BOUNDLESSNESS AND STABILITY OF SOLUTIONS TO SEMI-LINEAR EQUATIONS AND APPLICATIONS TO FLUID DYNAMICS

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Abstract. For an exterior domain $\Omega \subset \mathbb{R}^d$ with smooth boundary, we study the existence and stability of bounded mild solutions in time $t$ to the abstract semi-linear evolution equation $u_t + Au = \text{Pdiv}(G(u) + F(t))$ where $-A$ generates a $C_0$-semigroup on the solenoidal space $L^d_{\sigma,w}(\Omega)$ (known as weak-$L^d$), $\text{P}$ is Helmholtz projection; $G$ is a nonlinear operator acting from $L^d_{\sigma,w}(\Omega)$ into $L^{d/2}_{\sigma,w}(\Omega)^2$, and $F(t)$ is a second-order tensor in $L^{d/2}_{\sigma,w}(\Omega)^2$. Our obtained abstract results can be applied not only to reestablish the known results on Navier-Stokes flows on exterior domains and/or around rotating obstacles, but also to obtain a new result on existence and polynomial stability of bounded solutions to Navier-Stokes-Oseen equations on exterior domains.

1. Introduction. In the paper [17] when looking for periodic solutions to Navier-Stokes equations on the whole space $\mathbb{R}^d$, Maremonti announced an important problem related to bounded solutions of Navier-Stokes equations in unbounded (in all directions) domains which he called Theorem A saying that

"Denote by $f(t,x)$ the body force and $u(t,x)$ a solution to the Navier-Stokes equations $u_t - \Delta u + (u \cdot \nabla)u + \nabla p = f; X$ and $Y$ two Banach spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$ respectively. If $f(t,\cdot) \in X$ with $\|f(t,\cdot)\|_X$ uniformly bounded in time, then $u(t,\cdot) \in Y$ with $\|u(t,\cdot)\|_Y$ uniformly bounded in the time."

If the domain $\Omega$ is bounded (in some directions), then using the Poincaré inequality and some compact embeddings it is convenient to prove the validity of Theorem A. The situation becomes more complicated when one considers the unbounded domain $\Omega$ in all directions since the Poincaré inequality is no longer true and compact
embeddings are not valid. Therefore, some new approaches have been introduced to overcome this difficulty.

Maremonti [17, 18] and Maremonti-Padula [19] used some geometric properties of the domains such as the symmetry of Ω and/or the smallness of the complement \( \mathbb{R}^d \setminus \Omega \) to show the validity of Theorem A. Galdi and Sohr [7] discovered the fact that the specific structures of the phase-spaces \( X \) and \( Y \) played important roles when looking for bounded solutions (and also periodic ones) to Navier-Stokes equations in exterior domains. Consequently, they introduced in [7] some relevant function spaces featuring the decay of the solutions at spatial infinity to prove Theorem A on an exterior domain without restricted conditions on the domain. The last approach that we would like to mention was given by Yamazaki [23], and exploited the interpolation features of the weak-\( L^d \) spaces to prove the existence of bounded (in time) weak mild solutions of Navier-Stokes equations on exterior domains for each bounded external force. This approach has then been extended by Huy [12] to obtain bounded strong mild solutions in weak-\( L^3 \) spaces of Navier-Stokes equations around rotating obstacles. These latter two works have inspired us to write this present paper.

More precisely, in this paper, we will introduce a general framework to study the Theorem A on an exterior domain \( \Omega \), namely, we consider the general semi-linear equations on \( \Omega \) of the form

\[
\begin{cases}
    u_t + Au = \mathbb{P} \text{div}(G(u) + F(t)) \\
    u(0) = u_0,
\end{cases}
\]  

(1.1)

where \(-A\) generates a \( C_0 \)-semigroup \( (e^{-tA})_{t \geq 0} \) on \( L^d_{\sigma,w}(\Omega) \), \( \mathbb{P} \) is Helmholtz projection; \( G \) is a nonlinear operator acting from \( L^d_{\sigma,w}(\Omega) \) into \( L^{d/2}_{\sigma,w}(\Omega)^d \), and \( F(\cdot) \) is a time-dependent second-order tensor in \( L^{d/2}_{\sigma,w}(\Omega)^d \). Under assumptions on \( L^p - L^q \) smoothing properties of \( (e^{-tA})_{t \geq 0} \) and local Lipschitz properties of \( G \), and using the interpolation techniques (see [23, 12] for the origin of the approach) combined with differential inequalities (see [11]) and fixed point arguments we are able to prove the existence of bounded (in time \( t \)) solutions to (1.1) for each bounded tensor \( F(\cdot) \). Moreover, our methods can be extended to obtain the stability of such bounded solutions. Our main results are contained in Theorems 2.4 and 2.5. We then apply our abstract results to concrete fluid flows and rediscover the existence and stability of bounded motions of Navier-Stokes flows on exterior domains (essentially obtained by Yamazaki [23]) and of Navier-Stokes flows around rotating obstacles (originally obtained in Huy [12]). Moreover, using our abstract results we prove new results on existence and stability of bounded solutions to Navier-Stokes-Oseen equations on an exterior domain (see Theorem 3.6).

2. Semi-linear evolution equations in weak-\( L^d \) Spaces. We start by recalling some notions. Let \( \Omega \) be an exterior domain of \( C^3 \)-class in \( \mathbb{R}^d \) with \( d \geq 3 \). Throughout this paper, the following spaces will be used

\[
\begin{align*}
    C_0^{\infty}_{\sigma,\Omega} &= \{ v \in C_0^{\infty}(\Omega) : \text{div} v = 0 \text{ in } \Omega \}, \\
    L^p_{\sigma}(\Omega) &= \left\{ v \in C_0^{\infty}(\Omega) : \|v\|_{L^p(\Omega)} \right\}.
\end{align*}
\]  

(2.1)

We also need the notion of Lorentz space \( L^{r,q}(\Omega) \), \( (1 < r < \infty, 1 \leq q \leq \infty) \), defined as in [1, 22] and note that \( L^{r,1}(\Omega) = L^r(\Omega) \) and that for \( q = \infty \) the space \( L^{r,\infty}(\Omega) \)
is called the weak-$L^r$ space and is denoted by $L^r_w(\Omega) := L^{r,\infty}(\Omega)$. The following weak Hölder inequality is known (see [2, Lemma 2.1]):

**Lemma 2.1.** Let $1 < p \leq \infty$, $1 < q \leq \infty$ and $1 < r < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $f \in L^p_w, g \in L^q_w$ then $fg \in L^r_w$ and

$$
\|fg\|_{r,w} \leq C \|f\|_{p,w} \|g\|_{q,w}
$$

with constant $C$ depending only on $p$ and $q$. Here, we understand that $L^\infty_w = L^\infty$.

