Metastable two-component solitons near an exceptional point

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We consider a two-dimensional nonlinear waveguide with distributed gain and losses. The optical potential describing the system consists of an unperturbed complex potential depending only on one transverse coordinate, i.e., corresponding to a planar waveguide, and a small non-separable perturbation depending on both transverse coordinates. It is assumed that the spectrum of the unperturbed planar waveguide features an exceptional point (EP), while the perturbation drives the system into the unbroken phase. Slightly below the EP, the waveguide sustains two-component envelope solitons. We derive one-dimensional equations for the slowly varying envelopes of the components and show their stable propagation. When both traverse directions are taken into account within the framework of the original model, the obtained two-component bright solitons become metastable and persist over remarkably long propagation distances.

I. INTRODUCTION

Stable propagation of linear waves in either conservative or dissipative systems requires reality of the spectrum of the governing evolution operator. When this operator is non-Hermitian and depends on control parameters, its spectrum can undergo qualitative changes upon variation of these parameters. In particular, the spectrum can change from purely real to a complex one. This (phase) transition between real and complex spectra typically occurs either through an exceptional point (EP) in the discrete spectrum [1–3] or through a spectral singularity in the continuous spectrum [4, 5]. Although EPs as well as spectral singularities are introduced as characteristics of linear spectral problems [6], they also impact propagation of nonlinear waves. First of all, stability of linear waves of a given nonlinear system is a necessary (although not yet sufficient) condition for stability of localized nonlinear waves, for example, of bright solitons (see [7, 8] for review). An EP in the spectrum of the underlying linear system affects the equations governing weakly nonlinear waves having propagation constants in the vicinity of the EP [9, 10]. If parameters of a nonlinear medium are close to an EP locally, i.e., only in a given spatial domain, a soliton interacting with such domain can be scattered according to different scenarios [11]. In a waveguide geometry characterized by a separable optical potential (created by modulation of the dielectric permittivity) the existence of EP in the linear spectrum of the carrying transverse modes of a separable optical potential can change the sign of the effective Kerr nonlinearity felt by a wavepacket propagating along the waveguide [12].

In a more general context of nonlinear systems, an EP is sometimes introduced as a point of coalescence of the eigenvalues and eigenvectors of a nonlinear eigenvalue problem. Location of such EP in the parameter space depends on the nonlinearity, i.e., on the amplitude of the field. This was particularly well studied for models with double-well potentials [13–15], and it was also found that the nonlinearity may have significant impact on the spectrum of the system in the vicinity of an EP. Thus, considering a nonlinear system with parameters, at which its linear limit is close enough to an EP, one can expect fragility of the stability that may be destructively affected by the nonlinearity. Therefore, one may expect considerable constraints on stable propagation of nonlinear waves in such systems. In this paper, we show that this is not necessarily so: metastable solitons can exist even when the parameters of the underlying linear system are in close proximity of an EP.

The organization of our paper is as follows. In Sec. II we introduce a nonlinear planar waveguide, which is described by an optical potential depending on one of the transverse directions and features an EP. Then in Sec. III we develop a perturbation theory for the spectrum of the corresponding non-Hermitian evolution operator near the EP in the presence of a nonseparable perturbation depending on both transverse coordinates. In Sec. IV we derive the two-component system of equations governing evolution of the slowly varying amplitudes of the guided modes having the propagation constants in close proximity of the second-order EP. We employ the method of multiple-scale expansion where two coupled modes have to be accounted for self-consistency of theory. The obtained system for slowly varying amplitudes features high-order dispersion, resembling (although not coinciding with) dispersion that may lead to pyramid diffraction, considered previously in [11]. In Sec. V we describe solitons of the derived effective one-dimensional (1D) model. Such solitons are stable in the 1D model and they become metastable in the full 2D model governing light propagation in the dissipative waveguide (Sec. VI).
II. THE MODEL

We consider propagation of a paraxial beam along the \( z \)-direction in a medium with gain and losses modulated along \( x \)- and \( y \)-directions. The amplitude of the field \( \Psi \) in dimensionless units is governed by the nonlinear Schrödinger equation

\[
\frac{i}{\hbar} \frac{\partial \Psi}{\partial z} = -\frac{1}{2} \nabla^2 \Psi - V(x,y)\Psi + \chi |\Psi|^2 \Psi,
\]

where \( \nabla = (\partial_x, \partial_y) \), the complex-valued optical potential \( V(x,y) \) is parameterized by a real control parameter \( \delta \), and the real coefficient \( \chi \) characterizes Kerr nonlinearity of the medium. Considering \( 0 \leq \delta \ll 1 \), we assume that the potential \( V(x,y) \) can be represented in the form

\[
V_b(x,y) = V_0(y) + \delta V_1(x,y).
\]

Here \( V_0(y) \) is a complex-valued potential for which the spectrum of the linear non-Hermitian Hamiltonian \( H_0 := -\frac{1}{2} \partial_y^2 - V_0(y) \)

has an EP \(-b_0\) on the real axis, \( b_0 \in \mathbb{R} \), with \( \phi_0(y) \) being the respective eigenfunction:

\[
H_0 \phi_0 = -b_0 \phi_0.
\]

Schematics of the described waveguide is illustrated in Fig. 1. Since \( x \) does not enter Eq. (4), we can say that the waveguide has the EP at each value of \( x \).

