The pedagogical value of the four-dimensional picture: II. Another way of looking at the electromagnetic field

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Abstract
A definition of the electromagnetic field can be neatly formulated by recognizing that the simplest form of the four-force is indeed feasible. We show that Maxwell’s equations almost entirely stem from the properties of spacetime, notably from the fact that our world has dimension \(d = 4\). Their complete reconstruction requires three additional assumptions that are seemingly divorced from spacetime properties but which may, in fact, have much to do with their geometry.

Keywords: 4D picture, a definition of the electromagnetic field, the origin of Maxwell’s equations

1. Introduction

In this paper, the second of a series of papers initiated by [1], we review the utility of some basic four-dimensional concepts in classical electrodynamics.

In section 2, we discuss the question of whether the idea of an electromagnetic field can be converted into a neat theoretical concept. The student embarking on a study of classical electrodynamics on a course of theoretical physics may be surprised that the textbooks do not begin with a definition of the electromagnetic field. Strange as it may seem, this definition is also missing from the later stages of the physics curriculum. For example, a graduate course in electromagnetism covered by Jackson’s book [2], which is informally accepted as the standard textbook of classical electrodynamics in Western universities, lacks a well defined concept of the electromagnetic field. To introduce the electric and magnetic field strengths \(E\) and \(B\), Jackson turns to the Lorentz force equation

\[
F = q(E + v \times B),
\]  

(1)

1 In this paper, we use the Gaussian system of units and put the speed of light to be 1.
and notes that this equation was originally discovered experimentally and then justified in numerous experimental studies [3]. The impression gained from such an introduction of $E$ and $B$ is that the defining equation (1) is well substantiated phenomenologically but has no theoretical motivation. Why is the theoretical motivation necessary? A phenomenological equation may be found to be merely an approximation for a fundamental relationship, and if one day we had to correct the analytic form of equation (1) then the very sense of what we held to be the electromagnetic field may drastically change along with it. A strongly motivated and uniquely formulated definition is necessary because otherwise we discuss the properties of a physical object while being in the dark about its essence and, hence, run the risk of confusing it with objects of a different nature.

The desired definition of the electromagnetic field can be found by recognizing that the simplest analytic form of the four-force $f^\mu$ is indeed feasible. Furthermore, we will see that the associated three-force $F$ eludes the simplicity argument, which offers an explanation for why equation (1) fails to be the defining equation of the electromagnetic field.

Section 3 is devoted to the dynamical law for the electromagnetic field. We show that Maxwell’s equations are almost entirely derivable from the properties of spacetime. The complete reconstruction of Maxwell’s equations requires three additional assumptions, which are seemingly divorced from geometry. However, a closer inspection of symmetries peculiar to electrodynamics in the spacetime dimension $d = 4$ makes it clear that these allegedly non-geometric assumptions may have much to do with geometry.

There are many suggestions as to how Maxwell’s equations can be derived with the use of plausible, easy and elegant heuristic considerations (see, e.g., [4] and references therein). However, the plausibility, ease and elegance by themselves are of little concern in the current approach. An effort is made to understand to what extent the form of the dynamical law governing the electromagnetic field is ordered by geometrical features of our world, in particular by the fact that spacetime has four dimensions.

**2. What is the electromagnetic field?**

Let us clarify what is meant by the definition of the electromagnetic field. If the concept of a particle is primary, then the definition should display how the field exerts on particles. The key observation is the fact that the four-force $f^\mu$ is orthogonal to the four-velocity of a particle $v^\mu$ at the point where this four-force is applied and, hence, $f^\mu$ cannot be independent of $v^\mu$. It is reasonable to begin by looking for the simplest form of this dependence. Let $f^\mu$ be linear in $v^\mu$. The case $f^\mu = \alpha v^\mu$ is of no interest because $f^\mu$ is orthogonal to $v^\mu$ only for $\alpha = 0$. Hence, we turn to the construction

$$f^\mu = \beta^{\mu\nu} v^\nu.$$  \hfill (2)

By the condition of orthogonality

$$f \cdot v = 0,$$  \hfill (3)

we come to the equation

$$\beta^{\mu\nu} v_\mu v_\nu = 0,$$  \hfill (4)

which is obeyed by an arbitrary $v^\mu$ provided that $\beta^{\mu\nu} = -\beta^{\nu\mu}$. Therefore, if there exists a physical object that is distributed over all space and affects any particle through a four-force linear in the four-velocity, then this object is characterized by an antisymmetric tensor $\beta^{\mu\nu}$ at each point. Such objects are collectively known as force fields, or simply fields. Note, however, that $\beta^{\mu\nu}$ contains information on both the state of the field and how it affects the particle.
Let us separate these concepts. Consider the simplest case that the coupling of a particle with the field is given by a real scalar parameter $q$. Then (2) becomes

$$f^\mu = q v^\nu F^\mu\nu.$$  \hspace{1cm} (5)

