Limiting absorption principle and radiation conditions for Schrödinger operators with long-range potentials

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LIMITING ABSORPTION PRINCIPLE AND RADIATION CONDITIONS FOR SCHröDINGER OPERATORS WITH LONG-RANGE POTENTIALS

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ABSTRACT. We show Rellich’s theorem, the limiting absorption principle, and a Sommerfeld uniqueness result for a wide class of one-body Schrödinger operators with long-range potentials, extending and refining previously known results. Our general method is based on elementary commutator estimates, largely following the scheme developed recently by Ito and Skibsted.

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1. Introduction

We shall study the one-body Schrödinger operator

\[ H = H_0 + V, \quad H_0 = p^*p = -\Delta, \quad p = -i\nabla \]

on \( L^2(\mathbb{R}^d), d \geq 1 \), for a real bounded potential \( V \) under natural minimal decay assumptions. We employ the notation

\[ r = r(x) = \langle x \rangle = (1 + |x|^2)^{1/2}, \quad \omega = \nabla r = \frac{x}{r}, \]

and use the convention that big- and small-O notation include a local boundedness assumption. Here are our assumptions on \( V \):

**Condition 1.1.** There is a radial function \( W_0 = W_0(r) \geq 0 \) and a splitting \( V = V^{sr} + V^{lr} \) into real bounded functions such that:

- \( W_0 = o(r^{-1}) \) as \( |x| \to \infty \) and \( W_0 \in L^1([1, \infty), dr) \).
- \( V^{lr} \) has first order distributional derivatives in \( L^1_{loc} \), and

\[ V^{lr} = o(1) = o(r^0) \text{ as } |x| \to \infty, \quad \omega \cdot \nabla V^{lr} \leq W_0. \]
c) $V^{sr}$ satisfies the short-range condition

$$|V^{sr}| \leq W_0.$$  

When studying the limiting resolvent a slightly stronger long-range condition is required:

**Condition 1.2.** In addition to Condition 1.1 the long-range part satisfies $|\nabla V^{lr}| \leq W_0$.

It is of course possible to exchange the boundedness assumptions on $V^{sr}$ and $V^{lr}$ with suitable relative $H$ boundedness and compactness assumptions. We stick to the bounded case for simplicity.

These conditions are natural in the general scattering theory of Schrödinger operators. For instance, the condition $W_0 = o(r^{-1})$ exactly excludes positive eigenvalues of $H$ (cf. the Wigner-von-Neumann potential), and an $L^1$ type condition appears in the WKB-construction. The $L^1$ and $o(r^{-1})$ conditions have also been studied together. Agmon [3], extending the work of Kato [11] and Rellich [16], showed that under Condition 1.1 (with a few extra regularity assumptions), any $C^2$ solution $\phi$ to the homogeneous Helmholtz equation

$$(H - \lambda)\phi = 0, \quad \lambda > 0$$

that satisfies the Rellich decay condition (cf. [16])

$$\liminf_{R \to \infty} R^{-1} \int_{|x| \leq R} |\phi|^2 \, dx = 0$$

must vanish identically. Yafaev [22, Chapter 11] (see also [20]) gave a new proof of a version of Agmon’s result assuming Condition 1.2 (essentially), and furthermore proved the following Kato type smoothness estimate: For any $\lambda_0 > 0$ the operators $W_0^{1/2} \delta_{\varepsilon}(H - \lambda)W_0^{1/2}, \lambda \geq \lambda_0, \varepsilon > 0$, are uniformly bounded on $L^2$. Here

$$\delta_{\varepsilon}(H - \lambda) = \frac{1}{2\pi i} (R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)), \quad R(z) = (H - z)^{-1}.$$  

The main goal of the present paper is to extend these results. Concretely, we show Rellich’s theorem (a version of Agmon’s result), give uniform bounds on individual resolvents in the optimal Besov setting, establish the limiting absorption principle (LAP), and prove a Sommerfeld type uniqueness criterion (radiation condition) for the inhomogeneous Helmholtz equation. These are fundamental ingredients in the development of stationary scattering theory. Our theorems (and their proofs) are very explicit in nature, and in particular make no use of distinguished functional analysis (such as Mourre estimates). We remark that these results are well known under the extensively studied assumption $W_0 = r^{-1-\varepsilon}, \varepsilon > 0$. See for instance [2, 7, 8, 12, 17, 19] in various settings. As such, the achievement of the present paper is to extend a series of ‘sharp’ results, which usually require $O(r^{-1-\varepsilon})$ decay on the potential in the literature, to potentials with decay dictated by Conditions 1.1 and 1.2. The formulations of our theorems below should be compared with that of [10].

All of our main results follows entirely from elementary commutator estimates, largely based on the general scheme developed in [10] by Ito and Skibsted. They study the limiting absorption principle and radiation conditions under ‘classical’ $O(r^{-1-\varepsilon})$ long- and short-range conditions, although in a highly geometric setting. We trim away all unnecessary geometry in our flat Euclidean setting, which allows a simplified exposition. Secondly, we tweak the ‘commutators with weight inside’ in a subtle yet very powerful way to avoid higher order derivatives and archive stronger commutator lower bounds. Finally, the radiation condition estimates in [10] (and elsewhere in the literature) heavily exploits the ‘wiggle room’ in the $O(r^{-1-\varepsilon})$ condition, that is, multiplicative factors of $r^{\varepsilon}$ for $\varepsilon' < \varepsilon$ can be handled without difficulty. We shall rediscover this wiggle room in Condition 1.2.

Before moving on we remark that an $L^1$ type short-range condition also appears in [6, Chapter 14]. Hörmander assumes that $V = V^{sr}$ maps the open unit ball of

$$B^*_0 = \{ \psi \in H^2_{koc} \mid \|\psi\|_{B^*_0} < \infty \}, \quad \|\psi\|_{B^*_0} = \|\psi\|_{B^*_0} + \|p\psi\|_{B^*_0} + \|H_0\psi\|_{B^*_0}.$$
into a relatively compact subset of $B$, where $B$ is the Besov (or Agmon-Hörmander) space and $B^*$ its dual. Assuming $V$ is relatively compact this is essentially equivalent to
\[
\sum_{k=1}^{\infty} 2^k \| V 1_{\{2^{k-1} \leq |x| \leq 2^{k+1}\}} \|_{H^2 \to L^2} < \infty
\]
cf. [6, Theorem 14.4.2], which in particular holds if $V$ is dominated by a radial decreasing $L^1$ function. A similar short-range condition is also used in [14]. The compactness of $V : B^*_{H_0} \to B$ allows Hörmander to derive spectral properties of $H$ from $H_0$ using perturbative arguments. In the follow up [7, Chapter 14] on long-range scattering, only potentials with $O(r^{-1-\varepsilon})$ decay are considered. We remark that in the setting of the present paper, $W_0$ might not even map $B^*_{H_0}$ into $B$, let alone compactly. It is of course also well known that perturbative arguments fail for long-range potentials.

1.1. Basic setting. We denote by $L(X,Y)$ the space of bounded operators $X \to Y$ and write $L(X) = L(X,X)$.

First note that $V$ is relatively $H_0$ compact under Condition 1.1 since $V = o(1)$, so the prescription $H = H_0 + V$ defines a self-adjoint operator on $L^2$ with domain $D(H) = H^2$. Here $H^2 = W^{2,2}$ is the standard $L^2$ order two Sobolev space. We denote other Sobolev spaces similarly. The compactness also ensures that $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) = [0, \infty)$.

Recall the Besov spaces (also known as Agmon-Hörmander spaces cf. [2]) mentioned above: Set $F_k = 1_{\{2^{k-1} \leq r < 2^k\}}$ for $k \geq 1$. Then define
\[
B = \{ \psi \in L^2_{\text{loc}} \mid \| \psi \|_B < \infty \}, \quad \| \psi \|_B = \sum_{k=1}^{\infty} 2^{k/2} |F_k \psi|,
\]
\[
B^* = \{ \psi \in L^2_{\text{loc}} \mid \| \psi \|_{B^*} < \infty \}, \quad \| \psi \|_{B^*} = \sup_{k \geq 1} 2^{-k/2} |F_k \psi|,
\]
\[
B_{0}^* = \{ \psi \in L^2_{\text{loc}} \mid \lim_{k \to \infty} 2^{-k/2} |F_k \psi| = 0 \}.
\]

We equip the spaces with their natural norm, and identify the dual space of $B$ with $B^*$ through the $L^2$-pairing. It is well known that $B$ and $B^*$ are non-reflexive Banach spaces with $C^\infty_c(\mathbb{R}^d)$ dense in $B$. The space $B_{0}^*$ is exactly the closure of $C^\infty_c$ in $B^*$, hence the notation. In particular, $B_{0}^*$ is a closed subspace of $B^*$. Also remark that $B$ is separable. This is easily verified using the fact that $B$ isometrically embeds into the separable sequential space $\ell^1(\mathbb{N}; L^2(\mathbb{R}^d))$ of $L^2$ functions (which can be viewed as a Bochner space). This in particular means that any norm bounded subset of $B^*$ is sequentially weak-star relatively compact since it is metrizable. See [18, Theorem 3.16].

Denote by $L^2_s(\mathbb{R}^n)$, $s \in \mathbb{R}$, the weighted space of all $\psi \in L^2_{\text{loc}}$ such that $r^s \psi \in L^2$ with the natural norm. These are reflexive Banach (or Hilbert) spaces with $C^\infty_c$ dense. The following elementary continuous inclusions are well-known:
\[
L^2_s \subseteq B \subseteq L^{1/2}_{1/2} \subseteq L^2 \subseteq L^{1/2}_{-1/2} \subseteq B_0 \subseteq B^* \subseteq L^2_{-s}, \quad s > 1/2.
\]

Finally introduce the weighted Sobolev spaces $H^s_s = r^{-s} H^1$ and $H^s = r^{-s} H^2$ with their natural norms. Clearly $C^\infty_c$ is dense in any such space.

