UNIPOTENT REDUCTION AND THE POINCARÉ PROBLEM

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1. Introduction

Let $S$ be a complex projective surface. A foliation on $S$ is an exact sequence:

\[(1.1) \quad 0 \to L \xrightarrow{F} TS \to N_{\mathcal{F}} \to O_{\Gamma} \to 0,\]

where $L$ and $N_{\mathcal{F}}$ are invertible sheaves and $\Gamma$ is a zero-dimensional subscheme of $S$, called the singular locus of $\mathcal{F}$.

In fact, the entire exact sequence 1.1 is determined by the inclusion $\mathcal{F}$. Indeed, we can consider the quotient of $\mathcal{F}$ to be:

\[0 \to L \xrightarrow{F} TS \to C \to 0.\]

The sheaf $C$ will not be locally free in general, but it will be the case for its double dual sheaf $N_{\mathcal{F}} := C^{**}$. Completing the exact sequence

\[0 \to L \xrightarrow{F} TS \to N_{\mathcal{F}},\]

we obtain 1.1 ([4], [14]).

The aim of this paper is to give some results concerning the so called Poincaré Problem. Roughly speaking, this problem asks for a numerical criteria to identify when a morphism $F : L \to TS$, defining a foliation, is equal to another of the form $T_f \to TS$, with $f : S \to \mathbb{P}^1$ a holomorphic map and $T_f$ (that must be isomorphic to our original $L$) the sheaf of relative vector fields of $f$.

The original statement of the problem is on $\mathbb{P}^2$ ([11], [12]), in this case $f$ must be a rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, undefined in a finite number of points (Bezout’s Theorem). Section 2 of this paper is devoted to an explanation of how this situation can be modified to the case of a holomorphic map $f : S \to \mathbb{P}^1$, ($S$ will be the blowing-up of $S$ in the indetermination locus of the original rational map). The classical formulation of the Poincaré Problem is explained there. This expository section (which includes, moreover, several results on foliation theory used below) does not contains any original result and is intended as an effort to fill a hypothetical gap between the specialists in foliation theory and those in fibration theory. Once the bridge between both theories is constructed we work almost completely with the language of fibration theory.

Section 3 studies the problem of bounding the genus of a fibration:

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in terms of numerical information of $K_F$, the canonical sheaf of the associated foliation. In the notation introduced in 1.1, $K_F = L^{-1} (\simeq T_f^{-1})$. We apply the technique of unipotent reduction of a fibration to obtain the bounds. In the results proved in that section some hypothesis on the asymptotical behavior of the canonical sheaf with respect to the fibers $C$ is made. In this sense, they are not authentic solutions to the Poincaré Problem, as they assume some knowledge about the fibers $C$. This is an example of the kind of results we prove in section 3:

**Theorem 3.1.** Assume $K_F$ is a nef divisor. If, for some $N >> 0$ and some $n \in \mathbb{N}$

$$h^0(K^n_C) > h^0(K^n_F),$$

then $h^0(K^n_C(-C)) \neq 0$, and consequently

$$nK^n_F \geq 2(g - 1).$$

In this statement $K^n_C = \alpha^* K^n_F$, $\alpha$ standing for the unipotent reduction of degree $N$ (see section 3.1).

Our purpose is twofold: on one hand, to show how positive conditions on $K_F$ (i.e. nefness or ampleness) make the problem easier to handle; on the other hand, to put the general problem (under the positive assumptions) within the framework of a well developed theory.

Finally, section 4 is of a more elementary nature and is devoted to prove the following result:

**Theorem 4.6.** Let $\mathcal{F}$ be a foliation defined on $S$ admitting a holomorphic first integral $f : S \to \mathbb{P}^1$ of genus $g > 0$. Assume $\mathcal{F}$ is non-degenerated. If for every singular point of $\mathcal{F}$, the eigenvalues of the linear part of $\mathcal{F}$ has absolute value greater than 3; then,

$$(g - 1) \leq 4(-\chi(K_F) + \chi(\mathcal{O}_S)).$$

We prove a number of auxiliary results, each with its own independent interest. Almost all the arguments used in the proofs are really simple, but in a crucial step (Lemma 4.4) we use the unipotent reduction.

This Theorem gives us a complete answer to Poincaré Problem when the eigenvalues associated to the singularities of the reduced foliation of $\mathcal{F}$ has absolute values greater that 3.

