GRUSON-SERGANOVA CHARACTER FORMULAS AND THE DUFLO-SERGANOVA COHOMOLOGY FUNCTOR

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Abstract. We establish an explicit formula for the character of an irreducible finite-dimensional representation of \( \mathfrak{gl}(m|n) \). The formula is a finite sum with integer coefficients in terms of a basis \( \mathcal{E}_\mu \) (Euler characters) of the character ring. We prove a simple formula for the behaviour of the “superversion” of \( \mathcal{E}_\mu \) in the \( \mathfrak{gl}(m|n) \) and \( \mathfrak{osp}(m|2n) \)-case under the map \( ds \) on the supercharacter ring induced by the Duflo-Serganova cohomology functor \( DS \). As an application we get combinatorial formulas for superdimensions, dimensions and \( g_0 \)-decompositions for \( \mathfrak{gl}(m|n) \) and \( \mathfrak{osp}(m|2n) \).

1. Introduction

Let \( \mathfrak{g} \) be a finite-dimensional Kac-Moody superalgebra. Denote by \( W \) the Weyl group of \( \mathfrak{g} \).

1.1. A brief history of character formulas. Let \( L(\lambda) \) be a simple finite-dimensional \( \mathfrak{g} \)-module. In 1977 V. Kac \[K1\] showed that the Weyl character formula

\[
R e^\rho \text{ch} L(\lambda) = \sum_{w \in W} \text{sgn}(w) w(e^{\lambda + \rho}),
\]

where \( R \) is the Weyl denominator and \( \rho \) is the Weyl vector holds if \( L(\lambda) \) is typical. In 1980 I. Bernstein and D. Leites \[BL\] established for \( \mathfrak{g} = \mathfrak{gl}(1|n) \) the character formula

\[
R e^\rho \text{ch} L(\lambda) = \sum_{w \in W} \text{sgn}(w) w(e^{\lambda} - e^{-\beta}),
\]

where \( \beta \in \Delta^+_1 \) satisfies \( (\beta|\lambda) = 0 \). This formula was extended to the \( \mathfrak{osp}(2|2n) \)-case in \[vdJ3\] and to \( \mathfrak{gl}(m|n) \)-modules of atypicality one in \[HKTMvdJ\]. In 1998 J. Germoni produced similar character formulas for the cases \( \mathfrak{osp}(3|2) \) and \( D(2|1; a) \); except for the case of the standard \( \mathfrak{osp}(3|2) \)-module Germoni’s formula is very similar to (1): the only difference is the factor \( 1/2 \) appearing in the right-hand side in certain cases (other formulas were obtained earlier by van der Jeugt in \[vdJ1\],\[vdJ2\]). In 1990 J. van der Jeugt, J. Hughes, R.C. King and J. Thierry-Mieg \[HKTMvdJ\] suggested to write the character
formula in the general $\mathfrak{gl}(m|n)$-case as a sum of terms \[ \sum_{w \in W} \text{sgn}(w) w \left( \prod_{\beta \in U} e^{\lambda + \rho} \right) \] for some $U \subset \Delta$, satisfying $|U| = 0$.

In 1994 V. Kac and M. Wakimoto [KW] conjectured that

\[ Re^\rho \text{ch} L(\lambda) = j^{-1} \sum_{w \in W} \text{sgn}(w) w \left( \prod_{\beta \in S} e^{\lambda + \rho} \right) \]

if the following conditions (which we will call KW-conditions) hold: $\lambda$ is the highest weight of $L$ with respect to a base $\Sigma$ containing $S$, $(S|\lambda + \rho) = (S|S) = 0$ and the cardinality of $S$ is equal to the atypicality of $L$. This conjecture was established in [CHR1], [CK] and [GK]. For each $S \subset \Delta$ satisfying $(S|\lambda) = (S|S) = 0$ we set

\[ \text{KW}(\lambda, S) := \sum_{w \in W} \text{sgn}(w) w \left( \prod_{\beta \in S} e^{\lambda} \right). \]

(We call the above terms “Kac-Wakimoto terms” since the condition $(S|S) = 0$ is crucial for our argument).

Notice that Germoni’s formulas demonstrate that the KW-conditions are not necessary for (2) to be valid: the only atypical $\mathfrak{osp}(3|2)$-module satisfying the KW-condition is trivial whereas (1) holds for each $L \not\cong V_{st}$.

The first general formula for $\text{ch} L$ was discovered by V. Seganova [S3] in the $\mathfrak{gl}(m|n)$-case by expressing the character as an infinite sum over characters of Kac modules. This algorithmic solution was enhanced by J. Brundan [Br1] who showed that the values of the coefficients in the infinite sum can be computed in terms of weight diagrams. This description was then used Y. Su and R. Zhang [SuZh1] to establish a finite character formula for $\mathfrak{gl}(m|n)$. For the $\mathfrak{osp}$-case a finite character formula was produced by C. Gruson and V. Serganova in [GST1]. For the exceptional Lie superalgebras the character formulas were proven by J. Germoni ([G1]) and L. Martirosyan ([M]). For $\mathfrak{q}_n$ an implicit finite character formula was given by I. Penkov and V. Serganova in [PS2]; Y. Su and R. B. Zhang [SuZh2] wrote this formula explicitly using [Br2]; for the $\mathfrak{p}_n$-case an infinite Serganova-type character formula was recently proven by B.-H. Hwang and J.-H. Kwon [HK].

The Kac-Wakimoto character formula suggests the following two refinements of van der Jeugt-Hughes-King-Thierry-Mieg’s proposal: to present $Re^\rho \text{ch} L$ as a linear combinations of $\text{KW}(\nu, S(\nu))$ where

I. $S_\nu$ is maximal, i.e. $|S_\nu|$ equals to the atypicality of $L(\nu)$ or

II. $S_\nu$ can be embedded into a certain base $\Sigma_L$.

The Kac-Wakimoto is of both types. Germoni’s formula and the Su-Zhang formula for $\mathfrak{gl}(m|n)$ are of type I; the Gruson-Serganova formula for $\mathfrak{osp}$ and the Su-Zhang formula for $\mathfrak{q}_n$ are of type II. For the exceptional algebras the character formulas in [G1], [M] can be rewritten as type I or as type II formulas.
1.2. The Gruson-Serganova type character formula. In this paper we obtain a formula of type II for $\mathfrak{gl}(m|n)$ (by above, such formulas were obtained early for all other cases). The main difference of the $\mathfrak{gl}(m|n)$-case is that $\Sigma_L$ depends on $L$.

Let $\text{Irr}$ be the set of isomorphism classes of the finite-dimensional irreducible $g$-modules. The terms $\{\text{ch} L, \ L \in \text{Irr}\}$ form a natural basis of the character ring. We say $\text{ch} L$ is given by a Gruson-Serganova type character formula if it can be written as a sum

$$\text{Re}^\rho \text{ch} L = \sum_{L' \in \text{Irr}} b_{L,L'} \text{KW}(L'),$$

where the Kac-Wakimoto terms $\text{KW}(L)$ have the following properties:

(i) $\{\mathcal{E}_L := (\text{Re}^\rho)^{-1} \text{KW}(L), \ L \in \text{Irr}\}$ form a basis of the character ring (where $R$ is the Weyl denominator). The terms $\mathcal{E}_L$ are equal to the Euler characteristics $\mathcal{E}_\lambda$ (for a suitable choice of parabolic) of Penkov-Serganova and hence we can equally write the character of $L(\lambda)$ as a finite sum with integral coefficients in the Euler characters;  
(ii) the character formula $\text{Re}^\rho \text{ch} L = \sum_{L' \in \text{Irr}} b_{L,L'} \text{KW}(L)$ is finite;  
(iii) the matrix $B := (b_{L,L'})$ is a lower triangular matrix with integral entries and 1s on the main diagonal; moreover, there exists a diagonal matrix $D$ with $D^2 = I_d$ such that the entries of $DBD^{-1}$ are non-negative (the entries of $DBD^{-1}$ can be interpreted as a number of certain paths in a directed graph).  
(iv) $\text{KW}(L) := j(L)^{-1} \text{KW}(\lambda^l, S_L)$, where $j(L)$ is a scalar, $\lambda^l$ is the $\rho$-shifted highest weight of $L$ with respect to a certain base $\Sigma_L$ containing $S_L$.

We call the cardinality of $S(L)$ the tail of $L$ ($\text{tail}(L)$); this is a non-negative integer which is less than or equal to the atypicality of $L$. If $b_{L,L'} \neq 0$, then $L$ and $L'$ lie in the same block and $\text{tail}(L') \leq \text{tail}(L)$. We call the highest weight of $L$ a Kostant weight if $\text{tail}(L)$ is equal to the atypicality of $L$; in this case $b_{L,L'} = \delta_{L,L'}$. From [CHR1], [CK] it follows that for $\mathfrak{gl}(m|n), \mathfrak{osp}(2m+1|2n), L(\lambda)$ satisfies the Kac-Wakimoto character formula if and only if its highest weight is a Kostant weight; this also holds for the $\mathfrak{osp}(2m|2n)$-modules of atypicality greater than one, see [GSI].

In the $\mathfrak{osp}$-case [GSI] was obtained in [GSI] in a slightly different form (property (iv) is established in Proposition 4.3 below). In this case $\Sigma_L = \Sigma$ is the usual “mixed” base and $\lambda^l = \lambda + \rho$. In this paper we will establish [GSI] in the $\mathfrak{gl}$-case. In this case $\Sigma_L$ depends on $L$; in Section 3 we describe the assignment $L \mapsto \lambda^l$; the image of this assignment can be naturally described in terms of weight diagrams.

For a finite-dimensional module $N$ define the $\xi$-character

$$\text{ch}_\xi N := \dim(N_0)e^\nu + \xi \dim(N_1)e^{\nu'}$$
(where $\xi$ is a formal variable with $\xi^2 = 1$); clearly, the $\mathbb{Z}$-span of $\xi$-characters form a ring, which we denote by $\text{Ch}_\xi(\mathfrak{g})$. The character ring $\text{Ch}(\mathfrak{g})$ (resp., the supercharacter ring $\text{Sch}(\mathfrak{g})$) is a factor of $\text{Ch}_\xi(\mathfrak{g})$ by $\xi = 1$ (resp., $\xi = -1$); these rings were explicitly described by A. N. Sergeev and A. P. Veselov [SerV]. Several important notions (for instance, $\text{dim} N$ and $\text{sdim} N$) can be viewed as linear maps from $\text{Ch}_\xi(\mathfrak{g})$. The Gruson-Serganova formula give an expression of $\text{Ch}_\xi L$ see (26) and a formula for $\text{sch} L$ in terms of $E_L^+$ (which are “superanalogues” of $E_L^-$).

An important example is the map $\text{mult}_{L'} : \text{Ch}(\mathfrak{g}) \to \mathbb{Z}$ which assigns to $\text{ch} N$ the (non-graded) multiplicity $[N : L']$, where $L'$ is a $\mathfrak{g}_0$-module; in Corollary A.5.3 we give formulas for $\text{mult}_{L'}(E_L)$ and for $\text{dim}(E_L)$.

Another important linear map is induced by the Duflo-Serganova monoidal functor $DS_x : \mathcal{F}\text{in}(\mathfrak{g}) \to \mathcal{F}\text{in}(\mathfrak{g}_x)$ where $\mathfrak{g}_x$ is a smaller rank Lie superalgebra. C. Hoyt and Sh. Reif [HR] showed that $DS_x$ induces a ring homomorphism

$$ds_x : \text{Sch}(\mathfrak{g}) \to \text{Sch}(\mathfrak{g}_x)$$

given by $ds_x : \text{sch} N \mapsto \text{sch} DS_x(N)$; moreover, $ds_x$ coincides with the evaluation of $\text{sch}$ to a subalgebra $\mathfrak{h}_x \subset \mathfrak{h}$. In Theorem 7.2 we show that $ds_x(E_L^-)$ is given by a simple formula (for the exceptional Lie superalgebras a similar formula follows from [Gor2]). Since $DS_x$ preserves $\text{dim}$, this gives a formula for $\text{dim} E_L^-$, see Corollary 7.2.8. It turns out that $\text{dim} E_L^- = 0$ except for the case when $\text{tail}(L)$ equals to the defect of $\mathfrak{g}$ (by above, $\text{tail}(L)$ is less than or equals to the defect of $\mathfrak{g}$).

Using (3) and the forementioned formulas one obtains the expressions for $\text{sch} DS_x(L)$, $\text{dim} L$, $[L : L']$ and $\text{dim} L$ for $L \in \text{Irr}$ (we do not write this long expressions). Note that $DS_x(L)$ is described in [HW], [GH] and [Gor2]; various formulas for $\text{dim} L$ and $\text{sdim} L$ appeared in [SuZh1], [M], [HW], and [GH].

1.3. Euler characters. The Euler characters were originally defined as Euler characteristics of the cohomology of vector bundles on a super flag variety [S3]. They were first introduced in [P] [PS1] [PS2] and play also a crucial role in Brundan’s work on characters in the $\mathfrak{g}(n)$-case [Br2] [Br3]. We will describe the Euler characters for the “core-free case”, i.e. for the principal block of $\mathfrak{gl}(d|d)$ or $\mathfrak{osp}(2d + t|2d)$, where $t = 0, 1, 2$. (A similar description works for all $\mathfrak{osp}$-weights and for the “stable weights” in $\mathfrak{gl}$-case.)

Fix a flag of parabolic subalgebras

$$\mathfrak{g} = \mathfrak{p}^{(d)} \supset \mathfrak{p}^{(d-1)} \supset \ldots \supset \mathfrak{p}^{(0)} = \mathfrak{b},$$

where $d$ is the defect of $\mathfrak{g}$ and $\mathfrak{l}^{(i)} := \mathfrak{p}^{(i)}\mathfrak{p}$ is of defect $i$ (one has $\mathfrak{l}^{(i)} = \mathfrak{gl}(i|i)$ for $\mathfrak{g} = \mathfrak{gl}(d|d)$, $\mathfrak{l}^{(i)} = \mathfrak{osp}(2i+t|2i)$, for $\mathfrak{g} = \mathfrak{osp}(2d+t|2d)$). For a pair of parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ containing a fixed Borel $\mathfrak{b}$ let $\Gamma_{\mathfrak{p} \mathfrak{q}}(V)$ denote the maximal finite dimensional quotient of the induced module $\mathcal{U}(\mathfrak{p}) \otimes_{\mathcal{U}(\mathfrak{q})} V$. Then $\Gamma_{\mathfrak{p} \mathfrak{q}}$ defines a functor from the category of finite-dimensional $\mathfrak{q}$-modules $\mathcal{F}\text{in}(\mathfrak{q})$ to $\mathcal{F}\text{in}(\mathfrak{p})$; and we denote by $\Gamma_{\mathfrak{p} \mathfrak{q}}^i$ its derived functors as in
For $\lambda, \mu \in \Lambda^+_m$, we consider the Poincaré polynomial in the variable $z$

$$K^{\lambda,\mu}_{p,q}(z) := \sum_{i=0}^{\infty} [\Gamma^i_{p,q}(L_q(\lambda)) : L_p(\mu)] z^i,$$

where $L_q(\lambda)$ (resp. $L_p(\mu)$) stands for the corresponding simple $q$ (resp. $p$) module. Prop. 1 in [GS1] expresses the Euler characteristic

$$E_{\lambda,p} := \sum_{\mu \in \Lambda^+_m} K^{\lambda,\mu}_{0,p}(-1) \text{ch} L(\mu)$$

in terms of $\text{ch} L_p(\lambda)$. The polynomials $K^{\lambda,\mu}_{p,q}(z)$ for the “neighbouring parabolics” were computed in [S2] in the $\mathfrak{gl}$-case and in [GS1] in the $\mathfrak{osp}$-case. These polynomials can be conveniently described in terms of so-called “arc diagrams”, see Section A.5.1. The coefficients $K^{\lambda,\mu}_{0,p}(-1)$ can be computed iteratively via the formula

$$K^{\lambda,\mu}_{0,q}(-1) = \sum_{\nu} K^{\lambda,\nu}_{p,q}(-1) K^{\nu,\mu}_{0,p}(-1)$$

established in [GS1], Thm. 1.

In the $\mathfrak{gl}$-case the matrix $A_{ab} := (K^{\lambda,\mu}_{0,p}(-1))$ is invertible and the inverse matrix can be explicitely described; this gives the Serganova character formula [S2]; this is an “infinite formula”: some rows of $A_{ab}^{-1}$ have infinitely many non-zero entries. For the $\mathfrak{osp}$-case one has $E_{\lambda,ab} = 0$ for some $\lambda$’s, so the matrix $A_{ab}$ is not invertible. In order to obtain the Gruson-Serganova character formula we take for each $\lambda$ the “maximal suitable parabolic”, setting $p_\lambda := p_{\text{tail}(\lambda)}$, where $\text{tail}(\lambda)$ the maximal $i$ such that $\lambda_{h\cap l(i)} = 0$. (One has $\text{tail}(\lambda) = \text{tail}(L(\lambda))$ for $\mathfrak{osp}$-weights and for the “stable” $\mathfrak{gl}$-weights). The iterative formula (5) allows to interpret $K^{\lambda,\mu}_{0,p}(-1)$ in terms of “decreasing paths” in a certain directed graph. This graph has several nice properties, which allow to express the inverse matrix $(K^{\lambda,\mu}_{0,p}(-1))^{-1}$ in terms of the paths in this graph; using (4) we obtain

$$\text{ch} L(\lambda) = \sum_{\mu \in \Lambda^+_m} (-1)^{||\lambda||-||\mu||} d^{\lambda,\mu}_{\lessdot} E_{\mu,p_\mu}$$

for stable weights $\lambda$ where $d^{\lambda,\mu}_{\lessdot}$ is the number of “increasing paths” from $\mu$ to $\lambda$ in the directed graph and $||\lambda||$ is defined in [GS5]. Since $\dim L_{p_\mu}(\mu) = 1$, Prop. 1 in [GS1] gives an explicit formula for $E_{\mu,p_\mu}$; using the denominator identity from [KW] this formula can be rewritten as $Re^{\rho} E_{\mu,p_\mu} = KW(L)$. This gives (3).

For the $\mathfrak{osp}$-case this program was executed in [GS1]. In the first part of our paper we execute a similar program for $\mathfrak{gl}$. The graph in the $\mathfrak{gl}$-case has an easier description than in the $\mathfrak{osp}$-case, but its structure is more complicated: by contrast with the $\mathfrak{osp}$-case each component has infinitely many sources. We show that in the $\mathfrak{gl}$-case each vertex has finitely many predecessors and this property allows us to obtain (6). We will reveal some additional details in 1.4 below.
To link (6) and (3) we show in Proposition 4.3 that the Euler characters are proportional to Kac-Wakimoto terms. This result is also fundamental when we study the effect of $d$s on Euler characters in the second part of the paper.

A similar approach works for the exceptional Lie superalgebras and for $q_n$; in these cases each component of the graph has a unique source. To the best of our knowledge, the Gruson-Serganova type character formula is not known for the cases each component of the graph has a unique source. To the best of our knowledge, the Gruson-Serganova type character formula is not known for the $p_n$-case; we expect that in this case each component has infinitely many sources as for the $\mathfrak{gl}(m|n)$-case.

1.4. Method of Proof. The proof of (6) uses iterated parabolic induction. We will outline this proof for the principal block of $\mathfrak{gl}(d|d)$. A similar proof works for all $\mathfrak{osp}$-weights and for the “stable weights” in $\mathfrak{gl}$-case. On the other hand, the character formula for a simple module of atypicality $d$ can be reduced to this case.

The graphs $\hat{\Gamma}^\chi$ and $\Gamma^\chi$ are directed graphs with the same set of vertices enumerated by the highest weights of the irreducible modules in the principal block. In $\Gamma^\chi$ the vertices $\mu, \lambda$ are joined by the edge $\mu \xrightarrow{e} \lambda$ if $K_{p^{(s)}_\mu, p^{(s-1)}_\lambda}^{\lambda, \mu} \neq \delta_{\lambda, \mu}$ for such edge we set $b(e) := s$. For $\Gamma^\chi$ we require that $p^{s-1}_\lambda \supset p_\mu$ (in other words, $\Gamma^\chi$ can be obtained from $\hat{\Gamma}^\chi$ by deleting the edges with $b(e) \leq \text{tail}(\lambda)$). By [MusS], for the $\mathfrak{gl}(m|n)$-case $K_{p^{(s)}_\mu, p^{(s-1)}_\lambda}^{\lambda, \mu}$ is either $\delta_{\lambda, \nu}$ or $z^i$ with $i \equiv ||\lambda|| - ||\mu||$. In particular, if $\mu$ and $\lambda$ are connected by an edge in $\hat{\Gamma}^\chi$, then $K_{p^{(0)}_\mu, p^{(0)}_\lambda}^{\lambda, \mu} = (-1)^{||\lambda|| - ||\mu||}$.

We define a “decreasing path” in $\hat{\Gamma}^\chi$, $\Gamma^\chi$ as a path with a decreasing function $b$. The formula (8) allows to express $K_{\mu, \lambda}^{\lambda, \mu}(-1)$ can be written as a sum of $(-1)^{\text{length}(P) + ||\lambda|| - ||\mu||}$, where $P$ runs through the decreasing paths from $\mu$ to $\lambda$ in $\hat{\Gamma}^\chi$. We show that $K_{\mu, \lambda}^{\lambda, \mu}(-1)$ has the similar formula in terms of the decreasing paths in $\Gamma^\chi$ (the proof uses the fact that a path in $\Gamma^\chi$ lies in $\Gamma^\chi$ if the last edge of this path lies in $\Gamma^\chi$).

The matrix $A_{\succ} := (K_{\mu, \lambda}^{\lambda, \mu}(-1))$ is invertible; by above, its entries can be written in terms of the decreasing paths from $\mu$ to $\lambda$ in $\Gamma^\chi$, where the decreasing path is defined in terms of the function $b$. We change $b$ to another function $b'$ such in such a way that the function $b(e)$ is substituted by another function $b'(e)$ such that a path is decreasing for $b$ if and only if it is decreasing for $b'$. The function $b'$ has the following advantage:

(*) if $\nu$ is the start of an edge $e_1$ and the end of an edge $e_2$, then $b'(e_1) \neq b'(e_2)$

(this property does not hold for the function $b$). By above, $K_{\mu, \lambda}^{\lambda, \mu}(-1)$ can be written as a sum of $(-1)^{\text{length}(P) + ||\lambda|| - ||\mu||}$, where $P$ runs through the decreasing paths from $\mu$ to $\lambda$ in $\Gamma^\chi$. The property (*) implies that the entries of $A_{\prec}^{-1}$ can be written as a sum of $(-1)^{||\lambda|| - ||\mu||}$, where $P$ runs through the increasing paths with respect to $b'$; this gives the formula (6) ($d_{\lambda, \mu}^{\lambda, \mu}$ stands for the number of increasing paths from $\mu$ to $\lambda$ with respect to $b'$). The graph $\hat{\Gamma}^\chi$ and the functions $b, b'$, $\text{deg}(e)$ can be naturally described in terms of arc diagrams, (see 4.5.3).
In order to prove the finiteness of the formula (6) we show that each vertex \( \lambda \) has a finite set of predecessors in the \( \Gamma^x \) (for \( \mathfrak{gl} \)-case this property does not hold for \( \hat{\Gamma}^x \); for \( \mathfrak{osp} \) and \( \mathfrak{q} \)-cases the property hold in both cases, since \( \{ \mu \in \Lambda_{m|n} | \mu \leq \lambda \} \) is finite).

1.5. Modified superdimensions. The Kac-Wakimoto conjecture states that \( \text{sdim} L(\lambda) \neq 0 \) if and only if \( L(\lambda) \) has a maximal atypicality; this conjecture was proven by V. Serganova in [S4]. In the \( \mathfrak{gl}(m|n) \)-case \( \text{sdim} L(\lambda) \) was computed in [HW].

