Contact Problem for Layered Medium Supported by a Wedge

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Abstract. In this paper the plane contact problem for a layered medium supported by an elastic wedge is considered. A compressive load is applied to the top of the layered medium through a frictionless rigid punch. It is assumed that all contact areas are frictionless and only compressive normal tractions can be transmitted through the interfaces. Hence, the length of the contact region along the layers and layer-wedge interface is finite and are also unknown. The problem is formulated in terms of a system of singular integral equations in which the unknown functions are the pressure between layers in layered medium and between layered medium and the wedge. The analytical formulation of layered medium supported by wedge is given. As a numerical example, contact length and stress intensity factor at the tip of rails are calculated and the result are given by diagrams.

1. Introduction

The contact problem in theory of elasticity was first investigated in the pioneering work of the problems by Boussinesq [1] and Hertz [2]. The results obtained from these studies have encouraged many researchers for studying the contact problem. Leading studies about the three dimensional contact problems except the problems defined previously can be found in the works of Love [3], Harding and Sneddon [4]. The solution of the anisotropic semi plane with friction forces and many punches has been obtained by Galin [5]. Some contact problems for elastic layer were given by Alexandrov [6], Alexandrov and Vorovich [7], Wu and Chiu [8]. For some contact problem contact area is constant. If the contact area is finite and unknown, the problem is named “receding contact” problem. Problem involving receding contact is given by Keer et al. [9]. An elastic layer lying on the half plane and an elastic layer supported by two quarter planes solved by Erdogan and Ratwani [10], Ratwani and Erdogan [11]. For the existence of residual stresses in crack problem, one of the crack tips may be close, so closing crack tip has a cusp shape rather than a standard parabolic form. It is considered as a smooth contact problem and this kind of problem is named as “crack contact problem” [12-13]. Contact problem related to half space and half plane were given by Fabrickant and Sanker [14], Gecit [15] and Cakiroglu et al. [16].

In this paper an attempt is made to develop a general technique of the solution for a layered medium which is supported by an elastic wedge. It will be assumed that contact between the layer and the wedge is frictionless and only compression is permitted in the contact area. The contact problem between the layer and the wedge is one of the receding contact problems. In this study, it will be assumed that loads that are acting to elastic layer directly. In this case, an elastic wedge will be chosen for the support of the layer. The results are also the same for the case of assuming the problem symmetric and considering...
the head angle as $\pi$ with directly acting loads [11]. The results are also the same for the case of assuming the problem symmetric and considering the head angle as $\pi/2$ with directly acting loads [12].

2. Formulation of the mixed boundary value problem for elastic wedges

Let us consider an elastic wedge subjected to traction shown in figure 1. To obtain displacement formulation for wedge, Mellin transformation can be applied to the Airy stress function in polar coordinates. In the absence of body forces, Airy stress function $\varphi$ satisfies the following equation.

$$\nabla^4 \varphi = 0 \quad \nabla^2 = \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

(1)

Performing Mellin transforms on Eq. (1) we can obtain an ordinary differential equation. After solving the equation, referring to the derivation given in [17], one can find

$$M[\varphi] = A(s)e^{as} + \bar{A}(s)e^{-as} + B(s)e^{i(s+2)\theta} + \bar{B}(s)e^{-i(s+2)\theta}$$

(2)

Here the complex transform parameters $A$, $\bar{A}$, $B$ and $\bar{B}$ are the functions of $s$ and conjugate of each other respectively. Let us define complex stress and complex displacement as:

$$\sigma = \sigma_{\theta\theta} + i \sigma_{\theta\phi}; \quad u = u_r + i u_\theta$$

(3)

Performing Mellin transforms of $r^2$ times of complex stress and complex displacement respectively one can obtain following expression [17].

$$M[r^2 \sigma] = 2i(s + 1)[Ase^{as} + B(s + 1)e^{i(s+2)\theta} - \bar{B}e^{-i(s+2)\theta}]$$

$$M[r^2 u] = -\frac{s + 1}{\mu} [Ase^{as} + B(s + 1)e^{i(s+2)\theta} + \kappa \bar{B}e^{-i(s+2)\theta}]$$

(4)

To obtain the displacement equations of a wedge, we consider a wedge with the head angle $\theta_0$ and the traction at $\theta = 0$ as a single load, shown in figure 1. Boundary conditions of this wedge can be expressed as:

