MODEL STRUCTURES FOR CORRESPONDENCES AND BIFIBRATIONS

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ABSTRACT. We study the notion of a bifibration in simplicial sets which generalizes the classical notion of two-sided discrete fibration studied in category theory. If $A$ and $B$ are simplicial sets we equip the category $(\text{Set}_\Delta)_{/(A \times B)}$ of simplicial sets over $A \times B$ with the structure of a model category for which the fibrant objects are the bifibrations from $A$ to $B$. We also equip the category $\text{Corr}(A, B)$ of correspondences of simplicial sets from $A$ to $B$ with the structure of a model category. We describe several Quillen equivalences relating these model structures with the covariant model structure on $(\text{Set}_\Delta)_{/(B^{\text{op}} \times A)}$.

1. INTRODUCTION

A useful concept from ordinary category theory is the notion of profunctor. This has several incarnations. If $A$ and $B$ are categories, then a profunctor from $A$ to $B$ may be viewed as a functor $F: B^{\text{op}} \times A \to \text{Set}$, or equivalently as a colimit preserving functor $P(A) \to P(B)$ between the categories of presheaves on $A$ and $B$ respectively. There is an equivalence of categories

$$[B^{\text{op}} \times A, \text{Set}] \xrightarrow{\sim} \text{Corr}(A, B)$$

between the category of profunctors from $A$ to $B$, and the category $\text{Corr}(A, B)$ of correspondences from $A$ to $B$, i.e. functors $p: C \to [1]$ such that $p^{-1}(0) = B$ and $p^{-1}(1) = A$.

There is also an equivalence of categories

$$\text{Corr}(A, B) \xrightarrow{\sim} \text{DFib}(A, B)$$

between the category of correspondences from $A$ to $B$ and the category $\text{DFib}(A, B)$ of two-sided discrete fibrations from $A$ to $B$. A two-sided discrete fibration $(p, q): X \to A \times B$ is, roughly speaking, a functor whose fibers $X(a, b)$ are covariant in $a \in A$ and contravariant in $b \in B$. The concept was exploited by Street in [19, 20].

There are analogues for the notions of profunctor, correspondence and two-sided discrete fibration at the level of simplicial sets. These notions have been studied in [2, 4, 9, 13, 10, 15]. If $A$ and $B$ are simplicial sets, then a profunctor from $A$ to $B$ may be thought of as a simplicial map $B^{\text{op}} \times A \to S$, where $S$ denotes the $\infty$-category of spaces (Definition 1.2.16.1 of [13]), alternatively we may replace such a map with the left fibration over $B^{\text{op}} \times A$ that it classifies.

The notion of correspondence has a straightforward interpretation at this level also: if $A$ and $B$ are simplicial sets we shall say that a simplicial map $p: X \to \Delta^1$ is a

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correspondence from $A$ to $B$ if there are isomorphisms $p^{-1}(0) \simeq B$ and $p^{-1}(1) \simeq A$ (see Definition 3.1). The correspondences from $A$ to $B$ form the objects of a category $\text{Corr}(A, B)$, which is a certain subcategory of the category $(\text{Set}_\Delta)/\Delta^1$ of simplicial sets over $\Delta^1$ (see Remark 3.3). Correspondences of simplicial sets feature prominently in Lurie’s discussion of adjoint functors in [13]; they also play a role in [4].

The notion of two-sided discrete fibration also extends to the context of simplicial sets. In [13], Lurie introduced the notion of a bifibration $(f, g): X \to A \times B$ which is an inner fibration together with a condition which encodes the idea that the fibers $X(a, b)$ of the map $(f, g)$ depend covariantly on $a$ and contravariantly on $b$ (this notion is also considered in [10]). Bifibrations are the analog for simplicial sets of the notion of two-sided discrete fibration in category theory. For brevity we shall use Lurie’s terminology of ‘bifibration’ rather than ‘two-sided discrete fibration’. In the paper [15] Riehl and Verity refer to bifibrations as modules; they play a key role in their study of the formal category theory of $\infty$-categories.

One of our aims in this paper is to exhibit the bifibrations in simplicial sets from $A$ to $B$ as the fibrant objects of a model category. In Section 4.4 we shall prove the following result (see Theorem 4.19):

**Theorem A.** Let $A$ and $B$ be simplicial sets. There is the structure of a left proper, combinatorial model category on $(\text{Set}_\Delta)/(A \times B)$ for which the cofibrations are the monomorphisms and the fibrant objects are the bifibrations from $A$ to $B$.

The existence of this model structure was known to Joyal (see [10]) but a construction of it has not appeared in the literature to date. Following Joyal we call this model structure the bivariant model structure to reflect the covariant and contravariant nature of bifibrations.

To establish the existence of this model structure we study bifibrations in some detail, replicating many properties of left and right fibrations established by Joyal and Lurie. For instance we study the behaviour of bifibrations under exponentiation (Section 4.2), and we introduce the concept of bivariant anodyne map in $(\text{Set}_\Delta)/(A \times B)$ (see Section 4.2). We introduce the notion of bivariant equivalence (Section 4.6) and prove that a map $X \to Y$ between bifibrations in $(\text{Set}_\Delta)/(A \times B)$ is a bivariant equivalence if and only if it is a fiberwise homotopy equivalence, generalizing the corresponding facts for left and right fibrations (Remark 2.2.3.3 of [13]). We also prove that a bifibration $X \to A \times B$ is a trivial Kan fibration if and only if its fibers are contractible Kan complexes. Again, this is a generalization of the corresponding facts for left and right fibrations (see Lemma 2.1.3.4 of [13]).

In addition to the model structure for bifibrations, we also construct a model structure for correspondences. In Section 3.2 we prove the following result (see Theorem 3.9):

**Theorem B.** Let $A$ and $B$ be $\infty$-categories. There is the structure of a left proper, combinatorial model category on $\text{Corr}(A, B)$ for which the cofibrations are the monomorphisms and the fibrant objects are the correspondences $X \to \Delta^1$ in $\text{Corr}(A, B)$ for which $X$ is an $\infty$-category.

The model structure for correspondences is left induced (in the sense of [5]) from the Joyal model structure on the slice category $(\text{Set}_\Delta)/B_{\ast A}$. Its existence is well-known to experts — it is stated, but not proved, in [10] and it is alluded to in [13] for instance.
Our other objective in this paper is to describe a series of Quillen equivalences linking
the covariant model structure on \((\text{Set}_\Delta)/\!(B^{op} \times A)\), the correspondence model structure
on \(\text{Corr}(A,B)\), and the bivariant model structure on \((\text{Set}_\Delta)/\!(A \times B)\), which generalize the
 equivalences \((1)\) and \((2)\). Such a description has recently been given by Ayala and
Francis in \([2]\) at the level of \(\infty\)-categories. We shall refine the equivalences between
\(\infty\)-categories that are established in \([2]\) to Quillen equivalences between the model
categories above. In fact, we shall describe some additional Quillen equivalences, one
of which is of a rather surprising nature.

The twisted arrow category construction (see Construction 5.2.1.1 of \([14]\)) associates
to a simplicial set \(X\) a new simplicial set \(\text{Tw}(X)\), equipped with a canonical map
\(\text{Tw}(X) \to X^{op} \times X\) which is a left fibration if \(X\) is an \(\infty\)-category. If \(X\) is a correspon-
dence from \(A\) to \(B\), then base change along the map \(B^{op} \times A \to X^{op} \times X\) induced by
the inclusions \(A \subseteq X\) and \(B \subseteq X\) induces a functor \(a^*: \text{Corr}(A,B) \to (\text{Set}_\Delta)/\!(B^{op} \times A)\)
which participates in a series of adjunctions

\[
\begin{array}{c}
\text{Set}_\Delta)/\!(B^{op} \times A) \xrightarrow{a_!} \text{Corr}(A,B) \\
\text{Corr}(A,B) \xleftarrow{a^*} (\text{Set}_\Delta)/\!(B^{op} \times A)
\end{array}
\]

In Section 3.3 we shall prove the following result (see Theorem 3.23 and Theorem 3.25):

**Theorem C.** Let \(A\) and \(B\) be \(\infty\)-categories. Then the adjoint pairs

\[a_!: (\text{Set}_\Delta)/\!(B^{op} \times A) \rightleftharpoons \text{Corr}(A,B): a^*\]

and

\[a^*: \text{Corr}(A,B) \rightleftharpoons (\text{Set}_\Delta)/\!(B^{op} \times A): a_*\]

are both Quillen equivalences for the correspondence model structure and the covariant
model structure on \((\text{Set}_\Delta)/\!(B^{op} \times A)\).

Of note is the fact that the functor \(a^*\) appears as both a left and right Quillen equivalence. There is a similar series of adjunctions

\[
\begin{array}{c}
(\text{Set}_\Delta)/\!(A \times B) \xrightarrow{d_!} \text{Corr}(A,B) \\
\text{Corr}(A,B) \xleftarrow{d^*} (\text{Set}_\Delta)/\!(A \times B)
\end{array}
\]

connecting the categories \((\text{Set}_\Delta)/\!(A \times B)\) and the category \(\text{Corr}(A,B)\), which is described
in terms of the edgewise subdivision functor \(sd_2\) from \([6]\). In Section 4.7 we prove (see
Theorem 4.40)

**Theorem D.** Let \(A\) and \(B\) be \(\infty\)-categories. Then the adjoint pair

\[d^*: \text{Corr}(A,B) \rightleftharpoons (\text{Set}_\Delta)/\!(A \times B): d_*\]

is a Quillen equivalence for the correspondence model structure on \(\text{Corr}(A,B)\) and the
bivariant model structure on \((\text{Set}_\Delta)/\!(A \times B)\).

The adjoint pair \((d_!, d^*)\) is not a Quillen pair for these model structures; there is
however another Quillen equivalence relating these model categories (see Theorem 4.42).
In summary then the contents of this paper are as follows. In Section 2 we review some facts about the covariant model structure and Joyal’s notion of dominant map that we will need later in the paper. In Section 3 we describe the model structure for correspondences and prove the existence of the Quillen equivalences from Theorem C above. In Section 4 we study the notion of a bifibration in simplicial sets; we introduce the concept of a bivariant anodyne map and bivariant equivalence. We describe the bivariant model structure on $(\text{Set}_\Delta)/(A\times B)$ and prove the existence of the Quillen equivalence from Theorem D above.

Finally, we point out that several of the results in this paper seem to be known to experts, but equally proofs of them are missing from the literature; in this paper we fill these gaps.

**Notation:** for the most part we use the notation and terminology from Lurie’s books \[13\] and \[14\], except where we have indicated. Thus $\text{Set}_\Delta$ denotes the category of simplicial sets, $\text{h}(S)$ denotes the homotopy category of a simplicial set $S$, etc. Following the convention in \[14\], we will say that a *left cofinal* map of simplicial sets is what is called a *cofinal* map in \[13\] and that a map of simplicial sets is *right cofinal* if and only if its opposite is left cofinal.

## 2. The covariant model structure

Let $S$ be a simplicial set. We recall some features of the covariant model structure on the category $(\text{Set}_\Delta)/S$ of simplicial sets over $S$ from \[10\] and \[13\].

**Notation 2.1.** Recall that the category $(\text{Set}_\Delta)/S$ is canonically enriched over $\text{Set}_\Delta$. If $X \to S$ and $Y \to S$ are objects of $(\text{Set}_\Delta)/S$ then the simplicial mapping space $\text{map}_S(X,Y)$ is the simplicial set defined by the pullback diagram

$$
\begin{array}{ccc}
\text{map}_S(X,Y) & \longrightarrow & Y^X \\
\downarrow & & \downarrow \\
\Delta^0 & \longrightarrow & S^X
\end{array}
$$

where the lower horizontal map corresponds to the structure map $X \to S$.

### 2.1. Covariant equivalences

Recall that a map $f: X \to Y$ in $(\text{Set}_\Delta)/S$ is said to be a *covariant equivalence* if the induced map

$$
\text{map}_S(Y,L) \to \text{map}_S(X,L)
$$

is a weak homotopy equivalence for every left fibration $L \to S$. The covariant equivalences are the weak equivalences for the *covariant* model structure on $(\text{Set}_\Delta)/S$ introduced by Joyal and Lurie.

**Theorem 2.2** (Joyal/Lurie). There is the structure of a left proper, combinatorial model category on $(\text{Set}_\Delta)/S$ for which

- the weak equivalences are the covariant equivalences;
- the cofibrations are the monomorphisms; and
- the fibrant objects are the left fibrations.
Dually there is the contravariant model structure on $(\text{Set}_\Delta)/S$, described in terms of right fibrations on $S$.

The following theorem from [17] gives a very useful criterion for recognizing covariant equivalences.

**Theorem 2.3** ([17]). Let $S$ be a simplicial set and let $f: X \to Y$ be a map in $(\text{Set}_\Delta)/S$. The following statements are equivalent:

1. $f$ is a covariant equivalence;
2. the induced map $X \times_S R \to Y \times_S R$ is a weak homotopy equivalence for every right fibration $R \to S$;
3. for every vertex $s$, and for every factorization $\Delta^0 \to Rs \to S$ of the map $s: \Delta^0 \to S$ into a right anodyne map followed by a right fibration, the induced map $X \times_S Rs \to Y \times_S Rs$ is a weak homotopy equivalence.

2.2. **The right cancellation property.** Recall that a class of monomorphisms $A$ in a category $\mathcal{C}$ is said to satisfy the right cancellation property if the following condition is satisfied: if $u$ and $v$ are composable morphisms in $\mathcal{C}$ such that $u \in A$ and $vu \in A$, then $v \in A$ also. Left anodyne maps are an important example of a class of maps with this property.

**Proposition 2.4** (Joyal). The class of left anodyne maps satisfies the right cancellation property.

The following result from [17] gives a useful criterion for detecting when a given class of monomorphisms in $\text{Set}_\Delta$ satisfying the right cancellation property contains the class of left anodyne maps.

**Proposition 2.5** ([17]). Let $A$ be a saturated class of monomorphisms in $\text{Set}_\Delta$ which satisfies the right cancellation property. Then the following statements are equivalent:

1. $A$ contains the class of left anodyne morphisms;
2. $A$ contains the initial vertex maps $\Delta^\{0\} \to \Delta^n$ for all $n \geq 1$;
3. $A$ contains the horn inclusions $h^\{0\}_n: \Lambda^n_0 \subseteq \Delta^n$ for all $n \geq 1$.

**Remark 2.6.** A similar criterion, framed in terms of the spine inclusions $\Delta^\{0,1\} \cup \cdots \cup \Delta^\{n-1,n\} \subseteq \Delta^n$, appears in [13].

From [18] we have another very useful example of a class of monomorphisms satisfying the right cancellation property.

**Proposition 2.7** ([18]). The class of inner anodyne maps in $\text{Set}_\Delta$ satisfies the right cancellation property.

We will make use of this fact in the proof of Proposition 3.17.

2.3. **Dominant maps.** In this section we recall some facts about the notion of dominant maps of simplicial sets introduced by Joyal (we shall need some of the results from this section in the proof of Lemma 4.29 in Section 4.6).

**Definition 2.8** (Joyal). A map $u: A \to B$ in $\text{Set}_\Delta$ is said to be dominant if the right derived functor

$$Ru^*: \text{Ho}((\text{Set}_\Delta)/B) \to \text{Ho}((\text{Set}_\Delta)/A)$$

is fully faithful for the contravariant model structures on $(\text{Set}_\Delta)/A$ and $(\text{Set}_\Delta)/B$. 
Remark 2.9. The notion of dominant map is also studied by Gaitsgory and Rozenblyum, who use the term *contractible* map instead of dominant map (see Section 2.3 of [7]).

Remark 2.10. It follows immediately from Definition 2.8 that dominant maps are closed under retracts and invariant under categorical equivalences.

The following result is due to Joyal; we give a proof since we have not been able to find one in the literature to date.

**Lemma 2.11** (Joyal). If \( u : A \to B \) is dominant and \( R \to B \) is a right fibration then the induced map \( R \times_B A \to R \) is dominant.

**Proof.** Suppose given a dominant map \( u : A \to B \) and suppose that \( p : R \to B \) is a right fibration. Form the pullback diagram

\[
\begin{array}{ccc}
A \times_B R & \xrightarrow{v} & R \\
\downarrow{q} & & \downarrow{p} \\
A & \xrightarrow{u} & B
\end{array}
\]

We need to prove that the right derived functor

\[
\mathbf{R}u^* : \text{Ho}((\mathbf{Set}_\Delta)/(A \times_B R)) \to \text{Ho}((\mathbf{Set}_\Delta)/(A \times_B R))
\]

is fully faithful, where \((\mathbf{Set}_\Delta)/(A \times_B R)\) and \((\mathbf{Set}_\Delta)/R\) are equipped with the contravariant model structures. We will prove that the counit

\[
\epsilon_v : Lp! \mathbf{R}u^* \to \text{id}
\]

is an isomorphism. Since \( p : R \to B \) is a right fibration, the left derived functor

\[
Lp! : \text{Ho}((\mathbf{Set}_\Delta)/R) \to \text{Ho}((\mathbf{Set}_\Delta)/B)
\]

is conservative (Corollary 10.15 of [10]), and hence it suffices to prove that the image \( Lp! \epsilon \) is an isomorphism in \( \text{Ho}((\mathbf{Set}_\Delta)/B) \). We have a natural isomorphism \( Lp! Lp^! \simeq Lu_! Lq! \).

