Nonlinear Stochastic Attitude Filters on the Special Orthogonal Group 3: Ito and Stratonovich

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Abstract—This paper formulates the attitude filtering problem as a nonlinear stochastic filter problem evolved directly on the Special Orthogonal Group $SO(3)$. One of the traditional potential functions for nonlinear deterministic complimentary filters is studied and examined against angular velocity measurements corrupted with noise. This work demonstrates that the careful selection of the attitude potential function allows to attenuate the noise associated with the angular velocity measurements and results into superior convergence properties of estimator and correction factor. The problem is formulated as a stochastic problem through mapping $SO(3)$ to Rodriguez vector parameterization. Two nonlinear stochastic complimentary filters are developed on $SO(3)$. The first stochastic filter is driven in the sense of Ito and the second one considers Stratonovich. The two proposed filters guarantee that errors in the Rodriguez vector and estimates are semi-globally uniformly ultimately bounded in mean square, and they converge to a small neighborhood of the origin. Simulation results are presented to illustrate the effectiveness of the proposed filters considering high level of uncertainties in angular velocity as well as body-frame vector measurements.

Index Terms—Attitude estimates, Nonlinear stochastic filter, Stochastic differential equations, Brownian motion process, Ito, Stratonovich, Wong-Zakai, Rodriguez vector, Special orthogonal group, rotational matrix, SDEs, SO(3).

I. INTRODUCTION

This paper concerns the problem of attitude estimation of a rigid-body rotating in 3D space. In fact, attitude estimation is one of the major sub-tasks in the field of robotics. The attitude can be constructed from a set of vector measurements made on body-frame and reference-frame as it acts as a linear transformation of one frame to the other. Generally, the attitude estimation problem aims to minimize the cost function such as Wahba’s Problem [1]. The earliest work in [1] was purely algebraic. Several alternative methods attempted to reconstruct the attitude simply and statically by solving a set of simultaneous known inertial and body-frame measurements, for instance, TRIAD or QUEST algorithms [2,3] and singular value decomposition (SVD) [4]. However, vectorial measurements are subject to significant noise and bias components. Therefore, the category of static estimation in [2–4] gives poor results in this case. Consequently, the attitude estimation problem can be tackled either by Gaussian filter or nonlinear deterministic filter.

In the last few decades, several Gaussian filters have been developed mainly to obtain higher estimation performance with noise reduction. Many attitude estimation algorithms are based on optimal stochastic filtering for linear systems known as Kalman filter (KF) [5]. The linearized version of KF can be modified in a certain way for nonlinear systems to obtain the extended Kalman filter (EKF) [6]. An early survey of attitude observers was presented in [7] and a more recent overview on attitude estimation was introduced in [8]. Over the last three decades, several nonlinear filters have been proposed to estimate the attitude of spacecrafts. However, EKF and especially the multiplicative extended Kalman filter (MEKF) is highly recommended and considered as a standard in most spacecraft applications [8]. Generally, the covariance of any noise components introduced in angular velocity measurements is taken into account during filter design. The family of KFs parameterize the global attitude problem using unit-quaternion. The unit-quaternion provides a nonsingular attitude parameterization of attitude matrix [9]. Also, the unit-quaternion kinematics and measurement models of the attitude can be defined by a linear set of equations dependent on the quaternion state through EKF. This advantage motivated researchers to employ the unit-quaternion in attitude representation (for example [7,10]). Although EKF is subject to theoretical and practical problems, the estimated state vector with the approximated covariance matrix gives a reasonable estimate of uncertainties in the dynamics. In general, a four-dimensional vector is used to describe a three-dimensional one. Since, the covariance matrix associated with the quaternion vector is $4 \times 4$, whereas the noise vector is $3 \times 1$, the covariance is assumed to have rank 3. Generally, the state vector is $7 \times 1$ as it includes the four quaternion elements and the three bias components. One of the earliest detailed derivations of EKF attitude design was presented in [7]. However, the unit-quaternion kinematics and measurement models can be modified to suit KF with a linear set of equations [11]. The KF in [11] has the same state dimensions as EKF and to some degree, it can outperform the EKF. MEKF [10] is the modified version of EKF and it is highly recommended for spacecraft applications [8]. In MEKF, the true attitude state is the product of reference and estimated error quaternion. The estimated error in quaternion is parameterized from a three-dimensional vector in the body-frame, and the error is estimated using EKF. Next, the MEKF is used to multiply the estimated error and the reference quaternion. The estimated error should be selected in such a way that it yields identity when multiplied by the reference quaternion. The EKF can be modified into invariant extended Kalman filter (IEKF), which has two groups of operations. The right IEKF considers the errors modeled in the inertial-frame and the left IEKF matches with the MEKF [12]. IEKF has autonomous error and its evolution error does not depend on
the system trajectory. A recently proposed attitude filtering solution known as geometric approximate minimum-energy filter (GAMEF) approach [13] is based on Mortensen’s deterministic minimum-energy [14]. Unlike KF, EKF, IEKF, and MEKF, the GAMEF kinematics are driven directly on \( SO(3) \). In addition, KF, EKF, and IEKF are based on first order optimal minimum-energy which makes them simpler in computation and implementation. In contrast, MEKF and GAMEF are second order optimal minimum-energy, and therefore they require more calculation steps and more computational power. The Unscented Kalman filter (UKF) uses the unit-quaternion kinematics, and its structure is nearly similar to KF, however, UKF utilizes a set of sigma points to enhance the probability distribution [15]. In spite of the fact that UKF requires less theoretical knowledge and outperforms EKF in simulations, it requires more computational power, while the sigma points could add complexity to the implementation process [16].

Particle filters (PFs) belong to the family of stochastic filters, but they do not follow the Gaussian assumptions [17]. The main idea of PFs is the use of Monte-Carlo simulations for the weighted particle approximation of the nonlinear distribution. In fact, PFs outperform EKF, however, they have higher computational cost, and they do not fit small scale systems [8]. Moreover, they do not have a clear measure of how close the solution is to the optimal one [13]. Quaternion based attitude PF showed a better performance than UKF with higher processing calculations [18]. All the Gaussian filters described above as well as PFs are based on unit-quaternion, where the main advantage is non-singularity in attitude parameterization, while the main drawback is non-uniqueness in representation.

Aside from Gaussian filtering methods, nonlinear deterministic filters provide an alternative solution of attitude estimation which aims to establish convergence bounds with stable performance. Indeed, inertial measurement units (IMUs) have a prominent role in enriching the research of attitude estimation [19–21]. IMUs fostered researchers to propose nonlinear deterministic complementary filters on \( SO(3) \) using vectorial measurements with the need of attitude reconstruction [19,22] or directly from vectorial measurements without attitude reconstruction [19,23]. Also, the work done in [19] provides the filter kinematics in quaternion representation. In general, nonlinear deterministic filters achieve almost global asymptotic stability as they disregard the noise impact in filter derivation.

Nonlinear deterministic attitude filters have three distinctive advantages, such as better tracking performance, less computational power, and simplicity in derivation when compared to Gaussian filters or PFs [8]. Furthermore, no sensor knowledge is required in nonlinear deterministic filters, due to the fact that they omit the noise component in filter derivation. Overall, nonlinear deterministic attitude filters outperform Gaussian filters [19]. Observers play a crucial role in different control applications, especially for nonlinear stochastic systems (for example [24–26]). Aside from attitude observers, control applications are utilized for nonlinear systems with uncertain components [27,28]. These applications could include robust stabilization [29], control of uncertain nonlinear multi-agent systems [30,31], and stochastic nonlinear control for time-delay systems [32].

Two major challenges have to be taken into account when designing the attitude estimator, 1) the attitude problem of the rigid-body, modeled on the Lie group of \( SO(3) \), is naturally nonlinear; and 2) the true attitude kinematics rely on angular velocity. Therefore, successful attitude estimation can be achieved by nonlinear filter design relying on angular velocity measurements which are normally contaminated with noise and bias components. Likewise, it is essential that the estimator design considers any noise and/or bias components introduced during the measurement process. Furthermore, any noise component is characterized by randomness and irregular behavior. Having this in mind, one of the traditional potential functions of nonlinear deterministic complimentary filters evolved on \( SO(3) \) is studied (for example [8,19]) taking into consideration angular velocity measurements corrupted with bias and noise components. This study established that selecting the potential function in an alternative way allowed to diminish the noise. Hence, two nonlinear stochastic complementary filters on \( SO(3) \) are proposed here to improve the overall estimation quality. The first stochastic filter is driven in the sense of Ito [33] and the second one is developed in the sense of Stratonovich [34]. In case when angular velocity measurement is contaminated with noise, as far as the Rodriguez vector(\( SO(3) \)) is concerned, the proposed filters are able to 1) steer the error vector towards an arbitrarily small neighborhood of the origin/(identity) in probability; 2) attenuate the noise impact to a very low level for known or unknown bounded covariance; and 3) make the error semi-globally/(almost semi-globally) uniformly ultimately bounded in mean square.

The rest of the paper is organized as follows: Section II presents an overview of mathematical notation, \( SO(3) \) to Rodriguez vector parameterization, and some helpful properties of the nonlinear stochastic attitude filter design. Attitude estimation dynamic problem in Rodriguez vector with Gaussian noise vector which satisfies the Brownian motion process is formulated in Section III. The nonlinear stochastic filters on \( SO(3) \) and the stability analysis are presented in Section IV. Section V shows the output performance and discusses the simulation results of the proposed filters. Finally, Section VI draws a conclusion of this work.

