Coherent Free-Space Optical Communication
Using Non-mode-Selective Photonic Lantern

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Abstract

A coherent free-space optical communication system based on non-mode-selective photonic lantern is studied. Based on simulation of photon distribution, the power distribution at single-mode fiber end of the photonic lantern is quantitatively described as a truncated Gaussian distribution over a simplex. The signal-to-noise and the outage probability are analyzed for the communication system using photonic lantern based receiver with equal-gain combining, and they are compared with those of the single-mode fiber receiver and multimode fiber receiver. The scope of application of the communication system is provided. It is shown that the signal-to-noise ratio gain of the photonic lantern based receiver over single-mode fiber receiver and multimode fiber receiver can be greater than 7 dB. The integral solution, series lower bound solution and asymptotic solution are presented for bit-error rate of photonic lantern based receiver, single-mode fiber receiver and multimode fiber receiver over the Gamma-Gamma atmosphere turbulence channels. Simulation results show that for the considered system the power distribution of the photonic lantern has limited influence on the outage probability and the bit-error rate performance.

Index Terms

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I. INTRODUCTION

In satellite communication, coherent free-space optical communication (FSOC) technology is attractive for its high sensitivity and ability to obtain a high data rate [1]–[3]. Recently, researchers have focused on designing coherent optical communication systems using fiber-based transmitters and receivers. Because the fiber-based receiver can make full use of the commercial available components from fiber-optic communication systems, such as fiber transmitter and receiver, erbium-doped fiber amplifiers (EDFAs), and fiber multiplexer and demultiplexer units [4], [5]. However, such implementation has its limitations because the overall efficiency (which will be defined in the sequel) is low.

In a coherent optical communication system using fiber-based receiver, when the signal beam reaches the receiver aperture plane, it is first coupled into the fiber and then mixed with the local oscillator (LO) beam to obtain the mixed signal. There are three important parameters associated with this process: coupling efficiency, mixing efficiency, and overall efficiency. The coupling efficiency is defined as the ratio of the average power coupled into the fiber to the average power in the receiver’s aperture plane [4], [6]–[11]. The mixing efficiency is defined as the ratio of the amplitude of the obtained mixed signal to the amplitude of theoretically mixed signal [12]–[16]. The overall efficiency is defined as the product of the coupling efficiency and the mixing efficiency, which embodies the extent to which the signal beam can be fully utilized. Low overall efficiency can typically degrade signal-to-noise ratio (SNR) [13], [14].

There are two commonly used fiber-based receiver schemes for coherent optical communication systems. The first receiver scheme is the single-mode fiber (SMF) receiver with SMF mixing. The SMF only propagates one field mode. Because the received signal beam and the LO beam propagate in the same SMF, their field modes are the same, i.e., the received signal beam and LO beam fields are matched both spatially and temporally at the detector. Then the mixing efficiency between the LO beam and the signal beam approaches 100% [14]. However, the core diameter of SMF is small (~10 µm), which limits achievable fiber coupling efficiency, especially in the presence of atmosphere turbulence in free-space channels [4], [6], [7]. [12]. For example, the maximum coupling efficiency is 81% in the absence of atmosphere turbulence [6]. For a moderate strength turbulence \( C_n^2 = 10^{-13} \text{m}^{-2/3} \), where \( C_n^2 \) is refractive-index structure constant), the coupling efficiency is less than 5% [4]. The second receiver scheme is the multimode fiber
TABLE I
COMPARISON BETWEEN SMF RECEIVER AND MMF RECEIVER.

|                        | SMF receiver | MMF receiver |
|------------------------|--------------|--------------|
| The coupling efficiencies | low          | high         |
| The mixing efficiencies  | high         | low          |
| The overall efficiencies | low          | low          |

(MMF) receiver with MMF mixing [8], [9], [13]. The coupling efficiency of MMF, whose core diameter is $\sim 50 \mu m$ [9], is much higher than that of the SMF [10], [13], [17]–[19]. However, only the portion of the signal beam that is in the same temporal and spatial mode of the LO beam can produce high mixing efficiency [13]. The MMF contains not only the fundamental mode component, but also high-order mode components. Then the mixing efficiency between the LO beam and the signal beam will be degraded [13], [19], [20]. For example, the coupling efficiency of MMFs tested in [9] is greater than 95%; for asymmetric square waveguide supporting seventy-five distinct modes tested in [13], the coupling efficiency for MMF receiver is 75-78%; the mixing efficiency is 21-23%, and the overall efficiency becomes only 11-17%. The properties of the SMF receiver and MMF receiver are summarized in Table I. From Table I, we can conclude that both SMF receiver and MMF receiver have low overall efficiency.

Recently, a non-mode-selective photonic lantern (PL) based coherent optical receiver has been proposed, and the overall efficiency of this receiver can be improved [20]–[24]. Fig. 1 shows the schematic diagram of a PL [25], [26]. In this diagram, one end of the PL is a relatively-large multimode core, and the other end is an array of several relatively-small single-mode cores. In between is a transition region. Fig. 2 shows a structural diagram of a complete coherent FSOC system based on non-mode-selective PL. The signal beam is transmitted from the transmitter, and is coupled into the receiver after passing through the atmosphere turbulence. In the receiver, the large-core MMF end of the PL is placed behind the receiver len to collect the multimode signal beam. Then the PL converts the multimode signal beam into $N$ single-mode signal beams. The single-mode LO beam is split into $N$ equal parts by a fiber beam splitter (FBS). Each single-mode

$^1$There are two types of PLs [26], [27]: mode-selective PL [28], [29] and non-mode-selective PL [20]–[23]. In a mode-selective PL, the single-mode cores are designed for transmitting light with different electromagnetic wave modes. While in a non-mode-selective PL, the single-mode cores are designed for transmitting light having the same electromagnetic wave mode. This paper focuses on the non-mode-selective PL.
signal beam of the PL is mixed with a single-mode LO beam in an optical hybrid. After that, each mixed signal is converted into an electrical signal by the corresponding balanced photodetector. All the electrical signals are sent to the combiner and the demodulator for processing. The system can fully take advantage of the MMF, which has higher coupling efficiency compared with the SMF, and can take advantage of the SMF that has nearly 100% mixing efficiency with the single-mode LO beam [21].

A recent work [30] investigated the performance of coherent FSOC receiver under moderate-to-strong turbulence. However, the effect of the power distribution at SMF end of the PL on SNR was not studied. From [26], we know that the power distribution at SMF end of non-mode-selective PL varies according to the input mode profile, temperature or pressure variations on the MMF section of the PL. In FSOC, the signal beam impaired by atmospheric turbulence contains not only the fundamental mode component but also the higher-order mode components, and the influence of atmospheric turbulence on signal beam changes with time and space. Then the mode profile of the signal beam coupled into MMF end of the PL will change with time, resulting
in the power distribution variation at SMF end of a non-mode-selective PL. As a result, the SNR of the coherent optical receiver based on PL will change [24]. Therefore, it is necessary to study the power distribution at SMF end of the PL. In [24], we proposed two different distributions: the multivariate Gaussian distribution over a simplex for small power fluctuation case and the uniform distribution over a simplex for large power fluctuation case, to describe the power distribution at SMF end of non-mode selective PL. It was found that different power distributions can have different effects on SNR, and when the number of single-mode fibers of the PL is equal to the number of guided modes at multimode end of the PL, the average SNR attains its maximum value [24].

