A NOTE ON COMMUTATORS OF THE FRACTIONAL SUB-LAPLACIAN ON CARNOT GROUPS

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Abstract. In this manuscript, we provide a point-wise estimate for the 3-commutators involving fractional powers of the sub-Laplacian on Carnot groups of homogeneous dimension $Q$. This can be seen as a fractional Leibniz rule in the sub-elliptic setting. As a corollary of the point-wise estimate, we provide an $(L^p, L^q) \to L^r$ estimate for the commutator, provided that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\alpha}{Q}$ for $\alpha \in (0, Q)$.

1. Introduction and main results. In this paper we propose to investigate commutator type estimates for fractional powers of the sub-Laplacian on a Carnot group $G$. In fact, we will be focusing mainly on a fractional type Leibniz rule. These kind of estimates are very effective in studying the regularity properties of certain fractional PDEs. The study of these type of PDEs can be of much interests, namely, from an analytical point of view, but also from a geometric point of view, after the paper [12], where a family of sub-elliptic fractional conformally invariant operators were exhibited. These operators are the CR parallel to the GJMS operators [13] defined in the Riemannian setting.

Because of this geometric motivation, the Riemannian GJMS operators where intensively investigated and in the special case of the Euclidean space $\mathbb{R}^n$, the fractional powers of the Laplacian became a main research focus. For instance, fractional type Leibniz rules were first investigated in [14] then for the case of the half-Laplacian, more estimates were proved in [5, 6] in the study of Half-Harmonic maps. Then these results were generalized to the full spectrum of fractional powers in [17] using potential analysis and then another alternative proof was provided using harmonic extensions in [15]. As is the case of local operators, the main tool to study regularity properties of certain PDEs is localizing the functions with a cut-off. For non-local operators this procedure becomes more complicated. That is why a Leibniz-type rule is valuable in this case. In our work, we will follow the potential analysis approach developed in [17] in order to provide a fractional sub-elliptic Leibniz-type rule in a general Carnot group. Then, in Section 6 we discuss the case of the Heisenberg group and the two kinds of fractional type operators related to

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the sub-Laplacian. Finally, we will present a few applications of these estimates to provide a regularity result for solutions to the fractional CR-Yamabe problem on the sphere. Then we will present an \( \varepsilon \)-regularity type estimate for sequences of almost solutions. This last result is of crucial importance to the study of bubbling phenomena in the fractional setting.

Indeed, when dealing with equations that are invariant under the action of a non-compact group (such as the conformal group in the Euclidean and the CR settings), a concentration phenomena occurs, mainly leading to a non-compact Sobolev-type embedding. These type of concentrations occur frequently around isolated points of the ambient space, hence, in order to understand how they occur, a localization procedure is required. We refer the reader for instance to [2] for the bubbling phenomena in a local sub-elliptic setting under and [7] for the fractional case in the Euclidean setting and the references therein. Since this bubbling phenomenon is more of a geometric result, it will be discussed and presented in a forthcoming work.

We start here by defining the commutators and introducing the main results (see Section 2 for all the notations). Given a Carnot group \( G \) we denote by \( Q \) its homogeneous dimension. For this purpose, we fix \( 0 < \alpha < Q \), \( u, v \in S(G) \), the 3-commutator \( H_\alpha(\cdot, \cdot) \) is then defined by

\[
H_\alpha(u, v) = (-\Delta_b)\frac{\alpha}{\alpha + \beta - \delta} (uv) - u(-\Delta_b)\frac{\alpha}{\alpha + \beta - \delta} v - v(-\Delta_b)\frac{\alpha}{\alpha + \beta - \delta} u.
\]

Also, given \( 0 < \tau < Q \) and \( \beta, \delta > 0 \) such that \( \beta + \alpha < \min\{1, \tau\} \) we define the operator \( T_{\tau, \beta, \delta} \) by

\[
T_{\tau, \beta, \delta}(u, v) = \left[ \left( (-\Delta_b)\frac{\alpha}{\alpha + \beta - \delta} - \frac{\beta}{\beta + \delta} \right) \frac{\alpha}{\alpha + \beta - \delta} u \right] \left( (-\Delta_b)\frac{\alpha}{\alpha + \beta - \delta} v \right) - \left( (-\Delta_b)\frac{\alpha}{\alpha + \beta - \delta} - \frac{\beta}{\beta + \delta} \right) \frac{\alpha}{\alpha + \beta - \delta} \left( (-\Delta_b)\frac{\alpha}{\alpha + \beta - \delta} u \right) v.
\]

Our first result, involves a point-wise estimate, similar to the one in [17], that can be seen as a fractional type Leibniz rule for \( 0 < \alpha < 2 \).

**Theorem 1.1.** Assume that \( u \) and \( v \) are two Schwartz functions on \( G \) and let \( 0 < \alpha < 2 \) and \( \epsilon > 0 \). Given \( \tau_1 \) and \( \tau_2 \) in \( \max\{0, \alpha - 1\}, \alpha \) such that \( \tau_1 + \tau_2 > \alpha \), there exists \( L \in \mathbb{N} \), \( s_{j,1} \in (0, \tau_1) \), \( s_{j,2} \in (0, \tau_2) \), for \( j = 1, \cdots, L \), satisfying \( \tau_1 + \tau_2 - s_{j,1} - s_{j,2} - \alpha \in (0, \epsilon) \) such that for all \( x \in G \),

\[
|H_\alpha(u, v)|(x) \lesssim \sum_{j=1}^{L} R_{\tau_1 + \tau_2 - s_{j,1} - s_{j,2} - \alpha} \left( R_{s_{j,1}} (-\Delta_b)^{\frac{\alpha}{\alpha + \beta - \delta} u} R_{s_{j,2}} (-\Delta_b)^{\frac{\alpha}{\alpha + \beta - \delta} v} \right)(x).
\]

One of the tools that allow us to extend Theorem 1.1 to higher values of \( \alpha \), are the following commutator estimates involving the Riesz potential. This type of estimates can be compared to the result of Chanillo [3], though, the estimates that were provided there for the Euclidean case are sharp.

**Theorem 1.2.** Assume that \( u \) and \( v \) are two Schwartz functions. We fix \( \tau > 0 \) and choose \( \beta \) and \( \delta \) two non-negative numbers such that \( \beta + \delta < \min\{\tau, 1\} \). Then for a \( \epsilon > 0 \), there exist \( L \in \mathbb{N} \), \( s_{j,1} \), \( s_{j,2} \) such that \( s_{j,1} + s_{j,2} = \tau - \beta - \delta \) and \( s_{j,1} < \epsilon \), such that

\[
|T_{\tau, \beta, \delta}(u, v)|(x) \lesssim \sum_{j=1}^{L} R_{s_{j,1}} |u|(x) R_{s_{j,2}} |v|(x) + R_{\hat{s}_{j,1}}(|u| R_{s_{j,1}} |u|)(x), \forall x \in G.
\]

A combination of Theorem 1.1 and 1.2 provides the following
Corollary 1. Under the assumptions of Theorem 1.1, we fix $0 < \alpha < Q$ and $\epsilon > 0$. Given $\tau_1$ and $\tau_2$ in $\{\max\{0, \alpha - 1\}, \alpha\}$ such that $\tau_1 + \tau_2 > \alpha$, there exists $L \in \mathbb{N}$, $s_j, 1 \in (0, \tau_1)$, $s_j, 2 \in (0, \tau_2)$, for $j = 1, \ldots, L$, satisfying $\tau_1 + \tau_2 - s_j, 1 - s_j, 2 - \alpha \in [0, \epsilon)$ such that for all $x \in \mathbb{G}$,

$$|H_\alpha(u, v)(x)| \lesssim \sum_{j=1}^{L} R_{\tau_1 + \tau_2 - s_j, 1 - s_j, 2 - \alpha} \left( R_{s_j, 1} \left| (-\Delta_b)^{\frac{\alpha}{2}} u \right| R_{s_j, 2} \left| (-\Delta_b)^{\frac{\alpha}{2}} v \right| \right)(x).$$  \hfill (4)

Using the $\lambda$-kernel estimates in [10], we easily show that

Corollary 2. Let $u$ and $v$ be two functions in the Schwartz class of $\mathbb{G}$, and $\alpha \in (0, Q)$, then

$$\|H_\alpha(u, v)\|_p \lesssim \|(-\Delta_b)^{\frac{\alpha}{2}} u\|_{q_1} \|(-\Delta_b)^{\frac{\alpha}{2}} v\|_{q_2}$$  \hfill (5)

for $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{\alpha}{Q}$.

We believe that these estimates can be extended to involve the case $p = 1$, but naturally with $L^1$ replaced by the Hardy space $H^1$ as in the Euclidean setting, [17, 15]. This case cannot be reached by our method, instead it would require a deeper analysis involving a Littlewood-Paley type decomposition. However, the main gain in these results is that the estimates are actually point-wise instead of global and this provides more flexibility in terms of their use.

