Chapter 7

On $gs\Lambda$-Homeomorphism in topological spaces

Introduction

In chapter 7 the author studies with the important concept $gs\Lambda$-homeomorphisms by using $gs\Lambda$-open sets. It is weaker than homeomorphisms. The notion of homeomorphisms plays a dominant role in topology and so many authors introduced varies types of homeomorphisms in topological spaces. In 1995, Maki, Devi and Balachandran introduced the concepts of semi-generalized homeomorphisms and generalized semi-homeomorphisms and studied some semi topological properties. Devi and Balachandran introduced a generalization of $\alpha$-homeomorphism in 2001. $gs\Lambda^*$-homeomorphism in topological spaces is also introduced and the group structure of the set of all $gs\Lambda^*$-homeomorphisms are also investigated the group structure of the set of all $gs\Lambda^*$-homeomorphisms.
7.1 Properties of $gs\Lambda$-Homeomorphism

Some of the important properties are listed below:

**Definition 7.1.1** A bijection $f:(X,\tau) \rightarrow (Y,\sigma)$ is called $gs\Lambda$-homeomorphism if $f$ is both $gs\Lambda$ open and $gs\Lambda$-continuous.

In other words a bijection $f:(X,\tau) \rightarrow (Y,\sigma)$ is called $gs\Lambda$-homeomorphism if both $f$ and $f^{-1}$ are $gs\Lambda$-continuous functions. We denote the family of all $gs\Lambda$-homeomorphisms of a topological space $(X,\tau)$ onto itself by $gs\Lambda H(X,\tau)$.

**Theorem 7.1.2** Every homeomorphism is $gs\Lambda$-homeomorphism.

**Proof:** Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be a homeomorphism. Then $f$ is bijective, open and continuous. Let $U$ be a open set in $(X,\tau)$. Since $f$ is open, $f(U)$ is open in $(Y,\sigma)$. As every open set is $gs\Lambda$-open set by (Theorem 2.3.5) $U$ is $gs\Lambda$-open in $(Y,\sigma)$. Thus $f$ is a $gs\Lambda$-open map. Let $G$ be a open set in $(Y,\sigma)$. Since $f$ is continuous, $f^{-1}(U)$ is open in $(X,\tau)$. As every open set is $gs\Lambda$-open set by Theorem 2.3.5 $U$ is $gs\Lambda$-open in $(X,\tau)$. Thus $f$ is a $gs\Lambda$-continuous map. Thus $f$ is $gs\Lambda$-homeomorphism.

**Remark 44** Converse of the above Theorem 7.1.2 need not be true as seen from the following example.

**Example 51** Let $X = Y = \{a,b,c,d,e\}, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}, \{c,d,e\}, \{a,b,c,d\}, \{a,c,d,e\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{d\}, \{a,d\}, \{c,d,e\}, \{a,c,d,e\}\}$. The identity function $f:(X,\tau) \rightarrow (Y,\sigma)$ is a $gs\Lambda$-homeomorphism but not homeomorphism. Since $A = \{a,d\}$ is open in $(Y,\sigma)$ but $f^{-1}(A) = \{a,d\}$ is not open in $(X,\tau)$. Thus $f$ is not continuous. Since $B = \{b\}$ is open in $(X,\tau)$ but $f(B) = \{b\}$ is not open in $(Y,\sigma)$. Thus $f$ is not open. Hence $f$ is not homeomorphism.
**Theorem 7.1.3** Every g.homeomorphism is gsΛ-homeomorphism if both Domain and Co-Domain are partition space.

**Proof:** Let \( f: (X, \tau) \longrightarrow (Y, \sigma) \) is g.homeomorphism where \((X, \tau)\) and \((Y, \sigma)\) are partition spaces. Thus \( f \) is bijective, g.open map and g.continuous. Since \((X, \tau)\) and \((Y, \sigma)\) are partition spaces, it is easy to observe that \( f \) is gs\( \Lambda \)-open and gs\( \Lambda \)-continuous function. Hence the proof follows.

**Theorem 7.1.4** If a bijective function \( f \) is both M.gs\( \Lambda \)-open and continuous then \( f \) is a gs\( \Lambda \)-homeomorphism.

**Proof:** Since every M.gs\( \Lambda \)-open map is gs\( \Lambda \)-open map, and every continuous function is gs\( \Lambda \)-continuous function, the proof follows.

**Remark 45** Converse of the above Theorem 7.1.4 need not be true, as seen from the following example.

**Example 52** Let \( X = Y = \{a,b,c,d,e\} \) and \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}, \{c,d,e\}, \{a,b,c,d\}, \{a,c,d,e\}\} \) and \( \sigma = \{\emptyset, Y, \{a\}, \{d\}, \{a,d\}, \{c,d\}, \{d,e\}, \{c,d,e\}, \{a,c,d\}, \{a,d,e\}, \{a,c,d,e\}\} \). The identity function \( f: (X, \tau) \longrightarrow (Y, \sigma) \) is a gs\( \Lambda \)-homeomorphism, but neither M.gs\( \Lambda \)-open nor continuous. Since \( A = \{a,d\} \) is open in \((Y, \sigma)\) but \( f^{-1}(A) = \{a,d\} \) is not open in \((X, \tau)\). Thus \( f \) is not continuous. Since \( B = \{a,e\} \) is a gs\( \Lambda \)-open set in \((X, \tau)\) but \( f(B) = \{a,e\} \) is not gs\( \Lambda \)-open in \((Y, \sigma)\). Thus \( f \) is not M.gs\( \Lambda \)-open.

**Theorem 7.1.5** If a bijection \( f \) is both contra open and Contra continuous then \( f \) is a gs\( \Lambda \)-homeomorphism.

