Abstract: Group theory, the ultimate theory for symmetry, is a powerful tool that has a direct impact on research in robotics, computer vision, computer graphics and medical image analysis. Symmetry is very important in chemistry research and group theory is the tool that is used to determine symmetry. Usually, it is not only the symmetry of molecule but also the symmetries of some local atoms, molecular orbitals, rotations and vibrations of bonds, etc. that are important. Harada-Norton group is an example of a sporadic simple group. There are 14 maximal subgroups of Harada-Norton group. Generators (also known as words) of 11 maximal subgroups are already known. The aim of this note is to give generators of the remaining 3 maximal subgroups, which is an open problem mentioned on A World-wide-web Atlas of Group Representations [1]. In this report we compute the generators of $A_6 \times A_6, D_8, 2^{3+2} \times (3 \times L_3(2))$ and $3^4 : 2.(A_4 \times A_4).4$. Moreover we also compute the generators for the Maximal subgroups of some linear groups.

Keywords: Harada-Norton group, maximal subgroup, generators, finite group, normalizer

1 Introduction

Group theory is important in organic chemistry in studying symmetry of molecules [2]. Usually, all the molecules are symmetric and rotations and vibrations of bonds are important [3]. For example, from the symmetries of molecular orbital wave functions one can figure out the information about the binding [4]. From the symmetries, we can explain the transition and change the bands [3, 4]. Symmetry elements and symmetric operations are important concepts in group theory and if we apply any operation on a molecule and the molecule remains unchanged we call it symmetry operation. That means, the molecule remains same after applying any symmetric operation [5–7]. When we apply symmetric operation on a molecule, the position of items and bounds get changes but the appearance of molecule remains unchanged [9]. With the help of group theory and using the symmetry of molecule, we can decide physical properties of molecule [10]. The symmetry of a molecule provides with the information of what energy levels the orbitals will be, what the orbitals symmetries are, what transitions can occur between energy levels, even bond order to name a few can be found, all without rigorous calculations. The fact that so many important physical aspects of molecules can be derived from symmetry is a very profound statement and this is what makes group theory so powerful [11]. The study of the maximal subgroups of sporadic simple groups began in the 1960s. Chang Choi [12, 13] found all the maximal subgroups of $M_{24}$. In literature, the maximal subgroups of $HS$ and $McL$ groups, $HN$ and fisher groups $Fi_{22}$ and $Fi_{23}$ are known. The local and non-local subgroups of $Fi_{22}, Fi_{23}$ and $Fi_{24}$ are given in [16–18]. In 1979, R.A. Wilson discovered the maximal subgroups of Suzuki group [19] and Rudvís group [20]. In 1990, Steve Linton determined the maximal subgroups of $Th$, $Fi_{24}$ and its automorphism groups. He completely discussed the maximal subgroups in [21, 22]. In 1999, R.A. Wilson constructed the maximal subgroups of $B$. Recently Wilson has updated the list of maximal subgroups of the Monster Group. There are still some undetermined cases. To date there are 44 maximal subgroups of Monster. Its standard generators are given in [1]. Fur-
ther, the sporadic groups can be classified into three generations. The first generation contains the Mathieu groups. The second generation contains $Co_1$, $Co_2$, $Co_3$, $Suz$, $McI$, $HS$ and $J_2$. The third generation contains the remaining 8 groups: $Fi_{22}$, $Fi_{23}$, $Fi_{24}$, $Th$, $HN$, $He$ and $B$. Finally, the Monster group itself is considered to be in this generation.

The concept of standard generators for sporadic simple groups was introduced by R. A. Wilson. He started a project known as an online version of Atlas, which would provide not only representations (matrix and permutation) but also words for the maximal subgroups of simple and almost simple groups. The words for the maximal subgroups of $M_{12}.2$, $M_{22}.2$, $HS.2$, $McL.2$, $J_2.2$, $Suz.2$, $He.2$, $Fi_{22}.2$, $HN.2$ and $Fi_{24}$ are discussed by Simon in [23]. In 2001, John N. Bray worked on the maximal subgroups of sporadic simple groups of order less than $10^{16}$. He presents a complete list by providing words for the maximal subgroups of 17 sporadic simple groups which includes $M_{11}$, $M_{12}$, $J_1$, $M_{22}$, $J_2$, $J_3$, $Ru$, $O’N$, $Co_3$, $HS$, $McL$, $Suz$, $He$, $Fi_{22}$, $Co_2$, $M_{26}$, $M_{2}$ and $Fi_{22}$. Words for maximal subgroups of these groups are given on the world-wide-web. However, there are still some cases to be dealt with. We pursue the work initiated by R.A. Wilson of finding words for maximal subgroups of certain sporadic simple groups. Thus, the only cases on the list of the world-wide-web Atlas of finite group representations which need to be solved are $HN$, $Fi_{23}$, $Co_1$, $B$ and $M$.

