Eigenfunction Expansion for the Elastic Rectangle

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Abstract. In the paper, we construct an exact solution to a boundary value problem of the theory of elasticity for a rectangle in which the longitudinal sides are free, while normal and tangential stresses are given at the ends (even-symmetric deformation with respect to the central axes). The solution is represented in the form of series in Papkovich–Fadle eigenfunctions. The coefficients are determined explicitly by using functions biorthogonal to the Papkovich–Fadle eigenfunctions. We give the final formulas which have a simple appearance and can easily be used in engineering practice. The obtained solution is compared with the solution to the corresponding boundary value problem for a half-strip.

1. Introduction

Despite the increasing role of numerical methods for solving boundary value problems arising in the mathematical modeling of engineering problems, the development of analytical approaches to solving complex boundary value problems remains relevant in engineering mathematics.

There are many mechanical problems associated with the equilibrium of an elastic rectangle under the conditions of plane deformation or plane stress state, the bending of a clamped thin elastic rectangular isotropic plate and the flow of a viscous incompressible fluid in a rectangular cavity with a given wall motion, which can be formulated in terms of the biharmonic equation with given boundary conditions.

The biharmonic problem for a rectangle has been and remains complex in various sections of linear elasticity theory. In mathematics, it is a reference problem for various analytical and numerical methods. This is an excellent testing problem for verifying already existing methods and developing new numerical ones.

The historical aspect of the biharmonic problem is also of great interest. Numerous data on the solution to the biharmonic problem are widely presented in the literature on elasticity theory, plate theory and the creeping flow of a viscous fluid [1–10]. The last review of 2003 [11] includes more than 750 references to the most significant works on the biharmonic problem in almost 200 years. It contains a complete review of the history of different approaches to solving biharmonic problems.
In this paper, on the basis of the authors’ method of solving boundary value problems in a rectangular domain, we give an exact solution to a boundary value problem of the theory of elasticity for a rectangle whose horizontal sides are free, while normal and tangential stresses are given at the ends. Only the even-symmetric deformation of the rectangle with respect to the horizontal and vertical coordinate axes is considered. This study does not rely on any previous methods of approximate analytical solutions. It is based on the theory of the expansion of functions into series in Papkovich–Fadle eigenfunctions, developed by the authors earlier for a half-strip with homogeneous boundary conditions on two opposite sides [12–15]. The coefficients of the series are determined exactly by using functions biorthogonal to the Papkovich–Fadle eigenfunctions. The biorthogonal functions are found with the help of the Borel transform in the class of quasi-entire functions of exponential type [16].

2. Statement of the boundary value problem
Let us consider the rectangle \( P: |y| \leq 1, |x| \leq d \), the deformation of which is symmetric with respect to the longitudinal axis \( x \) and transverse axis \( y \). We will assume that the longitudinal sides \( y = \pm 1 \) are free, i.e.

\[
\sigma_y(x, \pm 1) = r_y(x, \pm 1) = 0. \tag{1}
\]

Let normal and tangential stresses be given at the ends \( x = \pm d \) of the rectangle:

\[
\sigma_x(\pm d, y) = \sigma(y), \quad r_x(\pm d, y) = r(y). \tag{2}
\]

Denote by \( G \) the shear modulus of the plate. Also, let \( U(x, y) = Gu(x, y), \quad V(x, y) = Gv(x, y), \) where \( u(x, y) \) and \( v(x, y) \) are longitudinal and transverse displacements in the plate, respectively. Then, the solution in the rectangle can be written in the form of expansions in Papkovich–Fadle eigenfunctions (only formulas for the stresses are given):

\[
U(x, y) = C_0 + C_1 x + \sum_{k=1}^{\infty} a_k \xi(\lambda_k, y) \sinh \lambda_k x + a_k \xi(\bar{\lambda}_k, y) \sinh \bar{\lambda}_k x,
\]

\[
V(x, y) = -\nu C_1 y + \sum_{k=1}^{\infty} a_k \chi(\lambda_k, y) \cosh \lambda_k x + a_k \chi(\bar{\lambda}_k, y) \cosh \bar{\lambda}_k x,
\]

\[
\sigma_x(x, y) = 2(1 + \nu) C_1 + \sum_{k=1}^{\infty} a_k s_x(\lambda_k, y) \cosh \lambda_k x + a_k s_x(\bar{\lambda}_k, y) \cosh \bar{\lambda}_k x,
\]

\[
\sigma_y(x, y) = \sum_{k=1}^{\infty} a_k s_y(\lambda_k, y) \cosh \lambda_k x + a_k s_y(\bar{\lambda}_k, y) \cosh \bar{\lambda}_k x,
\]

\[
r_y(x, y) = \sum_{k=1}^{\infty} a_k t_y(\lambda_k, y) \sinh \lambda_k x + a_k t_y(\bar{\lambda}_k, y) \sinh \bar{\lambda}_k x.
\]

