Tracial Rokhlin Property for Actions of Amenable Groups

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Abstract

Tracial Rokhlin property was introduced by Chris Phillips to study the structure of crossed product of actions on simple C*-algebras. It was originally defined for actions of finite groups and group of integers. Matui and Sato generalized it to actions of amenable groups ([11] and [10]). In this paper, we give a further generalization of Matui and Sato’s definition. We shall show that many known results about tracial Rokhlin property could be generalized to actions of amenable groups under this definition, and some natural examples has the (weak) tracial Rokhlin property, at least for a special class of groups called ST-tileable groups.

1 Preliminary

Let $A$ be a C*-algebra in the following.

For $a, b \in A$, we mean by $[a, b]$ the commutator $ab - ba$.

For $\varepsilon > 0$, we write $a = _{\varepsilon} b$ to mean $\|a - b\| < \varepsilon$. For $B \subset A$, we write $a \in _{\varepsilon} B$ if there is some $b \in B$ such that $a = _{\varepsilon} b$.

If $h$ is a real function, then $h_{+}$ is the function defined by $h_{+}(t) = \text{Max}\{0, h(t)\}$.

If $a \in A$ is self-adjoint, then $a_{\pm} = \iota_{\pm}(a)$, where $\iota$ is the identity function.

The set of tracial states on $A$ is denoted by $T(A)$. For $a \in A$ and $\tau \in T(A)$, we define

\[
\|a\|_{2,\tau} = \|\tau(a^{*}a)^{1/2}\| \quad \|a\|_{2} = \sup_{\tau \in T(A)} \|a\|_{2,\tau}
\]

If $T(A)$ is non-empty, then $\|\cdot\|_{2}$ is a semi-norm. For $\tau \in T(A)$, we let $\pi_{\tau}, H_{\tau}$ denote the GNS representation of $A$ associated with $\tau$. The dimension function $d_{\tau}$ associated with $\tau$ is given by

\[
d_{\tau}(a) = \lim_{n \to \infty} \tau(a^{1/n}),
\]

for a positive element $a \in A$.

We write $V(A)$ for the Murray-von Neumann semigroup and $W(A)$ for the Cuntz semigroup. (See section 2 of [1] for an introduction of the Cuntz semigroup).
The space of states on $W(A)$ is denoted by $DF(A)$, where $DF$ stands for dimension functions. For any $\tau \in A$, $d_\tau$ give rise to a lower semicontinuous dimension functions on $A$.

Set

$$c_\infty(A) = \{(a_n) \in \ell^\infty(N, A) \mid \lim_{n \to \infty} \|a_n\| = 0\}, \quad A^\infty = \ell^\infty(N, A)/c_\infty(A).$$

Identify $A$ with the subalgebra of $A^\infty$ consists of constant sequences. Let

$$A_\infty = A^\infty \cap A'$$

and call it the central sequence algebra of $A$. A sequence $(x_n)_n \in \ell^\infty(N, A)$ is called a central sequence if it is a representative of an element in $A_\infty$.

For a sequence $x = (x_i)_{i \in \mathbb{N}}$, define $\|x\|_2 = \limsup_{n \to \infty} \|x_n\|_2$. This defines a seminorm in $A^\infty$. Let

$$J_A = \{x \in A^\infty \mid \|x\|_2 = 0\}.$$  \hspace{1cm} (1)

It’s easy to see that $J_A$ is a well-defined two-sided close ideal in $A^\infty$.

**Warning 1.1.** Some literature use $A_\infty$ to mean our $A^\infty$ and use $A_\infty \cap A'$ to mean the central sequence algebra. Our convention here makes the notation a little bit shorter.

The cardinality of a set $F$ is written as $|F|$.

**Definition 1.2.** Let $G$ be a countable discrete group.

(1) For a finite subset $K \subset G$ and $\varepsilon > 0$, we say that a finite subset $T \subset G$ is $(K, \varepsilon)$-invariant if $|T \cap \cap_{g \in F} gT| \geq (1 - \varepsilon)|T|$.

(2) The group $G$ is amenable if for any finite subset $K \subset G$ and $\varepsilon > 0$ there exists an $(K, \varepsilon)$-invariant finite subset $T \subset G$.

Let $G$ be any group, We write $\text{Act}_G(A)$ to be the set of all actions $\alpha : G \to \text{Aut}(A)$.

When $\alpha$ is an automorphism or an action of $A$, we can consider its natural extension on $A^\infty$ and $A_\infty$. We shall denote it by the same symbol $\alpha$.

For $\alpha \in \text{Aut}(A)$, we let

$$T^\alpha(A) = \{\tau \in T(A) \mid \tau \circ \alpha = \tau\}.$$  \hspace{1cm} (2)

We introduce the following comparison for the convenience of studying tracial Rokhlin property.

**Definition 1.3.** Let $f \in (A^\infty)_+$ and $a$ be an element of $A_+$. We say $f$ is pointwisely Cuntz subequivalent to $a$ and write $f \lesssim_{p.w.} a$, if $f$ has a representative $(f_n)_{n \in \mathbb{N}} \in \ell^\infty$, such that each $f_n$ is positive and $f_n \lesssim a$ in $A$, for all $n \in \mathbb{N}$.
2 Equivalent definitions of the tracial Rokhlin property

Throughout this paper, we assume that all C*-algebras are unital and separable, all groups are discrete, countable, unless otherwise specified.

**Definition 2.1.** Let $A$ be a C*-algebra, $G$ be an amenable group (Recall our convention at the beginning of this paper) and $\alpha \in \text{Act}_G(A)$. We say $\alpha$ has the tracial Rokhlin property, if for any finite subset $K$ of $G$, any $\varepsilon > 0$, there exists an $(K, \varepsilon)$-invariant finite subset $T \subseteq G$, such that for any nonzero positive element $z \in A$, there is a projection $f$ in the central sequence algebra $A_\infty$ satisfying:

(1) $\alpha_g(f)\alpha_h(f) = 0$, for any $g, h \in T$ such that $g \neq h$.

(2) With $e = \sum_{g \in T} \alpha_g(f)$, $1 - e \lesssim_p w$. (See Definition 1.3)

If $f$ as above is weakened to be a positive contraction, then we say that $\alpha$ has the weak tracial Rokhlin property.

**Remark 2.2.** It is immediate from the definition that only amenable groups can admit actions with (weak) tracial Rokhlin property. Matui and Sato also has a definition of (weak) tracial Rokhlin property (See Definition 2.5 of [11]), where the comparison condition (2) above is replaced by:

$$\lim_{n \to \infty} \max_{\tau \in T(A)} |\tau(\alpha_g(f_n)) - 1| = 0, \text{ where } f = (f_n)_{n \in \mathbb{N}}.$$  

We shall see that their definition is formally stronger than our definition here, and coincide if $T(A)$ has finitely many extreme points.

**Proposition 2.3.** Let $\alpha \in \text{Act}_G(A)$, where $G$ is amenable. Then $\alpha$ has the weak tracial Rokhlin property, if and only if for any finite subset $K$ of $G$, any $\varepsilon_0 > 0$, there is a $(K, \varepsilon_0)$-invariant subset $T$ of $G$, such that for any finite subset $F$ of $A$, any $\varepsilon_1 > 0$, and any non-zero positive element $z \in A$, there exists mutually orthogonal positive contractions $\{e_g\}_{g \in T}$ with the following properties:

(1) $||e_g, f|| < \varepsilon_1$, for any $g \in T$ and any $f \in F$.

(2) $||\alpha_{hg^{-1}}(e_g) - e_h|| < \varepsilon_1$, for any $g$ and $h$ in $T$.

(3) With $e = \sum_{g \in T} e_g$, we have $1 - e \lesssim z$.

We say that $\{e_g\}_{g \in T}$ satisfies the relation $R(T, F, \varepsilon_1, z)$. Furthermore, if $\alpha$ has the tracial Rokhlin property, then the positive contractions $e_g$ could always be chosen to be non-zero projections.
\textbf{Proof.} We shall only prove the weak tracial Rokhlin property case. The tracial Rokhlin property case is essentially the same.

For one direction, assume \( \alpha \) is an action with the stated property. Since \( A \) is separable, we can choose an increasing sequence of finite subsets \( F_n \) of \( A \) whose union is dense in \( A \). Given finite subset \( K \) of \( G \) and \( \varepsilon_0 > 0 \), we can find a fixed \((K, \varepsilon_0)\)-invariant subset \( T \) of \( G \) such that for any non-zero positive element \( a \in A \) and any \( n \in \mathbb{N} \), there exists mutually orthogonal positive contractions \( \{e_g^{(n)}\} \) satisfy the relation \( R(T, F_n, \frac{1}{n}, z) \). Let \( e_g \) be the image of the sequence \( (e_g^{(n)}) \) in \( A^\infty \). Let \( e_0 = \alpha_k^{-1}(e_k) \) for some \( k \in T \) (which is independent of \( k \) by Condition (2)), then it’s easy to see that \( e_0 \) is a positive contraction in \( A^\infty \) with properties required in Definition \( 2.1 \). Conversely, suppose \( \alpha \) has the weak tracial Rokhlin property. Then by definition, given finite subset \( K \) of \( G \) and \( \varepsilon_0 > 0 \), we can find a \((K, \varepsilon_0)\)-invariant subset \( T \) of \( G \), such that for any non-zero positive contraction \( z \in A \), there is a positive contraction \( f = (f^{(n)})_{n \in \mathbb{N}} \) in the central sequence algebra \( A_\infty \) such that:

1. \( \alpha_g(f)\alpha_h(f) = 0 \).
2. With \( r = \sum_{g \in T} \alpha_g(f) \), there is a representative \((d^{(n)})_{n \in \mathbb{N}} \) of \( 1 - r \) such that \( d^{(n)} \preceq z \), for each \( n \).

Now let \( \varepsilon_1 > 0 \) and \( F \subset_f A \) be given. Using projectivity of \( \oplus |T| C_0(0,1) \), we can lift the set \( \{\alpha_g(f)\}_{g \in T} \) to a set of mutually orthogonal positive contractions \( \{f_g = (f_g^{(n)})_{n \in \mathbb{N}}\}_{g \in T} \) in \( \ell^\infty(\mathbb{N}, A) \). Let \( M = \max\{||a|| \mid a \in F\} \) and let \( \delta = \frac{\varepsilon_1}{2M+1} \). We can choose \( n \) large enough so that

1. \( ||f_g^{(n)} - \alpha_g(f^{(n)})|| < \delta \).
2. \( ||f_g^{(n)} - a|| < \delta \), for any \( g \in T \) and \( a \in F \).
3. \( ||1 - \sum_{g \in T} f_g^{(n)} - d^{(n)}|| < \delta \).

