Isometric flows of $G_2$-structures

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Abstract

We survey recent progress in the study of flows of isometric $G_2$-structures on 7-dimensional manifolds, that is, flows that preserve the metric, while modifying the $G_2$-structure. In particular, heat flows of isometric $G_2$-structures have been recently studied from several different perspectives, in particular in terms of 3-forms, octonions, vector fields, and geometric structures. We will give an overview of each approach, the results obtained, and compare the different perspectives.

1 Introduction

One of the most challenging problems in differential geometry is the question of existence conditions for torsion-free $G_2$-structures on smooth 7-dimensional manifolds. Such $G_2$-structures are precisely the ones that correspond to metrics with holonomy contained in $G_2$. One approach that has been pioneered by Robert Bryant [4] is to consider heat-like flows of $G_2$-structures with the hope that under certain conditions they may converge to a torsion-free $G_2$-structure. A difficulty that is encountered in such an approach is that in general, deformations of a $G_2$-structure also affect the corresponding metric, and so any heat equation for the $G_2$-structure becomes nonlinear. This is not unlike the situation for the Ricci flow, where the underlying geometry changes along the flow, however in the $G_2$ case, we have two separate but closely related objects, the $G_2$-structure and the metric, both of which vary along the flow. Given a Riemannian metric on a 7-manifold that admits $G_2$-structures, there is a family of $G_2$-structures that correspond to it, so a possible approach could be to separate as much as possible the deformations of the metric from the deformations of $G_2$-structures that preserve the metric. Indeed, as was shown by Karigiannis [13], given a decomposition of 3-forms according to representations of $G_2$, the deformations of the $G_2$-structure 3-form that preserve the metric are precisely the ones that lie in the 7-dimensional representation $\Lambda^3_7$. Bryant’s
original Laplacian flow of closed $G_2$-structures has no component in $\Lambda^3_7$, and as such is transverse to directions that preserve the metric. This allowed for more tractable analytic properties. In contrast, a similar flow for co-closed $G_2$-structures that was proposed in [15] does have a component in $\Lambda^3_7$, which, as shown in [9], causes non-parabolicity of the flow. This suggests that the freedom of $G_2$-structures to move in directions that preserve the metric is some kind of degeneracy and thus suitable gauge-fixing conditions within the metric class are needed to address it.

These considerations show that it is necessary to have a clearer picture of $G_2$-structures within a fixed metric class. In [4], Bryant observed that such $G_2$-structures are parametrized by sections of an $\mathbb{RP}^7$-bundle, or more concretely, by pairs $(a, \alpha)$ where $a$ is a real-valued function and $\alpha$ is a vector field such that $a^2 + |\alpha|^2 = 1$, and $\pm (a, \alpha)$ define the same $G_2$-structure. If $\varphi$ is a fixed $G_2$-structure, then any other $G_2$-structure $\sigma_{(a, \alpha)}(\varphi)$ within the same metric class is given by:

$$
\sigma_{(a, \alpha)}(\varphi) = \left(a^2 - |\alpha|^2\right) \varphi - 2a\alpha \wedge \psi + 2\alpha \wedge (\alpha \wedge \varphi), \tag{1}
$$

where $\psi = \ast \varphi$.

Given that the group $G_2$ may be defined as the automorphism group of the octonions, a $G_2$-structure defines an octonion structure on the manifold, and in [10], this observation was used to interpret the above pair $(a, \alpha)$ as a unit octonion $V$, and then (1) is just the 3-form that corresponds to a modified octonion product defined by $V$. Thus, a flow of isometric $G_2$-structures can be interpreted as a flow of the unit octonion section $V$. In particular, a natural heat flow of isometric $G_2$-structures was introduced in [10]. Given an octonionic covariant derivative $D$, constructed from the Levi-Civita connection and the torsion of the initial $G_2$-structure $\varphi$, the heat flow of isometric $G_2$-structures is then the semilinear, parabolic equation

