Global existence for vector valued fractional reaction-diffusion equations

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Abstract

In this paper, we study the initial value problem for infinite dimensional fractional non-autonomous reaction-diffusion equations. Applying general time-splitting methods, we prove the existence of solutions globally defined in time using convex sets as invariant regions. We expose examples, where biological and pattern formation systems, under suitable assumptions, achieve global existence. We also analyze the asymptotic behavior of solutions.

Keywords— Fractional diffusion, global existence, Lie–Trotter method.

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1 Introduction

In this paper, we prove global existence of solutions for vector valued fractional non-autonomous reaction-diffusion equations. That is, we study the non autonomous system

\[ \partial_t u + \sigma(-\Delta)^\beta u = F(t, u), \]  

where \( u(x, t) \in Z \) for \( x \in \mathbb{R}^n, t > 0, \sigma \geq 0 \) and \( 0 < \beta \leq 1, \) \( F: \mathbb{R} \times Z \to Z \) a continuous map and \( Z \) a Banach space. We consider the initial problem \( u(x, 0) = u_0(x). \)

The aim of this paper is to develop a new method to obtain behavioral results on the fractional reaction diffusion equation, using recent numerical splitting techniques ([6], [14]) introduced for other purposes. The main results of this paper are to obtain general conditions for well posedness of the fractional reaction diffusion equation in Banach spaces.

Fractional reaction-diffusion equations are commonly used on many applications such as biological models, population dynamics models, nuclear reactor models, just to name a few (for references to examples see [4]). The difference between classical and fractional diffusion is that the classical Laplacian term associated with classical diffusion implies a Gaussian dispersal kernel in the corresponding equation, which does not represent all possible models in practice. The fractional model captures the faster...
spreading rates and power law invasion profiles observed in many applications. The main reason for this behavior is given by the fractional Laplacian. There are many different equivalent definitions of the fractional Laplacian and its behavior is well understood (see [7], [15], [21], [23], [30], [27] and [22]).

The non-autonomous nonlinear reaction diffusion equation dynamics were studied by [28] and others, analyzing the stability and evolution of the problem. Global existence in reaction-diffusion equations in bounded sets were studied in the book by Smoller [32] and in [12] where it is considered the $n$-dimensional case with classical diffusion and the intersection of half spaces as invariant regions in $\mathbb{R}^n$, in which the equation evolves. The case of taking a convex set as an invariant set, is considered only when the diffusion coefficients $\sigma_{ij} \in \mathbb{R}^{n \times n}$ is the identity matrix (see Corollary 14.8 (b) in [32]). Morgan [25] considered a similar case in which all the coefficients $\sigma_{ij}$ are different, but other conditions are needed over the system to achieve the result. These techniques have been used recently many times to obtain global well posedness for different classical diffusion problems (see [1], [5], [24], and [31]). Motivated by this, in this paper, we study global existence of fractional-diffusion equations using a completely different approach. We use time splitting methods in Banach spaces taking closed convex sets as invariant regions.

As an example, we explore the scalar system where the nonlinearity is given by $F(u) = (1 + ia)u - (1 + ib)|u|^2u$ with $a, b \in \mathbb{R}$ (see [12], [34] and [10]). For particular nonlinearities exact solutions are known, for example, in [20] was studied the existence of scalar traveling waves for the quadratic, cubic and quartic cases by the tanh method. We also explore a FitzHugh Nagumo pattern formation system in $\mathbb{R}^2$ and a population dynamic system in a Banach space. In both cases we found an appropriate invariant region that allows us to prove global existence in each case. Finally, we also analyze the asymptotic behavior of solutions in the real line.

This paper is organized as follows:

- **Section 2** We introduce the notation and prove some preliminary results concerning the linear and non linear parts of the fractional reaction diffusion equation.

- **Section 3** We introduce the propagators, allowing us to construct a splitting reaction diffusion equation. This is important to build up the linear part.

- **Section 4** We obtain several results for finally proving that the "splitting" equation converges to the "original" equation. This allow us to study the splitting equation, that is, to study separately the linear and non linear parts, in order to obtain interesting results on the original equation.

- **Section 5** We prove global well posedness for invariant closed convex sets of a Banach space. We prove that the linear and non linear parts of the splitting equation, independently maintain the solution inside the convex set. The results from section 4 extend this result to the "original" equation. We give some examples such us the Ginzburg-Landau equation and the Fisher-Kolmogorov equation.

- **Section 5.1** We expose an interesting example, a population dynamics model, where we have a trait variable in a Banach Space. This will show the importance of extending the results to Banach Spaces.

- **Section 5.2** We generalize the results in the beginning of section 5 by proving well posedness for products of Banach Spaces.
• Section 6 We show how powerful splitting methods are, by analyzing a completely different problem, the asymptotic behavior of a solution. The strategy is again, to split the linear and nonlinear parts, and analyze them separately, for then finally use the results in section 4 and 5.

2 Notations and Preliminaries.

We are interested in continuous functions to vectorial values, that is to say, whose evaluations take values in Banach Spaces. The main reason for this, is to analyze well posedness of population dynamics problems with discrete or continuous traits, that distinguish the population components (See subsection 5.1).

Let $Z$ be a Banach space, we define $C_u^\infty(\mathbb{R}^d, Z)$ as the set of uniformly continuous and bounded functions on $\mathbb{R}^d$ with values in $Z$. Taking the norm $\|u\|_{\infty, Z} = \sup_{x \in \mathbb{R}^d} |u(x)|_Z$, $C_u^\infty(\mathbb{R}^d, Z)$ is a Banach space. We denote by $C_u^k(\mathbb{R}^d, Z)$ the subspace of all $k$–times continuously differentiable functions such that $\partial^\gamma_x u \in C_u^\infty(\mathbb{R}^d, Z)$ with the norm $\|u\|_{k, \infty, Z} = \sum_{|\gamma| \leq k} \|\partial^\gamma_x u\|_{\infty, Z}$.

It is easy to see that if $g \in L^1(\mathbb{R}^d)$ and $u \in C_u^\infty(\mathbb{R}^d, Z)$ the Bochner integral is the defined the following way,

$$(g * u)(x) = \int_{\mathbb{R}^d} g(y)u(x-y)dy$$

This defines an element of $C_u(\mathbb{R}^d, Z)$ (See [11]). Additionally, the linear operator $u \mapsto g * u$ is continuous.

The following results show that the operator $-(-\Delta)^\beta$ defines a continuous contraction semigroup in the set $C_u(\mathbb{R}^d, Z)$. The following lemma is a consequence of Lévy–Khinchine formula for infinitely divisible distributions and the properties of the Fourier transform.

Lemma 1. Let $0 < \beta \leq 1$ and $g_\beta \in C_0(\mathbb{R}^d)$ such that $\hat{g}_\beta(\gamma) = e^{-|\gamma|^2 \beta}$, it holds $g_\beta$ is positive, invariant under rotations of $\mathbb{R}^d$, integrable and $\int_{\mathbb{R}^d} g_\beta(x)dx = 1$.

Proof. The first statement follows from Theorem 14.14 of [29], the remaining claims are immediate from the definition of $\hat{g}_\beta$. 

Based on the previous lemma, we study Green’s function associated to the linear problem and $-\sigma(-\Delta)^\beta$.

Proposition 2. Let $\sigma > 0$ and $0 < \beta \leq 1$, the function $G_{\sigma, \beta}$ given by

$$G_{\sigma, \beta}(x,t) = (\sigma t)^{-\frac{d}{2\beta}} g_\beta((\sigma t)^{-\frac{1}{2\beta}} x),$$

for any $t > 0$, it holds:
Proof. The first and second statements are a consequence of the definition of \( \dot{g}_\beta \). The third and fourth statements are immediate applying Fourier transform.

In the following proposition, we show that the linear operator \(-\sigma(-\Delta)^\beta\) defines a contraction continuous semigroup in the set \( C_u(\mathbb{R}^d, Z) \).

**Proposition 3.** For any \( \sigma > 0 \) and \( 0 < \beta \leq 1 \), the map \( S: \mathbb{R}_+ \to \mathcal{B}(C_u(\mathbb{R}^d, Z)) \) defined by \( S(t)u = G_{\sigma, \beta}(\cdot, t) * u \) is a continuous contraction semigroup.

