On Queries Determined by a Constant Number of Homomorphism
Counts

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Abstract

It is well known [16] that up to isomorphism a graph $G$ is determined by the homomorphism counts $\text{hom}(F,G)$, i.e., the number of homomorphisms from $F$ to $G$, where $F$ ranges over all graphs. Moreover, it suffices that $F$ ranges over the graphs with at most as many vertices as $G$. Thus in principle we can answer any query concerning $G$ with only accessing the $\text{hom}(\cdot,G)$'s instead of $G$ itself. In this paper, we zoom in on those queries that can be answered using a constant number of $\text{hom}(\cdot,G)$ for every graph $G$. We observe that if a query $\varphi$ is expressible as a Boolean combination of universal sentences in first-order logic, then whether a graph $G$ satisfies $\varphi$ can be determined by the vector

$$\longrightarrow \text{hom}_{F_1,\ldots,F_k}(G) := (\text{hom}(F_1,G),\ldots,\text{hom}(F_k,G)),$$

where the graphs $F_1,\ldots,F_k$ only depend on $\varphi$. This leads to a query algorithm for $\varphi$ that is non-adaptive in the sense that those $F_i$ are independent of the input $G$. On the other hand, we prove that the existence of an isolated vertex, which is not definable by such a $\varphi$ but in first-order logic, cannot be determined by any $\longrightarrow \text{hom}_{F_1,\ldots,F_k}(\cdot)$. These results provide a clear delineation of the power of non-adaptive query algorithms with access to a constant number of $\text{hom}(\cdot,G)$'s.

For adaptive query algorithms, i.e., algorithms that might access some $\text{hom}(F_{i+1},G)$ with $F_{i+1}$ depending on $\text{hom}(F_1,G),\ldots,\text{hom}(F_i,G)$, we show that three homomorphism counts $\text{hom}(\cdot,G)$ are both sufficient and in general necessary to determine the graph $G$. In particular, by three adaptive queries we can answer any question on $G$. Moreover, adaptively accessing two $\text{hom}(\cdot,G)$'s is already enough to detect an isolated vertex.

1. Introduction

In [16], one of the first papers on graph homomorphisms, Lovász proved that graphs $G$ and $H$ are isomorphic if and only if for all graphs $F$ the number $\text{hom}(F,G)$ of homomorphisms from $F$ to $G$ is equal to the number $\text{hom}(F,H)$ of homomorphisms from $F$ to $H$. Recently, this result has attracted a lot of attention in various contexts, e.g., algorithms and complexity [9, 13], machine learning [14, 2], and logic [15, 1]. Among others, it provides a powerful reduction of problems
concerning graph structures to questions on the number of homomorphisms, while homomorphisms have been the subject of extensive study in the last few decades.

We can rephrase Lovász’ result by saying that the infinite vector

\[ \overrightarrow{\text{hom}}(G) := \{ \text{hom}(F,G) \}_{F \text{ a graph}} \]

determines the graph \( G \) up to isomorphism. For a class \( C \) of graphs we consider the vector

\[ \overrightarrow{\text{hom}}_C(G) := \{ \text{hom}(F,G) \}_{F \in C}. \]

Using Lovász’ Cancellation Law [17] (see Theorem 7.7) it is easy to see that for some \( C \), including the class of 3-colorable graphs and the class of graphs that can be mapped homomorphically to an odd cycle, \( \overrightarrow{\text{hom}}_C(G) \) already determines \( G \) up to isomorphisms. A further example: the class of 2-degenerate graphs has this property [12].

For other natural classes of graphs, \( \overrightarrow{\text{hom}}_C(G) \) does not have the full power of distinguishing non-isomorphic graphs but characterizes interesting graph properties. For instance, let \( TW_k \) be the class of graphs of tree-width bounded by \( k \). It is shown in [10] that graphs \( G \) and \( H \) can be distinguished by the \( k \)-dimensional Weisfeiler-Leman algorithm if and only if \( \overrightarrow{\text{hom}}_{TW_k}(G) \neq \overrightarrow{\text{hom}}_{TW_k}(H) \).

Now we turn to results more relevant for the algorithmic problems we are interested in. Actually Lovász’ proof shows that in order to determine the isomorphism type of \( G \) it is sufficient to consider the homomorphism counts \( \text{hom}(F,G) \) for the graphs \( F \) whose number of vertices is bounded by that of \( G \). As a consequence, given an oracle to \( \overrightarrow{\text{hom}}(G) \), we might answer any query by first recovering the graph \( G \) and then computing the query on \( G \). However, such a naive algorithm requires exponentially many entries in \( \overrightarrow{\text{hom}}(G) \), i.e., \( \text{hom}(F,G) \) for all isomorphism types of graphs \( F \) with \( |V(F)| \leq |V(G)| \), rendering any practical implementation beyond reach.

There are queries that can be answered very easily using \( \overrightarrow{\text{hom}}(G) \), e.g., to decide whether \( G \) has a clique of size \( k \), all we need to know is \( \text{hom}(K_k,G) \) where \( K_k \) is the complete graph on \( k \) vertices. So ideally, one would hope that to answer a query on \( G \) it suffices to access a constant number of entries in \( \overrightarrow{\text{hom}}(G) \).

The question of using \( \overrightarrow{\text{hom}}(G) \) to answer queries algorithmically has been raised before. In [9] Curticapean et al. observed that counting (induced) subgraphs isomorphic to a fixed graph \( F \) can be reduced to computing appropriate linear combinations of sub-vectors of \( \overrightarrow{\text{hom}}(G) \). Thereby they introduced the so-called graph motif parameters. Using this framework, they were able to design some algorithms to count various specific subgraphs and induced subgraphs faster than the known ones. These results can be understood as answering counting queries using \( \overrightarrow{\text{hom}}(\cdot,G) \)’s. More explicitly, Grohe [15] asked whether it is possible to answer any \( C^{k+1} \)-query in polynomial time by accessing \( \text{hom}(F,G) \) for graphs \( F \) of tree-width bounded by \( k \). Here, \( C^{k+1} \) denotes counting first-order logic with \( k + 1 \) variables [5]. Observe that without the polynomial time constraint such an algorithm exists because graphs \( G \) and \( H \) cannot be distinguished by \( C^{k+1} \) if and only if \( \text{hom}(F,G) = \text{hom}(F,H) \) for finitely many graphs \( F \) of tree-width bounded by \( k \) [12] (see also [10]).

**Our contributions.** In this paper we study what Boolean queries (equivalently, graph properties) can be answered using a constant number of homomorphism counts. More precisely, let \( C \) be a class of graphs closed under isomorphism. We ask: are there graphs \( F_1, \ldots, F_k \) such that for any graph \( G \) whether \( G \in C \) can be decided by the finite vector

\[ \overrightarrow{\text{hom}}_{F_1, \ldots, F_k}(G) := \{ \text{hom}(F_1,G), \ldots, \text{hom}(F_k,G) \}. \]
In Section 4 we prove that this is the case if $C$ can be defined by a first-order logic (FO) sentence that is a Boolean combination of universal sentences. For $d \geq 1$ this includes the class of graphs of maximum degree $d$, of tree-depth $d$ exactly $d$, and the class of graphs of SC-depth $d$ exactly $d$ (but also the classes where we replace “exactly $d$” by “at most $d$”). On the negative side, in Section 5 we show that for any $k \geq 1$ and any $F_1, \ldots, F_k$ there are graphs $G$ and $H$ such that

- $G$ contains an isolated vertex and $H$ does not,
- $\hom_{F_1, \ldots, F_k}(G) = \hom_{F_1, \ldots, F_k}(H)$.

As a consequence, any $\hom_{F_1, \ldots, F_k}(G)$ is not sufficient to detect the existence of an isolated vertex in $G$. This is our technically most challenging result, which requires some non-trivial argument using linear algebra. Note that a graph of $G$ has an isolated vertex if and only if $G$ satisfies the first-order sentence $\exists x \forall y \neg Exy$. From a logic perspective, we now have an exact classification of the quantifier-prefix classes of FO-sentences that can or cannot be answered by $\hom_{F_1, \ldots, F_k}(G)$ for some $F_1, \ldots, F_k$ independent of $G$.

Answering a query using $\hom_{F_1, \ldots, F_k}(\cdot)$ can be phrased as an algorithm checking this query with non-adaptive access to the vector $\hom(G)$ on entries $F_1, \ldots, F_k$. It is also very natural to allow access to $\hom(G)$ to be adaptive. Informally, on input $G$ an adaptive algorithm still queries some $\hom(F_1, G), \ldots, \hom(F_k, G)$, but for $i = 0, \ldots, k - 1$ the choice of $F_{i+1}$ might depend on $\hom(F_1, G), \ldots, \hom(F_i, G)$ (see Definition 4.1 for a precise description). It turns out that adaptive query algorithms are extremely powerful. We first present an adaptive algorithm with two accesses to $\hom(G)$ that can decide whether $G$ contains an isolated vertex (see Section 6). Even more, the algorithm is able to compute all the information on the degrees of vertices in $G$. So in particular, it can decide whether $G$ is regular. In Section 7 we provide an adaptive algorithm that queries three entries in $\hom(G)$ that completely determines the graph $G$. As a consequence, it can answer any question on $G$. The downside of this algorithm is its superpolynomial running time, while all the aforementioned query algorithms run in polynomial time. We conjecture that there is no polynomial time algorithm that can reconstruct an input graph $G$ with access to $\hom(G)$ (even without the requirement of constant number of accesses).

2. Preliminaries

We denote by $\mathbb{N}$ the set of natural numbers greater than or equal to 0. For $n \in \mathbb{N}$ let $[n] := \{1, 2, \ldots, n\}$.

For graphs we use the notation $G = (V(G), E(G))$ common in graph theory. Here $V(G)$ is the nonempty set of vertices of $G$ and $E(G)$ is the set of edges. We only consider finite, simple and undirected graphs but briefly speak of graphs. To express that there is an edge connecting the vertices $u$ and $v$ of the graph $G$, we use (depending on the context) one of the notations $uv \in E(G)$ and $\{u, v\} \in E(G)$. For graphs $G$ and $H$ with disjoint vertex sets we denote by $G \cup H$ the disjoint union of $G$ and $H$, i.e., the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If the vertex sets are not disjoint, we tacitly pass to isomorphic copies with disjoint vertex sets.

For $n \geq 1$ we denote by $K_n$ a clique with $n$ vertices, by $P_n$ a path of $n$ vertices, and by $C_n$ a cycle of $n$ vertices.

For graphs $G$ and $H$ by $G \cong H$ we express that $G$ and $H$ are isomorphic. All classes of graphs considered in this paper are closed under isomorphism.

**Definition 2.1.** Let $G$ and $H$ be graphs and $f : V(G) \to V(H)$. The function $f$ is a homomorphism if $uv \in E(G)$ implies $f(u)f(v) \in E(H)$ for all $u, v \in V(G)$. It is an embedding if $f$ is a homomorphism that is one-to-one. We call $f$ an epimorphism if $f$ is a homomorphism, the range of $f$ is $V(H)$,
and for every $u'v' \in E(H)$ there are $u, v \in V(G)$ with $uv \in E(G)$ and with $f(u) = u'$, $f(v) = v'$. We get the definitions of strong homomorphism, of strong embedding, and of strong epimorphism by additionally requiring in the previous definitions that $(uv \in E(G) \iff f(u)f(v) \in E(H))$ for all $u, v \in V(G)$.

