ON THE SPECTRAL PROBLEM ASSOCIATED WITH THE TIME-PERIODIC NONLINEAR SCHRODINGER EQUATION

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Abstract. According to its Lax pair formulation, the nonlinear Schrödinger (NLS) equation can be expressed as the compatibility condition of two linear ordinary differential equations with an analytic dependence on a complex parameter. The first of these equations—often referred to as the $x$-part of the Lax pair—can be rewritten as an eigenvalue problem for a Zakharov-Shabat operator. The spectral analysis of this operator is crucial for the solution of the initial value problem for the NLS equation via inverse scattering techniques. For space-periodic solutions, this leads to the existence of a Birkhoff normal form, which beautifully exhibits the structure of NLS as an infinite-dimensional completely integrable system. In this paper, we take a first few steps towards developing an analogous picture for time-periodic solutions by performing a spectral analysis of the $t$-part of the Lax pair with a periodic potential.

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1. Introduction

The nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} - 2\sigma |u|^2 u = 0, \quad \sigma = \pm 1,$$

(1.1)

is one of the most well-studied nonlinear partial differential equations. As a universal model equation for the evolution of weakly dispersive wave packets, it arises in a vast number of applications, ranging from nonlinear fiber optics and water waves to Bose-Einstein condensates. Many aspects of the mathematical theory for (1.1) are well-understood. For example, for spatially periodic solutions (i.e., $u(x,t) = u(x+1,t)$), there exists a normal form theory for (1.1) which beautifully exhibits its structure as an infinite-dimensional completely integrable system (see [12] and references therein). This theory takes a particularly simple form in the case of the defocusing (i.e., $\sigma = 1$) version of (1.1). Indeed, for $\sigma = 1$, the normal form theory ascertains the existence of a single global system of Birkhoff coordinates (the cartesian version of action-angle coordinates) for (1.1). For the focusing (i.e., $\sigma = -1$) NLS, such coordinates also exist, but only locally [19]. The existence of Birkhoff coordinates has many implications. Among other things, it provides an explicit decomposition of phase space into invariant tori, thereby making it evident that an $x$-periodic solution of the defocusing NLS is either periodic, quasi-periodic, or almost periodic in time. The construction of Birkhoff coordinates for (1.1) is a major achievement which builds on ideas going back all the way to classic work of Gardner, Greene, Kruskal and Miura on the KdV equation [10, 11] and of Zakharov and Shabat on the NLS equation [28]. Early works on the (formal) introduction of action-angle variables include [26, 27]. More recently, Kappeler and collaborators have developed powerful methods which have led to a rigorous construction of Birkhoff coordinates for both KdV [14, 15, 17] and NLS [12, 19] in the spatially periodic case.
The key element in the construction of Birkhoff coordinates is the spectral analysis of the Zakharov-Shabat operator \( L(u) \) defined by
\[
L(u) = i\sigma_3 \left( \frac{d}{dx} - U \right), \quad \text{where} \quad U = \begin{pmatrix} 0 & u \\ \sigma u & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
In particular, the periodic eigenvalues of this operator are independent of time if \( u \) evolves according to (1.1) and thus encode the infinite number of conservation laws for (1.1). The time-independence is a consequence of the fact that equation (1.1) can be viewed as the compatibility condition \( \phi_{xt} = \phi_{tx} \) of the Lax pair equations \([20, 23]\)
\[
\begin{align*}
\phi_x + i\lambda \sigma_3 \phi &= U \phi, \\
\phi_t + 2i\lambda^2 \sigma_3 \phi &= V \phi,
\end{align*}
\]
where \( \lambda \in \mathbb{C} \) is the spectral parameter, \( \phi(x, t, \lambda) \) is an eigenfunction,
\[
V = \begin{pmatrix} -i\sigma|u|^2 & 2\lambda u + iu_x \\ 2\lambda \sigma u - i\sigma u_x & i\sigma |u|^2 \end{pmatrix},
\]
and we note that (1.2) is equivalent to the eigenvalue problem \( L(u)\phi = \lambda \phi \). Strangely enough, although the spectral theory of equation (1.2) (or, equivalently, of the Zakharov-Shabat operator) has been so thoroughly studied, it appears that no systematic study of the spectral theory of equation (1.2) has yet been carried out (there only exist a few studies of the NLS equation on the half-line with asymptotically time-periodic boundary conditions which touch tangentially on the issue \([4, 23]\)). The purpose of the present paper is to perform such a study.

For the spectral analysis, it is appropriate (at least initially) to treat the four functions \( u, \sigma u, u_x, \sigma u_x \) in the definition of \( V \) as independent. We will therefore consider the spectral problem (1.3) with potential \( V \) given by
\[
V = V(\lambda, \psi) = \begin{pmatrix} -i\psi^1 \psi^2 & 2\lambda \psi^1 + i\psi^3 \\ 2\lambda \psi^2 - i\psi^4 & i\psi^1 \psi^2 \end{pmatrix},
\]
where \( \psi = \{\psi^j(t)\}_{j=1}^4 \) are periodic functions of \( t \in \mathbb{R} \) with period one.

Apart from the purely spectral theoretic interest of studying (1.3), there are at least three other reasons motivating the present study:

- First, in the context of fiber optics, the roles of the variables \( x \) and \( t \) in equation (1.1) are interchanged, see e.g. [2]. In other words, in applications to fiber optics, \( x \) is the temporal and \( t \) is the spatial variable. Since the analysis of (1.3) plays the same role for the \( x \)-evolution of \( u(x, t) \) as the analysis of the Zakharov-Shabat operator plays for the \( t \)-evolution, this motivates the study of (1.3).

- Second, one of the most important problems for nonlinear integrable PDEs is to determine the solution of initial-boundary value problems with asymptotically time-periodic boundary data \([3, 6, 24]\). For example, consider the problem of determining the solution \( u(x, t) \) of (1.1) in the quarter-plane \( \{x > 0, t > 0\} \), assuming that the initial data \( u(x, 0), x \geq 0 \), and the boundary data \( u(0, t), t \geq 0 \) are known, and that \( u(0, t) \) approaches a periodic function as \( t \to \infty \). The analysis of this problem via Riemann-Hilbert techniques relies on the spectral analysis of (1.3) with a periodic potential determined by the asymptotic behavior of \( u(0, t) \) \([4, 23]\).

- Third, at first sight, the differential equations (1.2) and (1.3) may appear unrelated. However, the fact that they are connected via equation (1.1) implies that they can be viewed as different manifestations of the same underlying mathematical structure. Indeed, for the analysis of elliptic equations and boundary value problems, a coordinate-free intrinsic approach in which the two parts of the Lax pair are combined into a single differential form has proved the most fruitful \([9, 13]\). In such a formulation, eigenfunctions which solve both the \( x \)-part (1.2) and the \( t \)-part (1.3) simultaneously play a central role. It is therefore natural to investigate how the spectral properties of (1.2) are related to those of (1.3). Since the NLS equation is just one
example of a large number of integrable equations with a Lax pair formulation, the present work can in this regard be viewed as a case study with potentially broader applications.

1.1. Comparison with the analysis of the $x$-part. Compared with the analysis of the $x$-part (1.2), the spectral analysis of the $t$-part (1.3) presents a number of novelties. Some of the differences are:

- Whereas equation (1.2) can be rewritten as the eigenvalue equation $L(u\phi) = \lambda\phi$ for an operator $L(u)$, no (natural) such formulation is available for (1.3) due to the more complicated $\lambda$-dependence. Nevertheless, it is possible to define spectral quantities associated with (1.3) in a natural way.

- Asymptotically for large $|\lambda|$, the periodic and antiperiodic eigenvalues of (1.2) come in pairs which lie in discs centered at the points $\pi n$, $n \in \mathbb{Z}$, along the real axis [12]. In the case of (1.3), a similar result holds, but in addition to discs centered at points on the real axis, there are also discs centered at points on the imaginary axis (see Lemma 3.12). Moreover, the spacing between these discs shrinks to zero as $|\lambda|$ becomes large.

- For so-called real type potentials (the defocusing case), the Zakharov-Shabat operator is self-adjoint, implying that the spectrum associated with (1.2) is real. No such statement is true for the $t$-part (1.3). This is clear already from the previous statement that there exist pairs of eigenvalues tending to infinity contained in discs centered on the imaginary axis. However, it is also true that the eigenvalues of (1.3) near the real axis need not be purely real and the eigenvalues near the imaginary axis need not be purely imaginary. This can be seen from the simple case of a single-exponential potential. Indeed, consider the potential

\[(\psi^1(t), \psi^2(t), \psi^3(t), \psi^4(t)) = (\alpha e^{i\omega t}, \sigma\alpha e^{-i\omega t}, c e^{i\omega t}, \sigma c e^{-i\omega t}),\]

where $\alpha, c \in \mathbb{C}$, $\omega \in 2\pi\mathbb{Z}$, and $\sigma = \pm 1$. For potentials of this form, equation (1.3) can be solved explicitly (see Section 5) and Fig. 1 shows the periodic and antiperiodic eigenvalues of (1.3) for two choices of the parameters.

- Whereas the matrix $U$ in (1.2) is off-diagonal and contains only the function $u$ and its complex conjugate $\bar{u}$, the matrix $V$ in (1.3) is neither diagonal nor off-diagonal and involves also $u_x$ and $\bar{u}_x$. This has implications for the spectral analysis—an obvious one being that (1.5) involves four instead of two scalar potentials $\psi^j(t)$.

- The occurrence of the factor $\lambda^2$ in (1.3) implies that the derivation of the asymptotics of the fundamental solution as $|\lambda| \to \infty$ requires new techniques (see the proof of Theorem 2.7). For the $x$-part, the analogous result can be established via an application of Gronwall’s lemma [12]. This approach does not seem to generalize to the $t$-part, but instead we are able to perform an asymptotic analysis inspired by [7, Chapter 6] (see also [22]).

1.2. Outline of the paper. In order to facilitate comparison with the existing literature on the $x$-part (1.2), our original intention was to follow [12] in terms of notation and exposition. However, as it became more and more evident that the analysis of (1.3) is quite different from that of (1.2), we were forced to deviate from this plan. Nevertheless, some resemblance to the first two chapters of [12] remains.

In Section 2 we define and study the fundamental matrix solution of (1.3). The main result is Theorem 2.7, which establishes the asymptotic behavior of the fundamental solution for large $\lambda$. In Section 3 we consider the spectrum and derive asymptotic localization results for the eigenvalues. The main result is the Counting Lemma (Lemma 3.12). In Section 4 potentials of real and imaginary type (corresponding to the defocusing and focusing NLS, respectively) are investigated. The main results are Theorem 4.4 and Corollaries 4.9 and 4.10. In Section 5
we consider the special, but important, case of single-exponential potentials. In Section 6 we derive useful formulas for the gradients of the fundamental solution and the discriminant.

Figure 1. Plots of the periodic and antiperiodic eigenvalues for two single exponential potentials with different sets of parameters $\sigma$, $\omega$, $\alpha$ and $c$; cf. (1.6). Fig. 1a shows the periodic and antiperiodic eigenvalues for the real type potential given by $\sigma = 1$, $\omega = -2\pi$, $\alpha = \frac{6}{15} + \frac{11}{4}i$, $c = \frac{1}{10}$; Fig. 1b shows the spectrum of the imaginary type potential with $\sigma = -1$, $\omega = -2\pi$, $\alpha = \frac{1}{2}$, $c = i\alpha\sqrt{2\alpha^2 - \omega}$, which arises from an exact plane wave solution of the focusing NLS.

2. Fundamental solution

In Section 2.1 we introduce the framework for the study of (1.3) and establish basic properties of the fundamental solution. In Section 2.2 we derive estimates for the fundamental matrix solution and its $\lambda$-derivative for large $|\lambda|$. These estimates will be used in in Section 3 to asymptotically localize the Dirichlet, Neumann and periodic eigenvalues and the critical points of the discriminant of (1.3).

2.1. Framework and basic properties. The potential matrix $V$ in (1.3) depends on the spectral parameter $\lambda \in \mathbb{C}$ and the potential $\psi = (\psi^1, \psi^2, \psi^3, \psi^4)$ taken from the space

$$X := H^1(T, \mathbb{C}) \times H^1(T, \mathbb{C}) \times H^1(T, \mathbb{C}) \times H^1(T, \mathbb{C}),$$

where $H^1(T, \mathbb{C})$ denotes the Sobolev space of complex absolutely continuous functions on the one-dimensional torus $T = \mathbb{R}/\mathbb{Z}$ with square-integrable weak derivative, which is equipped with the usual norm induced by the $H^1$-inner product

$$\langle \cdot, \cdot \rangle : H^1(T, \mathbb{C}) \times H^1(T, \mathbb{C}) \to \mathbb{C}, \quad \langle u, v \rangle \mapsto \int_0^1 (u\bar{v} + u_t\bar{v}_t) \, dt.$$

We endow the space $X$ with the inner product

$$\langle \psi_1, \psi_2 \rangle := (\psi^1_1, \psi^2_1) + (\psi^1_2, \psi^2_2) + (\psi^3_1, \psi^3_2) + (\psi^4_1, \psi^4_2),$$

which induces the norm $\|\psi\| = \sqrt{\langle \psi, \psi \rangle}$ on $X$. Likewise we consider the spaces

$$X_{\tau} := H^1([0, \tau], \mathbb{C}) \times H^1([0, \tau], \mathbb{C}) \times H^1([0, \tau], \mathbb{C}) \times H^1([0, \tau], \mathbb{C})$$
on the interval $[0, \tau]$ for fixed $\tau > 0$, where the Sobolev spaces $H^1([0, \tau], \mathbb{C})$ are equipped with the inner product

$$(\cdot, \cdot)_\tau : H^1([0, \tau], \mathbb{C}) \times H^1([0, \tau], \mathbb{C}) \to \mathbb{C}, \quad (u, v) \mapsto \int_0^\tau (u\overline{v} + u_t\overline{v_t}) \, dt.$$ 

We set

$$(\psi_1, \psi_2)_\tau := (\psi_1, \psi_2)_\tau + (\psi_2, \psi_2)_\tau + (\psi_1, \psi_3)_\tau + (\psi_4, \psi_4)_\tau,$$

which makes $X_\tau$ an inner product space and induces the norm $\|\psi\|_\tau := \sqrt{\langle \psi, \psi \rangle_\tau}$. For the components $\psi^i$ of $\psi \in X$ or $\psi \in X_\tau$ respectively, we write

$$\|\psi^i\| = \sqrt{\langle \psi^i, \psi^i \rangle}, \quad \|\psi^i\|_\tau = \sqrt{\langle \psi^i, \psi^i \rangle_\tau}, \quad i = 1, 2, 3, 4.$$ 

Since not every $\psi \in X_1$ is periodic, $X$ is a proper closed subspace of $X_1$. The spaces $X$ and $X_\tau$ inherit completeness from $H^1(T, \mathbb{C})$ and $H^1([0, \tau], \mathbb{C})$ respectively, hence they are Hilbert spaces.

On the space $M_{2 \times 2}(\mathbb{C})$ of complex valued $2 \times 2$-matrices we consider the norm $| \cdot |$, which is induced by the standard norm in $\mathbb{C}^2$, also denoted by $| \cdot |$, i.e.

$$|A| := \max_{z \in \mathbb{C}^2, |z| = 1} |Az|.$$ 

The norm $| \cdot |$ is submultiplicative, i.e. $|AB| \leq |A| |B|$ for $A, B \in M_{2 \times 2}(\mathbb{C})$.

For given $\lambda \in \mathbb{C}$ and $\psi \in X$, let us write the initial value problem corresponding to (1.3) as

$$D\phi = R\phi + V\phi, \quad \phi(0) = \phi_0, \quad \lambda \equiv R(\lambda) := -2i\lambda^2\sigma_3, \quad \phi = \left(\begin{array}{c} \phi^1 \\ \phi^2 \end{array}\right) : \mathbb{T} \to \mathbb{C}^2.$$ 

Equation (2.1) reduces to (1.3) if we identify $(\psi^1, \psi^2, \psi^3, \psi^4) = (u, \sigma \dot{u}, u_x, \sigma \dot{u}_x)$.

In analogy to the conventions for the eigenvalue problem (1.2) for the $x$-part of the NLS Lax pair, we say that the spectral problem (2.1) is of Zakharov-Shabat (ZS) type. The corresponding equation written in AKNS [1] coordinates $(q_0, p_0, q_1, p_1)$ reads

$$D\phi = -2\lambda^2 \left(\begin{array}{cc} 1 & -1 \\ 1 & \lambda^2 \end{array}\right) \phi + \left(\begin{array}{ccc} 2\lambda q_0 - p_1 \\ 2\lambda p_0 - (p_0^2 + q_0^2) + q_1 \\ -2\lambda q_0 + p_1 \end{array}\right) \phi. \quad (2.3)$$

It is obtained by multiplying the operator equation $D = R + V$ from the right with $T$ and from the left with $T^{-1}$, where

$$T = \left(\begin{array}{cc} 1 & i \\ 1 & -i \end{array}\right), \quad T^{-1} = \frac{1}{2} \left(\begin{array}{cc} 1 & 1 \\ -i & i \end{array}\right), \quad (2.4)$$

and by writing

$$\psi^1 = q_0 + ip_0, \quad \psi^2 = q_0 - ip_0, \quad \psi^3 = q_1 + ip_1, \quad \psi^4 = q_1 - ip_1,$$

that is,

$$q_0 = \frac{1}{2}(\psi^1 + \psi^2), \quad p_0 = -\frac{i}{2}(\psi^1 - \psi^2), \quad q_1 = \frac{1}{2}(\psi^3 + \psi^4), \quad p_1 = -\frac{i}{2}(\psi^3 - \psi^4).$$

In what follows we show the existence of a unique matrix-valued fundamental solution $M$ of (2.1), that is, a solution of

$$DM = RM + VM, \quad M(0) = I, \quad (2.5)$$
where $I \in M_{2 \times 2}(\mathbb{C})$ denotes the identity matrix. The proof relies on a standard iteration technique. We first observe that the fundamental matrix solution for the zero potential $\psi = 0$ is given by

$$ E_\lambda(t) := e^{-2 \lambda^2 t} = \begin{pmatrix} e^{-2 \lambda^2 t} \\ e^{2 \lambda^2 t} \end{pmatrix}, \quad t \geq 0. $$

Indeed, $E_\lambda$ solves the initial value problem

$$ DE_\lambda = RE_\lambda, \quad E_\lambda(0) = I. $$

For $\lambda \in \mathbb{C}$, $\psi \in X$ and $0 \leq t < \infty$ we inductively define

$$ M_0 := E_\lambda(t), \quad M_{n+1}(t) := \int_0^t E_\lambda(t-s)V(s)M_n(s) \, ds, \quad n \geq 0. \quad (2.6) $$

For each $n \geq 1$, $M_n$ is continuous on $[0, \infty) \times \mathbb{C} \times X$ and satisfies

$$ M_n(t) = \int_{0 \leq s_n \leq \cdots \leq s_1 \leq t} E\lambda(t) \prod_{i=1}^n E\lambda(-s_i)V(s_i)E\lambda(s_i)^t \, ds_n \cdots ds_1. $$

Using that $|E\lambda(t)| = e^{2(\Re \lambda^2)t}$ for $t \geq 0$, we estimate

$$ |M_n(t)| \leq e^{2(2n+1)\Re \lambda^2 t} \int_{0 \leq s_n \leq \cdots \leq s_1 \leq t} \prod_{i=1}^n |V(s_i)| \, ds_n \cdots ds_1 $$

$$ \leq \frac{e^{2(2n+1)\Re \lambda^2 t}}{n!} \int_{[0,t]^n} \prod_{i=1}^n |V(s_i)| \, ds_n \cdots ds_1 $$

$$ \leq \frac{e^{2(2n+1)\Re \lambda^2 t}}{n!} \left( \int_0^t |V(s)| \, ds \right)^n $$

$$ \leq \frac{e^{2(2n+1)\Re \lambda^2 t}}{n!} t^{n/2} (2 \max(1,|\lambda|))^n |C(\psi, t)|^n, $$

where one can choose

$$ C(\psi, t) := \| \max(|\psi_1\psi_2|, |\psi_1| + |\psi_3|, |\psi_2| + |\psi_4|) \|_t $$

as a uniform bound for bounded sets of $[0, \infty) \times X$. Therefore the matrix

$$ M(t) := \sum_{n=0}^\infty M_n(t) \quad (2.7) $$

exists and converges uniformly on bounded subsets of $[0, \infty) \times \mathbb{C} \times X$. By construction, $M$ solves the integral equation

$$ M(t, \lambda, \psi) = E_\lambda(t) + \int_0^t E_\lambda(t-s)V(s, \lambda, \psi)M(s, \lambda, \psi) \, ds, \quad (2.8) $$

hence $M$ is the unique matrix solution of the initial value problem $(2.5)$. Since each $M_n$, $n \geq 0$ is continuous on $[0, \infty) \times \mathbb{C} \times X$ and moreover analytic in $\lambda$ and $\psi$ for fixed $t \in [0, \infty)$, $M$ inherits the same regularity due to uniform convergence. Thus we have proved the following result.