2. Preliminaries and notations. A connected and simply connected stratified nilpotent Lie group $(\mathbb{G}, \cdot)$ is said to be a Carnot group of step $k$, if its Lie algebra $\mathfrak{g}$ admits a step $k$ stratification, namely, there exist linear subspaces $V_1, \ldots, V_k$ such that

$$\mathfrak{g} = V_1 \oplus \ldots \oplus V_k, \quad [V_1, V_i] = V_{i+1}, \quad V_k \neq \{0\}, \quad V_i = \{0\} \text{ if } i > k,$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. Set $m_i = \dim(V_i)$, for $i = 1, \ldots, k$ and $h_i = m_1 + \cdots + m_i$, so that $h_k = n$. For the sake of simplicity, we also write $h_0 = 0$, $m := m_1$. We denote by $Q$ the homogeneous dimension of $\mathbb{G}$, i.e., we set

$$Q := \sum_{i=1}^{k} i \dim(V_i).$$

Given $\lambda > 0$, the dilation $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ is defined as

$$\delta_\lambda(x_1, \ldots, x_n) = (\lambda y_1, \ldots, \lambda^k y_k),$$

where $(x_1, \ldots, x_n) = (y_1, \ldots, y_k) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k} = \mathbb{G}$.

Then, we consider a family of left invariant vector fields $\{X_1, \ldots, X_m\}$ that coincides with the Euclidean bases $\{e_1, \ldots, e_m\}$ at zero, i.e., $X_i(0) = e_i$, the horizontal bundle $H\mathbb{G}$ is defined to be the span of this family of vectors. Thus one provides it with an inner product making the basis $\{X_1, \ldots, X_m\}$ orthonormal.

Once an orthonormal basis $X_1, \ldots, X_m$ of the horizontal bundle is fixed, we define, for any function $f : \mathbb{G} \to \mathbb{R}$ for which the partial derivatives $X_j f$ exist, the horizontal gradient of $f$, denoted by $\nabla f$, as the horizontal section

$$\nabla f := \sum_{i=1}^{m} (X_i f) X_i,$$
The sub-Laplacian $\Delta_b$ is then defined by

$$\Delta_b := -\sum_{j=1}^{m} X_j^2.$$ 

If we identify $G$ with its Lie algebra $\mathfrak{g}$ via the exponential map and consider $\{X_1, \cdots, X_n\}$ a basis of left invariant vectors of $\mathfrak{g}$, then we can define the Schwartz class of functions (see [10, Section D]), by

$$S(G) := \{ f \in C^\infty(G); X^I f \in L^\infty(G) \text{ for all multi-index } I \},$$

where $X^I = X_{i_1}^{i_1} \cdots X_{i_n}^{i_n}$, for $I = (i_1, \cdots, i_n) \in \mathbb{N}^n$.

The reader is referred to [8, 10, 1] for more details and examples regarding Carnot groups.

We recall now some definitions and properties of the fractional sub-Laplacian and adopt the following comparison notations: We write

$$f(x) \approx g(x),$$

if there exists a constant $C > 0$ such that

$$\frac{1}{C} f(x) \leq g(x) \leq C f(x), \forall x \in G.$$ 

Also, we will write

$$f(x) \lesssim g(x),$$

if there exists a constant $C > 0$ such that

$$f(x) \leq C g(x), \forall x \in G.$$ 

Recall that there exists a gauge $|\cdot|$ on $G$ that we can assume symmetric, that induces a quasi-distance (see [1] Chap 5, for more details). Moreover we have the existence of two positive constants $c$ and $C$ such that

$$c|x| - |y| \leq |yx| \leq C(|x| + |y|).$$

We can assume without loss of generality that $c < 1$ and $C > 1$. We provide now a very useful lemma, that we will be using later on in our computations

**Lemma 2.1 ([8]).** Let $f$ be a $\lambda$-homogeneous function. If we assume that $f$ is smooth on $G \setminus \{0\}$, then there exists a constant $C > 0$ such that if $|xy| \approx |x|$ we have

$$|f(xy) - f(x)| \leq C \max\{|xy|^\lambda - 1, |x|^{\lambda - 1}\}|y|.$$ 

**Proof.** A version of this Lemma was proved in [8] for $2|y| < |x|$. In fact, using the homogeneity of $f$, we can rescale and assume that $|x| = 1$ and $|y| \leq \frac{1}{2}$, therefore $|xy| \neq 0$ and hence, the proof follows from the regular intermediate value theorem, leading to

$$|f(xy) - f(x)| \leq C|x|^{\lambda - 1}|y|.$$ 

It remains then to prove the case when $|y| \approx |x|$. Again, if $|xy| > \frac{1}{2}$, then the proof is done, or else, we have that $|xy| < \frac{1}{4}$, then $|y| > \frac{1}{2}$, hence, we have that

$$|f(xy) - f(x)| \leq C|x|^{\lambda - 1}|y|.$$ 

We move now to the definition of the fractional sub-Laplacian and its inverse. For this purpose, we consider the heat semi-group defined by $H_t = e^{t \Delta_b}$, then one can define as in [8, 10], the following fractional operators for $\alpha > 0$:

$$(-\Delta_b)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha - 1}H_t dt$$
and
\[ (-\Delta_b)^\alpha = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-k-1} (-\Delta_b)^k H_t dt, \]
for any integer \( k > \alpha \). We have now the following:

**Proposition 1** ([8, 10]). Let \( 0 < \alpha < Q \) and consider \( h(t, x) \) the fundamental solution of the operator \( \Delta_b + \frac{\partial}{\partial t} \), then the integral
\[ R_\alpha(x) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha}{2} - 1} h(t, x) dt \]
converges absolutely and it satisfies the following properties:

- \( R_\alpha \) is a kernel of type \( \alpha \). In particular it is homogeneous of degree \( \alpha - Q \)
- \( R_2 \) is the fundamental solution of \( -\Delta_b \)
- \( R_\alpha \ast R_\beta = R_{\alpha+\beta} \) for \( \alpha \) and \( \beta > 0 \).
- For \( f \in L^p(\mathbb{G}) \) and \( 1 < p < \infty \), we have that
\[ (-\Delta_b)^{-\frac{\alpha}{2}} f = f \ast R_\alpha. \]

As a corollary of this proposition one has
\[ R_\alpha(x) \approx |x|^{-Q+\alpha}, \]
and \( \rho(x) = (R_\alpha(x))^{\frac{\alpha}{\alpha-\beta}} \) defines a \( \mathbb{G} \)-homogeneous norm, smooth away from the origin and it induces a quasi-distance that is equivalent to the left-invariant Carnot-Carathéodory distance. In a similar way, one can define the function \( \tilde{R}_\alpha \), introduced in [4], for \( \alpha < 0 \) and \( \alpha \not\in \{0, -2, -4, \ldots \} \) by
\[ \tilde{R}_\alpha(x) = \frac{\alpha}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha}{2} - 1} h(t, x) dt. \]
Again, it is easy to see that \( \tilde{R}_\alpha \) is \( \mathbb{G} \)-homogeneous of degree \( \alpha - Q \) and
\[ \tilde{R}_\alpha(x) \approx |x|^\alpha - Q. \]

Using this function, one can define another representation for the fractional sub-Laplacian, that will be fundamental in the proof of our results.

**Theorem 2.2** ([4]). If \( u \) is a Schwartz function on \( \mathbb{G} \), then for \( 0 < \alpha < 2 \) one has
\[ (-\Delta_b)^{-\frac{\alpha}{2}} u(x) = PV \int_\mathbb{G} (u(y) - u(x)) R_{-\alpha}(y^{-1}x) dy. \]

For the sake of notation, we will use in this paper the following notation:
\[ R_\alpha u := (-\Delta_b)^{-\frac{\alpha}{2}} u = u \ast R_\alpha. \]

3. **Proof of Theorem 1.1.** We are interested in the study of the commutator
\[ H_\alpha(u, v) = (-\Delta_b)^{-\frac{\alpha}{2}} uv - u(-\Delta_b)^{-\frac{\alpha}{2}} v - v(-\Delta_b)^{-\frac{\alpha}{2}} u. \]

Notice that
\[ H_\alpha(u, v)(x) = \int_\mathbb{G} [u(x) - u(y)] [v(x) - v(y)] \tilde{R}_{-\alpha}(y^{-1}x) dy. \]
We take now \( u = (-\Delta_b)^{-\frac{\alpha}{2}} a = R_{\tau_1} a \) and \( v = (-\Delta_b)^{-\frac{\alpha}{2}} b = R_{\tau_2} b \), then we can write
\[ H_\alpha(u, v)(x) = \int \int \left[ R_{\tau_1}(\eta^{-1}y) - R_{\tau_1}(\eta^{-1}x) \right] \left[ (R_{\tau_2}(\xi^{-1}y) - R_{\tau_2}(\xi^{-1}x)) \right] \times \tilde{R}_\alpha(y^{-1}x) a(\eta) b(\xi) d\eta d\xi dy. \]
Therefore, one needs to study the kernel $k$ defined by:

$$k(x, y, \eta, \xi) := \left[ R_{\tau_1}(\eta x^{-1}y) - R_{\tau_1}(\eta^{-1}x) \right] \left[ (R_{\tau_2}(\xi^{-1}y) - R_{\tau_2}(\xi^{-1}x)) \right] \tilde{R}_\alpha(y^{-1}x).$$

In order to simplify the computations, we introduce the kernel $k_\tau$ defined by

$$k_\tau(x, y, \eta) = |R_{\tau}(\eta^{-1}y) - R_{\tau}(\eta^{-1}x)|,$$

and we split the space into three parts using the following characteristic functions:

$$\{ \chi_1 = \chi_{|y^{-1}x|<2|\eta^{-1}y|} \chi_{|y^{-1}x|<2|\eta^{-1}x|} \chi_{|y^{-1}x|>2|\eta^{-1}x|} \}
\{ \chi_2 = \chi_{|y^{-1}x|<2|\eta^{-1}y|} \chi_{|y^{-1}x|>2|\eta^{-1}x|} \}
\{ \chi_3 = \chi_{|y^{-1}x|>2|\eta^{-1}y|} \chi_{|y^{-1}x|<2|\eta^{-1}x|} \}.$$

