Weighted vertex cover on graphs with maximum degree 3

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Abstract

We give a parameterized algorithm for weighted vertex cover on graphs with maximum degree 3 whose time complexity is \(O^*(1.402^t)\), where \(t\) is the minimum size of a vertex cover of the input graph.

Keywords graph algorithms, parameterized complexity.

1 Introduction

For an undirected graph \(G\), a vertex cover of \(G\) is a set of vertices \(S\) such that every edge of \(G\) is incident on at least one vertex in \(S\). In the vertex cover problem the input is an undirected graph \(G\) and the goal is to find a vertex cover of \(G\) with minimum size. In the weighted vertex cover problem the input is an undirected graph \(G\) and a weight function \(w : V(G) \rightarrow \mathbb{R} \geq 0\). The goal is to find a vertex cover of \(G\) with minimum weight.

The parameterized complexity of the vertex cover problem have been studied extensively. For the unweighted problem, the first parameterized algorithm was given in [2]. Improved algorithms were given in [3,4,6,8,9,11,12,15]. The unweighted problem was also studied on graphs with maximum degree 3 [5,7,13,16]. For the weighted vertex cover problem, Niedermeier et al. [12] gave an algorithm with time complexity \(O^*(1.396^W)\), where \(W\) is the minimum weight of a vertex cover of the input graph. They also gave an algorithm with exponential space whose running time is \(O^*(1.379^W)\). Fomin et al. [10] gave an algorithm with exponential space whose running time is \(O^*(1.357^W)\). Shachnai and Zehavi [14] gave a polynomial space algorithm with time complexity \(O^*(1.381^s)\), where \(s \leq W\) is the minimum size of a minimum weight vertex cover of the input graph, and an exponential space algorithm with time complexity \(O^*(1.363^s)\). Additionally, they gave an algorithm with time complexity \(O^*(1.443^t)\), where \(t \leq s\) is the minimum size of a vertex cover of the input graph, and an algorithm for graphs with maximum degree 3 whose time complexity is \(O^*(1.415^t)\).

In this paper, we give an algorithm for weighted vertex cover on graph with maximum degree 3 whose time complexity is \(O^*(1.402^t)\).

2 Preliminaries

For a graph \(G\), let \(V(G)\) and \(E(G)\) denote the sets of vertices and edges of \(G\), respectively. For a graph \(G\) and a vertex \(v \in V(G)\), \(N(v) = \{u \in V(G) : (u, v) \in E(G)\}\) and \(N[v] = N(v) \cup \{v\}\). For a set of vertices \(S\), \(N(S) = (\bigcup_{v \in S} N(v)) \setminus S\).

For a graph \(G\) and a set of vertices \(S\), \(G[S]\) is the subgraph of \(G\) induced by \(S\) (namely, \(G[S] = (S, E(G) \cap (S \times S))\)). We also define \(G - S = G[V \setminus S]\). For a vertex \(v\), we write \(G - v\) instead of \(G - \{v\}\).

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For a graph $G$ and a set $U \subseteq V(G)$, let $C_i(G,U)$ be the set of connected components $C$ of $G[U]$ such that $|C| = i$ and $G[C]$ is a complete graph. Let $C_i^*(G,U)$ be the set of all $C \in C_i(G,U)$ such that there is no $v \in V(G) \setminus U$ that is adjacent to all the vertices of $C$. Let $V_i(G,U) = \bigcup_{C \in C_i(G,U)} C$, $V_i^*(G,U) = \bigcup_{C \in C_i^*(G,U)} C$, and $V_{\geq 2}(G,U) = U \setminus V_1(G,U)$.

3 The algorithm

Our algorithm is based on the algorithm of Shachnai and Zehavi [14]. We first describe the algorithm of Shachnai and Zehavi (we note that we describe the algorithm slightly differently than [14]). Given an instance $G^*,w^*$ of weighted vertex cover, the algorithm first finds a vertex cover $U^*$ of $G^*$ with minimum size (using a fixed parameter algorithm for the unweighted problem). Additionally, $U^*$ has the property that there is a mapping $f : C_3^*(G^*,U^*) \to C_2(G^*,U^*)$ such that for every $C \in C_3^*(G^*,U^*)$ there is a vertex $v \in V(G^*) \setminus U^*$ that is adjacent to a vertex in $C$ and to the two vertices of $f(C)$. The algorithm then calls WVCAlg($G^*,U^*,w^*,f$), where WVCAlg($G,U,w,f$) is a recursive procedure that returns a minimum weight vertex cover of $G$. The parameter $U$ is a vertex cover of $G$ (not necessarily a vertex cover with minimum size). In particular, $V(G) \setminus U$ is an independent set of $G$.

