CHIRAL CORRECTIONS TO THE S–WAVE PION–NUCLEON SCATTERING LENGTHS *

Véronique Bernard
Centre de Recherches Nucléaires et Université Louis Pasteur de Strasbourg
Physique Théorique, Bat. 40A, BP 20, 67037 Strasbourg Cedex 2, France

Norbert Kaiser
Physik Department T30, Technische Universität München
James Franck Straße, W-8046 Garching, Germany

Ulf-G. Meißner†
Universität Bern, Institut für Theoretische Physik
Sidlerstr. 5, CH–3012 Bern, Switzerland

ABSTRACT
We calculate the chiral corrections to Weinberg’s prediction for the S–wave \( \pi N \) scattering lengths up–to–and–including order \( \mathcal{O}(M_{\pi}^3) \) making use of heavy baryon chiral perturbation theory. For the isospin–odd scattering length \( a^- \) these corrections are small and bring the prediction closer to the empirical value. In the case of the isospin–even \( a^+ \) large cancellations appear so that the \( \mathcal{O}(M_{\pi}^3) \) result depends sensitively on certain resonance parameters which enter the calculation of the contact terms present at next–to–leading order.

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† Heisenberg Fellow.
1. One of the most spectacular successes of current algebra in the sixties was Weinberg’s prediction [1] for the S–wave pion–nucleon scattering lengths, \(a_{1/2} = M_\pi/4\pi F_\pi^2 = -2a_{3/2} = 0.175 M_\pi^{-1}\), with \(M_\pi\) the physical pion mass and \(F_\pi\) the pion decay constant. Tomozawa [2] also derived the sum rule \(a_{1/2} - a_{3/2} = 3M_\pi/8\pi F_\pi^2 = 0.263 M_\pi^{-1}\). Empirically, the combination \((2a_{1/2} + a_{3/2})/3\) is best determined. The Karlsruhe–Helsinki group gives \(0.083 \pm 0.004 M_\pi^{-1}\) [3] consistent with the pionic atom measurement [4] of \(0.086 \pm 0.004 M_\pi^{-1}\). The value of \(a_{1/2} - a_{3/2}\) is more uncertain. The KH analysis leads to \(0.274 \pm 0.005 M_\pi^{-1}\) [5] whereas the VPI group has recently given a larger value [6]. Their analysis was critically reexamined by Höhler [7]. In what follows, we will use the central values from the work of Koch [3], namely \(a_{1/2} = 0.175 M_\pi^{-1}\) and \(a_{3/2} = -0.100 M_\pi^{-1}\). The agreement of the current algebra predictions with these numbers is rather spectacular. However, in the last decade it has become clear that current algebra is only the first term in a systematic expansion of the QCD Green functions in powers of (small) external momenta and the (light) quark masses [8,9]. Therefore, one would like to know what the next–to–leading order corrections to the original predictions are. This is exactly the question we will address here. The basic framework to perform the calculation of these corrections is baryon chiral perturbation theory which makes use of an effective Lagrangian of the asymptotically observed fields. In particular, we use the heavy fermion formulation [10] in which the nucleons are considered as static sources and one has a one–to–one correspondence between the loop and the small momentum expansion. We will work out one loop and counter term contributions up–to–and–including order \(M_\pi^3\) based on the power counting scheme developed by Weinberg [8] and later extended to the baryon sector by Gasser et al. [11]. For a review on these methods, the reader is referred to ref.[12].

2. Consider the \(\pi N\) forward scattering amplitude for a nucleon at rest with its four–velocity given by \(v_\mu = (1, 0, 0, 0)\).* Denoting by \(b\) and \(a\) the isospin of the outgoing and incoming pion, in order, the scattering amplitude takes the form

\[
T^{ba} = T^+ (\omega) \delta^{ba} + T^- (\omega) i\epsilon^{bac} \tau^c
\]

with \(q\) the pion four–momentum and \(\omega = v \cdot q = q_0\). Under crossing \((a \leftrightarrow b, q \rightarrow -q)\) the functions \(T^+\) and \(T^-\) are even and odd, respectively, \(T^\pm (\omega) = \pm T^\pm (-\omega)\). At threshold one has \(\vec{q} = 0\) and the pertinent scattering lengths are defined by

\[
a^\pm = \frac{1}{4\pi} \left(1 + \frac{M_\pi}{m}\right)^{-1} T^\pm (M_\pi)
\]

