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Integral formulas for a Dirichlet series

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Abstract

We present an integral representation formula for a Dirichlet series whose coefficients are the values of the Liouville’s arithmetic function.
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1 Introduction

Let $\sum_{n=1}^{\infty} a(n) \frac{n}{n^s}$ be a Dirichlet series such that:
- its analytic continuation is a meromorphic function with only one pole at $s = 1$
- there is a functional equation looking like:

$$\sum_{n=1}^{\infty} a(n) \frac{n}{n^s} = \varphi(s) \sum_{n=1}^{\infty} \frac{b(n)}{n^{1-s}}$$

This gives a sequence $(b(n))$ allowing us to write a pseudo-cotangent or a pseudo-tangent function similar to a cotangent or a tangent function:

$$\sum_{n=0}^{\infty} \frac{b(n)}{z^2 + (2n + 1)^2 \pi^2}$$

We prefer to choose a tangent function because the cotangent function has a singularity at the origin, hence some trouble to get a power series.

It may be possible to deduce from the functional equation an integral formula for the starting Dirichlet series. Now we can hope to find a sequence $(c(n))$ allowing us to get an extension of this integral formula by a modification of the pseudo-tangent such as:

$$\sum_{n=1}^{\infty} c(n) \frac{1}{e^{z/n} + 1}$$

An easy example is the Riemann’s $\zeta$ function itself, the three sequences are:

$(a(n)) = (1, 1, 1, 1, ...)$ \hspace{1cm} $(b(n)) = (1, 1, 1, 1, ...)$ \hspace{1cm} $(c(n)) = (1, 0, 0, 0, ...)$

Another simple example is the Dirichlet series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$:

$(a(n)) = (1, -1, 1, -1, ...)$ \hspace{1cm} $(b(n)) = (1, 0, 1, 0, ...)$ \hspace{1cm} $(c(n)) = (1, 0, 0, 0, ...)$

This is a general program. Here we take a particular case: a Dirichlet series equivalent to the Dirichlet series whose coefficients are the values of the Liouville arithmetic function. We obtain a representation integral formula.

2 Some functions associated with the Riemann’s $\zeta$ function

2.1 The functions $\zeta$, $\zeta_a$, $\zeta_{imp}$.

The following Dirichlet functions are well known:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \Re(s) > 1$$ (1)

$$\zeta_a(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \quad \Re(s) > 1$$ (2)

$$\zeta_{imp}(s) = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^s} \quad \Re(s) > 1$$ (3)
The links with the $\zeta$ function are easy:

$$\zeta(s) = \frac{1}{1 - 2^{-s}} \zeta_{a}(s)$$  \hspace{1cm} (4)$$

$$\zeta(s) = \frac{1}{1 - 2^{-s}} \zeta_{imp}(s)$$  \hspace{1cm} (5)$$

Cf, for example [6] .

### 2.2 The functions $\zeta_{\lambda}$, $\zeta_{\mu}$, $\zeta_{\alpha}$.

Let $\lambda$ be the Liouville’s arithmetic function: $\lambda(1) = 1$; for a prime $p$, $\lambda(p) = -1$; for all $a$ et $b$, $\lambda(ab) = \lambda(a)\lambda(b)$.

Let $\zeta_{\lambda}$ be the corresponding Dirichlet function:

$$\zeta_{\lambda}(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \quad \Re(s) > 1$$  \hspace{1cm} (6)$$

$$\zeta_{\lambda}(s) = \frac{\zeta(2s)}{\zeta(s)}$$  \hspace{1cm} (7)$$

$\zeta_{\lambda}$ is a meromorphic function on $\mathbb{C}$.

Let $\mu$ be the Möbius arithmetic function:

$$\zeta_{\mu}(s) = \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad \Re(s) > 1$$  \hspace{1cm} (8)$$

$\zeta_{\lambda}$ and $\zeta_{\mu}$ are also well known, cf [6] .

