EMERGENT UNIVERSE IN STAROBINSKY MODEL

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Abstract

We present an emergent universe scenario making use of a new solution of the Starobinsky model. The solution belongs to a one parameter family of solutions, where the parameter is determined by the number and the species (spin-values) of primordial fields. The general features of the model have also been studied.

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1 INTRODUCTION

Recently Ellis and Maartens [1] reconsidered the possibility of a cosmological model in which there is no big-bang singularity, no beginning of time and the universe effectively avoids a quantum regime for space-time by staying large at all times. The universe starts out in the infinite past as an almost static universe and expands slowly, eventually evolving into a hot big-bang era. An interesting example of this scenario is given by Ellis, Murugan and Tsagas [2], for a closed universe model with a minimally coupled scalar field $\phi$, which has a special form of interaction potential $V(\phi)$. It was pointed out that this potential is similar to what one obtains from a $R+\alpha R^2$ theory after a suitable conformal transformation and identifying $\phi = -\sqrt{3} \ln(1+2\alpha R)$ with a negative $\alpha$. Although the probability of these solutions is not high, the emergent universe scenario nevertheless merits attention as it solves many conceptual and technical problems of the big bang model. In this paper, we point out that the Starobinsky model, the original as well as the modified version, permit solutions describing an emergent universe. The solution may be used to model varieties of cosmological scenarios consistent with the observational results, available at present and expected in the near future. Thus, it may be possible to build models which avoid the quantum regime for space-time but share the good features of the standard big bang model.

In the models considered in Ref. [1] and [2], a closed universe was considered. However, recent results from BOOMERANG and WMAP indicate that the universe is most likely to be spatially flat. If the universe has always been large enough, the field equations become simpler. In fact, in the Starobinsky model, the field equations can be written as a second order differential equation for the Hubble parameter $H$, vide equation (7).

2 STAROBSKIN Model :

In the Starobinsky model, one considers the semi-classical Einstein equation,

$$ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G < T_{\mu\nu} > $$

(1)

where $< T_{\mu\nu} >$ is the vacuum expectation value of the energy momentum tensor. In the case of free, massless, conformally invariant fields the vacuum expectation value $< T_{\mu\nu} >$
can be written, with the Robertson Walker metric as,

\[ < T_{\mu\nu} > = K_1 (1) H_{\mu\nu} + K_3 (3) H_{\mu\nu} \]  

where \( K_1 \) and \( K_3 \) are numbers and

\[ (1) H_{\mu\nu} = 2 R_{\mu\nu,\sigma} - 2 g_{\mu\nu} R^{\sigma}_{;\sigma} + 2 R R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^2, \]  

\[ (3) H_{\mu\nu} = 2 R^\sigma_{\mu} R_{\nu;\sigma} - \frac{2}{3} R R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{\sigma\tau} R_{\sigma\tau} + \frac{1}{4} g_{\mu\nu} R^2. \]

Equation (2), written for \( < T^{\mu}_{\mu} > \), gives the well-known trace-anomaly, indicating that the conformal invariance is broken by the regularization process. Note that the two tensors \( (1) H_{\mu\nu} \) and \( (3) H_{\mu\nu} \) have different features. The tensor \( (1) H_{\mu\nu} \) is identically conserved and can be obtained by varying a local action \( A \sim \int \sqrt{-g} R^2 \, d^4x \). However, one of the counter terms to be added to the lagrangian to regularise \( < T^{\mu}_{\mu} > \) has the form \( \mu \sqrt{-g} R^2 \), where \( \mu \) is a logarithmically divergent constant, thus permitting the possibility of an addition of any finite number to \( \mu \). Hence, \( K_1 \) is not determined and can be given any arbitrary value including a negative one. The tensor \( (3) H_{\mu\nu} \) is conserved only in a conformally flat universe and it cannot be obtained by variation of any local action. The constant \( K_3 \) is fully determined, once the fields are specified, e.g.,

\[ K_3 = \frac{1}{1440\pi^2} \left( N_o + \frac{11}{2} N_{1/2} + 31 N_1 \right), \]  

where \( N_I \) gives the number of quantum fields of spin \( I \). The freedom of choosing \( K_1 \) arbitrarily has led to the modified Starobinsky [3,4] model, where one adds a counter term \( \sim \sqrt{-g} R^2 \), and chooses \( K_1 \gg K_3 \). The theory then becomes essentially a typical \( R^2 \) -theory. However, the earlier work on Starobinsky models were done in the context of a big bang model. The emergent universe model needs a different scenario and one makes different choices of \( K_1 \) and \( K_3 \), as will be shown below.

