COMPLETE HOLOMORPHIC VECTOR FIELDS ON $\mathbb{C}^2$
WHOSE UNDERLYING FOLIATION IS POLYNOMIAL

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Abstract. We extend the classification of complete polynomial vector fields on $\mathbb{C}^2$ given by Marco Brunella in [1] to cover the case of holomorphic (non-polynomial) vector fields whose underlying foliation is however still polynomial.

1. Introduction and Statement of Results

Given a holomorphic vector field $X$ on $\mathbb{C}^2$ one knows that the associated ordinary differential equation

$$\dot{z} = X(z), \ z(0) = z_0 \in \mathbb{C}^2,$$

has a unique local solution $t \mapsto \varphi_z(t)$, that can be extended by analytic continuation along paths in $\mathbb{C}$, with origin at $t = 0$, to a maximal connected Riemann surface $\pi_z : \Omega_z(X) \to \mathbb{C}$ which is spread as a Riemann domain over $\mathbb{C}$. The projection $\pi_z$ permits to lift this extension as a well-defined holomorphic function $\varphi_z : \Omega_z(X) \to \mathbb{C}^2$ (see [5, p. 126]). This map is the solution of $X$ through $z$ and its image $C_z = \varphi_z(\Omega_z(X))$ is the trajectory of $X$ through $z$. $X$ is complete when $\Omega_z(X) = \mathbb{C}$ for every $z \in \mathbb{C}^2$. In this case the flow $\varphi(t, z) = \varphi_z(t)$ of $X$ defines an action of $(\mathbb{C}, +)$ on $\mathbb{C}^2$ by holomorphic automorphisms, and each trajectory of $X$, as Riemann surface uniformized by $\mathbb{C}$ in a Stein manifold $\mathbb{C}^2$, is analytically isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$(will be said of type $\mathbb{C}$ or $\mathbb{C}^*$). As an important property, we remark that the trajectories of type $\mathbb{C}^*$ of a complete holomorphic vector field on $\mathbb{C}^2$ are proper (see [11]). Let us recall that a trajectory $C_z$ is said to be proper if its topological closure defines an analytic curve in $\mathbb{C}^2$ of pure dimension one.

1.1. Suzuki’s Classification. In his pioneering work M. Suzuki classified on $\mathbb{C}^2$ [11]: (a) complete holomorphic vector fields whose time $t$ maps $\varphi(t, \cdot)$ of the flow $\varphi$ are polynomials (algebraic flows), modulo polynomial automorphism, and (b) complete holomorphic vector fields whose trajectories are all

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proper (proper flows), modulo holomorphic automorphism. The vector fields $X$ of the two classifications together are of the form [10, Théorèmes 2 et 4]:

1) 

$$[a(x)y + b(x)] \frac{\partial}{\partial y},$$

with $a(x)$ and $b(x)$ entire functions in one variable.

2) 

$$\lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y},$$

with $\lambda, \mu \in \mathbb{C}$.

3) 

$$\lambda x \frac{\partial}{\partial x} + (\lambda ny + x^m) \frac{\partial}{\partial y},$$

with $\lambda \in \mathbb{C}^*$, $m \in \mathbb{N}$.

4) 

$$\lambda (x^m y^n) \cdot \left\{ n x \frac{\partial}{\partial x} - m y \frac{\partial}{\partial y} \right\},$$

with $m, n \in \mathbb{N}^*$, $(m, n) = 1$, and $\lambda$ an entire function in $z$ ($z = x^m y^n$).

5) 

$$\lambda (x^m(x^\ell y + p(x))^n) \cdot \left\{ n x^{\ell+1} \frac{\partial}{\partial x} - [(m + n\ell)x^\ell y + mp(x) + nxp(x)] \frac{\partial}{\partial y} \right\},$$

where $m, n, \ell \in \mathbb{N}^*$, $(m, n) = 1$, $p \in \mathbb{C}[x]$ of degree $< \ell$ with $p(0) \neq 0$, and $\lambda$ an entire function in $z$ ($z = x^m(x^\ell y + p(x))^n$) with a zero of order $\geq \ell/m$ at $z = 0$.

We will refer to the above list as Suzuki’s list. Let us comment some aspects of it. In case 5), the condition of $\lambda$ at $z = 0$ guarantees that the vector field thus defined is holomorphic. Without this restriction, the vector field is complete on $x \neq 0$ but it has a pole along $x = 0$. Algebraic flows arise only for 1), 2) and 3), while proper flows are defined by vector fields of 1), 2) if $\lambda/\mu \in \mathbb{Q}$, 4) and 5). In this latter situation, there exists always a rational first integral, given by $x$ in the first case, $y^p/x^q$ in the second one ($p, q \in \mathbb{Z}$ with $p/q = \lambda/\mu \in \mathbb{Q}$), $x^m y^n$ in the third one, and $x^m(x^\ell y + p(x))^n$ in the fourth one. Therefore, modulo holomorphic automorphism, having a proper flow, which is equivalent to the existence of a meromorphic first integral [12], is equal to having a rational first integral of one of the four types above. Flows occurring in 2), with $\lambda/\mu \notin \mathbb{Q}$, and 3) with $m \neq 0$ are never proper.
1.2. Brunella’s Classification. The classification of complete polynomial vector fields on \( \mathbb{C}^2 \), modulo polynomial automorphism, has been recently obtained by M. Brunella in the outstanding work [1]. This classification is given by the following vector fields (expressed in terms of Suzuki’s list):

I) \[ (cx + d) \frac{\partial}{\partial x} + Z, \]
where \( c, d \in \mathbb{C} \), and \( Z \) is as 1) with \( a, b \in \mathbb{C}[x] \).