The singular points for $\zeta_{\mu}$ are the zeros of $\zeta$. But $\zeta_{\lambda}$ has no singular points outside the domain $0 \leq \Re(s) \leq 1$. In that domain, its singular points are the zeros of the $\zeta$ function, except for $s = 1$.

Let $\zeta_{\alpha}$ be:

$$\zeta_{\alpha}(s) = \frac{\zeta_{a}(2s)}{\zeta_{a}(s)}$$  \hspace{1cm} (9)$$

With (4) we get:

$$\zeta_{\lambda}(s) = \frac{1 - 2^{1-s}}{1 - 2^{1-2s}} \zeta_{\alpha}(s)$$  \hspace{1cm} (10)$$

We do not need the arithmetic function $\alpha$ such that:

$$\zeta_{\alpha}(s) = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s}$$

### 2.3 The function $\zeta_{\beta}$.

Let $\zeta_{\beta}$ be:

$$\zeta_{\beta}(s) = \frac{\zeta_{imp}(2s-1)}{\zeta_{imp}(s)}$$  \hspace{1cm} (11)$$

With (5) we have:

$$\zeta_{\beta}(s) = \frac{(1 - 2^{1-2s})\zeta(2s-1)}{(1 - 2^{-s})\zeta(s)}$$  \hspace{1cm} (12)$$
ζ_β is the generating function of an arithmetic function β:
\[
\zeta_\beta(s) = \sum_{n=1}^{\infty} \frac{\beta(n)}{n^s} \quad \Re(s) > 1
\] (13)

According to a theorem of Newman, cf [3], the following series is convergent and its value is:
\[
\sum_{n=0}^{\infty} \frac{\mu(2n+1)}{2n+1} = 0
\] (14)

Let (2n+1) be an odd number. There is a unique decomposition in a factor without square and a square:
\[
\begin{align*}
2n+1 &= kh^2 \\
\beta(2n+1) &= \mu(k)h \\
\beta(2n+1) &= h
\end{align*}
\] (15)

We have the estimate:
\[
-1 < \frac{\beta(2n+1)}{\sqrt{2n+1}} \leq 1
\] (16)

The equality is true if and only if 2n + 1 is a square.

The Dirichlet series:
\[
\zeta_\beta(s + 1/2) = \sum_{n=0}^{\infty} \frac{\beta(n)}{n^{s+1/2}} = \sum_{n=1}^{\infty} \frac{\beta(n)}{\sqrt{n} \cdot n^s}
\] (17)

is, following (13) convergent for \( \Re(s) > 1/2 \).

The following inequality is useful:
\[
\sum_{n=1}^{\infty} \frac{|\beta(n)|}{n^{3/2}} \leq \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \sum_{h=1}^{\infty} \frac{1}{h^2}
\] (18)

### 2.4 The function \( \zeta_\nu \).

Let \( \zeta_\nu \) be:
\[
\zeta_\nu(s) = \frac{1}{\zeta_{\text{imp}}(s+1)} \zeta_{\text{imp}}(2s+2) = \frac{\zeta_\beta(s+3/2)}{\zeta_{\text{imp}}(s+1)}
\] (19)

It is the generating function of an arithmetic function \( \nu \):
\[
\zeta_\nu(s) = \sum_{n=1}^{\infty} \frac{\nu(n)}{n^s} \quad \Re(s) > 0
\] (20)

Of course, \( \nu \) is zero on even integers: \( \nu(2m) = 0 \).
\[
\sum_{l|(2n+1)} l \nu(l) = \frac{\beta(2n+1)}{\sqrt{2n+1}}
\] (21)

the Möbius formula gives:
\[
(2n+1)\nu(2n+1) = \sum_{kl=2n+1} \mu(k)\frac{\beta(l)}{\sqrt{l}}
\] (22)

Let \( d \) be the arithmetic function \( d(n) = \) number of divisors of \( n \). We have an estimate for all integers \( m \):
\[
|\nu(m)| \leq \frac{d(m)}{m}
\] (23)
Theorem 1. The series whose terms are $\nu(n)$ converge and the value is 0.

$$\sum_{n=1}^{\infty} \nu(n) = 0.$$ (24)

Proof. Use (22), (18) and (14). $\square$

This result is very important for the present work.

