Effect of discontinuity in threshold distribution on the critical behaviour of a random fiber bundle

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The critical behaviour of a Random Fiber Bundle Model with mixed uniform distribution of threshold strengths and global load sharing rule is studied with a special emphasis on the nature of distribution of avalanches for different parameters of the distribution. The discontinuity in the threshold strength distribution of fibers non-trivially modifies the critical stress as well as puts a restriction on the allowed values of parameters for which the recursive dynamics approach holds good. The discontinuity leads to a non-universal behaviour in the avalanche size distribution for smaller values of avalanche size. We observe that apart from the mean field behaviour in the avalanche size distribution, a new behaviour for smaller avalanche size is observed as a critical threshold distribution is approached. The phenomenological understanding of the above result is provided using the exact analytical result for the avalanche size distribution. Most interestingly, the prominence of non-universal behaviour in avalanche size distribution depends on the system parameters.

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I. INTRODUCTION

Breakdown phenomena in nature has captured the attention of scientists for years. A study of this phenomena plays a major role for the prediction of failure and design of materials and structures. One of the paradigmatic model mimicking the fracture processes is random fiber bundle model (RFBM) which is simple yet subtle enough to capture the essential physics of the breakdown phenomena.

Random fiber bundle models have been studied extensively in recent years. Typically a RFBM consists of $N$ parallel fibers with randomly distributed threshold strength ($\sigma_{th}$) taken from a given distribution. If the stress generated due to an external force is greater than $\sigma_{th}$ of a fiber, it breaks. The dynamics of the model is initiated by applying a small external force just enough to break the weakest fiber present in the bundle. The load carried by this broken fiber is shared amongst the remaining intact fibers following a load sharing rule causing further failures. When no further failure takes place, the external force is once again increased quasistatically to break the weakest intact element present in the bundle and the process continues till the bundle breaks down completely at an external stress called the critical stress. Even though the threshold distribution of real materials may not be known exactly, in theoretical models the distributions are usually approximated by either a uniform distribution or a Weibull distribution.

The avalanche size is defined as the number of broken fibers between two successive loadings. The distribution of avalanche size turns out to be a key factor in characterising any breakdown phenomena. Hemmer and Hansen studied the avalanche size distribution $D(\Delta)$ of an avalanche of size $\Delta$ in a RFBM with the global load sharing (GLS) scheme, in which the additional stress due to a broken fiber is distributed equally to the remaining intact fibers. They established a universal power-law distribution in the large $\Delta$ limit given as $D(\Delta) \propto \Delta^{-\xi}$ with $\xi = 5/2$.

![Mixed Uniform Distribution](image)

RFBM with threshold strengths which are continuous and are uniformly distributed between 0 to $\sigma_1$ and also between $\sigma_2$ to 1, have been studied separately. But what happens when two such bundles are merged is not known, especially when $\sigma_1 < \sigma_2$ such that there exists a discontinuity in the threshold strength distribution. In this paper, we investigate the role of such a discontinuity on the critical behaviour of a RFBM. The distribution of threshold strength of fibers used in the present work is given as (See Fig. 1)

$$
\rho(\sigma_{th}) = \begin{cases} 
1 & 0 < \sigma_{th} \leq \sigma_1 \\
1 - (\sigma_2 - \sigma_1) & \sigma_1 < \sigma_{th} < \sigma_2 \\
0 & \sigma_2 \leq \sigma_{th}
\end{cases}
$$
\[ \sigma_2 \leq \sigma_{th} \leq 1. \] (1)

The discontinuity as defined above introduces dilution in the model in the sense that two types of fibers separated by a gap in their threshold distributions coexist in the same bundle. It is shown below that this discontinuity plays a crucial role in the dynamics of the model. Here, a fraction \( f \) of fibers belong to the weaker threshold distribution with strengths uniformly lying between 0 and \( \sigma_1 \) (Class A) whereas the remaining fraction of fibers have stronger threshold strengths between \( \sigma_2 \) and 1 (Class B). Clearly, \( (\sigma_2 - \sigma_1) \) is the measure of the discontinuity which vanishes in the limit of purely uniform distribution.