**Proof:** Every contra open map is gs\( \Lambda \)-open map by Theorem 3.4.3, and every contra continuous function is gs\( \Lambda \)-continuous function by Theorem 4.1.7. Hence the proof follows.
Remark 46 Converse of the above Theorem 7.1.5 need not be true, as seen from the following example.

Example 53 Let $X = Y = \{a,b,c,d,e\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}, \{a,c,d,e\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{d\}, \{a,d\}, \{c,d\}, \{c,d,e\}, \{a,c,d\}, \{a,d,e\}, \{a,c,d,e\}\}$. The identity function $f:(X,\tau)\rightarrow(Y,\sigma)$ is a $gs\Lambda$-homeomorphism, but neither contra open nor Contra continuous. Since $A=\{a,d\}$ is open in $(Y,\sigma)$ but $f^{-1}(A)=\{a,d\}$ is not closed in $(X,\tau)$. Thus $f$ is not contra continuous. Since $B=\{a,c,d\}$ is open in $(X,\tau)$ but $f(B) = \{a,c,d\}$ is not closed in $(Y,\sigma)$. Thus $f$ is not contra open.

Theorem 7.1.6 If a bijective function $f$ is both open (contra open) and $gs\Lambda$-irresolute function then $f$ is a $gs\Lambda$-homeomorphism.

Proof: Every open(contra open) map is $gs\Lambda$-open map by Theorem 2.3.4, and every $gs\Lambda$-irresolute function is $gs\Lambda$-continuous function by Theorem 6.1.5, hence the proof follows.

Remark 47 Converse of the above Theorem 7.1.6 need not be true, as seen from the following example.

Example 54 Let $X = Y = \{a,b,c,d,e\}$ and $\tau = \{\emptyset, X, \{a\}, \{d\}, \{a,d\}, \{c,d\}, \{d,e\}, \{c,d,e\}, \{a,c,d\}, \{a,c,d,e\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a,b\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}, \{c,d,e\}, \{a,b,c,d\}, \{a,c,d,e\}, \{b,c,d,e\}\}$. The identity function $f:(X,\tau)\rightarrow(Y,\sigma)$ is a $gs\Lambda$-homeomorphism, but neither open(contra open) nor $gs\Lambda$-irresolute. Since $A=\{d,e\}$ is open in $(X,\tau)$ but $f(A) = \{d,e\}$ is not open(closed) in $(Y,\sigma)$. Thus $f$ is not open(contra open). Since $B=\{b,d\}$ is $gs\Lambda$-open in $(Y,\sigma)$ but $f(B) = \{b,d\}$ is not $gs\Lambda$-open in $(X,\tau)$. Thus $f$ is not $(X,\tau)$ irresolute.

Theorem 7.1.7 If a bijective function $f$ is $\lambda$-open and $\lambda$-continuous then $f$ is $gs\Lambda$-homeomorphism.
**Proof:** Let a bijective function \( f:(X,\tau) \longrightarrow (Y,\sigma) \) be \( \lambda \)-open and \( \lambda \)-continuous function. Let \( U \) be a open set in \( (Y,\sigma) \). Since \( f \) is \( \lambda \)-continuous map \( f^{-1}(U) \) is \( \lambda \)-open in \( (X,\tau) \). As every \( \lambda \)-open set is \( gs\Lambda \)-open set by(Theorem 2.3.5) \( U \) is \( gs\Lambda \)-open in \( (X,\tau) \). Thus \( f \) is a \( gs\Lambda \)-continuous map. Let \( G \) be a open set in \( (X,\tau) \). Since \( f \) is \( \lambda \)-open, \( f(G) \) is \( \lambda \)-open in \( (Y,\sigma) \). By Theorem 2.3.5 \( f(G) \) is a \( gs\Lambda \)-open set in \( Y \). Thus \( f \) is a \( gs\Lambda \)-open map. Hence \( f \) is \( gs\Lambda \)-homeomorphism.

**Remark 48** Converse of the above Theorem 7.1.7 need not be true as seen from the following example.

**Example 55** Let \( X = Y = \{a,b,c,d,e\} \), \( \tau = \{\emptyset,X,\{a\}, \{b\}, \{a,b\}, \{c\}, \{a,c\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}, \{c,d,e\} \) and \( \sigma = (Y,\sigma) = \{\emptyset, Y, \{a\}, \{d\}, \{a,d\}, \{c,d\}, \{d,e\}, \{c,d,e\}, \{a,c,d\}, \{a,d,e\}, \{a,c,d,e\}\} \).

The identity function \( f:(X,\tau) \longrightarrow (Y,\sigma) \) is a \( gs\Lambda \)-homeomorphism, but neither \( \lambda \)-open nor \( \lambda \)-continuous. Since \( A=\{d\} \) is open in \( (Y,\sigma) \) but \( f^{-1}(A) = \{d\} \) is not \( \lambda \)-open in \( (X,\tau) \). Thus \( f \) is not \( \lambda \)-continuous. Since \( B=\{e\} \) is \( \lambda \)-open in \( (X,\tau) \) but \( f(B) = \{e\} \) is not \( \lambda \)-open in \( (Y,\sigma) \). Thus \( f \) is not \( \lambda \)-open.

**Theorem 7.1.8** If a bijective function \( f \) is \( \lambda \)-open and \( \lambda \)-irresolute then \( f \) is a \( gs\Lambda \)-homeomorphism.

**Proof:** The proof follows as every \( \lambda \)-open map is \( gs\Lambda \)-open map, and every \( \lambda \)-irresolute function is \( gs\Lambda \)-continuous function.

