POISSON ALGEBRAS OF ADMISSIBLE FUNCTIONS ASSOCIATED TO TWISTED DIRAC STRUCTURES

ALEXANDER CARDONA

Abstract. We define algebras of admissible functions associated to twisted Dirac structures, and we show that they are Poisson algebras. We study the standard cases associated to Dirac structures defined by graphs of non-degenerate 2-forms.

MSC(2000): 53C57, 53D17.
Keywords: Twisted Dirac structures, Poisson brackets, twisted symplectic graphs.

1. Introduction

Poisson algebras of admissible functions associated to non-twisted Dirac structures have been studied by Courant and Weinstein (see [7][8]). A Dirac structure on a manifold $M$ is a maximally isotropic sub-bundle $L$ of the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$, which is involutive under the Courant bracket on $\mathbb{T}M$

$$[X \oplus \xi, Y \oplus \eta]_C = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2}d(\iota_X \eta - \iota_Y \xi),$$

where $X \oplus \xi, Y \oplus \eta \in \Gamma(\mathbb{T}M)$. The isotropy condition here is given with respect to the natural symmetric pairing

$$\langle X \oplus \xi, Y \oplus \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X))$$

in $\mathbb{T}M$, and the bracket (1) can be seen as the skew-symmetrization of the Dorfman bracket [9]

$$[X \oplus \xi, Y \oplus \eta]_D = [X, Y] + \mathcal{L}_X \eta - i_Y d\xi,$$

which coincides with (1) on sections of $L$. Particular cases of Dirac manifolds are Poisson and symplectic manifolds (which correspond to graphs, in the generalized tangent bundle $\mathbb{T}M$, of the corresponding Poisson bi-vector and symplectic form, respectively). In the symplectic case, for example, if $h \in \Omega^2(M)$ denotes the symplectic form,

$$\mathbb{L}_h = \{(X, i_X h) \in \Gamma(\mathbb{T}M) \mid X \in \mathfrak{X}(M)\}$$

(4)

defines a Dirac structure on $M$, and many features of the symplectic geometry associated to $h$ are captured by this Dirac structure. In particular, the Poisson algebra on $\mathcal{C}^\infty(M)$ defined by the action of Hamiltonian vector fields on smooth functions appears here as the algebra of admissible functions associated to the Dirac structure (see e.g.[7][8]).

In general, given a Dirac structure $L$ on $M$, it is possible to associate to it a Poisson algebra of smooth functions on $M$, which is usually a subalgebra of $\mathcal{C}^\infty(M)$, and is called the algebra of admissible functions [7]. A smooth function $f$ on a manifold $M$ with a Dirac structure $L$ is called
admissible if there exists a smooth vector field \( X_f \) on \( M \) such that \( (X_f, df) \in \Gamma(\mathbb{L}) \). We will denote by \( C^\infty_\mathbb{L}(M) \) the set of \( \mathbb{L} \)-admissible functions on \( M \). In the Poisson and symplectic cases the set of admissible functions is all of \( C^\infty(M) \), but in general it is not the case. If a function \( f \) is admissible, we will call a vector field \( X_f \) such that \( (X_f, df) \) is a section of \( \mathbb{L} \) a Hamiltonian vector field associated to \( f \). In [7] it is shown that, in spite of the fact that Hamiltonian vector fields are not unique in general, the bracket

\[
\{f, g\} = X_f(g)
\]

defines a Poisson algebra structure on the space \( C^\infty_\mathbb{L}(M) \) of \( \mathbb{L} \)-admissible functions on \( M \) (see also [4]).

The Courant bracket (1) can be twisted by an extra term given by a 3-form on \( M \). In [12] it is pointed out that brackets of the form

\[
[X \oplus \xi, Y \oplus \eta]_H = [X \oplus \xi, Y \oplus \eta]_C - \iota_Y \iota_X H,
\]

where \( X \oplus \xi, Y \oplus \eta \in \Gamma(\mathbb{T}M) \) and \( H \in \Omega^3(\mathbb{M}) \) is a closed 3-form on \( M \), called the twisting, give rise to the same kind of structure as before. A maximal isotropic sub-bundle \( \mathbb{L} \) of \( T \mathbb{M} = T M \oplus T^* M \), which is involutive under the twisted Courant bracket (6) on \( T \mathbb{M} \) is called a \( H \)-twisted Dirac structure. We will denote such structures by \( \mathbb{L}_H \)—although the twisting actually appears on the bracket and not on the sub-bundle \( \mathbb{L} \)—to distinguish the twisted and non-twisted cases. Twisted Dirac structures appear naturally in Poisson geometry when, for example, a reduction of a (twisted or non-twisted) Dirac structure is performed [2]. In quantum field theory and superstring theory, the form \( H \) has an interpretation as the Neveu-Schwarz 3-form [10].