$$
\frac{\partial V}{\partial t} = \Delta_D V + |DV|^2 V \tag{2}
$$

with some initial condition $V(0) = V_0$ and where $\Delta_D = -D^*D$ is the Laplacian operator corresponding to $D$. This was obtained as the negative gradient flow of an energy functional with respect to $D$. The critical points of the flow (2) correspond to $G_2$-structures for which the torsion tensor is divergence-free, i.e. satisfies $\text{div}T = 0$, where divergence is taken with respect to the Levi-Civita connection. This is significant for several reasons. The divergence of torsion is precisely the term that causes the non-parabolicity of the Laplacian flow of co-closed $G_2$-structures from [15] as mentioned above, and $\text{div}T = 0$ for closed $G_2$-structures. Thus, closed $G_2$-structures are automatically critical points of (2). Secondly, $T$ has been interpreted in [10] as an imaginary octonion-valued 1-form, which is added to the Levi-Civita connection to obtain the octonionic covariant derivative $D$, hence the condition $\text{div}T = 0$ is precisely analogous to the Coulomb gauge condition in gauge theory. This analogy makes this condition a reasonable candidate for a gauge-fixing condition within a fixed metric class.
Soon after the introduction of the flow (2) in [10], it was further studied from different perspectives by several authors: Bagaglini in [1]; Dwivedi, Gianniotis, and Karigiannis in [8]; the author in [11]; Loubeau and Sá Earp in [17].

Equivalently to the flow of octonions (2), one can consider directly the evolution of the 3-form $\varphi$ via the equation

$$\frac{\partial \varphi}{\partial t} = 2 (\text{div} T) \cdot \varphi$$

where $T$ is the torsion tensor that corresponds to the $G_2$-structure 3-form at time $t$. This is the way the flow was formulated in [1] and in [8] (although here we are following [10, 11] and added a factor of 2 in (3). In [17], a more general approach is taken and a harmonic heat flow of geometric structures is considered. In the case of $G_2$-structures, it is shown to reduce to (3). In this survey we will review the above approaches to the flow of isometric $G_2$-structures and outline the key analytic results.

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2 Isometric $G_2$-structures

A $G_2$-structure on a 7-manifold is defined by a smooth positive 3-form $\varphi$ [3, 12]. This is a nowhere-vanishing 3-form that defines a Riemannian metric $g_\varphi$, such that for any vectors $u$ and $v$, the following holds

$$g_\varphi (u, v) \text{vol}_\varphi = \frac{1}{6} (u \cdot \varphi) \wedge (v \cdot \varphi) \wedge \varphi. \quad (4)$$

At any point, the stabilizer of $g_\varphi$ (along with orientation) is $SO(7)$, whereas the stabilizer of $\varphi$ is $G_2 \subset SO(7)$. This shows that at a point, positive 3-forms forms that correspond to the same metric, i.e., are isometric, are parametrized by $SO(7)/G_2 \cong \mathbb{RP}^7 \cong S^7/\mathbb{Z}_2$. Therefore, on a Riemannian manifold, metric-compatible $G_2$-structures are parametrized by sections of an $\mathbb{RP}^7$-bundle, or alternatively, by sections of an $S^7$-bundle, with antipodal points identified. This is precisely the parametrization given by [11].

Alternatively, a $G_2$-structure in a fixed metric class can be interpreted as a reduction of the principal $SO(7)$-bundle $P$ of orthonormal frames to a principal $G_2$-subbundle, and hence each such reduction corresponds to a section $\sigma$ of an $SO(7)/G_2$-bundle $N$ and equivalently, an $SO(7)$-equivariant map $s : P \rightarrow SO(7)/G_2 \cong S^7/\mathbb{Z}_2$. This is the picture used in [17].