We denote by $\text{Hom}(G, H)$ the set of homomorphisms from $G$ to $H$, thus $\text{hom}(G, H) := |\text{Hom}(G, H)|$ is the number of homomorphisms from $G$ to $H$. Similarly, we define $\text{s-Hom}(G, H)$ and $\text{s-hom}(G, H)$ for strong homomorphisms and use corresponding notations for the other notions of morphisms. Finally, $\text{Aut}(G)$ and $\text{aut}(G)$ denote the set of automorphisms of $G$ and its number, respectively.

The following equalities can easily been verified and will often tacitly been used later. For graphs $F_1, F_2$, and $G$,

\[ \text{hom}(F_1 \cup F_2, G) = \text{hom}(F_1, G) \cdot \text{hom}(F_2, G), \]

if $G$ is a connected graph, then $\text{emb}(G, F_1 \cup F_2) = \text{emb}(G, F_1) + \text{emb}(G, F_2)$.

Once and for all we fix an enumeration

\[ F_1^0, F_2^0, \ldots \]

of graphs such that each graph is isomorphic to exactly one graph in the list and such that $i \leq j$ implies $F_i^0 \leq F_j^0$. Here for graphs $F$ and $G$ by $F \leq G$ we mean that

\[ |V(F)| < |V(G)| \quad \text{or} \quad (|V(F)| = |V(G)| \quad \text{and} \quad |E(F)| \leq |E(G)|). \]

We call $F_1^0, F_2^0, \ldots$ the basic enumeration. In particular, $F_1^0$ is a graph whose vertex set is a singleton. We repeatedly use:

**Theorem 2.2** (Lovász Isomorphism Theorem [10]). Let $G$ and $H$ be graphs. If $\text{hom}(F, G) = \text{hom}(F, H)$ for all graphs $F$ with $|V(F)| \leq \min\{|V(G)|, |V(H)|\}$, then $G$ and $H$ are isomorphic. Hence, the finite vector

\[ \left(\text{hom}(F, G)\right)_F \quad \text{a graph with } |V(F)| \leq |V(G)| \]

determines $G$ up to isomorphism.

### 3. Query algorithms

For what classes $C$ of graphs is there a finite set $F$ of graphs such that the membership of any graph $G$ in $C$ is determined by the values $\text{hom}(F, G)$, where $F$ ranges over $F$? This question leads to the following definition.

**Definition 3.1.** Let $C$ be a class of graphs. A hom-query algorithm for $C$ (with a constant number of non-adaptive accesses to homomorphism counts) consists of a $k \geq 1$, graphs $F_1, \ldots, F_k$, and a $X \subseteq \mathbb{N}^k$ such that for all $G$,

\[ G \in C \iff (\text{hom}(F_1, G), \ldots, \text{hom}(F_k, G)) \in X. \]

We then say that the hom-query algorithm decides $C$. Analogously we define the notions of emb-query algorithm, s-hom-query algorithm, and s-emb-query algorithm.

We often will use the following fact whose proof is immediate: A class $C$ can be decided by a hom-query algorithm if and only if there is a finite set $\{F_1, \ldots, F_k\}$ of graphs such that for all $G$ and $H$ (recall that $\text{hom}_{F_1, \ldots, F_k}(G) = (\text{hom}(F_1, G), \ldots, \text{hom}(F_k, G))$),

if $\text{hom}_{F_1, \ldots, F_k}(G) = \text{hom}_{F_1, \ldots, F_k}(H)$, then $(G \in C \iff H \in C)$.
Remark 3.2. If the set $X$ in Definition 3.1 is decidable, then we easily extract an actual algorithm with an oracle to $\text{hom}(G)$. However the above equivalence only holds for arbitrary $X$. Nevertheless, all our positive results have decidable $X$’s. We use the current definition to ease presentation, and also to make our negative result, i.e., Theorem 5.1, stronger.

Examples 3.3. (a) By taking $k = 1$, a graph $F$ whose vertex set is a singleton, and $X := \{2n + 1 \mid n \geq 1\}$, we get a hom-query algorithm for the class of graphs with an odd number of vertices.

(b) Theorem 2.2 shows that every class that only contains finitely many graphs up to isomorphism can be decided by a hom-query algorithm.

(c) By passing from $k \geq 1, F_1, \ldots, F_k$, and $X \subseteq \mathbb{N}^k \setminus X$, we see that with every class $C$ also the class $C^{\text{comp}} := \{G \mid G \notin C\}$ has a corresponding query algorithm.

By Definition 3.1 we have four types of query algorithms. The following proposition, the main result of this section, shows that a class has a query algorithm of one type if and only if it has a query algorithm of any other type. This will allow formulations like “there is a query algorithm for the class $C$.” The equivalences (i) $\iff$ (ii) and (iii) $\iff$ (iv) are known, e.g. see [15, 14].

Proposition 3.4. For a class $C$ of graphs the following are equivalent.

(i) There is a hom-query algorithm for $C$.

(ii) There is an emb-query algorithm for $C$.

(iii) There is an s-hom-query algorithm for $C$.

(iv) There is an s-emb-query algorithm for $C$.

(v) There is a hom-query algorithm for $C^c$, the class of graphs that are complements of graphs in $C$ (the complement of a graph $G$ is the graph $G^c = (V(G), \{uv \mid u \neq v \text{ and } uv \notin E(G)\}$).

Proof: (i) $\iff$ (ii): We sketch the proof of the Lovász Isomorphism Theorem in [15], which also leads to the desired equivalence.

Every $h \in \text{Hom}(F,G)$ can be written as $h = f \circ g$, where for some graph $F'$ we have $g \in \text{Epi}(F,F')$ and $f \in \text{Emb}(F',G)$. Clearly, $F' \leq F$ (as otherwise $\text{Epi}(F,F') = \emptyset$). Hence.

$$\text{hom}(F,G) = \sum_{F' \leq F} \frac{1}{\text{aut}(F')} \cdot \text{epi}(F,F') \cdot \text{emb}(F',G),$$

where the sum ranges over all isomorphism types of graphs $F'$ with $F' \leq F$.

Let $m \geq 1$. Let $\text{HOM}^m$ be the square matrix with $m$ rows and with entries $\text{HOM}^m_{ij} := \text{hom}(F_i^0, F_j^0)$ for $i, j \in [m]$ (recall that the sequence of graphs $F_1^0, F_2^0, \ldots$ was fixed in (3)). Define analogously $\text{EMB}^m$ by setting $\text{EMB}^m_{ij} := \text{emb}(F_i^0, F_j^0)$. By (3), $\text{EMB}^m$ is an upper triangular matrix with positive diagonal entries and thus is invertible. An analysis of (5) leads to the equality (see [15])

$$\text{HOM}^m = M \cdot \text{Emb}^m$$

for an invertible matrix $M$. Thus, the entries of $\text{EMB}^m$ determine the entries of $\text{HOM}^m$, and the other way around holds as well. Thus, for example, if we have an emb-query algorithm with graphs $F_1, \ldots, F_k$ and $F_1, \ldots, F_k$ occur in the basic enumeration among $F_1^0, F_2^0, \ldots, F_m^0$ (up to isomorphism), then there is a hom-query algorithm with graphs $F_1^0, F_2^0, \ldots, F_m^0.$
(iii) \(\Leftrightarrow\) (iv): Let \(h \in s\text{-HOM}(F,G)\), in particular \(h \in \text{HOM}(F,G)\). We consider the decomposition \(h = f \circ g\) outlined above. One easily verifies that \(g\) is a strong epimorphism and \(f\) a strong embedding, i.e., \(g \in s\text{-EPI}(F,F')\) and \(f \in s\text{-EMB}(F',G)\). Thus, we get
\[
s\text{-hom}(F,G) = \sum_{F' \leq F} \frac{1}{\text{aut}(F')} \cdot s\text{-epi}(F,F') \cdot s\text{-emb}(F',G),
\]
where again the sum ranges over all isomorphism types of graphs \(F'\) with \(F' \leq F\). If \(F' \leq F\), \(|V(F')| = |V(F)|\), and \(s\text{-epi}(F,F') \neq 0\), then \(F\) and \(F'\) are isomorphic and \(\text{Aut}(F') = s\text{-EPI}(F,F')\).

Hence
\[
s\text{-hom}(F,G) = s\text{-emb}(F,G) + \sum_{|V(F')|<|V(F)|} \frac{1}{\text{aut}(F')} \cdot s\text{-epi}(F,F') \cdot s\text{-emb}(F',G),
\]
and therefore
\[
s\text{-emb}(F,G) = s\text{-hom}(F,G) - \sum_{|V(F')|<|V(F)|} \frac{1}{\text{aut}(F')} \cdot s\text{-epi}(F,F') \cdot s\text{-emb}(F',G),
\]

These last two equations show how by induction on \(i\) we get \(s\text{-hom}(F_i^0,G)\) if we have \(s\text{-emb}(F_i^0,G)\) and the other way around.

(i) \(\Leftrightarrow\) (iii): We start with some simple observations (a) to (c).

(a) If \(F\) and \(F'\) are distinct graphs with the same vertex set and \(f \in s\text{-HOM}(F,G)\), then \(f \notin s\text{-HOM}(F',G)\) (look at an edge in \(E(F)\triangle E(F')\)).

(b) If \(f \in \text{HOM}(F,G)\), then there is a unique graph \(F'\) with the same vertex set as \(F\) such that \(f \in s\text{-HOM}(F',G)\). In fact, set
\[
E(F') := \{uv \mid u,v \in V(F), u \neq v, \text{ and } f(u)f(v) \in E(G)\};
\]
clearly, \(E(F') \supseteq E(F)\) and thus, \(F \leq F'\).

(c) Let \(F\) and \(G\) be graphs and \(\ell := |V(F)|\). Let \(F_1, \ldots, F_\ell\) be an enumeration of pairwise non-isomorphic graphs with exactly \(\ell\) vertices. Then we can assign to every \(f \in \text{HOM}(F,G)\) a unique pair \((i,\xi)\) with \(i \in [\ell]\), \(\xi \in \text{EMB}(F,F_i)\), and such that \(f \circ \xi^{-1} \in s\text{-HOM}(F_i,G)\). Hence
\[
\text{hom}(F,G) = \sum_{i \in [\ell] \text{ with } \text{emb}(F,F_i) \neq 0} s\text{-hom}(F_i,G).
\]

To get part (c), by the previous results it suffices to show that if \(f \in \text{HOM}(F,G)\), \(i,j \in [\ell]\), \(\xi \in \text{EMB}(F,F_i)\), \(\eta \in \text{EMB}(F,F_j)\), \(f \circ \xi^{-1} \in s\text{-HOM}(F_i,G)\), and \(f \circ \eta^{-1} \in s\text{-HOM}(F_j,G)\), then \(\eta \circ \xi^{-1}\) is an isomorphism between \(F_i\) and \(F_j\) (in particular, \(i = j\)). Clearly, \(\eta \circ \xi^{-1} : V(F_i) \rightarrow V(F_j)\) is a bijection. For \(u,v \in F_i\) we know (as \(f \circ \xi^{-1} \in s\text{-HOM}(F_i,G)\))
\[
\{u,v\} \in E(F_i) \iff \{(f \circ \xi^{-1})(u),(f \circ \xi^{-1})(v)\} \in E(G),
\]
and (as \(f \circ \eta^{-1} \in s\text{-HOM}(F_j,G)\))
\[
\{(\eta \circ \xi^{-1})(u),(\eta \circ \xi^{-1})(v)\} \in E(F_j) \iff \{(f \circ \eta^{-1} \circ \eta \circ \xi^{-1})(u),(f \circ \eta^{-1} \circ \eta \circ \xi^{-1})(v)\} \in E(G).
\]
Hence,
\[ \{u, v\} \in E(F_i) \iff \{(\eta \circ \xi^{-1})(u), (\eta \circ \xi^{-1})(v)\} \in E(F_j) \]
and thus, \( \eta \circ \xi^{-1} : F_i \cong F_j \).

