A causal viscous cosmology without singularities

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Abstract  
An isotropic and homogeneous cosmological model with a source of dark energy is studied. That source is simulated with a viscous relativistic fluid with minimal causal correction. In this model the restrictions on the parameters coming from the following conditions are analyzed: a) energy density without singularities along time, b) scale factor increasing with time, c) universe accelerated at present time, d) state equation for dark energy with \( w \) bounded and close to -1. It is found that those conditions are satisfied for the following two cases. i) When the transport coefficient (\( \tau_\Pi \)), associated to the causal correction, is negative, with the additional restriction \( \zeta |\tau_\Pi| > 2/3 \), where \( \zeta \) is the relativistic bulk viscosity coefficient. The state equation is in the “phantom” energy sector. ii) For \( \tau_\Pi \) positive, in the “k-essence” sector. It is performed an exact calculation for the case where the equation of state is constant, finding that option (ii) is favored in relation to (i), because in (ii) the entropy is always increasing, while this does not happen in (i).

1 Introduction  
As stressed in ref. [1], there is relevant experimental evidence about the acceleration of the universe. That effect can be attributed to the so-called dark energy, which can be interpreted as a term of “negative pressure” in the Einstein equations, as remarked in ref. [2].
The most cited experimental observations related to universe acceleration are due to Perlmutter et al. [3] and Riess et al. [4], which include measures of the redshift of supernovas. The results are compatible with the addition of the cosmological constant to the Einstein equations. This cosmological constant would be analogous to a negative pressure that leads the universe to an accelerated expansion.

The dark energy can be described by the state equation $p = w \rho$, with $w < -1/3$, in order to have an accelerated universe [5]. In agreement with experimental observations [6], the value of $w$ would be very close to $-1$ ($w = -1.04^{+0.09}_{-0.10}$). From a theoretical point of view, the different values of $w$ considered in the literature come from some Lagrangian formulations of field theories (for a review, see ref. [7]). It is so for a field with minimal coupling, i.e., $-1 \leq w \leq 1$, which is known as "quintessence". Also for potentials coming from string theory as, for example, the formulation known as "k-essence", with $-1 \leq w \leq -1/3$. From s-brane, in superstring theory, it is obtained for the "phantom" field $w < -1$. The last case corresponds to a minimal coupling field, but with contrary sign in the kinetic term with respect to an ordinary field. This model of dark energy shows an anomalous behavior of difficult justification [8], for example, it leads to a singularity in the density of energy at finite time, known as "Big Rip" [9]. Moreover, it describes energy density as an increasing function of time, with the consequence of the violation of the dominant energy condition [10] and, in addition, it leads to negative values of entropy.

Another used approach is based on considering a relativistic viscous fluid as a matter source of the Einstein equations. For example, in ref. [11] the matter of the universe is a viscous fluid; moreover, the approximation introduced originally by Eckart [12] and later improved by Landau-Lifshitz (L-L) [13] is used. When this model is applied to an isotropic and homogeneous universe, the acceleration in the expansion is produced by the bulk viscosity. The works [14] and [15] follow this line of research. More recently, ref. [16] shows that, if the bulk viscosity coefficient is negative, then the expansion decreases. On the other hand, ref. [17] proves the equivalence between the dynamics coming from a macroscopic approach based on the bulk viscosity term and a particle creation model that describes the phase of a slow-roller inflation.

Nevertheless, the application of the causal approach in the matter source is usually not found in the literature. As it is well known, the L-L theory has problems of stability and admits propagation of superluminal signals [18]. This last fact, as shown in papers [19] and [20], implies violations of causality. In order to avoid causality violation problems, some theories of viscous fluid, that incorporate second order gradients in the velocities, have been developed [21], [22].

However, when the purpose is the study of the dynamics of the universe, the theories with causal correction give very complex expressions for the energy-momentum tensor. It is for that reason that, in the present work, a much simpler approach is proposed, in line with ref. [24], where the anomalies of the L-L theory are avoided with a minimum of complexity.

In ref. [24] it is shown how the parabolic equation for the perturbation
in the velocities, with the causality violation problem, can be turned into an
hyperbolic equation without that problem. In order to give a causal behaviour, a
time delay in the propagation of information, which the authors call "relaxation
time", is introduced. This is implemented by means of a memory function
(Green function). This formulation introduces a correction in the bulk viscosity,
in agreement with the correction coming from the Israel-Stewart theory \[21\], or
from the more general proposal of ref. \[24\] or \[25\] at the low order, necessary
to avoid anomalies with causality.

The aim of this work is to perform an analysis of the cosmological equations
for a viscous fluid model, in order to understand the importance of the causal
correction in the dynamics of the universe. The purpose is also to determine
the influence of the correction term mentioned above over the entropy of the
universe.