A straightforward argument, using the fact that \( p \) is a right fibration, shows that the canonical natural transformation

\[
Lq! \mathbf{R}u^* \to \mathbf{R}u^* Lp!
\]

is a natural isomorphism. Therefore \( Lp! \epsilon \) is isomorphic to the natural transformation

\[
\epsilon_u Lp! : Lu_! \mathbf{R}u^* Lp! \to Lp!
\]

which is itself a natural isomorphism since \( u \) is dominant. \( \square \)

We state the following result which appears in [10] and [7]. We first need some notation.

**Notation 2.12.** If \( f : b \to b' \) is an edge of a simplicial set \( B \) then we will write \( B_{b'/b} \) for the double slice \((B_{b'})_{/f}\). If \( B \) is the nerve of a category, then the simplicial set \( B_{b'/b} \) is the nerve of the category of factorizations of the arrow \( f : b \to b' \).

**Lemma 2.13.** Suppose that \( A \) and \( B \) are \( \infty \)-categories. A map \( u : A \to B \) is dominant if and only if the \( \infty \)-category \( A \times_B B_{b'/b} \) is weakly contractible for every edge \( f : b \to b' \) in \( B \).
The proof of this statement is reasonably straightforward and is left to the reader. We note the following consequences.

**Remark 2.14.** It follows easily that a map \( u: A \to B \) of simplicial sets is dominant if and only if the opposite map \( u^{\text{op}}: A^{\text{op}} \to B^{\text{op}} \) is dominant (recall that dominant maps are invariant under categorical equivalences).

**Remark 2.15.** It follows, using Theorem 4.1.3.1 from [13], that every dominant map is left cofinal and right cofinal.

**Lemma 2.16.** If \( u: A \to B \) and \( v: C \to D \) are dominant maps of simplicial sets, then the product \( u \times v: A \times C \to B \times D \) is also dominant.

**Proof.** It suffices to prove that if \( u: A \to B \) is dominant then \( u \times \text{id}: A \times C \to B \times C \) is dominant for any simplicial set \( C \). Since dominant maps are invariant under categorical equivalence, we may suppose without loss of generality that \( A, B \) and \( C \) are \( \infty \)-categories. This follows immediately from Lemma 2.13, using the fact that for any vertices \( b \in B \) and \( c \in C \) we have a pullback diagram

\[
\begin{array}{ccc}
A \times B_{b/} \times C_{c/} & \longrightarrow & B_{b/} \times C_{c/} \\
\downarrow & & \downarrow \\
A \times C & \longrightarrow & B \times C
\end{array}
\]

involving the undercategories \( B_{b/} \) and \( C_{c/} \), and similarly for overcategories. \( \square \)

We conclude this section with the following useful example of a dominant map.

**Lemma 2.17.** For every \( n \geq 0 \) the diagonal map \( \Delta^n \to \Delta^n \times \Delta^n \) is dominant.

**Proof.** The diagonal map \( \Delta^n \to \Delta^n \times \Delta^n \) is a retract of the diagonal map \( (\Delta^1)^n \to (\Delta^1)^n \times (\Delta^1)^n \). Therefore, since dominant maps are closed under retracts (Remark 2.10) and products (Lemma 2.16) we are reduced to proving that \( \Delta^1 \to \Delta^1 \times \Delta^1 \) is dominant. This can be proven using Lemma 2.13 and a case by case analysis. \( \square \)

2.4. Inner anodyne maps and inner fibrations. We close this section by recording a couple of straightforward results about inner fibrations and inner anodyne maps that we will need later in the paper.

**Lemma 2.18.** Suppose that \( p: S \to T \) is an inner fibration, where \( S \) and \( T \) are Kan complexes. If \( p \) has the right lifting property against the map \( \Delta^0 \to \Delta^1 \) then \( p \) is a Kan fibration.

**Proof.** It suffices to prove that \( p \) is a left fibration, since \( T \) is a Kan complex. Every edge of \( S \) is an equivalence and hence is \( p \)-cocartesian (Proposition 2.4.1.5 of [13]). Therefore \( p \) has the right lifting property against every horn inclusion of the form \( \Lambda^0_i \subseteq \Delta^n, n \geq 2 \) (Remark 2.4.1.4 of [13]). Therefore, invoking the assumption that \( p \) has the right lifting property against the map \( \Delta^0 \to \Delta^1 \), it follows that \( p \) is a left fibration. \( \square \)

**Lemma 2.19.** Let \( B \) be an \( \infty \)-category. Suppose that \( i: A \to B \) is an acyclic cofibration in the Joyal model structure on \( \text{Set}_\Delta \), such that \( i \) is a bijection on 0-simplices. Then \( i \) is inner anodyne.
Proof. Factor \( i = pj \), where \( j : A \to B' \) is inner anodyne and \( p : B' \to B \) is an inner fibration. Then \( p \) is a categorical fibration, since \( p \) is bijective on objects and \( B \) is an \( \infty \)-category. Therefore \( p \) is a trivial Kan fibration and hence has a section \( s : B \to B' \), which exhibits \( i \) as a retract of \( j \). Hence \( i \) is inner anodyne. \( \square \)

3. Correspondences

3.1. The category of correspondences from \( A \) to \( B \). We recall the notion of a correspondence between simplicial sets from Section 2.3.1 and Section 5.2.1 of [13].

**Definition 3.1** (Lurie). Let \( A \) and \( B \) be simplicial sets. A correspondence from \( A \) to \( B \) is a map \( p : X \to \Delta^1 \) with \( p^{-1}(0) = B \) and \( p^{-1}(1) = A \).

**Remark 3.2.** We do not require that the map \( p \) in the above definition is an inner fibration: we will reserve the term fibrant correspondence to describe such a map (see Section 3.2 below). Note also that we call a correspondence from \( A \) to \( B \) what is called a correspondence from \( B \) to \( A \) in [13].

**Remark 3.3.** We write \( \text{Corr}(A,B) \) for the subcategory of \( (\text{Set}_\Delta)/\Delta^1 \) whose objects are the correspondences from \( A \) to \( B \) and where a map \( f : X \to Y \) is a map in \( (\text{Set}_\Delta)/\Delta^1 \) such that \( f|A = \text{id}_A \) and \( f|B = \text{id}_B \).

**Remark 3.4.** Clearly \( B \sqcup A \), equipped with the canonical map \( B \sqcup A \to \partial\Delta^1 \to \Delta^1 \) is an initial object of \( \text{Corr}(A,B) \). If \( p : X \to \Delta^1 \) is a correspondence in \( \text{Corr}(A,B) \) and \( u : \Delta^n \to X \) is a simplex, then the composite map \( pu : \Delta^n \to \Delta^1 \) has a unique decomposition \( pu = i \star f \), where \( i : \Delta^k \to \Delta^0 \) and \( f : \Delta^{n-k-1} \to \Delta^0 \). It follows that \( ui \) factors through \( B \), and \( uf \) factors through \( A \). Therefore \( ui \star uf \) is an \( n \)-simplex of \( B \star A \). This defines a unique map \( X \to B \star A \), from which it follows that \( B \star A \) is a terminal object of \( \text{Corr}(A,B) \).

**Remark 3.5.** There is a canonical full inclusion \( i : \text{Corr}(A,B) \hookrightarrow (\text{Set}_\Delta)/B \star A \). The inclusion \( i \) has a left adjoint \( L : (\text{Set}_\Delta)/B \star A \to \text{Corr}(A,B) \) which exhibits \( \text{Corr}(A,B) \) as a full reflective subcategory of \( (\text{Set}_\Delta)/B \star A \). The reflector \( L \) is defined on objects as follows: if \( X \in (\text{Set}_\Delta)/B \star A \) with structure map \( p \) then \( L(X) \) is the correspondence defined by the pushout diagram

\[
p^{-1}(B \sqcup A) \quad \longrightarrow \quad X \\
\longdownarrow \quad \quad \quad \longdownarrow \\
B \sqcup A \quad \longrightarrow \quad L(X)
\]

**Remark 3.6.** As a full reflective subcategory of the presentable category \( (\text{Set}_\Delta)/B \star A \), it follows (see Corollary 6.24 of [13]) that \( \text{Corr}(A,B) \) is presentable (here presentable is understood in the sense of Definition A.1.1.2 of [13]). In particular it follows that \( \text{Corr}(A,B) \) has all limits and colimits.

3.2. The model structure on correspondences. Suppose now that \( A \) and \( B \) are \( \infty \)-categories. The Joyal model structure on \( \text{Set}_\Delta \) induces a model structure on the slice category \( (\text{Set}_\Delta)/B \star A \) in the usual way. By definition, a map \( X \to B \star A \) is a fibrant object in this model structure if and only if it is a categorical fibration.
Let us say that an object in $\mathcal{C}_{\text{orr}(A,B)}$ is fibrant if and only if the canonical map $X \to B \star A$ is fibrant in the induced model structure on $(\text{Set}_\Delta)_{/B \star A}$. We have the following result.

**Lemma 3.7.** Let $X \in \mathcal{C}_{\text{orr}(A,B)}$, where $A$ and $B$ are $\infty$-categories. The following statements are equivalent:

1. $X$ is fibrant
2. the canonical map $p: X \to B \star A$ is an inner fibration
3. the canonical map $X \to \Delta^1$ is an inner fibration
4. $X$ is an $\infty$-category.

**Proof.** The equivalence of statements (3) and (4) is clear. It is also clear that (1) $\Rightarrow$ (2). We prove that (2) $\Rightarrow$ (1). Suppose that the canonical map $p: X \to B \star A$ is an inner fibration. Since $B \star A$ is an $\infty$-category (Proposition 1.2.8.3 of [13]), it follows that $X$ is an $\infty$-category. Hence $p: X \to B \star A$ is a categorical fibration if and only if $h(X) \to h(B \star A)$ is an isofibration, i.e. has the right lifting property against the inclusion $\{0\} \to J$, where $J$ denotes the groupoid interval. We have $h(B \star A) = h(B) \star h(A)$, where the right hand side denotes the join of the categories $h(B)$ and $h(A)$. Therefore the only isomorphisms in $h(B \star A)$ are represented by equivalences in $B$ or equivalences in $A$. Since $X \in \mathcal{C}_{\text{orr}(A,B)}$ these equivalences lift automatically to equivalences in $X$.

Finally, to complete the proof, we shall prove that (2) $\iff$ (4). The implication (2) $\Rightarrow$ (4) is immediate from the fact that $B \star A$ is an $\infty$-category. To prove the converse, assume that $X$ is an $\infty$-category and consider a commutative diagram

$$
\begin{array}{ccc}
\Lambda^n_i & \xrightarrow{u} & X \\
\downarrow & & \downarrow^p \\
\Delta^n & \xrightarrow{v} & B \star A
\end{array}
$$

We will prove that this map is compatible with the projection to $B \star A$. The map $v: \Delta^n \to B \star A$ decomposes as $v = x \star y$, where $x: \Delta^k \to B$ and $y: \Delta^{n-1-k} \to A$. If $k = -1$ or $k = n$ then we can find a diagonal filler for the diagram above since both $A$ and $B$ are $\infty$-categories. Otherwise, we have $\Delta^k \subseteq \Lambda^n_k$ and $\Delta^{n-1-k} \subseteq \Lambda^n_n$. Since $X$ is an $\infty$-category, we may extend the map $u$ along the inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ to obtain a map $w: \Delta^n \to X$. It follows that $w|\Delta^k = x$ and $w|\Delta^{n-1-k} = y$ and hence $pw = v$. □

More generally, we have

**Lemma 3.8.** Suppose that $f: X \to Y$ is a map between fibrant objects in $\mathcal{C}_{\text{orr}(A,B)}$. Then the underlying map of simplicial sets is a categorical fibration if and only if it is an inner fibration.

**Proof.** As in the proof of Lemma 3.7 above, we need to check that the induced map $h(X) \to h(Y)$ is an isofibration of categories. An isomorphism in $h(Y)$ maps to an isomorphism in $h(B \star A)$ and hence is represented by either an equivalence in $B$ or an equivalence in $A$. The result then follows since $f$ is a map in $\mathcal{C}_{\text{orr}(A,B)}$. □

**Theorem 3.9.** Let $A$ and $B$ be $\infty$-categories. There exists the structure of a left proper, combinatorial model category on $\mathcal{C}_{\text{orr}(A,B)}$ for which a map $X \to Y$ is a
• cofibration if the underlying map of simplicial sets is a monomorphism;
• weak equivalence if the underlying map of simplicial sets is a categorical equivalence

and for which the fibrant objects are the correspondences \( X \) whose underlying simplicial set is an \( \infty \)-category.

**Proof.** We use Proposition A.2.6.13 from \([13]\). To begin with, as observed in Remark 3.6 above, \( \mathrm{Corr}(A,B) \) is presentable. We verify the three conditions (1), (2) and (3) from op. cit. Let \( C \) denote the class of cofibrations in \( \mathrm{Corr}(A,B) \) and let \( W \) denote the class of weak equivalences in \( \mathrm{Corr}(A,B) \). The weakly saturated class of monomorphisms in \((\Set_\Delta)_{/B \star A}\) is generated by the set of boundary inclusions \( \partial \Delta^n \subseteq \Delta^n \) in \((\Set_\Delta)_{/B \star A}\) for \( n \geq 0 \). The simplices in \( B \star A \) are of the following three types: \( x \star \emptyset : \Delta^n \star \emptyset \to B \star A \), \( x \star y : \Delta^m \star \Delta^n \to B \star A \), and \( \emptyset \star y : \emptyset \star \Delta^n \to B \star A \). It follows that \( C \) is generated as a weakly saturated class by the set \( C_0 \) of monomorphisms in \( \mathrm{Corr}(A,B) \) of the form

\[
(\emptyset \star \partial \Delta^n) \cup_{\emptyset \cup \emptyset} (B \sqcup A) \to (\emptyset \star \Delta^n) \cup_{\emptyset \cup \Delta^n} (B \sqcup A)
\]

\[
(\partial \Delta^m \star \emptyset) \cup_{\partial \Delta^m \sqcup \emptyset} (B \sqcup A) \to (\Delta^m \star \emptyset) \cup_{\Delta^m \sqcup \emptyset} (B \sqcup A)
\]

\[
(\partial \Delta^m \star \Delta^n \cup \partial \Delta^n) \cup_{(\partial \Delta^m \star \Delta^n \cup \partial \Delta^n)} (B \sqcup A) \to (\Delta^m \star \Delta^n) \cup_{\Delta^m \sqcup \Delta^n} (B \sqcup A)
\]

For (1), observe that the class \( W \) is the inverse image of the class of categorical equivalences of simplicial sets under the forgetful functor \( \mathrm{Corr}(A,B) \to \Set_\Delta \). It follows that \( W \) is perfect by Corollary A.2.6.12 of \([12]\).

For (2), observe that if \( f : X \to Y \) is a map in \( \mathrm{Corr}(A,B) \) which has the right lifting property with respect to every morphism in \( C_0 \), then \( f \) is a trivial Kan fibration. For then \( f \) has the right lifting property with respect to every monomorphism in \( \Set_\Delta \).

For the characterization of the fibrant objects, observe that \( X \to B \star A \) has the right lifting property with respect to all maps in \( C \cap W \) if and only if the underlying map of simplicial sets is a categorical fibration. We then apply Lemma 3.7. \( \square \)

**Remark 3.10.** The model structure for correspondences is the left induced model structure (in the sense of \([5]\)) on \( \mathrm{Corr}(A,B) \) associated to the adjoint pair \((L,i)\) and the Joyal model structure on \((\Set_\Delta)_{/B \star A}\).

**Remark 3.11.** The category \((\Set_\Delta)_{/B \star A}\) has a natural structure as a simplicial category which is tensored and cotensored over \( \Set_\Delta \). This structure induces on \( \mathrm{Corr}(A,B) \) the structure of a simplicial category which is tensored and cotensored over \( \Set_\Delta \). If \( X \in \mathrm{Corr}(A,B) \) and \( K \) is a simplicial set, then the cotensor \( X \otimes K \) is defined by the pushout diagram

\[
(B \times K) \sqcup (A \times K) \longrightarrow X \times K
\]

\[
\downarrow
\]

\[
B \sqcup A \longrightarrow X \otimes K
\]

where the left hand vertical map is induced by the canonical projections \( B \times K \to B \) and \( A \times K \to A \). The construction \( X \otimes K \) extends to define a functor \( \otimes : \mathrm{Corr}(A,B) \times \Set_\Delta \to \Set_\Delta \). Observe that for a fixed correspondence \( X \in \mathrm{Corr}(A,B) \), the induced functor \( X \otimes (\cdot) : \Set_\Delta \to \mathrm{Corr}(A,B) \) commutes with colimits.
If $X$ is again a correspondence in $\text{Corr}(A, B)$ and $K$ is a simplicial set, then the cotensor $K^Y$ is defined by the pullback diagram

$$
\begin{array}{ccc}
K^Y & \rightarrow & Y^K \\
\downarrow & & \downarrow \\
B \star A & \rightarrow & (B \star A)^K
\end{array}
$$

where the lower horizontal map is conjugate to the canonical projection $(B \star A) \times K \rightarrow B \star A$. Note that $K^Y$ so defined is a correspondence: the two squares in the commutative diagram

$$
\begin{array}{ccc}
B \sqcup A & \rightarrow & B^K \sqcup A^K \\
\downarrow & & \downarrow \\
B \sqcup A & \rightarrow & B^K \sqcup A^K & \rightarrow & (B \star A)^K
\end{array}
$$

are both pullbacks, and the composite map $B \sqcup A \rightarrow (B \star A)^K$ factors as $B \sqcup A \rightarrow B \star A \rightarrow (B \star A)^K$, where the second map is the canonical map above. The adjointness $(-) \otimes K \dashv K(-)$ is clear. It follows (Lemma II 2.2 of [8]) that $\text{Corr}(A, B)$ has the structure of a simplicial category, tensored and cotensored over $\text{Set}$.

