On a two-phase Hele-Shaw problem with a time-dependent gap and distributions of sinks and sources

Tatiana Savina, Lanre Akinyemi and Avital Savin

1 Department of Mathematics, Ohio University, Athens, OH 45701, United States of America
2 Condensed Matter and Surface Science Program, Ohio University, Athens, OH 45701, United States of America
3 Nanoscale & Quantum Phenomena Institute, Ohio University, Athens, OH 45701, United States of America
4 Honors Tutorial College, Ohio University, Athens, OH 45701, United States of America

E-mail: savin@ohio.edu (T.V. Savina)

Received 3 July 2017, revised 22 October 2017
Accepted for publication 15 November 2017
Published 20 December 2017

Abstract
A two-phase Hele-Shaw problem with a time-dependent gap describes the evolution of the interface, which separates two fluids sandwiched between two plates. The fluids have different viscosities. In addition to the change in the gap width of the Hele-Shaw cell, the interface is driven by the presence of some special distributions of sinks and sources located in both the interior and exterior domains. The effect of surface tension is neglected. Using the Schwarz function approach, we give examples of exact solutions when the interface belongs to a certain family of algebraic curves and the curves do not form cusps. The family of curves are defined by the initial shape of the free boundary.

Keywords: Muskat problem, generalized Hele-Shaw flow, Schwarz function

(Some figures may appear in colour only in the online journal)

1. Introduction
Free boundary problems have been a significant part of modern mathematics for more than a century. They were influenced by the celebrated Stefan problem, which describes solidification, that is, an evolution of the moving front between liquid and solid phases. Free boundary problems also appear in fluid dynamics, geometry, finance, and many other applications (see https://people.ohio.edu/savin/)
[1] for a detailed discussion). Recently, they started to play an important role in modeling of biological processes involving moving fronts of populations or tumors [2]. These processes include cancer, biofilms, wound healing, granulomas, and atherosclerosis [2]. Biofilms are defined as communities of microorganisms, typically bacteria, that are attached to a surface. The biofilms motivated Friedman et al [3] to consider a two-phase free boundary problem, where one phase is an incompressible viscous fluid, and the other phase is a mixture of two incompressible fluids, which represent the viscous fluid and the polymeric network (with bacteria attached to it) associated with a biofilm. Free boundary problems are also used in modeling of a tumor growth with one phase to be the tumor region, and the other phase to be the normal tissue surrounding the tumor [4].

A Muskat problem is a free boundary problem related to the theory of flows in porous media [5]. It describes an evolution of an interface between two immiscible fluids, ‘oil’ and ‘water’, in a Hele-Shaw cell or in a porous medium. Here we study a two-phase Hele-Shaw flow assuming that the upper plate uniformly moves up or down changing the gap width of a Hele-Shaw cell. Hele-Shaw free boundary problems have been extensively studied over the last century (see [6, 7] and references therein). There are two classical formulations of the Hele-Shaw problems: the one-phase problem, when one of the fluids is assumed to be viscous while the other is effectively inviscid (the pressure there is constant), and the two-phase (or Muskat) problem. A statement of the problem with a time-dependent gap between the plates was mentioned in [8] among other generalized Hele-Shaw flows. The one-phase (interior) version of this problem was considered in [9], where conditions of existence, uniqueness, and regularity of solutions were established under assumption that surface tension effects on the free boundary are negligible; some exact solutions were constructed as well. An interior problem with a time-dependent gap and a non-zero surface tension was considered in [10], where asymptotic solutions were obtained for the case when initial shape of the droplet is a weakly distorted circle. Note also that the mathematical formulation of the interior problem with a time-dependent gap is similar to the problem of evaporation of a thin film [18]. When the surface tension is negligible, the pressure in both formulations can be obtained as a solution to the Poisson’s equation in a bounded domain with homogeneous Dirichlet data on the free boundary.

Much less progress has been made for the Muskat problem. Regarding the problem with a constant gap width, we should mention works [11–17]. Specifically, Howison [11] has obtained several simple solutions including the traveling-wave solutions and the stagnation point flow. In [11], an idea of a method for solving some two-phase problems was proposed and used to reappraise the Jacquard-Séguier solution [12]. Global existence of solutions to some specific two-phase problems was considered in [13–15]. Crowdy [16] presented an exact solution to the Muskat problem for the elliptical initial interface between two fluids of different viscosity. In [16], it was shown that an elliptical inclusion of one fluid remains elliptical when placed in a linear ambient flow of another fluid. In [17], new exact solutions to the Muskat problem were constructed, extending the results obtained in [16], to other types of inclusions. This paper is concerned with a two-phase Hele-Shaw problem with a variable gap width in the presence of sinks and sources.

Let \( \Omega_2(t) \subset \mathbb{R}^2 \) with a boundary \( \Gamma(t) \) at time \( t \) be a simply-connected bounded domain occupied by a fluid with a constant viscosity \( \nu_2 \), and let \( \Omega_1(t) \) be the region \( \mathbb{R}^2 \setminus \Omega_2(t) \) occupied by a different fluid of viscosity \( \nu_1 \). To consider a two-phase Hele-Shaw flow forced by a time-dependent gap, we start with the Darcy’s law

