TOPOLOGICAL CORRELATION FUNCTIONS IN MINKOWSKI SPACETIME

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Abstract

We consider a class of non-unitary Toda theories based on the Lie superalgebras $A^{(1)}(n,n)$ in two-dimensional Minkowski spacetime, which can be twisted into a topological sector. In particular we study the simplest example with $n = 1$ and real fields, and show how this theory can be treated as an integrable perturbation of the $A(1,0)$ superconformal model. We construct the conserved currents and the vertex operators which are chiral primary fields in the conformal theory. We then define the chiral ring of the $A^{(1)}(1,1)$ Toda theory and compute topological correlation functions in the twisted sector. The calculation is performed using a $N = 2$ off-shell superspace approach.
1 Introduction

Toda field theories in two spacetime dimensions are quantum integrable systems whose affine version realizes an off–critical deformation of the corresponding conformal non–affine model. The on–shell properties of these theories are well understood by now: the exact S–matrices have been constructed \[1\], being elastic and two–body factorizable as a consequence of the existence of higher–spin conserved currents \[2\].

Off–shell the situation is not so clear: form factors and correlation functions have been computed only for few specific models \[3\] and in this case integrability doesn’t seem to play the same relevant role as in the on–shell counterpart.

The addition of supersymmetry might provide better hopes for the construction of completely solvable models. In particular it is well known that chiral and antichiral Green’s functions of any \(N = 2\) supersymmetric theory are spacetime independent, a drastic simplification if one were to attempt their complete determination. These classes of Green’s functions are the ones which appear as physical correlators in the twisted topological version \[4\] of the \(N = 2\) supersymmetric theory and for some specific choices of the superpotential they have been computed exactly \[5, 6\].

In this paper we focus on the calculation of chiral correlation functions for the class of \(N = 2\) supersymmetric Toda theories in Minkowski spacetime which are based on the \(A^{(1)}(n, n)\) superalgebras \[7, 8, 9\]. These are nonunitary systems that can be twisted into a topological, BRST invariant sector \[10\] where ghost–like fields are eliminated from the physical spectrum. There survives instead an infinite number of solitonic configurations \[10\] which at the quantum level can be realized as BRST invariant vertex operators. The identification of the chiral ring and the study of topological correlators are the main issues that we address.

In section 2 we briefly present the \(N = 2\) supersymmetric formulation of the \(A^{(1)}(n, n)\) Toda field theory action. Then we concentrate on the simplest example with \(n = 1\) and real fields. The conserved currents and the corresponding charges are computed in Section 3. There we show how the \(A^{(1)}(1, 1)\) affine Toda theory can be obtained adding relevant perturbation terms to the superconformal \(A(1, 0)\) model. Section 4 is devoted to the definition and construction of the chiral ring: it consists in an infinite set of chiral primary vertex operators which generate physical states acting on the vacuum. The spectrum is infinite since we are dealing with a nonunitary theory. Local and nonlocal primary fields are present corresponding to classical solutions with trivial boundary conditions or solitonic ones respectively. They are constructed perturbatively with respect to the superconformal \(A(1, 0)\) model and classified by their scale dimensions, the \(O(1, 1)\) charge and the solitonic charge. The topological version of the theory is reviewed in Section 5, while the computation of topological correlation functions is the subject of Section 6. We analyze how the requirement of \(O(1, 1)\) charge conservation can be implemented consistently while maintaining background charge balance. We also show that spacetime indendence of the topological correlators reduces the computation of all of them to the calculation of one–point
correlation functions. In Section 7 we perform the explicit calculation for the \( c = 1 \) model to zero order in the relevant perturbation. We use superspace techniques which not only simplify the algebra but are actually necessary for the correct introduction of screening operators which are marginal perturbations of the free field theory. Finally in Section 8 we present our conclusions. Notations and conventions are listed in Appendix A, while some details of the computation of the correlation function are collected in Appendix B.

2 The \( A^{(1)}(n, n) \) affine Toda field theory

Among the class of \( N = 1 \) supersymmetric Toda field theories the ones associated to the \( A^{(1)}(n, n) \) superalgebras admit a second supersymmetry \[1\] and they can be formulated in terms of \( 2n \), \( N = 2 \) complex superfields \( \Psi_i^{(+)}, \bar{\Psi}_i^{(+)}, \Psi_i^{(-)}, \bar{\Psi}_i^{(-)} \) whose components are \( \Psi_i^+ \to (\phi_i^+, \psi_i^+, \bar{\psi}_i^+, F_i^+) \), \( \Psi_i^- \to (\phi_i^-, \psi_i^-, \bar{\psi}_i^-, F_i^-) \) and \( \Psi_i^{+*}, \Psi_i^{-*} = \Psi_i^{*-} \) (conventions on \( N = 2 \) superspace are listed in Appendix A). The superspace action in two-dimensional Minkowski spacetime is given by

\[
S = \frac{1}{2\pi} \left\{ \int d^2z d^2\theta \, K_{ij} \left[ \Psi_i^+ \bar{\Psi}_j^- + \Psi_i^- \bar{\Psi}_j^+ \right] + \frac{1}{\beta^2} \int d^2z d^2\theta \, W(\Psi) + \frac{1}{\beta^2} \int d^2z d^2\theta \, W(\bar{\Psi}) \right\}
\]

(2.1)

where the superpotential is

\[
W(\Psi) = \sum_{i=1}^n e^{\beta \Psi_i^+} + g_+ e^{-\beta} \sum_{i=1}^n \Psi_i^+ + \sum_{i=1}^n e^{\beta \Psi_i^-} + g_- e^{-\beta} \sum_{i=1}^n \Psi_i^-
\]

(2.2)

\( \beta \) is a coupling constant and \( K_{ij} \) is the \( n \times n \) matrix

\[
K = \begin{pmatrix}
1 & 1 & 1 & 1 & \cdots \\
1 & 0 & 0 & 0 & \cdots \\
1 & 1 & 1 & \cdots \\
1 & 0 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]

(2.3)

\[
K^{-1} = \begin{pmatrix}
0 & 1 & 0 & \cdots \\
1 & 0 & -1 & 0 & \cdots \\
0 & -1 & 0 & 1 & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & (-1)^n & \cdots & (-1)^n + 1
\end{pmatrix}
\]

We have included two coupling constants \( g_+ \) and \( g_- \) in the superpotential for later convenience. The \( A^{(1)}(n, n) \) theory is obtained by setting \( g_+ = g_- = 1 \), whereas for \( g_- = 1, \ g_+ = 0 \) the superpotential reduces to the one for the \( N = 1 \) superconformal invariant \( A(n, n) \) theory and for \( g_+ = g_- = 0 \) one obtains the \( N = 2 \) superconformal \( A(n, n - 1) \) Toda. At the component level, after elimination of the auxiliary fields, the action can be reexpressed as

\[
S = \frac{1}{2\pi} \int d^2z \left\{ K_{ij} \left[ \partial \phi_i^- \bar{\partial} \phi_j^+ + \frac{i}{2} \psi_i^- \bar{\partial} \psi_j^+ - \frac{i}{2} \bar{\psi}_i^- \partial \psi_j^+ \right] - \frac{\partial V}{\partial \phi_i^+} K_{ij}^{-1} \frac{\partial V}{\partial \phi_j^-} + \frac{1}{2} \bar{\psi}_i^+ \psi_j^+ \frac{\partial^2 V}{\partial \phi_i^+ \partial \phi_j^+} + \frac{1}{2} \bar{\psi}_i^- \psi_j^- \frac{\partial^2 V}{\partial \phi_i^- \partial \phi_j^-} + \text{h.c.} \right\}
\]

(2.4)
where
\[ V(\phi^\pm) = \frac{1}{\beta^2} \left[ \sum_{i=1}^n e^{i\beta\phi^+_i} + g_\pm e^{-\beta \sum_{i=1}^n \phi^+_i} \right] \]  
(2.5)

These theories are integrable and in Refs. [8, 9] the first non trivial higher–spin quantum currents were constructed to all orders in perturbation theory. Moreover they are not unitary since the matrix \( K_{ij} \) is not positive definite.

The action in eq. (2.1) is invariant under a complexified \( N = 2 \) algebra generated by the supersymmetry operators
\[
Q^+ = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \theta^+} - i \bar{\theta}_+ \partial \right) \quad Q^- = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \theta^-} + i \bar{\theta}_- \partial \right)
\]
\[
\bar{Q}^+ = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \bar{\theta}^+} - i \theta_- \partial \right) \quad \bar{Q}^- = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \bar{\theta}^-} + i \theta_+ \partial \right)
\]  
(2.6)

which satisfy \( \{Q^+, Q^-\} = -i \partial \), \( \{\bar{Q}^+, \bar{Q}^-\} = i \bar{\partial} \). In these models the usual \( U(1) \) invariance of \( N = 2 \) supersymmetry is extended to a \( U(1) \otimes O(1,1) \) invariance. As shown in Ref. [10] this allows to perform a topological twist of the theory directly in Minkowski space by combining the \( O(1,1) \) Lorentz group with this extra \( O(1,1) \) invariance.

When the coupling constant \( \beta \) is imaginary the classical equations of motion from the action (2.4) with \( g_+ = g_- = 1 \) admit solitonic solutions which have been studied in Ref. [10]. In particular the soliton spectrum survives the twist and the topological sector has the same solitonic content as the bosonic \( a_n(1) \) theory.