Before describing procedure WVCAlg, we define two base branching rules:

(B1) Let $v \in U$. Return a set of minimum weight among WVCAlg($G-v,U\setminus\{v\},w,f$) and WVCAlg($G-N[v],U\setminus N[v],w,f$) $\cup$ $N(v)$.

(B2) Let $C \in C_3(G,U)$. Return a set of minimum weight among WVCAlg($G-A,U\setminus A,w,f$) $\cup$ $A$, where the minimum is taken over every set $A \subset C$ of size 2.

Procedure WVCAlg is composed of reduction and branching rules. The procedure applies the first applicable rule from the following rules (the last rule is numbered [13] in order to leave space for the rules of our algorithm).

1. If $G$ is bipartite, compute a minimum weight vertex cover $S$ of $G$ and return $S$.
2. If there is a connected component $S$ in $G$ of size at most 10, compute a minimum weight vertex cover $S$ of $G[C]$ and return WVCAlg($G-C,U\setminus C,w,f$) $\cup$ $S$.
3. If there is $v \in U$ such that $v$ has no neighbors in $V(G) \setminus U$, return WVCAlg($G,U\setminus\{v\},w,f$).
4. If there is $C \in C_3(G,U)$ such that $f(C') \neq f(C)$ for every $C' \in C_3(G,U) \setminus \{C\}$, choose $v \in f(C)$. Apply Rule (B1) on $v$. In the branch $G-v$ apply Rule (B2) on $C$.
5. If $C_3(G,U) \neq \emptyset$, choose distinct $C,C' \in C_3(G,U)$ such that $f(C) = f(C')$, and choose $v \in f(C)$. Apply Rule (B1) on $v$. In the branch $G-v$ apply Rule (B2) on $C$, and in each resulting branch, apply Rule (B2) on $C'$.
6. If there is $u \in U$ such that $|N(u) \cap U| = 1$ and $|N(v) \cap U| = 2$, where $v$ is the unique neighbor of $u$ in $G[U]$, apply Rule (B1) on $v$.
7. If there is $v \in U$ such that $|N(v) \cap U| = 2$, apply Rule (B1) on $v$.
13. Choose $v \in U$ such that $|N(v) \cap U| = 1$, and apply Rule (B1) on $v$.

The analysis of the algorithm above uses the measure and conquer technique, using the measure function $m(G,U) = |V_{\geq 2}(G,U)|$. The analysis shows that the number of leaves in the branching tree of $G^*$ is at most $1.415^m(G^*,U^*) \leq 1.415^t$, where $t = |U^*|$ is the minimum size of a vertex cover of $G^*$.

We now describe our algorithm. A good vertex cover of a graph $G$ is a vertex cover $U$ of $G$ such that every connected component in $G[U]$ has size at most 2. Suppose that $U$ is a good vertex cover of $G$. We say that a vertex $x \in V(G) \setminus U$ is bad if $|N(x) \cap V_1(G,U)| \geq$
1 and either \(|N(x) \cap V_2(G, U)| = 1\) or \(x\) is adjacent to both vertices of a connected component in \(C_2(G, U)\). We say that \(x\) is semi-bad if \(x\) is not bad, \(|N(x) \cap V_1(G, U)| = 1\) and \(|N(x) \cap V_2(G, U)| = 2\). If \(x\) is not bad or semi-bad, we say that \(x\) is good.

We define a mapping \(h_{G,U} : V_2(G, U) \to \{0, 1, 1/2, 3/2, 1\}\) as follows. If \(v \in V_2(G, U)\), \(h_{G,U}(v) = \max\{0, 1 - b_1 - b_2/2\}\), where \(b_1\) (resp., \(b_2\)) is the number of bad (resp., semi-bad) neighbors of \(v\). Now consider a vertex \(v \in V_2(G, U) \setminus V_2\)\((G, U)\), and let \(v'\) be the unique neighbor of \(v\) in \(G[U]\). Then, \(h_{G,U}(v) = h_{G,U}(v') = \max\{0, 1 - b_1/2 - b_2/4\}\), where \(b_1\) (resp., \(b_2\)) is the number of bad (resp., semi-bad) vertices in \(N\{v, v'\}\).

Our algorithm is based on the algorithm of Shachnai and Zehavi. We make two changes to procedure WVCAlg. First, the procedure receives an additional parameter \(q \in \{0, 1, 2\}\), which is initially 0. Additionally, the following rules are added.