\[
v_j = -k_j \nabla p_j \quad \text{in} \quad \Omega_j(t), \quad j = 1, 2,
\]
where $v_j$ and $p_j$ are a two-dimensional gap-averaged velocity vector and a pressure of fluid $j$ respectively, $k_j = h^2(t)/12e_j$, and $h(t)$ is the gap width of the Hele-Shaw cell. Equation (1.1) is complemented by the volume conservation, $\dot{A}(t) h(t) = A(0)h(0)$ for any time $t$, where $A(t)$ and $A(0)$ are the areas of $\Omega_2(t)$ and $\Omega_2(0)$ respectively. The conservation of volume for a time-dependent gap may be written as a modification of the usual incompressibility condition
\[ \nabla \cdot \mathbf{v}_2 = 0, \]
where $\mathbf{v}_2 = (u, v, w)$ is a three-dimensional velocity vector of the fluid occupying the domain $\Omega_2(t)$. Indeed, the averaging of the three-dimensional incompressibility condition across the gap gives [9]:
\[ 0 = \int_0^{h(t)} (u_a + v_y + w_z) dz/h(t) = u_{av}^w + v_{av}^w + (w(h(t)) - w(0))/h(t) = u_{av}^w + v_{av}^w + \frac{\dot{h}(t)}{h(t)}. \]
Here $z = 0$ corresponds to the lower plate and $z = h(t)$ corresponds to the upper plate, and $h(t)$ and $\dot{h}(t)$ are assumed to be small enough to avoid any inertial effects as well as to keep the large aspect ratio. The latter implies [9]
\[ \nabla \cdot \mathbf{v}_2 = \frac{\dot{h}(t)}{h(t)} \quad \text{in} \quad \Omega(t). \tag{1.2} \]
Note that similar consideration may be applied to any finite part of the region $\Omega_1(t)$. Thus, equations (1.1) and (1.2) suggest to formulated the problem in terms of the pressure $p_j$ as a solution to Poisson’s equation,
\[ \Delta p_j = \frac{1}{k_j} \frac{\dot{h}(t)}{h(t)}, \tag{1.3} \]
almost everywhere in the region $\Omega_j(t)$, satisfying boundary conditions
\[ p_1(x, y, t) = p_2(x, y, t) \quad \text{on} \quad \Gamma(t), \tag{1.4} \]
\[ -k_1 \frac{\partial p_1}{\partial n} = -k_2 \frac{\partial p_2}{\partial n} = v_n \quad \text{on} \quad \Gamma(t). \tag{1.5} \]
We remark that when sinks and sources are present in $\Omega_j(t)$, equation (1.3) has an additional term, $\Delta p_j = \frac{1}{k_j} \frac{\dot{h}(t)}{h(t)} + \mu_j$, describing the corresponding distribution. Equation (1.4) states the continuity of the pressure under the assumption of negligible surface tension. Equation (1.5) means that the normal velocity of the boundary itself coincides with the normal velocity of the fluid at the boundary.

The free boundary $\Gamma(t)$ moves due to a change of the gap width as well as the presence of sinks and sources located in both regions. The supports of the sinks and sources, specified in section 2, are either points or lines/curves. The presence of sinks and sources obviously changes the dynamics of the evolution of the interface between the fluids, which is shown for an elliptical interface in section 3.

For what follows, it is convenient to reformulate the problem in terms of harmonic functions $\tilde{p}_j$, where
\[ p_j(x, y, t) = \tilde{p}_j(x, y, t) + \frac{1}{4k_j h(t)} (x^2 + y^2). \tag{1.6} \]
Then the problem (1.3)–(1.4) reduces to
\[ \Delta \tilde{p}_j = \chi_j \mu_j \quad \text{in} \quad \Omega_j(t), \quad (1.7) \]

where \( \chi_j = 0 \) or \( \chi_j = 1 \) in the absence or presence of sinks and sources in \( \Omega_j(t) \) respectively,

\[ \tilde{p}_1(x, y, t) = \tilde{p}_2(x, y, t) + \frac{k_1 - k_2}{4k_1 k_2} \frac{\dot{h}(t)}{h(t)} (x^2 + y^2) \quad \text{on} \quad \Gamma(t), \quad (1.8) \]

\[ -k_1 \frac{\partial \tilde{p}_1}{\partial n} = -k_2 \frac{\partial \tilde{p}_2}{\partial n} = v_n + \frac{1}{4} \frac{\dot{h}(t)}{h(t)} \frac{\partial}{\partial n} (x^2 + y^2) \quad \text{on} \quad \Gamma(t). \quad (1.9) \]

The main difficulty of the two-phase problems is the fact that the pressure on the interface is unknown. However, if we assume that the free boundary remains within the family of curves, specified by the initial shape of the interface separating the fluids (which is feasible if the surface tension is negligible), the problem is drastically simplified.

In this paper, using reformulation of the Muskat problem with the time-dependent gap in terms of the Schwarz function equation, we describe a method of constructing exact solutions, and using this method we consider examples in the presence and in the absence of additional sinks and sources.

The structure of the paper is as follows. In section 2 we describe the method of finding exact solutions. Examples of the exact solutions are given in section 3, and concluding remarks are given in section 4.

2. The method of finding exact solutions for a Muskat problem with a time-dependent gap

Consider a problem

\[ \Delta \tilde{p}_j = \chi_j \mu_j \quad \text{in} \quad \Omega_j(t), \quad (2.10) \]

\[ \tilde{p}_1(x, y, t) + \Psi_1(x, y, t) = \tilde{p}_2(x, y, t) + \Psi_2(x, y, t) \quad \text{on} \quad \Gamma(t), \quad (2.11) \]

\[ -k_1 \frac{\partial \tilde{p}_1}{\partial n} = -k_2 \frac{\partial \tilde{p}_2}{\partial n} = v_n + \Phi(x, y, t) \quad \text{on} \quad \Gamma(t). \quad (2.12) \]

In the case when

\[ \Psi_j = \frac{1}{4k_j} \frac{\dot{h}(t)}{h(t)} (x^2 + y^2), \quad j = 1, 2, \quad (2.13) \]

\[ \Phi = \frac{1}{4} \frac{\dot{h}(t)}{h(t)} \frac{\partial}{\partial n} (x^2 + y^2), \quad j = 1, 2, \quad (2.14) \]

the problem (2.10)–(2.12) coincides with (1.7)–(1.9).

As stated before, the evolution of the interface separating the fluids is forced by the change in the gap width and the presence of sinks and sources. In the absence of the surface tension, there is a possibility to control the interface by keeping \( \Gamma(t) \) within a family of curves defined by \( \Gamma(0) \). For what follows, it is convenient to reformulate problem (2.10)–(2.12) in terms of the Schwarz function \( S(z, t) \) of the curve \( \Gamma(t) \) [19–22]. This function for a real-analytic curve \( \Gamma := \{ g(x, y, t) = 0 \} \) is defined as a solution to the equation \( g ((z + \bar{z})/2, (z - \bar{z})/2i, t) = 0 \) with respect to \( \bar{z} \). This (regular) solution exists in some neighborhood \( U_\Gamma \) of the curve \( \Gamma \), if the assumptions of the implicit function theorem are satisfied [19]. Note that if \( g \) is a polynomial,
then the Schwarz function is continuable into $\Omega_j$, generally as a multiple-valued analytic function with a finite number of algebraic singularities (and poles). In $U_T$, the normal velocity, $v_n$, of $\Gamma(t)$ can be written in terms of the Schwarz function $[23], v_n = -i\dot{S}(z, t)/\sqrt{4\partial_z S(z, t)}$.