Different from [24], this paper proposes a more accurate power distribution: truncated Gaussian distribution over a simplex, and this proposal is based on the simulation results of photon distribution in Section II-B. In addition, the SNR and outage probability of the communication system using PL based receiver with equal-gain combining (EGC) are analyzed, and they are compared with traditional SMF and MMF receivers in Section III. The bit-error rate (BER) performance of a binary phase-shift keying (BPSK) system is analyzed. The integral solution, series solution and asymptotic solution of the BER are presented. The BER performance of the system is compared with the traditional SMF and MMF receivers over the Gamma-Gamma atmosphere turbulence channels in Section IV. Simulation results show that the signal-to-noise ratio gain of the PL based receiver with EGC over the single-mode fiber receiver and multimode fiber receiver can be greater than 7 dB; and the power distribution of the PL has only limited influence on the bit-error rate of the coherent communication system using PL based receiver with EGC. To the authors’ best knowledge, this is the first work that analytically quantifies the error performances of a non-mode-selective PL based receiver over SMF and MMF receivers.

II. System Model

A. Free-Space Atmosphere Channel

In FSOC, atmospheric turbulence introduces fluctuation of irradiance, which results in fluctuation of SNR. The probability density function (PDF) of the received signal irradiance can be modeled as a Gamma-Gamma distribution [31], [32], which emerges as a useful turbulence model as it has excellent fit with measurement data over a wide range of turbulence conditions
The PDF of the received signal irradiance $I$ ($I > 0$) is given by

$$f(I) = \frac{2(\alpha \beta)^{\frac{\alpha + \beta}{2}}}{\Gamma(\alpha) \Gamma(\beta)} I^{\alpha + \beta - 1} K_{\alpha - \beta} \left( \frac{2\sqrt{\alpha \beta I}}{\alpha \beta} \right),$$

(1)

where $\Gamma(\cdot)$ is the Gamma function; $K_{\alpha - \beta}(\cdot)$ is the modified Bessel function of the second kind with order $\alpha - \beta$. The parameters $\alpha$ and $\beta$ are directly related to the atmospheric conditions [31], and they respectively denote the effective numbers of large-scale and small-scale cells of the scattering process, respectively. Without loss of generality, the received signal irradiance $I$ is normalized, i.e., $E[I] = 1$, where $E[\cdot]$ denotes the mathematical expectation.

**B. Power Distribution in PL**

When the signal beam transmitted from the transmitter reaches the receiver system after passing through the atmosphere turbulence, it is coupled into the MMF end of the PL. We assume the power received at MMF end of the PL is $P_M$, then we have [33]

$$P_M = \zeta_M AI,$$

(2)

where $\zeta_M$ is the coupling efficiency of MMF, and $A$ is the area of receiving aperture of the len. When the PL converts the multimode signal beam into $N$ single-mode signal beams, loss will be introduced [26]. If we denote the loss factor of the PL by $\xi_{PL} (0 < \xi_{PL} \leq 1)$, then the output optical power of the PL is $P_S = \xi_{PL} P_M$.

For a PL with $N$ SMFs, if we denote the power distributed at each SMF end by $P_{S,i}$ ($i = 1, 2, \cdots, N$) and denote the ratio of $P_{S,i}$ to $P_S$ by $a_i$, then we have

$$P_{S,i} = a_i P_S = a_i \xi_{PL} \zeta_M AI,$$

(3)

where random variables (RVs) $a_i$ ($i = 1, 2, \cdots, N$) satisfy

$$a_1 + a_2 + \cdots + a_N = 1, \quad 0 \leq a_i \leq 1, \quad i = 1, 2, \cdots, N,$$

(4)

where the set of $\{a_1, a_2, \cdots, a_N\}$ that satisfies (4) is called a standard unit simplex [34].

The exact power distribution at SMF end of the PL is not known. Because the optical power is proportional to the photon number, the ratios $\{a_1, a_2, \cdots, a_N\}$ for the optical power is identical to the ratios for the photon numbers. Therefore, we can simulate the photon distribution to obtain the power distribution.
Fig. 3. The obtained ratio distribution for one SMF end of the PL with $N = 5$, $M = 500$, and $L = 10^7$ (The range of $a_i$ is between 0 and 1. For simplicity, we only plot the range of $a_i$ from 0 to 0.4.)

I) Simulation Model For Photon Distribution: Here, we use a Monte-Carlo method to simulate the photon distribution at SMF end of the PL. We denote the number of SMF of the PL by $N$. Because the loss of PL has no effect on the power distribution at SMF end of the PL, we do not consider the loss of PL in the simulation of photon distribution. Because this work assumes non-mode-selective PL, it is reasonable to assume that each SMF of a PL is exactly the same. Then the probability of each photon at MMF end assigned to any SMF of PL is assumed the same. Therefore, the explicit Monte-Carlo process can be summarized as follows: Step 1, we first generate $M$ photons and assign each photon into one SMF end randomly; Step 2, we calculate and record the ratio of the photon number $m_i$ of $i$th SMF end to the total photon number $M$ as $a_i = m_i/M$, where $i = 1, 2, \cdots, N$; Step 3, repeat Step 1 and Step 2 $L$ times. Then we can obtain the distribution of $a_i$ from its $L$ samples for the $i$th SMF and obtain the correlation coefficient between $a_i$ and $a_j$ for $i \neq j$.

The obtained distribution of the ratio $a_i$ for some SMF end is shown in Fig. 3. We find that the photon number distribution at the $i$th SMF end of the PL has excellent fit with the truncated
Gaussian distribution with mean value $1/N$. The obtained correlation coefficients between the ratios of two distinct SMFs over the number of SMF ends are shown in Fig. 4. We can see that the correlation coefficients between the ratios of two distinct SMFs are always negative, which is due to the constraint (4). Besides, we can see that the correlation coefficient between two SMFs increases as $N$ increases. For example, when $N = 2$, according to the constraint (4), the correlation coefficient between two SMFs is $-1$. As $N$ approaches $\infty$, the correlation coefficient between two SMFs should approach 0. We also perform the curve fitting on the simulation results and find that the correlation coefficients can be fitted as $-\frac{1}{N-1}$, which coincides to the analytical result obtained in Section II-B2.

2) Truncated Multivariate Gaussian Model For Power Distribution: According to above simulation results, it is reasonable to assume that the ratios $\mathbf{a} = [a_1, a_2, \cdots, a_N]^T$ for the optical power satisfies a truncated multivariate Gaussian distribution [35] over the simplex defined in (4). The mathematical expectation of this truncated multivariate Gaussian distribution

\(^2\)We remark that the obtained variance of $a_i$ can vary as the number of simulation repeating times varies due to the converging property of the Monte-Carlo method. A large number of repeating times results in a small variance. However the correlation coefficient between $a_i$ and $a_j$ is independent of the number of repeating times
is \( E[a] = \mu_a = [\frac{1}{N}, \frac{1}{N}, \cdots, \frac{1}{N}]^T \), where \([\cdot]^T\) represents the transpose operator 3. Here we derive the PDF of this truncated multivariate Gaussian distribution analytically.

