Supersymmetry for disordered systems with interaction

G. Schwiete\(^1\) and K. B. Efetov\(^{1,2}\)
\(^1\) Theoretische Physik III, Ruhr-Universität Bochum, 44780 Bochum, Germany
\(^2\) L. D. Landau Institute for Theoretical Physics, 117940 Moscow, Russia

(Dated: March 23, 2022)

Considering disordered electron systems we suggest a scheme that allows us to include an electron-electron interaction into a supermatrix \(\sigma\)-model. The method is based on replacing the initial model of interacting electrons by a fully supersymmetric model. Although this replacement is not exact, it is a good approximation for a weak short range interaction and arbitrary disorder. The replacement makes the averaging over disorder and further manipulations straightforward and we come to a supermatrix \(\sigma\)-model containing an interaction term. The structure of the model is rather similar to the replica one, although the interaction term has a different form. We study the model making perturbation theory and renormalization group calculations. We check the renormalizability of the model in the first loop approximation and in the first order in the interaction. In this limit we reproduce the renormalization group equations known from earlier works. We hope that the new supermatrix \(\sigma\)-model may become a new tool for nonperturbative calculations for disordered systems with interaction.

PACS numbers: 71.23.An, 72.10.Bg

I. INTRODUCTION

Nonlinear \(\sigma\)-models serve as an important tool for the theoretical description of disordered and chaotic systems. These models efficiently describe physics at large distances and times, which is the most important limit in the theory of localization, mesoscopic fluctuations, etc. The main idea of the approach is to integrate out fast “electronic degrees of freedom” and to reduce calculations to solving a model containing only slow “diffusion modes”. This is the way how nonlinear \(\sigma\)-models are derived. The initial model of Ref.\(^4\) was not completely satisfactory because it contained formally divergent integrals. In order to avoid such integrals one should either shift contours of integration in a complicated way or start with a representation of the Green functions in terms of functional integrals over anticommuting (Grassmann) variables\(^5\).

Although these replica \(\sigma\)-models proved to be very convenient for writing perturbation expansions in diffusion modes and renormalization group equations, it became clear soon that they were not very useful in nonperturbative calculations. As an attempt to improve the method, a supersymmetric version of the field theory was formulated by Efetov\(^6\). The supersymmetry method combines both the integration over commuting and anticommuting variables and there is no need to use the replica trick. It turned out that the supersymmetry method was very efficient for studying various problems of disorder and chaos (for a review, see, e.g., Ref.\(^6\)).

The next step was to include electron-electron interactions and this was done by Finkel'stein\(^7\) who generalized the \(\sigma\)-model of Ref.\(^3\) to interacting systems using the replica approach and obtained a set of renormalization group equations. It is interesting to mention that, for electron systems, the interaction can only be included using the representation of Green functions in terms of integrals over the anticommuting variables suggested in Ref.\(^3\). The alternative representation of Refs.\(^8,9\) may work for interacting bosons only. Needless to say that the \(\sigma\)-model for interacting systems is considerably more complicated than that suggested for noninteracting ones.

One more possibility to derive a \(\sigma\)-model for disordered systems is based on the Keldysh technique\(^7\). As both the replica and supersymmetry approaches, the Keldysh formalism enables one to average over the disorder in the beginning of all calculations, which is the main step when deriving the \(\sigma\)-models. The special form of the Keldysh time contour automatically ensures that the weight denominator in the representation of the Green functions via functional integrals is equal to unity, which is the basis for performing the averaging over the disorder. The Keldysh formalism for the disorder problems was first suggested in Ref.\(^7\) and later used in Ref.\(^7\) for derivation of a \(\sigma\)-model for non-interacting systems. More recently, \(\sigma\)-models based on the Keldysh formalism and the fermionic representation of Ref.\(^7\) were derived for interacting systems\(^10,11\). These models, as well as the previous ones, are applicable in the diffusive regime.

Although all the \(\sigma\)-models listed above are very similar as far as their structure and methods for perturbative expansions are concerned, they are quite different with respect to nonperturbative calculations. The simplest case where this difference can be traced is the problem of the level statistics in a disordered metal grain. In the \(\sigma\)-model language the level statistics problem is equivalent to solving a zero-dimensional \(\sigma\)-model. For the supersymmetric \(\sigma\)-model, one can reduce the calculation of the level-level correlation function to the computation of a definite integral over 2 or 3 variables (depending on the symmetries of the model). The latter can be calcu-
lated without big difficulties and one comes to the famous Wigner-Dyson statistics (see, e.g., Ref. 14). The method of calculations based on supersymmetry can also be used for calculations for random matrices.

The situation is quite different when trying to approach these problems with either replica or Keldysh $\sigma$-models. The first published observation of difficulties encountered when calculating a proper integral over $N \times N$ matrices ($N$ is the replica number must be put to zero at the end) was presented in Ref. 14. Computation with the Keldysh $\sigma$-model is not easier because one has time (or energy) as an additional variable and integrals that appear for the zero-dimensional $\sigma$-model are functional integrals rather than definite integrals over several variables. Recently, several attempts have been undertaken to improve this situation and to develop methods of nonperturbative calculations for noninteracting replica- and Keldysh $\sigma$-models. Taking into account nontrivial saddle points the authors of these works succeeded in reproducing oscillating asymptotics of the Wigner-Dyson formulas. The latest achievement in this direction is apparently the publication 15 where the level-level correlation function was explicitly derived for the unitary ensemble using the replica approach.

In spite of the rather complicated calculations and limited success, the attempts to perform nonperturbative calculations with the replica and Keldysh $\sigma$-models were motivated by a probable extension to interacting systems. Incorporation of interaction into the supersymmetry scheme has not been considered and was generally believed to be impossible. This opinion is, of course, not groundless. As we have mentioned, in the presence of interaction, it becomes crucial whether one considers fermions or bosons. The one particle picture that allows one to consider the fermionic and bosonic representations on equal footing is applicable for noninteracting particles only. In the supersymmetry method one uses integrations over both fermionic and bosonic variables to describe, e.g., a system of electrons and it is completely unclear how one can apply this method for interacting particles.

There is no canonical way to incorporate interaction into a supersymmetric model. In particular, it is hard to see how the normalization of the partition function to unity could be achieved by adding bosonic variables while keeping the full information encoded in the theory. In reality, we cannot include an arbitrary interaction into the supersymmetric $\sigma$-model exactly. Therefore our goal is more modest and we restrict our consideration to weak interactions.

Instead of trying to include interaction into the supersymmetry scheme we propose an artificial model with interaction that is fully supersymmetric and allows the standard treatment within the supersymmetry scheme (the supersymmetry is violated only by source terms that allow us to calculate different correlation functions). We justify the usefulness of this model by comparing it with the initial electron model and find that, for a weak short range interaction and an arbitrary disorder, the models are close to each other. The comparison is performed by writing diagrams perturbative in the interaction but exact in disorder. It is shown that on the level of a generalized Hartree-Fock-approximation in the interaction (but exact in disorder) the supersymmetric model we suggest and the initial electron model are equivalent to each other.

Having demonstrated this equivalence we derive the $\sigma$-model using integration over $4M$-component supervec-tors ($M$ is the number of Matsubara frequencies, spin is included). Subsequently a supersymmetric $\sigma$-model is derived for systems with unitary symmetry using the standard scheme of derivation based on the averaging of the disorder, integration over supermatrices $Q$ and calculating the integrals in the saddle-point approximation, which fixes the eigenvalues of $Q$. In the limit of vanishing interaction the resulting model reduces to a form similar to the noninteracting $\sigma$-model, the main difference being the size of the supermatrices that carry two additional discrete frequency indices. For noninteracting systems the supersymmetry technique proved to be a very powerful tool for the analysis of nonperturbative effects related to disorder. The interesting question arises how a weak interaction influences the physics in this regime. The present model is intended to provide a starting point for tackling problems of this kind.

Before addressing new problems, one should make sure that well-known perturbative results of the theory of interacting electrons can be reproduced with the supersymmetric $\sigma$-model derived. We address this question in the present paper by studying the renormalization group procedure introduced by Finkel'stein 6 in the framework of the replica $\sigma$-model. Of course, as our model is written for a weak interaction, we cannot make a comparison for an arbitrary strength of the interaction. Therefore, we restrict ourselves to writing renormalization group equations in the first order in interaction. Under this restriction we are able to follow the scheme of the renormalization of Ref. 6 and demonstrate the renormalizability of the model. As a result, we obtain renormalization group equations that agree with those written in Ref. 6 in the limit of small effective interaction amplitudes.

This paper is organized as follows. Introducing the partition function of an interacting electron gas with disorder in the Matsubara technique we first consider the derivation of a $\sigma$-model for the theory without interaction (Sec. III). Here we point out some technical differences to the standard approach to this problem. Then we turn to the interaction part in Sec. III motivating the introduction of the supersymmetric model that is then studied in the remainder of the paper. First we present a diagrammatic analysis (exact in disorder), by which the model may be compared to the exact theory and the range of applicability can be assessed. Then, a supersymmetric $\sigma$-model is derived in Sec. IV. Contraction rules useful for perturbation theory are given in Sec. V and we consider the density-density correlation function as well
as Altshuler-Aronov corrections to conductivity. On these examples we show how one can perform perturbative calculations for the model derived. We demonstrate that in certain cases one can include also higher order interaction effects. Next we derive the renormalization group equations at lowest order in the interaction amplitudes (Sec. VII) and also obtain the Altshuler-Aronov corrections to the density of states.

II. SUPERSYMMETRIC MODEL WITHOUT INTERACTION

The conventional supermatrix $\sigma$-model for noninteracting particles is written for two different frequencies. The interaction makes all Matsubara frequencies important and we adjust the supersymmetry formalism to this case. In order to understand the scheme better we consider first the noninteracting case. Of course, the final results of calculations with the effective model derived here must be the same as those obtained in Ref. within the conventional technique but there are some peculiarities in the intermediate steps.

The partition function of a disordered interacting fermion gas may be written as a functional integral in the imaginary-time formalism

$$Z = \int D(\chi^*, \chi) e^{-S}.$$  \hspace{1cm} (2.1)

Here the integration is over Grassmann fields $\chi_{\sigma}(r, \tau)$ and $\chi_{\sigma}^* (r, \tau)$. The variables $\chi$ and $\chi^*$ depend on the position $r$, imaginary time $\tau$ and the spin index $\sigma$ and obey the conditions

$$\chi_{\sigma}(r, \tau) = -\chi_{\sigma}(r, \tau + \beta), \quad \chi_{\sigma}^*(r, \tau) = -\chi_{\sigma}^*(r, \tau + \beta).$$

$\beta = T^{-1}$ is the inverse temperature. These constraints should be respected when introducing the Fourier transform and we write

$$\chi(r, n) = \sqrt{T} \int_0^\beta d\tau e^{i\omega_n \tau} \chi(r, \tau),$$

where $\omega_n = (2n + 1)\pi T$ are fermionic Matsubara frequencies.

