Theory of coherent active convolved illumination for superresolution enhancement

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Recently an optical amplification process called the plasmon injection (Π) scheme was introduced as an effective solution to overcoming losses in metamaterials. Implementations with near-field imaging applications have indicated substantial performance enhancements even in the presence of noise. This powerful and versatile compensation technique, which has since been renamed to a more generalized active convolved illumination (ACI), offers new possibilities of improving the performance of many previously conceived metamaterial based devices and conventional imaging systems. In this work, we present the first comprehensive mathematical breakdown of ACI for coherent illumination. Our analysis highlights the distinctive features of ACI and provides its rigorous understanding during the loss compensation process. The theory paves the way for enhanced superresolution imaging and can be generalized to the compensation of information loss in other coherent and incoherent linear systems.

I. INTRODUCTION

Metamaterials which are artificial, inhomogeneous structures designed with subwavelength metallic or dielectric building blocks, rose to prominence nearly two decades ago as an appealing direction for designing materials with unprecedented electromagnetic properties previously considered difficult, if not impossible, to realize. Invisibility cloaks [1], ultra-high-resolution imaging [2–4] and photonics [5], enhanced photovoltaics [6, 7], miniaturized antennas [8], ultrafast optical modulation [9], and metasurfaces [10, 11] are few of the multitude of applications which have been envisioned. Supported by parallel efforts in micro and nanofabrication, metamaterials are anticipated to have broad impact on many technologies employing electromagnetic radiation. However, despite enormous theoretical and experimental progress, numerous lingering problems [12] require diligent consideration. Optical losses continue to be one of the greatest threats to the viability of many of the metamaterial-based devices proposed to date. Mitigation of losses remains a challenging problem for the metamaterial community. Gain medium was initially proposed [13–16] as a potential solution. However, later studies showed that stability and gain saturation issues as a result of stimulated emission near the field enhancement regions lead to intense noise generation [17–19]. Due to these concerns and other associated complexities such as pump requirement, progress towards the development of a robust loss compensation scheme has been somewhat sluggish even after nearly two decades of efforts. Dielectric metasurfaces have also been proposed to alleviate some of these concerns [10, 11].

A recent theoretical study [20] investigated a unique and alternative solution to manage the losses in metamaterials. The compensation process, designated plasmon injection (PI or Π) scheme, employs an additional source to modify the field incident on a lossy metamaterial. This auxiliary source is designed to adequately amplify an arbitrary field thereby enhancing its transmission through the metamaterial. Subsequent theoretical studies with previous metamaterial near-field imaging systems employing negative index materials (NIMs) [21], superlenses [22], and hyperlenses [23, 24] produced promising results. Implementing the Π scheme with the above systems resulted in performance improvements. The distinguishing feature of the Π scheme is the auxiliary source. The earlier variants of the Π scheme were shown to emulate linear deconvolution [21].

The physical generation of the auxiliary source requires some considerations. It was shown [25] that the auxiliary source can be generated through a convolution process with the original object field incident at the detector while selectively providing amplification to a controllable band of spatial frequencies. As a result of this process, the auxiliary source becomes correlated with the original object field [25, 26]. A near-field spatial filter designed with hyperbolic metamaterials (HMMs) was proposed to physically generate the auxiliary source [27] with the above properties. The filter was integrated with a 50 nm silver film to illustrate the overall loss compensation process. This was the first potential application of the spatial filtering properties [25, 26] of HMMs in the context of loss compensation. Later studies with coherent [33] and incoherent [34] illumination produced favourable results. An improvement in the resolution limit of a near-field silver superlens elevated the viability of the Π scheme as an effective alternative to previously conceived loss mitigation approaches [7, 13–16, 35, 36] including dielectric metasurfaces [10, 11]. Even though the techniques presented in [33, 34] possess similar properties to the original concept of the Π scheme in [20], the scheme was generalized to a more encompassing term active convolved illumination (ACI) in [34].

In this paper we construct a theoretical framework to provide the first comprehensive mathematical exposition of the fundamental concept of ACI for coherent illumination. Pendry’s classic setup [2] of a silver superlens...
operating at a wavelength of 365 nm is adopted since it is the simplest configuration which broadly exemplifies the rudimentary impact of material losses affecting most metamaterial systems. This model is employed to accentuate how the ACI permits recovery of attenuated spatial frequencies with minimal noise amplification. Specific attention is drawn towards the required mechanisms, such as selective amplification and physical convolution. This study strengthens, from a mathematical perspective, the previous results and associated assertions made with numerical simulations. The contents of this paper are organized into three sections. Section II introduces the model for a two dimensional lossy near-field imaging system followed by an overview of ACI and an implementation with coherent illumination systems. Section III presents a mathematical formulation which is used in section IV to show how compensation of losses is performed while inhibiting noise amplification.

