An iteration regularizaion method with general convex penalty for nonlinear inverse problems in Banach spaces

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Abstract. In this paper, we discuss the construction, analysis and implementation of a novel iterative regularization scheme with general convex penalty term for nonlinear inverse problems in Banach spaces based on the homotopy perturbation technique, in an attempt to detect the special features of the sought solutions such as sparsity or piecewise constant. By using tools from convex analysis in Banach spaces, we provide a detailed convergence and stability results for the presented algorithm. Numerical simulations for one-dimensional and two-dimensional parameter identification problems are performed to validate that our approach is competitive in terms of reducing the overall computational time in comparison with the existing Landweber iteration with general convex penalty.

1. Introduction

In this paper, we will consider the nonlinear ill-posed operator equation

\begin{equation}
F(x) = y,
\end{equation}

where $F : D(F) \subset \mathcal{X} \to \mathcal{Y}$ is a nonlinear operator between the Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ with norms $\| \cdot \|$, whose topological dual spaces are denoted by $\mathcal{X}^*$ and $\mathcal{Y}^*$, respectively. Instead of the right hand side $y$, only a noisy data $y^\delta$ is available such that

$$
\| y - y^\delta \| \leq \delta,
$$

with a small known noise level $\delta > 0$. Due to the ill-posedness of equation (1.1), a direct inversion of noise-contaminated data $y^\delta$ would not lead to a meaningful solution. Consequently, to find the stable and desired approximations of solutions of (1.1), we have to apply some regularization strategy. Tikhonov regularization is certainly the most popular stabilization approach for solving nonlinear ill-posed problems \cite{9, 21} and generalized in Banach spaces \cite{25}. The minimization of Tikhonov-type functional is usually realized via optimization schemes, in which the good choice of the regularization parameter is crucial for the quality of the reconstructed solution.
which often leads to increasing numerical stabilities and costs. On the contrary, due to the straightforward implementation, iterative regularization methods \[17] seem to be a promising and attractive alternative, in which the iterative steps plays the role of the regularization parameters.

Here the focus is on the generalization of the Landweber iteration method regarding Banach spaces. For given parameter \( r > 1 \), by making use of a gradient method for solving the minimization problem

\[
\min \frac{1}{r} \| F(x) - y^\delta \|_r,
\]

we therefore consider the following iteration

\[
\xi_{n+1} = \xi_n - \mu_n F'(x_n)^* J_r^\delta (F(x_n) - y^\delta), \quad x_{n+1} = J_{s^*}^\delta(\xi_{n+1}),
\]

(1.3)

together with suitably chosen step length \( \mu_n \), where \( F'(x_n)^* : Y^* \rightarrow X^* \) denotes the adjoint of \( F'(x_n) \), \( J_r^\delta : Y \rightarrow Y^* \) and \( J_{s^*}^\delta : X^* \rightarrow X \) denote the corresponding duality mapping with gauge function \( t \mapsto t^{-r} \) and \( t \mapsto t^{s^*-1} \) respectively. The choice of the parameter \( s^* \in [1, 2] \) is determined by the supposed smoothness of the dual space \( X^* \).

Starting with [23] for linear problems, many publications have been concerned with an iteration of this type for nonlinear problems, see [12, 16, 18]. In [16], introducing a uniformly convex penalty \( \Theta : X \rightarrow (-\infty, \infty] \) chosen with desired features, the Landweber-type iteration with general convex penalty (named by LICP later) was proposed:

\[
\xi_{n+1} = \xi_n - \mu_n F'(x_n)^* J_r^\delta (F(x_n) - y^\delta), \quad x_{n+1} = \nabla \Theta^\delta(\xi_{n+1}),
\]

(1.4)

where \( \Theta^\delta : X^* \rightarrow (-\infty, \infty] \) is the convex conjugate of \( \Theta \) and \( \nabla \Theta^\delta : X^* \rightarrow X \) denotes its gradient. The function \( \Theta \) can be chosen as the hybrid terms combining two very powerful features: \( L^2 + L^1 \) function known to promote sparsity and \( L^2 + TV \) function allowing for detecting the sharp edges. Iteration (1.4) can be interpreted as linearized Bregman iteration see [19, 26] in the linear case. A general convergence analysis and regularization results on (1.4) is given in [16, 20] under the termination with the discrepancy principle.

Homotopy perturbation iteration for nonlinear ill-posed problems in Hilbert spaces was first constructed by Li Cao, Bo Han and Wei Wang in [3, 4]. Its essential idea is to introduce an embedding homotopy parameter and combine the traditional perturbation method with the homotopy technique. Using the notation \( T_n := F'(x_n) \), N-order homotopy perturbation iteration method can be formulated by

\[
x_{n+1} = x_n - \sum_{j=1}^{N} (I - T_n^* T_n)^{j-1} T_n^* (F(x_n) - y^\delta).
\]

(1.5)

It is noteworthy that [13] also can be explained as the N-steps classical Landweber iteration for solving the linearized problem [13]: \( F(x_n) + T_n(x - x_n) = y^\delta \). With the one-order approximation truncation \( (N = 1) \), it can yield the classical Landweber iteration [11]:

\[
x_{n+1} = x_n - T_n^* (F(x_n) - y^\delta).
\]

(1.6)

With the two-order approximation truncation \( (N = 2) \), the homotopy perturbation iteration [3] can be obtained:

\[
x_{n+1} = x_n - T_n^* (2I - T_n T_n^*) (F(x_n) - y^\delta).
\]

(1.7)
It is shown that only half-time for (1.7) is needed with the same accuracy compared with (1.6). Subsequently, it was successfully applied to the well log constrained seismic waveform inversion [11]. Nevertheless, both (1.6) and (1.7) mainly restricted to the case of quadratic penalty terms may be no longer available for detecting the specific solutions with the discontinuity of sharp points or edges.

Inspired by the homotopy perturbation iteration in Hilbert space, in this paper we propose a novel iteration regularization method with general uniformly convex penalty for nonlinear inverse problems in Banach spaces. The approach (homotopy perturbation iteration with general convex penalty, named by HPICP later) generalizes the two-order homotopy perturbation iteration (1.7) in Banach spaces, in which the duality mappings \( J^Y \) and \( J^{\ast} \) : \( X \to Y \) with suitably chosen step length \( \mu_n \) are used and general uniformly convex penalty \( \Theta : X \to (-\infty, \infty] \) is introduced. For simplicity of exposition, we here write the notation \( r_n := J^Y_r(F(x_n) - g^\delta) \) and the proposed HPICP method can be formulated as follows:

\[
\xi_{n+1} = \xi_n - \mu_n T_n^s (2r_n - \nu_n J^Y_r(T_n J^s_r T_n^s r_n)), \quad x_{n+1} = \nabla^* \Theta^\ast(\xi_{n+1}). \tag{1.8}
\]

with suitably chosen step length \( \mu_n, \nu_n \). In contrast to LICP method [16], our proposed approach has the advantage of improving the calculation speed by strongly reducing the iteration numbers. As an iterative regularization method, the discrepancy principle is used to terminate the iteration. Due to the non-smooth convex penalty, which may include \( L^1 \) penalty or TV penalty, the iteration can produce good results in applications, where the sought solution is sparse or discontinuous. Moreover, iterative regularization in Banach spaces can be used to the non-Gaussian noisy data. We expect that our method can become favorable by using other accelerated versions.

The outline of this paper is as follows. In Section 2, we give some preliminary results from convex analysis in Banach spaces. In Section 3, we present the detailed convergence analysis and regularization results of our method combined with the discrepancy principle as stooping rule. In section 4 we report some numerical simulations to test the performance of the method. Finally, a short conclusion is drawn in Section 5.

2. Preliminaries

Throughout this paper \( X \) is supposed to be uniformly smooth and uniformly convex, hence it is reflexive and the dual \( X^\ast \) has the same properties [7]. Let \( 1 < s, s^\ast < \infty \) denote conjugate exponents, i.e., \( 1/s + 1/s^\ast = 1 \). For any \( x \in X \) and \( \xi \in X^\ast \), we write \( \langle x, \xi \rangle = \langle \xi, x \rangle \) for the duality pairing. We use \( A : X \to Y \) to denote a bounded linear operator and \( A^\ast : Y^\ast \to X^\ast \) to denote its adjoint, i.e. \( \langle A^\ast \varsigma, x \rangle = \langle \varsigma, Ax \rangle \) for any \( x \in X, \varsigma \in Y^\ast \), and \( \|A\| = \|A^\ast\| \) for the operator norm of \( A \). Let \( \mathcal{N}(A) = \{x \in X : Ax = 0\} \) be the null space of \( A \), and let

\[
\mathcal{N}(A)^\perp := \{\xi \in X : \langle \xi, x \rangle = 0, \text{ for all } x \in \mathcal{N}(A)\}.
\]

be the annihilator of \( \mathcal{N}(A) \).

