GENERALIZATIONS OF THE IMAGE CONJECTURE
AND THE MATHIEU CONJECTURE

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Abstract. We first propose a generalization of the image conjecture \[Z3\] for the commuting differential operators related with classical orthogonal polynomials. We then show that the non-trivial case of this generalized image conjecture is equivalent to a variation of the Mathieu conjecture \[Ma\] from integrals of $G$-finite functions over reductive Lie groups $G$ to integrals of polynomials over open subsets of $\mathbb{R}^n$ with any positive measures. Via this equivalence, the generalized image conjecture can also be viewed as a natural variation of Duistermaat and van der Kallen’s theorem \[DK\] on Laurent polynomials with no constant terms. To put all the conjectures above in a common setting, we introduce what we call the Mathieu subspaces of associative algebras. We also discuss some examples of Mathieu subspaces from other sources and derive some general results on this newly-introduced notion.

1. Introduction

1.1. Background and Motivation. The main motivations and contents of this paper are as follows.

First, in \[Z3\] a so-called image conjecture (IC) on images of commuting differential operators of polynomial algebras of order one with constant leading coefficients has been proposed. It has also been shown there that the well-known Jacobian conjecture proposed by O. H. Keller \[Ke\] (See also \[BCW\] and \[E\]) and, more generally, the vanishing conjecture \[Z1, Z2\] on differential operators (of any order) with constant coefficients, are actually equivalent to some special cases of the IC.

Second, as pointed out in \[Z2\], all classical orthogonal polynomials in one or more variables can be obtained from some commuting differential operators of order one with constant leading coefficients.

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Unfortunately, most of these differential operators are not differential operators of polynomial algebras. Instead, they are differential operators of some localizations of polynomial algebras such as Laurent polynomial algebras, etc.

Note that, due to their applications in many different areas of mathematics such as in ODE, PDE, the eigenfunction problems and representation theory, orthogonal polynomials have been under intense study by mathematicians in the last two centuries. For example, in [SHW] published in 1940, about 2000 published articles on orthogonal polynomials mostly in one variable had been included. Therefore it will also be interesting to consider the IC for the commuting differential operators related with classical orthogonal polynomials.

Unfortunately, the straightforward generalization of the IC from polynomial algebras to their localizations does not hold in general. In this paper, we propose another generalization of the IC (See Conjecture 3.1) for the commuting differential operators related with classical orthogonal polynomials.

We will also show that, under certain conditions, the new generalization is actually equivalent to a conjecture (See Conjecture 3.2) on integrals of polynomials over open subsets of \( B \subset \mathbb{R}^n \) with any (positive) measures. The latter conjecture turns out to be a natural variation of the Mathieu conjecture [Ma] (See Conjecture 1.2) from \( G \)-finite functions on reductive Lie groups \( G \) to polynomial functions over the open subsets \( B \subset \mathbb{R}^n \) above. It also can be viewed as a natural variation of Duistermaat and van der Kallen’s theorem [DK] (See Theorem 1.3) on Laurent polynomials with no constant terms.

To be more precise, let us first introduce the following notion which will provide a common ground for all the results and conjectures to be discussed in this paper.

**Definition 1.1.** Let \( R \) be any commutative ring and \( A \) a commutative \( R \)-algebra. We say that a \( R \)-subspace \( M \) of \( A \) is a Mathieu subspace of \( A \) if the following property holds: for any \( a, b \in A \) with \( a^m \in M \) for any \( m \geq 1 \), we have \( a^m b \in M \) when \( m >> 0 \), i.e. there exists \( N \geq 1 \) (depending on \( a \) and \( b \)) such that \( a^m b \in M \) for any \( m \geq N \).

Note that, any ideal of \( A \) is automatically a Mathieu subspace of \( A \). But conversely, not all Mathieu subspaces are ideals. Actually, many Mathieu subspaces are not even closed under the product of the ambient algebra \( A \). So the new notion can be viewed as a generalization of the notion of ideals. For more examples and general results on Mathieu subspaces, see Section 4.
The notion is named after Olivier Mathieu due to his following conjecture proposed in [Ma], 1995.

**Conjecture 1.2. (The Mathieu Conjecture)** Let $G$ be a compact connected real Lie group with the Haar measure $\sigma$. Let $f$ a complex-valued $G$-finite function over $G$ such that $\int_G f^m d\sigma = 0$ for any $m \geq 1$. Then, for any $G$-finite function $g$ over $G$, $\int_G f^m g d\sigma = 0$ when $m \gg 0$.

Note that, in terms of the newly introduced notion of Mathieu subspaces, the Mathieu conjecture just claims that the $\mathbb{C}$-subspace of complex-valued $G$-finite functions $f$ with $\int_G f d\sigma = 0$ is a Mathieu subspace of the $\mathbb{C}$-algebra $A$ of complex-valued $G$-finite functions over $G$.

One of the motivations of the Mathieu conjecture is its connection with the Jacobian conjecture (See [BCW] and [E]). Actually, Mathieu also showed in [Ma] that his conjecture implies the Jacobian conjecture.

For later purposes, here we also point out that J. Duistermaat and W. van der Kallen [DK] in 1998 had proved the Mathieu conjecture for the case of tori, which now can be re-stated as follows.

**Theorem 1.3. (Duistermaat and van der Kallen)** Let $z = (z_1, z_2, ..., z_n)$ be $n$ commutative free variables and $M$ the subspace of the Laurent polynomial algebra $\mathbb{C}[z^{-1}, z]$ consisting of the Laurent polynomials with no constant terms. Then $M$ is a Mathieu subspace of $\mathbb{C}[z^{-1}, z]$.

Another main motivation behind the new notion of Mathieu subspaces is the following so-called image conjecture (IC) proposed recently by the author in [Z3] on the images of commuting differential operators of polynomial algebras of order one with constant leading coefficients.

Let $z = (z_1, z_2, ..., z_n)$ be $n$ commutative free variables and $\mathbb{C}[z]$ the algebra of polynomials in $z$ over $\mathbb{C}$. For any $1 \leq i \leq n$, set $\partial_i := \partial / \partial z_i$. We say a differential operator $\Phi$ of $\mathbb{C}[z]$ is of order one with constant leading coefficients if $\Phi = h(z) + \sum_{i=1}^n c_i \partial_i$ for some $h(z) \in \mathbb{C}[z]$ and $c_i \in \mathbb{C}$. We denote by $D[z]$ the subspace of all differential operators of order one with constant leading coefficients. For any subset $\mathcal{C} = \{\Phi_i | i \in I\}$ of differential operators of $\mathbb{C}[z]$, we set $\text{Im} \mathcal{C} := \sum_{i \in I}(\Phi_i \mathbb{C}[z])$ and call it the image of $\mathcal{C}$. We say $\mathcal{C}$ is commuting if, for any $i, j \in I$, $\Phi_i$ and $\Phi_j$ commute with each other.

With the notation fixed above, the IC can be re-stated as follows.

**Conjecture 1.4. (The Image Conjecture)** For any commuting subset $\mathcal{C} \subset D[z]$, $\text{Im} \mathcal{C}$ is a Mathieu subspace of $\mathbb{C}[z]$.

Note that the IC, the Mathieu conjecture and also Conjectures 3.1–3.2 mentioned at the beginning of this subsection are all problems on
whether or not certain subspaces are Mathieu subspaces. It is also the case for the Jacobian conjecture and, more generally, the vanishing conjecture $[Z1]$, $[Z2]$ on differential operators (of any order) with constant coefficients via their connections with the IC (See $[Z3]$). Furthermore, we can also include the well-known Dixmier conjecture $[D]$ in the list since it has been shown, first by Y. Tsuchimoto $[T]$ in 2005 and later by A. Belov and M. Kontsevich $[BK]$ and P. K. Adjamagbo and A. van den Essen $[AE]$ in 2007, that the Dixmier conjecture is actually equivalent to the Jacobian conjecture. The implication of the Jacobian conjecture from the Dixmier conjecture was actually proved much earlier by V. Kac (unpublished but see $[BCW]$) in 1982.

Therefore, it is interesting and important to study Mathieu subspaces separately in a general and abstract setting. So we will also discuss more examples of Mathieu subspaces from other sources and derive some general results on this newly introduced notion (See Section 4).

1.2. Arrangement. In Subsection 2.1 we first recall some classical orthogonal polynomials and their related commuting differential operators (See Examples 2.2 and 2.4). We also fix some notations and summarize some facts that will be needed for the rest of this paper.

In Subsection 2.2, we consider the straightforward generalization of the IC for the commuting differential operators related with some multi-variable Jacobi orthogonal polynomials but without the constraints on the parameters required by the Jacobi polynomials. We will show in Proposition 2.6 that the straightforward generalization of the IC does not hold for these differential operators. But, if we generalize the IC in a different way, we will have a positive answer for these differential operators under the constraints on the parameters required by the Jacobi polynomials (See Corollary 2.10).

Another purpose of this subsection is to explain in a concrete setting the main ideas behind the generalization of the IC that will be formulated and discussed in Section 3. Some of the results of this subsection will also be needed later in Subsection 3.2.

In Subsection 3.1 we first formulate a generalization (See Conjecture 3.1) of the IC for the differential operators related with orthogonal polynomials, and also a conjecture (See Conjecture 3.2) on integrals of polynomials over open subsets $B \subset \mathbb{R}^n$ with any positive measures. We show in Proposition 3.3 that the non-trivial case of Conjecture 3.1 is actually equivalent to some special cases of Conjecture 3.2. We also point out that Conjecture 3.2 in some sense can be viewed as a natural variation of the Mathieu conjecture (See Conjecture 1.2) and Duistermaat and van der Kallen’s theorem (See Theorem 1.3).
In Subsection 3.2, we prove some cases of Conjectures 3.1 and 3.2. We also discuss a connection of Conjecture 3.2 with the polynomial moment problem which was first proposed by M. Briskin, J.-P. Francoise and Y. Yomdin in the series paper [BFY1]-[BFY5] and recently was solved by F. Pakovich and M. Muzychuk [PM].

In Section 4, we discuss Mathieu subspaces in the most general setting. Some examples of Mathieu subspaces from other sources will be given and some general results on this newly introduced notion will also be derived.

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2. Differential Operators Related with Classical Orthogonal Polynomials

In this section, we first recall in Subsection 2.1 some classical orthogonal polynomials in one or more variables and their related differential operators. We also summarize in Lemma 2.5 some facts that will be needed in later sections.

The classical reference for one-variable orthogonal polynomials is [Sz] (see also [AS], [C], [Si]). For multi-variable orthogonal polynomials, see [DX], [Ko] and references therein. But here we will essentially follow the presentations given in [Z2] in terms of differential operators of certain localizations of polynomial algebras, and emphasize that the related differential operators are all commuting differential operators of order one with constant leading coefficients. The presentation in [Z2] for multi-variable orthogonal polynomials will also be simplified here.

In Subsection 2.2, we consider the straightforward generalization of the image conjecture (IC) for some commuting differential operators of the Laurent polynomials. Up to changes of variables, these differential operators are related with some multi-variable Jacobi orthogonal polynomials (See Example 2.4). We show in Proposition 2.6 that the IC does not hold for these differential operators. But it does hold for
the same differential operators if we generalize the IC in a different way (See Corollary 2.10).

2.1. Differential Operators Related with Classical Orthogonal Polynomials. First, let us recall the definition of classical orthogonal polynomials. In order to be consistent with the traditional notations of orthogonal polynomials, in this subsection we will use \(x = (x_1, x_2, \ldots, x_n)\) instead of \(z = (z_1, z_2, \ldots, z_n)\) to denote free commutative variables.

**Definition 2.1.** Let \(B\) be a non-empty open subset of \(\mathbb{R}^n\) and \(w(x)\) a real valued function defined over \(B\) such that \(w(x) \geq 0\) for any \(x \in B\) and \(0 < \int_B w(x)dx < \infty\). Assume further that \(\int_B f(x) \bar{g}(x)w(x)dx\) is finite for any \(f(x) \in C[x]\). A sequence of polynomials \(\{u_\alpha(x) \mid \alpha \in \mathbb{N}^n\}\) is said to be orthogonal over \(B\) if

(a) \(\deg u_\alpha = |\alpha| := \sum_{i=1}^n k_i\) for any \(\alpha = (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n\).