We decompose the action $S$ into three parts,

$$S = S_f + S_{dis} + S_{int}.$$

The first term $S_f$ describes free electrons with a chemical potential $\mu$,

$$S_f = \int_0^\beta d\tau dr \chi_{\sigma}^*(r, \tau) \left( \partial_\tau + \frac{\hat{p}_r^2}{2m} - \mu \right) \chi_{\sigma}(r, \tau),$$

where $m$ is the electron mass and summation over the spin index is implied.

The disorder part $S_{dis}$ describes the coupling to a static disorder potential $U(r)$

$$S_{dis} = \int_0^\beta d\tau dr \chi_{\sigma}^*(r, \tau) U(r) \chi_{\sigma}(r, \tau).$$

When deriving the $\sigma$-model we will average over this potential assuming that it is Gaussian and $\delta$-correlated

$$\langle U(r) \rangle = 0, \quad \langle U(r) U(r') \rangle = \frac{1}{2\pi T \nu} \delta(r - r'),$$

where $\nu$ is the bare density of states at the Fermi surface and $\tau$ is the elastic scattering time.

By $S_0$ we denote the non-interacting part of the action

$$S_0 = S_f + S_{dis}.$$  \hspace{1cm} (2.3)

The interaction part $S_{int}$ is chosen in a conventional form

$$S_{int} = \frac{1}{2} \int_0^\beta d\tau dr dr' \chi_{\sigma}^*(r, \tau) \chi_{\sigma}^*(r', \tau) \times V_0(r - r') \chi_{\sigma}(r', \tau) \chi_{\sigma}(r, \tau).$$  \hspace{1cm} (2.4)

where $V_0(r - r')$ is the bare potential of the electron-electron interaction and the subscripts $\alpha, \beta$ stand for spin components.

In order to calculate correlation functions one should add to the action $S$ sources of a proper symmetry.

The aim of this section is to generalize the supersymmetry technique to all Matsubara frequencies and express the partition function in terms of a functional integral over supervectors. In order to arrive at a supersymmetric form we should add bosonic variables. Some caution is necessary to ensure convergence of the resulting Gaussian integrals, which is very important for consideration of nonperturbative effects. To this end we introduce new fermionic variables

$$\chi_{\sigma}^* \rightarrow i\chi_{\sigma}^* \text{sgn} (\omega_n),$$  \hspace{1cm} (2.5)

so that now $S_0$ takes the form

$$S_0 = -i \sum_n \int dr \chi_{\sigma}^*(r, n) \text{sgn} (\omega_n)$$  \hspace{1cm} (2.6)

$$\times \left[ -i\omega_n + \frac{\hat{p}_r^2}{2m} - \mu + U(r) \right] \chi_{\sigma}(r, n).$$

To simplify the notation and to arrive at a familiar structure we arrange the fermionic fields $\chi(r, n)$ with different Matsubara indices $n$ into vectors $\chi(r)$ in the following way

$$\chi(r) = \begin{pmatrix} \vdots \\ \chi(r, -2) \\ \chi(r, -1) \\ \chi(r, 0) \\ \chi(r, 1) \\ \chi(r, 2) \\ \vdots \end{pmatrix}. \hspace{1cm} (2.7)$$

With the help of

$$\Lambda_{nk} = \text{sgn} (-\omega_n) \delta_{nk} \hspace{1cm} (2.8)$$
the conjugate vector is defined as $\tilde{\chi}(r) = \chi^\dagger(r)\Lambda$, $\tilde{\chi}(r, n) = \chi^\dagger(r, n)\text{sgn}(\omega_n)$. We also use the frequency matrix $E_{nk} = \omega_k\delta_{nk}$.

With these notations we rewrite the action $S_0$ in the form

$$S_0 = -i \int dr \bar{\chi}_\alpha(r) \left[ iE + \hat{\xi}_p + U(r) \right] \chi_\alpha(r), \quad (2.9)$$

where $\hat{\xi}_p = \frac{p^2}{m} - \mu$.

Of course, when writing integrals over the anticommuting variables $\chi$ a change of variables does not bring anything new. However, it is quite important for introducing supervectors $\psi$ because one gets convergent integrals over bosonic variables. We write these supervectors as follows

$$\psi(r) = \begin{pmatrix} \psi(r, -2) \\
\psi(r, -1) \\
\psi(r, 0) \\
\psi(r, 1) \\
\vdots \end{pmatrix}, \quad \psi(r, n) = \begin{pmatrix} \chi(r, n) \\
S(r, n) \end{pmatrix},$$

where $S(r, n), S^\dagger(r, n)$ are bosonic variables. Now we replace

$$\chi \rightarrow \psi, \quad \tilde{\chi} \rightarrow \tilde{\psi} = \psi^\dagger\Lambda$$

in $S_0$ to find the supersymmetric action $S'_0$

$$S'_0 = -i \int dr \tilde{\psi}_\alpha(r) \left[ iE + \hat{\xi}_p + U(r) \right] \psi_\alpha(r). \quad (2.10)$$

With this preparation a $\sigma$-model can be constructed for the noninteracting theory following standard considerations as described, for example, in Ref. In Ref., for simplicity, we consider the ensemble with the unitary symmetry, assuming that a small magnetic field suppresses Cooperon modes. Then, it is not necessary to further double the size of the supervectors. As a consequence, after the disorder averaging the resulting quartic term may be decoupled with the help of a $4M \times 4M$ ($M$ is the number of Matsubara frequencies) supermatrix fields $Q(r)$ with two spin and two Matsubara indices

$$e^{-\frac{T}{2\pi\nu}} \int dr \left( \tilde{\psi}(r)\psi(r) \right)^2 \quad (2.11)$$

$$= \int DQ e^{-\frac{T}{2\pi\nu}} \int dr \tilde{\psi}(r)Q(r)\psi(r) - \frac{T}{2\pi\nu} \int dr \text{Str}Q^2(r).$$

For the definition of the supertrace operation “Str” we refer to Ref. In contrast to the orthogonal case no special caution is necessary for the decoupling, since Cooperon modes are frozen. The saddle point equation reads

$$Q(r) = \frac{1}{\pi\nu} \left( \psi(r)\tilde{\psi}(r) \right) = \frac{1}{\pi\nu} g(r, r), \quad (2.12)$$

$$\left[ iE + \hat{\xi}_p + \frac{1}{2\tau}Q(r) \right] g(r, r') = i\delta(r - r'), \quad (2.13)$$

and is solved by $Q = \Lambda$. In complete analogy to the standard case one derives the $\sigma$-model

$$Z = \int DQ e^{-F} \quad (2.14)$$

$$F = \frac{\pi\nu}{4} \int dr \text{Str}[D(\nabla Q)^2 - 4EQ]. \quad (2.15)$$

The supermatrix field $Q(r)$ obeys the nonlinearity constraint $Q^2(r) = 1$ and further $Q^4(r) = KQ(r)K$. $D = \psi^2/\tau/d$, where $d$ is the dimension, is the classical diffusion coefficient. Here and in the following we adopt the convention that all internal indices that are not displayed explicitly are summed over as part of the supertrace operation. Repeated indices are also summed over. The validity of the model is restricted to the diffusive regime, momenta are restricted by the condition $ql \ll 1$ and temperature by $T\tau \ll 1$.

III. INTERACTION PART

In the previous derivation of the $\sigma$-model for noninteracting particles containing all Matsubara frequencies we closely followed a corresponding part of the paper. Now we are to include the interaction. Unfortunately, unlike the fermionic replica approach of Ref., the supersymmetry does not allow to include the interaction in a simple way and one cannot further follow the scheme of Ref.

Instead of trying to derive a $\sigma$-model from the original electron model with interaction we follow another root, namely, we replace the original electron model with interaction by an artificial supersymmetric one. There are two requirements for this new model: it should not be very different from the original one and it should be treatable within the supersymmetry scheme. We have succeeded in constructing such a model that gives the same correlation functions as the original one in the limit of a weak interaction. Due to its supersymmetric form the derivation of the $\sigma$-model can be performed using the standard scheme.

We start our discussion writing the original interaction term $S_{\text{int}}$, Eq. 2.4, in a slightly different form using the transformation, Eq. 2.5, and the vectors $\chi, \tilde{\chi}$, Eq. 2.6,

$$S_{\text{int}} = -\frac{T}{2} \int dr dr' \left( \tilde{\chi}^\alpha(r)\Delta^j\chi^\alpha(r') \right) \times V_0(r - r') \left( \tilde{\chi}^\beta(r')\Delta^{-j}\chi^\beta(r') \right), \quad (3.1)$$

where the frequency matrix $\Delta^j$ was introduced,

$$\left( \Delta^j \right)_{nm} = \delta_{j,n-m},$$

and $\alpha$ and $\beta$ stand for spin components.

The integrand in the interaction term, Eq. 3.1, contains a product of two “scalar products” of the vectors $\chi$. Using the anticommutation relations for the Grassmann variables $\chi$ we can also rewrite Eq. 3.1 in an equivalent
We have understood that for the noninteracting part $S$ that just replacing the
vectors $\chi$ by the supervectors $\psi$, we could simply substitute the vectors
$\chi$ by the supervectors $\psi$ in Eq. (2.10). It is clear
that just replacing the vectors $\chi$ by the supervectors $\psi$ in
Eq. (3.1) cannot be the resolution of the problem, since in
this way one takes into account “Hartree” diagrams in
the first order but cannot obtain the “Hartree” ones. In
contrast, replacing the vectors $\chi$ by the supervectors $\psi$ in
Eq. (3.2) one can reproduce the “Hartree” type diagrams
but the “Fock” diagrams are zero.

Actually, this property is the key to the construction
of the new supersymmetric model. In order to take into
account both “Hartree” and “Fock” type diagrams we
write the new interaction term $S'_{int}$ in the following form

$$S'_{int} = -\frac{T}{2} \int d\vec{r} d\vec{r}' \left[ \left( \psi^\alpha (\vec{r}) \Delta^j \psi^\beta (\vec{r}') \right) V_0 (\vec{r} - \vec{r}') \left( \bar{\psi}^\beta (\vec{r}') \Delta^{-j} \psi^\alpha (\vec{r}) \right) \right]$$

The structure of the two terms in the integrand in Eq.
(3.3) repeats the structure of Eqs. (3.1) and (3.2). However,
due to the supervector structure of $\psi$ these terms are not
equal to each other, which contrast the case with the
anticommuting vectors $\chi$. Moreover, if $\psi$ contained
only bosonic variables these two terms would cancel each
other completely.