**Example 56** In example 55, the identity function \( f:(X,\tau) \longrightarrow (Y,\sigma) \) is a \( gs\Lambda \)-homeomorphism but but neither \( \lambda \)-open nor \( \lambda \)-irresolute. Since \( A=\{a,e\} \) is \( \lambda \)-open in \( (Y,\sigma) \) but \( f^{-1}(A) = \{a,e\} \) is not \( \lambda \)-open in \( (X,\tau) \). Thus \( f \) is not \( \lambda \)-irresolute. Since \( B=\{a,d\} \) is \( \lambda \)-open in \( (X,\tau) \) but \( f(B) = \{b,d\} \) is not \( \lambda \)-open in \( (Y,\sigma) \). Thus \( f \) is not \( \lambda \)-open.
Theorem 7.1.9 Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a bijective function. If \( f \) and \( f^{-1} \) are \( gs\Lambda \)-irresolute then \( f \) is \( gs\Lambda \)-homeomorphism.

**Proof:** Let a bijective function \( f: (X, \tau) \rightarrow (Y, \sigma) \) be such that \( f \) and \( f^{-1} \) are \( gs\Lambda \)-irresolute. Let \( U \) be a open set in \( (X, \tau) \). As every open set is \( gs\Lambda \)-open set by (Theorem 2.3.5) \( U \) is \( gs\Lambda \)-open in \( (X, \tau) \). Since \( f^{-1} \) is \( gs\Lambda \)-irresolute \( f^{-1}(U) \) is \( gs\Lambda \)-open in \( (Y, \sigma) \). Thus \( f^{-1} \) is a \( gs\Lambda \)-continuous map. Let \( G \) be a open set in \( (Y, \sigma) \), by Theorem 2.3.5 \( G \) is a \( gs\Lambda \)-open set in \( Y \). Since \( f \) is \( gs\Lambda \)-irresolute, \( f^{-1}(U) \) is \( gs\Lambda \)-open in \( (X, \tau) \). Thus \( f \) is a \( gs\Lambda \)-continuous map. Hence we get \( f \) is a \( gs\Lambda \)-homeomorphism.

**Remark 49** Converse of the above Theorem 7.1.9 need not be true as seen from the following example.

**Example 57** Let \( X=Y=\{a,b,c,d,e\} \), \( \tau=\{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,b,e\}, \{a,b,d,e\}, \{a,b,c,e\}\} \) and \( (Y, \sigma)=\{\emptyset, Y, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{b,c\}, \{a,b,c,d\}, \{b,c,d\}, \{b,c,d,e\}\} \). The identity function \( f: (X, \tau) \rightarrow (Y, \sigma) \) is a \( gs\Lambda \)-homeomorphism, but neither \( f \) nor \( f^{-1} \) are \( gs\Lambda \)-irresolute. Since \( A=\{b,e\} \) is \( gs\Lambda \)-open in \( (X, \tau) \) but \( f(A)=\{b,e\} \) is not \( gs\Lambda \)-open in \( (Y, \sigma) \). Thus \( f^{-1} \) is not \( gs\Lambda \)-irresolute. Since \( B=\{a,b,c\} \) is \( gs\Lambda \)-open in \( (Y, \sigma) \) but \( f^{-1}(B)=\{b,d\} \) is not \( gs\Lambda \)-open in \( (X, \tau) \). Thus \( f \) is not \( gs\Lambda \)-irresolute.

**Theorem 7.1.10** Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a bijective function. If \( f \) and \( f^{-1} \) are contra \( gs\Lambda \)-irresolute then \( f \) is \( gs\Lambda \)-homeomorphism.

**Proof:** Let a bijective function \( f: (X, \tau) \rightarrow (Y, \sigma) \) be such that \( f \) and \( f^{-1} \) are contra \( gs\Lambda \)-irresolute. Let \( U \) be a open set in \( (X, \tau) \). As every open set is \( gs\Lambda \)-closed set, \( U \) is \( gs\Lambda \)-closed in \( X \). Since \( f^{-1} \) is contra \( gs\Lambda \)-irresolute, we get \( (f^{-1})^{-1}(U)=f(U) \), which is \( gs\Lambda \)-open in \( (Y, \sigma) \). Thus \( f^{-1} \) is a \( gs\Lambda \)-continuous map. Let \( G \) be a open set in \( (Y, \sigma) \), by Theorem 2.3.4, \( G \) is a \( gs\Lambda \)-closed set in \( Y \). Since \( f \) is contra \( gs\Lambda \)-irresolute, \( f^{-1}(U) \) is \( gs\Lambda \)-open in \( (X, \tau) \). Thus \( f \) is a \( gs\Lambda \)-continuous map. Hence, \( f \) is \( gs\Lambda \)-homeomorphism.
Remark 50  Converse of the above Theorem 7.1.10 need not be true as seen from the following example.

Example 58  Let \( X = Y = \{a,b,c,d,e\} \) and \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,b,e\}, \{a,b,d,e\}, \{a,b,c,e\}\} \) and \( \sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}, \{a,b,c,d\}, \{b,c,d,e\}, \{b,c,d,e\}\} \). The identity function \( f : (X, \tau) \longrightarrow (Y, \sigma) \) is a \( gs\Lambda \)-homeomorphism, but neither \( f \) nor \( f^{-1} \) are \( gs\Lambda \)- irresolute. Since \( A = \{b,e\} \) is \( gs\Lambda \)-open in \( (X, \tau) \) but \( f(A) = \{b,e\} \) is not \( gs\Lambda \)- open in \( (Y, \sigma) \). Thus \( f^{-1} \) is not \( gs\Lambda \)- irresolute. Since \( B = \{a,b,c\} \) is \( gs\Lambda \)- open in \( (Y, \sigma) \) but \( f^{-1}(B) = \{b,d\} \) is not \( gs\Lambda \)- open in \( (X, \tau) \). Thus \( f \) is not \( gs\Lambda \)- irresolute.

