Dissipative Control and Observation of Linear Time-Delay Systems: Part I (Complete Version)

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Abstract—The paper consists of two parts. Part I develops optimization-based methods for the dissipative state-feedback control of linear systems with general delays of finite length via the Krasovskii functional (KF) approach. Our model imposes no limits to the delay numbers at the states, inputs, regulated outputs or controller, where the distributed delays (DDs) can contain any number of square-integrable functions. We introduce the notion of equivalent decompositions which can simultaneously factorize or approximate any function in the DDs. Moreover, the equivalent decomposition approach allows the construction of complete-type KFs whose integral kernels can contain any number of differentiable and linearly independent functions. The solutions to the problem are summarized in several theorems/corollaries and iterative algorithms, which can be used together to compute controller gains without requiring nonlinear solvers. Two numerical examples are tested to show the effectiveness of the proposed methodologies.

Index Terms—Linear Time-Delay Systems; Distributed Delays; Dissipative Controller Design, Krasovskii Functionals.

I. INTRODUCTION

Generally speaking, two types of delays, pointwise and distributed delays (DDs), have been utilized to model transport, propagation or aftereffects in practical dynamical systems. The nature of a pointwise-delay $x(t - r)$ is explained in [35] which can be denoted by a transport equation with boundary conditions. Meanwhile, delays can be created by transporting media with more complex structures. A DD is denoted by an integral $\int_{-\tau}^{0} F(\tau)x(t + \tau)d\tau$ over a delay interval $[-\tau, 0]$ with a matrix-valued function $F(\cdot)$, which takes into account a segment of the past dynamics’s information. Systems with both pointwise and DDs have many applications such as modeling biological processes [8] and chemical reaction networks [25], etc. In fact, the general form of a linear delay operator, denoted via a Lebesgue-Stieltjes integral [19], can be expressed as a summation of pointwise and distributed delays in general. Therefore, it is ideal to consider both types of delays in a method for time-delay systems.

Most methodologies for linear time-delay systems (LTDSs) are carried out in the time or frequency domain, where real or complex analysis is applied. To the best of the author’s knowledge, the newest trend of frequency-domain-based methods can be found in [27] and [1]. These works are predominantly nourished by the recent development of non-smooth optimization algorithms [31, 24]. However, no DDs have been considered by the above works, which is attributable to the obstacles in handling the Laplace transform of $\int_{-\tau}^{0} F(\tau)x(t + \tau)d\tau$ since $\int_{-\tau}^{0} F(\tau)e^{\tau\sigma}d\tau$ may not have a closed-form expression in the frequency domain.

For time-domain approaches [5, 23], the construction of Krasovskii functionals (KF) has been proven as an effective solution for the stability analysis and stabilization of LTDSs [37, 11, 9, 34] supported by efficient numerical algorithms for semidefinite programming (SDPs) [3, 33]. For a comprehensive collection of the existing literature on this topic, see the monographs in [5]. In contrast to the Lyapunov approach for an LTI system, the KF approach could only establish sufficient conditions where the induced conservatism is largely based on the generality of the predetermined form of KFs [23] and the integral inequalities [11] utilized to construct them. Because more general KFs [37, 9] have been increasingly adopted for the reduction of conservatism, we may not directly use congruent transformations to form convex controller synthesis conditions from the original stability analysis condition. Finally, a very interesting method is proposed in [41] for the stabilization of LTDSs with DDs, which can be considered as a combination of both
time and frequency domain approaches based on the concept of smoothed spectral abscissa [40, 18] and delay Lyapunov matrix [17, 16].

Nevertheless, it is safe to say that there are no effective solutions in the literature for the control and observation of LTDSs with both pointwise and general DDs, especially if there is an unlimited number of delays. Even if we only consider the case of stability analysis, most existing KF approaches impose restrictions on the structure of state space parameters [30] or DD kernels [11] or the number of DD kernels and delays [37, 9]. The method in [41] requires the computation of the delay Lyapunov matrix and its derivatives. It has not been elaborated in [41] how the computation can be carried out for an LTDS with general DDs or non-commensurate delays. Finally, the linear quadratic optimal control [6, 14] approach (infinite time horizon) for the stabilization of LTDSs require finding solutions for infinite-dimensional algebraic Riccati equations. However, the Riccati equations are solved via finite dimensional approximations [21], whose numerical results do not guarantee the stability the closed-loop system mathematically.

The aim of this work is to establish an efficient optimization-based solution to the dissipative controller synthesis problem of a very general class of LTDSs via the KF approach. The system’s model contains an unlimited number of pointwise and general DDs at the states, inputs, outputs, where the DDs can contain any number of $L^2$ functions over bounded intervals. The solutions to the synthesis problem are obtained by solving convex SDPs without asking for nonlinear solvers. Finally, the method is extended to construct controllers with general delays, when no delays exist at the input.

The main items and contributions of this paper are summarized as follows:

- The dissipative control problem in this work has not been investigated in the literature according to the author’s best knowledge. The generality of our LTDS with a quadratic supply rate function has important research significance as it can cover many LTDSs related control problems in an engineering setting. This is largely due to the incorporation of an unlimited number of pointwise and $L^2$ DDs.
- The notion of equivalent decomposition is proposed to handle the $L^2$ DD kernels, which significantly generalizes the approximation scheme in [9]. Unlike [9] where all $L^2$ DD kernels are approximated by a restricted class of differentiable functions, the proposed approach allows users to decide which $L^2$ kernels are factorized directly and which are approximated by any number of differentiable and linearly independent functions (DALIFs) with their derivatives. Moreover, it also allows one to construct KFs with integral kernels containing any number of DALIFs, which can be totally independent of the functions inside of the DDs.
- Two theorems and an iterative algorithm are proposed as the solution to the dissipative synthesis problem. The first theorem is derived from convexifying the bilinear matrix inequality (BMI) in the first theorem via Projection Lemma [13], without weakening the matrix parameters of the KFs. Moreover, the first theorem can be solved by the proposed iterative algorithms initiated by a feasible solution of the second theorem. Hence our method does not require the use of nonlinear SDP solvers.
- The method has also been extended to compute controllers with general delays when delays do not exist at the system’s input. This is a crucial contribution since static state controllers may not be sufficient for LTDSs in terms of stability or performance.
- Due to the connection between LTDSs and other types of systems, it has been shown that advanced treatment of LTDSs with general DDs [37, 11, 9] can lead to efficient solutions of many engineering problems such as networked control system [43] and PDE-ODE coupled system [2]. As a result, the proposed methods can serve as a blueprint for the future development of new solutions of real-world and engineering problems.

The organization of the rest of the paper is outlined as follows. Preliminaries are first presented in Section II concerning the derivation of the closed-loop system and the notion of equivalent decomposition. The main results concerning dissipative static controller synthesis are set out in Sections III, whereas Section IV summarizes the extended method for designing controllers with delays. Finally, the computation results of two numerical examples are presented in Section V prior to the final conclusion. Note that we place some important lemmas and proofs in the appendices.
NOTATION

Throughout this work, the notations $\mathcal{Y}^X := \{ f(\cdot) : X \ni x \mapsto f(x) \in \mathcal{Y}\}$ and $\mathbb{R}_{\geq a} := \{ x \in \mathbb{R} : x \geq a \}$ and $\mathbb{S}^n := \{ X \in \mathbb{R}^{n \times n} : X = X^\top \}$ Moreover, $\mathcal{C}(\mathcal{X}; \mathbb{R}^n) := \{ f(\cdot) \in (\mathbb{R}^n)^\mathcal{X} : f(\cdot) \text{ is continuous on } \mathcal{X} \}$ and $\mathcal{C}^k(\mathcal{Z}; \mathbb{R}^n) := \{ f(\cdot) \in \mathcal{C}(\mathcal{X}; \mathbb{R}^n) : \frac{df}{dx}(x) \in \mathbb{C}(\mathcal{Z}; \mathbb{R}^n) \}$. $\mathcal{M}(\mathcal{X}; \mathbb{R}) := \{ f(\cdot) \in \mathcal{L}^p : \forall y \in \mathcal{B}(\mathbb{R}), f^{-1}(y) \in \mathcal{L}(\mathcal{X}) \}$ denotes all measurable functions from Lebesgue measurable set $\mathcal{X}$ to $\mathbb{R}$. In addition, $\mathcal{L}^p(\mathcal{X}; \mathbb{R}^n) := \{ f(\cdot) \in \mathcal{M}(\mathcal{X}; \mathbb{R}^n) : \| f(\cdot) \|_p < +\infty \}$ with $\mathcal{X} \subseteq \mathbb{R}^n$ and the semi-norm $\| f(\cdot) \|_p := (\int_\mathcal{X} |f(x)|^p \, dx)^{\frac{1}{p}}$. $\mathbf{Sy}(X) := X + X^\top$ for any square matrix. Notations $\mathbf{Col}^n_{i=1} x_i := [x_i]_{i=1}^n := [x_1^\top \ldots x_i^\top \ldots x_n^\top]^\top$ denotes a column vector containing a sequence of mathematical objects (scalars, vectors, matrices etc.), whereas $\mathbf{Row}^n_{i=1} x_i := [x_i]_{i=1}^n := [x_1 \ldots x_i \ldots x_n]$ stands for the row vector of mathematical objects. Moreover, we use $\sqrt{X}$ to denote the meaning for almost all $x \in \mathcal{X}$ with respect to the Lebesgue measure. The symbol $*$ is used as abbreviations for $[\mathcal{Y}]^X = X^\top Y X$ or $X^\top Y X^{\top}$ for any $X \in \mathbb{C}^{n \times n}$, $Y \in \mathbb{C}^{p \times q}$ and its n-ary form $\bigotimes^n_{i=1} X_i = X_1 \otimes X_2 \otimes \cdots \otimes X_n$ to denote the diagonal sum of matrices $X_i \in \mathbb{C}^{n_i \times m_i}$. $\otimes$ stands for the Kronecker product. We use $\sqrt{X}$ to represent the unique square root of $X \succ 0$. The order of matrix operations in this paper is matrix (scalars) multiplications $\otimes > \div > + >$.

Finally, we use $[.]_{n,m}$, to represent empty matrices [38, See I.7] in this paper which follow the same definition and rules in the programming language of Matlab®. We assume $I_0 = [0,0]$, $O_{0,m} = [0,m]$ and $\mathbf{Col}^n_{i=1} x_i = [0,m]$, $\mathbf{Row}^n_{i=1} x_i = [m,0]$ if $n < 1$, where $[0,m]$, $[m,0]$ are empty matrices with an appropriate column dimension $m \in \mathbb{N}$ based on specific contexts.

II. PRELIMINARIES

A. Open-Loop LTDS

In this paper, we deal with an LTDS in the form of
\[ \dot{x}(t) = \sum_{i=0}^\nu A_i x(t - r_i) + \sum_{i=1}^\nu \int_{-r_i}^{t-r_i} A_i(\tau) x(t + \tau) \, d\tau. \]

$^a$Note that $\sqrt{X^{-1}} = (\sqrt{X})^{-1}$ for any $X \succ 0$ based on the application of eigendecomposition of $X \succ 0$.

\[ z(t) = \sum_{i=0}^\nu B_i x(t - r_i) + \sum_{i=1}^\nu \int_{-r_i}^{t-r_i} C_i(t) x(t + \tau) \, d\tau. \]

\[ \forall \theta \in [-r_0, 0], \quad x(t_0 + \theta) = \psi(\theta), \]

where the equation is defined $\forall t \geq t_0 \in \mathbb{R}$ with $\psi(\cdot) \in \mathcal{C}(\mathbb{R}; \mathbb{R}^n)$, and $\mathbb{R} := [-r_0, 0]$ and delays $r_\nu > r_{\nu-1} > \cdots > r_2 > r_1 > r_0 = 0$ are of known values with $\nu \in \mathbb{N}$. Moreover, $x(\tau) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the control input, $w(\cdot) \in \mathbb{L}^2(\mathbb{R}_{\geq t_0}; \mathbb{R}^q)$ represents an exogenous disturbance, and $z(\cdot) \in \mathbb{R}^m$ is the regulated output. The dimensions in (1) are determined by $n \in \mathbb{N}$ and $m; p; q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Finally, the DDs in (1) satisfy
\[ \tilde{A}_i(\cdot) \in \mathbb{L}^2(\mathcal{I}_i; \mathbb{R}^{n_i \times n_i}), \quad \tilde{C}_i(\cdot) \in \mathbb{L}^2(\mathcal{I}_i; \mathbb{R}^{m_i \times n_i}), \quad \tilde{B}_i(\cdot) \in \mathbb{L}^2(\mathcal{I}_i; \mathbb{R}^{n_i \times p}) \]
for all $i \in \mathbb{N}_0 : \{1, \ldots, \nu\}$ with $\mathcal{I}_i = [-r_i, -r_{i-1}]$.

