Global simulations of Tayler instability in stellar interiors: a long-time multi-stage evolution of the magnetic field.

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ABSTRACT
Magnetic fields are observed in massive Ap/Bp stars and are presumably present in the radiative zone of solar-like stars. To date, there is no clear understanding of the dynamics of the magnetic field in stably stratified layers. A purely toroidal magnetic field configuration is known to be unstable, developing mainly non-axisymmetric modes. Rotation and a poloidal field component may lead to stabilization. Here we perform global MHD simulations with the EULAG-MHD code to explore the evolution of a toroidal magnetic field located in a layer whose Brunt-Väisälä frequency resembles the lower solar tachocline. Our numerical experiments allow us to explore the initial unstable phase as well as the long-term evolution of such field. During the first Alfvén cycles, we observe the development of the Tayler instability with the prominent longitudinal wavenumber, \( m = 1 \). Rotation decreases the growth rate of the instability and eventually suppresses it. However, after a stable phase, energy surges lead to the development of higher-order modes even for fast rotation. These modes extract energy from the initial toroidal field. Nevertheless, our results show that sufficiently fast rotation leads to a lower saturation energy of the unstable modes, resulting in a magnetic topology with only a small fraction of poloidal field, which remains steady for several hundreds of Alfvén traveltimes. The system then becomes turbulent and the field is prone to turbulent diffusion. The final toroidal–poloidal configuration of the magnetic field may represent an important aspect of the field generation and evolution in stably stratified layers.

Key words: Sun: magnetic fields, Stars: magnetic field, instabilities, MHD.

1 INTRODUCTION
The mean-field dynamo theory (Parker 1955; Steenbeck et al. 1966) is the most comprehensible framework to explain the origin of large-scale magnetic fields throughout the cosmos. Comparison between observations, theoretical and numerical models allows for some constraining in the determination of the dynamo parameters. Nonetheless, explaining some cases of stellar magnetism with the mean-field theory is challenging. Such is the situations of the ~ 10% chemically peculiar stars of types A and B, called Ap/Bp stars. These stars are characterized by magnetic fields with simple tilted dipoles with amplitudes above 300 G (see e.g., Donati & Landstreet 2009; Kochukhov et al. 2019) sustained in a radiative zone. Concurrently, A-type stars like Vega (Lignières et al. 2009; Petit et al. 2010) and Sirius A (Petit et al. 2011) depict fields with less than 1 G, leaving a gap, also called magnetic desert, between 1 and 300 G. The origin of this dichotomy is yet to be understood (see for instance Aurière et al. 2007; Szklarski & Arlt 2013; Cantiello & Braithwaite 2019; Jermyn & Cantiello 2020), but any explanation certainly relies on the evolution of magnetic fields in stably stratified layers (hereafter SSL). The dynamo mechanism faces also controversy in late type stars, like the Sun, having a convective envelope and a radiative interior. After the discovery of the solar internal differential rotation (Schou et al. 1998), the dynamo models have included the strong radial shear at the base of the convection zone as a main constituent of the generation process (see Charbonneau 2020, for a review). Nevertheless, observations of stellar magnetic fields suggest that a general dynamo mechanism may exist for partially convective stars, where this shear layer is present as well as for fully convective stars (Wright & Drake 2016). Global simulations have shown that large-scale magnetic fields can be generated in convection zones only, without the need for radial shear.

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These simulations have obtained both steady (Brown et al. 2010) and oscillatory dynamo solutions (Käpylä et al. 2012). Both kind of solutions are also possible in dynamo simulations including the tachocline (Ghizaru et al. 2010; Guerrero et al. 2016). Because in the latter simulations the field evolves in a region where the turbulent diffusivity is reduced, the oscillatory solutions exhibit cycles with periods closer to the solar one.

Lawson et al. (2015); Guerrero et al. (2019b) have proposed that magnetic instabilities may play a relevant role in the field generation by inducing an $a$-effect in the radiative zone and ultimately defining the cycle period. One class of dynamo operating in SSLs was heuristically described by Spruit (2002), revisited by Zahn et al. (2007) and implemented in a mean-field model by Bonanno (2013). Thus, the evolution of magnetic fields in radiative zones is instrumental for the understanding of the origin of magnetic fields in early and late type stars.

Thanks to the energy principle of Bernstein et al. (1958), the spectral theory of static MHD plasma is rather well understood. Using this principle Tayler (1973) demonstrated that a toroidal field, $B_{\phi}$, can be unstable close to the axis of symmetry of a star. The stability conditions against axisymmetric and non-axisymmetric perturbations are $d(B_{\phi}/s)/ds < 0$, and $d(sB_{\phi}^2)/ds < 0$, where $s$ is the cylindrical radius, respectively. Markey & Tayler (1973a, 1974) studied the stability of purely poloidal fields with field lines closing inside the star. They found that this field configuration is also likely unstable. In the case of cylindrical geometry, Wright (1973) and Tayler (1980) argued that mixed poloidal and toroidal field configurations may be stable against adiabatic perturbations and could therefore exist inside stellar interiors. This conclusion can be violated in the presence of resonant modes with $k \cdot B = 0$, where $k$ is the wavevector of the perturbation, making any mixed field configuration unstable to high longitudinal wave numbers, as discussed in Bonanno & Urpin (2012).

For systems with rigid rotation, Pitts & Tayler (1985) found that rotation may help to stabilize the toroidal field. Complete stabilization, however, can only be reached in the adiabatic case and considering unrealistic fast rotation. The effects of the Coriolis force opposing unstable perturbations may be different at equator and poles. Therefore, a latitudinal dependence is expected Goossens & Tayler (1980) (in spherical geometry this dependence may exists even in the non-rotating situation because distance to the symmetry axis). A general analytical study regarding the influence of rotation on the Tayler instability is complicated and has been restricted to simplified models.

The first MHD numerical experiments designed to address this problem where performed by Braithwaite (2006). They consisted of a local compressible model in Cartesian coordinates resembling a fraction of a star located at the north pole. The initial field was torus circulating along the axis of rotation. The time evolution of the magnetic field was restricted to the linear phase of the Tayler instability (TI) before reaching non-linear stages. The results confirmed the theoretical findings. Particularly important for the present work is the verification that rotation stabilizes the magnetic field whenever the rotational frequency is larger than the Alfvén frequency (AF), $\omega_0 > \omega_A$. Another important result is the confirmation that the Brunt-Väisälä frequency ($\omega_B$), defined by the gravity acceleration and the thermal stratification, imposes a lower limit for the radial wave number, i.e., the larger the $\omega_B$, the smaller the wavelength of the unstable modes. This also exposes the dependency of the TI on the magnetic diffusivity, as magnetic fields with small spatial scales may diffuse before the instability develops (see also Acheson & Gibbons 1978; Spruit 1999). Braithwaite (2006) found that even at large values of magnetic diffusivity the instability is not fully suppressed but its growth rate decreases. Similarly, his analysis demonstrated that thermal diffusion may diminish the effects of buoyancy allowing for larger wavelengths to be unstable for a given value of $\omega_B$. Understanding the interplay between all these physical mechanisms is not straightforward.

