AN ANISOTROPIC INVERSE MEAN CURVATURE FLOW FOR
SPACELIKE GRAPHIC CURVES IN LORENTZ-MINKOWSKI PLANE $\mathbb{R}^2_1$

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Abstract. In this paper, we consider the evolution of spacelike graphic curves defined over a piece of hyperbola $\mathcal{H}^1(1)$, of center at origin and radius 1, in the 2-dimensional Lorentz-Minkowski plane $\mathbb{R}^2_1$ along an anisotropic inverse mean curvature flow with the vanishing Neumann boundary condition, and prove that this flow exists for all the time. Moreover, we can show that, after suitable rescaling, the evolving spacelike graphic curves converge smoothly to a piece of hyperbola of center at origin and prescribed radius, which actually corresponds to a constant function defined over the piece of $\mathcal{H}^1(1)$, as time tends to infinity.

Keywords: Anisotropic inverse mean curvature flow, spacelike curves, Lorentz-Minkowski space, Neumann boundary condition.

MSC 2020: Primary 53E10, Secondary 35K10.

1. Introduction

Throughout this paper, let $\mathbb{R}^2_1$ be the 2-dimensional Lorentz-Minkowski space with the following Lorentzian metric

$$\langle \cdot, \cdot \rangle_L = dx_1^2 - dx_2^2.$$  

In fact, $\mathbb{R}^2_1$ is an 2-dimensional Lorentz manifold with index 1. Denote by 

$$\mathcal{H}^1(1) = \{ (x_1, x_2) \in \mathbb{R}^2_1 \mid x_1^2 - x_2^2 = -1 \text{ and } x_2 > 0 \},$$

which is exactly the hyperbola of center $(0,0)$ (i.e., the origin of $\mathbb{R}^2$) and radius 1 in $\mathbb{R}^2$. Clearly, from the Euclidean viewpoint, $\mathcal{H}^1(1)$ is one component of a hyperbola of two arms.

In this paper, we consider the evolution of spacelike curves (contained in a prescribed convex sector domain) along an anisotropic inverse mean curvature flow (IMCF for short), and can prove the following main conclusion.

Theorem 1.1. Let $\alpha < 0$, $M \subset \mathcal{H}^1(1)$ be some convex curve segment of the hyperbola $\mathcal{H}^1(1) \subset \mathbb{R}^2_1$, and $\Sigma := \{ rx \in \mathbb{R}^2_1 \mid r > 0, x \in \partial M \}$. Let $X_0 : M \to \mathbb{R}^2_1$ such that $M_0 := X_0(M)$ is a compact, strictly convex spacelike $C^{2,\gamma}$-curve ($0 < \gamma < 1$) which can be written as a graph over $M$. Assume that

$$M_0 = \text{graph}_M u_0$$

is a graph over $M$ for a positive map $u_0 : M \to \mathbb{R}$ and

$$\partial M_0 \subset \Sigma, \quad \langle \mu \circ X_0, \nu_0 \circ X_0 \rangle_L \mid_{\partial M} = 0,$$

where $\nu_0$ is the past-directed timelike unit normal vector of $M_0$, $\mu$ is a spacelike vector field defined along $\Sigma \cap \partial M = \partial M$ satisfying the following property:

- For any $x \in \partial M$, $\mu(x) \in T_x M$, and moreover, $\mu(x) = \mu(rx)$.

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Then we have:
(i) There exists a family of strictly convex spacelike curves \( M_t \) given by the unique embedding
\[
X \in C^{2+\gamma,1+\gamma/2}(M \times [0, \infty), \mathbb{R}^2_1) \cap C^\infty(M \times (0, \infty), \mathbb{R}^2_1)
\]
with \( X(\partial M, t) \subset \Sigma \) for \( t \geq 0 \), satisfying the following system
\[
\begin{aligned}
\frac{\partial}{\partial t} X &= \frac{1}{|X|^\alpha} \kappa' \\
\langle \mu \circ X, \nu \circ X \rangle_L &= 0 \\
X(\cdot, 0) &= M_0
\end{aligned}
\]
where \( k \) is the curvature of \( M_t := X(M, t) = X_t(M) \), \( \nu \) is the past-directed timelike unit normal vector of \( M_t \), and \( |X| := |\langle X, X \rangle_L|^{1/2} \) is the norm of \( X \) induced by the Lorentzian metric of \( \mathbb{R}^2_1 \).
(ii) The leaves \( M_t \) are spacelike graphs over \( M \), i.e.,
\[
M_t = \text{graph}_M u(x, t).
\]
(iii) Moreover, the evolving spacelike curves converge smoothly after rescaling to a piece of \( \mathcal{H}^1(r_\infty) \), where \( r_\infty \) satisfies
\[
(1.2) \quad \frac{1}{\sup_M u_0} \cdot \frac{\mathcal{L}(M_0)}{\mathcal{L}(M)} \leq r_\infty \leq \frac{1}{\inf_M u_0} \cdot \frac{\mathcal{L}(M_0)}{\mathcal{L}(M)},
\]
where \( \mathcal{H}^1(r_\infty) := \{ r_\infty x \in \mathbb{R}^2_1 | x \in \mathcal{H}^1(1) \} \), \( \mathcal{L}(M) \) and \( \mathcal{L}(M_0) \) stand for the length of spacelike curves \( M, M_0 \) respectively.

**Remark 1.1.**
(1) This work was firstly announced by us in a previous work \[5\], which actually corresponds to the higher dimensional case of the work here. Besides, we also mentioned this work in series works \[3, 6\] later. As pointed out in \[6\], Section 1, our work here can be seen as the anisotropic version of the lower dimensional case of the IMCF in \[4\] and the inverse Gauss curvature flow (IGCF for short) in \[6\] simultaneously. Of course, the IMCF in \[4\] and the IGCF in \[6\] should be imposed the zero Neumann boundary condition (NBC for short). Based on this fact and the experience on the study of the anisotropic IMCF with zero NBC in the \( (n + 1) \)-dimensional Lorentz-Minkowski space \( \mathbb{R}^{n+1}_1 \) (both the lower dimensional and the higher dimensional cases), we proposed\[4\] in \[6\], Remark 1.1 that one can also consider the anisotropic version of the IGCF with zero NBC in \( \mathbb{R}^{n+1}_1, n \geq 2 \).
(2) As explained clearly and mentioned in \[3\], Remark 1.1], we prefer to use López’s setting introduced in \[13\] to deal with the geometric quantities of spacelike curves in \( \mathbb{R}^2_1 \) for the purpose of convenience. One can have a glance at this setting through the computation of curvature of spacelike graphic curves in \( \mathbb{R}^2_1 \) (defined over \( M \subset \mathcal{H}^1(1) \)) shown in the proof of Lemma 2.1 below.
(3) In fact, in Theorem \[11\], \( M \) is some convex curve segment of the spacelike curve \( \mathcal{H}^1(1) \) implies that the curvature of \( M \) is positive everywhere w.r.t. the vector field \( \mu \) (provided its direction is suitably chosen).
(4) Of course, the notation \( T_x M \) means the tangent space at the point \( x \in \partial M \), which in the situation here is a 1-dimensional vector space diffeomorphic to the Euclidean 1-space \( \mathbb{R} \). In order to clearly comprehend the property (for the vector field \( \mu \)) required in Theorem \[11\], we suggest readers to check the 3rd footnote given in our previous work \[4\], where the detailed explanation can be found.

\[1\] Using a similar analytical technique introduced in \[6\], the long-time existence and related asymptotical behavior of the anisotropic version of the IGCF with zero NBC in \( \mathbb{R}^{n+1}_1 (n \geq 2) \) can be expected. As we said in \[6\], Remark 1.1], we left this as an exercise for readers who are interested in this topic.
(5) It is easy to check that all the arguments in the sequel are still valid for the case \( \alpha = 0 \) except some minor changes should be made. For instance, if \( \alpha = 0 \), then the expression (3.1) below becomes \( \varphi(t) = -t + c \). However, in this setting, one can also get the \( C^0 \) estimate as well. Therefore, Theorem 1.1 also holds for \( \alpha = 0 \), and in this situation, the homogeneous anisotropic factor \( |X|^{-\alpha} \) equals 1, and consequently the flow in Theorem 1.1 degenerates into the classical IMCF with zero NBC in \( \mathbb{R}_1^2 \). This phenomenon also happens in the higher dimensional case – see Remark 1.1 for details.

