Abstract: We exploit the power and potential of the (anti-)chiral superfield approach (ACSA) to Becchi-Rouet-Stora-Tyutin (BRST) formalism to derive the nilpotent (anti-)BRST symmetry transformations for any arbitrary $D$-dimensional interacting non-Abelian 1-form gauge theory where there is an $SU(N)$ gauge invariant coupling between the gauge field and the Dirac fields. We derive the conserved and nilpotent (anti-)BRST charges and establish their nilpotency and absolute anticommutativity properties within the framework of ACSA to BRST formalism. The clinching proof of the absolute anticommutativity property of the conserved and nilpotent (anti-)BRST charges is a novel result in view of the fact that we consider, in our present endeavor, only the (anti-)chiral super expansions of the superfields that are defined on the $(D,1)$-dimensional super-submanifolds of the general $(D,2)$-dimensional supermanifold on which our $D$-dimensional ordinary interacting non-Abelian 1-form gauge theory is generalized. To corroborate the novelty of the above result, we apply the ACSA to an $\mathcal{N} = 2$ supersymmetric (SUSY) quantum mechanical (QM) model of a harmonic oscillator and show that the nilpotent and conserved $\mathcal{N} = 2$ super charges of this system do not absolutely anticommutate.

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1 Introduction

The abstract mathematical properties (i.e. off-shell nilpotency and absolute anticommutativity) associated with the Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST symmetries find their geometrical basis within the framework of the usual superfield approach (USFA) to BRST formalism. One of the key concepts behind USFA is the idea of horizontality condition (HC) where a particular geometrical quantity (i.e. an exterior derivative) plays a decisive role. The central outcome of the HC is the observation that it leads to the derivation of (anti-)BRST symmetries for only the gauge field and associated (anti-)ghost fields of a given (anti-)BRST invariant theory. It does not shed any light on the derivation of the (anti-)BRST symmetries associated with the matter fields in an interacting gauge theory. The USFA has been systematically and consistently extended so as to derive the (anti-)BRST symmetry transformations for the gauge, (anti-)ghost and matter fields together. The extended version of the USFA has been christened as the augmented version of superfield approach (AVSA) to BRST formalism [9-12] where, in addition to the HC, the gauge invariant restrictions (GIRs) have also been invoked. The latter are consistent with the HC and both of them complement each-other in a meaningful manner (within the framework of AVSA) where we precisely derive the nilpotent symmetries for all the fields of an interacting gauge theory.

The key feature of the above superfield approaches [1-12] is the fact that all the superfields of the \((D, 2)\)-dimensional supermanifold (on which a given \(D\)-dimensional ordinary gauge theory is generalized) are expanded along all the possible Grassmannian directions of the supermanifold. This supermanifold is parameterized by the superspace coordinates \(Z^M = (x^\mu, \theta, \bar{\theta})\) where \(x^\mu (\mu = 0, 1, ..., D - 1)\) are the \(D\)-dimensional bosonic coordinates and a pair of Grassmannian variables satisfy: \(\theta^2 = \bar{\theta}^2 = 0, \, \theta \bar{\theta} + \bar{\theta} \theta = 0\). The purpose of our present endeavor is to apply a simpler version of the above superfield approaches [1-12] where only the (anti-)chiral superfields are taken into account for the derivation of the proper (anti-)BRST symmetry transformations. In a recent set of papers [13-15], we have exploited the (anti-)chiral superfield/supervariable approach (ACSA) to BRST formalism \(\ast\) to obtain the (anti-)BRST symmetries for the \(D\)-dimensional (non-)interacting Abelian 1-form and 1D toy models of gauge theories. We have been able to establish that, despite the (anti-)chiral superfield/supervariable considerations, the (anti-) BRST charges turn out to be absolutely anticommuting in nature. This observation is a completely novel and surprising result within the framework of ACSA to BRST formalism. So far, we have not applied the ACSA to BRST formalism in the case of any arbitrary \(D\)-dimensional (non-)interacting non-Abelian 1-form gauge theory (with or without matter fields).

In our present investigation, we apply, first of all, the ACSA to BRST formalism in the case of a \(D\)-dimensional interacting non-Abelian gauge theory and show that the expressions for the (anti-)BRST charges for the interacting non-Abelian theory are exactly the same as in the case of non-Abelian theory without matter fields. In other words, as can be seen in the expressions (see, e.g. Eq. (12) below), there is a possibility of no presence of the matter fields in the expressions for the (anti-)BRST charges. As a consequence, the nilpotency

\(\ast\)The beauty of the (anti-)chiral superfield/supervariable approach is the observation that we obtain the (anti-)BRST symmetries for all the fields/variables of the theory from the (anti-)BRST (i.e. quantum gauge) invariant restrictions on the (anti-)chiral superfields/supervariables.
and absolute anticommutativity of the (anti-)BRST charges can be captured within the framework of ACSA to BRST formalism, too, as has been done in our previous work [16]. In fact, we have been able to demonstrate the above mathematically elegant properties within the framework of AVSA to BRST formalism where the full expansions of the superfields have been taken into account [16]. In the proof of the absolute anticommutativity property, we have been forced to invoke the CF-condition [17] to recast the expressions for the nilpotent (anti-)BRST charges in their appropriate forms (see, Sec. 6) as has been also done in our earlier works [16, 18]. Thus, we adopt here the same theoretical trick for the proof of the absolute anticommutativity of the nilpotent (anti-)BRST charges for our interacting non-Abelian 1-form theory with Dirac fields.

One of the highlights of our present investigation is the theoretical material contained in Sec. 6 where we have captured the nilpotency and absolute anticommutativity properties of the (anti-)BRST charges within the framework of ACSA to BRST formalism. We have been able to express these expressions in the ordinary space where the explicit (anti-)BRST symmetry transformations for the ordinary fields and their off-shell nilpotency properties have been taken into account in a judicious manner. It is pertinent, at this stage, to pinpoint the fact that our knowledge of the ordinary and superspace formulations have helped each-other in a beautiful and complementary fashion in our theoretical discussions. Sometimes our knowledge, in the ordinary space, has helped us in our theoretical discussions in the context of superspace formulation and, at other times, our understanding of the superspace formulation has come in handy for our theoretical discussions in the ordinary space. Thus, the contents of Sec. 6 (which are one of the highlights of our present endeavor) are the outcome of our understandings of the BRST formalism in the ordinary space and superspace and their inter-connections†.

Against the backdrop of the above discussions, we would like to lay emphasis on the fact that we have also applied the (anti-)chiral supervariable approach (ACSA) to the \( \mathcal{N} = 2 \) SUSY quantum mechanical models of various kinds [19-22] and derived the \( \mathcal{N} = 2 \) SUSY symmetry transformations and corresponding conserved and nilpotent super charges. In this derivation, the role of SUSY invariant quantities has been very decisive because we have demanded that such quantities should not depend on the Grassmannian variables \((\theta, \bar{\theta})\). We have been able to prove the nilpotency of these \( \mathcal{N} = 2 \) super charges. However, it has been found that the ACSA (applied to the \( \mathcal{N} = 2 \) SUSY QM models) does not lead to the derivation of the absolute anticommutativity for the \( \mathcal{N} = 2 \) conserved and nilpotent super charges. This observation, for obvious reasons, is consistent with the basic tenets of \( \mathcal{N} = 2 \) SUSY QM systems. In our Appendix B, we demonstrate this fact in an explicit fashion so that novelty of our observation of the absolute anticommutativity property for the (anti-)BRST charges could be corroborated within the framework of ACSA. Thus, it is crystal clear that ACSA does not lead to the absolute anticommutativity of the conserved and nilpotent charges everywhere.

The following key factors have propelled our curiosity to pursue our present investi-

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†We have established that the nilpotent and absolutely anticommuting (anti-)BRST transformations (and their corresponding conserved charges) are deeply connected with the translational generators \((\partial_{\theta}, \partial_{\bar{\theta}})\) along the Grassmannian directions of the \((D, 1)\)-dimensional (anti-)chiral super-submanifolds (of the general \((D, 2)\)-dimensional supermanifold on which our \(D\)-dimensional ordinary non-Abelian 1-form interacting gauge theory is generalized).
First, we have captured the nilpotency and absolute anticommutativity of the fermionic (anti-)BRST charges for the interacting Abelian 1-form gauge theories with Dirac and complex scalar fields \cite{18} as well as 4D Abelian 2-form gauge theory \cite{15} within the framework of ACSA to BRST formalism. Thus, it is very important for us to prove the same in the context of an interacting non-Abelian 1-form gauge theory with Dirac fields so that our ideas, connected with the ACSA to BRST formalism, could be firmly established. Second, the ideas of ACSA to BRST formalism are simple and straightforward and they lend support to the augmented version of superfield approach (AVSA) to BRST formalism which is based on the more formal and precise mathematical foundations (see, e.g. \cite{4, 5, 9-12, 16}). Our present work, once again, establishes the validity of this observation where there is a complete agreement between our results (with ACSA) and that of the AVSA to BRST formalism. Third, to establish the novelty of our observation of the absolute anticommutativity property in the context of the (anti-)BRST charges, we have concentrated on the application of ACSA to an \(\mathcal{N}=2\) SUSY QM model of harmonic oscillator and shown that the \(\mathcal{N}=2\) super charges do not absolutely anticommute (cf. Appendix B). Finally, our present endeavor is also our modest step forward towards our central objective of applying the theoretical techniques of ACSA to BRST formalism in the context of higher \(p\)-form (\(p = 2, 3, \ldots\)) gauge theories as well as other physically interesting \(\mathcal{N}=2\) SUSY QM models (which are popular in literature).

The theoretical materials of our present investigation are organized as follows. First of all, we discuss in Sec. 2, the bare essentials of the nilpotent and absolutely anticommuting (anti-)BRST symmetries within the framework of Lagrangian formulation. The subject matter of Sec. 3 concerns itself with the derivation of BRST symmetries of our theory by exploiting the anti-chiral superfields and their super expansions. Our Sec. 4 is devoted to the discussion of anti-BRST symmetries which are derived by using the anti-BRST invariant restrictions on the chiral superfields. Sec. 5 of our paper contains the discussion about the invariance of the Lagrangian densities within the framework of (anti-)chiral superfield formalism. In Sec. 6, we deal with the discussion of nilpotency and absolute anticommutativity properties of the conserved (anti-)BRST charges within the framework of ACSA to BRST formalism. Finally, we summarize our key results in Sec. 7 and point out a few possible future theoretical directions for further investigation(s).

In our Appendix A, we concisely discuss about the novelty of our key observation of the absolute anticommutativity property (associated with the (anti-)BRST charges) within the framework of ACSA to BRST formalism. Our Appendix B is devoted to the application of ACSA to the \(\mathcal{N}=2\) SUSY QM model of a 1D harmonic oscillator where we demonstrate that the \(\mathcal{N}=2\) nilpotent and conserved super charges do not absolutely anticommute.

Convention and Notations: We adopt the convention of taking the metric tensor \(\eta_{\mu\nu}\) for the background \(D\)-dimensional flat Minkowskian spacetime as: \(\eta_{\mu\nu}=\text{diag}\,(+1,-1,-1,\ldots)\) so that \(\partial_{\mu}A^\mu = \partial_0 A_0 - \partial_i A_i\) where the Greek indices \(\mu, \nu, \lambda \ldots = 0, 1, 2, \ldots D-1\) represent the time and space directions and Latin indices \(i, j, k \ldots = 1, 2, 3, \ldots D-1\) correspond to the space directions only. We choose the convention of dot and cross products in the Lie algebraic space as: \(P \cdot Q = P^a Q^a\) and \((P \times Q)^a = f^{abc} P^b Q^c\) between a set of two non-null vectors \((P^a, Q^a)\) where \(a, b, c, \ldots = 1, 2, 3, \ldots \mathcal{N}^2-1\) and \(f^{abc}\) are the totally antisymmetric structure constants for the \(SU(N)\) Lie algebra. We have also adopted the convention of left-derivative
in all our relevant computations with respect to the fermionic fields \((\psi, \bar{\psi}, C, \bar{C})\) which obey: 
\[ \psi \psi + \bar{\psi} \bar{\psi} = 0, \psi^2 = 0, \bar{\psi}^2 = 0, (C)^2 = 0, (\bar{C})^2 = 0, C^a \bar{C}^b + \bar{C}^a C^b = 0, C^a \bar{C}^b + \bar{C}^b C^a = 0, C \psi + \psi \bar{C} = 0, \]
etc. We denote the (anti-)BRST transformations and corresponding conserved charges by the notations \(s_{(a)b}\) and \(Q_{(ab)}\) in the whole body of our text. In the context of the \(\mathcal{N} = 2\) SUSY QM model, we have adopted the notations \(s_1\) and \(s_2\) for the SUSY transformations and corresponding conserved charges have been denoted by \(Q\) and \(\bar{Q}\).

