Degeneration of trigonal curves and solutions of the KP-hierarchy

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Abstract

It is known that soliton solutions of the KP-hierarchy correspond to singular rational curves with only ordinary double points. In this paper we study the degeneration of theta function solutions corresponding to certain trigonal curves. We show that, when the curves degenerate to singular rational curves with only ordinary triple points, the solutions tend to be some intermediate solutions between solitons and rational solutions. They are considered as certain limits of solitons.

The Sato Grassmannian is extensively used here to study the degeneration of solutions, since it directly connects solutions of the KP-hierarchy to the defining equations of algebraic curves. We define a class of solutions in the Wronskian form which contain soliton solutions as a subclass and prove that, using the Sato Grassmannian, the degenerate trigonal solutions are connected to those solutions by certain gauge transformations.

Keywords: KP-hierarchy, soliton solution, algebraic curves, theta function, Sato Grassmannian, Boussinesq equation
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(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper we study the limits of theta function solutions of the KP-hierarchy corresponding to certain trigonal curves when they degenerate to singular rational curves with only ordinary triple points as their singularities. There are two major motivations in current research. Recently, soliton solutions of the KP equation are extensively studied in connection with the various wave patterns of line solitons, total positivity of the Grassmannians and cluster
algebras [11, 12, 16, 17, 23]. From the viewpoint of spectral curves soliton solutions correspond to singular rational curves with only ordinary double points [1, 2, 24, 26, 35, 36]. Therefore, it is interesting to study how the structure of soliton solutions studied in [16] is reflected in that of theta function solutions corresponding to non-singular algebraic curves. To study this problem it is necessary to connect theta function solutions to soliton solutions explicitly. In general to compute explicitly the limits of theta function solutions is a non-trivial but curious problem in itself. The case of hyperelliptic curves is relatively well studied [4, 26]. In this case, soliton solutions are obtained. However, not all soliton solutions are obtained as limits of hyperelliptic solutions, so it is important to study the case of non-hyperelliptic curves. But the problem is, to compute a limit of a non-hyperelliptic solution is not easy. The reason for this is that it is difficult to find a canonical homology basis explicitly for non-hyperelliptic curves in general.

In the last two decades the theory of multi-variate sigma functions associated with higher genus algebraic curves have been developed [6-8, 22, 27-29]. The higher genus sigma function is a certain modification of Riemann’s theta function. The most important property of it is the modular invariance. It means that, although the sigma function is defined by specifying a canonical homology basis, it does not depend on the choice of it. For some class of algebraic curves, such as \((n, s)\) curves, the modular invariance is expressed in a stronger form. Namely the Taylor coefficients of the sigma function become polynomials of coefficients of the defining equations of an algebraic curve. It means that the sigma function has a limit when the coefficients of the defining equations of a curve are specialized in any way. So it is an intriguing and a well-defined problem to establish the higher genus generalization of the ‘elliptic-trigonometric-rational degeneration’ of the Weierstarss elliptic function. The paper [3] studies some problems in this direction.

In the present paper we study the degeneration of the theta function solutions corresponding to certain non-hyperelliptic curves. To this end we extensively use the Sato Grassmannian [34]. It is the moduli space of the formal power series solutions of the KP-hierarchy (see theorem 1). Therefore a theta function solution corresponds to some point of the Sato Grassmannian. The importance of the Sato Grassmannian here is the fact that it connects directly solutions of the KP-hierarchy to the defining equations of the algebraic curves. It means that one can compute the limit of a theta solution without information about canonical homology bases, which is the main idea in the present paper.

Now let us explain the content of the paper in more detail. We consider the trigonal curve, which is a special case of the so called a \((3, 3m + 1)\) curve [7, 9], given by the equation

\[
y^3 = x^{3m} \prod_{j=1}^{3m} (x - \lambda_j^3),
\]

and its degeneration to the singular rational curve with only ordinary triple points:

\[
y^3 = x^m \prod_{j=1}^{m} (x - \lambda_j^3)^3.
\]

A solution of the KP-hierarchy corresponding to (1) can be expressed by the higher genus sigma function [28]. As mentioned above, the sigma function has a well-defined limit when the curve (1) degenerates to the curve (2). Unfortunately we do not yet know the explicit form of the limit of the sigma function. The point is that we can compute the limit of the solution independently of the sigma function using the Sato Grassmannian. Then, in turn, it will be possible to determine the limit of the sigma function using it.
The strategy to determine the limits of solutions is as follows. We first define a class of solutions in the Wronskians form which contain soliton solutions as a subclass. We call a solution in this class a generalized soliton (GS). We embed GSs to the Sato Grassmannian. Next we embed theta function solutions corresponding to the curves (1) to the Sato Grassmannian. We then compare them with the GSs. We prove that a solution in the former is connected to a solution in the latter by a certain gauge transformation. In this way the degenerate trigonal solutions are shown to be expressed by Wronskians of functions determined from the algebraic curves (2). They are some intermediate solutions between solitons and rational solutions which can be considered as certain degenerate limits of solitons.

The organization of the present paper is as follows. In section 2 after a brief explanation on the KP-hierarchy, the Sato Grassmannian and the correspondence of points of it with solutions of the KP-hierarchy are reviewed. The Wronskian construction of \((n, k)\) solitons are reviewed in section 3. In section 4, generalized soliton solutions are introduced and the corresponding points in the Sato Grassmannian are determined. An alternative description of GSs is given for the sake of applications in section 5. In section 6, an example of a frame of a GS which is relevant to the description of degenerate trigonal solutions is given. The map from the set of the affine rings of non-singular algebraic curves to the Sato Grassmannian is summarized in section 7. In section 8, the definition of the higher genus sigma function and the description of the solution corresponding to the affine ring of the curve (1) in terms of the sigma function are reviewed. The degeneration of the sigma function solution is determined in section 9. In section 10, an example of the solution in the simplest case \(m = 1\) is given in detail. Finally, physical properties of the solution of section 10 are discussed in section 11.

### 2. KP-hierarchy and the Sato Grassmannian

The KP-hierarchy in the bilinear form [13] is the equation for the function \(\tau(x)\), \(x = (x_1, x_2, x_3, \ldots)\) given by

\[
\int e^{-\sum_{i=1}^{\infty} y_i \lambda^i} \tau(x - y - [\lambda^{-1}]) \tau(x + y + [\lambda^{-1}]) \frac{d\lambda}{2\pi i} = 0,
\]

where \(y = (y_1, y_2, y_3, \ldots)\), \([\lambda^{-1}] = (\lambda^{-1}, \lambda^{-2}/2, \lambda^{-3}/3, \ldots)\) and the integral signifies taking the coefficient of \(\lambda^{-1}\) in the series expansion. By expanding in \(y\) it is equivalent to the infinite system of Hirota’s bilinear equations. The first of which is the KP equation in Hirota’s bilinear form:

\[
(D_{1}^{4} - 4D_{1}D_{3} + 3D_{2}^{2})\tau \cdot \tau = 0.
\]

Here the Hirota derivatives \(D_{1}^{2}\) etc are defined by

\[
\tau(x + y)\tau(x - y) = \sum_{\alpha_i \geq 0} (D_{1}^{\alpha_1}D_{2}^{\alpha_2}\cdots)(\tau \cdot \tau) \frac{y_{1}^{\alpha_1}y_{2}^{\alpha_2}\cdots}{\alpha_1!\alpha_2!\cdots}.
\]

If we set \(u = 2\partial_{x}^{2} \log \tau(x), (x = x_1, y = x_2, t = x_3)\) we have the KP equation

\[
3u_{yy} + (-4u_t + 6uu_x + uu_{xx})_{x} = 0.
\]

In this paper we mean by the KP hierarchy, the equation (3) for \(\tau(x)\).

Next, we recall the definition of the Sato Grassmannian [34] which we denote by the universal Grassmann manifold (UGM) [32, 33] (see also [15, 25]) and the correspondence between solutions of the KP-hierarchy and points of the UGM.
Let \( V = \mathbb{C}((z)) \) be the vector space of formal Laurent series in the variable \( z \) and \( V_0 = \mathbb{C}[z^{-1}] \), \( V_0 = \mathbb{C}[z] \) subspaces of \( V \). Then
\[
V = V_0 \oplus V_0, \quad V/V_0 \simeq V_0.
\]
Let \( \pi : V \rightarrow V_0 \) be the projection map. The Sato Grassmannian UGM is defined by
\[
\text{UGM} = \{ \text{a subspace } U \text{ of } V | \dim \ker(\pi|_U) = \dim \text{coker}(\pi|_U) < \infty \}.
\]

An ordered basis of a point \( U \) of the UGM is called a frame of \( U \). We can express a frame of \( U \) by an infinite matrix as follows.