Let \( s^{(n)} = \sum_{g \in T} f_g^{(n)} \), then \( (1 - s^{(n)} - \delta)_+ \preceq d^{(n)} \preceq z \), by Lemma 2.5 of \( 4 \). Define a non-negative continuous function \( h \) on \([0,1]\) by

\[ h(x) = 1 - \frac{1}{1 - \delta}(1 - x - \delta)_+ \quad \text{(4)} \]

Then \( h(0) = 0 \) and \( h(x) - x \leq \delta \), for any \( x \in [0,1] \). Let \( e_g = h(f_g^{(n)}) \), we have \( ||e_g - f_g^{(n)}|| < \delta \). We claim that \( \{e_g\}_{g \in T} \) are mutually orthogonal positive contractions satisfying relation \( R(T, \varepsilon_1, z, F) \). To show this, we compute:

1. \( e_g e_k = h(\alpha_g f^{(n)} h(f_k^{(n)})) = h(f_g^{(n)} f_k^{(n)}) = 0 \), for any \( g \neq h \) in \( T \).
2. \( ||\alpha_g e_k - e_g|| \leq ||\alpha_g \alpha_k^{-1}(e_k) - e_g|| + 4\delta = 4\delta < \varepsilon_1 \).
3. \( ||e_g, a|| \leq ||f_g^{(n)}, a|| + 2\delta M < (2M + 1)\delta < \varepsilon_1 \), for any \( g \in T \) and \( b \in F \).
(4) With \( e = \sum_{g \in T} e_g \), we have
\[
1 - e = 1 - \sum_{g \in T} h(f_g^{(n)}) = 1 - h(\sum_{g \in T} f_g^{(n)}) = \frac{1}{1 - \delta}(1 - \sum_{g \in T} f_g^{(n)} - \delta) + \varepsilon d^{(n)} \lesssim z. 
\]

\[ \square \]

Recall the definition of \( \| \cdot \|_2 \) and \( J_A \) in Section [1]. In the rest of the paper we shall always use \( \pi: A^\infty \to A^\infty/J_A \) to denote the quotient map.

**Proposition 2.4.** Let \( \alpha \in \text{Act}_G(A) \) be an action with the weak tracial Rokhlin property, where \( A \) is simple. Then for any finite subset \( K \) of \( G \) and any \( \varepsilon_0 > 0 \), there is a \( (K, \varepsilon_0) \)-invariant subset \( T \) of \( G \) and positive contraction \( f \in A^\infty \), such that

1. \( \alpha_g(f)\alpha_h(f) = 0 \), for \( g, h \in T \) with \( g \neq h \).
2. \( \pi(\sum_{g \in T} \alpha_g(f)) = 1 \) in \( A^\infty/J_A \).

Moreover, suppose \( A \) has strict comparison, then the converse is true.

If \( \alpha \) has the tracial Rokhlin property, one could replace positive contraction by non-zero projection in the above statement.

**Proof.** We shall only prove the weak tracial Rokhlin property case. The tracial Rokhlin property case is essentially the same.

Suppose \( \alpha \) has the only weak tracial Rokhlin property. A unital simple C*-algebra \( A \) is not of type I. By Proposition 4.10 of [4], for any \( n \in \mathbb{N} \) we can find an embedding \( \theta_n : C_0(0, 1] \otimes M_n \to A \). Let \( a_n = \theta_n(\iota \otimes e_1) \), where \( \iota \) is the identity function. It’s easy to see that \( \lim_{n \to \infty} \max_{\tau \in T(A)} d_\tau(a_n) = 0 \).

Let \( T \) be some \( (K, \varepsilon) \)-invariant subsets of \( G \). Let \( \{F_n\} \) be a sequence of finite subsets of \( A \) whose union is dense. Find positive contractions \( \{e_g^{(n)}\} \subset A \) satisfying the relation \( R(T, F_n, 1/n, a_n) \) as in Proposition 2.3. Pick some \( k \in T \) and define \( f = (\alpha_{k^{-1}}(e_k^{(n)}))_{n \in \mathbb{N}} \). Use the fact that if \( a, b \) are positive contractions such that \( a \lesssim b \), then \( \tau(a^2) \leq d_\tau(a^2) \leq d_\tau(b) \), we can see that \( f \) has the desired property.

For the converse, suppose \( \alpha \) has the stated property. Given finite subset \( K \) of \( G \) and \( \varepsilon_0 > 0 \), we can find a \( (K, \varepsilon_0) \)-invariant subset \( T \) of \( G \) and \( f = (f^{(n)})_{n \in \mathbb{N}} \) in \( A^\infty \), such that

1. \( \alpha_g(f)\alpha_h(f) = 0 \), for \( g, h \in T \) with \( g \neq h \).
2. \( \lim_{n \to \infty} \max_{\tau \in T(A)} \tau(1 - \sum_{g \in T} \alpha_g(f^{(n)})) = 0. \)

Note that the second condition above is implied by \( \|1 - \sum_{g \in T} \alpha_g(f)\|_2 = 0. \)

Now let \( z \) be a non-zero positive element in \( A \). Our goal here is to show that \( 1 - \sum_{g \in T} \alpha_g(f) \) has a representative \( (d^{(n)})_{n \in \mathbb{N}} \) such that \( d^{(n)} \lesssim z \).

First we can lift the set \( \{\alpha_g(f)\}_{g \in T} \) to a set of mutually orthogonal positive
contractions \( \{ f_g = (f_g^{(n)})_{n \in \mathbb{N}} \}_{g \in T} \) in \( \ell^\infty(\mathbb{N}, A) \). Then \( 1 - \sum_{g \in T} f_g \) is a representative of \( 1 - \sum_{g \in T} \alpha_g(f) \) and:

\[
\lim_{n \to \infty} \max_{\tau \in \mathcal{T}(A)} \tau(1 - \sum_{g \in T} f_g^{(n)}) = 0.
\]

Let \( \delta_n = \max_{\tau \in \mathcal{T}(A)} \tau(1 - \sum_{g \in T} f_g^{(n)}) \). Let \( \eta = \min_{\tau \in \mathcal{T}(A)} \{ d_\tau(z) \} \), which is positive since \( A \) is simple. Choose \( N \) large enough such that for any \( n > N \), \( \sqrt{\delta_n} < \eta \). Let \( (a_n)_{n \in \mathbb{N}} \) be the sequence such that \( a_n = 0 \) for any \( n \leq N \) and \( a_n = (1 - \sum_{g \in T} f_g^{(n)} - \sqrt{\delta_n})_+ \) otherwise. For \( n > N \) and any \( \tau \in \mathcal{T}(A) \), we can estimate:

\[
d_\tau(a_n) \leq \frac{1}{\sqrt{\delta_n}} \tau(1 - \sum_{g \in T} f_g^{(n)}) \leq \frac{\delta_n}{\sqrt{\delta_n}} < \eta \leq d_\tau(z).
\]

Since \( A \) has strict comparison, \( a_n \preceq z \), for any \( n > N \). Since \( \lim_{n \to \infty} \delta_n = 0 \), \( (a_n)_{n \in \mathbb{N}} \) is also a representative of \( 1 - \sum_{g \in T} \alpha_g(f) \), and \( a_n \preceq z \) for any \( n \). This shows that \( \alpha \) has the weak tracial Rokhlin property.

\[\square\]

**Corollary 2.5.** Let \( \alpha \in \text{Act}_G(A) \), where \( A \) is infinite dimensional simple with strict comparison. If \( \alpha \) has the weak tracial Rokhlin property in the sense of Definition 2.7 of [11], then \( \alpha \) has the weak tracial Rokhlin property defined in this paper.

In the following we shall show that, if \( A \) has tracial rank zero, then weak tracial Rokhlin property actually implies tracial Rokhlin property. The case \( G = \mathbb{Z} \) has been proved by Phillips and Osaka (See Theorem 2.14 of [12] and Proposition 1.3 of [13]). We shall see that the proof can be simplified by working with sequence algebras.

**Lemma 2.6.** Let \( A \) be a C*-algebra and \( B \) be finite dimensional subalgebra. Let \( \{ e_{i,j} \} \) be the standard matrix units of \( B \). Then for any \( \varepsilon > 0 \), there is a \( \delta > 0 \), such that whenever a projection \( p \in A \) satisfies \( \| [p, e_{i,j}] \| < \delta \) for all \( i, j, l \), there is a projection \( q \) in the relative commutant \( A \cap B' \), such that \( \| p - q \| < \varepsilon \).

**Proof.** Fix some \( \varepsilon > 0 \). Choose \( \delta_0 \) according to \( \varepsilon/2 \) as in Lemma 2.5.10 of [9]. Choose \( \delta_1 \) according to \( \delta_0 \) as in Theorem 2.5.9 of [9] (It’s easy to see that this lemma generalize to finite dimensional C*-algebras). Let \( p \in A \) be a projection satisfying \( \| [p, e_{i,j}] \| < \delta_1 \). Identify \( B = pA_p \oplus (1 - p)A(1 - p) \) as a subalgebra of \( A \). Let \( a_{i,j}^l = pe_{i,j}^l p + (1 - p)e_{i,j}^l (1 - p) \in B \). Then \( \| a_{i,j}^l - e_{i,j}^l \| < \delta_0 \).

Hence by Theorem 2.5.9 of [9], there are matrix units \( \{ f_{i,j}^l \} \subset B \) such that \( \| f_{i,j}^l - e_{i,j}^l \| < \delta_1 \). By Lemma 2.5.10 of [9], there is a unitary \( u \in A \) such that \( uf_{i,j}^l u^* = e_{i,j}^l \) and \( \| u - 1 \| < \varepsilon/2 \). Now let \( q = upu^* \), then \( \| q - p \| < \varepsilon \). We shall show that \( q \) commutes with \( B \) by showing that \( q \) commutes with each \( e_{i,j}^l \).

Since \( \{ f_{i,j}^l \} \subset B \), we have \( f_{i,j}^l = (1 - p)f_{i,j}^l (1 - p) + pf_{i,j}^l p \), for any \( i, j, l \). Hence

\[
qe_{i,j}^l = upf_{i,j}^l u^* = upf_{i,j}^l pu^* = uf_{i,j}^l pu^* = e_{i,j}^l q
\]  \hspace{1cm} (5)
Theorem 2.7. Let \( \alpha \in \text{Act}_G(A) \) be an action with the weak tracial Rokhlin property. If \( A \) is a simple C*-algebra with tracial rank zero, then \( \alpha \) actually has the tracial Rokhlin property.

Proof. If \( A \) has tracial rank zero, then \( A \) is tracially approximately divisible, and therefore has strict comparison. By Proposition 2.4 for any finite subset \( K \in G \) and any \( \varepsilon > 0 \), there is a \( (K, \varepsilon) \)-invariant set \( T \) and a positive contraction \( f \in A_\infty \) such that

1. \( \alpha_g(f)\alpha_h(f) = 0 \), for all \( g, h \in T \).
2. \( \pi(\sum_{g \in T} \alpha_g(f)) = 1 \).

