On isometric embeddings and continuous maps onto the irrationals

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Abstract  Let $f : E \to F$ be a continuous map of a complete separable metric space $E$ onto the irrationals. We shall show that if a complete separable metric space $M$ contains isometric copies of every closed relatively discrete set in $E$, then $M$ contains also an isometric copy of some fiber $f^{-1}(y)$. We shall show also that if all fibers of $f$ have positive dimension, then the collection of closed zero-dimensional sets in $E$ is non-analytic in the Wijsman hyperspace of $E$. These results, based on a classical Hurewicz’s theorem, refine some results from Pol and Pol (Isr J Math 209:187–197, 2015) and answer a question in Banakh et al. (in: Pearl (ed) Open problems in topology II. Elsevier, Amsterdam, 2007).

Keywords  Isometric embeddings · Effros Borel spaces · Zero-dimensional sets

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1 Introduction

In [13] we proved that each complete separable metric space containing isometric copies of every countable complete metric space contains isometric copies of every separable metric space.

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We shall refine this result to the following effect.

**Theorem 1.1** Let \( f : E \to F \) be a continuous map of a complete separable metric space onto a non-\( \sigma \)-compact metric space. Then there exists a relatively discrete set \( S \) in \( E \) such that, for any complete separable metric space \( M \) containing isometric copies of every subset of \( S \) closed in \( E \), some fiber \( f^{-1}(y) \) embeds isometrically in \( M \).

The result from [13] follows from this theorem, if we consider the restriction map \( f : C[0, 1] \to C[\frac{1}{2}, 1] \) (recall that by the Banach–Mazur theorem, cf. [8, Theorem 5.17], the space \( (C(I), d_{\text{sup}}) \) of all real-valued continuous functions on the interval \( I = [0, 1] \), equipped with the metric \( d_{\text{sup}}(f, g) = \sup \{|f(t) - g(t)| : t \in I\} \), is isometrically universal for all separable metric spaces).

Also, as in [13], one can replace in this theorem isometries by uniform homeomorphisms.

The proofs will go along the same lines as in [13], and an essential part of the reasonings can be taken directly from [13], cf. Sect. 4.

However, a classical Hurewicz’s theorem on non-analyticity of the set of compact subsets of the rationals is applied in a different way than in [13]. We shall prove a result based on the Hurewicz theorem in a slightly more general form than needed for Theorem 1.1 in Sect. 3, to establish a link with some questions concerning the dimension, discussed in Sect. 5.

2 Preliminaries

2.1 The Effros Borel spaces

Our terminology related to the descriptive set theory follows [7,9]. An analytic space is a metrizable continuous image of the irrationals.

Given an analytic space \( E \), we denote by \( F(E) \) the space of closed subsets of \( E \) and \( \mathcal{B}_{F(E)} \)—the Effros Borel structure in \( F(E) \), is the \( \sigma \)-algebra in \( F(E) \) generated by the sets \( \{A \in F(E) : A \cap U \neq \emptyset\} \), where \( U \) is open in \( E \).

We shall say that \( A \subseteq F(E) \) is a Souslin set in the Effros Borel space \( (F(E), \mathcal{B}_{F(E)}) \) if \( A \) is a result of the Souslin operation on sets from \( \mathcal{B}_{F(E)} \).

If \( X \) is a compact metrizable space, we shall consider the hyperspace \( F(X) \) with the Vietoris topology and then \( \mathcal{B}_{F(X)} \) coincides with the \( \sigma \)-algebra of Borel sets in the compact metrizable space \( F(X) \).

If \( X \) is a compact metrizable extension of an analytic set \( E \subset X \), the map \( A \to \overline{A} \) (the closure is taken in \( X \)) from \( F(E) \) to \( F(X) \) is a Borel isomorphism, with respect to the Effros Borel structures, onto the analytic subspace \( \{\overline{A} : A \in F(E)\} \) of the hyperspace \( F(X) \) and hence Souslin sets in \( F(E) \) are mapped onto analytic sets in \( F(X) \), cf. [7, Section 2]. In particular, if \( E \subset G \subset X \) and \( G \) is analytic, the collection of closures of elements of \( F(E) \) in \( G \) is a Souslin set in \( F(G) \).

2.2 The Hurewicz theorem

Let \( I = [0, 1] \) and let \( \mathbb{Q} \) be the set of rationals in \( I \).
The classical Hurewicz theorem asserts that any Souslin set in $F(I)$ containing all compact subsets of $\mathbb{Q}$, contains an element intersecting $I\setminus\mathbb{Q}$.