Equations (2.10) and (3.3) give the action

$$S' = S'_0 + S'_{int}$$

of the new supersymmetric model we want to use. The
introduction of this model is the main step toward the
derivation of the supermatrix $\sigma$-model.

Before starting the derivation of the $\sigma$-model it is very
important to understand which important features of the
original model are kept in the supersymmetric one and
what is lost. Of course, the partition function $Z'$

$$Z' = \int \exp (-S') D\psi$$

of the supersymmetric model is exactly equal to unity
for any parameters of the Hamiltonian, which is a conse-
quence of the supersymmetry. Therefore, a comparison
of the partition functions of the models does not make
sense. At the same time, the partition function of the
initial model does not contain any singularities in any
dimension and this is not a quantity we want to study.

Interesting properties of the model can be seen in such
quantities as conductivity, tunnelling density of states,
etc. These quantities are expressed in terms of one-
and two-particle Green functions. So, we compare the
initial electron model with the supersymmetric one by
writing diagrams describing lowest orders of the pertur-
bation theory in interaction for Green functions. When
discussing one particle Green functions we compare the
self-energies of the Green functions, whereas vertices can
be calculated for two particle correlation functions. The
following correlation functions are most interesting for
us:

The imaginary time Green function,

$$\mathcal{G}_{\sigma \sigma'} (\vec{r}, \tau, \tau') = -\langle \chi_{\sigma} (\vec{r}, \tau) \chi_{\sigma'}^* (\vec{r}', \tau') \rangle_S$$

and calculated with the supersymmetric model

$$\langle \ldots \rangle_S' = \int D(\bar{\psi}, \psi) (\ldots) e^{-S'}.$$  (3.10)

The correlation function $\Pi(\vec{r}, \tau, \tau')$.

$$\Pi(\vec{r}, \tau, \tau') = \langle \rho (\vec{r}, \tau) \rho (\vec{r}', \tau') \rangle$$

$$\rho (\vec{r}, \tau) = \sum_\sigma \chi^\sigma_\sigma (\vec{r}, \tau)$$

$$\langle \ldots \rangle$$

By $\langle \ldots \rangle$ we denote the connected part of the averages,
$$\langle AB \rangle = \langle AB \rangle + \langle A \rangle \langle B \rangle.$$ We rewrite this correlation function in the form

$$\Pi(\vec{r}, \tau, \tau') = T^2 \sum_{kl} \Pi(\vec{r}, \Omega_k, \Omega_l) e^{-i\Omega_k \tau + i\Omega_l \tau'},$$

$$\Pi(\vec{r}, \tau, \Omega_k, \Omega_l) = -\langle \bar{\chi} (\vec{r}) \Delta^{-k} \chi (\vec{r}) \bar{\psi} (\vec{r}') \Delta^l \psi (\vec{r}') \rangle_S'.$$  (3.14)

The average $\langle \ldots \rangle_S'$ was defined in Eq. (3.10) above.

Equation (3.3) gives the one particle Green function for
the supersymmetric model, while Eq. (3.11) introduces
the density-density correlation function. In both cases
the action $S'$ is given by Eqs. (2.10) and (3.3). The
presence of the variables $\chi$ in Eqs. (3.10) and (3.11)
breaks the supersymmetry and allows us to have nonzero
results. Expanding the exponentials in $S'_0$ and calculating
the functional integrals over $\psi$ in Eqs. (3.3) and
(3.13) we obtain contributions that should be compared with
proper terms of the perturbation theory for the initial
model. Performing this calculation it is convenient to use
a diagrammatic language, which will be introduced now.

We have three building blocks:
An interaction line corresponding to the second term potential. All of them vanish due to supersymmetry.

An interaction line corresponding to the first term in Eq. (3.3) ("Fock interaction") is denoted by a single dotted line:

An interaction line corresponding to the second term in Eq. (3.3) ("Hartree interaction") is denoted by a double dashed line:

Of course, both types of the interaction introduced in (b) and (c) correspond to the bare interaction $V_0(r - r')$. The difference between them is related to how they are connected to the superfield $\psi$.

We can check easily doing the perturbation theory that the partition function $Z'$ is normalized to unity. This property will be used in the next section when performing the disorder averaging as for the noninteracting case. The normalization is obvious for the noninteracting part. On the other hand, for a perturbative expansion in $\delta$, since the resulting expressions contain products of terms of the form $\text{Str}(G^n) = 0$, where here $G$ denotes the average $\langle \psi \bar{\psi} \rangle$ taken with $S'_0$. Diagrammatically, in the first order in the interaction one finds four diagrams depicted in Fig. 1. Figures (1b) and (1d) can be written as a single supertrace, Figs. (1a) and (1c) as the product of two supertraces. Closed loops in the diagrams correspond to supertraces, if we agree that each line of the interaction (c) may be used to close a loop. Now we see why it is convenient to use the single and double lines for the interaction. With this convention all the four diagrams in Fig. 1 contain closed "superparticle" lines and vanish due to the supersymmetry. According to our convention, Figs. (1b) and (1d) contain one loop, while Figs. (1a) and (1c) contain two loops.

With this intuition we can discuss interaction corrections to Green functions depicted in Fig. 2. The first [Fig. (2a)] and fourth [Fig. (2d)] diagrams vanish, while the second [Fig. (2b)] and third [Fig. (2c)] give the Fock and Hartree contributions. Note however, that screening of the interaction lines inside these diagrams is not correctly reproduced. While the first [Fig. (3a)] and second [Fig. (3b)] diagrams vanish, the third [Fig. (3c)] diagram gives twice the desired contribution. The reason is a simple combinatorial factor related to the fact that we work with two interaction terms.

This shows that our replacement of the initial electron model by the supersymmetric one is not exact and its accuracy is restricted by the lowest order in the interaction for the self-energy. The screening of the interaction lines in the self-energy is described by higher orders in the interaction and is not given correctly by the supersymmetric model.

At the same time, the density-density correlation function $\Pi$, Eq. (3.14), can also be correctly described by the supersymmetric model. In addition to the corrections to the self-energy of the Green functions the density-density correlation function contains vertex corrections represented in Fig. 4. In order to write correctly this function in terms of the integral over the supervectors $\psi$ one should have both supersymmetric and nonsupersymmetric vertices, which corresponds to the definition, Eq. (3.14). Again, using our two types of interaction lines we can check that the ladder diagrams for the density-density correlation function are reproduced correctly. Diagrams containing closed supersymmetric loops (the second and the fourth diagrams in the first line and the first diagram in the second line) are equal to zero but the remaining diagrams (e.g., the first and the third diagram in the first line of Fig. 4 and the second diagram of the second line) do the job properly. One can easily extend this to higher order ladder diagrams.

Thus, we see that the replacement of the initial electron
model by the supersymmetric one allows us to reproduce reasonably the main order in the interaction. Let us emphasize that the consideration of the perturbation theory in the interaction presented in this section was performed using Green functions for a given disorder potential. No averaging over the disorder or making approximations with respect to it was necessary.

At the same time, our aim is to derive a $\sigma$-model, which implies an averaging over the disorder. The form of the supersymmetric $\sigma$-model allows us to perform the averaging over the disorder in exactly the same way as it has been done for the noninteracting case. The scheme of the derivation is similar to the one used by Finkelstein and we can simply follow his procedure.

**IV. $\sigma$-MODEL WITH INTERACTION**

Now we construct a low energy theory for the model introduced in the preceding section. After the disorder averaging and the decoupling of the quartic term by integration over the supermatrix field $Q(r)$ in complete analogy with what has been done for the noninteracting case one finds

$$Z' = \int D(\psi, \bar{\psi}) DQ \exp \left( -\frac{\nu}{4\tau} \int d\tau \text{Str} \, Q^2(r) \right) \times \exp \left( i \int d\tau \, \bar{\psi}(r) \left[ iE + \hat{\xi}_p + \frac{\nu}{2\tau} Q(r) \right] \psi(r) \right) \times \exp \left( -S_{\text{int}}'[\psi, \bar{\psi}] \right).$$

(4.1)

where the interaction term $S_{\text{int}}'$ is given by Eq. [3].

As in Ref. [2], we should single out in the quartic interaction $S_{\text{int}}'[\psi, \bar{\psi}]$ pairs slowly varying in space and time. However, in contrast to the replica approach, our supersymmetric interaction, Eq. [3], contains two terms. Hence, after singling out the slow pairs we obtain four terms instead of the two terms in Ref. [2]. At the same time, one obtains as usual only two interaction amplitudes $\Gamma_1$ and $\Gamma_2$ describing the small and large angle scattering. These amplitudes can be obtained by the replacement $V_0(\mathbf{q}) \to \Gamma_1/\nu$ and $V_0(p_1 - p_2 - \mathbf{q}) \to \Gamma_2/\nu$ for small momenta $\mathbf{q}$. The quantities $\Gamma_1$ and $\Gamma_2$ allow to make contact with the Fermi liquid theory. In particular, the singlet and triplet amplitudes $\Gamma_s = \Gamma_1 - \frac{1}{2} \Gamma_2$ and $\Gamma_t = \frac{1}{2} \Gamma_2$ are related to the Fermi-liquid constants $F^0_s$ and $F^0_t$.

$$\Gamma_s = \frac{1}{2} \frac{F^0_s}{1 + F^0_s}, \quad \Gamma_t = -\frac{1}{2} \frac{F^0_s}{1 + F^0_s}. \quad (\text{We remind, however, that we may work only in the limit of small } \Gamma_1 \text{ and } \Gamma_2).$$

When integrating over the fields $\psi, \bar{\psi}$ we consider the simplest possible approximation replacing $\exp(-S_{\text{int}}')$ by $\exp(-\langle S_{\text{int}}' \rangle)$ for the interaction term, where averaging is with respect to the quadratic form in $\psi$ in Eq. [3]. Using the relation $\langle \psi(\mathbf{r}) \bar{\psi}(\mathbf{r}') \rangle = g(\mathbf{r}, \mathbf{r}')$, where $g(\mathbf{r}, \mathbf{r}')$ is the Green function corresponding to the quadratic in $\psi$ form of Eq. [3], we get

$$\langle S_{\text{int}}' \rangle = -\frac{T}{2} \int (dp_1)(dp_2)(dq) \text{Str} \left[ \Delta^j g^{\alpha\alpha}(p_1 - q, p_1) \right] V_0(q) \text{Str} \left[ \Delta^{-j} g^{\beta\beta}(p_2 + q, p_2) \right]$$

$$+ \frac{T}{2} \int (dp_1)(dp_2)(dq) \text{Str} \left[ \Delta^j g^{\beta\alpha}(p_1 - q, p_1) \Delta^{-j} g^{\alpha\beta}(p_2 + q, p_2) \right] V_0(p_1 - p_2 - q)$$

$$+ \frac{T}{2} \int (dp_1)(dp_2)(dq) \text{Str} \left[ \Delta^j g^{\beta\alpha}(p_1 - q, p_1) \right] V_0(p_1 - p_2 - q) \text{Str} \left[ \Delta^{-j} g^{\alpha\beta}(p_2 + q, p_2) \right]$$

$$- \frac{T}{2} \int (dp_1)(dp_2)(dq) \text{Str} \left[ \Delta^j g^{\alpha\alpha}(p_1 - q, p_1) \Delta^{-j} g^{\beta\beta}(p_2 + q, p_2) \right] V_0(q). \quad (4.2)$$