We first notice that

$$|\eta^{-1}y| \chi_1 \approx |\eta^{-1}x| \chi_1.$$  

Therefore, using Lemma 2.1, we have, for $0 < \delta < \min\{\tau, 1\}$, that

$$k_\tau(x, y, \eta) \chi_1 \lesssim \min\{|\eta^{-1}x|, |\eta^{-1}y|\} |y^{-1}x| \chi_1 \lesssim |\eta^{-1}x|^{\tau - \delta} |y^{-1}x|^{\delta} \chi_1 \lesssim |\eta^{-1}y|^{\tau - \delta} |y^{-1}x|^{\delta} \chi_1.$$  

Moreover,

$$k_\tau(x, y, \eta) \chi_2 \lesssim |\eta^{-1}x|^{\tau - \delta} \chi_2 \lesssim |\eta^{-1}x|^{\tau - \delta} |y^{-1}x|^{\delta} \chi_2,$$

and similarly

$$k_\tau(x, y, \eta) \chi_3 \lesssim |\eta^{-1}y|^{\tau - \delta} \chi_3 \lesssim |\eta^{-1}y|^{\tau - \delta} |y^{-1}x|^{\delta} \chi_3.$$  

Therefore, since $k(x, y, \eta, \xi) = k_{\tau_1}(x, y, \eta)k_{\tau_2}(x, y, \xi)\tilde{R}_\alpha(y^{-1}x)$ we have, for $0 < \delta_1 < \min\{\tau_1, 1\}$ and $0 < \delta_2 < \min\{\tau_2, 1\}$,

$$k(x, y, \eta, \xi) \lesssim |y^{-1}x|^{-Q - \alpha + \delta_1 + \delta_2} \left[ |\eta^{-1}x|^{-Q + \tau_1 - \delta_1 |\xi^{-1}y|^{-Q + \tau_2 - \delta_2} \right. $$

$$+ |\eta^{-1}y|^{-Q + \tau_1 - \delta_1 |\xi^{-1}x|^{-Q + \tau_2 - \delta_2} + |\eta^{-1}y|^{-Q + \tau_1 - \delta_1 |\xi^{-1}y|^{-Q + \tau_2 - \delta_2} $$

$$+ |y^{-1}x|^{-\delta_1 - \delta_2} |\eta^{-1}x|^{-Q + \tau_1 |\xi^{-1}x|^{-Q + \tau_2} \chi_{|y^{-1}x|>2\max\{|\eta^{-1}x|, |\xi^{-1}x|\}} \right].$$

Now notice that using polar coordinates\(^2\), if one takes $0 < \delta_1 + \delta_2 - \alpha < \varepsilon$, where $\varepsilon$ is positive and small, we have for $\varepsilon > 0$ that

$$\int_{|y^{-1}x|>\varepsilon} |y^{-1}x|^{-Q - \alpha} \chi_{|y^{-1}x|>2\max\{|\eta^{-1}x|, |\xi^{-1}x|\}} dy \lesssim \max\{|\eta^{-1}x|, |\xi^{-1}x|\}^{-\alpha} \lesssim |y^{-1}x|^{-\delta_1 |\xi^{-1}x|^{-\alpha + \delta_1}. $$

This yields

$$|H_\alpha(u, v)(x)| \lesssim \int \int k(x, y, \eta, \xi)|a(\eta)||b(\xi)|d\eta d\xi dy \lesssim \left( R_{\delta_1 + \delta_2 - \alpha} (R_{\tau_2 - \delta_2} |b|)(x) R_{\tau_1 - \delta_1} |a|)(x) \right).$$

\(^2\)Naturally, these are not the Euclidean polar coordinates, instead one can use the dialation of the group $G$ in order to mimic the Euclidean ones. We refer the reader for instance to [1, Proposition 5.5.4].
Proof of Theorem 1.2.

proving the desired result.

Notice that

\[ T = \tau, \beta, \delta \]

Hence

\[ |H_\alpha(u, v)(x)| \lesssim R_{\tau_1 - \delta_1} |a|(x) R_{\tau_2 + \delta_1 - \alpha} |b|(x) + R_{\tau_1 + \delta_2 - \alpha} |a|(x) R_{\tau_2 - \delta_1} |b|(x) \]

Therefore, one can state that

\[ |H(u, v)(x)| \leq C \sum_{j=1}^{L} R_{\tau_1 + \tau_2 - \tau_{j,1} - s_{j,2} - \alpha} (R_{\tau_{j,1}} |a|R_{\delta_{j,2}} |b|)(x), \]

proving the desired result.

4. Proof of Theorem 1.2. From now on, we will let \( \tau > 0 \) and two non-negative numbers \( \beta \) and \( \delta \) so that \( \beta + \delta < \min(\tau, 1) \). We then, define the commutator \( T_{\tau, \beta, \delta}(u, v) \) by

\[
T_{\tau, \beta, \delta}(u, v) = \left[ R_{\tau} u, (-\Delta_b)^{\frac{\delta}{2}} v \right] - \left[ (-\Delta_b)^{\frac{\beta}{2}} v, R_{\tau} u \right] + \int \left[ R_{\tau} u(x) (-\Delta_b)^{\frac{\delta}{2}} v(x) - R_{\tau} u(y) (-\Delta_b)^{\frac{\beta}{2}} v(y) \right] \times \tilde{R}_{-\beta}(y^{-1}x) dy .
\]
As in the previous proof, we propose to study the kernel $k$ defined by

$$k(x, y, z, w) = \tilde{R}_{-\delta}(w) \left( R_\tau(z^{-1}x) - R_\tau(z^{-1}w^{-1}y) \right) \tilde{R}_{-\beta}(wy^{-1}x)$$

$$- \left( R_\tau(z^{-1}x) - R_\tau(z^{-1}y) \right) \tilde{R}_{-\beta}(y^{-1}x).$$

But, in this situation, we need a better splitting of the space, adapted to the different quantities involved. So we define the sets

- $A_1 = \{(x, y, z, w) \in (\mathbb{H}^n)^4; |y^{-1}x| \leq 2|y^{-1}z| \text{ and } |y^{-1}x| \leq 2|z^{-1}x|\}$,
- $A_2 = \{(x, y, z, w) \in (\mathbb{H}^n)^4; |y^{-1}x| \leq 2|y^{-1}z| \text{ and } |y^{-1}x| > 2|z^{-1}x|\}$,
- $A_3 = \{(x, y, z, w) \in (\mathbb{H}^n)^4; |y^{-1}x| > 2|y^{-1}z| \text{ and } |y^{-1}x| \leq 2|z^{-1}x|\}$,

and

- $B_1 = \{(x, y, z, w) \in (\mathbb{H}^n)^4; |wy^{-1}x| \leq 2|wy^{-1}z| \text{ and } |wy^{-1}x| > 2|z^{-1}x|\}$,
- $B_2 = \{(x, y, z, w) \in (\mathbb{H}^n)^4; |wy^{-1}x| \leq 2|wy^{-1}z| \text{ and } |wy^{-1}x| \leq 2|z^{-1}x|\}$,
- $B_3 = \{(x, y, z, w) \in (\mathbb{H}^n)^4; |wy^{-1}x| > 2|wy^{-1}z| \text{ and } |wy^{-1}x| \leq 2|z^{-1}x|\}$,

and

- $C_1 = \{(x, y, z, w) \in (\mathbb{H}^n)^4; |w| \geq 4|y^{-1}x|\}$,
- $C_2 = \{(x, y, z, w) \in (\mathbb{H}^n)^4; \frac{|y^{-1}x|}{4} \leq |w| \leq 4|y^{-1}x|\}$,
- $C_3 = \{(x, y, z, w) \in (\mathbb{H}^n)^4; 4|w| < |y^{-1}x|\}$.