**Remark 3.12.** Let $A$ and $B$ be $\infty$-categories. The model structure for correspondences on $\text{Corr}(A, B)$ is enriched over the Joyal model structure via the simplicial enrichment described in Remark 3.11.

### 3.3. Distributors to correspondences, and back again.

Recall that the *edgewise subdivision* of a simplicial set $X$ (in the sense of Segal [16]) is defined by composing the functor $X : \Delta^{\text{op}} \rightarrow \text{Set}$ with the opposite of the ‘doubling functor’

$$
d : \Delta \rightarrow \Delta \\
[n] \mapsto [n]^{\text{op}} \star [n]
$$

This construction can be used to relate the category $\left(\text{Set}_\Delta\right)/_{\left(B^{\text{op}} \times A\right)}$ with the category of correspondences from $A$ to $B$. The relation is as follows.

Observe that the doubling functor above induces a functor between simplex categories

$$
\sigma : \Delta/_{\left(B^{\text{op}} \times A\right)} \rightarrow \Delta/_{(B \star A)}
$$

which sends a pair $(u, v) : \Delta^n \rightarrow B^{\text{op}} \times A$ to

$$
\sigma(u, v) = u^{\text{op}} \star v : (\Delta^n)^{\text{op}} \star \Delta^n \rightarrow B \star A.
$$

The functor $\sigma$ induces an adjunction

$$
\sigma_1 : \left(\text{Set}_\Delta\right)/_{\left(B^{\text{op}} \times A\right)} \rightleftarrows \left(\text{Set}_\Delta\right)/_{B \star A} : \sigma^*
$$

and in fact the functor $\sigma^*$ has a further right adjoint $\sigma_* : \left(\text{Set}_\Delta\right)/_{(B^{\text{op}} \times A)} \rightarrow \left(\text{Set}_\Delta\right)/_{B \star A}$.

We make the following observations about the functors $\sigma_1$ and $\sigma^*$.

**Lemma 3.13.** The functor $\sigma_1 : \left(\text{Set}_\Delta\right)/_{\left(B^{\text{op}} \times A\right)} \rightarrow \left(\text{Set}_\Delta\right)/_{B \star A}$ sends monomorphisms to monomorphisms.
Proof. Let \( f: X \to Y \) be a monomorphism in \((\text{Set}_\Delta)/(B^{op} \times A)\). The map \( \sigma_!(f): \sigma_!(X) \to \sigma_!(Y) \) is a monomorphism if and only if the underlying map of simplicial sets is a monomorphism; therefore it suffices to check that for every \( n \geq 0 \) the induced map \( \sigma_!(f)_n: \sigma_!(X)_n \to \sigma_!(Y)_n \) is a monomorphism of sets. But \( \sigma_!(f)_n \) is easily seen to be the map \( f_{2n+1}: X_{2n+1} \to Y_{2n+1} \) which is a monomorphism by hypothesis. \( \square \)

**Lemma 3.14.** If \( f: C \to A \) and \( g: D \to B \) are maps determining objects \( D \times C \) and \( D^{op} \times C \) of \((\text{Set}_\Delta)/B^{op}A\) and \((\text{Set}_\Delta)/(B^{op} \times A)\) respectively, then

\[
\sigma^*(D \times C) = D^{op} \times C.
\]

Proof. If \( \phi: (\Delta^n)^{op} \times \Delta^n \to D \times C \) is a map in \((\text{Set}_\Delta)/B^{op}A\) such that \( (g \times f)\phi = u^{op} \times v \) where \( u: \Delta^n \to B^{op} \) and \( v: \Delta^n \to A \), then \( \phi \) is necessarily of the form \( \phi = x^{op} \times y \) for unique simplices \( x: \Delta^n \to D^{op} \) and \( y: \Delta^n \to C \) with \( x^{op} = \phi|((\Delta^1)^{op}) \) and \( y = \phi|\Delta^n \).

The key point here is the canonical functor \( \times: \text{Set}_\Delta \times \text{Set}_\Delta \to (\text{Set}_\Delta)/\Delta_1 \), defined by the join operation, is fully faithful (see Proposition 3.5 in [10]). \( \square \)

We write \( a_1: (\text{Set}_\Delta)/(B^{op} \times A) \to \text{Corr}(A, B) \) for the composite functor \( a_1 := L\sigma_1 \), and we write \( a^* : \text{Corr}(A, B) \to (\text{Set}_\Delta)/(B^{op} \times A) \) for the composite functor \( a^* := \sigma^* i \).

The functors \( a_1 \) and \( a^* \) form an adjoint pair \((a_1, a^*)\).

**Remark 3.15.** Observe that if \( f: C \to A \) and \( g: D \to B \) are maps then the functor \( \sigma^* : (\text{Set}_\Delta)/B^{op}A \to (\text{Set}_\Delta)/(B^{op} \times A) \) sends the object \( D \sqcup C \) to the initial object \( \emptyset \) of \((\text{Set}_\Delta)/(B^{op} \times A)\). It follows that the endo-functor \( a^* a_1 \) of \((\text{Set}_\Delta)/(B^{op} \times A)\) is isomorphic to the endo-functor \( \sigma^* \sigma_1 \). It follows that \( a^* a_1 \) preserves all colimits and hence is determined by its value on the \( n \)-simplices \( \Delta^n \to B^{op} \times A \) for \( n \geq 0 \). A short calculation, using Lemma 3.14, shows that in fact the unit map \( \Delta^n \to a^* a_1(\Delta^n) \) is isomorphic to the diagonal map \( \Delta^n \to \Delta^n \times \Delta^n \), where \( \Delta^n \times \Delta^n \) is regarded as an object of \((\text{Set}_\Delta)/(B^{op} \times A)\) via the map \( f \times g: \Delta^n \times \Delta^n \to B^{op} \times A \), where \( (f, g): \Delta^n \to B^{op} \times A \).

**Remark 3.16.** We observe that the functor \( a^* : \text{Corr}(A, B) \to (\text{Set}_\Delta)/(B^{op} \times A) \) has a right adjoint \( a_* : (\text{Set}_\Delta)/(B^{op} \times A) \to \text{Corr}(A, B) \). To see this, it suffices to prove that the functor \( a^* : \text{Corr}(A, B) \to (\text{Set}_\Delta)/(B^{op} \times A) \) preserves colimits. This follows from the fact that \( \sigma^* \) preserves colimits and the fact that \( \sigma^* i L = \sigma^* \) (this last fact can easily be seen using the observation made in Remark 3.15). If \( X \to B^{op} \times A \) is an object of \((\text{Set}_\Delta)/(B^{op} \times A)\) then \( a_*(X) \) is the correspondence from \( A \) to \( B \) such that

\[
(\text{Set}_\Delta)/(B^{op}A)(\Delta^n, ia_*(X)) = (\text{Set}_\Delta)/(B^{op} \times A)(a^* L(\Delta^n), X)
\]

for every \( n \)-simplex \( \Delta^n \to B \times A \).

**Proposition 3.17.** The functor \( a_1 : (\text{Set}_\Delta)/(B^{op} \times A) \to \text{Corr}(A, B) \) sends left anodyne morphisms in \((\text{Set}_\Delta)/(B^{op} \times A)\) to inner anodyne morphisms in \( \text{Corr}(A, B) \).

Proof. From Lemma 3.13 we have that \( a_1 \) sends monomorphisms to monomorphisms. Let \( A \) denote the class of all monomorphisms \( v \) in \((\text{Set}_\Delta)/(B^{op} \times A)\) such that the underlying map of simplicial sets \( a_1(v) \) is inner anodyne. We need to prove that every left anodyne morphism in \((\text{Set}_\Delta)/(B^{op} \times A)\) is contained in \( A \). Therefore, by Proposition 2.7 it is sufficient to prove that \( A \) is saturated, satisfies the right cancellation property, and that the initial vertex maps \( i_n: \Delta^0 \to \Delta^n \) are contained in \( A \) for all \( n \geq 0 \). By Proposition 2.7 the class of inner anodyne maps in \( \text{Set}_\Delta \) has the right cancellation

[10]
property; the functoriality of $a!$ then implies that $\mathcal{A}$ also has the right cancellation property. Likewise it is clear that $\mathcal{A}$ is a saturated class of monomorphisms since the inner anodyne maps in $\Set_{\Delta}$ form a saturated class and $a!$ is a left adjoint.

Let $n \geq 0$; we show that $i_n: \Delta^0 \to \Delta^n$ is contained in $\mathcal{A}$. The map $a!(i_n)$ is a pushout of the map

$$(\Delta^0)^{op} \star \Delta^0 \cup (\Delta^0)^{op} \cup \Delta^0 \to (\Delta^0)^{op} \star \Delta^n,$$

and hence it suffices to prove that this last map is inner anodyne. This map factors as

$$(\Delta^0)^{op} \star \Delta^0 \cup (\Delta^n)^{op} \cup \Delta^n \to (\Delta^n)^{op} \star \Delta^n.$$

The first map in this composite is a pushout of the map $(\Delta^0)^{op} \cup (\Delta^n)^{op} \to (\Delta^n)^{op} \star \Delta^0$ which is inner anodyne by another application of this lemma. Hence $\mathcal{A}$ contains the left anodyne morphisms in $\Set_{\Delta}$, which completes the proof of the proposition. \hfill $\square$

The following corollary is straightforward.

**Corollary 3.18.** The functor $a^*: \Corr(A,B) \to (\Set_{\Delta})/(B^{op} \times A)$ sends inner fibrations in $\Corr(A,B)$ to left fibrations in $(\Set_{\Delta})/(B^{op} \times A)$.

**Remark 3.19.** If $A$ is an $\infty$-category then the image of the functor $a^*$ on the correspondence $A \times I$ in $\Corr(A,A)$ is precisely the canonical map $\Tw(A) \to A^{op} \times A$, where $\Tw(A)$ denotes the twisted arrow category of $A$ (see Construction 5.2.1.1. of [14]; note also that $\Tw(A)$ is precisely the Segal edge-wise subdivision of $A$ from [16]). Thus Corollary 3.18 gives an alternative proof that this canonical map is a left fibration (we hasten to point out that this proof proceeds along similar lines to the proof of Proposition 1.1 in [3]).

**Remark 3.20.** Suppose that $X$ is an $\infty$-category and that $x$ is an object of $X$. Observe that $\Tw(X)|\{x\} \times X$ may be described as the diagonal of the bisimplicial set $X_{\Delta^{op}}: \Delta^{op} \to \Set_{\Delta}$ defined by

$$[n] \mapsto X_{(\Delta^n)^{op}/},$$

where the slice $X_{(\Delta^n)^{op}/}$ is defined by the map $(\Delta^n)^{op} \to X$ given as the composite

$$(\Delta^n)^{op} \to (\Delta^0)^{op} \to X.$$
Remark 3.21. If \( X \in \text{Corr}(A, B) \) is a correspondence, then \( a^* X \to B^{op} \times A \) is the left hand vertical map in the pullback diagram

\[
\begin{array}{ccc}
a^* X & \longrightarrow & \text{Tw}(X) \\
\downarrow & & \downarrow \\
B^{op} \times A & \longrightarrow & X^{op} \times X
\end{array}
\]

where the lower horizontal map is induced by the inclusions \( A \subseteq X \) and \( B \subseteq X \).

Proposition 3.22. Let \( A \) and \( B \) be \( \infty \)-categories. The adjunction

\[
a_! : (\text{Set}_\Delta)/(B^{op} \times A) \xrightarrow{\rightleftharpoons} \text{Corr}(A, B) : a^*
\]

is a Quillen adjunction for the covariant model structure on \((\text{Set}_\Delta)/(B^{op} \times A)\) and the model structure for correspondences on \(\text{Corr}(A, B)\).

Proof. The functor \( a_! \) sends monomorphisms to monomorphisms, and hence \( a^* \) sends trivial fibrations to trivial fibrations. We prove that \( a^* \) sends fibrations between fibrant objects in \(\text{Corr}(A, B)\) to covariant fibrations in \((\text{Set}_\Delta)/(B^{op} \times A)\). By Corollary 3.18 the functor \( a^* \) sends inner fibrations in \(\text{Corr}(A, B)\) to left fibrations in \((\text{Set}_\Delta)/(B^{op} \times A)\). It follows that \( a^* \) sends fibrations between fibrant objects in \(\text{Corr}(A, B)\) to covariant fibrations in \((\text{Set}_\Delta)/(B^{op} \times A)\). This completes the proof of the proposition.

In [2] Ayala and Francis prove that there is a categorical equivalence between the \( \infty \)-category \( \text{Fun}(B^{op} \times A, \mathcal{S}) \) and an \( \infty \)-category of correspondences from \( A \) to \( B \). The following theorem refines their result to a statement at the level of model categories (this latter statement is also certainly well-known; it is stated without proof in [10] and it is also stated as Remark 2.3.1.4 in [13]). We shall give a proof, since one has not appeared in the literature so far, and since we shall need some results obtained in the course of the proof for the proof of Theorem 3.25.

Theorem 3.23. Let \( A \) and \( B \) be \( \infty \)-categories. Then the Quillen adjunction

\[
a_! : (\text{Set}_\Delta)/(B^{op} \times A) \xrightarrow{\rightleftharpoons} \text{Corr}(A, B) : a^*.
\]

extends to a Quillen equivalence for the covariant model structure on \((\text{Set}_\Delta)/(B^{op} \times A)\) and the model structure for correspondences on \(\text{Corr}(A, B)\).

Proof. We prove that (i) \( a^* \) reflects weak equivalences between fibrant objects, and (ii) if \( X \in (\text{Set}_\Delta)/(B^{op} \times A) \), then \( X \to a^* \mathcal{R} a_! X \) is a covariant equivalence in \((\text{Set}_\Delta)/(B^{op} \times A)\), where \( \mathcal{R}a_! X \) denotes a fibrant replacement of \( a_! X \) in the model structure for correspondences on \( \text{Corr}(A, B) \).

We prove (i). Suppose that \( f : X \to Y \) is a map between fibrant objects in \(\text{Corr}(A, B)\) such that \( a^* X \to a^* Y \) is a covariant equivalence in \((\text{Set}_\Delta)/(B^{op} \times A)\). We need to prove that \( f \) is a categorical equivalence. Therefore, we need to prove that \( f \) is essentially surjective and fully faithful. The essential surjectivity is immediate since \( f \) is a map between correspondences in \(\text{Corr}(A, B)\). To prove that \( f \) is fully faithful it suffices to prove that the induced map on mapping spaces \( \text{Hom}^L_X(a, b) \to \text{Hom}^L_Y(a, b) \) is a weak homotopy equivalence for each pair of objects \( a \in A \) and \( b \in B \). This follows immediately from Remark 3.20 and Remark 3.21.
Now we prove (ii). Let \( X \to B^{\text{op}} \times A \) be an object of \((\text{Set}_\Delta)/(B^{\text{op}} \times A)\); then we may factor \( a_1X \to B \times A \) in \( \text{Corr}(A, B) \) as \( a_1X \to R a_1X \to B \times A \), where \( a_1X \to R a_1X \) is a categorical equivalence and \( R a_1X \to B \times A \) is an inner fibration. Thus \( R a_1X \) is a fibrant replacement of \( a_1X \) in the model structure for correspondences on \( \text{Corr}(A, B) \) (Theorem 3.9). We claim that \( a^* \) sends categorical equivalences in \( \text{Corr}(A, B) \) to covariant equivalences in \((\text{Set}_\Delta)/(B^{\text{op}} \times A)\). To see this we argue as follows: suppose that \( f : X \to Y \) is a categorical equivalence in \( \text{Corr}(A, B) \) and choose a fibrant replacement \( Y' \to Y' \) of \( Y \) in \( \text{Corr}(A, B) \). Then \( Y' \) is an \( \infty \)-category, and \( j : Y \to Y' \) is an acyclic fibration which is a bijection on objects. It follows that \( j \) is inner anodyne (Lemma 2.19). We may factor the composite map \( jf \) in \((\text{Set}_\Delta)/(B^{\text{op}} \times A)\) as \( jf = f' j' \), where \( f' : X' \to Y' \) is an inner fibration, and where \( j' : X \to X' \) is inner anodyne. We observe that the underlying simplicial map \( f' \) is a bijection on vertices, and hence is a categorical fibration. Therefore, since \( f' \) is a categorical equivalence, \( f' \) is a trivial Kan fibration. Hence \( \sigma^*(f') \) is a trivial Kan fibration, since \( \sigma_! \) preserves monomorphisms (Lemma 3.13). It now suffices to prove the following claim: the functor \( \sigma^* : (\text{Set}_\Delta)/(B_*A) \to (\text{Set}_\Delta)/(B^{\text{op}} \times A) \) sends inner anodyne maps in \((\text{Set}_\Delta)/(B_*A)\) to left anodyne maps in \((\text{Set}_\Delta)/(B^{\text{op}} \times A)\).