We first remove the constraint \( a_1 + a_2 + \cdots + a_N = 1 \), then the joint PDF of the truncated multivariate Gaussian distribution has the following form:

\[
   f(a) = \frac{1}{C_1} \exp \left\{ -\frac{1}{2}[a - \mu_a]^T \Sigma_a^{-1} [a - \mu_a] \right\},
   \quad 0 \leq a_i \leq 1, \quad i = 1, 2, \cdots, N,
\]

where \( C_1 = \int_V \exp \left\{ -\frac{1}{2}[a - \mu_a]^T \Sigma_a^{-1} [a - \mu_a] \right\} dV \) is a constant number for normalization; \( V \) is the domain defined as \( V = \{0 \leq a_i \leq 1, i = 1, 2, \cdots, N\} \); \( \Sigma_a \) is the covariance matrix of \( a \). Because this work assumes non-mode-selective PL, it is reasonable to assume that \( a_1, a_2, \cdots, a_N \) have the same Gaussian variance 4 \( \text{var}(a_i) = \sigma^2, \quad i = 1, 2, \cdots, N \); and the Gaussian covariances \( \text{cov}(a_i, a_j) \) for any \( a_i \) and \( a_j \), when \( i \neq j, \quad i, j = 1, 2, \cdots, N \), are the same. Then the \( N \times N \) dimensional Gaussian covariance matrix \( \Sigma_a \) can be written as

\[
   \Sigma_a = \sigma^2 \begin{bmatrix}
   1 & \rho & \cdots & \rho \\
   \rho & 1 & \cdots & \rho \\
   \vdots & \vdots & \ddots & \vdots \\
   \rho & \rho & \cdots & 1
   \end{bmatrix},
\]

where \( \rho = \frac{\text{cov}(a_i, a_j)}{\sigma^2} \) is the correlation coefficient between \( a_i \) and \( a_j \) when \( i \neq j, \quad i, j = 1, 2, \cdots, N \). Then inverse matrix \( \Sigma_a^{-1} \) in (5) can be obtained as

\[
   \Sigma_a^{-1} = \frac{1}{[1 + (N - 1)\rho](1 - \rho)\sigma^2} \times \begin{bmatrix}
   1 + (N - 2)\rho & -\rho & \cdots & -\rho \\
   -\rho & 1 + (N - 2)\rho & \cdots & -\rho \\
   \vdots & \vdots & \ddots & \vdots \\
   -\rho & -\rho & \cdots & 1 + (N - 2)\rho
   \end{bmatrix}.
\]

However, when the constraint \( a_1 + a_2 + \cdots + a_N = 1 \) is considered, the covariance matrix \( \Sigma_a \) becomes a rank-deficient matrix and it has no inverse matrix. We first derive the correlation

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3Our analysis can be easily extended to the cases where different SMF ends have different mean values by replacing \( \mu_a \) with the actual mean values.

4Note that the Gaussian variance \( \text{var}(a_i) \) here is not the actual variance of \( a_i \). This is because the multivariate Gaussian distribution characterized is truncated by the definition domain \( V \). Then the actual variance \( \text{var}_{\text{Actual}}(a_i) \) is defined as \( \text{var}_{\text{Actual}}(a_i) = \int_V (a_i - 1/N)^2 f(a) dV \), which is smaller than the Gaussian variance \( \text{var}(a_i) \).
The constraint \( a_1 + a_2 + \cdots + a_N = 1 \) can be rewritten as \([a - \mu_a]^T \mathbf{1} = 0\), where \( \mathbf{1} = [1, 1, \cdots, 1]^T \) is an \( N \times 1 \) dimensional vector. Then we have [24]

\[
E \left[ (a - \mu_a)(a - \mu_a)^T \right] = \Sigma_a \mathbf{1}
\]

\[
= \sigma^2 (1 + (N - 1) \rho) \mathbf{1}
\]

\[
= 0,
\]

where \( \mathbf{0} = [0, 0, \cdots, 0]^T \) is an \( N \times 1 \) dimensional zero vector. Therefore, the correlation coefficient \( \rho \) can be obtained from (8) as \( \rho = -\frac{1}{N-1} \), which is the same as the correlation coefficient obtained from the simulation result in Fig. 4. This correlation coefficient is also consistent with the inverse matrix in (7) because the numerator of \( \Sigma_a^{-1} \) becomes zero when \( \rho = -\frac{1}{N-1} \), and thus the inverse matrix does not exist.

To obtain the explicit form of the joint PDF, we generalize a generalized inverse matrix of \( \Sigma_a \), and let \( \rho \to -\frac{1}{N-1} \) when the constraint \( a_1 + a_2 + \cdots + a_N = 1 \) is considered. Then the joint PDF of \( a \) can be obtained by substituting (7) into (5) and letting \( \rho \to -\frac{1}{N-1} \). After some algebra (see Appendix A), the joint PDF can be obtained as

\[
f(a) = \frac{1}{C_2} \exp \left\{ -\frac{1}{2} [a^* - \mu_{a^*}]^T \Sigma_{a^*}^{-1} [a^* - \mu_{a^*}] \right\}
\]

\[
\times \delta \left( a_1 + a_2 + \cdots + a_{N-1} \right),
\]

\[
0 \leq a_i \leq 1, \ i = 1, 2, \cdots, N,
\]

where

\[
C_2 = \int_V \exp \left\{ -\frac{1}{2} [a^* - \mu_{a^*}]^T \Sigma_{a^*}^{-1} [a^* - \mu_{a^*}] \right\}
\]

\[
\times \delta \left( a_1 + a_2 + \cdots + a_{N-1} \right) \, dV
\]

is a constant normalization factor; \( a^* = [a_1, a_2, \cdots, a_{N-1}]^T \) is an \((N-1) \times 1\) dimensional vector; \( \mu_{a^*} = [\frac{1}{N}, \frac{1}{N}, \cdots, \frac{1}{N}]^T \) is an \((N-1) \times 1\) dimensional vector; and the covariance matrix \( \Sigma_{a^*} \) for \( a^* \) is the first \((N-1) \times (N-1)\) dimensional submatrix of \( \Sigma_a \). Then the inverse of \( \Sigma_{a^*} \) can be obtained as

\[
\Sigma_{a^*}^{-1} = \frac{N-1}{N \sigma^2} \begin{bmatrix}
2 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 2
\end{bmatrix}
\]

(11)
C. Two Extreme Cases

Here we present two extreme cases of the truncated Gaussian distribution: the (joint) degenerate distribution and the (joint) uniform distribution, corresponding to the cases of minimum Gaussian variance $\sigma^2 = 0$ and maximum Gaussian variance $\sigma^2 = \infty$, respectively.

1) Degenerate Distribution Case: For a degenerate distribution, $a_i = \frac{1}{N}, i = 1, 2, \cdots, N$ with probability one. Therefore, the joint PDF can be expressed as

$$f(a) = \prod_{i=1}^{N} \delta(a_i - \frac{1}{N}).$$  \hspace{1cm} (12)

2) Uniform Distribution Case: For an uniform distribution, $a_i, i = 1, 2, \cdots, N$ is uniformly distributed on $[0, 1]$. The explicit joint PDF can be obtained by letting $\sigma^2 \to \infty$ in (9), i.e.,

$$f(a) = \lim_{\sigma^2 \to \infty} \frac{\exp\left\{ \frac{1}{2} (a^* - \mu_a^*)^T \Sigma^{-1}_a (a^* - \mu_a^*) \right\}}{C_2} \delta(a_1 + a_2 + \cdots + a_N - 1).$$ \hspace{1cm} (13)

From the expression of $\Sigma^{-1}_a$ in (11), we can find that the exponential term in (13) approaches one when $\sigma^2 \to \infty$. Similarly, for the denominator $C_2$, we have

$$\lim_{\sigma^2 \to \infty} C_2 = \int_{V} \delta(a_1 + a_2 + \cdots + a_N - 1) \, dV$$

$$= \int_{V_s} \, dV_s$$

$$= \frac{1}{(N-1)!},$$ \hspace{1cm} (14)

where $V_s = \frac{1}{(N-1)!}$ is the volume of the standard simplex defined in (4). Substituting (14) into (13), we can obtain the joint PDF as

$$f(a) = (N-1)! \delta(a_1 + a_2 + \cdots + a_N - 1).$$ \hspace{1cm} (15)

III. Signal-to-Noise Ratio and Outage Probability

A. Signal-to-Noise Ratio

When the output signals of the SMF ends are combined using EGC \(^6\) and the shot noise is the dominated noise source, the instantaneous SNR can be obtained as [24]

$$\gamma_{PL} = \frac{R_{\eta S} \left( \sum_{i=1}^{N} \sqrt{P_{S,i}} \right)^2}{NqB},$$ \hspace{1cm} \hspace{1cm} (16)

\(^5\)The actual variance of $a_i$ is $\frac{1}{12}$, because $a_i$ is uniformly distributed on $[0, 1]$.