2 Preliminaries: Lagrangian Formulation

We begin with the (anti-)BRST invariant coupled (but equivalent) Lagrangian densities (see, e.g. [23] for details) for the \(D\)-dimensional non-Abelian 1-form gauge theory where there is a coupling between the gauge field \((A_\mu)\) and Dirac fields \((\bar{\psi}, \psi)\), in the Curci-Ferrari gauge [24, 25], as

\[
\mathcal{L}_B = -\frac{1}{4} F_{\mu \nu} \cdot F_{\mu \nu} + \bar{\psi} (i \gamma^\mu D_\mu - m) \psi + B \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i \partial_\mu \bar{C} \cdot D^\mu C, \\
\mathcal{L}_{\bar{B}} = -\frac{1}{4} F^\mu_{\nu} \cdot F^\mu_{\nu} + \bar{\psi} (i \gamma^\mu D_\mu - m) \psi - \bar{B} \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i D_\mu \bar{C} \cdot \partial^\mu C, \tag{1}
\]

where the covariant derivatives \(D_\mu \psi = \partial_\mu \psi + i (A_\mu \cdot T) \psi\) and \(D_\mu C = \partial_\mu C + i (A_\mu \times C)\) are in the fundamental and adjoint representations of the \(SU(N)\) Lie algebra, respectively. This algebra is generated by the operators \((T^a)\) that satisfy: \([T^a, T^b] = f^{abc} T^c\) where \(f^{abc}\) are the structure constants that can be chosen to be totally antisymmetric in indices \(a, b, c = 1, 2...N^2 - 1\) for the semi-simple Lie group \(SU(N)\) (see, e.g. [26] for details).

In the above, the Nakanishi-Lautrup type auxiliary fields \(B(x)\) and \(\bar{B}(x)\) satisfy the Curci-Ferrari (CF)-condition \(B + \bar{B} + (C \times \bar{C}) = 0\) [17] which emerges from the equivalence requirement of the coupled (but equivalent) Lagrangian densities \(\mathcal{L}_B\) and \(\mathcal{L}_{\bar{B}}\) that mathematically implies the following

\[
B \cdot (\partial_\mu A^\mu) - i \partial_\mu \bar{C} \cdot D^\mu C \equiv -\bar{B} \cdot (\partial_\mu A^\mu) - i D_\mu \bar{C} \cdot \partial^\mu C, \tag{2}
\]

modulo a total spacetime derivative. It turns out, the following infinitesimal, continuous, off-shell nilpotent \((s_{(a)b}^2 = 0)\) and absolutely anticommuting \((s_b s_{ab} + s_{ab} s_b = 0)\) (anti-)BRST symmetry transformations \((s_{(a)b})\)

\[
\begin{align*}
  s_{ab} A_\mu &= D_\mu \bar{C}, & s_{ab} \bar{C} &= -\frac{i}{2} (\bar{C} \times \bar{C}), & s_{ab} C &= i \bar{B}, & s_{ab} \bar{B} &= 0, \\
  s_{ab} F_{\mu \nu} &= i (F_{\mu \nu} \times \bar{C}), & s_{ab} (\partial_\mu A^\mu) &= \partial_\mu \bar{D}^\mu \bar{C}, & s_{ab} \psi &= -i \bar{C} \psi, \\
  s_{ab} \bar{\psi} &= -i \bar{\psi} \bar{C}, & s_{ab} B &= i (B \times \bar{C}), \\
  s_b A_\mu &= D_\mu C, & s_b C &= -\frac{i}{2} (C \times C), & s_b \bar{C} &= i B, & s_b \bar{B} &= 0.
\end{align*}
\]
leave the action integrals \((S_1 = \int d^Dx \mathcal{L}_B)\) and \((S_2 = \int d^Dx \mathcal{L}_B)\) invariant. In fact, we also note here that the Lagrangian densities transform to the total spacetime derivatives (plus extra terms) under \(s_{(a)b}\) as given below:

\[
\begin{align*}
\mathcal{L}_B & = \partial_\mu (B \cdot D^\mu C), \quad s_{ab} \mathcal{L}_B = - \partial_\mu (\bar{B} \cdot D^\mu \bar{C}), \quad s_b \mathcal{L}_B = - \partial_\mu (B \cdot D^\mu \bar{C}), \\
\mathcal{L}_B & = \partial_\mu [(B + (C \times \bar{C}) \cdot \partial^\mu C] - (B + \bar{B} + (C \times \bar{C}) \cdot D_\mu \partial^\mu C), \\
s_{ab} \mathcal{L}_B & = - \partial_\mu [(\bar{B} + (C \times \bar{C})) \cdot \partial^\mu \bar{C}] + \{(B + \bar{B} + (C \times \bar{C})) \cdot D_\mu \partial^\mu \bar{C}. \quad (4)
\end{align*}
\]

We point out that the above extra pieces, besides the total spacetime derivative terms, are deeply connected with the CF-condition: \(B + \bar{B} + (C \times \bar{C}) = 0\).

At this juncture, a few comments are in order. First of all, we note that both the Lagrangian densities respect both (BRST and anti-BRST) symmetry transformations \(s_{(a)b}\) if the whole interacting non-Abelian theory is confined to be defined on a hypersurface in the \(D\)-dimensional Minkowski space where the CF-condition \([B + \bar{B} + (C \times \bar{C}) = 0]\) is satisfied. In other words, we note that we have the following

\[
\begin{align*}
\mathcal{L}_B & = - \partial_\mu \bar{B} \cdot \partial^\mu C, \\
s_{ab} \mathcal{L}_B & = \partial_\mu [B \cdot \partial^\mu \bar{C}], \quad \{s_b, s_{ab}\} A_\mu = 0, \\
\{s_b, s_{ab}\} F_{\mu\nu} = 0, \\
\{s_b, s_{ab}\} \psi = 0, \\
\{s_b, s_{ab}\} \bar{\psi} = 0, \quad \{s_b, s_{ab}\} \end{align*}
\]

(5)

are satisfied only when the CF-condition is taken into account.

According to the Noether theorem, the invariance of the action integrals \((S_1 = \int d^Dx \mathcal{L}_B, S_2 = \int d^Dx \mathcal{L}_B)\), under the BRST and anti-BRST transformations (i.e. \(s_b \mathcal{L}_B = \partial_\mu [B \cdot D^\mu \bar{C}], s_{ab} \mathcal{L}_B = - \bar{\partial}_\mu [\bar{B} \cdot D^\mu \bar{C}]\), leads to the following Noether conserved currents:

\[
\begin{align*}
J^\mu_{(ab)} & = - \bar{B} \cdot D^\mu \bar{C} - F_{\mu\nu} \cdot D_\nu C - \bar{\psi} \gamma^\mu \bar{C} \psi - \frac{1}{2} (\bar{C} \times C) \cdot \partial^\mu C, \\
J^\mu_{(b)} & = B \cdot D^\mu C - F_{\mu\nu} \cdot D_\nu C - \psi \gamma^\mu C \psi + \frac{1}{2} \bar{\partial}^\mu \bar{C} \cdot (C \times C). \quad (7)
\end{align*}
\]

The conservation laws \((\partial_\mu J^\mu_{(ab)} = 0)\) can be proven by using the following Euler-Lagrange (EL) equations of motion (EOM) which emerge from the Lagrangian density \(\mathcal{L}_B\), namely:

\[
\begin{align*}
B & = - (\partial_\mu A^\mu), \\
\partial_\mu (D^\mu C) & = 0, \\
D_\mu (\partial^\mu \bar{C}) & = 0, \\
(i \gamma^\mu \partial_\mu - m) \psi & = \gamma^\mu A_\mu \psi, \\
(i (\partial_\mu \bar{\psi}) \bar{\gamma}^\mu + m \bar{\psi} & = - \bar{\psi} \gamma^\mu A_\mu, \\
D_\mu F_{\mu\nu} - \partial^\nu B & = \bar{\psi} \gamma^\mu \psi + (\partial^\nu \bar{C} \times C). \quad (8)
\end{align*}
\]
In an exactly similar fashion, one could compute the EL-EOM from the Lagrangian density $\mathcal{L}_B$ which turn out to be similar to (8) except the following:

$$D_\mu F^{\mu\nu} + \partial^\nu \bar{B} = \bar{\psi}_i \gamma^\nu \psi_i - (\bar{C} \times \partial^\nu C),$$
$$\partial_\mu (D_\mu \bar{C}) = 0, \quad D_\mu (\partial^\mu C) = 0, \quad \bar{B} = \partial_\mu A^\mu. \quad (9)$$

The conserved (anti-)BRST charges $(Q_{(a)b} = \int d^D x J^0_{(a)b})$ can be derived from the above conserved currents (7) as:

$$Q_{ab} = \int d^{D-1} x \left[ - \bar{B} \cdot D_0 \bar{C} \left( \frac{1}{2} (\bar{C} \times \bar{C}) \cdot C - \bar{\psi} \gamma^0 \bar{C} \psi - F^0_i \cdot D_i \bar{C} \right) \right],$$
$$Q_b = \int d^{D-1} x \left[ B \cdot D_0 C + \frac{1}{2} \dot{C} \cdot (C \times C) - \bar{\psi} \gamma^0 C \psi - F^0_i \cdot D_i C \right]. \quad (10)$$

It can be explicitly checked that the terms $(-F^0_i \cdot D_i C)$ and $(-F^0_i \cdot D_i \bar{C})$ can be written, in terms of total space derivatives, as:

$$-F^0_i \cdot D_i C = - \partial_i \left[ F^0_i \cdot C \right] + (D_i F^0_i) \cdot C,$$
$$-F^0_i \cdot D_i \bar{C} = - \partial_i \left[ F^0_i \cdot \bar{C} \right] + (D_i F^0_i) \cdot \bar{C}. \quad (11)$$

Applying the Gauss divergence theorem and using the EL-EOM w.r.t. gauge field, it is straightforward to check that the above charges can be re-expressed in the following concise forms:

$$Q_{ab} = \int d^{D-1} x \left[ \bar{B} \cdot \bar{C} - B \cdot D_0 \bar{C} + \frac{1}{2} (\bar{C} \times \bar{C}) \cdot \dot{C} \right],$$
$$Q_b = \int d^{D-1} x \left[ B \cdot D_0 C - \bar{B} \cdot C - \frac{1}{2} \dot{C} \cdot (C \times C) \right]. \quad (12)$$

To be more precise, we have used the following EL-EOM

$$D_i F^{i0} - \dot{B} = \bar{\psi}_i \gamma^0 \psi_i + (\dot{\bar{C}} \times C),$$
$$D_i \bar{F}^{i0} - \dot{\bar{B}} = \bar{\psi}_i \gamma^0 \psi_i - (\dot{\bar{C}} \times \bar{C}), \quad (13)$$

which have emerged out from Eqs. (8) and (9).

It is interesting to point out that the matter fields disappear from the final expressions for the above (anti-)BRST charges and these expressions appear as if there were no matter fields in the theory. This has happened because of the fact that we have used the EOM (13) and the theoretical technique elaborated in (11). We shall concentrate on these concise expressions (i.e. Eq. (12)) of the conserved charges for our further discussions within the framework of ACSA to BRST formalism and capture their nilpotency and absolute anticommutativity properties.

### 3 Off-Shell Nilpotent BRST Symmetries: Anti-Chiral Superfields and Their Super Expansions

We exploit, in this section, the strength of BRST (i.e. quantum gauge) invariant restrictions on the anti-chiral superfields to derive the off-shell nilpotent BRST symmetry transformations. Towards this goal in mind, first of all, we generalize the ordinary $D$-dimensional fields...
(A_\mu, C, \bar{C}, \bar{B}, \psi, \bar{\psi}) of the Lagrangian density (1) onto a (D, 1)-dimensional anti-chiral super-submanifold (parameterized by the superspace coordinates x^\mu and \bar{\theta}) as

\begin{align*}
A_\mu(x) &\rightarrow B_\mu(x, \bar{\theta}) = A_\mu(x) + \bar{\theta} \, R_\mu(x), & C(x) &\rightarrow F(x, \bar{\theta}) = C(x) + i \, \bar{\theta} \, B_1(x), \\
\bar{C}(x) &\rightarrow \bar{F}(x, \bar{\theta}) = \bar{C}(x) + i \, \bar{\theta} \, B_2(x), & B(x) &\rightarrow \bar{B}(x, \bar{\theta}) = B(x) + i \, \bar{\theta} \, f_1(x), \\
\bar{B}(x) &\rightarrow \tilde{B}(x, \bar{\theta}) = B(x) + i \, \bar{\theta} \, f_2(x), & \psi(x) &\rightarrow \Psi(x, \bar{\theta}) = \psi(x) + i \, \bar{\theta} \, b_1(x), \\
\bar{\psi}(x) &\rightarrow \bar{\Psi}(x, \bar{\theta}) = \bar{\psi}(x) + i \, \bar{\theta} \, b_2(x),
\end{align*}

(14)

where the secondary fields (B_1, B_2, b_1, b_2), on the r.h.s., are bosonic in nature and the set of secondary fields (R_\mu, f_1, f_2) is fermionic. We further note that all the fields are defined as: A_\mu = A_\mu \cdot T, \quad C = C \cdot T, \quad b_1 = b_1 \cdot T, \quad R_\mu = R_\mu \cdot T, \text{ etc.}

We have to derive the explicit form of the above secondary fields in terms of the basic and auxiliary fields of our starting Lagrangian densities (1) for our non-Abelian 1-form interacting gauge theory in D-dimensions of spacetime.

