Research on the reversal of hard-ball system in two-dimensional bowl  
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Abstract: Putting several hard balls into a two-dimensional bowl can form a very basic two-dimensional model of hard-ball system. When the two-dimensional bowl has a parallel-rotation at a uniform speed around a center, when the number of balls is small, the direction of motion of the small balls in the two-dimensional bowl is consistent with that of the two-dimensional bowl; However, when we increase the number of balls one by one, the two will seem to show the opposite movement trend. We analyze this problem comprehensively through physical modeling and inspire readers to think about the significance of the model for analyzing complex physical problems. Finally, because this article involves many aspects of knowledge, college physics teachers can choose the whole problem or part of the problem to introduce it into the classroom.  

Key words: physical model; hard-ball reversal; two-dimensional bowl; numerical solution; computer simulation

1. Introduction  
Tip: see the appendix for some conventions, parameter, letters and data values, which are suggested to be understood in advance for reading.  
The first is the concept of parallel-rotation and the phenomenon of reversal. For example, figure 1 (a) and (b) represent parallel-rotation and rotational-rotation respectively. For parallel-rotation, it means that every point of the object rotates around a point in space, but the object itself does not rotate; When the object has rotational-rotation in space, it will rotate. As for the phenomenon of reversal, figure 1 (c) and (d) give a rough schematic diagram. The so-called rotation direction of multiple balls is a statistical trend relative to the two-dimensional bowl. We only pay attention to the movement trend of a single ball when the number of balls is extremely sparse.  
The second is the general analysis idea of the phenomenon. The collision of balls in the bowl in two-dimensional space involves the collision of rigid bodies and their own rotation, but also involves the collision between balls and the friction between balls and the bowl wall. It is very complex to analyze, and it is not easy to distinguish the main effects. In general research and discussion, this problem is mainly divided into two aspects, namely, the angle of momentum and the angle of angular momentum. The momentum angle mainly focuses on the collision between the ball and other balls or the bowl wall, while the angular momentum angle focuses on the rolling effect caused by the friction between the ball and the bowl wall. Finally, combine the two to get a reasonable physical image.  

Finally, the previous experiments were carried out in the three-dimensional space, but we are studying the two-dimensional projection of the three-dimensional space. However, due to projection, a series of physical quantity (such as moment of inertia) use variables in three-dimensional space. If readers are still confused about the experimental phenomenon, the online video[1] in the references is a good summary of the previous research and phenomenon explanation of this problem, which can be used for reference. However, the reasons and explanations proposed are not completely correct, which we will discuss in 2.3.  
The focus of this article is to inspire readers. We will rationally analyze the conclusion of the video and give a more reasonable explanation through mathematical modeling and calculation. This process is very suitable as a case of classroom teaching and helps to inspire students to think about unknown physical phenomena.  

In fact, in class, we should also guide students to clarify the problem and find a good analytical perspective like the above behaviors. And you will see that in the later analysis, it is
very effective to divide the problem into angular momentum and momentum to analyze.

![Diagram](image1.png)

**FIG 1** The red arrow is the direction of rotation of the two-dimensional bowl, and the blue arrow reflects the overall movement trend of the balls in the two-dimensional bowl. Among them, the movement mode of (a) is desirable, while (b) does not conform to our description.

2. Aspect of angular momentum

Strictly speaking, the essence of the angular momentum effect is the rotation caused by the friction between the ball and the two-dimensional bowl wall. We take the two-dimensional bowl as the reference system, so the ball will constantly collide with the bowl wall during the rotation, and finally attach to the bowl wall. After that, the movement is determined by the boundary friction. When the friction coefficient $\mu$ is large, the ball will roll around the bowl wall, and when the $\mu$ is reduced, the ball will gradually roll and slide. If the $\mu$ is very small, the ball will slide for a long time. Now let's solve it in combination with the above physical model.

2.1 Single ball solution

As shown in Figure 1(b), some balls cannot reach the boundary. When we consider the problem again, we naturally deal with the balls in contact with the boundary for a long time, and then analyze the balls that do not meet this condition.

2.1.1 Static friction

We set the angle $\alpha$ as a known function of time as $\alpha(t)$, that is. But even so, when the ball rolls and slides along the bowl wall, the degree of freedom of the system increases greatly, which still brings difficulties to the analysis. So, we first consider the case of pure rolling.

![Diagram](image2.png)

**FIG 2** The physical model of the ball moving against the wall in a two-dimensional bowl. The radius of the ball and the radius of the two-dimensional bowl are $l$ and $r$ respectively. The rotation angles of their centroids relative to $O$ and $O'$ are $\phi$ and $\alpha$. The rotation radius of the centroid of the two-dimensional bowl around $O'$ is $R$, and the direction is counterclockwise by default.