\(^6\)Here EGC method is used to combine the output signals of all SMF ends of the PL. Because there is only one receiving port and no diversity technique is introduced, the name “EGC” should not be confused with the diversity combining technique EGC in wireless communications.
where $R$ is the responsivity of the photodiode; $\eta_S$ is the mixing efficiency of SMF; $q$ is the electronic charge and $B$ is the noise equivalent bandwidth of the detector. Substituting $P_{S,i}$ into $\gamma_{PL}$, we can obtain

$$\gamma_{PL} = K \left( \sum_{i=1}^{N} \sqrt{a_i} \right)^2 I,$$

where $K = \frac{R\xi_{PL} \xi_{PL} B}{N q B}$.

Then the average SNR of coherent FSOC system using PL based receiver with EGC is

$$\bar{\gamma}_{PL} = E[\gamma_{PL}]$$

$$= KE \left[ \left( \sum_{i=1}^{N} \sqrt{a_i} \right)^2 \right] E[I]$$

$$= KE \left[ \left( \sum_{i=1}^{N} \sqrt{a_i} \right)^2 \right],$$

where we have used the assumption $E[I] = 1$ and $E \left[ \left( \sum_{i=1}^{N} \sqrt{a_i} \right)^2 \right]$ can be obtained as

$$E \left[ \left( \sum_{i=1}^{N} \sqrt{a_i} \right)^2 \right] = 1 + N(N - 1)E[\sqrt{a_1 a_2}].$$

It is challenging to obtain an analytical expression for $E \left[ \left( \sum_{i=1}^{N} \sqrt{a_i} \right)^2 \right]$ or $E[\sqrt{a_1 a_2}]$ for a general truncated multivariate Gaussian distribution. However, it is still meaningful to consider the extreme cases defined in II-C, because the degenerate case and the uniform case correspond to the best and the worst average SNR performances, respectively.

1) Degenerate Distribution Case: For the degenerate distribution, $a_i = \frac{1}{N}, (i = 1, 2, \cdots, N)$ and we have $E \left[ \left( \sum_{i=1}^{N} \sqrt{a_i} \right)^2 \right] = N$. Then the average SNR becomes

$$\bar{\gamma}_{PL,\text{Deg}} = KN.$$  

2) Uniform Distribution Case: For the uniform distribution, $a_i, (i = 1, 2, \cdots, N)$ is uniformly distributed in $[0, 1]$ and we can obtain $E[\sqrt{a_1 a_2}] = \frac{\pi}{4N}$ (see Appendix B). Then the average SNR is

$$\bar{\gamma}_{PL,\text{Uni}} = K \frac{\pi N + 4 - \pi}{4}.$$
For a general truncated multivariate Gaussian distribution, i.e., $0 < \sigma^2 < \infty$, the average SNR is between the SNR in (21) and (20). An important observation is that the average SNR ratio of the best case (degenerate distribution) over the worst case (uniform distribution) is

$$\frac{\bar{\gamma}_{PL,Deg}}{\bar{\gamma}_{PL,Uni}} = \frac{4N}{\pi N + 4 - \pi},$$

(22)

which is a function of the number of SMF $N$, and it is between $\frac{8}{4+\pi} \approx 1.12$ when $N = 2$ and $\frac{4}{\pi} \approx 1.27$ when $N = \infty$. This implies that the influence of the power distribution of PL on the average SNR is relatively small when EGC method is used for signal combining.

For comparison, we also present the SNR of the SMF receiver and MMF receiver here. When shot noise is the dominated noise, the instantaneous SNR of the SMF receiver is

$$\gamma_{SMF} = \frac{\zeta_S \eta_S RA}{qB} I,$$

(23)

where $\zeta_S$ is the coupling efficiency of SMF; and the average SNR of SMF is

$$\bar{\gamma}_{SMF} = E[\gamma_{SMF}] = \frac{\zeta_S \eta_S RA}{qB}.$$

(24)

Similarly, the instantaneous SNR of the MMF receiver is

$$\gamma_{MMF} = \frac{\zeta_M \eta_M RA}{qB} I,$$

(25)

where $\eta_M$ is the mixing efficiency of MMF mixer; and the average SNR of MMF is

$$\bar{\gamma}_{MMF} = E[\gamma_{MMF}] = \frac{\zeta_M \eta_M RA}{qB}.$$

(26)

B. Outage Probability

Given a threshold SNR $\gamma_{th}$, the outage probability for coherent communication using PL based receiver with EGC can be obtained as

$$P_{outage,PL} = P_r[\gamma_{PL} < \gamma_{th}]$$

$$= \int K(\sum_{i=1}^{N} \sqrt{a_i})^2 I < \gamma_{th} f(a) f(I) da dI$$

(27)

$$= \int_V f(a) \left[ \int_0^{\gamma_{th}/K(\sum_{i=1}^{N} \sqrt{a_i})^2} f(I) dI \right] da.$$

1) Degenerate Distribution Case: For the degenerate distribution, $a_i = \frac{1}{N}, (i = 1, 2, \cdots, N)$, then the outage probability becomes

$$P_{outage,PL,Deg} = \int_0^{\gamma_{th}/\bar{\gamma}_{PL,Deg}} f(I) dI.$$

(28)
2) Uniform Distribution Case: For the uniform distribution, by substituting (15) into (27), we can obtain the outage probability as

\[
P_{\text{outage},\text{PL,Uni}} = (N - 1)! \int_V \delta \left( a_1 + a_2 + \cdots + a_{N-1} \right) \left( \int_0^{\frac{\gamma_{th}}{K}} f(I) dI \right) d\mathbf{a}.
\]

(29)

Similarly, the outage probabilities for SMF receiver \(P_{\text{outage,SMF}}\) and MMF receiver \(P_{\text{outage,MMF}}\) can be obtained as

\[
P_{\text{outage,SMF}} = \int_0^{\frac{\gamma_{th}}{\bar{\gamma}_{\text{SMF}}}} f(I) dI
\]

(30)

and

\[
P_{\text{outage,MMF}} = \int_0^{\frac{\gamma_{th}}{\bar{\gamma}_{\text{MMF}}}} f(I) dI,
\]

(31)

respectively.

IV. Bit-Error Rate

A. Integral Expression of BER

The BER conditioned on received signal irradiance \(I\) and power distribution \(\mathbf{a}\) for an FSOC BPSK system \(^7\) using PL based receiver with EGC is given by [36]

\[
P_{e,\text{PL}}(I, \mathbf{a}) = Q(\sqrt{\gamma_{PL}}),
\]

(32)

where \(Q(\cdot)\) is the Gaussian Q-function; and \(\gamma_{PL}\) is the instantaneous SNR. Then the unconditional BER for PL based receiver with EGC can be obtained as the following integral form

\[
P_{e,\text{PL}} = \int_0^{\infty} \int_0^{\frac{\gamma_{th}}{\bar{\gamma}_{\text{SMF}}}} f(I) f(\mathbf{a}) Q(\sqrt{\gamma_{PL}}) d\mathbf{a} dI,
\]

(33)

where \(f(I)\) is the PDF of the signal irradiance \(I\) given in (1), and \(f(\mathbf{a})\) is the joint PDF of power ratios \(\mathbf{a}\) given in (9).

