Abstract—Age of Information (AoI) has proven to be a useful metric in networked systems where timely information updates are of importance. In the literature, minimizing “average age” has received considerable attention. However, various applications pose stricter age requirements on the updates which demand knowledge of the AoI distribution. Furthermore, the analysis of AoI distribution in a multi-hop setting, which is important for the study of Wireless Networked Control Systems (WNCS), has not been addressed before. Toward this end, we study the distribution of AoI in a WNCS with two hops and devise a problem of minimizing the tail of the AoI distribution with respect to the frequency of generating information updates, i.e., the sampling rate of a process, under first-come-first-serve (FCFS) queuing discipline. We argue that computing an exact expression for the AoI distribution may not always be feasible; therefore, we opt for computing upper bounds on the tail of the AoI distribution. Using these upper bounds we formulate Upper Bound Minimization Problems (UBMP), namely, Chernoff-UBMP and $\alpha$-relaxed Upper Bound Minimization Problem ($\alpha$-UBMP), where $\alpha > 1$ is an approximation factor, and solve them to obtain “good” heuristic rate solutions. We demonstrate the efficacy of our approach by solving the proposed UBMPs for three service-time distributions: geometric, exponential, and Erlang. Simulation results show that the rate solutions obtained are near optimal for minimizing the tail of the AoI distribution for the considered distributions.

I. INTRODUCTION

In the recent past, there has been an ever increasing interest in studying Wireless Networked Control Systems (WNCS) that support time-critical-control applications which include, among many others, autonomous vehicle systems, automation of manufacturing processes, smart grids, Internet-of-Things (IoT), sensor networks and augmented reality. A basic building block in WNCS is depicted in Figure 1. A sensor samples a plant/process of interest and transmits the status update or packets over a wireless channel (link 1) to a controller. The controller computes a control input using the received status update and transmits it to an actuator, using another communication channel (link 2). A status update that is received at the controller after a certain duration of its generation time may become stale, and the control decision taken based on this stale sample may result in untimely actuation affecting the performance of a time-critical-control application in a WNCS. Similarly, the same effect could result from a control decision (based on a fresh status update) reaching the actuator after a delay deadline. In this respect, the traditional goal of maximizing throughput becomes less relevant as freshness of the status updates not only depends on queuing and transmission delays in the network, but also on the frequency of generating updates at the source.

Age of Information (AoI), proposed in [1], has emerged as a relevant performance metric in quantifying the freshness of the status updates at a destination. It is defined as the time elapsed since the generation of the latest status update received at the destination. AoI accounts for the frequency of generation of updates by the source, since it linearly increases with time until a status update with latest generation time is received at the destination. Whenever such an update is received, AoI resets to the system delay of the update indicating its age. Motivated by the fact that having access to fresher status updates improves the control performance in a WNCS, we pursue a problem of optimizing the end-to-end AoI in a two-hop network. In particular, we study a queuing system with two queues in tandem, under First-Come-First-Serve (FCFS) scheduling. We allow the service times to be heterogeneous, i.e., servers at the first queue and the second queue may have different service-time distributions. Motivated by the fact that sensors in practice are typically configured to generate samples periodically, we consider periodic arrivals at the first queue

Fig. 1: A networked control system with a remote controller.

1 A preliminary version of this work considering single-hop scenario appeared in [2].

$^1$One may consider Last-Come-First-Serve (LCFS) queuing discipline as it was shown to minimize AoI process (in stochastic sense) for arbitrary arrival sequence $^2$. However, we believe analyzing the AoI violation probability under FCFS is a first and important step.
with input rate $R$. Assuming that the processing time at the controller is negligible, we aim to compute $R$ that minimizes the end-to-end AoI violation probability, i.e., the probability that AoI at the service end of the second queue violates a given age limit $d$. The AoI violation probability metric represents, for instance, a QoS guarantee that is required at the actuator such that state of the plant in WNCS is within a safety boundary.

In the recent past, several research works addressed the problem of input rate selection for optimizing AoI in a system. However, as we explain in Section II, these works either consider a single-hop system or memoryless arrivals or some form of “average age” function. In contrast, we study AoI violation probability in a two-hop queuing system with heterogeneous servers. As we will see in a while, computing an exact expression for the end-to-end AoI violation probability in a two-hop network is not straightforward. Therefore, we resort to working with tractable upper bounds which facilitate the computation of “good” heuristic solutions. In particular, we first compute the upper bounds for the single-hop case, i.e., the D/G/1 queue, due to its relevance in applications where both controller and actuator are collocated. We formulate the Upper Bound Minimization problems (UBMP) for computing the heuristic rate solutions. We then extend the results for two-hop and N-hop tandem queuing systems using max-plus convolution for the service processes.

The main contributions of this work are summarized below:

- We characterize, for the first time, the probability that AoI violates a given age limit $d$ for a network having arbitrary topology and any number of nodes between the source and the destination, given that the packets are input to the network by the source at a constant rate $R$.
- We formulate the AoI violation probability minimization problem $P$, and show that it is equivalent to minimizing the violation probability of the departure instant of a certain packet over the rate region $\{R, \mu\}$, where $\mu$ is the service capacity of the network.
- Using the above characterization, we first propose a UBMP for the single-hop scenario, i.e., the D/G/1 queue. Noting that the objective function in the UBMP can be intractable, we propose a Chernoff-UBMP, that has a closed-form objective, and an $\alpha$-relaxed UBMP the solution of which has $\alpha > 1$ approximation ratio with respect to UBMP.
- We extend the derived results and formulations for the two-hop queuing system and $N$-hop tandem queuing system, and present example computation of the expressions for the case of two-hop for geometric, exponential, and Erlang service-time distributions.
- We demonstrate the efficacy of the heuristic solutions provided by Chernoff-UBMP and $\alpha$-relaxed UBMP using simulation for different service-time distributions.

The rest of the paper is organized as follows. In Section II we present the related work. In Section III we present the problem formulation. Analysis on the AoI violation probability is presented in Section IV. The UBMP formulations for single-hop, two-hop, and $N$-hop scenarios are presented in Sections V and VI respectively. We present the computation of the upper bounds for different service-time distributions in Section VII. Numerical results are presented in Section VIII and we finally conclude in Section IX.

II. RELATED WORK

Several works in the AoI literature have focused on analyzing and providing expressions for average AoI statistics in different queuing systems, e.g., see [3–5]. In contrast, the authors in [9], [10] provided expressions for the distribution of AoI. However, for the case of periodic arrivals, closed-form expressions are available only for single-hop scenario and for exponential service times in [9], and for the case of no queue in [10]. Next, we summarize works that consider optimizing AoI under different system settings.

In [13], the authors have addressed the problem of computing the optimal input rate to minimize the time-average age for M/M/1, M/D/1 and D/M/1 queueing systems. This problem was addressed for M/M/1 with multiple sources in [14]. The authors in [15] studied the M/M/1/1 and M/M/1/2 systems, and computed the average AoI and the distribution of the peak AoI. In [16], the authors have studied the problem of computing optimal arrival rate for minimizing average peak age for a multi-class M/G/1 system and proposed an approximate solution. In contrast to the above works, the generate-at-will source model was studied in [17], [18] for a single-source-single-server system. While the authors in [17] solved for optimal-waiting times to minimize the average AoI, the authors in [18] solved the problem for any non-decreasing function of AoI. Given the arrival process, optimal scheduling policies are studied in [3] for multi-hop networks.

Optimizing AoI was also extensively studied for the systems with energy-harvesting source, e.g., see [17], [19], [20]. In the context of a cloud gaming system the authors in [21] used the D/G/1 system model to study the effect of freshness on video frame rendering to the client. Specifically, they have analyzed the average age by considering the aspect of missing frames. In contrast to above works, with motivations from the sensor-controller-actuator system in WNCS we study the AoI violation probability minimization in a two-hop queuing system with periodic arrivals.

III. SYSTEM MODEL AND PROBLEM STATEMENT

Motivated by the sensor-controller-actuator communicating over wireless channels, we study a two-hop queuing system, shown in Figure 2 under FCFS scheduling. The source generates packets (status updates) at a constant rate $R$. Thus, $R$ models the sampling rate of a process under observation. Let $T = \frac{1}{\lambda}$ denote the inter-arrival time between any two packets. We index the nodes by $k \in \{1, 2\}$, and the packets by $n \in \{0, 1, 2, \ldots\}$. Let $A_k(n, R)$ denote the arrival instant of packet $n$ and $D_k(n, R)$ the corresponding departure instant at node $k$. For notational simplicity, we use

\[ P \begin{cases} 1 & \text{if } T \leq d \\
\end{cases} 
\]
Fig. 2: Model of the two-hop network.

\[ A(n, R) = A_1(n, R) \] and \( D(n, R) = D_2(n, R) \) to denote the arrivals and departures of the system, respectively. Also, we have \( A_2(n, R) = D_1(n, R) \). The arrival time of packet \( n \) to the system is given by \( A(n, R) \). The service time for packet \( n \) at node \( k \) is given by a random variable \( X^k_n \). For \( k \in \{1, 2\} \), we assume \( X^k_n \) are i.i.d., for all \( n \), with mean service rate \( \mu_k = \frac{1}{\mathbb{E}[X^k_n]} > 0 \). Also, we assume \( X^1_n \) and \( X^2_n \) are independent, for all \( n \), but may have non-identical distributions, i.e., the servers could be heterogeneous.

