Interaction-induced decoherence of atomic Bloch oscillations

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We show that the energy spectrum of the Bose-Hubbard model amended by a static field exhibits Wigner-Dyson level statistics. In itself a characteristic signature of quantum chaos, this induces the irreversible decay of Bloch oscillations of cold, interacting atoms loaded into an optical lattice, and provides a Hamiltonian model for interaction induced decoherence.

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The Bose-Hubbard Hamiltonian serves as a paradigm in the field of quantum phase transitions [1]. Recently, this model was realized in experiments on ultracold atoms loaded into a three-dimensional optical lattice [2], opening new perspectives for the laboratory study of correlated bosonic systems. Consequently, new theoretical work on the Bose-Hubbard model was stimulated, which, in particular, addresses the response to a static field [3, 4] – a question which shifts the focus from the Stark problem for interacting bosons. Our analysis is formulated in a spirit close to ongoing experiments on cold atoms in optical lattices [2, 7], and we assume that the Bose-Hubbard ground state (which is mostly studied in the literature) to dynamical and spectral properties of the system. In single-particle quantum mechanics, these are associated with Bloch oscillations, in the time domain, and with the emergence of a Wannier-Stark ladder, in the energy domain [2].

The present Letter is devoted to the spectral properties of the Bose-Hubbard Hamiltonian under the additional action of a static field or, equivalently, to the Wannier-Stark problem for interacting bosons. Our analysis is formulated in a spirit close to ongoing experiments on cold atoms in optical lattices [2, 7], and we assume that the atoms are in the “super-fluid phase”, i.e. they are delocalized over the lattice in the absence of any external perturbation. This latter assumption distinguishes the present work from previous contributions [3, 4] devoted to the Mott insulator phase, and restricts the values of the interaction energy \( W \) to the range \( 0 \leq W \leq 0.1 \). Recently, it is well known that, for an infinite lattice, there is no irreversible decay of Bloch oscillations of cold, interacting atoms loaded into an optical lattice, and provides a Hamiltonian model for interaction induced decoherence.

where the strength of the static field \( F \) or, more precisely, the Stark energy (the period of lattice is set to unity) will be our free parameter.

Let us first address the issue of boundary conditions. It is well known that, for an infinite lattice, there is no smooth transition between the spectrum at \( F = 0 \) and \( F \neq 0 \). Formally, this is due to the fact that for any non-vanishing value of \( F \) the Hamiltonian (1) is an unbounded operator, whereas it is bounded for \( F = 0 \). However, for a lattice of finite size, \(-L \leq l \leq L\), the operator \( H \) is always bounded and, hence, the spectrum of the system changes continuously as a function of \( F \), as illustrated by the numerically generated level dynamics in the top left panel of Fig. 1 for \( N = 1 \) and Dirichlet (i.e., vanishing) boundary conditions. As \( F \) is increased, the spectrum evolves from a Bloch spectrum with energies \( E(k) = -J \cos[\pi k/(2L + 2)] \), \( k = -L, \ldots, L \), into a Wannier-Stark ladder \( E_l \approx Fl, l = -L, \ldots, L \). The other panels in Fig. 1 show the evolution of the field-free \( k = 0 \) eigenstate in the basis of the Wannier states, with in-
creasing $F$. The progressive localisation of the atomic wave function in $l$ with $F$ is known as Stark localisation.

When discussing the time evolution of a wave function governed by Eq. (1), it is preferable to use periodic boundary conditions instead of Dirichlet. To do so, one first eliminates the static term in Eq. (1) by transforming to the interaction representation, where the hopping and the on-site term in Eq. (1) define the unperturbed Hamiltonian, hence

$$
\hat{H} \rightarrow \hat{H}(t) = \frac{J}{2} \left( \exp\left( -i \frac{F}{\hbar} t \right) \sum_l \hat{a}_{l+1}^\dagger \hat{a}_l + \text{h.c.} \right) + \frac{W}{2} \sum_l \hat{a}_l (\hat{n}_l - 1),
$$

and then identifies the site $l = L + 1$ of the lattice with $l = -L$. This choice has the advantage that the time evolution operator of a system of non-interacting atoms over one Bloch period $T_B = 2\pi \hbar / F$ coincides with the unit matrix, independently of the size of the system. This facilitates the analysis of the dynamics in the thermodynamic limit $N, L \rightarrow \infty$. In what follows, we shall use Dirichlet boundary conditions when calculating eigenvalues, and periodic boundary conditions when simulating the dynamics.