Another way of exploring the evolution of magnetic fields in SSLs is solving numerically the evolution of the linearized MHD equations for different Fourier modes. The behavior of a more realistic distribution of toroidal field in spherical geometry can be solved in rotating systems in the purely adiabatic case or in the presence of dissipative terms. Kitchatinov & Rüdiger (2008) used this WKB approach in radius, while Bonanno & Urpin (2012, 2013) used it in latitude. These studies found that the growth of the instability is dependent on the topology of the initial magnetic field. Also, Bonanno & Urpin (2012) found that the instability grows faster at the equator than at the poles, yet the most unstable mode is always $m = 1$. According to their results, corresponding to adiabatic cases, solid body rotation may stabilize the magnetic field.

Publications on global MHD simulations of this instability are scarce and have focused mainly in the evolution of an initial poloidal magnetic field in the presence of shear (e.g., Szklarski & Arit 2013; Jouve et al. 2015, 2020). For the most fundamental case of a toroidal field evolving in a SSL in spherical geometry, Guerrero et al. (2019a) presented anelastic non-linear MHD simulations performed with the EULAG-MHD code. They explored the development of the Tayler instability of a toroidal field consisting of two bands of opposite polarity across the equator. The authors solved the inviscid MHD equations such that the dissipation is minimal and delegated to the proven implicit-large-eddy simulation (ILES) property of the numerical algorithm (Smolkarkiewicz & Charbonneau 2013; Guerrero et al. 2022). These global simulations were able to follow the evolution of the magnetic field beyond the linear phase. The authors explored the role of $\omega_B$ in the stabilization of the initial magnetic field. Their results indicated that increasing $\omega_B$, amounting to stronger buoyancy force, leads to growth rates that decrease following a power law, yet the instability is never fully suppressed. Importantly, the number of radial modes also increases for large values of $\omega_B$, as well as for large numerical resolution, i.e., less dissipation, in agreement with the results of Braithwaite (2006); Kitchatinov & Rüdiger (2008). Thus, in simulations with higher resolution and large $\omega_B$, the vertical extent of the unstable modes is so small that the instability becomes two-dimensional.

This work is a continuation of Guerrero et al. (2019a). However, in the models presented here the values of the Brunt-Väisälä frequency resembles the ones at the lower part of the solar tachocline. Our main goal is exploring the stabilizing effect of solid body rotation, yet we also get insights on fundamental properties of the TI. For instance, our setup allows us to verify the latitudinal dependence of the instability and the possible generation of helicity as a consequence of the growing of unstable modes (see, e.g., Stefan et al. 2019). In addition, our simulations allow us to explore the long term evolution and final morphology of the resulting magnetic field.

This paper is organized as follows. The description of the numerical model is presented in § 2. Our results and analysis for non-rotating and rotating cases are presented in § 3. Finally, in § 4, we present the conclusions and discuss the implication of the results for the understanding of magnetic fields in solar-like and Ap/Bp stars.
2 NUMERICAL SIMULATIONS

The numerical model solves the Lipsps-Hemler anelastic system of equations (Lipps & Hemler 1982; Lipps 1990) extended for the MHD case (Smolarkiewicz & Charbonneau 2013):

\[ \nabla \cdot (\rho \mathbf{u}) = 0, \]

\[ \frac{Du}{Dt} + 2\Omega \times \mathbf{u} = -\nabla \left( \frac{p'}{\rho_{ad}} \right) + \frac{g}{\Theta_{ad}} + \frac{1}{\mu_{ad}} (\mathbf{B} \cdot \nabla) \mathbf{B}, \]

\[ \frac{D\Theta}{Dt} = -\mathbf{u} \cdot \nabla \Theta_{amb} - \frac{\Theta'}{\tau}, \]

\[ \frac{DB}{Dt} = (\mathbf{B} \cdot \nabla) \mathbf{u} - B (\nabla \cdot \mathbf{u}). \]

Here, \( D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla \) is the total time derivative, \( \mathbf{u} \) is the velocity field in a rotating frame with \( \Omega = \Omega_0 (\cos \theta, -\sin \theta, 0) \), where \( \Omega_0 = 2\pi/T \) is the solid body angular velocity and \( T \) is the rotational period. The pressure-perturbation variable, \( p' \), accounts for both the gas and magnetic pressure; i.e., \( p' = p'_{\text{g}} + p'_{\text{m}} \), with \( \mathbf{B} \) and \( \mu_0 \) denoting the magnetic field and permeability of free space. The potential temperature perturbation, \( \Theta' \), enters both the buoyancy term in the momentum Eq. 2 and the entropy Eq. 3; \( \Theta \) is related to the specific entropy via \( s = c_s \ln \Theta + \text{const} \), where \( c_s \) is the specific heat. The \( \Theta' \) perturbations are defined with respect to a presumed ambient state, \( \Theta_{amb} \), typically based on the stellar structure models. The last term on the rhs of Eq. 3 relaxes \( \Theta' \) in a timescale \( \tau = 5.184 \times 10^4 \), unless stated otherwise (see Cossette et al. 2017, for a substantive discussion). The variables \( \rho_{ad} \) and \( \Theta_{ad} \) are, respectively, the density and potential temperature of the hydrostatic adiabatic reference state, whereas \( g = g \theta_E \) is the gravity acceleration of the solar interior adjusted from the solar model of Christensen-Dalsgaard et al. (1996); hereafter JCD.

The model corresponds to a spherical shell with \( 0 \leq \phi \leq 2\pi \), \( 0 \leq \theta \leq \pi \), and ranging from \( r_s = 0.6R_s \) to \( r_i = 0.76R_s \) in radius. Eqs. (1-4) are solved with the EULAG-MHD code (Smolarkiewicz & Charbonneau 2013) \(^1\). The majority of the simulations are performed with a grid resolution of \( 126 \times 64 \times 28 \) grid points in longitude (\( \phi \)), latitude (\( \theta \)) and radius (\( r \)), respectively. Parameters and results of this simulations are presented in Table 1, set A. The equations are solved in their inviscid form, therefore, viscosity, thermal conduction and magnetic diffusivity are accounted by the truncation terms of the multidimensional positive-definite advection transport algorithm (MPDATA; Smolarkiewicz 2006). Thus, the dissipation of the physical quantities changes when the resolution is increased or decreased. We evaluate how the grid resolution affects the results by performing simulations with double resolution in all directions \( (252 \times 128 \times 56) \), or with double resolution in the radial direction only \( (126 \times 64 \times 56) \), Table 1 sets B and C, respectively.