In this case, as shown in equations (1), we can write the condition that the ball is pure rolling in a two-dimensional bowl. Then the configuration of the whole system at any time can only be expressed by $\phi$ and $t$, and then the Lagrangian of the system can be written. We set the coordinates of the centroid of the ball as $x$ and $y$, and the rotation angle of the small ball as. Then there is the relationship in equations (2) and (3).

\[
\begin{align*}
\frac{r - l}{l} \, d\theta &= d\phi \quad (1) \\
(r - l) &= L \\
x &= R\sin \alpha + L\sin \theta \\
y &= -R\cos \alpha - L\cos \theta \quad (2) \\
L(\phi, t) &= T - V = T \\
&= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 \quad (3)
\end{align*}
\]

Let's discuss the determination of $\alpha(t)$ below. At the beginning of rotation, the two-dimensional bowl will always make a very short acceleration under the action of human hands or external forces for a duration of $\tau$. And then rotate at a uniform speed $\omega$. So, our motion process is decomposed into two parts, one is the acceleration part of $\alpha$, The other is the constant velocity part of $\alpha$. Because $\tau$ is very short, we
approximate that $\alpha = (1/2) \beta^2$ and after the acceleration we have $\alpha = \alpha_0 + \omega t$. Based on this, we can write the Lagrangian of these two parts as equations (4) and (5).

$$ L = \frac{1}{2} m R^2 \beta^2 t^2 + \frac{7}{10} m L^2 \dot{\phi}^2 + m R L \beta t \cos \left( \frac{1}{2} \beta t^2 - \phi \right) $$

$$ L = \frac{1}{2} m R^2 \omega^2 + \frac{7}{10} m L^2 \dot{\phi}^2 + m R L \omega \cos (\alpha_0 + \omega t - \phi) $$  (4)

Bringing the above 2 equations into the Euler Lagrange equation (6) to obtain equations (7) and (8).

$$ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} $$

$$ \ddot{\phi} = \frac{5R}{7L} \beta^2 t^2 \sin \left( \frac{1}{2} \beta t^2 - \phi \right) + \frac{5R}{7L} \beta \cos \left( \frac{1}{2} \beta t^2 - \phi \right) $$

$$ \ddot{\phi} = \frac{5R}{7L} \omega^2 \sin (\alpha_0 + \omega t - \phi) $$  (7)

$$ \ddot{\phi} = \frac{5R}{7L} \omega^2 \sin (\alpha_0 + \omega t - \phi) $$  (8)

The initial condition of equation (7) is $\dot{\phi} = 0$ and $\phi = 0$; The initial condition in equation (8) is $\dot{\phi} = \dot{\phi}(\tau)$ and $\phi = \phi(\tau)$. Of course, the values of $\dot{\phi}(\tau)$ and $\phi(\tau)$ are derived from those in equation (7). And we also have $\alpha_0 = (1/2) \beta^2$ and $\omega = \beta t$.

Obviously, equations (7) and (8) contain nonlinear terms, which makes it very difficult to obtain the analytical solutions of the two equations. For equation (8), we can do the substitution as equation (9) and then solve it.

$$ \phi = \alpha_0 + \omega t - \varphi $$  (9)

The solution process is as follows:

$$ \dot{\phi} = \frac{5R}{7L} \omega^2 \sin (\phi) $$

$$ t = C_2 \pm \int \frac{d\phi}{\sqrt{\left(2C_1 + \frac{10R}{7L} \omega^2 \cos \phi \right)^{1/2}}} $$

It contains elliptic integral, Though we can express the relationship as shown in equation (10), it is difficult to analyze the properties of the solution and get an intuitive image. Using the difference method to simplify equation (7) as (12), and compared with the sine map (11), chaos will appear in the long-term evolution probability. Although we only care about the behavior in a short time, it is still difficult to find an analytical solution.

$$ t = \lambda (\alpha_0 + \omega t - \varphi) $$

$$ \omega_{k+1} = \frac{4}{a} \sin (\pi \omega_k) \quad (0 < a \leq 4) $$

$$ \omega_{k+1} = \Pi (R, L, \beta, t) \sin (\varphi_k + \omega_k \Delta t) \Delta t + \cdots + \omega_k \Delta t $$  (12)

In classroom teaching, equation (7)(8) is a good case of using the Fourth Runge-Kutta method (RK4) to solve differential equations. We use this to deal with it, and the result is shown in Figure 3.
FIG 3 (a)(b) are the results of using RK4 to solve φ in equations (7) and (8), respectively. The figure shows the changes of the image after changing some parameters. (c)(d) the center lines are the time-varying curve of the φ angle of the ball after shortening τ with equal spacing (time step of 0.016s) and the trajectory of the centroid the ball in the O’ reference system. The blue line is the original τ value, the red line is the final τ value (0.12s), and the data always keeps the φ direction negative within this range. (e)(f) are the trajectory of the ball centroid in the O’ reference system when changing some parameters. (h) Is the graph of φ and ϕ changing with time, and (g) is the graph of φ in phase space.

In class, we usually use the method in Figure 3, that is, changing some parameters and observing the results of numerical solution. So, let's briefly analyze the phenomenon, as shown in Figure 3(a). First, the overall trend of φ in the acceleration process is slightly reduced first. When the oscillation increases, the oscillation intensity is positively correlated with β, and considering that the value of τ is very small, that is, when entering uniform motion in general, the motion direction of the ball's centroid will be opposite to the overall parallel-rotation direction of the two-dimensional bowl, but after entering the uniform rotation, this trend will generally be corrected. As shown in Figure 3(b), φ or ϕ, which are negative at the beginning of time evolution, show positive growth, and increase periodically in a non 2π, and the size of the cycle decreases with the increase of ω or β. At the same time, the increase of R will accelerate the change of φ angle, and the increase of L will slow down this process. Figure 3 (c) shows that the centroid of the ball can be reversed, which mainly depends on the initial value of ϕ. if the negative velocity of the ball is too large at the beginning, its stable state centroid rotation direction is opposite to the rotation direction of the two-dimensional bowl!

