Arc domination in digraphs

1 R.Anbunathan, 2 R.Rajeswari

1 Research scholar, Sathyabama Institute of Science and Technology, Chennai, India.  
2 Professor, Sathyabama Institute of Science and Technology, Chennai, India

anbun.adhavan@gmail.com, rajeswarivel1998@gmail.com

Abstract. Let \( D = (V, A) \) be a digraph. A subset \( S \) of arc set in a digraph \( D \) is called an arc dominating set of \( D \) if for every \( (v, w) \in A \setminus S \), there exists an \( (u, v) \in S \) such that \( \{(u, v), (v, w)\} \in A \). The minimum cardinality of an arc dominating set of \( D \) is called the arc domination number of \( D \) and is denoted by \( \gamma'(D) \). In this paper, arc domination number for various digraphs were determined and also derived a characterization for minimal arc dominating sets of digraphs.

Keywords: Minimal arc domination, Arc domination, Domination number, Neighbourhood, Digraphs

1. Introduction

A graph \( G(V(G), E(G)) \) is a non-empty finite set with \( V(G) \) is a vertex set and \( E(G) \) is unordered edge set. Let \( u \) and \( v \) be vertices of a graph \( G \). A \( u - v \) walk of \( G \) is an alternating sequence of vertices and edges of \( G \) beginning with the vertex \( u \) and ending with vertex \( v \) such that every edge is incident with two vertices immediately preceding and following it. Graph theory started with Euler who was requested to find a nice path across the seven Köningsberg bridges. Graph Theory and its procedures can be started not only in discrete mathematics, but also in scientific disciplines such as engineering, operational research, computer science, the life sciences and management sciences. Oystein Ore first defined the domination number of a graph in 1962[1], Then Teresa W. Haynes, Stephen T. Hedetniemi and Peter J Slater [4] and others were developed this concept. H. B. Walikar, B. D. Acharya and E. Sampathkukmar [2] were introduced some concepts in domination theory. Dominator coloring of various graphs were studied by Manjula et.al. [7][9]. Domination in digraphs were also studied by Thamizh selvam et al. and Rajeswari et al. [8][10]. Arc Domination in digraphs was defined by V.R.Kulli[5]. A subset \( S \) of arcs in a digraph \( D \) is called an arc dominating set of \( D \) if every arc \( (v, w) \) in \( A - S \), there exists an \( (u, v) \in S \) such that \( \{(u, v), (v, w)\} \in A \). The minimum cardinality of an arc dominating set of \( D \) is called the arc domination number of \( D \) and is denoted by \( \gamma'(D) \). Motivated by the above survey this study focuses on the arc domination some standard digraphs.

2. Main results

Theorem 2.1. Let \( P_{n+1} \), be a directed path with \( n \geq 2 \), then the arc domination number of \( P_{n+1} \) is \( \left\lceil \frac{n}{2} \right\rceil \)

i.e., \( \gamma'(P_{n+1}) = \left\lceil \frac{n}{2} \right\rceil \), where \( \left\lceil \cdot \right\rceil \) is a least integer function.

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Proof Let $P_{n+1}$ be a directed Path with vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$ and arc set $A = \{e_i = V_i V_{i+1} / 1 \leq i \leq n\}$

\[\text{Figure 1. Directed path } P_{n+1}\]

Let $S = \{e_1, e_3, e_5, \ldots, e_{n-1}\}$ if $n$ is even (or) $S = \{e_1, e_3, e_5, \ldots, e_n\}$ if $n$ is odd

Since $e_i$ is incident with $e_{i+1}$, Every arc in $A\setminus S$ is incident with some inarc of $S$. That is every arc of $P_{n+1}$ is uniquely dominated by the arc proceeds it, implying that arc domination number of $P_{n+1}$ is $\left\lceil \frac{n}{2} \right\rceil$. Therefore $\gamma'(P_{n+1}) = \left\lceil \frac{n}{2} \right\rceil$.