II. NEAR-FIELD IMAGING SYSTEM WITH ACI

As Pendry pointed out [2], the properties of a NIM necessary for superresolution imaging far beyond the diffraction limit, can be attained for transverse magnetic (TM) polarized light at a wavelength $\lambda = 365$ nm by a thin silver film embedded inside a dielectric. Under such conditions, resonant excitation of surface plasmons at the silver interface provides satisfactory amplification to high spatial frequencies which can then be focused assuming that the thickness of the silver film, object and image plane distances are much smaller than the incident wavelength. The configuration of such an imaging system is shown in figure 1(a), where the silver film with thickness $d$ is embedded inside a dielectric and positioned symmetrically between the object and image planes indicated by solid and dashed black lines, respectively. A TM field distribution on the object plane is detected from the image plane after propagating through the silver film. During this propagation process, material losses progressively degrade the transmission of high spatial frequencies with increasing transversal wavenumber $k_y$. Therefore, the ultimate performance of the system is limited to the highest spatial frequency whose attenuated amplitude is sufficiently strong enough to be accurately detected from the image plane amid noise. An ideal loss compensation scheme should extend this limit by intelligently providing adequate amounts of power to those previously undetectable spatial frequencies to allow them to survive the lossy transmission process by ensuring minimal noise amplification.

In ACI, loss compensation can be performed by introducing an additional material between the object plane and the lens as shown in figure 1(b). This material should behave as a tunable active band-pass spatial filter [25, 33]. We write the transfer function of the spatial filter as [27]

$$a(k_y) = b + G(k_y),$$

where we set $b$ as a real constant corresponding to a uniform low background transmission. $G(k_y) = |G(k_y)|e^{i\varphi(k_y)}$ is a complex band-limited function with phase $\varphi(k_y)$ and describes the pass-band of the passive filter. If the amplitude of the wave illuminating the system is increased by a factor $A_0 = b^{-1}$, the resulting transmitted spectrum is

$$A(k_y) = 1 + A_0 G(k_y).$$

Eq. 2 is defined as the transfer function of the active spatial filter [27]. The term “active spatial filter” simply refers to the process of physically providing increased energy to the passive filter with the transfer function in Eq. 1. In other words, linear transmission through passive materials is considered. The word active also distinguishes ACI from purely deconvolution based methods [21, 24] where no additional energy is provided to the system.

The response of the active spatial filter should be shift invariant along the object plane and integrating the filter with the lens should allow the entire system to be described with an active transfer function [25, 33] written as

$$T_A(k_y) = T(k_y)[1 + A_0 G(k_y)],$$

where $T(k_y)$ is the passive transfer function of the silver lens.

The Eqs. 2 and 3 are central to the ACI loss compensation process. The physical picture is best illustrated with the aid of the schematic shown in figure 3. An arbitrary field incident on the system, shown in figure 3(a) can be described as a linear, weighted superposition of harmonics or spatial frequencies shown in figure 3(b).
The active spatial filter in figure 2(c) is inserted between the lossy metamaterial in figure 2(e) and the incident field. The transfer function of the active filter has the form of Eq. 2 and is depicted by the inset in figure 2(c). The transmission amplitude of a lossy metamaterial, which deteriorates for increasing spatial frequencies, is illustrated by the inset in figure 2(e). For example, assume that the spatial frequencies $k_7$ and $k_8$ will be compensated. ACI achieves this by tuning the center frequency, $k_c$, of the active spatial filter such that $|A(k_y)| > 1$ over the identified spatial frequencies. This is illustrated by the inset in figure 2(c). After the harmonics of the incident field propagate through the active spatial filter, the amplitudes of the identified spatial frequencies are amplified relative to the other harmonics. Therefore, the field exiting the active filter contains the original harmonics of the object superimposed with the selectively amplified harmonics $k_7$ and $k_8$ (see figure 2(d)). The amplification provided to these harmonics (controlled by $A_0$) is adjusted to ensure that they survive the lossy transmission process through the metamaterial. The selectively amplified spatial frequencies at the exit of the filter constitute the auxiliary source as discussed in [27] and is conceptually similar to [20]. With the only difference being the generation process. It is important to emphasize that since the active spatial filter simply modifies the original field incident on the metamaterial by a convolution operation, it was deemed prudent to refer to the overall compensation scheme more broadly as ACI rather than active II scheme [20], which otherwise narrows down the process to only plasmons.

Using numerical simulations, ACI was implemented with an experimentally realized silver superlens [34] at the wavelength $\lambda = 365 nm$. A physical system approximating the properties of the active spatial filter was designed with aluminium-dielectric multilayered structures which exhibit hyperbolic dispersion [27]. The above theoretical formulation was tested by integrating the multilayered structure with the lens as shown in figure 1(b). Imaging results with coherent [33] and incoherent [34] illumination showed an improvement in the resolution limit of the lens even under the presence of noise. The HMMs used in [33] [34] were designed such that their transmission properties closely approximate Eq. 2 under high intensity illumination.

