Distinguishing quantum measurements of observables in terms of state transformers

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The modern framework of state transformers, i.e., the first Kraus representation of quantum measurement, is introduced and related both to the known textbook concepts and to measurement-interaction evolution (the second Kraus representation). In this framework the known kinds of measurements of ordinary (as distinct from generalized) observables are distinguished by necessary and sufficient conditions. Thus, repeatable, nonrepeatable, and ideal measurements are characterized both algebraically and geometrically utilizing polar factorization of state transformers.

I. INTRODUCTION

Quantum mechanics is no longer an esoteric scientific discipline solely about microscopic objects far beyond human perception. Much progress has been made in demonstrating the macroscopic quantum behavior of various systems such as superconductors, superfluids, nanoscale magnets, laser-cooled trapped ions, photons in a microwave cavity etc. On the fundamental level quantum mechanics is expected to underlie classical physics.

Measurement theory is a central part of quantum mechanics. Connection of the quantum formalism with laboratory practice hinges on measurement, which gives results and thus enables one to draw information from an observed specimen.

The newest and, perhaps, most promising applications of quantum mechanics - quantum information theory and quantum computation - also require quantum measurement as a basic notion.\textsuperscript{1,2} Nowadays the concepts of quantum measurement and of the ensuing change of state are far from those found in some older textbooks. \textit{The aim of this article} is to give a simple presentation of the modern framework of quantum measurement theory, and to discuss the basic kinds of measurements of ordinary discrete observables in it.

In textbooks one often gives a discrete observable (Hermitian operator) $M$ with possibly degenerate (distinct) eigenvalues in spectral form

$$M = \sum_i m_i \sum_j |i\rangle\langle i|$$
The range of values of $j$ depends on the degeneracy of $m_i$. The spectral form in terms of the eigenprojectors $P_i$ is uniquely determined by $M$. The one in terms of the eigenbasis $\{|ij\rangle: \forall ij\}$ is, on account of the degeneracies, nonunique. (Throughout, by "basis" is meant an orthonormal (ON) set of vectors spanning the entire space.)

Some standard textbooks\textsuperscript{3,4} then claim that the eigenvalues $m_i$ are the only possible results of the measurement of $M$, and, if the result $m_i$ is obtained, then its probability is, of course, positive, and an arbitrary state vector $|\psi\rangle$ undergoes the change:

$$|\psi\rangle \rightarrow |\psi\rangle_i \equiv p_i^{-1/2} P_i |\psi\rangle,$$ \hspace{1cm} (2a)

where

$$\forall i: \quad p_i \equiv \langle \psi | P_i | \psi \rangle$$ \hspace{1cm} (2b)

are the probabilities for the changes (2a).

One often considers the change occurring to the entire ensemble described by $|\psi\rangle$ when an observable $M$ is measured:

$$|\psi\rangle\langle\psi| \rightarrow \sum_i (p_i |\psi\rangle_i \langle\psi|_i) = \sum_i (P_i |\psi\rangle \langle\psi| P_i).$$ \hspace{1cm} (2c)

Finally, one has the completeness relation

$$\sum_i P_i = 1.$$ \hspace{1cm} (2d)

(Note that if $p_i = 0$, $|\psi\rangle_i$ cannot be defined as a unique state vector, and the corresponding term $(p_i |\psi\rangle_i \langle\psi|_i)$ in (2c) is understood to be zero.)

To avoid physically inessential mathematical intricacies, we confine ourselves throughout to finite-dimensional state spaces.

Following the practical terminology used in the textbook of Kaempffer,\textsuperscript{5} we call (2a) selective measurement, whereas we refer to (2c) as nonselective measurement. Note that one is dealing with two aspects of measurement. (Sometimes one objects to the use of the word "measurement" for the nonselective aspect because there is no definite result. But this lack is only apparent due to suppression of the role of the measuring instrument. Nonsselective measurement is the totality of all selective measurements, and the individual instruments keep track of the various definite results.)

It is often believed that formulae (2a)-(2c) originated in the monumental book of von Neumann.\textsuperscript{6} Actually, formulae (2a) and (2c) are due to Lüders.\textsuperscript{7} There is a precise sense in which formula (2c) can be interpreted as describing minimal change of state in measurement.\textsuperscript{8} (See also the references cited in ref. 8.)

Von Neumann discussed only measurement of discrete observables with all eigenvalues nondegenerate. If a discrete observable is not of this kind, it can always be refined into such an observable. To see this, one has to consider the first (nonunique) spectral form in (1) and replace $m_i$ by distinct $m_{ij}$ in it to obtain
an observable $M'$ that has all eigenvalues nondegenerate. $M'$ is a maximal refinement of $M$. The operator $M$ is a function of $M'$: $M = f(M')$ meaning that the eigenvectors of $M'$ are eigenvectors also of $M$, and $\forall ij : m_i = f(m_{ij})$.