Remark 1. The generality of the TDS in (1) is obvious. In General, a general LTDS in the form of Lebesgue-Stieltjes integrals [19] can be denoted by (1).

Remark 2. Many practical systems can be modeled by (1). For examples, we have the SIR model in [8, eq.(7)] or the networked control system in [15], or the chemical reaction networks in [25, eq.(30)], etc.

B. The Notion of Equivalent Decompositions of DDs

Since the DD matrices in (2) are infinite-dimensional, including them in a synthesis (stability) condition will lead to infinite-dimensional optimization constraints. To circumvent this issue, we introduce the notion of equivalent decompositions in this paper, which can parameterized any DD in (2) with matrices of finite dimensions.

Proposition 1. The conditions in (2) are true if and only if there exist $f_i(\cdot) \in \mathcal{C}(\mathcal{I}_i; \mathbb{R}^{d_i})$, $\varphi_i(\cdot) \in \mathbb{L}^2(\mathcal{I}_i; \mathbb{R}^{d_i})$, $\phi_i(\cdot) \in \mathbb{L}^2(\mathcal{I}_i; \mathbb{R}^{m_i})$ and $M_i \in \mathbb{R}^{n_i \times n_i}$, $A_i \in \mathbb{R}^{n_i \times n_i}$, $\tilde{B}_i \in \mathbb{R}^{n_i \times p}$, $\tilde{C}_i \in \mathbb{R}^{m_i \times n_i}$, $\tilde{B}_i \in \mathbb{R}^{m_i \times p}$ such that
\[ A_i(\tau) = \tilde{A}_i(\varphi_i(\tau) \otimes I_n), \quad B_i(\tau) = \tilde{B}_i(\varphi_i(\tau) \otimes I_p), \]
\[ C_i(\tau) = \tilde{C}_i(\varphi_i(\tau) \otimes I_n), \quad \tilde{B}_i(\tau) = \tilde{B}_i(\varphi_i(\tau) \otimes I_p), \]
\[ d \frac{df_i(\tau)}{d\tau} = M_i f_i(\tau), \quad \dot{f}_i(\tau) = \begin{bmatrix} \varphi_i(\tau) \\ \tilde{f}_i(\tau) \end{bmatrix}, \]
\[ G_i := \int_{\mathcal{I}_i} \varphi_i(\tau) \, d\tau \succ 0, \quad g_i(\tau) = \begin{bmatrix} \phi_i(\tau) \\ \tilde{f}_i(\tau) \end{bmatrix}. \]
hold for all \( i \in \mathbb{N}_p \) and \( \tau \in \mathcal{I}_i \), where \( \kappa_i = d_i + \delta_i + \mu_i \), 
\( \nu_i = d_i + \delta_i + \mu_i \) with \( d_i \in \mathbb{N} \) and \( \delta_i; \mu_i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).
Moreover, the derivatives in (5) at the boundaries of \( \mathcal{I}_i \) are one-sided. Finally, the conclusion of this proposition is always true for the case of \( \mu_i = 0 \).

**Proof:** See Appendix A.

**Remark 3.** The matrix inequalities in (6) indicate that the functions at each row of \( g_i(\cdot) \) are linearly independent \([20]\) in a Lebesgue sense over \( \mathcal{I}_i \) for all \( i \in \mathbb{N}_p \), where \( G_i \) in (6) are the Gramian matrices of \( g_i(\cdot) \).

**Remark 4.** Proposition 1 suffers no conservatism. From Appendix A, we know that any \( f_i(\cdot) \in C^1(\mathcal{I}_i; \mathbb{R}^d) \) can be used for the conditions in (5) even if no functions in \( f_i(\cdot) \) are included by the DDs in (2). This is because one can add an unlimited number of new functions to \( f_i(\cdot) \) or \( \varphi_i(\cdot) \) and (5) can still be satisfied by some \( M_i \). As long as all the kernels in (2) are “covered” by some functions in \( g_i(\cdot) \), then the constant matrices in Proposition 1 can always be constructed accordingly.

### III. Dissipative State Feedback Controller (DSFC) Design

#### A. Problem Formulation

Let \( \chi(t, \theta) = [x(t + r_i \theta - r_{i-1})]_{i=1}^\nu \in \mathbb{R}^{nu} \) with \( \theta \in [-1, 0] \). Now employ a static static controller \( u(t) = Kx(t), K \in \mathbb{R}^{p \times n} \) to (1) with Proposition 1, then the closed-loop system (CLS) is denoted by

\[\begin{align*}
\dot{x}(t) = (A_0 + B_0 K) x(t) + \left[\begin{array}{c}(A_i + B_i K) \\
\vdots
\end{array}\right] \chi(t, -1) + D_1 w(t) \\
\quad + \sum_{i=1}^\nu \int_{\mathcal{I}_i} \left( \left( A_i + B_i (I_{\kappa_i} \otimes K) \right) (g_i(\tau) \otimes I_n) \right) (x(t + \tau)d\tau) \\
\dot{z}(t) = (C_0 + \mathcal{B}_0 K) x(t) + \left[\begin{array}{c}(C_i + \mathcal{B}_i K) \\
\vdots
\end{array}\right] \chi(t, -1) + D_2 w(t) \\
\quad + \sum_{i=1}^\nu \int_{\mathcal{I}_i} \left( \left( C_i + \mathcal{B}_i (I_{\kappa_i} \otimes K) \right) (g_i(\tau) \otimes I_n) \right) (x(t + \tau)d\tau)
\end{align*}\]

\(\forall \theta \in \mathcal{R}_-, \quad x(t_0 + \theta) = \psi(\tau)\)  \hspace{1cm} (7)

where the decompositions of DDs are attained via

\[\forall i \in \mathbb{N}_p, \quad (g_i(\tau) \otimes I_p) K = (g_i(\tau) \otimes I_p) (1 \otimes K) = I_{\kappa_i} g_i(\tau) \otimes K I_n = (I_{\kappa_i} \otimes K) (g_i(\tau) \otimes I_n).\]  \hspace{1cm} (8)

The functions \( \varphi_i(\cdot) \) and \( \phi_i(\cdot) \) are separated in \( g_i(\tau) \) because they will receive different mathematical treatment. Specifically, \( \phi_i(\tau) \) is always approximated by \( \hat{f}_i(\tau) \) via

\[\phi_i(\tau) = \Gamma_i f_i^{-1} \hat{f}_i(\tau) + \varepsilon_i(\tau), \quad \tau \in [\tau_{i-1}, -\tau_{i-1}]\] \hspace{1cm} (9)

with \( F_i = \int_{\mathcal{I}_i} \hat{f}_i(\tau) \hat{f}_i^T(\tau)d\tau > 0 \) and \( \mathbb{R}^{\mu_i \times \kappa_i} \ni \Gamma_i := \int_{\mathcal{I}_i} \phi_i(\tau) \hat{f}_i^T(\tau)d\tau \)  \hspace{1cm} (10)

for all \( i \in \mathbb{N}_p \) based on the application of least-square approximation. (See [29, page 182] for the expression of the approximation). Note that \( \varepsilon_i(\tau) = \phi_i(\tau) - \Gamma_i f_i^{-1} \hat{f}_i(\tau) \) defines the error, and \( F_i > 0 \) always holds because of (6). Moreover, we utilize \( \mathbb{S}^{\mu_i} \ni E_i := \int_{\mathcal{I}_i} \varepsilon_i(\tau)\varepsilon_i^T(\tau)d\tau = \int_{\mathcal{I}_i} \left( \phi_i(\tau) - \Gamma_i f_i^{-1} \hat{f}_i(\tau) \right) \left( \phi_i(\tau) - \Gamma_i f_i^{-1} \hat{f}_i(\tau) \right)^T d\tau \)

\[= \int_{\mathcal{I}_i} \phi_i(\tau) \phi_i^T(\tau)d\tau - \mathbb{S} Y \left( \int_{\mathcal{I}_i} \phi_i(\tau) \hat{f}_i^T(\tau)d\tau \right) F_i^{-1} \Gamma_i^T + \Gamma_i F_i^{-1} \int_{\mathcal{I}_i} \hat{f}_i(\tau) \hat{f}_i^T(\tau)d\tau F_i^{-1} \Gamma_i^T \]

\[= \int_{\mathcal{I}_i} \phi_i(\tau) \phi_i^T(\tau)d\tau - \Gamma_i F_i^{-1} \Gamma_i^T \] \hspace{1cm} (11)

to measure the approximation error, where \( E_i > 0 \) always holds due to [9, eq.(18)].

**Remark 5.** The equations in (9)–(11) are well defined with \( \mu_i = 0 \), \( \phi_i(\tau) = \left[0_{0 \times 1}\right] \) which corresponds to the case that no functions are approximated in \( g_i(\cdot) \). Such a case is always usable with Proposition 1 since an unlimited number of linearly independent \( \mathbb{L}^2 \) functions can be added to \( \varphi_i(\cdot) \). This shows the advantage of using empty matrices, as the cases of \( \phi_i(\tau) = \left[0_{0 \times 1}\right] \) and \( \phi_i(\tau) \neq \left[0_{0 \times 1}\right] \) can be treated with a unified framework. In conclusion, Proposition 1 allows users to decide which \( \mathbb{L}^2 \) functions in (2) are approximated (the ones in \( \phi_i(\cdot) \)) by \( \hat{f}_i(\cdot) \) and which are factorized directly (the ones in \( \varphi_i(\cdot) \)).

**Remark 6.** Let \( \delta_i = 0 \) in Proposition 1, then (9)–(12) with \( \nu = 1 \) becomes identical to the approximation scheme in [9]. However, the absence of \( \varphi_i(\cdot) \) severely limits the generality of the approximator \( f_i(\cdot) \). This is because \( \frac{df_i(\tau)}{d\tau} = M_i f_i(\tau) \) cannot be satisfied by all differentiable functions for some \( M \in \mathbb{R}^{d_i \times d_i} \). As a result, Proposition 1 is significantly more general than the approach in [9].

Now by (5) and the approximation in (9), we have

\[\begin{align*}
g_i(\tau) &= \left[\begin{array}{c} \phi_i(\tau) \\
\hat{f}_i(\tau) \end{array}\right] = \left[\begin{array}{c} \Gamma_i f_i^{-1} \hat{f}_i(\tau) \\
\hat{f}_i(\tau) \end{array}\right] + \left[\begin{array}{c} \varepsilon_i(\tau) \\
0_{\kappa_i} \end{array}\right] \\
\hat{\Gamma}_i &= \left[\begin{array}{c} \Gamma_i f_i^{-1} \\
I_{\kappa_i} \end{array}\right] \in \mathbb{R}^{\kappa_i \times \kappa_i}, \quad \hat{\Gamma}_i = \left[\begin{array}{c} I_{\mu_i} \\
0_{\kappa_i \times \mu_i} \end{array}\right] \in \mathbb{R}^{\kappa_i \times \mu_i}, \quad \hat{\Gamma}_i \in \mathbb{R}^{\kappa_i \times \kappa_i}
\end{align*}\]

which further gives the identity

\[\forall i \in \mathbb{N}_p, \quad (I_{\kappa_i} \otimes K) (g_i(\tau) \otimes I_n)\]
\[ g_i(\tau) I_n = \left( \hat{F}_i(\tau) + \bar{I}_i \varepsilon_i(\tau) \right) \otimes I_n \]

\[
\begin{aligned}
&= \left( I_{\kappa_i} \otimes K \right) \left( \hat{F}_i + \bar{I}_i \varepsilon_i(\tau) \right) \otimes I_n \\
&= \left( I_{\kappa_i} \otimes K \right) \left( \bar{I}_i \varepsilon_i(\tau) \right) \otimes I_n \\
&= \hat{A}_i \bar{I}_i \varepsilon_i(\tau) \otimes I_n + \bar{H}_i \left( \hat{F}_i(\tau) \otimes I_n \right) E_i(\tau) \\
&= \hat{A}_i \bar{I}_i \varepsilon_i(\tau) \otimes I_n + \bar{H}_i \left( \hat{F}_i(\tau) \otimes I_n \right) E_i(\tau)
\end{aligned}
\]

Now using (17)–(18) to (7) produces

\[ x(t) = (A + B_1 \left[ (I_{\beta} \otimes K) \otimes O_q \right] \theta(t), \]

\[ z(t) = (C + B_2 \left[ (I_{\omega} \otimes K) \otimes O_q \right] \varphi(t), \quad \forall t \geq t_0 \]

with \( t_0 \) and \( \varphi(\cdot) \) in \( (1) \), where \( \beta = 1 + \nu + \kappa \) with \( \kappa = \sum_{i=1}^p \kappa_i \) and \( \kappa_i = \delta_i + \delta_i + \mu_i + \mu_\ell \).

\[ A = \left[ \left[ A_{i} \right]_{i=0}^\nu \right]^{\nu} \\
B_1 = \left[ \left[ B_{i} \right]_{i=0}^\nu \right]^{\nu} \\
C = \left[ \left[ C_{i} \right]_{i=0}^\nu \right]^{\nu} \\
B_2 = \left[ \left[ B_{i} \right]_{i=0}^\nu \right]^{\nu}
\]

B. Main Results on the Dissipative Controller Design

To verify the stability of the CLS in \( (19) \), a KF based stability criterion is presented as follows.