We define \( \mu \triangleq \min(\mu_1, \mu_2) \). Later, in Section VI.C, we show how the results can be extended to \( N \)-hop tandem queuing network.

At the destination, we are interested in maintaining timely state information of the process. We are thus interested in the AoI metric, denoted by \( \Delta(t, R) \), which is defined as:

\[ \Delta(t, R) \triangleq t - \max\{A(n, R) : D(n, R) \leq t\}. \tag{1} \]

For a given age limit requirement \( d > 0 \), in the following we study the distribution of AoI by characterizing its violation probability, i.e., \( \mathbb{P}(\Delta(t, R) > d) \), both in the transient and the steady states of the system. Given the age limit \( d \), we are interested in solving the following problem \( \mathcal{P} \):

\[ \min_{R} \lim_{t \to \infty} \mathbb{P}(\Delta(t, R) > d). \]

Let \( R^*(d) \) denote an optimal rate solution for \( \mathcal{P} \).

Henceforth, we drop \( R \) from the notation when it is obvious from the context, for the sake of notation simplicity. For \( k \in \{1, 2\} \), the moment generating function of \( X^k_n \) is given by \( M_k(s) = \mathbb{E}[e^{sX^k_n}] \).

We now state the Chernoff bound, which will be used extensively to formulate the upper bound minimization problems in the sequel.

**Definition 1.** Assuming that the moment generating function of a random variable \( Y \) exists, the Chernoff bound for its distribution is given by

\[ \mathbb{P}(Y > y) \leq \min_{s > 0} e^{-sy} \mathbb{E}[e^{sy}]. \]

Note that the upper bounds derived using the Chernoff bound involves minimization over the parameter \( s \). We shall see that, for the two-hop network, these bounds attain finite values only when there exists \( s > 0 \) such that \( \max(M_1(s), M_2(s)) < e^{s/R} \). To this end, we formulate the minimization problems over the set \( \mathcal{S} \subseteq \mathbb{R}^+ \) which characterizes \( s \) values for which \( \max(M_1(s), M_2(s)) < e^{s/R}, \) i.e.,

\[ \mathcal{S} \triangleq \{ s > 0 : \max(M_1(s), M_2(s)) < e^{s/R} \}. \tag{2} \]

We assume that \( \mathcal{S} \) is non-empty. In the following lemma we show that this assumption is in fact a sufficient condition for the stability of the system.

**Lemma 1.** If there exists \( s > 0 \) such that

\[ \max(M_1(s), M_2(s)) < e^{s/R}, \]

then the queues are stable.

**Proof.** Recall that the queues are stable if \( \min(\mu_1, \mu_2) > R \). Consider the case \( M_1(s) < e^{s/R} \), which implies

\[ \mathbb{E}[e^{sX^1_n}] < e^{s/R} \Rightarrow \mathbb{E}[e^{sX^1_n}] < e^{s/R} \Rightarrow \mu_1 > R, \]

for any \( s > 0 \). In the second step above we have used Jensen’s inequality. Similarly, if \( M_2(s) < e^{s/R} \), then \( \mu_2 > R \). Therefore, for any \( s > 0 \), \( \max(M_1(s), M_2(s)) < e^{s/R} \) implies \( \min(\mu_1, \mu_2) > R \), and the lemma follows. \( \square \)

We define

\[ \beta_k(s) \triangleq \frac{M_k(s)}{e^{s/R}}, \quad k \in \{1, 2\}. \tag{3} \]

By definition, for all \( s \in \mathcal{S} \), \( \beta_k(s) < 1 \).

**IV. AoI Violation Probability Analysis**

In this section, we study the properties of the distribution of AoI – the results derived are valid for arbitrary network topology with any number of nodes between the source and the destination, given that the packets are input to the network by the source at a constant rate \( R \).

We start by investigating structural characteristics of the stochastic behaviour of AoI. Toward this end, we use the max-plus representation of Reich’s equation to model the evolution of the queues. For any realization of the service times at node \( k \), the relation between \( D_k(n, R), A_k(n, R) \) and \( \{X^k_n\} \), is given by \( \mathbb{E}[e^{sX^k_n}] \).

\[ D_k(n, R) = \max_{0 \leq v \leq n} \left\{ A_k(n - v, R) + \sum_{i=0}^{v} X^k_{n-i} \right\}. \tag{4} \]

Consider the definition in \( \{1\} \), for \( \Delta(t, R) \) not to exceed the age limit \( d \), the latest departure at \( t \) must have arrived no earlier than \( t - d \). Therefore, to study the distribution of \( \Delta(t, R) \), we tag the packet arriving on or immediately after \( t - d \) and use it to characterize this process. Given rate \( R \), let \( \hat{n}_R \) denote the first arrival on or immediately after time \( t - d \), given by

\[ \hat{n}_R \triangleq \lfloor R(t - d) \rfloor. \tag{5} \]

The tagged packet \( \hat{n}_R \) plays a key role in characterizing the violation probability as we will show next.

In the following lemma we present a key insight regarding the transient characterization of the AoI violation probability.

**Lemma 2.** Given the input arrival rate \( R \), age limit \( d \), and \( t < \infty \), if there exists \( n \) such that \( t - d \leq \frac{n}{R} < t \), then

\[ \mathbb{P}(\Delta(t, R) > d) = \mathbb{P}(D(\hat{n}_R) > t), \]

otherwise, \( \mathbb{P}(\Delta(t, R) > d) = 1 \).

**Proof.** Let \( n^*_R \) be the latest packet departure at \( t \), i.e., \( n^*_R = \arg \max_n \{D(n, R) \leq t\} \). Thus, \( \Delta(t, R) = t - A(n^*_R) \).

\( \hat{n}_R \) is a function of \( t - d \) as well. We omit \( t - d \) from the notation here for ease of exposition.
Therefore, to ensure the equivalence holds and the result is proven.

\[ A(\hat{n}_R) = \frac{[R(t-d)]}{R} \leq \frac{R(t-d) + 1}{R} < t. \]

Furthermore, since \( t - d \leq A(\hat{n}_R) \) for any \( t \) by definition, the claim holds at least for \( \hat{n}_R \), for \( R > \frac{1}{d} \). To prove the claim for \( R = \frac{1}{d} \), we consider

\[ t - d \leq \frac{n}{R} < t \]

\[ \Leftrightarrow \frac{n}{R} < t - d + \frac{n}{R} \]

\[ \Leftrightarrow n < Rt \leq n + 1. \]

Note that for any \( R \) and \( t \) there always exists an \( n \) such that the last inequality above holds. Therefore, the claim is true and Case 1 follows from Lemma 2 by letting \( t \) go to infinity.

**Case 2** (\( R < \frac{1}{d} \)): In this case, the existence of \( n \) such that \( t - d \leq \frac{n}{R} < t \) depends on \( t \). Again, using Lemma 2 for all \( t \) where this condition is satisfied we have \( \mathbb{P}(\Delta(t, R) > d) = \mathbb{P}(D(\hat{n}_R) > t) \). For all other values of \( t \), we have \( \mathbb{P}(\Delta(t, R) > d) = 1 \). This implies that as \( t \) goes to infinity the violation probability oscillates between \( \mathbb{P}(D(\hat{n}_R) > t) \) and 1. Thus, we obtain the limit supremum and the limit infimum.

Intuitively, given \( R \), the support of the steady state AoI distribution should be \( \left[ \frac{1}{R}, \infty \right) \), because AoI cannot be less than \( \frac{1}{R} \) when the samples are generated at rate \( R \). Not only Theorem 1 asserts this intuitive reasoning, but also characterizes the limit infimum and limit supremum for the region \( d < \frac{1}{R} \), where the AoI violation probability does not exist. Therefore, to ensure the existence of the AoI violation probability we consider the feasible rate region \( \left[ \frac{1}{R}, \mu \right) \), where \( \mu = \min(\mu_1, \mu_2) \), and \( R < \mu \) ensures queue stability. In light of this, and using (7) from Theorem 1 we formulate an equivalent problem \( \mathcal{P} \) as follows:

\[ \min_{R \to \infty} \lim_{t \to \infty} \mathbb{P}(D(\hat{n}_R) > t), \]

s.t. \( \frac{1}{d} \leq R < \mu \). \hspace{1cm} (8)

**Remark 1:** The results in Lemma 2 and Theorem 1 are valid for arbitrary network topology with any number of nodes between the source and the destination, given that the packets are input to the network by the source at a constant rate \( R \). For arbitrary network topology, one can formulate problem \( \mathcal{P} \) given in (8) with the following constraints on \( R \): 1) \( R \geq \frac{1}{d} \), and 2) \( R \) belongs to the rate region in which the network is stable.