The Papkovich–Fadle eigenfunctions \( \xi(\lambda_k, y), \chi(\lambda_k, y), s_x(\lambda_k, y), s_y(\lambda_k, y) \) and \( t_y(\lambda_k, y) \) have the following form:

\[
\xi(\lambda_k, y) = \left( \frac{1 - \nu}{2} \sin \lambda_k - \frac{1 + \nu}{2} \lambda_k \cos \lambda_k \right) \cos \lambda_k y - \frac{1 + \nu}{2} \lambda_k y \sin \lambda_k \sin \lambda_k y,
\]

\[
\chi(\lambda_k, y) = \left( \frac{1 + \nu}{2} \lambda_k \cos \lambda_k + \sin \lambda_k \right) \sin \lambda_k y - \frac{1 + \nu}{2} \lambda_k y \sin \lambda_k \cos \lambda_k y,
\]

\[
s_x(\lambda_k, y) = (1 + \nu) \lambda_k \left( \sin \lambda_k \cos \lambda_k - \lambda_k \cos \lambda_k y - \lambda_k y \sin \lambda_k \cos \lambda_k y \right),
\]

\[
s_y(\lambda_k, y) = (1 + \nu) \lambda_k \left( \sin \lambda_k + \lambda_k \cos \lambda_k \right) \cos \lambda_k y + \lambda_k y \sin \lambda_k \sin \lambda_k y,
\]

\[
t_y(\lambda_k, y) = (1 + \nu) \lambda_k \left( \cos \lambda_k \sin \lambda_k y - y \sin \lambda_k \cos \lambda_k y \right),
\]

where the numbers \( \lambda_k, \bar{\lambda}_k \) \((\text{Re} \lambda_k < 0)\) are the complex zeros of the entire function \( L(\lambda) = \lambda + \sin \lambda \cos \lambda \). \( \nu \) is Poisson’s ratio.
3. Solution to the boundary value problem

We will assume that the elementary solution corresponding to the constants $C_0$ and $C_1$ is known, and we will take (for simplicity) $C_0 = C_1 = 0$. Satisfying the boundary conditions (2) at the ends of the rectangle, we come to the problem of determining the coefficients $a_k, a_k^*$ from the expansions

$$
\sigma(y) = \sum_{k=1}^{\infty} a_k s_k(\lambda_k, y) \cosh \lambda_k d + \bar{a}_k s_k(\lambda_k, y) \cosh \bar{\lambda}_k d,
$$

$$
\tau(y) = \sum_{k=1}^{\infty} a_k t_{xy}(\lambda_k, y) \sinh \lambda_k d + \bar{a}_k t_{xy}(\lambda_k, y) \sinh \bar{\lambda}_k d.
$$

The coefficients $\{a_k\}_{k=1}^{\infty}$ and $\{\bar{a}_k\}_{k=1}^{\infty}$ are determined from here explicitly by using the functions $\{X_k(y)\}_{k=1}^{\infty}$ and $\{T_{xy}(y)\}_{k=1}^{\infty}$ biorthogonal to the Papkovich–Fadle eigenfunctions (4). The equations for their determination are as follows [12–15]:

$$
\int_{-\infty}^{\infty} s_k(\lambda, y) X_k(y) dy = \frac{L(\lambda)}{\lambda^2 - \lambda_{\lambda_k}^2},
$$

$$
\int_{-\infty}^{\infty} t_{xy}(\lambda, y) X_k(y) dy = \frac{\lambda L(\lambda)}{\lambda^2 - \lambda_{\lambda_k}^2}.
$$

We give the final formulas for the stresses in the rectangle. Regardless of the kind of homogeneous boundary conditions on its longitudinal sides, these formulas will always have the same form.

