CONTROLLABILITY OF NETWORKS OF ONE-DIMENSIONAL SECOND ORDER P.D.E. – AN ALGEBRAIC APPROACH

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Abstract. We discuss controllability of systems that are initially given by boundary coupled p.d.e. of second order. Those systems may be described by modules over a certain subring $\mathcal{R}$ of the ring $\mathcal{M}_0$ of Mikusiński operators with compact support. We show that the ring $\mathcal{R}$ is a Bézout domain. This property is utilized in order to derive algebraic and trajectory related controllability results.

1. Introduction

The solution of control design problems is, in general, preceded by a controllability analysis of the system under consideration. While for linear finite dimensional systems, both algebraic and analytic controllability notions are used in parallel, the analysis of infinite dimensional systems is dominated by (functional) analytic methods [9]. The latter approach has proven to be useful, in particular for the analysis of state space controllability, i.e., the possibility of steering the system under consideration from a given initial state to a desired final state. For example, controllability of the same class of systems as considered in the present contribution has been analyzed this way in [10,20]. However, when focusing on trajectory tracking problems, the behavioural controllability notion due to Willems [45] is an interesting alternative to classical state space controllability. The connections between this approach and the algebraic system properties have been pointed out in the past for several classes of distributed parameter systems, as for so called multidimensional systems [33, 49], for delay systems [17, 30], and more general convolutional systems [43, 44]. In addition, the algebraic viewpoint is closely related to parametrization of trajectories of the system under consideration. This constructive nature makes the approach very attractive for applications, in particular for open loop control design. Finally, paying attention to particular structural properties of the models under consideration may result in a deeper understanding of the respective class of systems.

From the algebraic (module theoretic) viewpoint, as used within this contribution, a linear system is a finitely presented module. This notion was first introduced by Fliess for linear finite dimensional systems in [11]. For this class of systems, the freeness of the module corresponds to the flatness of the system under consideration in the sense of the theory of nonlinear finite dimensional systems while its basis corresponds to a flat output [12]. Moreover, torsion freeness, i.e., the absence of autonomous subsystems, is equivalent to freeness. The module theoretic approach is applicable to systems with distributed parameters and lumped controls as well: The convolutional equations associated with a given boundary value problem serve as defining relations for the system module, the latter defined over a certain ring of generalized functions. Such coefficient rings are generally not principal ideal domains. For this reason, the two basic controllability related properties, torsion freeness and freeness, are not necessarily equivalent. An approach to circumvent the problems caused by this “lack of structure” is the concept of $\pi$-freeness, which relies on localization and was at first developed for linear delay systems [13]. This way a basis can be introduced at least within an appropriate extension of the module under consideration. The approach has been proven to be very useful for both trajectory planning and open loop control design [21, 31, 36, 37, 39, 46, 47]. Nevertheless it seems to be difficult to compare such purely algebraic controllability notions to the behavioural ones. For this reason, within the present contribution we do not use localization.

This paper addresses the development of an algebraic approach to the controllability of networks of spatially one-dimensional parabolic and hyperbolic constant coefficient p.d.e. of second order. Here, by a network we understand a system consisting of several branches each of which is governed

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by a system of p.d.e. and which are coupled via the boundary conditions. We investigate, through algebraic properties of the coefficient rings obtained for considered class of p.d.e., the related controllabilities of the associated system module and establish some controllability results including module theoretic and behavioral ones. In accordance with [38, 39, 46], we use the general solution of the Cauchy problem with respect to space in order to rewrite the given model as a linear system of convolutional equations. The latter are regarded as the defining relations of a finitely presented module. It turns out that the coefficient ring of this module, a subring of the ring of Mikusiński operators with compact support introduced in [1], is a Bézout domain, i.e., every finitely generated ideal is principal. An algorithm enabling us to calculate the generator of a finitely generated ideal is presented within this paper. This latter result is strongly inspired by those derived in [2, 17] for particular rings of distributed delay operators which in our setting may arise from the wave equation. The derived properties of the coefficient ring allow us to decompose the systems of p.d.e. which are coupled via their boundary conditions. We show how to pass from this model to a system of convolution equations giving rise to our module theoretic setting. Section 3 is devoted to the study of the coefficient ring of this module. In Section 4 we obtain several controllability results for the systems under consideration. Finally, in Section 5 we apply the method to a system example of two boundary coupled p.d.e.

2. Boundary value problems as convolutional systems

2.1. Models considered. We assume, that the model equations for the distributed variables in $w_1, \ldots, w_l$ and the lumped variables in $u = (u_1, \ldots, u_m)$ are given by

\begin{align}
\partial_x w_i &= A_i w_i + B_i u, \quad w_i : \Omega_i \to \mathcal{B}^2, \quad u \in \mathcal{B}^m \\
A_i &\in (\mathbb{R}[s])^{2 \times 2}, \quad B_i \in (\mathbb{R}[s])^{2 \times m}
\end{align}

where $\mathcal{B}$ denotes an appropriate space of Boehmians (see, e.g., [6, 25, 27] and App. 2) and $s$ is the differentiation operator with respect to time. The assumptions which are crucial for the applicability of our approach are twofold. First, we assume that all the matrices $A_1, \ldots, A_l$ give raise to the same characteristic polynomial, namely,

\begin{equation}
\det(\lambda I - A_i) = \lambda^2 - \sigma, \quad \sigma = as^2 + bs + c \neq 0, \quad a, b, c \in \mathbb{R}, \quad a \geq 0.
\end{equation}

Additionally, we require the intervals $\Omega_1, \ldots, \Omega_l$ of definition of the above differential equations to be rationally dependent. More precisely, we assume the $\Omega_i$ ($i = 1, \ldots, l$) to be given by an open neighbourhood of

\begin{equation}
\bar{\Omega}_i = [x_{i,0}, x_{i,1}], \quad \ell_i = x_{i,1} - x_{i,0} = q_i \ell, \quad q_i \in \mathbb{Q}, \quad \ell \in \mathbb{R}.
\end{equation}

In the following, and without further loss of generality, we assume $x_{i,0} = 0$. The model is completed by the boundary conditions

\begin{equation}
\sum_{i=0}^l L_i w_i(0) + R_i w_i(\ell_i) + Du = 0
\end{equation}

where $D \in (\mathbb{R}[s])^{q \times m}$ and $L_i, R_i \in (\mathbb{R}[s])^{q \times 2}$.

**Remark 2.1.** In a more general setting, instead of the boundary conditions (1d), one could consider auxiliary conditions of the form

\begin{equation}
\sum_{i=0}^l Q_i(w_i) + Du = 0.
\end{equation}
Here,
\[ Q_i(w_i) = \sum_{j=0}^{\nu} L_{i,j} w_i(\alpha_{i,j} x) + \sum_{j=1}^{\nu} \int_{\Omega_{i,j}} Q^*_{i,j}(x) w_i(x) dx \]
with \( L_{i,j} \in (\mathbb{R}[s])^{q \times 2}, \) \( Q^*_{i,j} \in (\mathbb{R}[s,x])^{q \times 2}, \) \( \Omega_i \supset \Omega_{i,j} = [\beta_{i,j,1}, \beta_{i,j,2}], \) \( \alpha_{i,j}, \beta_{i,j,k} \in \mathbb{Q} \cap \Omega_i, \) \( \mu, \nu \in \mathbb{N}. \)

### 2.2. Solution of the initial value problem.
This section deals with the solution of a single initial value problem of the form (1a) with initial conditions given at \( x = x_2. \) Solution of the initial value problem (2) with \( A \) having the same properties as \( A_i, \) \( B_i \) \((i = 1, \ldots, l)\) introduced within the previous section. To this end, we start with the initial value problem
\[ \partial_x w = Aw + Bu, \quad w(\xi) = w_\xi \]
with \( A, B \) having the same properties as \( A_i, B_i \) \((i = 1, \ldots, l)\) introduced within the previous section.

This yields in particular
\[ (\partial^2_x - \sigma) S(x) = 0, \quad S(x) = 0, \quad (\partial_2 S)(x) = 1, \]
associated with the characteristic equation (1a). It is well known that this equation has a unique operational solution as long as the principal part \( \partial^2_x - \alpha^2 \) of the differential operator \( \partial^2_x - \sigma \) is hyperbolic w.r.t. the parallels of \( x = 0. \) This was implicitly required above by assuming \( \alpha \geq 0 \) in (1b). Moreover, under these assumptions the operator \( S \) as well as its derivative \( C = \partial_x S \) correspond to infinitely differentiable functions mapping \( \Omega \) to the ring \( \mathcal{M}_0 \) of Mikusiński operators with compact support \( \mathcal{M}_0 \) (see App. B and [1] for results related to the support of Mikusiński operators and App. [A] for explicit expressions for \( C(x) \) and \( S(x) \)).