There are still some hard cases which must be solved in order to have a complete list. In this paper we provide words for the maximal subgroups of the Harada-Norton Group. Moreover, we provide words of some linear groups i.e., $L_2(8)$, $L_2(8) : 2$, $L_2(13)$, $L_2(13) : 2$, $L_2(16)$, $L_2(17)$, $L_2(17) : 2$, $L_3(19)$, $L_3(19) : 2$, $L_3(23)$, $L_3(29)$, $L_3(31)$, $L_3(3)$, $L_3(3) : 2$, $L_3(5)$. Ideally the words should be as short as possible. We use extensively GAP [26] and MAGMA [25] for group theoretic calculations.

Our notation follows [26]. In particular, $a^b = b^{-1}ab$ and $[a, b] = a^{-1}b^{-1}ab$.

## 2 Main Results

In this section we give generators for the maximal subgroups of Harada-Norton and some Linear Groups.

### 2.1 Harada-Norton Group

In modern algebra, more precisely in group theory, an example of a sporadic simple group is the Harada-Norton group denoted by $HN$ having order $2^{14}.3^6.5^6.7.11.19 = 273030912000000 \cong 3 \times 10^{16}$. There are total 26 sporadic groups and Harada-Norton group is one of them founded in 1976 by Harada and in 1975 by Norton. By observing that the Harada-Norton group has a trivial Schur multiplier and has an order 2 outer automorphism group. Let the Higman-Sims group $HS$, then the Harada-Norton group has involution whose centralizer is of the form $2.HS.2$.

The prime 5 assumes an exceptional part in the group. For instance, it centralizes an element of order 5 in the Monster group (which is the manner by which Norton thought that it was), and thus acts normally on a vertex operator algebra over the field with 5 element [27]. This infers it follows up on a 133 dimensional algebra over $F_5$, with a commutative however nonassociative product, practically equivalent to the Griess algebra [28].

Conway and Norton proposed in their 1979 paper [29] that monstrous moonshine isn’t constrained to the monster, yet comparative wonders might be found for different groups. Larissa Queen [30] and others in this manner found that one can develop the extensions of numerous Hauptmoduln from simple combinations of dimensions of sporadic group. For $HN$, the pertinent McKay-Thompson series is $T_{5A}(r)$ where one can set the constant $a(0) = -6,

$$
\eta(5) = 5 \eta(5) - 6 = \left( \frac{\eta(r)}{\eta(5)} \right)^6 + 3 \left( \frac{\eta(5r)}{\eta(r)} \right)^6
$$

where $\eta(t)$ denotes Dedekind eta function.

The Harada-Norton group has been studied extensively in recent years and many papers are written on this group, here we mention a few [31–39]. Monomial modular representations and symmetric generation of the Harada-Norton group. The uniqueness of this group was proved in [32]. Ryba et al. [41] found matrix generators for this group and in [42] Norton and Wilson found all maximal subgroups of the Harada-Norton group in 1986. The following 14 are the maximal subgroups of Harada-Norton group.

1. $A_{12}$
2. $2.HS.2$
3. $U_3(8) : 3$
4. $2^{1+8}.(A_5 \times A_5).2$
5. $(D_{10} \times U_3(5)).2$
6. $5^{1+6}, 2^{1+4}, 5.4$
7. $2^6.U_4(2)$
8. $A_6 \times A_6, D_8$
9. $2^{3+2+6}.(3 \times L_2(2))$
10. $5^{2+1+2}.4.A_5$

### References

[1] A
[2] 2.HS.2
[3] $U_3(8) : 3$
[4] $2^{1+8}.(A_5 \times A_5).2$
[5] $(D_{10} \times U_3(5)).2$
[6] $5^{1+6}, 2^{1+4}, 5.4$
[7] $2^6.U_4(2)$
[8] $A_6 \times A_6, D_8$
[9] $2^{3+2+6}.(3 \times L_2(2))$
[10] $5^{2+1+2}.4.A_5$
We want to work inside the subgroups as much as possible. It is an interesting problem to find the generators of a group. The Atlas of group representations contains the words for 11 maximal subgroups of $HN$ except the 3 cases marked by asterisk. In this report we determine the generators for the above mentioned subgroups as words in the generators of $HN$. It is well known that if $G$ is a simple group, $M$ is the maximal subgroup of $G$ and $K$ is the minimal normal subgroup of $M$, then $M = N_G(K)$. The cases we have dealt with, occur as normalizers of elementary abelian groups and the required information is provided in [26]. Thus we see that $N(2B^3) = 2^{2+2+6}.(3 \times L_5(2))$ and $N(3^4) = 3^4 \times (A_6 \times A_6).4$. The normalizers were computed by the methods given in [23]. We have used GAP [24] and MAGMA [25] for computations.

### 2.1.1 Generators of $(A_6 \times A_6) : D_8$

We want to work inside the subgroups as much as possible. We see that $H = (A_6 \times A_6) : 2^2 \times A_{12} < HN$, so all we need is to construct $H$ inside $A_{12}$ and then find an involution inside $HN$ which extends $H$ to $(A_6 \times A_6) : D_8$. The details are as follows.