Similarly, the unconditional BERs for SMF receiver and MMF receiver are obtained as

\[
P_{e,\text{SMF}} = \int_0^{\infty} f(I) Q(\sqrt{\gamma_{\text{SMF}}}) dI
\]

(34)

and

\[
P_{e,\text{MMF}} = \int_0^{\infty} f(I) Q(\sqrt{\gamma_{\text{MMF}}}) dI,
\]

(35)

respectively.

\(^7\) Although we only present the BER for BPSK scheme here, the BER and symbol error rate (SER) for other coherent modulation schemes can be easily found in a similar way.
B. Analytical Lower Bound For BER

Substituting (17) into (33), we can obtain the unconditional BER as

$$P_{e,PL} = \int_0^\infty \int_V f(I)f(a)Q\left(\sqrt{K\left(\sum_{i=1}^N \sqrt{a_i}\right)^2 I}\right) da dI.$$  \hfill (36)

Note that \(\left(\sum_{i=1}^N \sqrt{a_i}\right)^2 \leq N\), where the equal sign is obtained when \(a_i = \frac{1}{N}, i = 1, 2, \cdots, N\). Then we can obtain a lower bound for \(P_{e,PL}\) as

$$P_{lower} = \int_0^\infty \int_V f(I)f(a)Q\left(\sqrt{KNI}\right) da dI \hfill (37)$$

which is also the unconditional BER of PL based receiver for the degenerate distribution case.

Then we can obtain an analytical expression of the lower bound (37) by using a series expansion of the modified Bessel function of the second kind in (1) as [36]

$$K_v(x) = \frac{\pi}{2\sin(\pi v)} \sum_{p=0}^\infty \left[ \frac{(x/2)^{2p-v}}{\Gamma(p-v+1)p!} - \frac{(x/2)^{2p+v}}{\Gamma(p+v+1)p!} \right],$$  \hfill (38)

\(v \notin Z, |x| < \infty\)

and an alternative expression of the \(Q\)-function [37]

$$Q(x) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(\frac{-x}{2\sin^2 \theta}\right) d\theta.$$  \hfill (39)

Substituting (1), (38), and (39) into (37), and after some algebra (see Appendix C), we can obtain an analytical lower bound in series form as

$$P_{lower} = \frac{\Lambda(\alpha, \beta)}{2} \sum_{p=0}^\infty \left\{ a_p(\alpha, \beta) \left(\bar{\gamma}_{PL, Deg} / 2\right)^{(p+\beta)} B\left(\frac{1}{2}, p + \beta + \frac{1}{2}\right) - a_p(\beta, \alpha) \left(\bar{\gamma}_{PL, Deg} / 2\right)^{(p+\alpha)} B\left(\frac{1}{2}, p + \alpha + \frac{1}{2}\right) \right\},$$  \hfill (40)

where \(B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt\) is the Beta function, and

$$\Lambda(\alpha, \beta) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\sin[(\alpha - \beta)\pi]},$$

$$a_p(x, y) = \frac{(xy)^{p+y}\Gamma(p+y)}{\Gamma(p-x+y+1)p!}.$$  \hfill (41)

In addition, by replacing \(\bar{\gamma}_{PL, Deg}\) in (40) with \(\bar{\gamma}_{SMF}\) and \(\bar{\gamma}_{MMF}\), we can obtain the unconditional BER for SMF receiver and MMF receiver, respectively.
C. Truncation Error Analysis

To implement the series form lower bound BER in (40), we have to truncate the summation of infinite terms into a summation of finite terms. Therefore, it is necessary to analyze the truncation error. For simplicity, in the following we use $\bar{\gamma}$ to represent $\bar{\gamma}_{PL,\text{Deg}}, \bar{\gamma}_{\text{SMF}}$, and $\bar{\gamma}_{\text{MMF}}$. Substituting $B(x, y) = \Gamma(x)\Gamma(y) / \Gamma(x+y)$ [38, 8.384(1)] and $\Gamma(x+1) = x\Gamma(x)$ [38, 8.331(1)] into (40), we obtain the error probability as

$$P_e = \frac{\sqrt{\pi}}{2} \Lambda(\alpha, \beta) \sum_{p=0}^{\infty} \frac{1}{p!} \left( \frac{2\alpha\beta}{\bar{\gamma}} \right)^p \{ G_p(\alpha, \beta) - G_p(\beta, \alpha) \},$$

(42)

where $G_p(x, y)$ is defined as

$$G_p(x, y) = \frac{\Gamma(p + y + \frac{1}{2})}{(p + y)\Gamma(p - x + y + 1)} \left( \frac{2xy}{\bar{\gamma}} \right)^y.$$  

(43)

Now we can estimate the truncation error caused by eliminating the infinite terms after the first $J$ terms in (42). This truncation error can be defined as

$$\epsilon_J = \frac{\sqrt{\pi}}{2} \Lambda(\alpha, \beta) \sum_{p=0}^{\infty} \frac{1}{p!} \left( \frac{2\alpha\beta}{\bar{\gamma}} \right)^p \{ G_p(\alpha, \beta) - G_p(\beta, \alpha) \}.  

(44)

When $p \to \infty$, we have $G_p(\alpha, \beta) \to 0$ and $G_p(\beta, \alpha) \to 0$. Then we can obtain an upper bound of the truncation error as

$$\epsilon_J < \frac{\sqrt{\pi}}{2} \Lambda(\alpha, \beta) \sum_{p=0}^{\infty} \frac{1}{p!} \left( \frac{2\alpha\beta}{\bar{\gamma}} \right)^p \max_{p \geq L} \{ G_p(\alpha, \beta) - G_p(\beta, \alpha) \}$$

$$< \frac{\sqrt{\pi}}{2} \Lambda(\alpha, \beta) \left( \frac{2\alpha\beta}{\bar{\gamma}} \right)^J \exp \left( \frac{2\alpha\beta}{\bar{\gamma}} \right)$$

$$\times \max_{p \geq L} \{ G_p(\alpha, \beta) - G_p(\beta, \alpha) \},$$

(45)

where in the last inequality we have used the Lagrange form for the remainder term of Taylor series expansion for the exponential function. Note that when $J$ approaches $\infty$, the term $J! \left( \frac{2\alpha\beta}{\bar{\gamma}} \right)^J$ approaches zero. Therefore, truncation error $\epsilon_J$ diminishes to zero with increasing index $J$. Besides, we can also observe that $\epsilon_J$ diminishes rapidly with the average SNR $\bar{\gamma}$. This suggests that the series lower bound solution is highly accurate in the large SNR regimes. We can therefore perform an asymptotic BER analysis.