Following the idea of von Neumann, one can measure $M$ by measuring $M'$. But it is not a minimal measurement, because (2a) is replaced by
\[
|\psi\rangle\langle\psi| \rightarrow \sum_j (p_{ij}/p_i) |ij\rangle\langle ij|,
\]
where the probabilities are now
\[
\forall ij : p_{ij} \equiv \|\langle ij | \psi\rangle\|^2.
\]
In this way one performs an excessive measurement of $M$. (One overdoes the measurement not only regarding the change of state, but also regarding the information obtained in the measurement because one learns about an observable $M'$ that is a refinement of $M$.)

It is perhaps interesting to point out that the mentioned excessive measurement of the state can always be avoided if one chooses a suitable maximal refinement $M'$ of $M$. But then the choice of $M'$ must depend on the state $\rho$ in which the measurement is performed.

Owing to (2d), one can write $|\psi\rangle\langle\psi| = \sum_i \sum_j P_i |\psi\rangle\langle\psi| P_j$. Viewing this in matrix form in the representation of the eigenbasis $\{|ij\rangle : \forall ij\}$ of $M'$, and comparing this with (2c), one can see that all off-diagonal submatrices are replaced by zero. Thus $|\psi\rangle$ looses coherence. By "coherence" one, actually, means interference experiments, in which one measures in $|\psi\rangle$ some observable incompatible with $M'$. The same measurement performed after that of $M$ gives quite different results. Thus, in (2c) coherence is lost or decoherence (with respect to the eigenprojectors of $M$) sets in. (I recommend as further reading ref. 10. It gives a short and clear presentation of the widely accepted environment-induced decoherence theory in measurement.)

Besides the quantum system, which we denote as subsystem 1, there is a measuring apparatus, subsystem 2, with an initial state $|0\rangle_2$, and a certain interaction between system and apparatus expressed as a unitary evolution operator $U_{12}$ that takes the initial state $(|\psi\rangle_1 \otimes |0\rangle_2)$ into a suitably correlated final composite-system state vector
\[
|\Psi\rangle_{12}^f = U_{12} (|\psi\rangle_1 \otimes |0\rangle_2). \quad (3a)
\]

The measuring apparatus contains one more entity: an observable $B_2 = \sum_i (b_i |\chi_i\rangle_2 \langle \chi_i |_2)$ with all eigenvalues nondegenerate, called the pointer observable. (For simplicity the dimension of the state space of the measuring apparatus is assumed to be equal to the number of distinct eigenvalues of $M$.)

The mentioned "suitability" of correlations in the final state means that if one expands the final state vector $|\Psi\rangle_{12}^f$ in the eigenbasis $\{|\chi_i\rangle_2 : \forall i\}$ of the pointer observable, one obtains
\[
|\Psi\rangle_{12}^f = \sum_i (P_i^\dagger |\psi\rangle_1 \otimes |\chi_i\rangle_2). \quad (3b)
\]
(Note that $P_i^\dagger$ is the same as $P_j$. The
change of notation is due to the need to deal with two subsystems.)

Expansion of a bipartite vector in a factor basis is always possible and it gives unique (generalized) "expansion coefficients" (vectors in the opposite factor space). One can see this by taking a basis also in the opposite factor space, expanding the bipartite vector in the product basis, and, finally, by grouping the opposite-space vectors that go with one and the same basis vector (in the factor space in which the expansion is performed).*

One says that one "reads the pointer position" when one measures \( B_2 \) (instantaneously). If the result is \( b_i \), then, of course, \( p_i > 0 \), and one applies the Lüders projection (2a) \textit{mutatis mutandis}, which gives

\[
p_i^{-1/2} (1 \otimes |\chi_i\rangle_2 \langle \chi_i |_2) |\Psi\rangle_{12} = (p_i^{-1/2} P_i |\psi\rangle_1 \otimes |\chi_i\rangle_2
\]

in this case. Thus one arrives at (the literal form of) (2a).

The measurement of the pointer observable is accompanied by collapse or objectification\textsuperscript{13} of the composite-system state. These terms are closely connected with the selective version of decoherence, which takes place in nonselective measurement. The physical source of the phenomenon is a controversial point. But it is beyond dispute that this problem of measurement theory arises from the incompatibility of the unitary linear dynamics of the composite system plus apparatus (cf (3a)) with the transition from a superposition of pointer states (cf (3b)) to a definite pointer state (cf (2a)). We will not discuss the problem of collapse in this study.