Let $P = P_r$ be the Helmholtz projection on $L^r(\Omega)$. Then, $P$ defines a bounded projection on each $L^{r,q}(\Omega)$ ($1 < r < \infty$, $1 \leq q \leq \infty$) which is also denoted by $P$. We have the following notations of solenoidal Lorentz spaces:

$$
L^{r,q}_\sigma(\Omega) := P(L^{r,q}(\Omega)).
$$

Then we can see (see [2, Thm. 5.2]) that

$$
L^{r,q}(\Omega) = L^{r,q}_\sigma(\Omega) \oplus G^{r,q}(\Omega)
$$

where $G^{r,q} = \{\nabla p \in L^{r,q} : p \in L^{r,q}_{\text{loc}}(\Omega)\}$. We also have

$$
L^{r,q}_\sigma(\Omega) = \{L^{r'_q}_\sigma(\Omega), L^{r'_q}_\sigma(\Omega)\}_{\theta,q}
$$

where $1 < r_0 < r < r_1 < \infty$, $1 \leq q \leq \infty$, $\frac{1}{r} = \frac{1}{r_0} + \frac{q}{r_1}$ and $(\cdot,\cdot)_{\theta,q}$ denotes the real interpolation functor.

Furthermore, if $1 \leq q < \infty$, then

$$
(L^{r,q}_\sigma(\Omega))' = L^{r',q'}(\Omega)
$$

where, as usual $r' = \frac{r}{r-1}$, $q' = \frac{q}{q-1}$ and $q' = \infty$ if $q = 1$.

When $q = \infty$ let $L^{r}_\sigma,w(\Omega) = L^{r,\infty}(\Omega)$ and denote by $\| \cdot \|_{s,w}$ the norm in $L^{r}_\sigma,w(\Omega)$.

We also need the following space of bounded continuous functions on $\mathbb{R}^+ := (0,\infty)$ with values in $L^{r}_\sigma,w(\Omega)$:

$$
C_b(\mathbb{R}^+, L^{r}_\sigma,w(\Omega)) := \{f : \mathbb{R}^+ \to L^{r}_\sigma,w(\Omega) \mid f \text{ is continuous and } \sup_{t \in \mathbb{R}^+} \|f(t)\|_{s,w} < \infty\}
$$

endowed with the norm

$$
\|v\|_{\infty,s,w} := \sup_{t \in \mathbb{R}^+} \|v(t)\|_{s,w} \text{ for } v \in C_b(\mathbb{R}^+, L^{r}_\sigma,w(\Omega)).
$$

In this paper we will consider a linear operator $A$ defined on a subspace of $L^{r}_\sigma,w(\Omega)$, and suppose the following assumption.

**Assumption 2.2.** We suppose that the operator $-A$ and its dual $-A'$ generate bounded $C_0$-semigroups $(e^{-tA})_{t \geq 0}$ and $(e^{-tA'})_{t \geq 0}$ (respectively) satisfying the following $L^p - L^q$ smoothing estimates.

1. For some $r > d$:

$$
\|e^{-tA}x\|_{r,w} \leq Mt^{-\frac{d}{2}(\frac{1}{r} - \frac{1}{2})}\|x\|_{d,w}.
$$

2. For all $1 < p < \frac{d}{d-2}$:

$$
\|\nabla e^{-tA'}x\|_{\frac{d}{d-2},1} \leq Mt^{-\frac{d}{2} - \frac{d}{d-2}}\|x\|_{p,\infty}.
$$

3. For the number $r > d$ appearing in Item (1):

$$
\|\nabla e^{-tA'}x\|_{\frac{d}{r-1},1} \leq Mt^{-\frac{d}{2} + \frac{d}{r-2}}\|x\|_{\frac{r}{r-1},1}.
$$
Now, for a given time-dependent, second-order tensor field $F : \mathbb{R}^+ \to L_w^{d/2}(\Omega)^{d^2}$ we consider the unknown vector field $u = u(t, x)$ on $\mathbb{R}^d$ satisfying
\[
\begin{cases}
  u_t + Au = \nabla \div (G(u) + F(t)) \\
  u|_{t=0} = u_0 \in L^d_{\sigma, u}(\Omega),
\end{cases} \tag{2.8}
\]
where the linear operator $-A$ generates a bounded $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $L^d_{\sigma, u}(\Omega)$, and the nonlinear operator $G : L^d_{\sigma, u}(\Omega) \to L^{d/2}_w(\Omega)^{d^2}$ satisfies
\begin{enumerate}
  \item $G(0) = 0$, and
  \item $\|G(v_1) - G(v_2)\|_{d/2, w} \leq (\kappa + \|v_1\|_{d, w} + \|v_2\|_{d, w})\|v_1 - v_2\|_{d, w}$ \tag{2.9}
\end{enumerate}
for all $v_1, v_2 \in L^d_{\sigma, u}(\Omega)$, where $\kappa \geq 0$ is a constant.

By a \textit{mild solution} to (2.8) we mean a continuous function $u$ defined on $\mathbb{R}^+$ satisfying the following equation
\[
u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A} \nabla \div (G(u) + F(\tau))d\tau \] 
for $t \geq 0$. \tag{2.10} 

We then need the following preparatory lemma for latter use.

\begin{lemma}
Let the operator $-A$ and its dual $-A'$ satisfy Assumption 2.2. Then, the following inequality holds.
\[
\int_0^\infty \|\nabla e^{-tA'}\phi\|_{\frac{d}{p_j}, 1}d\xi \leq \hat{M}\|\phi\|_{\frac{d}{p_j}, 1} \text{ for } \phi \in L^d_{\sigma, 1}(\Omega). \tag{2.11}
\]
\end{lemma}

\begin{proof}
To prove inequality (2.11) we chose real numbers $p_1$ and $p_2$ such that $1 < p_1 < \frac{d}{2-d} < p_2 < \frac{d}{d-2}$. Then, we consider the sublinear operator $T$ which maps a function $\phi \in L^d_{\sigma, u}(\Omega) + L^d_{\sigma, u}(\Omega)$ to a function $v(\cdot)$ defined on $(0, \infty)$ by $v(t) = \|\nabla e^{-tA'}\phi\|_{\frac{d}{p_j}, 1}$ for $t > 0$. By (2.6) we have
\[
u(t) \leq M t^{-\frac{d}{2} - \frac{d}{2p_1} + \frac{d}{p_2} - \frac{d}{2}} \|\phi\|_{p_j, \infty} = Mt^{\frac{d}{p_2} - \frac{d}{p_j}} \|\phi\|_{p_j, \infty} \text{ for } j = 1, 2. \tag{2.12}
\]