Using the above defined operational functions one easily verifies that the (unique) solution \( x \mapsto \Phi(x, \xi) \) of the initial value problem
\[ \partial_x \Phi(x, \xi) = A\Phi(x, \xi), \quad \Phi(\xi) = I, \]
with \( I \) denoting the identity, is given by
\[ \Phi(x, \xi) = A S(x - \xi) + I C(x - \xi). \]
From the uniqueness of the solution one deduces the addition formula
\[ \Phi(x, \xi) \Phi(\xi, \xi) = \Phi(x, \xi). \]
For \( A \) the companion matrix of the characteristic polynomial, \( i.e., \)
\[ A = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix}, \quad \Phi(x, \xi) = \begin{pmatrix} C(x) & S(x) \\ \sigma S(x) & C(x) \end{pmatrix}, \]
this yields in particular
\[ C(x + y) = C(x) C(y) + \sigma S(x) S(y), \quad S(x + y) = C(x) S(y) + S(x) C(y). \]
The solution of the initial value problem associated with the inhomogeneous equation
\[ \partial_x \Psi(x, \xi) = A \Psi(x, \xi) + B \]
with homogeneous initial conditions, prescribed at \( x = \xi, \) is obtained using the well known variation of constants method. This yields
\[ \Psi(x, \xi) = \int_\xi^x \Phi(x, \zeta) d\zeta B. \]
Thus, the general solution of the initial value problem (2) reads
\[ w(x) = \Phi(x, \xi) w_\xi + \Psi(x, \xi) u. \]
The entries of the matrix \( \Phi \) belong clearly to \( \mathbb{C}[s, C, S]. \) Contrary, according to (4), the entries of \( \Psi \) may contain also the integrals of \( S \) and \( C. \) However, if \( \sigma \neq 0 \) then \( A \) is invertible over \( \mathbb{C}(s). \)

Thus, using the fact that \( \partial_x \Phi(x, z) = -\Phi(x, z) A \) those integrals can be expressed as
\[ \int_\xi^x \Phi(x, \zeta) d\zeta = -\int_\xi^x \partial_x \Phi(x, \zeta) A^{-1} d\zeta = (\Phi(x, \xi) - I) A^{-1}. \]

More specifically, instead of stating that \( S(x), C(x) \in \mathcal{M}_0, \) one could distinguish the cases \( a > 0 \) and \( a = 0. \) In the first case both operators, \( S(x) \) and \( C(x), \) correspond to distributions of order zero with compact support, while in the latter case they correspond to ultra-distributions of Gevrey order \( 2 \) and support in \( 0. \) Both of these spaces may be embedded in \( \mathcal{M}_0. \)
Choosing \( A \) as in (4) one obtains in particular
\[
\int_0^x C(\zeta) \, dx = S(x), \quad \int_0^x S(\zeta) \, dx = (C(x) - 1)/\sigma.
\]
Later on, the latter equations will essentially ease our controllability analysis.

2.3. A module presented by a system of convolution equations. In the previous section we have discussed the solutions of the initial value problems associated with the equations \((1a)\). In the sequel these results are used in order to define an algebraic structure representing the model under consideration.

Substituting the general solutions of the initial value problems into the boundary conditions, one obtains the following linear system of equations:
\[
W(x) = W_\xi(x) \xi_c, \quad P_\xi \xi_c = 0.
\]
Here, \( \xi = (\xi_1, \ldots, \xi_n), \ c_\xi^T = (w^T(\xi_1) \cdots w^T(\xi_l), u^T), \)
\[
W_\xi = \begin{pmatrix}
\Phi_1(x, \xi_1) & 0 & 0 & \Psi_1(x, \xi_1) \\
0 & \ddots & 0 & \\
0 & \cdots & \Phi_l(x, \xi_l) & \Psi_l(x, \xi_l)
\end{pmatrix}, \quad P_\xi = (P_{\xi,1} \cdots P_{\xi,l+1})
\]
with
\[
P_{\xi,i} = L_i \Phi_i(0, \xi_i) + R_i \Phi_i(\ell, \xi_i), \quad i = 1, \ldots, l
\]
\[
P_{\xi,l+1} = D + \sum_{i=1}^l L_i \Psi_i(0, \xi_i) + R_i \Psi_i(\ell, \xi_i).
\]
A suitable algebraic object for the representation of the model under consideration should contain all the system variables, i.e., the distributed variables in \( w \), their values as well as their (spatial and temporal) derivatives. Moreover, it should reflect not only the structure imposed by the original boundary value problem \((1a)\) but also that imposed by the solution of the according initial value problems, i.e., by equation \((8)\). In order to analyze the model we will, therefore, use a module generated by the variables collected in \( c_\xi \) with the presentation given in \((8)\) \([13,14,16,29]\). The choice of the coefficient ring, which has to contain at least the entries of \( W_\xi \) and \( P_\xi \), is discussed below.

According to the previous section, the entries of \( W_\xi \) and those of \( P_\xi \) are composed of the functions \( C \) and \( S \) mapping \( \mathbb{R} \) to \( M_0 \) and all the values of these functions. Moreover, the matrices may involve values of the spatial integrals of \( C \) and \( S \), too. Thus a possible choice for the coefficient ring is the ring \( \mathcal{R}_\mathbb{R}^l[s, \mathcal{G}, \mathcal{G}^f] \). Here, for any \( \mathcal{X} \subseteq \mathbb{R} \), \( \mathcal{R}_\mathbb{R}^l = [\mathcal{G}_\mathcal{X}, \mathcal{G}_\mathcal{X}^f] \), with
\[
\mathcal{G} = \{C, S\}, \quad \mathcal{G}_\mathcal{X} = \{C(\zeta), S(\zeta)|z \in \mathcal{X}\},
\]
\[
\mathcal{G}^f = \{C^f, S^f\}, \quad \mathcal{G}_\mathcal{X}^f = \{C^f(\zeta), S^f(\zeta)|z \in \mathcal{X}\},
\]
\( \ell \) defined as in \((16)\), and
\[
S^f(x) = \int_0^x S(\zeta) \, d\zeta, \quad C^f(x) = \int_0^x C(\zeta) \, d\zeta.
\]
Inspired by the results given in \([2,17,30]\), and in view of the simplification of the analysis of the module properties, instead of the ring \( \mathcal{R}_\mathbb{R} \), we will use a slightly larger ring, given by \( \mathcal{R}_\mathbb{R} = \mathbb{C}(s)[\mathcal{G}_\mathbb{R}] \cap M_0 \).

Definition 2.1. The convolutional system \( \Sigma \) associated with the boundary value problem \((1)\) is the module generated by the elements of \( c_\xi \) over \( \mathcal{R} = \mathcal{R}_\mathbb{R} [\mathcal{G}, \mathcal{G}^f] \) with presentation matrix \( P_\xi \). By \( \Sigma_{\mathcal{R}} \) (resp. \( \Sigma_Q \)) we denote the same system but viewed as a module over \( \mathcal{R}_\mathbb{R} \) (resp. \( \mathcal{R}_Q \)).

One easily verifies that \( \Sigma \) does not depend on the choice of \( \xi \) (cf. \([39,46]\)). In view of the assumed mutual rational dependence of the lengths \( \ell_1, \ldots, \ell_l \), for the analysis of the system properties, we will start with the system \( \Sigma_Q \), i.e., a system containing the values of the distributed variables at rational multiples of \( \ell \) only. However, having analyzed the properties of \( \Sigma_Q \), we may pass to \( \Sigma_{\mathcal{R}} \) (resp. \( \Sigma \)) by an extension of scalars, i.e., \( \Sigma_{\mathcal{R}} = \mathcal{R}_\mathbb{R} \otimes_{\mathcal{R}_Q} \Sigma_Q \) (resp. \( \Sigma = \mathcal{R} \otimes_{\mathcal{R}_Q} \Sigma_Q \)).
3. The operator ring $\mathcal{R}_Q$ is a Bézout domain

In this section we study the structures of the ideals within the rings $\mathcal{R}_Q$. To this end, we first establish some results on the ideals in $C(s)[\mathcal{S}_Q]$ and $C(s)[\mathcal{S}_N]$.