This possibility is actually realized in nature. The field whose state at each point $x^\mu$ of Minkowski space is specified by an antisymmetric tensor $F^\mu\nu$ and whose influence on particles is represented by the force equation (5) is called the electromagnetic field. The tensor $F^\mu\nu$ is referred to as the electromagnetic field tensor, or field strength. The scalar quantity $q$ will be provisionally called the electric charge-coupling.

The identity of a given classical particle is preserved in time. We assume that there are particles such that their coupling with the electromagnetic field $q$ does not vary with time, \[ \dot{q} = 0. \hspace{1cm} (6) \]

In a particular frame of reference, two three-dimensional vectors, $E$ and $B$, can be always defined in terms of six components of the antisymmetric tensor $F^\mu\nu$,

$$E_i = F_{0i} = F^{0i},$$  \hspace{1cm} (7)

$$B_k = -\frac{1}{2} \varepsilon_{klm} F^{lm},$$  \hspace{1cm} (8)

where $\varepsilon_{klm}$ is the three-dimensional Levi-Civita symbol. Combining (7) and (8) with the relations $f^\mu = \gamma (F \cdot v, F)$ and $v^\mu = \gamma (1, -v)$, we return to equation (1). Both quantities $f^\mu$ and $F$, expressed respectively by (5) and (1), go under the general name of Lorentz force.

We are thus led to the unambiguous theoretical concept of an electromagnetic field. The adoption of this concept is tantamount to stating that the simplest analytic form of the four-force $f^\mu$, linear in the four-velocity $v^\mu$ and proportional to a scalar coupling $q$, is indeed feasible in our world, from whence follow the existence of a field whose states are specified by an antisymmetric tensor $F^\mu\nu$ in every point of spacetime.

The reader may wonder whether simplicity is a serious theoretical argument. To answer this question, we recall that many fundamental physical principles are statements about extreme values of some physical quantities, as exemplified by the principle of least action in mechanics or the second law of thermodynamics selecting the state of maximal entropy. Simplicity is a kind of extremal quality, and so the search for the simplest form of $f^\mu$ is fully justified.

On the other hand, if we would look for the simplest form of the three-force $F$ defined in a particular frame of reference by the conventional decomposition $f^\mu = \gamma (F \cdot v, F)$, then this attempt would be foiled. Indeed, $F$ need not be velocity-dependent, so that the simplest form of this quantity is $F = \text{constant}$. Next in order of simplicity to a constant is an $F$ which only depends on space: $F = F(x)$. Thus, we see that equation (1) defies all attempts of elevating it to a satisfactory definition of the electromagnetic field. The concept of the electromagnetic field, naturally arising in the Minkowski paradigm of spacetime, seems to be devoid of theoretical motivations beyond this context.

Consider a more complex case where the interaction of a particle and electromagnetic field is specified by a pseudoscalar coupling $q^\star$. Let us define the field $^*F^{\mu\nu}$ dual to $F^{\mu\nu}$ as

$$^*F^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma},$$  \hspace{1cm} (9)

where $\varepsilon^{\mu\nu\rho\sigma}$ is the four-dimensional Levi-Civita symbol, and adopt the convention that the three- and four-dimensional Levi-Civita symbols are related by

$$\varepsilon_{ijk} = \varepsilon^{0ijk}. \hspace{1cm} (10)$$

The four-force $f^\mu$ linear in $v^\mu$, which depends on $^*F^{\mu\nu}$, rather than $F^{\mu\nu}$, is

$$f^\mu = q^\star v^\nu ^*F^{\mu\nu}. \hspace{1cm} (11)$$
Using (7)–(10), we obtain from (11)

$$\mathbf{F} = q^*(\mathbf{B} - \mathbf{v} \times \mathbf{E}).$$

(12)

A comparison of (1) and (12) shows that the parameter $q^*$ transforms as a pseudoscalar under space reflections $x \rightarrow -x$. The quantity $q^*$ is called the magnetic charge-coupling. Particles that are affected by $\mathbf{F}$ of the form (12) are referred to as magnetic monopoles. An important point is that equation (11) is not designed to give an alternative definition of the electromagnetic field, but is used to define a particle carrying the magnetic charge-coupling $q^*$. Despite the prodigious experimental effort that went into searching for magnetic monopoles, no manifestation of particles with magnetic charges $q^*$ has been found. At present, such particles have the status of hypothetical objects.