3 The model

We restrict now our attention to the simplest model in the above mentioned class, the \( A^{(1)}(1,1) \) theory with coupling constant \( i \beta \) in the exponentials. Furthermore we choose the scalar fields \( \phi^\pm \) to be real and \( \psi^\pm, \bar{\psi}^\pm \) to be the chiral and antichiral components of two Majorana fermions \( (\psi^{\pm*} = \psi^\pm, \bar{\psi}^{\pm*} = -\bar{\psi}^\pm) \). As discussed in Ref. [10] these reality conditions do not affect the \( O(1,1) \) invariance which allows to perform the twist into the topological sector. However the \( U(1) \) invariance is lost and the theory does not possess a genuine \( N = 2 \) supersymmetry. The action becomes
\[
S = \frac{1}{2\pi} \int d^2z \left[ \partial \phi^\pm \partial \phi^- + \frac{i}{2} \psi^+ \partial \psi^- - \frac{i}{2} \bar{\psi}^+ \partial \bar{\psi}^- \right.
\]
\[
+ \frac{1}{\beta^2} e^{i\beta(\phi^+ + \phi^-)} - \frac{1}{\beta^2} g_- e^{i\beta(\phi^+ - \phi^-)} - \frac{1}{\beta^2} g_+ e^{i\beta(\phi^- - \phi^+)} + \frac{1}{\beta^2} g_+ e^{-i\beta(\phi^+ + \phi^-)}
\]
\[
- \frac{1}{2} \bar{\psi}^\pm \psi^\pm \left( g_- e^{i\beta \phi^+} + g_+ e^{-i\beta \phi^-} \right) - \frac{1}{2} \psi^- \bar{\psi}^- \left( g_+ e^{i\beta \phi^-} + g_- e^{-i\beta \phi^+} \right) \left] \right.
\]
\[
\equiv S_0 + S_{\text{pert}} \]  
(3.1)

where
\[
S_0 = \frac{1}{2\pi} \int d^2z \left[ \partial \phi^\pm \partial \phi^- + \frac{i}{2} \psi^+ \partial \psi^- - \frac{i}{2} \bar{\psi}^+ \partial \bar{\psi}^- + V_0 \right]
\]  
(3.2)
with
\[ V_0 = \frac{1}{\beta^2} e^{i\beta(\phi^+ + \phi^-)} - \frac{1}{2} \psi^+ e^{i\beta\phi^+} - \frac{1}{2} \psi^- e^{i\beta\phi^-} \] (3.3)

is the action of the \( A(1,0) \) theory. The bosonic and fermionic fields satisfy equal–time commutation relations
\[ [\phi^\pm(x^0, x^1), \phi^\mp(y^0, y^1)]_{x^0 = y^0} = 4\pi i \delta(x^1 - y^1) \]
\[ \{\psi^\pm(x^0, x^1), \psi^\mp(y^0, y^1)\}_{x^0 = y^0} = 4\pi \sqrt{2} \delta(x^1 - y^1) \]
\[ \{\bar{\psi}^\pm(x^0, x^1), \bar{\psi}^\mp(y^0, y^1)\}_{x^0 = y^0} = -4\pi \sqrt{2} \delta(x^1 - y^1) \] (3.4)

In the following we will drop the subscript \( x^0 = y^0 \), all commutators being evaluated at equal times.

Despite the lack of \( U(1) \) invariance the theory can be formulated in \( N = 2 \) superspace. In terms of two superfields \( \Psi^+ \) and \( \Psi^- \), chiral and antichiral respectively, with components \( \Psi^\pm \rightarrow (\phi^\pm, \psi^\pm, \bar{\psi}^\pm, F^\pm) \) the action in eq. (3.1) can be written as
\[ S = \frac{1}{2\pi} \left\{ \int d^2 z d^4 \theta \Psi^+ \Psi^- + \frac{1}{\beta^2} \int d^2 z d^2 \theta \left[ e^{i\beta\phi^+} + g_+ e^{-i\beta\phi^-} \right] \right. \\
+ \left. \frac{1}{\beta^2} \int d^2 z d^2 \bar{\theta} \left[ e^{i\beta\phi^-} + g_- e^{-i\beta\phi^+} \right] \right\} \] (3.5)

Later we will use the superspace formulation as a suitable device to compute correlation functions.

The symmetries of the model are generated by the following conserved currents: the spin–2 stress–energy tensor, the spin–1 \( O(1,1) \) generator and the two spin \( \frac{3}{2} \) supersymmetries. According to the definitions given in Appendix A, we have
\[ T = -\partial \phi^+ \partial \phi^- - \frac{i}{4} \psi^- \partial \psi^+ - \frac{i}{4} \psi^+ \partial \psi^- - \frac{i}{2\beta} \partial^2 \phi^+ - \frac{i}{2\beta} \partial^2 \phi^- \]
\[ J = \frac{i}{2} \psi^+ \psi^- + \frac{i}{\beta} \partial \phi^+ - \frac{i}{\beta} \partial \phi^- \]
\[ G^+ = -\psi^+ \partial \phi^- - \frac{i}{\beta} \partial \phi^+ \]
\[ G^- = i \psi^- \partial \phi^+ - \frac{1}{\beta} \partial \psi^- \] (3.6)

which satisfy the following conservation equations
\[ \bar{\partial} J = \partial \left[ \frac{i}{2} \bar{\psi}^- \psi^+ + \frac{i}{\beta} \bar{\partial} \phi^+ - \frac{i}{\beta} \bar{\partial} \phi^- \right] \equiv -\bar{\partial} \bar{J} \]
\[ \bar{\partial} G^+ = \partial \left( \frac{2}{\beta} g_+ \bar{\psi}^- e^{-i\beta\phi^-} \right) - \frac{1}{\beta} \partial \left[ i \bar{\partial} \psi^+ + \bar{\psi}^- \left( e^{i\beta\phi^-} + g_- e^{-i\beta\phi^-} \right) \right] \equiv -\partial \bar{G}^+ \]
\[ \bar{\partial} G^- = \partial \left( -\frac{2}{\beta} i g_+ \bar{\psi}^- e^{-i\beta\phi^-} \right) + \frac{i}{\beta} \partial \left[ i \bar{\partial} \psi^+ + \bar{\psi}^- \left( e^{i\beta\phi^-} + g_+ e^{-i\beta\phi^-} \right) \right] \equiv -\partial \bar{G}^- \]
\[\partial T = \partial \left[ \frac{1}{\beta^2} \left( g_+ e^{i\beta (\phi^+ - \phi^-)} + g_+ e^{i\beta (\phi^+ - \phi^-)} - 2g_+ g_- e^{-i\beta (\phi^+ + \phi^-)} \right) \right. \]

\[\left. + \frac{1}{2} g_- \bar{\psi}^- \psi^- e^{-i\beta \phi^-} + \frac{1}{2} g_+ \bar{\psi}^+ \psi^+ e^{-i\beta \phi^+} \right] \]

\[ - \frac{i}{4} \partial \left[ \psi^+ \bar{\partial} \psi^- - i \psi^+ \bar{\psi}^+ \left( e^{i\beta \phi^+} + g_+ e^{-i\beta \phi^+} \right) \right] \]

\[ - \frac{i}{4} \partial \left[ \psi^- \bar{\partial} \psi^+ - i \psi^- \bar{\psi}^- \left( e^{i\beta \phi^-} + g_- e^{-i\beta \phi^-} \right) \right] \]

\[ - \frac{i}{2\beta} \partial \bar{\partial} \phi^+ - \frac{i}{\beta} \bar{\psi}^+ \psi^+ \left( e^{i\beta \phi^+} - g_+ e^{-i\beta \phi^+} \right) \]

\[ + \frac{i\beta}{2} \bar{\psi}^+ \psi^+ \left( e^{i\beta \phi^+} - g_+ e^{-i\beta \phi^+} \right) \]

\[ - \frac{i}{2\beta} \partial \bar{\partial} \phi^- - \frac{i}{\beta} \bar{\psi}^- \psi^- \left( e^{i\beta \phi^-} - g_- e^{-i\beta \phi^-} \right) \]

\[ + \frac{i\beta}{2} \bar{\psi}^- \psi^- \left( e^{i\beta \phi^-} - g_- e^{-i\beta \phi^-} \right) \]

\[ \equiv - \partial \bar{T} \]  \hspace{1cm} (3.7)

We also have

\[ \bar{T} = -\bar{\partial} \phi^+ \bar{\partial} \phi^- + \frac{i}{4} \bar{\psi}^- \bar{\partial} \psi^+ + \frac{i}{4} \bar{\psi}^+ \bar{\partial} \psi^- - \frac{i}{2\beta} \bar{\partial}^2 \phi^+ - \frac{i}{2\beta} \bar{\partial}^2 \phi^- \]

\[ \bar{J} = \frac{i}{2} \bar{\psi}^+ \psi^+ - \frac{i}{\beta} \bar{\partial} \phi^+ + \frac{i}{\beta} \bar{\partial} \psi^- \]

\[ \bar{G}^+ = -\bar{\psi}^+ \bar{\partial} \phi^- - \frac{i}{\beta} \bar{\partial} \psi^+ \]

\[ \bar{G}^- = i \bar{\psi}^- \bar{\partial} \phi^+ - \frac{1}{\beta} \bar{\partial} \psi^- \]  \hspace{1cm} (3.8)

for which similar conservation equations hold. (We note that \( \bar{J} = J \) and \( \bar{J} = J \)).

Setting \( g_+ = g_- = 0 \) in eq. \( (3.7) \), the right-hand sides vanish on-shell since the \( A(1,0) \) theory is \( N = 2 \) superconformal invariant and on-shell conservations hold separately in the holomorphic and antiholomorphic sectors. In any case terms proportional to the equations of motion are necessary whenever the off-shell invariance of the action needs be exhibited.

We note that in eqs. \( (3.7), (3.8) \) improvement terms, i.e. total derivative terms, have been included in order to implement the correct holomorphic currents in the \( A(1,0) \) theory. In particular the term \( -\frac{i}{\beta^2} (\partial^2 \phi^+ + \partial^2 \phi^-) \) in \( T \) and the corresponding one in \( \bar{T} \) signal the presence of a background charge \( \left( \frac{1}{4}, \frac{1}{4} \right) \) coupled to the \( (\phi^+, \phi^-) \) fields at infinity. Moreover it is easy to check that the central charge for the improved stress–energy tensor is \( c = 3 - \frac{6}{\beta^2} \).

In general for a spin–s current \( J^{(s)} \) which satisfies the conservation equation \( \partial J^{(s)} + \partial \bar{J}^{(s)} = 0 \), the corresponding charge is \( \int \frac{dx}{2\pi i \sqrt{2}} (J^{(s)} + \bar{J}^{(s)}) \). Therefore from eq. \( (3.7) \) and its counterpart in the antiholomorphic sector we can construct the conserved charges of the model. In particular, the supersymmetry charges are

\[ G^+_{-\frac{s}{2}} = \int \frac{dx}{2\pi i \sqrt{2}} \left[ -\psi^+ \partial \phi^- + \frac{1}{\beta} \bar{\psi}^- \left( e^{i\beta \phi^-} - g_- e^{-i\beta \phi^-} \right) \right] \]
\[ G^{-\frac{1}{2}} = \int \frac{dx}{2\pi i\sqrt{2}} \left[ i\psi^- \partial \phi^+ - \frac{i}{\beta} \bar{\psi}^+ \left( e^{i\beta \phi^+} - g_+ e^{-i\beta \phi^+} \right) \right] \]

\[ \bar{G}^{+\frac{1}{2}} = \int \frac{dx}{2\pi i\sqrt{2}} \left[ -\bar{\psi}^+ \partial \phi^- + \frac{1}{\beta} \psi^- \left( e^{i\beta \phi^-} - g_- e^{-i\beta \phi^-} \right) \right] \]

\[ G^{-\frac{1}{2}} = \int \frac{dx}{2\pi i\sqrt{2}} \left[ i\bar{\psi}^- \partial \phi^+ - \frac{i}{\beta} \psi^+ \left( e^{i\beta \phi^+} - g_+ e^{-i\beta \phi^+} \right) \right] \quad (3.9) \]

In terms of the charges the conservation equations satisfied by the supersymmetry currents can be written as follows

\[ \partial G^+ = -\frac{2}{\beta^2} \frac{g_-}{\beta} \partial \left[ G^{-\frac{1}{2}}, e^{-i\beta \phi^-} \right] \]

\[ \partial G^- = \frac{2}{\beta^2} \frac{g_+}{\beta} \partial \left[ G^{+\frac{1}{2}}, e^{-i\beta \phi^+} \right] \]

\[ \partial \bar{G}^+ = -\frac{2}{\beta^2} \frac{g_-}{\beta} \bar{\partial} \left[ G^{-\frac{1}{2}}, e^{-i\beta \phi^-} \right] \]

\[ \partial \bar{G}^- = \frac{2}{\beta^2} \frac{g_+}{\beta} \bar{\partial} \left[ G^{+\frac{1}{2}}, e^{-i\beta \phi^+} \right] \quad (3.10) \]

These relations are exact to all orders in perturbation theory. For \( g_+ = 0 \) they agree with the result given in Ref. [13] where the perturbation corresponding to antichiral fields was neglected.