3.1. Ideals in $C(s)[\mathcal{S}_Q]$ and $C(s)[\mathcal{S}_N]$. This section is devoted to the analysis of the ideals in $C(s)[\mathcal{S}_Q]$ and $C(s)[\mathcal{S}_N]$. The obtained results will be the key for the analysis of the properties of $\mathcal{R}_Q$ and $\mathcal{R}_N$ introduced in Section 2.3.

In the following, we will replace $C(s)$ by any field $k$. Moreover, the generating set $\mathcal{S}_X$ may be replaced by any set $\mathcal{S}$ the elements of which satisfy the addition formulas derived in Section 2.2.

**Definition 3.1.** Let $k$ be a field and $\mathcal{X}$ an additive subgroup of $\mathbb{R}$. By $\mathcal{S}_X$ we denote the set $\{S_a, C_{\alpha}|a \in \mathcal{X}\}$ the elements of which are subject to the following relations ($\sigma \in k$):

\[
\begin{align}
(9a) & \quad C_a C_b \pm \sigma S_a S_b = C_{a \pm b}, \quad S_a C_b \pm C_a S_b = S_{a \pm b} \\
(9b) & \quad C_0 = 1, \quad S_0 = 0.
\end{align}
\]

From the above definition one easily deduces

\[
\begin{align}
(10a) & \quad C_a = C_{-a}, \quad S_a = -S_{-a} \\
(10b) & \quad 2C_a C_b = C_{a+b} + C_{a-b}, 2\sigma S_a S_b = C_{a+b} - C_{a-b}, 2C_a S_b = S_{a+b} - S_{a-b}.
\end{align}
\]

Moreover, any element $r \in k[\mathcal{S}_X]$ can be written in the form

\[
r = \sum_{i=0}^{n} a_{\alpha_i} C_{\alpha_i} + b_{\alpha_i} S_{\alpha_i}, \quad n \in \mathbb{N}, \quad a_{\alpha_i}, b_{\alpha_i} \in k, \quad \alpha_i \in \mathcal{X}^+
\]

where $\mathcal{X}^+ = \{|\alpha| : \alpha \in \mathcal{X}\}$. Finally the units in $k[\mathcal{S}_X]$ belong to $k$.

In the following, it is necessary to distinguish the cases where the equation $\lambda^2 - \sigma = 0$ has a solution over $k$ or not. For our application this is clearly equivalent to the question whether the roots of the characteristic equation (11) belong to $\mathbb{R}[s]$. The necessity to distinguish these cases is explained by the following simple example which, in addition, shows that the cases $\mathcal{X} = \mathbb{N}$ and $\mathcal{X} = \mathbb{Q}$ need to be analyzed separately.

**Example 3.1.** Consider the ideal $\mathcal{I} = (a, b)$, $a = S_1$, $b = C_1 + 1$. Over $k[\mathcal{S}_Q]$ we have

\[
a = S_1 = 2C_{1/2}S_{1/2}, \quad b = C_1 + 1 = 2C_{1/2}^2.
\]

Thus, both generators belong to $(C_{1/2})$ which, conversely, belongs to $\mathcal{I}$ since $2C_{1/2} = -\sigma S_{1/2}a + C_{1/2}b$. The ideal $\mathcal{I}$ is, therefore, generated by $C_{1/2}$ which does not belong to $k[\mathcal{S}_N]$ if $\lambda^2 - \sigma$ is irreducible over $k$. However, the situation is different if $\sqrt{\sigma}$ belongs to $k$. From the relations given in (11) and (13), it follows immediately that

\[
(C_{1/2} + \sqrt{\sigma} S_{1/2})(C_{1/2} - \sqrt{\sigma} S_{1/2}) = 1.
\]

Over $k[\mathcal{S}_Q]$, $C_{1/2}$ can be factorized as

\[
C_{1/2} = (C_{1/2} + \sqrt{\sigma} S_{1/2})(1 + C_1 - \sqrt{\sigma} S_1)/2.
\]

The element $C_{1/2}$ is, thus, associated with $1 + C_1 - \sqrt{\sigma} S_1$ which indeed belongs to $k[\mathcal{S}_N]$.

3.1.1. The polynomial $\lambda^2 - \sigma$ is reducible over $k$.

**Proposition 3.1.** The ring $k[\mathcal{S}_N]$ is a PID.

**Proof.** From the addition formulas given in (11) and (13), it follows that $k[\mathcal{S}_N]$ is isomorphic to $k[S_1, C_1]$ which, in turn, is isomorphic to $k[z^{-1}, z]$ by

\[
S_1 \mapsto z^{-1} - z^{-1}/\lambda, \quad C_1 \mapsto z^{-1} + z^{1}.
\]

The latter ring is Euclidean with the norm function given by the difference of the degrees of the monomials of maximal and minimal degree w.r.t. $z$. \(\square\)

**Corollary 3.1.** The ring $k[\mathcal{S}_Q]$ is a Bézout domain.
Lemma 3.1. Let \( \{a_n\} \) be written as \( \{b_n\} \) where the coefficients with odd or those with even indices vanish. More precisely, any element \( \alpha \in \mathcal{Q} \) of \( \mathbb{C} \) is obtained from polynomials which is constructed from point delays: From \( \lambda = s\sqrt{\alpha} - \alpha, \alpha \in \mathbb{C} \) we obtain \( 2\mathcal{C}(x) = e^{s\sqrt{\alpha}} + e^{-(s\sqrt{\alpha})}. \)

Remark 3.1. Note that, for the same reason as given in Remark 3.3, \( k[\mathcal{Q}] \) is not a PID.

Remark 3.2. Having in mind that in our application \( \sigma \) is given according to equation (10) and \( k = \mathbb{C}(s) \), for \( \sqrt{\sigma} \in k \), say \( \sqrt{\sigma} = \lambda \), the operators \( C(x) \) and \( S(x) \) introduced in Section 2.2 are constructed from point delays: From \( \lambda = s\sqrt{\alpha} - \alpha, \alpha \in \mathbb{C} \) we obtain \( 2\mathcal{C}(x) = e^{s\sqrt{\alpha}} + e^{-(s\sqrt{\alpha})}. \)

Note that in this case our results are simply a restatement of those presented in [2, 17].

3.1.2. The polynomial \( \lambda^2 - \sigma \) is irreducible over \( k \). As indicated by Example 3.1, the second case, i.e., that \( \lambda^2 = 0 \), has no solutions over \( k \), is much more challenging than the first one. There, the ring \( k[\mathcal{Q}] \) corresponds basically to the ring \( \mathbb{Q}[x, y]/(x^2 + y^2 - 1) \) of trigonometric polynomials which is obtained from \( k[\mathcal{Q}] \) for \( \sigma = -1 \) and \( k = \mathbb{Q} \). The latter ring is lacking the pleasing properties of a PID or even a Bézout domain. However, Example 3.1 suggests, that the difficulties can be circumvented when allowing to halve the argument, i.e., working with \( k[\mathcal{Q}] \) instead of \( k[\mathcal{Q}] \).

Definition 3.2. For any nonzero \( r \in k[\mathcal{Q}] \) the norm \( \nu(r) \) is defined as the highest \( \alpha \in \mathbb{X}^+ \) such that at least one of the coefficients \( a_\alpha \) and \( b_\alpha \) in (11) is nonzero.

Lemma 3.1. Let \( S \) the multiplicative subset of \( k[\mathcal{Q}] \) consisting of all the elements such that either the coefficients with odd or those with even indices vanish. More precisely, any element \( s \) of \( S \) can be written as

\[
s = \sum_{i \in I_s} a_{s,i}C_i + b_{s,i}S_i,
I_s = \left\{ \nu(s) - 2i \mid i \in \mathbb{Z}, 0 \leq i \leq \frac{\nu(s)}{2} \right\}
\]

Let \( p, q \in S \) the norms of which are strictly positive. Without loss of generality assume \( \nu(p) \geq \nu(q) \). Consider the ideal \( \mathfrak{I} = (p, q) \) generated by \( p \) and \( q \). Then there exists \( \bar{p}, \bar{q} \in S \) with \( \mathfrak{I} = (\bar{p}, \bar{q}) \) and either \( \nu(p) > \nu(\bar{p}) \geq \nu(\bar{q}) \) or \( \bar{q} = 0 \).

Proof. In the following, three different cases are considered.

Case 1. If \( \nu(p) > \nu(q) \) one can apply a division step similar to that of polynomials. More precisely, we will show that there exists \( r, h \in S \) with either \( r = 0 \) or \( \nu(r) < \nu(p) \) such that \( p = qh + r \). Then we may set \( \bar{p} = q, \bar{q} = r \) (or vice versa) to complete the discussion of the first case.