III. FORMULATION OF NOISE IN THE FOURIER DOMAIN

In this section, the imaging system configuration in figure 1 is used to develop a mathematical framework which will provide an insight into key features of ACI. The effect of noise in the system is incorporated to elaborate the efficacy of ACI from the perspective of loss compensation with minimal noise amplification. Let the image planes (shown in figure 1) have a length $L$ along the $y$-axis. A continuous signal $i(y)$, along the image planes is measured by a detector which can be an array of pixels or a scanning near-field probe (see figure 3(a)). An arbitrary spatial field distribution is decomposed into $M$ discrete samples at intervals of $\Delta y$ where $M$ is an even integer. The above spatial decomposition is represented by the segmented line in figure 3(a) where each segment is defined as a pixel. A dimensionless integer $p$ satisfying $-\frac{M}{2} \leq p \leq \frac{M}{2} - 1$ uniquely identifies each pixel centered at $y(p) = p\Delta y \equiv \xi$. The signal sampled by the $p^{th}$ pixel is denoted by $i(\xi)$. In subsequent calculations we will set $L = 80 \lambda$ with $\lambda = 365 nm$ and $M = 5840$. Therefore, the sampling interval is $\Delta y = 5nm$ which is slightly larger than previously demonstrated apertureless probes which can achieve resolutions down to $1nm$ [38]. The resulting noisy image at each pixel is described with the popular signal-modulated noise model [39–43]. The image plane is thought of as an array of statistically independent random variables (RV) and the subsequent noisy image at the $p^{th}$ pixel is denoted by

\[ i_n(\xi) = \left[ i(\xi) + n_{sd}(\xi) + n_{si}(\xi) \right] e^{i\theta(\xi)}. \]  \hspace{1cm} (4)

$i(\xi)$ is the noiseless image and is corrupted by two statistically independent signal-dependent (SD) and signal-independent (SI) noise processes $n_{sd}(\xi)$ and $n_{si}(\xi)$, respectively. The RVs $n_{sd}(\xi)$ and $n_{si}(\xi)$ have zero mean, Gaussian probability density functions (PDFs) and standard deviations $\sigma_1$ and $\sigma_2$, respectively. $\theta(\xi)$ is the phase of the noiseless coherent field at the $p^{th}$ pixel.
(i.e., $\xi = p\Delta y$). $f\{|i(\xi)|\}^\gamma$ is a function of the ideal image amplitude and is referred to as the modulation function \[10\] acting on the RV $n_{sd}(\xi)$. $\gamma$ is a parameter satisfying $\gamma \leq 1$ \[29\]. The overall variance of noise at each pixel is the sum of the variances of the SD and SI noise terms, 

$$\sigma_{\zeta_p}^2 = f\{|i(\xi)|\}^{2\gamma} \sigma_z^2 + \sigma_{\zeta}^2. \quad (5)$$

The modulation function and the value of $\gamma$ are selected to best mimic the behavior of SD noise which affects the system. In the upcoming calculations we drop the contribution of SI noise \[40\] \[43\].

**FIG. 3.** Illustration of an image measurement process in a detector system. (a) The continuous field $i(y)$ along the image plane of length $L$ is decomposed into $M$ samples at intervals of $\Delta y$. Each sample is identified by an integer $p$. During the detection process, noise degrades the ideal image and the standard deviation of the overall noise at each sample is $\sigma_{\zeta_p}^2 |i_n(\xi)|$ is the magnitude of the recorded image at the $p$th pixel or $y = p\Delta y \equiv \xi$ spatial coordinate. (b) The Fourier spectrum of $i(\xi)$ is similarly a decomposition into $M$ spatial frequencies and $\sigma_{\zeta_q}$ is the standard deviation of noise at the $q$th frequency.

The above detection process results in a similar decomposition of the continuous Fourier spectrum of $i(y)$ into $M$ discrete spatial frequencies as illustrated by the segmented line in figure 3(b). Two adjacent frequencies are separated by $\Delta k_y$ and the individual spatial frequencies are referenced by $k_y = q\Delta k_y \equiv \xi$, where $-\frac{M}{2} \leq q \leq \frac{M}{2} - 1$. The Fourier transform of the discretized noise-free image, $i(\xi)$, is denoted by $I(\xi)$, and the standard deviation of noise at the $q$th spatial frequency is $\sigma_{\zeta_q}^2$. Knowledge of $\sigma_{\zeta_q}^2$ is particularly useful in determining the maximum achievable limiting resolution for optical systems where transmission progressively worsens for high spatial frequencies. For example, the Fourier components with transmitted amplitudes comparable to, or less than $\sigma_{\zeta_q}$ will be indiscernible from random noise fluctuations within the measured signal. Therefore, $\sigma_{\zeta_q}$ allows us to identify the spatial frequencies whose Fourier domain information is effectively lost due to noise effects. Additionally, the effectiveness of a loss compensation technique can also be evaluated by monitoring its effect on $\sigma_{\zeta_q}$. Thus, a formulation of $\sigma_{\zeta_q}$ is important for our understanding of the underlying mechanism of ACI and its capacity at compensating losses while minimizing noise amplification.