Next, we present our solution approach for solving \( \mathcal{P} \) for a single-hop case and then show how the approach can be extended for the two-hop system in Section VII.

**V. SINGLE-HOP SCENARIO**

In this section we solve \( \mathcal{P} \) by assuming that \( X^*_2 = 0 \) for all \( n \). This implies that \( D(n) = D_1(n) \), \( \mu_2 = \infty \) and the system is equivalent to the D/GI/1 system. Our motivation for presenting the single-hop case is because of its importance in solving the two-hop case, and also due to its relevance to practical scenarios, where only estimation of the processes...
is required, or both controller and actuator are collocated. In order to find a solution for $\tilde{P}$, we must first evaluate the probability $P\{D(\hat{n}_R) > t\}$, where $D(n)$ is given by (4). Note that $D(n)$ is random, since the service process $\{X^n_t, n \geq 0\}$ is random, and is given in terms of the maximum of $n + 1$ random variables. Hence, obtaining an exact expression is tedious. Therefore, we opt for a more tractable approach by using probabilistic inequalities to obtain bounds on the distribution of $D(\hat{n}_R)$. Consequently, we propose the Upper Bound Minimization Problem (UBMP) and its more computationally tractable counterparts $\alpha$-UBMP and Chernoff-UBMP to obtain near optimal heuristic solutions for $\tilde{P}$.

A. A Bound for the Distribution of $D$

As mentioned earlier, the evaluation of the distribution function of $D(n)$ requires the computation of the distribution of the maximum of random variables. Fortunately, there are several approaches that have been used in the literature to estimate this probability. One such approach approximates the probability of the maximum by the maximum probability, i.e., $P\{\max Y_i > y\} \approx \max P\{Y_i > y\}$. However, this approximation is not always accurate and in some cases may result in very large deviation from the actual distribution. Hence, it cannot be used when reliability of the solution must be well defined as it is the case here. An alternative approach is to use extreme value theorem. However, the obtained extreme value distributions are not always tractable. A more promising approach is to use Boole’s inequality, commonly known as the “union bound,” where the probability of a union of events is bounded by the sum of their probabilities. The bound obtained in our case is not only tractable, but also provides good heuristic solutions for $\tilde{P}$. In the following lemma, we present this upper bound for the distribution function $\lim_{t \to \infty} P\{D(\hat{n}_R) > t\}$.

Lemma 3. Given $d$, we have

$$\lim_{t \to \infty} P\{D(\hat{n}_R) > t\} \leq \sum_{v=0}^{\infty} \Phi(v, R),$$

where

$$\Phi(v, R) \equiv P\{\sum_{i=0}^{v} X^n_1 > d + \frac{v - 1}{R}\}. \quad (9)$$

Proof. Using (4), we have

$$P\{D(\hat{n}_R) > t\} = P\left\{\max_{0 \leq v \leq \hat{n}_R} \left(A(\hat{n}_R - v) + \sum_{i=0}^{v} X^n_{\hat{n}_R - i}\right) > t\right\}$$

$$\leq \hat{n}_R \sum_{v=0}^{\hat{n}_R} P\left\{\sum_{i=0}^{v} X^n_{\hat{n}_R - i} > t - \frac{\hat{n}_R - v}{R}\right\}$$

$$\leq \hat{n}_R \sum_{v=0}^{\hat{n}_R} P\left\{\sum_{i=0}^{v} X^n_{\hat{n}_R - i} > t - R(t - d) + 1 - \frac{v}{R}\right\}$$

$$= \sum_{v=0}^{\hat{n}_R} P\left\{\sum_{i=0}^{v} X^n_1 > d + \frac{v - 1}{R}\right\}.$$

In step 2 above we have applied the union bound, and used $\hat{n}_R = \lfloor R(t - d) \rfloor \leq R(t - d) + 1$ in step 3. The result follows by noting that $\hat{n}_R$ goes to infinity as $t$ goes to infinity.

B. UBMP Formulations

Using (8), Lemma 3 and $\mu_2 = \infty$, we obtain the following UBMP problem.

$$\min_{R} \sum_{v=0}^{\infty} \Phi(v, R) \quad \text{s.t.} \quad \frac{1}{d} \leq R < \mu_1. \quad (10)$$

It is worth noting that the function $\Phi(0, R)$ is non-increasing in $R$ while the functions $\{\Phi(v, R) : v > 1\}$ are non-decreasing in $R$. This indicates that, for any service time distribution, the objective function of UBMP will potentially have at least one stationary point for $R$ in the interval $[\frac{1}{d}, \mu_1]$.

A shortcoming of UBMP is that its objective function is intractable, in general, as it involves computation of a sum of infinite terms and each term requires computation of the distribution of sum of service times. To this end, we formulate Chernoff-UBMP obtained by using Chernoff bound for $\Phi(v, R)$ in Lemma 3.\n
1) Chernoff-UBMP: Since $X^n_1$ are i.i.d, the Chernoff bound for $\Phi(v, R)$, defined in (9), is given by

$$\Phi(v, R) \leq \min_{s \in S} e^{-s(d + \frac{v}{R})} \mathbb{E}[e^{\sum_{i=0}^{v} X^n_1}]$$

$$= \min_{s \in S} e^{-s(d + \frac{v}{R})} M_1^1(s)$$

$$= \min_{s \in S} e^{-s(d + \frac{v}{R})} M_1(s) \beta^v(s), \quad (11)$$

where $\beta(s)$ is defined in (8). Recall that, $\beta_1(s) < 1$ for all $s \in S$. Therefore, using (11) in the result of Lemma 3 we obtain

$$\sum_{v=0}^{\infty} \Phi(v, R) \leq \sum_{v=0}^{\infty} \min_{s \in S} e^{-s(d + \frac{v}{R})} M_1(s) \beta^v(s)$$

$$\leq \min_{s \in S} e^{-s(d + \frac{v}{R})} M_1(s) \sum_{v=0}^{\infty} \beta^v(s)$$

$$= \min_{s \in S} e^{-s(d + \frac{v}{R})} \frac{M_1(s)}{1 - \beta_1(s)}. \quad (12)$$

Even though the Chernoff bound relaxes the upper bound in Lemma 3, its objective function has a closed-form expression and can be computed numerically. The following theorem immediately follows from (12) and Lemma 3.

Theorem 2. Given $d$, an upper bound for the violation probability for a single hop is given by

$$\lim_{t \to \infty} P\{D(\hat{n}_R) > t\} \leq \min_{s \in S} \Psi_1(s, d, R),$$

where $\Psi_1(s, d, R)$ is defined in (12).

With a slight abuse in the usage, we refer to the bound given in Theorem 2 as Chernoff bound. In the following we formulate the Chernoff-UBMP for the single-hop scenario.

$$\min_{R} \min_{s \in S} \Psi_1(s, d, R) \quad \text{s.t.} \quad \frac{1}{d} \leq R < \mu_1. \quad (13)$$
Lemma 4. The function $\Psi_1(s, d, R)$ is strictly convex with respect to $\frac{1}{R}$.

Proof. Recall that $T = \frac{1}{R}$. We prove that $\frac{\partial^2 \Psi_1(s, d, T)}{\partial T^2} > 0$ for all $s \in S$. Let us define $f(T)$ as follows.

$$f(T) = \frac{e^{2sT}}{(e^{sT} - M_1(s))}.$$ 

Then, we rewrite $\Psi_1(s, d, T)$ as follows.

$$\Psi_1(s, d, T) = e^{-sd} [M_1(s)] \cdot f(T).$$

From the above equation we infer that it is sufficient to prove $\frac{\partial^2 f(T)}{\partial T^2} > 0$. Taking first derivative $f'(T) = \frac{\partial f(T)}{\partial T}$, we obtain

$$f'(T) = \frac{2se^{2sT}}{(e^{sT} - M_1(s))} - \frac{e^{2sT} 2se^{2sT}}{(e^{sT} - M_1(s))^2} = sf(T) \left[ 1 - \frac{M_1(s)}{e^{sT} - M_1(s)} \right].$$ 

(14)

Taking the second derivative $f''(T) = \frac{\partial^2 f(T)}{\partial T^2}$, we obtain

$$f''(T) = sf'(T) \left[ 1 - \frac{M_1(s)}{e^{sT} - M_1(s)} \right] + s^2 f(T) \left[ 1 - \frac{M_1(s)}{e^{sT} - M_1(s)} \right]^2 > 0.$$

In the second step above we have used (14). The last step follows by noting that $e^{sT} > M_1(s)$ for all $s \in S$, $M_1(s) > 0$ for all $s$, and $f(T) > 0$.

