Strict positive definiteness on products of compact two-point homogeneous spaces

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ABSTRACT
We present an explicit characterization for the real, continuous, isotropic and strictly positive definite kernels on a product of compact two-point homogeneous spaces, covering almost all possible choices for the spaces. The result complements similar characterizations previously obtained for products of high-dimensional spheres.

1. Introduction
Let $\mathbb{M}^d$ denote a $d$-dimensional compact two-point homogeneous space. A real and continuous kernel $K$ on $\mathbb{M}^d$ is positive definite (PD) if it is a symmetric function, that is,

$$K(x, y) = K(y, x), \quad x, y \in \mathbb{M}^d,$$

and satisfies the following condition: if $n$ is a positive integer and $x_1, x_2, \ldots, x_n$ are distinct points on $\mathbb{M}^d$, then the $n \times n$ matrix $[K(x_i, x_j)]$ is non-negative definite. A positive definite kernel on $\mathbb{M}^d$ is strictly positive definite (SPD) if all the matrices in the previous definition are in fact positive definite. Positive definite kernels and functions on metric spaces are an important subject in many areas of classical and modern mathematics, such as radial basis function interpolation and approximation, geomathematics, geostatistics, Fourier analysis, etc [1–3].

Throughout the paper, we will assume that the geodesic distance on $\mathbb{M}^d$ fulfills the following requirement: all geodesics have the same length $2\pi$. In addition to continuity, the geodesic distance on $\mathbb{M}^d$, here denoted by $|xy|$, $x, y \in \mathbb{M}^d$, allows the introduction of the notion of isotropy of a positive definite kernel $K$ on $\mathbb{M}^d$, in the same way I. J. Schoenberg did for positive definite kernels on spheres [4]. It demands that $K(x, y) = K_d^i (\cos |xy|/2), x, y \in \mathbb{M}^d$, for some function $K_d^i : [-1, 1] \rightarrow \mathbb{R}$, here called the isotropic part of $K$. According to
Gangolli [5], a continuous and isotropic kernel $K$ on $\mathbb{M}^d$ is PD if and only if

$$K_i^d(t) = \sum_{k=0}^{\infty} a_k^{(d-2)/2, \beta} F_k^{(d-2)/2, \beta}(t), \quad t \in [-1, 1], \quad (1.1)$$

in which $a_k^{(d-2)/2, \beta} \in [0, \infty), k \in \mathbb{Z}_+$ and $\sum_{k=0}^{\infty} a_k^{(d-2)/2, \beta} F_k^{(d-2)/2, \beta}(1) < \infty$. Here, $\beta = (d - 2)/2, -1/2, 0, 1, 3$, depending on the respective category $\mathbb{M}^d$ belongs to, among the following ones [6]: the unit spheres $S^d, d = 1, 2, \ldots$, the real projective spaces $\mathbb{P}^d(\mathbb{R}), d = 2, 3, \ldots$, the complex projective spaces $\mathbb{P}^d(\mathbb{C}), d = 4, 6, \ldots$, the quaternionic projective spaces $\mathbb{P}^d(\mathbb{H}), d = 8, 12, \ldots$, and the Cayley projective plane $\mathbb{P}^d(Cay), d = 16$. The symbol $F_k^{((d-2)/2, \beta)}$ stands for the Jacobi polynomial of degree $k$ associated with the pair $((d - 2)/2, \beta)$ [7].

This is the point where we can explain what the intentions in this paper are. The first target is to present a characterization for the real, continuous and isotropic kernels which are PD on a cartesian product of compact two-point homogeneous spaces. If $\mathbb{M}^d$ and $\mathbb{H}^d$ are the spaces, the isotropy of a kernel $K: \mathbb{M}^d \times \mathbb{H}^d \to \mathbb{R}$ corresponds to isotropy in both spaces, that is, $K$ is of the form

$$K((x, w), (y, z)) = K_i^{d, d'}(\cos(|xyz|/2), \cos(|wz|/2)), \quad x, y \in \mathbb{M}^d, \quad w, z \in \mathbb{H}^d,$$

for some function $K_i^{d, d'}: [-1, 1]^2 \to \mathbb{R}$ (the isotropic part of $K$). Since we believe such a characterization can be deduced via classical results from harmonic analysis, the proposal here is to obtain the characterization using an alternative procedure involving the Gauss hypergeometric function $2 F_1$ and basic convergence arguments. The characterization itself can be described as follows: if $K$ is a real, continuous and isotropic kernel on $\mathbb{M}^d \times \mathbb{H}^d$, then it is PD if and only if its isotropic part has a series representation in the form

$$K_i^{d, d'}(t, s) = \sum_{k,l=0}^{\infty} a_{k,l}(K_i^{d, d'})(F_k^{((d-2)/2, \beta)}(t) P_l^{((d'-2)/2, \beta')}(s), \quad t, s \in [-1, 1]^2,$$

in which $a_{k,l}(K_i^{d, d'}) \geq 0, k, l \in \mathbb{Z}_+$ and $\sum_{k,l=0}^{\infty} a_{k,l}(K_i^{d, d'})(F_k^{((d-2)/2, \beta)}(1) P_l^{((d'-2)/2, \beta')}(1) < \infty$. The numbers $\beta$ and $\beta'$ have to agree with $d$ and $d'$, respectively, respecting Wang’s classification in [6]. This result will be deduced in Section 2.

As explained in [8], positive definiteness on a product of spaces allows intermediate notions of strict positive definiteness. In the case of a positive definite kernel $K: \mathbb{M}^d \times \mathbb{H}^d \to \mathbb{R}$, one of them reads like this: $K$ is DC-strictly positive definite (DC-SPD) if the strict positive definiteness condition previously introduced holds for points of $\mathbb{M}^d \times \mathbb{H}^d$ having distinct components. Equivalently, all matrices of the form $K((x_i, w_i), (x_j, w_j))$ are positive definite if the $x_i$ are distinct in $\mathbb{M}^d$ and the $w_i$ are distinct in $\mathbb{H}^d$. Clearly, DC-strict positive definiteness is a weaker notion in the sense that it demands plain strict positive definiteness for just some of the distinct points in $\mathbb{M}^d \times \mathbb{H}^d$. In the case $\mathbb{M}^d = S^d$ and $\mathbb{H}^d = S^{d'}, d, d' \geq 2$, DC-strict positive definiteness of a real, continuous, isotropic and PD kernel $K$ was shown to be equivalent to the following condition [9]: the set $\{k + l: a_{k,l}(K_i^{d, d'}) > 0\}$ contains infinitely many even and infinitely many odd integers. In Section 3, we will complete this line of investigation, presenting a characterization for DC-strict
positive definiteness in all the other products involving compact two-point homogeneous spaces, except when one of the spaces is either a circle or \( \mathbb{P}^2(\mathbb{R}) \).

The third target in the paper is to present a characterization for plain strict positive definiteness of a real, continuous, isotropic and PD kernel on \( \mathbb{M}^d \times \mathbb{H}^{d'} \), in the same cases mentioned at the end of the previous paragraph. That is a counterpart of similar results for products of spheres previously obtained in [9–11]. The details for that will appear in Section 4.

2. Positive definiteness

In this section, we will provide a characterization for the real, continuous, isotropic and PD kernels on \( \mathbb{M}^d \times \mathbb{H}^{d'} \). No additional assumption on either \( d \) or \( d' \) will be made. The characterization is an extension to all the compact two-point homogeneous spaces of that one previously obtained in [8] in the case in which both spaces are spheres (see also [12] for an alternative proof in that case).