Like in the abstract Mourre theory, our commutator estimates are based on a well chosen ‘conjugate operator’. We shall consider the symmetric ‘radial’ differential operator
\[
A = \frac{1}{2}(p^* \omega + \omega^* p) = \frac{1}{2}(p \cdot \omega + \omega \cdot p) = p \cdot \omega + \frac{1}{2} \Delta r = \omega \cdot p - \frac{1}{2} \Delta r \tag{1.1}
\]
with domain $D(A) = H^1$. Note that the vector field $\omega$ is smooth and bounded. This in turn makes $A : H^1 \to L^2$ bounded, in particular relatively $H$-bounded, which is in contrast to the usual generator of radial dilations. The operator $A$ is also used in [10] and [1], who also provide a self-adjoint realization. Symmetry on $H^1$ suffices in the present paper. Note the basic commutator identity
\[
i[A,f] = i(Af - fA) = \omega \cdot \nabla f, \quad f \in H^1_{\text{loc}}.
\]
as quadratic forms on $C_c^\infty$. Using the operator $A$ we can decompose $H = H_0 + V$ into 'radial' and 'spherical' components in the following way. Consider the Laplace-Beltrami type operator associated to $A$:

$$L = p^*M p \quad M = 1 - \omega \omega^* = r \nabla^2 r,$$

where $\nabla^2 r$ is the Hessian of $r$. We can then write

$$H = H_0 + V = L + A^2 + V + E_1,$$

where

$$E_1 = \frac{1}{4}(\Delta r)^2 + \frac{1}{2}\omega \cdot \nabla \Delta r = \frac{1}{2}((d-1)(d-3)r^{-2} + (4d-10)r^{-4} + 7r^{-6}).$$

Essentially the same decomposition is used in [10], although they absorb $E_1$ and $V$ into the 'effective potential' $q = E_1 + V$, and they also divide out (for $r$ large) by $|\omega|^2 = 1 - r^{-2}$ in the matrix $\omega \omega^*$ in the definition of $L$. The explicit nature of $r$ in the present paper renders this unnecessary.

Finally introduce a function $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(t) = \begin{cases} 1, & t \leq 1 \\ 0, & t \geq 2 \end{cases}, \quad \chi' \leq 0,$$

and fix the smooth cut-offs $\chi_n = \chi_n(r) = \chi(r/2^n)$, $n \geq 0$. Here and throughout we use primes to denote derivatives of radial functions, i.e. if $f = f(r) \in C^1([1, \infty))$, for instance, then $\nabla f = \omega f'$.

1.2. Results.

1.2.1. Rellich type theorem. We start with a uniqueness criterion to the homogeneous Helmholtz equation

$$(H - \lambda)\phi = 0, \quad \lambda > 0. \tag{1.4}$$

If $\phi \in H^2_{\text{loc}}$ and $\phi$ satisfies (1.4) in the distributional sense, we call $\phi$ a generalized eigenfunction with eigenvalue $\lambda$. If in addition $\phi \in L^2$, then $\phi$ is naturally a genuine eigenfunction. We denote by $\mathcal{E}_+$ the set of positive eigenvalues of $H$.

**Theorem 1.3.** Suppose Condition 1.1. Let $I \subseteq (0, \infty)$ be a compact set and fix $\alpha_0 \in \mathbb{R}$. Then there exists a constant $C > 0$ such that

$$\|r^{\alpha_0} \phi\| \leq C\|r^{-3/4} \phi\|$$

for all generalized eigenfunctions $\phi \in B^s_0$ with eigenvalue $\lambda \in I$. In particular, $\phi$ is a genuine eigenfunction and $\phi \in H^2_s = r^{-s}H^2$ for all $s \in \mathbb{R}$. Moreover, the set $\mathcal{E}_+$ of positive eigenvalues is discrete and can accumulate only at zero, and each such eigenvalue is of finite multiplicity.

The sharp $B^s_0$ condition is crucial in the proofs of all theorems below. As such, the key implication in Theorem 1.3 is that $\phi \in B^s_0$ automatically dictates $\phi \in H^2$. The formulation above is similar to that of [6, Theorem 14.5.2].

Ideally we would like to conclude that any generalized eigenfunction $\phi \in B^s_0$ with positive eigenvalue must vanish identically, or equivalently given Theorem 1.3 that $\mathcal{E}_+ = \emptyset$. Absence of positive eigenvalues have been extensively studied under various conditions on the potential. We highlight [5] which shows that $\mathcal{E}_+ = \emptyset$ under Condition 1.2. We have not been able to find a similar result under Condition 1.1 without additional regularity assumptions, although [15, Theorem XIII.58] comes close. We also highlight the following result of Hörmander [6, Theorem 14.7.2] with a very short proof: If $V = V^{sr} = O(r^{-1})$ and $\phi \in \cap_{s \in \mathbb{R}} H^2_s$ is an eigenfunction with positive eigenvalue, then $\phi = 0$. This of course also shows that $\mathcal{E}_+ = \emptyset$ when $V$ is short-range using Theorem 1.3. In the interest of keeping the present paper self-contained we make no assumption that $\mathcal{E}_+ = \emptyset$. 


1.2.2. LAP bounds. For a set \( I \subseteq (0, \infty) \), let
\[
I_\pm = \{ \lambda \pm i \varepsilon \mid \lambda \in I, \, 0 < \varepsilon < 1 \}.
\]
This should be read as two separate sets \( I_+ \) and \( I_- \).

**Theorem 1.4.** Suppose Condition 1.1 and let \( I \subseteq (0, \infty) \setminus \mathcal{E}_+ \) be a compact set. There exist \( C > 0 \) so that uniformly for all \( \psi \in \mathcal{B} \) and \( z \in I_\pm \)
\[
\|R(z)\psi\|_{\mathcal{B}^*}^2 + ||AR(z)\psi||_{\mathcal{B}^*}^2 + \langle \psi^* \nabla^2 r \psi, R(z)\psi \rangle \leq C\|\psi\|_{\mathcal{B}^*}^2.
\]
Since \( A \) and \( p^* \nabla^2 r \) essentially act orthogonally, Theorem 1.4 can be extended to also include \( p\psi \). Indeed, simply write
\[
2^{-k}\|F_k p R(z)\psi\|_2^2 \leq \langle p^* F_k \nabla^2 r, R(z)\psi \rangle + 2^{-k}\|F_k \omega^* p R(z)\psi\|_2^2
\]
and take sup over \( k \in \mathbb{N} \). As such, Theorem 1.4 in particular states that \( R(z), \, z \in I_\pm \), is uniformly bounded in \( \mathcal{L}(\mathcal{B}, \mathcal{B}_{H_0}^*) \), where \( \mathcal{B}_{H_0}^* \) is the space mentioned in the introduction. Theorem 1.4 is complemented by the following (non-exhaustive) list of smoothness bounds. These are stated in terms of radial functions \( W = W(r) \) which behave like \( W_0 \), that is
\[
W \geq 0, \quad W = o(r^{-1}), \quad W \in L^1([1, \infty), dr).
\]
Throughout \( W^{1/2} \) will play the role usually taken by \( r^{-1/2-\varepsilon} \) in the literature.

**Theorem 1.5.** Suppose Condition 1.1. Fix functions \( W_1, W_2 \) that satisfy (1.5) and consider a compact set \( I \subseteq (0, \infty) \setminus \mathcal{E}_+ \). Then the following are bounded uniformly for \( z \in I_\pm : \)

a) The quadratic form \( \psi \rightarrow \langle p^* \nabla^2 r, R(z)\psi \rangle_{L^2} \) on \( L^2 \),
b) \( W^{1/2} R(z) \) and \( W^{1/2} p R(z) \) in \( \mathcal{L}(\mathcal{B}, L^2) \),
c) \( R(z) W^{1/2} \) and \( p R(z) W^{1/2} \) in \( \mathcal{L}(L^2, \mathcal{B}^*) \),
d) \( W^{1/2} R(z) W^{1/2} \) and \( W^{1/2} p R(z) W^{1/2} \) in \( \mathcal{L}(L^2) \).

We shall use practically all stated bounds in the proofs of Theorem 1.6 and Corollary 1.8 stated below. Assertion d) of Theorem 1.5 should be compared to [22, Theorem 11.1.1]. Standard applications of LAP bounds include absence of singular continuous spectrum and asymptotic completeness in the short range case. See [21, Chapter 4].

1.2.3. Limiting absorption principle and radiation conditions. Radiation condition bounds hinge on the observation that \( (\partial_s - \sqrt{s}) R(z)\psi \) inherits in the limit \( \text{Im} \, z \rightarrow 0 \) regularity from \( \psi \). This is in contrast to \( R(z)\psi \). As such, it should be possible to commutate \( (\partial_s - \sqrt{s}) R(z)h \) into \( h(\partial_s - \sqrt{s}) R(z) \) with controllable error for certain functions \( h \). We essentially prove that this is the case for any \( h = h(r) \in C^1([1, \infty)) \) such that
\[
h > 0, \quad 0 \leq k' \leq \beta_0 r^{-1} h, \quad h^2 W_0 = o(r^{-1}), \quad h^2 W_0 \in L^1([1, \infty), dr),
\]
for some \( \beta_0 < 1 \). We remark that there exists such a function \( h \) for which \( h \rightarrow \infty \) as \( r \rightarrow \infty \) since there are some ‘wiggle-room’ in the \( L^1 \) and \( o(r^{-1}) \) conditions of \( W_0 \). Note also that (1.6) automatically requires \( h \leq Cr^{\beta_0} \) by Grönwall’s inequality, or more precisely
\[
(s/t)^{\beta_0} \leq h(s)/h(t) \leq (t/s)^{\beta_0}, \quad 1 \leq s \leq t.
\]
The upper bound \( h = O(r^{\beta_0}) = O(r) \) can be relaxed somewhat in certain situations (Theorem 1.7 a) and c) stated below, for instance). Similar functions are considered in [6], although only for the free case \( V = 0 \).