3 Integral formula for the $\zeta_a$ function

3.1 The kernel $\frac{1}{e^z+1}$.

The kernel $\frac{1}{e^z+1}$ is better than $\frac{1}{e^z-1}$ because we do not have a singularity at the origin. We have some classical expansions:

$$\frac{1}{e^t+1} = \sum_{n=1}^{\infty} (-1)^{n-1} e^{-nt} \quad (t > 0)$$

$$\frac{1}{e^z+1} = \frac{1}{2} - 2z \sum_{n=0}^{\infty} \frac{1}{z^2 + (2n+1)^2 \pi^2}$$ (25)

Now, we take $|z| < \pi$ for convergence.

$$\frac{1}{e^z+1} = \frac{1}{2} - 2\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{\pi^{2k+2}} \zeta_{imp}(2k+2)$$ (26)

$$\frac{1}{e^z+1} = \frac{1}{2} - 2\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{\pi^{2k+2}} \zeta_{imp}(2k+2)$$ (27)

3.2 Integral representation formula for $\zeta_a$.

It is well known that:

$$\Gamma(s)\zeta_a(s) = \int_{0}^{\infty} \frac{1}{e^t+1} t^{s-1} dt \quad \Re(s) > 0$$ (28)

The continuation of this integral representation is possible by taking:

$$\frac{1}{e^t+1} - \frac{1}{2}$$

4 Functional equations

4.1 Functional equation between $\zeta_a$ and $\zeta_{imp}$.

For $\zeta$, the following functional equation of Riemann is well known cf [6]:

$$\zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi}{2} s \right) \Gamma(1-s) \zeta(1-s)$$ (29)

thence a functional equation between $\zeta_a$ and $\zeta_{imp}$:

$$\zeta_a(s) = -2\pi^{s-1} \sin \left( \frac{\pi}{2} s \right) \Gamma(1-s) \zeta_{imp}(1-s)$$ (30)
4.2 Functional equation between $\zeta_\alpha$ and $\zeta_\beta$.

From (30):

$$\frac{\zeta_\alpha(2s)}{\zeta_\alpha(s)} = \frac{-2\pi^{2s-1}\sin(\pi s)\Gamma(1-2s)\zeta_{\text{imp}}(1-2s)}{-2\pi^{s-1}\sin(\frac{\pi}{2} s)\Gamma(1-s)\zeta_{\text{imp}}(1-s)}$$

Hence (cf [2] for the duplication formula) a first form for the functional equation between $\zeta_\alpha$ and $\zeta_\beta$:

$$\zeta_\alpha(s) = 2^{1-2s} \pi^{s-1/2} \cos\left(\frac{\pi}{2} s\right) \Gamma\left(\frac{1}{2} - s\right) \zeta_\beta(1-s)$$

(31)

And a second form, but only for $\Re(s) < 0$:

$$\zeta_\alpha(s) = 2^{1-2s} \cos\left(\frac{\pi}{2} s\right) \Gamma\left(\frac{1}{2} - s\right) \sum_{m=0}^{\infty} \frac{\beta(2m+1)}{\sqrt{2m+1} (\pi(2m+1))^{1/2-s}}$$

(32)

5 Integral formulas

5.1 Integral formula for $\zeta_\alpha$ in the domain $-3/2 < \Re(s) < -1/2$.

The functional equation (32) gives an integral for $\zeta_\alpha$ in the domain $-3/2 < \Re(s) < 0$:

$$\zeta_\alpha(s) = 2^{1-2s} \cos\left(\frac{\pi}{2} s\right) \Gamma\left(\frac{1}{2} - s\right) \frac{2}{\pi} \cos\left(\frac{\pi}{2} s + \frac{\pi}{4}\right) \sum_{m=0}^{\infty} \frac{\beta(2m+1)}{\sqrt{2m+1}} \int_0^{\infty} \frac{x^{s+1/2}}{x^2 + \pi^2(2m+1)^2} dx$$