(8) If there is \(v \in V(G)\) with degree 1, let \(u\) be the neighbor of \(v\). If \(w(v) \geq w(u)\), return WVCAlg\((G - \{u, v\}, U \setminus \{u, v\}, w, f, q) \cup \{u\}\). Otherwise, let \(w' : V(G - v) \to \mathbb{R}^{\geq 0}\) be a function in which \(w'(u) = w(u) - w(v)\) and \(w'(x) = w(x)\) for every \(x \neq u\). Let \(S = WVCAlg(G - v, U \setminus \{v\}, w', f, q)\). If \(u \in S\) return \(S\) and otherwise return \(S\cup\{v\}\).

(9) If there is a triangle \(v_1, v_2, v_3\) such that either \(v_1\) and \(v_2\) have degree 2, or there is a vertex \(v_4\) such that \(N(v_4) = \{v_1, v_2\}\), return WVCAlg\((G - v_4, U \setminus \{v_1, v_2\}, w, f, q) \cup \{v_4\}\), where \(v_4\) is the vertex with minimum weight among \(v_1\) and \(v_2\).

(10) If \(q = 0\), then \(q' \leftarrow 1\) if \(|V_1(G, U)| \geq \beta \cdot |V_2(G, U)|\) and \(q' \leftarrow 2\) otherwise, where \(\beta = 0.175\). Return WVCAlg\((G, U, w, f, q')\).

(11) If \(q = 2\) and there is \(v \in V_2^*(G, U)\) with \(h_{G,U}(v) > 0\) then let \(v'\) be the unique neighbor of \(v\) in \(G[U]\). If \(|N(v) \setminus \{v'\}| = 1\) denote \(N(v) = \{v', x_1\}\), and otherwise denote \(N(v) = \{v', x_1, x_2\}\) where \(x_2\) is a good vertex. Choose a vertex \(u_1 \in (N(x_1) \setminus \{v\}) \cap V_2(G, U)\), and let \(u_1'\) be the unique neighbor of \(u_1\) in \(G[U]\). If \(N(v) = \{v', x_1, x_2\}, u_1 \notin N(x_2)\), and \(N(x_2) \setminus \{v\} \neq \{u_1'\}\), choose a vertex \(u_2 \in N(x_2) \setminus \{v, u_1'\}\). Now, apply Rule (B1) on \(u_1\). If \(u_2\) is defined, in each of the two branches obtained by the application of Rule (B1), apply Rule (B1) on \(u_2\). This gives four branches whose graphs are \(G - \{u_1, u_2\}, G - (N[u_1] \cup \{u_2\}), G - \{u_1\} \cup N[u_2]\), and \(G - (N[u_1] \cup N[u_2])\).

We now show the correctness of the rule. We have that \(|N(v) \setminus \{v'\}| \in \{1, 2\}\) (this follows from the assumption that \(G\) has maximum degree 3, and the assumption that Rule \([\dag]\) cannot be applied). Therefore, either \(N(v) = \{v', x_1\}\) or \(N(v) = \{v', x_1, x_2\}\). In the latter case, at least one of the vertices of \(N(v) \setminus \{v\}\) is good since \(h_{G,U}(v) > 0\). In both cases, the vertex \(x_1\) is not bad (since \(h_{G,U}(v) > 0\)). The vertex \(u_1\) exists since the assumption that Rule \([\dag]\) cannot be applied implies that \(N(x_1) \setminus \{v\} \neq \emptyset\), and at least one of the vertices in \(N(x_1) \setminus \{v\}\) is in \(V_2(G, U)\) (since \(x_1\) is not bad). Note that \(v \in V_2^*(G, U)\) implies that \(u_1 \neq v'\).

If \(N(v) = \{v', x_1, x_2\}\), \(u_1 \notin N(x_2)\), and \(N(x_2) \setminus \{v\} \neq \{u_1'\}\), then the vertex \(u_2\) exists: if \(u_1' \notin N(x_2)\) then \(u_2\) exists due to the assumption that Rule \([\dag]\) cannot be applied, and if \(u_1' \notin N(x_2)\) then \(u_2\) exist due to the assumption that \(N(x_2) \setminus \{v\} \neq \{u_1'\}\). Since \(x_2\) is good, \(u_2 \in V_2(G, U)\). By definition, \(u_2 \notin \{v, u_1, u_1'\}\). Additionally, since \(v \in V_2^*(G, U)\), \(u_2 \neq v'\).

Since Rule (B1) is correct, and since the vertices \(u_1, u_1', u_2, u_2'\) are distinct, it follows that Rule \([\ddagger]\) is correct.