Let $\tau$ be an arclength along $\Gamma(t)$, $\psi_j$ be a stream function, and $W_j = \bar{p}_j - i\bar{\psi}_j$ be the complex potential, that is defined on $\Gamma(t)$ and in $\Omega_j(t) \cap U_T, j = 1, 2$. Following [24–27], taking into account the Cauchy-Riemann conditions in the $(n, \tau)$ coordinates, for the derivative of $W_j(z, t)$ with respect to $z$ on $\Gamma(t)$ we have

$$\frac{\partial}{\partial \tau} W_j = \frac{\partial_{\tau} W_j}{\partial z} = \frac{\partial_{\tau} \bar{p}_j + i\partial_{\tau} \bar{\psi}_j}{\partial z} = \frac{\partial_{\tau} \bar{p}_j - i(\tau_n + \Phi)/k_j}{\partial z}. \quad (2.15)$$

Expressing $\partial_{\tau} z$ in terms of the Schwarz function, $\partial_{\tau} z = (\partial_z S(z, t))^{-1/2}$, we obtain

$$\frac{\partial}{\partial \tau} W_j = \frac{\partial_{\tau} \tilde{p}_j}{\partial S} - \frac{\dot{S}}{2k_j} - \frac{i\Phi}{k_j} \sqrt{\partial_z S}. \quad (2.16)$$

Here $\partial_z W_j \equiv \frac{\partial W_j}{\partial z}, \partial_{\tau} \equiv \frac{\partial}{\partial \tau}$. Equation (2.11) implies that $\tilde{p}_j + \Psi_j = \tilde{p}_2 + \Psi_2 = f$ on $\Gamma(t)$, where $f$ is an unknown function. To keep $\Gamma(t)$ in a certain family of curves defined by $\Gamma(0)$, for example, in a family of ellipses, we assume that $f$ on $\Gamma(t)$ is a function of time only. This possibility is shown in section 3, where specific examples are discussed. In that case the problem is simplified drastically, and on $\Gamma(t)$ we have

$$\frac{\partial}{\partial \tau} W_j = -\frac{\dot{S}}{2k_j} - \partial_{\tau}(\Psi_j(z, S(z, t))) - \frac{i\Phi}{k_j} \sqrt{\partial_z S} \quad j = 1, 2. \quad (2.17)$$

For the special case when $\Psi_j$ and $\Phi$ are given by (2.13) and (2.14), the last equation reduces to

$$\frac{\partial}{\partial \tau} W_j = -\frac{1}{2k_j} (\frac{\dot{S}}{S} + \frac{\dot{h}}{h}) \quad j = 1, 2. \quad (2.18)$$

Remark that each equation (2.18) can be continued off of $\Gamma$ into the corresponding $\Omega_j$, where $W_j$ is a multiple-valued analytic function. The equations (2.17) and (2.18) imply that the singularities of $W_1, W_2,$ and the singularities of the Schwarz function are linked. As such, the singularities of the Schwarz function play the crucial role in the construction of solutions in question.

To find the exact solutions, suppose that at $t = 0$ the interface is an algebraic curve, $\sum_{k=0}^n a_k(0)x^{n-k}y^k = 0,$ with the Schwarz function $S(z, a_k(0)).$ Assume that during the course of evolution the Schwarz function of the interface $S(z, a_k(t)) \equiv S(z, t)$ is such that $S(z, a_k(0)) = S(z, a_k(0))$, which leads us to the following six steps method:

1. Compute $\dot{S}(z, t)$, locate its singularities, and define their type.
2. Using equations (2.18) find preliminary expressions for $\partial_{\tau} W_j.$
3. By putting restrictions on the coefficients $a_k(t)$ in the preliminary expressions for $\partial_{\tau} W_j$ eliminate the terms involving undesirable singularities (if possible).
4. Integrate (2.18) with respect to $z$ in order to find $W_j$ up to an arbitrary function of time.
5. Take the real part of $W_j$ in order to obtain $p_j$ up to an arbitrary function of time.
6. Evaluate the quantities $p_j$ on the interface to determine the independent of $z$ function of integration from the steps 3 and 4.
7. Locate the supports and compute the distributions of sinks and sources.

Before describing how to locate the supports, we remark that the distributions in step 7 are related to the two-phase mother body [17]. The notion of a mother body arises from the potential theory [28–32] and was adopted to the one-phase Hele-Shaw problem in [33].
As mentioned above, generally, the complex potentials $W_j$ are multiple-valued functions in $\Omega_j$. For instance, if $\Gamma(t)$ is an algebraic curve, then the singularities of $W_j$ are either poles or algebraic singularities. To choose a branch of $W_j$, one has to introduce the cuts, $\gamma_j(t)$, that serve as supports for the distributions of sinks and sources, $\mu_j(t)$, $j = 1, 2$. Thus, each cut originates from an algebraic singularity $z_a(t)$ of the potential $W_j$. The supports consist of those cuts and/or points and do not bound any two-dimensional subdomains in $\Omega_j(t)$, $j = 1, 2$. Each cut included in the support of $\mu_j(t)$ is contained in the domain $\Omega_j(t)$, and the limiting values of the pressure on each side of the cut are equal. The value of the density of sinks and sources located on the cut is equal to the jump of the normal derivative $\partial_n p_j$ of the pressure $p_j$. In order for the total flux through the sinks and sources to be finite, all of the singularities of the function $W_j$ must have no more than the logarithmic growth.

The location of $z_a(t)$, as well as the directions of the cuts emanating from $z_a(t)$, are determined by the Schwarz function via (2.18). In the examples considered below, the Schwarz function has the following two representations near its singular points. The first representation being the square root (general position)

$$S^g(z,t) = \xi^g(z,t) \sqrt{z - z_a(t)} + \zeta^g(z,t),$$

(2.19)

where $z_a(t)$ is a non-stationary singularity, that is $\dot{z}_a \neq 0$. The second being the reciprocal square root

$$S^r(z,t) = \frac{\xi^r(z,t)}{\sqrt{z - z_a(0)}} + \zeta^r(z,t),$$

(2.20)

where $z_a(0)$ is a stationary singularity, that is $\dot{z}_a = 0$. Here $\xi^g(z,t)$ and $\xi^r(z,t)$ are regular functions of $z$ in a neighborhood of the point $z_a(t)$, and $\xi^g(z_a(t),t) \neq 0$.