The term \( S_{\text{pert}} \) in eq. (3.1) which must be added to the \( A(1,0) \) superconformal action \( S_0 \) in order to obtain the affine \( A^{(1)}(1,1) \) theory, can be reexpressed in terms of the supersymmetry charges defined in eq. (3.9). More precisely, starting from \( S_0 \) in eq. (3.2), one obtains the \( A(1,1) \) Toda theory by adding

\[ S_- = \frac{g_-}{4\pi \beta^2} \int d^2z \left[ G^{-\frac{1}{2}}, G^{+\frac{1}{2}}, e^{-i\beta \phi^-(z,\bar{z})} \right] \quad (3.11) \]

where \( G^{-\frac{1}{2}}, G^{+\frac{1}{2}} \) are the \( A(1,0) \) supersymmetry charges. As can be seen from eq. (3.9) the presence of the \( S_- \) perturbation leaves \( G^{-\frac{1}{2}}, G^{+\frac{1}{2}} \) unchanged whereas a \( g_- \)-dependence arises in \( G^{+\frac{1}{2}}, G^{-\frac{1}{2}} \). Then the action for the affine \( A^{(1)}(1,1) \) Toda is constructed by perturbing the \( A(1,1) \) theory with

\[ S_+ = -\frac{g_+}{4\pi \beta^2} \int d^2z \left[ G^{+\frac{1}{2}}, G^{-\frac{1}{2}}, e^{-i\beta \phi^+(z,\bar{z})} \right] \quad (3.12) \]

where in eq. (3.12) \( G^{+\frac{1}{2}}, G^{-\frac{1}{2}} \) are the perturbed \( A(1,1) \) charges. The \( S_+ \) perturbation modifies the \( G^{-\frac{1}{2}}, G^{+\frac{1}{2}} \) charges as in eq. (3.9), leaving \( G^{+\frac{1}{2}}, G^{-\frac{1}{2}} \) unchanged in this second step.

Classical solutions of the equations of motion from the action (3.1) are classified by the value of the topological charges

\[ T^\pm = \frac{\beta}{2\pi} \int dx \left( \frac{\partial \phi^\pm}{\partial x} \right) = \frac{\beta}{2\pi} \left[ \phi^+(+\infty) - \phi^+(-\infty) \right] \quad (3.13) \]
For \( g_+, g_- \neq 0 \) solitonic sectors are characterized by different, non zero values of \( T^+ \) and \( T^- \). At the quantum level we promote the topological charges \( T^\pm \) to be operators which, acting on physical states, give the solitonic number. States with different solitonic content are orthogonal.

We turn now to the definition of the chiral ring of the \( A^{(1)}(1,1) \) theory.

4 The chiral ring

We study first the spectrum of primary fields for the affine Toda theory. It consists in a set of local and nonlocal vertex operators which are classified by three quantum numbers: the scale dimensions, the \( O(1,1) \) charge and the topological charge. We construct them perturbatively in \( g_+ \) and \( g_- \) treating the affine \( A^{(1)}(1,1) \) theory as an integrable perturbation of the \( N = 2 \) superconformal \( A(1,0) \) model. Therefore the quantum numbers of the vertex operators are computed with respect to the \( A(1,0) \) charges

\[
\begin{align*}
L_0 &= \int \frac{dx}{2\pi i\sqrt{2}} \frac{x}{\sqrt{2}} T \\
\bar{L}_0 &= \int \frac{dx}{2\pi i\sqrt{2}} \frac{x}{\sqrt{2}} \bar{T} \\
J_0 &= \int \frac{dx}{2\pi i\sqrt{2}} J \\
\bar{J}_0 &= \int \frac{dx}{2\pi i\sqrt{2}} \bar{J}
\end{align*}
\]

(4.1)

with \( \partial T = \partial J = 0 \) and \( \partial \bar{T} = \partial \bar{J} = 0 \) on–shell.

The local operators are given by

\[
\Phi^\pm_a =: e^{ia\beta\phi^\pm} : \text{ any } a \in \mathcal{R}
\]

(4.2)

(In the following vertex operators are always normal ordered even when not explicitly indicated.)

Due to the lack of unitarity the spectrum is infinite and we do not have any restriction on the value of \( a \). Computing the commutators with the holomorphic and antiholomorphic components of the stress–energy tensor, the \( O(1,1) \) and the topological charges we obtain

\[
\begin{align*}
[L_0, \Phi^\pm_a] &= [\bar{L}_0, \Phi^\pm_a] = \frac{a}{2} \Phi^\pm_a \\
[J_0, \Phi^\pm_a] &= -[\bar{J}_0, \Phi^\pm_a] = \mp a \Phi^\pm_a \\
[T^\pm, \Phi^\pm_a] &= 0
\end{align*}
\]

(4.3)

This shows that the local operators in eq. (12) are characterized by conformal weight \( a \) and spin zero. Moreover the holomorphic and antiholomorphic conformal dimensions are always half of the \( O(1,1) \) charges for any \( a \).

In order to construct the non–local sector of primary fields it is convenient to introduce four non–local fields \( 14, 15 \)

\[
\rho^\pm = \frac{1}{2} \left[ \phi^\pm + \int_{-\infty}^x dy \, \dot{\phi}^\pm \right]
\]

\[
\bar{\rho}^\pm = \frac{1}{2} \left[ \phi^\pm - \int_{-\infty}^x dy \, \dot{\phi}^\pm \right]
\]

(4.4)
They satisfy
\[
\bar{\partial} \rho^\pm = \frac{1}{\sqrt{2}} \int_{-\infty}^{x} dy \frac{\partial V_0}{\partial \phi^+} \\
\partial \bar{\rho}^\pm = -\frac{1}{\sqrt{2}} \int_{-\infty}^{x} dy \frac{\partial V_0}{\partial \phi^+}
\]
(4.5)
where \(V_0\) is the \(A(1,0)\) potential in eq. (3.3). From eq. (3.4) it is easy to derive the equal–time commutators for the creation and annihilation components \(\rho^\pm\)
\[
\left[ \rho^\pm(x^0,x^1) , \bar{\rho}^\mp(y^0,y^1) \right]_{|x^0=y^0} = -\log[i\mu(x^1-y^1-\epsilon)]
\]
\[
\left[ \bar{\rho}^\pm(x^0,x^1) , \rho^\mp(y^0,y^1) \right]_{|x^0=y^0} = -\log[-i\mu(x^1-y^1+i\epsilon)]
\]
\[
\left[ \rho^\pm(x^0,x^1) , \bar{\rho}^\mp(y^0,y^1) \right]_{|x^0=y^0} = \left[ \rho^\pm(x^0,x^1) , \rho^\mp(y^0,y^1) \right]_{|x^0=y^0} = -i\frac{\pi}{2}
\]
(4.6)
where \(\epsilon\) and \(\mu\) are ultraviolet and infrared cutoffs respectively. In terms of the fields in eq. (4.4) we can write the most general non–local vertex operator as
\[
\Phi^\pm_{(a,b)} = e^{i\alpha \beta \rho^\pm + ib\beta \bar{\rho}^\pm}
\]
(4.7)
It reduces to the local vertex in eq. (4.2) for \(a = b\). Using the equal–time commutation relations in eq. (4.6) we can easily find the conditions that the non–local operators need satisfy to be primary fields and then compute their scale dimensions, the \(O(1,1)\) and topological charges. They have well–defined conformal dimensions if
\[
[V_0, \Phi^\pm_{(a,b)}] = 0
\]
(4.8)
i.e. they must commute with \(e^{i\beta \phi^+}\). Using results in Ref. [13] one shows that this amounts to
\[
a - b = \frac{n}{\beta^2} \quad n \in \mathbb{Z} \quad \text{and} \quad \beta^2 a > \frac{n-1}{2} \quad n \text{ odd}
\]
\[
\beta^2 a > \frac{n-2}{2} \quad n \text{ even}
\]
(4.9)
Finally we obtain the conformal dimensions, the \(O(1,1)\) and the topological charges
\[
\left[ L_0, \Phi^\pm_{(a,b)} \right] = \frac{a}{2} \Phi^\pm_{(a,b)} \quad \left[ \bar{L}_0, \Phi^\pm_{(a,b)} \right] = \frac{b}{2} \Phi^\pm_{(a,b)}
\]
\[
\left[ J_0, \Phi^\pm_{(a,b)} \right] = \mp a \Phi^\pm_{(a,b)} \quad \left[ \bar{J}_0, \Phi^\pm_{(a,b)} \right] = \pm b \Phi^\pm_{(a,b)}
\]
\[
\left[ T^+, \Phi^\pm_{(a,b)} \right] = \begin{cases} 0 & \beta^2 (a-b) \Phi^-_{(a,b)} \\ \beta^2 (a-b) \Phi^+_{(a,b)} & \end{cases}
\]
\[
\left[ T^-, \Phi^\pm_{(a,b)} \right] = \begin{cases} 0 & \beta^2 (a-b) \Phi^+_{(a,b)} \\ \beta^2 (a-b) \Phi^-_{(a,b)} & \end{cases}
\]
(4.10)
with \(\beta^2 (a-b) = n\). Thus the conformal weight of the operator is \(\frac{a+b}{2}\) and the spin \(\frac{n}{2\beta^2}\). For \(n > 0\) it describes solitons, whereas for \(n < 0\) it creates antisolitonic states.
Given the spectrum of primary fields we can define now the chiral ring. Since the theory we are dealing with is nonunitary, the definition of the chiral ring is not unique (see Ref. [16]). We choose to define it in the following way: left, right–chiral primaries satisfy the condition

$$G^+_{-\frac{1}{2}} \Phi = \tilde{G}^+_{-\frac{1}{2}} \Phi = 0$$ (4.11)

where $\Phi$ is either a local or a non–local primary vertex operator. Correspondingly left, right–antichiral fields are realized imposing

$$G^-_{-\frac{1}{2}} \Phi = \tilde{G}^-_{-\frac{1}{2}} \Phi = 0$$ (4.12)

We note that since the theory is interacting the rings are not factorized into holomorphic and antiholomorphic sectors. From the explicit expression of the supersymmetry charges it follows that chiral primary fields are of the form $\phi_{(a,b)}^-$ and antichiral fields are $\phi_{(a,b)}^+$ with $a$ and $b$ satisfying (4.9).

