On $k$-Fibonacci Numbers with Applications to Continued Fractions

Julius Fergy T. Rabago
Department of Mathematics and Computer Science
College of Science
University of the Philippines Baguio
Governor Pack Road, Baguio City 2600
PHILIPPINES
E-mail: jfrabago@gmail.com; jtrabago@upd.edu.ph

Abstract. Let $(\varpi_n)_{n \in \mathbb{N}}$ be the sequence of $k$-Fibonacci numbers recursively defined by

$$\varpi_1 = 1, \quad \varpi_2 = 1, \quad \varpi_{n+2} = k\varpi_{n+1} + \varpi_n, \quad \forall n \in \mathbb{N},$$

and $m$ be a fixed positive integer. In this work we prove that, for almost every $x \in (0, 1)$, the pattern $k, k, \ldots, k$ (comprising of $m$-digits) appears in the continued fraction expansion $x = [0; a_1, a_2, \ldots]$ with frequency

$$\hat{f}(k, m) := (-1)^m \frac{k}{\log 2} \log \left( \frac{\varphi_{m+1}^{-1} + 1}{\varphi_m^{-1} + 1} \right),$$

where $\varphi_m = \varpi_{m+1}/\varpi_m$, i.e.,

$$\lim_{n \to \infty} \frac{1}{n} \# \{ j \in \Omega_n : a_{j+i} = k \text{ for all } i \in \Omega_{m-1} \cup \{0\} \} = \hat{f}(k, m),$$

where $\Omega_n := \{1, 2, \ldots, n\}$.

1. Introduction
Consider the sequence of $k$-Fibonacci numbers $(F_{k,n})_{n \in \mathbb{N}}$ generated by the recurrence relation

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad n \in \mathbb{N}$$

with initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$. The sequence was first introduced by Falcón and Plaza in 2007 and was initially originated on their study of a recursive partition of triangles in the context of the finite element method and triangular refinements. Particularly, they showed in [5] an intriguing relation between the 4-triangle longest-edge partition and the $k$-Fibonacci numbers. The sequence, however, is actually a particular case of the widely known fundamental Lucas sequence $(u_n(p, q))_{n \in \mathbb{N}}$ (with $p, q \in \mathbb{R}^+$) described by the recurrence equation

$$u_0 = 0, \quad u_1 = 1, \quad u_{n+1} = pu_n - qu_{n-1}, \quad \forall n \in \mathbb{N}. \quad (1)$$
The above sequence, on the other hand, appears to be a mere instance of a more general sequence of second-order \((w_n(w_0, w_1; p, q))_{n \in \mathbb{N}}\) defined by the same recurrence relation \((1)\), but now with arbitrary initial values \(w_0\) and \(w_1\). The two sequences were extensively studied by Lucas \([10]\) in 1878 and Horadam \([8]\) in 1965, respectively. Since then, various extensions and generalizations of these sequences have been introduced and investigated. To date, one can find a huge amount of studies scattered in literature about Fibonacci sequence and its generalizations.

The Fibonacci sequence, and the Horadam sequence in general, have many interesting properties. For instance, it is well-known that the ratio of consecutive terms of the usual Fibonacci numbers \(F_{n+1}/F_n\) converges to the widely studied golden ratio (see, e.g., \([3\) and \([23]\)). In fact, in general, the consecutive terms of Horadam numbers \(w_{n+1}(w_0, w_1; p, -q)/w_n(w_0, w_1; p, -q) =: \Phi_n\) converges to the positive root of the quadratic equation \(x^2 - px - q = 0\), regardless of the initial conditions \(w_0\) and \(w_1\). Recently, numerous studies have been made concerning Horadam numbers. In fact, in some earlier paper, these numbers have been applied in the study of difference, differential and functional equations (see, e.g., \([18\] [19] [20], and the references therein). In this work, however, we are interested in the application of the integer sequence \((\varphi_n)_{n \in \mathbb{N}} := (w_n(1 - k, 1; k, -1))_{n \in \mathbb{N}}\) in the study of continued fractions. For more results regarding Horadam numbers, we refer the readers to a survey paper of Larcombe et al. \([13]\) (see also \([14]\) for a survey update and extensions).

