Trotter product formula and linear evolution equations on Hilbert spaces

On the occasion of the 100th birthday of Tosio Kato

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Abstract The paper is devoted to evolution equations of the form

$$\frac{\partial}{\partial t}u(t) = -(A + B(t))u(t), \quad t \in \mathbb{I} = [0, T],$$

on separable Hilbert spaces where $A$ is a non-negative self-adjoint operator and $B(\cdot)$ is family of non-negative self-adjoint operators such that $\text{dom}(A^\alpha) \subseteq \text{dom}(B(t))$ for some $\alpha \in [0, 1)$ and the map $A^{-\alpha}B(\cdot)A^{-\alpha}$ is Hölder continuous with the Hölder exponent $\beta \in (0, 1)$. It is shown that the solution operator $U(t, s)$ of the evolution equation can be approximated in the operator norm by a combination of semigroups generated by $A$ and $B(t)$ provided the condition $\beta > 2\alpha - 1$ is satisfied. The convergence rate for the approximation is given by the Hölder exponent $\beta$. The result is proved using the evolution semigroup approach.

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1 Introduction

A closer look to Kato’s work shows that abstract evolution equations and Trotter product formula were topics of high interest for Kato. Already at the beginning of his scientific career Kato was interested in evolution equations \[16, 17\]. This interest has lasted a lifetime \[18, 19, 20, 21, 26, 27, 29\]. Another topic of great interest for him was the so-called Trotter product formula \[23, 24, 22, 28\]. Even the paper \[23\] has inspired further developments in this field \[15\].

The topic of the present paper is to link evolution equations with the Trotter product formula. To this end we consider an abstract evolution equation of type

\[
\frac{\partial u(t)}{\partial t} = C(t)u(t), \quad u(s) = x_s, \quad s \in [0,T), \quad t \in I := [0,T],
\]

(1.1)
on the separable Hilbert space \(H\). Evolution equations of that type on Hilbert or Banach spaces are widely investigated, cf. \[1, 2, 4, 3, 7, 29, 31, 32, 43, 44, 45, 46, 47, 48, 51, 52, 53, 55, 56\] or the books \[5, 49, 54\]. We consider the equation (1.1) under the following assumptions.

Assumption 1.1

(S1) The operator \(A\) is self-adjoint in the Hilbert space \(H\) such that \(A \geq\ I\). Let \(\{B(t)\}_{t \in I}\) be a family of non-negative self-adjoint operators in \(H\) such that the function \((I + B(t))^{-1} : I \rightarrow \mathcal{L}(H)\) is strongly measurable.

(S2) There is an \(\alpha \in [0, 1)\) such that for a.e. \(t \in I\) the inclusion \(\text{dom}(A^\alpha) \subseteq \text{dom}(B(t))\) holds. Moreover, the function \(B(\cdot)A^{-\alpha} : I \rightarrow \mathcal{L}(H)\) is strongly measurable and essentially bounded, i.e.

\[
C_\alpha := \text{ess sup}_{t \in I} \|B(\cdot)A^{-\alpha}\| < \infty.
\]

(1.2)

(S3) The map \(A^{-\alpha}B(\cdot)A^{-\alpha} : I \rightarrow \mathcal{L}(H)\) is Hölder continuous, i.e, for some \(\beta \in (0, 1)\) there is a constant \(L_{\alpha, \beta} > 0\) such that the estimate

\[
\|A^{-\alpha}(B(t) - B(s))A^{-\alpha}\| \leq L_{\alpha, \beta}|t - s|^\beta, \quad (t, s) \in I \times I,
\]

(1.3)

holds. \(\triangle\)

Notice that under the assumption (S2) the operator \(C(t)\) is also an invertible non-negative self-adjoint operator for each \(t \in I\). Assumptions of that type were made in \[13, 14, 53, 55, 56\]. One checks that the assumptions (S1)-(S3) and the additional assumption \(\beta > \alpha\) imply the assumptions (I), (VI) and (VII) of \[56\] for the family \(\{C(t)\}_{t \in I}\). Hence, Proposition 3.1 and Theorem 3.2 of \[56\] yield the existence of a so-called solution (or evolution) operator for the evolution equation \[1.1\], i.e., a strongly continuous, uniformly bounded family of bounded operators \(\{U(t, s)\}_{(t, s) \in \Delta}, \Delta := \{(t, s) \in I \times I : 0 \leq s \leq t \leq T\}\), such that the conditions
Trotter product formula and evolution semigroups

\[ U(t,t) = I, \quad \text{for} \quad t \in \mathcal{I}, \]
\[ U(t,r)U(r,s) = U(t,s), \quad \text{for} \quad t, r, s \in \mathcal{I} \quad \text{with} \quad s \leq r \leq t, \quad \text{(1.4)} \]

are satisfied and \( u(t) = U(t,0)x \) is for every \( x \in \mathcal{H} \) a strict solution of \( (1.1) \), see Definition 1.1 of [56]. Because the involved operators are self-adjoint and non-negative one checks that the solution operator consists of contractions.

The aim of the present paper is to analyze the convergence of the following approximation to the solution operator \( \{U(t,s)\}_{(t,s) \in \mathcal{A}} \). Let

\[ s =: t_0 < t_1 < \ldots < t_n =: t, \quad t_j =: s + j\frac{r-s}{n}, \quad \text{(1.5)} \]

\( j = \{0,1,2,\ldots,n\}, n \in \mathbb{N}, \) be a partition of the interval \([s, t]\). Let

\[ G_j(t,s;n) := e^{-\frac{r-s}{n}A}e^{-\frac{s-t}{n}B(t)}, \quad j = 0, 1, 2, \ldots, n, \]
\[ V_n(t,s) := G_{n-1}(t,s;n)G_{n-2}(t,s;n) \cdots G_1(t,s;n)G_0(t,s;n), \quad \text{(1.6)} \]

\( n \in \mathbb{N} \). The main result in the paper is the following. If the assumptions (S1)-(S3) are satisfied and in addition the condition \( \beta > \alpha \) holds, then the solution operator \( \{U(t,s)\}_{(t,s) \in \mathcal{A}} \) of [56] admits the approximation

\[ \text{ess sup}_{(t,s) \in \mathcal{A}} \|V_n(t,s) - U(t,s)\| \leq \frac{R_\beta}{n^\gamma}, \quad n \in \mathbb{N}, \quad \text{(1.7)} \]

with some constant \( R_\beta > 0 \). The result shows that the convergence of the approximation \( \{V_n(t,s)\}_{(t,s) \in \mathcal{A}} \) is determined by the smoothness of the perturbation \( B(\cdot) \).

If the map \( A^{-\alpha}B(\cdot)A^{-\alpha} : \mathcal{I} \to \mathcal{L}(\mathcal{H}) \) is Lipschitz continuous, then the map is of course Hölder continuous with any exponent \( \gamma \in (\alpha, 1) \). Hence from (1.7) it immediately follows that for any \( \gamma \in (\alpha, 1) \) there is a constant \( R_\gamma \) such that

\[ \text{ess sup}_{(t,s) \in \mathcal{A}} \|V_n(t,s) - U(t,s)\| \leq \frac{R_\gamma}{n^{\gamma}}, \quad n \in \mathbb{N}, \quad \text{(1.8)} \]

In particular, for any \( \gamma \) close to one the estimate (1.8) holds.

In [14] the Lipschitz case was considered. It was shown that there is a constant \( \gamma_0 > 0 \) such that the estimate

\[ \text{ess sup}_{t \in \mathcal{I}} \|V_n(t,0) - U(t,0)\| \leq \gamma_0 \frac{\log(n)}{n}, \quad n = 2, 3, \ldots, \quad \text{(1.9)} \]

holds. It is obvious that the estimate (1.9) is stronger than

\[ \text{ess sup}_{t \in \mathcal{I}} \|V_n(t,0) - U(t,0)\| \leq \frac{R_\gamma}{n^{\gamma}}, \quad n \in \mathbb{N}, \]

(which follows from (1.8)) for any \( \gamma \) independent of how close it is to one.
To prove (1.7) we use the so-called evolution semigroup approach which allows not only to verify the estimate (1.7) but also to generalise it. The approach is quite different from the technique used in [14, 56]. We have successfully applied this approach already in [33] and [35]. The key idea is to forget about the evolution equation (1.1) and to consider instead of it the operators $K_0$ and $K$ on $H = L^2(I, \mathcal{F})$. The operator $K_0$ is the generator of the contraction semigroup $\{U_0(\tau)\}_{\tau \in \mathbb{R}_+}$,

$$\lim_{\tau \to 0^+} (U_0(\tau)f)(t) = e^{-\tau A} \chi_I(t-t_0)f(t), \quad f \in L^2(I, \mathcal{F}), \quad (1.10)$$

and $K$ is given by

$$K = K_0 + B, \quad \text{dom}(K) = \text{dom}(K_0) \cap \text{dom}(B),$$

where $B$ is the multiplication operator induced by the family $\{B(t)\}_{t \in I}$ in $L^2(I, \mathcal{F})$ which is self-adjoint and non-negative, for more details see Section 2. It turns out that under the assumptions (S1) and (S2) the operator $K_0$ is the generator of a contraction semigroup $\{U(\tau)\}_{\tau \in \mathbb{R}_+}$ on $L^2(I, \mathcal{F})$. For the pair $\{K_0, B\}$ we consider the Lie-Trotter product formula. From the original paper of Trotter [50] one gets that

$$\lim_{n \to \infty} \left( e^{-\frac{\tau}{n} K_0} e^{-\frac{\tau}{n} B} \right)^n = e^{-\tau K}, \quad \tau \in \mathbb{R}_+ := [0, \infty), \quad (1.11)$$

holds uniformly in $\tau$ on any bounded interval of $\mathbb{R}_+$. Since $e^{-\tau K_0} = 0$ and $e^{-\tau K} = 0$ for $\tau \geq T$ one gets even uniformly in $\tau \in \mathbb{R}_+$.