Setting now $\frac{1}{s_j} := \frac{1}{2} + \frac{d}{2} \left(\frac{1}{p_j} - \frac{d}{2} - \frac{d}{2} \right)$ we obtain $v(\cdot) \in L_{\sigma}^{s_j}(0, \infty)$ and $\|v\|_{s_j, u} \leq C_j \|\phi\|_{p_j, u}$ for $j = 1, 2$. Moreover, the constant $\theta \in (0, 1)$ such that $\frac{d-1}{\theta} = \frac{1}{p_j} + \frac{\theta}{2}$ also satisfies the equality $1 = \frac{1-\theta}{\theta} + \frac{\theta}{2}$. Therefore, we have the real interpolation relations
\[
(L^d_{\sigma, u}(\Omega), L^d_{\sigma, u}(\Omega))_{\theta, 1}^{s_j} = L^d_{\sigma} \tag{2.11}
\]
and
\[
(L^d_{\sigma}((0, \infty)), L^d_{\sigma}((0, \infty)))_{\theta, 1} = L^1((0, \infty)).
\]
Thus, applying real interpolation theorem for the operator $T$ (see e.g., [1, §5.3], [15]) we obtain that $v \in L^1((0, \infty))$ and $\|v\|_{L^1} \leq \hat{M}\|\phi\|_{\frac{d}{p_j}, 1}$ for $\phi \in L^d_{\sigma} \tag{2.11}$ yielding (2.11).
\end{proof}

We now state and prove a theorem on the existence and uniqueness of the mild solution of (2.8) which is bounded on $\mathbb{R}^+$ as follows.

\begin{theorem}
Let $F \in C_b(\mathbb{R}^+, L_w^{d/2}(\Omega)^{d^2})$. Suppose that $G : L^d_{\sigma, u}(\Omega) \to L^{d/2}_w(\Omega)^{d^2}$ satisfies conditions in (2.9), $-A$ satisfies Assumption 2.2, and $u_0 \in L^d_{\sigma, u}(\Omega)$.

Then, if $\kappa, \|u_0\|_{d, w}, \|F\|_{\infty, \frac{d}{2}, w}$ and $\rho$ are small enough, the problem (2.8) has a unique mild solution $\bar{u}$ in the ball $B_{\rho} := \{v \in C_b(\mathbb{R}^+, L^d_{\sigma, u}(\Omega)) : \|v\|_{\infty, d, w} \leq \rho\}$.
\end{theorem}
Proof. Firstly, for \( u_0 \in L_{\sigma,w}^d(\Omega) \) we prove that the function \( u \) defined by

\[
u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A} \text{div} F(\tau) d\tau \tag{2.13}\]

belongs to \( C_b(\mathbb{R}^+, L_{\sigma,w}^d(\Omega)) \) and satisfies

\[
\|u\|_{\infty,d,w} \leq M\|u_0\|_{d,w} + \tilde{M}\|F\|_{\infty,\frac{d}{2},w} \tag{2.14}
\]

for positive constants \( M \) and \( \tilde{M} \) independent of \( u_0 \) and \( F \).

Indeed, for each \( \phi \in L_{\sigma,w}^{d+1}(\Omega) \) using the boundedness of \( (e^{-tA})_{t \geq 0} \) and (2.11) we have

\[
|\langle u(t), \phi \rangle| \leq |\langle e^{-tA}u_0, \phi \rangle| + \int_0^t |\langle e^{-(t-\tau)A} \text{div} F(\tau), \phi \rangle| d\tau \\
\leq \|e^{-tA}u_0\|_{d,w}\|\phi\|_{\frac{d}{2},1} + \int_0^t |\langle -F(\tau), \nabla e^{-(t-\tau)A}\phi \rangle| d\tau \\
\leq \|e^{-tA}u_0\|_{d,w}\|\phi\|_{\frac{d}{2},1} + \|F\|_{\infty,\frac{d}{2},w} \int_0^t \|\nabla e^{-(t-\tau)A}\phi\|_{\frac{d}{2},1} d\tau \\
\leq M\|u_0\|_{d,w}\|\phi\|_{\frac{d}{2},1} + \tilde{M}\|F\|_{\infty,\frac{d}{2},w}\|\phi\|_{\frac{d}{2},1}. \tag{2.15}
\]

This implies that

\[
\|u\|_{\infty,d,w} \leq M\|u_0\|_{d,w} + \tilde{M}\|F\|_{\infty,\frac{d}{2},w}. \tag{2.16}
\]

Clearly, \( u(t) \) is a continuous function with respect to \( t \). Therefore, we obtain that \( u \in C_b(\mathbb{R}^+, L_{\sigma,w}^d(\Omega)) \) and (2.14) holds true.

Then, to prove the assertion of the theorem we define the transformation \( \Phi \) as follows: For \( v \in B_\rho \) we set \( \Phi(v) = u \) where \( u \in C_b(\mathbb{R}^+, L_{\sigma,w}^d(\Omega)) \) is given by

\[
u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A} \text{div}(G(v) + F(\tau)) d\tau. \tag{2.17}\]

Next, applying (2.14) for \( G(v) + F \) instead of \( F \) we obtain

\[
\|u\|_{\infty,d,w} \leq M\|u_0\|_{d,w} + \tilde{M}\|G(v)\|_{\infty,\frac{d}{2},w} \\
\leq M\|u_0\|_{d,w} + \tilde{M}\left(\|F\|_{\infty,\frac{d}{2},w} + \|G(v)\|_{\infty,\frac{d}{2},w}\right) \\
\leq M\|u_0\|_{d,w} + \tilde{M}\left(\|F\|_{\infty,\frac{d}{2},w} + (\kappa + \|v\|_{d,w})\|v\|_{d,w}\right) \\
\leq M\|u_0\|_{d,w} + \tilde{M}\left(\|F\|_{\infty,\frac{d}{2},w} + \kappa \rho + \rho^2\right) < \rho. \tag{2.18}\]

Therefore, for sufficiently small \( \|u_0\|_{d,w},\|F\|_{\infty,\frac{d}{2}} \) and \( \rho \), the transformation \( \Phi \) acts from \( B_\rho \) into itself. Moreover, the map \( \Phi \) can be expressed as

\[
\Phi(v)(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} \text{div}(G(v) + F(s)) ds.
\]

Therefore, for \( v_1, v_2 \in B_\rho \) we obtain that the difference \( \Phi(v_1) - \Phi(v_2) \) satisfies

\[
(\Phi(v_1) - \Phi(v_2))(t) = \int_0^t e^{-(t-s)A} \text{div}(G(v_1) - G(v_2)) ds.
\]
Applying again (2.14) we arrive at
\[ \|\Phi(v_1) - \Phi(v_2)\|_{\infty,d,w} \leq \tilde{M}(\|G(v_1) - G(v_2)\|_{\infty,d,w} + \|v_1\|_{\infty,d,w} + \|v_2\|_{\infty,d,w}) \leq \tilde{M}(\kappa + 2\rho)\|v_1 - v_2\|_{\infty,d,w}. \] (2.19)

Hence, if \( \kappa \) and \( \rho \) are sufficiently small the map \( \Phi \) is a contraction. Then, there exists a unique fixed point \( \hat{u} \) of \( \Phi \). By definition of \( \Phi \), the function \( \hat{u} \) is the unique mild solution to (2.8) and the proof is complete.

