Discrete sums of classical symbols on $\mathbb{Z}^d$
and zeta functions associated with
Laplacians on tori

Sylvie PAYCHA
March 12, 2008

Abstract

We prove the uniqueness of a translation invariant extension to non integer order classical symbols of the ordinary discrete sum on $L^1$-symbols, which we then describe using an Hadamard finite part procedure for sums over integer points of infinite unions of nested convex polytopes in $\mathbb{R}^d$. This canonical regularised sum is the building block to construct meromorphic extensions of the ordinary sum on holomorphic symbols. Explicit formulae for the complex residues at their poles are given in terms of non-commutative residues of classical symbols, thus extending results of Guillemin, Sternberg and Weitsman. These formulae are then applied to zeta functions associated with quadratic forms and with Laplacians on tori.

Introduction

Just as the ordinary $L^2$-trace on trace-class classical pseudodifferential operators on a closed manifold is known not to extend to a trace to the whole algebra of classical operators, one does not expect the Riemann integral (resp.discrete sum) on $L^1$ classical pseudodifferential symbols on $\mathbb{R}^d$ to extend to an $\mathbb{R}^d$ (resp. $\mathbb{Z}^d$) -translation invariant linear form on the whole algebra of symbols.

We show that these nevertheless have a canonical $\mathbb{R}^d$ (resp. $\mathbb{Z}^d$) -translation invariant linear extension to the set of non integer order classical symbols on $\mathbb{R}^d$ (see Theorem [1]), namely the canonical regularised integral $\int_{\mathbb{R}^d}$ (resp. the canonical regularised discrete sum $\sum_{\mathbb{Z}^d}$).

It is known (see e.g. [L]) that the canonical regularised integral $\int_{\mathbb{R}^d}$ can be expressed as an Hadamard finite part of ordinary integrals over euclidean balls of radius tending to infinity and that its translation invariance is related to a Stokes property [P2]. Integrating this canonical regularised integral over a closed manifold gives rise to the unique extension of the $L^2$-trace defined on trace-class classical pseudodifferential operators to the set of non integer order classical pseudodifferential operators [MSS], namely to the canonical trace introduced by Kontsevich and Vishik [KV].

1 AMS classification: 11E45, 58C40, 47G30
2 By linear we mean that it preserves linear combinations that lie in the set.
3 We borrow the notation $\int_{\mathbb{R}^d}$ from [L] and transpose it to the discrete sum but warn the reader that the same notation is used for the Dixmier trace by A.Connes.
Less is known about extensions of the ordinary discrete sum. Whereas one does not expect the canonical regularised discrete sum to vanish on derivatives, its value on derivatives is entirely determined by its restriction to $L^1$ symbols (see Proposition 1). We express the canonical regularised sum as an Hadamard finite part for expanded convex polytopes with increasing size, using a generalisation of the Euler-MacLaurin formula in order to compare these with the corresponding integrals over these polytopes.

The Khovanskii-Pukhlikov formula \[KP\] generalises the Euler-MacLaurin formula to polynomials in higher dimensions in as far as it compares the discrete sum over integer points with the integral of a polynomial function on expanded convex polytopes which are integral and regular. Guillemin, Sternberg and Weitsman \[GSW\] extended the Khovanskii-Pukhlikov formula to classical symbols. Whereas it is an exact formula for polynomials, it becomes an asymptotic formula in the case of classical symbols and a new constant term $C(\sigma)$ depending on the symbol $\sigma$ arises, which vanishes for polynomials. For gauged symbols $z \mapsto \sigma(z)$, the authors showed that $z \mapsto C(\sigma(z))$ is holomorphic and interpreted $C(\sigma)$ as a "regularised version" of the difference between the infinite sum $\sum_{n \in \mathbb{Z}^d} \sigma(n)$ of the symbol over integer points of $\mathbb{R}^d$ and the infinite integral $\int_{\mathbb{R}^d} \sigma(x) \, dx$.

In this paper, using the canonical regularised sum $\sum_{n \in \mathbb{Z}^d}$, we refine and generalise some of their results to any local holomorphic perturbation $\sigma(z)$ of $\sigma = \sigma(0)$ with non constant affine order $\alpha(z)$ showing the following statements.

1. The map $z \mapsto \sum_{n \in \mathbb{Z}^d} \sigma(z)(\underline{n})$ which is holomorphic on a half plane $\text{Re}(\alpha(z)) < -d$, extends to a meromorphic map $z \mapsto \sum_{n \in \mathbb{Z}^d} \sigma(z)(\underline{n})$ on the whole complex plane with simple poles. The pole at $z = 0$ is proportional to the noncommutative residue of $\sigma$ (see Theorem 3)

$$\text{Res}_{z=0} \sum_{n \in \mathbb{Z}^d} \sigma(z)(\underline{n}) = -\frac{1}{\alpha'(0)} \text{res}(\sigma).$$

2. When $\sigma$ has non integer order, the constant term in the Laurent expansion $\sum_{n \in \mathbb{Z}^d} \sigma(z)(\underline{n})$ around $z = 0$ coincides with the canonical regularised sum $\text{fp}_{z=0} \sum_{n \in \mathbb{Z}^d} \sigma(z)(\underline{n}) = \sum_{n \in \mathbb{Z}^d} \sigma(n)$.

3. In general, the finite parts $\text{fp}_{z=0} \sum_{n \in \mathbb{Z}^d} \sigma(z)(\underline{n})$ for different holomorphic perturbations $\sigma \mapsto \sigma(z)$ only differ if the perturbations do not coincide on the unit sphere, in which case they differ by a term proportional to some noncommutative residue (see Corollary 1).

These results strongly rely on corresponding known results \[KV\] for regularised integrals of symbols which we recall in an Appendix, namely the fact that the ordinary integral $z \mapsto \int_{\mathbb{R}^d} \sigma(z)(\underline{x}) \, dx$ has a meromorphic extension $z \mapsto \int_{\mathbb{R}^d} \sigma(z)(\underline{x}) \, dx$ to the whole complex plane with simple poles and that the pole at $z = 0$ is proportional to the noncommutative residue of $\sigma$ (see Theorem 5 in the Appendix).

\[4\] It was then generalised to simple integral polytopes by Cappell and Shaneson \[CS\] and subsequently by Guillemin \[G2\] and Brion and Vergne \[BV\]; these generalisations involve corrections to the Khovanskii-Pukhlikov formula when the simple polytope is not regular.
In the last part of the paper (sections 6 and 7), we apply our results to the zeta function

\[ Z_q(s) := \sum_{z \neq 0} q(u)^{-s} |u|^{-z} \]

associated with a positive definite quadratic form \( q \) on \( \mathbb{R}^d \) (see Theorem 4), resp. to the zeta function

\[ \zeta_{\Delta_d}(s) := \sum_{z \neq 0} |u|^{-2s+z} \]

associated with the Laplacian \( \Delta_d \) on the \( d \)-dimensional torus (see Proposition 5). Here \( |.| \) stands for the euclidean norm on \( \mathbb{R}^d \).

At arguments \( s \) with non positive real part, \( Z_q(s) \), resp. \( \zeta_{\Delta_d}(s) \) coincides with the constant \( C(q^{-s}) \) (see Proposition 4), resp. \( C(x \mapsto |x|^{-2s}) \), leading to a formula for \( \zeta \)-determinants of Laplacians on tori (see Proposition 6)

\[ \det_{\zeta}(\Delta_d) = \exp \left( -\partial_{s}|_{s=0} C(x \mapsto |x|^{-2s}) \right) \]

where the \( \partial_{s}|_{s=0} \) stands for the derivative in the half plane \( \text{Re}(s) \leq 0 \).

The paper is organised as follows.

1. Sums of classical symbols on positive integers
2. A canonical regularised integral and discrete sum on non integer order symbols
3. Concrete realisations of the canonical regularised integral and discrete sum
4. Sums of holomorphic symbols on \( \mathbb{Z}^d \)
5. Zeta functions associated with quadratic forms
6. Zeta functions associated with Laplacians on tori
7. Appendix: Prerequisites on regularised integrals of symbols

Acknowledgements

I would like to thank Victor Guillemin, Shlomo Sternberg and Jonathan Weitsman for their comments on a preliminary version of this paper. Let me also thank Viatcheslav Kharlamov for his comments following a talk at the Max Planck institute in Bonn during which I reported on these results as well as the institute itself, where this article was completed.
1 Discrete sums of classical symbols on integers

Before investigating discrete sums on polytopes, let us first consider discrete sums of classical symbols on ordinary integers. Along the lines of \cite{MP} we use the Euler-MacLaurin formula which relates a sum to an integral. It involves the Bernoulli numbers defined by the generating series:

\[
\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}.
\]  

Bernoulli polynomials are defined similarly:

\[
\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t e^{tx} - t}{e^t - 1},
\]

so that in particular, \(B_n(0) = B_n\). For example \(B_1(x) = -\frac{1}{2} + x\).

The Euler-MacLaurin formula states that for any \(f \in C^\infty(\mathbb{R})\) and any two integers \(M < N\) (see e.g. \cite{Ha}):

\[
\sum_{n=M}^{N} f(n) = f(M) + f(N) + \int_{M}^{N} f(x) \, dx + \sum_{k=2}^{K} (-1)^{k-1} \frac{B_k}{k!} \left( f^{(k-1)}(N) - f^{(k-1)}(M) \right) + \frac{(-1)^{K-1} K!}{K!} \int_{M}^{N} B_K(x) f^{(K)}(x) \, dx.
\]

with \(B_k(x) = B_k(x - \lfloor x \rfloor)\) and where \(K\) is any positive integer larger than 1.

When \(f\) is polynomial of degree \(d\), this reduces to

\[
\sum_{n=M}^{N} f(n) = \frac{f(M) + f(N)}{2} + \int_{M}^{N} f(x) \, dx + \sum_{k=2}^{d+1} (-1)^{k-1} \frac{B_k}{k!} \left( f^{(k-1)}(N) - f^{(k-1)}(M) \right).
\]

As it was shown in \cite{GSW}, see also \cite{MP}, this formula gives interesting information when applied to a symbol \(\sigma\) in the algebra \(\mathbb{C}S_{\text{c, c}}(\mathbb{R})\) of classical pseudodifferential symbols with constant coefficients defined in formula (25) of the Appendix. In particular, the Euler-MacLaurin formula provides a control on the asymptotics of \(\sum_{n=M}^{N} \sigma(n)\) as \(M = -N \to -\infty\) with \(N \to \infty\):

\[
\sum_{n=-N}^{N} \sigma(n) \sim_{N \to \infty} \frac{\sigma(-N) + \sigma(N)}{2} + \int_{-N}^{N} f(x) \, dx + \sum_{k=2}^{K} (-1)^{k} \frac{B_k}{k!} \left( \sigma^{(k-1)}(N) - \sigma^{(k-1)}(-N) \right) + \frac{(-1)^{K-1} K!}{K!} \int_{\mathbb{R}} B_K(x) \sigma^{(K)}(x) \, dx.
\]
Picking the constant term we define an Hadamard finite part \( \text{fp}_{N \to \infty} \sum_{n=-N}^{N} \sigma(n) \) (see also [MP]) which relates to the finite part \( \text{fp}_{N \to \infty} \int_{-N}^{N} \sigma(x) \, dx \) using (3):

\[
\text{fp}_{N \to \infty} \sum_{n=-N}^{N} \sigma(n) = \text{fp}_{N \to \infty} \int_{-N}^{N} \sigma(x) \, dx + \frac{\text{fp}_{N \to \infty} \sigma(-N) + \text{fp}_{N \to \infty} \sigma(N)}{2} + \sum_{k=2}^{K} (-1)^{k} \frac{B_{k}}{k!} \left( \text{fp}_{N \to \infty} \sigma^{(k-1)}(N) - \text{fp}_{N \to \infty} \sigma^{(k-1)}(-N) \right) + \frac{(-1)^{K-1}}{K!} \int_{R} B_{K}(x) \sigma^{(K)}(x) \, dx
\]

where this last term is actually a convergent integral.

The difference \( \text{fp}_{N \to \infty} \sum_{n=-N}^{N} \sigma(n) - \text{fp}_{N \to \infty} \int_{-N}^{N} \sigma(x) \, dx \) involves finite parts \( \text{fp}_{N \to \infty} \sigma^{(j)}(N) \) and \( \text{fp}_{N \to \infty} \sigma^{(j)}(-N) \) which vanishes when \( \sigma \) has non integer order. On non integer order symbols, the Hadamard finite parts \( \text{fp}_{N \to \infty} \int_{-N}^{N} \sigma(x) \, dx \) turn out to be independent of the choice of parameter \( N \) and for any complex number \( s \) with non negative real part, we have \( \text{fp}_{N \to \infty} \sum_{n=-N}^{N} \sigma(n) = \text{fp}_{N \to \infty} \int_{-N}^{N} \sigma(x) \, dx \) -translation invariant extensions of the map \( \mapsto \int R \sigma(x) \, dx \).

We apply this to the symbol \( \sigma \) which vanishes in a small neighborhood of 0 and is identical outside the unit interval. This symbol is of non integer order when \( s \notin Z \) and we have

\[
\sum_{n \in Z \setminus \{0\}} n^{-s} = \int_{R} \sigma_{s}(x) \, dx + C(\sigma_{s}) \quad \forall s \notin Z.
\]

The map \( s \mapsto C(\sigma_{s}) \) turns out to be holomorphic so that \( \lim_{s \to s_{0}} C(\sigma_{s}) = C(\sigma_{s_{0}}) \) and the Riemann zeta function reads:

\[
\zeta(s_{0}) := \text{fp}_{s \to s_{0}} \sum_{n=1}^{\infty} n^{-s} = \frac{1}{2} \text{fp}_{s \to s_{0}} \sum_{n \in Z \setminus \{0\}} \sigma_{s}(n) = \frac{1}{2} \text{fp}_{s \to s_{0}} \int_{R} \sigma_{s}(x) \, dx + \frac{1}{2} C(\sigma_{s_{0}}) \quad \forall s_{0} \in C.
\]

For \( \text{Re}(s) \leq 0 \), the expression \( \overline{\sigma}_{s}(x) = x^{-s} \) is defined on non non negative real numbers and for any complex number \( s_{0} \) with non negative real part, we have \( \zeta(s_{0}) = \frac{1}{2} C(\overline{\sigma}_{s_{0}}) \).
Remark 1 Note that at integer points $s_0$, one is to expect that
\[
\mathcal{F}_{p=s_0} \sum_{n=1}^{\infty} n^{-s} \neq \sum_{n=1}^{\infty} n^{-s_0}.
\]

In particular, taking $s_0 = -(2k+1)$ with $k$ a non negative integer we have
\[
\zeta(-(2k+1)) = -\frac{B_{2k+2}}{2k+2} \neq \sum_{n=1}^{\infty} n^{2k+1} = 0.
\]

The Laplacian $\Delta_{S^1}$ on the unit circle $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ induced by the Laplacian $-\frac{d^2}{dt^2}$ on $\mathbb{R}$ has purely discrete spectrum given by $\{\lambda_n = n^2, n \in \mathbb{Z}\}$ and for $\Re(s) > 1$ we have
\[
\zeta(s) = \frac{1}{2} \sum_{n \in \mathbb{Z} - \{0\}} \lambda_n^s = \frac{1}{2} \text{tr} \left( \Delta_{S^1}^{-\frac{s}{2}} \right).
\]

