A NOTE ON THE REGULARITY OF PRODUCTS

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Abstract. Let \( S = \mathbb{K}[x_1, \ldots, x_n] \) denote a polynomial ring over a field \( \mathbb{K} \). Given a monomial ideal \( I \) and a finitely generated multigraded \( M \) over \( S \), we follow Herzog’s method to construct a multigraded free \( S \)-resolution of \( M/IM \) by using multigraded \( S \)-free resolutions of \( S/I \) and \( M \). The complex constructed in this paper is used to prove the inequality \( \text{reg}(IM) \leq \text{reg}(I) + \text{reg}(M) \) for a large class of ideals and modules. In the case where \( M \) is an ideal, under one relative condition on the generators which specially does not involve the dimensions, the inequality \( \text{reg}(IM) \leq \text{reg}(I) + \text{reg}(M) \) is proven.

Introduction

Throughout this paper \( S = \mathbb{K}[x_1, \ldots, x_n] \) is a polynomial ring over a field \( \mathbb{K} \). The Castelnuovo-Mumford regularity, \( \text{reg}(M) \), is one of the most important invariants of a finitely generated graded module \( M \) over a polynomial ring \( S \). Despite in general the regularity of a module can be doubly exponential in the degrees of the minimal generators and in the number of the variables, \([3]\) and \([10]\), there are several descriptions of the regularity of sum, intersection and products of ideals in term of each factor. A look on the enormous works in this topic, for example \([4]\), \([5]\), \([11]\), \([14]\), \([7]\), \([6]\) shows the importance of finding a neat formula for the regularity of a combination of two ideals.

Let \( I \) and \( J \) be two monomial ideals of \( S \) and let \( F \) and \( G \) be the multigraded free \( S \)-resolutions of \( S/I \) and \( S/J \). In \([8]\) Herzog constructs a multigraded free \( S \)-resolution of \( S/(I + J) \). This resolution generalizes the Taylor resolution \([13]\). The complex constructed in this way is used to generalize results on the Castelnuovo-Mumford regularity that were obtained for square-free monomial ideals by G. Kalai and R. Meshulam \([9]\). More precisely, Herzog declares the expected formula for the sum and intersection of monomial ideals \( I, J \) of the polynomial ring \( S \),

\[
\text{reg}(I + J) \leq \text{reg}(I) + \text{reg}(J) - 1,
\]

\[
\text{reg}(I \cap J) \leq \text{reg}(I) + \text{reg}(J).
\]

The problem on the regularity of products of homogeneous ideals, even monomial ideals, is more complicated. There are several counterexamples, \([12]\), \([14]\), \([6]\), which show that the inequality \( \text{reg}(IJ) \leq \text{reg}(I) + \text{reg}(J) \) does not hold in general. The regularity of two ideals or an ideal and an \( R \)-module is

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related to the regularity of tensor product of two modules, the work started by Sidman \[11\] and continued by Conca and Herzog \[5\] who showed that $\text{reg}(IM) \leq \text{reg}(I) + \text{reg}(M)$ for a finitely generated graded $R$-module $M$ and a homogeneous ideal $I$ in the case where $\dim(S/I) \leq 1$. In \[2\] Caviglia showed that $\text{reg}(M \otimes N) \leq \text{reg}(M) + \text{reg}(N)$ whenever $\dim(\text{Tor}_1^S(M,N)) \leq 1$, the regularity of Tor modules was subsequently studied in detail by Eisenbud, Huneke, and Ulrich in \[7\].

The aim of this paper is to determine some cases in which the inequality $\text{reg}(IM) \leq \text{reg}(I) + \text{reg}(M)$ is valid. By changing the point of view, instead of considering the codimension of the homogeneous ideal $I$ or $\dim(\text{Tor}_1^S(S/I,M))$, a relation between the variables participate in the minimal generating set of $I$ and those correspond to the minimal generating set of $M$ is studied.