For further consideration, we define the associated (generalized) eigenfunction \( \tilde{\phi}_0 \) and generalized eigenfunction \( \tilde{\phi}_0 \) of the Hermitian conjugate \( H_0^\dagger \):

\[
H_0^\dagger \tilde{\phi}_0 = -b_0 \tilde{\phi}_0, \quad (H_0^\dagger + b_0)\tilde{\phi}_0 = \phi_0.
\]

Hereafter we use tildes for the spectral characteristics of the adjoint eigenvalue problem. In our case \( H^\dagger = H^* \), and hence \( \phi_0 = \phi_0^* \), \( \tilde{\phi}_0 = \tilde{\phi}_0^* \), where asterisks mean complex conjugation. We emphasize that neither \( \phi_0 \) nor \( \tilde{\phi}_0 \) depend on \( x \); they depend only on \( y \).

A signature of the EP is the orthogonality condition

\[
\langle \tilde{\phi}_0, \phi_0 \rangle = 0.
\]

Hereafter we use the inner product defined as

\[
(f, g) := \int_{-\infty}^{\infty} f^\ast(x,y)g(x,y)dy.
\]

Generally, the eigenfunction \( \phi_0 \) is defined up to an arbitrary nonzero coefficient, and the generalized eigenfunction \( \tilde{\phi}_0 \) is defined up to the addition of an arbitrary multiple of \( \phi_0 \). It will be convenient to fix two corresponding constants by imposing the following normalization [16]:

\[
\langle \tilde{\phi}_0, \phi_0 \rangle = \langle \tilde{\phi}_0, \tilde{\phi}_0 \rangle = 1,
\]

and the orthogonality condition

\[
\langle \tilde{\phi}_0, \tilde{\phi}_0 \rangle = 0.
\]

III. PERTURBATION THEORY FOR THE SPECTRUM NEAR EP

Suppose, that \( V_1(x,y) \) is chosen such that the spectrum of the perturbed linear Hamiltonian

\[
H_\delta := H_0 - \delta V_1(x,y)
\]

is all-real and there exist two eigenvalues \( b_{1,2}(x) \), i.e.,

\[
H_\delta \phi_j(x,y) = -b_j(x)\phi_j(x,y), \quad j = 1,2,
\]

satisfying the condition

\[
b_1(x) > b_0 > b_2(x)
\]

for all \( x \). We recall that \( x \) plays the role of a parameter in eigenvalue problem [12]. We therefore can say that in the two-dimensional plane of parameters \( (\delta, x) \) the Hamiltonian \( H_\delta \) has an exceptional line \( \delta = 0 \).

We also define adjoint operator and eigenfunctions:

\[
H_\delta^\dagger = -\frac{1}{2} \partial_y^2 - [V_\delta(x,y)]^\ast = H_0^\dagger - \delta [V_1(x,y)]^\ast
\]

and

\[
H_\delta^\dagger \tilde{\phi}_j(x,y) = -b_j(x)\tilde{\phi}_j(x,y), \quad j = 1,2.
\]

Furthermore, we assume that \( H_\delta \) does not have EPs when \( \delta > 0 \) [this condition can be relaxed: what is really important for our consideration is the condition \( \lvert \delta \rvert \ll 1 \)]. Then the following biorthogonality relations hold:

\[
\langle \tilde{\phi}_1, \phi_2 \rangle = \langle \tilde{\phi}_2, \phi_1 \rangle = 0.
\]

The structure of the eigenfunctions in the vicinity of the EP can be described in terms of the following asymptotic expansions \( (j = 1,2) \):

\[
\phi_j = \phi_0 + (-1)^j \delta \frac{1}{\rho_j + \delta \rho_{j1} + \delta^2/2 \rho_{j2} + O(\delta^2)}, \quad \tilde{\phi}_j = \tilde{\phi}_0 + (-1)^j \delta \frac{1}{\tilde{\rho}_j + \delta \tilde{\rho}_{j1} + \delta^2/2 \tilde{\rho}_{j2} + O(\delta^2)},
\]

