1}), i = 1, \ldots, n-1, \text{and } t(f_n) = v\). We set

\[
H_n(u) = \bigcup_{v \in V} H_n(u, v), \quad H^n_{\mathcal{G}} = \bigcup_{u \in V} H_n(u), \quad H_{\mathcal{G}} = \bigcup_{n=1}^{\infty} H^n_{\mathcal{G}}.
\]

Then \(\gamma = (f_1, \ldots, f_n) \in H_n(v_i, v_j)\) yields a homeomorphism \(\lambda(\gamma)\) from \(X_i\) to \(X_j\) by setting

\[
\lambda(\gamma)(x) = \lambda(f_n) \circ \cdots \circ \lambda(f_1)(x) \quad \text{for } x \in X_i.
\]

For \(x \in X_k\) with \(k \neq i\), \(\lambda(\gamma)(x)\) is not defined. We set for \(x \in X_i\)

\[
\text{orb}_n(x) = \{\lambda(\gamma)(x) \mid \gamma \in H_n(v_i)\} \subset \bigcup_{j=1}^{N_0} X_j, \quad \text{orb}(x) = \bigcup_{n=0}^{\infty} \text{orb}_n(x),
\]

where \(\text{orb}_0(x) = \{x\}\).

**Definition.** A family \((h_1, \ldots, h_N)\) of homeomorphisms on \(X\) is called \(\mathcal{G}\)-minimal if for any \(x \in \bigcup_{j=1}^{N_0} X_j\), the orbit \(\text{orb}(x)\) is dense in \(\bigcup_{j=1}^{N_0} X_j\).

**Lemma 7.1.** The following conditions are equivalent:

(i) \((h_1, \ldots, h_N)\) is \(\mathcal{G}\)-minimal;

(ii) There exists no proper closed subset \(F \subset \bigcup_{j=1}^{N_0} X_j\) such that \(\lambda(e_i)(F) \subset F\) for all \(i = 1, \ldots, N_1\);

(iii) There exists no proper closed subset \(F \subset \bigcup_{j=1}^{N_0} X_j\) such that \(\bigcup_{i=1}^{N_1} \lambda(e_i)(F) = F\).

**Proof.**

(i)\(\Rightarrow\)(ii) If there exists a closed subset \(F \subset \bigcup_{j=1}^{N_0} X_j\) such that \(\lambda(e_i)(F) \subset F\) for all \(i = 1, \ldots, N_1\), take \(x \in F \cap X_j\) for some \(j\). Then \(\text{orb}(x)\) is not dense in \(\bigcup_{j=1}^{N_0} X_j\).

(ii)\(\Rightarrow\)(i) For \(x \in \bigcup_{j=1}^{N_0} X_j\), let \(F\) be the closure of \(\text{orb}(x)\). Then we have \(\lambda(e_i)(F) \subset F\) for all \(i = 1, \ldots, N_1\), and hence \(F = \bigcup_{j=1}^{N_0} X_j\).

(ii)\(\Rightarrow\)(iii) This implication is trivial.

(iii)\(\Rightarrow\)(ii) Suppose that there exists a closed subset \(F \subset \bigcup_{j=1}^{N_0} X_j\) such that \(\lambda(e_i)(F) \subset F\) for all \(i = 1, \ldots, N_1\). Put \(\tilde{F}_n = \bigcup_{\gamma \in H^n_{\mathcal{G}}} \lambda(\gamma)(F)\) a closed subset of \(F\). Since \(\tilde{F}_{n+1} \subset \tilde{F}_n\) and \(\bigcup_{j=1}^{N_0} X_j\) is compact, the set \(E := \bigcap_{n=1}^{\infty} \tilde{F}_n\) is a nonempty
closed subset of \( \bigcup_{j=1}^{N_0} X_j \). Since \( \bigcup_{i=1}^{N_1} \lambda(e_i)(\tilde{F}_n) = \tilde{F}_{n+1} \), one has \( \bigcup_{i=1}^{N} \lambda(e_i)(E) \subset E \). On the other hand, take \( s(i) = 1, \ldots, N_0 \) such that \( v_{s(i)} = s(e_i) \). Then we have

\[
\bigcap_{n=1}^{\infty} \lambda(e_i)(\tilde{F}_n) = \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{N_0} \lambda(e_i)(\tilde{F}_n \cap X_j) = \bigcap_{n=1}^{\infty} \lambda(e_i)(\tilde{F}_n \cap X_{s(i)}) \\
\subset \bigcup_{j=1}^{N_0} \bigcap_{n=1}^{\infty} \lambda(e_i)(\tilde{F}_n \cap X_j) = \lambda(e_i)(E).
\]

For \( x \in \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{N_1} \lambda(e_i)(\tilde{F}_n) \) and \( n \in \mathbb{N} \), there exists \( i_n = 1, \ldots, N_1 \) such that \( x \in \lambda(e_{i_n})(\tilde{F}_n) \). Find \( i(x) = 1, \ldots, N_1 \) such that \( i(x) \) appears in \( \{i_n \mid n \in \mathbb{N}\} \) infinitely many times. Since \( \tilde{F}_n, n \in \mathbb{N} \) are decreasing subsets, one has \( x \in \lambda(e_{i(x)})(\tilde{F}_n) \) for all \( n \in \mathbb{N} \). Hence \( x \in \bigcup_{i=1}^{N_1} \bigcap_{n=1}^{\infty} \lambda(e_i)(\tilde{F}_n) \) so that we have \( \bigcup_{i=1}^{N_1} \bigcap_{n=1}^{\infty} \lambda(e_i)(\tilde{F}_n) \supset \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{N_1} \lambda(e_i)(\tilde{F}_n) \). Thus we have

\[
\bigcup_{i=1}^{N_1} \lambda(e_i)(E) \supset \bigcap_{n=1}^{\infty} \lambda(e_i)(\tilde{F}_n) \supset \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{N_1} \lambda(e_i)(\tilde{F}_n) = \bigcap_{n=1}^{\infty} \tilde{F}_{n+1} = E.
\]

\( \square \)

The following lemma is direct.

