BASIC COHOMOLOGY GROUP DECOMPOSITION OF K-CONTACT 5-MANIFOLDS

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Abstract. In this paper, we consider decompositions of basic degree 2 cohomology for a compact K-contact 5-manifold \((M, \xi, \eta, \Phi, g)\), and conclude the pureness and fullness of \(\Phi\)-invariant and \(\Phi\)-anti-invariant cohomology groups. Moreover, we discuss the decomposition of the complexified basic degree 2 cohomology group. This is an analogue problem when Draghici, Li and Zhang [4] considered the \(C^\infty\) pureness and fullness of \(J\)-invariant and \(J\)-anti-invariant subgroups of the degree 2 real cohomology group \(H^2(M, \mathbb{R})\) of any compact almost complex manifold \((M, J)\).

1. Introduction

Donaldson [3] posed a question: for an almost complex structure \(J\) on a compact 4-manifold \(M\) which is tamed by a symplectic form \(\omega\), is there a symplectic form compatible with \(J\)? In order to study this question, Li and Zhang [11], Draghici, Li and Zhang [4, 5] investigated the decomposition of the real degree two de Rham cohomology group \(H^2(M, \mathbb{R})\), and introduced \(J\)-invariant and \(J\)-anti-invariant subgroups \(H^+_J(M)\) and \(H^-_J(M)\). \(J\) is said to be \(C^\infty\) pure if \(H^+_J(M) \cap H^-_J(M) = \{0\}\), \(C^\infty\) full if \(H^+_J(M) + H^-_J(M) = H^2(M, \mathbb{R})\). Draghici, Li and Zhang [4] concluded that for a 4 dimensional almost complex manifold \((M, J)\), \(J\) is \(C^\infty\) pure and full, i.e.:

\[ H^2(M, \mathbb{R}) = H^+_J(M) \oplus H^-_J(M). \]

Moreover, they consider the complexified cohomology group \(H^2(M, \mathbb{C}) = H^2(M, \mathbb{R}) \otimes \mathbb{C}\), and get that if \(J\) is integrable,

\[ H^2(M, \mathbb{C}) = H^{1,1}_J \oplus H^{2,0}_J \oplus H^{0,2}_J. \]

For higher dimensional case, please refer to [7, 12] and references therein.

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As we all know almost complex manifolds are always of even real dimension. For odd dimensional case, we can consider contact manifolds. In this paper, we consider the decomposition of degree 2 basic cohomology group $H^2_B(F_\xi)$ of a compact K-contact 5-manifold $(M, \xi, \eta, \Phi, g)$. There are two subgroups $H^+_\Phi$ and $H^-_\Phi$ of $H^2_B(F_\xi)$ which are called $\Phi$-invariant and $\Phi$-anti-invariant basic cohomology group respectively. $\Phi$ is defined to be $C^\infty$ pure if $H^+_\Phi \cap H^-_\Phi = \{0\}$, $C^\infty$ full if $H^+_\Phi + H^-_\Phi = H^2_B(F_\xi)$. We conclude that $\Phi$ is $C^\infty$ pure and full, i.e. Theorem 2.3. Moreover, when $M$ is Sasakian, $\Phi$ is complex $C^\infty$ pure and full, i.e. Theorem 3.5.

2. 5 DIMENSIONAL K-CONTACT MANIFOLDS

Let us first recall some basic facts of K-contact and Sasakian manifolds. For details, please refer to [2, 8]. Suppose $(M, \xi, \eta, \Phi, g)$ is a $2n+1$ dimensional compact K-contact manifold, here $\eta$ is the contact 1-form satisfying $\eta \wedge (d\eta)^n \neq 0$ everywhere on $M$, $\xi$ is the Reeb vector field satisfying $\eta(\xi) = 1$ and $\iota_\xi d\eta = 0$, $\Phi \in \text{End}(TM)$ such that $\Phi \circ \Phi = -id + \xi \otimes \eta$. $(\xi, \eta, \Phi)$ is called an almost contact structure on $M$. $g$ is a Riemannian metric compatible with the almost contact structure $(\xi, \eta, \Phi)$ in the sense that $g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$ and $g(X, \Phi Y) = d\eta(X, Y)$. The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is called K-contact if $\xi$ is a Killing vector field of $g$, i.e., $L_\xi g = 0$, where $L$ stands for the Lie derivative.