From the momentum flow in the interaction potential one reads off processes with small and large angle scattering, which should be described by amplitudes $\Gamma_1$ and $\Gamma_2$, respectively. The singling out of the slow modes corresponds to a restriction of small momenta $\mathbf{q}$ and frequencies. Then, calculating the integral over $Q$ we may use the saddle point approximation that gives for the saddle point $g(\mathbf{r}, \mathbf{r}) = \pi \nu Q(\mathbf{r})$. Then, the effective free energy
can be written as
\[ F_{\text{int}} = \langle S_{\text{int}}^2 \rangle \]
\[ = -\frac{(\pi \nu)^2 T}{2} \int dr \Gamma_1 \left\{ \text{Str}[\Delta^0 Q^\alpha_\alpha] \text{Str}[\Delta^{-\delta} Q^\beta_\beta] + \text{Str}[\Delta^0 Q^\alpha_\alpha \Delta^{-\delta} Q^\beta_\beta] \right\} \]
\[ + \frac{(\pi \nu)^2 T}{2} \int dr \Gamma_2 \left\{ \text{Str}[\Delta^0 Q^\beta_\beta] \text{Str}[\Delta^{-\delta} Q^\alpha_\alpha] + \text{Str}[\Delta^0 Q^\beta_\beta \Delta^{-\delta} Q^\alpha_\alpha] \right\}. \quad (4.3) \]

Working with the saddle point of the noninteracting theory is a frequently used approximation in the context of \( \sigma \)-models for interacting disordered systems. The derivation for the remaining parts of the free energy can be performed exactly in the same way as in the replica approach. As a result, we come to the free energy functional \( F[Q] \) describing the supermatrix \( \sigma \)-model,
\[ F[Q] = \frac{\pi \nu}{4} \int d\sigma \left\{ D(\nabla Q)^2 - 4EQ \right\} \]
\[ + \frac{\pi \nu}{4} \int d\sigma \left[ \Gamma_2 Q\gamma_2 Q - \Gamma_1 Q\gamma_1 Q \right] \quad (4.4) \]
where
\[ Q\gamma_1 Q = 2\pi T \sum_{n, \alpha, \beta} \left( \text{Str}[Q^\alpha_{n,n_2}] \text{Str}[Q^\beta_{n_3,n_4}] \right) \]

In this way one can work with expressions including only supertraces and not single components. This makes the structure more transparent and additionally it has technical advantages when working with supermatrices, since anticommutativity of the fermionic components is taken care of automatically. The fields \( \phi_1, \phi_2 \) do not have a non-trivial structure in the \( AR \) space. Instead, this structure is given by
\[ \phi_1(r) = T \sum_j \phi_1(r,j) \Delta^j, \]
where \( j \) is a frequency index. The same definition is used for \( \varphi_1 \). The fields \( \varphi_1, \varphi_2 \) have only commuting entries and are proportional to the unit matrix, while \( \phi_1, \phi_2 \) are conventional supermatrices. For convenience, the explicit form together with the Gaussian integrals used to define the averaging are given in the appendix. Of particular importance are relations giving the averages of the fields
\[ \langle \varphi_1(r,j) \varphi_1(r',-j) + \phi_1(r,j) \phi_1(r',-j) \rangle_{\mu \nu}^{\alpha \delta} \]
\[ = \delta^{\alpha \delta}_{\mu \nu} \Gamma_1 \delta(r-r') \quad (4.8) \]
and
\[ \langle \varphi_2(r,j) \varphi_2(r',-j) + \phi_2(r,j) \phi_2(r',-j) \rangle_{\mu \nu}^{\alpha \delta} \]
\[ = 2 \delta^{\alpha \delta}_{\mu \nu} \Gamma_2 \delta(r-r'), \quad (4.9) \]
where upper indices refer to spin, lower indices to fermion-boson space. The importance of these relations will become obvious later.

Using the formalism developed in this section we can express the quantities like Green function in coinciding points determining the tunnelling density of states or the density-density correlation function in terms of a func-
tional integral over the supermatrices $Q$ with the free energy functional $F$, Eq. (13). This can be done using Eq. (1.11) and integrating over $\psi$ in Eqs. (1.9 and 1.12) with the effective free energy functional of Eq. (1.11) containing both $Q$ and $\psi$. The following simple formulas are obtained when averaging the interaction part and the pre-exponential factors in Eqs. (1.9) and (1.12) separately. Furthermore the saddle point equation $Q(r) = \pi \nu g(r, r)$ (2.10) is used,

$$
G_{\sigma\sigma'}(r, r', \omega_m, \omega_n) = \frac{i\pi \nu}{T} \int DQ \text{Str}[k_+ Q^{\sigma\sigma'}_{mn}(r)] e^{-F}.
$$

(4.10)

For the density-density correlation we consider only one out of the two possible pairings for the pre-exponential factors in Eq. (4.9), since we are interested in points $r, r'$ that are far apart and $g(r, r')$ falls off at distances $|r - r'|$ of the order of the mean free path,

$$
\Pi(r, r', \Omega_k, \Omega_l) = -(\pi \nu)^2 \int DQ \text{Str}[k_+ \Delta^{-k}Q(r)] \text{Str}[\Delta^l Q(r')] e^{-F}.
$$

Equations (4.5), (4.6), (4.9), (4.10), and (4.11) are the final results of the derivation of the $\sigma$-model with interaction. Any physical quantity we are interested in can be computed, at least in principle, using these formulas. Of course, our ultimate goal is to compute some new quantities in the nonperturbative regime. However, in order to see how the model works we should understand first how it can help to reproduce known perturbative results. Such calculations will be carried out in the next sections.

V. PERTURBATION THEORY IN DIFFUSION MODES

Disordered systems with interaction can be studied using a conventional diagrammatic technique (see, e.g., Ref. [22]). In low dimensions, the most important contributions come from a certain class of diagrams called “cooperons” and “diffusons”. Although these diagrams are important even without any interaction, the latter leads to new contributions. The $\sigma$-model formalism allows one to integrate over “electron degrees of freedom” in the beginning of all calculations and reduce them to study of the low lying diffusion modes. In the language of the $\sigma$-model summation of the diffusons and cooperons is equivalent to an expansion in small fluctuations near the ground state of the free energy functional $F$, Eq. (1.11). The minimum of $F$ is achieved setting $Q = \Lambda$, where $\Lambda$ is defined in Eq. (2.5).

For a perturbative expansion the matrix $Q$ may be parametrized in the vicinity of $\Lambda$ as

$$
Q = \Lambda (1 + iP)(1 - iP)^{-1},
$$

(5.1)

where $P = KP^\dagger K$,

$$
P = \begin{pmatrix} 0 & B \\ kB^\dagger & 0 \end{pmatrix},
$$

(5.2)

$$
B = \begin{pmatrix} a & i\sigma \\ \rho^\dagger & ib \end{pmatrix},
$$

(5.3)

Following the structure of $\psi$ all matrices in $AR$-space are now arranged in block form as

$$
\begin{pmatrix} M(k, l)|_{k<0, l<0} & M(k, l)|_{k<0, l\geq 0} \\ M(k, l)|_{k\geq 0, l<0} & M(k, l)|_{k\geq 0, l\geq 0} \end{pmatrix}.
$$

The perturbation theory in the diffusion modes is carried out by making expansion in the supermatrices $P$. Making expansion in $P$ we represent first the free energy functional $F$ in a form of a series in $P$. As the zero approximation we take the noninteracting quadratic part $F_0$

$$
F_0 = \pi \nu \int dr \text{Str}[D\nabla P(r)\nabla P(r) + 2E\Lambda P^2(r)].
$$

(5.4)

The rest of the functional consists of quadratic terms coming from the interaction, Eqs. (4.10) and (4.11), and higher order in $P$ terms coming from both noninteracting and interacting parts of the functional $F$. For explicit calculations we must calculate averages of different powers of $P$ with the bare functional $P_0$, Eq. (5.1). Calculating Gaussian integrals it is not difficult to derive the following contraction rules for arbitrary supermatrices $M, N$ (we remind the reader that the unitary case is considered)

$$
\langle \text{Str}[M_{k'\nu'} P_{k\nu}(r)] \text{Str}[N_{\nu'j'} P_{j}(r')] \rangle_0 = \text{Str}[M_{k'\nu'} N_{j\nu'}] \frac{1}{2\pi \nu} D(r - r', k - l) \times \delta_{jk} \delta_{ll'} \Theta(l, l') \Theta(k, k').
$$

(5.5)

and

$$
\langle \text{Str}[M_{\nu'j'} P_{jk} N_{k\nu'} P_{li}(r')] \rangle_0 = \text{Str}[M_{\nu'j'}] \text{Str}[N_{k\nu'} - \text{Str}M_{\nu'j'}] \text{Str}[N_{k\nu'}] \times \frac{1}{4\pi \nu} D(r - r', j - k) \delta_{k\nu} \delta_{j\nu} \Theta(k, k') \Theta(j, j').
$$

(5.6)

The subscript “0” will be omitted in the following. We denote $M^+ = (M - \Lambda M\Lambda)/2$, and $M^\dagger = (M + \Lambda M\Lambda)/2$. The diffusion propagator $D$ is given by

$$
D(r - r', l - k) = \int (dq) \frac{e^{iq(r-r')}}{Dq^2 + |\Omega_{l-k}|},
$$

(5.7)

and the symbol $\Theta(k, l)$ stands for

$$
\Theta(k, l) = \begin{cases} 1 & \text{for } k \geq 0, \ l \geq 0 \text{ or } k < 0, \ l < 0 \\ 0 & \text{otherwise} \end{cases}
$$

Correlation functions can be calculated expanding Eqs. (4.10) and (4.11) in $P$ and integrating over this
variable with the help of the contraction rules, Eqs. (5.5) and (5.6). A few elementary diagrams for the density-density correlation function are depicted in Fig. 4. As discussed in Sec. III one can read off from the diagrams in Fig. 4 that only one vertex term should distinguish between the fermionic and the bosonic sector, i.e., contain only fermionic variables, while the other remains supersymmetric. In this way no artificial contributions arise from the diagrams shown in Fig. 4. The same argument can be generalized straightforwardly to the ladder-type diagrams considered below. Using two vertex terms of the same kind would not lead to correct results. In this case either all diagrams vanish identically if both vertex terms are supersymmetric, or all diagrams in Fig. 4 give a contribution, if both vertex terms contain fermionic variables only.