Towards the above objective in mind, we list here the useful and interesting BRST invariant quantities for the Lagrangian density \( \mathcal{L}_B \) (of (1)) as:

\begin{align*}
& s_b (D_\mu C) = 0, \quad s_b B = 0, \quad s_b (C \times C) = 0, \quad s_b (C \psi) = 0, \\
& s_b \left[ A^\mu \cdot \partial_\mu B + i \, \partial_\mu \bar{C} \cdot D^\mu C \right] = 0.
\end{align*}

(15)

According to the basic tenets of ACSA to BRST formalism, the above set of quantities\(^4\) should be independent of the “soul” coordinate \( \bar{\theta} \) when these physically important quantities are generalized onto a (D, 1)-dimensional anti-chiral super-submanifold (of the general (D, 2)-dimensional supermanifold). For instance, we note that, the following are true\(^5\), namely;

\begin{align*}
s_b B = 0 &\quad \implies \bar{B}(x, \bar{\theta}) = B(x) \implies f_1(x) = 0 \\
&\quad \implies \bar{B}^{(b)}(x, \bar{\theta}) = B(x) + \bar{\theta} \, (0), \\
s_b (C \times C) = 0 &\quad \implies F(x, \bar{\theta}) \times F(x, \bar{\theta}) = C(x) \times C(x) \\
&\quad \implies B_1 \times C = 0.
\end{align*}

(16)

The latter condition \( B_1 \times C = 0 \) implies that one of the non-trivial solutions of this restriction is \( B_1 \propto (C \times C) \) because we know that: \( (C \times C) \times C = 0 \). Let us choose \( B_1 = \kappa \, (C \times C) \) where \( \kappa \) is some numerical constant. With the above choice, we have the reduced/modified form of the superfield \( F(x, \bar{\theta}) \) as:

\begin{equation}
F(x, \bar{\theta}) \rightarrow F^{(m)}(x, \bar{\theta}) = C(x) + i \, \bar{\theta} \, \kappa \, (C \times C).
\end{equation}

(17)

\(^4\)The analogues of the GIRs (at the classical level) are the (anti-)BRST invariant quantities (at the quantum level). Hence, they are physical and they are required to be independent of the “soul” coordinates when they are recast in the language of superfields within the framework of ACSA to BRST formalism. We would like to mention here that the BRST invariant quantities in (15) have been found by the method of trial and error as there is no definite rule/principle to obtain them.

\(^5\)We shall denote the anti-chiral superfields with superscript \( (b) \) whose super expansions would lead to the derivation of BRST symmetry transformations (for the corresponding ordinary fields) as the coefficient of \( \bar{\theta} \) in the anti-chiral super expansions of the superfields.
Here the superscript \((m)\) denotes the modified form of the superfield. Now we focus on 
\(s_b(D_\mu C) = 0\) which implies the following equality (with the input from (17)):
\[
\partial_\mu F^{(m)}(x, \bar{\theta}) + i B_\mu(x, \bar{\theta}) \times F^{(m)}(x, \bar{\theta}) = \partial_\mu C(x) + i A_\mu(x) \times C(x). \tag{18}
\]

The substitution of the super expansions from (14) and (17), in the above, leads to the following important relationship:
\[
R_\mu = -2 \kappa D_\mu C(x). \tag{19}
\]

As a consequence, we have the modified form of the superfield \(B_\mu(x, \bar{\theta})\) as:
\[
B_\mu(x, \bar{\theta}) \longrightarrow B^{(m)}_\mu(x, \bar{\theta}) = A_\mu(x) - 2 \kappa \bar{\theta} D_\mu C(x). \tag{20}
\]

We exploit now the BRST invariant quantity \(s_b(B \times \bar{C}) = 0\). This invariance leads to the following equality (with input from (16)), namely;
\[
\bar{B}^{(b)}(x, \bar{\theta}) \times F(x, \bar{\theta}) = B(x) \times C(x), \tag{21}
\]
and using (15) and (14), we obtain \(B_2(x) \propto B(x)\). For the sake of brevity\(^4\), however, we choose \(B_2(x) = B(x)\) so that we obtain
\[
\bar{F}^{(b)}(x, \bar{\theta}) = \bar{C}(x) + i \bar{\theta} B(x) = \bar{C}(x) + \bar{\theta} (s_b \bar{C}(x)), \tag{22}
\]
where the superscript \((b)\) denotes that the above superfield has been obtained after the BRST invariant restriction (21). We use now the following equality
\[
\begin{align*}
B^{(m)}_\mu(x, \bar{\theta}) \cdot \partial_\mu \bar{B}^{(b)}(x, \bar{\theta}) + i \partial_\mu F^{(b)}(x, \bar{\theta}) \cdot \partial^\mu F^{(m)}(x, \bar{\theta}) \\
- \partial_\mu \bar{F}^{(b)}(x, \bar{\theta}) \cdot \left[ B^{(m)}_\mu(x, \bar{\theta}) \times F^{(m)}(x, \bar{\theta}) \right] \\
= A^\mu(x) \cdot \partial_\mu B(x) + i \partial_\mu \bar{C}(x) \cdot \partial^\mu C(x) \\
- \partial_\mu \bar{C}(x) \cdot \left[ A^\mu(x) \times C(x) \right],
\end{align*}
\tag{23}
\]
which emerges from the BRST invariant quantity \(s_b \left[ A^\mu \cdot \partial_\mu B(x) + i \partial_\mu \bar{C}(x) \cdot D^\mu C(x) \right] = 0\). Substitutions of the super expansions from (14), (17), (20) and (22) lead to:
\[
\kappa = -\frac{1}{2} \implies B^{(b)}_\mu(x, \bar{\theta}) = A_\mu(x) + \bar{\theta} (D_\mu C(x)) \equiv A_\mu(x) + \bar{\theta} (s_b A_\mu(x)),
\]
\[
\kappa = -\frac{1}{2} \implies F^{(b)}(x, \bar{\theta}) = C(x) + \bar{\theta} \left[ -\frac{i}{2} (C \times C) \right] \\
\equiv C(x) + \bar{\theta} (s_b C(x)). \tag{24}
\]

We note (from (22), (24) and (14)) that we have already derived the BRST transformations:
\(s_b \bar{C} = iB, \ s_b A_\mu = D_\mu C, \ s_b C = -\left(\frac{i}{2}\right) (C \times C), \ s_b B = 0\) as the coefficients of \(\bar{\theta}\) in the

\(^4\)We have chosen \(B_2(x) = B(x)\) to be consistent with the BRST symmetry transformations (3). However, we have the freedom to choose \(B_2(x) = \alpha B(x)\) where \(\alpha\) is a constant numerical factor.
super expansions of the anti-chiral superfields with superscript \( (b) \) which are derived after the applications of BRST invariant restrictions.

We now focus on the derivation of \( s_b \tilde{B} = i (\tilde{B} \times C) \), \( s_b \psi = -i C \psi \), \( s_b \bar{\psi} = -i \bar{\psi} C \) from the BRST symmetry invariances: \( s_b (\tilde{B} \times C) = 0 \), \( s_b (C \psi) = 0 \) and \( s_b (\bar{\psi} C) = 0 \). These invariances lead to the following equalities and their consequences (i.e. the derivation of secondary fields), namely:

\[
\begin{align*}
\tilde{B}(x, \bar{\theta}) \times F^{(b)}(x, \bar{\theta}) &= B(x) \times C(x) \implies f_2(x) = (B \times C), \\
F^{(b)}(x, \bar{\theta}) \Psi(x, \bar{\theta}) &= C(x) \psi(x) \implies b_1(x) = -C(x) \psi(x), \\
\bar{\Psi}(x, \bar{\theta}) F^{(b)}(x, \bar{\theta}) &= \bar{\psi}(x) C(x) \implies b_2(x) = -\bar{\psi}(x) C(x).
\end{align*}
\]

We would like to mention here that we have used: \( C \ C = \frac{1}{2} \{C, C\} \equiv \frac{1}{2} (C \times C) \) in the derivation of the expressions for \( b_1(x) \) and \( b_2(x) \). The substitutions of these values into the super expansions (14) lead to the following expansions

\[
\begin{align*}
\tilde{B}^{(b)}(x, \bar{\theta}) &= \tilde{B}(x) + \bar{\theta} \left[ i (\tilde{B} \times C) \right] = \tilde{B}(x) + \bar{\theta} (s_b \tilde{B}(x)), \\
\Psi^{(b)}(x, \bar{\theta}) &= \psi(x) + \bar{\theta} (-i C \psi) \equiv \psi(x) + \bar{\theta} (s_b \psi(x)), \\
\bar{\Psi}^{(b)}(x, \bar{\theta}) &= \bar{\psi}(x) + \bar{\theta} (-i \bar{\psi} C) \equiv \bar{\psi}(x) + \bar{\theta} (s_b \bar{\psi}(x)),
\end{align*}
\]

where superscript \((b)\) denotes the anti-chiral superfields that have been obtained after the applications of BRST invariant restrictions.

We end this section with the remark that we have obtained \textit{all} the BRST symmetry transformations for \textit{all} the fields of our present non-Abelian 1-form gauge theory (where there is an \textit{interaction} between the gauge field and Dirac fields \( (\psi, \bar{\psi}) \)) as the coefficient of \( \bar{\theta} \) in the super expansions of the anti-chiral superfields with superscript \((b)\). We lay emphasis on the fact that our results lend support to the results obtained in \([4, 5]\) which are obtained by the formal application of the HC (that depends crucially on the exterior derivative \( d = dx^\mu \partial_\mu \) \((d^2 = 0)\) of the de Rham cohomological operators of differential geometry). In addition, we derive the off-shell nilpotent (anti-)BRST symmetry transformations for the \textit{matter} fields which has not been derived and discussed in \([4, 5]\).

## 4 Off-Shell Nilpotent Anti-BRST Symmetries: Chiral Superfields and Their Chiral Super Expansions

In this section, we derive the anti-BRST symmetry transformations for our interacting non-Abelian 1-form gauge theory by invoking the anti-BRST (i.e. quantum gauge) invariant restrictions on the \textit{chiral} superfields. Towards this goal in mind, we generalize the ordinary fields \((A_\mu, C, \bar{C}, B, \bar{B}, \psi, \bar{\psi})\) of our \textit{D}-dimensional \textit{ordinary} non-Abelian 1-form gauge theory onto the \textit{chiral} \((D, 1)\)-dimensional super-submanifold (of the \textit{general} \((D, 2)\)-dimensional supermanifold) as

\[
\begin{align*}
A_\mu(x) &\rightarrow B_\mu(x, \theta) = A_\mu(x) + \theta \tilde{R}_\mu(x), & C(x) &\rightarrow F(x, \theta) = C(x) + i \theta \tilde{B}_1(x), \\
\bar{C}(x) &\rightarrow \bar{F}(x, \theta) = \bar{C}(x) + i \theta \tilde{B}_2(x), & B(x) &\rightarrow \tilde{B}(x, \theta) = B(x) + i \theta \tilde{f}_1(x), \\
\bar{B}(x) &\rightarrow \bar{\tilde{B}}(x, \theta) = \bar{B}(x) + i \theta \bar{f}_2(x), & \psi(x) &\rightarrow \Psi(x, \theta) = \psi(x) + i \theta \bar{b}_1(x), \\
\bar{\psi}(x) &\rightarrow \bar{\Psi}(x, \theta) = \bar{\psi}(x) + i \theta \bar{b}_2(x),
\end{align*}
\]

(27)
where we point out explicitly that the \((D, 1)\)-dimensional \emph{chiral} super-submanifold is parameterized by the bosonic coordinates \(x^\mu\) \((\mu = 0, 1, \ldots, D - 1)\) and fermionic \((\theta^2 = 0)\) Grassmannian variable \(\theta\). To obtain the secondary fields \((\bar{R}_\mu(x), \bar{B}_1(x), \bar{B}_2(x), \bar{f}_1(x), \bar{f}_2(x), \bar{b}_1(x), \bar{b}_2(x))\) in terms of the basic and auxiliary fields of the Lagrangian densities (1), we have found out (by the method of trial and error) the following useful and interesting anti-BRST invariant quantities

\[
\begin{align*}
    s_{ab}(\bar{C} \psi) &= 0, \quad s_{ab}(D_\mu \bar{C}) = 0, \quad s_{ab}(\bar{B}) = 0, \quad s_{ab}(\bar{C} \times \bar{C}) = 0, \\
    s_{ab}(B \times \bar{C}) &= 0, \quad s_{ab}[A^\mu \cdot \partial_\mu \bar{B} + i D^\mu \bar{C} \cdot \partial_\mu \bar{C}] = 0, \quad s_{ab}(\bar{\psi} \bar{C}) = 0,
\end{align*}
\]

(28)

where the fields (in the brackets) are present in the starting Lagrangian density \((\mathcal{L}_\theta)\) that respects \emph{perfect} anti-BRST symmetry (i.e. \(s_{ab} \mathcal{L}_\theta = - \partial_\mu[B \cdot D^\mu \bar{C}]\)) in the sense that the corresponding action integral \((\mathcal{S} = \int d^Dx \mathcal{L}_B)\) remains \emph{invariant} for the physically well-defined fields which vanish off at \(x = \pm \infty\).