In this paper we address the question of the definition of Poisson algebras of smooth functions on \( M \) associated to \( H \)-twisted Dirac structures (the non-twisted case has been studied in [7]). We will define in section 2 the set of \( H \)-admissible functions associated to a twisted Dirac structure \( \mathbb{L}_H \), and we will show in theorem 2.1 that this set has the structure of a Poisson algebra with the usual Poisson bracket. Our definition of admissible function in the twisted case was inspired by the notion of Hamiltonian symmetries given in [14], in the context of differential graded Lie algebras associated to dg-manifolds, so we will follow such a point of view. Many results proven in the case of admissible functions are also true in the case of admissible pairs, in the sense of definition 2.1 (see also [1][2][17]), but here we want to focus on the case of functions on \( M \). In section 2 we prove our main results and, in order to illustrate their significance in the case of twisted Dirac structures defined by graphs of non-degenerate 2-forms, as in [4], we give a characterization of our notion of \( H \)-admissibility in this situation. We end this paper by illustrating the non-triviality of the algebra of \( H \)-admissible functions associated to a twisted Dirac structure with a well-known example from the classical theory of dynamical systems, which arises naturally in this context.

2. The Poisson algebra associated to twisted Dirac structures

In this section we will describe the Poisson algebra associated to a \( H \)-twisted Dirac structure \( \mathbb{L}_H \) on a closed smooth manifold \( M \), namely the algebra \( C^\infty_\mathbb{L}(M) \) of \( \mathbb{L} \)-admissible functions associated to the Dirac structure. Even though many facts we want to show explicitly for Dirac structures in \( \Gamma(T M \oplus \Lambda^{n-1} T^* M) \), when \( n = 2 \), are true for any \( n \geq 0 \) (i.e. for any higher analogue in the sense of [2][17]), we will focus on this case, where the Poisson algebras of functions associated to twisted Dirac structures appear. Our definition of admissible function in the twisted
case was inspired by the notion of Hamiltonian symmetries given in \[14\], in the context of differential graded Lie algebras associated to dg-manifolds, so we will begin by a short recall of such a point of view in order to motivate our approach.

2.1. Derived Brackets and Hamiltonian Symmetries. Let us consider, for \( n \geq 0 \), the trivial \( \mathbb{R}^n \)-bundle \( P_n \) over the odd tangent bundle \( T^1[M] \), with the derivation given by

\[
Q_H = d + H \partial_t,
\]

where \( d \) denotes the de Rham differential and \( H \in \Omega^{n+1}(M) \). The derivation \( Q_H \) defines a homological vector field (i.e. it satisfies \([Q_H, Q_H] = 0\) if and only if \( H \) is closed, and isomorphism classes of such bundles over \( T^1[M] \) are in one to one correspondence with \( H^n_{\text{dR}}(M, \mathbb{R}) \) (see \[11\]). Smooth functions on the dg-manifold \((P_n, Q_H)\) correspond, locally, to the algebra \( C^\infty(P_n) = \Omega^0(M) \otimes S[t] \), where \( t \) denotes the coordinate on the fiber \( \mathbb{R} \), whose degree is defined to be \( n \), so that \( Q_H \) is actually a derivation of degree \( 1 \) (see \[9\] and \[13\] for the background, original references and notations related with the point of view of graded manifolds, and \[15\] for the relation between homological vector fields and Lie algebroids).

The twisted Courant bracket \([5]\) is known to be the derived bracket obtained from the complex of derivations \( \text{Der}^*(P_{2n}, Q_H) \) of the DGLA (differential graded Lie algebra) associated to the dg-manifold \((P_{2n} = T^1[M] \oplus \mathbb{R}^2, Q_H)\) (see \[11\] and \[13\]). Since the derived complex of derivations associated to the dg-manifold \((P_n, Q_H)\) is nothing but the extended de Rham complex \[14\]

\[
\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}(M) \xrightarrow{d} \mathfrak{X}(M) \oplus \Omega^{n-1}(M),
\]

and the corresponding derived brackets are

\[
[a \partial_t, b \partial_t] = [(d + H \partial_t)(a \partial_t), b \partial_t] = 0 \quad \forall a, b \in \Omega^{n-k}(M),
\]