The Reeb vector field $\xi$ is often called the characteristic vector field and uniquely determines a 1-dimensional foliation $\mathcal{F}_\xi$ on $M$. The line bundle $L_\xi$ consists of tangent vectors that are tangent to the leaves of $\mathcal{F}_\xi$, and the contact subbundle $D$ is a codimension 1 subbundle of $TM$ whose fibers are the kernel of $\eta$. Then we have:

$$TM = L_\xi \oplus D.$$ 

Consider the cone on $M$ as $C(M) = M \times \mathbb{R}^+$ with warped product metric $g_{C(M)} = dr^2 + r^2 g$. Let $\Upsilon = r \frac{\partial}{\partial r}$ be the Liouville vector field. For the almost contact structure $(\xi, \eta, \Phi)$ on $M$, an almost complex structure $J$ on $C(M)$ can be defined as a section of the endomorphism bundle of the tangent bundle $TC(M)$ satisfying:

$$JY = \Phi Y + \eta(Y)\Upsilon, J\Upsilon = -\xi.$$

$(\xi, \eta, \Phi)$ is said to be normal if the corresponding almost complex structure $J$ on $C(M)$ is integrable, and a normal contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on $M$ is called a Sasakian structure. Moreover, a pair $(M, \mathcal{S})$ is called a Sasakian manifold.
Suppose \((M, \xi, \eta, \Phi, g)\) is a K-contact manifold, a differential \(p\)-form \(\alpha\) on \(M\) is said to be basic if
\[
t_\xi \alpha = 0, L_\xi \alpha = 0.
\]

We denote by \(\Omega^p_B(F_\xi)\) the basic \(p\)-forms. It is easy to check that the exterior derivation \(d\) takes basic forms to basic forms, so the subalgebra \(\Omega_B(F_\xi) = \oplus_p \Omega^p_B(F_\xi)\) forms a subcomplex of the de Rham complex. Its cohomology ring \(H^*_B(F_\xi)\) is defined to be the basic cohomology ring of \(F_\xi\). In the following we set \(d_B = d|_{\Omega^*_B(F_\xi)}\). For any \(\alpha \in \Omega^p_B(F_\xi)\), the transverse Hodge star operator \(\bar{*}\) can be defined as follows:
\[
\bar{*}\alpha = *(\eta \wedge \alpha) = (-1)^p t_\xi * \alpha.
\]

The adjoint operator \(\delta_B : \Omega^p_B(F_\xi) \rightarrow \Omega^{p-1}_B(F_\xi)\) of the basic differential operator \(d_B\):
\[
\delta_B = -\bar{*}d_B\bar{*}.
\]

The basic Laplacian \(\Delta_B\):
\[
\Delta_B = d_B\delta_B + \delta_B d_B.
\]

Analogue to the Hodge decomposition of compact Riemannian manifolds, we have the transverse Hodge decomposition \([6, 10, 13]\):
\[
\Omega^p_B(F_\xi) = \mathcal{H}^p(F_\xi) \oplus \text{Im}(d_B) \oplus \text{Ker}(\delta_B),
\]
where \(\mathcal{H}^p(F_\xi)\) is the space of basic harmonic \(p\)-forms defined as the kernel of
\[
\Delta_B : \Omega^p_B(F_\xi) \rightarrow \Omega^p_B(F_\xi).
\]
Specifically, for five dimensional K-contact manifold \((M, \xi, \eta, \Phi, g)\), one considers the contact subbundle \(D\) with bundle metric \(g_D\) induced by \(g\). For simplicity, we still denote \(g_D\) by \(g\). The operators
\[
\frac{1}{2}(\text{id} + \bar{*}), \frac{1}{2}(\text{id} - \bar{*})
\]
induces a decomposition of the exterior bundle \(\Lambda_D\) of \(D\) by decompose any \(\alpha\) into \(\frac{1}{2}(\alpha \pm \bar{*}\alpha)\):
\[
\Lambda^2_D = \Lambda^+_g \oplus \Lambda^-_g.
\]
Denote by \(\Omega^p_\Phi(F_\xi)\) the relevant space of basic forms. Hence,
\[
\Omega^p_\Phi(F_\xi) = \Omega^+_\Phi(F_\xi) \oplus \Omega^-_\Phi(F_\xi).
\]
We call elements in \(\Omega^+_\Phi(F_\xi)\) and \(\Omega^-_\Phi(F_\xi)\) the basic self-dual and basic anti-self-dual forms. Moreover, \(\Phi\) acts on the bundle of \(\Lambda^2_D\) by \(\alpha(\cdot, \cdot) \rightarrow \alpha(\Phi \cdot, \Phi \cdot)\), so we have the splitting by decomposition \(\alpha(\cdot, \cdot) = \frac{1}{2}[\alpha(\cdot, \cdot) \pm \alpha(\Phi \cdot, \Phi \cdot)]\):
\[
\Lambda^2_D = \Lambda^+_\Phi \oplus \Lambda^-_\Phi.
\]
We denote by $\Omega^\pm_\Phi(\mathcal{F}_\xi)$ the space of $\Phi$-invariant basic 2-forms, $\Omega^-_\Phi(\mathcal{F}_\xi)$ the space of $\Phi$-anti-invariant basic 2-forms. Then the $\Phi$-invariant and $\Phi$-anti-invariant basic cohomology groups can be defined as follows respectively:

$$
H^\pm_\Phi(\mathcal{F}_\xi) = \{[\alpha] \in H^2_\Phi(\mathcal{F}_\xi) \mid \alpha \in \Omega^\pm_\Phi(\mathcal{F}_\xi)\};
$$

$$
H^-_\Phi(\mathcal{F}_\xi) = \{[\alpha] \in H^2_\Phi(\mathcal{F}_\xi) \mid \alpha \in \Omega^-_\Phi(\mathcal{F}_\xi)\}.
$$

For a basic form $\alpha$, we denote $\alpha_h$, $(\alpha)_g^+$ and $(\alpha)_g^-$ the relevant basic harmonic, basic (anti-)self-dual and $\Phi$(-anti)-invariant part of $\alpha$ respectively.

With the notations of basic (anti-)self-dual forms, we have the following refined transverse Hodge decomposition:

**Lemma 2.1.** If $\alpha \in \Omega^+_g$ and $\alpha = \alpha_h + d_B \theta + \delta_B \Psi$, then $(d_B \theta)_g^+ = (\delta_B \Psi)_g^+$ and $(d_B \theta)_g^- = -(\delta_B \Psi)_g^-$. In particular,

$$
\alpha - 2(d_B \theta)_g^+ = \alpha_h,
$$

and $\alpha + 2(d_B \theta)_g^- = \alpha_h + 2d_B \theta$ is closed.

**Proof.** By the basic Hodge decomposition: $\alpha = \alpha_h + d_B \theta + \delta_B \Psi$, there holds

$$
\ast \alpha = \ast \alpha_h + \ast d_B \theta + \ast \delta_B \Psi.
$$

Here $\ast \alpha_h$ is harmonic, since $\Delta_B \ast \alpha_h = \ast \Delta_B \alpha_h = 0$, and $\ast \delta_B \Psi = \ast (\ast d_B \ast) \Psi = d_B \ast \Psi$. Hence, $\ast \delta_B \Psi = d_B \theta$, and furthermore, $\ast d_B \theta = \delta_B \Psi$. Then,

$$
(d_B \theta)_g^+ = \frac{1}{2}(\text{id} + \ast)(d_B \theta) = \frac{1}{2}(\text{id} + \ast)(\ast \delta_B \Psi) = (\delta_B \Psi)_g^+;
$$

$$
(d_B \theta)_g^- = \frac{1}{2}(\text{id} - \ast)(d_B \theta) = \frac{1}{2}(\text{id} - \ast)(\ast \delta_B \Psi) = -(\delta_B \Psi)_g^-.
$$

Therefore,

$$
\alpha = \alpha_h + (d_B \theta)_g^+ + (d_B \theta)_g^- + (\delta_B \Psi)_g^+ + (\delta_B \Psi)_g^- \\
= \alpha_h + 2(d_B \theta)_g^+.
$$