Calculating the density-density correlation function \( \Pi(q, \Omega) \) we are interested in small frequencies and momenta. In the static limit it is related to the compressibility \( \partial_{\mu n} \) according to

\[
\partial_{\mu n} = \lim_{q \to 0} \Pi(q, 0).
\]

For systems without interaction the relation \( \partial_{\mu n} = 2\nu \) holds. One can calculate \( \Pi(q, \Omega) \) from the supersymmetric model representing this function as

\[
\Pi(q, \Omega) = \frac{\partial n}{\partial \mu} + \frac{1}{(1 + F_\Omega^2)^2} \hat{\Pi}(q, \Omega),
\]

(5.8)

\[
\langle F_\Omega F_\Omega \rangle \sim T \sum_k \int \varphi(-q_k - k) \hat{\Pi}(q, \Omega_k) \vartheta(q, k).
\]

(5.9)

The terms \( F_\Omega \) and \( F_\varphi \) are given by

\[
F_\varphi = i\pi \nu \int \vartheta \text{Str}[\hat{Q}],
\]

(5.10)

\[
F_\varphi = i\pi \nu \int \vartheta \text{Str}[k_+ \varphi Q],
\]

(5.11)

where \( k_+ = (1 + k)/2 \). The factor \( 1/(1 + F^2_\Omega) \) is due to vertex corrections taken from Fermi-liquid theory. It is related to the amplitudes \( \Gamma_i \) as \( 1/(1 + F^2_\Omega) = 1 - 2\Gamma_1 + \Gamma_2 \) and also determines \( \partial_{\mu n} = 2\nu/(1 + F^2_\Omega) \). We consider the Fermi-liquid approximation\( ^{21} \), for which the calculation amounts to the ladder summation depicted in Fig. 5. In the zero order one should expand \( Q \sim A + 2i\Delta \) in the interaction terms, Eqs. (5.5) and (5.6), in the free energy \( F \) and in the pre-exponential, Eq. (5.9). In both the cases the saddle point \( A \) does not contribute. Only the singlet interaction amplitude \( \Gamma_s \) enters and one must sum a geometric series that arises after expanding in the interaction terms written in the quadratic approximation in \( P \). As a result, we obtain

\[
\hat{\Pi}(q, \Omega_k) = -2\nu \frac{|\Omega_k|}{Dq^2 + z_1^{(0)}|\Omega_k|},
\]

where \( z_1^{(0)} = 1 + 2\Gamma_1 - \Gamma_2 \). Using this relation and Eq. (5.6) one comes to the result

\[
\Pi(q, \Omega_k) = \partial_{\mu n} \frac{D_{FL} q^2}{D_{FL} q^2 + |\Omega_k|},
\]

where \( D_{FL} = (1 + F^2_0)D \). The diffusion coefficient, which enters the \( \sigma \)-model, can be related to conductivity with the help of the equation of continuity

\[
\sigma = e^2 \lim_{\Omega \to 0} \lim_{p \to 0} \frac{\Omega}{p^2} \Pi(q, \Omega).
\]

(5.12)

This leads to the Einstein relation

\[
\sigma = e^2 \partial_{\mu n} D_{FL}.
\]

This relation can be seen to hold also after renormalization\( ^{6} \) (see Sec. VII below) and in this way interaction corrections to conductivity can be obtained.

In the remaining part of this section, in contrast, we describe simple perturbation theory for finding the first order interaction correction to conductivity from the \( \sigma \)-model. A general expression for the calculation of the linear response conductivity can be given with the help of the original fermionic partition function \( Z \) in Eq. (2.1),

\[
\frac{\sigma(\omega)}{e^2} = \frac{K(q \to 0, \Omega_n)}{\Omega_n} |_{\Omega_n \to \omega^+},
\]

(5.13)

\[
K(r, \tau, r', \tau') = \left( -\frac{\delta^2}{\delta A_{\mu}(r, \tau) \delta A_{\mu}(r', \tau')} \right) Z[A] |_{A=0}. \]

(5.14)

\[
K(q, \Omega_n) = \int_{r, \tau} K(r, \tau, 0, 0) e^{-iqr^\tau} e^{i\Omega_n \tau}.
\]

(5.15)

Analytic continuation is performed from positive Matsubara frequencies. No summation over \( \mu \) is implied. One comes to \( Z[A] \) after the substitution \( \hat{p} \to \hat{p} - A \) in \( Z = Z[0] \). The electron charge \( e \) appears in the prefactor. For the \( \sigma \)-model the corresponding replacement is \( \nabla Q \to \nabla Q - i[\hat{A}, Q] \) leading to a change in the free energy \( F = F[0] \to F[A] \). \( A \) is chosen supersymmetric because we should treat fermions and bosons on equal footing for the derivation of the \( \sigma \)-model. In order to obtain the analog of \( K \), however, it is necessary to break supersymmetry. To this end one should expand \( Z[A] = \int DQ e^{-F[A]} \) up to second order in \( A \) and multiply one of the fields \( A \)
by the projector onto the fermionic sector \( k_+ \). Since we are interested in the leading order correction in the interaction we also expand up to first order in \( F_{\text{int}} \). In order to express the result of this procedure in a compact form we define

\[
F_{A}^{(1)} = -\frac{\pi i D}{2} \int dr \text{Str} \left[ \nabla Q(r) [A(r), Q(r)] \right],
\]

\[
F_{A}^{(1)+} = -\frac{\pi i D}{2} \int dr \text{Str} \left[ \nabla Q(r) [k_+ A(r), Q(r)] \right],
\]

\[
F_{A}^{(2)+} = -\frac{\pi i D}{4} \int dr \text{Str} \left[ [k_+ A(r), Q(r)] [A(r), Q(r)] \right].
\]

We can now write the analog of \( K \) for the supersymmetric model,

\[
K'(r, \tau, \tau', r') = -\frac{\delta^2}{\delta A_{\mu}(r, \tau) \delta A_{\mu}(r', \tau')} \times \\
\left( -\left\langle F_{A}^{(2)+} \right\rangle_0 + \frac{1}{2} \left\langle F_{A}^{(1)+} F_{A}^{(1)} \right\rangle_0 \\
+ \left\langle F_{A}^{(2)+} F_{\text{int}} \right\rangle_0 - \frac{1}{2} \left\langle F_{A}^{(1)+} F_{A}^{(1)} F_{\text{int}} \right\rangle_0 \right) \bigg|_{A=0} \\
= \sum_{i=1}^{4} K_i(r, \tau, r', \tau').
\]

Averaging is with respect to the noninteracting part of the free energy. We expand the matrix \( Q \) in terms of \( P \) and use \( F_{\text{int}}[\hat{A}] \) for averaging. Classical conductivity can be found at the level of the saddle point \( Q = \Lambda \). The only contribution comes from \( K_1 \) and one finds \( \sigma_0 = 2e^2D/4 \), where the factor of 2 is due to the spin degree of freedom.

Next we turn to interaction corrections. At order \( P^4 \) \( K'_4 \) vanishes in the limit \( q \to 0 \). There are two contributions to \( K'_3 \),

\[
K'_3(r, \tau, \tau', r') \approx 2(\pi \nu)^2 D \partial_{A_{\mu}} \partial_{A'_{\mu}} \int_{r_i} \langle \text{Str} [k_+ [\hat{A}_{\mu}, \Lambda P_{\mu_1}] [\hat{A}_{\mu_1}, \Lambda P_{\mu_1}]] \\
\times \sum_{j=1,2} (-1)^j \langle \text{Str} [\hat{\phi}_{j r_2} \Lambda P_{r_2}] \text{Str} [\hat{\phi}_{j r_3} \Lambda P_{r_3}] \rangle \rangle_0 \\
+ 2(\pi \nu)^2 D \partial_{A_{\mu}} \partial_{A'_{\mu}} \int_{r_i} \langle \text{Str} [k_+ [\hat{A}_{\mu}, \Lambda] [\hat{A}_{\mu}, \Lambda P_{\mu_1}]] \\
\times \sum_{j=1,2} (-1)^j \langle \text{Str} [\hat{\phi}_{j r_2} \Lambda P_{r_2}] \text{Str} [\hat{\phi}_{j r_3} \Lambda P_{r_3}] \rangle \rangle_0.
\]

After further evaluation one finds a cancellation and thus there is no correction including only two diffusons. Considering now corrections with three diffusion modes the only nonvanishing contributions in the limit \( q \to 0 \) can be seen to originate from \( K'_4 \), where all \( Q \) matrices should be replaced by \( 2i \Lambda P \). At order \( P^6 \) we therefore have

\[
K'_4(r, \tau, \tau', r') = 4(\pi \nu)^4 D^2 \partial_{A_{\mu}} A'_{\mu}.
\]
can even diverge. In the original work the case of long range Coulomb interaction was considered and the Cooperon modes were suppressed (e.g., by a small magnetic field). Appropriate scaling equations for this case were derived. Later on the approach has been generalized to many different situations (for a review see Refs. and ). By a comparison to the field theoretic solution the scaling equations can also be written starting from conventional perturbation theory and for some calculations the diagrammatics can also be conveniently used (for a review see, e.g., Ref. ). The authors of Ref. also considered the case of short range interactions. A recent discussion in the context of a Keldysh σ-model can be found in Ref. .

The renormalization scheme for the interacting disordered electron gas is more complicated than for the noninteracting case. “Infrared” singularities arise in the theory due to both the frequency and momentum integration. It turns out that in addition to the diffusion coefficient and the interaction amplitudes a “charge” multiplying the energy matrix must be introduced. It describes the relative change of momentum and frequency scales. Our aim is to show that the supersymmetric model introduced above is capable of reproducing the correct RG equations in the limit of weak short range interactions, i.e., in the first order in $t$ and to leading order in the interaction amplitudes. Therefore the procedure is used as a test for the new model and it is the scheme introduced by Finkel’steins that serves this purpose best. As a by-product, the Altshuler-Aronov corrections to conductivity and the density of states can also be obtained in this way.