II. MATHEMATICAL NOTATION

Throughout this paper, \( \mathbb{R}_+ \) denotes the set of nonnegative real numbers. \( \mathbb{R}^n \) is the real \( n \)-dimensional space while \( \mathbb{R}^{n \times m} \) denotes the real \( n \times m \) dimensional space. For \( x \in \mathbb{R}^n \), the Euclidean norm is defined as \( ||x|| = \sqrt{x^\top x} \), where \( x^\top \) is the transpose of the associated component. \( C^n \) denotes the set of functions with continuous \( n \)th partial derivatives. \( K \) denotes a set of continuous and strictly increasing functions such that \( \gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) and vanishes only at zero. \( K_{\infty} \) denotes a set of continuous and strictly increasing functions which belongs to class \( K \) and is unbounded. \( \mathbb{P} \{ \cdot \} \) denotes probability, \( \mathbb{E} [\cdot] \) denotes an expected value, and \( \exp(\cdot) \) refers to an exponential of associated component. \( \lambda (\cdot) \) is the set of singular values of the associated matrix with \( \lambda (\cdot) \) being the minimum value.
\( I_n \) denotes identity matrix with dimension \( n \)-by-\( n \), and \( \mathbf{0}_n \) is a zero vector with \( n \)-rows and one column. \( V \) denotes a potential function and for any \( V(x) \) we have \( V_x = \partial V / \partial x \) and \( V_{xx} = \partial^2 V / \partial x^2 \).

Let \( \mathbb{GL}(3) \) denote the 3-dimensional general linear group which is a Lie group with smooth multiplication and inversion. \( \mathbb{SO}(3) \) denotes the Special Orthogonal Group and is a subgroup of the general linear group. The attitude of a rigid-body is defined as a rotational matrix \( R \):

\[
\mathbb{SO}(3) = \{ R \in \mathbb{R}^{3 \times 3} | R^T R = R R^T = I_3, \det(R) = 1 \}
\]

where \( I_n \) is the identity matrix with \( n \)-dimensions and \( \det(\cdot) \) is the determinant of the associated matrix. The associated Lie-algebra of \( \mathbb{SO}(3) \) is termed \( \mathfrak{so}(3) \) and is defined by

\[
\mathfrak{so}(3) = \{ A \in \mathbb{R}^{3 \times 3} | A^T = -A \}
\]

with \( A \) being the space of skew-symmetric matrices and define the map \([\cdot]_x : \mathbb{R}^3 \to \mathfrak{so}(3)\) such that

\[
A = [\alpha]_x = \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}
\]

For all \( \alpha, \beta \in \mathbb{R}^3 \), we have \([\alpha]_x \beta = \alpha \times \beta \) where \( \times \) is the cross product between two vectors. Let the vex operator be the inverse of \([\cdot]_x \), denoted by \( \text{vex} : \mathfrak{so}(3) \to \mathbb{R}^3 \) such that \( \text{vex}(A) = \alpha \in \mathbb{R}^3 \). Let \( \mathcal{P}_a \) denote the anti-symmetric projection operator on the Lie-algebra \( \mathfrak{so}(3) \), defined by \( \mathcal{P}_a : \mathbb{R}^{3 \times 3} \to \mathfrak{so}(3) \) such that

\[
\mathcal{P}_a (B) = \frac{1}{2} (B - B^T) \in \mathfrak{so}(3)
\]

for all \( B \in \mathbb{R}^{3 \times 3} \). The following two identities will be used in the subsequent derivations

\[
[\beta]_x [\alpha]_x = (\beta^T \alpha) I_3 - \alpha \beta^T, \quad \alpha, \beta \in \mathbb{R}^3
\]

\[
[R\alpha]_x = R [\alpha]_x R^T, \quad R \in \mathbb{SO}(3), \alpha \in \mathbb{R}^3
\]

The normalized Euclidean distance of a rotation matrix on \( \mathbb{SO}(3) \) is given by the following equation

\[
\| R \|_f = \frac{1}{4} \text{Tr} \{ I_3 - R \} \in [0, 1]
\]

where \( \text{Tr} \{ \cdot \} \) denotes the trace of the associated matrix and \( \| R \|_f \in [0, 1] \). The attitude of a rigid-body can be constructed knowing angle of rotation \( \alpha \in \mathbb{R} \) and axis parameterization \( u \in \mathbb{R}^3 \). This method of attitude reconstruction is termed angle-axis parameterization [9]. The mapping of angle-axis parameterization to \( \mathbb{SO}(3) \) is defined by \( \mathcal{R}_\alpha : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{SO}(3) \) such that

\[
\mathcal{R}_\alpha (\alpha, u) = I_3 + \sin(\alpha) [u]_x + (1- \cos(\alpha)) [u]^2
\]

From the other side, the attitude can be defined knowing Rodriguez parameters vector \( \rho \in \mathbb{R}^3 \). The associated map to \( \mathbb{SO}(3) \) is given by \( \mathcal{R}_\rho : \mathbb{R}^3 \to \mathbb{SO}(3) \) such that

\[
\mathcal{R}_\rho (\rho) = \frac{1}{1 + \| \rho \|^2} \left( I_3 + 2 \rho \rho^T + 2 |\rho|_x \right)
\]

Substituting for the rotation matrix in (5), one can further show that the normalized Euclidean distance in (3) can be expressed in terms of Rodriguez parameters:

\[
\| R \|_f = \frac{1}{4} \text{Tr} \{ I_3 - R \} = \frac{\| \rho \|^2}{1 + \| \rho \|^2}
\]

The anti-symmetric projection operator in square matrix space of the rotation matrix \( R \) in (5) can be obtained in the sense of Rodriguez parameters vector as

\[
\mathcal{P}_a (R) = 2 \frac{1}{1 + \| \rho \|^2} \rho_x \in \mathfrak{so}(3)
\]

It follows that the composition mapping \( \Upsilon_a (\cdot) \) is

\[
\Upsilon_a (R) = \text{vex} (\mathcal{P}_a (R)) = 2 \frac{\rho}{1 + \| \rho \|^2} \in \mathbb{R}^3
\]

where \( \Upsilon_a := \text{vex} \circ \mathcal{P}_a \).

### III. Problem Formulation in Stochastic Sense

Let \( R \in \mathbb{SO}(3) \) denote the attitude (rotational) matrix, which describes the relative orientation of the body-frame \( \{ B \} \) with respect to the inertial-frame \( \{ I \} \) as given in Fig. 1.

![Fig. 1. The orientation of a 3D rigid-body in body-frame relative to inertial-frame.](image)

The attitude can be extracted from \( n \)-known non-collinear inertial vectors which are measured in a coordinate system fixed to the rigid body. Let \( v_i^B \in \mathbb{R}^3 \) for \( i = 1, 2, \ldots, n \), be vectors measured in the body-fixed frame. Let \( R \in \mathbb{SO}(3) \), the body-fixed vector \( v_i^B \in \mathbb{R}^3 \) is defined by

\[
v_i^B = R^T v_i^I + b_i^B + \omega_i^B
\]

where \( v_i^I \in \mathbb{R}^3 \) denotes the inertial fixed-frame vector for \( i = 1, 2, \ldots, n \). \( b_i^B \) and \( \omega_i^B \) denote the additive bias and noise components of the associated body-frame vector, respectively, for all \( b_i^B, \omega_i^B \in \mathbb{R}^3 \). The assumption that \( n \geq 2 \) is necessary for instantaneous three-dimensional attitude determination. In case when \( n = 2 \), the cross product of the two measured vectors can be accounted as the third vector measurement such that \( v_3^B = v_2^B \times v_1^B \) and \( v_3^B = v_1^B \times v_2^B \). It is common to employ the normalized values of inertial and body-frame vectors in the process of attitude estimation such as

\[
v_i^I = \frac{v_i^I}{\| v_i^I \|}, \quad v_i^B = \frac{v_i^B}{\| v_i^B \|}
\]
In this manner, the attitude can be defined knowing \( v_l \) and \( v_b \). Gyroscope or the rate gyro measures the angular velocity vector in the body-frame relative to the inertial-frame. The measurement vector of angular velocity \( \Omega_m \in \mathbb{R}^3 \) is
\[
\Omega_m = \Omega + b + \omega
\]  
(10)
where \( \Omega \in \mathbb{R}^3 \) denotes the true value of angular velocity, \( b \) denotes an unknown constant (bias) or slowly time-varying vector, while \( \omega \) denotes the noise component associated with angular velocity measurements, for all \( b, \omega \in \mathbb{R}^3 \). The noise vector \( \omega \) is assumed to be Gaussian. The true attitude dynamics and the associated Rodriguez vector dynamics are given in (11) and (12), respectively, as
\[
\dot{\hat{R}} = R[\Omega]_x
\]
(11)
\[
\dot{\rho} = \frac{1}{2} (I_3 + [\rho]_x + \rho \rho^T) \Omega
\]
(12)
In general, the measurement of angular velocity vector is subject to additive noise and bias components. These components are characterized by randomness and unknown behavior. In view of the fact that any unknown components in angular velocity measurements may impair the estimation process of the true attitude dynamics in (11) or (12), it is necessary to assume that the attitude dynamics are excited by a wide-band of random Gaussian noise process with zero mean. Combining angular velocity measurement in (10) and the attitude dynamics in (12), the attitude dynamics can be expressed as follows
\[
\dot{\rho} = \frac{1}{2} (I_3 + [\rho]_x + \rho \rho^T) (\Omega_m - b - \omega)
\]
(13)
where \( \omega \in \mathbb{R}^3 \) is a bounded continuous Gaussian random noise vector with zero mean. The fact that derivative of any Gaussian process yields Gaussian process allows us to write the stochastic attitude dynamics as a function of Brownian motion process vector \( d\beta/dt \in \mathbb{R}^3 \). Let \( \{\omega, t \geq t_0\} \) be a vector process of independent Brownian motion process such that
\[
\omega = Q \int d\beta
\]
(14)
where \( Q \in \mathbb{R}^{3 \times 3} \) is an unknown time-variant matrix with only nonzero and nonnegative bounded components in the diagonal. The covariance component associated with the noise \( \omega \) can be defined by \( Q^2 = QQ^T \). The properties of Brownian motion process are defined as \([33,36,37] \)
\[
P\{\beta(0) = 0\} = 1, \quad \mathbb{E}[d\beta/dt] = 0, \quad \mathbb{E}[\beta] = 0
\]
Let the attitude dynamics of Rodriguez vector in (12) be defined in the sense of Ito [33]. Considering the attitude dynamics in (13) and substituting \( \omega \) by \( Qd\beta/dt \) as in (14), the stochastic differential equation of (12) in view of (13) can be expressed by
\[
d\rho = f(\rho, b) dt + g(\rho) Q d\beta
\]
(15)
Similarly, the stochastic dynamics of (11) become
\[
dR = R[\Omega_m - b]_x dt - R[Qd\beta]_x
\]
(16)
where \( b \) was defined in (10), \( g(\rho) := -\frac{1}{2} (I_3 + [\rho]_x + \rho \rho^T) \) and \( f(\rho, b) := -g(\rho) (\Omega_m - b) \) with \( g : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3} \) and \( f : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \). \( g(\rho) \) is locally Lipschitz in \( \rho \), and \( f(\rho, b) \) is locally Lipschitz in \( \rho \) and \( b \). Accordingly, the dynamic system in (15) has a solution for \( t \in [t_0, T] \forall t_0 \leq t < \infty \) in the mean square sense and for any \( \rho(t) \in \mathbb{R}^3 \) such that \( t \neq t_0, \rho - \rho_0 \) is independent of \( \{\beta(\tau), \tau \geq t\}, \forall t \in [t_0, T] \) (Theorem 4.5 [36]). Now the aim is to achieve adaptive stabilization of an unknown bias and unknown time-variant covariance matrix. Let \( \sigma \) be the upper bound of \( Q^2 \) such that
\[
\sigma = \max\{Q_{11}^2, \max\{Q_{22}^2\}, \max\{Q_{33}^2\}\} \in \mathbb{R}^3
\]
(17)
where \( \max\{\cdot\} \) is the maximum value of an element.

