1-String CZ-Representation of Planar Graphs

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September 23, 2014

Abstract

In this paper, we prove that every planar 4-connected graph has a CZ-representation—a string representation using paths in a rectangular grid that contain at most one vertical segment. Furthermore, two paths representing vertices \( u, v \) intersect precisely once whenever there is an edge between \( u \) and \( v \). The required size of the grid is \( n \times 2n \).

1 Preliminaries

A possible way of representing graphs is to assign to every vertex a curve so that two curves cross if and only if there is an edge between the respective vertices. Here, two curves \( u, v \) cross means that they share a point \( s \) internal to both of them and the boundary of a sufficiently small closed disk around \( s \) is crossed by \( u, v, u, v \) (in this order). The representation of graphs using crossing curves is referred to as a string representation, and graphs that can be represented in this way are called string graphs.

In 1976, Ehrlich, Even and Tarjan showed that every planar graph has a string representation \[8\]. It is only natural to ask if this result holds if one is restricted to using only some “nice” types of curves. In 1984, Scheinerman conjectured that all planar graphs can be represented as intersection graphs of line segments \[11\]. This was proved first for bipartite graphs \[10, 7\] with the strengthening that every segment is vertical or horizontal. The result was extended to triangle-free graphs, which can be represented by line segments with at most three distinct slopes \[6\].

Since Scheinerman’s conjecture seemed difficult to prove for all planar graphs, interest arose in possible relaxations. Note that any two line segments intersect at most once. Define 1-STRING to be the class of graphs that are intersection graphs of curves (of arbitrary shape) that intersect at most once. The

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original construction of string representation for planar graphs given in [8] requires curves to cross multiple times. In 2007, Chalopin, Gonçalves and Ochem showed that every planar graph is in 1-STRING [2, 4]. With respect to Scheinerman’s conjecture, while the argument of [2, 4] shows that the prescribed number of intersections can be achieved, it provides no idea on the complexity of curves that is required.

Another way of restricting curves in string representations is to require them to be orthogonal, i.e., to be paths in a grid. Call a graph a VPG-graph (as in “Vertex-intersection graph of Paths in a Grid”) if it has a string representation with orthogonal curves. It is easy to see that all planar graphs are VPG-graphs (e.g. by generalizing the construction of Ehrlich, Even and Tarjan). For bipartite planar graphs, curves can even be required to have no bends [10, 7]. For arbitrary planar graphs bends are required in orthogonal curves, and recently Chaplick and Ueckerdt showed that 2 bends per curve always suffice [5]. Let $B_2$-VPG be the graphs that have a string representation where curves are orthogonal and have at most 2 bends; the result in [5] then states that planar graphs are in $B_2$-VPG. Unfortunately, in Chaplick and Ueckerdt’s construction, curves may cross each other repeatedly, and so it does not prove that planar graphs are in 1-STRING.

The conjecture of Scheinerman remained open until 2009 when it was proved true by Chalopin and Gonçalves [2] who extended the technique used to prove their 1-STRING result [3].

Our results: In this paper, we show that every planar 4-connected graph has a string representation that simultaneously satisfies the requirements for 1-STRING (any two curves cross at most once) and the requirements for $B_2$-VPG (any curve is orthogonal and has at most two bends). Our result hence re-proves, in one construction, the results by Chalopin et al. [3] and the result by Chaplick and Ueckerdt [5], albeit only for 4-connected planar graphs. (We briefly discuss extensions in Section 4.)

In our construction all curves have one of four possible shapes: C-shape, Z-shape, or their mirror images. We call such a representation a CZ-representation (see Section 2 for the formal definition).

Theorem 1.1. Every 4-connected planar graph has a 1-string CZ-representation.

Since our construction has at most $n$ vertical and $2n$ horizontal line segments, and since any orthogonal grid can be deformed to be on integer coordinates without empty rows or columns, our construction can be embedded to a rectangular grid of size $n \times 2n$. Note that none of the previous results provided an intuition of the required size of the grid.