Consider the case \( \mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(M|N) \). Fix a triangular decomposition in the usual way (see [GS1], [GS2], etc.), i.e. a distinguished base for \( \mathfrak{gl}(m|n) \) and the mixed base for \( \mathfrak{osp}(m|2n) \). Let \( L(\lambda) \) be a finite-dimensional simple module of atypicality \( k \). In this case applying DS\(_x\) with \( rk(x) = k \) to \( L(\lambda) \) gives by Theorem 7.2 an isotypic representation \( L(\lambda') \oplus m(\lambda) \) of \( \mathfrak{g}_x = \mathfrak{gl}(m - k|n - k) \) in the \( \mathfrak{gl} \)-case and in the \( \mathfrak{osp} \)-case either an isotypic representation \( L(\lambda') \oplus m(\lambda) \) of \( \mathfrak{g}_x = \mathfrak{osp}(m - 2k|2n - 2k) \) (if \( L(\lambda') \) is \( \sigma \) invariant for the involution \( \sigma \) of \( OSp \), see Section 2.2) or \( (L(\lambda') \oplus L(\lambda')^\sigma) \oplus m(\lambda) \) else. We put \( L^\text{core} = L(\lambda') \) in the \( \mathfrak{gl} \) and \( \mathfrak{osp}(2m + 1|2n) \)-case and

\[
L^\text{core} := \begin{cases} L(\lambda') & \text{if } \lambda' \text{ is } \sigma - \text{invariant} \\ L(\lambda') \oplus L(\lambda')^\sigma & \text{else} \end{cases}
\]

in the \( \mathfrak{osp}(2m|2n) \)-case. Then \( L^\text{core} \) only depends on the central character of \( \lambda \). Using this notation we obtain in case where the atypicality of \( L(\lambda) \) equals the rank of \( x \) the uniform formula

\[
\text{DS}_x(L(\lambda)) \cong \Pi^i(L^\text{core}) \oplus m(\lambda)
\]

for some parity shift \( \Pi^i \).

Identifying \( \mathfrak{g}_x \) with a subalgebra of \( \mathfrak{g} \) as in [DS], we can interpret the above formula as follows: for a simple \( \mathfrak{g}_x \)-module \( L' \) the “super multiplicity” of \( L' \) in \( L(\lambda) \) is zero if \( [L^\text{core} : L'] = 0 \) and is \( \pm m(\lambda) \) otherwise, see [2.3.1].

If \( L(\lambda) \) is maximal atypical, \( \mathfrak{g}_x \) is one of the algebras \( \mathfrak{gl}_k, \mathfrak{o}_k, \mathfrak{sp}_k, \mathfrak{osp}(1|2k) \).

The numbers \( m(\lambda) \) can be computed in the equal rank case. In this case \( m(\lambda) = \text{sdim} L(\lambda) \) is equal to the number of increasing paths from the Kostant weights, which are \( \mu \)'s with \( \dim L(\mu) = 1 \), to \( \lambda \). (In all cases it is easy to see that the existence of such paths is equivalent to the maximal atypicality condition). Therefore we reprove the Kac-Wakimoto conjecture and establish another combinatorial expressions for the superdimensions.

If \( L(\lambda) \) is not maximal atypical, one can introduce a modified superdimension \( \text{sdim}^k \) on the thick tensor ideal spanned by the irreducible representations of atypicality \( k \) instead. We show that the modified superdimension is given by the formula \( \text{sdim}^k(L(\lambda)) = \pm m(\lambda) \text{sdim}^0(L^\text{core}) \) for \( L(\lambda) \) of atypicality \( k \) where \( \text{sdim}^0 \) is the (unique up to a scalar) modified superdimension on the thick ideal of projective objects, reproving results of [S4] and [Ku].
By [GH] the isotypic multiplicity in \(\text{osp}\)-case can be expressed in terms of the arc diagram of \(\lambda\).

1.6. **Structure of the article.** We recall some backgrounds in Sections 2 and 3. In particular, in Section 3 we discuss stability, tail and weight diagrams. We define parabolic induction functors and their derived versions in Section 4. The main results - formula (3) for \(\text{gl}(m|n)\) and the behaviour of \(E^{-\mu}\)'s under \(ds\)- are proven in Section 5 and Theorem 7.2. For the relationship of (3) to other existing \(\text{gl}(m|n)\)-character formulas (notably the one from Su-Zhang) see 5.4. Section 6 deals with \(E^{\mu}, p^{\mu}\) and \(KW(L)\) in the \(\text{gl}(m|n)\)-case; in particular, we describe the assignment \(L \mapsto \lambda^\dagger\) and establish property (iv). The results on superdimensions and modified superdimensions are assembled in Section 8. We discuss properties of \(KW(L)\) in Appendix A. In A.5 we compute \(\dim E^{\lambda}\) and obtain a formula for the multiplicity of a \(\mathfrak{g}_0\)-module in \(L(\lambda)\), see Corollary A.5.3, A.5.5 and A.5.6 for the graded versions.

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1.7. **Index of definitions and notation.** Throughout the paper the ground field is \(\mathbb{C}\); \(\mathbb{N}\) stands for the set of non-negative integers. We will use the standard Kac notation for the root systems.
2. Preliminaries

We denote by \( \Pi \) the parity change functor. Throughout the Sections 2–8 \( g \) stands for one of the Lie superalgebras \( \mathfrak{gl}(m|n), \mathfrak{osp}(2m|2n) \) or \( \mathfrak{osp}(2m + 1|2n) \).

2.1. Notation. We use the standard notation: the root system \( \Delta \) lies in the lattice \( \Lambda_{m|n} \subset \mathfrak{h}^* \) spanned by \( \{\varepsilon_i\}_{i=1}^m \cup \{\delta_i\}_{i=1}^n \). We denote by \( \Lambda \) the lattice spanned by \( \{\varepsilon_i\}_{i=1}^\infty \cup \{\delta_i\}_{i=1}^\infty \) and view \( \Lambda_{m|n} \) as a subset of \( \Lambda \). We define the parity homomorphism \( p : \Lambda \rightarrow \mathbb{Z}_2 \) by \( p(\varepsilon_i) = 0, \) \( p(\delta_j) = 1 \) for all \( i, j \).

The category \( \mathcal{F}_{\text{in}} \) of finite dimensional representations of \( g \) with parity preserving morphisms is the direct sum of two categories: \( \tilde{\mathcal{F}} \) with the modules whose weights lie in \( \Lambda_{m|n} \) and \( \tilde{\mathcal{F}}^\perp \) with the modules whose weights lie in \( \mathfrak{h}^* \setminus \Lambda_{m|n} \). The category \( \tilde{\mathcal{F}}^\perp \) is semisimple and \( \text{DS}_x(\tilde{\mathcal{F}}^\perp) = 0 \) for \( x \neq 0 \); all simple modules in \( \tilde{\mathcal{F}}^\perp \) are typical, their characters are given by the Weyl-Kac character formula (in our notations \( \mathcal{S}L(\lambda) = E_{\lambda} \)).

The category \( \tilde{\mathcal{F}} \) is canonically isomorphic to the category of \( G \)-modules, where \( G \) is a classical supergroup corresponding to \( g \):

\[
G := GL(m|n) \quad \text{for} \quad \mathfrak{gl}(m|n) \quad \text{and} \quad G := SOSp(m|n) \quad \text{for} \quad g = \mathfrak{osp}(m|n).
\]

We fix the same triangular decomposition as in [GS1], [GS2], [MusS]: for \( \mathfrak{gl}(m|n) \) we choose the base \( \Sigma \) which contains only one odd root \( \varepsilon_m - \delta_1 \) and in the \( \mathfrak{osp} \)-case we choose a base \( \Sigma \) which contains a maximal possible number of odd roots; we always consider \( \Sigma \) as the ordered set with respect to the usual order (see examples in 3.1 below).

We denote by \( \Lambda^+_{m|n} \) the set of dominant weights in \( \Lambda_{m|n}^+ \):

\[
\Lambda^+_{m|n} := \{ \lambda \in \Lambda_{m|n} \mid \dim L(\lambda) < \infty \}.
\]

The simple modules in \( \tilde{\mathcal{F}} \) are of the form \( L(\lambda), \Pi(L(\lambda)) \) for \( \lambda \in \Lambda^+_{m|n} \).

The category \( \tilde{\mathcal{F}} \) decomposes into a direct sum two categories

\[
\tilde{\mathcal{F}} = \mathcal{F} \oplus \Pi \mathcal{F}
\]
such that the simple objects in $\mathcal{F}$ are labelled by the dominant integral weights. Note that $\tilde{\mathcal{F}}$ and $\mathcal{F}$ are tensor categories.

2.2. $Osp$, $SOsp$ and $\sigma$. One has $O(2r+1) = SO(2r+1) \times \mathbb{Z}_2$ and $O(2r) = SO(2r) \times \mathbb{Z}_2$; we can choose the subgroup $\mathbb{Z}_2$ in such a way that $\mathbb{Z}_2$ acts on $\mathfrak{osp}(2r|2n)$ by an involutive automorphism $\sigma$ which stabilizes the Cartan algebra $\mathfrak{h}$. For $r > 1$ (i.e., $\mathfrak{g} \neq \mathbb{C}$), $\sigma$ induces a Dynkin diagram involution given by

$$
\sigma(\delta_j) = \delta_j, \quad j = 1, \ldots, n; \quad \sigma(\varepsilon_i) = \varepsilon_i \quad \text{for } i = 1, \ldots, r-1; \quad \sigma(\varepsilon_r) = -\varepsilon_r.
$$

For odd $m = 2r + 1$ the orthosymplectic supergroup $Osp(m|2n)$ is a direct product $Osp(2r+1|2n) \cong SOsp(2r+1|2n) \times \mathbb{Z}_2$ where the non trivial element of $\mathbb{Z}_2$ acts as minus the identity. For even $m = 2r$ it is a semidirect product $Osp(2r|2n) \cong SOsp(2r|2n) \rtimes \mathbb{Z}_2$.

The underlying even group of $Osp(m|2n)$ is $O(m) \times Sp(2n)$ and $SO(m) \times Sp(2n)$ in the $SOsp$-case.

The automorphism $\sigma$ can be extended to $\mathfrak{osp}(2r|2n)$. For $m > 1$ the involution $\sigma$ on $\mathfrak{h}^*$ is given by

$$
\sigma(\delta_j) = \delta_j \quad j = 1, \ldots, n
$$

$$
\sigma(\varepsilon_i) = \varepsilon_i \quad i = 1, \ldots, r-1
$$

$$
\sigma(\varepsilon_r) = -\varepsilon_r
$$

A finite-dimensional $SO(2r)$-module $N$ can be extended to $O(2r)$ if and only if $N^\sigma \cong N$. Similarly, a finite-dimensional $SOsp(2r|2n)$-module $N$ can be extended to $Osp(2r|2n)$ if and only if $N^\sigma \cong N$. See also [ES] for more details.

2.3. The DS-functor. The DS-functor was introduced in [DS]. We recall the definition below.

For a $\mathfrak{g}$-module $M$ and $g \in \mathfrak{g}$ we set

$$
M^g := \ker_M g.
$$

We fix now an $x \in \mathfrak{g}_1$ with $[x, x] = 0$. We set $\mathfrak{g}_x := \mathfrak{g}^{ad,x}/[x, \mathfrak{g}]$; note that $\mathfrak{g}^{ad,x}$ and $\mathfrak{g}_x$ are Lie superalgebras. For a $\mathfrak{g}$-module $M$ we set

$$
DS_x(M) = M^x/xM.
$$

Observe that $M^x, xM$ are $\mathfrak{g}^{ad,x}$-invariant and $[x, \mathfrak{g}] M^x \subset xM$, so $DS_x(M)$ is a $\mathfrak{g}^{ad,x}$-module and $\mathfrak{g}_x$-module. Thus $DS_x : M \rightarrow DS_x(M)$ is a symmetric monoidal functor from the category of $\mathfrak{g}$-modules to the category of $\mathfrak{g}_x$-modules.
2.3.1. **Remark.** Notice that the action of $x$ provides a $\mathfrak{g}^{ad_x}$-isomorphism $M/M^x \overset{\sim}{\to} \Pi(xM)$. This implies that the “super multiplicity” of a simple $\mathfrak{g}^{ad_x}$-module $L'$ in a $\mathfrak{g}$-module $M$ equals to the “super multiplicity” of $L'$ in the $\mathfrak{g}^x$-module $DS_x(M)$:

$$[M : L'] - [M : \Pi(L')] = [DS_x(M) : L'] - [DS_x(M) : \Pi(L')].$$

In many examples $\mathfrak{g}_x$ can be identified with a subalgebra of $\mathfrak{g}$; in this case the same holds for a simple $\mathfrak{g}_x$-module $L'$. The examples of such situation includes the cases when $\mathfrak{g}$ is a finite-dimensional Kac-Moody algebra (and $x$ is arbitrary), see [DS].

2.3.2. Let $G_0$ denote $GL(m) \times GL(n)$ (the $\mathfrak{gl}$-case) or $O(m) \times Sp(2n)$ (the $\mathfrak{osp}$-case). Then there exists $g \in G_0$ and isotropic mutually orthogonal linearly independent roots $\alpha_1, \ldots, \alpha_j$ such that

$$Ad_g(x) = x_1 + \ldots + x_j, \quad \text{where } x_i \in \mathfrak{g}_{\alpha_i}.$$  

The number $j$ does not depend on the choice of $g$ and is denoted by rank $x$ (or $rk(x)$) [DS]. The Lie superalgebra $\mathfrak{g}_x$ depends only on the rank of $x$. For $rk(x) = k$ we have

$$\mathfrak{g}_x \cong \begin{cases} \mathfrak{gl}(m-k|n-k) & g = \mathfrak{gl}(m|n) \\ \mathfrak{osp}(m-2k|2n-2k) & g = \mathfrak{osp}(m|2n). \end{cases}$$

Take $x := x_1 + \ldots + x_j$ as above. Then the algebra $\mathfrak{h}_x := \mathfrak{h}^{ad_x}/([x, \mathfrak{g}] \cap \mathfrak{h})$ is a Cartan subalgebra of $\mathfrak{g}_x$. The functor $DS_x$ induces a ring homomorphism $ds_x : \text{Sch}(\mathfrak{g}) \to \text{Sch}(\mathfrak{g}_x)$ such that

$$\text{sch } DS_x(N) = ds_x(\text{sch } N)$$

for each $N \in \mathcal{F}\text{in}(\mathfrak{g})$. This homomorphism can be described as follows: the restriction $f \mapsto f|_{\mathfrak{h}_x}$ gives a ring homomorphism $\text{Sch}(\mathfrak{g}) \to \text{Sch}(\mathfrak{g}^{ad_x})$; the image of this map lies in $\text{Sch}(\mathfrak{g}_x)$ (which is a subring in $\text{Sch}(\mathfrak{g}^{ad_x})$) and $ds_x : \text{Sch}(\mathfrak{g}) \to \text{Sch}(\mathfrak{g}_x)$ is the corresponding map. If we choose $\mathfrak{h}_x \subset \mathfrak{h}^x$ such that $\mathfrak{h}^{ad_x} = \mathfrak{h}_x \oplus ([x, \mathfrak{g}] \cap \mathfrak{h})$, then $ds_x$ is given by $f \mapsto f|_{\mathfrak{h}_x}$, see [HR], Lemma 4.

2.3.3. In this paper we will describe the action of $ds_x$ on a certain basis of $\text{Sch}(\mathfrak{g})$. We do not use DS, but $ds$ only; and while $DS_x$ depends on $x$ (even for fixed rank [HW]), $ds_x$ depends only on the rank of $x$ and we simply write $ds_k$ for $ds_x$ with $rk(x) = k$. Then

$$(7) \quad ds_j = (ds_1)^j.$$

### 3. Weights, roots and diagrams

We use the standard notation for the roots of $\mathfrak{g}_0$ and denote by $\Pi_0$ a standard set of simple roots. In what follows we consider only bases $\Sigma$ of $\Delta$ which are compatible with $\Pi_0$, that is $\Delta^+(\Sigma)_0 = \Delta^+(\Pi_0)$. By [SI], all such bases are connected by chains of odd reflections. These bases can be encoded by words consisting of $m$ letters $\epsilon$ and $n$ letters $\delta$ (see examples below).
We fix a standard bilinear form on \( \mathfrak{h}^* \): \((\epsilon_i | \epsilon_j) = \delta_{ij} = - (\delta_i | \delta_j), (\epsilon_i | \delta_j) = 0\).

3.1. **The base \( \Sigma \) and the sets \( S_s \).** We set \( S_0 := \emptyset \) and introduce the sets \( S_s \) for \( s = 1, \ldots, \min(m,n) \) as follows.

\[
S_s := \begin{cases} 
\{ \varepsilon_{m+1-i} - \delta_i \}_{i=1}^s & \text{for } \mathfrak{gl}(m|n) \\
\{ \delta_{n-i} - \varepsilon_{m-1} \}_{i=0}^{s-1} & \text{for } \mathfrak{osp}(2m|2n) \\
\{ \varepsilon_{m-i} - \delta_{n-1} \}_{i=0}^{s-1} & \text{for } \mathfrak{osp}(2m+1|2n) 
\end{cases}
\]

Notice that \( S_{\min(m,n)} \) is a basis of a maximal isotropic subspace of \( \mathfrak{h}^* \).

For \( \mathfrak{gl}(m|n) \) we take the base \( \Sigma \) corresponding to the word \( \varepsilon^m \delta^n \):

\[
\Sigma := \{ \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n \}.
\]

For \( \mathfrak{osp} \)-case we denote by \( \Sigma \) a base containing \( S_{\min(m,n)} \) (such a base is unique): this is the base \( \delta^{n-m}(\varepsilon \delta)^m \) (resp., \( \varepsilon^{m-n}(\varepsilon \delta)^n \)) for \( \mathfrak{osp}(2m|2n) \) with \( n \geq m \) (resp., \( n \leq m \)) and \( \delta^{n-m}(\varepsilon \delta)^m \) (resp., \( \varepsilon^{m-n}(\varepsilon \delta)^n \)) for \( \mathfrak{osp}(2m+1|2n) \) with \( n \geq m \) (resp., \( n \leq m \)).

For instance,

\[
\Sigma = \{ \delta_1 - \delta_2, \ldots, \delta_{n-m+1} - \varepsilon_1, \varepsilon_1 - \delta_{n-m+2}, \delta_{n-m+2} - \varepsilon_2, \ldots, \varepsilon_{m-1} - \delta_n, \delta_n \pm \varepsilon_m \}
\]

for \( \mathfrak{osp}(2m|2n) \) with \( n \geq m \).

3.1.1. Remark. In [GH] we used the same bases \( \Sigma \), but in the \( \mathfrak{osp}(2m|2n) \)-case we chose different \( S \) for different blocks. The formulas in Proposition 4.3 hold for both choices of \( S \).

3.1.2. We denote by \( \rho \) the Weyl vector of \( \mathfrak{g} \) (see \( \Delta_1 \)). Note that \( \rho \) is unique for \( \mathfrak{osp}(M|2n) \) with \( M \neq 2 \); for \( \mathfrak{osp}(2|2n) \) we take \( \rho = \sum_{i=1}^n (n-i) \delta_i \) and for \( \mathfrak{gl}(m|n) \) we take \( \rho = \sum_{i=1}^n (m+1-i) \varepsilon_i - \sum_{i=1}^n i \delta_i \). Note that \( (\rho|S_{\min(m,n)}) = 0 \); in the \( \mathfrak{osp}(2m|2n) \)-case one has \( \sigma(\rho) = \rho \).

3.2. **Atypicality, stability and tails.** We call \( S \subset \Delta_1 \) an iso-set if \( S \) forms a basis of an isotropic subspace in \( \mathfrak{h}^* \), i.e. \( S \) is linearly independent and \( (S|S) = 0 \). For instance, \( S_1 \) is an iso-set.

For \( \lambda \in \mathfrak{h}^* \) we denote by \( \text{at} (\lambda) \) the atypicality of \( \lambda \) (i.e. the maximal cardinality of an iso-set orthogonal to \( \lambda \)). The atypicality of \( L(\lambda) \) is equal to \( \text{at}(\lambda + \rho) \).

The stability is usually introduced for a weight diagram. Below we will introduce this notion for a weight (and a fixed base \( \Sigma \)).

We say that \( \mathfrak{g}_s \subset \mathfrak{g} \) is an equal rank subalgebra if \( \mathfrak{g}_s \) is of the following form: \( \mathfrak{g}_s = \mathfrak{gl}(s|s) \) for \( \mathfrak{gl} \)-case, \( \mathfrak{g}_s = \mathfrak{osp}(2s+1|2s) \) for \( \mathfrak{g} = \mathfrak{osp}(2m+1|2n) \), \( \mathfrak{g}_s = \mathfrak{osp}(2s|2s) \) or \( \mathfrak{osp}(2s+2|2s) \) for \( \mathfrak{g} = \mathfrak{osp}(2m|2n) \), and, in addition, \( \mathfrak{g}_s \) has a base \( \Sigma_s \subset \Sigma \). Note that \( \rho'_s := \rho|_{\mathfrak{g}_s|\mathfrak{h}} \) satisfies \( (\rho'_s|\alpha) = 2(\alpha|\alpha) \) for each \( \alpha \in \Sigma_s \), so \( \rho'_s \) is “a Weyl vector” for \( \mathfrak{g}_s \) (\( \rho'_s \) is the usual Weyl vector except for \( \mathfrak{gl} \)-case). Observe that \( \mathfrak{g} \) contains a unique copy of \( \mathfrak{g}_s \) for each \( s \) with \( 0 < s \leq \min(m,n) \).
3.2.1. **Definition.** In the $\mathfrak{gl}$-case we say that $\lambda \in \Lambda^+_{m|n}$ is a **stable** weight if there exists an equal rank subalgebra $\mathfrak{g}_s \subset \mathfrak{g}$ such that for

$$\text{at}(\lambda + \rho)|_{\mathfrak{h} \cap \mathfrak{g}_s} = \text{at}(\lambda + \rho) = s \ (= \text{defect} \ \mathfrak{g}_s).$$

3.2.2. **Definition.** Take $\lambda \in \Lambda^+_{m|n}$ which is assumed to be stable for $\mathfrak{gl}(m|n)$-case. We denote by $\mathfrak{g}_\lambda$ the maximal equal rank subalgebra of $\mathfrak{g}$ satisfying $\lambda|_{\mathfrak{h} \cap [\mathfrak{s}, \mathfrak{g}_\lambda]} = 0$; we call $\mathfrak{g}_\lambda$ the **tail subalgebra** of $\lambda$ and denote by tail($\lambda$) the defect of $\mathfrak{g}_\lambda$.

3.2.3. **Examples.** The $\mathfrak{gl}(3|3)$-weight $\lambda$ with $\lambda + \rho = 3\varepsilon_1 + 2\varepsilon_2 - 2\delta_2 - 5\delta_3$ is stable (with $\mathfrak{g}_s = \mathfrak{gl}(2|2)$), one has $\mathfrak{g}_\lambda = \mathfrak{gl}(1|1)$ and tail($\lambda$) = 1; the $\mathfrak{gl}(3|3)$-weight $\nu$ with $\nu + \rho = 3\varepsilon_1 + \varepsilon_2 - \delta_2 - 5\delta_3$ is stable (with $\mathfrak{g}_s = \mathfrak{gl}(2|2)$), one has $\mathfrak{g}_\nu = \mathfrak{gl}(2|2)$ and tail($\nu$) = 2.

3.3. **Weight diagrams.** Many properties of a finite dimensional representation $L(\lambda)$ can be better understood by assigning a **weight diagram** to the weight $\lambda$ (see e.g. [S4], [HW] [EAS]). These were first defined in [BS2] for $\mathfrak{gl}$ and then for $\mathfrak{osp}(m|2n)$ in [GS1] and for $OSp$ in [ES]. Note that the conventions how to draw these weight diagrams differ. The original weight diagrams of [BS2] use a different labeling of the vertices: Our > is a ×, our < a ◦ and our × a ∨. For the difference between the weight diagrams of [GS1] and [ES] in the $\mathfrak{osp}$-case see [ES] Proposition 6.1. We follow essentially [GS1].

We denote by $\Lambda^\geq_{m|n}$ the following subgroup of $\mathfrak{h}^*$:

$$\Lambda^\geq_{m|n} := \left\{ \sum_{i=1}^m a_i \varepsilon_i + \sum_{j=1}^n b_j \delta_j \mid a_1 \in \frac{1}{2} \mathbb{Z}, \ a_1 - b_1 \in \mathbb{Z}, \ a_i - a_j, b_i - b_j \in \mathbb{N}, \ \text{for} \ i < j \right\}. $$

The set $\Lambda^\geq_{m|n}$ contains $\rho, \Lambda^+_m$ and $\Lambda^+_n + \rho$.

3.3.1. We assign the weight diagram (a labeling of the real line $\mathbb{R}$ by certain symbols) to each weight $\sum_{i=1}^m a_i \varepsilon_i + \sum_{j=1}^n b_j \delta_j \in \Lambda^\geq_{m|n}$ using the following rules:

- For $\mathfrak{gl}(m|n)$ with odd $m - n$ we put $>$ (resp., $<$) at the position with the coordinate $j$ if $a_i = j$ (resp., $b_i = -j$) for some $i$;

- For $\mathfrak{gl}(m|n)$ with even $m - n$ we put $>$ (resp., $<$) at the position with the coordinate $j + \frac{1}{2}$ if $a_i = j$ (resp., $b_i = -j$) for some $i$;

- For $\mathfrak{osp}(2m|2n)$ we put $>$ (resp., $<$) at the position with the coordinate $t$ if $|a_i| = j$ (resp., $|b_i| = j$) for some $i$; if $a_m \neq 0$ we add the sign $+$ (resp., $-$) if $a_m > 0$ (resp., $a_m < 0$);
for $\mathfrak{osp}(2m + 1|2n)$ we put $>$ (resp., $<$) at the position with the coordinate $j - 1/2$ if $a_i = j$ (resp., $b_i = j$) for some $i$; we add the sign $+$ (resp., $-$) if the zero position is occupied by $\times^p$ for $p > 0$ and $(\lambda + \rho|\varepsilon_i) = 1/2$ for some $i$ (resp., $(\lambda + \rho|\varepsilon_i) \neq 1/2$ for each $i$).