With this aim, the present article is organized as follows. In the next section,
the cosmological model and the dynamical equations resulting from the causal
viscous correction are introduced. Section 3 is devoted to the analysis of the
restrictions on the parameters. In Section 4, the contribution of the causal
corrections to entropy is studied. Finally, in Section 5 the main conclusions are
presented.

2 The cosmological model

Let us consider an isotropic and homogeneous universe with matter modelled as
a relativistic fluid. Then, a spatially flat Friedmann-Robertson-Walker (FRW)
universe is considered. From here on, the units for which \( c = 8\pi G = k = 1 \)
and the signature for the metrics \( (+, -, -, -) \) are used, as in ref. \[26\].

2.1 Dynamic equations

The dynamics will be described by the Einstein equations with the energy-
momentum tensor (EMT) as a source of viscous fluid (see ref. \[26\] or \[27\]),
i.e.:

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu},
\]

where de EMT can be factorized as \[24\]:

\[
T_{\mu\nu} = \tilde{T}_{\mu\nu} + \Pi_{\mu\nu}.
\]

\( \tilde{T}_{\mu\nu} \) is the ideal fluid part, given by

\[
\tilde{T}_{\mu\nu} = (p + \rho) u_\mu u_\nu - pg_{\mu\nu},
\]

where \( \rho \) is the density of energy (dark energy in our case), \( p \) is the pressure, \( u_\mu \)
the four-velocity, and \( g_{\mu\nu} \) is the space-time metrics. Given the convention used
here, we have the normalization equation \( u_\mu u^\mu = 1 \).
The tensor $\Pi_{\mu\nu}$ in Eq. (2) is the viscosity term. This term can also be factorized into two tensors, one of them the traceless part ($\pi_{\mu\nu}$), related with the shear viscosity, and the other part with non-vanishing trace ($\Pi$), representing the bulk viscosity. Then, we have (as in [22])

$$\Pi_{\mu\nu} = \pi_{\mu\nu} + \Delta_{\mu\nu}\Pi,$$

with

$$\Delta_{\mu\nu} = g_{\mu\nu} - u_{\mu}u_{\nu}. \quad (5)$$

It is convenient to re-write $\tilde{T}_{\mu\nu}$ in the form:

$$\tilde{T}_{\mu\nu} = \rho u_{\mu}u_{\nu} - p\Delta_{\mu\nu}. \quad (6)$$

Then,

$$\tilde{T}_{\mu}^\mu = \rho - 3p. \quad (7)$$

By replacing Eq. (4) into Eq. (2), we obtain

$$T_{\mu\nu} = \rho u_{\mu}u_{\nu} - (p - \Pi) \Delta_{\mu\nu} + \pi_{\mu\nu}. \quad (8)$$

Therefore, now we have

$$T_{\mu}^\mu = \rho - 3(p - \Pi), \quad (9)$$

because

$$\pi_{\mu}^\mu = 0. \quad (10)$$

By comparing Eq. (7) with Eq. (9), we can see that the quantity $p - \Pi$ is equivalent to a corrected pressure. Then, we can say that $-\Pi$ is a kind of “negative pressure” that represents the viscosity effect. On this basis, it is convenient to define

$$p^\dagger \equiv p - \Pi \quad (11)$$

By taking the trace in Eq. (2), and by using Eq. (9) with the definition (11) and the Hubble coefficient $H = a/a$, with $a(t)$ the scale factor, one finds

$$H + 2H^2 = \frac{1}{6} (\rho - 3p^\dagger). \quad (12)$$

If the 00 component is taken from Eq. (11), and the term $\pi_{\mu\nu}$ is neglected in Eq. (5) because at lower order it goes as $\partial_{\alpha}u_{\beta}$ (see ref. [22]), then one obtains

$$H^2 = \frac{1}{3}\rho. \quad (13)$$

For the closed universe, a term of the form $a^{-2}$ must be added in Eqs. (12) and (13). Then, when $a >> 1$, the formulation is approximately that corresponding
to a flat universe. By derivation of Eq. (13), and by replacing the result in Eq. (12), the following well-known and useful expression can be obtained:

$$\dot{\rho} + 3H (\rho + p) = 0.$$  \hspace{1cm} (14)

It is easy to prove that the last equation is valid for both flat and closed universe in exact way.

2.2 Causal correction

The functional form of the quantity $\Pi$ was developed in the references [28], [21], [29], [25], and the generalization to second order in velocity gradients is also given by [24]. Hereafter, the approach by Koide et al. [23] will be used. Such proposal corresponds to the lowest order, in which the violation of causality does not occur. The quantity $\Pi$, in that reference, is approximated by:

$$\Pi \simeq \zeta \nabla_\mu u^\mu - \tau_\Pi u_\mu \nabla_\mu \Pi,$$  \hspace{1cm} (15)

where $\zeta$ is the bulk viscosity coefficient and $\tau_\Pi$, sometimes called "second viscosity coefficient", comes from the causal correction (see [24]). The product $\zeta \tau_\Pi \equiv \tau_R$ is called "relaxation time" (see [23]).