Similarly, $\alpha + 2(d_B \theta)_g^- = \alpha_h + 2d_B \theta$. \hfill $\Box$

According to He [9], choose a coordinate $\{x, y_1, y_2, y_3, y_4\}$ such that, the frame $\{e, e_1, e_2, \Phi e_1, \Phi e_2\}$ is an adapted orthonormal frame. Its dual frame is $\{\eta, \theta_1, \theta_2, \Phi \theta_1, \Phi \theta_2\}$. Then $\omega = \frac{1}{2}d\eta = \theta_1 \wedge \Phi \theta_1 + \theta_2 \wedge \Phi \theta_2$, and $\eta \wedge (\frac{1}{2}d\eta)^2 = 2\eta \wedge \theta_1 \wedge \Phi \theta_1 \wedge \theta_2 \wedge \Phi \theta_2$ is twice volume form.
Since $\Phi \omega = \omega$, $\bar{\ast} \omega = \omega$, and we have the following equalities:

$$\begin{align*}
\Lambda^+_{\Phi} &= \text{span}\{\theta_1 \wedge \Phi \theta_1, \theta_2 \wedge \Phi \theta_2, \theta_1 \wedge \theta_2 + \Phi \theta_1 \wedge \Phi \theta_2, \theta_1 \wedge \Phi \theta_2 - \Phi \theta_1 \wedge \theta_2\}; \\
\Lambda^I_{\Phi} &= \text{span}\{\theta_1 \wedge \theta_2 - \Phi \theta_1 \wedge \Phi \theta_2, \theta_1 \wedge \Phi \theta_2 + \Phi \theta_1 \wedge \theta_2\}; \\
\Lambda^+_{gD} &= \text{span}\{\theta_1 \wedge \Phi \theta_1 - \theta_2 \wedge \Phi \theta_2, \theta_1 \wedge \theta_2 - \Phi \theta_1 \wedge \Phi \theta_2, \theta_1 \wedge \Phi \theta_2 + \Phi \theta_1 \wedge \theta_2\}; \\
\Lambda^-_{gD} &= \text{span}\{\theta_1 \wedge \Phi \theta_1 - \theta_2 \wedge \Phi \theta_2, \theta_1 \wedge \theta_2 + \Phi \theta_1 \wedge \Phi \theta_2, \theta_1 \wedge \Phi \theta_2 - \Phi \theta_1 \wedge \theta_2\},
\end{align*}$$

there hold the following equalities:

$$\begin{align*}
\Lambda^+_{\Phi} &= \mathbb{R} \omega \oplus \Lambda^-_{gD}, \Lambda^+_{gD} = \mathbb{R} \omega \oplus \Lambda^-_{\Phi}; \\
\Lambda^+_{\Phi} \cap \Lambda^+_{gD} &= \mathbb{R} \omega, \Lambda^-_{\Phi} \cap \Lambda^-_{gD} = 0.
\end{align*}$$

We denote by $Z_{\Phi}^-$ the set of closed $\Phi$–anti-invariant 2-forms, $\mathcal{H}^+_g$ the set of basic harmonic self-dual 2-forms, and $\mathcal{H}^{+,-\omega \perp}_g$ the set of basic harmonic self-dual 2-forms which are perpendicular to $\omega$ with respect to the metric induced by $g$. Then we have:

**Lemma 2.2.** $Z_{\Phi}^- \subset \mathcal{H}^+_g$, and $Z_{\Phi}^- \subset H^-_{\Phi}$ is bijective. Furthermore, $H^-_{\Phi} = Z_{\Phi}^- = \mathcal{H}^{+,-\omega \perp}_g$.

**Proof.** If $\alpha \in Z_{\Phi}^-$, then $d\alpha = 0$. Since $\alpha$ is self dual, i.e., $\bar{\ast} \alpha = \alpha$, $\delta_B \alpha = \ast d_B \ast \alpha = \ast d_B \alpha = 0$, i.e., $\alpha \in \mathcal{H}^+_g$.

By $\Lambda^+_{gD} = \mathbb{R} \omega \oplus \Lambda^-_{gD}$ and $Z^-_{\Phi} = H^-_{\Phi}$, we have $H^-_{\Phi} = Z^-_{\Phi} = \mathcal{H}^{+,-\omega \perp}_g$.