We may also use the $G_2$-structure $\varphi$ and the metric to define the octonion bundle $\mathcal{O}M \cong \Lambda^0 \oplus TM$ on $M$ as a rank 8 real vector bundle equipped with an octonion product of sections given by

$$A \circ_\varphi B = (ab - g(\alpha, \beta), a\beta + b\alpha + \alpha \times_\varphi \beta) \quad (5)$$
for any sections \( A = (a, \alpha) \) and \( B = (b, \beta) \). We set the metric \( g = g_\varphi \), since we are fixing the metric, even though the \( G_2 \)-structure may change. Here we define \( \times_\varphi \) by \( g(\alpha \times_\varphi \beta, \gamma) = \varphi(\alpha, \beta, \gamma) \) and given \( A \in \Gamma(\mathbb{O}M) \), we write \( A = (\text{Re} \ A, \text{Im} \ A) \). The metric on \( TM \) is extended to \( \mathbb{O}M \) to give the octonion inner product \( \langle A, B \rangle = ab + g(\alpha, \beta) \), which is Hermitian with respect to the octonion product. In the formula (11), the pair \((a, \alpha)\) can now be interpreted as a unit octonion section.

The intrinsic torsion of a \( G_2 \)-structure is defined by \( \nabla_\varphi \), where \( \nabla \) is the Levi-Civita connection for the metric \( g \) that is defined by \( \varphi \). Following [14], we have

\[
\nabla_a \varphi_{bcd} = 2 T_a \psi_{ebcd} \quad \text{and} \quad \nabla_a \psi_{bcde} = -8 T_a [b \varphi_{cde}] \quad (6)
\]

where \( T_{ab} \) is the full torsion tensor, note that an additional factor of 2 is for convenience, and \( \psi = *\varphi \) is the 4-form that is the Hodge dual of \( \varphi \) with respect to the metric \( g \). The \( G_2 \)-structure is known as torsion-free if \( T = 0 \), and in that case \( \nabla \) has holonomy contained in \( G_2 \). Conversely, if \( \nabla \) has holonomy contained in \( G_2 \), then there exists a torsion-free \( G_2 \)-structure within the metric class. Let \( V = (a, \alpha) \) be a unit octonion section, then define \( \sigma_V(\varphi) = \sigma_{(a, \alpha)}(\varphi) \), as in (1).

It has been shown in [10] that the torsion of the \( G_2 \)-structure \( \varphi_V = \sigma_V(\varphi) \) is given by

\[
T(V) = VT - (\nabla V)^{-1} \quad (7)
\]

where \( T \) is the torsion of \( \varphi \), interpreted as a 1-form with values in the bundle of imaginary octonions \( \text{Im} \mathbb{O}M \). If we now define an octonion covariant derivative \( D \) on sections of \( \mathbb{O}M \) via

\[
DV = \nabla V - VT, \quad (8)
\]

the expression (7) simply becomes

\[
T(V) = -(DV)^{-1}. \quad (9)
\]

As shown in [10], the derivative \( D \) has other nice properties - it is metric-compatible, and satisfies a partial product rule with respect to octonion product on \( \mathbb{O}M \), that is, \( D(UV) = (\nabla U)V + U(DV) \). Now given (9), the divergence of \( T(V) \) can be expressed as

\[
\text{div} T(V) = -(\Delta_D V)^{-1} - |DV|^2. \quad (10)
\]

3 Energy functional

Given that the torsion varies across \( G_2 \)-structures within the same metric class, an obvious question is how to pick a representative of the class with the “best” torsion. A reasonable way to try and characterize the best torsion is to look for critical points of a functional. Therefore, given the set \( \mathcal{F}_g \) of all \( G_2 \)-structures that are compatible with a given metric \( g \), and assuming \( M \) is compact, define the functional \( \mathcal{E} : \mathcal{F}_g \rightarrow \mathbb{R} \) by

\[
\mathcal{E}(\varphi) = \int_M |T(\varphi)|^2 \text{vol}, \quad (11)
\]
where $T^{(\varphi)}$ is the torsion of a $G_2$-structure $\varphi$. This is the functional used by Dwivedi, Gianniotis, and Karigiannis in [8].

As we have seen in the previous section, given a $G_2$-structure $\varphi$, any other $G_2$-structure within the same metric class is given by $\sigma_V(\varphi)$ for a unit octonion section $V$. Therefore, the functional (11) is equivalent to the functional $\mathcal{E}_0 : \Gamma(S\Omega M) \rightarrow \mathbb{R}$ given by

$$\mathcal{E}_0(V) = \int_M |T^{(V)}|^2 \text{vol} = \int_M |DV|^2 \text{vol}$$

(12)

where we have also applied (9). Hence, in fact, the functional $\mathcal{E}_\varphi$ is equivalent to an energy functional with respect to the derivative $D$. This is the functional used in [10, 11].