Let EXT be the square matrix with \( \ell \) rows and with entries
\[ \text{EXT}_{ij} := \begin{cases} 1, & \text{if } \text{emb}(F_i, F_j) > 0 \\ 0, & \text{otherwise} \end{cases} \]
Let H(G) be the column with the entries \( \text{hom}(F_1, G), \ldots, \text{hom}(F_\ell, G) \) and s-H(G) the column with the entries \( \text{s-hom}(F_1, G), \ldots, \text{s-hom}(F_\ell, G) \). By (7), we have \( H(G) = \text{EXT} \cdot \text{s-H}(G) \). Clearly the matrix EXT is upper triangular and the elements of the diagonal are distinct from 0, hence it has an inverse. Thus not only the entries of s-H(G) determine the entries of H(G) but also the other way around.

(i) \( \iff \) (v): By the equivalence between (i) and (iv) it suffices to show that \( C \) has an s-emb-query algorithm if and only if \( C^c \) has an s-emb-query algorithm. However, this equivalence immediately follows from the equality
\[ \text{s-Emb}(F, G) = \text{s-Emb}(F^c, G^c). \]

Remark 3.5. The proofs of the equivalences of the statements (i) to (iv) of the previous proposition show the following: If for \( C \) we have a query algorithm of one type based on graphs \( F_1, \ldots, F_k \) and \( m := \max\{|V(F_i)| \mid i \in [k]\} \), then for any other type we can compute finitely many graphs, all with at most \( m \) vertices, that are the graphs of a query algorithm for \( C \) of this other type.

4. Some classes with query algorithms

We start by showing that every class of graphs that excludes a finite set of graphs as induced subgraphs has a query algorithm. Of course, the complement and the union of such classes again have such an algorithm. In terms of first-order logic this means that every class axiomatizable by a Boolean combination of universal sentences has a query algorithm.

Let \( F \) be a finite set of graphs. We set
\[ \text{Forb}(F) := \{G \mid \text{no induced subgraph of } G \text{ is isomorphic to a graph in } F\}. \]
We say that a class \( C \) of graphs is definable by a set of forbidden induced subgraphs if there is a finite class \( F \) with \( C = \text{Forb}(F) \).

If \( F := \{F\} \) with \( V(F) := [2] \) and \( E(F) := \{12\} \), then \( \text{Forb}(F) \) is the class of graphs without edges. Examples of classes definable by a set of forbidden induced subgraphs are classes of bounded vertex cover number (attributed to Lovász), of bounded tree-depth [11], or even of bounded shrub-depth [13]. By the next lemma all these classes have a query algorithm.

Lemma 4.1. Every class of graphs definable by a set of forbidden induced subgraphs can be decided by a query algorithm.

Proof: If \( C = \text{Forb}(\emptyset) \), we set \( k = 1 \), let \( F \) be an arbitrary graph, and take \( X := \mathbb{N} \). Assume now that \( C = \text{Forb}(F) \) with \( F = \{F_1, \ldots, F_k\} \) and \( k \geq 1 \). Then
\[ G \in C \iff \text{s-emb}(F_1, G) = \ldots = \text{s-emb}(F_k, G) = 0. \]
Hence, \( k, F_1, \ldots, F_k \), and \( X \subseteq \mathbb{N}^k \) with \( X = \{(0, 0, \ldots, 0)\} \) constitute an s-emb-algorithm for \( C \). \( \square \)

The following lemma shows that the universe of classes with query algorithms is closed under the Boolean operations. Part (a) was already mentioned as part (c) of Examples 3.3. We omit the straightforward proof.
Lemma 4.2. (a) If $C$ has a query algorithm, then so does $\{G \mid G \not\in C\}$.

(b) If $C$ and $C'$ have query algorithms, then $C \cap C'$ and $C \cup C'$ have query algorithms.

Recall that formulas $\varphi$ of first-order logic $FO$ for graphs are built up from atomic formulas $x_1 = x_2$ and $Ex_1x_2$ (where $x_1, x_2$ are variables) using the Boolean connectives $\neg$, $\land$, and $\lor$ and the universal $\forall$ and existential $\exists$ quantifiers. A sentence is a formula without free variables (i.e., all variables of $\varphi$ are in the scope of a corresponding quantifier). If $\varphi$ is a sentence, then we denote by $C(\varphi)$ the class of graphs that are models of $\varphi$.

An FO-formula is universal if it is built up from atomic and negated atomic formulas by means of the connectives $\land$ and $\lor$ and the universal quantifier $\forall$. If in the definition of universal formula we replace the universal quantifier by the existential one, we get the definition of an existential formula. The following result is well known (e.g., see [18]).

Lemma 4.3. Let $C$ be a class of graphs. Then

$$C \text{ is the class of graphs that are models of a universal sentence}$$

$$\iff C \text{ is definable by a set of forbidden induced subgraphs.}$$

By Lemma 4.1 – Lemma 4.3 we get:

Theorem 4.4. If the FO-sentence $\varphi$ is a Boolean combination of universal sentences, then there is a query algorithm for $C(\varphi)$.

In the next section we will see that the class $C(\text{isolated})$ of graphs that contain at least one isolated vertex has no query algorithm. Note that $C(\text{isolated}) = C(\exists x \forall y \neg Exy)$.

Remark 4.5. In part (a) of Examples 3.3 we have seen that there is a query algorithm for the class of graphs with an odd number of vertices, a class that is not definable in first-order logic.

The class $C(3)$ of 3-regular graphs is an example of a class decidable by a query algorithm that is definable in first-order logic but not by a Boolean combination of universal sentences.

Indeed, using the following facts we get a query algorithm deciding whether a graph $G$ belongs to $C(3)$.

- We check whether $G$ has degree bounded by 3 (note that this property is expressible by a universal sentence).

- If the degree of $G$ is bounded by 3, then we query $\text{hom}(K_1, G)$ in order to get $n := |V(G)|$.

- We query $\text{hom}(P_2, G)$, i.e., the number of homomorphisms from the path $P_2$ of two vertices to $G$. Then, $G$ is 3-regular if and only if $\text{hom}(P_2, G) = 3 \cdot n$.

It is easy to see that $C(3)$ is definable in first-order logic. If it would be definable by a Boolean combination of universal sentences, then it would be definable by a sentence $\varphi$ of the form

$$\varphi = \exists x_1 \ldots \exists x_m \forall y_1 \ldots \forall y_{\ell} \psi$$

with $m, \ell \in \mathbb{N}$ and with quantifier-free $\psi$. Let $G$ be a graph with more than $m + 1$ vertices that is the disjoint union of copies of the clique $K_4$. Of course, $G$ is 3-regular. Hence, $G$ is a model of $\varphi$. In particular, there are vertices $u_1, \ldots, u_m$ that satisfy in $G$ the formula $\forall y_1 \ldots \forall y_{\ell} \psi(x_1, \ldots, x_m)$ if we interpret $x_i$ by $u_1, \ldots, x_m$ by $u_m$. Choose a vertex $u \in V(G) \setminus \{u_1, \ldots, u_m\}$. Then, $G \setminus u$, the graph induced by $G$ on $V(G) \setminus \{u\}$, is still a model of $\varphi$ but not 3-regular.
Remark 4.6. In 1993 Chaudhuri and Vardi [6] (see also [1]) showed the analogue of the Lovász Isomorphism Theorem for the right-homomorphism sequence of a graph $G$. More precisely, the sequence of values $\text{hom}(G, F)$ where $F$ ranges over all graphs characterizes the isomorphism type of $G$. To what extent our results carry over to the sequence of right-hom values?

We show that already the question whether a graph $G$ contains a clique of size 3 cannot be decided by a right-hom query, i.e., there are no $k \geq 1$ and $F_1, \ldots, F_k$ such that for all graphs $G$ and $H$,

$$\text{hom}(G, F_1) = \text{hom}(H, F_1), \ldots, \text{hom}(G, F_k) = \text{hom}(H, F_k) \text{ imply } G \text{ contains a clique of size 3 } \iff H \text{ contains a clique of size 3.}$$

(8)

Note that the existence of a clique of size 3 can be expressed by the existential sentence $\exists x \exists y \exists z (Exy \land Eyz \land Exz)$.

In fact, for a graph $G$ we denote by $\chi(G)$ the chromatic number of $G$, i.e., the least $s$ such that $G$ is $s$-colourable. Clearly, for the clique $K_m$ with $m$ elements,

$$m < \chi(G) \iff \text{hom}(G, K_m) = 0,$$

and hence, for every graph $F$,

$$\text{if } |V(F)| < \chi(G), \text{ then } \text{hom}(G, F) = 0.$$

(9)

For a contradiction assume that $F_1, \ldots, F_k$ satisfy (8). Set

$$s := 1 + \max\{|V(F_i)| \mid i \in [k]\}.$$

Then $s \geq 3$. According to [19] there is a $G$ without a clique of size 3 such that $\chi(G) = s$. Thus by (9), we have

$$\text{hom}(G, F_1) = 0, \ldots, \text{hom}(G, F_k) = 0.$$

Clearly,

$$\text{hom}(K_s, F_1) = 0, \ldots, \text{hom}(K_s, F_k) = 0.$$

The graph $G$ contains no 3-clique, however, $K_s$ contains a 3-clique.

5. No query algorithm detects isolated vertices

Theorem 5.1. There is no query algorithm for the class $C(\text{isolated})$ of graphs containing at least one isolated vertex.

First we sketch the idea underlying the proof of Theorem 5.1. Recall that in (3) we fixed the enumeration $F^0_1, F^0_2, \ldots$ of graphs that contains a copy of every isomorphism type and respects the relation $\leq$.

In this section we let $F_1, F_2, \ldots$ be the subsequence of $F^0_1, F^0_2, \ldots$ consisting of the connected graphs.

For $i \geq 1$ let $\alpha_i := (\text{emb}(F_i, F_j))_{j \geq 1}$ be the vector containing the emb-values of $F_i$ for connected graphs. An observation, central for the proof, can be vaguely expressed by saying that for each $n \geq 1$ there is an $r_n \in \mathbb{N}$ such that appropriate sub-vectors of length $r_n$ of $\alpha_1, \ldots, \alpha_n$ are linearly independent vectors of the vector space $\mathbb{Q}^{r_n}$ and hence, a basis of $\mathbb{Q}^{r_n}$. In particular, every further vector of $\mathbb{Q}^{r_n}$ is a linear combination of these vectors. Furthermore, $r_n$ tends to infinity when $n$ increases.
Clearly (see (1))
\[
\text{hom}(H_1 \cup H_2, G) = \text{hom}(H_1, G) \cdot \text{hom}(H_2, G)
\]
holds for arbitrary graphs $H_1, H_2,$ and $G$. Hence, when assuming for a contradiction that there is a hom-query for the class $C$(isolated), we can require that the corresponding graphs are connected. Furthermore, we will see that it suffices to show that there is no emb-query with connected graphs.