We then show the polynomial stability of the bounded solutions to (2.8) in the following theorem.

**Theorem 2.5.** Under the conditions of Theorem 2.4 we consider Equation (2.8) on an exterior domain \( \Omega \subset \mathbb{R}^d \) \( (d \geq 3) \) with a \( C^3 \)-boundary. For the number \( r > d \) appearing in Assumption 2.2 we suppose that
(a) for all \( 1 < p < \frac{dr-d}{d-r-d} \):
\[ \|\nabla e^{-tA^*}x\|_{\frac{dr-d}{d-r-d},1} \leq M t^{-\frac{d-r}{2}} \|x\|_{p,\infty} \text{ for all } x \in L^d_{\sigma,w}(\Omega), \] (2.20)
(b) \( G \) satisfies
\[ \|G(v_1) - G(v_2)\|_{\frac{dr-d}{d-r-d},w} \leq (\kappa + \|v_1\|_{d,w} + \|v_2\|_{d,w})\|v_1 - v_2\|_{r,w} \]
for \( v_1, v_2 \in L^d_{\sigma,w}(\Omega) \cap L^r_{\sigma,w}(\Omega) \) (2.21)
with a small constant \( \kappa \geq 0 \).

Then, the small solution \( \hat{u} \) of (2.8) is stable in the sense that for any other solution \( u \in C_b(\mathbb{R}_+, L^d_{\sigma,w}(\Omega)) \) of (2.8) such that \( \|u(0) - \hat{u}(0)\|_{d,w} \) is small enough we have
\[ \|u(t) - \hat{u}(t)\|_{r,w} \leq \frac{C}{t^{\frac{r}{2}} - \frac{d}{p}} \text{ for all } t > 0, \] (2.22)
with the number \( r > d \) as in (2.20).

**Proof.** We first note that, in a same way as in the proof of Lemma 2.3 it can be proven that
\[ \int_0^\infty \|\nabla e^{-tA^*}\phi\|_{\frac{dr-d}{d-r-d},1} d\xi \leq M_1 \|\phi\|_{\frac{dr-d}{d-r-d},1} \text{ for } \phi \in L^{\sigma-1}_{\sigma}(\Omega). \] (2.23)

Putting \( v = u - \hat{u} \) we obtain that \( v \) satisfies the equation
\[ v(t) = e^{-tA}v(0) + \int_0^t e^{-(t-\tau)A}\text{div}(G(u) - G(\hat{u}))d\tau. \] (2.24)

For the \( r > d \) as in (2.20) we set
\[ \mathbb{M} := \{ v \in C_b(\mathbb{R}_+, L^d_{\sigma,w}(\Omega)) : \sup_{t \in \mathbb{R}_+} t^{\frac{r}{2} - \frac{d}{p}}\|v(t)\|_{r,w} < \infty \} \]
edowed with the norm \( \|v\|_{\mathbb{M}} = \|v\|_{\infty,d,w} + \sup_{t \in \mathbb{R}_+} t^{\frac{r}{2} - \frac{d}{p}}\|v(t)\|_{r,w} \). Then, we will prove that if \( \|v(0)\|_{d,w}, \|\hat{u}\|_{\infty,d,w} \) are small enough, then (2.24) has a unique solution in a small ball of \( \mathbb{M} \).

Indeed, for \( v \in \mathbb{M} \) we consider the mapping \( \Phi \) defined formally by
\[ \Phi(v)(t) := e^{-tA}v(0) + \int_0^t e^{-(t-\tau)A}\text{div}(G(v + \hat{u}) - G(\hat{u})). \]
Let $B_\rho$ be a ball in $\mathbb{M}$ of radius $\rho$ and centered at 0. We then prove that if $\|v(0)\|_{d,w}$ and $\rho$ are small enough, the transformation $\Phi$ acts from $B_\rho$ to itself and is a contraction. For $v \in \mathbb{M}$, arguing similarly as in the proof of theorem 2.4 we obtain $\Phi(v) \in C_b(\mathbb{R}_+, L^d_{r,w}(\Omega))$. Furthermore

$$t^{1/2} - \mathcal{F} \Phi(v)(t) = t^{1/2} - \mathcal{F} e^{-tA}v(0) + t^{1/2} \int_0^t e^{-\tau A} \text{div}(G(v + \hat{u}) - G(\hat{u})).$$

By the estimates in (2.5) for semigroup $(e^{-tA})_{t \geq 0}$ we obtain that

$$\left\| t^{1/2} - \mathcal{F} e^{-tA}v(0) \right\|_{r,w} \leq M \|v(0)\|_{d,w}.$$

We now estimate $\int_0^t e^{-(t-\tau)A} \text{div}(G(v + \hat{u}) - G(\hat{u})) = \int_0^t e^{-\xi A} \text{div}(G(v(t - \xi) + \hat{u}(t - \xi)) - G(\hat{u}(t - \xi)))$. For each $\phi \in C^\infty_0$, we have

$$\left\| \int_0^t e^{-\xi A} \text{div}(G(v(t - \xi) + \hat{u}(t - \xi)) - G(\hat{u}(t - \xi))) d\xi, \phi \right\|$$

$$\leq \int_0^t \left\| \langle -(G(v(t - \xi) + \hat{u}(t - \xi)) - G(\hat{u}(t - \xi)), \nabla e^{-\xi A} \phi \rangle \right\| d\xi$$

$$= \int_0^{t/2} \left\| \langle -(G(v(t - \xi) + \hat{u}(t - \xi)) - G(\hat{u}(t - \xi)), \nabla e^{-\xi A} \phi \rangle \right\| d\xi$$

$$+ \int_{t/2}^t \left\| \langle -(G(v(t - \xi) + \hat{u}(t - \xi)) - G(\hat{u}(t - \xi)), \nabla e^{-\xi A} \phi \rangle \right\| d\xi. \quad (2.25)$$

