REGULARITY AND KOSZUL PROPERTY OF SYMBOLIC POWERS OF MONOMIAL IDEALS

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Abstract. Let $I$ be a homogeneous ideal in a polynomial ring over a field. Let $I^{(n)}$ be the $n$-th symbolic power of $I$. Motivated by results about ordinary powers of $I$, we study the asymptotic behavior of the regularity function $\text{reg}(I^{(n)})$ and the maximal generating degree function $\omega(I^{(n)})$, when $I$ is a monomial ideal. It is known that both functions are eventually quasi-linear. We show that, in addition, the sequences $\{\text{reg}(I^{(n)}/n\}$ and $\{\omega(I^{(n)}/n\}$ converge to the same limit, which can be described combinatorially. We construct an example of an equidimensional, height two squarefree monomial ideal $I$ for which $\omega(I^{(n)})$ and $\text{reg}(I^{(n)})$ are not eventually linear functions. For the last goal, we introduce a new method for establishing the componentwise linearity of ideals. This method allows us to identify a new class of monomial ideals whose symbolic powers are componentwise linear.

1. Introduction

Let $R = k[x_1, \ldots, x_r]$ be a polynomial ring over a field $k$. In this paper we investigate the maximal generating degree and the regularity of symbolic powers of monomial ideals in $R$. Let $I$ be a homogeneous ideal of $R$. Then the $n$-th symbolic power of $I$ is defined by

$$I^{(n)} = \bigcap_{p \in \text{Min}(I)} I^n R_p \cap R,$$

where $\text{Min}(I)$ is as usual the set of minimal associated prime ideals of $I$.

Symbolic powers were studied by many authors. While sharing some similar features with ordinary powers, the symbolic powers are usually much harder to deal with. One difficulty lies in the fact that the symbolic Rees algebra, defined as

$$R_s(I) = R \oplus I^{(1)} \oplus I^{(2)} \oplus \cdots,$$

is not noetherian in general. Examples of non-noetherian symbolic Rees algebras were discovered by Roberts [32] and simpler examples were provided by Goto-Nishida-Watanabe [11].

Denote by $\text{reg}(I)$ and $\omega(I)$ to be the regularity of $I$ and the maximal degree of the minimal homogeneous generators $I$, respectively. By celebrated results by Cutkosky-Herzog-Trung [7] and Kodiyalam [23], we know that $\text{reg} I^n$ and $\omega(I^n)$ are eventually linear functions with the same leading coefficient. In particular, there exist the limits

$$\lim_{n \to \infty} \frac{\text{reg} I^n}{n} = \lim_{n \to \infty} \frac{\omega(I^n)}{n}.$$
and the common limit is an integer. On the other hand, by [3, Proposition 7], when \( I \) defines \( 2r + 1 \) points on a rational normal curve in \( \mathbb{P}^r \), where \( r \geq 2 \), then for all \( n \geq 1 \),

\[
\text{reg} I^{(n)} = 2n + 1 + \left\lfloor \frac{n - 2}{r} \right\rfloor.
\]

Hence the function \( \text{reg} I^{(n)} \) is not eventually linear in general. Cutkosky [6] could even construct a smooth curve in \( \mathbb{P}^3 \) whose homogeneous defining ideal \( I \) has the property that \( \lim_{n \to \infty} \text{reg} I^{(n)}/n \) is an irrational number. Another peculiar example is given in [7, Example 4.4]: given any prime number \( p \equiv 2 \) modulo 3, there exist some field \( k \) of characteristic \( p \), and some collection of 17 fat points in \( \mathbb{P}^2_k \) whose defining ideal \( I \) has the property that \( \text{reg} I^{(n)} \) is not eventually quasi-linear.

While the question about eventual quasi-linear behavior of \( \text{reg} I^{(n)} \) has a negative answer in general, various basic questions remain tantalizing. For example:

1. There was no known example of a homogeneous ideal \( I \) in a polynomial ring for which the limit \( \lim_{n \to \infty} \frac{\text{reg}(I^{(n)})}{n} \) does not exist (Herzog-Hoa-Trung [18, Question 2]);
2. It remains an open question whether for every such homogeneous ideal \( I \), the function \( \text{reg} I^{(n)} \) is bounded by a linear function;
3. Even an answer for the analogue of the last question for \( \omega(I^{(n)}) \) remains unknown.

In [21, Theorem 4.9], it is shown that \( \lim_{n \to \infty} \frac{\text{reg}(I^{(n)})}{n} \) exists if \( I \) is a squarefree monomial ideal (but a description of the limit was not provided). By [18, Section 2], Question (2) (and hence of course (3)) has a positive answer if either \( I \) is a monomial ideal, or \( \dim(R/I) \leq 2 \), or the singular locus of \( R/I \) has dimension at most 1. The general case remains open for all of these questions.

Symbolic powers of monomial ideals are simpler than that of general ideals because symbolic Rees algebras of monomial ideals are noetherian [24, Proposition 1], [17, Theorem 3.2]. In the present paper, we address the following questions for a monomial ideal \( I \) of \( R \).

**Question 1.1.** Does the limit \( \lim_{n \to \infty} \frac{\text{reg}(I^{(n)})}{n} \) exist? If it does, describe the limit in terms of \( I \). The same questions for \( \lim_{n \to \infty} \frac{\omega(I^{(n)})}{n} \).

**Question 1.2** (Minh-T.N. Trung [26, Question A, part (i)]). Is the function \( \text{reg} I^{(n)} \) eventually linear if \( I \) is squarefree?

A motivation for Question 1.2 is a result of Herzog, Hibi, Trung [17], that \( \text{reg} I^{(n)} \) is eventually quasi-linear. Another motivation is a recent result of Hoa et al. [20] on the existence of \( \lim_{n \to \infty} \text{depth} I^{(n)} \) when \( I \) is a squarefree monomial ideal. It is worth pointing out that Question 1.2 has a negative answer for non-squarefree monomial ideals; see Example 3.10.

Extending previous result of Hoa and T.N. Trung, our first main result answers Question 1.1 in the positive for both limits (they are actually the same). We also describe explicitly the limits in terms of certain polyhedron associated to \( I \). Our second main result answers the other question in the negative. In fact, a counterexample is given using equidimensional height 2 squarefree monomial ideals, in other
words, \textit{cover ideals} of graphs. Interestingly, at the same time, our counterexample also gives a negative answer for the analogue of Question 1.2 for the function \(\omega(I^{(n)})\).

In detail, the main tool for Question 1.1 comes from the theory of convex polyhedra. Assume that \(I\) admits a minimal primary decomposition \(I = Q_1 \cap \cdots \cap Q_s \cap Q_{s+1} \cap \cdots \cap Q_t\) where \(Q_1, \ldots, Q_s\) are all the primary monomial ideals associated to the minimal prime ideals of \(I\). We define certain polyhedron associated to \(I\) as follows:

\[
SP(I) = NP(Q_1) \cap \cdots \cap NP(Q_s) \subset \mathbb{R}^r,
\]

where \(NP(Q_i)\) is the Newton polyhedron of \(Q_i\). Then \(SP(I)\) is a convex polyhedron in \(\mathbb{R}^r\). For a vector \(v = (v_1, \ldots, v_r) \in \mathbb{R}^r\), denote \(|v| = v_1 + \cdots + v_r\). Let

\[
\delta(I) = \max\{|v| \mid v \text{ is a vertex of } SP(I)\}.
\]

Answering Question 1.1, our first two main results are:

\textbf{Theorem 1.3} (Theorems 3.3 and 3.6). For all monomial ideals \(I\), there are equalities

\[
\lim_{n \to \infty} \frac{\omega(I^{(n)})}{n} = \lim_{n \to \infty} \frac{\reg(I^{(n)})}{n} = \delta(I).
\]

While computing the regularity of the symbolic powers of \(I\) it is difficult, the computation of \(\delta(I)\) is fairly simple by linear programming technique.

It is not hard to show that \(\delta(I) \geq \omega(I)\) if \(I\) is a squarefree monomial ideal (Lemma 4.3). Moreover, there are many examples in which \(\delta(I) = \omega(I)\). This is the case when \(I\) is a quadratic squarefree monomial ideal (hence \(\omega(I) = 2\)), thanks to a result by Bahiano [2]; see Example 4.4. Let \(G\) be an arbitrary simple graph with the vertex set \(V(G) = \{1, \ldots, r\}\) and the edge set \(E(G)\). Recall that the \textit{cover ideal} of \(G\) is defined by

\[
J(G) = \bigcap_{\{i,j\} \in E(G)} (x_i, x_j).
\]

We also prove in Theorem 4.9 that \(\delta(J(G)) = \omega(J(G))\), if \(G\) is either bipartite, unmixed, or claw-free. An exact formula for \(\reg J(G)^{(n)}\) remains elusive even for such graphs; see, for example, [36, 37], for related work. Proposition 4.11 provides another large class of graphs for which the equality \(\delta(J(G)) = \omega(J(G))\) holds. Our main tool for proving Theorem 4.9 and Proposition 4.11 is a combinatorial formula for \(\delta(J(G))\) in Theorem 4.6. It looks challenging to interpret \(\delta(J(G))\) in terms of other known graph-theoretical invariants of \(G\).

We next study componentwise linear ideals in the sense of Herzog and Hibi [14] which are also known as \textit{Koszul} ideals [19]. Our main tool is the following new result on Koszul ideals, which is proved by the theory of linearity defect.

\textbf{Proposition 1.4} (See Theorem 5.1). Let \(R\) be a polynomial ring over \(k\) with the graded maximal ideal \(m\). Let \(x\) be a non-zero linear form, \(I'\) and \(T\) non-zero homogeneous ideals of \(R\) such that the following conditions are simultaneously satisfied:

(i) \(I'\) is Koszul;
(ii) \(T \subseteq mI'\);
(iii) \(x\) is a regular element with respect to \(R/T\) and \(\text{gr}_m T\), the associated graded module of \(T\) with respect to the \(m\)-adic filtration.

Denote \(I = xI' + T\). Then \(I\) is Koszul if and only if \(T\) is so.
A common method (among a dozen of others), to establish the Koszul property of an ideal is to show that it has linear quotients. Compared with this method, the criterion of Proposition 1.4 has the advantage that it does not require the knowledge of a system of generators of the ideal. It just asks for the knowledge of a decomposition which is in many cases not hard to obtain, the more so if we work with monomial ideals. Indeed, let $I$ be a monomial ideal of $R$, and $x$ one of its variables. Then we always have a decomposition $I = xI' + T$, where $I', T$ are monomial ideals, and $x$ does not divide any minimal generator of $T$. For such a decomposition, condition (iii) in Proposition 1.4 is automatic. Hence given conditions (i) and (ii), we can prove the Koszulness of $I$ by passing to $T$, which lives in a smaller polynomial ring.

Proposition 1.4 is interesting in its own and has further applications, which we hope to pursue in future work. The main application of this proposition in our paper is to study the Koszulness of symbolic powers of the cover ideal $J(G)$ of a graph $G$. By using Proposition 1.4, we prove:

**Theorem 1.5** (Theorem 5.7). Let $G$ be the graph obtained by adding to each vertex of a graph $H$ at least one pendant. Then all the symbolic powers of $J(G)$ are Koszul.

It is worth mentioning that, via Alexander duality, this can be seen as a generalization of previous work of Villarreal [41] and Francisco-Hà [9] on the Cohen-Macaulay property of graphs.

In order to give a counter-example to Question 1.2, we apply Theorem 1.5 for corona graphs.

**Theorem 1.6** (Theorem 5.15). For $m \geq 3$ and $s \geq 2$, let $G = \text{cor}(K_m, s)$ be the graph obtained from the complete graph on $m$ vertices $K_m$ by adding exactly $s$ pendants to each of its vertex. Let $J = J(G)$. Then for all $n \geq 0$,

1. $\text{reg}(J^{(2n)}) = \omega(J^{(2n)}) = m(s + 1)n$;
2. $\text{reg}(J^{(2n+1)}) = \omega(J^{(2n+1)}) = m(s + 1)n + m + s - 1$.

In particular, for all $n$,

$$\text{reg}(J^n) = \omega(J^n) = (m + s - 1)n + (m - 2)(s - 1)\left\lfloor \frac{n}{2} \right\rfloor,$$

which is not an eventually linear function of $n$.

Figure 1. The graph cor$(K_3, 2)$
Let us summarize the structure of this article. In Section 2, we recall some necessary background. In Section 3, we prove that for any monomial ideal \( I \), the limits \( \lim_{n \to \infty} \frac{\text{reg}(I^{(n)})}{n} \) and \( \lim_{n \to \infty} \frac{\omega(I^{(n)})}{n} \) exist and equal to each other. We identify them in terms of the afore-mentioned polyhedron associated to \( I \). In Section 4, we describe structural properties of the symbolic powers of a cover ideal \( J(G) \), and compute the function \( \omega(J(G)^{(n)}) \) in terms of the graph \( G \) in certain situations. We are able to show that \( \delta(J(G)) = \omega(J(G)) \) for graphs which are either bipartite, unmixed, or claw-free (Theorem 4.9). In Section 5, we first prove the Koszulness criterion of Proposition 1.4. The main result of this section is the Koszul property of the symbolic powers for certain class of cover ideals, stated in Theorem 5.7. Combining this with results in Section 4, Theorem 1.6 is deduced at the end of this section.

2. Preliminaries

For standard terminology and results in commutative algebra, we refer to the book of Eisenbud [8]. Good references for algebraic aspects of monomial ideals and simplicial complexes are the books of Herzog and Hibi [15], Miller and Sturmfels [25], and Villarreal [42].

2.1. Regularity. Let \( R \) be a standard graded algebra over a field \( k \). Let \( M \) be a finitely generated graded nonzero \( R \)-module. Let

\[
F: \cdots \rightarrow F_p \rightarrow F_{p-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow 0
\]

be the minimal graded free resolution of \( M \) over \( R \). For each \( i \geq 0, \ j \in \mathbb{Z} \), denote \( \beta_i^R(M) = \text{rank} F_i = \dim_k \text{Tor}^R_i(k, M) \) and \( \beta_{i,j}^R(M) = \dim_k \text{Tor}^R_i(k, M)_j \). We usually omit the superscript \( R \) and write simply \( \beta_i(M) \) and \( \beta_{i,j}(M) \) whenever this is possible. Let

\[
t_i(M) = \sup \{ j \mid \beta_{i,j}(M) \neq 0 \}
\]

where, by convention, \( t_i(M) = -\infty \) if \( F_i = 0 \). The Castelnuovo–Mumford regularity of \( M \) measures the growth of the generating degrees of the \( F_i \), \( i \geq 0 \). Concretely, it is defined by

\[
\text{reg}_R(M) = \sup \{ t_i(M) - i \mid i \geq 0 \}.
\]

In the remaining of this paper, we denote by \( \omega(M) \) the number \( t_0(M) \). Hence \( \omega(M) \) is the maximal degree of a minimal homogeneous generator of \( M \). The definition of the regularity implies

\[
\omega(M) \leq \text{reg}_R(M).
\]

If \( M \) is generated by elements of the same degree \( d \), and \( \text{reg}_R M = d \), we say that \( M \) has a linear resolution over \( R \). We also say \( M \) has a \( d \)-linear resolution in this case.

If \( R \) is a standard graded polynomial ring over \( k \), it is customary to denote \( \text{reg}_R M \) simply by \( \text{reg} M \).

2.2. Linearity defect, Koszul modules, Betti splittings. We use the notion of linearity defect, formally introduced by Herzog and Iyengar [19]. Let \( R \) be a standard graded \( k \)-algebra, and \( M \) a finitely generated graded \( R \)-module. The linearity defect of \( M \) over \( R \), denoted by \( \text{ld}_R M \), is defined via certain filtration of the minimal graded free resolution of \( M \). For details of this construction, we refer to [19, Section 1]. We say \( M \) is called a Koszul module if \( \text{ld}_R M = 0 \). Koszul
modules are those modules with a linear resolution in the terminology of Šega [35].
We say that $R$ is a Koszul algebra if $\operatorname{reg}_R k = 0$. As a matter of fact, $R$ is a Koszul algebra if and only if $k$ is a Koszul $R$-module [19, Remark 1.10].