This shows that \( \{ \pi(\alpha_g(f)) \}_{g \in T} \) are mutually orthogonal projections, in particular \( \pi(f) \) is a projection. In the following we shall lift \( \pi(f) \) to a projection in \( A_\infty \) with the desired properties.

List the elements in \( T \) by \( g_1, g_2, \ldots, g_n \). Let \( F \) be a finite subset of \( A \) and \( \delta > 0 \).

Since \( A \) has tracial rank zero, there is a finite dimensional subalgebra \( B \subset A \) with \( 1_B = p \), such that

1. \( \| [p, a] \| < \delta/4 \), for any \( a \in F \).
2. \( pa = \delta/4 B \).
3. \( \| 1 - p \|_2 < \delta/n \).

Consider \( C = A_\infty \cap B^\prime \supset A_\infty \). By Lemma 2.6 \( C = (A \cap B')^{\infty} \). The sequence algebra of a real rank zero C*-algebra is again real rank zero, hence \( C \) has real rank zero. Now \( \pi: JC^\prime_f \to \pi(JC^\prime_f) \) is a surjective map between two real rank zero C*-algebras, such that \( \pi(f) \) is a projection in the image. We can then lift \( \pi(f) \) to a projection \( f_0 \in JC^\prime_f \). Let \( \hat{f} = pf_0p \in JC^\prime_f \). Then \( \hat{f} \) is a projection in \( C \) and \( \alpha_g(\hat{f})\alpha_h(\hat{f}) = 0 \), for any \( g \neq h \in T \). Now for any \( a \in S \), find \( b \in B \) such that \( \| pap - b \| < \delta \). Then

\[
\| \hat{f}a - a\hat{f} \| = \| pf_0pa - apf_0p \|
\leq \| pf_0pa - papf_0p \| + \delta/4 + \delta/4
\leq \| pf_0b - bf_0p \| + \delta/2 + \delta/4 + \delta/4 = \delta
\]

Since \( \| 1 - \sum_{g \in T} \alpha_g(f_0) \|_2 = 0 \), we have

\[
\| 1 - \sum_{g \in T} \alpha_g(\hat{f}) \|_2 = \| 1 - \sum_{g \in T} \alpha_g(pf_0) \|_2
\leq \| 1 - \sum_{g \in T} \alpha_g(f_0) \|_2 + \| \sum_{g \in T} \alpha_g((1 - p)f_0(1 - p)) \|_2
\leq 0 + n\| (1 - p) \|_2 < \delta.
\]

Now we look at an increasing sequence of finite subsets \( \{ F_n \} \) whose union is dense in \( A \) and let \( \delta \) tends to 0, we can get a sequence of projections \( \hat{f}_k \) which
becomes more and more commutative with $F_n$ such that $\|1 - \sum_{g \in T} \alpha_g(f_k)\|_2$ tends to 0. We can then use a Cantor’s diagonal argument to select a projection $q$ in $A^\infty$ satisfying the conditions in Proposition 2.4, therefore $\alpha$ has the tracial Rokhlin property.

An action $\alpha \in \text{Act}_G(A)$ is called strongly outer, if and only if for any $g \neq 1$ and any $\tau \in T^{\alpha_g}(A)$, the weak extension of $\alpha_g$ on $\pi(\tau(A))''$ is not weakly inner.

**Proposition 2.8.** Let $G$ be a countable discrete amenable group, let $A$ be a unital simple infinite dimensional C*-algebra, and let $\alpha \in \text{Act}_G(A)$ be an action with the weak tracial Rokhlin property. Suppose that the tracial state space $T(A)$ has finitely many extreme points. Then $\alpha$ is strongly outer.

**Proof.** Let $1 \neq g \in G$ be given, and let $\tau$ be an $\alpha_g$-invariant trace. Let $E: A \rtimes_{\alpha_g} \mathbb{Z} \to A$ be the conditional expectation determined by $E(a_n U^n_g) = a_0$, where $a_n \in A$ and $U_g$ is the canonical unitary in $A \rtimes_{\alpha_g} \mathbb{Z}$ implementing the action. We will show that, for any trace $\Phi \in T(A \rtimes_{\alpha_g} \mathbb{Z})$, we have $\Phi(a U_g) = 0$. If this is done, then the proof of Lemma 4.4 of [8] shows that $\alpha_g$ is not weakly inner.

For any $\tau \in T(A)$ and $x = (x_n)_{n \in \mathbb{N}}$, let $\tau_\infty(x) = \limsup_{n \to \infty} \tau(x_n)$. Note that if $x$ is positive, then $\tau_\infty(x) = 0$ if and only if $\lim_{n \to \infty} \tau(x_n) = 0$. Let $\varepsilon > 0$ be arbitrary. Since $\alpha$ has the weak tracial Rokhlin property, by Proposition 2.4 we can find a $(\{g\}, \varepsilon)$-invariant subset $T$ of $G$, and a positive contraction $f = (f_n) \in A_\infty$, such that:

1. $\alpha_h(f) \alpha_k(f) = 0$, for $h, k \in T$ with $h \neq k$.
2. $\tau_\infty(1 - \sum_{h \in T} \alpha_k(f)) = 0$, for any $\tau \in T(A)$.

Now let $\tau_1, \tau_2, \ldots, \tau_n$ be the extreme tracial states of $A$. For any $h \in G$, $\tau \to \tau \circ \alpha_h$ defines an bijection of the set of extreme tracial states to itself. Hence for any $\tau \in T(A)$,

$$\tau_\infty(f) \leq \limsup_{n \to \infty} \sum_i \tau_i(f_n) = \frac{1}{|T|} \limsup_{n \to \infty} \sum_i \sum_{h \in T} \tau_i(\alpha_h(f_n)) = \frac{n}{|T|}.$$

Therefore, regarding $a U_g$ as equivalence class of constant sequence in $(A \rtimes_{\alpha_g}$
\(Z)\), without loss of generality assume \(\|a\| = 1\). We have

\[
|\Phi_\infty(aU_g)| \leq |\Phi_\infty(\sum_{h \in T} \alpha_h(f) aU_g)| + |\Phi_\infty((1 - \sum_{h \in T} \alpha_h(f)) aU_g)|
\]

\[
\leq |\Phi_\infty(\sum_{h \in T \cap gT} \alpha_h(f) aU_g)| + |\Phi_\infty((1 - \sum_{h \in T \setminus gT} \alpha_h(f)))^{1/2}\Phi_\infty(aU_g a^*)^{1/2}|
\]

\[
\leq |\Phi_\infty(\sum_{h \in T \cap gT} \alpha_h(f^{1/2}) aU_g \alpha_{g^{-1}}(f^{1/2}))| + |\sum_{h \in T \setminus gT} \Phi_\infty(\alpha_h(f)^2)\Phi_\infty(aU_g a^*)|
\]

\[
\leq |\Phi_\infty(\sum_{h \in T \cap gT} \alpha_{g^{-1}}(f^{1/2}) \alpha_h(f^{1/2}) aU_g)| + |T \setminus gT| n \varepsilon.
\]

We used inequalities about traces \(\tau\), and extend it to \(\tau_\infty\) in the above estimation. Since \(\varepsilon\) is arbitrary, this shows that \(\Phi_\infty(aU_g) = 0\), and therefore \(\Phi(aU_g) = 0\). \(\square\)

The assumption that \(A\) has finite many extreme tracial states is needed in the above proposition because \(T\) is not completely \(\{g\}\)-invariant, as we can see from the last two lines of estimation in the above proof. If we can always make \(T\) completely \(\{g\}\)-invariant, in particular when \(G\) is finite, this assumption could be dropped. The same proof shows that for any trace \(\Phi \in T(A \rtimes_\alpha G)\), we have \(\Phi(aU_g) = 0\) for any \(g \neq 1\). Hence we have the following corollary:

**Corollary 2.9.** Let \(\alpha \in \text{Act}_G(A)\) be an action with the weak tracial Rokhlin property. Suppose that the tracial state space \(T(A)\) has finitely many extreme points or \(G\) is finite. Then the canonical embedding \(A \to A \rtimes_\alpha G\) induces a bijection between \(T^a(A)\) and \(T(A \rtimes_\alpha G)\).

**Definition 2.10.** (Definition 2.4 of [10]) Let \(G\) be a countable discrete amenable group and let \(\alpha \in \text{Act}_G(R)\) be an outer action on the AFD II\(_1\) factor \(R\). We say that \(G\) has the property \((Q)\) if the following holds: For any finite subset \(K \subset G\) and \(\varepsilon > 0\), there exists an \((K, \varepsilon)\)-invariant finite subset \(T \in G\) and a sequence of projections \((p_n)_n\) in \(R\) such that

\[
\|1 - \sum_{g \in K} \alpha_g(p_n)\|_2 \to 0 \quad \text{and} \quad \|[x, p_n]\|_2 \to 0, \quad \forall x \in R.
\]

as \(n \to \infty\).

It is show in Theorem 3.6 of [10] that if \(A\) is a unital, simple, separable, nuclear, stably finite, infinite dimensional \(C^*\)-algebra with finitely many extremal tracial states, and if the group \(G\) has property \((Q)\), then an action \(\alpha \in \text{Act}_G(A)\) is strongly outer if and only if it has the weak tracial Rokhlin property defined by Matui and Sato.

**Corollary 2.11.** If \(A\) is a unital, simple, separable, nuclear, stably finite and infinite dimensional \(C^*\)-algebra with finitely many extreme tracial states, and if the group \(G\) has property \((Q)\), then our definition of weak tracial Rokhlin property coincide with that of Matui and Sato (Definition 2.7 of [10]).
In the following two sections we generalize results in [13]. The idea is adapted from there, but we reorganize the proof in a way that maybe used for further generalization.

3 The Murray-von Neumann semigroup

For a C*-algebra $A$, we let $V(A)$ be the Murray-von Neumann semigroup of $A$. We say that $V(A)$ has strict comparison if for any $p, q \in V(A)$, we have that $\tau(p) < \tau(q)$ for any $\tau \in T(A)$ implies $p \preceq q$. Note that such C*-algebra is said to satisfy Blackadar’s Second Fundamental Comparability Question, or that the order of projections is determined by traces in different literature.

We say $V(A)$ is almost divisible, if for any $p \in V(A)$ and any $n \in \mathbb{N}$, there is some $q \in V(A)$ such that $nq \leq p \leq (n + 1)q$.

Note that if $A$ is simple infinite dimensional with real rank zero, then $V(A)$ is almost divisible, by Lemma 2.3 of [13].