We do not elaborate here on \emph{all} the step-by-step computations (as we have done in the previous section). The algebraic computations are exactly on the similar lines as in the previous section. Thus, we collect here \emph{all} the key results that emerge out by demanding the validity of the basic tenets of ACSA to BRST formalism where the anti-BRST (i.e. quantum gauge) invariant quantities are required to remain independent of the “soul” coordinate \(\theta\), namely;

\[
\begin{align*}
    s_{ab} \bar{B}(x) = 0 & \implies \bar{f}_2(x) = 0 \implies \bar{B}^{(ab)}(x, \theta) = \bar{B}(x) + \theta \left( \bar{B}(x) \right), \\
    s_{ab}(\bar{C} \times \bar{C}) = 0 & \implies \bar{B}_2 = \kappa (\bar{C} \times \bar{C}) \implies \bar{F}(m)(x, \theta) \equiv \bar{C}(x) + i \theta \kappa (\bar{C} \times \bar{C}), \\
    s_{ab}(\bar{B} \times \bar{C}) = 0 & \implies \bar{B}_1(x) = \bar{B}(x) \implies \bar{F}(ab)(x, \theta) \equiv C(x) + \theta (i \bar{B}(x)), \\
    s_{ab}(B \times \bar{C}) = 0 & \implies \bar{f}_1(x) = B \times \bar{C} \implies \bar{B}^{(ab)}(x, \theta) = B(x) + (i \bar{B} \times \bar{C}), \\
    s_{ab}(D_\mu \bar{C}) = 0 & \implies \bar{R}_\mu = - 2 \kappa (D_\mu \bar{C}) \implies \bar{B}(m)(x, \theta) = A_\mu(x) + \theta (- 2 \kappa D_\mu \bar{C}),
\end{align*}
\]

(29)

where the superscripts \((m)\) and \((ab)\) denote the modified form of the superfields and the superfields that have been obtained after the anti-BRST invariant restrictions which lead to the derivation of anti-BRST symmetry transformations as the coefficients of \(\theta\), respectively. We now take up the anti-BRST invariant quantity: \(s_{ab}[A^\mu \cdot \partial_\mu \bar{B} + i D_\mu \bar{C} \cdot \partial_\mu \bar{C}] = 0\) which leads to the following restriction on the \emph{chiral} superfields, namely;

\[
\begin{align*}
    B^{(m)}(x, \theta) \cdot \partial_\mu \bar{B}^{(ab)}(x, \theta) + i \partial_\mu \bar{F}(m)(x, \theta) \cdot \partial^\mu \bar{F}(ab)(x, \theta) \\
    - (B^{(m)}(x, \theta) \times \bar{F}(m)(x, \theta)) \cdot \partial_\mu \bar{F}(ab)(x, \theta) \\
    \equiv A^\mu(x) \cdot \partial_\mu \bar{B}(x) + i D_\mu \bar{C}(x) \cdot \partial^\mu \bar{C}(x),
\end{align*}
\]

(30)

where the explicit super expansions from Eq. \((29)\) have to be used. Once it is done, we
obtain $\kappa = -\frac{1}{2}$ (cf. Sec. 3) which leads to the following:

$$
\bar{F}^{(m)}(x, \theta) \rightarrow \bar{F}^{(ab)}(x, \theta) = \bar{C}(x) + \theta \left( -\frac{i}{2} \bar{C} \times \bar{C} \right) \\
\equiv \bar{C}(x) + \theta \left( s_{ab} \bar{C}(x) \right),
$$

$$
B^{(m)}_{\mu}(x, \theta) \rightarrow B^{(ab)}_{\mu}(x, \theta) = A_{\mu}(x) + \theta \left( D_{\mu} \bar{C}(x) \right) \\
\equiv A_{\mu}(x) + \theta \left( s_{ab} A_{\mu}(x) \right). \tag{31}
$$

Thus, we note from Eqs. (29) and (31), that we have already derived the anti-BRST symmetry transformations for $\bar{B}(x), \bar{C}(x), B(x), \bar{C}(x)$ and $A_{\mu}(x)$ as the coefficients of the Grassmannian variable $\theta$ in the chiral supermanifold. By contrast, the anti-BRST symmetry transformation is intimately connected with the translational generator $\partial_{\theta}$ along the $\bar{C}$-direction of the super-submanifold of the $\bar{C}$ supermanifold. Finally, the ACSA to BRST formalism produces the (anti-)BRST formalism for the gauge, (anti-)ghost and matter fields together which is not the case with the application of HC alone.

We are in the position now to derive the anti-BRST symmetry transformations that are associated with matter fields ($\bar{\psi}, \bar{\psi}$). Towards this goal in mind, we focus on the anti-BRST invariant quantities $s_{ab}(\bar{C} \bar{\psi}) = 0$, $s_{ab}(\bar{\psi} \bar{C}) = 0$ and demand the following restrictions on the composite chiral superfields, namely;

$$
\bar{F}^{(ab)}(x, \theta) \Psi(x, \theta) = \bar{C}(x) \psi(x), \quad \bar{\Psi}(x, \theta) \bar{F}^{(ab)}(x, \theta) = \bar{\psi}(x) \bar{C}(x), \tag{32}
$$

which lead to the expressions for the secondary fields $\bar{b}_1(x) = - C(x) \psi(x)$ and $\bar{b}_2(x) = - \bar{\psi}(x) \bar{C}(x)$, respectively. It is pertinent to point out that we have used here the theoretical trick: $\bar{C} \bar{C} = \frac{1}{2} \{\bar{C}, \bar{C}\} = \frac{1}{2}(\bar{C} \times \bar{C})$ in the determination of $b_1(x)$ and $b_2(x)$. These values, ultimately, lead to the following super expansions:

$$
\Psi^{(ab)}(x, \theta) = \psi(x) + \theta \left( -i \bar{C} \psi \right) \equiv \psi(x) + \theta \left( s_{ab} \psi \right), \\
\bar{\Psi}^{(ab)}(x, \theta) = \bar{\psi}(x) + \bar{\theta} \left( -i \bar{\psi} \bar{C} \right) \equiv \bar{\psi}(x) + \bar{\theta} \left( \bar{s}_{ab} \bar{\psi} \right). \tag{33}
$$

Thus, we have computed all the secondary fields of the super expansions (27) in terms of the basic and auxiliary fields of the Lagrangian densities (1) and derived the off-shell nilpotent anti-BRST symmetry transformations ($s_{ab}$) for all the fields that are present in the Lagrangian densities (1).

We end our present discussion with some remarks connected with Sec. 3. as well as Sec. 4. First, we note that we have not utilized the idea of HC anywhere. Rather, we have used only the (anti-)BRST (i.e. quantum gauge) invariant restrictions on the superfields to derive the proper (anti-)BRST symmetry transformations in Sec. 3 and 4. Second, a close look at Eqs. (16), (22), (24) and (26) demonstrate that $s_0 \leftrightarrow \partial_{\theta}$ (i.e. the BRST symmetry transformation is intimately connected with the translational generator $\partial_{\theta}$ along the $\bar{C}$-direction of the $\bar{C}$ super-submanifold of the $\bar{C}$ supermanifold). By contrast, the anti-BRST symmetry transformation is deeply connected (i.e. $s_{ab} \leftrightarrow \partial_{\theta}$) with the translational generator $\partial_{\theta}$ along the $\theta$-direction of the chiral super-submanifold of the $\bar{C}$ supermanifold. Finally, the ACSA to BRST formalism produces the (anti-)BRST symmetry transformations for the gauge, (anti-)ghost and matter fields together which is not the case with the application of HC alone.
5 Invariance of $\mathcal{L}_B$ and $\mathcal{L}_{\bar{B}}$ Under the (Anti-)BRST Symmetries: ACSA to BRST Formalism

In this section, we capture the (anti-)BRST invariance of the Lagrangian densities $\mathcal{L}_B$ and $\mathcal{L}_{\bar{B}}$ (cf. Eq. (4)) in the language of ACSA to BRST formalism. Towards this objective in mind, we generalize, first of all, the Lagrangian density $\mathcal{L}_B$ to its counterpart anti-chiral super Lagrangian density $\tilde{\mathcal{L}}^{(ac)}_B$ (with $\tilde{B}^{(b)}(x, \bar{\theta}) = B(x)$) as:

$$
\mathcal{L}_B \rightarrow \tilde{\mathcal{L}}^{(ac)}_B = -\frac{1}{4} \tilde{F}^{\mu\nu}_B(x, \bar{\theta}) \cdot \tilde{F}^{(ac)}_{\mu\nu}(x, \bar{\theta}) + \bar{\Psi}^{(b)}(x, \bar{\theta}) (i \gamma^\mu \partial_\mu - m) \Psi^{(b)}(x, \bar{\theta}) - \bar{\Psi}^{(b)}(x, \bar{\theta}) \gamma^\mu B^{(b)}(x, \bar{\theta}) \Psi^{(b)}(x, \bar{\theta}) + B(x) \cdot \left[ \partial_\mu B^{(b)}(x, \bar{\theta}) \right] + \frac{1}{2} \left[ B(x) \cdot B(x) + \tilde{B}^{(b)}(x, \bar{\theta}) \cdot \tilde{B}^{(b)}(x, \bar{\theta}) \right] - i \partial_\mu F^{(b)}(x, \bar{\theta}) \cdot \partial^\mu F^{(b)}(x, \bar{\theta}) + \partial_\mu F^{(b)}(x, \bar{\theta}) \cdot \left[ B^{(b)}(x, \bar{\theta}) \times F^{(b)}(x, \bar{\theta}) \right].
$$

where all the symbols have been explained earlier in the super expansions (16), (22), (24) and (26) except $F_{\mu\nu}^{(ac)}(x, \bar{\theta})$ which we explain, namely:

$$
F_{\mu\nu}^{(ac)} = \partial_\mu B^{(b)}_{\nu}(x, \bar{\theta}) - \partial_\nu B^{(b)}_{\mu}(x, \bar{\theta}) + i \left[ B^{(b)}_{\mu}(x, \bar{\theta}) \times B^{(b)}_{\nu}(x, \bar{\theta}) \right] \equiv F_{\mu\nu}(x) + \bar{\theta} (i F_{\mu\nu} \times C) \equiv F_{\mu\nu}(x) + \bar{\theta} (s_b F_{\mu\nu}(x)),
$$

where the super expansion of $B^{(b)}_{\mu}(x, \bar{\theta})$ is given in Eq. (24). Substitutions of the super expansions for $\tilde{B}^{(b)}(x, \bar{\theta})$, $F^{(b)}(x, \bar{\theta})$, $\tilde{F}^{(b)}(x, \bar{\theta})$ and $F_{\mu\nu}^{(ac)}(x, \bar{\theta})$ into the above super Lagrangian density lead to the following

$$
\tilde{\mathcal{L}}^{(ac)}_B = \mathcal{L}_B + \bar{\theta} \partial_\mu [B \cdot D^\mu C],
$$

modulo some other explicit total spacetime derivative terms (without being the coefficient of $\bar{\theta}$). Thus, it is clear that we have the following:

$$
\frac{\partial}{\partial \bar{\theta}} \tilde{\mathcal{L}}^{(ac)}_B = \partial_\mu [B \cdot D^\mu C] \iff s_b \mathcal{L}_B = \partial_\mu [B \cdot D^\mu C].
$$

The above quantity is true because of the identification of BRST symmetry transformations ($s_b$) with the translational generator ($\partial_{\bar{\theta}}$) along the Grassmannian direction of the anti-chiral super-submanifold of the general $(D, 2)$-dimensional supermanifold. Geometrically, the super Lagrangian density $\tilde{\mathcal{L}}^{(ac)}_B$ is the sum of composite anti-chiral superfields (in addition to the ordinary fields) such that its translation along $\bar{\theta}$-direction of the anti-chiral super-submanifold produces a total spacetime derivative (i.e $s_b \mathcal{L}_B = \partial_\mu [B \cdot D^\mu C]$) in the ordinary spacetime thereby rendering the action integral ($S = \int d^D x \, \mathcal{L}_B$) invariant.