Lemma 5. For $s > 0$, the function $\Psi_1(s, d, R)$ is convex in $s$.

Proof. We have

$$\Psi_1(s, d, R) = e^{-s(d - \frac{1}{R})} \cdot \frac{M_1(s)}{(1 - \beta_1(s))}$$

$$= e^{-s(d - \frac{1}{R})} \cdot \sum_{v=0}^{\infty} M_1(s) \beta_1^v(s)$$

$$= \sum_{v=0}^{\infty} e^{-s(d + \frac{v+1}{R})} M_1^{v+1}(s)$$

$$= \sum_{v=0}^{\infty} \mathbb{E}[e^{-sX}]^{v+1},$$

where $\hat{X} = (d + \frac{v+1}{R})/(v+1) - X^1_1$. Recall that the sum of convex functions is a convex function. Therefore, from the last step above, we infer that $\Psi_1(s, d, R)$ is convex if $\mathbb{E}[e^{-sX}]$ is convex for $v \in \{0, 1, \ldots\}$. For $s > 0$, $e^{-sX}$ is convex in $s$ for any $v$ and any realization of $X^1_1$. Therefore, $\mathbb{E}[e^{-sX}]$ is convex, and since $e^{-x}$ is convex and increasing in $x$, we have that $\mathbb{E}[e^{-sX}]^{v+1}$ is convex. Hence the result is proven.

Both Lemmas 4 and 5 can be leveraged to efficiently solve (13). The heuristic solutions we obtain by solving the Chernoff-UBMP can be improved further for service distributions for which the distribution of a finite sum of service times can be computed exactly. Therefore, we next propose a relatively tight upper bound called $\alpha$-relaxed upper bound and formulate $\alpha$-UBMP.

2) $\alpha$-UBMP: In the upper bound provided in Lemma 3, we propose to compute first $K < \infty$ terms of the summation, and use Chernoff bound for the rest of the terms. In the following, we make this precise. We first present a bound on the summation starting from $K$.

Lemma 6. For any $K \geq 0$, we have

$$\sum_{v=K}^{\infty} \Phi(v, R) \leq \min_{s \in S} \Psi_1(s, d, R, K),$$

where

$$\Psi_1(s, d, R, K) = e^{-s(d - \frac{1}{R})} \cdot \frac{M_1(s) \beta_1^K(s)}{(1 - \beta_1(s))}.$$

Proof. The result follows by using the upper bound for $\Phi(v, R)$ given in (11) and repeating the steps in (12) for the summation over $v$ from $K$ to infinity.

For the single hop scenario we define $\alpha$ as follows.

$$\alpha = 1 + \frac{\min_{s \in S} \Psi_1(s, d, R, K)}{\sum_{v=0}^{K-1} \Phi(v, R)}.$$

Note that $\alpha$ depends on the value of $K$. Using Lemmas 3 and 6 we next state the $\alpha$-relaxed upper bound without proof.

Theorem 3. Given $d$, the $\alpha$-relaxed upper bound for the violation probability for a single hop is given by

$$\lim_{t \to \infty} \mathbb{P}\{D(\hat{n}_R) > t\} \leq \sum_{v=0}^{K-1} \Phi(v, R) + \min_{s \in S} \Psi_1(s, d, R, K).$$

Note that, by definition the $\alpha$-relaxed upper bound is at most $\alpha$ times worse than the upper bound $\sum_{v=0}^{\infty} \Phi(v, R)$. More precisely, the $\alpha$-relaxed upper bound has $\alpha$ approximation factor with respect to $\sum_{v=0}^{\infty} \Phi(v, R)$. To see this,

$$\sum_{v=0}^{K-1} \Phi(v, R) + \min_{s \in S} \Psi_1(s, d, R, K)$$

$$\leq \sum_{v=0}^{K-1} \Phi(v, R) \left( 1 + \frac{\min_{s \in S} \Psi_1(s, d, R, K)}{\sum_{v=0}^{K-1} \Phi(v, R)} \right)$$

$$\leq \alpha \sum_{v=0}^{\infty} \Phi(v, R).$$

Note that $\alpha > 1$, and it is easy to see that as $K$ increases, the value of $\alpha$ approaches 1 from above. In this work, we choose $K$ the largest value that is computationally tractable in numerical evaluations. Now, we formulate $\alpha$-UBMP as follows:

$$\min_{R} \sum_{v=0}^{K-1} \Phi(v, R) + \min_{s \in S} \Psi_1(s, d, R, K)$$

s.t. $$\frac{1}{d} \leq R < \mu_1.$$ (15)
VI. Extensions to Two-Hop and N-Hop Scenarios

In this section, we present Chernoff-UBMP and α-UBMP for the two-hop scenario and also present Chernoff-UBMP for N-hop tandem queuing network.

In the following we first focus on the two-hop scenario. Similar to the case of single-hop scenario, we use Reich’s equation and apply union bound to obtain an upper bound for the AoI violation probability which is presented in the following lemma.

Lemma 7. Given d, and \( \hat{n}_R \) as defined in (8), we have

\[
\lim_{t \to \infty} \mathbb{P}(D(\hat{n}_R) > t) \leq \lim_{\hat{n}_R \to \infty} \sum_{v_0=0}^{\hat{n}_R} \sum_{v_1=0}^{\hat{n}_R-v_0} \Phi(v_0, v_1, R),
\]

where

\[
\Phi(v_0, v_1, R) \triangleq \mathbb{P} \left\{ \sum_{i=0}^{v_0} X^i_2 + \sum_{i=0}^{v_1} X^i_1 > d + \frac{v_0 + v_1 - 1}{R} \right\}. \tag{16}
\]

Proof. The proof is given in Appendix A.

A. Chernoff-UBMP for Two-Hop Scenario

Theorem 4. For the two-hop network with deterministic arrivals, the violation probability is upper bounded as follows:

\[
\lim_{t \to \infty} \mathbb{P}(D(\hat{n}_R) > t) \leq \min_{s \in S} \Psi_2(s, d, R),
\]

where

\[
\Psi_2(s, d, R) = e^{-s(d-R)} \cdot \frac{M_1(s) \cdot M_2(s)}{(1-\beta_1(s))(1-\beta_2(s))}. \tag{17}
\]

Proof. We use the relation between departure times, arrival times and the service times given by (3) iteratively and apply union bound and Chernoff bound to obtain the result. The details of the proof are given in Appendix B.

The Chernoff-UBMP problem for the two-hop network is stated below.

\[
\begin{align*}
\min_{R} \min_{s \in S} & \quad \Psi_2(s, d, R), \\
\text{s.t.} & \quad \frac{1}{d} \leq R < \mu. \tag{18}
\end{align*}
\]

In the following lemmas we provide convexity properties of \( \Psi_2(s, d, R) \). Since the proofs of the lemmas are similar to that in the case of single-hop scenario (Lemmas 4 and 5), we omit them here.

Lemma 8. For the two-hop network with deterministic arrivals, given \( s \in S \) and \( d > 0 \), \( \Psi_2(s, d, R) \) is convex with respect to \( \frac{1}{R} \).

Lemma 9. For the two-hop network with deterministic arrivals, given \( s \in S \) and \( d > 0 \), \( \Psi_2(s, d, R) \) is convex with respect to \( s \).

B. α-UBMP for Two-Hop Scenario

In the following theorem we present the α-relaxed upper bound.

Theorem 5. For the two-hop network with deterministic arrivals, for any \( K \geq 1 \), the α-relaxed upper bounded is given by

\[
\sum_{v_0=0}^{K} \sum_{v_1=0}^{K} \Phi(v_0, v_1, R) + \min_{s \in S} \Psi(s, d, R, K).
\]

where

\[
\Psi(s, d, R, K) = e^{-s(d-R)} \cdot \frac{M_1(s) \cdot M_2(s) \cdot (\beta_1^K(s) + \beta_2^K(s) - \beta_1^K(s) \cdot \beta_2^K(s))}{(1-\beta_1(s))(1-\beta_2(s))}. \tag{19}
\]

Proof. The proof is given in Appendix C.

We note that the α-relaxed upper bound is computationally expensive when compared to that in the single-hop scenario because of the nested sum.