The boundary conditions are defined as follow: impenetrable and stress-free at the top and bottom surfaces of the shell for the velocity field; radial field and perfect conductor for the magnetic boundaries at the top and bottom boundaries, respectively; the boundary conditions for \( \Theta' \) assume zero normal flux at the top and bottom boundaries.

The ambient and adiabatic states are computed considering hydrostatic equilibrium in a non-rotating system, without magnetic field, for a polytropic gas. They are built by solving numerically the following equations for temperature, \( T \), and density, \( \rho \):

\[ \frac{\partial T_i}{\partial r} = -\frac{g}{R_i (m_i + 1)}; \]

\[ \frac{\partial \rho_i}{\partial r} = -\frac{\rho_i}{T_i} \frac{g}{R_i} + \frac{\partial T_i}{\partial r}. \]

The index \( i \) stands for "ad" (adiabatic) or "amb" (ambient), and \( R_s = 13732 \) is the gas constant of a monatomic gas. Density, temperature and pressure are related via the equation of state for the perfect gas: \( p_i = R_i \rho_i T_i \). The boundary values at \( r_s \), for solving Eqs. (5 and 6) are \( T_s = 3.12 \times 10^6 \) K and \( \rho_s = 506 \text{ kg m}^{-3} \).

In the simulations presented here we consider values of the Brunt-Väisälä frequency that are similar to those of the lower part of the solar tachocline. Thus, the adiabatic and ambient profiles assume polytropic indexes \( m_{ad} = 1.5 \) and \( m_{amb} = 2.5 \), respectively. Radial profiles of \( \rho_{amb} \) and \( \rho_{ad} \) are presented in Fig. 1(a) with continuous blue and dashed yellow lines, respectively. The radial profile of the square of the Brunt-Väisälä frequency,

\[ \omega_{BV}^2 = \frac{g}{\Theta_{amb}} \frac{\partial \Theta_{amb}}{\partial r}, \]

is presented in Fig. 1(b). For comparison, the red lines in both panels show profiles from the JCD model for the Sun.

\(^1\) The code is available at the dedicated website: http://www.astro.umontreal.ca/~paulchap/grps/eulag-mhd.html
The initial magnetic configuration is a toroidal field antisymmetric across the equator, i.e., \( B_{\theta 0} = B_{\phi 0} = 0 \) and
\[
B_{\phi 0}(r, \theta, \phi) = B_0 f(r) \sin \theta \cos \theta \psi,
\]
with
\[
f(r) = \exp \left[ -\frac{(r - r_c)^2}{d^2} \right],
\]
where \( r_c = 0.675 R_* \), \( d = 0.03 R_* \). This configuration is used in all simulations presented in this paper. \( B_0 \) is the magnitude of the initial magnetic field and is a free parameter of the model. The field lines distribution corresponding to this magnetic field is presented in the left panel of Fig. 2 (left panel). The profile of the magnetic field in the meridional plane is shown in the right panel of Fig. 2. The red and blue colors in the figure represent clockwise and counterclockwise orientation of the toroidal field. Notice that the radial profile considered here is different from the one used in previous studies (e.g., Braithwaite 2006; Kitchatinov & Rüdiger 2008; Bonanno & Urpin 2012). This initial field is a reasonable representation of a toroidal field resulting after the winding-up of a poloidal field due to differential rotation. For the velocity field and the thermal perturbations the initial conditions are zero.

3 RESULTS

The models presented here are similar to the ones studied by Guerrero et al. (2019a), except that in this paper the radial extent of the shell is thinner, with values of Brunt-Väisälä frequency that resembles the lower part of solar tachocline. The simulations are characterized by the non-dimensional parameters
\[
\delta^2 = \frac{\omega_{\phi \mu}^2}{\omega_A^2}, \quad \text{and} \quad \eta^2 = \frac{4\Omega^2}{\omega_A^2},
\]
where
\[
\omega_{\phi \mu}^2 = \frac{g}{\Theta_{\text{amb}}} \frac{\partial \Theta_{\text{amb}}}{\partial r},
\]
and
\[
\omega_A^2 = \frac{B_{\phi 0}^2}{\mu_0 \rho_{\text{amb}} \sigma_c^2}.
\]

are averages of the Brunt-Väisälä and Alfvén frequencies, in radius, and radius and latitude, respectively. The length scale considered in the Alfvén frequency is the level arm at 45°, \( \sigma_c = r_c \sin(\pi/4) \). The time units presented in this work are normalized with \( t_A = 1/\omega_A \). This quantity represents the number of cycles that an Alfvén wave travels along toroidal field lines with radius \( \sigma_c \).

3.1 Non-rotating cases

Ideally, simulations should start from a state in magnetohydrostatic equilibrium i.e., the gas pressure should balance the magnetic pressure. This basic state would then be perturbed to drive the onset of the instability. This was done, for instance, by Bonanno et al. (2012) to study the helical symmetry breaking occurring during the TI. However, this study uses a more complex setup than that of Bonanno et al. (2012), and finding this equilibrium state is difficult, especially in the rotating cases where the Coriolis force is also present. In this way, the hydrostatic equilibrium of the ambient state, Eqs. (5 and 6), is unbalanced by the initial magnetic field. After a few time steps, a new balance is achieved with the development of radial and latitudinal profiles of \( \Theta' \) and \( \mathbf{u} \). These profiles have most of their energy in mode \( m = 0 \), yet, due to numerical noise (i.e. round-off errors) other modes with \( m > 0 \) are also present through the domain. Although their energy is compatible with numerical precision, they are sufficient to trigger the TI (a view on this process can be observed in the movies A_R10B10.mp4 and B_R10B10.mp4 submitted as complementary material).

Fig. 3 shows the evolution of the toroidal magnetic field in latitude and time, at \( r = 0.7 R_* \) and \( \phi = 90° \). A noticeable decay of the initial field occurs after the unstable modes reach their maximum energy value. The decay follows a wave-like pattern, with minor changes for different longitudes (not shown). After the field decays it rapidly dissipates. These pattern resembles the results obtained by Guerrero et al. (2019a) (e.g., see their Fig. 7). The configuration of the magnetic field lines at the instant when \( B_\phi \) experiences a significant decay is presented in Fig. 4(a). The figure shows the tilt of the magnetic lines with respect to the symmetry axis, and the consequent displacement of magnetic axis at polar regions and the opening of the field lines at the equator. These are all manifestations of the hydromagnetic instability. Fig. 4(b) shows the field lines after at 300 Alfvén travel times. At this moment, the magnitude of the field has markedly decreased and the magnetic topology is substantially different from the initial one. The black dashed lines in Fig. 3 indicate the times where these snapshots were taken. Note that similar effects of the instability occur at the equator and the poles. Although they are not simultaneous (see below), the growth rate at both places is similar.

