ON REFLECTION REPRESENTATIONS OF COXETER GROUPS OVER
NON-COMMUTATIVE RINGS II

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Abstract. This paper generalizes an isomorphism between certain quotients of path algebras
and matrix rings over free products, used in the first paper in this sequence. Applications of the
isomorphism to certain path algebras associated to Coxeter groups and to Cohn path algebras
are given.

1. Introduction

Representations of a Coxeter system, \((W, S)\), over a number of quotients of a path algebra
related to the Coxeter graph, \(\Gamma_{W,S}\), of the system are studied in Dyer \[17\]. In Pilkington \[27\],
we studied one of these quotient algebras, \(R_{W,S}\), in detail. We showed that in the case of a Coxeter
system of finite rank, \(R_{W,S}\) is the direct limit of the corresponding algebras defined on the finite
rank parabolic subgroups. In that paper, we answered in the affirmative a question posed in Dyer
\[17\] about its multiplicative properties. In the course of that proof, we used an imbedding of the
quotient algebra, \(R_{W,S}\), for a Coxeter group of finite rank \(N\), into the ring of \((N + 1) \times (N + 1)\)
matrices over a free product of rings, related to the edges of the Coxeter graph. These rings are
either field extensions of \(\mathbb{Q}\), rings of Laurent polynomials over \(\mathbb{Q}\) or rings of polynomials in two
non-commuting variables over \(\mathbb{Q}\).

The above imbedding uses a special case of an isomorphism between the path algebra
\(\hat{R}_{W,S} = R_{W,S} \otimes_{\mathbb{Z}} \mathbb{Q}\) and the \(N \times N\) matrices over such a free product of rings, where \(N\) is the rank of
the system. This imbedding exists for graphs \(\Gamma_{W,S}\) with a spanning tree where the edges of the
spanning tree are associated to finite entries in the Coxeter Matrix. This applies in the case of the
Weyl groups and many commonly studied examples. In this paper, we present this more general
isomorphism. Since the imbedding is applicable to a wider range of path algebras, we pull out the
general principles and demonstrate applications to both Coxeter systems and Cohn path algebras
(see Abrahms et al. \[1\]).

Let \((W, S) = \{s_i\}_{1 \leq i \leq N}\) be a Coxeter system of finite rank with Coxeter matrix \((m_{ij})_{1 \leq i, j \leq N}\). The
Coxeter graph \(\Gamma_{W,S} = (S, Y)\) has vertices \(S\) and edges \(Y = \{y_{ij} = [s_is_j], 1 \leq i \leq N, 3 \leq m_{ij} \leq \infty\}\).
The associated path algebra \(Q \Gamma_{W,S}\) and its quotient \(\hat{R}_{W,S}\) are fully described in Section 5 of this
paper. Let \(K_n\) denote the field extension of \(\mathbb{Q}\) obtained by adjoining \(4 \cos^2 \frac{\pi}{n}\). Let \(\mathbb{Q}[x_{ij}, \bar{x}_{ij}]\) be the
ring of Laurent polynomials over \(\mathbb{Q}\) and \(\mathbb{Q}[x_{ij}, \bar{x}_{ij}]\) be the polynomial ring in the non-commuting
variables \(x_{ij}\) and \(\bar{x}_{ij}\). We show the following:

Theorem 1.1. Let \((W, S)\) be a finite rank Coxeter system, with Coxeter graph \(\Gamma_{W,S}\) as described
above. Let \(Y_1 = \{y_{ij} \in Y|3 \leq m_{ij} < \infty\}\). Let \(T\) be a spanning tree for \(\Gamma_{W,S}\), whose edges are denoted

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by \(Y^T\), with the property that \(Y^T \subseteq Y_1\). Then we have a unital ring isomorphism
\[
\hat{R}_{W,S} \otimes_{\mathbb{Z}} \mathbb{Q} \cong M_N(\mathbb{Q}),
\]
where \(\mathbb{Q}\) is isomorphic to the free product of the following rings;
\[
\mathbb{K}_{m_{ij}}, \quad 3 \leq m_{ij} < \infty, \quad i < j,
\]
\[
\mathbb{Q}[y_{ij}, y_{ij}^{-1}], \quad y_{ij} \in Y \setminus Y_T, \quad 3 \leq m_{ij} < \infty, i < j
\]
\[
\mathbb{Q}[x_{ij}, \bar{x}_{ij}], \quad y_{ij} \in Y \setminus Y_T, \quad m_{ij} = \infty, i < j
\]

To demonstrate the wider applicability of the imbedding described above, we apply the imbedding to the Cohn path algebras. In [2], Abrahms and Kanuni show that a Cohn path algebra has the IBN property. We give an alternative proof of this, in fact we show that a Cohn path algebra has a stronger property, namely the UGN property or in more modern terms; it satisfies the rank condition. In section [6] we show:

**Theorem 1.2.** A Cohn path algebra has the UGN property.

### 2. General Notation and Definitions

2.1. The notation \(X = \{a, b, c\}\) will be used to denote that \(X\) is a set with elements \(a, b\) and \(c\) and \(Y \subset X\) will indicate that \(Y\) is a subset of \(X\). The set \(X - Y\) will be the set of all elements in \(X\) but not in \(Y\) and the set \(X \cup Z\) will denote the disjoint union of the sets \(X\) and \(Z\).

Unless otherwise stated, all rings considered have an identity (are unital). If \(R\) is a ring, we will denote the identity of \(R\) by \(1_R\) in situations where the ring in question may not be clear from the context. The group of multiplicative units of \(R\) will be denoted by \(R^*\). If \(\phi : R_1 \rightarrow R_2\) is a homomorphism of rings and \(S \subset R_1\) is a subring of \(R_1\), we let \(\phi|_S\) denote the restriction of \(\phi\) to \(S\). If \(\psi : R_2 \rightarrow R_3\) is a homomorphism from \(R_2\) to the ring \(R_3\), we let \(\psi \circ \phi\) denote the composition map. We will use the notation \(R_1 \cong R_2\) to indicate that the rings \(R_1\) and \(R_2\) are isomorphic.

If \(X\) is a subset of a ring \(R\), the two sided ideal of \(R\) generated by the set \(X\) will be denoted by \((X)\). If \(I\) is an ideal of a ring \(R\), we write \(r_1 \equiv r_2 \mod I\) when \(r_1 - r_2 \in I\). We denote the ring of \(n \times n\) matrices over a ring \(R\) by \(M_n(R)\) and we will denote the set of \(m \times n\) matrices by \(M_{m \times n}(R)\) where \(m\) is possibly not equal to \(n\). We will also use the notation \((x \in X | r \in R)\) to denote a group with generators in the set \(X\) and relations in the set \(R\). In cases where the meaning of the notation is not clear from the context, we will add clarification.

The symbol \(\mathbb{N}\) will be used to denote the non-negative integers, \([0, 1, 2, 3, \ldots]\), and the symbol \(\mathbb{N}_{\geq k}\) will be used to denote the subset of \(\mathbb{N}\) consisting of numbers greater than or equal to \(k\). We let \([k]\) denote the set \([1, \ldots, k]\). The integers will be denoted by \(\mathbb{Z}\), and the rational numbers by \(\mathbb{Q}\). Polynomial rings \(\mathbb{Q}[\{x_i\}_{i \in I}]\) will have non-commuting variables \(\{x_i\}_{i \in I}\) unless stated otherwise.