The inner anodyne maps in \((\text{Set}_\Delta)/(B_*A)\) are a saturated class of monomorphisms, generated by the inner horn inclusions \( \Lambda^n_k \to \Delta^n \) in \((\text{Set}_\Delta)/(B_*A)\). We need to calculate the image \( \sigma^*(\Lambda^n_k) \to \sigma^*(\Delta^n) \) of such a horn inclusion under the functor \( \sigma^* \). The simplices in \( B \times A \) are of the following three types: \( \Delta^n, \Lambda^n_k, \partial\Delta^n \) and \( \Delta^n \times \Delta^n, \Lambda^n_k \times \Delta^n, \partial\Delta^n \times \Delta^n \) and \( \emptyset \times \Delta^n \). It follows that the inner horn inclusions in \((\text{Set}_\Delta)/(B_*A)\) are of the following types:

- \( \Lambda^n_k \to \Delta^n, 0 < k < n \),
- \( \Lambda^n_k \times \Delta^n \to \Delta^n, 0 < k \leq m, n > 0 \),
- \( \Delta^n \times \Lambda_k^n \to \Delta^n, 0 \leq k < n \),
- \( \emptyset \times \Lambda^n_k \to \emptyset \times \Delta^n, 0 < k < n \).

By Lemma 3.14 the image under \( \sigma^* \) of each of the first and last of these types of morphism is the empty map, while the image under \( \sigma^* \) of the second and third maps are respectively the left anodyne morphisms

- \( (\Lambda^n_k)^{\text{op}} \times \Delta^n \to (\Delta^n)^{\text{op}} \times \Delta^n, 0 < k \leq m, n > 0 \),
- \( (\Delta^n)^{\text{op}} \times \Lambda^n_k \to (\Delta^n)^{\text{op}} \times \Delta^n, 0 \leq k < n \).

This completes the proof of the claim.

To complete the proof of the theorem it suffices to prove that \( X \to a^*a_1X \) is a covariant equivalence in \((\text{Set}_\Delta)/(B^{\text{op}} \times A)\). Equivalently, by Remark 3.15 it suffices to prove that \( X \to \sigma^*\sigma_1X \) is a covariant equivalence in \((\text{Set}_\Delta)/(B^{\text{op}} \times A)\). We will prove that in fact this map is a left anodyne map in \((\text{Set}_\Delta)/(B^{\text{op}} \times A)\).

Using the skeletal filtration of \( X \), we see that by an induction argument we are reduced to the case where \( X \) is obtained from \( X' \) by adjoining a single \( n \)-simplex along an attaching map \( \partial\Delta^n \to X' \). We have a commutative diagram

\[
\begin{array}{ccc}
\Delta^n & \xrightarrow{\partial\Delta^n} & X' \\
\downarrow & & \downarrow \\
\sigma^*\sigma_1\Delta^n & \xrightarrow{\sigma^*\sigma_1\partial\Delta^n} & \sigma^*\sigma_1X'
\end{array}
\]
in which the two right hand vertical maps are left anodyne by the inductive hypothesis, and where the left hand vertical map is the diagonal inclusion \( \Delta^n \to \Delta^n \times \Delta^n \) (see Remark 3.15) and hence is left anodyne. Therefore it suffices by Lemma 3.24 below to prove that for any \( n \geq 0 \) the square

\[
\begin{array}{ccc}
\partial \Delta^n & \longrightarrow & \sigma^* \sigma \partial \Delta^n \\
\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & \sigma^* \sigma \Delta^n
\end{array}
\]

is a pullback. From Remark 3.15, the map \( \Delta^n \to \sigma^* \sigma \Delta^n \) is the diagonal inclusion \( \Delta^n \to \Delta^n \times \Delta^n \). Let us write \( \delta_n : \Delta^n \to \Delta^n \times \Delta^n \) for this map. Clearly the square

\[
\begin{array}{ccc}
\Delta^{n-1} & \delta_{n-1} \to & \Delta^{n-1} \times \Delta^{n-1} \\
\downarrow d_i & & \downarrow d_i \times d_i \\
\Delta^n & \delta_n \to & \Delta^n \times \Delta^n
\end{array}
\]

is a pullback for any \( 0 \leq i \leq n \). It follows that the square

\[
\begin{array}{ccc}
\partial_i \Delta^{n-1} & \longrightarrow & \partial_i \Delta^{n-1} \times \partial_i \Delta^{n-1} \\
\downarrow & & \downarrow \\
\Delta^n & \delta_n \to & \Delta^n \times \Delta^n
\end{array}
\]

(3)

is a pullback for any \( 0 \leq i \leq n \). Since

\[
\partial \Delta^n = \bigcup_{i=0}^n \partial_i \Delta^{n-1}
\]

is a union of the subobjects \( \partial_i \Delta^{n-1} \) of \( \Delta^n \) and the functor \( \sigma^* \sigma \) is a left adjoint, it follows that

\[
\sigma^* \sigma \partial \Delta^n = \bigcup_{i=0}^n \partial_i \Delta^{n-1} \times \partial_i \Delta^{n-1}
\]

is a union of the subobjects \( \partial_i \Delta^{n-1} \times \partial_i \Delta^{n-1} \) of \( \Delta^n \times \Delta^n \). The result then follows from the fact that the square (3) above is a pullback for every \( 0 \leq i \leq n \).

\[\square\]

**Lemma 3.24.** Suppose that

\[
\begin{array}{ccc}
A & \longleftarrow & B \longrightarrow C \\
\downarrow f & & \downarrow g \swarrow h \\
A' & \longleftarrow & B' \longrightarrow C'
\end{array}
\]

is a commutative diagram of maps of simplicial sets in which the left hand square is a pullback and in which the maps \( f, g \) and \( h \) are left anodyne. Then the induced map

\[ A \cup_B C \to A' \cup_{B'} C' \]

is also left anodyne.
Proof. The induced map factors as

\[ A \cup_B C \rightarrow A \cup_B C' \approx A \cup_B B' \cup_B C' \rightarrow A' \cup_{B'} C' \]

and, since left anodyne maps are preserved under pushouts, we see that it suffices to prove that \( A \cup_B B' \rightarrow A' \) is left anodyne. The map \( A \rightarrow A' \) factors as \( A \rightarrow A \cup_B B' \rightarrow A' \); therefore the result follows from the right cancellation property of left anodyne maps in \( \text{Set}_\Delta \) (Corollary 4.1.2.2 of [13]), since \( A \rightarrow A' \) is left anodyne by hypothesis and \( A \rightarrow A \cup_B B' \) is a pushout of the left anodyne map \( B \rightarrow B' \). \( \square \)

The functor \( a^* \) has the distinction of being simultaneously a left and right Quillen equivalence. Recall the adjoint pair \((a^*, a_*)\) (see Remark 3.16). We have the following theorem.

**Theorem 3.25.** Let \( A \) and \( B \) be \( \infty \)-categories. Then the adjoint pair

\[ a^*: \text{Corr}(A, B) \rightleftarrows (\text{Set}_\Delta)/(B^{op} \times A): a_* \]

is a Quillen equivalence for the model structure for correspondences on \( \text{Corr}(A, B) \) and the covariant model structure on \( (\text{Set}_\Delta)/(B^{op} \times A) \).

Proof. We show first that the pair \((a^*, a_*)\) is a Quillen adjunction. Clearly \( a^* \) sends monomorphisms to monomorphisms; and we have proved above (see the proof of Theorem 3.23) that \( a^* \) sends categorical equivalences to covariant equivalences.

To prove that the Quillen pair \((a^*, a_*)\) is a Quillen equivalence it suffices to prove that the Quillen pair \((a^* a_! a^*, a^* a_* a^! a_!)\) is a Quillen equivalence. We have proven above (see the proof of Theorem 3.23) that the natural transformation \( X \rightarrow a^* a_! X \) is left anodyne for every \( X \in (\text{Set}_\Delta)/(A^{op} \times B) \). It follows easily that \( a^* a_! \) reflects covariant equivalences. To complete the proof it suffices to prove that \( a^* a_* X \rightarrow X \) is a covariant equivalence for every left fibration \( X \rightarrow B^{op} \times A \). We will prove that in fact this map is a trivial Kan fibration. Suppose given a commutative diagram

\[
\begin{array}{ccc}
\partial \Delta^n & \rightarrow & a^* a_* X \\
\downarrow & & \downarrow \\
\Delta^n & \rightarrow & X
\end{array}
\]

where \( n \geq 0 \). By adjointness, the indicated diagonal filler in this diagram exists if and only if the indicated diagonal filler in the corresponding diagram

\[
\begin{array}{ccc}
a^* a_! \partial \Delta^n \cup \partial \Delta^n & \rightarrow & \Delta^n \\
\downarrow & & \downarrow \\
\Delta^n & \rightarrow & X
\end{array}
\]

exists. But the left hand vertical map is left anodyne by the right cancellation property of the class of left anodyne maps (Proposition 2.4). Therefore the indicated diagonal filler exists since \( X \rightarrow B^{op} \times A \) is a left fibration. \( \square \)
4. BIFIBRATIONS

4.1. The category of bifibrations. In this section we recall the definition of bifibration from Section 2.4.7 of [13].

**Definition 4.1.** Let $A$ and $B$ be simplicial sets. A map $(p, q): X \to A \times B$ in $\mathsf{Set}_{\Delta}$ is called a *bifibration* if the following conditions are satisfied:

1. $(p, q)$ is an inner fibration;
2. for every $n \geq 1$ and for every commutative diagram
   \[
   \begin{array}{ccc}
   \Lambda^n_0 & \to & X \\
   \downarrow & \nearrow & \downarrow (p,q) \\
   \Delta^n & \to & A \times B
   \end{array}
   \]
   such that $\pi_B f: \Delta^{\{0,1\}} \to B$ is a degenerate edge, the indicated diagonal filler exists;
3. for every $n \geq 1$ and for every commutative diagram
   \[
   \begin{array}{ccc}
   \Lambda^n & \to & X \\
   \downarrow & \nearrow & \downarrow (p,q) \\
   \Delta^n & \to & A \times B
   \end{array}
   \]
   such that $\pi_A f: \Delta^{\{n-1,n\}} \to A$ is a degenerate edge, the indicated diagonal filler exists.

**Remark 4.2.** Suppose that $p: X \to A$ is a left fibration where $X$ and $A$ are simplicial sets. Then for any simplicial set $B$, the induced map $p \times \text{id}_B: X \times B \to A \times B$ is a bifibration. The conditions (1) and (2) from Definition 4.1 are clearly satisfied. To see that condition (3) is satisfied, note that it suffices to prove that the indicated diagonal filler exists in every diagram of the form

\[
\begin{array}{ccc}
\Lambda_n & \to & X \\
\downarrow & \nearrow \scriptstyle p & \downarrow \\
\Delta_n & \to & A
\end{array}
\]

in which $v|_{\Delta^{\{n-1,n\}}} \to A$ is a degenerate edge of $A$. The existence of such a diagonal filler is clear when $n = 1$; if $n \geq 2$ the existence of such a diagonal filler is equivalent to the existence of the indicated diagonal filler in the induced diagram

\[
\begin{array}{ccc}
\Delta^{\{n\}} & \to & X_{\Delta^{n-2}/} \\
\downarrow & \nearrow & \downarrow \\
\Delta^{\{n-1,n\}} & \to & X_{\partial \Delta^{n-2}/ \times A_{\partial \Delta^{n-2}/} A_{\Delta^{n-2}/}}
\end{array}
\]

But the right hand vertical map in this diagram is a trivial Kan fibration (Proposition 2.1.2.5 of [13]) since $p: X \to A$ is a left fibration. The existence of the required diagonal
filler follows. Dually, if \( q: Y \to B \) is a right fibration, then for any simplicial set \( A \), the induced map \( \text{id}_A \times q: A \times Y \to A \times B \) is a bifibration.

**Lemma 4.3.** Let \((p,q): X \to A \times B\) be a bifibration, where \( A \) and \( B \) are simplicial sets. Then the fiber \( X_{(a,b)} \) of \((p,q)\) over \((a,b)\) is a Kan complex for every pair of vertices \((a,b)\) \(\in A \times B\).

**Proof.** Clearly bifibrations over \( A \times B \) are stable under base change along maps of the form \( f \times g: A' \times B' \to A \times B \). Therefore, pulling back along the map \( a \times b: \Delta^0 \times \Delta^0 \to A \times B \) induces a bifibration \( X_{(a,b)} \to \Delta^0 \times \Delta^0 \). It follows (Remark 2.4.7.4 of [13]) that \( X_{(a,b)} \to \Delta^0 \) is a right fibration. Hence \( X_{(a,b)} \) is a Kan complex. \(\square\)

### 4.2. Bivariant anodyne maps.

In this section we introduce the concept of bivariant anodyne maps and study some of their properties.

**Definition 4.4.** Let \( A \) and \( B \) be simplicial sets. Let us say that a map \( u: M \to N \) in \((\text{Set}_\Delta) / (A \times B)\) is a bivariant anodyne map if it belongs to the weakly saturated class generated by the following classes of maps in \((\text{Set}_\Delta) / (A \times B)\):

1. the inner horn inclusions

\[
\begin{array}{ccc}
\Lambda_i^n & \to & \Delta^n \\
\downarrow & & \downarrow \\
A \times B & &
\end{array}
\]

where \( 0 < i < n \);

2. the horn inclusions

\[
\begin{array}{ccc}
\Lambda_0^n & \to & \Delta^n \\
\downarrow & \searrow (f,g) \\
A \times B & &
\end{array}
\]

where \( f: \Delta^{0,1} \to B \) is a degenerate edge;

3. the horn inclusions

\[
\begin{array}{ccc}
\Lambda_n^n & \to & \Delta^n \\
\downarrow & \searrow (f,g) \\
A \times B & &
\end{array}
\]

where \( g: \Delta^{n-1,n} \to A \) is a degenerate edge.

We extend the original usage of the term bifibration in [13] to cover the following more general situation.

**Definition 4.5.** Let \( A \) and \( B \) be simplicial sets and let \( f: X \to Y \) be a map in \((\text{Set}_\Delta) / (A \times B)\). We say \( f \) is a bifibration if it has the right lifting property against all bivariant anodyne maps in \((\text{Set}_\Delta) / (A \times B)\).

**Remark 4.6.** Let \( X \) and \( Y \) be objects of \((\text{Set}_\Delta) / (A \times B)\) and suppose that \( X \) has structure map \((p,q): X \to A \times B\). A map \( f: X \to Y \) in \((\text{Set}_\Delta) / (A \times B)\) is a bifibration if and only if the following conditions are satisfied:
(1) $f$ is an inner fibration;
(2) for every commutative diagram
\[
\begin{align*}
\Lambda^n_0 & \overset{u}{\rightarrow} X \\
\downarrow & \quad \downarrow f \\
\Delta^n & \rightarrow Y
\end{align*}
\]
in which the edge $qu: \Delta^{0,1} \rightarrow B$ is degenerate, the indicated diagonal filler exists;
(3) for every commutative diagram
\[
\begin{align*}
\Lambda^n & \overset{v}{\rightarrow} X \\
\downarrow & \quad \downarrow f \\
\Delta^n & \rightarrow Y
\end{align*}
\]
in which the edge $pv: \Delta^{n-1,n} \rightarrow A$ is degenerate, the indicated diagonal filler exists.

To see the equivalence of these statements note that if $f: X \rightarrow Y$ is a bifibration then the conditions (i), (ii) and (iii) are automatically satisfied. Conversely, suppose that $f: X \rightarrow Y$ is a map in $(\text{Set}_\Delta)/(A \times B)$ such that (i), (ii) and (iii) are satisfied. The class of maps in $(\text{Set}_\Delta)/(A \times B)$ which have the right lifting property against $f$ is weakly saturated. By hypothesis it contains the classes of maps (1), (2) and (3) from Definition 4.4 hence it contains the weakly saturated class generated by these maps, in other words the class of bivariant anodyne maps.

**Remark 4.7.** Thus a map $X \rightarrow A \times B$ is a bifibration in the sense of Definition 4.5 if and only if it is a bifibration in the sense of Definition 4.1.