**Theorem 2.1** (Existence of the fundamental solution $M$). The power series $(2.7)$ with coefficients given by $(2.6)$ converges uniformly on bounded subsets of $[0, \infty) \times \mathbb{C} \times X$ to a continuous function denoted by $M$, which is analytic in $\lambda$ and $\psi$ for each fixed $t \geq 0$ and satisfies the integral equation $(2.8)$.

The fundamental solution $M$ is in fact compact:
Proposition 2.2 (Compactness of \( M \)). For any sequence \((\psi_k)_k\) in \( X \) which converges weakly to an element \( \psi \in X \) as \( k \to \infty \), i.e. \( \psi_k \to \psi \), one has
\[
| M(t, \lambda, \psi_k) - M(t, \lambda, \psi) | \to 0
\]
uniformly on bounded sets of \([0, \infty) \times \mathbb{C} \).

Proof. It suffices to prove the statement for each \( M_n \), since the series \([2.7]\) converges uniformly on bounded subsets of \([0, \infty) \times \mathbb{C} \times X \). The assertion is true for \( M_0 = E_\lambda \), which is independent of \( \psi \). To achieve the inductive step, we assume that the statement holds for \( M_n, n \geq 1 \), and consider an arbitrary sequence \( \psi_k \to \psi \) in \( X \). Then
\[
M_n(t, \lambda, \psi_k) \to M_n(t, \lambda, \psi)
\]
uniformly on bounded subsets of \([0, \infty) \times \mathbb{C} \). Thus
\[
M_{n+1}(t, \lambda, \psi_k) = \int_0^t E_\lambda(t-s)V(s, \lambda, \psi_k)M_n(s, \lambda, \psi_k) \, ds
\]
\[
\to \int_0^t E_\lambda(t-s)V(s, \lambda, \psi)M_n(s, \lambda, \psi) \, ds
\]
uniformly on bounded subsets of \([0, \infty) \times \mathbb{C} \).

Furthermore, \( M \) satisfies the Wronskian identity:

Proposition 2.3 (Wronskian identity). Everywhere on \([0, \infty) \times \mathbb{C} \times X \) it holds that
\[
\det M(t, \lambda, \psi) = 1.
\]
In particular, the inverse \( M^{-1} \) is given by
\[
M^{-1} = \begin{pmatrix} m_4 & -m_2 \\ -m_3 & m_1 \end{pmatrix} \text{ if } M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}.
\]

Proof. The fundamental solution \( M \) is regular for all \( t \geq 0 \). Therefore a direct computation yields
\[
\partial_t \det M = \text{tr}(\partial_t M \cdot M^{-1}) \det M.
\]
Since
\[
\text{tr}(\partial_t M \cdot M^{-1}) = \text{tr}(R + V) = 0
\]
it follows that \( \det M(t) = \det M(0) = 1 \) for all \( t \geq 0 \).

The solution of the inhomogeneous problem corresponding to the initial value problem \([2.1],[2.2]\) has the usual “variation of constants representation”:

Proposition 2.4. The unique solution of the inhomogeneous equation
\[
Df = (R + V)f + g, \quad f(0) = v_0\]
with \( g \in L^2([0, 1], \mathbb{C}) \times L^2([0, 1], \mathbb{C}) \) is given by
\[
f(t) = M(t) \left( v_0 + \int_0^t M^{-1}(s)g(s) \, ds \right) \quad (2.9)
\]

Proof. Differentiating \([2.9]\) with respect to \( t \) and using that \( M \) is the fundamental solution of \([2.5]\), we find that
\[
f'(t) = Df(t) = M'(t)v_0 + M'(t)\int_0^t M^{-1}(s)g(s) \, ds + M(t)M^{-1}(t)g(t)
\]
\[
= (R + V)M(t) \left( v_0 + \int_0^t M^{-1}(s)g(s) \, ds \right) + g(t)
\]
\[
= (R + V)f(t) + g(t)
\]
and \( f(0) = v_0 \).
As a corollary we obtain a formula for the $\lambda$-derivative $\dot{M}$ of $M$.

**Corollary 2.5.** The $\lambda$-derivative $\dot{M}$ of $M$ is given by

$$\dot{M}(t) = M(t) \int_0^t M^{-1}(s)N(s)M(s) \, ds,$$

where

$$N = 2 \begin{pmatrix} -2\lambda i & \psi_1 \\ \psi_2 & 2\lambda i \end{pmatrix}.$$  

In particular, $\dot{M}$ is analytic on $\mathbb{C} \times X$ and compact on $[0, \infty) \times \mathbb{C} \times X$ uniformly on bounded subsets of $[0, \infty) \times \mathbb{C}$.

**Proof.** Differentiation of $DM = (R + V)M$ with respect to $\lambda$ gives

$$\dot{M} = (R + V)M + \frac{d}{d\lambda} \left( R(\lambda) + V(\lambda) \right) M = (R + V)M + NM,$$

and Proposition 2.2 yields (2.10). The second claim is a consequence of Proposition 2.2. □

The fundamental solution $M$ of the ZS-system is related to the fundamental solution $K$ of the AKNS-system by

$$K = T^{-1}MT,$$

cf. (2.4). That is, if

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}, \quad K = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix},$$

then

$$k_1 = \frac{m_1 + m_2 + m_3 + m_4}{2}, \quad k_2 = \frac{m_1 - m_2 + m_3 - m_4}{-2i},$$

$$k_3 = \frac{m_1 + m_2 - m_3 - m_4}{2i}, \quad k_4 = \frac{m_1 - m_2 - m_3 + m_4}{2}.$$  

The fundamental solution for the zero potential in AKNS coordinates is therefore given by

$$e^{2i\lambda^2 t} = \begin{pmatrix} \cos 2\lambda^2 t & \sin 2\lambda^2 t \\ -\sin 2\lambda^2 t & \cos 2\lambda^2 t \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$  

**Remark 2.6.** It is obvious that all results in this section possess an analogous version in which the space $X$ of 1-periodic potentials is replaced by the space $X_\tau$ of potentials defined on the interval $[0, \tau]$, $\tau > 0$.

2.2. **Leading order asymptotics.** The results in this section hold for $0 \leq t \leq 1$ and hence apply to the time-periodic problem we are primarily interested in.

It was pointed out in [23] that the fundamental matrix solution $M$ of (2.5) for a potential with sufficient smoothness and decay admits an asymptotic expansion as $|\lambda| \to \infty$ of the form

$$M(\lambda, t) = \left( I + \frac{Z_1(t)}{\lambda} + \frac{Z_2(t)}{\lambda^2} + \cdots \right) e^{-2i\lambda^2 t} + \left( \frac{W_1(t)}{\lambda} + \frac{W_2(t)}{\lambda^2} + \cdots \right) e^{2i\lambda^2 t},$$

(2.12)

where the matrices $Z_i$, $W_i$, $i = 1, 2, \ldots$, can be explicitly expressed in terms of the potential and therefore only depend on the time $t \geq 0$, and satisfy $Z_i(0) + W_i(0) = 0$ for all integers $i \geq 1$. This suggests that $M$ satisfies

$$M(\lambda, t) = e^{-2i\lambda^2 t} + O(|\lambda|^{-1} e^{2|3\lambda^2|t}) \quad \text{as} \quad |\lambda| \to \infty$$

for $t$ within a given bounded interval. These considerations suggest the following result.

**Theorem 2.7** (Asymptotics of $M$ and $\dot{M}$ as $|\lambda| \to \infty$). **Uniformly on** $[0,1] \times \mathbb{C}$ **and on bounded sets in** $X_1$,

$$M(t, \lambda, \psi) = E_\lambda(t) + O(|\lambda|^{-1} e^{2|3\lambda^2|t})$$

in the sense that there exist constants $C > 0$ and $K > 0$ such that

$$|\lambda| e^{-2|3\lambda^2|t} |M(t, \lambda, \psi) - E_\lambda(t)| \leq C$$

(2.13)
uniformly for all \(0 \leq t \leq 1\), all \(\lambda \in \mathbb{C}\) with \(|\lambda| > K\) and all \(\psi\) contained in a bounded set in \(X_1\). Moreover, the \(\lambda\)-derivative of \(M\) satisfies
\[
\dot{M}(t, \lambda, \psi) = \dot{E}_\lambda(t) + \mathcal{O}(e^{2|3\lambda^2|t})
\]  
(2.14)
uniformly on \([0, 1] \times \mathbb{C}\) and on bounded sets in \(X_1\).

Theorem 2.7 will be established via a series of lemmas. We first introduce some notation and briefly discuss the idea of the proof.

For \(\lambda \in \mathbb{C}\) and \(\psi \in X\), let \(M\) be the fundamental solution of \((2.5)\), which will be considered on the unit interval \([0, 1]\). We set
\[
\theta := 2\lambda^2
\]
and define \(M^+\) and \(M^-\) by
\[
M^+(t, \lambda, \psi) := M(t, \lambda, \psi)e^{i\theta t \sigma_3}, \quad M^-(t, \lambda, \psi) := M(t, \lambda, \psi)e^{-i\theta t \sigma_3},
\]
For a given complex \(2 \times 2\)-matrix
\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
we denote by \(A^d\) its diagonal part and by \(A^{od}\) its off-diagonal part, i.e.
\[
A^d = \begin{pmatrix} a \\ c \end{pmatrix}, \quad A^{od} = \begin{pmatrix} b & c \end{pmatrix}.
\]
We will always identify a potential \(\psi \in X_t, t > 0\), with its absolutely continuous version. This allows us to evaluate \(\psi\) at each point; we set \(\psi_0 := \psi(0)\) and \(\psi_j := \psi^j(0)\) for \(j = 1, 2, 3, 4\). For a given potential \(\psi \in X_1, t \in [0, 1]\) and \(\lambda \in \mathbb{C}\) we define
\[
Z_p(t, \lambda, \psi) := I + \frac{Z_1(t, \psi)}{\lambda} + \frac{Z_2^{od}(t, \psi)}{\lambda^2},
\]
where
\[
Z_1(t, \psi) := -\frac{i}{2} \begin{pmatrix} \psi_1 & \psi_2 \\ -\psi_2 & \psi_1 \end{pmatrix} + \frac{1}{2} \Gamma \sigma_3, \quad Z_2^{od}(t, \psi) := \frac{1}{4} \begin{pmatrix} \psi^4 & i\psi^2 \Gamma \\ -i\psi^2 \Gamma & \psi^3 + i\psi^4 \end{pmatrix},
\]
with
\[
\Gamma := \Gamma(t, \psi) := \int_0^t \left( \psi^4 \psi^4 - \psi^3 \psi_0 \right) \mathrm{d}t.
\]
Furthermore we set
\[
W_p(t, \lambda, \psi) := \frac{W_1(t, \psi)}{\lambda} + \frac{W_2(t, \psi)}{\lambda^2} + \frac{W_3^{od}(t, \psi)}{\lambda^3},
\]
where
\[
W_1(t, \psi) = W_1(\psi) := \frac{i}{2} \begin{pmatrix} \psi_0^{\dagger} \\ -\psi_0 \end{pmatrix}, \quad W_2(t, \psi) := -\frac{1}{4} \begin{pmatrix} \psi_0^{\dagger} \psi_0^{\dagger} & -i\psi_0^{\dagger} \Gamma \\ -i\psi_0 \Gamma & \psi_0 \psi_0^{\dagger} \end{pmatrix},
\]
\[
W_3^{od}(t, \psi) := \frac{i}{8} \begin{pmatrix} -\psi_0^{\dagger} \psi_0 & i\psi_0^{\dagger} \Gamma \\ \psi_0 \psi_0^{\dagger} & \psi_0^{\dagger} \end{pmatrix}.
\]
We finally define \(M_p\) by
\[
M_p(t, \lambda, \psi) := Z_p(t, \lambda, \psi)e^{-i\theta t \sigma_3} + W_p(t, \lambda, \psi)e^{i\theta t \sigma_3}
\]
and set \(M_p^+ := M_p e^{i\theta t \sigma_3}, M_p^- := M_p e^{-i\theta t \sigma_3}\), i.e.
\[
M^+_p(t, \lambda, \psi) = Z_p(t, \lambda, \psi) + W_p(t, \lambda, \psi)e^{2i\theta t \sigma_3},
\]
\[
M^-_p(t, \lambda, \psi) = Z_p(t, \lambda, \psi)e^{-2i\theta t \sigma_3} + W_p(t, \lambda, \psi).
\]
Letting \(Q_j, j = 1, 2, 3, 4\), denote the four quadrants of the complex \(\lambda\)-plane, we set
\[
D_+ := Q_1 \cup Q_3 \quad \text{and} \quad D_- := Q_2 \cup Q_4.
\]
For an arbitrary complex number \( \lambda = x + iy \) with \( x, y \in \mathbb{R} \) and \( t > 0 \), it holds that
\[
|e^{2i\lambda^2 t}| = e^{-4xy} < 1 \iff \lambda \in D_+,
\]
\[
|e^{-2i\lambda^2 t}| = e^{4xy} < 1 \iff \lambda \in D_-.
\]

We will prove Theorem 2.7 by establishing asymptotic estimates for the distance between the fundamental solution \( M \) and the explicit expression \( M_p \) that approximates \( M \). For this purpose we will consider the columns of \( M^+ \) and \( M^- \) separately and compare them with the columns of \( Z_p \) and \( W_p \), respectively, after restricting attention to either \( \overline{D}_+ \) or \( \overline{D}_- \). By combining all possible combinations, we are able to infer asymptotic estimates for the full matrix \( M \) valid on the whole complex plane.

**Remark 2.8.** For a given smooth potential \( \psi \), the matrices \( Z_t \) and \( W_t \) can be determined recursively up to any order \( i \geq 0 \) by integration by parts. Indeed, note that \( V = V_0 + \lambda V_1 \) where
\[
V_0 := \begin{pmatrix} -i \psi^1 \psi^2 & i \psi^3 \\ -i \psi^4 & i \psi^1 \psi^2 \end{pmatrix}, \quad V_1 := \begin{pmatrix} 2 \psi^2 & 2 \psi^1 \end{pmatrix}.
\]
Assuming that the formal expression
\[
\left( \sum_{k=-\infty}^{\infty} \frac{Z_k(t, \psi)}{\lambda^k} \right) e^{-i \theta t \sigma_3} + \left( \sum_{k=-\infty}^{\infty} \frac{W_k(t, \psi)}{\lambda^k} \right) e^{i \theta t \sigma_3}
\]
with
\[
Z_0(t, \psi) = I, \quad Z_{-1}(t, \psi) = W_{-1}(t, \psi) = W_0(t, \psi) = 0
\]
solves (2.5), one infers the following recursive equations for the coefficients \( Z_k \) and \( W_k \):
\[
(Z_k)_t + 4i \sigma_3 Z_k^{(0)} = V_0 Z_k + V_1 Z_{k+1},
\]
\[
(W_k)_t + 4i \sigma_3 W_k^{(0)} = V_0 W_k + V_1 Z_{k+1}
\]
for all integers \( k \geq -1 \) and \( Z_k(0, \psi) + W_k(0, \psi) = 0 \) for all integers \( k \geq 1 \). For \( \psi \in X_1 \), the matrices \( Z_p \) and \( W_p \) satisfy
\[
Z_p(0, \lambda, \psi) + W_p(0, \lambda, \psi) = I + O(|\lambda|^{-2}),
\]
which turns out to be enough to prove the asymptotic estimates of \( M \) asserted in Theorem 2.7.

**Lemma 2.9.** Let \( \psi \in X_1 \) be an arbitrary potential. Then \( M \) is the fundamental matrix solution of the Cauchy problem (2.5) if and only if \( M^+ \) satisfies
\[
M^+ t + 2i \theta \sigma_3 (M^+)^{(0)} = V M^+, \quad M^+(0, \lambda) = I.
\]

**Proof.** By applying the product rule, assuming that (2.5) holds and noting that \( \sigma_3 \) commutes with diagonal matrices, we obtain
\[
M^+_t = (M e^{i \theta t \sigma_3})_t = M_t e^{i \theta t \sigma_3} + M e^{i \theta t \sigma_3} i \theta \sigma_3
\]
\[
= (V M - i \theta \sigma_3 M) e^{i \theta t \sigma_3} + i \theta M^+ \sigma_3
\]
\[
= VM^+ - i \theta [\sigma_3, M^+]
\]
\[
= VM^+ - 2i \theta [\sigma_3, M^+]^{(0)}.
\]
Conversely, if (2.15) holds, we similarly obtain
\[
M e^{i \theta t \sigma_3} = (V M - i \theta \sigma_3 M) e^{i \theta t \sigma_3},
\]
and a multiplication with \( e^{-i \theta \sigma_3} \) from the right yields that \( M \) satisfies the differential equation in (2.5). The statement concerning the initial conditions holds because \( M(0, \lambda) = M^+(0, \lambda) \).

The following lemma is concerned with the invertibility of \( Z_p \) and \( W_p \). We let \( \mathbb{C}^K := \{ \lambda \in \mathbb{C} : |\lambda| > K \} \) for \( K > 0 \) and let \( B_r(0, X_1) \) denote the ball of radius \( r > 0 \) in \( X_1 \) centered at 0. Furthermore, we define
\[
\tilde{X}_1 := \{ \psi \in X_1 : \psi^0_1 \psi^0_0 \neq 0 \}
\]
and
\[
B_r(0, \tilde{X}_1) := B_r(0, X_1) \cap \tilde{X}_1.
\]

**Lemma 2.10.** Let \( r > 0 \).
\[(1)\] There exists a constant \(K_r > 0\) such that \(Z_p\) is invertible on \([0,1] \times \mathbb{C}^{K_r} \times B_r(0, X_1)\) with
\[
Z_p^{-1}(t, \lambda, \psi) = \sum_{n=0}^{\infty} \left( -\frac{Z_1(t, \psi)}{\lambda} - \frac{Z_2^{\text{od}}(t, \psi)}{\lambda^2} \right)^n.
\tag{2.16}
\]

\[(2)\] There exists a constant \(K_r > 0\) such that \(W_p\) is invertible on \([0,1] \times \mathbb{C}^{K_r} \times B_r(0, \tilde{X}_1)\) with
\[
W_p^{-1}(t, \lambda, \psi) = \left[ \sum_{n=0}^{\infty} \left( -\frac{(W_1^{-1} W_2)(t, \psi)}{\lambda} - \frac{(W_1^{-1} W_3^{\text{od}})(t, \psi)}{\lambda^2} \right)^n \right] \lambda W_1^{-1}(t, \psi).
\tag{2.17}
\]

If \(\psi_0^1 = 0\) or \(\psi_0^2 = 0\), then \(W_p(t, \lambda, \psi)\) is not invertible.

**Proof.** We use the general fact that if an element \(A\) of a Banach algebra \((\mathbb{A}, \| \cdot \|)\) satisfies \(\|A\| < 1\), then \(I - A\) is invertible and its inverse is given by the Neumann series \(\sum_{n\geq 0} A^n\).