\textbf{Lemma 1.} Let \( w(t) \equiv 0_q \) in \( (19) \) and all delay values be given, then the trivial solution \( x(t) \equiv 0_n \) of \( (19) \) is uniformly asymptotically (exponentially) stable with any \( \psi(\cdot) \in C(\mathbb{R}; \mathbb{R}^n) \) if there exist \( \epsilon_1; \epsilon_2; \epsilon_3 > 0 \) and a differentiable functional \( v : C(\mathbb{R}; \mathbb{R}^n) \to \mathbb{R} \) with \( v(0_n(\cdot)) = 0 \) such that

\[ \epsilon_1 \left\| \psi(0) \right\|^2_2 \leq v(\psi(\cdot)) \leq \epsilon_2 \left\| \psi(\cdot) \right\|^2_\infty \]

\[ \forall t \geq t_0, \quad \frac{d}{dt} v(x_t(\cdot)) \leq -\epsilon_3 \left\| x(t) \right\|^2_2 \]

for any \( \psi(\cdot) \in C(\mathbb{R}; \mathbb{R}^n) \) in \( (19) \), where \( \left\| \psi(\cdot) \right\|^2_\infty := \sup_{-\tau_i \leq \xi \leq 0} \left\| \psi(\cdot) \right\|^2_2 \). The following definition of dissipativity is based on the original framework outlined in [42].

\textbf{Definition 1.} The system in \( (19) \) with \( s(z(t), w(t)) \) is said to be dissipative if there exists a differentiable functional \( v : C(\mathbb{R}; \mathbb{R}^n) \to \mathbb{R} \) such that

\[ \forall t \geq t_0, \quad \frac{d}{dt} v(x_t(\cdot)) - s(z(t), w(t)) \leq 0 \]
with \( t_0 \in \mathbb{R}, z(t) \) and \( w(t) \) in (19). Moreover, \( x_i(\cdot) \) in (27) is defined by the equality \( \forall t \geq t_0, \forall \theta \in \mathbb{R}, x_i(\theta) = x(t+\theta) \) with \( x(t) \) satisfying (19).

Note that (27) implies the original definition of dissipativity by taking Lebesgue integrations at both side. In this paper, the supply function is denoted as

\[
s(z(t), w(t)) = \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} J^T \end{bmatrix} J^{-1} \tilde{J} J_2 \begin{bmatrix} z(t) \\ w(t) \end{bmatrix},
\]

\( J_1, J_2 \in \mathbb{R}^{m \times m}, J_3 \in \mathbb{S}^{n} \)

whose structure is based on the quadratic constraints in [36]. The structure in (28) features numerous performance criteria such as

- \( L^2 \) gain performance: \( J_1 = -\gamma I_m, \tilde{J} = I_m, J_2 = O_{m,q}, J_3 = \gamma I_q \) with \( \gamma > 0 \)
- Passivity: \( J_1 < 0, \tilde{J} = O_m, J_2 = I_m, J_3 = O_m \)

Next, the main results on DSFC are presented in Theorem 1–2 and Algorithm 1, where Theorem 2 is proposed as a convexification of Theorem 1 which can be further solved by Algorithm 1.

**Theorem 1.** Let all the parameters in Proposition 1 be given, then the CLS in (19) with the supply rate function in (28) is dissipative, and the trivial solution of (19) with \( w(t) = 0_q \) is uniformly asymptotically (exponentially) stable if there exist a controller gain \( K \in \mathbb{R}^{p \times n} \) and matrix parameters \( P_1 \in \mathbb{S}^n, P_2 \in \mathbb{R}^{n \times q}, P_3 \in \mathbb{S}^q \) with \( q = nd, d = \sum \nu_i d_i \) and \( Q_i; R_i \in \mathbb{S}^q, i \in \mathbb{N}_\nu \) such that

\[
\begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} + \begin{bmatrix} O_n & \bigoplus_{i=1}^{\nu} I_d_i \otimes Q_i \end{bmatrix} > 0, \quad (29)
\]

\[
Q = \bigoplus_{i=1}^{\nu} Q_i > 0, \quad R = \bigoplus_{i=1}^{\nu} R_i > 0, \quad (30)
\]

\[
\begin{bmatrix} \Psi & \Sigma^T J^T \\ * & J_1 \end{bmatrix} = \begin{bmatrix} \Phi \otimes J_n \otimes O_{p,(n\mu+q+m)} \\ \Omega \otimes J_n \otimes O_{p,(n\mu+q+m)} \end{bmatrix} + \Xi \otimes (-J_1) > 0, \quad (31)
\]

where \( \Sigma = C + B_2 \left[ (I_\beta \otimes K) \otimes Q_q \right] \) and \( C, B_2 \) in (22)–(23)

\[
\Psi = \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} + \begin{bmatrix} O_n & \bigoplus_{i=1}^{\nu} I_d_i \otimes Q_i \end{bmatrix} > 0, \quad (29)
\]

\[
Q = \bigoplus_{i=1}^{\nu} Q_i > 0, \quad R = \bigoplus_{i=1}^{\nu} R_i > 0, \quad (30)
\]

\[
\begin{bmatrix} \Psi & \Sigma^T J^T \\ * & J_1 \end{bmatrix} = \begin{bmatrix} \Phi \otimes J_n \otimes O_{p,(n\mu+q+m)} \\ \Omega \otimes J_n \otimes O_{p,(n\mu+q+m)} \end{bmatrix} + \Xi \otimes (-J_1) > 0. \quad (31)
\]

Finally, the number of unknown variables is \((0.5d^2 + 0.5d + \nu + 0.5)n^2 + (0.5d + 0.5 + \nu + p)n \in \mathcal{O}(d^2n^2)\).

**Proof:** The proof of Theorem 1 is based on the construction of the complete type Krasovskii functional

\[
\nu(x_i(\cdot)) = \eta^T(t) \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} \eta(t)
\]

\[
+ \sum_{i=1}^{\nu} \int_{\mathcal{I}_i} x^T (t+\tau) Q_i x(t+\tau) d\tau
\]

\[
+ \sum_{i=1}^{\nu} \int_{\mathcal{I}_i} x^T (t+\tau) R_i x(t+\tau) d\tau, \quad (40)
\]

where \( x_i(\cdot) \) follows the same definition in (27), and \( P_1 \in \mathbb{S}^n, P_2 \in \mathbb{R}^{n \times q}, P_3 \in \mathbb{S}^q \) and \( Q_i; R_i \in \mathbb{S}^q, i \in \mathbb{N}_\nu \) and

\[
\eta(t) := \text{Col} \begin{bmatrix} x(t), \xi(t) \end{bmatrix} \quad (41)
\]

with \( F_i = \int_{\mathcal{I}_i} f_i(\tau) f_i^T(\tau) d\tau, \forall i \in \mathbb{N}_\nu \). Note that \( \sqrt{F_i^{-1}} \) are well defined and unique because of (6).

From \( \chi(t, \tau) = [x(t+\tau) - \dot{x}(t+\tau)]_i \), we have

\[
\sum_{i=1}^{\nu} \frac{d}{dt} \int_{\mathcal{I}_i} x^T (t+\tau) Q_i x(t+\tau) d\tau
\]
\[
= \sum_{i=1}^{\nu} x^T(t - r_{i-1})Q_i x(t - r_{i-1}) - \sum_{i=1}^{\nu} x^T(t - r_{i})Q_i x(t - r_{i})
\]
\[
= \chi^T(t, 0) Q \chi(t, 0) - \chi^T(t, -1) Q \chi(t, -1),
\]  
(42)
\[
\sum_{i=1}^{\nu} \frac{d}{dt} \int_{I_i} (\tau + r_{i}) x^T(t + \tau) R_i x(t + \tau) d\tau
\]
\[
= \sum_{i=1}^{\nu} \hat{r}_i x^T(t - r_{i-1}) R_i x(t - r_{i-1})
\]
\[
- \sum_{i=1}^{\nu} \int_{I_i} x^T(t + \tau) R_i x(t + \tau) d\tau = \chi^T(t, 0) R A \chi(t, 0) - \sum_{i=1}^{\nu} \int_{I_i} [\hat{r}_i x(t + \tau)] d\tau
\]  
(43)
where \( Q = \bigoplus_{i=1}^{\nu} Q_i \), and \( R = \bigoplus_{i=1}^{\nu} R_i \) with \( \Lambda \) in (36).

Given the relations in (41)–(43), differentiating (weak derivative) \( v(x(t)) \) along the trajectory of (19) and considering \( s(z(t), w(t)) \) in (28) produces

\[
\dot{v}(t) \leq \sum_{i=1}^{\nu} \left[ S^T \left[ \begin{array}{c}
S & P_1 & P_2, \\
\ast & P_3, \\
O_{n, (n, m)} & \end{array} \right] \right] \theta(t) + \chi^T(t, 0) (Q + R A) \chi(t, 0)
\]
\[
- \chi^T(t, -1) Q \chi(t, -1) - \sum_{i=1}^{\nu} \int_{I_i} x^T(t + \tau) R_i x(t + \tau) d\tau
\]
\[
- w^T(t) J_3 w(t) - \dot{\theta}^T(t) \Sigma \bar{J}_T J_1^{-1} J_\Sigma \theta(t)
\]  
(44)
where \( \chi(t, 0) = [x(t - r_{i-1})]^T \mid_{i=1}^{\nu} \) and \( \chi(t, -1) = [x(t - r_{i})]^T \mid_{i=1}^{\nu} \) and \( \theta(t) \) is given in (24) and \( \Sigma, \bar{J} \) and \( \bar{F} \) are defined in the statements of Theorem 1. Note that \( \bar{J} \) and \( \bar{F} \) in (36)–(37) are obtained by the identities

\[
\left[ \int_{I_i} \left[ F_i^{-1} f_i(\tau) \otimes I_n \right] x(t + \tau) d\tau \right]_{i=1}^{\nu}
\]
\[
= \int_{-1}^{0} \left( \bigoplus_{i=1}^{\nu} \sqrt{F_i^{-1}} f_i(\hat{r}_i \tau - r_{i-1}) \otimes I_n \right) \chi(t, \tau) d\tau
\]
\[
= \left( \bigoplus_{i=1}^{\nu} \sqrt{F_i^{-1}} \otimes I_n \right) \int_{-1}^{0} \left( \bigoplus_{i=1}^{\nu} \sqrt{F_i^{-1}} f_i(\hat{r}_i \tau - r_{i-1}) \otimes I_n \right) \chi(t, \tau) d\tau
\]
\[
= \left( \bigoplus_{i=1}^{\nu} \sqrt{F_i^{-1}} \otimes I_n \right) \int_{-1}^{0} \left[ \bigoplus_{i=1}^{\nu} \left[ O_{d, x, \delta_i} \otimes I_d_i \right] \otimes I_n \right) \chi(t, \tau) d\tau
\]
\[
\times \int_{-1}^{0} \left( \bigoplus_{i=1}^{\nu} \sqrt{F_i^{-1}} f_i(\hat{r}_i \tau - r_{i-1}) \otimes I_n \right) \chi(t, \tau) d\tau
\]
\[
= \left( \bigoplus_{i=1}^{\nu} \sqrt{F_i^{-1}} \otimes I_n \right) \int_{-1}^{0} \left[ \bigoplus_{i=1}^{\nu} \left[ O_{d, x, \delta_i} \otimes I_d_i \right] \otimes I_n \right) \chi(t, \tau) d\tau
\]
\[
\dot{\chi}^T(t, 0) R A \chi(t, 0) - \sum_{i=1}^{\nu} \int_{I_i} [\hat{r}_i x(t + \tau)] d\tau
\]  
(45)
Note that also the parameters \( A, B_1, C \) and \( B_2 \) in (44) are given in (20)–(23).