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2. FROM POINCARE’S PROBLEM TO FIBRATIONS

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$:

$$0 \to \mathcal{O}(-m + 1) \overset{\mathcal{F}}{\to} T\mathbb{P}^2.$$

Assume $\mathcal{F}$ admits a rational first integral. This means that there exists a rational map:
\[ f : \mathbb{P}^2 \to \mathbb{P}^1, \]
such that the set-theoretical fibers of \( f \) coincide with the leaves of \( F \) (in other words: \( F \) is algebraically integrable).

Rational maps of this kind are given by pencils of curves \( \lambda F + \mu G = 0 \) (\deg F = \deg G = d).

The classical Poincaré Problem consists of finding a function \( g \) such that \( d \leq g(m) \), i.e., to bound the degree of the pencil defining the first integral in terms of the degree of the foliation. ([11], [12], [15]).

The following basic formula was proved by Poincaré. The generic element of \( \lambda F + \mu G = 0 \) is an irreducible curve. If we denote by \( \sum n_{ij}F_{ij} \) the divisor of reducible components of the pencil, then:

\[ 2d - 2 = m + \sum d_{ij}(n_{ij} - 1), \quad d_{ij} := \deg F_{ij}, \]

Thus, Poincaré's Problem is equivalent to bounding the numbers \( n_{ij} \) and \( d_{ij} \). In section 4 we deal with the problem of bounding the multiplicities \( n_{ij} \).

A more general (and reasonable) version of the problem is to bound \( d \) in terms of some numerical information of the foliation, and not only its degree. ([6], [15]).

The aim of this section is to explain how this problem can be translated into the language of fibered surfaces. In what follows, we assume that the singularities of \( F \) are non-degenerated. This implies that the singular points of \( F \) are of two different classes:

a) Dicritical points: points in the clousure of infinitely many leaves of \( F \).

b) Saddle points: points in the clousure of exactly two leaves.

Taking into account that these singularities are points where the map \( f \) is not smooth, we obtain the following relations:

a') The dicritical points of \( F \) are the points where the map \( f \) is not defined; equivalently, they are the base locus of the pencil associated to \( f \). The local expression of the pencil associated to \( f \) around these points is of the form: \( x^p - \lambda y^q = 0 \). The linear part of the local vector field defining \( F \) around a dicritical point is related to the corresponding local form of \( f \): if \( f \) is locally \( x^p - \lambda y^q = 0 \), the linear part will be \( \text{diag}(p', q') \) with \( p = ap', q = aq' \) and \( a = \text{m.c.d}(p, q) \).

b') The saddle points of \( F \) are the singularities of the non-generic fibers of \( f \) (considered with its reduced structures) located away from the base locus. The local expression of the pencil associated to \( f \) is of the form: \( x^p y^q = \lambda \). The linear part of the local vector field defining \( F \) around a dicritical point is related to the corresponding local form of \( f \): if \( f \) is locally \( x^p y^q = \lambda \), the linear part will be \( \text{diag}(-p', q') \) with \( p = ap', q = aq' \) and \( a = \text{m.c.d}(p, q) \).

Now, we can proceed with our translation:

**Step 1** There are two parallel resolution theorems:

1.1. We can solve the indeterminacies of \( f \) ([3]): There exists a minimal chain of blowing-ups:

\[ \tilde{S} = S_n \to S_{n-1} \to ... \to \mathbb{P}^2, \]

and a commutative diagram:
such that \( \tilde{f} \) is a holomorphic map (i.e. defined in all the points of \( \tilde{S} \)).

Therefore, \( \tilde{f} : \tilde{S} \to \mathbb{P}^1 \), is a fibered surface. The meaning of fibration, as used in this paper, is a morphism \( f : S \to Y \) between a nonsingular projective surface \( S \) and a non singular projective curve \( Y \), with connected fibers. The general fiber \( \tilde{C} \) is a non-singular curve, birationally equivalent to \( C \), the clousure of the general fiber of \( f \). In particular, \( g(C) = g(\tilde{C}) \).

1.2. We can solve the dicritical singularities of \( \mathcal{F} \). This is a particular case of a celebrated theorem by Seindeberg ([13]). The result of this process is a minimal chain of blowing-ups:

\[
\tilde{S} = S_n \to S_{n-1} \to \ldots \to \mathbb{P}^2,
\]

such that the foliation \( \tilde{\mathcal{F}} = \pi^* \mathcal{F} \) has only saddle singularities.