For a homogeneous ideal $I$ (resp. a finitely generated multigraded $S$-module $M$) we define $\text{Gens}(I)$ (resp. $\text{Gens}(M)$) to be the variables participate in the minimal generating set of $I$ (resp. in the degrees of the minimal generating set of $M$). Using the techniques in \[8\], it is shown that in the case where $I$ is a monomial ideal and $\text{Gens}(I) \cap \text{Gens}(M) = \emptyset$ the Herzog’s complex (generalized Taylor complex) provides a free resolution for $M/IM$ in term of those of $S/I$ and $M$. This resolution in turn shows that the inequality $\text{reg}(IM) \leq \text{reg}(I) + \text{reg}(M)$ is valid in this case (see Theorem 1.2). For two homogeneous ideals $I$ and $J$ of $S$, we show that the condition $\text{Gens}(I) \cap \text{Gens}(J) = \emptyset$ implies that $I \cap J = IJ$. This is the case where the inequality of the regularity was already known. Trying to extend the desired inequality for ideals, it is shown that if $I$ and $J$ are two homogeneous (not necessarily monomial) ideals in which $|\text{Gens}(I) \cap \text{Gens}(J)| \leq 1$, then $\text{reg}(IJ) \leq \text{reg}(I) + \text{reg}(J)$ (see Theorem 1.6). Finally, an already known example of Conca and Herzog \[6, 2.1\] shows that the inequality $\text{reg}(IJ) \leq \text{reg}(I) + \text{reg}(J)$ is no longer generally valid if $|\text{Gens}(I) \cap \text{Gens}(J)| \geq 2$ (see Example 1.7).

1. Main results

Throughout $k$ is a field and $S = k[x_1, \ldots, x_n]$ is a polynomial ring, $M$ is a finitely generated multigraded ($\mathbb{N}^n$-graded) $S$-module. In his technical paper Herzog \[8\] defines a new product between free $S$-modules. For the sake of a ready to hand definition we restate the construction of this product.

For a homogeneous element $m \in M$ of degree $(a_1, \ldots, a_n) \in \mathbb{N}^n$ the unique monomial in $S$ which has the same degree as $m$ is denoted by $u_m$. We define the set of gens of $M$, $\text{Gens}(M)$, as the set of $x_i \in \{x_1, \ldots, x_n\}$ such that $x_i$ divides some $u_m$ where $m$ is a member of a minimal generating set of $M$. In addition, $\text{Gens}(M) = \emptyset$, if $M$ is generated by elements of degree zero.

**Definition 1.1.** Let $F$ and $G$ be free $S$-modules with homogeneous basis $B$ and $C$, respectively. The $*$-product of $F$ and $G$, $F * G$ is the multigraded free $S$-module with a basis given by the symbols $f * g$ where $f \in B$ and $g \in C$, the multidegree of $f * g$ is defined to be $|u_f u_g|$, the least common multiple of $u_f$ and $u_g$.

Comparing to the ordinary tensor product, $F \otimes G$ is a free $S$-module with the basis $f \otimes g$ where $f \in B$ and $g \in C$ and $\deg(f \otimes g) = \deg(u_f u_g)$. Hence, $F * G$ and $F \otimes G$ are free $S$-modules of a same rank. Keeping in mind that $S$ is a domain and the set $\{f * g : f \in B \text{ and } g \in C\}$ is a basis for $F * G$, one can
Therefore we may assume that

\[ I \]

\[ N \]

of \( T \) after specialization, the multigraded free

\[ j \]

see that the homogeneous multigraded map

\[ B \rightarrow \mathbb{C} \]

with basis \( S \)

Theorem 1.2.

Let \( I \) be a monomial multigraded ideal of \( S \) and \( M \) be a finitely generated multigraded \( S \)-module such that \( \text{Gens}(I) \cap \text{Gens}(M) = \emptyset \). Let \( F \) and \( G \) be the minimal multigraded free resolutions of \( S/I \) and \( M \), respectively. Then \( F \ast G \) is a multigraded free resolution of \( M/IM \).

Proof. The proof goes along the same lines as that of [8, 2.1], we just mention the slight modifications which have to be done. In the first step of the proof, we use polarization for the ideal \( I \) and assume that \( I \) is squarefree. As well by [4, Theorem 2.1], the multigraded \( S \)-module \( M \) can be lifted to a multigraded \( T \)-module \( N \) where \( T \) is a polynomial ring over \( S \), such that all shifts in the multigraded free \( T \)-resolution of \( N \) are squarefree. The shifts of this multigraded free \( T \)-resolution are of the expected form; so that after specialization, the multigraded free \( T \)-resolution becomes the multigraded free \( S \)-resolution of \( M \). Therefore we may assume that \( I \) and \( M \) have squarefree free resolution. We continue to the proof as in [8].