**Proof.** We first prove the semigroup property, which is deduced from iii of the previous proposition:

\[
S(t)S(t')u = G_{\sigma, \beta}(\cdot, t) * (G_{\sigma, \beta}(\cdot, t') * u) = (G_{\sigma, \beta}(\cdot, t) * (G_{\sigma, \beta}(\cdot, t') * u)) * u = (G_{\sigma, \beta}(\cdot, t + t') * u) = S(t + t')u
\]

We show that \( S(t)u \) converges to \( u \) for all \( u \in C_u(\mathbb{R}^d, Z) \) when \( t \to 0 \). Indeed, we have for \( \delta > 0 \),

\[
|S(t)u(x) - u(x)|_Z \leq \int_{\mathbb{R}^n} G_{\sigma, \beta}(y,t)|u(x-y) - u(x)|_Z \, dy
\]

\[
= \int_{|y|<\delta} G_{\sigma, \beta}(y,t)|u(x-y) - u(x)|_Z \, dy + \int_{|y|\geq\delta} G_{\sigma, \beta}(y,t)|u(x-y) - u(x)|_Z \, dy.
\]

The first integral of the right side of the equality can be estimated as follows:

\[
\int_{|y|<\delta} G_{\sigma, \beta}(y,t)|u(x-y) - u(x)|_Z \, dy \leq \int_{\mathbb{R}^n} G_{\sigma, \beta}(y,t) \max_{|y|<\delta} |u(x-y) - u(x)|_Z \, dy = \max_{|y|<\delta} |u(x-y) - u(x)|_Z
\]

This can be small enough because, \( |y| < \delta \) and \( u \) is uniformly continuous. For the second term we proceed in the following way,

\[
\int_{|y|\geq\delta} G_{\sigma, \beta}(y,t)|u(x-y) - u(x)|_Z \, dy = 2\|u\|_\infty(\sigma t)^{-\frac{\beta}{2}} \int_{|y|\geq\delta} g_\beta((\sigma t)^{-\frac{\beta}{2}} y) \, dy
\]

\[
= 2\|u\|_\infty \int_{|y|\geq\delta(\sigma t)^{-1/(2\beta)}} g_\beta(y) \, dy
\]
Since \( \delta(\sigma t)^{-1/(2\beta)} \to \infty \) when \( t \to 0^+ \) and \( g_\beta \in L^1(\mathbb{R}^d) \), the right side of the previous equality tends to 0. The next property proves that \( S \) is well defined, that is \( S u \in C_\mu(\mathbb{R}^d, Z) \).

\[
|\langle S(t)u(x_1) - S(t)u(x_2)\rangle| \leq \int_{\mathbb{R}^n} G_{\sigma,\beta}(y,t)|u(x_1 - y) - u(x_2 - y)| dy \\
\leq \varepsilon \int_{\mathbb{R}^n} G_{\sigma,\beta}(y,t) dy = \varepsilon,
\]

In the last inequality we used that \( \varepsilon \) is a solution of the integral equation

Moreover, there exists a non-increasing function \( T : [0, \infty) \to [0, \infty) \), such that \( T^*(z_0) \geq \hat{T}(\|z_0\|_Z) \). The solution of (2.3) is a solution of the integral equation

\[
z(t) = z_0 + \int_0^t F(t', z(t')) dt'.
\]

We denote by \( N : \mathbb{R} \times \mathbb{R} \times C_\mu(\mathbb{R}^d, Z) \to C_\mu(\mathbb{R}^d, Z) \) the flow generated by the ordinary equation, i.e.: for any \( x \in \mathbb{R}^d \), \( N(t, t_0, u_0)(x) \) is the solution of the problem (2.3) with initial data \( z_0 = u_0(x) \). Therefore, if \( u(t) = N(t, t_0, u_0) \)

\[
u(x, t) = u_0(x) + \int_0^t F(t', u(x, t')) dt'.
\]
Remark 5. All results can be extended to the set $C^k_u(\mathbb{R}^d, Z)$. If $u_0 \in C^k_u(\mathbb{R}^d, Z)$ and $F$ a sufficiently smooth map, then $S(t)u_0 \in C^k_u(\mathbb{R}^d, Z)$ and $F(t, u) \in C^k_u(\mathbb{R}^d, Z)$ therefore using (2.2) we have $u(t) \in C([0, T], C^k_u(\mathbb{R}^d, Z))$.

The following result relates the solutions of (2.3) with the problem (2.2) in the case of having constant initial data.

**Proposition 6.** If $u_0$ is a constant function, then $u(t) = N(t, t_0, u_0)$ is a solution of (2.2).

**Proof.** Since $u_0$ is a constant function, from the uniqueness of the problem (2.3), we have $u(t) = u_0$ for any $t > 0$ where the solution is defined. Therefore,

$$u(t) = u_0 + \int_0^t F(t', u(t'))dt' = S(t)u_0 + \int_0^t S(t-t')F(t', u(t'))dt',$$

which proves our assertion. \(\square\)

## 3 Propagators

To build the approximate solutions, we decompose the time variable in regular intervals and consider the evolution, in an alternate form, of the linear and non linear problem. To achieve this, we turn on and off each term of the equation. The first step, is to consider the abstract linear problem,

$$\partial_t u - \alpha(t)Au = 0,$$

$$u(s) = u_0,$$

with $\alpha(t) > 0$ and $A$ is the infinitesimal generator of $S$, a strongly continuous semigroup of operators defined in the Banach space $X$. The mild solution of the non autonomous problem can be written as $u(t) = S(\alpha)(t, s)u_0 = S(\tau(t, s))u_0$, where $\tau$ is defined by

$$\tau(t, s) = \int_s^t \alpha(t')dt'.$$

Formally, we have $\partial_t u = \partial_t S(\tau(t, s))u_0 = \partial_t \tau(t, s)A S(\tau(t, s))u_0$. To analyze the Lie-Trotter method, we define $\alpha : \mathbb{R} \to \mathbb{R}$ a periodic function of period 1 as:

$$\alpha(t) = \begin{cases} 2, & \text{if } k \leq t < k + 1/2, \\ 0, & \text{if } k - 1/2 \leq t < k, \end{cases} \quad (3.5)$$

for $k \in \mathbb{Z}$. Given $h > 0$, we define the function $\alpha_h : \mathbb{R} \to \mathbb{R}$ as $\alpha_h(t) = \alpha(t/h)$. Clearly $0 \leq \alpha_h \leq 2$, $\alpha_h$ is $h$-periodic and its mean value is 1. We consider $\tau_h : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\tau_h(t, t') = \int_{t'}^t \alpha_h(t'')dt''.$$

The following results show that $S_{\alpha_h}$ defines a propagator in $X$. We also obtain some estimates that we will use in the following section concerning the convergence. We can prove that

**Lemma 7.** The map $\tau_h$ is continuous in $\mathbb{R}^2$ and satisfies
A. Besteiro and D. Rial

Figure 1: Graph of $\alpha_h(t)$.

i. $0 \leq \tau_h(t, t') \leq 2(t - t')$, si $t' \leq t$,

ii. $\tau_h(t, t') + \tau_h(t', t'') = \tau_h(t, t'')$, si $t'' < t' < t$,

iii. $\tau_h(t + kh, t' + kh) = \tau_h(t, t')$, for $k \in \mathbb{Z}$,

iv. $\tau_h(t' + kh, t') = kh$, for $k \in \mathbb{Z}$,

v. $|t - t' - \tau_h(t, t')| \leq h$,

Proof. The first statement is a consequence of the inequality $0 \leq \alpha_h \leq 2$. The additivity is immediate from the definition. The third statement is a consequence of the $\alpha_h$ periodicity. As the mean value of $\alpha_h$ is $h$, then $\tau_h(t + kh, t + kh, t' + kh, t') = h$, and using additivity property we have,

$$\tau_h(t' + kh, t') = \sum_{j=1}^{k} \tau_h(t' + jh, t' + (j - 1)h) = kh.$$ 

For the last claim, we consider $t = t' + kh + sh$, with $k \in \mathbb{Z}$ y $0 \leq s < 1$, as $|1 - \alpha_h(t)| \leq 1$, then

$$|(t - t') - \tau_h(t, t')| = |kh + sh - \tau_h(t' + kh + sh, t')|$$

$$= |(kh + sh) - \tau_h(t' + kh + sh, t' + kh) - \tau_h(t' + kh, t')|$$

$$= |sh - \tau_h(t' + kh + sh, t' + kh)|$$

$$= \left| \int_{t' + kh}^{t' + kh + sh} (1 - \alpha_h(t''))dt'' \right|$$

$$\leq \int_{t' + kh}^{t' + kh + sh} |1 - \alpha_h(t'')|dt'' \leq h,$$

that proves the last assertion. 