We shall derive from this theorem the following observation, which we shall use in the next section.

Let us arrange points of $\mathbb{Q}$ into a sequence $q_1, q_2, \ldots$ (without repetitions), let

$$D = \left\{ \left( q_n, \frac{1}{m} \right) : n = 1, 2, \ldots, m \geq n \right\}, \quad L = (I\setminus\mathbb{Q}) \times \{0\},$$

(2.1)

let

$$T = L \cup D$$

(2.2)

be the subspace of the plane (notice that $D$ is relatively discrete in $T$), and let

$$D = \{ A \subset D : A \text{ is closed in } T \}. \quad (2.3)$$

Lemma 2.2.1 For any Souslin set $\mathcal{E}$ in $F(T)$ containing $D$, some element of $\mathcal{E}$ intersects $L$.

Proof For $A \in F(T)$, $\overline{A}$ will denote the closure in the plane. As was recalled in 2.1, the set $\{ \overline{A} : A \in \mathcal{E} \}$ is analytic in $F(\overline{T})$ (notice that $\overline{T} = (I \times \{0\}) \cup D$), hence the set $\{(K, \overline{A}) \in F(I) \times F(\overline{T}) : A \in \mathcal{E} \text{ and } K \times \{0\} \subset \overline{A}\}$ is analytic in the product of the hyperspaces, and so is its projection onto $F(I)$,

$$\mathcal{E}^* = \{ K \in F(I) : K \times \{0\} \subset \overline{A} \text{ for some } A \in \mathcal{E} \}. \quad (2.4)$$

If $K \subset \mathbb{Q}$ is compact, $A = D \cap (K \times I)$ is closed in $T$ and $K \times \{0\} \subset \overline{A}$, hence $K \in \mathcal{E}^*$, cf. (2.4). By the Hurewicz theorem, there is $A \in \mathcal{E}$ such that $A$ intersects $L$, cf. (2.1) and (2.4), and since $A$ is closed in $T$, $A$ intersects $L$. $\Box$

2.3 A remark on continuous maps onto the irrationals

We shall need the following observation. This is close to some well-known results, but for readers convenience, we shall provide a brief justification.

Lemma 2.3.1 Let $f : E \to F$ be a continuous map of an analytic space onto a non-$\sigma$-compact metrizable space. There is a closed copy of the irrationals $P$ in $F$ and continuous maps $g_n : P \to E$ such that, for each $t \in P$, $\{ g_n(t) : n = 1, 2, \ldots \}$ is a dense subset of $f^{-1}(t)$.

Proof Let $p : \mathbb{N}^\mathbb{N} \to E$ be a continuous surjection of the irrationals onto the analytic space $E$.

Then $u = f \circ p : \mathbb{N}^\mathbb{N} \to F$ is a continuous surjection onto a non-$\sigma$-compact metrizable space and one can find a closed copy of the irrationals $P$ in $F$ such that the restriction map $u \mid u^{-1}(P) : u^{-1}(P) \to P$ is open, cf. [14, proof of Theorem 3.1].
By a selection theorem of Michael [10], one can define a sequence of continuous selections \( w_n : P \to u^{-1}(P) \) for the lower-semicontinuous multifunction \( t \to u^{-1}(t) \) such that, for each \( t \in P \), the set \( \{ w_n(t) : n = 1, 2, \ldots \} \) is dense in \( u^{-1}(t) \).

Then the functions \( g_n = p \circ w_n : P \to f^{-1}(P) \) satisfy the assertion. \( \square \)

3 An application of the Hurewicz theorem

The following proposition strengthens a known fact that, for the irrationals \( \mathbb{N}^N \), any Souslin set in \( F(\mathbb{N}^N) \) containing all countable closed sets in \( \mathbb{N}^N \), contains also a non-\( \sigma \)-compact set (this is stated in [9, Exercises 27.8, 27.9]; to derive this fact from the proposition, notice that \( \mathbb{N}^N \) is homeomorphic to \( \mathbb{N}^N \times \mathbb{N}^N \) and consider the projection \( \mathbb{N}^N \times \mathbb{N}^N \to \mathbb{N}^N \)).

The setting is a bit more general than needed for Theorem 1.1, but it is useful to establish connections with some topics in the dimension theory, discussed in Sect. 5.