1) If only the normal stresses $\sigma_n(\pm d, y) = \sigma(y)$ are given at the ends $x = \pm d$ of the rectangle and the tangential ones are equal to zero, we obtain

$$
U(x, y) = \sum_{k=1}^{\infty} 2 \text{Re} \left\{ x_k \bar{\lambda}_k \xi(\lambda_k, y) \frac{\text{Im}(\bar{\lambda}_k \sinh \lambda_k d \sinh \lambda_k x)}{\lambda_k M_k} \right\},
$$

$$
V(x, y) = \sum_{k=1}^{\infty} 2 \text{Re} \left\{ x_k \bar{\lambda}_k \chi(\lambda_k, y) \frac{\text{Im}(\bar{\lambda}_k \sinh \lambda_k d \cosh \lambda_k x)}{\lambda_k M_k} \right\},
$$

$$
\sigma_n(x, y) = \sum_{k=1}^{\infty} 2 \text{Re} \left\{ x_k \frac{s_k(\lambda_k, y) \text{Im}(\bar{\lambda}_k \sinh \lambda_k d \cosh \lambda_k x)}{\lambda_k M_k} \right\},
$$

$$
\sigma_t(x, y) = \sum_{k=1}^{\infty} 2 \text{Re} \left\{ x_k \frac{t_{xy}(\lambda_k, y) \text{Im}(\bar{\lambda}_k \sinh \lambda_k d \cosh \lambda_k x)}{\lambda_k M_k} \right\}.
$$

2) In the case when only the tangential stresses $\tau_{xy}(\pm d, y) = \tau(y)$ are given at the ends $x = \pm d$ of the rectangle and the normal ones are equal to zero, we have

$$
U(x, y) = \sum_{k=1}^{\infty} 2 \text{Re} \left\{ t_k \bar{\lambda}_k \xi(\lambda_k, y) \frac{\text{Im}(\bar{\lambda}_k \cosh \lambda_k d \sinh \lambda_k x)}{\lambda_k M_k} \right\},
$$

$$
V(x, y) = \sum_{k=1}^{\infty} 2 \text{Re} \left\{ t_k \bar{\lambda}_k \chi(\lambda_k, y) \frac{\text{Im}(\bar{\lambda}_k \cosh \lambda_k d \cosh \lambda_k x)}{\lambda_k M_k} \right\},
$$

$$
\sigma_n(x, y) = \sum_{k=1}^{\infty} 2 \text{Re} \left\{ t_k \frac{s_k(\lambda_k, y) \text{Im}(\lambda_k^2 \cosh \lambda_k d \cosh \lambda_k x)}{\lambda_k^2 M_k} \right\},
$$

$$
\sigma_t(x, y) = \sum_{k=1}^{\infty} 2 \text{Re} \left\{ t_k \frac{t_{xy}(\lambda_k, y) \text{Im}(\lambda_k \cosh \lambda_k d \sinh \lambda_k x)}{\lambda_k M_k} \right\}.
$$
In formulas (7) and (8), $M_k = L'(\lambda_k) / 2\lambda_k = \cos^2 \lambda_k$. The numbers $x_k$ and $t_k$ are the Lagrange coefficients, which are determined with the help of the finite parts $\{x_k(y)\}_{k=1}^\infty$ and $\{t_k(y)\}_{k=1}^\infty$ of the biorthogonal functions $\{X_k(y)\}_{k=1}^\infty$ and $\{T_k(y)\}_{k=1}^\infty$, by the formulas [12–15]

$$x_k = \int_0^1 \sigma(y)x_k(y)dy, \quad t_k = \int_0^1 \tau(y)t_k(y)dy,$$

where

$$x_k(y) = \frac{\cos \lambda_k y}{2(\nu+1)\lambda_k \sin \lambda_k}, \quad t_k(y) = -\frac{\sin \lambda_k y}{2(\nu+1)\sin \lambda_k}.$$

The expanded functions $\sigma(y)$ and $\tau(y)$ must satisfy the following conditions. The function $\sigma(y)$ must be (1) self-balanced and (2) equal to zero in the neighborhood of the ends of the segment $|y| \leq 1$, while the function $\tau(y)$ must either satisfy condition (2) or be continuous in the neighborhood of the points $y = \pm 1$ and equal to zero at these points.

We also give the final formulas describing the solution to the boundary value problem in the half-strip $\{\Pi : x \geq 0, |y| \leq 1\}$ with free longitudinal sides, to the end of which the normal and tangential stresses $\sigma(y, 0) = \sigma(y)$, $\tau_y(0, y) = \tau(y)$ are applied.