Note, however, that the existence of just one magnetically charged particle would provide a plausible explanation for the quantization of electric charge [5]. Until now, there has been no other convincing explanation of why there is a minimal value of $q$, the elementary charge, and every charged particle carries a charge which is a multiple of this elementary charge.

3. Maxwell’s equations

In the preceding section we defined the electromagnetic field as a physical object that manifests itself through its influence on a particle by the four-force linear in the particle four-velocity, provided that the coupling between this object and the affected particle is given by a scalar parameter $q$. This definition implies that the state of the electromagnetic field at each spacetime point $x^\mu = (t, \mathbf{x})$ is characterized by an antisymmetric tensor $F_{\mu\nu}$. In particular Lorentz frame, this is equivalent to assigning the electric field intensity $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ to each point. Now let us discuss the law governing the electromagnetic field behaviour in space and time. It is well known that this law can be written as a system of partial differential equations, Maxwell’s equations. Our task is to understand the geometrical and physical contents of Maxwell’s equations. To achieve this aim, it would be reasonable to attempt to derive these equations from the geometrical properties of our world, in particular from the fact that space has three dimensions.

Let us fix a particular frame of reference and consider the spatial behaviour of $\mathbf{E}$ and $\mathbf{B}$. A smooth vector function $\mathbf{V}(\mathbf{x})$ can be reconstructed with the knowledge of nine components of its gradients $\nabla \mathbf{V}$. However, doing this actually requires much less information when the famous Helmholtz theorem is taken into account: if a smooth vector function $\mathbf{V}(\mathbf{x})$ disappears at infinity, it can be reconstructed from its curl $\mathbf{C} = \nabla \times \mathbf{V}$ and divergence $D = \nabla \cdot \mathbf{V}^2$. If we further assume that the evolutionary law for the electromagnetic field is given by a system of differential equations, then, in view of the Helmholtz theorem, these equations must contain only those linear combinations of spacetime derivatives $\partial_\mu F^{\mu\nu}$ that are expressed in terms of curls and divergences of $\mathbf{E}$ and $\mathbf{B}$. With some simple but lengthy algebra, taking into account the definitions (7)–(9), and the convention (10), we obtain the desired linear combinations

$$\partial_\mu F^{\mu\nu} = (\nabla \cdot \mathbf{E}, -\dot{\mathbf{E}} + \nabla \times \mathbf{B})$$

(13)

The proof of this theorem is simple. The relation $\nabla \times (\nabla \times \mathbf{V}) = \nabla (\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$, familiar from any course of the vector analysis, can be rewritten as the Poisson equation $\nabla^2 \mathbf{V} = \mathbf{S}$ with a computable function in the right-hand side $\mathbf{S} = \nabla D - \nabla \times \mathbf{C}$. The general solution to the Poisson equation is $\mathbf{V} = \mathbf{V}_s + \mathbf{V}_0$ where $\mathbf{V}_s$ is a particular solution of this equation and $\mathbf{V}_0$ the general solution of the associated homogeneous equation $\nabla^2 \mathbf{V} = 0$, the Laplace equation. If we require that $\mathbf{V}$ disappear at infinity, then $\mathbf{V}_0$ is identically zero because, in any spatial domain, $\mathbf{V}_0$ takes its maximal and minimal values on the boundary of this domain. In particular, given a solution of the Laplace equation $\mathbf{V}_0$ vanishing on the boundary, it is identically zero in this domain. Therefore, the Poisson equation has the unique solution $\mathbf{V}(\mathbf{r}) = \mathbf{V}_s(\mathbf{r}) = \int d^3 x \frac{\mathbf{S}(\mathbf{x})}{r^2}$, which goes to zero as $r \rightarrow \infty$. This completes the proof.
where the overdot denotes the time-derivative. It is easy to verify that there are no linear combinations of \( \partial_\lambda F_{\mu\nu} \) containing curls and divergences of \( E \) and \( B \) other than those shown in (13) and (14).

Thus, from essentially geometric considerations we come to the field equations

\[
\partial_\lambda F^{\lambda\mu} = j^\mu, \tag{15}
\]

\[
\partial_\mu \ast F_{\mu\nu} = m^\mu, \tag{16}
\]

where \( m^\mu \) and \( j^\mu \) stand for the sources of local variations of the field state. However, the meaning and construction of \( m^\mu \) and \( j^\mu \) are, so far, indeterminate.