Essentially following [10], in place of \( \sqrt{s} \) we consider a phase \( a = a_z \) constructed as follows: For each \( \lambda \in (0, \infty) \), pick \( r_\lambda \geq 1 \) such that
\[
\lambda - V^{tr} > \lambda/2 \text{ when } r \geq \frac{1}{2} r_\lambda.
\]
This is possible since \( V^{tr} = o(1) \). We may furthermore assume \( \lambda \rightarrow r_\lambda \) is decreasing. Then define
\[
\eta_\lambda(r) = 1 - \chi(2r/r_\lambda)
\]
which is supported in \( r \geq \frac{1}{2} r_\lambda \) and constantly equal to one for \( r \geq r_\lambda \). For \( z = \lambda \pm i \varepsilon \) with \( \varepsilon \geq 0 \) and \( \lambda > 0 \), set
\[
a = a_z = \pm \eta_\lambda \sqrt{|z - V|^r}
\]
if the underlying Euclidean space is of dimension \( d \neq 2 \). If \( d = 2 \), define instead
\[
a = \pm \eta \sqrt{|z - V|^r + \frac{1}{4} r^{-2}}.
\]
The extra term under the square root when \( d = 2 \) 'corrects' the leading \(-\frac{1}{4} r^{-2}\) term in (1.3), which is negative only when \( d = 2 \). In either case we choose the positive branch of the square root. The sign \( \pm \) in front is solely to ensure \( \text{Im}(a) \geq 0 \), while the cut-off \( \eta \) allows us to take derivatives of \( a \). If \( V = V^s \) is short-range, simply take \( \eta_\lambda = 1 \). Again, \( a = a_{\lambda \pm i \varepsilon} \) should be read as two separate identities (also when \( \varepsilon = 0 \)). As stated in [10], the phase \( a \) is an approximate solution to the Ricatti equation
\[
\omega \cdot \nabla a + a^2 - (z - V)^r = 0.
\]
A better approximation (with an extra term) is used in [10], although they do not use the \( \frac{1}{4} r^{-2} \) correction when \( d = 2 \).

**Theorem 1.6.** Suppose Condition 1.2 and let \( I \subseteq (0, \infty) \setminus \mathcal{E}_+ \) be compact. Consider a function \( h \) that satisfies (1.6). Then there exists \( C > 0 \) so that for any \( \psi \in h^{-1} \mathcal{B} \) and \( z \in \mathcal{I}_\pm \)
\[
\|h(A - a)R(z)\psi\|^2_{\mathcal{B}} + (p^2 h^2 \nabla^2 \omega R(z) \psi)_{\mathcal{B}} \leq C\|h\psi\|^2_{\mathcal{B}}.
\]
We remark that the strengthening from Condition 1.1 to Condition 1.2 is only used once to bound \( |M \nabla a| \leq CW_0 + Cr^{-3} \). We have the following radiation condition 'smoothness bounds':

**Theorem 1.7.** Suppose Condition 1.2. Consider functions \( W_1, W_2 \) that satisfy (1.5) and a function \( h \) that satisfies (1.6). Assume moreover that \( h^2 W_2 = o(r^{-1}) \) and \( h^2 W_2 \in L^1([1, \infty), dr) \). Then, for any compact set \( I \subseteq (0, \infty) \setminus \mathcal{E}_+ \),
\[
a) \ W_1^{1/2} h(A - a)R(z)W_2^{1/2} \text{ is uniformly bounded for } z \in \mathcal{I}_\pm \text{ in } \mathcal{L}(L^2),
b) \ h(A - a)R(z)W_2^{1/2} \text{ is uniformly bounded for } z \in \mathcal{I}_\pm \text{ in } \mathcal{L}(L^2, \mathcal{B}^*),
c) \text{If in addition } h' \geq \alpha r^{-1} h \text{ for some } \alpha > 0, \text{ then } r^{-1/2} h(A - a)R(z) r^{-1/2} h^{-1} \text{ is uniformly bounded for } z \in \mathcal{I}_\pm \text{ in } \mathcal{L}(L^2).
\]
As an application we archive the limiting absorption principle. We remind the reader that the function '\( h \)' below can be chosen so that \( h(r) \to \infty \) as \( r \to \infty \).

**Corollary 1.8.** Suppose Condition 1.2. Let \( I \subseteq (0, \infty) \setminus \mathcal{E}_+ \) be a compact set, and fix functions \( W, h \) such that \( W \) satisfies (1.5), \( h \) satisfies (1.6), \( h^2 W = o(r^{-1}) \), and \( h^2 W \in L^1([1, \infty), dr) \). Then there exists \( C > 0 \) such that
\[
\|W^{1/2}(R(z) - R(z'))W^{1/2}\|_{\mathcal{L}(L^2)} \leq C \frac{1}{h(|z - z'|^{-1})}
\]
for all \( z, z' \in \mathcal{I}_\pm \) with \( |z - z'| \leq 1 \). We conclude the LAP: Equip \( \mathcal{B}^* \) with the weak-star topology. For each fixed \( \psi \in \mathcal{B} \) and \( \lambda \in (0, \infty) \setminus \mathcal{E}_+ \), the boundary values
\[
R(\lambda \pm i 0)\psi = \varepsilon_{0+} \lim_{\varepsilon \to 0^+} R(\lambda \pm i \varepsilon)\psi \in \mathcal{B}^*
\]
exist, and the corresponding extensions
\[
\{ \lambda \pm i \varepsilon \mid \lambda > 0, \varepsilon \geq 0 \} \setminus \mathcal{E}_+ \ni z \to R(z)\psi \in \mathcal{B}^*
\]
are locally uniformly continuous.

We highlight the explicit nature of the bound (1.7). When \( W_0 = O(r^{-1-2\varepsilon}) \) with \( \varepsilon < 1 \), say, taking \( W = h^2 \), \( s > 1/2 \), and \( h = r^{\varepsilon} \) with \( \varepsilon < \min\{s - 1/2, \varepsilon\} \) shows that the resolvent \( z \to R(z) \in \mathcal{L}(L^2_\omega, L^2_{-\lambda}) \) is locally Hölder continuous with parameter \( \varepsilon' \). This is in agreement with
A slight refinement is possible by taking \( h = r^{\min\{s-1/2,s\}}(\log r)^{-1/2-\delta}, \delta > 0 \), for instance.

By uniform boundedness it is naturally also possible to take the weak-star limit \( \varepsilon \to 0^+ \) for \( pR(\lambda \pm i\varepsilon)\psi \) and \( H_{0}\!\!\!\!\!\!\!\!\!_{\text{loc}}R(\lambda \pm i\varepsilon)\psi \) in \( B^* \). In particular, the limiting resolvents \( R(\lambda \pm i0) \) map into \( H_{0}\!\!\!\!\!\!\!\!\!_{\text{loc}}^2 \), or more precisely the space \( B^*_{H_{0}} \) considered in the introduction (and in [6]). Weak-star continuity therefore allows us to extend the conclusions of Theorems 1, 1.5, 1.6, and 1.7 to include the case \( z = \lambda \pm i0 \) as well. Making use of this we can establish the following Sommerfeld uniqueness result which generalizes Theorem 1.3 to the inhomogeneous case.

**Corollary 1.9.** Suppose Condition 1.2 and fix \( \lambda \in (0,\infty) \setminus \mathcal{E}_+ \). The limiting resolvents \( R(\lambda \pm i0) \in \mathcal{L}(B,B^*) \) are uniquely characterized by the following: For any function \( h \) that satisfies (1.6) and any \( \psi \in h^{-1}B \),

\[
\begin{align*}
\text{a) } & R(\lambda \pm i0)\psi \in H_{0}\!\!\!\!\!\!\!\!\!_{\text{loc}}^2 \cap hB^* \text{ and } (H - \lambda)R(\lambda \pm i0)\psi = \psi \text{ in the distributional sense,} \\
\text{b) } & (A - a)R(\lambda \pm i0)\psi \in h^{-1}B_{0}^*.
\end{align*}
\]

2. LAP BOUNDS AND RELlich’S THEOREM

In this section we prove Theorems 1.3, 1.4, and 1.5. Inspired by [10], we first give a lower bound on certain commutators ‘with weights inside’. We impose Condition 1.1 throughout.

2.1. Commutator with weight inside. Following the recently developed method in [10], we need to bound commutators with ‘weights inside’ of the following type:

\[
(2\Im(AFH))\psi = (i(HfA - AFH))\psi = i(fA\psi,H\psi) - i(H\psi,fA\psi), \quad \psi \in C_c^\infty,
\]

where \( f = f(r) \) is a function. Contrary to [10], we may only assume that \( f \) is positive and locally Lipschitz as a function of \( r \) on \([1,\infty)\). It is well-known that any such function \( f \) is in \( W^{1,\infty}_0 \) and furthermore (strongly) differentiable almost everywhere. Moreover, since \( \nabla r = \omega \neq 0 \) for \( x \neq 0 \), the usual chain rule applies:

\[
\nabla(f \circ r) = (f' \circ r)\nabla r \quad a.e.
\]

both weakly and strongly, where the right hand side is well defined almost everywhere cf. [13, Theorem 6.19]. We refer to [4] for details on Sobolev and locally Lipschitz functions. With this in mind the computations below are routine.

We bound the commutator (2.1) using the decomposition (1.2). To this end, the following explicit computation is obviously useful.