C. N-hop Scenario

For an N-hop tandem network we have \( k \in \{1, 2, \ldots, N\} \) and \( D(n) = D_N(n) \). For simplicity of presentation, in this section, we assume that \( X^i_k \) are identically distributed. Therefore, we have \( \mu = \mu_k \) for all \( k \), and \( M_k(s) = M_1(s) \) for all \( k \). We now define the set \( S \) as follows.

\[
S = \{ s > 0 : M_1(s) < e^{s/R} \}.
\]

Lemma 10. Given \( d \), and \( \hat{n}_R \) as defined in (3), we have

\[
\lim_{t \to \infty} \mathbb{P}(D(\hat{n}_R) > t) \leq \lim_{\hat{n}_R \to \infty} \sum_{v_0=0}^{\hat{n}_R} \sum_{v_1=0}^{\hat{n}_R-v_0} \Phi(v_0^{N-1}, R),
\]

where

\[
\Phi(v_0^{N-1}, R) \triangleq \mathbb{P} \left\{ \sum_{k=0}^{N-1} \sum_{i=0}^{v_k} X^i_{N-k} > d + \frac{\sum_{k=0}^{N-1} v_k - 1}{R} \right\}. \tag{20}
\]

and \( v_0^{N-1} = (v_0, v_1, \ldots, v_{N-1}) \).

Proof. The proof follows similar steps as the proof of Lemma 7 and is omitted.

Theorem 6. For the N-hop network with deterministic arrivals, the violation probability is upper bounded as follows:

\[
\lim_{t \to \infty} \mathbb{P}(D(\hat{n}_R) > t) \leq \min_{s \in S} \Psi_N(s, d, R),
\]

where

\[
\Psi_N(s, d, R) = e^{-s(d-R)} \cdot \frac{[M_1(s)]^N}{[1 - M_1(s)]^N}. \tag{21}
\]

Proof. We use the relation between departure times, arrival times and the service times given by (3) recursively starting from the last node \( N \), and apply union bound and Chernoff bound to obtain the result. The proof follows similar steps as in the proof of Theorem 4 and therefore it is omitted.
Therefore, an upper bound minimization problem for the N-hop network can be stated as follows.

\[
\min_R \ min_{s > 0} \Psi_N(s, d, R),
\]
\[
s.t. \quad \frac{1}{d} \leq R < \mu. \tag{22}
\]

**Discussion:** We note that similar to the single-hop and two-hop scenario, \(\Psi_N(s, d, R)\) is also convex with respect to \(\frac{1}{R}\) and with respect to \(s\). One may also obtain \(\alpha\)-UBMP for the N-hop scenario. However, the \(\alpha\)-relaxed upper bound involves the nested sum which becomes computationally expensive as \(N\) increases. Furthermore, we note that as \(N\) increases the upper bounds become more relaxed and therefore the heuristic solutions provided by Chernoff-UBMP may not be close to optimal solution. Nevertheless, these heuristic solutions could potentially be used as starting points. For example, when the controller has non-negligible processing time, the sensor-controller-actuator can be modelled as a three-hop tandem queuing system and one may use the heuristic solutions provided by the Chernoff-UMBP for three-hop scenario.

Next, we present an independent result for service-time distributions with bounded support.

**Service Distributions with Bounded Support:** Note that in practice, the service time distributions typically have bounded support. For example, the channel capacity for transmissions is always upper bounded due to bandwidth limitation. Considering that the service time is upper bounded by \(b \in \mathbb{R}_{>0}\), in the following theorem we present a result for computing an optimal rate for age limits above certain threshold.

**Theorem 7.** For an \(N\)-hop network, if the support of the service time distribution is upper bounded by \(b < \infty\), then for all \(d \geq (N + 1)b\), the AoI violation probability is zero at \(R^* = (N + 1)/d\), i.e., \(R^*\) is an optimal solution for \(\Psi_N\).

**Proof.** We rewrite \(\Phi(v_0^{N-1}, R)\) (defined in (20)) as follows.

\[
\Phi(v_0^{N-1}, R) = \mathbb{P}\left\{ \sum_{k=0}^{N-1} \sum_{i=0}^{v_k} \left( X_{N-k} - \frac{1}{R} \right) > d - \frac{N + 1}{d} \right\}
\]

Substituting \(R^* = (N + 1)/d\) in the above equation, and noting that \(X_{N-k} \leq b \leq \frac{1}{R^*}\) for all \(k \geq 1\) and for all \(n\), we obtain

\[
\Phi(v_0^{N-1}, R^*) = \mathbb{P}\left\{ \sum_{k=0}^{N-1} \sum_{i=0}^{v_k} \left( X_{N-k} - \frac{1}{R^*} \right) > 0 \right\} = 0.
\]

Therefore, from Lemma 10 we conclude that the AoI violation probability \(\lim_{n \to \infty} \mathbb{P}(T_D(\hat{n}_{R^*}) > t)\) is equal to zero and thus \(R^* = (N + 1)/d\) is an optimal solution. \(\square\)

**VII. Application Examples: Geometric, Exponential and Erlang Service**

In the following we show the computation of the upper bounds for typical service distributions, namely, geometric, exponential and Erlang. These distributions are most commonly used in the queuing analysis, and also they serve as good models for several practical service-time processes. Note that for these distributions, the distribution of the sum of service times is known and thus the \(\alpha\)-relaxed upper bound can be computed. Later in Section VIII we will evaluate the performance of the the computationally efficient solutions for these service distributions. To shorten the expressions, in the sequel we denote

\[
Y_1 = \sum_{i=0}^{v_0} X_i^1, \quad Y_2 = \sum_{i=0}^{v_0} X_i^2, \quad \text{and} \quad \kappa = d + \frac{v_0 + v_1 - 1}{R}.
\]

**A. Geometric Service: Wireless Links with Packet Errors**

Consider that each packet generated by the sensor is of fixed length and the packets that carry actuator commands are also of fixed length, possibly different from sensor packet length. To accommodate for packet transmission errors in the wireless links, we use geometric distribution to model the number of time slots required for transmitting a packet successfully. In particular, we consider the service distributions at link 1 and link 2 are geometric with success probabilities \(p_1\) and \(p_2\), respectively. Given the age limit \(d\) at the actuator, we compute heuristic solutions for \(R\).

In the following we compute the first term of the \(\alpha\)-relaxed upper bound given in Theorem 5. Since \(Y_1\) and \(Y_2\) are integers, we have

\[
\sum_{v_0=0}^{K-1} \sum_{v_1=0}^{K-1} \Phi(v_0, v_1, R) = \sum_{v_0=0}^{K-1} \sum_{v_1=0}^{K-1} \mathbb{P}\{Y_1 + Y_2 > \kappa\}
\]

\[
= \sum_{v_0=0}^{K-1} \sum_{v_1=0}^{K-1} \mathbb{P}\{Y_1 + Y_2 > \lceil \kappa \rceil\}. \tag{23}
\]

Since for geometrical distribution \(X^i_k \geq 1\), for all \(i\) and \(k \in \{1, 2\}\), we have \(Y_1 \geq v_1 + 1\) and \(Y_2 \geq v_0 + 1\). Therefore, for \(\kappa \leq v_1 + v_2 + 1\), we have \(\mathbb{P}\{Y_1 + Y_2 > \kappa\} = 1\). For \(\kappa > v_1 + v_2 + 2\) we compute the probability by conditioning on \(Y_2 = y\) for positive integers \(y \geq v_0 + 1\).

\[
\mathbb{P}\{Y_1 + Y_2 > \kappa\} = \sum_{y=v_0+1}^{\infty} \mathbb{P}\{Y_1 + Y_2 > \kappa\mid Y_2 = y\} \mathbb{P}\{Y_2 = y\} = \sum_{y=v_0+1}^{\infty} \mathbb{P}\{Y_1 > \kappa - y\} \mathbb{P}\{Y_2 = y\} + \mathbb{P}\{Y_2 \geq \kappa - v_1\}.
\]

In the last step above we have used \(\mathbb{P}\{Y_1 > \kappa - y\} = 1\) for \(y \geq \kappa - v_1\). Noting that the sum of i.i.d. geometric random variables has a negative binomial distribution, we have

\[
\mathbb{P}\{Y_2 = y\} = \mathbb{P}\left\{ \sum_{i=0}^{v_0} X^2_i = y \right\} = \binom{y-1}{v_0} p_2^{v_0+1} (1 - p_2)^{y-v_0-1},
\]

and

\[
\mathbb{P}\{Y_1 > \kappa - y\} = \frac{B(1 - p_2; \kappa - y - v_1, v_1 + 1)}{B(\kappa - y - v_1, v_1 + 1)}.
\]

where \(B(\cdot)\) is the incomplete beta function given by

\[
B(z; a, b) = \int_0^z x^a (1 - x)^b dx, \quad B(a, b) = \int_0^1 x^a (1 - x)^b dx.
\]
Similarly, we compute \( P\{Y_2 > |\kappa| - v_1 \} \). Finally, using \( P\{Y_1 + Y_2 > |\kappa|\} \) we compute (23). For computing the Chernoff bound we require the moment generating function, which for geometric service is given below.

\[
M_k(s) = \frac{p_k e^{s}}{1-(1-p_k)e^{s}}.
\]

Since the Chernoff bound is convex in \( s \), we use bisection algorithm to compute the minimum value.