* Remember that we treat the nucleons as very heavy fields.
with $m$ the nucleon mass. The S–wave scattering lengths for the total $\pi N$ isospin 1/2 and 3/2 are related to $a^\pm$ via

$$a_{1/2} = a^+ + 2a^-, \quad a_{3/2} = a^+ - a^- \quad (3)$$

The abovementioned central empirical values translate into $a^+ = -0.83 \cdot 10^{-2} M^{-1}_\pi$ and $a^+ = 9.17 \cdot 10^{-2} M^{-1}_\pi$. In what follows, we will not exhibit the canonical units of $10^{-2} M^{-1}_\pi$. The benchmark values are therefore $a^+ = -0.83 \pm 0.38$ and $a^- = 9.17 \pm 0.17$ compared to the current algebra predictions of $a^+ = 0$ and $a^- = 8.76$ (using $M_\pi = 138$ MeV and $F_\pi = 93$ MeV).

3. To calculate the scattering lengths, we use the effective pion–nucleon Lagrangian. We work in flavor SU(2) and in the isospin limit $m_u = m_d = \hat{m}$. The pion fields are collected in the matrix $U(x) = \exp[i\vec{\tau} \cdot \vec{\pi}(x)/F_\pi] = u(x)$. The effective Lagrangian to order $O(q^3)$, where $q$ denotes a genuine small momentum or a quark mass, reads

$$L_{\pi N} = L_{\pi N}^{(1)} + L_{\pi N}^{(2)} + L_{\pi N}^{(3)}$$

$$L_{\pi N}^{(1)} = \bar{H}(iv \cdot D + g_A S \cdot u)H$$

$$L_{\pi N}^{(2)} = c_1 \bar{H}H \text{Tr}(\chi_+) + (c_2 - \frac{g_A^2}{8m}) \bar{H}v \cdot u v \cdot uH + c_3 \bar{H}u \cdot uH$$

$$L_{\pi N}^{(3)} = (b_1 - \frac{g_A^2}{32m^2}) v^\mu v^\lambda \bar{v}^\rho \bar{H}K_{\mu\lambda\rho}H + b_2 v^\rho \bar{H}K_{\mu}^\mu H + b_3 v^\rho \bar{H}K_{\rho}^\mu H$$

with

$$u_\mu = iu^\dagger \nabla_\mu Uu^\dagger$$

$$u_{\mu\lambda} = iu^\dagger \nabla_\mu \nabla_\lambda Uu^\dagger$$

$$K_{\mu\lambda\rho} = i[u_{\mu\lambda}, u_{\rho}]$$

where $H$ denotes the heavy nucleon field, $S_\mu$ the covariant spin–operator subject to the constraint $\nu \cdot S = 0$, $\nabla_\mu$ the covariant derivative acting on the pions and we adhere to the notations of ref.[13]. The superscripts (1,2,3) denote the chiral dimension. The lowest order effective Lagrangian is of order $O(q)$. The one loop contribution is suppressed with respect to the tree level by $q^2$ thus contributing at $O(q^3)$. In addition, there are contact terms of order $q^2$ and $q^3$ with coefficients not fixed by chiral symmetry. Notice furthermore that one in addition has to go further in the $1/m$ expansion of the relativistic tree level graphs as can be seen from the terms which come together with

* We have added the errors obtained for $a_{1/2}$ and $a_{3/2}$ in quadrature.
the ones proportional to $c_2$ and $b_1$. At next–to–leading order, all these terms have to be retained. Due to crossing symmetry, $L^{(2)}_{\pi N}$ contributes only to $T^+(\omega)$ whereas $L^{(3)}_{\pi N}$ solely enters $T^-(\omega)$. We have not exhibited the standard meson Lagrangian $L^{(2)}_{\pi\pi} + L^{(4)}_{\pi\pi}$. The contact terms appearing in $L^{(4)}_{\pi\pi}$ only contribute to the shift of the pion mass, the pseudoscalar coupling $G_\pi$ and the pion decay constant from their chiral limit to the physical values. For the following, we define

$$L = \frac{M_\pi}{8\pi F^2}, \quad \mu = \frac{M_\pi}{m}$$

(6)

The calculation of the scattering lengths is straightforward. For the isospin–odd $a^-$ one arrives at

$$a^- = a^-(M_\pi) + a^-(M^2_\pi) + a^-(M^3_\pi)$$

$$a^-(M_\pi) = L, \quad a^-(M^2_\pi) = -L\mu$$

$$a^-(M^3_\pi) = L\mu^2\left[1 + \frac{g^2 A}{4}\right] + \frac{L^2 M_\pi}{\pi}\left[1 - 2\ln\frac{M_\pi}{\lambda} - 64\pi L^2 M_\pi F^2_\pi \left[b^1_1(\lambda) + b_2 + b_3\right]\right]$$