Let us choose \( 8\pi G = 1 \) and write the evolution equation (1) of the flat FRW universe as

\[ \frac{\dot{a}^2}{a^2} = K_3 \frac{\dot{a}^4}{a^4} - 6K_1 \left[ 2 \frac{\ddot{a}}{2a^2} \frac{d}{dt} - \frac{\ddot{a}^2}{a^2} + 2 \frac{\dot{a}^2}{a^3} - 3 \frac{\dot{a}^2}{a^2} \right] \]  

\[ \frac{2}{a^2} \frac{d}{dt} - \frac{\ddot{a}^2}{a^2} + 2 \frac{\dot{a}^2}{a^3} - 3 \frac{\dot{a}^2}{a^2} \]
which can be written in terms of the Hubble parameter

\[ H^2 \left( \frac{1}{K_3} - H^2 \right) = -\frac{6K_1}{K_3} \left( 2H \ddot{H} + 6H^2 \dot{H} - \dot{H}^2 \right). \]  

(7)

Equation (7) will determine the evolution of the universe when the initial conditions are provided.

### 3 EMERGENT UNIVERSE:

We now look for a solution which describes a universe which exists eternally and always remains large so that a classical description is possible at all times. The matter, as indicated earlier, will be described by quantum field theories. We first note the following two exact solutions:

- **H = 0** is a solution but it describes an eternally static universe. It can be checked that the solution is stable against small linear perturbations, if \( K_1 \) is positive. However, with \( K_1 \) negative, the state is unstable.

- **H = \( \frac{1}{\sqrt{K_3}} \)** gives a de Sitter type solution, but the solution is not stable.

To get an emergent universe, we look for a solution of the suggestive form

\[ a(t) = a_o \left( \beta + e^{\alpha t} \right)^{\omega} \]  

(8)

where the constants \( \alpha \) and \( \omega \) will be determined from equation (7). The constants \( a_o \) and \( \beta \) may be determined from initial conditions. Since we have

\[ H = \frac{\omega \alpha e^{\alpha t}}{\beta + e^{\alpha t}}, \quad \dot{H} = \frac{\omega \alpha^2 e^{\alpha t}}{(\beta + e^{\alpha t})^2}, \quad \ddot{H} = \frac{\omega \alpha^3 e^{\alpha t}(1 - e^{\alpha t})}{(\beta + e^{\alpha t})^3} \]  

(9)

the function in (8) will be a solution, if \( \omega = \frac{2}{3} \), \( \alpha = \frac{3}{2} \sqrt{\frac{1}{K_3}} \) and \( K_1 = -\frac{2}{27}K_3 \). Thus \( K_1 \) will be chosen negative in this model.

The fact that the emergent universe scenario is indeed permitted by equation (7) comes as a surprise. Since \( K_1 \) and \( K_3 \) are related, we have a one parameter family of solutions specified by the value of \( K_3 \). The general features of the solutions are

1) The scale factor \( a(t) \) has a non-zero value \( a(t \to -\infty) = a_o \beta^{2/3} \) as \( t \to -\infty \), which may be chosen much larger than the Planck length. The Hubble parameter \( H \) and its
time derivatives $H$ and $\dot{H}$ all vanish in the limit $t \to -\infty$. Thus the solution describes an emergent universe with natural initial conditions.

2) If $\beta > 1$, the universe remains almost static during the period $-\infty < t < \frac{2}{3} \sqrt{K_3} \ln \beta = t_o$. Thus during the entire infinite past of $t = 0$, the universe expands only by a factor

$$\frac{a(t = 0)}{a(t \to -\infty)} = \left(1 + \frac{1}{\beta}\right)^{2/3} \sim 1$$

for a large $\beta$.

For $t > \frac{2}{3} \sqrt{K_3} \ln \beta$, the universe expands rapidly, eventually reaching an asymptotically de Sitter stage. Fig 1, gives the time variation of the Hubble parameter and its derivatives, for $K_3 = 1.8$ (which occurs if the particle number and species correspond to the minimal SU(5) model) and $\beta$ is given a value $10^4$.

3) It may be useful to check if the solution is stable under small linear perturbations. We write

$$H = H_s (1 + \delta)$$

Figure 1: Variation of $h = H X 10^{-2}$ and the derivatives, $\dot{H}$ and $\ddot{H}$ are represented by thin, broken and thick lines respectively ($\beta = 10000$ and $K_3 = 1.8$).
where $\delta$ is a small perturbation and $H_s = \frac{\dot{\phi} e^{\alpha t}}{(\beta + e^{\alpha t})}$ is the solution obtained above. Substituting this in equation (7), we obtain the equation for $\delta$:

$$\ddot{\delta} + A(t) \dot{\delta} + B(t) \delta = 0 \quad (12)$$

where

$$A(t) = \frac{\alpha (1 + 2 e^{\alpha t})}{\beta + e^{\alpha t}},$$

$$B(t) = \frac{2 \alpha^2 e^{\alpha t}}{(\beta + e^{\alpha t})^2}. \quad (13)$$

Note that $\frac{\alpha}{\beta} < A(t) < 2\alpha$ and $B(t) > 0$ for $-\infty < t < \infty$. It is difficult to solve the equation (12). However, if we assume that $\delta$ has a solution $\delta \sim e^{mt}$, we must have, for $t$ negative,

$$m^2 + \frac{\alpha}{\beta} m \sim 0 \quad (14)$$

and $m$ cannot be positive, since $\alpha$ and $\beta$ are both positive. Thus the solution seems to be stable under small perturbations at least in the negative $t$ regime.

