Reconstruction of the electric field of the Helmholtz equation in three dimensions

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Abstract

In this paper, the paper is devoted to study the Cauchy problem for the inhomogeneous Helmholtz equation that arises naturally in many physical applications related to the radiation field, vibration phenomena. Under specific assumptions, we prove the ill-posedness of the problem in a particular situation. Applying the truncation regularization method, we establish the regularized solution and show that it converges to the exact solution as the perturbation in given data tends to zero. Besides, the error estimate and regularization parameter are formulated. Our theoretical result is guaranteed by a numerical example.

Keywords and phrases: Helmholtz equation; Ill-posed problem; Convergence estimates.

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1 Introduction

The Maxwells equations with the following vectors of electric induction \( D(r,t) \) and of magnetic induction \( B(r,t) \) formed

\[
D(r,t) = e^{i\omega t} D(r), \quad B(r,t) = e^{i\omega t} B(r),
\]

where \( \omega \) is a constant frequency, lead to Helmholtz equations

\[
\Delta D(r) + k^2 D(r) = 0, \quad \Delta B(r) + k^2 B(r) = 0, \quad r \in \Omega \subset \mathbb{R}^3.
\]

In the case when boundary conditions for the fields \( D(r) \) and \( B(r) \) are linear, they formulated boundary value problems for each component of the electromagnetic field separately which connected with lasers beam. The problem is to determine a radiation field in optoelectronics is a difficult problem, because, as a rule, experimental determination of the whole radiation field is not possible. Practically, we are able to measure the field only on certain subsets of physical space (e.g. on some surfaces) surrounding the solid. More details in applications, the reader is referred to Reginska et al [1], page 977.

With assumption that the electromagnetic field vanishes on the side faces of the cuboid, we have a problem of finding approximate radiation field \( u \) satisfying the Helmholtz equation

\[
\Delta u + k^2 u = 0, \quad \text{in} \quad \Omega \subset \mathbb{R}^3.
\]
Here, the boundary condition is \( u(r) = 0 \) on side faces of \( \Omega \) and \( \Omega = (0, a) \times (0, b) \times (0, c) \). The related inverse problem consists in reconstruction of a radiation field \( u \), which is a solution of (3) from values of \( u \) and its normal derivative on the boundary.

In the present paper, we are interested in considering a more general physical situation similar to that presented above, but the mathematical model is slightly modified. The electromagnetic field is a solution of the following inhomogeneous Helmholtz equation

\[
\Delta u + k^2 u = f(r), \quad \text{in} \quad \Omega \subset \mathbb{R}^3,
\]

with the following conditions on a part of boundary

\[
u(r) = g(r), \quad \partial_z u(r) = h(r), \quad r \in \Gamma,
\]

for given functions \( f, g, h \). Here, we put \( \Omega = \mathbb{R}^2 \times (0, d) \subset \mathbb{R}^3, d > 0 \) and \( \Gamma_0 \) and \( \Gamma \) denote two parallel surfaces in the space \( \mathbb{R}^3 \)

\[
\Gamma_0 = \{ r \in \mathbb{R}^3 : z = 0 \}, \quad \Gamma_d = \{ r \in \mathbb{R}^3 : z = d \}.
\]

The reference Helmholtz problem in homogeneous case and boundary condition \( u(r) = g(r) \), and the Sommerfeld radiation condition

\[
\lim_{r \to \infty} r \left( \frac{\partial u}{\partial r} - iku \right) = 0,
\]

have been formulated in [1] only for comparison with Problem (3).

The aim of this paper is to present an approximation method for solving in a stable way the Cauchy problem for the Helmholtz equation with approximately given inexact data. Cauchy problems for elliptic equations are ill-posed [9], i.e. the solution does not depend continuously on the boundary data. Stability aspects of Cauchy problems were discussed for instance in [2, 7, 12, 14]. For the Helmholtz equation, an influence of the frequency on the stability of Cauchy problems was described in [8]. Moreover, other ill-posed problems for the Helmholtz equation were extensively studied in the literature, among others: the inverse problem of determining the shape of a part of a boundary [4] and the inverse problem of determination of sources [5, 11].

In this paper, we discuss the nature of the ill-posedness of the problem (4)-(5) and propose an approximation method based on regularization in the frequency space for solving the problem in stable way when the given functions \( f, g \) and \( h \) are perturbed. In particular, we point out the case that implies the ill-posedness under Fourier transform, superposition principle and Parseval’s identity. Then, a regularization method is used for solving this case only. A similar technique was used in [6] for a sideways heat equation in the case of one-dimensional space. We also study the error estimate between the approximate solution and the exact solution and show the regularization parameter with the order of convergence for our proposed method.

The paper is organized as follows. In Section 2, we consider the model problem and discuss its ill-posedness. The regularization method based on truncated Fourier transform is analysed in Section 3 where its convergence and stability are proved under a suitable choice of regularization parameter. Then, one numerical implementation is discussed in Section 4 to illustrate the proposed method.
2 Mathematical problem of 3-D Helmholtz equation and ill-posedness

Before presenting the main result, we introduce some details used throughout the whole sections.

In this paper, for any nonempty open set \( \Omega \subset \mathbb{R}^2 \), the space \( L^2(\Omega) \) is the usual space of square-measurable integrable functions on \( \Omega \), the norm of which is defined by

\[
\|u\|_{L^2(\Omega)} = \left( \int_{\Omega} |u(x)|^2 \, dx \right)^{1/2}.
\]

We also call \( H^2(\Omega) \) the space of functions and their derivatives up to order 2, in the distributional sense, belonging to \( L^2(\Omega) \).

Because of three-dimensional, we can have a new expression of \( \Gamma_0 \) and \( \Gamma_d \) as in (6). They have one variable in common by denoting \( \xi = (x, y) \in \mathbb{R}^2 \).

\[
\Gamma_0 = \{ (\xi, 0) : \xi \in \mathbb{R}^2 \}, \quad \Gamma_d = \{ (\xi, d) : \xi \in \mathbb{R}^2 \}.
\]

By the definition of Hadamard, the problem is well-posed if it has a solution for all admissible data, furthermore, this solution is unique and depends continuously on the given data. In this section, we are going to show ill-posedness of the problem, i.e. the solution does not depend continuously on the boundary data and small errors in the data can destroy the numerical solution. Therefore, a regularization method should be collected and applied to solve ill-posed problems.

Let us consider the model problem of reconstruction of the radiation field in the domain \( \Omega \) as introduced in the previous section, which is more generalized and considered as the inhomogeneous problem associated with the specific conditions and the given corresponding functions. The problem we consider can be written as follows.

\[
\begin{aligned}
\Delta u + k^2 u &= -f, \quad \text{in } \Omega, \\
u(\xi, d) &= g(\xi), \\
\partial_z u(\xi, d) &= h(\xi), \\
u(., z) &\in L^2(\mathbb{R}^2), \quad z \in [0, d],
\end{aligned}
\]

where \( g, h \in L^2(\mathbb{R}^2) \) are given data and \( f \in L^2(\Omega) \) plays a role as the given source term. In practice, we will look for an approximation solution inside domain \( \Omega \) in the case when the given data have their approximation. So, let us make the following assumption:

(A1) Let \( g_\delta, h_\delta \in L^2(\mathbb{R}^2) \) and \( f_\delta \in L^2(\Omega) \) play roles as the general measurement data with noise level \( \delta > 0 \) such that

\[
\|f_\delta - f\|_{L^2(\Omega)} \leq \delta, \quad \|g_\delta - g\|_{L^2(\mathbb{R}^2)} \leq \delta, \quad \|h_\delta - h\|_{L^2(\mathbb{R}^2)} \leq \delta.
\]

Moreover, we make another assumption on the exact data

(A2) \( g(\xi) f(\xi, z) \leq 0 \) for all \( (\xi, z) \in \Omega \).

Suppose that there exists uniquely a solution \( u \) in \( H^2(\Omega) \) of the problem [10] for the exact data. Then \( u \) is defined by sum of two functions \( u_1 \) and \( u_2 \) where \( u_1 \in H^2(\Omega) \) satisfies
\[
\begin{aligned}
&\Delta u_1 + k^2 u_1 = 0, \quad \text{in } \Omega, \\
u_1(\xi, 0) = 0, \\
\partial_z u_1(\xi, d) = h(\xi), \\
u_1(\cdot, z) \in L^2(\mathbb{R}^2), \quad z \in [0, d],
\end{aligned}
\]  
\tag{12}

and \( u_2 \in H^2(\Omega) \) satisfies

\[
\begin{aligned}
&\Delta u_2 + k^2 u_2 = -f, \quad \text{in } \Omega, \\
u_2(\xi, d) = g(\xi) - u_1(\xi, d), \\
\partial_z u_2(\xi, d) = 0, \\
u_2(\cdot, z) \in L^2(\mathbb{R}^2), \quad z \in [0, d].
\end{aligned}
\]  
\tag{13}

The following lemma concerning the solution \( u_1 \) of (12) is proved in \([1]\). It shows that the solution \( u_1 \) continuously depends on the given data \( h \), i.e. the problem (12) is well-posed. For short, we present the proof and give the summation for full details.