Assume (30) is true, apply (96) with \( \varpi(\tau) = 1 \) and \( g_i(\tau) = \phi_i(\tau) \), \( f_i(\tau) = \hat{f}_i(\tau), i \in \mathbb{N} \), to the integral terms \( \sum_{i=1}^{\nu} I_{x_i}(t + \tau) R_i x(t + \tau) d\tau \) in (44), then

\[
\sum_{i=1}^{\nu} \int_{I_i} x^T(t + \tau) R_i x(t + \tau) d\tau \geq \sum_{i=1}^{\nu} \int_{-1}^{0} \left( \bigoplus_{i=1}^{\nu} \sqrt{F_i^{-1}} f_i(\hat{r}_i \tau - r_{i-1}) \otimes I_n \right) \chi(t, \tau) d\tau
\]
\[
+ \sum_{i=1}^{\nu} \left[ \bigoplus_{i=1}^{\nu} \left[ O_{d, x, \delta_i} \otimes I_d_i \right] \otimes I_n \right) \chi(t, \tau) d\tau
\]
\[
= \xi^T(t) \left( \bigoplus_{i=1}^{\nu} \left[ O_{x, R_i} \otimes I_n \right) \xi(t) + \chi^T(t) \left( \bigoplus_{i=1}^{\nu} \left[ O_{d, x, \delta_i} \otimes I_d_i \right] \otimes I_n \right) \chi(t) \right).
\]  
(48)
Moreover, considering the structure of 
\[ \Psi - \Sigma^T \bar{J}^T J^{-1}_I \Sigma \prec 0 \] with \( \vartheta(t) \) in (24), (50) infers
\[ \exists \epsilon_3 > 0, \quad \frac{d}{dt} v(x_i(\cdot)) \bigg|_{t=t_0, x_i(\cdot) = \varphi(\cdot)} \leq -\epsilon_3 \| \varphi(0) \|_2 \] (51)
for any \( \varphi(\cdot) \in \mathbb{C}(\mathbb{R}; \mathbb{R}^n) \) in (19) with \( w(t) \equiv 0_q \). Note that \( x_i(\cdot) \) in (51) is defined in (26). As a result, if (30) and 
\[ \Psi - \Sigma^T \bar{J}^T J^{-1}_I \Sigma \prec 0 \] are feasible, then \( v(x_i(\cdot)) \) in (40) satisfies (26)–(27). Finally, applying the Schur complement to \( \Psi - \Sigma^T \bar{J}^T J^{-1}_I \Sigma \prec 0 \) with (30) and \( J^{-1}_I \prec 0 \) yields (31). Hence, we have proved that if (30)–(31) are feasible, then there exists \( \epsilon_3 > 0 \) such that \( v(x_i(\cdot)) \) in (40) satisfies (26)–(27).

Now we start to show that there exist \( \epsilon_1; \epsilon_2 > 0 \) such that \( v(x_i(\cdot)) \) in (40) satisfies (25) if (29)–(30) are feasible. Consider \( v(x_i(\cdot)) \) in (40) with \( t = t_0 \), it follows that
\[ \exists \lambda > 0, v(x_{i_0}(\cdot)) = v(\varphi(\cdot)) = \lambda \| \varphi(0) \|_2^2 \] (52)
for any \( \varphi(\cdot) \in \mathbb{C}(\mathbb{R}; \mathbb{R}^n) \) in (19), where (52) is derived via the property of \( \forall X \in \mathbb{S}^n, \exists \lambda > 0 : \forall x \in \mathbb{R}^n \setminus \{0\}, x^T (\lambda I_n - X) x > 0 \) and (96) with \( \omega(\tau) = 1 \) and \( f_i(\tau) = \sqrt{F_i^{-1}} f_i(\tau) \). Consequently, (52) shows that there exists \( \epsilon_2 > 0 \) such that \( v(x_i(\cdot)) \) in (40) satisfies (25).

Now we show that if (29)–(30) are feasible, then \( v(x_i(\cdot)) \) in (40) satisfies (25) with some \( \epsilon_1; \epsilon_2 > 0 \). Applying (96) to (40) with \( \omega(\tau) = 1 \), \( g_i(\cdot) = \| v_{0 \times 1} \) and \( f_i(\tau) = \sqrt{F_i^{-1}} f_i(\tau) \) produces
\[ \sum_{i=1}^{\nu} \int_{I_i} x^T (t + \tau) Q_i x(t + \tau) d\tau \geq \left( \sum_{i=1}^{\nu} \left( I_{i_0} \otimes Q_i \right) \int_{I_i} \left( \sqrt{F_i^{-1}} f_i(\tau) \otimes I_n \right) x(t + \tau) d\tau \right) \]
\[ = \left( \sum_{i=1}^{\nu} \left( I_{i_0} \otimes Q_i \right) \int_{I_i} \left( \sqrt{F_i^{-1}} f_i(\tau) \otimes I_n \right) x(t + \tau) d\tau \right) = \left( \sum_{i=1}^{\nu} \int_{I_i} \left( \sqrt{F_i^{-1}} f_i(\tau) \otimes I_n \right) x(t + \tau) d\tau \right) \]
provided that (30) holds. Moreover, by utilizing (53) to (40) with (30) and (52), it is clear to see that the existence of the feasible solutions of (29)–(30) infer that \( v(x_i(\cdot)) \) in (40) satisfies (25) with some \( \epsilon_1; \epsilon_2 > 0 \).

In conclusion, we have shown that feasible solutions of (29)–(31) infers the existence of the Krasovskii functional \( v(x_i(\cdot)) \) in (40) and \( \epsilon_1; \epsilon_2 > 0 \) satisfying the dissipative condition in (27) and the stability criteria in (25)–(26). As a result, the trivial solution of (19) with \( w(t) \equiv 0_q \) is uniformly asymptotically (exponentially) stable, and (19) with (28) is dissipative if (29)–(31) are true.

Remark 7. Because (19) is identical to the original system in (7), Theorem 1 is valid for the CLS in (7), not an approximated system. In fact, (31) would be infeasible if the eigenvalues of the approximation errors \( E_i \) are too large. As a result, a numerical solution of the synthesis condition in Theorem 1 always guarantees dissipative stabilization for the CLS in (7). This is fundamentally different from the methods in [6, 14, 21], whose numerical results do not guarantee the stability of the CLS.

Remark 8. The functional in (41) is a realization of the complete Krasovskii functional [23] defined for \( x(t) = \sum_{i=0}^{\nu} A_i x(t - r_i) \) via the "basis" functions \( f_i(\tau) \). As we have pointed out in Remark 4, \( f_i(\tau) \) in (3)–(4) and (41) can contain any number of DALIFs even if they are not included by any DD in (2). Hence the generality of the functional in (40) is substantially greater than the ones in [37, 11, 9] even for the case of \( \nu = 1 \). Meanwhile, we can select \( g_i(\tau) \) considering the functions inside of the DDs in (2). Since \( \varphi_i(\cdot) \) \( \varphi_i(\cdot) \) are not included by (40), hence the number of unknowns in Theorem 1 is of \( O(d^2n^2) \), which depends on the dimensions of \( f_i(\cdot) \).

The inequality in (31) is bilinear if we want to compute \( K \), which cannot be solved numerically via standard SDP solvers. In Theorem 2, we convexify the BMI in (31) via the application of Projection Lemma [13], which produces a convex dissipative synthesis condition.

Lemma 2 (Projection Lemma). [13] Given \( n; p; q \in \mathbb{N}, \Pi \in \mathbb{R}^{n \times n} \), \( Q \in \mathbb{R}^{p \times n} \), there exists \( \Theta \in \mathbb{R}^{p \times q} \) such that the following two propositions are equivalent:
\[ \Pi + P^T \Theta^T Q + Q^T \Theta P < 0, \] (54)
\[ P^T \Pi P \perp < 0 \] and \( Q^T \Pi Q \perp < 0, \] (55)
where the columns of \( P \perp \) and \( Q \perp \) contain bases of null space

bThe equivalent \( f_i(\cdot) \) functions in [37, 11, 9] are the special case of the proposed \( f_i(\cdot) \) in this work.
of matrix $P$ and $Q$, respectively, which means that $PP_\perp = 0$ and $QQ_\perp = 0$.

Proof: Refer to [13] and [5].

**Theorem 2.** Given $\{\alpha_i\}_{i=1}^\beta \subset \mathbb{R}$ and the functions and parameters in Proposition 1, then the CLS in (19) with the supply rate function in (28) is dissipative and the trivial solution of (19) with $\omega(t) \equiv 0$ is uniformly asymptotically (exponentially) stable if there exist $\hat{P}_1, X \in \mathbb{S}^n$, $\hat{P}_2 \in \mathbb{R}^{n \times \ell}$, $\hat{P}_3 \in \mathbb{S}^\ell$ and $\hat{Q}_1, \hat{R}_i \in \mathbb{S}^\ell$, $\beta = nd$ and $V \in \mathbb{R}^{n \times n}$ such that

$$
\begin{bmatrix}
\hat{P}_1 & \hat{P}_2 \\
\ast & \hat{P}_3
\end{bmatrix}
+ \begin{bmatrix}
O_n & \begin{pmatrix}
\nu & I_{d_i} \otimes \hat{Q}_i
\end{pmatrix}
\end{bmatrix} > 0, \quad (56)
$$

$$
\hat{Q} = \bigoplus_{i=1}^\nu \hat{Q}_i > 0, \quad \hat{R} = \bigoplus_{i=1}^\nu \hat{R}_i > 0 \quad (57)
$$

$$
\text{Sy} \left( \begin{bmatrix}
\hat{P}_1 & O_{n,\nu,\ell} & \hat{P}_2 \hat{I} & O_{n, (n\ell + q + m)}
\end{bmatrix} \begin{bmatrix}
-I_n & \Pi & \hat{I}
\end{bmatrix} \right) + \begin{bmatrix}
O_n & \hat{P} & \ast
\end{bmatrix} < 0 \quad (58)
$$

where $\hat{P} = \begin{bmatrix}
\hat{P}_1 & O_{n,\nu,\ell} & \hat{P}_2 \hat{I} & O_{n, (n\ell + q + m)}
\end{bmatrix}$ and $\hat{I} = \begin{bmatrix}
A & [I_\beta \otimes X] \oplus I_q & B_1 & \big( [I_\beta \otimes V] \oplus O_q \big) & O_{n,m}
\end{bmatrix}$

with $\hat{I}$ in (36) and matrices

$$
\hat{\Phi} = \text{Sy} \left( \begin{bmatrix}
\hat{P}_2 \\
O_{n,\ell,\nu} \hat{I}^\top \hat{P}_3 \\
O_{(n\ell + q + m),\ell}
\end{bmatrix} \begin{bmatrix}
\hat{F} \otimes I_n & O_{\ell, (n\ell + q + m)}
\end{bmatrix} \right)
+ \begin{bmatrix}
O_{(n\beta,\ell)} \\
-1 & \Sigma & 0 & m
\end{bmatrix}
+ \begin{bmatrix}
O_n & \hat{Q} & \bigoplus_{i=1}^\nu I_{x_i} \otimes \hat{R}_i
\end{bmatrix}
+ \begin{bmatrix}
O_n & \hat{Q} & \bigoplus_{i=1}^\nu I_{\mu_i} \otimes \hat{R}_i & J_3 & 0
\end{bmatrix}
+ \begin{bmatrix}
\hat{Q} & \Lambda \hat{R} & O_n & O_{n\kappa} & O_{q+m}
\end{bmatrix} \quad (59)
$$

with matrices $\hat{F}$ in (37) and $\Sigma = C \left( [I_\beta \otimes X] \oplus I_q \right) + B_2 \left( [I_\beta \otimes V] \oplus O_q \right)$ and the parameters $A, B_1, B_2, C$ in (20)–(23). The controller gain is calculated via $K = V X^{-1}$. Finally, the number of unknowns in this theorem is $(0.5d^2 + 0.5d + \nu + 1)n^2 + (0.5d + 1 + \nu + p)n \in \mathbb{O}(d^2n^2)$.