Figure 3 (c)(d)(e)(f) shows that the ball is always rotating and translating relative to O’, and whether the overall trend of the centroid in the O system is clockwise or counterclockwise, the centroid in each rotation in the O’ system maintains the same motion law (counterclockwise in the Figure 3).

Figure 3 (g)(h) shows that the period of φ and ϕ is not the same, and φ always increases, and the two naturally form a spiral line in the phase space.

Again, this part is helpful to inspire students to think about the relationship between the properties of the solution of a complex differential equation and its parameters.
2.1.2 Dynamic friction

The above analysis stems from the static friction of the bowl wall. When the static friction condition \( f \leq \mu N \) is no longer satisfied, the ball will slide. At this time, the condition of equation (1) is no longer satisfied, so we will pay attention to the movement of the centroid of the ball.

At this time, we change the reference system and take \( O \) as the reference system, as shown in Figure 4, so the centroid of the ball will be subjected to an inertial force that changes with time. It should be noted that when the rolling conditions of the small ball cannot be met, the sliding will occur again, and the only contribution to the moment of centroid is friction. Therefore, when sliding occurs, the tangential component of the inertia force on the ball must be opposite to the direction of friction. According to the above conditions, equation (13) can be listed, in which the sign before friction \( f \) is negative, which represents sliding in the process of gradual deceleration, and the plus sign represents sliding in the process of forward acceleration.

\[
\begin{align*}
-m\dot{\alpha}^2 R\cos(\varphi - \alpha) + N &= m\dot{\varphi}^2 L \\
m\dot{\varphi}L &= \pm\mu N - m\dot{\alpha}^2 R\sin(\varphi - \alpha) \\
\pm\mu N \cdot m\dot{\alpha}^2 R\sin(\varphi - \alpha) &\geq 0
\end{align*}
\]

Bring in \( \dot{\alpha} = \omega \) and equation (8) and find the stability condition of pure rolling all the time as.

\[
\mu \geq \frac{2}{7} \left| \frac{\dot{\varphi}^2 + \omega^2 R\cos(\alpha_0 + \omega t - \varphi)}{\omega^2 R\sin(\alpha_0 + \omega t - \varphi)} \right|
\]

The result of solving equation (13) by RK4 is shown in Figure 5.

![Figure 4](image)

**FIG 4** The ball is acted by the inertial force in the \( O \) reference system, and the acceleration process is not considered, so there is no tangential acceleration.

Strictly speaking, sliding can occur in the process of acceleration and uniform speed. However, the focus of our research is whether the sliding friction will cause the reversal of the ball center of mass. Therefore, the long-term effect of sliding friction should be considered. The movement trend in the acceleration process is that the ball moves in the opposite direction of the rotation of the two-dimensional bowl, which is very clear. Sliding friction is undoubtedly the case when the tangential acceleration is large. Because of the short-term effect, we ignore it. This is also an idea often used in teaching.

In order to simplify the process, we set the friction coefficient to be very weak and less than the minimum friction coefficient measured in Figure 5(c) for a long time. This will make the sliding happen for a long time, and then reflect
the role of sliding friction. Just like the third inequality in equation (13), in this case, the ball is always subject to sliding friction, but in fact, when the tangential force of the inertia force is small enough, stable rolling can occur. We ignore this process here and think that this is a local rolling phenomenon in long-term sliding. In addition, the friction $f$ direction under this model is controlled by the direction of the tangential component of the inertia force $m\dot{a}^2 \cos(\phi - \alpha)$. This will cause the change of $f$ not to be completely continuous, but it will not affect the movement trend of the overall model. At the same time, we believe that since the friction coefficient $\mu$ is very small, sliding will occur immediately after the acceleration phase of the two-dimensional bowl. That is, the final value of the physical quantity in the acceleration stage is the initial value of the sliding physical quantity.

Figure 5 (a)(b) shows that the inversion of the centroid can also occur under the action of sliding friction, but it is unstable (such as the thick green line in (a)) and will still turn into a positive rotation after a period. And the phenomenon of reverse rotation under the condition of stable rolling will also become positive rotation due to sliding. Figure 5(c) shows that the minimum friction coefficient required for rolling also changes periodically.

Overall, for a single ball, the forward rotation and reversal of its centroid are possible, which mainly depends on the angular velocity at the end of the two-dimensional bowl acceleration and the velocity of the ball centroid currently. If the ball can do stable pure rolling, the centroid will reverse when the initial speed is negative and the value is large; If the small ball slides, the long-term sliding will make the center of mass of the small ball gradually change the movement trend, and finally tend to turn forward. When pure rolling occurs, the rotation direction of the small ball angle will be opposite to the rotation direction of the center of mass; The problem of sliding is more complex. When the ball centroid changes direction or rotation direction, the rotation direction of the ball angle will be the same as the rotation direction of the centroid, considering that the tangential force direction is constantly changing. Theoretically, the two directions are always the same, always opposite or alternating all can happen.