To capture the anti-BRST invariance of the Lagrangian density $\mathcal{L}_B$ (cf. Eq. (4)), we generalize the ordinary fields of $it$ to the chiral super Lagrangian density $\tilde{\mathcal{L}}^{(c)}_B$ (with input
\[ \tilde{B}^{(ab)}(x, \theta) = \tilde{B}(x) \] as:

\[
\begin{align*}
\mathcal{L}_B \rightarrow \tilde{\mathcal{L}}^{(c)}_B &= -\frac{1}{4} \tilde{F}^{\mu\nu(c)}(x, \theta) \cdot \tilde{F}^{(c)}_{\mu\nu}(x, \theta) \\
&+ \tilde{\Psi}^{(ab)}(x, \theta) \left( i \gamma^\mu \partial_\mu - m \right) \tilde{\Psi}^{(ab)}(x, \theta) \\
&- \tilde{\Psi}^{(ab)}(x, \theta) \gamma^\mu B^{(ab)}_\mu(x, \theta) \tilde{\Psi}^{(ab)}(x, \theta) + \tilde{B}(x) \cdot \tilde{B}(x) \\
&+ \frac{1}{2} \left[ \tilde{B}^{(ab)}(x, \theta) \cdot \tilde{B}^{(ab)}(x, \theta) + \tilde{B}(x) \cdot \tilde{B}(x) \right] \\
&= B^{(ab)}(x, \theta) \cdot \tilde{F}^{(ab)}(x, \theta),
\end{align*}
\]

where all the symbols have been explained earlier in the super expansions (29), (31), (32) and (33) except \( F^{(c)}_{\mu\nu}(x, \theta) \) which we explicitly elaborate as follows

\[
F^{(c)}_{\mu\nu} = \partial_\mu B^{(ab)}_\nu(x, \theta) - \partial_\nu B^{(ab)}_\mu(x, \theta) + i \left[ B^{(ab)}_\mu(x, \theta) \times B^{(ab)}_\nu(x, \theta) \right] \\
\equiv F_{\mu\nu}(x) + \theta (F_{\mu\nu} \times \tilde{C}) \equiv F_{\mu\nu}(x) + \theta \left( s_{ab} F_{\mu\nu}(x) \right),
\]

where we have substituted the expansion for \( B^{(ab)}_\mu \) (cf. Eq. (31)). The substitutions of all the chiral superfields with superscript \((ab)\) into (38) lead to the following:

\[
\tilde{\mathcal{L}}^{(c)}_B = \mathcal{L}_B + \theta \partial_\mu [-\tilde{B} \cdot D^\mu \tilde{C}],
\]

modulo some other explicit total spacetime derivative terms (\textit{without} being the coefficient of \( \theta \)). Ultimately, we obtain the following mapping:

\[
\frac{\partial}{\partial \theta} \tilde{\mathcal{L}}^{(c)}_B = \partial_\mu [-\tilde{B} \cdot D^\mu \tilde{C}] \iff s_{ab} \mathcal{L}_B = -\partial_\mu [\tilde{B} \cdot D^\mu \tilde{C}].
\]

Geometrically, the above equation implies that the chiral super Lagrangian density is a very specific sum of the chiral superfields that have been obtained after the applications of anti-BRST invariant restrictions and some ordinary fields such that its translation along \( \theta \)-direction of the chiral super-submanifold produces a total spacetime derivative (i.e. \( s_{ab} \mathcal{L}_B = -\partial_\mu [\tilde{B} \cdot D^\mu \tilde{C}] \)) in the ordinary space thereby rendering the ordinary action integral \( (S = \int d^D x \mathcal{L}_B) \) invariant.

We can also capture the anti-BRST invariance of the Lagrangian density \( \mathcal{L}_B \) and BRST invariance of the Lagrangian density \( \mathcal{L}_B \) within the framework of ACSA to BRST formalism. In this context, we note that the Lagrangian density \( \mathcal{L}_B \) can be generalized to the chiral super Lagrangian density \( \tilde{\mathcal{L}}^{(c)}_B \) as:

\[
\mathcal{L}_B \rightarrow \tilde{\mathcal{L}}^{(c)}_B = -\frac{1}{4} \tilde{F}^{\mu\nu(c)}(x, \theta) \cdot \tilde{F}^{(c)}_{\mu\nu}(x, \theta) \\
+ \tilde{\Psi}^{(ab)}(x, \theta) \left( i \gamma^\mu \partial_\mu - m \right) \tilde{\Psi}^{(ab)}(x, \theta) \\
- \tilde{\Psi}^{(ab)}(x, \theta) \gamma^\mu B^{(ab)}_\mu(x, \theta) \tilde{\Psi}^{(ab)}(x, \theta) + \tilde{B}^{(ab)}(x, \theta) \cdot \tilde{B}^{(ab)}(x, \theta) \\
+ \frac{1}{2} \left[ \tilde{B}^{(ab)}(x, \theta) \cdot \tilde{B}^{(ab)}(x, \theta) + \tilde{B}(x) \cdot \tilde{B}(x) \right] \\
- i \partial_\mu \tilde{F}^{(ab)}(x, \theta) \cdot \partial^\mu \tilde{F}^{(ab)}(x, \theta) \\
+ \partial_\mu \tilde{F}^{(ab)}(x, \theta) \cdot \tilde{B}^{(ab)}(x, \theta) \times \tilde{F}^{(ab)}(x, \theta),
\]
where all the symbols have been explained in Sec. 4 and in present section. It is straightforward to check that the substitutions of all these super expansions into the above super Lagrangian density lead to the following explicit result, namely;

\[
\mathcal{L}_{B}^{(c)} = \mathcal{L}_{B} + \theta \left[ - \partial_{\mu} \left\{ \tilde{B} + (C \times \bar{C}) \right\} \cdot \partial^{\mu} \bar{C} + (B + \bar{B} + (C \times \bar{C})) \cdot D_{\mu} \partial^{\mu} \bar{C} \right] = \mathcal{L}_{B} + \theta \partial_{\mu} [B \cdot \partial^{\mu} \bar{C}], \tag{43}
\]

where the final/last expression has been obtained after the application of CF-condition: \( B + \bar{B} + (C \times \bar{C}) = 0 \). Now, it is crystal clear that we have the following mapping between the Grassmannian partial derivative of \textit{chiral} super submanifold and anti-BRST symmetry transformations \( s_{ab} \) in the ordinary space:

\[
\frac{\partial}{\partial \bar{\theta}} \mathcal{L}_{B}^{(c)} = \partial_{\mu} [B \cdot \partial^{\mu} \bar{C}] \quad \iff \quad s_{ab} \mathcal{L}_{B} = \partial_{\mu} [B \cdot \partial^{\mu} \bar{C}], \tag{44}
\]

which completely agrees with our observation in Eq. (5). Thus, we note that the Lagrangian density \( \mathcal{L}_{B} \) also respects the anti-BRST symmetry transformations on a constrained hyper-surface\(^\text{I} \) in the flat \( D \)-dimensional Minkowskian sapcetime manifold (which is defined by the CF-condition: \( B + \bar{B} + (C \times \bar{C}) = 0 \)). It is interesting to mention, in passing, that we also have the absolutely anticommuting \( s_{(a)b} \) on \textit{this} hypersurface where \( B + \bar{B} + (C \times \bar{C}) = 0 \) is true.

At this stage, we dwell a bit on the BRST invariance of the Lagrangian density \( \mathcal{L}_{B} \). Towards this objective in mind, first of all, we generalize \textit{this} Lagrangian density to the super \textit{anti-chiral} Lagrangian density \( \mathcal{L}_{B}^{(ac)} \) as

\[
\mathcal{L}_{B} \longrightarrow \tilde{\mathcal{L}}_{B}^{(ac)} = - \frac{1}{4} \tilde{F}_{\mu\nu}^{(ac)}(x, \bar{\theta}) \cdot \tilde{F}_{\mu\nu}^{(ac)}(x, \bar{\theta}) + \Psi^{(b)}(x, \bar{\theta}) (i \gamma^{\mu} \partial_{\mu} - m) \Psi^{(b)}(x, \bar{\theta}) - \bar{\Psi}^{(b)}(x, \bar{\theta}) \gamma^{\mu} B_{\mu}^{(b)}(x, \bar{\theta}) \bar{\Psi}^{(b)}(x, \bar{\theta}) - \bar{B}_{\mu}^{(b)}(x, \bar{\theta}) \cdot \partial_{\mu} B^{(b)}(x, \bar{\theta}) + \frac{1}{2} \left[ B(x) \cdot B(x) + \bar{B}^{(b)}(x, \bar{\theta}) \cdot \bar{B}^{(b)}(x, \bar{\theta}) \right] - i \partial_{\mu} F^{(b)}(x, \bar{\theta}) \cdot \partial^{\mu} F^{(b)}(x, \bar{\theta}) + \left[ B^{(b)}(x, \bar{\theta}) \times F^{(b)}(x, \bar{\theta}) \right] \cdot \partial_{\mu} F^{(b)}(x, \bar{\theta}), \tag{45}
\]

where all the symbols/notations have been explained earlier in Sec. 3 and in the present section. We note that the above super Lagrangian density is the \textit{sum} of the composite superfields (derived after the application of BRST invariant restrictions) \textit{and} ordinary fields. We are now in the position to operate a derivative w.r.t. \( \bar{\theta} \) on the above \textit{super} Lagrangian density as

\[
\frac{\partial}{\partial \bar{\theta}} \tilde{\mathcal{L}}_{B}^{(ac)} = \partial_{\mu} [- \bar{B} \cdot \partial^{\mu} C] \quad \iff \quad s_{ab} \mathcal{L}_{B} = \partial_{\mu} [- \bar{B} \cdot \partial^{\mu} C], \tag{46}
\]

\(^I\) It is obvious that, if the CF-condition is taken into account, the action integral \( S = \int d^{D} x \mathcal{L}_{B} \) remains invariant under the \textit{anti-BRST} symmetry transformations \( s_{ab} \), too.
where the mapping between the superspace and ordinary space has been taken into account. In fact, the above result has been obtained due to the fact that the super anti-chiral Lagrangian density (45) can be written in the following explicit form

\[ \tilde{\mathcal{L}}_{B}^{(ac)} = \mathcal{L}_{B} + \bar{\theta} \left[ \partial_{\mu} \left\{ B + (C \times \bar{C}) \right\} \cdot \partial^{\mu} C - (B + \bar{B} + (C \times \bar{C})) \cdot D_{\mu} \partial^{\mu} C \right] \equiv \mathcal{L}_{B} + \bar{\theta} \partial_{\mu} \left[ - \bar{B} \cdot \partial^{\mu} C \right], \tag{47} \]

where the final expression has been obtained after the application of CF-condition: \( B + \bar{B} + (C \times \bar{C}) = 0 \). It is worthwhile to mention here that the result of (46) has been obtained after the derivation of (47) and the application of the derivative w.r.t. \( \bar{\theta} \) on it (i.e. Eq. (47)). This statement is true because of our observations in Eq. (5) and Eqs. (37), (41), (44) and (46). In other words, there is a precise agreement between the results obtained in the ordinary space and superspace (with the help of the mappings: \( s_{b} \leftrightarrow \partial_{\bar{\theta}} \), \( s_{ab} \leftrightarrow \partial_{\theta} \)).

We end this section with the final remark that we have already captured the essential features of the (anti-)BRST invariance of the Lagrangian densities \( \mathcal{L}_{B} \) and \( \mathcal{L}_{\bar{B}} \) within the framework of ACSA to BRST formalism.

6 Nilpotency and Absolute Anticommutativity Properties of the Fermionic Conserved (Anti-)BRST Charges: ACSA

In this section, we capture the off-shell nilpotency and absolute anticommutativity properties of the (anti-)BRST charges within the framework of ACSA to BRST formalism. In the proof of absolute anticommutativity property, we invoke the CF-condition at appropriate places. At the very onset, we would like to lay emphasis on the fact that our knowledge of the ordinary space BRST formalism and its connection with the superspace/superfield approach to BRST formulation has helped us in all our theoretical discussions of this section. In other words, our understandings of BRST formalism in both the spaces is intertwined together in a beautiful and useful manner. It is because of this reason that we have been able to express the mathematical properties of \( Q_{ab} \) in the language of ACSA.