For each $d \in \mathbb{Z}$, denote by $M\langle d \rangle$ the submodule of $M$ generated by homogeneous elements of degree $d$. Following Herzog and Hibi [14], $M$ is called componentwise linear if for all $d \in \mathbb{Z}$, $M\langle d \rangle$ has a $d$-linear resolution. By results of Römer [33, Theorem 3.2.8] and Yanagawa [43, Proposition 4.9], if $R$ is a Koszul algebra, then $M$ is Koszul if and only if $M$ is componentwise linear.

Because of the last result and for unity of treatment, we use the terms Koszul modules throughout, instead of componentwise linear modules.

The following result is folklore; see for example [1, Proposition 3.4].

**Lemma 2.1.** Let $R$ be a standard graded $k$-algebra, and $M$ be a Koszul $R$-module. Then $\operatorname{reg}_R M = \omega(M)$.

We also recall the following base change result for the linearity defect.

**Lemma 2.2** (Nguyen and Vu [29, Corollary 3.2]). Let $R \to S$ be a flat extension of standard graded $k$-algebras. Let $I$ be a homogeneous ideal of $R$. Then $\operatorname{ld}_R I = \operatorname{ld}_S IS$.

Let $(R, m)$ be a noetherian local ring (or a standard graded $k$-algebra) and $P, I, J \neq (0)$ be proper (homogeneous) ideals of $R$ such that $P = I + J$.

**Definition 2.3.** The decomposition of $P$ as $I + J$ is called a Betti splitting if for all $i \geq 0$, the following equality of Betti numbers holds:

\[
\beta_i(P) = \beta_i(I) + \beta_i(J) + \beta_{i-1}(I \cap J).
\]

We have the following reformulations of Betti splittings.

**Lemma 2.4** ([29, Lemma 3.5]). The following are equivalent:

1. The decomposition $P = I + J$ is a Betti splitting;
2. The natural morphisms $\operatorname{Tor}^R(k, I \cap J) \to \operatorname{Tor}^R(k, I)$ and $\operatorname{Tor}^R(k, I \cap J) \to \operatorname{Tor}^R(k, J)$ are both zero;
3. The mapping cone construction for the map $I \cap J \to I \oplus J$ yields a minimal free resolution of $P$.

### 2.3. Symbolic powers of monomial ideals.

Let $R = k[x_1, \ldots, x_r]$ be a standard graded polynomial ring, and $I$ a monomial ideal of $R$. Let $\mathcal{G}(I)$ denotes the set of minimal monomial generators of $I$. In the present paper, when talking about minimal generators of a monomial ideal we mean minimal monomial generators of it. Let

\[
I = Q_1 \cap \cdots \cap Q_s \cap Q_{s+1} \cap \cdots \cap Q_t
\]

be a minimal primary decomposition of $I$, where $Q_i$ is a primary monomial ideal for $i = 1, \ldots, t$, and $P_j = \sqrt{Q_j}$ is a minimal prime of $I$ if and only if $1 \leq j \leq s$. (Hence $Q_{s+1}, \ldots, Q_t$ are embedded primary components.) For each $i = 1, \ldots, s$, the monomial ideal $Q_j$ is obtained from minimal generators of $I$ by setting $x_i = 1$ for all $i$ for which $x_i \notin P_j$, thus

\[
\omega(Q_j) \leq \omega(I), \quad \text{for } j = 1, \ldots, s.
\]

In the case of monomial ideals, we have a simple formula for the symbolic powers in terms of the minimal primary components. It is immediate from the definition
of symbolic power, and the fact that monomial primary ideals must be generated by powers of certain variables and monomials involving only of those variables.

**Fact 2.5.** With notation as above, for all \( n \geq 1 \), there is an equality

\[
I^{(n)} = Q_1^n \cap Q_2^n \cap \cdots \cap Q_s^n.
\]

A function \( f : \mathbb{N} \to \mathbb{N} \cup \{-\infty\} \) is called *quasi-linear* if there exist a positive integer \( N \) and rational numbers \( a_i \in \mathbb{Q} \) and \( b_i \in \mathbb{Q} \cup \{-\infty\} \), for \( i = 0, \ldots, N-1 \), such that

\[
f(n) = a_i n + b_i,
\]

for all \( n \in \mathbb{N} \) with \( n \equiv i \pmod{N} \).

In this case, the smallest such number \( N \) is called the period of \( f \).

Assume that \( f \) is not identically \(-\infty\). Then \( \lim_{n \to \infty} \frac{f(n)}{n} \) exists if and only if \( a_0 = \cdots = a_{N-1} \). In this case, we say that \( f \) has a constant leading coefficient.

**Lemma 2.6.** With notation as above, for every \( i \geq 0 \), \( t_i(I^{(n)}) \) is quasi-linear in \( n \) for \( n \gg 0 \). In particular, \( \omega(I^{(n)}) \) and \( \text{reg}(I^{(n)}) \) are quasi-linear in \( n \) for \( n \gg 0 \).

**Proof.** By [17, Theorem 3.2], the symbolic Rees ring \( R_s(I) = \bigoplus_{n=0}^{\infty} I^{(n)} \) is finitely generated. By the very same way as the proof of [7, Theorem 4.3], we obtain \( t_i(I^{(n)}) \) is quasi-linear in \( n \) for \( n \gg 0 \). \( \square \)

If \( I \) is a monomial ideal of \( R \), the minimal graded free resolution of \( I \) is \( \mathbb{Z}^r \)-graded. For each \( \alpha \in \mathbb{Z}^r \), we denote by \( \beta_{i,\alpha}(I) \) the number \( \dim_k \text{Tor}^i_R(k, I)_\alpha \). Clearly \( \beta_{i,\alpha}(I) = 0 \) if \( \alpha \notin \mathbb{N}^r \).

When we talk about a monomial \( x^\alpha \) of \( R \), we always mean \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r \) and \( x^\alpha = x_1^{\alpha_1} \cdots x_r^{\alpha_r} \). A vector \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r \) is called squarefree if for all \( i = 1, \ldots, r \), \( \alpha_i \) is either 0 or 1. Let \( e_1, \ldots, e_r \) be the canonical basis of the free \( \mathbb{Z} \)-module \( \mathbb{Z}^r \). For any \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r \) the upper Koszul simplicial complex associated with \( I \) at degree \( \alpha \) is defined by

\[
K^\alpha(I) = \{ \text{squarefree vector } \tau \mid x^{\alpha - \tau} \in I \},
\]

where we use the convention \( \alpha - \tau = \alpha - \sum_{i \in \tau} e_i \). The multigraded Betti numbers of \( I \) can be computed as follows.

**Lemma 2.7.** ([25, Theorem 1.34]) For all \( i \geq 0 \) and all \( \alpha \in \mathbb{N}^r \), there is an equality

\[
\beta_{i,\alpha}(I) = \dim_k H_{i-1}(K^\alpha(I); k).
\]

Let \( \overline{I} \) be the integral closure of the monomial ideal \( I \). To describe \( \overline{I} \) geometrically, for a subset \( A \) of \( R \), denote

\[
E(A) = \{ \alpha \mid \alpha \in \mathbb{N}^r \text{ and } x^\alpha \in A \}.
\]

The Newton polyhedron of \( I \) is the convex polyhedron in \( \mathbb{R}^r \) defined by \( NP(I) = \text{conv}(E(I)) \). Then \( \overline{I} \) is a monomial ideal determined by (see [8, Exercises 4.22 and 4.23]):

\[
(2.2) \quad E(\overline{I}) = NP(I) \cap \mathbb{N}^r.
\]

For each \( n \geq 1 \), let

\[
\mathcal{S}P_n(I) = \bigcap_{i=1}^s NP(Q_i^n),
\]

and

\[
J_n(I) = Q_1^n \cap \cdots \cap Q_s^n.
\]
Then from Equation (2.2) we have $E(J_n(I)) = SP(I) \cap \mathbb{N}^r$.

We will denote $SP(I)$ simply by $SP(I)$. This is the same as the symbolic polyhedron introduced in [4, Definition 5.3], if $I$ has no embedded primes. But in general, the two notions are different, since in contrast to our definition, the definition of symbolic power in [4] involves all the associated primes.

For subsets $X$ and $Y$ of $\mathbb{R}^r$ and a positive integer $n$, we denote

$$nX = \{ny \mid y \in X\},$$

$$X + Y = \{x + y : x \in X, y \in Y\}.$$  

Denote by $\mathbb{R}_+$ the set of non-negative real numbers. The following lemma gives us the structure of the convex polyhedron $SP_n(I)$.

**Lemma 2.8.** Let $\{v_1, \ldots, v_d\}$ be the set of vertices of $SP(I)$. Then

$$SP_n(I) = nSP(I) = n\text{conv}\{v_1, \ldots, v_d\} + \mathbb{R}^r_+.$$  

**Proof.** For each $i = 1, \ldots, s$, we have $NP(Q^n_i) = nNP(Q_i)$ by [31, Lemma 2.5]. It follows that $SP_n(I) = nSP(I)$.

For $v \in SP(I)$ and $u \in \mathbb{R}^r_+$, one has $v + u \in SP(I)$ again by [31, Lemma 2.5]. Combining this with [34, Formula (28), Page 106] we have

$$SP(I) = \text{conv}\{v_1, \ldots, v_d\} + \mathbb{R}^r_+.$$  

Thus, $SP_n(I) = nSP(I) = n\text{conv}\{v_1, \ldots, v_d\} + \mathbb{R}^r_+$, as required. \hfill \Box

The following result was proved in [39, Lemma 6].

**Lemma 2.9.** Let $Q$ be a monomial ideal of $R$. Then the Newton polyhedron $NP(Q)$ is the set of solutions of a system of inequalities of the form

$$\{x \in \mathbb{R}^r \mid \langle a_j, x \rangle \geq b_j, j = 1, \ldots, q\},$$

such that the following conditions are simultaneously satisfied:

(i) Each hyperplane with the equation $\langle a_j, x \rangle = b_j$ defines a facet of $NP(Q)$, which contains $s_j$ affinely independent points of $E(\mathcal{G}(Q))$ and is parallel to $r - s_j$ vectors of the canonical basis. In this case $s_j$ is the number of non-zero coordinates of $a_j$.

(ii) $0 \neq a_j \in \mathbb{N}^r, b_j \in \mathbb{N}$ for all $j = 1, \ldots, q$.

(iii) If we write $a_j = (a_{j,1}, \ldots, a_{j,r})$, then $a_{j,i} \leq s_j \omega(Q)^{s_j-1}$ for all $i = 1, \ldots, r$.

Using this, we can give information about facets of $SP_n(I)$, which will be useful to bound from below the maximal generating degree of $I^{(n)}$ by some linear function of $n$.

**Lemma 2.10.** The polyhedron $SP(I)$ is the solutions in $\mathbb{R}^r$ of a system of linear inequalities of the form

$$\{x \in \mathbb{R}^r \mid \langle a_j, x \rangle \geq b_j, j = 1, 2, \ldots, q\},$$

where for each $j$, the following conditions are fulfilled:

(i) $0 \neq a_j \in \mathbb{N}^r, b_j \in \mathbb{N}$;

(ii) $|a_j| \leq r^2 \omega(I)^{r-1};$

(iii) The equation $\langle a_j, x \rangle = b_j$ defines a facet of $SP(I)$.  

Proof. Note that \( S\mathcal{P}(I) \) is the solution in \( \mathbb{R}^r \) of the system of all linear inequalities that arise from those inequalities defining \( N\mathcal{P}(Q) \) where \( j = 1, \ldots, s \). Now combining Lemma 2.9 with the fact that \( \omega(Q_j) \leq \omega(I) \) (Inequality (2.1)), the lemma follows.

Let \( \Delta \) be a simplicial complex on \( \{1, \ldots, r\} \). For a subset \( F = \{i_1, \ldots, i_j\} \) of \( \{1, \ldots, r\} \), set \( x^F = x_{i_1} \cdots x_{i_j} \) and \( P_F = (x_i : i \notin F) \). Then the Stanley-Reisner ideal of \( \Delta \) is the squarefree monomial ideal

\[
I_\Delta = (x^G \mid G \notin \Delta) \subseteq R.
\]

Let \( F(\Delta) \) denote the set of all facets of \( \Delta \). If \( F(\Delta) = \{F_1, \ldots, F_m\} \), we write \( \Delta = \langle F_1, \ldots, F_m \rangle \). Then \( I_\Delta \) admits the primary decomposition

\[
I_\Delta = \bigcap_{F \in F(\Delta)} P_F.
\]

Thanks to Fact 2.5, for every integer \( n \geq 1 \), the \( n \)-th symbolic power of \( I_\Delta \) is given by

\[
I_\Delta^{(n)} = \bigcap_{F \in F(\Delta)} P_F^n.
\]

2.4. Graph theory. Let \( G \) be a finite simple graph. We use the symbols \( V(G) \) and \( E(G) \) to denote the vertex set and the edge set of \( G \), respectively. When there is no confusion, the edge \( \{u, v\} \) of \( G \) is written simply as \( uv \). Two vertices \( u \) and \( v \) are adjacent if \( \{u, v\} \in E(G) \).

For a subset \( S \) of \( V(G) \), we define

\[
N_G(S) = \{v \in V(G) \setminus S \mid uv \in E(G) \text{ for some } u \in S\}
\]

and \( N_G[S] = S \cup N_G(S) \). When there is no confusion, we shall omit \( G \) and write \( N(S) \) and \( N[S] \). If \( S \) consists of a single vertex \( u \), denote \( N_G(u) = N_G(S) \) and \( N_G[u] = N_G[S] \). Define \( G[S] \) to be the induced subgraph of \( G \) on \( S \), and \( G \setminus S \) to be the subgraph of \( G \) with the vertices in \( S \) and their incident edges deleted.

The degree of a vertex \( u \in V(G) \), denoted by \( \text{deg}_G(u) \), is the number of edges incident to \( u \). If \( \text{deg}_G(u) = 0 \), then \( u \) is called an isolated vertex; if \( \text{deg}_G(u) = 1 \), then \( u \) is a leaf. An edge emanating from a leaf is called a pendant.

A vertex cover of \( G \) is a subset of \( V(G) \) which meets every edge of \( G \); a vertex cover is minimal if none of its proper subsets is itself a cover. The cover ideal of \( G \) is defined by \( J(G) := (x^\tau \mid \tau \) is a minimal vertex cover of \( G \)\). Note that \( J(G) \) has the primary decomposition

\[
J(G) = \bigcap_{\{i,j\} \in E(G)} (x_i, x_j).
\]

An independent set in \( G \) is a set of vertices no two of which are adjacent to each other. An independent set in \( G \) is maximal (with respect to set inclusion) if the set cannot be extended to a larger independent set. The set of all independent sets of \( G \), denoted by \( \Delta(G) \), is a simplicial complex, called the independence complex of \( G \).
3. Asymptotic maximal generating degree and regularity

Let $I$ be a monomial ideal of $R = k[x_1, \ldots, x_r]$ and let

$$I = Q_1 \cap \cdots \cap Q_s \cap Q_{s+1} \cap \cdots \cap Q_l$$

be a minimal primary decomposition of $I$, where $Q_1, \ldots, Q_s$ are the components associated to the minimal primes of $I$. By Fact 2.5 we have

$$I^{(n)} = Q_1^n \cap Q_2^n \cap \cdots \cap Q_s^n.$$

Recall that

$$SP_n(I) = NP(Q_1^n) \cap NP(Q_2^n) \cap \cdots \cap NP(Q_s^n) = nSP(I),$$

and

$$J_n(I) = \overline{Q_1} \cap \overline{Q_2} \cap \cdots \cap \overline{Q_s}.$$ 

For simplicity, we denote $J_n = J_n(I)$ in the sequel. Observe that $x^\alpha \in J_n$ if and only if $\alpha \in SP_n(I) \cap \mathbb{N}^r$. We note two simple facts.