Based on the above lemmas, we conclude the following theorem:

**Theorem 2.3.** For a five dimensional closed K-contact manifold $(M, \xi, \eta, \Phi, g)$, $\Phi$ is $C^\infty$ pure and full in the following sense:

$$H^2_B(\mathcal{F}_\xi) = H^+_{\Phi} \oplus H^-_{\Phi}.$$
Proof. If \( a \in H_\Phi^+ \cap H_\Phi^- \), let \( \alpha' \in Z_\Phi^+ \), \( \alpha'' \in Z_\Phi^- \) be representatives for \( a \), \( \alpha' = \alpha'' + dB\gamma \) for some basic 1-form \( \gamma \). Then

\[
0 = \int_M \alpha' \wedge \alpha'' \wedge \eta
= \int_M (\alpha'' + dB\gamma) \wedge \alpha'' \wedge \eta
= \int_M \alpha'' \wedge \alpha'' \wedge \eta + \int_M dB\gamma \wedge \alpha'' \wedge \eta
= \int_M \alpha'' \wedge \alpha'' \wedge \eta + \int_M \gamma \wedge dB\alpha'' \wedge \eta - \int_M \gamma \wedge \alpha'' \wedge dB\eta
= \int_M \alpha'' \wedge \alpha'' \wedge \eta
= \int_M |\alpha''|^2 dvol,
\]

since \( \gamma \wedge \alpha'' \wedge dB\eta \) is a basic 5-form, it is zero, and \( \alpha'' \) is basic self-dual form, satisfies \( \bar{*}\alpha'' = \alpha'' \).

Hence, \( \alpha'' = 0 \), i.e., \( a = 0 \).

Next, we prove \( H^2(F_\xi) = H_\Phi^+ \oplus H_\Phi^- \). Suppose the contrary, then there exists \( b \in H^2(F_\xi) \setminus H_\Phi^+ \oplus H_\Phi^- \). Since \( H_g^- \subset H_g^+ \), assume \( b \in H_g^+ \). Let \( \beta \) be the basic harmonic, self-dual representative of \( b \), and denote \( f = \langle \beta, \omega \rangle \). Then \( f \neq 0 \). Otherwise, \( b \in H_\Phi^- \). Consider the basic self-dual form \( f\omega \). By Lemma 2.1, \( (f\omega)_h + 2(f\omega)^{exact} = f\omega + 2[(f\omega)^{exact}]_g \) is closed and \( \Phi \)-invariant. Thus, \( c = [(f\omega)_h + 2(f\omega)^{exact}] \in H_\Phi^+ \). However,

\[
\int \beta \wedge [(f\omega)_h + 2(f\omega)^{exact}] \wedge \eta
= \int \langle \beta, (f\omega)_h + 2(f\omega)^{exact} \rangle d\mu
= \int \langle \beta, f\omega + 2((f\omega)^{exact})_g \rangle d\mu
= \int f^2 d\mu
\neq 0.
\]

This contradicts the assumption that \( b \perp H_\Phi^+ \oplus H_\Phi^- \).

\( \Box \)
We consider the complex basic 2-forms in this section. There holds the following decomposition:

\[ \Lambda^2_{D,\mathbb{C}} = \Lambda^{2,0}_\Phi \oplus \Lambda^{1,1}_\Phi \oplus \Lambda^{0,2}_\Phi. \]

Let \( \omega^i = \theta^i + \sqrt{-1}\Phi\theta^i \). Then:

\[
\begin{align*}
(\Lambda^{1,1}_\Phi)_\mathbb{R} = & \text{span}\{\sqrt{-1}\omega^1 \wedge \bar{\omega}^1, \sqrt{-1}\omega^2 \wedge \bar{\omega}^2, \omega^1 \wedge \bar{\omega}^2 + \bar{\omega}^1 \wedge \omega^2, \\
& \sqrt{-1}(\omega^1 \wedge \bar{\omega}^2 - \bar{\omega}^1 \wedge \omega^2)\}, \\
(\Lambda^{2,0}_\Phi \oplus \Lambda^{0,2}_\Phi)_\mathbb{R} = & \text{span}\{\omega^1 \wedge \omega^2 + \bar{\omega}^1 \wedge \bar{\omega}^2, \sqrt{-1}(\omega^1 \wedge \omega^2 - \bar{\omega}^1 \wedge \bar{\omega}^2)\}.
\end{align*}
\]

By a direct calculation we have:

\( \Lambda^+_{\Phi} = (\Lambda^{1,1}_\Phi)_\mathbb{R}, \)

\( \Lambda^-_{\Phi} = (\Lambda^{2,0}_\Phi \oplus \Lambda^{0,2}_\Phi)_\mathbb{R}. \)

**Definition 3.1.** Let \( H^{p,q}_{\Phi} \) be the subspace of the complexified basic cohomology \( H^2_B(\mathcal{F}_\xi; \mathbb{C}) = H^2_B(\mathcal{F}_\xi; \mathbb{R}) \otimes \mathbb{C} \), consisting of classes which can be represented by a complex closed form of type \((p, q)\).