\[
[i_X + a \partial_t, i_Y + b \partial_t] = [\mathcal{L}_X + (d + i_X H) \partial_t, b \partial_t] = (\mathcal{L}_X b + i_X H) \partial_t \quad \forall a, b \in \Omega^{n-k}(M),
\]

it follows that, for any \( n \geq 0 \), we have a twisted Courant bracket on sections of the bundle \( TM \oplus \Lambda^{n-1} T^* M \) defined as the anti-symmetrization of \[9\]. A Dirac structure of type \( n \geq 1 \), is an isotropic sub-bundle \( \mathcal{L} \) of \( \Gamma(TM \oplus \Lambda^{n-1} T^* M) \cong \mathfrak{X}(M) \oplus \Omega^{n-1}(M) \) such that \( \mathcal{L}^\perp = \mathcal{L} \), with respect to the symmetric pairing

\[
\langle X \oplus a, Y \oplus b \rangle_+ = \frac{1}{2} (i_X b + i_Y a),
\]

for \( X \oplus a, Y \oplus b \in \mathfrak{X}(M) \oplus \Omega^{n-1}(M) \), and such that \( [\Gamma(L), \Gamma(L)] \subset \Gamma(L) \) with respect to the bracket defined by \[9\]. These higher analogues of Dirac structures and Courant algebroids have been studied in \[17\] and \[2\], respectively, and they are related with the \( n \)-plectic structures defined in \[1\]. It has been shown that, in the non-twisted case (i.e. when the \((n+1)\)-form \( H \) is zero) there are Poisson algebras of forms associated to them. When \( n = 2 \), these algebras correspond to the Poisson algebras of \textit{admissible functions} on \( M \), associated to Dirac structures whenever \( H = 0 \), defined by Courant in \[7\]. If the twisting \( H \in \Omega^3(M) \) is non-trivial, there is a stronger idea of admissibility from which a Poisson structure can be given to a subspace of \( C^\infty(M) \). This idea of admissibility is encoded in the sub-DGLA of derivations of \((P_n, Q_H)\) consisting of infinitesimal symmetries of the bundle \( P_n \) obtained from “geometric symmetries” of the twisting form \( H \), defined in \[13\]. In particular, the derivations in degree \(-1\) in such a complex are the given by

\[
\text{GDer}^{-1}(Q_H) = \{ i_X + a \partial_t \in \text{Der}^{-1}(Q_H) \mid da + ixH = 0 \}.
\]
and, since the condition $da + i_X H = 0$, for $\alpha \in \Omega^{n-1}(M)$, ensures that $\mathcal{L}_X H = 0$, it contains the symmetries of $H$ encoded by vector fields on $M$. The derived algebra of this sub-DGLA, called the Hamiltonian symmetries of the homological vector field $Q_H$ in [14], turns to be the extended de Rham complex with derived brackets [13]
\[
\begin{array}{ll}
[\alpha \partial_t, \beta \partial_t] = 0 & \forall \alpha, \beta \in \Omega^{n-k}(M), \\
[i_X + \alpha \partial_t, \beta \partial_t] = (\mathcal{L}_X \beta) \partial_t & \forall \beta \in \Omega^{n-k}(M),
\end{array}
\]
and
\[
[i_X + \alpha \partial_t, i_Y + \beta \partial_t] = i_{[X,Y]} + (\mathcal{L}_X \beta) \partial_t & \forall \alpha, \beta \in \Omega^{n-1}(M). \quad (12)
\]
We will use this dg-Leibniz algebra to give a meaning to admissible function, giving rise to a Poisson algebra of smooth functions on $M$ in the twisted case.

2.2. Admissible pairs associated to Dirac structures.

**Definition 2.1.** Let us consider $X \in \mathfrak{X}(M)$ and $\alpha \in \Omega^{n-1}(M)$ such that $i_X + \alpha \partial_t \in \text{Der}^{-1}(Q_H)$. We say that $(X, \alpha)$ is an admissible pair, and then $X$ is called a Hamiltonian vector field associated to $\alpha$, if $i_X + \alpha \partial_t \in \text{GDer}^{-1}(Q_H)$, i.e. if
\[
da + i_X H = 0, \quad (13)
\]
for the twisting form $H$.