FIG. 1: Schematics of the geometry. The unperturbed potential \( V_0(y) \) (left panel) features an EP at each value of \( x \). For the perturbed potential \( V_0(y) + \delta V_1(x,y) \) the degeneracy is lifted, and for each \( x \) the waveguide operates slightly below the EP, which means that for each \( x \) there are two close real-valued propagation constants \( b_{1,2}(x) \) (right panel) and two eigenfunctions \( \phi_{1,2}(x,y) \) which are about to merge but nevertheless are distinct.
at $\delta \to 0$. In Eqs. (17), the functions $\rho_j$ and $\tilde{\rho}_j$ are so far undetermined corrections that generically depend on $x$ and $y$. Since in the eigenvalue problem (12) $x$ is a parameter, we can represent

$$b_j(x) = b_0 - (1)^2 \delta^{1/2} \beta_j(x) + \delta \beta_j(x) + \mathcal{O}(\delta^2),$$

(18)

where the coefficients $\beta_j$, $\beta_1$ and $\beta_2$ are to be found. Substituting Eqs. (17) and (18) in (12), one observes that the obtained equation in the leading order $\delta^0$ is satisfied. The balance of $\delta^{1/2}$-order terms requires that $(j = 1, 2)$

$$(H_0 + b_0)\rho_j = \beta_j(x)\phi_0.$$

(19)

Comparing this equation with Eq. (5) we conclude that

$$\rho_j(x, y) = \beta_j(x)\phi_0(y), \quad \tilde{\rho}_j(x, y) = \beta_j(x)\tilde{\phi}_0(y).$$

(20)

Further, from the orthogonality condition (16) we obtain the relation $\delta^{1/2}(\beta_1 - \beta_2) + \mathcal{O}(\delta) = 0$. Therefore

$$\beta_1(x) = \beta_2(x) =: \beta(x),$$

(21)

and hence

$$\rho_1(x, y) = \rho_2(x, y) = \beta(x)\phi_0(y).$$

(22)

Now Eqs. (17a) and (17b) can be rewritten as

$$\phi_j = \phi_0 + (1)^2 \delta^{1/2} \beta(x)\phi_0 + \mathcal{O}(\delta),$$

(23a)

$$\tilde{\phi}_j = \tilde{\phi}_0 + (1)^2 \delta^{1/2} \beta(x)\tilde{\phi}_0 + \mathcal{O}(\delta),$$

(23b)

resulting in the normalization condition

$$\langle \tilde{\phi}_j, \phi_j \rangle = (1)^2 2\delta^{1/2} \beta(x) + \mathcal{O}(\delta).$$

(24)

To determine $\beta(x)$, we consider the next order, $\mathcal{O}(\delta)$, of equations (12):

$$(H_0 + b_0)\rho_j - V_1\phi_0 = \beta^2\phi_0 - \beta_j\phi_0.$$  

(25)

Applying $\langle \tilde{\phi}_j, \cdot \rangle$, either for $j = 1$ and $j = 2$ we obtain

$$\beta^2(x) = -\langle \tilde{\phi}_0, V_1\phi_0 \rangle.$$  

(26)

The function $f$ is defined up to the addition of an arbitrary multiple of $\phi_0$, but from the following analysis it will become evident that without loss of generality this multiple can be set to zero.

Proceeding to the $\mathcal{O}(\delta^{3/2})$-order, from equations (12) we obtain $(j = 1, 2)$

$$(H_0 + b_0)\rho_j = (1)^2 \beta(\rho_j_1 - \beta_j\phi_0 + V_1\phi_0) - \beta_j\phi_0.$$  

(29)

Solvability conditions for these equations read

$$\beta_j = \langle \tilde{\phi}_0, V_1\phi_0 \rangle + \langle \tilde{\phi}_0, \rho_j \rangle.$$  

(30)

Substituting here $\rho_{11}$ and $\rho_{21}$ from (27), we obtain the next-order coefficients of the expansion for the propagation constants in the form:

$$\beta_{11} = \beta_{21} = \frac{1}{2}\langle \tilde{\phi}_0, V_1\phi_0 \rangle + \frac{1}{2}\langle \tilde{\phi}_0, f \rangle =: \gamma(x).$$  

(31)

On the other hand, the orthogonality condition (16) in the $\mathcal{O}(\delta)$-order requires

$$\gamma = \langle \tilde{\phi}_0, f \rangle.$$  

(32)

Combining this expression with (31), we obtain

$$\gamma = \langle \tilde{\phi}_0, V_1\phi_0 \rangle.$$  

(33)

The requirement for the propagation constant to be real, implies that the perturbation $V_1$ should be chosen to ensure the reality of the right-hand side of (33). If this condition is satisfied, then (33) and the relations

$$\langle \rho_j, \tilde{\phi}_0 \rangle = \langle V_1\phi_0, \tilde{\phi}_0 \rangle = \beta_j,$$