**Remark 4.8.** If $X \rightarrow Y$ is a bifibration in $(\text{Set}_\Delta)/(A \times B)$ then for every vertex $b$ in $B$ the restriction $X|A \times \{b\} \rightarrow Y|A \times \{b\}$ is a left fibration in $(\text{Set}_\Delta)/A$. Similarly for every vertex $a$ in $A$ the restriction $X|\{a\} \times B \rightarrow Y|\{a\} \times B$ is a right fibration in $(\text{Set}_\Delta)/B$.

**Proposition 4.9.** Let $A$ and $B$ be simplicial sets. Consider the following classes of morphisms in $(\text{Set}_\Delta)/(A \times B)$:

1. all inner horn inclusions
\[
\begin{align*}
\Lambda^n_i & \rightarrow \Delta^n \\
\downarrow & \quad \downarrow \kappa \\
A \times B & 
\end{align*}
\]
for $0 < i < n$;
2. all horn inclusions
\[
\begin{align*}
\Lambda^n_0 & \rightarrow \Delta^n \\
\downarrow & \quad \downarrow (f,g) \\
A \times B & 
\end{align*}
\]
such that $g|\Delta^{\{0,1\}}$ is a degenerate edge of $B$;

$(2')$ all inclusions

$$
\begin{array}{ccc}
\Delta^1 \times \partial \Delta^n \cup \{0\} \times \Delta^n & \xrightarrow{(f,g)} & \Delta^1 \times \Delta^n \\
A \times B
\end{array}
$$

such that $g|\Delta^1 \times \{i\}$ is a degenerate edge of $B$ for every vertex $i$ in $\Delta^n$;

$(2'')$ all inclusions of the form

$$
\begin{array}{ccc}
\Delta^1 \times K \cup \{0\} \times \Delta^n & \xrightarrow{(f,g)} & \Delta^1 \times L \\
A \times B
\end{array}
$$

where $K \hookrightarrow L$ is an inclusion and where $g|\Delta^1 \times \{v\}$ is a degenerate edge of $B$ for every vertex $v$ of $L$.

Then the weakly saturated classes of morphisms in $(\text{Set}_\Delta)/(A \times B)$ generated by the classes $(1)$ and $(2)$, the classes $(1)$ and $(2')$, and the classes $(1)$ and $(2'')$ are all equal.

Proof. The proof of this proposition is essentially the same as the proof of Proposition 3.1.1.5 of [13]. We give the details. To begin with, the weakly saturated class generated by $(1)$ and $(2')$ is clearly contained in the weakly saturated class generated by $(1)$ and $(2'')$. As in the proof of op. cit., one easily proves that the weakly saturated class generated by $(1)$ and $(2'')$ is contained in the weakly saturated class generated by $(1)$ and $(2')$. It follows that the weakly saturated class generated by $(1)$ and $(2')$ is equal to the weakly saturated class generated by $(1)$ and $(2'')$.

We prove that every map in $(2)$ is a retract of a map in $(2'')$. Suppose given a map

$$
\begin{array}{ccc}
\Lambda^n_0 & \xrightarrow{(f,g)} & \Delta^n \\
A \times B
\end{array}
$$

such that $g|\Delta^{\{0,1\}}$ is a degenerate edge of $B$. Let $j: \Delta^n \hookrightarrow \Delta^n \times \Delta^1$ correspond to the inclusion $\Delta^n \times \{1\} \subseteq \Delta^n \times \Delta^1$. Define the retraction $r: \Delta^n \times \Delta^1 \rightarrow \Delta^n$ as the map induced by the map $r: [n] \times [1] \rightarrow [n]$ of partially ordered sets defined by

$$
r(m,0) = \begin{cases} 
m & \text{if } m \neq 1, \\
0 & \text{if } m = 1
\end{cases}
$$

and by $r(m,1) = m$ for all $m \in [n]$. Observe that the composite map $gr: \Delta^n \times \Delta^1 \rightarrow B$ restricts to a degenerate edge $gr|\Delta^n \times \{i\} \times \Delta^1$ of $B$ for every vertex $i$ of $\Delta^n$; this is clear from the definition of $r$ if $i \neq 1$ and follows from the assumption that $g|\Delta^{\{0,1\}}$ is degenerate when $i = 1$. The maps $j$ and $r$ exhibit the inclusion $\Lambda^n_0 \hookrightarrow \Delta^n$ as a retract of the map

$$
\begin{array}{ccc}
\Lambda^n_0 \times \Delta^1 \cup \Delta^n \times \{0\} & \hookrightarrow & \Delta^n \times \Delta^1 \\
A \times B
\end{array}
$$

in $(\text{Set}_\Delta)/(A \times B)$ with structure map $(f,g)r: \Delta^n \times \Delta^1 \rightarrow A \times B$. From the discussion above, this map belongs to the class of maps $(2'')$. It follows that the weakly saturated
class generated by (1) and (2) is contained in the weakly saturated class generated by (1) and (2′).

We now prove that the weakly saturated class generated by (1) and (2′) is contained in the weakly saturated class generated by (1) and (2). Suppose given a map in (2′) of the form

\[ \partial \Delta^n \times \Delta^1 \cup \Delta^n \times \{0\} \rightarrow A \times B \]

in which the structure map \((f, g): \Delta^n \rightarrow A \times B\) satisfies \(g|\{i\} \times \Delta^1\) is a degenerate edge of \(B\) for every vertex \(i\) of \(\Delta^n\). We have the standard filtration

\[ \partial \Delta^n \times \Delta^1 \cup \Delta^n \times \{0\} = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = \Delta^n \times \Delta^1 \]

in which the inclusion \(X^i \subseteq X^{i+1}\) fits into a pushout diagram of the form

\[ \Lambda^{n+1}_i \rightarrow X^i \]

\[ \downarrow \]

\[ \Delta^{n+1}_i \rightarrow X^{i+1} \]

for every \(i = 0, 1, \ldots, n\). The \((n+1)\)-simplex of \(X^n\) obtained from \(X^{n-1}\) via the attaching map \(\Lambda^{n+1}_0 \rightarrow X^{n-1}\) corresponds to the \((n+1)\)-chain

\[ (0, 0) \]

\[ \downarrow \]

\[ (0, 1) \rightarrow (1, 1) \rightarrow \cdots \rightarrow (n, 1) \]

of \([n] \times [1]\). By assumption the edge \(g|\{0\} \times \Delta^1\) of \(B\) is degenerate. It follows easily that the map above belongs to the weakly saturated class generated by the maps (1) and (2). □

4.3. Stability properties for bifibrations. In this section we prove some stability properties for bifibrations under exponentiation, analogous to the discussion in Section 2.1.2 of [13] for left fibrations.

**Proposition 4.10.** Let \(f: X \rightarrow Y\) be a bifibration in \((\text{Set}_\Delta)/(A \times B)\). Then for any monomorphism \(u: M \rightarrow N\) in \((\text{Set}_\Delta)/(A \times B)\), the induced map

\[ X^N \rightarrow X^M \times_{Y^M} Y^N \]

is a bifibration in \((\text{Set}_\Delta)/(A^N \times B^N)\).

**Proof.** The induced map \(X^N \rightarrow X^M \times_{Y^M} Y^N\) is an inner fibration by Corollary 2.3.2.5. of [13]. We prove that the induced map has the right lifting property against the class of maps (2) from Definition [13]. By Proposition [13] it suffices to prove that the indicated
diagonal filler exists in every commutative diagram of the form

\[
\begin{array}{ccc}
\Delta^1 \times \partial \Delta^n \cup \{0\} \times \Delta^n & \rightarrow & X^N \\
\uparrow & & \downarrow \\
\Delta^1 \times \Delta^n & \rightarrow & X^M \times_{Y^M} Y^N \\
\end{array}
\]

where, if \( u: \Delta^1 \times \Delta^n \rightarrow X^M \times_{Y^M} Y^N \rightarrow A^N \times B^N \) denotes the composite map, then \( \pi_{BN} u|\Delta^1 \times \{i\} \) is a degenerate edge in \( B^N \) for every vertex \( i \) of \( \Delta^n \). By adjointness, it is sufficient to prove that the indicated diagonal filler exists in the commutative diagram

\[
\begin{array}{ccc}
\Delta^1 \times (\partial \Delta^n \times N \cup \Delta^n \times M) \cup \{0\} \times \Delta^n \times N & \rightarrow & X \\
\uparrow & & \downarrow f \\
\Delta^1 \times \Delta^n \times N & \rightarrow & Y. \\
\end{array}
\]

For every vertex \( i \) of \( \Delta^n \) and for every vertex \( v \) of \( N \), the map \( \Delta^1 \times \Delta^n \times N \rightarrow Y \) restricts to an edge \( \Delta^1 \times \{i\} \times \{n\} \rightarrow Y \) of \( Y \) which is mapped to a degenerate edge in \( B \) by \( q: Y \rightarrow B \), where \( (p,q): Y \rightarrow A \times B \) denotes the structure map. Therefore the indicated diagonal filler exists by Proposition 4.9.

The proof that the induced map has the right lifting property against all maps in the class (3) of Definition 4.4 is completely analogous.

Let \( X \) and \( M \) be objects of \((\text{Set}_{\Delta})/(A \times B)\). Recall (Notation 2.1) that \( \text{map}_{A \times B}(M,X) \) denotes the simplicial mapping space for the simplicially enriched category \((\text{Set}_{\Delta})/(A \times B)\).

**Lemma 4.11.** Let \( p: X \rightarrow Y \) be a bifibration in \((\text{Set}_{\Delta})/(A \times B)\). Then for any monomorphism \( u: K \rightarrow L \) in \((\text{Set}_{\Delta})/(A \times B)\), the induced map

\[
\text{map}_{A \times B}(u,p): \text{map}_{A \times B}(L,X) \rightarrow \text{map}_{A \times B}(K,X) \times_{\text{map}_{A \times B}(K,Y)} \text{map}_{A \times B}(L,Y)
\]

is a Kan fibration between Kan complexes.

**Proof.** We first prove that \( \text{map}_{A \times B}(M,X) \) is a Kan complex for any bifibration \( X \rightarrow A \times B \) and any object \( M \) in \((\text{Set}_{\Delta})/(A \times B)\). By Proposition 4.10 the induced map \( X^M \rightarrow A^M \times B^M \) is a bifibration. We have a pullback diagram

\[
\begin{array}{ccc}
\text{map}_{A \times B}(M,X) & \rightarrow & X^M \\
\downarrow & & \downarrow \\
\Delta^0 & \rightarrow & A^M \times B^M \\
\end{array}
\]

where \( \phi \) corresponds to the structure map \( M \rightarrow A \times B \). Hence \( \text{map}_{A \times B}(M,X) \) is a Kan complex by Lemma 4.3.

Next we prove the assertion in the special case that \( p \) is a bifibration \( X \rightarrow A \times B \). Suppose that \( u: K \rightarrow L \) is a monomorphism in \((\text{Set}_{\Delta})/(A \times B)\). The induced map

\[
\text{map}_{A \times B}(L,X) \rightarrow \text{map}_{A \times B}(K,X)
\]

is an inner fibration. Since it is an inner fibration between Kan complexes it suffices by Lemma 2.18 to prove that it has the right lifting property with respect to the inclusion.
$\Delta^0 \subseteq \Delta^1$. By adjointness, the indicated diagonal filler exists in the diagram

$$
\begin{array}{ccc}
\Delta^0 & \longrightarrow & \text{map}_{A \times B}(L, X) \\
\downarrow & & \downarrow \\
\Delta^1 & \longrightarrow & \text{map}_{A \times B}(K, X)
\end{array}
$$

if and only if the indicated diagonal filler exists in the diagram

$$
\begin{array}{ccc}
L \times \Delta^0 \cup K \times \Delta^1 & \longrightarrow & X \\
\downarrow & & \downarrow \\
L \times \Delta^1 & \longrightarrow & A \times B
\end{array}
$$

But for any vertex $v$ of $L$, the map $\{v\} \times \Delta^1 \to A \times B$ is sent to a degenerate edge of $B$ under the projection $A \times B \to B$. The indicated diagonal fillers therefore exist by Proposition 4.9. It follows by Lemma 2.18 that the induced map above is a Kan fibration between Kan complexes.

Finally, we prove the general form of the assertion. Suppose that $p: X \to Y$ is a bifibration and that $u: K \to L$ is a monomorphism. We use Lemma 2.18 again. The map

$$
\text{map}_{A \times B}(L, X) \to \text{map}_{A \times B}(K, X) \times_{\text{map}_{A \times B}(K, Y)} \text{map}_{A \times B}(L, Y)
$$

is an inner fibration between Kan complexes by the results of the preceding paragraphs. Therefore we are reduced to proving that the indicated diagonal filler exists in any commutative diagram of the form

$$
\begin{array}{ccc}
\Delta^0 & \longrightarrow & \text{map}_{A \times B}(L, X) \\
\downarrow & & \downarrow \\
\Delta^1 & \longrightarrow & \text{map}_{A \times B}(K, X) \times_{\text{map}_{A \times B}(K, Y)} \text{map}_{A \times B}(L, Y)
\end{array}
$$

By adjointess, this is equivalent to proving that the indicated diagonal filler exists in the induced diagram

$$
\begin{array}{ccc}
L \times \Delta^0 \cup K \times \Delta^1 & \longrightarrow & X \\
\downarrow & & \downarrow \\
L \times \Delta^1 & \longrightarrow & Y
\end{array}
$$

By Proposition 4.9 using the fact that the composite map $L \times \Delta^1 \to Y \to A \times B$ factors through $L \times \Delta^0$ via the structure map $L \to A \times B$, we see that such a diagonal filler exists. This completes the proof of the Lemma.

4.4. The bivariant model structure. In this section we describe the model structure for bifibrations (see Theorem 4.19).

**Definition 4.12.** A map $Y \to Z$ in $(\text{Set}_\Delta)/(A \times B)$ is said to be a **bivariant equivalence** if the induced map

$$
\text{map}_{A \times B}(Z, X) \to \text{map}_{A \times B}(Y, X)
$$

is a homotopy equivalence between Kan complexes for every bifibration $X \to A \times B$. 
Lemma 4.13. Suppose that \( u: K \to L \) is a bivariant anodyne map. Then \( u \) is a bivariant equivalence.

Proof. Let \( \mathcal{A} \) denote the class of monomorphisms \( u: K \to L \) in \( \text{(Set}_\Delta)/(A \times B) \) such that the induced map

\[
\text{map}_{A \times B}(L, X) \to \text{map}_{A \times B}(K, X)
\]

is a homotopy equivalence for all bifibrations \( X \to A \times B \). By Lemma 4.11, the class \( \mathcal{A} \) is equivalently the class of monomorphisms \( u: K \to L \) in \( \text{(Set}_\Delta)/(A \times B) \) such that the induced map above is a trivial Kan fibration for every bifibration \( X \to A \times B \). It follows easily that \( \mathcal{A} \) is weakly saturated.

To complete the proof we will prove that \( \mathcal{A} \) contains the classes (1), (2) and (3) from Definition 4.4. It is clear that \( \mathcal{A} \) contains the class of inner anodyne maps in \( \text{(Set}_\Delta)/(A \times B) \). We prove that \( \mathcal{A} \) contains the class of maps (2) from Definition 4.4 (the proof that \( \mathcal{A} \) contains the class of maps (3) from Definition 4.4 is completely analogous).

It suffices to prove that \( \mathcal{A} \) contains the class of maps \((2'')\) from Proposition 4.9. Let \( X \to A \times B \) be a bifibration. Let \( u: M \to N \) belong to the class of maps \((2'')\) from Proposition 4.9. We prove that the induced map

\[
\text{map}_{A \times B}(N, X) \to \text{map}_{A \times B}(M, X)
\]

is a trivial Kan fibration. Consider a commutative diagram

\[
\begin{array}{ccc}
\partial \Delta^n & \to & \text{map}_{A \times B}(N, X) \\
\downarrow & & \downarrow \\
\Delta^n & \to & \text{map}_{A \times B}(M, X)
\end{array}
\]

To show that the indicated diagonal filler in this diagram exists, it suffices, by adjointness, to prove that \( X \to A \times B \) has the right lifting property against the canonical map

\[
\partial \Delta^n \times N \cup \Delta^n \times M \to \Delta^n \times N
\]

in \( \text{(Set}_\Delta)/(A \times B) \) where the structure map \( \Delta^n \times N \to A \times B \) factors as \( \Delta^n \times N \xrightarrow{p_2} N \to A \times B \), and where \( N \to A \times B \) is the given structure map of the object \( N \) of \( \text{(Set}_\Delta)/(A \times B) \). It follows easily that the map \((4)\) belongs to the class of maps \((2'')\) from Proposition 4.9 and hence the indicated diagonal filler can be found. \( \square \)

Recall that a fiberwise homotopy between maps \( f, g: X \to Y \) in \( \text{(Set}_\Delta)/(A \times B) \) is an edge in the simplicial set \( \text{map}_{A \times B}(X, Y) \) (see Notation 2.1) between the vertices \( f \) and \( g \). Recall that a map \( h: X \to Y \) in \( \text{(Set}_\Delta)/(A \times B) \) is said to be a fiberwise homotopy equivalence if there exists a map \( k: Y \to X \) in \( \text{(Set}_\Delta)/(A \times B) \) such that the maps \( hk, 1_Y \) and the maps \( kh, 1_X \) are fiberwise homotopic.