**Definition 1.** Consider the stochastic differential system in (15). For a given function \( V(\rho) \in C^2 \), the differential operator \( \mathcal{L}V \) is given by
\[
\mathcal{L}V(\rho) = V^T_\rho f(\rho, b) + \frac{1}{2} Tr \left\{ g(\rho) Q^2 g^T(\rho) V_{\rho \rho} \right\}
\]
such that \( V_\rho = \partial V/\partial \rho \), and \( V_{\rho \rho} = \partial^2 V/\partial \rho^2 \).

**Definition 2.** [38] The trajectory \( \rho \) of the stochastic differential system in (15) is said to be semi-globally uniformly ultimately bounded (SGUUB) if for some compact set \( \Delta \in \mathbb{R}^3 \) and any \( \rho_0 = \rho(t_0) \), there exists a constant \( \kappa > 0 \), and a time constant \( T = T(\kappa, \rho_0) \) such that \( \|\Delta\| < \kappa, \forall t > t_0 + T \).

**Lemma 1.** [37,39] Let the dynamic system in (15) be assigned a potential function \( V \in C^2 \) such that \( V : \mathbb{R}^3 \rightarrow \mathbb{R}_+ \), class \( K_{\infty} \) function \( \alpha_1(\cdot) \) and \( \alpha_2(\cdot) \), constants \( c_1 > 0 \) and \( c_2 \geq 0 \) and a nonnegative function \( Z(\|\rho\|) \) such that
\[
\alpha_1(\|\rho\|) \leq V(\rho) \leq \alpha_2(\|\rho\|)
\]
(18)
\[
\mathcal{L}V(\rho) = V^T_\rho f(\rho, b) + \frac{1}{2} \text{Tr} \left\{ g(\rho) Q^2 g^T(\rho) V_{\rho \rho} \right\}
\]
\[
\leq - c_1 Z(\|\rho\|) + c_2
\]
(19)
then for \( \rho_0 \in \mathbb{R}^3 \), there exists almost a unique strong solution on \([0, \infty)\) for the dynamic system in (15), the solution \( \rho \) is bounded in probability such that
\[
\mathbb{E}[V(\rho)] \leq V(\rho_0) \exp(-c_1 t) + \frac{c_2}{c_1}
\]
(20)
Furthermore, if the inequality in (20) holds, then \( \rho \) in (15) is SGUUB in the mean square. In addition, when \( c_2 = 0 \), \( f(0, b) = 0_3 \), \( g(0) = 0_3 \times 3 \), and \( Z(\|\rho\|) \) is continuous, the equilibrium point \( \rho = 0 \) is globally asymptotically stable in probability and the solution of \( \rho \) satisfies
\[
P\left\{ \lim_{t \rightarrow \infty} Z(\|\rho\|) = 0 \right\} = 1, \quad \forall \rho_0 \in \mathbb{R}^3
\]
(21)
The proof of this lemma and existence of a unique solution can be found in [37]. For a rotation matrix \( R \in SO(3) \), let us define \( \mathcal{U} \subseteq SO(3) \) by \( \mathcal{U} := \left\{ R | \text{Tr}(R) = -1, \mathcal{P}_a(R) = 0 \right\}. \) We have \(-1 \leq \text{Tr}(R) \leq 3 \) such that the set \( \mathcal{U} \) is forward invariant and unstable for the dynamic system in (11) which
implies that $\rho = \infty$. For almost any initial condition such that $R_0 \notin \mathcal{U}$ or $\rho_0 \in \mathbb{R}^3$, we have $-1 < \text{Tr} \{ R_0 \} \leq 3$ and the trajectory of $\rho$ is semi-globally uniformly ultimately bounded in mean square.

**Lemma 2. (Young’s inequality)** Let $x$ and $y$ be $x, y \in \mathbb{R}^n$. Then, for any $c > 1$ and $d > 1$ satisfying $(c - 1)(d - 1) = 1$ with a small positive constant $\varepsilon$, the following holds

$$x^\top y \leq (1/c) \varepsilon \|x\|^c + (1/d) \varepsilon^{-d} \|y\|^d \quad (22)$$

In the next section, the presence of noise will be examined in light of a traditional form of potential function. The concept of an alternate potential function with specific characteristics able to attenuate the noise behavior will be carefully elucidated.

**IV. Stochastic Complementary Filters On $\mathbb{SO}(3)$**

The main goal of attitude estimation is to derive the attitude estimate $\hat{R} \rightarrow R$. Let’s define the error in attitude estimate from the body-frame to estimator-frame by

$$\hat{R} = R^\top \hat{R} \quad (23)$$

Let $\hat{b}$ and $\hat{\sigma}$ be estimates of unknown parameters $b$ and $\sigma$, respectively. Define the error in vector $b$ and $\sigma$ by

$$\hat{b} = b - \hat{b} \quad (24)$$

$$\hat{\sigma} = \sigma - \hat{\sigma} \quad (25)$$

Thus, driving $\hat{R} \rightarrow R$ ensures that $\hat{R} \rightarrow I_3$ and $\hat{\rho} \rightarrow 0_3$ where $\hat{\rho}$ is Rodriguez error vector associated with $\hat{R}$. In this section, two nonlinear stochastic complementary filters are developed on the Special Orthogonal Group $\mathbb{SO}(3)$. These filters in the sense of Rodriguez vector guarantee that the error vector is SGUUB in mean square for the case of noise contamination of the angular velocity measurements.

**A. Nonlinear Deterministic Attitude Filter**

In this subsection, we aim to study the behavior of nonlinear deterministic filter on $\mathbb{SO}(3)$ with noise introduced in angular velocity measurements. The attitude $\hat{R}$ can be reconstructed through a set of measurements in (9) to obtain $R_y$, for instance [2–4]. $R_y$ is corrupted with noise and bias greatly increase the difference between $R_y$ and the true $R$. The filter design aims to use the angular velocity measurements and the given $R_y$ to obtain good estimate of $R$. Consider the following filter design

$$\hat{R} = \hat{R} \left[ \Omega_m - \hat{b} - W \right] \times, \quad \hat{R} (0) = \hat{R}_0 \quad (26)$$

$$\hat{b} = \gamma_1 \Psi_a (\hat{R}), \quad \hat{b} (0) = \hat{b}_0, \quad \hat{R} = R_y^\top \hat{R} \quad (27)$$

$$W = k_1 \Psi_a (\hat{R}), \quad \hat{R} = R_y^\top \hat{R} \quad (28)$$

where $\Omega_m$ is angular velocity measurement, $\hat{b} \in \mathbb{R}^3$ is the estimate of the unknown bias $b$, and $\Psi_a (\hat{R}) = \text{vex} \left( \Psi_a (\hat{R}) \right)$ was given in (7). Also, $\gamma_1 > 0$ is an adaptation gain and $k_1$ is a positive constant.

Let the error in vector $b$ be defined as in (24) and assume that no noise was introduced to the dynamics ($\omega = 0_3$). The derivative of attitude error in (23) can be obtained from (11) and (26) as

$$\dot{\hat{R}} = \dot{\hat{R}} \left[ \Omega - \hat{R}^\top \Omega + \hat{b} - W \right] \times \quad (29)$$

where $\hat{R}^\top \Omega = \hat{R}^\top [\Omega] \times \hat{R}$. Hence, in view of (16) and (15), the error dynamic in (29) can be expressed in Rodriguez error vector dynamic by

$$\dot{\hat{\rho}} = \frac{1}{2} (I_3 + [\hat{\rho}]_\times + \hat{\rho}_{\Omega}^\top) \left( \Omega - \hat{R}^\top \Omega + \hat{b} - W \right) \quad (30)$$

From literature, one of traditional potential functions for adaptive filter estimation is $V (\hat{R}, \hat{b}) = \frac{1}{4} \text{Tr} \{ I_3 - \hat{R} \} + \frac{1}{2 \gamma_1} \hat{b}^\top \hat{b}$ (for example [8,19]). The equivalent of the aforementioned function in form of Rodriguez error is

$$V (\hat{\rho}, \hat{b}) = \frac{\|\hat{\rho}\|^2}{1 + \|\hat{\rho}\|^2} + \frac{1}{2 \gamma_1} \hat{b}^\top \hat{b} \quad (31)$$

let $\hat{f} := \frac{1}{2} (I_3 + [\hat{\rho}]_\times + \hat{\rho}_{\Omega}^\top) \left( \Omega - \hat{R}^\top \Omega + \hat{b} - W \right)$. For $V := V (\hat{\rho}, \hat{b})$, the derivative of (31) is

$$\dot{V} = V_{\hat{\rho}}^\top \hat{f} - \frac{1}{\gamma_1} \hat{b}^\top \hat{b} = \Psi_a (\hat{R})^\top (b - W) - \frac{1}{\gamma_1} \hat{b}^\top \hat{b} \quad (32)$$

where $\frac{1}{2} V_{\hat{\rho}}^\top (I_3 + [\hat{\rho}]_\times + \hat{\rho}_{\Omega}^\top) \left( \Omega - \hat{R}^\top \Omega \right) = 0$ which was obtained by substitution of $\hat{R} = R \hat{\rho} (\hat{\rho})$ in (5). Substituting for $\hat{b}$ and $W$ in (27) and (28), respectively, yields

$$\dot{V} = -k_1 \left\| \Psi_a (\hat{R}) \right\|^2 = -4k_1 \frac{\|\hat{\rho}\|^2}{\left(1 + \|\hat{\rho}\|^2\right)^2} \quad (33)$$