If $>, <$ occupy the same position we write these symbols as $\times$ (stands for $s$ symbols $<$ and $s$ symbols $>$); $\overset{\times}{\varepsilon}$ stands for $s$ symbols $<$ and $s + 1$ symbols $>$). We put an “empty symbol” $\circ$ at the non-occupied positions with the coordinates in $a_0 + \mathbb{Z}$; sometimes instead of $\circ$ we put its coordinate (for instance, $0 \times \circ$ means that $\times$ has the coordinate 2). For a diagram $f$ we denote by $f(a)$ the symbols at the $a$-th position.

Note that for $\mathfrak{osp}(2m + 1|2n)$-case our diagram is obtained from the diagram used in $\mathfrak{gl}$ by the shift by $-1/2$.

3.3.2. Examples. The diagram of $\rho$ for $\mathfrak{osp}(2n + 1|2n)$ has the sign $-$ and contains $n$ symbols $\times$ in the zero position; we write this as $(\times)^n$; similarly, for $\mathfrak{osp}(2n|2n)$ the diagram of $\rho$ is $(\times)^n$ and for $\mathfrak{gl}(3|3)$ the diagram of $\rho$ is $\times \times \times$, where the rightmost symbol $\times$ appears in the position 0. For $\mathfrak{gl}(m|n)$ we sometimes add a coordinate of $\circ$ instead one empty symbol; for instance, for $\mathfrak{gl}(4|3)$ the diagram of $\rho$ can be written as

$$0 \times \times \times > \text{ or } -1 \circ \times \times \times > .$$

3.3.3. We assign to each $\lambda \in \Lambda_{m|n}^{\geq}$ the diagram of $\lambda + \rho$ constructed as above; this diagram will be denoted by $\text{diag}(\lambda)$. This procedure gives a one-to-one correspondence between $\Lambda_{m|n}^{\geq}$ and the diagrams containing $k$ symbols $\times$, $m - k$ symbols $>$ and $n - k$ symbols $<$ (where $k \leq \min(m, n)$) with the following additional properties:

- the atypicality of $L(\lambda)$ is equal to the number of symbols $\times$ in the diagram of $\lambda + \rho$;
- in the $\mathfrak{gl}$-case the coordinates of the occupied positions lie in $\mathbb{Z}$ and each occupied position contains exactly one of the signs $\{>, <, \times\}$;
- in the $\mathfrak{osp}$-case the coordinates of the occupied positions lie in $\mathbb{N}$ and each non-zero occupied position contains exactly one of the signs $\{>, <, \times\}$;
- in the $\mathfrak{osp}(2m|2n)$-case the zero position does not contain $<$, contains at most one symbol $>$ and an arbitrary number of $\times$; a diagram $f$ has a sign if and only if $f(0) = \circ$;
- in the $\mathfrak{osp}(2m + 1|2n)$-case the zero position contains at most one of the symbols $>$, $<$ and an arbitrary number of $\times$; a diagram $f$ has a sign if and only if $f(0) = \times^i$ for $i > 0$.

3.3.4. Remark: $OSp(2m|2n)$-modules. By 22 simple $OSp(2m|2n)$-modules are in one-to-one correspondence with the unsigned $\mathfrak{osp}(2m|2n)$-diagrams.

3.3.5. Tail in the diagramic language. It is easy to see that in the $\mathfrak{gl}$-case $\lambda \in \Lambda_{m|n}^{\geq}$ is stable if and only if all symbols $\times$ in $\text{diag}(\lambda)$ precede all symbols $<, >$. 
For $\lambda \in \Lambda_{m|n}^+$ we can easily express $\text{tail}(\lambda)$ in terms of $f := \text{diag}(\lambda)$:

- for $\text{osp}(2m|2n)$-case $\text{tail}(\lambda)$ equals to the number of symbols $\times$ in the zero position of $f$;
- for $\text{osp}(2m+1|2n)$-case $\text{tail}(\lambda)$ is the number of symbols $\times$ in the zero position of $f$ if $f$ does not have $(+)$ sign and is less by 1 if $f$ has the sign $(+)$;
- for a stable weight $\lambda$ in the $\text{gl}$-case $\text{tail}(\lambda)$ equals to the maximal length of the subdiagram $\times \times \cdots \times$ which starts from the first symbol $\times$ in $\text{diag}(\lambda)$.

For instance, in the $\text{osp}$-case $\text{tail}(\circ \times \times) = 0$ and $\text{tail}((+) \times^3 \times) = 2$; in the $\text{gl}$-case one has $\text{tail}(\circ \times \circ \times \circ \times) = 2$.

Note that in the $\text{gl}$-case $\text{tail}(\lambda) \neq 0$ if $\lambda$ is an atypical stable weight.

3.4. Cores and howls. We call the symbols $>$, $<$ the core symbols. A core diagram is a weight diagram which does not contain symbols $\times$ and does not have a sign.

For a weight diagram $f$ we denote by $\text{core}(f)$ the core diagram which is obtained from the diagram of $f$ by replacing all symbols $\times$ by $\circ$ and deleting the sign. For instance, $\text{core}(\circ \circ \times) = \circ \circ \circ$.

For a dominant central character $\chi$ we set $\chi_{\lambda} := \chi_{\text{diag}(\lambda)}$.

3.4.1. We say that a $g$-central character is dominant if $\mathcal{F}(g)$ contains modules with this central character. We denote by $\chi_{\lambda}$ the central character of $L(\lambda)$. For a dominant central character $\chi$ we set

$$\Lambda^\chi := \{ \lambda \in \Lambda_{m|n}^+ | \chi_{\lambda} = \chi \}.$$ 

For $\text{gl}(m|n)$, $\text{osp}(2m+1|2n)$-case the dominant central characters are parametrized by the core diagrams, i.e. for $\lambda, \nu \in \Lambda_{m|n}^+$

$$\chi_{\chi} = \chi_{\nu} \implies \text{core}(\lambda + \rho) = \text{core}(\nu + \rho);$$

for $\text{osp}(2m|2n)$ the same holds for the atypical dominant central characters and one has

$$\chi_{\nu} \in \{ \chi_{\nu}, \chi_{\nu^\sigma} \} \implies \text{core}(\lambda) = \text{core}(\nu).$$

For a dominant central character $\chi = \chi_{\lambda}$ we set $\text{core}(\chi) := \text{core}(\lambda)$. By above, a dominant central character is determined by its core for $g = \text{gl}(m|n)$, $\text{osp}(2m+1|2n)$; for $\text{osp}(2m|2n)$ this holds for atypical dominant central characters.

For $\text{osp}(2m|2n)$-case we introduce $t \in \{ 0, 2 \}$ for each dominant central character $\chi$ (resp., for each $\lambda \in \Lambda_{m|n}^+$) in the following way: $t = 0$ if $\text{core}(\chi)$ (resp., core($\lambda$)) has an empty zero position and $t = 2$ otherwise (i.e., the zero position is occupied by $>$); for $\text{osp}(2m+1|2n)$ we set $t = 1$ and for $\text{gl}(m|n)$ we set $t = 0$. We will sometimes use the notation $t(\chi)$ or $t(\lambda)$; one has $t(\lambda) := t(\chi_{\lambda})$. 
3.4.2. We say that a diagram $f$ is core-free if $\text{core}(f) = \emptyset$ or $g = \mathfrak{osp}(2m|2n)$ and $\text{core}(f) \Rightarrow (\rangle$ occupies the zero position.

3.4.3. By [GS1], the blocks in $\mathcal{F}(g)$ are parametrized by the dominant central characters; for $\mathfrak{gl}(m|n)$ the block of atypicality $s$ is equivalent to the block $\chi_0$ in $\mathfrak{gl}(s|s)$; for $\mathfrak{osp}(M|2n)$-case the block of atypicality $s$ is equivalent to the block $\chi_0$ in $\mathfrak{osp}(2s + t|2s)$. The equivalences are described in [GS1]. For $\lambda \in \Lambda^+_m$ let $\text{hwl}(\lambda)$ be the corresponding weight in $\chi_0$. Diagrammatically the passage from $\lambda$ to $\text{hwl}(\lambda)$ essentially amounts to removing the core symbols $\langle, >$ from $\text{diag}(\lambda)$ except for the symbol $\rangle$ at the zero position in $\mathfrak{osp}(2m|2n)$-case (see [GS1] [GH] for details). (In particular, $\text{hwl}(\lambda)$ has a core-free diagram.) If $\text{tail}(\lambda)$ is defined, then $\text{tail}(\lambda) = \text{tail}(\text{hwl}(\lambda))$.

For example, if $\text{diag}(\lambda) \Rightarrow \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times 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If \( \lambda + \rho \) is typical, we set \( ||\lambda||_{gr} \) (we do not define \( ||\lambda|| \) in this case).

3.5.1. **Remark.** For \( t \neq 2 \) one has \((-1)^{\rho(howl(\lambda))} = (-1)^{||\lambda||} \). If \( \lambda, \nu \) are stable \( gl \)-weights with \( \chi_\lambda = \chi_\nu \), then \((-1)^{p(\lambda) - p(\nu)} = (-1)^{||\lambda|| - ||\nu||} \).

3.5.2. **Remark.** The odd-looking formulas for \( ||\lambda|| \) with \( t(\lambda) = 2 \) and for \( ||\lambda||_{gr} \) with \( t(\lambda) = 1 \) can be interpreted as follows. Consider \( f' \) which is obtained from \( f \) by removing \( > \) from the zero position and then shifting all entries at the non-zero positions of \( f \) by one position to the left; then \( ||\lambda|| = \sum_{i=1}^{j} a'_i \), where \( a'_i \) are the coordinates of \( \times \) in \( f' \). The above operation induces a bijection \( \tau \) between the core-free \( osp(2m + 2|2m) \)-weights and the core-free \( osp(2m + 1|2n) \)-weights: this bijection, introduced in [CSI], assigns to \( f \) the diagram \( f' \) with the sign chosen in such a way that \( tail(f) = tail(\tau(f)) \). For instance,

\[
\tau(> \circ \times) = - \times \times, \quad \tau(> \circ) = - \times, \quad \tau(> \times) = + \times, \quad \tau(> \circ \circ \times) = \circ \times .
\]

One has

\[
||\lambda|| = ||\tau(\lambda)||, \quad ||\lambda||_{gr} = ||\tau(\lambda)||_{gr} .
\]

3.5.3. **Definition.** We call \( \lambda \in \Lambda^+_{m|n} \) a Kostant weight if \( dim (L(howl(\lambda)) = 1 \).

Note that \( dim (L(howl(\lambda)) = 1 \) means that \( howl(\lambda) = 0 \) (resp., \( howl(\lambda) \in \mathbb{Z} \ str \)) for the \( osp \)-case (resp., for the \( gl \)-case).

Observe that \( ||\lambda||_{gr} = 0 \) if and only if \( \lambda \) is a Kostant weight (\( ||\lambda||_{gr} \) can be seen as the “distance” to the nearest Kostant weight). For instance, for \( gl(3|3) \) and \( diag(\lambda) = \times \times \circ \times \) one has \( ||\lambda||_{gr} = 1 \).

For the \( gl \)-case this term was used in [BS]: in [CHR1] these weights are called totally connected. If in addition \( \lambda \) is stable, such weight is called a ground state in [HW] [W]. The Kostant weights are precisely the weights where all \( \times \) are adjacent to each other discounting possible core symbols.

3.5.4. **Remark.** For the \( gl(m|n) \)-case the modules satisfying the KW-conditions (see [1]) were classified in [CHR1]; for the \( osp(M|N) \)-case this was done in [CK]. The results of these classification can be formulated as follows. Except for the case \( g = osp(2m|2n) \) with \( t = 0 \) and atypicality 1, \( L(\lambda) \) satisfies the KW-conditions if and only if \( \lambda \) is a Kostant weight. For the case \( g = osp(2m|2n) \) with \( t = 0 \) all simple finite-dimensional modules of atypicality 1 satisfy the KW-conditions. The latter case has the following interpretation. Let \( \mathcal{F}(osp(2m|2n))^X \) be a block of atypicality 1 with \( t = 0 \). Since \( osp(2|2) = sl(1|2) \) we have

\[
\mathcal{F}(osp(2m|2n))^X \cong \mathcal{F}(osp(2|2))^X = \mathcal{F}(sl(1|2))^X \cong \mathcal{F}(sl(1|1))^X
\]

so the image of each simple module \( L \in \mathcal{F}(osp(2m|2n))^X \) is the trivial \( sl(1|1) \)-module (that is \( dim(howl(howl(\lambda)) = 1 \) even if \( dim(howl(\lambda)) \neq 1 \)).
From the above description, it follows that the KW-conditions are compatible with the equivalence of categories given by the translation functors $T_{a,a+1}$ described in 4.7. This is not true in general: the switch functor $\mathcal{F}^{\chi_0}(\mathfrak{osp}(2m+1|2n)) \rightarrow \mathcal{F}^{\chi_0}(\mathfrak{osp}(2m+1|2n))$ given by $N \mapsto (N \otimes V_{st})^{\chi_0}$ maps the trivial module (satisfying the KW-conditions) to the standard module, which does not satisfy these conditions.

4. PARABOLIC INDUCTION, EULER CHARACTERS AND CHARACTER FORMULAS

We define parabolic induction functors $\Gamma_{p,q}^i$ and the Poincaré polynomials $K_{p,q}^\lambda,\mu(z)$ in Section 4.1 and Euler characters $E_\lambda$ in Section 4.2. We give a diagrammatic description of the Poincaré polynomials in the $\mathfrak{gl}(m|n)$-case. This leads to a character formula for $\text{ch}L(\lambda)$.

4.1. The functors $\Gamma_{p,q}^i$. Let $q \subset p \subset g$ be a pair of parabolic subalgebras containing $b$ and let $V$ be a finite-dimensional $q$-module. We denote by $\Gamma_{p,q}(V)$ the maximal finite-dimensional quotient of the induced module $U(p) \otimes U(q)V$. View $\Gamma_{p,q}$ as a functor from the category of finite-dimensional $q$-modules to the category of finite-dimensional $p$-modules and define the derived functors $\Gamma_{p,q}^i$ as in [GS1] ($\Gamma_{p,q}^i := \Gamma_i(P/Q,\bullet)$ in the notations of [GS1]). By [GS2] for $\mathfrak{gl}(m|n)$ these functors coincide with the functors $\Gamma_{p,q}^i$ defined in [MusS].

For $\lambda, \mu \in \Lambda^+_m|n$ we consider the Poincaré polynomial in the variable $z$ as

$$K_{p,q}^{\lambda,\mu}(z) := \sum_{i=0}^{\infty} [\Gamma_{p,q}^i(L_q(\lambda)) : L_p(\mu)]z^i,$$

where $L_q(\lambda)$ (resp. $L_p(\mu)$) stands for the corresponding simple $q$ (resp. $p$) module.

4.1.1. Fix a central character $\chi$ and a flag of parabolic subalgebras

$$g = p^{(d)} \supset p^{(d-1)} \supset \ldots \supset p^{(0)} = b,$$

where $d$ is the defect of $g$ and $l(i) := [p^{(i)},p]$ is given by $l(i) = \mathfrak{gl}(i|i)$ for $g = \mathfrak{gl}(d|d)$, $l(i) = \mathfrak{osp}(2i+t|2i)$, for $g = \mathfrak{osp}(2d+t|2d)$.

The polynomials $K_{p,q}^{\lambda,\mu}(z)$ for the “neighbouring parabolics” were given in [S2] in the $\mathfrak{gl}$-case and in [GS1] in the $\mathfrak{osp}$-case. In the $\mathfrak{gl}$-case we will describe these polynomials in terms of so-called “arc diagrams” in 4.5 below. Using these polynomials the values $K_{p,q}^{\lambda,\mu}(-1)$ can be computed iteratively using the formula

$$(8) \quad K_{p,q}^{\lambda,\mu}(-1) = \sum_\nu K_{p,q}^{\lambda,\nu}(-1)K_{p,q}^{\nu,\mu}(-1)$$

established in [GS1], Thm. 1.
4.2. **The terms** $\mathcal{E}_\lambda$. Take $\lambda \in \Lambda^+_{m|n}$, which is assumed to be stable for $\mathfrak{gl}$-case. Let $\mathfrak{g}_\lambda$ be the tail subalgebra of $\lambda$ (see 3.2.2). As in [GS1] we introduce

\[ \mathcal{E}_\lambda := R^{-1}e^{-\rho}J_W\left(\prod_{\alpha \in \Delta(\mathfrak{g}_\lambda)_1^+} e^{\lambda+\rho}(1 + e^{-\alpha})\right), \]

see A.1 for notation. Clearly, $\mathcal{E}_\lambda \in \mathcal{R}$, see A.2 for notation. By [GS1], Prop.1 (Euler characteristic formula) one has

\[ \mathcal{E}_\lambda = \sum_{\mu \in \Lambda^+_{m|n}} K^{\lambda \mu}_{\mathfrak{g}_\lambda}(-1) \operatorname{ch} L(\mu), \quad \text{where } p_\lambda := \mathfrak{b} + \mathfrak{g}_\lambda. \]

The sum in the right-hand side of the formula is finite (see, for example, [GS1], Lemma 3).

4.2.1. **Remark.** The perspective of [GS1], [S3] is a bit different. The $\mathcal{E}_\lambda$’s are defined as actual Euler characters. It is important not to confuse the Euler character $\mathcal{E}_\lambda$ of [GS1] with the Euler character $\mathcal{E}_\lambda$ of [GS2]. In the latter case $\mathcal{E}_\lambda$ simply equals for $\mathfrak{gl}(m|n)$ the character of the Kac module $K(\lambda)$.

4.2.2. In the case when $\mathfrak{g}_\lambda = \mathfrak{g}$ the formula (A.2.2) gives $\mathcal{E}_\lambda = e^\lambda = \operatorname{ch} L(\lambda)$.

4.3. **Proposition.** Take $\lambda \in \Lambda^+_{m|n}$ which is stable in the $\mathfrak{gl}(m|n)$-case. Set $s := \operatorname{tail}(\lambda)$.

(i) In the $\mathfrak{osp}$-case we have

\[ j_s R e^{\rho} \mathcal{E}_\lambda = \operatorname{KW}(\lambda + \rho, S_s), \]

where $j_s = \max(1, 2^t s!)$ for $t = 0$ and $j_s = 2^t s!$ for $t = 1, 2$.

(ii) If $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $\lambda$ is stable, then

\[ j_s R e^{\rho} \mathcal{E}_\lambda = \operatorname{KW}(\lambda + \rho_L, S_s), \]

where $j_s = (-1)^{\lfloor \frac{s}{2} \rfloor} s!$ and $\rho_L$ is the Weyl vector for the base $\varepsilon^{n-s}(\varepsilon\delta)^s \delta^{n-s}$.

**Proof.** For $\mathfrak{osp}$-case set $\Sigma_L := \Sigma$; for $\mathfrak{gl}$-case let $\Sigma_L$ be the base $\varepsilon^{m-s}(\varepsilon\delta)^s \delta^{n-s}$. Denote by $\rho_\lambda$ (resp., $\rho'_\lambda$) the Weyl vector for $\Delta(\mathfrak{g}_\lambda)$ with respect to the base $\Sigma \cap \Delta(\mathfrak{g}_\lambda)$ (resp., $\Sigma_L \cap \Delta(\mathfrak{g}_\lambda)$). Let $W_\lambda \subset W$ be the Weyl group of $\mathfrak{g}_\lambda$. Note that $S_s$ is the maximal iso-set in $\Delta(\mathfrak{g}_\lambda)$. Combining (A.2.2) and (A.4.5) we obtain

\[ J_{W_\lambda} \left( \prod_{\alpha \in \Delta(\mathfrak{g}_\lambda)_1^+} e^{\rho_\lambda} (1 + e^{-\alpha}) \right) = j_s^{-1} J_{W_\lambda} \left( \prod_{\alpha \in S_s} e^{\rho'_\lambda} (1 + e^{-\alpha}) \right). \]
One has \( J_W = J_{W/W_\lambda} \cdot J_{W_\lambda} \), where \( W/W_\lambda \) stands for any set of representatives. Using \( W_\lambda \)-invariance of \( \lambda \) and \( \rho - \rho_\lambda \) we obtain

\[
Re^\rho E_\lambda = J_W \left( \prod_{\alpha \in \Delta(g_\lambda)_+^t} e^{\lambda + \rho}(1 + e^{-\alpha}) \right) = J_{W/W_\lambda}(J_{W_\lambda} \left( \prod_{\alpha \in \Delta(g_\lambda)_+^t} e^{\lambda + \rho}(1 + e^{-\alpha}) \right))
\]

\[
= j^{-1}_s J_{W/W_\lambda}(J_{W_\lambda} \left( \prod_{\alpha \in S_s} e^{\lambda + \rho - \rho_\lambda}(1 + e^{-\alpha}) \right)) = j^{-1}_s J_W \left( \prod_{\alpha \in S_s} e^{\lambda + \rho - \rho_\lambda}(1 + e^{-\alpha}) \right)
\]

\[
= KW(\lambda + \rho - \rho_\lambda + \rho_\lambda', S_s).
\]

For the \( \mathfrak{osp} \)-case one has \( \Sigma = \Sigma_L \), so \( \rho_\lambda = \rho_\lambda' \); this gives (i). For \( \mathfrak{gl} \)-case notice that \( \Sigma_L \) is obtained from \( \Sigma \) by the chain of odd reflections with respect to the roots in \( \Delta(g_\lambda) \); this gives \( \rho_L - \rho = \rho_\lambda' - \rho_\lambda \) and establishes (ii).

4.4. The \( \mathfrak{osp} \)-case. Consider the case \( g = \mathfrak{osp}(M|2n) \) \( (M = 2m \text{ or } M = 2m + 1) \). Theorems 3, 4 and Remark after Thm. 3 of \([GS1]\) imply that for \( \lambda \in \Lambda^+_{m|n} \) one has

\[
\text{ch } L(\lambda) = \sum_{\mu \in \Lambda^+_{m|n}} (-1)^{|\lambda||-|\mu|} d^\lambda_\mu_{\leq} E_\mu,
\]

where \( d^\lambda_\mu_{\leq} \) is the number of “increasing paths” from \( \text{diag}(\mu) \) to \( \text{diag}(\lambda) \) in the graph \( D_\mathfrak{g} \) described in \([GS1]\), Sect. 11; we will recall some properties of this graph below.

4.4.1. Properties of \( D_\mathfrak{g} \). The connected components of \( D_\mathfrak{g} \) correspond to the dominant central characters, so for each component \( D_\mathfrak{g}^\chi \) we can define \( t \in \{0, 1, 2\} \) via the corresponding central character. The map \( \lambda \rightarrow \text{howl}(\lambda) \) gives an isomorphism \( D_\mathfrak{g}^\chi \cong D_{\mathfrak{osp}(2k+1|2k)}^{\chi_0} \) for \( k : = \text{at}(\chi) \), \( t : = t(\chi) \); the map \( \tau \) induces an isomorphism \( D_{\mathfrak{osp}(2k+1|2k)}^{\chi_0} \cong D_{\mathfrak{osp}(2k+1|2k)}^{\chi_0} \).

Assume that \( \text{diag}(\mu) \) is a predecessor of \( \text{diag}(\lambda) \) in \( D_\mathfrak{g} \). From \([GS1]\), Sect. 11, we conclude that for the cases \( t = 0, 2 \) \( \text{diag}(\lambda) \) is obtained from \( \text{diag}(\mu) \) by moving several symbols \( \times \) to the right; moreover, if \( \text{diag}(\lambda) \) has a sign, then \( \text{diag}(\mu) \) has the same sign.