We then adopt three additional postulates of non-geometric nature, which lead directly and unambiguously to Maxwell’s equations.

(i) Linearity of the evolutionary law for the electromagnetic field.

(ii) Extended action–reaction principle.

(iii) Lack of magnetic monopoles.

From (i), together with Poincaré invariance, we conclude that the sources \( j^\mu \) and \( m^\mu \) in the field equations (15) and (16) are independent of \( F_{\mu\nu} \). They may depend on particle characteristics, such as coupling constants \( q \) and world line variables \( \xi^i(t) \). To find this dependence, we apply \( \partial_\mu \) to (15) and take into account the antisymmetry of \( F_{\lambda\mu} \) and symmetry of \( \partial_\mu \partial_\lambda \) in \( \lambda \) and \( \mu \), which yields the identity \( \partial_\mu \partial_\lambda F^{\lambda\mu} = 0 \). Therefore, the consistency of (15) is ensured by

\[
\partial_\mu j^\mu = 0. \tag{17}
\]

Integrating this equation over a domain bounded by two spacelike hypersurfaces \( \Sigma_1 \) and \( \Sigma_2 \) and using the Gauss–Ostrogradsky theorem, one can verify that the quantity

\[
Q = \int_{\Sigma} d\sigma_\mu j^\mu, \tag{18}
\]

called the total electric charge-source, is independent of the hypersurface \( \Sigma \). In particular, if the hypersurface \( \Sigma \) is shifted along the time axis, \( Q \) remains invariant. The relation \( Q = \text{constant} \), expressing the conservation of the total charge-source, is tempting to relate to the constancy of the charge-coupling \( q \), implied by equation (6). To do this requires postulate (ii). Indeed, let the hypersurface \( \Sigma \) be intersected by \( N \) world lines of charged particles. Then, by (ii),

\[
Q = \sum_{I=1}^{N} q_I. \tag{19}
\]

Imagine that only a single point particle with the coupling \( q \) is in the universe, then

\[
Q = q. \tag{20}
\]

Therein lies the extended action–reaction principle in electrodynamics: the charge-coupling measures the variation of the particle state for a given electromagnetic field state while the charge-source measures the variation of the electromagnetic field state for a given particle state. The four-dimensionality of spacetime is favourable to the action–reaction principle because the number of degrees of freedom of electromagnetic field at each spacetime point, given by six components of \( F_{\mu\nu} \), equals the phase space dimension of an interacting particle spanned by three coordinates of its position \( x \) and three components of conjugate momentum \( p \). For comparison, in a conceivable \((1 + 5)\)-dimensional pseudo-Euclidean spacetime in which the
defining equation (5) is taken to be valid, the number of degrees of freedom of electromagnetic field $F_{\mu\nu}$ is $\frac{1}{2}d(d-1) = 15$ while the conventional phase space has dimension $2(d-1) = 10$ and, hence, the action–reaction principle fails$^3$.

Both quantities, $Q$ and $q$, would be reasonable to lump together as the electric charge or briefly the charge.

How could we realize (19) mathematically? We assume that the charge is an inherent characteristic of point particles. Then, the source $j^\mu$ (called the four-current density of electric charges or simply the four-current), is given by

$$j^\mu(x) = \sum_{I=1}^{N} q_I \int_{-\infty}^{\infty} ds_I v^\mu_I(s_I) \delta^{(4)}[x - z_I(s_I)],$$

(21)

where $v^\mu_I(s_I)$ is the four-velocity of $I$th particle, and $\delta^{(4)}(x)$ is the four-dimensional Dirac delta-function. Because the hypersurface $\Sigma$ in (18) is arbitrary, we take $\Sigma$ such that all the world lines are perpendicular to it at intersection points. For a small vicinity of the intersection point, we have $ds_I d\sigma_{\mu} v^\mu_I = d^4x$, where $x^\mu$ are coordinates in the Lorentz frame with the time axis directed along $v^\mu_I$. Inserting (21) in (18), we arrive at (19).

We next turn to equation (16) and reiterate mutatis mutandis the above arguments. By the action–reaction principle, the total magnetic charge-source $Q^* = \int d\sigma_{\mu} m^\mu$ equals the sum of magnetic charge-couplings, $Q^* = \sum_{I=1}^{N} q^*_I$. With postulate (iii), we find

$$m^\mu = 0.$$  

(22)

Finally, the electromagnetic field is governed by the system of equations

$$\partial_\lambda F^{\lambda\mu} = 4\pi j^\mu,$$  

(23)

$$\partial_\nu F^{\lambda\mu} = 0,$$  

(24)

which are just Maxwell’s equations.