The physical states of the $A^{(1)}(1,1)$ theory in the chiral and antichiral rings are constructed by acting with the vertex operators $\Phi_{(a,b)}$ on the $N = 2$ supersymmetric vacuum. The states $|a,b\rangle = \Phi_{(a,b)}|0\rangle$ with $a \neq b$ correspond to classical solutions which exhibit a non–trivial behavior at infinity. In the $A(1,0)$ and $A(1,1)$ Toda theories where classical solitonic solutions are not present, these states are eliminated by the extra conditions

$$[T^+, \Phi^-_{(a,b)}] = 0 \quad \quad [T^-, \Phi^+_{(a,b)}] = 0$$ (4.13)

### 5 Topological $A^{(1)}(1,1)$ theory

The $O(1,1)$ symmetry of the affine Toda action in eq. (3.1) can be combined with the Lorentz invariance to twist the theory into the topological sector [10]. The spin content of the topological theory is determined by the new Lorentz group $L' = (L \otimes O(1,1))_{\text{diag}}$. Two equivalent twists are viable, the only difference being a spin interchange between $\psi$ and $\bar{\psi}$. We choose the new spin assignment

$$\begin{align*}
\psi^+ & \quad s' = 0 \\
\psi^- & \quad s' = 1
\end{align*} \quad \quad \begin{align*}
\bar{\psi}^+ & \quad s' = 0 \\
\bar{\psi}^- & \quad s' = -1
\end{align*}$$ (5.1)

which corresponds to a twist of the stress–energy tensor as $T' = T + \frac{1}{2}\partial J$, $\tilde{T}' = \tilde{T} - \frac{1}{2}\bar{\partial} \tilde{J}$ and $\tilde{T}' = \tilde{T} - \frac{1}{2}\bar{\partial} \tilde{J}$. It has central charge equal to zero. With respect to the new spin assignment the supersymmetry charges $G^+_{-\frac{1}{2}}$ and $\tilde{G}^+_{-\frac{1}{2}}$ have spin zero. Therefore the nilpotent combination

$$Q = iG^+_{-\frac{1}{2}} + \tilde{G}^+_{-\frac{1}{2}}$$ (5.2)

$$= \int dx \frac{i}{2\pi \sqrt{2}} \left[ (i\psi^+ \partial \phi^- + \bar{\psi}^+ \bar{\partial} \phi^-) - \frac{1}{\beta} (\psi^- + i\bar{\psi}^-) \left( e^{i\beta \phi^-} - g_- e^{-i\beta \phi^-} \right) \right]$$
is the BRST charge of the topological theory. Moreover it is easy to check that the twisted stress–energy tensor is $Q$–exact

\[
T' = \left\{ Q, \frac{1}{2} \psi^- \partial \phi^+ + \frac{i}{2 \beta} \bar{\partial} \psi^- \right\}
\]

\[
\bar{T}' = \left\{ Q, -\frac{i}{2} \bar{\psi}^- \bar{\partial} \phi^+ + \frac{1}{2 \beta} \bar{\partial} \bar{\psi}^- \right\}
\]

\[
\tilde{T}' = \left\{ Q, -\frac{i}{\beta} g^+ \psi^+ e^{-i \beta \bar{\phi}^+} \right\}
\]

(5.3)

The BRST transformations on the fields are

\[
\delta \phi^+ = \eta [i \psi^+ + \bar{\psi}^+]
\]

\[
\delta \phi^- = 0
\]

\[
\delta \psi^+ = -2 \eta \frac{\partial V}{\partial \phi^-}
\]

\[
\delta \bar{\psi}^- = -2 \eta \partial \phi^-
\]

\[
\delta \bar{\psi}^+ = 2 \eta i \frac{\partial V}{\partial \phi^-}
\]

\[
\delta \psi^- = -2 i \eta \bar{\partial} \phi^-
\]

(5.4)

where $\eta$ is a real spin–0 parameter and $V(\phi^-)$ is the potential in eq. (2.3) with coupling constant $i \beta$ in the exponentials. The physical spectrum of the topological theory is defined by the condition $Q|\text{phys}\rangle = 0$. This implies that only BRST–invariant vertex operators generate physical states when acting on the topological vacuum ($Q|0\rangle = 0$). From the explicit expression of $Q$ in terms of the supersymmetry charges it follows that the physical spectrum of the topological theory coincides with the chiral ring of the untwisted theory. The unperturbed physical states are then generated by local primaries $e^{i a \beta \phi^-}$ and $n$–soliton operators $e^{i a \beta \rho^- + i b \rho^+}$, where $a$, $b$ satisfy the conditions in eq. (4.9).

We note that $\frac{\partial V}{\partial \phi^-}$ is $Q$–exact, being proportional to the fermionic $Q$–transformations. Thus $\phi^-$ is cohomological trivial except at the critical points of the potential. We define our theory perturbatively around one of these points.

So far we have constructed the unperturbed physical spectrum of the topological theory. The perturbation modifies the chiral ring introducing a non–trivial dependence on the couplings. The perturbed chiral ring can be obtained by computing perturbatively the structure constants (three points correlation functions) of the ring. The computation of correlation functions will be the subject of the next section.
6 Topological correlation functions

We consider the most general correlation function in the topological sector of the $A^{(1)}(1,1)$ Toda theory

$$F_N \equiv \langle 0 | \Phi^{-}_{(a_1, b_1)}(x_1, t_1) \Phi^{-}_{(a_2, b_2)}(x_2, t_2) \cdots \Phi^{-}_{(a_N, b_N)}(x_N, t_N) |0\rangle_{\text{top}}$$

(6.1)

for a string of $N$ local or solitonic vertex operators where $|0\rangle$ is the topological vacuum. Exploiting the BRST invariance of the chiral fields at fixed $t$ and using the relation $[Q, H] = 0$ where $H$ is the hamiltonian of the theory, it is immediate to check that the expression (6.1) is invariant under BRST transformations. Consequently the topological correlation functions are numbers, independent of positions and times. Indeed, performing the calculation in the Heisenberg representation we can write

$$\frac{\partial}{\partial x_j} \langle \cdots \rangle = \langle \cdots \frac{\partial \Phi^{-}_{(a_j, b_j)}}{\partial x_j} \cdots \rangle = \langle \cdots \frac{1}{\sqrt{2}}[L_{-1} - L_{-1}, \Phi^{-}_{(a_j, b_j)}] \cdots \rangle$$

$$= \frac{i}{2\sqrt{2}} \langle \cdots [Q, G^{-}_{-\frac{1}{2}} + i\tilde{G}^{-}_{-\frac{1}{2}}], \Phi^{-}_{(a_j, b_j)} \cdots \rangle = 0$$

(6.2)

and

$$\frac{\partial}{\partial t_j} \langle \cdots \rangle = \langle \cdots \frac{1}{\sqrt{2}}[L_{-1} + \bar{L}_{-1}, \Phi^{-}_{(a_j, b_j)}] \cdots \rangle$$

$$= \frac{i}{2\sqrt{2}} \langle \cdots [Q, G^{-}_{-\frac{1}{2}} - i\tilde{G}^{-}_{-\frac{1}{2}}], \Phi^{-}_{(a_j, b_j)} \cdots \rangle = 0$$

(6.3)

Therefore we can compute correlation functions for vertex operators at the same position and time or, in light–cone coordinates, with $z_1 = z_2 = \cdots = z_N$, $\bar{z}_1 = \bar{z}_2 = \cdots = \bar{z}_N$.

Additional restrictions come from the requirement of topological charge conservation. Physically this condition means that the correlation function is non–vanishing only when solitons and antisolitons are present in equal number. From eq. (4.10) we have

$$[T^+, \Phi^{-}_{(a_i, b_i)}] = \beta^2 (a_i - b_i) \Phi^{-}_{(a_i, b_i)}$$

(6.4)

Thus we need impose

$$\sum_{i=1}^{N} T_i^+ \equiv \sum_{i=1}^{N} a_i - \sum_{i=1}^{N} b_i = 0$$

(6.5)

This constraint and space–time independence allow to reduce all topological correlators to one–point correlation functions

$$F_N = \langle 0 | \Phi^{-}_{a}(z_1, \bar{z}_1) |0\rangle_{\text{top}}$$

(6.6)

where we have set $a = \sum_{i=1}^{N} a_i$ and $(z_1, \bar{z}_1)$ is any point in the two dimensional space–time. Clearly it is the topological nature of the theory which prevents the existence of non–trivial solitonic asymptotic states.
We compute now $F_N$ using perturbation theory. As previously discussed the $A^{(1)}(1,1)$
theory is obtained from the $A(1,0)$ system by adding the perturbations in eqs. \eqref{eq:3.11}, \eqref{eq:3.12}. Since the $S_+$ term is $Q$–exact, perturbation theory in $g_+$ is trivial. The relevant perturbation is $S_-$ given in eq. \eqref{eq:3.11}. Thus we compute perturbatively in $g_-$ and write
\begin{equation}
F_N = \langle 0| \Phi^{-}_a(z_1, \bar{z}_1) e^{S_-} |0\rangle_{\text{top}}
\end{equation}
where now $|0\rangle$ is the topological $A(1,0)$ vacuum annihilated by the $A(1,0)$ BRST charge explicitly given in eq. \eqref{eq:5.2} with $g_- = 0$.