A general expression for the standard deviation of SD noise at the $q$th spatial frequency can be calculated from the Fourier transform relation written as a Riemann sum \[44\]. The Fourier transform of the noisy image $i_n(\xi)$ in Eq. 4 is written as

$$I_n(\xi) = \sum_{\xi=-\frac{M}{2}\Delta y}^{(\frac{M}{2}-1)\Delta y} |i_n(\xi)| e^{i\theta(\xi)} \exp \left( -i2\pi\xi\Delta y \right), \quad (6)$$

with real and imaginary parts $I'_n(\xi)$ and $I''_n(\xi)$, respectively. Note that the number of samples $M$, is related to $\Delta y$ and $\Delta k_y$ by $M = 1/\Delta y \Delta k_y$ \[44\]. We can substitute $|i_n(\xi)|$ from Eq. 4 into Eq. 6 to express the real and imaginary parts of $I_n(\xi)$ as

$$I'_n(\xi) = \sum_{\xi=-\frac{M}{2}\Delta y}^{(\frac{M}{2}-1)\Delta y} \left\{ |i(\xi)| + n_{sd}(\xi) \right\} \cos[\phi(\xi, \xi)] \Delta y, \quad (7)$$

and

$$I''_n(\xi) = \sum_{\xi=-\frac{M}{2}\Delta y}^{(\frac{M}{2}-1)\Delta y} \left\{ |i(\xi)| + n_{sd}(\xi) \right\} \sin[\phi(\xi, \xi)] \Delta y, \quad (8)$$

respectively, and $\phi(\xi, \xi) = 2\pi \xi \theta - \phi(\xi)$. Note that the contribution of SI noise has been discarded by setting $\sigma_2 = 0$.

Based on Eq. 6, we can write $I_n(\xi)$ as

$$I_n(\xi) = I(\xi) + N_{sd}(\xi), \quad (9)$$

where $I(\xi)$ and $N_{sd}(\xi)$ are the Fourier transforms of $i(\xi)$ and $n_{sd}(\xi)$ in Eq. 4 respectively. The real and imaginary parts of $I(\xi)$ and $N_{sd}(\xi)$ can also be expressed in terms of the sums of cosines and sines similar to $I_n(\xi)$ (see Eqs. 7 and 8). $N_{sd}(\xi)$ in Eq. 9 has a standard deviation $\sigma_{\zeta_q}$ describing SD noise at the $q$th Fourier component. The variance of the real and imaginary parts of $N_{sd}(\xi)$ are denoted by $\sigma_{\zeta_q}^2$, and $\sigma_{\zeta_q}^2$, respectively. According to Eqs. 6 and 9, $N_{sd}(\xi)$ is a weighted superposition of all the RVs in the spatial domain. Each RV involved in the summation is statistically independent with a Gaussian
PDF. Therefore, we can apply Bienaymé’s identity, to express $\sigma_{y,}\sigma$ and $\sigma_{z,}\sigma$ as an
\begin{equation}
\sigma_{x,}^2 = \sum_{\xi = -\frac{N}{2}}^{\frac{N}{2}-1} \sigma_{x,}^2 \cos^2[\phi(\xi, \zeta)](\Delta y)^2, \quad (10)
\end{equation}
and
\begin{equation}
\sigma_{y,}^2 = \sum_{\xi = -\frac{N}{2}}^{\frac{N}{2}-1} \sigma_{y,}^2 \sin^2[\phi(\xi, \zeta)](\Delta y)^2, \quad (11)
\end{equation}
respectively, and $\sigma_{y,}^2 = f\{|i(\xi)|\}^2\sigma_{y,}^2$. The overall variance of SD noise at each spatial frequency is simply the sum of the variances of the real and imaginary parts in Eqs. 10 and 11 that is
\begin{equation}
\sigma_{y,}^2 = \sum_{\xi = -\frac{N}{2}}^{\frac{N}{2}-1} f\{|i(\xi)|\}^2\sigma_{y,}^2(\Delta y)^2. \quad (12)
\end{equation}
Without loss of generality, the subsequent calculations can be simplified by assuming the modulation function in Eq. 12 is a linear function of $\{i(\xi)\}$ with $\gamma = 1$. This represents a worst case scenario to illustrate the same effects with other realistic values of $\gamma$ [34]. Note that practical detectors are approximated with the Poisson distribution of photon noise [30]. Substituting $f\{|i(\xi)|\} = |i(\xi)|$ and $\gamma = 1$ we can rewrite Eq. 12 as
\begin{equation}
\sigma_{y,}^2 = \left[\sum_{\xi = -\frac{N}{2}}^{\frac{N}{2}-1} |i(\xi)|^2 \sigma_{y,}^2 \Delta y\right] \Delta y. \quad (13)
\end{equation}
The summation enclosed inside brackets, is proportional to the optical power on the image plane. Therefore, we can employ the energy conservation theorem by using Parseval’s relation and rewrite $\sigma_{y,}^2$ in Eq. 13 as
\begin{equation}
\sigma_{y,}^2 = \left[\sum_{\xi = -\frac{N}{2}}^{\frac{N}{2}-1} |I(\xi)|^2 \sigma_{y,}^2 \Delta k_y\right] \Delta y
\end{equation}
\begin{equation}
= \frac{(\frac{N}{2}-1)\Delta k_y}{M} \sum_{\zeta = -\frac{M}{2}}^{\frac{M}{2}-1} |I(\zeta)|^2 \sigma_{y,}^2. \quad (14)
\end{equation}
Eq. 14 says that the variance of SD noise at the $q^{th}$ Fourier component is proportional to the total power contained by the signal. Additionally, $\sigma_{y,}^2$ is constant in the Fourier domain. However, the presence of an extra $\Delta y$ clearly makes $\sigma_{y,}^2$ dependent on the spatial discretization. Rescaling $\Delta y$ in Eq. 14 would result in well known effects of upsampling or downsampling of continuous signals extensively used in signal processing. Therefore, Eq. 14 cannot be readily generalized for an arbitrary detector system without considering the physical mechanism through which information is extracted. The variance of detected noise may not necessarily reduce with pixel miniaturization and multiple factors must also be considered when determining the overall effect on noise. The number of detected photons are also intimately related to the pixel active area, quantum efficiency, the pixel optical path, integration time, and sensitivity [15][18]. Additionally, it may be necessary to incorporate crosstalk effects between adjacent pixels to accurately model the effect of pixel scaling on $\sigma_{y,}^2$. However, the effects of pixel miniaturization on the detected noise are considered independent from ACI, which only deals with compensation of signal losses for fixed number of pixels.