The study of the time evolution of different longitudinal modes at different latitudes can be performed by computing the power spectrum of the magnetic and/or the kinetic energy integrated over latitudinal bands at one specific radius. The magnetic energy as function of the spectral index \( m \) and time is defined as,
\[
\tilde{E}_B(m, t) = \tilde{B}_B^m(m, t)^2 + \tilde{\dot{B}}_B(m, t)^2,
\]
where the tildes represent the quantities in the Fourier space. For example, \( \tilde{B}_B^m \) is computed as
\[
\tilde{B}_B^m = \frac{1}{2\pi} \left[ \mu_0 \rho_{\text{amb}} \right] \int_{0}^{2\pi} \int_{0}^{\sigma_c} B_{\phi 0}^m \exp(-im\phi) \mathrm{d}t \mathrm{d}\phi.
\]
Note that \( \tilde{B}_B^m \) is the residual,
\[
\tilde{B}_B^m(r, \theta, \phi, t) = B_{\phi 0}(r, \theta, \phi, t) - \hat{B}_0(r, \theta, \phi, t).
\]
Table 1. Parameters and results of the simulations. The rotational rate is given by $\Omega_0$, with $\omega_{BV}$, $\delta$ and $\eta$ defined in Eqs. 11, 10, respectively. $B_P/B_T$ shows the ratio between poloidal and toroidal fields at the end of linear phase. $\sigma$ is the growth of $m=1$ magnetic longitudinal mode on early Alfvén Cycles, while $|\Gamma|$ represents the calculated decrease rate of the $m=0$ longitudinal mode. Sets are separated by resolution, field intensity and/or relaxation time. Meanwhile the simulation acronym specifies set, rotation period in earth days and field intensity.

Figure 3. Time-latitude diagram presenting the evolution of $B_{\phi}$ for the non-rotating case A_NRB10. The field corresponds to $r=0.7 R_\star$ and $\phi = 90^\circ$. The red (blue) colors represents clockwise (counter clockwise) toroidal magnetic field. Time units are expressed in Alfvén travel times.

This subtraction is only needed for the toroidal component. Three different latitudinal intervals were considered, $[\theta_1, \theta_2]$, defined as $NP = [0, \pi/9]$, $EQ = [\pi/3, 2\pi/3]$, and $SP = [8\pi/9, \pi]$. These are the latitudes where changes in the field topology are initially observed. Similar equations are used to compute the spectral kinetic energy. Throughout the paper references to spectral energies corresponds to quantities evaluated using these transformations. For the TI, the development of all the modes is expected, yet $m=1$ is predicted to be the fastest growing one (Goedbloed et al. 2010).

As was demonstrated analytically by Zahn (1974), the resistance of the plasma in stellar interiors to dynamical instabilities depends strongly on the thermal timescale and/or on the BV frequency. In the context of the TI, such results have been confirmed by Braithwaite (2006) for changes in BV frequency and the thermal conductivity, and by Guerrero et al. (2019a) for changes in BV frequency only. The evolution of unstable parcels of the fluid depends...
on the thermal properties of the gas, represented in the prognostic equations by $\Theta'$. In these equations, the thermal timescale is defined through the relaxation time, $\tau$ (Eq. 3). Smaller values of $\tau$ result in small amplitudes of short lived thermal perturbations, and vice-versa. Thus, it is possible to explore the thermal effects on the TI by changing $\tau$. The set of simulations A2, together with simulation A_RNB10, encompass values of $\tau$ between 0.5 and 600 days. The temporal evolution of the mode $m=1$ for these simulations is presented in Fig. 5. The figure shows that the growth rate, $\sigma$, decreases with the increase of $\tau$ following a power law, $\sigma \propto \tau^{-0.32}$ (see red line in the inset panel). The response of the plasma to changes in $\tau$ is similar to the response when increasing the BV frequency (Guerrero et al. 2019a). In both situations the TI is not fully suppressed but grows slowly. Note also that for sufficiently large values of $\tau$, there is no change in $\sigma$. This happens because the amplitude of $\Theta'$ depends on the smaller timescale in the model. Thus, for a long relaxation timescale it will be independent of $\tau$ and rely on the Alfvénic timescale ($\sim 150$ days for the value of $B_0$ used in this set of simulations).

In Fig. 6(a) we present the time evolution of the modes $m=0$ (continuous lines), $m=1$ (dashed lines) and $m=2$ (dotted lines) of the magnetic (left) and kinetic (right) spectral energy density for the non-rotating simulation A_RNB10. The colors stand for polar regions (red), NP (blue), and EQ (thin green lines). At the beginning of evolution, after a transient phase where the fluid balances the magnetic forces, the energy of the longitudinal modes rises. The modes $m=1$ and $m=2$ grow faster than $m=0$. Similar behavior is observed in the kinetic energy spectra. The instability develops earlier at polar regions, nevertheless, there is no significant difference in the growth rate between pole and equator for modes $m=1$ and 2. The mode $m=0$ develops faster at the poles. The growth of the mode $m=0$ alongside $m=1$ is an expected result for a steeper radial profile of the magnetic field (Braithwaite 2006).

The magnetic field topology is substantially affected by the development of unstable modes only when the energy of these modes is comparable to the initial magnetic energy. The deformation of the field lines at this stage is displayed in Fig 4 (a). After further evolution, the field lines lose any resemblance with the initial configuration, see Fig 4 (b).

From the previous qualitative and quantitative analysis, three different stages of evolution are identified: (a) the linear phase, where the unstable modes grow exponentially; (b) the saturation phase, where the energy of these modes reaches similar values than the initial magnetic energy; (c) the decaying phase where the magnetic field diffuses. As it will be shown later, this phase is compatible to turbulent decay following the Kolmogorov scaling, $\dot{E}_B \propto m^{-5/3}$.

In the following sections we will study the behavior of the magnetic and velocity fields in these three phases for simulations with various rotation rates. We will also explore the effects of different resolution on the magnetic field evolution during these phases.

3.2 The stabilizing effect of rotation

Several authors have shown that rotation helps stabilizing toroidal fields in radiative zones whenever $\Omega_0$ is comparable to $\omega_A$ ($\eta \sim 1$) (e.g., Braithwaite 2006; Bonanno & Urpin 2013). Nevertheless, if the system has any finite diffusion, full stabilization will only be achieved if $\omega_A \ll \Omega_0$ ($\eta \gg 1$) (Kitchatinov & Rüdiger 2008). Otherwise, rotation will only decrease the growth rate of the instability. All these works have focused on the linear phase but have not explored the saturation nor the decaying phase of the field.