(34)

obtained by applying $\langle \cdot, \phi_0 \rangle$ to (25), yield

$$\langle \rho_j, \tilde{\phi}_0 \rangle = \langle \phi_j, \phi_0 \rangle = \langle \phi_j, \phi_0 \rangle = 0.$$  

(35)

Therefore, the estimate (24) can be improved as follows

$$\langle \tilde{\phi}_j, \phi_j \rangle = (1)^2 2\delta^{1/2} \beta(x) + \mathcal{O}(\delta^{3/2}).$$  

(36)

Finally, with the same accuracy we compute useful relations

$$\langle \tilde{\phi}_k, \phi_j \rangle = (1)^2 \beta_x \delta^{1/2} + \mathcal{O}(\delta^{3/2}),$$

(37)

$$\langle \tilde{\phi}_k, \phi_j \rangle = (1)^2 \beta_y \delta^{1/2} + \mathcal{O}(\delta^{3/2}).$$  

(38)

IV. MULTIPLE-SCALE EXPANSION

Now we turn to the nonlinear model and, using the multiple-scale expansion, look for the solution of Eq. (1) in the form

$$\Psi = \delta^{1/2}e^{ib_0z} \left[ \phi_1(x, y)U_1(x, z) + \phi_2(x, y)U_2(x, z) \right] + \mathcal{O}(\delta^{3/2}).$$  

(39)
where $U_{1,2}(x, z)$ are the envelopes of the two modes that coalesce in the EP in the limit $\delta = 0$. Thus, the field we are looking for is two-component. We substitute \ref{40a} into the main equation \ref{1} and apply $\langle \phi_1, \cdot \rangle$ and $\langle \phi_2, \cdot \rangle$ to the resulting expression. Using the results of Sec. \ref{III}, we arrive at a system of two coupled equations that govern the dynamics of $U_1$ and $U_2$:

$$i\partial_x U_1 = \mathcal{H}_0 U_1 - \delta^{1/2} \beta U_1 + \frac{\beta_{xx}}{4\beta} U_2 - \frac{\beta_x}{2\beta} \partial_x U_1 + \frac{\beta_x}{2\beta} \partial_x U_2 - \frac{\delta^{1/2} \chi_0}{\beta} (U_1^* + U_2^*)^2 (U_1 + U_2^2^2 (U_1^* + U_2^*), (40)$$

$$i\partial_x U_2 = \mathcal{H}_0 U_2 + \delta^{1/2} \beta U_2 + \frac{\beta_{xx}}{4\beta} U_1 - \frac{\beta_x}{2\beta} \partial_x U_2 + \frac{\delta^{1/2} \chi_0}{\beta} (U_1 + U_2^2^2 (U_1^* + U_2^*). (41)$$

Here

$$\mathcal{H}_0 = -\frac{1}{2} \partial_x^2 - \frac{\beta_{xx}}{4\beta}, (42)$$

the effective nonlinearity is determined as

$$\chi_0 = \chi(\phi_0, |\phi_0|^2 \phi_0), (43)$$

and all terms of the order of $\delta$ (and higher) are neglected. Upon derivation of the system \ref{40}–\ref{41} we used that

$$(i, j, k = 1, 2)$$

$$\langle \phi_1, |\phi_j|^2 \phi_k \rangle = \langle \phi_0^*, |\phi_0|^2 \phi_0 \rangle + O(\delta^{1/2}),$$

$$\langle \phi_1, \phi_j^* \phi_k \rangle = \langle \phi_0^*, |\phi_0|^2 \phi_0 \rangle + O(\delta^{1/2}).$$

Generally speaking, the effective nonlinearity coefficient $\chi_0$ obtained in \ref{43} is complex-valued. However, it is necessarily real if the unperturbed potential is $\mathcal{P}_y \mathcal{T}$ symmetric. Here, according to standard definitions, the operator $\mathcal{P}_y$ corresponds to the reversal of the y-axis, and the operator $\mathcal{T}$ corresponds to the complex conjugation, and thus $[\mathcal{P}_y \mathcal{T}, \mathcal{H}_0] = 0$, where the unperturbed operator $\mathcal{H}_0$ is defined in \ref{13}. Indeed, the $\mathcal{P}_y \mathcal{T}$ symmetry implies that $\mathcal{P}_y \mathcal{T} \phi_0 = e^{i\theta} \phi_0$, where $\theta$ is a constant phase. From \ref{3} and \ref{8} it follows that $e^{i2\theta} = 1$, i.e., possible values of $\theta$ are 0 and $\pi$. In either case $\mathcal{P}_y |\phi_0|^2 = |\phi_0|^2$, and one can verify that $\chi_0$ is real:

$$\langle \phi_0^*, |\phi_0|^2 \phi_0 \rangle = \mathcal{T}(\phi_0, |\phi_0|^2 \phi_0)$$

$$= \langle e^{-i\theta} \mathcal{P}_y \mathcal{T} \phi_0, |\phi_0|^2 e^{i\theta} \phi_0 \rangle^*$$

$$= e^{-i2\theta} \langle \mathcal{T} \phi_0, |\phi_0|^2 \phi_0 \rangle^* = \langle \phi_0^*, |\phi_0|^2 \phi_0 \rangle^*.$$ 