It is trivial to find $(A_6 \times A_6)$ inside $A_{12}$. Next we find an involution inside $N_{A_{12}}(A_6 \times A_6)$, which extends $(A_6 \times A_6)$ to $(A_6 \times A_6) : 2$. We find another involution which extends $(A_6 \times A_6) : 2$ to $H$. Now we want to use this working inside $HN$.

The standard generators of $A_{12}$ inside $HN$ can be constructed by observing that the 3A and 11A classes of $A_{12}$ fuse to 3A and 11A classes of $HN$. After obtaining the standard generators of $A_{12}$, we lift $A_{12}$ inside $HN = \langle a, b \rangle$, where $a, b$ are as in [1]. As a final step, we find an involution inside $N_{HN}(2^2)$ which extends $H$ to $(A_6 \times A_6) : D_8$. The computational details are given below.

First we download the standard generators of $A_{12}$ from Atlas given by $c, d$. Then we find the Centralizer of $A_6$ inside $A_{12}$. We now give the details of computing the centralizer of $A_6$ inside $A_{12}$, for that first we consider the standard generators of $A_6$ given in Atlas $c, d$ next we convert the $c$ and $d$ in terms of standard generators of $A_{12}$ which is given by

\[
x_1 = c \\
x_2 = d^{-1}cd^2cd^3c
\]

Here $x_1, x_2$ are generators of $A_6$ inside $A_{12}$, now we find the centralizer of $A_6$ inside $A_{12}$ which includes the following computations given by

\[
b_6 = (cd^4cd^2cd)^5c(cd^3cd^2cd)^5 \\
i_1 = b_2 \\
c_3 = x_2 \\
e_3 = (i_1x_2^3i_1x_2^3)^2x_2i_1x_2^3i_1x_2^3i_1 \\
u_1 = b_6 \\
u_9 = c_3e_3 \\
u_{16} = u_1u_9
\]

Here $u_{16}$ and $u_6$ are generators of centralizer of $A_6$ inside $A_{12}$. Then the generators of $A_6$ plus the generators of Centralizer of $A_6$ inside $A_{12}$ gives us $A_6 \times A_6$ given by

\[
v_1 = u_1u_9x_2 \\
v_2 = x_2u_6x_1
\]

Here $v_1, v_2$ are generators of $A_6 \times A_6$.

Now we find the normalizer of $A_6 \times A_6$ inside $A_{12}$. This normalizer contain an involution given by

\[
v_3 = (c^3d^{11}c^3d^2c^1)^{15}
\]

which extends the group $A_6 \times A_6$ to $A_6 \times A_6 : 2$. Similarly in the same way we can find the normalizer of $A_6 \times A_6 : 2$ inside $A_{12}$ and this normalizer contains an involution given by

\[
v_4 = (c^3d^{10}c^2d^3c^3)^{15}
\]

which extends $A_6 \times A_6 : 2$ inside $A_6 \times A_6 : 2^2$. Here all the calculations are inside $A_{12}$ and $A_6 \times A_6 : 2^2$ is the maximal subgroup of $A_{12}$ and it is not possible to extend $A_6 \times A_6 : 2^2$ to $A_6 \times A_6 : D_8$ so our next target is to uplift the whole structure inside $HN$. Before uplifting we have to calculate the standard generators of $A_{12}$ inside $HN$. The generators of $A_{12}$ inside $HN$ are given in [1], now we use these generators to find the standard generators of $A_{12}$ inside $HN$. The words for the generators of $A_{12}$ are given by $c$ and $d$. Before this we will give some random elements.

\[
a_1 = (cd)^2 \\
a_2 = cd \\
a_3 = a_1^{a_1}a_1a_1a_2, a_2a_1a_1a_1 \\
a_4 = a_1^2
\]

with the help of a power maps search inside the 3A and 11A classes, we found the standard generators of $A_{12}$ are given by

\[
x = a_4^2
\]
We find the normalizer of $1512$ involution inside $A_{12}$. After lifting $A_6 \times A_6 : 2^2$ inside $HN$ we just need one more involution which gives us the required subgroup. It is not an easy task to find the last involution inside $HN$ by random searching. So first here we find the normalizer of $A_6 \times A_6 : 2^2$ inside $HN$, then searching an involution inside this normalizer such that this involution extends $A_6 \times A_6 : 2^2$ to $(A_6 \times A_6) : D_8$ and combining this involution with $w_3, w_4$ will give us $D_8$. The words for the normalizer of $A_6 \times A_6 : 2^2$ inside $HN$ are given below.