We then estimate the two integrals on the last line of (2.25). Using (2.21) and (2.23) for the first integral we have

$$\int_0^{t/2} \left\| \langle -(G(v(t - \xi) + \hat{u}(t - \xi)) - G(\hat{u}(t - \xi)), \nabla e^{-\xi A} \phi \rangle \right\| d\xi$$

$$\leq \int_0^{t/2} \left\| G(v(t - \xi) + \hat{u}(t - \xi)) - G(\hat{u}(t - \xi)) \right\|_{d\xi} \left\| \nabla e^{-\xi A} \phi \right\|_{\frac{1}{d\xi}} d\xi$$

$$\leq (t/2)^{-\frac{1}{2} + \frac{d}{2}} \left( \kappa + \|v\|_M + 2\|\hat{u}\|_{\infty,d,w} \|v\|_M \right) \int_0^{t/2} \left\| \nabla e^{-\xi A} \phi \right\|_{\frac{1}{d\xi}} d\xi$$

$$\leq M_1 (t/2)^{-\frac{1}{2} + \frac{d}{2}} \left( \kappa + \|v\|_M + 2\|\hat{u}\|_{\infty,d,w} \|v\|_M \right) \|\phi\|_{\frac{1}{r - 1}}. \quad (2.26)$$

We next estimate the second integral in the last line of (2.25). In fact, using (2.7) and (2.9) we have

$$\int_{t/2}^t \left\| \langle -(G(v(t - \xi) + \hat{u}(t - \xi)) - G(\hat{u}(t - \xi)), \nabla e^{-\xi A} \phi \rangle \right\| d\xi$$

$$\leq \int_{t/2}^t \left\| G(v(t - \xi) + \hat{u}(t - \xi)) - G(\hat{u}(t - \xi)) \right\|_{d\xi} \left\| \nabla e^{-\xi A} \phi \right\|_{\frac{1}{d\xi}} d\xi$$

$$\leq M (\kappa + \|v\|_M + 2\|\hat{u}\|_{\infty,d,w} \|v\|_M) \int_{t/2}^\infty \xi^{-\frac{1}{2} + \frac{d}{2}} \|\phi\|_{\frac{1}{r - 1}} d\xi$$

$$\leq M \left( \frac{t}{2} \right)^{-\frac{1}{2} + \frac{d}{2}} \left( \kappa + \|v\|_M + 2\|\hat{u}\|_{\infty,d,w} \|v\|_M \right) \|\phi\|_{\frac{1}{r - 1}}. \quad (2.27)$$
Combining now (2.25), (2.26) and (2.27) we obtain
\[
\left\| \int_0^t e^{-\xi A} \text{div}(G(v(t - \xi) + \hat{u}(t - \xi)) - G(\hat{u}(t - \xi)))d\xi \right\|_{r,w} \\
\leq \tilde{M}(\frac{t}{2})^{-\frac{1}{2}} \frac{2^d}{\nu_0} (\kappa + \|v\|_M + 2\|\hat{u}\|_{\infty,d,w})\|v\|_M \|\phi\|_{\frac{2}{d},1}
\]
for all \( t > 0 \) implying that
\[
\|\Phi(v)\|_M \leq M\|v(0)\|_{d,w} + \tilde{M}(\kappa + \|v\|_M + 2\|\hat{u}\|_{\infty,d,w})\|v\|_M.
\]
Similar calculations yield
\[
\|\Phi(v_1) - \Phi(v_2)\|_M \leq \tilde{M}(2\kappa + \|v_1\|_M + \|v_2\|_M + 4\|\hat{u}\|_{\infty,d,w})\|v_1 - v_2\|_M.
\]
Therefore, if \( \|v(0)\|_{d,w}, \|\hat{u}\|_{\infty,d,w} \) and \( \rho \) are small enough, then the transformation \( \Phi \) acts from \( B_\rho \) into itself and is a contraction. As the fixed point of \( \Phi \), the function \( v = u - \hat{u} \) belongs to \( \tilde{M} \). Inequality (2.22) hence follows, and we obtain the stability of the small solution \( \hat{u} \). The proof is complete. \( \square \)

3. Applications to fluid flows. In this section we will apply the abstract results obtained in the previous section to study the fluid flows on exterior domains. Concretely, we shall investigate the boundedness and stability of motions of Navier-Stokes and Navier-Oseen-Stokes flows on exterior domains and Navier-Stokes flows around rotating obstacles.

3.1. Navier-Stokes equations on exterior domains. We start with Navier-Stokes equations on an exterior domain \( \Omega \) with \( C^3 \)-boundary \( \partial \Omega \)
\[
\left\{ \begin{array}{c}
u_t + (u \cdot \nabla)u - \Delta u + \nabla p = \text{div} F \\
\nabla \cdot u = 0 \\
u|_{t=0} = u_0
\end{array} \right. \text{ in } \Omega \times (0, \infty), \quad \nabla \cdot u = 0 \quad \text{in } \Omega \times (0, \infty), \quad u|_{t=0} = u_0 \quad \text{in } \Omega. \tag{3.1}
\]

Applying Helmholtz projection \( P \) to the problem (3.1) and using the fact that \( (u \cdot \nabla)u = \text{div}(u \otimes u) \) for \( u \in L^d_{\sigma,w}(\Omega) \) we obtain the following operator version of Navier-Stokes equations
\[
\left\{ \begin{array}{c}
u_t + Au = \text{Pdiv}(-u \otimes u) + \text{Pdiv} F \\
u|_{t=0} = u_0 \in L^d_{\sigma,w}(\Omega). \tag{3.2}
\end{array} \right.
\]

We can see that this equation is a concrete form of (2.8) with concrete Stokes operator \( A = -P \Delta \) and operator \( G \) defined formally by \( G(u) = -u \otimes u \). It is evident that \( G : L^d_{\sigma,w}(\Omega) \rightarrow L^{d/2}_{\sigma,w}(\Omega)^d \) and \( G(0) = 0 \). We then check the condition (2) given in (2.9). Indeed, for \( u, v \in L^d_{\sigma,w}(\Omega) \) by the weak Hölder inequality (2.2) we have that \( u \otimes v \in L^{d/2}_{\sigma,w}(\Omega)^d \) and \( \|u \otimes v\|_{\frac{d}{2},w} \leq \|u\|_{d,w}\|v\|_{d,w} \). Therefore, for \( v_1, v_2 \in L^d_{\sigma,w}(\Omega) \) we have
\[
\|G(v_1) - G(v_2)\|_{\frac{d}{2},w} = \|v_1 \otimes v_1 - v_2 \otimes v_2\|_{\frac{d}{2},w} = \|(v_1 - v_2) \otimes v_1 + v_2 \otimes (v_1 - v_2)\|_{\frac{d}{2},w} \\
\leq \|(v_1 - v_2)\|_{d,w}\|v_1\|_{d,w}\|v_2\|_{d,w}. \tag{3.3}
\]
Therefore, we obtain that the operator \( G \) satisfies the conditions given in (2.9) with \( \kappa = 0 \). Moreover, the weak Hölder inequality (2.2) implies also that \( G \) satisfies condition given in (2.21).
From the well-known $L^{r,q} - L^{p,q}$ smoothing properties of the Stokes semigroup on an exterior domain with $C^3$-boundary (see [23, Thm. 2.2] and [2, Prop. 6.6]), we can see that Assumption 2.2 and Inequality (2.20) are fulfilled for any number $r > d$. Therefore, an application of Theorems 2.4 and 2.5 yields the following result on existence, uniqueness and stability of bounded solutions to Navier-Stokes equations on exterior domains.