Importantly, the effective nonlinearities for different components of the field, described by \ref{40}–\ref{41}, have opposite signs. This effect resembles the finding reported in \ref{12} where it has been shown that the presence of an EP in the spectrum of the underlying linear problem can change the sign of the effective nonlinearity.

Equations \ref{40}–\ref{41} acquire a more convenient form if one introduces new functions $U_\pm = U_2 \pm U_1$ which satisfy the following system:

$$i\partial_x U_+ = -\frac{1}{2} \partial_x^2 U_+ + \delta^{1/2} \beta U_-, (44a)$$

$$i\partial_x U_- = \left(-\frac{1}{2} \partial_x^2 - \frac{\beta_{xx}}{2\beta} \right) U_- - \frac{\beta_x}{\beta} \partial_x U_- + \delta^{1/2} \beta U_+ + \delta^{1/2} \chi_0 |U_+|^2 U_+. (44b)$$

The equations obtained for the envelopes make explicit the scaling of the solutions as well as constraints that should be imposed on the $\beta(x)$ dependence. Indeed, from Eq. \ref{44a} we conclude that the envelope is smooth, in the sense that it depends on the scaled variables $\delta^{1/2}$ and $\delta^{1/4}$. From Eq. \ref{44b} it follows that the consistency of the multiple-scale expansion requires $|\beta_x/\beta| \lesssim \delta^{1/4}$ and $|\beta_{xx}/\beta| \lesssim \delta^{1/2}$.

V. 1D SOLITONS AT CONSTANT $\beta$

Let us now consider solitons of the 1D model \ref{44} at a constant $\beta > 0$, when $\beta_x = \beta_{xx} = 0$. For stationary solutions, $U_\pm = u_\pm(x) e^{i\mu z}$, the system \ref{44} reduces to

$$\mu u_+ = \frac{1}{2} \frac{d^2 u_+}{dx^2} - \delta^{1/2} \beta u_-,$$ (45a)

$$\mu u_- = \frac{1}{2} \frac{d^2 u_-}{dx^2} - \delta^{1/2} \beta u_+ - \delta^{1/2} \chi_0 |u_+|^2 u_+. (45b)$$

The spectrum of the linear ($\chi_0 = 0$) limit of this system ($U_\pm \propto e^{i\mu z + ikx}$) has two branches

$$\mu = -\frac{k^2}{2} \pm \sqrt{\delta} \beta. (46)$$

System \ref{45} can be further reduced to a fourth-order nonlinear equation

$$-\frac{1}{4} \frac{d^4 u_+}{dx^4} + \mu \frac{d^2 u_+}{dx^2} + (\delta \beta^2 - \mu^2) u_+ + \delta \chi_0 |u_+|^2 u_+ = 0. (47)$$

Two comments are in order. First, one can see that in the vicinity of the EP the governing equation includes the fourth-order dispersion, that corroborates the previous results on linear diffraction \ref{9} \ref{10}. Second, the models similar to model \ref{47} can be encountered in fiber optics in description of evolution of pulses close to the zero-dispersion wavelength (see e.g. \ref{17} \ref{18}).

A. Two-component solitons

Equation \ref{47}, and hence system \ref{45}, allow for an exact solution. Indeed, for $\chi_0 > 0$ and the propagation constant

$$\mu = \frac{5}{3} \delta^{1/2} \beta > \mu_+ (48)$$
obtains [17]

\[ \beta = 1. \]

following eigenvalue problem after splitting into real and two-component system (44) with constant \( \beta \) curve) and corresponding exact solutions at \( \mu = 0.167, \delta = 0.1 \) (red curve) and \( \mu = 0.527, \delta = 0.1 \) (blue curve). Here \( \chi_0 = 1, \beta = 1 \).