On a Banach space \( Y \), we consider the convex function \( x \to \|x\|^r / r \) (\( 1 < r < \infty \)). Its subdifferential at \( x \) is given by

\[
J^Y_r(x) := \{\xi \in Y : \|\xi\| = \|x\|^{r-1} \text{ and } \langle \xi, x \rangle = \|x\|^r\}.
\]
which gives the duality mapping \( J^Y_\gamma : Y \to 2^{Y^*} \) of \( Y \) with gauge function \( t \mapsto t^{p-1} \). It is well known that the duality mapping \( J^Y_\gamma (1 < r < \infty) \) is single valued and uniformly continuous on bounded sets if \( Y \) is uniformly smooth. It is an in general nonlinear set-value mapping. For \( s^* > 1 \) with \( 1/s + 1/s^* = 1 \) we denote by \( J^*_s : X^* \to X \) the duality mapping of the \( X^* \) with gauge function \( t \mapsto t^{s^*-1} \).

Given a convex function \( \Theta : X \to (-\infty, \infty] \) we use \( D(\Theta) := \{ x \in X : \Theta(x) < +\infty \} \) to denote its effective domain. It is called proper if \( D(\Theta) \neq \emptyset \). The subdifferential of \( \Theta \) at \( x \in X \) is defined as

\[
\partial \Theta(x) := \{ \xi \in X^* : \Theta(z) - \Theta(x) - \langle \xi, z - x \rangle \geq 0, \text{ for all } z \in X \}.
\] (2.1)

The subdifferential mapping \( \partial \Theta : X \to 2^{X^*} \) is multi-valued and we set \( D(\partial \Theta) := \{ x \in D(\Theta) : \partial \Theta(x) \neq \emptyset \} \).

A proper convex function \( \Theta : X \to (-\infty, \infty] \) is called uniformly convex if there is a continuous increasing function \( h : [0, \infty] \to [0, \infty] \), with the property that \( h(t) = 0 \) implies \( t = 0 \), such that

\[
\Theta(\gamma \bar{x} + (1 - \gamma)x) + \gamma(1 - \gamma)h(||\bar{x} - x||) \leq \gamma \Theta(\bar{x}) + (1 - \gamma)\Theta(x).
\]

for all \( x, \bar{x} \in X \) and all \( \gamma \in [0, 1] \). If \( h \) can be taken as \( h(t) = c_0 t^p \) for some \( c_0 > 0 \) and \( p \geq 2 \), then \( \Theta \) is called \( p \)-convex. Any uniformly convex mapping is strictly convex.

In the convex analysis, the Legendre-Fenchel conjugate is an important notation. Given a proper, lower semi-continuous, convex function \( \Theta : X \to (-\infty, \infty] \), its Legendre-Fenchel conjugate \( \Theta^* : X^* \to [-\infty, \infty] \) is defined by

\[
\Theta^*(\xi) := \sup_{x \in X} \{ \langle \xi, x \rangle - \Theta(x) \}, \quad \forall \xi \in X^*.
\]

It is well known that \( \Theta^* \) is also proper, lower semi-continuous and convex. And as an immediate consequence of the definition, we will have the following lemma.

**Lemma 2.1.** For arbitrary \( x \in X, \xi \in X^* \), Young-Fenchel inequality holds as follows:

\[
\langle \xi, x \rangle \leq \Theta(x) + \Theta^*(\xi)
\]

and

\[
\xi \in \partial \Theta(x) \iff x \in \partial \Theta^*(\xi) \iff \langle \xi, x \rangle = \Theta(x) + \Theta^*(\xi).
\] (2.2)

We in Banach spaces introduce the Bregman distance with respect to convex function \( \Theta \), which for any \( x \in D(\partial \Theta(x)) \) and \( \xi \in \partial \Theta(x) \) is given by

\[
D_\xi \Theta(z, x) := \Theta(z) - \Theta(x) - \langle \xi, z - x \rangle, \quad \forall z \in X.
\]

It is clear that the Bregman distance is non-negative and it holds \( D_\xi \Theta(x, x) = 0 \). Bregman distance can be used to obtain important information under the Banach space norm when \( \Theta \) has stronger convexity.

**Lemma 2.2** ([27], Corollary 3.5.11). Let \( \Theta : X \to (-\infty, \infty] \) is proper, lower semi-continuous and \( p \)-convex for some \( p \geq 2 \). Then

1) there exists a constant \( c_0 > 0 \) such that

\[
D_\xi \Theta(\bar{x}, x) \geq c_0 ||\bar{x} - x||^p, \quad \forall \bar{x} \in X \text{ and } \xi \in \partial \Theta(x).
\] (2.3)
2) $D(\Theta^*) = \mathcal{X}^*$, \( \Theta^* \) is Fréchet differentiable and its gradient $\nabla \Theta^* : \mathcal{X}^* \rightarrow \mathcal{X}$ satisfies
\[
\| \nabla \Theta^*(\xi_1) - \nabla \Theta^*(\xi_2) \| \leq \left( \frac{\| \xi_1 - \xi_2 \|}{2c_0} \right)^{\frac{1}{s}}, \forall \xi_1, \xi_2 \in \mathcal{X}^*. \tag{2.4}
\]

In addition, by the subdifferential calculus there also holds
\[
x = \nabla \Theta^*(\xi) \iff x = \arg \min_{x \in \mathcal{X}} \{ \Theta(x) - \langle \xi, x \rangle \}. \tag{2.5}
\]

3. The method and its convergence

In this section we first formulate the novel iteration regularization method with the general uniformly convex penalty terms. And then we present the detailed convergence analysis. Throughout this section we will assume that $\mathcal{X}$ is $s$-convex for some $s \in [2, \infty)$, $(1/s + 1/s^* = 1)$ and $\Theta : \mathcal{X} \rightarrow (-\infty, \infty]$ is a proper, lower semi-continuous, $p$-convex function with $p \geq 2$. Assuming that $\mathcal{Y}$ is uniformly smooth Banach space so that the duality mapping $J^\mathcal{Y} : \mathcal{Y} \rightarrow \mathcal{Y}^*$ is single-valued and continuous for each $1 < r < \infty$. By picking $x_0 \in D(\partial \Theta)$ and $\xi_0 \in \partial \Theta(x_0)$ as the initial guess, we define $x^\dagger$ to be the solution of (1.1) with the property
\[
D_{\xi_0} \Theta(x^\dagger, x_0) := \min_{x \in D(\partial \Theta) \cap D} \{ D_{\xi_0} \Theta(x, x_0) : F(x) = y \}. \tag{3.1}
\]

We are interested in developing algorithms to find the solution $x^\dagger$ of (1.1). We will need to impose the following conditions on the nonlinear operator $F$ where $B_\rho(x_0) := \{ x \in \mathcal{X} : \| x - x_0 \| \leq \rho \}$.

**Assumption 3.1.**
(a) There is $\rho > 0$ such that $B_{2\rho}(x_0) \subset D(F)$ and (1.1) has a solution in $B_\rho(x_0) \cap D(\Theta)$.
(b) Operator $F$ is weakly closed on $D(F)$ and is Fréchet differentiable on $B_{2\rho}(x_0)$, and $F^* : x \rightarrow F^*(x)$ is continuous on $B_{2\rho}(x_0)$.
(c) Fréchet operator $F^*$ is locally uniformly bounded so that
\[
\| F^*(x) \| \leq B_0, \ \forall x \in B_{2\rho}(x_0).
\]
(d) There exists $0 \leq \eta < 1$ such that the tangential cone condition holds
\[
\| F(x) - F(\bar{x}) - F^*(\bar{x})(x - \bar{x}) \| \leq \eta \| F(x) - F(\bar{x}) \|, \ \forall x, \bar{x} \in B_{2\rho}(x_0).
\]

When $\mathcal{X}$ is a reflexive Banach space, by using the $p$-convexity and the weakly lower semi-continuity of $\Theta$ together with the weakly closedness of $F$, it is standard to show that $x^\dagger$ exists. The following result shows that $x^\dagger$ is in fact uniquely defined, and more detailed proof can be seen in [16].

**Lemma 3.2.** Let $\mathcal{X}$ be reflexive and $F$ satisfy Assumption 3.1. If $x^\dagger \in B_{\rho}(x_0) \cap D(\Theta)$, then $x^\dagger$ is the unique solution of (1.1) in $B_{2\rho}(x_0) \cap D(\Theta)$ satisfying (3.1).