Lyapunov’s direct method ensures that for $\text{Tr} \left\{ \hat{R}_0 \right\} \neq -1$, $\Psi_a (\hat{R})$ converges asymptotically to zero. As such, $(I_3, 0_3)$ is an isolated equilibrium point and $\left( \hat{R}, \hat{b} \right) \rightarrow (I_3, 0_3)$ for $\omega = 0_3$ [19]. If angular velocity measurements $\Omega_m$ are contaminated with noise ($\omega \neq 0_3$), it is more convenient to represent the differential operator in (32) in the form of Definition 1. Hence, the following extra term will appear

$$\frac{1}{2} \text{Tr} \left\{ \hat{g}^\top V_{\rho} \hat{g} Q^2 \right\} = \frac{1}{4 \left(1 + \|\hat{\rho}\|^2\right)^2} \text{Tr} \left\{ (I_3 - 3\hat{\rho}_{\Omega}^\top) Q^2 \right\}$$

In this case, the operator $\mathcal{L} V (0, 0) = \frac{1}{4} \text{Tr} \{ Q^2 \}$ which implies that the significant impact of covariance matrix $Q^2$ cannot be lessened. One way to attenuate the noise associated with the angular velocity measurements is to chose a potential function in the sense of Rodriguez error vector $\hat{\rho}$ of order higher than two. It is worth mentioning that the deterministic filter in (26), (27) and (28) is known as a passive complementary filter proposed in [19].
B. Nonlinear Stochastic Attitude Filter in Ito Sense

Generally, the assumption behind nonlinear deterministic filters is that angular velocity vector measurements are joined with constant or slowly time-variant bias [8,19]. However, angular velocity vector measurements are typically subject to additive noise components which may weaken the estimation process of the true attitude dynamics in (11). Therefore, we aim to design a nonlinear stochastic filter in Ito sense taking into consideration that angular velocity vector measurements are subject to a constant bias and a wide-band of Gaussian random with zero mean such that $E[\omega] = 0$. Let the true inertial vector $v^I$ and body-frame vector $v^B$ be defined as in (8). Let the error in attitude estimate be similar to (23).

Consider the nonlinear stochastic filter design

\[
\dot{\hat{R}} = \hat{R} \left[ \Omega_m - \tilde{b} - W \right] \times, \quad \hat{R}(0) = \hat{R}_0 \tag{34}
\]

\[
\dot{\hat{b}} = \gamma_1 ||\hat{R}|| I_3 \mathbf{a}(\hat{R}) - \gamma_2 k_b \hat{b}, \quad \hat{b}(0) = \hat{b}_0 \tag{35}
\]

\[
\dot{\hat{\sigma}} = k_1 ||\hat{R}|| D_T \mathbf{a}(\hat{R}) - 2 \gamma_2 k_\sigma \hat{\sigma}, \quad \hat{\sigma}(0) = \hat{\sigma}_0 \tag{36}
\]

\[
W = \frac{k_1}{\varepsilon} \frac{2 - ||\hat{R}||}{1 - ||\hat{R}||} \mathbf{a}(\hat{R}) + k_2 D_T \hat{\sigma} \tag{37}
\]

where $\Omega_m$ is angular velocity measurement defined in (10), $\tilde{b}$ is the estimate of the unknown bias $b$, $\hat{\sigma}$ is the estimate of $\sigma$ which includes the upper bound of $Q^2$ as given in (17), $\hat{R} = R_y^T \hat{R}$ with $R_y$ being the reconstructed attitude, $\mathbf{a}(\hat{R}) = \text{vex} \left( \mathbf{p}(\hat{R}) \right)$ as given in (7), $D_T = \left[ \mathbf{a}(\hat{R}), \mathbf{a}(\hat{R}) \times, \mathbf{a}(\hat{R}) \times \times \right]$, and $||\hat{R}||$ is the Euclidean distance of $\hat{R}$ as defined in (3). Also, $\gamma_1 > 0$ and $\gamma_2 > 0$ are adaptation gains, $\varepsilon > 0$ is a small constant, while $k_b$, $k_\sigma$, $k_1$, and $k_2$ are positive constants.

**Theorem 1.** Consider the rotation dynamics in (16), angular velocity measurements in (10) in addition to other given vectorial measurements in (9) coupled with the observer (34), (35), (36), and (37). Assume that two or more body-frame non-collinear vectors are available for measurements and the design parameters $\gamma_1$, $\gamma_2$, $\varepsilon$, $k_b$, $k_\sigma$, $k_1$, and $k_2$ are chosen appropriately with $\varepsilon$ being selected sufficiently small. Then, for angular velocity measurements contaminated with noise ($\omega \neq \mathbf{0}$), all the signals in the closed-loop system is semi-globally uniformly bounded in mean square. In addition, the observer errors can be minimized by the appropriate selection of the design parameters.

**Proof:** Let the error in vector $b$ be defined as in (24). Therefore, the derivative of attitude error in incremental form of (23) can be obtained from (15) and (34) by

\[
d\hat{R} = R^T \dot{\hat{R}} \left[ \Omega_m - \tilde{b} - W \right] dt + [\Omega]^T R^T \hat{R} dt
\]

\[
= \left( \dot{\hat{R}} [\Omega] + [\Omega]^T R^T \hat{R} \right) dt + \hat{R} [Qd\beta]_x
\]

\[
= \hat{R} \left[ \Omega - \hat{R}^T \hat{\Omega} + \tilde{b} - W \right] dt + \hat{R} [Qd\beta]_x \tag{38}
\]

Similar extraction of Rodriguez error vector dynamic in view of (16) to (15) can be expressed from (38) to (39) in Ito’s representation [33] as

\[
d\hat{\rho} = \hat{f} dt + \hat{g} Q d\beta \tag{39}
\]

where $\hat{\rho}$ is the Rodriguez error vector associated with $\hat{R}$. Let $\hat{g} = \frac{1}{2} \left( I_3 + [\hat{\rho}]_x + \hat{\rho} \hat{\rho}^T \right)$ and $\hat{f} = \hat{g} \left( \Omega - \dot{\hat{R}}^T \hat{\Omega} + \tilde{b} - W \right)$. Consider the following potential function

\[
V (\hat{\rho}, \hat{b}, \hat{\sigma}) = \left( \frac{||\hat{\rho}||^2}{1 + ||\hat{\rho}||^4} \right)^2 + \frac{1}{2\gamma_1} \hat{b}^T \hat{b} + \frac{1}{2\gamma_2} \hat{\sigma}^T \hat{\sigma} \tag{40}
\]

For $V := V (\hat{\rho}, \hat{b}, \hat{\sigma})$, the differential operator $\mathcal{L} V$ in Definition 1 for the dynamic system in (39) can be expressed as

\[
\mathcal{L} V = V_{\hat{\rho}} \hat{f} + \frac{1}{2} \text{Tr} \left\{ \hat{g}^T V_{\hat{\rho} \hat{\rho}} V_{\hat{\rho} \hat{\rho}} Q^2 \right\} - \frac{1}{\gamma_1} \hat{b}^T \hat{b} - \frac{1}{\gamma_2} \hat{\sigma}^T \hat{\sigma} \tag{41}
\]

where $V_{\hat{\rho}} = \partial V / \partial \hat{\rho}$ and $V_{\hat{\rho} \hat{\rho}} = \partial V^2 / \partial \hat{\rho}^2 \hat{\rho}$. The first and the second partial derivatives of (40) with respect to $\hat{\rho}$ can be obtained as follows

\[
V_{\hat{\rho}} = 4 \left( \frac{||\hat{\rho}||^2}{1 + ||\hat{\rho}||^2} \right)^3 \hat{\rho} \tag{42}
\]

\[
V_{\hat{\rho} \hat{\rho}} = 4 \left( \frac{1 + ||\hat{\rho}||^2}{1 + ||\hat{\rho}||^2} \right) \left( 2 - 4 \frac{||\hat{\rho}||^2}{1 + ||\hat{\rho}||^2} \right) \frac{\hat{\rho} \hat{\rho}^T}{1 + ||\hat{\rho}||^2} \tag{43}
\]

substituting $\hat{R} = R_y(\hat{\rho})$ in (5), one can verify that

\[
\frac{1}{2} V_{\hat{\rho} \hat{\rho}}^T \left( I_3 + [\hat{\rho}]_x + \hat{\rho} \hat{\rho}^T \right) \left( \Omega - \hat{R}^T \hat{\Omega} \right) = 0
\]

Hence, the first part of the differential operator $\mathcal{L} V$ in (41) can be evaluated by

\[
V_{\hat{\rho}} \hat{f} = 2 \left( \frac{||\hat{\rho}||^2}{1 + ||\hat{\rho}||^2} \right)^2 \hat{\rho}^T \left( \hat{b} - W \right) \tag{44}
\]

Keeping in mind the identity in (1) and $\hat{g}$ in (39) and combining them with (43), the component $\text{Tr} \left\{ \hat{g}^T V_{\hat{\rho} \hat{\rho}} Q^2 \right\}$ can be simplified and expressed as

\[
\frac{1}{2} \text{Tr} \left\{ \hat{g}^T V_{\hat{\rho} \hat{\rho}} Q^2 \right\} = \frac{1}{2} \frac{1}{1 + ||\hat{\rho}||^2} \text{Tr} \left\{ \left( 1 + ||\hat{\rho}||^2 \right) ||\hat{\rho}||^2 Q^2 \right. \\
+ \left. \left( 2 - ||\hat{\rho}||^2 - 3 ||\hat{\rho}||^4 \right) \hat{\rho} \hat{\rho}^T Q^2 \right\} \tag{45}
\]