**Lemma 1.** If \( u_1 \) is the solution of the problem (12) and \( k < \frac{\pi}{2d} \) holds, then there exists \( C > 0 \) such that

\[
\|u_1\|_{L^2(\Omega)} \leq C \|h\|_{L^2(\mathbb{R}^2)}.  
\]  
\tag{14}

**Proof.** Since \( u_1(\cdot, z) \in H^2(\mathbb{R}^2) \) for each \( z \in [0, d] \), we apply the Fourier transform on two-dimensional case with respect to variable \( \xi \in \mathbb{R}^2 \) as follows.

\[
\hat{u}_1(\rho, z) = \int_{\mathbb{R}^2} u_1(\xi, z) e^{-2\pi i \langle \rho, \xi \rangle} d\xi,  
\]  
\tag{15}

where \( \rho = (\rho_1, \rho_2) \in \mathbb{R}^2 \) and \( (\rho, \xi) = \rho_1 \xi_1 + \rho_2 \xi_2 \).

Therefore, in stead of study \( u_1 \), we will investigate its Fourier transform, \( \hat{u}_1 \). At this time, we totally construct the problem (12) in terms of Fourier transform. This means

\[
\begin{aligned}
&(\hat{u}_1)_{zz}(\rho, z) = \left( |\rho|^2 - k^2 \right) \hat{u}_1(\rho, z), \quad (\rho, k) \in \mathbb{R}^2, z \in (0, d), \\
\hat{u}_1(\rho, 0) = 0, \quad \rho \in \mathbb{R}^2, \\
\partial_z \hat{u}_1(\rho, d) = \hat{h}(\rho), \quad \rho \in \mathbb{R}^2.
\end{aligned}
\]  
\tag{16}

For simplification, we put

\[
\lambda_{\rho, k} = |\rho|^2 - k^2,
\]

and set

\[
A_1 = \{ (\rho, k) : \lambda_{\rho, k} > 0 \}, \quad A_2 = \{ (\rho, k) : \lambda_{\rho, k} = 0 \},
\]

\[
A_3 = \{ (\rho, k) : \lambda_{\rho, k} < 0 \}, \quad A = A_1 \cup A_2 \cup A_3.
\]  
\tag{17}

By an elementary calculation, the solution of the problem (16) can be found as follows.
the problem (13). For simplicity of notation, we write the solution
study the problem (10) with

We consider the case

Proof.

Applying the Fourier transform again, we have to consider the following problem instead of
In the sequel, the problem (10) can be reduced to the problem (13). It means that we shall

It is inclined to use the second condition of the problem (21), then we take its derivative as

Hence, we obtain the desired result with $C = \sqrt{d} \max \left\{ d, \frac{\tan (dk)}{k} \right\}$.

In the sequel, the problem (10) can be reduced to the problem (13). It means that we shall study the problem (10) with $h (\xi) = 0$.

Applying the Fourier transform again, we have to consider the following problem instead of the problem (13). For simplicity of notation, we write the solution $u$ instead of $w_2$.

Based on (17)-(18), we first consider the case $(\rho, k) \in A_1 \cup A_3$. In this case, the solution of (20) with respect to $z$ is found by using superposition, namely $u = w_1 + w_2$ where $w_1$ the complementary solution satisfies

and $w_2$ the particular solution satisfies

These solutions can be found in the following lemmas.

Lemma 2. The solution $w_1$ to (21) has the form

Proof. We consider the case $(\rho, k) \in A_1$. Solving the ordinary differential equation (21) in this case, we have

It is inclined to use the second condition of the problem (21), then we take its derivative as follows.

$$
\frac{\partial w_1}{\partial z} (\rho, z) = \sqrt{\lambda_{\rho, k}} \left[ C_1 (\rho) e^{z \sqrt{\lambda_{\rho, k}}} - C_2 (\rho) e^{-z \sqrt{\lambda_{\rho, k}}} \right].
$$
Because of $\partial_z w_1 (\rho, d) = 0$, it is easy to figure out $C_2 (\rho) = C_1 (\rho) e^{2d\sqrt{\lambda_{\rho,k}}}$. Thus, we obtain

$$w_1 (\rho, z) = C_1 (\rho) \left[ e^{z\sqrt{\lambda_{\rho,k}}} + e^{2(z-d)\sqrt{\lambda_{\rho,k}}} \right]. \quad (26)$$

Considering the remainder condition $w_1 (\rho, d) = \hat{g} (\rho)$, we shall get the first term $C_1 (\rho) = \frac{1}{2} \hat{g} (\rho) e^{-d\sqrt{\lambda_{\rho,k}}}$, and from $w_1 (\rho, z) = \frac{1}{2} \hat{g} (\rho) \left[ e^{(z-d)\sqrt{\lambda_{\rho,k}}} + e^{(d-z)\sqrt{\lambda_{\rho,k}}} \right]$, this follows that

$$w_1 (\rho, z) = \frac{1}{2} \hat{g} (\rho) \cos \left( (d - z) \sqrt{\lambda_{\rho,k}} \right). \quad (27)$$

Similarly, for the case $(\rho, k) \in A_3$, we obtain

$$w_1 (\rho, z) = \hat{g} (\rho) \cos \left( (d - z) \sqrt{-\lambda_{\rho,k}} \right).$$

Lemma 3. The solution $w_2$ to (24) has the form

$$w_2 (\rho, z) = \begin{cases} (\sqrt{\lambda_{\rho,k}})^{-1} \int_{\rho}^z \hat{f} (\rho, s) \sin \left( (z - s) \sqrt{\lambda_{\rho,k}} \right) \, ds, & (\rho, k) \in A_1, \\ (\sqrt{-\lambda_{\rho,k}})^{-1} \int_{\rho}^z \hat{f} (\rho, s) \sin \left( (z - s) \sqrt{-\lambda_{\rho,k}} \right) \, ds, & (\rho, k) \in A_3. \end{cases} \quad (28)$$

Proof. The particular solution $w_2$ in the case $(\rho, k) \in A_1$ is

$$w_2 (\rho, z) = F (\rho, z) e^{z\sqrt{\lambda_{\rho,k}}} + G (\rho, z) e^{-z\sqrt{\lambda_{\rho,k}}}.$$ \quad (29)

In order to define $F$ and $G$, we must solve the following system where $F$ and $G$ are its solutions

\[
\begin{aligned}
\frac{\partial F}{\partial z} (\rho, z) e^{z\sqrt{\lambda_{\rho,k}}} + \frac{\partial G}{\partial z} (\rho, z) e^{-z\sqrt{\lambda_{\rho,k}}} &= 0, \\
\frac{\partial F}{\partial z} (\rho, z) e^{z\sqrt{\lambda_{\rho,k}}} - \frac{\partial G}{\partial z} (\rho, z) e^{-z\sqrt{\lambda_{\rho,k}}} &= - (\sqrt{\lambda_{\rho,k}})^{-1} \hat{f} (\rho, z). \quad (30)
\end{aligned}
\]

Thus, by directly using primary computation, the system is equivalent to

\[
\begin{aligned}
\frac{\partial F}{\partial z} (\rho, z) &= - \frac{1}{2} (\sqrt{\lambda_{\rho,k}})^{-1} \hat{f} (\rho, z) e^{-z\sqrt{\lambda_{\rho,k}}}, \\
\frac{\partial G}{\partial z} (\rho, z) &= \frac{1}{2} (\sqrt{\lambda_{\rho,k}})^{-1} \hat{f} (\rho, z) e^{z\sqrt{\lambda_{\rho,k}}}. \quad (31)
\end{aligned}
\]

Then, we obtain both $F$ and $G$.

\[
\begin{aligned}
F (\rho, z) &= - \frac{1}{2} (\sqrt{\lambda_{\rho,k}})^{-1} \int_{\rho}^{z} \hat{f} (\rho, s) e^{-s\sqrt{\lambda_{\rho,k}}} \, ds + C_1 (\rho), \quad (32) \\
G (\rho, z) &= \frac{1}{2} (\sqrt{\lambda_{\rho,k}})^{-1} \int_{\rho}^{z} \hat{f} (\rho, s) e^{s\sqrt{\lambda_{\rho,k}}} \, ds + C_2 (\rho). \quad (33)
\end{aligned}
\]

From (32) and (33), we restate the finding solution and point out its derivative on the boundary $\Gamma_d$ with respect to variable $z$. 