To prove assertion \((1)\), we choose \(K_r > 0\) so large that
\[
\frac{Z_1(t, \psi)}{\lambda} + \frac{Z_2^{\text{od}}(t, \psi)}{\lambda^2} < \frac{1}{2}
\]
for all \(t \in [0,1]\), \(\lambda \in \mathbb{C}^{K_r}\) and \(\psi \in B_r(0, X_1)\). This can always be achieved, because the functions \(\{\psi^j\}_{j=1}^3\), and hence also the functions \(|Z_1(t, \psi)|\) and \(|Z_2^{\text{od}}(t, \psi)|\), are uniformly bounded on \([0,1] \times B_r(0, X_1)\).

To prove assertion \((2)\), we fix \(\psi \in X_1\) and consider the determinant of \(W_p\):
\[
\det W_p(t, \lambda, \psi) = \frac{1}{4\lambda^2} \left( 1 + \frac{1}{4\lambda^2} \Gamma^2 + \frac{1}{4\lambda^2} \psi^1 \psi^2 + \frac{i}{8\lambda^3} (\psi^2 \psi^3 - \psi^1 \psi^4) + \frac{1}{16\lambda^4} (\psi^3 \psi^4 + i(\psi^2 \psi^3 + \psi^1 \psi^4) \Gamma - \psi^1 \psi^2 \Gamma^2) \right).
\]

We see that both \(\psi_0^1\) and \(\psi_0^2\) have to be nonzero in order for \(W_p\) to be invertible. If \(\psi_0^1, \psi_0^2 \neq 0\), then \(W_p\) is invertible iff the expression within the square brackets is nonzero. This expression is of order 1 uniformly for \(t \in [0,1]\) and \(\psi \in X_1\); hence there exists a constant \(K_r > 0\) such that this expression (and hence also \(\det W_p(t, \lambda, \psi)\)) is nonzero for \(t \in [0,1], \lambda \in \mathbb{C}^{K_r}\) and \(\psi \in B_r(0, \tilde{X}_1)\). In this case, we can write
\[
W_p(t, \lambda, \psi) = W_1(t, \psi) \left( I + \frac{(W_1^{-1} W_2)(t, \psi)}{\lambda} + \frac{(W_1^{-1} W_3^{\text{od}})(t, \psi)}{\lambda^2} \right).
\]

Thus we can – by the argument used in the proof of \((1)\) – find a constant \(K_r > 0\) such that the inverse of \(W_p\) is given by (2.17) on \([0,1] \times \mathbb{C}^{K_r} \times B_r(0, \tilde{X}_1)\). \(\square\)

**Lemma 2.10** and its proof suggest the introduction of the following notation.

**Definition 2.11.** For each \(r > 0\), we define
\[
K_r := \inf_{|\lambda| > 1} \{ |1 - Z_p(t, \lambda, \psi)| < 1/2 \forall t \in [0,1] \forall \psi \in B_r(0, X_1) \},
\]
and, using the abbreviations \(W_1^{-1} = W_1^{-1}(t, \psi)\), \(W_2 = W_2(t, \psi)\) and \(W_3^{\text{od}} = W_3^{\text{od}}(t, \psi)\),
\[
K_r := \inf_{|\lambda| > 1} \left\{ \frac{|W_1^{-1} W_2|}{|\lambda|} + \frac{|W_1^{-1} W_3^{\text{od}}|}{|\lambda|^2} < 1/2 \forall t \in [0,1] \forall \psi \in B_r(0, \tilde{X}_1) \right\}.
\]

**Corollary 2.12.** Let \(r > 0\).

\[\text{1)}\] The matrix \(Z_p\) is invertible on \([0,1] \times \mathbb{C}^{K_r} \times B_r(0, X_1)\) and its inverse \(Z_p^{-1}\) is given by (2.16). Both \(Z_p\) and \(Z_p^{-1}\) are uniformly bounded on \([0,1] \times \mathbb{C}^{K_r} \times B_r(0, X_1)\). Furthermore,
\[
Z_p^{-1} = I - \frac{Z_1}{\lambda} + \frac{Z_2^2 - Z_2^{\text{od}}}{\lambda^2} + O(|\lambda|^{-3})
\tag{2.18}
\]
uniformly on $[0,1] \times C^{K'} \times B_r(0,X_1)$ as $|\lambda| \to \infty$.

(2) The matrix $W_p$ is invertible on $[0,1] \times C^{K'} \times B_r(0,X_1)$ and its inverse $W_p^{-1}$ is given by (2.17). Both $W_p$ and $W_p^{-1}$ are uniformly bounded on $[0,1] \times C^{K'} \times B_r(0,X_1)$. Furthermore,

$$W_p^{-1} = \lambda W_1^{-1} - W_1^{-1}W_2W_1^{-1} + \frac{(W_1^{-1}W_2)^2 W_1^{-1} - W_1^{-1}W_3^2 W_1^{-1}}{\lambda} + O(|\lambda|^{-2})$$

uniformly on $[0,1] \times C^{K'} \times B_r(0,X_1)$ as $|\lambda| \to \infty$.

Proof. The expansions (2.18) and (2.19) follow from (2.16) and (2.17), respectively.

For $t \in [0,1]$, $\lambda \in C$ and $\psi \in X_1$, we define

$$A := V - i\theta\sigma_3$$

$$A_{p,Z} := [(\partial_t Z_p) - i\theta Z_p\sigma_3]Z_p^{-1},$$

$$A_{p,W} := [(\partial_t W_p) + i\theta W_p\sigma_3]W_p^{-1},$$

whenever the inverses $Z_p^{-1}$ and $W_p^{-1}$ exist. By Lemma 2.10, the inverse of $Z_p$ exists uniformly on $[0,1]$ and on bounded sets in $X_1$ provided that $|\lambda|$ is large enough (in order for $W_p^{-1}$ to exist one also needs $\psi_1, \psi_3 \neq 0$). For a $t$-dependent matrix $A$ with entries in $L^p$, we define

$$\|A\|_{L^p(0,1)} := \left( \int_0^1 |A(t)|^p \, dt \right)^{1/p}, \quad 1 \leq p < \infty.$$

**Lemma 2.13.** Let $V$ be an arbitrary bounded subset of $X_1$ and let $1 \leq q \leq 2$. Then

$$\|\partial_t Z_1(\psi)\|_{L^q(0,1)} = O(1),$$

$$\|\partial_t Z_2^d(\psi)\|_{L^q(0,1)} = O(1),$$

$$\|\partial_t W_2(\psi)\|_{L^q(0,1)} = O(1),$$

$$\|\partial_t W_3^d(\psi)\|_{L^q(0,1)} = O(1)$$

uniformly on $V$.

Proof. The case $q = 2$ follows directly from the definitions of $Z_1, Z_2^d, W_2$ and $W_3^d$, the continuity of the operator

$$\| \cdot \|_{L^2(0,1)} \circ \partial_t : H^1(0,1) \to \mathbb{R}$$

and the fact that $H^1(0,1)$ is an algebra. The cases $1 \leq q < 2$ follow from the case $q = 2$ in view of the continuous embeddings $L^2(0,1) \to L^q(0,1), 1 \leq q < 2$.

**Lemma 2.14.** For $\lambda \in C$ and $\psi \in X_1$, let $M$ be the fundamental solution of (2.5) on the unit interval.

1. If $|\lambda|$ is so large that $Z_p^{-1}$ exists for all $t \in [0,1]$, then

$$(Z_p^{-1}M^+)t = Z_p^{-1}\Delta Z M^+ - i\theta[\sigma_3, Z_p^{-1}M^+].$$  \hfill (2.20)

2. If $\psi \in X_1$ and $|\lambda|$ is so large that $W_p^{-1}$ exists for all $t \in [0,1]$, then

$$(W_p^{-1}M^-)t = W_p^{-1}\Delta W M^- + i\theta[\sigma_3, W_p^{-1}M^-].$$  \hfill (2.21)

Proof. Equation (2.20) is obtained by a direct calculation:

$$(Z_p^{-1}M^+)t = -Z_p^{-1}(\partial_t Z_p)Z_p^{-1}M^+ + Z_p^{-1}M^+$$

$$= -Z_p^{-1}(A_{p,Z}Z_p + i\theta Z_p\sigma_3)Z_p^{-1}M^+ + Z_p^{-1}(AM^+ + i\theta M^+\sigma_3)$$

$$= Z_p^{-1}\Delta Z M^+ - i\theta[\sigma_3, Z_p^{-1}M^+].$$

The proof of (2.21) is similar.

For $z \in C$, we define the linear map $e^{iz\sigma_3}$ on the space of complex $2 \times 2$-matrices by

$$e^{iz\sigma_3}(A) := e^{z\sigma_3}Ae^{-z\sigma_3}.$$

Lemma 2.14 yields Volterra equations for $M^+$ and $M^-$. 

Lemma 2.15. For \( \lambda \in \mathbb{C} \) and \( \psi \in X_1 \), let \( M \) be the fundamental solution of (2.5) on the unit interval.

1. If \(|\lambda|\) is so large that \( Z_p^{-1} \) exists for all \( t \in [0,1] \), then \( M^+ \) satisfies
   \[
   M^+ (t, \lambda, \psi) = Z_p(t, \lambda, \psi) e^{-i\theta t\lambda} [Z_p^{-1}(0, \lambda, \psi)] + \int_0^t Z_p(t, \lambda, \psi) e^{-i\theta (t-\tau)\lambda} [(Z_p^{-1} \Delta Z M^+)(\tau, \lambda, \psi)] d\tau. \tag{2.22}
   \]

2. If \( \psi \in \tilde{X}_1 \) and \(|\lambda|\) is so large that \( W_p^{-1} \) exists for all \( t \in [0,1] \), then \( M^- \) satisfies
   \[
   M^- (t, \lambda, \psi) = W_p(t, \lambda, \psi) e^{i\theta t\lambda} [W_p^{-1}(0, \lambda, \psi)] + \int_0^t W_p(t, \lambda, \psi) e^{i\theta (t-\tau)\lambda} [(W_p^{-1} \Delta W M^-)(\tau, \lambda, \psi)] d\tau. \tag{2.23}
   \]

Proof. Using the first part of Lemma 2.14, we obtain
   \[
   (e^{i\theta(t-\lambda)}(Z_p^{-1}M^+))_t = e^{i\theta(t-\lambda)}(Z_p^{-1}\Delta Z M^+). \tag{2.24}
   \]
   In order to obtain (2.22), we first integrate (2.24) from 0 to \( t \) and use that \( M^+(0, \lambda) = I \) to determine the integration constant. Applying \( e^{-i\theta(t-\lambda)} \) to both sides of the resulting integral equation and multiplying by \( Z_p \) from the left, we find (2.22).

The Volterra equation for \( M^- \) follows in an analogous way from the equation
   \[
   (e^{-i\theta(t-\lambda)}(W_p^{-1}M^-))_t = e^{-i\theta(t-\lambda)}(W_p^{-1}\Delta W M^-),
   \]
   which is a consequence of the second part of Lemma 2.14. \( \square \)

Lemma 2.16. Let \( r > 0 \). There exists a constant \( C > 0 \) such that
   \[
   |\lambda| \parallel \Delta Z(\lambda, \psi) \parallel_{L^1(0,1)} \leq C \quad \text{uniformly for } (\lambda, \psi) \in \mathbb{C}^K \times B_r(0, X_1), \tag{2.25}
   \]
   \[
   \parallel \Delta W(\lambda, \psi) \parallel_{L^1(0,1)} \leq C \quad \text{uniformly for } (\lambda, \psi) \in \mathbb{C}^K \times B_r(0, \tilde{X}_1). \tag{2.26}
   \]

Proof. Note that
   \[
   4i\sigma_3 Z_1^{ad} = V_1, \quad 4i\sigma_3 Z_2^{ad} = V_0 + V_1 Z_1 \tag{2.27}
   \]
   for arbitrary \( \psi \in X_1 \). By Corollary 2.12, the asymptotic estimate (2.18) holds uniformly on \([0,1] \times \mathbb{C}^K \times B_r(0, X_1) \) as \(|\lambda| \to \infty \). In particular, \( \Delta Z \) is well-defined on \([0,1] \times \mathbb{C}^K \times B_r(0, X_1) \) and satisfies
   \[
   \Delta Z = V_0 + \lambda V_1 - 2i\lambda^2 \sigma_3 + 2i\lambda^2 \left( I + \frac{Z_1}{\lambda} + \frac{Z_2^{ad}}{\lambda^2} \right) \sigma_3 \left( I - \frac{Z_1}{\lambda} + \frac{Z_2^{ad}}{\lambda^2} \right) + O(|\lambda|^{-1}) \tag{2.28}
   \]
   in \( L^1(0,1) \) uniformly on \( \mathbb{C}^K \times B_r(0, X_1) \) as \(|\lambda| \to \infty \). We have used Lemma 2.13 to estimate the \( \partial_t Z_p \)-term. By keeping only the \( \lambda^k \)-terms for \( k = 0, 1, 2 \) in (2.28) and by employing (2.27), we obtain
   \[
   \Delta Z = V_0 + \lambda V_1 - 2i\lambda|\sigma_3, Z_1| + 2i[Z_2^{ad}, \sigma_3] + 2i[\sigma_3, Z_1] Z_1 + O(|\lambda|^{-1})
   \]
   \[
   = V_0 - 4i\sigma_3 Z_2^{ad} + 4i\sigma_3 Z_1^{ad} Z_1 + O(|\lambda|^{-1})
   \]
   in \( L^1(0,1) \) uniformly on \( \mathbb{C}^K \times B_r(0, X_1) \) as \(|\lambda| \to \infty \). This proves (2.25).

In order to prove (2.26), we fix \( \psi \in X_1 \) and note that
   \[
   W_1 \sigma_3 W_1^{-1} = -\sigma_3 W_1 W_1^{-1} = -\sigma_3, \quad W_1^{-1} = -\frac{2i}{\psi^0_1 \psi^0_0} \begin{pmatrix} 0 & -\psi^1_0 \\ \psi^0_0 & 0 \end{pmatrix}. \tag{2.29}
   \]
Direct but tedious computations then give
\[ 4i\sigma_3 W_2^d W_1^{-1} = V_1, \]  
(2.30)
\[ -4i\sigma_3 W_2^d W_1^{-1} W_2 W_1^{-1} = V_0^d + \frac{1}{2} \Gamma \sigma_3 V_1, \]  
(2.31)
\[ 4i\sigma_3 W_3^d W_1^{-1} = V_0^d - \frac{1}{2} \Gamma \sigma_3 V_1. \]  
(2.32)
In particular, (2.31) and (2.32) imply
\[ 4i\sigma_3 (W_2^d - W_2^d W_1^{-1} W_2) W_1^{-1} = V_0. \]  
(2.33)
By Corollary 2.12, the asymptotic estimate (2.19) holds uniformly on \([0, 1] \times C^K \times B_r(0, X_1)\) as \(|\lambda| \to \infty\). In particular, \(\Delta W\) is well-defined and satisfies
\[ \Delta W = V_0 + \lambda V_1 - 2i\lambda^2 \sigma_3 - (\partial_t W_p) W_p^{-1} - 2i\lambda^2 W_p \sigma_3 W_p^{-1} \]  
(2.34)
on \([0, 1] \times C^K \times B_r(0, X_1)\). Since \(W_1\) is constant, we infer from Lemma 2.13 and (2.19) that \(|(\partial_t W_p) W_p^{-1}_{L^1(0, 1)}| = O(|\lambda|^{-1})\) uniformly on \(C^K \times B_r(0, X_1)\) as \(|\lambda| \to \infty\). Moreover, by (2.19) and (2.29),
\[ \lambda^2 W_p \sigma_3 W_p^{-1} = -\lambda^2 \sigma_3 + 2\lambda^2 \sigma_3 W_2^d W_1^{-1} + 2\sigma_3 (W_3^d - W_2^d W_1^{-1} W_2) W_1^{-1} + O(|\lambda|^{-1}) \]  
(2.35)uniformly on \([0, 1] \times C^K \times B_r(0, X_1)\) as \(|\lambda| \to \infty\). Thus, the \(\lambda^2\)-terms in (2.34) cancel. In view of (2.30) and (2.33), the \(\lambda^1\) and \(\lambda^0\)-terms also cancel. This proves (2.26).

Let \([A]_1\) and \([A]_2\) denote the first and second columns of a \(2 \times 2\)-matrix \(A\). Let \(|[A]_j|, j = 1, 2\), denote the standard \(C^2\)-norm of the vector \([A]_j\).

**Lemma 2.17.** Let \(r > 0\).

1. There exists a constant \(C > 0\) such that
   \[ |\lambda| \left| [M^+(t, \lambda, \psi) - Z_0(t, \lambda, \psi)] e^{-i\theta \sigma_3} (Z_0^{-1}(0, \lambda, \psi))]_2 \right| \leq C \]  
(2.36)
   uniformly on \([0, 1] \times DK^r \times B_r(0, X_1)\) and
   \[ |\lambda| \left| [M^+(t, \lambda, \psi) - Z_0(t, \lambda, \psi)] e^{-i\theta \sigma_3} (Z_0^{-1}(0, \lambda, \psi))]_1 \right| \leq C \]  
(2.37)
   uniformly on \([0, 1] \times DK^r \times B_r(0, X_1)\).

2. There exists a constant \(C > 0\) such that
   \[ |\lambda| \left| [M^-(t, \lambda, \psi) - W_0(t, \lambda, \psi)] e^{i\theta \sigma_3} (W_0^{-1}(0, \lambda, \psi))]_2 \right| \leq C \]  
(2.38)
   uniformly on \([0, 1] \times DK^r \times B_r(0, X_1)\) and
   \[ |\lambda| \left| [M^-(t, \lambda, \psi) - W_0(t, \lambda, \psi)] e^{i\theta \sigma_3} (W_0^{-1}(0, \lambda, \psi))]_1 \right| \leq C \]  
(2.39)
   uniformly on \([0, 1] \times DK^r \times B_r(0, X_1)\).