Towards our main discussions, first of all, we discuss the nilpotency property of the conserved (anti-)BRST charges within the framework of ACSA to BRST formalism. It is straightforward to check that the above conserved charges (i.e. nilpotent (anti-)BRST charges) can be written as (cf. Eq. (12)).

\[
Q_{ab} = \frac{\partial}{\partial \bar{\theta}} \int d^{D-1}x \left[ i \hat{F}^{(ab)}(x, \theta) \cdot \hat{F}^{(ab)}(x, \theta) - i \hat{B}(x) \cdot \hat{B}^{(ab)}(x, \theta) \right] \\
\equiv \int d\theta \int d^{D-1}x \left[ i \hat{F}^{(ab)}(x, \theta) \cdot \hat{F}^{(ab)}(x, \theta) - i \hat{B}(x) \cdot \hat{B}^{(ab)}(x, \theta) \right].
\]

\[
Q_{b} = \frac{\partial}{\partial \bar{\theta}} \int d^{D-1}x \left[ B(x) \cdot \hat{B}^{(b)}(x, \bar{\theta}) + i \hat{F}^{(b)}(x, \bar{\theta}) \cdot \hat{F}^{(b)}(x, \bar{\theta}) \right] \\
\equiv \int d\bar{\theta} \int d^{D-1}x \left[ B(x) \cdot \hat{B}^{(b)}(x, \bar{\theta}) + i \hat{F}^{(b)}(x, \bar{\theta}) \cdot \hat{F}^{(b)}(x, \bar{\theta}) \right], \tag{48}
\]
where we have taken  \( \tilde{B}^{(ab)}(x,\theta) = \tilde{B}(x) \) and  \( \tilde{B}^{(b)}(x,\bar{\theta}) = B(x) \) which have been derived earlier (primarily due to: \( s_{ab} B(x) = 0, s_b B(x) = 0 \)). Rest of all the symbols have been explained earlier. We have also established that: \( s_b \leftrightarrow \partial_{\bar{\theta}} \) and \( s_{ab} \leftrightarrow \partial_{\theta} \). Thus, the above expressions for the (anti-)BRST charges (in the superspace \( B \) using the CF-condition \( s \)) earlierm primarily due to:

We recast the expression for the BRST charge principles (i.e. the continuous symmetries and their generators). In view of this, first of all, proof first

itive transformations (\( s \)) translations and

Grassmannian directions as well as the nilpotency (\( Q \)) ordinary transformations and ordinary fields as:

\[
Q_{ab} = s_{ab} \int d^{D-1}x \left[ i \dot{C}(x) \dot{C}(x) - \tilde{B}(x) \cdot A_0(x) \right],
\]

\[
Q_b = s_b \int d^{D-1}x \left[ B(x) \cdot A_0(x) + i \dot{C}(x) \cdot C(x) \right].
\]

(49)

It is now crystal clear that we have:

\[
\partial_{\theta} Q_{ab} = 0 \iff \partial_{\bar{\theta}}^2 = 0, \quad s_{ab} Q_{ab} = -i \{ Q_{ab}, Q_{ab} \} = 0 \iff s_{ab}^2 = 0, \\
\partial_{\bar{\theta}} Q_b = 0 \iff \partial_{\bar{\theta}}^2 = 0, \quad s_b Q_b = -i \{ Q_b, Q_b \} = 0 \iff s_b^2 = 0.
\]

(50)

In other words, the nilpotency of the (anti-)BRST charges (i.e. \( Q^2_{(a)b} = 0 \)) is deeply connected with the nilpotency (\( \partial_{\theta}^2 = \partial_{\bar{\theta}}^2 = 0 \)) of the translational generators (\( \partial_{\theta}, \partial_{\bar{\theta}} \)) along Grassmannian directions as well as the nilpotency (\( s_{ab}^2 = 0 \)) of the (anti-) BRST symmetry transformations (\( s_{(a)b} \)).

As far as the proof of absolute anticommutativity is concerned, we begin with such a proof first in the ordinary space** by exploiting the beauty and strength of the symmetry principles (i.e. the continuous symmetries and their generators). In view of this, first of all, we recast the expression for the BRST charge \( Q_b \) (cf. Eq. (12)) in an appropriate form by using the CF-condition \( B + \tilde{B} + (C \times \dot{C}) = 0 \). This suitable (i.e. modified but equivalent) form of \( Q_b \) is as follows

\[
Q_b = \int d^{D-1}x \left[ \tilde{B} \cdot C - \tilde{B} \cdot D_0 C - (C \times \dot{C}) \cdot D_0 C \\
+ \frac{1}{2} \dot{C} \cdot (C \times \dot{C}) + (\dot{C} \times C) \cdot C \right]
\]

\[
\equiv \int d^{D-1}x \left[ \tilde{B} \cdot C - \tilde{B} \cdot D_0 C + \frac{1}{2} \dot{C} \cdot (C \times C) \\
- i (C \times \dot{C}) \cdot (A_0 \times C) \right],
\]

(51)

where we have used \( D_0 C = \partial_{\theta} C + i (A_0 \times C) \equiv \dot{C} + i (A_0 \times C) \). The above final expression of \( Q_b \) can be written as an anti-BRST exact quantity:

\[
Q_b = s_{ab} \int d^{D-1}x \left[ i C \cdot \dot{C} - \frac{1}{2} C \cdot (A_0 \times C) \right].
\]

(52)

The expression for \( Q_b \), in the above form, proves the absolute anticommutativity of the (anti-)BRST charges in the following manner

\[
s_{ab} Q_b = -i \{ Q_b, Q_{ab} \} = 0 \iff s_{ab}^2 = 0,
\]

(53)

**We purposefully perform this exercise to demonstrate that our knowledge in the ordinary space and superspace is intertwined (for all the discussions contained in this section).
where we have used the idea of continuous symmetry generator and the off-shell nilpotency of the anti-BRST symmetry transformation \((s_{ab})\). Taking into account our knowledge of \(s_{ab} \leftrightarrow \partial_\theta\), the above expression (52) can be written, within the framework of ACSA to BRST formalism, as:

\[
Q_b = \frac{\partial}{\partial \bar{\theta}} \left[ \int d^{D-1}x \left\{ i F^{(b)}(x, \bar{\theta}) \cdot \dot{F}^{(b)}(x, \bar{\theta}) - \frac{1}{2} F^{(b)}(x, \bar{\theta}) \cdot \left[ B^{(b)}_0(x, \bar{\theta}) \times F^{(b)}(x, \bar{\theta}) \right] \right\} \right] = \int d\theta \int d^{D-1}x \left\{ i F^{(b)}(x, \bar{\theta}) \cdot \dot{F}^{(b)}(x, \bar{\theta}) - \frac{1}{2} F^{(b)}(x, \bar{\theta}) \cdot \left[ B^{(b)}_0(x, \bar{\theta}) \times F^{(b)}(x, \bar{\theta}) \right] \right\}.
\]

(54)

From the above expression, it is clear that:

\[
\partial_b Q_b = 0 \iff \partial_b^2 = 0 \iff s_{ab}Q_b = -i \{Q_b, Q_{ab}\} = 0.
\]

(55)

Thus, we note that the absolute anticommutativity of the BRST charge \((Q_b)\) with the anti-BRST charge \((Q_{ab})\) is deeply connected with the nilpotency \((\partial_b^2 = 0)\) of the translational generator \(\partial_b\) along the \(\theta\)-direction of the \textit{chiral} super-submanifold of the general \((D, 2)\)-dimensional gauge theory (on which the fields of our ordinary \((D, 2)\)-dimensional supermanifold). To accomplish this goal, we note that the anti-BRST charge can be written, in the \textit{BRST-exact} form, as

\[
Q_{ab} = s_b \int d^{D-1}x \left[ -i \bar{C} \cdot \dot{C} + \frac{1}{2} \bar{C} \cdot (A_0 \times \bar{C}) \right],
\]

(56)

where we have already used the CF-condition \(B + B + (C \times C) = 0\) to recast the expression for \(Q_{ab}\) in an appropriate form\(^\dagger\) so that it could be written as (56). Within the framework of ACSA to BRST formalism, the above form (56) can be written (on the \((D, 1)\)-dimensional \textit{anti-chiral} super-submanifold) as:

\[
Q_{ab} = \frac{\partial}{\partial \bar{\theta}} \left[ \int d^{D-1}x \left\{ -i \bar{F}^{(b)}(x, \bar{\theta}) \cdot \dot{F}^{(b)}(x, \bar{\theta}) + \frac{1}{2} \bar{F}^{(b)}(x, \bar{\theta}) \cdot (B^{(b)}_0(x, \bar{\theta}) \times \bar{F}^{(b)}(x, \bar{\theta})) \right\} \right]
\equiv \int d\theta \int d^{D-1}x \left\{ -i \bar{F}^{(b)}(x, \bar{\theta}) \cdot \dot{F}^{(b)}(x, \bar{\theta}) + \frac{1}{2} \bar{F}^{(b)}(x, \bar{\theta}) \cdot (B^{(b)}_0(x, \bar{\theta}) \times \bar{F}^{(b)}(x, \bar{\theta})) \right\}.
\]

(57)

\(^\dagger\)The algebraic computations are on exactly similar lines as we have done for the proof of the absolute anticommutativity of BRST charge \((Q_b)\) with the anti-BRST charge \((Q_{ab})\).
Thus, it is straightforward to note that we have:

\[ \partial_{\bar{\theta}} Q_{ab} = 0 \iff \partial_{\bar{\theta}}^2 = 0, \quad s_b Q_{ab} = -i \{ Q_{ab}, Q_b \} = 0 \iff s_b^2 = 0. \]  

(58)

Finally, we remark that the absolute anticommutativity of the anti-BRST charge with the BRST charge is connected with the nilpotency of the translational BRST generator along \( \bar{\theta} \)-direction of the \((D, 1)\)-dimensional anti-chiral super-submanifold (of the general \((D, 2)\)-dimensional supermanifold). This observation, in turn, implies that the above absolute anticommutativity is also deeply connected with the nilpotency \( s_b^2 = 0 \) of the BRST symmetry transformations \( (s_b) \).

7 Conclusions

In our present investigation, we have exploited the theoretical strength of ACSA to BRST formalism to derive the (anti-)BRST symmetry transformations by demanding that the (anti-)BRST (i.e. quantum gauge) invariant quantities must be independent of the “soul” coordinates. In terms of the geometrical quantities defined on the (anti-)chiral super-submanifolds and (anti-)chiral superfields (derived after the application of the (anti-)BRST invariant restrictions), we have been able to express the conserved (anti-)BRST charges of our theory in the language of ACSA to BRST formalism. This exercise, in turn, has helped us to capture the properties of the off-shell nilpotency and absolute anticommutativity of the conserved charges of our interacting \( D \)-dimensional non-Abelian 1-form gauge theory.

One of the novel observations of our present endeavor is the proof of absolute anticommutativity of the (anti-)BRST conserved charges despite the fact that we have considered only the (anti-)chiral super expansions of the (anti-)chiral superfields within the framework of ACSA to BRST formalism. In this proof, the celebrated CF-condition [17] has played a crucial role. In fact, our knowledge of various key aspects of the superfield approach to BRST formalism (on the suitably chosen supermanifolds) and their deep connection with the nilpotent (anti-)BRST symmetries in the ordinary space has helped us in accomplishing the above goal (which is one of the highlights of our present investigation). We have been able to express the nilpotency and absolute anticommutativity properties in the ordinary space, too. However, this has been possible because of our deep understanding of various aspects of the superfield approach to BRST formalism.

Within the framework of ACSA to BRST formalism, the observation of absolute anticommutativity of the (anti-)BRST charges is a completely novel result because we have studied various \( \mathcal{N} = 2 \) SUSY quantum mechanical models and applied the (anti-)chiral supervariable approach to derive the nilpotent \( \mathcal{N} = 2 \) supersymmetric transformations but the corresponding \( \mathcal{N} = 2 \) SUSY charges have been shown to be not absolutely anticommuting in nature [23-27]. In fact, it has been shown that the anticommutator of \( \mathcal{N} = 2 \) SUSY charges generates the time translation of the variable on which it operates. In other words, the anticommutator of the \( \mathcal{N} = 2 \) super charges leads to the derivation of Hamiltonian for the \( \mathcal{N} = 2 \) SUSY quantum mechanical model. Against this backdrop, it is

\[ \text{We take a simple example of the } \mathcal{N} = 2 \text{ SUSY quantum mechanical model of a } 1D \text{ harmonic oscillator to corroborate this statement in our Appendix B.} \]
clear that the observation of the absolute anticommutativity property of the (anti-)BRST charges is a completely novel result within the framework of ACSA to BRST formalism. We discuss briefly about this surprisingly novel result in our Appendix A.

The ideas of ACSA to BRST formalism are simple and straightforward and they lead to the derivation of (anti-)BRST symmetries for all the fields together. This should be contrasted with the HC which leads to the derivation of the off-shell nilpotent (anti-)BRST symmetries for the gauge and (anti-)ghost fields only. We plan to extend our ideas in the context of discussions for the higher $p$-form ($p = 2, 3, 4,...$) gauge theories (within the framework of (anti-)chiral superfield approach to BRST formalism) so that the ACSA to BRST formalism could be firmly established. In this context, it is gratifying to mention that we have already applied our ideas of ACSA to BRST formalism in the case of a free 4D Abelian 2-form gauge theory and have proven the absolute anticommutativity of the (anti-)BRST charges (see, e.g. [15] for details). In this proof, we have shown that the CF-type restriction (for the Abelian 2-form gauge theory) plays a decisive role.