For the situation that the data contains noise, we may imitate (1.8) to define an iterative sequence $\{(x^n_0, \xi^n_0)\}$ in $\mathcal{X} \times \mathcal{X}^*$. For simplicity of the presentation we set $T^n_0 := F'(x^n_0)$ and $r^n_0 := J^\mathcal{Y}(F'(x^n_0) - y^0)$. We denote the initial guess by
\( x_0^\delta := x_0 \in \mathcal{D}(\partial \Theta) \) and \( \xi_0^\delta := \xi_0 \in \partial \Theta(x_0) \). Once we have \( \{(x_n^\delta, \xi_n^\delta)\} \), we may define \( \{(x_{n+1}^\delta, \xi_{n+1}^\delta)\} \) by

\[
\begin{align*}
\xi_{n+1}^\delta &= \xi_n^\delta - \mu_n^\delta T_n^\delta \left( 2r_n^\delta - \nu_n^\delta J_n^\delta (T_n^\delta J_n^\delta)^* r_n^\delta \right), \\
x_{n+1}^\delta &= \arg\min_{x \in \mathcal{X}} \{\Theta(x) - \langle \xi_{n+1}^\delta, x \rangle\},
\end{align*}
\tag{3.2}
\]

with a proper choice of the step size \( \mu_n^\delta \). Note that by using (2.2) one can see that

\[
x_{n+1}^\delta = \arg\min_{x \in \mathcal{X}} \{\Theta(x) - \langle \xi_{n+1}^\delta, x \rangle\} \iff x_{n+1}^\delta = \nabla^\star(\xi_{n+1}^\delta),
\]

which will be used in the forthcoming theoretical analysis.

In case of noisy data, the iteration procedure (3.2) has to be coupled with a stopping rule in order to act as a regularization method. We will employ the discrepancy principle as a stopping rule, which determines the stopping index \( n_\delta = n_\delta(\delta, y^\delta) \) by

\[
\|F(x_n^\delta) - y^\delta\| \leq \tau \delta \leq \|F(x_n^\delta) - y^\delta\|, \quad 0 \leq n < n_\delta,
\]

for some sufficiently large \( \tau > 0 \), i.e., \( x_{n_\delta}^\delta \) is the calculated approximate solution.

Next we will show that (3.2) has well convergence under the discrepancy principle. In the following proposition we will first prove monotonicity of the errors.

**Proposition 3.3 (Error analysis).** Let Assumption 3.1 hold with \( 0 \leq \eta < 1 \) and let \( \Theta : \mathcal{X} \to (-\infty, \infty] \) be a proper, lower semi-continuous, \( p \)-convex function with \( p \geq 2 \) satisfying (2.3) for some \( c_0 > 0 \). Assume that

\[
D_{\xi_0^\delta} \Theta(x_0^\delta, x_0) \leq c_0 \rho^p. \tag{3.3}
\]

Let \( \{(x_n^\delta, \xi_n^\delta)\} \) be defined by iteration (3.2) with

\[
y_n^\delta = \frac{\|r_n^\delta\|}{\|J_n^\delta (T_n^\delta J_n^\delta)^* r_n^\delta\|}, \quad \mu_n^\delta = \mu_0 B_0^{-p} \|F(x_n^\delta) - y^\delta\|^{p-r},
\]

and the iteration is terminated by the discrepancy principle

\[
\|F(x_{n_\delta}^\delta) - y^\delta\| \leq \tau \delta \leq \|F(x_n^\delta) - y^\delta\|, \quad 0 \leq n < n_\delta
\]

with \( \tau > 1 \) and \( \mu_0 > 0 \) satisfies

\[
c_1 := 1 - 3\eta - \frac{3(\eta + 1)}{\tau} - 3 \left( \frac{3\mu_0}{2c_0} \right)^{\frac{1}{p-1}} > 0.
\]

Then \( n_\delta < \infty \) and \( x_n^\delta \in B_{2p}(x_0) \) for all \( n \geq 0 \). Moreover, for any solution \( \hat{x} \) of \( F(x) = y \) in \( B_{2p}(x_0) \cap \mathcal{D}(\Theta) \) and all \( n \) there hold

\[
D_{\xi_{n+1}^\delta} \Theta(\hat{x}, x_{n+1}^\delta) \leq D_{\xi_n^\delta} \Theta(\hat{x}, x_n^\delta), \tag{3.4}
\]

\[
c_1 \mu_n^\delta \|F(x_n^\delta) - y^\delta\|^p \leq D_{\xi_n^\delta} \Theta(\hat{x}, x_n^\delta) - D_{\xi_{n+1}^\delta} \Theta(\hat{x}, x_{n+1}^\delta). \tag{3.5}
\]

**Proof.** From the definition of Bregman distance and together with using (2.2), it follows that

\[
D_{\xi_{n+1}^\delta} \Theta(\hat{x}, x_{n+1}^\delta) - D_{\xi_n^\delta} \Theta(\hat{x}, x_n^\delta)
\]

\[
= \Theta(x_n^\delta) - \Theta(x_{n+1}^\delta) - \langle \xi_{n+1}^\delta, \hat{x} - x_{n+1}^\delta \rangle + \langle \xi_n^\delta, \hat{x} - x_n^\delta \rangle
\]

\[
= \Theta^\star(\xi_{n+1}^\delta) - \Theta^\star(\xi_n^\delta) - \langle \xi_{n+1}^\delta - \xi_n^\delta, \hat{x} \rangle.
\]
By introducing \( x_\delta_n = \nabla \Theta^*(\xi_\delta_n) \), we can write
\[
D_{\xi_\delta_{n+1}} \Theta(\hat{x}, x_{\delta_{n+1}}) - D_{\xi_\delta} \Theta(\hat{x}, x_\delta) = \Theta^*(\xi_{\delta_{n+1}}) - \Theta^*(\xi_\delta) - \langle \xi_{\delta_{n+1}} - \xi_\delta, \nabla \Theta^*(\xi_\delta) \rangle + \langle \xi_{\delta_{n+1}} - \xi_\delta, x_\delta - \hat{x} \rangle.
\]
Since \( \Theta \) is p-convex, we may use (3.2) to obtain
\[
\Theta^*(\xi_{\delta_{n+1}}) - \Theta^*(\xi_\delta) - \langle \xi_{\delta_{n+1}} - \xi_\delta, \nabla \Theta^*(\xi_\delta) \rangle = \int_0^1 \langle \xi_{\delta_{n+1}} - \xi_\delta, \nabla \Theta^*(\xi_\delta + t(\xi_{\delta_{n+1}} - \xi_\delta)) - \nabla \Theta^*(\xi_\delta) \rangle dt
\leq ||\xi_{\delta_{n+1}} - \xi_\delta|| \int_0^1 \|\nabla \Theta^*(\xi_\delta + t(\xi_{\delta_{n+1}} - \xi_\delta)) - \nabla \Theta^*(\xi_\delta)\| dt
\leq (2c_0)^{1-p^*} ||\xi_{\delta_{n+1}} - \xi_\delta||^{p^*},
\]
where \( p^* \) is the number conjugate to \( p \), i.e., \( 1/p + 1/p^* = 1 \).

By the definition of \( \xi_{\delta_{n+1}} \) in iteration (3.2), we then have
\[
D_{\xi_\delta_{n+1}} \Theta(\hat{x}, x_{\delta_{n+1}}) - D_{\xi_\delta} \Theta(\hat{x}, x_\delta) \leq (2c_0)^{1-p^*} ||\xi_{\delta_{n+1}} - \xi_\delta||^{p^*} + \langle \xi_{\delta_{n+1}} - \xi_\delta, x_\delta - \hat{x} \rangle,
\]
where
\[
\xi_{\delta_{n+1}} - \xi_\delta = -\mu^*_n T^\delta \left( 2\nu^*_n - \nu^*_n J^\delta r_n (T^\delta J^\delta r_n, T^\delta r_n) \right).
\]

By virtue of the property of the duality mapping \( J^\delta_r \), we have
\[
\| J^\delta_r (T^\delta J^\delta r_n, T^\delta r_n) \| \leq \| T^\delta J^\delta r_n \|^{\frac{r}{r-1}} \text{ and } \| r_n^\delta \| \leq \| F(x_n^\delta) - y^\delta \|^{-\frac{r}{r-1}}. \tag{3.7}
\]

Moreover, according to the scaling condition in Assumption 3.1(c) (i.e., \( T^\delta_n \| \leq B_0 \)), we have by taking \( \nu^*_n \leq \| T^\delta_n \|^{-\delta} \) that
\[
\nu^*_n \| T^\delta_n J^\delta_r (T^\delta J^\delta r_n, T^\delta r_n) \| \leq B_0 \| r_n^\delta \|. \tag{3.8}
\]