**Lemma 7.2.** Let \( J \) be an ideal of \( A_{\mathcal{G}, X} \). Denote by \( F \subset \bigcup_{j=1}^{N_0} X_j \) the closed subset such that \( J = \{f \in C(\bigcup_{j=1}^{N_0} X_j) \mid f(x) = 0 \text{ for } x \in F \} \). Then we have

(i) \( J \) is a \( \rho^{\mathcal{G}, X} \)-invariant ideal of \( A_{\mathcal{G}, X} \) if and only if \( \lambda(e_i)(F) \subset F \) for all \( i = 1, \ldots, N_1 \).

(ii) \( J \) is a saturated ideal of \( A_{\mathcal{G}, X} \) if and only if \( \bigcup_{i=1}^{N} \lambda(e_i)(F) \supset F \).

(iii) \( J \) is a \( \rho^{\mathcal{G}, X} \)-invariant saturated ideal of \( A_{\mathcal{G}, X} \) if and only if \( \bigcup_{i=1}^{N} \lambda(e_i)(F) = F \).

Hence we have

**Lemma 7.3.** The following conditions are equivalent:

(i) \((h_1, \ldots, h_N)\) is \( \mathcal{G} \)-minimal;

(ii) There exists no proper \( \rho^{\mathcal{G}, X} \)-invariant ideal of \( A_{\mathcal{G}, X} \);

(iii) There exists no proper \( \rho^{\mathcal{G}, X} \)-invariant saturated ideal of \( A_{\mathcal{G}, X} \).

A finite labeled graph \( \mathcal{G} \) is said to satisfy condition (I) if for every vertex \( v_i \) there exists distinct paths with distinct labeled edges both of whose sources and terminals are the vertex \( v_i \). We denote by \( \mathcal{O}_{\mathcal{G}, h_1, \ldots, h_N} \) the \( C^* \)-symbolic crossed product \( A_{\mathcal{G}, X} \rtimes_{\rho^{\mathcal{G}, X}} \Lambda_{\mathcal{G}} \) for the \( C^* \)-symbolic dynamical system \((A_{\mathcal{G}, X}, \rho^{\mathcal{G}, X}, \Sigma)\). Assume that there exists a faithful \( h_i \)-invariant probability measure on \( X \).

**Theorem 7.4.** Suppose that the labeled graph satisfies condition (I). \((h_1, \ldots, h_N)\) is \( \mathcal{G} \)-minimal if and only if the \( C^* \)-algebra \( \mathcal{O}_{\mathcal{G}, h_1, \ldots, h_N} \) is simple.

**Proof.** Suppose that there exists a proper ideal \( \mathcal{I} \) of \( \mathcal{O}_{\mathcal{G}, h_1, \ldots, h_N} \). Since the labeled graph \( \mathcal{G} \) satisfies condition (I), the \( C^* \)-symbolic dynamical system \((A_{\mathcal{G}, X}, \rho^{\mathcal{G}, X}, \Sigma)\) satisfies condition (I) ([Ma2;Section 4]), so that \((A_{\mathcal{G}, X}, \rho^{\mathcal{G}, X}, \Sigma)\) satisfies condition (I) by Theorem 6.5. Hence \( J := \mathcal{I} \cap A_{\mathcal{G}, X} \) is a nonzero \( \rho^{\mathcal{G}, X} \)-invariant saturated ideal of \( A_{\mathcal{G}, X} \). If \( J = A_{\mathcal{G}, X} \), then \( A_{\mathcal{G}, X} \subset \mathcal{I} \) and \( S_{\alpha}^* S_{\alpha} \in \mathcal{I} \) so that \( S_{\alpha} \in \mathcal{I} \). Hence \( \mathcal{I} = \mathcal{O}_{\mathcal{G}, h_1, \ldots, h_N} \). Therefore \( J \) is not a proper ideal of \( A_{\mathcal{G}, X} \), and by Lemma 7.3 \((h_1, \ldots, h_N)\) is not \( \mathcal{G} \)-minimal.
Next suppose that \((h_1, \ldots, h_N)\) is not \(G\)-minimal. By Lemma 7.3, there exists a proper \(\rho^G,X\)-invariant saturated ideal \(J\) of \(A_{G,X}\). The ideal \(\mathcal{I}_J\) of \(O_{G,h_1,\ldots,h_N}\) generated by \(J\) satisfies \(\mathcal{I}_J \cap A_{G,X} = J\) by Proposition 4.5. Hence \(\mathcal{I}_J\) is a proper ideal of \(O_{G,\gamma_1,\ldots,\gamma_N}\). □

In [KW; Corollary 33], Kajiwara-Watatani have proved a similar result for the \(C^*\)-algebras from circle bimodules.