\*When a composite state vector, e. g. \(|\Psi\rangle_{12}'\), is expanded in an arbitrary second-subsystem basis, the second tensor factors are orthonormal state vectors, but the first ones need not be; they need not even be orthogonal in general (cf (4) e. g.). In the special case when the composite-system state vector is expanded in an eigenbasis of its reduced density matrix, in our example, of \( \rho_2 \equiv \text{Tr}_1(|\Psi\rangle_1 \langle\Psi|_{12}) \), where the partial trace is taken over the first subsystem, and only in this case, also the first factors are orthogonal vectors. Then one can write the expansion with positive numerical coefficients and with ON vectors in both factors. This is then the Schmidt expansion.\textsuperscript{11,12} It can also be obtained by interchanging the role of subsystems 1 and 2. (Anyway, the Schmidt expansion is biorthogonal, i. e., orthogonal in both factors, and it is expansion in the eigenbases of both reduced density matrices simultaneously.) The Schmidt expansion is often made use of; but it will not be utilized in this article. (Though, expansion (3b) is obviously biorthogonal, and, as it has just been explained, it could be easily rewritten as a Schmidt expansion.)

II. THE FIRST AND THE SECOND REPRESENTATIONS OF KRAUS

Any choice of \( U_{12} \), \(|0\rangle_2 \), and a pointer observable \( B_2 = \sum_i b_i |i\rangle_2 \langle i |_2 \) lead, via (3a), to some final state vector

\[
|\Psi\rangle_{12}' \equiv \sum_i (M_i |\psi\rangle_1 \otimes |i\rangle_2, \quad (4)
\]

which defines linear operators \( \{M_i : \forall i\} \) in the state space of the system. Reading the pointer position \( b_i \) for \( p_i > 0 \) gives rise to the selective change of state

\[
(|\psi\rangle_1 \otimes |0\rangle_2) \rightarrow p_i^{-1/2} (1 \otimes |\chi_i\rangle_2 \langle \chi_i |_2) |\Psi\rangle_{12}',
\]
i. e., as seen from (4),

\[ |\psi\rangle \rightarrow p_i^{-1/2} M_i |\psi\rangle. \quad (5a) \]

The probabilities are given by

\[ \forall i : \quad p_i = \langle \psi | M_i^\dagger M_i |\psi\rangle, \quad (5b) \]

where \( M_i^\dagger \) is the adjoint of \( M_i \). The corresponding nonselective measurement produces the change of state

\[ |\psi\rangle \langle \psi | \rightarrow \text{Tr}_2 |\Psi\rangle|\Psi\rangle = \sum_i M_i |\psi\rangle \langle \psi | M_i^\dagger. \quad (5c) \]

Finally, one has

\[ \sum_i M_i^\dagger M_i = 1. \quad (5d) \]

(This is implied by the facts that the RHS of (5c) has trace one and that \( |\psi\rangle \) is an arbitrary state vector.)

The operators \( \{M_i : \forall i\} \) are sometimes called state transformers on account of the selective change of state (5a). They are also called measurement operators.\(^2\)

Evidently, putting \( \forall i : \quad M_i \equiv P_i \), relations (5a)-(5d) take on the special form (2a)-(2d). In this case one is dealing with the Lüders state transformers. (Note that measurement of each observable \( M \) has its own pointer basis \( \{\hat{\psi}_i : \forall i\} \). In the special case at issue one must put \( \forall i : \quad |\hat{\psi}_i\rangle \equiv |\chi_2\rangle \).

If the quantum state is a general (mixed or pure) one, then a density operator \( \rho \) takes the place of \( |\psi\rangle \). The reader can easily prove that relations (5a)-(5c) generalize into

\[ \forall i : \quad p_i \equiv \text{Tr}(\rho M_i^\dagger M_i), \quad (6a) \]

\[ \rho \rightarrow \sum_i M_i \rho M_i^\dagger. \quad (6c) \]

(Hint: Express \( \rho \) as a convex combination of pure states with statistical weights as the coefficients.)

To decide if one is now dealing with a general measurement, one wonders about the converse state of affairs: If \( \{N_l : \forall l\} \) is an arbitrary set of state transformers, i. e., if they are linear operators satisfying (5d) \textit{mutatis mutandis}, does there exist a unitary operator \( \tilde{U}_{12} \) that will take \(|\psi\rangle \otimes |0\rangle_2 \) into a final state \( |\Phi\rangle_12 \) implying (5a-c) with these linear operators \textit{mutatis mutandis}?: Affirmative answer requires that the number of values of \( l \) must not exceed the dimension of the second state space because otherwise the following relations (analogues of (4) and (3a) respectively) are not consistent:

\[ |\Phi\rangle_12 \equiv \sum_l (N_l |\psi\rangle_1 \otimes |l\rangle_2 \quad (7a) \]

\[ \equiv \tilde{U}_{12}(|\psi\rangle_1 \otimes |0\rangle_2). \quad (7b) \]

The basis \( \{|l\rangle_2 : \forall l\} \) can be the same as in (4).

**Exercise 1.** Show that if \( \langle \psi_i | \psi_j \rangle = \delta_{i,j} \), then also the corresponding composite-system vectors \( |\Phi\rangle_12 \) defined by relation (7a) are orthonormal. (Hint: Utilize (5d) \textit{mutatis mutandis}.)