![Magnitude of the analytical transfer function for a 50nm thick silver lens embedded inside a dielectric and symmetrically placed between the object and image planes as shown in figure 1(a). The required compensation at each Fourier component should ideally be the inverse of the transfer function and is shown by the green line.](image)

In subsequent discussions, an analytical equation [49] is used for the transfer function of the silver lens imaging system, which is configured similar to an experimental silver lens [50] as shown in the schematic in figure 1(a). The transfer function of the lens with $d = 50nm$ and embedded inside a background dielectric of relative permittivity $\epsilon_d = 2.5$ [27][33][34] is shown by the black line in figure 4. The relative permittivity of silver at $\lambda = 365nm$ is $\epsilon_{Ag} = -1.88+0.60i$, calculated from the Drude-Lorentz model [51]. The corresponding estimated compensation necessary for each spatial frequency is shown in figure 4 by the green line and is simply the inverse of the corresponding magnitude of transmission. In figure 4 and subsequent ones only positive spatial frequencies are shown, although the full spectrum is considered in all the calculations.
IV. EFFECT OF ACI ON SD NOISE

In this section, we provide a detailed analysis of ACI focusing on the aspect of minimizing noise amplification during the loss compensation process. Selective amplification, previously illustrated with figure 2, is shown to be the cornerstone of ACI [25]. The subsequent discussion is broken into four parts leading to the conclusion that ACI can provide adequate compensation to attenuated spatial frequencies while inhibiting noise amplification. This in turn results in a significant superresolution enhancement in the imaging system. The discussion is initiated with Eq. [4] to derive equations for the variance of SD noise for the passive and active imaging systems (see figure 1). The importance of selective amplification is shown by comparing the above variances. The notion that substantial amplification can be provided to a band of spatial frequencies with small increments in noise variance is illustrated. Third, the resulting effect on system resolution is explained by comparing the signal-to-noise ratio (SNR) between passive and active systems. A substantial improvement in SNR in the active system is shown specifically within the selectively amplified portion of the Fourier spectrum. An example object with a Gaussian spectrum is employed to solidify these discussions and defined as

\[ |O(\zeta)| = \exp\left(-\frac{\zeta^2}{2\alpha^2}\right), \quad (15) \]

where \( \alpha \) describes the full width at half maximum of \( |O(\zeta)| \) and is defined as

\[ \alpha = \frac{0.25M\Delta k_y}{2\sqrt{2\ln 2}}. \quad (16) \]

The above form ensures that \( |O(\zeta)| \) is negligibly small as \( q \to M \). This avoids aliasing effects in the discretized Fourier spectrum and ensures that the sampling theorem is satisfied. We will conclude this section with a numerical illustration of the above features of ACI on a more arbitrary object.