(b) the sequence \(\{u_\alpha(x) \mid \alpha \in \mathbb{N}^n\}\) forms an orthogonal basis of \(C[x]\) with respect to the Hermitian form defined by

\[
(f, g) := \int_B f \bar{g}w(x)dx
\]

for any \(f, g \in C[x]\), where \(\bar{g}\) denotes the complex conjugate of the polynomial \(g \in C[x]\).

The function \(w(x)\) is called the weight function. For all classical orthogonal polynomials, \(w(x)\) is smooth over \(B\) but might have some singular points over the boundary of \(B\) (See Examples 2.2 and 2.4 below). When the open set \(B \subset \mathbb{R}^n\) and \(w(x)\) are clear in the context, we simply call the polynomials \(u_\alpha(x) \ (\alpha \in \mathbb{N}^n)\) in the definition above orthogonal polynomials. If the orthogonal polynomials \(u_\alpha(x) \ (\alpha \in \mathbb{N}^n)\) also satisfy \(\int_B |u_\alpha|^2 w(x)dx = 1\) for any \(\alpha \in \mathbb{N}^n\), we call \(u_\alpha(x) \ (\alpha \in \mathbb{N}^n)\) orthonormal polynomials.

Note that, if \(u_\alpha(x) \ (\alpha \in \mathbb{N}^n)\) are orthogonal polynomials, say, as in Definition 2.1, then, for any \(c_\alpha \in \mathbb{C}^\times \ (\alpha \in \mathbb{N}^n), c_\alpha u_\alpha \ (\alpha \in \mathbb{N}^n)\) are also orthogonal polynomials over \(B\) with the same weight function \(w(x)\).

An obvious way to construct orthogonal polynomials is to apply the Gram-Schmidt process. But, surprisingly, most of classical orthogonal polynomials can also be obtained by the following so-called Rodrigues’ formulas which, in terms of the notation as in Definition 2.1 can be stated as follows.

**Rodrigues’ Formula:** There exist some non-zero constants \(c_\alpha \in \mathbb{R} \ (\alpha \in \mathbb{N}^n)\) and an \(n\)-tuple \(g(x) = (g_1(x), g_2(x), \ldots, g_n(x))\) of polynomials
in $x$ such that
\begin{equation}
(2.2) \quad u_\alpha(x) = c_\alpha w(x)^{-1} \frac{\partial^{|\alpha|}}{\partial x^\alpha}(w(x)g^\alpha(x)).
\end{equation}

Note that, not all orthogonal polynomials defined in Definition 2.1 can be obtained by Rodrigues’ formulas. For example, the weight function $w(x)$ of some orthogonal polynomials may not even be differentiable.

Now, for any $1 \leq i \leq n$, let
\begin{equation}
(2.3) \quad \Lambda_i := w(x)^{-1} \left( \frac{d}{dx_i} \right) w(x) = \frac{d}{dx_i} + w(x)^{-1} \frac{dw(x)}{dx_i}.
\end{equation}

and set $\Lambda := (\Lambda_1, \Lambda_2, ..., \Lambda_n)$. Then, by Rodrigues’ formula above, we see that the orthogonal polynomials $\{u_\alpha(x) \mid \alpha \in \mathbb{N}^n\}$ have the form
\begin{equation}
(2.4) \quad u_\alpha(x) = c_\alpha \Lambda^\alpha(g^\alpha(x))
\end{equation}
for any $\alpha \in \mathbb{N}^n$.

Note also that the differential operator $\Lambda_i$ in Eq. (2.3) is a differential operator of order one with constant leading coefficients. Furthermore, in the multi-variable case, the differential operators $\Lambda_i$ ($1 \leq i \leq n$) commute with one another since they are the conjugations of the commuting differential operators $\partial_i$ ($1 \leq i \leq n$) by the multiplication operator by $w^{-1}(z)$.

Let us look at the following classical orthogonal polynomials.

**Example 2.2.**

1. **Hermite Polynomials:**
   (a) $B = \mathbb{R}$ and the weight function $w(x) = e^{-x^2}$.
   (b) the differential operator $\Lambda$ and the polynomial $g(x)$:
   \begin{equation}
   (2.5) \begin{cases}
   \Lambda = \frac{d}{dx} - 2x, \\
g(x) = 1,
\end{cases}
   \end{equation}
   (c) the Hermite polynomials in terms of $\Lambda$ and $g(x)$:
   \[H_m(x) = (-1)^m \Lambda^m(g^m(x)).\]

2. **Laguerre Polynomials:**
   (a) $B = \mathbb{R}^+$ and $w(x) = x^\alpha e^{-x}$ ($\alpha > -1$).
   (b) the differential operator $\Lambda$ and the polynomial $g(x)$:
   \begin{equation}
   (2.6) \begin{cases}
   \Lambda = \frac{d}{dx} + (\alpha x^{-1} - 1), \\
g(x) = x,
\end{cases}
   \end{equation}
(c) the Laguerre polynomials in terms of $\Lambda$ and $g(x)$:

$$L_m(x) = \frac{1}{m!} \Lambda^m(g^m(x)).$$

(3) Jacobi Polynomials:
(a) $B = (-1, 1)$ and $w(x) = (1-x)\alpha(1+x)\beta$ with $\alpha, \beta > -1$.
(b) the differential operator $\Lambda$ and the polynomial $g(x)$:

$$\Lambda = \frac{d}{dx} - \alpha(1-x)^{-1} + \beta(1+x)^{-1},$$
$$g(x) = 1 - x^2.$$  

(2.7)

(c) the Jacobi polynomials in terms of $\Lambda$ and $g(x)$:

$$P_{\alpha,\beta}^m(x) = (-1)^m 2^m m! \Lambda^m g^m(x).$$

(2.8)

(4) Classical Orthogonal Polynomials over Unit Balls:
(a) $B = \mathbb{B}^n = \{ x \in \mathbb{R}^n \left| \|x\| < 1 \} \text{ and the weight function}$

$$w_\mu(x) = (1 - \|x\|)^{\mu-1/2},$$

where $\| \cdot \|$ denotes the usual Euclidean normal of $\mathbb{R}^n$ and $\mu > 1/2$.

(b) the differential operators $\Lambda$ and the polynomials $g(x)$:

$$\begin{align*}
\Lambda &= \frac{\partial}{\partial x_i} - \frac{(2\mu-1)x_i}{1-\|x\|^2}, \\
g(x) &= 1 - \|x\|^2.
\end{align*}$$

(2.9)

(c) the classical orthogonal polynomials $\{ U_\alpha \mid \alpha \in \mathbb{N}^n \}$ over the unite ball $\mathbb{B}^n$ in terms of $\Lambda$ and $g(x)$:

$$U_\alpha(x) = \frac{(-1)^{|\alpha|}(2\mu)^{|\alpha|}}{2^{|\alpha|}|\alpha|!(\mu+1/2)^{|\alpha|}} \Lambda^\alpha(g^\alpha(x)),$$

where, for any $c \in \mathbb{R}$ and $k \in \mathbb{N}$, $(c)_k = c(c+1) \cdots (c+k-1)$.

(5) Classical Orthogonal Polynomials over Simplices:
(a) $B = T^n = \{ x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i < 1; \ x_1, ..., x_n > 0 \}$ and the weight function

$$w_\kappa(x) = x_1^{\kappa_1} \cdots x_n^{\kappa_n} (1 - |x|_1)^{\kappa_{n+1}},$$

where $\kappa_i > -1$ (1 $\leq$ $i$ $\leq$ $n$ + 1) and $|x|_1 = \sum_{i=1}^n x_i$.

(b) the differential operators $\Lambda$ and the polynomials $g(x)$:

$$\begin{align*}
\Lambda_i &= \frac{\partial}{\partial x_i} + \frac{\kappa_i}{x_i} - \frac{\kappa_{n+1}}{1-|x|_1}, \\
g_i(x) &= x_i(1 - |x|_1)
\end{align*}$$

for any 1 $\leq$ $i$ $\leq$ $n$. 

(2.10)
(c) the classical orthogonal polynomials \( \{U_\alpha \mid \alpha \in \mathbb{N}^n\} \) over the simplex \( \mathbb{T}^n \) in terms of \( \Lambda \) and \( g(x) \):
\[
U_\alpha(x) = \Lambda^\alpha(g^\alpha(x)).
\]

Remark 2.3. (a) A very important special family of Jacobi polynomials are the Gegenbauer polynomials which are obtained by setting \( \alpha = \beta = \lambda - 1/2 \) for some \( \lambda > -1/2 \). The Gegenbauer polynomials are also called the ultraspherical polynomials in the literature.

(b) For the special cases with \( \lambda = 0, 1, 1/2 \), the Gegenbauer Polynomials are called the Chebyshev polynomial of the first kind, the second kind and the Legendre polynomials, respectively.

(c) When \( n = 2 \), up to some non-zero constants the orthogonal polynomials \( U_\alpha(x) \ (\alpha \in \mathbb{N}^2) \) in Eq. (2.10) are also called Appell polynomials.

Note that, one important way to construct multi-variable orthogonal polynomials is to take cartesian products of orthogonal polynomials in one variable.

More precisely, let \( x = (x_1, x_2, \ldots, x_n) \) and \( \{u_{i,m}(x_i) \mid m \geq 0\} \ (1 \leq i \leq n) \) be orthogonal polynomials over a subset \( B_i \subset \mathbb{R} \) with weight function \( w_i(x_i) \) over \( B_i \). Let
\[
B = B_1 \times B_2 \times \cdots \times B_n,
\]
(2.11)
\[
w(x) = w_1(x_1)w_2(x_2)\cdots w_n(x_n),
\]
(2.12)
\[
u_\alpha(x) = u_{i_1,k_1}(x_1)u_{i_2,k_2}(x_2)\cdots u_{i_n,k_n}(x_n)
\]
for any \( \alpha = (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n \).

Then it is easy to see that \( \{u_\alpha(x) \mid \alpha \in \mathbb{N}\} \) are orthogonal polynomials over \( B \subset \mathbb{R}^n \) with respect to the weight function \( w(x) \).

Furthermore, if, for any \( 1 \leq i \leq n \) and \( m \geq 0 \), the orthogonal polynomial \( u_{i,m}(x_i) = c_{i,m}A_i^m(g_i^m(x_i)) \) for some nonzero \( c_{i,m} \in \mathbb{R} \), \( g_i(x) \in \mathbb{C}[x_i] \) and a differential operator \( A_i \) of a localization of \( \mathbb{C}[x_i] \). Set \( \Lambda = (A_1, A_2, \ldots, A_n) \) and \( g(x) = (g_1(x_1), g_2(x_2), \ldots, g_n(x_n)) \). Then, it is easy to see that \( \Lambda \) is a commuting subset of differential operators of a localization of \( \mathbb{C}[x] \), and the orthogonal polynomials \( u_\alpha(x) \ (\alpha \in \mathbb{N}^n) \) over \( B \) are given by
\[
u_\alpha(x) = c_\alpha A^\alpha(g^\alpha(x)),
\]
where \( c_\alpha = \prod_{i=1}^n c_{i,\alpha_i} \) for any \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n \).