II) \[ ay \frac{\partial}{\partial y} + Z, \]
where \( a \in \mathbb{C} \), and \( Z \) is as 4) with \( \lambda \in \mathbb{C}[z] \).

III) \[ a \left( \frac{x^\ell y + p(x)}{x^\ell} \right) \frac{\partial}{\partial y} + Z, \]
where \( a \in \mathbb{C} \), and \( Z \) is as 5) with \( \lambda \in \mathbb{C}[z] \) which does not really verify any condition in the order at \( z = 0 \), but which nevertheless satisfies the following polynomial relation that guarantees that the sum is holomorphic:

\[ \lambda(x^m(x^\ell y + p(x))^n) [mp(x) + nxp(x)] - ap(x) \in x^\ell \cdot \mathbb{C}[x, y]. \]

These two classifications above are given in different contexts. While Suzuki works with holomorphic objects (holomorphic vector fields modulo holomorphic automorphism), Brunella is interested in polynomial ones (polynomial vector fields modulo polynomial automorphism). However, both are related. On one hand, each vector field in Suzuki’s list is multiple of a polynomial one by a holomorphic function. On the other hand, each polynomial field in Brunella’s classification can be decomposed in the sum of a complete vector field with a polynomial first integral in the form of Suzuki’s list and a vector field which preserves this integral, verifying moreover the necessary conditions to avoid the rationality of the sum: \( a, b \in \mathbb{C}[x] \) in I), \( \lambda \in \mathbb{C}[z] \) in II), and \( \lambda \in \mathbb{C}[z] \) and satisfying \((*)\) in III). Let us also remark that if one does not consider these restrictions the proofs of Propositions 1 and 2 in [1] also work to characterize the rational complete vector fields that preserve a polynomial of type \( \mathbb{C} \) or \( \mathbb{C}^* \).

1.3. Statement of the theorem. The result of this work is the extension of Brunella’s classification to cover the case of non-polynomial holomorphic vector fields whose associated foliation in \( \mathbb{C}^2 \) is still polynomial, that is, defined by a polynomial vector field. Let us observe that these vector fields admit an unique representation of the form \( f \cdot Y \), with \( Y \) a polynomial vector field with isolated singularities and \( f \) a transcendental function, up to multiplication by constants.
Theorem 1. Let $X$ be a complete vector field on $\mathbb{C}^2$ of the form $f \cdot Y$, where $Y$ is a polynomial vector field with isolated singularities and $f$ is a transcendental function. Then, all the trajectories of $X$ are proper and, up to a holomorphic automorphism, $X$ is in Suzuki’s list.

A more precise classification, up to polynomial automorphisms, will be stated in §5 and proved in the course of the proof (§2, §3 and §4).

1.4. About the Theorem and its proof. Let us comment some aspects of the proof. Although $Y$ is not necessarily complete, Brunella’s results [1] can be applied to the foliation $F$ generated by $Y$ on $\mathbb{C}^2$ extended to $\mathbb{CP}^2$.

Let us first remind some definitions. According to Seidenberg’s Theorem, the minimal resolution $\tilde{F}$ of $F$ is a foliation defined on a rational surface $M$ after pulling back $F$ by a birational morphism $\pi: M \to \mathbb{CP}^2$, that is a finite composition of blowing ups. Associated to this resolution one has: 1) The Zariski’s open set $U = \pi^{-1}(\mathbb{C}^2)$ of $M$, over which $Y$ can be lifted to a holomorphic vector field $\tilde{Y}$, 2) the exceptional divisor $E$ of $U$, and 3) the divisor at infinity $D = M \setminus U = \pi^{-1}(\mathbb{CP}^2 \setminus \mathbb{C}^2) = \pi^{-1}(L_\infty)$, that is a tree of a smooth rational curves. The vector field $\tilde{Y}$ can be extended to $M$, although it may have poles along one or more components of $D$. Let us still denote this extension by $\tilde{Y}$. In $\mathbb{C}^2$ one blows-up only singularities of the foliation, which are in the zero set of $X$, hence $\tilde{X}$ is holomorphic and complete on the full $U$, and its essential singularities are contained in $D$.

We start studying the cases in which $F$ has rational first integral (§2). The next step is the analysis of $F$ when Kodaira dimension $\text{kod}(\tilde{F})$ of $\tilde{F}$ is 1 or 0, which corresponds to the absence of rational first integral. The unique case in which $X$ can be determined using directly [1] is $\text{kod}(\tilde{F}) = 0$ and $Y$ of type $\mathbb{C}$ (1.- of §4). The remaining cases, that is, $\text{kod}(\tilde{F}) = 1$ (§3) and $\text{kod}(\tilde{F}) = 0$ and $Y$ of type $\mathbb{C}^*$ (2.- of §4), require to go a bit further on [1]. First we see that $\tilde{F}$ is a Riccati foliation adapted to a fibration $g: M \to \mathbb{P}^1$, whose projection to $\mathbb{C}^2$ by $\pi$ defines a rational function $R$ of type $\mathbb{C}$ or $\mathbb{C}^*$ (Lemma [1]). Let us denote the $\tilde{F}$-invariant components of $g$ by $\Gamma$. At this point, one could think as a strategy to continue that $X$ can be determined if one proves similarly as in [1] Lemma 3 the completeness of $\tilde{X}$ on $M \setminus \Gamma$ (and thus the completeness of its projection by $g$). Then it would be enough to see that the poles of $\tilde{X}$ together with its essential singularities must be contained in $\Gamma$. But this does not generally occur since $f$ can have poles and essential singularities which are transversal to the fibers of $g$. This was pointed out to me by the referee with Example 1.
Finally, we can avoid the previous obstacle. The principle idea is to decompose $X$ as a complete vector field multiplied by a second integral, what implies that all its trajectories are proper (Proposition 2). The case $R$ of type $\mathbb{C}$ is almost direct (§3.1, 1.- of §4). However, for the case $R$ of type $\mathbb{C}^*$ (§3.2, 2.- of §4) we need to prove the presence of an invariant line by $Y$ (Lemma 2). It allows to determine $X$.