For a vertex \(u \in V\) put \(H_n[u] = H_n(u, u)\). Then we have

**Proposition 7.5.** Suppose that \(G\) satisfies condition (I) and is irreducible. If there exists a path \((f_1, \ldots, f_n) \in H_n[v_i]\) for some vertex \(v_i \in V\) and \(n \in \mathbb{N}\) such that the homeomorphism \(\lambda(f_n) \circ \cdots \circ \lambda(f_1)\) on \(X_i\) is minimal, then \((h_1, \ldots, h_N)\) is \(G\)-minimal.

**Proof.** Put \(\xi = (f_1, \ldots, f_n)\). Then \(\lambda(\xi)\) is a minimal homeomorphism on \(X_i\). For vertices \(v_j, v_k \in V\), we may take paths \(\gamma \in \bigcup_{i=1}^{\infty} H_m(v_i, v_j)\) and \(\gamma' \in \bigcup_{i=1}^{\infty} H_m(v_k, v_i)\). Since for any \(x \in X_i\), the orbit \(\bigcup_{i=0}^{\infty} \lambda(\xi)^i(x)\) is dense in \(X_i\), the set for any \(y \in X_k\)
\(\bigcup_{i=0}^{\infty} \lambda(\gamma) \circ \lambda(\xi)^i \circ \lambda(\gamma')(y)\) is dense in \(X_j\). Thus \((h_1, \ldots, h_N)\) is \(G\)-minimal. □

The above discussions may be generalized to a \(\lambda\)-graph system with a family \(\{h_1, \ldots, h_N\}\) of homeomorphisms of a compact Hausdorff space \(X\).

# 8. Irrational Rotation Cuntz-Krieger Algebras

Let \(X\) be the circle \(\mathbb{T}\) in the complex plane. Take an arbitrary finite family of real numbers \(\{\theta_1, \ldots, \theta_N\}\) with \(\theta_i \in [0, 1)\). Let \(\alpha_i \in \text{Aut}(C(\mathbb{T}))\) be the automorphisms of \(C(\mathbb{T})\) defined by \(\alpha_i(f)(t) = f(e^{2\pi \sqrt{-1} \theta_i} t), f \in C(\mathbb{T}), t \in \mathbb{T}\) for \(i = 1, \ldots, N\). Put \(\Sigma = \{\alpha_1, \ldots, \alpha_N\}\). Let \(G\) be a finite directed labeled graph \((G, \lambda)\) over \(\Sigma\) with underlying finite directed graph \(G = (V,E)\) and left resolving labeling \(\lambda : E \to \Sigma\). We denote by \(\{v_1, \ldots, v_{N_0}\}\) the vertex set \(V\). In [KW], Kajiwara-Watatani have studied the \(C^*\)-algebras constructed from circle correspondences. Their situation is more general than ours.

Assume that each vertex of \(V\) has both an incoming edge and an outgoing edge. Then we have a \(C^*\)-symbolic dynamical system as in the preceding sections, which we denote by \((A_{G,T}, \rho_{\theta_1,\ldots,\theta_N}, \Sigma)\). Its \(C^*\)-symbolic crossed product is denoted by \(O_{G,\theta_1,\ldots,\theta_N}\). Let \(A^G\) be the matrix for \(G\) defined in (2.1).

**Proposition 8.1.** The \(C^*\)-algebra \(O_{G,\theta_1,\ldots,\theta_N}\) is the universal unital \(C^*\)-algebra generated by \(N\) partial isometries \(S_i, i = 1, \ldots, N\) and \(N_0\) partial unitaries \(U_j, j = 1, \ldots, N_0\) subject to the following relations:

\[
\sum_{m=1}^{N} S_m^* S_m = 1, \quad \sum_{j=1}^{N_0} U_j^* U_j = 1, \quad U_i^* U_i = U_i U_i^*
\]

\[
U_i S_n = \sum_{j=1}^{N_0} A^G(i, \alpha_n, j) e^{2\pi \sqrt{-1} \theta_n} S_n U_j,
\]

\[
S_n S_n^* U_i = U_i S_n S_n^* \quad \text{for } i = 1, \ldots, N_0, \quad n = 1, \ldots, N
\]

such that

\[
K_i(O_{G,\theta_1,\ldots,\theta_N}) = \mathbb{Z}^{N_0}/(1 - A_G)^{-1} \mathbb{Z}^{N_0} \oplus \text{Ker}(1 - A_G) \quad i = 0, 1,
\]

where \(A_G\) is the \(N_0 \times N_0\) matrix defined by \(A_G(i, j) = \sum_{\alpha \in \Sigma} A^G(i, \alpha, j)\).
Proof. It suffices to show the formulae of \(K\)-groups. Since \(K_1(A_G,T) = \mathbb{Z}^{N_0}, i = 0, 1\), by [Pim] (cf. [KPW]) the six term exact sequence of \(K\)-theory:

\[
\begin{array}{cccccc}
\mathbb{Z}^{N_0} & \xrightarrow{id - A_G} & \mathbb{Z}^{N_0} & \xrightarrow{id} & K_0(O_{G,\theta_1,\ldots,\theta_N}) \\
\uparrow & & & \downarrow & \\
K_1(O_{G,\theta_1,\ldots,\theta_N}) & \xleftarrow{id} & \mathbb{Z}^{N_0} & \xleftarrow{id - A_G} & \mathbb{Z}^{N_0},
\end{array}
\]

holds so that one has the short exact sequences for \(i = 0, 1\)

\[
0 \longrightarrow \mathbb{Z}^{N_0}/(1 - A_G)\mathbb{Z}^{N_0} \longrightarrow K_i(O_{G,\theta_1,\ldots,\theta_N}) \longrightarrow \text{Ker}(1 - A_G) \longrightarrow 0.
\]