As we can see, Eq. (15) is an implicit equation of $\Pi$; therefore, to calculate the source of the dynamical equation it is useless. However, we can deduce an approximate, but explicit, expression to determine $\Pi$. In order to do this, we can substitute Eq. (15) into itself and neglect the terms higher than the first order in $\tau_\Pi$. Then, one obtains

$$\Pi \simeq \zeta \nabla_\mu u^\mu - \zeta \tau_\Pi u_\mu \nabla_\mu \nabla_\alpha u^\alpha.$$  \hspace{1cm} (16)

The first term of Eq. (16) corresponds to the Landau-Lifshitz theory [13], while the second term is the minimum necessary in order to avoid causality violation.

Now we will express $\Pi$ as a function of the scale factor $a(t)$ and its derivatives. So, it is convenient to employ the continuity equation ($\nabla_\mu (nu^\mu) = 0$ with $n$ the density number) and to express the equations by means of the proper time. So, we finally get

$$\Pi = 3\zeta H - 3\zeta \tau_\Pi \dot{H}.$$  \hspace{1cm} (17)

This functional form for $\Pi$ is particularly convenient to solve the dynamical equation.

2.3 System of equations to be solved

The equation of state (EoS), which relates the pressure and the density of dark energy, is added to the equations given above. Following ref. [30], we can use the following expression for the dark energy

$$w = \frac{p}{\rho} = -1 - \lambda \rho^{\alpha - 1},$$  \hspace{1cm} (18)
with $\alpha$ an arbitrary parameter.

On the other hand, by replacing Eq. (14) into Eqs. (13) and (18) we obtain

$$
\left(1 + \frac{3}{2} \zeta \tau_\Pi \right) \dot{H} - \frac{3}{2} \lambda H^{2\alpha} - \frac{3}{2} \zeta H = 0. \tag{19}
$$

As we will see, by the resolution of Eq. (19), in some cases it is possible to obtain an exact expression for the dark energy density as a time dependent function.

The questions that we want to answer in this paper are referred to the conditions that the parameters must satisfy for the following requirements be fulfilled: i) energy density without singularities at finite time, ii) scale factor $a(t)$ as an increasing function of the time, iii) accelerated universe at present time, and iv) $w$ close to $-1$.

By operating with Eqs. (12)-(18), it is easy to obtain the following set of equations, useful to test the above requirements:

$$
\Delta t = \frac{1}{\sqrt{3}} \int_{\rho_P}^{\rho} \frac{M(\rho)}{\rho^{1/2}} d\rho, \tag{20}
$$

$$
\ln \left(\frac{a}{a_P}\right)^3 = \int_{\rho_P}^{\rho} M(\rho) d\rho, \tag{21}
$$

$$
\frac{\dot{a}}{a} = \frac{1}{3} \rho \left(1 + \frac{3}{2} \zeta \frac{1}{\rho M(\rho)}\right), \tag{22}
$$

with $\Delta t \equiv t - t_p$ (subindex “$P$” indicate Planck era), and

$$
M(\rho) \equiv \left(1 + \frac{3}{2} \zeta \tau_\Pi \right) / \left(\lambda \rho^\alpha + \sqrt{3} \zeta \rho^{1/2}\right). \tag{23}
$$

Eqs. (20) - (23), plus Eq. (18), are the set of equations that, in the next section, will be used to analyze the restrictions in the parameters necessary to avoid singularities in the physical quantities. In particular, the main interest is to determine the influence of the causal correction on the results.

3 System of equations to be solved

In this section, the conditions that the quantities on the left side of Eqs. (20) - (23) and the state equation (18) must satisfy will be formulated, i.e.:

i) The condition on $\Delta t = \Delta t(\rho)$. In this case, two possibilities are considered:

I) The dark energy density as a decreasing function of time, i.e.

$$
\lim_{\rho \to \rho_f} \Delta t = \infty, \text{ for } \rho_f < \rho_P, \text{ (in particular, } \rho_f \text{ can be zero).}
$$

II) The dark energy density as an increasing function of time, i.e.
\[
\lim_{\rho \to \rho_f} \Delta t = \infty, \text{ for } \rho_f > \rho_P. \text{ A particular case is } \rho_f = \infty, \text{ which is} \\
\text{known in the literature as } "\text{Little Rip}” \text{ [30].}
\]

ii) The scale factor \( a(t) \) as an increasing function of time, i.e.

\[
\lim_{\rho \to \rho_f} a(\rho) = a_f, \text{ such that } a_f > a_o, \text{ where } a_o \text{ is the scale factor observed} \\
at \text{ present }.
\]

iii) Accelerated universe (at least for \( t \sim t_o \)), i.e. \( \ddot{a} > 0 \).