Previously it was shown that under certain assumptions the strong convergence can be improved to operator-norm convergence on Hilbert spaces, see [9, 10, 15, 38, 42] as well as on Banach spaces, see [11]. For an overview the reader is referred to [57]. To consider the Trotter product formula for evolution equations is relatively new and was firstly realized in [34, 35] for Banach spaces.

In the following we improve the convergence (1.11) to operator-norm convergence. We show that under the assumptions (S1)-(S3) and $\beta > 2\alpha - 1$ there is a constant $R_\beta > 0$ such that

$$\sup_{t \in \mathbb{R}_+} \left\| \left( e^{-\frac{\tau}{n} K_0} e^{-\frac{\tau}{n} B} \right)^n - e^{-\tau K} \right\| \leq \frac{R_\beta}{n^\beta}, \quad n \in \mathbb{N}, \quad (1.12)$$

holds.

It turns out that $K$ is the generator of an evolution semigroup. This means, there is a propagator $\{U(t,s)\}_{(t,s) \in A_0}$, $A_0 := \{(t,s) \in I_0 \times I_0 : s \leq t\}$, $I_0 = (0, T)$, such that the contraction semigroup $\{U(\tau) = e^{-\tau K}\}_{\tau \in \mathbb{R}_+}$ admits the representation

$$(U(\tau)f)(t) = U(t, t-\tau) \chi_I(t-\tau)f(t), \quad f \in L^2(I, \mathcal{F}), \quad (1.13)$$

We recall that a strongly continuous, uniformly bounded family of bounded operators $\{U(t,s)\}_{(t,s) \in A_0}$ is called a propagator if (1.12) is satisfied for $I_0$ and $A_0$ instead of $I$ and $A$, respectively. Roughly speaking, a propagator is a solution operator
restricted to $\Delta_0$ where the assumption that $U(t,0)x$ should be a strict solution is dropped. Obviously, the notion of a propagator is weaker than that of a solution operator. For its existence one needs only the assumptions (S1) and (S2), see Theorem 4.4 and 4.5 in [34] or Theorem 3.3 [35]. Of course, the propagator coincides with the solution operator of [56] if the assumptions (S1)-(S3) are satisfied and $\beta > \alpha$.

By Proposition 3.8 of [37] and (1.12) we immediately get that under the assumptions (S1)-(S3) and $\beta > 2\alpha - 1$ the estimate

$$\text{ess sup}_{(t,s) \in \Delta_0} \| V_n(t,s) - U(t,s) \| \leq \frac{R_\beta}{n^\beta}, \quad n \in \mathbb{N},$$

holds, where the constant $R_\beta$ is that one of (1.12). Notice that the condition $\beta > 2\alpha - 1$ is weaker than $\beta > \alpha$, i.e., if $\beta > \alpha$, then $\beta > 2\alpha - 1$ holds. If $\alpha$ satisfies the condition $\frac{1+\beta}{2} > \alpha > \beta$, then the assumptions (I), (VI) and (VII) of [56] for the family $\{C(t)\}_{t \in I}$ are not valid but nevertheless we get an approximation of the corresponding propagator $\{U(t,s)\}_{(t,s) \in \Delta_0}$.

The results are stronger than those in [34, 35] for Banach spaces. In [34] a convergence rate $O(n^{-(\beta-\alpha)})$ was found, whereas in [35] the Lipschitz case has been considered and the rate $O(n^{-(1-\alpha)})$ for $\alpha \in (\frac{1}{2}, 1)$ was proved.

It turns out that the result (1.7) can be hardly improved. Indeed in [36] the simple case $\mathcal{H} := \mathbb{C}$ and $A = 1$ was considered. In that case the family $\{B(t)\}_{t \in I}$ reduces to a non-negative bounded measurable function: $b(\cdot) : I \to \mathbb{R}$ which has to be Hölder continuous with exponent $\beta \in (0, 1)$. For that case it was found in [36] that the convergence rate is $O(n^{-\beta})$ which coincides with (1.7). For the Lipschitz case it was found $O(n^{-1})$ which suggests that (1.8) and (1.9) might be not optimal.

The paper is organised as follows. In Section 2 we give a short introduction into evolution semigroups. For more details the reader is referred to [33, 34, 39, 40].

The results are proven in Section 3. In Section 3.1 we prove auxiliary results which are necessary to prove the main results of Section 3.2.

**Notation:** Spaces, in particular, Hilbert are denoted by Gothic capital letters like $\mathcal{H}$, $\mathcal{K}$, etc. Operators are denoted by Latin or italic capital letters. The Banach space of bounded operators on space is denoted by $L(\cdot)$, like $L(\mathcal{H})$. We set $\mathbb{R}_+ := [0, \infty)$. If a function is called measurable, then it means Lebesgue measurable. The notation “a.e.” means that a statement or relation fails at most for a set of Lebesgue measure zero. In the following we use the notation $\text{ess sup}_{P_{(t,s)} \in \mathcal{A}}$ or $\text{ess sup}_{P_{(t,s)} \in \Delta_0}$. In that case the Lebesgue measure of $\mathbb{R}^2$ is meant.

We point out that we call operator $K$ to be generator of a semigroup $\{e^{-\tau K}\}_{\tau \in \mathbb{R}_+}$, see e.g. [41], although in [12, 25] it is the operator $-K$, which is called the generator.
2 Evolution semigroups

Below we consider the Hilbert space $\mathcal{H} = L^2(\mathcal{I}, \mathcal{F})$ consisting of all measurable functions $f(\cdot) : \mathcal{I} \to \mathcal{F}$ such that the norm function $\|f(\cdot)\| : \mathcal{I} \to \mathbb{R}_+$ is square integrable. Further, let $D_0$ be the generator of the right-hand shift semigroup on $L^2(\mathcal{I}, \mathcal{F})$, i.e.

$$(e^{-\tau D_0}f)(t) = \chi_{\mathcal{I}}(t - \tau)f(t - \tau), \quad t \in \mathcal{I}, \quad \tau \in \mathbb{R}_+, \quad f \in L^2(\mathcal{I}, \mathcal{F}).$$

Notice that $e^{-\tau D_0} = 0$ for $\tau \geq T$. The generator $D_0$ is given by

$$(D_0f)(t) := \frac{\partial}{\partial t} f(t), \quad f \in \text{dom}(D_0) := W^{1,2}_0(\mathcal{I}, \mathcal{F}) = \{f \in W^{1,2}(\mathcal{I}, \mathcal{F}) : f(0) = 0\}.$$  \hfill (2.1)

We remark that $D_0$ is a closure of the maximal symmetric operator and its semigroup is contractive.

Further we consider the multiplication operator $A$ in $L^2(\mathcal{I}, \mathcal{F})$,

$$(Af)(t) := A f(t), \quad f \in \text{dom}(A) := \left\{ f \in L^2(\mathcal{I}, \mathcal{F}) : f(t) \in \text{dom}(A) \quad \text{for a.e.} \ t \in \mathcal{I} \right\} \quad A f(t) \in L^2(\mathcal{I}, \mathcal{F}).$$

If (S1) is satisfied, then $A$ is self-adjoint and $A \geq I_{L^2(\mathcal{I}, \mathcal{F})}$. For the resolvent one has the representation

$$((A - z)^{-1}f)(t) = (A - z)^{-1}f(t), \quad t \in \mathcal{I}_0, \quad f \in L^2(\mathcal{I}, \mathcal{F}), \quad z \in \rho(A) = \rho(A),$$

and the corresponding semigroup $\{e^{-\tau A}\}_{\tau \in \mathbb{R}_+}$ is given by

$$(e^{-\tau A}f)(t) = e^{-\tau A} f(t), \quad t \in \mathcal{I}, \quad f \in L^2(\mathcal{I}, \mathcal{F}), \quad \tau \in \mathbb{R}_+. \hfill (2.2)$$

Notice that the operators $e^{-\tau D_0}$ and $e^{-\tau A}$ commute. Let us consider the contraction semigroup

$$U_0(\tau) := e^{-\tau D_0} e^{-\tau A}, \quad \tau \in \mathbb{R}_+. \hfill (2.3)$$

Obviously, the semigroup $\{U_0(\tau)\}_{\tau \in \mathbb{R}_+}$ admits the representation (1.13). Due to the maximal $L^2$-regularity of $A$, cf. [6], its generator $K_0$ is given by

$$K_0 := D_0 + A, \quad \text{dom}(K_0) := \text{dom}(D_0) \cap \text{dom}(A).$$

Further we consider the multiplication operator $B$, defined as

$$(Bf)(t) := B(t)f(t) \quad f \in \text{dom}(B) := \left\{ f \in L^2(\mathcal{I}, \mathcal{F}) : f(t) \in \text{dom}(B(t)) \quad \text{for a.e.} \ t \in \mathcal{I} \right\} \quad B(t)f(t) \in L^2(\mathcal{I}, \mathcal{F}).$$ \hfill (2.4)
If (S1) is satisfied, then $B$ is self-adjoint and non-negative. For the resolvent we have the representation

$$((B - z)^{-1} f)(t) = (B(t) - z)^{-1} f(t), \quad f \in L^2(I, \mathcal{H}), \quad z \in \mathbb{C}_+,$$

for a.e. $t \in I$. The semigroup $\{e^{-\tau B}\}_{\tau \in \mathbb{R}^+}$, admits the representation

$$(e^{-\tau B} f)(t) = e^{-\tau B(t)} f(t), \quad f \in L^2(I, \mathcal{H}),$$

for a.e. $t \in I$.

By [34, Proposition 4.4] we get that under the assumptions (S1) and (S2) the operator $K := \mathcal{K}_0 + B, \quad \text{dom}(K) := \text{dom}(\mathcal{K}_0) \cap \text{dom}(B)$, is a generator of a contraction semigroup on $L^2(I, \mathcal{H})$. From [34, Proposition 4.5] we obtain that $K$ is the generator of an evolution semigroup. Because $K$ is a generator of a contraction semigroup it turns out that the corresponding propagator consists of contractions.

If $\{U(t,s)\}_{(t,s) \in \mathcal{D}_0}$ is a propagator, then by virtue of (1.13) it defines a semigroup, which by definition is an evolution semigroup. It turns out that there is a one-to-one correspondence between the set of evolution semigroups on $L^2(I, \mathcal{H})$ and propagators. It is interesting to note that evolution generators can be characterize quite independent from a propagator, see [33, Theorem 2.8] or [34, Theorem 3.3].