(12) If \(q = 2\) and there is \(v \in V_2(G, U) \setminus V_2^*(G, U)\) with \(h_{G,U}(v) > 0\), then let \(v'\) be the unique neighbor of \(v\) in \(G[U]\). Let \(x \in V(G) \setminus U\) be a vertex that is adjacent to both \(v\) and \(v'\). Let \(A = \{x \in N(v, v') : N(x) \setminus \{v, v'\} \neq \emptyset\}\). If \(|A| = 2\), choose a non-bad vertex \(x_1 \in A\). Otherwise, choose \(x_1, x_2 \in A\) such that \(x_1\) is not bad and \(x_2\) is good. Choose \(u_1 \in (N(x_1) \setminus \{v, v'\}) \cap V_2(G, U)\), and let \(u_1'\) be the unique neighbor of \(u_1\) in \(G[U]\). If
$|A| = 3$, $u_1 \notin N(x_2)$, and $N(x_2) \setminus \{v, v'\} \neq \{u'_2\}$, choose a vertex $u_2 \in N(x_2) \setminus \{v, v', u'_2\}$.

Now, apply Rule (B1) on $u_1$. If $u_2$ is defined, in each of the two branches obtained by the application of Rule (B1), apply Rule (B1) on $u_2$.

We now show the correctness of the rule. The existence of $x$ follows from the fact that $v \in V_2^*(G, U)$. Due to the assumption that Rule (8) and Rule (10) cannot be applied, $|A| \in \{2, 3\}$, where in the case $|A| = 3$ we have that $x \in A$. If $|A| = 2$, there is at most one bad vertex in $A$ (since $h_{G,U}(v) > 0$), and therefore $A$ contains at least one non-bad vertex. Otherwise ($|A| = 3$), from the assumption that $v \in V_2^*(G, U) \setminus V_2^*(G, U)$, we have that there is a vertex $x \in V(G) \setminus U$ that is adjacent to both $v$ and $v'$. Since $|A| = 3$, $x \in A$. By definition, $x$ is either bad or good, and therefore $A$ cannot contain three semi-bad vertices. It follows that $A$ contains a good vertex, and an additional vertex that is either good or semi-bad.

The vertex $u_1$ exists since $N(x_1) \setminus \{v, v'\} \neq \emptyset$ by the definition of $A$, and at least one of the vertices in $N(x_1) \setminus \{v, v'\}$ is in $V_2^*(G, U)$ (since $x_1$ is not bad). If $|A| = 3$, $u_1 \notin N(x_2)$, and $N(x_2) \setminus \{v, v'\} \neq \{u'_1\}$, the vertex $u_2$ exists (if $u'_1 \notin N(x_2)$ then $u_2$ exists due to the definition of $A$, and if $u'_1 \in N(x_2)$ then $u_2$ exist due to the assumption that $N(x_2) \setminus \{v, v'\} \neq \{u'_1\}$). Since $x_2$ is good, $u_2 \in V_2^*(G, U)$. By definition, $u_2 \notin \{v, v', u_1, u'_1\}$.

Since Rule (B1) is correct, and since the vertices $u_1, u'_1, u_2, u'_2$ are distinct, it follows that Rule (12) is correct.

Note that when the algorithm applies Rule (10) in some branch, $U$ is a good vertex cover of $G$. Therefore, the only rules that are applied afterward in the branch are Rules (1), (2), (3), (8), (9), (11), (12), and (13).

We now analyze the time complexity of the algorithm. We first analyze rules (1) to (13). We use the measure function $m_1(G, U) = |V_{22}(G, U)| + \alpha |V_1(G, U)|$, where $\alpha = 0.156$. The branching rule with the largest branching number is Rule (11). This rule generates four branches. In each of the three branches that are obtained from the branch $G - v$, 3 vertices of $V_{22}(G, U)$ are deleted from the graph, and 2 vertices are moved from $V_{22}(G, U)$ to $V_1(G, U)$. Therefore, the value of $m_1(G, U)$ decreases by $5 - 2\alpha$ in these branches. In the branch $G - N[v]$ the algorithm applies Rule (3) on a vertex of $C$. Therefore, in this branch one vertex of $V_{22}(G, U)$ is deleted from the graph, one vertex is moved out of $V_{22}(G, U)$, and one vertex is moved from $V_{22}(G, U)$ to $V_1(G, U)$. Therefore, the value of $m_1(G, U)$ decreases by $3 - \alpha$. Thus, the branching vector of Rule (11) is $(5 - 2\alpha, 3 - \alpha)$, and the branching number is 1.402.