Let \( S/I \) and \( M \) admit minimal multigraded free resolutions \( F : 0 → F_p → \cdots → F_1 → S → 0 \) and \( G : 0 → G_q → \cdots → G_1 → G_0 → 0 \), respectively, where \( F_i \), resp. \( G_i \), is a multigraded free \( S \)-module with basis \( B_i \), resp. \( C_i \), for all \( 0 \leq i \leq p \), resp. \( 0 \leq i \leq q \). The complex \( G \) arisen from the first spectral
sequence of the double complex $F_\bullet \ast G_\bullet$ is of the form:

$$\tilde{G}_\bullet : 0 \to \bigoplus_{g \in C_q} (S/I_g)g \to \cdots \to \bigoplus_{g \in C_1} (S/I_g)g \to \bigoplus_{g \in C_0} (S/I_g)g \to 0$$

where $I_g$ is an ideal generated by the monomials $[u, u_g]/u_g$ in which $u$ is a member of the generating set of $I$. Here is the point that makes this theorem more general. The fact that $\tilde{G}_\bullet$ is acyclic, [8], in conjunction with the fact that the second spectral sequence converges shows that to know what is resolved by $F_\bullet \ast G_\bullet$ we have to know what $H_0(\tilde{G}_\bullet)$ is. We consider the most right terms of $F_\bullet \ast G_\bullet$, that is $F_1 \ast G_0 \oplus S \ast G_1 \xrightarrow{\psi} S \ast G_0 \to 0$. Since $S$ is generated by 1, the map $j$ induces the isomorphisms $S \ast G_1 \cong S \otimes G_1$ and $S \ast G_0 \cong S \otimes G_0$. The assumption that $\text{Gens}(I) \cap \text{Gens}(M) = \emptyset$ implies that the homogenous homomorphism $j : F_1 \ast G_0 \to F_1 \otimes G_0$ is an isomorphism, since $j(f_1 \ast g_0) = \text{gcd}(u_{f_1}, u_{g_0})f_1 \otimes g_0 = f_1 \otimes g_0$ for all $f_1 \in B_1$ and $g_0 \in C_0$. Hence we have the following commutative diagram, where $\varphi$ is the map at the beginning of the complex $F_\bullet \otimes G_\bullet$.

$$
\begin{array}{c}
F_1 \ast G_0 \oplus S \ast G_1 \xrightarrow{\psi} S \ast G_0 \\
\cong \downarrow j \quad \cong \downarrow j \\
F_1 \otimes G_0 \oplus S \otimes G_1 \xrightarrow{\varphi} S \otimes G_0
\end{array}
$$

To see that this diagram is commutative, we just need to verify the image of $f_1 \ast g_0$ for $f_1 \in B_1$ and $g_0 \in C_0$.

$$\psi(f_1 \ast g_0) = a_{f_1, 1}u_{g_0}f_{1, 1} 1 \ast g_0 = a_{f_1, 1}([u_{g_0}, u_{f_1}]/[u_{g_0}, 1])1 \ast g_0 = a_{f_1, 1}u_{f_1} 1 \ast g_0,$$

recall that $\text{gcd}(u_{f_1}, u_{g_0}) = 1$. We then have $j(\psi(f_1 \ast g_0)) = a_{f_1, 1}u_{f_1} 1 \otimes g_0 = \varphi(j(f_1 \ast g_0)) = \varphi(f_1 \otimes g_0)$, which shows that the above diagram is commutative. Therefore, $H_0(F_\bullet \ast G_\bullet) = H_0(F_\bullet \otimes G_\bullet) = S/I \otimes_S M \cong M/IM$, as desired. \hfill \Box

Regarding the above theorem, the main theorem of [8] deals with the case where $M = S/J$ and $J$ is a monomial ideal. In this case $\text{Gens}(M) = \emptyset$, hence the condition $\text{Gens}(M) \cap \text{Gens}(I) = \emptyset$ is automatically satisfied and the above argument for determining $H_0(\tilde{G}_\bullet)$ becomes vacuous.

With a free resolution of the product in hand, we are now able to give an upper bound for the Castelnuovo-Mumford regularity and the projective dimension of products. For a graded $S$-module $L$, we set $M_i(L)$ to be the highest shifts appears in the graded minimal $S$-free resolution of $L$. The castelnuovo-Mumford regularity of $M$ is defined as $\text{reg}(L) := \max \{M_i(L) - i : i \geq 0\}$.

**Corollary 1.3.** Let $I$ be a monomial ideal of $S$ and $M$ be a f.g. multigraded $S$-module such that $\text{Gens}(M) \cap \text{Gens}(I) = \emptyset$. Then

(a) $\text{proj dim}(M/IM) \leq \text{proj dim}(M) + \text{proj dim}(I) + 1$; and

(b) $\text{reg}(IM) \leq \text{reg}(I) + \text{reg}(M)$.

**Proof.** Part (a) is due to the fact that $F_\bullet \ast G_\bullet$ is acyclic and has length $\text{proj dim}(M) + \text{proj dim}(S/I) + 1$. 