Laplacians $\Delta_{T^d}$ on $d$-dimensional tori $T^d \simeq \mathbb{R}^d/2\pi\mathbb{Z}^d$ induced by the Laplacian $\Delta_d := -\sum_{i=1}^{d} \frac{d^2}{dx_i^2}$ on $\mathbb{R}^d$ have purely discrete spectrum given by $\{\lambda_{\mathbf{u}} = |\mathbf{u}|^2, \mathbf{u} \in \mathbb{Z}^d\}$ where $|\cdot|$ is the euclidean norm on $\mathbb{R}^d$. Zeta functions associated with Laplacians on tori yield a natural generalisation of the Riemann zeta function:
\[
\zeta_{\Delta_d}(s) = \sum_{\mathbf{u} \in \mathbb{Z}^d - \{0\}} \lambda_{\mathbf{u}}^{-s} = \text{tr} \left( \Delta_{T^d}^{-\frac{s}{2}} \right) \text{ for } \Re(s) > \frac{d}{2}.
\]

The subsequent constructions lead to a generalisation of the one dimensional results described above to higher dimensions providing along the way a meromorphic extension $z \mapsto \sum_{\mathbf{u} \in \mathbb{Z}^d - \{0\}} \lambda_{\mathbf{u}}^{-s-z/2}$ of $\sum_{\mathbf{u} \in \mathbb{Z}^d - \{0\}} \lambda_{\mathbf{u}}^{-s}$ defined for $\Re(s) >> 0$. To carry out this generalisation we strongly rely on the symbolic feature of the eigenvalues $\lambda_{\mathbf{u}}$ of $\Delta_d$ which is rather specific to these operators; a natural question would be to generalise some of these results to any elliptic pseudodifferential operator on a closed manifold with positive order and positive leading symbol, which is probably a difficult issue to handle.
2 A canonical regularised integral and discrete sum on non integer order symbols

2.1 Translation invariant linear forms on non integer order symbols

Given a symbol $\sigma$ in the algebra $C_{\text{c}}(\mathbb{R}^d)$ of classical pseudodifferential symbols with constant coefficients defined in the Appendix, and a vector $p \in \mathbb{R}^n$, the map $p \mapsto t_p^* \sigma := \sigma (\cdot + p)$ has the following Taylor expansion at $p = 0$:

$$t_p^* \sigma (x) := \sum_{|\beta| \leq N} \frac{\partial^\beta \sigma (x)}{|\beta|!} p^\beta + \sum_{|\beta| = N+1} \frac{p^\beta}{|\beta|!} \int_0^1 (1 - t)^N \partial^\beta \sigma (x + tp) \, dt \quad \forall x \in \mathbb{R}^d.$$  \hspace{1cm} (5)

The Riemann integral and discrete sum induce two linear forms $\int_{\mathbb{R}^d} : C_{\text{s}}^{< -d}(\mathbb{R}^d) \rightarrow \mathbb{C}$ and $\sum_{\mathbb{Z}^d} : C_{\text{s}}^{< -d}(\mathbb{R}^d) \rightarrow \mathbb{C}$ on the subalgebra $C_{\text{s}}^{< -d}(\mathbb{R}^d)$ of classical pseudodifferential symbols the order of which has real part $<-d$ (defined in the Appendix) which satisfy the following translation invariance property:

$$\int_{\mathbb{R}^d} t_p^* \sigma = \int_{\mathbb{R}^d} \sigma \quad \forall p \in \mathbb{R}^d; \quad \sum_{\mathbb{Z}^d} t_p^* \sigma = \sum_{\mathbb{Z}^d} \sigma \quad \forall p \in \mathbb{Z}^d$$

so that

$$\sum_{0 < |\beta| \leq N} \left( \int_{\mathbb{R}^d} \partial^\beta \sigma (x) \, dx \right) \frac{p^\beta}{|\beta|!} + \sum_{|\beta| = N+1} \frac{p^\beta}{|\beta|!} \int_0^1 (1 - t)^N \left( \int_{\mathbb{R}^d} \partial^\beta \sigma (x + tp) \, dx \right) \, dt = 0 \quad \forall p \in \mathbb{R}^d,$$

and similarly replacing the integral by a discrete sum over $\mathbb{Z}^d$ and letting $p$ vary in $\mathbb{Z}^d$. Differentiating this identity in the components of $p$ at $p = 0$ we infer that

$$\int_{\mathbb{R}^d} \partial^\beta \sigma (x) \, dx = 0 \quad \forall \beta \neq 0,$$

and thereby recover the fact that the ordinary integral vanishes on derivatives of symbols of order $<-d$.

Let us further extend the notion of translation invariant linear form to the set $C_{\text{s}}^{< -d}(\mathbb{R}^d)$ of non integer order symbols defined in formula (26) of the Appendix. To do so, we observe that for $|\beta|$ large enough, the map $x \mapsto \partial^\beta \sigma (x + tp)$ lies in $L^1(\mathbb{R}^d)$.

**Definition 1** Given a linear form $\rho : L^1(\mathbb{R}^d) \rightarrow \mathbb{C}$, such that

$$\rho \left( t_p^* \sigma \right) = \sum_{|\beta| \leq N} \rho (\partial^\beta \sigma) \frac{p^\beta}{|\beta|!} + \sum_{|\beta| = N+1} \frac{p^\beta}{|\beta|!} \int_0^1 (1 - t)^N \rho (\partial^\beta \sigma (\cdot + tp)) \, dt \quad \forall p \in \mathbb{R}^d,$$  \hspace{1cm} (6)

a linear form $\lambda : C_{\text{s}}^{< -d}(\mathbb{R}^d) \rightarrow \mathbb{C}$ which restricts to $\rho$ on $C_{\text{s}}^{< -d}(\mathbb{R}^d) \cap C_{\text{s}}^{< -n}(\mathbb{R}^d)$.

---

| 7 | }
• is extended to translated symbols by:

\[ t_\mathbf{L}^* \lambda (\sigma) := \lambda (t_\mathbf{L}^* \sigma) = \sum_{|\beta| \leq N} \lambda (\partial^\beta \sigma) \frac{\rho^\beta}{\beta!} + \sum_{|\beta| = N+1} \frac{\rho^\beta}{\beta!} \int_0^1 (1 - t)^N \rho (\partial^\beta \sigma(\cdot + t\sigma)) \, dt, \]

for any large enough integer \( N \).

• The linear form \( \lambda \) is said to be \( \mathbb{R}^d \) (resp. \( \mathbb{Z}^d \)) -translation invariant whenever for any large enough integer \( N \), for any \( \mathbf{p} \in \mathbb{R}^d \) (resp. for any \( \mathbf{p} \in \mathbb{Z}^d \)) we have:

\[ \sum_{0 < |\beta| \leq N} \lambda (\partial^\beta \sigma) \frac{\rho^\beta}{\beta!} + \sum_{|\beta| = N+1} \frac{\rho^\beta}{\beta!} \int_0^1 (1 - t)^N \rho (\partial^\beta \sigma(\cdot + t\sigma)) \, dt = 0. \]

**Remark 2** These definitions are independent of the choice of the integer \( N \) as a result of \( \mathbb{R}^d \) and the fact that \( \lambda \) coincides with \( \rho \) on \( CS_{\mathbb{Z}_c}^\leq (\mathbb{R}^d) \).

The following proposition shows that an \( \mathbb{R}^d \) or \( \mathbb{Z}^d \)-translation invariant linear extension of a linear form \( \rho \) on \( L^1 \)-symbols to non integer order symbols, is entirely determined by \( \rho \).

**Proposition 1** Let \( \lambda : CS_{\mathbb{Z}_c}^\geq (\mathbb{R}^d) \rightarrow \mathbb{C} \) be a linear form which restricts on \( CS_{\mathbb{Z}_c}^\geq (\mathbb{R}^d) \cap CS_{\mathbb{Z}_c}^\leq (\mathbb{R}^d) \) to a linear form \( \rho : L^1(\mathbb{R}^d) \rightarrow \mathbb{C} \) satisfying \( \Box \).

1. If \( \lambda \) is \( \mathbb{R}^d \)-translation invariant, it vanishes on partial derivatives.

2. If \( \lambda \) is \( \mathbb{Z}^d \)-translation invariant, its value on derivatives \( \lambda (\partial^\beta \sigma) \), \( \beta \neq 0, \sigma \in CS_{\mathbb{Z}_c}^\leq (\mathbb{R}^d) \) is entirely determined by the restriction \( \rho \) to \( CS_{\mathbb{Z}_c}^\leq (\mathbb{R}^d) \).

**Proof:**

1. Differentiating the identity

\[ \sum_{0 < |\beta| \leq N} \lambda (\partial^\beta \sigma) \frac{\rho^\beta}{\beta!} + \sum_{|\beta| = N+1} \frac{\rho^\beta}{\beta!} \int_0^1 (1 - t)^N \rho (\partial^\beta \sigma(\cdot + t\sigma)) \, dt = 0 \]

with respect to the coordinates of \( \mathbf{p} \) at \( \mathbf{p} = 0 \) yields the first part of the assertion.

2. Let \( \lambda_1 \) and \( \lambda_2 \) be two \( \mathbb{Z}^d \)-translation invariant linear forms on \( CS_{\mathbb{Z}_c}^\geq (\mathbb{R}^d) \) satisfying the assumptions of the theorem with the same restriction \( \rho \). The Taylor formula \( \Box \) applied to \( \sigma \in CS_{\mathbb{Z}_c}^\geq (\mathbb{R}^d) \) yields by linearity of \( \lambda_i \) and for \( N \) chosen large enough:

\[ \lambda_i (t_\mathbf{L}^* \sigma) = \sum_{|\beta| \leq N} \lambda_i (\partial^\beta \sigma) \frac{\rho^\beta}{\beta!} + \sum_{|\beta| = N+1} \frac{\rho^\beta}{\beta!} \int_0^1 (1 - t)^N \rho (\partial^\beta \sigma(\cdot + t\sigma)) \, dt, \]

so that \( t_\mathbf{L}^* \lambda_1 (\sigma) - \sum_{|\beta| \leq N} \lambda_1 (\partial^\beta \sigma) \frac{\rho^\beta}{\beta!} = t_\mathbf{L}^* \lambda_2 (\sigma) - \sum_{|\beta| \leq N} \lambda_2 (\partial^\beta \sigma) \frac{\rho^\beta}{\beta!} \quad \forall \mathbf{p} \in \mathbb{Z}^d \)

since \( \lambda_1 \) and \( \lambda_2 \) both coincide with \( \rho \) on \( CS_{\mathbb{Z}_c}^\leq (\mathbb{R}^d) \cap CS_{\mathbb{Z}_c}^\geq (\mathbb{R}^d) \). Since \( t_\mathbf{L}^* \lambda_i = \lambda_i \) for any \( \mathbf{p} \in \mathbb{Z}^d \), this implies that the polynomial expressions \( \sum_{0 < |\beta| \leq N} \lambda_1 (\partial^\beta \sigma) \frac{\rho^\beta}{\beta!} \)
and \( \sum_{0<|\beta|\leq N} \lambda_2(\partial^\beta\sigma) \frac{p^\beta}{|p|} \) in the coordinates of \( p \) coincide for all \( p \in \mathbb{Z}^d \) and hence that their coefficients coincide \( \lambda_1(\partial^\beta\sigma) = \lambda_2(\partial^\beta\sigma) \) when \( 0 < |\beta| < N \). Since this holds for any large enough \( N \), we conclude that \( \lambda_1(\partial^\beta\sigma) = \lambda_2(\partial^\beta\sigma) \) when \( \beta \neq 0 \). It follows that its value on derivatives \( \lambda(\partial^\beta\sigma) \), \( \beta \neq 0, \sigma \in CS_{\mathbb{Z}_{\mathbb{C}}}(\mathbb{R}^d) \) is entirely determined by the restriction \( \rho \) to \( CS_{\mathbb{Z}_{\mathbb{C}}}(\mathbb{R}^d) \).

\[ \blacksquare \]

### 2.2 The canonical regularised integral and discrete sum

In order to determine all translation invariant linear forms on \( CS_{\mathbb{Z}_{\mathbb{C}}}(\mathbb{R}^d) \) that extend the integral and discrete sum on \( L^1(\mathbb{R}^d) \) we need the following lemma which collects results from [FGLS] (see Lemma 1.3); see also [P2].

**Lemma 1** Any symbol \( \sigma \in CS_{\mathbb{Z}_{\mathbb{C}}}(\mathbb{R}^d) \) is up to some smoothing symbol, a finite sum of partial derivatives, i.e. there exist symbols \( \tau_i \in CS_{\mathbb{Z}_{\mathbb{C}}}(\mathbb{R}^d), i = 1, \cdots, n \) such that

\[ \sigma \sim \sum_{i=1}^{n} \partial_i \tau_i. \quad (7) \]

The following theorem shows that translation invariant linear forms on non integer order symbols are entirely determined by their restriction to \( L^1 \)-symbols.

**Theorem 1** Let \( \lambda : CS_{\mathbb{Z}_{\mathbb{C}}}(\mathbb{R}^d) \rightarrow \mathbb{C} \) be a linear form which coincides on \( CS_{\mathbb{Z}_{\mathbb{C}}}(\mathbb{R}^d) \cap CS_{\mathbb{Z}_{\mathbb{C}}}(\mathbb{R}^d) \) with a linear form \( \rho : L^1(\mathbb{R}^d) \rightarrow \mathbb{C} \) satisfying \([0]\). If \( \lambda \) is either \( \mathbb{Z}^d \) or \( \mathbb{R}^d \)-translation invariant, then it is entirely determined by its restriction \( \rho \).

**Remark 3** \( \mathbb{R}^d \)-translation invariance and its relation to Stokes’ property had already been investigated in [P2].

**Proof:** Let \( \lambda \) be an \( \mathbb{R}^d \) or \( \mathbb{Z}^d \)-translation invariant linear form on \( CS_{\mathbb{Z}_{\mathbb{C}}}(\mathbb{R}^d) \) satisfying the assumptions of the theorem. Since a symbol \( \sigma \in CS_{\mathbb{Z}_{\mathbb{C}}}(\mathbb{R}^d) \) has vanishing residue by Lemma 1 up to a smoothing symbol, it is a finite linear combination of partial derivatives of symbols in \( CS_{\mathbb{Z}_{\mathbb{C}}}(\mathbb{R}^d) \). With the notations of the lemma we write \( \sigma = \sum_{i=1}^{n} \partial_i \tau_i + r \) for some smoothing symbol \( r \) on which by assumption, the linear form \( \lambda \) coincides with its restriction \( \rho \) to \( CS_{\mathbb{Z}_{\mathbb{C}}}(\mathbb{R}^d) \). Thus, by linearity

\[ \lambda(\sigma) = \sum_{i=1}^{n} \lambda(\partial_i \tau_i) + \rho(r). \]

But by Proposition 1 the value of \( \lambda \) on derivatives is entirely determined by its restriction \( \rho \). It follows that \( \lambda \) on \( CS_{\mathbb{Z}_{\mathbb{C}}}(\mathbb{R}^d) \) is entirely determined by its restriction \( \rho \) as announced in the theorem. \( \blacksquare \)

On the grounds of Theorem 1 we define the canonical regularised integral over \( \mathbb{R}^d \) and the canonical regularised sum over \( \mathbb{Z}^d \) on non integer order symbols.