The simulations presented in this section include the Coriolis term in Eq. 2 to explore if the TI can be stabilized by rotation. In the set of simulations A1, $\omega_A$ is kept constant and the rotation rate, $\Omega_0$ varies between $0.24 \times 10^{-6}$ Hz and $7.27 \times 10^{-6}$ Hz, corresponding to rotation periods between 300 and 10 days, and $6.7 \leq \eta \leq 197.8$. Alternatively, in the set of simulations A3 the rotation is kept constant, $\Omega_0 = 7.27 \times 10^{-6}$ Hz (10 days), and $\omega_A$ varies in the range $3.68 \times 10^{-6}$ Hz $\leq \omega_A \leq 58.81 \times 10^{-6}$ Hz. This corresponds to values of $B_0$ between 0.65 and 10.4 T, and $24.7 \leq \eta \leq 395.7$. The simulations evolve for several hundreds of Alfvén travel times to consider the three phases of evolution described above.

3.2.1 Linear phase

Exploring the effects of changing $\Omega_0$ and $\omega_A$ in the linear phase of the TI allows for comparison between global non-linear simulations and the earlier theoretical predictions (see e.g. Pitts & Tayler 1985) and numerical results (see e.g. Braithwaite 2006). Figure 6 (panels b to d) shows the time evolution the magnetic (left) and kinetic (right) energy densities in simulations A_R300B10, A_R150B10 and A_R10B10, respectively. The effects of including rotation in the simulation A_R300B10 ($\eta \sim 7$) can be observed in Fig. 6(b). The linear evolution of mode $m=1$ in the log-normal plot is an important indication that this value of $\eta$ is not sufficient to suppress the TI. It is interesting to notice that no significant difference are observed between lower (see green dashed lines) and high latitudes (red and blue dashed lines). Nevertheless, other modes, such as $m=0$ and $m=2$, do remain stable until $t \sim 75\Omega_0$. After this point, the energy of these modes continues rising with the same growth rate as that of $m=1$. The kinetic energy modes $m=1$ and $m=2$ evolve in a similar way. The mode $m=0$ has a larger energy which is roughly constant along the linear phase.

Fig. 6(c) corresponds to simulation A_R150B10, with the rotation rate increased by a factor of two (i.e. $\eta \sim 14$). The magnetic mode $m=1$ grows linearly until $t \sim 40\Omega_0$, yet the same mode of the kinetic energy seem to be stable. Interestingly, the magnetic and kinetic modes $m=1$ and 2 (and other high order modes not presented in the figure) at both, poles and equator reach the energy of saturation following two subsequent energy jumps. The same is observed in the evolution of $\dot{E}_B$ and $\dot{E}_k$ for simulations with $\eta \geq 14$, i.e., a
Figure 6. Evolution of the modes $m = 0$ (Eq. 14, continuous), $m = 1$ (dashed), and $m = 2$ (dotted) of the magnetic (left panels) and kinetic (right panels) energies. Panels (a) to (d) correspond to simulations A_NRB10 (no rotation), A_R300B10 ($P_{\text{rot}} = 300$ days), A_R150B10 ($P_{\text{rot}} = 150$ days), and A_R10B10 ($P_{\text{rot}} = 10$ days), respectively. Different colors correspond to the latitudes where the energy of the unstable modes is computed, the red and blue lines correspond to the north and south poles, and the thin green lines correspond to the equator. The black dot-dashed lines indicate the saturation phase (see the text).
slow initial linear growth, with a growth rate that depends on the rotation rate (see Table 1), followed by energy jumps. The panel (d) of Fig. 6, shows the evolution of simulation A10B10, with \( \eta = 197 \), where the initial growing is almost entirely suppressed.

Fig. 7 displays a comparison between the initial temporal evolution of the mode with magnetic \( m = 1 \) for the simulations of the set A1 (Table 1). The thicker lines correspond to the simulations presented in Fig. 6. The thin lines correspond to the remaining simulations as seen in the legend. The first \( \sim 15\Omega \) of the non-rotating simulation (A_NRB10) have been disregarded because they correspond to initial transient period described above. Note the decreasing growth rate of the simulations as a function of \( \eta \), in this case varied through the rotation rate. The jumps of energy occur after \( \sim 40\Omega \) for \( \eta \gtrsim 13 \). For intermediate values of \( \eta \) the TI coexists with the process generating these energy jumps. In the fast rotating simulations, \( \eta \gtrsim 100 \), the TI seems to be suppressed but the surges still occur. These results partially agree with the local simulations performed by Braithwaite (2006) which focus only in the early stages of the linear phase. Yet, these energy jumps are not observed in the time evolution of their models.

Fig. 8 shows the temporal evolution of simulations where \( \Omega_0 \) is kept constant and \( \eta \) is changed by considering different values of \( B_0 \) (set A3). The results confirm the same pattern discussed above, i.e., for large \( \eta \) (panel a) the TI is suppressed and the energy grows through consecutive jumps. For \( \eta \lesssim 100 \), the TI coexists with these energy surges. Note that these cases cannot be directly compared with the simulations of set A1, because varying \( B_0 \) also changes the value of \( \delta \). And both quantities influence the growth rate.

In EULAG-MHD the dissipation coefficients depend on the resolution of the model. Generally, it is expected that viscosity, thermal conductivity, and magnetic diffusivity decrease as resolution increases. As previous studies have demonstrated, changes in these coefficients also affects the TI.

Finally, Fig. 9 shows the time evolution of simulations in set B, where the grid resolution is increased by a factor of two in all directions. Panels (a) to (c) correspond to different value of \( \eta \) obtained with different rotation rates. Line styles and colors are the same as in Fig. 6. The resolution increase implies the reduction of thermal and magnetic diffusion coefficients, which have concurrent effects. A lower thermal diffusion makes the system more stable to TI, while lower magnetic diffusion makes it more unstable. Panel (a) shows that this did not significantly changed the non-rotating case displayed in Fig. 6(a). The major differences being the lower growth rate and consequent delayed saturation phase. This decrease in \( \sigma \) suggest a dominance of the thermal effects in this case. This result agrees with previous analytical (see e.g. Spruit 1999) and numerical (Braithwaite 2006) results. Meanwhile, for rotating simulations the change in \( \sigma \) is insignificant. For Panel (b) even the moment of saturation phase is similar to its counterpart in Fig. 6(b). Meanwhile, the fast rotating case demonstrates a tenfold increase in the stabilization during linear phase.

From all the sets of simulations presented in this section, it is possible to conclude that rotation stabilizes the development of the Tayler instability. Larger values of \( \eta \) and \( \delta \) are able to fully suppress the development of the \( m = 1 \) mode for a few tens of Alfvén travel times. Afterwards, a new instability occurs showing sudden surges of energy. Note that for very slow rotation, e.g., case A300B10, these jumps are not observed. While to this point we are not able to explain the physics of this second instability, the sets of simulations A1, A3 and B, show that its development depends on the magnetic field strength and the dissipation coefficients. The stronger the field, and the smaller the dissipation coefficients, the longer it takes for the jumps to occur. Independently of the nature of these energy surges, in all the cases the final energy of the unstable modes never surpasses the initial magnetic energy. Since the model solves for the full system of non-linear equations, it is possible to explore the long term behavior of the magnetic field in the simulations. The next sections presents the results concerning to the subsequent saturation and decaying stages.