**Proposition 3.1** Let \( f : E \to F \) be a continuous map of an analytic space onto a non-\( \sigma \)-compact metrizable space. Then there exists a relatively discrete set \( S \) in \( E \) such that for any Souslin set \( A \) in \( F(E) \) containing all subsets of \( S \) closed in \( E \), there are \( A \in A \) and \( y \in F \) with \( f^{-1}(y) \subset A \).

**Proof** Let \( P \) be a closed copy of the irrationals in \( F(\mathbb{N}^N) \) and \( g_n : P \to E \) continuous maps described in Lemma 2.3.1, and let \( T = L \cup D \) be the subspace of the plane defined in (2.1) and (2.2).

Since \( T \) is a zero-dimensional \( G_\delta \)-subset of the plane, there is a homeomorphic embedding

\[
 h : T \to P, \ h(T) \text{ closed in } P. \tag{3.1}
\]

Let us arrange points of \( D \) into a sequence without repetitions

\[
 D = \{ d_1, d_2, \ldots \} \text{ and } c_n = h(d_n). \tag{3.2}
\]

We shall check that, cf. (3.2),

\[
 S = \{ g_m(c_n) : n = 1, 2, \ldots, m \leq n \} \subset E \tag{3.3}
\]

satisfies the assertion of the proposition.

Since \( g_m(c_n) \in f^{-1}(c_n) \), \( f(S) = h(D) \) is relatively discrete and \( S \) intersects each fiber of \( f \) in at most finite set, cf. (3.3). Therefore \( S \) is relatively discrete.

Let, for \( X \in F(T) \),

\[
 \varphi(X) = f^{-1}(h(X \cap L)) \cup (S \cap f^{-1}(h(X \cap D))). \tag{3.4}
\]

Since all accumulation points of \( S \) in \( E \) are in \( f^{-1}(h(L)) \) and \( h(X) \) is closed in \( F \), cf. (3.1), we have \( \varphi(X) \in F(E) \).
We shall check that
\[ \varphi : F(T) \to F(E) \text{ is Borel,} \] (3.5)
with respect to the Effros Borel structure.

To that end, let us fix an open set \( U \) in \( E \), and let
\[ U = \{ X \in F(T) : \varphi(X) \cap U \neq \emptyset \}. \] (3.6)

Let \( X \in U \). If for some \( m \leq n \), \( d_n \in X \) and \( g_m(c_n) \in U \), cf. (3.2), (3.3), (3.4), the element \( \{ Y \in F(T) : d_n \in Y \} \) of \( \mathcal{B}_{F(T)} \) contains \( X \) and is contained in \( U \).

Let \( a \in X \cap L \) and \( f^{-1}(h(a)) \cap U \neq \emptyset \). Since the points \( g_m(h(a)) \) are dense in \( f^{-1}(h(a)) \), there is \( m \) such that \( g_m(h(a)) \in U \). Let \( V \) be a neighbourhood of \( h(a) \) in \( F \) such that \( g_m(V) \subset U \), and let us pick a rectangle \( J = (r, s) \times [0, \frac{1}{p}] \) disjoint from \( \{ d_1, \ldots, d_m \} \) with \( r, s \in \mathbb{Q} \), containing \( a \), such that \( h(J \cap T) \subset V \). If \( Y \in F(T) \) hits \( J \), there is either \( b \in Y \cap L \) with \( h(b) \in V \) and then \( f^{-1}(h(b)) \subset \varphi(Y) \) intersects \( U \), or there is \( d_n \in Y \cap J \) with \( n > m \) and then, cf. (3.3), (3.4), \( g_m(c_n) \in \varphi(Y) \cap U \).

Therefore the element \( \{ Y \in F(T) : Y \cap J \neq \emptyset \} \) of \( \mathcal{B}_{F(T)} \) contains \( X \) and is contained in \( U \).

We demonstrated that \( U \) is a countable union of elements of \( \mathcal{B}_{F(T)} \), hence belongs to the Effros Borel structure of \( F(T) \).