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\[ w_2(\rho, z) = F(\rho, z) e^{z \sqrt{\lambda_{\rho,k}}} + G(\rho, z) e^{-z \sqrt{\lambda_{\rho,k}}}, \]  

(34)

\[ \frac{\partial w_2}{\partial z}(\rho, z) = \left[ \frac{\partial F}{\partial z}(\rho, z) + \sqrt{\lambda_{\rho,k}} F(\rho, z) \right] e^{z \sqrt{\lambda_{\rho,k}}} + \left[ \frac{\partial G}{\partial z}(\rho, z) - \sqrt{\lambda_{\rho,k}} G(\rho, z) \right] e^{-z \sqrt{\lambda_{\rho,k}}}. \]  

(35)

We now look back two specific conditions of the problem \((22)\) in order to calculate exactly two terms \(C_1(\rho)\) and \(C_2(\rho)\). Due to \(w_2(\rho, d) = 0\) and \(\partial_z w_2(\rho, d) = 0\), we will have the following system.

\[ \begin{cases} 
F(\rho, d) e^{d \sqrt{\lambda_{\rho,k}}} + G(\rho, d) e^{-d \sqrt{\lambda_{\rho,k}}} = 0 \\
F(\rho, d) e^{d \sqrt{\lambda_{\rho,k}}} - G(\rho, d) e^{-d \sqrt{\lambda_{\rho,k}}} = 0 
\end{cases} \]  

(36)

At this point, \(C_1(\rho)\) and \(C_2(\rho)\) will be determined directly.

\[ C_1(\rho) = \frac{1}{2} \left( \sqrt{\lambda_{\rho,k}} \right)^{-1} \int_0^d \hat{f}(\rho, s) e^{-s \sqrt{\lambda_{\rho,k}}} ds, \]  

(37)

\[ C_2(\rho) = -\frac{1}{2} \left( \sqrt{\lambda_{\rho,k}} \right)^{-1} \int_0^d \hat{f}(\rho, s) e^{s \sqrt{\lambda_{\rho,k}}} ds. \]  

(38)

Finally, from \((32), (33), (34), (37)\) and \((38)\), the solution \(w_2\) is formed below by a simple calculation.

\[ w_2(\rho, z) = \frac{1}{2} \left( \sqrt{\lambda_{\rho,k}} \right)^{-1} \int_0^d \hat{f}(\rho, s) e^{(z-s) \sqrt{\lambda_{\rho,k}}} ds \]
\[ + \frac{1}{2} \left( \sqrt{\lambda_{\rho,k}} \right)^{-1} \int_d^z \hat{f}(\rho, s) e^{(s-z) \sqrt{\lambda_{\rho,k}}} ds \]
\[ = \left( \sqrt{\lambda_{\rho,k}} \right)^{-1} \int_0^d \hat{f}(\rho, s) \sinh \left( (z-s) \sqrt{\lambda_{\rho,k}} \right) ds. \]  

(39)

Similarly, for the case \((\rho, k) \in A_3\), we obtain

\[ w_2(\rho, z) = \left( \sqrt{-\lambda_{\rho,k}} \right)^{-1} \int_0^d \hat{f}(\rho, s) \sin \left( (z-s) \sqrt{-\lambda_{\rho,k}} \right) ds. \]

(40)

From \((23)\) and \((28)\), the solution of the problem \((20)\) in the case \((\rho, k) \in A_1 \cup A_3\) is formulated. Then, by proving the following lemma, we obtain the solution of \((20)\) in all cases, \(A = A_1 \cup A_2 \cup A_3\).

**Lemma 4.** Let \(w\) be the solution of the following problem

\[ \begin{cases} 
\hat{u}_{zz}(\rho, z) = -\hat{f}(\rho, z), & \rho \in \mathbb{R}^2, z \in (0, d), \\
\hat{u}(\rho, d) = \hat{g}(\rho), & \rho \in \mathbb{R}^2, \\
\partial_z \hat{u}(\rho, d) = 0, & \rho \in \mathbb{R}^2. 
\end{cases} \]  

(40)
Then,
\[ w(\rho, z) = \hat{g}(\rho) + \int_{s}^{d} \int_{s}^{d} \hat{f}(\rho, \gamma) \, d\gamma ds. \]  

(41)

**Proof.** By using Newton–Leibniz formula twice, we will have
\[ \partial_{z} \hat{u}(\rho, z) = \partial_{z} \hat{u}(\rho, d) + \int_{s}^{d} \hat{f}(\rho, s) \, ds = \int_{s}^{d} \hat{f}(\rho, s) \, ds. \]  

(42)

Therefore, the solution is in the form of
\[ \hat{u}(\rho, z) = \hat{g}(\rho) + \int_{s}^{d} \int_{s}^{d} \hat{f}(\rho, \gamma) \, d\gamma ds. \]  

(43)

**Remark 5.** Combining (23), (28) and (44), the solution of the problem (20) is shown.
\[
\hat{u}(\rho, z) = \begin{cases} 
\hat{g}(\rho) \cosh \left( (d - z) \sqrt{\lambda_{\rho,k}} \right) \\
+ \frac{1}{\sqrt{\lambda_{\rho,k}}} \int_{z}^{d} \hat{f}(\rho, s) \sinh \left( (z - s) \sqrt{\lambda_{\rho,k}} \right) \, ds, & \text{if } (\rho, k) \in A_{1}, \\
\hat{g}(\rho) + \int_{s}^{d} \int_{s}^{d} \hat{f}(\rho, \gamma) \, d\gamma ds, & \text{if } (\rho, k) \in A_{2}, \\
\hat{g}(\rho) \cos \left( (d - z) \sqrt{-\lambda_{\rho,k}} \right) \\
+ \frac{1}{\sqrt{-\lambda_{\rho,k}}} \int_{z}^{d} \hat{f}(\rho, s) \sin \left( (z - s) \sqrt{-\lambda_{\rho,k}} \right) \, ds, & \text{if } (\rho, k) \in A_{3}. 
\end{cases}
\]  

(44)

From now on, we will prove that the problem (20) (also the problem (10)) is ill-posed in two lemmas below.

**Lemma 6.** If \( w \) is the solution of the problem (20) in the case \( (\rho, k) \in A_{2} \cup A_{3} \), then \( w \) is continuously dependent of the given data \( g \) and \( f \) in the sense of
\[
\| w \|_{L^{2}(\Omega)}^{2} \leq C \left( \| g \|_{L^{2}(\mathbb{R}^{2})}^{2} + \| f \|_{L^{2}(\Omega)}^{2} \right).
\]  

(45)

**Proof.** For \( (\rho, k) \in A_{2} \), it is not hard to derive the result from (41). Indeed, by using Holder’s inequality, we have
\[
|w(\rho, z)| \leq |\hat{g}(\rho)| + \int_{s}^{d} \int_{s}^{d} \left| \hat{f}(\rho, \gamma) \right| \, d\gamma ds \\
\leq |\hat{g}(\rho)| + \int_{s}^{d} (d - s) \left( \int_{s}^{d} \left| \hat{f}(\rho, \gamma) \right|^{2} \, d\gamma \right)^{1/2} \, ds \\
\leq |\hat{g}(\rho)| + d \int_{s}^{d} \left( \int_{s}^{d} \left| \hat{f}(\rho, \gamma) \right|^{2} \, d\gamma \right)^{1/2} \, ds.
\]  

(46)
Thus, by using the Cauchy-Schwarz inequality and Holder’s inequality, we deduce that

\[
\int_{\mathbb{R}^2} |w(\rho, z)|^2 \, d\rho \leq 2 \int_{\mathbb{R}^2} \left[ |\hat{g}(\rho)|^2 + d^2 \left( \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} |f(\rho, \gamma)|^2 \, d\gamma \right)^{1/2} \, ds \right)^2 \right] \, d\rho \\
\leq 2 \left[ \|g\|_{L^2(\mathbb{R}^2)}^2 + d^2 (d - z) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |f(\rho, \gamma)|^2 \, d\gamma \, ds \, d\rho \right] \\
\leq 2 \left[ \|g\|_{L^2(\mathbb{R}^2)}^2 + d^3 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |f(\rho, \gamma)|^2 \, d\gamma \, ds \, d\rho \right] \\
\leq 2 \left[ \|g\|_{L^2(\mathbb{R}^2)}^2 + d^3 (d - z) \|f\|_{L^2(\Omega)}^2 \right], \quad (47)
\]

which completes the proof.