**Theorem 3.1.** Consider Equation (3.2) on an exterior domain $\Omega \subset \mathbb{R}^d$ ($d \geq 3$) with a $C^3$-boundary. Suppose that $F \in C_b(\mathbb{R}_+, L^d_{w}(\Omega)^{d^2})$. The following assertions hold true.

1. If $\|u_0\|_{d,w}, \|F\|_{\infty, \frac{d}{2}, w}$ and $\rho$ are small enough, then Equation (3.2) has a unique mild solution $\hat{u}$ in the ball $B_\rho$ of $C_b(\mathbb{R}_+, L^d_{\sigma,w}(\Omega))$.

2. The above solution $\hat{u}$ is stable in the sense of Theorem 2.5 in which the number $r > d$ can be chosen arbitrarily on the interval $(d, \infty)$.

We would like to note that, using interpolation techniques and Kato’s iteration, Yamazaki [23] has proved the existence and stability of bounded mild solutions as well as periodic mild solutions in weak sense. Here in the above theorem we obtain the existence and stability of bounded mild solutions in strong sense. We are inspired by the above-mentioned work of Yamazaki. In fact, the interpolation techniques used to obtain (2.11) and (2.23) are similar to some techniques in the work of Yamazaki [23].

### 3.2. Navier-Stokes flows around rotating obstacles.

In this subsection we consider the following Navier-Stokes flows around rotating obstacles in $\mathbb{R}^3$:

\[
\begin{cases}
  u_t + (u \cdot \nabla) u - \Delta u + \nabla p = (\zeta \times x) \cdot \nabla u - \zeta \times u + \text{div} F & \text{in } \Omega \times (0, \infty), \\
  \nabla \cdot u = 0 & \text{in } \Omega \times (0, \infty), \\
  u = \zeta \times x & \text{on } \partial \Omega \times (0, \infty), \\
  u|_{t=0} = u_0 & \text{in } \Omega,
\end{cases}
\]

(3.4)

where $\zeta = (0, 0, a)$ is a constant vector representing angular velocity of the rotation around $x_3$-axis of an obstacle $D \subset \mathbb{R}^3$ with complement $\Omega = \mathbb{R}^3 \setminus D$.

To handle with the solutions to Problem (3.4) we first consider the stationary problem

\[
\begin{cases}
  (\nu \cdot \nabla) \nu - \Delta \nu + \nabla p = (\zeta \times x) \cdot \nabla \nu - \zeta \times \nu & \text{in } \Omega \times (0, \infty), \\
  \nabla \cdot \nu = 0 & \text{in } \Omega, \\
  \nu = \zeta \times x & \text{on } \partial \Omega, \\
  \lim_{|x| \to \infty} \nu(x) = 0.
\end{cases}
\]

(3.5)

By [4] we have that for small $|\zeta|$ there exists a unique small solution $\nu$ of (3.5) satisfying

\[ \|\nu\|_{3,w} \leq C|\zeta|. \]
Next, by setting \( u = z + \nu \) where \( \nu \) is the above-mentioned solution of (3.5), we obtain that \( u \) satisfies (3.4) if and only if \( z \) satisfies the following equation:

\[
\begin{cases}
z_t - \Delta z - (\zeta \times x) \cdot \nabla z + \zeta \times z + (z \cdot \nabla)z + \nu \cdot \nabla z + \nabla p = \text{div} F & \text{in } \Omega \times (0, \infty), \\
\nabla \cdot z = 0 & \text{in } \Omega \times (0, \infty), \\
z|_{t=0} = u_0 - \nu & \text{in } \Omega,
\end{cases}
\]

\[ z|_{t=0} = u_0 - \nu \quad \text{in } \Omega. \]

We then define the operator \( \mathcal{L} \) by

\[
D(\mathcal{L}) := \{ u \in L^r_\sigma(\Omega) \cap W^{2,r}(\Omega) : u|_{\partial \Omega} = 0 \text{ and } (\zeta \times x) \cdot \nabla u \in L^r(\Omega) \}
\]

\[
\mathcal{L}u := -\mathbb{P}[\Delta u + (\zeta \times x) \cdot \nabla u - \zeta \times u] \quad \text{for } u \in D(\mathcal{L}).
\]

It is known that \(-\mathcal{L}\) is a generator of a bounded \( C_0 \)-semigroup \((e^{-t\mathcal{L}})_{t \geq 0}\) on \( L^r_\sigma(\Omega) \) for each \( 1 < r < \infty \) (although not analytic) (see [8]). Then, \((e^{-t\mathcal{L}})_{t \geq 0}\) is extended to the bounded \( C_0 \)-semigroup on the space \( L^{r,q}_\sigma(\Omega) \) using interpolation relations. Applying the Helmholtz projection \( \mathbb{P} \) to this equation we obtain the following operator form

\[
\begin{cases}
z_t + \mathcal{L}z = \mathbb{P}\text{div}(-z \otimes z - \nu \otimes z - z \otimes \nu + F) & \\
z|_{t=0} = z_0 \in L^3_{3,\nu}(\Omega),
\end{cases}
\]

where \( z_0 = u_0 - \nu \).

We again obtain a concrete form of (2.8) with concrete operators \( \mathcal{L} \) defined as above and \( G \) defined as \( G(z) := -z \otimes z - \nu \otimes z - z \otimes \nu \) which is the same operator as in the previous Subsection 3.2. Similarly as in Subsection 3.2 we can check that the operator \( G \) satisfies the conditions given in (2.9) with \( \kappa = 2C|\zeta| \). Moreover, the weak Hölder inequality (2.2) implies that \( G \) satisfies conditions given in (2.21).

Since the \( L^{r,q} - L^{p,q} \) estimates are valid for \((e^{-t\mathcal{L}})_{t \in \mathbb{R}^+}\) (see [10, Thm. 2.3], [12, Prop. 1.2]), applying Theorem 2.5 we obtain the following results on the existence, uniqueness and stability of bounded Navier-Stokes flows around rotating obstacles.

**Theorem 3.2.** Let \( \Omega \subset \mathbb{R}^3 \) be an exterior domain of class \( C^3 \). Suppose that \( \zeta \) is constant vector in \( \mathbb{R}^3 \) and \( F \in C_b(\mathbb{R}_+, L^{3/2}_{3,\nu}(\Omega)^{3 \times 3}) \). Then, the following assertions hold true.