(6) As shown in Lemma 2.1 below, one can use a single parameter \( \xi \) to describe any point \( x \in M \subset \mathcal{H}^1(1) \), and moreover, under this parametrization, the component of the Riemannian metric on \( \mathcal{H}^1(1) \) is \( \sigma_{\xi\xi} = g_{\mathcal{H}^1(1)}(\partial_{\xi}, \partial_{\xi}) = 1 \). Without loss of generality, there should exist an interval \([c, d]\) such that \( x(\xi), \xi \in [c, d], \) runs over the whole curve segment \( M \) and \( \Sigma \cap \partial M = \{x(c), x(d)\} \). Furthermore, the lengths \( \mathcal{L}(M) \) and \( \mathcal{L}(M_0) \) can be computed as follows:

\[
\mathcal{L}(M) = \int_M d\mathcal{H}^1 = \int_c^d d\xi = d - c,
\]

with \( \mathcal{H}^1(\cdot) \) the 1-dimensional Hausdorff measure of a prescribed Riemannian curve, and

\[
\mathcal{L}(M_0) = \int_{M_0} d\mathcal{H}^1 = \int_c^d \sqrt{u_0^2(\xi) - |Du_0(\xi)|^2} d\xi,
\]

where as in Lemma 2.1 \( D \) is the covariant connection on \( \mathcal{H}^1(1) \). Then the estimate for the radius \( r_\infty \) in (iii) of Theorem 1.1 becomes

\[
\frac{1}{\sup_M u_0} \frac{\int_c^d \sqrt{u_0^2(\xi) - |Du_0(\xi)|^2} d\xi}{d - c} \leq r_\infty \leq \frac{1}{\inf_M u_0} \frac{\int_c^d \sqrt{u_0^2(\xi) - |Du_0(\xi)|^2} d\xi}{d - c}.
\]

BTW, the formula for the component of the induced metric on \( M_0 \) (see (ii) of Lemma 2.1), and the length formula of curves have been used directly in the computation of \( \mathcal{L}(M) \) and \( \mathcal{L}(M_0) \).

This paper is organized as follows. In Section 2, we will prove the short-time existence of the flow discussed in Theorem 1.1 (i.e., the IMCF with zero NBC in \( \mathbb{R}_1^2 \)). In Section 3, several estimates, including \( C^0 \), time-derivative and gradient estimates, of solutions to the flow equation will be shown in details. Estimates of higher-order derivatives of solutions to the flow equation, which naturally leads to the long-time existence of the flow, will be investigated in Section 4. In the end, we will clearly show the convergence of the rescaled flow in Section 5.

2. The scalar version of the flow equation

Since the spacelike \( C^{2,\gamma} \)-curve \( M_0 \) can be written as a graph of \( M \subset \mathcal{H}^1(1) \), there exists a function \( u_0 \in C^{2,\gamma}(M) \) such that \( X_0 : M \to \mathbb{R}_1^2 \) has the form \( x \mapsto G_0 := (x, u_0(x)) \). The curve \( M_t \) given by the embedding

\[
X(\cdot, t) : M \to \mathbb{R}_1^2,
\]
at time \( t \) may be represented as a graph over \( M \subset \mathcal{H}^1(1) \), and then we can make ansatz

\[
X(x, t) = (x, u(x, t))
\]
for some function \( u : M \times [0, T) \to \mathbb{R} \). The following formulae are needed.

Lemma 2.1. Define \( p := X(x, t) \) and assume that a point on \( \mathcal{H}^1(1) \) is parameterized by the coordinate \( \xi \), that is, \( x = x(\xi) \). By the abuse of notations, let \( \partial_{\xi} \) be the corresponding coordinate field on \( \mathcal{H}^1(1) \) and \( \sigma_{\xi\xi} = g_{\mathcal{H}^1(1)}(\partial_{\xi}, \partial_{\xi}) = 1 \) be the Riemannian metric on \( \mathcal{H}^1(1) \). Denote by \( u_{\xi} := D_{\partial_{\xi}} u \), and \( u_{\xi\xi} := D_{\partial_{\xi}} D_{\partial_{\xi}} u \) the covariant derivatives of \( u \) w.r.t. the metric \( g_{\mathcal{H}^1(1)} \), where \( D \) is the covariant connection on \( \mathcal{H}^1(1) \). Let \( \nabla \) be the Levi-Civita connection of \( M_t \) w.r.t. the
metric \( g := u^2 g_{\mathbb{R}^1} - dr^2 \) induced from the Lorentzian metric \( \langle \cdot, \cdot \rangle_L \) of \( \mathbb{R}_1^2 \). The following formulae hold:

(i) The tangential vector on \( M_t \) is

\[
X_\xi = \partial_\xi + u_\xi \partial_r,
\]
and the corresponding past-directed timelike unit normal vector is given by

\[
\nu = -\frac{1}{u} \left( \frac{u_\xi}{u^2} \partial_\xi + \partial_r \right),
\]
where \( u^\xi = \sigma^\xi_\xi u_\xi = u_\xi, \ |Du|^2 = u_\xi u^\xi = |u_\xi|^2, \) and \( v = \sqrt{1 - u^{-2} |Du|^2} \).

(ii) The induced metric \( g \) on \( M_t \) has the form

\[
g_{\xi\xi} = u^2 \sigma^\xi_\xi - u_\xi^2 = u^2 - u_\xi^2,
\]
and its inverse is given by

\[
g^{\xi\xi} = \frac{1}{u^2} \left( \sigma^{\xi\xi} + \frac{u_\xi^2}{u^2 v^2} \right) = \frac{1}{u^2} \left( 1 + \frac{u_\xi^2}{u^2 v^2} \right).
\]

(iii) The curvature of \( M_t \) is given by

\[
k = \frac{u_\xi u_\xi + u^2 - 2 u_\xi^2}{v^2 u^2}.
\]

(iv) Let \( p = X(x, t) \in \Sigma \) with \( x \in \partial M, \ \hat{\mu}(p) \in T_p M_t, \ \mu = \mu^\xi(x) \partial_\xi(x) \) at \( x \), with \( \partial_\xi \) the basis vector of \( T_x M \). Then

\[
\langle \hat{\mu}(p), \nu(p) \rangle_L = 0 \iff \mu^\xi(x) u_\xi(x, t) = 0.
\]

Proof. The proof (except the calculation of curvature of spacelike graphic curves in \( \mathbb{R}_1^2 \)) is very similar to that of [4, Lemma 3.1], and we prefer to omit this part.

As mentioned in (2) of Remark 1.1, now, we prefer to use López’s setting introduced in [13] to compute the curvature. Clearly, the tangential vector on \( M_t \) can be rewritten as

\[
X_\xi = (1, u_\xi),
\]
the Minkowski arc-length parameter is defined as

\[
ds = \sqrt{|\langle X_\xi, X_\xi \rangle_L|} \, d\xi,
\]
and the unit tangent vector is

\[
T = X_s := \frac{1}{\sqrt{|\langle X_\xi, X_\xi \rangle_L|}} X_\xi.
\]
Hence, we have

\[
T := (T^1, T^2) = \frac{1}{\sqrt{u^2 - u_\xi^2}} (1, u_\xi),
\]
which implies

\[
\nabla_{X_\xi} T = T_{X_\xi} := T^1_\xi \partial_\xi + T^2_\xi \partial_r.
\]
so we get

\[ T_\xi = \left( \frac{uu_\xi u_{\xi\xi}^2 + u_\xi u^2 - 2u_\xi^3}{u(u^2 - u_\xi^2)^2}, \frac{u^2 u_{\xi\xi} + u^3 - 2u_\xi^2 u}{(u^2 - u_\xi^2)^2} \right), \]

and

\[ T_s = T_\xi \frac{d\xi}{ds} = \left( \frac{uu_\xi u_{\xi\xi} + u_\xi u^2 - 2u_\xi^3}{u(u^2 - u_\xi^2)^2}, \frac{u^2 u_{\xi\xi} + u^3 - 2u_\xi^2 u}{(u^2 - u_\xi^2)^2} \right). \]

(2.1)

Since \( T \) is spacelike, the curvature \( k \) can be defined as (see [13, pp. 14-16])

\[ k = |T_s| = \sqrt{|\langle T_s, T_s \rangle_L|}, \]

which, together with (2.1), gives

\[ k = \left| \frac{uu_\xi u_{\xi\xi} + u^2 - 2u_\xi^2}{u^3 v^3} \right|. \]

Without loss of generality, assume that \( u_{\xi\xi} + u^2 - 2u_\xi^2 > 0 \), and then

\[ k = |T_s| = \frac{uu_\xi u_{\xi\xi} + u^2 - 2u_\xi^2}{u^3 v^3}, \]

Choosing the unit normal \( N \) as follows

\[ N = \frac{u}{\sqrt{u^2 - u_\xi^2}} \left( \frac{u_\xi}{u^2}, 1 \right), \]

which is future-directed, and clearly it satisfies

\[ T_s = kN. \]