For \((\rho, k) \in A_3\), under the assumption \(k < \frac{\pi}{2d}\), we have the estimation

\[
0 < (s - z) \sqrt{\lambda_{\rho, k}(s)} \leq (d - z) \sqrt{-\lambda_{\rho, k}} \leq dk < \frac{\pi}{2}, \quad (48)
\]

for \(z < s \leq d\). Then, there follows the following inequalities.

\[
|\sin \left( (z - s) \sqrt{-\lambda_{\rho, k}} \right)| \leq \sin \left( (s - z) \sqrt{-\lambda_{\rho, k}} \right) \leq \sin \left( (d - z) \sqrt{-\lambda_{\rho, k}} \right). \quad (49)
\]

\[
\tan \left( (d - z) \sqrt{-\lambda_{\rho, k}} \right) \leq \tan \left( d \sqrt{-\lambda_{\rho, k}} \right). \quad (50)
\]

Moreover, we notice that the function \(\frac{\tan x}{x}\) is increasing where \(x \in \left(0, \frac{\pi}{2}\right)\). From the expression \((44)\), we use \((48), (49)\) and \((50)\) to obtain

\[
|w(\rho, z)| \leq \cos \left( (d - z) \sqrt{-\lambda_{\rho, k}} \right) |\hat{g}(\rho)| + \frac{1}{\sqrt{-\lambda_{\rho, k}}} \int_{\mathbb{R}^2} |f(\rho, s)| \sin \left( (d - z) \sqrt{-\lambda_{\rho, k}} \right) \, ds \\
\leq |\hat{g}(\rho)| + \frac{\tan \left( (d - z) \sqrt{-\lambda_{\rho, k}} \right)}{\sqrt{-\lambda_{\rho, k}}} \int_{\mathbb{R}^2} |f(\rho, s)| \, ds \\
\leq |\hat{g}(\rho)| + \frac{\tan \left( d \sqrt{-\lambda_{\rho, k}} \right)}{\sqrt{-\lambda_{\rho, k}}} \int_{\mathbb{R}^2} |f(\rho, s)| \, ds \\
\leq |\hat{g}(\rho)| + \frac{\tan \left( dk \right)}{k} \int_{\mathbb{R}^2} |f(\rho, s)| \, ds \\
\leq C \left( |\hat{g}(\rho)| + \int_{\mathbb{R}^2} |f(\rho, s)| \, ds \right), \quad (51)
\]

where \(C = \max \left\{ \frac{\tan \left( dk \right)}{k}, 1 \right\}\).

Next, using Holder’s inequality, we have

\[
\int_{\mathbb{R}^2} |f(\rho, s)| \, ds \leq \sqrt{d} \left( \int_{\mathbb{R}^2} |f(\rho, s)|^2 \, ds \right)^{1/2}. \quad (52)
\]
We tend to find the boundedness of $w$ in the $L^2$ norm. We first apply the Cauchy-Schwarz inequality and (52) to the inequality (51) to get

\[
\int_{\mathbb{R}^2} |w(\rho, z)|^2 d\rho \leq C^2 \int_{\mathbb{R}^2} \left( |\hat{g}(\rho)| + \int_0^t |\hat{f}(\rho, s)| ds \right)^2 d\rho \\
\leq 2C^2 \left[ \int_{\mathbb{R}^2} |\hat{g}(\rho)|^2 d\rho + d \int_{\mathbb{R}^2} \int_0^t |\hat{f}(\rho, s)|^2 ds d\rho \right] \\
\leq 2C^2 \left( \|g\|_{L^2(\mathbb{R}^2)}^2 + d \|f\|_{L^2(\Omega)}^2 \right).
\]

(53)

Hence, we obtain

\[
\|w\|_{L^2(\Omega)}^2 \leq 2dC^2 \left( \|g\|_{L^2(\mathbb{R}^2)}^2 + d \|f\|_{L^2(\Omega)}^2 \right),
\]

(54)

which completes the proof of the lemma.

**Lemma 7.** The problem (20) in the case $(\rho, k) \in A_1$ is ill-posed.

**Proof.** We have the following estimation.

\[
\cosh \left( (d - z) \sqrt{\lambda_{\rho,k}} \right) > \frac{1}{4} e^{2(d-z)\sqrt{\lambda_{\rho,k}}} > \frac{e^{2|\rho|(d-z)}}{4e^{2k(d-z)}}.
\]

(55)

Then, from the first assumption (A2), it follows that

\[
\|u(., z)\|_{L^2(\mathbb{R}^2)}^2 = \|\hat{u}(., z)\|_{L^2(\mathbb{R}^2)}^2 \geq \int_{(\rho,k) \in A_1} |w_1(\rho, z) + w_2(\rho, z)|^2 d\rho \\
\geq \int_{(\rho,k) \in A_1} |w_1(\rho, z)|^2 d\rho \\
\geq \frac{1}{4e^{2kd}} \int_{(\rho,k) \in A_1} |\hat{g}(\rho)|^2 e^{2|\rho|(d-z)} d\rho,
\]

(56)

and

\[
\|u(., z)\|_{L^2(\mathbb{R}^2)}^2 \geq \int_{(\rho,k) \in A_1} |w_2(\rho, z)|^2 d\rho \\
\geq \int_{(\rho,k) \in A_1} \left| \int_z^d \hat{f}(\rho, s) \frac{\sinh \left( (s - z) \sqrt{\lambda_{\rho,k}} \right)}{\sqrt{\lambda_{\rho,k}}} ds \right|^2 d\rho,
\]

(57)

for any fixed $z \in (0, d)$.

Therefore, let us observe (56) and (57). When $|\rho| \to \infty$, the decay of $\hat{g}(\rho)$ and $\hat{f}(\rho, s)$ is simultaneously derived from the boundedness of $u(., z)$ in $L^2(\mathbb{R}^2)$ and the rapid escalation of exponent function and $\sinh x$ function, then this is the reason why the main problem is ill-posed in the Hadamard sense with $L^2$ norm. It is clear to realize that in general, we cannot expect $\hat{g}\delta$ and $\hat{f}\delta$ which respectively resemble the exact data $\hat{g}$ and $\hat{f}$ in the meaning of the same decay. Thus, the solution $u_\delta(., z)$ corresponding to the measurement alternative data $g\delta$ and $f\delta$ of the problem (10) with the boundary condition $u(\xi, d) = g\delta(\xi)$ does not exist. \qed
3 The truncation regularization method

In the previous section, we prove that the problem (10) with $h \equiv 0$ under some specific assumptions is ill-posed. We then use the truncation method in order to stabilize the problem. From the assumption (A2), without loss of generality, we will choose $f (\xi, z) \leq 0$ and $g (\xi) \geq 0$ in the whole section.

Applying the truncation regularization, we will construct a regularized solution of the considered problem. Based on the lemmas, assumptions and remarks we discussed in the Section 2, it is rational if for $\delta > 0$ we define stability terms $\hat{f}_\delta$ and $\hat{g}_\delta$. They only get values in a bounded set controlled and parameterized by the so-called parameter $\varepsilon > 0$ depending on $\delta$ and disappears outside. Therefore, for any fixed $\varepsilon$ we put

\[
\left[ \hat{f}_\delta (\rho, z) \quad \hat{g}_\delta (\rho) \right] = \begin{cases} 
\left[ \hat{f}_\delta (\rho, z) \quad \hat{g}_\delta (\rho) \right], & \rho \in \Theta_\varepsilon, \\
0, & \rho \notin \Theta_\varepsilon,
\end{cases}
\]  

and $\left[ \hat{f}^\varepsilon \quad \hat{g}^\varepsilon \right] := \left[ \hat{f}_0 \quad \hat{g}_0 \right]$ where the bounded set is

\[
\Theta_\varepsilon := \left\{ \rho \in \mathbb{R}^2 : |\rho|^2 \leq \frac{1}{\varepsilon} \right\}.
\]

We restate $\hat{f}_\delta (\rho, z)$, the Fourier transform of $f$ corresponding with $f_\delta$ and $\hat{g}_\delta (\rho)$, the Fourier transform of $g$ corresponding with $g_\delta$.

\[
\hat{f}_\delta (\rho, z) = \int_{\mathbb{R}^2} f_\delta (\xi, z) e^{-2\pi i (\rho, \xi)} d\xi,
\]

\[
\hat{g}_\delta (\rho) = \int_{\mathbb{R}^2} g_\delta (\xi) e^{-2\pi i (\rho, \xi)} d\xi,
\]

where $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$ and $(\rho, \xi) = \rho_1 \xi_1 + \rho_2 \xi_2$.

Then, $\hat{u}_\delta$ is given by the following formula

\[
\hat{u}_\delta (\rho, z) = \hat{g}_\delta (\rho) \cosh \left( (d - z) \sqrt{\lambda_{\rho,k}} \right) + \frac{1}{\sqrt{\lambda_{\rho,k}}} \int z \hat{f}_\delta (\rho, s) \sinh \left( (z - s) \sqrt{\lambda_{\rho,k}} \right) ds,
\]

is the solution of the problem (20) with the condition $\hat{u}_\delta (\rho, d) = \hat{g}_\delta (\rho)$. Let us consider $u_\varepsilon (\xi, z)$ as the inverse Fourier transform of $\hat{u}_\delta (\rho, z)$, as follows

\[
u_\varepsilon (\xi, z) = \int_{\mathbb{R}^2} \left( \hat{g}_\delta (\rho) \cosh \left( (d - z) \sqrt{\lambda_{\rho,k}} \right) + \frac{1}{\sqrt{\lambda_{\rho,k}}} \int z \hat{f}_\delta (\rho, s) \sinh \left( (z - s) \sqrt{\lambda_{\rho,k}} \right) ds \right) e^{2\pi i (\xi, \rho)} d\rho.
\]

Thus, the function (63) shall be considered as a regularized solution to the problem (10) with $h \equiv 0$ where $\varepsilon$ is the regularization parameter which depends on the common error bound $\delta$. This parameter shall be estimated in the main result. Besides, the regularized solution for the exact data will be denoted by $u_\varepsilon (\xi, z)$.