2.2 Multiple balls solution

Above, we mainly discussed the process of a single ball rolling and sliding in the bowl. In fact, its physical equation is complex enough, but if we consider multiple balls, strict solution is very unrealistic. We should first study the statistical law of multiple balls colliding with each other in the two-dimensional bowl, that is, how the distribution of multiple balls colliding with each other in the two-dimensional bowl is formed.

2.2.1 general equation for solving multiple balls

We still take $o$ as the reference system. In this reference system, the elastic collision between the ball and the ball is realized, and the collision between the ball and the two-dimensional bowl wall is the collision with the radial velocity declining. So, after many collisions, the balls will be concentrated on the boundary of the two-dimensional bowl. The reader can analogy a hard ball system that collides freely in a uniformly accelerated box. Over time, the balls concentrate on the box wall to obtain support, and then uniformly accelerate with the box. Considering that the collision of multi ball system is very common, in the subsequent analysis, we uniformly believe that every collision between the ball and the bowl wall will lose the same radial velocity.

However, if we do not consider the collision loss velocity between the ball and the bowl wall, but only consider the elastic collision between the balls, this is a standard hard-ball model. For such a system, it can be proved that it meets the hybrid hypothesis and global chaos\[4\]. In the case of long-term collision, according to the analysis in reference\[3\], the two-dimensional bowl area is divided into free zone, no-fly zone and correlation zone. We will find that when the radius of the ball increases, it is more likely to collide at the boundary; When the radius of the ball is small,
the probability of collision in the center of the bowl will be larger.

Now consider the above together. Due to the momentum loss caused by the collision with the bowl wall, the balls concentrate on the boundary to form clusters after long-term movement, and the clusters like a ball with larger radius, which are more likely to collide at the boundary. So, if there are enough balls and the size of the two-dimensional bowl is appropriate, almost all the balls will form a stable pellet structure at the boundary. This is also reflected in the video\textsuperscript{1}. Therefore, the small collision friction inside the pellet can be ignored, and the pellet becomes a new “ball” and is analyzed, which goes back to the problem discussed in 2.1, but the equation needs to be slightly modified, and the new physical image is shown in Figure 6(a).

We also often use the method of transforming complex problems into known problems in class like this.

\[ L(\theta, t) = M R^2 \dot{\alpha}^2 + M \rho^2 (\varphi) \dot{\varphi}^2 + 2 MR \rho (\varphi) \dot{\alpha} \dot{\varphi} \cos (\alpha - \varphi) + M R \rho (\varphi) \dot{\alpha} \dot{\varphi} \cos (\alpha - \varphi) (\sin \varphi - \cos \varphi) \]
\[ + \frac{1}{2} I (\dot{\theta} (\varphi)) \dot{\varphi}^2 \]
\[ M [\ddot{\varphi} - \dot{\varphi}^2 \rho] = M \ddot{\alpha} R \cos (\varphi - \alpha) \]
\[ - [N \cos \delta - \mu N \sin \delta] \]
\[ M [\dot{\varphi} \rho + 2 \dot{\varphi} \dot{\rho}] = \pm [\mu N \cos \delta - N \sin \delta] \]
\[ (f \cdot \mathbf{k}) F (M \ddot{\alpha} R) \cdot \mathbf{k} \leq 0 \]

In equation (14), \( M \) is the total mass of the pellet and the number of rotation proportions. \( \theta \) reflects the relationship between the pellet total-rotation angle and the pellet rotation angle. And when the pellet system is relatively simple, it is a constant. And \( I \) is the moment of inertia of the pellet relative to the centroid.

In equation (15), \( \mathbf{k} \) is the unit vector perpendicular to the connection between the contact point between the pellet and the bowl wall (the point of friction or support force) and the pellet centroid, \( F (M \ddot{\alpha} R) \) is the inertial centrifugal force, and the third inequality in equation (15) is the same as the inequality in equation (13), which must be satisfied when sliding.

2.2.2 Pellet analysis example and pellet mode

We briefly analyze a simplest 7-ball system, and assume its initial position as shown in Figure 7(a).

If the pellet is pure rolling along the bowl wall, it is obvious that if the system still has only one degree of freedom, at this time, \( \delta \) and \( \rho \) and the pellet total-rotation angle \( \theta \) can be reduced to a function of \( \varphi \). The system can still be solved by simple Lagrangian. The Lagrangian is more complex, as shown in equation (14). If the pellet slides for a long time, the degree of freedom of the system increases, and the dissipative generalized force is involved, we still write the equation of motion of the centroid in accordance with equation (15) in 2.1.2.

\[ L(\varphi, t) = MR^2 \dot{\alpha}^2 + M \rho^2 (\varphi) \dot{\varphi}^2 + 2 MR \rho (\varphi) \dot{\alpha} \dot{\varphi} \cos (\alpha - \varphi) + M R \rho (\varphi) \dot{\alpha} \dot{\varphi} \cos (\alpha - \varphi) (\sin \varphi - \cos \varphi) \]

From the geometric relationship in Figure 7(b), the relationship between physical quantities and \( \varphi \) in pure rolling can be derived as follows.