The above equation is actually the Frenet formula for spacelike curves \( M_t \) in \( \mathbb{R}^3_1 \). The proof is finished. \( \square \)

Using techniques as in Ecker [2] (see also [7, 8, 14]), the problem (1.1) can be reduced to solve the following scalar equation with the corresponding initial data and the corresponding NBC

(2.2)

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -\frac{v}{u^\alpha k} & \text{in } M \times (0, \infty) \\
\nabla_\mu u &= 0 & \text{on } \partial M \times (0, \infty) \\
u(., 0) &= u_0 & \text{in } M.
\end{align*}
\]

\[ ^3 \text{Clearly, } k \neq 0, \text{ which implies } u_{\xi\xi} + u^2 - 2u_\xi^2 \neq 0, \text{ otherwise, the IMCF equation in (1.1) would degenerate.} \]

If \( u_{\xi\xi} + u^2 - 2u_\xi^2 < 0 \), one has \( k = \frac{2u_\xi^2 - uu_{\xi\xi} - u^2}{u^3 v^3} \), and then one can choose the future-directed timelike unit normal vector \( N \) such that the flow equation becomes \( \frac{\partial N}{\partial t} = \frac{1}{N^\alpha k} \) \( N \), which is still parabolic. Then similar argument can be made to get the main conclusions in Theorem (1.1).
By Lemma 2.1, define a new function \( \varphi(x,t) = \ln u(x,t) \) and then the curvature can be rewritten as
\[
k = \frac{e^{-\varphi}}{v} \left( 1 + \frac{1}{v^2} \varphi_{xx} \right).
\]
Hence, the evolution equation in (2.2) can be rewritten as
\[
\frac{\partial}{\partial t} \varphi = -e^{-\alpha \varphi} \left( 1 - |D\varphi|^2 \right) \frac{1}{1 + \frac{1}{v^2} \varphi_{xx}} := Q(\varphi, D\varphi, D^2 \varphi).
\]
Thus, the problem (1.1) is again reduced to solve the following scalar equation with the NBC and the initial data
\[
\begin{aligned}
\frac{\partial \varphi}{\partial t} &= Q(\varphi, D\varphi, D^2 \varphi) \quad \text{in } M \times (0,T) \\
\nabla^\mu \varphi &= 0 \quad \text{on } \partial M \times (0,T) \\
\varphi(\cdot, 0) &= \varphi_0 \quad \text{in } M,
\end{aligned}
\]
where
\[
\left( 1 + \frac{1}{v^2} \varphi_{0,xx} \right)
\]
is positive on \( M \), since \( M_0 \) is convex. Clearly, for the initial spacelike graphic curve \( M_0 \),
\[
\left. \frac{\partial Q}{\partial \varphi_{xx}} \right|_{\varphi_0} = \frac{1}{u^{2+\alpha}k^2v^2}
\]
is positive on \( M \). Based on the above facts, as in [7, 8, 14], we can get the following short-time existence and uniqueness for the parabolic system (1.1).

**Lemma 2.2.** Let \( X_0(M) = M_0 \) be as in Theorem 1.1. Then there exist some \( T > 0 \), a unique solution \( u \in C^{2+\gamma,1+\gamma/2}(M \times [0,T]) \cap C^\infty(M \times (0,T]) \), where \( \varphi(x,t) = \log u(x,t) \), to the parabolic system (2.3) with the coefficient
\[
\left( 1 + \frac{1}{v^2} \varphi_{0,xx} \right)
\]
positive on \( M \). Thus there exists a unique map \( \psi : M \times [0,T] \to M \) such that \( \psi(\partial M, t) = \partial M \) and the map \( \tilde{X} \) defined by
\[
\tilde{X} : M \times [0,T] \to \mathbb{R}^2 : (x,t) \mapsto X(\psi(x,t), t)
\]
has the same regularity as stated in Theorem 1.1 and is the unique solution to the parabolic system (1.1).

Let \( T^* \) be the maximal time such that there exists some \( u \in C^{2+\gamma,1+\gamma/2}(M \times [0,T^*)) \cap C^\infty(M \times (0,T^*)) \) which solves (2.3). In the sequel, we shall prove a priori estimates for those admissible solutions on \([0,T]\) where \( T < T^* \).

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4 One might find that our previous works on inverse curvature flows (see, e.g., [1-3, 4, 5, 6, 9]) have used \( \varphi = \log u \) to represent the logarithmic relation between functions \( \varphi \) and \( u \). However, we did not prescribe a base for the logarithmic function \( \log u \). This is because of two reasons. First, in most of non-Chinese mathematical literatures, there is no strict difference between \( y = \log x \) and \( e \)-base logarithmic function \( y = \ln x \), that is, in those literatures, \( y = \log x \) was treated as \( y = \ln x \). Second, in nearly all the analysis process, the non \( e \)-base logarithmic function has almost the same function with the \( e \)-base logarithmic function. Based on these two reasons, there is not necessary to give a base for the logarithmic function when doing changes of variable. Hence, by abuse of notations, we will use \( \log \), in simultaneously, and will give a prescribed base clearly if necessary.
3. $C^0$, $\dot{\varphi}$ and Gradient Estimates

Lemma 3.1 ($C^0$ estimate). Let $\varphi$ be a solution of (2.3), and then for $\alpha < 0$, we have

c_1 \leq u(x,t)\Theta^{-1}(t,c) \leq c_2, \quad \forall x \in M, \, t \in [0,T]

for some positive constants $c_1, c_2$, where $\Theta(t,c) \coloneqq \{-\alpha t + e^{\alpha c}\}^{\frac{1}{\alpha}}$ with

$$\inf_M \varphi(\cdot,0) \leq c \leq \sup_M \varphi(\cdot,0)$$

Proof. Let $\varphi(x,t) = \varphi(t)$ (independent of $x$) be the solution of (2.3) with $\varphi(0) = c$. In this case, the first equation in (2.3) reduces to an ODE

$$\frac{d}{dt} \varphi = -e^{-\alpha \varphi}.$$ 

Therefore,

$$\varphi(t) = \frac{1}{\alpha} \ln(-\alpha t + e^{\alpha c}), \quad \text{for } \alpha < 0.$$ 

Using the maximum principle, we can obtain that

$$\frac{1}{\alpha} \ln(-\alpha t + e^{\alpha \varphi_1}) \leq \varphi(x,t) \leq \frac{1}{\alpha} \ln(-\alpha t + e^{\alpha \varphi_2}),$$

where $\varphi_1 := \inf_M \varphi(\cdot,0)$ and $\varphi_2 := \sup_M \varphi(\cdot,0)$. The estimate is obtained since $\varphi = \ln u$. \hfill \qed

Lemma 3.2 ($\dot{\varphi}$ estimate). Let $\varphi$ be a solution of (2.3) and $\Sigma$ be the boundary of a smooth, convex cone defined as in Theorem 1.1, then for $\alpha < 0$,

$$\min\left\{\inf_M (\dot{\varphi}(\cdot,0) \cdot \Theta(0)^{\alpha}), -1\right\} \leq \dot{\varphi}(x,t) \Theta(t)^{\alpha} \leq \max\left\{\sup_M (\dot{\varphi}(\cdot,0) \cdot \Theta(0)^{\alpha}), -1\right\}.$$ 

Proof. Set $M(x(\xi),t) = \dot{\varphi}(x(\xi),t)\Theta(t)^{\alpha}$. Differentiating both sides of the first evolution equation of (2.3), it is easy to get that

$$\begin{cases}
\frac{\partial M}{\partial t} = Q^\xi M_{\xi} + Q^\xi M_\xi - \alpha \Theta^{-\alpha} (1 + M) M & \text{in } M \times (0,T) \\
\nabla_\mu M = 0 & \text{on } \partial M \times (0,T) \\
M(\cdot,0) = \varphi_0 \cdot \Theta(0)^{\alpha} & \text{in } M,
\end{cases}$$

where $Q^\xi := \frac{\partial Q}{\partial \varphi_\xi}$ and $Q^\xi := \frac{\partial Q}{\partial \varphi_\xi}$. Then the result follows from the maximum principle. \hfill \qed

Lemma 3.3 (Gradient estimate). Let $\varphi$ be a solution of (2.3) and $\Sigma$ be the boundary of a smooth, convex cone described as in Theorem 1.1. Then for $\alpha < 0$, we have

$$|D\varphi| \leq \sup_M |D\varphi(\cdot,0)| < 1, \quad \forall x \in M, \, t \in [0,T].$$

Proof. Set $\psi = \frac{|D\varphi|^2}{2}$. By differentiating $\psi$, we have

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial t} \varphi_\xi \cdot \varphi_\xi = Q^\xi \varphi_\xi.$$ 