For (b), since $F_{\bullet} \ast G_{\bullet}$ is not probably the minimal free resolution of $M/IM$ we have that $M_i(M/IM) \leq$ the highest shift in $(F_{\bullet} \ast G_{\bullet})$. The highest shift in $(F_{\bullet} \ast G_{\bullet})$; is less than or equal to $\max_{j+k=i} \{M_j(M), M_k(S/I)\}$, and so

$$\text{reg}(M/IM) = \max\{M_i(M/IM) - i : i \geq 0\} \leq \max_{j+k=i} \{M_j(M) - j, M_k(S/I) - k\} \leq \text{reg}(M) + \text{reg}(S/I).$$

Now, the exact sequence $0 \to IM \to M \to M/IM \to 0$ implies that

$$\text{reg}(IM) \leq \max\{\text{reg}(M), \text{reg}(M/IM) + 1\} \leq \text{reg}(M) + \text{reg}(S/I) + 1 = \text{reg}(M) + \text{reg}(I).$$

\[\Box\]

One may apply Corollary [13] for the case where $M = J$ is a monomial ideal to obtain the formula $\text{reg}(IJ) \leq \text{reg}(I) + \text{reg}(J)$ provided that $\text{Gens}(J) \cap \text{Gens}(I) = \emptyset$. Although this is the desired formula for the regularity of product of ideals, it is shown in Corollary [15] of the following general proposition that under the condition $\text{Gens}(J) \cap \text{Gens}(I) = \emptyset$ one has $IJ = I \cap J$. Hence to make the inequality $\text{reg}(IJ) \leq \text{reg}(I) + \text{reg}(J)$ valuable, we will later reduce the condition on $\text{Gens}$ (c.f. Theorem [16]).

**Proposition 1.4.** Consider the polynomial ring $S = \mathbb{k}[x_1, \cdots, x_n]$. Let $1 \leq k < n$ be an integer, $R = \mathbb{k}[x_1, \cdots, x_k]$ and $R' = \mathbb{k}[x_{k+1}, \cdots, x_n]$. Suppose that $M$ and $N$ are two extended modules, that is, there are graded (not necessarily multigraded) $R$-modules $M_1$ and $R'$-module $N_1$ such that $M = M_1 \otimes_R S$ and $N = N_1 \otimes_{R'} S$. Then

(a) $\text{Tor}_i^S(M, N) = 0$ for all $i \geq 1$; and
(b) $\text{reg}(M \otimes_S N) \leq \text{reg}(M) + \text{reg}(N)$.

**Proof.** To prove (a), let $F_{\bullet}$ be a $R$-free resolution of $M_1$. $F_{\bullet} \otimes_R S$ provides a $S$-free resolution for $M_1 \otimes_RS = M$. To compute $\text{Tor}_i^S(M, N)$, we consider the homology of the complex $(F_{\bullet} \otimes_R S) \otimes_S N$. Considering the natural isomorphisms $(F_{\bullet} \otimes_R S) \otimes_S N \cong (F_{\bullet} \otimes_R S) \otimes_S (N_1 \otimes_{R'} S) \cong F_{\bullet} \otimes_R (N_1 \otimes_{R'} S) \cong F_{\bullet} \otimes_R (S \otimes_{R'} N_1) \cong F_{\bullet} \otimes_R (R \otimes_{\mathbb{k}} R') \otimes_{R'} N_1 \cong F_{\bullet} \otimes_{\mathbb{k}} N_1$, we have

$$\text{Tor}_i^S(M, N) = H_i((F_{\bullet} \otimes_R S) \otimes_S N) = H_i(F_{\bullet} \otimes_{\mathbb{k}} N_1) = \text{Tor}_{i}^k(M_1, N_1) = 0$$

the last equality holds, since $k$ is a field.

For (b), it is enough to notice that if $F_{\bullet}$ and $G_{\bullet}$ are $R$-free resolution and $R'$-free resolution of $M_1$ and $N_1$, respectively, then by part (a) $F_{\bullet} \otimes_{\mathbb{k}} G_{\bullet}$ is a $S$-free resolution for $M \otimes_S N$. Now a similar computation as in the proof of the Corollary [13] yields the assertion. \[\Box\]

**Corollary 1.5.** Let $I$ and $J$ be two homogeneous ideals of $S$ with $\text{Gens}(J) \cap \text{Gens}(I) = \emptyset$, then $I \cap J = IJ$.