Let \(A\) be a commutative unital ring. An (associative, possibly non-unital) \(A\)-algebra is a (possibly non-unital, associative) ring \(B\) with an additional structure of unital \(A\)-module for which the multiplication map \(B \times B \rightarrow B\) is \(A\)-bilinear (i.e. \((\alpha_1 b_1 + \alpha_2 b_2)(\alpha_3 b_3 + \alpha_4 b_4) = \sum_{i=1}^{2} \sum_{j=1}^{2} \alpha_i \alpha_j b_i b_j\), for all \(\alpha_i \in A, b_j \in B\)). Morphisms of \(A\)-algebras are \(A\)-linear ring homomorphisms. We call \(A\) the coefficient ring of the \(A\)-algebra \(B\). We regard rings as \(\mathbb{Z}\)-algebras. We say \(B_1\) is a sub-(\(A\))-algebra of the \(A\)-algebra \(B\) if \(B_1\) is a subring of \(B\) which inherits its \(A\)-algebra structure from \(B\).

### 3. Graphs and Spanning Trees

As in Pilkington [27], we use the notation and terminology of Serre [28] for graphs and trees.
Definition 1. A graph $\Gamma = (S, Y)$ consists of a set of vertices $S = \text{vert } \Gamma$, a set of edges $Y = \text{edge } \Gamma$ and two maps

$$Y \to S \times S, \quad y \to (o(y), t(y))$$

and

$$Y \to Y, \quad y \to \bar{y},$$

which satisfy the following condition: for each $y \in Y$, we have $\bar{\bar{y}} = y, \bar{\bar{y}} \neq y$ and $o(y) = t(\bar{y})$.

Each $s \in S$ is called a vertex of $\Gamma$ and each $y \in Y$ is called an edge of $\Gamma$. If $y \in Y$, $o(y)$ is called the origin of $y$ and $t(y)$ is called the terminus of $y$, together $o(y)$ and $t(y)$ are called the extremities of $y$. We say that two vertices are adjacent if they are the extremities of some edge. If $S = \{s_i\}_{i \in I}$ and $y \in Y$, we let $o'(y) = i$, when $o(y) = s_i$, and we let $t'(y) = j$ when $t(y) = s_j$.

Definition 2. An orientation of a graph $\Gamma$ is a subset $Y_\circ$ of $Y = \text{edge } \Gamma$ such that $Y$ is the disjoint union $Y = Y_\circ \cup \bar{Y}_\circ$.

3.1. Paths. A path of length $n \geq 1$ in a graph $\Gamma$ is a sequence of edges $p = (y_1, \ldots, y_n)$ such that $t(y_i) = o(y_{i+1})$, $1 \leq i \leq n - 1$. We will denote such a path in what follows as $p = y_1 \ldots y_n$. Paths of length zero are just single vertices and will be denoted by $p = [s]$ for $s \in S$. The length of a path $p$ will be denoted by $\ell(p)$. Let $\mathfrak{P}$ denote the set of all paths in $\Gamma$, we can extend the maps $o$ and $t$ to $\mathfrak{P}$ by letting $o(y_1, \ldots, y_n) = o(y_1) = s_0$ and $t(y_1, \ldots, y_n) = t(y_n) = s_n$ and $o([s]) = s = t([s])$. If $s_i = t(y_i)$ and $s_{i-1} = o(y_i)$, we say that $p$ is a path from $s_0$ to $s_n$ and that $s_0$ and $s_n$ are the extremities of the path.

A graph is connected if any two vertices are the extremities of at least one path. If $p = y_1 \ldots y_n$ and $y_{i+1} = \bar{y}$ for some $1 \leq i \leq n - 1$, then the pair $(y_i, y_{i+1})$ is called a backtracking. A path $p$ is a circuit if it is a path without backtracking such that $o(p) = t(p)$ and $\ell(p) \geq 1$. A circuit of length one is called a loop. A graph $\Gamma$ is called combinatorial if it has no circuit of length less than or equal to two. If a graph $\Gamma$ is combinatorial, a path $p = y_1 \ldots y_n$ is determined by the extremities of its edges and can be characterized and denoted by its vertices as

$$p = y_1 \ldots y_n = \{s_0s_1 \ldots s_n\}, \text{ where } s_i = t(y_i), s_{i-1} = o(y_i).$$

A geometric edge of a combinatorial graph, $\Gamma$, is a set $\{s_1, s_2\}$ of extremities of an edge in $\Gamma$. Each such geometric edge corresponds to a pair of edges $(y, \bar{y})$ in the edge set of $\Gamma$. A combinatorial graph is determined by its vertices and geometric edges.

3.2. Trees. If $T \not\subseteq \emptyset$ is a connected graph without circuits, $T$ is called a tree. It is a combinatorial graph. A geodesic in a tree is a path without backtracking. By Proposition 8, Serre [28], if $s_1$ and $s_2$ are two vertices in a tree, $T$, there is exactly one geodesic from $s_1$ to $s_2$ and it has distinct vertices. The length of the geodesic from $s_1$ to $s_2$ is called the distance from $s_1$ to $s_2$ and we will denote it by $d_T(s_1, s_2)$.

Given a non-empty connected graph $\Gamma$, the set of subgraphs (defined in the obvious way) which are trees are ordered by inclusion. The union of a chain of trees is a tree and hence the set has a maximal element, by Zorn’s lemma. Such a maximal element is called a maximal tree of $\Gamma$. By Proposition 11, Serre [28], if $T$ is a maximal tree of a connected non-empty graph $\Gamma$, then $T$ contains all of the vertices of $\Gamma$. Furthermore, if $\Gamma$ has a finite number of vertices with $\text{Card}(S) = s$, then the number of edges of the spanning tree $2(s - 1)$ (the number of geometric edges is $s - 1$). If the graph $\Gamma$ is not a tree, then, adding an edge (from the edge set of $\Gamma$) to $T$, creates a cycle, called a fundamental cycle. There are $e - s + 1$ distinct fundamental cycles in $\Gamma$ where $e$ is the number of edges in $\Gamma$. 
3.3 Path Algebras. Let $\Gamma = (S, Y)$ be a graph with vertices $S = \{s_i \mid i \in I\}$, edges $Y$ and paths $\mathfrak{P}$. Given a commutative ring $A$ with identity, the path algebra of $\Gamma$ over $A$, denoted $A\Gamma$, is an associative $A$-algebra, with an $A$-basis of paths in $\mathfrak{P}$. Multiplication of paths is given by concatenation:

$$(y_1 y_2 \ldots y_n)(y'_1 y'_2 \ldots y'_m) = \begin{cases} 0 & \text{if } t(y_n) \neq o(y'_1) \\ y_1 y_2 \ldots y_n y'_2 \ldots y'_m & \text{if } t(y_n) = o(y'_1) \end{cases}$$

The paths of length 0, $\{[s_i] \mid s_i \in S\}$ form a set of orthogonal idempotents with this multiplication and if $S$ is finite, $A\Gamma$ has an identity given by

$$1 = \sum_{s \in S} [s].$$

Let $\mathfrak{F}_\Gamma = \{f_y(t) \mid y \in Y_1\}$ be a set of polynomials in the polynomial ring $A[t]$ indexed by $Y_1 \subseteq Y$.

**Definition 3.1.** We say that $\mathfrak{F}_\Gamma$ has property $F$ if the following conditions are satisfied:

1. $Y_1 = Y_1$ and for each $y \in Y_1$, $f_y$ is non-constant, $f_y = f_y$ and the constant term of $f_y$ is a unit in $A$.