We define $\Omega = \{(t, t') \in \mathbb{R}^2 : 0 \leq t' \leq t\}$ and the application $S_h: \Omega \to \mathcal{B}(X)$ defined by $S_h(t, t') = S(\tau_h(t, t'))$, from the previous lemma have:

Corollary 8. Let $S: [0, \infty) \to \mathcal{B}(X)$ a strongly continuous one-parameter semigroup of operators, we have that $S_h$ satisfies:

i. $S_h(t, t) = 1$.

ii. $S_h(t, t'') = S_h(t, t')S_h(t', t'')$, if $0 \leq t'' \leq t' \leq t$.

iii. There exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|S_h(t, t')\|_{\mathcal{B}(X)} \leq Me^{2\omega(t - t')}$, for $(t, t') \in \Omega$. 
iv. If $u \in X$, the map $(t,t') \mapsto S_h(t,t')u$ is continuous.

v. If $u \in D = \text{Dom}(A)$ and $t' \leq t \neq kh/2$ with $k \in \mathbb{Z}$, then the map $t \mapsto S_h(t,t')u$ is differentiable and we have

$$\partial_t S_h(t,t')u = \begin{cases} 2AS_h(t,t')u, & \text{if } kh < t < (k + 1/2)h, \\ 0, & \text{if } (k - 1/2)h < t < kh, \end{cases}$$

Figure 2: Graph of $\tau_h(t,t')$, the steps are in the semintegers multiples of $h$.

4 Approximate solutions

In this section we develop the basic tools (Proposition 10 and Theorem 11) that allow us to obtain some properties of the solutions of the problem (2.4) from the approximations obtained with the Lie-Trotter method. Theorem 11 is an extension of theorem 3.9 in [14] to the non autonomous case. We define the system

$$\begin{cases} \partial_t u_h - \alpha_h(t)Au_h = (2 - \alpha_h(t))F(t,u_h), \\ u_h = u_{h,0}, \end{cases} \tag{4.6}$$

with $\alpha_h(t)$ as in (3), $u \in X$, $t > 0$, $F: \mathbb{R} \times X \to X$ is a continuous function and $X$ is a Banach space. Similarly with define the integral equation:

$$u_h(t) = S_h(t,0)u_{h,0} + \int_0^t (2 - \alpha_h(t'))S_h(t,t')F(t',u_h(t'))dt' \tag{4.7}$$

Proposition 9. Let $u_{h,0} \in \text{Dom}(A)$, if $u$ is solution of the system (4.6) then $u_h$ is solution of (4.7) for $t \in [0,T]$.

Proof. The procedure is similar to [11], Lemma 4.1.1.

In the following proposition, we show that the solution of the integral problem (4.7) corresponds to the approximations obtained with the Lie-Trotter method.
Let $u_h$ the solution of \eqref{eq:4.7}, if $U_{h,k} = u_h(kh)$ y $V_{h,k} = u_h(kh - h/2)$, then

\begin{align}
V_{h,k+1} &= S(h)U_{h,k}, \quad \text{(4.8a)} \\
U_{h,k+1} &= N(kh + h, kh + h/2, V_{h,k+1}), \quad \text{(4.8b)}
\end{align}

where $N$ is the flux associated to $2F$, that is $w(t) = N(t, s, w_0)$ where $w$ is the solution of

\[
\begin{cases}
\dot{w} = 2F(t, w(t)), \\
w(s) = w_0.
\end{cases}
\]

**Proof.** For $t_1 \in (0, t)$ it verifies

\[
u_h(t) = S_h(t, t_1)u_{h,0}(t_1) + \int_{t_1}^{t} (2 - \alpha_h(t'))S_h(t, t')F(t', u_h(t'))dt',
\]

using that $t_1 = kh$ y $t = kh + h/2$, we have

\[
V_{h,k+1} = S_h(kh + h/2, kh)U_{h,k} + \int_{kh}^{kh+h/2} (2 - \alpha_h(t'))S_h(kh + h/2, t')F(t', u_h(t'))dt',
\]

given that $\alpha_h(t) = 2$ for $t \in [kh, kh + h/2)$, we have $\tau_h(kh + h/2, kh) = h$ and therefore (4.8a). Similarly, $\alpha_h(t) = 0$ for $t \in [kh + h/2, kh + h)$, then $\tau_h(t, kh + h/2) = 0$ and therefore

\[
u_h(t) = V_{h,k+1} + 2\int_{kh+h/2}^{t} F(t', u_h(t'))dt',
\]

evaluating in $t = kh + h$, we obtain (4.8b).

**Theorem 11.** Let $u \in C([0, T^*), X)$ the solution of the integral problem \eqref{eq:2.2}

\[
u(t) = S(t)u_0 + \int_{0}^{t} S(t - t')F(t', u(t'))dt',
\]

$T \in (0, T^*)$ y $\varepsilon > 0$. There exists $h^* > 0$ such that if $0 < h < h^*$, then $u_h$ the solution of \eqref{eq:4.7} with $u_h = u_0$, is defined in $[0, T]$ and verifies $\|u(t) - u_h(t)\|_X \leq \varepsilon$ for $t \in [0, T]$.

To prove the theorem, we need two previous lemmas. We follow the procedure of Theorem 3.9 in [14] (see also [6]).

**Lemma 12.** Let $f \in C([0, T], X)$, if

\[
I_h(t, t') = (S(t - t') - S_h(t, t'))f(t'),
\]

then $\lim_{h \to 0^+} \sup_{(t, t') \in \Omega_T} \|I_h(t, t')\|_X = 0$, where $\Omega_T = \{(t, t') \in \mathbb{R}^2 : 0 \leq t' \leq t \leq T\}$.

**Proof.** Given $\varepsilon > 0$, there exists $g \in C([0, T], X)$ such that $g(t) \in D$ for $t \in [0, T]$, $Ag \in C([0, T], X)$ and $\max_{t \in [0, T]} \|f(t) - g(t)\|_X < \varepsilon$.

\[
\|S(t - t') - S_h(t, t')\|_X \leq 2Me^{2\omega T} \max_{t \in [0, T]} \|f(t) - g(t)\|_X \leq 2Me^{2\omega T} \varepsilon.
\]

\[
\|S(t - t') - S_h(t, t')\|_X \leq 2Me^{2\omega T} \varepsilon.
\]
On the other hand, we can write

\[
S(t-t')g(t') = g(t') + \int_0^{t-t'} S(\xi)Ag(t')d\xi,
\]

\[
S_h(t,t')g(t') = g(t') + \int_0^{\tau_h(t,t')} S(\xi)Ag(t')d\xi,
\]

subtracting both equations we obtain

\[
(S(t-t') - S_h(t,t'))g(t') = \pm \int_{J(t,t')} S(\xi)Ag(t')d\xi,
\]

where \( J(t,t') \) is the interval \( J(t,t') = [\min\{(t-t'),\tau_h(t,t')\},\max\{(t-t'),\tau_h(t,t')\}] \), then

\[
\| (S(t-t') - S_h(t,t'))g(t')\|_X \leq \|(t-t') - \tau_h(t,t')\| \max_{t \in [0,T]} \| Ag(t)\|_X \\
\leq h \max_{t \in [0,T]} \| Ag(t)\|_X. \tag{4.11}
\]