We have used the same notation for the resulting morphism \( \tilde{S} \to \mathbb{P}^1 \) of both processes. They are, in fact, the same:

**Proposition 2.1.** Let \( \mathcal{F} \) be a non-degenerated foliation on \( \mathbb{P}^2 \) admitting a rational first integral

\[
f : \mathbb{P}^2 \to \mathbb{P}^1.
\]

The resolution process of Seindenberl coincides with the resolution of indeterminacies of \( f \). In particular, \( \tilde{f} : \tilde{S} \to \mathbb{P}^1 \), is a holomorphic first integral for \( \tilde{\mathcal{F}} \) and the geometric genus of the general leaves of \( \mathcal{F} \) and \( \tilde{\mathcal{F}} \) are the same.

The proof of this fact is, under the non-degenerated hypothesis, an elementary calculus based on the local equations of the blowing-up.

**Step 2.** Poincaré ([11], see also [9] and [15]) proved the following inequality: if \( \mathcal{F} \) is non-degenerated, of degree \( d \) and admits a rational first integral; then, assuming our previous notation:

\[
(2.1) \quad \frac{m - 4}{4} d + 1 \leq g.
\]

In virtue of this relation and Step 1, the problem can be reformulated as: given a fibration \( f : S \to \mathbb{P}^1 \) of genus \( g \), find an upper bound for \( g \) in terms of numerical invariants of the foliation associated to \( f \).

The last phrase needs some explanation. Associated to \( f \) there is an invertible sheaf \( K_f := K_S \otimes f^* K_{\mathbb{P}^1} \), the so called relative dualizing sheaf. On the other hand, the foliation associated to \( f \) is given by an inclusion:

\[
0 \to \Theta_f \xrightarrow{\mathcal{F}} TS \xrightarrow{df} f^* T\mathbb{P}^1 \to \mathcal{I} \to 0.
\]

The sheaf \( \mathcal{I} \) is supported on the singular points of \( f \). The sheaf \( (\Theta_f)^{-1} \) is denoted by \( K_{\mathcal{F}} \) and called the canonical sheaf of \( \mathcal{F} \) ([4], [8]).
We introduce the following notation, used systematically in the remaining part of this paper. Denote by

\[ \Delta = \sum_{ij} n_{ij} F_{ij}, \]

the divisor defined as the sum of all the non-reduced fibers of \( f \). Hence, if we fix the index \( i \), \( \sum_j n_{ij} F_{ij} \) will denote a single non-reduced fiber of \( f \).

Moreover, we write \( \Delta = \Delta_{\text{red}} + \Delta_0 \), where \( \Delta_{\text{red}} = \sum_{ij} F_{ij} \) is the reduced divisor associated to \( \Delta \). Note that \( \Delta_0 = \sum_{ij} (n_{ij} - 1) F_{ij} \). The relation between \( K_f \) and \( K_F \) is:

\[ K_F = K_f(-\Delta_0). \]

In [14] the following was proved:

**Theorem 2.2.** Let \( f : S \to Y \) be a fibration of genus \( g \geq 2 \). The following are equivalent:

- a) \( K_F \) is big and nef,
- b) \( f \) is relatively minimal (no exceptional curves of the first kind contained in the fibers) and non-isotrivial (two general fibers are not isomorphic).

Furthermore, if a) or b) holds, then the only irreducible curves \( D \subset S \) such that \( K_F \cdot D = 0 \) are the \((-2)\)-vertical curves (rational curves with self-intersection \(-2\) contained in some fiber of \( f \)).

The proof of this theorem uses the unipotent reduction of \( f \). The next section is devoted to the exploitation of the same technique in connection with Poincaré’s problem.

### 3. Unipotent reduction and the Poincaré Problem

#### 3.1 Unipotent reduction of a fibration.

In this section we apply a useful technique in fibration theory, namely the unipotent reduction, to the study of the Poincaré Problem. We first recall the basic principles of unipotent reduction.

Let \( f : S \to Y \) be a fibration, then there exists a commutative diagram:

\[ \begin{array}{ccc}
\tilde{S} & \xrightarrow{\alpha} & S \\
\downarrow f & & \downarrow f \\
X & \xrightarrow{\pi} & Y \\
\end{array} \]

such that \( \tilde{S} \) is a non-singular projective surface and \( \tilde{f} \) is a reduced fibration, i.e. all its fibers are reduced. The ramification values of \( \pi \) contain the critical values of \( f \) as a subset.