The analysis of the other branching rules is similar to the analysis of these rules in [13]. One exception is Rule (7), for which we need a more careful analysis. Due to the previous rules, when this rule is applied, the connected component of $v$ in $G[U]$ is a chordless cycle with at least 4 vertices, and denote the vertices of this cycle by $v, v_1, v_2, \ldots, v_{|G|-1}$. Therefore, after applying Rule (7), Rule (13) can be applied in the branch $G - v$ on $v_2$. Therefore, there are three branches: $G - \{v, v_2\}$, $G - (\{v\} \cup N[v_2])$, and $G - N[v]$. If the size of the cycle is 4, in the first and third branches, 2 vertices of the $V_{22}(G, U)$ are deleted from the graph, and 2 vertices of $V_{22}(G, U)$ are moved to $V_1(G, U)$. In the second branch, 3 vertices of $V_{22}(G, U)$ are deleted from the graph, and one vertex of $V_{22}(G, U)$ is moved to $V_1(G, U)$. It follows that the branching vector is $(4 - 2\alpha, 4 - \alpha, 4 - 2\alpha)$. Similarly, the branching vector is $(3 - \alpha, 5 - 2\alpha, 3 - \alpha)$ if the size of the cycle is 5. If the size of the cycle is at least 6, Rule (8) can be applied in the branch $G - \{v, v_2\}$ on $v_4$, and in the branch $G - N[v]$ on $v_3$. Therefore, there are five branches: $G - \{v, v_2, v_4\}$, $G - (\{v, v_2\} \cup N[v_4])$, $G - (\{v\} \cup N[v_2])$, $G - (N[v] \cup \{v_3\})$, and $G - (N[v] \cup N[v_3])$. The branching vector is at least as good as $(5 - 2\alpha, 6 - 2\alpha, 4 - \alpha, 5 - 2\alpha, 6 - 2\alpha)$. The branching numbers of the branching vectors above are 1.342, 1.389, and 1.395, respectively.
We now consider a recursive call WVCAlg($G', U'$, $w', f, q$) in which Rule 10 is applied and $|V_1(G', U')| \geq \beta \cdot |V_2(G', U')|$. For the analysis of this branch, we switch from the measure function $m_1(G, U)$ to a measure function $m_2(G, U) = (1 + \alpha \beta) \cdot |V_2(G, U)|$. Note that $m_2(G', U') \leq m_1(G', U')$, so the switch is correct. After Rule 10 is applied, only Rules (1), (2), (3), (8), (9), and (13) are applied, and the only branching rule among these rules is Rule 13. When Rule 13 is applied, the value of $m_2(G, U)$ decreases by $2(1 + \alpha \beta)$ in each branch (since in each branch, one vertex in $V_2(G, U)$ is deleted from the graph, and one vertex in $V_2(G, U)$ is moved to $V_1(G, U)$). The branching vector is $(2(1 + \alpha \beta), 2(1 + \alpha \beta))$ and the branching number is 1.402.

We now consider a recursive call WVCAlg($G', U'$, $w', f, q$) in which Rule 10 is applied and $|V_1(G', U')| < \beta \cdot |V_2(G', U')|$. In order to show that our algorithm has $O^*(1.402^n)$ running time, it suffices to show that the number of leaves in the branching tree of this call is at most $1.402^{m_1(G', U')}$. To show this we will use the following lemmas.

**Lemma 1.** For a recursive call WVCAlg($G, U, w, f, q$) in which $q = 2$, the number of leaves in the branching tree of the call is at most $(\sqrt{2})^{m(G, U)} \cdot 0.9808^{M(G, U)}$, where $m(G, U) = |V_2(G, U)|$ and $M(G, U) = \sum_{v \in V_2(G, U)} h_{G, U}(v)$.

**Lemma 2.** If $U$ is a good vertex cover of $G$, $M(G, U) \geq |V_2(G, U)| - 3|V_1(G, U)|$.

**Proof.** We prove the lemma by induction on $|V_1(G, U)|$. The base of the induction is true since $h_{G, U}(v) = 1$ for all $v$ if $|V_1(G, U)| = 0$, so $M(G, U) = |V_2(G, U)|$. If $|V_1(G, U)| > 0$, pick $v \in V_1(G, U)$ and let $G' = G - v$ and $U' = U \setminus \{v\}$. By the induction hypothesis, $M(G', U') \geq |V_2(G', U')| - 3|V_1(G', U')| = |V_2(G, U)| - 3|V_1(G, U)| + 3$. We will show that $M(G', U') \geq M(G', U') - 3$ which will prove the lemma. $v$ has at most 3 neighbors, and we will show that each neighbor decreases the value of $M(G', U')$ by at most 1 compared to $M(G', U')$.