Using the isomorphism induced by \( \tau \), we conclude that for \( t = 1 \), \( \text{diag}(\lambda) \) is obtained from \( \text{diag}(\mu) \) by moving several symbols \( \times \) to the right or by changing the sign to the sign \( \times \). This implies

\[
\lambda > \mu, \quad \text{howl}(\lambda) > \text{howl}(\mu), \quad ||\lambda|| > ||\mu||, \quad ||\lambda||_{gr} > ||\mu||_{gr}, \quad \text{tail}(\mu) \geq \text{tail}(\lambda)
\]

and that if \( \lambda \) is stable, then \( \mu \) is stable. Moreover,

\[
\lambda - \mu \in \begin{cases} \frac{1}{2} \mathbb{N} \Pi_0 + \mathbb{N}(\delta_n - \varepsilon_m) + \mathbb{N}(\delta_n + \varepsilon_m) & \text{for } t = 0 \\ \frac{1}{2} \mathbb{N} \Pi_0 & \text{for } t = 1, 2. \end{cases}
\]

By above, \( D_\mathfrak{g} \) is \( \mathbb{N} \)-graded with respect to \( ||\,||_{gr} \) (if \( \text{diag}(\mu) \) is a predecessor of \( \text{diag}(\lambda), \) then \( ||\mu||_{gr} < ||\lambda||_{gr} \)). In particular, each vertex in \( D_\mathfrak{g} \) has finitely many predecessors.
The map $\tau$ described in 3.5.2 gives an isomorphism of the graph $D_{\text{osp}(2m+1|2n)}$ and the subgraph of $D_{\text{osp}(2m+2|2n)}$ which correspond to the union of connected components with $\ell = 2$.

4.4.2. We conclude that for $\text{osp}(M|N)$ we have

$$d^\lambda_\mu = \lambda, d^\lambda_\mu = \mu, d^\mu_\mu = 0 \implies \chi_\lambda = \chi_\mu, \|\mu\|_D < \|\lambda\|_D.$$

Moreover the sum in the right-hand side of the character formula is finite and the terms $\{E_\lambda\}_{\lambda \in \Lambda^+_{m|n}}$ form a basis in the character ring of $F$.

Using 3.5.1 we obtain

$$(-1)^{\mu(\lambda)} \text{ch} L(\lambda) = \sum_{\mu \in \Lambda^+_{m|n}} (-1)^{\mu(\mu)} d^\lambda_\mu E_\mu \quad \text{if } \lambda \text{ is stable and } t(\lambda) \neq 2.$$

4.4.3. Remark. The $q(n)$-case can be treated with the same methods. The character of $L(\lambda)$ can be written as a finite sum in the Euler characters where the coefficients are again given by the number of increasing paths in a certain bimarked graph. As for the $\text{osp}$-case the finiteness is automatic since each vertex $\lambda$ in this graph has a finite number of predecessors. However Y. Su and R. B. Zhang already obtained in [SuZh2] a similar character formula based on earlier work of J. Brundan [Br2], so that we have refrained from including this case.

4.5. The Poincaré polynomials in the $\text{gl}$-case. Let $k$ be the degree of atypicality of $\lambda + \rho$. For $i = 0, \ldots, k - 1$ the polynomials $K^{\lambda, \mu}_{p(i), p(i+1)}(z)$ were computed in [S2] (see also Cor. 3.8 in [MusS]). We will recall the diagramic interpretation (which was provided by Serganova in one of her wonderful talks) in 4.5.3.

Let $\mathfrak{g} := \text{gl}(m|n)$. We identify a weight $\lambda \in \Lambda^+_{m|n}$ and the diagram of $\lambda + \rho$, which we denote by $\text{diag} (\lambda)$.

4.5.1. Arc diagrams. Take a weight $\nu \in \Lambda^+_{m|n}$. Denote by $\text{diag}(\nu)$ the weight diagram of $\nu + \rho$. The arc diagram $\text{Arc}(\nu)$ consists of the arcs $\text{arc}(a; a')$, where $a < a'$ and $\text{diag}(\nu)$ has $\times$ (resp., $\circ$) at the position $a$ (resp., $a'$). These arcs satisfy the following properties:

| each symbol $\times$ is connected by an arc to exactly one symbol $\circ$; |
| each symbol $\circ$ is connected to at most one symbol $\times$; |
| the arcs do not intersect; |
| each symbols $\circ$ situated under an arc is connected to a symbol $\times$. |

The arc diagram $\text{Arc}(\nu)$ is unique and can be constructed in the following way: we pass from right to left through the weight diagram and connect each of the finitely many crosses $\times$ with the next empty symbol to the right by an arc (ignoring core symbols).
4.5.2. Example.

![Arc diagram for \( \times \times \circ \circ \times \times \circ \circ \circ \circ \times \).]

4.5.3. Definition. For a weight diagram \( f \) we denote by \( f^u_a \) the weight diagram obtained from \( f \) by interchanging the symbols at the positions \( u \) and \( a \).

Let \( \lambda, \nu \in \Lambda^+_{m/n} \) be such that \( (\lambda) = \text{diag}(\nu)^u_a \). We say that \( \lambda \) is obtained from \( \nu \) by a move if \( \text{diag}(\nu) \) has \( \times \) at the position \( a \), \( \circ \) at the position \( u \) and \( u \) lies under the arc originated at \( a \), that is \( \text{Arc}(\nu) \) contains \( \text{arc}(a; a') \) with \( a < u \leq a' \). For such a move we define the weight as the number of arcs in \( \text{Arc}(\nu) \) which are “strictly above” \( u \) (for instance, if \( u = a' \), then the move has zero weight).

Observe that if \( \lambda \) can be obtained from \( \nu \) by a move as above, then such move is unique. In this case we set \( b'(\nu; \lambda) := u \). Note that \( \text{diag}(\lambda) \) has \( \times \) at the \( u \)-th position; we set \( b(\nu; \lambda) := i + 1 \), where \( i \) is the number of the symbols \( \times \) with the coordinates less than \( u \) in \( \text{diag}(\lambda) \).

We will consider only the case of stable \( \lambda \), so the symbols \( \times \) in \( \text{diag}(\lambda) \) precede the core symbols. If \( \lambda \) is obtained from \( \nu \) by a move, then \( \nu \) is stable. A move is called a non-tail move if \( b(\nu; \lambda) > \text{tail}(\lambda) \).

4.5.4. Example. Take \( \nu \) with \( \text{diag}(\nu) = 0 \times \times \times \) with arc diagram

![Arc diagram for 0 \( \times \times \times \).]

There are 6 weights \( \lambda_1, \ldots, \lambda_6 \) which can be obtained from \( \nu \); in all cases \( b(\nu; \lambda) = 3 \). For instance, \( \lambda_1 \) with \( \text{diag}(\lambda_1) = 1 \times \times \times \) can be obtained from \( \nu \) by a move of weight 2 with \( b'(\nu; \lambda_1) = 4 \). Similarly, \( \lambda_2 \) with \( \text{diag}(\lambda_2) = 1 \times \circ \times \times \) can be obtained from \( \nu \) by a move of weight 1 with \( b'(\nu; \lambda_1) = 5 \). Another example is \( \lambda_3 \) with \( \text{diag}(\lambda_3) = 0 \times \circ \times \times \) can be obtained from \( \nu \) by a move of weight 1 with \( b'(\nu; \lambda_1) = 4 \).

From the weight \( \lambda_2 \) we can obtain \( \mu \) with \( \text{diag}(\mu) = 1 \times \circ \times \times \) by a move of weight 0 with \( b'(\lambda_2; \mu) = 4 \) and \( b(\lambda_2; \mu) = 2 \).

Among the above examples only the first move is a tail move.
4.5.5. Let \( \lambda \) be a stable weight and \( k := \text{at}(\lambda + \rho) \).

For \( i = 1, \ldots, k \) the results of \([S2]\) give

\[
K^{\lambda,\lambda}_{p(i),p(i-1)}(z) = 1 \\
K^{\lambda,\mu}_{p(i),p(i-1)}(z) = z^s 
\]

if \( \mu \) is obtained from \( \lambda \) by a move of weight \( s \) and \( b(\lambda; \mu) = i \). In all other cases \( K^{\lambda,\lambda}_{p(i),p(i-1)}(z) = 0 \). Moreover, \( K^{\lambda,\mu}_{\mathfrak{g},p(k)}(z) = \delta_{\lambda,\mu} \) (see, Lemma 5 (a “Typical Lemma”) in \([GS1]\)) if \( p(k) \neq \mathfrak{g} \).

4.5.6. Lemma. Take a stable weight \( \lambda \in \Lambda^+_{m|n} \). Assume that \( \lambda \) is obtained from \( \nu \) by a move of weight \( w \).

(i) Then \( \nu \) is stable, \( \text{core}(\lambda) = \text{core}(\nu) \) and

\[
\lambda > \nu, \quad ||\lambda|| - ||\nu|| - (w + 1) \in 2\mathbb{Z}, \quad \text{tail}(\nu) \leq b(\nu; \lambda)
\]

(ii) If the move is a non-tail move, then

\[
||\lambda||_{gr} > ||\nu||_{gr}, \quad \text{tail}(\lambda) \leq \text{tail}(\nu).
\]

Proof. The first assertion immediately follows from the formula \( \text{diag}(\lambda) = \text{diag}(\nu)^a_u \) for \( a < u \). Consider the case of a non-tail move, i.e. \( b(\nu; \lambda) \leq \text{tail}(\lambda) \).

Since \( \lambda, \nu \) are stable, their diagrams start from the subdiagrams \( \times \ldots \times \) containing, respectively, \( \text{tail}(\lambda) \) and \( \text{tail}(\nu) \) symbols \( \times \). Let \( A_{\lambda} \) (resp., \( A_{\nu} \)) be the coordinates of symbols \( \times \) in these subdiagrams (one has \( A(\lambda) = \{u_{\lambda} + i\lambda^{\text{tail}(\lambda) - 1}\}_{i=0}^{\lambda^{\text{tail}(\lambda)}} \), where \( u_{\lambda} \) is the minimal coordinate of the non-empty symbol in \( \text{diag}(\lambda) \)). The inequality \( b(\nu; \lambda) \leq \text{tail}(\lambda) \) means that \( u > \max A_{\lambda} \). This gives \( A_{\lambda} \subset A_{\nu} \) and implies (ii). \( \square \)

4.6. Graph \( D_{\mathfrak{g}} \). Let \( D_{\mathfrak{g}} \) be a graph with the set of vertices enumerated by \( \Lambda^+_{m|n} \). We identify the weight \( \lambda \) with \( \text{diag}(\lambda) \). We join \( f, g \) by the edge \( f \rightarrow g \) if \( \text{core}(f) = \text{core}(g) \) and \( \text{howl}(g) \) is obtained from \( \text{howl}(f) \) by a non-tail move described in \([1.5]\).

Recall that \( f \rightarrow g \) implies that \( \text{howl}(g) \) is obtained from \( \text{howl}(f) \) by moving \( \times \) from a position \( a \) to an empty position \( u > a \). We mark each edge by the corresponding \( u \).

4.6.1. Subgraphs \( D^{\chi}_{\mathfrak{g}} \). Clearly, if \( \lambda \) and \( \nu \) lie in the same connected component of \( D_{\mathfrak{g}} \), then \( \chi_{\lambda} = \chi_{\nu} \). Denote by \( D^{\chi}_{\mathfrak{g}} \) the full subgraph with the vertices \( \lambda \) such that \( \chi_{\lambda} = \chi \). If \( \chi \) has atypicality \( s \), then the map \( f \mapsto \text{howl}(f) \) gives an isomorphism of \( D^{\chi}_{\mathfrak{g}} \) and \( D^{\chi_0}_{\mathfrak{g}(s|s)} \).

If at \( \lambda \leq 1 \), then the corresponding vertex is isolated. It is not hard to see that \( D^{\chi}_{\mathfrak{g}} \) is connected for at \( \chi > 1 \).
4.6.2. **Corollary.**

(i) Let $\nu$ be a predecessor of $\lambda$. Then $\text{diag}(\lambda)$ is obtained from $\text{diag}(\nu)$ by moving some symbols $\times$ to the right. In particular, $\text{core}(\lambda) = \text{core}(\nu)$, $\lambda > \nu$ and\[\text{howl}(\lambda) > \text{howl}(\nu), \quad ||\lambda|| > ||\nu||, \quad ||\lambda||_{gr} > ||\nu||_{gr}, \quad g_{\text{howl}(\lambda)} \subset g_{\text{howl}(\nu)}\]

(ii) Any vertex in $D_g$ has a finite number of predecessors.

**Proof.** Let $\nu$ be a predecessor of $\lambda$. Then $\text{diag}(\lambda)$ is obtained from $\text{diag}(\nu)$ by moving several symbols $\times$ to the right; this gives $\lambda > \nu$ and $||\lambda|| > ||\nu||$; the rest of the formulas in (i) follow from Lemma 4.5.6. By above, for (ii) it is enough to consider the case $g = gl(s|s)$ and $\text{core}(\lambda) = \emptyset$. By Lemma 4.5.6 the set of predecessors of $\lambda$ in $D_g$ lie in the following set\[
\{ \nu \in A^+_s | \text{core}(\nu) = \emptyset, \quad ||\nu||_{gr} < ||\lambda||_{gr}, \quad A_\lambda \subset A_\nu \}\,

where $A_\lambda, A_\nu$ as in the proof of Lemma 4.5.6. In particular, the coordinates of all non-empty symbols in $\text{diag}(\nu)$ lie between $u_\lambda - s$ and $u_\lambda - s + ||\lambda||_{gr}$, where $u_\lambda$ is the minimal coordinate of the non-empty symbol in $\text{diag}(\lambda)$. This gives (ii). $\square$

4.6.3. We call a path in $D_g$ *increasing* (resp., *decreasing*) if the marks strictly increase (resp., decrease) along the path.

4.6.4. **Example.** If $\lambda$ is a Kostant weight, then $E_\lambda = e^\lambda$.

For $gl(n|n)$ the adjoint representation $\textbf{Ad}$ has a three step Loewy filtration\[
\textbf{Ad} = \left( \begin{array}{c} \mathbb{C} \\ \Pi(L(\epsilon_1 - \delta_n)) \end{array} \right) \mathbb{C}
\]
The middle term with highest weight $\lambda = \epsilon_1 - \delta_n$ corresponds to the diagram\[
-n \times \times \times \cdots \times \times ; \quad n \text{ times}
\]
this diagram is connected to the Kostant weights\[
-n \times \times \times \quad -n - 1 \times \times \times
\]
the corresponding weights 0 and $\mu = \sum_{i=1}^{n} (\delta_i - \epsilon_i)$. This gives\[
\text{ch} L(\epsilon_1 - \delta_3) = E_{\epsilon_1 - \delta_n} - 1 - e^\mu, \quad \text{sch} L(\epsilon_1 - \delta_3) = E_{\epsilon_1 - \delta_n} + 1 + e^\mu.
\]
Notice that $\text{sdim}(\textbf{Ad}) = 0$, hence $\text{sdim} L(\epsilon_1 - \delta_n) = 2$. 
4.6.5. Examples. For $\mathfrak{gl}(1|1)$ the graph $D_\mathfrak{g}$ does not have edges.

For $\mathfrak{gl}(2|2)$ the vertices $\lambda$ with $\text{core}(\lambda) \neq \emptyset$ are isolated; the vertices with $\text{core}(\lambda) = \emptyset$ form a connected component $D_{\mathfrak{g}0}^\chi$ of the following form

\[
\begin{array}{cccccc}
0 \times \times  & 3 \rightarrow & 0 \times \circ \times  & 4 \rightarrow & 0 \times \circ \circ \times & 5 \rightarrow 0 \times \circ \circ \circ \times \cdots \\
1 \times \times  & 4 \rightarrow & 1 \times \circ \times  & 5 \rightarrow & 1 \times \circ \circ \times & 6 \rightarrow 1 \times \circ \circ \circ \times \cdots \\
\end{array}
\]

The left column corresponds to the Kostant weights ($||\lambda||_{gr} = 0$); the next column to the $\lambda$s with $||\lambda||_{gr} = 1$ and so on.

4.6.6. Proposition. (i) Each vertex is connected to a Kostant weight by an increasing path.

(ii) The Kostant weights are the sources of the graph $D_\mathfrak{g}$.

Proof. For (i) take $f_0$ with $||f_0||_{gr} \neq 0$ and let $u$ be the coordinate of the rightmost symbol $\times$ (which is not in the tail, since $f$ is not a Kostant weight) and $a$ be maximal such that $a < u$ and $f(a) = \circ$. Set $f_1 = (f)^a_u$. Then $f = (f_1)^a_u$ and $f$ is obtained from $f_1$ by a non-tail move with $b'(f_1, f) = u$. If $||f_1||_{gr} \neq 0$, we construct $f_2$ by the same rule. Continuing this process we obtain an increasing path

\[f_r \rightarrow f_{r-1} \rightarrow \ldots \rightarrow f_1\]

with $||f_{i+1}||_{gr} < ||f_i||_{gr}$; thus for some $r$ one has $||f_r||_{gr} = 0$. This gives (i). Now (ii) follows from (i) and the inequality $||\lambda||_{gr} > ||\nu||_{gr}$ if $\nu$ is a predecessor of $\lambda$. \[\square\]

4.7. Character formula for $\mathfrak{gl}(m|n)$. Take $\mathfrak{g} = \mathfrak{gl}(m|n)$ with a distinguished base $\Sigma$. For $\mu, \lambda \in \Lambda^+_{m|n}$ denote by $\mathcal{P}^>(\mu, \lambda)$ the set of decreasing paths from $\mu$ to $\lambda$ and by $d_{\lambda,\mu}^<$ the number of increasing paths from $\mu$ to $\lambda$ in the graph $D_\mathfrak{g}$. Set

\[d_{\lambda,\mu}^\prime = (-1)^{||\lambda||-||\mu||} \sum_{P \in \mathcal{P}^>(\mu, \lambda)} (-1)^{\text{length}P}.
\]

By above, $d_{<,\mu}^\lambda = d_{<,\lambda,\text{howl}(\mu)}$ and $d_{\lambda,\mu}^\prime = d_{\lambda,\text{howl}(\lambda),\text{howl}(\mu)}^\prime$.

4.7.1. Let $\Lambda_{st}^X$ be the set of stable weights in $\Lambda^X$. In the next section we will prove the following formulas for $\lambda \in \Lambda_{st}^X$:

\[\mathcal{E}_\lambda = \sum_{\mu \in \Lambda^X} d_{\lambda,\mu}^\prime \text{ch } L(\mu),
\]

\[\text{ch } L(\lambda) = \sum_{\mu \in \Lambda_{st}^X} (-1)^{||\lambda||-||\mu||} d_{\lambda,\mu}^\prime \mathcal{E}_\mu.
\]
Notice that, by Corollary [4.6.2], the right-hand sides of the above formulas have finite number of non-zero summands.

4.7.2. For a non-stable weight $\lambda$ we introduce $E_\lambda$ by the first formula in (12), i.e.

\[(13)\quad E_\lambda := \sum_{\mu \in \Lambda^x} d^\lambda_{\lambda,\mu} \text{ch} \ L(\mu).\]

If $\lambda$ is stable and $\mu$ is not stable, then $d^\lambda_{\lambda,\mu} < 0$ (by Corollary 4.6.2). Therefore the second formula in (12) can be rewritten as

\[(14)\quad \text{ch} \ L(\lambda) = \sum_{\mu \in \Lambda^x} (-1)^{|\lambda| - |\mu|} d^\lambda_{\lambda,\mu} E_\mu = \sum_{\mu \in \Lambda^+_{\lambda}} (-1)^{|\lambda| - |\mu|} d^\lambda_{\lambda,\mu} E_\mu\]

if $\lambda$ is stable. By (12) the matrices $(d^\lambda_{\lambda,\mu}) = (d'_{\text{howl}(\lambda),\text{howl}(\nu)})$ and $((-1)^{|\lambda| - |\mu|} d^\lambda_{\lambda,\mu})$ are mutually inverse. Using (13) we deduce (14) for each $\lambda \in \Lambda^+_{m|n}$.

4.7.3. We retain notation of A.3.1. Fix a central character $\chi$ and denote by $\mathcal{F}_{\text{in}}^x$ the full subcategory of $\mathcal{F}_{\text{in}}$ of the modules with the central character $\chi$. We will consider translation functors $T_{V,\chi,\chi'}^V$ for special cases when these functors are equivalence of categories and $V$ is either the standard representation or its dual. These functors can be described as follows.

Recall that for a weight diagram $f$ denote by $(f)^{a+1}_a$ the diagram $f'$ obtained from $f$ by interchanging the symbols in the positions $a$ and $a + 1$. We denote by $T_{a,a+1}$ the corresponding operations on $\Lambda^x_{m|n}$ and on the central characters: $T_{a,a+1}(\nu) = \nu'$ such that $\text{diag}(\nu') = T_{a,a+1}(\text{diag}(\nu))$ and $T_{a,a+1}(\chi) = \chi'$ such that $\text{core}(\chi') = T_{a,a+1}(\text{core}(\chi))$.

For $V = V_{st}, V_{st}^*$ the translation functor $T_{V,\chi,\chi'}^V : \mathcal{F}_{\text{in}}^x \to \mathcal{F}_{\text{in}}^{x'}$ is an equivalence of categories if $\chi' = T_{a,a+1}(\chi)$ for some $a$ and exactly one of the positions $a, a + 1$ in $\text{core}(\chi)$ is empty (so for $\lambda \in \Lambda^x$ exactly one of the positions $a, a + 1$ in $\text{diag}(\lambda)$ is occupied by a core symbol and $T_{a,a+1}$ interchanges this core symbol with $\circ$ or $\times$ respectively).

One has

\[T_{V,\chi,\chi'}^V(L(\lambda)) = L(T_{a,a+1}(\lambda)).\]

Note that $\text{howl}(\lambda) = \text{howl}(T_{a,a+1}(\lambda))$. By a slight abuse of notation, we denote such functor by $T_{a,a+1}$.

4.7.4. Lemma. For $\lambda' := T_{a,a+1}(\lambda)$ one has

\[\text{Re} \theta \mathcal{E}_\lambda = \Theta_{\chi,\chi'}(\text{Re} \theta \mathcal{E}_{\lambda'})\]

where $\Theta_{\chi,\chi'} : \mathcal{R}_\Sigma \to \mathcal{R}_\Sigma$ is the ring homomorphism corresponding to $T_{a,a+1}$ (see A.3.1).
Proof. For each $\mu \in \Lambda^x$ set $\mu' := T_{a,a+1}(\mu)$. By (13) $Re^\rho \mathcal{E}_\lambda = \sum_{\mu' \in \Lambda^x} d'_{\lambda,\mu} Re^\rho \text{ch} L(\mu)$. Using [A.3.1] we get
\[
\Theta_{\lambda,\Lambda'}(Re^\rho \mathcal{E}_\lambda) = \sum_{\mu \in \Lambda^x} d'_{\lambda,\mu} Re^\rho \text{ch} L(\mu').
\]
Since $\text{hwl}(\mu') = \text{hwl}(\mu)$ one has $d'_{\lambda,\mu} = d'_{\lambda',\mu'}$, so
\[
Re^\rho \mathcal{E}_\lambda = \sum_{\mu' \in \Lambda^x} d'_{\lambda',\mu'} Re^\rho \text{ch} L(\mu') = \sum_{\mu' \in \Lambda^x} d'_{\lambda',\mu'} Re^\rho \text{ch} L(\mu')
\]
as required. \qed

4.8. Another form of the character formula. In the $\mathfrak{osp}$-case we retain the notation of Proposition 4.3 and set
\[
\text{KW}(\nu) := \text{KW}(\nu + \rho, S_{\text{tail}(\nu)}), \quad j(\nu) := j_{\text{tail}(\nu)}.
\]
For $\mathfrak{gl}$-case we will introduce $\text{KW}(\nu)$ in 6.1.2 and set $j(\nu) := \text{tail}(\nu)!$.

4.8.1. Corollary.
\[
Re^\rho \text{ch} L(\lambda) = \sum_{\mu \in \Lambda^x} (-1)^{|\lambda| - |\mu|} d_{\lambda,\mu} \frac{\text{KW}(\mu)}{j(\mu)}.
\]

Proof. Combining Proposition 4.3 and 4.4 we obtain the assertion for the $\mathfrak{osp}$-case. For the $\mathfrak{gl}$-case we combine (14), Lemma 4.7.4 and Corollary 6.4 (ii). \qed

4.8.2. Remark. Setting $\text{KW}(L(\nu)) := j(\nu)^{-1} \text{KW}(\nu)$ we obtain the formula (3).

4.8.3. Remark. The graph $D_{\theta}$ is an oriented graph; this graph does not have multi-edges for the $\mathfrak{gl}$-case and for the $\mathfrak{osp}$-case with $t = 1, 2$; for $t = 0$ the graph has double edges.