Proof: First of all, note that the inequality $\text{Sy} \left( P^\top \Pi \right) + \Phi \prec 0$ in (31) can be rewritten as

$$
\text{Sy} \left( P^\top \Pi \right) + \Phi = \begin{bmatrix}
O_n & P & \Pi & I_{n\beta + q + m}
\end{bmatrix} \prec 0. \quad (60)
$$

It is easy to see that the structure of (60) is similar to one of the inequalities in (55). Given that two matrix inequalities are presented in (55), thus a new matrix inequality must be constructed to utilize Lemma 2. Now by considering the structure of $\Phi$, we have

$$
\begin{bmatrix}
O_n & P & \Pi & I_{n\beta + q + m}
\end{bmatrix} \prec 0. \quad (61)
$$

where $\text{Sy} \left( \begin{bmatrix}
O_{n(q+m),(n+n\beta)} & I_{q+m}
\end{bmatrix} \right)$. Since the matrix in (61) corresponds to the $2 \times 2$ block matrix at the bottom-right corner of the matrices $\text{Sy} \left( P^\top \Pi \right) + \Phi$ or $\Phi$, hence the inequality in (61) is implied by (60) or (31). On the other hand, the following identities

$$
\begin{bmatrix}
-I_n & \Pi
\end{bmatrix} = \begin{bmatrix}
O_{n, (n\beta + q + m)}
\end{bmatrix}, \quad (62)
$$

which satisfy $\text{rank} \left( \begin{bmatrix}
I_n & \Pi
\end{bmatrix} \right) = n + \beta$ and $\text{rank} \left( \begin{bmatrix}-I_n & \Pi
\end{bmatrix} \right) = n$ imply that $\begin{bmatrix}-I_n & \Pi
\end{bmatrix}$ and $\begin{bmatrix}I_n & \Pi
\end{bmatrix}$ can be utilized by Lemma 2 given the rank nullity theorem.

Applying Lemma 2 to (60)–(62) yields the conclusion that (60)–(61) are true if and only if $\exists W \in \mathbb{R}^{(n + \beta n) \times n}$

$$
\text{Sy} \left( \begin{bmatrix}I_{n\beta,\ell} \\Omega_{(q+m),(n+\beta n)}
\end{bmatrix} W \left[ \begin{bmatrix}-I_n & \Pi
\end{bmatrix} \right] + \begin{bmatrix}O_n & P \\ast & \Phi
\end{bmatrix} \right) < 0. \quad (63)
$$

Now (63) is still bilinear due to the product between $W$ and $\Pi$. To convexify (63), let

$$
W = \begin{bmatrix}W & \text{Col}_i\alpha_i W
\end{bmatrix} \quad (64)
$$

with $W \in \mathbb{S}^n$ and $\{\alpha_i\}_{i=1}^\beta \subset \mathbb{R}$. With (64), the inequality in (63) becomes

$$
\Theta = \text{Sy} \left( \begin{bmatrix}W & \text{Col}_i\alpha_i W
\end{bmatrix} \left[ \begin{bmatrix}-I_n & \Pi
\end{bmatrix} \right] + \begin{bmatrix}O_n & P \\ast & \Phi
\end{bmatrix} \right) \prec 0. \quad (65)
$$

which infers (60). Note that (65) is only a sufficient condition implying (60) or (31) due to the structural constraints in (64). Note that also the invertibility of $W \in \mathbb{S}^n$ is guaranteed by (65) since $-2W$ is the only element at the first diagonal block of $\Theta$. 


Let $X^T = W^{-1}$, apply congruent transformations to the matrix inequalities in (29)–(30) and (65). Then

$$(I_\nu \otimes X) Q (I_\nu \otimes X) > 0, (I_\nu \otimes X) R (I_\nu \otimes X) > 0,$$

$$(I_{1+\beta} \otimes X^T) \oplus I_{q+m} \Theta [(I_{1+\beta} \otimes X) \oplus I_{q+m}] < 0,$$

$$(I_{1+d} \otimes X^T) \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} (I_{1+d} \otimes X) > 0$$

(66)

hold if and only if (29)–(30) and (65) hold. Moreover, with (91) and the definitions

$$\hat{P}_i = \left[ \begin{array}{cc} P_1 & P_2 \\ * & P_3 \end{array} \right] (I_{1+d} \otimes X),$$

$$\hat{Q} := \bigoplus_{i=1}^N XQ_i X, \quad \hat{R} := \bigoplus_{i=1}^N X R_i X$$

the inequalities in (66) can be rewritten as (56)–(57) and

$$\left( [I_{1+\beta} \otimes X^T] \oplus I_{q+m} \right) \Theta \left( [I_{1+\beta} \otimes X] \oplus I_{q+m} \right) =$$

$$\Theta = S \left( \begin{array}{cc} X & \Theta \end{array} \right) \left( \begin{array}{cc} O_n & \hat{P} \\ \hat{Q} & \Phi \end{array} \right) < 0$$

(68)

where $\hat{P} = XP \left( [I_{1+\beta} \otimes X] \oplus I_{q+m} \right) =

$$\left[ \begin{array}{cc} P_1 & O_n \nu \\ P_2 & O_{n,\nu} \end{array} \right]$$

(69)

and $\Theta = \Pi \left( [I_{1+\beta} \otimes X] \oplus I_{q+m} \right) = [A] \left( [I_{1+\beta} \otimes X] \oplus I_{q+m} \right) B_1 \left( [I_{1+\beta} \otimes X] \oplus I_{q+m} \right) O_{n,m}

(70)

with $V = KX$ and $\Phi$ in (59). Note that (68) is the same as (58), and the form of $\Phi$ in (59) is derived via the relations

$$\hat{I}(I_\nu \otimes X) = (I_\nu \otimes X) \hat{I}$$

(71)

and

$$\hat{F} \oplus I_n \left( [I_{1+\beta} \otimes X] \oplus I_{q+m} \right) =

$$

which are derived from (91) and (93). Furthermore, since $-2X$ is the only term at the first diagonal block of $\Theta$ in (58), thus $X$ is invertible if (58) holds. This is consistent with the invertibility of $W$ implied by (65).

As a result, we have shown the equivalence between (29)–(30) and (56)–(57). Meanwhile, it has been shown that (58) is equivalent to (65) which infers (31). Consequently, (29)–(31) are satisfied if (56)–(58) hold for some $W \in \mathbb{S}^n$ and $\{\alpha_i\}_{i=1}^\beta \subset \mathbb{R}$. This finishes the proof.

Theorem 2 is proposed to compute $K$. If one wants to analyze the stability of the open-loop system in (1) with $B_i = \ddot{B}_i(\tau) = O_{n,p}$ and $\mathcal{H}_i = \ddot{B}_i(\tau) = O_{m,p}$, $i \in \mathbb{N}_\nu$, then Theorem 1 should be applied which is convex in this case. Note that the slack variables in Theorem 2 do not make it more feasible compared to Theorem 1.

Remark 9. Though (64) can introduce conservatism, the structure in (56) remains identical to (29). As a result, the use of Lemma 2 at (63) does not degenerate the matrix parameters of the KF in (40), thereby creating less conservatism compared to simplifying $P_1$, $P_2$ to convexify (31) via congruent transformations.

Remark 10. For $\{\alpha_i\}_{i=1}^\beta \subset \mathbb{R}$, we can assume $\alpha_i = 0$ for $i = 2 \cdots \beta$ with an adjustable scalar $\alpha_1 \in \mathbb{R} \setminus \{0\}$. Note that $\alpha_1 \neq 0$ is necessary since $\alpha_1 = 0$ will make the $A_0$ related-diagonal-block in (58) infeasible.

C. An Inner Convex Approximation Solution of Theorem 1

The step at (64) can introduce conservatism. In this section, Algorithm 1 is proposed based on the idea outlined in [7] to solve the BMI in Theorem 1, which can be initiated by a feasible solution of Theorem 2. Thus the advantage of both Theorem 1 and 2 are combined together without requesting a nonlinear SDP solver.

First of all, we want to point out that the inequality in (31) is nonconvex in general whereas (29)–(30) remain convex even when a synthesis problem is considered. Now we reformulate the inequality in (31) as

$$\mathcal{U}(H, K) := S \left( \begin{array}{cc} P^T \Pi \end{array} \right) + \Phi$$

$$= S \left( \begin{array}{cc} \vec{P}^T \left[ A \quad O_{n,m} \right] + \vec{\Phi} \end{array} \right)$$

(72)

with $B := \left[ B_1 \quad O_{n,m} \right]$ and $\vec{\Phi} := \left[ \begin{array}{cc} \vec{P}^T \left[ A \quad O_{n,m} \right] + \vec{\Phi} \end{array} \right]$, where $\vec{P}$ is given in (38), and $A$ and $B_1$ are given in (20)–(21), and $H := \left[ P_1 \quad P_2 \right]$ with $P_1$ and $P_2$ in Theorem 1. It is important to stress that $\vec{\Phi}$ contains no non-convexities. Utilizing the results of Example 3 in [7], one can conclude that $\Delta \left( \cdot, \vec{G}, \cdot, \vec{N} \right)$, which is defined as

$$S(\alpha, \beta, \gamma, \delta) \supset \Delta \left( \vec{G}, \vec{G}, \vec{N}, \vec{N} \right) := \left[ \begin{array}{cc} [Z \oplus (I_n - Z)]^{-1} \left[ G - \vec{G} \right] \\
N - \vec{N} \end{array} \right]$$

(73)

with $Z \oplus (I_n - Z) \succ 0$ satisfying

$$\forall G, \vec{G} \in \mathbb{R}^{n \times \ell}, \forall N, \vec{N} \in \mathbb{R}^{n \times \ell}, \quad T + S \left( \vec{G}^T \vec{N} \right)$$

(74)

is a psd-convex overestimate of $\Delta(G, N) = T + S \left( \vec{G}^T \vec{N} \right)$ with respect to the parameterization

$$\text{Col} \left( \vec{G}, \vec{N} \right) = \text{Col} \left( \text{vec}(G), \text{vec}(N) \right).$$

(75)
Now let $T = \hat{\Phi}$, $G = P$ and $\bar{P}_1 \in \mathbb{S}^n$, $\bar{P}_2 \in \mathbb{R}^{n,d_n}$
\[ \tilde{G} = \bar{P}_1 \begin{bmatrix} O_{n,m} & \bar{P}_2 \end{bmatrix} \begin{bmatrix} O_{n,(\mu+q+m)} \end{bmatrix}, \]
\[ H = \begin{bmatrix} P_1 & P_2 \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} \bar{P}_1 & \bar{P}_2 \end{bmatrix}, \]
\[ N = B K, \quad K = (I_b \otimes K) \oplus O_{p+m}, \]
\[ \tilde{N} = B \tilde{K}, \quad \tilde{K} = \left( I_b \otimes \tilde{K} \right) \oplus O_{p+m} \]
in (73) with $\ell = n\beta + q + m$ and $Z \oplus (I_n - Z) \succ 0$ and $\hat{\Phi}$, $H$ and $K$ are defined in (72). Then one can obtain
\[ \mathcal{U}(H, K) = \hat{\Phi} + \text{Sy} \left[ P^T B \left( I_b \otimes K \right) \right] \]
\[ \leq \mathcal{S} \left( H, \tilde{H}, K, \tilde{K} \right) := \hat{\Phi} + \text{Sy} \left( P^T N + P^T \tilde{N} \right) \]
\[ + \left[ P^T - P^T N \otimes -N^T \right] [Z \oplus (I_n - Z)]^{-1} [\ast] \]
by (74), where $\mathcal{S} \left( \ast, \tilde{H}, \ast, \tilde{K} \right)$ is a psd-convex overestimate of $\mathcal{U}(H, K)$ in (72) with respect to the parameterization
\[ y = \text{Col} \left[ \text{vec}(H), \text{vec}(K) \right] = \text{Col} \left[ \text{vec}(H), \text{vec}(K) \right] = \tilde{y}. \]

From (77), it is obvious that $\mathcal{S} \left( H, \tilde{H}, K, \tilde{K} \right) \prec 0$ infers (72). Moreover, $\mathcal{S} \left( H, \tilde{H}, K, \tilde{K} \right) \prec 0$ holds if and only if
\[ \left[ \hat{\Phi} + \text{Sy} \left( P^T N + P^T \tilde{N} \right) \right] \]
\[ \left[ \begin{array}{cc} \ast & -Z \otimes O_n \\ -Z & \ast \end{array} \right] \]
\[ \left[ \begin{array}{cc} -Z & \ast \end{array} \right] \]
holds with $N, \tilde{N}$ in (76) based on the application of the Schur complement given $Z \oplus (I_n - Z) \succ 0$. As a result, (72) is inferred by (78) which can be computed by standard SDP solvers if $\hat{H}$ and $\tilde{K}$ are known.