Theorem 2.2 Let $C_n$, be a directed cyclic graph $D(V,A)$ with $n \geq 3$ arcs, then the arc domination number of $C_n$ is $\left\lceil \frac{n}{2} \right\rceil$ i.e., $\gamma'(C_n) = \left\lceil \frac{n}{2} \right\rceil$, where $\left\lceil \cdot \right\rceil$ is a least integer function.

Proof Let $C_n$ be a directed cycle with vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$ and arc set $A = \{e_i = V_i V_{i+1} / 1 \leq i \leq n-1\} \cup \{V_n V_1\}$.

\[\text{Figure 2. Directed cycle } C_n\]

Let $S = \{e_1, e_3, e_5, \ldots, e_{n-1}\}$ if $n$ is even (or) $S = \{e_1, e_3, e_5, \ldots, e_n\}$ if $n$ is odd

Since $e_i$ incident with $e_{i+1}$. Every arc of $A\setminus S$ is incident with some inarc of $S$. That is every arc of $C_n$ is uniquely dominated by the arc follows it, implying that arc domination number of $(C_n)$ is $\left\lceil \frac{n}{2} \right\rceil$. Therefore $\gamma'(C_n) = \left\lceil \frac{n}{2} \right\rceil$.
Result 2.3 Number of edges in perfect direct binary tree \( T_{2n} \) with 2n length level.

Number of arc in \( T_{2n} = 2^1 + 2^2 + 2^3 + \ldots + 2^{2n} = \frac{2^{2n} - 1}{2-1} = 2^{2n - 1} = n(A(T_{2n})) \)

Number of arcs in odd length levels in \( T_{2n} = 2^1 + 2^3 + 2^5 + \ldots + 2^{2n-1} = \frac{2^{2n} - 1}{4-1} = \frac{2^{2n} - 1}{3} = n(A(T_{2n})) \)

Number of arcs in even length levels in \( T_{2n} = \frac{2n(A(T_{2n}))}{3} \)

![Level one perfect binary directed tree](image1)
Level one perfect binary directed tree \( T_1 \)

![Level two perfect binary directed tree](image2)
Level two perfect binary directed tree \( T_2 \)

![Level three perfect binary directed tree](image3)
Level three perfect binary directed tree \( T_3 \)

![Level four perfect binary directed tree](image4)
Level four perfect binary directed tree \( T_4 \)

Figure 3. Number of edges in perfect directed binary tree

\textbf{Theorem 2.4 Let} \( T_n \) be a perfect binary directed tree with length of \( n \geq 2 \) levels. then the arc domination number of \( T_n \) is

\[ \gamma'(T_n) = \begin{cases} \left\lfloor \frac{n(A)}{3} \right\rfloor + 1 & \text{for length of even level trees} \\ \left\lfloor \frac{n(A)}{3} \right\rfloor + 1 & \text{for length of odd level trees} \end{cases} \]

\textbf{Proof}

We consider two cases

\textbf{case 1:} Perfect binary tree with an even levels. \((T_{2n})\)

\( S = \{ \text{Set of arcs in length of odd levels} \} \) and \( A \setminus S = \{ \text{Set of arcs in length of even levels} \} \), from the diagram of perfect binary tree every arc in \( A \setminus S \) is dominated by exactly one arc in length of odd level in \( S \), which is the minimum possibility.

\( \gamma'(T_{2n}) = \frac{n(A(T_{2n})))}{3} \)
Theorem 2. Consider a directed path \( P_{11} \) with vertex set \( V = \{v_1, v_2, \ldots, v_{11}\} \) and arc set \( A = \{e_i \mid v_i V v_i+1, 1 \leq i \leq n\} \). Let \( S = \{e_1, e_3, e_5, \ldots, e_9\} \) and \( A \setminus S = \{e_4, e_6, e_8, e_{10}\} \) then by Theorem 2.1 \( S \) is an arc dominating set. But \( A \setminus S \) is not an arc dominating set because it is not satisfying the definition of arc domination of digraphs, i.e., \( e_i \) has no out arc.