Thus, relation (7a) determines \( \tilde{U}_{12} \) incompletely as an (incomplete) isometry, i. e., a linear map taking a subspace onto another equally dimensional one, preserving the scalar product.
Any incomplete isometry can be extended, though nonuniquely, into a unitary operator in the entire space. (This is seen by completing the pair of orthonormal sets of vectors that determine the partial isometry into bases.)

Thus, the answer to the above question is affirmative, and relation (7b) makes sense in terms of a unitary operator $U_{12}$. The first relation defines the composite state vector in terms of the operators $N_i$, and the second determines $U_{12}$, though incompletely.

Therefore, one speaks of (5a)-(5d) (or of (6a)-(6c) and (5d)) as a general quantum measurement or the measurement of a general quantum observable.\(^{14}\) (For details see the standard textbook of quantum information theory\(^2\), or the enlightening review\(^{15}\) and the references therein, or ref. 16.) General quantum measurements comprise, besides measurements of ordinary observables, also measurements of generalized observables (see Definition 4 below). The latter are closely connected with so-called positive-operator-valued measures (POVM). We discuss them below.

A set of state transformers on the one hand, and the initial state of the measuring apparatus, the unitary interaction operator, and the pointer observable on the other hand are two sides of a coin. Sometimes one calls them "instrument".\(^{17}\)

Putting an ordinary or generalized measurement in the form of linear operators $\{M_i : \forall i\}$ satisfying (5d), i.e., expressing it in terms of state transformers, is called the first Kraus representation or the operator-sum representation. The use of composite-system unitary operators via (3a) and (4) is called the second Kraus representation.\(^{16}\) The state transformers are sometimes called Kraus operators.

Instead of the considerably intricate (but also rather general) exposition of Kraus in ref. 16, one can read the simpler and more modern presentation in ref. 18 or that in ref. 1.

One should note that $\Pi_i \equiv M_i^\dagger M_i$ are positive operators associated with the selected measurement results. A positive operator $\Pi_i$ satisfies, by definition, the inequality $\langle \psi | \Pi_i | \psi \rangle \geq 0$ for every vector $|\psi\rangle$. It is necessarily Hermitian. It has the important property of possessing a unique positive (operator) square root.

On account of the positivity of the $\Pi_i$, one speaks, in general, of positive-operator-valued measures (POVM) as generalizations of observables (Hermitian operators). The latter represent the special case of projector-valued measures (see Definition 1 below).\(^{1}\)

**Lemma:** Whenever a POVM $\{\Pi_i : \forall i\}$ is given, there exist linear operators $\{M_i : \forall i\}$ such that $\forall i : M_i^\dagger M_i = \Pi_i$.

**Proof:** Let $\{U_i : \forall i\}$ be arbitrary unitary operators, and let us define $\forall i : M_i \equiv U_i(\Pi_i)^{1/2}$. Then $M_i^\dagger M_i = \Pi_i$.\(^{1}\)

\(^{1}\)In the mathematical literature a positive-operator- (or the special case of a projector-) valued measure is defined on the set of all Borel sets (generalizations of intervals) of the real line, and then it is equivalent to a general (or ordinary) observable. In the case of discrete observables, to which we are confined in this article, besides $\{\Pi_i : \forall i\}$ (or $\{P_i : \forall i\}$), it is also indispensable to specify the (real) eigenvalues $\{m_i : \forall i\}$.\(^{6}\)
(\Pi_i)^{1/2}U_i^{-1}U_i(\Pi_i)^{1/2} = \Pi_i. \quad (\text{Naturally, one can take all } U_i \text{ equal to } 1.)\quad \Box

Remark 1: Thus, any POVM defines a whole family of state transformers, all of which reproduce the same POVM via \( \Pi_i \equiv M_i^\dagger M_i \). The probabilities, which equal

\[ \forall i : \quad p_i \equiv \langle \psi | \Pi_i | \psi \rangle, \quad (8) \]

depend only on the positive operators \( \Pi_i \) associated with the individual results. But the changes of state, e. g., the selective ones, depend on the linear operators \( M_i \) (cf (5a)), and these are associated with the \( \Pi_i \) in a nonunique way. One should keep in mind that it is actually the interaction between system and measuring apparatus (plus the pointer basis) that selects the \( M_i \), because it determines the unitary evolution operator \( U_{12} \), and this, in turn, determines the linear operators \( M_i \) via (3a) and (4).

General measurement or unitary evolution, i. e., any quantum mechanical change of state, is expressible in the form of action of a trace preserving completely positive superoperator on the density operator of the system. (A positive superoperator, by definition, preserves positivity of the operator on which it acts. Complete positivity means that even when the positive superoperator is tensor multiplied by the identity superoperator, the resulting composite-system superoperator is also positive.) Every such superoperator is amenable to both the first and the second Kraus representations.