From the equations (4.10) y (4.11) we obtain the result. \( \square \)

**Lemma 13.** Let \( f \in C(\Omega_T, X) , \) with \( \Omega_T \) as in the previous lemma, if

\[
I_h(t) = \int_0^t (\alpha_h(t') - 1) f(t,t')dt',
\]

then \( \lim_{h \to 0^+} \sup_{t \in [0,T]} \| I_h(t)\|_X = 0. \)

**Proof.** From the uniform continuity \( f \), we can see that exists \( \delta > 0 \) such that if \( 0 \leq t', t'' \leq t \leq T \) and \( |t' - t''| < \delta \), then \( \| f(t,t') - f(t,t'')\|_X \leq \varepsilon. \) Let \( k = \lceil t/h \rceil \), we can write

\[
I_h(t) = \sum_{j=1}^{k} \int_{(j-1)h}^{jh} (\alpha_h(t') - 1) f(t,t')dt' + \int_{kh}^{t} (\alpha_h(t') - 1) f(t,t')dt'.
\]

As the mean value of \( \alpha_h \) is 1 in intervals of length \( h \), for \( f_j \in X \) we have

\[
0 = \int_{(j-1)h}^{jh} (\alpha_h(t') - 1) f_j dt', \tag{4.12}
\]

therefore

\[
\int_{(j-1)h}^{jh} (\alpha_h(t') - 1) f(t,t')dt' = \int_{(j-1)h}^{jh} (\alpha_h(t') - 1) (f(t,t') - f_j)dt'.
\]

If \( h < \delta \) and \( f_j = f(t,jh) \), then \( \| f(t,t') - f_j\|_X \leq \varepsilon \) for \( t \in [(j-1)h,jh] \) and therefore

\[
\left\| \int_{(j-1)h}^{jh} (\alpha_h(t') - 1) (f(t,t') - f_j)dt' \right\|_X \leq \varepsilon h. \tag{4.13}
\]

If \( M = \max_{(t',t') \in \Omega_T} \| f(t,t')\|_X \), then we have

\[
\left\| \int_{kh}^{t} (\alpha_h(t') - 1) f(t,t')dt' \right\|_X \leq \int_{kh}^{t} \| f(t,t')\|_X dt' \leq Mh. \tag{4.14}
\]
From (4.13), (4.12) y (4.14), we can obtain
\[ \| I_h(t) \|_X \leq \sum_{j=1}^{k} \varepsilon h + Mh \leq T\varepsilon + Mh, \]
from where we get the result. \(\square\)

**Proof of theorem 11.** If \([0,T^*_h]\) is the interval of existence of the integral equation (4.7), for \(0 \leq t < \min \{T,T^*_h\}\) the subtraction \(u(t) - u_h(t)\) satisfies
\[ u(t) - u_h(t) = (S(t) - S_h(t,0))u_0 + \int_0^t S(t-t') F(t',u(t')) dt' \\
- \int_0^t (2 - \alpha_h(t')) S_h(t,t') F(t',u_h(t')) dt'. \]

If we define
\[
I_{1,h}(t) = (S(t) - S_h(t,0))u_0, \\
I_{2,h}(t) = \int_0^t (2 - \alpha_h(t')) (S(t-t') - S_h(t,t')) F(t',u(t')) dt', \\
I_{3,h}(t) = \int_0^t (\alpha_h(t') - 1) S(t-t') F(t',u(t')) dt',
\]
then
\[
u(t) - u_h(t) = I_{1,h}(t) + I_{2,h}(t) + I_{3,h}(t) \\
+ \int_0^t (2 - \alpha_h(t')) S_h(t,t') (F(t',u(t')) - F(t',u_h(t'))) dt'.
\]

Using the lemma 12, using \(f(t) = u_0\), we can see that \(\lim_{h \to 0} \sup_{t \in [0,T]} \| I_{1,h}(t) \|_X = 0.\) Given that,
\[
\| I_{2,h}(t) \|_X \leq 2 \int_0^t \| (S(t-t') - S_h(t,t')) F(t',u(t')) \|_X dt' \leq 2T \sup_{(t,t') \in \Omega_T} \| (S(t-t') - S_h(t,t')) F(t',u(t')) \|_X,
\]
using once again the lemma 12 for \(f(t) = F(t,u(t))\), we obtain \(\lim_{h \to 0} \sup_{t \in [0,T]} \| I_{2,h}(t) \|_X = 0.\) The map \(f(t,t') = S(t-t') F(t',u(t'))\) is continuous in \(\Omega_T\), therefore from lemma 12, we can deduce \(\lim_{h \to 0} \sup_{t \in [0,T]} \| I_{3,h}(t) \|_X = 0.\)

We consider \(R = \max_{t \in [0,T]} \| u(t) \|_X + \varepsilon\) and \(L\) the Lipschitz constant of \(F\) for \(B_R(0) \subset X\), if we define
\[ J_{\varepsilon} = \{0 \leq t < \min \{T,T^*_h\} : \| u_h(t') \|_X < R + \varepsilon, 0 \leq t' \leq t\}, \]
from the estimate (4.15) we obtain for \(t \in J_{\varepsilon}:
\[
\| u(t) - u_h(t) \|_X \leq \| I_{1,h}(t) \|_X + \| I_{2,h}(t) \|_X + \| I_{3,h}(t) \|_X.
\]
\[ + 2Me^{2\omega T} L \int_0^t \|u(t) - u_h(t')\|_X \, dt', \]

and from Gronwall’s lemma

\[ \|u(t) - u_h(t)\|_X \leq e^{CT} \left( \sup_{t \in [0,T]} \|I_1, h(t)\|_X + \sup_{t \in [0,T]} \|I_2, h(t)\|_X + \sup_{t \in [0,T]} \|I_3, h(t)\|_X \right) \]

where \( C = 2Me^{2\omega T} L \). Taking \( h^* > 0 \) sufficiently small, we have that \( \|u(t) - u_h(t)\|_X < \varepsilon/2 \) for \( t \in J_\varepsilon \ \forall \ 0 < h < h^* \). Therefore \( \sup J_\varepsilon = \min \{T, T^*_h\} \), but as \( \|u_h(t)\| \leq R + \varepsilon < \infty \), it verifies \( T < T^*_h \), that proves the theorem. \( \square \)

## 5 Well posedness of the Cauchy problem

In this section, we analyze the well posedness for the problem (2.4) for different interesting cases. The local case can be analyzed using standard methods, so we refer the reader to the bibliography. We address the global problem, \( t \in [0, \infty) \), by the notion of positively invariant convex families. For classical diffusion (\( \beta = 1 \)), similar ideas can be found in chapter 14 of [32]. But this method presents two problems, the operator must be a differential elliptic operator and \( u(x) \) belongs to a space of finite dimension. Both difficulties are overcome considering the Lie-Trotter approximations and then passing to the limit. We take advantage of this, to study the evolution of a population model, where individuals have a characteristic trait that differentiates them. In [2] the existence of stationary solutions is studied, for a scalar characteristic trait. In order not to limit a priori the possibilities of modeling this problem, we consider the abstract case, where the characteristic trait is an element in a measure space.