**Remark 3.1.** Let $J$ be a monomial ideal and $x^\alpha \in J$, with $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$. Then $x^\alpha \in G(J)$ if and only if for every $i$ with $\alpha_i \geq 1$, we have $x^{\alpha_i e_i} \notin J$.

**Lemma 3.2.** Let $J$ be a monomial ideal and $x^\alpha \in J$. For $i = 1, \ldots, r$, let $m_i \in \mathbb{N}$ be an integer such that $x^{\alpha - m_i e_i} \notin J$ if $\alpha_i \geq m_i$. Then there are integers $0 \leq n_i \leq m_i - 1$ such that $x^{\alpha - (n_1 e_1 + \cdots + n_r e_r)} \in G(J)$.

**Proof.** Just choose $0 \leq n_i < m_i$ for $i = 1, \ldots, r$ such that

$$x^{\alpha - (n_1 e_1 + \cdots + n_r e_r)} \in J$$

and $n_1 + \cdots + n_r$ is as large as possible. \hfill $\square$

The first main result of this paper is

**Theorem 3.3.** There is an equality $\lim_{n \to \infty} \omega(I^{(n)}) n = \delta(I)$.

In fact, setting $\rho = r^2 \omega(\omega)r^{-1}$, we will prove that for all $n \geq 1$, the following bounds for $\omega(I^{(n)})$ hold:

$$\delta(I)n - \rho(1 + s(r - 1)\omega(I)) \leq \omega(I^{(n)}) \leq \delta(I)n + r(r - 1)\omega(I).$$

This clearly implies the conclusion of Theorem 3.3.

For the upper bound, we need the following auxiliary statements.

**Lemma 3.4.** Let $x^\alpha \in I^{(n)}$ be a monomial. Assume that for some $1 \leq i \leq r$, we have $x^{\alpha - e_i} \notin I^{(n)}$. Denote $m = (r - 1)\omega(I) + 1$. If $\alpha_i \geq m$, then $x^{\alpha - m e_i} \notin J_n$.

**Proof.** Since $x^{\alpha - e_i} \notin I^{(n)}$, $x^{\alpha - e_i} \notin Q_j^n$ for some $1 \leq j \leq s$. By [40, Theorem 7.58], we have $Q_j = Q_j^{n-p}Q_j^n$ for some $0 \leq p \leq r - 1$.

Since $x^\alpha \in Q_j^n$ and $x^{\alpha - e_i} \notin Q_j^n$, it follows that $x_i$ divides some generator of $Q_j^n$. As the monomial ideal $Q_j$ is primary, $x_i^{\omega(Q_j)} \in Q_j$. In particular, $x_i^{\omega(I)} \in Q_j$ because $\omega(Q_j) \leq \omega(I)$.

We now assume on the contrary that $x^{\alpha - m e_i} \in J_n$. Then $x^{\alpha - m e_i} \notin Q_j^n$. Since $Q_j = Q_j^{n-p}Q_j^n$, there are two monomials $m_1 \in Q_j^{n-p}$ and $m_2 \in Q_j^n$ such that $x^{\alpha - m e_i} = m_1 m_2$. It follows that $x^{\alpha - e_i} = (m_1 x_i^{m-1}) m_2$. Observe that $x_i^{n-1} \in Q_j^{r-1}$ as $m - 1 = (r - 1)\omega(I)$. Thus $x^{\alpha - e_i} = (m_1 x_i^{m-1}) m_2 \in Q_j^{n-p}Q_j^{r-1} \subseteq Q_j^n$, a contradiction. The lemma follows. \hfill $\square$
Lemma 3.5. There is an inequality $\omega(J_n) < \delta(I)n + r$.

Proof. Let $x^\alpha \in \mathcal{G}(J_n)$, $v_1, \ldots, v_d$ be all the vertices of $\mathcal{SP}(I)$. By Lemma 3.2, we can represent $\alpha$ as

$$\alpha = n(\lambda_1 v_1 + \cdots + \lambda_d v_d) + u$$

where $\lambda_i \geq 0$, $\lambda_1 + \cdots + \lambda_d = 1$, and $u = (u_1, \ldots, u_r) \in \mathbb{R}_+^r$.

Since $x^\alpha$ is a minimal generator of $J_n$, necessarily $u_i < 1$ for every $i$. Therefore,

$$|\alpha| \leq \delta(I)n + (u_1 + \cdots + u_r) < \delta(I)n + r.$$ 

It follows that $\omega(J_n) < \delta(I)n + r$, as required. \qed

Now we are ready for the

Proof of the inequality on the right of (3.1). Let $x^\alpha$ be a minimal generator of $I^{(n)}$. By Remark 3.1 we have $x^{\alpha-e_i} \notin I^{(n)}$ for each $i = 1, \ldots, r$, whenever $\alpha_i \geq 1$. For $1 \leq i \leq r$, set $m_i = (r - 1)\omega(I) + 1$. By Lemma 3.4, $x^{\alpha-m_i e_i} \notin J_n$ if $\alpha_i \geq (r - 1)\omega(I) + 1$.

By Lemma 3.2, there are integers $0 \leq n_i \leq (r - 1)\omega(I)$ such that the monomial $x^{\alpha-(n_1 e_1 + \cdots + n_r e_r)}$ is a minimal generator of $J_n$. Thus

$$\omega(J_n) \geq |\alpha| - (n_1 + \cdots + n_r) \geq |\alpha| - r(r - 1)\omega(I),$$

and hence $|\alpha| \leq \omega(J_n) + r(r - 1)\omega(I)$. It follows that $\omega(I^{(n)}) \leq \omega(J_n) + r(r - 1)\omega(I)$. Together with Lemma 3.5, we obtain

$$\omega(I^{(n)}) \leq \delta(I)n + r + r(r - 1)\omega(I).$$

This is the desired inequality. \qed

For the remaining inequality in (3.1), we will make some use of Lemma 2.10.

Proof of the inequality on the left of (3.1). Let $v = (v_1, \ldots, v_r)$ be a vertex of the polyhedron $\mathcal{SP}(I)$ such that $\delta(I) = |v|$. Let $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ where $\alpha_i = \lceil nv_i \rceil$. Because $nv$ is a vertex of $\mathcal{SP}(nI)$, $x^\alpha \in J_n$.

For each $i = 1, \ldots, s$, we have $x^\alpha \in Q_i^n = Q_i^{n-(r-1)Q_i^{r-1}}$ by [40, Theorem 7.58], so we can write

$$x^\alpha = m_{i,1}m_{i,2}m_{i,3}$$

where $m_{i,1} \in Q_i$, $m_{i,2} \in Q_i^{n-r}$ and $m_{i,3} \in Q_i^{r-1}$. Let $f_i = m_i^{r-1}$ so that $\deg(f_i) \leq (r - 1)\omega(Q_i) \leq (r - 1)\omega(I)$. We have $x^\alpha f_i = (m_{i,1}m_{i,2})m_{i,3} \in Q_i^n$.

Let $x^\beta = f_1 \cdots f_s$ and $x^\gamma = x^\alpha x^\beta$. Then $x^\gamma \in Q_i^n$ for all $i$, consequently $x^\alpha x^\beta \in I^{(n)}$. Moreover, $\gamma_i = 0$ if and only if $\alpha_i = 0$, if and only if $v_i = 0$. Note that $|\beta| = \deg(f_1) + \cdots + \deg(f_s) \leq s(r - 1)\omega(I)$.

By Lemma 2.10, the convex polyhedron $\mathcal{SP}(I)$ is the solutions in $\mathbb{R}^r$ of a system of linear inequalities of the form

$$\{x \in \mathbb{R}^r \mid \langle a_j, x \rangle \geq b_j, \ j = 1, 2, \ldots, q\},$$

such that:

1. each equation $\langle a_j, x \rangle = b_j$ defines a facets of $\mathcal{SP}(I)$,
2. $a_j \in \mathbb{N}^r$, $b_j \in \mathbb{N}$, and,
3. $|a_j| \leq r^2 \omega(I)^{r-1}$ for any $j$. 

Therefore, the system of linear inequalities (3.1) is satisfied for every $x \in \mathcal{SP}(I)$, and the proof is complete.
Let \( \rho = r^2 \omega(I)^{-1} \) so that \( |a_j| \leq \rho \) for every \( j = 1, \ldots, q \).

Since \( v \) is a vertex of \( SP(I) \), by [34, Formula 23 in Page 104], we may assume that \( v \) is the unique solution of the following system

\[
\{ x \in \mathbb{R}^r \mid \langle a_i, x \rangle = b_i, i = 1, \ldots, r \}.
\]

For an index \( i \) with \( \gamma_i \geq 1 \), since the last system has a unique solution, we deduce that \( a_{j,i} \neq 0 \) for some \( 1 \leq j \leq r \). For simplicity, we denote \( a = a_j = (a_1, \ldots, a_r) \) so that \( a_i \geq 1 \).

Let \( m = \rho(1 + s(r - 1)\omega(I)) + 1 \). If \( \gamma_i \geq m \), we have

\[
\langle \alpha, \gamma - me_i \rangle = \langle \alpha, \alpha \rangle + \langle a, \beta \rangle - a_i m \leq \langle a, n v + e_i + \cdots + e_r \rangle + \langle a, \beta \rangle - a_i m = \langle a, n v \rangle + |a| + \langle a, \beta \rangle - a_i m
\]

\[
\leq nb_j + |a| + |a||\beta| - m < nb_j
\]

since \( m = \rho(1 + s(r - 1)\omega(I)) + 1 > |a| + |a||\beta| \). Consequently, \( x^{\gamma - me_i} \notin J_n \), and hence \( x^{\gamma - me_i} \notin I(n) \).

By Lemma 3.2, there are non-negative integers \( n_i \leq \rho(1 + s(r - 1)\omega(I)) \) for \( i = 1, \ldots, r \) such that \( x^{\gamma - (n_1 e_1 + \cdots + n_r e_r)} \) is a minimal generator of \( I(n) \). Therefore

\[
\omega(I(n)) \geq |\gamma| - (n_1 + \cdots + n_r) \geq |\alpha| + |\beta| - r \rho(1 + s(r - 1)\omega(I))
\]

\[
\geq |\alpha| - r \rho(1 + s(r - 1)\omega(I)) \geq |nv| - r \rho(1 + s(r - 1)\omega(I))
\]

\[
= \delta(I)n - r \rho(1 + s(r - 1)\omega(I)).
\]

This finishes the proof of (3.1) and hence that of Theorem 3.3. \( \square \)

The second main result of this paper is

**Theorem 3.6.** There is an equality \( \lim_{n \to \infty} \frac{\text{reg}(I(n))}{n} = \delta(I) \).

Recall that for any finitely generated graded \( R \)-module \( M \), and for any \( i \geq 0 \), we have the notation

\[
t_i(M) = \sup \{ j : \text{Tor}_j^R(k, M) \neq 0 \}.
\]

From Theorem 3.3 and the fact that \( \omega(M) \leq \text{reg} M \), we see that Theorem 3.6 will follow from a suitable linear upper bound for \( \text{reg} I(n) \). This is accomplished by

**Lemma 3.7.** For all \( i \geq 0 \), there is an inequality

\[
t_i(I(n)) \leq \delta(I)n + 2r + r(r^2 \omega(I)^{-1}) + (r - 1)\omega(I).
\]

**Proof.** By Lemma 2.10, the convex polyhedron \( SP(I) \) is the solutions in \( \mathbb{R}^r \) of a system of linear inequalities of the form

\[
\{ x \in \mathbb{R}^r \mid \langle a_j, x \rangle \geq b_j, j = 1, 2, \ldots, q \},
\]

where for each \( j \), the equation \( \langle a_j, x \rangle = b_j \) defines a facets of \( SP(I) \), \( a_j \in \mathbb{N}^r \), \( b_j \in \mathbb{N} \), and \( |a_j| \leq r^2 \omega(I)^{-1} \).

Let \( \rho = r^2 \omega(I)^{-1} \) so that \( |a_j| \leq \rho \) for every \( j \).

Take \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r \) such that \( \beta_i, \alpha(I^\Delta(n)) \neq 0 \). Since \( \beta_i, \alpha(I^\Delta(n)) = \dim_k \tilde{H}_{i-1}(K^\alpha(I^\Delta(n)); k) \neq 0 \) by Lemma 2.7, we have \( K^\alpha(I^\Delta(n)) \) is not a cone. Hence, for each \( j = 1, \ldots, r \), we have \( j \notin \tau \) for some \( \tau \in F(K^\alpha(I^\Delta(n))) \).

Since \( \tau \cup \{ j \} \notin K^\alpha(I^\Delta(n)) \), we have \( x^{\alpha - \tau - e_j} \notin I(n) \). Let \( m = (r - 1)\omega(I) + 1 \).

Claim: If \( \alpha_j \geq \rho + m \), then \( x^{\alpha - (\rho + m)e_j} \notin J_n \).
Indeed, by Lemma 3.4, $x^{\alpha - \tau - me_j} \notin J_n$. Therefore, $\langle a_i, \alpha - \tau - me_j \rangle < nb_i$ for some $1 \leq i \leq q$. Since $x^{\alpha - \tau} \in I^{(n)} \subseteq J_n$, we have $\langle a_i, \alpha - \tau \rangle \geq nb_i$. It follows that $a_{i,j} \geq 1$. Thus

$$\langle a_i, \alpha - (\rho + m)e_j \rangle = \langle a_i, \alpha - \tau - me_j \rangle + \langle a_i, \tau - \rho e_j \rangle < nb_j + \langle a_i, \tau - \rho e_j \rangle$$

$$= nb_j + \langle a_i, \tau \rangle - \langle a_i, \rho e_j \rangle = nb_j + \langle a_i, \tau \rangle - a_{i,j}\rho$$

$$\leq nb_j + \langle a_i, \tau \rangle - \rho \leq nb_j.$$

The last inequality holds since $\rho \geq |a_i| \geq \langle a_i, \tau \rangle$. Consequently, $x^{\alpha - (\rho + m)e_j} \notin J_n$, as desired.