In what follows, if $f \in A[t]$ and $y \in Y$, we let $f(y\bar{y})$ denote the element of the subring $[o(y)]A\Gamma[o(y)] \subset A\Gamma$ (where the imbedding is non-unital) that we get by replacing $t$ by $y\bar{y}$ and $1_A$ by $[o(y)]$ in the polynomial $f$ (this is $f$ evaluated at $y\bar{y}$ in the subring $[o(y)]A\Gamma[\{o(y)\}]$). Let $I_{\mathfrak{F}_\Gamma}$ denote the two sided ideal of $A\Gamma$ generated by the set of elements $\{f_y(y\bar{y}) \mid y \in Y_1\}$, and we let $R_{\mathfrak{F}_\Gamma}$ denote the quotient algebra $R_{\mathfrak{F}_\Gamma} = A\Gamma/I_{\mathfrak{F}_\Gamma}$.

Note that each pair $(y, \bar{y}) \in Y_1$ gives two relations in $R_{\mathfrak{F}_\Gamma}$, namely $f_y(y\bar{y}) = 0$ and $f_y(\bar{y}y) = f_y(y\bar{y}) = 0$. If $x \in A\Gamma$, we let $\bar{x}$ denote the element $x + I_{\mathfrak{F}_\Gamma}$ of $R_{\mathfrak{F}_\Gamma}$, and if $X \subset A\Gamma$, we let $\bar{X} = \{\bar{x} \mid x \in X\}$.

Using the above notation, we have the following:

**Lemma 3.2.** Let $A$ be a commutative ring and $\Gamma = (S, Y)$ a graph. Let $\mathfrak{F}_\Gamma = \{f_y(t) \mid y \in Y_1 \subseteq Y\}$ be a set of polynomials in $A[t]$ with property $F$. For each $y \in Y_1$, there exists $y' \in \{t(y)\}A\Gamma[o(y)]$ such that $y\bar{y}' = [o(y)]$ and $\bar{y}'y = [t(y)]$.

**Proof.** Since $f_y(t)$ is non-constant and has a non-zero invertible constant term, it is of the form $f_y(t) = \kappa h(t) - \kappa$, for some $h(t) \in A[t]$ and $\kappa \in A^*$. I claim that $y\bar{y}' = \frac{1}{\kappa}y\bar{y}h(y\bar{y})$ satisfies the equalities given in the statement of the lemma. Since

$$yy' = \frac{1}{\kappa}y\bar{y}h(y\bar{y}) = \frac{1}{\kappa}f_y(y\bar{y}) + [o(y)],$$

we see that $\overline{(yy')} = [o(y)]$. Likewise,

$$y'\bar{y} = \frac{1}{\kappa}\bar{y}h(\bar{y}y)y = \frac{1}{\kappa}y\bar{y}h(\bar{y}y) = \frac{1}{\kappa}f_y(\bar{y}y) + \overline{[t(y)]} = \frac{1}{\kappa}f_y(\bar{y}y) + \overline{[t(y)]}.$$ 

Therefore $\overline{(y'\bar{y})} = \overline{[t(y)]}$ and this proves the lemma. $\square$

4. A faithul Representation of $R_{\mathfrak{F}_\Gamma}$.

We restrict our attention to a graph with a finite number of vertices and a finite number of edges. Let $\Gamma = (S, Y)$ be a connected graph with vertices $S = \{s_1, \ldots, s_N\}$ edges $Y$, where $Y$ is finite. Let $K$ be a field and let $\mathfrak{F}_\Gamma = \{f_y \mid y \in Y_1\} \subset K[t]$ be a set of polynomials with property $F$. We assume that $\Gamma$ has a maximal tree $T = (S, Y^T)$, where the edges $Y^T$ of $Y$ have the property that $Y^T \subseteq Y_1$. 

4.1. Fixing such a maximal tree $T$ with the above property, we choose a fixed vertex $r \in S$ and define an orientation on $T$ by letting

$$Y^T_+ = \{ y \in Y^T | d_T(r, o(y)) < d_T(r, t(y)) \}.$$

Clearly $Y^T = Y^T_+ \cup \overline{Y^T_+}$. We extend this to an orientation on $\Gamma$ by choosing an orientation on the edges associated to the fundamental cycles $B = Y \setminus Y^T = \{ b_1, \ldots, b_m \}$ for some choice of $B_+ = \{ b_1, \ldots, b_m \}$. Letting $Y_+ = Y^T_+ \cup B_+$ we get an orientation of $\Gamma$; $Y = Y_+ \cup \overline{Y_+}$.

4.2. As defined in the previous section, $R_{\overline{\Gamma}}$ denotes the quotient of the path algebra $K\Gamma$ associated to the set of polynomials $\overline{\Gamma}$. We define a special set of elements of $R_{\overline{\Gamma}}$ associated to the geodesics in $T$, which will help us to define an isomorphism between $R_{\overline{\Gamma}}$ and a matrix ring over a non-commutative ring.

Let $G^i_j$ denote the geodesic in $T$ starting at $s_i$ and ending at $s_j$ for $s_i, s_j \in S$. If $i \neq j$ and $G^i_j = y_1 y_2 \ldots y_n$, we let $P^i_j = \alpha(y_1) \alpha(y_2) \ldots \alpha(y_n) \in KT$, where $\alpha : Y^T \to KT$ is given by

$$\alpha(y) = \begin{cases} y & \text{if } y \in T_+ \\ \overline{y} & \text{if } y \in \overline{T_+} \end{cases}$$

We let $P^i_i = [s_i]$. Note that by Lemma 5.2 $\alpha(y) \in [\alpha(y)]KT[t(y)]$ for all $y \in Y^T$.

**Lemma 4.1.** We have the following identities in $KT$, where $s_i, s_j, s_k, s_l \in S$:

- $P^i_i = [s_i]$
- $P^i_j P^j_i = 0$ if $j \neq k$
- $P^i_j P^j_k P^k_i$ mod $I_{\overline{\Gamma}}$ if $j = k$

**Proof.** By definition, $P^i_i = [s_i]$. Suppose that $i \neq j$ and $k \neq l$, with $G^i_j = y_1 y_2 \ldots y_n$ and $G^k_l = y'_1 y'_2 \ldots y'_m$, where $y_1, y'_1 \in Y^T$, $1 \leq i \leq n, 1 \leq j \leq m$. Clearly, if $j \neq k$, then $P^i_j \in [s_i]KT[s_j]$ and $P^k_l \in [s_k]KT[s_l]$, hence $P^i_j P^k_l = 0$. It is also clear that $P^i_j P^j_k = 0$ if $j \neq k$ and either $i = j$ or $k = l$, we leave the details to the reader.

To show that $P^i_j P^j_k P^k_l \equiv P^i_l P^l_i$ mod $I_{\overline{\Gamma}}$, we proceed by induction on $\ell(G^i_j) + \ell(G^j_k) = n + m$.