In this construction \( N \) is a common multiple of the multiplicities of the non reduced components of the fibers of \( f \), and \( \pi \) is a totally ramified cyclic covering. We assume that \( N \) is sufficiently large and \( \pi \) has \( N \) ramification values (see [2], sections III.9 and III.10, for the details concerning this construction). The genus of \( X \) can be computed easily, for sufficiently large \( N \), using Riemann-Hurwitz and the fact that \( \pi \) is totally ramified on \( N \) points:
\[ g_X - 1 = \frac{N(N - 1)}{2} + N(g_Y - 1). \]

In particular, if \( Y = \mathbb{P}^1 \), \( g_X - 1 = \frac{N(N-3)}{2} \).

The other fundamental fact that we shall use systematically is due to Serrano:

\[ K_{\tilde{f}} := \alpha^* K_F \]

is the relative dualizing sheaf of \( \tilde{f} \). Thus \( K_{\tilde{f}} = K_{\tilde{f}} = K_{\tilde{S}} \otimes \tilde{f}^* K_X^{-1} \) ([14], claim in the proof of Prop. 2.1), we shall call this result Serrano’s Lemma. Thus, \( K_{\tilde{f}} \mathcal{C} = 2g - 2 \) where \( \mathcal{C} \) denotes a fiber of \( \tilde{f} \).

### 3.2 Positivity properties of \( K_{\tilde{f}} \) and explicit bounds for \( g \).

Let \( f : S \to \mathbb{P}^1 \) be a fibration with fibers supported on normal crossing divisors (i.e., the associated foliation is non-degenerated). As explained in section 2, it is an important problem in foliations theory to find a upper bound for the genus \( g \) of the fibres of \( f \) in terms of the invariants of \( S \) and \( K_F \). The aim of this section is to show how positivity conditions on \( K_{\tilde{f}} \) allows us to solve this problem. In the remains of this paper we use the notation introduced in section 3.1.

**Theorem 3.1.** Assume \( K_{\tilde{f}} \) is a nef divisor. If, for some \( N \gg 0 \) and some \( n \in \mathbb{N} \),

\[ h^0(K_{\tilde{f}}^n) > h^0(K_{\tilde{f}}^n), \]

then \( h^0(K_{\tilde{f}}^n(-C)) \neq 0 \), and consequently

\[ nK_{\tilde{f}}^2 \geq 2(g - 1). \]

**Proof.** The direct image sheaf \( \alpha_* \mathcal{O}_{\tilde{S}} \) can be computed explicitly ([5]):

\[ \alpha_* \mathcal{O}_{\tilde{S}} = \oplus_{i=0}^{N-1} \mathcal{L}^{(i)}. \]

As \( f \) takes values in \( \mathbb{P}^1 \) the first terms in this decomposition are: \( \mathcal{L}^{(0)} = \mathcal{O}_S \), \( \mathcal{L}^{(1)} = \mathcal{O}_S(-C) \). In general, \( \mathcal{L}^{(i)} \) is an invertible sheaf contained in \( \mathcal{L}^{(i-1)} \):

\[ (3.2) \quad 0 \to \mathcal{L}^{(i)} \to \mathcal{L}^{(i-1)}. \]

Until this point, all the considerations have been general. Now, in order to simplify the notation, we develop the complete argument in the case \( n = 1 \). The general case follows after an obvious modification.

Using projection formula and Serrano’s Lemma we obtain:

\[ \alpha_* K_{\tilde{f}} \simeq \alpha_* (\alpha^* K_{\tilde{f}}) = K_{\tilde{f}} \otimes \alpha_* \mathcal{O}_{\tilde{S}} \]

\[ = K_{\tilde{f}} \otimes (\mathcal{O}_S \oplus \mathcal{O}_S(-C) \oplus ... ) \]

\[ = K_{\tilde{f}} \oplus K_{\tilde{f}}(-C) \oplus ... \]

If \( h^0(K_{\tilde{f}}) > h^0(K_{\tilde{f}}) \), then, by inclusion 3.2, \( h^0(K_{\tilde{f}}(-C)) > 0 \). The Theorem follows from the fact that \( K_{\tilde{f}} \) is nef:
\[ K_F \cdot (K_F(-C)) = K_F^2 - 2(g - 1) \geq 0. \]