iv) \( w \sim -1 \pm 0.1 \)

### 3.1 General restrictions

In this subsection, the restrictions on the parameters due to the above conditions are analyzed in general. In particular, the two possibilities I and II are considered:

#### 3.1.1 I) \( \rho \) as a decreasing function

Condition (i) tells us that \( \Delta t \) is a decreasing function of \( \rho \), because \( \rho \) decreases with \( t \). Then, the following inequality must be satisfied: \( d\Delta t/d\rho < 0 \). As a consequence, from Eq. (20) the implication is

\[
M(\rho) < 0. \tag{24}
\]

From condition (ii), \( \ln \left( \frac{a}{a_P} \right)^3 \) is a decreasing function of \( \rho \). Therefore, the inequality (24) is implied again, i.e., condition (ii) does not introduce a new restriction. It is noteworthy that the condition (24) may be satisfied by negative numerator or denominator of \( M(\rho) \). In the first case, it should be \( \tau_\Pi < 0 \), with the additional condition

\[
\zeta |\tau_\Pi| > 2/3. \tag{25}
\]

In the second case, we should make \( \lambda < 0 \) (quintessence sector, k-essence, tachyon field, etc.), with the additional condition

\[
|\lambda| > \sqrt{3\zeta} \rho_f^{1-\alpha}. \tag{26}
\]

Since in model I \( \rho(t) \) decreases with time, the above condition requires \( \alpha \leq 1/2 \) in order to avoid the indefinite increase of the boundary of \( |\lambda| \), or that \( \rho(t) \) tends to a finite value, as we shall see in the next subsection.

From condition (iii), the following inequality (besides that of (24)) must be satisfied:

\[
\rho > \frac{3}{2} |M(\rho)|^{-1}, \forall \rho \leq \rho_P. \tag{27}
\]
This implies that there is a density value, let’s call it $\rho_c$, below which the condition (iii) is not met. This value will depend on how close to 1 the value of $\frac{3}{2} |\eta_0|\tau$ is, and how small $\lambda$ is. This will be clear when, in the next subsection, a specific case will be analyzed.

Condition (iv) implies that

$$|\lambda| \rho^{\alpha - 1} \lesssim 0.1. \quad (28)$$

Then, for $w$ limited, it must be $\alpha \geq 1$, which is in contradiction with the condition coming from (24).

### 3.1.2 II) $\rho$ as an increasing function

Condition (i) in this case tells us that $\Delta t$ is an increasing function of $\rho$, i.e.: $d\Delta t/d\rho > 0$. Therefore,

$$M(\rho) > 0. \quad (29)$$

Condition (ii) $\Rightarrow d\ln(a/a_p)/d\rho > 0 \Rightarrow (29)$.

Condition (iii) is directly satisfied.

Condition (iv) is the same as in the case of (28), but with $\lambda = |\lambda|$ (phantom energy sector). But, since $\rho$ can grow indefinitely, in order to keep $w$ bounded, in this case it should hold that $\alpha \leq 1$.

So, as we see, both cases (I and II) share the same condition $\alpha = 1$. Hence, it is interesting to analyze this case in more detail, a task that we will undertake in the next subsection.

### 3.2 The particular case $\alpha = 1$

Why to study the detail of a particular case, if the general conditions were just given? The reason is that the general conditions are valid for functions with monotonic behavior, i.e., functions that increase or decrease along all the time interval. However, this analysis is beyond that case. For example, $a(t)$ is an increasing function of time in a range of time, but in another range it is decreasing function, in such a way that in the present time the universe is accelerated, consistent with observations, but in a far later time, slowdown occurs. Then, this example is not covered by the criterion given above. Moreover, this case satisfies automatically one of the required conditions: the EoS remains bounded during the whole evolution of the universe.

Then we start with Eq. (19), which is solved exactly by means of the methodology of ref. [16], we obtain the following solution:

$$H(t) = -\frac{\zeta}{2\lambda} \left[1 + \coth (\gamma t)\right], \quad (30)$$

with $\gamma \equiv (3\zeta/4)/(1 + 3\zeta\tau/2)$. 

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The functional form of this equation shows us, that for any natural number \( n \),
\[
d^nH(t)/dt^n \neq \infty \quad \forall \ t \neq 0
\]
then, the studied case, does not present the singularity, identified as Type IV in the literature [31, 32].