By plugging (2.19) and (2.20) into (2.18), in a small neighborhood of $z_a(t)$ we have

$$W_j^g(z,t) = \frac{1}{2k_j} \dot{z}_a \xi^g(z_a(t),t) \sqrt{z - z_a(t)} + \ldots,$$

(2.21)

$$W_j^r(z,t) = \frac{1}{k_j} C_0(t) \sqrt{z - z_a(0)} + \ldots,$$

(2.22)

where the dots correspond to the smaller and regular terms that do not affect the computation of the directions of the cuts. The quantity $C_0(t)$ is defined by

$$C_0(t) = \frac{\xi^r(z_a(0),t)}{\dot{h}(t)} \frac{\dot{h}(t)}{h(t)} \xi^r(z_a(0),t).$$

Formulas (2.21) and (2.22) along with the substitutions $z = z_a + \rho \exp(i\varphi^g)$ (with small $\rho$), imply that

$$p_j^g(z,t) = \frac{\sqrt{\rho}}{2k_j} \Re[\dot{z}_a \xi^g(z_a(t),t) \exp(\frac{i\varphi^g}{2})] + \ldots,$$

(2.23)

$$p_j^r(z,t) = -\frac{\sqrt{\rho}}{k_j} \Re[C_0(t) \exp(\frac{i\varphi^r}{2})] + \ldots.$$

(2.24)

Computing the zero level of a variation of $p_j$ along a small loop surrounding the singular point, we finally obtain the following directions of the cuts: for the general position

$$\varphi^g = \pi - 2(\arg[\xi^g(z_a(t),t)] + \arg[\dot{z}_a(t)]) + 2\pi k, \quad k = 0, \pm 1, \pm 2...$$

(2.25)
and for the reciprocal square root

$$\varphi' = \pi - 2 \text{arg}[C_0(t)] + 2\pi k, \quad k = 0, \pm 1, \pm 2 \ldots$$  \hspace{1cm} (2.26)

In the next section, we use the described method to construct exact solutions to the Muskat problem. In the considered examples, the evolution of the interface is driven by the change in the gap width of the Hele-Shaw cell. The examples include the elliptical shape with and without sinks and sources in the finite domain as well as the Cassini’s oval in the presence of sinks and sources.

3. Examples of specific initial interfaces

3.1. Circle

To illustrate the method, we start with the simplest example for which the solution is known. Suppose that the initial shape of the interface is a circle with the equation

$$x^2 + y^2 = a_0^2,$$

and during the evolution the boundary remains circular,

$$x^2 + y^2 = a(t)^2.$$

The corresponding Schwarz function is

$$S = a^2(t)/z.$$ Taking into account the volume conservation, equation (2.18) in this case reads as

$$\partial_z W_j = 0,$$

which implies that

$$\tilde{p}_j$$

is a function depending on \( t \) only,

$$\tilde{p}_j = -\frac{a_0^2 b_0 h(t)}{4k_j h^2} + f(t),$$  \hspace{1cm} (3.27)

due to the volume conservation of the fluid occupying \( \Omega_2(t) \), the product of functions \( a(t) \) and \( b(t) \) is linked to the gap width, \( h(t) \), via the equation

$$h(t) = a_0 b_0 h_0/(a(t)b(t)),$$

where \( a_0 = a(0), \ b_0 = b(0), \) and \( h_0 = h(0) \). Therefore, \( h(t)/h(t) = -\partial_t(ab)/(ab) \), and the equation (2.18) could be rewritten as

$$\partial_t W_j = -\frac{1}{4k_j} \left( \partial_z S + \frac{\dot{h}}{h} S \right).$$

Due to the volume conservation of the fluid occupying \( \Omega_2(t) \), the product of functions \( a(t) \) and \( b(t) \) is linked to the gap width, \( h(t) \), via the equation

$$h(t) = a_0 b_0 h_0/(a(t)b(t)),$$
\[
\partial_t W_j = - \frac{1}{2kj} \left( \partial_t S - \frac{\partial_t (ab)}{ab} S \right),
\] (3.29)

which results in

\[
\partial_t W_j = - \frac{z}{2kj} \left\{ \frac{\partial}{\partial t} \left( \frac{a^2 + b^2}{d^2} \right) - \frac{(a^2 + b^2)}{ab d^2} \frac{\partial}{\partial t} (ab) \right\}
- \frac{(2z^2 - d^2)}{\sqrt{z^2 - d^2}} \frac{ab}{2kd^2} \frac{\partial}{\partial t} \left( d^2 \right)
\] (3.30)

and

\[
W_j = - \frac{z^2}{4kj} \left\{ \frac{\partial}{\partial t} \left( \frac{a^2 + b^2}{d^2} \right) - \frac{(a^2 + b^2)}{ab d^2} \frac{\partial}{\partial t} (ab) \right\}
- \frac{abz}{2kd^2} \sqrt{z^2 - d^2} \frac{\partial}{\partial t} (d^2) + C_j(t),
\] (3.31)

where \(C_j(t)\) is an arbitrary function of time.

(a) Evolution with constant inter-focal distance. To obtain an exact solution in the absence of sinks and sources in the finite part of the plane, we set \(d(t) = d(0)\). Then, the second term in the formula (3.31) vanishes, which implies the following expression for the pressure

\[
\bar{p}_j = \Re \{W_j\} = \frac{1}{4kj} \left( (x^2 - y^2) \frac{\dot{a} d_0^2}{a(a^2 - d_0^2)} + 2aa \right) + f(t),
\] (3.32)

therefore,

\[
\bar{p}_j = \frac{\dot{a}}{2kd^2(a^2 - d_0^2)} \left( d_0^2 x^2 - d^2 (x^2 + y^2) + a^2 (a^2 - d_0^2) \right) + f(t)
\] (3.33)

is the solution to the problem (1.3)–(1.5). Note that when \(d_0 = 0\), this formula coincides with formula (3.28) related to the circular interface. Hence, \(\Gamma(t)\) is a family of co-focal ellipses,

\[
\frac{x^2}{a^2(t)} + \frac{y^2}{b^2(t)} = 1,
\]

controlled by one of the functions \(a(t), b(t)\) or \(h(t)\). If \(h(t)\) is given, then

\[
a^2(t) = \frac{1}{2} \left( a_0^2 - b_0^2 + \sqrt{(a_0^2 - b_0^2)^2 + 4a_0^2b_0^2h_0^2/h^2(t)} \right),
\] (3.34)

\[
b^2(t) = \frac{1}{2} \left( b_0^2 - a_0^2 + \sqrt{(a_0^2 - b_0^2)^2 + 4a_0^2b_0^2h_0^2/h^2(t)} \right).
\] (3.35)

An example of such an evolution with a linear function \(h(t)\) is shown in figure 1(a).