**Theorem 14.** There exists a function \( T^*: C_u(\mathbb{R}^d, Z) \rightarrow \mathbb{R}_+ \) such that for \( u_0 \in C_u(\mathbb{R}^d, Z) \), exists a unique \( u \in C([0, T^*(u_0)], C_u(\mathbb{R}^d, Z)) \) mild solution of (1.1) with \( u(0) = u_0 \). Moreover, one of the following alternatives holds:

- \( T^*(u_0) = \infty; \)
- \( T^*(u_0) < \infty \) and \( \lim_{t \uparrow T^*(u_0)} \|u(t)\|_{C_u(\mathbb{R}^d, Z)} = \infty. \)

*Proof.* See Theorem 4.3.4 in [11]. \( \square \)

**Proposition 15.** Under conditions of theorem above, then

1. \( T^*: C_u(\mathbb{R}^d, Z) \rightarrow \mathbb{R}_+ \) is lower semi-continuous;
2. If \( u_{0,n} \rightarrow u_0 \) in \( C_u(\mathbb{R}^d, Z) \) and \( 0 < T < T^*(u_0) \), then \( u_n \rightarrow u \) in the Banach space \( C([0,T], C_u(\mathbb{R}^d, Z)). \)

*Proof.* See Proposition 4.3.7 in [11]. \( \square \)

Let \( \{K(t)\}_{t \in \mathbb{R}_+} \) be a family of convex closed sets of \( Z \), we define that \( \{K(t)\}_{t \in \mathbb{R}_+} \) is positively \( F \)-invariant if \( z \in K(t') \) then \( z(t) \in K(t' + t) \) for any \( t, t' \in \mathbb{R}_+ \), where \( z(t) \) is the solution of (2.3). In particular, if \( K(t) \) is constant, then we use the notation \( K = K(t) \) and it is positively \( F \)-invariant, i.e. the solution of (2.3) verifies \( z(t) \in K \), for \( t \in (0, T^*(z_0)) \).
Remark 16. If $|F(t,z)|_Z \leq a(t) + b(t)|z|_Z$, we can see that the family of closed balls given by

$$B(t) = \{z \in Z : |z|_Z \leq \phi(t)\}$$

and

$$\phi(t) = e^{f_0^t b(\xi) d\xi} \times (e^{-f_0^t b(\xi) d\xi})^t \int_0^t a(\zeta)e^{-f_0^t b(\xi) d\xi} d\zeta + r$$

is positively $F$–invariant with $z(0) = r$.

Lemma 17. Let $\sigma \geq 0$, $0 < \beta \leq 1$ and let $K$ be a closed convex set of $Z$, for any $t > 0$ and $u \in C_u(\mathbb{R}^d,K)$, it holds $S(t)u \in C_u(\mathbb{R}^d,K)$.

Proof. Suppose, contrary to the claim, that the assertion of the lemma is false. Then, there exists $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$ such that $v = (S(t)u(x)) \notin K$. Using Hahn-Banach separation theorem, we take a separating hyperplane; i.e., $\omega \in Z^*$ and $\lambda \in \mathbb{R}$ verifying $\langle \omega, z \rangle \leq \lambda$ for any $z \in K$ and $\langle \omega, v \rangle > \lambda$, but

$$\langle \omega, v \rangle = \int_{\mathbb{R}^d} G_{\sigma,\beta}(x-y,t) \langle \omega, u(y) \rangle dy \leq \lambda \int_{\mathbb{R}^d} G_{\sigma,\beta}(x-y,t) dy = \lambda,$$

a contradiction. $\square$

Proposition 18. Let $\{K(t)\}_{t \in \mathbb{R}^+}$ be a family of closed convex sets of $Z$, then the following conditions are equivalent:

- $\{K(t)\}_{t \in \mathbb{R}^+}$ is positively $F$–invariant;
- If $u_0 \in C_u(\mathbb{R}^d,K(0))$, then $u(t) \in C_u(\mathbb{R}^d,K(t))$ for $t \in (0,T^*(u_0))$, where $u$ is the solution of (2.2).

Proof. For $t \in (0,T^*(u_0))$ and $n \in \mathbb{N}$, we define $h = t/n$ and $\{U_{n,j}\}_{0 \leq j \leq n}$, the sequence given by

$$U_{n,0} = u_0,$$

$$U_{n,j} = S(h)N(jh, jh - h/2, U_{n,j-1}), \quad j = 1, \ldots, n. \quad (5.16a)$$

From Proposition 10 and Theorem 11, it may be concluded that $U_{n,j}$ is defined and $\|u(t) - U_{n,n}\|_C_u(\mathbb{R}^d, Z) \to 0$ when $n \to \infty$. We claim that $U_{n,j} \in C_u(\mathbb{R}^d,K(jh))$, the proof is by induction on $j$. Suppose that $U_{n,j-1} \in C_u(\mathbb{R}^d,K((j-1)h))$, using that $\{K(t)\}_{t \in \mathbb{R}^+}$ is positively $F$–invariant, we have $N(jh,jh - h/2, U_{n,j-1}) \in C_u(\mathbb{R}^d,K(jh))$. From Lemma 17, $S(h)N(jh,jh - h/2, U_{n,j-1}) \in C_u(\mathbb{R}^d,K(jh))$ and our claim follows. Since $K(t)$ is closed, $C_u(\mathbb{R}^d,K(t))$ is a closed set of $C_u(\mathbb{R}^d,Z)$, using that $u(t) = \lim_{n \to \infty} U_{n,n}$, we get $u(t) \in C_u(\mathbb{R}^d,K(t))$. On the other hand using proposition 6, if $z_0 \in K(0)$, defining $u_0(x) = z_0$, we have $u_0 \in C_u(\mathbb{R}^d,K(0))$ and then $z(t) = u(t) \in C_u(\mathbb{R}^d,K(t))$ for $t \in (0,T^*(u_0))$. $\square$

Theorem 19. Let $\{K(t)\}_{t \in \mathbb{R}^+}$ be a family of bounded convex closed sets of $Z$. Suppose that $\{K(t)\}_{t \in \mathbb{R}^+}$ is positively $F$–invariant, then for any $u_0 \in C_u(\mathbb{R}^d,K(0))$, it holds $T^*(u_0) = \infty$ and $u(t) \in C_u(\mathbb{R}^d,K(t))$ for $t > 0$. 

Proof. From Proposition 18, we have \( u(t) \in C_u(\mathbb{R}^d, K(t)) \) for \( t \in (0, T^*(u_0)) \). Therefore, \( \|u(t)\|_{\infty, z} \leq \max \{ |z| : z \in K(t) \} < \infty \), provided \( K(t) \) is a bounded set. Therefore, Theorem 14 shows \( T^*(u_0) = \infty \). \( \Box \)

Example 20 (Ginzburg-Landau equation). The Ginzburg-Landau equation is given by (1.1), where \( \beta = 1, \sigma > 0 \) (Re(\( \sigma > 0 \)) and \( F(u) = (1 + ia)u - (1 + ib)|u|^2u \) with \( a, b \in \mathbb{R} \) (see [10], [12] and [34]). In general, we can consider \( F(u) = f_R(|u|^2)u + tf_1(|u|^2)u \), where \( f_R, f_1 : \mathbb{R}^+ \to \mathbb{R} \) are smooth functions. If \( f_R(\eta) \leq 0 \) for \( \eta > 0 \), then \( K = B(0, \eta) \) is a bounded convex positively \( F \)-invariant set of \( \mathbb{C} \). For \( \sigma \geq 0 \) and \( 0 < \beta \leq 1 \), from Theorem 19, we have fractional Ginzburg-Landau equation is globally well posed for \( u_0 \in C_u(\mathbb{R}^d, K) \).

Example 21 (Fisher–Kolmogorov equation). Fisher [17] and Kolmogorov et al [19] introduced a classical model to describe the propagation of an advantageous gene in a one-dimensional habitat. We consider the generalized non-linear reaction-diffusion equation

\[
\partial_t u + \sigma(-\Delta)\beta u = \chi u(1-u),
\]

where \( u \) is the chemical concentration, \( \sigma \) is the diffusion coefficient and the positive constant \( \chi \) represents the growth rate of the chemical reaction. Since then a great deal of work has been carried out to extend their model to take into account the other biological, chemical and physical factors. This equation is also used in flame propagation ( [18]), nuclear reactor theory ( [8]), autocatalytic chemical reactions ( [13] and [16]), logistic growth models ( [26]) and neurophysiology ( [33]). Consider \( b_0 > 1 \) and \( K(t) = [0, b(t)] \), with

\[
b(t) = \frac{b_0 e^{\chi t}}{1 + b_0 (e^{\chi t} - 1)},
\]

we can see that \( \{K(t)\}_{t \in \mathbb{R}^+} \) is a family of compact intervals, positively \( F \)-invariant for \( F(z) = \chi z(1-z) \). In particular, for any \( u_0 \in C_u(\mathbb{R}) \) with \( u_0(x) \geq 0 \), taking \( b_0 = \sup_{x \in \mathbb{R}^d} u(x) \), we can see that \( T^*(u_0) = \infty \) and \( \limsup_{t \to \infty} |u(x, t)| \leq 1 \) for any \( x \in \mathbb{R}^d \). In the case \( 0 < a_0 = \inf_{x \in \mathbb{R}^d} u(x) < 1 \), we have that \( K(t) = [a(t), b(t)] \) with

\[
a(t) = \frac{a_0 e^{\chi t}}{1 + a_0 (e^{\chi t} - 1)},
\]

is \( F \)-positive. Therefore, \( \lim_{t \to \infty} \|u(t) - 1\|_{\infty} = 0 \).