In order to show that \( r, h \) with the claimed properties exist set

\[
h = a_hC_\Delta + b_hS_\Delta,
\Delta = \nu(p) - \nu(q)
\]

where the coefficients \( a_h, b_h \in k \) have to be determined appropriately. It follows

\[
s = \sum_{i \in I_q} \left((a_{h,i}a_{q,i}C_i + b_{h,i}b_{q,i}C_iS_i) + (a_hb_{q,i}S_iC_\Delta + b_hq_iS_iS_\Delta)\right)
\]

\[
= \frac{1}{2\sigma} \sum_{i \in I_q} \left((\sigma a_ha_{q,i} + b_hb_{q,i})C_{\Delta+i} + (\sigma a_ha_{q,i} - b_hb_{q,i})C_{\Delta-i}\right)
+ \sigma(b_ha_{q,i} + a_hb_{q,i})S_{\Delta+i} + \sigma(b_ha_{q,i} - a_hb_{q,i})S_{\Delta-i}
\]

\[
= \sum_{i \in I_p} a_{s,i}C_i + b_{s,i}S_i
\]

where the leading coefficients are given by

\[
a_{s,\nu(p)} = \frac{1}{2\sigma}(\sigma a_ha_{q,\nu(q)} + b_hb_{q,\nu(q)}),\quad b_{s,\nu(p)} = \frac{1}{2}(b_ha_{q,\nu(q)} + a_hb_{q,\nu(q)}).
\]

From this equation and from \( r = hq - p \) the norm of \( r \) is smaller than that of \( p \) if and only if \( a_h, b_h \) satisfy

\[
\begin{pmatrix} a_{q,\nu(q)} & \sigma^{-1}b_{q,\nu(q)} \\ b_{q,\nu(q)} & a_{q,\nu(q)} \end{pmatrix} \begin{pmatrix} a_h \\ b_h \end{pmatrix} = 2 \begin{pmatrix} a_{p,\nu(p)} \\ b_{p,\nu(p)} \end{pmatrix}.
\]

By the definition of the norm at least one of the coefficients \( a_{q,\nu(q)} \) and \( a_{q,\nu(q)} \) is nonzero. Since, additionally, \( \sqrt{\sigma} \notin k \) it follows \( \sigma a_{q,\nu(q)}^2 - b_{q,\nu(q)}^2 \neq 0 \) and \( a_h, b_h \) can be always chosen according to (12).

\[\text{Remark 3.1.} \text{Actually, the trigonometric ring is a Dedekind domain [7, 8, 32].}\]
Case 2. If $\nu(p) = \nu(q)$ and for some $c \in k$ the equations $a_{p,q}(\nu(q)) = ca_{p,q}(p)$, $b_{p,q}(\nu(q)) = cb_{p,q}(p)$ hold, the ideal $\mathcal{I}$ is generated by $p = p$, $q = q - cp$ where $\nu(q) < \nu(p)$. If $\bar{q} = 0$ the proof is complete otherwise we can proceed according to the first case with the pair $\bar{p}, \bar{q}$ instead of $p, q$.

Case 3. If $\nu(p) = \nu(q)$ but we are not in the second case set

\begin{align}
(13a) & \quad (p \quad q)^T = A_1 (\bar{p} \quad \bar{q})^T \\
(13b) & \quad (\bar{p} \quad \bar{q})^T = A_2 (\bar{p} \quad \bar{q})^T
\end{align}

with

$$A_1 = \begin{pmatrix} a_{p,n} & b_{p,n} \\ a_{q,n} & b_{q,n} \end{pmatrix}, \quad A_2 = \begin{pmatrix} C_1 & \sigma S_1 \\ S_1 & C_1 \end{pmatrix}, \quad n = \nu(q) = \nu(p)$$

Obviously, $p, q$ belong to the ideal generated by $\bar{p}, \bar{q}$. Both matrices, $A_1$ and $A_2$, are invertible, the first one since otherwise we would be in the second case, the latter one since, by (10), its determinant equals 1. Thus, $(\bar{p}, \bar{q}) = (p, q)$.

It remains to show that the norms of $\bar{p}$ and $\bar{q}$ are both smaller than $n$. From equation (13a) one obtains $\nu(\bar{p}) = \nu(\bar{q}) = n$ with $\bar{p}, \nu(\bar{q}) = b_{\bar{p}}, \nu(\bar{q}) = a_{\bar{q}}, \nu(\bar{q}) = 0$. From (13b) one has

$$\bar{p} = C_1 C_n - \sigma S_1 S_n + \sum_{i \in I_p} C_1 (a_{\bar{p},i} C_i + b_{\bar{p},i} S_i) - \sigma S_1 (a_{\bar{q},i} C_i + b_{\bar{q},i} S_i)$$

$$\bar{q} = C_1 S_n - S_1 C_n + \sum_{i \in I_p} C_1 (a_{\bar{q},i} C_i + b_{\bar{q},i} S_i) - S_1 (a_{\bar{p},i} C_i + b_{\bar{p},i} S_i)$$

with $I_p^c = I_p \setminus \{n\}$. The norms of the sums in the above expression are at most $n - 1$ while for the leading terms one obtains according to (13a)

$$C_1 C_n - \sigma S_1 S_n = C_{n-1}, \quad C_1 S_n - S_1 C_n = S_{n-1}.$$

Thus, the norms of $\bar{p}, \bar{q}$ cannot exceed $n - 1$.□

Lemma 3.2. Let $p, q \in S \subseteq k[\mathcal{G}]$ with $\nu(p) \geq \nu(q)$. Then there exists $\bar{p}, \bar{q} \in S \cap (p, q)$ such that $(p, q) = (\bar{p}, \bar{q})$ and $\nu(\bar{q}) < \nu(q)$, $\nu(p) \leq \nu(q)$ or $\bar{q} = 0$.

Proof. By Lemma 3.1 $(p, q) = (p^*, q^*)$ with $\nu(p) > \nu(p^*) \geq \nu(q^*)$ or $q^* = 0$. In the latter case the claim has been proved. Otherwise, repeat the above argument $p^*, q^*$ until we are in the claimed situation which happens after at most $\nu(p) - \nu(q) + 1$ steps.□

Proposition 3.2. Any ideal $\mathcal{I}$ in $k[\mathcal{G}]$ generated by a subset $\mathcal{G}$ of $S$ is principal.

Proof. Step 1. We show that up to multiplication with units there is only one element $q$ of lowest norm $\nu(q) = n$ in $S \cap \mathcal{I}$. To this end, assume there are at least two such elements, say $p$ and $q$. By Lemma 3.2 there exist $\bar{p}, \bar{q} \in S$ with $(\bar{p}, \bar{q}) = (p, q)$ where $n > \nu(\bar{p}) \geq \nu(\bar{q})$ or $n \geq \nu(\bar{p})$ and $\bar{q} = 0$. Since $n$ is the lowest possible norm for an element of $\mathcal{I} \cap S$, only the case $n = \nu(\bar{p})$ and $\bar{q} = 0$ remains. But this can happen only if we are in case 2 of Lemma 3.1 having $\bar{p} = p$ and $q = cp$, $c \in k^\times$.

Step 2. We now show that any element of $\mathcal{G}$ belongs to $(q)$ where $q$ is defined as in the first step. To this end choose any element $p$ from $\mathcal{G}$. Applying case 1 of Lemma 3.1 several times one gets $p = h q + r$, $\nu(r) \leq n$, $r \in S$. Since, by assumption, $q$ has the smallest possible norm, it follows $\nu(r) = n$ or $r = 0$. This in turn yields $r = cq$, $c \in k$ according to Step 1. Finally, we have $p = (h + c)q$ and, therefore, $\mathcal{I} = (q)$. □

Proposition 3.3. Any finitely generated ideal $\mathcal{I}$ in $k[\mathcal{G}]$ is principal, i.e., $k[\mathcal{G}]$ is a Bézout domain.