**Example 2.1.** The first non-trivial case, our motivating example, is given by a twisting by a closed 2-form $h$ (i.e. when $n = 1$). In this case the pairing (12) is identically zero on any pair of sections of $TM \oplus 1$, and (13) says that a function $f \in \mathcal{C}^\infty(M)$ is admissible, and has Hamiltonian vector field $X$, if
\[
df + i_X h = 0. \quad (14)
\]
Thus, if $h$ is also non-degenerate (i.e. a symplectic form), for every smooth function on $M$ there exists a vector field $X$ such that $(X, f)$ is an admissible pair, namely the Hamiltonian vector field given by (14), in agreement with the classical setting of symplectic geometry. Notice that, restricting the derived bracket (12) to pairs of Hamiltonian vector fields and admissible functions, we find
\[
i_{[X^f, i_X g + \alpha \partial_t]} + \beta \partial_t = i_{[X^f, X^g]} + (\mathcal{L}_{X^f} g) \partial_t
\]
and thus, if $h$ is non-degenerate,
\[
i_{X^f + \alpha \partial_t, i_X g + \beta \partial_t} = i_{[X^f, X^g]} + \{f, g\} \partial_t,
\]
where
\[
\{f, g\} = \mathcal{L}_{X^f}(g) = X^f(g)
\]
is the usual Poisson bracket on functions associated to the symplectic form $h$.

**Example 2.2.** Replacing the closed non-degenerate 2-form of the preceding example by a closed non-degenerate $(n + 1)$-form on $M$, we naturally get the corresponding notion of Hamiltonian vector fields. Thus, we compare our approach to admissible functions to the one used for Dirac structures in the literature (see [7]). Consider the image under $1 \oplus d$ of the space of admissible sections sections of $TM \oplus 1$ in $\mathfrak{X}(M) \oplus \Omega^1(M)$, i.e. the set of pairs $(X, df)$ such that (14) follows for the closed form $h \in \Omega^2(M)$. If $h$ is non-degenerate, such an image defines a non-twisted Dirac structure on $M$, which is actually the graph (4) of the isomorphism induced by $h$ between tangent and cotangent spaces of $M$ point by point. Thus, the image under the exterior differential of the
symplectic model at the level \( n = 1 \) is the Dirac symplectic model \([4]\) at the level \( n = 2 \) without twisting. In this case it is easy to see that, if we consider \( \alpha = df \), the condition \([13]\) is empty, so that any exact 1-form is admissible and the bracket \([5]\) defines a Poisson bracket on \( C^\infty(M) \), as proven in \([7]\). Moreover, restricting the derived bracket \([2]\) to admissible pairs gives

\[
[i_X + df, i_Y + dg] = \{X, Y\} + df\partial_i.
\]

(15)

We will see later that the same can be done in the twisted case, asking not only \( f \), but also \( df \), to be admissible in the sense of Definition 2.1. These facts are also true for any non-degenerate form. We will now concentrate in the case \( n = 2 \), for \( \alpha = X, d\alpha \). In this case it is easy to see that, if we consider \( \alpha \), in \( \Gamma(TM) \) in order to show that the Courant bracket between \( \alpha, \beta \) is skew-symmetric and has other nice properties. Actually, the twisting by \( H \) here corresponds to the \( n \)-plectic structures discussed in \([1]\), when \( H \) is a non-degenerate form. We will now concentrate in the case \( n = 2 \), for \( \alpha = X, d\alpha \).

By the de Rham differential of the set of admissible pairs \((X, \alpha)\) of a \( H \)-twisted Dirac structure in \( \mathfrak{X}(M) \oplus \Omega^{n-1}(M) \), for \( H \) non-degenerate, defines a non-twisted Dirac structure in \( \mathfrak{X}(M) \oplus \Omega^n(M) \).

**Proposition 2.1.** The image by the de Rham differential of the set of admissible pairs \((X, \alpha)\) of a \( H \)-twisted Dirac structure in \( \mathfrak{X}(M) \oplus \Omega^{n-1}(M) \), for \( H \) non-degenerate, defines a non-twisted Dirac structure in \( \mathfrak{X}(M) \oplus \Omega^n(M) \).