5.1 Population dynamics with a continuous trait

In [2], Arnold et al. consider a model of population dynamics in which the population is structured with respect to the space variable \( x \) and a trait variable denoted by \( \theta \). The distribution function \( u(x, \theta, t) \geq 0 \) denote the number density of individuals at time \( t \in \mathbb{R}_+ \), position \( x \in \mathbb{R}^d \), and whose trait is \( \theta \in X \). The evolution of \( u \) is governed by an integro-PDE model of reaction-diffusion type in infinite (continuous) dimension in which selection, mutations, competition, and migrations are taken into account. The modeling assumptions are the following: migration is described by a (normal or anomalous) diffusion operator \( -\sigma(-\Delta)^2 \); mutations are described by a linear kernel \( M(\theta, \vartheta) \) which is related to the probability that individuals with trait \( \vartheta \) have offsprings with trait \( \theta \); selection is implemented in the model, thanks to a fitness function \( k \) which may depend
on trait $\theta$; finally a logistic term involving a kernel $C(\theta, \vartheta)$ models the competition (felt by individuals of trait $\theta$) due to individuals of trait $\vartheta$. Under those assumptions, the evolution of the population is governed by the following integro-PDE:

$$\partial_t u(x, \theta, t) + \sigma(-\Delta_x)u(x, \theta, t) = F(u(x, \theta, t))$$  \hspace{1cm} (5.17)

with initial condition $u(x, \theta, 0) = u_0(x, \theta)$. The map $F$ is given by

$$F(z)(\theta) = k(\theta)z(\theta) + \int_X M(\theta, \vartheta)z(\vartheta)d\mu(\vartheta)$$

$$- \left( \int_X C(\theta, \vartheta)z(\vartheta)d\mu(\vartheta) \right)z(\theta),$$

Let $X$ be a compact Hausdorff space, $\mathcal{B}$ the $\sigma$-algebra of Borel sets and $\mu$ a regular Borel probability, we set the problem on $Z = L^1(X) = L^1(X, \mathcal{B}, \mu)$. Following [2], we assume $k \in C(X)$, $M, C \in C(X \times X)$ verifying $M \geq 0$ and $C > 0$, we define

$$k_+ = \max_{\theta \in X} \left( k(\theta) + \int_X M(\theta, \vartheta)d\mu(\vartheta) \right),$$

$$c_- = \min_{\theta, \vartheta \in X} C(\theta, \vartheta), \quad c_+ = \max_{\theta, \vartheta \in X} C(\theta, \vartheta).$$

The lower bound for $C$ means that all individuals are in competition. To obtain well-posedness of (5.17), we prove some previous results.

**Lemma 22.** The map $F:L^1(X) \to L^1(X)$ is locally Lipschitz.

**Proof.** Let $z, \tilde{z} \in L^1(X)$, we have

$$|F(z) - F(\tilde{z})|_{L^1} \leq \int_X |k(\theta)||z(\theta) - \tilde{z}(\theta)|d\mu(\theta)$$

$$+ \int_{X \times X} M(\theta, \vartheta)|z(\vartheta) - \tilde{z}(\vartheta)|d\mu(\vartheta)d\mu(\theta)$$

$$+ \int_{X \times X} C(\theta, \vartheta)|z(\theta)|d\mu(\theta)d\mu(\vartheta)$$

$$+ \int_{X \times X} C(\theta, \vartheta)|\tilde{z}(\vartheta)||z(\theta) - \tilde{z}(\vartheta)|d\mu(\vartheta)d\mu(\theta).$$

Using $k, M, C$ are bounded, we get

$$|F(z) - F(\tilde{z})|_{L^1} \leq (|k|_{L^\infty} + |M|_{L^\infty} + |C|_{L^\infty}(|z|_{L^1} + |\tilde{z}|_{L^1}))|z - \tilde{z}|_{L^1},$$

which complete the proof.

We have the same result for continuous functions:

**Lemma 23.** The map $F:C(X) \to C(X)$ is locally Lipschitz.

**Proof.** The proof is similar to the above lemma.

The nonnegativity of density $z(\theta,t)$ is established by the next proposition (and corollary below).
**Proposition 24.** Let $z$ be the solution of (2.3) with $z(0) = z_0 \in C(X)$. If $z_0 > 0$ then $z(t) > 0$ for any $t \in [0, T^*(u_0))$.

**Proof.** For any $(\theta, t) \in X \times [0, T^*(u_0))$, we define
\[
g(\theta, t) = \int_X M(\theta, \vartheta)z(\vartheta, t)d\mu(\vartheta),
\]
\[
a(\theta, t) = \int_X C(\theta, \vartheta)z(\vartheta, t)d\mu(\vartheta)
\]
then $g(\theta, .), a(\theta, .)$ are continuous, the solution verifies $z(\theta, .) \in C^1([0, T^*(u_0))]$ and
\[
\partial_t z(\theta, t) = (k(\theta) - a(\theta, t))z(\theta, t) + g(\theta, t)
\]
with $z(\theta, 0) = z_0(\theta)$. Then
\[
z(\theta, t) = e^{A(\theta, t, 0)}z_0(\theta) + \int_0^t e^{A(\theta, t, t')}g(\theta, t')dt',
\]
(5.19)
where
\[
A(\theta, t, t') = k(\theta)(t - t') - \int_{t'}^t a(\theta, t'')dt''.
\]
Let $t_0 = \sup\{t \in [0, T^*(u_0)) : \min_{X \times [0, t]} z > 0\}$. Suppose $t_0 < T^*(u_0)$, there exists $\theta_0 \in X$ such that $\min_{\theta \in X} z(\theta, t_0) = z(\theta_0, t_0) = 0$. But
\[
u(\theta_0, t_0) = e^{A(\theta_0, t_0, 0)}u_0(\theta_0) + \int_0^{t_0} e^{A(\theta_0, t_0, t')}g(\theta_0, t')dt' > 0,
\]
a contradiction. \qed

**Corollary 25.** Let $z$ be the solution of (2.3) with $z(0) = z_0 \in C(X)$. If $z_0 \geq 0$ then $z(t) \geq 0$ for any $t \in [0, T^*(u_0))$.

**Proof.** Consider $z_{0, n} = z_0 + 1/n$, for any $0 < T < T^*(z_0)$, there exists $n_0 \in \mathbb{N}$ such that $T < T^*(z_{0, n})$ if $n \geq n_0$. Since $z_{0, n} > 0$, using proposition 24 we have $z_{n}(t) > 0$ for $t \in [0, T]$. As $z_n$ converges to $z$ in $C(X \times [0, T])$, we see that $z \geq 0$. Since $T$ is arbitrary, we obtain the result. \qed

We now show global well-posedness in $C(X)$. From corollary above $a(\theta, t), g(\theta, t) \geq 0$, then $A(\theta, t, t') \leq |k|_{L^\infty}(t - t')$. Integrating (5.19) on $X$, we obtain
\[
\int_X z(\theta, t)d\mu(\theta) \leq e^{\int_X z_0(\theta)d\mu(\theta)} + \int_0^t \int_X e^{\int_X z(\theta, t')M(\theta, \vartheta)z(\vartheta, t')d\mu(\vartheta)}d\mu(\theta)dt' \\
\leq e^{\int_X z_0(\theta)d\mu(\theta)} + e^{\int_X |M|_{L^\infty} \int_0^t e^{-|k|_{L^\infty} t'} z(\theta, t')d\mu(\theta)}dt'.
\]
using Gronwall’s lemma, we obtain
\[
\int_X z(\theta,t)d\mu(\theta) \leq e^{(|k|L_\infty+|M|L_\infty)t} \int_X z_0(\theta)d\mu(\theta) \leq e^{(|w|L_\infty+|M|L_\infty)t} |z_0|_{L_\infty}
\]
which implies
\[
0 \leq g(\theta,t) \leq |M|L_\infty e^{(|k|L_\infty+|M|L_\infty)t} |z_0|_{L_\infty}
\]
From (5.19), we get
\[
|z(t)|_{L_\infty} \leq e^{(|k|L_\infty+|M|L_\infty)t} (1+|M|L_\infty)|z_0|_{L_\infty}.
\]
In particular, \(T^*(z_0) = \infty\) for any \(z_0 \in C(X)\) with \(z_0 \geq 0\). In order to apply theorem 19, we construct a positive \(F\)-invariant convex set of \(L^1(X)\).