Let $\tilde{q} = [Q_{1,1}, Q_{2,2}, Q_{3,3}]^T$ and $\sigma$ be similar to (17). From (44) and (45), one can write the operator $\mathcal{L} V$ in (41) as

\[
\mathcal{L} V = 2 ||\hat{\rho}||^2 \hat{\rho}^T (\hat{b} - W) + \text{Tr} \left\{ \left( 2 - ||\hat{\rho}||^2 - 3 ||\hat{\rho}||^4 \right) \hat{\rho} \hat{\rho}^T Q^2 \right\} \\
+ \frac{1}{2 (1 + ||\hat{\rho}||^2)^2} + \frac{1}{\gamma_1} \hat{b}^T \hat{b} - \frac{1}{\gamma_2} \hat{\sigma}^T \hat{\sigma} \tag{46}
\]
Since $\|q\|^2 = \text{Tr} \{Q^2\}$ and $\text{Tr} \{\hat{\rho}^T Q^2\} \leq \|\hat{\rho}\|^2 \|q\|^2$, we have

$$L\dot{V} \leq 2 \frac{\|\hat{\rho}\|^2 \hat{\rho}^T (\dot{b} - W) + \|\hat{\rho}\|^4 \text{Tr} \{Q^2\} + 3 \|\hat{\rho}\|^2 \|q\|^2}{(1 + \|\hat{\rho}\|^2)^2} - \frac{\|\hat{\rho}\|^2 \|\rho\|^2 \rho^T Q^2 \hat{\rho}}{2 (1 + \|\hat{\rho}\|^2)^3} - \frac{1}{\gamma_1} \frac{\dot{b}^T \dot{b}}{b} - \frac{1}{\gamma_2} \hat{\sigma}^T \hat{\sigma}$$

(47)

According to Lemma 2, the following equation holds

$$\frac{3 \|\hat{\rho}\|^2 \|q\|^2}{2 (1 + \|\hat{\rho}\|^2)^3} \leq \frac{9 \|\hat{\rho}\|^4}{8 (1 + \|\hat{\rho}\|^2)} + \frac{\varepsilon}{2} \frac{\|q\|^2}{\|\hat{\rho}\|^4}$$

where $\varepsilon$ is a sufficiently small positive constant. Combining (48) with (47) yields

$$L\dot{V} \leq 2 \frac{\|\hat{\rho}\|^2 \hat{\rho}^T (\dot{b} - W) + \frac{3}{2} \sum_{i=1}^3 \sigma_i + \frac{9 \|\hat{\rho}\|^4}{8 (1 + \|\hat{\rho}\|^2)} \|\hat{\rho}\|^4}{(1 + \|\hat{\rho}\|^2)^2} - \frac{\|\hat{\rho}\|^2 \|\rho\|^2 \rho^T Q^2 \hat{\rho}}{2 (1 + \|\hat{\rho}\|^2)^3} + \frac{\varepsilon}{2} \left(\sum_{i=1}^3 \sigma_i\right)^2$$

(49)

Define $\hat{\sigma} = \sum_{i=1}^3 \sigma_i$. Substitute $\hat{b}$, $\hat{\sigma}$, and $W$ from (35), (36), and (37), respectively, in (49). Also, $\|R\|_F = \|\hat{\rho}\|^2 / (1 + \|\hat{\rho}\|^2)$ and $\text{Tr} \{Q^2\} = 2\|\rho\|^2 / (1 + \|\hat{\rho}\|^2)$ as defined in (6) and (7), respectively. Hence, (49) yields

$$L\dot{V} \leq -4 \left(\frac{8k_2 - 1}{8} - \frac{32k_1 - 9}{32\varepsilon}\right) \|\hat{\rho}\|^4 \left(1 + \|\hat{\rho}\|^2\right)^3 \|\hat{\rho}\|^4 - \frac{4k_1}{\varepsilon} \left(1 + \|\hat{\rho}\|^2\right)^3 - \frac{1}{\gamma_1} \frac{k_\rho \|\hat{b}\|^2}{b} + \frac{k_\sigma \|\sigma\|^2}{\sigma}$$

(50)

from (50) $k_3 \hat{b}^T \hat{b} = -k_\rho \|\hat{b}\|^2 + k_3 \hat{b}^T \hat{b}$ and $k_\sigma \sigma^T \sigma = -k_\rho \|\sigma\|^2 + k_\sigma \sigma^T \sigma$. Combining this result with Young’s inequality yields

$$k_3 \hat{b}^T \hat{b} \leq -k_\rho \|\hat{b}\|^2 + k_3 \frac{\|b\|^2}{2}$$

$$k_\sigma \sigma^T \sigma \leq -k_\rho \|\sigma\|^2 + k_\sigma \frac{\|\sigma\|^2}{2}$$

(51)

thereby, the differential operator in (50) results in

$$L\dot{V} \leq -4 \left(\frac{8k_2 - 1}{8} - \frac{32k_1 - 9}{32\varepsilon}\right) \|\hat{\rho}\|^4 \left(1 + \|\hat{\rho}\|^2\right)^3 - \frac{1}{\gamma_1} \frac{k_\rho \|\hat{b}\|^2}{b} - \frac{k_\sigma \|\sigma\|^2}{2}$$

(52)

such that (51) in $\mathcal{G}(3)$ form is equivalent to

$$L\dot{V} \leq -4 \left(\frac{8k_2 - 1}{8} - \frac{32k_1 - 9}{32\varepsilon}\right) \|\hat{R}\|_F \|\lambda_\sigma(R)\|^2 - \frac{1}{\gamma_1} \frac{k_\rho \|\hat{b}\|^2}{b} - \frac{k_\sigma \|\sigma\|^2}{2}$$

(53)

Setting $\gamma_1 \geq 1$, $\gamma_2 \geq 1$, $k_1 \geq \frac{9}{32}$, $k_2 \geq \frac{1}{8}$, $k_\rho \geq 0$, $k_\sigma > 0$, and the positive constant $\varepsilon$ sufficiently small with $Q^2 : \mathbb{R}^2 \rightarrow \mathbb{R}^{3 \times 3}$ being bounded, the operator $L\dot{V}$ in (51) becomes similar to (19) in Lemma 1. Define $c_2 = \frac{k_\rho}{2} \|b\|^2 + \frac{1}{2} (k_\rho + \varepsilon) \sigma^2$ which is governed by the unknown constant bias $b$ and the the upper bound of covariance $\sigma$. Let $\tilde{X} = \left[\|\hat{\rho}\|^2 / (1 + \|\hat{\rho}\|^2), \frac{1}{\sqrt{2\gamma_1}}, \frac{1}{\sqrt{2\sigma}} \hat{\sigma}^T\right] \in \mathbb{R}^7$ and

$$\mathcal{H} = \left[\begin{array}{ccc}
4k_1 / \varepsilon & 0_3 & 0_3 \\
0_3 & \gamma_1 k_\rho I_3 & 0_{3 \times 3} \\
0_3 & 0_{3 \times 3} & \gamma_2 k_\sigma I_3
\end{array}\right] \in \mathbb{R}^{7 \times 7}
$$

(54)

Hence, the differential operator in (51) can be expressed as

$$L\dot{V} \leq -4 \left(\frac{8k_2 - 1}{8} - \frac{32k_1 - 9}{32\varepsilon}\right) \|\hat{\rho}\|^4 \left(1 + \|\hat{\rho}\|^2\right)^3 - \frac{1}{\gamma_1} \frac{k_\rho \|\hat{b}\|^2}{b} - \frac{k_\sigma \|\sigma\|^2}{2} - \tilde{X}^T \mathcal{H} \tilde{X} + c_2$$

(55)

or more simply

$$L\dot{V} \leq -h (\|\hat{\rho}\|) - \lambda (\mathcal{H}) V + c_2$$

(56)

such that $h (\cdot)$ is a class $\mathcal{K}$ function which includes the first two components in (53), and $\lambda (\cdot)$ denotes the minimum eigenvalue of a matrix. Based on (54), one easily obtains

$$\frac{d \mathbb{E} [V]}{dt} = \mathbb{E} [L\dot{V}] \leq -\lambda (\mathcal{H}) \mathbb{E} [V] + c_2$$

(57)

Consider $K = \mathbb{E} [V (t)]$; thus $\frac{d \mathbb{E} [V]}{dt} \leq 0$ for $\lambda (\mathcal{H}) > c_2 / K$. Hence, $V \leq K$ is an invariant set and for $\mathbb{E} [V (0)] \leq K$ there
is $\mathbb{E} [V(t)] \leq K \forall t > 0$. Based on Lemma 1, the inequality in (55) holds for $V(0) \leq K$ and for all $t > 0$ such that

$$0 \leq \mathbb{E} [V(t)] \leq V(0) \exp (-\lambda (t) t) + \frac{c_2}{\lambda (t)} \forall t \geq 0$$

(56)

The above-mentioned inequality implies that $\mathbb{E} [V(t)]$ is eventually bounded by $c_2 / \lambda (t)$ indicating that $X$ is SGUUB in the mean square. Let us define $\hat{Y} = \left[ \hat{\rho}^T, \hat{b}^T, \hat{\sigma}^T \right]^T \in \mathbb{R}^3$. Since $\hat{X}$ is SGUUB, $\hat{Y}$ is SGUUB in the mean square. For a rotation matrix $R \in SO(3)$, let us define $U_0 \subseteq \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ as $U_0 = \left\{ (\hat{R}_0, \hat{b}_0, \hat{\sigma}_0) \right\}$. The set $U_0$ is forward invariant and unstable for the dynamic system in (11). From almost any initial condition such that $R_0 \notin U_0$ or, equivalently, $\hat{R}_0 \in \mathbb{R}^3$, the trajectory of $\hat{X}$ is SGUUB in the mean square.