By Lemma 3.4, there are integers $0 \leq n_i \leq \rho + m - 1$ for $i = 1, \ldots, r$, for which

$$x^{\alpha - n_i e_i - \cdots - n_r e_r} \in \mathcal{G}(J_n).$$

It follows that

$$\omega(J_n) \geq |\alpha| - |\tau| - (n_1 + \cdots + n_r) \geq |\alpha| - r(\rho + m - 1),$$

and hence

$$|\alpha| \leq \omega(J_n) + r + r(\rho + m - 1) = \omega(J_n) + r + r(r^2 \omega(I)^{r-1} + (r - 1)\omega(I)).$$

Together with Lemma 3.5, this yields

$$t_i(I^{(n)}) \leq \omega(J_n) + r + r(r^2 \omega(I)^{r-1} + (r - 1)\omega(I))$$

$$\leq \delta(I)n + 2r + r(r^2 \omega(I)^{r-1} + (r - 1)\omega(I)),$$

and the proof is complete. \hfill \Box

**Proof of Theorem 3.6.** By Lemma 3.7 we have

$$\text{reg } I^{(n)} = \max\{t_i(I^{(n)}) - i \mid i \geq 0\} \leq \delta(I)n + 2r + r(r^2 \omega(I)^{r-1} + (r - 1)\omega(I)).$$

On the other hand, by the proof of Theorem 3.3 (more precisely (3.1)), there exists $c \in \mathbb{R}$ such that

$$\omega(I^{(n)}) \geq \delta(I)n + c \text{ for all } n \geq 1.$$

In particular, $\text{reg } I^{(n)} \geq \omega(I^{(n)}) \geq \delta(I)n + c \text{ for all } n \geq 1$. Thus,

$$\delta(I)n + c \leq \text{reg } I^{(n)} \leq \delta(I)n + 2r + r(r^2 \omega(I)^{r-1} + (r - 1)\omega(I))$$

for all $n \geq 1$. It follows that

$$\lim_{n \to \infty} \frac{\text{reg}(I^{(n)})}{n} = \delta(I),$$

as required. \hfill \Box

**Remark 3.8.** Although the limits $\lim_{n \to \infty} \frac{\omega(I^{(n)})}{n}$ and $\lim_{n \to \infty} \frac{\text{reg}(I^{(n)})}{n}$ do exist, it is not true that the limit

$$\lim_{n \to \infty} \frac{t_i(I^{(n)})}{n}$$

exists for all $i \geq 0$. 
Example 3.9. In the polynomial ring $R = \mathbb{Q}[x, y, z, u, v]$, consider the ideal $I$ with the primary decomposition

\[ I = (x^2, y^2, z^2)^2 \cap (x^3, y^3, u) \cap (z, v). \]

By [28, Lemma 4.5] we have

\[ \operatorname{depth}(R/I(n)) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases} \]

From the Auslander-Buchsbaum formula, we get

\[ \operatorname{pd} I(n) = \begin{cases} 3 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases} \]

In particular, $t_3(I(n)) = -\infty$ if $n$ is even, and $t_3(I(n)) > 0$ if $n$ is odd. Since $t_3(I(n))$ is a quasi-linear function in $n$ for $n \gg 0$, we deduce that

\[ \liminf_{s \to \infty} \frac{t_3(I(2s+1))}{2s+1} \geq 0. \]

So the limit $\lim_{n \to \infty} \frac{t_3(I(n))}{n}$ does not exit.

Example 3.10. Let $p, q \geq 1$ be integers. Let $R = k[x, y, z, t], I = I_{p,q} = (x, y) \cap (x, z^p) \cap (y^p, t^q)$. Explicitly

\[ I = (xy^p, xt^q, y^p z^p, y z^p t^q). \]

On the one hand, it is not hard to compute $\delta(I)$. Indeed, $\mathcal{SP}(I)$ is defined by linear inequalities

\[ \begin{cases} x + y \geq 1, \\ px + z \geq p, \\ qy + pt \geq pq, \\ x, y, z, t \geq 0. \end{cases} \]

Subtracting $z, t$ (and also $x$ and $y$ if necessary), to suitable non-negative numbers, we see that any vertex of $\mathcal{SP}(I)$ must satisfy the following system of equalities and inequalities

\[ \begin{cases} x + y \geq 1, \\ px + z = p, \\ qy + pt = pq, \\ x, y, z, t \geq 0. \end{cases} \]

This has solution $z = p(1-x), t = q(p-y)/p$ and

\[ \begin{cases} x + y \geq 1, \\ 0 \leq x \leq 1, \\ 0 \leq y \leq p. \end{cases} \]

The last system yields a trapezoid in the $xy$ plane with vertices

\[[x, y] \in \{(1, 0), (0, 1), (0, p), (1, p)\}. \]

Hence $\mathcal{SP}(I)$ has the following vertices

\[(x, y, z, t) \in \{(1, 0, 0, q), (0, 1, p, q(p-1)/p), (0, p, p, 0), (1, p, 0, 0)\}. \]
In particular, \( \delta(I) \in \{ q + 1, p + q + 1 - q/p, 2p \} \), and concretely
\[
\delta(I) = \begin{cases} 
q + 1, & \text{if } q \geq p^2 + 1, \\
p + q + 1 - q/p, & \text{if } p^2 \geq q \geq p, \\
2p, & \text{if } q \leq p - 1.
\end{cases}
\]

Therefore
\[
\lim_{n \to \infty} \frac{\reg I^{(n)}}{n} = \begin{cases} 
q + 1, & \text{if } q \geq p^2 + 1, \\
p + q + 1 - q/p, & \text{if } p^2 \geq q \geq p, \\
2p, & \text{if } q \leq p - 1.
\end{cases}
\]

On the other hand, the regularity function \( I^{(n)} \) can be rather complicated in certain cases. For example, the above asymptotic formula suggests that when \( q = p + 1 \), \( \reg I^{(n)}_{p,p+1} \) seems to be eventually quasi-linear of periodic \( p \). Experiments with Macaulay2 [12] also suggest that \( I^{(n)}_{p,p+1} \) is not Koszul for all \( p \geq 2, n \geq 1 \), hence the techniques developed in the present paper do not apply to the computation of \( \reg I^{(n)}_{p,p+1} \). We also see from the asymptotic formula that for \( p \geq 2 \), \( \reg I^{(n)}_{p,p+1} \) is not eventually linear. Hence Question 1.2 has a negative answer in embedding dimension 4 if we also take non-squarefree monomial ideals into account.

4. Cover ideals

In this section we investigate the symbolic powers of cover ideals of graphs. Our main results in this section are:

1. Theorem 4.6, which determines explicitly the invariant \( \delta(J(G)) \) in terms of the combinatorial data of \( G \);
2. Theorem 4.9, which provides large families of graphs \( G \) such that \( \delta(J(G)) \) attains its minimal value \( \omega(J(G)) \);
3. Theorem 4.13, which computes the maximal generating degrees of the symbolic powers of \( J(G) \).

Combining Theorems 4.6 and 4.13 with a result on the Koszul properties of the symbolic powers of some cover ideals, we construct in Theorem 5.15 a family of graphs \( G \) for which both \( \reg J^{(n)}(G) \) and \( \omega(J^{(n)}(G)) \) are not eventually linear function of \( n \).

Let \( \Delta \) be a simplicial complex on the vertex set \( \{1, \ldots, r\} \) and \( n \geq 1 \). We first describe \( SP_n(I_\Delta) \) in a more specific way. For \( F \in \mathcal{F}(\Delta) \), \( NP(P^n_F) \) is defined by the system

\[
\sum_{i \in F} x_i \geq n, x_1 \geq 0, \ldots, x_r \geq 0,
\]

so that \( SP_n(I_\Delta) \) is determined by the following system of inequalities:

\[
(4.1) \quad \begin{cases} 
\sum_{i \notin F} x_i \geq n, & \text{for } F \in \mathcal{F}(\Delta), \\
x_1 \geq 0, \ldots, x_r \geq 0.
\end{cases}
\]

From this, one has

**Remark 4.1.** Let \( x^\alpha \in I^{(n)}_\Delta \) be a monomial. The following are equivalent:

1. \( x^\alpha \in \mathcal{G}(I^{(n)}_\Delta) \);
2. for every \( i \) such that \( \langle \alpha, e_i \rangle \geq 1 \), we have \( x^{\alpha-e_i} \notin I^{(n)}_\Delta \),
(3) for every \( i \) such that \( \langle \alpha, e_i \rangle \geq 1 \), there exists \( F \in \mathcal{F}(\Delta) \) such that \( i \not\in F \) and \( \langle \alpha, \sum_{j \not\in F} e_j \rangle = n \).

The following lemma is a consequence of the last remark.

**Lemma 4.2.** Let \( p \geq 1, m_1, \ldots, m_p \geq 0 \) be integers and \( x^{\alpha^j} \in I_{\Delta}^{(m_j)} \) be monomials for \( j = 1, \ldots, p \). Assume that \( x^{\alpha^1 + \cdots + \alpha^p} \in I_{\Delta}^{(m_1)} \cdots I_{\Delta}^{(m_p)} \subseteq I_{\Delta}^{(m_1 + \cdots + m_p)} \) is a minimal generator of \( I_{\Delta}^{(m_1 + \cdots + m_p)} \). Then for all \( n_1, \ldots, n_p \geq 0 \), \( x^{n_1 \alpha^1 + \cdots + n_p \alpha^p} \) is a minimal generator of \( I_{\Delta}^{(m_1 n_1 + \cdots + m_p n_p)} \).

In particular:

(i) For every subset \( W \subseteq [p] \), \( x^{\sum_{i \in W} \alpha^i} \) is a minimal generator of \( I_{\Delta}^{(\sum_{i \in W} m_i)} \).

(ii) If \( x^\alpha \in G(I_{\Delta}) \) then \( x^n \alpha \in G(I_{\Delta}^{(n)}) \) for all \( n \geq 1 \).

**Proof.** We claim that for every \( 1 \leq i \leq p \), if \( m_i = 0 \) then \( \alpha^i = 0 \). Indeed, for example, assume \( m_p = 0 \) and \( \alpha^p \neq 0 \). Then

\[
x^{\alpha^1 + \cdots + \alpha^p} = x^{\alpha^p} x^{\alpha^1 + \cdots + \alpha^{p-1}} \not\in G(I_{\Delta}^{(m_1 + \cdots + m_{p-1})}) = G(I_{\Delta}^{(m_1 + \cdots + m_p)}),
\]

a contradiction. Hence the claim is true. In view of the desired conclusion, we can assume that \( m_i \geq 1 \) for all \( i = 1, \ldots, p \).

Take arbitrary \( i \) such that \( (n_1 \alpha^1 + \cdots + n_p \alpha^p, e_i) \geq 1 \). Then \( \langle \alpha^1 + \cdots + \alpha^p, e_i \rangle \geq 1 \). Since \( x^{\alpha^1 + \cdots + \alpha^p} \in G(I_{\Delta}^{(m_1 + \cdots + m_p)}) \), by Remark 4.1, there exists \( F \in \mathcal{F}(\Delta) \) such that \( i \notin F \) and

\[
\langle \alpha^1 + \cdots + \alpha^p, \sum_{j \notin F} e_j \rangle = m_1 + \cdots + m_p.
\]

For all \( u = 1, \ldots, p \), since \( x^{\alpha^u} \in I_{\Delta}^{(m_u)} \),

\[
\langle \alpha^u, \sum_{j \notin F} e_j \rangle \geq m_u.
\]

Thus the equality actually happens for all \( u = 1, \ldots, p \). This implies that

\[
\langle n_1 \alpha^1 + \cdots + n_p \alpha^p, \sum_{j \notin F} e_j \rangle = m_1 n_1 + \cdots + m_p n_p.
\]

Hence by Remark 4.1, \( x^{n_1 \alpha^1 + \cdots + n_p \alpha^p} \) is a minimal generator of \( I_{\Delta}^{(m_1 n_1 + \cdots + m_p n_p)} \).

The proof is concluded.

**Lemma 4.3.** For all \( n \geq 1 \), there is an inequality \( \omega(I_{\Delta}^{(n)}) \leq \delta(I_{\Delta}) n \).

**Proof.** For simplicity, denote \( \delta = \delta(I_{\Delta}) \). Let \( x^\alpha \) be a minimal generator of \( I_{\Delta}^{(n)} \).

We may assume that \( \alpha_i \geq 1 \) for \( i = 1, \ldots, p \) and \( \alpha_i = 0 \) for \( i = p+1, \ldots, r \) for some \( 1 \leq p \leq r \).

For each \( i = 1, \ldots, p \), there is a facet \( F_i \in \mathcal{F}(\Delta) \) which does not contain \( i \) such that \( \alpha \) lies in the hyperplane \( \sum_{j \not\in F_i} x_j = n \). From the system (4.1) we deduce that the intersection of \( SP_n(I_{\Delta}) \) with the set

\[
\begin{cases}
\sum_{j \not\in F_i} x_j = n & \text{for } i = 1, \ldots, p, \\
x_s = 0 & \text{for } s = p + 1, \ldots, r,
\end{cases}
\]

is a compact face of \( SP_n(I_{\Delta}) \).
Example 4.4. Let \( G \) be a graph on the vertex set \( \{1, \ldots, r\} \). Let 
\[
I(G) = (x_ix_j \mid \{i, j\} \in E(G)) \subseteq k[x_1, \ldots, x_r]
\]
be the edge ideal of \( G \). Then \( \omega(I(G)^{(n)}) = 2n \) for all \( n \geq 1 \). In particular, by Theorem 3.3, \( \delta(I(G)) = 2 \).

Indeed, for any \( n \geq 1 \) we have \( \omega(I(G)^{(n)}) \leq 2n \) by [2, Corollary 2.11]. On the other hand, if \( x_ix_j \) is a minimal generator of \( I(G) \), then \( (x_ix_j)^n \) is a minimal generator of \( I(G)^{(n)} \), and so \( \omega(I(G)^{(n)}) \geq 2n \). Hence, \( \omega(I(G)^{(n)}) = 2n \).

Of course, \( I(G)^{(n)} \) need not be generated in degree \( 2n \). For example, if \( I(G) = (xy, xz, yz) \) then 
\[
I(G)^{(2)} = (x, y)^2 \cap (x, z)^2 \cap (y, z)^2 = (x^2y^2, x^2z^2, y^2z^2, xyz).
\]

We do not know whether for any graph \( G \), \( \text{reg } I(G)^{(n)} \) is asymptotically linear in \( n \). This is the case when \( G \) is a cycle (see [13, Corollary 5.4]).

Let \( G \) be a graph on \( [r] = \{1, \ldots, r\} \). Then the polyhedron \( SP(J(G)) \) is defined by the following system of inequalities:
\[
\begin{align*}
&x_i + x_j \geq 1, \text{ for } \{i, j\} \in E(G), \\
x_1 \geq 0, \ldots, x_r \geq 0.
\end{align*}
\]

The following lemma is quite useful to identifying the vertices of \( SP(J(G)) \).

Lemma 4.5. Let \( G \) be a graph on \([r]\) with no isolated vertex, and \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r \). Assume that \( \alpha \) is a vertex of \( SP(J(G)) \). Then \( \alpha_i \in \{0, 1/2, 1\} \) for every \( i = 1, \ldots, r \). Denote \( S_0 = \{i : \alpha_i = 0\} \), \( S_1 = \{i : \alpha_i = 1\} \) and \( S_{1/2} = \{i : \alpha_i = 1/2\} \). Then the following statements hold:

(i) \( S_0 \) is an independent set of \( G \).
(ii) \( S_1 = N(S_0) \).
(iii) The induced subgraph of \( G \) on \( S_{1/2} \) has no bipartite component.
(iv) If \( v \) is a leaf not lying in \( S_0 \) and \( N(v) = \{u\} \) then \( u \notin S_1 \).

Proof. Since \( \alpha \) is a vertex of \( SP(J(G)) \), by [34, Formula (23), Page 104], \( \alpha \) is the unique solution of a system
\[
\begin{align*}
x_i + x_j = 1, & \text{ for } \{i, j\} \in E_1, \\
x_i = 0, & \text{ for } i \in V_1,
\end{align*}
\]
of exactly \( r \) linearly independent equations, where \( E_1 \subseteq E(G) \) and \( V_1 \subseteq \{1, \ldots, r\} \) with \( |E_1| + |V_1| = r \).

Step 1: Let \( H \) be the subgraph of \( G \) with the same vertex set and \( E(H) = E_1 \). Let \( H_1, \ldots, H_s \) be connected components of \( H \). Assume that \( V(H_i) \cap V_1 \neq \emptyset \) for \( i = 1, \ldots, t \); and \( V(H_t) \cap V_1 = \emptyset \) for \( i = t + 1, \ldots, s \) for some \( 0 \leq t \leq s \). We show that \( \alpha_j \in \{0, 1\} \) if \( j \in \bigcup_{i=1}^t V(H_i) \) and \( \alpha_j = 1/2 \) if \( j \in \bigcup_{i=t+1}^s V(H_i) \).