**Definition 2** Let the canonical regularised integral \( \int_{\mathbb{R}^d} \) (resp. the canonical regularised sum \( \sum_{\mathbb{Z}^d} \)) be the unique \( \mathbb{R}^d \) (resp. \( \mathbb{Z}^d \)) -translation invariant extension of the Riemann integral \( \int_{\mathbb{R}^d} \) (resp. sum \( \sum_{\mathbb{Z}^d} \)) to \( CS_{\mathbb{Z}_{\mathbb{C}}}(\mathbb{R}^d) \).
3 Concrete realisations of the canonical regularised integral and discrete sum

3.1 Integrals of symbols over infinite convex polytopes

We recall from [P2] that the Hadamard finite part integral

$$\sigma \mapsto \lim_{R \to \infty} \int_{B(0,R)} \sigma(x) \, dx$$

defines an \(\mathbb{R}^d\)-translation invariant linear form on \(CS_{\mathbb{Z}^c}(\mathbb{R}^d)\) which extends the Riemann integral on \(L^1\) symbols.

By Theorem 1 it therefore coincides with the canonical integral

$$\int_{\mathbb{R}^n} \sigma = \lim_{R \to \infty} \int_{B(0,R)} \sigma(x) \, dx \quad \forall \sigma \in CS_{\mathbb{Z}^c}(\mathbb{R}^d).$$

**Remark 4** For general symbol \(\sigma \in CS_{\mathbb{Z}^c}(\mathbb{R}^d)\), the expression \(\lim_{R \to \infty} \int_{B(0,R)} \sigma(x) \, dx\) depends on the choice of the parameter \(R\) but for convenience and following the notations of [L] we extend this notation to all symbols setting

$$\int_{\mathbb{R}^n} \sigma = \lim_{R \to \infty} \int_{B(0,R)} \sigma(x) \, dx$$

for any \(\sigma \in CS_{\mathbb{Z}^c}(\mathbb{R}^d)\), which we refer to as the cut-off regularised integral of \(\sigma\) as recalled in formula (28) of the Appendix.

In the following, we show that the canonical regularised integral on non integer order symbols can also be realised as a finite part as \(R\) tends to infinity of an integral over an \(R\)-expanded polytope.

Let us first introduce some notations, following those of [GSW], see also [KSW2].

A compact convex polytope \(\Delta\) in \(\mathbb{R}^d\) is a compact set which can be obtained as the intersection of finitely many half-spaces \(\Delta = \bigcap_{i=1}^m H_i\) where

$$H_i = \{x \in \mathbb{R}^d, \langle u_i, x \rangle + a_i \geq 0\}$$

for some vectors \(u_i \in \mathbb{R}^d, i = 1, \cdots, m\) and some positive real numbers \(a_i, i = 1, \cdots, m\).

Assuming that \(m\) is the smallest possible number of such half spaces, the facets or codimension one faces of \(\Delta\) are the \(\sigma_i = \Delta \cap \partial H_i, i = 1, \cdots, m\). The vector \(u_i\) can be thought of as an inward normal vector to the \(i\)-th facet \(\sigma_i\).

Alternatively, a compact convex polytope is the convex hull of a finite set of points in \(\mathbb{R}^d\) and when this set is minimal, its elements are called vertices of \(\Delta\).

A compact convex polytope is *simple* if each vertex is the intersection of exactly \(d\) facets. It is called *integral* or *lattice polytope* if its vertices lie in the lattice \(\mathbb{Z}^d\). It is *regular* if moreover, the edges emanating for each vertex lie along lines generated by a \(\mathbb{Z}\)-basis of the lattice \(\mathbb{Z}^d\) or equivalently if each local cone at any of its vertices can be transformed by an integral unimodular affine transformation to a neighborhood of 0.

Let \(\Delta \subset \mathbb{R}^d\) be a simple, integral and regular compact convex polytope and such that the origin 0 lies in the interior of \(\Delta\). Since \(\Delta\) is integral, the vectors \(u_i, i = 1, \cdots, m\) can be chosen to belong to the dual lattice \(\mathbb{Z}^d\) and they can be normalized to be *primitive lattice elements* i.e. not be a multiple of a lattice element by an integer larger than one.

As pointed out in [KSW2], the fact that a normal vector \(\nu\) to a facet \(\sigma\) can be chosen to be integral is a consequence of Cramer’s rule; indeed, choosing integral edge vectors \(\beta_1, \cdots, \beta_{d-1}\) that emanate from a vertex on the facet \(\sigma\) and such that \(\beta_1, \cdots, \beta_{d-1}\) span
the tangent plane to the facet and $\beta_i$ is transverse to $\sigma$, solving the linear equations
\[(u, \beta_i) = \cdots = (u, \beta_{d-1}) = 0 \text{ and } (u, \beta_d) = 1,\]
we get an inward normal vector $u$ with rational entries, from which it is easy to build an integral inward normal vector.

The expanded polytope $R \cdot \Delta$ for any positive real number $R$ is defined by:
\[x \cdot u_i + Ra_i \geq 0, \quad i = 1, \cdots, m.\]

Note that $\bigcup_{N \in \mathbb{N}} N \cdot \Delta = \mathbb{R}^d$.

**Example 1** Let $|\mathbf{x}|_{\text{sup}} := \sup_{i=1}^{d}|x_i|$ denote the supremum norm of $\mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$. The balls $B_{|\mathbf{x}|_{\text{sup}}}(0, R) = \{\mathbf{x} \in \mathbb{R}^d, |\mathbf{x}|_{\text{sup}} \leq R\}$ for the supremum norm are defined by the inequalities:
\[-R \leq \langle \mathbf{x}, e_i \rangle \leq R, \quad i = 1, \cdots, d\]
where $(e_1, \cdots, e_d)$ is the canonical orthonormal basis in $\mathbb{R}^d$. Setting $m = 2d$, $u_i = e_i, i = 1, \cdots, d$, $a_i = 1, i = 1, \cdots, d$, $u_{i+d} = -e_i, i = 1, \cdots, d$, $a_{i+d} = 1, i = 1, \cdots, d$, we recover the above set of inequalities which define an expanded simple integral regular (compact convex) polytope.

For a continuous function $f$ on $\mathbb{R}^d$ and any $R > 0$ we set:
\[\tilde{P}_R \Delta(f) := \int_{R \cdot \Delta} f(\mathbf{x}) d\mathbf{x}.\]

**Proposition 2** Let $\sigma \in CS_{C_c}(\mathbb{R}^d)$ for some complex number $a$.

1. The map $R \mapsto \tilde{P}_R \Delta(\sigma)$ defines a log-polynomial symbol of order $a + d$ and log-type 1 on $\mathbb{R}^+$ which differs from the log-polynomial symbol $R \mapsto \int_{B(0,R)} \sigma(\mathbf{x}) d\mathbf{x}$ by a classical symbol of order $a + d$.

2. The constant term $fp_{R \to \infty} \tilde{P}_R \Delta(\sigma)$ in its asymptotic expansion relates to the finite part $fp_{R \to \infty} \int_{B(0,R)} \sigma(\mathbf{x}) d\mathbf{x}$ as follows
\[fp_{R \to \infty} \tilde{P}_R \Delta(\sigma) - fp_{R \to \infty} \int_{B(0,R)} \sigma(\mathbf{x}) d\mathbf{x} = \int_{\Delta \cap B(0,1)} \sigma_{-d}(\mathbf{x}) d\mathbf{x} \quad \forall \sigma \in CS_{C_c}(\mathbb{R}^d),\]
with $A \cap B := (A - (A \cap B)) \cup (B - (A \cap B))$ the symmetric difference.

**Remark 5** In particular for any $\sigma \in CS_{C_c}(\mathbb{R}^d)$ the map $R \mapsto \int_{|\mathbf{x}|_{\text{sup}} \leq R} \sigma(\mathbf{x}) d\mathbf{x}$ defines a log-polynomial symbol of order $a + d$ and log-type 1 and if $\sigma$ has non integer order we have
\[fp_{R \to \infty} \int_{|\mathbf{x}|_{\text{sup}} \leq R} \sigma(\mathbf{x}) d\mathbf{x} = fp_{R \to \infty} \int_{B(0,R)} \sigma(\mathbf{x}) d\mathbf{x} \quad \forall \sigma \in CS_{C_c}^{\mathbb{Z}}(\mathbb{R}^d). \quad (8)\]

**Proof:** By formula (27) of the Appendix, the map $R \mapsto \int_{B(0,R)} \sigma(\mathbf{x}) d\mathbf{x}$ defines a log-polynomial symbol of order $a + d$ and log-type 1. Let us estimate the difference.

---

6 We do not use the usual notation $A \Delta B$ to avoid a clash of notations with the polytope commonly denoted in the literature by $\Delta$. 

11
defines a classical symbol by (9). Its finite part differs from for large $R$:

$$
\int_{R} \sigma(x) \, dx - \int_{B(0, R)} \sigma(x) \, dx
$$

$$
= \sum_{j=0}^{N} \int_{R \Delta \ominus B(0, R)} \chi(x) \sigma_{a-j}(x) \, dx + \int_{R \Delta \ominus B(0, R)} \sigma_{(N)}(x) \, dx
$$

$$
= \sum_{j=0}^{N} \int_{R \Delta \ominus B(0, R)} \sigma_{a-j}(x) \, dx + \int_{R \Delta \ominus B(0, R)} \sigma_{(N)}(x) \, dx
$$

(9)

where we use the fact that $\chi$ is one outside the unit euclidean ball. Since $\sigma_{(N)}$ is a symbol of order $a - N$ we have:

$$
|\sigma_{(N)}(x)| \leq C(1 + |x|^{2})^{|\text{Re}(a) - N|}
$$

for some constant $C$. Hence, for large enough $N$

$$
\left| \int_{R \Delta \ominus (R \Delta \cap B(0, R))} \sigma_{(N)}(x) \, dx \right| \leq C (1 + R^{2})^{|\text{Re}(a) - N|} \text{Vol}(R \Delta - (R \Delta \cap B(0, R)))
$$

$$
\leq C R^{d}(1 + R^{2})^{|\text{Re}(a) - N|} \text{Vol}(\Delta)
$$

$$
\leq C (1 + R^{2})^{|\text{Re}(a) - N|} \text{Vol}(\Delta).
$$

Using once more the fact that $\sigma_{(N)}$ is a symbol of order $a - N$ combined with the equivalence of the supremum and the euclidean norms we also have:

$$
|\sigma_{(N)}(x)| \leq C' (1 + \|x\|_{\text{sup}})^{\text{Re}(a) - N}
$$

for some constant $C'$ and

$$
| \int_{B(0, R) \ominus (R \Delta \cap B(0, R))} \sigma_{(N)}(x) \, dx | \leq C' (1 + d R)^{|\text{Re}(a) - N|} \text{Vol}(B(0, R) - (R \Delta \cap B(0, R)))
$$

$$
\leq C' R^{d}(1 + R)^{|\text{Re}(a) - N|} \text{Vol}(B(0, 1))
$$

$$
\leq C' (1 + R)^{|\text{Re}(a) + d - N|} \text{Vol}(B(0, 1))
$$

Consequently, we can choose $N$ sufficiently large so that

$$
\int_{R \Delta \ominus B(0, R)} \sigma_{(N)}(x) \, dx = O((1 + R)^{|\text{Re}(a) + d - N|}).
$$

This settles the case of integrals involving the remainder term $\sigma_{(N)}$. As for integrals of homogeneous symbols $\int_{R \Delta \ominus B(0, R)} \sigma_{a-j}(x) \, dx$, we have

$$
\int_{R \Delta \ominus B(0, R)} \sigma_{a-j}(x) \, dx = \int_{\Delta \ominus B(0, 1)} \sigma_{a-j}(R y) \, dy = R^{a-j+d} \int_{\Delta \ominus B(0, 1)} \sigma_{a-j}(y) \, dy,
$$

which shows they are homogeneous of degree $a - j + d$.

Combining these results shows that $R \mapsto \int_{R \Delta \ominus B(0, R)} \sigma(x) \, dx$ defines a classical symbol of order $a + d$ with constant term given by

$$
\text{fp}_{R \rightarrow \infty} \int_{R \Delta \ominus B(0, R)} \sigma(x) \, dx = \int_{\Delta \ominus B(0, 1)} \sigma_{-d}(y) \, dy.
$$

Since $R \mapsto \int_{B(0, R)} \sigma(x) \, dx$ defines a log-polyhomogeneous symbol of order $a + d$ and log-type 1, so does the map $R \mapsto \int_{R \Delta} \sigma(x) \, dx$ by \[\Box\]. Its finite part differs from $\text{fp}_{R \rightarrow \infty} \int_{B(0, R)} \sigma(x) \, dx$ by $\text{fp}_{R \rightarrow \infty} \int_{R \Delta \ominus B(0, R)} \sigma(x) \, dx = \int_{\Delta \ominus B(0, 1)} \sigma_{-d}(y) \, dy$. \(\square\)
3.2 Discrete sums of classical symbols on infinite convex polytopes

With notations similar to those of [GSW] and [KSW2], for any $R > 0$ we define the dilated polytope $\Delta_{R,h} \subset \mathbb{R}^d$ obtained by shifting the $i$-th facet of $\Delta_R$ outwards by a distance $h_i$. It is described by the inequalities:

$$\langle x, u_i \rangle + Ra_i + h_i \geq 0, \quad i = 1, \ldots, m$$

with $h = (h_1, \ldots, h_m) \in \mathbb{R}^m$.

When $\Delta$ is simple, so is $\Delta(h)$ simple for $h$ small enough. For $h = 0$, $\Delta_{R,0}$ yields back the expanded polytope $\mathcal{R}_R$ and for $R = 0$ we set $\Delta_h := \Delta_{0,h}$. For a function $f$ on $\mathbb{R}^d$ and any $R > 0$, any $h \in \mathbb{R}^2d$ we further set:

$$P_{R,\Delta}(f)(h) := \sum_{\Delta_{R,h} \cap \mathbb{Z}^d} f(\tilde{x}), \quad \tilde{P}_{R,\Delta}(f)(h) := \int_{\Delta_{R,h}} f(x) \, dx.$$ 

For $h = 0$, the integral, resp. the sum are taken over the expanded polytope $\mathcal{R}_R$, resp. integer points in $\mathcal{R}_R$ and we set $P_{R,\Delta} := P_{R,\Delta}(0)$ similarly to $\tilde{P}_{R,\Delta}(f)(0)$.

Let us first establish a technical lemma which will be useful for what follows.

**Lemma 2** Given a symbol $\sigma \in CS^a_{c,c}(\mathbb{R}^d)$ with complex order $a$ and any positive integer $\gamma$, the map $R \mapsto \partial_h^{\gamma}_i \left( \tilde{P}_{R,\Delta}(\sigma)(h) \right) |_{h=0}$ defines a log-polyhomogeneous symbol on $\mathbb{R}^+$ of order $a + d - \gamma$. Its finite part given by the constant term in the symbolic asymptotic expansion in $R$ as $R \to \infty$ reads:

$$fp_{R \to \infty} \left( \partial_h^{\gamma}_i \tilde{P}_{R,\Delta}(\sigma)(h) \right) |_{h=0} = \left( \partial_h^{\gamma}_i \tilde{P}_{\Delta}(\sigma-d+\gamma_i-1)(h) \right) |_{h=0}$$

and therefore vanishes whenever the order of $\sigma$ is non integer.