Therefore,
\[
\| \xi_{\delta_{n+1}} - \xi_\delta \|^{p^*} = (\mu^*_n)^p \| 2T^\delta r_n - \nu^*_n J^\delta r_n (T^\delta J^\delta r_n, T^\delta r_n) \|^{p^*} \leq (3B_0)^p (\mu^*_n)^p \| r_n^\delta \|^{p^*}.
\]

Furthermore, we estimate
\[
\langle \xi_{\delta_{n+1}} - \xi_\delta, x_\delta - \hat{x} \rangle = -\mu_n (2\nu_n - \nu_n J^\delta (T_n J^\delta r_n, T_n r_n), T_n (x_\delta - \hat{x}))
= -2\mu_n (r_n (T_n (x_\delta - \hat{x})), + \mu_n \nu_n (J^\delta (T_n J^\delta r_n, T_n r_n), T_n (x_\delta - \hat{x})),
\]

where
\[
-\langle r_n, -T_n (x_\delta - \hat{x}) \rangle = -\langle r_n, F(\hat{x}) - F(x_\delta - T_n (\hat{x} - x_\delta)) \rangle - \langle r_n, F(x_\delta - y^\delta + y^\delta) - y^\delta \rangle
\leq \eta \| r_n^\delta \| \| F(\hat{x}) - F(x_\delta) \| - \langle r_n, F(x_\delta - y^\delta) + \| r_n^\delta \| \delta
\leq (1 + \eta) \| F(x_\delta) - y^\delta \|^{\frac{r}{r-1}} \delta - (1 - \eta) \| F(x_\delta) - y^\delta \|^{\frac{r}{r-1}}
\]
and
\[
\nu_n (J^\delta (T_n J^\delta r_n, T_n r_n), T_n (x_\delta - \hat{x})) \leq \nu_n \| J^\delta (T_n J^\delta r_n, T_n r_n) \| \| T_n (x_\delta - \hat{x}) \|
\leq (1 + \eta) \| F(x_\delta) - y^\delta \|^{\frac{r}{r-1}} \| F(\hat{x}) - F(x_\delta) \|
\leq (1 + \eta) \| F(x_\delta) - y^\delta \|^{\frac{r}{r-1}} (\delta + \| F(x_\delta) - y^\delta \|).
Hence, by the stopping rule we have

\[
\langle \xi_{n+1}^{\delta} - \xi_n^{\delta}, x_n^{\delta} - \hat{x} \rangle \leq -\mu_n^{\delta}(1 - 3\eta)\|F(x_n^{\delta}) - y^{\delta}\|r + 3(1 + \eta)\mu_n^{\delta}\|F(x_n^{\delta}) - y^{\delta}\|r - 1\delta \\
\leq -\mu_n^{\delta}(1 - 3\eta - (1 + \eta)/\tau)\|F(x_n^{\delta}) - y^{\delta}\|r.
\]

In addition, by the definition of \(\mu_n^{\delta}\) it is easy to see that

\[
B_0^r (\mu_n^{\delta})^{-p-1} \|F(x_n^{\delta}) - y^{\delta}\|p(r-1) \leq \mu_n^{\delta}^{-p-1}\|F(x_n^{\delta}) - y^{\delta}\|r.
\]

Then combining with these two inequalities with (3.6), we thus obtain

\[
D_{\xi_{n+1}}^\delta \Theta(\hat{x}, x_{n+1}^{\delta}) - D_{\xi_n}^\delta \Theta(\hat{x}, x_n^{\delta}) \\
\leq (2c_0)^{1-p'}(3B_0)p\mu_n^{\delta}r\|F(x_n^{\delta}) - y^{\delta}\|p(r-1) - \mu_n^{\delta}(1 - 3\eta - 3(1 + \eta)/\tau)\|F(x_n^{\delta}) - y^{\delta}\|r \\
\leq 3p'(2c_0)^{1-p'}\mu_n^{\delta}(\mu_0)p^{-1}\|F(x_n^{\delta}) - y^{\delta}\|r - \mu_n^{\delta}(1 - 3\eta - 3(1 + \eta)/\tau)\|F(x_n^{\delta}) - y^{\delta}\|r \\
\leq -c_1\mu_n^{\delta}\|F(x_n^{\delta}) - y^{\delta}\|r,
\]

i.e., the error is decreasing. To show \(x_{n+1}^{\delta} \in B_2(p, x_0)\), we first use the above inequality with \(\hat{x} = x^1\) and (3.9) to obtain

\[
D_{\xi_{n+1}}^\delta \Theta(x^1, x_{n+1}^{\delta}) - D_{\xi_n}^\delta \Theta(x^1, x_n^{\delta}) \leq c_0\rho.
\]

In view of (2.3), we then have \(\|x_{n+1}^{\delta} - x^1\| \leq \rho\) and \(\|x^1 - x_0\| \leq \rho\). Consequently \(x_{n+1}^{\delta} \in B_2(p, x_0)\).

We next show \(n_\delta < \infty\). According to the definition of \(n_\delta\), for any \(n < n_\delta\) such that \(\|F(x_n^{\delta}) - y^{\delta}\| > \tau\delta\). Then there holds

\[
\mu_n^{\delta} = \mu_0B_0^{-p}\|F(x_n^{\delta}) - y^{\delta}\|p-r,
\]

and

\[
\sum_{n=0}^{n_\delta} \mu_n^{\delta}\|F(x_n^{\delta}) - y^{\delta}\|r \geq \mu_0B_0^{-p}\|F(x_n^{\delta}) - y^{\delta}\|p > \mu_0B_0^{-p}(\tau\delta)^p.
\]

By summing (3.9) over \(n\) from \(n = 0\) to \(n = m\) for any \(m < n_\delta\) and using the above inequality we obtain

\[
c_1\mu_0B_0^{-p}(\tau\delta)^p(m + 1) \leq D_{\xi_0} \Theta(\hat{x}, x_0).
\]

Since this is true for any \(m < n_\delta\), it follows that \(n_\delta < \infty\). \(\square\)

When the iteration (1.8) is applied to the exact data, i.e., using \(y\) instead of \(y^{\delta}\) in (1.8), we will drop the superscript \(\delta\) in all the quantities involved, for instance, we will write \(\xi_n\) as \(\xi_n^{\delta}\), \(x_n^{\delta}\) as \(x_n\), and so on. Observing that

\[
\mu_n\|F(x_n) - y\|r = \mu_0B_0^{-p}\|F(x_n) - y\|p.
\]

The proof of Proposition 3.3 in fact shows that, under Assumption (3.1) if

\[
c_2 := 1 - 3\eta - 3 \left( \frac{3\mu_0}{2c_0} \right)^{\frac{1}{r'}} > 0,
\]
then

\[ x_n \in B_{2\rho}(x_0) \quad \forall n \geq 0, \]

and for any solution \( \hat{x} \) of (1.1) in \( B_{2\rho}(x_0) \cap D(\Theta) \) and all \( n \) there hold

\[ D_{\xi_{n+1}} \Theta(\hat{x}, x_{n+1}) \leq D_{\xi_n} \Theta(\hat{x}, x_n), \quad (3.9) \]

\[ c_{2\rho_0} B_0^{-p}\|F(x_n) - y\|^p \leq D_{\xi_n} \Theta(\hat{x}, x_n) - D_{\xi_{n+1}} \Theta(\hat{x}, x_{n+1}). \quad (3.10) \]

These two inequalities imply immediately that

\[ \lim_{n \to \infty} \|F(x_n) - y\|^p = 0. \quad (3.11) \]

As the first step toward the proof of convergence on \( x^*_{\delta n} \), we need to derive some convergence results on the sequences \( \{x_n\} \) and \( \{\xi_n\} \). This will be achieved by the following proposition which gives a general convergence criterion on any sequences \( \{x_n\} \subset X \) and \( \{\xi_n\} \subset X^* \) satisfying certain conditions.

**Lemma 3.4.** Let all the conditions in Proposition 3.3 hold. For the sequences \( \{\xi_n\} \) and \( \{x_n\} \) defined by iteration (1.8) with exact data, there exists a solution \( x^* \in B_{2\rho}(x_0) \cap D(\Theta) \) of (1.1) such that

\[ \lim_{n \to \infty} \|x_n - x^*\| = 0 \quad \text{and} \quad \lim_{n \to \infty} D_{\xi_n} \Theta(x^*, x_n) = 0. \]

If in addition \( \mathcal{N}(F'(x^*)) \subset \mathcal{N}(F'(x)) \) for all \( x \in B_{2\rho}(x_0) \), then \( x^* = x^1 \).