Now, we want to permute the summation and the integral. To do this we have only to prove absolute integrability, but for $-3/2 < \Re(s) < -1/2$. Let $\sigma = \Re(s)$, take the inequality (16):

$$\int_0^{\infty} 2x \sum_{n=1}^{N} \frac{\beta(n)}{\sqrt{n(x^2 + \pi^2 n^2)}} |x^{s-1/2}| dx \leq \int_0^{\infty} \left(\frac{1}{2} - \frac{1}{e^x + 1}\right)x^{\sigma-1/2} dx \quad -3/2 < \Re(s) < -1/2$$

We get an integral formula for $\zeta_\alpha$ in $-3/2 < \Re(s) < -1/2$:

$$\zeta_\alpha(s) = 2^{1-2s} \pi \cos\left(\frac{\pi}{2} s\right) \cos\left(\frac{\pi}{2} s + \frac{\pi}{4}\right) \Gamma(1/2 - s) \int_0^{\infty} 2x \sum_{m=0}^{\infty} \frac{\beta(2m+1)}{\sqrt{2m+1}(x^2 + \pi^2(2m+1)^2)} x^{s-1/2} dx$$

(33)

Let:

$$\varphi(s) = \frac{2^{1-2s}}{\pi} \cos\left(\frac{\pi}{2} s\right) \cos\left(\frac{\pi}{2} s + \frac{\pi}{4}\right) \Gamma(1/2 - s)$$

$$N(x) = 2x \sum_{m=0}^{\infty} \frac{\beta(2m+1)}{\sqrt{2m+1}(x^2 + \pi^2(2m+1)^2)}$$

Write (33) as:

$$\zeta_\alpha(s) = \varphi(s) \int_0^{\infty} N(x)x^{s-1/2} dx \quad -3/2 < \Re(s) < -1/2$$

(34)

Now, the aim is to prove this formula for a greater domain.
5.2 The meromorphic function $\mathcal{N}$.

$$\mathcal{N}(z) = 2z \sum_{m=0}^{\infty} \frac{\beta(2m+1)}{\sqrt{2m+1}(z^2 + \pi^2(2m+1)^2)}$$

is a meromorphic function in $\mathbb{C}$.

All the poles are simple at $i\pi(2m+1)$ for $m \in \mathbb{Z}$. The residu is:

$$\frac{\beta(2m+1)}{\sqrt{2m+1}}$$

The expansion of $\mathcal{N}$ in a power series, in a neighborhood of zero is:

$$\mathcal{N}(z) = 2 \sum_{k=0}^{\infty} \frac{\zeta(2k+5/2)}{\pi(2k+2)^{2k+1}}$$

(36)

5.3 Definition of the meromorphic function $\mathcal{M}$.

Let $\mathcal{M}$ be the following meromorphic function in $\mathbb{C}$.

$$\mathcal{M}(z) = \sum_{m=0}^{\infty} \nu(2m+1) \left( \frac{1}{2} - \frac{1}{e^{z/(2m+1)}+1} \right)$$

(37)

All the poles are simple at $i(2l+1)\pi$ for $l \in \mathbb{Z}$. The residu of $\mathcal{M}$ is:

$$\sum_{(2m+1)(2l+1)} (2m+1)\nu(2m+1) = \frac{\beta(2l+1)}{\sqrt{2l+1}}$$

because of (21). We obtain the same poles and the same residus. Of course, this does not give the equality between $\mathcal{N}$ and $\mathcal{M}$.