Our analysis of the spectrum of the multi-particle system follows the one for the single-particle problem. Let us assume for the moment that there are no atom-atom interactions, i.e. $W = 0$. As already mentioned, for large values of $F$ the single-atom energy levels form a Wannier ladder, and the energies of an $N$-atom system are consequently given by

$$
E_m = F \sum_{l=-L}^L m_l \equiv F l_{tot}, \quad \text{for } |l_{tot}| \leq LN,
$$

where $m_l$ ($m = m_{-L}, \ldots, m_L, \sum_{-L}^L m_l = N$) are the occupation numbers of the Wannier-Stark states. Note that, in general, many different sets $m$ correspond to the same total energy, and the $N$-particle Wannier ladder levels $E_m = F l_{tot}$ are, thus, typically degenerate. The $N$-particle wave function associated with a given level $E_m$ can be constructed from single-particle Wannier-Stark states $|\psi \rangle$ by an appropriate symmetrisation procedure. In the basis of Fock states (symmetrised products of Wannier functions $|n \rangle$), an arbitrary Wannier-Stark state, at finite $F$, is given by the sum

$$
|\psi_m \rangle = \sum_n c_n^{(m)} |n \rangle, \quad |n \rangle = |n_{-L}, \ldots, n_L \rangle,
$$

and in the limit $F \rightarrow \infty$ only one coefficient $c_n^{(m)}$ with $n = m$ differs from zero in Eq. (4). On the contrary, in the opposite limit $F \rightarrow 0$, almost all expansion coefficients are non-zero and the Wannier-Stark states approach $N$-particle Bloch states with (once again, degenerate) energies

$$
E(k) = -J \sum_{k=-L}^L \cos \left( \frac{\pi k}{2L+2} \right) n_k,
$$

the straightforward $N$-particle generalization of the above one-particle result.

Let us now include the effect of atom-atom interactions. Figure 2 shows the energy levels of the Hamiltonian as a function of $F$, for $N = 3$ atoms loaded into a lattice with 11 sites (i.e., $L = 5$). As expected, the atom-atom interactions remove the above-mentioned degeneracy – for small $F$ the spectrum appears dense (almost continuous), and for large $F$ the degenerate levels of the Wannier ladder split into “Wannier-ladder energy bands” (see Eq. (6) below). In this latter limit, the spectrum and the associated Wannier-Stark states can still be found analytically. Indeed, since the hopping term in Eq. (1) couples only those Fock states separated by one single quantum in the Stark excitation, one has

$$
E_m \simeq F \sum_{l=-L}^L m_l + \frac{W}{2} \sum_{l=-L}^L m_l(m_l - 1),
$$

and

$$
|\psi_m \rangle \simeq |m \rangle - \frac{J}{2} \left( \sum_{l=-L}^{L-1} \sum_{m'} \frac{\langle m'| \hat{a}_{l+1}^\dagger \hat{a}_l |m \rangle}{E_m - E_{m'}} |m' \rangle + \text{h.c.} \right),
$$

where $|E_{m'} - E_m| \sim F$.

The perturbative results cannot hold when $F < J$. Moreover, the complex level dynamics which are borne out for small $F$ in the inset in Fig. 2 indicate
In the absence of atom-atom interactions, the observed rather easily in state of the art experiments peculiar spectrum of (1), at intermediate field strengths?

Hamiltonians statistics, the nearest neighbour distribution tends towards Poissonian statistics, 

\[ \langle s \rangle = \int_0^\infty P(s) ds \]

for normalised spacings \( s = \Delta E / \Delta E \) is the average level spacing in the central part of the spectrum, with \( F = 0.01 \) (top) and \( F = 0.04 \) (bottom). \( N = 4 \) atoms loaded into a lattice of size \( 2L + 1 = 11 \). The dashed and dash-dotted lines indicate GOE and Poisson cumulative distributions, respectively.