### B. Exponential Service

In this subsection, we study the two-hop system with exponentially distributed service times with rates \( \mu_1 \) and \( \mu_2 \) at links 1 and 2, respectively. For this case, \( Y_1 \) is a sum of \( v_1 + 1 \) i.i.d. exponential random variables, which is given by the Gamma distribution with shape parameter \( v_1 + 1 \) and rate parameter \( \mu_1 \). Similarly, \( Y_2 \) has Gamma distribution with shape parameter \( v_2 + 1 \) and rate parameter \( \mu_2 \). Therefore, we compute \( \Phi(v_0, v_1, R) \) as follows.

\[
\Phi(v_0, v_1, R) = \int_0^\infty P\{Y_1 > \kappa - y\} f_{Y_2}(y)dy,
\]

where \( f_{Y_2}(\cdot) \) is the PDF of \( Y_2 \), given by

\[
f_{Y_2}(y) = \frac{\mu_2^{v_2+1}y^{v_2}e^{-\mu_2 y}}{v_2!},
\]

and \( \Gamma(x, a) \) is the upper incomplete gamma function:

\[
\Gamma(x, a) = \int_a^\infty y^{x-1}e^{-y}dy.
\]

Further, if \( \mu_1 = \mu_2 = \mu \), then

\[
\Phi(v_0, v_1, R)\big|_{\mu_1=\mu_2} = \frac{\Gamma(v_0 + v_1 + 2, \mu \kappa)}{(v_0 + v_1 + 1)!}.
\]

For computing the Chernoff bound we use the MGF of the exponential distribution which is given below.

\[
M_k(s) = \frac{\mu_k}{\mu_k - s}, \text{ for } s < \mu_k.
\]

### C. Erlang Service

Consider the Erlang service distribution at link \( k \) has shape parameter \( b_k \) and rate \( \lambda_k \). This implies \( \mu_k = b_k \lambda_k \). We note that, in this case, \( Y_k \) has Gamma distribution with shape parameter \((v_k + 1)b_k\) and rate parameter \( \lambda_k \). Therefore, we compute the bounds using similar expressions given in the previous subsection.

**Remark 2**: We note that the Chernoff upper bound and the \( \alpha \)-relaxed upper bound presented above may take values greater than 1. It is natural to cap the values of these upper bounds by 1 because for probability values an upper bound greater than 1 is not of any use, in general. However, somewhat to our surprise, in our simulations we found that allowing the values of the proposed bounds greater than 1 provides good heuristic solutions for the sampling rate, especially for parameter setting where the upper bounds are always greater than 1. Since our primary objective is to find upper bounds that can provide good heuristic solutions, but need not necessarily be tight upper bounds, we consider values greater than 1 for the bounds in our numerical evaluation. However, this should not be confused with the violation probability which does not exceed 1 at all times.

### VIII. Numerical Evaluation

In this section, we evaluate the performance of \( \alpha \)-UBMP solutions and Chernoff-UBMP solutions for geometric, exponential and Erlang service distributions. We first study the trends of the proposed upper bounds in comparison to the AoI violation probability obtained using simulation for both single-hop and two-hop scenarios. We then evaluate the quality of numerically computed solutions using the UBMPs in comparison with that of the simulation-based estimate of the optimum violation probability. The numerical computations are done using MATLAB, and the simulation is implemented in C where we run \( 10^{10} \) iterations for each data point. The default parameters are as follows. For exponential distribution \( \mu_1 \) and \( \mu_2 \) equal 1 packet/ms; for Erlang distribution we use shape parameters \( b_1 = b_2 = 3 \) and rate parameters \( \lambda_1 = \lambda_2 = 3 \), and therefore the mean rates \( \mu_1 \) and \( \mu_2 \) equal one packet/ms; for geometric service we choose success probabilities \( p_1 = 0.85 \) and \( p_2 = 0.9 \). The minimum value for \( R \) is chosen to be 0.2 packets/ms and its maximum value is chosen to be 0.75 \( \min(\mu_1, \mu_2) \) packets/ms. The minimum value for \( d \) is chosen to be 5 ms and its maximum value is chosen to be 15 ms. We use \( K = 30 \) for computing \( \alpha \)-relaxed upper bound for all the distributions because for Geometric service MATLAB does not provide precision guarantees for higher \( K \) values for computing \( \Phi(v_0, v_1, R) \), and for other service distributions, choosing \( K = 30 \) is sufficient to obtain \( \alpha \) values close to 1.

### A. Properties of Upper Bounds

1) Single Hop: In Figures 4 and 5 we present the upper bounds and the simulated AoI violation probability for varying arrival rate \( R \) and varying age limit \( d \) for different distributions for the single-hop scenario. From Figure 4 we observe that the upper bounds and the violation probability have convex nature and a global minimum in the chosen range of \( R \). Further, observe that the curvature of the upper bounds approximately follow the curvature of the simulated violation probability around its minimum value and only deviates at higher sampling rate. This is an interesting property as it suggests that a rate that minimizes the upper bound also minimizes the violation probability. We note that the \( \alpha \)-relaxed upper bound curves are not continuous because the probability terms \( \Phi(v_0, v_1, R) \) involves a floor function, namely, \([\beta]\). From Figure 5 we observe that the decay rates of the upper bounds match closely the decay rate of the violation probability. This further strengthens our statement above that minimizing the upper bounds results in good heuristic rate solutions for the considered range of age limits.
2) Two Hop: In Figures 6 and 7 we present the upper bounds and the simulated AoI violation probability for varying arrival rate $R$ and varying age limit $d$ for different distributions for the two-hop scenario. We observe similar trends as in the case of single-hop scenario. Nevertheless, the bounds become relatively looser. This can be attributed to the fact that the union bound is applied twice for the two-hop scenario.

Note that for both single-hop and two-hop scenarios $\alpha$-relaxed bound is much lower than the Chernoff bound. Nevertheless, Chernoff bound can be useful for the cases where the exact distribution of the summation of service times is intractable.
stated before. Although \(\alpha\) be attributed to the fact that the upper bounds have decay probabilities achieved by the heuristic rate solutions and the solutions of the UBMPs are near optimal for optimum violation probability is negligible. This suggests that the solutions of the UBMPs are near optimal for hop scenarios. Note that the difference between the violation probabilities obtained by solving the UBMPs and the estimated bounds the solutions of \(\text{Chernoff-bound}\) the solutions of \(\text{UBMP}\) provide only slightly lower violation probability than that of the \(\text{Chernoff-bound}\), which is minimized at \(\frac{1}{\lambda_1}, \mu_1\). Noting that computing an exact expression for the violation probability is hard, we propose an Upper Bound Minimization Problem (UBMP) and its more computationally tractable versions Chernoff-UBMP and \(\alpha\)-UBMP, which result in heuristic rate solutions. We also present the Chernoff-UBMP for N-hop tandem queuing system. We solve Chernoff-UBMP and \(\alpha\)-UBMP for single-hop and two-hop scenarios for three service-time distributions, namely, geometric, exponential and Erlang. Numerical results suggest that the rate solutions provided by \(\alpha\)-UBMP are near optimal for \(P\), demonstrating the efficacy of our method.

**B. Quality of the Heuristic Solution**

In Figure 8, we compare the violation probabilities for rate solutions obtained by solving the UBMPs and the estimated minimum/optimum violation probability obtained by exhaustive search using simulation, for both single-hop and two-hop scenarios. Note that the difference between the violation probabilities achieved by the heuristic rate solutions and the optimum violation probability is negligible. This suggests that the solutions of the UBMPs are near optimal for \(P\). This can be attributed to the fact that the upper bounds have decay rate that matches the decay rate of the violation probability as stated before. Although \(\alpha\)-relaxed upper bound is much lower than Chenoff bound the solutions of \(\alpha\)-UBMP provide only slightly lower violation probability than that of the Chernoff-UBMP solutions. Thus, Chernoff-UBMP is relatively tractable and the rate solutions provided can be used as first step toward computing close-to-optimal solutions by utilizing additional information about the service distributions.

**Remark 3**: We note that unlike the time-average age objective, which is minimized at 0.515 utilization factor (\(\lambda_1/\mu_1\)) for the D/M/1 queue [12], the optimal rate solution and in turn the utilization factor that minimizes AoI violation probability depends on age limit \(d\). For a comparison, in Figure 8(b) the single-hop scenario is equivalent to D/M/1 system and in this case the the optimal utilization factors are \{0.425, 0.4, 0.4, 0.35, 0.35, 0.35, 0.35, 0.35\}.