On the other hand, following the approach in [17], recall that a principal $H$-subbundle of a principal $G$-bundle $P$ may be characterized by an equivariant map $s : P \rightarrow G/H$, or equivalently, as a section $\sigma$ of the associated bundle $N = P \times_G (G/H) \cong P/H$. Assuming that $G$ is semi-simple, so that it admits a bi-invariant metric, we may define a metric $\eta$ on $N$, together with the corresponding Levi-Civita connection $\nabla^\eta$. Moreover, given a metric $g$ on the base manifold, we may induce a metric on $T^*M \otimes \sigma^*TN$, which is compatible with the splitting $TN = V_N \oplus H_N$ induced by $\nabla^\eta$. Using this metric, we may then define an energy functional $\mathcal{E}_\Gamma : \Gamma(N) \rightarrow \mathbb{R}$ on sections of $N$:

$$\mathcal{E}_\Gamma(\sigma) = \int_M |d\sigma|^2 \text{vol}. \quad (13)$$

Alternatively, suppose that moreover $G$ is compact, so that $P$ is compact. Then, let us define an energy functional on $G$-equivariant maps $s : P \rightarrow G/H$:

$$\mathcal{E}_G(s) = \int_P |ds|^2 \text{vol}_P$$

(14)

where an induced metric on $T^*P \otimes s^*T(G/H)$ is used. It is then shown in [17], that for any section $\sigma \in \Gamma(N)$ and its corresponding $G$-equivariant map $s \in C_G^\infty(P,G/H)$, $\mathcal{E}_G(s) = c_1 \mathcal{E}_\Gamma(\sigma) + c_2$ where $c_1$ and $c_2$ are uniform constants.

Consider the orthogonal splitting $d\sigma = d^V\sigma + d^H\sigma$ into horizontal and vertical parts. Since the horizontal component of the metric is given by $\pi^*g$, where $\pi : N \rightarrow M$ is the bundle projection map, we find that for any $X \in TM$,

$$|d^H\sigma(X)|^2 = (\pi^*g)(d\sigma(X),d\sigma(X)) = g((\pi \circ \sigma)_*X,(\pi \circ \sigma)_*X) = g(X,X).$$

Thus, the horizontal part of $d\sigma$ contributes only a constant term to (13), and it is thus sufficient to consider just the vertical component

$$\mathcal{E}^V_\Gamma(\sigma) = \int_M |d^V\sigma|^2 \text{vol}. \quad (15)$$

In the $G_2$ case, Loubeau and Sá Earp show in [17] that this functional is equivalent to (11).
It should be emphasized that the reason that the critical points of the functional \( (\sigma) \), if and only if the corresponding \( G \equiv \sigma \), is constructed from any compatible metric on \( M \), then \( |dV| \sigma |^2 = \frac{2}{3} |T(\sigma) |^2 \) where \( T(\sigma) \) is the torsion tensor of the \( G_2 \)-structure defined by the section \( \sigma \).

### 4 Gradient flow

Given the functionals defined in the previous section, we may consider critical points and negative gradient flows of the functionals. This is summarized below.

| Space | Functional | Critical points | Negative gradient flow |
|-------|------------|-----------------|------------------------|
| \( F_0 \) | \( \mathcal{E}(\varphi) \) | \( \text{div} T(\varphi) = 0 \) | \( \frac{\partial E}{\partial t} = 2 \text{div} T(\varphi), \forall \varphi \) |
| \( \Gamma (SO M) \) | \( \mathcal{E}_G (V) \) | \( \Delta_0 V + |DV|^2 V = 0 \) | \( \frac{\partial E}{\partial t} = \Delta_0 V + |DV|^2 V \) |
| \( \Gamma (N) \) | \( \mathcal{E}_G (\sigma) \) | \( \tau^V (\sigma) = 0 \) | \( \frac{\partial E}{\partial t} = \tau^V (\sigma) \) |
| \( C^\infty (P, G/H) \) | \( \mathcal{E}_G (s) \) | \( \tau^H (s) = 0 \) | \( \frac{\partial E}{\partial t} = \tau^H (s) \) |