**Proof.** We first show that there is a strictly increasing subsequence \( \{n_k\} \) of integers such that \( \{x_{n_k}\} \) is convergent. To this end, let

\[ R_n := \|y - F(x_n)\|^p. \]

It follows from (3.11) that

\[ \lim_{n \to \infty} R_n = 0. \quad (3.12) \]

Moreover, if \( R_n = 0 \) for some \( n \), then \( y = F(x_n) \). Consequently it follows from the definition of the method that \( x_m = x_n \) for all \( m \geq n \). Therefore

\[ R_n = 0 \implies R_m = 0 \quad \text{for all} \quad m \geq n. \quad (3.13) \]

In view of (3.12) and (3.13), we can introduce a subsequence \( \{n_k\} \) by setting \( n_0 = 0 \) and letting \( n_k \), for each \( k \geq 1 \), be the first integer satisfying

\[ n_k \geq n_{k-1} + 1 \quad \text{and} \quad R_{n_k} \leq R_{n_{k-1}}. \]

For such chosen strictly increasing sequence \( \{n_k\} \) we have

\[ R_{n_k} \leq R_n, \quad 0 \leq n < n_k. \quad (3.14) \]

Now for any \( l < k \), we consider \( D_{\xi_{n_l}} \Theta(x_{n_k}, x_{n_l}) \) as

\[ D_{\xi_{n_l}} \Theta(x_{n_k}, x_{n_l}) = D_{\xi_{n_l}} \Theta(\hat{x}, x_{n_l}) - D_{\xi_{n_k}} \Theta(\hat{x}, x_{n_k}) + (\xi_{n_k} - \xi_{n_l}, x_{n_k} - \hat{x}). \quad (3.15) \]
Therefore, by using the property of the duality mapping $J_p^n$, we have
\[
|\langle \xi_{n+1} - \xi_n, x_{n_k} - \hat{x} \rangle| \leq \mu_n \langle 2r_n - \nu_n J_p^n (T_n J_s^* T_n^* r_n), T_n (x_{n_k} - \hat{x}) \rangle
\]
Using the nonlinear condition on $F$ it is easy to obtain
\[
||T_n (x_{n_k} - \hat{x})|| \leq ||T_n (x_n - \hat{x})|| + ||T_n (x_{n_k} - x_n)||
\]
\[
\leq (1 + \eta) (||F(x_n) - y|| + ||F(x_{n_k}) - F(x_n)||)
\]
\[
\leq (1 + \eta) (2||F(x_n) - y|| + ||F(x_{n_k}) - y||).
\]
Therefore, by using the property of the duality mapping $J_p^n$, we have
\[
|\langle \xi_{n+1} - \xi_n, x_{n_k} - \hat{x} \rangle| \leq \mu_n ||2r_n - \nu_n J_p^n (T_n J_s^* T_n^* r_n), T_n (x_{n_k} - \hat{x})||
\]
\[
\leq 2\mu_n r_n ||T_n (x_{n_k} - \hat{x})|| + \mu_n \nu_n ||J_p^n (T_n J_s^* T_n^* r_n)|| ||T_n (x_{n_k} - \hat{x})||
\]
\[
\leq 3\mu_n ||F(x_n) - y|| + p^{-1} ||T_n (x_{n_k} - \hat{x})||
\]
\[
\leq 9(1 + \eta)\mu_0 B_0^{-p} ||F(x_n) - y||^p.
\]
Then combining this with the inequality (3.10), we can derive that
\[
|\langle \xi_{n} - \xi_{n_l}, x_{n_k} - \hat{x} \rangle| \leq \sum_{n=nl}^{n_k-1} |\langle \xi_{n+1} - \xi_n, x_{n_k} - \hat{x} \rangle|
\]
\[
\leq 9(1 + \eta)\mu_0 B_0^{-p} \sum_{n=nl}^{n_k-1} ||F(x_n) - y||^p
\]
\[
\leq C \left( D_{\xi_{n_l}} \Theta(\hat{x}, x_{n_l}) - D_{\xi_{n_k}} \Theta(\hat{x}, x_{n_k}) \right)
\]
with $C = 9(1 + \eta)B_0^{-p}/c2 > 0$. Thus we have from (3.15) and (3.17) that
\[
D_{\xi_{n_l}} \Theta(x_{n_k}, x_{n_l}) \leq (1 + C) \left( D_{\xi_{n_l}} \Theta(\hat{x}, x_{n_l}) - D_{\xi_{n_l}} \Theta(\hat{x}, x_{n_l}) \right).
\]
By the monotonicity of $D_{\xi} \Theta(\hat{x}, x_n)$, we obtain that $D_{\xi_{n_l}} \Theta(x_{n_k}, x_{n_l}) \to 0$ as $k, l \to \infty$.

By the p-convexity of $\Theta$ we can conclude that $\{x_{n_k}\}$ is a Cauchy sequence in $\mathcal{X}$ and thus $x_{n_k} \to x_\ast$ as $k \to \infty$ for some $x_\ast \in B_{2p}(x_0) \subset \mathcal{X}$.

Next we show that $x_\ast \in D(\Theta)$. We use $\xi_{n_l} \in \partial \Theta(x_{n_l})$ to obtain
\[
\Theta(x_{n_k}) = \Theta(x_{n_l}) + \langle \xi_{n_l}, x_{n_k} - x_{n_l} \rangle + D_{\xi_{n_l}} \Theta(x_{n_k}, x_{n_l})
\]
\[
\leq \Theta(x_{n_l}) + \langle \xi_{n_l}, x_{n_k} - x_{n_l} \rangle + (1 + C) D_{\xi_{n_l}} \Theta(\hat{x}, x_{n_l})
\]
(3.19)
Since $x_{n_k} \to x_\ast$ as $k \to \infty$, by using the lower semi-continuity of $\Theta$ we obtain
\[
\Theta(x_\ast) \leq \liminf_{k \to \infty} \Theta(x_{n_k}) \leq \Theta(x_{n_l}) + \langle \xi_{n_l}, x_\ast - x_{n_l} \rangle + (1 + C) D_{\xi_{n_l}} \Theta(\hat{x}, x_{n_l}) < \infty
\]
This implies that $x_\ast \in D(\Theta)$. 

where
\[
|\langle \xi_{n_k} - \xi_n, x_{n_k} - \hat{x} \rangle| = \sum_{n=nl}^{n_k-1} |\langle \xi_{n+1} - \xi_n, x_{n_k} - \hat{x} \rangle|
\]
\[
= \sum_{n=nl}^{n_k-1} \mu_n \langle 2r_n - \nu_n J_p^n (T_n J_s^* T_n^* r_n), T_n (x_{n_k} - \hat{x}) \rangle
\]
Furthermore, in order to derive the convergence in Bregman distance, we take $k \to \infty$, and use (3.18) and $x_{n_k} \to x_*$ to derive for $l < k$ that

$$D_{\xi_0} \Theta(x_*, x_{n_l}) \leq \liminf_{k \to \infty} D_{\xi_0} \Theta(x_{n_k}, x_{n_l}) \leq \liminf_{k \to \infty} (1 + C)(D_{\xi_0} \Theta(x_*, x_{n_k}) - \epsilon_0)$$

where $\epsilon_0 := \lim_{n \to \infty} D_{\xi_0} \Theta(x_*, x_n)$, whose existence is guaranteed by the monotonicity of $D_{\xi_0} \Theta(x_*, x_n)$. Since the above inequality holds for all $l$, by letting $l \to \infty$ we can obtain $\epsilon_0 \leq 0$. Therefore, we derive $\lim_{n \to \infty} D_{\xi_0} \Theta(x_*, x_n) = \epsilon_0 = 0$.