### 3.2.2 Saturation phase

Once the unstable modes reach saturation energy, a change in the topology of the initial magnetic field is observed. The saturation, decay, as well as the final configuration of the magnetic field depend on the rotation rate. Figs. 10 and 11 show the magnetic field lines from the simulation with the slowest (A300B10, \( \eta = 6.7 \)) and fastest rotation (A10B10, \( \eta = 197 \)), respectively. On each figure the panels (a) depict the saturation stage where the topology still contains a prominent toroidal field, yet the displacement of the
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Figure 9. Evolution of the energy of initial modes of the doubled resolution simulations. Line style and color properties are the same as the ones used on Fig. 6.

Field lines show the imprint of the unstable modes. In the panels (b), the snapshots are taken during the decaying phase. The magnetic field strength is about one order of magnitude smaller than its initial strength, and the field lines are deformed mostly at equatorial latitudes. However, a toroidal structure is still observed. This is in contrast with the results obtained for the non-rotating case ($A_{NRB10}$, $\eta = 0$), presented in Fig. 4, and demonstrates the stabilizing effects of rotation. Also, note that the magnetic field contains a relevant poloidal component (vertical field lines) which may contribute to preserve a stable configuration.

The same behavior in the saturation stage is observed in the high resolution simulations (set B). A comparison between cases $B_{NRB10}$ ($\eta = 0$), $B_{R300B10}$ ($\eta = 6.7$) and $B_{R10B10}$ ($\eta = 197$) is presented in Fig. 12(a)-(c), respectively. The color coding in this figure is the same as in Fig. 2. The upper (lower) row shows projections of the north pole (equator). In panels (a) and (b), the mode $m = 1$ appears at the poles as a misaligned structure that wobbles around the rotation axis. At the equator, it appears as the opening of the field lines forming a clamshell-like structure. Note also the large inclination of the magnetic axis resulting in the non-rotating case. On the other hand, the field lines obtained in the fast rotating simulation (panel c), do not display relevant tilt of the magnetic axis, and the field lines at the equator are barely open, depicting only high order modes.

The qualitative description of the results in Figs. 11 and 12 can be quantified through the spectrum of the magnetic energy density. In Fig. 13 the energy spectrum, $\tilde{E}_B$, is calculated by decomposing the total vector magnetic field on its spherical harmonics representation using the SHTns library (Schaeffer 2013), at the radius $r = r_c$. Note that in this case the spectrum averages all the available latitudinal modes. The upper and lower panels correspond to characteristic simulations from set A1, $A_{NRB10}$ (blue line),
The magnetic energy is mostly in the modes $m = 0$ (the initial magnetic field) and $1$ (the fastest growing mode). Conversely, in the simulations with fast rotation (green) modes up to $m \sim 35$ develop similar energy than the mode $m = 1$. These results are independent of the grid resolution. The energy growth of the high order modes occurs during the energy surges discussed above. At the decaying phase, the profile of the energy density spectra is similar for all the simulations. They have a scaling law compatible with decay-turbulence, where $E_B \propto m^{-3/2}$ (Kolmogorov 1941). Note that the energy of the $m = 0$ mode decays substantially only for the non-rotating simulations (see continuous blue lines). This implies that even slow rotation is sufficient to conserve the initial magnetic field.

The effects of rotation during the non-linear phase of evolution can be explored through the temporal evolution of the energy contained in the toroidal component of the magnetic field, $E_{B\phi} = \frac{\mathbf{B}_\phi \cdot \mathbf{B}_\phi}{2}$. Here, the average in $\mathbf{B}$ is taken in the directions $\phi$, $\theta$, and $r$ between $0.65R_\star$ and $0.70R_\star$, corresponding to the full width at half maximum (FWHM) of the initial Gaussian profile of Eq. (8). Fig. 14(a) depicts the temporal evolution of $E_{B\phi}$, the continuous (dashed) lines correspond to simulations from the set A1 (B). Lines with different colors correspond to various rotation rates. The figure encompasses the linear, saturation, and the decaying phases until $t \sim 800t_\lambda$. Note that the decay of $E_{B\phi}$ during the linear phase is not evident in this representation because it is of the order of $10^{-6}$ (energy of the residual toroidal field). The graph demonstrates that rotation prevents the decay of the toroidal magnetic field. For no rotation the decay is sharp (see black lines). Progressively increasing the rotation rate results in slower and slower decay. The decay rate of the toroidal magnetic energy, $\Gamma$, can be estimated in time segments where the time derivative of $E_{B\phi}$ is constant (see the values of $\Gamma$ in Table 1). The black dotted lines are eye guides exemplifying these segments.

In addition, $\Gamma$ also depends on the numerical resolution, i.e., on the effective numerical magnetic diffusion. This is evident by comparing the continuous with the dashed lines corresponding to the sets of simulations A1 and B, respectively. A matter of fact, increasing the resolution only in the radial coordinate (set C, not presented in the figure), does not show the significant change in the decay rate observed when the resolution is increased twofold in the three coordinates (set B). This results confirms that the evolution of the field happens mainly in the horizontal directions whenever $\delta$ is sufficiently high.

In panel (b) of Fig. 14 we present $|\Gamma|$ as a function of $\eta$ for simulations from the sets A1 (blue circles), B (black triangles), C (green diamonds). The value of $|\Gamma|$ changes by a factor of 2.15 from the lowest to the highest rotating rates, also lowest to highest values of $\eta$ (simulations A_R300B10 to A_R10B10, blue circles), with a scaling that may be fitted with the power law $|\Gamma| \propto \eta^{-0.19}$. Extrapolating these law for $|\Gamma| \to 0$ (fully stable situation) implies unrealistic fast rotation. Nevertheless, the black triangles, corresponding to the set B, show that this result is sensitive to the resolution. For instance, the decay rate changes by a factor of 1.8 between simula-
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3.3 Long term evolution

After the sharp decay observed during the saturation phase, \( E_{B_\delta} \) continues decreasing. Interestingly, during this phase the slope of the curves changes and ultimately tends to have a similar slope, independent of the rotation rate or even of the numerical resolution (see \( t > 700 \tau_s \) in Fig. 14(a)). This may be interpreted as a turbulent diffusive decay also confirmed by the \( m^{-5/2} \) scaling observed in the magnetic power spectra (the same scaling is obtained for the kinetic power spectra, not presented in the paper). Therefore, by multiplying the \( |\Gamma| \) estimated for this stage (see red dotted line if Fig. 14(a)) by \( \sigma r^2 / 2 \), it is possible to evaluate the turbulent magnetic diffusivity, \( \beta_t \). For the set A1 of simulations \( \beta_t \approx 8 \times 10^7 \text{ m}^2/\text{s} \); for set B, \( \beta_t \approx 5 \times 10^7 \text{ m}^2/\text{s} \). This may have consequences regarding the life time of magnetic fields in SSLs.