For later purposes, let us consider the multi-variable Jacobi orthogonal polynomials which, by Remark 2.3, also cover the multi-variable Gegenbauer, Chebyshev and Legendre orthogonal polynomials.
Example 2.4. Let $B = (-1, 1)^n \subset \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}^n$ with all the components $\alpha_i, \beta_i > -1$ ($1 \leq i \leq n$). For any $1 \leq i \leq n$, set

$$\Lambda_i := \frac{d}{dx_i} - \alpha_i (1 - x_i)^{-1} + \beta_i (1 + x_i)^{-1},$$

$$g_i(x) := 1 - x_i^2.$$  \hspace{1cm} (2.15)

Furthermore, set

$$w(x) := \prod_{i=1}^{n} (1 - x_i)(1 + x_i)^{\beta_i},$$

$$\Lambda_{\alpha, \beta} := (\Lambda_1, \Lambda_2, ..., \Lambda_n),$$

$$g(x) := (g_1(x), g_2(x), ..., g_n(x)).$$ \hspace{1cm} (2.17)

For any $1 \leq i \leq n$ and $m \geq 0$, let $P_{m_i}^{\alpha_i, \beta_i}(x)$ be the $m_i$th one-variable Jacobi polynomial in $x_i$ (See Example 2.2 (3)) with $\alpha = \alpha_i$ and $\beta = \beta_i$. For any $m = (m_1, m_2, ..., m_n) \in \mathbb{N}^n$, set

$$P_m^{\alpha, \beta}(x) := \prod_{i=1}^{n} P_{m_i}^{\alpha_i, \beta_i}(x).$$ \hspace{1cm} (2.18)

Then, for any fixed $\alpha, \beta \in (\mathbb{R}^{>1})^n$, the sequence $\{P_m^{\alpha, \beta}(x) | m \in \mathbb{N}^n\}$ forms a sequence of orthogonal polynomials over $B$ with the weight function given by Eq. (2.17). From Eq. (2.8), it is easy to see that the relation of $\{P_m^{\alpha, \beta}(x) | m \in \mathbb{N}^n\}$ with the commuting differential operators $\Lambda$ in Eq. (2.18) and the polynomial $g(x)$ in Eq. (2.19) is given by

$$P_m^{\alpha, \beta}(x) = \frac{(-1)^{|m|}}{2^{|m|} m!} \Lambda_{\alpha, \beta}^m g^m(x).$$ \hspace{1cm} (2.20)

Finally, let us summarize the relations of orthogonal polynomials with commuting differential operators of order one with constant leading coefficients in the following lemma.

Lemma 2.5. (a) Up to some nonzero multiplicative scalars, all classical orthogonal polynomials $\{u_\alpha(x) | \alpha \in \mathbb{N}^n\}$ above including those obtained by Cartesian products of classical orthogonal polynomials have the form in Eq. (2.4) for some $g(x) = (g_1(x_1), g_2(x_2), ..., g_n(x_n)) \in \mathbb{C}[x]^n$ and differential operators $\Lambda = (\Lambda_1, \Lambda_2, ..., \Lambda_n)$ of some localizations $B$ of $\mathbb{C}[x]$.

(b) The set $\Lambda$ is a commuting subset of differential operators of $B$ of order one with constant leading coefficients.

(c) For any nonzero $\alpha \in \mathbb{N}^n$, the orthogonal polynomials $u_\alpha(x) \in \text{Im}'\Lambda := \mathbb{C}[x] \cap \sum_{i=1}^{n} (\Lambda_i \mathbb{C}[x])$.  \hspace{1cm} (2.21)
Note that (a) and (b) follow immediately from the discussion in this subsection. (c) follows from the fact that \( \Lambda^\alpha(g_i(x)g^\alpha) \) for any \( 1 \leq i \leq n \) and \( \alpha \in \mathbb{N}^n \), which can also be easily checked directly.

2.2. The Image Conjecture for the Differential Operators Related with the Multi-Variable Jacobi Orthogonal Polynomials.

Considering the important roles of orthogonal polynomials played in so many different areas, it will be interesting to see if the image conjecture (IC), Conjecture 1.4, also holds for the commuting differential operators related with orthogonal polynomials.

In this subsection, we consider the straightforward generalization of the IC for the following family of commuting differential operators of Laurent polynomial algebras in a slightly more general setting, namely, with the base field \( \mathbb{C} \) replaced by integral domains over \( \mathbb{C} \). As we will see that the straightforward generalization of the IC is false for these differential operators (See Propositions 2.6 and 2.9). But another generalization of the IC for these differential operators under the constraints from the multi-variable Jacobi orthogonal polynomials actually holds (See Corollary 2.10).

Let \( \mathcal{A} \) be any integral domain over \( \mathbb{C} \) and \( \mathcal{A}[z^{-1}, z] \) the algebra of Laurent polynomials with coefficients in \( \mathcal{A} \). For any \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{C}^n \), we set \( \Phi_{\lambda_i} := \partial_i + \lambda_i z_i^{-1} \) \((1 \leq i \leq n)\) and \( \Phi_{\lambda} := (\Phi_{\lambda_1}, \Phi_{\lambda_2}, ..., \Phi_{\lambda_n}) \). We will also view \( \Phi_{\lambda} \) as a commuting subset (instead of just an \( n \)-tuple) of differential operators of \( \mathcal{A}[z^{-1}, z] \) of order one with constant leading coefficients.

Note that, the differential operators \( \Phi_{\lambda} \) are essentially the differential operators related with the multi-variable Jacobi orthogonal polynomials \( P_{m}^{\alpha,\beta}(x) \) \( (m \in \mathbb{N}^n) \) in Eq. (2.20) with \( \alpha = 0 \) or \( \beta = 0 \). For example, by setting \( \alpha = 0 \) and \( \beta = \lambda \), and changing the variables \( x_i \rightarrow z_i - 1 \) \((1 \leq i \leq n)\), from Eqs. (2.15) and (2.18) we see that the differential operators \( \Lambda_{\alpha,\beta} \) related with the Jacobi polynomials will coincide with the differential operators \( \Phi_{\lambda} \). Similarly, this is also the case when \( \beta = 0 \) if we set \( \alpha = \lambda \) and apply the changing of variables \( x_i \rightarrow z_i + 1 \) \((1 \leq i \leq n)\).

But we emphasize that, unlike for the parameters \( \alpha \) and \( \beta \) of the Jacobi polynomials, here we do not require \( \lambda_i > -1 \) \((1 \leq i \leq n)\) nor even \( \lambda \in \mathbb{R}^n \) unless stated otherwise.

Now we fix any \( \lambda \in \mathbb{C}^n \) and the differential operators \( \Phi_{\lambda} \) as above, and set \( \text{Im} \Phi_{\lambda} := \sum_{i=1}^{n} (\Phi_{\lambda_i} \mathcal{A}[z^{-1}, z]) \). We also fix the following notation that will be used throughout the rest of this paper.

For any \( \gamma \in \mathbb{Z}^n \) and \( g(z) \in \mathcal{A}[z^{-1}, z] \), we denote by \([z^\gamma]g(z)\) the coefficient of the monomial \( z^\gamma \) in \( g(z) \). For convenience, we also allow
in the notation above to be any element of \( \mathbb{C}^n \), i.e. we set \([z^{\gamma}]g(z) = 0\) for any \( g(z) \in A[z^{-1}, z] \) and \( \gamma \in \mathbb{C}^n \setminus \mathbb{Z}^n \). In the case that \([z^{\gamma}]g(z) = 0\), we also say that \( g(z) \) has no \( z^{\gamma} \) term.

With all the notations fixed above, we have the following proposition.

**Proposition 2.6.** For any \( \lambda \in \mathbb{C}^n \), denote by the abusing notation \( -\lambda - 1 \) the \( n \)-tuple \( (-\lambda - 1, 1, ..., 1) \). Then, we have

(a) \( \text{Im} \Phi_{\lambda} \) is the \( \lambda \)-subspace of \( A[z^{-1}, z] \) consisting of the Laurent polynomials \( g(z) \in A[z^{-1}, z] \) with \( [z^{-\lambda-1}]g(z) = 0 \). In particular, \( \text{Im} \Phi_{\lambda} = A[z^{-1}, z] \) if \( \lambda \notin \mathbb{Z}^n \).

(b) \( \text{Im} \Phi_{\lambda} \) is a Mathieu subspace of \( A[z^{-1}, z] \) iff \( \lambda \notin \mathbb{Z}^n \) or \( \lambda = (-1, -1, ..., -1) \).

**Proof:** We first prove the proposition for the case \( n = 1 \), i.e. for the one-variable case.

For any \( g(z) \in A[z^{-1}, z] \), consider the ordinary differential equation with the unknown function \( f(z) \in A[z^{-1}, z] \):

\[
\Phi_{\lambda} f = f' + \lambda z^{-1} f = g.
\]

(2.22)

The equation above can be solved by the following standard trick in ODE. First, we view the equation as a differential equation for elements of \( A[z^{\pm \lambda}, z^{\pm 1}] \), and set \( \tilde{f}(z) := z^\lambda f(z) \in A[z^{\pm \lambda}, z^{\pm 1}] \). Then \( f(z) = z^{-\lambda} \tilde{f}(z) \). Plug this expression of \( f(z) \) in Eq. (2.22), it is easy to check that \( \tilde{f}(z) \) satisfies the following equation:

\[
z^{-\lambda} \tilde{f}' = g.
\]

Therefore, we have \( \tilde{f}(z) = \int z^\lambda g(z) \, dz \) and

\[
f(z) = z^{-\lambda} \int z^\lambda g(z) \, dz.
\]

(2.23)

From the arguments, we see that any solution \( f \in A[z^{-1}, z] \) of Eq. (2.22) must be given by Eq. (2.23) up to a \( z^{-\lambda} \) term. But, conversely, the RHS of Eq. (2.23) does not necessarily produce an element of \( A[z^{-1}, z] \) unless \( z^\lambda g(z) \) has no \( z^{-1} \) term, i.e. the residue \( \text{Res} z^\lambda g(z) = 0 \).

Therefore, the differential equation Eq. (2.22) has a Laurent polynomial solution \( f(z) \in A[z^{-1}, z] \) iff \( \text{Res} z^\lambda g(z) = 0 \) iff \([z^{-\lambda-1}]g(z) = 0\). Hence, we have (a) of the proposition for the case \( n = 1 \).

To show (b) for the case \( n = 1 \), let us look at all the values of \( \lambda \in \mathbb{C} \) such that \( \text{Im} \Phi_{\lambda} \) is a Mathieu subspace of \( A[z^{-1}, z] \).

First, if \( \lambda \notin \mathbb{Z} \), by (a) \( \text{Im} \Phi_{\lambda} = A[z^{-1}, z] \) which is obviously a Mathieu subspace of \( A[z^{-1}, z] \).
Consider the case $\lambda \in \mathbb{Z}$. If $-\lambda - 1 \neq 0$, i.e. $\lambda \neq -1$, then, by statement (a) for the case $n = 1$, we have, $1 \in \text{Im} \Phi_\lambda$ and $\text{Im} \Phi_\lambda \neq \mathcal{A}[z^{-1}, z]$. By the general property of Mathieu subspaces given in Lemma 4.5 in Section 4, Im $\Phi_\lambda$ is not a Mathieu subspace of $\mathcal{A}[z^{-1}, z]$.

A more convincing counter-example for this case can be constructed as follows. Set

\begin{align}
(2.24) \\
v(z): = z^{-\lambda - 1} & \\
(2.25) \\
u(z): = \begin{cases} 
1 + z^{-\lambda} & \text{if } \lambda < -1; \\
1 + z^{-\lambda - 2} & \text{if } \lambda > -1.
\end{cases}
\end{align}

Then, it is easy to check that, for any $m \geq 1$, we have $[z^{-\lambda - 1}]u^m = 0$ and $[z^{-\lambda - 1}](u^m v) = z^{-\lambda - 1}$. By statement (a) for the case $n = 1$, we have that, for any $m \geq 1$, $u^m \in \text{Im} \Phi_\lambda$ but $u^m v \notin \text{Im} \Phi_\lambda$.

Next, consider the case $\lambda = -1$. By (a), we know that Im $\Phi_{\lambda = -1}$ is the $\mathcal{A}$-subspace of $\mathcal{A}[z^{-1}, z]$ consisting of Laurent polynomials with no constant terms. In the case that $\mathcal{A} = \mathbb{C}$, (b) follows directly from Duistermaat and van der Kallen’s theorem, Theorem 1.3.

In the case that $\mathcal{A} \neq \mathbb{C}$, (b) also follows from Theorem 1.3 via Lefschetz’s principle since, whenever we fix $a(z), b(z) \in \mathcal{A}[z^{-1}, z]$ with $a^m(z)$ has no constant term for any $m \geq 1$, to show that $a^m b$ has no constant term when $m \gg 0$, we may replace $\mathcal{A}$ by the field $K$ generated by the (finitely many) coefficients of $a(z)$ and $b(z)$ over $\mathbb{Q}$, which can be embedded in $\mathbb{C}$ as a subfield.