For each \( i \in \{1, \ldots, t\} \) and each \( j \in H_i \), we take \( p \in V(H_i) \cap V_1 \). Then \( \alpha_p = 0 \) by the assumption. Since \( H_i \) is connected, there is a path from \( p \) to \( j \) in \( H_i \), say
\[
p = j_0, j_1, \ldots, j_m = j.
\]
Since $\alpha_{j_0} + \alpha_{j_{m+1}} = 1$ for $u = 0, \ldots, m-1$, we deduce that $\alpha_{j_m} = \begin{cases} 0, & \text{if } m \text{ is even}, \\ 1, & \text{if } m \text{ is odd}. \end{cases}$

For each $u = t + 1, \ldots, s$, from the above discussion, the system

$$\begin{align*}
\begin{cases}
x_i + x_j = 1, \\
\{i,j\} \in E(H_u),
\end{cases}
\end{align*}$$

also has a unique solution. As $V(H_u) \cap V_1 = \emptyset$, $H_u$ cannot be an isolated vertex, so $E(H_u) \neq \emptyset$. Since $x_i = 1/2$ for all $i \in V(H_u)$ is a solution of the last system, it is the unique one. Hence we see that $\alpha_i \in \{0, 1/2\}$ for all $i$.

**Step 2:** If there are adjacent vertices $i, j \in S_0$ then as $\alpha \in SP(J(G))$, we get $0 = \alpha_i + \alpha_j \geq 1$. This is a contradiction. Hence $S_0$ is an independent set, proving (i).

**Step 3:** Similarly there can be no edge connecting any $i \in S_0$ with some $j \in S_{1/2}$. Hence $N(S_0) \subseteq S_{1}$. Now assume that $S_1$ has a vertex, say $i$, that is not adjacent to any vertex in $S_0$. Then $\gamma = \alpha - \frac{1}{2} e_i$ is a point of $SP(J(G))$. On the other hand, $\alpha + \frac{1}{2} e_i$ is obviously a point of $SP(J(G))$. Hence we have a convex decomposition

$$\alpha = \frac{1}{2}(\alpha - e_i/2) + \frac{1}{2}(\alpha + e_i/2),$$

contradicting the fact that $\alpha$ is a vertex of $SP(J(G))$. Thus, as $G$ has no isolated vertex, every vertex in $S_1$ is adjacent to one in $S_0$, and thus $S_1 \subseteq N(S_0)$. In particular, $S_1 = N(S_0)$, proving (ii).

**Step 4:** Next we show (iii). Assume the contrary, the induced subgraph of $G$ on $S_{1/2}$ has a bipartite component $G_1$. Let $(A, B)$ be the bipartition of $G_1$. Construct the vectors $\alpha', \alpha''$ as follows: $\alpha_i' = \begin{cases} \alpha_i & \text{if } i \notin A \cup B, \\ 0 & \text{if } i \in A, \\ 1 & \text{if } i \in B. \end{cases}$

$$\alpha'' = \begin{cases} \alpha_i & \text{if } i \notin A \cup B, \\ 1 & \text{if } i \in A, \\ 0 & \text{if } i \in B. \end{cases}$$

We show that $\alpha', \alpha'' \in SP(J(G))$. Indeed, take an edge $\{i, j\} \in E(G)$. If neither $i$ nor $j$ belong to $A \cup B$, then $\alpha_i' + \alpha_j' = \alpha_i + \alpha_j \geq 1$. If exactly one of $i$ and $j$ belongs to $A \cup B$, we can assume that $i$ does. Then $j \in S_1$, since by (ii), $V(G_1) \subseteq S_{1/2} \subseteq V(G) \setminus N(S_0)$. In this case $\alpha_i' + \alpha_j' = \alpha_i' + \alpha_j = 1 + \alpha_i' \geq 1$. If both $i$ and $j$ belong to $A \cup B$, then we can assume that $i \in A, j \in B$, so $\alpha_i' + \alpha_j' = 1$. Hence in any case $\alpha' \in SP(J(G))$, and the same argument works for $\alpha''$.

But then the convex decomposition $\alpha = (\alpha' + \alpha'')/2$ shows that $\alpha$ is not a vertex of $SP(J(G))$, a contradiction. Thus (iii) is true.

**Step 5:** Assume that $u \in S_1$. By part (ii), we get $S_0 \neq \emptyset$. Since $v \notin S_0$, either $v \in S_1$ or $v \in S_{1/2}$. If $v \in S_1$ then by (ii), $v \in N(S_0)$, a contradiction with $v$ is a leaf and its unique neighbor is $u \in S_1$. Hence $v \in S_{1/2}$. Define the vectors $\alpha^1, \alpha^2$ as follows:

$$\alpha^1_i = \begin{cases} \alpha_i & \text{if } i \neq v, \\ 0 & \text{if } i = v. \end{cases}$$
Theorem 4.6. Let $\alpha^2 \in \mathcal{SP}(J(G))$. We show that $\alpha^2 \in \mathcal{SP}(J(G))$.

Take any edge $\{i,j\} \in E(G)$. If $i \neq v$ or $j \neq v$, then $\alpha_i^1 + \alpha_j^1 = \alpha_i + \alpha_j \geq 1$. If say $i = v$, then necessarily $j = u$, and

$$\alpha_i^1 + \alpha_j^1 = \alpha_i^1 + \alpha_j^1 = 0 + \alpha_u = 1,$$

noting that $u \in S_1$. Hence $\alpha^2 \in \mathcal{SP}(J(G))$. But then the convex decomposition $\alpha = (\alpha^1 + \alpha^2)/2$ shows that $\alpha$ is not a vertex of $\mathcal{SP}(J(G))$, a contradiction. Thus (iv) is true and the proof is concluded. \hfill $\square$

The first main result of this section is

**Theorem 4.6.** Let $G$ be a graph on $|r|$ without isolated vertices, and $J = J(G)$. Then there are equalities

\[(4.2)\]

$$\delta(J) = \max \left\{ \frac{|N(S)| + |G \setminus N[S]|}{2} \mid S \in \Delta(G) \text{ and } G \setminus N[S] \text{ has no bipartite component} \right\} = \frac{r}{2} + \max \left\{ \frac{|N(S)| - |S|}{2} \mid S \in \Delta(G) \text{ and } G \setminus N[S] \text{ has no bipartite component} \right\}.$$ 

**Proof.** Let $d$ be the expression in the last line of (4.2). Clearly $d$ equals the expression on the second line of (4.2), as $r = |G| = |S| + |N(S)| + |G \setminus N[S]|$.

**Step 1:** We show that $d \leq \delta(J)$.

Let $S$ be an independent set of $G$ such that $d = \frac{r}{2} + (|N(S)| - |S|)/2$ and $G \setminus N[S]$ has no bipartite component.

For $i = 1, \ldots, r$, define $\gamma_i$ as follows

$$\gamma_i = \begin{cases} 0 & \text{if } i \in S, \\ 1 & \text{if } i \in N(S), \\ \frac{1}{2} & \text{if } i \in V(G) \setminus N[S]. \end{cases}$$

Let $\gamma = (\gamma_1, \ldots, \gamma_r)$. Then $\gamma$ is a point of $\mathcal{SP}(J)$. Since $2\gamma \in \mathbb{N}^r$, $x^2 \gamma \in J(G)^{2\gamma}$. Observe that $x^2 \gamma$ is a minimal generator of $J(G)^{2\gamma}$, since $G$ has no isolated vertex. Hence $|2\gamma| \leq 2\delta(J)$ by Lemma 4.3, namely $\delta(J) \geq |\gamma| = d$.

**Step 2:** To prove the reverse inequality, let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be any vertex of $\mathcal{SP}(J)$. By Lemma 4.5, $\alpha_i \in \{0, 1/2, 1\}$ for every $i$. Let $S = S_0 = \{i \mid \alpha_i = 0\}$, $S_1 = \{i \mid \alpha_i = 1\}$ and $S_{1/2} = \{i \mid \alpha_i = 1/2\}$. By the same lemma, $S \in \Delta(G)$ and $G \setminus N[S]$ has no bipartite component.

Thus

$$|\alpha| = |S_2| + \frac{|S_{1/2}|}{2} = \frac{|S| + |S_1| + |S_{1/2}|}{2} = \frac{|S_1| - |S|}{2} = \frac{r}{2} + \frac{|N(S)| - |S|}{2} \leq d.$$ 

Choosing the vertex $\alpha$ such that $|\alpha| = \delta(J)$, we deduce $\delta(J) \leq d$, as required. \hfill $\square$

Denote by $\tau_{\max}(G)$ the maximal cardinality of a minimal vertex cover of $G$. Since the minimal monomial generators of $J(G)$ correspond to the minimal vertex covers of $G$, there is an equality $\omega(J(G)) = \tau_{\max}(G)$. 

\[\]
Corollary 4.7. Let $G$ be a graph on $[r]$ without isolated vertices. Then there are inequalities

$$
\max \left\{ \tau_{\text{max}}(G), \frac{r}{2} \right\} \leq \delta(J(G)) \leq \max \left\{ \tau_{\text{max}}(G), \frac{r + \tau_{\text{max}}(G) - 3}{2} \right\}.
$$

Proof. By Lemma 4.3 for $n = 1$, $\tau_{\text{max}}(G) = \omega(J(G)) \leq \delta(J(G))$.

We note that $x_1x_2 \cdots x_r \in J(G)^{(2)}$. It is a minimal generator of $J(G)^{(2)}$, since $G$ has no isolated vertex. Hence again by Lemma 4.3,

$$
r \leq \omega(J(G)^{(2)}) \leq 2\delta(J(G)),
$$

namely $r/2 \leq \delta(J(G))$. This yields the inequality on the left.

For the inequality on the right, take any independent set $S$ of $G$ such that $G \setminus N[S]$ has no bipartite component. We have to show that

$$
(4.3) \quad \frac{r}{2} + \frac{|N(S)| - |S|}{2} \leq \max \left\{ \tau_{\text{max}}(G), \frac{r + \tau_{\text{max}}(G) - 3}{2} \right\}.
$$

If $S = \emptyset$ then the left-hand side is $r/2$. If $G \setminus N[S] = \emptyset$, then $r = |N(S)| + |S|$. In this case the left-hand side of (4.3) is $|N(S)| \leq \tau_{\text{max}}(G)$, since $N(S)$ is now a minimal vertex cover of $G$.

Assume that $S$ and $G \setminus N[S]$ are both non-empty. Let $H$ be a connected component of $G \setminus N[S]$, then by the assumption on $S$, $H$ is neither an isolated point, nor bipartite. Thus $|V(H)| \geq 3$. As a connected graph, $H$ has then a minimal vertex cover $W$ of size at least 2. For this, note that if $H$ has a minimal vertex cover of a singleton $u$, then $H = N_H[u]$, and $N_H(u)$ is an independent set. In turn, this implies that $N_H(u)$ is a minimal vertex cover of $H$ of size $|V(H)| - 1 \geq 2$.

Let $W'$ be a minimal vertex cover of $G \setminus N[S]$ containing $W$. Then $N(S) \cup W'$ is a minimal vertex cover of $G$ (the minimality holds since $S$ is an independent set). Thus

$$
|N(S)| \leq \tau_{\text{max}}(G) - |W'| \leq \tau_{\text{max}}(G) - 2.
$$

Consequently, using the fact that $S \neq \emptyset$,

$$
\frac{r}{2} + \frac{|N(S)| - |S|}{2} \leq \frac{r}{2} + \frac{\tau_{\text{max}}(G) - 2 - 1}{2} = \frac{r + \tau_{\text{max}}(G) - 3}{2}.
$$

This finishes the proof of (4.3), and that of the corollary.

Remark 4.8. Computations with Macaulay2 [12] show that for any graph $G$ without isolated vertex on $r \leq 8$ vertices, the equality $\delta(J(G)) = \tau_{\text{max}}(G)$ holds. In particular, for such graphs, $\delta(J(G)) = \max\{\tau_{\text{max}}(G), |V(G)|/2\}$.

The corona graph $G = \text{cor}(K_3, 2)$ in Figure 1 has 9 vertices, and $\delta(J(G)) = 9/2 > \tau_{\text{max}}(G) = 4$, hence again $\delta(J(G)) = \max\{\tau_{\text{max}}(G), |V(G)|/2\}$.

In general, both inequalities in Corollary 4.7 are strict, see Example 4.16.

We will see later in Lemma 5.14 a family of graphs for which the difference $\delta(J(G)) - \tau_{\text{max}}(G)$ can be arbitrarily large. On the other hand, for large classes of graphs, the equality $\delta(J(G)) = \tau_{\text{max}}(G)$ does hold. We say that $G$ is an unmixed graph if every minimal vertex cover of $G$ has the same size. Equivalently, $G$ is unmixed if and only if every associated prime ideal of $J(G)$ has the same height.

We say that $G$ is claw-free if it does not contain the complete bipartite graph $K_{1,3}$ as an induced subgraph.

The second main result of this section is
Figure 2. The graph $K_{3,3}$ is bipartite, unmixed, but not claw-free

**Theorem 4.9.** Let $G$ be a graph without isolated vertices, that satisfies either of the following properties:

1. bipartite;
2. unmixed;
3. claw-free.

Then there are equalities $\delta(J(G)) = \omega(J(G)) = \tau_{\text{max}}(G)$.

The proof uses the following lemma, that is inspired by work of Seyed Fakhari [37, Theorem 3.2].

**Lemma 4.10.** Let $\mathcal{H}$ be a family of graphs with the following properties:

(i) for every $G \in \mathcal{H}$ and every vertex $x \in G$, the graph $G \setminus N_G[x]$ also belongs to $\mathcal{H}$;

(ii) for every $G \in \mathcal{H}$ without isolated vertices, the inequality $\tau_{\text{max}}(G) \geq \frac{|V(G)|}{2}$ holds.

Then for every $G \in \mathcal{H}$ without isolated vertices, the equality $\delta(J(G)) = \tau_{\text{max}}(G)$ holds.

**Proof.** By Corollary 4.7, it remains to show that for any $G \in \mathcal{H}$ without isolated vertices, $\delta(J(G)) \leq \tau_{\text{max}}(G)$. We use the formula of Theorem 4.6.

Let $S$ be an independent set of $G$ such that $S' = G \setminus N[S]$ has no bipartite component. Applying successively property (i) of $\mathcal{H}$, $S'$ also belongs to $\mathcal{H}$. Since $S'$ has no bipartite component, it has no isolated vertices. In particular, property (ii) guarantees the existence of a minimal vertex cover $W$ of $S'$ with cardinality $\geq |S'|/2$.

It is a general fact that when $S$ is an independent set, and $W$ is a minimal vertex cover of $G \setminus N[S]$, $W \cup N(S)$ is a minimal vertex cover of $G$. In our case, $W \cup N(S)$ has cardinality $\geq |N(S)| + |S'|/2$. This shows that

$$\tau_{\text{max}}(G) \geq |N(S)| + |S'|/2 = |N(S)| + \frac{|G \setminus N[S]|}{2}.$$ 

Taking supremum over all $S$, we deduce $\tau_{\text{max}}(G) \geq \delta(J(G))$. The proof of the lemma is concluded. $\Box$

**Proof of Theorem 4.9.** We check that each of the families of graphs in Theorem 4.9 satisfies the two conditions in Lemma 4.10. This was done respectively in the proofs of Theorems 3.4, 3.6 and 3.7 in [37]. $\Box$

The following result gives a family of graphs $G$ with the property $\delta(J(G)) = \tau_{\text{max}}(G)$ not covered by Theorem 4.9. Recall that for two graphs $G$ and $H$, their join
Remark 4.12. The hypothesis $r$ of Proposition 4.11 says that for “most” sparse enough graphs $G$, $G$ is an induced subgraph of a graph $G_m$ having $\delta(J(G_m)) = \tau_{\text{max}}(G_m)$, but having none of the properties bipartite, unmixed, and claw-free.