Taking into account that \( H = d\ln a/dt \), we can integrate in time to obtain
\[
a = a(t),
\]
which results:
\[
a(t) = a_P \left[ \frac{1 - e^{-2|\gamma|t}}{1 - e^{-2|\gamma|t_P}} \right]^{-\zeta/2\lambda|\gamma|}. \tag{31}
\]

From Eq. (31) we can see that, for \( a(t) \) be an increasing function of \( t \), as \( \zeta > 0 \)
(see ref. [13]), there are the following possibilities:

### 3.2.1 Phantom energy sector: \( \lambda > 0 \).

a) \( \gamma < 0 \) Then \( \tau_P < 0 \) with the condition [25] is required in order to satisfy the inequality on \( \gamma \).

Then, we can write
\[
a(t) = a_P \left[ \frac{1 - e^{-2|\gamma|t}}{1 - e^{-2|\gamma|t_P}} \right]^{-\zeta/2\lambda|\gamma|}, \tag{32}
\]
whereupon
\[
\lim_{t \to \infty} a(t) = a_P \left[ 1 - e^{-2|\gamma|t_P} \right]^{-\zeta/2\lambda|\gamma|}, \tag{33}
\]

As we can see from Eq. (33), \( a(\infty) \neq \infty \), but it can be as large as we want. In fact, if for example \( |\gamma| \sim 1 \), as \( t_P \ll 1 \), developing the exponential up to linear term, we obtain \( a(\infty) \simeq (2t_P)^{-\zeta/2\lambda} \): also the exponent can be in absolute value as large as we want, if \( \lambda \ll 1 \). The last requirement is consistent with the fact that \( w \sim -1 \) in the phantom energy region.

Moreover, by using Eq. (30) and Eq. (13), the energy density results
\[
\rho(t) = \frac{3}{4} \left( \frac{\zeta}{\lambda} \right)^2 \left[ 1 - \coth (|\gamma|t)^2 \right], \tag{34}
\]
Clearly \( \lim_{t \to \infty} \rho(t) = 0 \) (is included in the family of models labeled by I). This result creates some conceptual conflict in relation with the above result: if the scale factor reaches a finite value in infinite time when the density is null, what happened with dark energy? Did it disappear? Nevertheless, as we saw above, this would not have a “noticeable” effect because \( a(\infty) \) would be a power of the inverse of Planck time as large as we would like.

Now we can analyze if this model gives us an accelerated universe. If the two derivatives of Eq. (32) are performed, we obtain:
\[
\ddot{a}(t) = \frac{2\zeta |\gamma|}{\lambda} Q(t_P) \left( 1 - e^{-2|\gamma|t} \right) \left( 1 - e^{-4|\gamma|t} \right) \left( \frac{\zeta}{2\lambda |\gamma|} - e^{2|\gamma|t} \right), \tag{35}
\]
with \( Q(t_P) \equiv a_P \left( 1 - e^{-2|\gamma|t_P} \right)^{-\frac{1}{2|\gamma|}} \). So, for \( \dot{a}(t) > 0 \), the following inequality must hold:

\[
\frac{\zeta}{2|\lambda|\gamma} > e^{2|\gamma|t}.
\]

The two quantities of the above inequality would be equal for a "change time" \( t_c \) given by

\[
t_c = \ln \left( \frac{\zeta \tau_{\Pi} - 2/3}{\frac{\zeta |\tau_{\Pi}|-2/3}{\lambda}} \right).
\]

Then, the value of \( \lambda \) can be set so the universe will slow at \( t_c > t_a \), where \( t_a \) is the current observation time. Again, we obtain \( \lambda << 1 \), whereby the value of \( w \) should be very close to \(-1\), in agreement with the observation.

b) \( \gamma > 0 \) As is easily seen, Eq. (31) gives us, in this case, a decreasing evolution, so this possibility is discarded.

### 3.2.2 K-essence sector: \( \lambda < 0 \)

a) \( \gamma > 0 \) Now we start with

\[
a(t) = a_P \left( \frac{e^{2\gamma t} - 1}{e^{2\gamma t} - 1} \right)^{\frac{\zeta/2|\lambda|\gamma}{\lambda}}
\]

Then \( \lim_{t \to \infty} a(t) = \infty \). The energy density is in this case is

\[
\rho(t) = \frac{3}{4} \left( \frac{\zeta}{\lambda} \right)^2 \left[ 1 + \coth (\gamma t) \right]^2.
\]

As we see from Eq. (39), a function that converges to a finite value is obtained, i.e.: \( \lim_{t \to \infty} \rho(t) = 3 \left( \zeta/\lambda \right)^2 \). It is worth to notice that, for a very small value of \( \lambda \), i.e., \( |\lambda| < \sqrt{3\zeta/\rho_P}^{1/2} \), the density could grow beyond the density at the Planck time; in that case, the model would be included in the family that we called II. We can say that, density of dark energy will grow up to a finite value and therefore does not become a "Little Rip" type singularity [30],[33] for any time, provided \( \lambda \neq 0 \).