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Appendix A: Absolute Anticommutativity and Full Super Expansion

To highlight the novel observation of the absolute anticommutativity of the (anti-)BRST charges (in our present investigation), we discuss here concisely the connection of this property with the full super expansion of the superfields along all the Grassmannian direction, of the $(D, 2)$-dimensional supermanifold on which the fields of a given $D$-dimensional ordinary gauge theory are generalized. As we know, one of the decisive features of BRST and anti-BRST symmetry transformations is the absolute anticommutativity property which primarily captures the linear independence of these symmetry transformations. Expressed in terms of the translational generators $(\partial_\theta, \partial_{\bar{\theta}})$ along the Grassmannian directions of the $(D, 2)$-dimensional supermanifold, we observe the following (with inputs $\partial_\theta \longleftrightarrow s_b$, $\partial_{\bar{\theta}} \longleftrightarrow s_{ab}$):

$$s_b \ s_{ab} + s_{ab} \ s_b = 0 \quad \iff \quad \partial_\theta \partial_{\bar{\theta}} + \partial_{\bar{\theta}} \partial_\theta = 0.$$  \hspace{1cm} (A.1)

Since the continuous symmetry transformations $s_{(a)b}$ are deeply connected with (and generated by) the Noether conserved charges $Q_{(a)b}$, the above relationship (A.1) can be also translated into the following:

$$Q_b \ Q_{ab} + Q_{ab} \ Q_b = 0 \quad \iff \quad \partial_\theta \partial_{\bar{\theta}} + \partial_{\bar{\theta}} \partial_\theta = 0.$$ \hspace{1cm} (A.2)

We claim, in this Appendix, that (A.1) and (A.2) become quite obvious and transparent when we take the full super expansions of the superfields defined on the $(D, 2)$-dimensional
supermanifold. However, this is not the case when we take only the (anti-)chiral super expansions of the (anti-)chiral superfields that are defined on the (\(D, 1\))-dimensional super-submanifolds of the general \((D, 2)\)-dimensional supermanifold (on which a given \(D\)-dimensional ordinary gauge theory is generalized). For instance, we have applied the (anti-)chiral supervariable approach to \(\mathcal{N} = 2\) SUSY quantum mechanical models where the property of absolute anticommutativity is not satisfied (see, e.g. \[19-22\] for details).

To corroborate the above statement, we begin with a generic superfield \(\Omega (x, \theta, \bar{\theta})\) which has the following full super expansion along all the Grassmannian directions of the \((D, 2)\)-dimensional supermanifold, namely;

\[
\Omega (x, \theta, \bar{\theta}) = \omega (x) + \theta \bar{P}(x) + \bar{\theta} P(x) + i \theta \bar{\theta} Q(x), \tag{A.3}
\]

where \(\omega (x)\), on the r.h.s., is the basic field of the \(D\)-dimensional gauge theory and the set \((P(x), \bar{P}(x), Q(x))\) represents the existence of secondary fields. The fermionic \((\theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} + \bar{\theta} \theta = 0)\) nature of the Grassmannian variables \((\theta, \bar{\theta})\) demonstrate that, if \(\Omega (x, \theta, \bar{\theta})\) were fermionic in nature, the pair \((P(x), \bar{P}(x))\) would be bosonic and \(Q(x)\) would be fermionic. On the other hand, if \(\Omega (x, \theta, \bar{\theta})\) were bosonic, the pair \((P(x), \bar{P}(x))\) would be fermionic and \(Q(x)\) would be bosonic. It is elementary to check that the following are true for the super expansion (A.3), namely;

\[
\frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}} \Omega (x, \theta, \bar{\theta}) = -i Q(x), \quad \frac{\partial}{\partial \bar{\theta}} \frac{\partial}{\partial \theta} \Omega (x, \theta, \bar{\theta}) = +i Q(x). \tag{A.4}
\]

The above relationship automatically establishes the following

\[
\left( \frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}} + \frac{\partial}{\partial \bar{\theta}} \frac{\partial}{\partial \theta} \right) \Omega (x, \theta, \bar{\theta}) = 0, \tag{A.5}
\]

which leads to the operator form of the relationship: \(\partial_{\theta} \partial_{\bar{\theta}} + \partial_{\bar{\theta}} \partial_{\theta} = 0\). This relationship, when translated into the ordinary space, leads to the connections that have been expressed in (A.1) and (A.2). Thus, the property of absolute anticommutativity, at the level of symmetry operators and conserved charges, becomes very natural, automatic and transparent when we consider the full super expansions of the superfields defined on the \((D, 2)\)-dimensional supermanifold. It is crystal clear that when we take only the (anti-)chiral super expansions of the superfields, the above relationship (A.4) and (A.5) do not become transparent and obvious.

In our present investigation, we have taken only the truncated version of the super expansion (A.3). In other words, we have considered only the (anti-)chiral version of (A.3). Thus, the absolute anticommutativity \((Q_b Q_{ab} + Q_{ab} Q_b = 0)\) of the charges \(Q_{(a)b}\) is not obvious. In fact, for the \(\mathcal{N} = 2\) SUSY quantum mechanical models, it has been shown \[19-22\] that the conserved nilpotent super charges do not absolutely anticommutate within the framework of (anti-)chiral supervariable approach to these models (see, Appendix B). Thus, the observation of absolute anticommutativity of the (anti-)BRST charges, within the framework of ACSA to BRST formalism, is a completely novel result. Now, with the back up from our earlier works \[13-16, 18\] and very recent work \[28\], we have been able to establish that the absolute anticommutativity of the (anti-)BRST charges is a universal truth when we apply the ACSA to BRST formalism in the cases of \(p\)-form \((p = 1, 2, 3,...)\) gauge theories as well as the reparameterization invariant theories \[28\].
Appendix B: $\mathcal{N} = 2$ SUSY QM Model of a Harmonic Oscillator and Its Nilpotent and Conserved Charges

In this Appendix, we discuss, in a bit elaborate fashion, our earlier work [19] on the $\mathcal{N} = 2$ SUSY QM system of a harmonic oscillator. However, our present discussions are somewhat different from [19]. This very interesting model is described by the following Lagrangian (with mass $m = 1$ and natural frequency $\omega$), namely;

$$L_0 = \frac{1}{2} \dot{x}^2 - \frac{1}{2} \omega^2 x^2 + i \bar{\psi} \dot{\psi} - \omega \bar{\psi} \psi,$$  \hfill (B.1)

where $\dot{x} = (dx/dt)$ and $\dot{\psi} = (d\psi/dt)$ are the “generalized” velocities for the bosonic variable $x$ and its fermionic SUSY counterpart $\psi$ w.r.t. the evolution parameter $t$ that characterizes the trajectory of the $\mathcal{N} = 2$ toy model of a 1D SUSY harmonic oscillator. In fact, we have a pair of fermionic variables $\psi$ and $\bar{\psi}$ (i.e. $\mathcal{N} = 2$ SUSY partners) which obey: $\psi^2 = 0, \bar{\psi}^2 = 0, \psi \bar{\psi} + \bar{\psi} \psi = 0$. The above Lagrangian (B.1) respects the following two $\mathcal{N} = 2$ SUSY transformations (see, e.g. [19] for details)

$$s_1 \Phi = i \psi, \quad s_1 \bar{\psi} = 0, \quad s_1 \bar{\psi} = - (\dot{x} + i \omega x),$$

$$s_2 \Phi = - i \bar{\psi}, \quad s_2 \bar{\psi} = 0, \quad s_2 \psi = (\dot{x} - i \omega x),$$  \hfill (B.2)

where $s_1$ and $s_2$ are nilpotent ($s_1^2 = 0, s_2^2 = 0$) of order two provided we use the EL-EOM ($\dot{\psi} + i \omega \psi = 0$ and $\dot{\bar{\psi}} - i \omega \bar{\psi} = 0$) but they do not absolutely anticommute (i.e. $s_1 s_2 + s_2 s_1 \neq 0$) even if we use the on-shell conditions (i.e. EL-EOM). The above continuous symmetries, according to Noether's theorem, lead to the derivation of the following conserved $\mathcal{N} = 2$ SUSY charges $Q$ and $\bar{Q}$

$$Q = (i p - \omega x) \psi \equiv (i \dot{x} - \omega x) \psi,$$

$$\bar{Q} = - \bar{\psi} (i p + \omega x) \equiv - \bar{\psi} (i \dot{x} + \omega x),$$  \hfill (B.3)

where $p = \dot{x}$ is the momentum corresponding to the bosonic variable $x$. Using the following EL-EOMs, derived from the Lagrangian $L_0$, namely;

$$\ddot{x} + \omega^2 x = 0, \quad \ddot{\psi} + \omega^2 \psi = 0, \quad \ddot{\bar{\psi}} + \omega^2 \bar{\psi} = 0,$$

$$\dot{\psi} + i \omega \psi = 0, \quad \dot{\bar{\psi}} - i \omega \bar{\psi} = 0,$$  \hfill (B.4)

it is straightforward to check that both the above $\mathcal{N} = 2$ SUSY charges are conserved (i.e. $\dot{Q} = 0, \dot{\bar{Q}} = 0$). In other words, these charges remain constant w.r.t. the evolution parameter $t$ which parametrizes the trajectory of an $\mathcal{N} = 2$ SUSY QM model of a 1D harmonic oscillator.

These conserved charges are also the generators for the $\mathcal{N} = 2$ SUSY transformations $s_1$ and $s_2$ because it can be checked that the following relationships lead to the derivations of the symmetry transformations (B.2), namely;

$$s_1 \Phi = - i [\Phi, Q]_\pm, \quad s_2 \Phi = - i [\Phi, \bar{Q}]_\pm,$$  \hfill (B.5)
where \( \Phi = x, \psi, \bar{\psi} \) is the generic variable of our theory and \((\pm)\) signs, as the superscripts on the square bracket, denote the (anti)commutator for the generic variable \( \Phi \) being fermionic/bosonic in nature. The basic canonical brackets: \([x, p] = 1, i \{\psi, \bar{\psi}\} = 1\) have to be exploited in the verification of \((B.5)\) as far as the derivations of the \(\mathcal{N} = 2\) SUSY transformations \(s_1\) and \(s_2\) are concerned. The above charges \(Q\) and \(\bar{Q}\) are nilpotent \((Q^2 = \bar{Q}^2 = 0)\) of order two because

\[
s_1 Q = -i \{Q, Q\} = 0 \Rightarrow Q^2 = 0, \quad s_2 \bar{Q} = -i \{\bar{Q}, \bar{Q}\} = 0 \Rightarrow \bar{Q}^2 = 0, \quad (B.6)
\]

where the l.h.s. can be computed easily by the direct applications of \(s_1\) and \(s_2\) on the charges \((B.3)\). The nilpotency property \((B.6)\) can also be proven by the following very useful and interesting observations

\[
Q = s_1 (-i \bar{\psi} \psi), \quad \bar{Q} = s_2 (i \bar{\psi} \psi), \quad (B.7)
\]

which demonstrate that \(s_1 Q = 0, s_2 \bar{Q} = 0\) due to the nilpotency property: \(s_1^2 = s_2^2 = 0\). In other words, we find that the nilpotency \((s_1^2 = s_2^2 = 0)\) of \(s_1\) and \(s_2\) are deeply connected with the nilpotency \((Q^2 = \bar{Q}^2 = 0)\) of \(Q\) and \(\bar{Q}\). It is worthwhile to mention here that the expressions for \(Q\) and \(\bar{Q}\) (cf. \((B.3)\)) can never ever be expressed as some kind of variations w.r.t. \(s_2\) and \(s_1\), respectively. In other words, these observations certify that the conserved and nilpotent charges \(Q\) and \(\bar{Q}\) do not absolutely anticommute (i.e. \(s_2 Q = -i \{Q, \bar{Q}\} \neq 0, s_1 \bar{Q} = -i \{\bar{Q}, Q\} \neq 0\)) as can be checked, in a straightforward manner, by using the basic principles behind the relationships between the symmetry transformations \(s_1\) and \(s_2\) and their generators (cf. \((B.5)\)). The above observations are very important for our discussions on the (anti-)chiral supervariable approach (i.e. ACSA) to our specific \(\mathcal{N} = 2\) QM model of a 1D SUSY harmonic oscillator.