**Proof.** For \(\lambda \in C^K\), the functions
\[ \mathcal{M}(t, \lambda, \psi) := [M^+(t, \lambda, \psi)]_2, \]
\[ \mathcal{M}_0(t, \lambda, \psi) := [Z_0(t, \lambda, \psi)] e^{-i\theta \sigma_3} (Z_0^{-1}(0, \lambda, \psi))]_2; \]
\[ E(t, \tau, \lambda, \psi) := Z_0(t, \lambda, \psi) \left( e^{-2i\theta (t-\tau)} \right) Z_0^{-1}(\tau, \lambda, \psi), \]
are well-defined on their domains \([0, 1] \times C^K \times B_r(0, X_1)\) and \([0, 1]^2 \times C^K \times B_r(0, X_1)\) respectively, where the inverse \(Z_0^{-1}\) is given by (2.16) and is uniformly bounded on \([0, 1] \times C^K \times B_r(0, X_1)\)
by Lemma 2.10 and Corollary 2.12. Due to Lemma 2.15, \( \mathcal{M} \) satisfies the following Volterra equation for \( t \in [0,1] \), \( \lambda \in \mathbb{C}^{K_r} \) and \( \psi \in B_r(0, X_1) \):

\[
\mathcal{M}(t, \lambda) = \mathcal{M}_0(t, \lambda) + \int_0^t E(t, \tau, \lambda) \Delta_g(\tau, \lambda) \mathcal{M}(\tau, \lambda) \, d\tau,
\]

where the \( \psi \)-dependence has been suppressed for simplicity. Thus \( \mathcal{M} \) admits the power series representation

\[
\mathcal{M}(t, \lambda) = \sum_{n=0}^{\infty} \mathcal{M}_n(t, \lambda)
\]

which converges (pointwise) absolutely and uniformly on \([0,1] \times \mathbb{C}^{K_r} \times B_r(0, X_1)\), where

\[
\mathcal{M}_n(t, \lambda) := \int_0^t E(t, \tau, \lambda) \Delta_g(\tau, \lambda) \mathcal{M}_{n-1}(\tau, \lambda) \, d\tau \quad (n \geq 1)
\]

satisfies the estimate

\[
|\mathcal{M}_n(t, \lambda)| \leq \frac{C^n}{n!|\lambda|^n}
\]

uniformly on \([0,1] \times \mathbb{C}^{K_r} \times B_r(0, X_1)\). The functions \( E \) and \( \mathcal{M}_0 \) satisfy

\[
|M_0(t, \lambda)| \leq |Z_p(t, \lambda)| |Z_p^{-1}(0, \lambda)|,
\]

\[
|E(t, \tau, \lambda)| \leq |Z_p(t, \lambda)| |Z_p^{-1}(\tau, \lambda)|,
\]

for \( 0 \leq \tau \leq t \leq 1 \) and \((\lambda, \psi) \in \mathbb{D}^{K_r}_- \times B_r(0, X_1)\). Therefore, in view of Corollary 2.12 and Lemma 2.16, there exists a constant \( C > 0 \) such that

\[
|\mathcal{M}_n(t, \lambda)| \leq \frac{C^n}{n!|\lambda|^n}
\]

uniformly on \([0,1] \times \mathbb{D}^{K_r}_+ \times B_r(0, X_1)\), and thus

\[
|\mathcal{M}(t, \lambda) - \mathcal{M}_0(t, \lambda)| \leq \sum_{n=1}^{\infty} \frac{C^n}{n!|\lambda|^n} = \frac{Ce^\frac{|\lambda|}{\sqrt{K_r}}}{|\lambda|}
\]

uniformly on \([0,1] \times \mathbb{D}^{K_r}_+ \times B_r(0, X_1)\). This proves (2.36); the proofs of (2.37)-(2.39) are similar. \( \square \)

**Lemma 2.18.** Let \( r > 0 \).

1. There exists a constant \( C > 0 \) such that

\[
|\lambda|^2 \left| \left[ Z_p(t, \lambda, \psi) e^{-i \theta t \lambda} (Z_p^{-1}(0, \lambda, \psi)) - M_p^+(t, \lambda, \psi) \right] \right|_2 \leq C
\]

uniformly on \([0,1] \times \mathbb{D}^{K_r}_- \times B_r(0, X_1)\) and

\[
|\lambda|^2 \left| \left[ Z_p(t, \lambda, \psi) e^{-i \theta t \lambda} (Z_p^{-1}(0, \lambda, \psi)) - M_p^+(t, \lambda, \psi) \right] \right|_1 \leq C
\]

uniformly on \([0,1] \times \mathbb{D}^{K_r}_+ \times B_r(0, X_1)\).

2. There exists a constant \( C > 0 \) such that

\[
|\lambda|^2 \left| \left[ W_p(t, \lambda, \psi) e^{i \theta t \lambda} (W_p^{-1}(0, \lambda, \psi)) - M_p^-(t, \lambda, \psi) \right] \right|_2 \leq C
\]

uniformly on \([0,1] \times \mathbb{D}^{K_r}_- \times B_r(0, \hat{X}_1)\) and

\[
|\lambda|^2 \left| \left[ W_p(t, \lambda, \psi) e^{i \theta t \lambda} (W_p^{-1}(0, \lambda, \psi)) - M_p^-(t, \lambda, \psi) \right] \right|_1 \leq C
\]

uniformly on \([0,1] \times \mathbb{D}^{K_r}_+ \times B_r(0, \hat{X}_1)\).
Proof. Since
\[
\left[e^{i\theta t\partial_x} (Z_1(0, \psi)) \right]_2 = e^{2i\theta t} \left(-\frac{1}{2} \psi_0^1 \mathbf{0} \right) = e^{2i\theta t} [Z_1(0, \psi)]_2 = e^{2i\theta t} [W_1(0, \psi)]_2
\]
for \(\psi \in X_1\), Corollary 2.12 yields
\[
\left[ Z_p(t, \lambda, \psi) e^{-i\theta t\partial_x} (Z^{-1}_p(0, \lambda, \psi)) \right]_2 = \left[ Z_p(t, \lambda, \psi) e^{-i\theta t\partial_x} \left(1 - \frac{Z_1(0, \psi)}{\lambda} + O(|\lambda|^{-2})\right) \right]_2
\]
\[
= \left[ Z_p(t, \lambda, \psi) \right]_2 - \frac{e^{2i\theta t}}{\lambda} \left[ Z_1(0, \psi) \right]_2 + O\left( \frac{e^{2i\theta t}}{|\lambda|^2} \right)
\]
\[
= \left[ Z_p(t, \lambda, \psi) \right]_2 + \frac{e^{2i\theta t}}{\lambda} \left[ W_1(0, \psi) \right]_2 + O\left( \frac{e^{2i\theta t}}{|\lambda|^2} \right)
\]
uniformly on \([0,1] \times C^K_r \times B_r(0, X_1)\) as \(|\lambda| \to \infty\). On the other hand, we have
\[
\left[ M_p^+(t, \lambda, \psi) \right]_2 = \left[ Z_p(t, \lambda, \psi) \right]_2 + \frac{e^{2i\theta t}}{\lambda} \left[ W_1(0, \psi) \right]_2 + O\left( \frac{e^{2i\theta t}}{|\lambda|^2} \right)
\]
uniformly on \([0,1] \times C^K_r \times B_r(0, X_1)\) as \(|\lambda| \to \infty\). Since \(\Im \theta \leq 0\) for \(\lambda \in D_-\), the estimate (2.40) follows. The estimates (2.41)–(2.43) are proved in a similar way.

Proof of Theorem 2.7. The first assertion of the theorem follows by combining Lemma 2.17 and Lemma 2.18. Indeed, suppose \(\psi_1^1, \psi_2^2 \neq 0\) (i.e. \(\psi \in X_1\)) and let \(r > 0\). By (2.36) and (2.40), there exists a \(C > 0\) such that
\[
|\lambda| \left|[M^+(t, \lambda, \psi) - M^+_p(t, \lambda, \psi)]_2 \right| \leq C
\]
uniformly on \([0,1] \times D^K_r \times B_r(0, \hat{X}_1)\). Thus,
\[
|\lambda| \left| e^{-2|3\lambda^2|t} \left|[M(t, \lambda, \psi) - M^+_p(t, \lambda, \psi)]_2 \right| \right| \leq C
\]
uniformly on \([0,1] \times D^K_r \times B_r(0, \hat{X}_1)\). Since \(M^+_p(t, \lambda, \psi) = E_\lambda(t) + O(|\lambda|^{-1} e^{2(3\lambda^2|t})\) uniformly on \([0,1] \times D^K_r \times B_r(0, \hat{X}_1)\) as \(|\lambda| \to \infty\), we infer that there exists a \(C > 0\) such that
\[
|\lambda| \left| e^{-2|3\lambda^2|t} \left|[M(t, \lambda, \psi) - E_\lambda(t)]_2 \right| \right| \leq C
\]
uniformly on \([0,1] \times D^K_r \times B_r(0, \hat{X}_1)\). Analogously, by using (2.37) and (2.41), one infers the existence of a constant \(C > 0\) such that
\[
|\lambda| \left| e^{-2|3\lambda^2|t} \left|[M(t, \lambda, \psi) - E_\lambda(t)]_1 \right| \right| \leq C
\]
uniformly on \([0,1] \times D^K_r \times B_r(0, \hat{X}_1)\). The estimates (2.38) and (2.42) (resp. (2.39) and (2.43)) yield the same asymptotic estimates for \([M - E_\lambda]_2\) (resp. \([M - E_\lambda]_1\)) for \(\lambda\) restricted to \(D^K_r\) (resp. \(\overline{D^K_r}\)). In summary, this yields the existence of constants \(C, K > 0\) such that
\[
|\lambda| \left| e^{-2|3\lambda^2|t} \left|[M(t, \lambda, \psi) - E_\lambda(t)]_1 \right| \right| \leq C
\]
uniformly on \([0,1] \times C^K \times B_r(0, \hat{X}_1)\). By abstract continuity arguments, the same estimate holds for all \(\psi \in B_r(0, X_1)\). This proves (2.13).

To prove (2.14), we recall Cauchy’s inequality: the derivative \(f'\) of a holomorphic function \(f: C \supseteq G \to C\) satisfying \(|f(z)| \leq C\) on a disc \(D(r, a) \subseteq G\) of radius \(r\) centered at \(a\) in the open domain \(G\) can be estimated at the point \(a\) by \(|f'(a)| \leq Cr^{-1}\). By using that
\[
|\lambda| \left| e^{-2|3\lambda^2|t} \left|[M(t, \lambda, \psi) - E_\lambda(t)]_1 \right| \right| = O(1)
\]
uniformly for \(t \in [0,1]\) and locally uniformly for \(\psi \in X\) as \(|\lambda| \to \infty\), and by applying Cauchy’s inequality, we obtain (2.14). □
Theorem 2.19. For any potential \( \psi \in X_1 \) and any sequence of complex numbers \( z_n = \pm \sqrt{n\pi/2} + O(|n|^{-1/2}) \) with \( n \in \mathbb{Z} \),

\[
\sup_{0 \leq t \leq 1} |M(t, z_n) - E_{z_n}(t)| = O(|n|^{-1/2}), \tag{2.44}
\]

\[
\sup_{0 \leq t \leq 1} |\dot{M}(t, z_n) - \dot{E}_{z_n}(t)| = O(1) \tag{2.45}
\]

as \( |n| \to \infty \). If \( z_n^2 = n\pi/2 + O(|n|^{-1/2}) \), then

\[
\sup_{0 \leq t \leq 1} |M(t, z_n) - E_{\pm \sqrt{n\pi/2}(t)}| = O(|n|^{-1/2}). \tag{2.46}
\]

The estimates (2.44) and (2.45) hold uniformly on bounded subsets of \( X_1 \) and on sequences \( (z_n)_{n \in \mathbb{Z}} \) satisfying \( |z_n + \sqrt{n\pi/2}| \leq C\sqrt{1/|n|} \) for all \( n \in \mathbb{Z} \) for some constant \( C > 0 \); estimate (2.46) holds uniformly on bounded subsets of \( X_1 \) and on sequences \( (z_n)_{n \in \mathbb{Z}} \) where \( |z_n^2 + n\pi/2| \leq C\sqrt{1/|n|} \) for all \( n \in \mathbb{Z} \) for some constant \( C > 0 \).

Proof. The estimates (2.44) and (2.45) follow directly from Theorem 2.7, because \( \Im z_n^2 = O(1) \) as \( |n| \to \infty \) by assumption, and therefore \( e^{2(\Im z_n^2)|t|} = O(1) \) uniformly in \( t \in [0, 1] \) as \( |n| \to \infty \).

To prove (2.46) we note that \( |e^z - 1| \leq |z| |e^{|z|}| \) for arbitrary \( z \in \mathbb{C} \), thus the additional restriction on \( z_n \) implies that

\[
|e^{2z_n^2|t| - e^{n\pi it}|t|} - 2|z_n^2 - n\pi/2| e^{3z_n^2}|t| = O(|n|^{-1/2}) \tag{2.47}
\]

uniformly for \( t \in [0, 1] \) as \( |n| \to \infty \). The triangle inequality implies that

\[
|M(t, z_n) - E_{\pm \sqrt{n\pi/2}(t)}| \leq |M(t, z_n) - E_{z_n}(t)| + |E_{z_n}(t) - E_{\pm \sqrt{n\pi/2}(t)}|
\]

for \( t \in [0, 1] \), and hence (2.46) follows from (2.44) and (2.47) yield.

\[\square\]

Remark 2.20. For convenience, the above results are stated for the space \( X_\tau \) with \( \tau = 1 \) (which contains the periodic space \( X \) as a subspace). It is clear that analogous results hold for an arbitrary fixed \( \tau > 0 \).

3. Spectra

We will consider three different notions of spectra associated with the spectral problem (2.1): the Dirichlet, Neumann and periodic spectrum. These spectra are the zero-sets of certain spectral functions, which are defined in terms of the entries of the fundamental solution \( M \) evaluated at time \( t = 1 \). We introduce the following notation:

\( \dot{M} := M|_{t=1}, \dot{m}_i := m_i|_{t=1}, i = 1, 2, 3, 4. \)

3.1. Dirichlet and Neumann spectrum. We define the Dirichlet domain \( A_D \) of the AKNS-system (2.3) by

\[
A_D := \{ f \in \dot{H}_C^1 : f_2(0) = 0 = f_2(1) \},
\]

where \( \dot{H}_C^1 := H^1([0, 1], \mathbb{C}) \times H^1([0, 1], \mathbb{C}) \). The Dirichlet domain \( D_D \) of the corresponding ZS-system (2.1) is then given by

\[
D_D := \{ g \in \dot{H}_C^1 : (g_1 - g_2)(0) = 0 = (g_1 - g_2)(1) \},
\]

as \( A_D \) corresponds to \( D_D \) under the transformation \( T \). For a given potential \( \psi \in X \), we say that \( \lambda \in \mathbb{C} \) lies in the Dirichlet spectrum if there exists a \( \phi \in D_D \) which solves (2.1).
Theorem 3.1. Fix \( \psi \in X \). The Dirichlet spectrum of (2.1) is the zero-set of the entire function
\[
\chi_D(\lambda, \psi) := \frac{\hat{m}_4 + \hat{m}_3 - \hat{m}_2 - \hat{m}_1}{2i} \bigg|_{(\lambda, \psi)}.
\]
(3.1)
In particular, \( \chi_D(\lambda, 0) = \sin 2\lambda^2 \).

Proof. Due to the definition of \( D_D \), a complex number \( \lambda \) lies in the Dirichlet spectrum of (2.1) if and only if the fundamental solution \( M \) maps the initial value \((1, 1)\) to a collinear vector at \( t = 1 \). That is, if and only if \( \hat{m}_1 + \hat{m}_2 = \hat{m}_3 + \hat{m}_4 \).

By Theorem 3.1, the characteristic function \( \chi_D \) satisfies
\[
\chi_D(\lambda, \psi) = \sin 2\lambda^2 + O(|\lambda|^{-1} e^{2|\Im \lambda|^2})
\]
uniformly on bounded sets in \( X \). For \( \psi \in X \), we set
\[
\sigma_D(\psi) := \{ \lambda \in \mathbb{C} : \chi_D(\lambda, \psi) = 0 \}.
\]

Lemma. If \( \lambda \in \mathbb{C} \) satisfies \( |\lambda - n\pi| \geq \pi/4 \) for all integers \( n \), then
\[
4 |\sin \lambda| > e^{2|\lambda|^2}.
\]

Proof. See [12, Appendix F].

As a direct consequence of the previous lemma we obtain:

Lemma. If \( \lambda \in \mathbb{C} \) satisfies \( |2\lambda^2 - n\pi| \geq \pi/4 \) for all integers \( n \), then
\[
4 |\sin 2\lambda^2| > e^{2|\lambda|^2}.
\]

Let us introduce the following notation:
\[
\mathbb{C}_+ := \{ z \in \mathbb{C} : \Re z > 0 \}, \quad \mathbb{C}_- := \{ z \in \mathbb{C} : \Re z < 0 \},
\]
\[
\mathbb{C}^+ := \{ z \in \mathbb{C} : \Im z > 0 \}, \quad \mathbb{C}^- := \{ z \in \mathbb{C} : \Im z < 0 \}.
\]

For \( |n| \geq 1 \), we set
\[
D_n := \{ \lambda \in \mathbb{C} : |2\lambda^2 - n\pi| < \frac{\pi}{4} \},
\]
and define
\[
D_n^1 := \begin{cases}
D_n \cup \mathbb{C}_+ & n \geq 1 \\
D_n \cup \mathbb{C}_- & n \leq -1
\end{cases}, \quad D_n^2 := \begin{cases}
D_n \cup \mathbb{C}^+ & n \geq 1 \\
D_n \cup \mathbb{C}^- & n \leq -1
\end{cases}.
\]

We also set \( D_0^1 := D_0^2 := \{ \lambda \in \mathbb{C} : |\lambda| < \sqrt{\pi/8} \} \) and define the disc \( B_N \) centered at the origin by (see Fig. 2)
\[
B_N := \{ \lambda \in \mathbb{C} : |\lambda| < \sqrt{(N + 4)\pi} \}, \quad N \geq 1.
\]

Lemma 3.2 (Counting lemma for Dirichlet eigenvalues). Let \( B \) be a bounded subset of \( X \). There exists an integer \( N \geq 1 \), such that for every \( \psi \in B \), the entire function \( \chi_D(\lambda, \psi) \) has exactly one root in each of the two discs \( D_n^i, i = 1, 2 \), for \( n \in \mathbb{Z} \) with \( |n| > N \), and exactly \( 2(2N + 1) \) roots in the disc \( B_N \) when counted with multiplicity. There are no other roots.

Proof. Outside of the set
\[
\Pi := \bigcup_{n \in \mathbb{Z}} D_n^i
\]
we have \( \frac{e^{2|\lambda|^2}}{|\sin 2\lambda^2|} < 4 \) by the previous lemma. Therefore we obtain from (3.2) that
\[
\chi_D(\lambda, \psi) = \sin 2\lambda^2 + o(e^{2|\lambda|^2}) = \chi_D(\lambda, 0)(1 + o(1))
\]
for $|\lambda| \to \infty$ with $\lambda \notin \Pi$ uniformly for $\psi \in B$. More precisely, this means that there exists an integer $N \geq 1$ such that, for all $\psi \in B$,

$$|\chi_D(\lambda, \psi) - \chi_D(\lambda, 0)| < |\chi_D(\lambda, 0)|$$

on the boundaries of all discs $D_n$ with $|n| > N$, $i = 1, 2$, and also on the boundary of $B_N$. (Note that $|\chi_D(\lambda, 0)| > \delta$ on these boundaries for some $\delta > 0$ which can be chosen independently of $|n| > N$.) Then Rouché's theorem tells us that the analytic functions $\chi_D(\cdot, \psi)$ possess the same number of roots inside these discs as $\chi_D(\cdot, 0)$. This proves the first statement, because $\chi_D(\cdot, 0) : \lambda \mapsto \sin 2\lambda^2$ has exactly one root in each $D_n$ for $|n| > N$, $i = 1, 2$, and $2(2N + 1)$ roots in the disc $B_N$.

It is now obvious that there are no other roots, because the number of roots of $\chi_D(\cdot, \psi)$ in each of the discs $B_{N+k}$, $k \geq 1$, is exactly $2(N + k + 1)$ due to the same argument as we used before. But these roots correspond to the $2(2N + 1)$ roots of $\chi_D(\cdot, \psi)$ inside $B_N \subseteq B_{N+k}$ plus the $2k$ roots inside the discs $D_l \subseteq B_{N+k}$ with $N < |l| \leq N + k$ that we have already identified earlier in the proof.

We will in the sequel denote the Dirichlet eigenvalues, i.e. the roots of $\chi_D(\psi)$, by $\mu_n^i = \mu_n^i(\psi)$, $i = 1, 2$, $n \in \mathbb{Z}$. The Dirichlet spectrum of the zero potential $\psi = 0$ consists of two bi-infinite sequences

$$\mu_n^i(0) = \operatorname{sgn}(n)\sqrt{\frac{(-1)^{i-1}|n|\pi}{2}}, \quad i = 1, 2 \tag{3.4}$$

on the real and imaginary axes in the complex plane; here sgn denotes the sign function for integers $n \in \mathbb{Z}$:

$$\operatorname{sgn}(n) := \begin{cases} 1 & n > 0 \\ 0 & n = 0 \\ -1 & n < 0. \end{cases}$$

Lemma 3.2 tells us that $\mu_n^i \in D_n^i$ for sufficiently large $|n|$.

**Proposition 3.3.** Uniformly on bounded subsets of $X$,

$$\mu_n^i(\psi) = \mu_n^i(0) + \ell_n^i, \quad i = 1, 2$$

for $2 < p \leq \infty$, where $\mu_n^i(0)$ is given by (3.4).