**Lemma 2.1.** Let \( f = f(r) \) be locally Lipschitz as a function of \( r \) on \([1,\infty)\). Then, as forms on \( C_c^\infty \),

\[
2\Im(AF) = -\Im(\Delta r f'r^{-2}\omega^*p) - \frac{1}{2}\div(f r^{-2}(\Delta r)'\omega) + p^*(2fr^{-1} - f')Mp + p^*r^{-2}f'(1 + \omega\omega^*)p.
\]

**Proof.** Note that \( M\omega = -r^{-2}\omega \). This will be used repeatedly. First rewrite

\[
2\Im(AF) = 2\Im(Ap^*fMp) - 2\Im([p^*,f]Mp) = 2\Im(Ap^*fMp) + 2\Re(Af'\omega^*Mp).
\]

We compute each term separately. For the second:

\[
2\Re(Af'\omega^*Mp) = 2\Re((p^* + \frac{1}{2}\Delta r)f'r^{-2}\omega^*p) = 2p^*\omega r^{-2}f'\omega^*p - \Im(\Delta r f'r^{-2}\omega^*p).
\]

For the first term, rewrite \( 2\Im(Ap^*fMp) = 2\Im(p^*\omega p^*fMp) + \Re(\Delta r p^*Mp) \) and then compute

\[
\Re(\Delta r p^*Mp) = p^*f\Delta r Mp - \Im((\Delta r)'f\omega^*Mp) = p^*f\Delta r Mp - \frac{1}{2}\div(r^{-2}(\Delta r)'f\omega).
\]
and

\[
2 \text{Im}(p^* \omega p^* f Mp) = i \sum_{i,k,j} (p_i f M_{ik} p_k \omega_j p_j - p_j \omega_j p_k M_{ik} f p_i) \\
= i \sum_{i,j,k} (p_i f M_{ik} \omega_j p_k - p_j p_k \omega_j M_{ik} f p_i) \\
+ \sum_{i,k,j} (p_i f M_{ik} (\nabla^2 r)_{kj} p_j + p_j (\nabla^2 r)_{jk} M_{ki} f p_i) \\
= i \sum_{i,j,k} (p_j p_i f M_{ik} \omega_j p_k - p_j p_k M_{ik} \omega_j f p_i) - i \sum_{i,j,k} p_i [f M_{ik} \omega_j, p_j] p_k \\
+ \sum_{i,j} p_i f (M \nabla^2 r)_{ij} p_j + p_j (\nabla^2 r M)_{ji} f p_i.
\]

Exchanging the roles of \(i\) and \(k\) in the second term of the first sum, this term vanishes since \(M\) is symmetric. The third sum is equal to \(2 p^* f M \nabla^2 r p\) since \(M\) and \(\nabla^2 r\) commutes. Finally, we can explicitly compute the second sum as

\[-i \sum_{i,j,k} p_i [f M_{ik} \omega_j, p_j] p_k = -p^* f \Delta r M p - p^* M f' |\omega|^2 p + 2 p^* \omega r^{-3} f \omega^* p.\]

Collecting similar terms we conclude

\[
2 \text{Im}(AfL) = 2 p^* \omega r^{-2} f' \omega^* p - \text{Im}(\Delta r f' r^{-2} \omega^* p) - \frac{1}{2} \text{div}(fr^{-2} (\Delta r)' \omega) + 2 p^* f M \nabla^2 r p \\
- p^* M f'|\omega|^2 p + 2 p^* \omega r^{-3} f \omega^* p \\
= - \text{Im}(\Delta r f' r^{-2} \omega^* p) - \frac{1}{2} \text{div}(fr^{-2} (\Delta r)' \omega) + p^* (2f r^{-1} - |\omega|^2 f') M p \\
+ 2 p^* r^{-2} f' \omega \omega^* p.
\]

The lemma follows.

We now turn to our main estimate used in the proofs on Theorems 1.3 and 1.4. We shall consider functions \(f = f(r) \in C^2([1, \infty))\) such that

\[
f \geq 0, \quad 0 \leq f' \leq \beta_1 r^{-1} f, \quad |f''| \leq \beta_2 r^{-2} f, \quad \beta_1, \beta_2 \geq 0, \tag{2.2}
\]

and a fixed function \(W = W(r)\) that satisfies (1.5) with \(W \geq W_0\). With this is place, set

\[
\bar{f} = \bar{f}_{K,W}(r) = f(r) \exp \left( K \int_1^r W(s) \, ds \right), \quad K \geq 0, \tag{2.3}
\]

and note \(\bar{f}\) is locally Lipschitz as a function of \(r\), whence \(\bar{f} \in W^{1,\infty}_{\text{loc}}\) and the usual chain rule applies cf. the discussion above. Moreover, the \(o(r^{-1})\) assumption on \(W\) also ensures that

\[0 \leq \bar{f}' \leq C r^{-1} \bar{f}\]

for a constant \(C\) dependent only on \(\beta_1\) and \(K\).

\textbf{Lemma 2.2.} For any compact set \(I \subseteq (0, \infty), \beta_1, \beta_2 \geq 0,\text{ and any function } W \text{ that satisfies (1.5) with } W \geq W_0,\text{ we can choose a parameter } K > 0 \text{ and constants } C, c > 0 \text{ such that the following holds: Let } f \text{ be a bounded function that satisfies (2.2). Then there is a real function } \gamma \text{ such that}

\[
2 \text{Im}(A\bar{f}_{K,W}(H - \lambda)) \geq cf' + cfW + cAf'A + cAfWA + cp^* f \nabla^2 rp \\
- Cr^{-2} f - \text{Re}(\gamma(H - \lambda)) \tag{2.4}
\]

for all \(\lambda \in I\), as forms on \(H^2\). We have the bound \(|\gamma| \leq Cr^{-1} f\).
Proof. We prove the bound (2.4) as forms on $C_c^\infty$ and then extend $H^2$ by continuity. Consider first $\tilde{f}$ with arbitrary $K > 0$ to be fixed later. Our choice of $K$ only depends on $\inf I$. All constants $C, C_1, C_2, \ldots$ and $c_1, c_2, \ldots$ below are positive and allowed only to depend on $\beta_1, \beta_2, W,$ the compact set $I,$ along with any auxiliary data such as the dimension $d$ and $W_0$. Introduce the error

$$Q = fr^{-2} + p^*fr^{-2}p.$$ 

First note the simple bound $AgA \leq Cg + p^*gp$ for $g \geq 0$ locally bounded. This follows from (1.1). Then expand the left hand side of (2.4) according to (1.2) using the Cauchy-Schwarz and Young inequalities

\begin{align}
2\text{Im}(A\tilde{f}(H - \lambda)) &\geq \lambda |\omega|^2\tilde{f}'t + A|\omega|^2\tilde{f}'A + 2\text{Im}(A\tilde{f}V^{lr}) + 2\text{Im}(A\tilde{f}V^{sr}) \\
&\quad + 2\text{Im}(A\tilde{f}L) - C_1Q \\
&\geq |\omega|^2\tilde{f}'(\lambda - V^{lr}) + A|\omega|^2\tilde{f}'A - \tilde{f}\omega \cdot \nabla V^{lr} - |\tilde{f}V^{sr}| - A|\tilde{f}V^{sr}|A \\
&\quad + 2\text{Im}(A\tilde{f}L) - C_1Q \\
&\geq \tilde{f}'(V^{lr} - \lambda) + A\tilde{f}A - 2\tilde{f}W - A\tilde{f}WA + 2\text{Im}(A\tilde{f}L) - C_2Q.
\end{align}

We removed the $|\omega|^2$ factor by noting $|\omega|^2 = 1 - r^{-2}$. Let us bound the $L$ term. Writing $\theta = \theta(r) = \exp(K\int_I Wds)$, it follows by Lemma 2.1 and the $o(r^{-1})$ condition on $W$ that

$$2\text{Im}(A\tilde{f}L) \geq p^*(2r^{-1}\tilde{f} - KW\tilde{f})Mp - p^*f'\theta Mp - C_3Q \geq \frac{3}{2}p^*\tilde{f}\nabla^2rp + p^*\omega f'\theta\omega s p - p^*f'\theta p - C_4Q,$$

and noting the general identity

$$ppp = \text{Re}(gH_0) + \text{Im}((\nabla g \cdot p), \quad g \in H^1_{loc},$$

we proceed using the Cauchy-Schwarz inequality and (2.2):

\begin{align}
p^*\tilde{f}\nabla^2rp + p^*\omega f'\theta\omega s p - p^*f'\theta p &\geq p^*(\tilde{f}r^{-1} - (r^{-1}\tilde{f} - \frac{1}{\beta_1 + 1}f')\omega s p) - p^*f'\theta p \\
&\geq \frac{1}{\beta_1 + 1}p^*|\omega|^2f'\theta p - p^*f'\theta p \\
&\geq -(1 - \frac{1}{\beta_1 + 1})\text{Re}(f'\theta(H - \lambda)) \\
&\quad - (1 - \frac{1}{\beta_1 + 1})(\lambda - V^{lr})f'\theta - C_5Q.
\end{align}

Here the $C^2$ condition on $f$ was necessary to bound $\text{Im}(\nabla f'\theta \cdot p)$. We conclude

$$2\text{Im}(A\tilde{f}L) \geq \frac{3}{2}p^*\tilde{f}\nabla^2rp - (1 - \frac{1}{\beta_1 + 1})\text{Re}(f'\theta(H - \lambda)) - (1 - \frac{1}{\beta_1 + 1})(\lambda - V^{lr})f'\theta - C_6Q,$$

or in combination with (2.5):

$$2\text{Im}(A\tilde{f}(H - \lambda)) \geq \frac{3}{2}p^*\tilde{f}\nabla^2rp + \frac{1}{\beta_1 + 1}p^*f'\theta(\lambda - V^{lr}) + \tilde{f}W(K(\lambda - V^{lr}) - 2) + A\tilde{f}W(K - 1)A + A\tilde{f}'\theta A - C_7Q - \text{Re}(\gamma_1(H - \lambda)),$$

where $\gamma_1 = (1 - \frac{1}{\beta_1 + 1})f'\theta$. It follows from the property $V^{lr} = o(1)$ that $\lambda - V^{lr} \geq \frac{1}{2}\inf I$ for $r$ large enough, whence we can choose $K$ sufficiently large (only dependent on $\inf I$) so that

$$2\text{Im}(A\tilde{f}(H - \lambda)) \geq c_1f' + c_1Wf + c_1Af'fA + c_1AWfA + c_1p^*\tilde{f}\nabla^2rp - C_8Q - \text{Re}(\gamma_1(H - \lambda)).$$

It remains to bound the $p$ term in $Q$. This is elementary:

$$p^*r^{-2}fp = \text{Re}(r^{-2}f(H - \lambda)) + (\lambda - V)r^{-2}f + \frac{1}{2}\Delta(r^{-2}f) \leq \text{Re}(r^{-2}f(H - \lambda)) + C_9r^{-2}f.$$ 

This proves (2.4) as forms on $C_c^\infty$, and the bound obviously extends to $H^2$ by continuity and density since $f$ is assumed bounded.
A stronger lower bound with $O(r^{-3})$ errors is possible under the assumption $\beta_1 < 2$. It would then also be possible to relax $f \in C^2$ to $f \in C^1$. See Lemma 3.1 below. We remark that since the choice of $K$ above only depends on $I$, the $o(r^{-1})$ conditions on $W$ and $W_0$ can be relaxed to $O(r^{-1})$ by only considering inf $I$ sufficiently large relative to the value of $\lim_{r \to \infty} rW_0$. We shall not expand on this here, however.