**A. Renormalization group procedure**

In this section we work in two spatial dimensions and rewrite the model of Eq. in the following form:

$$F[Q] = \frac{\pi \nu}{4} \int dr \text{Str} \left[ D(\nabla Q(r)) - 4zEQ(r) \right] + \frac{(\pi \nu)^2}{2} \int dr dr' \left\langle \text{Str} \left( \hat{\phi}_1(r)Q(r) \right) \text{Str} \left( \hat{\phi}_2(r')Q(r') \right) \right\rangle_{\phi_1} - \frac{(\pi \nu)^2}{2} \int dr dr' \left\langle \text{Str} \left( \hat{\phi}_1(r)Q(r) \right) \text{Str} \left( \hat{\phi}_1(r')Q(r') \right) \right\rangle_{\phi_1}. \quad (6.1)$$

In order to reduce the number of terms we use the fields $\phi_e = \phi_1 + \phi_2$ together with the symbolic notation $\left\langle \hat{\phi}_1 \hat{\phi}_2 \right\rangle = \left\langle \phi_1 \phi_2 \right\rangle + \left\langle \phi_2 \phi_1 \right\rangle$. The “charge” $z$ has been introduced as discussed before. We start the renormalization group procedure separating slow and fast modes

$$Q = UQ_0 \hat{U}, \quad Q_0 = U_0 \Lambda \hat{U}_0, \quad U \hat{U} = U_0 \hat{U}_0 = 1.$$  

Fluctuations described by the supermatrix $U_0$ are fast while those described by $U$ are slow. As a result of this separation we write the free energy $F[Q]$ in the form

$$F[Q] = F^{(0)} + F_{\text{int}},$$

where

$$F^{(0)} = \frac{\pi \nu}{4} \int dr \text{Str} \left[ D(\nabla Q_0)^2 + 2DQ_0 \right] + \frac{(\pi \nu)^2}{2} \int dr dr' \left\langle \left( \text{Str} \left[ \hat{\phi}_1U_0Q_0 \hat{U}_0 \right] \right) \right\rangle_{\phi_1} - \frac{(\pi \nu)^2}{2} \int dr dr' \left\langle \left( \text{Str} \left[ \hat{\phi}_1U_0Q_0 \hat{U}_0 \right] \right) \right\rangle_{\phi_1}. \quad (6.2)$$

$$F_{\text{int}} = \frac{(\pi \nu)^2}{2} \int dr dr' \left\langle \left( \text{Str} \left[ \hat{\phi}_1U_0Q_0 \hat{U}_0 \right] \right) \right\rangle_{\phi_1} - \frac{(\pi \nu)^2}{2} \int dr dr' \left\langle \left( \text{Str} \left[ \hat{\phi}_1U_0Q_0 \hat{U}_0 \right] \right) \right\rangle_{\phi_1}. \quad (6.3)$$

and $F = \hat{U} \nabla U$.

Next, we parametrize fast modes $Q_0$ in the following way:

$$Q_0 = \Lambda(1 + iP)(1 - iP)^{-1}, \quad \{P, \Lambda\} = 0.$$  

One must specify the precise momentum range related to the fast and slow modes and additionally account for the frequency dependence. This can be done in the following way:

1. Frequencies in the interval $\lambda < \omega < \tau^{-1}, 0 < \lambda < 1$ and momenta in the shell $\lambda < \tau^{-1} E/k^2/z < \tau^{-1}$ are referred to as fast.

2. If slow variables $U$ have at least one Matsubara index corresponding to the fast frequencies, they must be set equal to $1$.

3. If fast variables $P$ do not have at least one fast frequency or fast momentum, they vanish.

The separation of modes in the energy part requires some care. The energy matrix must be split into a fast and a slow part: $E = E^f + E^s$. Then

$$\int dr \text{Str} \left[ EUAP^2 \hat{U} \right] = \int dr \text{Str} \left[ E^s UAP^2 \hat{U} \right] + \int dr \text{Str} \left[ E^f A^2 \right]. \quad (6.4)$$

Using the parametrization introduced above the free energy splits into several parts. First, the original free energy is recovered with $Q$ replaced by $\hat{Q} = U \Lambda \hat{U}$ and with the slow energy matrix $E^s$. In Finkel’steins scheme it is sufficient to keep terms of the second order in $P$ for the remaining parts. As a result one finds

$$F^{(0)} = F_0 + F_1 + F_2 + F_E, \quad (6.5)$$

$$F_0 = \pi \nu \int dr \text{Str} \left[ D \nabla P \nabla P + 2zE^f A^2 \right], \quad (6.6)$$

$$F_1 = 2\pi \nu D \int dr \text{Str} \left[ \left( \Phi P, \nabla P \right) \right], \quad (6.7)$$

$$F_2 = -2\pi \nu D \int dr \text{Str} \left[ \left( \Phi A \right)^2 P^2 + \left( A \Phi A \right)^2 \right], \quad (6.8)$$

$$F_E = 2\pi \nu \int dr \text{Str} \left[ E^s UAP^2 \hat{U} \right]. \quad (6.9)$$
The interaction part $F_{\text{int}} = F_{\text{int},1} + F_{\text{int},2}$ can be written as follows:
\begin{align}
F_{\text{int},1} &= 2(\pi \nu)^2 \int d\mathbf{r} d\mathbf{r}' \left( \left( \text{Str} [\hat{\phi}_{\mathbf{r}} U_{\mathbf{r}} A P_r U_{\mathbf{r}'}] \right) \times \text{Str} [\hat{\phi}_{\mathbf{r}'} U_{\mathbf{r}'} A P_{r'} U_{\mathbf{r}''}] \right)_{\phi_1} - \left( \phi_1 \leftrightarrow \phi_2 \right), \tag{6.10} \\
F_{\text{int},2} &= 2(\pi \nu)^2 \int d\mathbf{r} d\mathbf{r}' \left( \left( \text{Str} [\hat{\phi}_{\mathbf{r}} Q_r] \right) \times \text{Str} [\hat{\phi}_{\mathbf{r}'} U_{\mathbf{r}'} A P_r^2 U_{\mathbf{r}''}] \right)_{\phi_1} - \left( \phi_1 \leftrightarrow \phi_2 \right). \tag{6.11}
\end{align}

The fast modes can be integrated out in the Gaussian approximation and an effective free energy $F$ is generated in this way
\[ \tilde{F}[\bar{Q}] = -\ln \left( \int DP \ e^{-F_1-F_2-F_E-F_{\text{int}} \ e^{-F_0}} + F[\bar{Q}] \right). \tag{6.12} \]

The functional $\tilde{F}$ is the appropriate free energy for the slow modes, taking into account the influence of the fast modes. It turns out that this influence can effectively be expressed by a change of $D, z, \Gamma_1,$ and $\Gamma_2$ with the scale determined by $\lambda$ and this is what demonstrates the renormalizability. From now on it will be clear from the context whether we refer to slow $\bar{Q}$ or fast $Q$, so that we can simply use a common symbol $Q$.

**B. Diffusion coefficient**

Here we calculate the corrections to the diffusive part
\[ F_D = \frac{\pi \nu}{4} \int \text{Str}\left[ D(\nabla Q)^2 \right]. \]

The energy part does not contribute and the renormalization of the diffusive part can be written in the form
\[ \delta F_D = \langle F_{\text{int}} \rangle - \langle F_1 F_{\text{int}} \rangle - \langle F_2 F_{\text{int}} \rangle + \frac{1}{2} \langle F_1^2 F_{\text{int}} \rangle. \tag{6.13} \]

By $\langle \ldots \rangle$ we denote the connected part of the averages.

The term $F_{\text{int}}$ itself does not contain any gradients. It is therefore not obvious, how $\langle F_{\text{int}} \rangle$ can contribute to the renormalization of the diffusion coefficient. Therefore we discuss this calculation in some detail.

According to the contraction rule $\langle \ldots \rangle$ we have $\langle PP \rangle = 0$. As a direct consequence of this property the average $\langle F_{\text{int},2} \rangle$ vanishes. For the part $F_{\text{int},1}$ we should distinguish between several possible cases.

(a) The matrices $P$ can have two fast frequency indices. Then the $U$-modes completely vanish from the expression. It is quadratic in $P$ and therefore contributes together with $F_0$ to the quadratic form in $P$. This leads to the dressing of the diffusion modes in Finkel’stein’s approach. In our model these higher order interaction effects can only be taken into account in special cases (as in Sec. IV), while in general we have to restrict ourselves to the lowest order in the interaction amplitudes for the renormalization.

(b) The matrices $P$ have two slow frequency indices, but the momentum is fast. It turns out that integrating over $P$ the term is quadratic in the slow fields $Q$ and contributes to the renormalization of the interaction part (see Sec. V ID below).

(c) The matrices $P$ have one fast and one slow frequency index. The diffusion propagator, arising after calculating Gaussian integrals in $P$ can be expanded in slow momenta, which leads to a correction to the diffusion coefficient. If it is expanded in small frequencies, it contributes to the renormalization of $z$. The latter case becomes important only if the diffusion modes are dressed and we refer to Ref. 22 for details. The former case will be discussed next.

We consider the case (c). After integration over $P$-modes one obtains
\begin{align*}
\langle F_{\text{int}} \rangle &= -2\pi \nu T^2 \int d\mathbf{r} d\mathbf{r}' D(r-r', k-l) \\
&\times \text{Str} \left[ \left( \langle \hat{\phi}_{\mathbf{r}}^j \hat{\phi}^j_{\mathbf{r}'} \rangle - \langle \hat{\phi}_{\mathbf{r}}^{j2} \hat{\phi}^j_{\mathbf{r}'} \rangle \right) \times (\delta^{lj} U_{\mathbf{r}}^k) \right], \tag{6.14}
\end{align*}
where $k$ is a fast frequency index while $l$ is slow. In Sec. IV it has been stressed that Eqs. 15 and 16 are crucial for the model. It is in fact very important for the consistency of the theory, that $\langle \hat{\phi} \hat{\phi} \rangle \propto 1$. Integrating over $\hat{\phi}$ we find
\begin{align*}
\langle F_{\text{int}} \rangle &= -2\pi \nu T (\Gamma_1 - 2\Gamma_2) \int (d\mathbf{p}) (d\mathbf{q}) D(\mathbf{p} + \mathbf{q}, k-l) \\
&\times \text{Str} \left[ U_{\mathbf{q},m}^\perp \bar{U}_{\mathbf{q},m}^\perp \Theta_{k,m} + U_{\mathbf{q},m}^\parallel \bar{U}_{\mathbf{q},m}^\parallel \Theta_{k-1,m} \right]. \tag{6.15}
\end{align*}

Here $\mathbf{p}$ is a fast momentum while $\mathbf{q}$ is slow. Keeping only fast $k$ and $\mathbf{p}$ in the diffusion propagator the $U$ dependence would drop completely. Therefore one expands the diffusion propagator up to second order in $\mathbf{q}$. Using the identity $\text{Str}[\nabla U \nabla U] = -\text{Str}[\hat{\Phi}]^2$ and approximating the frequency summation by an integral, we come to the following expression
\begin{align*}
\langle F_{\text{int}} \rangle &= (\Gamma_1 - 2\Gamma_2) \int d\mathbf{r} \text{Str} [\hat{\Phi}]^2 \int (d\mathbf{p}) \\
&\times \int d\Omega (2D^2 \mathbf{p}^2 D^3(\mathbf{p}, \Omega) - D D^2(\mathbf{p}, \Omega)) \tag{6.16}
\end{align*}
The momenta and frequencies in the region of the integration in Eq. (6.16) are large and correspond to the fast variables. We use the symbol $f'$ to indicate that integration is carried out over positive frequencies only.