**Remark 1.** We emphasize that the sequence \((\Phi_n)_{n \in \mathbb{N}}\) oscillates at the positive root of the quadratic equation \(x^2 - px - q = 0\) as \(n\) increases. To see this, we prove that \(\Phi_{2s-1} < \Phi_{2s}\) and \(\Phi_{2s} > \Phi_{2s+1}\) for all \(s \in \mathbb{N}\) via induction principle. Note that for \(w_2/w_1 = (p^2 + q)/q = p + q/p > p = w_2/w_1\) and \(w_3/w_2 = (p^3 + 2pq)/(p^2 + q) = p + (pq)/(p^2 + q) > p + p/q = w_3/w_2\). Hence, the case \(s = 1\) is already verified. To proceed with the method, we assume \(\Phi_{2s-1} < \Phi_{2s}\) and \(\Phi_{2s} > \Phi_{2s+1}\) is true for some \(s \in \mathbb{N}\) and then show that the inequalities \(\Phi_{2s+1} < \Phi_{2s+2} + \Phi_{2s+3}\) holds true for \(s \geq 1\). Now, since \(\Phi_n = w_{n+1}(w_0, w_1; p, -q)/w_n(w_0, w_1; p, -q) =: w_{n+1}/w_n\), then we have

\[
\Phi_n = \frac{w_{n+1}}{w_n} = \frac{pw_n + qw_{n-1}}{w_n} = p + q\frac{w_{n-1}}{w_n} = p + q\Phi_{n-1}.
\]

So it follows that

\[
\Phi_{2s+2} - \Phi_{2s+1} = q\frac{\Phi_{2s} - \Phi_{2s+1}}{\Phi_{2s+1}\Phi_{2s}} > 0 \iff \Phi_{2s+2} > \Phi_{2s+1}.
\]

Using this last inequality, we get

\[
\Phi_{2s+2} - \Phi_{2s+3} = q\frac{\Phi_{2s+2} - \Phi_{2s+1}}{\Phi_{2s+2}\Phi_{2s+1}} > 0 \iff \Phi_{2s+2} > \Phi_{2s+3}.
\]

By principle of mathematical induction, conclusion follows.

As an immediate consequence of this result, we see that the sequence of ratios \(\varphi_{n+1}/\varphi_n =: \varphi_n\) of consecutive terms of \(k\)-Fibonacci sequence (with initial conditions \(\varphi_1 = \varphi_2 = 1\)) oscillate at \(\varphi := (k + \sqrt{k^2 + 4})/2\) as \(n\) increases without bound. It is also worth noting that the transformation \(\varphi_n = \varphi_{n+1}/\varphi_n\), transforms the recurrence relation \(\varphi_{n+2} = k\varphi_{n+1} + \varphi_n\) to the nonlinear difference equation \(\varphi_{n+1} = k + 1/\varphi_n\), for all \(n \in \mathbb{N}\).

**Remark 2.** Obviously, the number \(\Phi := (p + \sqrt{p^2 + 4q})/2\), from which the expression \(\Phi_n\) converges as \(n\) approaches infinity, can be written in the form of continued fractions as follows:
Since $\Phi$ satisfies $\Phi^2 - p\Phi - q = 0$, then we get the relation $\Phi = p + q/\Phi$. Iteratively applying this relation to the left hand side of the equation itself, we obtain the continued fraction expansion

$$\Phi = p + \frac{q}{p + \frac{q}{p + \frac{q}{p + \ldots}}}.$$ 

In particular, we see that $\lim_{n \to \infty} \{\varpi_{n+1}/\varpi_n\} = \varphi = [k; k, k, \ldots]$. 