We can verify that each collection of sets exhaust $(G)^4$. In a first step, we will split the kernel $k$ so that $k \leq k_1 + k_2$ where

$$k_1(x, y, z, w) = \tilde{R}_{-\delta}(w) \left| R_\tau(z^{-1}x) - R_\tau(z^{-1}w^{-1}y) \right| \tilde{R}_{-\beta}(wy^{-1}x),$$

and

$$k_2(x, y, z, w) = \tilde{R}_{-\delta}(w) \left| R_\tau(z^{-1}x) - R_\tau(z^{-1}y) \right| \tilde{R}_{-\beta}(y^{-1}x).$$

**Step I:** Estimate on $C_1 \cup C_2$.

Notice that on $C_1 \cup C_2$ we have that

$$\int \tilde{R}_{-\delta}(w) \chi_{C_1 \cup C_2} dw \lesssim \int |w|^{-Q+\delta} \chi_{C_1 \cup C_2} dw \lesssim |y^{-1}x|^{-\delta}.$$

Therefore,

$$\int k_2 \chi_{C_1 \cup C_2} dw \lesssim |R_\tau(z^{-1}x) - R_\tau(z^{-1}y)||y^{-1}x|^{-Q-\beta-\delta}.$$
For instance, in the set \((C_1 \cup C_2) \cap B_1\), we have that
\[
|wy^{-1}z|\chi_{(C_1 \cup C_2) \cap B_1} \leq C(|z^{-1}x| + |wy^{-1}x|)\chi_{(C_1 \cup C_2) \cap B_1} \leq C|z^{-1}x|\chi_{(C_1 \cup C_2) \cap B_1},
\]
and
\[
2|wy^{-1}z|\chi_{(C_1 \cup C_2) \cap B_1} \geq |wy^{-1}x|\chi_{(C_1 \cup C_2) \cap B_1} \geq c(|z^{-1}x| - |yw^{-1}z|)\chi_{(C_1 \cup C_2) \cap B_1}.
\]
Therefore, \(|z^{-1}x|\) and \(|wy^{-1}z|\) are comparable and we can use Lemma 2.1, to get for any \(\varepsilon \in [0,1]\)
\[
k_1(x,y,z,w)\chi_{(C_1 \cup C_2) \cap B_1} \lesssim |w|^{-Q-\delta} \max\{|z^{-1}x|,|wy^{-1}z|\}^{-Q+\tau-\varepsilon} \\
\times |wy^{-1}x|^{-Q-\beta+\varepsilon}\chi_{(C_1 \cup C_2) \cap B_1}.
\]
Now notice that on \(C_1\), we have that
\[
|y^{-1}x|\chi_{C_1} \lesssim |wy^{-1}x|\chi_{C_1},
\]
and on \(C_2\),
\[
|w|\chi_{C_2} \approx |y^{-1}x|\chi_{C_2}.
\]
Thus,
\[
k_1(x,y,z,w)\chi_{(C_1 \cup C_2) \cap B_1} \lesssim |w|^{-Q-\delta}|z^{-1}x|^{-Q+\tau-\varepsilon}|y^{-1}x|^{-Q-\beta+\varepsilon}\chi_{C_1} \\
+ |y^{-1}x|^{-Q-\delta}|z^{-1}x|^{-Q+\tau-\varepsilon}|wy^{-1}x|^{-Q-\beta+\varepsilon}\chi_{C_2}.
\]
So if we take \(\varepsilon \in (\beta,1)\) and integrate on \(w\), we get
\[
\int \chi_{(C_1 \cup C_2) \cap B_1} k_1(x,y,z,w)dw \lesssim |y^{-1}x|^{-Q-\beta-\delta+\varepsilon}|z^{-1}x|^{-Q+\tau-\varepsilon}.
\]
Therefore, we have the desired inequality if one chooses \(\varepsilon \in (\delta + \beta, \tau)\), \(s = \tau - \varepsilon\) and \(t = \varepsilon - \beta - \delta\).

The case of \(C_1 \cup C_2 \cap B_2\): Notice here that
\[
|wy^{-1}z|\chi_{B_2} \geq c(|wy^{-1}x| - |x^{-1}z|)\chi_{B_2} \geq C|z^{-1}x|\chi_{B_2}.
\]
Therefore, we have that
\[
\chi_{(C_1 \cup C_2) \cap B_2} k_1(x,y,z,w) \lesssim |w|^{-Q-\delta}|z^{-1}x|^{-Q+\tau}|wy^{-1}x|^{-Q-\beta}.
\]
Again, on \(B_2\) we have that \(|wy^{-1}x|\chi_{B_2} \geq 2|z^{-1}x|\chi_{B_2}\). Hence,
\[
\chi_{(C_1 \cup C_2) \cap B_2} k_1(x,y,z,w) \lesssim |w|^{-Q-\delta}|z^{-1}x|^{-Q+\tau-\varepsilon}|wy^{-1}x|^{-Q-\beta+\varepsilon}.
\]
This yields, for \(D = \{|wy^{-1}x| \lesssim |y^{-1}x|\}\),
\[
\chi_{(C_1 \cup C_2) \cap B_2} k_1(x,y,z,w) \lesssim |w|^{-Q-\delta}|z^{-1}x|^{-Q+\tau-\varepsilon}|y^{-1}x|^{-Q-\beta+\varepsilon}\chi_{C_1} \\
+ |y^{-1}x|^{-Q-\delta}|z^{-1}x|^{-Q+\tau-\varepsilon}|wy^{-1}x|^{-Q-\beta+\varepsilon}\chi_{D}.
\]
So if we integrate over \(w\) while taking \(\varepsilon > \beta\), we get
\[
\int \chi_{(C_1 \cup C_2) \cap B_2} k_1(x,y,z,w)dw \lesssim |y^{-1}x|^{-Q-\beta-\delta+\varepsilon}|z^{-1}x|^{-Q+\tau-\varepsilon},
\]
and therefore, we have the same conclusion as in \(B_1\).

Case of \((C_1 \cup C_2) \cap B_3\): Notice that on \(B_3\) we have that \(|wy^{-1}x|\chi_{B_3} \leq 2|z^{-1}x|\chi_{B_3}\) and \(|wy^{-1}x|\chi_{B_3} \geq 2|wy^{-1}z|\chi_{B_3}\), hence
\[
k_1 \chi_{(C_1 \cup C_2) \cap B_3} \lesssim |w|^{-Q-\delta}|wy^{-1}z|^{-Q+\tau}|wy^{-1}x|^{-Q-\beta} \\
\lesssim |w|^{-Q-\delta}|wy^{-1}z|^{-Q+\tau-\varepsilon}|wy^{-1}x|^{-Q-\beta+\varepsilon}.
\]
Multiplying by $u$ and integrating with respect to $z$ yields, for $D_1 = C_1 \cup C_2 \cap B_3$,
\[
\int \chi_{D_1} k_1(x,y,z,w)u(z)dz \lesssim |w|^{-Q-\delta}|wy^{-1}x|^{-Q-\beta+\epsilon} \int |wy^{-1}z|^{-Q+\tau-\epsilon}|u|(z)dz \\
\lesssim |w|^{-Q-\delta}|wy^{-1}x|^{-Q-\beta+\epsilon} R_{\tau-\epsilon}|u|(w^{-1}y).
\]
But on $C_1 \cup C_2$ we have that
\[
\max\{|y^{-1}x|, |wy^{-1}x|\} \chi_{C_1 \cup C_2} \lesssim \chi_{C_1 \cup C_2}|w|.
\]
Thus, if we take $\epsilon = \epsilon_1 + \epsilon_2$, where $\epsilon_1 > \delta$ and $\epsilon_2 > \beta$, we have that
\[
\int \int \chi_{D_1} k_1(x,y,z,w)u(z)dzdw \lesssim \left( \int |wy^{-1}x|^{-Q-\beta+\epsilon} R_{\tau-\epsilon}|u|(w^{-1}y)dw \right) \\
\times |y^{-1}x|^{-Q-\delta+\epsilon_1} \lesssim |y^{-1}x|^{-Q-\delta+\epsilon_1} R_{\tau-\beta-\epsilon_1}|u|(x),
\]
which finishes the proof for $s = \tau - \beta - \epsilon_1$ and $t = \epsilon_1 - \delta$.

**Step 2:** Estimate on $C_3$.

In this step we will rewrite the kernel $k$ by noticing that
\[
k = \tilde{R}_{\delta}(w) \left( [R_\tau(z^{-1}x) - R_\tau(z^{-1}w^{-1}y)] \tilde{R}_{\beta}(wy^{-1}x) \right) \\
- [R_\tau(z^{-1}x) - R_\tau(z^{-1}y)] \tilde{R}_{\beta}(wy^{-1}x) \\
= \tilde{R}_{\delta}(w) \left( [R_\tau(z^{-1}x) - R_\tau(y^{-1}z)] [\tilde{R}_{\beta}(y^{-1}x) - \tilde{R}_{\beta}(wy^{-1}x)] \right) \\
\times \tilde{R}_{\beta}(wy^{-1}x) \tilde{R}_{\beta}(y^{-1}x) + [R_\tau(z^{-1}y) - R_\tau(w^{-1}z^{-1}y)] \tilde{R}_{\beta}(wy^{-1}x),
\]
and we will be using the splitting induced by the $A_i$, $i = 1, 2, 3$. So first we will look at $k$ on the set $C_3 \cap (A_1 \cup A_2)$.