First, \( \theta \) changes periodically with respect to \( \varphi \)
As shown in Figure 8(a), in a single period, it is a symmetric function, and the expression of half period is as follows

\[ \varphi = \arcsin \left( \frac{2l \sin \vartheta}{(r-l)^2 + 4l^2 - 4(r-l)lc \cos \vartheta} \right) \]

\[ + \frac{l}{r} \vartheta \approx \left( \frac{2l}{r-3l} + \frac{l}{r} \right) \vartheta = k \vartheta \]

\[ \vartheta(\varphi) = \frac{1}{k} \varphi = k \varphi \]

On the one hand, due to \( \vartheta = \arccos \left( \frac{l}{r-l} \right) - \pi/3 \) and \( r > 3l \), the closer \( r \) is to \( 3l \), the larger the two-dimensional bowl space occupied by the pellet, \( \vartheta \) can be approximated by small angle (“Approximately equal sign” in the above equation). It is easy to deduce \( \vartheta(\varphi) \) after approximation. On the other hand, small \( \vartheta_0 \) means close \( r \) and \( 3l \), which will increase the error. Therefore, it is not that the closer the two are, the more suitable the small angle approximation is.

However, whether it is similar or not, we have the following relationships.

\[ I = \frac{134}{35} Ml^2 \]

\[ \Theta = \left( \frac{r-l}{r} \right)^2 \]

\[ \delta(\varphi) = \frac{l}{r} \vartheta(\varphi) \]

\[ \rho(\varphi) = \left[ (r-l)^2 + 4l^2 - 4(r-l)lc \cos \vartheta(\varphi) \right]^{1/2} \]

The above relationship can be brought into equation (14) for solution.

As shown in Figure 8(a), it can be seen that \( l=0.01m \) is better than \( l=0.015m \) (red line group is better than blue line group) when the \( \varphi \) angle is small.

This also shows that the small angle approximation should be considered comprehensively. However, the two groups of lines in the trajectory diagram in Figure 8(b) almost coincide, it shows that the reflection on the trajectory is consistent in a short time. Although there are differences in the solution. It can be used to simplify the analysis of problems.

For the sliding problem, the equation involves many unknowns, and the solution is very complex, so it will not be discussed in detail here. The general analysis method is the same as above, that is, after changing the reference system, consider the whole as a pellet, especially consider its centroid motion equation, and then solve it. As for forward rotation or reverse rotation, in the case of stable pellet, the pellet is like a ball with more complex geometric properties, and the macroscopic property of the pellet is similar to a single ball, so the conclusion is consistent with the results discussed in 2.1.

The following is a brief list of several pellet accumulations and related data. It is not difficult
for readers to find that these accumulations are dense accumulations in two-dimensional space. The phenomenon of multiple balls in the video\textsuperscript{[1]} is also the densest accumulation formed by seven balls.

\begin{align*}
\text{n} = 7 & \quad I = \frac{121}{30} M R^2 \\
\text{n} = 13 & \quad I = \frac{906}{65} M R^2 \\
\text{n} = 19 & \quad I = \frac{908}{66} M R^2 \\
\text{n} = 31 & \quad I = \frac{2702}{185} M R^2 \\
\text{n} = 37 & \quad I = \frac{3794}{185} M R^2 \\
\text{n} = 49 & \quad I = \frac{5846}{245} M R^2 \\
\text{n} = 55 & \quad I = \frac{7298}{275} M R^2 \\
\text{n} = 61 & \quad I = \frac{1846}{61} M R^2
\end{align*}

FIG 9 Some common pellet accumulation is related to their number of balls and the moment of inertia relative to their own centroid. $M$ is its total mass, and $R$ is the radius of each ball.

2.3 summary of angular momentum

When the number of balls is scarce, it is easy for the ball to contact the boundary of the two-dimensional bowl, and we can analyze the ball in isolation; When the number of balls is enough, the balls will gradually gather to form pellet because of the effect of the collision and the limitation of the two-dimensional bowl. At this time, it can still be analyzed as a single ball with complex geometric properties.

As for forward rotation and reversal, when the friction coefficient $\mu$ is large, the ball or pellet can do stable pure rolling, and the positive reversal of the centroid is related to the initial velocity of the centroid, when the two-dimensional bowl starts to rotate at a uniform speed (when solving relevant problems, the initial conditions of equations like formula (7)(8) can be flexibly determined according to the situation), and both may occur. The rotation angle of the ball itself is fixed opposite to the rotation direction of the centroid; When the friction coefficient $\mu$ is small, the ball or pellet slides for a long time. At this time, even if there is an initial reversal of the centroid relative to the reference system, after a period, it will change the direction of motion and become a positive rotation. However, there is no fixed relationship between its rotation direction and the rotation direction of the centroid.

Now let's refer to the phenomenon in the video\textsuperscript{[1]}, in which the so-called positive reversal is ambiguous, which also leads to a seemingly strange phenomenon that few small balls - the same direction; Many small balls - the opposite directions. In fact, through careful observation, it can be found that the motion of a single ball and the centroid of the pellet are consistent with the rotation direction of the two-dimensional bowl and can be regarded as the pure rolling situation due to the large friction coefficient $\mu$. However, with the increase of the pellet volume, the audience's attention is more focused on the rotation of the pellet itself than the movement of the pellet centroid. From the above analysis, in the case of pure rolling, the rotation angle of the pellet itself is fixed opposite to the macroscopic rotation direction of the centroid. Then the illusion that the ball is more inverted is produced. The fact is that the center of mass is always forward, and the ball or ball itself is always reversed. This is exactly what is mentioned in the introduction is not completely correct. At the same time, it also explains that the video\textsuperscript{[1]} later said that the small ball in the center of the pellet is rotating forward, and the whole looks like reversal.