If $x$ is bad than either $x$ has a neighbor $u$ in $V_2(G, U)$, and let $u'$ be the unique neighbor of $u$ in $G[U]$. If $u \in V_2^*(G, U)$, $u$ is the only neighbor of $x$ in $V_2(G, U)$. Therefore, $x$ decreases the value of $h_{G, U}(u)$ by at most 1 compared to $h_{G', U'}(u)$ (namely, $h_{G, U}(u) \geq h_{G', U'}(u) - 1$), and does not change the $h_{G, U}$-values of the other vertices. If $u \in V_2(G, U) \setminus V_2^*(G, U)$, $x$ can be also adjacent to $u'$, but it does not have neighbors in $V_2(G, U) \setminus \{u, u'\}$. Therefore, $x$ decreases the values of $h_{G, U}(u)$ and $h_{G, U}(u')$ by at most $\frac{1}{2}$, and does not change the $h_{G, U}$-values of the other vertices. Therefore, in this case we also have that $x$ decreases the value of $M(G', U')$ by at most 1.

If $x$ is semi-bad then $x$ has two neighbors $u_1, u_2 \in V_2(G, U)$. If $u_1, u_2 \in V_2^*(G, U)$ then $x$ decreases the values of $h_{G, U}(u_1)$ and $h_{G, U}(u_2)$ by at most $\frac{1}{2}$, and does not change the $h_{G, U}$-values of the other vertices. Therefore, $x$ decreases the value of $M(G', U')$ by at most 1. It is also easy to verify that this is also true when one or two vertices from $u_1, u_2$ are in $V_2(G, U) \setminus V_2^*(G, U)$.

By Lemma 1, the number of leaves in the branching tree of WVCAlg($G', U'$, $w', f, q$) is at most $(\sqrt{2})^{m(G', U')} \cdot 0.9808^{M(G', U')}$. By Lemma 2 and since $|V_1(G', U')| < \beta \cdot |V_2(G', U')|$, we have that $M(G', U') \geq |V_2(G', U')| - 3|V_1(G', U')| > (1 - 3\beta) \cdot m(G', U')$. Therefore, the number of leaves in the branching tree of the call is at most $(\sqrt{2}) \cdot 0.9808^{1 - 3\beta} m(G', U') \leq 1.402^{m(G', U')} \leq 1.402^{m_1(G', U')}$. We now prove Lemma 1. The proof uses induction on the height of branching tree of the call. Consider a call WVCAlg($G, U, w, f, q$). If Rule 13 is applied in this call then $M(G, U) = 0$. The application of Rule 13 decreases $m(G, U)$ by 2 in each branch. Therefore, by the induction hypothesis, the number of leaves in the branching tree of the call is at most $2 \cdot (\sqrt{2})^{m(G, U)} - 2 = (\sqrt{2})^{m(G, U)}$.
Now consider a call in which Rule (11) is applied. Suppose that \( u_2 \) is defined, and let \( u'_2 \) be the unique neighbor of \( u_2 \) in \( G[U] \). Rule (11) generates four branches. In the first three branches, 2 vertices from \( \{ u_1, u'_1, u_2, u'_2 \} \) are deleted from the graph, and the remaining 2 vertices are moved from \( V_2(G, U) \) to \( V_1(G, U) \). Therefore, the value of \( m(G, U) \) decrease by 4 in these branches. Moreover, in the branch \( G_2 = G - (N[u_1] \cup N[u_2]) \), \( v \) has no neighbors in \( V(G_2) \setminus U \), so Rule (3) is applied on \( v \), and then Rule (3) is applied on \( v' \). Thus, the value of \( m(G, U) \) decreases by 6 (\( v, u'_1, u'_2 \) are deleted from the graph and \( v', u_1, u_2 \) are moved from \( V_2(G, U) \) to \( V_1(G, U) \)).

We now bound the decrease in \( M(G, U) \) in each of the four branches. Consider the branch \( G - \{ u_1, u_2 \} \). Since \( u'_1 \) is moved from \( V_2(G, U) \) to \( V_1(G, U) \), the value \( h_{G, U}(u'_1) \) will no longer be included in \( M(G, U) \), so this causes a decrease of at most 1 in \( M(G, U) \). Additionally, \( u'_1 \) can have two neighbors in \( V(G) \setminus U \). Since \( u'_1 \) is moved from \( V_2(G, U) \) to \( V_1(G, U) \), each neighbor of \( u'_1 \) can cause a decrease of at most 1 in \( M(G, U) \) (see the proof of Lemma [2]). Therefore, \( u'_1 \) causes a decrease of at most 3 in \( M(G, U) \). Similarly, \( u'_2 \) causes a decrease of at most 3 in \( M(G, U) \).