Suppose first that $n + m = 0$, then we must have $i = j = l$ and $P^i_j P^j_k = [s_i] = P^i_i$. If $n + m = 1$, then either $n = 0$ or $m = 0$. If $n = 0$, then $i = j$ and $P^i_j P^j_k = [s_i] P^i_k = P^i_k$. The argument is similar if $m = 0$. Now suppose that $P^i_j P^j_k \equiv P^i_i$ mod $I_{\overline{\Gamma}}$, if $\ell(G^i_j) + \ell(G^j_k) < n + m$ and consider a product $P^i_j P^j_k$, where $\ell(G^i_j) + \ell(G^j_k) = n + m$. If the path $G^i_j G^j_k = y_1 y_2 \ldots y_n y'_1 y'_2 \ldots y'_m$ in $T$ has no backtracking, it is the geodesic from $s_i$ to $s_l$ in $T$. In that case, we have $P^i_j P^j_k = \alpha(y_1) \alpha(y_2) \ldots \alpha(y_n) \alpha(y'_1) \alpha(y'_2) \ldots \alpha(y'_m) \equiv P^i_i$. If the path $G^i_j G^j_k = y_1 y_2 \ldots y_n y'_1 y'_2 \ldots y'_m$ has a backtracking, it must be $(y_n, y'_1)$, since neither $G^i_j$ nor $G^j_k$ have backtracking. Thus, $\alpha(y_n) \alpha(y'_1) = \overline{y}_n y'_1$ or $y_n y'_1$. By Lemma 3.2, we have $\alpha(y_n) \alpha(y'_1) \equiv [\alpha(y_n)]$ mod $I_{\overline{\Gamma}}$ and

$$P^i_j P^j_k = \alpha(y_1) \alpha(y_2) \ldots \alpha(y_n-1) \alpha(y'_1) \ldots \alpha(y'_m) \mod I_{\overline{\Gamma}}.$$

Since $y_1 y_2 \ldots y_{n-1}$ and $y'_2 y'_3 \ldots y'_m$ have no backtracking and their edges are in $T$, they must be equal to $G^i_j$ and $G^j_k$ respectively, where $s_i = t(y_{n-1}) = \alpha(y'_1)$. Now $\ell(G^i_j) + \ell(G^j_k) = n + m - 2 < n + m$, therefore, by induction, $P^i_j P^j_k \equiv P^i_i P^i_j \equiv P^i_i$ mod $I_{\overline{\Gamma}}$. \qed
4.3. Using the orientation $Y = Y_1 \cup T_1$ defined above, we can define an algebra $Q$ associated to our graph $\Gamma$ and set of polynomials $\mathcal{F}_1 = \{f_y|y \in Y_1\}$. We let

$$Q = K[\{s_y|y \in Y_1\}, \{x_y, \bar{x}_y\}_{y \in B_1}] / J,$$

where $Q_1 = K[\{s_y|y \in Y_1\}, \{x_y, \bar{x}_y\}_{y \in B_1}]$ denotes the free $K$-algebra generated by the non-commuting variables $\{s_y|y \in Y_1\}, \{x_y, \bar{x}_y\}_{y \in B_1}$ and $J$ is the ideal of $Q_1$ generated by the polynomials

$$\{f_y(t_y)|y \in Y_1, f_y(x_y \bar{x}_y), f_y(x_y \bar{x}_y)\}_{y \in B_1 \cap Y_1}.$$

4.4. Since $Q_1$ is a free $K$-algebra on the generators $\{s_y|y \in Y_1\}, \{x_y, \bar{x}_y\}_{y \in B_1}$, for each $i, 1 \leq i \leq N$, there is a unique homomorphism of $K$-algebras, $\hat{\psi}_i: Q_1 \rightarrow [s_i] \Gamma_1[s_i] \in K \Gamma$, such that

$$\hat{\psi}_i(1_{Q_1}) = [s_i],$$

$$\hat{\psi}_i(t_y) = P_{s_i}(y) y \bar{s}_i P_{o(y)}, \quad y \in Y_1,$$

$$\hat{\psi}_i(x_y) = P_{s_i}(y) y P_{o(y)}, \quad y \in B_1,$$

$$\hat{\psi}_i(\bar{x}_y) = P_{s_i}(y) y \bar{s}_i P_{o(y)}, \quad y \in B_1.$$

**Lemma 4.2.** Let $\hat{\psi}_i: Q_1 \rightarrow K \Gamma$ be as described above. Then $\hat{\psi}_i(J) \subseteq \mathcal{I}_{\mathcal{G}_1}$ and $\hat{\psi}_i$ gives a well defined (non-unital) $K \Gamma$-algebra homomorphism from $\psi_i: Q \rightarrow K \mathcal{G}_1$.

**Proof.** It suffices to show that $\hat{\psi}_i$ sends the generators of $J$ to $I_{\mathcal{G}_1}$. Let $y \in Y_1$, then

$$\hat{\psi}_i(f_y(t_y)) = [s_i] f_y(\hat{\psi}_i(t_y))[s_i]$$

$$= [s_i] f_y(P_{s_i}(y) y \bar{s}_i P_{o(y)})[s_i]$$

$$= P_{s_i}(y) f_y(y \bar{s}_i) P_{o(y)} \in I_{\mathcal{G}_1}.$$

For $y \in B_1$, note that

$$\hat{\psi}_i(x_y \bar{x}_y) = \hat{\psi}_i(x_y) \hat{\psi}_i(\bar{x}_y) = P_{s_i}(y) y P_{o(y)} \in I_{\mathcal{G}_1}.$$

Thus, we have

$$\hat{\psi}_i(f_y(x_y \bar{x}_y)) = [s_i] f_y(\hat{\psi}_i(x_y \bar{x}_y))[s_i]$$

$$= [s_i] f_y(P_{s_i}(y) y \bar{s}_i P_{o(y)})[s_i]$$

$$= P_{s_i}(y) f_y(y \bar{s}_i) P_{o(y)} \mod I_{\mathcal{G}_1}.$$

Since $f_y(y \bar{s}_i) \in I_{\mathcal{G}_1}$, we have $\hat{\psi}_i(f_y(x_y \bar{x}_y)) \in I_{\mathcal{G}_1}$. Similarly

$$\hat{\psi}_i(\bar{x}_y x_y) \equiv P_{s_i}(y) y \bar{s}_i P_{o(y)} \mod I_{\mathcal{G}_1}$$

and

$$\hat{\psi}_i(f_y(x_y \bar{x}_y)) \equiv P_{s_i}(y) f_y(y \bar{s}_i) P_{o(y)} \mod I_{\mathcal{G}_1},$$

giving $\hat{\psi}_i(f_y(x_y \bar{x}_y)) \in I_{\mathcal{G}_1}$. This proves the lemma. \qed
Thus, for each \( i, 1 \leq i \leq N \), we have a well defined homomorphism of the quotient rings \( \psi_i : Q \to R_{\tilde{\mathcal{S}}_r} \), given by
\[
\psi_i(q + J) = \psi(q) + I_{\tilde{\mathcal{S}}_r}, \quad q \in Q_1.
\]

**Lemma 4.3.** For \( 1 \leq i, j \leq N \) and \( q \in Q \), we have
\[
\psi_j(q) = P_{s_j}^i \psi_i(q) P_{s_j}^i.
\]

**Proof.** It is enough to show that this is true on the generators of \( Q \). Using Lemma 4.4 and the definition of \( \psi_i \), it is easy to check that for each
\[
g \in \{(t_y)_{y \in \mathcal{Y}^T}, \{x_y, \bar{x}_y\}_{y \in \mathcal{B}_+} \},
\]
we have \( P_{s_j}^i \psi_i(g) P_{s_j}^i = \psi_j(g) \). The details are left to the reader. \( \square \)

4.5. Let \( \Psi : M_N(Q) \to R_{\tilde{\mathcal{S}}_r} \) be the homomorphism of \( K \)-vector spaces from the matrix ring \( M_N(Q) \) to \( R_{\tilde{\mathcal{S}}_r} \) which acts on basis elements of \( M_N(Q) \) of the form \( qe_{ij} \) as follows:
\[
\Psi(qe_{ij}) = \psi_i(q) P_{s_i}^j, \quad q \in Q,
\]
where \( e_{ij} \) is the \( N \times N \) matrix with 1 in the \((i, j)\) position and zeros elsewhere.

**Lemma 4.4.** The map \( \Psi : M_N(Q) \to R_{\tilde{\mathcal{S}}_r} \) defined above is a homomorphism of rings.

**Proof.** It suffices to check that \( \Psi((q_1e_{ij})(q_2e_{kl})) = \Psi(q_1e_{ij})\Psi(q_2e_{kl}) \), \( q_1, q_2 \in Q, 1 \leq i, j, k, l \leq N \).