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = 0 \quad (\theta = \theta_0, \ 0 < r < \infty)$$

$$\sigma_{r\theta} = 0 \quad (\theta = 0, \ 0 < r < \infty)$$

$$\sigma_{\theta\phi} = \frac{f \delta(r - t)}{\mu} \quad (\theta = 0, \ 0 < r < \infty)$$

(5)

Where, $f$ is the single load acting on the edge to the point $t$. Above boundary conditions can be rewritten by using Eq. (3) as:
\( \sigma(r, \theta_o) = 0, \quad \sigma(r, 0) = i f \cdot \delta(r - t) \) \hspace{1cm} (6)

One can obtain \( A(s) \) and \( B(s) \) by using the conditions in Eq. (6) and the help of first equation in Eq. (4) as follows:

\[
sA(s) = \frac{\int_{t_{s+1}} f(t)}{2(s+1)} - s b_1(s) - i(s+2)b_2(s)
B(s) = b_1(s) + ib_2(s)
\]  

(7)

Here:

\[
b_1(s) = \frac{\int_{t_{s+1}} f(t)}{4(s+1)} D(s) \left( s + 1 \right) \left( 1 - \cos 2 \theta_o \right) + 1 - \cos 2(s+1)\theta_o
b_2(s) = \frac{\int_{t_{s+1}} f(t)}{4(s+1)} D(s) \left( s + 1 \right) \sin 2 \theta_o + \sin 2(s+1)\theta_o
D(s) = \left( s + 1 \right) \left( 1 - \cos 2 \theta_o \right) - \left[ 1 - \cos 2(s+1)\theta_o \right]
\]  

(8)

We can obtain the solution of the wedge by substituting this expression in Eq. (4) and performing inverse transform of the new equation. For the calculation of \( \partial u_\theta / \partial r \) over \( \theta = 0 \) line, after making necessary manipulation and also omitting details, one can find following integral equation:

\[
\frac{4\mu}{1 + \kappa} \frac{\partial u_\theta}{\partial r} = \int_{b}^{c} f(t) \left[ \frac{2\theta_o + \sin \theta_o}{2(2\theta_o - 1 + \cos 2\theta_o)} - \frac{1}{\pi} \left( y \sin 2\theta_o + \sinh 2\theta_o y \right) \sin \delta y dy \right] dt
\]  

\( b < r < c \quad \delta = \ln(t / r) \)  

(9)

Letting \( \theta_o \rightarrow \pi/2 \), above equation reduces to equation which is given in [10]. The internal integral expression in Eq. (9) is divergent for \( t \rightarrow r \), although the integral is continuous and definite in the interval of \( 0 \leq y < \infty \). The reason of divergence is because of the behavior of the integral at the infinity. Thus adding and subtracting the asymptotic part of the integrand to and from the integrand and taken over the related integrals, one can obtain:

\[
\frac{4\mu}{1 + \kappa} \frac{\partial u_\theta}{\partial r} = \frac{1}{\pi} \int_{b}^{c} f(t) \left[ \frac{1}{r} \frac{\pi(2\theta_o + \sin 2\theta_o)}{2(2\theta_o - 1 + 2\cos 2\theta_o)} + \right.
\left. \int_{0}^{\infty} \frac{y \sin 2\theta_o + sh 2\theta_o y}{\cosh 2\theta_o y - 1 - y^2(1 - \cos 2\theta_o)} \right) \sin \delta y dy dt \quad ; \quad b < r < c
\]  

(10)

The singularity in above equation is a Cauchy type singularity. The type of singularity can be seen by writing the first term in the following form.

\[
\frac{1}{r} = \frac{1}{r(t / r - 1)} \left[ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} \left( \frac{t}{r} - 1 \right)^n \right] = \frac{1}{t-r} \left[ 1 + O\left( \frac{t}{r} - 1 \right) \right]
\]  

(11)

One of the special case for Eq. (10) can be obtained by letting \( \theta_o \rightarrow \pi \). This case is the solution for half plane. For this case the internal integral can be expressed as:

\[
\int_{0}^{\infty} \frac{\sinh 2\pi y}{\cosh 2\pi y - 1} \sin \delta y dy = \frac{1}{2} \cosh \frac{\delta}{2} - \frac{1}{\delta}
\]  

(12)
Substituting this result in Eq. (10) and making necessary manipulations one can obtain well known following result:

\[
\frac{4\mu}{1+\kappa} \frac{\partial u_\theta}{\partial r} = -\frac{1}{\pi b} \int_b^c f(t) \, dt \quad b < r < c
\]  