Lemma 3.1. Let $A$ be a simple C*-algebra with Property (SP). Suppose that $V(A)$ has strict comparison and is almost divisible. Let $\alpha \in \text{Act}_G(A)$ be an action with the tracial Rokhlin property. Then for every finite $F \subset A \rtimes_\alpha G$, every $\varepsilon > 0$, and every nonzero $z \in (A \rtimes_\alpha G)_+$, there exist a projection $f \in A$, two finite subsets $T_0 \subset T \subset G$ with $\frac{|T_0|}{|T|} > 1 - \varepsilon$, an embedding $\phi: M_{|T|} \otimes fAf \to A \rtimes_\alpha G$ whose image shall be called $D$, a projection $p = \phi(\sum_{g \in T_0} e_{g,g} \otimes f)$, where $\{e_{g,h}\}_{g,h \in T}$ are the standard system of matrix units for $M_{|T|}$, and some $g_0 \in T$ such that

1. $\phi(e_{g_0,g_0} \otimes a) = \alpha_{g_0}(a)$, for any $a \in fAf$.
2. $\phi(e_{g,g} \otimes f) \in A$, for any $g \in T$.
3. $||\phi(e_{g,h} \otimes a) - u_g u_h^*|| \leq \varepsilon ||a||$, for any $g \in T$ and $a \in fAf$.
4. $pb \subset_\varepsilon D$ and $bp \subset_\varepsilon D$, for any $b \in F$.
5. if $T_0 = T$, then $pb = \varepsilon bp$, for any $b \in F$.
6. $1 - p \preceq z$.

Proof. We first choose two nonzero orthogonal positive elements $z_0, z_1 \in A_+$ such that $z_0 \oplus z_1 \preceq z$ according to Lemma 5.1 of [3]. Since $A$ has property (SP), we could assume that $z_0$ and $z_1$ are projections. Let $\eta = \min_{\tau \in T(A)} \tau(z_0) > 0$. Let $\varepsilon_0 = \min\{\frac{\eta}{2}, \varepsilon\}$. Without loss of generality assume that there is a finite set $K \subset f G$ such that elements of $F$ are all of the form $\sum_{g \in K} a_g u_g$, where $a_g \in A$ and $u_g$ the canonical unitaries implementing the action.

By Definition 2.1 we can find a $(K, \varepsilon_0)$-invariant subset $T \subset f G$ and a central projection $q \in A_\infty$ such that

1. $\{a_g(q)\}$ are pairwise orthogonal.
2. $1 - \sum_{g \in T} a_g(q) \preceq_{p.w.} z_1$. 

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By the universal property of matrix algebra, there is an embedding $\psi: M_{|T|} \to (A \rtimes_\alpha G)\sim$ such that $\psi(e_{g,h}) = u_g q u_h^*$. Let $F_0 = \{a_g \mid \sum_{g \in K} q_g u_g \in F\}$. Using semiprojectivity of $M_{|T|}$, we can lift $\psi$ to a sequence of embeddings $\psi_n: M_{|T|} \to (A \rtimes_\alpha G)$. We could further assume that $\psi_n(e_{g,h}) \in A$, for $g \in T$ by standard perturbation argument (See Lemma 2.5.7 of [9]).

Now fix some $g_0 \in T$. Let $q_n = \alpha_{g_0}^{-1}(\psi_n(e_{g_0,g_0}))$. It's clear that $(q_n)_{n \in \mathbb{N}}$ is a representative of $q$. Let $M = \text{Max}\{\|a\| \mid a \in F_0\}$. Define

$$\delta = \text{Min}\{1/2, \frac{\varepsilon}{|K||T|M + 2|K||T|}, \varepsilon/2\}$$

We can find some large enough $n$ such that:

(1') Let $e_g = \psi_n(e_{g,g}) \in A$, we have $\|e_g, a\| < \delta$, for any $g \in T$ and any $a \in F_0$.

(2') Let $f = q_n$, we have $\|\psi_n(e_{g,h}) - u_g f u_h^*\| < \delta$, for any $g \in T$.

(3') With $e = \sum_{g \in T} e_g$, we have $1 - e \gtrsim z_1$.

The last condition comes from the fact that if two projections are close enough, then they are unitarily equivalent.

We now define an embedding $\phi: M_{|T|} \otimes fA f$ by

$$\phi(e_{g,h} \otimes a) = \psi_n(e_{g,g}) \alpha_{g_0}(a) \psi_n(e_{g_0,h}).$$

and extend linearly.

Let $D = \phi(M_{|T|} \otimes fA f)$ be the image of $\phi$. Let $T_0 = \cap_{g \in K} g T$ and

$$p = \phi\left(\sum_{g \in T_0} (e_{g,g} \otimes f)\right) = \sum_{g \in T_0} e_g.$$  

We now verify the conditions required in this lemma. Condition (1) and (2) follows for the definition.

For condition (3), we have the following estimation:

$$\phi(e_{g,h} \otimes a) = \frac{\delta\|a\|}{2} u_g f u_{g_0}^* \alpha_{g_0}(a) u_{g_0} f u_h^* = u_g f a u_h^* = u_g a u_h^*.$$ 

Hence $\|\phi(e_{g,h} \otimes a) - u_g a u_h^*\| \leq 2\delta\|a\| \leq \varepsilon\|a\|$.

For condition (4), Let $b = \sum_{h \in K} b_h u_h \in F$, we have:

$$pb = \sum_{g \in T_0, h \in K} e_g h_k u_h = \delta |K||T_0|M \sum_{g \in T_0, h \in K} u_g f u_{g'}^* h_k u_h$$

$$= \delta |K||T_0| \sum_{g \in T_0, h \in K} u_g f a_{g^{-1}}(b_h) f u_{h^{-1} g}$$

$$= \phi\left(\sum_{g \in T_0, h \in K} e_{g,h^{-1} g} \otimes f a_{g^{-1}}(b_h)f\right).$$
Since $\delta |K| |T_0| + 2\delta |K| |T_0| < \varepsilon$, this shows $pb \subset \varepsilon D$. The proof that $bp \subset \varepsilon D$ is similar.

For condition (5), if $T_0 = T$, then $hT = T$, for any $h \in K$. Hence

$$\|pb - bp\| = \| \sum_{g \in T_0, h \in K} (e_gb_hu_h - b_hu_he_g) \| \leq \delta |T_0| |K| |T_0| |M| + \| \sum_{g \in T_0, h \in K} (b_he_gu_h - b_he_hu_h) \| \leq \varepsilon.$$

For the last condition, we write $1 - p = (1 - \sum_{g \in T} e_g) + \sum_{g \in T \setminus T_0} e_g$. Since $\|e_g - \alpha_g(f)\| < \delta < 1$, the two projections are unitarily equivalent in $A$. Hence for any $\alpha$-invariant trace $\tau$ and any $g, h \in T$, we have $\tau(e_g) = \tau(\alpha_g(f)) = \tau(f) = \tau(e_h)$. Therefore

$$\tau(\sum_{g \in T \setminus T_0} e_g) = \frac{|T \setminus T_0|}{|T|} \tau(e) \leq \xi_0 < \tau(z_0)$$

By Proposition 2.4 of [13] (Although it’s stated for real rank zero C*-algebra, but all is needed is that $V(A)$ is almost divisible), we have $\sum_{g \in T \setminus T_0} e_g \precsim z_0$ in $A \rtimes_\alpha G$. Hence

$$1 - p = (1 - \sum_{g \in T} e_g) + \sum_{g \in T \setminus T_0} e_g \precsim z_1 \oplus z_0 \precsim z. \quad (11)$$

**Theorem 3.2.** Let $A$ be a simple C*-algebra with real rank zero. Let $\alpha \in \text{Act}_G(A)$ be an action of a countable discrete amenable group with the tracial Rokhlin property. Then $V(A \rtimes_\alpha G)$ has strict comparison.

**Proof.** In the proof of Theorem 3.5 of [13], if we replace Lemma 2.5 of [13] by Lemma [3.1] everywhere, we see that the proof works well for actions of general discrete amenable groups. Hence $V(A \rtimes_\alpha G)$ has strict comparison. 

\[ \square \]

### 4 Real and stable rank of the crossed product

The following lemma says that any single self-adjoint element of the crossed product could be 'tracially' approximated by subalgebras with real rank zero. It is weaker than tracial approximation formulated in (Definition 2.2, [2]), but its good enough to deduce that the crossed product has real rank zero, at least when we know the crossed product has strict comparison for projections.

**Lemma 4.1.** Let $A$ be a simple infinite dimensional C*-algebra with real rank zero and has strict comparison for projections. Let $\alpha \in \text{Act}_G(A)$ have the tracial Rokhlin property, where $G$ is a countable discrete amenable group. Then for any self-adjoint element $a \in A \rtimes_\alpha G$, any $\varepsilon > 0$ and any nonzero positive element $z \in A \rtimes_\alpha G$, there is a C*-subalgebra $D$ of $A \rtimes_\alpha G$ with real rank zero and a projection $p \in D$ such that:
(1) \( \|pa - ap\| < \varepsilon \).

(2) \( pap \in \varepsilon D \).

(3) \( 1 - p \preceq z \).

Proof. Let \( a, \varepsilon \) and \( z \) are given as in this lemma. Without loss of generality assume \( \|a\| \leq 1 \). Choose two nonzero orthogonal projections \( z_0 \) and \( z_1 \) in \( A \) such that \( z_0 + z_1 \preceq z \). Let \( \eta = \min_{r \in T(A)T(z_0)} \). Let

\[
\delta = \min \{ \varepsilon/4, 1/6, 1 - \eta/2, \, \frac{\eta\varepsilon}{8 + 4\varepsilon} \} \tag{12}
\]

By Lemma 6.1 there exist a projection \( f \in A \), two finite subsets \( T_0 \subset T \subset G \) with \( \frac{|T_0|}{|T|} > 1 - \delta \), an embedding \( \phi: M_{|T|} \otimes fAf \to A \otimes G \) whose image shall be called \( D \), and a projection \( q = \sum_{g \in T_0} \phi(e_{g,g} \otimes f) \) such that

1. Let \( e_g = \phi(e_{g,g} \otimes f) \), we have \( e_g \in A \).

2. There exist \( d_1 \) and \( d_2 \) in \( D \) such that \( \|qa - d_1\| < \delta \) and \( \|aq - d_2\| < \delta \).

3. If \( T_0 = T \), then \( \|qa - aq\| < \delta \), for any \( a \in F \).

4. \( 1 - q \preceq z_1 \).

If \( T_0 = T \), then we are done. So assume \( T_0 \neq T \).

Let \( q_0 = \sum_{g \in T_0} e_{g,g} \otimes f \), \( e_0 = \sum_{g \in T} e_{g,g} \otimes f \) and \( e = \phi(e_0) \). Note that \( q = \phi(q_0) \). We can write

\[
a = qa + (1 - q)aq + (1 - q)a(1 - q) = 2\delta \quad d_1 + (1 - q)d_2 + (1 - q)a(1 - q). \tag{13}
\]

Let \( d = d_1 + (1 - q)d_2 \) and \( \tilde{d} = \frac{d + d^*}{2} \). Then \( \tilde{d} \) is a self-adjoint element in \( D \) such that \( \|a - (\tilde{d} + (1 - q)a(1 - q))\| < 2\delta \). Let \( d_0 \) be a self-adjoint element in \( M_{|T|} \otimes fAf \) such that \( \tilde{d} = \phi(d_0) \).