## IX. Conclusion and Future Work

We provide a general characterization of AoI violation probability for a network with periodic input arrivals. Using this characterization, we formulate an optimization problem \(P\) to find the optimal input rate which minimizes the AoI violation probability. Further, we show that \(P\) is equivalent to the problem of minimizing the violation probability of the departure time of a tagged arrival \(\hat{h}_R\) over the rate region \([\frac{1}{\lambda_1}, \mu_1]\). Noting that computing an exact expression for the violation probability is hard, we propose an Upper Bound Minimization Problem (UBMP) and its more computationally tractable versions Chernoff-UBMP and \(\alpha\)-UBMP, which result in heuristic rate solutions. We also present the Chernoff-UBMP for N-hop tandem queuing system. We solve Chernoff-UBMP and \(\alpha\)-UBMP for single-hop and two-hop scenarios for three service-time distributions, namely, geometric, exponential and Erlang. Numerical results suggest that the rate solutions provided by \(\alpha\)-UBMP are near optimal for \(P\), demonstrating the efficacy of our method.
For future work, we are investigating the extension of our results to stochastic arrivals. We are also studying the computational complexity for solving $\alpha$-UBMP and investigating more efficient solution methods, i.e., by identifying the range of $\alpha$ for which a good heuristic solution for $P$ can be obtained. Finally, we would like to study the problem under different queuing disciplines, including LCFS.

REFERENCES

[1] S. Kaul, M. Gruteser, V. Rai, and J. Kenney, “Minimizing age of information in vehicular networks,” in Proc. IEEE SECON, 2011.

[2] J. P. Champati, H. Al-Zubaidy, and J. Gross, “Statistical guarantee optimization for age of information for the D/GI/1 queue,” in IEEE INFOCOM 2018 - IEEE Conference on Computer Communications Workshops (INFOCOM WKSHPS), April 2018, pp. 130–135.

[3] A. M. Bedewy, Y. Sun, and N. B. Shroff, “Age-optimal information updates in multihop networks,” in Proc. IEEE ISIT, 2017.

[4] S. Kaul, R. Yates, and M. Gruteser, “Status updates through queues,” in Proc. Conference on Information Sciences and Systems (CISS), 2012.

[5] K. Chen and L. Huang, “Age of information in the presence of error,” CoRR, vol. abs/1605.00559, 2016.

[6] E. Najm, R. D. Yates, and E. Soljanin, “Status updates through M/GI/1 queues with HABQ,” in Proc. IEEE ISIT, June 2017, pp. 131–135.

[7] A. Soysal and S. Ulukus, “Age of information in G/GI/1 systems: Age expressions, bounds, special cases, and optimization,” CoRR, vol. abs/1905.13743, 2019.

[8] Y. Inoue, H. Masuyama, T. Takine, and T. Tanaka, “A general formula for the stationary distribution of the age of information and its application to single-server queues,” CoRR, vol. abs/1804.06139, 2018.

[9] J. P. Champati, H. Al-Zubaidy, and J. Gross, “On the distribution of AoI for the GI/GI/1 and GI/GI/1/2 systems: Exact expressions and bounds,” in IEEE INFOCOM 2019 - IEEE Conference on Computer Communications, April 2019, pp. 37–45.

[10] A. Kosta, N. Pappas, and V. Angelakis, “Age of information: A new concept, metric, and tool,” Foundations and Trends in Networking, vol. 12, no. 3, pp. 162–259, 2017.

[11] S. Kaul, R. Yates, and M. Gruteser, “Real-time status: How often should we update?” in Proc. IEEE INFOCOM, 2012.

[12] R. D. Yates and S. Kaul, “Real-time status updating: Multiple sources,” in Proc. IEEE ISIT, 2012.

[13] M. Costa, M. Codreanu, and A. Ephremides, “On the age of information in status update systems with packet management,” IEEE Transactions on Information Theory, vol. 62, no. 4, pp. 1897–1910, April 2016.

[14] L. Huang and E. Modiano, “Optimizing age-of-information in a multi-class queueing system,” in Proc. IEEE ISIT, 2015.

[15] R. D. Yates, “Lazy is timely: Status updates by an energy harvesting source,” in Proc. IEEE ISIT, 2015.

[16] Y. Sun, E. Uysal-Biyikoglu, R. D. Yates, C. E. Koksal, and N. B. Shroff, “Update or wait: How to keep your data fresh,” IEEE Transactions on Information Theory, vol. 63, no. 11, pp. 7492–7508, Nov 2017.

[17] B. T. Bacinoglu, E. T. Ceran, and E. Uysal-Biyikoglu, “Age of information under energy replenishment constraints,” in Proc. Information Theory and Applications Workshop (ITA), 2015.

[18] B. T. Bacinoglu, Y. Sun, E. Uysal, and V. Mutlu, “Optimal status updating with a finite-battery energy harvesting source,” Journal of Communications and Networks, vol. 21, no. 3, pp. 280–294, June 2019.

[19] R. D. Yates, M. Tavan, Y. Hu, and D. Raychaudhuri, “Timely cloud gaming,” in IEEE INFOCOM 2017 - IEEE Conference on Computer Communications, May 2017, pp. 1–9.

[20] J. Liebeherr, “Duality of the max-plus and min-plus network calculus,” Found. and Trends in Networking, vol. 11, no. 3-4, pp. 139–282, 2017.

APPENDIX

A. Proof of Lemma A

The violation probability is given by

\[
\mathbb{P}\{D(\hat{n}_R) > t\} = \mathbb{P}\left\{ D_2(\hat{n}_R) > t \right\}
\]

\[
= \mathbb{P}\left\{ \max_{0 \leq t_0 \leq \hat{n}_R} \left( A_2(\hat{n}_R - t_0) + \sum_{i=0}^{v_0} X_{2i}^{\hat{n}_R-t_0} \right) > t \right\}
\]

\[
= \mathbb{P}\left\{ \sum_{t_0=0}^{\hat{n}_R} \left( A_2(\hat{n}_R - t_0) + \sum_{i=0}^{v_0} X_{2i}^{\hat{n}_R-t_0} \right) > t \right\}
\]

Further, we have

\[
A_2(\hat{n}_R - t_0) = D_1(\hat{n}_R - t_0) - v_0 R
\]

Substituting $A_2(\hat{n}_R - t_0)$ and $A_1(\hat{n}_R - v_0 - 1) = \frac{\hat{n}_R - v_0 - t_0}{R}$ in (25) we obtain

\[
\mathbb{P}\{D(\hat{n}_R) > t\} \leq \sum_{t_0=0}^{\hat{n}_R} \mathbb{P}\left\{ \max_{0 \leq t_1 \leq \hat{n}_R-t_0} \left( \frac{\hat{n}_R - v_0 - t_1}{R} + \sum_{i=0}^{v_0} X_{1i}^{\hat{n}_R-t_0-t_1} \right) + \sum_{i=0}^{v_0} X_{2i}^{\hat{n}_R-t_0} > t \right\}
\]

In the second step above, we have used the union bound. In the third step we have used $\hat{n}_R \leq R(t-d)+1$. Also, since $X_1$ and $X_2$ are i.i.d., we re-indexed the superscripts of $X_1$ and $X_2$ in the summations. The result follows from the fact that as $t$ goes to infinity $\hat{n}_R$ goes to infinity.

B. Proof of Theorem 2

We first obtain Chernoff bound for $\Phi(v_0, v_1, R)$. We have

\[
\Phi(v_0, v_1, R) = \mathbb{P}\left\{ \sum_{i=0}^{v_1} X_i^1 + \sum_{i=0}^{v_0} X_i^0 > d + \frac{v_0 + v_1 - 1}{R} \right\}
\]

\[
\leq \min_{s > 0} e^{-s(d-v_0-v_1-1)/R} \mathbb{E}[e^{s(\sum_{i=0}^{v_1} X_i^1 + \sum_{i=0}^{v_0} X_i^0)}]
\]

\[
= \min_{s > 0} e^{-s(d-v_0-v_1-1)} [M_1(s)]^{v_1+1}[M_2(s)]^{v_0+1}
\]
Lemma 11. Provide a closed form expression for $\min_{s>0} e^{-s/(d-\frac{1}{2})} M_1(s)M_2(s)\beta_1^{\nu_1}(s)\beta_2^{\nu_0}(s)$.