Finally we show that $x_* = x^\dagger$. We have

$$D_{\xi_0} \Theta(x_{n_k}, x_0) = D_{\xi_0} \Theta(x^\dagger, x_0) - D_{\xi_0} \Theta(x^\dagger, x_{n_k}) + \langle \xi_{n_k} - \xi_0, x_{n_k} - x^\dagger \rangle$$

By using (3.17), for any $\epsilon > 0$ we can find $k_0$ such that

$$\left| \langle \xi_{n_k} - \xi_{n_{k_0}}, x_{n_k} - x^\dagger \rangle \right| < \frac{\epsilon}{2}, \quad k \geq k_0.$$ 

We next consider $\langle \xi_{n_{k_0}} - \xi_0, x_{n_k} - x^\dagger \rangle$. By the definition of $\xi_n$, we have

$$\xi_{n+1} - \xi_n = -\mu_n (2T_n^* r_n - v_n \beta_n T_n^* J_n T_n^* r_n)$$

where $\mu_n > 0$. By the definition of $\beta_n$, we have

$$\xi_{n+1} - \xi_n = F'(x^\dagger)^* v_n + \beta_n \quad \text{and} \quad \| \beta_n \| \leq \frac{\epsilon}{3B_1 n_{k_0}}, \quad 0 \leq n < n_{k_0},$$

where $B_1 > 0$ is a constant such that $\| x_n - x^\dagger \| \leq B_1$ for all $n$. Consequently

$$\left| \langle \xi_{n_{k_0}} - \xi_0, x_{n_k} - x^\dagger \rangle \right| = \sum_{n=0}^{n_{k_0}-1} \left| \langle \xi_{n+1} - \xi_n, x_{n_k} - x^\dagger \rangle \right|$$

Using (3.20) we obtain

$$\langle \xi_{n_k} - \xi_0, x_{n_k} - x^\dagger \rangle \leq (1 + \eta) \sum_{n=0}^{n_{k_0}-1} \left\| \xi_{n_k} - \xi_0 \right\| \left\| x_{n_k} - x^\dagger \right\| \leq (1 + \eta) \sum_{n=0}^{n_{k_0}-1} \left\| \xi_{n_k} - \xi_0 \right\| \left\| x_{n_k} - x^\dagger \right\| \leq \frac{\epsilon}{3}.$$ 

Since $\| F(x_{n_k}) - y \| \to 0$ as $n \to \infty$, we can find $k_1 \geq k_0$ such that

$$\left| \langle \xi_{n_{k_0}} - \xi_0, x_{n_k} - x^\dagger \rangle \right| < \frac{\epsilon}{2}, \quad \forall k \geq k_1.$$ 

Therefore $\left| \langle \xi_{n_k} - \xi_0, x_{n_k} - x^\dagger \rangle \right| < \epsilon$ for all $k \geq k_1$. Since $\epsilon > 0$ is arbitrary, we obtain $\lim_{k \to \infty} \langle \xi_{n_k} - \xi_0, x_{n_k} - x^\dagger \rangle = 0$. By taking $k \to \infty$ in (3.20) we obtain

$$D_{\xi_0} \Theta(x_*, x_0) \leq D_{\xi_0} \Theta(x^\dagger, x_0).$$

According to the definition of $x^\dagger$ we must have $D_{\xi_0} \Theta(x_*, x_0) = D_{\xi_0} \Theta(x^\dagger, x_0)$. A direct application of Lemma 3.2 gives $x_* = x^\dagger$. 

\[\square\]
In order to use the above result to establish the convergence of Algorithm 1, we also need the following stability result.

**Theorem 3.5** (Stability analysis). Let \( X \) be reflexive and let \( Y \) be uniformly smooth. Let all the conditions in Proposition 3.3 hold. Then for all \( n \geq 0 \) there hold

\[
\xi_n^\delta \rightarrow \xi_n \quad \text{and} \quad x_n^\delta \rightarrow x_n, \quad \text{as} \quad \delta \rightarrow 0.
\]

**Proof.** The result is trivial for \( n = 0 \). We next assume that the result is true for some \( n \geq 0 \) and show that \( \xi_{n+1}^\delta \rightarrow \xi_{n+1} \) and \( x_{n+1}^\delta \rightarrow x_{n+1} \) as \( \delta \rightarrow 0 \). We consider two cases.

**Case 1:** \( F(x_n) = y \). In this case we have \( \mu_n = 0 \) and \( \lim_{\delta \to 0} \| F(x_n^\delta) - y^\delta \| = 0 \) by the continuity of \( F \). Thus

\[
\xi_{n+1}^\delta - \xi_{n+1} = \xi_n - \xi_n - \mu_n^\delta \nabla F_n^\delta \left( 2r_n^\delta - r_n^\delta J_n^\delta(T_n^\delta J_n^\delta r_n^\delta) \right)
\]

which implies that

\[
\| \xi_{n+1}^\delta - \xi_{n+1} \| \leq \| \xi_n - \xi_n \| + 3B_0 \mu_n^\delta \| F(x_n^\delta) - y^\delta \|^{r-1} \\
\leq \| \xi_n - \xi_n \| + 3B_0 \mu_n^\delta \| F(x_n^\delta) - y^\delta \|^{r-1}.
\]

By the induction hypotheses, we then have \( \xi_{n+1}^\delta \rightarrow \xi_{n+1} \). Consequently, by using the continuity of \( \nabla \Theta^* \), we have \( x_{n+1}^\delta = \nabla \Theta^*(\xi_{n+1}^\delta) \rightarrow \nabla \Theta^* (\xi_{n+1}) = x_{n+1} \) as \( \delta \rightarrow 0 \).

**Case 2:** \( F(x_n) \neq y \). In this case we have \( \| F(x_n^\delta) - y^\delta \| > \tau \delta \) for small \( \delta \rightarrow 0 \). Therefore

\[
\mu_n^\delta = \mu_0 B_0^{-p} \| F(x_n^\delta) - y^\delta \|^{p-r} \rightarrow \mu_n = \mu_0 B_0^{-p} \| F(x_n) - y \|^{p-r}
\]

as \( \delta \rightarrow 0 \). By Assumption 3.1(b) and the uniform smoothness of \( Y \), we know that \( F \), \( F' \) and \( J_n^\delta \) are continuous. It then follows from the induction hypotheses that \( \xi_{n+1}^\delta \rightarrow \xi_{n+1} \) and \( x_{n+1}^\delta \rightarrow x_{n+1} \) as \( \delta \rightarrow 0 \) using again the continuity of \( \nabla \Theta^* \).

We now apply the above results for proving the following convergence result, which shows the iteration (1.3) in combination with the discrepancy principle is a regularization method.

**Theorem 3.6** (Convergence analysis). Let \( X \) be reflexive and let \( Y \) be uniformly smooth. Let Assumption 3.3 hold. Let \( \Theta : X \rightarrow (-\infty, \infty] \) be proper, lower semicontinuous, and \( p \)-convex function satisfies (2.3). Assume that initial value \( x_0 \) and \( \xi_0 \) satisfies

\[
D_{\xi_0} \Theta(x^1, x_0) \leq \rho_0 \rho^p.
\]

Then for \( \{ \xi_n^\delta \} \) and \( \{ x_n^\delta \} \) defined by Algorithm 1 with \( \tau > 1 \) and \( \mu_0 > 0 \) satisfying

\[
c_1 := 1 - 3\eta - \frac{3(\eta + 1)}{\tau} - 3 \left( \frac{3\mu_0}{2\rho_0} \right)^{\frac{1}{p-1}} > 0,
\]

there is a solution \( x_* \in B_{2\rho}(x_0) \cap D(\Theta) \) of (2.3) such that

\[
\lim_{\delta \to 0} \| x_n^\delta - x_* \| = 0 \quad \text{and} \quad \lim_{\delta \to 0} D_{\xi_n^\delta} \Theta(x_*, x_{n\delta}) = 0.
\]

If in addition \( N(F'(x^\delta)) \subset N(F'(x)) \) for all \( x \in B_{2\rho}(x_0) \cap D(F) \), then \( x_* = x^1 \).

**Proof.** See Theorem 3.9 in [16].
4. Numerical examples

In this section we present some numerical simulations with one-dimensional and two-dimensional cases to test the good performance of the proposed HPICP method with various choices of the convex function $\Theta$, in comparison with the existing Landweber iteration method with convex penalty (LICP). Our simulations were done by using MATLAB R2010a on a Lenovo laptop with Intel Core i5–4200U CPU 2.30 GHz and 4.00 GB memory.

A key ingredient for HPICP method is the resolution of the minimization problem

$$x = \arg\min_{z \in \mathcal{X}} \{ \Theta(z) - \langle \xi, z \rangle \}, \forall \xi \in \mathcal{X}^*.$$  \hspace{1cm} (4.1)

Next we give some discussion on the resolution of (4.1) for various choices of $\Theta$ with $p = 2$ as follows:

**Case I:** Let $\mathcal{X} = L^2(\Omega)$ and the sought solution is partly sparse, we may consider the 2-convex function

$$\Theta(x) := \frac{1}{2\beta} \int_{\Omega} |x(\omega)|^2 d\omega + \int_{\Omega} |x(\omega)| d\omega$$  \hspace{1cm} (4.2)

with $\beta > 0$. The minimization of (4.1) for this case can be given explicitly by the following soft thresholding:

$$x = \arg\min_{x \in L^2} \left\{ \|x\|_{L^1} + \frac{1}{2\beta} \|x\|_{L^2}^2 - \langle \xi, x \rangle \right\}$$

$$= \begin{cases} 
\beta(\xi(\omega) - 1), & \text{if } \xi(\omega) > 1, \\
0, & \text{if } |\xi(\omega)| \leq 1, \\
\beta(\xi(\omega) + 1), & \text{if } \xi(\omega) < -1.
\end{cases}$$

**Case II:** Let $\mathcal{X} = L^2(\Omega)$ and the sought solution is piecewise constant, we may consider the total variation like function

$$\Theta(x) := \frac{1}{2\beta} \int_{\Omega} |x(\omega)|^2 d\omega + TV(x),$$  \hspace{1cm} (4.3)

with $\beta > 0$. The minimization of (4.1) for this case can be given explicitly as following:

$$x = \arg\min_{x \in L^2} \left\{ \beta TV(x) + \frac{1}{2} \|x - \beta\xi\|_{L^2(\Omega)}^2 \right\},$$

which is the well-known ROF model (see [23]) in image denoising. There are many efficient numerical solvers developed in the literature [1, 2, 6, 22, 28]; we use the fast iterative shrinkage-thresholding algorithm (FISTA) introduced from [1, 2] in our numerical simulations.