The imaging systems (see figures 1(a) and (b)) are illuminated with a TM-polarized source from the object plane. The spatially coherent discretized complex magnetic field distribution along the object plane is denoted as \( o(\zeta) \). The Fourier transforms of the noiseless image for the passive and active imaging systems are

\begin{align*}
    I_P(\zeta) &= O(\zeta)T(\zeta), \quad (17a) \\
    I_A(\zeta) &= O(\zeta)T(\zeta)[1 + A_0G(\zeta)], \quad (17b)
\end{align*}

respectively, where \( O(\zeta) = F(o(\zeta)) \) and \( F \) is the Fourier transform operator. The subscripts “\( P \)” and “\( A \)” refer to the passive and active imaging systems, respectively. We point out that \( O(\zeta)A_0G(\zeta) \) in Eq. (17b) is the selectively amplified portion of the incident field illustrated in the schematic in figure 2 and is defined as the auxiliary source [25, 26, 33]. Therefore, \( O(\zeta)T(\zeta)A_0G(\zeta) \) is the residual auxiliary source which survived the lossy transmission process through the lens. It is important to note that the auxiliary source is correlated with the object field \( \varphi \).

The standard deviations of SD noise at the \( q^{th} \) Fourier component corresponding to \( I_P(\zeta) \) and \( I_A(\zeta) \) in Eq. (17) are denoted by \( \sigma_{\zeta, P}^2 \) and \( \sigma_{\zeta, A}^2 \), respectively. Their expressions are determined by substituting \( |I(\zeta)|^2 \) in Eq. (14) with \( |I_P(\zeta)|^2 \) and \( |I_A(\zeta)|^2 \), respectively. That is

\[ \sigma_{\zeta, P}^2 = \frac{1}{M} \sum_{\zeta=-\frac{M}{2}}^{\frac{M}{2}-1} |O(\zeta)T(\zeta)|^2 \sigma_1^2, \quad (18) \]

and

\[ \sigma_{\zeta, A}^2 = \frac{1}{M} \sum_{\zeta=-\frac{M}{2}}^{\frac{M}{2}-1} |O(\zeta)T(\zeta)|^2 \sigma_1^2[1 + A_0G(\zeta)]^2 \\
= \sigma_{\zeta, P}^2 + \sigma_{\zeta, Aux}^2. \quad (19) \]

Note that \( \sigma_{\zeta, Aux}^2 \) can be split into its contributing parts. \( \sigma_{\zeta, Aux}^2 \) describes the contribution to the SD noise from the residual auxiliary source and is given by

\[ \sigma_{\zeta, Aux}^2 \approx \frac{1}{M} \sum_{\zeta=-\frac{M}{2}}^{\frac{M}{2}-1} |O(\zeta)T(\zeta)|^2 A_0^2 \sigma_1^2 \frac{G'(\zeta)}{A_0} + |G(\zeta)|^2, \quad (20) \]

where \( G'(\zeta) \) is the real part. Eqs. (18) and (19) say that integrating the active spatial filter with the imaging system produces an additional source of SD noise whose standard deviation \( \sigma_{\zeta, Aux}^2 \) is dependent on the filter parameters.

Eq. (20) shows how the active filter parameters, such as \( A_0 \), the center frequency \( q_c \), and the width of \( G(\zeta) \) contribute to the noise at each Fourier component. Before proceeding further, we reduce Eq. (20) to

\[ \sigma_{\zeta, Aux}^2 \approx \frac{1}{M} \sum_{\zeta=-\frac{M}{2}}^{\frac{M}{2}-1} |O(\zeta)T(\zeta)|^2 A_0^2 \sigma_1^2 |G(\zeta)|^2, \quad (21) \]

since the summation of the first term inside the brackets in Eq. (20) can be generally dropped. For example, consider compensating the spatial frequencies \( 10k_0 \leq \zeta \leq 12k_0 \). According to the green line in figure 3, the estimated value for \( A_0 \) is approximately within the order \( 10^6 - 10^8 \).

The ratio \( R_\sigma \) of \( \sigma_{\zeta, Aux}^2 \) in Eq. (21) to \( \sigma_{\zeta, P}^2 \) in Eq. (18)
is written as
\[
R_\sigma = \frac{\sum_{\zeta = -\frac{N}{2}\Delta k_y}^{\frac{N}{2}-1}\Delta k_y |O(\zeta)T(\zeta)|^2 A_0^2 |G(\zeta)|^2}{\sum_{\zeta = -\frac{N}{2}\Delta k_y}^{\frac{N}{2}-1}\Delta k_y |O(\zeta)T(\zeta)|^2}.
\] (22)

Large values of \(R_\sigma\) indicate substantial SD noise addition from the auxiliary source. For simplicity, we rewrite the transfer function of the active spatial filter in Eq. 2 as
\[
A_R(\zeta) = 1 + A_0 G_R(\zeta),
\] (23)
where \(G_R(\zeta) = |G_R(\zeta)|e^{i\varphi_R(\zeta)}\) is a unit magnitude rectangular function of width \(W k_0\) and centered at \(\zeta_c\). That is
\[
|G_R(\zeta)| = \begin{cases} 
1 & \left|\frac{\zeta - \zeta_c}{W k_0}\right| \leq \frac{1}{2} \\
0 & \text{otherwise},
\end{cases}
\] (24)