(7)

with $\lambda$ the scale of dimensional regularization. While the constants $b_2$ and $b_3$ are finite, $b_1$ has to be renormalized as follows to render the isospin–odd scattering amplitude $T^-(\omega)$ finite,

$$b_1 = b^1_1(\lambda) - \frac{K}{2F^2}$$

$$K = \frac{\lambda^{d-4}}{16\pi^2} \left[\frac{1}{d-4} + \frac{1}{2} \left(\gamma_E - 1 - \ln 4\pi\right)\right]$$

(8)

Here, $F$ is the pion decay constant in the chiral limit. In what follows, we will use $\lambda = m_\Delta = 1.232$ GeV, motivated by the resonance saturation principle. Of course, physical observables do not depend on this particular choice. Notice that the contact term contributions are suppressed by a factor $M^2_\pi$ with respect to the leading current algebra term. Matters are different for the isospin–even scattering length $a^+$. It consists of contributions of order $M^2_\pi$ and $M^3_\pi$,

$$a^+ = a^+(M^2_\pi) + a^+(M^3_\pi)$$

$$a^+(M^2_\pi) = 32\pi F^2_\pi L^2 (c_2 + c_3 - 2c_1 - \frac{g^2 A}{8m})$$

$$a^+(M^3_\pi) = \frac{3}{4} g^2 A L^2 M_\pi - 32\pi F^2_\pi L^2 \mu (c_2 + c_3 - 2c_1 - \frac{g^2 A}{8m})$$

(9)

The coefficients $c_{1,2,3}$ are all finite. From the form of eq.(9) it is obvious that the contact terms play a more important role in the determination of $a^+$ than for $a^-$. 

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4. The most difficult task is to pin down the various low-energy constants appearing in eqs. (7) and (9). Let us first consider $c_{1,2,3}$. The coefficient $c_1$ can be unambiguously fixed from the pion–nucleon $\sigma$–term [13],

$$c_1 = -\frac{1}{4M_\pi^2} \left[ \sigma_{\pi N}(0) + \frac{9g_A^2M_\pi^3}{64\pi F_\pi^2} \right] = -0.87 \pm 0.11 \text{ GeV}^{-1}$$

(10)

using $g_A = 1.26$ and $\sigma_{\pi N}(0) = 45 \pm 8$ MeV [14]. To estimate the remaining constants, we make use of the principle of resonance saturation [15]. It states that to a high degree of accuracy the low–energy constants can be calculated from resonance exchanges by integrating out the heavy fields from an effective Lagrangian of the pions chirally coupled to the various resonances. In the meson sector, this has been shown to work very well. We extend this method to the baryon sector since it is essentially the only method of estimating the unknown coefficients. From the meson sector, scalar meson exchange can contribute to $c_1$ and $c_3$,

$$c_1 - \frac{1}{2}c_3|_S = c_1 - c_1 \frac{c_d}{c_m}$$

(11)

with $2c_d/c_m = L_5/L_8$ [15]. The central values for the parameters $c_d$ and $c_m$ given in ref.[15] are $c_m = 42$ MeV and $c_d = 32$ MeV, i.e. $2c_d/c_m = 1.56$. However, within the uncertainty of $L_5$ and $L_8$, this ratio can vary between 0.75 and 2.25. In addition, intermediate $\Delta(1232)$ states give a contribution to $c_2 + c_3$. The general $\pi\Delta N$–vertex can be written as

$$\mathcal{L}_{\pi\Delta N} = \frac{g_{\pi\Delta N}}{2m} \Delta^\mu [g_{\mu\nu} - (Z + \frac{1}{2})\gamma_\mu \gamma_\nu] \partial^\nu \pi^a N + \text{h.c.}$$

(12)

where $\Delta^\mu$ denotes the Rarita–Schwinger field and $Z$ parametrizes the off–shell behaviour of the spin–3/2 field.* While Peccei [16] fixed $Z = -1/4$, a more recent phenomenological analysis of Benmerrouche et al.[17] gives a rather wide range, $-0.8 \leq Z \leq 0.3$. Using the empirically well–fulfilled SU(4) coupling constant relation $g_{\pi\Delta N} = 3g_{\pi N}/\sqrt{2}$ together with the Goldberger–Treiman relation, we can cast the $\Delta(1232)$ contribution to $c_2 + c_3$ into the form

$$c_2 + c_3|_\Delta = -\frac{g_A^2}{2m_\Delta^2} \left( \frac{1}{2} - Z \right) \left[ 2m_\Delta (1 + Z) + m (\frac{1}{2} - Z) \right]$$