**Lemma 3.2.** There hold the following properties of the subgroups \( H^{p,q}_{\Phi} \):

\[
\begin{align*}
(3.3) & \quad H^{p,q}_{\Phi} = \overline{H^{q,p}_{\Phi}}; \\
(3.4) & \quad H^{p,p}_{\Phi} = (H^{p,p}_{\Phi} \cap H^2_B(\mathcal{F}_\xi; \mathbb{R})) \otimes \mathbb{C}; \\
(3.5) & \quad (H^{p,q}_{\Phi} + H^{q,p}_{\Phi}) = (H^{p,q}_{\Phi} + H^{q,p}_{\Phi}) \cap H^{p+q}_B(\mathcal{F}_\xi; \mathbb{R}) \otimes \mathbb{C}.
\end{align*}
\]

**Proof.** Choose a complex form \( \Psi \), then (3.3) follows from the fact that \( \Psi \) is closed if and only if its conjugate \( \overline{\Psi} \) is closed. (3.4) and (3.5) follow from a fact in linear algebra:

Let \( V \) be a real vector space and \( W \) a complex subspace of \( V \otimes_\mathbb{R} \mathbb{C} \), and \( W \) is invariant under conjugation as a subspace. Then \( W = (W \cap V) \otimes \mathbb{C} \). See [1].

**Lemma 3.3.** For a compact 5-dimensional K-contact manifold, there hold the following:

\[
\begin{align*}
(3.6) & \quad H^+_\Phi = H^{1,1}_\Phi \cap H^2(\mathcal{F}_\xi; \mathbb{R}); \\
(3.7) & \quad H^{1,1}_\Phi = H^+_\Phi \otimes_\mathbb{R} \mathbb{C}.
\end{align*}
\]

**Proof.** We first prove (3.6). By (3.2) we have \( H^+_\Phi \subseteq H^{1,1}_\Phi \cap H^2(\mathcal{F}_\xi; \mathbb{R}) \). For the converse inclusion, we choose an element \( \rho \in H^{1,1}_\Phi \cap H^2(\mathcal{F}_\xi; \mathbb{R}) \), such that \( \rho \) is a \( d_B \) closed basic \((1, 1)\) form of the form...
\[ \rho = \sigma + d_B \tau, \text{ where } \sigma \text{ a } d_B \text{ closed basic real form. Hence, } [\rho] \text{ is also represented by the real } d_B \text{ closed basic } (1,1) \text{ form } \frac{1}{2}(\rho + \bar{\rho}) = \sigma + d_B(\frac{\tau + \bar{\tau}}{2}). \]

This shows that \( H_\Phi^+ \supseteq H_\Phi^{1,1} \cap H^2(\mathcal{F}_\xi; \mathbb{R}) \).

The relation (3.7) is a direct consequence of (3.4) with \( p = 1 \) and (3.6). \( \square \)

**Lemma 3.4.** Suppose that \( M \) is a compact 5-dimensional K-contact manifold. If the contact metric structure \( S = (\xi, \eta, \Phi, g) \) is normal, i.e., \( (M,S) \) is Sasakian, there hold the following:

\[
\begin{align*}
(3.8) & \quad (H^2_{\Phi} + H_{\Phi}^{0,2}) = H_\Phi^- \otimes \mathbb{C}; \\
(3.9) & \quad H_\Phi^- = (H^2_{\Phi} + H_{\Phi}^{0,2}) \cap H^2(\mathcal{F}_\xi; \mathbb{R}).
\end{align*}
\]

**Proof.** Choose a complex form \( \Theta = \alpha + i\Phi \alpha \in \Omega^2(\mathbb{C}) \), where \( \alpha \in \Omega^1 \). Since \( d_B = \partial_B + \overline{\partial}_B \) and \( 2\alpha = \Theta + \overline{\Theta} \), we have

\[ 2d_B \alpha = (\partial_B + \overline{\partial}_B)(\Theta + \overline{\Theta}) = \partial_B \overline{\Theta} + \overline{\partial}_B \Theta. \]