**Example 1.** Let us consider the complete polynomial vector field

$$Y = x \frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$

and its holomorphic first integral $f = xe^{-y}$. The foliation $\tilde{\mathcal{F}}$ generated by $Y$ in $M = \mathbb{C}P^1 \times \mathbb{C}P^1$ is Riccati with respect to $g(x, y) = y$, it has a semidegenerate fibre over $y = \infty$, with a saddle-node singularity at $x = \infty$, $y = \infty$, and such that the flow of $Y$ preserves $g$.

On the other hand, $\Gamma = \{y = \infty\}$ and the complete vector field $X = f \cdot Y$ does not project via $g$ to a complete vector field on $g(M \setminus \Gamma)(=\mathbb{C})$. After a change of coordinates $x \mapsto 1/x, y \mapsto 1/y$, $X$ becomes

$$X = e^{-\frac{1}{xy}} \left\{ x \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y} \right\},$$

and $f$ has a first order pole along the weak separatrix $C = \{x = 0\}$, that is transversal to $g$, and an essential singularity along the strong separatrix $\{y = 0\} \subset \Gamma$. In this example the essential singularity is contained in a fiber of $g$. However, by multiplying $Y$ by a transcendental first integral $e^{xe^{-y}}$, we obtain another complete holomorphic vector field with the essential singularity transversal to $g$.

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## 2. Some properties of $X$ and Rational first integrals

**Property 1.** Types of $X$ and $Y$. A complete holomorphic vector field on $\mathbb{C}^2$ is either of type $\mathbb{C}$ or $\mathbb{C}^*$, depending on the type of its generic (in sense of logarithmic capacity) trajectory. Moreover in the latter case there is a meromorphic first integral (see [12, Théorème II]). For $Y$, which can be uncomplete, the same occurs: if $f = 0$ is empty or invariant by $Y$, the types of $X$ and $Y$ coincide. When $f = 0$ is not invariant by $Y$, and due to the steiness of $\mathbb{C}^2$, $X$ must be of type $\mathbb{C}^*$ and $Y$ of type $\mathbb{C}$.

**Property 2.** If $Y$ is complete $f$ is affine along its trajectories. It is a consequence of [13 Proposition 3.2]: Let us take a point $z$ with $Y(z) \neq 0$. 


and the solution \( \varphi_z : \mathbb{C} \to C_z \) of \( Y \) through it. The restriction of \( X \) to \( C_z \), \( X_{|C_z} \), is complete since \( C_z \) outside the zeros of \( X \) is a trajectory of this vector field. As \( \varphi_z \) is a holomorphic covering map (see [5, Proposition 1.1]) \( \varphi_z^*(X_{|C_z}) \) is complete and then affine. Therefore

\[
\varphi_z^*(X_{|C_z}) = (f \circ \varphi_z(t)) \cdot \varphi_z^*(Y_{|C_z}) = (f \circ \varphi_z(t)) \frac{\partial}{\partial t},
\]

and \( (f \circ \varphi_z)(t) = at + b \), for \( a, b \in \mathbb{C} \). In particular \((Yf)(\varphi_z(t)) = (f \circ \varphi_z)'(t)\) is constant and hence \( Y^2f = 0 \). Such a function \( f \) is called a second integral of \( Y \). In complex geometry is important to study these integrals. The main reason is that they are the natural tool to produce new complete vector fields. While holomorphic first integrals of a complete \( Y \) were described in Suzuki’s work, the second ones had not been extensively studied until the recent work of D. Varolin [13].

**Property 3.** If \( Y \) is complete, it has a holomorphic first integral, and then its trajectories are proper. Therefore, \( X \) is in Suzuki’s list. There are two cases:

1.- \( Yf \) is not constant. Then \( Yf \) is an holomorphic first integral.

2.- \( Yf \) is constant. We observe that if \( C_z \) is of type \( \mathbb{C}^* \), \( f \circ \varphi_z \) is not only affine but even constant, because of \( \varphi_z^*(f \cdot Y) \) is invariant by a group of translations, and hence \( Yf = 0 \) along it. \( Yf = 0 \) on \( C_z \) implies \( Yf = 0 \) everywhere, for \( Yf \) is a constant, hence \( f \) itself is a first integral. Then we can assume that all the trajectories of \( Y \) must be of type \( \mathbb{C} \). This last property together with the fact of being \( f \circ \varphi_z \) linear implies that each trajectory \( C_z \) of \( Y \) (a copy of \( \mathbb{C} \)) meets all the fibres of \( f \) in an unique point. Then \( f \) must define a (global) fibration over \( \mathbb{C} \) which is trivialized by the trajectories of \( Y \), and hence these trajectories are proper. Moreover, according to Suzuki (see [10, p. 527]), there is a holomorphic first integral, which can be reduced to a coordinate after a holomorphic automorphism.

2.1. Rational first integrals.

**Proposition 1.** If \( \mathcal{F} \) has a rational first integral, up to a polynomial automorphism, \( X \) is as 1), 4), or 5) of Suzuki’s list. In fact, the first integral is polynomial.

*Proof.* Let us recall from the introduction that any complete holomorphic vector field with a meromorphic first integral (i.e. with a proper flow) can be transformed by a holomorphic automorphism in 1), 2) if \( \lambda/\mu \in \mathbb{Q} \), 4) or 5) of Suzuki’s list, with respectively \( x, y^p/x^q \) \( (p, q \in \mathbb{Z} \) with \( p/q = \lambda/\mu \in \mathbb{Q} \), \( x^m y^n \), and \( x^m(x^p y + p(x))^n \) as first integral. But still more, as \( X \) has rational first integral, the reduction to one of these possible forms can be obtained by
a polynomial automorphism (see proof of [10, Théorème 4]). What excludes 2), since \( f \) is transcendental and it cannot be transformed by a polynomial automorphism in a constant map.