Therefore, we have proved the proposition for the case $n = 1$. Now we assume $n \geq 2$.

For convenience, throughout the rest of the proof, we denote by $\mathcal{U}$ the subalgebra of $\mathcal{A}[z^{-1}, z]$ of Laurent polynomials in $z_i$ ($2 \leq i \leq n$) with coefficients in $\mathcal{A}$. Note that $\mathcal{A}[z^{-1}, z]$ may be viewed as the Laurent polynomial algebra in $z_1$ over $\mathcal{U}$, i.e. we have $\mathcal{A}[z^{-1}, z] = \mathcal{U}[z_1^{-1}, z_1]$.

First, assume that $\lambda \notin \mathbb{Z}$. Then there exists $1 \leq i \leq n$ such that $\lambda_i \notin \mathbb{Z}$. Without losing any generality, we assume $\lambda_1 \notin \mathbb{Z}$. By statement (a) for the one-variable case with $\mathcal{A}$ replaced by $\mathcal{U}$ and $\lambda$ by $\lambda_1$, we have

$$\Phi_{\lambda_1}(\mathcal{U}[z_1^{-1}, z_1]) = \mathcal{U}[z_1^{-1}, z_1] = \mathcal{A}[z^{-1}, z].$$

Since $\Phi_{\lambda_1}(\mathcal{U}[z_1^{-1}, z_1]) = \Phi_{\lambda_1}(\mathcal{A}[z^{-1}, z]) \subset \text{Im} \Phi_\lambda$, from the equation above, we have $\text{Im} \Phi_\lambda = \mathcal{A}[z^{-1}, z]$.

Now assume $\lambda \in \mathbb{Z}^n$. We first show that $z^\beta \in \text{Im} \Phi_\lambda$ for any $\beta \in \mathbb{Z}^n$ with $\beta \neq -\lambda - 1$.

Pick up any $\beta \neq -\lambda - 1 \in \mathbb{Z}^n$, there exists $1 \leq i \leq n$ such that the $i^{th}$ component $\beta_i$ of $\beta$ is different from $-\lambda_i - 1$. We assume $\beta_i \neq -\lambda_i - 1$
(the proof for other cases is similar). Then, by statement (a) for the one-variable case with \( A \) replaced by \( U \) and \( \lambda \) by \( \lambda_1 \), we have
\[
z^\beta = (z_1^{\beta_1} z_3^{\beta_3} \cdots z_n^{\beta_n}) z_1^{\beta_1} \in \Phi_{\lambda_1}(U[z_1^{-1}, z_1]) = \Phi_{\lambda_1}(A[z^{-1}, z]) \subset \text{Im} \Phi_\lambda.
\]

Consequently, we see that any \( g(z) \in A[z^{-1}, z] \) with \([z^{-\lambda-1}]g(z) = 0\) lies in \( \text{Im} \Phi_\lambda \). Conversely, for any \( g(z) \in \text{Im} \Phi_\lambda \), we write \( g(z) \) as
\[
g(z) = \sum_{i=1}^{n} \Phi_{\lambda_i} f_i(z)
\]
for some \( f_i(z) \in A[z^{-1}, z] \) (1 \( \leq i \leq n \)).

By statement (a) for the one-variable case with \( A \) replaced by \( U \) and \( \lambda \) by \( \lambda_1 \), we see that \( \Phi_{\lambda_1} f_1(z) \) in Eq. (2.26) has no \( z^\gamma \) term for any \( \gamma \in \mathbb{Z}^n \) with the first component \( \gamma_1 = -\lambda_1 - 1 \). Hence, for the similar reason, for any \( 2 \leq i \leq n \), \( \Phi_{\lambda_i} f_i(z) \) in Eq. (2.26) cannot have the \( z^\gamma \) term for any \( \gamma \in \mathbb{Z}^n \) with the \( i \)th component \( \gamma_i = -\lambda_i - 1 \). Therefore, by Eq. (2.26) we see that \( g(z) \) cannot have any \( z^{-\lambda-1} \) term.

Combining the results in the last two paragraphs, we have statement (a) of the proposition.

Next we show statement (b). First, if \( \lambda \not\in \mathbb{Z}^n \), by (a) we have \( \text{Im} \Phi_\lambda = A[z^{-1}, z] \) which is obviously a Mathieu subspace of \( A[z^{-1}, z] \).

Assume that \( \lambda \in \mathbb{Z}^n \) but \( \lambda \not\in (-1, -1, ..., -1) \). Since \( -\lambda - 1 \not\in 0 \in \mathbb{N}^n \), by statement (a), we have, 1 \( \in \text{Im} \Phi_\lambda \) and \( \text{Im} \Phi_\lambda \not\in A[z^{-1}, z] \). By Lemma 4.5 in Section 4 \( \text{Im} \Phi_\lambda \) is not a Mathieu subspace of \( A[z^{-1}, z] \).

Finally, consider the case \( \lambda = (-1, -1, ..., -1) \). Similarly as for the one variable case, by Duistermaat and van der Kallen’s theorem, Theorem 1.3, and Lefschetz’s principle, it is easy to see that \( \text{Im} \Phi_\lambda \) is a Mathieu subspace of \( A[z^{-1}, z] \) in this case. \( \square \)

**Remark 2.7.** From Proposition 2.6 we see that (the straightforward generalization of) the IC for the Laurent polynomial algebras does not always hold.

But, on the other hand, it is still interesting to see, for which commuting differential operators of \( \mathbb{C}[z^{-1}, z] \) or any localization of \( \mathbb{C}[z] \), the IC holds. For example, for the differential operators \( \Phi_\lambda \) with \( \lambda = (-1, -1, ..., -1) \), the IC is actually equivalent to Duistermaat and van der Kallen’s theorem, Theorem 1.3. Even more mysteriously, among all the cases that \( \text{Im} \Phi_\lambda \not\in A[z^{-1}, z] \), this is the only case that the IC holds.

Next let us consider the IC for the polynomial algebra \( A[z] \) (instead of \( A[z^{-1}, z] \)) and the differential operators \( \Phi_\lambda \) (even though \( A[z] \) is not
closed under the action of $\Phi_\lambda$). But, first, we need prove the following lemma.

**Lemma 2.8.** Let $\mathcal{A}$, $\Phi_\lambda$ ($\lambda \in \mathbb{C}^n$) and $\text{Im} \Phi_\lambda$ as in Proposition 2.6. Then, for any $g(z) \in \mathcal{A}[z]$, $g(z) \in \text{Im} \Phi_\lambda$ iff $g(z) \in \sum_{i=1}^n(\Phi_\lambda, \mathcal{A}[z])$.

**Proof:** Note that the ($\Leftarrow$) part of the lemma is trivial. We use induction on $n \geq 1$ to show the other part.

First, assume $n = 1$ and $g(z) \in \text{Im} \Phi_\lambda$. Then, by Proposition 2.6, we have $[z^{-\lambda_1-1}]g(z) = 0$. By writing $g(z)$ as a linear combination of monomials $z^k$ ($k \in \mathbb{N}$) over $\mathcal{A}$, it is easy to check that Eq. (2.27) has a polynomial solution $f(z) \in \mathcal{A}[z]$. Since $\Phi_\lambda f(z) = g(z)$ as shown in the proof of Proposition 2.6, the lemma holds in this case.

Next assume the lemma holds for the $n-1$ case and consider the $n$-variable case.

Note first that, for any $\lambda \in \mathbb{C}^n$ and $g(z) \in \mathcal{A}[z]$, there exist $u(z) \in \mathcal{A}[z_2, z_3, ..., z_n]$ and $v(z) \in \mathcal{A}[z]$ such that $[z^\gamma]v(z) = 0$ for any $\gamma \in \mathbb{N}^n$ with the first component $\gamma_1 = -\lambda_1 - 1$ and

$$g(z) = z_1^{-\lambda_1-1}u(z) + v(z).$$

Now further assume $g(z) \in \text{Im} \Phi_\lambda$. Then, by Proposition 2.6 (a), we have $[z^{-\lambda_1-1}]g(z) = 0$. We show below that both terms on the right hand side of Eq. (2.27) lie in $\sum_{i=1}^n(\Phi_\lambda, \mathcal{A}[z])$.

First, if $-\lambda_1 - 1 \not\in \mathbb{N}$, we have $u(z) = 0$. Otherwise, set $\mathcal{A}' := \mathcal{A}[z_1]$ and $z'' := (z_2, ..., z_n)$. We view $u(z)$ and also $z_1^{-\lambda_1-1}u(z)$ as polynomials in $z''$ over the integral domain $\mathcal{A}'$. Note that, the coefficient of the monomial $z_2^{-\lambda_2-1} \cdots z_n^{-\lambda_n-1}$ in $u(z) \in \mathcal{A}'[z'']$ is same as $[z^{-\lambda_1-1}]g(z)$ which is equal to zero. Hence the coefficient of $z_2^{-\lambda_2-1} \cdots z_n^{-\lambda_n-1}$ in $z_1^{-\lambda_1-1}u(z) \in \mathcal{A}'[z'']$ is also equal to zero.

Apply Proposition 2.6 (a) to $z_1^{-\lambda_1-1}u(z) \in \mathcal{A}'[z'']$ with $\mathcal{A}$ replaced by $\mathcal{A}'$, we know that $z_1^{-\lambda_1-1}u(z) \in \sum_{i=2}^n(\Phi_\lambda, \mathcal{A}'[z_2^\pm, ..., z_n^\pm])$. Then, by applying the induction assumption to $z_1^{-\lambda_1-1}u(z)$ with $\mathcal{A}$ replaced by $\mathcal{A}'$, we have, $z_1^{-\lambda_1-1}u(z) \in \sum_{i=2}^n(\Phi_\lambda, \mathcal{A}'[z_2, ..., z_n]) = \sum_{i=2}^n(\Phi_\lambda, \mathcal{A}[z])$.

Second, set $\mathcal{A}'' := \mathcal{A}[z_2, ..., z_n]$. Then, viewing $v(z)$ as a polynomial in $z_1$ over the integral domain $\mathcal{A}'$, we have $[z_1^{-\lambda_1-1}]v(z) = 0$. By Proposition 2.6 (a) with $\mathcal{A}$ replaced by $\mathcal{A}''$, $v(z) \in \Phi_\lambda(\mathcal{A}''[z_1^{-1}, z_1])$. Applying the lemma for the case $n=1$ to $v(z)$ with $\mathcal{A}$ replaced by $\mathcal{A}''$, we have $v(z) \in \Phi_\lambda(\mathcal{A}''[z_1]) = \Phi_\lambda(\mathcal{A}[z])$. Then, by Eq. (2.27), we have $g(z) \in \sum_{i=1}^n(\Phi_\lambda, \mathcal{A}[z])$, and the lemma follows.
Proposition 2.9. Let $A$ and $\Phi_\lambda$ ($\lambda \in \mathbb{C}^n$) as in Proposition 2.6. Set

\begin{equation}
\text{Im}'\Phi_\lambda := A[z] \cap \sum_{i=1}^{n}(\Phi_\lambda A[z]).
\end{equation}

Then, (a) $\text{Im}'\Phi_\lambda = A[z]$ iff $\lambda \notin (\mathbb{Z}^{<0})^n$, where $\mathbb{Z}^{<0}$ denotes the set of all negative integers.

(b) $\text{Im}'\Phi_\lambda$ is a Mathieu subspace of $A[z]$ iff $\lambda \notin (\mathbb{Z}^{<0})^n$ or $\lambda = (-1, -1, ..., -1)$.

Proof: Note first that, by Eq. (2.28) and Lemma 2.8, it is easy to see that

\begin{equation}
\text{Im}'\Phi_\lambda = A[z] \cap \text{Im} \Phi_\lambda.
\end{equation}

If $\lambda \notin \mathbb{Z}^n$, by Proposition 2.6 (a) and the equation above, we have that $\text{Im}'\Phi_\lambda = A[z]$.