4.9. Highest weights of $L$ with respect to different bases. Fix any base $\hat{\Sigma}$ compatible with $\Pi_0$ (i.e. $\Delta^+(\Sigma) \cap \Delta_0 = \Delta^+(\Pi_0)$) and denote the Weyl vector by $\hat{\rho}$. For a simple finite-dimensional module $L$ denote by $\text{hwt} L$ the “$\rho$-twisted highest weight of $L$” i.e. $\text{hwt}_{\hat{\Sigma}} L = \nu + \hat{\rho}$, where $\nu$ is the highest weight of $L$ with respect to $\hat{\Sigma}$. If $\beta \in \hat{\Sigma}$ is isotropic and $r_\beta$ is the corresponding odd reflection, then $\text{hwt}_{r_\beta \hat{\Sigma}} L = \text{hwt}_{\hat{\Sigma}} L$ if $(\text{hwt}_{\hat{\Sigma}} L|\beta) \neq 0$ and $\text{hwt}_{r_\beta \hat{\Sigma}} L = \text{hwt}_{\hat{\Sigma}} L + \beta$ otherwise. Using this procedure one can compute $\text{hwt}_{\hat{\Sigma}} L(\lambda)$ recursively. The character formula in Corollary 4.8.1 allows to give the following formula for $\text{hwt}_{\hat{\Sigma}} L(\lambda)$ for $t(\lambda) = 1, 2$. 
4.9.1. **Corollary.** Consider the partial order $\gg$ on $h^*$ given by $\nu \gg \mu$ if $\nu - \mu \in \mathbb{N}\tilde{\Sigma}$. View $KW(\lambda)$ as an element of $R_{\tilde{\Sigma}}$.

(i) In the $osp$-case with $t(\lambda) = 1, 2$ the weight $hwt_{\tilde{\Sigma}} L(\lambda)$ is a unique maximal element in supp $KW(\lambda)$ with respect to the partial order $\gg$.

(ii) In the $osp$-case with $t(\lambda) = 0$ the same holds if $\delta_n \pm \epsilon_m \in \tilde{\Sigma}$.

(iii) In the $gl$-case $\lambda + \rho$ is a unique maximal element in supp $KW(\lambda)$ with respect to the partial order $\gg$.

**Proof.** The assertions (i), (ii) follow by induction on $||\lambda||_{gr}$ if we combine Corollary 4.8.1 with (11). Similarly, (iii) follows by induction on $||\lambda||_{gr}$ from Corollary 4.8.1 and the fact that $d_{\leq}^{\lambda, \mu} \neq 0$ implies $\mu < \lambda$. □

5. **Proof of the formulas (12)**

The proof of (12) follows the plan explained in [Gor2], Section 3. In 5.1, 5.2 below we recall the main constructions of [Gor2].

5.1. **Marked graphs.** Consider a directed graph $(V, E)$ where $V$ and $E$ are at most countable, where the set of edges $E$ is equipped by two functions: $b : E \to \mathbb{Z}$ and a function $\kappa$ from $E$ to a commutative ring.

We say that $\iota : V \to \mathbb{N}$ (resp., $\iota : V \to \mathbb{Z}$) defines a $\mathbb{N}$-grading (resp., $\mathbb{Z}$-grading) on this graph if for each edge $\nu \xleftarrow{e} \lambda$ one has $\iota(\nu) < \iota(\lambda)$.

5.1.1. For a path

$$P := \nu_1 \xleftarrow{e_1} \nu_2 \xrightarrow{e_2} \nu_3 \ldots \xrightarrow{e_s} \nu_{s+1}$$

we define

$$\text{length}(P) := s, \quad \kappa(P) := \prod_{i=1}^{s} \kappa(e_i).$$

We call the path $P$ decreasing (resp., increasing) if $b(e_1) > b(e_2) > \ldots > b(e_s)$ (resp., $b(e_1) < \ldots < b(e_s)$). We consider a path $P = \nu$ (with one vertex and zero edges) as a decreasing/increasing path of zero length with $\kappa(P) = 1$.

5.1.2. **Definition.** We call two functions $b, b' : E \to \mathbb{Z}$ decreasingly-equivalent if for each path $\nu_1 \xrightarrow{e_1} \nu_2 \xrightarrow{e_2} \nu_3$ one has

$$b(e_1) > b(e_2) \iff b'(e_1) > b'(e_2).$$

5.1.3. Observe that two decreasingly-equivalent graphs have the same set of decreasing paths.
5.1.4. We denote the set of decreasing (resp., increasing) paths from \( \nu \) to \( \lambda \) by \( \mathcal{P}^>(\nu, \lambda) \) (resp., \( \mathcal{P}^< (\nu, \lambda) \)).

Let \((V, E)\) be a \( \mathbb{Z} \)-graded graph with a finite number of edges between any two vertices. Notice that in this case the number of paths between any two vertices is finite.

We introduce the square matrices \( A^<(\kappa) = (a^<_{\lambda,\nu})_{\lambda,\nu \in V} \) and \( A^>(\kappa) = (a^>_{\lambda,\nu})_{\lambda,\nu \in V} \) by

\[
a^<_{\lambda,\nu} := \sum_{P \in \mathcal{P}^>(\nu, \lambda)} \kappa(P), \quad a^>_{\lambda,\nu} := \sum_{P \in \mathcal{P}^< (\nu, \lambda)} (-1)^{\text{length}(P)} \kappa(P).
\]

Since the graph is a \( \mathbb{Z} \)-graded, these matrices are lower-triangular with \( a^>_{\lambda,\lambda} = a^<_{\lambda,\lambda} = 1 \). The following lemma is proven in [Gor2, Section 3.4] (the proof is similar to one in [GS1, Thm. 4]).

5.1.5. Lemma. Let \((V, E)\) be a \( \mathbb{Z} \)-graded graph with a finite number of edges between any two vertices. Assume that \( b : E \to \mathbb{Z} \) satisfies the property

\[(BB) \quad \text{for each path } \nu_1 \xleftarrow{e_1} \nu_2 \xleftarrow{e_2} \nu_3 \text{ one has } b(e_1) \neq b(e_2).
\]

Then \( A^>(\kappa) \cdot A^<(\kappa) = A^<(\kappa) \cdot A^>(\kappa) = \text{Id} \).

5.2. Graphs \( \hat{\Gamma}^\chi_{st} \) and \( \Gamma^\chi_{st} \). We take \( \mathfrak{g} := \mathfrak{g}(m|n) \) and fix a central character \( \chi \). We define \( \hat{\Gamma}^\chi_{st} \) and its subgraph \( \Gamma^\chi_{st} \) similarly to [Gor2].

5.2.1. Graph \( \hat{\Gamma}^\chi_{st} \). Let \( \hat{\Gamma}^\chi_{st}(z) \) be a graph with the set of vertices \( V := \Lambda^\chi_{st} \) and the following edges: if \( K^\chi_{p(i), p(i-1)} \neq \delta_{\nu, \lambda} \) (where \( \delta_{\nu, \lambda} \) is the Kronecker symbol) we join \( \nu, \lambda \) by the edge of the form

\( \nu \xleftarrow{e} \lambda \) with \( b(e) = i \).

The graph \( \hat{\Gamma}^\chi_{st} \) is obtained from \( \hat{\Gamma}^\chi_{st}(z) \) by removing the edges of the form \( \nu \xleftarrow{e} \lambda \) with \( b(e) \leq \text{tail}(\lambda) \). For the core-free case \( \Lambda^\chi_{st} = \Lambda^\chi \) and we denote the resulting graphs by \( \hat{\Gamma}^\chi \) and \( \Gamma^\chi \) respectively.

We denote by \( \mathcal{P}^>(\nu, \lambda) \) the set of decreasing paths from \( \nu \) to \( \lambda \) in the graph \( \Gamma^\chi_{st} \).

By [4.5.5] if \( \nu \xleftarrow{e} \lambda \) is an edge in \( \hat{\Gamma}^\chi_{st} \) (resp., in \( \Gamma^\chi_{st} \)) then \( \lambda \) is obtained from \( \nu \) by a move (a non-tail move) of weight \( s \) and for \( i := b(\lambda; \mu) = b(e) \) one has \( K^\chi_{p(i), p(i-1)} = z^s \). In particular, \( \hat{\Gamma}^\chi_{st} \) does not have multi-edges and is \( \mathbb{Z} \)-graded with respect to \( \|\| \).

5.2.2. Take \( \nu, \lambda \in \Lambda^\chi \) with \( \nu \neq \lambda \). By Lemma [4.5.6] (i)

\[
K^\chi_{p(i), p(i-1)} \cdot (-1) = (-1)^{\|\|\|\| + 1}
\]

if \( \Gamma^\chi_{st} \) contains an edge \( \nu \xleftarrow{e} \lambda \) with \( b(e) = i \) and \( K^\chi_{p(i), p(i-1)} = 0 \) otherwise. By Lemma [4.5.6] (ii) the graph \( \Gamma^\chi_{st} \) is \( \mathbb{N} \)-graded with respect to \( \|\| \) and satisfies the following condition

(Tail) for each edge \( \nu \xleftarrow{e} \lambda \) in \( \Gamma^\chi_{st} \) one has \( \text{tail}(\nu) \leq b(e) \).
This condition implies the following important property: a decreasing path $P$ in $\Gamma_{st}^\chi$ lies in $\Gamma_{st}^\chi$ if and only if the last edge of $P$ lies in this graph. Using this property, (8) and (15) we obtain for $\lambda \in \Lambda_{st}^\chi$ and $\mu \in \Lambda_{m|n}^+$

$$K_{b,b'}^{\lambda,\nu}(-1) = (-1)^{|\lambda|-|\mu|} \sum_{P \in P^>(\nu,\lambda)} (-1)^{\text{length}(P)}.$$  

(see Cor. 3.6 in [Gor2] for details). For $K_{b,b'}^{\lambda,\nu}(-1)$ one has the similar formula in terms of the decreasing paths in $\Gamma_{st}^\chi$.

5.2.3. Let $E$ be the set of edges in $\Gamma_{st}^\chi$. We introduce $b' : E \to \mathbb{Z}$ by

$$b'(\mu \xrightarrow{e} \lambda) := b'(\nu, \lambda).$$

One readily sees that $b$ and $b'$ are decreasingly equivalent. We denote by $P_{\nu,\lambda}^>$ the number of paths from $\nu$ to $\lambda$ in $\Gamma_{st}^\chi$ which are increasing with respect to $b'$.

Moreover, $b'$ satisfies the property (BB). Using Lemma 5.1.5 we conclude that for a stable weight $\lambda$ one has

$$\text{ch } L(\lambda) = \sum_{\mu \in \Lambda_{m|n}^+} (-1)^{|\lambda|-|\mu|} d_{\lambda,\mu}^{b'} \mathcal{E}_\mu,$$

where $d_{\lambda,\mu}^{b'}$ is the cardinality of $P_{\nu,\lambda}^>$.

5.2.4. Notice that $\Gamma_{st}^\chi$ coincides with the “stable part” (the full subgraph corresponding to the stable vertices) of the component $D_\chi^\lambda$ and that $b'$ corresponds to the marking in this graph. This completes the proof of (12).

5.3. Examples. Consider the core-free case: $\mathfrak{g} = \mathfrak{gl}(r|r)$ and $\chi_\lambda = \chi_0$.

5.3.1. Case $r = 1$. In this case $\mathcal{E}_\lambda = \text{ch } L(\lambda) = e^{\lambda}$.

5.3.2. Case $r = 2$. Set $\beta_1 := \varepsilon_1 - \delta_2$, $\beta_2 := \varepsilon_2 - \delta_1$.

The weights $\lambda \in \Lambda_{2|2}^+$ with $\chi_\lambda = \chi_0$ are of the form $s(\beta_1 + \beta_2) + i \beta_1$ for $s \in \mathbb{Z}$, $i \in \mathbb{Z}_{\geq 0}$; we denote such weight by $(s; i)$. The diagram of $(s; i)$ has symbols $\times$ at the positions $s$ and $s + i + 1$. One has $||(s; i)||_{gr} = i$ and tail$(s; i) = 1 + \delta_i$.

The graph $D_\phi$ is described in [4.6.5]. The decreasing paths are the paths of length at most 1; combining (10) and (16) we obtain

$$\mathcal{E}_{s;i} = \begin{cases} 
\text{ch } L(s; 0) & \text{if } i = 0, \\
\text{ch } L(s; 1) + \text{ch } L(s; 0) + \text{ch } L(s - 1; 0) & \text{if } i = 1, \\
\text{ch } L(s; i) + \text{ch } L(s; i - 1) & \text{if } i > 1.
\end{cases}$$
For $j > 0$ a vertex $(s; j)$ can be reached by increasing paths from vertices $(s; i)$ for $0 \leq i \leq j$ and from a vertex $(s - 1; 0)$; in both cases the path is unique; this gives

$$
\text{ch } L(s; j) = (-1)^j E_{s-1;0} + (-1)^{j-i} \sum_{i=0}^{j} E_{s;i}, \quad \text{sch } L(s; j) = E_{s-1;0} + \sum_{i=0}^{j} E_{s;i},
$$

5.4. Comparison with other character formulas. For the $\text{gl}(2|2)$-case a weight $(s, i)$ is a Kostant weight only if $i = 0$; thus the Kac-Wakimoto character formula does not hold for $L(s, i)$ with $i \neq 0$. By [Dr] the restriction of any $L(s, i)$ for $i > 0$ is a sum of four simple $\text{gl}_0$-modules, so the character of $L(s, i)$ is a sum over four Weyl character formula terms for $\text{gl}_0$. Any $\text{gl}(2|2)$-module is always partially disconnected (PDC) in the sense of [CHR2]. For PDC weights the authors establish the following character formula

$$
e^p R \cdot \text{ch } L(\lambda) = \frac{(-1)^{|(\lambda^p) - \lambda^0|} s^\lambda}{t_\lambda} J_W \left( \frac{e^{(\lambda^p)^0}}{\prod_{\beta \in S^\lambda} (1 + e^{-\beta})} \right),$$

where we refer to loc. cit for the definitions. The number $t_\lambda$ is two for $(s, 0)$ and one for $(s, i)$, $i > 0$. However already for $\text{gl}(3|3)$ there are simple modules which are not PDC.

The Su-Zhang formula [SuZh1] expresses the character in terms of $KW(\lambda, S)$ where $S$ is chosen to be maximal (the cardinality of $S$ is equal to the atypicality of $L$) whereas we take $S$ with cardinality equal to tail($L$).

6. Euler characters for $\text{gl}(m|n)$

In this section we define tail($\lambda$) and the $\lambda^\dagger$ which appeared in (iv) in 1.2 for the $\text{gl}(m|n)$-case. In addition, in Corollary 6.4 we deduce the property (i) of Section 1.2.

Recall that in this case tail($\lambda$) was introduced only for stable weights; the stable weight diagram has a “horizontal tail”, that is the leftmost part of the diagram is of the form $\times \ldots \times$. In 6.1.1 below we introduce tail($\nu$) for non-stable weights and assign to each tail($\lambda$) times weight $\nu \in \Lambda_{m|n}^+$ a weight diagram $f^\dagger$ with a “vertical tail” of size tail($\nu$); the weight $\nu^\dagger \in \Lambda_{m|n}^{\geq}$ is defined by diag($\nu^\dagger - \rho$) = $f^\dagger$. The set $\Lambda_{m|n}^+$ consists of the diagrams where at most one position contains more than one of the symbols $\circ, >, <, \times$ and, if such a position exists, it contains $\times^i$ for $i > 1$ with no symbols $\times$ which precede this position. The correspondence $f \mapsto f^\dagger$ gives a bijection between $\Lambda_{m|n}^+$ and $\Lambda_{m|n}^{\dagger}$, see 6.1 for details and examples.

In this section $\mathfrak{g} := \text{gl}(m|n)$.

6.1. Straightening the tail. Take $\nu \in \Lambda_{m|n}^{\geq}$ and set $f := \text{diag}(\nu)$.
6.1.1. Recall that the atypicality of $\nu$ is equal to the number of the symbols $\times$ in $f$. We assign to $f$ tail($f$) and the position $y_0(f)$ as follows.

If $\nu$ has atypicality zero ($f$ does not contain $\times$) we set tail($f$) = 0 and $y_0(f) = \infty$.

If $\nu$ is atypical we let tail($f$) := $s$, where $s$ is the maximal number such that $f$ does not have empty symbols between the first and the $s$th coordinate of following procedure: all symbols $\times$.

Moreover, ($\nu$) the rightmost $\epsilon$ $y$ in each position. For instance, for $f$ in the above example $\nu_0(f_1) = 6$ and $\nu_p - \delta_q = \nu_2 - \delta_3$. One has

$$0 = (\nu|\epsilon_p - \delta_q) = (\nu|\epsilon_p - \delta_q).$$

Moreover, $(\nu|\epsilon_i) = (\nu|\delta_j)$ for $i = p, p + 1, \ldots, p'$ and $j = q', q' + 1, \ldots, q$ with $p' - p = q - q' = s - 1$. (In the above example, $(\nu|\epsilon_i) = (\nu|\delta_j) = 6$ for $i = 2, 3, 4$ and $j = 1, 2, 3$). We set

$$S_{\nu'} := \{\epsilon_{p+i} - \delta_{q'+i})_{i=0}^{s-1}.\$$

By above, $(\nu|S_{\nu'}) = 0$. (In the above example, $S_{\nu'} = \{\epsilon_2 - \delta_1, \epsilon_3 - \delta_2, \epsilon_4 - \delta_3\}$). For $\nu \in \Lambda_{m|n}$ we set

$$KW(\nu) := KW(\nu|S_{\nu}).$$
6.1.3. Remark. Clearly, $S_{\lambda^\dagger}$ can be embedded to a base compatible with $\Pi_0$:

$$S_{\lambda^\dagger} \subset \delta^{q'-1} \varepsilon^{p-1} (\varepsilon \delta)^{s} \delta^{n-s-q'+1} \varepsilon^{m-s-p+1}.$$ 

Using 4.9 one can show that for $\lambda \in \Lambda_{m|n}^+$ the weight $\lambda^\dagger$ is the $\rho$-twisted highest weight of $L := L(\lambda)$ with respect to this base, which we denote by $\Sigma_L$. If $\lambda$ is stable, then $\Sigma_L$ is introduced in Proposition 4.3, see 6.1.5 below.

6.1.4. Remark. In the $\mathfrak{osp}$-case the weight $\lambda^\dagger = \lambda + \rho$ has a ”vertical tail” (diag($\lambda$) has $\times$ in the zero position); in the $\mathfrak{gl}$-case diag($\lambda$) has a ”horizontal tail” and the diagram of $\lambda^\dagger$ has a ”vertical tail” of the same size.

6.1.5. Example: stable weight $\lambda$. Let $\lambda \in \Lambda_{m|n}^+$ be a stable weight. In this case the weight diagram of $\lambda^\dagger$ starts from $\times$ and $S_{\lambda^\dagger} = \{ \varepsilon_{m-s+i} - \delta_i \}_{i=1}^s$.

Set $L := L(\lambda)$. As in Proposition 4.3 we denote by $\Sigma_L$ the base corresponding to the word $\varepsilon^{m-s}(\varepsilon \delta)^{s} \delta^{n-s}$ and by $\rho_L$ the Weyl vector corresponding to $\Sigma_L$. Notice that $S_{\lambda^\dagger} \subset \Sigma_L$. Observe $\Sigma_L$ is obtained from $\Sigma$ by odd reflections with respect to the roots of $\mathfrak{g}_\lambda$; these reflections do not change the highest weight of $L$, so the highest weight of $L$ with respect to $\Sigma_L$ is $\lambda$. It is easy to see that

$$\lambda^\dagger = \lambda + \rho_L.$$ 

Note that $(\lambda^\dagger | S_\times) = 0$. Using Proposition 4.3 we get

$$s! Re^\sigma \mathcal{E}_\lambda = KW(\lambda) = (-1)^{|\Sigma|} KW(\lambda^\dagger, S_\times).$$

6.2. Remark. Consider the set of $\rho$-twisted highest weights of $L := L(\lambda)$:

$$Hwt(L) := \{ hwt_{\Sigma'} L | \text{ where } \Sigma' \text{ is a base of } \Delta \}.$$ 

For each $\nu \in \Lambda_{m|n}$ let $s(\nu)$ be the maximal cardinality of an iso-set orthogonal to $\nu$ which can be embedded to a base compatible with $\Pi_0$. The number $s(\nu)$ can be visualized if we draw the weight diagram of $\nu$ using the same rules as in 3.3 (eventhough $\nu$ is not always in $\Lambda_{m|n}^\geq$): $s(\nu)$ is equal to the maximal number of symbols $\times$ which occupy the same position in the weight diagram. In particular, for $\nu \in \Lambda^1$, $s(\nu)$ is the size of the “vertical tail” of the diagram.

By above, $\lambda^\dagger \in Hwt(L)$. One has $s(\lambda^\dagger) = \text{tail}(\lambda)$ (and $S_{\lambda^\dagger}$ is an iso-set of the maximal cardinality which is orthogonal $\lambda^\dagger$).

6.2.1. Conjecture. $\text{tail}(\lambda) \leq \max_{\nu \in Hwt(L(\lambda))} s(\nu)$.

6.2.2. Remark: the $\mathfrak{osp}$-case. The formula in 6.2.1 does not hold for the $\mathfrak{osp}$-case: if $\lambda$ is a singly atypical $\mathfrak{osp}(2m|2n)$-weight satisfying the Kac-Wakimoto conditions which is not a Kostant weight, then the right-hand side of the formula is 1, whereas the left-hand side is 0. A natural question is whether the formula holds for other weights.
6.3. Translation functors and $KW(\lambda)$. Retain notation of L7.

**Lemma.** Take $\lambda \in \Lambda_{m|n}$. Let $a \in \mathbb{Z}$ be such that exactly one of the positions $\alpha, \alpha + 1$ in $\text{core}(\lambda)$ is empty.

(i) $(T_{a,a+1}(\lambda))^{\dagger} = T_{a,a+1}(\lambda^{\dagger})$;
(ii) For $\chi' = T_{a,a+1}(\lambda \chi)$ one has $\Theta^{V}_{\chi',\lambda \chi}(KW(\lambda)) = KW(T_{a,a+1}(\lambda))$.

**Proof.** For (i) note that $(T_{a,a+1}(f))^{\dagger} = T_{a,a+1}(f^{\dagger})$ except for the case $a = y_{0}(f)$ and $f(a + 1) = \circ$; in the latter case the positions $a, a + 1$ in core($\alpha$) are empty. Combining (i) and the formula $(\lambda^{\dagger})^{\dagger} = (\lambda^{\dagger})$ we reduce (ii) to the case when $\lambda = \lambda^{\dagger}$ (i.e., $\lambda \in \Lambda^{\dagger}$). We set $S := S_{\lambda}$. We will consider the case when

$$\text{core}(\lambda)(\alpha) = >, \quad \text{core}(\lambda)(\alpha + 1) = \circ$$

(other cases are similar). In this case $V = V_{st}$.

Using the formula $P_{\chi}(KW(\lambda)) = KW(\lambda)$ and $W$-invariance of $\text{ch} V$ we get

$$\Theta^{V}_{\chi',\lambda \chi}(KW(\lambda)) = P_{\chi'}(\text{ch} V \cdot J_{W}(\frac{e^{\lambda}}{\prod_{\beta \in S}(1 + e^{-\beta})})) = P_{\chi'}(J_{W}(\frac{e^{\lambda} \text{ch} V}{\prod_{\beta \in S}(1 + e^{-\beta})}))$$

which allows to rewrite (ii) in the following form

$$P_{\chi'}(J_{W}(\frac{e^{\lambda} \text{ch} V}{\prod_{\beta \in S}(1 + e^{-\beta})})) = KW(T_{a,a+1}(\lambda)).$$

Recall that $S = \{\varepsilon_{p+i} - \delta_{q+i}\}_{i=1}^{a}$ for $s := \tau(\lambda)$ and some $p, q$. Set

$$A := \{\gamma \in \{\varepsilon_{i}\}_{i=1}^{m} \cup \{\delta_{j}\}_{j=1}^{n} | (\gamma, S) = 0\}$$

and $S_{i} := S \setminus \{\varepsilon_{p+i} - \delta_{q+i}\}$ for $i = 1, \ldots, s$. Using $\text{ch} V = \sum_{i=1}^{m} e^{\varepsilon_{i}} + \sum_{j=1}^{n} e^{\delta_{j}}$ we get

$$J_{W}(\frac{e^{\lambda} \text{ch} V}{\prod_{\beta \in S}(1 + e^{-\beta})}) = \sum_{\gamma \in A} J_{W}(\frac{e^{\lambda+\gamma}}{\prod_{\beta \in S}(1 + e^{-\beta})}) + \sum_{i=1}^{s} J_{W}(\frac{e^{\lambda+\varepsilon_{p+i}}}{\prod_{\beta \in S_{i}}(1 + e^{-\beta})})$$

$$= \sum_{\gamma \in A} KW(\lambda + \gamma; S) + \sum_{i=1}^{s} KW(\lambda + \varepsilon_{p+i}; S_{i}).$$

Using (24) we obtain

$$P_{\chi'}(J_{W}(\frac{e^{\lambda} \text{ch} V}{\prod_{\beta \in S}(1 + e^{-\beta})})) = \sum_{\gamma \in B} KW(\lambda + \gamma; S) + \sum_{\gamma \in B_{0}} KW(\lambda + \gamma, S_{i}),$$

where $B := \{\gamma \in A | \text{core}(\lambda + \gamma) = T_{a,a+1}(\text{core}(\lambda))\}$ and

$$B_{0} := \{\varepsilon_{p+i} | i = 1, \ldots, s \text{ s.t. } \text{core}(\lambda + \varepsilon_{i}) = T_{a,a+1}(\text{core}(\lambda))\}.$$

Denote by $f$ the weight diagram of $\lambda$. Recall that $f(\alpha) = >$ and $f(\alpha + 1) = \times^{j}$ for some $i$. Since $\lambda \in \Lambda^{\dagger}$, $f$ has a “vertical tail” at the position $y := y_{0}(f)$ (so $j = 1$
if \( a + 1 \neq y \). Since \( f(a) \Rightarrow \) there exists a unique \( k \) such that \((\lambda, \varepsilon_k) = a\). Note that \( \text{core}(\lambda + \varepsilon_i) = T_{a,a+1}(\text{core}(\lambda)) \) implies \( i = k \) and \( \text{core}(\lambda + \delta_i) = \text{core}(\lambda) \) implies \((\lambda, \delta_i) = -a - 1\). Note that \( \varepsilon_k \in A \), so \( B_0 = \emptyset \). We get

\[
P_{\lambda'}(J_W\left(\frac{e^\lambda \text{ch} V}{\prod_{\beta \in S}(1 + e^{-\delta})}\right)) = \begin{cases} \text{KW}(\lambda + \varepsilon_k, S) & \text{if } a + 1 = y \\ \text{KW}(\lambda + \varepsilon_k, S) + \sum_{\varepsilon(\lambda, \delta_i) = -a - 1} \text{KW}(\lambda + \delta_i, S) & \text{otherwise} \end{cases}
\]

If \( f(a + 1) = \circ \), then \((\lambda, \delta_i) \neq -a - 1 \) for all \( i \) and \( \lambda + \varepsilon_k = T_{a,a+1}(\lambda) \) with \( S_{\lambda + \varepsilon_k} = S \); thus \( \text{KW}(\lambda + \varepsilon_k, S) = \text{KW}(T_{a,a+1}(\lambda)) \) and this gives (17).