where \( \tau^V (\sigma) := \text{Tr}_G (\nabla^V dV \sigma) \) is the vertical tension field of the functional \( \mathcal{E}_G (\sigma) \) and \( \tau^H (s) := \text{Tr}_G (\nabla^H ds) \) is the horizontal tension field of the functional \( \mathcal{E}_G (s) \).

It is proved in [20, Theorem 1] that \( \sigma \in \Gamma (N) \) is a harmonic section, i.e. a critical point of the functional \( \mathcal{E}_G \), if and only if the corresponding \( G \)-equivariant map \( s \in C^\infty (P, G/H) \) is a horizontally harmonic map, that is \( \tau^H (s) = 0 \). In the expression for \( \tau^H (s) \), the trace is just over the horizontal distribution in \( TP \). It should be emphasized that the reason that the critical points of \( \mathcal{E}_G \) are not exactly harmonic maps is that we are varying over only the equivariant maps, rather than arbitrary maps. On the other hand, Wood does prove in [20, Theorem 3], that if \( G/H \) is a normal \( G \)-homogeneous manifold and the metric on \( P \) is constructed from any compatible metric on \( G \), then \( \sigma \) is a harmonic section if and only if the corresponding \( s \) is a harmonic map, that is, \( \tau (s) := \text{Tr}_G (\nabla^H ds) = 0 \). Crucially, these conditions are satisfied for \( G = SO(7), H = G_2 \), and \( P \) the orthonormal frame bundle on \( M \). Moreover, as shown in [17], given these conditions, a family \( \sigma_t \in \Gamma (N) \) satisfies the harmonic section flow \( \frac{\partial \sigma_t}{\partial t} = \tau^V (\sigma_t) \) if and only if there is a corresponding family \( s_t \in C^\infty (P, G/H) \) that satisfies the harmonic map flow \( \frac{\partial s_t}{\partial t} = \tau (s_t) \). Also, Wood has shown in [13] that equivariance is preserved along the harmonic map flow, so that if the initial condition is equivariant, then the flow will continue to be equivariant. This shows a close relationship between harmonic map theory and the theory of harmonic sections, and hence the flow of isometric \( G_2 \)-structures.

On the other hand, one must be careful when applying harmonic map results. In particular, the energy \( \mathcal{E}_G (s) \) contains a topological term that can never be arbitrarily small, and thus standard small initial energy long-time existence results [5] for harmonic maps cannot be applied. Similarly, while a constant map is always harmonic, an equivariant map \( s : P \rightarrow G/H \) can never be constant (if \( H \neq G \)). Thus existence of non-trivial harmonic equivariant maps and hence harmonic sections is not guaranteed, as expected.
Some results from the theory of harmonic maps do carry over, at least in the $G_2$-case. It was shown in [8] [11] that almost monotonicity and $\varepsilon$-regularity results similar to the harmonic map heat flow [5] [6] [18] hold for the flow (3).

Let $p_{x_0,t_0}(x,t)$ be the backward heat kernel on $M$, that is, the solution of the backward heat equation for $0 \leq t \leq t_0$ that converges to a delta function at $(x,t) = (x_0,t_0)$. Then, given a time-dependent octonion section $V_t$ or equivalently, a 3-form $\varphi_t = \sigma_{V(t)}(\varphi)$ for some fixed $G_2$-structure $\varphi$, define the $\mathcal{F}$-functional [11]

$$
\mathcal{F}(x_0,t_0,t) = (t_0-t) \int_M \left| T^{(V_t)}(x) \right|^2 p_{x_0,t_0}(x,t) \text{vol}(x),
$$