We also need Lemma 2.2 with complex spectral parameter.

**Lemma 2.3.** For any compact set $I \subseteq (0, \infty)$, $\beta_1, \beta_2 \geq 0$, and any function $W$ that satisfies (1.5) with $W \supseteq W_0$, we can choose a parameter $K > 0$ and constants $C, c > 0$ such that the following holds: Let $f$ be a bounded function that satisfies (2.2). For all $z \in I_\pm$, there exists a complex function $\gamma = \gamma_z$ such that
\[
2 \text{Im}(A\tilde{f}_K(W(H-z))) \geq cf' + cfW + cAf'A + cAfWA + c\gamma H^{-2}r^p \\
- cr^{-2}f - \text{Re}(\gamma(H-z)) \tag{2.6}
\]
as forms on $H^2$. We have the uniform bound $|\gamma| \leq C\|f\|_\infty$.

**Proof.** Write $z = \lambda \pm i\varepsilon$ for $\lambda \in I$ and $\varepsilon > 0$. Noting that
\[
2 \text{Im}(A\tilde{f}(H-z)) = 2 \text{Im}(A\tilde{f}(H-\lambda)) + 2\varepsilon \text{Re}(A\tilde{f}),
\]
we only need to bound the second term by Lemma 2.2 (here it is important that the function $\gamma$ in (2.4) is real). Noting the simple identity $\varepsilon = \pm \text{Re}(i(H-z))$ and using that $\tilde{f}$ and $V$ are bounded,
\[
\pm 2\varepsilon \text{Re}(A\tilde{f}) \geq -\varepsilon \tilde{f} - \varepsilon \tilde{A}fA \geq -C_1\varepsilon - C_1\varepsilon H_0 \geq -C_2\varepsilon - \text{Re}(C_1\varepsilon(H-z)) = \pm C_2 \text{Im}(H-z) - \text{Re}(C_1\varepsilon(H-z)) = \text{Re}((C_1\varepsilon \mp iC_2)(H-z)).
\]
The lemma follows.

The appearance of $\|f\|_\infty$ in the bound of $\gamma$ is (sadly) necessary, and it explains why the resolvent $R(z)\psi$ does not inherit good decay from $\psi$ in the limit $\text{Im} z \to 0$. This should be compared to Lemma 3.1 below.

### 2.2. Proof of Rellich type theorem.

We prove Theorem 1.3 using Lemma 2.2. Our proof loosely follows [9]. For a generalized eigenfunction $\phi$, denote $\phi_n = \chi_n \phi \in H^2$. Given a bounded function $f$, it is of course very tempting to formally write
\[
0 = (\text{Re}(f(H-\lambda)))_{\phi} = \lim_{n \to \infty} \text{Re}(f(H-\lambda))_{\phi_n}.
\]
The situation here is a little subtle, though, since $\phi$ is not assumed $L^2$. This is where the $B_0^s$ condition enters the picture. The formal computation above turns out to be valid when $\phi \in B_0^s$, yet can fail for $\phi \in B^s$. The failure is easy to see: Consider $\phi(x) = \exp(-i\lambda^{-1/2}x)$ and $V = 0$ in dimension one.

In the proof below, note that $|\chi'_n| \leq C2^{-n}F_{n+1}$ for a constant $C$ not dependent of $n$.

**Lemma 2.4.** Let $\phi \in B_0^s$ be a generalized eigenfunction with eigenvalue $\lambda$ and $f : \mathbb{R}^d \to \mathbb{R}$ a bounded function. Then
a) $(\lambda - i)R(i)\phi_n \to \phi$ as $n \to \infty$ in $L^2_{s,s}$, $s > 1/2$,

b) $(\text{Im}(Af(H-\lambda)))_{R(i)\phi_n} \to 0$ and $(f(H-\lambda))_{R(i)\phi_n} \to 0$ as $n \to \infty$.

**Proof.** First note the useful identities
\[
R(i) = (\lambda - i)^{-1}(1 - R(i)(H-\lambda))
\]
on $H^2$, and
\[
(H-\lambda)\phi_n = [H_0, \chi_n]\phi_{n+1}
\]
as functions in $L^2$. We proceed with a), so let $s > 1/2$. Writing
\[
\phi - (\lambda - i)R(i)\phi_n = (\phi - \phi_n) + R(i)[H_0, \chi_n]\phi_{n+1},
\]
it suffices to show that \( R(i)[H_0, \chi_n] \phi_{n+1} \to 0 \) in \( L^2_s \) since \( \phi \in L^2_s \). This follows at once by expanding
\[
 r^{-s} R(i)[H_0, \chi_n] \phi_{n+1} = -2i (r^{-s} R(i) A r^s) r^{-s} \chi'_n \phi_{n+1} + (r^{-s} R(i) r^s) r^{-s} |\omega|^2 \chi''_n \phi_{n+1}
\]
and then noting that \( r^{-s} R(i) A r^s \) and \( r^{-s} R(i) r^s \) extend from \( C_c^\infty \) to bounded operators on \( L^2 \) (see [15, XIII.8 Lemma 1] and maybe take adjoints for the first operator). This shows a).

The proof of b) is entirely similar. Expanding \([H_0, \chi_n]\) like above we see
\[
\langle Af(H - \lambda) R(i) \phi_n, R(i) \phi_n \rangle = \langle [H_0, \chi_n] \phi_{n+1}, R(-i) f AR(i) \phi_n \rangle = \langle -2i \chi'_n \phi_{n+1}, AR(-i) f AR(i) \phi_n \rangle + \langle |\omega|^2 \chi''_n \phi_{n+1}, R(-i) f AR(i) \phi_n \rangle.
\]
Hence, for constants \( C_1, C_2 > 0 \) independent of \( n \), the Cauchy-Schwarz inequality gives
\[
|\langle \text{Im}(Af(H - \lambda)) \rangle_{R(i) \phi_n}| \leq C_1 \|\chi'_n \phi_{n+1}\|_B \|AR(-i) f AR(i) \phi_n\|_B + C_1 \|\chi''_n \phi_{n+1}\|_B \|R(-i) f AR(i) \phi_n\|_B
\]
\[
\leq C_2 2^{-n} \|F_{n+1} \phi\|_B \|\phi\|_B^*,
\]
and
\[
2^{-n} \|F_{n+1} \phi\|_B = 2^{-n} 2^{n+1} \|F_{n+1} \phi\| \to 0
\]
since \( \phi \in B_0^* \). We used along the way that \( AR(-i) f AR(i) \) extend from \( C_c^\infty \) to bounded operators \( B^* \to B^* \). This follows from the fact that they extend as bounded operators \( L^2_s \to L^2_s \) for all \( s \in \mathbb{R} \) (cf. the argument above) and the simple interpolative property from [6, Theorem 14.1.4]. The boundedness of \( f \) is here important. The second limit in b) follows by an identical argument. 

**Proof of Theorem 1.3.** Fix a compact set \( I \subseteq (0, \infty) \) and let \( \alpha_0 \geq 1/2 \). Consider the family of functions
\[
f = f_{k, \alpha}(r) = \Theta_k^\alpha(r), \quad \Theta_k(r) = \frac{r}{1 + kr}, \quad k > 0, \quad 1/2 \leq \alpha \leq \alpha_0.
\]
The same family is used in [9] and [6, Theorem 14.5.2]. Clearly any such \( f \) is bounded with
\[
f \geq 0, \quad 0 \leq f' \leq \alpha_0 r^{-1} f, \quad |f''| \leq \alpha_0(\alpha_0 + 3) r^{-2} f
\]
uniformly for all \( k > 0 \) and \( 1/2 \leq \alpha \leq \alpha_0 \). We implement Lemma 2.2 with this family of functions and \( W = W_0 \). Evaluate in the states \( R(i) \phi_n, n \in \mathbb{N} \), where \( \phi \in B_0^* \) is a generalized eigenfunction with eigenvalue \( \lambda \in I \), and then take the limit \( n \to \infty \) using Lemma 2.4 to conclude:
\[
\|f^{1/2} \phi\|^2 \leq C \|f^{1/2} r^{-1} \phi\|^2, \quad (2.7)
\]
where the constant \( C > 0 \) does not depend on \( k > 0, 1/2 \leq \alpha \leq \alpha_0 \), or the generalized eigenfunction \( \phi \). Bounding \( f^{1/2} \leq r^{\alpha/2} \), it follows by (2.7) and monotone convergence that
\[
\|r^{(\alpha-1)/2} \phi\|^2 \leq 2C \|r^{(\alpha-2)/2} \phi\|^2, \quad (2.8)
\]
for all \( 1/2 \leq \alpha \leq \alpha_0 \) and any generalized eigenfunction \( \phi \in B_0^* \) with eigenvalue in \( I \).

The result now follows from a bootstrap argument: First pick \( \alpha = 1/2 \). (2.8) then shows \( \phi \in L^2_n \). We can thus choose \( \alpha = 3/2 \) to conclude \( \phi \in L^2_{1/2} \), and so on. Repeating the argument finitely many times proves the first half of the theorem. The statements on \( E_\alpha \) follow by a simple compactness argument. See e.g. the proof of [15, Theorem XIII.33] where similar conclusions are reached.

\[\square\]
2.3. Proof of LAP bounds. The proof of Theorem 1.4 is almost identical to that of [10, Theorem 1.7] or [1, Theorem 1.7]. We include it since we shall reuse some arguments later and for the readers convenience.