Here we have used the fact that \( \partial_B \Theta = 0 = \overline{\partial}_B \overline{\Theta} \), since \( M \) is 5-dimensional and \( \partial_B \Theta \) is a basic \((3,0)\) form, \( \overline{\partial}_B \overline{\Theta} \) is a basic \((0,3)\) form. Therefore,

\[ d_B \alpha = 0 \iff \partial_B \overline{\Theta} = 0 \iff \overline{\partial}_B \Theta = 0. \]

Similarly,

\[ d_B(\Phi \alpha) = 0 \iff \partial_B(\overline{\Theta}) = 0 \iff \overline{\partial}_B(i\Theta) = 0. \]

Moreover, \( \overline{\partial}_B \Theta = 0 \) if and only if \( \overline{\partial}_B(i\Theta) = 0 \). Then it follows that \( d_B \alpha = 0 \) if and only if \( \partial_B(\Phi \alpha) = 0 \). Therefore, \( (H^2_{\Phi} + H_{\Phi}^{0,2}) = H_\Phi^- \otimes \mathbb{C} \).

The relation (3.9) follows from (3.5) with \( (p,q) = (2,0) \) and (3.8). \( \square \)

Next we suppose the contact metric structure \( S = (\xi, \eta, \Phi, g) \). Combining with Lemma 3.3 and Lemma 3.4 there holds the following:

**Theorem 3.5.** For a compact 5-dimensional K-contact manifold \((M,S)\), \( \Phi \) is always complex \( C^\infty \) pure in the sense:

\[ H_\Phi^{1,1} \cap H_{\Phi}^{2,0} \cap H_{\Phi}^{0,2} = \{0\}. \]

Moreover, if \( S \) is normal, then \( \Phi \) is also complex \( C^\infty \) full in the sense:

\[ H^2(\mathcal{F}_\xi; \mathbb{C}) = H_\Phi^{1,1} \oplus H_{\Phi}^{2,0} \oplus H_{\Phi}^{0,2}. \]

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References

[1] W. Barth; K. Hulek; C. Peters; A. Van de Ven. Compact complex surfaces. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 4. Springer-Verlag, Berlin, 2004.

[2] C.P. Boyer; K. Galicki. Sasakian Geometry. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2008.

[3] S. K. Donaldson. Two-forms on four-manifolds and elliptic equations. Inspired by S. S. Chern, 153–172, Nankai Tracts Math., 11, World Sci. Publ., Hackensack, NJ, 2006.

[4] T. Draghici; T.J. Li; W. Zhang. Symplectic forms and cohomology decomposition of almost complex four-manifolds. Int. Math. Res. Not. IMRN (2010), no. 1, 1–17.

[5] T. Draghici; T.J. Li; W. Zhang. On the J-anti-invariant cohomology of almost complex 4-manifolds. Q. J. Math. 64 (2013), no. 1, 83–111.

[6] A. El Kacimi-Alaoui; G. Hector. Décomposition de Hodge basique pour un feuilletage riemannien. Ann. Inst. Fourier (Grenoble) 36 (1986), no. 3, 207–227.

[7] A. Fino; A. Tomassini. On some cohomological properties of almost complex manifolds. J. Geom. Anal. 20 (2010), 107–131.

[8] A. Futaki; H. Ono; G. Wang. Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds. J. Differential Geom. 83 (2009), no. 3, 585–635.

[9] Z. He. Odd dimensional symplectic manifolds. MIT Ph.D thesis, 2010.

[10] F.W. Kamber; P. Tondeur. de Rham-Hodge theory for Riemannian foliations. Math. Ann. 277 (1987), no. 3, 415–431.

[11] T.J. Li; W. Zhang. Comparing tamed and compatible symplectic cones and cohomological properties of almost complex manifolds. Comm. Anal. Geom. 17 (2009), no. 4, 651–683.

[12] Q. Tan; H. Wang; J. Zhou. Primitive cohomology of real degree two on compact symplectic manifolds. Manuscripta Math. (online), DOI:10.1007/s00229-015-0761-7.

[13] P. Tondeur. Geometry of foliations. Monographs in Mathematics, vol. 90, Birkhäuser Verlag, Basel, 1997.