The vertex \( u_1 \) is deleted from the graph, so the value \( h_{G, U}(u_1) \) will no longer be included in \( M(G, U) \). This decreases \( M(G, U) \) by at most 1. Since \( u_1 \) is deleted from the graph, one or two neighbors of \( u_1 \) in \( V(G) \setminus U \) can change from semi-bad to bad vertices. Suppose that exactly one neighbor \( x \) changes from a semi-bad to a bad vertex. This change can cause a decrease of at most \( \frac{3}{2} \) in \( M(G, U) \), either by decreasing the \( h_{G, U} \)-value of one neighbor of \( x \) by \( \frac{1}{2} \), or by decreasing the \( h_{G, U} \)-values of a neighbor \( u \) of \( x \) and of its unique neighbor \( u' \) in \( G[\bar{U}] \) by \( \frac{1}{4} \) each. However, if \( x \) change from a semi-bad to a bad vertex, then before the application of the rule, \( h_{G, U}(u_1) \leq \frac{1}{2} \) (Note that if \( u_1 \in V_2(G, U) \setminus V'_2(G, U) \), then we actually have \( h_{G, U}(u_1) \leq \frac{3}{4} \) and \( h_{G, U}(u'_1) \leq \frac{3}{4} \). For the sake of the proof, we increase \( h_{G, U}(u'_1) \) by \( \frac{1}{4} \) and decrease \( h_{G, U}(u_1) \) by \( \frac{1}{4} \). After this change, \( h_{G, U}(u_1) \leq \frac{1}{2} \). Therefore, the decrease in \( M(G, U) \) due to not including \( h_{G, U}(u_1) \) is at most \( \frac{3}{2} \), and the total decrease in \( M(G, U) \) due to the deletion of \( u_1 \) is at most \( \frac{3}{2} + \frac{1}{2} = 1 \). The decrease in \( M(G, U) \) is also at most 1 in the case in which two neighbors of \( u_1 \) change from semi-bad to bad vertices. Similarly, \( u_2 \) causes a decrease of at most 1 in \( M(G, U) \). Therefore, the value of \( M(G, U) \) decreases by at most 8 in this branch.

Next consider the branch \( G - (N[u_1] \cup \{ u_2 \}) \). Since now \( u'_1 \) is deleted from the graph, the decreases in \( M(G, U) \) due to \( u_1, u'_1, u_2, u'_2 \) are at most 1, 1, 1, 3, respectively. Thus, the value of \( M(G, U) \) decreases by at most 6 in this branch. Symmetrically, the value of \( M(G, U) \) decreases by at most 6 in the branch \( G - (\{ u_1 \} \cup N[u_2]) \). In the branch \( G - (N[u_1] \cup N[u_2]) \), each vertex from \( u_1, u'_1, u_2, u'_2 \) decreases the value of \( M(G, U) \) by at most 1. As described above, in this branch \( v \) is deleted from the graph, and \( v' \) is moved from \( V_2(G, U) \) to \( V_1(G, U) \). These changes decrease \( M(G, U) \) by at most 1 and 3, respectively. The total decrease in \( M(G, U) \) in this branch is at most 8.

By the induction hypothesis, the number of leaves in the branching tree of the call \( WVCAlg(G, U, w, f, q) \) is at most \((\sqrt{2})^{m(G, U)-4} \cdot 0.9808^{M(G, U)-8} + 2(\sqrt{2})^{m(G, U)-4} \cdot 0.9808^{M(G, U)-6} + (\sqrt{2})^{m(G, U)-6} \cdot 0.9808^{M(G, U)-8} \leq (\sqrt{2})^{m(G, U)} \cdot 0.9808^{M(G, U)} \).

In the case when \( u_2 \) is not defined, there are two branches: \( G - u_1 \) and \( G - N[u_1] \). The decreases in \( m(G, U) \) in these branches are 2 and 4, respectively, and the decreases in \( M(G, U) \) are at most 4 and 6, respectively. By the induction hypothesis, the number of leaves in the branching tree of the call \( WVCAlg(G, U, w, f, q) \) is at most \((\sqrt{2})^{m(G, U)-2} \cdot 0.9808^{M(G, U)-4} + (\sqrt{2})^{m(G, U)-4} \cdot 0.9808^{M(G, U)-6} \leq (\sqrt{2})^{m(G, U)} \cdot 0.9808^{M(G, U)} \).