4) The evolution can be described alternatively in terms of a scalar field by substituting $H = \phi^2$ in equation (7). This gives

$$\ddot{\phi} + 3H \dot{\phi} + \frac{dV}{d\phi} = 0 \quad (15)$$

where $V = -m^2 \phi^2 + \lambda \phi^6$ with $m^2 = \frac{1}{48|K_1|}$ and $\lambda = \frac{K_3}{144|K_1|}$. Note that $V(\phi)$ has the shape of a double well potential with a maximum at $\phi = 0$ and two minima for $\phi_m = \pm (m^2)_{3/4}$, giving $V_{min} = -\frac{1}{72|K_1|V K_3}$. Also note that $\phi = 0$ gives an unstable static universe while the evolution of the emergent universe is given by the path from $V(\phi) = 0$ to $V_{min}$. However, $H$ here is time dependent and is determined self-consistently by the instantaneous value of $\phi$. The equation (15) does not make the problem simpler, although it looks very familiar and attractive.

5) The prescribed evolution of the universe gets modified when particles are produced due to the expansion of the universe. We have considered above only massless, conformally coupled fields in conformally flat spacetimes, i.e., the conformally trivial situation where
no particles are produced. The perturbative calculations of particle production make use of small deviations from this conformal triviality. One considers either (i) a small mass and/or (ii) deviations from a conformally flat space time, say by assuming a Bianchi I spacetime deviating a little from a FRW spacetime or (iii) assuming a small non conformal coupling with $|\xi - \frac{1}{6}| = \epsilon$. For simplicity, we shall consider here the case of non conformal coupling. Let us consider a real massive scalar field satisfying the equation

$$\ddot{\phi} + 3H\dot{\phi} + (m^2 + \xi R)\phi = 0$$  \hspace{1cm} (16)$$

with $|\xi - \frac{1}{6}| = \epsilon << 1$. As the scale factor changes, the changing gravitational field feeds energy into the perturbed scalar field modes. As long as the mode frequency of the field is greater than the Hubble expansion rate, a co-moving detector will not respond. Modes with lower frequencies will, however, be excited. Thus the presence of mass makes the particle production process less efficient. To begin with we consider the case $m = 0$ and $\epsilon \neq 0$. Zeldovich and Starobinsky, Starobinsky [3], Birrell and Davies [3] and Vilenkin [4] have studied this case. The rate of particle production per unit volume per unit conformal time $\eta$, ($d\eta = \frac{dt}{a(t)}$), is given by

$$\frac{dn}{d\eta} = \frac{1}{2} \epsilon^2 R^2$$  \hspace{1cm} (17)$$

where $R$ is the scalar curvature. This gives

$$Y = \frac{2a_o}{\epsilon^2} \frac{dn}{dt} = \left[ \frac{36\omega^2 \alpha^4 e^{2\omega t}(1 + 2\omega e^{\omega t})^2}{(\beta + e^{\omega t})^{14/3}} \right]$$  \hspace{1cm} (18)$$

In fig. 2, we have plotted $Y$ against $t$ for $\beta = 10,000$ and $K_3 = 1.8 and 2.5$. We note that the rate of particle production peaks around $t \sim t_o$, as expected. Increase in the number of species of particles i.e., a higher value of $K_3$ will indicate a delay in particle production and also a wider peak. The dependence of the rate of particle production on the parameter $\beta$ is shown in fig. 3. The absolute value of the peaks depends on the value of the scale factor $a(t)$ and the constant $a_o$ may be determined from the rate of particle production. Although we have considered zero mass particle we expect similar results even when $m$ is nonzero but small ( see Vilenkin [4]). During the subsequent evolution the particles produced thermalize and the universe enters eventually into a radiation dominated stage
of the standard hot bigbang model. The details of the process of particle production and
the evolution of the universe will be considered elsewhere.

4 DISCUSSION

We have presented here a one parameter family of solutions of the Starobinsky model which
describes an emergent universe. The earlier work on Starobinsky model was done in the
context of big bang singularity and the particle production was achieved by its oscillatory
solution. The present solution on the other hand describes the universe which is almost
dormant during the infinite past period $-\infty < t < \frac{2}{3}\sqrt{K_3} \ln \beta$, after which it undergoes a
rapid expansion, see fig. 1. The parameters $a_o$ and $\beta$ are related with initial conditions.
They also introduce a new mass scale in the process of particle production. We have
not considered here any specific theory of particle interactions which will determine the
subsequent evolution of the universe as well as the details of large scale structure formation.
Whether the present scenario can successfully explain the present observational data needs
further study. However, the fact that one encounters solutions describing an emergent
universe in different contexts may be a good reason for taking such solutions seriously,
although the probabilities for such solutions may not be very high.
Figure 3: Dependence of rate of particle production on time for $K_3 = 1.8$ and $\beta = 10000$ (thin line) and $\beta = 20,000$ (broken line)

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