**Proof.** This result follows directly from Lemma 3.2 because the radius of the disc $D_n^i$ centered at $\operatorname{sgn}(n)\sqrt{\frac{(-1)^{i-1}n\pi}{2}}, i = 1, 2$, which contains the Dirichlet eigenvalue $\mu_n^i$, is of order $O(|n|^{-1/2})$ as $|n| \to \infty$.

**Corollary 3.4.** There exists a neighborhood $W$ of the zero potential in $X$ such that for every $\psi \in W$ and $n \in \mathbb{Z}$,

$$\sigma_D(\psi) \cap D_n^i = \{\mu_n^i(\psi)\}, \quad i = 1, 2.$$

We define the Neumann domain $A_N$ of the AKNS-system (2.3) by

$$A_N := \{ f \in H_c^1 : f_1(0) = 0 = f_1(1) \}.$$ 

The Neumann domain $D_N$ of the corresponding ZS-system (2.1) is then given by

$$D_N := \{ g \in H_c^1 : (h_1 + h_2)(0) = 0 = (h_1 + h_2)(1) \},$$

as $A_N$ corresponds to $D_N$ under the transformation $T$. For a given potential $\psi \in X$, we say that $\lambda \in \mathbb{C}$ lies in the Neumann spectrum if there exists a $\phi \in D_N$ which solves (2.1).

**Theorem 3.5.** Fix $\psi \in X$. The Neumann spectrum related to (2.1) is the zero-set of the entire function

$$\chi_N(\lambda, \psi) := \frac{\hat{m}_4 - \hat{m}_3 + \hat{m}_2 - \hat{m}_1}{2i}|_{(\lambda, \psi)} \tag{3.5}.$$

In particular, $\chi_N(\lambda, 0) = \chi_D(\lambda, 0) = \sin 2\lambda^2$. 
We denote the Neumann eigenvalues, i.e. the roots of uniformly on bounded subsets of $X$. For \( \psi \) \( \text{Proposition 3.7.} \) Uniformly on bounded subsets of \( \nu_m \) \( \text{Lemma 3.6.} \) As in the Dirichlet case, we obtain the following results for the Neumann spectrum.

The Neumann spectrum of the zero potential \( \psi \) \( \text{Corollary 3.8.} \) There exists a neighborhood \( \text{Theorem 3.9.} \) The discriminant \( \text{Theorem 3.10.} \) Fix \( \psi \in X. \) A complex number \( \lambda \) is a periodic eigenvalue if and only if it is a zero of the entire function

\[
\chi_P(\lambda, \psi) := \Delta^2(\lambda, \psi) - 4. \tag{3.9}
\]

Note that \( D_P \) consists of both periodic and antiperiodic functions.
\textbf{Proof.} Since \( M \) is the fundamental solution of (2.1), a complex number \( \lambda \) is a periodic eigenvalue if and only if there exists a nonzero element \( f \in D_p \) with
\[
 f(1) = M(1, \lambda)f(0) = \pm f(0),
\]
hence if and only if 1 or \(-1\) is an eigenvalue of \( M(1, \lambda) \). As \( \det M(1, \lambda) = 1 \) by Proposition 2.3, the two eigenvalues of \( M(1, \lambda) \) are either both equal to 1 or both equal to \(-1\). Therefore we either have \( \Delta(\lambda) = 2 \) or \( \Delta(\lambda) = -2 \), which is equivalent to (3.9). \( \square \)

For \( \psi \in X \), we set
\[
 \sigma_p(\psi) := \{ \lambda \in \mathbb{C} : \chi_p(\lambda, \psi) = 0 \}.
\]
For the zero potential \( \psi = 0 \), we obtain
\[
 \chi_p(\lambda, 0) = -4 \sin^2 2\lambda^2,
\]
where each root has multiplicity two, except the root \( \lambda = 0 \) which has multiplicity four. Thus the periodic spectrum of the zero potential consists of two bi-infinite sequences of double eigenvalues
\[
 |\lambda|^i \pm(0) = \text{sgn}(n) \sqrt{\frac{(1)^{i-1} |n| \pi}{2}}, \quad i = 1, 2
\]
on the real and imaginary axes in the complex plane. The \( \lambda \)-derivative of the discriminant at the zero potential is given by
\[
 \hat{\Delta}(\lambda, 0) = -8 \lambda \sin 2\lambda^2,
\]
and its roots, denoted by \( \hat{\lambda}_n^i(0) \), \( i = 1, 2, n \in \mathbb{Z} \), coincide with the periodic eigenvalues in the zero potential case:
\[
 \hat{\lambda}_n^i(0) = \text{sgn}(n) \sqrt{\frac{(1)^{i-1} |n| \pi}{2}}, \quad i = 1, 2. \tag{3.11}
\]
Note that \( \lambda = 0 \) has multiplicity three; all the other roots of \( \hat{\Delta}(\cdot, 0) \) are single roots.

\textbf{Lemma 3.11.} Fix \( \psi \in X \). As \(|\lambda| \to \infty \) with \( \lambda \notin \Pi = \bigcup_{n \in \mathbb{Z}} D_n \),
\[
 \chi_p(\lambda, \psi) = (-4 \sin^2 2\lambda^2)(1 + O(|\lambda|^{-1})), \tag{3.12}
\]
\[
 \Delta(\lambda, \psi) = (-8 \lambda \sin 2\lambda^2)(1 + O(|\lambda|^{-1})). \tag{3.13}
\]
These asymptotic estimates hold uniformly on bounded subsets of \( X \). For the zero potential they hold without the error terms.

\textbf{Proof.} By Theorem 2.7 we have \( \Delta(\lambda, \psi) = 2 \cos 2\lambda^2 + O(|\lambda|^{-1}e^{2|3\lambda^2|}) \) uniformly on bounded subsets of \( X \), and thus
\[
 \chi_p(\lambda, \psi) = (-4 \sin^2 2\lambda^2) \left[ 1 + \frac{O(|\lambda|^{-1} e^{2|3\lambda^2|}) \cos 2\lambda^2}{\sin^2 2\lambda^2} + \frac{O(|\lambda|^{-1} e^{4|3\lambda^2|})}{\sin^2 2\lambda^2} \right].
\]
For \( \lambda \notin \Pi \), we have \( 4|\sin 2\lambda^2| > e^{2|3\lambda^2|} \) and so
\[
 \frac{|\cos 2\lambda^2|}{|\sin 2\lambda^2|} \leq \frac{e^{2|3\lambda^2|}}{|\sin 2\lambda^2|} < 4, \quad \lambda \notin \Pi. \tag{3.14}
\]
The estimate (3.12) follows. Moreover, by Theorem 2.7
\[
 \hat{\Delta}(\lambda, \psi) = (-8 \lambda \sin 2\lambda^2) \left[ 1 + \frac{O(e^{2|3\lambda^2|})}{\lambda \sin 2\lambda^2} \right]
\]
uniformly on bounded subsets of \( X \) and thus (3.14) yields (3.13). \( \square \)

The following result provides an asymptotic localization of the periodic eigenvalues.

\textbf{Lemma 3.12 (Counting Lemma).} Let \( B \) be a bounded subset of \( X \). There exists an integer \( N \geq 1 \), such that for every \( \psi \in B \), the entire function \( \chi_p(\lambda, \psi) \) has exactly two roots in each of the two discs \( D_n^i \), \( i = 1, 2 \), and exactly \( 4(2N + 1) \) roots in the disc \( B_N \), when counted with multiplicity. There are no other roots.
Proof. Let $B \subseteq X$ be bounded. By Lemma 3.11,
\[ \chi_P(\lambda, \psi) = \chi_P(\lambda, 0)(1 + o(1)) \]
for $|\lambda| \to \infty$ with $\lambda \notin \Pi$ uniformly for $\psi \in B$. Hence there exists an integer $N \geq 1$ such that, for all $\psi \in B$,
\[ |\chi_P(\lambda, \psi) - \chi_P(\lambda, 0)| < |\chi_P(\lambda, 0)| \]
on the boundaries of all discs $D_n^i$ with $|n| > N$, $i = 1, 2$, and also on the boundary of $B_N$. As in the proof of Lemma 3.2, the result follows by an application of Rouché’s theorem. \( \square \)

The discs $B_N$ and $D_n^i$, $i = 1, 2$, $|n| > N$, are illustrated in Fig. 2 (see also Fig. 5).

**Figure 2.** Localization of the periodic eigenvalues according to the Counting Lemma. The first $4(2N + 1)$ periodic eigenvalues lie in the large disc $B_N$ in the center. The remaining periodic eigenvalues lie in discs centered at the corresponding double eigenvalues $\lambda_n^{\pm}(0), i = 1, 2, |n| > N$, of the zero potential. The radii of these discs shrink to zero at order $O(|n|^{-1/2})$ as $|n| \to \infty$.

**Remark 3.13.** The Counting Lemma allows us to determine the sign of the discriminant at periodic eigenvalues with sufficiently large absolute value. Indeed, recall that $|\Delta(\lambda, \psi)| = 2$ when $\lambda$ is a periodic eigenvalue, cf. Theorem 3.10. Fix $\psi \in X$ and choose $N > 0$ according to the Counting Lemma so that each of the two discs $D_n^i$, $i = 1, 2$, for $n \in \mathbb{Z}$ with $|n| > N$ contains exactly two periodic eigenvalues. We denote these eigenvalues by $\lambda_n^{\pm}(\psi)$. In fact, we can without loss of generality assume that $D_n^i$ contains exactly two periodic eigenvalues $\lambda_n^{\pm}(s\psi)$ for each potential $s\psi$ belonging to the line segment $S := \{s\psi: 0 \leq s \leq 1\}$. Since $S \subseteq X$ is compact, we can choose $N$ uniformly with respect to $S$. Let us now consider the continuous paths $\rho_n^{i\pm}[0,1] \to \mathbb{C}, s \mapsto \lambda_n^{i\pm}(s\psi)$, $i = 1, 2$. Since $\Delta$ is continuous and $\Delta(\lambda_n^{i\pm}(s\psi), s\psi) \in \{-2, 2\}$ for $s \in [0, 1]$, we conclude that either
\[ \Delta(\rho_n^{i\pm}(s), s\psi) \equiv 2 \text{ on } [0, 1] \quad \text{ or } \quad \Delta(\rho_n^{i\pm}(s), s\psi) \equiv -2 \text{ on } [0, 1]. \]
Thus
\[ \Delta(\lambda_n^{i\pm}(\psi), \psi) = \Delta(\lambda_n^{i\pm}(0), 0) = 2 \cos n\pi = 2(-1)^n \quad \text{ for } |n| > N. \]

Lemma 3.11 yields a counting lemma also for the critical points of $\Delta$:

**Lemma 3.14.** Let $B$ be a bounded subset of $X$. There exists an integer $N \geq 1$, such that for every $\psi \in B$, the entire function $\Delta(\lambda, \psi)$ has exactly one root in each of the two discs $D_n^i$, $i = 1, 2$ for all $|n| > N$, and exactly $4N + 3$ roots in the disc $B_N$, when counted with multiplicity. There are no other roots.
Lemma 3.17. Anti-discriminant at a Dirichlet or a Neumann eigenvalue. We say that a potential \( \psi \) are relevant for the defocusing real \( \nu \equiv \nu(\psi) \) of the ZS-operator is of real type potentials that are of real type iff all coefficients of the corresponding AKNS system are real-valued. The subspace of \( X \) of all real type potentials will be denoted by \( \{ \psi \in X \mid \nu(\psi) \in \mathbb{R} \} \).

Proof. This result follows directly from Lemma 3.2 since the radius of the disc \( D^i_n \) centered at \( \operatorname{sgn}(n)\sqrt{(-1)^{r-1}n\pi/2} \), \( i = 1, 2 \), which contains the two periodic eigenvalues \( \lambda_n^{i,\pm}(\psi) \) and the critical point \( \lambda_n^i(\psi) \) for sufficiently large \( |n| \) uniformly on bounded subsets of \( X \), shrinks to zero at order \( O(|n|^{-1/2}) \) as \( |n| \to \infty \).

Corollary 3.16. There exists a neighborhood \( W \) of the zero potential in \( X \) such that for every \( \psi \in W \) and \( n \in \mathbb{Z} \) the following assertions hold:

1. \( \sigma_p(\psi) \cap D^i_n = \{ \lambda_n^{i,-}(\psi), \lambda_n^{i,+}(\psi) \} \), \( i = 1, 2 \);
2. \( \Delta(\lambda_n^{i,\pm}(\psi), \psi) = 2(-1)^n \), \( i = 1, 2 \);
3. \( \{ \lambda \in \mathbb{C} : \Delta(\lambda, \psi) = 0 \} \cap D^i_n = \{ \lambda_n^i(\psi) \} \), \( i = 1, 2 \).

We close this section with an identity that relates the values of the discriminant and the anti-discriminant at a Dirichlet or a Neumann eigenvalue.

Lemma 3.17. If \( \psi \in X \) and \( \mu_n^i := \mu_n^i(\psi) \) is any Dirichlet eigenvalue of \( \psi \), then
\[
\Delta^2(\mu_n^i, \psi) = 4 = \delta^2(\mu_n^i, \psi).
\]
This identity holds also at any Neumann eigenvalue \( \nu_n^i = \nu_n^i(\psi) \) of \( \psi \).

Proof. We recall that \( \dot{m}_1 \dot{m}_4 - \dot{m}_2 \dot{m}_3 = 1 \) by Proposition 2.3. Therefore,
\[
\Delta^2 - 4 = (\dot{m}_1 + \dot{m}_4)^2 - 4
= (\dot{m}_1 + \dot{m}_4)^2 - 4(\dot{m}_1 \dot{m}_4 - \dot{m}_2 \dot{m}_3)
= (\dot{m}_1 - \dot{m}_4)^2 + 4\dot{m}_2 \dot{m}_3.
\]

Let \( \mu_n^i \) be a Dirichlet eigenvalue of \( \psi \in X \), \( i = 1, 2 \). Then \( \mu_n^i \) is a root of \( \dot{m}_4 - \dot{m}_2 \dot{m}_3 = \dot{m}_1 \), that is
\[
(\dot{m}_1 - \dot{m}_4)|_{(\mu_n^i, \psi)} = (\dot{m}_3 - \dot{m}_2)|_{(\mu_n^i, \psi)}.
\]
Therefore,
\[
\Delta^2(\mu_n^i, \psi) = 4 = (\dot{m}_2(\mu_n^i, \psi) + \dot{m}_3(\mu_n^i, \psi))^2 = \delta^2(\mu_n^i, \psi).
\]
For Neumann eigenvalues \( \nu_n^i(\psi) \), \( i = 1, 2 \), we have
\[
(\dot{m}_1 - \dot{m}_4)|_{(\nu_n^i, \psi)} = (\dot{m}_2 - \dot{m}_3)|_{(\nu_n^i, \psi)},
\]
which again yields the desired identity.

4. Potentials of Real and Imaginary Type

For potentials \( \psi \in X \), we define
\[
\psi^* := P \tilde{\psi} := (\tilde{\psi}^2, \tilde{\psi}^1, \tilde{\psi}^4, \tilde{\psi}^3),
\]
where
\[
P := \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
We say that a potential \( \psi \) of the ZS-operator is of real type if \( \psi^* = \psi \). In this case, \( \psi^2 = \tilde{\psi}^1 \) and \( \psi^4 = \tilde{\psi}^3 \). That is, \( \psi = (q_0 + ip_0, q_0 - ip_0, q_1 + ip_1, q_1 - ip_1) \) for some real-valued functions \( \{q_j, p_j\}_{j=0}^1 \). Hence a potential is of real type iff all coefficients of the corresponding AKNS system are real-valued. The subspace of \( X \) of all real type potentials will be denoted by
\[
X_R := \{ \psi \in X \mid \psi^* = \psi \}.
\]
Note that this is a real subspace of \( X \), not a complex one; it consists of those potentials that are relevant for the defocusing NLS.
We can write the Zakharov-Shabat $t$-part \((1.3)\) as
\[
\begin{pmatrix}
-i\sigma^3\partial_t + 2k^2I + \left(\begin{array}{cc}
\psi^1\psi^2 & i(2\lambda\psi^1 + i\psi^3) \\
-2i\lambda\psi^2 & -\psi^4
\end{array}\right)
\end{pmatrix}\phi = 0,
\]
or, in other words,
\[
L(\psi)\phi = R(\lambda, \psi)\phi \tag{4.1}
\]
with
\[
L(\psi) \equiv -i\sigma_3\partial_t + \begin{pmatrix}
\psi^1\psi^2 & -\psi^3 \\
-\psi^4 & \psi^1\psi^2
\end{pmatrix}, \quad R(\lambda, \psi) \equiv -2\lambda^2I - 2i\begin{pmatrix}
0 & \psi^1 \\
-\psi^2 & 0
\end{pmatrix} \tag{4.2}
\]
For \(v = (v_1, v_2)\) and \(w = (w_1, w_2)\), let
\[
\langle v, w \rangle = \int_0^1 (v_1\bar{w}_1 + v_2\bar{w}_2) \, dt.
\]
If the eigenfunctions \(v, w\) lie in the periodic domain \(D_P\), we can integrate by parts without boundary terms and find that
\[
\langle w, L(\psi)v \rangle = \langle L(\psi^*)w, v \rangle.
\]
Therefore, if the potential \(\psi\) is of real type and \(v\) is a periodic eigenfunction with eigenvalue \(\lambda\),
\[
\langle R(\lambda, \psi)v, v \rangle = \langle L(\psi)v, v \rangle = \langle v, L(\psi)v \rangle = \langle v, R(\lambda, \psi)v \rangle
\]
and thus we find that
\[
\Re \lambda = 0 \quad \text{or} \quad \Im \lambda = \frac{\Im \left( \int_0^1 \psi^1v\bar{w} \, dt \right)}{\langle v, v \rangle}.
\]
According to the Counting Lemma, the periodic eigenvalues of type \(\lambda^2_n(\psi)\) for arbitrary \(\psi \in X\) necessarily possess non-vanishing imaginary parts for sufficiently large \(|n|\). In analogy with the \(x\)-part \((1.2)\), one might expect that \(\Im \lambda^1\pm(\psi) = 0\) for real type potentials. However, we will see in Section 5 that this is not the case: there are single exponential potentials of real type for which some \(\lambda^1\pm\) are nonreal.

**Lemma 4.1.** Let \(\psi\) be of real type and let \(\lambda \in \mathbb{R}\). Then \(m_4 = \bar{m}_1\) and \(m_3 = \bar{m}_2\). In particular, \(\Delta\) is real-valued on \(\mathbb{R} \times X_{\mathbb{R}}\). Moreover, if a solution \(v\) of \(L(\psi)v = R(\lambda, \psi)v\) is real in AKNS-coordinates, then \(v = \sigma_1\bar{v}\).

**Proof.** Since \(\psi = \psi^*\), the AKNS coordinates \((p_i, q_i), i = 0, 1\), are real. If in addition \(\lambda \in \mathbb{R}\), the system \((2.3)\) has real coefficients, so its fundamental solution \(K\) is real-valued. The relation \(M = TK\bar{T}^{-1}\), cf. \((2.11)\), then implies that \(m_4 = \bar{m}_1\) and \(m_3 = \bar{m}_2\). To prove the second claim, we note that \(\bar{T} = \sigma_1T\) and hence
\[
\bar{M} = TK\bar{T}^{-1} = \sigma_1TK\bar{T}^{-1}\sigma_1 = \sigma_1M\sigma_1.
\]
If \(v\) is real in AKNS coordinates, it has real initial data \(v_0\) and \(v = MTv_0\). Therefore \(\bar{v} = MT\bar{v}_0 = \sigma_1MTv_0 = \sigma_1v\).

We say that a potential \(\psi \in X\) is of imaginary type if \(\psi^* = -\psi\). The subspace
\[
X_I := \{ \psi \in X: \psi^* = -\psi \}
\]
of potentials of imaginary type is relevant for the focusing NLS.