But if we consider a directed cycle \( C_{11} \) with eleven vertex set \( V = \{v_1, v_2, \ldots, v_{11}\} \) and arc set \( A = \{e_i \mid v_i V v_i+1, 1 \leq i \leq n\} \) then \( S = \{e_1, e_3, e_5, \ldots, e_9\} \) and \( A \setminus S = \{e_2, e_4, e_6, e_8, e_{10}\} \) then by Theorem 2.2 \( S \) is an arc dominating set and \( A \setminus S \) is also an arc dominating set.

Observation 2.5. An arc domination set \( S \) is a minimal dominating set of a directed graph \( D(V, A) \) with \( |A| = 0 (\text{mod} 2) \) if and only if for each \( e \) in \( S \), one of the following conditions are satisfied

(i) \( N^{+}(e) \cap S = \Phi \)

(ii) There exists an \( e_i \) in \( A \setminus S \) such that \( N^{-}(e_i) \cap S = \{e\} \)

Proof Suppose \( S \) is a minimal arc dominating set of a directed graph \( D(V, A) \). Then every arc \( e \) in \( S \), \( S \setminus \{e\} \) is not an arc dominating set, thus some arc \( e_i \) in \( A \setminus (S \cup \{e\}) \) is not an arc domination by any arc in \( S \setminus \{e\} \). Now either \( N^{+}(e) \cap S = \Phi \) or \( e_i \) in \( A \setminus S \).

If \( e_i \) in \( A \setminus S \), then \( e_i \) is not indomincating arc in \( S \setminus \{e\} \). Therefore, \( e_i \) must be inarc to \( e \). Therefore \( N^{-}(e_i) \cap S = \{e\} \).

Conversely, Suppose \( S \) is an arc dominating set, for each arc \( e \) in \( S \), one of the two stated conditions holds. Now we prove that \( S \) is a minimal arc dominating set. Suppose \( S \) is not a minimal arc dominating set. Then there exists an arc \( e_i \) in \( S \) such that \( S \setminus \{e_i\} \) is an arc dominating set. Thus \( e_i \) has at least one inarc in \( S \setminus \{e_i\} \). Therefore, condition (i) does not hold.

Also if \( S \setminus \{e_i\} \) is an arc dominating set, then every arc in \( A \setminus S \) has at least one outarc in \( S \setminus \{e_i\} \). Therefore, for \( e_i \), condition (ii) does not hold. Which is a contradiction.

Observation 2.6. Every non trivial connected digraph \( G(V, A) \) has an arc dominating set \( S \) whose complement \( A \setminus S \) need not be an arc dominating set.

Proof To prove this result by counter examples.
Consider a directed path \( P_{11} \) with vertex set \( V = \{v_1, v_2, v_3, \ldots, v_{11}\} \) and arc set \( A = \{e_i \mid v_i V v_i+1, 1 \leq i \leq n\} \). Let \( S = \{e_1, e_3, e_5, \ldots, e_9\} \) and \( A \setminus S = \{e_2, e_4, e_6, e_8, e_{10}\} \) then by Theorem 2.1 \( S \) is an arc dominating set. But \( A \setminus S \) is not an arc dominating set because it is not satisfying the definition of arc domination of digraphs, i.e., \( e_i \) has no out arc.
Theorem 2.7 Let $G (V, A)$ be a regular directed graph then 

$$\left\lfloor \frac{n(A)}{\Delta^+(v) + 1} \right\rfloor \leq \gamma'(G) \leq n(A) - 1,$$

where $v$ is any vertex of $G$.

3. Conclusion

In this paper, found arc domination number for some standard directed graphs and also analyzed how domination of directed graph is varied from undirected graphs. Observed a characterization for arc domination of directed graphs and bounds for arc domination of digraphs. In future, the arc domination number and arc independence number of various directed graphs and its applications can be analysed.

Acknowledgment

The authors are grateful to the referees for their helpful comments.

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