In a recent paper, Pradhan, Hansen and Hemmer [7] showed that for a bundle which is close to the complete break down (i.e., imminent failure), a crossover in \( \xi \) from a value 5/2 to 3/2 is observed when the threshold distribution approaches the critical distribution. Critical distribution in their case is the distribution in which the lowest threshold of the remaining intact fibers is equal to half that of the strongest. This crossover has also been observed with other load sharing rules [8]. We are however interested in looking at the distribution of to half that of the strongest. This crossover has also been observed with other load sharing rules [8].

\[ U_{t+1} = 1 - P(\sigma_t) = 1 - P\left(\frac{\sigma}{U_t}\right) \]
and
\[ \sigma_{t+1} = \frac{\sigma}{U_{t+1}} = \frac{\sigma}{(1 - P(\sigma_t))} \]

where \( P(\sigma_t) \) is the fraction of broken fibers with the redistributed stress \( \sigma_t \), and is given as
\[ P(\sigma_t) = \int_0^{\sigma_t} \rho(\sigma_{th}) d\sigma_{th}. \]

This dynamics propagates until no further breaking takes place. It should be emphasised that the initial load is so small that the redistributed stress is always less than \( \sigma_2 \) and therefore fibers from class B cannot fail. Thus to initiate the breaking of class B fibers, the redistributed stress at a later time \( t \) must exceed \( \sigma_2 \). The fixed point solution for \( U (= U^*) \) and \( \sigma (= \sigma^*) \) at which no further failure takes place can be obtained using the standard technique of solving the above recursive relations. By substituting \( P(\sigma_t) \) in Eq. (3) we get
\[ U_{t+1} = 1 - P\left(\frac{\sigma}{U_t}\right) \]
\[ = 1 - \left[ \frac{\sigma_1}{1 - (\sigma_2 - \sigma_1)} + \frac{1}{1 - (\sigma_2 - \sigma_1)} \left( \frac{\sigma}{U_t} - \sigma_2 \right) \right]. \]

At the fixed point,
\[ U^* = 1 - \left[ \frac{\sigma_1}{1 - (\sigma_2 - \sigma_1)} + \frac{1}{1 - (\sigma_2 - \sigma_1)} \left( \frac{\sigma}{U^*} - \sigma_2 \right) \right] \]
giving the following stable fixed point solutions
\[ U^* = \frac{1}{2(1 - (\sigma_2 - \sigma_1))} \left[ 1 + \sqrt{1 - \frac{\sigma}{\sigma_c}} \right] \]
and
\[ \sigma^* = \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{\sigma}{\sigma_c}} \]
with
\[ \sigma_c = \frac{1}{4(1 - (\sigma_2 - \sigma_1))}. \]

If the external applied stress is less than \( \sigma_c \), the system reaches a fixed point. For \( \sigma > \sigma_c \), the bundle breaks down completely as both the \( U^* \) and \( \sigma^* \) become imaginary.

\[ \sigma^* = \frac{1}{4(1 - (\sigma_2 - \sigma_1))}. \]

The critical stress of the mixed model varies with the gap \( (\sigma_2 - \sigma_1) \) and reduces to the value \( \sigma_c = 1/4 \) for the uniform distribution as \( (\sigma_2 - \sigma_1) \rightarrow 0 \). In deriving Eq. (4), \( \sigma_t \) is assumed to be greater than \( \sigma_2 \) while the applied stress when

II. RESULTS AND DISCUSSIONS

A. Critical stress and exponents

To study the dynamics of failure of fibers, we use the recursive dynamics approach [4]. If a fraction \( f \) of the total fibers belong to Class A and the remaining \( 1 - f \) to class B, then the uniformity of the distribution demands
\[ f = \frac{1}{1 - (\sigma_2 - \sigma_1)} \int_{\sigma_1}^{\sigma_2} d\sigma; \]
so that
\[ \sigma_1 = \frac{f}{1 - f}(1 - \sigma_2). \] (2)

The above equation provides a relationship between \( f \), \( \sigma_1 \) and \( \sigma_2 \) and at the same time puts a restriction on the allowed values of the parameter \( \sigma_1 \) as shown below. Any value of \( \sigma_1 > f \) leads to a value of \( \sigma_2 \) smaller than \( \sigma_1 \) which is not an acceptable distribution (see Eq. 1).