Our approach is inspired by the construction of 1-string representations from 2007 [2, 4]. The authors proved the result in two steps. First, they showed that triangulations without separating triangles admit 1-string representations. By induction on the number of separating triangles, they then showed that 1-string
Figure 1: Every curve in a CZ-representation has one of the depicted shapes.

representation exists for any planar triangulation, and consequently for any planar graph.

In order to show that triangulations without separating triangles have 1-string representation, Chalopin et al. \cite{4} used a method inspired by Whitney’s proof that 4-connected planar graphs are Hamiltonian \cite{12}. Asano, Saito and Kikuchi later improved Whitney’s technique and simplified his proof \cite{1}. Our paper uses the same approach as \cite{4}, but borrows ideas from \cite{1} and develops them further to reduce the number of cases and hence simplify the proof.

2 CZ-Representation of 4-Connected Planar Graphs

Let us begin with a formal definition of a CZ-representation.

**Definition 2.1 (CZ-representation).** A planar graph $G$ has a 1-string CZ-representation if every vertex $v$ of $G$ can be represented by a curve $v$ such that:

1. Curve $v$ is orthogonal, i.e., it consists of horizontal and vertical segments.
2. Curve $v$ has at most two bends and at most one vertical segment (see Figure 1).
3. Curves $u$ and $v$ intersect at most once, and $u$ intersects $v$ if and only if $(u, v)$ is an edge of $G$.

A partial 1-string CZ-representation is a 1-string CZ-representation of a subgraph of $G$.

For brevity, we use “CZ-representation” to mean “partial 1-string CZ-representation”. Our technique for constructing a CZ-representation of a graph uses an intermediate step referred to as “an $(\text{int} \cup F)$-CZ-representation$^1$ of a W-triangulation that satisfies the chord condition with respect to three chosen corners”. We define these terms first.

A **triangulated disk** is a 2-connected planar graph $G$ such that every interior face is a triangle. A **separating triangle** is a cycle of length 3 which contains vertices both inside and outside. Following the notation of \cite{3}, a **W-triangulation** is a triangulated disk which does not contain a separating triangle. A **chord** of

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\footnote{Here, int is used as an abbreviation of interior.}
a triangulated disk is an interior edge for which both endpoints are on the outer face.

For two vertices $X, Y$ on the outer face of a connected planar graph, define $P_{XY}$ to be the counter-clockwise (ccw) path on the outer face from $X$ to $Y$ ($X$ and $Y$ inclusive). We will often study triangulated disks with three specified distinct vertices $A, B, C$ called the corners which must appear on the outer face in ccw order. We denote $P_{AB} = (a_1, a_2, \ldots, a_r)$, $P_{BC} = (b_1, b_2, \ldots, b_s)$ and $P_{CA} = (c_1, c_2, \ldots, c_t)$, where $c_t = a_1 = A$, $a_r = b_1 = B$ and $b_s = c_1 = C$.

**Definition 2.2** (Chord condition). A W-triangulation $G$ satisfies the chord condition with respect to the corners $A, B, C$ if $G$ has no chord within $P_{AB}, P_{BC}$ or $P_{CA}$, i.e., no interior edge of $G$ has both ends on $P_{AB}$, or both ends of $P_{BC}$, or both ends on $P_{CA}$.

**Definition 2.3** ($(\text{int} \cup F)$-CZ-representation). Let $G$ be a connected planar graph with corners $A, B, C$. Let $F$ be a set of outer face edges incident to $C$. An $(\text{int} \cup F)$-CZ-representation of $G$ is a CZ-representation of $G$ for which curves $u, v$ cross if and only if $(u, v)$ is an interior edge of $G$ or $(u, v) \in F$. Furthermore, the CZ-representation must satisfy that:

1. There exists a rectangle $\Theta$ containing all intersections of curves so that the top of $\Theta$ is intersected, from right to left in order, by the curves of the vertices of $P_{AB}$, and the bottom of $\Theta$ is intersected, from left to right in order, by the curves of the vertices of $P_{BA}$.