For a graph $G$ and $p \in \mathbb{N}$ denote by $pG$ the disjoint union of $p$ copies of $G$. By (2)
\[
\text{emb}\left(F, \bigcup_{i \in [r]} p_i G_i\right) = \sum_{i \in [r]} p_i \cdot \text{emb}(F, G_i)
\]
for connected graphs $F$, $p_i \in \mathbb{N}$, and graphs $G_i$. For an arbitrary hom-query algorithm we must show the existence of graphs $G$ and $H$, one with isolated vertices the other one without, that cannot be distinguished by this hom-query algorithm. The last equality is essential to construct such graphs using the knowledge about the linear independence or linear dependence of some tuples of vectors obtained in the first step of the proof.

We turn to a proof of the result telling us that “for each $n \geq 1$ appropriate finite sub-vectors of $\alpha_1, \ldots, \alpha_n$, whose length tends to infinity when $n$ increases, are linearly independent.”

**Expressive graphs.** We start with a definition.

**Definition 5.2.** By induction on $s \geq 1$, we define whether $F_s$ is *expressive*.

- $F_1$ is expressive (note that $F_1 = K_1$).
- Let $s \geq 2$. We set
  \[
  I_{s-1} := \{i \mid 2 \leq i \leq s - 1 \text{ and } F_i \text{ is expressive}\}. \tag{10}
  \]

Then $F_s$ is expressive if the matrix
\[
\left(\text{emb}(F_i, F_j)\right)_{i \in \{1\} \cup I_{s-1}, j \in I_{s-1} \cup \{s\}}
\]
is of full rank.

For example, as $I_1 = \emptyset$, $F_2 = P_2$ (where $P_n$ denotes a path with $n$ vertices), and $(\text{emb}(F_1, F_2)) = (2)$, we see that $F_2$ is expressive. The relevant matrices for $F_3 (= P_3)$ and $F_4 (= K_3)$ are
\[
\begin{pmatrix}
2 & 3 \\
2 & 4 \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
2 & 3 & 3 \\
2 & 4 & 6 \\
0 & 2 & 6 \\
\end{pmatrix}.
\]

As the determinant of the latter matrix is zero, $F_4$ is not expressive.

**Lemma 5.3.** Assume $F_s$ is not expressive. Then there are
\[
(p_j)_{j \in I_{s-1} \cup \{s\}} \in \mathbb{Z}^{|I_{s-1} \cup \{s\}|}
\]
with $p_s \neq 0$ such that for every $i \in \{1\} \cup I_{s-1}$,
\[
\sum_{j \in I_{s-1} \cup \{s\}} p_j \cdot \text{emb}(F_i, F_j) = 0.
\]
Proof : As $F_s$ is not expressive, we have $s \geq 4$. Let $t := \max I_{s-1}$. Then $I_{s-1} = I_{t-1} \cup \{t\}$. As $F_t$ is expressive, the matrix

$$M_t := (\text{emb}(F_i, F_j))_{i \in \{1\} \cup I_{t-1}, j \in I_{t-1} \cup \{t\}}$$

has maximal rank. As $F_s$ is not expressive, the matrix $M_s$, i.e.,

$$(\text{emb}(F_i, F_j))_{i \in \{1\} \cup I_{s-1}, j \in I_{s-1} \cup \{s\}} = (\text{emb}(F_i, F_j))_{i \in \{1\} \cup I_{t-1} \cup \{t\}, j \in I_{t-1} \cup \{t, s\}}$$

is not of full rank. Let $m := |\{1\} \cup I_{s-1}|$. We consider the columns of $M_s$ as vectors of the vector space $\mathbb{Q}^m$ over the rationals. As the matrix $M_t$ has full rank, the first $m - 1$ columns of $M_t$ are linearly independent. This yields the claim of the lemma. \qed

Lemma 5.4. There are infinitely many expressive graphs.

Proof : Towards a contradiction, let $F_t$ be the expressive graph with maximum index. Hence:

(S1) The matrix

$$(\text{emb}(F_i, F_j))_{i \in \{1\} \cup I_{t-1}, j \in I_{t-1} \cup \{t\}}$$

is of full rank.

(S2) For any $s > t$

$$(\text{emb}(F_i, F_j))_{i \in \{1, t\} \cup I_{t-1}, j \in I_{t-1} \cup \{t, s\}}$$

is not of full rank.

For every $i \geq 1$, let

$$\alpha_i := (\text{emb}(F_i, F_j))_{j \geq 1}$$

be an infinite (row) vector. For a nonempty finite set $J$ of positive natural numbers let

$$\alpha_i | J$$

denote the sub-vector of $\alpha_i$ obtained by restricting to the coordinates with index in $J$. Set

$$I^t := \{1\} \cup I_{t-1} \quad \text{and} \quad J^t := I_{t-1} \cup \{t\}.$$ 

Furthermore, assume

$$I^t = \{i_1, \ldots, i_r\} \text{ with } i_1 < \ldots < i_r.$$ 

Claim 1.

$$(\text{emb}(F_i, F_j))_{i \in I^t, j \in J^t} = \begin{pmatrix} \alpha_{i_1} | J^t \\ \vdots \\ \alpha_{i_r} | J^t \end{pmatrix}.$$ 

Hence, by (S1), $\alpha_{i_1} | J^t, \ldots, \alpha_{i_r} | J^t$ are linearly independent vectors in $\mathbb{Q}^r$ and hence are a basis of $\mathbb{Q}^r$. 

Now consider

$$\alpha_t | J^t = (\text{emb}(F_t, F_j))_{j \in J^t},$$

\[11\]
also a vector in $\mathbb{Q}^r$. Hence,

there are unique $c_1, \ldots, c_r \in \mathbb{Q}$ such that $\alpha_t \mid_{J^t} = \sum_{\ell \in [r]} c_\ell \cdot \alpha_{i_\ell} \mid_{J^t}$. \hfill (11)

**Claim 2.** For every $s > t$

$$\alpha_t \mid_{J^t \cup \{s\}} = \sum_{\ell \in [r]} c_\ell \cdot \alpha_{i_\ell} \mid_{J^t \cup \{s\}}.$$ 

In particular,

$$\text{emb}(F_t, F_s) = \sum_{\ell \in [r]} c_\ell \cdot \text{emb}(F_{i_\ell}, F_s).$$

**Proof of Claim 2:** The $(r + 1) \times (r + 1)$ matrix in (S2) is precisely

$$
\begin{pmatrix}
\alpha_{i_1} \mid_{J^t \cup \{s\}} \\
\vdots \\
\alpha_{i_r} \mid_{J^t \cup \{s\}} \\
\alpha_t \mid_{J^t \cup \{s\}}
\end{pmatrix}.
$$

By (S2), it has rank $\leq r$. Claim 1 implies that

$$\alpha_{i_1} \mid_{J^t \cup \{s\}}, \ldots, \alpha_{i_r} \mid_{J^t \cup \{s\}}$$

are linearly independent as well. Therefore, for some $c'_1, \ldots, c'_r \in \mathbb{Q}$ we have

$$\alpha_t \mid_{J^t \cup \{s\}} = \sum_{\ell \in [r]} c'_\ell \cdot \alpha_{i_\ell} \mid_{J^t \cup \{s\}}. \hfill (12)$$

This further implies

$$\alpha_t \mid_{J^t} = \sum_{\ell \in [r]} c'_\ell \cdot \alpha_{i_\ell} \mid_{J^t}.$$ 

By the uniqueness claim in (11) we conclude

$$c'_\ell = c_\ell$$

for all $\ell \in [r]$. With (12) we have shown the claim. \hfill ⊣

**Claim 3.** $c_1 \neq 0$.

**Proof of Claim 3:** Otherwise, by Claim 2, the rank of

$$
\begin{pmatrix}
\alpha_{i_2} \mid_{J^t \cup \{s\}} \\
\vdots \\
\alpha_{i_r} \mid_{J^t \cup \{s\}} \\
\alpha_t \mid_{J^t \cup \{s\}}
\end{pmatrix}
$$

is at most $r - 1$. However, this matrix contains as a submatrix

$$(\text{emb}(F_i, F_j))_{i,j \in [t-1] \cup \{t\}},$$
which has rank $r$ as it is upper triangular with positive elements in the diagonal. By Claim 2 and Claim 3, there are 

$$ (c'_i)_{i \in \{1, t\} \cup I_{t-1}} \in \mathbb{Q}^{\{1, t\} \cup I_{t-1}} $$

with $c'_1 \neq 0$ such that (by $\{1, t\} \cup I_{t-1} = \{i_1, \ldots, i_r\} \cup \{t\}$)

$$ \sum_{i \in \{1, t\} \cup I_{t-1}} c'_i \cdot \text{emb}(F_i, F_s) = \sum_{i \in \{i_1, \ldots, i_r\} \cup \{t\}} c'_i \cdot \text{emb}(F_i, F_s) = 0 $$

for all $s > t$.

Finally we show that this equality cannot hold if $F_s$ is a sufficiently large clique. Assume $F_s = K_m$ for some $m \geq 1$ to be determined later. Then for every $i \geq 1$

$$ \text{emb}(F_i, F_s) = |V(F_i)|! \cdot \binom{m}{|V(F_i)|} $$

Therefore,

$$ 0 = \sum_{i \in \{1, t\} \cup I_{t-1}} c'_i \cdot \text{emb}(F_i, F_s) = \sum_{n \geq 1} \sum_{i \in \{1, t\} \cup I_{t-1}} c'_i \cdot n! \cdot \binom{m}{n} $$

$$ = \sum_{n \geq 1} n! \cdot \binom{m}{n} \cdot \sum_{i \in \{1, t\} \cup I_{t-1}} c'_i = \sum_{n \geq 1} n! \cdot \binom{m}{n} \cdot e_n, $$

where

$$ e_n := \sum_{i \in \{1, t\} \cup I_{t-1}} c'_i. $$

Note that $e_1 = c'_1 \neq 0$. Hence there exists a maximum $n \geq 1$ with $e_n \neq 0$. Note that $n \leq |V(F_t)|$.

Thus,

$$ n! \cdot \binom{m}{n} \cdot e_n = - \sum_{n>\ell \geq 1} \ell! \cdot \binom{m}{\ell} \cdot e_\ell $$

As $e_n \neq 0$, we conclude

$$ \Theta(m^n) = O(m^{n-1}), $$

which cannot hold for sufficiently large $m$, a contradiction. \hfill \Box

**The class $C(\text{isolated})$ of graphs with isolated vertices.** We turn to the proof of Theorem 5.1. First we show that it suffices to show there is no emb-query algorithm that detects isolated vertices and only uses connected graphs; more precisely we show:

**Lemma 5.5.** Assume that for every finite set $K'$ of connected graphs there are graphs $G$ and $H$ such that (a) and (b) hold.

(a) $G$ has an isolated vertex and $H$ does not.

(b) For all $F' \in K'$, we have $\text{emb}(F', G) = \text{emb}(F', H)$.
Then for every finite set $K$ of graphs there are graphs $G$ and $H$ such that (c) and (d) hold (which means that there is no hom-query algorithm for the class $C$(isolated).)