We set \( f_i = z^i, \quad i \in \mathbb{Z} \), and write an element \( f \) of \( V \) as
\[
f = \sum_{i \in \mathbb{Z}} X f_i.
\]
(5)

We associate the infinite column vector \((X_i)_{i \in \mathbb{Z}}\) to \( f \). Then a matrix \( X = (X_{ij})_{i \in \mathbb{Z}, j \in \mathbb{N}} \) is a frame of \( U \) if \((X_{ij})_{i \in \mathbb{Z}, j \in \mathbb{N}}\), \( j \in \mathbb{N} \) is a basis of \( U \). We write it as
\[
X = \begin{pmatrix}
\ddots & \ddots \\
\vdots \quad X_{-2,1} & X_{-2,2} \\
\vdots \quad X_{-1,1} & X_{-1,2} \\
\cdots & \cdots \\
X_{0,1} & X_{0,2} \\
\cdots & \cdots \\
X_{1,1} & X_{1,2} \\
\ddots & \ddots \\
\end{pmatrix},
\]
(6)

where the columns are labeled from right to left as \( 1, 2, 3, ... \) and rows are labeled from up to down as \( ..., -1, 0, 1, ... \).

For a point \( U \) of the UGM there exists a frame \( X = (X_{ij})_{i \in \mathbb{Z}, j \in \mathbb{N}} \) of \( U \) satisfying the following conditions: there exists a non-negative integer \( l \) such that
\[
X_{ij} = \begin{cases} 
1 & \text{if } j > l \text{ and } i = -j \\
0 & \text{if } (j > l \text{ and } i < -j) \text{ or } (j \leq l \text{ and } i < -l).
\end{cases}
\]
(7)

It means that \( X \) is of the form
\[
X = \begin{pmatrix}
\ddots & O \\
\vdots & 1 \\
\vdots & \ast & 1 \\
\vdots & \ast & \ast & B \\
\end{pmatrix},
\]
where \( B \) is an \( \infty \times l \) matrix and its first row is placed at the \( -l \)th row of \( X \). In the following a frame of a point of the UGM is always assumed to satisfy the condition (7).

Here we introduce the notion of a Maya diagram. A Maya diagram of charge \( p \) is a sequence of integers \( M = (m_1, m_2, ...) \) such that \( m_1 > m_2 > \cdots \) and, for some \( l \), \( m_i = -l + p, \quad i \geq l \). In this paper we consider only a Maya diagram of charge 0 and call them simply a Maya diagram. With each Maya diagram \( M \) is associated to the partition \( \lambda \) by
\[ \lambda = (m_1 + 1, m_2 + 2, \ldots), \]

and vice versa.

Let \( \lambda \) be an arbitrary partition and \( M = (m_1, m_2, m_3, \ldots) \) the Maya diagram corresponding to it. The Plücker coordinate \( X_\lambda \) of a frame \( X \) is defined as

\[ X_\lambda = \det(X_{m_i,j})_{i,j \in \mathbb{N}}, \tag{8} \]

which is also denoted by \( X_M \) using the Maya diagram \( M \).

The infinite determinant (8) is actually defined as a finite determinant as follows. Let \( l \) be an integer in the condition (7) and \( l_1 \) an integer such that \( m_i = -i, i \geq l_1 \). Take an integer \( l_2 \) such that \( l_2 \geq \max(l + 1, l_1) \). Then

\[ X_\lambda = \det(X_{m_i,j})_{1 \leq i,j \leq l_2}. \tag{9} \]

It does not depend on the choice of \( l, l_1, l_2 \).

For a frame \( X \) satisfying (7) we define the \( \tau \) function by

\[ \tau(x; X) = \sum_\lambda X_\lambda s_\lambda(x), \tag{10} \]

where the summation is taken over all partitions and \( s_\lambda(x) \) is the Schur function corresponding to the partition \( \lambda \) (see the next section).

A frame \( X \) satisfying the condition (7) is not unique for \( U \). If \( X \) is replaced with another frame \( \tau(x; X) \) is multiplied by a non-zero constant.

**Theorem 1** ([34]). For a frame \( X \) of a point \( U \) of the UGM, \( \tau(x; X) \) is a solution of the KP-hierarchy. Conversely, for any formal power series solution \( \tau(x) \) of the KP-hierarchy there exists a unique point \( U \) of the UGM such that \( \tau(x) = c(X)\tau(x; X) \) for a frame \( X \) of \( U \) and some constant \( c(X) \) which depends on \( X \).

### 3. \((n, k)\)-soliton

Let us recall soliton solutions of the KP-hierarchy.

We first recall the general Wronskian construction of solutions of the KP-hierarchy [14, 32]. Let \( f_1, \ldots, f_m \) be functions which satisfy

\[ \frac{\partial f_l}{\partial x_l} = \frac{\partial f_l}{\partial x_1}, \quad l \geq 1. \tag{11} \]

Then

\[ \tau(x) = \operatorname{Wr}(f_1, \ldots, f_m) = \det(f_j^{(l-1)})_{1 \leq i,j \leq m}, \quad f_j^{(i)} = \frac{\partial f_j}{\partial x_1^i}, \]

is a solution of the KP-hierarchy.

Using the Wronskian method the soliton solutions can be constructed in the following way. Let \( n \) and \( k \) be positive integers such that \( n > k, \lambda_1, \ldots, \lambda_n \) mutually distinct non-zero complex numbers and \( A = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq k} \) an \( n \times k \) matrix of rank \( k \). We set

\[ f_j = \sum_{i=1}^n a_{ij} e^{\xi_i}, \quad \xi_i = \xi(x, \lambda_i), \quad \xi(x, \lambda) = \sum_{j=1}^\infty x_j \lambda^j. \tag{12} \]
Since $e^{\xi_i}$ satisfies (11) so does $f_j$. Therefore
\[ \tau(x) = \mathrm{Wr}(f_1, \ldots, f_k) = \sum_{I=(i_1, \ldots, i_k), 1 \leq i < \cdots < i_k \leq n} \Delta_I(\lambda) A_I e^{\xi_{i_1} + \cdots + \xi_{i_k}}, \quad (13) \]
is a solution of the KP-hierarchy, where, for $I = (i_1, \ldots, i_k)$,
\[ \Delta_I(\lambda) = \prod_{p < q} (\lambda_{i_p} - \lambda_{i_q}), \quad A_I = \det(a_{i_p,q})_{1 \leq p,q \leq k}. \]

**Definition 1.** We call the solution (13) an $(n,k)$ soliton.

Sato determined the point of the UGM corresponding to a $(n,k)$ soliton.

**Theorem 2 ([32]).** The point of the UGM corresponding to (13) is given by the frame which consists of the following functions:
\[ z^{-(l-1)} \sum_{i=1}^{n} \frac{a_{ij}}{1 - \lambda_i z} \quad (1 \leq j \leq k), \quad z^{-l} \quad (l \geq k), \quad (14) \]
where $1/(1 - \lambda_i z)$ is considered as a power series in $z$ by
\[ \frac{1}{1 - \lambda_i z} = \sum_{r=0}^{\infty} \lambda_i^r z^r. \]

4. **Generalization of $(n,k)$-soliton**

In this section we define a class of solutions of the KP-hierarchy in Wronskian form which contains $(n,k)$ solitons.