Notice that on $C_3$, $|y^{-1}x|$ and $|wy^{-1}x|$ are comparable. Moreover,
\[
\chi_{C_3 \cap (A_1 \cup A_2)}|wy^{-1}z| \lesssim \chi_{C_3 \cap (A_1 \cup A_2)}(|w| + |y^{-1}z|) \\
\lesssim \chi_{C_3 \cap (A_1 \cup A_2)}(|y^{-1}x| + |y^{-1}z|) \lesssim \chi_{C_3 \cap (A_1 \cup A_2)}|y^{-1}z|,
\]
and
\[
|wy^{-1}z| \geq c(|y^{-1}z| - |w|) \geq c(|y^{-1}z| - \frac{1}{4}|y^{-1}x|) \geq c|y^{-1}z|.
\]
So again $|wy^{-1}z|$ and $|y^{-1}z|$ are comparable. Therefore, by Lemma 2.1, we have for $\varepsilon \in [0, 1]$ and $D_3 = C_3 \cap (A_1 \cup A_2)$ that
\[
\chi_{D_3} \lesssim \left( |w|^{-Q-\delta}|y^{-1}x|^{-2Q-2\beta} \min\{|z^{-1}x|, |z^{-1}y|\}^{-Q+\tau-\epsilon}|y^{-1}x|^q|y^{-1}x|Q+\beta-\epsilon \right) |w|^\varepsilon \chi_{D_3}.
\]
Taking $\varepsilon = \epsilon_1 + \epsilon_2$, we find that
\[
\chi_{D_3} \lesssim |w|^{-Q-\delta+\epsilon_1}|z^{-1}x|^{-Q+\tau-\epsilon}|y^{-1}x|^{-Q-\beta+\epsilon_2} \\
+ |w|^{-Q-\delta+\epsilon_1}|z^{-1}y|^{-Q+\tau-\epsilon}|y^{-1}x|^{-Q-\beta+\epsilon_2}.
\]
Once again, we choose $\epsilon_1 > \delta$ and $\epsilon_2 > \beta$, and $\varepsilon = \epsilon_1 + \epsilon_2 < \min\{1, \tau\}$, to have after integration over $w$ on $C_3$
\[
\int \chi_{D_3}kdw \lesssim \left( |z^{-1}x|^{-Q+\tau-\epsilon}|y^{-1}x|^{-Q+\tau-\epsilon} + |z^{-1}y|^{-Q+\tau-\epsilon}|y^{-1}x|^{-Q+\tau-\epsilon} \right).
\]
Thus,
\[
\int \int \int \chi_{C_3 \cap (A_1 \cup A_2)} k(x, y, z, w) u(z) v(y) dw dy dz \lesssim R_s |u| R_t |v| + R_t (|v| R_s |u|),
\]
for \( t = \varepsilon - \delta - \beta \) and \( s = \tau - \varepsilon \).

We focus now on the last part, that is \( C_3 \cap A_3 \). Notice that since \(|z^{-1} x| > |y^{-1} z|\) on \( A_3 \), and \(|y^{-1} x| \approx |w y^{-1} x|\) on \( C_3 \), we have from (20), that
\[
\chi_{C_3 \cap A_3} k \lesssim |w|^{-Q-\delta} |y^{-1} z|^{-Q+\tau} |y^{-1} x|^{-Q-\beta} \chi_{C_3 \cap A_3} \chi_{C_3 \cap A_3}
\]
\[+ |w|^{-Q-\delta} R_\tau (z^{-1} y) - R_\tau (w^{-1} z^{-1} y) |y^{-1} x|^{-Q-\beta} \chi_{C_3 \cap A_3}
\]
\[\lesssim k_3 + k_4.
\]
(23)

We will estimate \( k_3 \) and \( k_4 \) separately. For \( k_3 \) we have that
\[
k_3 \lesssim |w|^{-Q-\delta+\varepsilon_1} |y^{-1} z|^{-Q+\tau} |y^{-1} x|^{-Q-\beta} \chi_{C_3 \cap A_3}
\]
\[+ |w|^{-Q-\delta+\varepsilon_1} |y^{-1} z|^{-Q+\tau-\varepsilon} |y^{-1} x|^{-Q-\beta+\varepsilon_2} \chi_{C_3 \cap A_3},
\]
(24)

which is similar to case \( C_3 \cap (A_1 \cup A_2) \). Hence, the desired inequality holds.

We move now to the term \( k_4 \). We have that
\[
k_4 \lesssim |w|^{-Q-\delta-\varepsilon_2} R_\tau (z^{-1} y) - R_\tau (w^{-1} z^{-1} y) |y^{-1} x|^{-Q-\beta+\varepsilon_2} \chi_{C_3 \cap A_3}.
\]
So in the set \( \{2|w| < |y^{-1} z|\} \), we have by Lemma 2.1
\[
\chi_{2|w|<|y^{-1} z|} k_4 \lesssim |w|^{-Q-\delta+\varepsilon_1} |z^{-1} y| |y^{-1} x|^{-Q-\beta+\varepsilon_2} \chi_{C_3 \cap A_3} \chi_{2|w|<|y^{-1} z|}
\]
and we are in the same situation as \( k_3 \) above. So we treat the estimate in \( 2|w| \geq |y^{-1} z| \).

Notice that, if \( D_4 = C_3 \cap A_3 \), then
\[
\chi_{2|w|\geq|y^{-1} z|} k_4 \lesssim \chi_{C_3 \cap A_3} \chi_{2|w|\geq|y^{-1} z|} |w|^{-Q-\delta+\varepsilon_1} |z^{-1} y| |y^{-1} x|^{-Q-\beta+\varepsilon_2}
\]
\[+ \chi_{|w^{-1} z^{-1} y|\leq|w|} |w|^{-Q-\delta-\varepsilon_2} |w^{-1} z^{-1} y|^{-Q+\tau-\varepsilon} |y^{-1} x|^{-Q-\beta+\varepsilon_2} \chi_{D_4}.
\]
(25)

Therefore we have first that
\[
J_1 := \int \chi_{2|w|\geq|y^{-1} z|} |w|^{-Q-\delta+\varepsilon_1} |z^{-1} y| |y^{-1} x|^{-Q-\beta+\varepsilon_2} dw
\]
\[\lesssim \chi_{D_4} |z^{-1} y|^{-Q+\tau-\delta-\varepsilon_1} |y^{-1} x|^{-Q-\beta+\varepsilon_2},
\]
which provides us with the desired estimate for \( \varepsilon_1 > \delta, \varepsilon_2 > \beta \) and \( \varepsilon_1 + \varepsilon_2 = \varepsilon \in (\beta + \delta, \min\{1, \tau\}) \). On the other hand, since
\[
J_2 := \int |w|^{-Q-\delta-\varepsilon_2} |w^{-1} z^{-1} y|^{-Q+\tau-\varepsilon} |y^{-1} x|^{-Q-\beta+\varepsilon_2} u(z) dz
\]
\[\lesssim |w|^{-Q-\delta+\varepsilon_2} |y^{-1} x|^{-Q-\beta+\varepsilon_2} R_{\tau-\varepsilon} |u|(w^{-1} y),
\]
integrating again with respect to \( w \) yields
\[
\int |w|^{-Q-\delta+\varepsilon_2} |y^{-1} x|^{-Q-\beta-\varepsilon_2} R_{\tau-\varepsilon} |u|(w^{-1} y) dw \lesssim |y^{-1} x|^{-Q-\beta+\varepsilon_2} R_{\tau-\varepsilon} |u|(y)
\]
and in the end,
\[
\int |y^{-1} x|^{-Q-\beta+\varepsilon_2} R_{\tau-\varepsilon} |u|(y) v(y) dy \lesssim R_{\varepsilon_2-\beta} ((R_{\tau-\varepsilon_2} |u|) v)(x).
\]
Which finishes the proof.
5. **Proof of Corollary 1.1.** In this case, we will write $\alpha = 2k + s$ with $s \in (0, 2)$, then the commutator $H_\alpha$ can be written as

\[
H_\alpha(u, v) = (-\Delta_b)^k(-\Delta_b)^2(\nabla H u v) - u(-\Delta_b)^k(-\Delta_b)^2 v - v(-\Delta_b)^k(-\Delta_b)^2 u \\
= (-\Delta_b)^2 H_k(u, v) + (-\Delta_b)^2 (v(-\Delta_b)^k u) - v(-\Delta_b)^k(-\Delta_b)^2 u \\
+ (-\Delta_b)^2 (u(-\Delta_b)^k u) - u(-\Delta_b)^k(-\Delta_b)^2 u.
\] (26)

Again, in order to simplify the notations, given $m \in \mathbb{N}$, we let $\nabla_H^m$ be a combination of $\nabla_H$ and $-\Delta_b$ of differentiation order $m$. Therefore, we have that

\[
H_k(u, v) = \sum_{i=1}^{2k-1} \nabla_H^i u \nabla_H^{2k-i} v.
\]

So we want to estimate the term $(-\Delta_b)^2 (\nabla_H^i u \nabla_H^{2k-i} v)$. But first we notice that

\[
H_s(\nabla_H^i u, \nabla_H^{2k-i} v) = H_s(\nabla_H^i u, \nabla_H^{2k-i} v) + \nabla_H^{2k-i} v(-\Delta_b)^2 \nabla_H^i u + \nabla_H^i u(-\Delta_b)^2 \nabla_H^{2k-i} v.
\]

Using Theorem 1.1, we have the existence of $s_{j,1}$, $s_{j,2}$, so that

\[
H_s(\nabla_H^i u, \nabla_H^{2k-i} v) \leq \sum_{j=1}^{2} R_{s_{j,1}-s_{j,2}} (R_{s_{j,1}} |(-\Delta_b)^2 \nabla_H^i u| R_{s_{j,2}} |(-\Delta_b)^2 \nabla_H^{2k-i} v|) (x) \\
\leq \sum_{j=1}^{2} R_{s_{j,1}-s_{j,2}} (R_{s_{j,1} + \tau_1 - s_{j,2} - i} |a| R_{s_{j,2} + \tau_2 - a + i} |b|) (x),
\] (27)

where we used here the fact that for $\alpha_1 > \alpha_2 > 0$ we have

\[
|R_{\alpha_1} (u)| = |R_{\alpha_1 - \alpha_2} (-\Delta_b)^{\frac{\alpha_2}{2}} u| \lesssim R_{\alpha_1 - \alpha_2} (-\Delta_b)^{\frac{\alpha_2}{2}} u|.
\]