Let us begin with some basics on Jacobi polynomials. For \( \alpha, \beta > -1 \), the set \( \{ P^k_{(\alpha, \beta)} : k \in \mathbb{Z}_+ \} \) of Jacobi polynomials associated to the pair \( (\alpha, \beta) \) is orthogonal on \([-1, 1]\) in the sense that

\[
\int_{-1}^{1} P^k_{(\alpha, \beta)}(t) P^l_{(\alpha, \beta)}(t) (1-t)^\alpha (1+t)^\beta \, dt = \delta_{k,l} h^\alpha_1^\beta ,
\]

where

\[
h^\alpha_1^\beta = \frac{2^{\alpha+\beta+1} \Gamma(k+\alpha+1) \Gamma(k+\beta+1)}{2k+\alpha+\beta+1 \Gamma(k+1) \Gamma(k+\alpha+\beta+1)}, \quad k \in \mathbb{Z}_+ .
\]

Here, and in many other places in the paper, \( \Gamma \) will stand for the usual gamma function. An immediate consequence is the orthogonality of the family \( \{ (t,s) \in [-1,1] \mapsto P^k_{(\alpha, \beta)}(t) P^l_{(\alpha', \beta')}(s) : k, l \in \mathbb{Z}_+ \} \) with respect to the weight function \( \sigma^d_{d'}(t,s) := (1-t)^\alpha (1+t)^\beta (1-s)^{\alpha'} (1+s)^{\beta'} \), \( t, s \in [-1,1] \).

There exists a generating formula for Jacobi polynomials via the \textit{Gauss hypergeometric function} \( _2F_1 \) [7,13–15]. As a regular solution of the hypergeometric differential equation, the hypergeometric function has a representation in the form \( _2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{[(c)_n n!]^{-1} (a)_n (b)_n z^n}{(c)_n n!} \), in which \( a,b \) and \( c \) are generic parameters, \( z \) is a complex variable, and \( (\lambda)_n \) is the Pochhammer symbol. The convergence holds for \( |z| < 1 \) if \( c \) is not a negative integer and for \( |z| = 1 \) if \( \text{Re}(c-a-b) > 0 \) [7, p.63].

For simplicity’s sake, we will write \( F(a,b;c;z) := _2F_1(a,b;c;z) \). The hypergeometric function \( F \) is differentiable with respect to \( z \) in \( \{ z \in \mathbb{C} : |z| < 1 \} \) [16, p.281] and

\[
\frac{d}{dz} F(a,b;c;z) = \frac{ab}{c} F(a+1,b+1;c+1;z).
\]

In particular,

\[
\int_{z_1}^{z_2} F(a,b;c;z) \, dz = \frac{c-1}{(a-1)(b-1)} \left[ F(a-1,b-1;c-1;z_2) - F(a-1,b-1;c-1;z_1) \right].
\]
A generating formula for Jacobi polynomials based on the function $F$ is the content of the following Poisson formula [13, p. 21]:

$$\sum_{n=0}^{\infty} \frac{P_n^{(\alpha, \beta)}(1)P_n^{(\alpha, \beta)}(t)}{h_n^{\alpha, \beta}} r^n = G^{\alpha, \beta}(r)F\left(\frac{\alpha + \beta + 2}{2}, \frac{\alpha + \beta + 3}{2}; \beta + 1; \frac{2r(1 + t)}{(1 + r)^2}\right),$$

in which $G^{\alpha, \beta}(r) := [\Gamma(\alpha + 1)\Gamma(\beta + 1)(1 + r)^{\alpha + \beta + 2}]^{-1} 2^{-(\alpha + \beta + 1)}\Gamma(\alpha + \beta + 2)(1 - r)$.

We now detach a consequence of the results described above to be used ahead.

**Lemma 2.1:** If $\alpha \geq \beta \geq -\frac{1}{2}$ and $r \in (-1, 1)$, then $\sum_{n=0}^{\infty} \frac{1}{h_n^{\alpha, \beta}} P_n^{(\alpha, \beta)}(1)P_n^{(\alpha, \beta)}(t) r^n < \infty$, for $[-1, 1]$.

**Proof:** If $r = 0$, the result is obvious. Otherwise, taking into account the convergence of the series representation for $F$, it is promptly seen that the series in the statement of the lemma will be convergent as long as $2|r||1 + t| < (1 + r)^2$. However, a simple calculation reveals that $(1 + r)^2 > 4|r|$ whenever $r \in (-1, 1) \setminus \{0\}$. Thus, since $t \in [-1, 1]$, the convergence follows in this case as well.

Lemma 2.2 is a critical step towards the desired characterization in this section. It follows from the definition of positive definiteness along with Gangolli’s characterization for positive definiteness on a single compact two-point homogeneous space and the Schur product theorem for non-negative definite matrices. Once again, $\beta$ and $\beta'$ have to agree with Wang’s classification for the compact two-point homogeneous spaces.

**Lemma 2.2:** For fixed non-negative integers $k$ and $l$, the function $(t, s) \in [-1, 1] \mapsto P_k^{((d-2)/2, \beta)}(t)P_l^{((d-2)/2, \beta')}(s)$ is the isotropic part of a PD kernel on $\mathbb{M}^d \times \mathbb{H}^d$.

In the next proposition, $dx$ and $dy$ will denote the volume elements on $\mathbb{M}^d$ and $\mathbb{H}^d$, respectively (dimensions will be omitted). We will require the expansion of a function $f$ from $L_1([-1, 1], \sigma_d^d)$:

$$f(t, s) \sim \sum_{k,l=0}^{\infty} a_{k,l}(f) P_k^{((d-2)/2, \beta)}(t)P_l^{((d-2)/2, \beta')}(s)$$

in which

$$a_{k,l}(f) = \frac{1}{h_k^{(d-2)/2, \beta} h_l^{(d-2)/2, \beta'}} \int_{[-1,1]^2} f(t, s) P_k^{((d-2)/2, \beta)}(t)P_l^{((d-2)/2, \beta')}(s) \, d\sigma_d^d(t, s),$$

for $k, l \in \mathbb{Z}_+$. The dependence of $a_{k,l}$ upon $d, d', \beta, \beta'$ will be omitted.

In the case $f$ is the isotropic part of a kernel belonging to the setting of the paper, the following formula holds.
**Proposition 2.3:** Let \( k \) and \( l \) be non-negative integers. If \( K \) is a real, continuous and isotropic kernel on \( \mathbb{M}^d \times \mathbb{H}^d \), then there exists a positive constant \( C \), that depends upon \( d, d', \beta \) and \( \beta' \), so that

\[
    a_{k,l}(K^d d') = C \int_{\mathbb{M}^d \times \mathbb{H}^d} \left[ \int_{\mathbb{M}^d \times \mathbb{H}^d} K((x, y), (w, z)) F_k^{(d-2)/2, \beta}(\cos(|xy|/2)) \right.
    \times P_{l}^{((d-2)/2, \beta')}(\cos(|wz|/2)) \, dy \, dz \, dx \, dw.
\]

**Proof:** The first step in the proof is to observe that the integral

\[
    \int_{\mathbb{M}^d \times \mathbb{H}^d} f(\cos(|xy|/2), \cos(|wz|/2)) P_k^{((d-2)/2, \beta)}(\cos(|xy|/2)) P_{l}^{((d-2)/2, \beta')}(\cos(|wz|/2)) \, dy \, dz
\]

equals to a positive multiple of \( a_{k,l}(f) \). Indeed, this follows after two applications of the Funk–Hecke formula for harmonics on compact two-point homogeneous spaces (see Proposition 2.8 and Remark 2.9 in [17]) coupled with Fubini’s theorem. The multiple itself depends upon \( d, d', \beta \) and \( \beta' \). Calling it \( C \) and integrating leads to the formula in the statement of the proposition.

If we introduce the positive definiteness of \( K \) among the assumptions, we obtain the following expected consequence.

**Proposition 2.4:** If \( K \) is a real, continuous, isotropic and PD kernel on \( \mathbb{M}^d \times \mathbb{H}^d \), then \( a_{k,l}(K^d d') \geq 0, k, l \in \mathbb{Z}_+ \).