In an earlier paper, Hakami [6] found an application of Fibonacci numbers in the study of continued fractions (see, for instance, [9], [10], [11] and [15] for detailed discussions of these numbers). More precisely, he proved that for a fixed positive integer $m$, and for almost every number $x \in (0, 1)$, the pattern $1, 1, \ldots, 1$ ($m$-digits) appears in the continued fraction expansion $x = [0; a_1, a_2, a_3, \ldots]$ with frequency

$$\hat{f}(k, m) := (-1)^m \frac{k}{\log 2 \log \left\{1 + (-1)^m F_{m+2}^{-1}\right\}},$$ 

where $F_m$ denotes the $m$th Fibonacci number (cf. [6] Theorem 1). By substituting $m = 1$ in [2], we see that the digit 1 appears in the $x$'s continued fraction expansion with density $-(\log 3/4)/\log 2$, and we mention that this result agrees with that given in [4, Corollary 3.8, Equation 3.25]. In fact a well-known result reads as follows: given the uniform distribution of the reals on the unit interval, the Gauss-Kuzmin distribution gives the probability $\pi_k := \Pr(a_n = k)$ of an integer $k$ appearing in any given place $a_n$ of the expansion by

$$\pi_k = -\frac{1}{\log 2} \log \left\{1 - \frac{1}{(k+1)^2}\right\} \quad (\text{cf. [11]}) .$$ 

This probability distribution has been famously studied by Kuz'min, Levy, Khinchin and Wirsing. Here, as motivated by the result delivered by Hakami in [6], we establish the following theorem:

**Theorem 1.** Let $(\varpi_n)_{n \in \mathbb{N}}$ be the sequence of $k$-Fibonacci numbers with initial conditions $\varpi_1 = \varpi_2 = 1$ and let $m$ be a fixed positive integer. Then, for almost every $x \in I := (0, 1)$, the pattern $k, k, \ldots, k$ (comprising of $m$-digits) appears in the continued fraction expansion $x = [0; a_1, a_2, \ldots]$ with frequency

$$\hat{f}(k, m) := (-1)^m \frac{k}{\log 2 \log \left\{1 + \varphi_{m+1}^{-1} + 1\right\}},$$ 

i.e.,

$$\lim_{n \to \infty} \frac{1}{n} \#\{j \in \Omega_n : a_{i+j} = k \text{ for all } i \in \Omega_{m-1} \cup \{0\}\} = \hat{f}(k, m),$$ 

where $\Omega_n := \{1, 2, \ldots, n\}$. 
As an immediate consequence of the above result, we see that when \( k = 1 \), we'll obtain

\[
\hat{f}(1, m) = \left(\frac{-1}{\log 2}\right) \log \left\{ \frac{\varpi_{m+1}/\varpi_{m+2} + 1}{\varpi_m/\varpi_{m+1} + 1} \right\}.
\]

The expression on the right hand side of the above equation can be further simplified using the recursion for the usual Fibonacci sequence and Simson’s identity. Upon simplification, the expression for \( \hat{f}(1, m) \) will eventually be transformed into \( (2) \).

Another interesting result which follows as a special case of Theorem (1) is given in the next corollary.

**Corollary 2.** Let \((Q_n)_{n \in \mathbb{N}} = \{1, 1, 3, 7, 17, 41, \ldots, P_n+2 = 2P_{n+1} + P_n, \ldots\}\) be a sequence of Pell-like numbers and \( m \) be a fixed positive integer. Then, for almost every number \( x \in (0, 1) \), the pattern \( 2, 2, \ldots, 2 \) (comprising of \( m \)-digits) appears in the continued fraction expansion \( x = [0; a_1, a_2, \ldots] \) with frequency

\[
\left(\frac{-1}{\log 2}\right) \frac{2}{\log 2} \log \left\{ 1 + \left(\frac{-1}{\log 2}\right)^m Q_{m+2}^{-2} \right\}.
\]

The rest of the paper is organized as follows. In the next section (Section 2) we present some basic concepts about Gauss measure and Gauss map which is discussed more detailedly in [4]. Section 3 is devoted on the complete proof of Theorem (1) while in Section 4, a short conclusion is provided. Throughout this work, we assume that the reader has some basic knowledge of elementary number theory (see, e.g., Borevich and Shafarevich [1], Hardy and Wright [7] and Niven, Zuckerman and Montgomery [17]), ergodic theory (see, e.g., Einsiedler and Ward [4]), analysis and measure theory (see, e.g., Rudin [21, 22]).