\[ w_8 = (w_4^2 b (w_1 b w_4^2 a)^2 (w_4^2 b)^2 (w_3)^2 \]
\[ w_9 = w_4 w_5 b (w_3 a)^3 w_2 b^3 w_2 b^2 \]
\[ w_{10} = a (b a^{-1}) (b a)^2 b^3 (w_4 b^2)^2 \]
\[ w_{11} = w_4 a^3 b^3 a^4 (w_3 b^2)^2 (w_4 b^2)^2 \]
\[ w_{12} = w_7 w_2 b^2 a^2 (w_6 a)^3 b^2 a (w_3)^2 a^2 \]
\[ w_{13} = (a^2 b)^2 b^5 w_3 (w_1)^2 (w_4)^3 w_3 b^{-1} \]
\[ w_{14} = a^{-1} w_4 (a (w_3)^2 b (w_1)^2 b^2 w_3 a^{-1} b)^3 \]
\[ w_{15} = a^{-1} (b a^2 b)^2 w_6 w_3 a^4 (a^2 b)^2 \]
\[ w_{16} = a^2 b (b^2 a)^2 w_6 b^2 a^3 (w_3 b^3)^2 \]
\[ w_{17} = a^3 b a^6 (w_3 b^2)^2 (w_4 b^2 a^2)^2 \]
\[ w_{18} = w_4^2 (w_3 b)^2 w_3 (w_1 b^2 a)^2 (w_2 b)^2 \]
\[ w_{19} = b w_2 b^2 a b^2 a^6 (w_6 a b^2)^2 w_3 a^3 b^3 \]
\[ w_{20} = a^6 (a w_2 b^2)^2 (w_4 b^2 a)^2 w_7 w_3^2 \]
\[ w_{21} = b^2 a^2 w_6 b^2 b^2 a (w_4 b^2)^2 \]
\[ w_{22} = a b^5 w_3 w_1^2 w_4 w_5 b^{-1} (w_3 a)^3 \]
\[ w_{23} = w_3 b^3 w_5 b^2 a b^{-1} b_3 a^2 \]
\[ w_{24} = (w_4 b^2)^2 w_3 a b b^6 a (w_3 b)^2 b^2 \]
\[ w_{25} = (b a^2)^2 w_7 (w_3)^2 b^2 b^2 (w_6 a)^6 b^2 \]
\[ w_{26} = a b w_2 a^2 (w_2 b)^2 a b^5 w_3 (w_1)^2 \]
\[ w_{27} = w_4 w_5 b^{-1} \]
\[ y_1 = (w_2 a)^2 w_2 b^3 b^2 a (w_3 b)^2 w_6 \]
\[ y_2 = b (w_2 a)^3 b w_6 a b^3 b^2 w_3 a b^{-1} b \]
\[ w_{28} = ((w_3 b)^2 a)^3 w_2 b^3 w_1 b^3 w_3 a^{-1} y y_1 \]
\[ w_{29} = (w_3 a)^3 a^3 b w_7 b a b (b a^{-1})^2 \]
\[ w_{30} = w_4 (w_3 b)^2 a b w_2 b^3 w_2 a (w_3 a)^2 \]
\[ w_{31} = a^2 b (b a^{-1}) (b a)^2 \]
\[ w_{32} = a^2 b (b a b)^2 w_2 b^2 a^3 (w_3 b a)^2 \]
\[ w_{33} = (w_4 b)^3 a^{-1} b (w_3 b a)^2 b w_2 b (b a^2)^2 \]
\[ w_{34} = (w_3 b)^3 a^{-1} b a b^3 a (w_4 b)^2 \]
\[ w_{35} = (w_3 b a)^2 b a (w_3 b a)^2 \]
\[ w_{36} = w_3 b^2 a^2 (w_6 a)^6 b^2 a (w_3 a)^2 \]
\[ w_{37} = a^{-1} (b a^2 b)^2 a b^5 w_3 (w_4 b)^3 \]
\[ w_{38} = (w_5 b)^2 w_5 b a^3 (w_4 b)^2 \]
\[ w_{39} = b^3 a^6 (a w_2 b^2)^2 (b a^2)^2 w_7 \]
\[ w_{40} = (w_3 b a)^2 b^2 (w_4 b)^2 a (w_3 a)^2 \]
\[ w_{41} = (a b a)^2 \]
\[ w_{42} = (w_3 b)^2 a b^5 w_3 (w_4 b)^3 (w_3 a)^2 b a \]
\[ w_{43} = \]
\[ w_{44} = (w_3 a)^2 a b w_2 b^3 w_2 w_3 a b^{-1} b a^{-1} w_4 \]
\[ y_3 = w_8 w_9 w_{10} w_{11} w_{12} w_{13} w_{14} w_{15} w_{16} w_{17} \]
\[ y_4 = w_{18} w_{19} w_{20} w_{21} w_{22} w_{23} w_{24} w_{25} w_{26} w_{27} w_{28} w_{29} \]
\[ y_5 = w_{30} w_{31} w_{32} w_{33} w_{34} w_{35} w_{36} w_{37} w_{38} w_{39} w_{40} w_{41} \]
\[ y_6 = w_{42} w_{43} w_{44} \]
\[ y_7 = y_3 y_4 y_5 y_6 \]

The words for $(A_6 \times A_6) : D_8$ are $f_1, f_2$ and $y_7$ and these three generators can be converted into the two generators given below.

\[ f_1 y_7, f_2 y_7 \]

We use an orbit shape to search for a conjugate of the subgroup we just found to reduce the word length of the
generators. Thus we have

\[(A_6 \times A_6) : D_8 = \langle a, d \rangle,\]

where \(d = ((ab)^2a^2(ba)^6(ab)^6a((ba)^3)^9)^5\).