Therefore, we have that

\[
|(-\Delta_b)^2 (\nabla_H^i u \nabla_H^{2k-i} v)| \lesssim \sum_{j=1}^{2} R_{s_{j,1}-s_{j,2}} (R_{s_{j,1} + \tau_1 - s_{j,2} - i} |a| R_{s_{j,2} + \tau_2 - a + i} |b|) (x) \\
+ R_{\tau_1 - s_{j,2}} |a| R_{\tau_2 - 2k+i} |b| + R_{\tau_1 - i} |a| R_{\tau_2 - a + i} |b|.
\] (28)

It follows then

\[
|(-\Delta_b)^2 H_k(u, v)| \leq C \sum_{i=1}^{2k-1} \sum_{j=1}^{2} R_{s_{j,1}-s_{j,2}} (R_{s_{j,1} + \tau_1 - s_{j,2} - i} |a| R_{s_{j,2} + \tau_2 - a + i} |b|) \\
+ R_{\tau_1 - s_{j,2}} |a| R_{\tau_2 - 2k+i} |b| + R_{\tau_1 - i} |a| R_{\tau_2 - a + i} |b|.
\] (29)

It remains now to estimate the terms $(-\Delta_b)^2 (v(-\Delta_b)^k u) - v(-\Delta_b)^k(-\Delta_b)^2 u$ and $(-\Delta_b)^2 (u(-\Delta_b)^k u) - u(-\Delta_b)^k(-\Delta_b)^2 u$. Notice that

\[
(-\Delta_b)^2 (v(-\Delta_b)^k u) - v(-\Delta_b)^k(-\Delta_b)^2 u = H_s((-\Delta_b)^k u, v) + (-\Delta_b)^k u(-\Delta_b)^2 v.
\]

Since $\tau_1$ and $\tau_2$ are bigger than $\alpha - 1 = 2k + s - 1$, we have that $\tau_2 > s$. We distinguish two cases.

**Case $\tau_1 > 2k$:**
In this case we have that

\[
|(-\Delta_b)^2 u| \lesssim R_{\tau_1 - 2k} |(-\Delta_b)^2 u|
\]
and
\[ |(\Delta_b)^{s} v| \lesssim R_{\tau_2-s}|(-\Delta_b)^{\frac{s}{2}} v| \]
Combining this with Theorem 1.1, we have the desired estimate.

**Case** \(\tau_1 \leq 2k\):
We choose \(\delta \in (2k - \tau_1, 1 - s)\), knowing that \(s < 1\), we have that
\[ (-\Delta_b)^{\frac{s}{2}}(v(-\Delta_b)^{k} u) - v(-\Delta_b)^{k}(-\Delta_b)^{\frac{s}{2}} u = (-\Delta_b)^{\frac{s}{2}}(v(-\Delta_b)^{\frac{s}{2}} R_{\tau_1-2k+\delta}(-\Delta_b)^{\frac{s}{2}} u)
\]
\[ - v(-\Delta_b)^{\frac{s}{2}}(\Delta_b)^{\frac{s}{2}} R_{\tau_1-2k+\delta}(-\Delta_b)^{\frac{s}{2}} u
\]
\[ = T_{\tau_2, s, \delta}(V, U) \quad (30)\]
where \(V = (-\Delta_b)^{\frac{s}{2}} v\) and \(U = R_{\tau_1-2k+\delta}(-\Delta_b)^{\frac{s}{2}} u\). Hence, one can apply Theorem 1.2 to conclude.

6. The case of the Heisenberg group. In the case where \(\mathbb{G} = \mathbb{H}^n\), and therefore \(Q = 2n + 2\), there are two kinds of fractional powers of the sub-Laplacian. Indeed, there is the usual fractional sub-Laplacian as defined above in this manuscript, and the conformally invariant one that we will denote by \(L_\alpha\), we will refer to this last one as the geometric fractional sub-Laplacian. As in [16], these two operators can be distinguished in terms of their spectral Fourier multipliers (we refer the reader to [18] for the definition of the Fourier transform and Fourier spectral multipliers on the Heisenberg group). Indeed, we have that the operator \((-\Delta_b)\) corresponds to the spectral multiplier
\[ A(k, \lambda) := (2k + n)|\lambda|. \]
Therefore, \((-\Delta_b)^{\frac{s}{2}}\) corresponds to the multiplier
\[ A(k, \lambda, \alpha) := ((2k + n)|\lambda|)^{\frac{s}{2}}. \]
On the other hand, the spectral multiplier of \(L_\alpha\) is
\[ \hat{A}(k, \lambda, \alpha) := (2|\lambda|)^{\frac{s}{2}} \frac{\Gamma\left(\frac{2k+n}{2} + \frac{2+\alpha}{4}\right)}{\Gamma\left(\frac{2k+n}{2} + \frac{2-\alpha}{4}\right)}. \]
In particular, we have that
\[ \hat{A}(k, \lambda, 2) = A(k, \lambda, 2) = A(k, \lambda), \]
and therefore \(L_2 = -\Delta_b\).

As opposed to the case of a general Carnot group and the potential \(R_\alpha\), the fundamental solution \(G_\alpha\) of \(L_\alpha\) can be computed explicitly. Indeed, as shown in [16] extending the result in [9] for the case of the sub-Laplacian, there exists a constant \(c_{n, \alpha}\) such that
\[ G_\alpha(x) = c_n \frac{1}{|x|^{Q-\alpha}}. \]
Also, we have a similar integral representation of \(L_\alpha\) where \(\tilde{R}_{\alpha}\) can be computed explicitly. Indeed, there exists a constant \(\tilde{c}_{n, \alpha}\), such that for \(u \in C_0^\infty(\mathbb{H}^n)\) we have
\[ L_\alpha u(x) = \tilde{c}_{n, \alpha} \int_{\mathbb{H}^n} \frac{u(x) - u(y)}{|y-x|^{Q+\alpha}} dy. \]
These two facts regarding the fundamental solution and the integral representation allow us then to have the same 3-commutator result for \(L_\alpha\) that is if we define \(H_\alpha^L\) by
\[ H_\alpha^L(u, v) = L_\alpha(uv) - uL_\alpha v - vL_\alpha u, \]
then we have
Proposition 2. Given \( u, v \in C_0^\infty(\mathbb{H}^n) \), \( 0 < \alpha < Q \) and \( \epsilon > 0 \). Given \( \tau_1 \) and \( \tau_2 \) in \( \{\max\{0,\alpha - 1\}, \alpha\} \) such that \( \tau_1 + \tau_2 > \alpha \), there exists \( L \in \mathbb{N} \), \( s_j, 1 \in (0, \tau_1) \), \( s_j, 2 \in (0, \tau_2) \), for \( j = 1, \ldots, L \), satisfying \( \tau_1 + \tau_2 - s_j, 1 - s_j, 2 - \alpha \neq 0 \) such that

\[
|H_\alpha^L(u, v)|(x) \lesssim \sum_{j=1}^L R_{\tau_1 + \tau_2 - s_j, 1 - s_j, 2 - \alpha}^2 \left| R_{s_j, 1}^2 L_{\tau_1} u R_{s_j, 2}^2 L_{\tau_2} v \right|(x). \tag{31}
\]

7. Application. In this section, we provide some applications of the use of the commutator estimate in the study of the fractional CR-Yamabe equation on the sphere. First, we consider the intertwining operator \( A_\alpha \) on the CR sphere \( S^{2n+1} \) defined, up to multiplicative constants, by

\[
\text{Jac}_{\frac{2Q-2}{Q}}(A_\alpha u) \circ \tau = A_\alpha \left( \text{Jac}_{\frac{2Q-2}{Q}}(u \circ \tau) \right) \quad \forall \tau \in \text{Aut}(S^{2n+1}), \ u \in C_0^\infty(S^{2n+1}).
\tag{32}
\]

This operator is related to the fractional sub-Laplacian \( L_\alpha \) on the Heisenberg group \( \mathbb{H}^n \) via the Cayley transform \( C : \mathbb{H}^n \to S^{2n+1} \setminus \{(0, \ldots, -1)\} \) as follows: If we denote by \( J \) the Jacobian of \( C \), \( p^* = \frac{2Q}{Q-\alpha} \) the critical exponent for the Sobolev embedding and \( q^* = \frac{2Q}{Q-\alpha} \) its dual, then

\[
L_\alpha(J^{\frac{1}{p^*}}(u \circ C)) = J^{\frac{1}{q^*}}(A_\alpha u) \circ C, \quad \text{for all } u \in H^{\frac{Q}{Q}}(S^{2n+1}). \tag{33}
\]

The two numbers \( q^* \) and \( p^* \) correspond to the Sobolev type embedding

\[
\|u\|_{p^*} \lesssim \|u\|_{H^{\frac{Q}{Q}}}
\]

and its dual

\[
\|u\|_{H^{-\frac{Q}{Q}}} \lesssim \|u\|_{q^*}.
\]

Then, we consider the following equation:

\[
A_\alpha u = |u|^{p^* - 2} u. \tag{34}
\]

One can easily check that from (33) the \( u \in H^{\frac{Q}{Q}}(S^{2n+1}) \) if and only if \( v = J^{\frac{1}{p^*}}(u \circ C) \in D^\alpha(\mathbb{H}^n) \) is a solution to

\[
L_\alpha v = |v|^{p^* - 2} v. \tag{35}
\]

This problem is variational and has a corresponding energy functional defined on the fractional Sobolev type space \( H^{\frac{Q}{Q}}(S^{2n+1}) \) by

\[
E(u) = \frac{1}{2} \int_{S^{2n+1}} u A_\alpha u \, dv_{S^{2n+1}} - \frac{1}{p^*} \int_{S^{2n+1}} |u|^{p^*} \, dv_{S^{2n+1}}.
\]

We refer the reader to [11, 16] and the references therein for more details regarding the problem.