**Lemma 26.** Let \(w \in C^1([0,T]), \, w \geq 0\), such that \(\dot{w} \leq kw - cw^2\), with \(k,c > 0\). If \(\lambda \geq k/c\) and \(0 \leq w(0) \leq \lambda\), then \(0 \leq w(t) \leq \lambda\).

**Proof.** Suppose \(w(t_1) > \lambda\) with \(0 < t_1 \leq T\), consider \(t_0 = \sup\{t \in [0,t_1] : w(t) \leq \lambda\}\). Using the mean value theorem, there exists \(t_* \in (t_0,t_1)\) such that
\[
w(t_1) - w(t_0) = \dot{w}(t_*)(t_1 - t_0),
\]
then \(\dot{w}(t_*) > 0\). But \(w(t_*) > \lambda\), which implies \(kw(t_*) - cw^2(t_*) < 0\), a contradiction. \(\Box\)

**Proposition 27.** Let \(z_0 \in C(X), \, z_0 \geq 0\). If \(\lambda \geq \max\{k_+/c_-, |u_0|_{L^1}\}\), then the solution of (2.3) \(z \in C(X \times [0,\infty))\) verifies \(z(t) \geq 0\) and \(|z(t)|_{L^1} \leq \lambda\) for any \(t \geq 0\).

**Proof.** From corollary 25, we can see that \(z(t) \geq 0\). Integrating \(F\) on \(X\), we get
\[
\frac{d}{dt} \int_X z(\theta,t)d\mu(\theta) = \int_X k(\theta)z(\theta,t)d\mu(\theta) + \int_{X \times X} M(\theta,\vartheta)z(\theta,t)d\mu(\vartheta)d\mu(\theta)
\]
\[
- \int_{X \times X} C(\theta,\vartheta)z(\theta,t)d\mu(\vartheta)d\mu(\theta)
\]
\[
= \int_X \left( k(\theta) + \int_X M(\theta,\vartheta)d\mu(\vartheta) \right) z(\theta, t)d\mu(\theta)
\]
\[
- \int_{X \times X} C(\theta,\vartheta)z(\theta,t)d\mu(\vartheta)d\mu(\theta).
\]
Inequalities (5.18) implies
\[
\frac{d}{dt} \int_X z(\theta,t)d\mu(\theta) \leq k_+ \int_X z(\theta,t)d\mu(\theta) - c_- \left( \int_X z(\theta,t)d\mu(\theta) \right)^2.
\]
From lemma 26, we have \(|z|_{L^1} \leq \lambda\). \(\Box\)

**Proposition 28.** For any \(\lambda \geq 0\), \(K_\lambda = \{z \in L^1(X) : z \geq 0 \text{ a.e., } |z|_{L^1} \leq \lambda\}\) is a bounded convex closed set. If \(\lambda \geq k_+/c_-\) then \(K_\lambda\) is positively \(F\)-invariant.

**Proof.** Let \(z_0 \in K_\lambda\), taking \(\{z_{0,n}\}_{n \in \mathbb{N}} \subset C(X) \cap K_\lambda\) such that \(|z_0 - z_{0,n}|_{L^1} \to 0\), from proposition 27 we can see that \(T^*(z_{0,n}) = \infty\) and \(z_{n}(t) \in K_\lambda\), for \(t \geq 0\). Using continuous dependence on initial data, we can see that \(|z(t) - z_n(t)|_{L^1} \to 0\) for any \(t \in [0,T^*(z_0))\), since \(K_\lambda\) is closed, we obtain \(z(t) \in K_\lambda\). \(\Box\)
Theorem 29. Let \( u_0 \in C_u(\mathbb{R}^d, L^1(X)) \), with \( u_0(x) \geq 0 \) a.e. in \( X \), the mild solution of equation (5.17) is globally well-posed. Moreover, \( u(x,t) \geq 0 \) a.e. in \( X \) and \( |u(x,t)|_{L^1(X)} \leq \lambda \) for \( \lambda = \max\{|u_0|_{\infty, L^1(X)}, k_+ / c_-\} \).

Proof. From Proposition 28, \( K_\lambda \) is positively \( F \)-invariant. Using the theorem 19, we see that \( T^*(u_0) = \infty \) and \( u(t) \in C_u(\mathbb{R}^d, K_\lambda) \) for \( t > 0 \). □

5.2 Global existence for products of Banach Spaces

We generalize the previous results by proving global existence for products of Banach Spaces. Proposition 30 proves that the semigroup operator maintains the solution inside the invariant region. Following that, Theorem 31 proves that if \( u_0 \) is inside the invariant region, then \( u(t) \) remains in it for all \( t > 0 \). Let \( \{Z_j\}_{1 \leq j \leq m} \) be Banach spaces and \( Z = Z_1 \times \cdots \times Z_m \) with the usual norm, we denote \( \pi_j : Z \rightarrow Z_j \) the projection map. If \( \sigma_j > 0 \), \( 0 < \beta_j \leq 1 \), and \( S_j(t)u = G_{\sigma_j,\beta_j}(.t) u \) for \( u \in C_u(\mathbb{R}^d, Z_j) \), then \( S : \mathbb{R} \rightarrow B(C_u(\mathbb{R}^d, Z)) \) given by

\[
S(t)u = (S_1(t)\pi_1 u, \ldots, S_m(t)\pi_m u)
\]

is a continuous contraction semigroup.

Proposition 30. Let \( K_j \subset Z_j \) be a closed convex set and \( K = K_1 \times \cdots \times K_m \subset Z \). If \( u \in C_u(\mathbb{R}^d, K) \), then \( S(t)u \in C_u(\mathbb{R}^d, K) \), for any \( t > 0 \).

In the case that \( \sigma_j \) or \( \beta_j \) are different, we can prove the following result:

Theorem 31. Let \( K_j(t) \subset Z_j \) be bounded closed convex sets, if \( K(t) = K_1(t) \times \cdots \times K_m(t) \) is a positively \( F \)-invariant set, then for any \( u_0 \in C_u(\mathbb{R}^d, K(0)) \), it holds \( T^*(u_0) = \infty \) and \( u(t) \in C_u(\mathbb{R}^d, K(t)) \) for \( t > 0 \).