This redefinition conveniently emphasizes the effect of selective amplification without loss of generality. We substitute \(|G(\zeta)|^2\) in Eq. 22 with \(|G_R(\zeta)|^2\) and rewrite the equation as
\[
R_\sigma = A_0^2 \frac{\sum_{\zeta = \zeta_L}^{\zeta_U} |O(\zeta)T(\zeta)|^2}{\sum_{\zeta = -\frac{N}{2}\Delta k_y}^{\frac{N}{2}-1}\Delta k_y |O(\zeta)T(\zeta)|^2}.
\] (25)

Note that the summation in the numerator is now confined within the lower and upper bounds of the rectangle function in Eq. 24 which are denoted as \(\zeta_L\) and \(\zeta_U\), respectively, and are given as
\[
\zeta_L = \zeta_c - \frac{W k_0}{2},
\] (26a)
\[
\zeta_U = \zeta_c + \frac{W k_0}{2}.
\] (26b)

Eq. 25 is important since it emphasizes the significance of selective amplification. Eq. 25 can be interpreted by first noting that \(|I_P(\zeta)|^2 = |O(\zeta)T(\zeta)|^2\) is the power spectral density (PSD) of the passive image. The summation of \(|O(\zeta)T(\zeta)|^2\) over all \(\zeta\) is proportional to the total power contained within the passive image. The PSD plots for \(|O(\zeta)|^2\) and the corresponding \(|I_P(\zeta)|^2\) are shown by black and red lines, respectively, in figure 3. Since \(|T(\zeta)|\) decays with increasing \(q\) (see black line in figure 4), the PSD of \(I_P(\zeta)\) clearly follows a similar trend. Most of the power contained within the image is distributed over a small portion of the Fourier spectrum. Eq. 25 can be rewritten as
\[
R_\sigma = A_0^2 \frac{P_{I_P, W}}{P_{I_P}},
\] (27)
where \(P_{I_P, W}\) is the portion of the total power contained by \(I_P(\zeta)\) distributed over bandwidth \(W k_0\) and centered at \(\zeta_c\), and \(P_{I_P}\) is the total power contained by \(I_P(\zeta)\). Note that the spatial frequencies over which \(P_{I_P, W}\) is distributed are the ones being amplified with ACI by a factor \(A_0\). Consider, for example, the case where the Fourier components within \(6 k_0 \leq \zeta \leq 8 k_0\) are selected band is modified to \(8 k_0 \leq \zeta \leq 10 k_0\). \(I_P(\zeta)\) in Eq. 25 will decrease as can be seen from figure 5. However, \(P_{I_P, W}\) will remain the same and therefore, the ratio \(R_\sigma\) from Eq. 25 will decrease. This conveniently restricts \(R_\sigma\) from becoming large even though the Fourier components within \(8 k_0 \leq \zeta \leq 10 k_0\) require larger amplification compared to the previous example.

The above inhibition of noise amplification during the compensation process results in substantial improvement in system performance [25, 33, 34]. This is investigated by comparing between the spectral SNR of the passive and active systems. A general expression for the spectral SNR is
\[
SNR(\zeta) = \frac{|I(\zeta)|}{\sigma_{\zeta}}.
\] (28)

FIG. 5. The power spectral densities of the object \(|O(\zeta)|^2\) and the passive image \(|I_P(\zeta)|^2\) showing how the total power contained within the object and image is distributed throughout the Fourier spectrum.

The SNR of the passive and active imaging systems are denoted as \(SNR_P(\zeta)\) and \(SNR_A(\zeta)\), respectively, which are determined from Eqs. \[17\text{-}19\]. Substituting the constant \(A_0\) in Eq. 23 with a functional form \(A_0(\zeta) = |T(\zeta)|^{-1}\) allows optimal amplification within \(W k_0\) bandwidth and is adopted below to emphasize the relative importance of selective amplification rather than
the exact functional form. Alternatively, a Gaussian or log-normal form of $|G_R(\zeta)|$ can also be used to better describe the previously considered metamaterial spatial filters [27, 33]. Additionally, for the remainder of this work we will use $\sigma_1 = 10^{-3}$ in the signal-modulated noise model in Eq. 4 for consistency with our previous works [25, 27, 33, 64], where an experimental imaging system detector [52] is considered.