C. Nonlinear Stochastic Attitude Filter in Stratonovich Sense

Stochastic differential equations can be defined and solved in the sense of Ito integral [33]. Alternatively, Stratonovich integral [34] can be employed for solving stochastic differential equations. The common feature between Stratonovich and Ito integral is that if the associated function multiplied by $d\beta$ is continuous and Lipschitz, the mean square limit exists. The Ito integral is defined for functional on $\{ \beta (t), t \leq t \}$ which is more natural but does not obey the chain rule. Conversely, Stratonovich is a well-defined Riemann integral for the sampled function, it has a continuous partial derivative with respect to $\beta$, it obeys the chain rule and it is more convenient for colored noise [34,36]. Hence, the Stratonovich integral is defined for explicit functions of $\beta$. In case when angular velocity measurements are contaminated with a wideband of random colored noise process, the solution of (13) for $\rho(t_0) = 0$ is obtained by

$$\rho(t) = \int_{t_0}^{t} f(\rho(\tau), b(\tau)) \, d\tau + \int_{t_0}^{t} g(\rho(\tau)) \, Q d\beta$$

(57)

according to subsection IV-B, the expected value of (57) is

$$\mathbb{E} [\rho] \neq \int_{t_0}^{t} \mathbb{E} [f(\rho(\tau), b(\tau))] \, d\tau$$

Thus, Stratonovich introduced the Wong-Zakai correction factor which can help in designing an adaptive estimate for the covariance component. Let us assume that the attitude dynamic in (15) was defined in the sense of Stratonovich [34], hence, its equivalent Ito [33,35,36] can be defined by

$$\lceil d\rho \rceil_i = [f(\rho, b)]_i dt + \sum_{k=1}^{3} \sum_{j=1}^{3} \frac{Q_{ij}^j}{2} g_{kj}(\rho) \frac{\partial g_{ij}(\rho)}{\partial \rho_k} dt + [g(\rho) \, Q d\beta]_i$$

(58)

where both $f(\rho, b)$ and $g(\rho)$ are defined in (15), $i, j, k = 1, 2, 3$ denote $i$th, $j$th and $k$th element components of the associate vector or matrix. The term $\sum_{k=1}^{3} \sum_{j=1}^{3} \frac{Q_{ij}^j}{2} g_{kj}(\rho) \frac{\partial g_{ij}(\rho)}{\partial \rho_k}$ denotes the Wong-Zakai correction factor of stochastic differential equations (SDEs) in the sense of Ito's representations [40]. Let $\mathcal{W}_i(\rho) = \sum_{k=1}^{3} \sum_{j=1}^{3} \frac{Q_{ij}^j}{2} g_{kj}(\rho) \frac{\partial g_{ij}(\rho)}{\partial \rho_k}$, accordingly, one can find that for $i = 1$

$$\sum_{k=1}^{3} \sum_{j=1}^{3} \frac{Q_{ij}^j}{2} g_{kj}(\rho) \frac{\partial g_{ij}(\rho)}{\partial \rho_k} = \frac{1}{4} \left( 1 + \rho_1^2 \right) \rho_1 \mathcal{Q}_{11} + (\rho_1 \rho_2 - \rho_3) \rho_2 \mathcal{Q}_{22} + (\rho_2 + \rho_1 \rho_3) \rho_3 \mathcal{Q}_{33}$$

Hence, $\mathcal{W}(\rho)$ for $i = 1, 2, 3$ is

$$\mathcal{W}(\rho) = \frac{1}{4} \left( I_3 + [\rho]_x + \rho \rho^T \right) \mathcal{Q}^2 \rho$$

Manipulating equations (58) and (59), the stochastic dynamics of the Rodriguez vector can be expressed as

$$d\rho = \mathcal{F}(\rho, b) dt + g(\rho) \, Q d\beta$$

(60)

where $g(\rho) := -\frac{1}{2} \left( I_3 + [\rho]_x + \rho \rho^T \right)$ and $\mathcal{F}(\rho, b) := -g(\rho) \left( \Omega_m - b \right)$. Define the error in attitude estimate similar to (23). Also, assume that the elements of covariance matrix $\mathcal{Q}^2$ are upper bounded by $\sigma$ as given in (17) such that the bound of $\sigma$ is unknown with nonnegative elements.

Consider the following nonlinear stochastic filter design

$$\dot{\hat{R}} = \hat{R} \left[ \Omega_m - \hat{b} - \frac{1}{2} \mathcal{D}_\Gamma \left( \mathcal{Y}_a(\hat{R}) \right) \dot{\sigma} - W \right]$$

(61)

$$\dot{\hat{b}} = \gamma_1 ||\hat{R}||_I \mathcal{Y}_a(\hat{R}) - \gamma_1 k_3 \hat{b}, \quad \hat{b}(0) = \hat{b}_0$$

(62)

$$\dot{\hat{\sigma}} = \gamma_2 ||\hat{R}||_I \left( k_1 \mathcal{D}_\Gamma + \frac{1}{2} \mathcal{D}_\Gamma \left( \mathcal{Y}_a(\hat{R}) \right) \mathcal{Y}_a(\hat{R}) \right)$$

(63)

$$W = \frac{k_1}{2} - ||\hat{R}||_I \mathcal{Y}_a(\hat{R}) + k_2 \mathcal{D}_\Gamma \hat{\sigma}$$

(64)

where $\hat{R}(0) = \hat{R}_0, \Omega_m$ is the angular velocity measurement as defined in (10), $\dot{b}$ and $\dot{\sigma}$ are estimates of the unknown parameters $b$ and $\sigma$, respectively. $\hat{R} = R_T^T \hat{R}$ with $R_T$ being the reconstructed attitude, $\mathcal{Y}_a(\hat{R}) = \text{vech} \left( \mathcal{P}_a(\hat{R}) \right)$ was given in (7), $||\hat{R}||_I$ is the Euclidean distance of $\hat{R}$, and $\mathcal{D}_\Gamma = \left[ \mathcal{Y}_a(\hat{R}), \mathcal{Y}_a(\hat{R}), \mathcal{Y}_a(\hat{R}) \right]$. $\gamma_1$ and $\gamma_2$ are positive adaptation gains, $\varepsilon > 0$ is a small constant, while $k_1, k_\sigma, k_1$ and $k_2$ are positive constants.

Theorem 2. Consider the rotation kinematics in (16) with angular velocity measurements and given vector measurements in (10) and (9), respectively, being coupled with the observer in (61), (62), (63) and (64). Assume that two or more body-frame non-collinear vectors are available for measurements. Then, for angular velocity measurements contaminated with noise ($\omega \neq \mathcal{O}_3$), $\left[ \rho^T, b^T, \sigma^T \right]^T$ is semi-globally uniformly ultimately bounded in mean square. Moreover, the observer errors can be made sufficiently small by choosing the appropriate design parameters.