Having insured the conservation of the topological charge, we still need impose that the $O(1,1)$ charge and the background charge $(\frac{1}{\beta}, \frac{1}{\beta})$ be balanced in the correlation function. We examine first the $O(1,1)$ charge. As it usually happens in standard $N=2$ topological theories \cite{5,17,13}, the twisting procedure generates a $O(1,1)$ anomaly, since we have now
\begin{equation}
J_0 = [J_1, L'_{-1}] \quad J^\dagger_0 = - [J_{-1}, L'_1] = J_0 + \frac{c}{3}
\end{equation}
\begin{equation}
\bar{J}_0 = [\bar{J}_1, \bar{L}'_{-1}] \quad \bar{J}^\dagger_0 = - [\bar{J}_{-1}, \bar{L}'_1] = \bar{J}_0 - \frac{c}{3}
\end{equation}
where $c$ is the central charge of the $A(1,0)$ theory. It follows that non zero contributions to the topological correlation function in eq. \eqref{eq:6.7} at order $M$ in the perturbative expansion only arise if
\begin{equation}
a - 2M = \frac{c}{3}
\end{equation}
where $(a, -a)$ and $(-2, 2)$ are the $O(1,1)$ charges of the vertex operator and the perturbation respectively.

It is convenient to compute the topological correlation functions as $N=2$ correlators in the untwisted theory and implement the $O(1,1)$ anomaly with a vertex carrying $(-\frac{c}{3}, \frac{c}{3})$ charges at infinity. The simplest operator which realizes this and is not BRST trivial, is the local antichiral primary field
\begin{equation}
\Phi^+ \equiv e^{i\beta \bar{\Phi}^+}
\end{equation}
Obviously its insertion in the correlation function breaks BRST invariance. As discussed in Ref. \cite{13} this can be cured defining
\begin{equation}
\langle 0| \Phi^{-}_a(z_1, \bar{z}_1)|0\rangle_{\text{top}} = \lim_{z_2, \bar{z}_2 \to \infty} \langle 0| [(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)]^a \Phi^{-}_a(z_1, \bar{z}_1) \Phi^+_\beta(z_2, \bar{z}_2) e^{S_-} |0\rangle
\end{equation}
where
\begin{equation}
\hat{S}_- = \frac{g_-}{2\pi \beta^2} \int d^2z d^2\bar{\theta} \ [(z - z_2)(\bar{z} - \bar{z}_2)]^{-2} e^{-i\beta \Psi}\n\end{equation}
Indeed it is easy to verify that the introduction of the $(z_2, \bar{z}_2)$ factors in the chiral vertex $\Phi^-_a$ and in the $S_-$ perturbation are such that the modified correlation function $\langle \cdots \rangle$ is still $(z_1, \bar{z}_1)$–independent. This can be shown explicitly using a modified charge
\begin{equation}
\hat{Q} = \int \frac{dz}{2\pi i} (z - z_2) [iG^+(z, \bar{z}) + G^+(z, \bar{z})]
\end{equation}
which commutes with $\Phi_+$. Taking the limit $z_2, \bar{z}_2 \to \infty$ at the end of the calculation one restores the standard BRST invariance of the topological correlation function.

At this stage the calculation of $\mathcal{F}_N$ has been reduced to the evaluation of a two–point function with the vertices $\Phi^+_a, \Phi^+_b$ inserted at $(z_1, \bar{z}_1), (z_2, \bar{z}_2)$. In general a two–point correlation function is invariant under Moebius transformations that leave the points $z_1$ and $z_2$ fixed. Thus in order to obtain a finite result we have to factor out the infinite Moebius volume given by

$$ V_\infty \bar{V}_\infty = \int dz \frac{z_1 - z_2}{|(z - z_1)(z - z_2)|} \int d\bar{z} \frac{\bar{z}_1 - \bar{z}_2}{|(|z - \bar{z}_1)(\bar{z} - \bar{z}_2)|} \quad (6.14) $$

We turn now to the problem of background charge balance which we need impose in order to cancel the dependence on the infrared cutoff $\mu$ and obtain non–zero results in the thermodynamic limit. To this end screening operators must be inserted in eq. (6.11). For the $A(1,0)$ theory there are one “bosonic” $U$ and two “fermionic” $U_{\pm}$ screening operators \cite{17, 18} which in terms of $N = 2$ superfields are given by

$$ U = \int d^2 z d^4 \theta \, e^{-\frac{i}{\beta}(\Psi^+ + \Psi^-)} \quad (6.15) $$

$$ U_+ = \int d^2 z d^2 \theta \, e^{i\beta \Psi^+} \quad U_- = \int d^2 z d^2 \bar{\theta} \, e^{i\beta \Psi^-} \quad (6.16) $$

The sum of the two fermionic screenings is the marginal perturbation of the $A(1,0)$ theory (see eq. (5.5)). Since even the bosonic screening is a marginal operator one could add $U$ to the lagrangian without affecting its physical content \cite{19}. Thus the insertion of a given number of screening operators corresponds to the computation of the correlation function at a given order in the marginal perturbations. Inserting $(p - 1)$ bosonic $U$ screenings and $q_{\pm}$ fermionic $U_{\pm}$ ones, at order $M$ in $g_-$, the request of background charge conservation gives

$$ \beta a - \beta M - (p - 1) \frac{1}{\beta} + q_- \beta = -\frac{1}{\beta} $$

$$ \beta c - (p - 1) \frac{1}{\beta} + q_+ \beta = -\frac{1}{\beta} \quad (6.17) $$

The two equations are consistent with the $O(1,1)$ charge balance (5.3) only for $q_+ - q_- = M$. Moreover setting $c = 3 - \frac{4}{\beta^2}$ in the second equation we find $p = \beta^2 (q_- + M + 1)$. Therefore non–zero correlation functions are obtained only for Toda theories with $\beta^2$ a positive integer. Writing $\beta^2 = k + 2$, $k = 1, 2, \cdots$ we obtain $c = \frac{3k}{k + 2}$, namely the central charge of the $A_{k+1}$ $N = 2$ minimal models \cite{20}.

We summarize here the results presented in this Section: the calculation of the $N$–point topological correlation function, at order $M$ in perturbation theory with respect to $g_-$, is given by

$$ \mathcal{F}_N^{(M)} = \langle 0 | \Phi^-_{(a_1, b_1)}(x_1, t_1) \Phi^-_{(a_2, b_2)}(x_2, t_2) \cdots \Phi^-_{(a_N, b_N)}(x_N, t_N) S_-^M | 0 \rangle_{\text{top}} $$

13
\[ = \lim_{z_2, \bar{z}_2 \to \infty} [V_\infty \tilde{V}_\infty]^{-1} \langle 0 | \left( (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \right)^a \Phi_-^a(z_1, \bar{z}_1) \Phi_+^a(z_2, \bar{z}_2) \] 

\[ \hat{S}_- \mathcal{U}_-^{-1} \mathcal{U}_+^a \mathcal{U}_-^c | 0 \rangle \] 

(6.18)

with \( \hat{S}_- \), \( \mathcal{U}_- \), \( \mathcal{U}_+ \) and \( V_\infty \tilde{V}_\infty \) in eqs.(6.12, 6.15, 6.16, 6.14) respectively and

\[ c = 3 - \frac{6}{\beta^2} = \frac{3k}{k + 2} \quad \quad k = 1, 2, \ldots \]

\[ \sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i = a = 2M + \frac{k}{k + 2} \]

\[ q_+ - q_- = M \quad \quad p - 1 = (k + 2)(q_+ + 1) - 1 \] (6.19)

7 The superspace calculation

In this section we use the \( N = 2 \) superspace formalism to compute the simplest example of topological correlation functions in eq. (6.18): we choose the model with \( c = 1 \) and perform the calculation at zero order in the \( g_- \) perturbation. Then from eq. (6.14) we have \( k = 1, \beta^2 = 3, a = \frac{3}{4} \). Moreover the minimum number of screening operators we need insert in order to obtain a non-vanishing correlation function is two bosonic and zero fermionic ones. Thus we want to evaluate

\[ \mathcal{F} = \lim_{z_2, \bar{z}_2 \to \infty} [V_\infty \tilde{V}_\infty]^{-1} \langle (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \rangle \frac{1}{\beta^2} \langle : e^{\frac{1}{4} \beta [\phi^- (z_1, \bar{z}_1) + \phi^+ (z_2, \bar{z}_2)]} : \mathcal{U} \mathcal{U} \rangle \] (7.1)

For notational convenience we leave \( \beta \) unspecified, inserting the value \( \beta^2 = 3 \) in the final result. We perform first the contraction of the two normal ordered exponentials

\[ \langle : e^{\frac{1}{4} \beta [\phi^- (z_1, \bar{z}_1) + \phi^+ (z_2, \bar{z}_2)]} : \mathcal{U} \mathcal{U} \rangle = e^{\frac{3}{2} \log |2(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)|} \langle : e^{\frac{1}{4} \beta [\phi^- (z_1, \bar{z}_1) + \phi^+ (z_2, \bar{z}_2)]} : \mathcal{U} \mathcal{U} \rangle \] (7.2)

We note that being the theory interacting the screenings have a nontrivial dependence on the auxiliary fields. If we were to use a component description and eliminate the auxiliary fields via their field equations, the calculation of the correlation function would become hardly manageable. A way to overcome this problem is to perform the computation off-shell directly in \( N = 2 \) superspace. We write then

\[ \langle : e^{\frac{1}{4} \beta [\phi^- (z_1, \bar{z}_1) + \phi^+ (z_2, \bar{z}_2)]} : \mathcal{U} \mathcal{U} \rangle = \langle : e^{\frac{1}{4} \beta [\Psi^- (Z_1, \bar{Z}_1) + \Psi^+ (Z_2, \bar{Z}_2)]} : \mathcal{U} \mathcal{U} \rangle \] (7.3)

where we have introduced the notation \( (Z, \bar{Z}) \equiv (z, \theta, \bar{z}, \bar{\theta}) \) and \( | \) means evaluating the final result at \( \theta_1 = \bar{\theta}_1 = \theta_2 = \bar{\theta}_2 = 0 \). The \( N = 2 \) superspace conventions we will adopt in the course of the calculation are collected in Appendix A. There the reader can also find the expression of the superfield propagators with ultraviolet and infrared cutoffs explicitly shown. Here we
Using the explicit expression of the propagators as given in eq. (A.13) one can perform the calculation in eq. (7.3) as follows: first we compute the contraction of the two screenings and then contract the result with the normal ordered exponential. Using the quantum–background method, we define quantum fields $\xi$ as $\Psi \rightarrow \Psi + \xi$ and compute

$$
\mathcal{U}^{(2)} = \int d^2 z d^4 \theta \int d^2 z' d^4 \theta' : e^{-\frac{i}{\beta}(\xi^+ + \xi^-)(Z, Z')} : e^{-\frac{i}{\beta}(\xi^+ + \xi^-)(Z', Z')} : e^{-\frac{i}{\beta}[(\Psi^+ + \Psi^-)(Z, Z')+(\Psi^+ + \Psi^-)(Z', Z')]}:
$$