From now on we will assume the absence of rational first integrals for \( \mathcal{F} \), and then for \( X \). Thus \( \tilde{\mathcal{F}} \) admits lots of tangent entire curves; one for each trajectory of \( X \), and most of them are Zariski dense in \( M \) by Darboux’s Theorem. What implies that the Kodaira dimension \( \text{kod}(\tilde{\mathcal{F}}) \) of \( \tilde{\mathcal{F}} \) is either 0 or 1 [7]. We will study these two possibilities as in [11, p. 437].

3. \( \text{kod}(\tilde{\mathcal{F}}) = 1 \)

According to McQuillan (see [7, Section IV]) \( \tilde{\mathcal{F}} \) must be a Riccati or a Turbulent foliation, that is, there is a fibration \( g : M \to B \) (maybe with singular fibres) whose generic fibre is respectively a rational or an elliptic curve transverse to \( \tilde{\mathcal{F}} \). We will say that \( g \) is adapted to \( \tilde{\mathcal{F}} \).

**Lemma 1.** \( \tilde{\mathcal{F}} \) is a Riccati foliation. In fact, \( g|_U \) is projected by \( \pi \) as a rational function \( R \) on \( \mathbb{C}^2 \) of type \( \mathbb{C} \) or \( \mathbb{C}^* \).

**Proof.** It follows from Property 1 that \( Y \) is of type \( \mathbb{C} \) or \( \mathbb{C}^* \):

1. - \( Y \) of type \( \mathbb{C} \). Then \( \mathcal{F} \) has only non-dicritical singularities. Otherwise we had infinitely many separatrices through a singularity, and infinitely many of them would define algebraic trajectories by Chow’s Theorem, which would give us a rational first integral for \( Y \) according to Darboux’s Theorem. Therefore both \( E \) and \( D \) are \( \tilde{\mathcal{F}} \)-invariant. But this implies that \( \tilde{\mathcal{F}} \) is a Riccati foliation because in this situation we can always construct a rational integral for a Turbulent \( \tilde{\mathcal{F}} \) [1, Lemma 1].

On the other hand, after contracting \( \tilde{\mathcal{F}} \)-invariant curves contained in fibres of \( g \) (rational curves), we can assume that \( g \) has no singular fibres and that around each \( \tilde{\mathcal{F}} \)-invariant fibre of \( g \), \( \tilde{\mathcal{F}} \) must follow one of the models described in [2, p. 56] and [11, p. 439]: nondegenerate, semidegenerate, or nilpotent. If we now analyze the proof of [11, Lemma 2], we see that it is enough to have that \( \text{kod}(\tilde{\mathcal{F}}) = 1 \), and that most of the leaves of \( \tilde{\mathcal{F}} \) are uniformized by \( \mathbb{C} \), what one knows by Property 1 to conclude that at least one of the \( \tilde{\mathcal{F}} \)-invariant fibres of \( g \) is semidegenerate, or nilpotent. But this fact and the invariancy of \( E \) and \( D \) by \( \tilde{\mathcal{F}} \) imply that the generic fibre of \( g \) must cut \( D \cup E \) in one or two points (see proof of [11, Lemma 5]). Hence the projection \( R \) is of type \( \mathbb{C} \) or \( \mathbb{C}^* \).

2. - \( Y \) of type \( \mathbb{C}^* \). The leaves of \( \mathcal{F} \) are proper and then they are properly embedded in \( \mathbb{C}^2 \) [11]. As \( \mathcal{F} \) has not rational first integrals, at least one leaf of \( \mathcal{F} \) defines a planar isolated end which is properly embedded in \( \mathbb{C}^2 \) and transcendental. It follows from [3] that \( \mathcal{F} \) is \( P \)-complete with \( P \) a
polynomial of type $\mathbb{C}^*$ or $\mathbb{C}$. But still more, as consequence of the proof of [3, Théorème], $P$ is obtained as the projection by $\pi$ of $g|_U$, that is, $R = P$ (see also [1, Proposition 3]).

Remark 1. We observe from 2.– of Lemma [1] that if the leaves of $\mathcal{F}$ are proper $R$ is a polynomial according to [3, Théorème].

We will study the two possibilities after the previous lemma.

3.1. $R$ of type $\mathbb{C}$.
By Suzuki (see [9]), up to a polynomial automorphism, we may assume that $R = x$. Hence $\mathcal{F}$ is a Riccati foliation adapted to $x$. Moreover, as the solutions of $X$ are entire maps, they can only avoid at most one vertical line by Picard’s Theorem. In particular $Y$ must be of the form

$$ Cx^N \frac{\partial}{\partial x} + [A(x)y + B(x)] \frac{\partial}{\partial y}, $$

with $C \in \mathbb{C}$, $N \in \mathbb{N}$, and $A, B \in \mathbb{C}[x]$ (see also [4, pp. 652-656]).

Let us take $G = f \cdot x^{N-1+\varepsilon}$ and $F = 1/x^{N-1+\varepsilon}$, with $\varepsilon = 0$ if $N \geq 1$, or $\varepsilon = 0$ or 1 if $N = 0$. Then $X$ is decomposed as the rational complete $F \cdot Y$ of the form (i) but with $a = A/x^{N-1+\varepsilon}$ and $b = B/x^{N-1+\varepsilon} \in 1/x^{N-1+\varepsilon} \cdot \mathbb{C}[x]$, where $d = 0$ and $c = C$, if $N \geq 1$ or $N = \varepsilon = 0$, or $c = 0$ and $d = C$, if $N = 0$ and $\varepsilon = 1$, multiplied by $G$. Let us observe that $dR(F \cdot Y) = c.R$ or $d$, and we conclude that $X$ has the form $i)$ of Theorem 2.