By compiling all the aforementioned procedures according to the expositions in [7], Algorithm 1 is established where $x$ consists of all the variables in $P_3, P_1, Q_2, R_1, R_2$ in Theorem 1 and $Z$ in (78). Furthermore, $\rho_1, \rho_2$ and $\varepsilon$ are given constants for regularizations and indicating error tolerance, respectively.

### IV. A variant scheme of DSFC

If there are no delays at the control input of (1), we can modify Theorem 1–2 and Algorithm 1 to solve a different DSFC problem where the controller contains $\nu$ pointwise and distributed delays.

Specifically, let $B_i = \tilde{B}_i(\tau) \equiv 0_{n,p}, \mathcal{B}_i = \tilde{\mathcal{B}}(\tau) = 0_{m,p}, i \in N_\nu$ in (1), which corresponds to a distributed-delay system without input delays. Now we want to construct
\[ u(t) = \sum_{i=0}^{\nu} \sum_{r_i=0}^{\nu} K_i x(t-r_i) + \sum_{i=1}^{\nu} \int_{-r_i-1}^{r_i} \tilde{K}_i(\tau) x(t+\tau)d\tau \]
\[ K_i \in \mathbb{R}^{p \times n}, \tilde{K}_i(\tau) \in \mathbb{L}_2(I; \mathbb{R}^{p \times n}). \]

**Algorithm 1:** An iterative solution for Theorem 1

```plaintext
begin
solve Theorem 2 return $K$
solve Theorem 1 with $K$ return $P_1, P_2$
solve Theorem 1 with $P_1, P_2$ return $K$
update $\tilde{H} \leftarrow H = [P_1, P_2], \quad \tilde{K} \leftarrow K$
while $\|y - \tilde{y}\|_{\infty} \geq \varepsilon$
do
update $\tilde{H} \leftarrow H, \quad \tilde{K} \leftarrow K,$
solve $\min \{ \text{tr} \left[ \rho_1(\tau) (H - \tilde{H}) \right] + \text{tr} \left[ \rho_2(\tau) (K - \tilde{K}) \right] \}
subject to (29)–(30), (78) with (76) and the parameters in Theorem 1, return $H$ and $K$
end
end
```

for the research on LTDSs, as early results in [32] have indicated that a controller with delays could be necessary in order to stabilize certain unstable LTDS.

By the proof of Proposition 1 in A, one can conclude that (2) and $\tilde{K}_i(\cdot) \in L^2(I; \mathbb{R}^{p \times n})$ are true if and only if (3)–(5) holds and there exist $\mathcal{X}_i \in \mathbb{R}^{p \times n}$ such that
\[ \forall i \in N_\nu, \forall \tau \in I_i, \quad \tilde{K}_i(\tau) = \mathcal{X}_i (g_i(\tau) \otimes I_n). \]

Now by using the above conclusion with (9)–(12) and (80), the CLS with (79) is
\[ \dot{x}(t) = (A + B_0 K) \vartheta(t), \quad z(t) = (C + \mathcal{B}_0 K) \vartheta(t), \]
\[ \forall \vartheta \in \mathcal{R}, \quad x(t_0 + \theta) = \psi(\theta), \quad \psi(\cdot) \in \mathcal{C}(\mathcal{R}; \mathbb{R}^n) \]
\[ K = \left[ K_i \right]_{i=0}^{\nu} \left[ \mathcal{X}_i \left( I_i \otimes I_n \right) \right]_{i=1}^{\nu} \cdots \]
\[ \cdots \left[ \mathcal{X}_i \left( \tilde{I}_i \sqrt{E_i} \otimes I_n \right) \right]_{i=1}^{\nu} O_{p,q} \]
where $A, C, \vartheta(t)$ are given. Then the CLS (81) with the supply rate function in (28) is dissipative and the trivial solution of (81) with $\omega(t) \equiv 0_q$ is uniformly asymptotically (exponentially) stable if there exist $P_1 \in \mathbb{S}^n, P_2 \in \mathbb{R}^{n \times e}, P_3 \in \mathbb{S}^e$ with $q = n \sum_{i=1}^{\nu} d_i$, and $Q_i; R_i \in \mathbb{S}^n, K_0; K_i \in \mathbb{R}^{p \times n}, X_i \in \mathbb{R}^{p \times n}, i \in N_{\nu}$, such that (29)–(31) hold with $\Omega = A + B_0 K$ and $\Sigma = C + \mathcal{B}_0 K$.

**Corollary 1.** Let all the parameters in Proposition 1 and (80) be given. Then the CLS (81) with the supply rate function in (28) is dissipative and the trivial solution of (81) with $\omega(t) \equiv 0_q$ is uniformly asymptotically (exponentially) stable if there exist $P_1 \in \mathbb{S}^n, P_2 \in \mathbb{R}^{n \times e}, P_3 \in \mathbb{S}^e$ with $q = n \sum_{i=1}^{\nu} d_i$, and $Q_i; R_i \in \mathbb{S}^n, K_0; K_i \in \mathbb{R}^{p \times n}, X_i \in \mathbb{R}^{p \times n}, i \in N_{\nu}$, such that (29)–(31) hold with $\Omega = A + B_0 K$ and $\Sigma = C + \mathcal{B}_0 K$.
Finally, the number of unknown variables in Corollary 2 is given by:

\[ \sum_{i=1}^{n} d_i \]

Proof: The corollary is proved via the substitutions

\[ \Omega = A + B_0 K \text{ and } \Sigma = C + B_0 K \text{ in (31)}. \]

Corollary 2. Given the conditions in Proposition 1 with (80) and known parameters \( \{\alpha_i\} \), then the CLS (81) with the supply rate function in (28) is dissipative and the trivial solution of (81) with \( w(t) \equiv 0 \) is uniformly asymptotically (exponentially) stable if there exist \( \hat{P}_i; \hat{Q}_i; \hat{R}_i \in \mathbb{S}^n \) and \( V_0; \hat{V}_i \in \mathbb{R}^{p \times n}; V_i \in \mathbb{R}^{p \times n}, i \in \mathbb{N}_0 \), such that (56)–(58) hold with

\[
\hat{H} = \begin{bmatrix}
A \left( (I_\beta \otimes X) + I_q \right) + B_0 V & O_{n,m} \\
C \left( (I_\beta \otimes X) + I_q \right) + B_0 V
\end{bmatrix}
\]

where \( A \) and \( C \) are given in (20), (22) and

\[
V = \left[ \begin{bmatrix} V_{i} \end{bmatrix} \right]^{\nu} \left[ \begin{bmatrix} V_{i} \end{bmatrix} \right]^{\nu} \cdots \left[ \begin{bmatrix} V_{i} \end{bmatrix} \right]^{\nu} \in \mathbb{R}^{p \times n}, V_i \in \mathbb{R}^{p \times n}, i \in \mathbb{N}_0
\]

Moreover, the controller gains are calculated via \( K_0 = V_0 X^{-1} \) and \( K_i = V_i X^{-1} \) and \( \kappa_i \in \mathbb{R} \) for all \( i \in \mathbb{N}_0 \), with

\[ \kappa_i \in \mathbb{R} \text{, } V_i \in \mathbb{R}^{p \times n}, \text{ and } V_i \in \mathbb{R}^{p \times n}, i \in \mathbb{N}_0 \]

Finally, the number of unknowns in Corollary 2 is given by:

\[ (0.5d^2 + 0.5d + \nu + 1)n^2 + (0.5d + 0.5 + \nu + p + \nu p + \nu p + \kappa p)n \in \mathbb{O}(d^2n^2) \]

Proof: The proof is obtained based on the proof of Theorem 2. Note that the corresponding step at (70) is

\[
\hat{H} = \begin{bmatrix}
A \left( (I_\beta \otimes X) + I_q \right) + B_0 K \left( (I_\beta \otimes X) + I_q \right) & O_{n,m} \\
C \left( (I_\beta \otimes X) + I_q \right) + B_0 V & O_{n,m}
\end{bmatrix}
\]

with \( V \) in (82) where \( V_0 = K_0 X, V_i = K_i X \) and \( V_i = \kappa_i \left( I_{n_i} \otimes X \right) \) for all \( i \in \mathbb{N}_0 \). Note that the equality \( K \left( (I_\beta \otimes X) + I_q \right) = V \) with \( K \) in (81) and \( V \) in (82) can be proved by the application of (91).

Corollary 1 can be solved by a modified version of Algorithm 1, as summarized in using the substitutions

\[ \begin{bmatrix} K & O_{p \times m} \end{bmatrix}, K = B_0 \begin{bmatrix} K & O_{p \times m} \end{bmatrix}, K \leftarrow \bar{K}, \bar{K} \leftarrow \bar{K} \text{ for the condition in (78) with the parameters in Corollary 1 and the parameterization}
\]

\[ y = \text{Col} \left[ \text{vec}(\bar{K}), \text{vec}(\bar{R}) \right] = \text{Col} \left[ \text{vec}(K), \text{vec}(R) \right] = \bar{y}, \]

where \( K \) is given in (81) and

\[
\begin{bmatrix} K \end{bmatrix}^{\nu} \left[ \begin{bmatrix} K \end{bmatrix}^{\nu} \right]^{\nu} \cdots \left[ \begin{bmatrix} K \end{bmatrix}^{\nu} \right]^{\nu} \in \mathbb{R}^{p \times n}, K \in \mathbb{R}^{p \times n}, i \in \mathbb{N}_0
\]

Algorithm 2: An iterative solution for Corollary 1

begin
solve Corollary 2 return \( \bar{K} \);
solve Corollary 1 with \( \bar{K} \) return \( P_1, P_2 \);
solve Corollary 1 with \( P_1, P_2 \) return \( \bar{K} \);
update \( H \leftarrow H \), \( \bar{K} \leftarrow \bar{K} \);
solve \[ \min_{x, H, K} \text{tr} \left[ \rho_1 [\bar{K} (H - \bar{K})] + \text{tr} \left[ \rho_2 [\bar{K} (K - \bar{K})] \right] \right] \]
subject to (29)–(30), (78) with (76) and
\[ N = B_0 \begin{bmatrix} K & O_{p \times m} \end{bmatrix}, \bar{N} = B_0 \begin{bmatrix} \bar{K} & O_{p \times m} \end{bmatrix}; \]
with the parameters in Theorem 1,
return \( H \) and \( \bar{K} \);
while \[ \frac{\|y - \bar{y}\|}{\|y\|} + 1 \geq \epsilon \]
do
update \( H \leftarrow H \), \( \bar{K} \leftarrow K \);
solve \[ \min_{x, H, K} \text{tr} \left[ \rho_1 [\bar{K} (H - \bar{K})] + \text{tr} \left[ \rho_2 [\bar{K} (K - \bar{K})] \right] \right] \]
subject to (29)–(30), (78) with (76) and
\[ N = B_0 \begin{bmatrix} K & O_{p \times m} \end{bmatrix}, \bar{N} = B_0 \begin{bmatrix} \bar{K} & O_{p \times m} \end{bmatrix}; \]
with the parameters in Theorem 1,
return \( H \) and \( \bar{K} \);
end
\]

Finally, the following diagram can intuitively explain the relations between the proposed theorems (corollaries) and iterative algorithms, which can be used as a single package to solve the DSFC problem for (1).