**Proof.** Let \((X, \alpha), (Y, \beta)\) be an admissible pair in \( \mathfrak{X}(M) \oplus \Omega^{n-1}(M) \), for a \( H \)-twisted Dirac structure \( L_H \subseteq \Gamma(TM) \oplus \Lambda^{n-1}T^*M \). The sections \((X, \alpha), (Y, \beta)\) belong to the Dirac structure \( L_H \), so that the pairing \((10)\) on \((X, \alpha), (Y, \beta)\) is zero, and \( i_X\beta = -i_Y\alpha \). Since both pairs are admissible, we have that \( da = -i_X H \) and \( d\beta = -i_Y H \). Then, the sections \((X, da), (Y, d\beta)\) in the image under \( 1 \oplus d \) of \( L_H \) satisfy

\[
\langle(X, da), (Y, d\beta)\rangle_+ = i_X d\beta + iy da = 0.
\]

On the other hand, computing the non-twisted Courant bracket,

\[
[(X, da), (Y, d\beta)]_C = ([X, Y], L_X d\beta) = ([X, Y], -L_X iy H) = ([X, Y], -i_{[X,Y]} H + i_Y L_X H) = ([X, Y], -i_{[X,Y]} H)
\]

so that \([X, -i_X H], (Y, -i_Y H)]_C = ([X, Y], -i_{[X,Y]} H) \) and, provided \( H \) is non-degenerate, the result is proven \( \square \)

Notice that, in general, for \((X, \alpha), (Y, \beta)\) admissible pairs, we have a natural candidate to define the Poisson bracket between \( \alpha, \beta \in \Omega^{n-1}(M) \), namely \( \mathcal{L}_X(\beta) \); it is elementary to show that, on admissible elements, this “bracket” is skew-symmetric and has other nice properties. Actually, the twisting by \( H \) here corresponds to the \( n \)-plectic structures discussed in \([1]\), when \( H \) is a non-degenerate form. We will now concentrate in the case \( n = 2 \), for \( \alpha = X, d\alpha \).

### 2.3. Admissible functions in the twisted case.

Let us now turn to the case of a \( H \)-twisted Dirac structures \( L_H \) in \( \Gamma(TM) \), where \( H \in \Omega^3(M) \) is closed (see \([4]\) and \([12]\)). Let us denote by \( T_{L_H} \) the tensor defined on sections of \( TM \) by

\[
T_{L_H}(A \otimes B \otimes C) = ([A, B], H, C)_+,
\]

(16)

where the pairing \([2]\) and the twisted Courant bracket \([6]\) are used. This tensor, sometimes called the Courant tensor, was defined in the non-twisted case in \([7]\) in order to show that the bracket defined by \([5]\), in the case of a non-twisted Dirac structure \( L \), defines a Poisson algebra on the algebra \( C^\infty(M) \) of admissible functions on \( M \). Indeed, it is clear from the definition that for any Dirac structure \( L \) we have \( T_L(A \otimes B \otimes C) = 0 \) whenever \( A, B, C \in \Gamma(L) \) and since, given admissible pairs \((X_f, df), (X_g, dg)\) and \((X_h, dh) \in \Gamma(L) \), it is shown in \([7]\) that

\[
T_L((X_f, df) \otimes (X_g, dg) \otimes (X_h, dh)) = \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\},
\]

then Jacobi identity follows.
Proposition 2.2. Let $T_{L_H}$ denote the Courant tensor associated to a twisted Dirac structure $L_H$ given by (19). Then
\[ T_{L_H}((X_1, \alpha_1) \otimes (X_2, \alpha_2) \otimes (X_3, \alpha_3)) = T_{L}((X_1, \alpha_1) \otimes (X_2, \alpha_2) \otimes (X_3, \alpha_3)) - i_{X_2}i_{X_1}H, \]
for any $(X_1, \alpha_1), (X_2, \alpha_2), (X_3, \alpha_3) \in \Gamma(L_H)$, where $T_{L}$ denotes the tensor (16) associated to the non-twisted Courant bracket.

Proof. Let us take sections $A_i = (X_i, \alpha_i)$, for $i = 1, 2, 3$, in $\Gamma(L_H)$. By definition
\[ T_{L_H}(A_1 \otimes A_2 \otimes A_3) = \langle ([X_1, \alpha_1], [X_2, \alpha_2])_H, (X_3, \alpha_3) \rangle, \]
so that, from (19),
\[ T_{L_H}(A_1 \otimes A_2 \otimes A_3) = \langle ([X_1, X_2], \mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 + d(\alpha_1(X_2)) - i_{X_2}i_{X_1}H), (X_3, \alpha_3) \rangle, \]
and then
\[ T_{L_H}(A_1 \otimes A_2 \otimes A_3) = i_{X_3}(\mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 + d(\alpha_1(X_2)) - i_{X_2}i_{X_1}H) + i_{[X_1, X_2]}\alpha_3 \]
\[ = i_{X_3}(\mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 + d(\alpha_1(X_2))) + i_{[X_1, X_2]}\alpha_3 - i_{X_3}i_{X_2}i_{X_1}H \]
\[ = T_{L}((X_1, \alpha_1) \otimes (X_2, \alpha_2) \otimes (X_3, \alpha_3)) - i_{X_3}i_{X_2}i_{X_1}H. \]