We apply now the (anti-)chiral supervariable approach (ACSA) to derive the on-shell nilpotent \((s_1^2 = s_2^2 = 0)\) symmetry transformations \((B.2)\) for the \(\mathcal{N} = 2\) QM model of a 1D harmonic oscillator. Towards this goal in mind, first of all, we derive \(s_1\), for which, we generalize the 1D basic variables \(x(t), \psi(t), \bar{\psi}(t)\) to their counterparts supervariables \(X(t, \bar{\theta}), \Psi(t, \bar{\theta}), \bar{\Psi}(t, \bar{\theta})\) which are defined on the \((1, 1)\)-dimensional anti-chiral super-submanifold of the general \((1, 2)\)-dimensional supermanifold (on which our 1D \(\mathcal{N} = 2\) SUSY QM model is generalized). The super expansions of the supervariables are

\[
x(t) \longrightarrow X(t, \bar{\theta}) = x(t) + \bar{\theta} f(t), \quad \psi(t) \longrightarrow \Psi(t, \bar{\theta}) = \psi(t) + \bar{\theta} (i b_1),
\]

\[
\bar{\psi}(t) \longrightarrow \bar{\Psi}(t, \bar{\theta}) = \bar{\psi}(t) + \bar{\theta} (i b_2), \quad (B.8)
\]

where the secondary variables \((b_1, b_2)\) are bosonic and \(f\) is fermionic due to the fermionic \((\bar{\theta}^2 = 0)\) nature of \(\bar{\theta}\). In fact the \((1, 1)\)-dimensional anti-chiral super-submanifold is parameterized by the superspace coordinates \(Z^M = (t, \bar{\theta})\). The following SUSY invariant quantities under \(s_1\), namely,

\[
s_1 \psi = 0, \quad s_1 (x \psi) = 0, \quad s_1 (\dot{x} \psi) = 0, \quad (B.9)
\]

lead to the derivation of the secondary variables \(b_1 = 0\) and \(f = i \psi\) (see, e.g. [19] for details). In other words, we have the following expansions

\[
X^{(b_1)}(t, \bar{\theta}) = x(t) + \bar{\theta} (i \psi) \equiv x(t) + \bar{\theta} (s_1 x(t)),
\]
\[ \Psi^{(h_1)}(t, \bar{\theta}) = \psi(t) + \bar{\theta}(0) \equiv \psi(t) + \bar{\theta}(s_1\psi(t)), \quad (B.10) \]

denotes the supervariables defined on the anti-chiral supermanifold which have been derived after the applications of the SUSY invariant restrictions (cf. (B.9)). With the helps from the expansions \( X^{(h_1)}(t, \bar{\theta}) \) and \( \Psi^{(h_1)}(t, \bar{\theta}) \), we can compute the expression for \( b_2(t) \) in the expansion for \( \bar{\psi}(t, \bar{\theta}) \). For this purpose, we note that the following quantity

\[ s_1 \left[ \frac{\dot{x}^2}{2} + i \bar{\psi} \ddot{\psi} + \frac{\omega^2 x^2}{2} \right] = 0, \quad (B.11) \]

is on-shell (i.e. \( \dot{\psi} + i\omega\psi = 0 \)) invariant. Hence, according to the basic tenets of ACSA, we demand the following

\[
\frac{\dot{X}^{(h_1)}(t, \bar{\theta}) X^{(h_1)}(t, \bar{\theta})}{2} + i \left[ \bar{\psi}(t) + i \bar{\theta}b_2(t) \right] \dot{\Psi}^{(h_1)}(t, \bar{\theta}) + \frac{\omega^2}{2} X^{(h_1)}(t, \bar{\theta}) X^{(h_1)}(t, \bar{\theta})
= \frac{\dot{x}^2}{2} + i \bar{\psi}(t) \dot{\psi}(t) + \frac{\omega^2}{2} x^2(t),
\]

which lead to the determination of \( b_2 = i \dot{x} - \omega x \). Thus, we obtain the following super expansion for the supervariable \( \bar{\Psi}(t, \bar{\theta}) \), namely;

\[ \bar{\Psi}^{(h_1)}(t, \bar{\theta}) = \bar{\psi}(t) - \bar{\theta}(\dot{x} + i\omega x) \equiv \bar{\psi}(t) + \bar{\theta}(s_1\bar{\psi}(t)). \quad (B.13) \]

It is evident that we have derived the on-shell nilpotent transformation \( s_1 \) for the variable \( \bar{\psi}(t) \) as the coefficient of \( \bar{\theta} \) in the expansion (B.13). It is self-evident that the superscript \( (h_1) \) on the supervariable \( \bar{\Psi}^{(h_1)}(t, \bar{\theta}) \) denotes the expansion for this supervariable which has been obtained after the application of the on-shell SUSY invariant restriction (B.11). A close and careful look at (B.10) and (B.13) demonstrates that we have derived all the symmetry transformation \( s_1 \) of (B.2) as the coefficients of \( \bar{\theta} \). Hence, we also note that we have the mapping: \( \partial_{\theta} \leftrightarrow s_1 \).

In order to derive the other symmetry transformations \( s_2 \) of (B.2) by using the ACSA to our \( N = 2 \) SUSY QM model of 1D harmonic oscillator, we generalize to basic 1D variables \( x(t), \psi(t), \bar{\psi}(t) \) to the (1, 1)-dimensional chiral supervariables \( X(t, \theta), \Psi(t, \theta), \bar{\Psi}(t, \theta) \) defined on the (1, 1)-dimensional chiral super-submanifold of the general (1, 2)-dimensional supermanifold, as

\[
x(t) \rightarrow X(t, \theta) = x(t) + \theta \bar{f}(t), \quad \psi(t) \rightarrow \Psi(t, \theta) = \psi(t) + i \theta \bar{b}_1(t),
\]

\[
\bar{\psi}(t) \rightarrow \bar{\Psi}(t, \theta) = \bar{\psi}(t) + i \theta \bar{b}_2(t), \quad (B.14) \]

where \( (\bar{b}_1, \bar{b}_2) \) are the secondary bosonic variables and \( \bar{f}(t) \) is a fermionic secondary variable. These secondary variables are to be determined in terms of the basic variables of the Lagrangian (B.1) by using the on-shell SUSY invariant restrictions. In this context, we observe that \( s_2(\bar{\psi}) = 0, s_2(x \bar{\psi}) = 0, s_2(\dot{x} \bar{\psi}) = 0 \) imply the following restrictions

\[ \bar{\Psi}(t, \theta) = \bar{\psi}(t), \quad X(t, \theta) \bar{\psi}(t) = x(t) \bar{\psi}(t), \quad \dot{X}(t, \theta) \bar{\psi}(t) = \dot{x}(t) \bar{\psi}(t), \quad (B.15) \]
which lead to $\ddot{b}_2(t) = 0, \ddot{f}(t) = i \bar{\psi}(t)$ (cf. [19] for details). Thus, we have the following super expansions

$$X^{(h_2)}(t, \theta) = x(t) + \theta (-i \bar{\psi}(t)) \equiv x(t) + \theta (s_2 x(t)),$$

$$\bar{\Psi}^{(h_2)}(t, \theta) = \bar{\psi}(t) + \theta (0) \equiv \bar{\psi}(t) + \theta (s_2 \bar{\psi}),$$

(B.16)

where the superscript $(h_2)$ on the supervariables, on the l.h.s., denote the expansions of the supervariables after the SUSY invariant restrictions $(B.15)$ and we note that we have already derived the transformations $s_2 x = -i \bar{\psi}, \ s_2 \bar{\psi} = 0$ as the coefficients in the super expansions $(B.16)$. This observation establishes that $\partial_\theta X^{(h_2)}(t, \theta) = s_2 x(t)$ and $\partial_\theta \bar{\Psi}^{(h_2)}(t, \theta) = s_2 \bar{\psi}(t)$ which, in turn, imply $\partial_\theta \leftrightarrow s_2$.

We have to now compute the expression for $b_1(t)$. In this context, we note that the following useful and interacting quantity is on-shell ($\dot{\psi} - i \omega \bar{\psi} = 0$) SUSY invariant, namely;

$$s_2 \left[ \frac{x^2}{2} - i \dot{x} \bar{\psi} + \frac{\omega^2 x^2}{2} \right] = 0. \tag{B.17}$$

The basic tenets of ACSA to $\mathcal{N} = 2$ SUSY QM system requires that the following SUSY restriction is true, namely:

$$\frac{\dot{X}^{(h_2)}(t, \theta)}{2} \dot{X}^{(h_2)}(t, \theta) - i \dot{\bar{\Psi}}^{(h_2)}(t, \theta) \left[ \psi(t) + i \theta b_1(t) \right] + \frac{\omega^2}{2} X^{(h_2)}(t, \bar{\theta}) X^{(h_2)}(t, \bar{\theta})$$

$$= \frac{\dot{x}^2(t)}{2} - i \dot{x} \bar{\psi}(t) \psi(t) + \frac{\omega^2}{2} x^2(t), \tag{B.18}$$

where the expansions of $X^{(h_2)}(t, \theta)$ and $\bar{\Psi}^{(h_2)}(t, \theta)$ have been listed in $(B.16)$. The above restriction leads to $b_1(t) = -i (\dot{x} - i \omega x)$. Hence, we have the following super expansion:

$$\bar{\Psi}^{(h_2)}(t, \theta) = \psi(t) + \theta (\dot{x} - i \omega x) \equiv \psi(t) + \theta (s_2 \psi(t)). \tag{B.19}$$

It is evident that we have derived the transformation $s_2 \psi = (\dot{x} - i \omega x)$ as the coefficient of $\theta$ in the above expansion where the superscript $(h_2)$ denotes that the supervariable $\bar{\Psi}^{(h_2)}(t, \theta)$ has been derived after the application of $(B.18)$. We note, once again, that we have: $\partial_\theta \leftrightarrow s_2$.

Against the backdrop of our derivation of $s_1$ and $s_2$ from the applications of the on-shell SUSY invariant restrictions, we are in the position to expresses the conserved charges $Q$ and $\bar{Q}$ in terms of the ACSA (to $\mathcal{N} = 2$ SUSY QM model of a 1D harmonic oscillator) as follows

$$Q = \frac{\partial}{\partial \bar{\theta}} \left[ -i \bar{\Psi}^{(h_1)}(t, \bar{\theta}) \Psi^{(h_1)}(t, \bar{\theta}) \right] \equiv \int d\bar{\theta} (-i \bar{\Psi}^{(h_1)}(t, \bar{\theta}) \Psi^{(h_1)}(t, \bar{\theta})),$$

$$\bar{Q} = \frac{\partial}{\partial \theta} \left[ i \bar{\Psi}^{(h_2)}(t, \theta) \Psi^{(h_2)}(t, \theta) \right] \equiv \int d\theta (i \bar{\Psi}^{(h_2)}(t, \theta) \Psi^{(h_2)}(t, \theta)), \tag{B.20}$$

where all the symbols have been clarified earlier. It is straightforward to note that the nilpotency ($Q^2 = \bar{Q}^2 = 0$) of the charges $Q$ and $\bar{Q}$

$$\partial_\theta Q = 0 \iff s_1 Q = -i \{Q, \bar{Q}\} = 0 \implies Q^2 = 0,$$
\[
\partial_\theta \bar{Q} = 0 \iff s_2 \bar{Q} = -i \{ \bar{Q}, Q \} = 0 \implies \bar{Q}^2 = 0,
\]

are deeply connected with the nilpotency \((\partial_\theta^2 = \partial_\theta = 0)\) of the translational generators \((\partial_\theta, \partial_\theta)\) along the \((1, 1)\)-dimensional (anti-)chiral super-submanifolds (of the general \((1, 2)\)-dimensional supermanifold on which the \(\mathcal{N} = 2\) SUSY QM model is generalized). The above observations, in turn, are very beautifully connected with the nilpotency \((s_1^2 = s_2^2 = 0)\) of the \(\mathcal{N} = 2\) SUSY symmetry transformations \((B.2)\). Thus, we observe that nilpotency properties of \((Q, \bar{Q}), (s_1, s_2)\) and \((\partial_\theta, \partial_\theta)\) are intertwined together in a meaningful and beautiful fashion.

We end this Appendix with the following concluding remarks which capture the key differences between the applications of ACSA to BRST formalism and \(\mathcal{N} = 2\) SUSY QM models. First and foremost, we observe that there is no way to express (cf. \((B.20)\)) the conserved and nilpotent \(\mathcal{N} = 2\) SUSY charges \(Q\) and \(\bar{Q}\) as the derivatives w.r.t. \(\partial_\theta\) and \(\partial_\theta\), respectively, which is not the case with the (anti-)BRST charges \(Q^{(a)\theta}\) that can be expressed as the derivatives w.r.t. both the translational generators \((\partial_\theta, \partial_\theta)\) along the (anti-)chiral super-submanifolds. Hence, within the framework of ACSA, the nilpotent and conserved \(\mathcal{N} = 2\) SUSY charges \(Q\) and \(\bar{Q}\) do not absolutely anticommute (i.e. \(Q \bar{Q} + \bar{Q} Q \neq 0\)) but the nilpotent and conserved (anti-)BRST charges \(Q^{(a)\theta}\) absolutely anticommute (i.e. \(Q \bar{Q} + \bar{Q} Q = 0\)). Finally, we also note that ACSA to BRST formalism is applicable to any arbitrary \(D\)-dimensional gauge as well as reparameterization invariant theories but ACSA to \(\mathcal{N} = 2\) SUSY QM models is applicable to 1D systems only.

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