Proof. Let \( u_0 \in C_u(\mathbb{R}^d, K(0)) \) and \( T^*(u_0) \) maximal time of existence of the solution \( u \) of (2.2). Let \( t \in (0, T^*(u_0)) \), \( n \in \mathbb{N} \) and \( \{U_{n,j}\}_{0 \leq j \leq n} \), as defined in Proposition 18. Suppose that \( U_{n,j-1} \in C_u(\mathbb{R}^d, K((j-1)h)) \), using that \( \{K(t)\}_{t \in \mathbb{R}^+} \) is positively \( F \)-invariant, we have \( N(jh, jh - h/2, U_{n,j-1}) \in C_u(\mathbb{R}^d, K(jh)) \). From Proposition 30 it follows that, \( S(h)N(jh, jh - h/2, U_{n,j-1}) \in C_u(\mathbb{R}^d, K(jh)) \). Using the same reasoning as in Proposition 18, we have that \( U_{n,j} \rightarrow u(jh) \) in \( C_u(\mathbb{R}^d, Z) \) when \( n \rightarrow \infty \) and \( u(t) \in C_u(\mathbb{R}^d, K(t)) \). Since \( K(t) \) is bounded, we obtain the result. □

Example 32. In [3] a FHN Model for pattern formation is presented:

\[
\begin{align*}
\partial_t u &= D_u \Delta u + (a - u)(u - 1)u - v \\
\partial_t v &= D_v \Delta v + e(bu - v)
\end{align*}
\]

(5.20)

with \( 0 < a < 1 \), \( e > 0 \) and \( b \geq 0 \). A similar example is analyzed in [32]. To apply theorem 31, we need to find positive \( F \)-invariant rectangle \( K = K_1 \times K_2 \), \( K_j = [-R_j, R_j] \), where \( F \) is given by

\[
F(u,v) = (au^2 - u^3 - au + u^2, e(bu - v)).
\]

Let \( R_1 > \max\{4, \sqrt{2b}\} \) and \( 2bR_1 < 2R_2 < R_1^2 \), we can see that the rectangle with \( R_1 \) and \( R_2 \) is \( F \)-invariant:

\[
\begin{align*}
F_1(R_1, v) &\leq a(R_1^2 - R_1) - R_1^2 + R_1^2 + |v| \leq a(R_1^2 - R_1) - R_1^3 + R_1^2 + R_2 < 0, \\
F_1(-R_1, v) &\geq a(R_1^2 + R_1) + R_1^2 + R_1^2 - |v| \geq a(R_1^2 + R_1) + R_1^3 + R_1^2 - R_2 > 0,
\end{align*}
\]
Lemma 34. \[
F_2(u, R_2) \leq e(b|u| - R_2) \leq e(bR_1 - R_2) < 0,
F_2(u, -R_2) \geq e(-b|u| + R_2) \geq e(-bR_1 + R_2) > 0.
\]

Then the field evaluated at the border of $K$ points inward. By theorem 31 the equation (5.20) is globally well posed.

6 Asymptotic behavior

We analyze the situation in which, if $u_0$ has a horizontal asymptote at $z_0$ then using the same methods as before, we prove that $u(t)$ approaches asymptotically to the time evolution of $z_0$. We consider the 1-dimensional real case. We first show in lemma 34 that if $u_0$ has a horizontal asymptote at $z_0$ then $S(t)u_0$ remains with the same horizontal asymptote. Next, we prove in lemma 35 that $N(t, t_0, u_0)(x)$ has a time dependent horizontal asymptote, which is the solution of the equation (2.3) with $z_0$ as an initial condition. Finally, we combine both results and a continuous dependence argument in lemma 36 to achieve proposition 33, the solution $u(t)$ of (1.1) has the same time dependent horizontal asymptote $z(t)$.

These results can be applied, for example, to the Fisher-Kolmogorov equation. Specifically in [19] solutions with the mentioned asymptotic behavior are analyzed.

Proposition 33. Let $u_0 \in C_u(\mathbb{R}, Z)$ such that $\lim_{x \to \pm \infty} u_0(x) = z_0^\pm \in Z$, if $u(t)$ is the solution of (2.4) then $\lim_{x \to \pm \infty} u(x, t) = z_0^\pm (t)$.

Lemma 34. Let $u_0 \in C_u(\mathbb{R}, Z)$ such that $\lim_{x \to \pm \infty} u_0(x) = z_0^\pm \in Z$. If $u(t) = S(t)u_0$, then $\lim_{x \to \pm \infty} u(x, t) = z_0^\pm$.

Proof. We only prove for $z_0^+$, the $z_0^-$ case is similar. Let $\varepsilon > 0$, there exists $x_+ > 0$ such that $|u_0(x) - z_0^+| < \varepsilon$ for $x > x_+$. Before proving the limit, we need an estimate of $g_\beta(z)$. Taking $r > 0$ large enough, we have

$$\int_{|z| > (\sigma t)^{-1/(2\beta)}} g_\beta(z)dz < \varepsilon/(2|u_0|_{\infty, \mathbb{Z}}).$$

(6.21)

Next, to study the asymptotic convergence, we analyze two cases, if $x > x_+ + r$ then,

$$|u(t, x) - z_0^+| \leq \int_{\mathbb{R}} G_{\sigma, \beta}(t, x - y)|u_0(y) - z_0^+|dy \leq \int_{y > x - r} G_{\sigma, \beta}(t, x - y)|u_0(y) - z_0^+|dy + \int_{y < x - r} G_{\sigma, \beta}(t, x - y)|u_0(y) - z_0^+|dy = I_1 + I_2.
$$

Since $y > x - r > x_+^+$, we have that $|u_0(y) - z_0^+| < \varepsilon$ and therefore we can bound the first integral,

$$I_1 \leq \varepsilon \int_{\mathbb{R}} G_{\sigma, \beta}(t, x - y)dy = \varepsilon.
$$

For the second integral, we will use estimate (6.21), and the norm of the initial condition $u_0$,

$$I_2 \leq 2|u_0|_{\infty, \mathbb{Z}} \int_{y < x - r} G_{\sigma, \beta}(t, x - y)dy = 2|u_0|_{\infty, \mathbb{Z}} \int_{z > r} G_{\sigma, \beta}(t, z)dz$$
= 2|u_0|_{\infty,Z} \int_{|z'|>(\sigma t)^{-1/2\beta}} g_{\beta}(z')\,dz' < \varepsilon

Bounding both integrals we prove the result.

Lemma 35. Let $u_0 \in C_u(\mathbb{R},Z)$ such that $\lim_{x \to \pm \infty} u_0(x) = z_0^\pm \in Z$. If $u(t) = N(t, t_0, u_0)$, then $\lim_{x \to \pm \infty} u(x, t) = z^\pm(t)$, where $z^\pm(t)$ is the solution (2.3) with initial data $z^\pm(0) = z_0^\pm$.

Proof. We again consider only the $z^+$ case. We use the continuous dependence of the initial data. Let $\varepsilon > 0$, there exists $\delta > 0$ such that if $|z_0^+ - z_0^+| \in \delta$, then $|z^+(t) - z(t)| \in \varepsilon$. Let $x_0^+ \in \mathbb{R}$ such that if $x > x_0^+$, $|u_0(x) - z_0^+| \in \delta$, then $|u(x, t) - z^+(t)| \in \varepsilon$.

Lemma 36. Let $\{u_n\}_{n \in \mathbb{N}} \subset C_u(\mathbb{R}^d, Z)$ such that $u_n \to u$ in $C_u(\mathbb{R}^d, Z)$. If for $n \in \mathbb{N}$, it holds $\lim_{x \to \pm} u_n(x) = z^\pm$, then $\lim_{x \to \pm} u(x) = z^\pm$.

Proof. Let $\varepsilon > 0$, we can take $n \in \mathbb{N}$ such that $||u - u_n||_{\infty,Z} < \varepsilon/2$. Then there exists $x_0^+ \in \mathbb{R}$ such that $|u_n(x) - z^+| \in \varepsilon/2$ if $x > x_0^+$. Therefore,

$$|u(x) - z^+| \leq |u(x) - u_n(x)|_Z + |u_n(x) - z^+|_Z < \varepsilon.$$  

Proof. (Proof of Proposition 33) Let $n \in \mathbb{N}$, $h = t/n$ and $\{U_{n,j}\}_{0 \leq j \leq n}$ the sequence defined by (5.16). We claim that $z^\pm(jh) = \lim_{x \to \pm \infty} U_{n,j}(x)$ for $j = 0, \ldots, n$. Clearly, the assertion is true for $j = 0$. If $z^\pm((j-1)h) = \lim_{x \to \pm \infty} U_{n,j-1}(x)$, from Lemma 35, we see that $z^\pm(jh) = \lim_{x \to \pm \infty} (N(jh, jh - h/2, U_{n,j-1}))(x)$, and using Lemma 34 we obtain $z^\pm(jh) = \lim_{x \to \pm \infty} U_{n,j}(x)$. We conclude $z^\pm(t) = z^\pm(nh) = \lim_{x \to \pm \infty} U_{n,n}(x)$ and, since $U_{n,n} \to u(t)$, lemma 36 implies the result.

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