Let $n, k$ as before, $N$, $r_j$ $(0 \leq j \leq N)$ non negative integers such that
\[ r_0 + \cdots + r_N = n, \]
\[ \lambda_{ij}, \ 0 \leq i \leq N, \ 1 \leq j \leq r_i \text{ non-zero complex numbers such that, for each } i, \lambda_{ij} \neq \lambda_{i,j'} \text{ if } j \neq j', \text{ and } A = (a_{ij}) \text{ an } n \times k \text{ matrix of rank } k. \]
We set $r_{-1} = 0$ for convenience. We also use $\lambda_i, 1 \leq i \leq n,$ defined by
\[ (\lambda_1, \ldots, \lambda_n) = (\lambda_{0,1}, \ldots, \lambda_{0,n}, \ldots, \lambda_{N,1}, \ldots, \lambda_{N,N}). \]

Set
\[ f_j = \sum_{i=0}^{N} \sum_{r_0 + \cdots + r_i = j} a_{ij} \left( \frac{d^r}{d\lambda^r} e^{\xi(x,\lambda)} \right)_{\lambda=\lambda_i}, \quad 1 \leq j \leq k. \quad (15) \]

Since the derivatives in $x_I$ and $\lambda$ commute, $f_j$ still satisfies equation (11). Therefore the Wronskian of the functions $f_j$ becomes a solution of the KP-hierarchy.

**Definition 2.** The solution $\tau(x) = \mathrm{Wr}(f_1, \ldots, f_k)$ with $f_j$ given by (15) is called an $(n,k)$ GS.

**Remark 1.** The above construction of $(n,k)$ GSs is similar to that of rational solutions of KP equation in [20, 21, 37].

The point of the UGM corresponding to an $(n,k)$ GS can be determined as follows. Let

\[ \frac{1}{1 - \lambda_i z} = \sum_{r=0}^{\infty} \lambda_i^r z^r. \]
\[
v^{(i)}(\lambda, z) = \frac{d^i}{d\lambda^i} \frac{1}{1 - \lambda z} = \frac{\delta_{z}^{(i)} - \lambda z}{(1 - \lambda z)^{i+1}}, \quad i \geq 0. \tag{16}
\]

**Theorem 3.** The point of the UGM corresponding to an \((n,k)\) GS specified by (15) is given by the following frame:

\[
z^{-(k-1)} \left( \sum_{s=0}^{N} \sum_{i=r_{-1}+\cdots+r_{-1}+1}^{n+r_{s}} a_{ij}v^{(i)}(\lambda, z) \right), \quad 1 \leq j \leq k,
\]

\[
z^{-j}, \quad j \geq k. \tag{17}
\]

**Proof.** The proof is similar to Sato’s proof of theorem 2.

The linear independence of the functions (17) is equivalent to \(\text{rank} A = k\). Therefore they define the frame of a point of the UGM. We shall show that the solution (10) corresponding to the frame (17) is equal to the Wronskian of the set of functions \(f_1, \ldots, f_k\). To this end we first express the frame (17) in the matrix form and compute the Plücker coordinates of it.

Let \(X = (X_{ij})_{i \in \mathbb{Z}_{\geq 0}, j \in \mathbb{N}}\) be the infinite matrix representing the frame (17). It is described in the following way.

We define the vectors \(u^{(i)}(\lambda), i \geq 0\) with the infinite components by

\[
u^{(i)}(\lambda) = \frac{d^i}{d\lambda^i} \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \end{bmatrix}.
\]

The components of \(u^{(i)}(\lambda)\) are labeled from up to down as 0, 1, 2, \ldots. Let \(C\) and \(B\) be the \(\mathbb{Z}_{\geq 0} \times n\) and \(\mathbb{Z}_{\geq 0} \times k\) matrices defined, respectively, by

\[
C = \left( u^{(0)}(\lambda_{0,1}), \ldots, u^{(0)}(\lambda_{0,n}), \ldots, u^{(N)}(\lambda_{N,1}), \ldots, u^{(N)}(\lambda_{N,n}) \right),
\]

\[
B = CA = (b_{ij})_{i \in \mathbb{Z}_{\geq 0}, 1 \leq j \leq k}.
\]

Then \(X\) is given by

\[
X_{ij} = \begin{cases} 
\delta_{i,-j} & \text{if } j \geq k + 1 \\
\delta_{i+k,k+1-j} & \text{if } 1 \leq j \leq k \text{ and } i \geq -k \\
0 & \text{if } 1 \leq j \leq k \text{ and } i < -k
\end{cases}.
\]

Namely \(X\) is of the form

\[
X = \begin{bmatrix} 
\ddots & 1 \\
& 1 \\
& B
\end{bmatrix},
\]

where the 0th row of \(B\) sits at the \(-k\)th row of \(X\).

We consider the Plücker coordinate \(X_M\) of \(X\) corresponding to a Maya diagram \(M\). The form of \(X\) given above implies that \(X_M = 0\) unless \(M\) is written in the form

\[
M = (m_1, \ldots, m_k, -k - 1, -k - 2, \ldots), \quad m_1 > \cdots > m_k \geq -k. \tag{18}
\]
Let
\[ \mathcal{L} = \{(l_1, \ldots, l_k) | 0 \leq l_1 < \cdots < l_k\}. \]
We associate the element \((l_1, \ldots, l_k)\) of \(\mathcal{L}\) with \(M\) by
\[ (l_1, \ldots, l_k) = (m_1 + k, \ldots, m_k + k). \]

By this map the set of Maya diagrams of the form (18) and \(\mathcal{L}\) bijectively correspond. For \((l_1, \ldots, l_k) \in \mathcal{L}\) we set \(B_{l_1, \ldots, l_k} := \text{det}(h_{i,j})_{1 \leq i, j \leq k}\). If \((l_1, \ldots, l_k) \in \mathcal{L}\) corresponds to \(M\) then it is easy to see that
\[ X_M = B_{l_1, \ldots, l_k}. \]
Let \(\mu\) be the partition corresponding to \(M\):
\[ \mu = (m_1 + 1, \ldots, m_k + k), \]
which is related with \((l_1, \ldots, l_k)\) by
\[ \mu = (l_k - (k - 1), \ldots, l_2 - 1, l_1). \tag{19} \]

Let \(p_j(x)\) be the polynomial of \(x = (x_1, x_2, \ldots)\) defined by
\[ e^\xi(x, \lambda) = \sum_{j=0}^{\infty} p_j(x) \lambda^j. \]
We set \(p_j(x) = 0\) for \(j < 0\). In terms of \((l_1, \ldots, l_k)\) the Schur function \(s_\mu(x)\) is given by
\[ s_\mu(x) = \text{det}(p_{\lambda-i}(x))_{0 \leq \lambda \leq \mu-1, 1 \leq i \leq k}. \tag{20} \]

We sometimes denote \(s_\mu(x)\) by \(s_{\mu}(x)\). For \((l_1, \ldots, l_k) \in \mathcal{L}\) and \(1 \leq r_1 < \cdots < r_k \leq n\) we set
\[ C_{r_1, \ldots, r_k}^n = \text{det}(C_{l,i})_{1 \leq i, j \leq k}; \]
\[ A_{r_1, \ldots, r_k} = \text{det}(a_{r,i})_{1 \leq i, j \leq k}. \]

Then we have, by (10) and the Binet–Cauchy formula for the determinant,
\[ \tau(x) = \sum_{0 \leq l_1 < \cdots < l_k} B_{l_1, \ldots, l_k} s_{l_1, \ldots, l_k}(x) \]
\[ = \sum_{0 \leq l_1 < \cdots < l_k} \sum_{1 \leq r_1 < \cdots < r_k \leq n} C_{r_1, \ldots, r_k}^n A_{r_1, \ldots, r_k} s_{l_1, \ldots, l_k}(x). \tag{21} \]

Let us rewrite the last expression. Let \(P = (p_{j-i}(x))_{0 \leq i, j \leq k-1, j \in \mathbb{Z}^p}\). We denote the \(r\)th column vector of \(C\) by \(C_r\).