**Proposition 4.2.** For \(\psi \in X_{\mathbb{R}}\) the fundamental solution \(M\) satisfies
\[
M(t, \lambda, \psi) = \sigma_1\bar{M}(t, \lambda, \psi)\sigma_1, \quad \lambda \in \mathbb{C}, \ t \in [0, 1]; \tag{4.3}
\]
if \(\psi \in X_I\) then \(M\) satisfies
\[
M(t, \lambda, \psi) = \sigma_1\sigma_3\bar{M}(t, \lambda, \psi)\sigma_3\sigma_1, \quad \lambda \in \mathbb{C}, \ t \in [0, 1]. \tag{4.4}
\]
In particular,
\[
\Delta(\bar{\lambda}, \psi) = \bar{\Delta}(\lambda, \psi) \quad \text{and} \quad \bar{\Delta}(\bar{\lambda}, \psi) = \bar{\Delta}(\lambda, \psi) \tag{4.5}
\]
for all $\psi \in X_R \cup X_T$ and $\lambda \in \mathbb{C}$, so that $\Delta$ and $\dot{\Delta}$ are real-valued on $\mathbb{R} \times (X_R \cup X_T)$.

**Proof.** Let us first assume that $\psi \in X_R$ and $\lambda \in \mathbb{C}$. Then a computation using (4.2) shows that

$$L(\psi^*)v = R(\lambda, \psi)v \quad \iff \quad L(\psi^*)v = R(\lambda, \psi)v,$$

where $v := \sigma_1 v = (\bar{v}_2, \bar{v}_1)$. The symmetry (4.3) follows from uniqueness of the solution of (4.1) and the initial condition $M(0, \lambda, \psi) = I$. Evaluation of (4.3) at $t = 1$ gives (4.5). This finishes the proof for the case of real type potentials. If $\psi \in X_T$, we instead have

$$L(\psi)v = R(\lambda, \psi)v \quad \iff \quad L(\psi)v = R(\lambda, \psi)v,$$

where $\dot{v} := \sigma_1 \sigma_3 \bar{v} = (-\bar{v}_2, \bar{v}_1)$, which leads to (4.4) and (4.5). \hfill $\square$

**Corollary 4.3.** There exists a neighborhood $W$ of the zero potential in $X$ such that for each $\psi \in W \cap (X_R \cup X_T)$ and each $n \in \mathbb{Z}$,

$$\{ \lambda \in \mathbb{C} : \Delta(\lambda, \psi) = 0 \} \cap \mathbb{D}_n = \{ \hat{\lambda}_n^1(\psi) \} \quad \text{and} \quad \hat{\lambda}_n^2(\psi) \in \mathbb{R}.$$

**Proof.** We already know from Corollary 3.16 that there exists a neighborhood $W$ of the zero potential such that, for all general potentials $\psi \in W$ and all $n \in \mathbb{Z}$,

$$\{ \lambda \in \mathbb{C} : \Delta(\lambda, \psi) = 0 \} \cap \mathbb{D}_n = \{ \hat{\lambda}_n^1(\psi) \}.$$

Due to the symmetry (4.5), we infer that, for all potentials $\psi \in W \cap (X_R \cup X_T)$ and $n \in \mathbb{Z}$,

$$0 = \Delta(\hat{\lambda}_n^1(\psi), \psi) = \Delta(\hat{\lambda}_n^2(\psi), \psi) = \Delta(\bar{\lambda}_n^2(\psi), \psi).$$

Since $\hat{\lambda}_n^1(\psi)$ is the only root of $\Delta(\cdot, \psi)$ in $\mathbb{D}_n^1$, we conclude that $\hat{\lambda}_n^1(\psi)$ is real. \hfill $\square$

**Theorem 4.4.** There exists a neighborhood $W$ of the zero potential in $X$ and a sequence of nondegenerate rectangles

$$R_n^{\varepsilon, \delta} := \{ \lambda \in \mathbb{C} : |\Re \lambda - \hat{\lambda}_n^1(\psi)| < \delta_n, |\Im \lambda| < \varepsilon_n \}$$

with $\varepsilon, \delta \in \varepsilon_n^{p,1/2}$, $2 < p < \infty$, such that for all $\psi \in W \cap (X_R \cup X_T)$ and all $n \in \mathbb{Z} \setminus \{0\}$,

$$\{ \lambda \in \mathbb{C} : \Delta(\lambda, \psi) \in \mathbb{R} \} \cap R_n^{\varepsilon, \delta} = \gamma_n^1(\psi) \cup (R_n^{\varepsilon, \delta} \cap \mathbb{R}).$$

The sets $\gamma_n(\psi)$ are analytic arcs transversal to the real axis, which cross the real line in the critical points $\hat{\lambda}_n^1(\psi)$ of $\Delta(\cdot, \psi)$. Moreover, these arcs are symmetric under reflection in the real axis and the orthogonal projection of $\gamma_n(\psi)$ to the imaginary axis is a real analytic diffeomorphism onto its image.

We refer to Fig. 4.4 for an illustration of the analytic arc $\gamma_n(\psi)$ within the rectangle $R_n^{\varepsilon, \delta}$ centered at the critical point $\hat{\lambda}_n^1(0)$ of the discriminant $\Delta(\cdot, 0)$. The $p$-based spaces $\ell^p_R$ are defined below in (4.11).

**Remark 4.5.** For potentials $\psi$ in $X_R \cup X_T$ near 0, Theorem 4.4 improves the asymptotic localization of the critical points $\hat{\lambda}_n^1(\psi)$ established in Lemma 3.14 cf. Remark 4.8.

The proof of Theorem 4.4 is based on an application of the implicit function theorem for real analytic mappings in an infinite dimensional setting. This level of generality is necessary in order to treat the arcs $\gamma_n$ in a uniform way. Let us first briefly discuss the strategy of the proof. Writing $\lambda = x + iy$ with $x, y \in \mathbb{R}$, we split $\Delta(\lambda; \psi) = \Delta(x, y; \psi)$ into its real and imaginary parts and write $\Delta = \Delta_1 + i\Delta_2$ with

$$\Delta_1(x, y; \psi) := \Re(\Delta(\lambda; \psi)), \quad \Delta_2(x, y; \psi) := \Im(\Delta(\lambda; \psi)).$$

The problem is then transformed into the study of the zero level set of $\Delta_2(x, y; \psi) = \Delta_2(x, y; \psi)$. For the zero potential, $\Delta(\lambda, 0)$ is given by (3.8); thus

$$\Delta_1(x, y; 0) = 2 \cos(2(x^2 - y^2)) \cosh(4xy), \quad \Delta_2(x, y; 0) = -2 \sin(2(x^2 - y^2)) \sinh(4xy).$$

We want to use the implicit function theorem to study the zero-set of $\Delta_2$ for $\lambda$ close to $\lambda_n^{1, \pm}(0)$ and for $\psi$ close to the zero potential. Let $\hat{\lambda}_n^1 := \hat{\lambda}_n^1(0)$ denote the critical points of $\Delta(\cdot, 0)$ given
in (3.11) and recall from (3.11) that \( \lambda_{n}^{\pm}(0) = \hat{\lambda}_{n}^{1} \). We clearly have \( \frac{\partial \Delta_{2}}{\partial x}(\hat{\lambda}_{n}^{1}, 0; 0) = 0 \) for all \( n \in \mathbb{Z} \) (“\( \mathbb{R} \) bifurcates into the arcs \( \gamma_{n} \)”). Therefore we introduce a function \( \hat{F} \) with the same zeros on \( \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \times (X_{\mathcal{R}} \cup X_{\mathcal{I}}) \) as \( \Delta_{2} \) by

\[
\mathbb{R} \times (\mathbb{R} \setminus \{0\}) \times (X_{\mathcal{R}} \cup X_{\mathcal{I}}) \rightarrow \mathbb{R},
\]

\[
(x, y, \psi) \mapsto \hat{F}(x, y; \psi) := \frac{\Delta_{2}(x, y; \psi)}{y}.
\]

We observe that \( \hat{F} \) has a real analytic extension

\[
F: \mathbb{R} \times \mathbb{R} \times (X_{\mathcal{R}} \cup X_{\mathcal{I}}) \rightarrow \mathbb{R}.
\]

Indeed, since \( \Delta \) is analytic on \( \mathbb{C} \times X \), \( \Delta_{2} \) is real analytic on \( \mathbb{R} \times \mathbb{R} \times X \), and therefore \( \hat{F} \) is real analytic on \( \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \times (X_{\mathcal{R}} \cup X_{\mathcal{I}}) \). By Proposition 4.2, \( \Delta_{2} \) is odd in the \( y \)-argument for each \( \psi \in X_{\mathcal{R}} \cup X_{\mathcal{I}} \); hence the constant term in the Taylor expansion of \( \Delta_{2} \) at \( y = 0 \) vanishes. Therefore \( \Delta_{2}(x, y; \psi)/y \) admits a Taylor series representation at \( y = 0 \), which converges absolutely locally around \( y = 0 \) to an analytic extension \( F \).

For \( \psi \in X_{\mathcal{R}} \cup X_{\mathcal{I}} \) and real sequences \( u = (u_{n})_{n \in \mathbb{Z}} \) and \( v = (v_{n})_{n \in \mathbb{Z}} \), we define the map \( \mathcal{F} = (\mathcal{F}_{n})_{n \in \mathbb{Z}} \) by

\[
\mathcal{F}_{n}(u, v; \psi) := F(\hat{\lambda}_{n}^{1} + u_{n}, v_{n}; \psi).
\]

For the zero potential and the zero sequence, both denoted by \( 0 \), we calculate \( \mathcal{F}(0, 0; 0) = -2 \sin(2(\hat{\lambda}_{n}^{1})^{2}) \frac{4\hat{\lambda}_{n}^{1}}{n} \sin |n| \pi )_{n \in \mathbb{Z}} = 0 \).

In order to determine \( \frac{\partial \mathcal{F}}{\partial u} \) at the origin \( (0, 0, 0) \), we first observe that \( \frac{\partial \mathcal{F}}{\partial u} \) has diagonal form because \( \mathcal{F}_{j} \) is independent of \( u_{n} \) for \( j \in \mathbb{Z} \) with \( j \neq n \). On the diagonal, we obtain

\[
\frac{\partial \mathcal{F}_{n}}{\partial u_{n}}(u, v; 0) = \frac{\partial}{\partial u_{n}} \left[ \frac{\Delta_{2}(\hat{\lambda}_{n}^{1} + u_{n}, v_{n}; 0)}{v_{n}} \right]
\]

\[
= \frac{2}{v_{n}} \left\{ 4(\hat{\lambda}_{n}^{1} + u_{n}) \cos[2((\hat{\lambda}_{n}^{1} + u_{n})^{2} - v_{n}^{2})] \sinh[4(\hat{\lambda}_{n}^{1} + u_{n})v_{n}]
\]

\[
+ \sin[2((\hat{\lambda}_{n}^{1} + u_{n})^{2} - v_{n}^{2})] 4v_{n} \cosh[4(\hat{\lambda}_{n}^{1} + u_{n})v_{n}] \right\} ,
\]

thus

\[
\frac{\partial \mathcal{F}_{n}}{\partial u_{n}}(0, 0; 0) = -32(\hat{\lambda}_{n}^{1})^{2} \cos[2((\hat{\lambda}_{n}^{1})^{2})],
\]

and therefore

\[
\frac{\partial \mathcal{F}}{\partial u}(0, 0; 0) = (16\pi |n| \text{ diag}((-1)^{n+1}))_{n \in \mathbb{Z}}.
\]

Consequently, \( \frac{\partial \mathcal{F}}{\partial u}(0, 0; 0) \) is at least formally bijective in a set-theoretic and algebraic sense, for example as a mapping from the space of real sequences \( \{u = (u_{n})_{n \in \mathbb{Z}}: \mathbb{Z} \rightarrow \mathbb{R} | u_{0} = 0\} \) to itself. In order to give these formal considerations a rigorous justification, we need to consider appropriate subspaces of sequences equipped with appropriate topologies.

The proof of Theorem (4.4) uses techniques from the theory of analytic maps between complex Banach spaces. We therefore review some aspects of this theory. Let \( (E, \| \cdot \|_{E}), (F, \| \cdot \|_{F}) \) be complex Banach spaces. Furthermore, we denote by \( \mathcal{L}(E, F) \) the Banach space of bounded (complex) linear operators \( E \rightarrow F \) endowed with the operator norm \( \| \cdot \|_{\mathcal{L}(E, F)} \), where

\[
\|L\|_{\mathcal{L}(E, F)} = \sup_{0 \neq h \in E} \frac{\|Lh\|_{F}}{\|h\|_{E}} < \infty \text{ for } L \in \mathcal{L}(E, F).
\]

In the special case \( F = \mathbb{C} \), we denote by \( E' = \mathcal{L}(E, \mathbb{C}) \) the topological dual space of \( E \). Let \( O \subseteq E \) be an open subset. A map \( f: O \rightarrow F \) is called analytic on \( O \), if it is differentiable at every \( u \in O \), i.e., if for all \( u \in O \) there exists a bounded linear operator \( A(u) \in \mathcal{L}(E, F) \) such that

\[
\lim_{\|h\|_{E} \rightarrow 0} \frac{\|f(u + h) - f(u) - A(u)h\|_{F}}{\|h\|_{E}} = 0.
\]
In this case we call $A(u)$ the derivative of $f$ at $u$ and write $df(u)$ for $A(u)$. In the special case $E = F = \mathbb{C}$, we simply write $df(u) = f'(u) \in \mathbb{C} \cong \mathbb{C}'$. We call $f$ weakly analytic on $O$ if for every $u \in O$, $h \in E$ and $L \in F'$ the function

$$z \mapsto Lf(u + zh)$$

is analytic in some neighborhood of zero.

**Lemma 4.6.** [17, Theorem A.3] Let $E$ and $F$ be complex Banach spaces, let $O \subseteq E$ be open and let $f : O \to F$ be a mapping. The following statements are equivalent.

1. $f$ is analytic in $O$.
2. $f$ is weakly analytic and locally bounded on $O$.
3. $f$ is infinitely many differentiable on $O$ and for all $u \in O$ the Taylor series of $f$ at $u$ converges to $f$ uniformly in a neighborhood of $u$.

Let us consider the following $\ell^p$-based spaces of (bi-infinite) sequences, which we will use to prove Theorem 4.4. For $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, we introduce the linear spaces

$$\ell^{p,s}_{\mathbb{R}} := \left\{ u = (u_n)_{n \in \mathbb{Z}} \mid (1 + n^2)^{\frac{s}{2}} |u_n|^p \in \ell^p_{\mathbb{R}} \right\}$$

(4.11)

endowed with the norms

$$|u|_{p,s} := \left( \sum_{n=-\infty}^{\infty} (1 + n^2)^{\frac{s}{2}} |u_n|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty; \quad |u|_{\infty,s} := \sup_{n \in \mathbb{Z}} \{(1 + n^2)^{\frac{s}{2}} |u_n|\}.$$ 

One easily checks that these spaces are Banach spaces. Furthermore, defining

$$\Lambda_n := (1 + n^2)^{\frac{s}{2}}, \quad n \in \mathbb{Z},$$

the map

$$\Lambda^r : \ell^{p,s}_{\mathbb{R}} \to \ell^{p,s-r}_{\mathbb{R}}, \quad u_n \mapsto \Lambda_n^r u_n,$$

is an isometric isomorphism for each $r \in \mathbb{R}$. In particular $\Lambda^s$ maps $\ell^{p,s}_{\mathbb{R}}$ isometrically onto $\ell^p_{\mathbb{R}}$. For $s \in \mathbb{R}$ and $1 < p < \infty$, the topological dual of $\ell^{p,s}_{\mathbb{R}}$ is isometrically isomorphic to $\ell^{q,-s}_{\mathbb{R}}$, i.e., $(\ell^{p,s}_{\mathbb{R}})' \cong \ell^{q,-s}_{\mathbb{R}}$, where $q$ is the Hölder conjugate of $p$ defined by $1/p + 1/q = 1$. The isomorphism is given by the dual pairing

$$\langle \cdot, \cdot \rangle_{p,s,q,-s} : \ell^{p,s}_{\mathbb{R}} \times \ell^{q,-s}_{\mathbb{R}} \to \mathbb{R}, \quad \langle u, v \rangle_{p,s,q,-s} := \sum_{n=-\infty}^{\infty} u_n v_n,$$

and can be deduced directly from the well-known $\ell^p$-$\ell^q$-duality. Henceforth, we will identify the dual of $\ell^{p,s}_{\mathbb{R}}$ with $\ell^{q,-s}_{\mathbb{R}}$ by means of $\langle \cdot, \cdot \rangle_{p,s,q,-s}$. In particular, $\ell^{p,s}_{\mathbb{R}}$ is a reflexive Banach space for $1 < p < \infty$. We will also consider the corresponding complex versions $\ell^{p,s}_{\mathbb{C}}$ based on $\ell^p_{\mathbb{C}}$. All the above properties hold true for $\ell^{p,s}_{\mathbb{C}}$ in an analogous way.

We will also use the closed subspaces

$$\tilde{\ell}^{p,s}_{\mathbb{R}} := \{ u \in \ell^{p,s}_{\mathbb{R}} : u_0 = 0 \}, \quad \tilde{\ell}^{p,s}_{\mathbb{C}} := \{ u \in \ell^{p,s}_{\mathbb{C}} : u_0 = 0 \},$$

which inherit reflexivity for $1 < p < \infty$:

\[(\tilde{\ell}^{p,s}_{\mathbb{R}})' \cong \tilde{\ell}^{q,-s}_{\mathbb{R}}, \quad (\tilde{\ell}^{p,s}_{\mathbb{C}})' \cong \tilde{\ell}^{q,-s}_{\mathbb{C}}.\]

The linear operator $T$ defined by

$$T_n u_n \mapsto |n| u_n, \quad T : \ell^{p,s}_{\mathbb{C}} \to \ell^{p,s-1}_{\mathbb{C}}$$

is a topological isomorphism. Likewise, $T^r : u_n \mapsto T_n u_n = |n|^r u_n$ is an isomorphism $\ell^{p,s}_{\mathbb{C}} \to \ell^{p,s-r}_{\mathbb{C}}$ for real $r$.

**Lemma 4.7.** Let $\psi \in X_{\mathbb{R}} \cup X_{\mathbb{C}}$, let $\lambda = x + iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$, and let $\Delta_1$ and $\Delta_2$ denote the real and imaginary parts of $\Delta = \Delta_1 + i\Delta_2$. 

special case. The proof relies on the asymptotic estimate (4.12). First note that (part of (4.15).

\[
\partial_y \Delta_2(x, y; \psi) = -8(x \sin[2(x^2 - y^2)] \cosh[4xy] - y \cos[2(x^2 - y^2)] \sinh[4xy])
- 4i\Gamma(\psi) \cos[2(x^2 - y^2)] \cosh[4xy] + O\left(\frac{e^{4xy}}{\sqrt{x^2 + y^2}}\right)
\]
(4.12)

uniformly for \(\psi\) in bounded subsets of \(X_R \cup X_T\), where

\[
\hat{\Gamma}(\psi) := \Gamma(1, \psi) = \int_0^1 (\psi^4 \psi^4 - \psi^2 \psi^3) \, dt.
\]

(2) For each \(p > 2\), the mapping

\[
(x_n, y_n) \mapsto \partial_y \Delta_2(\lambda_n t + x_n, y_n; \psi)
\]
(4.13)
maps bounded sets in \(\ell_p^{1/2} \times \ell_p^{1/2}\) to bounded sets in \(\ell_p^{p-1/2}\), where the bound in \(\ell_p^{p-1/2}\)
can be chosen uniformly for \(\psi\) in bounded subsets of \(X_R \cup X_T\). The analogous assertion
remains true when considering (4.13) as a map \(\ell_p^{1/2} \times \ell_p^{1/2} \to \ell_p^{p-1/2}\).