(c) $G$ has an isolated vertex and $H$ does not.

(d) For all $F \in K$, we have $\text{hom}(F, G) = \text{hom}(F, H)$.

Proof: Let $K$ be any finite set of graphs. By (1), $\text{hom}(F^1 \cup F^2, F^3) = \text{hom}(F^1, F^3) \cdot \text{hom}(F^2, F^3)$. Therefore, if the class $K$ satisfies (d) for some graphs $G$ and $H$, then the class of connected components of graph in $K$ satisfy (d), too. Hence, we can assume that the graphs in $K$ are connected. Let $n := \max\{|V(F)| \mid F \in K\}$ and let $K' := \{F_i \mid i \geq 1 \text{ and } |V(F_i)| \leq n\}$ (recall that $F_1, F_2, \ldots$ is the enumeration of connected graphs introduced at the beginning of this section). By assumption we know that there are graphs $G$ and $H$ such that (a) and (b) hold for $K'$. Now we recall (5), i.e.,

$$\text{hom}(F, G) = \sum_{F^4 \leq F} \frac{1}{\text{aut}(F^4)} \cdot \text{epi}(F, F^4) \cdot \text{emb}(F^4, G)$$

If $F$ is connected, then $\text{epi}(F, F^4) > 0$ implies that $F^4$ is connected. That is, the values $\text{hom}(F, G)$ for $F \in K$ are determined by the values of $\text{emb}(F^4, G)$ for $F^4 \in K'$. Therefore, (d) holds by (c). $\blacksquare$

Hence to finish the proof of Theorem 5.1 we have to show:

**Lemma 5.6.** For every finite set $K$ of connected graphs there are graphs $G$ and $H$ such that (a) and (b) hold.

(a) $G$ has an isolated vertex and $H$ does not.

(b) For all $F \in K$ we have $\text{emb}(F, G) = \text{emb}(F, H)$.

Proof: Clearly, we can assume that $K$ only contains graphs of the enumeration $F_1, F_2, \ldots$ of connected graphs. We prove the claim by induction on the number $k$ of graphs in $K$. First assume that $k = 1$, i.e., $K = \{F\}$ for some graph $F$. If $F = K_1$, then we can take $G := K_1 \cup K_1$ and $H := P_2$. Otherwise, $F$ contains at least one edge since it is connected. Then let $G := K_1 \cup P_2$ and $H := P_2$.

Now let $k \geq 2$. We distinguish two cases.

**Case 1: All graphs in $K$ are expressive.** By Lemma 5.4 we can find an $s$ such that $F_s$ is expressive and $I_{s-1}$ (as defined in (10)) contains all the indices of the graphs in $K$.

As the matrix $(\text{emb}(F_i, F_j))_{i,j \in I_{s-1}}$ is upper triangular with positive diagonal elements, it has maximal rank. Hence, for some

$$\left(p_j\right)_{j \in I_{s-1} \cup \{s\}} \in \mathbb{Z}^{\mid I_{s-1}\mid + 1}$$

with $p_s \neq 0$ we have for every $i \in I_{s-1}$,

$$\sum_{j \in I_{s-1} \cup \{s\}} p_j \cdot \text{emb}(F_i, F_j) = 0. \quad (13)$$

As $F_s$ is expressive, by Definition 5.2

$$(\text{emb}(F_i, F_j))_{i \in \{1\} \cup I_{s-1}, j \in I_{s-1} \cup \{s\}}$$

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is of full rank, or equivalently, its column vectors are linearly independent. Combined with (13) we get (recall $1 \notin I_{s-1}$)

$$\sum_{j \in I_{s-1} \cup \{s\}} p_j \cdot \text{emb}(F_1, F_j) \neq 0.$$  

As $F_1 = K_1$, this is equivalent to

$$\sum_{j \in I_{s-1} \cup \{s\}} p_j \cdot |V(F_j)| \neq \sum_{j \in I_{s-1} \cup \{s\}} -p_j \cdot |V(F_j)|. \quad (14)$$

We construct the graphs $G_0$ and $H_0$ as follows:

$$G_0 := \bigcup_{j \in I_{s-1} \cup \{s\}} \underbrace{F_j \cup \ldots \cup F_j}_{p_j \text{ times}}, \quad H_0 := \bigcup_{j \in I_{s-1} \cup \{s\}} \underbrace{F_j \cup \ldots \cup F_j}_{-p_j \text{ times}}.$$ 

Since $1 \notin I_{s-1} \cup \{s\}$, neither $G_0$ nor $H_0$ contains an isolated vertex. Furthermore, by (14)

$$|V(G_0)| \neq |V(H_0)|.$$ 

Without of loss of generality, assume $d := |V(G_0)| - |V(H_0)| > 0$. Set

$$G := G_0 \cup \bigcup_{d \text{ times}} K_1 \cup \ldots \cup K_1, \quad H := H_0.$$ 

Then $G$ and $H$ have the same number of vertices. Clearly $G$ contains isolated vertices, while $H$ does not; hence, (a) holds.

For (b) let $F \in K$, say $F = F_i$. As $s > i$ and $F_i$ is expressive, either $i = 1$ or $i \in I_{s-1}$. For $i = 1$,

$$\text{emb}(F_1, G) = |V(G)| = |V(H)| = \text{emb}(F_1, H).$$

For $i \in I_{s-1}$,

$$\text{emb}(F_i, G) = \text{emb}(F_i, G_0) \quad \text{(as $F_i$ is connected)}$$

$$= \sum_{j \in I_{s-1} \cup \{s\}, p_j \geq 0} p_j \cdot \text{emb}(F_i, F_j) \quad \text{(again as $F_i$ is connected and (2))}$$

$$= \sum_{j \in I_{s-1} \cup \{s\}, p_j < 0} -p_j \cdot \text{emb}(F_i, F_j) \quad \text{(by (13))}$$

$$= \text{emb}(F_i, H_0) = \text{emb}(F_i, H).$$

Thus, (b) holds too.

**Case 2: Some $F \in K$ is not expressive.** Let $s$ be the minimum index $i$ such that $F_i \in K$ and $F_i$ is not expressive. Recall that we prove our claim by induction on the number $k$ of graphs in $K$. Hence there are

$$i_1, i_2, \ldots, i_{t-1}, i_{t+1}, \ldots, i_k \quad \text{with} \quad i_1 < i_2 < \cdots < i_{t-1} < s < i_{t+1} < \cdots < i_k$$

such that $F_{i_1}, F_{i_2}, F_s, F_{i_{t+1}}, \ldots, F_{i_k}$ are the graphs in $K$. By induction hypothesis, there are two graphs $G_0$ and $H_0$ such that
(E1) $G_0$ has isolated vertices and $H_0$ does not.

(E2) For all $r \in [k] \setminus \{t\}$

\[ \text{emb}(F_i, G_0) = \text{emb}(F_i, H_0). \]

If $\text{emb}(F_s, G_0) = \text{emb}(F_s, H_0)$, then we are already done. Otherwise,

(E3) $g := \text{emb}(F_s, G_0) \neq \text{emb}(F_s, H_0) = : h$.

In addition, we observe that

(E4) $F_{i_1}, \ldots, F_{i_{t-1}}$ are all expressive.

Since $F_s$ is not expressive, by Lemma 5.3 there exist

\[ (p_j)_{j \in I_{s-1} \cup \{s\}} \in \Z^{I_{s-1} \cup \{s\}} \]

with $p_s \neq 0$ such that for every $i \in \{1\} \cup I_{s-1}$,

\[ \sum_{j \in I_{s-1} \cup \{s\}} p_j \cdot \text{emb}(F_i, F_j) = 0. \quad (15) \]

Let

\[ G_1 := \bigcup_{j \in I_{s-1} \cup \{s\}} \left\{ F_j \cup \ldots \cup F_j \right\}_{p_j \text{ times}} \]

\[ H_1 := \bigcup_{j \in I_{s-1} \cup \{s\}} \left\{ F_j \cup \ldots \cup F_j \right\}_{-p_j \text{ times}} \]

Without loss of generality, we assume $p_s > 0$. Hence $G_1$ contains $p_s$ disjoint copies of $F_s$ while $H_1$ contains none.

Claim 1. Neither $G_1$ nor $H_1$ contains an isolated vertex.

Proof of the claim: For each $j \in I_{s-1} \cup \{s\}$ the graph $F_j$ is connected and contains at least two vertices.

Claim 2. For every $r \in [k] \setminus \{t\}$, we have $\text{emb}(F_i, G_1) = \text{emb}(F_i, H_1)$.

Proof of the claim: First, consider the case $r < t$. Hence, $i_r < s$. Then (E4) implies that $i_r \in \{1\} \cup I_{s-1}$. It follows that

\[ \text{emb}(F_i, G_1) = \sum_{j \in I_{s-1} \cup \{s\}} p_j \cdot \text{emb}(F_i, F_j) \quad \text{as } F_i \text{ is connected and } (2) \]

\[ = \sum_{j \in I_{s-1} \cup \{s\}} -p_j \cdot \text{emb}(F_i, F_j) \quad \text{by } (15) \]

\[ = \text{emb}(F_i, H_1). \]

Now let $t < r \leq k$. Then $s < i_r$ and hence, $\text{emb}(F_i, F_j) = 0$ for $j \in [s]$. Therefore,

\[ \text{emb}(F_i, G_1) = \text{emb}(F_i, H_1). \]

This proves our claim.
Claim 3. \( \text{emb}(F_s, G_1) = p_s \cdot \text{aut}(F_s) > 0 \) and \( \text{emb}(F_s, H_1) = 0 \).

Proof of the claim: Note that \( j < s \) for \( j \in I_{s-1} \). Thus

\[
\text{emb}(F_s, G_1) = \sum_{j \in I_{s-1} \cup \{s\}} p_j \cdot \text{emb}(F_s, F_j) = p_s \cdot \text{emb}(F_s, F_s) = p_s \cdot \text{aut}(F_s).
\]

Arguing similarly we get \( \text{emb}(F_s, H_1) = 0 \).