(13)

Let us consider the contact problem between a wedge of the angle \( \gamma \) and a half-plane. The stresses at the top of the wedge for very small value of \( r \), are the form:

\[
\sigma_{ij} \approx r^{p-1} \quad (i, j = r, \theta) \quad (p \text{ real})
\]  

(14)

where; \(-(p-1)\) is the strength of singularity. \( p \) can be calculated, in following characteristic equation [18]

\[
(1+\alpha)\cos p\pi(p^2 \sin^2 \gamma - \sin^2 \gamma p\gamma) - (1/2)(1-\alpha)\sin p\pi(p\sin 2\gamma + \sin 2p\gamma) = 0
\]  

(15)

here:

\[
\alpha = \frac{\mu_2 / \mu_1 (\kappa_1 + 1) - (\kappa_2 + 1)}{(\mu_2 / \mu_1) (\kappa_1 + 1) + (\kappa_2 + 1)}
\]  

(16)

and \( \kappa_1, \mu_1, \kappa_2, \mu_2 \) are elastic constants of the wedge and half-plane respectively. When semi plane is rigid \( \alpha = -1 \) and when wedge is rigid the \( \alpha = +1 \) If the head of the wedge is smoothed, there will not be singularity at the head of the wedge. So the stresses will be zero at the head of the wedge.

3. The elastic layered medium which is supported by an elastic wedge

Here, we will consider a frictionless layered medium which is composed of \( S \) layers with different heights and elastic properties. This layered medium is, also considered, supported by an elastic wedge which head angel is \( \theta_0 \), shown in Figure 2.

Figure 2. Frictionless layered medium supported by an elastic wedge with head angel \( \theta_0 \).

There are \((S+1)\) unknown contact stresses in the problem, including punch stresses at the top of the medium. The unknowns can be calculated by using the expressions of the derivatives of displacement written in the \((S+1)\) contact regions. The problem will be layered medium lying on the half plane when \( \theta_0 \to \pi \).

At the \( n^th \) layer, elastic constants, height of the layer, contact stresses at the bottom of the layer, the boundaries of the contact region of bottom surface and displacements are denoted by \( \kappa_n, \mu_n, h_n, p_n, b_1^n, \)
$b^n$, $u_n$, and $v_n$ respectively. The following equation can be written in the contact region between $n$. layer and $(n-1)$. layer,

$$\frac{\partial v_n}{\partial x}(x,0) = \frac{\partial v_{n-1}}{\partial x}(x,h)$$

(17)

With necessary operations on the above equation one can obtain:

$$\begin{align*}
\int_{x_n}^{x_{n+1}} p_n(t)dt + \frac{1-\delta_n}{2} \int_{x_{n+1}}^{x_{n+1}} K_{2n}(x,t,h_n) p_{n+1}(t)dt + \frac{1+\delta_n}{2} \int_{x_{n+1}}^{x_{n+1}} K_{1n}(x,t,h_{n-1}) p_{n-1}(t)dt \\
+ \frac{1-\delta_n}{2} \int_{x_{n+1}}^{x_{n+1}} K_{1n}(x,t,h_n) p_{n+1}(t)dt + \frac{1+\delta_n}{2} \int_{x_{n+1}}^{x_{n+1}} K_{2n}(x,t,h_{n-1}) p_{n-1}(t)dt = 0
\end{align*}$$

(18)

where $p_n(t)$ is contact stresses between $n$th layer and $(n-1)$th. $\delta_n$ is an elastic constant and can be defined as:

$$\delta_n = \frac{(1+\kappa_{n-1})\mu_n - (1+\kappa_n)\mu_{n+1}}{(1+\kappa_{n+1})\mu_n + (1+\kappa_n)\mu_{n+1}}$$

(19)

After making some manipulations, following equation is obtained:

$$\begin{align*}
\int_{x_n}^{x_{n+1}} p_n(t)dt + \int_{x_{n+1}}^{x_{n+1}} K_{2n}(x,t,h_n) p_{n+1}(t)dt + \int_{x_{n+1}}^{x_{n+1}} K_{2n}(x,t,h_n) p_{n+1}(t)dt + \frac{1+\delta_n}{1-\delta_n} \int_{x_{n+1}}^{x_{n+1}} K(x-c,t-c) p_{n+1}(t)dt = 0
\end{align*}$$

(20)