We can estimate that

\[
|T_0| \geq \frac{\eta}{2}|T| \geq (4/\varepsilon + 2)|T\setminus T_0| \geq (4/\varepsilon + 1)|T\setminus T_0| + 1. \tag{14}
\]

Hence there exists \( n \in \mathbb{N} \) such that

\[
|T_0| \geq n \geq (4/\varepsilon + 1)|T\setminus T_0|. \tag{15}
\]

Choose a subset \( T_1 \subset T_0 \) such that \( |T_1| = n \). Let \( r_0 = \sum_{g \in T_1} e_{g,g} \otimes f \). By Lemma 4.4 of [13], there is a projection \( s_0 \) in \( M_{|T|} \otimes fAf \) such that

\[
e_0 - q_0 \leq s_0 \preceq r_0, \quad \|s_0d_0 - d_0s_0\| < \frac{2}{|T\setminus T_0| - 1} \leq \varepsilon/2 \tag{16}
\]

Let \( s = \phi(s_0) \geq e - q \). Let \( p = e - s \leq q \). For Condition (1), we have:

\[
\|pa - ap\| = \|p(a - (1 - q)a(1 - q)) - (a - (1 - q)a(1 - q))p\|
\leq 2\delta + \|p\tilde{d} - \tilde{d}p\|
= 2\delta + \|\phi((1 - s_0)d_0 - d_0(1 - s_0))\| < \varepsilon.
\]
Since \( p \leq q \), we have \( pap = pqaqp \in \varepsilon D \), this proves Condition (2).

Finally, for any \( \tau \in T(A \times_a G) \), we have

\[
\tau(s) \leq \tau(\phi(r_0)) \leq \frac{|T|}{T} < \eta \leq \tau(z_0). \tag{17}
\]

By Theorem 3.2 this shows that \( s \not\succ z_0 \). Hence

\[
1 - p = (1 - e) + s \leq (1 - p) \oplus s \not\succ z_1 \oplus z_0 \not\succ z. \tag{18}
\]

\( \square \)

**Proposition 4.2.** Let \( A \) be a unital simple C*-algebra with strict comparison for projections. Suppose for any self-adjoint element \( a \in A \), any \( \varepsilon > 0 \) and any nonzero positive element \( z \in A \), there is a unital C*-subalgebra \( D \) of \( A \) with real rank zero and \( 1_D = p \) such that:

1. \( \|pa - ap\| < \varepsilon \),
2. \( pap \in \varepsilon D \),
3. \( 1 - p \not\succ z \).

Then \( A \) has real rank zero.

**Proof.** Let \( a \) be a self-adjoint element in \( A \) and \( \varepsilon > 0 \) be given. Without loss of generality assume \( \|a\| = 1 \). Assume that \( a \) is not invertible, otherwise there is nothing to prove. Let \( \varepsilon_0 = \frac{\varepsilon}{26} \). Let \( g: [-1,1] \to [0,1] \) be a continuous function such that

\[
\text{supp } g = [-\varepsilon_0, \varepsilon_0] \text{ and } g(0) = 1. \tag{19}
\]

Let

\[
\varepsilon_1 = \text{Min}\{\varepsilon_0, \frac{1}{4} \text{Min}_{\tau \in T(A)} \{\tau(g(a))\}\} > 0. \tag{20}
\]

Choose \( \delta > 0 \) such that whenever \( a, b \) are normal elements with norm less or equal to 1 and \( \|a - b\| < \delta \), then \( \|g(a) - g(b)\| \leq \varepsilon_1 \), according to Lemma 2.5.11 of [9]. We further require that \( \delta \leq \varepsilon_1 \). Since \( A \) has strict comparison, we can find a C*-subalgebra \( D \) of \( A \) with real rank zero and a projection \( p \in D \) such that:

1. \( \|pa - ap\| < \delta/2 \).
2. there is some self-adjoint element \( d \in D \) such that \( \|pap - d\| < \delta \).
3. \( \tau(1 - p) < \delta/2 \), for any \( \tau \in T(A) \).

Replacing \( d \) by \( pdp \), we may assume that \( d \in pDp \). We may also assume that \( \|d\| \leq 1 \). Since \( pDp \) has real rank zero for being a corner of real rank zero C*-algebra, there is a projection \( r \in g(d)Dg(d) \) such that \( \|rg(d)r - g(d)\| < \delta \).

In the following, We shall show that \( 1 - p \not\succ r \leq p \) and, for any projection \( s \leq r \),
we have \( \|sa\| < \varepsilon, \|as\| < \varepsilon. \)

The choice of \( \delta \) shows that

\[
g(a) = \varepsilon_1, g(pap + (1 - p)a(1 - p)) = g(pap) + g((1 - p)a(1 - p))
\]

and \( g(pap) = \varepsilon_1 g(d) \). Hence for any \( \tau \in T(A), \) we can compute:

\[
\tau(r) \geq \tau(rgd(r) - \varepsilon_1 - \varepsilon_1 \\
\geq \tau(g(pap)) - \varepsilon_1 - \varepsilon_1 \\
\geq \tau(g(a)) - \tau((1 - p)a(1 - p)) - 3\varepsilon_1 \\
\geq \tau(g(a)) - \tau(1 - p) - 3\varepsilon_1 > \tau(1 - p).
\]

Since \( A \) has strict comparison, this shows that \( 1 - p \not\precsim r \).

Next, since \( r \in g(d)Dg(d) \), by the choice of \( g \), it’s easy to see that \( \|rd\| \leq \|g(d)\| \leq \varepsilon_0 \). Hence for any projection \( s \leq r \), \( \|sd\| = \|sr\| \leq \varepsilon_0 \). Similarly \( \|ds\| \leq \varepsilon_0 \). Now combine the fact that \( \|pa - pa\| < \delta/2 < \varepsilon_0 \) and \( \|pap - d\| < \varepsilon_0 \), we can get

\[
\|sa\| = \|s(pap) + spa(1 - p) + s(1 - p)a\| \leq (\|sd\| + \varepsilon_0) + \varepsilon_0 \leq 3\varepsilon_0
\]

And similarly \( \|as\| \leq \varepsilon \).

Now since \( 1 - p \not\precsim r \), let \( v \) be a partial isometry such that \( vv^* = 1 - p \) and \( v^*v = s \leq r \leq p \). Using the decomposition \( 1 = (p - s) \oplus s \oplus (1 - p) \), we may write \( a \) in the matrix form:

\[
a = \begin{pmatrix}
(p - s)a(p - s) & (p - s)as & (p - s)a(1 - p) \\
sa(p - s) & sas & sa(1 - p) \\
(p - s)a(p - s) & (1 - p)as & (1 - p)a(1 - p)
\end{pmatrix}
\]

Now \( (p - s)a(p - s) = \varepsilon_0 (p - s)d(p - s) \in (p - s)D(p - s) \). Since \( (p - s)D(p - s) \)
has real rank zero, there is an invertible self-adjoint element \( d_1 \in (p - s)D(p - s) \)
such that \( \|(p - s)a(p - s) - d_1\| < \varepsilon_0 \). Hence

\[
a = 2\varepsilon_0 \begin{pmatrix}
(p - s)d(p - s) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & (1 - p)a(1 - p)
\end{pmatrix}
\]

\[
= 2\varepsilon_0 \begin{pmatrix}
d_1 & 0 & 0 \\
0 & 0 & \varepsilon_0 v^* \\
0 & \varepsilon_0 v & (1 - p)a(1 - p)
\end{pmatrix}
\]

The last matrix corresponds to a invertible self-adjoint element \( a_0 \) in \( A \). By our choice of \( \varepsilon_0 \), we have \( \|a - a_0\| < \varepsilon. \)

\[\square\]

Combine Theorem \[3.2\] Lemma \[4.1\] and Proposition \[4.2\] we get the following:

**Theorem 4.3.** Let \( A \) be a simple unital C*-algebra with real rank zero and has strict comparison for projections. Let \( \alpha \in \text{Act}_G(A) \) be an action with the tracial Rokhlin property, where \( G \) is amenable group. Then \( A \rtimes_\alpha G \) has real rank zero.
Now let’s turn to the case of stable rank one. We first see that Lemma 5.2 of [13] could be generalized to actions of general amenable groups, because its proof depends only on Lemma 2.5 and Lemma 2.6 of [13] and some other lemmas unrelated to crossed product. We could use Lemma 3.1 to replace the first one, and Lemma 2.6 of [13] could be generalized to actions of amenable groups with the same proof. Hence we have:

**Lemma 4.4.** Let $A$ be a simple C*-algebra with real rank zero and strict comparison for projections. Let $\alpha \in \text{Act}_G(A)$ with the tracial Rokhlin property. Then for any nonzero projections $p_1, \ldots, p_n \in A \rtimes_\alpha G$ and arbitrary elements $a_1, \ldots, a_n \in A \rtimes_\alpha G$, any $\varepsilon > 0$, there exist a unital subalgebra $A_0 \subset A \rtimes_\alpha G$ which is stably isomorphic to $A$, a projection $p \in A_0$ and subprojections $r_1, \ldots, r_n$ of $p$ such that:

1. $pa \in \varepsilon A_0$, $ap \in \varepsilon A_0$.
2. $p_k r_k = \varepsilon r_k$, for any $k$.
3. $1 - p \precsim r_k$, for any $k$.

**Proposition 4.5.** Let $A$ be a unital simple stably finite C*-algebra with Property (SP). If for any $x \in A$, any $\varepsilon > 0$ and any projection $p_1, \ldots, p_n$, there is a unital simple subalgebra $D$ with stable rank one and Property (SP), a projection $p \in D$ and subprojections $r_1, \ldots, r_n$ of $p$ such that:

1. $pxp \in \varepsilon D$.
2. $r_k p_k = \varepsilon r_k$.
3. $1 - p \precsim r_k$.

Then $A$ has stable rank one.

**Proof.** Let $x$ be an arbitrary element of $A$ and let $\varepsilon > 0$ be given. Without loss of generality assume $\|x\| = 1$. Since $A$ is stably finite, every one sided invertible elements is two sided invertible, hence by Theorem 3.3(a) of [16], we may assume that $x$ is a two-sided zero divisor. Since $A$ has property (SP), we can find nonzero projections $e, f$ such that $ex = xf = 0$. Let $\varepsilon_0 = \varepsilon/11$. We can then find a unital simple subalgebra $D$ with stable rank one and Property (SP), a projection $p \in D$ and sub-projections $e_0, f_0$ of $p$, such that

$$e_0 e = \varepsilon_0 e_0, \quad f_0 f = \varepsilon_0 f_0$$

Consider $x_0 = (1 - e_0) x (1 - f_0)$. Then

$$x_0 = 2 \varepsilon_0 (1 - e_0) x (1 - f_0) = x$$

Since $D$ is a simple C*-algebra with Property (SP), there is a nonzero projection $r \leq e_0$ and $r \precsim f_0$. Since $D$ has stable rank one, there exists some unitary $u$ such that $uru^* \leq f_0$. Hence $r(x_0 u) = (x_0 u) r = 0$. 