At present, we have made it clear that the so-called inversion in the video\textsuperscript{[1]} is more like a kind of visual illusion, which has no direct relationship with the number of balls.

But our analysis is not over. Now there are mainly the following two problems:

1. When there are too few or too many balls, the role of the ball and the bowl wall can be analyzed as a single ball with the bowl wall. However, when the number of balls gradually increases and is not dense (hereinafter referred to as loose system), we will have to consider the frequent collisions between balls and will not take balls as a pellet. So,
what is the situation currently?

2. In 2.1, it is required that balls or pellet can be in long-term contact with the boundary. When there are many balls, regardless of the initial position of each ball in the bowl, pellet will be formed near the two-dimensional bowl wall due to collision, and the pellet meet the long-term contact with the boundary. However, a single ball or a small number of balls do not necessarily meet the above conditions at the beginning. So, what is the situation currently?

These will be discussed in the next section.

3. Aspect of momentum

In this part, collision will become the main factor affecting the ball movement. Currently, it is better to take the two-dimensional bowl as the reference system. We first consider the problem that a single ball does not touch the wall for a long time. In the analysis, we first remove the constraint of the ball, that is, the ball does not have to be in the two-dimensional bowl all the time and analyze the trajectory of the centroid in this case; Then a single-sided constraint is applied, in which case the ball will collide with the bowl wall and lose the radial velocity (in the view of the two-dimensional bowl reference system). Considering multiple collisions, we believe that each radial velocity loss is uniform. Considering the collision of multiple balls, the unilateral constraint of the two-dimensional bowl wall must be imposed at this time. This system can be analyzed as a system like the hard-ball system.

In this part, we omit the tedious mathematical derivation and adopt a more intuitive simulation method to enable readers to grasp the general laws of the loose system. The physical model we determined at this time is Figure 10. Of course, in classroom teaching, this example can well cooperate with the teaching of v-verlet algorithm in computational physics. We also use this method in the following simulation.

![Diagram](image.png)

**FIG 10** The physical model of the motion of a ball in a two-dimensional bowl with \( O \) as the reference system. (At this time, the two-dimensional bowl rotates in parallel at a uniform speed)

3.1 Analysis of single ball

(16) is the equation when the ball moves freely in a two-dimensional bowl without contacting the bowl wall with \( O \) as the reference system. At every moment, the inertial centrifugal force can be regarded as a conservative force. In this case, we can write the potential field (17) in the two-dimensional bowl according to \( F(t) = -\nabla \Gamma(t) \), and the potential field changes with time. This potential field is caused by non-conservative force, so Lagrangian quantities cannot be written, and we also use \( \Gamma(t) \) to distinguish.

\[
\begin{align*}
  F_\rho &= -m\dot{\alpha}^2 R \sin(\theta - \alpha) \\
  F_\theta &= -m\dot{\alpha}^2 R \cos(\theta - \alpha)
\end{align*}
\]  

(16)

\[
\Gamma(t) = m\dot{\alpha}^2 R \left[ y \cos(\alpha(t)) - x \sin(\alpha(t)) \right]
\]  

(17)

In fact, \( \Gamma(t) \) can add or subtract any constant, and we choose to make the potential energy of center point \( O \) constant to 0. The Lagrangian and Hamiltonian of the system at each time \( t_n \) are equations(18) and (19) respectively.

In teaching, it should be emphasized to students that equation(18)(19) describes the Lagrangian and Hamiltonian of the system at each moment \( t_n \). When they are brought into the Lagrangian equation and Hamiltonian equation, the \( t_n \) needs to be regarded as a constant, so it cannot well reflect the real operation of the centroid. The commonly used method to solve this problem is to introduce non-potential generalized force to solve it or induct the solution results at each time \( t_n \).
\[ L_{t_n}(x, y) = \frac{1}{2} m(x^2 + y^2) \]
\[ -m\dot{a}^2 R[y \cos(a(t_n)) - x \sin(a(t_n)) ] \]
\[ H_{t_n}(p_x, p_y, x, y) = \frac{1}{2} \left( \frac{p_x^2}{m} + \frac{p_y^2}{m} \right) \]
\[ + m\dot{a}^2 R[y \cos(a(t_n)) - x \sin(a(t_n)) ] \]  \hspace{1cm} (18)

In order to simplify the analysis, we set \( a(t) = \omega t \). at this time, and the initial value of \( \theta \) is \( \frac{3\pi}{2} \) and bring equation(19) into the Hamilton canonical equation, and after induction, we can get the dynamic equation of the centroid about the Cartesian coordinates in the two-dimensional bowl, such as equation(20). However, the mechanical energy of this system is not conserved. Can we calculate the conserved quantity of the system by Hamilton canonical transformation of a set of coordinates? According to the Noether theorem of unilateral constraint and the method in reference[6], we solve the conservation relationship of the ball when it does not interact with the bowl wall in a two-dimensional bowl, such as equation(21) and the corresponding infinitesimal coordinate transformation, such as equation(22).