### 2.1.2 Generators of \(2^{3+2+6}.(3 \times L_3(2))\)

Here our required subgroup is the \(N_{HN}(2B^3)\) [26]. From [42] we know that there are two classes of \(2B\)-pure subgroups inside \(HN\). The first class is generated by the center and any other \(2B\)-involutions such that these two involutions are taken from the extra special group \(2^{1+8}\) inside the centralizer of \(2B\)-involutions [42].

The group \(2^{1+8}\) can be constructed by finding the centralizer of a \(2B\) element. Then inside this centralizer, search for elements of order 4, 8, 12, 16, 24 or 32. Then power up these elements to obtain involutions which generate \(2^{1+8}\). Now searching inside \(2^{1+8}\), one can easily find a \(2B^3\). The details of computing the \(2B^3\) are given below:

The element of \(2B\) is given by \(a_1 = ((ba)^6b(ba)^3b(ba)^6b^2ab^2)^6\). The generators of centralizer of \(a_1\) inside \(HN\) are given by: \(b_1 = [a_1, a]\)

\[b_2 = [a_1, b]^{10}\]

\[b_3 = ab[a_1, ab]^5\]

\[b_4 = (ab^2aba) \left[ a_1, ab^2aba \right] \]

\[b_5 = ab^2(ab)^3 \left[ a_1, ab^2(ab)^3a \right]^{12}\]

\[b_6 = \left( b \ (ab)^2 \ (ab)^2 \ bab^2ab \right) \left[ a_1, b \ (ab)^2 \ (ab)^2 \ bab^2ab \right]^7\]

Thus generators of \(2B^3\) are \(b_7 = b_6^{10}\), \(b_8 = (b_1b_2b_3)^8\) and \(b_9 = (b_1b_2b_3)^6\). The generators for normalizer of \(2B^3\) inside \(HN\) are given below.

\[c_1 = (ab)^2 \left[ b_{29}, (ab)^2 \right]^7\]

\[c_2 = \left[ b_{29}, (ab)^2a \right]^7\]

\[d_1 = [b_{36}, a]^2\]

\[d_2 = [b_{36}, b]^{11}\]

\[d_3 = (ab) \left[ b_{34}, ab \right]^7\]

The generators for \(2^{3+2+6}.(3 \times L_3(2))\) are:

\[k_1 = (c_6c_3)^6c_2^2c_6c_2^2\]

\[k_2 = (d_1d_3)^3 \left( d_1d_3 \right)^2.\]

is in the following chain of subgroups.

\[3^4 < 3^{1+4} : 4A_5 < HN.\]

The generators for \(3^{1+4} : 4A_5\) which were copied from [1] are given below.

\[c_1 = (ab)^9 \left( (ab)^3b \right)^{10} (ab)^9\]

\[d_1 = (ab)^{10}b(ab)^8\]

Now we give some random elements of \(3^{1+4} : 4A_5\).

\[e_2 = (c_1d_1)^2 \ d_1c_1d_1\]

\[b_1 = c_1e_2c_1e_3^2(c_1e_7)^2c_1e_7^1\]

\[b_2 = c_1e_2c_1e_3^4c_1e_7^2 \ \left( c_1e_2^4 \right)^2\]

\[b_3 = c_1e_2c_1e_3^2c_1e_7 c_1e_3^3\]

\[b_4 = c_1e_2c_1e_3^2c_1e_7^1c_1e_3^3c_1e_3^2\]

The generators of \(3^4\) are \(b_1, b_2, b_3\) and \(b_5\). Before computing the normalizer we give some random elements of \(3^{1+4} : 4A_5\).

\[c_2 = c_1d_1\]

\[c_3 = ab_1\]

\[k_3 = (ac_3)^4c_3^4ac_3^4\]

\[c_4 = bb_1\]

\[c_5 = b_1c_4\]

\[k_4 = (b_1c_3^2b_1c_6^2)^2b_1c_5\]

\[c_6 = k_3k_4\]

\[k_1 = (c_1c_2)^5\]

\[k_2 = c_1c_2c_1c_2^2c_1c_2^2c_1c_2^2c_1c_2^2\]

\[k_5 = (k_3c_6)^6c_6k_3c_6\]

The generators of the normalizer of \(3^4\) inside \(HN\) are given below.

\[k_1 = (c_1c_2)^5\]

\[k_2 = c_1c_2c_1c_2^2c_1c_2^2c_1c_2^2c_1c_2^4\]

\[k_5 = (k_3c_6)^6c_6k_3c_6\]

Thus it turns out that \(3^4 : 2.(A_4 \times A_4).4 = \langle k_2, k_1k_5 \rangle.\)

The orbit shapes and order of the above subgroups of \(HN\) are given below.

### 2.1.3 Generators of \(3^4 : 2.(A_4 \times A_4).4\)

Following [26], we see that the required subgroup is the normalizer of \(3^4\) inside \(HN\). It turns out that \(3^4\) we seek
2.2 Linear Groups

In this section we provide words for the maximal subgroups of $L_2(8)$, $L_2(8) : 2$, $L_2(13)$, $L_2(13) : 2$, $L_2(16)$, $L_2(17)$, $L_2(17) : 2$, $L_2(19)$, $L_2(19) : 2$, $L_2(23)$, $L_2(29)$, $L_2(31)$, $L_3(3)$, $L_3(3) : 2$, $L_5(5)$ ideally the words should be as short as possible.