Theorem 7.1.11  The composition of homeomorphisms is \( gs\Lambda \)-homeomorphism.

Proof: Let the functions \( f : (X, \tau) \longrightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \longrightarrow (Z, \psi) \) be homeomorphisms. Let \( U \) be a open set in \( (Z, \psi) \). Since \( g \) is homeomorphism, \( g \) is continuous and so \( g^{-1}(U) \) is open in \( (Y, \sigma) \). Since \( f \) is a homeomorphism, \( f \) is continuous, and we get \( f^{-1}(g^{-1}(U)) = (gof)^{-1}(U) \) is open which is by Theorem 2.3.5 \( gs\Lambda \)-open in \( (X, \tau) \). This implies that \( gof \) is \( gs\Lambda \)-continuous. Again, let \( G \) be open in \( (X, \tau) \). Since \( f \) is a homeomorphism, \( f \) is open. Hence we get \( f(G) \) is open in \( (Y, \sigma) \). Since \( g \) is homeomorphism, \( g \) is open and hence we have \( g(f(G)) = (gof)(G) \) is open in \( (Z, \psi) \), which is \( gs\Lambda \)-open in \( (Z, \psi) \) by Theorem 2.3.5. This implies that \( gof \) is \( gs\Lambda \)-open. Since \( f \) and \( g \) are bijective, \( gof \) is also bijective. This completes the proof.
Theorem 7.1.12 If \( f:(X, \tau) \longrightarrow (Y, \sigma) \) and \( g:(Y, \sigma) \longrightarrow (Z, \psi) \) are bijective functions such that \( f \) is open, \( gs\Lambda \)-continuous and \( g \) is a homeomorphism then \( gof \) is a \( gs\Lambda \)-homeomorphism.

Proof: Let the bijections \( f:(X, \tau) \longrightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \longrightarrow (Z, \psi) \) be such that \( f \) is open, \( gs\Lambda \)-continuous and \( g \) is a homeomorphism. Let \( U \) be a open set in \((Z, \psi)\). Since \( g \) is homeomorphism, \( g \) is continuous and so \( g^{-1}(U) \) is open in \((Y, \sigma)\). Since \( f \) is \( gs\Lambda \)-continuous, \( f^{-1}(g^{-1}(U)) = (gof)^{-1}(U) \) is \( gs\Lambda \)-open in \((X, \tau)\). This implies that \( gof \) is \( gs\Lambda \)-continuous. Again, let \( G \) be open in \((X, \tau)\). Since \( f \) is open, \( f(G) \) is open in \((Y, \sigma)\). Since \( g \) is homeomorphism, \( g \) is open and hence we have \( g(f(G)) = (gof)(G) \) is open in \((Z, \psi)\), which is \( gs\Lambda \)-open in \((Z, \psi)\) by Theorem 2.3.5. This implies that \( gof \) is \( gs\Lambda \)-open. Since \( f \) and \( g \) are bijective, \( gof \) is also bijective. This completes the proof.

Theorem 7.1.13 If \( f:(X, \tau) \longrightarrow (Y, \sigma) \) and \( g:(Y, \sigma) \longrightarrow (Z, \psi) \) are such that \( f \) is open and \( gs\Lambda \)-irresolute, \( g \) is a \( gs\Lambda \)-homeomorphism then \( gof \) is a \( gs\Lambda \)-homeomorphism.

Proof: Let the bijective functions \( f:(X, \tau) \longrightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \longrightarrow (Z, \psi) \) are such that \( f \) is open and \( gs\Lambda \)-irresolute, \( g \) is a \( gs\Lambda \)-homeomorphism. Let \( U \) be a open set in \((Z, \psi)\). Since \( g \) is \( gs\Lambda \)-homeomorphism, \( g \) is \( gs\Lambda \)-continuous and so \( g^{-1}(U) \) is \( gs\Lambda \)-open in \((Y, \sigma)\). Since \( f \) is \( gs\Lambda \)-irresolute, \( f^{-1}(g^{-1}(U)) = (gof)^{-1}(U) \) is \( gs\Lambda \)-open in \((X, \tau)\). This implies that \( gof \) is \( gs\Lambda \)-continuous. Again, let \( G \) be open in \((X, \tau)\). Since \( f \) is open, \( f(G) \) is open in \((Y, \sigma)\). Since \( g \) is \( gs\Lambda \)-homeomorphism, \( g \) is \( gs\Lambda \)-open and hence we have \( g(f(G)) = (gof)(G) \) is \( gs\Lambda \)-open in \((Z, \psi)\). This implies that \( gof \) is \( gs\Lambda \)-open. Since \( f \) and \( g \) are bijective, \( gof \) is also bijective. This completes the proof.

Theorem 7.1.14 If \( f:(X, \tau) \longrightarrow (Y, \sigma) \) and \( g:(Y, \sigma) \longrightarrow (Z, \psi) \) are such that \( f \) is \( gs\Lambda \)-homeomorphism, \( g \) is continuous and \( M. \) \( gs\Lambda \)-open then \( gof \) is a \( gs\Lambda \)-homeomorphism.
Proof: Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \psi) \) are such that \( f \) is gs\( \Lambda \)-homeomorphism, \( g \) is continuous and \( M \). gs\( \Lambda \)- open. Let \( U \) be a open set in \((Z, \psi)\). Since \( g \) is continuous, \( g^{-1}(U) \) is open in \((Y, \sigma)\). Since \( f \) is gs\( \Lambda \)-continuous, \( f^{-1}(g^{-1}(U)) = (gof)^{-1}(U) \) is gs\( \Lambda \)-open in \((X, \tau)\). This implies that \( gof \) is gs\( \Lambda \)-continuous. Again, let \( G \) be open in \((X, \tau)\). Since \( f \) is gs\( \Lambda \)- open, \( f(G) \) is gs\( \Lambda \)- open in \((Y, \sigma)\). Since \( g \) is \( M \), \( g(f(G)) = (gof)(G) \) is gs\( \Lambda \)- open in \((Z, \psi)\). This implies that \( gof \) is gs\( \Lambda \)- open. Since \( f \) and \( g \) are bijective, \( gof \) is also bijective. This completes the proof.