Assume $\lambda \in \mathbb{Z}^n$ and set $\alpha := -\lambda - (1, 1, ..., 1)$. By Proposition 2.6 (a) and Eq. (2.29), we know that $\text{Im}'\Phi_\lambda$ is the $A$-subspace of polynomials with no $z^\alpha$ term.

If $\lambda \notin (\mathbb{Z}^{<0})^n$, i.e. $\lambda_i \geq 0$ for some $1 \leq i \leq n$, then we have $\alpha \notin \mathbb{N}^n$ and hence $\text{Im}'\Phi_\lambda = A[z]$.

Now assume $\lambda \in (\mathbb{Z}^{<0})^n$ but $\lambda \neq (-1, -1, ..., -1)$. Then there exists $1 \leq j \leq n$ such that $\lambda_j \leq -2$. Note that, in this case $\alpha \in \mathbb{N}^n$ but $\alpha \neq 0$. Consequently, we have, $1 \in \text{Im}'\Phi_\lambda$ and $\text{Im}'\Phi_\lambda \neq A[z]$. Then, by Lemma 4.5 in Section 4, $\text{Im}'\Phi_\lambda$ is not a Mathieu subspace of $A[z]$.

Finally, if $\lambda = (-1, -1, ..., -1)$, then $\alpha = 0$ and $\text{Im}'\Phi_\lambda$ is the ideal of all polynomials with no constant terms. So in this case $\text{Im}'\Phi_\lambda \neq A[z]$ but it is the ideal of $A[z]$ generated by $z_i$ ($1 \leq i \leq n$). Hence it is a Mathieu subspace of $A[z]$.

So we have exhausted all possible choices of $\lambda \in \mathbb{C}^n$. Combining all the results above, it is easy to see that both (a) and (b) of the proposition hold. □

As pointed out at the beginning of this subsection, up to some changes of variables, the differential operators $\Phi_\lambda$ are same as the differential operators $\Lambda_{\alpha,\beta}$ in Eq. (2.18) related with the multi-variable Jacobi orthogonal polynomials $P_{\alpha,\beta}^m (m \in \mathbb{N}^n)$ in Eq. (2.20) with $\alpha = \lambda$ and $\beta = 0$ or $\alpha = 0$ and $\beta = \lambda$.

Note that, the constraints on the parameters $\alpha$ and $\beta$ of the Jacobi polynomials are $\alpha, \beta \in \mathbb{R}^n$ and the components $\alpha_i, \beta_i > -1$ ($1 \leq i \leq n$). Now, if we put the same constraints on $\lambda$, then, by Proposition 2.9 it is easy to see that we have the following corollary.
Corollary 2.10. Let $A$ and $\Phi_\lambda$ as in Proposition 2.6. Assume further that $\lambda \in \mathbb{R}^n$ and $\lambda_i > -1$ for any $1 \leq i \leq n$. Then, $\text{Im}'\Phi_\lambda = A[z]$ and hence is a Mathieu subspace of $A[z]$.

3. Generalizations of the Image Conjecture

Motivated by the discussions in Subsection 2.2, we first formulate in Subsection 3.1 a generalization (See Conjecture 3.1) of the IC for the commuting differential operators related with orthogonal polynomials. We also show that, beside a trivial case, Conjecture 3.1 is actually equivalent to a special case of another conjecture, Conjecture 3.2, on integrals of polynomials over open subsets of $\mathbb{R}^n$ with any positive measures. As we will see that Conjecture 3.2 can also be viewed as a natural variation of the Mathieu conjecture, Conjecture 1.2 and Duistermaat and van der Kallen’s theorem, Theorem 1.3.

In Subsection 3.2 we will prove some cases of Conjectures 3.1 and 3.2. We will also discuss a connection of Conjecture 3.2 with the so-called polynomial moment problem.

3.1. The Generalized Image Conjecture. First, we propose the following generalization of the IC for the commuting differential operators related with classical orthogonal polynomials.

Conjecture 3.1. Let $B \subset \mathbb{R}^n$, $w(z)$ and $\{u_\alpha | \alpha \in \mathbb{N}^n\}$ as in Definition 2.7 with $x$ replaced by $z$. Assume further that the orthogonal polynomials $\{u_\alpha | \alpha \in \mathbb{N}^n\}$ can be obtained via Eq. (2.4) for some commuting differential operators $\Lambda = (\Lambda_1, \Lambda_2, ..., \Lambda_n)$ of a localization of the polynomial algebra $\mathbb{C}[z]$. Set

$$\text{Im}'\Lambda := \mathbb{C}[z] \cap \bigcap_{i=1}^n (\Lambda_i \mathbb{C}[z]).$$

Then, $\text{Im}'\Lambda$ is a Mathieu subspace of $\mathbb{C}[z]$.

Note first that, when $\Lambda$ are differential operators of $\mathbb{C}[z]$ (instead of a localization of $\mathbb{C}[z]$), we have, $\text{Im}'\Lambda = \text{Im}\Lambda$. Therefore, the conjecture above can be viewed as a generalization of the image conjecture, Conjecture 1.2, to the differential operators related with classical orthogonal polynomials.

Note also that, by Lemma 2.5 (c), $\text{Im}'\Lambda$ has co-dimension in $\mathbb{C}[z]$ zero or one depending on whether $u_0(z)$ lies in $\text{Im} \Lambda$ or not. Since $u_0(z)$ is a nonzero constant, we have $\text{Im}'\Lambda = \mathbb{C}[z]$ iff $1 \in \text{Im}'\Lambda$.

Therefore, if $1 \in \text{Im}'\Lambda$, we have $\text{Im}'\Lambda = \mathbb{C}[z]$, and hence Conjecture 3.1 holds trivially in this case. If $1 \notin \text{Im}'\Lambda$, we have

$$\text{Im}'\Lambda = \text{Span}_\mathbb{C}\{u_\alpha(z) | \alpha \neq 0\}. $$
In this case, Conjecture 3.1 turns out to be equivalent to a special case of the following conjecture.

**Conjecture 3.2.** Let $B$ be any non-empty open subset of $\mathbb{R}^n$ and $\sigma$ any positive measure such that $\int_B g(z) \, d\sigma$ is finite for any $g(z) \in \mathbb{C}[z]$. Let $M_B(\sigma)$ be the subspace of all polynomials $f(x) \in \mathbb{C}[z]$ such that $\int_B f(x) \, d\sigma = 0$. Then $M_B(\sigma)$ is a Mathieu subspace of $\mathbb{C}[z]$.

**Proposition 3.3.** Let $B \subset \mathbb{R}^n, w(z)$ and $\Lambda$ be as in Conjecture 3.1. Let $\sigma$ be the measure on $B$ such that $d\sigma = w(z) \, dz$. Assume further that $1 \not\in \text{Im}' \Lambda$. Then, we have

(a) $\text{Im}' \Lambda = M_B(\sigma)$.

(b) Conjecture 3.1 for the differential operators $\Lambda$ is equivalent to Conjecture 3.2 for the open subset $B \subset \mathbb{R}^n$ with the measure $\sigma$.

**Proof:** First it is easy to see that (b) follows directly from (a).

To show (a), choose any $f(z) \in \mathbb{C}[z]$ and write it (uniquely) as $f(z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha u_\alpha(z)$ with $c_\alpha \in \mathbb{C}$. Then, by Eq. (3.2) and the assumption that $1 \not\in \text{Im}' \Lambda$, we have that, $f(z) \in \text{Im}' \Lambda$ iff $c_0 = 0$.

On the other hand, since $\{u_\alpha \mid \alpha \in \mathbb{N}^n\}$ is an orthogonal basis of $\mathbb{C}[z]$ with respect to the Hermitian form defined in Definition 2.1, for any $\alpha \not= 0$, we have

$$\int_B u_\alpha(z) w(z) \, dz = \bar{u}_0^{-1} \int_B u_\alpha(z) \bar{u}_0 w(z) \, dz = 0. \quad (3.3)$$

Therefore, we have

$$\int_B f(z) \, d\sigma = \int_B f(z) w(z) \, dz = c_0 \int_B u_0(z) w(z) \, dz = c_0 u_0 \int_B w(z) \, dz.$$

Since $u_0$ and $\int w(z) \, dz$ are nonzero constants, we have that, $f(z) \in M_B(\sigma)$ iff $c_0 = 0$.

Combining the results above, we have $\text{Im}' \Lambda = M_B(\sigma)$ which is (a) of the proposition. $\square$

**Remark 3.4.** For the special case that $1 \in \text{Im}' \Lambda$, as pointed out above Conjecture 3.1 holds trivially. But in this case, Conjecture 3.2, even for the measures $d\sigma = w(z)\, dz$ given by the weight functions $w(z)$ of classical orthogonal polynomials, can still be highly non-trivial. See Subsection 3.2 for more discussions.

Several more remarks on Conjecture 3.2 are as follows.

First, Conjecture 3.2 can be viewed as a variation of the Mathieu conjecture, Conjecture 1.2, with the reductive Lie group $G$ replaced by an open subset $B \subset \mathbb{R}^n$; the Haar measure by any positive measure $\sigma$;
and $G$-finite functions by polynomials. Furthermore, by Eq. (3.2) and Proposition 3.3 (a), we see that Conjecture 3.2 with $d\sigma = w(z)dz$ can also be viewed as a natural variation of Duistermaat and van der Kallen’s theorem, Theorem 1.3, with the basis of $\mathbb{C}[z]$ formed by monomials $z^\alpha$ ($\alpha \in \mathbb{N}^n$) replaced by the basis formed by the orthogonal polynomials $u_\alpha(z)$ ($\alpha \in \mathbb{N}^n$). More precisely, Conjecture 3.2 with $d\sigma = w(z)dz$ can be re-stated as follows.

**Conjecture 3.5.** Let $\{u_\alpha \mid \alpha \in \mathbb{N}^n\}$ be a sequence of orthogonal polynomials over $B \subset \mathbb{R}^n$ with the weight function $w(z)$. Let $M$ be the subspace of $f(z) \in \mathbb{C}[z]$ whose constant term (the coefficient of $u_0$) in the unique expansion of $f(z)$ in terms of $u_\alpha(z)$’s is equal to zero. Then $M$ is a Mathieu subspace of the polynomial algebra $\mathbb{C}[z]$.

Of course, a shorter way to state the conjecture above is that the subspace $M$ spanned by the orthogonal polynomials $u_\alpha(z)$ ($\alpha \neq 0$) over $\mathbb{C}$ is a Mathieu subspace of the polynomial algebra $\mathbb{C}[z]$.

The second remark is that Conjecture 3.2 in general does not hold for analytic functions.

**Example 3.6.** Let $B = (0, 1) \subset \mathbb{R}$ and $d\sigma = dz$. Let $f(x) = e^{2\pi\sqrt{-1}z}$ and $g(z) = z$. Then, for any $m \geq 1$, we have

$$\int_0^1 f^m(x) \, d\sigma = \int_0^1 e^{2m\pi\sqrt{-1}z} \, dz = 0.$$But

$$\int_0^1 f^m(z)g(z) \, d\sigma = \int_0^1 ze^{2m\pi\sqrt{-1}z} \, dz$$

$$= \frac{1}{2m\pi\sqrt{-1}} \left( ze^{2m\pi\sqrt{-1}z} \bigg|_0^1 - \int_0^1 e^{2m\pi\sqrt{-1}z} \, dz \right)$$

$$= \frac{1}{2m\pi\sqrt{-1}} \neq 0.$$The third remark is that Conjecture 3.2 does not hold without the positivity assumption on the measure $\sigma$.