Now, from Eq. (38) the acceleration can be computed, and the following expression is obtained:

\[
\ddot{a}(t) = f(t) \left( \frac{\zeta}{2|\lambda|\gamma} - e^{-2\gamma t} \right),
\]

where \( f(t) > 0 \ \forall \ t \). Therefore, the condition for accelerated expansion is obtained from the condition that the quantity into the parenthesis in Eq. (40) be positive. As a consequence, the following condition on time results:
This means that the condition $|\lambda| < \frac{2}{3} + \zeta \tau_\Pi$, is sufficient for an accelerated regimen during all the time interval.

b) $\gamma < 0$ In order to complete the analysis, it can be stressed that the case in which both coefficients, $\lambda$ and $\gamma$ ($\tau_\Pi < 0$), are negative does not give a reasonable result, because it leads to a scale factor decreasing over time.

4 Causal viscosity contribution to the change of entropy

In this section, the following question will be addressed: which is the contribution to the entropy due to the causal corrective term in the viscosity? From Eq. (12) we can see that the quantity $p^\dagger$ is equivalent to an “effective pressure”. On the other hand, we can write the Gibbs equation in the form

$$dE = -\left( p - T \frac{dS}{dV} \right) dV.$$ (42)

The second term in the parenthesis can be interpreted as the heat per unit of volume absorbed by the system as the result of viscosity. It can be conceived as a negative pressure, analogous to the correction to the pressure introduced in Eq. (11), which we identified with $-\Pi$, associated to the bulk viscosity. Then, it is natural to propose the identification

$$T \frac{dS}{dV} \equiv \Pi,$$ (43)

with $\Pi$ given by Eq. (17). Then, there are two viscosity contributions to the change of entropy, indicated as

$$dS = dS_1 + dS_2,$$ (44)

with

$$dS_1 = \frac{3}{T} \zeta H dV,$$ (45)

$$dS_2 = -\frac{3}{T} \zeta \tau_\Pi H dV,$$ (46)

Eq. (45) gives the bulk viscosity contribution, and Eq. (46) supplies the second bulk viscosity contribution, related with the causal correction.

The first term of Eq. (44) is always positive, since we assume that, as in most physical systems, $T > 0$. According to the theory of fluids [13], it should hold that $\zeta > 0$, since an expansion stage with $dV > 0$ and $\dot{a} > 0$ is considered.
Also in the contraction phase the product $HdV$ remains positive. Therefore, the first term of Eq. (44) does not break time invariance (see also ref. [34] for the case where particle creation is considered).

The analysis of the influence due to the second term of Eq. (44) is easier when the equation is rewritten in the following convenient form:

$$dS = dS_1 \left[ 1 + \tau_\Pi \left( \frac{\dot{a}}{a} - \frac{\ddot{a}}{a} \right) \right].$$

(47)

A first remark is that the term into the parentheses that multiplies $\tau_\Pi$ becomes zero for an evolution of the form $a(t) \propto \exp(\alpha t)$. As we did above, we will separate the analysis of the contribution to $dS$, according $\tau_\Pi$ is greater or smaller than zero.

We see that, when $\tau_\Pi > 0$, the sufficient condition for the term of entropy associated with the coefficient $\tau_\Pi$ gives a positive contribution is

$$\frac{\dot{a}}{a} > \frac{\ddot{a}}{a}.$$  

(48)

It has to be noticed that a solution of the form $a(t) \propto t^\beta$ satisfies the above condition for any $\beta$. Another possibility would be $\dot{a}/a < 0$, but this does not agree with observations since it would lead to a deceleration of the universe.

When $\tau_\Pi < 0$, the sufficient condition is

$$\frac{\dot{a}}{a} < \frac{\ddot{a}}{a}.$$  

(49)

Since $\dot{a}$ and $a$ are positive, this condition also implies that $\ddot{a} > 0$.

4.1 Analysis for case $\alpha = 1$

To analyze the details of an exact calculation example, we can see again the case in which the state equation is constant (i.e. with $\alpha = 1$). In order to simplify notation, it is convenient to define

$$\delta \equiv \frac{\dot{a}}{a} - \frac{\ddot{a}}{a}. $$

(50)

Two subclasses are considered:

4.1.1 a) K-essence sector: $\lambda < 0$ and $\tau_\Pi > 0$.

By performing the derivatives of Eq. (38), we can calculate $\delta$:

$$\delta = \frac{2\gamma}{(e^{2\gamma t} - 1)},$$

(51)

with $\gamma$ defined as in Eq. (30). Then, $\delta > 0 \forall t$ and, therefore, $dS > 0 \forall t$. We can see also that $\lim_{t \to \infty} dS = dS_1$. This means that, when high values of $t$,
and hence of $a(t)$, are reached, the effect due to the causal correction becomes negligible.