V. NUMERICAL EXAMPLES

In this section, we present two numerical examples to show the effectiveness of our proposed methodologies. The examples involve using Algorithm 1 to solve the DSFC problem for LTDSs, thereby involving all the components of the proposed methods. All the computations are carried out in Matlab® using Yalmip [26] as the optimization interface, and SDPT3, Mosek [39, 28] as the numerical solvers for SDPs.
A. DSFC of an LTDSs with Multiple Delays

Consider a system in the form of (1) with \( r_1 = 1, r_2 = 1.7 \) and the state space matrices

\[
A_0 = \begin{bmatrix} -2 & 0.1 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.1 & 1 \\ 0.2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0.1 \\ 0 & -0.2 \end{bmatrix},
B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},
\]

\[
\tilde{A}_1(\tau) = \begin{bmatrix} 0.1 + 3 \sin(20\tau) & 0.8e^{\sin(20\tau) - 0.3e^{\sin(20\tau)}} \\ 0.3 + e^{2(i+1)\tau} + 0.6 & 3 \sin(20\tau) \end{bmatrix},
\tilde{A}_2(\tau) = \begin{bmatrix} -10 \cos(18\tau) & 3e^{\cos(18\tau)} - \frac{1}{\cos(0.7\tau + 1)} \\ 0.1e^{\sin(18\tau)} & 0.2 - 10 \cos(18\tau) \end{bmatrix},
\]

\[
\tilde{B}_1(\tau) = \begin{bmatrix} 0.01 & 0.1 \sin(1.2\tau) + 1 + 0.1 \\ 0.1 + \sin^2(1.2\tau) + 1 & 0.2 \end{bmatrix},
\tilde{B}_2(\tau) = \begin{bmatrix} 0.2e^{\cos(18\tau)} + 0.01e^{\sin(18\tau)} + \frac{0.1}{\cos(0.7\tau + 1)} \\ 0.1e^{\cos(18\tau)} + 0.02e^{\sin(18\tau)} \end{bmatrix},
\]

\[
C_0 = \begin{bmatrix} -0.1 & 0.2 \\ 0 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -0.1 & 0.2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0.1 \end{bmatrix},
\]

\[
\tilde{C}_1(\tau) = \begin{bmatrix} 0.7 + \cos(20\tau) \\ 0.4 - 0.5e^{\sin(20\tau)} \sin 18\tau \end{bmatrix},
\tilde{C}_2(\tau) = \begin{bmatrix} 0.2 + \sin(18\tau) \\ 0 \end{bmatrix},
\]

\[
\tilde{D}_1(\tau) = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \quad \tilde{D}_2(\tau) = \begin{bmatrix} 0.12 \\ 0.1 \end{bmatrix},
\]

(85)

with \( n = m = 2, p = q = 1 \). By using the numerical toolbox of the spectral method proposed in [4], it shows the nominal system is unstable. Moreover, we utilize

\[
\gamma > 0, \quad J_1 = -\gamma I_2, \quad \tilde{J} = I_2, \quad J_2 = O_2, \quad J_3 = \gamma
\]

for the supply function in (28) where \( \gamma \) is the \( L^2 \) gain to be minimized.

Remark 11. The parameters in (85) are chosen with sufficient degree of mathematical complexity in order to illustrate the strength of the proposed method. It is important to point out that our approach can handle many practical examples such as the ones in Remark 2, where the DDs therein are usually much simpler than the DDs in (85). Note that no existing methods can effectively solve the DSFC problem of an LTDS in (1) with (85) due to the complexity of the DDs with multiple non-commensurate delays and a non-Hurwitz \( A_0 \).

Assuming all the states in \( x(t) \) can be measured, we want to find a controller gain of \( u(t) = K x(t) \) to stabilize the open-loop system (1) while minimizing the \( L^2 \) gain. Observing the functions inside of the DDs, let \( \varphi_1(\tau) = 1/(\sin^2 1.2\tau + 1) \) and \( \varphi_2(\tau) = 1/(\cos^2 0.7\tau + 1) \) and

\[
\phi_1(\tau) = \begin{bmatrix} e^{\sin(20\tau)} \\ e^{\cos(20\tau)} \end{bmatrix}, \quad \phi_2(\tau) = \begin{bmatrix} e^{\sin(18\tau)} \\ e^{\cos(18\tau)} \end{bmatrix},
\]

\[
f_1(\tau) = \begin{bmatrix} \sin 20\tau \\ \sin 18\tau \end{bmatrix}, \quad f_2(\tau) = \begin{bmatrix} \sin 20\tau \lambda \end{bmatrix},
\]

for the parameters in Proposition 1 with

\[
M_1 = \begin{bmatrix} 0 & O_{d_1} & 0 & O_{\lambda_1} & O_{\lambda_1} \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & O_{d_2} & 0 & O_{\lambda_2} & O_{\lambda_2} \end{bmatrix}
\]

in (5). By (87) and (91), we can construct

\[
\hat{A}_1 = \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.8 & 0 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.8 & 0 \end{bmatrix},
\]

\[
\hat{B}_1 = \begin{bmatrix} 0.2 + \sin(18\tau) \\ 0 \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} 0.2e^{\cos(18\tau)} + 0.01e^{\sin(18\tau)} + \frac{0.1}{\cos(0.7\tau + 1)} \\ 0.1e^{\cos(18\tau)} + 0.02e^{\sin(18\tau)} \end{bmatrix},
\]

\[
\hat{D}_1 = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \quad \hat{D}_2 = \begin{bmatrix} 0.12 \\ 0.1 \end{bmatrix},
\]

(89)

satisfying the decompositions in (3)–(4).

Remark 12. The functions in (87) are chosen based on several reasons. First of all, the functions in \( \phi_i(\cdot) \) can be well approximated via appropriate trigonometric functions together with polynomials, some of which exist in the DDs. On the other hand, the functions in \( \varphi_i(\cdot) \) are directly factorized since it is very difficult to approximate them with \( d_i, \lambda_i \) of manageable values. As a result, the choice for (87) involves all the components proposed in Proposition 1, which balances feasibility and the implied computational complexity \( O(d^2n^2) \) affected by the dimensions of \( f_i(\tau) \). This serves as a good example.
showing the advantage of Proposition 1 over the existing approaches in [37, 11, 9].

Now we want to find a controller gain $K$ stabilizing the nominal system in (19) with (85) and simultaneously minimizing $\gamma$. Firstly, apply Theorem 2 to (19) with $d_1 = d_2 = \lambda_1 = \lambda_2 = 1$ and $\alpha_i = 0$, $i = 2 \cdots \beta$, $\alpha_1 = 5$ and the parameters in (85)–(89), where the matrices in (9)–(12) are computed via the $\text{vpaintegral}$ function in Matlab©. The numerical program yields $K = \begin{bmatrix} -1.3794 & -1.8608 \end{bmatrix}$ with min $\gamma = 0.8986$, where the controller gain is used for initializing Algorithm 1.

After running Algorithm 1 for the same system with the parameters in (85)–(89), it produces the results in Table I–II, where SA stands for the Spectral Ablcissa of the resulting CLSs with $w(t) \equiv 0$, and Nols for the number of iteration in the while loop. The results clearly show that adding more functions (larger $\lambda_1, \lambda_2$) to $f_i(\cdot)$ may increase the feasibility of the synthesis conditions leading to smaller min $\gamma$. Moreover, it shows that using Algorithms 1 can produce controller gains with significantly better performance (min $\gamma$) than Theorem 2 alone. Thus illustrates the contribution of Algorithms 1.

![Fig. 1: The close-loop system's trajectories $x(t)$](image1.png)

![Fig. 2: The trajectory of the control action $u(t)$](image2.png)

For numerical simulation, we consider the CLSs in (19) with $K = \begin{bmatrix} -1.5810 & -1.9805 \end{bmatrix}$ in Table II, and the state space parameters in (85). Moreover, let $t_0 = 0$, $z(t) = 0$, $t < 0$, and $\psi(\tau) = \begin{bmatrix} 50 & 30 \end{bmatrix}^T$, $\tau \in [-2, 0]$ for the initial condition, and $w(t) = 10 \sin(10t(I(t) - I(t-5)))$ as the disturbance where $I(t)$ is the Heaviside step function. The simulation is performed in Simulink using the ODE solver $\text{ode8}$ with 0.002 as the fundamental sampling time. The results are summarized in Figures 1–3 including the trajectories of the states $x(t)$, control action $u(t)$ and regulated outputs $z(t)$ of the CLS. Note that all the DDs are discretized in simulation via the trapezoidal rule with 200 sample points.

| Controller gain $K$ | $[-1.5456]$$^T$ | $[-1.5365]$$^T$ | $[-1.5180]$$^T$ | $[-1.5033]$$^T$ |
|---------------------|-----------------|-----------------|-----------------|-----------------|
| $\min \gamma$      | 0.6573          | 0.6542          | 0.6523          | 0.6509          |
| SA                  | -0.7223         | -0.7214         | -0.7224         | -0.7233         |
| Nols                | 5               | 10              | 15              | 20              |

**TABLE I**: Controller gains with $\min \gamma$ produced with $d_1 = d_2 = \lambda_1 = \lambda_2 = 1$

| Controller gain $K$ | $[-1.5538]$$^T$ | $[-1.5848]$$^T$ | $[-1.5870]$$^T$ | $[-1.5810]$$^T$ |
|---------------------|-----------------|-----------------|-----------------|-----------------|
| $\min \gamma$      | 0.6443          | 0.6398          | 0.6376          | 0.6361          |
| SA                  | -0.7223         | -0.7214         | -0.7224         | -0.7233         |
| Nols                | 5               | 10              | 15              | 20              |

**TABLE II**: Controller gains with $\min \gamma$ produced with $d_1 = d_2 = 1, \lambda_1 = \lambda_2 = 2$

### B. DSFC for an LTDS with Controllers Delays

This subsection aims to show the advantage of adding delays to controllers when the system in (1) has no delays at its input.

Consider a system with the same parameters in subsection V-A except for $B_i = B_i(\tau) = O_{n,p}$ and $B_i = B_i(\tau) = O_{m,n}, \forall i \in \mathbb{N}_p$. Then the controller defined in (79)–(80) can be utilized for stabilizing the open-loop system while minimizing $\gamma$ in (86).
The procedures of computing controller gains here are entirely identical to the previous subsection apart from utilizing [10, Algorithm 2] to the CLS in (81) supported by Corollary 1–2 instead of Theorem 1–2. Specifically, we apply Proposition 1 with (80) for the DDs in (85), (79) using the same parameters in (87) for $g_i(\cdot)$ and $M_i$. This leads to the same parameters $\tilde{A}_i$, $\tilde{C}_i$, in (89), whereas $\mathcal{K}_i$ in (80) are the gains to be computed.

Next, we assume $\alpha_i = 0$, $i = 2, \cdots, \beta$, $\alpha_1 = 5$ and $\alpha_1 = 50$, respectively, when Corollary 2 is applied. The numerical results produced by [10, Algorithm 2] are summarized in Table III–VI where the resulting $\mathcal{K}_i$ are omitted due to limit space. Note that the results in Table III–IV and Table V–VI shows that using the delay structures in (79) can materially improve the performance of min $\gamma$ compared to the use of a static controller $u(t) = Kx(t)$. This justifies the use of the delay structures in (79) even though it requires more resources for the controller realization.

Since the CLSs in (81) belong to the retarded type, their nominal stability is guaranteed [22] with the numerical implementation of the DDs in (80) as long as the accuracy reach certain degree. This property ensures that the resulting controllers in (81) can be materialized numerically for real-world applications.

![Fig. 3: The regulated output $z(t)$](image)

![Fig. 4: The Close-Loop System’s Trajectories $x(t)$](image)

| $\min \gamma$ | 0.5242 | 0.524 | 0.5238 | 0.5237 |
|---------------|--------|-------|--------|--------|
| SA            | −0.6983| −0.6979| −0.6976| −0.6989|
| Nols          | 5      | 10    | 15     | 20     |

**TABLE III:** $\min \gamma$ produced by the [10, Algorithm 2] with $d_1 = d_2 = \lambda_1 = \lambda_2 = 1$

| $\min \gamma$ | 0.5785 | 0.5760 | 0.5736 | 0.5714 |
|---------------|--------|-------|--------|--------|
| SA            | −0.7259| −0.7319| −0.7358| −0.7385|
| Nols          | 5      | 10    | 15     | 20     |

**TABLE IV:** $\min \gamma$ produced by the [10, Algorithm 2] with $d_1 = d_2 = \lambda_1 = \lambda_2 = 2$

| $\min \gamma$ | 0.5723 | 0.5669 | 0.5626 | 0.5590 |
|---------------|--------|-------|--------|--------|
| SA            | −0.7179| −0.7113| −0.7099| −0.7099|
| Nols          | 5      | 10    | 15     | 20     |

**TABLE V:** $\min \gamma$ produced by the [10, Algorithm 2] with $d_1 = d_2 = \lambda_1 = \lambda_2 = 1$ and $K_i = \bar{K}_i(\tau) = O_{p,n}$

For numerical simulation, we use the CLS with $K_0 = \begin{bmatrix} -9.1247 & -22.9729 \end{bmatrix}$, $K_1 = \begin{bmatrix} 0.2429 & -0.117 \\ 0.0972 & 0.1237 \end{bmatrix}$

$\mathcal{K}_1 = \begin{bmatrix} -0.2358 & -0.3997 & -0.3585 & 0.1859 & -0.4752 \\ -0.8404 & 4.0466 & -2.6485 & -0.9318 & -3.3031 & -1.2266 \\ -2.4909 & 0.1010 & -0.5549 & 0.80751 & 0.1430 & 0.0620 & 0.148 \\ -0.0276 & -0.0464 & 0.2099 & -0.3101 & -0.8961 & -0.9972 \\ -6.3285 & -1.1248 & 5.0129 & -1.1893 & 1.1018 & 1.5506 \\ -0.3628 & -0.0725 & 4.3602 & 6.5996 & 0.0282 & 0.6417 \end{bmatrix}$

(90)

corresponding to $\min \gamma = 0.523$ in Table IV. The rest of the setting is identical to subsection V-A, and the simulation results are summarized in Figures 4–6.
allows us to construct the KF in (for LTDSs similar to the SDP approach for an LTI delay-function over variable, it could be more general than existing ones, thereby leading to

Moreover, \( \forall X \in \mathbb{R}^{n \times m}, \forall Y \in \mathbb{R}^{m \times p}, \forall Z \in \mathbb{R}^{q \times r} \),

\[
(X \otimes I_q)(Y \otimes Z) = XY \otimes Z = XY \otimes ZI_r = (X \otimes Z)(Y \otimes I_r). \tag{91}
\]

\[
(X \otimes I_q)(Y \otimes Z) = XY \otimes Z = I_mXY \otimes (ZI_r) = (I_m \otimes Z)(XY \otimes I_r). \tag{92}
\]

Moreover, \( \forall X \in \mathbb{R}^{n \times m} \), we have

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \otimes X = \begin{bmatrix} A \otimes X & B \otimes X \\ C \otimes X & D \otimes X \end{bmatrix}, \quad I_n \otimes X = \bigoplus_{i=1}^n X \tag{93}
\]

for any \( A, B, C, D \) with appropriate dimensions.