**Lemma 1.** The following equation is valid:
\[ \sum_{0 \leq l_1 < \cdots < l_k} C_{l_1, \ldots, l_k}^{r_1, \ldots, r_k} s_{l_1, \ldots, l_k}(x) = \text{det}(P \cdot (C_{r_1}, \ldots, C_{r_k})). \tag{22} \]

**Proof.** By the Binet–Cauchy formula and (20) the right hand side of (22) is written as
\[ \sum_{0 \leq l_1 < \cdots < l_k} \text{det}(p_{j-i}(x))_{0 \leq i \leq k-1, 1 \leq j \leq k} C_{l_1, \ldots, l_k}^{r_1, \ldots, r_k} = \sum_{0 \leq l_1 < \cdots < l_k} s_{l_1, \ldots, l_k}(x) C_{l_1, \ldots, l_k}^{r_1, \ldots, r_k} \]
which shows (22).
Let us compute the product of matrices inside the determinant in the right hand side of (22). Since
\[
\left( P \mathbf{u}^{(0)}(\lambda) \right)_s = \sum_{j=0}^{\infty} p_{j-s}(x) \lambda^j = \lambda^s \sum_{j=0}^{\infty} p_j(x) \lambda^{-j} = \lambda^s e^{x(\lambda)},
\]
we have
\[
\left( P \mathbf{u}^{(i)}(\lambda) \right)_s = \left( P \frac{d^i}{d\lambda^i} \mathbf{u}^{(0)}(\lambda) \right)_s = \left( \frac{d^i}{d\lambda^i} P \mathbf{u}^{(0)}(\lambda) \right)_s = \frac{d^i}{d\lambda^i} \left( \lambda^s e^{x(\lambda)} \right).
\]
Let
\[
\tilde{u}^{(i)}(x, \lambda) = \frac{d^i}{d\lambda^i} \left( \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{k-1} \end{bmatrix} e^{x(\lambda)} \right),
\]
and
\[
\tilde{C}(x) = \left( \tilde{u}^{(0)}(x, \lambda_{01}), ..., \tilde{u}^{(0)}(x, \lambda_{0n}), ..., \tilde{u}^{(N)}(x, \lambda_{N1}), ..., \tilde{u}^{(N)}(x, \lambda_{Nr}) \right).
\]
We denote the \( r \)th column vector of \( \tilde{C}(x) \) by \( \tilde{C}(r) \). Then
\[
P \cdot (C_{r1}, ..., C_{rn}) = (\tilde{C}(x)_{r1}, ..., \tilde{C}(x)_{rn}).
\]
Exchanging the order of the summation in (21), using lemma 1 and the Binet–Cauchy formula again we get
\[
\tau(x) = \sum_{1 \leq r_1 < ... < r_n \leq N} \det \left( \tilde{C}(r_1), ..., \tilde{C}(r_n) \right) = \det \left( \tilde{C}(x)A \right).
\]
The \( j \)th column vector of \( \tilde{C}(x)A \) is given by
\[
(\tilde{C}(x)A)_j = \sum_{r=0}^{N} \sum_{j=r_1+...+r_{r-1}+1}^{r_1+...+r_{r-1}+1} a_{ij} \tilde{u}^{(i)}(x, \lambda_i)
\]
\[
= \sum_{r=0}^{N} \sum_{j=r_1+...+r_{r-1}+1}^{r_1+...+r_{r-1}+1} a_{ij} \left( \frac{d^i}{d\lambda^i} \begin{bmatrix} e^{x(\lambda)} \\ \lambda e^{x(\lambda)} \\ \vdots \\ \lambda^{k-1} e^{x(\lambda)} \end{bmatrix} \right)_{\lambda=\lambda_i}
\]
\[
= \sum_{r=0}^{N} \sum_{j=r_1+...+r_{r-1}+1}^{r_1+...+r_{r-1}+1} a_{ij} \left( \frac{d^i}{d\lambda^i} \begin{bmatrix} \frac{d}{d\lambda} e^{x(\lambda)} \\ \frac{d^2}{d\lambda^2} e^{x(\lambda)} \\ \vdots \\ \frac{d^{k-1}}{d\lambda^{k-1}} e^{x(\lambda)} \end{bmatrix} \right)_{\lambda=\lambda_i}.
\]
\[ = \sum_{s=0}^{N} \sum_{i=r_{i-1}+1}^{r_{i}} a_{ij} \begin{pmatrix} \frac{d}{dx} e^{\xi(x, \lambda)} \bigg|_{\lambda=\lambda_i} \\ \vdots \\ \frac{d^{r_i-1}}{dx^{r_i-1}} e^{\xi(x, \lambda)} \bigg|_{\lambda=\lambda_i} \end{pmatrix} \]

Thus

\[ \tau(x) = \det (\tilde{C}(x)A) = W(\tilde{f}_1, \ldots, \tilde{f}_k). \]

5. Change of a basis

For applications it is necessary to write \(1/(1 - \lambda z)^r\) as a linear combination of \(\{v^{(i)}(\lambda, z)\}\).

Lemma 2. For \(m \geq 0\)

\[
\frac{1}{(1 - \lambda z)^{m+1}} = \sum_{i=0}^{m} \frac{1}{i!} \left( \begin{array}{c} m \\ i \end{array} \right) \lambda^{i} v^{(i)}(\lambda, z). 
\]  (23)

Proof. By differentiating the equation

\[
\frac{1}{1 - w} = \sum_{n=0}^{\infty} w^n \\
\]

\(m\) times we get

\[
\frac{m!}{(1 - w)^{m+1}} = \sum_{n=0}^{\infty} (n + m)(n + m - 1) \cdots (n + 1) w^n.
\]

Substituting \(w = \lambda z\) and dividing by \(m!\) we obtain

\[
\frac{1}{(1 - \lambda z)^{m+1}} = \frac{1}{m!} \sum_{n=0}^{\infty} (n + m) \cdots (n + 1) \lambda^n z^n
\]

\[
= \frac{1}{m!} \frac{d^m}{d\lambda^m} \sum_{n=0}^{\infty} \lambda^{n+m} z^n
\]

\[
= \frac{1}{m!} \frac{d^m}{d\lambda^m} \left( \lambda^m \frac{1}{1 - \lambda z} \right).
\]

By computing the last expression using the Leibniz rule we get the desired result. \(\square\)
The case of $m = 0, 1$ of (23) is

$$\frac{1}{1 - \lambda z} = \psi^{(0)}(\lambda, z),$$
$$\frac{1}{(1 - \lambda z)^2} = \psi^{(0)}(\lambda, z) + \lambda \psi^{(1)}(\lambda, z).$$

(24)

Corollary 1. The set of frames of $(n, k)$ generalized solitons, $n \geq 1$, coincide with the set of frames of the following form:

$$z^{-(k-1)} \frac{g_i(z)}{f_i(z)}, \quad 1 \leq i \leq k,$$
$$z^{-i}, \quad i \geq k,$$

where $f(z), g(z)$ are polynomials such that $f_i(0) \neq 0$, $\deg f_i(z) > \deg g_i(z)$ for any $i$ and $\{g_i(z)/f_i(z) \mid 1 \leq i \leq k\}$ is linearly independent.

The number $n$ in the corollary is determined from the structure of the roots of $f_i(z)$ as one sees in the example in the next section.

6. Example of GS

Let $a_{ij}, b_{ij}, 1 \leq i \leq n, 1 \leq j \leq k$ be complex numbers.

Lemma 3. The following set of functions is a frame of a $(2n, k)$ GS if the first $k$ functions are linearly independent.

$$z^{-k+1} \sum_{i=1}^{n} \left( \frac{a_{ij}}{1 - \lambda_i z} + \frac{b_{ij}}{(1 - \lambda_i z)^2} \right), \quad 1 \leq j \leq k$$
$$z^{-i}, \quad i \geq k.$$

(25)

The corresponding matrix $A$ is given by

$$A = \begin{pmatrix}
(a_{11} + b_{11}) & \cdots & (a_{1k} + b_{1k}) \\
\vdots & \ddots & \vdots \\
(a_{nk} + b_{nk}) & \cdots & (a_{nk} + b_{nk}) \\
\lambda_1 b_{11} & \cdots & \lambda_1 b_{1k} \\
\vdots & \ddots & \vdots \\
\lambda_n b_{nk} & \cdots & \lambda_n b_{nk}
\end{pmatrix}.$$  

(26)

Proof. By corollary 1 the set of functions (25) is a frame of a GS under the assumption of the lemma. Let us calculate the type of the GS and the matrix $A$ associated with it.

By (24) we get

$$\sum_{i=1}^{n} \left( \frac{a_i}{1 - \lambda_i z} + \frac{b_i}{(1 - \lambda_i z)^2} \right)$$

$$= \left[ \psi^{(0)}(\lambda_1, z), \ldots, \psi^{(0)}(\lambda_n, z), \psi^{(1)}(\lambda_1, z), \ldots, \psi^{(1)}(\lambda_n, z) \right]$$

$$\begin{pmatrix}
a_1 + b_1 \\
\vdots \\
(a_n + b_n) \\
\lambda_1 b_1 \\
\vdots \\
\lambda_n b_n
\end{pmatrix}.$$
Therefore, by theorem 3, (25) is a frame of a \((2n,k)\) GS with \(N=1, r_0 = r_1 = n, \lambda_0 = \lambda, \lambda_1 = \lambda, \) and the \(2n \times k\) matrix \(A\) is given by (26).