Proof. Let $\mathcal{I} = (r_1, \ldots, r_m)$ for some $m \in \mathbb{N}$. Write the generators according to (11), i.e.,

\begin{equation}
(14) \quad r_j = \sum_{i=0}^{n_j} a_{\alpha_{i,j}} c_{\alpha_{i,j}} + b_{\alpha_{i,j}} s_{\alpha_{i,j}}, \quad n_j \in \mathbb{N}, \quad a_{\alpha_{i,j}}, b_{\alpha_{i,j}} \in k, \quad \alpha_{i,j} \in Q^+.
\end{equation}

Let $d$ be a common denominator of all the $\alpha_{i,j}$ occurring in these equations. Then the generators of $\mathcal{I}$ can be identified with elements of the subset $d$ defined in Lemma 3.1 of the Ring $R_d$ via the embedding $E : R_d \rightarrow R_Q$ which is defined by $C_2 \mapsto C_{1/d}$, $S_2 \mapsto S_{1/d}$. Let $\tilde{r}_1, \ldots, \tilde{r}_m$ elements
of $R_N$ the images of which are $r_1, \ldots, r_m$. The ideal $\bar{J}$ generated by $\bar{r}_1, \ldots, \bar{r}_m$ is principal by Proposition 3.2. Consequently, $J$ is generated by the image of the generator of $\bar{J}$ under $E$. □

Remark 3.3. Note that neither $k[\mathfrak{S}_Q]$ nor $k[\mathfrak{S}_N]$ are principal ideal domains (PID). The first is not Noetherian: As an example for an ideal that is not finitely generated take $\langle \{S_{1/2^n} | n \in \mathbb{N} \} \rangle$. Moreover, $k[\mathfrak{S}_N]$ is not a PID since there are finitely generated ideals that cannot be generated by one single element: The ideal $(S_1, C_1 + 1)$ viewed as an element of $k[\mathfrak{S}_Q]$ is generated by $C_{1/2}$ which does not belong to $k[\mathfrak{S}_N]$.

3.2. $R_Q$ is a Bézout domain. We are now in position to prove that $R_Q$ is a Bézout domain. After the preparation done in the previous subsection the remaining steps are very similar to those given in [2, 17]. In particular, the proof of Lemma 3.3 which prepares Theorem 3.1 is strongly inspired by [2, Theorem 1].

Lemma 3.3. For two coprime elements $p, q \in R_Q$ there exist $a, b \in R_Q$ such that $ap + bq = 1$.

Proof. By Prop. 3.3 (resp. Cor. 3.3) $C(s)[\mathfrak{S}_Q]$ is a Bézout domain. Thus, there exist $a, b \in C[s, \mathfrak{S}] \subset R_Q$ such that $ap + bq = h$ where $h \in C[s]$. Write $h$ as product $h = \prod_{i=1}^{N} (s - s_i)$ and proceed by induction (we do not assume $s_i \neq s_j$).

Assume there exist $a, b \in R_Q$ with $ap + bq = \prod_{i=1}^{N} (s - s_i)$. In the following, for any $\gamma \in M_0$ we set $\bar{\gamma} = \mathcal{L}(\gamma)(s_{\mathbb{N}})$, with the entire function $\mathcal{L}(\gamma)$ being the Laplace transform of $\gamma$ given according to App. B.3. By the coprimeness of $p, q$, and by Lemma B.3.3 in the Appendix, $a^*\, b^*$, defined by

$$
a^* = \begin{cases}
\frac{\bar{a}q - q\bar{a}}{q(s - s_{\mathbb{N}})} & (s - s_{\mathbb{N}}) \nmid q \\
\frac{\bar{a}}{s - s_{\mathbb{N}}} & (s - s_{\mathbb{N}}) \nmid q
\end{cases}, \\
b^* = \begin{cases}
\frac{\bar{p}b - pb\bar{e}}{p(s - s_{\mathbb{N}})} & (s - s_{\mathbb{N}}) \nmid p \\
\frac{b}{s - s_{\mathbb{N}}} & (s - s_{\mathbb{N}}) \nmid p
\end{cases}
$$

belong to $R_Q$. One easily verifies that $pa^* + qa^* = \prod_{i=1}^{N} (s - s_i)$. Applying this step $N$ times completes the proof. □

Theorem 3.1. The ring $R_Q$ is a Bézout domain, i.e., any finitely generated ideal is principal.

Proof. We show that any two elements $p, q \in R_Q$ possess a common divisor $\bar{c}$ that can be written as linear combination of $p, q$. (It is then the unique greatest common divisor of $p$ and $q$.)

According to section B.3.3 the ring $C(s)[\mathfrak{S}_Q]$ is a Bézout domain. Consequently, there are elements $a, b \in C[s, \mathfrak{S}_Q]$ such that

$$
eq \frac{c}{p} = \frac{a}{b} \quad (s - s_{\mathbb{N}}) \nmid q
$$

is a $\text{g.c.d.}$ in $C(s)[\mathfrak{S}_Q]$. Hence, $p/c$ and $q/c$ belong to $C(s)[\mathfrak{S}_Q]$. In particular, there are $n_i \in R_Q$, $d_i \in C[s]$ with $\text{gcd}_{R_Q}(n_i, d_i) = 1$ ($i = 1, 2$) such that $p/c = n_1/d_1$ and $q/c = n_2/d_2$. It follows $pd_1 = cn_1$, $qd_2 = cn_2$. Consequently, both $d_1$ and $d_2$ divide $c$ in $R_Q$. Since $d_1$ and $d_2$ are polynomials, they possess a least common multiple $h = d_1d_2/\text{gcd}(d_1, d_2)$, and it follows $c = h \in R_Q$. Clearly, $\bar{c}$ divides both, $p$ and $d$. Dividing (15) by $\bar{c}$ yields the equation

$$
\frac{a}{p} n_1 d_2 / \text{gcd}(d_1, d_2) + b n_2 d_1 / \text{gcd}(d_1, d_2) = d_1d_2 / \text{gcd}(d_1, d_2).
$$

By the coprimeness of $n_1$ and $d_1$, resp. $n_2$ and $d_2$, it follows $\text{gcd}(\bar{p}, h) = d_2/\text{gcd}(d_1, d_2)$, resp. $\text{gcd}(\bar{q}, h) = d_1/\text{gcd}(d_1, d_2)$. Thus, $\text{gcd}(\bar{q}, h)$ and $\text{gcd}(\bar{p}, h)$ are coprime and, since by equation (16) any common divisor of $\bar{p}$ and $\bar{q}$ divides $h$, we can finally conclude the coprimeness of $\bar{p}$ and $\bar{q}$. Thus, by Lemma 3.3 there are $a^*, b^* \in R_Q$ such that $a^*\, \bar{p} + b^*\, \bar{q} = 1$. The claim follows directly by multiplying this equation by $\bar{c}$. □
4. Controllability analysis

4.1. Systems and Dynamics.

Definition 4.1. An $R$-system $\Lambda$, or a system over $R$, is an $R$-module.

Definition 4.2. A presentation matrix of a finitely presented $R$-system $\Sigma$ is a matrix $P$ such that $\Sigma \cong [v]/[Pv]$ where $[v]$ is free with basis $v$.

Definition 4.3. An $R$-dynamics, or a dynamics over $R$, is an $R$-system $\Lambda$ equipped with an input, i.e., a subset $u$ of $\Lambda$ which may be empty, such that the quotient $R$-module $\Lambda/[u]$ is torsion. The input $u$ is independent if the $R$-module $[u]$ is free, with basis $u$.

Definition 4.4. An output $y$ is a subset, which may be empty, of $\Lambda$. An input-output $R$-system, or an input-output system over $R$, is an $R$-dynamics equipped with an output.

Definition 4.5. Let $A$ be an $R$-algebra and $\Lambda$ be an $R$-system. The $A$-module $A \otimes_R \Lambda$ is an $A$-system, which extends $\Lambda$.

4.2. System controllabilities. In this section we emphasize several controllability notions which are defined directly on the basis of the above system definition without referring to a solution space. For the latter we refer to the next subsection. Let us start with some purely algebraic definitions:

Definition 4.6 (see [13]). Let $A$ be an $R$ algebra. An $R$-system $\Lambda$ is said to be $A$-torsion free controllable (resp. $A$-projective controllable, $A$-free controllable) if the $A$-module $A \otimes_R \Lambda$ is torsion free (resp. projective, free). An $R$-torsion free (resp. $R$-projective, $R$-free) controllable $R$-system is simply called torsion free (resp. projective, free) controllable.

Elementary homological algebra (see, e.g., [35]) yields

Proposition 4.1. $A$-free (resp. $A$-projective) controllability implies $A$-projective (resp. $A$-torsion free) controllability.

Proposition 4.2. $R$-free controllability implies $A$-free controllability for any $R$-algebra $A$. More generally, given any $R$-system $\Sigma$ that is a direct sum of a torsion module $t\Sigma$ and a free module $\Lambda$, the extended system $A \otimes_R \Sigma$ is a direct sum of the torsion module $A \otimes_R t\Sigma$ and the free module $A \otimes_R \Lambda$.