Most often, the subgroups have been generated by two elements by using random searching in [44]. This method is quite successful if one of the short words is $a$ and is in a very small conjugacy class. One can then search by generating subgroups using those short words. The generators for the maximal subgroup $2^3 : 7$ of $L_2(8)$, $9 : 6$ and $L_2(8)$ of the group $L_2(8)$ : 2, $D_{14}$ of the group $L_2(13)$, $D_{28}$ of the group $L_2(13) : 2$, $D_{32}$ and $D_{16}$ of the group $L_2(17)$ : 2, $D_{20}$ of the group $L_2(19)$, $D_{16}$ and $19 : 18$ of the group $L_2(19) : 2$, $D_{24}$ of the group $L_2(23)$, $A_5$ of the group $L_2(29)$, $D_{30}$, $D_{32}$ and $S_4$ of the group $L_2(31)$, $3^2 : 2S_4$ of the group $L_3(3)$ are computed by random searching. There are still some hard cases in which this method is not of much use.

The next method depends on the information given in Atlas of Finite Simple Groups [26]. Following Atlas, the maximal subgroup $31 : 15$ of $L_2(31)$ is computed by taking the normalizer of $31A B$ i.e., $N(31AB) = 31 : 15$. Similarly $N(2A) = D_{12}$ and $N(3A) = D_{30}$. The normalizer here is computed by the methods given in [43] and the programmes given by simon [23] with a little change in them.

### Table 1: Maximal subgroups of $L_2(8)$.

| SubGroups | 1st generator | 2nd generator |
|-----------|---------------|---------------|
| $D_{18}$  | $b^4(ab)^{-1}(ab)^{-1}aba$ | $(ba)^2(b^3a)^2b^{-1}a$ |
| $D_{16}$  | $ab$          | $(ba^{-1}ab)^{-1}aba$ |
| $2^3 : 7$ | $a$           | $(ab(ab)^{-1})^2a$ |

Following Atlas, the subgroup $D_{18}$ is the normalizer of $3A$. i.e., $D_{18} = N(3A)$. Similarly $D_{14} = N(7ABC)$ and $3^3 : 7 = N(2A^2)$.

The subgroups $9 : 6$, $7 : 6$ and $L_2(8)$ were computed by random searching, while $2^3 : 7$ is computed by the information given in Atlas [34], i.e., $3^3 : 7 : 3 = N(2A^2)$.

Following Atlas, $D_{14} = N(7ABC)$, $D_{12} = N(2A)$, $A_6 = N(2A^2)$ and $13 : 6 = N(13AB)$.

### Table 2: Maximal subgroups of $L_2(8) : 2$.

| SubGroups | 1st generator | 2nd generator |
|-----------|---------------|---------------|
| $D_{14}$  | $a$           | $(ab)^2b$    |
| $D_{12}$  | $(bab^{-1})^2(ba)^2$ | $(ab)^{-1}aba$ |
| $A_{12}$  | $(bab^{-1})^2(ba)^2$ | $(b^{-1}abab^{-1})^2a$ |
| $13 : 6$  | $ab$          | $b(ab)^{-1}aba$ |

The maximal subgroup $D_{28}$ is computed by random searching, while the remaining four were constructed by the information given in Atlas $D_{28} = C(2B)$, $D_{24} = N(3A)$ and $S_4 = N(2A^2)$.

### Table 3: Maximal subgroups of $L_2(13)$.

| SubGroups | 1st generator | 2nd generator |
|-----------|---------------|---------------|
| $S_4$     | $ba$          | $(ba)^3(b^{-1}a)^{-1}b$ |
| $D_{28}$  | $a$           | $(ab)^{-1}aba$ |
| $D_{26}$  | $b^{-1}a(bab^{-1}ab)^{-2}ab^2a$ | $(bab^{-1})^3(ab)^2a$ |
| $L_{2}(13)$ | $bab^{-1}$  | $bab(ab)^{-1}(ab)^2b$ |

The maximal subgroup $A_5$ is computed by random searching, while the remaining maximal subgroups were constructed by the information given in Atlas $D_{34} = N(17A - H)$, $D_{30} = N(3A)$ and $2^4 : 15 = N(2A^4)$.