3.2. $R$ of type $\mathbb{C}^*$.
By Suzuki (see [10]), up to a polynomial automorphism, we may assume that $R = x^m(x^\ell y + p(x))^n$, where $m \in \mathbb{N}^*$, $n \in \mathbb{Z}^*$, with $(m, n) = 1$, $\ell \in \mathbb{N}$, $p \in \mathbb{C}[x]$ of degree $< \ell$ with $p(0) \neq 0$ if $\ell > 0$ or $p(x) \equiv 0$ if $\ell = 0$.

New coordinates. According to relations $x = u^n$ and $x^\ell y + p(x) = v u^{-m}$, it is enough to take the rational map $H$ from $u \neq 0$ to $x \neq 0$ defined by

$$ (u, v) \mapsto (x, y) = (u^n, u^{-(m+n\ell)} [v - u^m p(u^n)]) $$

in order to get $R \circ H(u, v) = v^n$.

Although $R$ is not necessarily a polynomial ($n \in \mathbb{Z}$), it follows from the proof of [4, Proposition 3.2] that $H^* \mathcal{F}$ is a Riccati foliation adapted to $v^n$ having $u = 0$ as invariant line. Thus

$$ H^* X = (f \circ H) \cdot H^* Y $$

$$ = (f \circ H(u, v)) \cdot u^k \cdot Z $$

$$ = (f \circ H(u, v)) \cdot u^k \cdot \left\{ a(v)u \frac{\partial}{\partial u} + c(v) \frac{\partial}{\partial v} \right\}, $$

(2)
where \( k \in \mathbb{Z} \), and \( a, c \in \mathbb{C}[v] \).

Our goal now is to prove that in (2) the polynomial \( c(v) \) is a monomial \( cv^N \). It will be a consequence of the following lemma.

**Lemma 2.** The line \( x = 0 \) is invariant by \( Y \).

*Proof.* Take the Riccati foliation \( \tilde{F} \) on \( M \), and let \( F \) be the fibre over 0. It follows from the local study of [1] or [2] that at most one irreducible component of \( F \) can be non-invariant by \( \tilde{F} \) (just look at the blow-up of models of [1]). Moreover, if such a non-invariant component exists, then it is everywhere transverse to the foliation. This settles immediately the case \( \ell = 0 \) in \( R \) since at least one irreducible component of \( \{xy = 0\} \) must be invariant.

In the case \( \ell > 0 \), \( \{R = 0\} \) has two disjoint components, one (the axis \( \{x = 0\} \)) isomorphic to \( \mathbb{C} \) and another isomorphic to \( \mathbb{C}^* \). We want to prove that the first is necessarily invariant. Let us assume the contrary. Let \( C \) be the irreducible component of \( F \) corresponding to \( \{x = 0\} \) and assume that it is transverse to \( \tilde{F} \). There is one and only one point \( p \in C \) which belongs to the divisor at infinity \( D \). This point is also the unique intersection point between \( C \) and the other components of \( F \). Because \( D \) and \( F \setminus C \) are invariant, and the foliation is regular at \( p \), we see that there exists a common irreducible component \( E \subset D \cap F \) such that, on a neighborhood \( U \) of \( C \), we have

\[
D \cap U = E \cap U \quad \text{and} \quad F \cap U = (E \cap U) \cup C.
\]

Now, by contracting components of \( F \) different from \( C \) we get a model \( C_0 \) like (a) of [1] (not like (b), which contains two quotient singularities). The direct image \( D_0 \) of \( D \) is then an invariant divisor which cuts \( C_0 \) at a single point \( p_0 \). Hence it cuts a generic fibre also at a single point, which contradicts that \( R \) is of type \( \mathbb{C}^* \). \( \square \)

By Lemma 2 as \( H \) is a finite covering map from \( u \neq 0 \) to \( x \neq 0 \), \( H^*X \) is complete on \( u \neq 0 \). Thus according to Picard’s Theorem its solutions are entire maps which can avoid at most one horizontal line, and hence \( c(v) \) in (2) is of the form \( cv^N \) with \( c \in \mathbb{C}, N \in \mathbb{N} \).

We can write \( H^*X \) as the product of the complete field \( 1/v^{N-1+\varepsilon} \cdot Z \) in \( u \neq 0 \) by the function \( f \circ H(u,v) \cdot u^k \cdot v^{N-1+\varepsilon} \), where \( \varepsilon = 0 \) if \( N \geq 1 \), or \( \varepsilon = 0 \) or 1 if \( N = 0 \).

**Proposition 2.** \( Y \) has proper trajectories

*Proof.* If \( X \) is of type \( \mathbb{C}^* \) it follows by [11]. If \( X \) is of type \( \mathbb{C} \), with the notations of §3.1 an §3.2, we distinguish two cases:
of type $\mathbb{C}$: Assume that $F \cdot Y$ is of type $\mathbb{C}$ by [11]. As $F \cdot Y$ is complete the restriction of $G$ to each solution $\varphi_z$ of that field is constant (Property 2), and hence $G$ is a meromorphic first integral of $Y$.

$R$ of type $\mathbb{C}^*$: Assume that $1/v^{N-1+\varepsilon} \cdot Z$ is of type $\mathbb{C}$ by [11]. One sees that $(f \circ H(u, v) \cdot u^k \cdot v^{N-1+\varepsilon}) \circ \varphi_z$ must be constant for each entire solution $\varphi_z$ of that field through points $z$ in $u \neq 0$ (Property 2). Then, according to (1), $(f \circ H(u, v) \cdot u^k \cdot v^{N-1+\varepsilon})^{mn}$ is projected by $H$ as

$$f^{\eta n} \cdot x^{\eta m} \cdot (x^\ell \cdot (x^\ell \cdot p(x))^n \cdot m^{N-1+\varepsilon})$$

thus obtaining a meromorphic first integral of $Y$.