3 Results

We start with a general observation concerning the conditions (S1)-(S3).

**Remark 3.1** If the conditions (S1)-(S3) are satisfied for some $\alpha \in [0, 1)$, then they are also satisfied for each $\alpha' \in (\alpha, 1]$. Indeed, the condition (S1) is obviously satisfied. To show (S2) we note that $\text{dom}(A^{\alpha'}) \subseteq \text{dom}(A^\alpha) \subseteq \text{dom}(B(t))$ for a.e. $t \in I$. Using the representation

$$B(t)A^{-\alpha'} = B(t)A^{-\alpha}A^{-(\alpha'-\alpha)} \quad (3.1)$$

for a.e. $t \in I$ we get that the map $B(\cdot)A^{-\alpha'} : I \rightarrow \mathcal{L}(\mathcal{H})$ is strongly measurable. Further, from (3.1)

$$C_{\alpha'} := \text{ess sup}_{t \in I} \|B(t)A^{-\alpha'}\| \leq \text{ess sup}_{t \in I} \|B(t)A^{-\alpha}\| = C_\alpha < \infty.$$

Moreover we have

$$\|A^{-\alpha'}(B(t) - B(s))A^{-\alpha'}\| \leq \|A^{-\alpha}(B(t) - B(s))A^{-\alpha}\| \leq L_{\alpha, \beta}|t - s|^{\beta},$$

$(t, s) \in I \times I$, which shows that there is a constant $L_{\alpha', \beta} \leq L_{\alpha, \beta}$ such that
\|A^{-\alpha'}(B(t) - B(s))A^{-\alpha'}\| \leq L_{\alpha',\beta}|t - s|^\beta \quad (t, s) \in I \times I.

holds for \((t, s) \in I \times I\).

Since \(A\) is self-adjoint and non-negative, one has \(\|A^\gamma e^{-\tau A}\| \leq 1/\tau^\gamma\) for any \(\tau \in \mathbb{R}_+\) and \(\gamma \in [0, 1]\). Then by virtue of (2.2) and of (1.10), (2.3) one gets the estimates

\[\|A^\gamma e^{-\tau A}\| = \|e^{-\tau A}A^\gamma\| \leq \frac{1}{\tau^\gamma}\]

(3.2)

### 3.1 Auxiliary estimates

In this section we prove a series of estimates necessary to establish (1.12). The following lemma can be partially derived from [34, Lemma 7.4].

**Lemma 3.2** Let the assumptions (S1) and (S2) be satisfied. Then for any \(\gamma \in [\alpha, 1)\) there is a constant \(\Lambda_\gamma \geq 1\) such that

\[\|A^\gamma e^{-\tau A}\| \leq \Lambda_\gamma \tau^\gamma\]

(3.3)

holds.

**Proof.** The proof of the first estimate follows from Lemma 7.4 of [34] and Remark 3.1. The second estimate can be proved similarly as the first one. One has only to modify the proof of Lemma 7.4 of [34] in a suitable manner and to apply again Remark 3.1. \(\square\)

**Remark 3.3** Lemma 2.1 of [14] claims that for the Lipschitz case the solution operator \(\{U(t, s)\}_{(t, s) \in I}\) of (1.1) admits the estimates

\[\sup_{(t, s) \in I} (t - s)^\gamma \|A^\gamma U(t, s)\| < \infty \quad \text{and} \quad \sup_{(t, s) \in I} (t - s)^\gamma \|U(t, s)A^\gamma\| < \infty\]

for \(\gamma \in [0, 1]\). Proposition 2.1 of [36] immediately yields that the corresponding evolution semigroup \(\{U(t) = e^{-\tau K}\}_{t \in \mathbb{R}_+}\) satisfies the estimates (3.3) for \(\gamma = 1\). \(\triangle\)

Now we set

\[T(\tau) = e^{-\tau K_0}e^{-\tau B}, \quad \tau \in \mathbb{R}_+.

(3.4)

Notice that \(T(\tau) = 0\) for \(\tau \geq T\).

**Lemma 3.4** Let the assumptions (S1) and (S2) be satisfied. Then for any \(\gamma \in [\alpha, 1)\) the estimates

\[\|A^{-\gamma}(T(\tau) - U(\tau))\| \leq 2C_\gamma \tau\quad \text{and} \quad \|(T(\tau) - U(\tau))A^{-\gamma}\| \leq 2C_\gamma \tau,\]

(3.5)

hold for \(\tau \geq 0\), where

\[C_\gamma := \text{ess sup}_{t \in I} \|B(t)A^{-\gamma}\|.

(3.6)
Hence, we obtain the identity
\[ \kappa C_3.1. \] The specific constant 2Cγ is obtained following carefully the proof of Lemma 7.6 of [34]. The second estimate can be proved modifying the proof of the first estimate in an obvious manner. □

**Lemma 3.5** Let the assumptions (S1)-(S3) be satisfied. Then for any \( \gamma \in [\alpha, 1) \) and \( \beta \in (0, 1) \) there is a constant \( Z(\gamma, \beta) > 0 \) such that
\[
\| A^{-\gamma}(T(\tau) - U(\tau))A^{-\gamma} \| \leq Z(\gamma, \beta) \tau^{1 + \varkappa}, \quad \tau \in \mathbb{R}^+, \quad (3.7)
\]
holds where \( \varkappa := \min\{\gamma, \beta\} \).

**Proof.** We use the representation:
\[
\frac{d}{d\sigma} e^{-(\tau-\sigma)K} e^{-\sigma A} e^{-\sigma B} = e^{-(\tau-\sigma)K} \left\{ K e^{-\sigma B} - e^{-\sigma K} e^{-\sigma B} \right\} e^{-\sigma B} = e^{-(\tau-\sigma)K} \left\{ B e^{-\sigma K} e^{-\sigma B} \right\} e^{-\sigma B}
\]
which yields
\[
e^{-(\tau-\sigma)K} \left\{ B e^{-\sigma K} e^{-\sigma B} \right\} e^{-\sigma B} = \left( e^{-(\tau-\sigma)K} - e^{-(\tau-\sigma)K_0} \right) \left\{ B e^{-\sigma K_0} e^{-\sigma B} \right\} (e^{-\sigma B} - I) + \]
\[
ee^{-(\tau-\sigma)K_0} \left\{ B e^{-\sigma K_0} - e^{-\sigma K} e^{-\sigma K_0} B \right\} (e^{-\sigma B} - I) + \]
\[
\left( e^{-(\tau-\sigma)K} - e^{-(\tau-\sigma)K_0} \right) \left\{ B e^{-\sigma K_0} - e^{-\sigma K_0} B \right\} + \]
\[
ee^{-(\tau-\sigma)K_0} \left\{ B e^{-\sigma K_0} - e^{-\sigma K} e^{-\sigma K_0} B \right\} .
\]
Hence, we obtain the identity
\[
A^{-\gamma} e^{-(\tau-\sigma)K} \left\{ B e^{-\sigma K_0} - e^{-\sigma K_0} B \right\} e^{-\sigma B} A^{-\gamma} =
\]
\[
A^{-\gamma} \left( e^{-(\tau-\sigma)K} - e^{-(\tau-\sigma)K_0} \right) \left\{ B e^{-\sigma K_0} e^{-\sigma B} \right\} (e^{-\sigma B} - I)A^{-\gamma} + \]
\[
e^{-(\tau-\sigma)K_0} A^{-\gamma} \left\{ B e^{-\sigma K_0} - e^{-\sigma K} e^{-\sigma K_0} B \right\} (e^{-\sigma B} - I)A^{-\gamma} + \]
\[
A^{-\gamma} \left( e^{-(\tau-\sigma)K} - e^{-(\tau-\sigma)K_0} \right) \left\{ B e^{-\sigma K_0} - e^{-\sigma K_0} B \right\} A^{-\gamma} + \]
\[
e^{-(\tau-\sigma)K_0} A^{-\gamma} \left\{ B e^{-\sigma K_0} - e^{-\sigma K_0} B \right\} A^{-\gamma}
\]
which leads to the estimate
where (3.13) was used for the last inequality.

Since the fundamental properties of semigroups and (3.6) yield

we get for (3.8) the estimate

Note that by (3.2) and (3.6) one gets

Due to (3.12) one estimates (3.8) as

for \( \sigma > 0 \). Due to (3.12) one estimates (3.8) as

Since the fundamental properties of semigroups and (3.6) yield

and

we get for (3.8) the estimate

To estimate (3.9) we recall that \( \mathcal{A} \) and \( \mathcal{K}_0 \) commute. Then by (3.6) one gets

where (3.15) was used for the last inequality.