The importance of the notions of torsion free and free controllability is intuitively clear: While the first one refers to the absence of a nontrivial subsystem which is governed by an autonomous system of equations, the latter refers to the possibility to freely express all system variables in terms of a basis of the system module. For this reason, and, secondarily, in reminiscence to the theory of nonlinear finite dimensional systems, we have the following:

Definition 4.7. Take an $A$-free controllable $R$-system $\Lambda$ with a finite output $y$. This output is said to be $A$-flat, or $A$-basic, if $y$ is a basis of $A \otimes_R \Lambda$. If $A \cong R$ then $y$ is simply called flat, or basic.

In finite dimensional linear systems theory, the so called Hautus criterion is a quite popular tool for checking controllability. This criterion has been generalized to delay systems (see, e.g., [30]) and to the more general convolutional systems defined over $\mathcal{E}'$ [42] and $\mathcal{M}_0$ [46]. All those rings may be embedded into the ring of entire functions via the Laplace transform. This motivates the following quite general definition:

Definition and Proposition 4.1. Let $R$ be any ring that is isomorphic to a subring of the ring $\mathcal{O}$ of entire functions with pointwise defined multiplication. Denote the embedding $R \to \mathcal{O}$ by $\mathcal{L}$. A finitely presented $R$-system with presentation matrix $P$ is said to be spectrally controllable if one of the following equivalent conditions holds:

((i)) The $\mathcal{O}$-matrix $\hat{P} = \mathcal{L}(P)$ satisfies $\exists k \in \mathbb{N} : \forall \sigma \in \mathbb{C} : \text{tr} \hat{P}(\sigma) = k$.

((ii)) The module $\Sigma_\mathcal{O} = \mathcal{O} \otimes_R \Sigma$ is free.
Proof. The ring $\mathcal{O}$ is an elementary divisor domain [19]. As a consequence, over $\mathcal{O}$, any matrix admits a Smith normal form by left and right multiplication with unimodular matrices. Since the units in $\mathcal{O}$ are just the functions which possess no zeros in $\mathbb{C}$, the rank of the Smith normal form equals the rank of $\hat{P}(\sigma)$ for any $\sigma \in \mathbb{C}$. Thus, the rank of $\hat{P}(\sigma)$ remains constant if and only if the non-zero entries of the Smith normal form possess no zeros in $\mathbb{C}$ which, in turn, is equivalent to the absence of non-trivial torsion elements in $[\nu]/[\nu \hat{P}]$, i.e., in $\Sigma_\mathcal{O}$. \hfill $\square$

Proposition 4.3. Let $R$ be any Bézout domain that is isomorphic to a subring of $\mathcal{O}$ with the embedding $R \to \mathcal{O}$ denoted by $\mathcal{L}$. Then the notions of spectral controllability and $R$-torsion free controllability are equivalent if and only if $\mathcal{L}$ maps non-units in $R$ to non-units in $\mathcal{O}$.

Proof. Since $R$ is a Bézout domain, torsion freeness of $\Sigma$ implies freeness. Tensoring with the free module $\mathcal{O}$ yields another free module $\Sigma_\mathcal{O}$, and, thus, by Definition and Proposition 4.1 spectral controllability. Again, since $R$ is a Bézout domain, any presentation matrix admits a Hermite form. Thus, the torsion submodule $t\Sigma$ of $\Sigma$ can be presented by a triangular square matrix $tP$ of full rank. If $\Sigma$ is not torsion-free, at least one diagonal entry of this matrix is not a unit in $R$. If this entry is mapped to a non-unit in $\Sigma_\mathcal{O}$ by $\mathcal{L}$, it admits a complex zero $\sigma_0$. Thus, $\mathcal{L}(tP)$ has a loss off rank at $\sigma = \sigma_0$. Contrary, if there is a non unit $r \in R$ which corresponds to a unit $\hat{r} \in \mathcal{O}$, consider $\Sigma \cong \llbracket \tau \rrbracket / \llbracket \tau \hat{r} \rrbracket$. Obviously, the image of $\tau$ in $\Sigma_\mathcal{O}$ is zero. Thus, the trivial module $\Sigma_\mathcal{O}$ is torsion free. \hfill $\square$

Remark 4.1. Note that, under the additional assumption that $\Sigma$ admits a presentation matrix of full row-rank, the assumption of $R$ being a Bézout domain may be replaced by a less restrictive one. In this case, equivalence of $(\mathcal{Q} \otimes_R R) \cap \mathcal{O}$-torsion free controllability, with $\mathcal{Q}$ the ring of rational functions in one complex variable, and spectral controllability may be established (see, e.g., [30, 46] for different examples).

We are now able to state the main result of our paper:

Theorem 4.1. The convolutional system $\Sigma$ defined in Definition 2.1 is free if and only if it is torsion free. More generally $\Sigma = t\Sigma \oplus \Sigma/t\Sigma$ where $t\Sigma$ is torsion and $\Sigma/t\Sigma$ is free. Moreover, $\Sigma$ is spectrally controllable if and only if it is torsion free.

Proof. Recall that, according to Definition 2.1 $\Sigma \cong \mathcal{R} \otimes_{\mathcal{R}_\mathcal{Q}} \Sigma_\mathcal{Q}$ and $\mathcal{R}_\mathcal{Q}$ is a Bézout domain by Proposition 4.1. Since the first assertion holds for finitely presented modules over any Bézout domain, it holds for $\Sigma_\mathcal{Q}$. The second assertion follows from Proposition 4.3 (The fact that the Laplace transform maps any non-unit of $\mathcal{R}_\mathcal{Q}$ to a non-unit in $\mathcal{O}$ is obvious.) Clearly, both results hold as well for $\Sigma$, which is obtained by an extension of scalars. \hfill $\square$

4.3. Trajectorian controllability. In this section we will give two different interpretations of our algebraic controllability results that directly refer to trajectories of the system, i.e., to (generalized) functions which may be assigned to the system variables. To this end we need to introduce the notions of a solution space and a trajectory.

Definition 4.8. Let $\Sigma$ be an $R$-system and $\mathcal{F}$ a space of generalized functions. The space $\mathcal{F}$ is called a solution space of $\Sigma$ if it can be equipped with the structure of an $R$-module.

Definition 4.9 (see [15]). Let $\mathcal{F}$ be a solution space of an $R$-system $\Sigma$. An $\mathcal{F}$-trajectory of $\Sigma$ is an element of $\text{Hom}_R(\Sigma, \mathcal{F})$.

The crux of the first controllability notion (Def. 4.10) is the possibility to assign an arbitrary (generalized) function from $\mathcal{F}$ to any system variable.

Definition 4.10 (see [15]). An $R$-system is called $\mathcal{F}$-trajectory controllable if for any element $a \in \Sigma$ and any $b \in \mathcal{F}$ there exists a trajectory $f$ with $f(a) = b$.

The following result is borrowed from [15] and applies to any torsion-free controllable $R$-system where $R \subset \mathcal{M}$.

Proposition 4.4. The system $\Sigma_\mathcal{R}/t\Sigma_\mathcal{R}$ is $\mathcal{M}$-trajectory-controllable.

Another controllability notion is the following due to [45]. As the above it relies on the notion of a trajectory. However, since it refers to the possibility of connecting trajectories, the notions of future and past come into play. Thus, an appropriate solution space should allow the definition of
such local properties. This is not possible for the field of Mikusiński operators in general but for its subring \( \mathcal{M}_R \) and more generally for the space \( \mathcal{B} \) of Boehmians \([1,26]\). The controllability criterion in the behavioural framework is the possibility of concatenating trajectories. In our algebraic setting we may formulate this criterion as follows.

**Definition 4.11** (cf. \([34,45]\)). Let \( \Sigma \) be an \( R \)-system and \( \mathcal{F} \) a solution space of \( \Sigma \) that possesses the structure of a sheaf on \( R \). Then \( \Sigma \) is called \( \mathcal{F} \)-behavioral-controllable if for any two trajectories \( f_1, f_2 \in \text{Hom}(\Sigma, \mathcal{F}) \) there exists \( f \in \text{Hom}(\Sigma, \mathcal{F}) \) such that for any \( a \in \Sigma \) there are \( t_1^a, t_2^a \in R \) with 
\[
|f(a)|(-\infty, t_1^a) = f_1(a)|(-\infty, t_1^a) \quad \text{and} \quad |f(a)|(t_2^a, \infty) = f_2(a)|(t_2^a, \infty).
\]

**Theorem 4.2.** The system \( \Sigma_R/t\Sigma_R \), where \( \Sigma_R \) is defined in Definition 2.1, is \( B \)-behavioral controllable.