**Remark 6** The case $\gamma_i = 0$ was dealt with in Proposition [2] where we showed that $R \mapsto P_{R,\Delta}(\sigma)$ defines a log-polyhomogeneous symbol of order $a + d$ and log-type $1$.

**Proof:**

1. Let us first carry out the proof in the case of a hypercube defined by the following set of inequalities

$$-\beta_i \leq \langle x, e_i \rangle \leq \alpha_i, \quad i = 1, \ldots, d$$

for given positive numbers $\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_d$. Here as before, $\{e_1, \ldots, e_d\}$ stands for the canonical orthonormal basis of $\mathbb{R}^d$. Setting $m = 2d$, we rewrite the inequalities defining $\Delta$ as follows:

$$\langle x, u_i \rangle + a_i \geq 0; \quad \langle x, u_{i+d} \rangle + a_{i+d} \geq 0,$$

where we have set $u_i = -e_i, a_i = \alpha_i, u_{i+d} = e_i, a_{i+d} = \beta_i$. 


For any positive real number \( R \),

\[
\left( \partial_{h_i} \tilde{P}_R \Delta (\sigma)(h) \right)_{h=0} = \left( \int_{-h_i-R_{\beta_1} \leq x_1 \leq h_{1+d}+R_{\alpha_1}} dx_1 \cdots \int_{-h_i-R_{\beta_i} \leq x_i \leq h_{i+d}+R_{\alpha_i}} dx_i \cdots \int_{-h_d-R_{\alpha_d} \leq x_d \leq h_{2d}+R_{\alpha_d}} dx_d \sigma(x) \right)_{h=0}
\]

\[
= (-1)^{\gamma_i} \prod_{j=1, j \neq i}^d \int_{-R_{\beta_j} \leq x_j \leq R_{\alpha_j}} dx_j \partial_i^{\gamma_i-1} \sigma(x_1, \cdots, x_{i-1}, -R_{\beta_i}, x_{i+1}, \cdots, x_d)
\]

is the integral over the face \( x_i = -R_{\beta_i} \) of the expanded polytope \( R \Delta \).

If \( \sigma \in C^a_{\text{c.c.}}(\mathbb{R}^d) \) then \( \tau_i := \partial_i^{\gamma_i-1} \sigma \) defines a classical symbol of order \( a_i = a - \gamma_i + 1 \) on \( \mathbb{R}^d \). With the notations of (24) in the Appendix, let us write

\[
\tau_i(x) = \sum_{k=0}^N \chi(x) \tau_{i,a_i-k}(x) + \tau_{i,N}(x). \quad (10)
\]

Provided \( R \) is chosen large enough, on the face \( x_i = -R_{\beta_i} \) of the expanded polytope we have

\[
\tau_i \left( x_1, \cdots, x_{i-1}, -R_{\beta_i}, x_{i+1}, \cdots, x_d \right)
\]

\[
= \sum_{k=0}^N \tau_{i,a_i-k} \left( x_1, \cdots, x_{i-1}, -R_{\beta_i}, x_{i+1}, \cdots, x_d \right) + \tau_{i,N} \left( x_1, \cdots, x_{i-1}, -R_{\beta_i}, x_{i+1}, \cdots, x_d \right).
\]

Hence,

\[
\prod_{j=1, j \neq i}^d \int_{-R_{\beta_j} \leq x_j \leq R_{\alpha_j}} dx_j \partial_i^{\gamma_i-1} \sigma(x_1, \cdots, x_{i-1}, -R_{\beta_i}, x_{i+1}, \cdots, x_d)
\]

\[
= \sum_{k=0}^N \prod_{j=1, j \neq i}^d \int_{-R_{\beta_j} \leq x_j \leq R_{\alpha_j}} dx_j \tau_{i,a_i-k} \left( x_1, \cdots, x_{i-1}, -R_{\beta_i}, x_{i+1}, \cdots, x_d \right)
\]

\[
+ \prod_{j=1, j \neq i}^d \int_{-R_{\beta_j} \leq x_j \leq R_{\alpha_j}} dx_j \tau_{i,N} \left( x_1, \cdots, x_{i-1}, -R_{\beta_i}, x_{i+1}, \cdots, x_d \right). \quad (11)
\]

Since \( \tau_{i,N} \) is a symbol of order \( a_i - N \), we have

\[
|\tau_{i,N}(x)| \leq C \left( 1 + |x|^2 \right)^{\text{Re}(a_i)-N} \]

for some constant \( C \). On the face of the polytope defined by equation \( x_i = -R_{\beta_i} \), choosing \( N \) large enough for \( \text{Re}(a_i) - N \) to be negative we write

\[
|\tau_{i,N}(x)| \leq C \left( 1 + \beta_i^2 R^2 \right)^{\text{Re}(a_i)-N}.
\]
It follows that there is some large enough $N$ such that

$$\left| \prod_{j=1, j \neq i}^{d} \int_{-R \beta_j \leq x_j \leq R \alpha_j} dx_j \tau_{i,N}(x_1, \ldots, x_{i-1}, -R \beta_i, x_{i+1}, \ldots, x_d) \right|$$

$$\leq C R^d (1 + \beta_i^2 R^2)^{\frac{\text{Re}(a_i) - N}{2}} \prod_{j=1, j \neq i}^{d} \int_{-\beta_j \leq x_j \leq a_j} dx_j$$

$$\leq C' (1 + R^2)^{\frac{\text{Re}(a_i) + d - N}{2}}$$

for some constant $C'$.

Combining these results shows that the map $R \mapsto (\partial_{\bar{h}}^\gamma \tilde{P}_R \Delta(\sigma)(h))_{|h=0}$ defines a classical symbol of order $a_i + d = a + d - \gamma_i + 1$.

Using (11) we see that its finite part as $R$ tends to infinity reads

$$\text{fp}_{R \to \infty} \left( \partial_{\bar{h}}^\gamma \tilde{P}_R \Delta(\sigma)(h) \right)_{|h=0}$$

$$= \prod_{j=1, j \neq i}^{d} \int_{-\beta_j \leq x_j \leq \alpha_j} dx_j \tau_{i,-d}(x_1, \ldots, x_{i-1}, -\beta_i, x_{i+1}, \ldots, x_d)$$

$$= \prod_{j=1, j \neq i}^{d} \int_{-\beta_j \leq x_j \leq \alpha_j} dx_j \left( \partial_{\bar{h}}^{\gamma - 1} \right)^{-d} \partial_{\bar{h}}^\gamma \tilde{P}_R \Delta(\sigma_{d - \gamma_i - 1})(h)_{|h=0}$$

and therefore vanishes if $\sigma$ has non integer order.

2. The proof generalises to any general simple regular integral convex polytope. Indeed, using (10), we split as before the derivative $\left( \partial_{\bar{h}}^\gamma \tilde{P}_R \Delta(\sigma)(h) \right)_{|h=0}$ into a finite sum of homogeneous terms and a remainder term. The faces of the polytope moving out to infinity as $R \to \infty$, the behaviour as $R \to \infty$ is controlled using

- on the one hand, the homogeneity of the derivatives of the integrals of the homogeneous terms which turn out to be integrals of derivatives of the homogeneous components of the symbol over faces of the polytope and

- on the other hand, the symbolic behaviour of the integrand in the remainder term which is an integral over faces of the polytope of a symbol the order of which has arbitrarily negative real part.

\[ \square \]

We now want to compare $P_N \Delta(\sigma)$ and $\tilde{P}_N \Delta(\sigma)$ as $N$ goes to infinity.

Following the notations of [GSW], let

$$\text{Todd}(\partial) := \sum_{\alpha} b_{\alpha} \partial^\alpha$$
More precisely, there are polynomials $M_C$ for some constant $R_N$ where the integral
\[\sum_{\alpha} b_{\alpha} \partial^{\alpha} \]
comparing the discrete sum $P_N \Delta(f)$ with the integral $\tilde{P}_N \Delta(f)$ over the expanded convex polytope $N \Delta$ for polynomial functions $f$:

\[P_N \Delta(f) - \tilde{P}_N \Delta(f) = \left( (\text{Todd} \circ \partial_{h}) - \text{Id} \right) \tilde{P}_N \Delta(f) |_{h=0}.\]

The following proposition is a reformulation of results of \[GSW\] (formula (15), see also \[AW\], \[KSW1\] and \[KSW2\] for previous results along these lines) where the authors generalise the Khovanskii-Pukhlikov formula to classical symbols, in which case the formula is not exact anymore but only holds asymptotically.

**Proposition 3** \[GSW\] Given a symbol $\sigma \in CS^{\alpha}_{\infty}(\mathbb{R}^d)$ with complex order $a$, the discrete map $N \mapsto P_N \Delta(\sigma)$ can be interpolated by a log-polyhomogeneous symbol on $\mathbb{R}^+$ of order $a + d$ and log-type 1:

\[R \mapsto \tilde{P}_R \Delta(\sigma) + (\text{Todd} \circ \partial_{h}) - \text{Id} \right) \tilde{P}_R \Delta(\sigma) |_{h=0} + C(\sigma)\]

for some constant $C(\sigma)$ independent of the choice of expanded polytope, i.e.

\[P_N \Delta(\sigma) - \tilde{P}_N \Delta(\sigma) \sim_{N \to \infty} (\text{Todd} \circ \partial_{h}) - \text{Id} \right) \tilde{P}_N \Delta(\sigma) |_{h=0} + C(\sigma).\]

More precisely, there are polynomials $M^{[j]}$, $j \in \mathbb{N}$ on $\mathbb{R}^d$ such that

\[P_N \Delta(\sigma) - \tilde{P}_N \Delta(\sigma) = \left( (M^{[j]} \circ \partial_{h}) - \text{Id} \right) \tilde{P}_N \Delta(\sigma) |_{h=0} + R^j(\sigma)(N)\]

where

\[R^j(\sigma)(N) := \sum_p (-1)^p \int_{C_{p,N}} \sum_{|\alpha|=j} \phi^p_{\alpha,j}(x) \partial^{\alpha} \sigma(x) dx\]

tends to $C(\sigma)$ as $N \to \infty$.

The $C_{p,N}$ are convex polytopes growing with $N$ and $\phi^p_{\alpha,j}$ bounded piecewise smooth periodic functions as described in \[KSW2\] and \[AW\].

**Remark 7** Let us comment on the interpolation \[12\] which is a reinterpretation of the statements of \[GSW\].

Since we know by Lemma 2 that for any positive $\gamma_i$, the maps $R \mapsto \left( \partial_{h}^{\gamma_i} \tilde{P}_R \Delta(\sigma)(h) \right) |_{h=0}$ define classical symbols on $\mathbb{R}^+$ of order $a + d - \gamma_i$ if $\sigma$ is of order $a$, it follows that the maps $R \mapsto \left( (M^{[j]} \circ \partial_{h}) - \text{Id} \right) \tilde{P}_R \Delta(\sigma)(h) |_{h=0}$ give rise to classical symbols on $\mathbb{R}^+$ of order $a + d - k_j$ for some integer $k_j$ and log-type 1. This combined with \[13\] tells us that \[12\] indeed provides an interpolation of $N \mapsto P_N \Delta(\sigma)$ by a log-polyhomogeneous symbol of order $a + d - \gamma_i$ and log-type 1.

Since the expression $P_N \Delta(\sigma) = \tilde{P}_N \Delta(\sigma) + \left( (M^{[j]} \circ \partial_{h}) - \text{Id} \right) \tilde{P}_N \Delta(\sigma)(h) |_{h=0} + R^j(\sigma)(N)$ has an asymptotic expansion as $N \to \infty$ of the same type as $\tilde{P}_N \Delta(\sigma)$, we can extract the constant term in the expansion and consider the Hadamard finite part $\text{fp}_{N \to \infty} \sum_{N \in \mathbb{Z}^d} \sigma(n)$. 

16
Remark 8 Clearly, when $\sigma \in CS^{<d}(\mathbb{R}^d)$ then $f_{\mathbb{P}}_{N \to \infty} \sum_{N \Delta \cap \mathbb{Z}^d} \sigma(n) = \lim_{N \to \infty} \sum_{N \Delta \cap \mathbb{Z}^d} \sigma(n) = \sum_{\mathbb{Z}^d} \sigma(n)$ is an ordinary limit.

Example 2 Given a polynomial $Q(x) = \sum_{|\alpha| \leq K} a_\alpha x^\alpha$ on $\mathbb{R}^d$, $a_\alpha \in \mathbb{C}$ then:

$$f_{\mathbb{P}}_{N \to \infty} \sum_{|\alpha| \leq K} a_\alpha x^\alpha = 0.$$

Indeed, we have

$$\sum_{|\alpha| \leq K} a_\alpha \prod_{i=1}^{d} n_i^{|\alpha_i|}.$$

Since the one dimensional sum $N \to \sum_{N} n_i^{|\alpha_i|}$ is known to be polynomial in $N$ (this easily follows from the Euler-MacLaurin formula, see e.g. [MP]), its finite part as $N \to \infty$ coincides with its value at $N = 0$:

$$f_{\mathbb{P}}_{N \to \infty} \sum_{|\alpha| \leq K} a_\alpha \prod_{i=1}^{d} \left( \lim_{N \to \infty} \sum_{N} n_i^{|\alpha_i|} \right).$$

One easily checks that this last expression $f_{\mathbb{P}}_{N \to \infty} \sum_{N} n_i^{|\alpha_i|}$ vanishes as a result of the fact that $f_{\mathbb{P}}_{N = 0} \sum_{N = -N} P(n) = 0$ for any polynomial $P$ [MP].

Theorem 2 The Hadamard finite part $\sigma \mapsto f_{\mathbb{P}}_{N \to \infty} \sum_{N \Delta \cap \mathbb{Z}^d} \sigma(n)$ coincides with the canonical regularised sum on non integer order symbols and we have:

$$f_{\mathbb{P}}_{N \to \infty} \sum_{N \Delta \cap \mathbb{Z}^d} \sigma(n) = \int_{\mathbb{R}^d} \sigma + C(\sigma) \quad \forall \sigma \in CS_{c.c}^{\mathbb{Z}}(\mathbb{R}^d). \quad (14)$$

In particular, it is translation invariant and independent of the choice of the convex polytope $\Delta$ on non integer order symbols. Consequently, the map $\sigma \mapsto C(\sigma) = \sum_{\mathbb{Z}^d} \sigma - f_{\mathbb{R}^d} \sigma$ is also translation invariant on non integer order symbols i.e.

$$C(t_{\mathbb{Z}}^\sigma) = C(\sigma) \quad \forall t \in \mathbb{Z}^d, \quad \forall \sigma \in CS_{c.c}^{\mathbb{Z}}(\mathbb{R}^d).$$

Remark 9 For a general symbol $\sigma \in CS_{c.c}(\mathbb{R}^d)$ $f_{\mathbb{P}}_{N \to \infty} \sum_{N \Delta \cap \mathbb{Z}^d} \sigma(n)$ depends on the choice of the polytope $\Delta$. As in [MP] we choose the polytope to be a hypercube and set

$$\sum_{\mathbb{Z}^d} \sigma := f_{\mathbb{P}}_{N \to \infty} \sum_{|n|_{\sup} \leq N} \sigma(n)$$

which we refer to as the regularised cut-off sum of $\sigma$ on $\mathbb{Z}^d$ in analogy with the similar terminology used for integrals. Note that in [MP] this notation had been introduced only for radial functions $f(x) = \sigma(|x|_{\sup})$ where $\sigma$ is a classical symbol on $\mathbb{R}^d$.