\[
\begin{align*}
    x &= -R\sin(\omega t) + v'(x)t + x_0 \\
    y &= R\cos(\omega t) + v'(y)t + y_0 - R \\
    \sin(\omega t) \left( \frac{\dot{x}x + \dot{y}y}{x} \right) + \cos(\omega t) \left( \frac{\dot{x}x - \dot{y}y}{y} \right) &= \text{const}
\end{align*}
\]  \hspace{1cm} (20)

\[
\begin{align*}
    t^* &= t + \epsilon \\
    x^* &= x - \epsilon \left( \frac{\dot{y}y}{x} \right) \\
    y^* &= y + \epsilon \left( \frac{\dot{x}x}{y} \right)
\end{align*}
\]  \hspace{1cm} (21)

In equation(20), \( v'(x) \) and \( v'(y) \) are the initial velocities of the centroid of the ball relative to the reference system \( O' \). \( x_0 \) and \( y_0 \) are the initial positions of the centroid of the ball relative to the \( O \) reference system. From equation(21), the conservation of physical quantities in this system is relatively complex.

We make the trajectory of the ball when the collision of the container wall is not considered, as shown in Figure 11. When the ball collides with the bowl wall, we believe that it satisfies the uniform loss of radial velocity, that is, the velocity change of each collision satisfies equation(23). Where \( \dot{r}_B \) and \( \dot{r}_A \) are the radial velocities before and after collision with the bowl wall respectively.

\[
\begin{align*}
    -\dot{r}_A &= e\dot{r}_B \quad e \in (0,1) \\
    \dot{r} &= \frac{\dot{x}x + \dot{y}y}{\sqrt{x^2 + y^2}}
\end{align*}
\]  \hspace{1cm} (23)
FIG 11 The four lines of blue, orange, green and red in (a) are the trajectories of the centroid of the ball in the $O'$ system when there are different initial velocities in the $O'$ reference system; The purple line has no initial velocity, and all the trajectories in (a) has no bowl wall constraint. (b)(c)(d)(e) is to simulate the trajectory of a single ball in the $O$ reference system at different time. At the same time, the radius of the ball is changed from 0.015m to 0.005m to facilitate visualization. Similarly, after changing the radius of ball, Adams modeling is used. (i)(f)(g)(h) are the images of the evolution of the distance between the ball centroid and $O'$ point with time after changing the stiffness coefficient (5000, 10000and15000 in turn).

As can be seen from Figure 11(a), when the centroid has an initial velocity relative to the $O'$ reference system without the constraint of the bowl wall, the motion mode of the centroid is the superposition of its own rotation and the uniform linear motion of the initial velocity. And its own rotation is always positive; If there is no initial velocity, the trajectory is like the purple line in the figure, which is a closed circle obviously. Figure 11(b)(c)(d)(e) show that when there is unilateral constraint on the boundary, the ball will convert to close to the wall after a period of evolution due to radial velocity loss, accompanied by smaller and smaller rotation. This is also proved by Adams’ simulation as shown in Figure 11(f)(g)(h). The ball periodically moves away from point $O$, but with the time evolution and the reduction of stiffness coefficient, the range of the small ball away from point $O$ becomes smaller and smaller, and finally tends to close to the wall.

3.2 Analysis of multiple balls

We assume that the system has $N$ balls, which form a hard-ball system under a container reference system that is not subject to any action other than the collision contact force and all collisions are completely elastic collisions. Such a system satisfies the hybrid hypothesis$^{[4,5]}$, but now in the system we are considering ($O$ reference system), each ball is subject to an inertial centrifugal force with a direction varying with time, relatively speaking, the centroids of these balls will form some curves with certain trajectories in the two-dimensional bowl space. Considering the two-dimensional bowl, the phase space of this system is a $4N$ space of dimension, so the path they form in the phase space is $\mathcal{F}(x) = 0$. The density function of these points in the phase space can be expressed as $ho(x)=C(x)\delta[\mathcal{F}(x)]$, and the coefficient $C(x)$ can be determined by the normalization relationship. The average value of the field of physical quantity $x_i(x)$ in phase space under long-term evolution can also be simply calculated by equation(24).

$$\langle x_i(x) \rangle = \int \rho(x)x_i(x)d^{4N}x$$ (24)

However, when we analyze the collision of multiple balls, the number of balls will not be too large (otherwise, it will return to the situation in 2.2). The balls can also be regarded as discrete particles. Therefore, we can directly use the analysis method of discrete system.

We now consider the collision between balls and consider it to be a completely elastic collision, which satisfies the following expression.

$$\begin{cases}
v_{1A} = v_{1B} + \frac{2m_2}{m_1 + m_2} \left[(v_{2B} - v_{1B}) \cdot \frac{r}{|r|^2} \right]r \\
v_{2A} = v_{2B} - \frac{2m_1}{m_1 + m_2} \left[(v_{1B} - v_{2B}) \cdot \frac{r}{|r|^2} \right]r
\end{cases}$$

Where $v_{1A}$, $v_{2A}$, $v_{1B}$ and $v_{2B}$ are the velocity of two balls before and after collision respectively($A$ represents before the collision, $B$ represents after the collision). $r$ is the vector connecting the centroids of the two spheres, with the direction from 1 to 2.