It follows that the Jacobi identity, for brackets (5) of admissible functions, has an obstruction in the twisted case given by
\[ T_{L_H}((X_f, df) \otimes (X_g, dg) \otimes (X_h, dh)) = \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} - H(X_f, X_g, X_h). \]
We will see that this obstruction disappears if we restrict the space of admissible functions, as defined in (17), to a smaller set in which the admissibility in the sense of definition 2.1 plays a central role.

Definition 2.2. Let $L_H$ be a Dirac structure on a manifold $M$, twisted by a 3-form $H$. A function $f$ is $H$-admissible if it is admissible in the sense of Courant and $(X_f, df)$ is an admissible pair in $\Gamma(L_H)$. We will denote by $C^\infty_{H}(M)$ the set of $H$-admissible functions on $M$.

Notice that if $H = 0$, i.e. if there is no twisting, the definition of admissible function coincides with the one of Courant. On the other hand, if the twisting is non-trivial, the set of $H$-admissible functions may be smaller than the space of admissible functions in the usual sense but, as we will see, it is still a Poisson algebra. First, we show that we recover the usual bracket relation (19), but this time with $H$-admissible functions and the twisted bracket:

Proposition 2.3. Restricting the twisted Courant bracket (7) to admissible pairs $(X_f, df)$ and $(X_g, dg)$ gives
\[ ([X_f, df], (X_g, dg)]_H = ([X_f, X_g], d\{f, g\}). \]

Proof. Since $(X_f, df), (X_g, dg) \in \Gamma(L_h)$ are $H$-admissible,
\[ ([X_f, df], (X_g, dg)]_H = ([X_f, X_g], \mathcal{L}_{X_f}(dg) - i_{X_g}(d^2 f + i_{X_f}H)) \]
\[ = ([X_f, X_g], d\mathcal{L}_{X_f}(g)) \]
\[ = ([X_f, X_g], d\{f, g\}) \]
\[ \square \]
Theorem 2.1. Let \( f, g \) be \( H \)-admissible functions on \( M \) with respect to the twisted Dirac structure \( \mathbb{L}_H \), where \( H \in \Omega^3(M) \) is closed. Then the product \( fg \) and the bracket \( \{ f, g \} \) defined by (5) are \( H \)-admissible functions. Moreover, such a bracket satisfies both Leibniz and Jacobi identities, and defines a Poisson algebra structure on the space \( C^\infty_{L,H}(M) \).

Proof. Let \( f, g, h \) be \( H \)-admissible functions, and let us denote by \( (X_f, df), (X_g, dg), (X_h, dh) \in \Gamma(\mathbb{L}_H) \) the corresponding admissible pairs in \( \mathbb{G} \text{Der}^{-1}(Q_H) \). Let \( X_{fg} = gX_f + fX_g \), then
\[
g(X_f, df) + f(X_g, dg) = (gX_f + fX_g, gdf + dfd) = (X_{fg}, d(fg)) \in \Gamma(\mathbb{L}_H)
\]
and
\[
i_x f_x H = g i_x i_x f_x H + f i_x i_x H = 0,
\]
so that \( fg \) is also \( H \)-admissible. Notice that both antisymmetry and Leibniz identity are independent of the twisting. Indeed, since \( i_x df = -i_x df \) for admissible pairs \( (X_f, df) \) and \( (X_g, dg) \),
\[
\{ f, g \} = X_f(g) = L_{X_f}(g) = i_x dg + di_x g = -i_x df = -L_{X_f}(f) = -\{ f, g \}
\]
and, second,
\[
\{ f, g, h \} = X_{fg}(h) = gX_f(h) + fX_g(h) = g\{ f, h \} + f\{ g, h \}.
\]
Next, since
\[
i_{[X_f, X_g]} H = L_{X_f} i_x i_x H - i_x L_{X_f} H = -i_x di_x H + i_x i_x dH = 0,
\]
propagation 2.3 implies that \( \{ f, g \} \) is \( H \)-admissible and
\[
X_{\{ f, g \}} = [X_f, X_g].
\]
Finally, definition 2.4 implies that \( i_x H = i_x i_x H = i_x i_x H = 0 \), so that by Proposition 2.3 there is no obstruction to the Jacobi identity. \( \square \)