The decay of toroidal field observed in Fig. 14(a) is followed by an increase of the poloidal component of the magnetic field which ultimately leads to a stable configuration. Figure 15 displays the ratio between the poloidal and the toroidal field components, \( \bar{B}_p/\bar{B}_\phi \), in terms of the Alfvén travel time. Here, \( \bar{B}_p = \sqrt{B_p^2 + B_\phi^2} \) and \( \bar{B}_\phi \) are the volume averaged poloidal and toroidal magnetic field components, respectively. The average is performed as described above. Continuous and dashed lines correspond to simulations of sets A1 and B, respectively. The black lines represent the non-rotating cases, while different colors correspond to different rotation rates.

A significant increase in the poloidal field is observed when the simulations reach the saturation phase (Fig. 15). The non-rotating cases, as well as the cases with slow rotation, \( \eta = 6.7 \), display a rapid increase of the poloidal component, followed by a decrease when the simulation advances towards the decaying phase. For simulations with \( \eta \gtrsim 30 \), the ratio \( \bar{B}_p/\bar{B}_\phi \) converges to a plateau at \( \sim 0.3 \). The same ratio is also observed for the set of simulations A3 where both parameters \( \delta \) and \( \eta \) change. In conclusion, both, the rotation and the poloidal field contribute to sustain the remnant toroidal field stable for several hundreds of Alfvén travel times. Nevertheless, this field is embedded in a turbulent region and decays accordingly. For the simulation with higher resolution B_R10B10, the final value of \( \bar{B}_p/\bar{B}_\phi \) is smaller than for their low resolution counterpart, A_R10B10 (see dashed light-blue line in Fig. 15). Therefore, although the initial field is unstable, the growth of the unstable modes is prevented by rotation. Furthermore, only...
Figure 13. Distribution of the spectral magnetic energy density, $\tilde{E}_B$, in the longitudinal modes, $m$, for simulations (a) of set A, $A_{NR10}$ (blue line), $A_{R10B10}$ (green), and $A_{R300B10}$ (red); and (b) of set B, $B_{NR10}$ (blue line), $B_{R10B10}$ (green), and $B_{R300B10}$ (red). The dashed lines correspond to the linear phase in a stage with similar energy between simulations. The continuous lines correspond to the decaying phase. In this phase we observe the slope typical of fully developed homogeneous and isotropic turbulence. Animations showing the temporal evolution of the magnetic and kinetic energy spectra, for simulations $A_{R10B10}$ and $B_{R10B10}$, are available as supplementary material.

Figure 14. (a) Evolution of $E_B$, as a function of time. The continuous (dashed) lines correspond to simulations from the set A1 (B). Lines with different colors correspond to various rotation rates as indicated in the legend. The black dotted lines show the time segments where the decay rate, $\Gamma$, is calculated (for the sake of clarity of the figure, only the segments corresponding to three characteristic simulations are shown). The turbulent magnetic diffusivity, $\eta_t$, is estimated in the late phase of evolution ($t \geq 500 [1/\omega_A]$) where the energy decays roughly at the same rate for all simulations (see red dotted line as an example). (b) Distribution of $\Gamma$ as a function of $\eta$ (Eq. 10) for simulations from the sets A1 (blue circles), B (black triangles), C (green diamonds). The simulations from set A2, with fixed rotation and varying magnetic fields, are presented with red squares (see Table 1). The inset presents the same results but with $\Gamma$ expressed in terms of the Alfvén travel time.

A small fraction of poloidal field leads to a configuration that remains stable, yet, this field is prone to turbulent diffusion. With the increase of resolution, the saturation and decay stages occur at a much longer time (compare solid and dashed light blue lines in Fig. 14(a)). This makes higher resolution simulations highly time consuming and prohibitive for the available supercomputer time. Thus, it remains uncertain how the ratio $B_p/B_\phi$ will change for effective diffusion approaching the values of the Ohmic magnetic diffusivity inside radiative zones. For the radiative zone below the solar tachocline, $\rho_{\text{Alfv}} \sim 10^3$ m$^2$/s (Zahn et al. 2007).

3.4 Helicities

The generation of the poloidal magnetic field observed in Fig. 15 must originate from the processes discussed in the previous sections. This field component develops as the magnetic field attempts to achieve a stable configuration (Tayler 1973). As seen, the rotating cases reach a steady state with a magnetic field mostly toroidal with a small fraction of the poloidal component. On the other hand, the poloidal field amplification is associated with the development of helical motions or currents in dynamo theory (Steenbeck et al. 1966). Note, however, that the fastest growing mode of the magnetic field associated to the Tayler instability is $m = 1$, whereas in the mean-field dynamo theory the large scale is normally considered axisymmetric. Therefore, mean-field dynamo coefficients require some axisymmetric contribution. By computing the small-scale kinetic and current helicities from the simulated data we can infer if such contribution exists.

In Fig. 16 we depict the time evolution of both the small scale kinetic and current helicities,

\[ H_u = \frac{1}{2} \left( \nabla \times \dot{u} \right) \cdot \dot{u} \]  

\[ H_c = \frac{1}{2} \left( \nabla \times \dot{B} \right) \cdot \dot{B} \]
This time correlates well with the increase of 
$B_{NR10}$, $B_{R300B10}$ and $B_{R10B10}$. The intensities of the helicities were multiplied by 
a factor of 5 on column a for better visualization.

Figure 15. Temporal evolution of the ratio $B_p/B_0$ for simulations in the 
sets A (continuous lines) and B (dashed lines). Note that rapidly rotating 
simulations seem to converge to a constant value of this ratio.

Figure 16. Temporal evolution of $H_u$ (blue) and $H_f$ (green) for simulations 
in sets A (column a) and B (column b), with each row corresponding to the 
same rotational period. The intensities of the helicities were multiplied by 
a factor of 5 on column a for better visualization.

$$H_u = \frac{1}{2} (\nabla \times B') \cdot B'/\mu_0 \rho \omega_0,$$

where the prime denotes perturbations of the velocity and 
magnetic fields with respect to the longitudinal mean, i.e., all 
modes but $m = 0$. The overline here has the same meaning previ-
ously defined, except that now the latitudinal average is done 
in the Northern hemisphere only. In Fig. 16, the green and blue 
lines corresponds to the $H_u$ and $H_f$, respectively. The left and right 
columns show three characteristic simulations from set A1, namely 
A$_{NRB10}$, A$_{R300B10}$ and A$_{R10B10}$, and three from the set B, 
B$_{NRB10}$, B$_{R300B10}$ and B$_{R10B10}$.