Then using the evolution equation of $\varphi$ in (2.3) yields

$$\frac{\partial \psi}{\partial t} = Q^{\xi\xi} \varphi_{\xi\xi} \varphi_\xi + Q^{\xi} \varphi_{\xi\xi} \varphi_\xi - \alpha Q^{\xi} \varphi_\xi.$$
Therefore, we can express $\varphi_{\xi\xi\xi\xi\xi} \varphi_{\xi}$ as
\[
\varphi_{\xi\xi\xi\xi\xi} \varphi_{\xi} = \psi_{\xi} - \varphi_{\xi\xi}^2.
\]
Also, we can express $\varphi_{\xi\xi} \varphi_{\xi}$ as
\[
\varphi_{\xi\xi} \varphi_{\xi} = \psi_{\xi}.
\]
Then, we have
\[
\frac{\partial \psi}{\partial t} = Q_{\xi\xi}^\xi \psi_{\xi\xi} + Q_{\xi}^\xi \psi_{\xi} - Q_{\xi\xi} \varphi_{\xi\xi}^2 - \alpha Q \varphi_{\xi}^2
\]  
(3.3)
Since $Q_{\xi\xi}^\xi$ and $\alpha Q$ is positive definite, the third and fourth terms in the RHS of (3.3) are non-positive. Since $\varphi$ is a solution of (2.3), that is,
\[
\nabla \mu \varphi = \mu^\xi \varphi_{\xi} = 0 \quad \text{on } \partial M \times (0, T),
\]
where $\mu = \mu^\xi \partial_{\xi}$, we have
\[
\nabla \mu \psi = \mu^\xi \psi_{\xi} = \mu^\xi \varphi_{\xi\xi} \varphi_{\xi} \quad \text{on } \partial M \times (0, T).
\]
So, we can get
\[
\begin{cases}
\frac{\partial \psi}{\partial t} \leq Q_{\xi\xi}^\xi \psi_{\xi\xi} + Q_{\xi}^\xi \psi_{\xi} & \text{in } M \times (0, T) \\
\nabla \mu \psi = 0 & \text{on } \partial M \times (0, T) \\
\psi(\cdot, 0) = \frac{|\varphi_{\xi}(\cdot, 0)|^2}{2} & \text{in } M.
\end{cases}
\]  
(3.4)
Using the maximum principle, we have
\[
|D \varphi| \leq \sup_M |D \varphi(\cdot, 0)|.
\]
Since $G_0 = \{(x(\xi), u(x(\xi), 0)) | x \in M\}$ is a spacelike graph of $\mathbb{R}^2_1$, so we have
\[
|D \varphi| \leq \sup_M |D \varphi(\cdot, 0)| < 1, \quad \forall \ x \in M, \ t \in [0, T].
\]
Our proof is finished. \qed

**Remark 3.1.** The gradient estimate in Lemma 3.3 makes sure that the evolving graphs $G_t := \{(x, u(x, t)) | x \in M, 0 \leq t \leq T\}$ are spacelike graphs.

Combing the gradient estimate with $\dot{\varphi}$ estimate, we can obtain:

**Corollary 3.4.** If $\varphi$ satisfies (2.3), then we have
\[
0 < c_3 \leq k \Theta \leq c_4 < +\infty
\]
where $c_3$ and $c_4$ are positive constants independent of $\varphi$.

### 4. Hölder estimates and the long-time existence

Set $\Phi = \frac{1}{|X|^{1/2}}, \ w = \langle X, \nu \rangle_L$ and $\Psi = \frac{\Phi}{w}$. We can get the following evolution equations:

**Lemma 4.1.** Under the assumptions of Theorem 1.1, we have
\[
\frac{\partial}{\partial t} g_{\xi\xi} = -2 \Phi k g_{\xi\xi},
\]
\[
\frac{\partial}{\partial t} g^{\xi\xi} = 2 \Phi k g^{\xi\xi},
\]
\[ \frac{\partial}{\partial t} \nu = \nabla \Phi X_{\xi}, \]

\[ \partial_{t} k = \Phi k^{2} - \alpha(\alpha + 1)\Phi u^{-2}|\nabla u|^{2} + \alpha u^{-1}\Phi \Delta u - 2\Phi k^{-2}|\nabla k|^{2} + \Phi k^{-1}\Delta k \]

and

\[ \frac{\partial \Psi}{\partial t} = \text{div}_{g}(u^{-\alpha}k^{-2}\nabla \Psi) - 2k^{-2}u^{-\alpha}\Psi^{-1}|\nabla \Psi|^{2} + \alpha\Psi^{2} - \alpha u^{-\alpha-1}k^{-2}u_{\xi} \nabla^{\xi} \Psi + \alpha\Psi^{2}u^{-1}\nabla^{\xi}u \langle X, X_{\xi} \rangle_{L}. \]

Proof. It is easy to get the first three evolution equations, and we omit here. Direct calculation results in

\[ \Phi_{\xi} = -\alpha\Phi u^{-1}u_{\xi} - \Phi k^{-1}k_{\xi}, \]

and

\[ \Phi_{\xi \xi} = \alpha(\alpha + 1)\Phi u^{-2}u_{\xi}^{2} + 2\alpha u^{-1}\Phi k^{-1}u_{\xi}k_{\xi} - \alpha u^{-1}\Phi u_{\xi \xi} - \Phi k^{-1}k_{\xi \xi} + 2\Phi k^{-2}k_{\xi \xi}^{2}. \]

According to the definition of curvature, we have

\[ k = g^{\xi \xi} \langle X_{\xi \xi}, \nu \rangle_{L}, \]

and then

\[ \partial_{t} k = \frac{\partial}{\partial t}g^{\xi \xi} \cdot \langle X_{\xi \xi}, \nu \rangle_{L} + g^{\xi \xi} \cdot \frac{\partial}{\partial t} \langle X_{\xi \xi}, \nu \rangle_{L}. \]

Since

\[ \frac{\partial}{\partial t} \langle X_{\xi \xi}, \nu \rangle_{L} = -\Phi \xi \xi - \Phi k^{2}g_{\xi \xi}, \]

so

\[ \partial_{t} k = -\Delta \Phi + \Phi k^{2}. \]

Thus,

\[ \partial_{t} k = -\alpha(\alpha + 1)\Phi u^{-2}|\nabla u|^{2} + \alpha u^{-1}\Phi \Delta u + \Phi k^{2} - 2\Phi k^{-2}|\nabla k|^{2} - 2\alpha u^{-1}\Phi k^{-1}u_{\xi} \nabla^{\xi} k + \Phi k^{-1}\Delta k. \]

Clearly,

\[ \partial_{t} w = -\Phi - \alpha\Phi u^{-1}\nabla^{\xi} u \langle X, X_{\xi} \rangle_{L} - \Phi k^{-1}\nabla^{\xi} k \langle X, X_{\xi} \rangle_{L}, \]

by the calculation, we have

\[ w_{\xi} = -k \langle X, X_{\xi} \rangle_{L}, \]

\[ w_{\xi \xi} = -kg_{\xi \xi} - k_{\xi} \langle X, X_{\xi} \rangle_{L} + k^{2}g_{\xi \xi}w. \]

Thus,

\[ \Delta w = -k - \nabla^{\xi} k \langle X, X_{\xi} \rangle_{L} + k^{2}w, \]

and

\[ \partial_{t} w = -u^{-\alpha}w - \alpha u^{-\alpha-1}k^{-1}\nabla^{\xi} u \langle X, X_{\xi} \rangle_{L} + u^{-\alpha}k^{-2}\Delta w. \]
Hence
\[
\frac{\partial \Psi}{\partial t} = \frac{1}{u^{1+\alpha} k w^\alpha} \frac{1}{w^\alpha k w} - \frac{1}{w^2} \frac{\partial k}{\partial t} - \frac{1}{w^2} \frac{\partial w}{\partial t}
\]
\[
= \alpha u^{-2\alpha} k^{-2} w^{-2} + \alpha (\alpha + 1) u^{-2\alpha} k^{-3} w^{-1} |\nabla u|^2 + 2 u^{-2\alpha} k^{-5} w^{-3} |\nabla k|^2 + 2 \alpha u^{-2\alpha} k^{-4} w^{-2} |\nabla u|^2
\]
\[
+ 2 \alpha u^{-2\alpha} k^{-4} w^{-1} u_\xi \nabla \xi k - \alpha u^{-2\alpha} k^{-3} w^{-1} \Delta u - u^{-2\alpha} k^{-4} w^{-1} \Delta k
\]
\[
- u^{-2\alpha} k^{-3} w^{-2} \Delta w + \alpha u^{-2\alpha} k^{-2} w^{-2} \nabla \xi u \langle X, \xi \rangle_L.
\]