Using Theorem 3.1 we can deduce another condition ensuring the existence of a bound for \( g \):

**Theorem 3.2.** Assume \( K_F \) is an ample line bundle. Let \( n \) be a natural number such that \( K_F^n \) is very ample. If, for \( N > 0 \), \( h^0(K_F^n(-C)) > 0 \), then

\[ n^2K_F^2 \geq 2(g - 1). \]

**Proof.** Once again, we concentrate on the case \( n = 1 \). We can assume that \( h^0(K_F) = h^0(K_F) \), otherwise, we obtain the conclusion from Theorem 3.1 (obviously, \( K_F \) very ample is more than enough to guarantee it is nef).

Fix the embedding \( S \to \mathbb{P}^k \) given by the linear system \( K_F \). If we apply unipotent reduction, we obtain a commutative diagram:

\[
\begin{array}{ccc}
\tilde{S} & \overset{\alpha}{\longrightarrow} & S \\
|K_f| & & |K_F|, \\
S_0 & \longrightarrow & S
\end{array}
\]

(we abuse the notation slightly by using the same symbol for a linear system and the rational map it defines).

The bottom arrow is simply the projection on the subspace \( \alpha^*|K_F| \). Under our hypothesis this last morphism must be the identity and \( \alpha = |K_f| \).

Let \( C_1 \) be a fiber such that \( h^0(K_f(-C_1)) \neq 0 \), call \( C_2, ..., C_N \), the set of fibers of \( \tilde{f} \) such that \( \alpha(C_1) = \alpha(C_1) \). Then, since \( \alpha = |K_f| \), we must have \( h^0(K_f(-C_1 - C_2 - ... - C_N)) \neq 0 \). Let \( \tilde{B} \in |K_f| \) be an irreducible and nonsingular curve (the existence of such a curve is guaranteed by Bertini’s Theorem). Restricting a nonzero section of \( K_f(-C_1 - C_2 - ... - C_N)|_{\tilde{B}} \), we obtain a nonzero section of \( K_f(-\tilde{C}_1 - \tilde{C}_2 - ... - \tilde{C}_N)|_{\tilde{B}} \). Thus this sheaf is of positive degree and:

\[ \tilde{B} \cdot \left( \sum_{i=1}^{N} C_i \right) = N(2g - 2) \leq K_f \cdot \tilde{B} = K_f^2 \leq NK_f^2. \]

\[ \square \]

We don’t know of any argument ensuring the existence of global sections of \( h^0(K_f^n(-C)) \). However, we can, at least, prove the following:

**Lemma 3.3.** Assume \( K_f^n \) is big and nef. Then, for all \( N >> 0 \) and any fiber \( C \) of \( \tilde{f} \), \( K_f^n(-C) \) is big and nef.

**Proof.** This lemma is valid even if the original fibration \( f : S \to Y \) is defined over a curve \( Y \) of genus greater than zero. As previously, we make the proof for \( n = 1 \).

Obviously \( K_f(-C) \) is big for \( N \) large enough. Let \( D \) be any irreducible curve on \( \tilde{S} \). We must to prove \( K_f(-C) \cdot D \geq 0 \). We can assume \( D \) is not contained in a fiber of \( \tilde{f} \), because for any irreducible component \( F \) of a fiber, \( F \cdot C = 0 \) and \( K_f \) is nef.
Let \( \alpha(D) = D_0 \). Note that \( D_0 \) is an irreducible curve in \( S \), as all the exceptional curves of \( \alpha \) are contained in fibers of \( \bar{f} \). We have a commutative diagram:

\[
\begin{array}{ccc}
D & \xrightarrow{\alpha|_D} & D_0|_{D_0} \\
\downarrow{f|_D} & & \downarrow{\bar{f}} \\
X & \xrightarrow{\pi} & Y
\end{array}
\]

The degree of \( \bar{f}|_D \) is \( \tilde{C}.F \) and the degree of \( f|_{D_0} \) is \( C.D_0 \), whereas the degree of \( \alpha|_D \) is a divisor \( n \) of \( N \). Thus, \( C.D_0 = \frac{N}{n} \tilde{C}.D \). Moreover, \( \tilde{C}.D \geq C.D_0 \), so we conclude \( n = N \) and \( \tilde{C}.D = C \cdot D_0 \). From this it follows that \( \alpha^*D_0 = D + E \) where all the irreducible components of \( E \) are exceptional curves of \( \alpha \).