Similarly we can prove that

**Theorem 7.1.15** If \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \psi) \) are such that \( f \) is a homeomorphism, \( g \) is continuous and gs\( \Lambda \)- open then \( gof \) is a gs\( \Lambda \)-homeomorphism.

**Theorem 7.1.16** Let \( f : (X, \tau) \to (Y, \sigma) \) be bijective gs\( \Lambda \)-continuous map. Then the following are equivalent.

1. \( f \) is gs\( \Lambda \)- open map.
2. \( f \) is gs\( \Lambda \)- homeomorphism
3. \( f \) is gs\( \Lambda \)-closed map.

**Proof:** Proofs are clear from definitions.

**Theorem 7.1.17** Let \( f : (X, \tau) \to (Y, \sigma) \) be a gs\( \Lambda \)- homeomorphism. Let \( A \) be a open and gs\( \Lambda \)-closed subset of \( X \) and \( B \) be a closed subset of \( Y \) such that \( f(A) = B \). Assume \((X, \tau) \) to be a gs\( \Lambda \)- space. Then the restriction \( f_A : (A, \tau_A) \to (B, \sigma) \) gs\( \Lambda \)-homeomorphism.

**Proof:** We have to show that \( f_A \) is a bijection, gs\( \Lambda \)- open map and gs\( \Lambda \)-continuous map.
1. Since \( f \) is one-one \( f_A \) is also one-one. Also since \( f(A)=B \) we have \( f_A(A)=B \), hence we have \( f_A \) is also onto. Thus \( f_A \) is bijective.

2. Let \( U \) be a open set of \((A,\tau_A)\). Then \( U = A \cap H \), for some open set \( H \) in \((X,\tau)\). Since \( f \) is one-one \( f(U)=f(A \cap H) = f(A) \cap f(H)= B \cap f(H) \). Since \( f \) is \( gs\Lambda \)- open and \( H \) is an open set in \( X \), we have \( f(H) \) is \( gs\Lambda \)- open set in \( Y \). Since \( B \) is also \( gs\Lambda \)- open in \( Y \), \( f(U) \) is \( gs\Lambda \)- open in \( B \).

   Hence \( f_A \) is a \( gs\Lambda \)- open map.

3. Let \( V \) be a closed set of \((B,\sigma)\). Then \( V = K \cap B \), for some closed subset \( K \) of \( Y \). Since \( B \) is a closed set of \((Y,\sigma)\), \( V \) is also a closed set of \((Y,\sigma)\).

   By hypothesis and assumption \( f^{-1}(V) \cap A = H_1 \) (say) is a closed subset in \( X \). Since \( f_A^{-1}(V)= H_1 \), it is sufficient to show that \( H_1 \) is a \( gs\Lambda \)-closed subset in \((A,\tau_A)\). Let \( G \) be a semi open set in \((A,\tau_A)\), such that \( H_1 \subseteq G \).

   Then by hypothesis \( G \) is semi open in \( X \). Since \( H_1 \) is \( gs\Lambda \)-closed in \( X \), we have \( Cl_{\lambda}(H_1) \subseteq G \). Since \( A \) is open in \( X \), \( Cl_{\lambda}(H_1) \cap A \subseteq G \cap A \subseteq G \). Hence \( H_1 = f_A^{-1}(V) \) is \( gs\Lambda \)-closed set in \((A,\tau_A)\). Therefore \( f_A^{-1} \) is a \( gs\Lambda \)-continuous map. Therefore \( f_A^{-1} \) is a homeomorphism.

### 7.2 Properties of \( gs\Lambda^* \)-Homeomorphism

In this section, we introduce the concepts \( gs\Lambda^* \)-homeomorphisms into topological spaces and we investigate the group structure of the set of all \( gs\Lambda^* \)-homeomorphism.

**Definition 7.2.1** A bijection \( f:(X,\tau)\rightarrow(Y,\sigma) \) is called \( gs\Lambda^* \)-homeomorphism if \( f \) is both \( M.gsl \)-open and \( gs\Lambda \)- irresolute functions.

That is if \( f \) and \( f^{-1} \) preserve \( gs\Lambda \)- open sets (\( gs\Lambda \)-closed sets). In other words a bijection \( f:(X,\tau) \rightarrow(Y,\sigma) \) is said to be
gsΛ*-homeomorphism if both f and f⁻¹ are gsΛ-irresolute. We say that spaces (X,τ) and (Y,σ) are gsΛ*-homeomorphic if there exists a gsΛ*-homeomorphism from (X,τ) onto (Y,σ). We denote the family of all gsΛ*-homeomorphisms of a topological space (X,τ) onto itself by gsΛ*H(X,τ).

**Theorem 7.2.2** Every gsΛ*-homeomorphism is a gsΛ-homeomorphism.

**Proof:** Let f:(X,τ) → (Y,σ) be a gsΛ*-homeomorphism. Then f is bijective, gsΛ-irresolute and f⁻¹ is gsΛ-irresolute. Since every gsΛ-irresolute function is gsΛ-continuous, f and f⁻¹ are gsΛ-continuous and hence by definition f is a gsΛ-homeomorphism.