Based on Eq. 28, $SNR_P(\zeta)$ and $SNR_A(\zeta)$ are written as

$$SNR_P(\zeta) = \frac{|O(\zeta)T(\zeta)|}{\sigma_{c_{q,P}}}$$  \hspace{1cm} (29)

and

$$SNR_A(\zeta) = \frac{|O(\zeta)T(\zeta)|(1 + A_0(\zeta)G_R(\zeta))}{\sigma_{c_{q,A}}}$$  \hspace{1cm} (30)

respectively. $SNR_P(\zeta)$ is plotted by the black line in figure 6 and $SNR_A(\zeta)$ for filters with $W = 1, 3, 4,$ and 6 by the pink, green, blue and purple lines, respectively. Note that $\zeta_c$ is kept constant at $10k_0$ and the dashed yellow line marks $SNR = 0$ dB. The intersection of $SNR_P(\zeta)$ with the dashed line marks the resolution limit of the passive system since larger Fourier components will be indistinguishable from noise in the detected signal. However, $SNR_A(\zeta)$ shows a remarkable improvement especially within the regions where compensation is provided. We point out that $SNR_A(\zeta)$ is less than $SNR_P(\zeta)$ outside the selected bands as expected, since the additional noise from $\sigma_{c_{q,A_{ux}}}$ affects the entire spectrum (see Eq. 14). This contribution increases with $W$ as is evident from figure 6. Nevertheless, the additional increment in noise variance is significantly smaller than the amplification provided to each Fourier component inside the selected bands, which results in an impressive enhancement in SNR. The purple line is particularly interesting since it encapsulates the remarkable power of ACI. The rectangle function in Eq. 24 spans a fairly broad $6k_0$ bandwidth and has essentially extended the resolution limit of the system close to double compared to the passive system.

In general, the theory of ACI can be expanded to arbitrary objects. This is illustrated with figure 7 where the Fourier spectrum of an arbitrary object is plotted by the black line. The corresponding noise-free passive image spectrum is calculated from Eq. 17a and corrupted with noise in the spatial domain according to the signal-modulated noise model in Eq. 4. The noisy image is then Fourier transformed to obtain $I_{n,P}(\zeta)$. The magnitudes of $I_{P}(\zeta)$ and $I_{n,P}(\zeta)$ are shown in figure 7 by pink and light green lines, respectively. The standard deviation of the SD noise which has degraded the passive image spectrum is determined from Eq. 18 and is shown by the dashed dark green line. We can see how $|I_{P}(\zeta)|$ progressively worsens with increasing $\zeta$. Eventually, $\sigma_{c_{q,P}}$ becomes comparable to $|I_{P}(\zeta)|$ at approximately $\zeta = 7k_0$ after which $|I_{n,P}(\zeta)|$ is overwhelmed by noise, similar to

FIG. 6. SNRs of the passive (black line) and active imaging systems (pink, green, blue and purple lines) with different $W$. The effective resolution limit of the passive imaging system is approximately $\zeta = 7k_0$. In contrast, the SNR of the active imaging systems incorporating ACI is increased within the selected bands of each filter. Slightly reduced SNR outside the selected bands indicate the noise contribution from the auxiliary source.

FIG. 7. Generalization of the ACI to an arbitrary object. The amplitude of the Fourier transforms of the object, the corresponding noise-free and noisy passive images, and the active image are shown by the black, pink, light green and light blue lines, respectively. The standard deviations of SD noise for the noisy passive image and the active image are shown by the dashed dark green and dark blue lines, respectively. The active image is then also corrupted with noise and Fourier transformed to obtain $I_{n,A}(\zeta)$, magnitude of which is shown by the light blue line in figure 7. The standard deviation of the SD noise for the active system $\sigma_{c_{q,A}}$ is calculated from Eq. 19 and is shown by the dashed dark blue line. Figure 7 clearly maintains the noise-resistant effect of selective amplification. Note that the missing nodes on the object spectrum are accurately recovered
inside the band $8k_0 \leq \zeta \leq 12k_0$ where the selective amplification is provided. The inhibition of noise amplification with ACI’s selective amplification is therefore applicable for arbitrary objects. Additionally, we point out that prior studies [25, 26, 33] with HMM spatial filters with transfer functions emulating Eq. 2 also produced similar results and were shown to enhance the resolution limit of a silver superlens.

V. CONCLUSION

We have presented a mathematical analysis of the conceptual framework of ACI. We showed that selective amplification of a controllable band of spatial frequencies with an auxiliary source can provide sufficient amplification to previously attenuated spatial frequencies with minimal amplification of noise. We also provided a detailed analytical explanation of the role and importance of the various aspects of ACI for greater insights into the previous results [25, 33]. We believe the same mathematical framework can be further expanded to include incoherent illumination [34]. The Wiener-Khinchin theorem can be used to formulate an incoherent version of the active filter transfer function similar to Eq. 2. We believe that this work can also theoretically explain the other numerical and experimental results presented in independent works including pattern uniformity in lithography [31], high-resolution Bessel beam generation [30], and hyperbolic dark-field lens [53], and fosters further explanation of recent simulation and experimental results in far-field imaging [54, 55]. Also, it is important to note that correlations play an important role in ACI [25, 26]. The theoretical concepts of ACI presented here can be applied to numerous imaging and communications scenarios (e.g., atmospheric imaging [56–58], biomaging [59], deep-learning based imaging [60], structured-light imaging [54], tomography [61], free space optical communications [62, 65], etc.) at different frequencies. Since ACI operates down at the physical layer, all of these scenarios should benefit from ACI for improved performance.

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