FIG. 2: (a) Families of solitons, in terms of the dependencies of powers \( P_\pm \) on \( \mu \), for \( \delta = 0.01 \) (red curves) and \( \delta = 0.1 \) (blue curves). Solid and dashed lines correspond to \( P_+ \) and \( P_-, \) respectively. Gray vertical lines denote values of \( \mu \) that correspond to the exact solution (48)–(49). (b) Profiles of corresponding exact solutions at \( \mu = 0.167, \delta = 0.01 \) (red curve) and \( \mu = 0.527, \delta = 0.1 \) (blue curve). Here \( \chi_0 = 1, \beta = 1 \).

belonging to the semi-infinite gap of linear spectrum, one obtains [17]

\[ u_+(x) = \beta \sqrt{\frac{10}{3\chi_0}} \text{sech}^2 \xi, \quad (49a) \]
\[ u_-(x) = -\beta \sqrt{\frac{10}{3\chi_0}} (\text{sech}^2 \xi + \text{sech}^4 \xi), \quad (49b) \]

where for compactness we have introduced

\[ \xi = \sqrt{\beta} \frac{1}{3} \xi^{1/4}. \quad (50) \]

The exact soliton given by Eq. (49) belongs to a continuous family parameterized by the propagation constant \( \mu \). Examples of such families are shown in Fig. 2(a) in the form of dependencies \( P_\pm (\mu) \), where \( P_\pm \) are the powers carried by each of the two components:

\[ P_\pm = \int_{-\infty}^{\infty} |u_\pm(x)|^2 dx. \quad (51) \]

B. Linear stability of two-component solitons

Now we proceed to the linear-stability analysis of the found solitons in the framework of the two-component model (44). For perturbed solutions in the form \( U_\pm = e^{i\eta z} [u_\pm(x) + q_\pm(x, z)] \), where \( q_\pm \) are small perturbations of real-valued solutions \( u_\pm(x) \), the linearization of the two-component system (44) with constant \( \beta \) gives the following eigenvalue problem after splitting into real and imaginary parts:

\[ \begin{align*}
\partial_x q_+ &= L_0 \text{Im} q_+ + \delta^{1/2} \beta \text{Im} q_- , \\
\partial_x q_- &= -L_0 \text{Re} q_+ - \delta^{1/2} \beta \text{Re} q_- , \\
\partial_x q_- &= L_0 \text{Im} q_- + \delta^{1/2}(\beta + \chi_0 \beta^{-1} u_0^2) \text{Im} q_+ , \\
\partial_x q_- &= -L_0 \text{Re} q_- - \delta^{1/2}(\beta + 3\chi_0 \beta^{-1} u_0^2) \text{Re} q_+ ,
\end{align*} \]

(52)

where \( L_0 = -(1/2) \partial_x^2 + \mu \). Making the substitution (Re \( q_+, \text{Im} q_+, \text{Re} q_-, \text{Im} q_- ) = e^{i\omega z}(Q_+, R_+, Q_-, R_-) \), where \( \omega \) is the linear stability eigenvalue (positive real part of \( \omega \) corresponds to the exponential growth of the perturbation along the propagation distance), we then eliminate \( Q_- \) and \( R_- \) from the linear stability equations. This leads to a quadratic eigenvalue problem

\[ (\omega^2 \mathbb{1} + \omega \mathcal{C} + \mathcal{K}) \Delta = 0 , \quad (53) \]

where

\[ \Delta = (Q_+, R_+)^T , \quad (54) \]
\[ \mathcal{C} = \begin{pmatrix} 0 & -2\mu + \partial_x^2 \\ 2\mu - \partial_x^2 & 0 \end{pmatrix} , \quad \mathcal{K} = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix} , \quad (55) \]
\[ L_+ = -\frac{1}{4} \partial_x^4 + \mu \partial_x^2 + (\beta^2 \delta - \mu^2) + (2 \pm 1) \delta \chi_0 u_0^2 , \quad (56) \]

and \( \mathbb{1} \) is the identity operator. Quadratic eigenvalue problem (53) can be further converted into the generalized eigenvalue problem (57)

\[ (\Delta - \omega \mathcal{B}) \mathcal{Z} = 0 , \quad (57) \]

where the augmented matrices read

\[ \mathcal{A} = \begin{pmatrix} \mathcal{K} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} , \quad \mathcal{B} = \begin{pmatrix} \mathcal{C} & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} , \quad \mathcal{Z} = \begin{pmatrix} \Delta \\ \omega \Delta \end{pmatrix} . \]

Numerical solution of the generalized eigenvalue problem (57) indicates that the soliton families with \( \chi_0 > 0 \) are entirely stable.