The first application that we present here, is the regularity of solutions to the fractional CR Yamabe problem on \( S^{2n+1} \). Namely, the regularity of weak solutions of (34).

**Theorem 7.1.** Let \( u \) be a weak solution of (34), then there exists \( \gamma \in (0, 1) \) such that \( u \in C^{0,\gamma}(S^{2n+1}) \).

**Proof.** We assume for the sake of simplicity that the injectivity radius \( \iota > 2 \) and we fix \( 1 > r > 0 \). Then we consider the cut-off function \( \rho(x) = 1 \) for \( |x| \leq \frac{1}{2} \) and \( \rho(x) = 0 \) for \( |x| > 1 \). We define \( \rho_r(x) = \rho(\frac{x}{r}) \). Let \( \eta \) be a smooth function on \( S^{2n+1} \) such that \( \eta = 1 \) on \( \text{supp}(\rho_r) \) and \( B = \text{supp}(\eta) \). First, we notice that

\[
A_\alpha(\rho_r u) = J^{-\frac{1}{p^*}}[L_\alpha(J^{\frac{1}{p^*}}(\rho_r u) \circ C)] \circ C^{-1}.
\]
Using Corollary 1.1, we have that
\[ L_\alpha(J^\frac{1}{p} \rho u) \circ \mathcal{C} = \rho \circ \mathcal{C} L_\alpha(J^\frac{1}{p} u) + J^\frac{1}{p} u \circ \mathcal{C} L_\alpha(r \circ \mathcal{C}) + H_\alpha^L(J^\frac{1}{p} u \circ \mathcal{C}, \rho \circ \mathcal{C}). \]
Hence,
\[ A_\alpha(r \circ \mathcal{C}) = |u|^{p-2} \rho u + F_1 + F_2 = \eta |u|^{p-2} \rho u + F_1 + F_2. \]
\[
\begin{align*}
|L_\alpha(r \circ \mathcal{C})(x)| &\lesssim \frac{1}{1 + |x|^{Q+\alpha}} \quad \text{and} \quad J(x) \approx \frac{1}{1 + |x|^{2Q}}, \forall x \in \mathbb{H}^n. \quad (36)
\end{align*}
\]
We let \( v = J^\frac{1}{p} u \circ \mathcal{C} \), then we have
\[
\int_{S^{2n+1}} |F_1|^p \, dv_{S^{2n+1}} = \int_{\mathbb{H}^n} \left| J^{1-\frac{1}{p^*}} v L_\alpha(r \circ \mathcal{C}) \right|^p \, dx 
\leq ||v||^{p*}_{L^p(\mathbb{H}^n)} ||J^{1-\frac{1}{p^*}} L_\alpha(r \circ \mathcal{C})||_{L^p(\mathbb{H}^n)} < +\infty.
\]
The last inequality follows from the fact that if \( u \in H^{\frac{1}{p^*}}(S^{2n+1}) \) then \( v \in D^\alpha(\mathbb{H}^n) \).
Next, we move to the second term, namely, \( F_2 \). In this case we have

**Lemma 7.2.** There exists \( \bar{\eta} > q^* \) such that \( F_2 \in L^q(S^{2n+1}) \).

**Proof of Lemma 7.2.** Notice first that it is easy to show that \( F_2 \in L^q(S^{2n+1}) \). In fact, we have
\[
\int_{S^{2n+1}} |F_2|^q \, dv_{S^{2n+1}} = \int_{\mathbb{H}^n} |H_\alpha^L(v, \rho \circ \mathcal{C})|^q \, dx.
\]

Using Corollary 1.1, we have that
\[
\|H_\alpha^L(v, \rho \circ \mathcal{C})\|_{L^q(\mathbb{H}^n)} \lesssim \|L_\alpha(v)\|_{L^{q*}(\mathbb{H}^n)} \|L_\alpha(\rho \circ \mathcal{C})\|_{L^{\frac{1}{q^*}}(\mathbb{H}^n)} 
\lesssim \|v\|^{p*-2} \|L^p(\mathbb{H}^n)\| L^\frac{1}{q^*}(\mathbb{H}^n) 
\lesssim \|v\|^{p*-1} \|L^q(\mathbb{H}^n)\| L^{\frac{1}{q^*}}(\mathbb{H}^n) < +\infty. \quad (37)
\]
But in our case, we require a slightly stronger estimate. For this purpose, we mimic the same proof, though, we will have an extra weight.
\[
\int_{S^{2n+1}} |F_2| \, dv_{S^{2n+1}} = \int_{\mathbb{H}^n} \left| J^{1-\frac{1}{p}} H_\alpha^L(v, \rho \circ \mathcal{C}) \right| \, dx.
\]
One then, can take \( \bar{\eta} \) to be close to \( q^* \) so that \( w = J^{1-\frac{1}{p}} \) is an \( A_\rho \)-weight. Again, as in Corollary 1.2, using weighted \( \lambda \)-kernel estimates, one has
\[
\|L_\alpha(\rho \circ \mathcal{C})\|_{L^q(\mathbb{H}^n, \omega dx)} \lesssim \|v\|^{p*-1} \|L^p(\mathbb{H}^n)\| L^\frac{1}{q^*}(\mathbb{H}^n, \omega dx),
\]
where \( \omega \) is such that \( \frac{1}{q} = \frac{1}{\bar{q}^*} + \frac{1}{p} - \frac{1}{q*} \). But if we use (36), we have
\[
\|L_\alpha(\rho \circ \mathcal{C})\|_{L^q(\mathbb{H}^n, \omega dx)} = \|w^{\frac{1}{q^*}} L^\alpha(\rho \circ \mathcal{C})\|_{L^q(\mathbb{H}^n)} < +\infty.
\]

From now on, we will set \( f = F_1 + F_2 \). We define the operator \( P_\eta : H^{\alpha,q}(S^{2n+1}) \to L^q(S^{2n+1}) \) by
\[
P_\eta(g) = \eta |u|^{p-2} g.
\]
Then \( P_\eta \) is well defined and bounded for \( 1 < q < \frac{Q}{\alpha} \). Indeed, we have \( H^{\alpha,q}(S^{2n+1}) \hookrightarrow L^{\frac{Q}{2\alpha}}(S^{2n+1}) \) and \( u \in L^p(S^{2n+1}) \), so the claim follows from Hölder’s inequalities. Moreover,

\[
\|P_\eta\|_{op} \lesssim \|u\|_{L^p(B)}^{p-2},
\]

where \( \|\cdot\|_{op} \) is the operator norm. So clearly the operator \( A_\alpha - P_\eta \) continuously maps \( H^{\alpha,q}(S^{2n+1}) \) to \( L^q(S^{2n+1}) \) for \( 1 < q < \frac{Q}{\alpha} \) and is invertible if \( \|u\|_{L^p(B)} \) is small. Of course, this can be assumed to be true by taking \( r \) smaller and thus \( B \) smaller. In particular, for \( q = \frac{Q}{\alpha} \), there exists a unique solution \( g \in H^{\alpha,q}(S^{2n+1}) \) to the equation

\[
A_\alpha g - P_\eta g = f.
\]

On the other hand, if we consider the operator \( \hat{P}_\eta \) defined by \( \hat{P}_\eta g = \eta|u|^{p^*-2}\hat{g} \), then it is well defined and continuous from \( L^p(S^{2n+1}) \to H^{-\alpha,p^*}(S^{2n+1}) \) and

\[
\|\hat{P}_\eta\|_{op} \lesssim \|u\|_{L^p(B)}^{p-2}.
\]

Indeed, if \( \hat{g} \in L^{2^*}(S^{2n+1}) \), then by Hölder’s inequalities \( \eta|u|^{p^*-2}\hat{g} \in L^{q^*}(S^{2n+1}) \).

But,

\[
L^{q^*}(S^{2n+1}) \hookrightarrow H^{-\frac{Q}{\alpha}}(S^{2n+1}) \hookrightarrow H^{-\alpha,p^*}(S^{2n+1}).
\]

Again, this implies that the operator \( A_\alpha - \hat{P}_\eta : L^{p^*}(S^{2n+1}) \to H^{\alpha,2^*}(S^{2n+1}) \) is invertible if \( \|u\|_{L^{p^*}(B)} \) is small. Therefore, since \( f \in H^{-\alpha,p^*}(S^{2n+1}) \), we have the existence of a unique \( \hat{g} \in L^{p^*}(S^{2n+1}) \) such that

\[
A_\alpha \hat{g} - \hat{P}_\eta \hat{g} = f.
\]

To finish, we notice that \( H^{\alpha,q}(S^{2n+1}) \hookrightarrow L^{2^*} \) for \( q^* < q < \frac{Q}{\alpha} \). We have then, by uniqueness of the solution, that \( \hat{g} = g = \rho u \in H^{\alpha,q}(S^{2n+1}) \). Therefore \( u \in L^{q^*}(S^{2n+1}) \) and \( q^* = \frac{Q^*}{Q - \alpha} > 2^* \). So now, we can re-iterate the process using a boot-strap argument to get the desired result.

\[\text{Corollary 3.} \] Any solution \( v \in D^\alpha(H^n) \) of (35) has the following decay

\[
|v(x)| \lesssim \frac{1}{1 + |x|^{Q - \alpha}}, \forall x \in H^n.
\]

\[\text{Proof.} \] The first claim follows from the fact that if \( v \in D^\alpha(H^n) \) is a solution of (35), then \( u = J^{-\frac{Q}{n}}v \circ C^{-1} \) is a weak solution of (34). Hence, from Theorem (39), \( v \) is bounded and so the decay follows. 