We here consider the nonlinear model problem which consists of recovering the potential term in an elliptic equation. Let $\Omega \subset \mathbb{R}^d (d = 1, 2)$ be an open bounded domain with a Lipschitz boundary $\Gamma$ and $f \in L^2(\Omega)$. We consider the identification of the parameter $c$ in the equation

$$\begin{cases} 
-\Delta u + cu = f, & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = 0, & \text{on } \Gamma.
\end{cases}$$  \hspace{1cm} (4.4)
We assume that the true potential $c^\dagger$ is in $L^2(\Omega)$. For each $c$ in the domain $D(F) := \{ c \in L^\infty(\Omega) : c \geq \bar{c} \text{ for some } \bar{c} \geq 0 \}$, there has a unique solution $u = u(c) \in H^1(\Omega)$. By the Sobolev embedding $H^1(\Omega) \to L^r(\Omega)$, we can define the nonlinear operator $F : \mathcal{X} = L^2(\Omega) \to \mathcal{Y} = L^r(\Omega)$ with $F(c) = u(c)$ for any $1 < r < \infty$. Hence we identify $c$ in the admissible set $D(F)$ from an $L^r$ measurement of $u$. Recall that in the Banach space $L^r(\Omega)$ with $1 < r < \infty$, the duality mapping $J_r : L^r(\Omega) \to L^{r^\ast}(\Omega)$ is given by

$$J_r(v) := |v|^{r-1}\text{sign}(v), \quad v \in L^r(\Omega).$$

We next will report numerical results to indicate the performance of HPICP method with various choices of the convex function $\Theta$ and the Banach spaces $\mathcal{Y}$. The main computational cost stems from the numerical solutions of differential equations related to calculating the Fréchet derivatives and their adjoint. In order to carry out the computation, the forward operator was discretized using finite elements on a uniform grid (triangular, in the case of two dimensions), which is based on the shared Matlab code by Bangti Jin of [5]. Given the true parameter $c^\dagger(x)$, the simulated noise data $u^\delta$ is generated by adding noise to the synthetic exact data $u^\ast = F(c^\dagger)$ as follows

$$u^\delta = u^\ast + \delta \cdot n,$$

here $\delta$ is the noise level and $n$ is the random variable obeying the standard normal distribution. In addition, in order to measure the accuracy of solution more quantitatively, we employ the following relative error

$$\text{RE} = \frac{\|c - c^\dagger\|}{\|c^\dagger\|},$$

where $c$ represents the approximate solution.

In the following we implement the HPICP method using $c_0 = \xi_0 = 0$ as the initial guess. We take the step length $\nu_n = \|T_n^{\delta^\ast}r_n^\delta\|^2_{L^2}/\|T_n^{\delta^\ast}J_r^T(T_n^{\delta^\ast}J_r^T r_n^\delta)\|^2_{L^2}$ with

$$\mu_n^\delta = \frac{\mu_0 \|F(x_n^\delta) - y^\delta\|_Y^{(r-1)r}}{\|T_n^{\delta^\ast}r_n^\delta\|^2_{L^2}},$$

here $\mu_0 = (1 - 1/\tau)/\beta$. To test the effects of $\beta$ for given convex penalty (4.2) and (4.3), we apply different choice for $\beta$ to perform the numerical computation. We will later report the detailed numerical results recovered by our proposed method (HPICP) and the current existing method (LICP), respectively, including the required iteration number (i.e., $n_\delta$), the computational time (i.e., time(s)) as well as the relative error (i.e., RE) between the true solutions and the regularized solutions.

**Example 4.1 One-dimensional example.** We consider the one-dimensional problem (4.3) on the interval $\Omega = [-1, 1]$ with the source term $f(x) = 1$. The mesh size is $h = 1/N$ with the grid points number $N = 256$. Figure 1(a) shows the situation that the sought solution is piecewise constant, which is given by

$$c^\dagger = 2 + \frac{3}{4}\chi_{[-0.5,-0.3]} + \frac{3}{2}\chi_{[-0.1,0.1]} + \frac{1}{2}\chi_{[0.3,0.5]}.$$ 

For this case, we take $\Theta$ to be $L^2 + TV$ regularization functional defined in (4.3).

We identify the true parameter $c^\dagger(x)$ given in Figure 1(a) using Gaussian noisy data with $\delta = 0.1\%$ noise level shown in Figure 1(b). We choose $\tau = 1.1$ in the
discrepancy principle. The comparison of reconstructed results by HPICP and LICP with $r = 2$ (i.e., $Y = L^2([-1,1])$) are summarized in Table 1. It can be seen from Table 1 that the regularized solutions by HPICP have the similar relative error qualities to those by LICP, but in less iteration number and computational time. That is to say, our proposed HPICP method leads to a strongly decrease of the iteration numbers and the overall computational time can be significantly reduced. In particular, for smaller $\beta$, the calculation times shows better performance of HPICP method under consideration. On the other hand, it is clear that when relatively larger $\beta$ is used, more accurate reconstructed results can be obtained; however the computation could take longer time because the convexity of the minimization problem involved becomes weaker and hence more iteration steps are required to obtain an approximate minimizer within a certain accuracy.

In order to visibly illustrate the convergence behavior of both methods, we draw the curves from RE vs. time(s) with various parameter $\beta$ in Figure 1(c). It is clear that the relative errors of HPICP are consistently lower than those of LICP. As can be expected, the convergence rate of the proposed HPICP is significantly accelerated compared to that of LICP. We then in Figure 1(d)-(i) plot the corresponding regularized solutions by HPICP for some selected values of $\beta$ to further visualize the
Table 1. Comparison of numerical results for Example 4.1 by HPICP and LICP with four different values of \( \beta \) under the noise level \( \delta = 0.1\% \).

| \( \beta \)   | \( n_5 \) | RE    | time(s) |
|-------------|----------|-------|---------|
| 0.025       | 24903    | 0.0638| 35.7858 |
| 1           | 1676     | 0.0643| 9.3851  |
| 5           | 9931     | 0.0564| 40.5642 |
| 10          | 3460     | 0.0404| 30.8745 |
| 20          | 10072    | 0.0298| 88.0081 |
| 50          | 14573    | 0.00299| 78.9212|

Since the results by LICP have the similar qualities to those by HPICP, we here do not list them. We observe that all the locations of the bumps are correctly identified with appropriate value of \( \beta \), and their magnitudes are also reasonable.

In addition, in order to test the robustness of the proposed method (HPICP) to noise, four various noise level are added to the generated exact data, respectively. For each noise level, we summarized detailed computational results in Table 2. As can be expected, HPICP shows the favorable robustness. We observe from Table 2 that with the increase of the noise levels the relative errors increase, which indicates that the accuracy of the measurement data has an effect on the reconstructed solution quality.

Table 2. Comparison of numerical results for Example 4.1 by HPICP with \( \beta = 20 \) at four different noise levels.

| Noise level | \( n_5 \) | RE    | Time(s) |
|-------------|----------|-------|---------|
| \( \delta = 1\% \) | 1744 | 0.0699 | 28.3991 |
| \( \delta = 0.5\% \) | 3001 | 0.0760 | 46.5127 |
| \( \delta = 0.1\% \) | 12137 | 0.0189 | 174.4382 |
| \( \delta = 0.05\% \) | 19550 | 0.0178 | 267.8776 |

Finally, we present a graphical demonstration of the effect of using Banach spaces setting in our considerations. Therefore we choose the Banach space \( Y = L^r[-1,1] \). Figure 2(a) shows the plot of the noisy data that contains a few data points, called outliers, which are highly inconsistent with other data points. \( L^r \) misfit data terms with \( r > 1 \) close to 1 are especially suitable for the outliers noise, see [8, 13]. For fixed \( \beta = 20 \) and under the above outliers data, Figure 2(b)-(d) present the reconstruction results with \( r = 1.05 \), \( r = 1.5 \) and \( r = 2 \), respectively. It can be seen that the method with small \( r \) is robust enough to prevent being affected by outliers.