The global one form of times. Let us take the one-form $\eta$ obtained when we remove the codimension one zeros and poles of $dR(x, y)$. The contraction of $\eta$ by $Y$, $\eta(Y)$, is a polynomial, which vanishes only on components of fibres of $R$ since $Y$ has only isolated singularities. In fact, the number of these fibres over nonzero values is at most one. Otherwise the entire solutions of $X$ would be projected by $R$ avoiding at least two points, which is impossible by Picard’s Theorem. Then, up to multiplication by constants:

$$\eta(Y) = \begin{cases} x^\alpha \cdot (x^\ell \cdot p(x))^\beta \cdot (x^m(x^\ell \cdot p(x))^n - s)^\gamma, & \text{if } n > 0; \\ x^\alpha \cdot (x^\ell \cdot p(x))^\beta \cdot (x^m - s(x^\ell \cdot p(x))^{-n})^\gamma, & \text{if } n < 0. \end{cases}$$

where $\alpha, \beta, \gamma \in \mathbb{N}$, and $s \in \mathbb{C^*}$.

Let us define $\tau = [1/(f \cdot \eta(Y))] \cdot \eta$. This one-form on $\{f \cdot \eta(Y) \neq 0\}$ coincides locally along each trajectory of $X$ with the differential of times given by its complex flow. It is called the global one-form of times for $X$. Moreover $\tau$ can be easily calculated attending to (3) as

$$\tau = \frac{x(x^\ell \cdot p(x))}{f \cdot \eta(Y)} \cdot \frac{dR}{R}. \tag{4}$$

In $(u, v)$ coordinates we then get

$$\varrho = H^* \tau = \begin{cases} \frac{u^{m(\beta-1)-n(\alpha-1)}}{(f \circ H) \cdot v^{\beta-1} \cdot (v^n - s)^\gamma} \cdot \frac{dv^n}{v^n}, & \text{if } n > 0; \\ \frac{u^{m(\beta-1)-n(\alpha-1)-mn\gamma}}{(f \circ H) \cdot v^{\beta-1} \cdot (1 - sv^n)^\gamma} \cdot \frac{dv^n}{v^n}, & \text{if } n < 0. \end{cases} \tag{5}$$

It holds that $\varrho(H^* X) = 1$. Since $\varrho - 1/[(f \circ H(u, v) \cdot u^k \cdot cv^N)] \cdot dv$ contracted by $H^* X$ is identically zero and we are assuming that there is no rational first integral,

$$\varrho = 1/[(f \circ H(u, v) \cdot u^k \cdot cv^N)] \cdot dv. \tag{6}$$
Therefore (5) and (6) must be equal and \( k \) of (2) can be explicitly calculated.

Finally, let us observe that for any path \( \epsilon \) contained in a trajectory of \( X \) from \( p \) to \( q \) that can be lifted by \( H \) as \( \tilde{\epsilon} \), \( \int_{\epsilon} \varrho \) represents the complex time required by the flow of \( X \) to travel from \( p \) to \( q \).

We may assume that \( \gamma = 0 \), \( \beta = N \) and \( \alpha > 0 \) in (5). Moreover according to Remark 1 we can also assume that \( R \) is a polynomial and that \( n > 0 \).

Let us observe that \( Y \) can be explicitly calculated as

\[
Y = u^k \cdot H_s(a(v)u \frac{\partial}{\partial u} + cv^N \frac{\partial}{\partial v}) =
\]

\[
u u^{n-1} \quad \begin{pmatrix} 0 \\ n\ell u^m p(u^n) - u^n + m \ell v - (m + n\ell)v \end{pmatrix} \cdot \begin{pmatrix} a(v)u \\ cv^N \end{pmatrix}
\]

where \( u = x^{1/n} \) and \( v = x^{m/n} (x^\ell y + p(x)) \).

We analyze two cases:

- \( N \geq 1 \). We show that each term \( 1/v^{N-1} \cdot Z \) and \( f \circ H(u, v) \cdot u^k \cdot v^{N-1} \) of the decomposition of \( H \cdot X \) can be separately projected by \( H \).

Let us observe that \( a(0) \neq 0 \). Otherwise \( Y \) had not isolated singularities since \( N > 0 \). The first component

\[
n x^{(k+n)/n} a(x^{m/n} (x^\ell y + p(x)))
\]

of (7) must be a polynomial. Since \( k = n(\alpha - 1) - m(N - 1) \) by (5) and (6), \( k = n \cdot \delta \) with \( \delta \in \mathbb{Z} \). On the other hand \( (m, n) = 1 \), and it implies that \( N - 1 = n \cdot \kappa \) with \( \kappa \in \mathbb{Z} \). Using (1), one gets

\[
H_s(f \circ H(u, v) \cdot u^k \cdot v^{N-1}) = G = f \cdot x^\delta \cdot (x^m (x^\ell y + p(x))^n)\kappa
\]

(8)

\[
H_s(1/v^{N-1} \cdot Z) = F \cdot Y = 1/((x^m (x^\ell y + p(x))^n)\kappa) \cdot Y.
\]

Finally, as \( dv^n(1/v^{N-1} \cdot Z) = nc \cdot v^n \), \( dR(F \cdot Y) = nc \cdot R \). If one now defines \( G \) and \( F \) according to (8), and \( \Omega = nc \), \( X \) is as in ii) and iii) of A) in Theorem 2.