We have the following result.

Lemma 4.14. If \( f: X \to Y \) is a fiberwise homotopy equivalence in \( \text{(Set}_\Delta)/(A \times B) \) then \( f \) is a bivariant equivalence. If \( X \to A \times B \) and \( Y \to A \times B \) are bifibrations then the converse is also true.
Proof. To prove the first statement it suffices to prove that if \(h: X \times \Delta^1 \to Y\) is a fiberwise homotopy between maps \(f, g: X \to Y\) in \((\text{Set}_\Delta)/(A \times B)\), then \(h\) induces a homotopy between the maps \(f^*: \text{map}_{A \times B}(Y,Z) \to \text{map}_{A \times B}(X,Z)\) for any bifibration \(Z \to A \times B\). This follows easily from the fact that \(\text{map}_{A \times B}(M \times \Delta^1,Z) = \text{map}_{A \times B}(M,Z)\Delta^1\) for any object \(M\) in \((\text{Set}_\Delta)/(A \times B)\).

We prove the second statement. Suppose that \(f: X \to Y\) is a bivariant equivalence between bifibrations. Observe that the map \(f^*: \text{map}_{A \times B}(Y,X) \to \text{map}_{A \times B}(X,X)\) is a homotopy equivalence between Kan complexes and hence there exists a map \(g: Y \to X\) in \((\text{Set}_\Delta)/(A \times B)\) and an edge \(h\) in \(\text{map}_{A \times B}(X,X)\) between \(gf\) and \(1_X\). Hence \(gf\) and \(1_X\) are fiberwise homotopic. The vertices \(fgf\) and \(f\) belong to the same path component in \(\text{map}_{A \times B}(X,Y)\). Therefore, by the assumption on \(f\), there exists an edge \(k\) in \(\text{map}_{A \times B}(Y,Y)\) between the vertices \(fg\) and \(1_Y\). Hence \(fg\) and \(1_Y\) are fiberwise homotopic. Hence \(f\) is a fiberwise homotopy equivalence. \(\Box\)

**Lemma 4.15.** A trivial fibration in \((\text{Set}_\Delta)/(A \times B)\) is a bivariant equivalence.

Proof. This follows immediately from Lemma 4.14 using the fact that a trivial fibration in \((\text{Set}_\Delta)/(A \times B)\) is a fiberwise homotopy equivalence. \(\Box\)

**Definition 4.16.** A map \(X \to Y\) in \((\text{Set}_\Delta)/(A \times B)\) is said to be a bivariant fibration if it has the right lifting property against all monic bivariant equivalences.

**Proposition 4.17.** Let \(f: X \to Y\) be a map in \((\text{Set}_\Delta)/(A \times B)\) between bifibrations \(X \to A \times B\) and \(Y \to A \times B\). Then \(f\) is a bifibration if and only if \(f\) is a bivariant fibration.

Proof. If \(f: X \to Y\) is a bivariant fibration then it has the right lifting property against every bivariant anodyne map in \((\text{Set}_\Delta)/(A \times B)\) since a bivariant anodyne map is a bivariant equivalence (Lemma 4.13).

We prove the converse. Suppose that \(f: X \to Y\) has the right lifting property against every bivariant anodyne map and that \(X \to A \times B\), \(Y \to A \times B\) are bifibrations. Let \(M \to N\) be a monic bivariant equivalence. We need to show that we can find the indicated diagonal filler in any commutative diagram of the form

\[
\begin{array}{ccc}
M & \longrightarrow & X \\
\downarrow & & \downarrow f \\
N & \longrightarrow & Y \\
\end{array}
\]

From such a diagram we obtain the commutative diagram

\[
\begin{array}{ccc}
\text{map}_{A \times B}(N,X) & \longrightarrow & \text{map}_{A \times B}(N,Y) \\
\downarrow & & \downarrow \\
\text{map}_{A \times B}(M,X) & \longrightarrow & \text{map}_{A \times B}(M,Y)
\end{array}
\]

in which each of the horizontal maps and vertical maps are Kan fibrations between Kan complexes by Lemma 4.11. Since \(M \to N\) is a bivariant equivalence, the vertical maps are in fact trivial Kan fibrations. It follows from Lemma 4.11 that the induced map \(\text{map}_{A \times B}(N,X) \to \text{map}_{A \times B}(M,X) \times_{\text{map}_{A \times B}(M,Y)} \text{map}_{A \times B}(N,Y)\)
is a trivial Kan fibration. In particular it is surjective on vertices which implies the existence of the sought-after diagonal filler in the diagram above.

**Proposition 4.18.** Let $A$ and $B$ be simplicial sets. The subcategory of bivariant equivalences in the category of morphisms $((\text{Set}_\Delta)_{/(A \times B)})^{[1]}$ is an accessible subcategory.

**Proof.** We first prove that if a map $f: X \to Y$ in $(\text{Set}_\Delta)_{/(A \times B)}$ is a bivariant fibration and a bivariant equivalence then it is a trivial fibration. Given such a map $f$, we factor it as $f = pi$ where $i: X \to X'$ is a monomorphism and where $p: X' \to Y$ is a trivial fibration in $(\text{Set}_\Delta)_{/(A \times B)}$. Then we have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i} & X' \\
\downarrow^f & & \downarrow^p \\
Y & & \\
\end{array}
$$

By Lemma 4.15 the map $p$ is a bivariant equivalence, hence $i$ is a bivariant equivalence by 2-out-of-3. Therefore the indicated diagonal filler exists, and hence $f$ is a retract of a trivial fibration. It follows that $f$ is a trivial fibration.

The remainder of the proof proceeds in exactly the same fashion as the proof of Corollary A.2.6.6 of [13]; the small object argument shows the existence of a functor $T: ((\text{Set}_\Delta)_{/(A \times B)})^{[1]} \to ((\text{Set}_\Delta)_{/(A \times B)})^{[1]}$ together with a natural transformation $1 \to T$ such that for any morphism $f: X \to Y$ in $(\text{Set}_\Delta)_{/(A \times B)}$, in the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
T(X) & \xrightarrow{T(f)} & T(Y) \\
\end{array}
$$

the vertical maps are bivariant anodyne maps in $(\text{Set}_\Delta)_{/(A \times B)}$, $T(f): T(X) \to T(Y)$, $T(X) \to A \times B$ and $T(Y) \to A \times B$ are bifibrations. Therefore $T(f)$ is a bivariant fibration. It follows that $f$ is a bivariant equivalence if and only if $T(f)$ is a trivial Kan fibration. Hence the bivariant equivalences in $((\text{Set}_\Delta)_{/(A \times B)})^{[1]}$ form an accessible subcategory since the trivial Kan fibrations form an accessible subcategory.

**Theorem 4.19.** Let $A$ and $B$ be simplicial sets. Then there is the structure of a left proper, combinatorial model category on $(\text{Set}_\Delta)_{/(A \times B)}$ for which

- the class of cofibrations is the class of monomorphisms in $(\text{Set}_\Delta)_{/(A \times B)}$;
- the class of fibrations is the class of bivariant fibrations in $(\text{Set}_\Delta)_{/(A \times B)}$.

The fibrant objects are precisely the bifibrations $X \to A \times B$.

**Proof.** We use Proposition A.2.6.8 from [13]. The category $(\text{Set}_\Delta)_{/(A \times B)}$ is presentable, so therefore we need to verify the conditions (1)–(5) from the statement of that proposition. The conditions (1) and (4) are clear; the condition (3) is Proposition 4.18 and condition (5) follows from Lemma 4.15. Therefore it remains to prove that condition (2) holds. It suffices to prove that if $i: M \to N$ is a morphism in $(\text{Set}_\Delta)_{/(A \times B)}$ which
has the left lifting property against every bivariant fibration in \((\text{Set}_\Delta)/(A \times B)\), then \(i\) is a monic bivariant equivalence. Via the small object argument we can find a commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{f} & X \\
\downarrow i & & \downarrow p \\
N & \xrightarrow{g} & Y
\end{array}
\]

in which \(f\) and \(g\) are bivariant anodyne maps, and \(p: X \to Y\), \(X \to A \times B\), and \(Y \to A \times B\) are bifibrations. It follows that \(p\) is a bivariant fibration (Proposition 4.17) and hence the indicated diagonal filler \(d: N \to X\) exists. Since \(di = f\) is monic it follows that \(i\) is monic. Since \(f\) and \(g\) are bivariant equivalences, \(d\) is also a bivariant equivalence (this follows easily from the fact that if \(u\) is a map of Kan complexes such that there exist maps \(v\) and \(w\) such that \(vu\) and \(uw\) are homotopy equivalences, then \(u\) is also a homotopy equivalence). Thus the class of monic bivariant equivalences forms a weakly saturated class, completing the verification of condition (2). The result follows. \(\square\)

4.5. **Comparison with the covariant and contravariant model structures.** In this section we study some relationships between the bivariant model structure on \((\text{Set}_\Delta)/(A \times B)\) and the contravariant model structure on \((\text{Set}_\Delta)/A\) and the covariant model structure on \((\text{Set}_\Delta)/B\).

Let \(\pi_A: A \times B \to A\) and \(\pi_B: A \times B \to B\) be the canonical projections. Observe that the functor \(\pi_A^*: (\text{Set}_\Delta)/A \to (\text{Set}_\Delta)/(A \times B)\) admits a left adjoint \((\pi_A)!: (\text{Set}_\Delta)/(A \times B) \to (\text{Set}_\Delta)/A\) and a right adjoint \((\pi_A)_*: (\text{Set}_\Delta)/(A \times B) \to (\text{Set}_\Delta)/A\). Similarly the functor \(\pi_B^*: (\text{Set}_\Delta)/B \to (\text{Set}_\Delta)/(A \times B)\) admits a left adjoint \((\pi_B)!: (\text{Set}_\Delta)/(A \times B) \to (\text{Set}_\Delta)/B\) and a right adjoint \((\pi_B)_*: (\text{Set}_\Delta)/(A \times B) \to (\text{Set}_\Delta)/B\). We have the following proposition.

**Proposition 4.20.** Let \(A\) and \(B\) be simplicial sets. Then the following statements are true:

1. The adjunction

\[
(\pi_B)!: (\text{Set}_\Delta)/(A \times B) \rightleftarrows (\text{Set}_\Delta)/B: \pi_B^*
\]

is a Quillen adjunction for the bivariant model structure on \((\text{Set}_\Delta)/(A \times B)\) and the contravariant model structure on \((\text{Set}_\Delta)/B\);

2. The adjunction

\[
(\pi_A)!: (\text{Set}_\Delta)/(A \times B) \rightleftarrows (\text{Set}_\Delta)/A: \pi_A^*
\]

is a Quillen adjunction for the bivariant model structure on \((\text{Set}_\Delta)/(A \times B)\) and the covariant model structure on \((\text{Set}_\Delta)/A\).

**Proof.** We prove statement (1), the proof of statement (2) follows by duality. It is clear that \(\pi_B^*\) sends trivial Kan fibrations in \((\text{Set}_\Delta)/B\) to trivial Kan fibrations in \((\text{Set}_\Delta)/(A \times B)\). Therefore it suffices by Proposition 4.17 and Remark 4.2 to prove that if \(X \to Y\) is a right fibration in \((\text{Set}_\Delta)/B\) between right fibrations \(X \to B\) and \(Y \to B\), then \(A \times X \to A \times Y\) is a bifibration in \((\text{Set}_\Delta)/(A \times B)\). Clearly \(A \times X \to A \times Y\) satisfies (1) and (3) of Remark 4.14 therefore it suffices to prove that the indicated diagonal filler
exists in every diagram of the form

\[
\begin{array}{ccc}
\Lambda^n_0 & \longrightarrow & A \times X \\
\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & A \times Y \\
(f,g) & \longleftarrow & A \times B
\end{array}
\]

in which \( g: \Delta^{\{0,1\}} \to B \) is a degenerate edge of \( B \). Clearly it suffices to show that the indicated diagonal filler exists in the induced diagram

\[
\begin{array}{ccc}
\Lambda^n_0 & \longrightarrow & X \\
\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & Y \\
\end{array}
\]

in \((\mathbb{S}et_\Delta)/B\). If \( n = 1 \) then the indicated diagonal filler in (6) exists since the arrow \( v: \Delta^1 \to Y \) factors through the Kan complex \( Y_{g(n)} \). If \( n \geq 2 \), an argument analogous to the one used in Remark 4.2 gives the existence of the required diagonal filler. □

Remark 4.21. There is a canonical functor \((\mathbb{S}et_\Delta)/A \times (\mathbb{S}et_\Delta)/B \to (\mathbb{S}et_\Delta)/(A \times B)\) defined on objects by sending a pair \((S,T)\) to the product \( S \times T \to A \times B \). This functor is divisible on the right and left (see Section 7 of [12]). If \( T \to B \) is an object of \((\mathbb{S}et_\Delta)/B\), then we denote the functor right adjoint to \((-) \times T: (\mathbb{S}et_\Delta)/A \to (\mathbb{S}et_\Delta)/(A \times B)\) by \((-)/T: (\mathbb{S}et_\Delta)/(A \times B) \to (\mathbb{S}et_\Delta)/A\). The functor \((-)/T\) is defined on objects as follows. If \( X \to A \times B \) is an object of \((\mathbb{S}et_\Delta)/(A \times B)\), then \( X/T \) is defined by the pullback diagram

\[
\begin{array}{ccc}
X/T & \longrightarrow & X^T \\
\downarrow & & \downarrow \\
A & \longrightarrow & A^T \times B^T,
\end{array}
\]

where the map \( A \to A^T \times B^T \) is isomorphic to the product of the diagonal map \( A \to A^T \) and the constant map \( \Delta^0 \to B^T \) given by the structure map \( T \to B \).

Remark 4.22. Suppose that \( X \to Y \) is a map in \((\mathbb{S}et_\Delta)/(A \times B)\) and that \( S \to T \) is a map in \((\mathbb{S}et_\Delta)/B\). There is a canonical commutative diagram in \((\mathbb{S}et_\Delta)/A\) of the form

\[
\begin{array}{ccc}
X/T & \longrightarrow & Y/T \\
\downarrow & & \downarrow \\
X/S & \longrightarrow & Y/S
\end{array}
\]

and an induced map

\[
X/T \to X/S \times_{Y/S} Y/T
\]

in \((\mathbb{S}et_\Delta)/A\).

The following lemma gives a sufficient criterion for the induced map from Remark 4.22 to be a right fibration.
Lemma 4.23. Let \( X \to Y \) be a bifibration in \((\text{Set})_{/(A \times B)}\) and \( S \to T \) be a monomorphism in \((\text{Set})_{/B}\). Then the induced map
\[
X/T \to X/S \times_{Y/S} Y/T
\]
is a left fibration in \((\text{Set})_{/A}\).

Proof. This follows from Remark 4.8 using the fact that
\[
X/T \to X/S \times_{Y/S} Y/T
\]
is a bifibration in \((\text{Set})_{/(AT \times BT)}\) by Proposition 4.10. \(\square\)

As an application of this lemma we have the following useful proposition.

Proposition 4.24. Let \( T \to B \) be an object of \((\text{Set})_{/B}\). Then the adjoint pair
\[
(-) \times T : (\text{Set})_{/A} \rightleftarrows (\text{Set})_{/(A \times B)} : (-)/T
\]
is a Quillen adjunction for the covariant model structure on \((\text{Set})_{/A}\) and the bivariant model structure on \((\text{Set})_{/(A \times B)}\).

Proof. It is clear that the functor \((-)/T\) preserves trivial fibrations. Therefore it suffices to prove that \((-)/T\) preserves fibrations between fibrant objects. Hence it suffices to prove that if \( X \to Y \) is a bifibration in \((\text{Set})_{/(A \times B)}\), then \( X/T \to Y/T \) is a left fibration in \((\text{Set})_{/A}\). This follows immediately from Lemma 4.23, taking \( S = \emptyset \). \(\square\)

In particular, taking \( T \to B \) to be the identity map \( \text{id}_B : B \to B \), we see that the functor \( \pi_A^* : (\text{Set})_{/A} \to (\text{Set})_{/(A \times B)} \) is left Quillen for the covariant model structure on \((\text{Set})_{/A}\) and the bivariant model structure on \((\text{Set})_{/(A \times B)}\). An analogous statement is true for the functor \( \pi_B^* \). We record this observation in the following proposition.

Proposition 4.25. Let \( A \) and \( B \) be simplicial sets. Then the following statements are true:

1. The adjunction
\[
\pi_B^* : (\text{Set})_{/B} \rightleftarrows (\text{Set})_{/(A \times B)} : (\pi_B)^*
\]
is a Quillen adjunction for the bivariant model structure on \((\text{Set})_{/(A \times B)}\) and the contravariant model structure on \((\text{Set})_{/B}\):

2. The adjunction
\[
\pi_A^* : (\text{Set})_{/A} \rightleftarrows (\text{Set})_{/(A \times B)} : (\pi_A)^*
\]
is a Quillen adjunction for the bivariant model structure on \((\text{Set})_{/(A \times B)}\) and the covariant model structure on \((\text{Set})_{/A}\).