Proof of Theorem 1.4. Fix a compact set \( I \subseteq (0, \infty) \setminus \mathcal{E}_+ \). Consider the family of functions
\[
f = f_k(r) = 1 - (1 + r/2^k)^{-1}, \quad k \geq 1.
\]
Note
\[
0 \leq f \leq 1, \quad 0 \leq f' \leq r^{-1}f, \quad |f''| \leq 2r^{-2}f
\]
uniformly for \( k \geq 1 \). We implement Lemma 2.3 with this family of functions and \( W = W_0 \). Evaluate in the states \( R(z)\psi, \psi \in \mathcal{B}, z \in I_\pm \), and use Cauchy-Swarz to conclude
\[
\|f^{1/2}R(z)\psi\|^2 + \|f^{1/2}AR(z)\psi\|^2 + (p^sf^2r^p)R(z)\psi \leq C_1 \left( \|\psi\|_B \|R(z)\psi\|_{B^*} \right) + \|f^{1/2}r^{-1}R(z)\psi\|^2 + \|\psi\|_B \|AR(z)\psi\|_{B^*}
\]
for a constant \( C_1 \) uniform in \( k, \psi \) and \( z \). From now on all constants in \( k, \psi \) and \( z \). We conclude
\[
\|f^{1/2}_k\phi\|^2 \geq \|F_kf^{1/2}_k\phi\|^2 \geq \frac{1}{2}2^{-k}\|F_k\phi\|^2
\]
for all \( \phi \in L^2_{\text{loc}} \) and \( k \in \mathbb{N} \). Taking sup over all \( k \) furnishes the \( B^* \) norm on the right.

Employing this observation along with Young’s inequality in (2.9) yields
\[
\|R(z)\psi\|^2_{B^*} + \|AR(z)\psi\|^2_{B^*} + (p^sf^2r^p)R(z)\psi \leq C_2 \|r^{-1}R(z)\psi\|^2
\]
for all \( \psi \in \mathcal{B} \) and \( z \in I_\pm \). It therefore suffices to show that \( R(z) : \mathcal{B} \to \mathcal{B}^* \) is uniformly bounded on \( I_\pm \).

Now assume for a contradiction that \( R(z) : \mathcal{B} \to \mathcal{B}^* \) is not uniformly bounded on \( I_\pm \) (for some fixed sign \( \pm \)). We can then take sequences \((z_n) \subseteq I_\pm \) and \((\psi_n) \subseteq \mathcal{B} \) so that
\[
\|R(z_n)\psi_n\|_{B^*} = 1 \quad \text{for all } n \in \mathbb{N}, \quad \lim_{n \to \infty} \|\psi_n\|_B = 0.
\]
(2.10) then shows that \( AR(z_n)\psi_n \) is uniformly bounded as well. We may assume by compactness that \( z_n \to \lambda \in \Gamma I_\pm \) and that \( R(z_n)\psi \to \phi \) in the weak-star topology of \( \mathcal{B}^* \). Note that \( \text{Im} \lambda \neq 0 \) would immediately lead to a contradiction, so we can furthermore assume \( \lambda \in I \).

We now claim \( R(z_n)\psi_n \to \phi \) strongly in \( L^2_s \) for \( s > 1/2 \). Indeed, let \( s > t > 1/2 \), and note that \( R(z_n)\psi_n \) converges weakly to \( \phi \) in \( L^2_s \) and \( L^2_t \). A rewrite
\[
r^{-s}R(z_n)\psi_n - r^{-s}R(i)\psi_n - (i - z_n)(r^{-s}R(i)r^t)(r^{-t}R(z_n)\psi_n)
\]
reveals that the convergence is actually strong since \( r^{-s}R(i)r^t \) extends to a compact operator on \( L^2 \). The compactness follows from [15, XIII.8 Lemma 1] and the basic fact that \( r^{d-s} \) is a relatively compact operator. It is then immediately obvious that \( (pR(z_n)\psi_n) \) and \( (H_0R(z_n)\psi_n) \) are Cauchy in \( L^2_s \), and both \( p \) and \( H_0 \) are continuous on distributions, whence
\[
R(z_n)\psi_n \to \phi, \quad pR(z_n)\psi_n \to p\phi, \quad H_0R(z_n)\psi_n \to H_0\phi \quad \text{in } L^2_s
\]
for all \( s > 1/2 \). Here \( p\phi \) and \( H_0\phi \) are understood in the distributional sense. We conclude in particular that \( \phi \in H^2_{\text{loc}} \), and
\[
\langle (H - \lambda)\phi, \rho \rangle = \lim_{n \to \infty} \langle (H - z_n)R(z_n)\psi_n, \rho \rangle = 0
\]
for all \( \rho \in C_0^\infty \) by (2.11), so \( \phi \) is a generalized eigenfunction with eigenvalue \( \lambda > 0 \).

We now show \( \phi \in \mathcal{B}^* \) to derive a contradiction. Recall the argument that led to (2.10). The bound \( 0 \leq f \leq r^{1/2}2^{-k/2} \) then leads to
\[
2^{-k}\|F_kR(z_n)\psi_n\|^2 \leq C_3 \left( 2^{-k/2}\|r^{-3/4}R(z_n)\psi_n\|^2 + \|\psi_n\|_B \|R(z_n)\psi_n\|_{B^*} + \|\psi_n\|_B \|AR(z_n)\psi_n\|_{B^*} \right)
\]
for all \( k, n \in \mathbb{N} \). Take the limit \( n \to \infty \) using (2.11), (2.12), and the fact that \( AR(z_n)\psi_n \) is uniformly bounded:
\[
2^{-k}\|F_k\phi\|^2 \leq C_42^{-k/2}\|r^{-3/4}\phi\|^2.
\]
Letting \( k \to \infty \) we conclude \( \phi \in B^0_\infty \) and therefore that \( \phi = 0 \) by Theorem 1.3 since \( \lambda \) is not an eigenvalue by assumption. This is a contradiction, for (2.10) clearly shows

\[
1 = \| R(z_n) \psi_n \| \leq C_2 \| r^{-1} R(z_n) \psi_n \|^2
\]

for all \( n \in \mathbb{N} \), and the right hand side goes to zero for \( n \to \infty \). \qed

**Proof of Theorem 1.5.** Fix a compact set \( I \subseteq (0, \infty) \setminus \mathcal{E}_+ \). First b). Consider Lemma 2.3 with \( f = 1 \) and \( W = W_0 + W_1 \). We conclude by Theorem 1.4 and the Cauchy-Schwarz inequality that

\[
\| W_1^{1/2} R(z) \psi \|^2 + \| W_1^{1/2} AR(z) \psi \|^2 \leq C_1 \| \psi \|^2_B,
\]

for a constant \( C_1 > 0 \), uniformly for all \( \psi \in B \) and \( z \in I_\pm \). Another application of Theorem 1.4 then shows

\[
\| W_1^{1/2} p R(z) \psi \|^2 = \langle p^* W_1 p \rangle_{R(z)} \psi = \langle p^* W_1 p \rangle_{R(z)} \psi + \| W_1^{1/2} \omega^* p R(z) \psi \|
\]

\[
\leq C_2 \| p^* \nabla^2 r p \|_{R(z)} + \| W_1^{1/2} (\frac{1}{2} \Delta r + A) R(z) \psi \|^2 \leq C_3 \| \psi \|^2_B,
\]

for suitable constants \( C_2, C_3 \), using here that \( W_1 = O(r^{-1}) \). This shows b).

We turn to a) and d). Consider Lemma 2.3 with \( f = 1 \) and \( W = W_0 + W_1 + W_2 \). Evaluate in \( R(z) W_2^{1/2} \psi \) for \( \psi \in L^2 \), \( z \in I_\pm \), and use Cauchy-Schwarz to conclude

\[
\| (W_1 + W_2)^{1/2} R(z) W_2^{1/2} \psi \|^2 + \| (W_1 + W_2)^{1/2} AR(z) W_2^{1/2} \psi \|^2 + \| p^* \nabla^2 r p \|_{R(z)} W_2^{1/2} \psi \|^2
\]

\[
\leq C \| r^{-1} R(z) W_2^{1/2} \psi \|^2 + \| \psi \| \| W_2^{1/2} R(z) W_2^{1/2} \psi \| + \| \psi \| \| W_2^{1/2} AR(z) W_2^{1/2} \psi \|.
\]

Using b) to bound \( \| r^{-1} R(z) W_2^{1/2} \psi \|^2 \leq C \| \psi \|^2 \) (take adjoints), it follows from Young’s inequality that

\[
\| (W_1 + W_2)^{1/2} R(z) W_2^{1/2} \psi \|^2 + \| (W_1 + W_2)^{1/2} AR(z) W_2^{1/2} \psi \|^2 + \| p^* \nabla^2 r p \|_{R(z)} W_2^{1/2} \psi \|^2 \leq C_4 \| \psi \|^2.
\]

We can bound the \( p \) term like above:

\[
\| W_1^{1/2} p R(z) W_2^{1/2} \psi \|^2 = \langle p^* W_1 M p \rangle_{R(z)} W_2^{1/2} \psi + \| W_1^{1/2} \omega^* p R(z) W_2^{1/2} \psi \|^2 \leq C_5 \| \psi \|^2,
\]

which finishes a) and d).