Similarly to (17)-(18), we put

\[
\lambda_{\rho,\varepsilon} = |\rho|^2 - \frac{1}{\varepsilon},
\]
Lemma 8. Let $u_{\varepsilon}$ be the exact solution of the problem (10) in the case of $h \equiv 0$ and let $u_{\varepsilon}$ be the function defined in (63). If the condition $\varepsilon \left( k^2 - \frac{\pi^2}{4d^2} \right) < 1$ holds, then for $z \in (0, d)$ we obtain the estimation below:

$$
\| u(., z) - u_{\varepsilon}(., z) \|_{L^2(\mathbb{R}^2)} \leq \left[ e^{-z\sqrt{\frac{1}{2} - k^2}} \left( 1 + e^{-2(d-z)\sqrt{\frac{1}{2} - k^2}} \right) + e^{-z\sqrt{\frac{1}{2} - k^2}} \right] M,
$$

where $M = \| u(., 0) \|_{L^2(\mathbb{R}^2)}$. As a consequence,

$$
\| u(., z) - u_{\varepsilon}(., z) \|_{L^2(\mathbb{R}^2)} \to 0, \quad as \varepsilon \to 0.
$$

Proof. The first goal is to force $\cosh \left( d\sqrt{\frac{\pi^2}{\lambda_{\rho,k}}} \right) \neq 0$ for all possible situations. We look for the case $(\rho, k) \in A_1 \cup A_2$, then it is easy to see the goal. Furthermore, if $(\rho, k) \in A_3$, then $\cosh \left( d\sqrt{\frac{\pi^2}{\lambda_{\rho,k}}} \right) \neq 0$ when $|\rho|^2 > k^2 - \frac{\pi^2}{4d^2}$. Thus, for $1 > \varepsilon \left( k^2 - \frac{\pi^2}{4d^2} \right)$ we have

$$
\int_{(\rho, z) \in B_1} |\hat{u}(\rho, z) - \hat{u}_{\varepsilon}(\rho, z)|^2 \rho \, d\rho = \int_{(\rho, z) \in B_1} |\hat{u}(\rho, z) - \hat{u}_{\varepsilon}(\rho, z)| |\hat{u}(\rho, z)| \rho \, d\rho
$$

$$
= \int_{(\rho, z) \in B_1} |\hat{u}(\rho, z) - \hat{u}_{\varepsilon}(\rho, z)| |\hat{g}(\rho) \cosh \left( (d - z) \sqrt{\lambda_{\rho,k}} \right) |
p
$$

$$
+ \frac{1}{\sqrt{\lambda_{\rho,k}}} \int_{0}^{d} \hat{f}(\rho, s) \sinh \left( (z - s) \sqrt{\lambda_{\rho,k}} \right) ds \rho \, d\rho. \quad (67)
$$

To obtain our purpose, we now have two estimations. More precisely, we consider the integrations,

$$
A = \int_{(\rho, z) \in B_1} |\hat{u}(\rho, z) - \hat{u}_{\varepsilon}(\rho, z)| |\hat{g}(\rho) \cosh \left( d\sqrt{\lambda_{\rho,k}} \right) \cosh \left( (d - z) \sqrt{\lambda_{\rho,k}} \right) \rho \, d\rho, \quad (68)
$$

$$
B = \int_{(\rho, z) \in B_1} |\hat{u}(\rho, z) - \hat{u}_{\varepsilon}(\rho, z)| \left( \frac{1}{\sqrt{\lambda_{\rho,k}}} \int_{0}^{d} \hat{f}(\rho, s) \sinh \left( (s - z) \sqrt{\lambda_{\rho,k}} \right) ds \right) \rho \, d\rho. \quad (69)
$$

First, by using Holder’s inequality for (68), we get

$$
A \leq A_{\varepsilon}(z, d) M \| u(., z) - \hat{u}_{\varepsilon}(., z) \|_{L^2(\mathbb{R}^2)}, \quad (70)
$$

where $A_{\varepsilon}(z, d)$ is defined and estimated, as follows.
\[ A_e (z, d) = \sup_{(\rho, \varepsilon) \in B_1} \left| \frac{\cosh \left( (d - z) \sqrt{\lambda_{\rho, k}} \right)}{\cosh (d \sqrt{\lambda_{\rho, k}})} \right| \]
\[ = \sup_{(\rho, \varepsilon) \in B_1} \frac{e^{(d-z)\sqrt{\lambda_{\rho, k}}} + e^{-(d-z)\sqrt{\lambda_{\rho, k}}}}{e^{d\sqrt{\lambda_{\rho, k}}} + e^{-d\sqrt{\lambda_{\rho, k}}}} \]
\[ = \sup_{(\rho, \varepsilon) \in B_1} e^{-z\sqrt{\lambda_{\rho, k}}} \left( 1 + e^{-2(d-z)\sqrt{\lambda_{\rho, k}}} \right) \]
\[ \leq \sup_{(\rho, \varepsilon) \in B_1} e^{-z\sqrt{\frac{1}{2} - k^2}} \left( 1 + e^{-2(d-z)\sqrt{\frac{1}{2} - k^2}} \right). \quad (71) \]

It is because that \( e^{-z\sqrt{\frac{1}{2} - k^2}} \left( 1 + e^{-2(d-z)\sqrt{\frac{1}{2} - k^2}} \right) \) is a decreasing function with respect to variable \( s \). Thus, the estimation \((70)\) and \((71)\) imply the boundedness of \( A \).

\[ A \leq Me^{-z\sqrt{\frac{1}{2} - k^2}} \left( 1 + e^{-2(d-z)\sqrt{\frac{1}{2} - k^2}} \right) \|\hat{u} (., z) - \hat{u}^\varepsilon (., z)\|_{L^2(\mathbb{R}^2)}. \quad (72) \]

Second, due to \( \frac{1}{\varepsilon} > k^2 - \frac{\pi^2}{4d^2} \), we have \( \sinh (z\sqrt{\lambda_{\rho, k}}) \neq 0 \), then the estimation of \( B \) is obtained.

\[ B \leq \int_{(\rho, \varepsilon) \in B_1} |\hat{u} (\rho, z) - \hat{u}^\varepsilon (\rho, z)| \frac{1}{\sqrt{\lambda_{\rho, k}}} \int_z^d \left| \hat{f} (\rho, s) \right| \sinh \left( s \sqrt{\lambda_{\rho, k}} \right) \sinh \left( (s - z) \sqrt{\lambda_{\rho, k}} \right) dsd\rho \]
\[ \leq B_e (z, s) M \left\| \hat{u} (., z) - \hat{u}^\varepsilon (., z) \right\|_{L^2(\mathbb{R}^2)}, \quad (73) \]

where \( B_e (z, s), z \leq s \leq d \) is defined and estimated, as follows.

\[ B_e (z, d) = \sup_{(\rho, \varepsilon) \in B_1} \left| \frac{\sinh \left( (s - z) \sqrt{\lambda_{\rho, k}} \right)}{\sinh \left( s \sqrt{\lambda_{\rho, k}} \right)} \right| \]
\[ = \sup_{(\rho, \varepsilon) \in B_1} \frac{e^{(s-z)\sqrt{\lambda_{\rho, k}}} - e^{-(s-z)\sqrt{\lambda_{\rho, k}}}}{e^{s\sqrt{\lambda_{\rho, k}}} - e^{-s\sqrt{\lambda_{\rho, k}}}} \]
\[ = \sup_{(\rho, \varepsilon) \in B_1} e^{-z\sqrt{\lambda_{\rho, k}}} \left( 1 - e^{-2(s-z)\sqrt{\lambda_{\rho, k}}} \right) \]
\[ \leq e^{-z\sqrt{\frac{1}{2} - k^2}}. \quad (74) \]

Combine all of the estimations \((72), (73)\) and \((74)\), we thus obtain the result.

\[ \left\| u (., z) - u^\varepsilon (., z) \right\|_{L^2(\mathbb{R}^2)}^2 \leq \left[ e^{-z\sqrt{\frac{1}{2} - k^2}} \left( 1 + e^{-2(d-z)\sqrt{\frac{1}{2} - k^2}} \right) + e^{-z\sqrt{\frac{1}{2} - k^2}} \right] \times M \left\| \hat{u} (., z) - \hat{u}^\varepsilon (., z) \right\|_{L^2(\mathbb{R}^2)}. \quad (75) \]

Hence, after simplifying the term which can be eliminated, the proof is completed. \[ \square \]
Remark 9. Under the assumption \( k < \frac{\pi}{2d} \) (see Theorem 1), the condition \( \frac{1}{\varepsilon} > k^2 - \frac{\pi^2}{4d^2} \) in the lemma above holds for all \( \varepsilon > 0 \).