If \( j \neq k \), \( \Psi((q_1e_{ij})(q_2e_{kl})) = \Psi(O) = 0 \), where \( O \) denotes the \( N \times N \) zero matrix. On the other hand, \( \Psi(q_1e_{ij})\Psi(q_2e_{kl}) = \psi_i(q_1) P_{s_i}^j \psi_k(q_2) P_{s_k}^l = 0 \), since \( \psi_k(q_2) \in \left[ \tilde{s}_k \right] R_{\tilde{\mathcal{S}}_r} \left[ s_k \right] \).

If \( j = k \), we have
\[
\Psi((q_1e_{ij})(q_2e_{ji})) = \Psi((q_1q_2e_{ij})) = \psi_i(q_1q_2) P_{s_j}^i \psi_j(q_2) P_{s_j}^i = \psi_i(q_1) P_{s_j}^i \psi_j(q_2) P_{s_j}^i = \psi_i(q_1) P_{s_j}^i \psi_j(q_2) P_{s_j}^i,
\]
by lemmas 4.1 and 4.3. Thus \( \Psi((q_1e_{ij})(q_2e_{ji})) = \Psi(q_1e_{ij})\Psi(q_2e_{ji}) \).

To show that \( \Psi \) is an isomorphism, we construct an inverse homomorphism \( \Phi : R_{\tilde{\mathcal{S}}_r} \to M_N(Q) \). Let \( \phi : K\Gamma \to M_N(Q) \) be defined as follows on the vertices and edges of \( \Gamma \) (which generate \( K\Gamma \) as a \( K \)-algebra):
\[
\phi([s_i]) = e_{ii}, \quad s_i \in S
\]
\[
\phi(y) = \begin{cases} 
  e_{\sigma(y)'}(y), & y \in Y_T^+ \\
  t^e_{\sigma(y)'}(y), & y \in Y_T^- \\
  x^e_{\sigma(y)'}(y), & y \in B_T^+ \\
  \bar{x}^e_{\sigma(y)'}(y), & y \in B_T^-
\end{cases}
\]

Since \( e_{ij}e_{kl} \neq 0 \) if and only if \( j = k \), it is not difficult to see that \( \phi \) can be extended to a well defined homomorphism of \( K \)-algebras:
\[
\phi : K\Gamma \to M_N(Q).
\]
Lemma 4.5. Let $\phi : K\Gamma \to M_N(Q)$ be the $K$-algebra homomorphism defined above, then $I_{3r} \subseteq \ker \phi$.

Proof. It is enough to show that the generators of $I_{3r}$, $\{f_y(y\bar{y}), y \in Y_1\}$, are in $\ker \phi$. If $y \in Y^T \subseteq Y_1$, then $\phi(y\bar{y}) = \phi(y)\phi(\bar{y})$ is either $t_y e_{\alpha(y)\alpha'(y)}$ or $\bar{y} t_y e_{\alpha(y)\alpha'(y)}$. Thus $\phi(f_y(y\bar{y})) = f_y(t_y) e_{\alpha(y)\alpha'(y)} = O$ or $\phi(f_y(y\bar{y})) = f_y(t_y) e_{\alpha(y)\alpha'(y)} = f_y(t_y) e_{\alpha(y)\alpha'(y)} = O$ where $O$ is the zero matrix in $M_N(Q)$.

If $y \in B_+ \cap Y_1$, then

$$\phi(y\bar{y}) = x_y e_{\alpha(y)\alpha'(y)} = x_y e_{\alpha(y)\alpha'(y)} = O \text{ where } O \text{ is the zero matrix in } M_N(Q).$$

Thus, as before, $\phi(f_y(y\bar{y})) = f_y(x_y e_{\alpha(y)\alpha'(y)}) = O$ where $O$ is the zero matrix in $M_N(Q)$.

For $y \in \overline{B}_+ \cap Y_1$, we have

$$\phi(y\bar{y}) = \bar{x}_y x_y e_{\alpha(y)\alpha'(y)} = \bar{x}_y x_y e_{\alpha(y)\alpha'(y)}.$$

Thus $\phi(f_y(y\bar{y})) = f_y(x_y \bar{x}_y e_{\alpha(y)\alpha'(y)}) = O$, where $O$ is the zero matrix in $M_N(Q)$.

By the above lemma, the quotient map

$$\Phi : R_{3r} \to M_N(Q)$$

is a well defined ring homomorphism, with $\Phi(p) = \phi(p)$, for $p \in K\Gamma$.

We now show that $\Phi = \Psi^{-1}$ and thus we have that $R_{3r}$ is isomorphic to $M_N(Q)$ as a $K$-algebra.

Lemma 4.6. Let $\phi : K\Gamma \to M_N(Q)$ be as defined above and let $y \in Y^+_T$. Then $\phi(y^+) = e_{\alpha(y)\alpha'(y)}$ and $\phi(P_{s_i}) = e_{k,l}, 1 \leq k, l \leq N$.

Proof. If $y \in Y^+_T$, then (as pointed out in the proof of Lemma 3.2), $f_y(t) = th(t) - \kappa$ for some $\kappa \neq 0 \in K$ and $h(t) \in K[t]$. We had $y^+ = \frac{1}{\kappa} h(y\bar{y})$.

Thus,

$$\phi(y^+) = \frac{1}{\kappa} t_y h(t_y) e_{\alpha(y)\alpha'(y)} = \frac{1}{\kappa} (f_y(t_y) + \kappa) e_{\alpha(y)\alpha'(y)} = \frac{1}{\kappa} Q e_{\alpha(y)\alpha'(y)} = e_{\alpha(y)\alpha'(y)}.$$

Recall that for $1 \leq i, j \leq N, P_{s_i} = \alpha(y_1)\alpha(y_2)\ldots\alpha(y_n)$, where $y_1 y_2 \ldots y_n$ is the geodesic in $T$ from $s_i$ to $s_j$. Now $\alpha(y_k) = y_k, \alpha(y_k) = y_k^\ast$ if $y_k \in Y^+_T$ and $\alpha(y_k) = y_k^\ast$ if $y_k \in Y^+_T$. Thus since $\phi(y) = e_{\alpha(y)\alpha'(y)}$ and $\phi(y^+) = e_{\alpha(y)\alpha'(y)}$ for $y \in Y^+_T$, the result follows.

Lemma 4.7. Let $\Phi : R_{3r} \to M_N(Q)$ and $\Psi : M_N(Q) \to R_{3r}$ be as defined above. Then, the composition $\Phi \circ \Psi$ is the identity map on $M_N(Q)$.

Proof. It suffices to show that $\Phi \circ \Psi$ is the identity on the following set of generators of $M_N(Q)$:

$$\{ e_{ij} \}_{1 \leq j \leq N} \cup \{ t_y e_{ij} \}_{y \in Y^+_T} \cup \{ x_y e_{ij} \}_{y \in B_+}.$$ 

We have $\Phi(\Psi(e_{ij})) = \Phi(\psi_1(1_q) P_{s_i}^j) = \Phi(\tilde{s}_i) e_{ij} = e_{ij}$ for $1 \leq i, j \leq N$.