In order to prove (4.1), we calculate
\[
\Psi_\xi = -\alpha u^{-\alpha-1} k^{-1} w^{-1} u_\xi - u^{-\alpha} k^{-1} w^{-1} k_\xi - u^{-\alpha} k^{-1} w^{-2} w_\xi.
\]

and
\[
\Psi_{\xi \xi} = \alpha (\alpha + 1) u^{-2\alpha-2} k^{-1} w^{-1} u_\xi^2 + 2 u^{-\alpha} k^{-3} w^{-1} k_\xi^2 + 2 u^{-\alpha} k^{-1} w^{-3} w_\xi^2
\]
\[
+ 2 \alpha u^{-2\alpha-1} k^{-2} w^{-1} u_\xi k_\xi + 2 u^{-\alpha} k^{-2} w^{-2} k_\xi w_\xi + 2 \alpha u^{-2\alpha-1} k^{-1} w^{-2} w_\xi u_\xi
\]
\[
- \alpha u^{-\alpha-1} k^{-1} w^{-1} u_{xx} - u^{-\alpha} k^{-2} w^{-1} k_{\xi \xi} - u^{-\alpha} k^{-1} w^{-2} w_{\xi \xi}.
\]

Therefore,
\[
u^{-\alpha} k^{-2} \Delta \Psi = \alpha (\alpha + 1) u^{-2\alpha-2} k^{-3} w^{-1} |\nabla u|^2 + 2 u^{-2\alpha} k^{-5} w^{-1} |\nabla k|^2 + 2 u^{-2\alpha} k^{-3} w^{-3} |\nabla w|^2
\]
\[
+ 2 \alpha u^{-2\alpha-1} k^{-4} w^{-1} u_\xi \nabla \xi k + 2 u^{-\alpha} k^{-4} w^{-2} w_\xi \nabla \xi k + 2 \alpha u^{-2\alpha-1} k^{-3} w^{-2} w_\xi \nabla \xi u
\]
\[
- \alpha u^{-2\alpha-1} k^{-3} w^{-1} \Delta u - u^{-2\alpha} k^{-4} w^{-1} \Delta k - u^{-2\alpha} k^{-3} w^{-2} \Delta w.
\]

So we have
\[
\text{div}(u^{-\alpha} k^{-2} \nabla \Psi) = \alpha (2\alpha + 1) u^{-2\alpha-2} k^{-3} w^{-1} |\nabla u|^2 + 2 u^{-2\alpha} k^{-5} w^{-1} |\nabla k|^2 + 2 u^{-2\alpha} k^{-3} w^{-3} |\nabla w|^2
\]
\[
+ 5 \alpha u^{-2\alpha-1} k^{-4} w^{-1} u_\xi \nabla \xi k + 2 u^{-\alpha} k^{-4} w^{-2} w_\xi \nabla \xi k + 3 \alpha u^{-2\alpha-1} k^{-3} w^{-2} w_\xi \nabla \xi u
\]
\[
- \alpha u^{-2\alpha-1} k^{-3} w^{-1} \Delta u - u^{-2\alpha} k^{-4} w^{-1} \Delta k - u^{-2\alpha} k^{-3} w^{-2} \Delta w.
\]

and
\[
2 k^{-1} w |\nabla \Psi|^2 = 2 \alpha u^{-2\alpha-2} k^{-3} w^{-1} |\nabla u|^2 + 2 u^{-2\alpha} k^{-5} w^{-1} |\nabla k|^2 + 2 u^{-2\alpha} k^{-3} w^{-3} |\nabla w|^2
\]
\[
+ 4 \alpha u^{-2\alpha-1} k^{-4} w^{-1} u_\xi \nabla \xi k + 4 u^{-2\alpha} k^{-4} w^{-2} w_\xi \nabla \xi k + 4 \alpha u^{-2\alpha-1} k^{-3} w^{-2} w_\xi \nabla \xi u.
\]

In sum, we have
\[
\frac{\partial \Psi}{\partial t} - \text{div}_g (u^{-\alpha} k^{-2} \nabla \Psi) + 2 k^{-1} w |\nabla \Psi|^2
\]
\[
= \alpha u^{-2\alpha} k^{-2} w^{-2} + \alpha u^{-2\alpha-1} k^{-4} w^{-1} u_\xi \nabla \xi k + \alpha u^{-2\alpha-1} k^{-3} w^{-2} w_\xi \nabla \xi u
\]
\[
+ \alpha^2 u^{-2\alpha-2} k^{-3} w^{-1} |\nabla u|^2 + \alpha u^{-2\alpha-1} k^{-2} w^{-2} \nabla \xi u \langle X, \xi \rangle_L
\]
\[
= \alpha \Psi^2 - \alpha u^{-\alpha-1} k^{-2} u_\xi \nabla \xi \Psi + \alpha \Psi^2 u^{-1} \nabla \xi u \langle X, \xi \rangle_L.
\]

The proof is finished. \(\square\)

Now, we define the rescaled flow by
\[
\tilde{X} = X \Theta^{-1}.
\]

Thus,
\[
\tilde{u} = u \Theta^{-1},
\]
\[
\tilde{\varphi} = \varphi - \ln \Theta,
\]
and the rescaled mean curvature equation takes the form
\[ \frac{\partial}{\partial t} \tilde{u} = -\frac{v}{\tilde{u}^\alpha k} \Theta^{-\alpha} + \tilde{u} \Theta^{-\alpha}. \]

Defining \( t = t(s) \) by the relation
\[ \frac{dt}{ds} = \Theta^\alpha \]
such that \( t(0) = 0 \) and \( t(S) = T \). Then \( \tilde{u} \) satisfies
\[
\begin{aligned}
\frac{\partial}{\partial s} \tilde{u} &= -\frac{v}{\tilde{u}^\alpha k} \tilde{u} + \tilde{u} \quad \text{in } \partial M \times (0, S) \\
\nabla_{\tilde{u}} \tilde{u} &= 0 \quad \text{on } \partial M \times (0, S) \\
\tilde{u}(\cdot, 0) &= \tilde{u}_0 \quad \text{in } M.
\end{aligned}
\]

Lemma 4.2. Let \( X \) be a solution of (1.1) and \( \tilde{X} = X \Theta^{-1} \) be the rescaled solution. Then
\[
\begin{aligned}
D\tilde{u} &= Du \Theta^{-1}, \\
D\tilde{\varphi} &= D\varphi, \\
\frac{\partial}{\partial s} \tilde{u} &= \frac{\partial u}{\partial t} \Theta^{-1} + u \Theta^{-1}, \\
\tilde{g}_{\xi \xi} &= \Theta^{-2} g_{\xi \xi}, \\
\tilde{g}^{\xi \xi} &= \Theta^2 g^{\xi \xi}, \\
k &= k \Theta.
\end{aligned}
\]

Proof. These relations can be computed directly. \qed

Lemma 4.3. Let \( u \) be a solution to the parabolic system (2.3), where \( \varphi(x, t) = \ln u(x, t) \), and \( \Sigma \) be the boundary of a smooth, convex cone described as in Theorem 1.1. Then there exist some \( 0 < \beta < 1 \) and some \( C > 0 \) such that the rescaled function \( \tilde{u}(x(\xi), s) := u(x(\xi), t(s)) \Theta^{-1}(t(s)) \) satisfies
\[
[D\tilde{u}]_\beta + \left[ \frac{\partial \tilde{u}}{\partial s} \right]_\beta + \left[ \tilde{k} \right]_\beta \leq C(||u_0||_{C^{2+\gamma, 1+\frac{\gamma}{2}}(M)}, \beta, M),
\]
where \([f]_\beta := [f]_{x, \beta} + [f]_{s, \frac{\beta}{2}}\) is the sum of the Hölder coefficients of \( f \) in \( M \times [0, S] \) with respect to \( x \) and \( s \).