Proof: Since the Hadamard finite part $f_{\mathbb{P}}_{N \to \infty} \sum_{N \Delta \cap \mathbb{Z}^d} \sigma(n)$ coincides with the ordinary sum $\sum_{n \in \mathbb{Z}^d} \sigma(n)$ whenever $\sigma \in CS_{c.c}^{<d}(\mathbb{R}^d)$, by Theorem 1 all we need to check is the $\mathbb{Z}^d$-translation invariance on non integer order symbols, i.e.

$$f_{\mathbb{P}}_{N \to \infty} \sum_{N \Delta \cap \mathbb{Z}^d} t_{\mathbb{Z}}^n \sigma(n) = f_{\mathbb{P}}_{N \to \infty} \sum_{N \Delta \cap \mathbb{Z}^d} \sigma(n) \quad \forall \sigma \in CS_{c.c}^{\mathbb{Z}}(\mathbb{R}^d) \quad \forall t \in \mathbb{Z}^d.$$
1. Let us first show that for a symbol $\sigma$ of non integer order, the Hadamard finite part $\text{fp}_{N \to \infty} P_{N \Delta}(\sigma)$ is independent of the polytope one is expanding. This follows from results of [GSW] recalled in Proposition [3] by which the constant $C(\sigma)$ is independent of the polytope one is expanding. Since by Lemma 2, the finite parts $\text{fp}_{R \to \infty} \left( (M[\delta])(\partial h) - Id \right) \hat{P}_R \Delta(\sigma)(h) \big|_{h=0}$ vanish whenever the symbol has non integer order, we infer from (13) that

$$\text{fp}_{R \to \infty} P_{R \Delta}(\sigma) = \text{fp}_{R \to \infty} \hat{P}_R \Delta(\sigma) + C(\sigma) = \int_{\mathbb{R}^d} \sigma + C(\sigma)$$

is independent of $\Delta$.

2. We now show translation invariance of the map $\sigma \mapsto \text{fp}_{\mathbb{R} \to \infty} P_{\mathbb{R} \Delta}(\sigma)$ on non integer order symbols. We write $\sum_{N \Delta \cap \mathbb{Z}^d} t^*_p \sigma(n) = \sum_{N \Delta \cap \mathbb{Z}^d} t^*_{-p}(N \Delta) \cap \mathbb{Z}^d \sigma(n)$. Let $N$ be an integer chosen large enough so that the translated expanded polytope $t_{-p}(N \Delta)$, which is a simple regular polytope contains 0 in its interior. As in the case of the original expanded polytope $N \Delta$, its faces move out to infinity as $N \to \infty$. We can therefore implement the same proof as in Lemma 2 using the symbolic properties of $\sigma$ as well as the fact that it has non integer order, we check that $\text{fp}_{N \to \infty} \partial_{h_0} \hat{P}_{t_{-p}(N \Delta)}(\sigma)(h) \big|_{h=0} = 0$. By the Khovanskii-Pukhlikov formula, we infer from there that

$$\sum_{\mathbb{Z}^d} t^*_p \sigma := \text{fp}_{N \to \infty} P_{t_{-p}(N \Delta)}(\sigma) = \text{fp}_{N \to \infty} \hat{P}_{t_{-p}(N \Delta)}(\sigma) + C(\sigma) = \int_{\mathbb{R}^d} \sigma + C(\sigma) = \sum_{\mathbb{Z}^d} \sigma.$$

3. Since $\int_{\mathbb{R}^d} \sigma + C(\sigma) = \sum_{\mathbb{Z}^d} \sigma$ and since both $\sum_{\mathbb{Z}^d}$ and $\int_{\mathbb{R}^d}$ are invariant under translation by $p \in \mathbb{Z}^d$ on non integer order symbols, so is the map $\sigma \mapsto C(\sigma)$, consequently

$$C \left( t^*_p \sigma \right) = C(\sigma) \quad \forall \sigma \in CS_{\mathbb{R}^d}^\mathbb{Z}(\mathbb{R}) \quad \forall p \in \mathbb{Z}^d$$

which shows [14].

\[\square\]

---

Note that it does not coincide with the expanded translated polytope $N \cdot t_{-p}(\Delta)$. 

18
4 Sums of holomorphic symbols on $\mathbb{Z}^d$

Following [KV] we slightly generalise the notion of gauged symbols used in [GSW] (who follow the terminology introduced by V. Guillemin) in as far as we allow any non constant affine order whereas gauged symbols have affine order with derivative equal to 1.

**Definition 3** We call local holomorphic regularisation of a symbol $\sigma \in CS_{c.c.}(\mathbb{R}^d)$ at zero any holomorphic family $\mathcal{R}(\sigma)(z) : z \mapsto \sigma(z) \in CS_{c.c.}(\mathbb{R}^d)$ in a neighborhood of zero such that $\sigma(0) = \sigma$ and $\sigma(z)$ has non constant affine order $\alpha(z)$.

**Example 3** A Riesz perturbation $\mathcal{R}(\sigma)(z)(x) := \chi(x) \sigma(x) \lvert x \rvert^z + (1 - \chi(x)) \sigma(x)$ for some smooth cut-off function $\chi$ which vanishes in a neighborhood of zero and is one outside the unit ball, is a local holomorphic regularisation at zero. Note that $\sigma'(z)(x) = \chi(x) \sigma(x) \log \lvert x \rvert \lvert x \rvert^z$ vanishes on the unit sphere.

**Theorem 3** Let $\mathcal{R}(\sigma)(z) := \sigma(z)$ be a local holomorphic regularisation of $\sigma \in CS_{c.c.}(\mathbb{R}^d)$ at zero with order $\alpha(z)$ then the map

$$z \mapsto \sum_{\mathbb{Z}^d} \sigma(z)(\mathbf{n})$$

is meromorphic with a discrete of simple poles in $\alpha^{-1}([-d, \infty) \cap \mathbb{Z})$ and complex residue at $z = 0$ given by:

$$\text{Res}_{z=0} \sum_{\mathbb{Z}^d} \sigma(z)(\mathbf{n}) = -\frac{1}{\alpha'(0)} \text{res}(\sigma(0)).$$

The constant term in the Laurent series at $z = 0$

$$\sum_{\mathbb{Z}^d} \sigma(\mathbf{n}) := \lim_{z \to 0} \sum_{\mathbb{Z}^d} \sigma(z)(\mathbf{n})$$

reads

$$\sum_{\mathbb{Z}^d} \sigma(\mathbf{n}) = \oint_{\mathbb{R}^d} \sigma(x) \, dx + C(\sigma),$$

where we have set $\oint_{\mathbb{R}^d} \sigma(x) \, dx = \lim_{z \to 0} \oint_{\mathbb{R}^d} \sigma(z)(x) \, dx$.

Whenever the order of $\sigma$ has real part $< -d$ (resp. is non integer), the map $z \mapsto \sum_{\mathbb{Z}^d} \sigma(z)(\mathbf{n})$ is holomorphic at $z = 0$ and converges to the ordinary sum $\sum_{\mathbb{Z}^d} \sigma(\mathbf{n})$ (resp. cut-off regularised sum $\sum_{\mathbb{Z}^d} \sigma(\mathbf{n})$) as $z \to 0$ so that in that case

$$\sum_{\mathbb{Z}^d} \sigma(\mathbf{n}) = \sum_{\mathbb{Z}^d} \sigma(\mathbf{n}), \quad \text{resp.} \quad \sum_{\mathbb{Z}^d} \sigma(\mathbf{n}) = -\sum_{\mathbb{Z}^d} \sigma(\mathbf{n}).$$

---

8. The definition of holomorphic families of symbols and operators is recalled in the Appendix section A.4.

9. This restriction is not strictly necessary but convenient to work with.
Remark 10 Here again, $C(\sigma)$ arises as a difference of regularised integrals, confirming a result of [GSW].

Proof: By Theorem 2 outside the set $\alpha^{-1}([-d, \infty[\cap \mathbb{Z})$ we have:

$$\sum_{\mathbb{Z}^d} \sigma(z) = \int_{\mathbb{R}^d} \sigma(z(x)) \, dx + C(\sigma(z)). \quad (17)$$

On the one hand, by results of [KV] (see Theorem 5 in the Appendix) we know that under the assumptions of the theorem, the map $z \mapsto \int_{\mathbb{R}^d} \sigma(z(x)) \, dx$ is meromorphic with a discrete set of simple poles in $\alpha^{-1}([-d, \infty[\cap \mathbb{Z})$ and that at zero

$$\text{Res}_{z=0} \int_{\mathbb{R}^d} \sigma(x) \, dx = -\frac{\sqrt{2\pi}}{\alpha'(0)} \text{res}(\sigma). \quad (18)$$

On the other hand, we know from [GSW] that $z \mapsto C(\sigma(z))$ is holomorphic\footnote{Their proof can easily be generalised to our more general setup of holomorphic families with any non constant affine order.}. It therefore follows from (17) that the map $z \mapsto -\sum_{\mathbb{Z}^d} \sigma(z) = \sum_{\mathbb{Z}^d} \sigma(z)$ is meromorphic with a discrete of simples poles in $\alpha^{-1}([-d, \infty[\cap \mathbb{Z})$ and complex residue at $z = 0$ given by

$$\text{Res}_{z=0} \sum_{\mathbb{Z}^d} \sigma(z) = -\frac{1}{\alpha'(0)} \text{res}(\sigma(0)).$$

Taking finite parts at $z = 0$ in (17) yields (16) since $\lim_{z \to 0} C(\sigma(z)) = C(\sigma)$. When the order of $\sigma$ has real part $<-d$ (resp. is non integer), the map $z \mapsto \int_{\mathbb{R}^d} \sigma(z(x)) \, dx$ is holomorphic at 0 since $\sigma$ has vanishing residue. Its limit at $z = 0$ coincides with the ordinary integral $\int_{\mathbb{R}^d} \sigma(x) \, dx$ (resp. the cut-off regularised integral $\int_{\mathbb{R}^d} \sigma(x) \, dx$). By (17) and since $z \mapsto C(\sigma(z))$ is known to be holomorphic ([GSW]), the map $z \mapsto \sum_{\mathbb{Z}^d} \sigma(z) = \sum_{\mathbb{Z}^d} \sigma(z)$ is also holomorphic at $z = 0$ and its limit reads:

$$\sum_{\mathbb{Z}^d} \sigma(z) = \lim_{z \to 0} \sum_{\mathbb{Z}^d} \sigma(z) = \int_{\mathbb{R}^d} \sigma(x) \, dx + C(\sigma) = \sum_{\mathbb{Z}^d} \sigma,$$

resp.

$$\sum_{\mathbb{Z}^d} \sigma(z) = \lim_{z \to 0} \sum_{\mathbb{Z}^d} \sigma(z) = \int_{\mathbb{R}^d} \sigma(x) \, dx + C(\sigma) = \sum_{\mathbb{Z}^d} \sigma,$$

where the last sum is an ordinary sum (resp. a cut-off regularised sum). $\Box$

Remark 11 Whereas regularised sums $\sum_{\mathbb{Z}^d} \sigma(z)$, which coincide with cut-off regularised sums $\sum_{\mathbb{Z}^d} \sigma(z)$ on non integer order symbols are translation invariant on such symbols, they are not translation invariant on integer order symbols. For example, in dimension $d = 1$, and for non positive negative integers $k$ we have

$$\text{fp}_{z=0} \sum_{n \in \mathbb{Z}} |n+p|^{k+z} \neq \text{fp}_{z=0} \sum_{n \in \mathbb{Z}} |n|^{k+z}.$$
Indeed, on the one hand,

\[ \sum_{n \in \mathbb{Z}} |n + p|^{k+z} \]

where \( \zeta(s, p) := \sum_{n=1}^{\infty} |n + p|^{-s+z} \) is the Hurwitz zeta function (see e.g. [C]).

On the other hand, we have:

\[ \sum_{n \in \mathbb{Z}} |n|^{k+z} = 2 \zeta(-k), \]

but \( \zeta(-k) = -\frac{B_{k+1}}{k+1} \neq \zeta(-k, p) = -\frac{B_{k+1}(p)}{k+1} \) where \( B_n(x) \) stands for the \( n \)-th Bernoulli polynomials, and \( B_n = B_n(0) \) for the \( n \)-th Bernoulli constants.

**Corollary 1** Given two local holomorphic regularisations of \( \sigma \), \( \mathcal{R} \) at zero which sends \( \sigma \) to \( \sigma(z) \) of order \( \alpha(z) \) and \( \tilde{\mathcal{R}} \) which sends \( \sigma \) to \( \tilde{\sigma}(z) \) of same affine order \( \alpha(z) \) we have:

\[ \sum_{\mathbb{Z}^d} \mathcal{R}(\mathbf{n}) - \sum_{\mathbb{Z}^d} \tilde{\mathcal{R}}(\mathbf{n}) = -\sqrt{2\pi} \int_{\mathbb{R}^d} \frac{\text{Res}(\sigma'(0) - \tilde{\sigma}'(0))}{\alpha'(0)}. \]  

(19)

In particular, whenever \( \mathcal{R}(\sigma) \) and \( \tilde{\mathcal{R}}(\sigma) \) coincide with \( \sigma \) on the unit sphere, their cut-off regularised sums coincide:

\[ \sum_{\mathbb{Z}^d} \mathcal{R}(\mathbf{n}) = \sum_{\mathbb{Z}^d} \tilde{\mathcal{R}}(\mathbf{n}). \]

**Remark 12** A priori, as explained in the Appendix, the first derivatives \( \sigma'(0) \) and \( \tilde{\sigma}'(0) \) are not classical but log-polyhomogeneous of log-type 1 (see e.g. [L], [PS]). However, since they have same order, their difference is classical, so that the noncommutative residue of their difference indeed makes sense.