Consider the case \( f(a + 1) = \times \) with \( a + 1 \neq y \). By Lemma A.4.4 (ii), \( \text{KW}(\lambda + \varepsilon_k, S) = 0 \) (since \((\lambda + \varepsilon_k, \varepsilon_{k-1} - \varepsilon_k) = (S, \varepsilon_{k-1} - \varepsilon_k) = 0\)). Since \( a + 1 \neq y \), there is a unique \( i \) such that \((\lambda, \delta_i) = -a - 1\). One has \( \lambda + \delta_i = T_{a,a+1}(\lambda) \) and \( S_{\lambda + \delta_i} = S \), so (17) holds.

In the remaining case \( a + 1 = y \). Then \( f(a + 1) = \times^s \) and \( k = p + s + 1 \). Note that \((\lambda + \varepsilon_k, \varepsilon_i) = a + 1 \) if and only if \( i = p + 1, \ldots, p + s + 1 \) and \((\lambda + \varepsilon_k, \delta_i) = -a - 1 \) if and only if \( i = q + 1, \ldots, q + s \). Set

\[
\mu := (a + 1)\left(\sum_{i=1}^{s+1} \varepsilon_{p+i} - \sum_{i=1}^s \delta_{q+i}\right).
\]

Let \( W_\mu \cong S_{s+1} \times S_q \subset W \) be the group of permutations of \( \varepsilon_{p+1}, \ldots, \varepsilon_{p+s+1} \) and of \( \delta_{q+1}, \ldots, \delta_{q+s} \). Notice that \( \lambda + \varepsilon_k \) is \( W_\mu \)-invariant. Choosing any set of representatives in \( W/W_\mu \) we have

\[
\text{KW}(\lambda + \varepsilon_k, S) = J_W\left(\frac{e^\lambda \text{ch} V}{\prod_{\beta \in S}(1 + e^{-\beta})}\right) = J_{W_\mu}\left(\frac{e^{\lambda + \varepsilon_k}}{\prod_{\beta \in S}(1 + e^{-\beta})}\right).
\]

Comparing the denominator identities for \( g(l(s + 1|s) \) with respect to the bases \((\varepsilon \delta) \varepsilon \) and \( \varepsilon(\varepsilon \delta) \varepsilon \) we get

\[
J_{W_\mu}\left(\frac{e^{-\sum_{\beta \in S'} \beta}}{\prod_{\beta \in S'}(1 + e^{-\beta})}\right) = J_{W_\mu}\left(\frac{e^{-\sum_{\beta \in S'} \beta}}{\prod_{\beta \in S'}(1 + e^{-\beta})}\right),
\]

where \( S' := \{\varepsilon_{p+i+1} - \delta_{q+i}\}_{i=1}^s \). This gives

\[
J_W\left(\frac{e^{\lambda + \varepsilon_k}}{\prod_{\beta \in S}(1 + e^{-\beta})}\right) = J_{W/W_\mu}\left(\frac{e^{\lambda + \varepsilon_k}}{\prod_{\beta \in S'(1 + e^{-\beta})}\right) = J_{W_\mu}\left(\frac{e^{\lambda'}}{\prod_{\beta \in S'}(1 + e^{-\beta})}\right),
\]

where \( \lambda' := \lambda + \varepsilon_k - \sum_{\beta \in S'} \beta \). One readily sees that \( \lambda' = T_{a,a+1}(\lambda) \) and \( S' = S_{\lambda'} \). This completes the proof. \( \square \)

### 6.4. Corollary

(i) Let \( T : F \text{In}^k \overset{\sim}{\longrightarrow} F \text{In}^{\lambda'} \) be a composition of the translation functors \( T^{V}_{\lambda,\lambda'} \), which are equivalence of categories and let \( \Theta_{\lambda,\lambda'} : R_{\Sigma} \rightarrow R_{\Sigma} \) be the corresponding composed map. If \( T(L(\lambda - \rho)) = L(\lambda' - \rho) \), then \( \Theta_{\lambda,\lambda'}(\text{KW}(\lambda)) = \text{KW}(\lambda') \).
(ii) For each \( \lambda \in \Lambda^+_{m|n} \) one has

\[
\text{tail}(\lambda)! \Re^{\rho} E_{\lambda} = KW(\lambda).
\]

6.4.1. Denote by \( K(\lambda) \) the Kac module of the highest weight \( \lambda \). Take \( \lambda' \) as in Corollary 6.4 (i). From [S2], Thm. 5.1 it follows that \( T(K(\lambda)) = K(\lambda') \), where \( \lambda' \) are as above. This gives the following formula

\[
(18) \Theta_{\chi, \chi'}(\Re^{\rho} K(\lambda - \rho)) = K(\lambda' - \rho)
\]

which will be used later (this formula can be also proven as Lemma 6.3 (ii)).

7. Euler supercharacters and the Duflo-Serganova functor

Let \( \text{Sch}(g) \) be the ring of supercharacters of \( g \). Recall that \( \text{DS}_x \) induces for any \( x \) a homomorphism \( \text{Sch}(g) \to \text{Sch}(g_x) \) which depends only on the rank of \( x \). We denote this homomorphism by \( \text{ds}_j \), where \( j \) is the rank of \( x \). We always assume that \( j > 0 \) (\( \text{ds}_0 = \text{Id} \)).

In this section \( g \) stands for \( \mathfrak{gl}(m|n) \) or \( \mathfrak{osp}(M|N) \).

7.1. Euler supercharacters. Recall that \( \pi \) is the involution of \( \mathbb{Z}[\Lambda_{m|n}] \) given by \( \pi(e^\lambda) := (-1)^p(\lambda)e^\lambda \); we extend this involution to the ring of fractions of \( \mathbb{Z}[\Lambda_{m|n}] \). Recall that \( \text{sch}_L(\lambda) = (-1)^p(\lambda)\pi(\text{ch}_L(\lambda)) \). For each \( \nu \in \Lambda^+_{m|n} \) set

\[
E^{-\nu} := (-1)^{p(\nu)}\pi(E_\nu).
\]

Using the character formulas and 3.5.1 we obtain the following formulas.

7.1.1. Corollary. For \( \lambda \in \Lambda^+_{m|n} \) one has

\[
\text{sch}_L(\lambda) = \sum_{\mu \in \Lambda^+_{m|n}} (-1)^{p(\lambda - \mu) + |\lambda| - |\mu|} d_{<}^{\lambda, \mu} E^{-\mu},
\]

where \( d_{<}^{\lambda, \mu} \) is the number of increasing paths from \( \mu \) to \( \lambda \) in \( D_g \).

If \( \lambda \) is stable and \( t \neq 2 \), then \( \text{sch}_L(\lambda) = \sum_{\mu \in \Lambda^+_{m|n}} d_{<}^{\lambda, \mu} E^{-\mu}. \)

The main result of this section is the following theorem, which will be proven in 7.3 below.

7.2. Theorem. Take \( \lambda \in \Lambda^+_{m|n} \). If \( \text{tail}(\lambda) < j \), then \( \text{ds}_j(E^{-\lambda}) = 0 \). If \( \text{tail}(\lambda) \geq j \) let \( \lambda' \in \Lambda^+_{m-j|n-j} \) be such that the weight diagram \( \lambda' + \rho' \) is obtained from the diagram of \( \lambda + \rho \) by the removal the first \( j \) leftmost symbols \( \times \). Then
where $K(\lambda')$ is the Kac $\mathfrak{g}'$-module with the even highest weight vector of weight $\lambda'$.

For $\mathfrak{osp}(2m|2n)$ with $m > j = \text{tail}(\lambda)$ one has

$$\text{ds}_x(\mathcal{E}_\lambda^-) = \begin{cases} 
\mathcal{E}_\lambda^- & \text{if } \text{tail}(\lambda) > j; \\
\mathcal{E}_\lambda^- + \mathcal{E}_{\lambda'}^- & \text{if } \text{tail}(\lambda) = j, \quad \mathfrak{g} = \mathfrak{osp}(2m+1|2n); \\
\mathcal{E}_\lambda^- & \text{if } \text{tail}(\lambda) = j, \quad \mathfrak{g} = \mathfrak{osp}(2j|2n); \\
\text{sch } K(\lambda') & \text{if } \text{tail}(\lambda) = j, \quad \mathfrak{g} = \mathfrak{gl}(m|n), 
\end{cases}$$

Remark. For a typical module $N$ one has $\text{DS}_x(N) = 0$ for each $x \neq 0$. If $L(\lambda)$ is typical, then $\text{Re}^\rho \text{ch} L(\lambda) = \text{KW}(\lambda + \rho, \emptyset)$ and

$$\text{sch } L(\lambda) = \mathcal{E}_\lambda^- = (\pi(R))^{-1} \sum_{w \in W} (-1)^{p(\lambda + \rho - w(\lambda + \rho))} \text{sgn } w \cdot e^{w(\lambda + \rho)}.$$

By above, $\text{ds}_x(\mathcal{E}_\lambda^-) = 0$ (since $\text{DS}_x(L(\lambda)) = 0$). In particular, $\text{ds}_x(\mathcal{E}_\lambda^-) = 0$ if $\lambda \notin \Lambda^+_m$.

7.2.2. Weight $\lambda'$. If $\text{tail}(\lambda) \geq j$ for $\mathfrak{osp}$ or $\text{tail}(\lambda) > j$ for $\mathfrak{gl}$, then

$$\text{tail}(\lambda') = \text{tail}(\lambda) - j.$$

Thus $\lambda \mapsto \lambda'$ corresponds to the “tail-cutting”. Moreover, for $\mathfrak{gl}$-case $y_0(\lambda^\dagger) = y_0((\lambda')^\dagger)$ (see 6.1.1 for notation); in other words, the diagram of $(\lambda')^\dagger$ is obtained from the diagram of $\lambda^\dagger$ by cutting $j$ elements from the “vertical tail”; for example,

$$f = > x x < x x > o \ldots \quad f^\dagger = > o o < o x^4 > o \ldots$$

$$f' = > o o < x x > o \ldots \quad (f')^\dagger = > o o < o x^2 > o \ldots$$

Remark. Let $j$ be the rank of $x$. We take $x \in \sum_{\beta \in S_j} \mathfrak{g}_\beta$ and identify $\mathfrak{g}_x$ with a subalgebra of $\mathfrak{g}$ as in [DS]. In the $\mathfrak{osp}$-case $\lambda' = \lambda|_{\mathfrak{h}_x}$; for $\mathfrak{gl}$ this holds if $\lambda$ is stable.

7.2.4. Remark. In the $\mathfrak{osp}$-case the “tail-cutting” is “reversible” (we can reconstruct the tail if our dog is still alive): the weight diagram of $\lambda + \rho$ is obtained by adding $j$ symbols $\times$ to the zero position in the diagram of $\lambda' + \rho'$. Therefore $\text{ds}_j(\mathcal{E}_\lambda^-) = \text{ds}_j(\mathcal{E}_\nu^-) \neq 0$ implies $\lambda = \nu$. (The same holds for $\mathfrak{gl}$-case if $\text{tail}(\lambda) > j$.)

This gives the following corollary.

7.2.5. Corollary. In the $\mathfrak{osp}$-case $\{\mathcal{E}_\lambda^- | \text{tail}(\lambda) \leq j\}$ is a basis of the kernel of $\text{ds}_j$. 
7.2.6. **Remark.** Take \( \lambda \in \Lambda^+_{m|n} \) which is assumed to be stable for \( \mathfrak{gl} \)-case. Using the notation of [4,2] we introduce

\[
E_{\lambda,i} := R^{-1} e^{-\rho} J_W \left( \frac{e^{\lambda+\rho}}{\prod_{a \in \Delta(p^{(i)})} (1 + e^a)} \right), \quad E_{\lambda,i}^{-} := (-1)^{p(\lambda)} \pi(E_{\lambda,i}).
\]

Note that \( E_{\lambda} = E_{\lambda,\text{tail}(\lambda)} \). Take \( \lambda \in \Lambda^+_{m|n} \) which is assumed to be stable in the \( \mathfrak{gl} \)-case and retain notation of Theorem [7,2] If \( \text{tail}(\lambda) \geq j \) for \( \mathfrak{osp} \) or \( \text{tail}(\lambda) > j \) for \( \mathfrak{gl} \), then \( E_{\lambda,\text{tail}(\lambda)-j}^{-} = E_{\lambda}^{-} \). In the \( \mathfrak{gl} \)-case with \( \text{tail}(\lambda) = j \) one has \( E_{\lambda,0}^{-} = \text{sch} \, K(\lambda) \).

7.2.7. Let \( \chi \) be a central character of atypicality \( j = \text{rank} \, x \). Let \( \nu \) be the weight of \( \mathfrak{g}_x \) with the diagram equal to the diagram of \( \chi \). This means that for each \( \lambda \in \Lambda^+_{m|n} \) with \( \chi_{\lambda} = \chi \) one has \( \nu = \lambda|_{\mathfrak{h}_x} \) (the diagram of \( \nu \) is obtained from the diagram of \( \lambda \) by removing all symbols \( \times \)). Note that \( \nu \) is a typical weight, so \( E_{\nu}^{-} = \text{sch} \, L(\nu) \). We put \( L^{\text{core}} = L(\nu) \) in the \( \mathfrak{gl} \) and \( \mathfrak{osp}(2m+1|2n) \)-case and

\[
L^{\text{core}} := \begin{cases} L(\nu) & \text{if } \nu \text{ is } \sigma - \text{invariant} \\ L(\nu) \oplus L(\nu)^\sigma & \text{else} \end{cases}
\]

in the \( \mathfrak{osp}(2m|2n) \)-case. The notion of \( L^{\text{core}} \) was first introduced in [GST1], but there \( L^{\text{core}} \) always equals \( L(\nu) \) and therefore differs from ours in the \( \mathfrak{osp}(2m|2n) \) in case \( \nu \) is not \( \sigma \)-invariant.

7.2.8. Extend \( \text{sdim} \) to a linear function on the Grothendieck ring \( \text{Ch}_x(\mathfrak{g}) \). Clearly, \( \text{sdim} \) gives a linear function on \( \text{Sch}(\mathfrak{g}) \).

**Corollary.** For \( \lambda \in \Lambda^+_{m|n} \) \( \text{sdim} \, E_{\lambda}^{-} = \text{sdim}(L(\lambda))^{\text{core}} \) if \( \lambda \) is a Kostant weight and \( L(\lambda) \) has the maximal atypicality; \( \text{sdim} \, E_{\lambda}^{-} = 0 \) for other weights.

**Proof.** Since \( \text{sdim}(DS_x(N)) = \text{sdim} \, N \) (see [DS], [S1]), the homomorphism \( d_x \) preserves \( \text{sdim} \). Take \( x \) of the maximal rank (= \( \min(m,n) \)). By Theorem [7,2] \( d_x(E_{\lambda}^{-}) = 0 \) if \( \text{tail}(\text{howl}(\lambda)) = \text{tail}(\lambda) < \min(m,n) \). Hence \( d_x(E_{\lambda}^{-}) \neq 0 \) implies \( \text{tail}(\text{howl}(\lambda)) = \min(m,n) \) which means that \( \lambda \) is a Kostant weight and \( L(\lambda) \) has the maximal atypicality. Now let \( \lambda \) be a Kostant weight and \( L(\lambda) \) has the maximal atypicality. The algebra \( \mathfrak{g}_x \) is either a Lie algebra \( (\mathfrak{gl}_{m-n-1}, \mathfrak{o}_{2(m-n)}, \mathfrak{o}_{2(m-n)}, \mathfrak{s}p_{2m-2n}, \mathfrak{osp}(1|2(n-m))) \) and \( L^{\text{core}} \) is a \( \mathfrak{g}_x \)-module with \( \text{sch} \, L^{\text{core}} = E_{\lambda}^{-} \) except for the case when \( \mathfrak{g}_x = \mathfrak{o}_{2(m-n)} \neq 0 \) and \( \text{sch} \, L^{\text{core}} = E_{\lambda}^{-} + E_{(\lambda)\sigma}^{-} \). \( \square \)

7.2.9. **Corollary.** Take \( \lambda \in \Lambda^+_{m|n} \). If the rank of \( x \) is equal to the atypicality of \( \chi_{\lambda} \), then

\[
d_x \, L(\lambda) = m(\lambda) \, [L^{\text{core}}]
\]

where \( m(\lambda) \) is equal to the number of increasing paths from \( \lambda \) to the weights with adjacent \( \times \)'s (see Section [7,3,5]) and \( L^{\text{core}} \) denotes the class of \( L^{\text{core}} \) in the supercharacter ring.
7.3. **Proof of Theorem 7.2.** Using (7) we reduce the assertions to the case \( j := 1 \). Set \( s := \text{tail}(\lambda) \).

7.3.1. First, we consider the \( \mathfrak{osp} \)-case and the \( \mathfrak{gl} \)-case with a stable weight \( \lambda \).

Take \( \beta_0 \in S_1 \) (\( \beta_0 = \pm (\varepsilon_m - \delta_n) \) for \( \mathfrak{osp} \)-case and \( \beta_0 = \varepsilon_m - \delta_1 \) for \( \mathfrak{gl}(m|n) \)).

We take \( x \in \mathfrak{g}_{\beta_0} \). Set \( \mathfrak{g}' := DS_x(\mathfrak{g}) \). By [DS] (and [S1]) we can identify \( \mathfrak{g}' \) with a subalgebra of \( \mathfrak{g} \) such that \( \mathfrak{h} \cap \mathfrak{g}' \) is a Cartan subalgebra of \( \mathfrak{g}' \) and a base \( \Sigma' \) for \( \Delta(\mathfrak{g}') \) satisfies
\[
\Delta^+ = \Delta^+(\Sigma') \prod \{\beta_0\} \prod B \prod \{\alpha + \beta_0 | \alpha \in B\}
\]
for some \( B \subset \Delta^+ \). Let \( \rho' \) (resp., \( R' \)) be the Weyl vector (resp., denominator) for \( \mathfrak{g}' \) with respect to \( \Sigma' \). As in [A.6] we define
\[
\text{pr}(e^\nu) = c_\nu e^{\nu | \rho'},
\]
where \( c_\nu := e^{-\pi_i(\nu | \delta_i)} \) with \( q = n \) for \( \mathfrak{osp} \)-case, \( q = 1 \) for \( \mathfrak{gl} \)-case (\( \beta_0 | \delta_q \neq 0 \)). By (20) one has
\[
\rho - \rho' \in \mathbb{Z}_{\beta_0}, \quad \text{pr}(R(1 + e^{-\beta_0})) = R'.
\]

By [A.6.1] we have
\[
ds_x(\mathcal{E}_\lambda^-) = (\pi \text{pr} \pi) (\mathcal{E}_\lambda^-) = (-1)^{p(\lambda)} (\pi \text{pr}) (\mathcal{E}_\lambda),
\]
which allows to rewrite the required formula as follows: \( \text{pr}(\mathcal{E}_\lambda) = 0 \) if \( s = 0 \) and
\[
(21) \quad (-1)^{p(\lambda) - p(\lambda')} \text{pr}(\mathcal{E}_\lambda) = \begin{cases} 
\mathcal{E}_\lambda & \text{if } s > 1; \\
\mathcal{E}_\lambda' & \text{if } s = 1, \quad \mathfrak{g} = \mathfrak{osp}(2m + 1|2n); \\
\mathcal{E}_\lambda & \text{if } s = 1, \quad \mathfrak{g} = \mathfrak{osp}(2|2n); \\
\text{ch } K(\lambda') & \text{if } s = 1, \quad \mathfrak{g} = \mathfrak{gl}(m|n), \\
\mathcal{E}_\lambda & \text{for } \mathfrak{osp}(2m|2n), m > s = 1, \quad (\lambda')^\sigma = \lambda', \\
\mathcal{E}_\lambda + \mathcal{E}_{(\lambda')^\sigma} & \text{for } \mathfrak{osp}(2m|2n), m > s = 1, \quad (\lambda')^\sigma \neq \lambda'.
\end{cases}
\]

In the \( \mathfrak{gl} \)-case take \( \lambda^\dagger \) as in 6.1.1; in the \( \mathfrak{osp} \)-case we have \( \lambda^\dagger = \lambda + \rho \). Using Proposition 4.3 we get
\[
(22) \quad j_s \text{pr}(\mathcal{E}_\lambda) = \text{pr}\left(R^{-1} \rho e^{-\rho} \text{KW}(\lambda^\dagger, S_s)\right) = c_{-\rho} \cdot (R' e^{\rho'})^{-1} \text{pr}\left((1 + e^{-\beta_0}) \text{KW}(\lambda^\dagger, S_s)\right).
\]

For \( s = 0 \) the formula (27) gives \( \text{pr}(\mathcal{E}_\lambda) = 0 \) as required. From now on we assume \( s > 0 \). The pair \( (\lambda^\dagger, S_s) \) satisfies the assumptions of Proposition A.6.3; using this proposition and taking into account that for \( \mathfrak{gl} \)-case
\[
R' e^{\rho'} \text{ch } K(\lambda') = \text{KW}(\lambda' + \rho', \emptyset)
\]
we see that (21) holds up to a non-zero scalar \( a_\lambda \) which can be computed directly. Instead of performing such computation we can employ the following reasoning. One has
\[
\text{sch } DS_x(L(\lambda)) = ds_x(\mathcal{E}_\lambda^-) + \sum_{\nu < \lambda} d^\lambda_\nu ds_x(\mathcal{E}_\nu^-).
\]
By above, \( ds_x(\mathcal{E}_\nu^-) \) is proportional to \( \mathcal{E}_\nu^- \) (or to \( \mathcal{E}_\nu^- + \mathcal{E}_{(\nu')^\sigma} \) for \( \mathfrak{osp}(2m|2n) \)), where \( \nu' := \nu|_{\nu'} \). By Corollary 4.9.1
\[
\text{supp}(\mathcal{E}_\nu^-) \subset \nu' - \mathbb{N}\Sigma',
\]
where \( \mathcal{E}_\nu^- \) is viewed as element of \( \mathcal{R}_{\Sigma'} \), see [A2]. The inequality \( \nu < \lambda \) means that \( \lambda - \nu \in \mathbb{N}\Sigma \) which implies \( \nu' \in \lambda' - \mathbb{N}\Sigma' \) by (20). Hence the coefficient of \( e^{\lambda'} \) in \( ds_x(\mathcal{E}_\lambda^-) \) is equal to \( \text{sdim} \text{DS}_x(L(\lambda))_{\lambda'} \). Using the same reasoning for the formula
\[
\text{sdim} L(\lambda') = \sum_{\nu'} d_{<\nu'}^{\lambda',\nu'} \mathcal{E}_{\nu}'
\]
we conclude the coefficient of \( e^{\lambda'} \) in \( \mathcal{E}_\lambda^- \) is 1. Combining \( (\lambda, \beta_0) = 0 \) and \( \beta_0 \in \Sigma \) one readily sees that \( \text{DS}_x(L(\lambda))_{\lambda'} = \mathbb{C} \), so \( \text{sdim} \text{DS}_x(L(\lambda))_{\lambda'} = 1 \). Hence the coefficients of \( e^{\lambda'} \) in \( ds_x(\mathcal{E}_\lambda^-) \) and in \( \mathcal{E}_\lambda^- \) are equal, so \( a_{\lambda} = 1 \).