We finally consider c). First note the uniform bound on \( R(z) W_1^{1/2} \) follows from b) after taking adjoints. Now consider Lemma 2.3 with \( f = f_k = 1 - (1 + r/2^k)^{-1} \) like in the proof of Theorem 1.4. Evaluate in \( R(z) W_1^{1/2} \psi \) and use d) to see

\[
\| f_k^{1/2} AR(z) W_1^{1/2} \psi \|^2 \leq C_6 \| \psi \|^2
\]

uniformly for all \( k \geq 1 \) and \( \psi \in L^2 \). Restrict the integral region on the left to \( 2^{k-1} \leq r \leq 2^k \) and take sup over all \( k \) to conclude \( \| AR(z) W_1^{1/2} \psi \|^2_B \leq C_6 \| \psi \|^2 \). Using this, the uniform bound on \( p R(z) W_1^{1/2} \) can be established as follows: For any \( k \in \mathbb{N} \)

\[
2^{-k} \| F_k p R(z) W_1^{1/2} \psi \|^2 \leq \langle p^* F_k \nabla^2 r p \rangle_{R(z)} W_1^{1/2} \psi + 2^{-k} \| F_k \omega^* p R(z) W_1^{1/2} \psi \|^2.
\]

The assertion follows by taking sup over \( k \). \qed

### 3. Radiation Conditions

In this section we prove Theorems 1.6 and 1.7 and its corollaries. We impose Condition 1.2 throughout, which in particular means that \( \mathcal{E}_+ = \emptyset \) by Theorem 1.3. Inspired by [10], we first prove a radiation condition 'commutator' lower bound.
3.1. Radiation condition commutator estimate. Fix a compact set $I \subseteq (0, \infty)$. For $z = \lambda \pm i\varepsilon$, recall the phase $\sigma = \sigma_z$ and set $E_2 = (1 - \eta_0^2)(V^{ir} - z) - i\omega \cdot \nabla a$. We can then refine the decomposition (1.2) as

$$ H - z = A^2 + L + V - z + E_1 = (A + a)(A - a) + L + V^{sr} + E_1 + E_2 $$

(3.1a)

when $d \neq 2$, and

$$ H - z = (A + a)(A - a) + L + V^{sr} + (E_1 + \eta_\frac{1}{2}r^{-2}) + E_2 $$

(3.1b)

when $d = 2$.

The setup for our commutator estimate is similar to that of Lemma 2.2. Concretely, we shall this time around consider functions $f = f(r) \in C^1([1, \infty))$ such that

$$ f \geq 0, \quad 0 \leq f' \leq \beta r^{-1}f, \quad 0 \leq \beta \leq 2 $$

(3.2)

The restriction that $\beta \leq 2$ is important. Contrary to Lemma 2.2, we need no modification $\hat{f}$ of $f$, and $f$ will not be assumed bounded. We shall employ the following properties of the phase $a$ below:

$$ \text{Im}(a) \geq 0, \quad |a| \leq C, \quad |\nabla a| \leq CW_0 + Cr^{-3} $$

(3.3)

for a constant $C > 0$ uniform in $z \in \mathbb{I}_+$.  

**Lemma 3.1.** For any compact set $I \subseteq (0, \infty)$ and any $0 \leq \beta \leq 2$, there exist constants $C, c > 0$ such that the following holds: Let $f$ be a function that satisfies (3.2). Then for all $z \in \mathbb{I}_+$, as forms on $H^1 = r^{-1}H^2$,

$$ 2\text{Im}((A - a)^*f(H - z)) \geq c(A - a)^*f'(A - a) + (2 - \beta)p^*f\nabla^2rp $$

$$ - CfW_0 - Cp^*fW_0p - Cr^{-3}f - Cp^*fr^{-3}p. $$

**Proof.** The proof is essentially simpler than that of Lemma 2.2. We again initially consider all computations as forms on $C_c^\infty$. All constants below will only depend on $I$ and $\beta$, along with any auxiliary data such as $W_0$ and the dimension $d$. Collect errors in the term

$$ Q = r^{-3}f + p^*r^{-3}fp + W_0f + p^*W_0fp $$

Imagine writing out $2\text{Im}((A - a)^*f(H - z))$ according to (3.1b) or (3.1a). If $d = 2$,

$$ 2\text{Im}((A - a)^*f(E_1 + \eta_\frac{1}{2}r^{-2})) \geq -CQ $$

since $E_1 + \eta_\frac{1}{2}r^{-2} = O(r^{-4})$. When $d \neq 2$,

$$ 2\text{Im}((A - a)^*fE_1) = \text{Im}(a)fE_1 - |\omega|^2(fE_1)' \geq -CQ $$

as well, since $\text{Im}(a) \geq 0$ and the leading $O(r^{-2})$ term in $E_1$ is non-negative with $O(r^{-3})$ derivative. We can then fully expand according to (3.1a) or (3.1b):

$$ 2\text{Im}((A - a)^*f(H - z)) \geq (A - a)^*(|\omega|^2f' + f\text{Im}(a))(A - a) + 2\text{Im}((A - a)^*fL) $$

$$ + 2\text{Im}((A - a)^*fV^{sr}) + 2\text{Im}((A - a)^*fE_2) - C_1Q $$

(3.4)

$$ \geq (A - a)^*f'(A - a) + 2\text{Im}((A - a)^*fL) $$

$$ + 2\text{Im}((A - a)^*fE_2) + 2\text{Im}((A - a)^*fV^{sr}) - C_2Q. $$

Expand the $L$ term according to Lemma 2.1 using the bounds (3.3):

$$ 2\text{Im}((A - a)^*fL) \geq 2\text{Im}((La)f) + p^*(2fr^{-1} - f')M_p - C_3Q $$

$$ \geq 2\text{Im}(p^*Mfap) - 2\text{Re}(p^*M(\nabla a)f) - 2\text{Re}(p^*\omega r^{-2}f'a) $$

$$ + p^*f((2 - \beta)r^{-1})M_p - C_3Q $$

$$ \geq (2 - \beta)p^*f\nabla^2rp - 2\text{Re}(p^*M(\nabla a)f) - C_4Q $$

$$ \geq (2 - \beta)p^*f\nabla^2rp - C_5Q. $$
Similarly expand the third and fourth terms of (3.4) using Cauchy-Schwarz
\[ 2 \mathbf{Im}(\langle A - a \rangle \ast f \langle V \rangle) + 2 \mathbf{Im}(\langle A - a \rangle \ast f E_2) \geq -C_6 Q. \]
Combining everything in (3.4), conclude
\[ 2 \mathbf{Im}(\langle A - a \rangle \ast f (H - z)) \geq (A - a) \ast f' (A - a) + (2 - \beta) p^s f \nabla^2 r p - C_7 Q, \]
which establishes the claim on \( C_\infty^r \). Noting that \( f = O(r^2) \) by Grönwall’s inequality, the bound naturally extends to \( H_1^2 \) by continuity and density. \( \blacksquare \)

3.2. Proof of radiation condition bounds.

Proof of Theorem 1.6. Suppose the function \( h \) satisfies (1.6) with the constant \( \beta_0 < 1 \) appearing there. Consider the family of functions
\[ f = f_k(r) = h^2(r) \Theta_k^\beta_1(r), \quad \Theta_k = 1 - (1 + \frac{r}{2})^{-1}, \quad k \geq 1, \]
where \( \beta_1 > 0 \) is chosen such that \( 2\beta_0 + \beta_1 < 2 \). Clearly \( f \in C^1([1, \infty)) \) with
\[ 0 \leq f' = 2h^2 \Theta_k^\beta_1 + \beta_1 h^2 \Theta_k^\beta_1 (1 - \Theta) \leq (2\beta_0 + \beta_1) r^{-1} f \]
for any \( k \geq 1 \). Note that \( h^2(r-3 + W_0) = o(r^{-1}) \) and \( h^2(r^{-3} + W_0) \in L^1([1, \infty), dr) \) by assumption and Grönwall’s inequality. It thus follows from Lemma 3.1 with this family of functions applied to the states \( R(z) \psi, \psi \in C_\infty^r, z \in I_\pm \), the Cauchy-Schwarz inequality, and Theorem 1.5 that
\[
\| f^{1/2}(A - a) R(z) \psi \|^2 + \langle p^s f \nabla^2 p \rangle_{R(z) \psi} \leq C_1 (\| (W_0 + r^{-3})^{1/2} f^{1/2} R(z) \psi \|^2 \\
+ \| (W_0 + r^{-3})^{1/2} f^{1/2} p R(z) \psi \|^2 \\
+ \| f^{1/2} \psi \| B \| f^{1/2} (A - a) R(z) \psi \| B^*) \\
\leq C_2 (\| h \psi \|^2_B + \| h \psi \| B \| h (A - a) R(z) \psi \| B^*) ,
\]
for some constants \( C_1, C_2 > 0 \), uniformly for \( k \geq 1, z \in I_\pm \), and \( \psi \in C_\infty^r \). Here we used the trivial bound \( \Theta \leq 1 \). Restrict the integral region in the first term on the left to \( 2^{k-1} \leq r \leq 2^k \) and take sup over all \( k \geq 1 \) to conclude
\[
\| h(A - a) R(z) \psi \|^2_B + \langle p^s h^2 \nabla^2 r p \rangle_{R(z) \psi} \leq 4C_2 (\| h \psi \|^2_B + \| h \psi \| B \| h (A - a) R(z) \psi \| B^*)
\]
for all \( \psi \in C_\infty^r \) and \( z \in I_\pm \). An application of Young’s inequality shows that
\[
\| h(A - a) R(z) \psi \|^2_B + \langle p^s h^2 \nabla^2 r p \rangle_{R(z) \psi} \leq C_3 \| h \psi \|^2_B
\]
for some \( C_3 > 0 \) for all \( \psi \in C_\infty^r \). Here it is important that \( h(A - a) \psi \in B^* \), but this is obvious since \( \psi \in C_\infty^r \). Formally commutating the factor \( h \) in \( h(A - a) R(z) \psi \) to the right (using the \( C^1 \) condition on \( h \)) shows that the bound extends to all \( \psi \in h^{-1} B \). \( \blacksquare \)

Proof of Theorem 1.7. The proof is similar in spirit to that of Theorem 1.5 (using Lemma 3.1 instead of Lemma 2.3), so we skip some details. The assertion a) follows by considering Lemma 3.1 with \( f = h^2 \exp(\alpha \int_1^r h^2 W_2 + W_1 dr) \) with \( \alpha > 0 \) sufficiently small such that \( 2\beta_0 + \alpha \leq 2 \), evaluating in \( R(z) W_2^{1/2} \psi \) for \( \psi \in L^2 \), and using the two-sided estimates from Theorem 1.5 to bound some terms. The statement b) follows from a) by considering \( f = h^2 \Theta^\beta_1 \) like in the proof of Theorem 1.6 above, again evaluating the forms in \( R(z) W_2^{1/2} \psi \).