It is widely accepted that POVM and the two Kraus representations, i. e., the language of state transformers and composite-system unitary operators, is the modern framework of quantum measurement theory.

III. THE SPECIAL CASE OF ORDINARY OBSERVABLES

Let the measurement of a general observable, i. e., that of an ordinary or a generalized one, be given in terms of state transformers \( \{M_i' : \forall i'\} \), which imply the POVM \( \{\Pi_i' \equiv M_i^\dagger M_i' : \forall i'\} \). On the other hand, let a resolution of the identity \( \sum_p P_p = 1 \) enumerated by the same index be given. (The \( P_p \) are orthogonal projectors; physically: disjoint events, alternatives in suitable measurement, altogether making up the certain event.)

Definition 1: Let the POVM for a given index value \( i \) equal \( P_i \). Then the state transformers \( \{M_i' : \forall i'\} \) imply the selective measurement of an ordinary discrete observable (Hermitian operator) \( M = \sum_{i'} m_{i'} P_{i'} \) (cf (1)), in particular, of the eigenevent \( P_i \). (By "eigenevent" is meant the physical interpretation of the eigenprojector.) If the POVM reduces to a projector-valued measure for all values of \( i' \), then the state transformers represent a nonselective measurement of \( M \).

If one deals with the selective measurement of an observable \( M \) with the eigenprojector \( P_i \), then the probability relation (6b) takes on the familiar form:

\[ p_i = \text{Tr}(P_i \rho). \quad (9) \]

One should note that one is dealing with a class of discrete Hermitian operators de-
fined by a common resolution of the identity \( \sum_i P_i = 1 \). An arbitrary element \( M \) of the class is obtained by associating a distinct real number \( m_i \) with each value of \( i \) (cf (1)).

**Definition 2**: If the measurement of \( m_i \) of \( M \) has the property that its repetition necessarily gives the same result, then we have repeatable measurement. More precisely, if the measurement of the value \( m_i \) of an observable \( M \) given by (1) is expressed by the state transformer \( M_i \) from a set of state transformers \( \{ M_i' : \forall i' \} \), and the probability of the same value \( m_i \) of \( M \) in the transformed state \( M_i \rho M_i^\dagger / p_i \) is 1 for every state \( \rho \), then the (selective) measurement is called repeatable. Otherwise, it is called nonrepeatable. If the selective measurements are repeatable for all values \( m_i \), then the nonselective measurement is said to be repeatable.

Synonyms for "repeatable measurement" are "predictive", and "first-kind measurement". Nonrepeatable measurements are also said to be retrodictive or retrospective or of the second kind. (These concepts are not equivalent in the case of generalized observables.) Then one has ideal measurement of \( M \).

**Theorem 1**: (i) Let a general measurement \( \{ M_i : \forall i \} \) be the measurement of the value \( m_i \) of an ordinary discrete observable given by (1), i. e., let
\[
M_i^\dagger M_i = P_i. \tag{10}
\]
be valid. (Operator form of (6b) and (9) together.) Then one has
\[
M_i = M_i P_i. \tag{11}
\]
(ii) If and only if besides (10) also
\[
M_i = P_i M_i \tag{12}
\]
is valid, the selective measurement is repeatable. Otherwise, it is nonrepeatable.

(iii) The nonselective repeatable measurement of \( M \) is ideal if and only if
\[
\forall i : M_i = P_i, \tag{13}
\]
Proof:  (i) If (10) is valid, then the square norm of the vector $M_i (P_i^\dagger |\psi\rangle)$, where " $\perp$" denotes the orthocomplement, is zero: $\langle \psi | P_i^\dagger M_i^\dagger M_i P_i^\dagger | \psi \rangle = \langle \psi | P_i^\dagger P_i P_i^\dagger | \psi \rangle = 0$. Hence, also the vector itself is zero, and $M_i | \psi \rangle = M_i (P_i + P_i^\perp) | \psi \rangle = M_i P_i | \psi \rangle$. Since this is valid for an arbitrary vector $| \psi \rangle$, the claimed relation (11) follows.

(ii) As it is well known, an event (projector) $P$ is predicted with certainty in a state $| \psi \rangle$ if and only if $P | \psi \rangle = | \psi \rangle$. Hence, according to Definition 2, the measurement at issue is repeatable if and only if $P_i M_i | \psi \rangle = M_i | \psi \rangle$ (cf (5a)). Since the state vector $| \psi \rangle$ is arbitrary, the claimed criterion (12) ensues.

(iii) It has been proved in the detailed modern measurement theory given in ref. 13 (chapter III, subsection 3.7) that the state transformers of ideal measurement are those given by (13).

Corollary: If a state $\rho$ has a sharp value $m_i$ of an ordinary observable $M$ (given by (1)), then ideal nonselective measurement of $M$ does not change this state at all.