1) At the end of the half-strip $\sigma(0, 0) = \sigma(y)$, $\tau_y(0, 0) = 0$:

$$U(x, y) = \sum_{k=1}^\infty 2\Re \left\{ x_k \frac{\xi(\lambda_k, y) \text{Im}(-\lambda_k e^{i\lambda_k})}{\lambda_k M_k} \right\}, \quad V(x, y) = \sum_{k=1}^\infty 2\Re \left\{ x_k \frac{\lambda_k \text{Im}(\lambda_k e^{i\lambda_k})}{\text{Im}(\lambda_k)} \right\},$$

$$\sigma(0, y) = \sum_{k=1}^\infty 2\Re \left\{ \frac{x_k}{\lambda_k \lambda_k^2} \frac{s_k(\lambda_k, y) \text{Im}(\lambda_k e^{i\lambda_k})}{\text{Im}(\lambda_k)} \right\}, \quad \tau_y(0, y) = \sum_{k=1}^\infty 2\Re \left\{ \frac{\lambda_k^2 M_k}{\lambda_k^2} \frac{t_k(\lambda_k, y) \text{Im}(\lambda_k e^{i\lambda_k})}{\text{Im}(\lambda_k)} \right\}.$$

(11)

2) At the end of the half-strip $\tau_y(0, 0) = \tau(y)$, $\sigma_y(0, 0) = 0$:

$$U(x, y) = \sum_{k=1}^\infty 2\Re \left\{ t_k \frac{\lambda_k^2 M_k}{\lambda_k^2} \frac{s_k(\lambda_k, y) \text{Im}(\lambda_k e^{i\lambda_k})}{\text{Im}(\lambda_k)} \right\}, \quad V(x, y) = \sum_{k=1}^\infty 2\Re \left\{ t_k \frac{\lambda_k^2 M_k}{\lambda_k^2} \frac{\xi(\lambda_k, y) \text{Im}(\lambda_k e^{i\lambda_k})}{\text{Im}(\lambda_k)} \right\},$$

$$\sigma_y(x, y) = \sum_{k=1}^\infty 2\Re \left\{ t_k \frac{\lambda_k^2 M_k}{\lambda_k^2} \frac{s_k(\lambda_k, y) \text{Im}(\lambda_k e^{i\lambda_k})}{\text{Im}(\lambda_k)} \right\}, \quad \tau_y(x, y) = \sum_{k=1}^\infty 2\Re \left\{ t_k \frac{\lambda_k^2 M_k}{\lambda_k^2} \frac{\lambda_k^2 \text{Im}(\lambda_k e^{i\lambda_k})}{\text{Im}(\lambda_k)} \right\}.$$

(12)

In formulas (11) and (12), the Papkovich–Fadle eigenfunctions and numbers $\lambda_k, M_k, x_k, t_k$ are determined by the same formulas as in the corresponding problem for the rectangle.

4. Examples of solving boundary value problems

Example 1. To the ends of the rectangle, self-balanced normal stresses are applied

$$\sigma(y) = \begin{cases} y^4 - \frac{6\alpha^2}{5} y^2 + \frac{\alpha^4}{5} & (|y| \leq \alpha, \alpha < 1), \\ 0 & (\alpha < |y| \leq 1). \end{cases}$$

By formula (9) we find

$$x_k = \frac{8(15-6\alpha^2 \lambda_k^6) \sin \alpha \lambda_k + (\alpha^2 \lambda_k^2 - 15) \alpha \lambda_k \cos \alpha \lambda_k}{5(1+\nu) \lambda_k^6 \sin \lambda_k}.$$


Substituting the found coefficient into formulas (7) and (11), we obtain the solutions to the boundary value problems in the rectangle and half-strip. Figures 1 and 2 show the graphs illustrating the solutions. It is assumed that $d = 1$, $\alpha = 1/2$ and $\nu = 1/3$.

![Figure 1.](image1.png)  
**Figure 1.** The normal stresses $\sigma_x(-d, y)$ (solid line) and $\sigma_x(-d, y)$ (dashed line) at the end $x = -d$ of the rectangle.

![Figure 2.](image2.png)  
**Figure 2.** The normal stresses $\sigma_x(x, 0)$ in the rectangle (solid line) and half-strip (dashed line) along the $x$ axis.

**Example 2.** To the ends of the rectangle, tangential stresses are applied:

$$\tau(y) = \begin{cases} y(\alpha^2 - y^2) & (|y| \leq \alpha, \alpha < 1), \\ 0 & (\alpha < |y| \leq 1). \end{cases}$$

Using formula (9), we find

$$t_k = \frac{2 \left( (\alpha^2 \lambda_k^2 - 3) \sin \alpha \lambda_k + 3 \alpha \lambda_k \cos \alpha \lambda_k \right)}{(1 + \nu) \lambda_k^2 \sin \lambda_k}.$$

Figures 3 and 4 show the graphs illustrating the solutions in the rectangle and half-strip. It is assumed that $d = 2$, $\alpha = 1/2$ and $\nu = 1/3$.