Using the graphs \( G_0 \) and \( H_0 \) satisfying (E1), finally we define the graphs \( G \) and \( H \) as follows. For \( g \) and \( h \) defined in (E3) we first assume \( g > h \).

\[
G := \bigcup_{\text{emb}(F_s, G_1) \text{ times}} G_0 \bigcup_{g - h \text{ times}} H_1 \bigcup_{\text{emb}(F_s, G_1) \text{ times}} G_1
\]

\[
H := \bigcup_{\text{emb}(F_s, G_1) \text{ times}} H_0 \bigcup_{g - h \text{ times}} G_1
\]

By (E1) and Claim 1, \( G \) contains at least one isolated vertex while \( H \) does not contain isolated vertices. This proves (a). To establish (b), let \( r \in [k] \setminus \{t\} \). Then

\[
\text{emb}(F_i, H) = \text{emb}(F_i, G)
\]

follows from (E2) and Claim 2. For \( F_s \) we get

\[
\text{emb}(F_s, G) = \text{emb}(F_s, G_1) \cdot \text{emb}(F_s, G_0) + (g - h) \cdot \text{emb}(F_s, H_1) \quad \text{(by the definition of } G) \\
= g \cdot \text{emb}(F_s, G_1) + 0 \quad \text{(by (E3) and Claim 3)} \\
= h \cdot \text{emb}(F_s, G_1) + (g - h) \cdot \text{emb}(F_s, G_1) \\
= \text{emb}(F_s, G_1) \cdot \text{emb}(F_s, H_0) + (g - h) \cdot \text{emb}(F_s, G_1) \quad \text{(by (E3))} \\
= \text{emb}(F_s, H) \quad \text{(by the definition of } H). 
\]

If \( g < h \), we let

\[
G := \bigcup_{\text{emb}(F_s, G_1) \text{ times}} G_0 \bigcup_{h - g \text{ times}} G_1
\]

\[
H := \bigcup_{\text{emb}(F_s, G_1) \text{ times}} H_0 \bigcup_{h - g \text{ times}} H_1
\]

and argue similarly. \( \square \)

6. On the way to adaptive query algorithms

By the Lovász Isomorphism Theorem, for a graph \( G \) the values \( \text{hom}(F, G) \) for the graphs \( F \) with \( F \leq G \) determine \( G \) (up to isomorphism) and thus we know whether \( G \) has an isolated vertex. The next result shows that it suffices to consider stars with at most as many vertices as \( G \) has. Let \( S_r \) denote the star of \( r \) vertices, i.e., a graph that consists of a vertex of degree \( r - 1 \) (the center of the star) and \( r - 1 \) vertices of degree 1, all neighbors of the center. For a vertex \( u \) of a graph we denote by \( \text{deg}(u) \) its degree. Note that \( \text{deg}(u) = 0 \) means that \( u \) is isolated. The proofs of the following two results built on the well known observation (see e.g., [4, 15]) that

\[
\text{hom}(S_r, G) = \sum_{v \in V(G)} \text{deg}(v)^{r-1}.
\]

Proposition 6.1. Let \( G \) be a graph, \( n := |V(G)| \) and \( d_i := |\{u \in V(G) \mid \text{deg}(u) = i\}| \) for \( i \geq 0 \). Then the sequence of values of \( \text{hom}(S_j, G) \) for \( j \in [n] \) determines \( d_0, \ldots, d_{n-1} \) (note that \( d_k = 0 \) for \( k \geq n \)).
**Proof:** By looking at the value of the center of a star under an homomorphism, we see that for \( j \in [n] \) we have

\[
\text{hom}(S_j, G) = \sum_{0\leq i \leq n-1} d_i \cdot i^{j-1}.
\]

We consider the equations for \( j = 2, \ldots, j = n \), they are linear equations in the unknowns \( d_1, \ldots, d_{n-1} \). Its matrix is the Vandermonde matrix

\[
\begin{pmatrix}
1 & 2^1 & \cdots & (n-1)^1 \\
1 & 2^2 & \cdots & (n-1)^2 \\
\vdots \\
1 & 2^{(n-1)} & \cdots & (n-1)^{(n-1)}
\end{pmatrix}
\]

As this matrix is invertible, the system determines \( d_1, \ldots, d_{n-1} \) and therefore, \( d_0 \).

**Proposition 6.2.** Let \( G \) be a graph and \( n := |V(G)| \). Then \( \text{hom}(S_{n \log n}, G) \) determines the degree sequence \( d_0, \ldots, d_{n-1} \) of \( G \).

**Proof:** We order the tuples in \( \mathbb{N}^n \) lexicographically, i.e., if \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) are in \( \mathbb{N}^n \), then \( x < y \) if for some \( i \in [n] \),

\[
x_1 = y_1, \ldots, x_{i-1} = y_{i-1} \text{ and } x_i < y_i.
\]

If \( \sum_{j \in [n]} x_j = \sum_{j \in [n]} y_j = n \) and \( x < y \), then the \( \ell \in [n] \) satisfying \( \ell = 0 \) cannot be \( n \) as then \( x_n = n - \sum_{j \in [n-1]} x_j = n - \sum_{j \in [n-1]} y_j = y_n \).

We set \( x(G) = (d_{n-1}, \ldots, d_0) \) and define \( x(H) \) for a graph \( H \) with \( n = |V(H)| \) analogously. It suffices to show that

\[
x(G) < x(H) \text{ implies } \text{hom}(S_{n \log n}, G) < \text{hom}(S_{n \log n}, H).
\]

The following claim yields the statement of the proposition.

**Claim:** If \( x(G) < y \) and \( \sum_{j \in [n]} y_j = n \), then \( \text{hom}(S_{n \log n}, G) \leq \sum_{1 \leq j \leq n-1} y_j \cdot j^{n \log n-1} \).

**Proof:** As \( x(G) < y \) and \( \sum_{j \in [n]} y_j = n \), choose \( \ell \in [n-1] \) such that (16) holds. Note that our \( x_j \) now is \( d_{n-j} \). Thus,

\[
d_{n-1} = y_1, \ldots, d_{n-(i-1)} = y_{i-1}, \text{ and } d_{n-i} < y_i
\]

\[
\text{hom}(S_{n \log n}, G) = \sum_{j \in [n-1]} d_j \cdot j^{n \log n-1}
\]

\[
= \sum_{i+1 \leq j \leq n-1} d_j \cdot j^{n \log n-1} + d_i \cdot i^{n \log n-1} + \sum_{1 \leq j \leq i-1} d_j \cdot j^{n \log n-1}
\]

\[
\leq \sum_{i+1 \leq j \leq n-1} y_{n-j} \cdot j^{n \log n-1} + (y_{n-i} - 1) \cdot i^{n \log n-1} + \sum_{1 \leq j \leq i-1} d_j \cdot j^{n \log n-1}
\]

\[
\leq \sum_{i+1 \leq j \leq n-1} y_{n-j} \cdot j^{n \log n-1} + y_{n-i} \cdot i^{n \log n-1} - i^{n \log n-1} + (n-1)(i-1)^{n \log n-1}
\]

\[
< \sum_{i+1 \leq j \leq n-1} y_{n-j} \cdot j^{n \log n-1} + y_{n-i} \cdot i^{n \log n-1}
\]

(by \( i^{n \log n-1} > (n-1)(i-1)^{n \log n-1} \), see below)

\[
\leq \sum_{1 \leq j \leq n-1} y_{n-j} \cdot j^{n \log n-1}.
\]
It remains to show for $0 < i < n$,
\[ i \cdot n - 1 > (n - 1) \cdot (i - 1) \cdot n - 1. \quad (17) \]
Clearly this holds for $i = 1$ and for $i = 2$. In fact, for $i = 2$ the right hand side is equal to $n - 1$. As then $n \geq 3$ and $\log 3 > 1$ the left hand side is greater than $2^{n-1}$. Clearly, for $n \geq 3$, we have $2^{n-1} > n - 1$. Hence, we assume $2 < i < n$. The inequality (17) is equivalent to
\[ 1 > (n - 1) \cdot \left(1 - \frac{1}{i}\right)^{n - \log n - 1}. \]
We show that the right hand side is less than 1
\[
(n - 1) \cdot \left(1 - \frac{1}{i}\right)^{n - \log n - 1} = (n - 1) \cdot \left(1 - \frac{1}{i}\right)^{i} \cdot e^{-\log n - 1} \\
< (n - 1) \cdot e^{-\log n - 1} \quad \text{(as } 1 - \frac{1}{i} < e^{-1}) \\
= \frac{(n - 1)}{e^{\log n - 1}} = \frac{(n - 1)}{e^{\log n - 1}} \\
< \frac{(n - 1)}{(n - 1)^{\frac{1}{i}} = \frac{e^{\frac{1}{i}}}{(n - 1)^{\frac{1}{i}} - 1} = \frac{e^{\frac{1}{i}}}{(n - 1)^{\frac{1}{i}}} = \left(\frac{e}{(n - 1)^{\frac{1}{i}}} \right)^{\frac{1}{i}} \\
< 1 \quad \text{(as } 2 < i < n, \text{ thus } n \geq 4, \text{ and hence, } \frac{e}{(n - 1)^{\frac{1}{i}}} < 1). \]

Remark 6.3. For $k, d \in \mathbb{N}$ there is a query algorithm deciding whether there are exactly $k$ elements of degree $\geq d$ (this might surprise in view of the result concerning isolated vertices presented in the previous section). In fact there is an existential FO-sentence $\varphi_{k,d}$ expressing “there are at least $k$ elements of degree $\geq d$” and thus the universal sentence $\neg \varphi_{k,d}$ expresses “there are less than $k$ elements of degree $\geq d$.” Hence $\varphi_{k,d} \land \neg \varphi_{k+1,d}$ expresses there are exactly $k$ elements of degree $\geq d$.

7. Adaptive query algorithms

In Section 5 we have seen that there is no query algorithm that decides whether a graph $G$ is in $C(\text{isolated})$, i.e., whether $G$ has at least one isolated vertex. On the other hand, we have shown that for a graph $G$ with $n$ elements this can be decided by querying $\text{hom}(S_{n \cdot \log n}, G)$. That is, we have an algorithm for $C(\text{isolated})$ consisting of two homomorphism counts:

- query $n := \text{hom}(F_{1}^{0}, G)$ ($= |V(G)|$, note that $F_{1}^{0}$ is a graph with a single vertex);
- query $\text{hom}(S_{n \cdot \log n}, G)$.

That is, the selection of the graph for the second homomorphism count, in our case $S_{n \cdot \log n}$, depends on the answer to the first query. This leads to the notion of adaptive query algorithm. Recall that $F_{1}^{0}, F_{2}^{0}, \ldots$ is an enumeration of all graphs (up to isomorphism) respecting $\leq$.

Definition 7.1. Let $C$ be a class of graphs and $k \geq 1$. A $k$ adaptive hom-query algorithm for $C$ consists of a function $g : \{\emptyset\} \cup \bigcup_{i \in [k-1]} \mathbb{N}^{i} \to \mathbb{N}$ and a subset $X$ of $\mathbb{N}^{k}$ such that for every graph $G$,
\[ G \in C \iff (n_{1}, \ldots, n_{k}) \in X, \]
where $n_{1} := \text{hom}(F_{1}^{0}, G)$, $n_{2} := \text{hom}(F_{g(n_{1})}^{0}, G)$, $\ldots$, and $n_{k} := \text{hom}(F_{g(n_{1}, n_{2}, \ldots, n_{k-1})}^{0}, G)$. We then say that $C$ can be decided by a $k$ adaptive hom-query algorithm.
The main result of this section:

**Theorem 7.2.** Every class $\mathcal{C}$ can be decided by a 3 adaptive hom-query algorithm.

To get this result it suffices to show:

**Theorem 7.3.** Let $n \geq 1$. Then there exist graphs $F_1$ ($= F_1(n)$) and $F_2$ ($= F_2(n)$) such that for all graphs $G$ and $H$ with $|V(G)| = |V(H)| = n$,

$$\text{hom}(F_1, G) = \text{hom}(F_1, H) \text{ and } \text{hom}(F_2, G) = \text{hom}(F_2, H) \text{ imply } G \cong H.$$ 

In fact, if we assume this result, then for an arbitrary class $\mathcal{C}$ of graphs, we get the 3 adaptive hom-query algorithm that for a graph $G$ queries

- $\text{hom}(F_1^0, G)$; set $n := \text{hom}(F_1^0, G)$
- $\text{hom}(F_1(n), G)$ and $\text{hom}(F_2(n), G)$

(where $F_1(n)$ and $F_2(n)$ are the graphs of Theorem 7.3) and has as set $X$ the set

$$X := \left\{ (n, \text{hom}(F_1(n), H), \text{hom}(F_2(n), H)) \mid n \geq 1, H \in \mathcal{C} \text{ and } |V(H)| = n \right\}.$$

**Corollary 7.4.** For all graphs $G$ and $H$,

if $n_0 := \text{hom}(F_1^0, G) = \text{hom}(F_1^0, H)$, $\text{hom}(F_1(n_0), G) = \text{hom}(F_1(n_0), H)$, $\text{and } \text{hom}(F_2(n_0), G) = \text{hom}(F_2(n_0), H)$, then $G \cong H$.