We define the weighted Lebesgue function space

\[
L_\infty^2(\mathbb{K};\mathbb{R}^d) := \left\{ \phi(\cdot) \in \mathbb{M}(\mathbb{K};\mathbb{R}^d) : \|\phi(\cdot)\|_{2,\infty} < \infty \right\} \tag{94}
\]

d with \( d \in \mathbb{N} \) and \( \|\phi(\cdot)\|_{2,\infty} := \int_{\mathbb{K}} \varpi(\tau)\phi(\tau)\phi(\tau)\tau \) where \( \varpi(\cdot) \in \mathbb{M}(\mathbb{K};\mathbb{R}_{\geq 0}) \) and the function \( \varpi(\cdot) \) has countably infinite or finite number of zero values. Furthermore, \( \mathbb{K} \subseteq \mathbb{R} \cup \{\pm \infty\} \) and its Lebesgue measure is non-zero.

\[
\text{Lemma 4. Given } \mathbb{K} \text{ and } \varpi(\cdot) \text{ in (94) and } U \in \mathbb{S}^n_{\geq 0} := \left\{ X \in \mathbb{S}^n : X \succeq 0 \right\} \text{ with } n \in \mathbb{N}. \text{ Let } f_i(\cdot) \in L_\infty^2(\mathbb{K};\mathbb{R}^{i_1}) \text{ and } g_i(\cdot) \in L_\infty^2(\mathbb{K};\mathbb{R}^{\lambda_i}) \text{ with } i_1 \in \mathbb{N} \text{ and } \lambda_i \in \mathbb{N}_0, i \in \mathbb{N}_n, \text{ in which the functions } f_i(\cdot) \text{ and } g_i(\cdot) \text{ satisfy}
\]

\[
\int_{\mathbb{K}} \varpi(\tau) \begin{bmatrix} g_i(\tau) \\ f_i(\tau) \end{bmatrix} \begin{bmatrix} g_i(\tau)^T \\ f_i(\tau)^T \end{bmatrix} \mathrm{d}\tau > 0, \quad i \in \mathbb{N}_n \tag{95}
\]

which implies \( \int_{\mathbb{K}} \varpi(\tau) f_i(\tau) f_i(\tau)^T \mathrm{d}\tau > 0, \quad i \in \mathbb{N}_n. \) Then

\[
\int_{\mathbb{K}} \varpi(\tau) x_i^T(\tau) \begin{bmatrix} \bigoplus_{i=1}^n U_i \end{bmatrix} x(\tau) \mathrm{d}\tau \\
\geq \left[ * \right] \begin{bmatrix} \bigoplus_{i=1}^n \mathbb{F}^{-1} \otimes \mathbb{U}_i \end{bmatrix} \times
\left( \int_{\mathbb{K}} \varpi(\tau) \begin{bmatrix} \bigoplus_{i=1}^n f_i(\tau) \otimes I_n \end{bmatrix} x(\tau) \mathrm{d}\tau \right)
\]

\[
+ \left[ * \right] \begin{bmatrix} \bigoplus_{i=1}^n \mathbb{E}^{-1} \otimes \mathbb{U}_i \end{bmatrix} \times
\left( \int_{\mathbb{K}} \varpi(\tau) \begin{bmatrix} \bigoplus_{i=1}^n e_i(\tau) \otimes I_n \end{bmatrix} x(\tau) \mathrm{d}\tau \right)
\]

VI. CONCLUSION

This work has set out an effective solution for the DSFC problem of a general LTDS in (1), where the number of delays \( \nu \) is unlimited and the DDs can contain any number of \( \mathbb{L}^2 \) function over \( \mathcal{I}_i \). A key conceptual contribution is the notion of equivalent decomposition in Proposition 1, which circumvents the infinite dimensionality of the DDs and gives users the liberty to use different ways to handle them without theoretical conservatism. Because of the generality of \( f_i(\cdot) \), it also allows us to construct the KF in (40) which is much more general than existing ones, thereby leading to less conservative synthesis conditions. Because of the generality and scope of the proposed method, it could be considered as a milestone of the SDP-based solutions for LTDSs similar to the SDP approach for an LTI delay-free system. In the second part of the paper, the proposed methodology are extended to achieve dissipative observer design and ultimately observer-based control.

APPENDIX

Some lemmas are presented here which are crucial for the derivations of the results in this paper. A novel integral inequality is proposed to construct lower bounds for integrals defined over \( \mathcal{I}_i \).

\[
\text{Lemma 3. } \forall X \in \mathbb{R}^{n \times m}, \forall Y \in \mathbb{R}^{m \times p}, \forall Z \in \mathbb{R}^{q \times r},
\]

\[
(X \otimes I_q)(Y \otimes Z) = XY \otimes Z
\]

\[
= XY \otimes ZI_r = (X \otimes Z)(Y \otimes I_r). \tag{91}
\]

\[
(X \otimes I_q)(Y \otimes Z) = XY \otimes Z
\]

\[
= I_mXY \otimes (ZI_r) = (I_m \otimes Z)(XY \otimes I_r). \tag{92}
\]
holds for all $x(\cdot) \in L^2_\mathcal{K}(\mathbb{K}; \mathbb{R}^{nu})$, where $\mathcal{F}_i = \int_{\mathcal{K}} \varphi(\tau)f_i(\tau)\tau d\tau > 0$. In addition, $e_i(\tau) = g_i(\tau) - A_i f_i(\tau) \in \mathbb{R}^{\lambda_i}$ and $A_i = \int_{\mathcal{K}} \varphi(\tau)g_i(\tau)\tau d\tau$. Now we start to prove the sufficiency part of the statement of the parameters in Proposition 2019. (1), 2019(1), 2019.

A. Proof of Proposition 1

Proof: First of all, it is obvious that (2) is implied by (3)–(6) because of the definitions of $\varphi_i(\cdot)$, $f_i(\cdot)$, $\phi_i(\cdot)$ and the fact that $C^1(I; \mathbb{R}^{d_i}) \subset L^2(I; \mathbb{R}^{d_i})$. So the necessity part of the statement is proved.

Now we start to prove the sufficiency part of the statement. Namely, the condition in (2) implies the existence of the parameters in Proposition 1 satisfying (3)–(6). Given any $f_i(\cdot) \in C^1(I; \mathbb{R}^{d_i})$, $i \in \mathbb{N}_\nu$, $d \in \mathbb{N}$ satisfying $\int_{\mathcal{K}} f_i(\tau)\tau d\tau > 0$, one can always construct appropriate $\phi_i(\cdot)$ and $\varphi_i(\cdot) \in L^2(I; \mathbb{R}^{d_i})$ with $M_i \in \mathbb{R}^{d_i \times \lambda_i}$ such that the conditions in (5)–(6) are satisfied. Note that $\int_{\mathcal{K}} f_i(\tau)\tau d\tau > 0$ is implied by the matrix inequalities in (6) which indicate that the functions in $g_i(\cdot)$ in (5) are linearly independent in a Lebesgue sense over $[-r_i, -r_i-1]$ for each $i \in \mathbb{N}_\nu$. The aforementioned conclusion is true because $\frac{d f_i(\tau)}{d\tau}(\cdot) \in C(I; \mathbb{R}^{d_i}) \subset L^2(I; \mathbb{R}^{d_i})$ for all $i \in \mathbb{N}_\nu$, and the dimensions of $\varphi_i(\tau)$ and $\phi_i(\tau)$, $i \in \mathbb{N}_\nu$ can be arbitrarily enlarged with more linearly independent functions. Note that $\varphi_i(\tau)$ or $\phi_i(\tau)$ can be an empty matrix.

Now since $\dim(g_i(\tau))$ in (5)–(6) can be arbitrarily increased, (appropriate new functions can always be added) there always exist $\hat{A}_{i,j} \in \mathbb{R}^{\lambda_i \times n}$, $\hat{C}_{i,j} \in \mathbb{R}^{\lambda_i \times n}$, $\hat{B}_{i,j} \in \mathbb{R}^{n \times p}$, $\hat{B}_{i,j} \in \mathbb{R}^{n \times p}$ and $g_i(\tau) = \text{Col}_{j=1}^{\lambda_i} g_{i,j}(\tau)$ in (6) for the distributed delay terms in (2) such that

$$\hat{A}_i(\tau) = \sum_{j=1}^{\lambda_i} A_{i,j} g_{i,j}(\tau), \quad \hat{C}_i(\tau) = \sum_{j=1}^{\lambda_i} C_{i,j} g_{i,j}(\tau),$$

$$\hat{B}_j(\tau) = \sum_{j=1}^{\lambda_i} B_{i,j} g_{i,j}(\tau), \quad \hat{B}_j(\tau) = \sum_{j=1}^{\lambda_i} B_{i,j} g_{i,j}(\tau)$$

\[(97)\]

\[\forall i \in \mathbb{N}_\nu, \forall \tau \in I_i \text{ with } \kappa_i \in \mathbb{N}_0 \text{ where } \varphi_i(\cdot) \in L^2([-r_i, -r_i-1]; \mathbb{R}^{d_i}), \quad f_i(\cdot) \in C^1(I; \mathbb{R}^{d_i}) \quad \text{and} \quad \phi_i(\cdot) \in L^2(I; \mathbb{R}^{d_i}) \text{ satisfy (5)–(6) for some } M_i \in \mathbb{R}^{d_i \times \lambda_i}, \quad i \in \mathbb{N}_\nu.\]

Moreover, (97) can be rewritten as

$$\hat{A}_i(\tau) = [\hat{A}_{i,j}]_{j=1}^{\lambda_i} (g(\tau) \otimes I_n), \quad \hat{C}_i(\tau) = [\hat{C}_{i,j}]_{j=1}^{\lambda_i} (g(\tau) \otimes I_n) \quad \text{and} \quad \hat{B}_i(\tau) = [\hat{B}_{i,j}]_{j=1}^{\lambda_i} (g(\tau) \otimes I_p) \quad \text{for all } i \in \mathbb{N}_\nu.$$}

\[\forall \tau \in I_i \text{ which are in line with the decompositions in (3)–(4) by letting } \hat{A}_i = [\hat{A}_{i,j}]_{j=1}^{\lambda_i}, \quad \hat{C}_i = [\hat{C}_{i,j}]_{j=1}^{\lambda_i}, \quad \hat{B}_i = [\hat{B}_{i,j}]_{j=1}^{\lambda_i} \quad \text{and} \quad \hat{B}_i = [\hat{B}_{i,j}]_{j=1}^{\lambda_i} \text{ for all } i \in \mathbb{N}_\nu. \text{ Finally, the conclusion in (97) is true for the case of } \mu_i = 0 \text{ or } \delta_i = 0. \text{ Given all the aforementioned statements we have presented, then Proposition 1 is proved.} \]

\[\text{\blacksquare}\]

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