Theorem 1, (24) and the previous definition (37) of $\mathcal{M}$ give us:

$$\mathcal{M}(z) = - \sum_{m=0}^{\infty} \nu(2m+1) \frac{1}{e^{z/(2m+1)}+1}$$

(38)

5.4 Behavior of $\mathcal{M}$ at infinity.

By Abel’s summation by parts on (37), and theorem 1, (24), we get:

$$\lim_{x \to \infty} \mathcal{M}(x) = 0$$

(39)

5.5 A bound for the derivative $\mathcal{M}'$.

We can derive term by term the series of $\mathcal{M}(z)$. From (38), we get:

$$\mathcal{M}'(z) = \sum_{m=0}^{\infty} \nu(2m+1) \frac{e^{z/(2m+1)}}{2m+1} \frac{1}{(e^{z/(2m+1)}+1)^2}$$

(40)

Now, take this for $x$ real positive. There exists a constant $C$ such that for all $x \in [0, +\infty[$:

$$| \mathcal{M}'(x) | \leq C$$

(41)

5.6 A better bound for $\mathcal{M}$ at infinity?

Is it true that there exits a constant $C$ such that:

$$| \mathcal{M}(x) | \leq \frac{1}{x} C$$

(42)
5.7 Identity between \( \mathcal{M} \) and \( \mathcal{N} \).

The purpose is to prove that in a neighborhood of 0, we have:

\[
\mathcal{M}(z) = \mathcal{N}(z)
\]

Starting from (27) we get for \(| z | < \pi\):

\[
\sum_{n=0}^{\infty} \nu(2n+1)\left(\frac{1}{2} - \frac{1}{e^{z/(2n+1)} + 1}\right) = \sum_{n=0}^{\infty} \nu(2n+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{\pi^{2k+2}} \zeta_{\text{imp}}(2k+2) \frac{z^{2k+1}}{(2n+1)^{2k+1}}
\]

We can switch the summations because we have absolute convergence. Hence:

\[
\sum_{n=0}^{\infty} \nu(2n+1)\left(\frac{1}{2} - \frac{1}{e^{z/(2n+1)} + 1}\right) = 2 \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{\pi^{2k+2}} \zeta_{\text{imp}}(2k+2) \zeta_{\nu}(2k+1)
\]

With (19), we get the power series of \( \mathcal{M} \) at the origin:

\[
\mathcal{M}(z) = 2 \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{\pi^{2k+1}} \zeta_{\beta}(2k+5/2) \quad | z | < \pi
\]

This is exactly (36) and we get that \( \mathcal{M} \) and \( \mathcal{N} \) are two expressions of the same function.

5.8 Integral representation formulas for \( \zeta_\lambda \) in \(-3/2 < \Re(s) < -1/2\)

From (34):

\[
\zeta_\alpha(s) = \varphi(s) \int_0^\infty \mathcal{N}(x)x^{s-1/2}dx \quad (-3/2 < \Re(s) < -1/2)
\]

and with (42) we have:

\[
| \mathcal{N}(x) | = | \mathcal{M}(x) | \leq C/x
\]

Hence the convergence of the integral at \( \infty \) for \( \Re(s) < 1/2 \).

**Theorem 2.** In the domain \(-3/2 < \Re(s) < -1/2\), we have the integral representation formula:

\[
\zeta_\lambda(s) = \frac{1 - 2^{1-s}}{1 - 2^{1-2s}} \varphi(s) \int_0^\infty \mathcal{N}(x)x^{s-1/2}dx
\]

More explicitly, with the values of \( \varphi, \mathcal{N} \) and \( \mathcal{M} \):

\[
\zeta_\lambda(s) = \frac{1 - 2^{1-s}}{(2^{2s-1} - 1)\pi} \cos\left(\frac{\pi s}{2}\right) \cos\left(\frac{\pi}{2} + \frac{\pi}{4}\right) \Gamma(1/2 - s) \int_0^\infty \sum_{m=0}^{\infty} \frac{2x \beta(2m + 1)}{\sqrt{2m + 1}(x^2 + \pi^2(2m + 1)^2)} x^{s-1/2} dx
\]

\[
\zeta_\lambda(s) = \frac{2^{1-s} - 1}{(2^{2s-1} - 1)\pi} \cos\left(\frac{\pi s}{2}\right) \cos\left(\frac{\pi}{2} + \frac{\pi}{4}\right) \Gamma(1/2 - s) \int_0^\infty \sum_{m=0}^{\infty} \frac{\nu(2m + 1)}{e^{x/(2m+1)}} x^{s-1/2} dx
\]

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