Proof: Let a pure state $| \psi \rangle$ have the sharp value $m_i$ of $M$. Then, $M | \psi \rangle = m_i | \psi \rangle$, i.e., $| \psi \rangle \langle \psi |$ is an eigenprojector of $M$, and hence it commutes with $M$. Its sharp value 1 has to be preserved in the corresponding state transformation (2c) (cf Theorem 1(iii)) on account of the definition of ideal measurement. Since $| \psi \rangle$ is the only state vector with this property, one has $\sum_i P_i | \psi \rangle \langle \psi | P_i = | \psi \rangle \langle \psi |$.

Let $\rho$ be a mixed state with the sharp value $m_i$ of $M$. Let us, further, perform ideal nonselective measurement of $M$ on $\rho$. According to (2c), this changes the state into $\sum_i (P_i \rho P_i)$. Writing $\rho$ in spectral form in terms of eigenprojectors $\rho = \sum_k (r_k | k \rangle \langle k |)$, the transformed state is $\sum_k (r_k \sum_i (P_i | k \rangle \langle k | P_i) \rho P_i)$. If $\rho$ has the claimed sharp value, then necessarily the same is true for each pure state $| k \rangle$ (see Appendix A). Hence, as shown in the proof in the preceding passage, they do not change. Therefore, neither does $\rho$. $\square$

On account of the no-change property expressed in the Corollary ideal measurement is hard to find in the laboratory. In practice the interactions that cause a change in the measuring apparatus (show a result) do change also the state of the quantum system. It is like a kind of action and reaction (like in the second law of Newton in classical mechanics). Theoretically there is no problem in writing down $U_{12}$ violating this principle (if we may call it so). (See ref. 1 for a modern presentation of the von Neumann procedure$^6$ of defining such a $U_{12}$.)

As it was seen in Definition 3, the concept of ideal measurement was restricted to the nonselective case. But, formally, if a state transformer is the Lüders one, we can speak of selective ideal measurement. For this there exists a laboratory realization: negative measurement. To give a simple example, let the Stern-Gerlach apparatus (cf ref. 3, p. 388), measuring the spin of a spin-one-half particle, be so adjusted that one knows when the particle enters it. Further, let the lower half of
the screen be a detector, and let the upper half be removed (letting the particle out of the apparatus without interacting with it). Then, if the particle is not detected when it is expected in the lower half space, it must be in the upper one, and therefore it must have spin up. This device is sometimes called the Stern-Gerlach preparator because it prepares the particle in the spin-up state.

One may wonder why most textbooks confine their presentation to just one kind of measurement, the ideal one, often disregarding the fact that it is almost impossible to realize it in practice. The answer, of course, is that until the last two decades (cf ref. 16) the only known state transformers were the Lüders ones. So, there was no framework for a more general theory of measurement.

**Definition 4:** If (10) does not hold, i.e., if $\Pi_i$ $(\equiv M_i^\dagger M_i)$ is a positive operator more general than a projector, then one speaks of a (selective) measurement of a generalized observable or, shortly, of a generalized measurement.

If one wonders why one needs generalized observables, which are mentioned but not actually discussed in this article, then ref. 14 may be a useful source of information.

**IV. POLAR FACTORIZATION OF STATE TRANSFORMERS**

Every linear operator $A$ can be written as a product of a unitary operator $U$ and a positive one $H$: $A = UH$. This is called the polar factorization of the operator (for proof and details see Appendix B).

**Theorem 2:** Let $\{U_i H_i : \forall i\}$ be a general measurement with the state transformers written in terms of their polar factors.

(i) The result corresponding to $i$ has the meaning of ordinary measurement if and only if there exists an event (projector) $P_i$ so that

$$H_i = P_i.$$  \hfill (14)

(ii) If the result corresponding to $i$ has the meaning of ordinary measurement, it is repeatable if and only if the unitary polar factor $U_i$ maps the range $R(P_i) \equiv \{P_i | \psi> : \forall | \psi>\}$ into itself.

(iii) A nonselective repeatable measurement of $M$ (given by (1)) is ideal if and only if

$$\forall i: \quad U_i = 1.$$  \hfill (15)

**Proof:** (i) Since $H_i^2 = M_i^\dagger M_i$ is an identity, the first claim is obvious in view of (10). (ii) The sufficiency of the claimed condition for repeatability is also obvious because, if valid, one can write $U_i P_i = (P_i U_i) P_i$, i.e., relation (12) follows. Conversely, if $| \psi> \in R(P_i)$, then $U_i | \psi> = U_i (P_i | \psi>)$. Further, (14) and (12) imply $U_i | \psi> = (P_i U_i P_i) | \psi> = P_i U_i | \psi>$. This amounts to $(U_i | \psi>) \in R(P_i)$ as claimed. (iii) The equivalence of (13) and (15) is obvious.