**Proof:** Due to Lemma 2.2, if \( K \) is PD on \( \mathbb{M}^d \times \mathbb{H}^d \), then the integrand in the double integral appearing in the statement of the previous proposition defines a real, continuous, isotropic and PD kernel on \( \mathbb{M}^d \times \mathbb{H}^d \). Hence, the proof of the proposition resumes itself to showing that if \( K \) is a real, continuous, isotropic and PD kernel on \( \mathbb{M}^d \times \mathbb{H}^d \), then the integral

\[
    I := \int_{\mathbb{M}^d \times \mathbb{H}^d} \left[ \int_{\mathbb{M}^d \times \mathbb{H}^d} K((x, y), (w, z)) \, dy \, dz \right] \, dx \, dw
\]

is non-negative. In order to achieve that, we will fix \( \epsilon > 0 \) and will show there exists a number \( \tilde{T} = \tilde{T}(\epsilon) \geq 0 \) so that \( |I - \tilde{T}| \leq \epsilon \text{Vol}(\mathbb{M}^d \times \mathbb{H}^d)^2 \). Indeed, if \( I \) were negative, then the information above with a convenient choice for \( \epsilon \) would produce a contradiction. Since \( \mathbb{M}^d \times \mathbb{H}^d \) is a compact metric space, the kernel \( K \) is actually uniformly continuous on \( \mathbb{M}^d \times \mathbb{H}^d \). In particular, we can select \( \delta > 0 \) so that \( |K((x, y), (w, z)) - K((x', y'), (w', z'))| < \epsilon \) whenever \( x, x', y, y' \in \mathbb{M}^d, w, w', z, z' \in \mathbb{H}^d \), \( |xx'| < \delta, |yy'| < \delta, |ww'| < \delta \) and \( |zz'| < \delta \). Since the metric spaces \( \mathbb{M}^d \) and \( \mathbb{H}^d \) are totally bounded, we can cover them with finitely many open balls of radius \( \delta/2 \). Likewise, we can cover \( \mathbb{M}^d \times \mathbb{H}^d \) with finitely many open balls. Using the covering, we can partition \( \mathbb{M}^d \times \mathbb{H}^d \) into finitely many Borel subsets, say, \( \mathbb{M}^d \times \mathbb{H}^d = \bigcup_{j=1}^{p} B(j) \), so that \( |xx'| < \delta \) and \( |ww'| < \delta \) whenever \( ((x, w), (x', w')) \in B(j) \), \( j = 1, 2, \ldots, p \). It is now clear that if,
Due to the positive definiteness of series in the form
\[ \int K((x, y), (w, z)) \, dx \, dw \, dy \, dz. \]

Next, for each \( j \in \{1, 2, \ldots, p\} \), choose \((x_j, w_j) \in B(j)\) and define \( \lambda_j := \) the volume of \( B(j) \). Clearly, the number \( I := \sum_{j,k=1}^{p} \lambda_j \lambda_k K((x_j, x_k), (w_j, w_k)) \) is non-negative due to the positive definiteness of the kernel \( K \). On the other hand
\[
|I - \bar{I}| = \left| \sum_{j,k=1}^{p} \int_{B(j)} \int_{B(k)} \left[ K((x, y), (w, z)) - K((x_j, x_k), (w_j, w_k)) \right] \, dx \, dw \, dy \, dz \right|
\]
and, consequently, \( |I - \bar{I}| \leq \varepsilon \sum_{j,k=1}^{p} \lambda_j \lambda_k \leq \varepsilon \text{Vol}(\mathbb{M}^d \times \mathbb{H}^d)^2 \).

Next, we move to convergence of double series defined by Jacobi polynomials.

**Lemma 2.5:** Let \( K \) be a real, continuous, isotropic and PD kernel on the product \( \mathbb{M}^d \times \mathbb{H}^d \). If \( r, \rho \in (-1, 1) \), then
\[
\sum_{k,l=0}^{\infty} a_{k,l}(K^{d,d'}_l) F_k((d-2)/2, \beta) F_k((d-2)/2, \beta') (1) I_k((d-2)/2, \beta) (1) r^k \rho^l < \infty.
\]

**Proof:** We will prove the lemma in the case in which \( d \neq 1 \) and \( d' \neq 1 \). The proof in the cases in which either \( d=1 \) or \( d'=1 \) can be adapted from the general proof presented below. However, the computation referring to the case in which the space is \( S^1 \) needs to be done directly without mentioning the hypergeometric function. The calculations made in [8] are very similar to what is needed in these specials cases. Otherwise, the general term of the series in the statement of the lemma is
\[
\int_{-1}^{1} \int_{-1}^{1} K^{d,d'}_l(t, s) \frac{F_k((d-2)/2, \beta)}{h_k((d-2)/2, \beta)} (1) F_k((d-2)/2, \beta') (1) r^k \rho^l \, \sigma^d_\rho(t, s).
\]

Introducing the Gauss hypergeometric function in the above expression leaves the double series in the form
\[
\int_{-1}^{1} \int_{-1}^{1} \left[ K^{d,d'}_l(t, s) G((d-2)/2, \beta)(r) F \left( \frac{d + 2 \beta + 2}{4}, \frac{d + 2 \beta + 4}{4}; \beta + 1; \frac{2r(1 + t)}{(1 + r)^2} \right) \times G((d-2)/2, \beta')(\rho) F \left( \frac{d' + 2 \beta' + 2}{4}, \frac{d' + 2 \beta' + 4}{4}; \beta' + 1; \frac{2\rho(1 + s)}{(1 + \rho)^2} \right) \right] \, \sigma^d_\rho(t, s) \]
\]

Due to the positive definiteness of \( K \) and the continuity of \( K^{d,d'}_l \) in \([-1, 1] \times [-1, 1]\), we can estimate the double integral from above by
\[
C^{r,\rho} \int_{-1}^{1} \int_{-1}^{1} \left[ F \left( \frac{d + 2 \beta + 2}{4}, \frac{d + 2 \beta + 4}{4}; \beta + 1; \frac{2r(1 + t)}{(1 + r)^2} \right) \times F \left( \frac{d' + 2 \beta' + 2}{4}, \frac{d' + 2 \beta' + 4}{4}; \beta' + 1; \frac{2\rho(1 + s)}{(1 + \rho)^2} \right) \right] \, \sigma^d_\rho(t, s),
\]
in which \( C^{r,\rho} \) is a positive multiple of \( G((d-2)/2, \beta)(r) G((d-2)/2, \beta')(\rho) \). The weight in the definition of \( \sigma^d_\rho \) can be bounded above by \( 2^{\beta + \beta' - 2 + (d + d')/2} \). Introducing this bound and
solving the resulting integrals, we conclude that the double series is at most
\[
CF \left( \frac{d + 2\beta - 2}{4}, \frac{d + 2\beta}{4} ; \beta; \frac{4r}{(1 + r)^2} \right) F \left( \frac{d' + 2\beta' - 2}{4}, \frac{d' + 2\beta'}{4} ; \beta'; \frac{4\rho}{(1 + \rho)^2} \right),
\]
in which
\[
C = G^{(d-2)/2,\beta}(r)G^{(d'-2)/2,\beta'}(\rho) \frac{\beta\beta'2\beta'+\beta'+4+(d+d')/2(1+r)^2(1+\rho)^2}{(d+2\beta-2)(d'+2\beta'-2)(d+\beta)(d'+\beta)r\rho}.
\]

The proof is complete. ■

**Proposition 2.6:** If \( K \) is a real, continuous, isotropic and PD kernel on \( \mathbb{M}^d \times \mathbb{H}^d \), then the double series \( \sum_{k,l=0}^{\infty} a_{k,l}(K_i^{d,d'}) P_k((d-2)/2,\beta) F_k((d'-2)/2,\beta') (1) P_l((d'-2)/2,\beta') (1) \) converges absolutely and uniformly for \((t,s) \in [-1,1]^2\).

**Proof:** Due to the Weierstrass \( M \)-test for double series, it suffices to show that \( \sum_{k,l=0}^{\infty} a_{k,l}(K_i^{d,d'}) P_k((d-2)/2,\beta) F_k((d'-2)/2,\beta') (1) P_l((d'-2)/2,\beta') (1) \) converges. In order to do that, consider the sequence \((s_{p,q})_{p,q \in \mathbb{Z}+} \) given by the partial sums
\[
s_{p,q} := \sum_{k=0}^{p} \sum_{l=0}^{q} a_{k,l}(K_i^{d,d'}) P_k((d-2)/2,\beta) F_k((d'-2)/2,\beta') (1) P_l((d'-2)/2,\beta') (1), \quad p, q \in \mathbb{Z}+.
\]
By Lemma 2.4, \( a_{k,l}(K_i^{d,d'}) \geq 0 \), for all \( k, l \in \mathbb{Z}+ \). In particular, \( s_{p,q} \leq s_{p',q'} \) when \( p \leq p' \) and \( q \leq q' \). On the other hand, by the previous lemma,
\[
\sum_{k=0}^{p} \sum_{l=0}^{q} a_{k,l}(K_i^{d,d'}) P_k((d-2)/2,\beta) F_k((d'-2)/2,\beta') (1) P_l((d'-2)/2,\beta') (1) r^k \rho^l \leq C, \quad p, q \in \mathbb{Z}+, \quad r, \rho \in (-1, 1),
\]
for some \( C > 0 \). Applying the limits when \( r, \rho \to 1^+ \), we deduce the sequence \((s_{p,q}) \) is bounded above. The convergence of \((s_{p,q}) \) follows. ■

The main result in the section is as follows.