Lemma 10. Let \( u^\varepsilon \) and \( u^\varepsilon_3 \) be regularized solutions respectively defined by (63) with pairs of corresponding data \((f, g) \) and \((f_3, g_3) \) where \( f_3 \) and \( g_3 \) satisfying the assumption (A1). Then for \( z \in [0, d] \), there exists function \( M(z) > 0 \) depending on \( k, \varepsilon \) and \( d \) such that

\[
\| u^\varepsilon (., z) - u^\varepsilon_3 (., z) \|_{L^2(\mathbb{R}^2)} \leq M(z) \delta. \tag{76}
\]

Proof. By using the Cauchy-Schwarz inequality and Holder’s inequality, we have

\[
\| u^\varepsilon (., z) - u^\varepsilon_3 (., z) \|_{L^2(\mathbb{R}^2)}^2 \leq 2 \int_{B_2 \cup B_3} \left| \hat{g} (\rho) - \hat{g}_3 (\rho) \right|^2 \cosh^2 \left( (d - z) \sqrt{\Lambda_{\rho, k}} \right) \d \rho
\]

\[
+ \frac{1}{\Lambda_{\rho, k}} (d - z) \int_{B_2 \cup B_3} \left| \hat{f} (\rho, s) - \hat{f}_3 (\rho, s) \right|^2 \sinh^2 \left( (s - z) \sqrt{\Lambda_{\rho, k}} \right) \d s \d \rho
\]

\[
\leq 2 \delta^2 \sup_{(\rho, s) \in B_2 \cup B_3} \left[ \cosh^2 \left( (d - z) \sqrt{\Lambda_{\rho, k}} \right) + (d - z) \frac{1}{\Lambda_{\rho, k}} \int_{B_2 \cup B_3} \sinh^2 \left( (s - z) \sqrt{\Lambda_{\rho, k}} \right) \d s \right]. \tag{77}
\]

At this point, we first estimate the supremum for \( (\rho, k) \in A_1 \). The function \( \cosh \left( (d - z) \sqrt{\Lambda_{\rho, k}} \right) \) is increasing, and the function \( \int_{B_2 \cup B_3} \sinh^2 \left( (s - z) \sqrt{\Lambda_{\rho, k}} \right) \d s \) is easy to be calculated, that is

\[
\int_{B_2 \cup B_3} \sinh^2 \left( (s - z) \sqrt{\Lambda_{\rho, k}} \right) \d s = \frac{z - d}{2} + \frac{\sinh \left( 2 \sqrt{\Lambda_{\rho, k}} (d - z) \right)}{4 \sqrt{\Lambda_{\rho, k}}}
\]

\[
= \frac{\sinh \left( 2 \sqrt{\Lambda_{\rho, k}} (d - z) \right)}{4 \sqrt{\Lambda_{\rho, k}}}. \tag{78}
\]

Then, we can see that

\[
\frac{\sinh \left( 2 \sqrt{s - k^2} (d - z) \right) - 2 \sqrt{s - k^2} (d - z)}{4 \left( \sqrt{s - k^2} \right)^3}
\]

is also the increasing function with respect to variable \( s \). Thus, the supremum in the right side (77) is attained when \( (\rho, \varepsilon) \in B_2 \) and it equals to the following function,

\[
\cosh^2 \left( (d - z) \sqrt{\frac{1}{\varepsilon} - k^2} \right) + (d - z) \frac{\sinh \left( 2 \sqrt{\frac{1}{\varepsilon} - k^2} (d - z) \right) - 2 \sqrt{\frac{1}{\varepsilon} - k^2} (d - z)}{4 \left( \sqrt{\frac{1}{\varepsilon} - k^2} \right)^3}. \tag{79}
\]

Secondly, we consider the case \( (\rho, k) \in A_3 \), then

\[
\left| \cosh \left( (d - z) \sqrt{\Lambda_{\rho, k}} \right) \right| = \left| \cos \left( (d - z) \sqrt{-\Lambda_{\rho, k}} \right) \right| \leq 1, \tag{80}
\]

14
\[ \left| \frac{1}{\lambda_{p,k}} \int_{z}^{d} \sinh^2 \left( (s - z) \sqrt{\lambda_{p,k}} \right) ds \right| = \left| \int_{z}^{d} \frac{\sin^2 \left( (s - z) \sqrt{-\lambda_{p,k}} \right)}{-\lambda_{p,k}} ds \right| \leq \frac{1}{\lambda_{p,k}} \int_{z}^{d} (s - z)^2 \lambda_{p,k} ds \leq \frac{(d - z)^3}{3}. \] (81)

So, the supremum in this situation is equal to \( 1 + \frac{(d - z)^3}{3} \).

Finally, we find the supremum in the case \((\rho, k) \in A_2\). From the expression (41), we have the following estimation directly by using the Cauchy-Schwarz inequality and Holder’s inequality.

\[
\| \hat{u}^\varepsilon (., z) - \hat{u}_\delta^\varepsilon (., z) \|^2_{L^2(\mathbb{R}^2)} \leq \int_{(\rho, \varepsilon) \in B_2 \cup B_3} \left[ |\hat{g}(\rho) - \hat{g}_\delta(\rho)| + \int_{z}^{d} \int_{s}^{d} \left| \hat{f}(\rho, \gamma) - \hat{f}_\delta(\rho, \gamma) \right| d\gamma ds \right]^2 d\rho \\
\leq 2 \int_{(\rho, \varepsilon) \in B_2 \cup B_3} \left[ |\hat{g}(\rho) - \hat{g}_\delta(\rho)|^2 + \left( \int_{z}^{d} \int_{s}^{d} \left| \hat{f}(\rho, \gamma) - \hat{f}_\delta(\rho, \gamma) \right| d\gamma ds \right)^2 \right] d\rho \\
\leq 2 \left[ \delta^2 + (d - z) \int_{(\rho, \varepsilon) \in B_2 \cup B_3} \int_{z}^{d} \left( \int_{s}^{d} \left| \hat{f}(\rho, \gamma) - \hat{f}_\delta(\rho, \gamma) \right| d\gamma \right)^2 ds d\rho \right] \\
\leq 2 \left[ \delta^2 + (d - z) \int_{(\rho, \varepsilon) \in B_2 \cup B_3} \int_{s}^{d} \left( \int_{s}^{d} \left| \hat{f}(\rho, \gamma) - \hat{f}_\delta(\rho, \gamma) \right| d\gamma ds \right) d\rho \right] \\
\leq 2 \delta^2 \left[ 1 + (d - z) \int_{z}^{d} (d - s) ds \right] \\
\leq 2 \delta^2 \left[ 1 + \frac{(d - z)^3}{2} \right]. \] (82)

Hence, we combine all of the estimations (77), (79), (80), (81), and (82), then conclude that, there exists \( M(z) > 0 \) depending on \( k, \varepsilon \) and \( d \) such that \( \| u^\varepsilon (., z) - u_\delta^\varepsilon (., z) \|_{L^2(\mathbb{R}^2)} \leq M(z) \delta \), for \( z \in [0, d] \). Thus, we choose \( M(z) \) as follows.

\[ M^2(z) = 8 \sup_{z \in [0, d]} \left\{ 1 + \frac{(d - z)^3}{2} \cosh^2 \left( \frac{1}{\varepsilon} - k^2 \left( d - z \right) \right) + \left( d - z \right) \sinh \left( 2 \sqrt{\frac{1}{\varepsilon} - k^2} \left( d - z \right) \right) - \frac{2 \sqrt{\frac{1}{\varepsilon} - k^2} \left( d - z \right)}{4 \left( \frac{1}{\varepsilon} - k^2 \right)^3} \right\}. \] (83)