**Proposition 4.11.** Let $G$ be a graph without isolated vertices on $[r]$, where $r \geq 8$. Assume that $\tau_{\text{max}}(G) \leq r - 4$, equivalently, every maximal independent set of $G$ has size at least 4. Let $m$ be any integer such that

$$3 \leq m \leq \min \left\{ \frac{r - \tau_{\text{max}}(G) + 3}{2}, \frac{r}{2} \right\}.$$ 

Let $O(m)$ be the graph on $\{y_1, \ldots, y_m\}$ with no edges. Denote by $G_m$ the join of $G$ and $O(m)$. Then:

1. $G_m$ is a connected graph, but has none of the properties bipartite, unmixed, and claw-free;
2. $\delta(J(G_m)) = \tau_{\text{max}}(G_m) = r$.

**Remark 4.12.** The hypothesis $\tau_{\text{max}}(G) \leq r - 4$ of Proposition 4.11 is satisfied, for example, if $r = 8$, and $G = K_{4,4}$, which has $\tau_{\text{max}}(K_{4,4}) = 4$. It is also satisfied if $r \geq 10$ and every vertex of $G$ has degree strictly less than $\frac{r}{3} - 1$ (more concretely, the cycle of length $r$ has this property). Indeed, in that case, we show that $G$ has no maximal independent set $S$ of size 3. Assume the contrary, that $S = \{x, y, z\}$ is a maximal independent set of size 3. Then as $G$ has no isolated vertex, each of the $r - 3$ vertices of $G \setminus S$ is adjacent to an element of $S$. This implies that $S$ has a vertex of degree at least $\frac{r}{3} - 1$, a contradiction.

**Proof of Proposition 4.11.** Denote $V(G) = \{x_1, \ldots, x_r\}$.

1. Clearly $G_m$ is connected as every two non-adjacent vertices of $G_m$ can be connected via either $y_i$ or $x_1$. Since $r \geq 2$, $G$ has at least one edge $x_i x_j$, $i \neq j$. Hence having the odd cycle $x_i, x_j, y_i$, $G$ is not bipartite.

2. Observation: $G_m$ has only two types of minimal vertex covers: $\{x_1, \ldots, x_r\}$, and $W \cup \{y_1, \ldots, y_m\}$, where $W$ is a minimal vertex cover of $G$.

3. The vertex cover $\{x_1, \ldots, x_r\}$ has cardinality $r > m + \tau_{\text{max}}(G)$. The last inequality holds since from the hypotheses

$$m + \tau_{\text{max}}(G) \leq \frac{r + \tau_{\text{max}}(G) + 3}{2} < \frac{r + (r - 3) + 3}{2} = r.$$ 

Hence $G_m$ is not unmixed and

$$(4.4) \quad \tau_{\text{max}}(G_m) = r.$$ 

As $m \geq 3$, $G_m$ contains the claw $x_1, y_1, y_2, y_3$, as desired.

2. The equality $\tau_{\text{max}}(G_m) = r$ is (4.4). By Corollary 4.7, it remains to show that $\delta(J(G_m)) \leq r$. Let $S$ be an independent set of $G_m$ such that $G_m \setminus N_{G_m}[S]$ has no bipartite component. Being independent, $S$ cannot intersect non-trivially with both $\{x_1, \ldots, x_r\}$ and $\{y_1, \ldots, y_m\}$. There are three cases.

**Case 1:** $S = \emptyset$. Using the formula of Theorem 4.6, we get the value

$$\frac{m + r}{2} + \frac{|N_{G_m}(S)| - |S|}{2} = \frac{m + r}{2} \leq r.$$ 

The inequality holds because $m \leq r/2 < r$.

**Case 2:** $\emptyset \neq S \subseteq \{y_1, \ldots, y_m\}$. In this case, $N_{G_m}(S) = \{x_1, \ldots, x_r\}$. So $G_m \setminus N_{G_m}[S]$ is a subset of $\{y_1, \ldots, y_m\}$, and it has no bipartite component. In
particular, it must be empty, so that \( S = \{ y_1, \ldots, y_m \} \). The formula of Theorem 4.6 yields the value

\[
\frac{m + r}{2} + \frac{|N_{G_m}(S)| - |S|}{2} = \frac{m + r}{2} + \frac{r - m}{2} = r.
\]

**Case 3:** \( \emptyset \neq S \subseteq \{ x_1, \ldots, x_r \} \). In this case \( S \) is an independent set of \( G \), \( N_{G_m}(S) = N_G(S) \cup \{ y_1, \ldots, y_m \} \). Hence \( G \setminus N_G[S] = G_m \setminus N_{G_m}[S] \) has no bipartite component. The formula of Theorem 4.6 yields the value

\[
\frac{m + r}{2} + \frac{|N_{G_m}(S)| - |S|}{2} = \frac{m + r}{2} + \frac{r - m}{2} = r.
\]

Using Corollary 4.7, we further get

\[
m + \delta(J(G)) \leq m + \max \left\{ \tau_{\text{max}}(G), \frac{r}{2}, \frac{r + \tau_{\text{max}}(G) - 3}{2} \right\}
\]

\[
\leq r,
\]

thanks to the hypothesis

\[
m \leq \min \left\{ \frac{r - \tau_{\text{max}}(G) + 3}{2}, \frac{r}{2} \right\}.
\]

Hence in any case, \( \delta(J(G_m)) \leq r \). The proof is concluded. \( \square \)

The third main result in this section is

**Theorem 4.13.** Let \( G \) be a graph on \( [r] \) with no isolated vertex, and \( J = J(G) \). Then

1. \( \omega(J^{(2s)}) = \delta(J)2s \) for every \( s \geq 1 \).
2. There exists \( m_2 \in \mathcal{G}(J) \) such that for some \( m_1 \in \mathcal{G}(J^{(2)}) \) satisfying \( \deg(m_1) = 2\delta(J) \), we have \( m_1m_2 \in \mathcal{G}(J^{(3)}) \). Let \( e \) be the maximal degree of such an \( m_2 \). Then
   \( \omega(J^{(2s+1)}) = \delta(J)2s + e \), for every \( s \geq \omega(J) - e \).
3. If \( \delta(J) = \omega(J) \) or \( \delta(J) = r/2 \), then with the notation of part (2), we have \( e = \omega(J) \). In particular, \( \omega(J^{(2s+1)}) = \delta(J)2s + \omega(J) \) for all \( s \geq 0 \).

It is crucial for the proof that the symbolic Rees algebras of cover ideals of graphs are generated in degree at most 2.

**Theorem 4.14** (Herzog-Hibi-Trung [17, Theorem 5.1]). Let \( G \) be a graph and \( J = J(G) \). Then for every \( s \geq 1 \),

1. \( J^{(2s)} = (J^{(2)})^s \).
2. \( J^{(2s+1)} = J(J^{(2)})^s \).

For ease of reference, we record here an immediate corollary of this theorem.

**Corollary 4.15.** Let \( G \) be a graph. Then \( \reg J(G)^{(n)} \) is a quasi-linear function of \( n \) of period at most 2 for \( n \) large enough.

**Proof.** Follows from Theorem 4.14 and [38, Theorem 3.2]. \( \square \)
Proof of Theorem 4.13. (1) By Lemma 4.3, \( \omega(J^{(2s)}) \leq \delta(J)2s \).

For the reverse inequality, let \( \alpha = (\alpha_1, \ldots, \alpha_r) \) be a vertex of \( SP(J) \) such that \( \delta(J) = |\alpha| \). By [34, Formula 23 in Page 104], \( \alpha \) is a unique solution of a system of the type

\[
\begin{align*}
  x_i + x_j &= 1, \text{ for } \{i, j\} \in E_1, \\
  x_i &= 0, \text{ for } i \in V_1,
\end{align*}
\]

where \( E_1 \subseteq E(G) \) and \( V_1 \subseteq \{1, \ldots, r\} \) with \( |E_1| + |V_1| = r \). By Lemma 4.5, \( \alpha_i \in \{0, 1/2, 1\} \) for every \( i \).

Since \( 2s\alpha \in \mathbb{N}^r \), we get \( x^{2s\alpha} \in J(G)^{(2s)} \). Note that \( 2s\alpha \) is a vertex of \( SP_{2s}(J) \), so \( x^{2s\alpha} \) is a generator of \( J(G)^{(2s)} \). It follows that \( \omega(J^{(2s)}) \geq 2s|\alpha| = \delta(J)2s \), as desired.

(2) Let \( I = J^{(2)} \). By Theorem 4.14 we have \( J^{(2s+1)} = I^sJ \). Note that \( \omega(I) = 2\delta(J) \) by part (1) above. Therefore, we can write \( I = I_1 + I_2 \) where \( I_2 \) is generated by elements of \( G(I) \) of degree exactly \( 2\delta(J) \) and \( I_1 \) is generated by the remaining elements.

Since \( J^{(3)} = IJ \), the first assertion of (2) reduces to the following

Claim: \( I_2J \not\subseteq I_1J \).

Indeed, if \( I_2J \subseteq I_1J \), we will derive a contradiction. Since \( IJ = (I_1 + I_2)J = I_1J + I_2J = I_1J \), for every \( n \geq 1 \), from this equality and Theorem 4.14 we get \( J^{(2n+1)} = I^nJ = I^nJ \). In particular, \( \omega(J^{(2n+1)}) \leq \omega(I_1) + \omega(J) \), so \( \omega(I_1) \geq 2\delta(J) \) by Theorem 3.3. On the other hand, \( \omega(I_1) < 2\delta(J) \) by the definition of \( I_1 \), a contradiction.

We now return to proving part (2). By the claim, there exist \( m_1 \in G(I) \) with \( \deg(m_1) = 2\delta(J) \) and \( m_2 \in G(J) \) such that \( m_1m_2 \in G(IJ) \). Among all such couples \((m_1, m_2)\), choose one such that \( e = \deg(m_2) \) is maximal. By Lemma 4.2, \( m_1^*m_2 \in G(J^{(2s+1)}) \). In particular,

\[
\omega(J^{(2s+1)}) \geq \deg(m_1)s + \deg(m_2) = \delta(J)2s + e.
\]

It remains to show that the equality occurs whenever \( s \geq \omega(J) - e \).

Fix an \( s \geq \omega(J) - e \). As mentioned above \( J^{(2s+1)} = I^sJ \), so any minimal generator of \( G(J^{(2s+1)}) \) must have the form \( g_1g_2\cdots g_sf \), where \( g_i \) is a minimal generator of \( I = J^{(2)} \) and \( f \) is a minimal generator of \( J \). Let \( g = g_1\cdots g_s \). We can choose \( g_i, f \) such that \( \deg(gf) = \omega(J^{(2s+1)}) \).

From Lemma 4.2, \( g_i, f \) is a minimal generator of \( J^{(3)} = IJ \) and \( g_i^*f \in G(J^{(2s+1)}) \) for all \( i \). Assume that \( \deg(g_1) \leq \cdots \leq \deg g_s \). Then

\[
\omega(J^{(2s+1)}) = \deg(gf) = \deg g_1 + \cdots + \deg g_s + \deg f \leq s \deg g_s + \deg f \leq \omega(J^{(2s+1)}),
\]

so that \( \deg g_1 = \cdots = \deg g_s \) and

\[
\omega(J^{(2s+1)}) = \deg(gf) = s \deg(g_i) + \deg(f) \quad \text{for } i = 1, \ldots, s.
\]

If \( \deg(g_1) = 2\delta(J) \), then thanks to (4.6), we get the inequality \( \deg(f) \geq e \). The latter is necessarily an equality by the definition of \( e \). In this case, \( \omega(J^{(2s+1)}) = \delta(J)2s + e \).

Assume that \( \deg(g_1) < 2\delta(J) \). Since \( s \geq \omega(J) - e \geq \deg(f) - \deg(m_2) \), we have

\[
\delta(J)2s + \deg(m_2) \geq (\deg(g_1) + 1)s + \deg(f) - s = \deg(g_1)s + \deg(f) = \omega(J^{(2s+1)}),
\]
so thanks to (4.5), $\omega(J^{(2s+1)}) = \delta(J)2s + e$, as required.

(3) If $\delta(J) = \omega(J)$, then there exists $m \in G(J)$ of degree $\delta(J)$. By Lemma 4.2, $m^3$ is a minimal generator of $J^{(3)}$, so we can choose $m_1 = m^2, m_2 = m$ and $e = \deg(m) = \delta(J) = \omega(J)$.

If $\delta(J) = r/2$, then for $\alpha = (1, \ldots, 1) \in \mathbb{N}^r$, $x^\alpha \in G(J)$ and $|\alpha| = r = 2\delta(J)$. Let $x^\gamma \in G(J)$ be such that $|\gamma| = \omega(J)$. Observe that $x^\alpha + \gamma \in G(J^{(3)})$. Clearly $x^\alpha + x^\gamma \in \mathcal{I}J \subseteq J^{(3)}$. For any $1 \leq i \leq r$, we need to show that $x^\alpha + x^\gamma - e_i \not\in J^{(3)}$. Note that $x^\gamma_j \in \{0, 1\}$ for all $j = 1, \ldots, r$.

If $\gamma_i = 1$, since $x^\gamma \in G(J)$, there is an edge $ij \in E(G)$ such that $\gamma_i + \gamma_j = 1$. But then $\alpha_i + \alpha_j + \gamma_i + \gamma_j = 3$, hence $x^\alpha + x^\gamma - e_i \not\in J^{(3)}$.

If $\gamma_i = 0$, for any edge $ij \in E(G)$, we get $\alpha_i + \alpha_j + \gamma_i + \gamma_j = 3$. Again $x^\alpha + x^\gamma - e_i \not\in J^{(3)}$. Therefore we always have $x^\alpha + x^\gamma \in G(J^{(3)})$. Hence we can choose $m_1 = x^\alpha, m_2 = x^\gamma$ and again $e = \deg(m_2) = \omega(J)$. □

The following example shows that $\omega(J^{(G^{(2n+1)})})$ need not be a linear function in $n$ from $n = 0$, and the number $e$ in Theorem 4.13(2) can be strictly smaller than $\omega(J)$.

**Example 4.16.** Let $G$ be a graph with the vertex set

$\{x_i, y_i, z_i | i = 1, \ldots, 5\} \cup \{u, v, w\}$

depicted in Figure 3. Using the `EdgeIdeals` package in Macaulay2 [12], the graph $G$ and its cover ideal are given as follows.

```plaintext
R=ZZ/32003[x_1..x_5,y_1..y_5,z_1..z_5,u,v,w];
G=graph(R,{x_1*x_2,x_1*x_3,x_1*x_4,x_1*x_5,x_2*x_3,x_2*x_4,
x_2*x_5,x_3*x_4,x_3*x_5,x_3*y_1,x_3*z_1,x_4*y_2,x_4*z_2,
x_3*y_3,x_3*z_3,x_4*y_4,x_4*z_4,x_5*y_5,x_5*z_5,x_5*u,x_4*u,y_5*u,
u*v,u*w,v*w});
J=dual edgeIdeal G
```

In particular, $G$ has 18 vertices and 26 edges.