To summarize, a major part of the information encoded in Maxwell’s equations (23) and (24) is taken from the global topological properties of spacetime, notably from the fact that our world has dimension $d = 4$, and the residual information, seemingly divorced from geometry, which represents the physical contents of equations (23) and (24), translates into the above assumptions (i)–(iii).

4. Conclusion and outlook (for the expert reader)

The idea of spacetime provides a satisfactory framework for the unique definition of the electromagnetic field. This definition rests on the theoretical appeal of the simplest form of the four-force $f^\mu$ and the empirical fact that this simplest option is indeed taken by nature, whence it follows that there exists a physical object whose states are specified by an antisymmetric tensor $F_{\mu\nu}$ in every point of spacetime. It is this object that is regarded as the electromagnetic field. It is remarkable that the same definition is applicable to worlds with space dimensions other than 3.

$^3$ The mathematically inclined reader will recognize that if we require that the electromagnetic sector of the action is preserved in the original Maxwellian form for every spacetime dimension $d$ then, in order for the theory as a whole to be consistent, the particle sector of the action must be supplemented by terms with higher derivatives. For example, in the above $d = 6$ realm, acceleration-dependent terms are required. Then the extended phase space, equipped with five constraints, has dimension $4(d-1) - 5 = 15$, and the number of components of $F_{\mu\nu}$ equals the number of mechanical degrees of freedom, so that the action–reaction principle holds. The necessity of amending the particle dynamics by the addition of higher derivative terms to the action to obtain a consistent theory in the case $d > 4$ was also argued in [6], which, however, used another line of reasoning.
One can extend the consideration of the four-force \( f^\mu \) linear in the four-velocity \( v^\mu \) to the case that the coupling \( Q^a \) is a vector in some internal \( N \)-dimensional vector space describing charge states of the particle, to yield the force equation \( [8] \)

\[
f^\mu = \sum_{a=1}^{N} Q^a v_\mu G^{\mu \nu}_a,
\]

(25)

where the quantity \( G^{\mu \nu}_a \) is called the Yang–Mills field strength. A refined version of this vector coupling, involving the Lie algebra structures, gains insight into the form of gauge fields mediating the weak and strong interactions.

If we want the gravitational interaction to be covered in this framework, we should turn to the four-force \( f^\mu \), which is quadratic in the four-velocity \( v^\mu \). The necessity of considering quadratic relationships is due to the empirical fact that identical gravitating particles attract each other. With reference to exercise 7.2 in [9], it transpires that if the interaction is carried by a field \( F_{\mu \nu} \) (or \( G^{\mu \nu}_a \)), then identical particles with a real coupling \( q \) (or \( Q^a \)) repel each other. In contrast, quadratic relationships between \( f^\mu \) and \( v^\mu \) are compatible with scalar and symmetric tensor fields mediating such interactions that ensure the attraction of identical particles (exercises 7.1 and 7.3 in [9]). A diligent student may wish to explore the general quadratic relationship

\[
f_\lambda = -\Gamma_{\lambda \mu \nu} v^\mu v^\nu,
\]

(26)

where \( \Gamma_{\lambda \mu \nu} \) is an arbitrary rank \((0,3)\) tensor, and show that \( \Gamma_{\lambda \mu \nu} \) is always identically vanishing, except for the case where spacetime is a curved pseudo-Riemannian manifold with the line element \( ds^2 \) given by

\[
d s^2 = g_{\mu \nu} (x) dx^\mu dx^\nu,
\]

(27)

where \( g_{\mu \nu} \) is a symmetric rank \((0,2)\) tensor interpreted as the metric of this manifold, and \( \Gamma_{\lambda \mu \nu} \) is expressed in terms of \( g_{\mu \nu} \) as

\[
\Gamma_{\lambda \mu \nu} = \frac{1}{2} (\partial_\mu g_{\nu \lambda} + \partial_\nu g_{\lambda \mu} - \partial_\lambda g_{\mu \nu}).
\]

(28)

Therefore, in an effort to describe the gravitational interaction, we are inevitably led to the idea of spacetime warping.