(16)

where $T^{(V_t)} = -(DV_t)V_t^{-1}$ is the torsion of the $G_2$-structure $\varphi_t$. In [8], the analogous quantity is denoted by $\Theta_{(x_0,t_0)}(\varphi(t))$. It is then shown in both [8] Theorem 5.3 and [11] Proof of Corollary 7.2 that $\mathcal{F}$ satisfies an almost monotonicity formula along the flow (2). Suppose $V_t$ is a solution of the flow (2) for $0 \leq t < t_0$ with initial energy $\mathcal{E}(0) = \mathcal{E}_0$. Then, there exists a constant $C > 0$, that only depends on the background geometry, such that for any $t$ and $\tau$ satisfying $t_0 - 1 \leq \tau \leq t < t_0$, $\mathcal{F}$ satisfies the following relation

$$
\mathcal{F}(x_0,t_0,t) \leq C \mathcal{F}(x_0,t_0,\tau) + C(t-\tau) \left( \mathcal{E}_0 + \mathcal{E}_0^\beta \right).
$$

(17)

In [8], the last term in (17) was $C(t-\tau)(\mathcal{E}_0 + 1)$, which of course follows from (17) for a different constant $C$. In both [8] and [11] similar versions of an $\varepsilon$-regularity result is proven for $\mathcal{F}$. We’ll state it as in [11].

**Theorem 4.1 ([8] Theorem 5.7] and [11] Theorem 7.1)** Given $\mathcal{E}_0$, there exist $\varepsilon > 0$ and $\beta > 0$, both depending on $M$ and $\beta$ also depending on $\mathcal{E}_0$, such that if $V$ is a solution of the flow (2) on $M \times [0,t_0]$ with energy bounded by $\mathcal{E}_0$, and if

$$
\mathcal{F}(x_0,t_0,t) \leq \varepsilon
$$

(18)

for $t \in [t_0-\beta,t_0)$, then $V$ extends smoothly to $U_{x_0} \times [0,t_0]$ for some neighborhood $U_{x_0}$ of $x_0$ with $|DV| = |T^{(V)}|$ bounded uniformly.

Then, Theorem 4.1 was used in [8] [11] to show long-time existence of the isometric heat flow and convergence to a $G_2$-structure with $\text{div}T = 0$ given sufficiently small initial pointwise torsion.

Given a $G_2$-structure 3-form $\varphi$, in [8] a concept of entropy was defined:

$$
\lambda(\varphi,\sigma) = \max_{(x,t) \in M \times [0,\sigma]} \left\{ t \int_M |T^{(\varphi)}(y)|^2 p_{x,t}(y,0) \text{vol}(y) \right\}.
$$

(19)

This mirrors similar entropy concepts defined for the mean curvature flow, Yang-Mills flow, and the harmonic map heat flow, in [7], [16], and [2], respectively. The quantity $\lambda(\varphi,\sigma)$ is shown in [8] to be invariant under the scaling $(\varphi,\sigma) \mapsto (e^3\varphi, e^2\sigma)$. While the same quantity could be defined for an octonion section.
V, if considered as a function of $V$, $\lambda$ would lose the scaling property for $V$. So in this case, using the 3-form has an advantage. Overall, one of the key results in [8] is long term existence and convergence of the flow (3) given sufficiently small entropy.

\textbf{Theorem 4.2 ([8, Theorem 5.15])} Let $\varphi_0$ be a $G_2$-structure on a compact 7-manifold $M$. For any $\delta, \sigma > 0$, there exists $\varepsilon > 0$, such that if $\lambda(\varphi_0, \sigma) < \varepsilon$, then the flow (3) with initial condition $\varphi(0) = \varphi_0$ exists for all time and converges smoothly to a $G_2$-structure $\varphi_\infty$ that satisfies $\text{div} T(\varphi_\infty) = 0$ and $|T(\varphi_\infty)| < \delta$.

Although good progress has been made on properties of the flows (2) and (3), many questions still remain. For example, is it possible to prove long-time existence given small initial energy, rather than entropy or pointwise torsion? If we combine the equivariant harmonic map approach with the octonion approach, then everything could be reformulated in terms of equivariant maps from the orthonormal frame bundle $P$ to $S^7$ equipped with the octonion product. It is likely that the additional algebraic structure could help achieve stronger results.

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