If one looks carefully at the proof of Theorem (39), then it is evident that one of the key claims is that \( \|u\|_{L^{p^*}(B)} \) gets small when we shrink \( B \). This is an obvious claim since we are dealing with a fixed function \( u \in H^{\frac{Q}{2\alpha}}(S^{2n+1}) \hookrightarrow L^{p^*}(S^{2n+1}) \). Our second application comes as a reformulation of Theorem (39) but for the case of a sequence of almost solutions. That is, given a sequence \( (u_k) \subset H^{\frac{Q}{2\alpha}}(S^{2n+1}) \) of almost solutions to (34), if we can make \( \|u_k\|_{L^{p^*}(B)} \) uniformly small, then we gain some regularity (in this sense it is an estimate). This type of estimate is fundamental in ruling out the energy concentration phenomena for Palais-Smale sequences (compare to [7, Lemma 2.14]). This heuristic idea can be stated as follows.
Proposition 3 ($\varepsilon$-regularity). Let $(u_k)$ be a sequence in $H^{\frac{1}{2}}(S^{2n+1})$ that satisfies
\[
A_\alpha u_k = |u_k|^{p^*-2}u_k + o(1),
\]
where $o(1) \to 0 \in H^{-\frac{1}{2}}(S^{2n+1})$. Fix $r > 0$, $x_0 \in S^{2n+1}$ and $\eta$ a smooth compactly supported function on $B_{2r}(x_0)$ with $\eta = 1$ on $B_r(x_0)$, then there exists $\varepsilon_0 > 0$ such that if
\[
\int_{B_{2r}(x_0)} |u_k|^{p^*} d\nu_{S^{2n+1}} < \varepsilon_0,
\]
we have for all $R > 1$
\[
\|\eta u_k\|_{H^{1/2}} \lesssim \|u_k\|_{L^2} + R^a \|u_k\|_{L^{p^*}}^{p^*-1} + R^{-b} \|u_k\|_{H^{1/2}}^{p^*-1} + o(1), \forall R > 1,
\]
where $a$ and $b$ are positive constants depending on $Q$.

Proof. In order to have the desired estimate, we need to localize the equation. So we let $\eta$ a cut-off function supported in $B_{2r}(x_0)$ such that $\eta = 1$ on $B_r(x_0)$, then using relation (33) we have
\[
A_\alpha(\eta u_k) = J^{-\frac{1}{p^*}}[L_\alpha(J^{\frac{1}{p^*}}(\eta u_k) \circ C)] \circ C^{-1}.
\]
But
\[
L_\alpha(J^{\frac{1}{p^*}}(\eta u_k) \circ C) = \eta \circ CL_\alpha(J^{\frac{1}{p^*}} u_k \circ C) + J^{\frac{1}{p^*}} u_k \circ CL_\alpha(\eta \circ C) + H_\alpha(J^{\frac{1}{p^*}} u_k \circ C, \eta \circ C).
\]
Hence,
\[
A_\alpha(\eta u_k) = \eta A_\alpha(u_k) + I_1 + I_2,
\]
where
\[
I_1 := J^{-\frac{1}{p^*}} J^{\frac{1}{p^*}} u_k L_\alpha(\eta \circ C) \circ C^{-1}
\]
and
\[
I_2 := J^{-\frac{1}{p^*}} H_\alpha(J^{\frac{1}{p^*}} u_k \circ C, \eta \circ C) \circ C^{-1}.
\]
Notice that
\[
\|I_1\|_{H^{-\frac{1}{2}}} \lesssim \|I_1\|_{q^*} \lesssim \|u_k\|_{2},
\]
Next we estimate the commutator:
\[
\|I_2\|_{H^{-\frac{1}{2}}} \lesssim \|H_\alpha(J^{\frac{1}{p^*}} u_k \circ C, \eta \circ C)\|_{q^*}.
\]
Now using Corollary 1.1, we see that we have an upper bound of $H_\alpha(u, v)$ using a finite sum of terms of the form $R_{a-s-t}(R_s(L_\alpha u)(R_tL_\alpha v))$. We set $v_k = |J^{\frac{1}{p}} u_k|^{p^*-2} J^{\frac{1}{p}} u_k \circ C$ then clearly
\[
\|v_k\|_{q^*} = \|u_k\|_{p^*}^{p^*-1}.
\]
Next, we want to bound
\[
\|R_{a-s-t}(R_s(v_k + (o(1) \circ C))(R_tL_\alpha(\eta \circ C)))\|_{H^{-\frac{1}{2}}}.
\]
The resulting term that contains $o(1)$ clearly is converging to zero. Hence we worry about the term containing $v_k$. Also one can easily see that
\[
R_t(L_\alpha(\eta \circ C)) \lesssim \frac{1}{1 + |x|^{Q+\alpha-t}}.
\]
Now we fix $R > 1$, and we want to estimate
\[
\|R_{a-s-t}(R_s(v_k \circ C(\chi_{B_{2R}} + \chi_{B_{2R}^c}))(R_tL_\alpha(\eta \circ C)))\|_{q^*} \lesssim I + II.
\]
For the first term, we have for $a = \frac{(p^*-1)}{2Q}$ and $1 < p < q^*$,
\[
I \lesssim \|\chi_{B_{2R}} v_k\|_p \lesssim R^a \|u_k\|_{p^*}^{p^*-1}.
\]
On the other hand, we have for $\frac{1}{q'} = \frac{1}{q} - \frac{\alpha - s - t}{Q}$,

$$II \lesssim \left\| R_s(u_k \chi_{B_R^+}) \left( \chi_{B_R^+} + \chi_{B_R^-} \right) \right\|_{q}.$$ 

Using Minkowski’s inequality and the fact that $|y| \lesssim |x^{-1}y|$ we have

$$\left\| R_s(u_k \chi_{B_R^+}) \frac{\chi_{B_R^+}}{1 + |x|^{Q + \alpha - t}} \right\| \lesssim \left\| \frac{u_k}{|y|^{Q - s}} \right\|_{p} \lesssim \| u_k \|_q \cdot R^{s - \frac{Q}{p}} \lesssim R^{s - \frac{Q}{p}} \| u_k \|_{p}^{s - 1}.$$ 

For the last term we have that

$$\left\| R_s(u_k \chi_{B_R^+}) \frac{\chi_{B_R^+}}{1 + |x|^{Q + \alpha - t}} \right\| \lesssim \left\| \frac{u_k}{|y|^{Q - s}} \right\|_{p} \lesssim \frac{\chi_{B_R^+}}{1 + |x|^{Q + \alpha - t}} \frac{\chi_{B_R^+}}{\| \chi_{B_R^+} \|_{\alpha}} \lesssim R^{-Q} \| u_k \|_{p}^{s - 1}.$$ 

Combining these past inequalities yields

$$\| I_2 \|_{H^{-\frac{s}{2}}} \lesssim R^{a} \| u_k \|_{p}^{s - 1} + (R^{s - \frac{Q}{p}} + R^{-Q}) \| u \|_{H^{\frac{s}{2}}} + o(1).$$

Now using sub-elliptic estimates, (38) yields

$$\| \eta u_k \|_{H^{\frac{s}{2}}} \lesssim \| \eta |u_k|^{p - 2} u_k + o(1) + I_1 + I_2 \|_{H^{-\frac{s}{2}}} \lesssim \| \eta |u_k|^{p - 2} u_k \|_{q} + \| u_k \|_{2} + R^{a} \| u_k \|_{p}^{s - 1}
\lesssim \| \eta u_k \|_{H^{\frac{s}{2}}} \| u_k \|_{L^{p^*}(B_{2r})} + \| u_k \|_{2} + R^{a} \| u_k \|_{p}^{s - 1}
\lesssim \left( R^{s - \frac{Q}{p}} + R^{-Q} \right) \| u_k \|_{H^{\frac{s}{2}}} + o(1).$$

Therefore, if $\| u_k \|_{L^{p^*}(B_{2r})} < \varepsilon_0$, where $\varepsilon_0$ small enough and independent of $k$, we have that

$$\| \eta u_k \|_{H^{\frac{s}{2}}} \lesssim \| u_k \|_{2} + R^{a} \| u_k \|_{p}^{s - 1} + (R^{s - \frac{Q}{p}} + R^{-Q}) \| u_k \|_{H^{\frac{s}{2}}} + o(1).$$

\[\square\]

**Corollary 4.** Under the assumptions of the previous proposition, if in addition we have that $(u_k)$ converges weakly to zero in $H^{\frac{s}{2}}(S^{2n+1})$, then we have that $(u_k)$ converges strongly in $H^{\frac{s}{2}}(B_r)$.

**Proof.** The weak convergence of $(u_k)$ in $H^{\frac{s}{2}}(S^{2n+1})$ provides a boundedness of $\| u \|_{H^{\frac{s}{2}}}$ and the strong convergence of $(u_k)$ in $L^2$. Hence letting first $k \to \infty$ then $R \to \infty$ in (39) yields

$$\| \eta u_k \|_{H^{\frac{s}{2}}} \to 0.$$

\[\square\]

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