**Theorem 2.7:** Let \( K \) be a real, continuous and isotropic kernel on \( \mathbb{M}^d \times \mathbb{H}^d \). It is PD on \( \mathbb{M}^d \times \mathbb{H}^d \) if and only if its isotropic part \( K_i^{d,d'} \) has a representation in the form
\[
K_i^{d,d'}(t,s) = \sum_{k,l=0}^{\infty} a_{k,l}(K_i^{d,d'}) P_k((d-2)/2,\beta) F_k((d'-2)/2,\beta') (1) P_l((d'-2)/2,\beta') (1), \quad t, s \in [-1,1]^2,
\]
where \( a_{k,l}(K_i^{d,d'}) \geq 0, k, l \in \mathbb{Z}+ \) and \( \sum_{k,l=0}^{\infty} a_{k,l}(K_i^{d,d'}) P_k((d-2)/2,\beta) F_k((d'-2)/2,\beta') (1) P_l((d'-2)/2,\beta') (1) < \infty \).

**Proof:** Consider the function \( g \) defined by the Fourier expansion
\[
g(t,s) \sim \sum_{k,l=0}^{\infty} a_{k,l}(K_i^{d,d'}) P_k((d-2)/2,\beta) F_k((d'-2)/2,\beta') (1) P_l((d'-2)/2,\beta') (1), \quad t, s \in [-1,1].
\]
If \( K \) is PD, then Proposition 2.5 guarantees the convergence of the series for \( t = s = 1 \). Proposition 2.6 implies convergence for all the other values of \( t \) and \( s \) while Proposition 2.4
yields that all the coefficients in the expansion are non-negative. Since $g$ is continuous and the Fourier coefficients of $K_i^{d,d'}$ coincide with those of $g$, it follows that $K_i^{d,d'} = g$. This takes care of one implication in the theorem. As for the other, it follows from Lemma 2.2 and the fact that the pointwise limit of PD kernels is itself PD.

3. DC-strict positive definiteness

Either one of the concepts of strict positive definiteness we have introduced so far demands considering $n \times n$ matrices $A = [A_{\mu\nu}]$ with

$$A_{\mu\nu} = K_i^{d,d'}(\cos(|x_\mu x_\nu|/2), \cos(|w_\mu w_\nu|/2)),$$

in which $K_i^{d,d'}$ is the isotropic part of the kernel and $(x_\mu, w_\mu)$, $\mu = 1, 2, \ldots, n$, are distinct points in $\mathbb{R}^d \times \mathbb{R}^{d'}$. Analysing the associated quadratic forms $c^\prime Ac := \sum_{\mu,\nu=1}^n c_\mu c_\nu K((x_\mu, w_\mu), (x_\nu, w_\nu))$, $c = (c_\mu) \in \mathbb{R}^n$, it is possible to obtain a quite more convenient formulation for either concept. We will proceed discussing DC-strict positive definiteness and will just mention the formulation for plain strict positive definiteness later. From now on, if $K$ is a real, continuous, isotropic and PD kernel $K$ on $\mathbb{R}^d \times \mathbb{R}^{d'}$, we will use the following notation attached to the series representation of its isotropic part: $J_K := \{(k,l) : a_{k,l}^{(d-2)/2,\beta} > 0\}$.

At this point, we need the addition formula demonstrated by Giné [18] and Koornwinder [19] that is,

$$\sum_{j=1}^{\delta(k,d)} S_j^d(x) S_j^d(y) = c_k^{d,\beta} P_k^{((d-2)/2,\beta)}(\cos(|xy|/2)), \quad x, y \in \mathbb{R}^d,$$

where

$$c_k^{d,\beta} := \frac{\Gamma(\beta + 1)(2k + (d - 2)/2 + \beta + 1)\Gamma(k + (d - 2)/2 + \beta + 1)}{\Gamma((d - 2)/2 + \beta + 2)\Gamma(k + \beta + 1)}.$$ 

The set $\{S_{k,1}^d, S_{k,2}^d, \ldots, S_{k,\delta(k,d)}^d\}$ denotes an orthonormal basis of the space $H^d_k$ of spherical harmonics of degree $k$ on $\mathbb{R}^d$.

If we consider the representation for $K$ provided by Theorem 2.7 and the addition formula above, then the equality $c^\prime Ac = 0$ corresponds to

$$\sum_{k,l=0}^{\infty} a_{k,l}(K_i^{d,d'}) \sum_{j=1}^{\delta(k,d)} \sum_{j=1}^{\delta(l,d')} \left| \sum_{\mu=1}^n c_\mu S_{k,j}^d(x_\mu) S_{l,j}^{d'}(w_\nu) \right|^2 = 0.$$

In particular, $c^\prime Ac = 0$ if, and only if,

$$\sum_{\mu=1}^n c_\mu S_{k,i}^d(x_\mu) S_{l,j}^{d'}(w_\nu) = 0, \quad (k, l) \in J_K, \quad i \in \{1, 2, \ldots, \delta(k,d)\}, \quad j \in \{1, 2, \ldots, \delta(l,d')\}.$$
Reintroducing the addition formula, now leaving a free variable \((x, w) \in \mathbb{M}^d \times \mathbb{H}^{d'}\), the previous assertion implies that

\[
\sum_{\mu=1}^{n} c_\mu P_k^{((d-2)/2, \beta)}(\cos(|x_\mu x|/2)) P_l^{((d'-2)/2, \beta')}(\cos(|w_\mu w|/2)) = 0,
\]

for \((x, w) \in \mathbb{M}^d \times \mathbb{H}^{d'}\) and \((k, l) \in J_K\). However, if this last assertion holds, it is promptly seen that

\[
\sum_{\mu=1}^{n} \sum_{i=1}^{\delta(k,d)} \sum_{j=1}^{\delta(l,d')} S_{k,i}(x_\mu) S_{d_{k,i}}(x) S_{l,j}(w_\mu) S_{d_{l,j}}(w) = 0, \quad (x, w) \in \mathbb{M}^d \times \mathbb{H}^{d'}, \quad (k, l) \in J_K.
\]

Using the fact that \(\{S_{d_{l,1}}, S_{d_{l,2}}, \ldots, S_{d_{l,\delta(l,d')}}\}\) and \(\{S_{d_{k,1}}, S_{d_{k,2}}, \ldots, S_{d_{k,\delta(k,d)}}\}\) are basis of \(\mathcal{H}_{l}^{d_{l}}\) and \(\mathcal{H}_{k}^{d_{k}}\) respectively, we are reduced to

\[
\sum_{\mu=1}^{n} c_\mu S_{k,i}(x_\mu) S_{d_{l,j}}(w_\mu) = 0, \quad (k, l) \in J_K, \quad i \in \{1, 2, \ldots, \delta(k,d)\}, \quad j \in \{1, 2, \ldots, \delta(l,d')\}.
\]

once again.

The discussion above justifies the following result.

**Proposition 3.1:** Let \(K\) be a real, continuous, isotropic and PD kernel on \(\mathbb{M}^d \times \mathbb{H}^{d'}\). The following assertions are equivalent:

(i) \(K\) is DC-SPD;

(ii) If \(n \geq 1, x_1, x_2, \ldots, x_n\) are distinct points on \(\mathbb{M}^d\) and \(w_1, w_2, \ldots, w_n\) are distinct points on \(\mathbb{H}^{d'}\), then the only solution of the system

\[
\sum_{\mu=1}^{n} c_\mu P_k^{((d-2)/2, \beta)}(\cos(|x_\mu x|/2)) P_l^{((d'-2)/2, \beta')}(\cos(|w_\mu w|/2)) = 0,
\]

\[(x, w) \in \mathbb{M}^d \times \mathbb{H}^{d'}, \quad (k, l) \in J_K,
\]

is the trivial one, that is, \(c_\mu = 0, \mu = 1, 2, \ldots, n\).