To estimate (3.10) we have

(3.16)
To estimate (3.11) we use the representation
\[
A^{-\gamma} \left\{ B e^{-\sigma K_0} - e^{-\sigma K_0} B \right\} A^{-\gamma} = A^{-\gamma} B (e^{-\sigma A} - I) A^{-\gamma} e^{-\sigma D_0} A^{-\gamma} B A^{-\gamma+} A^{-\gamma} \left\{ B e^{-\sigma D_0} - e^{-\sigma D_0} B \right\} A^{-\gamma},
\]
that yields
\[
\left\| A^{-\gamma} \left\{ B e^{-\sigma K_0} - e^{-\sigma K_0} B \right\} A^{-\gamma} \right\| \leq \left\| A^{-\gamma} B (e^{-\sigma A} - I) A^{-\gamma} \right\| + \left\| A^{-\gamma} (e^{-\sigma A} - I) B A^{-\gamma} \right\| + \left\| A^{-\gamma} \left\{ B e^{-\sigma D_0} - e^{-\sigma D_0} B \right\} A^{-\gamma} \right\|.
\]
Then by (3.6) and by semigroup properties one gets
\[
\left\| A^{-\gamma} B (e^{-\sigma A} - I) A^{-\gamma} \right\| \leq \frac{C_\gamma}{\gamma} \sigma, \quad \sigma \in \mathbb{R}^+,
\]
and
\[
\left\| A^{-\gamma} (e^{-\sigma A} - I) B A^{-\gamma} \right\| \leq \frac{C_\gamma}{\gamma} \sigma, \quad \sigma \in \mathbb{R}^+.
\]
The last term is obtained by using (S3) (for \( \alpha \) substituted by \( \gamma \)) and the definitions (2.1), (2.4):
\[
\left\| A^{-\gamma} \left\{ B e^{-\sigma D_0} - e^{-\sigma D_0} B \right\} A^{-\gamma} \right\| \leq \sigma^\beta L_{\gamma, \beta}, \quad \sigma \in \mathbb{R}^+.
\]
Summing up one finds that
\[
\left\| A^{-\gamma} \left\{ B e^{-\sigma K_0} - e^{-\sigma K_0} B \right\} A^{-\gamma} \right\| \leq \frac{2C_\gamma}{\gamma} \sigma^\gamma + L_{\gamma, \beta} \sigma^\beta, \quad \sigma \in \mathbb{R}^+. \tag{3.17}
\]
Using the estimates (3.14), (3.15), (3.16) and (3.17) we get
\[
\left\| A^{-\gamma} e^{-(\tau - \sigma) K} \left\{ B e^{-\sigma K_0} - e^{-\sigma K_0} B \right\} e^{-\sigma B} A^{-\gamma} \right\| \leq 2C_\gamma \sigma^\gamma (\tau - \sigma) + 2C_\gamma^2 \sigma + 2C_\gamma^2 (\tau - \sigma) + \frac{2C_\gamma}{\gamma} \sigma^\gamma + L_{\gamma, \beta} \sigma^\beta
\]
\[
= 2C_\gamma \sigma^\gamma (\tau - \sigma) + 2C_\gamma^2 \tau + \frac{2C_\gamma}{\gamma} \sigma^\gamma + L_{\gamma, \beta} \sigma^\beta,
\]
or returning back to its derivative
\[ \left\| A^{-\gamma} \frac{d}{d\sigma} e^{-(\tau-\sigma)K} e^{-\sigma K_0} e^{-\sigma B} A^{-\gamma} \right\| \leq 2C^3_\gamma \sigma^{-\gamma}(\tau - \sigma) \gamma^2 \right) \cdot \frac{2C^2_\gamma}{(1 + \gamma)^{1+\gamma}} \right) + L_{\gamma, \beta} \gamma^1 + \frac{L_{\gamma, \beta}}{1 + \beta} \gamma^{1+\beta}, \quad 0 \leq \sigma \leq \tau. \]

Since
\[ A^{-\gamma}(e^{-tK_0}e^{-tB} - e^{-tK})A^{-\gamma} = \int_0^\tau A^{-\gamma} \frac{d}{d\sigma} e^{-(\tau-\sigma)K} e^{-\sigma K_0} e^{-\sigma B} A^{-\gamma} d\sigma, \]
we find the estimate
\[ \left\| A^{-\gamma}(e^{-tK_0}e^{-tB} - e^{-tK})A^{-\gamma} \right\| \leq \int_0^\tau \left\| A^{-\gamma} \frac{d}{d\sigma} e^{-(\tau-\sigma)K} e^{-\sigma K_0} e^{-\sigma B} A^{-\gamma} \right\| d\sigma, \]
which yields the estimate
\[ \left\| A^{-\gamma}(e^{-tK_0}e^{-tB} - e^{-tK})A^{-\gamma} \right\| \leq 2C^3_\gamma \int_0^\tau \sigma^1-\gamma(d\sigma + 2C^2_\gamma \gamma^2 + \frac{2C^2_\gamma}{(1 + \gamma)^{1+\gamma}} \gamma^{1+\gamma} + \frac{L_{\gamma, \beta}}{1 + \beta} \gamma^{1+\beta}, \quad \tau \in \mathbb{R}^+ \]
or after integration:
\[ \left\| A^{-\gamma}(e^{-tK_0}e^{-tB} - e^{-tK})A^{-\gamma} \right\| \leq \frac{2C^3_\gamma}{(2 - \gamma)(3 - \gamma)} \gamma^{3-\gamma} + 2C^2_\gamma \gamma^2 + \frac{2C^2_\gamma}{(1 + \gamma)^{1+\gamma}} \gamma^{1+\gamma} + \frac{L_{\gamma, \beta}}{1 + \beta} \gamma^{1+\beta}, \quad \tau \in \mathbb{R}^+, \]
If \( \gamma \in [\alpha, 1) \) and \( \gamma \leq \beta < 1 \), then one gets
\[ \left\| A^{-\gamma}(e^{-tK_0}e^{-tB} - e^{-tK})A^{-\gamma} \right\| \leq \left( \left\| A^{-\gamma} \right\| + \frac{2C^3_\gamma}{(2 - \gamma)(3 - \gamma)} \gamma^{2-\gamma} + 2C^2_\gamma \gamma^{1-\gamma} + \frac{2C^2_\gamma}{(1 + \gamma)^{1+\gamma}} \gamma^{1+\gamma} + \frac{L_{\gamma, \beta}}{1 + \beta} \gamma^{1+\beta} \right) \tau^{1+\gamma}, \]
\[ \tau \in \mathbb{R}^+, \text{ which immediately yields } (3.7). \]
If \( \gamma \in [\alpha, 1) \) and \( 0 < \beta < \gamma \), then one can rewrite it as
\[ \left\| A^{-\gamma}(e^{-tK_0}e^{-tB} - e^{-tK})A^{-\gamma} \right\| \leq \left( \left\| A^{-\gamma} \right\| + \frac{2C^3_\gamma}{(2 - \gamma)(3 - \gamma)} \gamma^{2-\gamma} + 2C^2_\gamma \gamma^{1-\gamma} + \frac{2C^2_\gamma}{(1 + \gamma)^{1+\gamma}} \gamma^{1+\gamma} + \frac{L_{\gamma, \beta}}{1 + \beta} \gamma^{1+\beta} \right) \tau^{1+\beta}, \]
\[ \tau \in \mathbb{R}^+, \text{ which shows } (3.7) \text{ for this choice of } \gamma \text{ and } \beta. \]

**Remark 3.6** For \( \gamma \in [\alpha, 1) \) and \( \gamma \leq \beta < 1 \) we find from (3.18) that
Let the assumption (S1) be satisfied. If for each Lemma 3.8
Now the estimates (3.22) and (3.23) yield (3.21). □

Using (3.2) and (3.13) we estimate the second term as

For \( \gamma \in [\alpha, 1) \) and \( 0 < \beta < \gamma \) we get from (3.19) that

Here \( C_\gamma := \text{ess sup}_{t \in \mathbb{R}} \| BA^{-\gamma} \| \), see (3.3), and \( L_{T\beta} \) is the Hölder constant of the function \( A^{-\gamma}B(\cdot)A^{-\gamma} : \mathbb{R} \rightarrow \mathcal{L}(\mathcal{H}) \), see (S3).

**Lemma 3.7** Let the assumptions (S1) and (S2) be satisfied. Then

\[
\| \mathcal{A}^{\gamma}(U(\tau) - T(\tau))A^{-\gamma} \| \leq \left( \frac{\Lambda_\gamma}{\gamma} + 1 \right) C_\gamma \tau^{1-\gamma}, \quad \tau \in \mathbb{R}_+,
\]

for \( \gamma \in [\alpha, 1) \).

**Proof.** We use the representation

\[
U(\tau) - T(\tau) = e^{-\tau K} - e^{-\tau K_0} + e^{-\tau K_0}(I - e^{-\tau B})
\]

which yields

\[
\mathcal{A}^{\gamma}(U(\tau) - T(\tau))A^{-\gamma} = \mathcal{A}^{\gamma}(e^{-\tau K} - e^{-\tau K_0})A^{-\gamma} + \mathcal{A}^{\gamma}e^{-\tau K_0}(I - e^{-\tau B})A^{-\gamma}
\]

Using the semigroup property we obtain for the first term the representation:

\[
\mathcal{A}^{\gamma}(e^{-\tau K} - e^{-\tau K_0})A^{-\gamma} = -\int_0^\tau \mathcal{A}^{\gamma}e^{-(\tau - x)K}BA^{-\gamma}e^{-xK_0}dx.
\]

Hence, by (3.3) and (3.6) one gets

\[
\| \mathcal{A}^{\gamma}(e^{-\tau K} - e^{-\tau K_0})A^{-\gamma} \| \leq \int_0^\tau \| \mathcal{A}^{\gamma}e^{-(\tau - x)K} \| \| BA^{-\gamma} \| dx \leq \Lambda_\gamma C_\gamma \int_0^\tau \frac{1}{(\tau - x)^{\gamma}} dx = \frac{\Lambda_\gamma C_\gamma}{1 - \gamma} \tau^{1-\gamma}.
\]

To estimate the second term we use the inequality \( \| \mathcal{A}^{\gamma}e^{-\tau K_0}(I - e^{-\tau B})A^{-\gamma} \| \leq \| \mathcal{A}^{\gamma}e^{-\tau K_0} \| \| (I - e^{-\tau B})A^{-\gamma} \| \).

Using (3.2) and (3.13) we estimate the second term as

\[
\| \mathcal{A}^{\gamma}e^{-\tau K_0}(I - e^{-\tau B})A^{-\gamma} \| \leq C_\gamma \tau^{1-\gamma}.
\]

Now the estimates (3.22) and (3.23) yield (3.21). □
\[ \|A^\gamma T(\tau)^m\| \leq \frac{M_\gamma}{(m\tau)^{\gamma}}, \quad m \in \mathbb{N}, \quad \tau \in \mathbb{R}_+, \quad (3.24) \]

holds for \( T(\tau) \) defined in (3.4), then

\[ \|A^\sigma T(\tau)^m\| \leq \frac{M_\gamma^2}{(m\tau)^{2\gamma}}, \quad m \in \mathbb{N}, \quad (3.25) \]

holds for \( \sigma \in [0, \gamma] \) and \( \delta := \sigma / \gamma \).