### Table 4: Maximal subgroups of $L_2(16)$.

| SubGroups | 1st generator | 2nd generator |
|-----------|---------------|---------------|
| $D_{30}$  | $ab$          | $(ab^{-1})(ab)^{-1}aba$ |
| $D_{34}$  | $abab^{-1}$  | $(bab^{-1})^2bab^{-1}$ |
| $A_5$     | $b$           | $(bab^{-1})^2bab^{-1}$ |
| $2^4 : 15$ | $ab$          | $(bab^{-1})^4bab^{-1}$ |

### Table 5: Maximal subgroups of $L_2(17)$.

| SubGroups | 1st generator | 2nd generator |
|-----------|---------------|---------------|
| $D_{18}$  | $(bab^{-1})^2bab^{-1}$ | $(bab^{-1})^2b$ |
| $D_{16}$  | $(bab^{-1})^2(ba)^2$ | $(ba^{-1})^2b^{-1}ab$ |
| $S_4$     | $(ab(ab)^{-1})^2ab$ | $(ab(ab)^{-1})^2bab^{-1}$ |

The maximal subgroup $A_5$ is computed by random searching, while the remaining maximal subgroups were constructed by the information given in Atlas $D_{34} = N(17A - H)$, $D_{30} = N(3A)$ and $2^4 : 15 = N(2A^4)$.
The maximal subgroups were computed by the information given in Atlas, i.e., $17 : 8 = N(17AB)$, $S_4 = N(2A^2)$, $D_{18} = 3A$ and $D_{16} = N(2A)$.

Table 7: Maximal subgroups of $L_2(17) : 2$.

| SubGroups | 1st generator | 2nd generator |
|-----------|---------------|---------------|
| $D_{32}$  | $a$           | $((ab)^h b)^a$ |
| $D_{36}$  | $a$           | $((ba)^h b^{-1}a^2 b$ |
| $L_2(17)$ | $b$           | $(abab)^3$    |
|           | $17 : 16$     | $(ab)^6 ab^{-1} ab^{-1}$ |
|           |               | $(b^{-1})^3 (ba)^3 b^{-1} ab^{-1}$ |

The maximal subgroups $D_{32}$, $L_2 17$ and $D_{36}$ were computed by random searching, while the subgroup $17 : 16$ is constructed by the information given in Atlas i.e., $17 : 16 = N(17AB)$.

Table 8: Maximal subgroups of $L_2(19)$.

| SubGroups | 1st generator | 2nd generator |
|-----------|---------------|---------------|
| $19 : 9$  | $ab$          | $baba^{-1}(ab)^2 (abab)^{-1} (ab^{-1})^2$ |
| $D_{20}$  | $a$           | $(ab)^{-1} (ab)^3 ab^{-1}$ |
| $D_{18}$  | $abab^{-1}$   | $(abab)^{-1} a$ |
| $A_5$     | $b$           | $b(abab)^{-1} (ab)^3 ab^{-1}$ |

Following Atlas, $19 : 9 = N(19AB)$, $A_5 = N(2A, 3A, 5AB)$, $D_{20} = N(2A)$ and $D_{18} = N(3A)$.

Table 9: Maximal subgroups of $L_2(19) : 2$.

| SubGroups | 1st generator | 2nd generator |
|-----------|---------------|---------------|
| $L_2(19)$ | $b$           | $(ab)^2$     |
| $D_{40}$  | $ab$          | $(ab)(ab^{-1})^2 (ab^{-1})^2 ab^{-1} (ab)^2 a$ |
| $D_{16}$  | $a$           | $(ab)^2 (b^{-1})^2 (abab)^{-1} aba$ |
| $S_4$     | $b$           | $b^{-1} ab^{-1} (ab^{-1} b)^3 (ab)^3$ |
|           | $19 : 18$     | $(ab^{-1} (ab)^2) a baba$ |

The maximal subgroups $L_2(19)$, $D_{36}$, $S_4$ and $19 : 18$ were computed by random searching, while the subgroup $D_{40}$ is constructed by the information given in Atlas i.e., $D_{40} = N(5AB)$.

Table 10: Maximal subgroups of $L_2 (23)$.

| SubGroups | 1st generator | 2nd generator |
|-----------|---------------|---------------|
| $23 : 11$ | $ab$          | $baba^{-1} (abab)^{-1} (ab)^2 (ab)^2 a$ |
| $D_{24}$  | $a$           | $(ab)^{-1} a baba^{-1}$ |
| $S_4$     | $(ba)^3 (b^{-1} a)^2 (abab)^{-1} (ba)^{-1} a$ |
| $D_{22}$  | $(ab)^2 (abab)^{-1} (b^{-1} a)^2 (baba)^3 (ab)^{-1} a$ |

The maximal subgroups $D_{22}$, $D_{24}$, $S_4$ and $23 : 11$ was constructed by the information given in Atlas i.e., $D_{22} = C(2B)$, $D_{24} = N(2A)$, $S_4 = N(2A^2)$ and $23 : 11 = N(23AB)$.

Table 11: Maximal subgroups of $L_2 (29)$.

| SubGroups | 1st generator | 2nd generator |
|-----------|---------------|---------------|
| $29 : 14$ | $ab$          | $(ba)^2 (b^{-1} a)^2 (ba)^2 (abab)^{-1} (ab)^2 a$ |
| $D_{28}$  | $(ba)^2 (b^{-1} a)^2 (ba)^2 (abab)^{-1} (ab)^2$ |
| $D_{30}$  | $((ab)^2 (ab^{-1})^2 )^2 a$ |
| $A_5$     | $a$           | $b^{-1} ab^{-1} (ab)^{-1} ab^{-1} (ab)^2$ |

The maximal subgroup $A_5$ is computed by random searching, while the subgroups $D_{28}$, $D_{30}$ and $29 : 14$ were constructed by the information given in Atlas i.e., $D_{28} = N(2A)$, $D_{30} = N(3A)$ and $29 : 14 = N(29AB)$.