(7.4)

where $\langle\ldots\rangle$ indicates the complete contraction of the two operators. We obtain

$$
: e^{-\frac{i}{\beta}(\xi^+ + \xi^-)(Z, Z')} : e^{-\frac{i}{\beta}(\xi^+ + \xi^-)(Z', Z')} :
= 1 + \sum_{n=1}^{\infty} \left( \frac{i}{\beta} \right)^n \frac{2n}{(n!)} \theta_n \left( \sum_{p=0}^{n} \left( \begin{array}{c} n \\ p \end{array} \right) p! (n-p)! \right) \langle \xi^+(Z, Z') \xi^-(Z', Z') \rangle^n - \langle \xi^+(Z, Z') \xi^-(Z', Z') \rangle^{n-1} \langle \xi^+(Z, Z') \xi^-(Z', Z') \rangle^1
$$

(7.5)

Using the explicit expression of the propagators as given in eq. (A.13) one can perform the $D$–algebra in the $(n-1)$–loop order contribution which gives

$$
\langle \xi^+(Z, Z') \xi^-(Z', Z') \rangle^n - \langle \xi^+(Z, Z') \xi^-(Z', Z') \rangle^{n-1} \langle \xi^+(Z, Z') \xi^-(Z', Z') \rangle^1
= 4\delta^{(4)}(\theta - \theta') \left\{ -\frac{p(n-p)}{n-1} \left( -\log |2(z - z')(\bar{z} - \bar{z}')| \right)^{n-1} \partial \bar{\partial} \left( -\log |2(z - z')(\bar{z} - \bar{z}')| \right) \right\}

(7.6)

where the spacetime and covariant spinor derivatives act on the $(Z', \bar{Z}')$ variables. Inserting the above expression in eq. (7.3) one can easily perform the sum over $p$ and use the identities in eq. (A.16) to rewrite the result in the following form

$$
: e^{-\frac{i}{\beta}(\xi^+ + \xi^-)(Z, Z')} : e^{-\frac{i}{\beta}(\xi^+ + \xi^-)(Z', Z')} :
= 1 + \delta^{(4)}(\theta - \theta') \sum_{n=1}^{\infty} \left( \frac{2}{\beta^2} \right)^n \left\{ -\frac{1}{(n-1)!} \left( \log |2(z - z')(\bar{z} - \bar{z}')| \right)^{n-1} \partial \bar{\partial} \left( \log |2(z - z')(\bar{z} - \bar{z}')| \right) \right\}

+ \frac{1}{n!} \left( \log |2(z - z')(\bar{z} - \bar{z}')| \right)^n \left\{ D^2 \bar{D}^2 + \bar{D}^2 D^2 + D_+ \bar{D}_- + D_- \bar{D}_+ \right\}

(7.7)

We note that the term

$$
|2(z - z')(\bar{z} - \bar{z}') + e^2| \frac{\bar{z} \partial}{\bar{z}} \partial \bar{\partial} \left( \log |2(z - z')(\bar{z} - \bar{z}') + e^2| \right)
$$

(7.8)
has been obtained from the infinite sum of contributions of the form
\[(\log|2(z - z')(\bar{z} - \bar{z}') + \epsilon^2|)^\alpha \partial \bar{\partial} \log|2(z - z')(\bar{z} - \bar{z}') + \epsilon^2|\] (7.9)
where we have explicitly indicated the dependence on the ultraviolet cutoff $\epsilon$. Using the relation
\[\partial \bar{\partial} \log|2(z - z')(\bar{z} - \bar{z}')| = 2\pi i \delta^{(2)}(z - z')\] (7.10)
one can easily check that, while each term in eq. (7.9) is logarithmically ultraviolet divergent, the final sum in eq. (7.8) vanishes when the cutoff is removed. Going back to eq. (7.3) we can then write
\[\langle :e^{\frac{i}{\hbar}\beta[\psi^-(Z_1,\bar{Z}_1) + \psi^+(Z_2,\bar{Z}_2)]} : U U \rangle = \int d^2 z d^4 \theta f(Z, \bar{Z}) \int d^2 z' d^4 \theta' f(Z', \bar{Z}') + \int d^2 z d^2 z' d^4 \theta \left[2(z - z')(\bar{z} - \bar{z}')|\frac{\beta^2}{i}\right]
\]
\[f(Z, \bar{Z}) = \frac{1}{\beta} (\psi^+(Z, \bar{Z})\psi^-(Z_1, \bar{Z}_1)) + \frac{1}{\beta} (\psi^- (Z, \bar{Z})\psi^+(Z_2, \bar{Z}_2))\]
(7.11)
having defined
\[f(z, \theta, \bar{z}, \bar{\theta}) = \frac{1}{\beta} [2(z - z')(\bar{z} - \bar{z}')|\frac{\beta^2}{i}\left] 4D^2 \bar{D}^2 f(z, \theta, \bar{z}, \bar{\theta}) - 4\bar{D}^2 D^2 f(z', \theta, \bar{z}', \bar{\theta})\right|\]
(7.13)
From the definition in eq. (7.12), using the explicit expressions of the propagators as given in eq. (A.13) of the Appendix, we obtain
\[f(z, \theta, \bar{z}, \bar{\theta})|_{\theta_1 = \theta_2 = 0} = e^{-\frac{1}{\beta} [1 + i \theta_1 \theta_2] (1 - i \theta_1 \bar{\theta}_2) \log|2(z - z_1)(\bar{z} - \bar{z}_1)|}
\]
\[e^{-\frac{1}{\beta} [1 - i \theta_1 \theta_2] (1 + i \theta_1 \bar{\theta}_2) \log|2(z - z_2)(\bar{z} - \bar{z}_2)|}\]
(7.14)
It is now tedious but straightforward to evaluate the expressions $D^2 \bar{D}^2 f$ and $\bar{D}^2 D^2 f$ at $\theta = \bar{\theta} = 0$. Details and intermediate steps of the explicit calculation are given in Appendix B. Using the result in eq. (B.3), after a rescaling of all factors by $z_2 \bar{z}_2$ we obtain
\[\lim_{z_2, \bar{z}_2 \to \infty} \mathcal{I} \sim \frac{2}{81} \left[\frac{\beta^2 + \frac{\beta^4}{3}}{(z_2 \bar{z}_2)}\right]^{rac{1}{4}}\]
(7.15)
\[\int d^2 z d^2 z' |(z - z')(\bar{z} - \bar{z}')|^\frac{1}{\beta^2} \frac{|z z' \bar{z}(z - 1)(\bar{z} - 1)z' z' \bar{z}' (\bar{z}' - 1)(\bar{z}' - 1)|^\frac{1}{3}}{z z' \bar{z}(z - 1)(\bar{z} - 1)z' z' \bar{z}' (\bar{z}' - 1)(\bar{z}' - 1)}\]
We note that the integral rescales with the correct power in $z_2 \bar{z}_2$ in order to give a non–trivial contribution to the correlation function as defined in eq. (7.11). Therefore for $\beta^2 = 3$ we can finally write
\[\mathcal{F} = \frac{2}{81} \mathcal{J} \mathcal{\bar{J}} \left[\mathcal{V}_\infty \mathcal{\bar{V}}_\infty\right]^{-1}\]
(7.16)
where
\[ V_\infty \equiv \lim_{z \to \infty} V \]
and
\[ J = \int dz dz' \frac{(z - z')^2 |z(z - 1)z'(z' - 1)|^{-\frac{1}{2}}}{z(z - 1)z'(z' - 1)} \]
with \( J \) the same integral in the \( \bar{z}, \bar{z}' \)-variables. We would like to remark that the correlation functions are factorized as products of holomorphic and antiholomorphic terms in spite of the lack of explicit factorization in the chiral ring.

The last step is the evaluation of the integrals in eq. (7.18). The details are given in Appendix B. We report here the final result: inserting the value of \( J \) from eq. (B.11) into eq. (7.16) we obtain
\[ F = 2^{-\frac{1}{3}} \left[ \frac{4 \Gamma^2(\frac{2}{3})}{3 \Gamma(\frac{4}{3})} \right]^2 \] (7.19)

8 Conclusions

The affine Toda theories based on the Lie superalgebras \( A^{(1)}(n, n) \) in two–dimensional Minkowski spacetime are nonunitary models that can be twisted into a topological sector. They can be obtained in a two–step procedure as perturbations of the \( N = 2 \) superconformal \( A(n, n - 1) \) Toda systems.

In this paper we have studied in detail the case \( n = 1 \). The action of the \( A(1, 0) \) model, \( S_0 \) given in eq. (3.2), contains an interaction, \( V_0 \), which is a marginal perturbation of the free theory, a necessary condition since the complete interacting theory is still conformally invariant. The affine theory, \( A^{(1)}(1, 1) \), is constructed by adding to \( S_0 \) the perturbation terms \( S_- \) and \( S_+ \) in eqs. (3.11, 3.12). We have shown that \( S_- \) is a relevant perturbation while \( S_+ \) is \( Q \)-exact with respect to the BRST charge which defines the topological version of the theory.

Having identified the chiral and antichiral rings, we concentrated on the computation of the topological correlation functions. The physical, BRST–invariant operators are primary chiral vertices with well–defined scale dimensions, \( O(1, 1) \) charge and solitonic topological number. When inserted into a correlation function they give a nonzero result only if the various charges are conserved in the process. This requirement with the additional request of background charge conservation fixes the value of the central charge of the theory to be the one of the \( N = 2 \) \( A_{k+1} \) minimal models. Moreover it puts severe restrictions on the allowed quantum numbers of the vertices. In particular solitons and antisolitons are always present in equal number. In fact one finds that the calculation of an \( N \)-point correlation function reduces to the evaluation of a one–point topological correlator, independent of the solitonic number.

At this stage it is convenient to go back to the untwisted version of the theory, implement the topological \( O(1, 1) \) anomaly with a vertex carrying this anomalous charge at infinity and
compute the correlation function as in ordinary perturbation theory. Within this approach the
general form of the $N$–point correlator is given in eq. (6.18). We emphasize that $F^{(M)}_N$ is at
order $M$ in the relevant $g_-$ perturbation and at order $(p - 1)$, $q_\pm$ in the marginal perturbations
$\mathcal{U}, \mathcal{U}_\pm$ respectively. The simplest correlator for the $c = 1$ model and
$M = 0$ has been computed

We have considered a theory in Minkowski spacetime. Obviously the same procedure would
work equally well in standard $N = 2$ supersymmetric theories formulated in euclidean space.
Since our calculation consists essentially in the evaluation of a two–point correlator, it would
be interesting to apply the $N = 2$ superspace method to the calculation of the Zamolodchikov
chiral–antichiral metric [22].