In the lemma below, we use the symbol \(M_1 \hookrightarrow M_2\) to indicate the existence of an isometric embedding of a metric space \(M_1\) into a metric space \(M_2\). The result is a classical result in the theory of compact two-point homogeneous spaces (see [13]).

**Lemma 3.2:** There exists a chain of isometric embeddings as follows

\[
S^1 \hookrightarrow \mathbb{P}^2(\mathbb{R}) \hookrightarrow \mathbb{P}^d(\mathbb{R}) \hookrightarrow \mathbb{P}^{2d}(\mathbb{C}) \hookrightarrow \mathbb{P}^{4d}(\mathbb{H}) \hookrightarrow \mathbb{P}^{8d}(\text{Cay}), \quad d = 2, 3, \ldots.
\]

In particular, if the compact two-point homogeneous space \(\mathbb{H}^{d'}\) is neither a sphere nor \(\mathbb{P}^2(\mathbb{R})\), then the lemma guarantees the existence of an integer \(q \geq 2\) so that \(S^q \hookrightarrow \mathbb{H}^{d'}\). On
the other hand, this embedding justifies a decomposition of the form

$$P_l^{((d-2)/2,\beta)}(s) = \sum_{j=0}^l b_j^l P_{l-j}^{((q-2)/2,(q-2)/2)}(s), \quad l = 0, 1, \ldots,$$

with all coefficients $b_j^l$ positive.

In the lemma below we consider the normalized Jacobi polynomials $R_k^{(\alpha,\beta)} = [P_k^{(\alpha,\beta)}(1)]^{-1}P_k^{(\alpha,\beta)}$, with $\alpha$ and $\beta$ as in Wang’s classification. For a proof, see [7,20].

**Lemma 3.3:** The Jacobi polynomials have the following properties:

1. $P_k^{(\alpha,\beta)}(-t) = (-1)^k P_k^{(\beta,\alpha)}(t), \ t \in [-1, 1]$;
2. Except for the case $\alpha = \beta = -\frac{1}{2}$, it holds $\lim_{k \to \infty} R_k^{(\alpha,\beta)}(t) = 0, \ t \in (-1, 1)$;
3. If $\alpha > \beta$, then $\lim_{k \to \infty} P_k^{(\beta,\alpha)}(1)[P_k^{(\alpha,\beta)}(1)]^{-1} = 0$.

This is the first characterization for DC-strict positive definiteness we have found.

**Theorem 3.4:** Let $K$ be a real, continuous, isotropic and PD kernel on $S^d \times \mathbb{H}^d$. Assume that $d \geq 2$ and that $\mathbb{H}^d$ is neither a sphere nor $\mathbb{P}^d(\mathbb{R})$. In order that $K$ be DC-SPD it is necessary and sufficient that either $\{l : (k, l) \in J_K$ for some $k\}$ be infinite or $J_K$ contains two sequences $\{(k_r, l)\}$ and $\{(k_s, l')\}$ for which $\{k_r + l\} \subset 2\mathbb{Z}_+, \{k_s + l'\} \subset 2\mathbb{Z}_+ + 1$, and $\lim_{r \to \infty} k_r = \lim_{s \to \infty} k_s = \infty$.

**Proof:** Assume that $K$ is DC-SPD. Recalling Lemma 3.2, it is easily seen that $K$ is DC-SPD on $S^d \times S^q$ for some $q \geq 2$. Introducing the equalities presented right after Lemma 3.2 into the series representation for $K_i^{d,d'}$ and rearranging leads to $(t, s \in [-1, 1])

$$K_i^{d,d'}(t, s) = \sum_{k,l=0}^{\infty} \left( \sum_{j=0}^{\infty} a_{k,l+j}(K_i^{d,d'}) b_{l+j} \right) P_k^{((d-2)/2,(d-2)/2)}(t) P_l^{((q-2)/2,(q-2)/2)}(s).$$

In particular, $\{k + l : \sum_{j=0}^{\infty} a_{k,l+j}(K_i^{d,d'}) b_{l+j} > 0\} = \{k + l : \sum_{j=0}^{\infty} a_{k,l+j}(K_i^{d,d'}) > 0\}$ contains infinitely many even and infinitely many odd integers. However, it is not hard to see that if the above condition holds and $\{l : a_{k,l}(K_i^{d,d'}) > 0$ for some $k\}$ is finite, then $J_K$ must contain two sequences $\{(k_r, l)\}$ and $\{(k_s, l')\}$ for which $\{k_r + l\} \subset 2\mathbb{Z}_+, \{k_s + l'\} \subset 2\mathbb{Z}_+ + 1$, and $\lim_{r \to \infty} k_r = \lim_{s \to \infty} k_s = \infty$. Indeed, the inferring of this fact demands to observe that if $a_{k,l}(K_i^{d,d'}) > 0$ for some $(k, l)$, then all the integers $k, k + 1, \ldots, k + l$ belong to the set $\{k + l : \sum_{j=0}^{\infty} a_{k,l+j}(K_i^{d,d'}) > 0\}$. This shows the necessity of the condition. As for the sufficiency, let $n$ be a positive integer, $x_1, x_2, \ldots, x_n$ distinct points in $\mathbb{M}^d$ and $w_1, w_2, \ldots, w_n$ distinct points in $\mathbb{H}^d$. We will show that, under the condition on $J_K$
mentioned in the statement of the theorem, the only solution of the system

\[
\sum_{\mu=1}^{n} c_\mu P_k^{(d-2)/2,\beta} (\cos (|x_\mu x|/2)) P_l^{(d-2)/2,\beta'} (\cos (|w_\mu w|/2)) = 0,
\]

\((x, w) \in \mathbb{M}^d \times \mathbb{H}^d, \quad (k, l) \in J_K,\)

is the trivial one. In order to achieve that, we will fix \(\gamma \in \{1, 2, \ldots, n\}\) and will show that \(c_\gamma = 0\). That will be done through specific choices of points \(x \in \mathbb{M}^d\) and \(w \in \mathbb{H}^d\) in the equation defining the system. We also need to consider the antipodal index sets \(\Upsilon(\gamma) = \{ \mu : |x_\mu x_\gamma| = 2\pi \}\) and \(\Omega(\gamma) = \{ \mu : |w_\mu w_\gamma| = 2\pi \}\). Due to the basic assumptions of the theorem, we know that \(d - 2 = 2\beta\) and that \(\Upsilon\) is unitary, say, \(\Upsilon(\gamma) = \{ \delta \}\). The Jacobi polynomials \(P_k^{(d-2)/2,\beta}\) are then Gegenbauer polynomials and, in particular, they are even functions when \(k\) is even and odd functions otherwise. The equation defining the system, with the choice \(x = x_\delta\) and \(w = w_\delta\), can be put into the form

\[
c_\gamma + (-1)^k + k P_l^{(\beta',(d-2)/2)} (1) \sum_{\mu \in \delta \setminus \Upsilon(\gamma)} c_\mu + (-1)^k \times \sum_{\mu \in \delta \setminus \Omega(\gamma)} c_\mu R_l^{(d-2)/2,\beta'} (\cos (|w_\mu w_\gamma|/2))
\]

\[
+ (-1)^k + k P_l^{(\beta',(d-2)/2)} (1) \sum_{\mu \in \Omega(\gamma) \setminus \{ \delta \}} c_\mu R_k^{(d-2)/2,\beta} (\cos (|x_\mu x_\gamma|/2))
\]

\[
+ \sum_{\mu \notin \delta \setminus \Omega(\gamma)} c_\mu R_k^{(d-2)/2,\beta} (\cos (|x_\mu x_\gamma|/2)) R_l^{(d-2)/2,\beta'} (\cos (|w_\mu w_\gamma|/2)) = 0.
\]