The remaining terms in Eq. (6.19) can be considered in an analogous way. The results are

\[- \langle F_1 F_{\text{int}} \rangle = 2(\Gamma_1 - 2\Gamma_2) \int \, d\mathbf{p} \text{Str}[\Phi^2] \int \, d\Omega (DD^2(p, \Omega) - 2D^2 \mathbf{p}^2 D^3(p, \Omega)), \quad (6.17)\]

\[- \langle F_2 F_{\text{int}} \rangle = -(\Gamma_1 - 2\Gamma_2) \int \, d\mathbf{p} \text{Str}[\Phi^2 - \Phi^{1,2}] \int \, d\Omega D^2(p, \Omega), \quad (6.18)\]

\[\frac{1}{2} \langle F_1^2 F_{\text{int}} \rangle = 2(\Gamma_1 - 2\Gamma_2) \int \, d\mathbf{p} \text{Str}[\Phi^2] \int \, d\Omega D^2 \mathbf{p}^2 D^3(p, \Omega). \quad (6.19)\]

Following Finkel'stein, terms that are parametrically small in $tD^2/(z\Lambda \tau^{-1}) \ll 1$, where $q$ is a typical slow momentum, have not been taken into account. Such terms can arise from expressions in which all frequency indices are fixed to be slow by $\Phi$-modes. A comparison of the combined contribution of Eqs. (6.16) (6.19) to the original diffusive part $F_D$ gives

\[\delta D = -\frac{1}{(2\pi)^2} J_2, \quad (6.20)\]

\[J_2 = 8\pi(\Gamma_1 - 2\Gamma_2) \int \, d\mathbf{p} \int \, d\Omega \frac{D^2 \mathbf{p}^2}{(D\mathbf{p}^2 + z\Omega)^3}. \quad (6.21)\]

C. Effective charge $z$

As we have mentioned, in the $\sigma$-model for interacting systems a new effective charge $z$ appears as a coefficient in the original $\sigma$-model,

\[F_z = -\pi \nu z \int \text{Str}[E \mathbf{Q}]. \]

Expanding this term in $P$ we write the relevant term of the second order in $P$ as

\[F_E = 2\pi \nu z \int \text{Str} [EU\Lambda P^2 \bar{U}]. \]

It is understood that the energy matrix $E$ contains only slow indices here. In our case the correction to $z$ originates only from one term

\[\delta F_z = -\langle F_E F_{\text{int}} \rangle. \]

Working with dressed interaction lines as in Ref. 21, one more contribution from $\langle F_{\text{int}} \rangle$ should be taken into account. The final result, however, is the same in both cases and we find

\[- \langle F_E F_{\text{int}} \rangle \sim z(\Gamma_1 - 2\Gamma_2) \times \int \, d\mathbf{p} \text{Str} [E \mathbf{Q}] \int \, d\Omega D^2(p, \Omega). \quad (6.22)\]

Equations (6.18) and (6.19) were again crucial to arrive at this result. The relation $\partial_\Omega D(p, \Omega) = -zD^2(p, \Omega)$ can be used now to obtain

\[\delta z = \frac{1}{\pi \nu} \int \, d\mathbf{p} (\Gamma_1 - 2\Gamma_2) [D(p, \Omega)]_{\Omega=\lambda \tau^{-1}}^{-1}. \quad (6.23)\]

Equation (6.24) gives the logarithmic correction to the effective charge that will be used for writing renormalization group equations.

D. Interaction part

The leading order corrections in $\Gamma_1$, $\Gamma_2$ contributing to the renormalization of the interaction part come from the average $\langle F_{\text{int}} \rangle$. We will see that it is very important here to work with both fields $\phi$ and $\varphi$. As pointed out above in Sec. VI B, for this term both frequencies of $P$ should be slow, while the momentum is to be fast. After integration over $P$ we obtain

\[\langle F_{\text{int}} \rangle = \frac{\pi \nu}{2} \int \, d\mathbf{p} \, D(p, 0) \times \int \, d\mathbf{r}' \text{Str} [Q_\tau \hat{\phi_1} \phi_2 Q_\tau \hat{\phi_1} \phi_2 - Q_\tau \hat{\phi_2} \phi_2 Q_\tau \hat{\phi_2} \phi_2] \hat{\phi_1} \phi_2 \quad (6.24)\]

We write $D(p, 0)$ here to symbolize that only the momentum should be integrated, since frequencies of $P$-modes are slow. As a result, the corrections to $\Gamma_1$ and $\Gamma_2$ are

\[\delta \Gamma_1 = \frac{1}{\pi \nu} \Gamma_2 \int \, d\mathbf{p} \, D(p, 0), \quad (6.25)\]

\[\delta \Gamma_2 = \frac{1}{\pi \nu} \Gamma_1 \int \, d\mathbf{p} \, D(p, 0). \quad (6.26)\]

This agrees with the corresponding result obtained from the replica model, since the dressing of the diffusion mode is not effective here. We would like to stress that the correction to $\Gamma_1$ is proportional to $\Gamma_2$ and vice versa. From the technical point of view it is of particular importance here that the terms $Q_\gamma_1 Q_\gamma_2$ and $Q_\gamma_2 Q_\gamma_1$ in the supersymmetric model of Eq. (14) include both the single supertrace and the product of two supertraces. Only in this way the correct structure can be reproduced in the calculation of $\langle F_{\text{int}} \rangle$.

E. RG equations

From Eqs. (6.20), (6.23), (6.25), and (6.26) appropriate renormalization group equations may be derived. They
take the form

\[
\frac{1}{t} \frac{dt}{d \ln \lambda^{-1}} = \frac{t}{z} (\Gamma_1 - 2 \Gamma_2), \tag{6.27}
\]

\[
\frac{1}{t} \frac{dz}{d \ln \lambda^{-1}} = -(\Gamma_1 - 2 \Gamma_2),
\]

\[
\frac{1}{t} \frac{d \Gamma_1}{d \ln \lambda^{-1}} = \Gamma_2,
\]

\[
\frac{1}{t} \frac{d \Gamma_2}{d \ln \lambda^{-1}} = \Gamma_1.
\]

These equations are valid in the first order in \( t = (2\pi)^2 \nu D \) and in the leading order in the interaction amplitudes for \( \Gamma_i \ll z \). In the first order in \( \Gamma_{1,2} \) they coincide with the corresponding limit of the renormalization group equations obtained in Refs. \(^{21,23}\) and \(^{11}\) for short range interactions, where terms of all orders in the interaction amplitudes were included. As our supersymmetric model, Eqs. \(^{21,23}\) and \(^{11}\), can serve as a good approximation for the initial electron model only for a weak interaction, taking into account the terms of higher orders in interaction would lead to an overestimation. Therefore, we conclude that in the range of applicability the supersymmetric \( \sigma \)-model gives the correct renormalization group equations.

From these equations one can see that the combination \( z_1 = z - 2 \Gamma_1 + \Gamma_2 \) is invariant under renormalization.

\[
\frac{dz_1}{d \ln \lambda^{-1}} = \frac{d}{d \ln \lambda^{-1}} (z - 2 \Gamma_1 + \Gamma_2) = 0 \tag{6.28}
\]

This invariance is not incidental but protected by a Ward identity and therefore its validity is not restricted to the approximations studied here. For more details we refer to Refs. \(^{21,23,25}\), and \(^{11}\). Its importance for the consistency of the entire scheme will become obvious in the next section, when we discuss once more the density-density correlation function.

F. Conductivity

During the process of the renormalization the meaning of the coefficient \( D \) appearing in the free energy and its relation to conductivity remains very well understood. In order to clarify its meaning one can study the density-density correlation function \(^{21,23,25}\). In order to calculate the density-density correlation functions one can add to the free energy functional a source term of the form \( F_\theta = i \rho \nu \int d \vec{r} \text{Str}[\hat{Q}(\vec{r})] \), Eq. \(^{21,23}\). Then corrections to this term can be calculated. In our approximation one should consider the average

\[- \langle F_\theta F_{\text{int},1} \rangle .\]

This expression covers vertex corrections as well as corrections to the single particle Green function. These contributions exactly cancel, so that \( F_\theta \), Eq. \(^{21,23}\), is unchanged. [This is also true in all orders in \( \Gamma_{1,2} \)]. We would like to emphasize that one may not use \( F_\phi \), Eq. \(^{6,11}\), in this procedure because this would lead to spurious contributions.

Now the dynamic part of the polarization can be calculated from the renormalized \( \sigma \)-model as in Sec. \(^{21,23}\). It is generally expected that the thermodynamic density of states \( \partial_\mu n \) does not develop logarithmic singularities \(^{6,21,23}\). Then we come to the result

\[
\tilde{\Pi}(q, \Omega_k) = -2\nu \frac{|Dq|}{D^2q^2 + z_1|\Omega_k|},
\]

where \( D \) is the renormalized diffusion coefficient. The coefficient \( z_1 \) is a constant under renormalization as we have seen above, Eq. \(^{6,23}\), and using Eq. \(^{6,11}\) we find again

\[
\Pi(q, \Omega_k) = \partial_\mu n \frac{D_c q^2}{D_c q^2 + |\Omega_k|};
\]

\( D_c = (1 + F_\theta^0) D \). With the help of the equation of continuity \(^{6,11}\) one recovers the Einstein relation

\[
\sigma = e^2 \partial_\mu n D_c.
\]

So, this relation remains valid and one can determine the conductivity from the renormalized diffusion coefficient. As an example, one can make contact to the Altshuler-Aronov corrections to the conductivity for the case of short range interactions. To this end one should replace \( z \) by its bare value \( z = 1 \) as well as \( \Gamma_1 \rightarrow V_1 \) in Eq. \(^{6,23}\), where \( V_1 = \nu V_0(0) \) and \( V_2 = \nu V_0(|p - p'|) \) (the bar means averaging over the Fermi surface). As a result one comes to the well-known formula, Eq. \(^{6,23}\).