**Proof:** \( \tau(z) := \frac{\sigma(z) - \tilde{\sigma}(z)}{z} \) defines a local holomorphic family of classical symbols around zero of order \( \alpha(z) \) such that \( \tau(0) = \sigma'(0) - \tilde{\sigma}'(0) \). Applying (15) to this family \( \tau(z) \) yields

\[ \sum_{\mathbb{Z}^d} \mathcal{R}(\mathbf{n}) - \sum_{\mathbb{Z}^d} \tilde{\mathcal{R}}(\mathbf{n}) = -\sqrt{2\pi} \int_{\mathbb{R}^d} \frac{\text{Res}(\sigma'(0) - \tilde{\sigma}'(0))}{\alpha'(0)}. \]

If \( \sigma(z) \) and \( \tilde{\sigma}(z) \) restricted to the unit sphere are independent of \( z \), \( \text{Res}(\sigma'(0) - \tilde{\sigma}'(0)) \) vanishes and \( \sum_{\mathbb{Z}^d} \mathcal{R}(\mathbf{n}) = \sum_{\mathbb{Z}^d} \tilde{\mathcal{R}}(\mathbf{n}) \). \( \square \)
5 Zeta functions associated with quadratic forms

Cut-off regularised sums $\sum_{\mathbb{Z}^d}$ are useful to build meromorphic extensions of ordinary sums of holomorphic families of symbols; we recover this way the existence of meromorphic extensions of zeta functions associated with quadratic forms. To a positive definite quadratic form $q(x_1, \cdots, x_d)$ and a smooth cut-off function $\chi$ which vanishes in a small neighborhood of 0 and is identically one outside the unit euclidean ball, we assign the classical symbol

$$\sigma_{q,s} (x) := \chi(x) q(x)^{-s} \in \mathcal{CS}_{c.c}(\mathbb{R}^d).$$

(20)

Theorem 4 Given any complex number $s$ the map

$$z \mapsto \sum_{n \in \mathbb{Z}^d - \{0\}} \sigma_{q,s+z} = \sum_{n \in \mathbb{Z}^d - \{0\}} q(n)^{-(s+z)}$$

which is holomorphic on the half plane $\Re(s+z) > d/2$ extends to a meromorphic map

$$z \mapsto \sum_{n \in \mathbb{Z}^d - \{0\}} \sigma_{q,s+z} = \sum_{n \in \mathbb{Z}^d - \{0\}} q(n)^{-(s+z)}$$

with simple pole at $z = 0$ given by:

$$\text{Res}_{z=0} \sum_{n \in \mathbb{Z}^d - \{0\}} q(n)^{-(s+z)} = \delta_{2s=d} \int_{|\omega|=1} q(\omega)^{-d/2} \, d\mu_S(\omega)$$

and constant term at $z = 0$:

$$Z_q(s) := \text{fp}_{z=0} \sum_{n \in \mathbb{Z}^d - \{0\}} q(n)^{-(s+z)}.$$ 

(21)

Moreover,

$$Z_q(s) = \int_{\mathbb{R}^d} \sigma_{q,s} + C(\sigma_{q,s}).$$

Proof: Up to the pole which we compute separately, the result follows from Theorem 3 applied to $\sigma_{q,s}$ and Riesz regularisation $R : \sigma \mapsto \sigma(x) |x|^{-s}$ combined with the fact that Riesz regularised integrals coincide with ordinary cut-off regularised integrals (see e.g. [MP], [PS], [P1]):

$$\text{fp}_{z=0} \int_{\mathbb{R}} \sigma(x) |x|^{-z} \, dx = \int_{\mathbb{R}} \sigma(x) \, dx \quad \forall \sigma \in \mathcal{CS}_{c.c}(\mathbb{R}^d).$$
Now, by (17) the pole at \( z = 0 \) is given by the pole of \( \frac{1}{2 \pi i} \int_{\mathbb{R}^d} \sigma_{q,s}(x) \, |x|^{-z} \, dx \). We write
\[
\text{Res}_{z=0} \int_{\mathbb{R}^d} \chi(x) \, q(x)^{-s} \, |x|^{-z} \, dx = \text{Res}_{z=0} \int_{|x|=1} \chi(x) \, q(x)^{-s} \, dx + \text{Res}_{z=0} \left( \frac{1}{2 \pi i} \int_{|x|=1} \frac{1}{z} \, q(x)^{-s} \, dx \right) = \text{Res}_{z=0} \left( \frac{1}{2 \pi i} \int_{|x|=1} q(x)^{-s} \, dx \right) = \delta_{2s-d} \left( \int_{|x|=1} q(x)^{-s} \, dx \right).
\]

As announced, there is therefore a pole at \( z = 0 \) only if \( s = d/2 \) in which case the residue coincides with \( \int_{|x|=1} q(x)^{-s} \, dx \). \( \square \)

**Remark 13** For \( d = 2 \) and \( q(x,y) = ax^2 + bxy + cy^2 \) with \( 4ac - b^2 > 0 \), \( Z_q(s) \) yields a meromorphic extension of Epstein’s \( \zeta \)-function \( \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} (am^2 + bmn + cn^2)^{-s} \) (see e.g. [CS]) which is known to satisfy a functional equation similar to the one satisfied by the Riemann zeta function.

When \( a = c = 1, b = 0 \), \( Z_q(s) \) provides a meromorphic extension of the zeta function of \( \mathbb{Z}[i] \) given by (see e.g. [C])
\[
Z_q(s) := \sum_{z \in \mathbb{Z}[i] \setminus \{0\}} |z|^{-2s} = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} m^2 + n^2.
\]

When \( a = b = c = 1 \), \( Z_q(s) \) provides a meromorphic extension of the zeta function of \( \mathbb{Z}[j] \) given by (see e.g. [C])
\[
Z_q(s) := \sum_{z \in \mathbb{Z}[j] \setminus \{0\}} |z|^{-2s} = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} m^2 + mn + n^2.
\]

**Proposition 4** Whenever \( \text{Re}(s) \leq 0 \)

1. \( Z_q(s) = C(x \mapsto q(x)^{-s}) \),

2. Specifically, for any non negative integer \( k \)
\[
Z_q(-k) = 0.
\]

3. Moreover, \( Z_q \) is holomorphic at \( s = -k \) for any non negative integer \( k \) and \( Z'(s) = \partial_{s \mid s = -k} C(q^{-s}) \) where the derivative at \( k = 0 \) stands for the derivative of the map \( C(q^{-s}) \) restricted to the half plane \( \text{Re}(s) \leq 0 \). \[11\]

\[11\]In contrast to the value at \( s = -k \) which vanishes, one does not expect the derivative to vanish in general.
Proof:

1. When \( \text{Re}(s) \leq 0 \), the map \( \underline{x} \mapsto q(\underline{x})^{-s} \) can be extended by continuity to \( \underline{x} = 0 \) by

\[
\tilde{\sigma}_{q,s}(\underline{x}) := q(\underline{x})^{-s} \quad \forall \underline{x} \neq 0, \quad \tilde{\sigma}_{q,s}(0) = 0.
\]

In that case, there is no need to introduce a cut-off function \( \chi \) at 0 and we write:

\[
Z_{q}(s) = \text{fp}_{z=0, \text{Re}(s+z) \leq 0} \sum_{\underline{z} \in \mathbb{Z}^{d}} q(\underline{z})^{-(s+z)} = \int_{\mathbb{R}^{d}} q(\underline{x})^{-s} \, d\underline{x} + C(\underline{x} \mapsto q(\underline{x})^{-s})
\]

along the lines of the proof of the previous proposition. Here we take the finite part at \( z = 0 \) of the restriction \( z \mapsto \sum_{\underline{z} \in \mathbb{Z}^{d}} q(\underline{z})^{-(s+z)} \) to the half plane \( \text{Re}(z) \leq 0 \).

Using polar coordinates \( \underline{x} = r \omega \) with \( r > 0 \) and \( \omega \) in the unit sphere, the result then follows from the fact that the cut-off regularised integral vanishes if \( \text{Re}(z) \leq 0 \) since we have

\[
\int_{\mathbb{R}^{d}} q(\underline{x})^{-2s} \, d\underline{x} = \text{fp}_{R \to \infty} \int_{|\underline{x}| \leq R} q(\underline{x})^{-2s} \, d\underline{x}
\]

\[
= \text{fp}_{R \to \infty} \int_{|\omega|=1} \int_{0}^{R} r^{-2s+d-1} \, d\underline{x}
\]

\[
= \left( \text{fp}_{R \to \infty} \int_{0}^{R} r^{-2s+d-1} \, dr \right) \left( \int_{|\omega| \leq 1} q(\omega)^{-2s} \, d\mu_{S}(\omega) \right)
\]

\[
= \left( \text{fp}_{R \to \infty} \int_{-2s+d}^{R} r^{-2s+d} \, dr \right) \left( \int_{|\omega| \leq 1} q(\omega)^{-2s} \, d\mu_{S}(\omega) \right)
\]

if \( -2s + d \neq 0 \)

\[
= 0 \text{ if } -2s + d \neq 0
\]

where \( \mu_{S} \) is the volume measure on the unit sphere induced by the canonical measure on \( \mathbb{R}^{d} \).

2. When \( s = -k \), we also have \( C(\underline{x} \mapsto q(\underline{x})^{k}) = 0 \) since \( C \) vanishes on polynomials so that \( Z_{q}(-k) = 0 \).

3. By Theorem 4 there is no pole at \( s = -k \) (the presence of the cut-off function \( \chi \) does not affect poles) since the only pole corresponds to \( s = d/2 \). The map \( Z_{q} \) is therefore holomorphic at \( s = -k \) with derivative given by the derivative of the map \( C(\underline{x} \mapsto q(\underline{x})^{-s}) \) at \( s = -k \).

\[\square\]

\[\footnote{Note that this extension is not smooth at 0 so that it does not define a symbol. It nevertheless has the same asymptotic behaviour as \( |\underline{x}| \to \infty \) as \( \underline{x} \mapsto \chi(\underline{x}) q(\underline{x})^{-s} \) which is enough for our needs.}\]
6 Zeta functions associated with Laplacians on tori

The Laplacian \(-\sum_{j=1}^{d} \partial_j^2\) on \(\mathbb{R}^d\) induces a non-negative elliptic differential operator \(\Delta_d\) of order 2 with positive leading symbol on \(\mathbb{T}^d \cong \mathbb{R}^d/(2\pi \mathbb{Z})^d\), the spectrum of which reads:

\[
\text{Spec}(\Delta_d) = \{n_1^2 + \cdots + n_d^2, \ n_i \in \mathbb{Z}\}.
\]

Let \(\Delta_d^\perp\) denote the restriction of \(\Delta_d\) to the orthogonal of its kernel, which we recall is finite dimensional. Recall (see e.g. [KV]) that the map \(z \mapsto \text{tr}'(\Delta_d - s) := \text{tr}\left((\Delta_d^\perp)^{-s}\right)\) which is holomorphic on the half-plane \(\text{Re}(s) > \frac{d}{2}\), extends to a meromorphic map \(z \mapsto \text{TR}'(\Delta_d^{-s}) := \text{TR}\left((\Delta_d^\perp)^{-s}\right)\) on the whole complex plane (see Theorem 6 in the Appendix), where TR stands for Kontsevich and Vishik’s canonical trace [KV], the definition of which is given in the Appendix (see Definition 7). The zeta function associated with \(\Delta_d\) is defined by

\[
\zeta_{\Delta_d}(s) := \text{fp}_{z=0} \text{TR}'(\Delta_d^{-(s+z)}).
\]

Applying Theorem 4 to the quadratic form \(q(x) = \sum_{i=1}^{d} x_i^2\) leads to a description of the \(\zeta\)-function associated with the Laplacian as a regularised discrete sum of powers of its eigenvalues.

**Proposition 5**
\[
\zeta_{\Delta_d}(s) = \text{fp}_{z=0} \sum_{\mathbb{Z}^d-\{0\}} (n_1^2 + \cdots + n_d^2)^{-(s+z)} = \int_{\mathbb{R}^d} \sigma_s + C(\sigma_s)
\]

where we have set \(\sigma_s(x) = \chi(x)|x|^{-2s}\).

Here \(\chi\) is any smooth cut-off function which vanishes in a small neighborhood of 0 and is identically equal to 1 outside the unit euclidean ball.

**Proof:** For \(\text{Re}(z)\) sufficiently large, we have

\[
\text{tr}'\left(\Delta_d^{-(s+z)}\right) = \sum_{\mathbb{Z}^d-\{0\}} (n_1^2 + \cdots + n_d^2)^{-(s+z)}.
\]

By theorem 4 the map \(\sum_{\mathbb{Z}^d-\{0\}} (n_1^2 + \cdots + n_d^2)^{-(s+z)}\) extends to a meromorphic map \(\int_{\mathbb{R}^d} \sigma_s(x) d\mu + C(\sigma_s)\) which by the uniqueness of the meromorphic extension therefore coincides with \(\text{TR}'(\Delta_d^{-(s+z)})\).

Again by Theorem 4 we further have

\[
\zeta_{\Delta_d}(s) = \int_{\mathbb{R}^d} \sigma_s(x) d\mu + C(\sigma_s)
\]

as announced in the proposition. \(\Box\)

By Proposition 4 the \(\zeta\)-function \(\zeta_{\Delta_d}\) is holomorphic at any non positive integers \(-k\). In particular, the zeta determinant

\[
\det_{\zeta}(\Delta_d) := e^{-\zeta_{\Delta_d}(0)}
\]

is well-defined. The following proposition relates it to the derivative at zero of the map \(s \mapsto C(x \mapsto |x|^{-2s})\).
Proposition 6 Whenever \( \text{Re}(s) \leq 0 \)
\[
\zeta_{\Delta_{d}}(s) = C(x \mapsto |x|^{-2s}).
\]
Specifically, for any non negative integer \( k \)
\[
\zeta_{\Delta_{d}}(-k) = 0.
\]
Moreover,
\[
\det_{\zeta}(\Delta_{d}) = \exp\left(-\partial_{|s|_{z=0}} C(x \mapsto |x|^{-2s})\right)
\]
where the subscript \( \partial_{|s|_{z=0}} \) stands for the derivative of the map \( s \mapsto C(x \mapsto |x|^{-2s}) \) restricted to the half plane \( \text{Re}(s) \leq 0 \).

**Proof:** This follows from Proposition 4 applied to \( q(x) = |x|^2 \). \( \square \)

**Example 4** When \( d = 1 \) this yields:
\[
\zeta(s) = \frac{1}{2} \zeta_{\Delta_{1}}(-s/2) = \frac{C(x \mapsto |x|^{-s})}{2} \quad \text{for} \quad \text{Re}(s) \leq 0.
\]
In particular, for \( s = -k \) with \( k \) a non negative integer, we have that \( \zeta(-k) = \frac{1}{2} \zeta_{\Delta_{1}}(-k/2) = C(x \mapsto |x|^{k}) \) vanishes for even \( k \).

When \( \text{Re}(s) > d/2 \), the sum \( \sum_{\mathbb{Z}^{d}} |n|^{-2s} \log |n| \) converges and we have
\[
\zeta'_{\Delta_{d}}(s) = -2 \sum_{\mathbb{Z}^{d}-\{0\}} |n|^{-2s} \log |n| \quad \text{if} \quad \text{Re}(s) \leq d/2.
\]

Just as one can extend cut-off regularised integrals to log-polyhomogeneous symbols, cut-off regularised sums can be extended to log-polyhomogeneous symbols so that one can define \( \sum_{\mathbb{Z}^{d}-\{0\}} |n|^{-2s} \log |n| \) for general complex numbers \( s \). But in general,
\[
\zeta_{\Delta_{d}}(s) \neq -2 \sum_{\mathbb{Z}^{d}} |n|^{-2s} \log |n| \quad \text{when} \quad \text{Re}(s) \leq d/2.
\]

**Remark 14** A similar observation holds for the \( \zeta \) function itself for in general \( \zeta_{\Delta_{d}}(s) \neq \sum_{\mathbb{Z}^{d}} |n|^{-2s} \), as for example when \( s = -k \) is a non positive integer. In that case, the right hand side vanishes whereas the left hand side does not.

Consequently, even though the map \( z \mapsto C(\sigma(z)) \) is holomorphic at \( z = 0 \) for any local holomorphic regularisation \( \mathcal{R}(\sigma) : z \mapsto \sigma(z) \) of a symbol \( \sigma \in CS_{\infty}(\mathbb{R}^{n}) \) so that \( \lim_{z \to 0} C(\sigma(z)) = C(\sigma) \), one expects that for \( \sigma \) of order \( a \), in general
\[
\partial_{|z|_{z=0}} C(\sigma(z)) \neq C(\sigma'(0)) \quad \text{when} \quad \text{Re}(a) \geq -d.
\]

**Remark 15** Comparing the zeta-regularised determinant \( \zeta'_{\Delta_{d}}(0) \) with the “cut-off regularised” determinant \( \exp\left(-2 \sum_{\mathbb{Z}^{d}-\{0\}} \log |n|\right) \) is reminiscent of issues addressed in [FG] in the context of Szegö operators.