Obviously, some of the sets appearing in the sum decomposition above may be empty. Also, the first two sums cannot co-exist, that is, just one of them can appear in the expression. If \(J_K\) contains a sequence \((k_r, l_r)\) for which \(\lim_{r \to \infty} l_r = \infty\), we may substitute it in the previous equation and let \(r \to \infty\). Recalling Lemma 3.3(ii),(iii) in order to implement the computation of the resulting limits, we deduce that the limit of each summand, but the first, vanishes. In particular, \(c_\gamma = 0\). We now proceed assuming the existence of two sequences \((k_s, l_s)\) and \((k_s', l_s')\) in \(J_K\) for which \(\{ k_r + l_r \} \subset 2\mathbb{Z}_+, \{ k_s + l_s' \} \subset 2\mathbb{Z}_+ + 1,\) and \(\lim_{r \to \infty} k_r = \lim_{s \to \infty} k_s = \infty\). If the second summand in the expression occurs, we can substitute each of the sequences in the expression and deduce that \(P^{(d-2)/2,\beta} (1) c_\gamma + P_l^{(\beta',(d-2)/2)} (1) c_\delta = P_l^{(d-2)/2,\beta'} (1) c_\gamma - P_l^{(\beta',(d-2)/2)} (1) c_\delta = 0\), after letting \(r \to \infty\) and \(s \to \infty\). We observe that the limits of the two last summands in the original equation are equal to 0 in this case. Now, if \(c_\delta \neq 0\), the first equality above provides a contradiction with the positivity of the gamma function in \((0, \infty)\). Thus, \(0 = c_\delta = c_\gamma\). Finally, if the third summand is the one occurring in the original expression, we need an additional equation provided by a second choice of points in the equation defining the system. Choosing \(x = x_\delta\)
and \( w = w_δ \) leads to

\[
c_δ + (-1)^{k+l} \frac{P_k^{(\beta',(d-2)/2)}}{P_k^{(d-2)/2,\beta'}} (1) c_γ + (-1)^k \sum_{\mu \in [\gamma]} c_\mu R_k^{((d-2)/2,\beta')} (\cos (|w_\mu w_δ|/2)) \\
+ (-1)^l \frac{P_k^{(\beta',(d-2)/2)}}{P_k^{(d-2)/2,\beta'}} (1) \sum_{\mu \in \Omega(\delta) \setminus [\gamma]} R_k^{((d-2)/2,\beta')} (\cos (|x_\mu x_δ|/2)) \\
+ \sum_{\mu \notin \Omega(\delta) \cup [\gamma]} c_\mu R_k^{((d-2)/2,\beta')} (\cos (|x_\mu x_δ|/2)) R_k^{((d-2)/2,\beta')} (\cos (|w_\mu w_δ|/2)) = 0.
\]

Substituting one of the sequences, say, \( \{k, l\} \), and letting \( r \to \infty \) in both equations, we deduce that both sums \( c_γ + \sum_{\mu \in [\delta]} c_\mu R_k^{((d-2)/2,\beta')} (\cos (|w_\mu w_γ|/2)) \) and \( c_δ + \sum_{\mu \notin [\gamma]} c_\mu R_k^{((d-2)/2,\beta')} (\cos (|w_\mu w_δ|/2)) \) are zero. But that corresponds to the single equality \( c_δ [1 - (R_k^{((d-2)/2,\beta')} (\cos (|w_\mu w_δ|/2)))^2] = 0 \), in which \( \cos (|w_\mu w_δ|/2) \neq \pm 1 \). Thus, \( c_δ = 0 \), and consequently, \( c_γ = 0 \).

The next theorem takes care of almost all the cases in which no space involved is a sphere.

**Theorem 3.5:** Let \( K \) be a real, continuous, isotropic and PD kernel on \( \mathbb{M}^d \times \mathbb{H}^d \). Assume that both \( \mathbb{M}^d \) and \( \mathbb{H}^d \) are neither a sphere or \( \mathbb{P}^2(\mathbb{R}) \). In order that \( K \) be DC-SPD it is necessary and sufficient that \( \{k + l : a_{k,l}(K_i^{(d,d')}) > 0\} \) be infinite.

**Proof:** Since the proof is similar to the proof of the previous theorem, some details will be omitted. The necessity part is similar to that in the proof of Theorem 3.4, using the same trick twice. The resulting kernel is DC-SPD on some \( S^q \times S^q \), \( q \geq 2 \), and the set of indices pertaining to the final argument takes the form

\[
\left\{ k + l : \sum_{j=0}^{\infty} \sum_{j'=0}^{\infty} a_{k+j,l+j'} (K_i^{d,d'}) b_k^{k+j} c_l^{l+j'} > 0 \right\} = \left\{ k + l : \sum_{j=0}^{\infty} \sum_{j'=0}^{\infty} a_{k+j,l+j'} (K_i^{d,d'}) > 0 \right\},
\]

where all the constants \( b_k^{k+j} c_l^{l+j'} \) are positive. Since this set has infinitely many even and infinitely many odd integers, it follows that \( \{k + l : (k, l) \in J_K\} \) is infinite. The sufficiency part follows the steps of the corresponding part in the previous theorem. Due to the assumption on \( J_K \), we can select a sequence \( \{(k_r, l_r)\} \) in \( J_K \) so that either \( \lim_{r \to \infty} k_r = \infty \) or \( \lim_{r \to \infty} l_r = \infty \). Choosing \( x = x_γ \) and \( w = w_γ \) in the equation defining the system, we obtain

\[
c_γ + (-1)^{k+l} \frac{P_k^{(\beta,(d-2)/2)}}{P_k^{(d-2)/2,\beta}} (1) P_k^{(\beta',(d-2)/2)} (1) \sum_{\mu \in \Omega(\gamma) \cap [\gamma]} c_\mu \\
+ (-1)^l \frac{P_k^{(\beta',(d-2)/2)}}{P_k^{(d-2)/2,\beta}} (1) \sum_{\mu \in \Omega(\gamma) \setminus [\gamma]} c_\mu R_k^{((d-2)/2,\beta')} (\cos (|w_\mu w_γ|/2))
\]
\[+ (-1)^n \frac{P_l^{(\beta', (d-2)/2)}}{P_i^{(d-2)/2}} \sum_{\mu \in \Omega(\gamma)} c_{\mu} R_k^{(d-2)/2, \beta} \left( \cos \left( \frac{|x_{\mu} y_{\gamma}|}{2} \right) \right) + \sum_{\mu \notin \Upsilon(\gamma) \cup \Omega(\gamma)} c_{\mu} R_k^{(d-2)/2, \beta} \left( \cos \left( \frac{|x_{\mu} y_{\gamma}|}{2} \right) \right) R_{l_r}^{(d-2)/2, \beta'} \left( \cos \left( \frac{|w_{\mu} w_{\gamma}|}{2} \right) \right) = 0.\]

After substituting the sequence in the above expression, if \(\lim_{r \to \infty} k_r = \infty\), then the limit of each summand, but the first, vanishes. In particular, \(c_\gamma = 0\). If \(\lim_{r \to \infty} l_r = \infty\), a similar analysis produces the same conclusion. \(\blacksquare\)

4. Strict positive definiteness

The strict positive definiteness of a real, continuous, isotropic and PD kernel on a product of high-dimensional spheres was completely characterized in [9]. The characterization in the case of a product of circles was reached in [10] while the case \(S^1 \times S^m, m \geq 2\), was fully analysed in [11]. Thus, just like in the previous section, in the analysis here we will assume that at least one of the spaces involved is not a sphere.

The section begins with the obvious counterpart of Proposition 4.1 for plain strict positive definiteness on \(M^d \times H^d\).

**Proposition 4.1:** Let \(K\) be a real, continuous, isotropic and PD kernel on \(M^d \times H^d\). The following assertions are equivalent:

(i) \(K\) is SPD;

(ii) If \(n \geq 1\) and \((x_1, w_1), (x_2, w_2), \ldots, (x_n, w_n)\) are distinct points on \(M^d \times H^d\), then the only solution of the system

\[\sum_{\mu=1}^n c_{\mu} R_k^{(d-2)/2, \beta} \left( \cos \left( \frac{|x_{\mu} x|}{2} \right) \right) P_l^{(d-2)/2, \beta'} \left( \cos \left( \frac{|w_{\mu} w|}{2} \right) \right) = 0,\]

\[(x, w) \in M^d \times H^d, \quad (k, l) \in J_K,\]

is the trivial one, that is, \(c_{\mu} = 0, \mu = 1, 2, \ldots, n\).