They split because \(\text{Ker}(1 - A_G)\) is free so that the desired formulae hold. \(\square\)

We denote by \(O_G\) the \(C^*\)-algebra of the labeled graph \(G\). It is isomorphic to a Cuntz-Krieger algebra (cf. [BP], [Ca], [Ma2], [Tom]). For \(i, j = 1, \ldots, N_0\), let \(f_1, \ldots, f_m\) be the set of edges in \(G\) whose source is \(v_i\) and terminal is \(v_j\). Then we set \(A^G(i,j) = e^{2\pi \sqrt{-1} \gamma_{k_1}} + \cdots + e^{2\pi \sqrt{-1} \gamma_{k_m}}\) formal sums for \(\lambda(f_l) = \alpha_{k_l}, l = 1, \ldots, m\). We have \(N_0 \times N_0\) matrix \(A^G\) with entries in formal sums of nonnegative real numbers.

**Proposition 8.2.** Suppose that the labeled graph \(G\) satisfies condition (I) and is irreducible. If there exists \(n \in \mathbb{N}\) and \(i = 1, \ldots, N_0\) such that the \((i, i)\)-component \((A^G)^n(i, i)\) of the \(n\)-th power of the matrix \(A^G\) contains an irrational angle of rotation, then \((\alpha_1, \ldots, \alpha_N)\) is \(G\)-minimal, so that the \(C^*\)-algebra \(O_{G,\theta_1,\ldots,\theta_N}\) is simple, purely infinite.

**Proof.** One knows that \((\alpha_1, \ldots, \alpha_N)\) is \(G\)-minimal by Proposition 7.5. It is easy to see that \((A_G, \rho_{\theta_1,\ldots,\theta_N}, \Sigma)\) is effective. As the algebra \(O_G\) is purely infinite, so is \(O_{G,\theta_1,\ldots,\theta_N}\) by Theorem 5.5. \(\square\)

We will study the structure of both the algebra \(O_{G,\theta_1,\ldots,\theta_N}\) and the fixed point algebra \(F_{G,\theta_1,\ldots,\theta_N}\) of \(O_{G,\theta_1,\ldots,\theta_N}\) under the gauge action. We denote by \(F_G\) the fixed point algebra of \(O_G\) under the gauge action.

**Proposition 8.3.** Assume that the labeled graph \(G\) satisfies condition (I).

(i) \(O_{G,\theta_1,\ldots,\theta_N}\) is isomorphic to the crossed product \(O_G \rtimes \gamma_{\theta_1,\ldots,\theta_N} \mathbb{Z}\) of the Cuntz-Krieger algebra \(O_G\) of the labeled graph \(G\) by an automorphisms \(\gamma_{\theta_1,\ldots,\theta_N}\) of \(O_G\).

(ii) \(F_{G,\theta_1,\ldots,\theta_N}\) is an \(AT\)-algebra, that is isomorphic to the crossed product \(F_G \rtimes \gamma_{\theta_1,\ldots,\theta_N} \mathbb{Z}\) of the \(AF\)-algebra \(F_G\) by the automorphism defined by the restriction of \(\gamma_{\theta_1,\ldots,\theta_N}\) to \(F_G\).

**Proof.** (i) Put \(E_i = U_i^*U_i, i = 1, \ldots, N_0\). The relations

\[
\sum_{j=1}^{N_0} E_j = 1, \quad S_n^*E_iS_n = \sum_{j=1}^{N_0} A^G(i, \alpha_n, j)E_j
\]

hold for \(n = 1, \ldots, N, i = 1, \ldots, N_0\). Hence the \(C^*\)-subalgebra \(C^*(S_n, E_i : n = 1, \ldots, N, i = 1, \ldots, N_0)\) of \(O_{G,\theta_1,\ldots,\theta_N}\) generated by \(S_n, E_i : n = 1, \ldots, N, i = 1, \ldots, N_0\).
1, \ldots, N_0 \) is isomorphic to the Cuntz-Krieger algebra \( \mathcal{O}_G \) of the labeled graph \( G \).

Put \( U = \sum_{i=1}^{N_0} U_i \) a unitary. It is straightforward to see the following relations hold:

\[
US_n U^* = e^{2\pi \sqrt{-1} \theta_n} S_n, \quad UE_i = E_i U = U_i,
\]

for \( n = 1, \ldots, N, i = 1, \ldots, N_0 \). Since the algebra \( \mathcal{O}_{G, \theta_1, \ldots, \theta_N} \) is generated by \( S_n, E_i \) for \( n = 1, \ldots, N, i = 1, \ldots, N_0 \) and by putting

\[
\gamma_{\theta_1, \ldots, \theta_N}(S_n) = e^{2\pi \sqrt{-1} \theta_n} S_n, \quad \gamma_{\theta_1, \ldots, \theta_N}(E_i) = E_i
\]

one sees that \( \mathcal{O}_{G, \theta_1, \ldots, \theta_N} \) is the crossed product of \( C^*(S_n, E_i : n = 1, \ldots, N, i = 1, \ldots, N_0) \) by the automorphism \( \gamma_{\theta_1, \ldots, \theta_N} \).