### 4.1.2 b) Phantom energy sector: $\lambda > 0$ and $\tau_\Pi < 0$.

Now we derive Eq. (32) to calculate $\delta$. In this case we obtain

$$\delta = 2 |\gamma| / \left(1 - e^{-2|\gamma|t}\right).$$

(52)

Therefore $\delta > 0 \ \forall \ t$. However now $dS$ is

$$dS = dS_1 (1 - |\tau_\Pi| \delta).$$

(53)

If we also consider that $\zeta |\tau_\Pi| > 2/3$ (a restriction necessary to meet the conditions (i) - (iv)), then there is no $t > 0$ for which entropy increases. Moreover, $\lim_{t \to \infty} dS = -dS_1 / (\frac{2}{3} \zeta |\tau_\Pi| - 1)$. These are then good arguments against this case.

## 5 Conclusions

A cosmological model as an isotropic and homogeneous universe, with a source of matter that simulates dark energy, was proposed. The source consists of a relativistic viscous fluid with minimal causal correction. Due to the symmetry of the model, the only viscous contribution is the bulk viscosity, which provides the negative pressure necessary to maintain an accelerated expansion, consistent with the observations.

The constrains in the model parameters due to the following conditions, were studied: i) energy density tending to a finite value along the time, ii) scale factor $a(t)$ increasing function of time, iii) accelerated present universe, and iv) state equation for dark energy $p/\rho = w$, with $w$ close to $-1$.

A result obtained is that the energy density $\rho$ is a decreasing function, for all times, when the second viscosity coefficient $\tau_\Pi$ is negative. It is worth recalling that the purpose of the term of TEM associated with this coefficient is to correct the defects of the non causal theory of relativistic fluids [13], in which fluid disturbances are propagated at superluminal speeds. This term (as it can easily be verified from calculations ref. [22]) leads to a bound for the propagation velocity. This behavior is not affected by the change of sign in $\tau_\Pi$.

In particular, a detailed study of the case where the state equation is constant ($w = -1 - \lambda$) was performed. It was found that, in order to meet the imposed conditions, in particular $a(t)$ as an increasing function of time, assuming one of the two following restrictions was necessary: a) $\tau_\Pi < 0$ and $\lambda > 0$, or b) $\tau_\Pi > 0$ and $\lambda < 0$.

In the first case, it is necessary to add the condition $\zeta |\tau_\Pi| > \frac{2}{3}$, which leads to $a(t \to \infty) \simeq (2t_\rho)^{-\zeta / 2\lambda}$. So, the value of the scale factor is finite but as large as we want, provided that we make $\lambda$ small enough. It is interesting to
note that the latter requirement makes \( w \) to be very close to \(-1\). Moreover, \( \rho(t \to \infty) = 0 \). Under these conditions the universe is accelerated until a certain time \( t_c \), which can be as large as we want, for a \( \lambda \) close enough to zero. After that time, the universe begins to slow the velocity of expansion.

In the second case, \( a(t \to \infty) = \infty \) was obtained, but with \( \rho(t \to \infty) = 3(\zeta/\lambda)^2 \), which can give us a density that increases with time if \( |\lambda| < \sqrt{3\zeta/\rho_p^{1/2}} \). When it is taken into account that \( \rho_p \) is very large \((\sim 10^{100} g/m^3)\), this bound is extremely small. Whenever we consider small \( \lambda \), but not to the above value, the energy density is a decreasing function of time. Moreover, the universe is accelerated for all times provided it is \( |\lambda| < 2/3 + \zeta \tau_{\Pi} \), which is not a strong constraint because it leaves open an interval of physically reasonable values. In addition, \( w \) will be limited, since \( \alpha = 1 \), and \( \lambda \) could be set to a value small enough to make the deviation from \(-1\) to fall into the observation error. Finally, an argument in favor of this model is the fact that entropy increases for any time.

It is worth noting that, in the case previously resolved, we have started with a linear EoS in \( \rho \) and a EMT representing a viscous fluid with causal correction. However, this can also be seen as an ideal fluid source but with an effective EoS with \( \rho^\dagger \) including the viscous correction \( (w^\dagger = p^\dagger/\rho^\dagger \text{ with } p^\dagger \text{ given by Eq. (11)}) \), in this way it can be considered as a case particular of the inhomogeneous EoS proposed in ref. \[35\]. But this choice has not been arbitrary but comes from the viscous fluid model with causal correction first developed by Israel-Stewart \[21\] and later generalized, so as to consider non-linear terms in the velocity gradients \[22, 24\].

This work opens different lines for future research. On the one hand, to consider the causal relativistic theory of fluid in a more complete version, including nonlinear terms in the velocity gradients. This will require more computational effort, but it would be interesting to discover the new conditions on the more complete set of coefficients and to evaluate their influence on dynamics of the universe. On the other hand, it can be studied whether, at least at the level of approximation used in this work, it is possible to establish a correspondence with some formulation from an analysis at a more fundamental level, i.e. with a field theory formulation. It would also be interesting to consider the state equations used in recent papers \[36\] to describe halos of dark matter, which are inferred from some experiments \[37\].