2. The curve $v$ of an outer face vertex $v$ has at most one bend. (By (1), this implies that $A$ and $B$ have no bends.)

See Figure 2 for examples of an $(\text{int} \cup F)$-CZ-representation. In all our constructions, we have $|F| \leq 1$, i.e., $F$ consists of at most one edge that is on the outer face and incident to $C$. If $F = \{e\}$, then $e$ is called the special edge. We sometimes write $(\text{int} \cup e)$-CZ-representation rather than $(\text{int} \cup \{e\})$-CZ-representation, and int-CZ-representation rather than $(\text{int} \cup \emptyset)$-CZ-representation. Note that the roles of corners $A$ and $B$ in an $(\text{int} \cup F)$-CZ-representation are symmetric: we can exchange $A$ and $B$, as long as we also reverse all cyclic orders of edges around each vertex (to preserve the sense of counter-clockwise) and flip the resulting representation horizontally (to undo the reversal.) Corner $C$, on the other hand, is distinct from the other two, for example because the special edge must be incident to $C$. Our key result is the following:

**Lemma 2.4.** Let $G$ be a W-triangulation that satisfies the chord condition with respect to corners $A, B, C$. Then $G$ has an $(\text{int} \cup F)$-CZ-representation for any set $F$ of at most one outer face edge incident to $C$.

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2For readers familiar with [2] or [4]: A W-triangulation that satisfies the chord condition with respect to corners $A, B, C$ is called a W-triangulation with 3-boundary $P_{AB}, P_{BC}, P_{CA}$ in [2], and the chord condition is the same as Condition (W2b) in [4].

3An $(\text{int} \cup F)$-CZ-representation corresponds roughly to what Chalopin et al. [4] call Property 1, except that they do not restrict the shape of the curves, and they fix $F$ to be edge $(C, c_2)$. 

4
The proof of Lemma 2.4 will be given in Section 3. Here we show how it implies our main result.

**Proof of Theorem 1.1.** First assume that $G$ is a triangulation, which by 4-connectivity means that it has no separating triangles. Let $A, B, C$ be the vertices on the outer face in ccw order. As the outer face is a triangle, $G$ clearly satisfies the chord condition with respect to $A, B, C$. Thus, by Lemma 2.4, it has an $(\text{int} \cup (C, c_{2}))$-CZ-representation contained in a rectangular box $\Theta$. This CZ-representation has an intersection for every edge except for $(A, B)$ and $(A, C)$. The ends of curves $A$ and $B$ outside of $\Theta$ can be used to create intersections for these edges as follows. Bend and stretch the upper end of $B$ rightwards and the upper end of $A$ upwards so that both the curves cross. Bend and stretch the lower end of curve $A$ leftwards and stretch $C$ downwards so that the two curves cross. Recall that $A$ and $B$ initially did not have any bends, so they have each one bend in the constructed 1-string CZ-representation of $G$. See Figure 3 for an illustration.

Now assume that $G$ is a 4-connected planar graph. Then stellate the graph, i.e., insert a vertex into each non-triangulated face and connect it to all vertices on that face. The resulting graph is triangulated and has no separating triangles, so it has a 1-string CZ-representation by the above. Deleting the curves of added vertices produces the result.

\[ \square \]

### 3 \( (\text{int} \cup F)\)-CZ-representations

In this section, we provide the proof of Lemma 2.4. We proceed by induction on the number of edges. In the base case, $n = 3$, so $G$ is a triangle, and the three corners $A, B, C$ must be the three vertices of this triangle. The $(\text{int} \cup F)$-CZ-representations for $F = \emptyset, \{(A, C)\}, \{(B, C)\}$ are depicted in Figure 4.