**Example 59** Let X = Y = {a,b,c,d,e}, τ = {∅, X, {a}, {b}, {a,b}, {a,b,e}, {a,b,d,e}, {a,b,c,e}} and σ = {∅, Y, {a}, {b}, {a,b}, {a,b,c}, {b,c,d}, {a,b,c,d}, {b,c,d,e}, {b,c,d,e}}. The identity function f:(X,τ) → (Y,σ) is a gsΛ-homeomorphism, but not gsΛ*-homeomorphism. Here A={b,e} is gsΛ open in (X,τ) but f(A) = {b,e} is not gsΛ-open in (Y,σ). Thus f⁻¹ is not gsΛ-irresolute. Since B={a,b,c} is gsΛ-open in (Y,σ) but f⁻¹(B) = {b,d} is not gsΛ-open in (X,τ). Thus f is not gsΛ-irresolute.

**Remark 51** The concepts gsΛ*-homeomorphism and homeomorphism are in general independent.

**Theorem 7.2.3** If f:(X,τ) → (Y,σ) is a gsΛ*-homeomorphism, then

gsΛCl(f⁻¹(B)) = f⁻¹(gsΛCl(B)) for every subset B of Y.

**Proof:** Let f:(X,τ) → (Y,σ) is a gsΛ*-homeomorphism. Then by definition, both f and f⁻¹ are gsΛ-irresolute and f is bijective. Let B ⊆ Y. Since gsΛCl(B) is a gsΛ-closed set in (Y,σ), and since f is gsΛ-irresolute, we have f⁻¹(gsΛCl(B)) is gsΛ-closed in (X,τ). But gsΛCl(f⁻¹(B)) is the smallest gsΛ-closed set containing f⁻¹(B). Therefore gsΛCl(f⁻¹(B)) ⊆ f⁻¹(gsΛCl(B)). Also, gsΛCl(f⁻¹(B)) is gsΛ-closed in (X,τ). Since f⁻¹ is gsΛ-irresolute,
\( f(\Lambda Cl(f^{-1}(B))) \) is \( \Lambda \)-closed in \( (Y, \sigma) \). Now, \( B = f(f^{-1}(B)) \subseteq f(\Lambda Cl(f^{-1}(B))) \).

Since \( f(\Lambda Cl(f^{-1}(B))) \) is \( \Lambda \)-closed and \( \Lambda Cl(B) \) is the smallest \( \Lambda \)-closed set containing \( B \), \( \Lambda Cl(B) \subseteq f(\Lambda Cl(f^{-1}(B))) \) implies that \( f^{-1}(\Lambda Cl(B)) \subseteq f^{-1}(f(\Lambda Cl(f^{-1}(B)))) = \Lambda Cl(f^{-1}(B)) \). That is, \( f^{-1}(\Lambda Cl(B)) \subseteq \Lambda Cl(f^{-1}(B)) \). Thus we get \( \Lambda Cl(f^{-1}(B)) = f^{-1}(\Lambda Cl(B)) \).

**Theorem 7.2.4**  If \( f:(X, \tau) \rightarrow (Y, \sigma) \) is a \( \Lambda \)-homeomorphism, then \( \Lambda Cl(f(B)) = f(\Lambda Cl(B)) \) for every \( B \subseteq X \).

**Proof:** Let \( f:(X, \tau) \rightarrow (Y, \sigma) \) be a \( \Lambda \)-homeomorphism. Since \( f \) is \( \Lambda \)-homeomorphism, \( f^{-1} \) is also a \( \Lambda \)-homeomorphism. Therefore by Theorem 7.2.3, it follows that \( \Lambda Cl(f(B)) = f(\Lambda Cl(B)) \) for every \( B \subseteq X \).

**Theorem 7.2.5**  If \( f:(X, \tau) \rightarrow (Y, \sigma) \) is a \( \Lambda \)-homeomorphism, then \( f(\Lambda \text{-int}(B)) = \Lambda \text{-int}(f(B)) \) for every \( B \subseteq X \).

**Proof:** Let \( f:(X, \tau) \rightarrow (Y, \sigma) \) be a \( \Lambda \)-homeomorphism. For any set \( B \subseteq X \), \( \Lambda \text{-int}(B) = (\Lambda Cl(B^c))^c \). \( f(\Lambda \text{-int}(B)) = f((\Lambda Cl(B^c))^c) = (f(\Lambda Cl(B^c)))^c \). Then using Theorem, we see that \( f(\Lambda \text{-int}(B)) = (\Lambda Cl(f(B^c)))^c = \Lambda \text{-int}(f(B)) \).

**Theorem 7.2.6**  If \( f:(X, \tau) \rightarrow (Y, \sigma) \) is a \( \Lambda \)-homeomorphism, then for every \( B \subseteq Y \), \( f^{-1}(\Lambda \text{-int}(B)) = \Lambda \text{-int}(f^{-1}(B)) \).

**Proof:** Let \( f:(X, \tau) \rightarrow (Y, \sigma) \) be a \( \Lambda \)-homeomorphism. Since \( f \) is \( \Lambda \)-homeomorphism, \( f^{-1} \) is also a \( \Lambda \)-homeomorphism. Therefore by Theorem 7.2.4, \( f^{-1}(\Lambda \text{-int}(B)) = \Lambda \text{-int}(f^{-1}(B)) \) for every \( B \subseteq Y \).

**Theorem 7.2.7**  If \( f:(X, \tau) \rightarrow (Y, \sigma) \) and \( g:(Y, \sigma) \rightarrow (Z, \psi) \) are \( \Lambda \)-homeomorphisms, then the composition \( g \circ f:(X, \tau) \rightarrow (Z, \psi) \) is also \( \Lambda \)-homeomorphism.