Having checked (3.5), let us consider the set
\[ S = \{ A \subset S : A \in F(E) \} \] (3.7)
and let
\[ S \subset \mathcal{A}, \mathcal{A} \text{ is Souslin in } (F(E), \mathcal{B}_{F(E)}). \] (3.8)

By (3.5),
\[ \mathcal{E} = \varphi^{-1}(\mathcal{A}) \text{ is Souslin in } (F(T), \mathcal{B}_{F(T)}). \] (3.9)

If \( X \subset D \) is closed in \( T \), \( h(X) \) is closed in \( F \) and \( \varphi(X) \) is closed in \( E \), cf. (3.4), hence \( \varphi(X) \in S \), cf. (3.7). Therefore, by (3.8), for the set \( \mathcal{D} \) defined in (2.3), we have \( \mathcal{D} \subset \mathcal{E} \) and Lemma 2.2.1 provides \( X \in \mathcal{E} \) and a point \( a \in X \cap L \). By (3.4) and (3.9) we get \( A = \varphi(X) \in \mathcal{A} \) and \( f^{-1}(h(a)) \subset A \).

\[ \square \]

4 Proof of Theorem 1.1

We shall recall briefly some reasonings from [13] to derive this theorem from Proposition 3.1.

Given \( f : E \to F \) as in this theorem, let us pick \( S \) satisfying the assertion of Proposition 3.1.
Let $e$ be the complete metric on $E$ and let $(M, d)$ be any complete separable metric space, containing isometric copies of every subset of $S$ closed in $E$. Let

$$
\mathcal{H} = \{ T \in F(E \times M) : \text{for every } (x_1, y_1), (x_2, y_2) \in T, \ e(x_1, x_2) = d(y_1, y_2) \}.
$$

One checks, cf. [13, p. 193], that $\mathcal{H}$ is in $\mathcal{B}_{F(E \times M)}$ and the map $T \rightarrow \pi(T)$ associating to $T \in \mathcal{H}$ its projection onto $E$ is a Borel map $\pi : \mathcal{H} \rightarrow F(E)$.

Therefore $A = \pi(\mathcal{H})$ is a Souslin set in $(F(E), \mathcal{B}_{F(E)})$. If $X \subset S$ is closed in $E$, there is an isometry $f : X \rightarrow f(X) \subset M$ and the graph of $f$ is an element of $\mathcal{H}$.

It follows that the Souslin set $A$ contains all subsets of $S$ closed in $E$, and by the choice of $S$, some $A \in \mathcal{A}$ contains a fiber $f^{-1}(y)$.

Now, $A = \pi(T)$ and $T$ is the graph of an isometry that embeds $A$ in $M$. In effect, $f^{-1}(y)$ embeds isometrically in $M$.

5 The collections of zero-dimensional sets in Effros Borel spaces

Our terminology concerning the dimension theory follows [15].

Given an analytic space, we shall write

$$
F_0(E) = \{ A \in F(E) : \dim A = 0 \}.
$$

We shall derive from Proposition 3.1 the following result.

**Proposition 5.1** Let $E$ be an analytic space that admits a continuous map $f : E \rightarrow F$ onto a non-$\sigma$-compact metrizable space such that all fibers $f^{-1}(y)$ have positive dimension. Then for any analytic extension $G$ of $E$ with $\dim(G \setminus E) \leq 0$, the set $F_0(G)$ is not Souslin in the Effros Borel space $(F(G), \mathcal{B}_{F(G)})$.

**Proof** By Proposition 3.1, there is a relatively discrete set $S$ in $E$ such that for any Souslin set $A$ in $F(E)$ containing $S = \{ A \in F(E) : \emptyset \neq A \subset S \}$, some element of the set $\mathcal{A}$ contains a fiber of $f$ and hence has positive dimension.

Now, consider an analytic extension $G$ of $E$ with $\dim(G \setminus E) \leq 0$ and, aiming at a contradiction assume that $F_0(G)$ is Souslin in $F(G)$. As was recalled in Sect. 2.1, the map $A \rightarrow \overline{A}$ from $F(E)$ to $F(G)$ is Borel, and hence we would get that the set $\mathcal{A} = \{ A \in F(E) : \dim \overline{A} \leq 0 \}$ is Souslin in $F(E)$.

If $A \in \mathcal{S}$, then $A$ is a relatively discrete closed set in $E$, and hence $\overline{A} \setminus A$ is a closed subset of $G$ contained in $G \setminus E$. This implies that $\dim \overline{A} = 0$, i.e., $\mathcal{S} \subset \mathcal{A}$. However, all members of $\mathcal{A}$ are zero-dimensional, which contradicts properties of $\mathcal{S}$. □

In particular, if $\mathbb{P}$ is the set of irrationals in $I = [0, 1]$,

$$
F_0(\mathbb{P} \times I) \text{ is not Souslin in } F(\mathbb{P} \times I)
$$

(this rectifies a remark in [5, §3.A]).