1. If \( \|u_0\|_{3,\nu}, |\zeta|, \|F\|_{\infty, \frac{3}{2}, \nu} \) and \( \rho \) are small enough, then Equation (3.2) has a unique mild solution \( \hat{u} \) in the ball \( B_p \) of \( C_b(\mathbb{R}_+, L^3_{3,\nu}(\Omega)) \).
2. The above solution \( \hat{u} \) is stable in the sense of Theorem 2.5 in which the number \( r > 3 \) can be chosen arbitrarily.

**Remark 3.3.** The results in the above theorem have essentially been obtained in [12]. In fact, in present paper we generalize the methods in [12] and make them available to apply to other problems of fluid dynamic for which such \( L^p - L^q \) smoothing properties as in Assumption 2.2 hold.

### 3.3. Navier-Stokes-Oseen equations.

In this subsection we prove a new result on existence and polynomial stability of bounded solutions to Navier-Stokes-Oseen equations on an exterior domain \( \Omega \subset \mathbb{R}^3 \).
with $C^3$-boundary:

$$
\begin{align*}
\begin{cases}
  u_t + (u \cdot \nabla)u - \Delta u + \nabla p &= \text{div} F & \text{in } \Omega \times (0, \infty), \\
  \nabla \cdot u &= 0 & \text{in } \Omega \times (0, \infty), \\
  u &= 0 & \text{on } \partial \Omega \times (0, \infty), \\
  u|_{t=0} &= u_0 & \text{in } \Omega, \\
  \lim_{|x| \to \infty} u(t, x) &= u_\infty. 
\end{cases}
\end{align*}
$$

(3.10)

The reason that we can consider these equations only on a sub-domain $\Omega$ of $\mathbb{R}^3$ is that the $L^r - L^q$ estimates for Oseen operator (see Proposition 3.5) are available only for domains in $\mathbb{R}^3$.

To look for the bounded solutions to Navier-Oseen-Stokes equations (3.13) we next solve the following stationary equation in Lorentz spaces:

$$
\begin{align*}
\begin{cases}
  (\nu \cdot \nabla)\nu - \Delta \nu + (u_\infty \cdot \nabla)\nu + \nabla p &= 0 & \text{in } \Omega \times (0, \infty), \\
  \nabla \cdot \nu &= 0 & \text{in } \Omega, \\
  \nu &= -u_\infty & \text{on } \partial \Omega, \\
  \nu(x) &= 0. 
\end{cases}
\end{align*}
$$

(3.11)

To this purpose, throughout this subsection, we suppose that $0 \not\in \Omega$. Then, we have the following theorem on existence and uniqueness of the solutions to (3.11) which can be proved in the same way as in [2, Thm. 2.3] or [5].

**Theorem 3.4.** Let $\Omega \subset \mathbb{R}^3$ be an exterior domain of class $C^3$ and let $|u_\infty|$ be small enough. Then, Equation (3.11) has a unique solution $\nu$ such that $|\nu(x)| \leq \frac{C}{|x|^s}$, $|\nabla \nu| \leq \frac{C}{|x|^t}$. In particular, this solution $\nu \in L^3_{3,w}$ satisfies

$$
\|\nu\|_{3,w} \leq \tilde{C}|u_\infty|. 
$$

(3.12)

**Proof.** The proof is a minor modification of [2, Thm. 2.3] (see also [3]).

To handle with the solutions to Problem (3.10) we set $u = z + \nu + u_\infty$ where $\nu$ is a solution to the stationary problem (3.11).

Then, $u$ satisfies (3.10) if and only if $z$ satisfies the following equation known as Navier-Oseen-Stokes equation:

$$
\begin{align*}
\begin{cases}
  z_t - \Delta z + (u_\infty \cdot \nabla)z + (z \cdot \nabla)z + \\
  + (\nu \cdot \nabla)z + (z \cdot \nabla)\nu + \nabla p &= \text{div} F & \text{in } \Omega \times (0, \infty), \\
  \nabla \cdot z &= 0 & \text{in } \Omega \times (0, \infty), \\
  z &= 0 & \text{on } \partial \Omega \times (0, \infty), \\
  z|_{t=0} &= u_0 - \nu - u_\infty & \text{in } \Omega, \\
  \lim_{|x| \to \infty} z &= 0. 
\end{cases}
\end{align*}
$$

(3.13)

Let now $A$ be Stokes operator defined as in the previous subsection. Then, for a constant vector $u_\infty \in \mathbb{R}^3$ we denote by $\mathcal{O} := A + \mathcal{P}(u_\infty \cdot \nabla)$ called the Oseen operator with the same domain of definition as that of $A$. It is known that $-\mathcal{O}$ is generates a bounded analytic semigroup $(e^{-t\mathcal{O}})_{t \geq 0}$ called Oseen semigroup on $L^r_{\sigma}(\Omega)$ for each $1 < r < \infty$ (see [14, 20]). We note that the analyticity of the Oseen semigroup was proved in [20]. More than ten years later, the boundedness of such a semigroup, which was much more difficult to be proved, was obtained in [14] under the smallness of $u_\infty$. Then, $(e^{-t\mathcal{O}})_{t \geq 0}$ is extended to the bounded analytic semigroup on the space $L^\infty_{\sigma}(\Omega)$ using interpolation relations.
Moreover, the $L^r - L^p$ smoothing properties of the Oseen semigroup (see [14, Thm. 1.2]) and the interpolating of Lorentz spaces yield the following $L^{r,q} - L^{p,q}$ smoothing properties of Oseen semigroup.

**Proposition 3.5.** Let $(e^{-tO})_{t \geq 0}$ be the Oseen semigroup. Then, we have

1. for $r > 3$:
   \[
   \|e^{-tO}x\|_{r,w} \leq M t^{-\frac{1}{2} \left( \frac{1}{r} - \frac{1}{3} \right)} \|x\|_{3,w},
   \]
   (3.14)

2. for $1 < p < q \leq 3$:
   \[
   \|\nabla e^{-tO'}x\|_{q,1} \leq M t^{-\frac{1}{2} \left( \frac{1}{p} - \frac{1}{3} \right)} \|x\|_{p,1}
   \]
   \[
   \|\nabla e^{-tO'}x\|_{q,1} \leq M t^{-\frac{1}{2} \left( \frac{1}{q} - \frac{1}{3} \right)} \|x\|_{p,\infty}.
   \]
   (3.15)

**Proof.** Since the $L^p - L^q$ smoothing properties are valid for the couple $L^p_\sigma(\Omega)$ and $L^q_\sigma(\Omega)$, i.e., $\|e^{-tO}x\|_r \leq M t^{-\frac{1}{2} \left( \frac{1}{r} - \frac{1}{3} \right)} \|x\|_3$ (see [14, Thm. 1.2]), using the following interpolation relations for the operator $e^{-tO}$:

\[
(L^p_\sigma(\Omega), L^q_\sigma(\Omega))_{\theta,s} = L^{p,s}_\sigma(\Omega)
\]

\[
(L^q_\sigma(\Omega), L^p_\sigma(\Omega))_{\theta,s} = L^{p,s}_\sigma(\Omega)
\]

(with $1 < p_1 < p < p_2 < \infty$, $1 < q_1 < q < q_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{\theta}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{\theta}{q_2}$, $p < q$, $p_1 < q_1$, $i = 1, 2$, and $1 \leq s \leq \infty$) we obtain (3.14). To prove the the first inequality in (3.15) we note that, since the second interpolation relations in (3.16) and (3.17) are trivial, we can fix there the index $q$ in the range $q \leq 3$ for each space in the interpolation couple of Banach spaces.