Hence, by “three adaptive hom-queries” we can characterize the isomorphism type of any graph. In Theorem 7.9 we will see that it is not possible to do this by two queries in general.

We turn to a proof of Theorem 7.3. An important tool in the proof will be the following lemma.

**Lemma 7.5.** Let $n \geq 1$ and $\mathcal{K}$ be a finite set of graphs. We can construct a graph $F_\mathcal{K}$ such that for all graphs $G$ and $H$ with exactly $n$ vertices we have $\text{hom}(F_\mathcal{K}, G) = \text{hom}(F_\mathcal{K}, H)$ if and only if $G$ and $H$ satisfy at least one of the conditions (a) and (b).

(a) There exist $F, F' \in \mathcal{K}$ such that

$$\text{hom}(F, G) = 0 \text{ and } \text{hom}(F', H) = 0.$$ 

(b) For all $F \in \mathcal{K}$,

$$\text{hom}(F, G) = \text{hom}(F, H).$$ 

**Proof:** The idea of the construction is best seen by assuming $\mathcal{K} = \{F_1, F_2\}$ (iterating the following process one gets the general case). We set

$$r := n^{\lvert V(F_1) \rvert}.$$ 

As $|V(G)| = n$, we know that $\text{hom}(F_1, G) \leq r$. We set

$$F_\mathcal{K} := F_1 \cup F_2 \cup F_2 \cup \cdots \cup F_2,$$

$r$ times.
Note that for every graph $F$, 
\[
\text{hom}(F_K, F) = \text{hom}(F_1, F) \cdot \text{hom}(F_2, F)^r.
\]  
(18) 

Hence, if (a) or (b) hold, then $\text{hom}(F_K, G) = \text{hom}(F_K, H)$. Conversely, assume $z := \text{hom}(F_K, G) = \text{hom}(F_K, H)$. If $z = 0$, then (a) must hold by \((15)\). If $z = 1$, then $\text{hom}(F_i, G) = \text{hom}(F_i, H) = 1$ for $i \in [2]$ and (b) holds. Otherwise, $z \geq 2$. Let $F := G$ or $F := H$ and set $x := \text{hom}(F_1, F)$ and $y := \text{hom}(F_2, F)$. Let $p$ be a prime number with $p|z$ (i.e., $p$ divides $z$). Choose the maximum $k$ such that $p^k | z$. Write $k$ in the form $k = \ell \cdot r + m$ with $0 \leq m < r$. As $z = x \cdot y^r$ and $x \leq r$, the factor $p^m$ appears in $x$ and the factor $p^\ell$ in $y$. This determines $x$ and $y$ and they do not depend on the $F \in \{G, H\}$ chosen. 
\[\square\]

The previous lemma doesn’t help too much if the alternative (a) holds. To overcome this alternative essentially we consider the bipartite graphs and the non-bipartite graphs separately. For this purpose we recall the definition and some simple facts on bipartite graphs.

By definition a graph $G$ is bipartite if there is a partition $V(G) = X \cup Y$ such that each edge has one end in $X$ and one edge in $Y$. The following lemma contains some simple facts on bipartite graphs.

**Lemma 7.6.** Let $G$ be a graph. Then:

(a) $G$ is bipartite $\iff \text{hom}(G, P_2) \neq 0$.

(b) If $G$ is connected and bipartite, then $\text{hom}(G, P_2) = 2$.

(c) $G$ is bipartite $\iff \text{hom}(G, H) \neq 0$ for all graphs $H$ with at least one edge.

(d) $G$ is bipartite $\iff G$ does not contain a cycle of odd length.

(e) If $G$ is bipartite and $F$ is not, then $\text{hom}(F, G) = 0$.

(f) If $G$ is bipartite, then $G$ is determined (up to isomorphism) by the values $\text{hom}(F, G)$ for the bipartite graphs $F$ with $F \leq G$ (by the Lovász Isomorphism Theorem and part (e)).

Recall the definition of the (weak) product of graphs. Let $G$ and $H$ be graphs. Then $G \times H$, the product of $G$ and $H$ is the graph with $V(G \times H) := \{(u, v) \mid u \in V(G), v \in V(H)\}$ and 
\[
\{(u, v), (u', v')\} \in E(G \times H) \iff \{u, u'\} \in E(G) \text{ and } \{v, v'\} \in E(H).
\]

One easily verifies that for any graph $H$, 
\[
\text{hom}(F, G \times H) = \text{hom}(F, G) \cdot \text{hom}(F, H).
\]  
(19) 

Besides the simple facts on bipartite graphs mentioned above, we also need a deep result.

**Theorem 7.7** (Lovász Cancellation Law \([17]\)). Let $H$ be a graph. Then

$H$ is not bipartite $\iff$ for all graphs $F$ and $G$, $(F \times H \cong G \times H$ implies $F \cong G)$.

The following lemma contains a further step for the proof of Theorem 7.7

**Lemma 7.8.** Let $n \geq 2$ and let $t$ be the smallest natural number with $n \leq 2t$. We set 
\[
K_t := \{ F \mid \text{hom}(F, C_{2t+1}) > 0 \text{ and } |V(F)| \leq (2t + 1)^2\}.
\]

For graphs $G$ and $H$ with $|V(G)| = |V(H)| = n$, if $\text{hom}(F, G) = \text{hom}(F, H)$ for all $F \in K_t$, then $G \cong H$. 

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**Proof:** If $G \not\cong H$, then by the Lovász Cancellation Law

$$G \times C_{2t+1} \not\cong H \times C_{2t+1}. $$

As $G \times C_{2t+1}$ has at most $n \cdot (2t+1)$ vertices, by the Lovász Isomorphism Theorem (see Theorem 2.2), there is a graph $F$ with $|V(F)| \leq n \cdot (2t+1) \leq (2t+1)^2$ such that

$$\text{hom}(F, G \times C_{2t+1}) \neq \text{hom}(F, H \times C_{2t+1}). \quad (20)$$

If $F$ is not in $K_t$, then $\text{hom}(F, C_{2t+1}) = 0$ and thus by (19), $\text{hom}(F, G \times C_{2t+1}) = \text{hom}(F, H \times C_{2t+1}) = 0$. Hence, $F \in K_t$ and so by assumption, $\text{hom}(F, G) = \text{hom}(F, H)$. However, this contradicts (20) (use again (19)).

**Proof of Theorem 7.3:** The case $n = 1$ is trivial. So we assume $n \geq 2$.

Let again $t$ be the smallest natural number with $n \leq 2t$ and

$$K_t := \{ F \mid \text{hom}(F, C_{2t+1}) > 0 \text{ and } |V(F)| \leq (2t+1)^2 \}. $$

For the class $K_{\text{bip}}$ of bipartite graphs in $K_t$ let $F_1$ be the graph constructed in Lemma 7.5 for $K_{\text{bip}}$ (i.e., $F_1 = F_{K_{\text{bip}}}$). As the disjoint union of bipartite graphs is bipartite, the proof of Lemma 7.5 shows that $F_1$ is bipartite. Let $F_2$ be the graph constructed in Lemma 7.5 for the class $K_t \setminus K_{\text{bip}}$. We show that for graphs $G$ and $H$ with $|V(G)| = |V(H)| = n$,

$$\text{hom}(F_1, G) = \text{hom}(F_1, H) \text{ and } \text{hom}(F_2, G) = \text{hom}(F_2, H) \text{ imply } G \cong H. $$

If $G$ and $H$ both have no edge, then clearly $G \cong H$. If, say $G$ has an edge but $E(H) = \emptyset$, then, as $F_1$ is bipartite, $\text{hom}(F_1, G) \neq 0 = \text{hom}(F_1, H)$ by Lemma 7.6 (c) and $E(F_1) \neq \emptyset$. Hence we can assume that both graphs contain at least one edge. For a contradiction assume that $\text{hom}(F_1, G) = \text{hom}(F_1, H)$ and $\text{hom}(F_2, G) = \text{hom}(F_2, H)$ and that $G \not\cong H$. Then, by Lemma 7.8 there is a graph $F_0 \in K_t$ with

$$\text{hom}(F_0, G) \neq \text{hom}(F_0, H). \quad (21)$$

Assume first that $F_0 \in K_{\text{bip}}$. As $G$ and $H$ contain at least one edge, by Lemma 7.6 (c)

$$\text{hom}(F, G) > 0 \quad \text{and} \quad \text{hom}(F, H) > 0, $$

for every bipartite graph $F$. In particular, this holds for all graphs $F$ in $K_{\text{bip}}$. Thus, $\text{hom}(F_1, G) = \text{hom}(F_1, H)$ implies the second case in Lemma 7.5 i.e., for every $F \in K_{\text{bip}},$

$$\text{hom}(F, G) = \text{hom}(F, H). $$

In particular, this holds for $F = F_0$ contradicting (21).

Thus $F_0 \in K_t \setminus K_{\text{bip}}$. Then $\text{hom}(F_0, F) = 0$ for all bipartite graphs $F$ (by Lemma 7.6 (c)). Hence, by (21) at least one of $G$ and $H$ must be non-bipartite. By $\text{hom}(F_2, G) = \text{hom}(F_2, H)$, Lemma 7.5 and (21) there exist graphs $F^G$ and $F^H$ in $K_t \setminus K_{\text{bip}}$ with

$$\text{hom}(F^G, G) = \text{hom}(F^H, H) = 0. $$

Without loss of generality, suppose that $G$ is not bipartite and thus contains an odd cycle, say of length $\ell$. Since $\ell \leq n \leq 2t + 1$, we have

$$\text{hom}(C_{2t+1}, C_\ell) > 0. $$

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In fact, one easily verifies that $\text{hom}(C_k, C_m) > 0$ for odd $m$ and $k$ with $m < k$. As $F^G \in K_t$, we know that $\text{hom}(F^G, C_{2t+1}) > 0$. Therefore, $\text{hom}(F^G, C_{t}) > 0$, which implies $\text{hom}(F^G, G) > 0$, a contradiction. 

As already mentioned not every graph property can be decided by a two adaptive hom-query algorithm. More precisely:

**Theorem 7.9.** There is no $s_0 \in \mathbb{N}$ such that for some function $g : \mathbb{N} \to \mathbb{N}$ and all graphs $G$ and $H$,

$$
\text{if } n_0 := \text{hom}(F_{s_0}^0, G) = \text{hom}(F_{s_0}^0, H) \text{ and } \text{hom}(F_{g(n_0)}^0, G) = \text{hom}(F_{g(n_0)}^0, H), \text{ then } G \cong H. \quad (22)
$$

**Proof:** Towards a contradiction, assume that $s_0$ and $g$ as stated above exist. Let $k := |V(F_{s_0}^0)|$ and choose $\ell \in \mathbb{N}$ with $4\ell + 2 > k$, consider the following three non-isomorphic graphs:

- the non-bipartite graph $G := C_{6\ell+3} \cup C_{6\ell+3}$,
- the bipartite graph $H_1 := C_{12\ell+6}$,
- and the bipartite graph $H_2 := C_{4\ell+2} \cup C_{4\ell+2} \cup C_{4k+2}$.