2. Preliminaries

Here we consider and discuss some elementary properties of the Gauss Map which is formally defined as follows (cf. [2, 4]):

**Definition 1.** The Gauss map, which we denote here by \( T(x) \), is given by

\[
T(x) = \begin{cases} 
0, & \text{if } x = 0, \\
\frac{1}{x} \mod 1 = \left\{ \frac{1}{x} \right\}, & \text{if } 0 < x \leq 1,
\end{cases}
\]

where \( \{\frac{1}{x}\} \) denotes the fractional part of \( x \).

It is known that any number \( x \in (0, 1) \) can be written in terms of continued fraction \([0; a_1, a_2, \ldots]\) where each \( a_i \) is a positive integer and \( i \) is finite for rational numbers and infinite for irrational numbers. Hence, for \( \frac{1}{1+a_1} < x \leq \frac{1}{a_1} \),

\[
T(x) = \frac{1}{x} - a_1 = [0; a_2, a_3, \ldots]
\]

and therefore \( 0 \leq T(x) < 1 \). It follows that \( T(x) \) is continuous on the interval \((1/(1+a_1), 1/a_1]\). Observe that

\[
\lim_{x \to \frac{1}{1+a_1}^-} T(x) = 0
\]

whereas

\[
\lim_{x \to \frac{1}{1+a_1}^+} T(x) = 1.
\]

Thus, \( T \) is discontinuous at each of the points \( x = 1/i \) for all \( i = 1, 2, \ldots \).
Remark 3. It is in fact not hard to see that \(T^j(x) = [0; a_{j+1}, a_{j+2}, \ldots]\) for every \(j = 0, 1, 2, \ldots\) which in turn implies that \(T^j(x)\) is discontinuous only at its corresponding endpoints \(1/(1+a_j)\) and \(1/a_j\).

The following lemmata shall be central to our investigation.

Lemma 3 ([4, Theorem 2.30]). Let \((X, \mathcal{B}, \mu, T)\) be a measure preserving system. If \(f \in L^1_\mu\), then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = f^*(x)
\]

converges almost everywhere in \(L^1_\mu\) to a \(T\)-invariant function \(f^* \in L^1_\mu\), and

\[
\int f^* \, d\mu = \int f \, d\mu.
\]

If \(T\) is ergodic, then \(f^* = \int f \, d\mu\) almost everywhere.

Lemma 4 ([4, Theorem 3.7]). The continued fraction map \(T(x) = \{\frac{1}{x}\}\) on \(I\) is ergodic with respect to the Gauss measure \(\mu\).

The previous lemma, as we shall see later on, will play an important role in the proof of Theorem 4 as it will let us utilize Lemma 3.

Lemma 5. Let \(m \in \mathbb{N}\). For \(x \in Y := I \setminus \mathbb{Q}\), let \(a_1(x), a_2(x), \ldots\) be the digits of its continued fraction expansion. Further, define \(I_m \subset I\) as the interval

\[
I_m = \begin{cases} \left(\frac{1}{\varphi_m}, \frac{1}{\varphi_{m+1}}\right), & \text{if } m \text{ is even,} \\ \left(\frac{1}{\varphi_{m+1}}, \frac{1}{\varphi_m}\right), & \text{if } m \text{ is odd.} \end{cases}
\]

Then, \(a_i(x) = k\) for all \(i \in \Omega_m\) if and only if \(x \in I_m\).