The induction step for $n \geq 4$ is divided into five cases.
Figure 3: Completing the \((\text{int} \cup F)\)-CZ-representation of a triangulation \(G = (A, B, C; B)\).

Figure 4: \((\text{int} \cup F)\)-CZ-representations of a triangle.

**Case 1: G has a chord incident to C.** By the chord condition, this chord has the form \((C, a_i)\) for some \(1 < i < r\). The graph \(G\) can be split along the chord \((C, a_i)\) into two graphs \(G_1\) and \(G_2\). Both \(G_1\) and \(G_2\) are bounded by simple cycles, hence triangulated disks. No edges were added, so neither \(G_1\) nor \(G_2\) contains a separating triangle. We select \((A, a_i, C)\) as corners for \(G_1\) and \((B, C, a_i)\) as corners for \(G_2\) and can easily verify that with this \(G_1\) and \(G_2\) satisfy the chord condition:

- \(G_1\) has no chords on \(P_{Aa_i}\) or \(P_{CA}\) as they would violate the chord condition in \(G\). There is no chord on \(P_{a_iC}\) as it is a single edge.

- \(G_2\) has no chords on \(P_{a_iB}\) or \(P_{BC}\) as they would violate the chord condition in \(G\). There is no chord on \(P_{a_iC}\) as it is a single edge.

So, by induction, Lemma 2.4 holds for both \(G_1\) and \(G_2\).

For \(F = \emptyset\) or \(F = \{(C, c_2)\}\), an \((\text{int} \cup F)\)-CZ-representation of \(G\) can be constructed as follows. Inductively, construct an \((\text{int} \cup F)\)-CZ-representation of \(G_1\) and an \((\text{int} \cup (C, a_i))\)-CZ-representation of \(G_2\); note that the special edge of each indeed attaches at the required corner. Rotate the CZ-representation of \(G_2\) by \(180^\circ\), and translate it so that it is below the CZ-representation of \(G_1\) with the two copies of \(a_i\) in the same column. Stretch one of the CZ-representations horizontally as needed until the two copies of \(c_j\) are also in the same column; then \(a_i\) and \(c_j\) can each be unified without adding bends by adding vertical segments. Stretch the CZ-representation of \(G_2\) further so that everything to the left of \(a_i\) in \(G_2\) appears to the left of the entire CZ-representation of \(G_1\);
Case 2: $G$ has a chord in the form $(a_i, c_j)$, $1 \leq i \leq r$, $1 \leq j < t$. We may assume that $j > 1$ (otherwise we are in Case 1), and $i > 1$ and $j < t$ (otherwise the chord condition is violated).

Let $(a_i, c_j)$ be the chord that maximizes $i - j$ (i.e., it is the furthest chord from vertex $A$). Note that possibly $i = r$, i.e., $a_i = a_r = B$. Split the graph $G$ along this chord into graphs $G_1$ (which contains $A$) and $G_2$ (which contains $C$ and the special edge, if any). Select $(A, a_i, c_j)$ as corners for $G_1$ and $(c_j, B, C)$ as corners for $G_2$. As before both $G_1$ and $G_2$ are W-triangulations, and we can verify that they satisfy the chord condition:

- $G_1$ has no chords on path $P_{Aa_i} \subseteq P_{AB}$ and $P_{c_jA} \subseteq P_{CA}$ as they would contradict the chord condition in $G$. The remaining side $P_{c_ja_i}$ is a single edge, and so does not have any chords either.
Figure 7: The construction for Case 2 also covers borderline cases: (Left) The chord \((a_i, c_j)\) is incident with \(B\). (Middle) The chord is incident to the end \(c_2\) of the special edge. (Right) Both the incidences are present.