(b) Evolution with variable inter-focal distance. If we admit solutions with variable inter-focal distance by keeping all terms in (3.31), we must allow, in addition to the gap change, some sinks/sources located in \(\Omega_2\). In that case, the pressure is
\[ \tilde{p}_j = -\frac{(x^2 - y^2)}{4kj} \left\{ \frac{\partial}{\partial t} \left( \frac{a^2 + b^2}{d^2} \right) - \frac{(a^2 + b^2)}{ab} \frac{\partial}{\partial t}(ab) \right\} - \frac{ab}{2kjd^2} \frac{\partial}{\partial t}(d^2) \frac{x (\alpha^2 - y^2)}{\alpha} - \frac{ab (\dot{a}b - \dot{a}b)}{2kd^2} + f(t). \]  

where

\[ \alpha^2 = \frac{(x^2 - y^2 - d^2 + \sqrt{(x^2 - y^2 - d^2)^2 + 4x^2y^2})}{2}, \]

Therefore, making

\[ \tilde{p}_j = -\frac{(x^2 - y^2)}{4kj} \left\{ \frac{\partial}{\partial t} \left( \frac{a^2 + b^2}{d^2} \right) - \frac{(a^2 + b^2)}{ab} \frac{\partial}{\partial t}(ab) \right\} - \frac{ab (\dot{a}b - \dot{a}b)}{2kd^2} \]

Equation (3.31) implies that there are two singular points in the interior domain \( \Omega_2, z = \pm d \). The Schwarz function near those points has the square root representation (2.19) with

\[ \xi^2 = -\frac{2ab}{d^2} \sqrt{z^2 \pm d}. \]

The direction of the cut at each point is defined by formula (2.25), which implies that at the point \( z_0 = d \), the angle is \( \phi^2 = \pi + 2\pi k \) and at the point \( z_0 = -d \), the angle is \( \phi^2 = 2\pi k, k = 0, \pm 1, \pm 2, \ldots \). Thus, the cut \( \gamma_2(t) \) is located along the inter-focal segment \([-d, d]\). The density of the distribution of sinks and sources along that segment is given by the formula

\[ \rho_2 = \frac{ab \partial (d^2)}{kd^4} \frac{(2x^2 - d^2)}{\sqrt{d^2 - x^2}}. \]

Such a density changes its sign along the inter-focal segment, so its presence does not affect the area of the ellipse. \( \dot{A} = \int_0^d \rho_2(x,t) \, dx = 0 \). Figure 1 shows how the sinks and sources change the evolution of the interface with increasing (see figure 1(b)) and decreasing (see figure 1(c)) inter-focal distances.
3.3. The Cassini’s oval

Similar to the previous examples, assume that \( \Gamma(t) \) remains in the specific family of curves, the Cassini’s ovals, given by the equation

\[
(x^2 + y^2)^2 - 2b(t)^2 (x^2 - y^2) = a(t)^4 - b(t)^4,
\]

where \( a(t) \) and \( b(t) \) are unknown positive functions of time. This curve consists of one closed curve, if \( a(t) > b(t) \) (see figure 2), and two closed curves otherwise. Assume that at \( t = 0 \) \( a(0) > b(0) \). The Schwarz function of Cassini’s oval,

\[
S(z, t) = \sqrt{b^2 z^2 + a^4 - b^4} / \sqrt{z^2 - b^2},
\]

has two singularities in \( \Omega_1(t) \), \( z = \pm i \sqrt{(a^4 - b^4)/b^2} \), and two singularities in \( \Omega_2(t) \), \( z = \pm b \).

The corresponding complex velocities have singularities at the same points,

\[
\partial_t W_j = -\frac{1}{2k_j} \left( \frac{B_1 z^2 + B_2}{\sqrt{(b^2 z^2 + a^4 - b^4)(z^2 - b^2)}} + \frac{b b \sqrt{b^2 z^2 + a^4 - b^4}}{\sqrt{(z^2 - b^2)^2}} \right).
\] (3.38)

Here

\[
B_1 = b b + b^3 h / h, \quad B_2 = 2a^3 \dot{a} - 2b^3 \dot{b} + (a^4 - b^4) \dot{h} / h,
\]

and \( \dot{h} / h = -A / \dot{A} \) due to volume conservation.

The area of Cassini’s oval can be computed in polar coordinates, \( A = a^2 E(\pi, \frac{b^2}{a^2}) = 2a^2 E(\frac{b^2}{a^2}) \), where \( E(\phi, k) = \int_0^\phi \sqrt{1 - k^2 \sin^2 t} \, dt \) and \( E(k) = E(\pi / 2, k) \), resulting in

\[
\frac{\dot{A}}{A} = \frac{2 \dot{a}}{a} + \frac{\partial E(\phi, k)}{E(\pi, \frac{b^2}{a^2})}.
\] (3.39)

Taking into account ([34], p. 772),

\[
\frac{\partial E(\phi, k)}{\partial k} = \frac{1}{k} \left( E(\phi, k) - F(\phi, k) \right),
\]

where

\[
F(\phi, k) = \int_0^\phi \frac{1}{\sqrt{1 - k^2 \sin^2 t}} \, dt,
\] (3.40)

\[
F(\pi / 2, k) = K(k), \text{ and } \partial E(\pi, \frac{b^2}{a^2}) = \left( E(\pi, \frac{b^2}{a^2}) - F(\pi, \frac{b^2}{a^2}) \right) \frac{2ab - 2ba}{ab},
\]

we have

\[
B_1(t) = -\frac{b}{a E(\pi, \frac{b^2}{a^2})} \left( -ab E(\pi, \frac{b^2}{a^2}) + 2(\dot{a}b - \ddot{a}b) F(\pi, \frac{b^2}{a^2}) \right),
\] (3.41)

\[
B_2(t) = \frac{2(ab - ab)}{ab E(\pi, \frac{b^2}{a^2})} \left( a^4 E(\pi, \frac{b^2}{a^2}) - (a^4 - b^4) F(\pi, \frac{b^2}{a^2}) \right),
\] (3.42)

and

\[
W_j = -\frac{1}{2k_j} \left( B_1 I_1 + B_2 I_2 + bb I_3 \right).
\] (3.43)
Here

\[ I_1 = \frac{b^2}{a^2} F\left(\text{cos}^{-1}\left(\frac{b}{z}\right), \frac{\sqrt{a^2 - b^2}}{a^2}\right) - \frac{a^2}{b^2} E\left(\text{cos}^{-1}\left(\frac{b}{z}\right), \frac{\sqrt{a^2 - b^2}}{a^2}\right) + \frac{\sqrt{(z^2 b^2 + a^4 - b^4)(z^2 - b^2)}}{ab^2}, \]