The solution \(\tau(x)\) corresponding to the frame (25) is written in the Wronskian form by the definition of a GS. Let us compute it.

Let \(v_k, w_k\) be the column vectors with \(k\) components given by

\[ v_k(\lambda) = (\lambda^{i-1})_{1 \leq i \leq k} \]

\[ w_k(\lambda) = (i\lambda^{i-1} + \lambda' \xi'(x,\lambda))_{1 \leq i \leq k}, \quad \xi'(x,\lambda) = \frac{d\xi(x,\lambda)}{d\lambda} = \sum_{j=1}^{\infty} j\lambda^{j-1} x_j. \]

Define the \(k \times 2n\) matrix \(D\) by

\[ D = (v_k(\lambda_1), \ldots, v_k(\lambda_n), w_k(\lambda_1), \ldots, w_k(\lambda_n)). \] 

Then

\[ \tau(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq 2n} A_{i_1 \ldots i_k} D_{i_1 \ldots i_k} e^{\xi_{i_1} + \cdots + \xi_{i_k}} \]

(28)

where \(D_{i_1 \ldots i_k}\) is the minor determinants corresponding to columns \(i_1, \ldots, i_k\) of \(D\). Notice that \(\lambda_{n+j} = \lambda_j\) and \(\xi_{n+j} = \xi_j\) for \(1 \leq j \leq n\). It is different from a soliton solution, since \(D_{i_1 \ldots i_k}\) depends on \(x_j\)'s in general.

7. Affine rings and the Sato Grassmannian

In this section we review how to associate a point of the UGM with the affine ring of a compact Riemann surface.

Let \(X\) be a compact Riemann surface of genus \(g\), \(p_\infty\) a point of \(X\) and \(z\) a local coordinate at \(p_\infty\). We denote by \(U_a\) the vector space of meromorphic functions on \(X\) which are holomorphic on \(X\setminus\{p_\infty\}\). It is called the affine ring of \(X\setminus\{p_\infty\}\).

We define the map \(\iota : U_a \rightarrow \mathbb{C}(z)\) as follows.

Let \(F\) be an element of \(U_a\) and \(F(z)\) the Laurent expansion of \(F\) at \(p_\infty\) in \(z\). Then we define

\[ \iota(F) = z^g F(z). \]

**Theorem 4 [(25, 28, 35)]. The subspace \(\iota(U_a)\) belongs to the UGM.**

8. \((3, 3m + 1)\) Curves and the sigma function solution

Let \(m\) be a positive integer and \(\lambda_1, \ldots, \lambda_{3m}\) non-zero complex numbers such that \(\lambda_1, \ldots, \lambda_{3m}\) are mutually distinct. We consider the algebraic curve defined by

\[ y^3 = x^{3m} \prod_{j=1}^{3m} (x - \lambda_j^3). \] 

(29)

It can be compactified by adding one point at \(\infty\). We denote the compact Riemann surface by \(X\) which is a special case of a \((3, 3m + 1)\) curve [7–9]. The genus of \(X\) is \(g = 3m\).
We take \( p_\infty = \infty \). Then the affine ring \( U_\omega \) of \( X \setminus \{ \infty \} \) is the space of polynomials of \( x, y \) and a basis of it as a vector space is given by
\[
x^i, \quad x^i y, \quad x^i y^2, \quad i \geq 0.
\]
(30)

We take the local coordinate \( z \) around \( \infty \) such that
\[
x = z^{-3}, \quad y = z^{-3m-1} \hat{f}(z), \quad \hat{f}(z) := \prod_{j=1}^{3m} (1 - \lambda_j^3 z^3)^{1/3}.
\]

The function \( \hat{f}(z) \) is considered as a power series in \( z \) by the Taylor expansion. Expanding elements of \( (30) \) in \( z \) we get a frame of \( \iota(U_\omega) \):
\[
z^{3m-3i}, \quad z^{-1-\lambda_i} \hat{f}(z), \quad z^{-3m-2-\lambda_i} \hat{f}(z)^2, \quad i \geq 0.
\]
(31)

The solution corresponding to \( \iota(U_\omega) \) can be written in terms of the multi-variate sigma function [28]. Let us recall it. To this end we first review the construction of the sigma function of \( X \) following [27, 29].

Let \( f_i, 1 \leq i \leq g = 3m \) be the monomials of \( x, y \) defined by
\[
(f_1, \ldots, f_g) = (1, x, \ldots, x^m, y, x^{m+1}, xy, \ldots, x^{2m-1}, x^{m-1} y),
\]
and \( (w_1, \ldots, w_g) \) the gap sequence at \( \infty \):
\[
(w_1, \ldots, w_g) = (1, 2, 4, 5, \ldots, 3m - 2, 3m - 1, 3m + 2, 3m + 5, \ldots, 6m - 1).
\]
We define \( du_{w_i}, 1 \leq i \leq g \) by
\[
du_{w_i} = -f_{g+1-i} \frac{dx}{3^3}.
\]
Then \( \{ du_{w_i} \}_{i=1}^g \) becomes a basis of holomorphic one forms on \( X \). The one form \( du_{w_i} \) has a zero order of \( w_i - 1 \) at \( \infty \) and has the expansion at \( \infty \) of the form
\[
du_{w_i} = \sum_{j=1}^{\infty} b_{ij} z^{-j} dz, \quad b_{ij} = \begin{cases} 0 & \text{if } j < w_i, \\ 1 & \text{if } j = w_i. \end{cases}
\]
(32)

Let \( \{ \alpha_i, \beta_i \}_{i=1}^g \) be a canonical homology basis, \( \delta \) the Riemann’s constant with respect to the base point \( \infty \). We choose an algebraic fundamental form \( \hat{\omega}(p_1, p_2) \) of proposition 2 in [27].

We define the period matrices \( \omega_i, \, i = 1, 2 \) and \( \Omega \) by
\[
2\omega_1 = \left( \int_{\alpha_i} du_{w_j} \right)_{1 \leq i, j \leq g}, \quad 2\omega_2 = \left( \int_{\beta_i} du_{w_j} \right)_{1 \leq i, j \leq g}, \quad \Omega = \omega_1^{-1} \omega_2.
\]
The \( \alpha_j \) and \( \beta_j \) integrals of \( \hat{\omega}(p_1, p_2) \) with respect to the variable \( p_2 \) are shown to be holomorphic one forms (corollary 6 of [29]). With the help of this fact we define \( \eta_r = (\eta_{r,ij})_{1 \leq i, j \leq g} \) by
\[
\int_{\alpha_i} \hat{\omega}(p_1, p_2) = \sum_{i=1}^{g} du_{w_i}(p_1)(-2\eta_{ij}), \quad \int_{\beta_i} \hat{\omega}(p_1, p_2) = \sum_{i=1}^{g} du_{w_i}(p_1)(-2\eta_{ij}).
\]
The components \( \eta_{r,ij} \) are the periods of certain second kind differentials.

Let \( \theta[\varepsilon](z|\Omega) \) be Riemann’s theta function with the characteristics \( \varepsilon = \begin{pmatrix} \varepsilon' \\ \varepsilon'' \end{pmatrix}, \varepsilon', \varepsilon'' \in \mathbb{R}^g \).

We consider the function \( \theta[-\delta](\{ 2\omega_1 \}^{-1} u|\Omega) \) of \( u = (u_{w_1}, \ldots, u_{w_g}) \).
We define \( \gamma \in \{ w_1, \ldots, w_g \}^m \) by
\[
\gamma = (\gamma_1, \ldots, \gamma_m) = (6m - 1, 6m - 7, \ldots, 5),
\]
and set
\[
\partial_\gamma = \partial_{\gamma_1} \cdots \partial_{\gamma_m}, \quad \partial_{\gamma_i} = \frac{\partial}{\partial u_{\gamma_i}}.
\]
By corollary 3 of [29] we have
\[
\partial_\gamma \theta[\delta](0|\Omega) := \neq 0.
\]

**Definition 3.** The sigma function associated with \((\{du_{wi}\}, \hat{\omega}(p_1, p_2))\) is defined by
\[
\sigma(u) = (-1)^{n(d-1)} \frac{\theta[\delta](2\omega_1)^{-1}u(\Omega)}{\partial_\gamma \theta[\delta](0|\Omega)} \exp \left( \frac{1}{2} u(\omega_1^{-1}u) \right), \quad u = (u_{wi}, \ldots, u_{wg}).
\]

The sigma function has the following remarkable properties which are absent in Riemann’s theta function.