**Remark 11.** From Lemma [10] and Lemma [8], we obtain the main result of this paper. The result shows that the regularized solution of the problem (10) with \( h \equiv 0 \) will approach to the exact solution in \( L^2 \) norm under some assumptions.
Theorem 12. Let $u$ be the exact solution of the problem (14) with $h \equiv 0$ and let $u^\delta$ be the regularized solution (63) with noisy data $f_\delta$ and $g_\delta$ satisfying (A1). We denote $\|u(.,0)\|_{L^2(\mathbb{R}^2)} = M_1 > 0$ and assume that the noise level $\delta \leq M_1$. If we put $\kappa_\delta := \sqrt{\frac{3}{\varepsilon} - k^2}$ and $\varepsilon := \varepsilon(\delta)$ such that

$$
\kappa_\delta(\delta) = -\frac{1}{d} \ln \frac{\delta}{M_1},
$$

then for every $z \in (0,d]$, we obtain the estimation

$$
\|u(.,z) - u^\delta(.,z)\|_{L^2(\mathbb{R}^2)} \leq \left( \frac{4\sqrt{4\delta^2(1-\frac{2M_1}{\delta^2}) + 2^{2(d-1)} + 2 \ln^3 \left( \frac{4}{M_1} \right)}}{2 - d^3 (d - z)} + M_1 \frac{d^2}{\delta^2} \right) \delta^\frac{d}{3}.
$$

As a consequence, for each $z \in (0,d]$ we have

$$
\|u(.,z) - u^\delta(.,z)\|_{L^2(\mathbb{R}^2)} \to 0, \quad \text{as } \delta \to 0.
$$

Proof. By using triangle’s inequality, we have

$$
\|u(.,z) - u^\delta(.,z)\|_{L^2(\mathbb{R}^2)} \leq \left\| u(.,z) - u^\delta(.,z) \right\|_{L^2(\mathbb{R}^2)} + \left\| u^\delta(.,z) - u^\delta(.,z) \right\|_{L^2(\mathbb{R}^2)}
$$

$$
\leq \left[ e^{-\delta \kappa_\delta} \left( 1 + e^{-2(d-z)\kappa_\delta} \right) + e^{-\delta \kappa_\delta} \right] M_1 + M_2 \delta
$$

$$
\leq \left[ e^{-\delta \kappa_\delta} (e^{(d-z)\kappa_\delta} + e^{-(d-z)\kappa_\delta}) + e^{-\delta \kappa_\delta} \right] M_1 + M_2 \delta
$$

$$
\leq \left[ 2 e^{-\delta \kappa_\delta} \cosh ((d-z)\kappa_\delta) + e^{-\delta \kappa_\delta} \right] M_1 + M_2 \delta
$$

$$
\leq \left[ e^{-\delta \kappa_\delta} M_2 + e^{-\delta \kappa_\delta} \right] M_1 + M_2 \delta
$$

$$
\leq M_2 \left( M_1 e^{-\delta \kappa_\delta} + \delta \right) + e^{-\delta \kappa_\delta} M_1,
$$

where $M_1$ is defined above and $M_2$ is the function with respect to variable $z$ that is slightly stronger than the function we prove in Lemma 10.

$$
M_2^2(z) = 16 \sup_{z \in [0,d]} \left\{ \cosh^2 ((d-z)\kappa_\delta) + (d-z) \frac{\sinh (2\kappa_\delta (d-z)) - 2\kappa_\delta (d-z)}{4 (\kappa_\delta)^3}; 1 + \frac{(d-z)^3}{2} \right\}.
$$

For $\varepsilon = \varepsilon(\delta)$, we have $e^{-\delta \kappa_\delta} = \frac{\delta}{M_1}$ and

$$
e^{(d-z)\kappa_\delta} = \left( \frac{\delta}{M_1} \right)^{\frac{d}{\delta^2}}.
$$

Thus, we obtain

$$
\|u(.,z) - u^\delta(.,z)\|_{L^2(\mathbb{R}^2)} \leq 2M_2 \delta + \left( \frac{\delta}{M_1} \right)^{\frac{d}{\delta^2}} M_1.
$$
We see that if $M_2$ does not relate to $\kappa_\varepsilon$, then (86) holds. Besides, we are interested in $\delta \to 0$ which implies $\kappa_\varepsilon \to \infty$, so that we have

$$M_2 = 4 \sqrt{\cosh^2((d-z)\kappa_\varepsilon) + (d-z) \frac{\sinh(2\kappa_\varepsilon(d-z)) - 2\kappa_\varepsilon(d-z)}{4(\kappa_\varepsilon)^3}}.$$  

We estimate $M_2$ as follows by using supplement estimations.

$$\cosh^2((d-z)\kappa_\varepsilon) = \frac{1}{2} + \frac{\cosh(2(d-z)\kappa_\varepsilon)}{2}$$

$$= \frac{1}{2} \left[ 1 + \left( \frac{\delta}{M_1} \right)^{\frac{2(z-d)}{d}} + \left( \frac{\delta}{M_1} \right)^{\frac{2(d-z)}{d}} \right]$$

$$\leq \frac{1}{2} \left( 2 + M_1 \frac{2(d-z)}{d} \delta \frac{2(z-d)}{d} \right),$$

$$\sinh(2\kappa_\varepsilon(d-z)) - 2\kappa_\varepsilon(d-z) \leq \frac{e^{2\kappa_\varepsilon(d-z)} - e^{2\kappa_\varepsilon(z-d)}}{4(\kappa_\varepsilon)^3} \leq -\delta^3 \frac{M_1^{\frac{2(d-z)}{d}} \delta \frac{2(z-d)}{d}}{8 \ln \left( \frac{\delta}{M_1} \right)}.$$ 

Thus, we have

$$M_2 \leq 4 \sqrt{\frac{1}{2} \left( 2 + M_1 \frac{2(d-z)}{d} \delta \frac{2(z-d)}{d} \right) - d^3 \left( d-z \right) \frac{M_1^{\frac{2(d-z)}{d}} \delta \frac{2(z-d)}{d}}{8 \ln \left( \frac{\delta}{M_1} \right)}}$$

$$\leq 4 \left[ 1 + M_1 \frac{2(d-z)}{d} \delta \frac{2(z-d)}{d} \left[ \frac{1}{2} - \frac{d^3 \left( d-z \right)}{8 \ln \left( \frac{\delta}{M_1} \right)} \right] \right].$$

Hence, the result is

$$\left\| u(.,z) - u^\delta(.,z) \right\|_{L^2(\mathbb{R}^2)} \leq 8 \left[ 1 + M_1 \frac{2(d-z)}{d} \delta \frac{2(z-d)}{d} \left[ \frac{1}{2} - \frac{d^3 \left( d-z \right)}{8 \ln \left( \frac{\delta}{M_1} \right)} \right] \right] \delta + \delta^3 \frac{d^3}{M_1^2}$$

$$\leq 4 \left( 4\delta^2 \left( 1 - \frac{1}{\delta} \right) + M_1 \frac{2(d-z)}{d} \left[ 2 - \frac{d^3 \left( d-z \right)}{2 \ln \left( \frac{\delta}{M_1} \right)} \right] + M_1 \frac{d^3}{d^2} \right) \delta^2.$$

(89)

which completes the proof.
Remark 13. From Theorem 12, the method gives a convergent approximation of $u(\xi, z)$ and we can control the bounded error $\delta$ and noisy data $f_{\delta}, g_{\delta}$ as well. Moreover, for $z \in (0, d)$ the error estimation is of the highest order of infinitesimals $\Delta, \delta \Delta$ and $-\Delta \frac{1}{\ln^{2} \delta}$, that is $\Delta$ or of the order $O\left(\Delta^{2}\right)$ by limiting each ratio with $\delta \to 0$ ($\delta > -\Delta \frac{1}{\ln^{2} \delta} > \Delta \frac{1}{\ln^{2} \delta}$ in the meaning of the order).

Remark 14. The assumption of $\kappa_{e}$ in Theorem 12 gives us an explicit formula for the regularization parameter $\varepsilon$ which is dependent on the data error $\delta$, that is

$$\varepsilon(\delta) = \left(k^{2} + \frac{1}{d^{2}} \ln^{2} \left(\frac{\delta}{M_{1}}\right)\right)^{-1}.$$

The estimation (89) in Theorem 12 can be better if we do not eliminate some giant terms in $M_{2}$, especially $-2\kappa_{e}(d - z)$ and $-e^{2\kappa_{e}(z-d)}$. Then, we obtain a tight estimation of $M_{2}$.

$$4 \sqrt{\frac{1}{2}(2 + \alpha \beta^{-1}) - d^{3}(d - z)} - \frac{\alpha \beta^{-1} - \beta \alpha^{-1} + \frac{4(d-z)}{d} \ln \left(\frac{\delta}{M_{1}}\right)}{8 \ln^{3} \left(\frac{\delta}{M_{1}}\right)},$$

where $\alpha = M_{1}^{2(d-z)}$ and $\beta = \delta^{2(d-z)}$ for short.

Remark 15. In practice, we do not know exactly the norm $\|u(., 0)\|_{L^{2}(\mathbb{R}^{2})}$ (also $u(x, y, 0)$). However, we can choose a supremum of $u(x, y, 0)$ from the known phenomena.