7.3.2. Consider the case when \( j = 1 \) and \( g = \mathfrak{gl}(m|n) \). If \( \lambda \) is stable, the required formula is established in 7.3.1. Using the fact that \( \text{DS}_x \) commutes with translation functors, we deduce from the stable case the required formula for the non-stable case taking into account Corollary 6.4 and 6.4.1 for \( s > 1 \) and \( s = 1 \) respectively. \( \square \)

8. Superdimensions and modified Superdimensions

We discuss modified nontrivial trace and dimension functions on the thick ideal \( I_k \) generated by the irreducible representations of atypicality \( k \), and how they can be calculated explicitly by means of the Duflo-Serganova functor. We do this for the \( \mathfrak{osp}(m|2n) \) and the \( \text{OSp}(m|2n) \)-case. For the \( \mathfrak{gl} \)-case see [HW].

8.1. The core of a block. Recall that \( \mathcal{F} = \text{Rep}(SOSp(m|2n)) \) and that we have a decomposition \( \mathcal{F} = \mathcal{F}' \oplus \Pi \mathcal{F}' \) into two subcategories which are equivalent by the parity shift \( \Pi \). We use the notation:
\[
\mathcal{F}' = \mathcal{F}'(m|2n) = \text{Rep}(OSp(m|2n))
\]
for the finite dimensional algebraic representations of \( OSp(m|2n) \). As for \( \mathcal{F} \) the category decomposes \( \mathcal{F}' = \mathcal{F}' \oplus \Pi \mathcal{F}' \) into two equivalent subcategories.

The irreducible typical module \( L_{\text{core}} \) (as defined in 7.2.7) attached to a block of atypicality \( k \) in \( \mathcal{F} \) is both an \( \mathfrak{osp}(m-2k|2n-2k) \) and \( OSp(m-2k|2n-2k) \)-module. Therefore the core can be defined in the \( \mathcal{F}' \)-case as well.

The \( \text{DS}_x \) functor on \( \mathcal{F} \) induces a functor
\[
\text{DS}_x : \mathcal{F}'(m|2n) \to \mathcal{F}'(m-2k|2n-2k),
\]
where \( k = rk(x) \) (see [CH] for details).

If the rank of \( x \) equals \( \text{def}(\mathfrak{osp}(m|2n)) \) we obtain
— $\mathfrak{g}_x = o(m - 2n|0)$, $m > 2n$;
— $\mathfrak{g}_x = \mathfrak{sp}(0|2n - 2m)$, $2n > m$ even;
— $\mathfrak{g}_x = \mathfrak{osp}(1|2n - 2m)$, $2n - m$ odd.

In the $OSp$-case we obtain representations of the groups $G_x = O(m - 2n)$, $Sp(2n - 2m)$ (considered as odd) and $OSp(1|2n - 2m)$.

8.2. **Superdimensions.** If we apply $DS_x$ to an irreducible representation $L(\lambda)$ with atypicality equal to $rk(x)$, then $DS_x(L(\lambda))$ doesn’t depend on the choice of $x$. Indeed the induced morphism on the supercharacter ring doesn’t depend on $x$ and $DS_x(L(\lambda))$ is semisimple. We simply write $DS_k$ in this case.

The parity rule of [GH] yields

$$DS_k(L(\lambda)) \in \Pi^{[\text{howl}(\lambda)]}(\mathcal{F}(\mathfrak{g}_x))$$

and hence

$$DS_k(L(\lambda)) = \Pi^{[\text{howl}(\lambda)]}(L_{\text{core}})^{\pm m(\lambda)}.$$ for the positive integer $m(\lambda)$ defined in Corollary 7.2.9 (the number of increasing paths from $\lambda$ to the weights with adjacent $\times$’s).

8.2.1. **$OSp$-modules.** We first consider $\mathfrak{g} = \mathfrak{osp}(2m|2n)$. By [ES, Proposition 4.11] the simple $OSp(2m|2n)$-modules are either of the form $L(\lambda)$ if $\lambda \in \Lambda^+_{m|n}$ is $\sigma$-invariant or $L(\lambda) \oplus L(\lambda^\sigma)$. Thus the simple $OSp(2m|2n)$-modules are in one-to-one correspondence with the unsigned $\mathfrak{osp}(2m|2n)$-diagrams. For $\mathfrak{osp}(2m + 1|2n)$ and any $\lambda \in \Lambda^+_{m|n}$ there are two irreducible $OSp(2m + 1|2n)$-modules $L(\lambda, +)$ and $L(\lambda, -)$ which restrict to $L(\lambda)$. We will often simply write $L_{OSp}(\lambda)$ for an irreducible representation of $OSp$. The diagram

$$
\begin{array}{ccc}
\mathcal{F}'(G) & \xrightarrow{\text{Res}} & \tilde{\mathcal{F}}(\mathfrak{g}) \\
\downarrow DS_x & & \downarrow DS_x \\
\mathcal{F}'(G_x) & & \\
\end{array}
$$

commutes for any $x$ since $DS_x(L(\lambda))$ is in $\mathcal{F}'(G_x)$. It follows from this diagram that the multiplicity of $L_{\text{core}}$ in $DS_xL_{OSp}(\lambda)$ is the same as for $\text{Res}(L_{OSp}(\lambda))$ if the restriction is irreducible and is twice the multiplicity of $DS(L(\lambda))$ if the restriction decomposes into two irreducible summands.

Since $DS$ is a symmetric monoidal functor it preserves the superdimension.
8.2.2. **Corollary.** For $L(\lambda) \in \tilde{\mathcal{F}}$ of atypicality $k$

$$\text{sdim } L(\lambda) = (-1)^{||\text{howl}(\lambda)||} m(\lambda) \text{sdim } L^{\text{core}}.$$ 

In particular $\text{sdim } L(\lambda) \neq 0$ if and only if $\lambda$ is maximal atypical.

8.3. **Examples.** We list some low-rank examples of $DS(L(\lambda))$ (up to parity shift) as well as the superdimension of $L(\lambda)$ for $\mathfrak{osp}(4|4)$ and $\mathfrak{osp}(6|6)$. These can be computed by counting the increasing paths to the zero weight using Corollary 7.2.9. The number $m(\lambda)$ can also be computed via the formalism of arc diagrams in [GH].

8.3.1. **DS for $\mathfrak{osp}(4|4)$**.

| $\lambda$ | $(0,0)$; $(i-1,i)$; $(0,2)$; $(0,i)$; $2 < i$; $(i,j)$; $0 < i < j - 1$; | $\text{DS}_1(L(\lambda))$ | $0$; $(i)$; $(0) \oplus (2)$; $(0)^{\oplus 2} \oplus (i)$; $(i) \oplus (j)$ | $\text{sdim}$ | $1$ | $2$ | $3$ | $4$ | $4$ |
|------------|---------------------------------|-----------------|-----------------|----------|

8.3.2. **DS for $\mathfrak{osp}(6|6)$**.

| $\lambda$ | $(0,0,i)$; $i \leq 3$; $(0,0,4)$; $(0,0,i)$; $4 < i$ | $\text{DS}_1(L(\lambda))$ | $0$; $(i)$; $(0) \oplus (0,4)$; $(0,0)^{\oplus 2} \oplus (0,i)$; | $\text{sdim}$ | $i + 1$ | $5$ | $6$ |
|------------|---------------------------------|-----------------|-----------------|----------|

| $\lambda$ | $(0,1,2)$; $(0,1,3)$; $(0,1,4)$; $(0,1,i)$; $4 < i$ | $\text{DS}_1(L(\lambda))$ | $(1,2)$; $(1,3)$; $(0,1) \oplus (1,4)$; $(0,1)^{\oplus 2} \oplus (1,i)$ | $\text{sdim}$ | $4$ | $8$ | $10$ | $12$ |

| $\lambda$ | $(0,2,3)$; $(0,2,4)$; $(0,2,i)$; $4 < i$ | $\text{DS}_1(L(\lambda))$ | $(2,3) \oplus (0,3)$; $(2,4) \oplus (0,4) \oplus (0,2)$; $(2,i) \oplus (0,4) \oplus (0,2)^{\oplus 2}$ | $\text{sdim}$ | $8$ | $15$ | $18$ |

| $\lambda$ | $(0,3,4)$; $(0,4,5)$; $(0,3,i)$; $4 < i$ | $\text{DS}_1(L(\lambda))$ | $(3,4) \oplus (0,3) \oplus (0,4)$ | $\text{sdim}$ | $12$ | $12$ | $20$ |

For $i > 0$

| $\lambda$ | $(i,i+1,i+2)$; $(i,i+1,i+3)$; $(i,i+1,i+4)$; $4 < i$ | $\text{sdim}$ | $4$ | $8$ | $12$ |

8.3.3. **Other examples.** $DS_1(L(0,1,\ldots,k)) = L(1,\ldots,k)$ and $\text{sdim } L(0,1,\ldots,k) = 2^k$.

$DS_1(L(0^i,1,\ldots,k)) = L(0^{i-1},1,\ldots,k)$ and $\text{sdim } L(0,0,1,\ldots,k) = 2^k$. 
8.4. Modified Traces. In this section \( \tilde{\mathcal{F}} \) means either \( \tilde{\mathcal{F}} \) or \( \tilde{\mathcal{F}}' \) unless otherwise specified.

If \( \text{at}(L(\lambda)) < n \), \( \text{sdim}(L) = 0 \). However one can define a modified superdimension for \( L \) as follows. Recall that a thick (tensor) ideal \( I \) in \( \tilde{\mathcal{F}} \) is a subset of objects which is closed under tensor products with arbitrary objects and closed under direct summands. A trace on \( I \) is by definition a family of linear functions

\[
t = \{ t_V : \text{End}_\mathcal{F}(V) \to k \} \]

where \( V \) runs over all objects of \( I \) such that following two conditions hold.

(i) If \( U \in I \) and \( W \) is an object of \( \tilde{\mathcal{F}} \), then for any \( f \in \text{End}_\mathcal{F}(U \otimes W) \) we have

\[
t_{U \otimes W}(f) = t_U(t_R(f))
\]

for the right trace \( tr_R() \).

(ii) If \( U, V \in I \) then for any morphisms \( f : V \to U \) and \( g : U \to V \) we have

\[
t_V(g \circ f) = t_U(f \circ g).
\]

For such a trace on \( I \) we define

\[
\dim^I(X) = t_X(id_X), \quad X \in I,
\]

the modified dimension of \((I, t)\). For an object \( J \in \tilde{\mathcal{F}} \) let \( I_J \) be the thick ideal generated by \( J \). By Kujawa [Ku, Theorem 2.3.1] the trace on the ideal \( I_L, L \) irreducible, is unique up to multiplication by an element of \( \mathbb{C} \).

8.5. The generalized Kac-Wakimoto conjecture. Let \( I_k \) be the thick ideal generated by all irreducible representations of atypicality \( k \). The ideal \( I_0 \) coincides with \( \text{Proj} \). The following theorem was proven for \( \mathfrak{gl}(m|n) \) by Serganova [S4] and for \( \mathfrak{osp}(m|2n) \) by Kujawa [Ku]. We give a slightly different simplified proof. Moreover we explain how to compute these modified superdimensions.

8.5.1. Theorem. (Generalized Kac-Wakimoto conjecture) The ideal \( I_k \) admits a non-trivial modified trace function. For irreducible \( L(\lambda) \) the associated dimension function \( \dim^k \) satisfies \( \dim^k L(\lambda) \neq 0 \) if and only if the atypicality of \( L(\lambda) \) is \( k \).

It was shown in [GKPM, Theorem 1.3.1] that if an ideal \( I \) carries a modified trace function, all indecomposable objects in \( I \) are ambidextrous in the sense of [GKPM]. Since the \( I_k \) define an exhaustive filtration of \( \tilde{\mathcal{F}} \), the conjecture implies that every simple module in \( \tilde{\mathcal{F}} \) is ambidextrous.

8.6. A trace on \( I_k \). There are two different ways to see that \( \text{Proj} \subset \tilde{\mathcal{F}}' \) carries a nontrivial trace function. It was proven in [GKPM2, Theorem 4.8.2] that \( \text{Proj} \subset \tilde{\mathcal{F}} \) has such a trace function. This implies that \( \text{Proj} \subset \tilde{\mathcal{F}}' \) has one as well using the restriction rules of Ehrig-Stroppel and the argument of [Ku].
Alternatively it follows from [HW2] that $\text{Proj} \subset \tilde{\mathcal{F}}'$ carries such a trace function. Note that it is unique up to a scalar: Any $P \in \text{Proj}$ satisfies $<P> = \text{Proj}$. Indeed, $<P> \subset \text{Proj}$ is clear, and $\text{Proj} \subset P$ follows since $\text{Proj}$ is the smallest thick ideal [CH]. We denote any normalization of this trace function by $T_r^0$.

### 8.6.1. Proposition

The thick ideal $I_k \subset \tilde{\mathcal{F}}$ carries a nontrivial modified trace function $T_r^k$.

**Proof.** Let $L(\lambda) \in \tilde{\mathcal{F}}$. Then $DS_X(L(\lambda)) \in \tilde{\mathcal{F}}'$ for all $x$ by [GH]. Let $X \in I_k$ and $f \in \text{End}(X)$. Then we define

$$T_r^k(f) = T_r^0(DS_k(f)).$$

Then $DS_k(X)$ is typical and therefore projective. Since $DS_k$ is a symmetric monoidal functor this defines a trace function. We claim that it is nontrivial. For $X = L(\mu)$ we obtain

$$DS_k(f) \in \text{End}(\Pi[\text{howl}(\mu)][L]\text{core}(\Pi[\text{howl}(\mu)][L]\text{core})).$$

Since the parity is either even or odd and $T_r^0$ is nontrivial for any typical module, we compute for $f \in \text{End}(X)$

$$T_r^k(id_L) = T_r^0(DS_k(id_L)) = m(\lambda)T_r^0(id_{\Pi[\text{howl}(\mu)][L]\text{core}}(id_{\Pi[\text{howl}(\mu)][L]\text{core}}) \neq 0.$$

The same proof works for $L_{\text{OSp}}(\lambda)$. □

### 8.6.2. Remark

It can be shown [Ku] that $I_k$ is in fact generated by an arbitrary irreducible representation of atypicality $k$. Therefore the above trace is the unique modified trace up to a scalar.

Since $DS_k(L) = 0$ for any $L$ of atypicality $< k$, we obtain for the modified superdimension $sd\text{dim}^k(X) := T_r^k(id_X)$

### 8.6.3. Corollary

Let $L(\lambda)$ be a representation of atypicality $\leq k$. Then $sd\text{dim}^k(L(\lambda)) \neq 0$ if and only if $\text{at}(L(\lambda)) = k$.

### Appendix A. Kac-Wakimoto terms and the rings $\mathcal{R}, \mathcal{R}_\Sigma'$

In this section $\mathfrak{g}$ is $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(M|N)$ or one of the exceptional Lie superalgebras $F(4), G(3), D(2|1, a)$. We use the standard notation for the roots of $\mathfrak{g}_0$ and denote by $\Pi_0$ a standard set of simple roots. In what follows we consider only bases $\Sigma$ of $\Delta$ which are compatible with $\Pi_0$, that is $\Delta^+(\Sigma)_0 = \Delta^+(\Pi_0)$. By [SI], all such bases are connected by chains of odd reflections. In the $\mathfrak{gl}$ and $\mathfrak{osp}$-cases these bases can be encoded by words consisting of $m$ letters $\varepsilon$ and $n$ letters $\delta$. 
A.1. **Notation.** We denote by $W$ the Weyl group of $\mathfrak{g}_0$. We denote by $\Delta$ the set of roots of $\mathfrak{g}$ and set

$$
\mathfrak{h}_\text{int} := \{ \lambda \in \mathfrak{h}^* | \forall w \in W \; \lambda - w\lambda \in \mathbf{Z}\Delta \},
$$

$$
P(\mathfrak{g}_0) := \{ \lambda \in \mathfrak{h}^* | \forall w \in W \; \lambda - w\lambda \in \mathbf{Z}\Delta_0 \}.
$$

For each non-isotropic root $\alpha$ let $r_\alpha \in W$ be the reflection with respect to $\alpha$. For any subset $Y \subset W$ we denote by $J_Y$ the linear operator $P \mapsto \sum_{w \in Y} \text{sgn}(w)w(P)$, where $\text{sgn} : W \rightarrow \mathbf{Z}_2$ is the standard sign homorphism (given by $\text{sgn} r_\alpha = -1$).

A.1.1. **Choice of the Weyl vector.** We denote by $\rho_0$ a Weyl vector of $\mathfrak{g}_0$ which is an element of $\mathfrak{h}^*$ satisfying

$$
r_\alpha \rho_0 = \rho_0 - \alpha
$$

for each $\alpha \in \Pi_0$. Note that $\rho_0$ is unique if $\Delta_0$ spans $\mathfrak{h}^*$, i.e. for $\mathfrak{g} \neq \mathfrak{gl}(m|n), \mathfrak{osp}(2|2n)$.

We choose the Weyl vector $\rho$ by the rule

$$
\rho := \rho_0 - \rho_1, \quad \rho_1 = \frac{1}{2} \sum_{\alpha \in \Delta^+_1} \alpha.
$$

If $\beta \in \Sigma$ is isotropic and $\Sigma' = r_\beta \Sigma$ we have $\rho' := \rho + \beta$. Using [S5] (or a short case-by-case reasoning) we obtain $\rho \in \mathfrak{h}_{\text{int}}^*$. We introduce

$$
R_0 := \prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha}), \quad R_1(\Sigma) := \prod_{\alpha \in \Delta_1^+(\Sigma)} (1 + e^{-\alpha}), \quad R(\Sigma) := \frac{R_0}{R_1(\Sigma)}.
$$

Note that the term

$$
e^{\rho_0 - \rho} R_1(\Sigma) = \prod_{\alpha \in \Delta_1^+(\Sigma)} (e^{\alpha/2} + e^{-\alpha/2})
$$

is $W$-invariant and does not depend on the choice of $\Sigma$. Hence for each $\Sigma'$ satisfying $\Delta_0^+ \subset \Delta^+_1(\Sigma)$ we have

$$
R(\Sigma') e^{\rho'} = Re^\rho, \quad \text{where } R := R(\Sigma).
$$

A.2. **Rings $R$ and $R_{\Sigma}$.** For a sum of the form $\sum_{\nu \in \mathfrak{h}^*} a_\nu e^\nu$ with $a_\nu \in \mathbf{Q}$ we define the support by the formula

$$
\text{supp}(\sum a_\nu e^\nu) = \{ \nu \in \mathfrak{h}^* | a_\nu \neq 0 \}.
$$

Let $R_{\Sigma}$ be the set consisting of the sums $\sum_{\nu \in \mathfrak{h}^*} a_\nu e^\nu$ with $a_\nu \in \mathbf{Q}$ and such that

$$
\text{supp}(\sum a_\nu e^\nu) \subset \bigcup_{i=1}^k (\nu_i - N\Sigma')
$$

for some $k$. Clearly, $R_{\Sigma}$ is a ring. This ring contains $\text{ch} N$ and $\text{sch} N$ for any $N$ in the BGG-category $O$. 
A.2.1. Denote by \( R \) the ring of rational functions of the form \( \frac{P}{Q} \), where \( P \) lies in the group ring \( \mathbb{Q}[h^*] \) and \( Q \) is a product of the factors of the form \( 1 \pm e^{-\alpha} \) for \( \alpha \in \Delta \). Using the formula

\[
1 \pm e^{-\alpha} = 1 \mp e^{-\alpha} + e^{-2\alpha} \mp e^{-3\alpha} + \ldots
\]

we will view the element \( \frac{P}{Q} \in R \) as a series \( R(\Sigma) \); we will call this series the \( \Sigma \)-expansion of \( \frac{P}{Q} \). For instance, \( R(\Sigma'), R(\Sigma')^{-1} \in R \) for any base \( \Sigma' \) and \( \Sigma \)-expansion of \( R(\Sigma')^{-1} \) is equal to the character of a Verma module of the highest weight 0 (defined with respect to the base \( \Sigma \)).

A.2.2. **Lemma.** For any base \( \Sigma' \) satisfying \( \Delta^+(\Sigma')_0 = \Delta^+(\Sigma) \) one has

\[
J_W \left( \frac{e^{\rho'}}{\prod_{\alpha \in \Delta^+_+(\Sigma')} (1 + e^{-\alpha})} \right) = R\rho.
\]

**Proof.** By above, \( e^{\rho_0 - \rho'} R_1(\Sigma') \) is \( W \)-invariant so

\[
J_W \left( \frac{e^{\rho'}}{\prod_{\alpha \in \Delta^+_+(\Sigma')} (1 + e^{-\alpha})} \right) = J_W \left( \frac{e^{\rho'}}{R_1(\Sigma')} \right) = \frac{J_W(e^{\rho_0})}{e^{\rho_0 - \rho'} R_1(\Sigma')}.
\]

The Weyl character formula for the trivial \( g_0 \)-module gives \( J_W(e^{\rho_0}) = R_0 e^{\rho_0} \). Using the above identity \( R(\Sigma') e^{\rho'} = R e^{\rho} \) we obtain the required formula. \( \Box \)

A.3. **Projection \( P_\chi \).** Let \( \mathcal{O}^\chi \) be the full subcategory of the category \( \mathcal{O} \) corresponding to a central character \( \chi \). For \( N \in \mathcal{O} \) let \( N^\chi \) be the projection of \( N \) to \( \mathcal{O}^\chi \). The character of \( N^\chi \) can be expressed via \( \chi \) by the following procedure.

By above, \( R(\Sigma') \) contains the terms \( \chi \) for any module \( N \in \mathcal{O} \). It is well-known that for \( N \in \mathcal{O}^\chi \) the \( \Sigma' \)-expansion of \( Rt^\rho \chi \) satisfies

\[
\text{supp}(Rt^\rho \chi N) \subset \{ \mu + \rho | \chi_\mu = \chi_\lambda \}.
\]

Introducing a projection \( P_\chi : \mathcal{R}_{\Sigma'} \rightarrow \mathcal{R}_{\Sigma} \) by \( P_\chi(\sum a_\mu e^\mu) = \sum a_\mu e^\mu \) we get

\[
(23) \quad Rt^\rho \chi N^\chi = P_\chi(Rt^\rho \chi N).
\]

A.3.1. For a finite-dimensional module \( V \) a translation functor \( T^V_{\chi,\chi'} : \mathcal{O}^\chi \rightarrow \mathcal{O}^\chi' \) is given by \( T^V_{\chi,\chi'}(N) := (N \otimes V)^\chi' \). By above,

\[
\text{supp}(Rt^\rho \chi T^V) \subset \{ \mu + \rho | \chi_\mu = \chi_\lambda \}.
\]

Introducing a projection \( P_\chi : \mathcal{R}_{\Sigma'} \rightarrow \mathcal{R}_{\Sigma} \) by \( P_\chi(\sum a_\mu e^\mu) = \sum a_\mu e^\mu \) we get

\[
(23) \quad Rt^\rho \chi N^\chi = P_\chi(Rt^\rho \chi N).
\]
A.4. The terms \(\text{KW}(\lambda, S)\). We say that a subset \(S \subset \Delta_1\) is an iso-set if \(S\) is a basis of an isotropic subspace of \(\mathfrak{h}^*\), i.e. \(S\) is linearly independent and \((S|S) = 0\).

For \(\lambda \in \mathfrak{h}^*\) and an iso-set \(S \subset \Delta_1\) satisfying \((\lambda|S) = 0\) we set
\[
\text{KW}(\lambda, S) := J_W\left(\prod_{\beta \in S} \frac{e^\lambda}{(1 + e^{-\beta})}\right).
\]

A.4.1. Remark. For an arbitrary weight \(\lambda \in \mathfrak{h}^*\) the group \(W\) should be substituted by the “\(\lambda\)-integral” subgroup, see [GK], Section 11.)