**Example 3.7.** Let $B = (-1, 1) \subset \mathbb{R}$ and $d\sigma = zdz$. Let $f(z) = z^2$ and $g(z) = z$. Then, for any $m \geq 1$, we have

$$\int_{-1}^1 f^m(z) \, d\sigma = \int_{-1}^1 z^{2m+1} \, dz = 0.$$But

$$\int_{-1}^1 f^m(z)g(z) \, d\sigma = \int_{-1}^1 z^{2m+2} \, dz = \frac{2}{2m+3} \neq 0.$$
Finally, one interesting observation about the example above is as follows. Even though Conjecture 3.2 fails for this example, if we consider the differential operator \( \Lambda \) related with the “weight” function \( w(z) = z \) as in Eq. (2.3), namely,

\[
\Lambda = \frac{d}{dz} + w^{-1}(z)\frac{dw(z)}{dz} = \frac{d}{dz} + z^{-1},
\]

then, by Corollary 2.10, we see that Conjecture 3.1 (formally) for the differential operator \( \Lambda \) above still holds.

3.2. Some Cases of Conjectures 3.1 and 3.2

Despite the simple appearances of Conjectures 3.1 and 3.2, there are only few cases that are known for these two conjectures.

First, for the differential operators related with the multi-variable Jacobi orthogonal polynomials (See Example 2.4), we have the following proposition.

**Proposition 3.8.** Let \( \Lambda_{\alpha,\beta} \) be the commuting differential operators defined in Eq. (2.18) (with \( x \) replaced by \( z \)) related with the Jacobi orthogonal polynomials. Assume further that there exists \( 1 \leq i \leq n \) such that \( \alpha_i = 0 \) or \( \beta_i = 0 \). Then \( \Im' \Lambda_{\alpha,\beta} = \mathbb{C}[z] \), and hence Conjecture 3.1 holds for \( \Lambda_{\alpha,\beta} \).

**Proof:** Without losing any generality, we may assume that \( \alpha_1 = 0 \) or \( \beta_1 = 0 \). Here we only prove the case that \( \beta_1 = 0 \). The proof of the case that \( \alpha_1 = 0 \) is similar.

Under the assumption above, the first component \( \Lambda_1 \) of \( \Lambda_{\alpha,\beta} \) in Eq. (2.18) is the differential operator \( \Lambda_1 = \partial_1 - \alpha_1(1 - z_1)^{-1} \). Now we apply the change of variables \( z_1 \to z_1 + 1 \) and \( z_i \to z_i \) for any \( 2 \leq i \leq n \), then \( \Lambda_1 \) becomes the differential operator \( \Phi_{\alpha_1} = \partial_1 + \alpha_1 z_1^{-1} \).

Let \( \mathcal{A} := \mathbb{C}[z_2, z_3, ..., z_n] \) and view \( \Phi_{\alpha_1} \) as a differential operator of \( \mathcal{A}[z_1^{-1}, z_1] \). Since \( \alpha_1 > -1 \), by applying Corollary 2.10 to \( \Phi_{\alpha_1} \), we have

\[
\mathcal{A}[z_1] \cap \Phi_{\alpha_1}(\mathcal{A}[z_1]) = \mathcal{A}[z_1].
\]

Since \( \mathcal{A}[z_1] = \mathbb{C}[z] \), we have

\[
\Im' \Phi_{\alpha_1} = \mathbb{C}[z] \cap \Phi_{\alpha_1}(\mathbb{C}[z]) = \mathbb{C}[z].
\]

Hence we also have \( \Im' \Lambda_1 = \mathbb{C}[z] \). Since \( \Im' \Lambda_1 \subset \Im' \Lambda_{\alpha,\beta} \), we have \( \Im' \Lambda_{\alpha,\beta} = \mathbb{C}[z] \).

Next let us consider the differential operators \( \Lambda_{\alpha} (\alpha \in (\mathbb{R}^{>-1})^n) \) related with the multi-variable Laguerre orthogonal polynomials. Note
that, by Eq. (2.6) and a similar construction for the multi-variable Jacobi polynomials in Example 2.4, we know that \( \Lambda_\alpha \) is given by

\[
\Lambda_\alpha = (\Lambda_{\alpha_1}, \Lambda_{\alpha_2}, \ldots, \Lambda_{\alpha_n}),
\]

where, for any \( 1 \leq i \leq n \),

\[
\Lambda_{\alpha_i} = \partial_i + \alpha_i z_i^{-1} - 1.
\]

**Proposition 3.9.** Let \( \Lambda_\alpha \) (\( \alpha \in (\mathbb{R}^{>0})^n \)) be the commuting differential operators defined above. Then, \( \text{Im}' \Lambda_\alpha = \mathbb{C}[z] \) iff \( \alpha_i = 0 \) for some \( 1 \leq i \leq n \). Hence, Conjectures 3.7 holds for \( \Lambda_\alpha \) under this condition.

**Proof:** \((\Leftarrow)\) Without losing any generality, we may assume that \( \alpha_1 = 0 \). Then, by Eq. (2.6), we have \( \Lambda_{\alpha_1=0} = \partial_1 - 1 \). Since \( \Lambda_{\alpha_1=0}(-1) = 1 \), we have \( 1 \in \text{Im}'(\Lambda_{\alpha_1=0}) \subset \text{Im}' \Lambda_\alpha \). Then, by Lemma 2.5 (c), we have \( \text{Im}' \Lambda_\alpha = \mathbb{C}[z] \).

This result can also be proved by the following more straightforward argument (without using Lemma 2.5). Note that \( \Lambda_{\alpha_1=0} = \frac{d}{dz_1} - 1 \) is invertible as a linear operator of \( \mathbb{C}[z] \). Its inverse operator is given by

\[
\Lambda_{\alpha_1=0}^{-1} = (\partial_1 - 1)^{-1} = -1 - \sum_{k=1}^{+\infty} \partial_1^k.
\]

Note that the infinity sum on the right hand side of the equation above is a well-defined linear map of \( \mathbb{C}[z] \).

Since \( \lambda_{\alpha_1=0} \) is invertible, we have \( \text{Im}' \Lambda_{\alpha_1=0} = \mathbb{C}[z] \). Hence we also have \( \text{Im}' \Lambda_\alpha = \mathbb{C}[z] \).

\((\Rightarrow)\) Assume that \( \text{Im}' \Lambda_\alpha = \mathbb{C}[z] \) but \( \alpha_i \neq 0 \) for any \( 1 \leq i \leq n \). In particular, we have \( 1 \in \text{Im}' \Lambda_\alpha \). So there exist \( h_i(z) \in \mathbb{C}[z] \) \((1 \leq i \leq n)\) such that

\[
1 = \sum_{i=1}^{n} \Lambda_{\alpha_i} h_i(z) = \sum_{i=1}^{n} (\partial_i + \alpha_i z_i^{-1} - 1) h_i(z).
\]

Now we view the RHS of Eq. (3.6) above as a Laurent polynomial in \( z_1 \) with coefficients in \( \mathbb{C}[z_2^\pm, \ldots, z_n^\pm] \). Then the coefficient of \( z_1^{-1} \) of the RHS is given by \( \alpha_1 h_1(z) \big|_{z_1=0} \) which, by Eq. (3.6), must be zero. Hence we also have \( h_1(z) \big|_{z_1=0} = 0 \) since \( \alpha_1 \neq 0 \). Therefore, \( h_1(z) = z_1 \tilde{h}_1(z) \) for some \( \tilde{h}_1(z) \in \mathbb{C}[z] \). Apply similar arguments to \( h_i(z) \) for \( 2 \leq i \leq n \), we see that, for any \( 1 \leq i \leq n \), there exists \( \tilde{h}_i(z) \in \mathbb{C}[z] \) such that

\[
h_i(z) = z_i \tilde{h}_i(z).
\]

Next, for any fixed \( 1 \leq i \leq n \) and \( f(z) \in \mathbb{C}[z] \), it is easy to check that

\[
\partial_i (z^\alpha f(z)e^{-\sum_{i=1}^{n} z_i}) = z^\alpha (\Lambda_{\alpha_i} f(z))e^{-\sum_{i=1}^{n} z_i}.
\]
Consequently, by Eq. (3.7) we have
\begin{equation}
\int_{0}^{+\infty} (\Lambda; h_i) z^\alpha e^{-\sum_{i=1}^{n} z_i} \, dz_i = \int_{0}^{+\infty} \partial_i (z^\alpha h_i e^{-\sum_{i=1}^{n} z_i}) \, dz_i
\end{equation}
\begin{equation}
= (z^\alpha h_i (z) e^{-\sum_{i=1}^{n} z_i}) \bigg|_{z_i=+\infty}^{z_i=0} = (z^\alpha \tilde{h_i} (z) e^{-\sum_{i=1}^{n} z_i}) \bigg|_{z_i=+\infty}^{z_i=0}
\end{equation}
\begin{equation}
= 0,
\end{equation}
where the last equality above follows from the fact that \( \alpha_i + 1 > 0 \) since \( \alpha_i > -1 \).
Combining Eqs. (3.6)–(3.9), we have
\begin{equation}
\int (\mathbb{R} > 0) \times \sum_{i=1}^{n} (\Lambda; h_i) z^\alpha e^{-\sum_{i=1}^{n} z_i} \, dz
\end{equation}
\begin{equation}
= \sum_{i=1}^{n} \int (\mathbb{R} > 0) \times (n-1) \left( \int_{0}^{+\infty} (\Lambda; h_i) z^\alpha e^{-\sum_{i=1}^{n} z_i} \, dz_i \right) \, dz_1 \cdots \tilde{dz_i} \cdots dz_{n}
\end{equation}
\begin{equation}
= 0.
\end{equation}
But this is a contradiction since \( z^\alpha e^{-\sum_{i=1}^{n} z_i} \) is continuous and positive everywhere over \( (\mathbb{R} > 0) \times \).

Next, motivated by the differential operators related with the multi-variable Hermite polynomials and also the differential operators \( \Phi_\lambda (\lambda \in \mathbb{C}^n) \) in Subsection 2.2, we consider the following family of commuting differential operators.
For any \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n \), we set
\begin{equation}
\Psi_{\alpha_i} := \partial_i + \alpha_i z_i.
\end{equation}
and
\begin{equation}
\Psi_\alpha := (\Psi_{\alpha_1}, \Psi_{\alpha_2}, \ldots, \Psi_{\alpha_n}).
\end{equation}
Note that, when \( \alpha_i = -2 \) for any \( 1 \leq i \leq n \), the differential operators \( \Psi_\alpha \) becomes the differential operators related with the multi-variable Hermite polynomials. This can be seen from Eq. (2.5) and a similar construction in Example 2.4 for the multi-variable Jacobi polynomials.

Note also that the differential operators \( \Psi_\alpha (\alpha \in \mathbb{C}^n) \) are actually the differential operators of the polynomial algebra \( \mathbb{C}[z] \) itself. So in this case, we have \( \text{Im} \Psi_\alpha = \text{Im} \Psi_\alpha \) for any \( \alpha \in \mathbb{C}^n \).

**Proposition 3.10.** Let \( \Psi_\alpha (\alpha \in \mathbb{C}^n) \) be defined above. Then, \( \text{Im} \Psi_\alpha = \mathbb{C}[z] \) iff \( \alpha_i = 0 \) for some \( 1 \leq i \leq n \). Hence Conjectures 1.4 and 3.1 hold under this condition.

**Proof:** \( \left( \Leftarrow \right) \) Assume that \( \alpha_i = 0 \) for some \( 1 \leq i \leq n \). Then, by Eq. (3.11), we have \( \Lambda_{i=0} = \partial_i \) which is obviously a surjective linear
map from $\mathbb{C}[z]$ to $\mathbb{C}[z]$. Hence we have $\text{Im} \Psi_{\alpha_1=0} = \mathbb{C}[z]$ and $\text{Im} \Psi_{\alpha} = \mathbb{C}[z]$.

$(\Rightarrow)$ Assume that $\text{Im} \Psi_{\alpha} = \mathbb{C}[z]$ but $\alpha_i \neq 0$ for any $1 \leq i \leq n$. We derive a contradiction as follows.

First, by applying the change of variables $z_i \to \sqrt{-2/\alpha_i} z_i$ ($1 \leq i \leq n$), the differential operators $\Psi_{\alpha_i}$ in Eq. (3.11) becomes $\sqrt{-\alpha_i/2}(\partial_i - 2z_i)$. Set $\Lambda_i := \partial_i - 2z_i$ and $\Lambda = (\Lambda_1, \Lambda_2, ..., \Lambda_n)$. From the argument above, it is easy to see that $\text{Im} \Lambda = \text{Im} \Psi_{\alpha} = \mathbb{C}[z]$.