Proof. In order to prove part (1), we recall from Theorem 2.7 that

\[
\hat{M}(\lambda, \psi) = e^{-2i\lambda^2 \sigma_3} + O\left(\frac{e^{2|\lambda|^2}}{|\lambda|^2}\right).
\]

In the proof of Theorem 2.7 we gained additional information on the remainder term: it is of the form

\[
\frac{Z_1(\psi)}{\lambda} e^{-2i\lambda^2 \sigma_3} + \frac{W_1(\psi)}{\lambda} e^{2i\lambda^2 \sigma_3} + O\left(\frac{e^{2|\lambda|^2}}{|\lambda|^2}\right),
\]

where the diagonal part of the \(1/\lambda\)-terms is given by

\[
\frac{1}{2} \hat{\Gamma}(\psi) \sigma_3 e^{-2i\lambda^2 \sigma_3}.
\]

Thus, for any potential \(\psi \in X\),

\[
\Delta(\lambda, \psi) = 2 \cos 2\lambda^2 - \frac{i\Gamma}{\lambda} \sin 2\lambda^2 + O\left(|\lambda|^{-2} e^{2|\lambda|^2}\right),
\]
\[
\hat{\Delta}(\lambda, \psi) = -8 \lambda \sin 2\lambda^2 - 4i\Gamma(\psi) \cos 2\lambda^2 + O\left(|\lambda|^{-1} e^{2|\lambda|^2}\right).
\]

(4.14)
(4.15)

Since \(\hat{\Gamma}(\psi) \in i\mathbb{R}\) for \(\psi \in X_R \cup X_T\), the asymptotic estimate (4.12) follows by taking the real
part of (4.15).

We prove part (2) of the lemma in the complex setting; this includes the real setting as a special case. The proof relies on the asymptotic estimate (4.12). First note that \((x_n, y_n) \mapsto (\lambda_n + x_n) y_n\) clearly maps bounded sets in \(\ell_p^{1/2} \times \ell_p^{1/2}\) to bounded sets in \(\ell_p^{p}\). Thus the remainder term in (4.12) has the asserted properties: the set of image sequences \((e^{4(\lambda_n t + x_n) y_n})_{n \in \mathbb{Z}}\)
is bounded in \(\ell_p^{\infty} \subseteq \ell_p^{p-1/2}\), \(p > 2\), for bounded sets in \(\ell_p^{1/2} \times \ell_p^{1/2}\) and therefore

\[
(x_n, y_n) \mapsto \frac{e^{4(\lambda_n + x_n) y_n}}{\sqrt{\lambda_n^2 + x_n^2 + y_n^2}}
\]

maps bounded sets in \(\ell_p^{1/2} \times \ell_p^{1/2}\) to bounded sets in \(\ell_p^{p}\), hence in particular to bounded (even compact) sets in \(\ell_p^{p-1/2}\). Next we show that the second term in (4.12) maps bounded sets in \(\ell_p^{1/2}\) to bounded sets in \(\ell_p^{\infty} \subseteq \ell_p^{p-1/2}\). The map \((x_n, y_n) \mapsto \cosh[4(\lambda_n + x_n) y_n]\) maps bounded sets in \(\ell_p^{1/2} \times \ell_p^{1/2}\) to bounded sets in \(\ell_p^{\infty}\). Using that \(\sin[2(\lambda_n t)] = 0\) and \(\cos[2(\lambda_n t)] = (-1)^n\),
we write

\[
|\cosh[4(\lambda_n^2 + x_n^2 - y_n^2)]| = |\sin[4\lambda_n x_n] \cos[2(x_n^2 - y_n^2)] - \sin[4\lambda_n x_n] \sin[2(x_n^2 - y_n^2)]|
\]
and see that \((x_n, y_n) \mapsto \cos[2((\lambda_n + x_n)^2 - y_n^2)]\) maps bounded sets in \(ℓ^p_{\mathbb{C}} \times ℓ^q_{\mathbb{C}}\) to bounded sets in \(ℓ^p_{\mathbb{C}}\). Concerning the first term in (4.12), we keep in mind that 
\(|\sin \lambda_n y_n| \leq L|\lambda_n y_n|\), where the Lipschitz constant \(L\) depends only
on 
\(|(\lambda_n y_n)_{n \in \mathbb{Z}}|\), and can therefore be chosen uniformly on bounded sets in \(ℓ^p_{\mathbb{C}}\); analogously
\(|\sin \lambda_n y_n| \leq L|\lambda_n y_n|\).

Therefore,
\[
(x_n, y_n) \mapsto (\lambda_n + x_n) \sin[2((\lambda_n + x_n)^2 - y_n^2)] \cosh[4(\lambda_n + x_n)y_n]
\]
maps bounded sets in \(ℓ^p_{\mathbb{C}} \times ℓ^q_{\mathbb{C}}\) to bounded sets in \(ℓ^p_{\mathbb{C}}\); similarly,
\[
(x_n, y_n) \mapsto y_n \cos[2((\lambda_n + x_n)^2 - y_n^2)] \sin[4(\lambda_n + x_n)y_n]
\]
maps bounded sets in \(ℓ^p_{\mathbb{C}} \times ℓ^q_{\mathbb{C}}\) to bounded sets in \(ℓ^p_{\mathbb{C}}\).

**Proof of Theorem 4.4.** The real analytic extension \(F : \mathbb{R} \times \mathbb{R} \times (X_R \cup X_I) \to \mathbb{R}\) of \(\tilde{F}\), cf. (4.7) and (4.8), can be written as
\[
F(x, y; \psi) = \int_0^1 (\partial_y \Delta_2)(x, sy; \psi) \, ds.
\]
Indeed,
\[
\int_0^1 (\partial_y \Delta_2)(x, y; s) \, ds = \int_0^y (\partial_y \Delta_2)(x, y'; \psi) \, dy' = \Delta_2(x, y; \psi) - \Delta_2(x, 0; \psi),
\]
where \(\Delta_2(x, 0; \psi) = 0\) because, by Proposition 4.5 \(\Delta\) is real-valued on \(\mathbb{R} \times (X_R \cup X_I)\). We obtain from (4.16) that
\[
|F(x, y; \psi)| \leq \max_{s \in [0, 1]} |(\partial_y \Delta_2)(x, sy; \psi)|.
\]

In view of Lemma 4.7 and (4.17), the operator \(F\) given by (4.9) defines a well-defined map
\[
F : ℓ^p_{\mathbb{R}} \times ℓ^q_{\mathbb{R}} \times (X_R \cup X_I) \supseteq B^p_{\mathbb{R}} \times B^q_{\mathbb{R}} \times (X_R \cup X_I) \to ℓ^p_{\mathbb{R}} \times ℓ^q_{\mathbb{R}}, \quad p > 2,
\]
where \(B^p_{\mathbb{R}} \equiv B^{p,1/2}_{\mathbb{R}}(ℓ^p_{\mathbb{R}})\) denotes the unit ball in \(ℓ^p_{\mathbb{R}}\) centered at 0 and the space \(ℓ^p_{\mathbb{R}} \times ℓ^q_{\mathbb{R}} \times (X_R \cup X_I)\) is endowed with the usual product topology. Let us consider the complexification \((B^p_{\mathbb{R}} \times B^q_{\mathbb{R}} \times (X_R \cup X_I)) \otimes \mathbb{C}\) of \(B^p_{\mathbb{R}} \times B^q_{\mathbb{R}} \times (X_R \cup X_I)\). By Lemma 4.7 and (4.17), there exists an open set \(U_C \subseteq (ℓ^p_{\mathbb{R}} \times ℓ^q_{\mathbb{R}} \times (X_R \cup X_I)) \otimes \mathbb{C}\), which contains \(B^p_{\mathbb{R}} \times B^q_{\mathbb{R}} \times (X_R \cup X_I)\), and an extension \(F_C : U_C \to ℓ^p_{\mathbb{R}} \times ℓ^q_{\mathbb{R}}\) of \(F\), which is locally bounded. Furthermore \(F_C\) is weakly analytic. To see this, we note that for each \(u \in U_C, h \in ℓ^p_{\mathbb{R}} \times ℓ^q_{\mathbb{R}} \times (X_R \cup X_I)\) and every \(L \in ℓ^{p,1/2}_{\mathbb{R}}\), \(F_C(u + zh)\) is uniformly bounded in \(ℓ^{p,1/2}_{\mathbb{R}}\) for \(z\) locally around the origin in \(\mathbb{C}\), it follows that the infinite series \(L F_C(u + zh)\) converges absolutely and uniformly on some neighborhood of 0 in \(\mathbb{C}\). This shows that \(F_C\) is indeed weakly analytic. We conclude from Lemma 4.6 that \(F_C\) is analytic on \(U_C\); in particular \(F\) is real analytic.

The partial derivative \(\partial_u F(0, 0; 0)\), which is given by (4.10), is a topological isomorphism \(ℓ^p_{\mathbb{R}} \to ℓ^{p,1/2}_{\mathbb{R}}\) and \(F(0, 0; 0) = 0\). Thus we can apply the implicit function theorem for Banach
space valued real analytic functions. We infer the existence of an open neighborhood \( W \) of the zero potential in \( X_\mathbb{R} \cup X_T \), an open \( \varepsilon \)-ball \( B^{p,1/2}_\varepsilon \) and a \( \delta \)-ball \( B^{p,1/2}_\delta \) around the origin in \( \mathbb{R}^{p,1/2} \) and a real analytic function

\[
G : B^{p,1/2}_\varepsilon \times W \rightarrow B^{p,1/2}_\delta
\]

such that, for all \( v \in B^{p,1/2}_\varepsilon \) and \( \psi \in W \),

\[
\mathcal{F}(G(v, \psi), v, \psi) = 0,
\]

and such that the map

\[
(v, \psi) \mapsto (G(v, \psi), v, \psi), \quad B^{p,1/2}_\varepsilon \times W \rightarrow B^{p,1/2}_\delta \times B^{p,1/2}_\varepsilon \times W
\]

describes the zero level set of \( \mathcal{F} \) in \( B^{p,1/2}_\delta \times B^{p,1/2}_\varepsilon \times W \).

We may assume that the sequences \( \varepsilon = (\varepsilon_n)_{n \in \mathbb{Z}} \) and \( \delta = (\delta_n)_{n \in \mathbb{Z}} \) satisfy \( \varepsilon_n > 0 \) and \( \delta_n > 0 \) for \( n \in \mathbb{Z} \setminus \{0\} \) and \( \varepsilon_0 = \delta_0 = 0 \). Clearly, if \( -1 \leq \tau_n \leq 1 \) for each \( n \), then \( (\tau_n \varepsilon_n)_{n \in \mathbb{Z}} \in B^{p,1/2}_\varepsilon \) and \( (\tau_n \delta_n)_{n \in \mathbb{Z}} \in B^{p,1/2}_\delta \). Thus we can run through the intervals in each coordinate in a uniform way. Let \( R^\varepsilon_n, n \in \mathbb{Z} \setminus \{0\} \), be the associated sequence of nondegenerate rectangles defined in (4.6). Our considerations show that, for every \( \psi \in W \) and \( n \in \mathbb{Z} \setminus \{0\} \), the zero-set of \( F \) can be parametrized locally near \( \hat{\lambda}_n^1 \) by the real analytic function

\[
z_n(\psi) : (-\varepsilon_n, \varepsilon_n) \rightarrow R^\varepsilon_n, \quad y_n \mapsto \hat{\lambda}_n^1 + G_n(y_n, \psi) + iy_n.
\]

We set

\[
\gamma_n(\psi) := z_n(\psi)((-\varepsilon_n, \varepsilon_n)) \subseteq R^\varepsilon_n
\]

and denote the zero-set of \( \Delta_2(\cdot, \psi) \) by

\[
N_{\Delta_2}(\psi) := \{(x, y) \in \mathbb{R}^2 : \Delta_2(x, y, \psi) = 0\} \subseteq \mathbb{R}^2.
\]

By construction,

\[
\gamma_n(\psi) \cap \mathbb{R} = N_{\Delta_2}(\psi) \cap (R^\varepsilon_n \setminus \mathbb{R}),
\]

Figure 3. Fig. 3a shows an illustration of the path \( \gamma_n \) within the rectangle \( R^\varepsilon_n \) which is contained in the disc \( D^1_n \). The critical points \( \lambda^1_n = \hat{\lambda}_n^1(\psi) = \gamma_n \cap \mathbb{R} \) and \( \hat{\lambda}_n^1(0) \) are marked with dots. Fig. 3b shows a plot of the zero-set of \( \Delta_2(\cdot, 0) \) in the complex \( \lambda \)-plane; the boundaries of the discs \( D^1_n, i = 1, 2 \), are indicated by dashed circles, the periodic eigenvalues (which coincide with the critical points of \( \Delta(\cdot, 0) \) and the Dirichlet and Neumann eigenvalues) are indicated by dots.
and furthermore, since $\Delta$ is real-valued on $\mathbb{R} \times (X_R \cup X_I)$, cf. Proposition 4.2 we have
\[ N_{\Delta_n}(\psi) \cap R_n^{\epsilon,\delta} = \gamma_n(\psi) \cup (R_n^{\epsilon,\delta} \cap \mathbb{R}) = Z_n(\psi) \subseteq \mathbb{C}. \]

Thus for arbitrary $\psi \in W$ and every $n \in \mathbb{Z} \setminus \{0\}$, $\lambda \in R_n^{\epsilon,\delta}$ satisfies
\[ \Delta(\lambda, \psi) \in \mathbb{R} \iff \lambda \in Z_n(\psi). \quad (4.19) \]

The intersection $\gamma_n(\psi) \cap R_n^{\epsilon,\delta}$ consists of a single point which we denote by $\xi_n \equiv \xi_n(\psi) \in R_n^{\epsilon,\delta} \subseteq D_n^1$. We will show that $\xi_n = \dfrac{\lambda}{n}$. Since $\Delta_2$ vanishes on $\mathbb{R}$, $\partial_2 \Delta_2(\xi_n, \psi) = 0$. Furthermore, we know that $\Delta_2$ vanishes on $\mathbb{R}$, hence $\partial_2 \Delta_2(\xi_n, \psi) = 0$. The Cauchy-Riemann equations then imply that $\Delta(\xi_n, \psi) = \partial_2 \Delta_2(\xi_n, \psi) + i \partial_1 \Delta_2(\xi_n, \psi) = 0$; hence $\xi_n(\psi)$ is a critical point of $\Delta(\cdot, \psi)$. Since $\Delta(\cdot, \psi)$ has only one critical point in $D_n^1$, namely $\lambda_n(\psi)$ according to Corollary 4.3, we conclude that $\gamma_n(\psi)$ crosses the real line in the point $\lambda_n^{1}(\psi) \in R_n^{\epsilon,\delta}$.

**Remark 4.8.** The proof of Theorem 4.4 shows that the sequences $(\epsilon_n)_n \in \mathbb{Z}$ and $(\delta_n)_n \in \mathbb{Z}$ that represent the lengths of the sides of the rectangles $R_n^{\epsilon,\delta}$ can be chosen to lie in the ball $B_1^{1/2} \subseteq \mathbb{R}$ for any $p > 2$. In particular, each rectangle $R_n^{\epsilon,\delta}$ is contained in the disc $D_n^1$ and
\[ \lambda_n^{1}(\psi) = \lambda_n^{1}(0) + \epsilon_n^{p_1/2}, \quad p > 2 \]
uniformly in a small neighborhood of the zero potential in $X_R \cup X_I$. This improves the asymptotic localization of Lemma 3.1 considerably in the case of small potentials of real and imaginary type.

Theorem 4.4 shows that, for all small real and imaginary type potentials, there exists an analytic arc between the periodic eigenvalues $\lambda_n^{1-}(\psi)$ and $\lambda_n^{1+}(\psi)$ along which the discriminant is real-valued, at least for fixed $n \in \mathbb{Z} \setminus \{0\}$.

**Corollary 4.9.** Let $n \in \mathbb{Z} \setminus \{0\}$ be fixed. There exists a neighborhood $W^*$ of 0 in $X$ such that for all $\psi \in W^* \cap (X_R \cup X_I)$ there exists an analytic arc $\gamma_n^*(\psi) = \gamma_n^*(\psi) \subseteq \mathbb{C}$ connecting the two periodic eigenvalues $\lambda_n^{1-} \equiv \lambda_n^{1-}(\psi)$. Qualitatively we distinguish two different cases: either (i) $\gamma_n^* = [\lambda_n^{1-}, \lambda_n^{1+}] \subseteq \mathbb{R}$ or (ii) $\gamma_n^*$ is transversal to the real line and the orthogonal projection of $\gamma_n^*$ to the imaginary axis is a real analytic diffeomorphism onto its image. In both cases, it holds that
- (1) $\Delta(\gamma_n^*, \psi) \subseteq [-2, 2]$,
- (2) $\dot{\gamma}_n^* = \gamma_n^*$,
- (3) $\dot{\lambda}_n^{1}(\psi) \in \gamma_n^* \cap \mathbb{R}$,
- (4) For the parameterization by arc length $\rho_n = \rho_n(s)$ of $\gamma_n^*$ with $\rho_n(0) = \dot{\lambda}_n^{1}(\psi)$, the function $s \mapsto \Delta(\rho_n(s), \psi)$ is strictly monotonous along the two connected components of $\gamma_n^* \setminus \{\dot{\lambda}_n^{1}(\psi)\}$.

(We consider the possible scenario $\lambda_n^{1-}(\psi) = \lambda_n^{1+}(\psi) = \lambda_n^{1}(\psi)$, where the set $\gamma_n^*(\psi)$ consists of the single element $\dot{\lambda}_n^{1}(\psi)$, as a degenerate special case.)

Proof. According to Theorem 4.4 there exists a neighborhood $W$ of 0 in $X$ such that for $\psi \in W \cap (X_R \cup X_I)$ and arbitrary $n \in \mathbb{Z} \setminus \{0\}$, the analytic arc $\gamma_n(\psi)$ describes the preimage of $\mathbb{R}$ under $\Delta(\cdot, \psi)$ locally around $\dot{\lambda}_n^{1}(0)$ and this arc is transversal to the real line. Fix $n \in \mathbb{Z} \setminus \{0\}$. We have $\lambda_n^{1-}(0) = \lambda_n^{1-}(0) \in R_n^{\epsilon,\delta}$, where the open rectangle $R_n^{\epsilon,\delta}$ is contained in the disc $D_n^1$. By the continuity $\chi_p$ and $\Delta$, there exists a neighborhood $W^*$ of $\psi = 0$ in $X$ such that $W^* \cap (X_R \cup X_I) \subseteq W$ and such that $\lambda_n^{1-}(\psi), \lambda_n^{1+}(\psi) \in R_n^{\epsilon,\delta}$ for all $\psi \in W^* \cap (X_R \cup X_I)$. Furthermore, by Corollary 3.16
\[ \Delta(\lambda_n^{1-}(\psi), \psi) = 2(-1)^n. \quad (4.20) \]
We infer from (4.19) in combination with (4.20) and Corollary 4.3 that
\[ \lambda_n^{1-}(\psi) \in Z_n(\psi) \quad \text{and} \quad \lambda_n^{1+}(\psi) \in Z_n(\psi) \cap \mathbb{R} = R_n^{\epsilon,\delta} \cap \mathbb{R} \]
for all \( \psi \in W \cap (X_\mathcal{R} \cup X_\mathcal{T}) \). If both \( \lambda^{1,-}_n(\psi) \) and \( \lambda^{1,+}_n(\psi) \) are real, we set
\[
\gamma^*_n \equiv \gamma^*_n(\psi) := [\lambda^{1,-}_n(\psi), \lambda^{1,+}_n(\psi)] \subseteq \mathbb{R}^2,
\]
otherwise we set
\[
\gamma^*_n \equiv \gamma^*_n(\psi) := \gamma_n(\psi) \cap \{ \lambda \in \mathbb{C} : |\Delta(\lambda, \psi)| \leq 2 \}.
\]
In both cases, we have \( \Delta(\{\gamma^*_n\}, \psi) \subseteq [-2,2], \gamma^*_n = \check{\gamma}^*_n \) and \( \check{\lambda}_n(\psi) \in \gamma^*_n \cap \mathbb{R} \). Furthermore,
\[
\frac{d}{ds} [\Delta(\rho_n(s), \psi)] = 0 \iff \check{\Delta}(\rho_n(s), \psi) = 0 \iff s = 0,
\]
because \( \frac{d}{ds} \rho_n \equiv 1 \) by assumption and, by Corollary 4.3
\[
\rho_n(0) = \check{\lambda}_n(\psi) \text{ is the only root of } \check{\Delta}(\cdot, \psi) \text{ in } R^c_d \subseteq D^c_n.
\]

A potential \( \psi \in X \) is an \( N \)-gap potential if \( N \geq 0 \) is the largest integer such that \( \lambda^{i,\pm}_n(\psi) = \check{\lambda}_n(\psi), i = 1, 2 \), for all \( |n| > N \). We denote by \( X_{N} \subseteq X \) the subset of all \( k \)-gap potentials with \( 0 \leq k \leq N \). An easy application of Corollary 4.9 yields the following result.