We now define \( U_t \) as the fraction of unbroken fibers after a time step \( t \). Then the redistributed stress at the instant \( t \) is \( \sigma_t = F/N_t = \sigma/U_t \) where the applied force \( F = N\sigma \) and \( N_t = NU_t \). The recurrence relations between \( U_t, U_{t+1} \) and between \( \sigma_t, \sigma_{t+1} \) for the GLS are obtained as [4]:
\[ U_{t+1} = 1 - P(\sigma_t) = 1 - P\left(\frac{\sigma}{U_t}\right) \]
and
\[ \sigma_{t+1} = \frac{\sigma}{U_{t+1}} = \frac{\sigma}{(1 - P(\sigma_t))} \]

where \( P(\sigma_t) \) is the fraction of broken fibers with the redistributed stress \( \sigma_t \), and is given as
\[ P(\sigma_t) = \int_0^{\sigma_t} \rho(\sigma_{th}) d\sigma_{th}. \]

This dynamics propagates until no further breaking takes place. It should be emphasised that the initial load is so small that the redistributed stress is always less than \( \sigma_2 \) and therefore fibers from class B cannot fail. Thus to initiate the breaking of class B fibers, the redistributed stress at a later time \( t \) must exceed \( \sigma_2 \). The fixed point solution for \( U (= U^*) \) and \( \sigma (= \sigma^*) \) at which no further failure takes place can be obtained using the standard technique of solving the above recursive relations. By substituting \( P(\sigma_t) \) in Eq. (3) we get
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At the fixed point,
\[ U^* = 1 - \left[ \frac{\sigma_1}{1 - (\sigma_2 - \sigma_1)} + \frac{1}{1 - (\sigma_2 - \sigma_1)} \left( \frac{\sigma}{U^*} - \sigma_2 \right) \right] \]
giving the following stable fixed point solutions
\[ U^* = \frac{1}{2(1 - (\sigma_2 - \sigma_1))} \left[ 1 + \sqrt{1 - \frac{\sigma}{\sigma_c}} \right] \]
and
\[ \sigma^* = \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{\sigma}{\sigma_c}} \]
with
\[ \sigma_c = \frac{1}{4(1 - (\sigma_2 - \sigma_1))}. \] (5)
plotted against the redistributed stress shows a discontinuity at $\sigma_1$. Beyond $\sigma_1$, the external force is increased to break the fiber with threshold strength $\sigma_2$, i.e., the gap in the threshold distribution is also reflected in the constitutive behaviour of the model.

The discontinuity on the other hand imposes some restrictions on the parameters for this calculation to be valid. Since the maximum value of the redistributed stress is equal to 0.5, $\sigma_2$ must be less than 0.5 so that some fibers from class B also fail at the critical point. The condition $\sigma_2 < 0.5$, eventually restricts the value of critical stress to be less than 0.5 for any chosen distribution. However, $\sigma_2 = 0.5$ is a limiting case when the redistributed stress at the critical point marginally reaches class B fibers. In short, we must have

$$\sigma_1 < f, \sigma_2 < 0.5.$$ 

One can define an order parameter $O$ associated with the transition [4] as shown below:

$$O = 2[1 - (\sigma_2 - \sigma_1)]U^* - 1 = (\sigma_c - \sigma)^{1/2} = (\sigma_c - \sigma)^\beta.$$ 

The order parameter goes to 0 as $\sigma \to \sigma_c$ following a power law $(\sigma_c - \sigma)^{1/2}$. Susceptibility can be defined as the increment in the number of broken fibers for an infinitesimal increase of load. Therefore,

$$\chi = \frac{dm}{d\sigma} \quad \text{where} \quad m = N[1 - U^*(\sigma)].$$

Hence,

$$\chi \propto (\sigma_c - \sigma)^{-1/2} = (\sigma_c - \sigma)^{-\gamma}.$$ 

The exponents $\beta$ and $\gamma$ stick to their mean field values [4][5] i.e., $\beta = \gamma = 1/2$. We conclude that though the discontinuity alters the critical stress, the critical exponents remain unaltered. It is to be noted that the model reduces to the already obtained results in various limits. For example, if class A fibers are absent ($f = 0$), $\sigma_c = 1/4(1 - \sigma_2)$ and an elastic to plastic deformation is observed [4].