4.6. Bivariant equivalences. In this section we establish some useful facts about bivariant equivalences.

Proposition 4.26. Let \( A \) and \( B \) be simplicial sets. Let \( f : X \to Y \) be a bifibration in \((\text{Set})_{/(A \times B)}\). If the fibers of \( f \) are contractible then \( f \) is a trivial Kan fibration.
Proof. We need to prove that the map $f$ has the right lifting property against the inclusion $\partial \Delta^n \subseteq \Delta^n$ for all $n \geq 0$. This is clear when $n = 0$, since the fibers of $f$ are non-empty. Suppose $n > 0$. Consider a commutative diagram of the form

$$
\begin{array}{ccc}
\partial \Delta^n & \xrightarrow{\psi} & X \\
\downarrow & & \downarrow f \\
\Delta^n & \xrightarrow{\lambda} & Y
\end{array}
$$

We want to show that the dotted arrow exists making the diagram commute. By a base change we may suppose that $A = B = \Delta^n$, $Y = \Delta^n$ and that the structure map $Y \to A \times B$ is the diagonal inclusion $\Delta^n \to \Delta^n \times \Delta^n$.

Let $h : \Delta^n \times [0] \to \partial \Delta^n$ denote the canonical projection. Let $k : \Delta^n \times [1] \to \Delta^n$ be the canonical contraction of $\Delta^n$ onto its final vertex so that $k|\Delta^n \times \{0\} = \text{id}_{\Delta^n}$ and $k|\Delta^n \times \{1\}$ is the constant map on the final vertex.

The inclusion $\partial \Delta^n \times \{0\} \to \partial \Delta^n \times [1] \in (\text{Set}_\Delta)_{/A \times B}$ with structure map $\lambda := (h,k)|\partial \Delta^n \times [1]$ is a bivariant anodyne map in $(\text{Set}_\Delta)_{/A \times B}$ by Proposition 4.9 with $K = \emptyset$, $L = \partial \Delta^n$. Hence we can find the indicated diagonal filler in the diagram

$$
\begin{array}{ccc}
\partial \Delta^n \times \{0\} & \xrightarrow{\phi} & X \\
\downarrow & & \downarrow f \\
\partial \Delta^n \times [1] & \xrightarrow{\lambda} & Y
\end{array}
$$

Observe that $\phi|\partial \Delta^n \times \{1\}$ has image inside the contractible Kan complex $X|\{n\}$. Hence we may extend $\phi|\partial \Delta^n \times \{1\}$ to a map $\tilde{\phi} : \Delta^n \times \{1\} \to X$. We have a commutative diagram

$$
\begin{array}{ccc}
\partial \Delta^n \times [1] \cup \Delta^n \times \{1\} & \xrightarrow{\psi'} & X \\
\downarrow & & \downarrow f \\
\Delta^n \times [1] & \xrightarrow{\psi''} & \Delta^n \times \Delta^n
\end{array}
$$

in which $\psi' = \phi \cup \tilde{\phi}$. Observe that $\psi|\{n\} \times [1]$ is an equivalence in $X|\{n\}$ and hence in $X$. It follows from Proposition 2.4.1.5 and Proposition 2.4.1.8 of [13] that there exists a diagonal filler $\psi'' : \Delta^n \times [1] \to X$ for this diagram. The restriction $\psi''|\Delta^n \times \{0\}$ is then the desired extension of the original map $\psi$. \qed

Theorem 4.27. Let $A$ and $B$ be simplicial sets and let $X \to A \times B$ and $Y \to A \times B$ be bifibrations. Then a map $f : X \to Y$ in $(\text{Set}_\Delta)_{/A \times B}$ is a bivariant equivalence if and only if it is a pointwise weak homotopy equivalence.

Proof. ($\Rightarrow$) Suppose $f : X \to Y$ is a bivariant equivalence between bifibrations. By Lemma 4.14 $f : X \to Y$ is a fiberwise homotopy equivalence. Therefore $f$ is a pointwise weak homotopy equivalence, since every fiberwise homotopy equivalence is a pointwise homotopy equivalence.

($\Leftarrow$) Let $f : X \to Y$ be a pointwise weak homotopy equivalence. Suppose first that $f : X \to Y$ is a bivariant fibration. Then $f$ is a trivial fibration by Proposition 4.26 since the fibers of $f$ are contractible. Hence $f$ is a bivariant equivalence (Lemma 4.13).

Now suppose that $f$ is an arbitrary pointwise weak homotopy equivalence between
bifibrations. Via the small object argument, we may factor \( f = hg \), where \( h: X' \to Y \) is a bifibration and where \( g: X \to X' \) is a bivariant anodyne map. Then \( g \) is a bivariant equivalence by Lemma 4.13 since it is a bivariant equivalence between bifibrations it is a pointwise weak homotopy equivalence by the forward implication proved above. Hence \( h \) is a pointwise weak homotopy equivalence. By Proposition 4.17 we see that \( h \) is a bivariant fibration. Hence it is a bivariant equivalence by the special case we have proven above.

The following characterization of bivariant equivalences is anticipated by Theorem 2.8. This characterization is due to Joyal.

**Theorem 4.28 (Joyal).** Let \( A \) and \( B \) be simplicial sets and let \( f: X \to Y \) be a map in \((\text{Set}_\Delta)_{(A \times B)}\). The following statements are equivalent:

(i) the map \( f: X \to Y \) is a bivariant equivalence;
(ii) if \( R \to A \) is a right fibration and \( L \to B \) is a left fibration then the induced map
\[
R \times_A X \times_B L \to R \times_A Y \times_B L
\]
is a weak homotopy equivalence.
(iii) for every pair of vertices \( a \in A \) and \( b \in B \), if \( \{a\} \to A \) into a right anodyne map followed by a right fibration, and if \( \{b\} \to L \to B \) is a factorization of \( \{b\} \to B \) into a left anodyne map followed by a left fibration, then the induced map
\[
Ra \times_A X \times_B Lb \to Ra \times_A Y \times_B Lb
\]
is a weak homotopy equivalence.

**Proof.** The proof that (ii) implies (iii) is trivial. We prove that (i) implies (ii). Suppose that \( f: X \to Y \) is a bivariant equivalence in \((\text{Set}_\Delta)_{(A \times B)}\). As in the proof of Proposition 4.18 above, we can find a commutative diagram in \((\text{Set}_\Delta)_{(A \times B)}\) of the form
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow{f'} & Y'
\end{array}
\]
in which the vertical arrows are bivariant anodyne maps and \( f': X' \to Y' \) is a trivial Kan fibration. It follows that without loss of generality, we may suppose that \( f: X \to Y \) is a bivariant anodyne map.

Therefore we will prove that if \( f: X \to Y \) is a bivariant anodyne map then the induced map \( R \times_A X \times_B L \to R \times_A Y \times_B L \) is a weak homotopy equivalence. The class of all maps \( X \to Y \) in \((\text{Set}_\Delta)_{(A \times B)}\) with this property is weakly saturated. Therefore it suffices to show that this class contains all maps of the form (1), (2) and (3) from Definition 4.4. It is clear, using Proposition 3.3.1.3 from [13] that this class contains all maps of the form (1). By a duality argument, it suffices to prove this for all maps of the form (2) from Definition 4.4.

By a base-change argument we may suppose that \( A = \Delta^n \). Let \( R \to A \) be a right fibration. We will prove the following statement: if \( \Lambda^n_0 \to \Delta^n \) is a map of the form (2) in \((\text{Set}_\Delta)_{(A \times B)}\) from Definition 4.4 then the image of \( R \times_A \Lambda^n_0 \to R \times_A \Delta^n \) under
\((\pi_B)! : (\text{Set}_\Delta)/(R \times B) \to (\text{Set}_\Delta)/B\) is a contravariant equivalence in \((\text{Set}_\Delta)/B\). Applying Theorem 2.3 we deduce that \(R \times_A \Delta^{(0)} \times_B L \to R \times_A \Delta^1 \times_B L\) is a weak homotopy equivalence.

We use the theory of mapping simplexes (see Section 3.2.2 of [13]). There is a sequence \(\phi : A^n \to \cdots \to A^1 \to A^0\) of composable morphisms between Kan complexes and a quasi-equivalence \(M(\phi) \to R\) (see Definition 3.2.2.6 of [13]). We have a pullback diagram of the form

\[
\begin{array}{ccc}
M(\phi)|_{\Lambda_0^n} & \longrightarrow & R|_{\Lambda_0^n} \\
\downarrow & & \downarrow \\
M(\phi) & \longrightarrow & R
\end{array}
\]

in which the horizontal maps are categorical equivalences by Proposition 3.2.2.10 of [13]. Therefore, it suffices to prove that \((\pi_B)! M(\phi)|_{\Lambda_0^n} \to (\pi_B)! M(\phi)\) is a contravariant equivalence in \((\text{Set}_\Delta)/B\), since every categorical equivalence is a contravariant equivalence. But the map \(M(\phi)|_{\Lambda_0^n} \to M(\phi)\) forms part of a pushout diagram

\[
\begin{array}{ccc}
A^n \times \Lambda_0^n & \longrightarrow & M(\phi) \times_{\Delta^n} \Lambda_0^n \\
\downarrow & & \downarrow \\
A^n \times \Delta^n & \longrightarrow & M(\phi)
\end{array}
\]

and hence is bivariant anodyne, since it is the pushout of the bivariant anodyne map \(A^n \times \Lambda_0^n \to A^n \times \Delta^n\). This suffices to complete the proof, by (1) of Proposition 4.20.

To see that the diagram (7) above is a pushout, observe that from the proof of Proposition 3.2.2.10 from [13] we have a pushout diagram of the form

\[
\begin{array}{ccc}
A^n \times \Delta^{\{0,\ldots,n-1\}} & \longrightarrow & M(\phi') \\
\downarrow & & \downarrow \\
A^n \times \Delta^n & \longrightarrow & M(\phi)
\end{array}
\]

where \(\phi'\) denotes the composable sequence \(\phi' : A^{n-1} \to \cdots \to A^1 \to A^0\). It follows that the top square and the outer square in the composite diagram

\[
\begin{array}{ccc}
A^n \times \Delta^{\{0,\ldots,n-1\}} & \longrightarrow & M(\phi') \times_{\Delta^n} \Lambda_0^n \simeq M(\phi') \\
\downarrow & & \downarrow \\
A^n \times \Lambda_0^n & \longrightarrow & M(\phi) \times_{\Delta^n} \Lambda_0^n \\
\downarrow & & \downarrow \\
A^n \times \Delta^n & \longrightarrow & M(\phi)
\end{array}
\]

are pushouts, and hence so is the diagram (7).
Finally, suppose that (iii) holds; we will prove that (i) holds, i.e. \( f \) is a bivariant equivalence. Via the small object argument, we may find a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X' & \rightarrow & Y'
\end{array}
\]

in which the vertical maps are bivariant anodyne maps and \( X', Y' \) are bifibrations with structure maps \((p_{X'}, q_{X'}): X' \rightarrow A \times B, (p_{Y'}, q_{Y'}): Y' \rightarrow A \times B\) respectively. By Theorem [1.27] it suffices to show that \( X' \rightarrow Y' \) is a pointwise homotopy equivalence. Let \( a \in A \) and \( b \in B \) be vertices, and let \( \{ a \} \rightarrow Ra \rightarrow A, \{ b \} \rightarrow Lb \rightarrow B \) be factorizations of \( \{ a \} \rightarrow A \) and \( \{ b \} \rightarrow B \) into a right anodyne map followed by a right fibration, and a left anodyne map followed by a left fibration respectively. We claim that the canonical maps

\[
\{ a \} \times_A X' \times_B \{ b \} \rightarrow Ra \times_A X' \times_B Lb
\]

and

\[
\{ a \} \times_A Y' \times_B \{ b \} \rightarrow Ra \times_A Y' \times_B Lb
\]

are weak homotopy equivalences. The first map above factors as

\[
\{ a \} \times_A X' \times_B \{ b \} \rightarrow Ra \times_A X' \times_B \{ b \} \rightarrow Ra \times_A X' \times_B Lb
\]

The map \( \{ a \} \times_A X' \times_B \{ b \} \rightarrow Ra \times_A X' \times_B \{ b \} \) is right anodyne (and hence a weak homotopy equivalence) since \( X' \times_B \{ b \} \rightarrow A \times \{ b \} \rightarrow A \) is a bifibration and hence the composite map \( X' \times_B \{ b \} \rightarrow A \times \{ b \} \rightarrow A \) is smooth (Proposition 4.1.2.15 of [13]). Similarly, \( Ra \times_A X' \rightarrow Ra \times B \) is a bifibration, and hence a duality argument using Proposition 4.1.2.15 of [13] again shows that the induced map \( Ra \times_A X' \times_B \{ b \} \rightarrow Ra \times_A X' \times_B Lb \) is left anodyne (and hence is a weak homotopy equivalence). It follows that the first canonical map above is a weak homotopy equivalence. The proof that the second canonical map above is a weak homotopy equivalence is completely analogous.

Therefore, under the hypothesis that (iii) holds, we see that \( X' \rightarrow Y' \) is a pointwise weak homotopy equivalence (and hence \( f \) is a bivariant equivalence) if and only if the two vertical maps

\[
Ra \times_A X \times_B Lb \rightarrow Ra \times_A X' \times_B Lb \quad \text{and} \quad Ra \times_A Y \times_B Lb \rightarrow Ra \times_A Y' \times_B Lb
\]

are weak homotopy equivalences. This follows from the implication (i) implies (ii), which we have already proven. \( \square \)

The following is a very useful example of a bivariant equivalence.

**Lemma 4.29.** Let \( A \) and \( B \) be simplicial sets. For every \( n \geq 0 \) and for every map \( \Delta^n \rightarrow A \times B \) in \((\text{Set}_\Delta)/(A \times B)\), the diagonal map \( \Delta^n \rightarrow \Delta^n \times \Delta^n \) in \((\text{Set}_\Delta)/(A \times B)\) is a bivariant equivalence.
Proof. We use the characterization of bivariant equivalences from Theorem 4.28. Let $a \in A$ and $b \in B$ be vertices. We have a commutative diagram

$$
\begin{array}{ccc}
A_a \times_A \Delta^n \times B_b/ & \longrightarrow & A_a \times_A \Delta^n \times \Delta^n \times B_b/ \\
\downarrow & & \downarrow \\
A_a \times_A \Delta^n & \longrightarrow & A_a \times_A \Delta^n \times \Delta^n \\
\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & \Delta^n \times \Delta^n
\end{array}
$$

in which both squares are pullbacks. It suffices to prove that the middle horizontal map is right anodyne, since the base change of a right anodyne map along a left fibration is right anodyne, and hence a weak homotopy equivalence. Since a dominant map is both left and right cofinal (Remark 2.15), and dominant maps are preserved under base change along right fibrations (Lemma 2.11), the result follows from the fact that $\Delta^n \rightarrow \Delta^n \times \Delta^n$ is dominant (Lemma 2.17).

□

Remark 4.30. Note that in general a map in $(\text{Set}_\Delta)/(A \times B)$ of the form $\Delta^n \rightarrow \Delta^n \times \Delta^n$ is not bivariant anodyne.

Proposition 4.31. Suppose $X \rightarrow Y$ is a categorical equivalence in $(\text{Set}_\Delta)/(A \times B)$. Then $X \rightarrow Y$ is a bivariant equivalence.

Proof. We use Theorem 4.28. Let $R \rightarrow A$ be a right fibration and let $L \rightarrow B$ be a left fibration. Since $X \rightarrow Y$ is a categorical equivalence, it follows that $R \times_A X \rightarrow R \times_A Y$ is a categorical equivalence by Proposition 3.3.1.3 of [13]. The induced map $R \times_A Y \times_B L \rightarrow R \times_A Y$ is a left fibration and hence $R \times_A X \times_B L \rightarrow R \times_A Y \times_B L$ is a categorical equivalence by Proposition 3.3.1.3 of [13] again. Hence the map $R \times_A X \times_B L \rightarrow R \times_A Y \times_B L$ is a weak homotopy equivalence.

□

The following corollaries are straightforward and are left to the reader.

Corollary 4.32. Let $A$ and $B$ be simplicial sets and let $(p,q): X \rightarrow A \times B$ be a bifibration. Then $(p,q)$ is a bivariant equivalence.

Corollary 4.33. The bivariant model structure on $(\text{Set}_\Delta)/(A \times B)$ is a left Bousfield localization of the Joyal model structure on $(\text{Set}_\Delta)/(A \times B)$.

Remark 4.34. We could have obtained the bivariant model structure by taking a left Bousfield localization of the Joyal model structure on $(\text{Set}_\Delta)/(A \times B)$ at the set of horn inclusions $\Lambda^n_0 \subseteq \Delta^n$ and $\Lambda^n_n \subseteq \Delta^n$ in $(\text{Set}_\Delta)/(A \times B)$ of the form described in Definition 4.1. However, this approach would require us to prove that every bifibration is a categorical fibration. This is straightforward to prove if $A$ and $B$ are $\infty$-categories, but it is not a priori obvious for arbitrary simplicial sets $A$ and $B$.