**Theorem 5 ([27, 28]).**
(i) The function \( \sigma(u) \) does not depend on the choice of a canonical homology basis \( \{\alpha_i, \beta_i\} \).
(ii) The coefficients of the Taylor expansion of \( \sigma(u) \) are polynomials of \( \lambda_3^j, 1 \leq j \leq 3m, \) with the coefficients in \( \mathbb{Q} \).

In order to give the formula for the \( \tau \) function we need some more notation. Let us write the expansion of \( \hat{\omega}(p_1, p_2) \) as
\[
\hat{\omega}(p_1, p_2) = \left( \frac{1}{(z_1 - z_2)^2} + \sum_{i,j=1}^{\infty} \hat{q}_{ij} z_1^{i-1} z_2^{j-1} \right) dz_1 dz_2,
\]
where \( z_i = z(p_i) \). By (32) we see that it is possible to define \( \{c_i\}_{i=1}^{\infty} \) by
\[
\log \left( z^{-(s-1)} \sqrt{\frac{du_{wi}}{dz}} \right) = \sum_{i=1}^{\infty} c_i \frac{1}{i}.
\]

By lemma 15 of [27] we have

**Proposition 1.** The expansion coefficients \( \hat{q}_{ij} \) and \( c_i \) are polynomials of \( \{\lambda_3^j\} \).

We set
\[
B = (h_{ij})_{1 \leq i \leq g, 1 \leq j \leq l}, \quad \hat{q}(x) = \sum_{i,j=1}^{\infty} \hat{q}_{ij} x_i x_j.
\]

Then

**Theorem 6 ([28]).** A \( \tau \) function corresponding to \( i(U_a) \) is given by
\[
\tau(x) = \exp \left( - \sum_{i=1}^{\infty} c_i x_i + \frac{1}{2} \hat{q}(x) \right) \sigma(Bx),
\]
and it is a solution of the 3-reduced KP-hierarchy.
9. Degeneration of \((3, 3m + 1)\) curves

We consider the following limit of \(X\):
\[
\lambda_{j+m}, \lambda_{j+2m} \to \lambda_j, \quad 1 \leq j \leq m.
\] (33)
The equation of \(X\) tends to
\[
y^3 = x \prod_{j=1}^{m} (x - \lambda_j^3), \quad x = z^{-3}.
\] (34)

We assume that \(\lambda_j^3, 1 \leq j \leq m\), are non-zero and mutually distinct. It has singularities only at \(Q_j = (\lambda_j^3,0), 1 \leq j \leq m\), which is an ordinary singular point with the multiplicity 3 (ordinary triple point). In the following we denote this singular rational affine algebraic curve by \(X_{\text{aff}}^{\text{sing}}\) and by \(U_a(X_{\text{sing}}^{\text{aff}})\) the affine ring of \(X_{\text{sing}}^{\text{aff}}\), that is,
\[
U_a(X_{\text{sing}}^{\text{aff}}) = \mathbb{C}[x,y]/J,
\]
where \(J\) is the ideal generated by \(y^3 - x \prod_{j=1}^{m} (x - \lambda_j^3)^3\).

Similarly to the non-singular case, \(U_a(X_{\text{sing}}^{\text{aff}})\) can be defined and it belongs to the UGM. A basis of \(U_a(X_{\text{sing}}^{\text{aff}})\) is given by (30). Therefore we have the corresponding basis of \(U_a(X_{\text{sing}}^{\text{aff}})\) by expanding elements of it in \(z\) and multiplying them by \(z^{3m}\).

Let \(f(z)\) be the limit of \(\frac{d}{dz} f(z)\):
\[
f(z) = \prod_{j=1}^{m} (1 - \lambda_j^3 z^3).
\]
Then the limit of the frame (31) of \(\iota(U_a)\) becomes
\[
z^{3m-3i}, \quad z^{-3i-1} f(z), \quad z^{-3m-3i-2} f(z)^2, \quad i \geq 0,
\] (35)
which coincides with the frame of \(\iota(U_a(X_{\text{sing}}^{\text{aff}}))\) explained above. So (35) is a frame of \(\iota(U_a(X_{\text{sing}}^{\text{aff}}))\). We show that this frame can be transformed to a frame of a generalized soliton.

To this end we need the notion of a gauge transformation.

Let \(G(z)\) be a formal power series in \(z\) such that \(G(0) = 1\) and \(X = (u_i(z))_{i \geq 1}\) a frame of the UGM. Then it is easy to see that \(G(z)X = (G(z)u_i(z))_{i \geq 1}\) becomes a frame of a point of the UGM. The multiplication by \(G(z)\) is called a gauge transformation. Let
\[
\log G(z) = \sum_{i=1}^{\infty} g_i z^i.
\]
Then
\[
\tau(x, G(z)X) = e^{\sum_{i=1}^{\infty} g_i z^i} \tau(x, X).
\]
Let \(\omega = e^{2\pi i/3}\) and
\[
(\lambda_1, \ldots, \lambda_{3m}) = (\lambda_1, \ldots, \lambda_m, \omega \lambda_1, \ldots, \omega \lambda_m, \omega^2 \lambda_1, \ldots, \omega^2 \lambda_m).
\]
Notice that we redefine \(\lambda_i (m + 1 \leq j \leq 3m)\) as above. They are not relevant to \(\lambda_j (m + 1 \leq j \leq 3m)\) in (29).

**Theorem 7.** The following set of functions is a frame of \(f(z)^{-2} \iota(U_a(X_{\text{sing}}^{\text{aff}}))\). It is a frame of a \((6m, 3m)\) generalized soliton.
\[ z^{-3m+1} \sum_{i=1}^{3m} \left( \frac{a^{(0)}_i}{1 - \lambda_i z} + \frac{b^{(0)}_i}{(1 - \lambda_i z)^2} \right), \quad 1 \leq j \leq m, \]

\[ z^{-3m+1} \sum_{i=1}^{3m} \frac{a^{(r)}_i}{1 - \lambda_i z}, \quad r = 1, 2, \quad 1 \leq j \leq m, \]

\[ z^{-j}, \quad j \geq 3m. \tag{36} \]

Here, for \( 1 \leq i \leq m, \)

\[ a^{(0)}_i = - \left( 2m - j + 2 \sum_{\ell \neq i}^{m} \frac{\lambda_i^2}{\lambda_i^2 - \lambda_i^j} \right) \frac{\lambda_i^j - \lambda_i}{3 \prod_{\ell \neq i} (\lambda_i^j - \lambda_i^j)^2}, \]

\[ a_{i+m,j}^{(0)} = \omega a^{(0)}_i, \quad a_{i+2m,j}^{(0)} = \omega^2 a^{(0)}_i, \]

\[ b^{(0)}_i = \frac{\lambda_i^{j-8}}{9 \prod_{\ell \neq i} (\lambda_i^j - \lambda_i^j)^2}, \quad b_{i+m,j}^{(0)} = \omega b^{(0)}_i, \quad b_{i+2m,j}^{(0)} = \omega^2 b^{(0)}_i, \]

\[ a^{(r)}_i = \frac{\lambda_i^{j-3-r}}{3 \prod_{\ell \neq i} (\lambda_i^j - \lambda_i^j)^2}, \quad a_{i+m,j}^{(r)} = \omega^{-r} a^{(r)}_i, \quad a_{i+2m,j}^{(r)} = \omega^{-2r} a^{(r)}_i. \]

In order to prove the theorem we first make a linear change of the basis (35).