(ii) The AF-algebra \( \mathcal{F}_G \) is regarded as the \( C^* \)-subalgebra of \( \mathcal{O}_{G, \theta_1, \ldots, \theta_N} \) generated by the elements of the form: \( S_{\mu} E_i S_{\nu}, \mu, \nu \in \Lambda^*, |\mu| = |\nu|, i = 1, \ldots, N_0 \). Under the identification, the algebra \( \mathcal{F}_{G, \theta_1, \ldots, \theta_N} \) is generated by \( \mathcal{F}_G \) and the above unitary \( U \). By \( \gamma_{\theta_1, \ldots, \theta_N}(S_{\mu} E_i S_{\nu}) = e^{2\pi \sqrt{-1} (\theta_{\mu_1} + \cdots + \theta_{\mu_k} - \theta_{\nu_1} - \cdots - \theta_{\nu_k})} S_{\mu} E_i S_{\nu} \) for \( \mu = (\mu_1, \ldots, \mu_k), \nu = (\nu_1, \ldots, \nu_k) \in \Lambda^k \), one knows that \( \mathcal{F}_{G, \theta_1, \ldots, \theta_N} \) is isomorphic to the crossed product \( \mathcal{F}_G \rtimes \gamma_{\theta_1, \ldots, \theta_N} \mathbb{Z} \) of \( \mathcal{F}_G \) by \( \gamma_{\theta_1, \ldots, \theta_N} \). By Proposition 6.8, one sees that \( \mathcal{F}_{G, \theta_1, \ldots, \theta_N} \) is an \( \mathbb{AT} \)-algebra. \( \square \)

9. Irrational Rotation Cuntz Algebras

In this section, we treat special cases of the previous section. We consider a labeled graph of \( N \)-loops with single vertex. Let \( A = C(\mathbb{T}) \) and \( \Sigma = \{1, \ldots, N\}, N > 1 \). Take real numbers \( \theta_1, \ldots, \theta_N \in (0, 1) \). Define \( \alpha_i(f)(z) = f(e^{2\pi \sqrt{-1} \theta_i} z) \) for \( f \in C(\mathbb{T}), z \in \mathbb{T} \). We have a \( C^* \)-symbolic dynamical system \( (C(\mathbb{T}), \alpha, \Sigma) \). Since \( \alpha_i, i = 1, \ldots, N \) are automorphisms, the associated subshift is the full shift \( \Sigma^\mathbb{Z} \). We denote by \( \mathcal{O}_{\theta_1, \ldots, \theta_N} \) the \( C^* \)-symbolic crossed product \( C(\mathbb{T}) \rtimes_\alpha \Sigma^\mathbb{Z} \). As the algebra \( \mathcal{O}_{\theta_1, \ldots, \theta_N} \) is the universal \( C^* \)-algebra generated by \( N \) isometries \( S_i, i = 1, \ldots, N \) and one unitary \( U \) subject to the relations:

\[
\sum_{j=1} S_j S_j^* = 1, \quad S_i^* S_i = 1, \quad US_i = e^{2\pi \sqrt{-1} \theta_i} S_i U, \quad i = 1, \ldots, N,
\]

it is realized as the ordinary crossed product \( \mathcal{O}_N \rtimes \gamma_{\theta_1, \ldots, \theta_N} \mathbb{Z} \) of the Cuntz algebra \( \mathcal{O}_N \) by the automorphism \( \gamma_{\theta_1, \ldots, \theta_N} \) defined by \( \gamma_{\theta_1, \ldots, \theta_N}(S_i) = e^{2\pi \sqrt{-1} \theta_i} S_i \). The K-groups are

\[
K_0(\mathcal{O}_{\theta_1, \ldots, \theta_N}) \cong K_1(\mathcal{O}_{\theta_1, \ldots, \theta_N}) \cong \mathbb{Z}/(N-1)\mathbb{Z}.
\]

By Theorem 5.5 and Theorem 7.4, one sees

**Proposition 9.1.** The \( C^* \)-algebra \( \mathcal{O}_{\theta_1, \ldots, \theta_N} \) is simple if and only if at least one of \( \theta_1, \ldots, \theta_N \) is irrational. In this case, \( \mathcal{O}_{\theta_1, \ldots, \theta_N} \) is pure infinite.

**Remark.** The algebra \( \mathcal{O}_{\theta_1, \ldots, \theta_N} \) is the crossed product \( \mathcal{O}_N \rtimes \gamma_{\theta_1, \ldots, \theta_N} \mathbb{Z} \) of the Cuntz algebra \( \mathcal{O}_N \) by the automorphism \( \gamma_{\theta_1, \ldots, \theta_N} \). The condition that at least one of \( \theta_1, \ldots, \theta_N \) is irrational is equivalent to the condition that the automorphisms \( (\gamma_{\theta_1, \ldots, \theta_N})^n \) are outer for all \( n \in \mathbb{Z}, n \neq 0 \). Hence by [Ki], the assertion for the simplicity of \( \mathcal{O}_{\theta_1, \ldots, \theta_N} \) in Proposition 9.1 holds.