4.7. Quillen equivalences. Recall that there is another subdivision functor for simplicial sets introduced in [6]. This functor $sd_2: \text{Set}_\Delta \rightarrow \text{Set}_\Delta$ is defined by composing a functor $X: \Delta^{op} \rightarrow \text{Set}$ with the diagonal functor

$$\delta: \Delta \rightarrow \Delta, \quad [n] \mapsto [n] \ast [n].$$
Just as in the earlier case for Segal’s subdivision functor in Section 3.3, the functor $sd_2$ can be used to relate the category $(\text{Set}_{\Delta})/(A \times B)$ with the category of correspondences from $A$ to $B$. The functor $\delta$ induces a functor

$$\delta: \Delta_{\text{op}}/(A \times B) \to (\text{Set}_{\Delta})/B \ast A$$

which sends a pair $(u, v): \Delta^n \to A \times B$ to

$$\delta(u, v) = v \ast u: \Delta^n \ast \Delta^n \to B \ast A.$$

The functor $\delta$ induces an adjunction

$$\delta_1: (\text{Set}_{\Delta})/(A \times B) \rightleftarrows (\text{Set}_{\Delta})/B \ast A: \delta^*$$

and in fact the functor $\delta^*$ has a further right adjoint $\delta_*: (\text{Set}_{\Delta})/(A \times B) \to (\text{Set}_{\Delta})/B \ast A$.

Composing the functor $\delta_!$ with the reflector $L: (\text{Set}_{\Delta})/B \ast A \to \text{Corr}(A, B)$ (see Remark 4.35) gives rise to a functor denoted $d_! : (\text{Set}_{\Delta})/(A \times B) \to \text{Corr}(A, B)$. We have an adjoint pair

$$d_! : (\text{Set}_{\Delta})/(A \times B) \rightleftarrows \text{Corr}(A, B): d^*.$$

Similarly we have an adjoint pair

$$(8) \quad d^* : \text{Corr}(A, B) \rightleftarrows (\text{Set}_{\Delta})/(A \times B): d_*$$

where $d_*$ is the functor which sends an object $X \to A \times B$ in $(\text{Set}_{\Delta})/(A \times B)$ to the correspondence $d_*X$ whose set of $n$-simplices is

$$(d_*X)_n = (\text{Set}_{\Delta})/(A \times B)(d^*L(\Delta^n), X)$$

where $\Delta^n \to B \ast A$ is an $n$-simplex.

**Remark 4.35.** Analogous to Lemma 3.14 if $f: C \to A$ and $g: D \to B$ are maps of simplicial sets determining objects $D \ast C$ and $C \times D$ of $(\text{Set}_{\Delta})/B \ast A$ and $(\text{Set}_{\Delta})/(A \times B)$ respectively, then we have $\delta^*(D \ast C) = C \times D$. Note also that, analogous to Remark 3.15 the functor $\delta^*: (\text{Set}_{\Delta})/B \ast A \to (\text{Set}_{\Delta})/(A \times B)$ sends the object $D \sqcup C$ to the initial object $\emptyset$ of $(\text{Set}_{\Delta})/(A \times B)$. It follows that there is a natural isomorphism of functors $d^*d_! \simeq \delta^*\delta_!$.

**Remark 4.36.** The relationship between the subdivision functor $sd_2$ from [6] and the functor $d^*$ is as follows. If $X \in \text{Corr}(A, B)$ then there is a pullback diagram of the form

$$\begin{array}{ccc}
d^*X & \longrightarrow & sd_2X \\
| & | & |
\downarrow & & \downarrow \\
A \times B & \longrightarrow & B \times A & \longrightarrow & X \times X
\end{array}$$

where the map $A \times B \to B \times A$ is the switch map which interchanges the two factors and where the map $B \times A \to X \times X$ is induced by the inclusions $B \subseteq X$ and $A \subseteq X$. This is analogous to the relationship between $a^*X$ and the twisted arrow category, or Segal edgewise subdivision of $X$ (see Remark 3.21).

**Remark 4.37.** Recall (see Lemma 1.1 of [6] and Proposition (A.1) of [16]) that for any simplicial set $X$ there are natural isomorphisms $|sd_2X| \simeq |X|$ and $|\text{Tw}(X)| \simeq |X|$ on geometric realizations. In particular there is an isomorphism $|\text{Tw}(X)| \simeq |sd_2X|$, natural in $X$. Recall also that there is a canonical isomorphism $|X^{op}| \simeq |X|$ between the
geometric realization of a simplicial set, and the geometric realization of the opposite simplicial set. We claim that the following diagram commutes

\[
\begin{array}{ccc}
|Tw(X)| & \longrightarrow & |X^{op}| \times |X| \\
\downarrow & & \downarrow \\
|sd_2 X| & \longrightarrow & |X| \times |X|
\end{array}
\]

where the left hand vertical map is the isomorphism mentioned above, and the right hand vertical map is the product of the canonical isomorphism \(|X^{op}| \simeq |X|\) and the identity map on \(|X|\). The map \(|Tw(X)| \to |X^{op}| \times |X|\) is induced by the inclusions \([n]^{op} \subseteq [n]^{op} \times [n]\) and \([n] \subseteq [n]^{op} \times [n]\). Similarly the map \(|sd_2 X| \to |X| \times |X|\) is induced by the two canonical inclusions \([n] \subseteq [n] \times [n]\).

Since all of the functors involved commute with colimits, it suffices by naturality to prove the claim in the special case when \(X = \Delta^n\). Since all of the functors involved also commute with finite products, and \(\Delta^n\) is a retract of \((\Delta^1)^n\), it suffices to prove the statement when \(X = \Delta^1\).

Under the isomorphism \(|Tw(\Delta^1)| \to |\Delta^1| \simeq [0, 1]\) (Proposition (A.1) of [16]), the induced map \(|Tw(\Delta^1)| \to |(\Delta^1)^{op}| \times |\Delta^1|\) corresponds to the map \((f, g): [0, 1] \to [0, 1] \times [0, 1]\) where

\[
f(t) = \begin{cases} 1 - 2t & \text{if } 0 \leq t \leq 1/2, \\ 0 & \text{if } 1/2 \leq t \leq 1, \end{cases} \quad \text{and} \quad g(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1/2, \\ 2t - 1 & \text{if } 1/2 \leq t \leq 1. \end{cases}
\]

Under the isomorphism \(|sd_2 \Delta^1| \to |\Delta^1| \simeq [0, 1]\) (Lemma 1.1 of [6]), the induced maps \(|sd_2 \Delta^1| \to |\Delta^1| \times |\Delta^1|\) corresponds to the map \((f', g'): [0, 1] \to [0, 1] \times [0, 1]\) where

\[
f'(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq 1/2, \\ 1 & \text{if } 1/2 \leq t \leq 1, \end{cases} \quad \text{and} \quad g'(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1/2, \\ 2t - 1 & \text{if } 1/2 \leq t \leq 1. \end{cases}
\]

Under the identification \(|\Delta^1| \simeq [0, 1]\), the isomorphism \(|(\Delta^1)^{op}| \to |\Delta^1|\) corresponds to the automorphism of \([0, 1]\) which sends \(t\) to \(1 - t\). Clearly the composite of \(f\) with this automorphism is equal to \(f'\). The claim follows.

Our first aim is to prove that the adjunction \([\mathbb{S}]\) is a Quillen adjunction. We have the following result.

**Proposition 4.38.** Let \(A\) and \(B\) be \(\infty\)-categories. Then the adjunction

\[
d^*: \text{Corr}(A, B) \rightleftarrows (\text{Set}_\Delta)_{/(A \times B)}: d_*
\]

is a Quillen adjunction for the correspondence model structure on \(\text{Corr}(A, B)\) and the bivariant model structure on \((\text{Set}_\Delta)_{/(A \times B)}\).

**Proof.** Proving the proposition quickly reduces to proving the following claim: the functor \(\delta^*: (\text{Set}_\Delta)_{/B \times A} \to (\text{Set}_\Delta)_{/(A \times B)}\) sends inner anodyne maps in \((\text{Set}_\Delta)_{/B \times A}\) to bivariant anodyne maps in \((\text{Set}_\Delta)_{/(A \times B)}\). Recall from the proof of Theorem 3.23 that the inner anodyne maps in \((\text{Set}_\Delta)_{/B \times A}\) are of the following form:

- \(\Lambda_k^m \ast \emptyset \to \Delta^m \ast \emptyset\), \(0 < k < n\),
- \(\Delta^m \ast \Lambda_k^n \cup \partial \Delta^m \ast \Delta^n \to \Delta^m \ast \Delta^n\), \(m \geq 0, 0 \leq k < n\),
- \(\Lambda_k^m \ast \Delta^n \cup \Delta^m \ast \partial \Delta^n \to \Delta^m \ast \Delta^n\), \(0 < k \leq m, n \geq 0\),
for some simplices $x: \Delta^m \to B$ and $y: \Delta^n \to A$. It follows from Remark 4.35 that it suffices to prove the following two statements: the canonical map
\[ \Lambda^k_n \times \Delta^m \cup \delta \Delta^m \to \Delta^n \times \Delta^m \]
is a bivariant anodyne map in $(\text{Set}_\Delta)/(A \times B)$ if $m \geq 0$ and $0 \leq k < n$; and the canonical map
\[ \Delta^n \times \Lambda^m_k \cup \partial \Delta^n \times \Delta^m \to \Delta^n \times \Delta^m \]
is a bivariant anodyne map in $(\text{Set}_\Delta)/(A \times B)$ if $0 < k \leq m$ and $n \geq 0$. The first statement follows readily from Lemma 4.23 and the second statement follows by duality. \qed

Our next aim is to prove that the Quillen adjunction from Proposition 4.38 is in fact a Quillen equivalence. We first need a preliminary result.

**Lemma 4.39.** Let $A$ and $B$ be simplicial sets and let $X \to A \times B$ be a bifibration in $(\text{Set}_\Delta)/(A \times B)$. Then the counit map $d^*d_*X \to X$ is a trivial Kan fibration.

**Proof.** By adjointness it suffices to prove that the induced map
\[ d^*d_! \partial \Delta^n \to d^*d_! \Delta^n \]
is an acyclic cofibration in the bivariant model structure for every boundary inclusion $\partial \Delta^n \subseteq \Delta^n$ in $(\text{Set}_\Delta)/(A \times B)$.

The unit map $\Delta^n \to d^*d_! \Delta^n$ is a bivariant equivalence (Lemma 4.29). Therefore, by a 2-out-of-3 argument, it suffices to prove that the unit map $S \to d^*d_! S$ is a monic bivariant equivalence for every object $S \to A \times B$ in $(\text{Set}_\Delta)/(A \times B)$. Recall (Remark 4.35) that there is an isomorphism $d^*d_! S = \delta^* \delta_! S$.

Using the skeletal filtration of $S$ we see that by an induction argument we are reduced to the case where $S$ is obtained from $S'$ by adjoining a single $n$-simplex along an attaching map $\partial \Delta^n \to S'$ in $(\text{Set}_\Delta)/(A \times B)$. We have a commutative diagram in $(\text{Set}_\Delta)/(A \times B)$ of the form
\[ \begin{array}{ccc}
\Delta^n & \xleftarrow{\partial \Delta^n} & S' \\
\downarrow & & \downarrow \\
\delta^* \delta_! \Delta^n & \xleftarrow{\delta^* \delta_! \partial \Delta^n} & \delta^* \delta_! S'
\end{array} \]
in which the middle and right hand vertical maps are monic bivariant equivalences by the induction hypothesis. A straightforward argument, using the fact that the bivariant model structure is left proper, shows that the induced map
\[ S = \Delta^n \cup_{\partial \Delta^n} S' \to \delta^* \delta_! S = \delta^* \delta_! \Delta^n \cup_{\delta^* \delta_! \partial \Delta^n} \delta^* \delta_! S' \]
is a bivariant equivalence. To close the inductive loop we need to prove that $S \to d^*d_! S$ is monic. For this it suffices to prove that for any $n \geq 0$ the square
\[ \begin{array}{ccc}
\partial \Delta^n & \to & \delta^* \delta_! \partial \Delta^n \\
\downarrow & & \downarrow \\
\Delta^n & \to & \delta^* \delta_! \Delta^n
\end{array} \]
is a pullback in $(\text{Set}_\Delta)/(A \times B)$. The proof of this is completely analogous to the proof of the corresponding fact in the proof of Theorem 3.23 and is omitted. \qed
Theorem 4.40. Let $A$ and $B$ be ∞-categories. The Quillen adjunction
\[ \text{d}^! : \text{Corr}(A, B) \rightleftarrows (\text{Set}_\Delta)/(A \times B) : \text{d}_* \]
of Proposition 4.38 is a Quillen equivalence.

Proof. From Lemma 4.39 we have that the counit map $d^*d_*X \to X$ is a trivial Kan fibration whenever $X \to A \times B$ is a bifibration. Therefore it suffices to prove that $d^*$ reflects weak equivalences. Suppose then that $X \to Y$ is a map in $\text{Corr}(A, B)$ such that the image $d^*X \to d^*Y$ is a bivariant equivalence. Therefore, by Theorem 4.28 we have that the induced map $d^*X \times_{A \times B} \text{B}^* \to d^*Y \times_{A \times B} \text{B}^*$ is a weak homotopy equivalence for all vertices $a \in A$ and $b \in B$. From Remark 4.36 we see that there is an isomorphism
\[ d^*X \times_{A \times B} (A/a \times B/b) = \text{sd}_2 X \times_{X \times X} (A/a \times B/b) \]
Using Remark 4.37 together with the fact that there is an isomorphism $(B/b)^{\text{op}} \simeq (B/b^{\text{op}})^{\text{op}}$ we see that there is an isomorphism
\[ |d^*X \times_{A \times B} (A/a \times B/b)| \simeq |d^*X| \times |B/b| \times |A| |(B/b^{\text{op}}) \times A/a|] \]
natural in the correspondence $X$. It follows that for any vertices $a \in A$ and $b \in B$, the induced map
\[ d^*X \times_{A \times B} (A/a \times B/b) \to d^*Y \times_{A \times B} (A/a \times B/b) \]
is a weak homotopy equivalence if and only if the induced map
\[ a^*X \times B^{\text{op}} \times A (B/b^{\text{op}}) \times A/a \]
is a weak homotopy equivalence. Therefore, $d^*X \to d^*Y$ is a bivariant equivalence if and only if $a^*X \to a^*Y$ is a covariant equivalence, using Theorem 2.3. Therefore $X \to Y$ is a weak equivalence in the correspondence model structure since $a^*$ reflects weak equivalences (Theorem 3.23).

Remark 4.41. Unlike the case for the adjoint pair $(a_!, a^*)$, the adjoint pair $(d_!, d^*)$ is not a Quillen adjunction. It can be shown (with some work) that the functor $d_! : (\text{Set}_\Delta)/(A \times B) \to \text{Corr}(A, B)$ sends maps of the form (2) and (3) from Definition 1.4 to inner anodyne maps in $\text{Corr}(A, B)$, but it does not in general send inner horn inclusions in $(\text{Set}_\Delta)/(A \times B)$ to inner anodyne maps in $\text{Corr}(A, B)$.

Finally let us describe another Quillen equivalence relating the correspondence model structure with the bivariant model structure. Recall from [2] the functor
\[ \Gamma : \text{Corr}(A, B) \to (\text{Set}_\Delta)/(A \times B) \]
which sends a correspondence $X \in \text{Corr}(A, B)$ to its simplicial set of sections $\Gamma(X) = \text{map}_{\Delta^1}(\Delta^1, X)$. The structure map $\Gamma(X) \to A \times B$ is induced by the inclusion $\partial \Delta^1 \subseteq \Delta^1$. The functor $\Gamma$ has a left adjoint $C : (\text{Set}_\Delta)/(A \times B) \to \text{Corr}(A, B)$ which sends an object $(f, g) : X \to A \times B$ in $(\text{Set}_\Delta)/(A \times B)$ to the correspondence
\[ C(X) = (X \times \Delta^1) \cup_{X \times \partial \Delta^1} (B \times A) \]
where the map $X \times \partial \Delta^1 \to A \times B$ restricts to $g$ on $X \times \{0\}$ and restricts to $f$ on $X \times \{1\}$.

We then have the following result from [2].
Theorem 4.42 ([2]). Let $A$ and $B$ be $\infty$-categories. The adjoint pair
\[ C : (\text{Set}_\Delta)/(A \times B) \rightleftarrows \text{Corr}(A, B) : \Gamma \]
is a Quillen equivalence for the bivariant model structure on $(\text{Set}_\Delta)/(A \times B)$ and the correspondence model structure on $\text{Corr}(A, B)$.

We only sketch the proof, since an $\infty$-categorical version can be found in [2].

Sketch of proof. The proof that $\Gamma$ is a right Quillen functor is a straightforward modification of the proof of Proposition 2.4.7.10 of [13]. It can be shown that the functor $C$ reflects weak equivalences (see [2]). The result then follows from Proposition B.3.17 of [14]. □

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