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Next, we shall approximate \( x_1 = x_0 u \) by an invertible element. To this end we find a unital subalgebra \( D_1 \) of \( A \) with stable rank one, a projection \( p_1 \in D_1 \) and sub-projection \( r_1 \) of \( p_1 \), and an element \( d \in D_1 \) such that
\[
px_1 p = \varepsilon_0 d, \quad r_1 r = \varepsilon_0 r_1, \quad \text{and} \quad 1 - p \preceq r_1.
\] (25)

Choose a partial isometry \( v \) such that \( vv^* = 1 - p \) and \( v^* v = s \leq r_1 \leq p_1 \).

According to the decomposition \( 1 = (1 - p_1) \oplus (p_1 - s) \oplus s \), we can write \( x_1 \) in the matrix form:
\[
x_1 = \begin{pmatrix}
(1 - p_1)x_1(1 - p_1) & (1 - p_1)x_1(p_1 - s) & (1 - p_1)x_1 s \\
(p_1 - s)x_1(1 - p_1) & (p_1 - s)x_1(p_1 - s) & (p_1 - s)x_1 s \\
ax_1(1 - p_1) & bx_1(p_1 - s) & sx_1 s
\end{pmatrix}
\]

Since \((p_1 - s)D_1(p_1 - s)\) has stable rank one, there is an invertible element \( d_1 \in (p_1 - s)D_1(p_1 - s) \) such that
\[
d_1 = \varepsilon_0 (p_1 - s)d(p_1 - s) = \varepsilon_0 (p_1 - s)x_1(p_1 - s).
\] (26)

We also have \( sx_1 = sr_1 x_1 = \varepsilon_0 sr_1 r x_1 = 0 \), and similarly \( x_1 s = \varepsilon_0 0 \). Therefore
\[
x_1 = 7\varepsilon_0 \begin{pmatrix}
a & b & 0 \\
c & d_1 & 0 \\
0 & 0 & 0
\end{pmatrix} = \varepsilon_0 \begin{pmatrix}
a & b & \varepsilon_0 \\
c & d_1 & 0 \\
\varepsilon_0 & 0 & 0
\end{pmatrix}.
\] (27)

Let’s call the last matrix \( x_2 \), which is invertible. Then
\[
\|x - x_2 u^*\| \leq \|x - x_0\| + \|(x_0 u - x_2)u^*\| < 11\varepsilon_0 < \varepsilon
\] (28)

Hence \( A \) has stable rank one.

Combine Lemma 4.4 and Proposition 4.5 we get:

**Theorem 4.6.** Let \( A \) be a simple unital C*-algebra with real rank zero, stable rank one and has strict comparison for projections. Let \( \alpha \in \text{Act}_G(A) \) be an action with the tracial Rokhlin property, where \( G \) is an amenable group. Then \( A \rtimes_\alpha G \) has stable rank one.

**5 Examples of actions with tracial Rokhlin property**

We now construct actions with (weak) tracial Rokhlin property where the group \( G \) is a class of amenable groups satisfies suitable version of Rokhlin lemma. We give some definitions from [18] to begin with.

**Definition 5.1.** ([18]) A monotile \( T \) in a discrete group \( G \) is a finite set for which one can find a set \( C \) such that \( \{Tc : c \in C\} \) is a covering of \( G \) by disjoint sets.
Definition 5.2. ([18]) A set $T \subset G$ is called an $R$-set (Rokhlin set) if for any measure preserving free action of $G$ on a finite measure space $(X, \mathcal{B}, \mu)$ and any $\varepsilon > 0$ there is a set $B \in \mathcal{B}$ satisfying:

1. for $t, t' \in T, t \neq t'$, the sets $tB$ and $t'B$ are disjoint;
2. $\mu(\cup_{t \in T} tB) > 1 - \varepsilon$.

Theorem 5.3. ([18]) A set $T$ in an amenable group $G$ is an $R$-set if and only if it is a monotile.

Definition 5.4. An amenable group is called $ST$-tileable (Single Tower tileable) if for any finite $K$ in $G$ and any $\varepsilon > 0$, there is a monotile $T$ in $G$ which is $(K, \varepsilon)$-invariant.

Remark 5.5. There was no name for such group in literature. Although it has been studied in [18], the term monotileable group was defined to mean something else.

Theorem 5.6. ([18]) All residually finite amenable groups and finitely generated abelian groups are $ST$-tileable. The class of $ST$-tileable groups are closed under taking extensions and direct union. In particular, all elementary amenable groups are $ST$-tileable.

In the following, for any unital C*-algebra $A$, we use $\otimes_{i \in \mathbb{Z}} A$ to mean the tensor product of infinite many copies of $A$ indexed by integers. We embed $\otimes_{i=1}^{n} A$ into $\otimes_{n \in \mathbb{Z}} A$ in the natural way and identify it with its image. We use similar convention for infinite product of topological spaces.

Lemma 5.7. Let $G$ be a infinite $ST$-tileable group and $\sigma : G \curvearrowright \mathbb{Z}$ be a faithful action, viewing $\mathbb{Z}$ as a set. Let $([0, 1], \mathcal{B}, \mu)$ be the unit interval with a probability measure on borel measurable sets such that, for any $t \in [0, 1]$, we have $\mu(\{t\}) = 0$. Let $\alpha : G \to \prod_{i \in \mathbb{Z}} [0, 1]$ be the induced index-shift action defined by

$$
\alpha_g((x_i)_{i \in \mathbb{Z}}) = (y_i)_{i \in \mathbb{Z}}, \text{ where } y_i = x_{\sigma^{-1}(i)} \quad (29)
$$

Let $\mu_0$ be the product measure on $\prod_{i \in \mathbb{Z}} [0, 1]$. Then for any $K \subset_f G$, any $\varepsilon_1 > 0$, there is a $(K, \varepsilon_1)$-invariant subset $T$ such that for any $\varepsilon_2 > 0$, there exist an open set $U$ such that

1. $gU$ and $g'U$ are disjoint, for any $g, g' \in T$ such that $g \neq g'$,
2. $\mu_0(\prod_{i \in \mathbb{Z}} [0, 1] \setminus \cup_{g \in T} gU) < \varepsilon_2$, and
3. there is some $N \in \mathbb{Z}_+$ such that for each $g \in T$, we have

$$
gU = \prod_{i < -N} [0, 1] \times U_g \times \prod_{i > N} [0, 1] \quad (30)
$$

for some open set $U_g \in \prod_{i=-N}^{N} [0, 1]$.

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Proof. It’s easy to see that $\alpha$ is measure preserving. We now show it is free. Given $g \in G \setminus \{1\}$, let $S = \{ x \mid \sigma_g(x) = x \}$. Since the action $\sigma$ is faithful, there exist some $i \in \mathbb{Z}$ such that $\sigma_g(i) = j \neq i$. Without loss of generality assume $i < j$. Let $x = (x_n)_{n \in \mathbb{Z}} \in S$. Then $\sigma_g(x) = x$ implies $x_i = x_j$. Let

$$J = \{(x_i, x_{i+1}, \ldots, x_j) \mid x_i = x_j\} \subset \prod_{n=i}^{j}[0,1]$$

Then $S \subset \prod_{n \leq i}[0,1] \times J \times \prod_{n \geq j}[0,1]$. Use Fubini’s theorem plus the assumption that single point set in $[0,1]$ has zero measure, we get $\mu_0(J) = 0$. Hence $\mu_0(S) = 0$.

Let $K \subset_f G$ and $\varepsilon_1 > 0$ be given. We can find a $(K, \varepsilon_1)$-invariant subset $T$ which is also a monotile. Let $\delta = \frac{\varepsilon_1}{2^i \min(1,|T| + 1)}$, by Theorem 5.3 for any $\delta$, we find a measurable set $B$ such that

1. $gB$ and $g' B$ are disjoint, for any $g, g' \in T$ such that $g \neq g'$;

2. $\mu_0(\prod_{g \in T}[0,1]\cup_{g \in T} gB) < \delta$.

We shall perturb the set $B$ to get an open set with the desired form. Since any probability measure on a compact metric space is regular, we can find a closed subset $X \subset B$ such that $\mu_0(B \setminus X) < \delta$ and an open set $U \supset B$ such that $\mu_0(U \setminus B) < \delta$. Write $U = \cup \lambda U_\lambda$, where each $U_\lambda$ is an open set of the form

$$\cdots \times [0,1] \times (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_k, b_k) \times [0,1] \cdots$$

We note that $\mu_0(\partial U) = 0$, using the fact that single point set has zero measure. By compactness of $X$, we can select $U_{\lambda_1}, U_{\lambda_2}, \ldots, U_{\lambda_n}$ among the $U_\lambda$’s whose union still cover $X$. Let $U_0 = \cup_{i=1}^{n} U_{\lambda_i}$. We can estimate that, for any $g, h \in T$ where $g \neq h$:

$$\mu_0(\alpha_g(U_0) \cap \alpha_h(U_0)) \leq \mu_0(\alpha_g(U_0 \setminus B)) + \mu_0(\alpha_h(U_0 \setminus B)) < 2\delta.$$  

(33)

Let

$$U_1 = U_0 \setminus \cup_{g,h \in T, g \neq h} U_0 \cap \alpha_{g^{-1} h}(U_0)$$

(34)

We can see that for $g \in T$, the sets $\alpha_g(U_1)$ are mutually disjoint. Using the fact that $\partial(X \cap Y) \subset \partial(X) \cup \partial(Y)$, we see that the boundary of $U_0 \cap \alpha_g(U_0)$ has measure 0, for any $g \in G$. Now we can estimate the measure:

$$\mu_0(\cup_{g \in T} \alpha_g(U_1))$$

$$> \mu_0(\cup_{g \in T} \alpha_g(U_0)) - 2\delta |T|^2$$

$$\geq \mu_0(\cup_{g \in T} \alpha_g(B)) - \cup_{g \in T} \alpha_g(B \setminus U_0) - 2\delta |T|^2$$

$$> 1 - \delta - \delta |T| - 2\delta |T|^2 = 1 - \varepsilon/2.$$  

So that $\mu_0(\prod_{g \in T}[0,1] \cup_{g \in T} \alpha_g(U_1)) < \varepsilon$.