(13)

* It is mandatory to use here the relativistic formulation of the spin–3/2 field since otherwise one would miss a contribution of order $M_\pi^2$ to $a^+$. 4
Clearly, this dependence on $Z$ is one of the major sources of uncertainty in fixing the values of the contact terms $c_2$ and $c_3$. Furthermore, there is also a contribution from the Roper $N^*(1440)$ resonance to $c_2 + c_3$,

$$c_2 + c_3 \big|_{N^*} = -\frac{g_A^2 R}{16(m + m^*)}$$  \hspace{1cm} (14)

with $R = 1 \ldots 1/4$ for $g_{\pi N N^*} = (1/2 \ldots 1/4)g_{\pi N}$ [18]. In complete analogy, one also has a $\Delta$ and $N^*$ contribution to the low–energy constants $b_1^\prime, b_2$ and $b_3$. At $\lambda = m_\Delta$, it can be written as

$$b_1^\prime(\lambda) + b_2 + b_3 \big|_{\lambda=m_\Delta} = -g_A^2 \left[ \frac{(Z - \frac{1}{2})^2}{8m_\Delta^2} + \frac{R}{32(m + m^*)^2} \right]$$  \hspace{1cm} (15)

Finally, other baryon resonances have been neglected since their couplings are either very small or poorly (not) known.* It is interesting to note that for $Z = -1/4$ and $R = 1$, the $N^*(1440)$ contribution to $c_2 + c_3$ and to $b_1^\prime(\lambda) + b_2 + b_3$ is 4 and 12 per cent of the $\Delta$–contribution, respectively. However, in the latter case the contact terms play a much less pronounced role as already discussed.

5. We now present our numerical results. Consider first the scattering length $a^-$. Using $M_\pi = 138$ MeV, $F_\pi = 93$ MeV, $m = 938.9$ MeV, $Z = -1/4$ and $R = 1$, we find

$$a^- = (8.76 - 1.29 + 1.69) \cdot 10^{-2} M_\pi^{-1} = 9.16 \cdot 10^{-2} M_\pi^{-1}$$  \hspace{1cm} (16)

where we have explicitly shown the contributions of order $M_\pi, M_\pi^2$ and $M_\pi^3$ (and momentarily reinstated the units). The total result is in good agreement with the empirical value. The largest part of the $M_\pi^3$ term comes from the pion loop diagrams, it amounts to 1.31 for $\lambda = m_\Delta$. Varying $Z$ within its band of allowed values, $a^-$ varies by $\pm 0.15$. The dependence on $R$ shows up only in the third digit and is thus irrelevant. The one–loop corrections bring the the lowest order value closer to the empirical one. One might argue that since the $M_\pi^3$ contribution is even larger than the $M_\pi^2$ one, the chiral series has no chance of converging. However, the important one loop effect can only show up at order $M_\pi^3$ due to the chiral counting. The two–loop contribution carries an

* A remark on the $\rho$–meson is in order. The chiral power counting enforces a $\rho\pi\pi$ vertex of order $q^2$ of the form $\mathcal{L}_{\rho\pi\pi}^{(2)} = g_{\rho\pi\pi} \text{Tr}(\rho_{\mu\nu}[u^\nu, u^\mu])$ [15]. In forward direction the contraction of the $\rho$–meson propagator with the corresponding $\rho\pi\pi$ matrix element vanishes. Therefore, one has no explicit $\rho$–meson induced contributions to $a^-$ of order $q^2$ and $q^3$.  

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explicit factor $M_\pi^2$ and is therefore expected to be much smaller. Clearly, one would like to perform a calculation beyond $O(q^3)$ as done here, but this is beyond the scope of this paper. In the case of the isospin–even scattering length $a^+$, the situation is much less satisfactory. There are large cancellations between the loop contribution and the $1/m$ suppressed kinematical terms of order $M_\pi^2$ and $M_\pi^3$. For $M_\pi = 138$ MeV, these two amount to $0.91 - 0.87 = 0.04$ in the conventional units. Therefore, the role of the contact terms is even further magnified. In fig. 1, we show $a^+$ as a function of $Z$ for our standard input and $R = 1$. The empirical value of $a^+$ can be obtained for a small and positive value of $Z$, $Z \approx 0.15$. As shown in fig. 2, $a^+$ varies also strongly as the ratio $2c_d/c_m$ changes. Setting $Z = 0$, the empirical value results for $2c_d/c_m = 1.32$, not far from its central value of 1.56. Clearly, a better understanding of these resonance parameters is necessary before one can draw a final conclusion on the accuracy of the chiral expansion for $a^+$. It is, however, gratifying that slight variations of these resonance parameters allow one to obtain the empirical value.