**Lemma 4.** The following is a basis of \( \iota(U_{a}(X_{\text{sING}})^{\text{aff}})):

\[ z^{3m-3i} (0 \leq i \leq m-1), \quad z^{-3i}f(z) (0 \leq i \leq m-1), \]

\[ z^{-3m-i}f(z)^2 (i \geq 0). \tag{37} \]

**Proof.** It is easy to see that the set of elements \( z^{3m-3i}, i \geq 0 \) and the set of elements \( z^{3m-3i}, 0 \leq i \leq m-1, z^{-3i}f(z), 0 \leq i \leq m-1, z^{-3m-3i}f(z)^2, i \geq 0 \) are connected by a triangular matrix with 1 in the diagonal entries. The set of elements \( z^{-3i-1}f(z), i \geq 0 \) and the set of elements \( z^{-3m-3i-1}f(z)^2, 0 \leq i \leq m-1, z^{-3m-3i-1}f(z)^2, i \geq 0 \) are connected similarly. The remaining elements \( z^{-3m-3i-3}f(z)^2, i \geq 0 \) of (35) and (37) are the same. Thus (37) is a basis of \( \iota(U_{a}(X_{\text{sING}})^{\text{aff}})) \). \( \square \)

**Proof of theorem 7.** Multiplying elements in (37) by \( f(z)^{-2} \) and expanding them into partial fractions we get (36). Thus it is a frame of \( f(z)^{-2}\iota(U_{a}(X_{\text{sING}})^{\text{aff}})) \). It follows from lemma 3 that (36) is a frame of a \((6m, 3m)\) generalized soliton. \( \square \)

We set

\[ b^{(r)}_i = 0, \quad 1 \leq i \leq 3m, 1 \leq j \leq m, r = 1, 2. \]

The \( 6m \times 3m \) matrix \( A \) corresponding to the frame (36) is given by

\[ A = \begin{pmatrix} A^{(0)} & A^{(1)} & A^{(2)} \\ b^{(0)} & O & O \end{pmatrix}. \tag{38} \]
where
\[ A^{(r)} = \left( a_{ij}^{(r)} + b_{ij}^{(r)} \right)_{1 \leq i,j \leq 3m, 1 \leq r \leq m}, \quad B^{(0)} = \left( \lambda_i b_{ij}^{(0)} \right)_{1 \leq i,j \leq 3m, 1 \leq j \leq m}. \]

The solution corresponding to the frame (36) of \( f(z)^{-2} f(U_a(X^a_{\text{sing}})) \) can be computed from this matrix using the formula (28).

10. Solution in the case of \( m = 1 \)

Let us consider the case of \( m = 1 \) (\( g = 3 \)) of theorem 7 and write down the solution corresponding to \( f(U_a(X^a_{\text{sing}})) \) using (27), (28) and (38).

We have
\[ a_{1,1}^{(0)} = -\frac{\lambda_1^{-5}}{3}, \quad a_{1,1}^{(1)} = \frac{\lambda_1^{-1}}{3}, \quad a_{1,1}^{(2)} = \frac{\lambda_1^{-2}}{3}, \quad b_{1,1}^{(0)} = \frac{\lambda_1^{-5}}{9}. \]

The matrix \( A \) is a \( 6 \times 3 \) matrix given by
\[
A = \begin{pmatrix}
a_{1,1}^{(0)} + a_{1,1}^{(0)} & a_{1,1}^{(1)} & a_{1,1}^{(2)} \\
\omega(a_{1,1}^{(0)} + b_{1,1}^{(0)}) & \omega^2a_{1,1}^{(1)} & \omega^2a_{1,1}^{(2)} \\
\omega^2(a_{1,1}^{(0)} + b_{1,1}^{(0)}) & \omega^2a_{1,1}^{(1)} & \omega^2a_{1,1}^{(2)} \\
\lambda_1 b_{1,1}^{(0)} & 0 & 0 \\
(\lambda_1 \omega) b_{1,1}^{(0)} & 0 & 0 \\
(\lambda_1 \omega^2) b_{1,1}^{(0)} & 0 & 0
\end{pmatrix} = \frac{\lambda_1^3}{9} \begin{pmatrix}-2 & 3\lambda_1^4 & 3\lambda_1^3 \\
-2\omega & 3\lambda_1^4 & 3\omega^2\lambda_1^3 \\
-2\omega^2 & 3\omega^2\lambda_1^4 & 3\omega^2\lambda_1^3 \\
\lambda_1 & 0 & 0 \\
\lambda_1 \omega & 0 & 0 \\
\lambda_1 \omega^2 & 0 & 0
\end{pmatrix}.
\]

For \( r = 0, 1, 2 \) let
\[ \eta_r = \sum_{j \neq 1, j \equiv \text{mod.3}} x_j \lambda_1^j, \quad \eta'_r = \frac{dx_j}{d\lambda_1} = \sum_{j \neq 1, j \equiv \text{mod.3}} jx_j \lambda_1^{j-1}. \]

Then the \( \tau \) function \( \tau(x) = \tau(x, f(U_a(X^a_{\text{sing}}))) \) is given by
\[
\tau(x) = (-1)^{\left( \frac{3}{2} \right)} \lambda_1^{-5} e^{-3\eta_r} \left\{ 2e^{\frac{3}{2}(\eta_1 + \eta_2)} \sin \left( \frac{\sqrt{3}}{2} (\eta_1 - \eta_2) + \frac{2\pi}{3} \right) + 2e^{-\frac{3}{2}(\eta_1 + \eta_2)} \sin \left( \frac{\sqrt{3}}{2} (\eta_1 - \eta_2) - \frac{2\pi}{3} \right) - 2\sin \left( \frac{\sqrt{3}}{3} (\eta_1 - \eta_2) \right) \right\}.
\]

From this expression the following properties of the solution can be seen.

Firstly, if \( \lambda_1 \) and \( x_j, j \geq 1 \) are real then \( \tau(x) \) is real. Secondly, the variables \( x_{3j}, j \geq 1 \) only appear in \( e^{-3\eta_r} \) which means that \( \tau(x) \) is a solution of the 3-reduced KP-hierarchy. It is valid for any \( m \) since \( \tau(x) \) is a solution of the 3-reduced KP-hierarchy before taking the limit by theorem 6. Thirdly, \( \tau(0) = 0 \) which is also the case of any \( m \).

11. Discussion

In this paper we consider the trigonal curve (1) and the corresponding quasi-periodic solutions of the KP-hierarchy. We have computed the limits of the solutions exactly in the Wronskian.
form when the curve degenerates to the singular rational curve (2). The obtained solutions have similar forms to \((2g, g)\) solitons which are linear combinations of exponentials in space-time variables. But they are different from \((2g, g)\) solitons since the tau functions of them are the mixtures of exponentials and polynomials.

Here we examine the properties of the solutions obtained mainly according to the example (39) of \(g = 3\). The analysis for general \(g\) is a problem for the future.

The KP equation (4), which was originally introduced to study the stability of the one soliton solution of the KdV equation [18], describes shallow water waves in a two dimensional space (see [5, 17] for example). A solution of the KdV equation

\[-4u_t + 6uu_x + u_{xxx} = 0\]

is considered as a \(y\)-independent solution of the KP equation. The wave crest of an \(n\)-soliton solution of the KdV equation, if the time is sufficiently large, consists of \(n\) lines on the \(xy\)-plane which are parallel to the \(y\)-axis. In general, the wave crest of a \(n\)-soliton solution [30] of the KP equation forms intersecting \(n\) lines which are not necessarily parallel to the \(y\)-axis. Thereby soliton solutions of the KP equation are called line solitons. Recently, more general soliton solutions are actively studied [16]. They form not only intersecting lines but also various web patterns according to the values of parameters. In any case, soliton solutions, if time is fixed, decay exponentially as \(x, y\) tends to \(\pm \infty\) except along finite number of directions. This is the most fundamental property of soliton solutions of the KP equation.

Now let us examine our solution. We set

\[x = x_1, \quad y = x_2, \quad t = x_3, \quad x_j = 0 (j \geq 4), \quad \lambda = \lambda_1.\]

Then \(u = 2\partial_x^2 \log \tau\) is a solution of the KP equation (4) as mentioned in section 2. We assume \(\lambda > 0\) in the following.

The special properties of the solution (39) corresponding to the trigonal curve (2) are as follows.