G. Density of states

The main purpose of this section is to obtain an expression for interaction corrections to the density of states from the supersymmetric \( \sigma \)-model derived. The single particle density of states \( \sigma \) (per spin degree of freedom) can be written as

\[
\nu(\varepsilon) = -\frac{1}{\pi} \int (d\vec{p}) G^R_{\sigma\sigma}(\vec{p}, \varepsilon).
\]

No summation over spin index \( \sigma \) is implied. In order to calculate the density of states in the Matsubara formalism it is convenient to first define the function

\[
\tilde{\nu}(\omega_n) = -\frac{1}{\pi} \int d\tau \mathcal{G}_{\sigma\sigma}(\vec{r}, \tau, \tau, 0) e^{i\omega_n \tau}.
\]

where \( \mathcal{G} \) denotes the Matsubara Green function of the system. It is sufficient to know \( \tilde{\nu}(\omega_n) \) for positive frequencies. The density of states \( \nu \) can be obtained from \( \tilde{\nu} \) by an analytic continuation,

\[
\nu(\varepsilon) = \Im \tilde{\nu}(\omega_n)|_{\omega_n \rightarrow -i\varepsilon + \delta}.
\]

In our formalism we obtain (compare Eq. \(^{6,23}\))

\[
\tilde{\nu}(\omega_n) = -i\nu \langle \text{Str} [k^+ Q^\sigma_{mm}(\vec{r})] \rangle, \tag{6.29}
\]
assuming that the saddle point approximation has been employed. Again, we separate fast and slow modes in $Q$ and expand $Q_0$ up to second order in $P$. We also expand $\exp(-F)$ up to first order in $F_{int}$. We should consider the correction

$$\delta \nu = i\nu \left\langle \text{Str}[k_+Q^\sigma_m(r)]F_{int}\right\rangle. \quad (6.30)$$

The calculation is very similar to that of $z$ and one obtains the correction to the density of states in the form

$$\delta \nu = -\nu\frac{t}{z}(\Gamma_1 - 2\Gamma_2) \ln \lambda^{-1}. \quad (6.31)$$

Equation (6.31) allows to write a renormalization group equation for the density of states

$$\frac{1}{\nu} \frac{\partial \nu}{\partial \ln \lambda^{-1}} = -\frac{t}{z}(\Gamma_1 - 2\Gamma_2). \quad (6.32)$$

The validity is, of course, restricted to small effective interaction amplitudes $\Gamma_i/z \ll 1$. As in Sec. VI, one can recover the Altshuler-Aronov correction for the short range case that can be written in the standard form

$$\delta \nu/\nu = -t(V_1 - 2V_2) \ln(T\tau). \quad (6.33)$$

The same formulas Eqs. (6.31)–(6.33) for the density of states would be obtained with Replica or Keldysh models in the limit of weak interactions.

H. Discussion of the renormalization group

The renormalization analysis carried out in this section allows us to come to the conclusion that the supersymmetric $\sigma$-model is renormalizable at least in the first order in the interaction amplitudes and inverse conductance $t = ((2\pi)^2\nu D)^{-1}$. The structure of the interaction term, Eqs. (6.31) and (6.32) is crucial for the renormalizability and therefore the renormalization group treatment can serve as a very good check of the $\sigma$-model. The renormalization group equations derived above (Eq. (6.31)) agree with those given in Refs. 11, 21, 23, and 25 in the leading order in $\Gamma_i$ and for $\Gamma_i \ll z$.

We do not present here the solution of these equations referring the reader to Refs. 11, 21, 23, and 25 where solutions of more general equations have been discussed for arbitrary values of the amplitudes $\Gamma_i$. Unfortunately, even these more general equations do not allow for definite conclusions in two dimensions, since under renormalization the quotient $\Gamma_2/z$ diverges. This leads to a divergence of the spin susceptibility, which has been interpreted as some sort of ferromagnetic instability. However, it should be stressed once again that the renormalization group calculations were performed here only as a check of the new $\sigma$-model. For the more complete study of the scaling behavior in 2D replica or Keldysh models are more convenient. The remarkable point is that the nontrivial set of Eqs. (6.31) has been obtained from a model based entirely on the supersymmetry method.

VII. DISCUSSION

In this paper we constructed a supersymmetric $\sigma$-model for disordered fermion systems with interaction. Instead of trying to combine the supersymmetry technique with an electron-electron interaction for the initial electron model we proposed an artificial fully supersymmetric model with interaction. The derivation of the $\sigma$-model from the latter model is straightforward and valid for an arbitrary interaction. Unfortunately, the artificial supersymmetric model is equivalent to the electron one only in the limit of a weak short range interaction. Therefore the $\sigma$-model we obtained is less general than the replica or Keldysh $\sigma$-models. So, as concerns the perturbation theory in the diffusion modes or the renormalization group calculations, it is better to perform them with the replica or Keldysh $\sigma$-models and our goal was not to construct a more convenient tool for such calculations. Although we have demonstrated how to carry out perturbative and RG computations in the framework of our $\sigma$-model, the purpose of these calculations was merely to check the $\sigma$-model. The check is successful and our hope is that using the supersymmetric $\sigma$-model we derived one can perform nonperturbative calculations.

It is well known that for noninteracting systems the supersymmetry technique has real advantages with respect to the other methods and allows one to consider, e.g., localization in wires, level statistics in quantum dots, etc. So, our hope is that the supersymmetric $\sigma$-model with interaction derived in the present paper can help us to get results in this direction. The supermatrices $Q$ derived here should have the correct symmetry (a mixture between compact and noncompact sectors) and there is no reason to think that, when properly treated, the model would lead to wrong results.

Nevertheless, using the present $\sigma$-model is definitely more difficult than the conventional $\sigma$-model for noninteracting particles. The problem is that the supermatrix $Q$ is now not a $8 \times 8$ or $4 \times 4$ supermatrix as it was for the noninteracting systems. In contrast, the size of the supermatrix $Q$ is now $4M \times 4M$ (for the unitary ensemble), where $M$ is the number of the Matsubara frequencies involved in the calculation (actually, it is infinitely large). Therefore, we should be able to calculate integrals over the supermatrices of an arbitrary large size. Of course, the same problem exists for the replica and Keldysh $\sigma$-model. In the replica $\sigma$-model one should, in addition, calculate for an arbitrary number $N$ of the replicas and set at the end $N = 0$. In the Keldysh approach the matrix $Q$ depends on two energies and one should discretize the energy to get a reasonable result.

The advantage of the supersymmetry approach is that the supersymmetric $\sigma$-model is simpler and well defined.

In principle, the task of making explicit calculations with $4M \times 4M$ supermatrices does not seem hopeless. For calculations in the $0D$ situation one can try to use the Itzykson-Zuber integral$^{26,27}$ that has been general-
ized for the supersymmetric case by Guhr et al.\textsuperscript{25}. An integral with the zero-dimensional supermatrix free energy functional taken in the form of Eq. (4.7) has the proper form and one may hope to proceed in this way. Another problem is localization in wires. With the supermatrix σ-model one can write using the transfer matrix technique an effective “Schrödinger equation” and then try to solve it. These are the most evident examples. Probably, one can try other situations, although the proper calculations can be difficult.

Of course, at the moment we cannot guarantee the complete success in this direction. At the same time, it is difficult to imagine that either replica or Keldysh σ-models can be more helpful. Although there is a certain progress in reproducing some known results for noninteracting systems using these methods\textsuperscript{15,16,17,18}, the calculations are quite difficult and the results are not complete. We do not see how these schemes can be extended to interacting systems and believe that the supermatrix σ-model developed here is a better opportunity to attack the nonperturbative problems.

Acknowledgments

The authors gratefully acknowledge the financial support of SFB TR 12 and GRK 384. The authors would like to thank I. Aleiner, I. Beloborodov, A. Lopatin, and K. Takahashi for useful discussions at the initial stages of the work.

APPENDIX: PARAMETRIZATION OF $\phi_1, \phi_2$ AND $\varphi_1, \varphi_2$

Here the definition of the fields $\phi_i, \varphi_i$ is given. In the following we write supermatrices with $2 \times 2$ block structure describing spin space. $a^{ij}, b^{ij}$ are commuting while $\sigma^{ij}$ are anticommuting. Variables $a, b, \sigma$ for different $\phi_i, \varphi_i$ are not related to one another. $a(r, \tau), b(r, \tau)$ and $a^{ij}(r, \tau), b^{ij}(r, \tau)$ are real. We assume $\Gamma_i$ to be short range and usually set $\Gamma_i(r - r') = \Gamma_i \delta(r - r')$. $\mathcal{N}$ are normalization constants for the Gaussian integrals. Integration is over all independent components of the corresponding matrices.

\begin{align}
\varphi_1(r, n) &= \begin{pmatrix} a(r, n) & 0 \\ 0 & a^\dagger(r, n) \end{pmatrix}, \\
\varphi_2(r, n) &= \begin{pmatrix} a^{11}(r, n) & a^{12}(r, n) \\ a^{12\ast}(r, -n) & a^{22}(r, n) \end{pmatrix}, \\
\phi_1(r, n) &= \begin{pmatrix} a(r, n) & 0 \\ 0 & a^\dagger(r, n) \end{pmatrix}, \\
\phi_2(r, n) &= \begin{pmatrix} a^{11}(r, n) & a^{12}(r, n) \\ a^{12\ast}(r, -n) & a^{22}(r, n) \end{pmatrix}.
\end{align}

and

\begin{align}
\langle \ldots \rangle_{\varphi_1} &= \mathcal{N} \int D\varphi_1 \langle \ldots \rangle e^{-\frac{1}{g} \int drdr' \text{Tr}[\varphi_1(r, -n)G^{-1}_{11}(r-r')\varphi_1(r', n)]}, \\
\langle \ldots \rangle_{\varphi_2} &= \mathcal{N} \int D\varphi_2 \langle \ldots \rangle e^{-\frac{1}{g} \int drdr' \text{Tr}[\varphi_2(r, -n)G^{-1}_{22}(r-r')\varphi_2(r', n)]}, \\
\langle \ldots \rangle_{\phi_1} &= \mathcal{N} \int D\phi_1 \langle \ldots \rangle e^{-\frac{1}{g} \int drdr' \text{Str}[\phi_1(r, -n)G^{-1}_{11}(r-r')\phi_1(r', n)]}, \\
\langle \ldots \rangle_{\phi_2} &= \mathcal{N} \int D\phi_2 \langle \ldots \rangle e^{-\frac{1}{g} \int drdr' \text{Str}[\phi_2(r, -n)G^{-1}_{22}(r-r')\phi_2(r', n)]}.
\end{align}
\[ \langle \ldots \rangle_{\phi_2} = N \int D\phi_2 \langle \ldots \rangle e^{-\frac{1}{2} \int dr dr' \text{Str}[\phi_2(r,-)\phi_2^{\dagger}(r')\phi_2(r',n)]}. \]  

(A.8)