**Claim 1.** $\text{hom}(F_{s_0}^0, G) = \text{hom}(F_{s_0}^0, H_1) = \text{hom}(F_{s_0}^0, H_2)$.

**Proof of Claim 1:** If $F_{s_0}^0$ is non-bipartite, then $\text{hom}(F_{s_0}^0, H_1) = \text{hom}(F_{s_0}^0, H_2) = 0$ (by Lemma 7.6(e)). Any odd cycle in $F_{s_0}^0$ (at least one exists by Lemma 7.6(d)) has length at most $|V(F_{s_0}^0)| = k < 4\ell + 2 < 6\ell + 3$, thus there is no homomorphism from $F_{s_0}^0$ to $C_{6\ell+3}$. It implies $\text{hom}(F_{s_0}^0, G) = 0$ as well.

Assume that $F_{s_0}^0$ is a bipartite graph. Let $A_1, A_2, \ldots, A_p$ be the connected components of $F_{s_0}^0$. Note that each $A_i$ has at most $k$ vertices. We fix a vertex $v_i$ in each $V(A_i)$. For $t \geq 3$ consider a cycle $C_t$ and again fix an arbitrary $u_t \in V(C_t)$. We set

$$
\rho(i, t) := |\{h \in \text{Hom}(A_i, C_t) \mid h(v_i) = u_t\}|.
$$

Clearly,

$$
\text{hom}(A_i, C_t) = t \cdot \rho(i, t).
$$

The key observation is that for $t, t' > k$

$$
\rho(i, t) = \rho(i, t').
$$

This follows easily from the fact that each $A_i$ is connected and has at most $k = |V(F_{s_0}^0)|$ vertices and thus, for $h \in \text{Hom}(A_i, C_t)$ with $h(v_i) = u_t$ every vertex in the image of $h$ has distance $< k$ from $u_t$. Hence

$$
\text{hom}(A_i, G) = \text{hom}(A_i, H_1) = \text{hom}(A_i, H_2) = (12\ell + 6) \cdot \rho(i, k + 1)
$$

Then, by (11),

$$
\text{hom}(F_{s_0}^0, G) = \text{hom}(F_{s_0}^0, H_1) = \text{hom}(F_{s_0}^0, H_2) = \prod_{i \in [p]} (12\ell + 6) \cdot \rho(i, k + 1).
$$

\[\square\]
Let \( n_0 := \text{hom}(F^0_{s_0}, G) = \text{hom}(F^0_{s_0}, H_1) = \text{hom}(F^0_{s_0}, H_2) \). Moreover, let
\[
F := F^0_{g(s_0)}.
\]

If \( F \) is non-bipartite, since both \( H_1 \) and \( H_2 \) are bipartite, we conclude (by Lemma 7.6(c))
\[
\text{hom}(F, H_1) = \text{hom}(F, H_2) = 0.
\]

We get a contradiction to (22) for \( G := H_1 \) and \( H := H_2 \). Otherwise, \( F \) is bipartite. We show
\[
\text{hom}(F, G) = \text{hom}(F, H_1).
\]

Then we have a contradiction to (22) for \( G := G \) and \( H := H_1 \).

It is straightforward to verify that \( H_1 \cong C_{6k+3} \times K_2 \). Recall that \( G = C_{6k+3} \cup C_{6k+3} \). Assume that \( F \) has the connected components \( A_1, \ldots, A_p \). Since each \( A_i \) is bipartite and connected, we have \( \text{hom}(A_i, K_2) = 2 \) (by Lemma 7.6(b) as \( K_2 = P_2 \)). Then
\[
\text{hom}(F, G) = \prod_{i \in [p]} \text{hom}(A_i, C_{6k+3} \cup C_{6k+3}) \quad \text{(by (11))}
\]
\[
= \prod_{i \in [p]} 2 \cdot \text{hom}(A_i, C_{6k+3}) \quad \text{(as the A_i’s are connected)}
\]
\[
= \prod_{i \in [p]} \text{hom}(A_i, K_2) \cdot \text{hom}(A_i, C_{6k+3})
\]
\[
= \prod_{i \in [p]} \text{hom}(A_i, K_2 \times C_{6k+3}) \quad \text{(by (19))}
\]
\[
= \prod_{i \in [p]} \text{hom}(A_i, H_1) = \text{hom}(F, H_1) \quad \text{(by } H_1 \cong K_2 \times C_{6k+3} \text{ and } F = \bigcup_{i \in [p]} A_i \text{)}.
\]

This finishes our proof. \( \square \)

**Remark 7.10.** Recall Remark 4.6 where we considered right-hom-query algorithms. It should be clear how we define \( k \)-adaptive right-hom-query algorithms for a class \( C \) of graphs; just replace in Definition 7.1
\[
n_1 := \text{hom}(F^0_{g(\emptyset)}, G), \ n_2 := \text{hom}(F^0_{g(n_1)}, G), \ldots, \ n_k := \text{hom}(F^0_{g(n_1, n_2, \ldots, n_{k-1})}, G)
\]
by
\[
n_1 := \text{hom}(G, F^0_{g(\emptyset)}), \ n_2 := \text{hom}(G, F^0_{g(n_1)}), \ldots, \ n_k := \text{hom}(G, F^0_{g(n_1, n_2, \ldots, n_{k-1})}).
\]

We show:

*For all \( k \) the class \( C \) of graphs with a clique of size 3 has no \( k \)-adaptive right-hom-query algorithm.*

In fact, assume that \( g \) and \( X \) (compare Definition 7.1) witness the existence of a \( k \)-adaptive right-hom-query algorithm for \( C \). Then set
\[
n_1 := g(0), \ n_2 := g(0), \ n_3 := g(0, 0), \ldots, n_k := g(0, \ldots, 0), \quad \text{\( k-1 \) times}
\]
Let $s > 3$ be bigger than any of the $|V(F^0_{n_i})|$’s. According to [19] there is a graph $G$ without a clique of size 3 such that $\chi(G) = s$. Thus by (9), we have
\[ \text{hom}(G, F^0_{n_1}) = 0, \ldots, \text{hom}(G, F^0_{n_k}) = 0. \]
Thus, $(0, \ldots, 0) \notin X$. Clearly,
\[ \text{hom}(K_s, F^0_{n_1}) = 0, \ldots, \text{hom}(K_s, F^0_{n_k}) = 0. \]
But $K_s$ contains a 3-clique, thus $(0, \ldots, 0) \in X$, a contradiction.

8. Conclusions

To the best of our knowledge this is the first paper that systematically analyzes properties of graphs that can be decided by a constant number of homomorphism counts. We gain a quite satisfactory picture. Separately we consider non-adaptive hom-query algorithms and adaptive hom-query algorithms. We characterize those prefix classes of first-order logic with the property that all classes of graphs definable by a corresponding first-order sentence have a non-adaptive query-algorithm with a constant number of homomorphism counts. Furthermore, we show that every class of graphs can be recognized by an adaptive hom-query algorithm with three homomorphism counts. We present an example where two counts are not sufficient. In general, given a class of graphs, the three adaptive hom-query algorithm we get for this class needs for a graph $G$ homomorphism counts $\text{hom}(F, G)$ where the size of $F$ is superpolynomial in the size of $G$. We believe that this is necessary for some classes. It is a challenging task for graph classes that are relevant in applications to analyze the existence of an algorithm where the size of the corresponding $F$’s are polynomial in the size of $G$.

References

[1] A. Atserias, P. Kolaitis, and W. Wu. On the expressive power of homomorphism counts. In 36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021, pages 1–13. IEEE, 2021.

[2] J. Böker. Graph similarity and homomorphism densities. In 48th International Colloquium on Automata, Languages, and Programming, ICALP 2021, July 12-16, 2021, Glasgow, Scotland (Virtual Conference), volume 198 of LIPIcs, pages 32:1–32:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.

[3] J. Böker, Y. Chen, M. Grohe, and G. Rattan. The complexity of homomorphism indistinguishability. In 44th International Symposium on Mathematical Foundations of Computer Science, MFCS 2019, pages 54:1–54:13, 2019.

[4] C. Borgs, J. Chayes, L. Lovász, V.T. Sós, and K. Vesztergombi. Counting graph homomorphisms. In Topics in Discrete Mathematics, pages 315–371, 2006.

[5] J. Cai, M. Fürer, and N. Immerman. An optimal lower bound on the number of variables for graph identifications. Combinatorica, 12(4):389–410, 1992.

[6] S. Chaudhuri and M. Y. Vardi. Optimization of Real conjunctive queries. In Proceedings of the Twelfth ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems, 1993, pages 59–70. ACM Press, 1993.
[7] Y. Chen and J. Flum. Tree-depth, quantifier elimination, and quantifier rank. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018*, Oxford, UK, July 09-12, 2018, pages 225–234, 2018.

[8] Y. Chen and J. Flum. FO-definability of shrub-depth. In *28th EACSL Annual Conference on Computer Science Logic, CSL 2020, January 13-16, 2020, Barcelona, Spain*, pages 15:1–15:16, 2020.

[9] R. Curticapean, H. Dell, and D. Marx. Homomorphisms are a good basis for counting small subgraphs. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC*, pages 210–223. ACM, 2017.

[10] H. Dell, M. Grohe, and G. Rattan. Lovász meets Weisfeiler and Leman. In *45th International Colloquium on Automata, Languages, and Programming, ICALP 2018*, volume 107, pages 40:1–40:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018.

[11] G. Ding. Subgraphs and well-quasi-ordering. *Journal of Graph Theory*, 16(5):489–502, 1992.

[12] Z. Dvořák. On recognizing graphs by numbers of homomorphisms. *Journal of Graph Theory*, 64(4):330–342, 2010.

[13] R. Ganian, P. Hlinený, J. Nesetril, J. Obdrzálek, P. Ossona de Mendez, and R. Ramadurai. When trees grow low: Shrubs and fast MSO1. In *Mathematical Foundations of Computer Science 2012 - 37th International Symposium, MFCS 2012, Bratislava, Slovakia, August 27-31, 2012. Proceedings*, pages 419–430, 2012.

[14] M. Grohe. Counting bounded tree depth homomorphisms. In *LICS ’20: 35th Annual ACM/IEEE Symposium on Logic in Computer Science*, 2020, pages 507–520. ACM, 2020.

[15] M. Grohe. word2vec, node2vec, graph2vec, x2vec: Towards a theory of vector embeddings of structured data. In *Proceedings of the 39th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, PODS 2020*, pages 1–16. ACM, 2020.

[16] L. Lovász. Operations with structures. *Acta Mathematica Academiae Scientiarum Hungarica*, 18:321–328, 1967.

[17] L. Lovász. On the cancellation law among finite relational structures. *Periodica Mathematica Hungarica*, 1:145–156, 1971.

[18] T. A. McKee. Forbidden subgraphs in terms of forbidden quantifiers. *Notre Dame Journal of Formal Log.*, 19:186–188, 1978.

[19] J. Mycielski. Sur le coloriage des graphes. *Information Processing Letters*, 108(6):412–417, 2008.