Momentum is conserved before and after the collision. We assume that for a period after the beginning, ball $q$ collided with the two-dimensional bowl wall $j_q$ times, And the radial velocity still satisfies the relationship of equation(23). Then the mean square average value of the total radial velocity of the ball at time $t_0$ is equation(25).

$$\langle \dot{r}^2 \rangle |_{t_0} = \frac{1}{N} \cdot \sum_{q=1}^{N} j_q \cdot \left[ e^2 \cdot \left(\dot{x}x + \dot{y}y\right)^2 \right]_{t_0}$$ (25)

With the increase of time value, the number of collisions increases. It is obvious that the mean square average value of $\dot{r}^2$ will approach to 0, that is, the ball will lose its radial velocity. And the only way to lose the radial velocity is the collision with the bowl wall, so at this time, most of
the balls will be close to the boundary of the bowl, which is the same as the analysis of a single ball in 3.1, and the simulation results as shown in Figure 12 also prove this conclusion.

FIG 12 The image of multiple ball collisions evolving over time, from left to right, from top to bottom. At the same time, the radius of the small ball is changed from 0.015m to 0.005m to facilitate visualization.

With the evolution of time, the balls will slowly gather on the two-dimensional bowl wall and collide with each other. After that, the ball that basically loses its radial velocity can be approximately regarded as the motion in the case of 2.1. Currently, there is no clear positive and reverse trend, which is indeed consistent with the situation that there is no macro trend during the increase of the number of balls in the video[1].

When the size of the ball becomes larger, some balls are indeed unable to contact the boundary of the two-dimensional bowl. However, from the perspective of long-term evolution, the ball will fill the bowl wall as much as possible, and then the ball moving next to the wall will form a new bowl wall and repeat the above rule, so this does not affect our conclusion of analyzing this problem. That is, the ball will lose a clear macro positive and reverse trend.

4. Conclusion

Generally, the rotation of balls in the bowl is divided into three processes. The first is that there are only a few balls, the second is that there are more balls, and the third is that there are many balls that almost fill the whole container.

When the number of balls is very small, the balls can be regarded as independent of each other. Because of the collision with the two-dimensional bowl wall, the balls will slowly change into the wall adhering motion. At the beginning of the wall adhering motion, the tangential velocity of the centroid determines the positive and reverse of the centroid; At the same time, the size of the friction coefficient $\mu$ of the bowl wall determines whether it is rolling or sliding, which affects whether the rotation direction of the ball itself is consistent with the rotation direction of the centroid. See the discussion in 2.3 for details.

When there are more balls, there are frequent collisions between balls, and at the same time, the balls are also colliding with the bowl wall. Finally, the balls will move close to the bowl wall, and the collision between balls will offset the macro positive and reverse trend of the centroid of a single ball, and the balls as a whole show irregular movement.

When the number of balls is so large that it almost fills the whole container, the balls will gather into a pellet. The impact caused by the collision inside the pellet is very weak, so it can be analyzed again as the case when the number of balls is very small.

Based on the above phenomena, we find that the simple bowl-ball collision model is very viable. In the theoretical mechanics level, it can involve some more complex differential equations with nonlinear terms, such as equation(7)(8); It also provides a very good case in numerical calculation; At the level of statistical physics, if we can advance the hard-ball model or find a better
conserved quantity of the hard-ball system under external forces, we may be able to calculate a more detailed ensemble to describe it.

At the same time, from the superficial description to the detailed calculation and analysis of the ball inversion phenomenon in the video\cite{1}, we also mentioned that the simple explanation in the video is not completely correct, that is, it does not accurately describe the key factors of the positive reversal; We have established a variety of models to simulate and analyze, and jointly overcome this problem from the perspective of angular momentum and momentum. On the one hand, it explains the important value of the model in the analysis of physical problems; On the other hand, it shows that even a simple phenomenon, in order to understand its essential law, we often can't judge only by intuition, but also need to draw a rational conclusion through specific analysis. And these are enlightening for classroom teaching.

Appendix

1. In the discussion in the full text, if there is no special description, all the positive and negative rotation of the centroid take the center of the two-dimensional bowl as the reference system, and the counterclockwise is positive by default. That is, forward rotation is counterclockwise and reverse rotation is clockwise.

2. If there is no special description change in all simulations, the following data and meanings are used for physical quantities:
   - $R=0.02 \text{m}$
     Radius of the parallel-rotation of two-dimensional bowl
   - $r=0.065 \text{m}$
     Radius of two-dimensional bowl
   - $l=0.015 \text{m}$
     Radius of balls
   - $\beta=52.36 \text{rad/s}^2$
     Angular acceleration of two-dimensional bowl rotation in $O'$ system during acceleration
   - $\omega=\beta \tau$
     Angular velocity of a two-dimensional bowl when it starts to do parallel-rotation at a uniform speed
   - $e=0.02$
     Radial coefficient of restitution when a ball collides with a two-dimensional bowl in the $O$ reference system

   
   $O$: Take the center of the two-dimensional bowl as the reference system.
   
   $O'$: Taking the parallel-rotation center of the two-dimensional bowl as the reference system (or ground system).

reference

[1]\url{https://www.bilibili.com/video/BV1ms411s7jt?spm_id_from=333.999.}
( VPN may be required)

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