The derivation of the law governing the electromagnetic field from ‘next to nothing’ (or, more precisely, from the properties of four-dimensional spacetime) can be an enormously enlightening experience in the physics curriculum. When (13) and (14) are considered, and the four-current is decomposed as \( j^\mu = (\varrho, j) \), one recasts Maxwell’s equations (23) and (24) in the conventional three-dimensional vector form

\[
\nabla \cdot \mathbf{E} = 4\pi \varrho,
\]

(29)

\[
\nabla \times \mathbf{B} = 4\pi \mathbf{j} + \dot{\mathbf{E}},
\]

(30)

\[
\nabla \cdot \mathbf{B} = 0,
\]

(31)

\[
\nabla \times \mathbf{E} = -\mathbf{B}.
\]

(32)

One could hardly discern the geometric origin of these equations. This raises the question of whether there is a space dimension other than three for which the Lorentz force equation (1) and Maxwell’s equations (29)–(32) can be written in a similar vector form. Apart from \( \mathbb{E}_3 \), the cross product of two vectors can only be defined in six-dimensional Euclidean space \( \mathbb{E}_7 \). It is well known that the cross product in \( \mathbb{E}_3 \) is closely related to quaternion algebra. The opportunity to define the cross product in \( \mathbb{E}_3 \) arises from octonion algebra, the largest composition algebra.
Although this opportunity leads to the reproduction of Maxwell’s equations (29)–(32), the Lorentz force equation (1) is modified by the addition of an extra term [7], so that the geometric and physical contents of the resulting equations are quite different from what was just represented.

Separating the geometric content of Maxwell’s equations from their physical content allows us to appreciate features peculiar to electrodynamics in comparison with the three other fundamental interactions. Take, for example, the action–reaction principle. In general relativity this principle does not hold. Indeed, from the equation of motion for a point particle

$$\frac{d^2 z_\mu}{d\tau^2} + \Gamma_{\lambda\mu\nu} \frac{dz^\lambda}{d\tau} \frac{dz^\nu}{d\tau} = 0,$$

(33)

where $\Gamma_{\lambda\mu\nu}$ is the Christoffel symbol assigned to the curved manifold in which the particle is located, and $\tau$ is an appropriate evolution variable, we see that the gravitational field exerts on all particles in a uniform way no matter what their masses. On the other hand, consider the equation of gravitational field

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = 8\pi G_N T^{\mu\nu},$$

(34)

in which the source of the gravitational field is taken to be the stress–energy tensor of a single particle with gravitational mass $m_g$ moving along the world line $z^\mu(\tau)$,

$$T^{\mu\nu}(x) = m_g \int_{-\infty}^{\infty} d\tau \dot{z}^\mu(\tau) \dot{z}^\nu(\tau) \delta^{(4)}[x - z(\tau)].$$

(35)

A greater $m_g$ will make the spacetime warping stronger. Thus, the influence of particles with different gravitational masses on the state of gravitational field is different. This is contrary to the action–reaction principle.

Maxwell’s equations in covariant tensor form (23) and (24) offer a universal description of the behaviour of an electromagnetic field in a world with an arbitrary space dimension $d - 1 = n$. At first glance this is a disadvantage rather than an advantage because this writing of Maxwell’s equations does not suggest that the $n = 3$ stands out against other values of $n$ and, hence, the above argument regarding the three-dimensional origin of Maxwell’s equations seems to fail. But this is not the case. The ‘$n = 3$ birthmark’ is implicit in (23) and (24): these equations are invariant under the group of conformal transformations (the largest group of spacetime symmetry of electrodynamics [10, 11]) only in the case $n = 3$ [12].

The separation of the geometric and non-geometric constituents of electrodynamics makes it possible to test each constituent separately by looking into its modifications. For example, if we abandon linearity, then we come to nonlinear modifications of electrodynamics, such as the Born–Infeld theory [13]. However, among those nonlinear modifications of electrodynamics with a reasonable weak field limit yielding the Maxwell theory, none exhibit conformal invariance of the field equations. This fact appears to be the strongest argument in support of linearity.

Assumption (iii), reflecting the evidence that particles with magnetic charges $q^\star$ do not exist (or perhaps are extremely rare in nature), is also related to geometry but in a subtle way. To see this, we refer to the fact that the homogeneous equation (24) can be derived from the symplectic structure of Poisson manifolds [14, 15].

To summarize, the physical content of Maxwell’s equations can be reformulated in geometrical terms. However, this reformulation would tell us about structures that are not identical to those inherent in the four-dimensional geometry of our spacetime. In fact, these two geometries are superimposed in electrodynamics.
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