The characterization for strict positive definiteness in the case in which neither space is a sphere or \(P^2(\mathbb{R})\) is as follows.

**Theorem 4.2:** Let \(K\) be a real, continuous, isotropic and PD kernel on \(M^d \times H^d\). Assume that both \(M^d\) and \(H^d\) are neither a sphere nor \(P^2(\mathbb{R})\). In order that \(K\) be SPD it is necessary and sufficient that the set \(J_K\) contains a sequence \(\{(k_r, l_r)\}\) for which \(\lim_{r \to \infty} k_r = \lim_{r \to \infty} l_r = \infty\).

**Proof:** Assume that there exists a sequence \(\{(k_r, l_r)\}\) as described in the statement of the theorem. Let \((x_1, w_1), (x_2, w_2), \ldots, (x_n, w_n)\) be distinct points in \(M^d \times H^d\) and suppose
that
\[ \sum_{\mu=1}^{n} c_{\mu} P_{k}^{(d-2)/2, \beta} (\cos (|x_{\mu} w|/2)) P_{l}^{(d-2)/2, \beta'} (\cos (|w_{\mu} w|/2)) = 0, \]

for real scalars \( c_1, c_2, \ldots, c_n, (x, w) \in \mathbb{M}^d \times \mathbb{H}^d \) and \((k, l) \in J_K\). For \( \gamma \in \{1, 2, \ldots, n\} \) fixed, let us put \( x = x_{\gamma} \) and \( w = w_{\gamma} \) in the previous equation and split it into account the following index sets (recall the normalization we have adopted for the metric in the spaces involved): \( I_1 = \{ \mu : |x_{\mu} x_{\gamma}| = 2\pi = |w_{\mu} w_{\gamma}| \}, I_2 = \{ \mu : |x_{\mu} x_{\gamma}| = 2\pi \neq |w_{\mu} w_{\gamma}| \} \) and \( I_3 = \{ \mu : |x_{\mu} x_{\gamma}| \neq 2\pi = |w_{\mu} w_{\gamma}| \}. \) We observe that one or more of these sets may be empty. The outcome is

\[
\begin{align*}
&c_{\gamma} + (-1)^{k_r + l_r} \frac{P_{k_r}^{(\beta, (d-2)/2)}}{P_{l_r}^{((d-2)/2, \beta)}} (1 \frac{P_{l_r}^{(\beta', (d-2)/2)}}{P_{l_r}^{(d-2)/2, \beta'}} (1 \sum_{\mu \in I_1} c_{\mu} R_{l_r}^{(d-2)/2, \beta'} (\cos (|w_{\mu} w_{\gamma}|/2)) \\
&+ (-1)^{k_r} \frac{P_{l_r}^{(\beta', (d-2)/2)}}{P_{l_r}^{((d-2)/2, \beta)}} (1 \sum_{\mu \in I_2} c_{\mu} R_{l_r}^{(d-2)/2, \beta'} (\cos (|x_{\mu} x_{\gamma}|/2)) \\
&+ (-1)^{l_r} \frac{P_{l_r}^{(\beta', (d-2)/2)}}{P_{l_r}^{((d-2)/2, \beta)}} (1 \sum_{\mu \in I_3} c_{\mu} R_{l_r}^{(d-2)/2, \beta'} (\cos (|x_{\mu} x_{\gamma}|/2)) R_{l_r}^{(d-2)/2, \beta'}) \\
&\times (\cos (|w_{\mu} w_{\gamma}|/2)) = 0, \quad r = 1, 2, \ldots.
\end{align*}
\]

We observe that in the last summand, if \( \mu \) is fixed, either \( x_{\gamma} \neq x_{\mu} \) or \( w_{\gamma} \neq w_{\mu} \). Since \( \alpha > \beta \) and \( \alpha' > \beta' \), after substituting the sequence \( \{(k_r, l_r)\} \) in the equation we may let \( r \to \infty \) and apply Lemma 3.3 to conclude that \( c_{\gamma} = 0 \). In view of the previous proposition, the sufficiency part is resolved. Going the other way around, if \( K \) is SPD, we may repeat the procedure adopted in the first half of the proof of Theorem 3.5. The index set of the resulting PD kernel on \( S^q \times S^q \) \((q \geq 2)\) is \((k, l) : \sum_{j=0}^{\infty} \sum_{j'=0}^{\infty} \theta_{k+j, l+j'} (K_{d+q}^{d+q}) > 0 \). Since the characterization for strict positive definiteness on \( S^q \times S^q \) described in [9] implies that the set above must contain at least one sequence \((k_r, l_r)\) for which \( \lim_{r \to \infty} k_r = \lim_{r \to \infty} l_r = \infty \), the set \( J_K \) must contain a sequence of this same type.

In the case in which \( \mathbb{M}^d = S^d \), the following upgrade of the previous lemma will be more favourable.

**Lemma 4.3:** Let \( K \) be a real, continuous, isotropic and PD kernel on \( S^d \times \mathbb{H}^d \), in which \( d \geq 2 \) and \( \mathbb{H}^d \) is not a sphere. The following statements are equivalent:

(i) \( K \) is SPD on \( S^d \times \mathbb{H}^d \);
(ii) If \( n \geq 1, (x_1, w_1), (x_2, w_2) \ldots, (x_n, w_n) \) are distinct points on \( S^d \times \mathbb{H}^d \), and the set \( \{x_1, x_2, \ldots, x_n\} \) does not contain any pair of antipodal points, then the only solution of
the system
\[
\sum_{\mu=1}^{n} \left[ (-1)^k c'_\mu + c''_\mu \right] P_k^{((d-2)/2, \beta)}(x_\mu \cdot x) P_l^{((d-2)/2, \beta')}(\cos (|w_\mu w|/2)) = 0,
\]
\[(x, w) \in S^d \times \mathbb{H}^d, \quad (k, l) \in J_K,
\]

is \(c'_\mu = c''_\mu = 0, \mu = 1, 2, \ldots, n\).

**Proof:** Assume that (i) holds. Let \((x_1, w_1), (x_2, w_2), \ldots, (x_n, w_n)\) be distinct points in \(S^d \times \mathbb{H}^d\) and assume that \(\{x_1, x_2, \ldots, x_n\}\) does not contain pairs of antipodal points. Since \(2\beta = d - 2\), the system described in (ii) can be written in the form
\[
2n \sum_{\nu=1}^{c_v \nu} P_k^{((d-2)/2, \beta)}(\cos (|x'_\nu, x|/2)) P_l^{((d-2)/2, \alpha')}(\cos (|w'_\nu, w|/2)) = 0, \quad x \in S^d, \quad w \in \mathbb{H}^d,
\]
in which \((x'_\nu, w'_\nu) = (x_\nu, w_\nu)\) and \(c_v = c'_\nu\) if \(v \in \{1, 2, \ldots, n\}\) and \((x'_\nu, w'_\nu) = (-x_\nu, w_\nu)\) and \(c_v = c''_\nu\) if \(v \in \{n + 1, 2, \ldots, 2n\}\). Since the \(2n\) points \((x'_\nu, w'_\nu)\) are distinct, Proposition 4.1 implies that \(c_v = 0, v = 1, 2, \ldots, 2n\). In particular, \(c'_\nu = c''_\nu = 0, \mu = 1, 2, \ldots, n\).