- \(G_2\) has no chords on path \(P_{Cc_j} \subseteq P_{CA}\) and \(P_{BC}\) as they would contradict the chord condition for \(G\). Furthermore, \(G_2\) has no chord on path \(P_{c_jB}\) due to the selection of \((a_i, c_j)\) and since \(P_{AB}\) has no chords.

In order to construct an \((\text{int} \cup F)\)-CZ-representation of \(G\), apply induction to get an \((\text{int} \cup (a_i, c_j))\)-CZ-representations of \(G_1\) and an \((\text{int} \cup F)\)-CZ-representation of \(G_2\). Similarly as before, stretch the representations so that we can align and join the two curves \(a_i\) and \(c_j\) as shown in Figure 6. Since \(a_i\) has no bends in the CZ-representation of \(G_1\) and \(c_j\) has no bends in the CZ-representation of \(G_2\), the number of bends of those curves in the constructed representation is at most 1. The number of bends of any other curve representing an outer face vertex does not change, so it is also at most 1. Thus, the constructed representation is a valid \((\text{int} \cup F)\)-CZ-representation. Figure 6 shows the construction (with \(F = \emptyset\)) when \(a_i \neq B\) and \(2 < j\). Figure 7 shows the construction when \(a_i = B\) or \(c_j = c_2\).

Case 3: \(G\) has a chord in the form \((a_i, b_j)\), \(1 \leq i \leq r\), \(1 \leq j \leq s\). By interchanging the roles of corners \(A\) and \(B\), this case can be transformed into Case 2.
Case 4: $G$ has a chord in the form $(b_j, c_k), 1 \leq j \leq r, 1 \leq k \leq t$. Note that we may assume $1 < j < r$ and $1 < k < t$ as all other cases either violate the chord condition or were already covered. Let $(b_j, c_k)$ be the chord maximizing $j - k$ (i.e., furthest from $C$).

In order to construct an int-CZ-representation of $G$, split the graph along $(b_j, c_k)$ into two W-triangulations $G_1$ (which includes $C$ and the special edge, if any) and $G_2$ (which includes $A$). Set $(A, B, c_k)$ as corners for $G_1$ and $(c_k, b_j, C)$ as corners for $G_2$ and verify the chord condition:

- $G_1$ has no chords on either $P_{C_{ck}} \subseteq P_{CA}$ or $P_{b_jC} \subseteq P_{BC}$ as they would contradict the chord condition in $G$. The third side is a single edge $(b_j, c_k)$ and so it does not have any chords either.

- $G_2$ has no chords on either $P_{c_kA} \subseteq P_{CA}$ or $P_{AB}$ as they would violate the chord condition in $G$. It does not have any chords on the path $P_{Bc_k}$ due to the selection of the chord $(b_j, c_k)$.

Thus, by induction, $G_1$ has an $(int \cup F)$-CZ-representation and $G_2$ has an $(int \cup (b_j, c_k))$-CZ-representation. After horizontal deformation, the CZ-representations can be aligned so that the ends of $b_j$ and $c_k$ in $G_2$ can be connected to the upper ends of $b_j$ and $c_k$ in $G_1$. As $b_j$ and $c_k$ have no bends in the CZ-representation of $G_2$, the construction does not increase the number of bends on any curve and produces an $(int \cup F)$-CZ-representation of $G$. Figure 8 shows the construction.

Case 5: $G$ has no chords. Assume after possible exchange of $A$ and $B$ that the special edge, if it exists, is $(C, c_2)$. Let $u_1, \ldots, u_q$ be the neighbours of vertex $C$ in clockwise order, starting with $b_{r-1}$ and ending with $c_2$. We know that $\deg(C) \geq 2$, for otherwise the neighbours of $C$ would have a chord between them since $G$ is a triangulated disk. Therefore, $C$ has at least one neighbour $u_i, 1 < i < q$. We also know that $u_2, \ldots, u_{q-1}$ are not on the outer face, since $C$ is not incident to a chord.