Example 2.3. Let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \), and let \( \langle \cdot, \cdot \rangle \mathfrak{g} \) be a non-degenerate symmetric bilinear form on it. Then
\[
\mathbb{L}_G = \{(X_R - X_L), \frac{1}{2}(X_R - X_L))|X \in \mathfrak{g}\} \leq TG \oplus T^*G,
\]
where \( X_R \) and \( X_L \) denote the right-invariant and left-invariant vector fields associated to \( X \in \mathfrak{g} \), respectively, defines a twisted Dirac structure on \( G \) with twisting 3-form
\[
H_G = \frac{1}{2}([X, Y], Z)_{\mathfrak{g}},
\]
called the Cartan-Dirac structure on \( G \) (see [4]). Since, for an orthonormal basis \( \{ X_1, X_2, \ldots, X_n \} \) for \( \mathfrak{g} \),
\[
i_{X_i} i_{X_m} i_{X_n} H_G = C_{i_mn}^{\mathfrak{m}},
\]
the structure constants of the Lie algebra, it is clear that the space of \( H_G \)-admissible functions is completely determined by \( \mathfrak{g} \). In the case of \( G = SO(3) \), with the usual bi-invariant metric on it and the corresponding orthonormal basis for \( \mathfrak{so}(3) \), it is easy to see that \( i_{X_i} i_{X_m} i_{X_n} H_G = 1 \), so in this case the algebra of \( H_{SO(3)} \)-admissible functions associated to the Cartan-Dirac structure is trivial.
2.4. Twisted symplectic graphs and constants of motion. We will finish this paper considering the example given by a symplectic graph twisted by a closed 3-form $H$ (also called $H$-closed 2-forms in [12]). Consider the Dirac structure defined in (4), i.e. the graph

$$\mathbb{L}_H = \{(X, i_X h) \mid X \in \mathfrak{X}(M)\},$$

in $\mathcal{T} M$, of a non-degenerate 2-form $h$. It follows from the definition of the twisted bracket (8) that this Dirac structure is integrable if and only if $dh - H = 0$, so that $h$ cannot be closed and, as a consequence, the definition (5) gives a Poisson bracket for which, as follows from the remarks after Proposition 2.2 (see also [12]),

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = H(X_f, X_g, X_h).$$

(17)

Consider now functions $f, g, h \in C^\infty(M)$ which are $H$-admissible, then (13) implies that

$$i_{X_f} H = i_{X_g} H = i_{X_h} H = 0,$$

so that the Jacobi identity holds. In this case, being a graph of a symplectic form, the twisted Dirac structures associates to any function a Hamiltonian vector field $X_f$, but it is this vector field which makes $f$ an $H$-admissible function through the condition $i_{X_f} H = 0$. We can characterize such pairs with the following

**Proposition 2.4.** A pair $(X_f, df)$ in $\mathbb{L}_H$ is $H$-admissible if and only if $\mathcal{L}_{X_f} h = 0$.

**Proof.** Since $(X_f, df) \in \Gamma(\mathbb{L}_H)$, $df = -i_{X_f} h$ so that $\mathcal{L}_{X_f} h = di_{X_f} h + i_{X_f} dh = i_{X_f} H$, and the result follows $\square$

Finally, as a consequence of Proposition 2.3, we find back the usual bracket relation (15) for the $H$-twisted bracket:

$$[(X, df), (Y, dg)]_H = ([X_f, X_g], d\{f, g\}).$$

To see that the Poisson algebra of $H$-admissible functions is not trivial in general, let $(M, \omega)$ be a symplectic manifold and consider the 2-form $h = \varphi \cdot \omega$, where $\varphi \in C^\infty(M)$. Then the twisted Dirac structure (4) is integrable with respect to the twisted Courant bracket (6) if and only if $H = dh = d\varphi \wedge \omega$. Notice that, on the one hand, if $\varphi$ is chosen in such a way that $h$ is non-degenerate, any smooth function on $M$ is admissible in the sense of Courant, i.e. for any $f \in C^\infty(M)$ there exists a vector field $X_f \in \mathfrak{X}(M)$ such that $df = -i_{X_f} h$, $(X_f, -df) \in \Gamma(\mathbb{L}_H)$, where our notation means involutivity with respect to the twisted Courant bracket (6). On the other hand, by Proposition 2.3, a smooth function $f$ on $M$ is $H$-admissible if and only if

$$\mathcal{L}_{X_f} h = (\mathcal{L}_{X_f} \varphi)\omega + \varphi(\mathcal{L}_{X_f} \omega) = \{f, \varphi\}\omega = 0.$$

This means that, in the cases in which $\varphi$ is the Hamiltonian function for a dynamical system with phase space $(M, \omega)$, an observable $f \in C^\infty(M)$ is $H$-admissible if and only if it is a constant of motion.