A.4.2. Note that \(\text{KW}(\lambda, S)\) \cdot \(\prod_{\alpha \in \Delta_1^+} (1 + e^{-\alpha}) \in S(\mathfrak{h}^*)\), so \(\text{KW}(\lambda, S) \in \mathcal{R}\). One readily sees that for the \(\Sigma\)-expansion of \(\text{KW}(\nu, S)\) we have
\[
\text{supp} \text{KW}(\nu, S) \subset W(\nu + ZS).
\]
By [Ser1], [K2], \(\chi_{\mu - \rho} = \chi_{\nu - \rho}\) for each \(\mu \in \nu + ZS\). Thus for \(P_\chi\) introduced in A.3 we have
\[
(24) \quad P_\chi(\text{KW}(\lambda + \rho, S)) = \delta_{\chi,\chi_0} \text{KW}(\lambda + \rho, S).
\]

A.4.3. For \(\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(M|N)\) one has \(S = \{\pm \varepsilon_{p_i}, \pm \delta_{q_i}\}_{i=1}^t\), where \(p_i \neq p_j, q_i \neq q_j\) for \(i \neq j\). We denote the intersection of \(Z\Delta\) with the span of \(\varepsilon_{p_1}, \ldots, \varepsilon_{p_t}, \delta_{q_1}, \ldots, \delta_{q_t}\) by \(\mathfrak{h}(S)^*\). Notice that \(S\) spans a maximal isotropic subspace in \(\mathfrak{h}(S)^*\).

A.4.4. Lemma. (i) \(w \text{KW}(\lambda, S) = \text{sgn}(w) \text{KW}(\lambda, S) = \text{KW}(w\lambda, wS)\);
(ii) \(\text{KW}(\lambda, S) = 0\) if there exists \(\alpha \in \Delta_0\) such that \((\alpha|S) = (\alpha|\lambda) = 0\);
(iii) \(\text{KW}(\lambda - \beta, S) = \text{KW}(\lambda, (S \cup \{-\beta\}) \setminus \{\beta\})\) for each \(\beta \in S\);
(iv) in the \(\mathfrak{osp}\)-case if \((\lambda|\mathfrak{h}(S)^*) = 0\), then
\[
\text{KW}(\lambda - \beta, S) = - \text{KW}(\lambda, S) \quad \text{for each} \quad \beta \in S.
\]

Proof. (i), (iii) are straightforward and (ii) follows from (i) for \(w := r_\alpha\). For (iv) note that
\[
\text{KW}(\lambda, S) + \text{KW}(\lambda - \beta, S) = J_W\left(\frac{e^\lambda + e^{\lambda-\beta}}{\prod_{\beta \in S} (1 + e^{-\beta})}\right) = \text{KW}(\lambda, S \setminus \{\beta\}).
\]
Since \(\beta = \pm \varepsilon_i, \pm \delta_j\) for some \(i, j\) we have \((\lambda|\delta_j) = (S \setminus \{\beta\}|\delta_j) = 0\) so (iii) gives
\[
\text{KW}(\lambda; S \setminus \{\beta\}) = 0
\]
as required. \(\square\)
A.4.5. **Denominator identity.** Let \( S \) be an iso-set of the cardinality \( \min(m, n) \) and let \( \Sigma' \) be a base of \( \Delta \) containing \( S \) (for instance, \( \Sigma' = \Sigma \) for \( \mathfrak{osp} \)-case and \( \Sigma' \) corresponding to \( (\varepsilon \delta)^m \delta^n - m \) for \( n \geq m \)). By [Gor1] one has \( \operatorname{KW}(\rho'; S) = j Re^\rho \), where \( j \) is a certain integer (\( j \) is the order of the “smallest factor” in \( W \), for instance, \( j = m! \) for \( \mathfrak{gl}(m|n) \) with \( m \leq n \)).

Consider the case \( g = \mathfrak{gl}(s|s) \) or \( \mathfrak{osp}(2s+t|2s) \). Then \( j = s! \) for \( \mathfrak{gl}(s|s) \), \( j = \max(2^{s-1}!s!, 1) \) for \( \mathfrak{osp}(2s|2s) \), and \( j = 2^s s! \) for \( \mathfrak{osp}(2s+t|2s) \) with \( t = 1, 2 \). Let \( \Sigma' \) be the base corresponding to the word \( (\varepsilon \delta)^s \); this base contains an iso-set \( \{\varepsilon_i - \delta_i\}_{i=1}^s \). Note that \( w\rho' = \rho' \) for any \( w \in S_s \times S_s \); using Lemma [A.4.4] (i) we obtain

\[
j Re^\rho = \operatorname{KW}(\rho', \{\varepsilon_i - \delta_i\}_{i=1}^s) = (-1)^{\frac{s(s+1)}{2}} \operatorname{KW}(\rho', \{\varepsilon_i - \delta_{s+1-i}\}_{i=1}^s).
\]

A.5. **The term** \( \frac{\operatorname{KW}(\lambda, S)}{Re^\rho} \). Recall that \( L_{g_0}(\lambda - \rho_0) \) is finite-dimensional if and only if \( \lambda \in P^{++}(g_0) \), where

\[
P^{++}(g_0) := \{\lambda \in P(g_0) \mid \forall w \in W \lambda - w\lambda \in \mathbb{Z}_{\geq 0}\Delta^+, \lambda \neq w\lambda\}.
\]

The character ring \( \operatorname{Ch}(g_0) \) has a basis \( \{\operatorname{ch} L_{g_0}(\lambda - \rho_0)\}_{\lambda \in P^{++}(g_0)} \). This allows to extend \( \dim \) to the linear map \( \dim : \operatorname{Ch}(g_0) \rightarrow \mathbb{Z} \) having

\[
\dim(\operatorname{ch} N) = \dim N
\]

for any finite-dimensional module \( N \).

The Weyl character and the Weyl dimension formulas give the following.

A.5.1. **Lemma.** Take \( \lambda \in P(g_0) \). One has

(i) \( J_W(e^\lambda) = 0 \) if and only if \( \lambda \not\in WP^{++}(g_0) \);
(ii) \( \frac{J_W(e^\lambda)}{R_0 e^{\rho_0}} \in \operatorname{Ch}(g_0) \);
(iii) \( \dim(\frac{J_W(e^\lambda)}{R_0 e^{\rho_0}}) = \prod_{\alpha \in \Delta_0^+} \frac{(\lambda|\alpha)}{(\rho_0|\alpha)} \).

**Proof.** If \( \lambda \not\in WP^{++}(g_0) \), then \( r_\alpha \lambda = \lambda \) for some \( \alpha \in \Delta_0^+ \) and thus \( J_W(e^\lambda) = 0 \) and both sides of (iii) are equal to zero. Now take \( \lambda \in WP^{++}(g_0) \), that is \( \lambda = w\nu \) for \( \nu \in P^{++}(g_0) \). Then \( J_W(e^\lambda) = \operatorname{sgn}(w) J_W(e^\nu) \). Using the Weyl character formula we get

\[
(R_0 e^{\rho_0})^{-1} J_W(e^\lambda) = \operatorname{sgn}(w) (R_0 e^{\rho_0})^{-1} J_W(e^\nu) = \operatorname{sgn}(w) \operatorname{ch} L(\nu - \rho_0)
\]

which establishes (i). The Weyl dimension formula gives

\[
\dim(\frac{J_W(e^\lambda)}{R_0 e^{\rho_0}}) = \operatorname{sgn}(w) \dim L(\nu - \rho_0) = \operatorname{sgn}(w) \prod_{\alpha \in \Delta_0^+} \frac{\nu|\alpha}{\rho_0|\alpha} = \operatorname{sgn}(w) \prod_{\alpha \in w^{-1}\Delta_0^+} \frac{\lambda|\alpha}{\rho_0|\alpha}.
\]

One has \( (-1)^{\#\{\alpha \in w^{-1}\Delta_0^+ \cap (-\Delta_0^+)\}} = \operatorname{sgn}(w) \); this gives (iii) for \( \lambda \in WP^{++}(g_0) \). \( \square \)
A.5.2. For a subset \( U \subset \Delta \) we will use the notation
\[
\text{sum}(U) := \sum_{\beta \in U} \beta.
\]
Observe that all weights of a finite-dimensional \( \mathfrak{g} \)-module lie in \( P(\mathfrak{g}_0) \).

Take \( \lambda \in P(\mathfrak{g}_0) + \rho \). Recall that \( \rho_0 - \rho = \rho_1 = \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha \). One has
\[
\frac{KW(\lambda, S)}{Re^\rho} = \sum_{U \subset \Delta_1^+ \setminus S} \frac{J_W(e^{\lambda+\rho} \prod_{\beta \in \Delta_1^+ \setminus S} (1 + e^{-\beta}))}{R_0 e^{\rho_0}} = \sum_{U \subset \Delta_1^+ \setminus S} \frac{J_W(e^{\lambda+\rho - \text{sum}(U)})}{R_0 e^{\rho_0}}.
\]

A.5.3. Corollary. For each \( \lambda \in P(\mathfrak{g}_0) \) the term \( \frac{KW(\lambda, S)}{Re^\rho} \) lies in \( \text{Ch}(\mathfrak{g}_0) \) and
\[
\dim \left( \frac{KW(\lambda + \rho, S)}{Re^\rho} \right) = \sum_{U \subset \Delta_1^+ \setminus S} \prod_{\beta \in \Delta_0^+} \frac{(\lambda + \rho_0 - \text{sum}(U)|\alpha)}{(\rho_0|\alpha)}.
\]
Moreover,
\[
\frac{KW(\lambda + \rho, S)}{Re^\rho} = \sum n_{\lambda \mu} \text{ch } L_{\mathfrak{g}_0}(\mu)
\]
with the coefficients given by
\[
n_{\lambda \mu} = \sum_{U \subset \Delta_1^+ \setminus S} \sum_{w \in W} \text{sgn}(w) \delta_w(\mu + \rho_0), \lambda + \rho_0 - \text{sum}(U).
\]

A.5.4. Example. If \( \text{ch } L \) is given by the Kac-Wakimoto formula
\[
Re^\rho \text{ch } L = j^{-1} KW(\lambda + \rho, S),
\]
then
\[
\dim L = j^{-1} \prod_{U \subset \Delta_1^+ \setminus S} \frac{(\lambda + \rho_0 - \sum_{\beta \in U} \beta|\alpha)}{(\rho_0|\alpha)}.
\]

A.5.5. \( L(\lambda) \) as a \( \mathfrak{g}_0 \)-module. A Verma module \( M(\lambda) \) has a filtration with the factors of the form \( \{M_{\mathfrak{g}_0}(\lambda - \text{sum}(U))\}_{U \subset \Delta_1^+} \). Notice that if \( M_{\mathfrak{g}_0}(\lambda - \text{sum}(U)) \) has a finite-dimensional quotient, then this quotient is \( L_{\mathfrak{g}_0}(\lambda - \text{sum}(U)) \). Hence \([L(\lambda) : L_{\mathfrak{g}_0}(\lambda - \mu)] \neq 0 \) implies \( \mu = \text{sum}(U) \) for some \( U \subset \Delta_1^+ \). The multiplicity \( m_{\lambda, U} := [L(\lambda) : L_{\mathfrak{g}_0}(\lambda - \text{sum}(U))] \) can be computed using Corollary [A.5.3]:

\[
m_{\lambda, U} = \sum_{\mu} (-1)^{|\lambda| - |\mu|} d_{\lambda \mu}^{U} \sum_{U' \subset \Delta_1^+ \setminus S_\mu} j^{-1} \sum_{w \in W} \text{sgn}(w) \delta_w(\lambda + \rho_0 - \text{sum}(U'), \mu + \rho_1 - \text{sum}(U')).
\]

(25)
For the \textbf{osp}-case this gives
\[
m_{\lambda,U} = \sum_{\mu} (-1)^{|\lambda| - |\mu|} d_{\mu}^{\lambda} \sum_{U' \subset \Delta_1^+ \setminus S_\mu} j_{\mu}^{-1} \sum_{w \in W} \text{sgn}(w) \delta_w(\lambda + \rho_0 - \text{sum}(U), \mu + \rho_0 - \text{sum}(U')).
\]

A.5.6. Remark. A variation of the above reasoning allows to find the \textit{graded multiplicities}
\[
[L(\nu)_0 : L_{\emptyset_0}(\mu)] + \xi [L(\nu)_1 : L_{\emptyset_0}(\mu)]
\]
using the Gruson-Serganova character formula. In order to do this we define the graded version of KW(\lambda, S) by the following procedure.

Let \(\xi\) be a formal (even) variable satisfying \(\xi^2 = 1\). We denote by \(\text{Ch}_\xi(\mathfrak{g})\) the ring of \(\xi\)-characters of the finite-dimensional \(\mathfrak{g}\)-modules and view \(\text{Ch}_\xi(\mathfrak{g})\) as a subring of \(\mathcal{R}[\xi]\). For \(\nu \in \mathbb{Z}\Delta\) consider the map \(\Xi : e^\nu \mapsto e^{\mathcal{P}(\nu)} e^\nu\) and extend this map to the rational functions \(\frac{P}{Q}\), where \(P, Q\) are polynomials in \(e^\nu\) with \(\nu \in \mathbb{Z}\Delta\). This allows to define for \(\lambda \in \mathfrak{h}_{\text{int}}^*\) the term \(\text{KW}_\xi(\lambda, S)\) by the formula
\[
\text{KW}_\xi(\lambda, S) := e^\lambda \Xi (e^{-\lambda} \text{KW}(\lambda, S)).
\]

Note that \(\text{KW}_\xi(\lambda, S)\) and \(\Xi(R) e^\rho\) lie in the ring \(\mathcal{R}[\xi]\) and can be viewed as elements of \(\mathcal{R}_{\xi}[\xi]\). Taking \(\lambda \in P(\mathfrak{g}_0)\) we have
\[
\frac{\text{KW}_\xi(\lambda + \rho, S)}{\Xi(R) e^\rho} = \frac{J_W (e^{\lambda + \rho_0} \prod_{\beta \in \Delta_1^+ \setminus S} (1 + \xi e^{-\beta}) )}{R_0 e^{\rho_0}} \in \text{Ch}_\xi(\mathfrak{g}_0).
\]

The graded multiplicity \([\text{KW}(\lambda + \rho, S)]_{\Xi(R) e^\rho} : \text{ch} L_{\emptyset_0}(\mu)\] is given by
\[
\sum_{U \subset \Delta_1^+ \setminus S} \xi^{#U} \sum_{w \in W} \text{sgn}(w) \delta_w(\mu + \rho_0, \lambda + \rho_0 - \text{sum}(U)).
\]

The Gruson-Serganova formula \((3)\) gives the following formula for \(\text{ch}_L\):
\[
\Xi(R) e^\rho \text{ch}_L = \frac{\prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha})}{\prod_{\alpha \in \Delta_1^+} (1 + \xi e^{-\alpha})} e^\rho \text{ch}_L L = \sum_{L' \in \text{Irr} \text{KW}(L')} \pm_{L, L'} \text{KW}(L'),
\]
(where \(L = L(\lambda), L' = L(\nu)\) and the sign \(\pm\) is given by \((-1)^{p(\lambda' - \nu')})\); combining the above formulas one obtains an analogue of \((25)\) for the graded multiplicity of \(L_{\emptyset_0}(\mu)\) in \(L(\lambda)\).

A.6. The map \(\text{pr}\). Let \(\mathfrak{g}\) be \(\mathfrak{gl}(m|n)\) or \(\mathfrak{osp}(M|2n)\). Fix an odd root \(\beta_0\) of the form \(\beta_0 = \pm (\varepsilon_p - \delta_q)\).

Let \(e^\mu, \mu \in \mathfrak{h}^*\) be a basis of the group algebra \(\mathbb{C}[\mathfrak{h}^*]\). Consider a projection \(\text{pr} : \mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}[\mathfrak{h}^*]\) given by
\[
\text{pr}(e^{a\alpha_p}) := 1, \quad \text{pr}(e^{a\delta_q}) := e^{i\pi a}, \quad \text{pr}(e^{a\varepsilon_i}) := e^{a\varepsilon_i}, \quad \text{pr}(e^{a\delta_j}) := e^{a\delta_j}
\]
for any \( a \in \mathbb{C} \) and the indeces \( t \neq p, j \neq q \). Note that \( \text{pr} \) is an algebra homomorphism and \( \text{pr}(e^{\beta_0}) = -1 \). We extend \( \text{pr} \) to the rational functions of the form \( \frac{P}{Q} \), where \( P, Q \in \mathbb{C}[\mathfrak{h}^*] \) is such that \( \text{pr}(Q) \neq 0 \).

Since \( \text{pr} \) is an algebra homomorphism for each \( \lambda \in \mathfrak{h}^* \) one has

\[
(27) \quad \text{pr}(\text{KW}(\lambda, \emptyset)(1 + e^{-\beta_0})) = 0.
\]

A.6.1. Take a non-zero vector \( x \in \mathfrak{g}_{\beta_0} \). Identify \( \mathfrak{g}' := \text{DS}_x(\mathfrak{g}) \) with the subalgebra of \( \mathfrak{g} \) (recall that \( \mathfrak{g}' = \mathfrak{gl}(m-1|n-1) \) for \( \mathfrak{g} = \mathfrak{gl}(m|n) \), \( \mathfrak{g}' = \mathfrak{osp}(M-2|2n-2) \) for \( \mathfrak{g} = \mathfrak{osp}(M|2n) \)).

Recall that \( \mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h} \) is a Cartan subalgebra of \( \mathfrak{g}' \).

Observe that \( \text{pr}(e^{\lambda}) = c_\lambda e^{\lambda|_{\mathfrak{h}'}} \) for \( c_\lambda := e^{-\pi i (\lambda|_{\delta_q})} \) and that the restriction of \( \pi \text{pr} \pi \) to the supercharacter ring \( J(\mathfrak{g}) \) is equal to \( ds_e \) [CHR1].

A.6.2. Set

\[
\xi := \begin{cases} \frac{1}{2} (\sum_{i=1}^m \varepsilon_i - \sum_{i=1}^n \delta_i) & \text{for } \mathfrak{osp}(2m+1|2n), \\ 0 & \text{otherwise} \end{cases}
\]

and that the restriction of \( \pi \text{pr} \pi \) to the supercharacter ring \( J(\mathfrak{g}) \) is equal to \( ds_e \) [CHR1].

A.6.3. Proposition. Let \( S, \lambda \) be as in (28).

For \( \mathfrak{g} = \mathfrak{osp}(2m|2n) \) with \( m > 1 \) and \( |S| = 1 \) one has

\[
\text{pr}(\text{KW}(\lambda, S)(1 + e^{-\beta_0})) = \text{KW}(\lambda', \emptyset) + \text{KW}((\lambda')^s, \emptyset).
\]

For other cases

\[
\text{pr}(\text{KW}(\lambda, S)(1 + e^{-\beta_0})) = ae^{-\pi i (\lambda|_{\delta_q})} \text{KW}(\lambda', S'),
\]

where \( a = |S| \) for \( \mathfrak{g} = \mathfrak{osp}(2|2n), \mathfrak{gl}(m|n) \) and \( a = 2|S| \) for \( \mathfrak{osp}(2m+1|2n) \) and for \( \mathfrak{osp}(2m|2n) \) with \( m, |S| > 1 \).

Proof. Denote by \( W' \) the Weyl group of \( \mathfrak{g}' \) and notice that \( W' = \text{Stab}_W \beta_0 \). Set

\[
s := |S|, \quad c := e^{-\pi i (\lambda|_{\delta_q})}.
\]
One has
\[ pr(KW(\lambda, S)(1 + e^{-\beta_0})) = \sum_{w \in W} sgn(w)y(w), \]
where \[ y(w) := pr(\frac{e^\lambda - 1 + e^{-\beta_0}}{(1 + e^{-\beta_0})}). \]

Observe that \( pr(1 + e^\alpha) = 0 \) for \( \alpha \in \Delta \) is equivalent to \( \alpha = \pm \beta_0 \). Since \( pr \) is an algebra homomorphism, this gives \( y(w) = 0 \) if \( \pm \beta_0 \not\in wS \). Therefore
\[ pr(KW(\lambda, S)(1 + e^{-\beta_0})) = Y_+ + Y_- , \]
where
\[ Y_\pm := \sum_{w \in W; \pm \beta_0 \in wS} sgn(w)y(w). \]

Each \( \beta \in S' \) can be written as \( \beta_0 = w_\beta \beta \) for \( w_\beta := r_{\varepsilon_i - \varepsilon_j}r_{\delta_j - \delta_i} \). Setting \( w_{\beta_0} := Id \) we have \( sgn(w_{\beta_0}) = 1 \) and
\[ w_\beta \lambda = \lambda, \quad w_\beta \beta_0 = \beta, \quad w_\beta \beta = \beta_0, \quad w_\beta(\beta') = \beta' \quad \text{for} \quad \beta' \in S \setminus \{\beta, \beta_0\}. \]
for each \( \beta \in S \). The operator \( pr \) commutes with the action of \( w' \) for \( w' \in W' \). This gives
\[ y(w'w_\beta) = \frac{pr(e^{w'w_\beta})}{\prod_{\beta' \in S'}(1 + e^{-w'\beta'})} = w'(\frac{pr(e^\lambda)}{\prod_{\beta' \in S'}(1 + e^{-\beta'})}) \quad \text{for any} \quad w' \in W'. \]
Since \( W' = \text{Stab}_W \beta_0 \) one has \( \{w \in W; \beta_0 \in wS\} = W'w_{\beta_0} \).

By [A.6.2] \( pr(e^\lambda) = ce^\lambda \). Summarizing we obtain
\[ Y_+ = \sum_{\beta \in S} \sum_{w' \in W'} sgn(w')y(w'w_\beta) = csJ_{w'}\left(\frac{e^\lambda}{\prod_{\beta' \in S'}(1 + e^{-\beta'})}\right), \]
that is \( Y_+ = csKW(\lambda'; S') \).

For \( g = gl(m|n), \mathfrak{osp}(2|2n) \) the set \( WS \) does not contain \( -\beta_0 \) so \( Y_- = 0 \); this completes the proof for these cases.

For the remaining cases \( g = \mathfrak{osp}(M|N) \) with \( M > 2 \) the set \( WS \) contains \( -\beta_0 \). For \( g = \mathfrak{osp}(2m|2n) \) with \( s > 1 \) we fix \( \beta_1 := \pm(\varepsilon_i - \delta_j) \in S' \); for \( \mathfrak{osp}(2m|2n) \) with \( m > 1 \) and \( s = 1 \) we set \( i := m \) if \( p \neq m \) and \( i := m - 1 \) if \( p = m \). We set
\[ w_- := \begin{cases} r_{\varepsilon_p}r_{\delta_q} & \text{for} \quad g = \mathfrak{osp}(2m + 1|2n) \\ r_{\varepsilon_p}r_{\delta_q} r_{\varepsilon_p} r_{\delta_q} & \text{for} \quad g = \mathfrak{osp}(2m|2n), \quad s > 1 \\ r_{\varepsilon_p}r_{\delta_q} & \text{for} \quad g = \mathfrak{osp}(2m|2n), \quad s = 1. \end{cases} \]
Notice that \( w_- \in W \) and \( w_- \beta_0 = -\beta_0 \). Therefore
\[ \{w \in W; \beta_0 \in wS\} = \prod_{\beta \in S} W'w_-w_\beta \]
and thus
\[ Y_- = \sum_{\beta \in S} \sum_{w' \in W'} \text{sgn}(w'_w_-) y(w'_w_-w_\beta) \]

For \( w' \in W' \) we have
\[ y(w'_w_-w_\beta) = \text{pr} \left( \frac{e^{w'_w_-\lambda}(1 + e^{-w'_w_-\beta_0})}{\prod_{\beta \in S} (1 + e^{-w'_w_-\beta})} \right) = -w' \text{pr} \left( \frac{e^{w_-\lambda}}{\prod_{\beta \in S'} (1 + e^{-w_-\beta})} \right). \]

Therefore
\[ Y_- = -s \text{sgn}(w_-) J_{W'} \left( \text{pr} \left( \frac{e^{w_-\lambda}}{\prod_{\beta \in S'} (1 + e^{-w_-\beta})} \right) \right). \]

For \( \text{osp}(2m + 1|2n) \) one has \( w_-S' = S' \) and \( w_-\lambda = \lambda + \beta_0 \), that is \( \text{pr}(e^{w_-\lambda}) = -ce^{\lambda} \).

Therefore \( Y_- = cs \text{KW}(\lambda', S') \) as required.

For \( \text{osp}(2m|2n) \) with \( s > 1 \) one has
\[ w_-\lambda = \lambda, \quad w_-S' = (S' \cup \{-\beta_1\}) \setminus \{\beta_1\}. \]

Using Lemma A.4.4 we get
\[ Y_- = -s \text{KW}(\lambda', S' \cup \{-\beta_1\}) \setminus \{\beta_1\} = s \text{KW}(\lambda', S'). \]

For the remaining case \( \text{osp}(2m|2n) \) with \( m > 1 \) and \( S = \{\beta_0\} \) we have
\[ \text{sgn}(w_-) = -1, \quad \text{pr}(e^{w_-\lambda}) = e^{r_\epsilon \lambda'}, \]
that is \( Y_- = \text{KW}(r_\epsilon, \lambda', \emptyset) \). Since \( r_\epsilon, \lambda' = (\lambda')^s \) this completes the proof. \( \square \)

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