- \( N = 0 \). As \( Y \) is a polynomial vector field with isolated singularities, a simple inspection of the two components in (7) implies that \( k = m + n\ell \) and \( a \in (1/z) \cdot \mathbb{C}[z^n] \), with \( a(0) = 0 \) if \( n > 1 \). Finally, according to (7), one sees that \( 1/x^{(m+n\ell)/n} \cdot Y \) it is obtained as the projection of a complete vector field whose trajectories are of type C. Therefore \( X \) is as in B) of Theorem 2.

This finishes the part of \( \text{kod}(\tilde{F}) = 1 \).
4. \( \text{cod}(\mathcal{F}) = 0 \)

We may suppose in what follows that \( Y \) is of type \( \mathbb{C}^* \). If \( Y \) is of type \( \mathbb{C}^* \), as \( f = 0 \) is empty or invariant by \( Y \), \( X \) is also of type \( \mathbb{C}^* \) (Property \([1]\)). According to \([11]\) the leaves of type \( \mathbb{C}^* \) of \( \mathcal{F} \) are proper and they are properly embedded in \( \mathbb{C}^2 \). Moreover, if these leaves of \( \mathcal{F} \) are algebraic there exists a rational integral by Darboux’s Theorem. Therefore at least one leaf of type \( \mathbb{C}^* \) of \( \mathcal{F} \) defines a planar isolated end which is properly embedded in \( \mathbb{C}^2 \) and transcendental, and then \( \mathcal{F} \) is \( P \)-complete with \( P \) a polynomial of type \( \mathbb{C}^* \) or \( \mathbb{C} \) \([11\text{, Proposition 3}] \) (see also 2.- of Lemma \([1]\)). What is enough to apply the results of \( \S 3.1 \) and \( \S 3.2 \).

According to \([7\text{, Section IV}] \) we can contract \( \tilde{\mathcal{F}} \)-invariant rational curves on \( M \) (via the contraction \( s \)) to obtain a new surface \( \bar{M} \) (maybe singular), a reduced foliation \( \bar{\mathcal{F}} \) on this surface, and a finite covering map \( r \) from a smooth \( S \) to \( \bar{M} \) such that: 1) \( r \) ramifies only over (quotient) singularities of \( \bar{M} \) and 2) the foliation \( r^*(\bar{\mathcal{F}}) \) is generated by a holomorphic vector field \( Z_0 \) on \( S \) with isolated zeroes. It follows from \([11\text{, p. 443}] \) that the covering \( r \) can be lifted to \( M \) via a birational morphism \( g : T \to S \) and a ramified covering \( h : T \to M \) such that \( s \circ h = r \circ g \). So we have the following diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{h} & T \\
| \downarrow s & \searrow & \downarrow g \\
| \swarrow & \nearrow & \downarrow \pi \\
\bar{M} & \xleftarrow{r} & S
\end{array}
\]

This construction guarantees the existence of two open sets \( V, W \subset T \) with the property that the covering \( \pi \circ h : V \to \mathbb{C}^2 \setminus \pi(E) \) is either unramified \( (V = W) \) or it ramifies only over a line \( L \in \mathbb{C}^2 \setminus \pi(E) \) \((V \neq W) \). It allows to lift \( Z_0 \) via \( g \) as a rational vector field \( Z \) on \( T \) generating \( g^*(r^*(\bar{\mathcal{F}})) = h^*(\tilde{\mathcal{F}}) \) and verifying that it is holomorphic and complete on \( W \), and with a pole along \( V \setminus W \) \([11\text{, Lemma 7}] \). One analyzes the above two possibilities:

1. If \( V = W \), using the regular cover \( \pi \circ h : V \to \mathbb{C}^2 \setminus \pi(E) \), that is trivial, one can extend \( Z \) to a finite set of points to thus obtain a complete polynomial vector field that generates \( \mathcal{F} \). Therefore \( Y = Z \) and then \( f \) is a holomorphic second integral of \( Y \). According to Property \([8]\) \( Y \) has a holomorphic first integral and hence its trajectories are proper.

On the other hand, we know that the flow of \( Y \) is algebraic, since this vector field arises from \( Z_0 \) on \( S \) which generates algebraic automorphisms of \( S \). As a consequence, after a polynomial automorphism, \( Y \) has to be one
of these two vector fields (see [1]):

a) $\lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y}, \quad \lambda, \mu \in \mathbb{C}, \lambda/\mu \notin \mathbb{Q},$

b) $\lambda x \frac{\partial}{\partial x} + (\lambda my + x^m) \frac{\partial}{\partial y}, \quad \lambda \in \mathbb{C}, m \in \mathbb{N}.$

Cases a), or b) with $m > 0,$ never have proper trajectories. Therefore $Y$ is as b) with $m = 0,$ and $X$ has the form i) of Theorem 2.

Remark 2. Once $Y$ have been determined, $f$ can be easily obtained. As $Y_y = 1,$ $f = H \cdot y + G,$ with $H$ and $G$ first integrals of $Y.$ It is enough to define $H = Yf$ and $G = f - y \cdot Yf.$ On the other hand, computing the flow of $Y,$ we can see directly that its trajectories are contained in the level sets of $xe^{-\lambda y}.$ Finally, according to Stein Factorization Theorem, $H = h(xe^{-\lambda y})$ and $G = g(xe^{-\lambda y})$ with $h, g$ entire functions in one variable.

2. – If $V \neq W$ then $Y$ is of type $\mathbb{C}^* \begin{array}{l}[1, p. 445] \end{array}$ which contradicts our assumptions.