Signatures of the helicities are observed for all cases, starting 
to appear just before the saturation phase (see red vertical line). 
This time correlates well with the increase of $B_p/B_0$ observed in 
Fig. 15. Nevertheless, there is no preferred sign for the helicities,

but sign reversals occur in all cases. The amplitudes of $H_u$ and $H_f$ 
have the same order of magnitude, yet the signal of $H_f$ decays 
faster. It is clear, however, that the helicities from simulations of 
set B have higher amplitudes. Notably, the non-rotating cases (first 
row) display a stronger current helicity than the rotating counter-
parts. Meanwhile, in the rotating simulations the helicities last for 
several Alfvén travel times, depicting several sign reversals and a 
damped evolution.

Previous studies have demonstrated that large scale shear in 
combination with a sign changing may lead to incoherent $a$-effect 
and may be sufficient for sustaining the dynamo (e.g. Vishniac & 
Brandenburg 1997; Mitra & Brandenburg 2012). Thus, the results 
presented here might support the idea of a large-scale dynamo in a 
radiative layer. Simulations including shear are ongoing work.

4 SUMMARY AND CONCLUSIONS

Magnetic fields are presumably present in the radiative zone of 
solar-like stars and have also been observed at the surface of mas-
sive Ap/Bp objects. However, certain magnetic fields topologies in 
these layers are prone to different kinds of instabilities (e.g., Tayler 
1973; Markey & Tayler 1973b).

This study focuses on the Tayler instability (TI), where the 
decay of an initial axisymmetric toroidal field leads to the growing 
of other longitudinal modes, modifying the original configuration. 
Tayler (1973); Pitts & Tayler (1985) proposed that this instabil-
ity could be quenched by the influence of rotation and/or by 
the presence of a poloidal field whose forces opposes the displace-
ment caused by the unstable modes. In this work, we use non-linear 
MHD simulations, in spherical geometry, to explore the stabiliz-
ing role of rotation on a toroidal magnetic field, anti-symmetric 
about the equator, permeating a layer whose values of the Brunt-
Väisälä frequency are similar to those of the lower part of the so-
lar tachocline. The simulations were performed with the EULAG-
MHD code, which solves the anelastic MHD equations. To un-
derstand the role of rotation in the Tayler instability we consider non-
rotating and rotating cases, with rotational periods between 300 and 
10 days. Alternatively, we present cases with fixed rotation and vari-
ous initial magnetic field strengths. Our analysis was performed in 
terms of the non dimensional quantities, $\eta$ (Eq. 10), which mea-
sures the relative importance of the Coriolis and magnetic forces, 
and $\delta$, measuring the relative importance of the buoyant and 
magnetic forces.

Since the magnetic diffusivity, as well as other dissipative pro-
cesses, are expected to be insignificant in the solar radiative zone, 
we use the code without explicit dissipation terms. Nevertheless, 
there is always a numerical effective dissipation of all quantities 
which in the EULAG-MHD code is nonlinear and intermittent in 
space and time, and depends on the grid size. Thus, increasing the 
numerical resolution of the model results in less dissipation. This 
affects the dynamics of the magnetic field in different forms. We re-
mark that simultaneously diminishing thermal diffusion and mag-
netic diffusivity may have opposite effects regarding the TI (Zahn 
1974; Braithwaite 2006).

During the evolution of the magnetic field we identify three 
stages. First, the linear phase, where the unstable modes grow ex-
ponentially. Most of the previous work regarding TI has focused 
on this phase. Second, when the growing modes reach their maxi-
mum energy, the initial magnetic field exhibits a sharp decay. This 
is identified as the saturation phase. And third, a diffusive decay
phase. At this stage the energy spectrum follows a scaling law with $E \sim m^{-5/3}$. Thus, we identify this as turbulent decay.

For all sets of simulations presented in Table 1, we find that the cases without rotation or rotating slowly ($\eta < 6.7$) present a clear exponential growth of the longitudinal mode $m = 1$ during the linear phase. This is the canonical signature of the Taylor instability. Although the unstable modes develop first at polar regions, there is no significant difference in the growth rate between polar and equatorial latitudes.

For $\eta > 13.4$, the influence of rotation appears and the mode $m = 1$ is kept stable for several Alfvén travel times. However, after this stable phase, sudden energy surges occur for the modes $m \geq 1$, which subsequently reach energy levels close to the energy of $B_{s0}$. This is an unexpected result and, to our knowledge, not reported in the literature. We have been unable to identify the nature of these energy surges and they will be subject of follow up studies. Nevertheless, the results from simulations in sets A3 and B indicate that the time interval where the modes $m \geq 1$ remain stable depends on the amplitude of the initial magnetic field and on the resolution of the simulations, tantamount of the dissipative processes. These dependencies make us believe that this is a physical phenomenon rather than a numerical artifact.

Either because of the TI or the energy surges, the unstable modes reach energy levels comparable to that of the initial field, $B_{s0}$, i.e., in both cases the energy of the growing modes is extracted from the initial magnetic field which decays at this saturation stage. Consequently, a substantial change in the magnetic field topology is observed during the saturation phase. The decay rate of these changes, $|\Gamma|$ (eq. 7), depends on the value of $\eta$. The results demonstrate that the initial magnetic field topology is less affected by the TI whenever the relevant time scale is set by the rotation.

As the toroidal field decays, the formation of a poloidal component is observed in all simulations. The analysis shows that for simulations with $\eta \geq 30$, the ratio $B_\phi/B_0$ converges to a plateau at $\sim 0.3$. The same ratio is also observed for the set of simulations A3 where both parameters, $\delta$ and $\eta$, change. For the high resolution simulation B_R10B10, this ratio is smaller, $B_\phi/B_0 \sim 0.1$. The implication of this finding is that, despite the initial instability, the growing of unstable modes is effectively prevented by the Coriolis force. These modes transfer only a small fraction from the initial magnetic energy into a poloidal field. The resultant configuration remains topologically stable by hundreds of Alfvén travel times (see Fig. 12).

After reaching such a ratio, the simulations evolve with this equilibrium configuration prone only to turbulent diffusion. For these cases, despite the decay in amplitude, the field experiences minor topological changes. On the other hand, in cases without rotation or with lower $\eta$, the initial topology is mostly lost. The poloidal component grows rapidly and eventually leads the simulation to stop, requiring prohibitive smaller time steps.

During the generation of poloidal field, prior to the saturation phase, the simulations develop helical motions and currents. They do not show a preferred hemispheric sign, but several reversals with damping evolution. Interestingly, the amplitude of the helicities is larger for the non-rotating simulations. However, in these cases the helicities also decay faster. Such incoherent $\alpha$-effect may be sufficient for sustaining a dynamo. Thus, the results presented here might support the idea of a large-scale Tayler-Spruit dynamo in a radiative layer. To sustain this dynamo, however, a certain amount of shear is required to replenish the toroidal field. The evolution of the field under these conditions is left for a future study.

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**DATA AVAILABILITY STATEMENT**

Analysis results and data underlying this article will be shared on request to the corresponding author.

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