![Figure 3. The graph G](image)

Let $J = J(G)$. By using Macaulay2 [12] we get
(1) \( \omega(J) = 9, \omega(J^{(2)}) = 19, \) and \( \omega(J^{(3)}) = 27. \) By Theorem 4.13, \( \delta(J) = 19/2. \)

(2) The monomials \( m_1 = u^2v^2 \prod_{i=1}^5 (x_i y_i z_i) \in \mathcal{G}(J^{(2)}) \) and
\[
m_2 = x_2 x_3 x_4 x_5 y_1 z_1 w \in \mathcal{G}(J)
\]
satisfy \( m_1 m_2 \in \mathcal{G}(J^{(3)}) \). Note that \( \deg(m_1) = 19, \deg(m_2) = 8. \)

In the notation of Theorem 4.13, we deduce \( 8 \leq e \leq \omega(J) = 9. \) If \( e = 9, \) then by \emph{ibid.} we have \( \omega(J^{(2n+1)}) = \delta(J)2n + 9 = 19n + 9 \) for \( n \geq \omega(J) - 9 = 0. \) Setting \( n = 1, \) we get \( \omega(J^{(3)}) = 28, \) a contradiction. Hence \( e = 8 \) and \( \omega(J^{(2n+1)}) = 19n + 8 \) if (and only if) \( n \geq \omega(J) - 8 = 1. \)

Moreover, observe that both inequalities of Corollary 4.7 are strict in this case:
\[
\begin{align*}
\max \left\{ \tau_{\max}(G), \frac{|V(G)|}{2} \right\} = \tau_{\max}(G) &= 9, \\
\delta(J(G)) &= 19/2, \\
\max \left\{ \tau_{\max}(G), \frac{|V(G)|}{2}, \frac{|V(G)| + \tau_{\max}(G) - 3}{2} \right\} &= 12.
\end{align*}
\]

5. THE KOZUL PROPERTY OF SYMBOLIC POWERS OF COVER IDEALS

The following result is our main tool in the study of the Koszul property of symbolic powers.

**Theorem 5.1.** Let \((R, m)\) be a standard graded \( k\)-algebra. Let \( x \) be a non-zero linear form and \( I', T \) be non-trivial homogeneous ideals of \( R \) such that the following conditions are fulfilled:

(i) \( I' \) is a Koszul module and \( x \) is \( I' \)-regular (e.g. \( x \) is an \( R \)-regular element),

(ii) \( T \subseteq mI' \),

(iii) \( x \) is a regular element with respect to \( R/T \) and \( \text{gr}_m T \).

Denote \( I = xI' + T. \) Then the decomposition \( I = xI' + T \) is a Betti splitting, and there is a chain
\[
\text{ld}_R T \leq \text{ld}_R I = \text{ld}_R(T + (x)) = \text{ld}_R(T/TxT) \leq \text{ld}_R T + 1.
\]

Moreover, \( I \) is a Koszul module if and only if \( T \) is so.

Before proving Theorem 5.1, we recall the following result.

**Lemma 5.2** (Nguyen [27, Theorem 3.1]). Let \( 0 \to M' \to P' \to N' \to 0 \) be a short exact sequence of non-zero finitely generated \( R \)-modules where

(i) \( M' \) is a Koszul module;

(ii) \( M' \cap mP' = mM' \).

Then there are inequalities \( \text{ld}_R P' \leq \text{ld}_R N' \leq \max\{\text{ld}_R P', 1\}. \) In particular, \( \text{ld}_R N' = \text{ld}_R P' \) if \( \text{ld}_R P' \geq 1 \) and \( \text{ld}_R N' \leq 1 \) if \( \text{ld}_R P' = 0. \)

Moreover, \( \text{ld}_R N' = 0 \) if and only if \( \text{ld}_R P' = 0 \) and for all \( s \geq 1, \) we have \( M' \cap m^s P' = m^s M \).

We also have an easy observation.

**Lemma 5.3.** Let \((R, m)\) be a standard graded \( k\)-algebra, and \( x \in m \) a non-zero linear form. Let \( T \) be a homogeneous ideal of \( R \) such that \( x \) is \((R/T)\)-regular. Then the following are equivalent:

(1) \( x \) is \( \text{gr}_m T \)-regular,
Proof. Clearly $x$ is $\text{gr}_m T$-regular if and only if
\[(m^{s+2} T : x) \cap m^s T = m^{s+1} T, \quad \text{for all } s \geq 0.\]
Hence \((2) \implies (1)\).

Conversely, assume that \((1)\) is true. Since $m^{s-1} T \subseteq m^s T : x$, it suffices to show for all $s \geq 1$ that $m^s T : x \subseteq m^{s-1} T$. Induct on $s \geq 1$.

For $s = 1$,
$$mT : x \subseteq T : x = T,$$
where the equality follows from the hypothesis $x$ is $(R/T)$-regular.

Assume that the statement holds true for $s \geq 1$. Using the induction hypothesis, we have
$$m^{s+1} T : x \subseteq (m^{s+1} T : x) \cap (m^s T : x) \subseteq (m^{s+1} T : x) \cap m^{s-1} T = m^s T.$$
The equality in the chain follows from \((5.1)\). The proof is concluded.

\textbf{Proof of Theorem 5.1.} We proceed through several steps.

\textbf{Step 1:} First we establish the equalities $\text{ld}_R I = \text{ld}_R T/xT = \text{ld}_R (T + (x))$. Consider the short exact sequence
$$0 \to xI' \to I \to \frac{T}{xI' \cap T} \xrightarrow{T} 0.$$
The equality holds since $T : x = T \subseteq I'$.

We claim that
\[(5.2) \quad xI' \cap m^s I = m^s xI' \quad \text{for all } s \geq 1.\]
The inclusion “$\supseteq$” is clear. For the converse inclusion, take $a \in xI' \cap m^s I = xI' \cap m^s (xI' + T)$. Subtracting to an element in $m^s xI'$, we may assume that $a \in xI' \cap m^s T$. The last module is contained in
$$(x) \cap m^s T = x(m^s T : x) \subseteq xm^{s-1} T \subseteq xm^s I'.$$
In the last chain, the first inclusion follows from Lemma 5.3 and the hypothesis $x$ is $\text{gr}_m T$-regular. The second inclusion follows from the hypothesis $T \subseteq mI'$. Thus the claim follows.

Recall that $xI' \cong I'$ is Koszul by the hypothesis. Hence using Lemma 5.2 for the above exact sequence, and Equality \((5.2)\), we get
$$\text{ld}_R I = \text{ld}_R (T/xT).$$

Arguing similarly as above for the ideal $T + (x) = xR + T$, we have $\text{ld}_R (T + (x)) = \text{ld}_R T/xT$. Hence $\text{ld}_R I = \text{ld}_R (T/xT) = \text{ld}_R (T + (x))$, as claimed.

\textbf{Step 2:} Note that $x$ is $T$-regular, since it is $\text{gr}_m T$-regular. Let $K(x; T)$ denote the Koszul complex $0 \to T(-1) \xrightarrow{x} T \to 0$, then $K(x; T)$ is quasi-isomorphic to $T/xT$. Hence using (the graded analogue of) a result of Iyengar and Römer [22, Remark 2.12], we obtain
$$\text{ld}_R T \leq \text{ld}_R K(x; T) = \text{ld}_R (T/xT) \leq \text{ld}_R T + 1.$$
Hence we get the desired chain
$$\text{ld}_R T \leq \text{ld}_R I = \text{ld}_R (T/xT) \leq \text{ld}_R T + 1.$$

\textbf{Step 3:} For the assertion on Betti splitting, note that $xI' \cap T = xT$. Since $x$ is $T$-regular, then the morphism $\text{Tor}^R_i (k, xT) \to \text{Tor}^R_i (k, T)$ is the multiplication
by $x$ of $\operatorname{Tor}_i^R(k, T)$, which is trivial. Since $xT \subseteq \mathfrak{m}xT$ and $xT$ is Koszul, the map $\operatorname{Tor}_i^R(k, xT) \to \operatorname{Tor}_i^R(k, xI')$ is also trivial thanks to [29, Lemma 4.10(b1)]. Hence by Lemma 2.4, the decomposition $I = xI' + T$ is a Betti splitting.

**Step 4:** As shown above, $\text{ld}_R T \leq \text{ld}_R I$, hence if $I$ is Koszul then so is $T$. Conversely, assume that $T$ is Koszul. Now $x$ is $\text{gr}_m T$-regular, so by (the graded analogue of) [22, Theorem 2.13(a)], we deduce that $T/xT$ is also Koszul. It remains to use the equality $\text{ld}_R I = \text{ld}_R T/xT$. The proof is concluded. \hfill \Box

**Example 5.4.** The following example shows that the condition $x$ is $\text{gr}_m T$-regular in Theorem 5.1 is critical, even when the base ring is regular.

Let $R = k[a, b, c, d]$, $T = (a, c^2)(b, d^2) = (ab, ad^2, bc^2, c^2d^2)$. Let $x = a - b$, $I = (x) + T$, $\mathfrak{m} = R_+$. We claim that:

(i) $x$ is $(R/T)$-regular but not $\text{gr}_m T$-regular,
(ii) $T$ is Koszul but $I$ is not.

(i): We observe that $T : x = T$ because $T = (a, c^2) \cap (b, d^2)$. We also have $c^2d^2x = c^2(ad^2) - d^2(bc^2) \in \mathfrak{m}^2T$.

Hence $c^2d^2 \in (\mathfrak{m}^2T : x) \setminus (\mathfrak{m}T)$, thus $x$ is not $\text{gr}_m T$-regular.

(ii): Write $T = aJ + U$ where $J = (b, d^2), U = c^2(b, d^2)$. Then $J$ is Koszul, $U \subseteq \mathfrak{m}J$ and $U \cong J(\mathfrak{m}J)$ is Koszul. Applying Corollary 5.6, $T$ is also Koszul.

We have

$$I = T + (x) = (x) + (a^2, ac^2, ad^2, c^2d^2).$$

Denote $L = (a^2, ac^2, ad^2, c^2d^2) \subseteq S = k[a, c, d]$. Note that $R = S[x]$, so by Corollary 5.6 and Lemma 2.2, $\text{ld}_R I = \text{ld}_R(LR + (x)) = \text{ld}_S L$.

Assume that $I$ is Koszul, then so is $L$. Denote by $L_{\leq s}$ the ideal generated by homogeneous elements of degree at most $s$ of $L$. Then by [15, Lemma 8.2.11], we also have $L_{\leq 3} = (a^2, ac^2, ad^2) \cong (a, c^2, d^2)(-1)$ is Koszul. In particular, by Lemma 2.1, $\text{reg} L_{\leq 3} = 3$.

But then $\text{reg}(a, c^2, d^2) = 2$! This contradiction confirms that $I$ is not Koszul.

**Remark 5.5.** Example 5.4 also shows that even if $T$ is a Koszul ideal in a polynomial ring $R$, and $x$ is a regular linear form modulo $T$, the ideal $T + (x)$ need not be Koszul.

Nevertheless, it is not hard to see that this is true if moreover $T$ has a linear resolution. Indeed, in this case $T \cong \text{gr}_m T$ as $R$-modules, so $x$ is $\text{gr}_m T$-regular. Applying Theorem 5.1, we get $\text{ld}_R(T + (x)) = \text{ld}_RT = 0$.

The next consequence of Theorem 5.1 generalizes [29, Lemma 8.2].

**Corollary 5.6.** Let $(R, \mathfrak{m})$ be a polynomial ring over $k$. Let $x$ be a non-zero linear form, $I', T$ be non-trivial homogeneous ideals of $R$ such that the following conditions are satisfied:

(i) $I'$ is Koszul,
(ii) $T \subseteq \mathfrak{m}I'$,
(iii) there exists a polynomial subring $S$ of $R$ such that $R = S[x]$ and $T$ is generated by elements in $S$.

Denote $I = xI' + T$. Then the decomposition $I = xI' + T$ is a Betti splitting and $\text{ld}_R I = \text{ld}_RT$. 

Proof. First we verify that \( x, I' \), and \( T \) satisfy the hypotheses of Theorem 5.1. Note that condition (iii) ensures that \( x \) is \((R/T)\)-regular. Hence it remains to check that \( x \) is \( \text{gr}_m T \)-regular. By the proof of Lemma 5.3, we only need to show that for all \( s \geq 1 \),

\[
\mathfrak{m}^s T : x \subseteq \mathfrak{m}^{s-1} T.
\]

Take \( a \in \mathfrak{m}^s T : x \).

By change of coordinates, we can assume that \( x \) is one of the variables. Let \( n \) be the graded maximal ideal of \( S \) extended to \( R \). Then \( \mathfrak{m}^s = ((x) + n)^s = x \mathfrak{m}^{s-1} + n^s \), therefore

\[
xa \in \mathfrak{m}^s T = x \mathfrak{m}^{s-1} T + n^s T.
\]

So for some \( b \in \mathfrak{m}^{s-1} T \), \( x(a - b) \in n^s T \), namely

\[
a - b \in n^s T : x = n^s T.
\]

Therefore \( a \in \mathfrak{m}^{s-1} T + n^s T = \mathfrak{m}^{s-1} T \), as claimed.

That \( I = xI' + T \) is a Betti splitting follows from Theorem 5.1.

Regarding \( T \) as an ideal of \( S \), by Theorem 5.1, we also have

\[
\text{ld}_R I = \text{ld}_R T / xT = \text{ld}_R \left( T \otimes_k \frac{k[x]}{(x)} \right) = \text{ld}_S T,
\]

where the last equality holds because of [30, Lemma 2.3]. Hence \( \text{ld}_R I = \text{ld}_R T \), as desired. \(\square\)

The main result of this section is as follows.

**Theorem 5.7.** Let \( G \) be the graph obtained by adding to each vertex of a graph \( H \) at least one pendant. Then all the symbolic powers of the cover ideal \( J(G) \) of \( G \) are Koszul.

First, we need an auxiliary lemma. If \( m = x_1^{a_1} \cdots x_t^{a_t} \) is a monomial of \( R \), its support is defined by \( \text{supp}(m) = \{ x_i \mid \alpha_i \neq 0 \} \). For a set \( B \) of monomials in \( R \), set \( \text{supp} B = \bigcup_{m \in B} \text{supp}(m) \).

**Lemma 5.8.** Let \( I \) be a monomial ideal of \( S = k[x_1, \ldots, x_t] \). Let \( 1 \leq t \leq s \) be an integer. Assume that for every monomial \( m \in S \) with \( \text{supp}(m) \subseteq \{ x_{1, \ldots, x_t} \} \), the ideal \( (m) \cap I \) is Koszul. Denote \( R = S[y, z] \). Let \( m_1 = y^a f g \), where \( \alpha \geq 1 \) is an integer and \( f, g \) are monomials of \( S \) satisfying the following conditions:

(i) \( \text{supp}(g) \subseteq \{ x_1, \ldots, x_t \} \),

(ii) \( \text{supp} f \cap \left( \text{supp} G(I) \cup \{ x_1, \ldots, x_t \} \right) = \emptyset \).

Then for all monomials \( m \in R \) with \( \text{supp}(m) \subseteq \{ x_1, \ldots, x_t \} \) and all \( p, q \geq 0 \), the ideal \( (z, m_1)^p \cap (mz^q) \cap I \) is Koszul.

**Proof.** We prove by induction on \( p + q \). If \( p + q = 0 \), then \( p = q = 0 \). In this case, the conclusion holds true by the assumption.

Assume that \( p + q \geq 1 \). If \( p \leq q \), then we have

\[
(z, m_1)^p \cap (mz^q) \cap I = (mz^q) \cap I.
\]

Since \( mz^q \cap I = z^q((m) \cap I) \), we have \( (mz^q) \cap I \) is Koszul.

Assume that \( p > q \). Consider two cases.
Case 1: \( q = 0 \). We have
\[
(z, m_1)^p \cap (m) \cap I = (z, m_1)^p \cap ((m) \cap I)
\]
\[
= (z(z, m_1)^{p-1} + (m_1^{\ell})) \cap ((m) \cap I)
\]
\[
= zJ + L
\]
where \( J = (z, m_1)^{p-1} \cap (m) \cap I \) and \( L = (m_1^{\ell}) \cap (m) \cap I \).

Observe that \( J \) is Koszul by the induction hypothesis. From the assumptions,
\[
supp(lcm(y^a f)) \cap (supp(m) \cup supp(G(I))) = \emptyset,
\]
so
\[
L = (y^{pa} f^p g^p) \cap (m) \cap I = y^{pa} f^p ((g^p) \cap (m) \cap I)
\]
\[
= y^{pa} f^p (lcm(g^p, m) \cap I). 
\]

Since \( supp(lcm(g^p, m)) \subseteq \{x_1, \ldots, x_e\} \), the assumptions yields that \( lcm(g^p, m) \cap I \) is Koszul. Therefore \( L \) is Koszul.