Banakh et al. [1, Question 9.12], asked about the Borel type of the collection $F_0(E)$ in the space $CL(E) = F(E) \setminus \{ \emptyset \}$, when $E$ is a completely metrizable separable space,
and $CL(E)$ is considered with the Wijsman topology $\tau_W$, determined by some metric $d$ generating the topology of $E$ (i.e., $\tau_W$ is the weakest topology making all functionals $A \to \text{dist}(z, A), z \in E$, continuous), cf. [2,4].

The Wijsman hyperspace $(CL(E), \tau_W)$ is completely metrizable, separable, cf. [6], and the Borel sets with respect to $\tau_W$ coincide with the members of the Effros Borel structure in $CL(E)$. Therefore,

$$F_0(\mathbb{P} \times I) \text{ is not a Borel (or even Souslin) set in } CL(\mathbb{P} \times I). \quad (5.3)$$

One can check that its complement $F(\mathbb{P} \times I) \setminus F_0(\mathbb{P} \times I)$ is Souslin. Let us consider, however, the subspace $I^2 \setminus Q^2$ of the square, $Q = I \setminus \mathbb{P}$. Since $(I^2 \setminus Q^2) \setminus (\mathbb{P} \times I) = Q \setminus \mathbb{P}$ is zero-dimensional, also $F_0(I^2 \setminus Q^2)$ is not Souslin in $F(I^2 \setminus Q^2)$, by Proposition 5.1. But it is not clear to us whether $F(I^2 \setminus Q^2) \setminus F_0(I^2 \setminus Q^2)$ is Souslin.

In fact, we do not know an answer to the following general question.

**Question 5.2** Does there exist an analytic space $E$ such that $F(E) \setminus F_0(E)$ is not Souslin in the Effros Borel structure?

This question is related to the following problem, asked in [11], where countable-dimensional spaces are countable unions of zero-dimensional spaces.

**Problem 5.3** Is the collection $\mathcal{C}$ of all countable-dimensional compact sets in the Hilbert cube $I^\mathbb{N}$ coanalytic in the hyperspace $F(I^\mathbb{N})$ equipped with the Vietoris topology?

To see the link between these two questions, let us consider a Borel set $E \subset I^\mathbb{N}$ such that $I^\mathbb{N} \setminus E$ is countable-dimensional and all countable-dimensional subsets of $E$ are at most zero-dimensional, cf. [12]. We shall assume in addition that $E$ is disjoint from the set $\Sigma$ consisting of points in $I^\mathbb{N}$ with all but finitely many coordinates zero.

By [3, Ch.V, §5], there is a homeomorphism $h : I^\mathbb{N} \setminus \Sigma \to \mathbb{R}^N \times \mathbb{R}^N$ (where $\mathbb{R}$ is the real line), let $p : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ be the projection and let $f = p \circ h | E : E \to \mathbb{R}^N$. Then $f$ is a continuous surjection whose all fibers are uncountable-dimensional. Therefore, by Corollary 5.1, $F_0(E)$ is not Souslin in the Effros Borel space.

We do not know if the set $\bar{\mathcal{C}} = F(E) \setminus F_0(E)$ is Souslin. Let us show, however, that if this is the case, $\mathcal{C}$ in Problem 5.3 is coanalytic.

Suppose that $\mathcal{C}$ is Souslin in $(F(E), \mathcal{B}_{F(E)})$. Then, as was noticed in Sect. 2.1, the collection $\mathcal{E}^* = \{ \bar{A} : A \in \mathcal{E} \}$ of the closures in $I^\mathbb{N}$ is analytic in the hyperspace $F(I^\mathbb{N})$. Now, $K \in F(I^\mathbb{N}) \setminus \mathcal{C}$ if and only if $K \cap E$ is uncountable-dimensional, which is equivalent to $K \cap E \notin F_0(E)$. Therefore, $F(I^\mathbb{N}) \setminus \mathcal{C}$ is the projection of the analytic set $\{(K, L) \in F(I^\mathbb{N}) \times F(I^\mathbb{N}) : L \subset K \text{ and } L \in \mathcal{E}^* \}$, hence it is analytic.

**Added in the revision** Concerning Question 5.2, Debs and Saint Raymond gave in a recent paper “The descriptive complexity of the set of all closed zero-dimensional subsets of a Polish space” a subtle construction of a $G_\delta$-set $E$ in $I^3$ such that $F_0(E)$ is not even a $C$-set in $F(E)$ (in particular, $F(E) \setminus F_0(E)$ is not Souslin). The question concerning $F_0(I^2 \setminus Q^2)$ remains open.
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