Proof. We divide our proof into three steps.\footnote{In the proof of Lemma 4.3, the constant \( C \) may differ from each other. However, we abuse the symbol \( C \) for the purpose of convenience.}

Step 1: We need to prove that
\[
[D\tilde{u}]_{x, \beta} + [D\tilde{u}]_{s, \frac{\beta}{2}} \leq C(||u_0||_{C^{2+\gamma, 1+\frac{\gamma}{2}}(M)}, \beta, M).
\]

According to Lemmas 3.1, 3.2, and 3.3 it follows that
\[
|D\tilde{u}| + \left| \frac{\partial \tilde{u}}{\partial s} \right| \leq C(||u_0||_{C^{2+\gamma, 1+\frac{\gamma}{2}}(M)}, M).
\]

Then we can easily obtain the bound of \([\tilde{u}]_{x, \beta}\) and \([\tilde{u}]_{s, \frac{\beta}{2}}\) for any \( 0 < \beta < 1 \). Lemma 3.1 in [11, Chap. 2] implies that the bound for \([D\tilde{u}]_{s, \frac{\beta}{2}}\) follows from a bound for \([\tilde{u}]_{s, \frac{\beta}{2}}\) and \([D\tilde{u}]_{x, \beta}\). Hence it remains to bound \([D\tilde{\varphi}]_{x, \beta}\) since \( D\tilde{u} = \tilde{u} D\tilde{\varphi} \). For this, fix \( s \) and the equation (2.3) can be rewritten as an elliptic Neumann problem
\[
-\text{div}_\varphi \left( \frac{D\tilde{\varphi}}{\sqrt{1 - |D\tilde{\varphi}|^2}} \right) = \frac{1}{\sqrt{1 - |D\tilde{\varphi}|^2}} + e^{-\alpha \varphi} \sqrt{1 - |D\tilde{\varphi}|^2} \frac{1}{\varphi_s - 1}.
\]
In fact, the equation \([4.4]\) is of the form \(D_\xi(a^\xi(p) + a(\xi, s)) = 0\). Since \(\hat{\varphi}\) and \(|D\hat{\varphi}|\) are bounded, we know \(a\) is a bounded function in \(\xi\) and \(s\). We define \(a^\xi(p) := \frac{\partial a}{\partial \xi^p}\), the smallest and largest eigenvalues of \(a^\xi\) are controlled due to the estimate for \(|D\hat{\varphi}|\). By [10] Chap. 3; Theorem 14.1; Chap. 10, §2], we can get the interior estimate and boundary estimate of \([D\hat{\varphi}]_{x, \beta}\).

**Step 2:** The next thing to do is to show that

\[
\left[\frac{\partial \tilde{u}}{\partial s}\right]_{x, \beta} + \left[\frac{\partial \tilde{u}}{\partial s}\right]_{\frac{\partial}{\partial s}} \leq C(||u_0||_{C^{2+\gamma, 1+\frac{2}{\gamma}}(M)}, \beta, M).
\]

As \(\frac{\partial}{\partial s} \tilde{u} = \tilde{u}\left(-\frac{\nu}{\tilde{\nu} + 1}\right)\), it is enough to bound \(\left[\frac{\partial \tilde{u}}{\partial s}\right]_{\frac{\partial}{\partial s}}\). Set \(\tilde{w}(s) := \frac{\nu}{\tilde{\nu} + 1} = \Theta^\alpha \Psi\). Let \(\tilde{\nabla}\) be the Levi-Civita connection of \(\tilde{M}_s := \tilde{X}(M, s)\) w.r.t. the metric \(\tilde{g}\). Combining with \([4.1]\) and Lemma \([4.2]\), we get

\[
\frac{\partial \tilde{w}}{\partial s} = \text{div}_{\tilde{g}}(\tilde{w}^{-\alpha} k^{-2}\tilde{\nabla} \tilde{w}) - 2k^{-2} \tilde{w}^{-\alpha-1} |\nabla \tilde{w}|^2_{\tilde{g}} \frac{\nu}{\tilde{\nu} + 1} - \alpha \tilde{w} + \alpha \tilde{w}^2 + \alpha \tilde{w} P - \alpha \tilde{w}^{-\alpha-1} k^{-2} \nabla \xi \tilde{u} \nabla \xi \tilde{w},
\]

where \(P := u^{-1} \nabla \xi u(X, X_\xi)_L\). Applying Lemmas \([3.1]\) and \([3.3]\) we have

\[
|P| \leq |\nabla u|_g \leq C,
\]

where \(C\) depends only on \(\sup_M |D u(\cdot, 0)|, c_1\) and \(c_2\). The weak formulation of \([4.5]\) is

\[
\int_{s_0}^{s_1} \int_{\tilde{M}_s} \frac{\partial \tilde{w}}{\partial s} \eta d\mu ds = \int_{s_0}^{s_1} \int_{\tilde{M}_s} \text{div}_{\tilde{g}}(\tilde{w}^{-\alpha} k^{-2} \tilde{\nabla} \tilde{w}) \eta - 2k^{-2} \tilde{w}^{-\alpha-1} |\nabla \tilde{w}|^2_{\tilde{g}} \eta d\mu ds
\]

\[
+ \int_{s_0}^{s_1} \int_{\tilde{M}_s} (-\alpha \tilde{w} + \alpha \tilde{w}^2 + \alpha \tilde{w} P - \alpha \tilde{w}^{-\alpha-1} k^{-2} \nabla \xi \tilde{u} \nabla \xi \tilde{w}) \eta d\mu ds.
\]

Since \(\nabla_{\mu} \tilde{\varphi} = 0\), the boundary integrals all vanish, the interior and boundary estimates are basically the same. We define the test function \(\eta := \zeta^2 \tilde{w}\), where \(\zeta\) is a smooth function with values in \([0, 1]\) and is supported in a small parabolic neighborhood. Then

\[
\int_{s_0}^{s_1} \int_{\tilde{M}_s} \frac{\partial \tilde{w}}{\partial s} \zeta^2 \tilde{w} d\mu ds = \frac{1}{2} ||\tilde{w}\zeta||^2_{2, \tilde{M}_s} \bigg|_{s_0}^{s_1} - \int_{s_0}^{s_1} \int_{\tilde{M}_s} \zeta^2 \tilde{w}^2 d\mu ds,
\]

where \(\zeta := \frac{\partial}{\partial s}\). We have

\[
\int_{s_0}^{s_1} \int_{\tilde{M}_s} \text{div}_{\tilde{g}}(\tilde{w}^{-\alpha} k^{-2} \tilde{\nabla} \tilde{w}) \zeta^2 \tilde{w} d\mu ds
\]

\[
= \int_{s_0}^{s_1} \int_{\tilde{M}_s} \text{div}_{\tilde{g}}(\tilde{w}^{-\alpha} k^{-2} \tilde{\nabla} \tilde{w} \zeta^2 \tilde{w}) d\mu ds - \int_{s_0}^{s_1} \int_{\tilde{M}_s} \tilde{w}^{-\alpha} k^{-2} \zeta^2 \nabla \xi \tilde{w} \nabla \xi \tilde{w} d\mu ds
\]

\[
- 2 \int_{s_0}^{s_1} \int_{\tilde{M}_s} \tilde{w}^{-\alpha} k^{-2} \zeta \nabla \xi \tilde{w} \nabla \xi \tilde{w} \zeta d\mu ds.
\]

Using the divergence theorem, we have

\[
\int_{s_0}^{s_1} \int_{\tilde{M}_s} \text{div}_{\tilde{g}}(\tilde{w}^{-\alpha} k^{-2} \tilde{\nabla} \zeta^2 \tilde{w}) d\mu ds = - \int_{s_0}^{s_1} \int_{\partial \tilde{M}_s} \tilde{g}(\mu, \tilde{w}^{-\alpha} k^{-2} \nabla \zeta^2 \tilde{w}) d\mu ds = 0.
\]
Thus,
\[
\int_{s_0}^{s_1} \int_{\widetilde{M}_s} \nabla_s (\tilde{u}^{-\alpha} \tilde{k}^{-2} \bar{\nabla} \tilde{w}) \zeta^2 \tilde{w} d\mu_s ds \\
= - \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \tilde{u}^{-\alpha} \tilde{k}^{-2} \zeta^2 \bar{\nabla} \tilde{u} \bar{\nabla} \zeta \tilde{w} d\mu_s ds - 2 \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \tilde{u}^{-\alpha} \tilde{k}^{-2} \zeta \bar{\nabla} \tilde{u} \bar{\nabla} \zeta \tilde{w} d\mu_s ds.
\]
Since
\[
- \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \tilde{u}^{-\alpha} \tilde{k}^{-2} \bar{\nabla} \tilde{w} \zeta \left( \bar{\nabla} \tilde{w} + \tilde{w} \bar{\nabla} \zeta \right)^2 d\mu_s ds \\
= - \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \left( \tilde{u}^{-\alpha} \tilde{k}^{-2} \bar{\nabla} \tilde{u} \bar{\nabla} \zeta \tilde{w} + 2 \tilde{u}^{-\alpha} \tilde{k}^{-2} \zeta \bar{\nabla} \tilde{u} \bar{\nabla} \zeta \tilde{w} \right) d\mu_s ds
\]
is negative, so we can obtain
\[
\int_{s_0}^{s_1} \int_{\widetilde{M}_s} \nabla_s (\tilde{u}^{-\alpha} \tilde{k}^{-2} \bar{\nabla} \tilde{w}) \zeta^2 \tilde{w} d\mu_s ds \\
\leq \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \tilde{u}^{-\alpha} \tilde{k}^{-2} |\bar{\nabla} \zeta|^2 \tilde{w}^2 d\mu_s ds.
\]
(4.8)