The above arguments also give
\[
L \subseteq y^n \left( y^{(p-1)a} f^p g^p \cap (m) \cap I \right) \subseteq y \left( (m_1^{p-1}) \cap (m) \cap I \right) \subseteq yJ.
\]
Thus \((z, m_1)^p \cap (m) \cap I \) is Koszul by Corollary 5.6.

Case 2: \( q \geq 1 \). Then
\[
(z, m_1)^p \cap (mz^q) \cap I = z^q ((z, m_1)^{p-q} \cap (m) \cap I),
\]
which is Koszul by the induction hypothesis. The proof is complete. \( \square \)

Now we present the

**Proof of Theorem 5.7.** Assume that \( V(H) = \{x_1, \ldots, x_d\} \). Let \( R = k[x : x \in V(G)] \). In order to prove the theorem we prove the stronger statement that \( (m) \cap J(G)^{(n)} \) is Koszul for every monomial \( m \in R \) with \( supp(m) \subseteq \{x_1, \ldots, x_d\} \). Choosing \( m = 1 \), we get the desired conclusion.

Induct on \( d \).

**Step 1:** If \( d = 1 \), then \( G \) is a star with the edge set \( E(G) = \{x_1y_1, \ldots, x_1y_e\} \), where \( e \geq 1 \). In this case, \( J(G) = (x_1, y_1 \cdots y_e) \) and \( J(G)^{(n)} = (x_1, y_1 \cdots y_e)^n \).

Assume that \( m = x_1^n \). Since \( \quad (m) \cap J(G)^{(n)} = (x_1^n) \cap (x_1, y_1 \cdots y_e)^n = x_1^n (x_1, y_1 \cdots y_e)^{\max(0,n-p)} \)

it suffices to prove that \((x_1, y_1 \cdots y_e)^n \) is Koszul for all \( n \geq 0 \).

If \( n = 0 \), this is clear. Assume that \( n \geq 1 \) and the statement holds for \( n - 1 \). We write \( x = x_1, h = y_1 \cdots y_e \). Then
\[
(x_1, y_1 \cdots y_e)^n = (x, h)^n = x(x, h)^{n-1} + (h^n).
\]

By the induction hypothesis, \((x, h)^{n-1} \) is Koszul. Applying Corollary 5.6,
\[
\text{ld}_R(x, h)^n = \text{ld}_R(h^n) = 0.
\]

**Step 2:** Assume that \( d \geq 2 \). Let \( y_1, \ldots, y_e \) be the vertices of the pendants of \( G \) which are adjacent to \( x_d \), where \( e \geq 1 \). Let \( H' = H \setminus \{x_d\} \) and \( G' = G \setminus \{x_d, y_1, \ldots, y_e\} \). Then \( V(H') = \{x_1, \ldots, x_{d-1}\} \) and \( G' \) is obtained by adding to each vertex of \( H' \) at least one pendant.

Let \( S \) be the polynomial ring with variables being the vertices of \( G \setminus \{x_d, y_1\} \). Denote \( I = J(G')^{(n)} \). By the induction hypothesis and Lemma 2.2, \((m') \cap I \) is Koszul for every monomial \( m' \in S \) with \( supp(m') \subseteq \{x_1, \ldots, x_{d-1}\} \).

Denote \( y = y_1, z = x_d \), then \( R = S[y, z] \).
Let $m_1 = \prod_{x \in N_G(z)} x$, $f = y_2 \cdots y_e$, $g = \prod_{x \in N_H(z)} x$. Then

(i) $m_1 = yfg$,

(ii) $\text{supp}(g) \subseteq \{x_1, \ldots, x_{d-1}\}$,

(iii) $\text{supp}(f) \cap (\text{supp } G(I) \cup \{x_1, \ldots, x_{d-1}\}) = \emptyset$.

Moreover by Fact 2.5,

$$J(G^{(n)}) = (z, y_1)^n \cap \cdots \cap (z, y_e)^n \cap \bigcap_{x \in N_H(z)} (z, x)^n \cap J(G'^{(n)})$$

$$= (z, y_1 \cdots y_e g)^n \cap I = (z, m_1)^n \cap I.$$

The second equality holds by observing that $(z, y_1 \cdots y_e g)$ is a complete intersection, or by direct inspection.

Take any monomial $m \in R$ with $\text{supp}(m) \subseteq \{x_1, \ldots, x_{d-1}, z\}$. We can write $m = m'z^p$ where $\text{supp}(m') \subseteq \{x_1, \ldots, x_{d-1}\}$. Hence

$$(m) \cap J(G^{(n)}) = (z, m_1)^n \cap (m) \cap I = (z, m_1)^n \cap (m'z^p) \cap I.$$

By Lemma 5.8, the last ideal is Koszul. This finishes the induction on $d$ and the proof. \hfill \Box

The corona $\text{cor}(G)$ of a graph $G$ is the graph obtained from $G$ by adding a pendant at each vertex of $G$. More generally, the generalized corona $\text{cor}(G, s)$ is the graph obtained from $G$ by adding $s \geq 1$ pendant edges to each vertex of $G$ (see Figure 1).

By Alexander duality [15, Chapter 8], we know that the edge ideal $I(G)$ is Koszul (having a linear resolution) if and only if $J(G)$ is sequentially Cohen-Macaulay (respectively, Cohen-Macaulay). Combining this with work of Villarreal [41, Section 4], Francisco and Ha [9, Corollary 3.6], we know that $J(\text{cor}(G))$ has a linear resolution. We generalize this for all symbolic powers of $J(\text{cor}(G))$ as follows.

**Corollary 5.9.** Let $G$ be a simple graph. Then all the symbolic powers of the cover ideal $J(\text{cor}(G))$ have linear resolutions.

We introduce some more notation. Let $V(G) = \{x_1, \ldots, x_r\}$, where it is harmless to assume that $r \geq 1$. Let $y_1, \ldots, y_r$ be the new vertices in $V(\text{cor}(G))$, where $y_i$ is only adjacent to $x_i$ for all $i = 1, \ldots, r$.

**Convention 5.10.** We denote the coordinates of the ambient $\mathbb{R}^{2r}$ containing $SP(J(\text{cor}(G)))$ by $x_1, \ldots, x_r, y_1, \ldots, y_r$ instead of $x_1, \ldots, x_r, x_{r+1}, \ldots, x_{2r}$, thus $y_i = x_{r+i}$ for $i = 1, \ldots, r$.

The proof of Corollary 5.9 depends on the following lemma (where Convention 5.10 is in force).

**Lemma 5.11.** Denote $J = J(\text{cor}(G))$. Then for any vertex $\alpha \in \mathbb{R}^{2r}$ of $SP(J)$, up to a relabeling of the variables, there exist integers $0 \leq p \leq q \leq r$ such that $\alpha$ is a solution of the following system:

$$\begin{align*}
x_1 = \cdots = x_p = y_{p+1} = \cdots = y_q = 0, \\
y_1 = \cdots = y_p = x_{p+1} = \cdots = x_q = 1, \\
x_j = y_j = 1/2, & \quad \text{if } q + 1 \leq j \leq r.
\end{align*}$$

In particular, $|\alpha| = r$. 


Proof. For the first assertion, note that by Lemma 4.5, \( \alpha_i \in \{0, 1, 1/2\} \) for all \( i = 1, \ldots, 2r \). Denote \( S_0 = \{x_1 : \alpha_1 = 0\}, S_1 = \{x_1 : \alpha_1 = 1\}, S_{1/2} = \{x_1 : \alpha_1 = 1/2\} \).

By Lemma 4.5, we also have \( S_0 \) is an independent set of \( \text{cor}(G) \).

Without loss of generality, we can assume that \( S_0 = \{x_1, \ldots, x_p, y_{p+1}, \ldots, y_q\} \) for some \( 0 \leq p \leq q \leq r \) (recall Convention 5.10). We have to show that \( S_1 = \{y_1, \ldots, y_q, x_{p+1}, \ldots, x_q\} \).

By Lemma 4.5, \( \{y_1, \ldots, y_q, x_{p+1}, \ldots, x_q\} \subseteq N(S_0) = S_1 \). Clearly \( y_{q+1}, \ldots, y_r \notin S_1 \) since none of them belongs to \( N(S_0) \). Hence it remains to show that \( x_i \notin S_1 \) for \( q + 1 \leq i \leq r \).

By the definition of \( S_0 \), \( y_i \notin S_0 \). Now \( y_i \) is a leaf of \( \text{cor}(G) \) and \( N(y_i) = \{x_i\} \), so by Lemma 4.5, \( x_i \notin S_1 \), as desired.

The second assertion now follows from accounting. The proof is concluded. \( \square \)

Proof of Corollary 5.9. It is harmless to assume that \( r = |V(G)| \geq 1 \), as mentioned above. By Theorem 5.7 it suffices to show that \( J^{(n)} = J(\text{cor}(G))^{(n)} \) generated by monomials of degree \( rn \).

Step 1: Take any vertex \( v \in \mathbb{R}^{2r} \) of \( SP(J) \). By Lemma 5.11, it follows that \( |v| = r \); in particular \( \delta(J) = r \).

Step 2: Let \( x^\alpha \) be a minimal generator of \( J^{(n)} \). Since \( \alpha \in SP_n(J) \), we get
\[
\frac{1}{n} \alpha \in SP(J).
\]
Together with Step 1, it follows that
\[
\frac{1}{n} |\alpha| \geq \min\{|v| \mid v \text{ is a vertex of } SP(J)\} = r,
\]
namely \( |\alpha| \geq nr \).

On the other hand, by Lemma 4.3,
\[
|\alpha| \leq \omega(J^{(n)}) \leq \delta(J)n = rn.
\]
Thus \( |\alpha| = nr \), as required. \( \square \)

Remark 5.12. A graph \( G \) which contains no induced cycle of length at least 4 is called a chordal graph. We say that \( G \) is a star graph based on a complete graph \( K_m \) if \( G \) is connected and \( V(G) = \{1, \ldots, m, m+1, \ldots, m+g\} \) for some \( g \geq 0 \) such that:

1. the complete graph on \( \{1, \ldots, m\} \) is a subgraph of \( G \), and,
2. there is no edge in \( G \) connecting \( i \) and \( j \) for all \( m+1 \leq i < j \leq m + g \).

Any star graph based on a complete graph is chordal.

Let \( G \) be a chordal graph. Francisco and Van Tuyl [10, Proof of Theorem 3.2] showed that for such a \( G \), \( J(G) \) is Koszul\(^1\). In [16], Herzog, Hibi and Ohsugi conjectured that all the powers of \( J(G) \) are Koszul. Furthermore, in ibid., Theorem 3.3, they confirmed this in the case \( G \) is a star graph based on a complete graph \( K_m \). Hence it is natural to ask: If \( G \) is star graph based on a complete graph, is it true that \( J(G)^{(n)} \) Koszul for all \( n \geq 1 \)?

The answer is “No!” Here is a counterexample. Consider the graph \( G_2 \) in Figure 4. It is the complete graph on the vertices \( \{a, b, c, d\} \) with one edge removed. The corresponding cover ideal is
\[
J = J(G_2) = \langle ab, abd, acd \rangle.
\]

\(^1\)This result can be proved quickly using Corollary 5.6.
Since $G_2$ is a star graph based on $K_2$, $J$ and all of its ordinary powers are Koszul by [16, Theorem 3.3]. But $J^{(n)}$ is not Koszul for all $n \geq 2$ by [5, Page 186].

It is natural to ask

**Question 5.13.** Classify all star graphs based on a complete graph $G$ such that all the symbolic powers of $J(G)$ are Koszul.

Observe that a subset $\tau \subseteq V(G)$ is a minimal vertex cover of $G$ if and only if $V(G) \setminus \tau$ is a maximal independent set of $G$.

Let $K_m$ be a complete graph with $m$ vertices and $G = \text{cor}(K_m, s)$ where $m \geq 3$ and $s \geq 2$. In the rest of the paper we show that both $\omega(J(G)^{(n)})$ and $\text{reg}(J(G)^{(n)})$ are not necessarily asymptotic linear functions in $n$.

**Lemma 5.14.** For any $m \geq 3$ and $s \geq 2$, we have:

1. $\omega(J(\text{cor}(K_m, s))) = m + s - 1$.
2. $\delta(J(\text{cor}(K_m, s))) = \frac{1}{2}m(s + 1)$.

**Proof.** Let $G = \text{cor}(K_m, s)$. Then $G$ has $r = m(s + 1)$ vertices and $ms$ leaves.

1. Let $S$ be a maximal independent set of $G$. Then either $S$ is the set of leaves of $G$, or $S$ consists of a vertex of $K_m$ and $(m - 1)s$ leaves which are incident with the remaining vertices of $K_m$. Thus, the cover number of $G$ is

$$m(s + 1) - (1 + (m - 1)s) = m + s - 1,$$

and thus $\omega(J(G)) = m + s - 1$.

2. Since $|V(G)| = m(s+1)$, by Theorem 4.6 we only need to prove the following:

Let $S$ be an independent set of $G$ such that $G \setminus N[S]$ has no bipartite components. Then $|N(S)| \leq |S|$, with equality happens when $S = \emptyset$.

We consider three cases:

**Case 1:** $S = \emptyset$. Then $N(S) = \emptyset$ and $|N(S)| - |S| = 0$.

**Case 2:** $S$ contains a vertex of $K_m$, say $v$. Then $G \setminus N[S]$ is either empty or totally disconnected, in which case it is bipartite. Since $G \setminus N[S]$ has no bipartite component, the first alternative happens. It follows that $S$ consists of $v$ and all the leaves not adjacent to it. Thus, $|S| = 1 + (m - 1)s \geq |N(S)| = s + m - 1$, since

$$1 + (m - 1)s - (s + m - 1) = (m - 2)(s - 1) > 0.$$
**Case 3:** \( S \) contains only leaves of \( G \). Let \( x_1, \ldots, x_m \) be vertices of \( K_m \). Then \( N(S) \) consists only of vertices of \( K_m \), say \( x_1, \ldots, x_t \) for \( 1 \leq t \leq m \). Each \( x_i \) requires at least a leaf adjacent to it, so clearly \( |S| \geq t = |N(S)| \).

The proof is concluded. \( \square \)

Finally, we present a family of counterexamples to Question 1.2.

**Theorem 5.15.** Let \( G = cor(K_m, s) \) where \( m \geq 3 \) and \( s \geq 2 \). Let \( J = J(G) \) be its cover ideal. Then for all \( n \geq 0 \),

1. \( \text{reg}(J^{2n}) = \omega(J^{2n}) = m(s+1)n; \)
2. \( \text{reg}(J^{2n+1}) = \omega(J^{2n+1}) = m(s+1)n + m + s - 1. \)

In particular, for all \( n \),

\[ \text{reg}(J^n) = \omega(J^n) = (m + s - 1)n + (m - 2)(s - 1) \left\lfloor \frac{n}{2} \right\rfloor, \]

which is not an eventually linear function of \( n \).

**Proof.** By Theorem 5.7, \( J^n \) is Koszul for all \( n \geq 1 \). Hence by Lemma 2.1, \( \text{reg}(J^n) = \omega(J^n) \) for all \( n \).

Note that by Lemma 5.14, \( \delta(J) = \delta(J(G)) = m(s+1)/2 \), namely half the number of vertices of \( G \). Hence by Theorem 4.13, for all \( n \geq 0 \)

\[ \omega(J^{2n}) = 2n\delta(J), \]

\[ \omega(J^{2n+1}) = 2n\delta(J) + \omega(J). \]

From Lemma 5.14(1), \( \omega(J) = m + s - 1 \), so the desired formulas follow. \( \square \)

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