\[ I_2 = \frac{1}{a^2} \int \frac{dt}{\sqrt{1 - \frac{a^4 - b^4}{a^2} \sin^2 t}} \]

and the integral \( I_3 \) corresponds to the last term in (3.38). To ensure that the singularities of the complex potential have no more than the logarithmic type, we eliminate this term by setting \( b \) to zero. Thus, we have

\[ \dot{S}(z) = \frac{2a^3 \dot{a}}{\sqrt{b^2 z^2 + a^4 - b^4}} \]

and the equation (2.18) implies

\[ W_j = -\frac{a\dot{a}}{k E(\frac{b^2}{a^2})}\left[E\left(\frac{b^2}{a^2}\right) - K\left(\frac{b^2}{a^2}\right)\right] F\left(\xi, \frac{\sqrt{a^4 - b^4}}{a^2}\right) + K\left(\frac{b^2}{a^2}\right) E\left(\xi, \frac{\sqrt{a^4 - b^4}}{a^2}\right) - \frac{K\left(\frac{b^2}{a^2}\right)}{a^2 z} \sqrt{(z^2 b^2 + a^4 - b^4)(z^2 - b^2)} + C(t), \quad (3.44) \]

where \( \xi = \text{cos}^{-1}\left(\frac{b}{z}\right) \) and \( F(\alpha, \beta) \) is the incomplete elliptic integral of the first kind (3.40),

\[ F\left(\text{cos}^{-1}\left(\frac{b}{z}\right), \frac{\sqrt{a^4 - b^4}}{a^2}\right) = \int_0^{\text{cos}^{-1}(b/z)} \frac{dt}{\sqrt{1 - \frac{a^4 - b^4}{a^2} \sin^2 t}} = \int_0^{\text{cos}^{-1}(b/z)} \frac{dt}{\sqrt{1 - \frac{a^4 - b^4}{a^2} \sin^2 t}}. \]
Since $p_j = \Re \{ W_j \}$, we need to compute the real parts for each term in (3.44). Using the property $F(\alpha, \beta) = F(\pi, \beta)$ and the summation formula for the elliptic integrals of the first kind [35], we have

$$\frac{1}{2} \left[ F\left( \zeta, \frac{\sqrt{a^2 - b^2}}{a^2} \right) + \frac{\sqrt{a^2 - b^2}}{a^2} \right] = \frac{1}{2} F\left( \alpha, \frac{\sqrt{a^2 - b^4}}{a^2} \right),$$

where

$$\alpha = \sin^{-1} \frac{\cos \zeta \sin \zeta \sqrt{1 - \frac{a^2 - b^4}{a^2} \sin^2 \zeta} + \cos \zeta \sin \zeta \sqrt{1 - \frac{a^2 - b^4}{a^2} \sin^2 \zeta}}{1 - \frac{a^2 - b^4}{a^2} \sin^2 \zeta \sin^2 \zeta} \quad (3.45)$$

or

$$\alpha = \sin^{-1} \frac{a^2 z \sqrt{z^2 - b^2} \sqrt{b^2 z^2 + a^2 - b^4} + a^2 z \sqrt{z^2 - b^2} \sqrt{b^2 z^2 + a^2 - b^4}}{b^2 z^2 + 2a^4 (z^2 - b^2)} \quad (3.46).$$

Similarly, using the property $E(\alpha, \beta) = E(\pi, \beta)$ and the summation formula for the elliptic integrals of the second kind [35], we have

$$\frac{1}{2} \left[ E\left( \zeta, \frac{\sqrt{a^2 - b^2}}{a^2} \right) + \frac{\sqrt{a^2 - b^2}}{a^2} \right] = \frac{1}{2} E\left( \alpha, \frac{\sqrt{a^2 - b^4}}{a^2} \right) + \frac{(a^4 - b^4) \sqrt{(z^2 - b^2)[z^2 - b^2]}}{2a^4 z^2} \sin \alpha.$$

Consequently, the pressure is determined by

$$\tilde{p}_j = -\frac{a^2}{2k_B T} \frac{\left( E\left( \frac{b^2}{a^2} \right) - K\left( \frac{b^2}{a^2} \right) \right) F\left( \alpha, \frac{\sqrt{a^2 - b^4}}{a^2} \right)}{a^2} + k_B T \left( b^2 \right) \frac{2K\left( \frac{b^2}{a^2} \right)}{a^4 z^2} \Re \left\{ \frac{\sqrt{(z^2 - b^2)[z^2 - b^2]}}{z} \right\} + C_j(t). \quad (3.47)$$

Here

$$\Re \left\{ \frac{\sqrt{(z^2 - b^2)[z^2 - b^2]}}{z} \right\} = \frac{x(a_1^2 a_2^2 - x^2 y^2 b^2 + y^2(a_1^2 b^2 + a_2^2 2)}{(x^2 + y^2) a_1 a_2},$$

where

$$a_1^2 = (x^2 - y^2 - b^2 + \sqrt{(x^2 - y^2 - b^2)^2 + 4x^2 y^2})/2$$

and

$$a_2^2 = (x^2 - y^2 - b^2 + a^4 - b^4 + \sqrt{(x^2 - y^2) b^2 + a^4 - b^4)^2 + 4x^2 y^2 b^4})/2.$$ 

Taking into account the boundary condition to determine $C_j(t)$, we have

$$\tilde{p}_j = -\frac{a^2}{2k_B T} \frac{\left( E\left( \frac{b^2}{a^2} \right) - K\left( \frac{b^2}{a^2} \right) \right) F\left( \alpha, \frac{\sqrt{a^2 - b^4}}{a^2} \right)}{a^2} + k_B T \left( b^2 \right) \frac{2K\left( \frac{b^2}{a^2} \right)}{a^4 z^2} \Re \left\{ \frac{\sqrt{(z^2 - b^2)[z^2 - b^2]}}{z} \right\}$$

$$- \left( \frac{b^2}{a^2} - K\left( \frac{b^2}{a^2} \right) \right) K\left( \frac{\sqrt{a^2 - b^4}}{a^2} \right) - K\left( \frac{b^2}{a^2} \right) E\left( \frac{\sqrt{a^2 - b^4}}{a^2} \right) + f(t) \quad (3.48)$$
or

\[
\tilde{p}_j = -\frac{a\bar{a}}{2k_0E(\frac{b}{a})} \left[ \left( E(\frac{b^2}{a^2}) - K(\frac{b^2}{a^2}) \right) F(\alpha, \frac{\sqrt{a^2 - b^2}}{a^2}) + K(\frac{b^2}{a^2}) E(\alpha, \frac{\sqrt{a^2 - b^2}}{a^2}) \right. \\
+ K(\frac{b^2}{a^2}) \frac{(a^4 - b^4)\sqrt{(x^2 + y^2)^2 - 2b^2(x^2 - y^2) + b^4}}{a^4(x^2 + y^2)} \\
- \frac{2K(\frac{b^2}{a^2}) \sqrt{x^2y^2b^4 + y^2(\alpha_1^2b^2 + \alpha_2^2)}}{(x^2 + y^2)\alpha_1\alpha_2} - \frac{\pi}{2} \bigg] + f(t).
\]