We will study the fixed point algebra, denoted by \( \mathcal{F}_{\theta_1, \ldots, \theta_N} \), of \( \mathcal{O}_{\theta_1, \ldots, \theta_N} \) under the gauge action. It is generated by elements of the form \( S_{\mu} f S_{\nu}^* \) for \( f \in C(\mathbb{T}), |\mu| = |\nu| \).
Let $\mathcal{F}_{\theta_1, \ldots, \theta_N}^k$ be the $C^*$-subalgebra of $\mathcal{F}_{\theta_1, \ldots, \theta_N}$ generated by elements of the form $f \in C(\mathbb{T}), |\mu| = |\nu| = k$. The map

$$S_\mu f S_\nu^* \in \mathcal{F}_{\theta_1, \ldots, \theta_N}^k \rightarrow f \otimes S_\mu S_\nu^* \in C(\mathbb{T}) \otimes M_{N^k}$$

yields an isomorphism between $\mathcal{F}_{\theta_1, \ldots, \theta_N}^k$ and $C(\mathbb{T}) \otimes M_{N^k}$. Then the natural inclusion $\mathcal{F}_{\theta_1, \ldots, \theta_N}^k \hookrightarrow \mathcal{F}_{\theta_1, \ldots, \theta_N}^{k+1}$ through the identity $S_\mu f S_\nu^* = \sum_{i=1}^N S_{\mu_i} \alpha_i(f) S_{\nu_i}^*$ corresponds to the inclusion

$$f \otimes e_{i,j} \in C(\mathbb{T}) \otimes M_{N^k} \hookrightarrow \begin{bmatrix} \alpha_1(f) \otimes e_{i,j} & 0 \\ \alpha_2(f) \otimes e_{i,j} & \ddots \\ 0 & \alpha_N(f) \otimes e_{i,j} \end{bmatrix} \in C(\mathbb{T}) \otimes M_{N^{k+1}}.$$ 

For $\mu = (\mu_1, \ldots, \mu_k) \in \Sigma^k$, we set $\alpha_\mu = \alpha_{\mu_k} \circ \cdots \circ \alpha_{\mu_1}$. Since $\mathcal{F}_{\theta_1, \ldots, \theta_N}$ is an inductive limit of the inclusions $\mathcal{F}_{\theta_1, \ldots, \theta_N}^k \hookrightarrow \mathcal{F}_{\theta_1, \ldots, \theta_N}^{k+1}, k = 1, 2, \ldots$ as in Proposition 6.8, it is an AT-algebra.

**Proposition 9.2.** The $C^*$-algebra $\mathcal{F}_{\theta_1, \ldots, \theta_N}$ is simple if and only if $\theta_i - \theta_j$ is irrational for some $i, j = 1, \ldots, N$.

**Proof.** It is not difficult to prove the assertion directly by looking at the above inclusions $\mathcal{F}_{\theta_1, \ldots, \theta_N}^k \hookrightarrow \mathcal{F}_{\theta_1, \ldots, \theta_N}^{k+1}, k \in \mathbb{N}$. The following argument is a shorter proof by using [Ki]. Let $\mathcal{F}_N$ be the UHF-algebra of type $N^\infty$, that is the fixed point algebra of $\mathcal{O}_N$ by the gauge action. By Proposition 8.3, $\mathcal{F}_{\theta_1, \ldots, \theta_N}$ is the crossed product $\mathcal{F}_N \rtimes_{\gamma_{\theta_1, \ldots, \theta_N}} \mathbb{Z}$ where $\gamma_{\theta_1, \ldots, \theta_N}(S_\mu S_\nu^*) = e^{2\pi \sqrt{-1}(\theta_{\mu_1} + \cdots + \theta_{\mu_k} - \theta_{\nu_1} - \cdots - \theta_{\nu_k})} S_\mu S_\nu^*$ for $\mu = (\mu_1, \ldots, \mu_k), \nu = (\nu_1, \ldots, \nu_k) \in \Sigma^k$. Hence the automorphisms $\gamma_{\theta_1, \ldots, \theta_N}$ is the product type automorphism $\prod_{\theta \in \mathbb{T}} \text{Ad}(u_\theta) = \text{Ad}(u_\theta) \otimes \text{Ad}(u_\theta) \otimes \cdots$ for the unitary

$$u_\theta = \begin{bmatrix} e^{2\pi \sqrt{-1} \theta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{2\pi \sqrt{-1} \theta_N} \end{bmatrix} \in M_N(\mathbb{C})$$

under the canonical identification between $\mathcal{F}_N$ and $M_N \otimes M_N \otimes \cdots$. Then the condition that $\theta_i - \theta_j$ is irrational for some $i, j = 1, \ldots, N$ is equivalent to the condition that $(\text{Ad}(u_\theta))^n \neq \text{id}$ for all $n \in \mathbb{Z}, n \neq 0$. In this case, the product type automorphisms $(\prod_{\theta} \text{Ad}(u_\theta))^n$ are outer for all $n \in \mathbb{Z}, n \neq 0$. Hence by [Ki], the assertion holds.

For $\{\theta_1, \ldots, \theta_N\}$ and $n \in \mathbb{N}$, put

$$S_n(\theta_1, \ldots, \theta_N) = \{\theta_{i_1} + \cdots + \theta_{i_n} \mid i_1, \ldots, i_n = 1, \ldots, N\}.$$

then the sequence $\{S_n(\theta_1, \ldots, \theta_N)\}_{n \in \mathbb{N}}$ of finite sets is said to be uniformly distributed in $\mathbb{T}$ ([Ki2]) if

$$\lim_{n \to \infty} \frac{1}{N^n} \sum_{i_1, \ldots, i_n = 1}^N f(e^{2\pi \sqrt{-1}(\theta_{i_1} + \cdots + \theta_{i_n})}) = \int_{\mathbb{T}} f(t) dt \quad \text{for all } f \in C(\mathbb{T}).$$

The following lemma is easy
Lemma 9.3. \(\{S_n(\theta_1, \ldots, \theta_N)\}_{n \in \mathbb{N}}\) is uniformly distributed in \(T\) if and only if \(\theta_i - \theta_j\) is irrational for some \(i, j = 1, \ldots, N\).