We also have
\[
\int_{s_0}^{s_1} \int_{\widetilde{M}_s} (-\alpha \tilde{w} + \alpha \tilde{w}^2 + \alpha \tilde{w}^2 P - \alpha \tilde{u}^{-\alpha-1} \tilde{k}^{-2} \bar{\nabla} \tilde{u} \bar{\nabla} \zeta \tilde{w}) \zeta^2 \tilde{w} d\mu_s ds \\
\leq C |\alpha| \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \zeta^2 (\tilde{w}^2 + |\tilde{w}|^3) d\mu_s ds + \int_{s_0}^{s_1} \int_{\widetilde{M}_s} |\alpha| |\tilde{u}^{-\alpha-1} \tilde{k}^{-2} |\bar{\nabla} \tilde{u}| |\bar{\nabla} \tilde{w}| \zeta^2 |\tilde{w}| d\mu_s ds.
\]
Using Young’s inequality, we can obtain
\[
\int_{s_0}^{s_1} \int_{\widetilde{M}_s} |\alpha| |\tilde{u}^{-\alpha-1} \tilde{k}^{-2} |\bar{\nabla} \tilde{u}| |\bar{\nabla} \tilde{w}| \zeta^2 |\tilde{w}| d\mu_s ds \\
= \int_{s_0}^{s_1} \int_{\widetilde{M}_s} |\alpha| (\tilde{u}^{-\alpha-1} \tilde{k}^{-1} |\bar{\nabla} \tilde{w}| \zeta) \cdot (\tilde{u}^{-\alpha-1} \tilde{k}^{-1} |\bar{\nabla} \tilde{u}| |\tilde{w}|) d\mu_s ds \\
\leq \frac{|\alpha|}{2} \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \tilde{u}^{-\alpha-2} \tilde{k}^{-1} |\bar{\nabla} \tilde{w}|^2 \zeta^2 d\mu_s ds + \frac{|\alpha|}{2} \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \tilde{u}^{-\alpha-2} \tilde{k}^{-2} |\bar{\nabla} \tilde{u}|^2 \zeta^2 \tilde{w}^2 d\mu_s ds,
\]
and thus
\[
\int_{s_0}^{s_1} \int_{\widetilde{M}_s} (-\alpha \tilde{w} + \alpha \tilde{w}^2 + \alpha \tilde{w}^2 P - \alpha \tilde{u}^{-\alpha-1} \tilde{k}^{-2} \bar{\nabla} \tilde{u} \bar{\nabla} \zeta \tilde{w}) \zeta^2 \tilde{w} d\mu_s ds \\
\leq C |\alpha| \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \zeta^2 (\tilde{w}^2 + |\tilde{w}|^3) d\mu_s ds + \int_{s_0}^{s_1} \int_{\widetilde{M}_s} |\alpha| \tilde{u}^{-\alpha-1} \tilde{k}^{-2} |\bar{\nabla} \tilde{u}| |\bar{\nabla} \tilde{w}| \zeta^2 |\tilde{w}| d\mu_s ds \\
\leq C |\alpha| \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \zeta^2 (\tilde{w}^2 + |\tilde{w}|^3) d\mu_s ds + \frac{|\alpha|}{2} \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \tilde{u}^{-\alpha-2} \tilde{k}^{-2} |\bar{\nabla} \tilde{w}|^2 \zeta^2 d\mu_s ds \\
+ \frac{|\alpha|}{2} \int_{s_0}^{s_1} \int_{\widetilde{M}_s} \tilde{u}^{-\alpha-2} \tilde{k}^{-2} |\bar{\nabla} \tilde{u}|^2 \zeta^2 \tilde{w}^2 d\mu_s ds.
Combing (4.7), (4.8) and (4.9), we have
\[
\frac{1}{2} \int_{s_0}^{s_1} \left| w^\alpha \right|^2 + \frac{2 + \alpha}{2} \int_{s_0}^{s_1} \int_{M_s} \bar{u}^{-\alpha} \nabla \bar{w}^2 |w^\alpha|^2 d\mu ds + \left| \bar{w} \right|^2 d\mu ds + \int_{s_0}^{s_1} \int_{M_s} \left| \bar{w} \right|^2 d\mu ds,
\]
which implies
\[
\frac{1}{2} \int_{s_0}^{s_1} \left| \bar{w} \right|^2_2 ds + \frac{2 + \alpha}{2} \int_{s_0}^{s_1} \int_{M_s} \bar{u}^{-\alpha} \nabla \bar{w}^2 |w^\alpha|^2 d\mu ds + \left| \bar{w} \right|^2 d\mu ds + \int_{s_0}^{s_1} \int_{M_s} \left| \bar{w} \right|^2 d\mu ds.
\]
(4.10)

This means that \( \bar{w} \) belong to the De Giorgi class of functions in \( M \times [0, S) \). Similar to the arguments in [11, Chap. 5, §1 and §7], there exist constants \( 0 < \beta < 1 \) and \( C \) such that
\[
[\bar{w}]_\beta \leq C \left| \bar{w} \right|_{L^\infty(M \times [0, S])} \leq C (||u_0||_{C^{2+\gamma,1+\gamma'}(M)}, \beta, M).
\]

**Step 3:** Finally, we have to show that
\[
[\bar{k}]_{x,\beta} + [\bar{k}]_{s,\beta} \leq C (||u_0||_{C^{2+\gamma,1+\gamma'}(M)}, \beta, M).
\]

This follows from the fact that
\[
\bar{k} = \sqrt{1 - \frac{|D\varphi|^2}{\bar{u}^{\alpha+1} \bar{w}}}
\]

This implies the estimates for \( \bar{u}, \bar{w}, D\varphi \).

Then we can obtain the following higher-order estimates:

**Lemma 4.4.** Let \( u \) be a solution to the parabolic system (2.3), where \( \varphi(x, t) = \ln u(x, t) \), and \( \Sigma \) be the boundary of a smooth, convex cone described as in Theorem 1.1. Then for any \( s_0 \in (0, S) \), there exist some \( 0 < \beta < 1 \) and some \( C > 0 \) such that
\[
||\bar{u}||_{C^{2+\beta,1+\beta}(M \times [0, S])} \leq C (||u_0||_{C^{2+\gamma,1+\gamma'}(M)}, \beta, M)
\]
(4.11)
and for all \( \ell \in \mathbb{N} \),
\[
||\bar{u}||_{C^{2\ell+\beta,\ell+\beta}(M \times [s_0, S])} \leq C (||u_0(\cdot, s_0)||_{C^{2\ell+\beta,\ell+\beta}(M)}, \beta, M).
\]
(4.12)

**Proof.** By Lemma 2.1, we have
\[
uvk = 1 + \frac{1}{u^2} \varphi_k \varphi = 1 + u^2 \Delta_g \varphi.
\]

Since
\[
u^2 \Delta_g \varphi = \bar{u}^2 \Delta_g \bar{\varphi} = -|\nabla \bar{u}|^2 + \bar{u} \Delta_g \bar{u},
\]

Then we can obtain the following higher-order estimates.
then
\[
\frac{\partial \tilde{u}}{\partial s} = \frac{\partial u}{\partial t} \Theta^{a-1} + \tilde{u} = \frac{u v k}{u^{1+\alpha} k^2} \Theta^{a-1} - \frac{2 v}{u^\alpha k} \Theta^{a-1} + \tilde{u} = \Delta \tilde{u} - \frac{2 v}{\tilde{u}^\alpha k} + \tilde{u} + \frac{1 - \|\nabla \tilde{u}\|^2}{\tilde{u}^{1+\alpha} k^2},
\]
which is a uniformly parabolic equation with Hölder continuous coefficients. Therefore, the linear theory (see [12, Chap. 4]) yields the inequality (4.1.1).

Set \( \tilde{\varphi} = \ln \tilde{u} \), and then the rescaled version of the evolution equation in (4.2) takes the form
\[
\frac{\partial \tilde{\varphi}}{\partial s} = -e^{-\alpha \tilde{\varphi}} \frac{v^2}{1 + \frac{1}{v^2} \tilde{\varphi}_{\xi\xi}} + 1,
\]
where \( v = \sqrt{1 - |D \tilde{\varphi}|^2} \). According to the \( C^{2+\beta, \frac{3+\beta}{2}} \)-estimate of \( \tilde{u} \) (see Lemma [13]), we can treat the equations for \( \frac{\partial \tilde{\varphi}}{\partial s} \) and \( D\xi \tilde{\varphi} \) as second-order linear uniformly parabolic PDEs on \( M \times [s_0, S] \).