Let $u_j$ be a neighbour of $C$ that has at least one other neighbour on $P_{CA}$, and among all those, choose $j$ to be minimal. Such a $j$ exists and $j < q$ because

Figure 8: Case 4. Construction of an $(int \cup F)$-CZ-representation of $G = (A, B, C)$ with a chord $(b_j, c_k), 1 \leq j \leq r, 1 \leq k \leq t$. 

Case 5: $G$ has no chords. Assume after possible exchange of $A$ and $B$ that the special edge, if it exists, is $(C, c_2)$. Let $u_1, \ldots, u_q$ be the neighbours of vertex $C$ in clockwise order, starting with $b_{r-1}$ and ending with $c_2$. We know that $\deg(C) \geq 2$, for otherwise the neighbours of $C$ would have a chord between them since $G$ is a triangulated disk. Therefore, $C$ has at least one neighbour $u_i, 1 < i < q$. We also know that $u_2, \ldots, u_{q-1}$ are not on the outer face, since $C$ is not incident to a chord.

Let $u_j$ be a neighbour of $C$ that has at least one other neighbour on $P_{CA}$, and among all those, choose $j$ to be minimal. Such a $j$ exists and $j < q$ because
Figure 9: Decomposition into the top graph and bottom graph.

$G$ is triangulated and therefore $u_{q-1}$ is adjacent to both $C$ and $u_q$. We also know that $j > 1$, since otherwise there would be a chord from $u_j = u_1 = b_{x-1}$ to some vertex on $P_{CA}$.

Let the terminals be the neighbours of $u_j$ on $P_{CA}$; we denote these by $t_1, t_2, \ldots, t_x$ in the order in which they appear on $P_{CA}$. Separate $G$ into two graphs $G_T$ (the top graph) and $G_B$ (the bottom graph) as follows: $G_T$ is bounded by ($u_1, u_2, \ldots, u_j, t_x, P_{txA}, \ldots, A, P_{txB}, \ldots, B, P_{Bu_1}, \ldots, u_1$); and $G_B$ is bounded by ($C = t_1, P_{t_1tx}, t_x, u_j, t_1 = C$). See also Figure 9.

**Observation 3.1.** The top graph $G_T$ is a W-triangulation that satisfies the chord condition with respect to corners $A' := A, B' := B$ and $C' := u_1$.

**Proof.** Since $u_2, \ldots, u_j$ are interior vertices of $G$, the outer face of $G_T$ is a simple cycle, and so $G_T$ is a W-triangulation.

Since $G$ satisfies the chord condition, graph $G_T$ does not have any chords with both ends on $P_{AB'} = P_{AB}$ or $P_{BC'} \subseteq P_{BC}$. If there were any chords with both ends on $P_{CA'}$, then by $C' = u_1$ the chord would either connect two neighbours of $C$ (hence give a separating triangle of $G$), or connect some $u_i$ for $i < j$ to $P_{CA}$ (contradicting minimality of $j$), or connect $u_j$ to some other vertex on $P_{CA}$ (contradicting that $t_x$ is the last terminal), or have both ends on $P_{CA}$ (contradicting the chord condition for $G$). Hence no such chord can exist either.

So, we can apply induction on $G_T$ and obtain an $(\text{int} \cup \{u_1, u_2\})$-CZ-representation of $G_T$. Similarly as in previous cases the plan is to combine this with a representation of the rest. Define $G_Q := G_B - u_j$; we call this graph the chain graph. Unfortunately $G_Q$ is not necessarily 2-connected, and so we cannot apply induction to it directly, but we can obtain a CZ-representation for it by splitting it into smaller subgraphs.

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4 Comparing to the terminology of [1], our chain graph is similar to the graph $G'$ that is defined by the $Q$-chain and satisfies Property (A).

10
Figure 10: The chain graph. Blocks $G_1$ and $G_3$ are isolated edges, graphs $G_2$ and $G_4$ are W-triangulations. We illustrate the chosen corners for $G_2$.