**Example 4.2 Two-dimensional example.** Here, we consider the two-dimensional problem on the unit square, i.e., \( \Omega = [-1,1] \times [-1,1] \), \( f(x, y) = 1 \), and

\[
 c^4(x, y) = 1 + \cos(\pi x) \cos(\pi y) \chi_{\{|(x,y)|<1/2\}},
\]

where \( \chi_{\{|(x,y)|<1/2\}} \) is the characteristic function of the disk with radius 1/2, see Figure 3(a). For this situation, we take \( \Theta \) to be \( L^2 + L^1 \) regularization functional defined in [4, 12]. To obtain exact data \( u \) and noisy data \( u^\delta \), the mesh size for the forward solution is 7938 triangulation elements. We choose \( \tau = 2.1 \) in the discrepancy principle. The more detailed comparison of the solution by HPICP and LICP with
\(\beta = 1\) under five different noise levels are summarized in Table 3. It can be seen that both methods provide regularized solutions of similar quality, but the iteration number and the computational time can be significantly reduced when the homotopy perturbation technique are used. The results also indicate that both methods show the favorable robustness. Furthermore, we in Figure 3(b)-(d) depict the numerical results of HPICP with \(\beta = 1\) under three different noise levels, respectively. The numerical results of LICP have the similar qualities, and thus not shown here. It is clear that the solution accurately captures the shape as well as the magnitude of the potential \(c^\dagger\), and thus represents a good approximation.

**Table 3.** Comparison of numerical results for Example 4.2 by HPICP and LICP with \(\beta = 1\) under five different noise levels.

| Noise Level | LICP  | HPICP (\(r=1.5\)) | HPICP (\(r=2.0\)) | Time(s) |
|-------------|-------|--------------------|--------------------|---------|
| \(\delta = 1\%\) | 356   | 243                | 0.0195             | 0.0195  | 18.5048 |
| \(\delta = 0.5\%\) | 538   | 297                | 0.0149             | 0.0146  | 24.9653 |
| \(\delta = 0.1\%\) | 1398  | 960                | 0.0053             | 0.0052  | 79.5354 |
| \(\delta = 0.05\%\) | 1313  | 1213               | 0.0053             | 0.0040  | 111.0972|
| \(\delta = 0.01\%\) | 2439  | 6457               | 0.0040             | 0.0020  | 546.8676|
Figure 3. Results for the 2d inverse potential problem by HPICP with $\beta = 1$ at three different noise levels.

5. Conclusion

Motivated by chances of reducing numerical costs, this paper presented a novel iterative regularization approach with general uniformly convex penalty based on the homotopy perturbation technique for nonlinear ill-posed inverse problems in Banach spaces. Convergence and regularization properties were shown, as well as some numerical examples were performed to illustrate the feasibility and effectiveness. Compared with the existing Landweber iteration with general uniformly convex penalty, our approach reduces the overall computational time and improves the convergence rate. How to extend the approach for a novel class of reconstruction schemes is the next work.

Acknowledgement

The authors are grateful to Dr. Qinian Jin (Australian National University, Australia) for some useful comments. The work of Jing Wang is supported by the National Natural Science Foundation of China (NSFC) [grant number 11626092], Wei Wang by NSFC [grant number 11401257], Bo Han by NSFC [grant number 41474102].

References

[1] Beck A, Teboulle M. A fast iterative Shrinkage-Thresholding algorithm for linear inverse problems[3]. Siam Journal on Imaging Sciences, 2009, 2(1):183-202.
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[2] Beck A, Teboulle M. Fast gradient-based algorithms for constrained total variation image denoising and deblurring problems[J]. IEEE Transactions on Image Processing, 2009, 18(11): 2419-2434.

[3] Cao L, Han B, Wang W. Homotopy perturbation method for nonlinear ill-posed operator equations[J]. International Journal of Nonlinear Sciences and Numerical Simulation, 2009, 10(10):1319-1322.

[4] Cao L, Han B. Convergence analysis of the homotopy perturbation method for solving nonlinear ill-posed operator equations[J]. Computers and Mathematics with Applications, 2011, 61(8): 2058-2061.

[5] Chambolle A. An algorithm for total variation minimization and applications[J]. Journal of Mathematical Imaging and Vision, 2004, 20(1): 89-97.

[6] Chambolle A, Pock T. A first-order primal-dual algorithm for convex problems with applications to imaging[J]. Journal of Mathematical Imaging and Vision, 2011, 40(1): 120-145.

[7] Cioranescu I. Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems[M]. Kluwer Academic Pub, 1990.

[8] Clason C, Jin B. A semismooth newton method for nonlinear parameter identification problems with impulsive noise[J]. Siam Journal on Imaging Sciences, 2012, 5(2):505-536.

[9] Engl H W, Kunisch K, Neubauer A. Convergence rates for Tikhonov regularisation of non-linear ill-posed problems[J]. Inverse Problems, 1989, 5(4): 523.

[10] Fu H S, Cao L, Han B. A homotopy perturbation method for well log constrained seismic waveform inversion[J]. Chinese Journal of Geophysics-Chinese Edition, 2012, 55(9): 3173-3179.

[11] Hanke M, Neubauer A, Scherzer O. A convergence analysis of the Landweber iteration for nonlinear ill-posed problems[J]. Numerische Mathematik, 1995, 72(1):21-37.

[12] Hein T, Kazimierski K S. Accelerated Landweber iteration in Banach spaces. Inverse Problems, 2010, 26(26):1037-1050.

[13] Hohage T, Werner F. Convergence rates for inverse problems with impulsive noise[J]. Siam Journal on Numerical Analysis, 2014, 52(3):1203-1221.

[14] Hubmer S, Ramla R. Convergence analysis of a two-point gradient method for nonlinear ill-posed problems[J]. Inverse Problems, 2017, 33 (9).

[15] Jin Q. A general convergence analysis of some Newton-Type methods for nonlinear inverse problems.[J]. Siam Journal on Numerical Analysis, 2011, 49(49):549-573.

[16] Jin Q, Wang W. Landweber iteration of Kaczmarz type with general non-smooth convex penalty functionals[J]. Inverse Problems, 2013, 29(8): 085011.

[17] Kaltenbach A, Neubauer A, Scherzer O. Iterative regularization methods for nonlinear ill-posed problems[M]. Walter de Gruyter, 2008.

[18] Kaltenbach A, Schöpfer F, Schuster T. Iterative methods for nonlinear ill-posed problems in Banach spaces: convergence and applications to parameter identification problems[J]. Inverse Problems, 2009, 25(6):65003-65021(19).

[19] Lorenz D A, Schöpfer F, Wenger S. The linearized Bregman method via split feasibility problems: analysis and generalizations[J]. Siam Journal on Imaging Sciences, 2014, 7(7):1237-1262.

[20] Maaß P, Strehlow R. An iterative regularization method for nonlinear problems based on Bregman projections[J]. Inverse Problems, 2016, 32(11).

[21] Neubauer A. Tikhonov regularisation for non-linear ill-posed problems: optimal convergence rates and finite-dimensional approximation[J]. Inverse Problems, 1989, 5(4): 541.

[22] Ng M K, Qi L, Yang Y F, et al. On semismooth Newtons methods for total variation minimization[J]. Journal of Mathematical Imaging and Vision, 2007, 27(3): 265-276.

[23] Rudin L I, Osher S, Fatemi E. Nonlinear total variation based noise removal algorithms[J]. Physica D: Nonlinear Phenomena, 1992, 60(1-4): 259-268.

[24] Schöpfer F, Louis A K, Schuster T. Nonlinear iterative methods for linear ill-posed problems in Banach spaces. Inverse Problems, 2006, 22(1):311-329.

[25] Schuster T, Kaltenbacher B, Hofmann B, et al. Regularization methods in Banach spaces[M]. Walter de Gruyter, 2012.

[26] Yin W, Osher S, Goldfarb D, et al. Bregman iterative algorithms for l1-minimization with applications to compressed sensing.[J]. Siam Journal on Imaging Sciences, 2008, 1(1):143-168.

[27] Zălinescu C. Convex analysis in general vector spaces[M]. World Scientific, 2002.

[28] Zhu M, Chan T. An efficient primal-dual hybrid gradient algorithm for total variation image restoration[J]. UCLA CAM Report, 2008: 08-34.