Finally for c), note that \( h \geq c \alpha \) for some \( c > 0 \) since \( h' \geq \alpha r^{-1} h \), whence \( r^{-1} h^{-1/2} = o(r^{-1}) \) and \( r^{-1} h^{-1/2} \in L^1([1, \infty), dr) \). The claim c) is immediate by considering \( f = h^2 \) and evaluating the forms in \( R(z) r^{-1/2} h^{-1/2} \psi \). Note here that the lower bound \( h' \geq \alpha hr^{-1} \) is crucial to give a good lower bound on \( (A - a) \ast f' (A - a) \). \( \blacksquare \)
3.3. Proof of the LAP and corollaries.

Proof of Corollary 1.8. The proof is similar to that of [10, Corollary 1.11]. We first prove (1.7). Let \( z, z' \in I_+ \) for some fixed sign \( \varepsilon \), and take \( n \in \mathbb{N} \). Decompose

\[
R(z) - R(z') = \chi_n R(z) \chi_n - \chi_n R(z') \chi_n + (R(z) - \chi_n R(z) \chi_n - (R(z') - \chi_n R(z') \chi_n) \tag{3.5}
\]

We first bound the third and fourth terms above. By Theorem 1:

\[
\|W^{1/2}(R(z) - \chi_n R(z) \chi_n) W^{1/2}\| \leq \|W^{1/2}R(z) W^{1/2}(1 - \chi_n)\|
\]

\[
+ \|(1 - \chi_n) W^{1/2} R(z) \chi_n W^{1/2}\|
\]

\[
\leq C_1 \|h^{-1}(1 - \chi_n)\|_\infty
\]

for a constant \( C_1 \) uniform in \( n \in \mathbb{N} \) and \( z, z' \in I_+ \). All constants are uniform in these parameters from now on. We can bound the fourth term in (3.5) similarly, whence

\[
\|W^{1/2}(R(z) - \chi_n R(z) \chi_n) W^{1/2}\| + \|W^{1/2}(R(z') - \chi_n R(z') \chi_n) W^{1/2}\| \leq C_2 \|(1 - \chi_n) h^{-1}\|_\infty.
\]

For the first and second term in (3.5), rewrite

\[
\chi_n R(z) \chi_n - \chi_n R(z') \chi_n = \chi_n R(z) (\chi_{n+1}(H - z') - (H - z) \chi_{n+1}) R(z') \chi_n
\]

\[
= (z - z') \chi_n R(z) \chi_{n+1} R(z') \chi_n - \chi_n R(z)[H_0, \chi_{n+1}] R(z') \chi_n. \tag{3.6}
\]

We bound each term separately. For the first, further decompose

\[
\chi_{n+1} = (A - a_\varepsilon)^+ \chi_{n+1}(a_z + a_{z'})^{-1} - \chi_{n+1}(a_z + a_{z'})^{-1}(A - a_\varepsilon) - [A, \chi_{n+1}(a_z + a_{z'})^{-1}].
\]

Note here that \( \mathbf{a} = -a_z \) and that \( (a_z + a_{z'})^{-1} \) is uniformly bounded in \( I_\pm \) with \( |\nabla (a_z + a_{z'})| \leq C W_0 + C r^{-3} \). It thus follows from Theorems 1.7 and 1.5 that

\[
\|W^{1/2}(z - z') \chi_n R(z) \chi_{n+1} R(z') W^{1/2}\| \leq C_3 \|h^{-1}(z - z')\|_\mathcal{L}(B^*, B) + \|h^{-1}\|_\mathcal{L}(B^*, B).
\]

The second term in (3.6) is similar. This time decompose

\[
-H_0, \chi_{n+1}] = 2i \text{Re}(\chi_{n+1} A) = i(\chi_{n+1} (A - a_{z'}) + (A - a_z)^* \chi_{n+1} - (a_z - a_{z'}) \chi_{n+1}).
\]

Bounding \( |a_z - a_{z'}| \leq C \|z - z'| \), it follows by repeating the argument above that

\[
\|W^{1/2} \chi_n R(z)[H_0, \chi_{n+1}] R(z') \chi_n W^{1/2}\| \leq C_4 \|z - z'\|_\mathcal{L}(B^*, B) + \|h^{-1}\|_\mathcal{L}(B^*, B).
\]

Combining estimates we conclude

\[
\|W^{1/2}(R(z) - R(z')) W^{1/2}\| \leq C_5 \|(1 - \chi_n) h^{-1}\|_\infty + \|\chi_{n+1} h^{-1}\|_\mathcal{L}(B^*, B)
\]

\[
+ C_5 \|z - z'\|_\mathcal{L}(B^*, B) + \|\chi_{n+1} h^{-1}\|_\mathcal{L}(B^*, B).
\]

To make this bound more explicit, note the \( \chi_{n+1} \) terms are (relatively) negligible. Using that \( h \) is increasing along with the bounds

\[
(s/t)^{\beta_0} \leq h(s)/h(t) \leq (t/s)^{\beta_0} \quad 1 \leq s \leq t,
\]

valid since \( 0 \leq h' \leq \beta_0 h^{-1} \) with \( \beta_0 < 1 \), a Cauchy condensation type argument yields

\[
\|\chi_{n+1} h^{-1}\|_\mathcal{L}(B^*, B) \leq C_6 \int_1^{2^n} h^{-1} dr \leq C_T 2^n/h(2^n).
\]

We therefore conclude

\[
\|W^{1/2}(R(z) - R(z')) W^{1/2}\| \leq C_5 \frac{1}{h(2^n)} (1 + 2^n |z - z'|).
\]

Now suppose \( |z - z'| \leq 1 \). Take \( n \in \mathbb{N} \) such that \( 2^n \leq |z - z'|^{-1} \leq 2^{n+1} \). Then also \( h(2^n) \leq h(|z - z'|^{-1}) \leq C_9 h(2^n) \), whence

\[
\|W^{1/2}(R(z) - R(z')) W^{1/2}\| \leq C_{10} \frac{1}{h(|z - z'|^{-1})},
\]

which is (1.7).
We now prove the LAP. To this end, fix \( \psi \in \mathcal{B} \) and an open interval \( I \) with compact closure in \((0, \infty) \setminus \mathcal{E}_+ \). Such an interval \( I \) exists since \( \mathcal{E}_+ \) is discrete cf. Theorem 1.3. For a fixed sign \( \pm \), it suffices to show that the map

\[
\Phi_\pm : I_\pm \ni z \mapsto R(z)\psi \in \mathcal{B}^*
\]

is uniformly continuous where \( \mathcal{B}^* \) is equipped with the weak-star topology. Indeed, \( \Phi_\pm \) maps into a compact (completely metrizable) space by Theorem 1.4, so standard extension arguments applies. By uniform boundedness it therefore suffices to show that

\[
\langle (R(z) - R(z'))\psi, \rho \rangle \to 0 \text{ as } |z - z'| \to 0
\]

for all \( \rho \in C_0^\infty \), where \( z, z' \) are taken in \( I_\pm \). Consider the function

\[
W_\psi = \sum_{k=1}^{\infty} 2^{-k/2} \max\{\|F_k\psi\|, 2^{-k}\} F_k,
\]

and note \( W_\psi \) is chosen exactly such that \( W \) satisfies (1.5) and \( W^{-1/2}\psi \in L^2 \). Using the proved statement (1.7) with this \( W_\psi \) and a suitable function \( h(r) \to \infty \) as \( r \to \infty \), we conclude

\[
|\langle (R(z) - R(z'))\psi, \rho \rangle| \leq \|W^{-1/2}_\psi\rho\|\|W^{-1/2}_\psi\|\|W^{-1/2}_\psi(R(z) - R(z'))W^{-1/2}_\psi\| \to 0
\]

as \( |z - z'| \to 0 \), for any \( \rho \in C_0^\infty \), which finishes the proof.

**Proof of Corollary 1.9.** Fix a function \( h \) that satisfies (1.6), \( \lambda \in (0, \infty) \setminus \mathcal{E}_+ \), \( \psi \in h^{-1}\mathcal{B} \), and a sign \( \pm \). We first verify that \( R(\lambda \pm i0)h \) obeys a) and b) of the theorem. Using Theorem 1.4 and Corollary 1.8, a) follows easily. For b), take a function \( h_0 = h_0(r) \) so that \( h_0h \) satisfies (1.6), \( h_0(r) \to \infty \) as \( r \to \infty \), and \( h_0 \psi \in h^{-1}\mathcal{B} \). We remark that this is always possible. Applying Theorem 1.6 and taking a weak-start limit \( \varepsilon \to 0^+ \) we see \( h(A-a)R(\lambda \pm i0)\psi \in h_0^{-1}\mathcal{B}^* \subseteq \mathcal{B}_0^* \), showing b).

Conversely, suppose \( \phi_\pm \) satisfies a) and b) of the theorem. Consider \( \phi'_\pm = \phi_\pm - R(\lambda \pm i0)\psi \), and note \( \phi'_\pm \) is a generalized eigenfunction with eigenvalue \( \lambda \) by a). We show \( \phi'_\pm \in \mathcal{B}_0^* \) to conclude \( \phi = R(\lambda \pm i0)\psi \) by Theorem 1.3. To this end, note

\[
\text{Im}(\chi_n(H - \lambda)) = \text{Re}((A-a)\chi_n') + \chi_n'\text{Re}(a)
\]

for all \( n \in \mathbb{N} \) as forms on \( H^2_{loc} \). The left hand side vanishes on \( \phi'_\pm \), whence

\[
0 \leq \langle \pm \chi_n' \text{Re}(a) \rangle_{\phi'_\pm} = (\pm \text{Re}((A-a)\chi_n'))_{\phi'_\pm} \leq \|h^{-1}\phi_\pm\|_{\mathcal{B}}\|\chi_n'h(A-a)\phi'_\pm\|_{\mathcal{B}} \to 0
\]

as \( n \to \infty \) since \( h(A-a)\phi'_\pm \in \mathcal{B}_0^* \) by assumption. We conclude \( \phi_\pm = R(\lambda \pm i0)\psi \).

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