**Proof:** Let \( U \) be a \( \Lambda \)-open set in \( (Z, \psi) \). Since \( g \) is \( \Lambda \)-homeomorphism,
g is $g\Lambda$-irresolute and so $g^{-1}(U)$ is $g\Lambda$-open in $(Y,\sigma)$. Since $f$ is $g\Lambda^*$-homeomorphism, $f$ is $g\Lambda$-irresolute and so $f^{-1}(g^{-1}(U)) = (gof)^{-1}(U)$ is $g\Lambda$-open in $(X,\tau)$. This implies that $gof$ is $g\Lambda$-irresolute. Again, let $G$ be $g\Lambda$-open in $(X,\tau)$. Since $f$ is $g\Lambda^*$-homeomorphism, $f^{-1}$ is $g\Lambda$-irresolute and so $(f^{-1})^{-1}(G) = f(G)$ is $g\Lambda$-open in $(Y,\sigma)$. Since $g$ is $g\Lambda^*$-homeomorphism, $g^{-1}$ is $g\Lambda$-irresolute and so $(g)^{-1}(f(G)) = gof(G)) = (gof)(G) = ((gof)^{-1})^{-1}(G)$ is $g\Lambda$-open in $(Z,\psi)$. This implies that $(gof)^{-1}$ is $g\Lambda$-irresolute. Since $f$ and $g$ are $g\Lambda^*$-homeomorphism, $f$ and $g$ are bijective and so $gof$ is bijective. This completes the proof.

**Theorem 7.2.8** The set $g\Lambda^*H(X,\tau)$ is a group under composition of functions.

**Proof:** Define a binary operation $\Gamma: g\Lambda^*H(X,\tau) \times g\Lambda^*H(X,\tau) \rightarrow g\Lambda^*H(X,\tau)$, by $\Gamma(f,g) = gof$ for all $f,g \in g\Lambda^*H(X,\tau)$. Let $f, g \in g\Lambda^*H(X,\tau)$. Then $fog \in g\Lambda^*H(X,\tau)$ by Theorem 4.9. We know that the composition of maps are associative and the identity map $I: (X,\tau) \rightarrow (X,\tau)$ serves as identity element. Since $f$ is bijective, $f^{-1} \in g\Lambda^*H(X,\tau)$ whenever $f \in g\Lambda^*H(X,\tau)$ such that $fof^{-1} = f^{-1}of = I$ and so unique inverse element exists for each element of $g\Lambda^*H(X,\tau)$. This completes the proof.

**Theorem 7.2.9** If $f: (X,\tau) \rightarrow (Y,\sigma)$ is a $g\Lambda^*$-homeomorphism, then $f$ induces an isomorphism from the group $g\Lambda^*H(X,\tau)$ onto the group $g\Lambda^*H(Y,\sigma)$.

**Proof:** Let $f: (X,\tau) \rightarrow (Y,\sigma)$ is a $g\Lambda^*$-homeomorphism. Then define a map $\Psi_f : g\Lambda^*H(X,\tau) \rightarrow g\Lambda^*H(Y,\sigma)$ by $\Psi_f(h) = fohof^{-1}$ for every $h \in g\Lambda^*H(X,\tau)$. Let $h_1, h_2 \in g\Lambda^*H(X,\tau)$. Then $\Psi_f(h_1oh_2) = foh_1oh_2of^{-1} = fo(h_1of^{-1})ofoh_2of^{-1} = ((foh_1of^{-1})ofh_2of^{-1}) = \Psi_f(h_1)o\Psi_f(h_2)$. This proves that $\Psi_f$ is $g\Lambda^*$-homomorphism. Next let us prove that $\Psi_f$ is bijective. Since $\Psi_f(f^{-1}ohof) = h$, $\Psi_f$ is onto. Now, $\Psi_f(h) = I$ implies $fohof^{-1} = I$. That implies $h = I$. This proves that $\Psi_f$ is one-one. This shows that $\Psi_f$ is an
isomorphism.

**Definition 7.2.10** Let us define a function \( \Psi_f : g\Lambda^*H(X, \tau) \rightarrow g\Lambda^*H(Y, \sigma) \) by \( \Psi_f(h) = foh \circ f^{-1} \) for every \( h \in g\Lambda^*H(X, \tau) \). Let \( \Psi_f \) be a homomorphism. Let \( K = \{ h / h \in g\Lambda^*H(X, \tau); \Psi_f(h) = I_y \} \) where \( I_y \) is an identity element of \( g\Lambda^*H(Y, \sigma) \). Then \( K \) is called the kernel of \( \Psi_f \) and is denoted by \( \text{Ker}l \Psi_f \)

**Theorem 7.2.11** Let \( \Psi_f \) be a homomorphism. Then \( \Psi_f \) is one-one if and only if \( \text{Ker}l \Psi_f = \{ I_x \} \).

**Proof:** Suppose \( \Psi_f \) is one-one. Then clearly kernel of \( \Psi_f = \{ I_x \} \). Conversely suppose \( \text{Ker}l \Psi_f = \{ I_x \} \). Let \( \Psi_f(h_1) = \Psi_f(h_2) \), implies \( foh_1 \circ f^{-1} = foh_2 \circ f^{-1} \) implies \( foh_1 \circ f^{-1} \circ (foh_2 \circ f^{-1})^{-1} = I_y \). That is \( foh_1 \circ f^{-1} \circ f^{-1} \circ (oh_2 \circ f^{-1})^{-1} = I_y \). Hence we get \( foh_1 \circ h_2^{-1} \circ f^{-1} = I_y \). Thus we have \( h_1 \circ h_2^{-1} \in \text{Ker}l \Psi_f = \{ I_x \} \) and so \( h_1 = h_2 \). Therefore \( \text{Ker}l \Psi_f \) is one-one.

**Theorem 7.2.12** Let \( \Psi_f : g\Lambda^*H(X, \tau) \rightarrow g\Lambda^*H(Y, \sigma) \) be homomorphism. Then \( \text{Ker}l \Psi_f \) is a normal subgroup of \( g\Lambda^*H(X, \tau) \).