At the initial time \( s_0 \), all compatibility conditions are satisfied and the initial function \( u(\cdot, t_0) \) is smooth. We can obtain a \( C^{3+\beta, \frac{3+\beta}{2}} \)-estimate for \( D\xi \tilde{\varphi} \) and a \( C^{2+\beta, \frac{3+\beta}{2}} \)-estimate for \( \frac{\partial \tilde{\varphi}}{\partial s} \) (the estimates are independent of \( T \)) by Theorem 4.3 and Exercise 4.5 in [12, Chapter 4]. Higher regularity can be proven by induction over \( \ell \).

Theorem 4.5. Under the hypothesis of Theorem 1.1, we conclude
\[ T^* = +\infty. \]

Proof. The proof of this result is quite similar to the corresponding argument in [14, Lemma 8] and so is omitted.

5. CONVERGENCE OF THE RESCALED FLOW

We know that after the long-time existence of the flow has been obtained (see Theorem 4.5), the rescaled version of the system (2.3) satisfies
\[
\begin{cases}
\frac{\partial \tilde{\varphi}}{\partial s} = \tilde{Q}(\tilde{\varphi}, D\tilde{\varphi}, D^2\tilde{\varphi}) & \text{in } M \times (0, \infty) \\
D\mu \tilde{\varphi} = 0 & \text{on } \partial M \times (0, \infty) \\
\tilde{\varphi}(\cdot, 0) = \tilde{\varphi}_0 & \text{in } M,
\end{cases}
\]
where
\[
\tilde{Q}(\tilde{\varphi}, D\tilde{\varphi}, D^2\tilde{\varphi}) := -e^{-\alpha \tilde{\varphi}} \frac{v^2}{1 + \frac{1}{v^2} \tilde{\varphi}_{\xi\xi}} + 1
\]
and \( \tilde{\varphi} = \ln \tilde{u} \). Similar to what has been done in the \( C^1 \) estimate (see Lemma 3.3), we can deduce a decay estimate of \( \tilde{u}(\cdot, s) \) as follows.

Lemma 5.1. Let \( u \) be a solution of (2.2), then we have
\[
|D\tilde{u}(x(\xi), t)| \leq \lambda \sup_{M} |D\tilde{u}(\cdot, 0)|,
\]
where \( \lambda \) is a positive constant depending on \( c_1 \) and \( c_2 \).
Proof. Set $\tilde{\psi} = \frac{|D\tilde{\varphi}|^2}{2}$. Similar to the argument in Lemma 3.3, we can obtain
\[
\frac{\partial \tilde{\psi}}{\partial s} = \tilde{Q}^{\xi \xi} \tilde{\psi}_{\xi \xi} + \tilde{Q}^{\xi} \tilde{\psi}_{\xi} - \tilde{Q}^{\xi \xi} \tilde{\varphi}_{\xi \xi} + 2\alpha(1 - \tilde{Q})\tilde{\psi},
\]
with the boundary condition
\[
D_{\mu} \tilde{\psi} = 0.
\]
So we have
\[
\begin{cases}
\frac{\partial \tilde{\psi}}{\partial s} \leq \tilde{Q}^{\xi \xi} \tilde{\psi}_{\xi \xi} + \tilde{Q}^{\xi} \tilde{\psi}_{\xi} \quad \text{in } M \times (0, \infty) \\
D_{\mu} \tilde{\psi} = 0 \quad \text{on } \partial M \times (0, \infty) \\
\tilde{\psi}(\cdot, 0) = \frac{|D\tilde{\varphi}(\cdot, 0)|^2}{2} \quad \text{in } M.
\end{cases}
\]
Using the maximum principle and Hopf’s lemma, we can get the gradient estimates of $\tilde{\varphi}$, and then the inequality (5.2) follows from the relation between $\tilde{\varphi}$ and $\tilde{u}$. $\square$

Lemma 5.2. Let $u$ be a solution of the flow (2.2). Then,
\[
\tilde{u}(\cdot, s)
\]
converges to a real number as $s \to +\infty$.

Proof. Set $f(t) = \mathcal{H}^1(M_t)$, which, as before, represents the 1-dimensional Hausdorff measure of $M_t$ and is actually the length of $M_t$. The corresponding past-directed timelike unit normal vector is given by
\[
\nu = -\frac{1}{v} \left( \frac{u_{\xi} - u^2}{|Du|^2} \partial_1 + \partial_0 \right),
\]
where $v = \sqrt{1 - u^{-2}|Du|^2}$, and the unit normal vector $\nu$ can be written as $\nu = \nu^1 \partial_1 + \nu^0 \partial_0$ w.r.t. the basis $\{\partial_0 = \partial_r, \partial_1 = \partial_\xi\}$. Then
\[
-\text{div}_{M_t} \nu = -\left( \frac{\partial \nu^1}{\partial \xi} + \nu^1 \Gamma^1_{11} + \nu^0 \Gamma^1_{01} \right),
\]
with $\Gamma^K_{IJ}$ the Christoffel symbols of $\mathbb{R}^{n+1}_1$ w.r.t. the basis $\{\partial_0, \partial_1\}$. By Lemma 2.1, we can obtain
\[
-\text{div}_{M_t} \nu = \frac{u_{\xi \xi} u + u^2 - 2u_{\xi}^2}{u^3 \nu^3} = k,
\]
and according to the first variation of a submanifold (see, e.g., [16]), we have
\[
f'(t) = \int_{M_t} \text{div}_{M_t} \left( \frac{\nu}{|X|^a k} \right) d\mathcal{H}^1
\]
\[
= \int_{M_t} \left\langle \nabla_{\xi} \left( \frac{\nu}{|X|^a k} \right), e_{\xi} \right\rangle d\mathcal{H}^1
\]
\[
= -\int_{M_t} |u|^{-a} d\mathcal{H}^1,
\]
where $\{e_{\xi}\}$ is an orthonormal basis of the tangent bundle $TM_t$ (i.e., $e_{\xi} = X_\xi / |X_\xi|$ with $X_\xi$ defined as in Lemma 2.1). We know that (3.2) implies
\[
(-\alpha t + e^{\alpha \varphi_1})^{-1} \leq u^{-a} \leq (-\alpha t + e^{\alpha \varphi_2})^{-1},
\]
where \( \varphi_1 = \inf_{M^1} \varphi(\cdot, 0) \) and \( \varphi_2 = \sup_{M^1} \varphi(\cdot, 0) \). Hence
\[
-(\alpha t + e^{\alpha \varphi_2})^{-1} f(t) \leq f'(t) \leq -(\alpha t + e^{\alpha \varphi_1})^{-1} f(t).
\]
Combining this fact with (5.3) yields
\[
\frac{-\alpha t + e^{\alpha \varphi_2}}{e^{\varphi_2}} \leq f(t) \leq \frac{-\alpha t + e^{\alpha \varphi_1}}{e^{\varphi_1}}.
\]
Therefore, the rescaled hypersurface \( \widetilde{M}_s = M_t \Theta^{-1} \) satisfies the following inequality
\[
\frac{\mathcal{H}^1(M_0)}{e^{\varphi_2}} \leq \mathcal{H}^1(\widetilde{M}_s) \leq \frac{\mathcal{H}^1(M_0)}{e^{\varphi_1}},
\]
which implies that the area of \( \widetilde{M}_s \) is bounded and the bounds are independent of \( s \). Together with (4.11), Lemma [5.1] and the Arzelà-Ascoli theorem, we conclude that \( \bar{u}(\cdot, s) \) must converge in \( C^\infty(M) \) to a constant function \( r_\infty \) with
\[
\frac{1}{e^{\varphi_2}} \left( \frac{\mathcal{H}^1(M_0)}{\mathcal{H}^1(M)} \right) \leq r_\infty \leq \frac{1}{e^{\varphi_1}} \left( \frac{\mathcal{H}^1(M_0)}{\mathcal{H}^1(M)} \right),
\]
which implies the radius estimate (1.2). \( \square \)

So, we have

**Theorem 5.3.** The rescaled flow
\[
\frac{d\bar{X}}{ds} = \frac{1}{|\bar{X}|^{\alpha k}} \nu + \bar{X}
\]
events for all time and the leaves converge in \( C^\infty \) to a piece of hyperbolic plane of center at origin and radius \( r_\infty \), i.e., a piece of \( \mathcal{H}^1(r_\infty) \), where \( r_\infty \) satisfies (1.2).

**Acknowledgments**

This work is partially supported by the NSF of China (Grant Nos. 11801496 and 11926352), the Fok Ying-Tung Education Foundation (China) and Hubei Key Laboratory of Applied Mathematics (Hubei University).

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