Observe that any irreducible exceptional curve \( A \) of \( \alpha \) is a \( (-2) \) curve, so \( K_{\bar{f}}.A = 0 \) as a consequence of Serrano’s Lemma and adjunction formula. It follows that \( K_{\bar{f}}.E = 0 \). Applying the projection formula for the intersection of divisors we get:

\[
K_{\bar{f}}(-\tilde{C}).D = (D + E).K_{\bar{f}} - E.K_{\bar{f}} - D.\tilde{C} = \alpha^*D_0 \cdot \alpha^*K_F - D \cdot \tilde{C} = NK_F.D_0 - C.D_0.
\]

The only curves in \( S \) having zero intersection with \( K_F \) are \( (-2) \) curves contained in the fibers of \( f \) (Theorem 2.2). Therefore, \( K_F.D_0 > 0 \), and for \( N >> 0 \):

\[
K_{\bar{f}}(-\tilde{C}).D > 0.
\]

\( \square \)

A natural way to study the space of sections \( H^0(K^n_{\bar{f}}(-C)) \) is considering its direct image under \( \bar{f} \). We have another partial result:

**Proposition 3.4.** Assume \( K_F \) is big, nef and effective. If for some \( N >> 0 \) and \( n \geq 2 \), the direct image sheaf \( \bar{f}_*K^n_F \) splits in direct sum of invertible sheaves, then there exists a fiber \( C \) such that: \( H^0(K^n_{\bar{f}}(-C)) \neq 0 \).

**Proof.** Once again, we consider the case of the lowest possible power, i.e., \( n = 2 \). The proposition is a consequence of the positivity theorem of Arakelov, Parshin and Fujita ([1], [2]). In the first place, \( K^2_{\bar{f}} \) will be big and nef. Our assumption is:

\[
\bar{f}_*K^2_{\bar{f}} = \bigoplus_{i=1}^{3g-3} L_i,
\]

with \( L_i \) invertible.

Moreover, \( \deg L_i \geq 0 \), for all \( i \). Moreover, \( L_i \neq O_X \) for all \( i \), because \( h^1(-K^n_F) = 0 \) ([1]). Thus, for some \( i \) (say \( i = 1 \)), we must have \( h^0(L_1) > 0 \) and \( \deg L_1 > 0 \).

Write \( L_1 = O_X(p_1 + \ldots + p_r) \), then \( h^0(L_1(-p_1)) \neq 0 \). As \( L_i \) is a sumand of \( \bar{f}_*K^2_{\bar{f}} \) and using the projection formula, we conclude that \( h^0(\bar{f}_*K^2_{\bar{f}}(-C_{p_1})) \neq 0 \), where \( C_{p_1} \) denotes the fiber over \( p_1 \). \( \square \)

A last application of this circle of ideas is:
Proposition 3.5. If, for some \( N >> 0 \) and a natural number \( n \), the direct image sheaf \( f_*K_f^n \) is generated by global sections, then, either:

\[
(2n - 1)(g - 1) \leq h^0(K_f^n) \quad \text{or} \quad 2(g - 1) \leq nK_F^2.
\]

Proof. We explain the case \( n = 1 \). The hypothesis means that there exists a surjective morphism of locally free sheaves:

\[
H^0(f_*K_f) \otimes \mathcal{O}_X = H^0(K_f) \otimes \mathcal{O}_X \to f_*K_f \to 0.
\]

The rank of \( f_*K_f \) is \( g \) and the rank of \( H^0(f_*K_f) \otimes \mathcal{O}_X \) is \( h^0(K_f) \). It follows at once that \( g \leq h^0(K_f) \). By Theorem 3.1 we can assume that \( h^0(K_f) = h^0(K_F) \), and we obtain the result.

\[\square\]

4. Bounding the multiplicities of non-generic fibres

Let us start with a fibration \( f : S \to \mathbb{P}^1 \). Associated to \( f \) we have the relative duality sheaf

\[
(4.1) \quad K_f := K_S \otimes f^*K_{\mathbb{P}^1}^{-1} = K_S(2C),
\]

since \( K_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2) \). In the same way we have the canonical sheaf associated to the foliation \( \mathcal{F} \) defined by the fibres of \( f \). This sheaf, denoted by \( K_F \), is related to \( K_f \) by means of the formula

\[
(4.2) \quad K_f = K_F(\Delta_0)
\]

(see the notation introduced in section 2).