Proof: Let the error in vector $b$ and $\sigma$ be defined as in (24) and (25), respectively. Hence, the derivative of (23) in
incremental form can be obtained from (15) and (61) by
\[
d\hat{\rho} = \hat{F} dt + \hat{g} Q d\beta
\]  
where \( \hat{\rho} \) is Rodriguez error vector associated with \( \hat{R} \) with \( \hat{g} = \frac{1}{2} (I_3 + [\hat{\rho}]_\times + \hat{\rho} \hat{\rho} \top) \), \( \hat{F} = \hat{g} \left( \Omega - \hat{R} \top \Omega - \hat{b} - \frac{1}{2} \text{diag} (\hat{\rho}) \hat{\sigma} - W \right) \) and \( \hat{W}(\hat{\rho}) = \frac{1}{2} (I_3 + [\hat{\rho}]_\times + \hat{\rho} \hat{\rho} \top) Q^2 \hat{\rho} \). Consider the following potential function
\[
V (\hat{\rho}, \hat{b}, \hat{\sigma}) = \left( \frac{\|\hat{\rho}\|^2}{1 + \|\hat{\rho}\|^2} \right)^2 + \frac{1}{2\gamma_1} \hat{b} \top \hat{b} + \frac{1}{2\gamma_2} \hat{\sigma} \top \hat{\sigma}
\]  
For \( V := V (\hat{\rho}, \hat{b}, \hat{\sigma}) \), the differential operator \( \mathcal{L} V \) in Definition 1 for the dynamic system in (66) can be written as
\[
\mathcal{L} V = V^\top \hat{F} + \frac{1}{2} \text{Tr} \left\{ \hat{g} \hat{V} \hat{\rho} \hat{g} Q^2 \right\} - \frac{1}{2\gamma_1} \hat{b} \top \hat{b} - \frac{1}{2\gamma_2} \hat{\sigma} \top \hat{\sigma}
\]  
The first and the second partial derivatives of (67) with respect to \( \hat{\rho} \) are similar to (42) and (43), respectively. The first part of differential operator \( \mathcal{L} V \) in (68) can be evaluated by
\[
V^\top \hat{F} = 2 \frac{\|\hat{\rho}\|^2}{1 + \|\hat{\rho}\|^2} \hat{\rho} \top \left( \hat{b} - \frac{1}{2} \text{diag} (\hat{\rho}) \hat{\sigma} + \frac{1}{2} Q^2 \hat{\rho} - W \right)
\]  
where \( \frac{1}{2} V^\top \hat{F} (I_3 + [\hat{\rho}]_\times + \hat{\rho} \hat{\rho} \top) \left( \Omega - \hat{R} \top \Omega \right) = 0 \). The component \( \text{Tr} \left\{ \hat{g} \hat{V} \hat{\rho} \hat{g} Q^2 \right\} \) is similar to (45). Let \( \hat{q} = [\hat{Q}_{1,1}, \hat{Q}_{2,2}, \hat{Q}_{3,3}] \top \) and \( \sigma \) be similar to (17). The operator \( \mathcal{L} V \) in (67) becomes
\[
\mathcal{L} V \leq 2 \frac{\|\hat{\rho}\|^2}{1 + \|\hat{\rho}\|^2} \left( \hat{b} - \frac{1}{2} \text{diag} (\hat{\rho}) \hat{\sigma} + \frac{1}{2} Q^2 \hat{\rho} - W \right) + \frac{1}{2} \text{Tr} \left\{ \hat{\rho} \hat{\rho} \top Q^2 \right\} - \frac{1}{\gamma_1} \hat{b} \top \hat{b} - \frac{1}{\gamma_2} \hat{\sigma} \top \hat{\sigma}
\]  
Since \( \|\hat{q}\|^2 = \text{Tr} \left\{ Q^2 \right\} \) and \( \text{Tr} \left\{ \hat{\rho} \hat{\rho} \top Q^2 \right\} \leq \|\hat{\rho}\|^2 \|\hat{q}\|^2 \), we obtain
\[
\mathcal{L} V \leq 2 \left( \frac{\|\hat{\rho}\|^2}{1 + \|\hat{\rho}\|^2} \right)^2 \left( \hat{b} - \frac{1}{2} \text{diag} (\hat{\rho}) \hat{\sigma} + \frac{1}{2} Q^2 \hat{\rho} - W \right) + \frac{1}{2 \gamma_1} \hat{b} \top \hat{b} - \frac{1}{2 \gamma_2} \hat{\sigma} \top \hat{\sigma}
\]  
From the last result and taking into consideration the inequality in (48), according to Lemma 2, and (17), equation (70) becomes
\[
\mathcal{L} V \leq 2 \left( \frac{\|\hat{\rho}\|^2}{1 + \|\hat{\rho}\|^2} \right)^2 \left( \hat{b} - \frac{1}{2} \text{diag} (\hat{\rho}) \hat{\sigma} + \frac{1}{2} Q^2 \hat{\rho} - W \right) + \frac{1}{2 \gamma_1} \hat{b} \top \hat{b} - \frac{1}{2 \gamma_2} \hat{\sigma} \top \hat{\sigma}
\]  
with \( \mathcal{D} \hat{\rho} = [\hat{\rho}, \hat{\rho}, \hat{\rho}] \). From (71), we have \( \hat{\rho} \top \mathcal{D} \hat{\rho} \sigma = \left( \sum_{i=1}^{3} \sigma_i \|\hat{\rho}\|^2 \right) \|\hat{\rho}\|^2 \). Let us define \( \hat{\sigma} = \sum_{i=1}^{3} \sigma_i \). Substitute for the differential operators \( \hat{b}, \hat{\sigma} \) and the correction factor \( W \) from (62), (63) and (64), respectively, with \( \|\hat{R}\||1 = \|\hat{\rho}\|/ \left( 1 + \|\hat{\rho}\|^2 \right) \) and \( \textbf{Y}(\hat{R}) = 2 \rho / \left( 1 + \|\hat{\rho}\|^2 \right) \). Hence, the result in (71) is equivalent to
\[
\mathcal{L} V \leq 4 \left( \frac{8k_2 - 1}{8} \hat{\rho} - \frac{32k_1 - 9}{32} \hat{\rho} \right) \frac{\|\hat{\rho}\|^4}{(1 + \|\hat{\rho}\|^2)^3} - \left( \frac{1 + 3 \|\hat{\rho}\|^2}{2} \right) \frac{\|\hat{\rho}\|^2 \hat{\rho} \top Q^2 \hat{\rho}}{2 + \|\hat{\rho}\|^2} - \frac{4k_1}{\varepsilon} \frac{\|\hat{\rho}\|^4}{(1 + \|\hat{\rho}\|^2)^3} - k_b ||\hat{b}||^2 - k_\sigma ||\hat{\sigma}||^2
\]  
applying Young’s inequality, one has
\[
k_b \hat{b} \hat{b} \leq \frac{k_b}{2} ||\hat{b}||^2 + \frac{k_b}{2} ||\hat{b}||^2
\]  
\[
k_\sigma \hat{\sigma} \hat{\sigma} \leq \frac{k_\sigma}{2} ||\hat{\sigma}||^2 + \frac{k_\sigma}{2} ||\hat{\sigma}||^2
\]  
Consequently, (72) becomes
\[
\mathcal{L} V \leq 4 \left( \frac{8k_2 - 1}{8} \hat{\rho} - \frac{32k_1 - 9}{32} \hat{\rho} \right) \frac{\|\hat{\rho}\|^4}{(1 + \|\hat{\rho}\|^2)^3} - \left( \frac{1 + 3 \|\hat{\rho}\|^2}{2} \right) \frac{\|\hat{\rho}\|^2 \hat{\rho} \top Q^2 \hat{\rho}}{2 + \|\hat{\rho}\|^2} - \frac{4k_1}{\varepsilon} \frac{\|\hat{\rho}\|^4}{(1 + \|\hat{\rho}\|^2)^3} - \frac{k_b}{2} ||\hat{b}||^2
\]  
(73)
In other words, \((73)\) in \(\mathcal{S}(3)\) form is equivalent to
\[
\mathcal{L}V \leq - \left( \frac{1}{3} + \frac{3 \Vert \tilde{R} \Vert_t}{8 \left( 1 - \frac{1}{\Vert \tilde{R} \Vert_t} \right)} \right) \Vert \tilde{R} \Vert_t \mathbf{Y}_a(\tilde{R})^T \mathbf{Q}^2 \mathbf{Y}_a(\tilde{R})
- \left( \frac{8k_2 - 1}{2} \sigma + \frac{32k_1 - 9}{32e} \right) \frac{\Vert \tilde{R} \Vert_t}{\Vert \mathbf{T}_a(\tilde{R}) \Vert^2} - \frac{4k_1}{8} \Vert \tilde{R} \Vert_t^2
- \frac{k_0}{2} \Vert \tilde{b} \Vert^2 - \frac{k_\sigma}{2} \Vert \tilde{\sigma} \Vert^2 + \frac{k_0}{2} \Vert b \Vert^2 + \frac{1}{2} (k_{\sigma} + \varepsilon) \sigma^2
\]
\[
(74)
\]
Setting \(\gamma_1 \geq 1, \gamma_2 \geq 1, k_1 \geq \frac{2}{7}, k_2 \geq \frac{1}{4}, k_0 > 0, k_\sigma > 0, \) and the positive constant \(\varepsilon\) being sufficiently small, and defining \(c_2 = \frac{k_0}{2} \Vert b \Vert^2 + \frac{1}{2} (k_{\sigma} + \varepsilon) \sigma^2\), the operator \(\mathcal{L}V\) in \((73)\) becomes similar to \((4.16)\) in \([37]\) which is in turn similar to \((19)\) in Lemma 1. Define
\[
\tilde{X} = \begin{bmatrix}
\frac{\Vert \tilde{\rho} \Vert^2}{1 + \Vert \tilde{\rho} \Vert^2}, \frac{1}{\sqrt{2\gamma_1}} \tilde{b}, \frac{1}{\sqrt{2\gamma_2}} \tilde{\sigma}^T
\end{bmatrix}^T \in \mathbb{R}^7,
\]
\[
\mathcal{H} = \begin{bmatrix}
\gamma_1 k_1 \mathbf{I}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\
\mathbf{0}_{3 \times 3} & \gamma_2 k_\sigma \mathbf{I}_3
\end{bmatrix} \in \mathbb{R}^{7 \times 7}
\]
Thereby, the differential operator in \((73)\) is
\[
\mathcal{L}V \leq - \left( \frac{8k_2 - 1}{2} \sigma + \frac{32k_1 - 9}{32e} \right) \frac{\Vert \tilde{\rho} \Vert^4}{(1 + \Vert \tilde{\rho} \Vert^2)^3}
- \frac{(1 + 3 \Vert \tilde{\rho} \Vert^2) \Vert \tilde{\rho} \Vert^2 \mathbf{Q}^2 \tilde{\rho} - \tilde{X}^T \mathcal{H} \tilde{X} + c_2}{2 (1 + \Vert \tilde{\rho} \Vert^2)^3}
\leq - \mathcal{H} (\Vert \tilde{\rho} \Vert) - \Delta (\mathcal{H}) V + c_2
\]
\[
(75)
\]
such that \(h (\cdot)\) is a class \(K\) function which includes the first two components in \((75)\). Based on \((75)\), one easily obtains
\[
\frac{d (\mathbb{E} [V])}{dt} = \mathbb{E} [\mathcal{L}V] \leq - \Delta (\mathcal{H}) \mathbb{E} [V] + c_2
\]
\[
(76)
\]
Let \(K = \mathbb{E} [V (t)]:\) then \(\frac{d (\mathbb{E} [V])}{dt} \leq 0\) for \(\Delta (\mathcal{H}) > \frac{c_2}{K}\). Thereby, \(V (t) \leq K\) is an invariant set and for \(\mathbb{E} [V (t)] \leq K\) it follows that \(\mathbb{E} [V (t)] \leq K\forall t > 0\). Accordingly, the inequality in \((76)\) holds for \(V (0) \leq K\) and for all \(t > 0\) which means that
\[
0 \leq \mathbb{E} [V (t)] \leq V (0) \exp (- \Delta (\mathcal{H}) t) + \frac{c_2}{\Delta (\mathcal{H})}, \forall t \geq 0
\]
\[
(77)
\]
The above inequality entails that \(\mathbb{E} [V (t)]\) is eventually bounded by \(c_2 / \Delta (\mathcal{H})\) which implies that \(X\) is SGUUB in the mean square. For a rotation matrix \(R \in \mathbb{S}(3)\), define \(\mathcal{U}_0 \subseteq \mathbb{S}(3) \times \mathbb{R}^3 \times \mathbb{R}^3\) as \(\mathcal{U}_0 = \{ (\tilde{R}_0, \tilde{b}_0, \tilde{\sigma}_0) | \text{Tr} (\tilde{R}_0) = -1, \tilde{b}_0 = \mathbf{0}_3, \tilde{\sigma}_0 = \mathbf{0}_3 \}\). The set \(\mathcal{U}_0\) is forward invariant and unstable for the dynamic system in \((11)\). Therefore, for almost any initial condition such that \(\tilde{R}_0 \notin \mathcal{U}_0\) or, equivalently, for any \(\tilde{\rho}_0 \in \mathbb{R}^3, X\) is SGUUB in the mean square as in Definition 2.

Since, \(\mathbf{Q}^2 : \mathbb{R}^+ \rightarrow \mathbb{R}^{3 \times 3}\) is bounded, we have \(d (\mathbb{E} [V]) / dt < 0\) for \(V > c_2 / \Delta (\mathcal{H})\). Considering Lemma 1 and the design parameters of the stochastic observer in Theorem 1 or 2 and combining them with prior knowledge about the covariance upper bound, allows to make the error signal smaller if the design parameters are chosen appropriately.