**Example 2.4. Angular Momentum.** Consider $M = T_0^* \mathbb{R}^3$ with canonical Darboux coordinates $(q_1, q_2, q_3, p_1, p_2, p_3)$ and symplectic form $\omega = \sum_{i=1}^3 dp_i \wedge dq_i$. Let us take $\varphi(q_1, q_2, q_3, p_1, p_2, p_3) = \sum_{i=1}^3 \frac{p_i^2}{2} + V(r)$, where $r = (q_1^2 + q_2^2 + q_3^2)^{1/2}$ and $V(r)$ denotes a radial potential. Then the functions

$$L_1 = q_2 p_3 - q_3 p_2$$

$$L_2 = q_3 p_1 - q_1 p_3$$

$$L_3 = q_1 p_2 - q_2 p_1$$

are $H$-admissible.
are $H$-admissible for $H = d\varphi \wedge \omega$, and

$$\{L_1, L_2\} = L_3.$$  

Acknowledgements. The author is grateful to Henrique Bursztyn, Michel Cahen, Simone Gutt, Yoshiaki Maeda, Juan Camilo Orduz, Bernardo Uribe, and Alain Weinstein for many stimulating discussions on the geometry of Poisson and Dirac manifolds. The author also thanks the referee for pointing him reference [2], and for his very pertinent comments which helped to improve the exposition of this article. This research has been supported by the Vicerrectoría de Investigaciones and the Faculty of Sciences of the Universidad de los Andes.

References

[1] Baez, J., Hoffnung, A. and Rogers, C. Categorified symplectic geometry and the classical string. Comm. Math. Phys. 293, no. 3, pp. 701–725, 2010.
[2] Bi, YH. and Sheng, YH. On higher analogues of Courant algebroids. Sci. China Math. 54, no. 3, pp. 437–447, 2011.
[3] Bursztyn, H., Cavalcanti, G. and Gualtieri, M. Reduction of Courant algebroids and generalized complex structures. Adv. Math., 211, iss. 2, pp. 726–765, 2007.
[4] Bursztyn, H. and Weinstein, A. Poisson geometry and Morita equivalence. Poisson geometry, deformation quantisation and group representations, pp. 1–78, London Math. Soc. Lecture Note Ser., 323, Cambridge University Press, 2005.
[5] Cavalcanti, G. and Weinstein, A. Geometric models for noncommutative algebras. Berkeley Mathematics Lecture Notes, 10. American Mathematical Society, Providence, RI, 1999.
[6] Courant, T. Dirac manifolds. Trans. Amer. Math. Soc. 319, no. 2, pp. 631–661, 1990.
[7] Courant, T. and Weinstein, A. Beyond Poisson structures. Action hamiltoniennes de groupes. Troisième théorème de Lie (Lyon, 1986), pp. 39–49, Travaux en Cours, 27, Hermann, Paris, 1988.
[8] Dorfman I.Y. Dirac Structures and Integrability of Nonlinear Evolution Equations. Nonlinear Science Theory and Applications. Wiley, Chichester, 1993.
[9] Graña, M. Flux compactifications and generalized geometries. Classical Quantum Gravity 23, no. 21, pp. S883–S926, 2006.
[10] Roytenberg, D. On the structure of graded symplectic supermanifolds and Courant algebroids. Contemp. Math. 315, Amer. Math. Soc., Providence, RI, pp. 169–185, 2002.
[11] Severa, P. and Weinstein, A. Poisson geometry with a 3-form background. Noncommutative geometry and string theory (Yokohama, 2001). Progr. Theoret. Phys. Suppl. No. 144, pp. 145–154, 2001.
[12] Voronov, T. Graded manifolds and Drinfeld doubles for Lie bialgebroids. Preprint
[arXiv:1010.5413]

Mathematics Department, Universidad de Los Andes, A.A. 4976 Bogotá, Colombia.
E-mail address: acardona@uniandes.edu.co