Next consider $t_y e_{ij}$ where $y \in Y^+_T$ and $1 \leq i, j \leq N$. In this case, we have

$$\Phi(\Psi(t_y e_{ij})) = \Phi\left( \psi_1(t_y) P_{s_i}^j \right)$$

$$= \Phi\left( \frac{P_{1Q}^j(y \bar{y})}{P_{1Q}^j(y \bar{y})} P_{s_i}^j \right)$$

$$= \Phi\left( \frac{P_{s_i}^j(y \bar{y})}{P_{s_i}^j(y \bar{y})} \right)$$

$$= e_{ij} \Phi(\bar{y} \bar{y}) e_{ij}$$

$$= e_{ij} e_{ij} e_{ij} e_{ij} e_{ij} e_{ij} e_{ij} e_{ij} e_{ij} e_{ij} e_{ij} e_{ij} e_{ij} e_{ij} e_{ij} e_{ij} e_{ij} e_{ij}.$$
If \( y \in B_+ \), then
\[
\Phi(\Psi(x_y e_{ij})) = \Phi\left( \psi_i(x_y P_{s_i}^y) \right) = \Phi\left( P_{s_i}^{\psi(y)} \tilde{y} P_{s_i}^{o(y)} P_{s_i}^{t(y)} \right) = \Phi\left( P_{s_i}^{\psi(y)} \tilde{y} P_{s_i}^{o(y)} P_{s_i}^{t(y)} \right) = \epsilon_{i\alpha'(y)} \Phi(\tilde{y}) e_{i\alpha'(y)} e_{j\alpha'(y)} = \epsilon_{i\alpha'(y)} x_y e_{i\alpha'(y)} e_{j\alpha'(y)} = x_y e_{ij}.
\]

Finally, if \( y \in B_+ \), we have
\[
\Phi(\Psi(\tilde{x}_y e_{ij})) = \Phi\left( \psi_i(\tilde{x}_y P_{s_i}^y) \right) = \Phi\left( P_{s_i}^{\psi(y)} \tilde{y} P_{s_i}^{o(y)} P_{s_i}^{t(y)} \right) = \Phi\left( P_{s_i}^{\psi(y)} \tilde{y} P_{s_i}^{o(y)} P_{s_i}^{t(y)} \right) = \epsilon_{i\alpha'(y)} \Phi(\tilde{y}) e_{i\alpha'(y)} e_{j\alpha'(y)} = \epsilon_{i\alpha'(y)} x_y e_{i\alpha'(y)} e_{j\alpha'(y)} = x_y e_{ij}.
\]

This completes the proof of the lemma. \( \square \)

**Lemma 4.8.** Let \( \Phi : \bar{R}_{\Sigma} \to M_N(Q) \) and \( \Psi : M_N(Q) \to \bar{R}_{\Sigma} \) be as defined above. Then, then composition \( \Psi \circ \Phi \) is the identity map on \( \bar{R}_{\Sigma} \).

**Proof.** It is enough to demonstrate that this is true on the generators of the algebra \( \bar{R}_{\Sigma} : \{[s_i]_{1 \leq i \leq N} \} \) and \( \{ \tilde{y} \}_{y \in \gamma} \). First, we look at the paths of length 0. Let \( s_i \in S \), then
\[
\Psi\left( \Phi([s_i]) \right) = \Psi(\epsilon_{i}) = \psi_i(1_q) P_{s_i}^y = [s_i] = \tilde{s_i}.
\]

For \( y \in Y_+^L \), \( \Psi(\Phi(\tilde{y})) = \Psi(e_{\alpha(y)} e_{\alpha'(y)}) = \psi_{\alpha'(y)}(1_q) P_{\alpha(y)}^{\psi(y)} = [s_{\alpha'(y)}] \tilde{y} = \tilde{y}. \)

When \( y \in Y_+^L \), we have
\[
\Psi(\Phi(\tilde{y})) = \Psi(t_{\tilde{y}} e_{\alpha(y)} e_{\alpha'(y)}) = \psi_{\alpha'(y)}(t_{\tilde{y}}) P_{\alpha(y)}^{\psi(y)} = \psi_{\alpha'(y)}(t_{\tilde{y}}) P_{\alpha(y)}^{\psi(y)} = \psi_{\alpha'(y)}(t_{\tilde{y}}) \tilde{y} = \tilde{y},
\]
by Lemma 4.2 and the definition of \( P_{s_i}^y \).

Next we consider the case when \( y \in B_+ \). Here we have
\[
\Psi(\Phi(\tilde{y})) = \Psi(x_y e_{\alpha'(y)} e_{\alpha'(y)}) = \psi_{\alpha'(y)}(x_y) P_{\alpha(y)}^{\psi(y)} = \psi_{\alpha'(y)}(x_y) P_{\alpha(y)}^{\psi(y)} = \psi_{\alpha'(y)}(x_y) \tilde{t}(y) = \tilde{y}.
\]

Finally, we let \( y \in \mathcal{B}_+ \). In this case we get
\[
\Psi(\Phi(\tilde{y})) = \Psi(\tilde{x}_y e_{\alpha'(y)} e_{\alpha'(y)}) = \psi_{\alpha'(y)}(\tilde{x}_y) P_{\alpha(y)}^{\psi(y)} = \psi_{\alpha'(y)}(\tilde{x}_y) P_{\alpha(y)}^{\psi(y)} = \psi_{\alpha'(y)}(\tilde{x}_y) \tilde{t}(y) = \tilde{y}.
\]

This concludes the proof of the lemma. \( \square \)

**Definition 4.9.** (Cohn 3) Section 3) let \( K \) be a field. The \( K \)-algebra \( R \) is said to be the free product of the \( K \)-algebras \( \{ R_i \}_{i \in I} \) (over \( K \)) if \( \{ R_i \}_{i \in I} \) forms a family of \( R \) such that

(i) \( R_i \cap R_j = K \) for \( i \neq j \),
(ii) if \( X_i \) is a set of generators of \( R_i \), \( i \in I \), then \( \bigcup_i X_i \) is a set of generators of \( R \),
(iii) If \( C_i \) is a set of defining relations of \( R_i \), \( i \in I \) (in terms of the set of generators \( X_i \)) then
\[ \bigcup_i C_i \] is a set of defining relations of \( R \) (in terms of the set of generators \( \bigcup_i X_i \)).
Theorem 4.10. Let $K$ be a field and let $\Gamma = (S,Y)$ be a graph with a finite number of vertices and edges. Let $\mathcal{F}_Y = \{f_y(t)\}_{y \in Y \subseteq Y}$ be a set of polynomials in $K[t]$ with property $F$. If $\Gamma$ has a spanning tree $T = (S,Y^T)$ with edges $Y^T \subseteq Y_1$, then we can define an orientation, $Y = Y^+ \cup \overline{Y^+}$ on $\Gamma$, such that $R_{\mathcal{F}_Y}$ is isomorphic to the matrix ring $M_N(Q)$, where $N$ is the cardinality of $S$ and $Q$ is the free product of the following rings:

\[
\begin{align*}
K[t]/f_y(t) & \quad y \in Y_T \cap Y^+
\end{align*}
\]

\[
\begin{align*}
K[x,y,\overline{y}]/(f_y(x,y),f_y(\overline{x}y)y)) & \quad y \in (Y_1 \cap Y^+)\setminus(Y_T \cap Y^+)
\end{align*}
\]

\[
\begin{align*}
K[x,y,\overline{y}] & \quad y \in Y^+\setminus(Y_1 \cap Y^+)
\end{align*}
\]

The proof of the following lemma is identical to the proof of Theorem 6.6 in Pilkington [27], where a proof is given for the field $Q:

Lemma 4.11. let $K$ be a field and let $f(t)$ be a non-constant polynomial in $K[t]$ with a non-zero constant term. Let $K[x,\overline{x}]$ be the ring of polynomials over $K$ in the non-commuting variables $x$ and $\overline{x}$. Let $R_f$ denote the quotient ring

\[
R_f = K[x,\overline{x}]/(f(x\overline{x}),f(\overline{x}x)).
\]

Then $R_f$ is isomorphic to a free product

\[
K[z,z^{-1}] \ast_K K_f,
\]

where $K_f$ is the ring extension $K[t]/(f(t))$ of $K$.