### D. Stochastic Attitude Filters: Ito vs Stratonovich

In this work, the selection of potential functions in \((40)\) and \((67)\) contributes to attenuating and controlling the noise level associated with angular velocity measurements. Also, the selection of potential functions in \((40)\) and \((67)\) produced results analogous to those \((54)\) and \((75)\), respectively. This similarity in potential function selection and final results is critical as it guarantees fair comparison between the two proposed stochastic filters. The proposed stochastic filters are able to correct the attitude allowing the user to reduce the noise level associated with angular velocity measurements through \(\Delta (\mathcal{H})\) by setting the values of \(\varepsilon, k_1, k_\sigma, \gamma_1\) and \(\gamma_2\). These gains are non-linearly determinate attitude filters lack this advantage.

The main features of the nonlinear stochastic attitude filter in the sense of Ito can be listed as

1. The filter requires less computational power in comparison with the Stratonovich’s filter.
2. No prior information about the covariance matrix \(\mathbf{Q}^2\) is required.
3. This filter is applicable to white noise.

Whereas, the main characteristics of the nonlinear stochastic attitude filter in the sense of Stratonovich are

1. The filter demands more computational power in comparison with the Ito’s filter.
2. No prior information about the covariance matrix \(\mathbf{Q}^2\) is required.
3. The filter is applicable for white as well as colored noise.

### V. Simulations

This section presents the performance and comparison among the two proposed nonlinear stochastic filters on \(\mathbb{S}(3)\). The first nonlinear stochastic filter is driven in the sense of Ito and the second one considers Stratonovich. Consider the orientation matrix \(R\) obtained from attitude dynamics in equation \((11)\) with the following angular velocity input signal
\[
\Omega = \begin{bmatrix}
\sin (0.7t) \\
0.7 \sin (0.5t + \pi) \\
0.5 \sin (0.3t + \frac{\pi}{3})
\end{bmatrix} \text{ (rad/sec)}
\]
while the initial attitude is \(R (0) = \mathbf{I}_3\). Let the true angular velocity \((\Omega)\) be contaminated with a wide-band of random noise process with zero mean and standard deviation (STD) be equal to 0.5 (rad/sec) such that \(\Omega_m = \Omega + b + \omega\) with \(b = 0.1 [1, -1, 1]^T, \omega = 0.5 n (t),\) where \(n (t) = \text{randn (3,1)}\) is a \(\text{randn (3,1)}\) command, which refers to a normally distributed random vector at each time instant. Let non-collinear inertial-frame vector be obtained as \(\gamma^T = \frac{1}{\sqrt{\gamma^2}} [1, -1, 1]^T\) and \(v_f^T = [0, 0, 1]^T\), while body-frame vector \(v_B^i\) and \(v_B^j\) are obtained by \(v_B^i = R_T^T v_f^i + v_B^i + \omega_B^i\) and \(v_B^j = v_B^j \times v_B^j\) for \(i = 1, 2\). Also, suppose that an additional noise vector \(\omega_B^i\) with zero mean and STD of 0.15 corrupted the body-frame vector measurements and bias components \(b_B^i = 0.1 [-1, 1, 0.5]^T\) and \(b_B^j = 0.1 [0, 0, 1]^T\). The third vector of inertial-frame and body-frame is extracted by \(v_f^T = v_f^T \times v_f^T\) and \(v_B^T = v_B^T \times v_B^T\).
vectorial measurements, the corrupted reconstructed attitude \( R_y \) is obtained by SVD [4] with \( \hat{R} = R_y^\dagger R \), see Appendix A. The total simulation time is 15 seconds.

For a very large initial attitude error, the initial rotation of attitude estimate is given according to angle-axis parameterization in (4) by \( \hat{R}(0) = R_x\alpha (u/\|u\|) \) with \( \alpha = 179.9 \) (deg) and \( u = [1, 5, 3] \) being very close to the unstable equilibria such that \( \|\hat{R}(0)\|_I \approx 0.99999 \). The initial conditions are

\[
R(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \hat{R}(0) = \begin{bmatrix} -0.9429 & 0.2848 & 0.1729 \\ 0.2866 & 0.4286 & 0.8568 \\ 0.1700 & 0.8574 & -0.4857 \end{bmatrix}
\]

Initial estimates for both filters are \( \hat{b}(0) = [0, 0, 0]^T \) and \( \hat{\sigma}(0) = [0, 0, 0]^T \). The same notation is used in derivations of both nonlinear stochastic filters. The design parameters were chosen as \( \gamma_1 = 1, \gamma_2 = 1, k_3 = 0.5, k_\sigma = 0.5, k_1 = 0.5, k_2 = 0.5 \) and \( \varepsilon = 0.5 \). Additionally, the following color notation is used: green color demonstrates the true value, red illustrates the performance of Ito’s filter and blue represents the performance of Stratonovich stochastic filter. Also, magenta refers to a measured value.

The true angular velocity \( (\Omega) \) and the high values of noise and bias components introduced through the measurement process of \( \Omega_m \) plotted against time are depicted in Fig. 2. Also, Fig. 3 presents the true body-frame vectors and their uncertain measurements. Fig. 4 shows the tracked Euler angles \( (\phi, \theta, \psi) \) of Ito and Stratonovich stochastic attitude filters relative to true angles plotted against time. Fig. 4 presents impressive tracking performance of the proposed stochastic filters. The mapping from SO(3) implies that \( \rho \to \infty \) as \( \|\hat{R}\|_I \to 1 \). Accordingly, Fig. 5 demonstrates the convergence of the square error of Rodriguez vector \( \rho^2 \) from large error initialization to a very small value close to zero. Fig. 6 confirms all the previous discussion using normalized Euclidean distance \( \|\hat{R}\|_I = \frac{1}{4} \text{Tr} \left\{ I_3 - R^\top \hat{R} \right\} \) which shows remarkable stable and fast convergence to very small neighborhood of the origin. However, Ito stochastic filter is characterized by higher oscillatory performance compared to Stratonovich stochastic filter.

To further compare the steady-state performance of the proposed filters in terms of normalized Euclidean distance of the error \( \|\hat{R}\|_I \), Table I summarizes statistical details of the mean and the STD of \( \|\hat{R}\|_I \). Both filters showed very small mean error of \( \|\hat{R}\|_I \) with \( \|\hat{R}\|_I \) being regulated to close neighborhood of the origin however, Stratonovich’s filter showed a remarkable less mean errors and STD in comparison with Ito’s filter. Numerical results included in Table I proves that the proposed nonlinear stochastic filters are robust as illustrated in Fig. 4, 5, and 6.

Finally, Fig. 7 and 8 illustrate the estimates of the stochastic filters plotted against time. It can be concluded from Fig. 7 and 8 that the estimates of the proposed filter are stable and smooth.

| Filter     | Ito          | Stratonovich |
|------------|--------------|--------------|
| Mean       | \( 4.1 \times 10^{-3} \) | \( 2.8 \times 10^{-3} \) |
| STD        | \( 3 \times 10^{-3} \)   | \( 1.6 \times 10^{-3} \) |

**Fig. 2.** True and measured angular velocities.

**Fig. 3.** True and measured body-frame vectorial measurements.

**Fig. 4.** Tracking performance of Euler angles.

**Fig. 5.** Rodriguez vector square error \( \rho^2 \).

**Table I**

| Statistical analysis of \( \|\hat{R}\|_I \) of the two proposed filters. | \( \|\hat{R}\|_I \) over the period (1-15 sec) |
|------------------|-----------------|
| Filter           | Ito             | Stratonovich   |
| Mean             | \( 4.1 \times 10^{-3} \) | \( 2.8 \times 10^{-3} \) |
| STD              | \( 3 \times 10^{-3} \)   | \( 1.6 \times 10^{-3} \) |
Results show effectiveness and robustness of the two stochastic filters against bias and noise components contaminating angular velocity measurements, as well as uncertainty in vectorial measurements and large initial error. Stochastic filters have proven to be able to correct their attitude in a small amount of time requiring no prior information about the covariance matrix $Q^2$ in order to obtain impressive estimation performance. The main advantage of Stratonovich stochastic filter, as mentioned in Subsection IV-D, is that the filter is applicable to white as well as colored noise. In addition, it had smaller mean square error and STD to Ito’s filter as given in Table I. Nonetheless, Ito stochastic filter requires less computational power.

VI. CONCLUSION

Deterministic filters neglect the noise associated with the angular velocity measurements in filter derivation. This can be clearly noticed in the selection of the potential function. However, an alternate potential function which has not been considered in the literature is able to significantly attenuate the effects of noise in angular velocities to lower levels. As such, this paper reformulated the attitude problem to stochastic sense through Rodriguez vector parameterization. Two different non-linear stochastic attitude filters on the Special Orthogonal Group 3 ($SO(3)$) have been proposed. The first filter is developed in the sense of Ito and the second filter is driven in the sense of Stratonovich. The resulting estimators have proven to have fast convergence properties in the presence of high levels of noise in angular velocity and vectorial measurements.

ACKNOWLEDGMENT

The authors would like to thank University of Western Ontario for the funding that made this research possible. Also, the authors would like to thank Maria Shaposhnikova for proofreading the article.

APPENDIX A

An Overview on SVD in [4]

Let $R \in SO(3)$ be the true attitude. The attitude can be reconstructed through a set of vectors given in (9). Let $s_i$ be the confidence level of measurement $i$ such that for $n$ measurements we have $\sum_{i=1}^{n} s_i = 1$. In that case, the corrupted reconstructed attitude $\hat{R}_y$ can be obtained by

$$
\mathbf{J} (\hat{R}) = 1 - \sum_{i=1}^{n} s_i (v_i^B)^T R^T v_i^T = 1 - \text{Tr} \left( R^T B^T \right)
$$

$$
B = \sum_{i=1}^{n} s_i v_i^B (v_i^T)^T = U S V^T
$$

$$
U_+ = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det(U) \end{bmatrix}
$$

$$
V_+ = V \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 & \det(V) \end{bmatrix}
$$

$$
R_y = V_+ U_+^T
$$

For more details visit [4].

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