Table 12: Maximal subgroups of $L_2 (31)$.

| SubGroups | 1st generator | 2nd generator |
|-----------|---------------|---------------|
| $31 : 15$ | $ab$          | $ab^{-1} ab (ab)^{-1} (ab)^{-1} (ab)^2$ |
| $D_{30}$  | $a$           | $bab^{-1}$ |
| $D_{32}$  | $a$           | $b^{-1} (abab)^{-1} a (ab)^{-1} a baba$ |
| $A_5$     | $b$           | $b^{-1} (abab)^{-1} a (ab)^{-1} a baba$ |
| $S_4$     | $a$           | $b^{-1} (abab)^{-1} a (ab)^{-1} a baba$ |

The maximal subgroups $D_{30}$, $D_{32}$ and $S_4$ were computed by random searching, while the subgroup $A_5$, and $31 : 15$ were constructed by the information given in Atlas i.e., $A_5 = N(2A, 3A, 5AB)$, and $31 : 15 = N(31AB)$.

Table 13: Maximal subgroups of $L_3 (3)$.

| SubGroups | 1st generator | 2nd generator |
|-----------|---------------|---------------|
| $3^2 : 2S_4$ | $a$           | $(ab)^2$    |
| $S_6$     | $abab^{-1} (abab)^{-1} a (abab)^{-1} (ab)^{-1} a$ |
|           | $13 : 3$      | $ab$         |
|           |                | $(ba)^2 (b^{-1} a)^2 (ab)^{-1} a baba$ |

The maximal subgroups $3^2 : 2S_4$, $S_6$ and $13 : 3$ were constructed by the information given in Atlas i.e., $3^2 : 2S_4 = N(3A^2)$, $S_6 = N(2A^2)$ and $13 : 3 = N(13ABCD)$.

Table 14: Maximal subgroups of $L_3 (3) : 2$.

| SubGroups | 1st generator | 2nd generator |
|-----------|---------------|---------------|
| $L_3 (3)$ | $(b^{-1} a b a)^2$ | $(ab)^2 (ab)^{-1} (abab)^{-1} (ab)^{-1} a$ |
| $S_4 : 2$ | $(b^{-1} a b a)^2 (b^{-1} a b a)^2 (abab)^{-1} a$ |
| $2S_{1,2}$ | $(ab)^3 a b^2 (abab)^{-1} a baba$ |
| $3^2 : 2S_4$ | $a (abab)^{-1} a baba$ |
| $D_8$     | $(ab)^2 (b^{-1} a b a)^2 (ab)^{-1} a baba$ |
| $13 : 6$  | $(ab)^3 a (ab)^{-1} a baba$ |
|           | $(ab)^2 (b^{-1} a b a)^2 (ab)^{-1} a baba$ |
The maximal subgroups $L_3(3)$, $2S_4$, $3^{1+2}.D_8$ and $13 : 6$ were constructed by random searching, while the subgroup $S_4 : 2$ is constructed by the information given in Atlas i.e., $S_4 : 2 = C(2B)$.

Table 15: Maximal subgroups of $L_3(5)$.

| SubGroups | 1st generator | 2nd generator |
|-----------|---------------|---------------|
| $5^2 : GL_2(5)$ | $(ab)^2(a(ab))^5$ | $(ab)^3((ba)^2b)(ba)^3((ba)^3b)^5$ |
| $S_5$ | $bab(ba)^7a$ | $b^{-1}aba^{-1}b^{-1}a^{-1}(bab)^2(ab^{-1})^2b^{-3}a$ |
| $4^2 : S_3$ | $(ab)^2a(a(ab)^5$ | $(ab)^3((ba)^2b)(ba)^3((ba)^3b)^5$ |
| $S_5$ | $(ab)^2a(ab)^5$ | $(ab)^3((ba)^2b)(ba)^3((ba)^3b)^5$ |

The maximal subgroups $5^2 : GL_2(5)$, $S_5$, and $4^2 : S_3$ were constructed by the information given in Atlas i.e., $5^2 : GL_2(5) = N(5A^2)$, $S_5 = N(2A, 3A, 5AB)$, and $4^2 : S_3 = N(2A^2)$.

3 Conclusions

Mathematical tools help to solve many problems arising in chemistry and other areas of sciences [45–60], for example, graph theory help us to know about structural and physico-chemical properties of chemical compounds without using wet labs [50–54]. The finite groups are helpful in studying symmetry of molecules because almost all organic and inorganic compounds are symmetric about its center [56–60]. Our aim is to study some finite groups of higher order and find their words. In this paper we provide generators for the maximal subgroups of Harada-Norton and some linear groups. In the world-wide-web Atlas of Group Representations [1], there is only one copy of $S_5$ in the list of maximal subgroups of $L_3(5)$, but here we provide generators for two non-conjugate copies of $S_5$.

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