Conversely, if (i) does not hold, the previous proposition allows the selection of distinct point \((x_1, w_1), (x_2, w_2), \ldots, (x_n, w_n)\) in \(S^d \times \mathbb{H}^d\) so that the system
\[
\sum_{\mu=1}^{n} c_\mu P_k^{((d-2)/2, \beta)}(\cos (|x_\mu x|/2)) P_l^{((d-2)/2, \alpha')}(\cos (|w_\mu w|/2)) = 0,
\]
\[(x, w) \in S^d \times \mathbb{H}^d, \quad (k, l) \in J_K,
\]
has a non-trivial solution \(c_\mu, \mu = 1, 2, \ldots, c_n\). We can select \(p \leq (n)\) distinct points \((x'_1, w'_1), (x'_2, w'_2), \ldots, (x'_p, w'_p)\) in \(S^d \times \mathbb{H}^d\) in a such a way that \(\{x'_1, x'_2, \ldots, x'_p\}\) contains no pairs of antipodal points and \(\{(x_1, w_1), (x_2, w_2), \ldots, (x_n, w_n)\} \subseteq \{(x'_1, w'_1), (x'_2, w'_2), \ldots, (x'_p, w'_p)\}\). However, it is an easy matter to verify that the system
\[
\sum_{\mu=1}^{n} \left[ (-1)^k c'_\mu + c''_\mu \right] P_k^{((d-2)/2, \beta)}(\cos (|x'_\mu x|/2)) P_l^{((d-2)/2, \alpha')}(\cos (|w'_\mu w|/2)) = 0,
\]
\[(x, w) \in S^d \times \mathbb{H}^d, \quad (k, l) \in J_K,
\]
has a non-trivial solution as well. Thus, (ii) cannot hold. \(\blacksquare\)

The following proposition is an alternative to the previous lemma via the sets \(J_K^e := J_K \cap [2\mathbb{Z}_+ \times \mathbb{Z}_+]\) and \(J_K^d := J_K \cap [(2\mathbb{Z}_+ + 1) \times \mathbb{Z}_+]\).

**Proposition 4.4:** Let \(K\) be a real, continuous, isotropic and PD kernel on \(S^d \times \mathbb{H}^d\), in which \(d \geq 2\) and \(\mathbb{H}^d\) is not a sphere. The following statements are equivalent:

(i) \(K\) is SPD on \(S^d \times \mathbb{H}^d\);
(ii) If \( n \geq 1 \), \((x_1, w_1), (x_2, w_2), \ldots, (x_n, w_n)\) are distinct points on \( S^d \times \mathbb{H}^d \), and the set \( \{x_1, x_2, \ldots, x_n\}\) does not contain a pair of antipodal points, then the only solution of the system

\[
\sum_{\mu=1}^{n} c_{\mu}^l P_k^{((d-2)/2, \beta)} (|x_\mu x|/2)) P_l^{((d-2)/2, \beta)} (|w_\mu w|/2)) = 0, \quad (k, l) \in I_K^o,
\]

\[
\sum_{\mu=1}^{n} c_{\mu}^o P_k^{((d-2)/2, \beta)} (|x_\mu x|/2)) P_l^{((d-2)/2, \beta)} (|w_\mu w|/2)) = 0, \quad (k, l) \in I_K^o,
\]

\[(x, w) \in S^d \times \mathbb{H}^d,
\]
is \( c_{\mu}^c = c_{\mu}^o = 0, \mu = 1, 2, \ldots, n \).

**Proof:** If (ii) were not true, we could find distinct points \((x_1, w_1), (x_2, w_2), \ldots, (x_n, w_n)\) in \( S^d \times \mathbb{H}^d \), with \( \{x_1, x_2, \ldots, x_n\}\) containing no pair of antipodal points and either

\[
\sum_{\mu=1}^{n} c_{\mu}^l P_k^{((d-2)/2, \beta)} (|x_\mu x|/2)) P_l^{((d-2)/2, \beta)} (|w_\mu w|/2)) = 0, \quad (k, l) \in I_K^c,
\]

\[(x, w) \in S^d \times \mathbb{H}^d,
\]
or

\[
\sum_{\mu=1}^{n} c_{\mu}^o P_k^{((d-2)/2, \beta)} (|x_\mu x|/2)) P_l^{((d-2)/2, \beta)} (|w_\mu w|/2)) = 0, \quad (k, l) \in I_K^o,
\]

\[(x, w) \in S^d \times \mathbb{H}^d
\]

having a non-trivial solution. We proceed considering the first possibility that emerges from the conclusion above, being the other case similar. For each \( \mu \in \{1, 2, \ldots, n\} \), the system of equations \( c_{\mu}^l + c_{\mu}'' = c_{\mu}^c, -c_{\mu}^l + c_{\mu}''' = 0 \), has a unique solution \( c_{\mu}^l, c_{\mu}'' \). Since \( c_{\mu}'' \neq 0 \) for at least one \( \mu \), then \( (c_{\mu}^l, c_{\mu}''') \neq 0 \), for at least one \( \mu \). It is now clear that the system in Lemma 4.1(ii) would have a non-trivial solution for the selection of points \((x_1, w_1), (x_2, w_2), \ldots, (x_n, w_n)\). Thus, (i) implies (ii).

The converse will be justified as long as we show that if (ii) holds, then Lemma 4.1(iii) holds. But, if \( c_{\mu}^c \) and \( c_{\mu}^o \) are known, the system \( c_{\mu}^l + c_{\mu}'' = c_{\mu}^c, -c_{\mu}^l + c_{\mu}''' = c_{\mu}^o \), always has a unique solution. If \( c_{\mu}^l = c_{\mu}^o = 0 \) for all \( \mu \), then the solution of the corresponding system vanishes. Thus, if the system in (ii) has the trivial solution only, the same will be true of the system in Lemma 4.1(ii).

We are ready to state and prove the last main contribution of the paper.

**Theorem 4.5:** Let \( K \) be a real, continuous, isotropic and PD kernel on \( S^d \times \mathbb{H}^d \). Assume that \( d \geq 2 \) and that \( \mathbb{H}^d \) is neither a sphere nor \( \mathbb{R}^2(\mathbb{R}) \). In order that \( K \) be SPD it is necessary and sufficient that \( J_K \) contain sequences \( \{(k_r, l_r)\} \) and \( \{(k'_r, l'_r)\} \) with \( k_r \subset 2\mathbb{Z}_+, \{k'_r\} \subset 2\mathbb{Z}_+ + 1, \) and \( \lim_{r \to \infty} k_r = \lim_{r \to \infty} k'_r = \lim_{r \to \infty} l_r = \lim_{r \to \infty} l'_r = \infty \).
Proof: Let us assume that $J_K$ contains sequences as described in the statement of the theorem. We intend to use Proposition 4.4 in order to conclude that $K$ is SPD. Let $(x_1, w_1), (x_2, w_2), \ldots, (x_n, w_n)$ be distinct points in $S^d \times \mathbb{H}^d$, assume that $\{x_1, x_2, \ldots, x_n\}$ does not contain any pairs of antipodal points and that the system in Proposition 4.4(ii) holds. Fixing $\gamma$, introducing $x = x_\gamma$ and $w = w_\gamma$ in the first equation of the system and proceeding as in the proof of Theorem 4.2, we deduce that

$$c_\gamma^e + (-1)^{d(d-2)/2} \sum_{\mu \in \Omega(\gamma)} c_\mu^e R_k^{((d-2)/2, \beta)}(\cos(|x_\mu x_\gamma|/2)) + \sum_{\mu \in \Omega(\gamma)} c_\mu^e R_k^{((d-2)/2, \beta)}(\cos(|w_\mu w_\gamma|/2)) = 0.$$ 

Substituting the first double sequence guaranteed by our assumption in this equation and letting $r \to \infty$, Lemma 3.3 implies that $c_\gamma^e = 0$. A similar procedure employing the second equation of the system and the second double sequence from the assumption leads to $c_\gamma^o = 0$. Since $\gamma$ is arbitrary, the only solution of the system in Proposition 4.4-(ii) is the trivial one. Therefore, $K$ is SPD on $S^d \times \mathbb{H}^d$. In order to prove the condition is necessary, we need to imitate the corresponding part in the proof of Theorem 3.4. We get an SPD kernel on $S^d \times S^q$, $(q \geq 2)$, with corresponding index set $\{(k, l) : \sum_{j=0}^\infty a_{k, l+j}(K_{i, d}^{d, d}) > 0\}$. Once again, the characterization for strict positive definiteness on $S^d \times S^q$ described in [9] implies that $J_k$ must contain two sequences as quoted in the statement of the theorem being proved. 

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The first author was partially supported by CNPq, under grant [141908/2015-7]. The second one by FAPESP, under grant [2014/00277-5].

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