![Figure 3.](image3.png)  
**Figure 3.** The normal stress $\sigma_x(-d, y)$ (solid line) and tangential stress $\tau_{xy}(-d, y)$ (dashed line) at the end $x = -d$ of the rectangle.

![Figure 4.](image4.png)  
**Figure 4.** The normal stresses $\sigma_x(x, 0)$ in the rectangle (solid line) and half-strip (dashed line) along the $x$ axis.

5. **Conclusion**

In the paper, we have obtained simple and convenient formulas describing a solution to a boundary value problem of elasticity theory for a rectangle whose longitudinal sides are free, while normal and tangential stresses are given at the transverse sides. The solution is exact since it is not reduced to an infinite system of algebraic equations, but is represented as series in Papkovich–Fadle eigenfunctions whose coefficients are found by closed formulas. Only the even-symmetric deformation of the rectangle relative to the central axes is considered. Graphs are presented that illustrate the comparison
of the obtained solution with the solution to the corresponding boundary value problem in a half-strip. Other cases of parity with respect to the central axes are considered similarly.

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References
[1] Grinchenko V T 2003 The biharmonic problem and progress in the development of analytical methods for the solution of boundary-value problems J. Eng. Math. 46(3–4) pp 281–97
[2] Meleshko V V and Gomilko A M 1997 Infinite systems for a biharmonic problem in a rectangle Proc. R. Soc. Lond. A 453 pp 2139–60
[3] Dang Q A 2006 Iterative method for solving the Neumann boundary value problem for biharmonic type equation J. Comput. Appl. Math. 196(2) pp 634–43
[4] Ben–Artzi M, Chorev I, Croisille J–P and Fishelov D 2009 A compact difference scheme for the biharmonic equation in planar irregular domains SIAM J. Numer. Anal. 47(4) pp 3087–108
[5] Pestana J, Muddle R, Heil M, Tisseur F and Mihajlović M 2016 Efficient block preconditioning for a $C^1$ finite element discretization of the dirichlet biharmonic problem SIAM J. Sci. Comput. 38(1) pp A325–45
[6] Sidnyaev N I 2019 Analytical calculation for reliability validation of nuclear power plants At Energy 126(29) pp 29–33
[7] Sidnyaev N I 2018 A study of the destruction of spacecraft surfaces at contact interactions with microparticles of the space environment Cosmic Research 56(3) pp 213–22
[8] Zarubin V S, Kuvyrkin G N and Savelyeva I Y 2019 Variational estimates of the parameters of a thermal explosion of a stationary medium in an arbitrary domain Int. J. Heat Mass Transfer 135 pp 614–9
[9] Volkov A G, Dyugaeva N A, Kuvyrkin G N and Morozov A N 2017 Studying the change in characteristics of optical surfaces of a spacecraft Cosmic Res. 55(2) pp 124–7
[10] Manzhurov A V 2016 A mixed integral equation of mechanics and a generalized projection method of its solution Dokl. Phys. 61(10) pp 489–93
[11] Meleshko V V 2003 Selected topics in the history of two-dimensional biharmonic problem Appl. Mech. Rev. 56(1) pp 33–85
[12] Kovalenko M D and Shulyakovskaya T D 2011 Expansions in Fadle–Papkovich functions in a strip. Theory foundations Mech. Solids 46(5) pp 721–38
[13] Kovalenko M D, Menshova I V and Shulyakovskaya T D 2013 Expansions in Fadle–Papkovich functions: examples of solutions in a half-strip Mech. Solids 48(5) pp 584–602
[14] Kovalenko M D, Menshova I V and Kerzhaev A P 2018 On the exact solutions of the biharmonic problem of the theory of elasticity in a half-strip Z. Angew. Math. Phys. 69 121
[15] Kovalenko M D, Abrukov D A, Menshova I V, Kerzhaev A P and Yu G 2019 Exact solutions of boundary value problems in the theory of plate bending in a half-strip: basics of the theory Z. Angew. Math. Phys. 70 98
[16] Kerzhaev A P. Kovalenko M D and Menshova I V 2018 Borel transform in the class $W$ of quasi-entire functions Complex Anal. Oper. Theory 12(3) pp 571–87