D. Asymptotic Lower Bound For BER

We now examine the lower bound BER behavior in the large SNR regimes. When $\bar{\gamma} \to \infty$, we have $G_p(\alpha, \beta) \to 0$ and $G_p(\beta, \alpha) \to 0$. From (42) we know that the first term ($p = 0$)
of the series summation becomes the dominant term in the large SNR regimes. Therefore, the unconditional lower bound BER in high SNR regimes can be approximated by

$$P_e \approx \sqrt{\pi} \Lambda(\alpha, \beta) \left[ G_0(\alpha, \beta) - G_0(\beta, \alpha) \right].$$  \hspace{1cm} (46)

For typical turbulence conditions, we have $\alpha > \beta$. Then in high SNR regimes, we have

$$\frac{G_0(\beta, \alpha)}{G_0(\alpha, \beta)} = \frac{\beta \Gamma(\alpha + \frac{1}{2}) \Gamma(-\alpha + \beta + 1)}{\alpha \Gamma(\beta + \frac{1}{2}) \Gamma(-\beta + \alpha + 1)} \left( \frac{2\alpha\beta}{\bar{\gamma}} \right)^{\alpha - \beta} \ll 1.$$  \hspace{1cm} (47)

Therefore, we can omit the second term in (46) and obtain

$$P_e \approx H(\alpha, \beta) \left( \frac{1}{\bar{\gamma}} \right)^{\beta},$$  \hspace{1cm} (48)

where

$$H(\alpha, \beta) = \frac{\sqrt{\pi}}{2} \frac{(2\alpha\beta)^{\beta} \Gamma(\beta + \frac{1}{2})}{\Gamma(\alpha) \Gamma(\beta + 1) \Gamma(-\alpha + \beta + 1) \sin[(\alpha - \beta)\pi]}.$$  \hspace{1cm} (49)

This indicates that the asymptotic lower bound BERs in high SNR regimes of the coherent optical communication system based on PL receiver, SMF receiver, and MMF receiver are decayed exponentially by the average SNR with an exponential decay constant $\beta$.

V. Numerical Results

In Sections III and IV, we only present the integral form of the normalization constant $C_2$ in (10), the average SNR $\bar{\gamma}_{PL}$ in (18), the outage probability $P_{out, PL}$ in (27), and the unconditional BER $P_{e, PL}$ in (33) when the PL power distribution satisfies a general truncated multivariate Gaussian distribution. When $N$ is large, we have to count on the stochastic numerical integration methods. However, the generation of random numbers satisfying truncated multivariate Gaussian distribution is not trivial. Here we use the Monte-Carlo integration (MCI) method (see Appendix D) to calculate $C_2$, $\bar{\gamma}_{PL}$, $P_{out, PL}$, and $P_{e, PL}$. We set the number of SMF ends as $N = 10$ in the following simulations. The turbulence parameters are $(\alpha = 2.23, \beta = 1.54)$ for moderate turbulence condition and $(\alpha = 2.34, \beta = 1.02)$ for strong turbulence condition [31].

We first present the average SNR gain of the PL based receiver with EGC over the SMF receiver

$$\frac{\bar{\gamma}_{PL}}{\bar{\gamma}_{SMF}} = \frac{\xi_{PL}}{N} \frac{\xi_M}{\xi_S} E \left[ \left( \sum_{i=1}^{N} \sqrt{a_i} \right)^2 \right],$$

and over the MMF receiver

$$\frac{\bar{\gamma}_{PL}}{\bar{\gamma}_{MMF}} = \frac{\xi_{PL}}{N} \frac{\eta_M}{\eta_{SMF}} E \left[ \left( \sum_{i=1}^{N} \sqrt{a_i} \right)^2 \right].$$

We analyze the value of the coupling efficiencies of MMF, few-mode fiber and SMF in the literature [4], [6]–[11], [19], [20], and set the coupling efficiency gain of MMF over SMF as $\frac{\xi_M}{\xi_S} \in [0, 20]$. We analyze the value of the mixing efficiency of SMF and
MMF in the literature [12]–[16], [39], and set the mixing efficiency gain of SMF over MMF as $\frac{\eta_S}{\eta_M} \in [4, 8]$. The range of the PL loss is set as $\xi_{PL} \in [0, 1]$. For the PL based receiver, we consider three different Gaussian variances $\sigma^2 = 0$, $\sigma^2 = 0.01$, and $\sigma^2 = \infty$, which corresponds to the multivariate degenerate distribution, general truncated multivariate Gaussian distribution, and multivariate uniform distribution, respectively.

Figures 5 and 6 show the obtained average SNR gain $\bar{\gamma}_{PL}/\bar{\gamma}_{SMF}$ and $\bar{\gamma}_{PL}/\bar{\gamma}_{MMF}$, respectively. When $\bar{\gamma}_{PL}/\bar{\gamma}_{SMF} > 1$, we can choose to use the PL based receiver instead of SMF receiver for coherent FSOC systems. When $\bar{\gamma}_{PL}/\bar{\gamma}_{MMF} > 1$, we can choose to use the PL based receiver instead of MMF receiver for coherent FSOC systems. From Figs. 5 and 6, we can observe that the difference of the average SNR gain among three PL power distributions increases as average SNR gain increases. This indicates that the influence of the PL power distribution on the average SNR gain becomes significant in high SNR gain.

In addition, from Fig. 5 we can see that when $\xi_{PL} > 0.31$, or $\frac{\xi_M}{\xi_S} > 6$, the average SNR gain $\bar{\gamma}_{PL}/\bar{\gamma}_{SMF}$ can be larger than 5 even the PL power distribution subjects to uniform distribution, i.e., the SNR gain of PL based receiver over the SMF receiver for coherent FSOC system can be greater than 7 dB. Similarly, from Fig. 6 we can see that when $\xi_{PL} > 0.55$, or $\frac{\eta_S}{\eta_M} > 7.1$, the average SNR gain $\bar{\gamma}_{PL}/\bar{\gamma}_{MMF}$ can be larger than 5 even the PL power distribution subjects to
uniform distribution, i.e., the SNR gain of PL based receiver over the MMF receiver for coherent FSOC system can be greater than 7 dB.

Then we present the outage probability comparison between PL based receiver, SMF receiver, and MMF receiver for coherent FSOC systems under the moderate and strong turbulence conditions, shown in Fig. 7. The horizontal axis is set as average SNR of the SMF receiver, i.e., $\bar{\gamma}_0 = \bar{\gamma}_{SMF}$. The total loss of the PL tested in [20] is 1.3 dB, then we set $\xi_{PL} = 0.8$. In addition, we let $\frac{\eta_S}{\eta_M} = 5$ and $\frac{\zeta_M}{\zeta_S} = 6$ in the simulation. The SNR threshold is $\gamma_{th} = 2$. From Fig. 7, we can find that the outage probability for PL based receiver under different PL power distributions are close to each other. The ratio of outage probability when $\sigma^2 = \infty$ over outage probability when $\sigma^2 = 0$ is smaller than 1.5 for moderate turbulence and 1.3 for strong turbulence. This indicates that the PL power distribution has limited influence on the outage probability of the coherent FSOC system using PL based receiver with EGC.

Next we present the BER comparison between PL based receiver, SMF receiver, and MMF receiver for coherent FSOC systems under the moderate and strong turbulence conditions, shown in Fig. 8. From Fig. 8, we can find that the BER for PL based receiver under different PL power distributions are close to each other. The ratio of BER when $\sigma^2 = \infty$ over BER when $\sigma^2 = 0$ is smaller than 1.4 for moderate turbulence and 1.3 for strong turbulence. This indicates that
Fig. 7. The outage probability comparison between PL based receiver, SMF receiver, and MMF receiver for coherent FSOC system ($\xi_{PL} = 0.8$, $\frac{n_S}{n_M} = 5$, $\frac{\xi_M}{\xi_S} = 6$, $\gamma_{th} = 2$)

Fig. 8. The BER comparison between PL based receiver, SMF receiver, and MMF receiver for coherent FSOC system ($\xi_{PL} = 0.8$, $\frac{n_S}{n_M} = 5$, $\frac{\xi_M}{\xi_S} = 6$)
the PL power distribution has limited influence on the BER performance of the coherent FSOC system using PL based receiver.