From our construction, we can see that $U_1$ is an open set of the form $\prod_{i < -N}[0,1] \times V \times \prod_{i > N}[0,1]$, for some $N \in \mathbb{Z}^+$, and open set $V \in \prod_{i = -N}^{N} [0,1]$. Increasing $N$ if necessary, we can conclude that for all $g \in T$, the sets $\alpha_g(U_1)$ has the same form. \qed
Lemma 5.8. Let \( N \in \mathbb{Z}_+ \), and let \( F \) be a finite subset of \( \otimes_{i=-N}^N Z \). Let \( f_1, f_2, \ldots, f_n \) be \( n \) normalized continuous functions on \( \prod_{i=-N}^N [0,1] \) and \( \varepsilon > 0 \). Then there is an embedding \( \iota : C([0,1]) \to Z \), such that the induced embedding (still denoted by \( \iota \)) of \( C(\prod_{i=-N}^N [0,1]) \cong \otimes_{i=-N}^N C([0,1]) \) into \( \otimes_{i=-N}^N Z \) satisfies:

1. \( ||[a, \iota(f_i)]|| < \varepsilon \), for any \( a \in F \) and \( 1 \leq i \leq n \).
2. \( \tau(\iota(f_i)) = \int_{\prod_{i=-N}^N [0,1]} f_i \, d\mu \), where \( \tau \) is the unique tracial state on \( \otimes_{i=-N}^N Z \) and \( \mu \) is the Lebesgue measure.

Proof. Without loss of generality assume \( F \) consists of only elementary tensors of the form \( z^{(i)}_N \otimes z^{(i)}_{N+1} \otimes \cdots \otimes z^{(i)}_N \), for \( 1 \leq i \leq m \). Let

\[
\tilde{F} = \{ z^{(i)}_j | 1 \leq i \leq m, -N \leq j \leq N \}.
\]  

Let \( M = \max \{ ||z^{(i)}_j|| \} \). For each \( f_i \), let

\[
g_i = \sum_j c_{ij} g^{ij}_N \otimes g^{ij}_{N+1} \otimes \cdots \otimes g^{ij}_N
\]  

be a finite linear combination of elementary tensors such that \( ||f_i - g_i|| < \frac{\varepsilon}{M} \). We assume that each \( g^{ij}_k \in C([0,1]) \) has norm 1. Let \( C = \max_i \{ \sum_j \|c_{ij}\| \} \). Let \( \delta = \frac{\varepsilon}{2NM^2} \). By Lemma 2.4 of [4], we can find a unital embedding \( \iota : C([0,1]) \to Z \) such that

1. \( \|z^{(i)}_j, \iota(g)\| < \delta \), for any \( z^{(i)}_j \in \tilde{F} \) and any normalized \( g \in C([0,1]) \).
2. \( \tau(\iota(g)) = \int_{[0,1]} g \, d\mu \), for any \( g \in C([0,1]) \).

Consider the induced embedding \( \iota : \otimes_{i=-N}^N C([0,1]) \to \otimes_{i=-N}^N Z \). The first condition \( \|a, \iota(f_i)\| < \varepsilon \) in the lemma follows from Condition (1') above and the choice of \( \delta \). For the second condition in the lemma, we note that this is true for any elementary tensors \( g^{ij}_{N+1} \otimes \cdots \otimes g^{ij}_N \in \otimes_{i=-N}^N C([0,1]) \) by Condition (2') above and Fubini’s theorem. Therefore it’s true for any linear combinations of elementary tensors, and hence true for any continuous functions by continuity. \( \square \)

Theorem 5.9. Let \( G \) be a ST-tileable group, and let \( Z \) be the Jiang-Su algebra. Suppose \( \sigma \) is an action on \( Z \) (viewed as a set), then there is an induced action \( \alpha \) on \( \otimes_{i \in \mathbb{Z}} Z \) defined by

\[
\alpha_g(\otimes_i z_i) = \otimes_i z_{\sigma^{-1}(i)}.
\]  

If \( \sigma \) is faithful, the action \( \alpha \) has the weak tracial Rokhlin property. In particular, the Bernoulli shift has the weak tracial Rokhlin property.
Proof. There is a canonical isomorphism between $C(\prod_{i \in \mathbb{Z}} [0, 1])$ and $\otimes_{i \in \mathbb{Z}} C([0, 1])$. We abuse the notation $\alpha$ for all the induced actions on $\prod_{i \in \mathbb{Z}} [0, 1]$, on $C(\prod_{i \in \mathbb{Z}} [0, 1])$ and on $\otimes_{i \in \mathbb{Z}} C([0, 1])$. This should cause no confusion as long as the underlying space (or C*-algebra) is specified.

To verify that the action on $\otimes_{i \in \mathbb{Z}} Z$ has the weak tracial Rokhlin property, we let a finite subset $K \subset fG$ and $\varepsilon_1$ be given. We can find a $(K, \varepsilon_1)$-invariant $T \subset fG$ satisfies the conditions as in Lemma 5.7. Now let $F \subset f \otimes_{i \in \mathbb{Z}} Z$ and $\varepsilon_2 > 0$ be arbitrary. Without loss of generality we assume $F$ consists of elementary tensors $z^{-M} \otimes z^{-M+1} \otimes \cdots \otimes z^M$ in $\otimes_i^{M=-M} Z$, for some fixed $M \in \mathbb{Z}_+$. Let $\delta = \frac{\varepsilon_1}{|T|+1}$. By Lemma 5.7, we can find an open set $U \in \prod_{i \in \mathbb{Z}} [0, 1]$ such that:

1. $gU$ and $g'U$ are disjoint, for any $g, g' \in T$ s.t. $g \neq g'$.
2. $\mu_0(\prod_{i \in \mathbb{Z}} [0, 1]) \cup g \in T gU) < \delta$.
3. there exist some integer $N > M$ such that each $gU$ is of the form $\prod_{i < -N} [0, 1] \times V_g \times \prod_{i > N} [0, 1]$, for open sets $V_g \in \prod_{i = -N}^{N} [0, 1]$.

Using the regularity of $\mu_0$ on $\prod_{i = -N}^{N} [0, 1]$, we can find a closed subset $X \in V$ such that $\mu_0(U \setminus X) < \delta$. Let $f$ be a positive continuous function of norm 1 that is 1 on $X$ and 0 outside $V$, let $f_g$ be the function such that $f_g(x) = f(\alpha_g^{-1}(x))$.

Now by Lemma 5.8 we can choose an embedding $C([0, 1]) \rightarrow Z$ such that

1. $\|a, \iota(f_g)\| < \varepsilon$, for any $a \in F$ and $g \in T$.
2. $\tau(\iota(f_g)) = \int_{\prod_{i = -N}^{N} [0, 1]} f_g d\mu$, where $\tau$ is the unique tracial state on $\otimes_{i = -N}^{N} Z$ and $\mu$ is the Lebesgue measure.

Let $e_g = \iota(f_g)$. We see that $\{e_g\}_{g \in T}$ is a set of mutually orthogonal positive contractions, they commute with elements in $F$ up to an error of $\varepsilon$, and such that

$$\tau(1 - \sum_{g \in T} e_g) = \int_{\prod_{i = -N}^{N} [0, 1]} \left(1 - \sum_{g \in T} f_g\right) d\mu < \mu(\prod_{i = -N}^{N} (\cup_{g \in T} gX)) < \delta + |T|\delta < \varepsilon.$$ 

Since $Z$ has strict comparison, the action $\alpha$ has the weak tracial Rokhlin property by Proposition 2.4. \qed
Corollary 5.10. Any ST-tileable group has Property (Q) in the sense of Definition 2.10.

Proof. Let \( \tau \) be the unique tracial state on \( Z \) and \( \pi \) be the associated GNS representation. Then \( \pi(A)'' \) is the hyperfinite II\(_1\) factor \( R \). Let \( \alpha \) be an action of \( G \) on \( Z \) with the weak tracial Rokhlin property. Then it extend to an outer action on \( R \) by Proposition 2.8. For any \( K \subset_f G \) and any \( \varepsilon > 0 \), we can choose a \((K, \varepsilon)\)-invariant \( T \subset_f G \) and a positive contraction \( f = (f_n)_{n \in \mathbb{N}} \in A_\infty \) such that

1. \( \alpha_g(f)\alpha_h(f) = 0 \), for any \( g, h \in T \) such that \( g \neq h \).
2. \( \lim_{n \to \infty} \tau(\alpha_g(f_n)) = \frac{1}{|T|} \), for any \( g \in T \).

Using projectivity of \( \oplus^{|T|} C_0((0,1]) \), we can lift the set \( \{\alpha_g(f)\}_{g \in T} \) to a set of mutually orthogonal positive contractions \( \{f_g = (f_g^{(n)})\}_{g \in K} \subset \ell^\infty(\mathbb{N}, A) \). Let \( p_g^{(n)} \) be the support projection of \( f_g^{(n)} \), for \( g \in T \). We note that these projections are mutually orthogonal. Let \( p_g = (p_g^{(n)}) \). Then we have \( \tau(p_g^{(n)}) \geq \tau(f_g^{(n)}) \), for each \( n \in \mathbb{N} \) and \( g \in T \). Therefore

\[
\liminf_{n \to \infty} \tau(p_g^{(n)}) \geq \limsup_{n \to \infty} \tau(f_g^{(n)}) = \lim_{n \to \infty} \tau(\alpha_g(f_n)) = \frac{1}{|T|} \quad (40)
\]

But then

\[
\limsup_{n \to \infty} \tau(p_g^{(n)}) \leq \limsup_{n \to \infty} \tau(1 - \sum_{h \neq g} p_h^{(n)}) \leq 1 - \sum_{h \neq g} \liminf_{n \to \infty} p_h^{(n)} = \frac{1}{|T|} \quad (41)
\]

Hence \( \lim_{n \to \infty} \tau(p_g) = \frac{1}{|T|} \). Let \( \|(x_n)_{n \in \mathbb{N}}\|_2 = \limsup_{n \to \infty} \|x_n\|_2 \), which is well defined in \( R^\infty \). Therefore \( \|p_g - f_g\|_2 = 0 \), for any \( g \in T \). Choose some \( k \in T \) and let \( p = \alpha_{k-1}(p_k) \). Then

\[
\|1 - \sum_{g \in T} \alpha_g(p)\|_2 = \|1 - \sum_{g \in T} \alpha_g(f)\|_2 = 0 \quad (42)
\]

For any \( x \in R \), by Kaplansky density theorem, we can choose a bounded sequence \( x_n \) converges strongly to \( x \). Note that this is the same as saying \( \lim \|x_n - x\|_2 = 0 \). Embed \( R \) into \( R^\infty \) by constant sequences, we can compute:

\[
\|x, p\|_2 = \lim_{n \to \infty} \|(x_n, \alpha_k(p_k))\|_2 = \lim_{n \to \infty} \|(x_n, \alpha_k(f_k))\|_2 = 0. \quad (43)
\]

Hence \( \alpha \) has property (Q). \( \square \)

Another types of example comes from the product-type actions. We begin with the definition:

Definition 5.11. Let \( A = \otimes_{i=1}^\infty B(H_i) \), where \( H_i \) is a finite dimensional Hilbert space for each \( i \). An action \( \alpha \in \text{Act}_G(A) \) is called a product-type action if and only if for each \( i \), there exists a unitary representation \( \pi_i : G \to B(H_i) \), which induces an inner action \( \alpha_i : g \mapsto \text{Ad}(\pi_i(g)) \), such that \( \alpha = \otimes_{i=1}^\infty \alpha_i \).
Definition 5.12. Let \( \alpha \in \text{Act}_G(A) \) be a product-type action on a UHF-algebra \( A \). A telescope of the action is a choice of an infinite sequence of positive integers \( 1 = n_1 < n_2 < \cdots \) and a re-expression of the action, so that \( A = \otimes_{i=1}^{\infty} B(T_i) \) where \( T_i = \otimes_{j=n_i}^{n_{i+1}-1} H_j \), and the action on \( B(T_i) \) is \( \otimes_{j=n_i}^{n_{i+1}-1} \alpha_j \).