**Proof.** If (3.24) is satisfied, then

\[ \|(T(\tau)^* A^\gamma)^m A^\gamma\| \leq \frac{M_\gamma}{(m\tau)^{\gamma}}, \quad m \in \mathbb{N}, \]

holds, which is equivalent to

\[ A^\gamma T(\tau)^m (T(\tau)^*)^m A^\gamma \leq \frac{M_\gamma^2}{(m\tau)^{2\gamma}}, \quad m \in \mathbb{N}, \]

or

\[ T(\tau)^m (T(\tau)^*)^m \leq \frac{M_\gamma^2}{(m\tau)^{2\gamma}} A^{2\gamma}, \quad m \in \mathbb{N}. \]

Let \( \delta \in (0, 1) \). Using the Heinz inequality [8, Theorem X.4.2] we get

\[ \left( T(\tau)^m (T(\tau)^*)^m \right)^\delta \leq \frac{M_\gamma^2\delta}{(m\tau)^{2\delta\gamma}} A^{-2\gamma}, \quad m \in \mathbb{N}. \]

Since \( T(\tau)^m (T(\tau)^*)^m \) is a self-adjoint contraction we get

\[ T(\tau)^m (T(\tau)^*)^m \leq \left( T(\tau)^m (T(\tau)^*)^m \right)^\delta, \quad m \in \mathbb{N}, \]

which yields

\[ T(\tau)^m (T(\tau)^*)^m \leq \frac{M_\gamma^2\delta}{(m\tau)^{2\delta\gamma}} A^{-2\gamma}, \quad m \in \mathbb{N}, \]

or

\[ A^{\delta\gamma} T(\tau)^m (T(\tau)^*)^m A^{\delta\gamma} \leq \frac{M_\gamma^2\delta}{(m\tau)^{2\delta\gamma}}, \quad m \in \mathbb{N}. \]

Therefore, one gets

\[ \|(T(\tau)^* A^\gamma)^m A^\gamma\| \leq \frac{M_\gamma^\delta}{(m\tau)^{\delta\gamma}}, \quad m \in \mathbb{N}, \]

or

\[ \|A^{\delta\gamma} T(\tau)^m\| \leq \frac{M_\gamma^\delta}{(m\tau)^{\delta\gamma}}, \quad m \in \mathbb{N}. \]
Setting $\delta = \sigma / \gamma$ we obtain the proof of (3.25).

\[\Box\]

**Lemma 3.9** Let the assumptions (S1) and (S2) be satisfied and let $\gamma \in (\alpha, 1)$. Then there is a constant $M_\gamma > 0$ such that

\[\|A^\gamma T(\tau)^m\| \leq \frac{M_\gamma}{(m\tau)^\gamma}, \quad m = 1, 2, \ldots, n, \]

holds for any $T > 0$ if $\tau \in (0, T/n)$ and $n \geq n_0$ where $n_0 := \left\lceil (2(\frac{\Lambda_\gamma}{1-\gamma} + 1)C_\gamma)^{1/(1-\gamma)}T \right\rceil + 1$ and $\lfloor x \rfloor$ denotes the largest integer smaller than $x$.

\[\text{Proof.}\]

Let $M_\gamma > 0$ be a constant which satisfies the inequality

\[5\Lambda_\gamma + 2 \left( \frac{\Lambda_\gamma}{1 - \gamma} + 1 \right) C_\gamma M_\gamma T^{1-\gamma} \frac{1}{n^{1-\gamma}} + 4A_\gamma C_\gamma M_\gamma B(1 - \alpha, 1 - \gamma) T^{1-\alpha} \leq M_\gamma \]

for $n \geq n_0$. Here constants $\Lambda_\gamma$ and $C_\gamma$ are defined by Lemma 3.2 and Lemma 3.4, respectively, while $B(\cdot, \cdot)$ denotes the Euler Beta-function. (Note that such $M_\gamma > 0$ always exists, see Remark 3.10 below.)

Let $m = 1$. Then by (3.2) and (3.4) we get

\[\|A^\gamma T(\tau)\| \leq \|A^\gamma e^{-\tau K_0}\| \leq \frac{1}{\tau^\gamma} \leq \frac{\Lambda_\gamma}{\tau^\gamma} \leq \frac{M_\gamma}{\tau^\gamma}, \]

for $\tau > 0$ and, in particular, for $\tau \in (0, T/n)$. Hence (3.26) holds for $m = 1$.

Let us assume that (3.26) holds for $l = 1, 2, \ldots, m - 1$, with $m \leq n$, i.e.

\[\|A^\gamma T(\tau)^l\| \leq \frac{M_\gamma}{(l\tau)^\gamma}, \quad l = 1, 2, \ldots, m - 1, \]

for $\tau \in (0, T/n)$. We are going to show that (3.26) holds for $l = m$. To this aim we use the representation

\[\mathcal{U}(\tau)^m - T(\tau)^m = \sum_{k=0}^{m-1} \mathcal{U}(\tau)^{m-1-k}(\mathcal{U}(\tau) - T(\tau))T(\tau)^k, \quad m = 2, 3, \ldots, \]

which implies

\[T(\tau)^m = \mathcal{U}(\tau)^m - \sum_{k=0}^{m-1} \mathcal{U}(\tau)^{m-1-k}(\mathcal{U}(\tau) - T(\tau))T(\tau)^k, \quad m = 2, 3, \ldots.\]

Hence

\[A^\gamma T(\tau)^m = A^\gamma \mathcal{U}(\tau)^m - \sum_{k=0}^{m-1} A^\gamma \mathcal{U}(\tau)^{m-1-k}(\mathcal{U}(\tau) - T(\tau))T(\tau)^k\]

or
\[ A^{\gamma}T(\tau)^m = A^{\gamma}U(\tau)^m - A^{\gamma}U(\tau)^{m-1}(U(\tau) - T(\tau)) - A^{\gamma}(U(\tau) - T(\tau))T(\tau)^{m-1} - \sum_{k=1}^{m-2} A^{\gamma}U(\tau)^{m-1-k}(U(\tau) - T(\tau))T(\tau)^k. \]

for \( m = 3, 4, \ldots \). This yields the inequality
\[
\|A^{\gamma}T(\tau)^m\| \leq \|A^{\gamma}U(\tau)^m\| + \|A^{\gamma}U(\tau)^{m-1}(U(\tau) - T(\tau))\| + \sum_{k=1}^{m-2} \|A^{\gamma}U(\tau)^{m-1-k}(U(\tau) - T(\tau))T(\tau)^k\| \tag{3.29}
\]

for \( m = 3, 4, \ldots \). From Lemma 3.2 we get the estimates
\[
\|A^{\gamma}U(\tau)^m\| \leq \frac{\Lambda_1}{(m\tau)^{\gamma}}, \quad m = 2, 3, \ldots ,
\]

and consequently:
\[
\|A^{\gamma}U(\tau)^{m-1}(U(\tau) - T(\tau))\| \leq \frac{2\Lambda_1}{((m-1)\tau)^{\gamma}} \leq \frac{4\Lambda_1}{(m\tau)^{\gamma}}, \quad m = 2, 3, \ldots .
\]

Then summing up estimates for the first two terms in the right-hand side of (3.29) we obtain
\[
\|A^{\gamma}U(\tau)^m\| + \|A^{\gamma}U(\tau)^{m-1}(U(\tau) - T(\tau))\| \leq \frac{5\Lambda_1}{(m\tau)^{\gamma}}, \quad m = 2, 3, \ldots . \tag{3.30}
\]

Next we get for the third term in the right-hand side of (3.29) the estimate
\[
\|A^{\gamma}(U(\tau) - T(\tau))T(\tau)^{m-1}\| \leq \|A^{\gamma}(U(\tau) - T(\tau))A^{-\gamma}\| \|A^{\gamma}T(\tau)^{m-1}\|,
\]

\( m = 2, 3, \ldots \). Then using Lemma 3.7 we find that
\[
\|A^{\gamma}(U(\tau) - T(\tau))T(\tau)^{m-1}\| \leq \left( \frac{\Lambda_1}{1 - \gamma} + 1 \right) C_\gamma \tau^{1-\gamma} \|A^{\gamma}T(\tau)^{m-1}\|, \quad m = 2, 3, \ldots .
\]

By assumption (3.28) this yields
\[
\|A^{\gamma}(U(\tau) - T(\tau))T(\tau)^{m-1}\| \leq \left( \frac{\Lambda_1}{1 - \gamma} + 1 \right) M_\gamma C_\gamma \frac{1}{((m-1)\tau)^{\gamma}} \tau^{1-\gamma}, \quad m = 2, 3, \ldots , \quad \text{for} \quad \tau \in (0, T/n),
\]

which leads to
\[
\|A^{\gamma}(U(\tau) - T(\tau))T(\tau)^{m-1}\| \leq \left( \frac{\Lambda_1}{1 - \gamma} + 1 \right) M_\gamma C_\gamma \frac{2}{(m\tau)^{\gamma}} \tau^{1-\gamma}, \quad m = 2, 3, \ldots . \tag{3.31}
\]

Finally one gets for the sum in (3.29)
Then by Lemma 3.2 this implies
\[
\sum_{k=1}^{m-2} \| A^{\gamma} U^m (\tau)^{m-1-k} (U(\tau) - T(\tau)) T^k \| \\
\leq \sum_{k=1}^{m-2} \| A^{\gamma} U^m (\tau)^{m-1-k} \| \| (U(\tau) - T(\tau)) A^{-\alpha} \| \| A^\alpha T(\tau)^k \|, \quad m = 2, 3, \ldots .
\]

Taking into account Lemma 3.4 we get
\[
\sum_{k=1}^{m-2} \| A^{\gamma} U^m (\tau)^{m-1-k} (U(\tau) - T(\tau)) T^k \| \\
\leq A_f \sum_{k=1}^{m-2} \frac{1}{((m - 1 - k) \tau)^\gamma} \| (U(\tau) - T(\tau)) A^{-\alpha} \| \| A^\alpha T(\tau)^k \|, \quad m = 2, 3, \ldots .
\]