**B. Avalanche size distribution**

Let us now focus on the avalanche size distribution exponent $\xi$. Below is shown some of the allowed distributions which satisfy the above mentioned restrictions.

| Case | $f$  | $\sigma_1$ | $\sigma_2$ | $\sigma_c$ |
|------|------|------------|------------|------------|
| 1    | 0.10 | 0.08       | 0.28       | 0.31       |
| 2    | 0.20 | 0.19       | 0.24       | 0.26       |
| 3    | 0.20 | 0.15       | 0.40       | 0.33       |
| 4    | 0.30 | 0.25       | 0.42       | 0.30       |
| 5    | 0.30 | 0.29       | 0.33       | 0.26       |
| 6    | 0.40 | 0.35       | 0.47       | 0.29       |

We study the avalanche size distribution numerically by using the method of breaking of the weakest fiber [6]. In the simulation, the fibers are arranged in an increasing order of their threshold strengths. An external force sufficient to break the weakest fiber is applied and the load due to the breaking of this fiber gets redistributed among the remaining intact fibers following the GLS scheme. The number of failed fibers for a fixed external load is recorded till the dynamics reaches a fixed point. Thereafter, the external load is increased further and the above process is repeated till the critical stress is reached.

Following interesting observations (Fig. 2) are clearly highlighted. (i) For the cases 1, 2 and 5 of the table, the avalanche size exponent is 5/2. (ii) For the cases 3, 4 and 6, there is an apparent power law behaviour for smaller $\Delta$ with the exponent which is found to depend on the system parameters. In the examples chosen here the exponent happens to be close to 3. For larger $\Delta$, however, we retrieve the universal mean field behaviour with $\xi = 5/2$. Also, (iii) an increase in the region with $\xi \approx 3$ is seen as $\sigma_2 \to 0.5$. These observations establish the following: (i) there is a non-universal behaviour of $D(\Delta)$ in the small $\Delta$ limit, (ii) there is a crossover to the universal behaviour in large $\Delta$ limit, and (iii) the crossover behaviour is prominent as $\sigma_2 \to 0.5$.

The above mentioned results can be explained by extending the analytical result for the avalanche size distribution obtained by Hemmer and Hansen [6] to the mixed model. The general expression for the avalanche size distribution with GLS is given as

$$\frac{D(\Delta)}{N} = \frac{\Delta^{\Delta-1}}{\Delta!} \int_0^{\infty} dx \rho(x) \left(1 - \frac{x \rho(x)}{Q(x)}\right)^{\Delta-1} \exp(-\Delta \frac{x \rho(x)}{Q(x)}),$$

where $x$ is the redistributed stress and $Q(x)$ is the fraction of unbroken fibers at $x$. The upper limit of the integration ($x_c$) is the redistributed stress at the critical point. The right hand side of Eq. (6) is broken into two parts, $D_1(\Delta)$ and $D_2(\Delta)$ for the mixed model with any allowed values of $\sigma_1$ and $\sigma_2$ (where $\rho(x) = \rho = 1/(1 - \sigma_2 + \sigma_1)$):

As long as the redistributed stress is restricted to the class A fibers,

$$Q(x) = \frac{1 - \sigma_2 + \sigma_1 - x}{1 - \sigma_2 + \sigma_1},$$

we have

$$D_1(\Delta) = \frac{\Delta^{\Delta-1}}{\Delta!} \frac{1}{1 - \sigma_2 + \sigma_1} \int_0^{\sigma_1} dx \left(1 - \frac{x - \sigma_2 + \sigma_1 - 2x}{x}\right) \left(1 - \frac{x}{1 - \sigma_2 + \sigma_1 - x}\right)^{\Delta-1} \exp(-\Delta \frac{x}{1 - \sigma_2 + \sigma_1 - x}).$$

When the redistributed stress belongs to the second block (class B), we have

$$Q(x) = \frac{1 - x}{1 - \sigma_2 + \sigma_1},$$

and
for three different cases are shown in Fig. 3. Incidentally in the examples presented here, $\Delta_c$ increases with $\sigma_2$. If we look at a particular critical distribution (Fig. 4), a crossover from an apparent power-law behaviour of $D(\Delta)$ with exponent $\xi$ close to $3$ for small $\Delta$ to a universal behaviour with $\xi = 3/2$ is observed.

We now discuss the numerical integration and simulation results in the light of Eqs. (7) and (8). With a change of variable $x \to x/(1-\sigma_2+\sigma_1-x)$, we can rewrite Eq. (7) in the following form

$$D_1(\Delta) = \frac{\Delta^{\Delta-1}}{\Delta!} \int_0^{x_m} \frac{1-x}{x(1+x)^2} e^{(\log x - x)\Delta} dx$$