Theorem 5.13. Let \( \alpha \in \text{Act}_G(A) \) be a product-type action where \( G \) has property (Q) and \( A \) is UHF. Let \( H_i, \pi_i \) and \( \alpha_i \) be defined as in Definition 5.11. Let \( d_i \) be the dimension of \( H_i \) and \( \chi_i \) be the character of \( \pi_i \). We will use the same notations if we do a telescope to the action. Define \( \chi : G \to \mathbb{C} \) to be the characteristic function on \( 1_G \). Then the action \( \alpha \) has the tracial Rokhlin property if and only if there exists a telescope, such that for any \( n \in \mathbb{N} \), the infinite product

\[
\prod_{n \leq i < \infty} \frac{1}{d_i} \chi_i = \chi.
\]  

Proof. Any UHF algebra is \( \mathcal{Z} \)-absorbing and monracial. By Theorem 3.6 of [11], that \( \alpha \) has the tracial Rokhlin property is equivalent to that \( \alpha \) is strongly outer. In this case, \( \alpha \) has the tracial Rokhlin property if and only if \( \alpha|_H \) has tracial Rokhlin property for any cyclic subgroup \( H \subset G \).

Let \( \chi_{H,i} \) be the restriction of \( \chi_i \) to the subgroup \( H \), which is exactly the character of the restricted action \( \pi_i|_H \). We observe that \( \prod_{n \leq i < \infty} \frac{1}{d_i} \chi_i = \chi \) if and only if

\[
\prod_{n \leq i < \infty} \frac{1}{d_i} \chi_{H,i} = \chi, \forall \text{ cyclic subgroup } H \subset G.
\]  

Hence the theorem will be proved if we can show that it is true for any cyclic group \( G \). If \( G \) is finite, then it is proved in (17). If \( G \) is infinite, let \( x \) be a generator and \( U_i \) be the unitary in \( B(H_i) \) such that \( \pi_i(x) = \text{Ad} U_i \). Let \( S_{k,l} \) be a sequence consisting of eigenvalues of \( \otimes_{i=k}^l U_i \), repeated as often as multiplicity indicates. Kishimoto has shown that in case of infinite cyclic group acting on UHF algebra, the tracial Rokhlin property coincide with the Rokhlin property. He also show in Lemma 5.2 of [2] that the product-type action \( \alpha \) has the Rokhlin property if and only \{\( S_{k,l} \)\}_{l=k}^{\infty} \) is uniformly distributed, for any \( k \in \mathbb{N} \). Now fix some \( k \in \mathbb{N} \). For any sequence \( S = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) in \( T \), We let \( \mu_S \) be the measure on \( \mathbb{T} \) such that \( \mu_S = \frac{1}{n} \sum \delta_{\lambda_i} \), where \( \delta_{\lambda_i} \) is the Dirac measure concentrated at the point \( \lambda_i \in \mathbb{T} \). By definition, \{\( S_{k,l} \)\}_{l=k}^{\infty} \) is uniformly distributed if and only if

\[
\lim_{l \to \infty} \mu_{S_{k,l}}(f) = \int_{\mathbb{T}} f \, d\mu, \forall f \in C(\mathbb{T}),
\]  

where \( \mu \) is the normalized Haar measure. Now it’s not hard to see that

\[
\prod_{k \leq i < l} \frac{1}{d_i} \chi_i(n) = \mu(S_{k,l})(z^n), \forall n \in \mathbb{Z},
\]  

where \( z^n \in C(\mathbb{T}) \) stands for the function \( z \to z^n \).

Hence

\[
\prod_{k \leq i < \infty} \frac{1}{d_i} \chi_i = \chi.
\]
is equivalent to
\[
\lim_{n \to \infty} \mu_{S_{k,T}}(z^n) = \delta(n,0) = \int_T z^n \, d\mu, \forall n \in \mathbb{Z}
\] (49)

And therefore further equivalent to that \( \{S_{k,T}\}_{n=k}^{\infty} \) being uniformly distributed, since any continuous function in \( C(T) \) can be uniformly approximately by finite linear combinations of the functions \( z^n \).

Another types of example comes from actions on non-commutative tori. Let \( \theta \) be a non-degenerate anti-symmetric bicharacter on \( \mathbb{Z}^d \). We identify it with its matrix under the canonical basis of \( \mathbb{Z}^d \). Then the associated non-commutative tori \( A_\theta \) is simple. The commutative tori \( A_\theta \) is simple, unital \( \mathcal{AT} \) algebra with a unique trace. \( A_\theta \) is generated by unitaries \( U_x | x \in \mathbb{Z}^d \) subject to the relation
\[
U_y U_x = \exp(\pi i \langle x, \theta y \rangle) U_{x+y}, \forall x, y \in \mathbb{Z}^d.
\] (50)

**Proposition 5.14.** For any \( T \in M_n(\mathbb{Z}) \), the map \( U_x \to U_{T \cdot x} \) give rises to an automorphism \( \alpha_T \) of \( A_\theta \) if and only if \( \frac{1}{d} (T^t \theta T - \theta) \in M_d(\mathbb{Z}) \). It is an automorphism if and only if \( T \) is invertible. For any \( T \neq I \), \( \alpha_T \) is not weakly inner. If \( d = 2 \), then for any \( T \in GL_2(\mathbb{Z}) \), we have \( \frac{1}{d} (T^t \theta T - \theta) \in M_d(\mathbb{Z}) \).

**Corollary 5.15.** Let \( G \) be any ST-tileable subgroup of \( \{ T \in GL_d(\mathbb{Z}) | \frac{1}{d} (T^t \theta T - \theta) \in M_d(\mathbb{Z}) \} \). Then the action \( \alpha \in \text{Act}_G(A_\theta) \), defined by \( T \to \alpha_T \), has the tracial Rokhlin property.

**Proof.** Having tracial Rokhlin property is the same as being strongly outer here, since \( A_\theta \) has tracial rank zero and monotracial.

If we can find one example of actions with (weak) tracial Rokhlin property, we can actually find lots of them by forming inner tensors. More specifically, we have the following:

**Proposition 5.16.** Let \( \alpha \in \text{Act}_G(A) \) be an action with the weak tracial Rokhlin property and \( \beta \in \text{Act}_G(B) \) be arbitrary, where \( A, B \) are both simple. Then the inner tensor of these two actions \( \gamma = \alpha \otimes \beta \in \text{Act}_G(A \otimes_{\text{min}} B) \) has the the weak tracial Rokhlin property. If \( \alpha \) has the tracial Rokhlin property, then \( \gamma \) has the tracial Rokhlin property.

**Proof.** We first show that for any non-zero positive element \( x \in A \otimes_{\text{min}} B \), there is an non-zero positive element \( d \in A \) such that \( d \otimes 1 \not\precsim x \). By Kirchberg’s slice lemma (Lemma 2.7 of [15] or Lemma 4.1.9 of [15]), there is some non-zero positive elements \( a \in A_+ \) and \( b \in B_+ \) and some \( z \in A \otimes_{\text{min}} B \) such that \( zz^* = a \otimes b \) and \( z^*z \in \text{Her}(x) \). This in particular shows that \( a \otimes b \not\precsim x \). Since \( B \) is simple and unital, we can find elements \( \{ s_i \mid i = 1, 2, \ldots, n \} \subset B \) such that \( \sum_i s_i b s_i^* = 1 \). By Proposition 4.10 of [6], we can find some non-zero positive contraction \( d \in A \) such that \( d^\oplus_n \not\precsim a \). Hence
\[
d \otimes 1 = \sum_i (1 \otimes s_i)(d \otimes b)(1 \otimes s_i)^* \not\precsim (d \otimes b)^\oplus_n \sim d^\oplus_n \otimes b \not\precsim a \otimes b \not\precsim x.
\] (51)
For any finite set $K \subset fG$ and any $\varepsilon > 0$, we can find a $(K, \varepsilon)$-invariant $T \subset fG$ such that, for any $a \in A_+ \setminus \{0\}$, there is a positive contraction $f \in A_\infty$ such that:

1. $\alpha_g(f) \alpha_h(f) = 0$, for any $g, h \in T$ such that $g \neq h$.

2. With $e = \sum_{g \in G} \alpha_g(f)$, $1 - e \preceq_{p.w.} a$.

Now let $x \in (A \otimes_{\min} B)_+ \setminus \{0\}$. Choose a $d \in A_+ \setminus \{0\}$ such that $d \otimes 1 \preceq x$. Choose a positive contraction $f \in A_\infty$ with $d$ in place of $a$ as above. Now consider the positive contraction $f \otimes 1$, it’s clear that $f \otimes 1 \in (A \otimes_{\min} B)_\infty$ and:

1. $\gamma_g(f \otimes 1) \gamma_h(f \otimes 1) = (\alpha_g(f) \alpha_h(f)) \otimes 1 = 0$, for any $g, h \in T$ such that $g \neq h$.

2. With $\tilde{e} = \sum_{g \in T} \gamma_g(f \otimes 1)$, $1 - \tilde{e} \preceq_{p.w.} d \otimes 1 \preceq x$.

Hence $\gamma = \alpha \otimes \beta$ has the weak tracial Rokhlin property.

If $\alpha$ has the tracial Rokhlin property, then we can require $f$ to be a non-zero projection, then $f \otimes 1$ is also a projection, the above proof shows that $\gamma$ has the tracial Rokhlin property.

**Remark 5.17.** Let $G$ be any countable discrete group which admits an action on $Z$ with the weak tracial Rokhlin property, then we get lots of actions with the weak tracial Rokhlin property on any $Z$-stable C*-algebra $A$, by the above Proposition. In such cases, fixing $G$ and a $Z$-stable C*-algebra $A$, following the same argument as in [14], we can actually show that the set of actions with the weak tracial Rokhlin property is $G_\delta$-dense in $\text{Act}_G(A)$, where $\text{Act}_G(A)$ is endowed with the topology of pointwise convergence. In particular, by Theorem 2.7 if $G$ is ST-tileable and $A$ is simple with tracial rank zero, then actions with the tracial Rokhlin property forms a $G_\delta$-dense subset of $\text{Act}_G(A)$.

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