(3.49)

Thereby,

\[
p_j = \tilde{p}_j - \frac{aK(b/a)}{2k_0E(b/a)}(x^2 + y^2).
\]

To find the location of the sinks and sources in the interior domain \(\Omega_2\), note that the Schwarz function near its singular points \(z = \pm b\) has the reciprocal square root representation (2.20) with \(\xi(z, t) = \sqrt{b^2 + a^4 - b^4}/\pm b\). Formula (2.26) implies that \(\psi'(b) = \pi\) and \(\psi'(-b) = 0\). This results (taking into account the symmetry of the problem) in the segment \(x \in [-b, b]\) as a location of sinks and sources. The corresponding density is

\[
\mu_2 = \frac{B_1x^2 + B_2}{k_2\sqrt{(b^2x^2 + a^4 - b^4)(b^2 - x^2)}}.
\]

Note that \(\int_{-b}^{b} \mu_2(x) \, dx = 0\), which is consistent with the volume conservation.

To determine the location of the sinks and sources in domain \(\Omega_1\), we start with singular points \(z_0(t) = \pm i\sqrt{(a^4 - b^4)/b}\). The Schwarz function near these points has the square root representation (2.19), and the directions of the cuts are defined by formula (2.25).

In the neighborhood of the point \(z_0(t) = i\sqrt{(a^4 - b^4)/b}\), we have \(\arg[z_0] = \pi/2 + 2\pi k\) and \(\arg[\xi(z_0(t), t)] = -\pi/4 + \pi k\). Thus, according to (2.25) the direction of the cut is \(\psi^x = \pi/2 + 2\pi k, k = 0, \pm 1, \pm 2, \ldots\).

Similarly, at the point \(z_0(t) = -i\sqrt{(a^4 - b^4)/b}\), \(\arg[z_0] = -\pi/2 + 2\pi k\), \(\arg[\xi(z_0, t)] = -3\pi/4 + \pi k\). Therefore, the direction of the cut is \(\psi^x = -\pi/2 + 2\pi k\).

Taking into consideration symmetry with respect to the x-axis, we conclude that the support of \(\mu_1\) consists of two rays starting at the branch points and going to infinity (see the dashed lines in figure 2). The density of sinks and sources is defined by

\[
\mu_1 = \frac{B_1y^2 - B_2}{k_1\sqrt{(b^2y^2 + a^4 + b^4)(b^2 + y^2)}}.
\]

The evolution of the oval is controlled by a single function \(h(t)\), where \(b\) is constant and the parameter \(a(t)\) is defined by the equation:

\[
\frac{\dot{h}}{h} = -\frac{aK(b^2/a^2)}{aE(b^2/a^2)}.
\]

Figure 2 shows the evolution of the Cassini’s oval under squeezing with \(h(t) = h_0 - t\) at \(t = 0\) (see figure 2(a)) and \(t = 0.05\) (see figure 2(b)). The dots correspond to the singular points \(z_0\), the dashed lines correspond to the cuts.
4. Concluding remarks

We have studied a Muskat problem with a negligible surface tension and a gap width dependent on time. This study extended the results reported in [9, 10], and [17]. We suggested a method of finding exact solutions and applied it to find new exact solutions for initial elliptical shape and Cassini’s oval. The idea of the method was to keep the interface within a certain family of curves defined by its initial shape.

For the elliptical shape, we found two types of solutions: without sinks and sources in the interior domain, and with the presence of a special distribution of sinks and sources along the inter-focal distance. In the former solution, the inter-focal distance remains constant, while in the latter, it changes. In the case when the inter-focal distance decreases, the presence of the sink-source-distribution—since it does not change the area of the interior domain—could be possibly used to simulate the effect of surface tension. It will be studied elsewhere.

For the Cassini’s oval, we found a solution to the problem when both a gap change and special distributions of sinks and sources in both the interior and exterior domains are present.

Our mathematical model included an assumption that the volume of the bounded domain $\Omega_2(t)$ is conserved. To show other conserved quantities, we follow Richardson [8, 36, 37] deriving the moment dynamics equation,

$$\frac{d}{dt} \left[ h(t) \int_{\Omega_2(t)} u(x, y) \, dx \, dy \right] = -\chi_2 k_2(t) h(t) \int_{\gamma_2(t)} u(s) \mu_2(s, t) \, ds,$$

where $u(x, y)$ is a harmonic function in a domain $\Omega \supset \Omega_2(t)$. The latter follows from the chain of equalities:

$$\frac{d}{dt} \int_{\Omega_2(t)} u \, dx \, dy = \int_{\Gamma(t)} u \, v_n \, ds = -k_2 \int_{\Gamma(t)} u \frac{\partial \mu_2}{\partial n} \, ds,$$

$$= -\frac{h}{h} \int_{\Omega_2(t)} u \, dx \, dy - \chi_2 k_2 h \int_{\gamma_2(t)} u \mu_2 \, ds - k_2 f(t) \int_{\Gamma(t)} \frac{\partial u}{\partial n} \, ds.$$

By setting $f(t)$ to zero and rearranging the terms, we arrive at (4.50).

Equation (4.50) implies that in the absence of sinks and sources, $\chi_2 = 0$, the quantity $h \int_{\Omega_2(t)} u \, dx \, dy$ is conserved for any harmonic function $u(x, y)$ defined in $\Omega$. A special choice of $u(x, y)$ for $\chi_2 = 0, 1$, corresponds to the volume conservation—in that case, the integral on the right hand side is zero.

Remark that in the Saffman-Taylor formulation of the problem—where a viscous fluid occupying the gap between two plates is being displaced by a less viscous fluid, which is forced into the gap—unstable fingers are being formed. Similarly, a basic instability—a version of the Saffman-Taylor instability—was identified in [9] when a viscous circular bubble was surrounded by the air and the upper plate was lifting.