In above equation, $\delta_n$ can be obtained from Eq. (16) by letting $\kappa_n$ and $\mu_n$ are the elastic constants of the wedge. The equation for the top surface of the $S$. layer can be written, as:

$$\frac{2}{1+\delta_{n+1}} \int_{x_{n+1}}^{x_{n+1}} p_{n+1}(t)dt + \int_{x_{n+1}}^{x_{n+1}} K_{2n}(x,t,h_n) p_{n+1}(t)dt + \int_{x_{n+1}}^{x_{n+1}} K_{1n}(x,t,h_{n-1}) p_{n-1}(t)dt = \frac{4\mu_1}{1+\kappa_1} df$$

(21)

There are $S+1$ unknowns in Eq. (18),(20),(21). The contact of all the layers, are smooth contact and $p_n(t)$ $(n=2,S)$, are finite at the boundaries of the contact region. So unknown function $p_n(t)$ can be written as:

$$p_n(t) = g_n(t)(b^n_1 - t)^{1/2}(t-b^n_2)^{1/2} \quad (n=1,....,S) \quad b_1 > c$$

(22)

The boundaries $b^n_1$ and $b^n_2$ of the unknown contact regions can be found using the following expression

$$\int_{x_n}^{x_{n+1}} p_n(t)dt = -P \quad (n=1,....,S)$$

(23)

If $P$ is applied directly to the top surface, by defining the unknown $P(t)/P$ in Eq. (18),(20) and (23), it can easily be seen that the boundaries of the contact region are independent from amplitude of applied loads and depends on its application point.
4. Numerical example

In this section one of the railroad problems will be solved numerically as a simple application of general problem. The most critical part of the rails is the tips of the gap between two rails, as shown in Figure 3. The gap is considered smaller in comparison with the wheel. So the wheel is considered as upper half plane and the rails are considered as two elastic wedges with the head angles are equal to $\pi/2$.

![Figure 3. The tips of the gap between two rails.](image)

The derivative of the displacement of the wheel can be obtained with using the wheel elastic constants $\kappa_2, \mu_2$ and contact stresses $p(t)$, as:

$$\frac{4\pi\mu_2}{1 + \kappa_2} \frac{\partial v_2}{\partial x} = -\int_{t-x}^{t} \frac{p(t)}{v} dt$$

(24)

Due to the symmetry shown in Figure 3, above equation can be rewritten as:

$$\frac{4\pi\mu_2}{1 + \kappa_2} \frac{\partial v_2}{\partial x} = -\int_{t-x}^{t} \frac{p(t)}{v} dt - \int_{t-x}^{t} \frac{p(t)}{v} dt$$

(25)

Considering the elastic constants $\kappa_1, \mu_1$ for the wedge and using necessary equations, the derivative of the displacement of wedge at the boundary can be expressed as:

$$\frac{4\mu_1}{1 + \kappa_1} \frac{\partial u_r}{\partial r} (0, x-a) = -\frac{1}{\pi} \int_{a}^{b} p(t) K(x-a, t-a) dt$$

(26)

where;

$$K(x,t) = \frac{1}{x} - \frac{\pi (2\theta_y + \sin 2\theta_y)}{2x(2\theta_y^2 - 1 + \cos 2\theta_y)} + \int_{0}^{\delta} \left[ \frac{\sin 2\theta_y + sh 2\theta_y y}{ch 2\theta_y y - 1 - y^2(1 - \cos 2\theta_y)} - 1 \right] \frac{\sin \delta y}{x} dy$$

(27)

With the necessary operations one can obtain:

$$\int_{a}^{b} p(t) H(t, x) dt = \frac{4\pi\mu_1 x(x-a)}{1 + \kappa_1} \frac{R}{R}$$

(28)

where;
\[
H(t, x) = \frac{1 - \gamma}{1 + \gamma} \left( \frac{x - a}{t - x} - \frac{x - a}{t + x} \right) + \frac{1}{\delta - \pi^2} \left( \frac{\pi^2}{4} + k(x, t) \right)
\]

(29)

It should be mentioned here that the equation which determines the contact boundaries can be expressed as:

\[
\int_a^b p(t) dt = -Q
\]

(30)