**Corollary 4.10.** There exists a neighborhood \( W^* \) of 0 in \( X \) such that for all potentials \( \psi \in W^* \cap X_{N} \cap (X_\mathcal{R} \cup X_\mathcal{T}) \) and all \( n \in \mathbb{Z} \), there exists an analytic arc \( \gamma^*_n \equiv \gamma^*_n(\psi) \subseteq \mathbb{C} \) which connects the two periodic eigenvalues \( \lambda^{1,\pm}_n(\psi) \) and which satisfies all the properties listed in Corollary 4.9.

**Remark 4.11.** The Counting Lemma tells us that the eigenvalues \( \lambda^{1,\pm}_n \) lie in the disc \( D^c_n \) for all sufficiently large \( |n| \) uniformly on bounded sets in \( X \), where the radius of \( D^c_n \) is of order \( O(1/|n|) \). In other words,
\[
\lambda^{1,\pm}_n(\psi) = \lambda^{1,\pm}_n(0) + \epsilon^n_c/2\text{ uniformly on bounded sets in } X.
\]
If we had a stronger localization result, say of the form
\[
\lambda^{1,\pm}_n(\psi) = \lambda^{1,\pm}_n(0) + \epsilon^n_{p/2}\text{ uniformly for } \psi \text{ near the zero potential in } X_{\mathcal{R}} \cup X_{\mathcal{T}},
\]
then we could prove a uniform version (in \( n \in \mathbb{Z} \setminus \{0\} \)) of Corollary 4.9 valid for all real and imaginary type potentials near zero (not just for finite gap potentials). We note in this regard that for many spectral problems the localization of the eigenvalues can be improved when the potential possesses higher regularity, see e.g. [8] [10] [18] [25].

5. Example: single exponential potential

In this section, we consider single exponential potentials \( \psi \) of the form
\[
\psi(t) = (\alpha e^{i\omega t} - \sigma \alpha e^{-i\omega t}, \alpha e^{i\omega t} + \sigma \alpha e^{-i\omega t}), \quad \alpha, c \in \mathbb{C}, \omega \in \mathbb{R}, \sigma \in \{ \pm 1 \}.
\]
To ensure that \( \psi \in X \), i.e. that \( \psi \) has period one, we require that \( \omega \in 2\pi \mathbb{Z} \). If \( \sigma = 1 \), the potential \( \psi(t) \) in (5.1) is of real type and hence relevant for the defocusing NLS; if \( \sigma = -1 \), it is of imaginary type and hence relevant for the focusing NLS. A direct computation shows that the associated fundamental solution \( M(t, \lambda, \psi) \) is explicitly given by
\[
e^{\frac{i}{2} \omega \sigma \lambda t} \left( \cos(\Omega t) + \frac{4\lambda^2 + 2\sigma|\alpha|^2 + \omega}{2\Omega} \sin(\Omega t) \right) \cdot \left( \frac{2\lambda^{1,\pm}_n \sin(\Omega t)}{4\lambda^{1,\pm}_n + 2\sigma|\alpha|^2 + \omega} \sin(\Omega t) \right),
\]
where
\[
\Omega = \Omega(\lambda) = \sqrt{4\lambda^4 + 2\omega\lambda^2 + 4\sigma^2(\alpha^2 + \frac{\omega^2}{4} + \sigma|\alpha|^2)} - |\sigma|c^2.
\]
We fix the branch in (5.3) by requiring that
\[
\Omega(\lambda) = 2\lambda^2 + \frac{\omega}{2} + \mathcal{O}(\lambda^{-1}) \text{ as } |\lambda| \to \infty.
\]
Thus the discriminant \( \Delta \), i.e. the trace of (5.2), and the characteristic function for the periodic eigenvalues \( \chi_p \) defined in (3.9) are given by
\[
\Delta(\lambda, \psi) = -2\cos(\Omega t), \quad \chi_p(\lambda, \psi) = 4\sin^2(\Omega t).
\]
Fig. 4 shows the zero-set of $\Delta_2(\cdot, \psi) = \Im \Delta(\cdot, \psi)$ for four different choices of the parameters $\sigma$, $\alpha$, $c$ and $\omega$. All four choices correspond to exact plane wave solutions of the NLS. Indeed, if $\omega = -2\pi$, $\alpha > 0$, and $-\sigma^2 \alpha^2 - \omega > 0$ for $\sigma = \pm 1$, then $u(x, t) = \alpha e^{i\beta x + i\omega t}$ with $\beta = \sqrt{-\sigma^2 \alpha^2 - \omega}$, solves the defocusing (focusing) NLS if $\sigma = 1 \ (\sigma = -1)$. Moreover, $u(0, t) = \alpha e^{i\omega t}$ and $u_\tau(0, t) = ce^{i\omega t}$ with $c = i\alpha \beta$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Plots of the zero level sets of $\Delta_2(\cdot, \psi)$ in the complex $\lambda$-plane for single exponential potentials of real type (left column) and imaginary type (right column) which arise from exact plane wave solutions of the defocusing and focusing NLS respectively; periodic eigenvalues are indicated with dots and the dashed circles are the boundaries of the discs $D^\pm_n$.}
\end{figure}

The left and right columns of Fig. 4 show the defocusing and focusing cases, respectively. In the top row, the norm of the potential is small enough ($\sigma = 1/12$) that each periodic eigenvalue $\lambda_n^{1, \pm}$ is contained in the disc $D^\pm_n$, $i = 1, 2$, $n \in \mathbb{Z}$. In Fig. 4a, all periodic eigenvalues $\lambda_n^{1, \pm}$ are real and there is a spectral gap $[\lambda_n^{1, -}, \lambda_n^{1, +}]$; the remaining periodic eigenvalues satisfy $\lambda_n^{1, -} = \lambda_n^{1, +}$, $n \in \mathbb{Z} \setminus \{-1, 0\}$. The periodic eigenvalues $\lambda_n^{1, \pm}$, $n \in \mathbb{Z}$, lie on a curve that asymptotes to the imaginary axis. In Fig. 4b, $\lambda_{-1}^{1, -}$ and $\lambda_{-1}^{1, +}$ are not real but lie on the (global) arc $\gamma_{-1}$ which is symmetric with respect to the real axis and crosses the real line at the critical point $\lambda_{-1}^{1, -}$. 
In Fig. 4c and Fig. 4d the spectral gaps are larger than in Fig. 4a and Fig. 4b because the parameter $\alpha$ is larger ($\alpha = 1/2$).

Fig. 5 shows the zero-set of $\Delta_2(\cdot, \psi)$ for the real type single exponential potential (5.1) with parameters $\sigma = 1, \omega = -2\pi, \alpha = 6/15 + 11/4i$ and $c = 1/10$. This example clearly demonstrates that Corollaries 4.9 and 4.10 fail to remain true for potentials with sufficiently large X-norms. We further notice that some arcs $\gamma_n$ do not only “leave” the discs $D_i^n$ (and hence also the rectangles $R_{\epsilon, \delta}^i$), but the zero-set differs qualitatively from the previous examples: certain arcs “merge” with other arcs and subsequently split into new components. Such bifurcations yield discontinuities in the lexicographic ordering of the periodic eigenvalues. Note also that for this potential (and consequently all potentials in X with smaller X-norm), the assertion of the Counting Lemma holds for $N = 3$: there are $4(2 \cdot 3 + 1) = 28$ periodic eigenvalues contained in the disc $B_3$ (when counted with multiplicity) and each disc $D_i^n, i = 1, 2, |n| > 3,$ contains precisely one double eigenvalue.

**Figure 5.** A plot of the zero level set of $\Delta_2(\cdot, \psi)$ in the complex $\lambda$-plane for the real type single exponential potential (5.1) with $\sigma = 1, \omega = -2\pi, \alpha = 6/15 + 11/4i, c = 1/10$; periodic eigenvalues are indicated with dots, the large dashed circle is the boundary of the disc $B_3$ and the remaining dashed circles are the boundaries of the discs $D_i^n$.

### 6. Formulas for gradients

**6.1. Gradient of the fundamental solution.** Let $dF$ denote the Fréchet derivative of a functional $F: Y \rightarrow \mathbb{C}$ on a (complex) Banach space $Y$. If it exists, $dF: Y \rightarrow Y'$ is the unique map from $Y$ into its topological dual space $Y'$ such that

$$F(u + h) = F(u) + (dF)(u)h + o(h) \quad \text{as} \quad \|h\| \to 0$$

for $u \in Y$. The map $dFh: Y \rightarrow \mathbb{C}$ (also denoted by $\partial_h F$) is the directional derivative of $F$ in direction $h \in Y$. For any differentiable functional $F: X \rightarrow \mathbb{C}$ and $h \in X$, we have

$$dFh = \partial_h F = \int_0^1 (F_1h_1 + F_2h_2 + F_3h_3 + F_4h_4) \, dt$$

for some uniquely determined function $\partial F = (F_1, F_2, F_3, F_4): X \rightarrow X$. We denote the components of $\partial F$ by $\partial_i F$, $i = 1, 2, 3, 4$, and define the gradient $\partial F$ of $F$ by

$$\partial F = (\partial_1 F, \partial_2 F, \partial_3 F, \partial_4 F) = (F_1, F_2, F_3, F_4).$$
The following proposition gives formulas for the partial derivatives of the fundamental solution

$$M(t, \lambda, \psi) = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}.$$  

For fixed $t \geq 0$ and $\lambda \in \mathbb{C}$, we consider $M$ as a map $X \rightarrow M_{2 \times 2}(\mathbb{C})$. In particular, each matrix entry $m_i$, $i = 1, 2, 3, 4$ gives rise to a functional $X \rightarrow \mathbb{C}$. Let us set

$$\gamma \equiv \gamma(M) := \det M^d - \det M^{\text{od}} = m_1 m_4 + m_2 m_3.

**Proposition 6.1.** For any $t \geq 0$ and $0 \leq s \leq t$, the gradient of the fundamental solution $M$, defined on the interval $[0, t]$, is given by

$$
\begin{align*}
(\partial_1 M(t))(s) &= M(t) \begin{pmatrix} -i\gamma \psi_2 + 2\lambda_3 m_4 & -2i\psi_2 m_2 m_4 + 2\lambda_1 m_2^2 \\ 2i\psi_2 m_1 m_3 - 2\lambda m_3^2 & i\gamma \psi_2 - 2\lambda m_3 m_4 \end{pmatrix}(s), \\
(\partial_2 M(t))(s) &= M(t) \begin{pmatrix} -i\gamma \psi_1 - 2\lambda m_1 m_2 & -2i\psi_1 m_2 m_4 - 2\lambda_2 m_2^2 \\ 2i\psi_1 m_1 m_3 + 2\lambda_1 m_3^2 & i\gamma \psi_1 + 2\lambda m_1 m_2 \end{pmatrix}(s), \\
(\partial_3 M(t))(s) &= M(t) \begin{pmatrix} im_3 m_4 & im_4^2 \\ -im_3^2 & -im_3 m_4 \end{pmatrix}(s), \\
(\partial_4 M(t))(s) &= M(t) \begin{pmatrix} im_1 m_2 & im_2^2 \\ -im_1^2 & -im_1 m_2 \end{pmatrix}(s).
\end{align*}

Moreover, at the zero potential $\psi = 0$,

$$
\begin{align*}
(\partial_1 E_\lambda(t))(s) &= \begin{pmatrix} 0 & 2\lambda e^{-2\lambda^2(t-2s)} \\ 0 & 0 \end{pmatrix}, \\
(\partial_2 E_\lambda(t))(s) &= \begin{pmatrix} 0 & -2\lambda e^{2\lambda^2(t-2s)} \\ 0 & 0 \end{pmatrix}.
\end{align*}

Proof. By Theorem 2.1 the fundamental solution $M$ is analytic in $\psi$. It suffices therefore to verify the above formulas for smooth potentials $\psi$ for which the order of differentiation with respect to $t$ and $\psi$ can be interchanged. The general result then follows by a density argument.

Applying the directional derivative $\partial_h$ to both sides of equation \([2.5]\), we obtain

$$D \partial_h M = (R + V) \partial_h M + \partial_h (R + V) \cdot M.$$  

Since both $M(0)$ and $R$ are independent of $\psi$, Proposition 2.4 implies

$$\partial_h M(t) = M(t) \int_0^t M^{-1}(s) \partial_h V(s) M(s) \, ds.$$  

The integrand equals

$$\begin{pmatrix} m_4 & -m_2 \\ -m_3 & m_1 \end{pmatrix} \begin{pmatrix} -i(\psi_2 h_1 + \psi_1 h_2) & 2\lambda h_1 + ih_3 \\ 2i\lambda h_2 - ih_4 & i(\psi_2 h_1 + \psi_1 h_2) \end{pmatrix} \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix},$$

which can be rewritten as

$$
\begin{align*}
&\begin{pmatrix} -i\psi_2 (m_1 m_4 + m_2 m_3) + 2\lambda_3 m_4 & -2i\psi_2 m_2 m_4 + 2\lambda_1 m_2^2 \\ 2i\psi_2 m_1 m_3 - 2\lambda m_3^2 & i\psi_2 (m_1 m_4 + m_2 m_3) - 2\lambda m_3 m_4 \end{pmatrix} h_1 \\
+ \begin{pmatrix} -i\psi_1 (m_1 m_4 + m_2 m_3) - 2\lambda m_1 m_2 & -2i\psi_1 m_2 m_4 - 2\lambda_2 m_2^2 \\ 2i\psi_1 m_1 m_3 + 2\lambda_1 m_3^2 & i\psi_1 (m_1 m_4 + m_2 m_3) + 2\lambda m_1 m_2 \end{pmatrix} h_2 \\
+ \begin{pmatrix} im_3 m_4 & im_4^2 \\ -im_3^2 & -im_3 m_4 \end{pmatrix} h_3 + \begin{pmatrix} im_1 m_2 & im_2^2 \\ -im_1^2 & -im_1 m_2 \end{pmatrix} h_4.
\end{align*}

The expression for the gradient $\partial M(t)$ follows. In the case of the zero potential $\psi = 0$, we have $m_1 = e^{-2\lambda^2 u}$, $m_4 = e^{2\lambda^2 u}$ and $m_2 = m_3 = 0$, so the gradient $\partial E_\lambda(t)$ is easily computed. \qed
The following notation is useful to express the gradient of $M$ more compactly. Let $M_1$ and $M_2$ denote the first and second columns of $M$, and denote by $V^1$ the first two components, and by $V^2$ the last two components of the four-vector $\psi$:

$$V^1 := \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad V^2 := \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}.$$  

Analogously, let

$$\partial^1 := \begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix}, \quad \partial^2 := \begin{pmatrix} \partial_3 \\ \partial_4 \end{pmatrix}.$$  

The star product of two 2-vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ is defined by

$$a \ast b := \begin{pmatrix} a_2 b_2 \\ a_1 b_1 \end{pmatrix}.$$  

Moreover, recall that $\gamma = m_1 m_4 + m_2 m_3$. With this notation, we obtain

**Corollary 6.2.** For any $t \geq 0$, the gradient of the fundamental solution $M$ is given by

$$\partial^1 M(t) = M(t) \begin{pmatrix} -\gamma \sigma_1 V^1 + 2 \lambda \sigma_3 (M_1 \ast M_2) & -2i m_2 m_4 \sigma_1 V^1 + 2 \lambda \sigma_3 (M_2 \ast M_2) \\ 2i m_1 m_3 \sigma_1 V^1 - 2 \lambda \sigma_3 (M_1 \ast M_1) & i \gamma \sigma_1 V^1 - 2 \lambda \sigma_3 (M_1 \ast M_2) \end{pmatrix},$$

$$i \partial^2 M(t) = M(t) \begin{pmatrix} -M_1 \ast M_2 & -M_2 \ast M_2 \\ M_1 \ast M_1 & M_1 \ast M_2 \end{pmatrix}.$$  

In the special case when $\psi = 0$ and $\lambda$ is a periodic eigenvalue corresponding to the zero potential (i.e. $\lambda = \lambda_n^{\pm}(0)$, $i = 1, 2$, $n \in \mathbb{Z}$), we find

$$e_n^+ = M_1 \ast M_1, \quad e_n^- = M_2 \ast M_2, \quad n \in \mathbb{Z},$$

where

$$e_n^+ := \begin{pmatrix} 0 \\ e^{-2 \pi n} \end{pmatrix}, \quad e_n^- := \begin{pmatrix} e^{2 \pi n} \\ 0 \end{pmatrix}, \quad n \in \mathbb{Z}.$$  

6.2. Discriminant and anti-discriminant.

**Proposition 6.3.** The gradient of $\Delta$ is given by

$$\partial^1 \Delta = m_2[2i m_1 m_3 \sigma_1 V^1 - 2 \lambda \sigma_3 (M_1 \ast M_1)] - m_3[2i m_2 m_4 \sigma_1 V^1 - 2 \lambda \sigma_3 (M_2 \ast M_2)] + (\dot{m}_4 - \dot{m}_1)[i \gamma \sigma_1 V^1 - 2 \lambda \sigma_3 (M_1 \ast M_2)],$$

$$i \partial^2 \Delta = \dot{m}_2 M_1 \ast M_1 - \dot{m}_3 M_2 \ast M_2 + (\dot{m}_4 - \dot{m}_1) M_1 \ast M_2.$$

At the zero potential, $\partial \Delta(\lambda, 0) = 0$ for all $\lambda \in \mathbb{C}$.  

**Proof.** The formula for the gradient follows directly from Corollary 6.2. In the case of the zero potential, $m_2 = m_3 = 0$; hence $M_1 \ast M_2 = 0$ and therefore $\partial \Delta(\lambda, 0) = 0$ for all $\lambda \in \mathbb{C}$.  

The following formulas for the derivative of the anti-discriminant are derived in a similar way.

**Proposition 6.4.** The gradient of the anti-discriminant $\delta$ is given by

$$\partial^1 \delta = \dot{m}_4[2i m_1 m_3 \sigma_1 V^1 - 2 \lambda \sigma_3 (M_1 \ast M_1)] + (\dot{m}_2 - \dot{m}_3)[i \gamma \sigma_1 V^1 - 2 \lambda \sigma_3 (M_1 \ast M_2)] - \dot{m}_1[2i m_2 m_4 \sigma_1 V^1 - 2 \lambda \sigma_3 (M_2 \ast M_2)],$$

$$i \partial^2 \delta = \dot{m}_4 M_1 \ast M_1 + (\dot{m}_2 - \dot{m}_3) M_1 \ast M_2 - \dot{m}_1 M_2 \ast M_2.$$

In the special case when $\psi = 0$ and $\lambda$ is a periodic eigenvalue corresponding to the zero potential, i.e. $\lambda = \lambda_n^{\pm}(0)$, $i = 1, 2$, $n \in \mathbb{Z}$,

$$\partial^1 \delta = 2 \lambda_n^{\pm}(0)(-1)^n (e_n^+ + e_n^-), \quad i \partial^2 \delta = (-1)^n (e_n^+ - e_n^-).$$

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