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A

In this appendix we list our notations and conventions. We work in $N = 2$ superspace described
by light–cone coordinates

$$z = \frac{x^0 + x^1}{\sqrt{2}} \quad \bar{z} = \frac{x^0 - x^1}{\sqrt{2}}$$

(A.1)

where $x^0 \equiv t$ and $x^1$ are coordinates in Minkowski space with signature $g_{\mu\nu} = \text{diag}(1,-1)$, and
spinor coordinates $\theta_\pm, \bar{\theta}_\pm$ satisfying the following conjugation rules

$$\theta^*_+ = -\bar{\theta}_+ \quad \theta^*_- = \bar{\theta}_-$$

(A.2)

We define $N = 2$ supercovariant derivatives

$$D_+ = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \theta_+^*} + i \bar{\theta}_+ \partial \right) \quad \bar{D}_+ = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \theta_+} + i \theta_+ \bar{\partial} \right)$$

$$D_- = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \theta_-^*} - i \bar{\theta}_- \partial \right) \quad \bar{D}_- = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \theta_-} - i \theta_- \bar{\partial} \right)$$

(A.3)

where

$$\partial \equiv \partial_z = \frac{1}{\sqrt{2}} (\partial_0 + \partial_1) \quad \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{\sqrt{2}} (\partial_0 - \partial_1)$$

(A.4)

They satisfy the algebra

$$\{D_+, \bar{D}_+\} = i\partial \quad \{D_-, \bar{D}_-\} = -i\bar{\partial}$$

(A.5)

Often we make use of the following notation

$$\theta_+ \theta_- \equiv \theta^2 \quad \bar{\theta}_+ \bar{\theta}_- \equiv \bar{\theta}^2 \quad D_+ D_- \equiv D^2 \quad \bar{D}_+ \bar{D}_- \equiv \bar{D}^2$$

(A.6)
$N = 2$ chiral and antichiral superfields $\Psi$ and $\bar{\Psi} \equiv \Psi^*$ are subject to the constraints

$$\bar{D}_\pm \Psi = 0 \quad D_\pm \bar{\Psi} = 0$$

(A.7)

and they can be written as

$$\Psi(\theta_+, \theta_-, \bar{\theta}_+, \bar{\theta}_-) = e^{i\theta_+ \bar{\theta}_+ - i\theta_- \bar{\theta}_-} \Psi(\theta_+, \theta_-)$$

$$\bar{\Psi}(\theta_+, \theta_-, \bar{\theta}_+, \bar{\theta}_-) = e^{-i\theta_+ \bar{\theta}_+ + i\theta_- \bar{\theta}_-} \bar{\Psi}(\bar{\theta}_+, \bar{\theta}_-)$$

(A.8)

where

$$\Psi(\theta_+, \theta_-) = \phi + \frac{1}{\sqrt{2}} \theta_+ \bar{\psi} + \frac{1}{\sqrt{2}} \theta_- \psi + \theta_+ \theta_- F$$

$$\bar{\Psi}(\bar{\theta}_+, \bar{\theta}_-) = \bar{\phi} + \frac{1}{\sqrt{2}} \bar{\theta}_+ \bar{\psi}^* - \frac{1}{\sqrt{2}} \bar{\theta}_- \psi^* + \bar{\theta}_+ \bar{\theta}_- F$$

(A.9)

The convention for complex conjugation on the product of two fermions is $(\psi_1 \psi_2)^* = \psi_2^* \psi_1^*$. We have defined $\phi$ ($\bar{\phi} = \phi^*$) to be a complex scalar field, $F$ ($\bar{F} = F^*$) the auxiliary field and $\psi$, $\bar{\psi}$ the components of a Dirac spinor

$$\psi = \frac{1}{2^{1/4}} \left( \begin{array}{c} \psi \\ \bar{\psi} \end{array} \right) \quad \bar{\psi} \equiv \psi^* \gamma_0 = \frac{1}{2^{1/4}} (\bar{\psi}^* \psi^*)$$

(A.10)

with respect to the two dimensional Dirac basis

$$\gamma_0 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \quad \gamma_1 = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \quad C = \gamma_1$$

(A.11)

where $C$ is the charge conjugation matrix. Majorana–Weyl spinors satisfy the reality conditions $\psi^* = \psi$, $\bar{\psi}^* = -\bar{\psi}$.

We consider a general $N = 2$ superspace action for $n$ superfields

$$\frac{1}{2\pi} \int d^2 z d^4 \theta \ K_{ij} \Psi_i \bar{\Psi}_j + \frac{1}{2\pi} \int d^2 z d^2 \theta \ W(\Psi) + \frac{1}{2\pi} \int d^2 z d^2 \bar{\theta} \ W(\bar{\Psi})$$

(A.12)

where $K_{ij}$ is an invertible, symmetric, real $n \times n$ constant matrix and $W$ is the superpotential. The integration measure is defined as $d^2 z = d z d \bar{z}$, $d^2 \theta = d \theta_+ d \theta_- \rightarrow 2D^2$, $d^2 \bar{\theta} = d \bar{\theta}_+ d \bar{\theta}_- \rightarrow 2\bar{D}^2$ and $d^4 \theta = d^2 \theta d^2 \bar{\theta} \rightarrow 4D^2\bar{D}^2$. From the action in eq. (A.12) we obtain the chiral–antichiral superfield propagator (with $Z \equiv (z, \theta_+, \theta_-)$, $\bar{Z} \equiv (\bar{z}, \bar{\theta}_+, \bar{\theta}_-)$)

$$\langle \Psi_i(Z, \bar{Z}) \bar{\Psi}_j(Z', \bar{Z}') \rangle = -4 \ K_{ij}^{-1} D^2 \bar{D}^2 \delta^{(4)}(\theta - \theta') \log |\mu^2[2(z - z')(\bar{z} - \bar{z}') + \epsilon^2]|$$

(A.13)

where

$$\delta^{(4)}(\theta - \theta') = (\bar{\theta} - \bar{\theta}')^2(\theta - \theta')^2$$

(A.14)

and $\mu$, $\epsilon$ are infrared and ultraviolet cutoffs respectively. Standard superspace techniques greatly simplify the computation of Green’s functions. Typically the final result for a perturbative loop
contribution is always local in the $\theta$–variables once the $D$–algebra has been completed. In so doing one makes use of the commutation algebra in eq. (A.3) and of the relation

$$4\delta^{(4)}(\theta - \theta')D^2 \bar{D}^2 \delta^{(4)}(\theta - \theta') = \delta^{(4)}(\theta - \theta') \tag{A.15}$$

We list here some identities that we have employed repeatedly in the superspace calculation presented in Section 7:

$$D^2 \bar{D}^2 = \bar{D}^2 D^2 - i \partial \bar{\partial} \bar{D}^2 D^2 + D^2 \bar{D}^2 - \partial \bar{\partial}$$

$$D^2 \bar{D}^2 + \bar{D}^2 D^2 = \bar{D}^2 D^2 + D^2 \bar{D}^2 - \partial \bar{\partial} \tag{A.16}$$

Finally for a spin–1 conserved current $J_{\mu} (\partial^\mu J_{\mu} = 0)$ we define light–cone components

$$J = \frac{J_0 + J_1}{\sqrt{2}}, \quad \bar{J} = \frac{J_0 - J_1}{\sqrt{2}} \tag{A.17}$$

satisfying the conservation equations $\partial J = -\partial \bar{J}$. The corresponding conserved charge is \( \int \frac{dx}{2\pi i \sqrt{2}} (J + \bar{J}) \) and in general we define modes as

$$J_n = \int \frac{dx}{2\pi i \sqrt{2}} \left( \frac{x}{\sqrt{2}} \right)^n (J + \bar{J}) \quad n = 0, \pm 1, \pm 2, \ldots \tag{A.18}$$

These definitions are easily extended to higher spin–conserved currents. Therefore, for the supersymmetry currents $G_{\mu}$ and $\bar{G}_{\mu}$ we define the light–cone components as

$$G = \frac{G_0 + G_1}{\sqrt{2}}, \quad \tilde{G} = \frac{G_0 - G_1}{\sqrt{2}}$$

$$\bar{G} = \frac{\bar{G}_0 - \bar{G}_1}{\sqrt{2}}, \quad \tilde{\bar{G}} = \frac{\bar{G}_0 + \bar{G}_1}{\sqrt{2}} \tag{A.19}$$

and the corresponding modes

$$G_{n + \frac{1}{2}} = \int \frac{dx}{2\pi i \sqrt{2}} \left( \frac{x}{\sqrt{2}} \right)^{n + 1} (G + \tilde{G})$$

$$\bar{G}_{n + \frac{1}{2}} = \int \frac{dx}{2\pi i \sqrt{2}} \left( \frac{x}{\sqrt{2}} \right)^{n + 1} (\bar{G} + \tilde{\bar{G}}) \tag{A.20}$$

The supersymmetry charges are obtained setting $n = -1$. For the spin–2 stress energy tensor we define

$$T \equiv T_{++} = T_{00} + T_{11} + 2T_{01}$$

$$\bar{T} \equiv T_{--} = T_{00} + T_{11} - 2T_{01}$$

$$\tilde{T} \equiv T_{+-} = T_{00} - T_{11} \tag{A.21}$$
The modes corresponding to the $T$ component are

$$L_n = \int \frac{dx}{2\pi i\sqrt{2}} \left( \frac{x}{\sqrt{2}} \right)^{n+1} T$$

(A.22)

Analogous definitions hold for $\tilde{T}$ and $\check{T}$.