The reader is encouraged to show that commutation

$$[U_i, P_i] = 0$$  \hfill (16)

is necessary and sufficient for repeatability of selective measurement of an observable.
simple way. The various kinds of measurements in a transformers enables one to distinguish important from the laboratory point of view.

Secondly, the modern framework of state observables (not to mention generalized measurements). The latter are more im-

nately ideal; there are other, more complex, measurements of ordinary quantum observables (not to mention generalized measurements). The latter are more important from the laboratory point of view. Secondly, the modern framework of state transformers enables one to distinguish the various kinds of measurements in a simple way.

APPENDIX A

Proof of the general statement that if a mixed state $\rho$ has a sharp value $m_i$ of an observable $M$ (cf (1)) and a decom-

position of this state into pure ones $\rho = \sum_k w_k |k\rangle\langle k|$ is given ($\forall k : 0 < w_k \leq 1, \sum_k w_k = 1$), then also each pure state $| k \rangle$ has the sharp value of the same observable.

The state $\rho$ has the sharp value $m_i$ of $M = \sum_{i \neq j} m_i P_i$ (cf (1)) if $\text{Tr}\rho P_i = 1$. Substituting here the above decom-

position of $\rho$ into pure states and taking into account that $\sum_k w_k = 1$, one obtains $\sum_k w_k (1 - \langle k | P_i | k \rangle) = 0$. Then, in view of $\forall k : \langle k | P_i | k \rangle = ||P_i | k ||^2 \leq 1$, valid for every projector, one ends up with $\forall k : \langle k | P_i | k \rangle = 1$. Thus, the value $m_i$ is sharp also in every pure state $| k \rangle$.

APPENDIX B

The unique and the nonunique polar factorizations

Let $A$ be an arbitrary linear operator in a finite-dimensional linear unitary space $\mathcal{H}$. Clearly, $A^\dagger A$ is a positive operator. Let $Q$ be its range projector (taking $\mathcal{H}$ onto the range $\mathcal{R}(A^\dagger A)$).

Exercise 2: Show that one can write $A = AQ$. (Hint: Since $Q + Q^\perp = 1$ ($Q^\perp$ being the orthocomplementary projector), show the equivalent relation $AQ^{-1} = 0$, i. e., that $A^\dagger A | \psi \rangle = 0$ implies $A | \psi \rangle = 0$. To this purpose utilize the positive definiteness of the norm in $\mathcal{H}$.)

Utilizing (1) mutatis mutandis, one has $A^\dagger A = \sum_i m_i Q_i = \sum_i m_i \sum_j | ij \rangle \langle ij |$, a spectral form in terms of an eigenbasis $\{| ij \rangle : \forall ij \}$ in $\mathcal{R}(A^\dagger A)$, distinct and positive eigenvalues $m_i$ and eigenprojec-
tors $Q_i$, which are uniquely determined by the operator $A^\dagger A$. (Note that the ranges of the values of $j$ depend, in general, on the value of $i$.)

**Exercise 3:** Show that one can write

$$A = \left[A(\sum_i m_i^{-1/2} Q_i)\right] \times \left[\sum_i m_i^{1/2} Q_i\right] \equiv \tilde{U} H \quad (A.1)$$

in order to define $\tilde{U}$ and $H$.

**Exercise 4:** Show that the second factor on the RHS of (A.1) is $H = (A^\dagger A)^{1/2}$, and the first factor $\tilde{U}$ is a partial isometry, i.e., a linear map in $\mathcal{H}$ that takes a subspace of $\mathcal{H}$, in particular, $\mathcal{R}(A^\dagger A)$, onto another preserving the value of the scalar product, and, besides, it takes into zero the subspace orthocomplementary to $\mathcal{R}(A^\dagger A)$. (Hint: Show that, denoting by $(\ldots,\ldots)$ the scalar product, one has $(A | ij), A | i'j') = m_i^{1/2} m_{i'}^{1/2} \delta_{ii'} \delta_{jj'}$, and then $(\tilde{U} | ij), \tilde{U} | i'j') = \delta_{ii'} \delta_{jj'} = (|ij), |i'j'|$.)

The polar factors in (A.1) are evidently uniquely determined by $A$.

**Exercise 5:** Show that one can extend $\tilde{U}$ into a unitary operator $U$ in $\mathcal{H}$, and that $U$ is nonuniquely determined by $A$ unless the latter is nonsingular. (Hint: Utilize an arbitrary incomplete isometry mapping $\mathcal{R}(A^\dagger A)^\perp$ onto the subspace $[\tilde{U} \mathcal{R}(A^\dagger A)]^\perp$ preserving the scalar product in $\mathcal{R}(A^\dagger A)^\perp$ and taking $\mathcal{R}(A^\dagger A)$ into zero.) Further, show that one can rewrite (A.1) in the form

$$A = U H. \quad (A.2)$$

Polar factorization (A.2), though in general nonunique, can be more practical than (A.1), because a unitary operator is a concept that is simpler and more familiar than the notion of a partial isometry.