(9)

where $x_m = \sigma_1/(1-\sigma_2)$. The argument of the exponential term has a maximum at $x = 1$ which is outside the range of integration and hence the saddle point integration method cannot be applied [9]. Expanding the term $1/(1+x)^2$ in a power series and then using the incomplete gamma function [10], we arrive at the following result (see the Appendix)

$$D_1(\Delta) = \frac{e^{(1-x_m)\Delta}}{\Delta^{3/2}} x_m^{\Delta} \sum_{q=0}^{\infty} (-1)^q (q+1) x_m^q \times$$

$$\sum_{k=0}^{\infty} \frac{(x_m \Delta)^k}{(\Delta + q)(\Delta + q + 1)\ldots(\Delta + q + k)}$$
The leading behaviour of the infinite series (10) is $\Delta^{-5/2}e^{(1-x_m)\Delta x_m^2}$ which justifies the non-universality observed in numerical simulations in the small $\Delta$ limit. The question therefore remains why does the non-universal behaviour become prominent as $\sigma_2 \rightarrow 0.5$. The behaviour of $D_2(\Delta)$ [Eq. (8)] for smaller $\Delta (<< \Delta_c)$ is at the root of this. When $\sigma_2 \rightarrow 0.5$, $D_2(\Delta)$ goes as $\Delta^{-3/2}$ and hence the contribution from $D_1(\Delta)$ wins over to produce a prominent non-universal behaviour. The small $\Delta$ behaviour of $D(\Delta)$ is therefore non-mean-field and non-universal when $\sigma_2 \rightarrow 0.5$. On the other hand if $\sigma_2 << 0.5$, $x_m$ is relatively smaller and the contribution from class A fibers decays very fast as $\Delta$ increases and one observes a mean-field universal behaviour almost for the entire range of $\Delta$. We also observe a non-power-law behaviour for small $\Delta$ when $\sigma_1$ is very small and $\sigma_2$ is close to criticality as is expected from the analytical result. However for a given $\sigma_2 (\sim 0.5)$, larger $x_m$ leads to a non-universal behaviour up to larger $\Delta$. The crossover value $\Delta_c$ is thus roughly given by the value of $\Delta$ for which $D_2(\Delta)$ crosses over from a $\Delta^{-3/2}$ behaviour to $\Delta^{-5/2}$ behaviour.

III. CONCLUSIONS

In conclusion, we have studied a mixed fiber bundle with a discontinuous but uniform threshold distribution and GLS. Discontinuity leads to a functional dependence of the critical stress on the system parameters $\sigma_1, \sigma_2$ and $f$ and also imposes restrictions on the allowed values of these parameters. Although the critical exponents are unchanged, there is a non-trivial change in the burst avalanche distribution behaviour where discontinuity leads to a non-universal, non-mean field behaviour for small $\Delta$. We would like to emphasise that non-universality becomes prominent only when $\sigma_2 \rightarrow 0.5$. For large $\Delta$ limit, the behaviour is however universal and mean field. If $f = 0$ or $\sigma_2 << 0.5$, the non-universal behaviour completely disappears. The non-universality in $D(\Delta)$ is also seen for other distributions [9]. The beauty of our model is that the non-universal behaviour is tunable with the system parameters and there is a crossover from non-universality to universality in the limit of large $\Delta$. One should also note that the imminent failure of the bundle (i.e., final stages of the breakdown process) is the same as in Ref. [7] because the effect of class A fibers essentially vanishes in that limit.

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Appendix

In this appendix, we shall indicate how to arrive at Eq. (10) of the text starting from Eq. (9). Equation (9) can be written as

$$D_1(\Delta) = \frac{e^\Delta}{\Delta^{3/2}} \int_0^{x_m} \frac{1-x}{(1+x)^2} x^{\Delta-1} e^{-\Delta x} dx$$

Let $f(\Delta) = \int_0^{x_m} \frac{1-x}{(1+x)^2} x^{\Delta-1} e^{-\Delta x} dx$

$$= \int_0^{x_m} (1-x) x^{\Delta-1} e^{-\Delta x} \sum_{q=0}^\infty (-1)^q (q+1)x^q dx$$

$$= \sum_{q=0}^\infty (-1)^q (q+1) \int_0^{x_m} (1-x)x^{\Delta+q-1} e^{-\Delta x} dx.$$
Rearranging terms, we get

\[
f(\Delta) = e^{-x_m \Delta} x_m \sum_{q=0}^{\infty} (-1)^q (q + 1) x_m^q \times \left( \sum_{k=0}^{\infty} \frac{(x_m \Delta)^k}{(\Delta + q)(\Delta + q + 1) \ldots (\Delta + q + k)} \right).
\]

i.e., Eq. (10).

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