For the numerical solution of this problem Jacobi polynomials and Gauss-Jacobi integration formulas can be applied, the integral equations can be transformed to the linear equation system as (some necessary operations are not given here because of space limitations)

\[
\sum_{i=1}^{N} \phi(\tau_i) A_i H(\tau_i, \xi_j) = \frac{b - a}{2R} (1 + \xi_j) \left( \frac{a + b - a}{2R} + \frac{b - a}{2R} \xi_j \right)
\]

\[
\sum_{i=1}^{N} \phi(\tau_i) A_i = -\frac{Q + \kappa_i}{R \eta \mu_i} \quad (j=1, \ldots, N)
\]

(31)

where, \( \tau_i \) and \( \xi_j \) are the roots of Jacobi polynomials and can be expressed with respect to [19] as:

\[
P_N^{(0.5, \beta)}(\tau_i) = 0 \quad ; \quad P_N^{(-0.5, \beta+1)}(\xi_j) = 0
\]

(32)

and \( A_i \)’s in Eq. (31) are the weighting coefficients of the Jacobi polynomials and can be written as:

\[
A_i = -\frac{2N + \beta + 2.5}{(N + 1)!(N + \beta + 1.5)!} \frac{\Gamma(N + 1.5) \Gamma(N + \beta + 1)}{\Gamma(N + \beta + 1.5)} \frac{2^{\beta+0.5}}{P_N^{(0.5, \beta)}(\tau_i) P_N^{(-0.5, \beta+1)}(\xi_j)}
\]

(33)

Considering \( b/R \) is known, only unknowns are \( \phi(\tau_i) \) (\( i = 1, N \)) in the first expression of Eq. (31). So there are \( N \) unknowns and \( N \) equations. After finding \( \phi(\tau_i) \) function values, contact stresses can written as

\[
p(\tau_i) = \frac{2R}{b - a} \phi(\tau_i) (1 - \tau_i)^{1/2} (1 + \tau_i)^{\beta} \frac{4\eta \mu_i}{1 + \kappa_i}
\]

(34)

One of the most remarkable point is whether there is a separation at the point \( a \) between the wheel and the rail. The control of this case can be done by analyzing the sign of the internal stress function \( p(\tau) \). There will be separation in contact problem if \( p(\tau) \) is tension. If separation occurs, the tip point of rail will be safe against fracture because there will be finite stresses (actually zero stress) at the tip of the rail. Owing to calculations with various bi-elastic constants this case is not observed. Accordingly, the tip point of the rail is the most critical point. To investigate the safety of the tip of rail, we need to observe the singularity at the tip. As for the high singularities the stresses will go faster to infinity. For the materials with this characteristic tip of the wedge is more critical. Singularity, \( \beta \), changes between -0.5 and 0 with respect to bi-elastic constant \( \gamma \). The singularity is -0.5 for the rigid wedge and this is the most critical case of the tip of the wedge. If the wedge is rigid this is the sharp contact problem between the wedge and half plane, the singularity is -0.5; this is an expecting result. If the contacting materials are the same the singularity is -0.226. In practical this is the most common case as the material of the wheel and the rail are same mostly. Constructing the rails with softer material decreases the criticalness of the stresses. Another extreme case is the rigid circle case. For this case \( \beta \) is equal to zero. (It should be noted here that many necessary equations are not shown in this section because of space constraints).

One of the boundary of contact region, \( b \) is not known in the problem. In the numerical solution of problem, \( b \) is assumed to be known for a definite \( a \) value. After then \( Q \) has been calculated for this \( b \).
The diagram in figure 4, shows the $Q$ values with respect to contact region length for constant $a$ values. When $Q, R, a$ and elastic constants are given, the length of the contact region and $b$ value can be obtained from Figure 4.

$$\frac{Q (1 + \kappa)}{R} \times 10^j$$

5. Conclusion
Formulating the mixed boundary value problem of elastic layered medium with single integral equation systems has many advantages such as:
- Problem can easily be generalized, if there are two or more contact regions.
- The effect of these loads that are acted to the outside of the contact region can easily be counted in the problem.
- There is a reliable numerical method for solving the singular integral equation.
- When the boundaries of the contact region is at the right side of the head point $(c<b)$ then it is a smooth contact and the contact stresses are zero at end points. When $c=b$ there is a sharp contact at one of the ends and the stresses will have singularities at point $c$.

The numerical method that is used for the solution of integral equation which appears in the rail road problem can be also used for the integral equations of layered medium supported which is given by Eq. (18-21). In these problems important point is the singularity. For the case of singularity which occurs mainly in smooth contact and rigid punch case, Chebyshev polynomials can be use instead of Jacobi polynomials [19].

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