(i) The function \(u(x, y, t)\) does not depend on \(t\).
(ii) The parameters are \((\lambda_1, \ldots, \lambda_6) = (\lambda, \omega \lambda, \omega^2 \lambda, \lambda, \omega \lambda, \omega^2 \lambda)\).
(iii) There is the polynomial term \(-3\sqrt{3} \lambda_1 \eta_2 = -6\sqrt{3} \lambda x \) in \(\tau\).

The property (i) is a consequence of the fact that there exists a meromorphic function on the curve (1) which has a pole at \(\infty\) of order three and no poles elsewhere. This fact is characteristic for trigonal curves. The properties (ii) and (iii) are related to the special form of the curve (1) and the triple point singularities of (2).

By the property (i) \(u(x, y)\) becomes a solution of the Boussinesq equation

\[3u_{xy} + (6uu_x + u_{xxx})_x = 0,\]

where \(y\) is considered as a time variable. So let us examine the properties of \(u\) as a solution of the Boussinesq equation.

First of all it should be noted that \(u(x, y)\) is unbounded and is not a physical solution. However it has several interesting properties as we shall see below.

The asymptotics of the solution as \(x \to \pm \infty\) with the time fixed is as follows:

\[u(x, y) \to \frac{-\frac{1}{2} \lambda^2}{\cos^2 \left( \frac{x}{6} \pm \frac{\sqrt{3}}{2} (\lambda x - \lambda^2 y) \right)} \quad x \to \pm \infty, \quad y: \text{fixed}.\]

This formula suggests that \(u(x, y)\) is a kind of periodic solution (figure 1). This periodic nature of the solution is a consequence of the property (ii). On the other hand if we consider a
hyperelliptic curve and coalesce branch points in pairs, such a periodic function does not appear. We can see the formula (41) in such a way that there is a soliton at each zero of the denominator and \( u(x, y) \) is a superposition of those solitons. Actually we can observe collisions of solitons at finite values of \( x \) by the computer simulation. We shall discuss it in some detail below. The difference of the phases between the limits \( x \to \pm\infty \) can be considered as a consequence of the collisions.

Next, let us study the time evolution of those solitons, that is, the form of \( u(x, y) \) on the \( xy \)-plane. To this end we define lines \( \ell_c, \ell_{\pm,n}, c \in \mathbb{R}, n \in \mathbb{Z} \) by

\[
\ell : x + \lambda y = c, \quad \ell_{\pm,n} : \frac{\pi}{6} \pm \frac{\sqrt{3}}{2}(\lambda x - \lambda^2 y) = \frac{\pi}{2} + n\pi.
\]

Then we have

\[
u(x, y) \to 0 \quad x \to \pm\infty \text{ along the line } \ell_c,
\]

where \( u \) decays like \( 1/x^2 \). This is due to property (iii). The asymptotics of \( u \) along the lines \( \ell_{\pm,n} \) are

\[
u \to \begin{cases} -\infty & x \to \pm\infty \text{ along } \ell_{\pm,n} \\ -\lambda^2 & x \to \mp\infty \text{ along } \ell_{\pm,n}. \end{cases}
\]

With the aid of computer simulation we can observe that a soliton is associated to each of the lines \( \ell_{\pm,n}, n \in \mathbb{Z} \) and it usually moves to the positive direction of the \( x \)-axis by the velocity near \( \lambda \). However, near the time specified by the intersection of \( \ell_{\pm,n} \) and \( \ell_0 \), one of the solitons goes backward and the collision takes place (figure 2). After the collision two solitons disappear for some time and then they appear. Then, one of the solitons first moves to the negative direction and, after that, goes to the positive direction. Therefore, as a stationary solution of the KP equation, \( u(x, y) \) consists of an infinite number of parallel lines in the regions \( x, y \ll 0 \) and \( x, y \gg 0 \) on the \( xy \)-plane. These lines are disconnected along the line \( \ell_0 \) where pairs of lines form a ‘\( u \)-shape’ or ‘\( n \)-shape’ (figure 3(a)). The origin is the exception where three lines intersect (figure 3(b)).

Finally, we discuss some symmetry property that the solution (39) has. To see it we need to recall that there are two versions of the KP equation, KP1 and KP2 (see [4] for example). They are not transformed to each other by a real scale change of variables. The equation (4) is the KP2 equation.

The KP1 equation and the corresponding version of the Boussinesq equations are

Figure 1. \( -u(x, y) \). (a) \( \lambda = 0.5, y = -200, -10 \leq x \leq 40 \). (b) \( \lambda = 0.5, y = 200, -10 \leq x \leq 40 \).
\[ 3u_{yy} + (-4u_t + 6uu_x - u_{xxx})_x = 0, \]
\[ 3u_{yy} + (6uu_x - u_{xxx})_x = 0, \]

respectively. The KP2 equation is transformed to the KP1 equation by \( x \to ix, \ y \to iy, \ t \to it \).

A real valued solution of the KP2 equation is, in general, transformed to a complex valued solution of the KP1 equation by this change. However, for the present solution, \( u(ix, iy) \) becomes real again. This property can be easily checked if we rewrite \( \tau \) of (39) as

\[ \tau = c(x,y,t) + d(x,y,t), \]

where \( c(x,y,t) \) and \( d(x,y,t) \) are real valued functions. Note that this property is crucial for the construction of soliton solutions.

Figure 2. \( -u(x,y) \). (a) \( \lambda = 0.5, \ y = -50, \ -10 \leq x \leq 40 \). (b) \( \lambda = 0.5, \ y = -49, \ -10 \leq x \leq 40 \). (c) \( \lambda = 0.5, \ y = -48, \ -10 \leq x \leq 40 \). (d) \( \lambda = 0.5, \ y = -45.7, \ -10 \leq x \leq 40 \). (e) \( \lambda = 0.5, \ y = -45.5, \ -10 \leq x \leq 40 \). (f) \( \lambda = 0.5, \ y = -44, \ -10 \leq x \leq 40 \).
It is known that the characters of solutions of the KP1 and the KP2 equations are very different (see [5] and references therein). For example, the so called lump solutions, which decay algebraically in all directions and therefore are localized in both space and time directions, are found only for the KP1 equation. So it is interesting to see what $u(ix, iy)$ looks like.

First, we consider $u(ix, iy)$ as a solution of the Boussinesq equation (42). For each $y$ we have $u \to 0$ as $x \to \pm \infty$. We can see, by the computer simulation, that for each $y$ there is one value of $x$ for which $u$ diverges to $-\infty$ (figure 4). So it looks like one soliton solution.
Next, we examine the time evolution. We introduce the line \( \ell' : x - \lambda y = 0 \). The function \(|u(ix, iy)|\) decays exponentially as \( x, y \to \pm \infty \) along all lines passing through the origin except the line \( \ell' \) along which \(|u(ix, iy)|\) decays like \( 1/x^2 \). It reminds us of a lump solution. A lump solution is described by the polynomial tau function (see [5, 10] and references therein). The tau function of the present solution contains a polynomial term as mentioned in (iii). The algebraic decay of \( u \) along \( \ell' \) is a consequence of this term. So our solution looks like an intermediate solution between a line soliton and a lump solution (figure 5). Recently, lump solutions have attracted much attention as a model of rogue waves (see [10, 19] and references therein).

We end this discussion by a remark. As we have seen, our periodic-like solution of the KP2 equation is converted to one soliton-like solution of the KP1 equation. We can easily see the converse correspondence. Namely, the one soliton solution of the KP2 equation

\[
    u(x, y, t) = \frac{\frac{1}{2} \lambda^2}{\cosh^2 \left( \frac{1}{2} \left( \lambda x + \lambda^2 y + \lambda^3 t \right) \right)}
\]

is transformed to the periodic solution of the KP1 equation

\[
    u(ix, iy, it) = \frac{\frac{1}{2} \lambda^2}{\cos^2 \left( \frac{1}{2} \left( \lambda x + \lambda^2 y + \lambda^3 t \right) \right)},
\]

which is similar to (41). No collision of solitons occurs in this case.

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**Figure 5.** KP1. The horizontal axis = x, the vertical axis = y. (a) The contour plot: \( u(x, y, 0.5) = -0.5, -150 \leq x \leq 150, -200 \leq y \leq 200 \). (b) The contour plot: \( u(x, y, 0.5) = -10, -150 \leq x \leq 150, -200 \leq y \leq 200 \).
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