Moreover, the $L^r - L^p$ smoothing properties of the Oseen semigroup (see [14, Thm. 1.2]) and the interpolating of Lorentz spaces yield the following $L^{r,q} - L^{p,q}$ smoothing properties of Oseen semigroup.

Here, we emphasize that the inequality holds true also in the case $q = 3$.

To prove the second inequality in (3.15) we just use the interpolation relations (with $\frac{1}{p} = \frac{1}{p_1} + \frac{\theta}{p_2}$ and $1 < p_1 < p < p_2 < q \leq 3$):

\[
(L^p_\sigma(\Omega), L^{p,1}_\sigma(\Omega))_{\theta,1} = L^{p,1}_\sigma(\Omega)
\]

\[
(L^{q,1}_\sigma(\Omega), L^{q,1}_\sigma(\Omega))_{\theta,1} = L^{q,1}_\sigma(\Omega)
\]

(3.16)

for the operator $\nabla e^{-tO'}$ yields the first inequality in (3.15). Note that, as already mentioned above, the case $q = 3$ is also included.

To prove the second inequality in (3.15) we just use the interpolation relations (with $\frac{1}{p} = \frac{1}{p_1} + \frac{\theta}{p_2}$ and $1 < p_1 < p < p_2 < q \leq 3$)

\[
(L^p_\sigma(\Omega), L^{p,1}_\sigma(\Omega))_{\theta,\infty} = L^{p,\infty}_\sigma(\Omega)
\]

\[
(L^{q,1}_\sigma(\Omega), L^{q,1}_\sigma(\Omega))_{\theta,\infty} = L^{q,1}_\sigma(\Omega)
\]

(3.17)

for the operator $T = \nabla e^{-tO'}$.

Here, we note that, since the second interpolation relations in (3.16) and (3.17) are trivial, we can fix there the index $q$ in the range $q \leq 3$ for each space in the interpolation couple of Banach spaces.
Applying Helmholtz projection \( \mathbb{P} \) to the problem (3.13) and using the fact that 
\((v \cdot \nabla)u = \text{div}(v \otimes u)\) for \(v\) satisfying \(\text{div}v = 0\), we obtain the following operator version of Navier-Oseen-Stokes equations
\[
\begin{aligned}
\dot{z}_t + \nabla \cdot z &= \mathbb{P} \text{div}(-z \otimes z + v \otimes z - z \otimes \nu + F) \\
\dot{z}|_{t=0} &= z_0 \in L^3_{\sigma,w}(\Omega),
\end{aligned}
\]
where \(z_0 := u_0 - \nu - u_{\infty}\).

Theorems 2.4 and 2.5 yield the following result on existence, uniqueness and stability of bounded solutions to Navier-Oseen-Stokes equations on exterior domains.

**Theorem 3.6.** Let \(u_\infty\) be a constant vector in \(\mathbb{R}^3\) and \(F \in C_b(\mathbb{R}_+, L^{3/2}_w(\Omega)^{3 \times 3})\) and consider Equation (3.18) on an exterior domain \(\Omega \subset \mathbb{R}^3\) with a \(C^2\)-boundary. Then, the following assertions hold true.

1. If \(\|u_0\|_{d,w}, |u_\infty|, \|F\|_{\infty, \frac{3}{2}, w}\) and \(\rho\) are small enough, then Equation (3.18) has a unique mild solution \(\dot{u}\) in the ball \(B_\rho\) of \(C_b(\mathbb{R}_+, L^3_{\sigma,w}(\Omega))\).

2. The above solution \(\dot{u}\) is stable in the sense of Theorem 2.5 in which the number \(r > 3\) can be chosen arbitrarily on the interval \((3, \infty)\).

**Proof.** Equation (3.18) is again a concrete form of Equation (2.8) with concrete operators \(O\) (Oseen operator) and \(G\) defined formally as \(G(z) := -z \otimes z - \nu \otimes z - z \otimes \nu\). Proposition 3.5 implies that the Oseen semigroup \((e^{-tO})_{t \geq 0}\) satisfies Assumption 2.2 and Estimate (2.20).

The weak Hölder inequality (2.2) yields that \(G : L^3_{\sigma,w}(\Omega) \to L^{3/2}_w(\Omega)^{3 \times 3}\). Clearly, \(G(0) = 0\). We then check the condition (2) given in (2.9). Indeed, for \(v, \nu \in L^d_{\sigma,w}(\Omega)\) by the weak Hölder inequality (2.2) we have that \(\nu \otimes v \in L^{3/2}_w(\Omega)^{3 \times 3}\) and
\[
\|\nu \otimes v\|_{\frac{3}{2}, w} \leq \|\nu\|_{3,w} \|v\|_{3,w} \leq \hat{C}|u_\infty| \|v\|_{3,w}, \text{ also } \|v \otimes v\|_{\frac{3}{2}, w} \leq \|v\|_{3,w}^2.
\]

Therefore, for \(v_1, v_2 \in L^3_{\sigma,w}(\Omega)\) we have
\[
\|G(v_1) - G(v_2)\|_{\frac{3}{2}, w} = \|-(v_1 - v_2) \otimes v_1 - v_2 \otimes (v_1 - v_2) - \nu \otimes (v_1 - v_2) - (v_1 - v_2) \otimes \nu\|_{\frac{3}{2}, w} \\
\leq (\|v_1\|_{3,w} + \|v_2\|_{3,w} + 2\hat{C}|u_\infty|) \|v_1 - v_2\|_{3,w}. \tag{3.19}
\]

Therefore, the operator \(G\) satisfies the conditions given in (2.9) with \(\kappa = 2\hat{C}|u_\infty|\). Furthermore, the weak Hölder inequality (2.2) implies that \(G\) also satisfies conditions given in (2.21). Hence, the assertions of the theorem now follow from Theorems 2.4 and 2.5.

We would like to remark that our results here extend the results known for Navier-Stokes equations (i.e., the case \(u_\infty = 0\)) on exterior domains by Yamazaki [23] and for the “starting problem” by Galdi, Heywood and Shibata [6].

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