**Example 5** It follows from results of [MP] that the difference \( \zeta'_{\Delta_{1}}(-k/2) + 2 \sum_{Z} |n|^{k} \log |n| \) is a rational number for any positive integer \( k \).
Appendix : Prerequisites on regularised integrals of symbols

This appendix recalls known results on regularised integrals of classical symbols used in the main bulk of the paper.

A.1 Classical pseudodifferential symbols with constant coefficients

We only give a few definitions and refer the reader to [Sh, T, Tr] for further details on classical pseudodifferential symbols.

For any complex number $a$, let us denote by $S_{a,c}(\mathbb{R}^n)$ the set of smooth functions on $\mathbb{R}^n$ called symbols with constant coefficients, such that for any multiindex $\beta \in \mathbb{N}^n$ there is a constant $C(\beta)$ satisfying the following requirement:

$$\left| \partial^\beta a(x) \right| \leq C(\beta)(1 + |x|)^{\text{Re}(a) - |\beta|}$$

where $\text{Re}(a)$ stands for the real part of $a$, $|x|$ for the euclidean norm of $x$. We single out the subset $CS_{a,c}(\mathbb{R}^n) \subset S_{a,c}(\mathbb{R}^n)$ of symbols $\sigma$, called classical symbols of order $a$ with constant coefficients, such that

$$\sigma(x) = \sum_{j=0}^{N-1} \chi(x) \sigma_{a-j}(x) + \sigma(N)(x)$$

(24)

where $\sigma(N) \in S_{a-N}^a(\mathbb{R}^n)$ and where $\chi$ is a smooth cut-off function which vanishes in a small ball of $\mathbb{R}^n$ centered at 0 and which is constant equal to 1 outside the unit ball. Here $\sigma_{a-j}, j \in \mathbb{N}_0$ are positively homogeneous of degree $a - j$.

The ordinary product of functions sends $CS_{a,c}(\mathbb{R}^n) \times CS_{b,c}(\mathbb{R}^n)$ to $CS_{a+b,c}(\mathbb{R}^n)$ provided $b - a \in \mathbb{Z}$; let

$$CS_{c,c}(\mathbb{R}^n) = \left( \bigcup_{a \in \mathbb{C}} CS_{a,c}(\mathbb{R}^n) \right)$$

(25)

denote the algebra generated by all classical symbols with constant coefficients on $\mathbb{R}^n$.

Let

$$CS_{c,c}^{-\infty}(\mathbb{R}^n) = \bigcap_{a \in \mathbb{C}} CS_{a,c}(\mathbb{R}^n)$$

be the algebra of smoothing symbols. We write $\sigma \sim \sigma'$ for two symbols $\sigma, \sigma'$ which differ by a smoothing symbol.

We also denote by $CS_{c,c}^{P}(\mathbb{R}^n) := \bigcup_{\text{Re}(a) < p} CS_{a,c}(\mathbb{R}^n)$, the set of classical symbols of order with real part $< p$ and by

$$CS_{c,c}^{\mathbb{Z}}(\mathbb{R}^n) := \bigcup_{a \in \mathbb{C} - \mathbb{Z}} CS_{a,c}(\mathbb{R}^n)$$

(26)

the set of non integer order symbols.

A.2 The noncommutative residue and cut-off regularised integrals on symbols

We first recall the definition of the noncommutative residue of a classical symbol [G2, W1, W2].
**Definition 4** The noncommutative residue is a linear form on $CS_{c,c}(\mathbb{R}^n)$ defined by

$$\text{res}(\sigma) := \frac{1}{\sqrt{2\pi}} \int_{S^{n-1}} \sigma_n(x) \, d\mu_S(x)$$

where

$$d\mu_S(x) := \sum_{j=1}^{n} (-1)^{j-1} x_j \, dx_1 \wedge \cdots \wedge dx_j \wedge \cdots \wedge dx_n$$

denotes the volume measure on the unit sphere $S^{n-1}$ induced by the canonical measure on $\mathbb{R}^n$.

Let us now recall the construction of a useful linear extension of the ordinary integral. For any $R > 0$, $B(0,R)$ denotes the ball of radius $R$ centered at 0 in $\mathbb{R}^n$. We recall that given a symbol $\sigma \in CS_{c,c}(\mathbb{R}^n)$, the map $R \mapsto \int_{B(0,R)} \sigma(x) \, dx$ has an asymptotic expansion as $R \to \infty$ of the form (with the notations of (24)):

$$\int_{B(0,R)} \sigma(x) \, dx \sim_{R \to \infty} \alpha_0(\sigma) + \sum_{j=0, a-j+n \neq 0}^\infty \sigma_{a-j} R^{a-j+n} + \text{res}(\sigma) \cdot \log R. \quad (27)$$

**Definition 5** Given $\sigma \in CS_{c,c}(\mathbb{R}^n)$ with $a \in \mathbb{C}$, we call the constant term $\alpha_0(\sigma)$ the cut-off regularised integral of $\sigma$:

$$\int_{\mathbb{R}^n} \sigma(x) \, dx := \lim_{R \to \infty} \int_{B(0,R)} \sigma(x) \, dx.$$ 

The cut-off regularised integral $\int_{\mathbb{R}^n}$, which reads

$$\int_{\mathbb{R}^n} \sigma(x) \, dx = \int_{\mathbb{R}^n} \sigma_{(N)}(x) \, dx + \sum_{j=0}^{N-1} \int_{B(0,1)} \chi(x) \sigma_{a-j}(x) \, dx$$

$$- \sum_{j=0, a-j+n \neq 0}^{N-1} \frac{1}{a-j+n} \int_{S^{n-1}} \sigma_{a-j}(\omega) \, d\mu_S(\omega). \quad (28)$$

defines a linear form on $CS_{c,c}(\mathbb{R}^n)$ which extends the ordinary integral in the following sense; if $\sigma$ has complex order with real part smaller than $-n$ then $\int_{B(0,R)} \sigma(x) \, dx$ converges as $R \to \infty$ and

$$\int_{\mathbb{R}^n} \sigma(x) \, dx = \int_{\mathbb{R}^n} \sigma(x) \, dx.$$ 

A. 3 The noncommutative residue and canonical trace on operators

Let $U$ be a connected open subset of $\mathbb{R}^n$.

For any complex number $a$, let $S^a_{\text{cpt}}(U)$ denote the set of smooth functions on $U \times \mathbb{R}^n$ called symbols with compact support in $U$, such that for any multiindices $\beta, \gamma \in \mathbb{N}^n$, there is a constant $C(\beta, \gamma)$ satisfying the following requirement:

$$|\partial^\beta_x \partial^\gamma_\xi \sigma(x, \xi)| \leq C(\beta, \gamma)(1 + |\xi|)^{\text{Re}(a)-|eta|}$$
where \( \text{Re}(a) \) stands for the real part of \( a \), \( |\xi| \) for the euclidean norm of \( \xi \). We single out the subset \( \mathcal{CS}_\text{cpt}^a(U) \subset \mathcal{S}_\text{cpt}^a(U) \) of symbols \( \sigma \), called classical symbols of order \( a \) with compact support in \( U \), such that

\[
\sigma(x, \xi) = \sum_{j=0}^{N-1} \chi(\xi) \sigma_{a-j}(x, \xi) + \sigma_N(x, \xi)
\]

(29)

where \( \sigma_N \in \mathcal{S}_\text{cpt}^{-N}(U) \) and where \( \chi \) is a smooth cut-off function which vanishes in a small ball of \( \mathbb{R}^n \) centered at 0 and which is constant equal to 1 outside the unit ball. Here \( \sigma_{a-j}(x, \cdot), j \in \mathbb{N}_0 \) are positively homogeneous of degree \( a - j \).

Let

\[
\mathcal{CS}_\text{cpt}^{-\infty}(U) = \bigcap_{a \in \mathbb{C}} \mathcal{CS}_\text{cpt}^a(U)
\]

be the set of smoothing symbols with compact support in \( U \); we write \( \sigma \sim \tau \) for two symbols that differ by a smoothing symbol.

The star product

\[
\sigma \ast \tau \sim \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma \partial_x^\alpha \tau
\]

(30)

of symbols \( \sigma \in \mathcal{CS}_\text{cpt}^a(U) \) and \( \tau \in \mathcal{S}_\text{cpt}^b(U) \) lies in \( \mathcal{CS}_\text{cpt}^{a+b}(U) \) provided \( a - b \in \mathbb{Z} \).

Let

\[
\mathcal{CS}_\text{cpt}(U) = \langle \bigcup_{a \in \mathbb{C}} \mathcal{CS}_\text{cpt}^a(U) \rangle
\]

denote the algebra generated by all classical symbols with compact support in \( U \). We denote by \( \mathcal{CS}_\text{cpt}^p(U) := \bigcup_{\text{Re}(a) < p} \mathcal{CS}_\text{cpt}^a(U) \), the set of classical symbols of order with real part \( p \) with compact support in \( U \), by \( \mathcal{CS}_\text{cpt}^\mathbb{Z}(U) := \bigcup_{a \in \mathbb{Z}} \mathcal{CS}_\text{cpt}^a(U) \) the algebra of integer order symbols, and by \( \mathcal{CS}_\text{cpt}^{\mathbb{Z}}(U) := \bigcup_{a \in \mathbb{C} - \mathbb{Z}} \mathcal{CS}_\text{cpt}^a(U) \) the set of non integer order symbols with compact support in \( U \).

Both the noncommutative residue and the cut-off regularised integral extend to \( \mathcal{CS}_\text{cpt}(U) \).

**Definition 6**

1. The noncommutative residue of a symbol \( \sigma \in \mathcal{CS}_\text{cpt}(U) \) is defined by

\[
\text{res}(\sigma) := \frac{1}{(2\pi)^n} \int_U dx \int_{S^{n-1}} \sigma_{-n}(x, \xi) \mu_S(\xi) = \frac{1}{\sqrt{2\pi}} \int_U \text{res}_x(\sigma) \, dx
\]

where \( \text{res}_x(\sigma) := \frac{1}{\sqrt{2\pi}} \int_{S^{n-1}} \sigma_{-n}(x, \xi) \, d\mu_S(\xi) \) is the residue density at point \( x \) and where as before

\[
d\mu_S(\xi) := \sum_{j=1}^n (-1)^j \xi_j \, d\xi_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge d\xi_n
\]

denotes the volume measure on \( S^{n-1} \) induced by the canonical measure on \( \mathbb{R}^n \).

2. For any \( \sigma \in \mathcal{CS}_\text{cpt}(U) \) the cut-off regularised integral of \( \sigma \) is defined by

\[
\int_{T^*U} \sigma := \int_U dx \int_{T^*_U} \sigma(x, \xi) \, d\xi.
\]
Let $M$ be an $n$-dimensional closed connected Riemannian manifold (as before $n > 1$). For $a \in \mathbb{C}$, let $\mathcal{C}^{a}(M)$ denote the linear space of classical pseudodifferential operators of order $a$, i.e., linear maps acting on smooth functions $C^{\infty}(M)$, which using a partition of unity adapted to an atlas on $M$ can be written as a finite sum of operators

$$A = \text{Op}(\sigma(A)) + R$$

where $R$ is a linear operator with smooth kernel and $\sigma(A) \in C_{\text{cpt}}^{a}(U)$ for some open subset $U \subset \mathbb{R}^{n}$. Here we have set

$$\text{Op}(\sigma)(u) := \int_{\mathbb{R}^{n}} e^{i(x-y,\xi)} \sigma(x,\xi) u(y) \, dy \, d\xi$$

where $\langle \cdot , \cdot \rangle$ stands for the canonical scalar product in $\mathbb{R}^{n}$.

The star product (30) on classical symbols with compact support induces the operator product on (properly supported) classical pseudodifferential operators since $\text{Op}(\sigma \ast \tau) = \text{Op}(\sigma) \text{Op}(\tau)$. It follows that the product $AB$ of two classical pseudodifferential operators $A \in \mathcal{C}^{a}(M)$, $B \in \mathcal{C}^{b}(M)$ lies in $\mathcal{C}^{a+b}(M)$ provided $a - b \in \mathbb{Z}$.

Let us denote by $\mathcal{C}(M) = \bigcup_{a \in \mathbb{C}} \mathcal{C}^{a}(M))$ the algebra generated by all classical pseudodifferential operators acting on $C^{\infty}(M)$.

Given a finite rank vector bundle $E$ over $M$ we set $\mathcal{C}^{a}(M, E) := \mathcal{C}^{a}(M) \otimes \text{End}(E)$, $\mathcal{C}(M, E) := \mathcal{C}(M) \otimes \text{End}(E)$.

The sets $\mathcal{C}^{a\in\mathbb{Z}}(M, E)$ and $\mathcal{C}^{a\in\mathbb{Z}}(M, E)$ are defined similarly using trivialisations of $E$ from the sets $C_{\text{cpt}}^{a}(U)$ and $C_{\text{cpt}}^{a}(U)$.

Using a partition of unity, one can patch up the noncommutative residue, resp. the cut-off regularised integral of symbols with compact support to a noncommutative residue on all classical pseudodifferential operators [G1], [W1] [W2], resp. a canonical trace on non integer order classical pseudodifferential operators [KV].

**Definition 7**

1. The noncommutative residue is defined on $\mathcal{C}(M, E)$ by

$$\text{res}(A) := \frac{1}{(2\pi)^{n}} \int_{M} dx \int_{S^{n-1}M} \text{tr}_{x}(\sigma(A))_{-n}(x,\xi) \mu_{S}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{M} \text{res}_{x}(A) \, dx$$

where $\text{res}_{x}(A) := \frac{1}{\sqrt{2\pi}} \int_{S^{n-1}M} \text{tr}_{x}(\sigma(A))_{-n}(x,\xi) \mu_{S}(\xi)$ is the residue density at point $x$.

2. The canonical trace is defined on $\mathcal{C}^{a\in\mathbb{Z}}(M, E)$ by

$$\text{TR}(A) := \frac{1}{(2\pi)^{n}} \int_{M} dx \int_{T^{*}_{x}M} \text{tr}_{x} (\sigma(A)(x,\xi)) \, d\xi = \frac{1}{\sqrt{2\pi}} \int_{M} \text{TR}_{x}(A) \, dx$$

where $\text{TR}_{x}(A) := \frac{1}{\sqrt{2\pi}} \int_{T^{*}_{x}M} \text{tr}_{x} (\sigma(A)(x,\xi)) \, d\xi$ is the canonical trace density at point $x$.

A. 4 Holomorphic families of symbols and operators

The notion of holomorphic family of classical pseudodifferential operators used by Kontsevich and Vishik in [KV] generalises the notion of complex power $A^{b}$ of an elliptic operator developed by Seeley [Se], the derivatives of which lead to logarithms.
Definition 8 Let $\Omega$ be a domain of $\mathbb{C}$ and $U$ an open subset of $\mathbb{R}^d$. A family $(\sigma(z))_{z \in \Omega} \subset CS(U)$ is holomorphic when

(i) the order $\alpha(z)$ of $\sigma(z)$ is holomorphic on $\Omega$.