Let $h_i(z) \in \mathbb{C}[z]$ (1 $\leq$ i $\leq$ n) such that

$$1 = \sum_{i=1}^{n} \Lambda_i h_i.$$  

(3.13)

Note that, for any fixed $1 \leq i \leq n$, it is easy to check that

$$\partial_i(h_i e^{-\sum_{i=1}^{n} z_i^2}) = (\Lambda_i h_i) e^{-\sum_{i=1}^{n} z_i^2}.$$  

(3.14)

Consequently, we have

$$\int_{\mathbb{R}} (\Lambda_i h_i) e^{-\sum_{i=1}^{n} z_i^2} dz_i = \int_{\mathbb{R}} \partial_i(h_i e^{-\sum_{i=1}^{n} z_i^2}) dz_i$$

$$= (h_i(z) e^{-\sum_{i=1}^{n} z_i^2}) \bigg|_{z_i=-\infty}^{z_i=+\infty} = 0.$$  

(3.15)

Then, by using Eqs. (3.13)–(3.15) and applying a similar argument as in Eq. (3.10), we have

$$\int_{\mathbb{R}^n} e^{-\sum_{i=1}^{n} z_i^2} dz = \sum_{i=1}^{n} \int_{\mathbb{R}} (\Lambda_i h_i) e^{-\sum_{i=1}^{n} z_i^2} dz = 0,$$

which is a contradiction since $e^{-\sum_{i=1}^{n} z_i^2}$ is continuous and positive everywhere on $\mathbb{R}^n$. $\Box$

Now let us consider some cases of Conjecture 3.2.

**Proposition 3.11.** Conjecture 3.2 holds for any open $B \subset \mathbb{R}^n$ with any atomic measure $\sigma$ which is supported at finitely many points of $B$.

**Proof:** Let $S = \{u_1, u_2, ..., u_k\} \subset B$ be the support of $\sigma$, i.e. $\sigma(u_i) > 0$ ($1 \leq i \leq k$) and, for any measurable subset $U \subset B$, we have

$$\sigma(U) = \sum_{u \in S \cap U} \sigma(u).$$

Note first that, for any $f(z) \in \mathbb{C}[z], f(z) \in \mathcal{M}_B(\sigma)$ iff

$$\int_B f(z) d\sigma = \sum_{i=1}^{k} f(u_i)\sigma(u_i) = 0.$$  

(3.16)
Therefore, for any $f(z) \in \mathbb{C}[z]$ with $f^m(z) \in M_B(\sigma)$ for any $m \geq 1$, we have

\begin{equation}
\int_B f^m(z) d\sigma = \sum_{i=1}^k f^m(u_i)\sigma(u_i) = 0
\end{equation}

for any $m \geq 1$

If $f(u_i) = 0$ for all $1 \leq i \leq k$, then, for any $g(z) \in \mathbb{C}[z]$ and $m \geq 1$, we also have $(f^mg)(u_i) = 0$ for any $1 \leq i \leq k$. Hence, for any $m \geq 1$, $f^mg$ also satisfies Eq. (3.16) and lies in $M_B(\sigma)$. Therefore, Conjecture 3.2 holds for $f(z)$.

Assume $f(u_i)$ ($1 \leq i \leq k$) are not all zero. Let $\{c_1, c_2, \ldots, c_s\}$ be the set of all distinct nonzero values of $f(z)$ attained over $S$. For any $1 \leq j \leq s$, let $S_j$ be the subset of elements of $u \in S$ such that $f(u) = c_j$ and $k_j$ the cardinal number of $S_j$. Then, Eq. (3.17) can be re-written as

\begin{equation}
0 = \sum_{i=1}^k f^m(u_i)\sigma(u_i) = \sum_{j=1}^s c_j^m \sum_{u \in S_j} \sigma(u),
\end{equation}

Since the equation above holds for any $m \geq 1$, by using the invertibility of the Vandermonde matrices, it is easy to check that

\begin{equation}
0 = \sum_{u \in S_j} \sigma(u_j).
\end{equation}

for any $1 \leq j \leq s$.

But this is a contradiction since $\sigma(u_i) > 0$ for any $1 \leq i \leq k$. □

Next, let us consider Conjecture 3.2 for the Jacobi orthogonal polynomials. Contrast to Conjecture 3.1 for the Jacobi polynomials (See Proposition 3.8), the only case of Conjecture 3.2 that we know is the case of the one-variable Jacobi polynomials with $\alpha = \beta = 0$. In this case the weight function $w(z) \equiv 1$. Note that, by Remark 2.3 (b), the Jacobi polynomials in this case are actually the Legendre polynomials.

\textbf{Proposition 3.12.} Let $a, b \in \mathbb{R}$ with $a > b$ and $d\sigma = dz$. Then Conjecture 3.2 holds for open interval $B := (a, b) \subset \mathbb{R}$ with the Lebesgue measure $\sigma$.

The proposition above follows directly from the following theorem.

\textbf{Theorem 3.13.} Let $a < b \in \mathbb{R}$ and $f(z) \in \mathbb{C}[z]$. Assume that, there exists $N > 0$ such that $\int_a^b f^m(z) dz = 0$ for any $m \geq N$. Then $f(z) = 0$.

It seems that the theorem above is known but we failed to find any published proof in the literature. We did notice that Madhav V. Nori
\( \mathbb{N} \) has studied the problem above in a much more general setting. It is very possible that Theorem 3.13 will follow from some results obtained in \( \mathbb{N} \).

Jean-Philippe Furter [FZ] informed the author that he and his colleague Changgui Zhang have got an analytic proof for Theorem 3.13, which is under preparation. Mitya Boyarchenko [B] also sent the author a sketch of his brilliant but unpublished proof. Surprisingly, Boyarchenko’s proof is purely algebraic and uses only some results from algebraic number theory such as Dirichlet’s theorem on arithmetic progressions, etc.

Next, we end this section with a connection of the one-variable case of Conjecture 3.2 with the so-called polynomial moment problem proposed by M. Briskin, J.-P. Francoise and Y. Yomdin in the series of papers [BFY1]-[BFY5]. The polynomial moment problem is mainly motivated by the center problem for the complex Abel equation. The problem was recently solved by F. Pakovich and M. Muzychuk [PM] (See the theorem below). For more details on the polynomial moment problem, see the references quoted above and also citations therein. The author is very grateful to Harm Derksen, Jean-Philippe Furter, Jeffrey C. Lagarias, Leonid Makar-Limanov, Lucy Moser-Jauslin for communications and suggestions on this connection, and also to Fedor Pakovich for communications on his joint work [PM] with Mikhail Muzychuk.

Recall that the polynomial moment problem is the following problem: given any polynomial \( f(z) \in \mathbb{C}[z] \) and \( a \neq b \in \mathbb{C} \), find all polynomials \( q(z) \in \mathbb{C}[z] \) such that, for any \( m \geq 0 \),

\[
\int_a^b f^m(z)q(z)dz = 0. \tag{3.20}
\]

The problem above was solved recently by the following theorem obtained by F. Pakovich and M. Muzychuk [PM].

**Theorem 3.14. (Pakovich and Muzychuk)** Let \( a \neq b \in \mathbb{C} \) and \( f(z) \in \mathbb{C}[z] \). A non-zero polynomial \( q(z) \in \mathbb{C}[z] \) satisfies Eq. (3.20) for any \( m \geq 0 \) iff there exist some polynomials \( Q_j(z), f_j(z) \) and \( W_j(z) \) (\( j \in J \)) such that

1. \( W_j(a) = W_j(b) \) for any \( j \in J \);
2. \( q(z) = \sum_{j \in J} Q_j(W_j(z)) W_j'(z) \);
3. \( f(z) = f_j(W_j(z)) \) for any \( j \in J \).

Note that, with the same notation as in Pakovich and Muzychuk’s theorem above, if we choose \( a < b \), \( B = (a, b) \subset \mathbb{R} \) and the measure \( d\sigma = q(z)dz \) (ignoring the positivity requirement on the measure \( \sigma \) for
a moment), then Pakovich and Muzychuk’s theorem above gives all polynomials $f(z) \in \mathbb{C}[z]$ such that $\int_B f^m(z) \, d\sigma = 0$ for any $m \geq 0$.

But, unfortunately, Pakovich and Muzychuk’s theorem requires the integral in Eq. (3.20) vanish when $m = 0$, i.e. $\int_B d\sigma = \int_B q(z) \, dz = 0$. From this requirement, it is easy to see that $d\sigma = q(z) \, dz$ can not be a positive measure on the interval $B = (a, b)$. For example, $q(z)$ can not be any nonzero polynomial with real coefficients such that $q(c) \geq 0$ for any $c \in B$. Therefore, Pakovich and Muzychuk’s theorem can not be applied directly to approach Conjecture 3.2.

Nevertheless, it is still very interesting to see if some of the techniques (instead of the main theorem) in [PM] can somehow be applied to study Conjecture 3.2 for the cases when $q(z)$ are polynomials or analytic functions in one variable which are non-negative over some open intervals of the real line.

On the other hand, we see that Conjecture 3.2 also raises a new question on the polynomial moment problem, namely, what is the solution of the polynomial moment problem with the (slightly weaker) condition that the integrals in Eq. (3.20) vanish for any $m \geq 1$ but not necessarily for $m = 0$? We believe this question is also very interesting to investigate.

4. Some General Results on Mathieu Subspaces

Note that the Mathieu conjecture (Conjecture 1.2), the IC (Conjecture 1.4) and its generalizations (Conjectures 3.1 and 3.2) discussed in the previous sections are all about whether or not certain subspaces are Mathieu subspaces of their ambient commutative algebras. Furthermore, this is also the case for the well-known Jacobian conjecture and the Dixmier conjecture through their equivalences (See [Z3] for the discussions on these equivalences) to some special cases of the IC. Therefore, it is necessary and important to study Mathieu subspaces separately in a more general setting.

In this section, we give some examples of Mathieu subspaces from other sources and derive some general results on this newly introduced notion.

First, let us generalize the notion of Mathieu subspaces defined in Definition 1.1 to associative but not necessarily commutative algebras.

**Definition 4.1.** Let $A$ be an associative algebra over a commutative ring $R$ and $M$ a $R$-subspace of $A$. We say that $M$ is a left Mathieu subspace of $A$ if the following property holds: for any $a, b \in A$ with $a^m \in M$ for any $m \geq 1$, there exists $N \geq 1$ (depending on $a$ and $b$) such that $ba^m \in M$ for any $m \geq N$. 
We define right Mathieu subspaces and also (two-sided) Mathieu subspaces in the obvious ways.

It is easy to see that any left ideal is automatically a left Mathieu subspace. Similarly, this is also the case for right and two-sided ideals. Therefore, the notion of Mathieu subspaces can be viewed as a generalization of the notion of ideals even for noncommutative algebras. But, as we will see from examples to be discussed in this section, many Mathieu subspaces are not ideals. Actually they are not even closed under the product of the ambient algebras.

We start with the following noncommutative examples of Mathieu subspaces.

**Example 4.2.** For any \( n \geq 1 \) and integral domain \( R \) of characteristic zero, let \( A = M_{n \times n}(R) \) be the algebra of \( n \times n \) matrices with entries in \( R \) and \( M \) the subspace of trace-zero matrices.

Note that, for any \( A \in A \), \( A^m \in M \) for any \( m \geq 1 \) iff \( A \) is nilpotent. Then, for any \( B \in A \), we have \( BA^m = A^mB = 0 \in M \) for any \( m \geq n \). Therefore, \( M \) is a two-sided Mathieu subspace of \( A \) but certainly cannot be an ideal of \( A \) unless \( n = 1 \).

Next, from additive valuations on polynomial algebras, we can get the following family of Mathieu subspaces of polynomial algebras.