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References

[1] Sahni, V. (2004). Dark Matter and Dark Energy. arXiv preprint astro-ph/0403324; "The Physics of the Early Universe", E. Papantonopoulos (Ed.) (2005), Springer.
[2] Padmanabhan, T. (2005). Current Science, Vol. 88, No 7, 1057.
[3] Perlmutter, S. et al. (1999). Astrophys. J. 517, 565.
[4] Riess, A. G. et al. (1998). Astron. J., 116, 1009.
[5] Caldwell, R. R., Kamionkowski, M., and Weinberg, N. (2003). Phys. Rev. Lett. 91, 071301.
[6] Nakamura, K. et al.: (Particle Group Collaboration), J. Phys. G (2010), 37, 075021.
[7] Copeland, E.J., Sami, M., and Tsujikawa, S. (2006). Int. Journal of Modern Phys. D, Vol. 15, No. 11, pp 1753-1935.
[8] Nojiri, S., Odintsov, S.D., Tsujikawa, S. (2005). Phys. Rev. D, 71, 063004.
[9] Nojiri, S., and Odintsov, S.D. (2004). Phys. Rev. D, 70, 103522.
[10] Hawking, S.W., Ellis, G.F.R., "The Large Scale Structure of Space-Time", Cambridge Univ. Press 1973.
[11] Fabris, J.C., Gonçalves, S.V.B., and de Sá Ribero, R., (2006). Gen. Relativ. Grav. 38(3):495-506.
[12] Eckart, C. (1940). Phys. Rev. 58, 919.
[13] Landau, L.D. and Lifshitz, E.M., Course of Theoretical Physics. Volume 6, "Fluid Mechanics", Elsevier, 2nd edition (1987).
[14] Murphy, G.L. (1973). Phys. Rev. D 8, 4231.
[15] Zimdahl, W. (2014). International Journal of Geometric Methods in Modern Physics. Volume 11, No 2, 1460014.
[16] Brevik, I. and Grønn, Ø., (2013). Astrophys. Space Sci. 347: 399-404.
[17] Lima, J.A.S. and Germano, A.S.M. (1992). Phys. Lett. A, 170, 373-378.
[18] Hiscock, W.A. and Lindblom, L. (1983). Ann. of Phys. 151, 466-496.
[19] Dolgov, A.D. and Novikov, I.D. (1998). Phys. Lett. B, 442, Iss.1-4, 82-89.
[20] Liberati, S., Sonego, S. and Visser, M. (2002). Ann. of Phys. 298, 167-185
[21] Israel, W., and Stewart, J.M.(1979). Ann. of Phys. 118, 341-372.
[22] Romatschke, P. (2010). Int. J. Mod. Phys. E19: 1-53.
[23] Koide, T. et al. (2007), Phys. Rev. C, 75, 034909.
[24] Romatschke, P. (2010). Class. Quant. Grav. : 025006.
[25] Maartens, R. (1996). Causal Thermodynamics in Relativity. arXiv preprint astro-ph/9609119; Hanno Rund Workshop on Relativity and Thermodynamics, Natal, Brazil, Jun 1996.

[26] Landau, L.D. and Lifshitz, E.M., Course of Theoretical Physics. Volume 2, "The Classical Theory of Fields", Elsevier, 4th edition (1994).

[27] Misner, C.W., Thorne, K.S.and Wheeler, J.A. "Gravitation", W.H. Freeman and Company (1973), San Francisco.

[28] Israel, W. (1976). Ann. of Phys. 100, 310-331.

[29] Israel, W. and Stewart, J.M. (1976), Phys. Lett. A, Vol. 58, No 4, 213-215.

[30] Frampton, P.H., Ludwick, K.J., and Scherrer, R.J. (2011). Phys. Rev. D, 84, 063003.

[31] Beltran Jimenez, J., Lazkoz, R., Saez-Gomez, D., and Salzano, V. (2016), Eur. Phys. J. C76 no.11, 631.

[32] Nojiri, S., Odintsov, S. D., and Oikonomou, V. K. (2015), arXiv:1506.03307 [gr-qc].

[33] Brevik, I., Elizalde, E., Nojiri, S., and Odintsov, S. D. (2011), Phys. Rev. D 84, 103508.

[34] Castagnino, M. and Laciana, C. (2002), Class. Quant. Grav. 19, 2657-2670.

[35] Nojiri, S. and Odintsov, S. D. (2005), Phys. Rev. D 72:023003.

[36] Chavanis, P.H., Phys. Lett. B. (2016), 758, pp 59-66.

[37] Donato, F. et al. (2009). Mon. Not. R. Astron. Soc. 397, 1169.