Proof. We follow [4] for the proof of the above lemma. So let \(x \in Y\) and for simplicity, denote \(a_i := a_i(x)\). We have \(a_1 = \lfloor x \rfloor\). So \(a_1 = k\) provided \(1/x < k+1\) or equivalently, \(x > 1/(k+1)\), i.e. \(x\) must be in \(I_1 := (1/(k+1), 1)\) so that \(a_1 = k\). Hence, the lemma holds for \(m = 1\). Now, suppose the lemma is true for some \(m \in \mathbb{N}\). It was mentioned in [4, p. 79] that the Gauss map \(T(x)\) which sends \(x\) to \(\{x^{-1}\}\) in \(Y\) has the effect of shifting the continued fraction of \(x\) one step to the left. Hence, \(a_i = k\) for all \(i \in \Omega_{m+1}\) if and only if \(a_1 = k\) and \(\{x^{-1}\} \in I_m\) (viz., if and only if \(x \in I_1\) and \(\{x^{-1}\} = x^{-1} - k \in I_m\)). But, as we recall, \(I_m = (1/\varphi_m, 1/\varphi_{m+1})\) and that (by Remark 3) \(\{k+1/\varphi_{m+1}\}^{-1} = \varphi_{m+1}^{-1}\) and \(\{k+1/\varphi_{m+1}\}^{-1} = \varphi_{m+2}^{-1}\). Hence, \(x^{-1} - k \in I_m\). Now, by Remark 3, we see that \(1/\varphi_m < 1/\varphi_{m+2}\) whenever the inequality \(1/\varphi_m > 1/\varphi_{m+1}\) holds and vice versa. Therefore, the open interval just referred to is in fact \(I_{m+1}\), i.e. we have just shown that \(x^{-1} - k \in I_m\) if and only if \(x \in I_{m+1}\). Note also that, by construction, \(I_{m+1} \subset I_1\) for all \(m \in \mathbb{N}\) (this can also be seen from the fact \(\varphi_n \leq \varphi_{n+1}\) and \(\varphi_{n+1} = k\varphi_n + \varphi_{n-1} \leq (k+1)\varphi_n\)). Here follows the conclusion that \(a_i = k\) for all \(i \in \Omega_{m+1}\) if and only if \(x \in I_{m+1}\). This in turn proves that the lemma also holds for the case \(m + 1\). Thus, by principle of mathematical induction, the lemma holds for all \(m \in \mathbb{N}\).

Having these ideas understood, we are now in the position to prove our main result in the next section.
3. Proof of Theorem

Let $Y$ be defined as above, $\mu$ be the Gauss measure (i.e. $d\mu(x) = \frac{1}{\log 2} \frac{dx}{1 + x}$) and $T$ be the Gauss map, i.e.,

$$T(x) : Y \rightarrow Y$$

$$x \mapsto \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor .$$

By Lemma 4, $T$ is ergodic. Define $f : Y \rightarrow \mathbb{R}$ to be the characteristic function of the interval $I_m$ for fixed integer $m > 0$. Evidently, $f \in L^1(Y, \mu)$. So the point-wise Ergodic Theorem (see Lemma 3) applies. Therefore, we conclude that for $\mu$-almost all $x \in Y$ we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} f(T^{j-1}x) = \int_{Y} f \, d\mu. \tag{3}$$

However, from Lemma 5 and by the fact that the map $T$ corresponds to left shifting the continued fraction expansion of $x$ (see Remark 3), we have $f(T^{j-1}x) = k$ if and only if $a_{j+i}(x) = k$ for all $i \in \Omega_{m-1}$. Thus, the left hand side of equation (3) equates to

$$\lim_{n \to \infty} \frac{1}{n} \# \{ j \in \Omega_n : a_{j+i}(x) = k \text{ for all } i \in \Omega_{m-1} \cup \{0\} \}. \tag{4}$$

On the other hand, the integral $\int_{Y} f \, d\mu$ is computed as follows:

$$\int_{Y} f \, d\mu = \mu(I_m) = \frac{1}{\log 2} \int_{I_m} \frac{k}{1 + x} \, dx$$

$$= (-1)^m \frac{k}{\log 2} \left[ \log \left( \varphi^{-1}_{m+1} + 1 \right) - \log \left( \varphi^{-1}_m + 1 \right) \right]$$

$$= (-1)^m \frac{k}{\log 2} \log \left\{ \frac{\varphi^{-1}_{m+1} + 1}{\varphi^{-1}_m + 1} \right\}. \tag{5}$$

We have thus proved that for $\mu$-almost all $x \in Y$ (or equivalently, for Lebesgue almost all $x \in Y$), the limit in (4) equates to the expression in (5).

4. Conclusion

We have found that the pattern $k, k, \ldots, k$ (a string of $m$-digits of $k$) appears in the continued fraction expansion of $x \in (0, 1)$ with frequency $(-1)^m (k/ \log 2) \log \left\{ (\varphi^{-1}_{m+1} + 1)/(\varphi^{-1}_m + 1) \right\}$. This result was established through the Gauss map and Gauss measure together with the concept of Ergodic theory. The next step in this line of research is to consider the problem of finding the frequency of the string $\beta_1, \beta_2, \ldots, \beta_k$ in the continued fraction expansion of a number $x \in (0, 1)$. Consequently, this problem will be the subject of further discussion elsewhere.