Finally, using assumption (3.28) and Lemma 3.8 one obtains
\[
\sum_{k=1}^{m-2} \| A^{\gamma} U^m (\tau)^{m-1-k} (U(\tau) - T(\tau)) T^k \| \\
\leq 2A_f C_\alpha M_f \sum_{k=1}^{m-2} \frac{\tau}{((m - 1 - k) \tau)^\gamma (k \tau)^\alpha}, \quad m = 2, 3, \ldots ,
\]
or
\[
\sum_{k=1}^{m-2} \| A^{\gamma} U^m (\tau)^{m-1-k} (U(\tau) - T(\tau)) T^k \| \\
\leq 2A_f C_\gamma M_f \left( \sum_{k=1}^{m-2} \frac{1}{((m - 1 - k) \tau)^\gamma k^\alpha} \right) \tau^{1-\gamma-\alpha}, \quad m = 2, 3, \ldots ,
\]
for \( \tau \in (0, T/n) \). Since Lemma 3.11 below yields
\[
\sum_{k=1}^{m-2} \frac{1}{(m - 1 - k)^\gamma k^\alpha} \leq B(1, 1, \gamma)(m - 1)^{1-\gamma-\alpha}, \quad m = 2, 3, \ldots , \quad (3.32)
\]
where \( B(\cdot, \cdot) \) is the Euler Beta-function, we get
\begin{equation}
\sum_{k=1}^{m-2} \| A^k \mathcal{U}(\tau)^{m-1-k} (\mathcal{U}(\tau) - T(\tau)) T(\tau)^k \| \leq 2 \Lambda_\gamma C_\gamma M_\gamma^{- \frac{a}{\tau}} B(1 - \alpha, 1 - \gamma) \tau^{1 - \gamma} (m - 1)^{1 - \gamma - \alpha}, \quad m = 2, 3, \ldots,
\end{equation}
which in turn leads to
\begin{equation}
\sum_{k=1}^{m-2} \| A^k \mathcal{U}(\tau)^{m-1-k} (\mathcal{U}(\tau) - T(\tau)) T(\tau)^k \| \leq \frac{4 \Lambda_\gamma C_\gamma M_\gamma^{- \frac{a}{\tau}} B(1 - \alpha, 1 - \gamma) \tau^{1 - \gamma} m^{1 - \alpha}}{(m \tau)^{\gamma}},
\end{equation}
for \( m = 2, 3, \ldots \) and any \( \tau \in (0, T/n) \).

Now we take into account (3.29), (3.30), (3.31) and (3.33) to conclude that
\begin{equation}
\||A^l T(\tau)^m|| \leq \left\{ 5 \Lambda_\gamma + \frac{2 (\Lambda_\gamma + 1)}{n^{1 - \gamma}} + 4 \Lambda_\gamma C_\gamma M_\gamma^{- \frac{a}{\tau}} B(1 - \alpha, 1 - \gamma) \tau^{1 - \gamma} m^{1 - \alpha} \right\} \frac{1}{(m \tau)^{\gamma}},
\end{equation}
for \( m = 2, 3, \ldots \) and \( \tau \in (0, T/n) \). Then
\begin{equation}
\||A^l T(\tau)^m|| \leq \left\{ 5 \Lambda_\gamma + \frac{2 (\Lambda_\gamma + 1)}{n^{1 - \gamma}} + 4 \Lambda_\gamma C_\gamma M_\gamma^{- \frac{a}{\tau}} B(1 - \alpha, 1 - \gamma) T^{1 - \alpha} \right\} \frac{1}{(m \tau)^{\gamma}}.
\end{equation}
\( \Box \)

**Remark 3.10** One checks that condition (3.27) is always satisfied for sufficiently large \( M = M_\gamma \) and \( n \geq n_0 \). Indeed, after setting
\[
c_0 := 5 \Lambda_\gamma, \quad c_1 := 2 \left( \frac{\Lambda_\gamma + 1}{n^{1 - \gamma}} \right) C_\gamma T^{1 - \gamma}, \quad c_2 := 4 \Lambda_\gamma C_\gamma B(1 - \alpha, 1 - \gamma) T^{1 - \alpha}
\]
we get the condition
\[
c_0 + \frac{c_1}{n^{1 - \gamma}} M + c_2 M^{- \frac{a}{\tau}} \leq M
\]
which yields
\[
c_0 + c_2 M^{- \frac{a}{\tau}} \leq (1 - \frac{c_1}{n^{1 - \gamma}}) M
\]
or
\[
\frac{c_0}{M} + \frac{c_2}{M^{1 - \frac{\gamma}{\gamma}}} \leq 1 - \frac{c_1}{n^{1 - \gamma}}
\]

Since \( n > c_1 \) we have \( 1 - c_1/n^{1 - \gamma} > 0 \). The left-hand side tends to zero if \( M \to \infty \). Hence, choosing \( M \) sufficiently large we guarantee the existence of \( M_\gamma \) such that condition (3.27) is satisfied for any \( n \geq n_0 \). \( \square \)

It remains only to verify the following statement.

**Lemma 3.11** Let \( \alpha \in [0, 1) \) and \( \gamma \in [\alpha, 1) \). Then
\[
\sum_{k=1}^{n-1} \frac{1}{(n-k)^{\gamma} k^\alpha} \leq B(1 - \alpha, 1 - \gamma) n^{1-\gamma-\alpha}, \quad n \geq 2, 3, \ldots
\]

the estimate holds where \( B(\cdot, \cdot) \) is the Euler Beta-function.

**Proof.** If \( x \in (k-1, k] \), then
\[
\frac{1}{k^\alpha} \leq \frac{1}{x^\alpha} \quad \text{and} \quad \frac{1}{(n-k)^\gamma} \leq \frac{1}{(n-1-x)^\gamma}
\]

for \( k = 1, 2, \ldots, n-1 \). Hence
\[
\frac{1}{(n-k)^{\gamma} k^\alpha} \leq \frac{1}{(n-1-x)^{\gamma} x^\alpha}, \quad x \in (k-1, k].
\]

Therefore
\[
\frac{1}{(n-k)^{\gamma} k^\alpha} = \int_{k-1}^{k} \frac{1}{(n-k)^{\gamma} k^\alpha} dx \leq \int_{k-1}^{k} \frac{1}{(n-1-x)^{\gamma} x^\alpha} dx, \quad x \in (k-1, k],
\]

or
\[
\sum_{k=1}^{n-1} \frac{1}{(n-k)^{\gamma} k^\alpha} = \sum_{k=1}^{n-1} \int_{k-1}^{k} \frac{1}{(n-k)^{\gamma} k^\alpha} dx \leq \sum_{k=1}^{n-1} \int_{k-1}^{k} \frac{1}{(n-1-x)^{\gamma} x^\alpha} dx
\]
\[
= \int_{0}^{n-1} \frac{1}{(n-1-x)^{\gamma} x^\alpha} dx = B(1 - \alpha, 1 - \gamma) n^{1-\alpha-\gamma}
\]

\( \square \)
3.2 Main Results

In this section we collect our main results and their proofs. They are based on preliminaries established in Section 3.1.

**Theorem 3.12** Let the assumptions (S1)-(S3) be satisfied and let $\beta > 2\alpha - 1$. Then there is a constant $R_\beta > 0$ such that

$$\sup_{\tau \in \mathbb{R}_+} \| U(\tau) - T(\tau/n)^n \| \leq \frac{R_\beta}{n^\beta}$$

(3.34)

holds for $n \in \mathbb{N}$ and $\tau \in \mathbb{R}_+$.

**Proof.** Taking into account the representation

$$U(\tau/n)^n - T(\tau/n)^n = \sum_{m=0}^{n-1} U(\tau/n)^n - m^{-1} (U(\tau/n) - T(\tau/n))^m, \quad n \in \mathbb{N},$$

or, identically,

$$U(\tau/n)^n - T(\tau/n)^n = U(\tau/n)^n - 1(U(\tau/n) - T(\tau/n)) + (U(\tau/n) - T(\tau/n)) T(\tau/n)^n - 1 + \sum_{m=1}^{n-2} U(\tau/n)^n - m^{-1} (U(\tau/n) - T(\tau/n))^m, \quad n = 3, 4, \ldots,$$

we obtain the estimate

$$\| U(\tau/n)^n - T(\tau/n)^n \|$$

$$\leq \| U(\tau/n)^n - 1(U(\tau/n) - T(\tau/n)) \| A^{-\gamma} \| (U(\tau/n) - T(\tau/n)) \| A^\gamma T(\tau/n)^{n-1} \|$$

$$+ \sum_{m=1}^{n-2} \| U(\tau/n)^n - m^{-1} A^\gamma (U(\tau/n) - T(\tau/n)) A^{-\gamma} \| A^\gamma T(\tau/n)^m \|,$$

(3.35)

for $n = 3, 4, \ldots$

Note that using Lemma 3.2 and Lemma 3.4 one gets

$$\| U(\tau/n)^n - 1 A^\gamma (U(\tau/n) - T(\tau/n)) \| \leq 2 \frac{\Lambda_\gamma C_\gamma}{(\tau(n-1)/n)^2} \frac{\tau}{n},$$

which yields

$$\| U(\tau/n)^n - 1 A^\gamma (U(\tau/n) - T(\tau/n)) \| \leq 2^{1+\gamma} \Lambda_\gamma C_\gamma T^{1-\gamma} \frac{1}{n},$$