For $i = 1, \ldots, x - 1$, let $G_i^+$ be the graph bounded by $(t_i, P_t^{i:i+1}, t_{i+1}, u_j, t_i)$, and let the $i^{th}$ block be the graph $G_i := G_i^+ - u_j$. See also Figure 10

**Observation 3.2.** The $i^{th}$ block $G_i$ is either a single edge, or a W-triangulation that satisfies the chord condition with respect to corners $A' := t_{i+1}, B' := t_i$, and $C'$ the successor of $t_i$ on $P_{t_i, t_{i+1}}$.

**Proof.** Assume $G_i$ is not a single edge. First note that no vertex can appear twice on the outer face boundary of $G_i$, otherwise (since $G$ was a triangulated disk) there would be a double edge to $u_j$, or another terminal on $P_{t_i, t_{i+1}}$. So $G_i$ is bounded by a simple cycle and hence a W-triangulation.

Before we can argue the chord condition, we must see that the corners are distinct. Clearly $B' = t_i \neq t_{i+1} = A'$ by the definition of terminals. Also $C' \neq B' = t_i$ by definition of $C'$ as a neighbour of $t_i$. Finally, $C' \neq A' = t_{i+1}$, for otherwise $(t_i, t_{i+1})$ would be an edge and $\{u_j, t_i, t_{i+1}\}$ hence a separating triangle of $G$ (since $G_i$ is not a single edge). Now we can verify the chord condition:

- $G_i$ has no chord on $P_{A'B'}$, since all vertices on $P_{A'B'}$ are neighbours of $u_j$ and $G$ has no separating triangle.
- $G_i$ has no chord on $P_{B'C'}$ or $P_{C'A'}$, since both of these are sub-paths of $P_{CA}$ and $G$ satisfies the chord condition.

\[ \Box \]

Set $F_1 := F$ if $G_1$ is not a single edge and $F_1 = \emptyset$ otherwise. Set $F_i = \emptyset$ for $1 < i < x$. By induction, any $G_i$ that is not an edge has an \((\text{int} \cup F_i)\)-CZ-representation. If $G_i$ is a single edge $(t_i, t_{i+1})$, then we can represent it with two vertical segments for $t_i$ and $t_{i+1}$. We now merge these representations of $G_1, \ldots, G_{x-1}$ as illustrated in Figure 11, by merging for $i = 2, \ldots, x-1$ the two vertical segments $t_i$. The result satisfies all conditions for an \((\text{int} \cup F_i)\)-CZ-representation of $G_Q$ (for corners $B' := C$ and $A' := t_x$).

Now we merge this \((\text{int} \cup F_i)\)-CZ-representation of $G_Q$ with the \((\text{int} \cup \{u_1, u_2\})\)-CZ-representation of $G_T$ as illustrated in Figure 13. The neighbours of $G_Q$ in $G_T$ are vertices $u_1, u_2, \ldots, u_j$. The CZ-representation of $G_Q$ can be horizontally deformed and aligned so that it is below $G_T$ and the order of vertical segments is as in Figure 12 i.e., from left to right it is $u_1, C, u_2, u_3, \ldots, u_j$. 

11
{v for v ≠ C in G_Q’s boundary path P_{t_x,C} in reverse order}, {c for c ≠ t_x in G’s boundary path P_{t_x,A} in order}.

To create the required intersections, we first extend C upward and (still below G_T) rightward until it crosses u_2, …, u_j. Note that as a result, C receives its first bend.

The CZ-representation of G_T includes an intersection for the special edge (u_1, u_2), but does not include intersections for edges (u_i, u_{i+1}), 2 ≤ i < j. Since these are internal to G, such intersections need to be created. Do this for i = 2, …, j − 1, j ≥ 3 by extending u_i vertically below its crossing with C and u_{i−1} and then horizontally until it crosses u_{i+1} (which is adjacent due to the properties of a CZ-representation). We assume for this that u_{i+1} has been extended vertically suitably. This increases the number of bends of u_i to at most 2, which is allowed since u_i is an internal vertex of G.