Acknowledgments

The author would like to thank the anonymous reviewer for carefully handling and examining his manuscript. He is also grateful to the organizers of the conference The 2015 International Conference on Mathematics, its Applications, and Mathematics Education (ICMAME 2015) held at Sanata Dharma University, Yogyakarta, Indonesia, on 14-15 September 2015, for their invitation and for giving the opportunity to publish this paper in the proceedings of this conference.
References

[1] Borevich Z I, Shafarevich I R 1966 Number Theory vol 20 Series on Pure and Applied Mathematics (New York)

[2] Corless R M 1992 Continued fractions and chaos Amer. Math. Monthly 99 no 3 203–215

[3] Dunlap R A 1998 The Golden Ratio and Fibonacci Numbers (World Scientific)

[4] Einsiedler M, Ward T 2011 Ergodic Theory with a View Towards Number Theory Springer Graduate Text in Mathematics vol. 259 (London: Springer-Verlag London Ltd.)

[5] Falcón S, Plaza Á 2007 The k-Fibonacci sequence and the Pascal 2-triangle Chaos, Solitons & Fractals 33 no 1 38–49

[6] Hakami A 2015 An application of Fibonacci sequence on continued fractions Int. Math. Forum 10 no 2 69–74

[7] Hardy G H, Wright E M 1998 An Introduction to the Theory of Numbers (Oxford: Oxford Science Publications, Clarenden Press)

[8] Horadam A F 1965 Basic properties of a certain generalized sequence of numbers Fib. Quart. 3 161–176

[9] Jones W B, Thron W J 1980 Continued Fractions: Analytic Theory and Applications vol 11 Encyclopedia of Mathematics and its Applications (Massachusetts: Addison-Wesley Publishing Co.)

[10] Khovanskii A N 1963 The Application of Continued Fractions and Their Generalizations to Problems in Approximation Theory Translated by P. Wynn.. Noordhoff N. V. (Groningen)

[11] Khinchin A Y 1997 Continued Fractions with a Preface by B. V. Gnedenko (New York: Dover Publications, Inc.)

[12] Koshy T 2001 Fibonacci and Lucas Numbers with Applications Pure and Applied Mathematics (New York: Wiley-Interscience)

[13] Larcombe P J, Bagdasar O D, Fennessey E J 2013 Horadam sequences: a survey Bulletin of the I.C.A. 67 49–72

[14] Larcombe P J 2015 Horadam Sequences: a survey update and extension Submitted.

[15] Lorentzen L, Waadeland H 1992 Continued Fractions with Applications vol 3 Studies in Computational Mathematics (Amsterdam: North-Holland Publishing Co.)

[16] Lucas E 1878 Théorie des Fonctions Numériques Simplement Périodiques American Journal of Mathematics 1184–240, 289–321; reprinted as “The Theory of Simply Periodic Numerical Functions”, Santa Clara, CA: The Fibonacci Association, 1969.

[17] Niven I, Zuckerman H S, Montgomery H L 1991 An Introduction to the Theory of Numbers (New York: John Wiley and Sons)

[18] Rabago J F T 2015 On the closed-form solution of a nonlinear difference equation and another proof to Sroysang’s conjecture Submitted for publication

[19] Rabago J F T 2015 On second-order linear recurrent functions with period k and proofs to two conjectures of Sroysang Hacet. J. Math. Stat. To appear

[20] Rabago J F T 2014 On second-order linear recurrent homogenous differential equations with period k. Hacet. J. Math. Stat. 43 no 6 923–933

[21] Rudin W 1976 Principles of Mathematical Analysis 3rd Ed. (McGraw-Hill)

[22] Rudin W 1987 Real and Complex Analysis 3rd Ed. (McGraw-Hill)

[23] S. A. Vajda 1989 Fibonacci & Lucas Numbers and the Golden Section: Theory And Applications (Chishester: Ellis Horwood Ltd.)