Unstable fingers are subject to tip splitting and exhibit singularities in a finite time. In the present paper we did not consider neither formation of singularities, nor the ways of achieving a regularization. The aim of this study was, in contrast, to avoid formation of singularities by means of a special choice of sinks and sources. Note that linear stability results for the interior problem [9] indicate that a circular bubble is stable when the plate is moving down. The latter, together with the stability results for the Saffman-Taylor formulation in a radial flow geometry [38], suggest that the circular interface for the problem in question is expected to be linearly stable in two situations: (i) when a more viscous fluid occupies the interior domain and the upper plate is moving down or (ii) when a less viscous fluid is surrounded by a more viscous fluid and the upper plate is moving up.
Acknowledgments

The authors are grateful to the referees for helpful suggestions.

ORCID iDs

Tatiana Savina https://orcid.org/0000-0002-2351-3269

References

[1] Chen G-Q, Shahgholian H and Vazquez J-L 2015 Free boundary problems: the forefront of current and future developments Phil. Trans. R. Soc. A 373 20140285
[2] Friedman A 2015 Free boundary problems in biology Phil. Trans. R. Soc. A 373 20140368
[3] Friedman A, Hu B and Xue C 2014 On a multiphase multicomponent model of biofilm growth Arch. Ration. Mech. Anal. 211 257–300
[4] Friedman A and Chen D 2013 A two-phase free boundary problem with discontinuous velocity: application to tumor model J. Math. Anal. Appl. 399 378–93
[5] Muskat M 1934 Two-fluid systems in porous media. The encroachment of water into an oil sand Physics 5 250–64
[6] Vasil’ev A 2009 From the Hele-Shaw experiment to integrable systems: a historical overview Complex. Anal. Oper. Theory 3 551–85
[7] Gustafsson B, Teodorescu R and Vasil’ev A 2014 Classical and Stochastic Laplacian Growth p 315 (Berlin: Springer)
[8] Entov V M, Etingof P I and Kleinbock D Y 1995 On nonlinear interface dynamics in Hele-Shaw flows Eur. J. Appl. Math. 6 399–420
[9] Shelley M J, Tian F R and Wlodarski K 1997 Hele-Shaw flow and pattern formation in a time-dependent gap Nonlinearity 10 1471–95
[10] Savina T V and Nepomnyashchy A A 2015 On a Hele-Shaw flow with a time-dependent gap in the presence of the surface tension J. Phys. A: Math. Theor. 48 125501
[11] Howison S 2000 D A note on the two-phase Hele-Shaw problem J. Fluid Mech. 409 243–9
[12] Jacquard P and Ségui B 1992 Mouvement de deux fluides en contact dans un milieu poreux Journal de Mecanique 1 367–94
[13] Friedman A and Tao Y 2003 Nonlinear stability of the Muskat problem with capillary pressure at the free boundary Nonlinear Anal. 53 45–80
[14] Siegel M, Callis R E and Howison S 2004 Global existence, singular solutions, and ill-posedness for the Muskat problem Commun. Pure Appl. Math. LVII 0001–38
[15] Ye J and Tanveer S 2012 Global solutions for a two-phase Hele-Shaw bubble for a near-circle initial shape Complex Variables Elliptic Eq. 57 23–61
[16] Crowdy D 2006 Exact solutions to the unsteady two-phase Hele-Shaw problem Q. J. Mech. Appl. Maths 59 475–85
[17] Akinfeyev L, Savina T V and Nepomnyashchy A A 2017 Exact solutions to a Muskat problem with line distributions of sinks and sources Contemp. Math. 699 9–33
[18] Agam O 2009 Viscous fingering in volatile thin films Phys. Rev. E 79 021603
[19] Davis P 1979 The Schwarz Function and its Applications (Carus Mathematical Monographs) (Washington, DC: Mathematical Association of America)
[20] Khavinson D 1996 Holomorphic Partial Differential Equations and Classical Potential Theory (La Laguna, Spain: Universidad de La Laguna)
[21] Savina T 2012 On non-local reflection for elliptic equations of the second order in $\mathbb{R}^2$ (the Dirichlet condition) Trans. Am. Math. Soc. 364 2443–60
[22] Shapiro H S 1992 The Schwarz Function and its Generalization to Higher Dimensions (New York: Wiley)
[23] Howison S D 1992 Complex variable methods in Hele-Shaw moving boundary problems Eur. J. Appl. Math. 3 209–24
[24] Cummings I. J, Howison S D and King J R 1999 Two-dimensional Stokes and Hele-Shaw flows with free surfaces J. Appl. Math. 10 635–80
[25] Khavinson D, Mineev-Weinstein M and Putinar M 2009 Planar elliptic growth *Complex Anal. Oper. Theory* **3** 425–51
[26] Lacey A A 1982 Moving boundary problems in the flow of liquid through porous media *J. Aust. Math. Soc.* B **24** 171–93
[27] McDonald N R 2011 Generalized Hele-Shaw flow: a Schwarz function approach *Eur. J. Appl. Math.* **22** 517–32
[28] Gustafsson N 1998 On mother bodies of convex polyhedra *SIAM J. Math. Anal.* B **29** 1106–17
[29] Gustafsson B and Sakai M 1999 On potential theoretic skeletons of polyhedra *Geom. Dedicata* **76** 1–30
[30] Savina T V, Sternin B Y and Shatalov V E 2005 On a minimal element for a family of bodies producing the same external gravitational field *Appl. Anal.* **84** 649–68
[31] Emamizadeh B, Prajapat J V and Shahgholian H 2011 A two phase free boundary problem related to quadrature domains *Potential Anal.* **34** 119–38
[32] Gardiner S J and Sjödin T 2012 Two-phase quadrature domains *J. D’Anal. Math.* **116** 335–54
[33] Savina T V and Nepomnyashchy A A 2015 The shape control of a growing air bubble in a Hele-Shaw cell *SIAM J. Appl. Math.* **75** 1261–74
[34] Prudnikov A P, Brychkov Y A and Marichev O I 1986 Integrals and Series. Additional Chapters (Moscow: Nauka) p 800
[35] Bateman H and Erdélyi A 1955 *Higher Transcendental Functions* (New York: McGraw-Hill)
[36] Richardson S 1972 Some Hele-Shaw flows with a free boundary produced by the injection of fluid into a narrow channel *J. Fluid. Mech.* **56** 609–18
[37] Entov V M and Etingof P 2007 On generalized two-fluid Hele-Shaw flow *Eur. J. Appl. Math.* **18** 103–28
[38] Miranda J A and Widom M 1998 Radial fingering in a Hele-Shaw cell: a weakly nonlinear analysis *Physica D* **120** 315–28