C. Embedded soliton

For \( \chi_0 < 0 \), Eq. (47) admits another exact bright soliton solution which can be written down as [18]

\[ u_+(x) = \beta \sqrt{\frac{10}{3|\chi_0|}} \text{sech}^2 \xi \tanh \xi , \]
\[ u_-(x) = \beta \sqrt{\frac{10}{3|\chi_0|}} \text{sech} \xi \tanh^3 \xi , \]

(58a)
\[ (58b) \]

where \( \xi \) is given by (50). The propagation constant of this solution

\[ \mu = -\frac{5}{6} \sqrt{\delta \beta^2} \]

(59)
belongs to the continuous spectrum, i.e., this is an embedded soliton which can hardly be expected to be stable. The instability of this solution has indeed been confirmed by the linear-stability analysis (described in the previous section), as well as by using direct numerical simulations of one-dimensional propagation governed by vector model \[43\]. Nevertheless, the fact that the system derived here simultaneously supports bright solitons in both focusing and defocusing media is rather interesting.

VI. METAStABLE 2D SOLITONS

Now we turn to the two-dimensional solitons supported by the original model \[1\]. To find stationary solutions, we use the substitution \( \Psi(x,y,z) = e^{i\lambda z}w(x,y) \). Comparing this substitution with \[39\], we obtain the approximate relation \( \lambda \approx b_0 + \mu \), which connects the propagation constant of the 2D solitons (i.e., \( \lambda \)) with that of the 1D solitons considered above in Sec. VII. Using exact solutions obtained above for the 1D model, one can produce reasonable analytical approximation for the 2D soliton profile. At the same time, feasibility of the experimental observation of such 2D solutions depends on the existence of optical potentials \( V_δ(x,y) \) featuring an EP at \( \delta = 0 \) and purely real spectrum at \( 0 < \delta \ll 1 \). Examples of such potentials are well-known. We discuss two possible examples in the following subsections.

A. Exactly solvable \( PT \)-symmetric Scarf II potential

Let \( V_δ(x,y) \) be the \( PT \)-Scarf potential \[20\] \[22\]. For the analysis of its EP we use the representation \[2\] with

\[
V_0 = \left[ \frac{1}{4} (4\beta + 1)^2 - \frac{1}{8} \right] \text{sech}^2 y + \frac{i}{4} (4\beta + 1)^2 \text{sech} y \tanh y,
\]

\[
V_1 = \frac{\text{sech}^2 y}{2}. \tag{60}
\]

For any positive \( \beta > 0 \), the potential \( V_0(y) \) defined by \[60\] is exactly at the EP that corresponds to the coalescence of two eigenmodes at the propagation constant \( b_0 = 2\beta^2 \). The perturbation \( \delta V_1 \) increases the real part of total potential \( V_δ \) and therefore drives the system below the phase transition threshold.

Thanks to the solvability of the \( PT \)-symmetric Scarf II potential \[20\] \[22\], exact expressions for the propagation constants are available

\[
b_{1,2} = \frac{1}{8} \left[ \sqrt{(4\beta + 1)^2 + \delta \pm \sqrt{\delta}} - 1 \right]^2
= 2\beta^2 \pm \beta \sqrt{\delta} + \gamma \delta + O(\delta^{3/2}), \tag{62}
\]

where [see Eqs. (31) \[33\]]

\[
\gamma = \frac{\beta}{2(4\beta + 1)} + \frac{1}{8}. \tag{63}
\]

The eigenfunctions at the EP read

\[
\phi_0 = \frac{2^{2\beta + 1} i \beta}{\sqrt{\pi}} \exp \left\{ i \frac{4\beta + 1}{2} \arctan \sinh y \right\} \text{sech}^2 y, \tag{64}
\]

\[
\varphi_0 = \frac{\phi_0}{2\delta} \left( i \arctan \sinh y - \ln \sinh y - \psi(1) + \psi(4\beta) - \ln 2 \right), \tag{65}
\]

where \( \psi(z) = d(\ln \Gamma(z))/dz = \Gamma'(z)/\Gamma(z) \) is the digamma function \[23\]. One can check that the normalization conditions \[9\] are satisfied. The nonlinear coefficient defined by \[43\] is computed as

\[
\chi_0 = -\frac{\chi}{3\pi B(6\beta, 2\beta)}, \tag{66}
\]

where \( B(\cdot, \cdot) \) is the beta function. Thus \( \chi_0 > 0 (\chi_0 < 0) \) corresponds to the focusing (defocusing) nonlinearity of the physical model \[1\].

In Fig. 3(a, b) we compare analytical prediction for soliton shape (its cross-section at \( y = 0 \)) obtained using the combination of the exact 1D solution \[49\] and eigenfunctions given by \[61\] \[65\], with numerically obtained 2D soliton of Eq. \[1\] having the same propagation constant \( \lambda = 2\beta^2 + 5\delta^2 \beta/3 \). One can see that analytical and numerical solutions are very close at sufficiently small \( \delta \) values, while with increase of \( \delta \) the difference between them gradually increases.