Proof. \(\{S_n(\theta_1, \ldots, \theta_N)\}_{n \in \mathbb{N}}\) is uniformly distributed in \(T\) if and only if

\[
\lim_{n \to \infty} \frac{1}{N^n} \sum_{i_1, \ldots, i_N=1}^{N} e^{2\pi \sqrt{-1} \ell (\theta_{i_1} + \cdots + \theta_{i_N})} = 0 \quad \text{for all } \ell \in \mathbb{Z}, \ell \neq 0.
\]

Since \(\sum_{i_1, \ldots, i_N=1}^{N} e^{2\pi \sqrt{-1} \ell (\theta_{i_1} + \cdots + \theta_{i_N})} = (e^{2\pi \sqrt{-1} \ell \theta_1} + \cdots + e^{2\pi \sqrt{-1} \ell \theta_N})^n\), the condition (9.1) holds if and only if

\[
|e^{2\pi \sqrt{-1} \ell \theta_1} + \cdots + e^{2\pi \sqrt{-1} \ell \theta_N}| < N \quad \text{for all } \ell \in \mathbb{Z}, \ell \neq 0.
\]

The condition (9.2) is equivalent to the condition that \(\theta_i - \theta_j\) is irrational for some \(i, j = 1, \ldots, N\). □

Therefore we have

Theorem 9.4. For \(\theta_1, \ldots, \theta_N \in [0, 1)\), the following conditions are equivalent:

(i) \(\theta_i - \theta_j\) is irrational for some \(i, j = 1, \ldots, N\).

(ii) \(F_{\theta_1, \ldots, \theta_N}\) is simple.

(iii) \(F_{\theta_1, \ldots, \theta_N}\) has real rank zero.

Proof. The equivalence between (i) and (ii) follows from Proposition 9.2. It suffices to show the equivalence between (i) and (iii). Since

\[
\text{Sp}(u_\theta \otimes \cdots \otimes u_\theta) = S_n(\theta_1, \ldots, \theta_N)
\]

and \(\gamma_{\theta_1, \ldots, \theta_N}\) is a product type automorphism on \(\prod \otimes \text{Ad}(u_\theta)\) on the UHF-algebra \(F_N\), by [Ki2;Lemma 5.2] the crossed product \(F_N \rtimes_{\gamma_{\theta_1, \ldots, \theta_N}} \mathbb{Z}\) has real rank zero if and only if \(S_n(\theta_1, \ldots, \theta_N)\) is uniformly distributed in \(T\). □

We note that by [Ki;Lemma 5.2], the crossed product \(F_N \rtimes_{\gamma_{\theta_1, \ldots, \theta_N}} \mathbb{Z}\) has real rank zero if and only if \(F_{\theta_1, \ldots, \theta_N}\) has a unique trace.

Consequently we obtain

Theorem 9.5. For \(\theta_1, \ldots, \theta_N \in [0, 1)\), suppose that there exist \(i, j = 1, \ldots, N\) such that \(\theta_i - \theta_j\) is irrational. Then the C*-algebra \(F_{\theta_1, \ldots, \theta_N}\) is a unital simple \(\mathcal{A}\mathbb{T}\)-algebra of real rank zero with a unique tracial state such that

\[
K_0(F_{\theta_1, \ldots, \theta_N}) \cong \mathbb{Z}[\frac{1}{N}], \quad K_1(F_{\theta_1, \ldots, \theta_N}) \cong \mathbb{Z}.
\]

Hence \(F_{\theta_1, \ldots, \theta_N}\) is the Bunce-Deddens algebra of type \(N^\infty\).

Proof. Since \(K_i(C(C(T \otimes M_N) = \mathbb{Z}, i = 0, 1\) and the homomorphisms in Proposition 6.8 yield the \(N\)-multiplications on \(K_0(C(C(T \otimes M_N) = \mathbb{Z} \to K_0(C(C(T \otimes M_{N+1}) = \mathbb{Z}\) and the identities on \(K_1(C(C(T \otimes M_N) = \mathbb{Z} \to K_1(C(C(T \otimes M_{N+1}) = \mathbb{Z}\), we get the K-theory formulae by Proposition 6.8. The obtained isomorphism from \(K_0(F_{\theta_1, \ldots, \theta_N})\) to \(\mathbb{Z}[\frac{1}{N}]\) preserves their order and maps the unit 1 of \(F_{\theta_1, \ldots, \theta_N}\) to 1 in \(\mathbb{Z}[\frac{1}{N}]\). Hence \(F_{\theta_1, \ldots, \theta_N}\) is isomorphic to the Bunce-Deddens algebra of type \(N^\infty\). □
References

[BP] T. Bates and D. Pask, $C^*$-algebras of labeled graphs, J. Operator Theory 57 (2007), 207–226.

[BD] J. Bunce and J. Deddens, A family of simple $C^*$-algebras related to weighted shift operators, J. Funct. Anal. 19 (1975), 12–34.

[Ca] T. M. Carlsen, On $C^*$-algebras associated with sofic shifts, J. Operator Theory 49 (2003), 203–212.

[C] J. Cuntz, Simple $C^*$-algebras generated by isometries, Commun. Math. Phys. 57 (1977), 173–185.