The appearance of the new frequency \( \omega_W \) originates in the splitting of the Wannier ladder levels into “energy bands” – see our above discussion. It must be stressed that the result (5) is valid only for large values of the static field, where the spectrum is regular. Consequently, it is to be expected that for weak static fields the atomic Bloch oscillations will be qualitatively different, due to the irregular/chaotic structure of the spectrum. Indeed, numerical simulations of the dynamics indicate that, in the weak field regime, the Bloch oscillations decay irreversibly on rather short time scales. As an example, Fig. 4 shows the behaviour of the scaled (\( p \rightarrow p/NJ \)) momentum for \( F = 0.05 \), \( N = 7 \), \( L = 3 \) (top), and \( N = 9 \), \( L = 4 \) (bottom) [13]. After only few Bloch periods, the mean momentum has almost completely decayed but for feeble residual fluctuations, which rapidly vanish as the Hilbert space dimension is increased (compare top to bottom). Thus, for a weak static field, the envelope function in Eq. (8) approaches \( f(t) \sim \exp(-t/\tau) \) in the thermodynamic limit, which appears to be reached rather quickly with increasing system size, by virtue of the results shown in Fig. 4. Note that the decay of the Bloch oscillations is due to the decay of the off-diagonal elements of the one-particle density matrix [9], and that, hence, the time \( \tau \) can equally be considered as the decoherence time for a system of interacting bosons. The dependence of \( \tau \) on the system parameters hitherto remains an open problem.

Finally, let us briefly discuss the conditions for the observed chaos-transition in the Bose-Hubbard model. Our numerical simulations of the system dynamics, performed for fixed ratio \( W/J \) and different values of \( N \) and \( L (0.2 \leq \bar{n} \leq 1.2, L \leq 10, N \leq 10) \), suggest the condition

\[ \delta l \sim \bar{n}^{-1}, \]

as a criterion of the transition to chaos, where \( \delta l \) denotes the localization length of the single-particle wave function on the lattice (\( \delta l \simeq J/F \) for \( F < J \), and \( \delta l \simeq 1 \) for \( F > J \)), and \( \bar{n}^{-1} \) as the inverse filling factor has the meaning of an average particle distance. It is clear, however, that condition (10) cannot be universal, since it does not account for the on-site energy \( W \). Indeed, for \( W \rightarrow 0 \), the particle-particle interaction vanishes, and the system is integrable for arbitrary \( F \). On the other hand, when \( W \rightarrow \infty \), the Bose-Hubbard model
FIG. 4: Irreversible decay of atomic Bloch oscillations in the presence of a weak static field $F = 0.05$, for a filling factor $\bar{n} = 1$ and lattice sizes $2L + 1 = 7$ (top) and $2L + 1 = 9$ (bottom). Comparison of both plots suggests that the residual fluctuations of $p(t)$ rapidly die out as the system size is increased. (The respective dimensions of Hilbert space are $N = 1716$ (top) and $N = 24310$ (bottom).)

To conclude, we have shown that the spectrum of the Bose-Hubbard Hamiltonian amended by a static field (and at fixed particle-particle interaction corresponding to the “superfluid” regime in field-free case) is either regular or irregular, depending on the relative strength of the hopping matrix element and the external perturbation. In particular, we have seen that the irregular level structure at intermediate strengths of the static field manifests in a rapid decay of the Bloch oscillations of the mean atomic momentum, and that the time scale of this decay provides a direct measure for the decay of particle-particle coherences across the lattice. Hence, chaotic dynamics of cold, interacting atoms loaded into a one-dimensional optical lattice allow for experimental probing and control of interaction induced decoherence.

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[9] Note that the spectral properties determine the dynamics, and that the statistical features of the spectra we are analyzing here are robust when we choose periodic instead of vanishing boundary conditions. However, the spectral analysis for periodic boundary conditions is more involved, since we need to introduce the Floquet-Bloch operator generated by $\tilde{H}(t)$, and will therefore be presented in a separate contribution.
[10] For this choice of parameters, the dimension of the Hilbert space and, hence, the size of the matrix to be diagonalised is $N = (N + 2L)!/N!(2L)! = 1001$ (to be compared to $N = 286$ for $N = 3$ in Fig. 2—we chose $N = 4$ here to increase the statistical sample). Only the central part of the spectrum, with approximately constant density of states, is used for the statistical analysis.
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