**Proof:** Since \( \Psi_f(I_x) = I_y, I_x \in \text{Ker}l \Psi_f \) and hence \( \text{Ker}l \Psi_f \neq \emptyset \). Now let \( h_1, h_2 \in \text{Ker}l \Psi_f \), then \( \Psi_f(h_1) = \Psi_f(h_2) = I_y \). Therefore \( \Psi_f(h_1 \circ h_2^{-1}) = \Psi_f(h_1) \circ \Psi_f(h_2^{-1}) = I_y \). Thus \( h_1 \circ h_2^{-1} \in \text{Ker}l \Psi_f \) and hence \( \text{Ker}l \Psi_f \) is a subgroup of \( g\Lambda^*H(X, \tau) \). Now let \( h_1 \in \text{Ker}l \Psi_f \) and \( g \in g\Lambda^*H(X, \tau) \), then \( \Psi_f(goh_1 \circ g^{-1}) = go \circ (goh_1 \circ g^{-1}) \circ g^{-1} = go \circ goh_1 \circ g^{-1} = goh_1 \circ g^{-1} = \Psi_f(h_1) = I_y \) and so \( goh_1 \circ g^{-1} \in \text{Ker}l \Psi_f \). Therefore \( \text{Ker}l \Psi_f \) is a normal subgroup of \( g\Lambda^*H(X, \tau) \).

**Theorem 7.2.13** (Fundamental Theorem of homomorphism)

Let \( \Psi_f : g\Lambda^*H(X, \tau) \rightarrow g\Lambda^*H(Y, \sigma) \) be an epimorphism. Let \( K = \text{Ker}l \Psi_f \). Then \( g\Lambda^*H(X, \tau) / K \cong g\Lambda^*H(Y, \sigma) \).

**Proof:** Let us define \( \eta : g\Lambda^*H(X, \tau) / K \rightarrow g\Lambda^*H(Y, \sigma) \) by \( \eta(Kx) = \)
\[ \Psi_f(x). \text{ Clearly } \eta \text{ is a well defined bijection. Now since } \Psi_f \text{ is a homomorphism we get, } \eta(Kx Ky) = \eta(Kxy) = \Psi_f(xy) = \Psi_f(x)\Psi_f(y) = \eta(Kx)\eta(Ky). \]

Therefore \( \eta \) is a homomorphism. Thus \( \Psi_f \) induces an isomorphism \( \eta \) from \( gs\Lambda^*H(X,\tau) \) onto \( gs\Lambda^*H(Y,\sigma) \). Hence we get \( gs\Lambda^*H(X,\tau)/K \cong gs\Lambda^*H(Y,\sigma) \).

**Theorem 7.2.14** Let \((X,\tau)\) be a topological space. If \( f:X\rightarrow Y \) is \( gs\Lambda\)-irresolute injection and \( Y \) is \( gs\Lambda\)-Hausdroff space, then \( X \) is \( gs\Lambda\)-Hausdroff space.

**Proof:** Since \( f \) is injective, for any pair of distinct points \( x,y \in X \), \( f(x) \neq f(y) \). Let \( y_1 = f(x) \) and \( y_2 = f(y) \), implies \( x = f^{-1}(y_1) \) and \( y = f^{-1}(y_2) \). As \( Y \) is a \( gs\Lambda\)-Hausdroff space there exists \( U \in \text{gsL}_\text{O}(Y,y_1) \) and \( V \in \text{gsL}_\text{O}(Y,y_2) \) such that \( U \cap V = \emptyset \). Since \( f \) is \( gs\Lambda \) irresolute, by definition we have \( f^{-1}(U) \) and \( f^{-1}(V) \) are \( gs\Lambda \)-open sets in \( X \), with \( f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset \) and \( f^{-1}(y_1) \in f^{-1}(U) \), \( f^{-1}(y_2) \in f^{-1}(V) \). Thus it is shown that for every distinct points \( x,y \in X \) there exist distinct \( gs\Lambda \)-open subsets \( f^{-1}(U) \) and \( f^{-1}(V) \) of \( X \) containing \( x \) and \( y \) such that respectively, such that \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \). Hence we proved that \( X \) is a \( gs\Lambda\)-Hausdroff space.

**Theorem 7.2.15** A bijective \( gs\Lambda \) irresolute map \( f: (X,\tau) \rightarrow (Y,\sigma) \) of a \( gs\Lambda \)-compact space \( X \) onto a \( gs\Lambda \)-Hausdroff space \( Y \) is a \( gs\Lambda^* \) homeomorphism.

**Proof:** Let \( (X,\tau) \) be a \( gs\Lambda \)-compact space and \( (Y,\sigma) \) be a \( gs\Lambda \)-Hausdroff space. Let \( f:(X,\tau) \rightarrow (Y,\sigma) \) be a bijective \( gs\Lambda \) irresolute map. It is enough to prove that \( f^{-1} \) is \( gs\Lambda \)-irresolute map. Let \( F \) be \( gs\Lambda \)-closed subset of \( (X,\tau) \). Since \( (X,\tau) \) is a \( gs\Lambda \)-compact space \( F \) is a \( gs\Lambda \)-compact subset of \( X \). Since \( f \) is \( gs\Lambda \)-irresolute map \( f(F) \) is \( gs\Lambda \)-compact subset of \( Y \). Since \( Y \) is a \( gs\Lambda \)-Hausdroff space, \( f(F) \) is \( gs\Lambda \)-closed subset of \( Y \). Thus we proved \( f^{-1} \) is a \( gs\Lambda \)-irresolute map. Hence \( f \) is a \( gs\Lambda^* \) homeomorphism.