(ii) The map $z \mapsto \sigma^j(z)$ is holomorphic on $\Omega$ and $\forall k \geq 0, \partial^k_x \sigma(z) \in S^{\alpha(z)+\varepsilon}(U)$ for all $\varepsilon > 0$.

(iii) For any integer $j \geq 0$, the (positively) homogeneous component $\sigma_{\alpha(z)-j}(z)$ of degree $\alpha(z)-j$ of the symbol is holomorphic on $\Omega$.

The derivative of a holomorphic family $\sigma(z)$ of classical symbols yields a holomorphic family of symbols, the asymptotic expansions of which a priori involve a logarithmic term.

Lemma 3 The derivative of a holomorphic family $\sigma(z)$ of classical symbols of order $\alpha(z)$ defines a holomorphic family of symbols $\sigma'(z)$ of order $\alpha(z)$ with asymptotic expansion:

$$\sigma'(z)(x, \xi) \sim \sum_{j=0}^{\infty} \chi(\xi) \left( \log |\xi| \sigma'_{\alpha(z)-j,1}(z)(x, \xi) + \sigma'_{\alpha(z)-j,0}(z)(x, \xi) \right) \quad \forall (x, \xi) \in T^*_xU$$

for some smooth cut-off function $\chi$ around the origin which is identically equal to 1 outside the open unit ball and positively homogeneous symbols

$$\sigma'_{\alpha(z)-j,0}(z)(x, \xi) = |\xi|^{\alpha(z)-j} \partial_z \left( \sigma_{\alpha(z)-j}(z)(x, \frac{\xi}{|\xi|}) \right), \quad \sigma'_{\alpha(z)-j,1}(z) = \alpha'(z) \sigma_{\alpha(z)-j}(z)$$

of degree $\alpha(z)-j$.

Remark 16 If $\sigma(z)$ is independent of $z$ then $\sigma'_{\alpha(z)-j}$ restricted to the unit sphere vanishes.

Proof: We write

$$\sigma(z)(x, \xi) \sim \sum_{j=0}^{\infty} \chi(\xi) \sigma_{\alpha(z)-j}(z)(x, \xi).$$

Using the positive homogeneity of the components $\sigma_{\alpha(z)-j}$ we have:

$$\partial_z \left( \sigma_{\alpha(z)-j}(z)(x, \xi) \right)$$

$$= \partial_z \left( |\xi|^{\alpha(z)-j} \sigma_{\alpha(z)-j}(z)(x, \frac{\xi}{|\xi|}) \right)$$

$$= (\alpha'(z)|\xi|^{\alpha(z)-j} \sigma_{\alpha(z)-j}(z)(x, \frac{\xi}{|\xi|}) \log |\xi| + |\xi|^{\alpha(z)-j} \partial_z \left( \sigma_{\alpha(z)-j}(z)(x, \frac{\xi}{|\xi|}) \right)$$

$$= (\alpha'(z)\sigma_{\alpha(z)-j}(z)(x, \xi) \log |\xi| + |\xi|^{\alpha(z)-j} \partial_z \left( \sigma_{\alpha(z)-j}(z)(x, \frac{\xi}{|\xi|}) \right))$$

which shows that $\partial_z \left( \sigma_{\alpha(z)-j}(z)(x, \xi) \right)$ has order $\alpha(z)-j$. Thus

$$\partial_z (\sigma_N(z)(x, \xi)) = \sigma'(z)(x, \xi) - \sum_{j<N} \chi(\xi) \partial_z \left( \sigma_{\alpha(z)-j}(z)(x, \xi) \right)$$

lies in $S^\alpha(z)-N+\epsilon(U)$ for any $\epsilon > 0$ so that $\sigma'(z)$ is a symbol of order $\alpha(z)$ with asymptotic expansion:

$$\sigma'(z)(x, \xi) \sim \sum_{j=0}^{\infty} \chi(\xi) \sigma'_{\alpha(z)-j}(z) \quad \forall (x, \xi) \in T^*_xU$$

(33)
where
\[ \sigma'_{\alpha(z)}(z)(x, \xi) := \log |\xi| \sigma'_{\alpha(z)}(z)(x, \xi) + \sigma'_{\alpha(z)}(z)(x, \xi) \]
for some positively homogeneous symbols
\[ \sigma'_{\alpha(z)}(z)(x, \xi) := |\xi|^{\alpha(z) - j} \partial_z \left( \sigma_{\alpha(z)}(z)(x, \frac{\xi}{|\xi|}) \right) \]
and
\[ \sigma'_{\alpha(z)}(z)(x, \xi) := \alpha'(z) \sigma_{\alpha(z)}(z)(x, \xi) \]
of degree \( \alpha(z) - j \).

On the other hand, differentiating the asymptotic expansion \( \sigma(z)(x, \xi) \sim \sum_{j=0} \chi(\xi) \sigma_{\alpha(z)}(z)(x, \xi) \) w.r. to \( z \) yields
\[ \sigma'(z)(x, \xi) \sim \sum_{j=0} \chi(\xi) \partial_z (\sigma_{\alpha(z)}(z)(x, \xi)). \]

Hence,
\[ \partial_z (\sigma_{\alpha(z)}(z)(x, \xi)) = \sigma'_{\alpha(z)}(z)(x, \xi) = |\xi|^{\alpha(z) - j} \partial_z \left( \sigma_{\alpha(z)}(z)(x, \frac{\xi}{|\xi|}) \right) + \alpha'(z) \sigma_{\alpha(z)-j}(x, \xi) \log |\xi| \]
as announced. \( \square \)

The notion of holomorphic family extends to operators as follows.

**Definition 9** A family \( (A(z))_{z \in \Omega} \in \mathcal{C}(M, E) \) is holomorphic if in any local trivialisation we can write \( A(z) \) in the form \( A(z) = \text{Op}(\sigma(A(z))) + R(z) \), for some holomorphic family of symbols \( (\sigma(A(z)))_{z \in \Omega} \) and some holomorphic family \( (R(z))_{z \in \Omega} \) of smoothing operators i.e. given by a holomorphic family of smooth Schwartz kernels.

**A. 5 Defect formulae for regularised integrals and traces**

The noncommutative residue deserves its name since it is proportional to a complex residue as shows the following theorem. It also gives a “defect formula” which compares the finite part at the poles \( z_j \) of the meromorphic expansion given by \( z \mapsto \int_{\mathbb{R}^d} \sigma(z) \) with \( \int_{\mathbb{R}^d} \sigma(z_j) \).

**Theorem 5** Let \( U \) be an open subset of \( \mathbb{R}^n \) and let \( z \mapsto \sigma(z) \in \text{CS}^{\alpha(z)}(U) \) be a holomorphic family of classical pseudo-differential symbols parametrised by a domain \( \Omega \subset \mathbb{C} \) with non constant affine order \( \alpha(z) \). Then the map \( z \mapsto \int_{T^*_z U} \sigma(z) \) is meromorphic with poles of order 1 at points \( z_j \in \Omega \cap \alpha^{-1}([-n, +\infty[\cap \mathbb{Z}) \) and

1. \([KV]\]
\[ \text{Res}_{z=z_j} \int_{T^*_z U} \sigma(z)(x, \xi) d\xi = \frac{\sqrt{2\pi}^n}{\alpha'(z_j)} \text{res}(\sigma(z_j)). \]

2. Moreover \([ES]\),
\[ \text{fp}_{z=z_j} \int_{T^*_z U} \sigma(z)(x, \xi) d\xi = \int_{T^*_z U} \sigma(z_j) - \frac{\sqrt{2\pi}^n}{\alpha'(z_j)} \text{res}_x(\sigma'(z_j)). \]

**Remark 17** Here the noncommutative residue has been extended to the a priori log-polyhomogeneous symbol \( \sigma'(z_j) \) in a straightforward manner by the same formula as for classical symbols.
Definition 10 Given a symbol $\sigma \in CS(U)$ we call a local holomorphic family
\[ \sigma(z)(x, \xi) \sim \sigma(x, \xi) |\xi|^{-z} \]
such that $\sigma(0) = \sigma$ a Riesz regularisation of $\sigma$.

Example 6 $\sigma(z)(x, \xi) = (1 - \chi(\xi))\sigma(x, \xi) + \chi(\xi)\sigma(x, \xi) |\xi|^{-z}$ is a Riesz regularisation of the symbol $\sigma$, which depends on the choice of cut-off function $\chi$ around zero. However, as we shall see later on, this dependence does not affect the corresponding regularised integrals.

Specializing Theorem 5 to Riesz regularisations, we have:

Corollary 2 1. Let $\sigma(z)$ be a Riesz regularisation of a symbol $\sigma \in CS(U)$. The map
\[ z \mapsto \int_{\mathbb{R}^d} \sigma(z)(x, \xi) d\xi \]
is meromorphic with simple poles at $z = 0$.

2. The finite part at $z = 0$ and the “cut-off” finite part coincide:
\[ \int_{T^*_xU} \sigma(x, \xi) d\xi = \text{fp}_{z=0} \int_{T^*_xU} \sigma(z)(x, \xi) d\xi. \]

In particular, the finite part at $z = 0$ is independent of the choice of cut-off function involved in the definition of the Riesz regularisation of $\sigma$.

Theorem 6 extends to classical pseudodifferential operators.

Theorem 6 Let $z \mapsto A(z) \in C^{\alpha(z)}(M, E)$ be a holomorphic family of classical pseudo-differential operators with non constant affine order $\alpha(z)$. Then the map $z \mapsto \text{TR}(A(z))$ is meromorphic with poles of order 1 at points $z_j \in \Omega \cap \alpha^{-1}([-n, +\infty[ \cap \mathbb{Z})$ and

1. [KV]
\[ \text{Res}_{z=z_j} \text{TR}(A(z)) dz = -\frac{1}{\alpha'(z_j)} \text{res}(A(z_j)). \]

2. Moreover [PS],
\[ \text{fp}_{z=z_j} \text{TR}(A(z)) = \int_M dx \left( \int_{\mathbb{R}^d} \sigma(A(z_j)) - \frac{1}{\alpha'(z_j)} \text{res}(\sigma(A'(z_j))) \right). \]
References

[AW] J. Agapito, J. Weitsman, *The weighted Euler-MacLaurin formula for a simple integral polytope*, Asian Journ. Math. 9(2005) 199-212

[BV] M. Brion, M. Vergne, *Lattice points in simple polytopes*, Jour. Amer. Math. Soc. 10 (1997) 371-392

[C] P. Cartier, *An introduction to zeta functions*, in From Number theory to physics ed. M. Waldschmidt, P. Moussa, J.-M. Luck, C. Itzykson, Springer Verlag (1989)1-63

[CS] S. Cappell, J. Shaneson, *Genera of algebraic varieties for lattices above dimension one*, Bull. A.M.S. 30 (1994) 62-69 ; *Euler-Maclaurin expansions for lattices above dimension one*, C. R. Acad. Sci. Paris Sr. 1 Math. 321 (1995) 885-890

[ChS] S. Chowla, A. Selberg, *On Epstein’s zeta function*, Proc. Natl. Acad. Sci. USA 35 1949

[FG] L. Friedlander, V. Guillemin, *Determinants for zeroth order operators*, arXiv:math.SP/0601743 (2006)

[FGLS] B.V. Fedosov, F. Golse, E. Leichtnam, E. Schrohe, *The noncommutative residue for manifolds with boundary*, J. Funct. Anal. 142 (1996) 1-31

[G1] V. Guillemin, *A new proof of Weyl’s formula on the asymptotic distribution of eigenvalues*, Adv. Math. 55 (1985) 131–160

[G2] V. Guillemin, *Riemann-Roch for toric orbifolds*, J. Diff. Geom. 45 53-73

[GSW] V. Guillemin, S. Sternberg, J. Weitsman, *The Ehrart function for symbols*, Surveys in Differential Geometry, Special memorial volume dedicated to S.S. Chern (2006)

[Ha] G. Hardy, *Divergent series*, Oxford University Press, 1967

[Ho] L. Hörmander, *The analysis of linear partial differential operators. III. Pseudodifferential operators*. Grundlehren Math. Wiss. 274, Springer 1994

[KP] A.G. Khovanskii, A.V. Pukhlikov, *The Riemann-Roch theorem for integrals and sum of quasipolynomials on virtual polytopes*, Algebra i Analiz 4 (1992) 188-216, translation in St Petersburg Math. J. 4 (1993) 789-812

[KSW1] Y. Karshon, S. Sternberg, J. Weitsman, *The Euler-Maclaurin formula for simple integral polytopes*, Proc. Natl. Acad. Sci. USA 100, no.2 (2003). 426-433

[KSW2] Y. Karshon, S. Sternberg, J. Weitsman, *Euler-MacLaurin with remainder for a simple integral polytope*, Duke Math. Journ. 130, no.3 (2005) 401-434

[KV] M. Kontsevich, S. Vishik, *Geometry of determinants of elliptic operators*, Func. Anal. on the Eve of the XXI century, Vol I, Progress in Mathematics 131 (1994) 173–197 ; *Determinants of elliptic pseudo-differential operators*, Max Planck Preprint (1994)
[L] M. Lesch, *On the non commutative residue for pseudo-differential operators with log-polyhomogeneous symbols*, Ann. Global Anal. Geom. **17** (1998) 151–187

[MP] D. Manchon, S. Paycha, *Renormalised Chen integrals for symbols on $\mathbb{R}^n$ and renormalised polyzeta functions* (Preprint 2006)

[MMP] D. Manchon, Y. Maeda, S. Paycha, *Stokes’ formulae on classical symbol valued forms and applications*, [math.DG/0510454](http://arxiv.org/abs/math.DG/0510454) (2005)

[MSS] L. Maniccia, E. Schrohe, J. Seiler, *Uniqueness of the Kontsevich-Vishik trace*, arXiv:math.FA/0702250 (2007)

[P1] S. Paycha, *Regularised sums, integrals and traces; a pseudodifferential point of view*, Lecture Notes in preparation

[P2] S. Paycha, *The noncommutative residue and the canonical trace in the light of Stokes’ and continuity properties*, arXiv:0706.2552 (2007)

[PS] S. Paycha, S. Scott, *A Laurent expansion for regularised integrals of holomorphic symbols*, Geom. Funct. Anal., to appear. arXiv:math.AP/0506211 (2005)

[Se] R.T. Seeley, *Complex powers of elliptic operators, singular integrals*, Proc. Symp. Pure Math., Chicago, Amer. Math. Soc., Providence (1988) 288-307

[S] J. Shaneson, *Characteristic classes, Lattice points and Euler-Maclaurin formulae*, Proceedings of the International Congress of Mathematicians, Zurich, 1994, Basel Birkhäuser Verlag 1995

[Sh] M.A. Shubin, *Pseudo-differential operators and spectral theory*, Springer Verlag 1980

[T] M.E. Taylor, *Pseudo-differential operators*, Princeton Univ. Press 1981

[Tr] F. Trèves, *Introduction to Pseudo-differential and Fourier integral operators, Vol 1*, Plenum Press 1980

[W1] M. Wodzicki, *Spectral asymmetry and noncommutative residue (in Russian)*, Habilitation thesis, Steklov Institute (former) Soviet Academy of Sciences, Moscow 1984

[W2] M. Wodzicki, *Non commutative residue, Chapter I. Fundamentals, K-theory, Arithmetic and Geometry*, Springer Lecture Notes **1289** (1987) 320-399