B

In this appendix we present some details of the calculation of the $I$ and $J$ integrals in Section 7. Using the expression in eq. (7.14) we can compute the quantity in eq. (7.13)

$$I \equiv |2(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)|^{\frac{3}{2}} \int d^2 z d^2 z' |2(z - z')(\bar{z} - \bar{z}')(\bar{z} - z)(z - \bar{z})(z - z_2)(\bar{z} - \bar{z}_2)(z' - z_2)(\bar{z}' - \bar{z}_2)|^{-\frac{1}{4}}$$

$$= 2^{\frac{5}{2}} |(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)|^{\frac{3}{2}} \int d^2 z d^2 z'$$

$$\left(\frac{1}{3} \partial_{\bar{z}} \frac{1}{z - z_1} + \frac{1}{3} \partial_{\bar{z}} \frac{1}{z - z_2} + \frac{1}{9} \frac{1}{z - \bar{z}_1} \frac{1}{z - z_2} + \frac{1}{9} \frac{1}{z - z_1} \frac{1}{\bar{z} - \bar{z}_2} \right)$$

$$\left(\frac{1}{3} \partial_{\bar{z}'} \frac{1}{z' - z_1} + \frac{1}{3} \partial_{\bar{z}'} \frac{1}{z' - z_2} + \frac{1}{9} \frac{1}{z' - \bar{z}_1} \frac{1}{z' - z_2} + \frac{1}{9} \frac{1}{z' - z_1} \frac{1}{\bar{z}' - \bar{z}_2} \right)$$

(B.1)

We note that each term containing $\partial_{\bar{z}}$ and/or $\partial_{\bar{z}'}$ would lead to a divergent contribution and then should be regularized appropriately. Actually, by integration by parts, it is quite easy to show that these terms cancel out completely. Thus we end up with the following expression

$$I = \frac{2^{\frac{5}{2}}}{81} \left(\frac{1}{3} \partial_{\bar{z}} \frac{1}{z - z_1} + \frac{1}{9} \frac{1}{z - \bar{z}_1} \frac{1}{z - z_2} \right)$$

$$\left(\frac{1}{3} \partial_{\bar{z}'} \frac{1}{z' - z_1} + \frac{1}{9} \frac{1}{z' - \bar{z}_1} \frac{1}{z' - z_2} \right)$$

(B.2)

which can be rewritten as

$$I = \frac{2^{\frac{5}{2}}}{81} \left(\frac{1}{3} \partial_{\bar{z}} \frac{1}{z - z_1} + \frac{1}{9} \frac{1}{z - \bar{z}_1} \frac{1}{z - z_2} \right)$$

21
\[|(z - z')(\bar{z} - \bar{z}')|^{\frac{1}{2}} \text{ and evaluate the } z\text{-integrals. Therefore we write the integral for any value of } p, q \text{ by analytic continuation of the function } F\text{.}

Changing variables \(z \to z/z_2, \ z' \to z'/z_2\), it is now straightforward to obtain the asymptotic expression in eq. (7.15).

Finally we compute the integral in eq. (B.18). We start by considering the general expression

\[J = \int_{-\infty}^{+\infty} dz dz' \frac{|z(z - 1)|^b}{z(z - 1)z'(z' - 1)}\]

\[\text{(B.4)}\]

\(b\) being of the form \(b = \frac{p}{2q+1}\), \(p, q\) integers such that the integral is convergent. Then we define the integral for any value of \(p, q\) by analytic continuation.

We compute \(J\) trying to factorize the infinite volume \(V_\infty = \int \frac{dz}{z(z - 1)}\) of the one parameter \(\text{Moebius transformations. Therefore we write} J = \int_{-\infty}^{+\infty} dz dz' \frac{|z(z - 1)|^b}{z(z - 1)z'(z' - 1)}\]

\[\text{(B.5)}\]

and evaluate the \(z'\)-integral first. We split the integration in three intervals

\[J' = \int_{-\infty}^{+\infty} dz' (z - z')^{-2b} \frac{|z'(z' - 1)|^b}{z'(z' - 1)}\]

\[= \int_{-\infty}^{0} dz' (z - z')^{-2b} \frac{|z'(z' - 1)|^b}{z'(z' - 1)} - \int_{0}^{1} dz' (z - z')^{-2b} (z(1 - z))^{-b-1}\]

\(+ \int_{1}^{+\infty} dz' (z - z')^{-2b} \frac{|z'(z' - 1)|^b}{z'(z' - 1)}\]

\[\text{(B.6)}\]

Performing the change of variables \(z' = \frac{u}{u - 1}\) in the first integral and \(z' = \frac{1}{u}\) in the third one we write the three integrals in the form

\[\int_{0}^{1} dx \ x^{\lambda-1}(1 - x)^{\mu-1}(1 - \mathcal{H}(z)x)^{-\nu}\]

\[\text{(B.7)}\]

where \(\mathcal{H}\) indicates any function of \(z\). Using eq. (3.197.3) page 286 in Ref. [21] we obtain the following expression

\[J' = B(b, 1) F(2b, 1, 1 + b; 1 - z) - z^{-2b} B(b, b) F(2b, b, 2b; \frac{1}{z}) + B(1, b) F(2b, 1, 1 + b; z)\]

\[\text{(B.8)}\]

where \(B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}\) and \(F(a, b; c; z)\) is the hypergeometric function. Using the relations for analytic continuation of the function \(F\) we can write

\[F(2b, 1, 1 + b; 1 - z) = -F(2b, 1, 1 + b; z) + z^{-b} \frac{\Gamma(1 + b)\Gamma(b)}{\Gamma(2b)} F(1 - b, b, 1 - b; z)\]

\[\text{Equation (22)}\]
\[ F(2b, 1, 1 + b; z) + \frac{\Gamma(1 + b)\Gamma(b)}{\Gamma(2b)} [z(z - 1)]^{-b} \]

\[ F(2b, b, 2b; \frac{1}{z}) = z^b(z - 1)^{-b} \]  
(B.9)

Summing all the contributions, for the \( z' \) integration we obtain

\[ J' = [z(z - 1)]^{-b} \cdot (-2) \frac{\Gamma(b)^2}{\Gamma(2b)} \]  
(B.10)

Inserting the result in eq. (B.5) and simplifying the factor \([z(z - 1)]^b\) we can finally write

\[ J = -2 \frac{\Gamma(b)^2}{\Gamma(2b)} \cdot V_{\infty} \]  
(B.11)
References

[1] A.E. Arinshtein, V.A. Fateev and A.B. Zamolodchikov, Phys. Lett. B87 (1979) 389; J.L. Cardy and G. Mussardo, Phys. Lett. B225 (1989) 275; P.G.O. Freund, T.R. Klassen and E. Meltzer, Phys. Lett. B229 (1989) 243; P. Christe and G. Mussardo, Nucl. Phys. B330 (1990) 465; C. Destri and H.J. de Vega, Phys. Lett. B233 (1989) 336; G. Mussardo and G. Sotkov, in “Recent developments in Conformal Field Theories”, Trieste, Italy, October 2–4, 1989; T.R. Klassen and E. Meltzer, Nucl. Phys. B338 (1990) 485; H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, Phys. Lett. B227 (1989) 441; Nucl. Phys. B338 (1990) 689; G.W. Delius, M.T. Grisaru, S. Penati and D. Zanon, Phys. Lett. B256 (1991) 164; Nucl. Phys. B359 (1991) 125; C. Destri, H.J. De Vega and V.A. Fateev, Phys. Lett. B256 (1991) 173; G.W. Delius, M.T. Grisaru and D. Zanon, Phys. Lett. B277 (1992) 414; Nucl. Phys. B382 (1992) 365.

[2] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. 120 (1979) 253.

[3] F.A. Smirnov, J. Phys. A17 (1984) L873; A19 (1986) 575; A.N. Kirillov and F.A. Smirnov, Phys. Lett. B198 (1987) 506; Int. J. Mod. Phys. A3 (1988) 731; A. Fring, G. Mussardo and P. Simonetti, Nucl. Phys. B393 (1993) 413; Phys. Lett. B307 (1993) 83; A. LeClair, “Spectrum generating Affine Lie algebras and correlation functions in Massive Field Theory”, preprint CLNS–93–1220 (May 1993), hepth/9305110; A. LeClair and C. Efthimiou, “Particle–field duality and form–factors from vertex operators”, preprint CLNS–93–1263 (Dec 1993), hepth/9312121.

[4] E. Witten, Comm. Math. Phys. 117 (1988) 353; Comm. Math. Phys. 118 (1988) 411; Nucl. Phys. B340 (1990) 281.

[5] C. Vafa, Mod. Phys. Lett. A6 (1991) 337; R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B352 (1991) 59.

[6] B. Dubrovin, “Integrable systems and classification of 2–dimensional topological field theories”, S.I.S.S.A. preprint 162/92/FM, September 1992, hepth/9209040; S. Cecotti, L. Girardello and A. Pasquinucci, Nucl. Phys. B328 (1989) 701.

[7] M.A. Olshanetsky, Comm. Math. Phys. 88 (1983) 63.
[8] A. Gualzetti, S. Penati and D. Zanon, Nucl. Phys. B398 (1993) 622.

[9] S. Penati and D. Zanon, in “String theory, quantum gravity and the unification of the fundamental interactions”, eds. M. Bianchi et al., World Scientific (1993), 450.

[10] S. Penati, M. Pernici and D. Zanon, Phys. Lett. B309 (1993) 304.

[11] J. Evans and T. Hollowood, Phys. Lett. B293 (1992) 100; J. Evans, Nucl. Phys. B390 (1993) 225.

[12] H.C. Liao, D. Olive and N. Turok, Phys. Lett. B298 (1993) 95.

[13] N. Warner, “N=2 supersymmetric integrable models and topological field theories”, Proceedings of the Summer School on High Energy Physics and Cosmology, Trieste, Italy, June 15th–July 3rd, 1992, hep-th/9301088.

[14] D. Bernard and A. LeClair, Comm. Math. Phys. 142 (1991) 99.

[15] S.J. Chang and R. Rajaraman, Phys. Lett. B313 (1993) 59; “Chiral vertex operators in off–conformal theory: the sine–Gordon example”, hepth/9311107.

[16] M. Bershadsky, W. Lerche, D. Nemeschansky and N. Warner, Nucl. Phys. B401 (1993) 304.

[17] K. Li, Nucl. Phys. B354 (1991) 711.

[18] M. Yu and H.B. Zheng, Nucl. Phys. B288 (1987) 275; M. Kato and S. Matsuda, Phys. Lett. B184 (1987) 184; G. Mussardo, G. Sotkov and M. Stanishkov, Int. J. Mod. Phys. A4 (1989) 1135; K. Ito, Phys. Lett. B230 (1989) 71; Nucl. Phys. B332 (1990) 566.

[19] H.C. Liao and P. Mansfield, Phys. Lett. B255 (1991) 237.

[20] A.B. Zamolodchikov and V.A. Fateev, Sov. Phys. JETP 63 (1985) 913; P. di Vecchia, J.L. Petersen and H.B. Zheng, Phys. Lett. B162 (1986) 327; P. di Vecchia, J.L. Petersen and M. Yu, Phys. Lett. B172 (1986) 211; W. Boucher, D. Friedan and A. Kent, Phys. Lett. B172 (1986) 316; P. di Vecchia, J.L. Petersen, M. Yu and H.B. Zheng, Phys. Lett. B174 (1986) 280.

[21] I.S. Gradshteyn and I.M. Ryzhik, “Table of Integrals, Series and Products”, Academic Press (1965).

[22] S. Cecotti and C. Vafa, Nucl. Phys. B367 (1991) 359.