[C2] J. Cuntz, A class of $C^*$-algebras and topological Markov chains II: reducible chains and the Ext-functor for $C^*$-algebras, Invent. Math. 63 (1980), 25–40.

[CK] J. Cuntz and W. Krieger, A class of $C^*$-algebras and topological Markov chains, Invent. Math. 56 (1980), 251–268.

[DNNP] M. Dădărlat, G. Nagy, A. Némethi, C. Pasnicu, Reduction of topological stable rank in inductive limits of $C^*$-algebras, Pacific J. Math. 153 (1992), 267–276.

[De] V. Deaconu, Groupoids associated with endomorphisms, Trans. Amer. Math. Soc. 347 (1995), 1779–1786.

[De2] V. Deaconu, Generalized Cuntz-Krieger algebras, Proc. Amer. Math. Soc. 124 (1996), 3427–3435.

[De3] V. Deaconu, Generalized solenoids and $C^*$-algebras, Pacific J. Math. 190 (1999), 247–260.

[Ell] G. A. Elliott, Some simple $C^*$-algebras constructed as crossed products with discrete outer automorphisms groups, Publ. RIMS Kyoto Univ. 16 (1980), 299–311.

[EL] R. Exel and M. Laca, Cuntz-Krieger algebras for infinite matrices, J. reine. angew. Math. 512 (1999), 119–172.

[KPW] T. Kajiwara, C. Pinzari and Y. Watatani, Ideal structure and simplicity of the $C^*$-algebras generated by Hilbert modules, J. Funct. Anal. 159 (1998), 295–322.

[KW] T. Kajiwara and Y. Watatani, Hilbert $C^*$-bimodules and continuous Cuntz-Krieger algebras considered by Deaconu, J. Math. Soc. Japan 54 (2002), 35–60.

[Ka] Y. Katayama, Generalized Cuntz algebras $O_N^M$, RIMS kokyuroku 858 (1994), 87–90.

[Kat] T. Katsura, A class of $C^*$-algebras generalizing both graph graph algebras and homeomorphism $C^*$-algebras I, fundamental results, Trans. Amer. Math. Soc. 356 (2004), 4287–4322.

[Kat2] T. Katsura, A construction of $C^*$-algebras from $C^*$-correspondences, Advances in Quantum Dynamics, Contemporary Mathematics (AMS) 335 (2003), 173–182.

[Kat3] T. Katsura, Ideal structure of $C^*$-algebras associated with $C^*$-correspondences, to appear in Pacific J. Math..

[LM] D. Lind and B. Marcus, An introduction to symbolic dynamics and coding, Cambridge University Press., 1995.

[Ki] A. Kishimoto, Outer automorphisms and reduced crossed products of simple $C^*$-algebras, Commun. Math. Phy. 81 (1981), 429–435.

[Ki2] A. Kishimoto, The Rohlin property for automorphisms of UHF algebras, J. Reine Angew. Math. 465 (1995), 183–196.

[KPSS] A. Kumjian, D. Pask, I. Raeburn and J. Renault, Graphs, groupoids and Cuntz-Krieger algebras, J. Funct. Anal. 144 (1997), 505–541.

[Ma] K. Matsumoto, Presentations of subshifts and their topological conjugacy invariants, Doc. Math. 4 (1999), 285–340.

[Ma2] K. Matsumoto, $C^*$-algebras associated with presentations of subshifts, Doc. Math. 7 (2002), 1–30.

[Ma3] K. Matsumoto, Construction and pure infiniteness of the $C^*$-algebras associated with $\lambda$-graph systems, Math. Scand. 97 (2005), 73–89.

[Ma4] K. Matsumoto, $C^*$-algebras associated with presentations of subshifts II, -ideal structures and lambda-graph subsystems-, J. Australian Mathematical Society 81 (2006), 369–385.

[Ma5] K. Matsumoto, Symbolic dynamical systems and endomorphisms on $C^*$-algebras, RIMS Kokyuroku 1379 (2004), 26–47.

[Ma6] K. Matsumoto, Actions of symbolic dynamical systems on $C^*$-algebras, to appear in J. Reine Angew. Math.
[MS] P. S. Muhly and B. Solel, On the simplicity of some Cuntz-Pimsner algebras, Math. Scand. 83 (1998), 53–73.

[Pim] M. V. Pimsner, A class of $C^*$-algebras generalizing both Cuntz-Krieger algebras and crossed product by $\mathbb{Z}$, in Free Probability Theory, Fields Institute Communications 12 (1996), 189–212.

[PV] M. Pimsner and D. Voiculescu, Exact sequences for $K$-groups and Ext-groups of certain cross-products $C^*$-algebras, J. Operator Theory 4 (1980), 93–118.

[PWY] C. Pinzari, Y. Watatani and K. Yonetani, KMS states, entropy and the variational principle in full $C^*$-dynamical systems, Commun. Math. Phys. 213 (2000), 331–381.

[Re] J. N. Renault, A groupoid approach to $C^*$-algebras, Lecture Notes in Math. Springer 793 (1980).

[Sch] J. Schweizer, Dilations of $C^*$-correspondences and the simplicity of Cuntz-Pimsner algebras, J. Funct. Anal. 180 (2001), 404–425.

[Tom] M. Tomforde, A unified approach to Excel-Laca algebras and $C^*$-algebras associated to graphs, J. Operator Theory 50 (2003), 345–368.

[Tom2] M. Tomforde, Simplicity of ultragraph algebras algebras, Indiana Univ. Math. J. 52 (2003), 901–926.

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