6. To summarize, we have used heavy baryon chiral perturbation theory to calculate the corrections up–to–and–including order $M_\pi^3$ to Weinberg’s lowest order (current algebra) predictions for the two S–wave pion–nucleon scattering lengths, $a^\pm$. To estimate the strength of the various contact terms, we have made use of the principle of resonance saturation which is known to work very accurately in the meson sector. The main results of this investigation are:

- The chiral corrections to the isospin–odd scattering length $a^-$ are small and positive and move the lowest order prediction closer to the empirical value. The main effect comes from the pion loop diagrams. The contact term contribution is relatively small, thus masking the uncertainty in estimating the coefficients which appear together with these terms.

- The situation is very different for the isospin–even scattering length $a^+$. The contact term contribution completely dominates the chiral expansion since there is a large cancellation between the one–loop and the kinematical corrections. The total result for $a^+$ is very sensitive to some of the resonance parameters, the empirical value of $a^+$ can, however, be obtained by reasonable choices of these.

Evidently, further investigations have to go into two directions. First, a calculation beyond $O(q^3)$ has to be performed to find out how fast the chiral expansion of the scattering lengths converges. Second, a better understanding of the coefficients of the contact terms appearing at order $q^2$ and higher is necessary to further pin down the prediction for $a^+$. We hope to come back to these topics in the near future.
References

1. S. Weinberg, *Phys. Rev. Lett.* **17** (1966) 616.
2. Y. Tomozawa, *Nuovo Cim.* **46A** (1966) 707.
3. R. Koch, *Nucl. Phys.* **A448** (1986) 707.
4. W. Beer et al., *Phys. Lett.* **B261** (1991) 16.
5. G. Höhler, in Landölt–Börnstein, vol.9 b2, ed. H. Schopper (Springer, Berlin, 1983).
6. R.L. Workman, R.A. Arndt and M. Pavan, *Phys. Rev. Lett.* **68** (1992) 1653.
7. G. Höhler, Karlsruhe preprint TTP 92-21, presented at the Workshop on $\pi N$ Scattering, Blacksburg, Va., August 1992.
8. S. Weinberg, *Physica* **96A** (1979) 327.
9. J. Gasser and H. Leutwyler, *Ann. Phys. (N.Y.)* **158** (1984) 142; *Nucl. Phys.* **B250** (1985) 465.
10. E. Jenkins and A.V. Manohar, *Phys. Lett.* **B255** (1991) 558.
11. J. Gasser, M.E. Sainio and A. Švarc, *Nucl. Phys.* **B307** (1988) 779.
12. Ulf-G. Meißner, ”Recent Developments in Chiral Perturbation Theory”, Bern University preprint BUTP-93/01, 1993, to appear in *Rep. Prog. Phys.*
13. V. Bernard, N. Kaiser, J. Kambor and Ulf-G. Meißner, *Nucl. Phys.* **B388** (1992) 315.
14. J. Gasser, H. Leutwyler and M.E. Sainio, *Phys. Lett.* **B253** (1991) 252.
15. G. Ecker, J. Gasser, A. Pich and E. de Rafael, *Nucl. Phys.* **B321** (1989) 311.
16. R.D. Peccei, *Phys. Rev.* **176** (1968) 1812.
17. M. Bennmerrouche, R.M. Davidson and N.C. Mukhopadhyay, *Phys. Rev.* **C39** (1989) 2339.
18. T. Ericson and W. Weise, ”Pions and Nuclei”, Clarendon Press, Oxford, 1988.

Figure Captions

Fig.1 The scattering length $a^+$ as a function of $Z$ in units of $10^{-2} M_\pi^{-1}$. The input is specified in the text. The empirical range is also shown (the solid line gives the central value).

Fig.2 The scattering length $a^+$ as a function of $2c_d/c_m$ in units of $10^{-2} M_\pi^{-1}$. The input is specified in the text. For notations, see fig.1.