Besides, we can find that, when the BER is $10^{-6}$, the $\bar{\gamma}_0$ for PL based receiver, MMF receiver, and SMF under moderate turbulence are about 37 dB, 42.5 dB and 43 dB, respectively; and under strong turbulence are about 53 dB, 57.5 dB and 58 dB, respectively. This suggests that SMF receiver and MMF receiver require an additional 6 dB and 5.5 dB SNR to achieve the same BER as the PL based receiver with EGC under moderate turbulence; and require an additional 5 dB and 4.5 dB SNR to achieve the same BER as the PL based receiver with EGC under strong turbulence.

At last, we present the integral solution, series solution and the asymptotic solution of the unconditional lower bound BER in Fig. 9. The series solution is calculated by (40) with $J = 30$. We can see that the series lower bound solution is consistent with the integral solution, and the asymptotic lower bound BER approaches the exact BER curve in high SNR regimes ($\bar{\gamma} > 30$ dB).
VI. Conclusion

This paper proposed a truncated multivariate Gaussian distribution over a simplex for the power distribution at SMF ends of the PL. We analytically quantified the advantage of PL based receiver with EGC over SMF and MMF receivers for FSOC systems. Simulation results showed that the SNR gain of coherent FSOC systems using PL based receiver over SMF receiver and MMF receiver can be greater than 7 dB; and the power distribution of the PL has limited influence on the BER performance of FOSC systems using PL based receiver with EGC. In future works, we will study the influences of different combining methods, such as maximal ratio combining (MRC) and selection combining (SC), on the PL based receiver for FSOC systems. Besides, the PL with \( N \) SMF ends requires \( N \) balanced photodetectors to detect the received beams. Therefore, the cost and the complexity of the PL based receiver is higher than the SMF receiver. To reduce the number of balanced photodetectors and lower the complexity of the receiver, in the future work we will combine hybrid combining techniques, e.g., the hybrid-selection/equal-gain combining [40], with PL based receiver.

Acknowledgement

Bo Zhang acknowledges the support from the UCAS Joint PhD Training Program that allows her to conduct this research while she was visiting The University of British Columbia, Canada.

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APPENDIX A

DERIVATION OF $f(a)$

Substituting (7) into (5) and letting $\rho \to -\frac{1}{N-1}$, we can obtain

$$f(a) = \lim_{\rho \to \frac{1}{N-1}} \frac{1}{C_1} \exp \left\{ -\frac{1}{2} \left[ \frac{1 + (N-1)\rho}{1 + (N-2)\rho} \sum_{i=1}^{N-1} \frac{x_i}{\varepsilon^2} \right]^{2} \right\}$$

$$\times \lim_{\rho \to \frac{1}{N-1}} \exp \left\{ -\frac{1}{2} \left[ x_N - \frac{\rho}{1 + (N-2)\rho} \sum_{i=1}^{N-1} x_i \right]^{2} \right\}$$

(50)

where $x_i = a_i - \frac{1}{N}$ and $\varepsilon = \sigma \sqrt{\frac{1 + (N-1)\rho}{1 + (N-2)\rho}}$.

When $\rho \to -\frac{1}{N-1}$, we have $\varepsilon \to 0$. Because the limit of the Gaussian distribution can be expressed as the Dirac delta function, then we can simplify $H_2$ in (50) as

$$H_2 = \lim_{\rho \to \frac{1}{N-1}} \sqrt{2\pi \varepsilon^2} \times \delta \left( a_1 + a_2 + \cdots + a_N - 1 \right)$$

(51)

where $\delta(\cdot)$ is the Dirac delta function. Using the expression in (7), we can simplify $H_1$ in (50) as

$$H_1 = \frac{1}{C_1} \exp \left\{ -\frac{1}{2} [a^* - \mu_{a^*}]^T \Sigma_{a^*}^{-1} [a^* - \mu_{a^*}] \right\}$$

(52)

Substituting (51) and (52) into (50), we can obtain

$$f(a) = \lim_{\rho \to \frac{1}{N-1}} \frac{\sqrt{2\pi \varepsilon^2}}{C_1} \exp \left\{ -\frac{1}{2} [a^* - \mu_{a^*}]^T \Sigma_{a^*}^{-1} [a^* - \mu_{a^*}] \right\}$$

$$\times \delta (a_1 + a_2 + \cdots + a_N - 1)$$

(53)

We can find that there exists the same factor $\sqrt{2\pi \varepsilon^2}$ in the numerator and denominator $C_1$.

Finally, by eliminating the term $\sqrt{2\pi \varepsilon^2}$ in both numerator and denominator, the joint PDF in (9) can be obtained.
**APPENDIX B**

**DERIVATION OF \( E \left[ \sqrt{a_1 a_2} \right] \) FOR UNIFORM DISTRIBUTION**

The mathematical expectation of \( \sqrt{a_1 a_2} \) for a joint PDF \( f(a) \) in (15) is defined as

\[
E[\sqrt{a_1 a_2}] = (N - 1)! \int_0^1 \int_0^{1-a_1} \int_0^{1-a_1-a_2} \cdots \int_0^{1-a_1-\cdots-a_{N-1}} \sqrt{a_1} \, \sqrt{a_2} \, \delta(a_1 + a_2 + \cdots + a_N - 1) \, da_1 \, da_2 \cdots da_N
\]

By integrating \( a_{N-1} \) out, we can obtain

\[
E[\sqrt{a_1 a_2}] = (N - 1)! \int_0^1 \int_0^{1-a_1} \int_0^{1-a_1-a_2} \cdots \int_0^{1-a_1-\cdots-a_{N-2}} \sqrt{a_1} \, \sqrt{a_2} \, \delta(a_1 + a_2 + \cdots + a_{N-1} - 1) \, da_1 \, da_2 \cdots da_{N-1}.
\]  

(54)

By integrating \( a_{N-2} \) out, we can obtain

\[
E[\sqrt{a_1 a_2}] = (N - 1)! \int_0^1 \int_0^{1-a_1} \int_0^{1-a_1-a_2} \cdots \int_0^{1-a_1-\cdots-a_{N-3}} \sqrt{a_1} \, \sqrt{a_2} \, \delta(a_1 + a_2 + \cdots + a_{N-2} - 1) \, da_1 \, da_2 \cdots da_{N-2}.
\]  

(55)

Similarly, by successively integrating \( a_{N-3}, a_{N-4}, \ldots, a_3 \) out, we can obtain

\[
E[\sqrt{a_1 a_2}] = (N - 1)!(N - 2) \int_0^1 \int_0^{1-a_1} \int_0^{1-a_1-a_2} \cdots \int_0^{1-a_1-\cdots-a_3} \sqrt{a_1} \, \sqrt{a_2} \, \delta(a_1 + a_2 + \cdots + a_{N-3} - 1) \, da_1 \, da_2 \cdots da_{N-3}.
\]  

(56)

Using the relation of Beta function \( \int_a^b (t - a)^{x-1}(b - t)^{y-1} dt = (b - 1)^{x+y-1} B(x, y) \) [38, 3.196(3)], we can obtain

\[
E[\sqrt{a_1 a_2}] = (N - 1)(N - 2) B \left( \frac{3}{2}, N - \frac{1}{2} \right) B \left( \frac{3}{2}, N - 2 \right).
\]  

(57)

Using the equalities \( B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} \) [38, 8.384(1)], \( \Gamma\left(\frac{3}{2}\right) = \frac{\pi}{4}, \) and \( \Gamma(m) = (m-1)! \), we can obtain

\[
E[\sqrt{a_1 a_2}] = \frac{\pi}{4N}.
\]  

(58)

(59)
APPENDIX C

DERIVATION OF THE ANALYTICAL EXPRESSION FOR $P_{e,PL}^{lower}$

Substituting (1), (38), and (39) into (37), we can obtain

$$P_{e,PL}^{lower} = \Lambda(\alpha, \beta) \int_0^{\pi/2} \int_0^\infty \left\{ \sum_{p=0}^{\infty} \left[ \frac{a_p(\alpha, \beta) I^{p+\beta-1}}{\Gamma(p+\beta)} \exp\left( -\frac{\bar{\gamma}_{PL,Deg} I}{2 \sin^2 \theta} \right) \right] \right\} dI d\theta,$$

(60)

where $\Lambda(\alpha, \beta)$ and $a_p(x, y)$ are defined in (41).