B. Numerical results in the parabolic \( PT \)-symmetric potential

For a more systematic study of the families of the 2D solitons, we choose a less sophisticated potential in the form

\[
V_0 = -y^2 + i\gamma_0 ye^{-y^2/2}. \tag{67}
\]

One-dimensional nonlinear modes in a potential of similar form have been considered in \[24\]. This potential has an EP at \( \gamma_0 \approx 2.1684 \). Analytical expression for the eigenfunctions \( \phi_0 \) and \( \varphi_0 \) are not available in this case, but they can be found numerically. In order to drive potential \( V_0 \) to the unbroken \( PT \)-symmetric phase, we perturb it by decreasing the gain-and-loss amplitude by means of the following perturbation

\[
V_1 = -iy^2 + \frac{\beta^2}{0.5521} ye^{-y^2/2}. \tag{68}
\]

Notice that the numerical coefficient in the denominator is chosen to ensure \[26\].

A family of 2D solitons obtained numerically is shown in Fig. 4 where we present our results for the total power of solitons [cf. \[51\]]

\[
P = \iint |w(x,y)|^2 dx dy, \tag{69}
\]
FIG. 3: (a,b) Comparison between analytical (black solid line) and numerical (red dots) soliton profiles in the $\mathcal{PT}$-Scarf potential for two values of $\delta$. Here we show amplitudes of the solutions at fixed $y = 0$ as functions of $x$. (c) Three-dimensional plot of the numerical solution amplitude $|w(x,y)|$. In all panels $\beta = 0.4$, $\chi = -1$.

and for the soliton amplitudes and widths along the $x$ and $y$ axes. All these characteristics are functions of the propagation constant $\lambda$. While the analytical expression (39) is valid for soliton amplitudes $a \sim \delta^{1/2}$, the numerical continuation allows to obtain even large-amplitude solitons. The cutoff value [shown with dashed vertical line in Fig. 4(a)] corresponds to the edge of the continuum of two-dimensional scattering states. At the cutoff propagation constant the soliton amplitude and power vanish. At the same time, the widths $d_x$, $d_y$ illustrate the anisotropic nature of 2D solitons in our waveguide: as $\lambda$ approaches the cutoff, the soliton width in the $x$-direction diverges, while the width in the $y$-direction remains finite.

While stability analysis performed in Sec. VII in the frame of the reduced 1D model for slowly varying envelopes has indicated that the 1D solitons are stable, this result does not yet guarantee stable propagation of the respective 2D solitons constructed using (39). A systematic numerical study of soliton propagation governed by the (2 + 1)D equation (1) indicates that near the cutoff value the 2D solitons are robust and propagate over considerable distances without noticeable distortions even in the presence of input perturbations, but far from the cutoff the oscillatory instabilities come into play, whose strength gradually increases with the increase of soliton amplitude and propagation constant. The example of metastable evolution of the 2D soliton is presented in Fig. 5(a) showing that amplitude of such state remains practically unchanged with distance $z$, while cross-sections at different distances are shown in Fig. 5(c)-(e). The example of instability development for high-amplitude soliton stimulated by small input noise is presented in Fig. 5(b) and (f)-(h). As one can see, such unstable soliton starts radiating and at sufficiently large distance this radiation grows in amplitude and extends practically over the entire $x$ cross-section. We notice that weak oscillatory instabilities that affect propagation of the 2D solitons in our system can be possibly attributed to poorly localized (in the $x$ direction) unstable modes that bifurcate from the interior of the two-dimensional continuum [25]. However, an accurate analysis of this issue requires a separate and more detailed study.

VII. CONCLUSION

In this work, we have shown that in a waveguide with gain and loss it is possible to obtain propagation of metastable two-dimensional solitons with propagation constants in the vicinity of the exceptional point (but belonging to the unbroken phase). Such solitons are effectively two-component. Analytically they are described by the coupled linear and nonlinear Schrödinger equations which govern envelopes of the two carrier modes. The deviation of the propagation constant from the exceptional point is the small parameter of the multiple-scale expansion. The effective one-dimensional equations for the envelope allow for exact bright soliton solutions for either sign of the nonlinearity coefficient. The envelope solitons are stable in the one-dimensional setting, although they become metastable in the fully two-dimensional model. The lifetime of such solitons is very large making them feasible for the experimental observation.
FIG. 5: Peak amplitude versus distance and field modulus distributions at different propagation distances for stable 2D soliton with $\lambda = -1.2$ (a) and (c)-(e), and unstable 2D soliton with $\lambda = -0.8$ (b) and (f)-(h) in parabolic $\mathcal{PT}$ potential. In all cases $\beta = 1$ and $\delta = 0.1$. Panels (c-h) correspond to spatial window $(x, y) \in [-40, 40] \times [-8, 8]$. The focusing nonlinearity coefficient $\chi = -1$.

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