5. POLYNOMIAL VERSION OF THEOREM 1

Theorem 2. Let $X$ be a complete vector field on $\mathbb{C}^2$ of the form $f \cdot Y,$ where $Y$ is a polynomial vector field with isolated singularities and $f$ is a transcendental function. Then, all the trajectories of $X$ are proper and, up to a polynomial automorphism, $X$ can be decomposed as $G \cdot F \cdot Y$ in one of the two following cases:

A) $G$ is a meromorphic function that is affine along the trajectories of a rational complete vector field $F \cdot Y$ such that $dR(F \cdot Y) = \Omega \cdot R^j,$ where $\Omega \in \mathbb{C}, j = 0$ or $1,$ $R$ is a polynomial of type $\mathbb{C}$ or $\mathbb{C}^*,$ and $F$ is constant along the fibres of $R.$ Explicitly, $X$ is defined by the following forms:

i) Case $R = x,$ where

- $F \cdot Y$ is as in I),
- $a, b \in 1/x^{N-1+\varepsilon} \cdot \mathbb{C}[x], \quad$ where $d = 0$ if $N \geq 1$ or $N = \varepsilon = 0,$ and $c = 0$ if $N = 0, \varepsilon = 1.$
- $F = 1/x^{N-1+\varepsilon}, \quad G = f \cdot x^{N-1+\varepsilon},$

with $N \in \mathbb{N}, \varepsilon = 0$ if $N \geq 1$ or $\varepsilon = 0, 1$ if $N = 0.$

ii) Case $R = x^m y^n,$ where

- $F \cdot Y$ is as in II),
- $\lambda \in 1/x^\kappa \cdot \mathbb{C}[z], \quad z = x^m y^n,$
- $F = 1/(x^m y^n)^\kappa, \quad G = f \cdot x^\delta \cdot (x^m y^n)^\kappa,$

with $\kappa, \delta \in \mathbb{Z}, m, n \in \mathbb{N}^*, \ (m, n) = 1.$

iii) Case $R = x^m (x^l y + p(x))^n,$ where

- $F \cdot Y$ is as in III),
by the polynomial vector field

- \( \lambda \in 1/z^\kappa \cdot \mathbb{C}[z], z = x^m(x^\ell y + p(x))^n, \)
- \( F = 1/(x^m(x^\ell y + p(x))^n)^\kappa, G = f \cdot x^\delta \cdot (x^m(x^\ell y + p(x))^n)^\kappa, \)

with \( \kappa, \delta \in \mathbb{Z}, m, n, \ell \in \mathbb{N}^*, (m, n) = 1, p \in \mathbb{C}[x] \) of degree \( < \ell, p(0) \neq 0. \)

**B)** \( G = f \cdot x^{(m+n\ell)/n} \) is a multivaluated holomorphic function that is affine along the trajectories of \( F \cdot Y, \) which is a multivaluated complete vector field with all its trajectories of type \( \mathbb{C} \) defined by the product of \( F = 1/x^{(m+n\ell)/n} \) by the polynomial vector field

\[
Y = u^{m+n(\ell+1)}a(v)\frac{\partial}{\partial x} + [n\ell u^m p(u^n) - u^{n+m} p'(u^n) - (m + n\ell)v]a(v) + c\frac{\partial}{\partial y}
\]

where \( u = x^{1/n}, v = x^{m/n}(x^\ell y + p(x)), \) with \( m, n \in \mathbb{N}^*, (m, n) = 1, \ell \in \mathbb{N}, \)
\( p \in \mathbb{C}[x] \) of degree \( < \ell, p(0) \neq 0 \) if \( \ell > 0 \) or \( p(x) \equiv 0 \) if \( \ell = 0, c \in \mathbb{C}^*, \) and \( a \in (1/z) \cdot \mathbb{C}[z^n], \) with \( a(0) = 0 \) if \( n > 1. \)

**Remark 3.** Vector fields in A) are obtained by multiplication of a rational complete one \( F \cdot Y \) in (rational) Brunella’s classification and a meromorphic second integral \( G, \) that is, a function which is affine along the trajectories of \( F \cdot Y. \) It is important to remark that in Brunella’s list there are nonproper vector fields, which do not appear in our classification due to the existence of \( f. \)

On the other hand, any \( X \) of ii) and iii) can be expressed after the rational change of coordinates \( H \) given by (1) as

\[
H^*X = f \circ H(u, v) \cdot u^{nd} \cdot \left\{ a(v)u\frac{\partial}{\partial u} + cv^m\frac{\partial}{\partial v} \right\}.
\]

Let us also note that 1), 4) and 5) of Suzuki’s list define respectively cases i), ii) and iii) with polynomial first integral. However, A) contains other different vector fields since i), ii) and iii) can not be reduced in general to 1), 4) and 5) by a polynomial automorphism.

Vector fields \( X \) in B) can be expressed after \( H \) as

\[
H^*X = f \circ H(u, v) \cdot u^{m+n\ell} \cdot \left\{ a(v)u\frac{\partial}{\partial u} + c\frac{\partial}{\partial v} \right\}.
\]

Example 2 gives us one \( X \) in B) with an explicit \( f \) which is not in A).

**Example 2.** Let us consider

\[
X = f \cdot Y = e^{-(m/nc) \cdot z^m y^n} \left\{ z^{1+my^{n-1}}\frac{\partial}{\partial x} - (mx^m y^n - c)\frac{\partial}{\partial y} \right\}.
\]

We see that \( X \) is as in B) of Theorem 2 \( Y \) is as in B) with \( \ell = 0 \) and \( a(z) = (1/z) \cdot z^n = z^{n-1} \) and \( X \) is complete, since according to (7), \( H^*X \)
equals to
\[ e^{-(m/nc) \cdot v^n} \cdot H^*Y = (u \cdot e^{-(1/nc) \cdot v^n})^m \cdot \left\{ v^{n-1}u \frac{\partial}{\partial u} + c \frac{\partial}{\partial v} \right\} \]
is a complete polynomial vector field multiplied by a first integral.

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