**APPENDIX C**

An example of two spin-one-half particles

We consider the tensor product of two (two-dimensional) spin-one-half state spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$. We utilize the basis $\{|n\rangle_1 \otimes |n'\rangle_2 : n, n' = \uparrow, \downarrow\}$, which consists of tensor products of single-particle state vectors that are spin-up or spin-down along the $z$-axis (the standard basis). Denoting by $\sigma$ the Pauli matrix (in standard representation), we want to specify a simple observable:

$$M \equiv \sigma_1 \otimes 1_2 = P_\uparrow - P_\downarrow,$$

where both for $n = \uparrow$ and $n = \downarrow$

$$P_n \equiv |n\rangle_1 \langle n|_1 \otimes 1_2.$$

Show that: (i) If one defines the state transformers as $M_n \equiv P_n, \ n = \uparrow, \downarrow$, then one has ideal measurement of $M$. (ii) If one defines $M_n \equiv (1_1 \otimes U_2(n)) P_n, \ n = \uparrow, \downarrow$, where $U_2(n)$ is an arbitrary unitary operator in $\mathcal{H}_2$, then one deals with repeatable selective measurement of $M$. If all $M_n$ are defined in this way, and the $U_2(n)$ are chosen separately for each value of $n$, one has nonselective repeatable measurement. (iii) Finally, if one defines $M_n \equiv U_{12}(n) P_n, \ n = \uparrow, \downarrow$, where $U_{12}(n)$ is an arbitrary unitary operator in the composite space but such that it does not act as the
identity operator in $\mathcal{H}_1$, then one has non-repeatable selective measurement. If all $M_n$ are defined in this way, and the operators $U_{12}(n)$ are chosen separately for each $n$, we have nonselective nonrepeatable measurement of $M$.

1J. Preskill, *Quantum Computing Lectures* (http://www.theory.caltech.edu/people/preskill/ph229/) 1997, chapter 3.
2M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge Univ. Press, Cambridge UK) 2000, subsection 2.2.
3C. Cohen-Tannoudji, B. Diu and F. Laloe, *Quantum Mechanics* Volume 1 (John Wiley and Sons, New York) 1977, chapter III, section C.
4A. Messiah, *Quantum Mechanics* (North Holland, Amsterdam) 1961, vol. I, chapter VIII, subsection I.2.
5F. A. Kaempffer, *Concepts in Quantum Mechanics* (Academic Press, New York) 1965, chapters 5 and 6.
6J. von Neumann *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton) 1955.
7G. Lüders, "On the change of state in the process of measurement" (in German), Ann. der Physik, 8, 322-328 (1951).
8F. Herbut, "Derivation of the change of state in measurement from the concept of minimal measurement", Ann. Phys., 55, 271-300 (1969).
9F. Herbut, "Minimal-disturbance measurement as a specification in von Neumann’s quantal theory of measurement", Intern. J. Theor. Phys., 11, 193-204 (1974).
10W. H. Zurek, "Decoherence and the transition from quantum to classical", Physics Today 44, October, 36-44 (1991).
11A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Ac. Publ., Dordrecht) 1993, chapter 5, section 3.
12F. Herbut and M. Vujlič, "Distant measurement", Ann. Phys. (N. Y.), 96, 382-405 (1976).
13P. Busch, P. J. Lahti, and T. Mittelstaedt, *The Quantum Theory of Measurement*, Lecture Notes in Physics, M2 (Springer Verlag, Berlin) 1991.
14P. Mittelstaedt, *The Interpretation of Quantum Mechanics and the Measurement Process* (Cambridge University Press, Cambridge) 1998.
15P. Busch, M. Grabowski and P. J. Lahti, "Repeatable measurements in quantum theory: their role and feasibility", Found. Phys. 25 1239-1266 (1995).
16V. Vedral, "The role of relative entropy in quantum information theory", Rev. Mod. Phys., 74, 197-234 (2002).
17K. Kraus, *States, Effects, and Operations* (Springer-Verlag, Berlin) 1983.
18E. B. Davies, *Quantum Theory of Open Systems* (Academic Press, New York) 1976, chapter 4, section 1.
19R. Schumacher, "Sending entanglement through noisy quantum channels", Phys. Rev. A 54, 2614-2628 (1996).
20A. E. S. Green, *Nuclear Physics* (McGraw-Hill, New York) 1955.
21V. B. Braginsky, Y. I. Vorontsov, and K. S. Thorne, "Quantum nondemolition measurements", Science 269, 547-557 (1980). Reprinted in J. A. Wheeler and W. H. Zurek editors, *Quantum Theory and Measurement* (Princeton University Press, Princeton) 1983, p. 749.
22F. Harrison, "Measure for measure in quantum theory", Physics World March, 24-25 (1996).