Curve u_j already extends below its crossing with C and u_{j−1}. Now, extend it rightward so that it crosses all vertical segments of up to (and including) t_x. All these segments belong to vertices in G_Q’s boundary path P_{t_x,C}, which are indeed neighbours of u_j since G is a triangulated disk. Afterwards, u_j has up to two bends, which is allowed since it is not on the outer face of G.

It remains to create intersections for some of the edges in G_Q’s boundary path P_{t_x,C}. We can create these intersections similarly as for u_2, …, u_j by extending each curve upward and rightward until it hits the next one. This adds one bend in each curve, which is acceptable since the curve remains on the outer face only if it was a terminal before, hence had no bends previously. Note that an edge in P_{t_x,C} possible does not need an intersection (see e.g. edge (t_3, t_4) in Figure 12), namely, if the i\(^{th}\) block consists of a single edge (t_i, t_{i+1}) and this edge is not in F. In this case, simply stop t_i before it intersects t_{i+1}.

With this, all interior edges of G receive exactly one intersection of their corresponding curves. If F is non-empty, then the special edge (C, c_2) received its intersection either via the (int ∪ F_1)-CZ-representation of G_1 (if G_1 is a triangulated disk), or (C, c_2) = (t_1, t_2) and the intersection was created when handling P_{t_x,C}.

This ends the description of constructing an (int ∪ F)-CZ-representation in
Case 5 and hence proves Lemma 2.4 and Theorem 1.1.

4 Conclusions and Outlook

We showed that every 4-connected planar graph has a 1-string CZ-representation. A natural question is to extend this result to all planar graphs. We believe that this is possible, but currently cannot achieve fewer than \( k = 3 \) bends.

By again stellating the graph (possibly repeatedly if it was not 3-connected), it suffices to show that every 3-connected planar triangulation has a 1-string \( B_k \)-VPG representation. This statement can be proved by induction on the number of separating triangles by a technique used in [4] (and re-discovered in [9]). With every triangular face, create a “face region” (called “private region” in [9]) that intersects the curves of vertices of the face in a predefined way and does not intersect anything else. This is easy for W-triangulations by inspecting the constructions in Cases 1–5.

In the inductive step, find the smallest separating triangle \( T \) in \( \Delta \). By induction, the graph \( G_1 \) obtained by removing the inside of \( T \) has a 1-string \( B_k \)-VPG representation. The graph \( G_2 \) strictly inside \( T \) is either a single vertex or (as one can show) it is a W-triangulation that satisfies the chord condition for some suitably chosen corners, and hence has an int-CZ-representation. Place it inside the face region for \( T \), create the intersections needed for edges on the outer face of \( G_2 \) and edges between \( G_1 \) and \( G_2 \), and identify face regions for
newly created faces.

If one aims for a 1-string $B_3$-VPG representation, two bends per can be added to each curve of outer face vertices of $G_2$, and hence such a merge is easy (details are omitted).

If one aims for a 1-string $B_2$-VPG representation of planar graphs, only one bend can be added to each such curve, which seems impossible with our current geometric restrictions of $(\text{int} \cup F)$-CZ-representation. However, this might be feasible if we allow the outer face vertices to use rays in three directions. This is our ongoing research.

As for other future work, the CZ-representation constructed in this paper uses curves of four possible shapes. Is it possible to use fewer shapes or to restrict them further? Felsner et al. [9] asked the question whether every planar graph is the intersection graph of only two shapes, namely $\{L, \Gamma\}$. (This would also provide a different proof of Scheinerman’s conjecture.) Somewhat in between: is every planar graph the intersection graph of $xy$-monotone orthogonal curves, preferably in the 1-string model?

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