Lemma 4.11 and Theorem 4.10 give us the following:

Corollary 4.12. Let $K$ be a field and let $\Gamma = (S,Y)$ be an undirected graph with a finite number of vertices and edges. Let $\mathcal{F}_Y = \{f_y(t)\}_{y \in Y \subseteq Y}$ be a set of polynomials in $K[t]$ with property $F$. If $\Gamma$ has a spanning tree $T = (S,Y^T)$ with edges $Y^T \subseteq Y_1$, then we can define an orientation, $Y = Y^+ \cup \overline{Y^+}$ on $\Gamma$, such that $R_{\mathcal{F}_Y}$ is isomorphic to the matrix ring $M_N(Q)$, where $N$ is the cardinality of $S$ and $Q$ is the free product of the following rings:

\[
\begin{align*}
K[t]/f_y(t) & \quad y \in Y_1 \cap Y^+
\end{align*}
\]

\[
\begin{align*}
K[z,y,z^{-1}] & \quad y \in (Y_1 \cap Y^+)\setminus(Y_T \cap Y^+)
\end{align*}
\]

\[
\begin{align*}
K[x,y,\overline{y}] & \quad y \in Y^+\setminus(Y_1 \cap Y^+)
\end{align*}
\]

Note that Corollary 4.12 tells us that the free product $Q$ is independent of the spanning tree chosen. In particular, the free product $Q$ has one factor of the form $K[t]/f_y(t)$ where $y$ is a geometric edge in $Y_1$ and one ring corresponding to each fundamental loop (the number of which is an invariant). The ring corresponding to each fundamental loop is either a ring of Laurent polynomials or a ring of polynomials over two non-commuting indeterminates, depending on whether the edge defining it in in $Y_1$ or not. Since all edges which are not in $Y_1$, determine a fundamental loop, the number of polynomial rings is determined and the tree we choose does not affect the outcome.

5. Coxeter Systems and Associated Path Algebras

In this section, we will apply the results of our previous section to finite rank Coxeter Systems. In many cases of interest, including the irreducible Weyl groups, we will show that some of the algebras studied in Dyer [17] and Pilkington [27], are isomorphic to matrix rings over free products of rings.
5.1. A finite rank Coxeter matrix is a matrix \( M = (m_{ij})_{1 \leq i, j \leq N} \), with \( m_{ii} = 1 \) and \( m_{ij} = m_{ji} \in \mathbb{N}_{>2} \cup \{\infty\} \), \( 1 \leq i, j \leq N \), \( i \neq j \). For \( S = \{s_i\}_{1 \leq i \leq N} \), the corresponding Coxeter group is the group with presentation

\[
W = \langle s_i (1 \leq i \leq N) \mid (s_k s_l)^{m_{kl}} = 1 \rangle \text{ for } s_k, s_l \in S \text{ with } m_{kl} \neq \infty.
\]

The pair \((W, S)\) is called a Coxeter system with Coxeter matrix \( M \).

We can associate to a Coxeter system \((W, S)\) a graph \( \Gamma_{W,S} = (S, Y) \) with vertices \( s_i \in S \) and edges \( Y = \{y_{ij} = [s_i s_j] \mid i \neq j \text{ and } m_{ij} \geq 3\} \). Let \( \mathcal{P}_{W,S} \) denote the set of paths in \( \Gamma_{W,S} \). Since this graph does not have loops or multiple edges, one can describe edges and paths in the graph uniquely by the sequence of vertices that they pass through. We will also frequently use \( y_{ij} \) to denote the edge \([s_i, s_j] \in Y\). Let \( \tilde{P}_{W,S} \) denote the path algebra \( \mathbb{Q}_{\Gamma_{W,S}} \).

For \( m \in \mathbb{Z}, m \geq 3 \), let \( C_m(t) \) denote the minimum polynomial of \( 4 \cos^2 \frac{\pi}{m} \) in \( \mathbb{Z}[t] \). We have

\[
C_3(t) = t - 1, \quad C_4(t) = t - 2, \quad C_5(t) = t^2 - 3t + 1, \quad C_6(t) = t^3 - 3t^2 + 3t - 1, \ldots.
\]

Let \( Y_1 = \{y_{ij} \in Y \mid m_{ij} < \infty \} \subseteq Y \). Then \( Y_1 \). We let \( \mathbb{Q}_{\tilde{Y}_{W,S}} \) denote the set of all \( \mathbb{Q}_{\Gamma_{W,S}} \). Since \( C_y(t) = C_{\overline{y}}(t) \) for all \( y \in Y_1 \) and each polynomial in non-constant and has a non-zero constant term, \( \mathbb{Q}_{\tilde{Y}_{W,S}} \) has property \( F \). The definition of \( \mathbb{R}_{\tilde{Y}_{W,S}} \) (as a \( \mathbb{Q} \)-algebra) coincides with the definition of \( \mathbb{R}_{W,S} \) studied in Pilkington [27] and Dyer [17].

**Definition 5.1.** We let \( K_m = \mathbb{Q}[t]/(C_m(t)) \), denote the field extension \( \mathbb{Q} \left[ 4 \cos^2 \frac{\pi}{m} \right] \) of \( \mathbb{Q} \).

Applying the results of the previous section as summarized in Theorem 4.10 and Corollary 4.12 we get

**Theorem 5.2.** Let \((W, S)\) be a finite rank Coxeter system, with Coxeter graph \( \Gamma_{W,S} \) as described above. Let \( Y_1 = \{y_{ij} \in Y \mid m_{ij} < \infty \} \subseteq Y \). Suppose that there exists a spanning tree, \( T \), for \( \Gamma_{W,S} \), whose geometric edges are denoted by \( Y^T \), with the property that \( Y^T \subseteq Y_1 \). Then we have a unital ring isomorphism

\[
\tilde{R}_{W,S} \cong M_N(Q),
\]

where \( Q \) is isomorphic to the free product of the following rings:

\[
K_{m_{ij}}, \quad 3 \leq m_{ij} < \infty, \quad i < j, \\
\mathbb{Q}[y_{ij}, \overline{y}_{ij}], \quad y_{ij} \in Y \backslash Y^T, \quad m_{ij} < \infty, \\
\mathbb{Q}[x_{ij}, \overline{x}_{ij}], \quad y_{ij} \in Y \backslash Y^T, \quad m_{ij} = \infty.
\]

**Example 1.** Let \((W, S)\) be the Coxeter system with Coxeter matrix

\[
\begin{pmatrix}
1 & 3 & 2 & 4 & 2 \\
3 & 1 & 5 & 2 & 2 \\
2 & 5 & 1 & 6 & 5 \\
4 & 2 & 6 & 1 & \infty \\
2 & 2 & 5 & \infty & 1
\end{pmatrix}
\]

then \((W, S)\) be a Coxeter system with generators \( S = \{s_1 = r, s_2 = s, s_3 = t, s_4 = u, s_5 = v\} \) and relations

\[
r^2 = s^2 = t^2 = u^2 = v^2 = 1 = (rs)^3 = (st)^3 = (tu)^6 = (tv)^6 = (ru)^4 = (rt)^2 = (su)^2 = (sv)^2 = (vs)^2.
\]