(3.36)

for $n = 3, 4, \ldots$ and $\tau \in [0, T]$. 
Now using Lemma 3.4 and Lemma 3.9 for \( m = n - 1 \) we find
\[
\| (U(\tau/n) - T(\tau/n))A^{-\gamma}\| \| A^\gamma T(\tau/n)^{n-1} \| \leq 2C_\gamma \frac{\tau}{n} \frac{M_\gamma}{(\tau(n-1)/n)^\gamma},
\]
for \( n \geq n_0 \), where \( n_0 \) is defined in Lemma 3.9 and \( \tau \in [0, T] \). Hence,
\[
\| (U(\tau/n) - T(\tau/n))A^{-\gamma}\| \| A^\gamma T(\tau/n)^{n-1} \| \leq 2^{1+\gamma} C_\gamma M_\gamma T^{1-\gamma} \frac{1}{n}.
\]
(3.37)
Taking into account Lemma 3.2 Lemma 3.5 and Lemma 3.9 (for \( \kappa = \min\{\gamma, \beta\} \)) one gets
\[
\sum_{m=1}^{n-2} \| (U(\tau/n)^{n-m-1}A^\gamma)\| \| A^{-\gamma}(U(\tau/n) - T(\tau/n))A^{-\gamma}\| \| A^\gamma T(\tau/n)^m \|
\leq \sum_{m=1}^{n-2} \frac{\Lambda_\gamma Z_{\gamma, \beta}}{((n-m-1)/n)^\gamma} \left( \frac{\tau}{n} \frac{\gamma}{(m/\tau)^\gamma} \right) M_\gamma
\leq \frac{\Lambda_\gamma Z_{\gamma, \beta} M_\gamma T^{1+\gamma-2\gamma}}{n^{1+\gamma-2\gamma}} \sum_{m=1}^{n-2} \frac{1}{(n-m-1)/m^\gamma},
\]
for \( n > \max\{2, n_0\} \) and \( \tau \in [0, T] \). Then by (3.32) we obtain
\[
\sum_{m=1}^{n-2} \| (U(\tau/n)^{n-m-1}A^\gamma)\| \| A^{-\gamma}(U(\tau/n) - T(\tau/n))A^{-\gamma}\| \| A^\gamma T(\tau/n)^m \|
\leq \frac{\Lambda_\gamma Z_{\gamma, \beta} M_\gamma T^{1+\gamma-2\gamma}}{n^{1+\gamma-2\gamma}} B(1 - \gamma, 1 - \gamma) n^{1-2\gamma},
\]
or
\[
\sum_{m=1}^{n-2} \| (U(\tau/n)^{n-m-1}A^\gamma)\| \| A^{-\gamma}(U(\tau/n) - T(\tau/n))A^{-\gamma}\| \| A^\gamma T(\tau/n)^m \|
\leq \frac{\Lambda_\gamma Z_{\gamma, \beta} M_\gamma B(1 - \gamma, 1 - \gamma) T^{1+\gamma-2\gamma} \frac{1}{n^{\kappa}}}{n^{\kappa}}.
\]
(3.38)
Therefore, by virtue of (3.35), (3.36), (3.37) and (3.38) we get for \( n > \max\{2, n_0\} \) and \( \tau \in [0, T] \) the estimate
\[
\| U(\tau)^{n} - T(\tau/n)^n \|
\leq 2^{1+\gamma} A_\gamma C_\gamma T^{1-\gamma} \frac{1}{n} + 2^{1+\gamma} C_\gamma M_\gamma T^{1-\gamma} \frac{1}{n} + \Lambda_\gamma Z_{\gamma, \beta} M_\gamma B(1 - \gamma, 1 - \gamma) T^{1+\gamma-2\gamma} \frac{1}{n^{\kappa}}
\leq \left\{ 2^{1+\gamma} A_\gamma C_\gamma T^{1-\gamma} + 2^{1+\gamma} C_\gamma M_\gamma T^{1-\gamma} + \Lambda_\gamma Z_{\gamma, \beta} M_\gamma B(1 - \gamma, 1 - \gamma) T^{1+\gamma-2\gamma} \frac{1}{n^{\kappa}} \right\} \frac{1}{n^{\kappa}}.
\]
If \( \alpha < \beta < 1 \), then we choose \( \gamma = \beta \), i.e., \( \kappa = \beta \) and \( 1 + \kappa - 2\gamma = 1 - \beta \geq 0 \). Setting
\[ R'_\beta := 2^{1+\beta} \Lambda_\beta C_\beta T^{1-\beta} + 2^{1+\beta} C_\beta M_\beta T^{1-\beta} + \Lambda_\beta Z_{\beta,\beta} M_\beta B(1 - \beta, 1 - \beta) T^{1-\beta} \]

one obtains the estimate

\[ ||U(\tau)^n - T(\tau/n)^n|| \leq \frac{R'_\beta}{n^\beta}, \quad (3.39) \]

for \( n > \max \{2, n_0\} \) and \( \tau \in [0, T] \).

Now let \( 0 < \beta \leq \alpha \). Since \( 1 + \beta - 2\alpha > 0 \), there exists \( \gamma \in (\alpha, 1) \) such that \( 1 + \beta - 2\gamma \geq 0 \). Indeed, there is an \( \varepsilon > 0 \) verifying \( 1 + \beta - 2\alpha > 2\varepsilon \). Setting \( \gamma = \alpha + \varepsilon \) we get \( 1 + \beta - 2\gamma > 0 \). Notice that \( \varepsilon = \beta \). Then setting

\[ R'_\beta := 2^{1+\gamma} \Lambda_\gamma C_\gamma T^{1-\gamma} + 2^{1+\gamma} C_\gamma M_\gamma T^{1-\gamma} + \Lambda_\gamma Z_{\gamma,\gamma} M_\gamma B(1 - \gamma, 1 - \gamma) T^{1-\beta-2\gamma}, \]

we obtain (3.39) for \( n > \max \{2, n_0\} \).

Both results immediately imply that there is a constant \( R_\gamma \) such that \( (3.34) \) holds for \( \tau \in [0, T] \) and \( n \in \mathbb{N} \). Finally, using \( U(\tau) = 0 \) and \( T(\tau/n)^n = 0 \) for \( \tau \geq T \) we obtain (3.28) for any \( \tau \in \mathbb{R}_+ \). □

Now we set

\[ \tilde{T}(\tau) := e^{-\tau T} e^{-\tau K_0}, \quad \tau \in \mathbb{R}_+. \]

Corollary 3.13 Let the assumptions (S1)-(S3) be satisfied and \( \beta > 2\alpha - 1 \). Then there exists \( \tilde{R}_\beta > 0 \) such that estimate

\[ \sup_{\tau \in \mathbb{R}_+} ||U(\tau) - \tilde{T}(\tau/n)^n|| \leq \frac{\tilde{R}_\beta}{n^\beta} \quad (3.40) \]

holds for \( n \in \mathbb{N} \) and \( \tau \in \mathbb{R}_+ \).

Proof. Notice that

\[ \tilde{T}(\tau/n)^{n+1} = e^{-\tau T/n} T(\tau/n)^n e^{-\tau K_0/n}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}. \]

Hence

\[
\begin{align*}
U((n + 1)\tau/n) - \tilde{T}(\tau/n)^{n+1} &= e^{-(n+1)\tau K_0/n} - e^{-\tau T/n} T(\tau/n)^n e^{-\tau K_0/n} \\
&= e^{-(n+1)\tau K_0/n} - e^{-\tau T/n} e^{-\tau K_0/n} + e^{-\tau T/n} U(\tau) - T(\tau/n)^n e^{-\tau K_0/n} \\
&= (1 - e^{-\tau T/n}) e^{-\tau K_0/n} + e^{-\tau K_0/n} (e^{-\tau T/n} - e^{-\tau K_0/n})+ e^{-\tau T/n} (U(\tau) - T(\tau/n)^n) e^{-\tau K_0/n}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N},
\end{align*}
\]

which yields the estimate
\[ \| U((n+1)\tau_n) - \tilde{T}(\tau_n)^{n+1} \| \leq \| (I - e^{-\frac{\tau_n B}{n}}) e^{-\tau K} \| + \| e^{-\tau K} (e^{-\frac{\tau_n K}{n}} - e^{-\frac{\tau K_0}{n}}) \| + \| U(\tau) - T(\tau_n)^n \|, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}. \]  

(3.41)

Obviously, one has
\[ \| (I - e^{-\frac{\tau_n B}{n}}) e^{-\tau K} \| \leq \| (I - e^{-\frac{\tau_n B}{n}}) A^{-\alpha} \| \| A^{\alpha} e^{-\tau K} \|, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}. \]

Using
\[ (I - e^{-\frac{\tau_n B}{n}}) A^{-\alpha} = \int_0^{\tau_n} e^{-\sigma B} B e^{-\tau K} d\sigma, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}, \]

we get the estimate
\[ \| (I - e^{-\frac{\tau_n B}{n}}) A^{-\alpha} \| \leq C_\alpha \frac{\tau_n}{n}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}. \]

Taking into account condition (S2) and Lemma 3.2 we find
\[ \| (I - e^{-\frac{\tau_n B}{n}}) e^{-\tau K} \| \leq C_\alpha \Lambda_\alpha \frac{\tau_1}{n}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}, \]  

(3.42)

where we have used that \( e^{-\tau K} = 0 \) for \( \tau \geq T \).