The analysis of Rule (12) is similar to the analysis of Rule (11) (note that after the application of this rule, Rule (3) is applied in the branch \( G - (N[u_1] \cup N[u_2]) \) and we omit the details. We also need to consider Rules (2), (3), (8), and (9). Note that we
can ignore Rule 3 since at this stage of the algorithm, after the application of this rule on a vertex \( v \), Rule 8 is applied on \( v \). The application of Rules 2, 8, or 9 causes deletion of one or more vertices of the graph. The effect of deleting a single vertex \( v \) on the values \( m(G,U) \) and \( M(G,U) \) is as follows. If \( v \notin V_2(G,U) \) then \( m(G,U) \) does not change, and \( M(G,U) \) does not change or increases. If \( v \in V_2(G,U) \) then \( m(G,U) \) decreases by 2, and \( M(G,U) \) decreases by at most 4. Therefore, if the rule deletes one vertex \( v \in V_2(G,U) \), by the induction hypothesis the number of leaves in the branching tree of the call WVCAlg(\( G, U, w, f, q \)) is at most \( (\sqrt{2})^{m(G,U)−2} \cdot 0.9808^{M(G,U)} \). More generally, if the rule deletes incrementally \( l \) vertices from \( V_2(G,U) \), we have that the number of leaves is at most \( (\sqrt{2})^{m(G,U)−2l} \cdot 0.9808^{M(G,U)−4l} \leq (\sqrt{2})^{m(G,U)} \cdot 0.9808^{M(G,U)} \). This completes the proof of Lemma 1.

References

[1] R. Balasubramanian, M. R. Fellows, and V. Raman. An improved fixed-parameter algorithm for vertex cover. Information Processing Letters, 65(3):163–168, 1998.

[2] J. F. Buss and J. Goldsmith. Nondeterminism within \( P^* \). SIAM Journal on Computing, 22(3):560–572, 1993.

[3] L. S. Chandran and F. Grandoni. Refined memorisation for vertex cover. In Proc. 1st Workshop on Parameterized and Exact Computation (IWPEC), pages 61–70, 2004.

[4] J. Chen, I. A. Kanj, and W. Jia. Vertex cover: further observations and further improvements. Journal of Algorithms, 41(2):280–301, 2001.

[5] J. Chen, I. A. Kanj, and G. Xia. Labeled search trees and amortized analysis: improved upper bounds for NP-hard problems. Algorithmica, 43(4):245–273, 2005.

[6] J. Chen, I. A. Kanj, and G. Xia. Improved upper bounds for vertex cover. Theoretical Computer Science, 411(40-42):3736–3756, 2010.

[7] J. Chen, L. Liu, and W. Jia. Improvement on vertex cover for low-degree graphs. Networks: An International Journal, 35(4):253–259, 2000.

[8] R. G. Downey and M. R. Fellows. Parameterized computational feasibility. In Feasible mathematics II, pages 219–244. 1995.

[9] R. G. Downey, M. R. Fellows, and U. Stege. Parameterized complexity: A framework for systematically confronting computational intractability. In Contemporary trends in discrete mathematics: From DIMACS and DIMATIA to the future, volume 49, pages 49–99, 1999.

[10] F. V. Fomin, S. Gaspers, and S. Saurabh. Branching and treewidth based exact algorithms. In Proc. 17th International Symposium on Algorithms and Computation (ISAAC), pages 16–25, 2006.

[11] R. Niedermeier and P. Rossmanith. Upper bounds for vertex cover further improved. In Proc. 16th Symposium on Theoretical Aspects of Computer Science (STACS), pages 561–570, 1999.

[12] R. Niedermeier and P. Rossmanith. On efficient fixed-parameter algorithms for weighted vertex cover. Journal of Algorithms, 47(2):63–77, 2003.
[13] I. Razgon. Faster computation of maximum independent set and parameterized vertex cover for graphs with maximum degree 3. *Journal of Discrete Algorithms*, 7(2):191–212, 2009.

[14] H. Shachnai and M. Zehavi. A multivariate framework for weighted FPT algorithms. *Journal of Computer and System Sciences*, 89:157–189, 2017.

[15] U. Stege and M. R. Fellows. An improved fixed parameter tractable algorithm for vertex cover. *Technical report/Departement Informatik, ETH Zürich*, 318, 1999.

[16] M. Xiao. A note on vertex cover in graphs with maximum degree 3. In *International Computing and Combinatorics Conference*, pages 150–159, 2010.