**Proof.** By the same token as Proposition [14] suppose that $I = I_1S$ and $J = J_1S$ where $I_1$ and $J_1$ are graded ideals of $R$ and $R'$, respectively. Considering the natural maps $I_1 \to R$, $J_1 \to R'$, $R \otimes_R S \to S$
and \( R' \otimes_R S \rightarrow S \) and the fact that \( S \) is flat over \( R \) and \( R' \), one sees that \( S/I \cong R/I_1 \otimes_R S \) and \( S/J \cong R'/J_1 \otimes_R S \). Now the result follows from Proposition 1.4 in conjunction with the fact that \( \text{Tor}^S_1(S/I, S/J) = I \cap J/IJ \).

Theorem 1.6. Let \( I \) and \( J \) be two homogeneous ideals of \( S \) such that \( \text{Gens}(J) \cap \text{Gens}(I) \) consists of at most one element. Then \( \text{reg}(IJ) \leq \text{reg}(I) + \text{reg}(J) \).

Proof. Set \( A := \text{Gens}(J) \cap \text{Gens}(I) \). The case where \( A = \emptyset \) is an immediate consequence of Proposition 1.4 or Corollary 1.3.

With no loss of generality, assume that \( A = \{x_1\} \), that \( I = I_1S \) where \( I_1 \) is an ideal of \( R = k[x_1, \ldots, x_k] \) and that \( J = J_1S \) where \( J_1 \) is an ideal of \( R' = k[x_1, x_{k+1}, \ldots, x_n] \). Let \( F_* \) be a \( R \)-free resolution of \( I_1 \) and \( G_* \) be a \( R' \)-free resolution of \( R'/J_1 \). Then \( F_* \otimes_R S \) and \( G_* \otimes_{R'} S \) are \( S \)-free resolutions of \( I \) and \( S/J \), respectively. Hence for all integer \( i \),

\[
\text{Tor}^S_i(I, S/J) = H_i((F_* \otimes_R S) \otimes_S (G_* \otimes_{R'} S)) \cong H_i((F_* \otimes_R S) \otimes_{R'} G_*) \\
\cong H_i(F_* \otimes_R (R \otimes_{k[x_1]} R') \otimes_{R'} G_*) \cong H_i(F_* \otimes_{k[x_1]} G_*) = \text{Tor}^{k[x_1]}_i(I_1, R'/J_1)
\]

The fact that \( k[x_1] \) has global dimension 1 implies the vanishing of these Tor modules for all \( i \geq 2 \). To see the vanishing of the first Tor modules, notice that \( R \) is a flat \( k[x_1] \) module, hence the exact sequence \( 0 \rightarrow I_1 \rightarrow R \rightarrow R/I_1 \rightarrow 0 \) yields \( \text{Tor}^{k[x_1]}_1(I_1, R'/J_1) = \text{Tor}^{k[x_1]}_2(R/I_1, R'/J_1) = 0 \).

The vanishing of all Tor modules shows that \((F_* \otimes_R S) \otimes_S (G_* \otimes_{R'})S \) is a free resolution of \( I \otimes_S S/J = I/IJ \). Now, a similar calculation as in the proof of Corollary 1.3 shows the assertion.

The next example of Conca and Herzog [6] shows that the inequality \( \text{reg}(IJ) \leq \text{reg}(I) + \text{reg}(J) \) is no longer true if \( \text{Gens}(J) \cap \text{Gens}(I) \) consists of two elements.

Example 1.7. Let \( R = k[x_1, x_2, x_3, x_4] \), \( I = (x_2, x_3) \) and \( J = (x_2^2x_2, x_1x_2x_3, x_2x_3x_4, x_3x_3^2) \). The minimal free resolution of \( I, J \) and \( IJ \) are:

\[
0 \rightarrow R(-2) \rightarrow R^2(-1) \rightarrow 0, \quad 0 \rightarrow R^3(-4) \rightarrow R^4(-3) \rightarrow 0 \text{ and } 0 \rightarrow R(-8) \rightarrow R^5(-6) \oplus R^2(-7) \rightarrow R^{10}(-5) \oplus R(-6) \rightarrow R^8(-4) \rightarrow 0,
\]

respectively. Hence, we have \( \text{reg}(I) = 1, \text{reg}(J) = 3 \) and \( \text{reg}(IJ) = 5 > \text{reg}(I) + \text{reg}(J) \).

Notice that \( \text{Gens}(I) = \{x_2, x_3\} \) and \( \text{Gens}(J) = \{x_1, x_2, x_3, x_4\} \), thus \( \text{Gens}(J) \cap \text{Gens}(I) = \{x_2, x_3\} \).

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