Our first goal is the following:

Lemma 4.1. Let \( f : S \to \mathbb{P}^1 \) be a fibration; then:

\[
\Delta_0^2 = (K_F - K_S)^2.
\]

Proof. From relations 4.1 and 4.2 we obtain:

\[
K_F - K_S = (K_S + 2C - \Delta_0) - K_S = 2C - \Delta_0.
\]

Since all the irreducible components of \( \Delta_0 \) are contained in fibers of \( f \), we have \( C \cdot \Delta_0 = 0 \). Hence:

\[
(K_F - K_S)^2 = (2C - \Delta_0)^2 = \Delta_0^2.
\]

\[\square\]

In this way, we have determined the value of \(-\Delta_0\) in terms of numerical information of the foliation \( \mathcal{F} \). For what follows, denote by \( s \) the number of non-reduced fibers of \( f \). Note that
\[ \Delta_0^2 = (\Delta - \Delta_{red})^2 = (sC - \Delta_{red})^2 = \Delta_{red}^2. \]

Now, fix a critical value of \( f, t_i \in \mathbb{P}^1 \). Let \( C_i \) be the corresponding fibre, writing \( C_i = \sum_j n_{ij} F_{ij} \), we have:

\[ (\sum_j F_{ij})^2 = \sum_j F_{ij}^2 + 2 \sum_{j<k} F_{ij} \cdot F_{ik}. \]

We denote by \( c_2(\mathcal{F}) \) the number of singular points of \( \mathcal{F} \). Taking into account that the singularities of \( \mathcal{F} \) are precisely the intersections of the irreducible components of the non-generic fibers of \( f \) we obtain:

\[ \Delta_0^2 = \Delta_{red}^2 = \sum_{ij} F_{ij}^2 + 2c_2(\mathcal{F}). \]

Using this equality and Lemma 4.1 we can formulate our main conclusion to this point:

**Proposition 4.2.** With the same notation as before we have:

\[ -\sum_{ij} F_{ij}^2 = (K_F - K_S)^2 + 2c_2(\mathcal{F}). \]

Our next observation is that, if we fix a non-reduced fiber, say \( \sum_j n_{ij} F_{ij} \), and we know a bound for a single multiplicity \( n_{ij0} \), then we can obtain a bound for the remainder multiplicities \( n_{ij} \). In fact:

\[ 0 = F_{ij0} \cdot C = F_{ij0} \cdot (\sum_j n_{ij} F_{ij}) \]

\[ = n_{ij0} F_{ij0}^2 + \sum_{j \neq j0} n_{ij} F_{ij0} \cdot F_{i,j}. \]

Thus, the numbers \( F_{ij0}^2 \) are computed as:

\[ F_{ij0}^2 = -\frac{\sum_{j \neq j0} n_{ij} F_{ij0} \cdot F_{i,j}}{n_{ij0}}. \]

(4.3)

By Proposition 4.2, the number \(-F_{ij0}^2\) is bounded by \( \mathcal{F} \). Thus, a bound for \( n_{ij0} \) gives a bound for the multiplicities of the components intersecting positively to \( F_{ij0} \). Using the fact that all the fibers of \( f \) are 1-connected and iterating this process we obtain a bound for all the \( n_{ij} \) (\( i \) fixed).

This observation motivates the principal result of this section:

**Theorem 4.3.** Let \( f : S \to \mathbb{P}^1 \) be a fibration of genus \( g \geq 2 \), such that all its singular fibers have support on normal crossing divisors. Let \( n_{ij} \) be the multiplicities of the components of the singular fibers of \( f \). Then, either:

\( . \) there are at least \( s - 2 \) non-reduced fibers of \( f \), such that some of the associated multiplicities \( n_{ij} \) are less than or equal to \( 3 \), or

\( . (g - 1) \leq 4(-\chi(K_F) + \chi(\mathcal{O}_S)) \)
Before starting the proof we need some preliminaries. We can apply the unipotent reduction to $f$ (see section 3). Denote by $\bar{F}_{ij}$ the pre-image of the curves $F_{ij}$ under $\alpha$, the morphism defining the unipotent reduction. With this notation in mind we have the following lemma:

**Lemma 4.4.**

$$(s-2)(g-1) + \chi(K_F) - \chi(O_S) = \sum_{ij} \bar{F}_{ij} \cdot K_{\bar{f}} \cdot \frac{1}{2n_{ij}}.$$  

**Proof.** The proof makes use of the unipotent reduction of $f$, explained in the previous section. Let $\bar{f} : \bar{S} \to X$ be the unipotent reduction of $f$ associated to $N$, a common multiple of $n_{ij}$ for all $i$ and $j$. The number $K_{\bar{f}} \cdot K_S$ can be computed in two different ways:

$$K_{\bar{f}} \cdot K_S = K_{\bar{f}} \cdot (K_{\bar{f}} \otimes \bar{f}^* K_X)$$
$$= NK_F^2 + 4(g-1)(g_X-1),$$

and, using that $\alpha$ is a ramified covering away from some exceptional curves that are contracted to points, we can use the Riemann-Hurwitz formula ([2]):

$$K_{\bar{f}} \cdot K_S = \alpha^* K_F \cdot (\alpha^* K_S + \sum_{ij} (N/n_{ij} - 1) \bar{F}_{ij} + (N - 1) \sum_{k=1}^{N-s} C_k)$$
$$= NK_F \cdot K_S + \sum_{ij} (N/n_{ij} - 1) \bar{F}_{ij} \cdot K_{\bar{f}} + 2(N-s)(N-1)(g-1).$$

If we consider these equalities as polynomials in $N$, we obtain the result from the degree 1 term. □

Now, we can reach the proof of Theorem 4.3:

**Proof.** Denote by $l$ the number of non-reduced fibers with some associated multiplicity less than or equal to 3. After reordering the non-reduced fibers, we can assume that these $l$ fibers are exactly $C_1, \ldots, C_l$. Assume $l \leq s - 3$. Using the previous lemma write:

$$(s-2-l)(g-1) + l(g-1) - \sum_{i=1}^{l} \bar{F}_{ij} \cdot K_{\bar{f}} \cdot \frac{1}{2n_{ij}} = \sum_{i=l+1}^{s} \bar{F}_{ij} \cdot K_{\bar{f}} \cdot \frac{1}{2n_{ij}} - (\chi(K_F) - \chi(O_S))$$
$$\leq \frac{(s-l)(g-1)}{4} - (\chi(K_F) - \chi(O_S)).$$

The inequality uses the fact that, for any $i$, $\left(\sum_j \bar{F}_{ij}\right) \cdot K_{\bar{f}} = 2(g-1)$, as the first divisor is a fiber of $\bar{f}$. For the same reason:

$$l(g-1) - \sum_{i=1}^{l} \bar{F}_{ij} \cdot K_{\bar{f}} \cdot \frac{1}{2n_{ij}} \geq 0.$$  

Thus, we only need to prove that

$$4(s-l-2) - (s-l) \geq 1.$$  

This follows from $l \leq s - 3$. □
Corollary 4.5. Using the same notation as in Theorem 4.3 we have, either:

- for at least \( s - 2 \) non-reduced fibers of \( f \), its multiplicities are bounded by \( F \), or
- \((g - 1) \leq 4(-\chi(K_F) + \chi(O_S))\).

Proof. Simply combine Theorem 4.3 and the observation preceding it. \(\square\)

Theorem 4.3 allows us to obtain an answer to Poincaré’s Problem under some hypotheses on the singular points of \( F \) (compare with [10], Chapter 7, Theorem 15). We use the terminology explained in section 2.

Theorem 4.6. Let \( F \) be a foliation defined on \( S \) admitting a holomorphic first integral \( f : S \to \mathbb{P}^1 \) of genus \( g > 0 \). Assume \( F \) is non-degenerated. If for every singular point of \( F \), the eigenvalues of the linear part of \( F \) have absolute value greater than 3; then,

\[(g - 1) \leq 4(-\chi(K_F) + \chi(O_S)).\]

Proof. The condition \( p, q > 3 \) for all the singular points implies that all the multiplicities of non-reduced fibers of \( f \) satisfy \( n_{ij} > 3 \) (see item b’) in section 2). Thus, the theorem follows at once from theorem 4.3. \(\square\)

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