**Proof.** Since \( \Sigma_R/t\Sigma_R \) is free, any homomorphism is uniquely determined by the functions assigned to the basis. Thus, for the basis \( b = b_1, \ldots, b_n \) we may chose \( t_1^b > t_2^b \) and set
\[
f(b) = \begin{cases} 
  f_1(b), & t < t_1^b \\
  f_2(b), & t > t_2^b 
\end{cases}
\]

Moreover, any \( a \in \Sigma_R/t\Sigma_R \) is given by \( a = \sum_{i=0}^n a_i b_i \) where the \( a_i \) have compact support. This thus there exist \( T_1, T_2 \) such that \( sup\alpha_i \subseteq [T_1, T_2] \), \( i = 1, \ldots, n \). The claim follows by an application of the theorem of supports \( t_1 = t_1^b + T_1 \), \( t_2 = t_2^b + T_2 \) (see. [1]). \( \square \)

**Remark 4.2.** When distinguishing the cases \( a > 0 \) and \( a = 0 \) in \( 17 \) one could alternatively prove \( \mathcal{E} \)-behavioural controllability (resp. \( \mathcal{D}' \)-behavioural controllability) in the case \( a > 0 \) or \( \mathcal{E}_2 \)-behavioural controllability (resp. \( \mathcal{D}'_2 \)-behavioural controllability) in the case \( a = 0 \), where \( \mathcal{E} \) is the space of Schwartz-Distributions, \( \mathcal{F}_2 \) the space of Gevrey-Functions of order less than 2, and \( \mathcal{D}'_2 \) the space of Gevrey ultradistributions.

5. **An example: two boundary coupled equations**

In order to illustrate our results, in the following we discuss a simple example. Consider the system of two second order equations
\[
\partial_x^2 w_i(x) = \sigma w_i, \quad i = 1, 2,
\]
defined on an open neighbourhood \( \Omega_i \) of \([0, \ell_i] \subset R \), where \( \sigma = \alpha s^2 + \beta s + c \). Those equations are coupled via the boundary conditions \( i = 1, 2 \)
\[
\mu_{i1} w_i(\ell_i) + \mu_{i2} w'_i(\ell_i) = 0
\]
\[
w_i(0) = u.
\]

According to Section 2.2, the general solution of the initial value problems associated with \( 17a \) reads \( i = 1, 2 \)
\[
\begin{pmatrix} w_i(x) \\ w'_i(x) \end{pmatrix} = \begin{pmatrix} C(x-\ell_i) & S(x-\ell_i) \\ \sigma S(x-\ell_i) & C(x-\ell_i) \end{pmatrix} \begin{pmatrix} c_{i1} \\ c_{i2} \end{pmatrix},
\]
with \( c_{i1} = w_i(\ell_i), \ c_{i2} = \partial_x w_i(\ell_i) \). The boundary conditions at \( x = \ell_i \) yield
\[
\mu_{i1} c_{i1} + \mu_{i2} c_{i2} = 0
\]
\[
C(\ell_i)c_{i1} + S(\ell_i)c_{i2} = u.
\]
Here, the relations \( S(-\ell_i) = -S(\ell_i) \) and \( C(-\ell_i) = C(\ell_i) \), derived in Section 2.2 have already been incorporated.

Thus, according to Definition 2.1 the convolutional system \( \Sigma \) associated with the boundary value problem \( 17 \) is the \( \mathcal{R} \) module \( [c_{i1}, c_{i2}, c_{21}, c_{22}, u] \) the generators of which are subject to the equations \( 19 \).

In order to reduce the number of equations, we aim to introduce new variables \( \omega_1 \) and \( \omega_2 \) such that \( 19a \) is satisfied automatically, i.e.,
\[
\begin{align*}
  c_{i1} &= -\mu_{i2} \omega_i, & c_{i2} &= \mu_{i1} \omega_i, \quad i = 1, 2.
\end{align*}
\]
Indeed, since
\[
\omega_i = \frac{1}{\mu_{i1} + \mu_{i2}} (-\mu_{i2} c_{i1} + \mu_{i1} c_{i2}), \quad i = 1, 2,
\]
the new variables belong to $\Sigma$. Using the new generators $\omega_1, \omega_2$ and $u$, equation (19H) may be rewritten to obtain

$$u = -p_i \omega_i, \quad p_i = \mu_{12} C(\ell_i) + \mu_{11} S(\ell_i), \quad i = 1, 2.$$  

Thus, $p_1 \omega_1 - p_2 \omega_2 = 0$, and $\Sigma \cong [\omega_1, \omega_2]/[p_1 \omega_1 - p_2 \omega_2]$.

In accordance with Section 2.1 assume that $\ell_i = n_i \ell$, with $n_i \in \mathbb{N}$ and $i = 1, 2$. Thus, by Theorem 4.1 checking spectral, torsion free, and free controllability are equivalent. Since the aim of this section is not the presentation of a general controllability analysis for the boundary value problem (17) but rather to give an example for the application of the derived algebraic results, we shall restrict ourselves to particular values for $n_1$ and $n_2$. In order to avoid tedious computations, we chose simply $n_1 = 1$, $n_2 = 2$. Apart from that, we discuss the generic case only, i.e., we do not care about singularities which may occur for particular values of the $\mu_{ij}, i, j = 1, 2$.

Applying the algorithms of Section 3.1 we obtain $p_1 r_1 + p_2 r_2 = \epsilon$ with

$$r_1 = 2((\mu_{21} \mu_{11} - \mu_{22} \mu_{12}) \sigma C(\ell) + (\mu_{22} \mu_{11} - \mu_{21} \mu_{12}) \sigma S(\ell))$$

$$r_2 = \mu_{12} \sigma - \mu_{11}$$

$$\epsilon = -\mu_{22} \mu_{12} (\sigma - \bar{\sigma}), \quad \bar{\sigma} = \frac{2 \mu_{21} \mu_{11} \mu_{12} - \mu_{22} \mu_1^2}{\mu_2 \mu_1^2}.$$  

Following Section 3.2 it remains to modify $r_1, r_2$ in such a way that $\epsilon$ is replaced by a constant. This may be done by applying the induction step of Lemma 3.3 once. To this end, let $\bar{r}_1, \bar{r}_2, \bar{p}_1, \bar{p}_2$ be the complex numbers obtained by setting $\sigma = \bar{\sigma}$ in the Laplace transforms of $r_1, r_2, p_1, p_2$. Assume that neither $\bar{p}_1$ nor $\bar{p}_2$ are zero. Then the variables

$$q_1 = \frac{\bar{p}_2 r_1 - \bar{r}_1 p_2}{\bar{p}_2 \epsilon}, \quad q_2 = \frac{L_\sigma(p_1) r_2 - L_\sigma(r_2) p_1}{\bar{p}_1 \epsilon}$$

belong to $\mathcal{R}_Q$ and, therefore, to $\mathcal{R}$. Thus, we have the Bézout equation $p_1 q_1 + p_2 q_2 = 1$.

From the above results, one easily verifies that with

$$y = q_2 \omega_1 + q_1 \omega_2$$

one has $\omega_1 = p_2 y$ and $\omega_2 = p_1 y$. Hence, $y$ is a basis of the system under consideration.

6. Conclusion

For a class of convolutional systems associated with boundary coupled second order partial differential equations we have derived algebraic controllability results which translate directly into trajectory related controllability conditions. These results rely on a division algorithm for a particular ring of Mikusiński operators with compact support that is obtained from the operator solution of the Cauchy problem associated with the given system of partial differential equations. However, this means that our algebraic setting does not apply directly to the given boundary value problem but rather to a convolutional system arising from these solutions in connection with the boundary conditions. A promising approach allowing an algebraic treatment from the very beginning is currently under investigation.

The current work was motivated by previous contributions [2,17] in which similar results were presented for differential delay systems. Those approaches have been shown to be useful not only for controllability analysis but also for the design of closed loop control schemes using the factorization approach or the method of finite spectrum assignment [3,4,18]. This suggests the investigation of similar methods for the class of systems considered within this contribution.

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Appendix A. Representation of the operators $S(x)$ and $C(x)$

In this section we give interpretations of the operators $S(x)$ and $C(x)$ introduced in Section 2. Actually, we restrict to $S(x)$ from which $C(x)$ can be easily deduced by differentiation w.r.t. $x$.