Further, we have
\[ \| e^{-\tau K} (e^{-\frac{\tau_n K}{n}} - e^{-\frac{\tau K_0}{n}}) \| \leq \| e^{-\tau K} A^{\alpha} \| \| A^{-\alpha} (e^{-\frac{\tau_n K}{n}} - e^{-\frac{\tau K_0}{n}}) \|, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}. \]

Then using
\[ A^{-\alpha} (e^{-\frac{\tau_n K}{n}} - e^{-\frac{\tau K_0}{n}}) = - \int_0^{\tau_n} e^{-\sigma} e^{-\tau K_0} A^{-\alpha} B e^{-\sigma} e^{-(\tau - \sigma) K} d\sigma, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}, \]

we find the estimate
\[ \| A^{-\alpha} (e^{-\frac{\tau_n K}{n}} - e^{-\frac{\tau K_0}{n}}) \| \leq C_\alpha \frac{\tau_n}{n}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}. \]

Applying again Lemma 3.2 one gets
\[ \| e^{-\tau K} (e^{-\frac{\tau_n K}{n}} - e^{-\frac{\tau K_0}{n}}) \| \leq C_\alpha \Lambda_\alpha \frac{\tau_1}{n}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}. \]  

(3.43)

The insertion of (3.42) and (3.43) into (3.41) yields
\[ \| U((n+1)\tau_n) - \tilde{T}(\tau_n)^{n+1} \| \leq 2C_\alpha \Lambda_\alpha \frac{\tau_1}{n} + \| U(\tau) - T(\tau_n)^n \|, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}. \]

Then by Theorem 3.12 we obtain
Let the assumptions (S1)-(S3) be satisfied. Further, let 

\[ \mathcal{U}((n+1) \frac{j}{n}) - \mathcal{T}(\frac{j}{n})^{n+1} \| \leq 2C_a \Lambda_a \frac{1}{n} + R_\gamma \frac{1}{n^\gamma}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}. \]

Therefore, by setting \( R'_\gamma := 2C_a \Lambda_a + R_\gamma \) we obtain

\[ \| \mathcal{U}((n+1) \frac{j}{n}) - \mathcal{T}(\frac{j}{n})^{n+1} \| \leq \frac{R'_\gamma}{n^\gamma}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}. \]

which yields

\[ \sup_{\tau \in \mathbb{R}_+} \| \mathcal{U}((n+1) \frac{j}{n}) - \mathcal{T}(\frac{j}{n})^{n+1} \| \leq \frac{R'_\gamma}{n^\gamma}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}. \]

Let \( \tau = \tau' n / (n+1) \) for \( \tau' \in \mathbb{R}_+ \). Then

\[ \sup_{\tau \in \mathbb{R}_+} \| \mathcal{U}((n+1) \frac{j}{n}) - \mathcal{T}(\frac{j}{n})^{n+1} \| = \sup_{\tau' \in \mathbb{R}_+} \| \mathcal{U}(\tau') - \mathcal{T}(\frac{\tau'}{n+1})^{n+1} \| \leq \frac{R'_\gamma}{\tau'^\gamma}, \]

or

\[ \sup_{\tau' \in \mathbb{R}_+} \| \mathcal{U}(\tau') - \mathcal{T}(\frac{\tau'}{n+1})^{n+1} \| \leq 2 \frac{R'_\gamma}{(n+1)^\gamma}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}. \]

Setting \( R'_\gamma := \max\{2, 2^\gamma R'_\gamma\} \) we prove (3.40).

These results can be immediately extended to propagators. To this end we set

\[ \begin{aligned}
G_j(t,s,n) &:= e^{-\frac{j-t}{n} \beta(t)} e^{-\frac{j-s}{n} \lambda}, \quad j = 0, 1, 2, \ldots, n, \\
V_n(t,s) &:= G_n(t,s;n) G_{n-1}(t,s;n) \times \cdots \times G_2(t,s;n) G_1(t,s;n), 
\end{aligned} \]  

(3.44)

\[ \begin{aligned}
t_j &= s + j \frac{t-s}{n}, \quad j = 0, 1, 2, \ldots, n, \text{ in analogy to (1.6).}
\end{aligned} \]

**Theorem 3.14** Let the assumptions (S1)-(S3) be satisfied. Further, let \( \{ \mathcal{U}(t,s) \}_{(t,s) \in \mathcal{D}_0} \) be the propagator corresponding to the evolution generator \( \mathcal{K} \) and let \( \{ V_n(t,s) \}_{(t,s) \in \mathcal{D}_0} \) and \( \{ \tilde{V}_n(t,s) \}_{(t,s) \in \mathcal{D}_0} \) be defined by (1.6) and (3.44), respectively. If \( \beta > 2 \alpha - 1 \), then the estimates

\[ \begin{aligned}
\text{ess sup}_{(t,s) \in \mathcal{D}_0} \| U(t,s) - V_n(t,s) \| \leq \frac{R_\beta}{n^\beta} \quad \text{and} \quad \text{ess sup}_{(t,s) \in \mathcal{D}_0} \| U(t,s) - \tilde{V}_n(t,s) \| \leq \frac{R_\beta}{n^\beta} \quad (3.45)
\end{aligned} \]

hold for \( n \in \mathbb{N} \), where the constants \( R_\gamma \) and \( R'_\gamma \) are those of Theorem 3.12 and Corollary 3.13.

**Proof.** Note that Proposition 2.1 of [36] yields

\[ \begin{aligned}
\sup_{\tau \in \mathbb{R}_+} \| \mathcal{U}(\tau) - \mathcal{T}(\frac{\tau}{n})^{n+1} \| = \text{ess sup}_{(t,s) \in \mathcal{D}_0} \| U(t,s) - V_n(t,s) \|, \quad n \in \mathbb{N}.
\end{aligned} \]
Then applying Theorem 3.12 we prove (3.35).
To prove the second estimate we use Proposition 3.8 of [37] where the relation
\[ \sup_{\tau \in \mathbb{R}^+} ||U(\tau) - \overline{T}(\frac{L}{n})\overline{n}|| = \text{ess sup}_{(t,s) \in A_0} ||U(t,s) - \overline{V}_n(t,s)||, \quad n \in \mathbb{N}. \]
was shown. Applying Corollary 3.13 we complete the proof. \( \square \)

4 Example

As an example we consider the diffusion equation perturbed by a time-dependent scalar potential. For this aim let \( \Sigma = L^2(\Omega) \), where \( \Omega \subset \mathbb{R}^3 \) is a bounded domain with sufficiently smooth boundary. Domains in higher dimension can be treated analogously. The equation reads as
\[ \dot{u}(t) = \Delta u(t) - B(t)u(t), \quad u(s) = u_0, \quad t, s \in [0,T], \quad (4.1) \]
where \( \Delta \) denotes the Laplace operator in \( L^2(\Omega) \) with Dirichlet boundary conditions, i.e. \( \Delta : \text{dom}(\Delta) = H^2(\Omega) \cap H^2_0(\Omega) \rightarrow L^2(\Omega) \) and \( H^2_0(\Omega) \) denotes the subset of functions that vanish at the boundary. Then operator \( -\Delta \) is self-adjoint on \( \Sigma \) and positive. For any \( \alpha \in (0, 1) \) the fractional power of operator \( -\Delta \) is defined on the domain \( \text{dom}(( -\Delta )^\alpha) \), i.e. \( ( -\Delta )^\alpha : \text{dom}(( -\Delta )^\alpha) \rightarrow L^2(\Omega) \). The domain is given by a fractional Sobolev space and for \( \alpha > 1/2 \), we have \( \text{dom}(( -\Delta )^\alpha) = H^{2\alpha}_0(\Omega) \subset H^{2\alpha}(\Omega) \) (see [30] for more information).

Moreover let \( B(t) \) denote a time-dependent scalar-valued multiplication operator given by
\[ (B(t)f)(x) = V(t,x)f(x), \quad \text{dom}(B(t)) = \{ f \in L^2(I,\Sigma) : V(\cdot,x)f(x) \in L^2(I,\Sigma) \} \quad (4.2) \]
where \( V : I \times \Omega \rightarrow \mathbb{R} \) is measurable. We assume that the potential \( V(\cdot, \cdot) \) is real and non-negative. Then \( B(t) \) is obviously self-adjoint and non-negative on \( \Sigma \).

**Theorem 4.1** Let \( A \) be the Laplacian operator \( -\Delta \) with Dirichlet boundary conditions in \( L^2(\Omega) \), see above. Further, let \( \{B(t)\}_{t \in I} \) be the family of multiplication operators defined by (4.2). If \( V(\cdot, \cdot) : I \times \Omega \rightarrow \mathbb{R} \) is measurable, real, non-negative with regularity \( V \in L^\infty(I, L^{2+\varepsilon}(\Omega)) \cap C^\beta(I, L^{1+\varepsilon}(\Omega)) \) for \( \beta \in (0, 1) \) and some \( \varepsilon > 0 \), then the assumptions (S1)-(S3) are satisfied with \( \alpha \in [3/4, 1) \). Moreover, if \( \beta > 2\alpha - 1 \) then the converging rates of Theorem 3.12 Corollary 3.13 and Theorem 3.14 hold.

**Proof.** Since \( \Omega \) is bounded there one has \( \inf \sigma(A) > 0 \) which does not satisfy \( A \succeq I \) in general and, hence, assumption (S1) is not satisfied. Nevertheless \( \inf \sigma(A) > 0 \) is sufficient to prove the converging results. So we can believe that (S1) is satisfied.
Let $\alpha \geq 3/4$. Using the Sobolev space embeddings, we get that $H^{2\alpha}(\Omega) \subset L^2(\Omega)$ for any $\gamma \in [2, \infty]$. Hence, if $V \in L^\infty(I, L^{2+\epsilon}(\Omega))$, we conclude that the function $[0, T] \ni t \mapsto B(t)(-\Delta)^{-\alpha}$ is essentially operator-norm bounded in $t \in I$ and thus, (S2) is satisfied. Now, we consider

$$F(t) := (-\Delta)^{-\alpha}B(t)(-\Delta)^{-\alpha} : L^2(\Omega) \to H^{2\alpha}(\Omega) \subset L^2(\Omega).$$

The function $F(\cdot) : I \to \mathcal{L}(\mathcal{H})$ is bounded for fixed $t \in [0, T]$ if for any $f, g \in H^{2\alpha}(\Omega)$ the function $\langle f, B(t)g \rangle$ is bounded. This holds since $V(t, \cdot) \in L^{1+\epsilon}(\Omega)$ and $H^{2\alpha}(\Omega) \subset L^\gamma(\Omega)$ for any $\gamma \in [2, \infty]$. Hence we conclude that (S3) is satisfied and the claim is proved.

Theorem 4.1 provides a convergence rate of an approximation of the solution of (4.1) by the time-ordered product

$$\tilde{V}_n(t, s) = \prod_{j=1}^n e^{-\frac{t-s}{n}} e^{-\frac{\tau}{n} V(jt + (n-j)s, \cdot)} e^{-\frac{\tau}{n} \Delta} \quad (4.3)$$

This looks elaborate, but is indeed simple. There are strategies to compute the semigroup of the Laplace operator for bounded domains and there are also explicit formulas on special domains like disks etc. The factors $e^{-\tau V(jt)}$, $j = 1, 2, \ldots, n$ are scalar valued and can be easily computed.

Acknowledgment

We thank Takashi Ichinose and Hideo Tamura for the explanation of details of the proof of Theorem 1.1 of [14], which makes possible to prove Lemma 3.8 and Lemma 3.9.

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