4 Numerical implementation

In this section, we are going to show a simple implementation of reconstruction in order to validate the main result. As we know, the core of three-dimensional inhomogeneous Helmholtz equation in free space are presented in the following form.

$$\Delta u + k^{2}nu = -f,$$

where the coefficient $n$, in general, known in [6, 7, 9] as the index of refraction is equal to 1 and $k = 1/\lambda > 0$ plays a role as the wavenumber defined by wavelength $\lambda$. In our implementation, we will take the equation associated with the Cauchy data prescribed on the boundary $\Gamma_{d}$. On the other hand, data functions are almost zero outside the given domain. More precisely, assume that we consider the problem in a mock-up which is a nearly rectangular box with the approximate size $1.0 \text{ m} \times 1.0 \text{ m} \times 0.5 \text{ m}$. Then, the data function $g$ and source term $f$ are almost zero outside the top and bottom and $h = 0$ in the whole field. In the case of electromagnetic radiation in free space, the speed of light is about $3 \times 10^{8} \text{ m} \cdot \text{s}^{-1}$. Thus the wavelength $\lambda$ of a 100 MHz electromagnetic (radio) wave is nearly 3.0 m. This leads to the wavenumber $k = 1/3 \text{ m}^{-1}$. Also, we define some given functions as follows.

$$f(x, y, z) = -xy \left(\frac{z - \frac{1}{2}}{2}\right)^{2} \left(12 + \frac{1}{9} \left(\frac{z - \frac{1}{2}}{2}\right)^{2}\right), \quad g(x, y) = 0.$$  (90)
Therefore, the exact solution is \( xy \left( z - \frac{1}{2} \right)^4 \). We approximate the regularized solution by taking noise level \( \delta \) in given functions by

\[
f_\delta(x, y, z) = f(x, y, z) \left( 1 + \frac{\delta}{0.3167506677} \right), \quad g_\delta(x, y) = \delta.
\]  

(91)

Let us denote \( (0, c)^2 = (0, c) \times (0, c) \) where \( c > 0 \). In this example, we aim to consider solutions in Fourier transform in the situation \( A_1 \) because of Lemma 6 and Lemma 7. Then, from (60)-(62) and (90)-(91), we have the following representation.

\[
\hat{u}_\delta(\rho, z) = \hat{g}_\delta(\rho) \cosh \left( \left( \frac{1}{2} - z \right) \sqrt{\lambda_{\rho, 1}} \right) - \frac{1}{\sqrt{\lambda_{\rho, 1}}} \int_0^{1/2} f_\delta(\rho, s) \sinh \left( (s - z) \sqrt{\lambda_{\rho, 1}} \right) ds,
\]  

(92)

where \( \Theta_\varepsilon^1 = \Theta_\varepsilon \cap A_1 \) and

\[
\hat{g}_\delta(\rho) = \delta \int_{(0,1)^2} e^{-2\pi i (\rho, \xi)} d\xi, \quad f_\delta(\rho, s) = \int_{(0,1)^2} f(\xi, s) \left( 1 + \frac{\delta}{0.3167506677} \right) e^{-2\pi i (\rho, \xi)} d\xi,
\]  

(93)

for \( \rho \in \Theta_\varepsilon^1 \).

Simultaneously, we show the representation of exact solution.

\[
\hat{u}(\rho, z) = \left( z - \frac{1}{2} \right)^4 \int_{(0,1)^2} \xi_1 \xi_2 e^{-2\pi i (\rho, \xi)} d\xi, \quad \rho \in A_1.
\]  

(94)

Therefore, instead of considering the difference between the regularized solution and exact solution, we may estimate their Fourier transform. In order to define \( A_1 \) in terms of computation, we will truncate the set at a fixed \( L \). Moreover, we do not pay more attention to the integrations in (93), but the integration in (92) should be solved numerically. From the point of view, we apply Gauss-Legendre quadrature method to this. We will compute the Legendre-Gauss nodes and weights on an interval \([z_0, 1/2]\) with truncation order \( N \). In general, the regularization parameter \( \varepsilon \) and the truncation Fourier transform \( L \) are quite different. In practice, we do not know the exact solution, then the truncation Fourier transform is unknown. Thus, it is changeable to be the parameter. On the other hand, in this example, we aim to consider \( \rho \in \mathbb{Z}^2 \) in numerical sense, so we have \( \Theta_\varepsilon^1 = \Theta_\varepsilon \cap A_1 \) and \( \rho \in \Theta_\varepsilon^1 \) in (94) with \( \rho \) shall be chosen uniformly by the partition \( \frac{1}{30\sqrt{\varepsilon}} \) in the set \( \Theta_\varepsilon^1 \).

In common, the whole process is established as follows.

**Step 1.** Define a priori constant \( M_1 \). For the example, we have \( M_1 = \frac{1}{48} \) because of finding \( \|u(., 0)\|_{L^2(\mathbb{R}^2)} \), then the regularization parameter is

\[
\varepsilon(\delta) = \left( \frac{1}{9} + 4\ln^2(48\delta) \right)^{-1}.
\]

**Step 2.** Determine the set \( \Theta_\varepsilon^1 \) and consider a fixed \( z_0 \) and choose a truncation order \( N \) to approximate the integration in (92).
Figure 1: Modulus of Fourier transform of exact solution from Eq. (94) and regularized solution from Eq. (92) with noise amplitude $\delta = 10^{-1}$.

Figure 2: Modulus of Fourier transform of exact solution from Eq. (94) and regularized solution from Eq. (92) with noise amplitude $\delta = 10^{-2}$.
Figure 3: Modulus of Fourier transform of exact solution from Eq. \((94)\) and regularized solution from Eq. \((92)\) with noise amplitude \(\delta = 10^{-3}\).

Figure 4: Modulus of Fourier transform of exact solution from Eq. \((94)\) and regularized solution from Eq. \((92)\) with noise amplitude \(\delta = 10^{-4}\).
| $\delta$  | $E$ (0.4)           | $E$ (0.25)           | $E$ (0.1)           | $E_2$ (0.05)          |
|---------|---------------------|----------------------|---------------------|-----------------------|
| 1.0E-01 | 1.40421792E-02      | 1.44104551E-02      | 1.53538523E-02      | 1.58767301E-02        |
| 1.0E-02 | 2.91444011E-03      | 2.94877381E-03      | 3.05224512E-03      | 3.11545162E-03        |
| 1.0E-03 | 7.61081237E-05      | 8.11322809E-05      | 1.06859908E-04      | 1.32533217E-04        |
| 1.0E-04 | 4.95852862E-06      | 6.51466288E-06      | 3.95372158E-05      | 7.97083661E-05        |

Table 1: The error between the regularized solution and exact solution in Fourier transform at $z_0 = 0.4; 0.25; 0.1$ and $z_0 = 0.05$ with $\delta = 10^{-r}, r = 1, 4$.

**Step 3.** Generate a vector including $\rho = (\rho_1, \rho_2)$ in $\Theta_1^1$. Then, calculate $\hat{u}_{\epsilon}^\delta(\rho, z_0)$ and $\hat{u}(\rho, z_0)$.

**Step 4.** Compute the errors.

$$E(\tau_0) = \sqrt{\frac{1}{\text{card}(\Theta_1^1)} \sum_{\rho \in \Theta_1^1} |\hat{u}(\rho, z_0) - \hat{u}_{\epsilon}^\delta(\rho, z_0)|^2}.$$  

**Step 5.** Represent 3-D graphs of the modulus of Fourier transform of regularized solution and exact solution in situation $A_1$.

Table 1 shows the error between the regularized solution and exact solution in the meaning of Fourier transform at $z_0 = 0.4; 0.25; 0.1; 0.05$, respectively. In computational process, we choose $N = 5$. The comparison between the exact solution and the regularized solution at $z_0 = 0.05$ are shown in Figure 1-Figure 4 in terms of the modulus of Fourier transform. When $\delta$ becomes smaller, the $\frac{1}{\text{card}(\Theta_1^1)}$ then becomes bigger. So, we pay attention to the figures that the region becomes greater in width. It also explains how the reconstruction of the exact solution is by the proposed method. To sum up, the numerical result is reasonable for our theoretical result.

The numerical implementation is coded with MATLAB and the computations are done on a computer equipped with processor Pentium(R) Dual-Core CPU 2.30 GHz and having 3.0 GB total RAM.

### 5 Conclusion

In this paper, we apply the truncation regularization method to reconstruct the solution of three-dimensional inhomogeneous Helmholtz equation associated with Cauchy data in stable way. The simple and easy numerical example is devoted to solve the obstacle in Lemma 7 then verifies that the proposed method is efficient and accurate.

For further study, we intend to investigate another methods as in [2, 11, 12, 13], then lead to comparison of those regularization methods and analyse numerical solution and simulation as in [3, 4, 9, 10]. Also, various types of Helmholtz equation (see e.g. [14, 15]) will be studied.

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