**Example 4.3.** For any \( n \geq 1 \) and any integral domain \( R \), let \( A = R[z] \) be the polynomial algebra over \( R \) in \( n \) variables \( z \). For any linear functional \( \nu : \mathbb{R}^n \to \mathbb{R} \), we define an additive valuation \( \text{ord}_\nu : A \to \mathbb{R} \cup \{+\infty\} \) by setting, \( \text{ord}_\nu(0) := +\infty \) and, for any \( 0 \neq f(z) \in A \),

\[
\text{ord}_\nu(f) := \min\{\nu(\alpha) \mid \text{the coefficient of } z^\alpha \text{ in } f(z) \text{ is not zero}\}.
\]

For any \( c \in \mathbb{R} \), let \( M_c \) be the subspace of polynomials \( f \in A \) such that \( \text{ord}_\nu(f) \geq c \).

Then it is easy to check that, for any \( c > 0 \), \( M_c \) is a Mathieu subspace of \( A \) but not necessarily an ideal of \( A \) if \( \text{ord}_\nu(f) < 0 \) for some \( f \in A \).

More generally, we have the following family of examples from commutative rings (viewed as \( \mathbb{Z} \)-algebras) with valuations.

**Example 4.4.** Let \( A \) be any commutative ring and \( \nu \) a real-valued (additive) valuation \((\text{[AM]}, \text{Mat})\) of \( A \), i.e. \( \mu : A \to \mathbb{R} \cup \{+\infty\} \) such that, for any \( x, y \in A \), we have

\[
\nu(x) = +\infty \text{ iff } x = 0, \tag{4.2}
\]
\[
\nu(xy) = \nu(x) + \nu(y), \tag{4.3}
\]
\[
\nu(x + y) \geq \min\{\nu(x), \nu(y)\}. \tag{4.4}
\]
For any $c \in \mathbb{R}$, let $M_c$ be the subspace of elements $x$ of $A$ with $\nu(x) \geq c$. Then it is easy to check that, for any $c > 0$, $M_c$ is a Mathieu subspace of $A$ but not necessarily an ideal of $A$.

Similar as for ideals, we say a Mathieu subspace $M$ of an algebra $A$ is proper if $M \neq 0$ and $M \neq A$. From the example above, we see that some fields may actually have some proper Mathieu subspaces. Also, unlike proper ideals, proper Mathieu subspaces may contain some invertible elements. But, as we will see below, proper left or right Mathieu subspaces can not contain the identity element of the ambient algebras.

**Lemma 4.5.** Let $A$ be any algebra and $M$ a proper left or right Mathieu subspace of $A$. Then, $1 \notin M$.

**Proof:** Assume otherwise. Then, for any $m \geq 1$, $1^m = 1 \in M$. Then, for any $b \in A$, $b = 1^m b = b1^m \in M$ when $m \gg 0$. Hence, we have $M = A$ which is a contradiction. \(\square\)

The next proposition will give us more examples of Mathieu subspaces of polynomial algebras, which are not necessarily ideals.

**Proposition 4.6.** Let $K$ be a field of any characteristic. For any finite subset $S = \{u_1, u_2, ..., u_k\} \subset K^n$ and $\sigma = \{a_1, a_2, ..., a_k\} \subset K^\times$, denote by $M(\sigma)$ the subspace of polynomials $f(z) \in K[z]$ such that

$$\sum_{i=1}^{k} a_if(u_i) = 0. \tag{4.5}$$

Then, $M(\sigma)$ is a Mathieu subspace of $K[z]$ iff, for any non-empty subset $S' \subset S$,

$$\sum_{u \in S'} u \neq 0. \tag{4.6}$$

**Proof:** First, the ($\Leftarrow$) part of the proposition can be proved by similar arguments as in the proof of Proposition 3.11. So we skip it here.

To show the ($\Rightarrow$) part, assume that there exists a non-empty $S' \subset S$ such that Eq. (4.6) holds.

Let $f(z) \in K[z]$ such that

$$f(u_i) = \begin{cases} 1 & \text{if } u_i \in S' \\ 0 & \text{if } u_i \notin S' \end{cases} \tag{4.7}$$
Note that, such a polynomial \( f(z) \) always exists. For example, when \( n = 1 \) (the idea for \( n > 1 \) is similar), we may choose

\[
f(z) = \sum_{u \in S'} \frac{\prod_{c \in S, c \neq u} (z - c)}{\prod_{c \in S, c \neq u} (u - c)}
\]

For any \( m \geq 1 \), we have

\[
\sum_{i=1}^{k} a_i f^m(u_i) = \sum_{i \in S'} a_i = 0.
\]

Therefore, we have that \( f^m(z) \in M(\sigma) \) for any \( m \geq 1 \).

Now fix an \( u_j \in S' \) and \( g(z) \in K[z] \) such that \( g(u_j) = 1 \) and \( g(u_i) = 0 \) for any \( 1 \leq i \neq j \leq k \). Then, for any \( m \geq 1 \), we have

\[
\sum_{i=1}^{k} a_i (f^m g)(u_i) = \sum_{i=1}^{k} a_i f^m(u_i) g(u_i) = a_j g(u_j) = a_j \neq 0.
\]

Therefore, \( f^m g \notin M(\sigma) \) for any \( m \geq 1 \). Hence \( M(\sigma) \) is not a Mathieu subspace of \( K[z] \). \( \square \)

**Remark 4.7.** Note that, when \( k = 1 \), \( M(\sigma) \) in Proposition 3.11 is always an ideal of \( \mathbb{C}[z] \). But if \( k > 1 \), \( M(\sigma) \) in general is not an ideal. For example, for any \( k \geq 2 \), choose \( K = \mathbb{C} \), \( a_i = 1 \) and any distinct \( u_i \in \mathbb{C} \) (\( 1 \leq i \leq k \)).

For Laurent polynomial algebras \( \mathbb{C}[z^{-1}, z] \), by Duistermaat and van der Kallen’s theorem, Theorem 1.3, we see that the subspace of Laurent polynomials with no constant terms is a Mathieu subspace of \( \mathbb{C}[z^{-1}, z] \). Another example of Mathieu subspaces of \( \mathbb{C}[z^{-1}, z] \) is given by the following theorem which was first conjectured in [Z2] and later was proved in [EWZ].

**Theorem 4.8.** Let \( \mathcal{M} \) be the subspace of \( \mathbb{C}[z^{-1}, z] \) of Laurent polynomials \( f(z) \) with no holomorphic part, i.e. \( [z^\alpha]f(z) = 0 \) for any \( \alpha \in \mathbb{N}^n \). Then, \( \mathcal{M} \) is a Mathieu subspace of \( \mathbb{C}[z^{-1}, z] \).

Next we show that some properties of ideals are also shared by Mathieu subspaces.

**Proposition 4.9.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be any algebras over a commutative ring \( R \), and \( \varphi : \mathcal{A} \to \mathcal{B} \) an \( R \)-algebra homomorphism. Then,

(a) for any left Mathieu subspaces \( \mathcal{M}_i \) (\( 1 \leq i \leq m \)) of \( \mathcal{A} \), \( \cap_{1 \leq i \leq m} \mathcal{M}_i \) is also a left Mathieu subspace of \( \mathcal{A} \).

(b) for any left Mathieu subspace \( \mathcal{N} \) of \( \mathcal{B} \), \( \varphi^{-1}(\mathcal{N}) \) is also a left Mathieu subspace of \( \mathcal{A} \).
Proof: (a) Set $M := \cap_{1 \leq i \leq m} M_i$. It is easy to see that $M$ is also a $R$-subspace of $A$.

Let $a, b \in A$ with $a^m \in M$ for any $m \geq 1$. For any $1 \leq i \leq m$, let $N_i \in \mathbb{N}$ such that $ba^m \in M_i$ for any $m \geq N_i$. Let $N := \max\{N_i | 1 \leq i \leq m\}$. Then, for any $m \geq N$, we have $ba^m \in M_i$ for any $1 \leq i \leq m$. Hence $ba^m \in M$ for any $m \geq N$.

(b) Again, it is easy to see that $\varphi^{-1}(N)$ is a $R$-subspace of $A$ since $\varphi$ is an $R$-algebra homomorphism.

Let $a, b \in A$ with $a^m \in \varphi^{-1}(N)$ for any $m \geq 1$. Set $x := \varphi(a)$ and $y := \varphi(b)$. Then, for any $m \geq 1$, we have $x^m = \varphi^m(a) = \varphi(a^m) \in N$. Since $N$ is a left Mathieu subspace of $B$, there exists $N \in \mathbb{N}$ such that $yx^m \in N$ for any $m \geq N$. But $yx^m = \varphi(b)\varphi^m(a) = \varphi(ba^m)$, so we have $ba^m \in \varphi^{-1}(N)$ for any $m \geq N$. Therefore, $\varphi^{-1}(N)$ is a left Mathieu subspace of $A$. □

Remark 4.10. From the proof above, it is easy to see that Proposition 4.9 also holds for right or two-sided Mathieu subspaces.

Proposition 4.11. Let $K$ be any field with uncountably many elements and $A$ a commutative $K$-algebra. Let $z = (z_1, z_2, ..., z_n)$ be $n$ free commutative variables. Then, for any Mathieu subspace $M$ of $A$, $M[z] \subset A[z]$ is a Mathieu subspace of $A[z]$.

Proof: Note first that, by using induction on the number of free variables $z$, it will be enough to show the proposition for the one-variable case. So we assume $n = 1$.

We use the contradiction method. Assume the proposition is false. Then there exist $f(z), g(z) \in K[z]$ with $f^m \in M[z]$ for any $m \geq 1$ but $f^k(z)g(z) \notin M[z]$ for infinitely many positive integers $k$. Let $\{m_i \in \mathbb{N}\}$ be a strictly increasing sequence of positive integers such that $f^{m_i}g \notin M[z]$ for any $i \geq 1$.

Note that, for any $h(z) \in A[z]$ of degree $d := \deg h(z) \geq 0$ and $h(z) \notin M[z]$, by using the invertibility of the Vandermonde matrices, it is easy to check that there are at most $d$ distinct elements $c \in K^\times$ such that $h(c) \notin M$. Otherwise, all the coefficients of $h(z)$ would be in $M$ and $h(z) \in M[z]$.

Therefore, for each fixed $m_i$, there are only finitely many $c \in K$ such that $f^{m_i}(c)g(c) \in M$. Since $K$ has uncountably many distinct elements, there exists $b \in K$ such that $f^{m_i}(b)g(b) \notin M$ for all $i \geq 1$.

But, on the other hand, since $f^{m}(z) \in M[z]$ for any $m \geq 1$, all the coefficients of $f^m(z)$ are in $M$. Since $M$ is a $K$-subspace of $A$, we have $f^m(b) \in M$ for any $m \geq 1$. Furthermore, since $M$ is a Mathieu subspace of $A$ and $f(b)^m = f^m(b) \in M$ for any $m \geq 1$, we have $f^m(b)g(b) \in M$.
when \( m \gg 0 \). In particular, \( f^{m\iota}(b)g(b) \in \mathcal{M} \) when \( i \gg 0 \). Hence we get a contradiction. \( \square \)

Finally, one remark on the sums of Mathieu subspaces is as follows. Note that, by Proposition 4.9 (a), the intersection of any finitely many Mathieu subspaces is always a Mathieu subspace. Naturally, one may wonder if the sum of any finitely many Mathieu subspaces is also a Mathieu subspace. But this is not true in general.

**Example 4.12.** Let \( A = \mathbb{C}[z] \) in one variable \( z \). Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be the one-dimensional subspace spaces of \( \mathbb{C}[z] \) spanned by \( 1 + z \) and \( 1 - z \), respectively. Then, it is easy to check that both \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are Mathieu subspaces of \( \mathbb{C}[z] \) and \( \mathcal{M} := \mathcal{M}_1 + \mathcal{M}_2 = \mathbb{C} \cdot 1 + \mathbb{C} \cdot z \). But, since \( 1 \in \mathcal{M} \) and \( \mathcal{M} \neq \mathbb{C}[z] \), by Lemma 4.9, \( \mathcal{M} \) is not a Mathieu subspace of \( \mathbb{C}[z] \).

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