Let \( \Gamma_{W,S} = (S, Y) \) be the associated graph. Below, on the right, we show the geometric edges of \( \Gamma_{W,S} \), each geometric edge \( \{s_i, s_j\} \) is labelled with corresponding \( m_{ij} \). On the left we label a set of directed edges \( Y^+ = \{y_{ij}, y_{ji} \mid y_{ij} \in Y^+ \} \). Clearly \( Y = Y^+ \cup \overline{Y^+} \).
With $Y_1 = \{ y_{ij} \mid 3 \leq m_{ij} < \infty \}$, we have $\mathfrak{F}_{\Gamma_{W,S}} = \{ C_{m_{ij}}(t) \mid y_{ij} \in Y_1 \} \subset \mathbb{Q}[t]$. Thus $R_{\mathfrak{F}_{\Gamma_{W,S}}} = \mathbb{Q}[\Gamma_{W,S}] / I_{\mathfrak{F}_{\Gamma_{W,S}}}$, where $I_{\mathfrak{F}_{\Gamma_{W,S}}}$ is the two sided ideal of $\mathbb{Q}[\Gamma_{W,S}]$ generated by the following:

\begin{align*}
C_5(y_0, \overline{y}) &= [rsr] - [r], \\
C_4(y_7, \overline{y}) &= [ru] - 2[r], \\
C_6(y_3, \overline{y}) &= [utu] - 3[u], \\
C_3(y_7, \overline{y}) &= [srs] - [s], \\
C_4(y_7, \overline{y}) &= [uru] - 2[u], \\
C_6(y_3, \overline{y}) &= [tut] - 3[t], \\
C_5(y_t, \overline{y}) &= [ststs] - 3[sts] + [s], \\
C_5(y_t, \overline{y}) &= [tvvt] - 3[tvt] + [t], \\
C_5(y_t, \overline{y}) &= [tstst] - 3[tst] + [t], \\
C_6(y_t, \overline{y}) &= [tvvtv] - 3[vvtv] + [v].
\end{align*}

Below, we show a maximal tree $T = (S, Y^T)$ where $Y^T = Y \setminus \{ y_5, \overline{y_5}, y_6, \overline{y_6} \}$.

\[ \begin{array}{ccc}
  s & t & v \\
  3 & 5 & v \\
  r & 4 & u
\end{array} \]

Clearly $Y^T \subset Y_1$, hence, we can apply Corollary \[ \] to get $R_{\mathfrak{F}_{\Gamma_{W,S}}} \cong M_5(Q)$, where $Q$ is a free product of the following rings over $\mathbb{Q}$:

\[ K_3 \cong \mathbb{Q}, K_4 \cong \mathbb{Q}, K_5, K_6 \cong \mathbb{Q}, \mathbb{Q}[x_{y_5}, x_{\overline{y_5}}], \mathbb{Q}[x_{y_6}, x_{\overline{y_6}}] \]

6. Cohn Path Algebras

In this section, we give an application of Theorem \[ \] by giving a proof of the fact that Cohn path algebras satisfy the rank condition. This implies that they have invariant basis number, which has already been proven in Abrahms and Kanuni \[ \].

\textbf{Definition 6.1.} The Cohn Path algebra $C(\Gamma)$ (relative to the orientation $Y = Y^+ \cup \overline{Y}^+$ over a field $K$ is defined as the quotient of the path algebra $\mathcal{K}\Gamma$ by the ideal generated by the set \{ $yy - [t(t)]y \in Y^+$ \}.

It can be shown that for each such Cohn path algebra of a graph $\Gamma$, there exists a graph $\Gamma'$ such that the Leavitt Path Algebra of $\Gamma'$ is $C(\Gamma)$, see Abrahms et al. \[ \].

Consider the set of polynomials $\mathfrak{F}_\Gamma = \{ f_y(t) = t^2 - 1 \mid y \in Y \} \subset K[t]$. It is not difficult to see that $\mathfrak{F}_\Gamma$ has property $F$. We see also that $R_{\mathfrak{F}_\Gamma}$ is isomorphic to $M_N(Q)$ where $Q$ is a free product of Laurent polynomial rings over $K$, and $N$ is the cardinality of $S$.\[ \]
Proof. Since any maximal spanning tree of $\Gamma$ must have edges in $Y_1 = Y$, we can apply Theorem 4.10 and Corollary 4.12 for any such tree $T = (S, Y^T)$. Thus we have that $R_{\mathcal{S}_r}$ is isomorphic to $M_N(Q)$ where $N$ is the cardinality of $S$ and $Q$ is a free product over $K$ of the following rings:

$$K_y = K[t]/(t-1), \quad y \in Y \cap Y^+ = Y^+$$

$$K[x_y, x_y^{-1}], \quad y \in Y^+ \setminus (Y^T \cap Y^+)$$

Since $K_y = K[t]/(t-1) \cong K$, $Q$ is just the free product of rings of Laurent polynomials over $K$. □

Below, we give definitions of three successively stronger ring conditions. See Cohn [10], section 0.2 for more details.

**Definition 6.3.** A ring $R$ is said to have the invariant basis number (IBN) property if every free $R$ module has unique rank.

The following are Equivalent:

- $R$ has the IBN property,
- For all $m, n \in \mathbb{N}_{\geq 0}$, $R^n \cong R^m$ implies $m = n$,
- If $A \in M_{n \times m}(R)$ and $B \in M_{m \times n}(R)$ with $AB = I_n$ and $BA = I_m$, then $m = n$.

**Definition 6.4.** A ring $R$ is said to have the unbounded generator number property (UGN) (or satisfies the rank condition) if for every $n \in \mathbb{N}_{>0}$ there is a finitely generated left $R$ module which cannot be generated by $n$ elements.

The following are Equivalent:

- $R$ has the UGN property,
- If $R^n \cong R^m \oplus K$, for some $R$ module $K$, we must have $n \geq m$,
- For any $A \in M_{n \times m}(R)$, $B \in M_{m \times n}(R)$, if $AB = I_m$, then $n \geq m$.

**Definition 6.5.** A ring $R$ is said to be weakly $n$-finite if every generating set of $n$ elements in $R^n$ is free. If this holds for all $n \in \mathbb{N}_{>0}$, then $R$ is said to be weakly finite (WF) (also called stably finite).

The following are Equivalent:

- $R$ is weakly finite,
- If $R^n \cong R^m \oplus K$, for some $R$ module $K$, we must have $K = 0$,
- For any $A, B \in M_{n \times m}(R)$, if $AB = I_n$, then $BA = I_n$.

**Definition 6.6.** Cohn [10] (Sections 1.1 and 1.2) A free right ideal ring is an integral domain $R$ with invariant basis number, such that all right ideals of $R$ are free $R$-modules. A free left ideal ring is defined similarly and a free ideal ring (fir) is a ring which is both a left and a right free ideal ring. A ring $R$ is a semifir if $R$ has IBN and every finitely generated left ideal is free (equivalently every finitely generated right ideal is free). Every fir is a semifir.

By Cohn [10], Corollary 1.2, If $R$ is a semifir, then $R$ is weakly finite.

**Theorem 6.7.** A Cohn path algebra has the UGN property, that is; it satisfies the rank condition.

Proof. Let $\Gamma = (S, Y)$ be a graph with a finite number of vertices and edges and $K$ a field. Let $C(\Gamma)$ be Cohn path algebra defined over $K$ with respect to a given orientation on $\Gamma$. From Theorem 0.2.2 and the observations made prior to it, we have a surjective homomorphism $f : C(\Gamma) \to M_n(Q)$ where $Q$ is a free product of rings of Laurent polynomials of the form $K[x, x^{-1}]$. By Cohn [9] Corollary 3 to Theorem 4.3, $Q$ is a free ideal ring and hence is weakly finite. Thus $M_n(Q)$ is also weakly finite by the matrix characterization of the property. By Proposition 0.2.2 in Cohn [10].
if a ring $R$ has a homomorphic image which is weakly finite, then it has the UGN property. This concludes the proof. □

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