(26)

Assuming the moment generating function of $X$ exists, in the second step above we have used the Chernoff bound. In the third step above we have used the fact that $X_n^k$ are i.i.d. for all $k$ and $n$, and in the last step we have used (3). Using (26) in Lemma 7 we obtain

$$\lim_{t \to \infty} \mathbb{P}\{D(\hat{n}_R) > t\} \leq \lim_{\hat{n}_R \to \infty} \sum_{n=0}^{\hat{n}_R} \sum_{v_1=0}^{\hat{n}_R-v_0} \Phi(v_0, v_1, R) \leq \min_{s>0} e^{-s/(d-\frac{1}{2})} M_1(s)M_2(s)\phi(s, \beta_1(s), \beta_2(s)),$$

(27)

where

$$\phi(s, \beta_1(s), \beta_2(s)) = \lim_{\hat{n}_R \to \infty} \sum_{n=0}^{\hat{n}_R} \sum_{v_1=0}^{\hat{n}_R-v_0} \beta_1^{\nu_1}(s)\beta_2^{\nu_0}(s).$$

(28)

Note that in the second step of (27) we have used the fact that for positive quantities sum over minimum is less than or equal to minimum over the sum. In the following lemma we provide a closed form expression for $\phi(s, \beta_1(s), \beta_2(s))$.

**Lemma 11.** For $s \in S$,

$$\phi(s, \beta_1(s), \beta_2(s)) = \frac{1}{(1-\beta_1(s))(1-\beta_2(s))}.$$

**Proof.** Recall that $\beta_1(s) < 1$ and $\beta_2(s) < 1$, for all $s \in S$. Using this, we obtain

$$\phi(s, \beta_1^{\nu_1}, \beta_2^{\nu_0}) = \lim_{\hat{n}_R \to \infty} \sum_{n=0}^{\hat{n}_R} \sum_{v_1=0}^{\hat{n}_R-v_0} \beta_1^{\nu_1}(s)\beta_2^{\nu_0}(s)$$

$$= \lim_{\hat{n}_R \to \infty} \sum_{n=0}^{\hat{n}_R} \beta_2^{\nu_0}(s) \sum_{v_1=0}^{\hat{n}_R-v_0} \beta_1^{\nu_1}(s)$$

$$= \lim_{\hat{n}_R \to \infty} \sum_{n=0}^{\hat{n}_R} \beta_2^{\nu_0} \cdot \left(1 - \beta_1^{\nu_1+1}(s)\right)$$

$$= \lim_{\hat{n}_R \to \infty} \sum_{n=0}^{\hat{n}_R} \frac{\beta_2^{\nu_0}}{1-\beta_1(s)} - \lim_{\hat{n}_R \to \infty} \sum_{n=0}^{\hat{n}_R} \frac{\beta_2^{\nu_0}(s)\beta_1^{\nu_1+1}(s)}{1-\beta_1(s)}$$

$$= \frac{1}{(1-\beta_1(s))(1-\beta_2(s))} - \lim_{\hat{n}_R \to \infty} \sum_{n=0}^{\hat{n}_R} \frac{\beta_2^{\nu_0}(s)\beta_1^{\nu_1+1}(s)}{1-\beta_1(s)}$$

It is now sufficient to show that the summation term above is equal to zero. First we note that the summation is non-negative since $0 \leq \beta_1(s) < 1$ and $0 \leq \beta_2(s) < 1$. Let $\beta(s) = \min(\beta_1(s), \beta_2(s))$, then we have

$$\lim_{\hat{n}_R \to \infty} \frac{\beta_2^{\nu_0}(s)\beta_1^{\nu_1+1}(s)}{1-\beta_1(s)} \leq \lim_{\hat{n}_R \to \infty} \sum_{n=0}^{\hat{n}_R} \beta_1^{\nu_1+1}(s)$$

$$= \lim_{\hat{n}_R \to \infty} \frac{\beta_1^{\nu_1+1}(s)(1-\beta_1(s))}{\beta_1^{\nu_1+1}(s)(1-\beta_1(s))}$$

$$= \lim_{\hat{n}_R \to \infty} \frac{1}{\beta_1^{\nu_1+1}(s)(1-\beta_1(s))}$$

$$= \lim_{\hat{n}_R \to \infty} \frac{1}{\beta_1^{\nu_1+1}(s)(-\log(\beta_1(s))(1-\beta_1(s))}$$

$$= 0.$$

In the third step above we have used L’Hospital’s Rule. Since the summation is non-negative and is less than or equal to zero, it should be equal to zero.

It is easy to see that if $s \notin S$, then $\phi(s, \beta_1(s), \beta_2(s))$ will be equal to infinity. Therefore, using (27) and Lemma 11, we obtain

$$\lim_{t \to \infty} \mathbb{P}\{D(\hat{n}_R) > t\} \leq \min_{s>0} e^{-s/(d-\frac{1}{2})} M_1(s)M_2(s)\phi(s, \beta_1(s), \beta_2(s)).$$

Hence the result is proven.

**C. Proof of Theorem 3**

From Lemma 7 we have

$$\lim_{t \to \infty} \mathbb{P}\{D(\hat{n}_R) > t\} \leq \lim_{\hat{n}_R \to \infty} \sum_{n=0}^{\hat{n}_R} \sum_{v_1=0}^{\hat{n}_R-v_0} \Phi(v_0, v_1, R) = \lim_{\hat{n}_R \to \infty} \sum_{n=0}^{\hat{n}_R} \sum_{v_1=0}^{\hat{n}_R-v_0} \Phi(v_0, v_1, R),$$

(29)

where

$$\Phi_1(K) = \lim_{\hat{n}_R \to \infty} \sum_{n=0}^{\hat{n}_R} \sum_{v_1=0}^{\hat{n}_R-v_0} \Phi(v_0, v_1, R),$$

$$\Phi_2(K) = \lim_{\hat{n}_R \to \infty} \sum_{n=0}^{\hat{n}_R} \sum_{v_1=K}^{\hat{n}_R-v_0} \Phi(v_0, v_1, R).$$

In the following we use the Chernoff bound for $\Phi(v_0, v_1, R)$, given in (26), to derive bounds for $\Phi_1(K)$ and $\Phi_2(K)$.

$$\Phi_1(K) \leq \min_{s>0} e^{-s/(d-\frac{1}{2})} M_1(s)M_2(s)\phi_1(s, \beta_1(s), \beta_2(s))$$

(30)

where

$$\phi_1(s, \beta_1(s), \beta_2(s)) = \lim_{\hat{n}_R \to \infty} \sum_{n=0}^{\hat{n}_R} \sum_{v_1=0}^{\hat{n}_R-v_0} \beta_1^{\nu_1}(s)\beta_2^{\nu_0}(s)$$

$$\phi_1(s, \beta_1(s), \beta_2(s)) = \lim_{\hat{n}_R \to \infty} \sum_{n=0}^{\hat{n}_R} \sum_{v_1=0}^{\hat{n}_R-v_0} \beta_1^{\nu_1}(s)\beta_2^{\nu_0}(s) \frac{\beta^K(s)(1-\beta_1^{\nu_1+1}(s))}{1-\beta_1(s)}$$

$$= \frac{1}{(1-\beta_1(s))(1-\beta_2(s))} - \lim_{\hat{n}_R \to \infty} \sum_{n=0}^{\hat{n}_R} \frac{\beta_2^{\nu_0}(s)\beta_1^{\nu_1+1}(s)}{1-\beta_1(s)}$$

$$= \frac{(1-\beta_2^K(s))(1-\beta_1^{\nu_1+1}(s))}{(1-\beta_1(s))(1-\beta_2(s))}, \text{ for } s \in S.$$ (31)

The second term in the third step above vanishes as $\beta_1(s) < 1$ for $s \in S$. Using (31) in (30), we obtain

$$\Phi_1(K) \leq \min_{s \in S} e^{-s/(d-\frac{1}{2})} M_1(s)M_2(s)\phi_1(s, \beta_1(s), \beta_2(s)).$$

(32)

Again, substituting (26) in $\Phi_2(K)$, we obtain

$$\Phi_2(K) \leq \min_{s>0} e^{-s/(d-\frac{1}{2})} M_1(s)M_2(s)\phi_2(s, \beta_1(s), \beta_2(s)),$$

(33)
where
\[ \phi_2(s, \beta_1(s), \beta_2(s)) = \lim_{\hat{n}_R \to \infty} \sum_{v_0=K}^{\hat{n}_R} \sum_{v_1=0}^{\hat{n}_R-v_0} \beta_1^{v_1}(s)\beta_2^{v_0}(s). \]

Using similar analysis as in Lemma 11, we obtain
\[ \phi_2(s, \beta_1(s), \beta_2(s)) = \beta_2^K(s) \frac{(1-\beta_1(s))(1-\beta_2(s))}{(1-\beta_1(s))(1-\beta_2(s))}, \text{ for } s \in \mathcal{S}. \]

Using (34) in (33), we obtain
\[ \Phi_2(K) \leq \min_{s \in \mathcal{S}} e^{-s(d+\frac{1}{\hat{p}})} M_1(s)M_2(s) \beta_2^K(s) \frac{(1-\beta_1(s))(1-\beta_2(s))}{(1-\beta_1(s))(1-\beta_2(s))}. \]

Finally, substituting (32) and (35) in (29) we obtain the result.