If $a > 0$ in equation (11) we may rewrite $\sigma$ as

$$\sigma = \frac{\tau^2 ((s + \alpha)^2 - \beta^2)}{4a^2}, \quad \tau = \sqrt{a}, \quad \alpha = \frac{b}{2a}, \quad \beta = \sqrt{\frac{b^2 - c}{4a^2} - \frac{c}{a}}.$$

According to [23] the operator $S(x)$ corresponds to the compactly supported function

$$S(x, t) = \left\{ (h(t + x\tau) - h(t - x\tau)) \frac{e^{-\alpha t}}{2\tau} J_0(\beta \sqrt{\tau^2 x^2 - t^2}) \right\},$$

where $J_0$ denotes the Bessel function of order zero. Thus, for any approximate identity $(\varphi_n)$, the operator $S(x)$ possesses a regular representation given by (cf. also App. [B.23])

$$(20) \quad S(x) = f_n(x)/\varphi_n, \quad f_n(x, t) = \int_{-x\tau}^{x\tau} e^{-\alpha \xi} \frac{2\tau}{\tau^2} J_0(\beta \sqrt{\tau^2 x^2 - \xi^2}) \varphi_n(t - \xi) d\xi.$$

Contrary, if $a = 0$ in (11), $S(x)$ can be written as power series in the differentiation operator:

$$S(x) = \sum_{k=0}^{\infty} \frac{(s - a)^k, x^{2k+1}}{(2k + 1)!}.$$

The convergence of this series is verified directly from the regular representation

$$S(x) = \sum_{k=0}^{\infty} \frac{2^{2k+1} \psi_{k,n}}{(2k + 1)!} \varphi_n, \quad \psi_{k+1,n} = \psi_{k,n} - a \psi_{k,n}, \quad \psi_0,n = \varphi_n$$

where $(\varphi_n)$ is any approximate identity of Gevrey order less than 2.

Appendix B. Mikusiński Operators and Boehmians

B.1. Generalized quotients. The set of locally integrable functions with left-bounded support forms a commutative ring $L_+$ with respect to the pointwise addition and the convolution product. A celebrated theorem of Titchmarsh ([40, Theorem VII], [41, Theorem 15]) states the following:

**Theorem B.1.** Assume that the convolution product of two locally integrable functions $f$ and $g$ the support of which is contained in $\mathbb{R}^+$ vanishes on $[0, T]$. Then there exist nonnegative real numbers $T_g, T_f$ with $T_g + T_f \geq T$ such that both, $f$ and $g$, vanish identically on $[0, T_f]$ and $[0, T_g]$ respectively.

**Corollary B.1.** The ring $L_+$ is free of divisors of zero.

**Definition B.1.** The field $\mathcal{M}$ of Mikusiński operators is the quotient field of complex-valued locally integrable functions on $\mathbb{R}$ with left bounded support$^5$.

**Definition B.2.** Consider a commutative ring $R$ together with an $R$-module $M$. Let $\Delta$ be a family of sequences of $R$ such that:

1. If $(\varphi_n), (\psi_n) \in \Delta$ then $(\varphi_n \psi_n) \in \Delta$.
2. For $f, g \in M$ and $(\varphi_n) \in \Delta$ the equality of sequences $(f \varphi_n) = (g \varphi_n)$ implies $f = g$.

Then the elements of $\Delta$ will be called $\Delta$-sequences in $M$ [25, 28].

**Definition B.3.** Consider an $R$-module $M$ with $R$ a commutative ring and $\Delta$ a family of $\Delta$-sequences in $M$. Let $\mathcal{M}(M, \Delta)$ the set of all pairs of sequences $(f_n) \in M^\Delta$ and $(\varphi_n) \in \Delta$ satisfying $\varphi_i f_j = \varphi_j f_i$ for all $i, j \in \mathbb{N}$. With the (equivalence)-relation $\sim$ defined by

$$((f_n), (\varphi_n)) \sim ((g_n), (\psi_n)) \leftrightarrow \varphi_i g_j = \psi_j f_i \text{ for all } i, j \in \mathbb{N}$$

Then the space $\mathcal{M}(M, \Delta)$ of Boehmians on $M$ is defined as $\mathcal{M}(M, \Delta)/\sim$. For notational simplicity a Boehmian is simply denoted as $f_n/\varphi_n$. The addition on $\mathcal{M}(M, \Delta)$ may be defined according to $f_n/\varphi_n + g_n/\psi_n = (\psi_n f_n + \varphi_n g_n)/(\varphi_n \psi_n)$ [25, 28].

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4For several alternative proofs see also [22–24, 48].

5Contrary to this definition $\mathcal{M}$ is sometimes defined as the quotient field of the convolution ring of continuous functions with support in $\mathbb{R}^+$. However, in both cases the obtained fields of fractions are isomorphic.
In the following \( \mathcal{L}(\mathbb{R}) \) denotes the space of locally integrable functions on \( \mathbb{R} \) and \( \mathcal{L}_0(\mathbb{R}) \) (resp. \( \mathcal{L}_+(\mathbb{R}) \)) the subset containing the functions with bounded (resp. left-bounded) support. With the pointwise addition and the convolution product \( \mathcal{L}_0 \) (resp. \( \mathcal{L}_+ \)) form a commutative ring and \( \mathcal{L} \) (resp. \( \mathcal{L}_+ \)) is a \( \mathbb{L}_0 \)-module (resp. \( \mathbb{L}_+ \)-module).

**Theorem and Definition B.1.** [25,26] Let \( \Delta \) be the family of all sequences \( (\varphi_n) \) in \( \mathcal{L}_0 \) satisfying
1. \( \exists C \in \mathbb{R} : \forall n \in \mathbb{N} : \int_\mathbb{R} |\varphi_n(t)|dt \leq C \)
2. \( \forall n \in \mathbb{N} : \int_\mathbb{R} \varphi_n(t)dt = 1 \)
3. \( \forall \epsilon \in \mathbb{R} \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : \text{supp} \varphi_n \subseteq [-\epsilon, \epsilon] \).

Then we may define the following spaces of Boehmians:

- The ring \( \mathcal{M}_0 \) of Mikusiński operators with compact support is defined as \( \mathcal{B}([\mathcal{L}_0], \Delta) \) [1].
- The ring \( \mathcal{M}_R \supset \mathcal{M}_0 \) of regular Mikusiński operators is defined to be \( \mathcal{B}([\mathcal{L}_+], \Delta) \).
- The elements of the space \( \mathcal{B}([\mathcal{L}], \Delta) \), which for short is denoted by \( \mathcal{B} \), are simply called Bohemians.

Obviously, the spaces \( \mathcal{M}_R \) and \( \mathcal{B} \) possess the structure of \( \mathcal{M}_0 \) modules.

**Remark B.1.** The sets \( \mathcal{M}_0 \) and \( \mathcal{M}_R \) regarded as commutative rings are clearly isomorphic to subrings of \( \mathcal{M} \). Since the rings \( \mathcal{L}_+ \) and \( \mathcal{L}_0 \) are free of divisors of zero, for these spaces the equivalence relation in [5,6] could be replaced by:

\[
(\alpha_n, \beta_n) \sim (\gamma_n, \delta_n) \iff (\varphi_i \gamma_j = \psi_j \alpha_i \text{ for some } i, j \in \mathbb{N})
\]

**B.2. Divisibility in the ring of Mikusiński operators with compact support.** In this section we shall state some divisibility properties of the ring \( \mathcal{M}_0 \). For proofs we refer to the cited literature.

**Proposition B.1.** The mapping \( T^\alpha \) defined by the pointwise multiplication of a function \( f \in \mathcal{L}_0 \) with an exponential function \( t \mapsto e^{\alpha t} \) defines an isomorphism on \( \mathcal{L}_0 \) which extends to an isomorphism on \( \mathcal{M}_0 \) [23,46].

**Proposition B.2.** The mapping \( L : \mathcal{L}_0 \rightarrow \mathbb{C} \) assigning to every element of \( \mathcal{L}_0 \) the value of its integral can be shown to be a homomorphism. Its unique extension to \( \mathcal{M}_0 \) is denoted by the same symbol.

**Proposition B.3.** An operator \( a \in \mathcal{M}_0 \) divisible by \( (s+\alpha) \), \( \alpha \in \mathbb{C} \) if and only if \( L \circ T^\alpha(a) = 0 \).

**Remark B.2.** Note that the function \( a \) : \( \mathbb{C} \rightarrow \mathbb{C} \) given by \( \hat{a}(\sigma) = L \circ T^{-\sigma}(a) \) is the Laplace transform \( \mathcal{L}(a) \) of \( a \). It is an entire function which satisfies a growth condition on the imaginary axis derived in [5]. Within this context, Proposition [13] means that \( s - \alpha \) divides \( a \) in \( \mathcal{M}_0 \) if and only if \( \sigma - \alpha \) divides the Laplace transform of \( a \) within the space of entire functions.

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