Using $\int_0^\infty x^m \exp(-\beta x^n) dx = \frac{\Gamma\left(\frac{m+1}{n}\right)}{n^\frac{m+1}{n}} [38, 3.326(2)]$, we can obtain

$$P_{e,PL,Deg} = \Lambda(\alpha, \beta) \sum_{p=0}^{\infty} \int_0^{\pi/2} \left\{ a_p(\alpha, \beta) \left( \frac{\bar{\gamma}_{PL,Deg}}{2} \right)^{-(p+\beta)} \sin^{2p+2\beta} \theta \right\} d\theta.$$

(61)

Using the Beta function $B(x, y) = 2 \int_0^{\pi/2} \sin^{2x-1} \psi \cos^{2y-1} \psi d\psi [38, 8.380(2)]$ and $B(x, y) = B(y, x) [38, 8.384(1)]$ into (61), we can obtain the series solution to the unconditional BER as (40).

APPENDIX D

MCI METHOD FOR CALCULATING $C_2$, $\bar{\gamma}_{PL}$, $P_{out,PL}$, AND $P_{e,PL}$

In an MCI method, to obtain the integral result of $\int_x g(x) dx$, we first choose a PDF $f(x)$, which is referred as the sampling function, and rewrite the integral as $\int_x f(x) g(x) f(x) dx$. Then the integral can be viewed as the mathematical expectation of the objective function $O(x) \triangleq \frac{g(x)}{f(x)}$ when $x$ subjects to a PDF $f(x)$, i.e., $\int_x f(x) O(x) dx = \mathbb{E}[\frac{g(x)}{f(x)}]$. Therefore, we can generate $M$ samples $\{x_1, x_2, \ldots, x_M\}$ of $x$ according to the PDF $f(x)$ and use the average value of the objective function $\frac{1}{M} \sum_{m=1}^{M} O(x_m)$ to estimate the mathematical expectation [41].

A. Calculating $C_2$

Specifically, to obtain the normalization constant $C_2$, we can first rewrite $C_2$ as

$$C_2 = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{MG}(\mathbf{a}) C_3 I_n(\mathbf{a}) da_1 \cdots da_N,$$

(62)
where $C_3 = \left[2\pi \sigma^2 N/(N - 1)\right]^{\frac{N+1}{2}} / \sqrt{N}$ [24]; $f_{MG}(a)$ is the PDF of multivariate Gaussian variables $a = [a_1, a_2, \cdots, a_N]^T$ satisfying $a_1 + a_2 + \cdots + a_N = 1$, and $f_{MG}(a)$ is given by [24]

$$f_{MG}(a) = \frac{1}{C_3} \exp \left\{ -\frac{1}{2} [a^* - \mu_a^*]^T \Sigma_a^{-1} [a^* - \mu_a^*] \right\} \times \delta (a_1 + a_2 + \cdots + a_N - 1) \quad (63)$$

and $I_n(a)$ is an indicator function defined as

$$I_n(a) = \begin{cases} 
1, & a \in V \\
0, & a \notin V.
\end{cases} \quad (64)$$

Then we can choose $f_{MG}(a)$ as the sampling function and the objective function becomes $O(a) = C_3 I_n(a)$. The generation of random numbers $\{a_1, a_2, \cdots, a_N\}$ satisfying PDF $f_{MG}(a)$ can be achieved by two steps: first generate $\{a_1, a_2, \cdots, a_{N-1}\}$ according to the $N-1$ dimensional multivariate Gaussian PDF

$$f_{MG}(a^*) = \frac{1}{C_3} \exp \left\{ -\frac{1}{2} [a^* - \mu_a^*]^T \Sigma_a^{-1} [a^* - \mu_a^*] \right\}; \quad (65)$$

then $a_N$ is obtained as $a_N = 1 - a_1 - a_2 - \cdots - a_{N-1}$.

**B. Calculating $\bar{\gamma}_{PL}$**

Similarly, to obtain the average SNR, we rewrite (18) as

$$\bar{\gamma}_{PL} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{MG}(a) \frac{K f(a)}{f_{MG}(a)} \left( \sum_{i=1}^{N} \sqrt{a_i} \right)^2 I_n(a) da_1 \cdots da_N. \quad (66)$$

Then we can choose $f_{MG}(a)$ as the sampling function and the objective function becomes

$$O(a) = K \frac{f(a)}{f_{MG}(a)} \left( \sum_{i=1}^{N} \sqrt{a_i} \right)^2 I_n(a) \quad (67)$$

$$= K \frac{C_3}{C_2} \left( \sum_{i=1}^{N} \sqrt{a_i} \right)^2 I_n(a).$$

**C. Calculating $P_{outage,PL}$**

To obtain the outage probability, we rewrite (27) as

$$P_{outage,PL} = \int_{0}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(I) f_{MG}(a) \frac{f(a)}{f_{MG}(a)} I_n'(a, I) da_1 \cdots da_N dI, \quad (68)$$

where $I_n'(a, I)$ is an indicator function defined as

$$I_n'(a, I) = \begin{cases} 
1, & a \in V \quad \text{and} \quad K \left( \sum_{i=1}^{N} \sqrt{a_i} \right)^2 I < \gamma_{th} \\
0, & \text{otherwise}.
\end{cases} \quad (69)$$
Then we can choose \( f(I)f_{MG}(\alpha) \) as the sampling function and the objective function becomes

\[
O(\alpha, I) = \frac{f(\alpha)}{f_{MG}(\alpha)} I_n(\alpha, I)
\]

\[
= \frac{C_3}{C_2} I_n'(\alpha, I).
\] (70)

D. Calculating \( P_{\text{e,PL}} \)

To obtain the unconditional BER for coherent FSOC system using PL based receiver with EGC, we rewrite (33) as

\[
P_{\text{e,PL}} = \int_0^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty f(I)f_{MG}(\alpha) \frac{f(\alpha)}{f_{MG}(\alpha)}
\]

\[
\times Q\left( \sqrt{K \left( \sum_{i=1}^{N} \sqrt{\alpha_i} \right)^2 I} I_n(\alpha) \right) \int_{a_1} \cdots \int_{a_N} dI.
\] (71)

Then we can choose \( f(I)f_{MG}(\alpha) \) as the sampling function and the objective function becomes

\[
O(\alpha, I) = \frac{f(\alpha)}{f_{MG}(\alpha)} Q\left( \sqrt{K \left( \sum_{i=1}^{N} \sqrt{\alpha_i} \right)^2 I} I_n(\alpha) \right)
\]

\[
= \frac{C_3}{C_2} Q\left( \sqrt{K \left( \sum_{i=1}^{N} \sqrt{\alpha_i} \right)^2 I} I_n(\alpha) \right).
\] (72)

In addition, to calculate the outage probability \( P_{\text{outage,PL,uni}} \) and the unconditional BER \( P_{\text{e,PL,uni}} \) for uniform distribution case, the key is to generate the random numbers \( \{a_1, a_2, \ldots, a_N\} \) satisfying multivariate uniform distribution over the standard simplex. This can be achieved by the Algorithm 2 given in [34].