The Fitting height of finite groups with a fixed-point-free automorphism satisfying an identity

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Abstract

Motivated by classic theorems of Thompson and Berger on the Fitting height of finite groups with a fixed-point-free automorphism of coprime order, we conjecture that, for every non-zero polynomial \( f(x) = a_0 + a_1 x + \cdots + a_d x^d \in \mathbb{Z}[x] \), there is an integer \( k > 0 \) with the following property. Let \( G \) be a finite (solvable) group with a fixed-point-free automorphism \( \alpha \) satisfying \( \gcd(|G|, k) = 1 \) and

\[
\{ g^{a_0} \cdot \alpha(g)^{a_1} \cdot \alpha^2(g)^{a_2} \cdots \alpha^d(g)^{a_d} \mid g \in G \} = \{1\}.
\]

Then the Fitting height of \( G \) is at most the number of irreducible factors of \( f(x) \). We confirm the conjecture for a large family of polynomials with explicit constants \( k \).

1 Introduction

Classic results. Let \( G \) be a finite group admitting a fixed-point-free automorphism \( \alpha \). Such a group \( G \) is necessarily solvable, according to the classification of the finite simple groups [21]. It is not known whether the derived length of \( G \) can be bounded from above by a function that depends only on the order \( n := |\alpha| \) of the automorphism. But Dade [5] was able to find such a function bounding the Fitting height of \( G \) and, in doing so, solved a conjecture of Thompson [24]. Jabara [13] has recently lowered the bound on the Fitting height to \( 7 \cdot \Omega(n^2) \), where \( \Omega(n) \) counts the number of prime divisors of \( n \) with multiplicity.

Under the additional assumption that \( \gcd(|G|, n) = 1 \), even stronger results have been obtained. A theorem of Berger [3], refining an earlier result of Thompson [24], states that the Fitting height of \( G \) is then at most \( \Omega(n) \). Examples of Gross [9] show moreover that this bound is optimal. In the special case that the automorphism has prime order \( n \), Berger’s theorem states that the group \( G \) is nilpotent. This result, also originally due to Thompson [23], gave a positive answer to the so-called Frobenius conjecture. In this case, a theorem by Higman [11] shows that even the nilpotency class of \( G \) can be bounded from above by a function that depends only on \( n \). Kreknin and Kostrikin [18] later found the explicit upper bound \( (n - 1)2^{\Omega(n)} \).

In order to discuss various analogues of these results, we recall a definition from [19]. Let \( H \) be a group and let \( \gamma \) be an endomorphism of \( H \). The polynomial \( f(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n \cdot x^n \in \mathbb{Z}[x] \) is an ordered identity of \( \gamma \) if the map

\[
f(\gamma) : H \longrightarrow H : h \mapsto h^{a_0} \cdot \gamma(h)^{a_1} \cdot \gamma^2(h)^{a_2} \cdots \gamma^n(h)^{a_n}
\]

vanishes identically.

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Example 1.1. Let $G$ be a finite group with automorphism $\alpha$. If the polynomial $-1 + x^n$ is an ordered identity of $\alpha$, then the order of $\alpha$ divides $n$. And vice-versa.

So we conclude that the Fitting height of a finite group admitting a fixed-point-free automorphism with ordered identity $-1 + x^n$ is bounded from above by $7 \cdot \Omega(n)^2$ and that it is bounded by $\Omega(n)$ if $\gcd(|G|, n) = 1$.

Example 1.2. Let $G$ be a finite group with endomorphism $\gamma$. If the constant polynomial $n$ is an ordered identity of $\gamma$, then the exponent of $G$ divides $n$. And vice-versa.

Finite groups of bounded exponent have been studied extensively in the context of the restricted Burnside problem, most notably by Hall—Higman [10], Kostrikin [17], and Zel’manov [27, 28]. It is known, in particular, that the Fitting height of a finite solvable group with ordered identity $n \in \mathbb{Z} \setminus \{0\}$ is bounded from above by a function that depends only on the prime factorization $\prod i p_i^{k_i}$ of $n$. In Shalev’s note [22], one can find the upper bound $\prod i (2k_i + 1)$, which readily implies the upper bound $3^{\Omega(n)}$.

Example 1.3. Let $G$ be a finite group with endomorphism $\gamma$. If the linear polynomial $-n + t$ is an ordered identity of $\gamma$, then $G$ is an $n$-abelian group. And vice-versa.

These $n$-abelian groups were studied by Baer [2] and classified by Alperin [1]. The classification implies that, for $n \notin \{0, 1\}$, the Fitting height of a finite, $n$-abelian group $G$ is at most $\max\{3^{\Omega(n)}, 3^{\Omega(n-1)}\}$. If we further assume that $\gcd(|G|, n(n - 1)) = 1$, then $G$ is known to be abelian, so that the Fitting height of $G$ is at most 1.

Example 1.4. Let $G$ be a finite group with automorphism $\alpha$. The automorphism $\alpha$ is said to be $n$-split if and only if $1 + x + x^2 + \cdots + x^{n-1}$ is an ordered identity of $\alpha$.

It is not difficult to verify that the automorphism in Example 1.1 is $n$-split if it is fixed-point-free. And, conversely, if the automorphism in Example 1.4 is fixed-point-free, then its order divides $n$. But finite groups $G$ with an $n$-split automorphism that is not necessarily fixed-point-free have also been studied extensively in the literature. Ersoy [8] has shown that such a group $G$ is solvable, provided that $n$ is odd. If $n$ is a prime, then $G$ is even nilpotent, according to a theorem of Hughes—Thompson [12] and Kegel [14]. We refer to the work of Khukhro [15, 16] and Zel’manov [29] for more results involving $n$-split automorphisms. We also highlight Zel’manov’s recent generalization of these results to torsion groups with identities [30].

More theorems in this general spirit can be found in the literature. We refer, in particular, to Turull’s classic results [25] on the Fitting height of finite groups with a fixed-point-free group of coprime operators, and to the recent results in [6, 7].

A change in perspective. For any given polynomial $f(x) \in \mathbb{Z}[x] \setminus \{0\}$, we now consider all the finite groups admitting a fixed-point-free automorphism with this ordered identity $f(x)$. We claim that it is possible to find a uniform upper bound on the Fitting height of such groups, and that we can find a particularly good bound if we further exclude finitely-many primes from the torsion in these groups.

Conjecture 1.5. For every $f(x) \in \mathbb{Z}[x] \setminus \{0\}$, there exist integers $k(f(x)) > 0$ and $m(f(x))$ with the following property.

Let $G$ be a finite (solvable) group admitting a fixed-point-free automorphism $\alpha$ and let $f(x)$ be an ordered identity of $\alpha$.

(a) Then the Fitting height of $G$ is at most $m(f(x))$.

(b) If $\gcd(|G|, k(f(x))) = 1$, then the Fitting height of $G$ is at most the number of irreducible factors of $f(x)$. 

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The classic theorems outlined above settle the conjecture for all polynomials of the form $-1 + x^n$, for all constant polynomials $n$, for all polynomials of the form $-n + x$, and for all polynomials of the form $(x^n - 1)/(x - 1)$. Moreover, in [19], we settled claim (b) of the conjecture for all irreducible polynomials. The results in [19] also extend the already-mentioned results of Alperin [1], Thompson [23], Higman [11], Kreknin—Kostrikin [18], Hughes—Thompson [12], Kegel [14], and Khukhro [15] in various ways.

**Main result.** In this paper, we will confirm claim (b) of the conjecture for the family of all Higman-solvable polynomials. These polynomials are formally introduced in Definition 3.3 and we postpone the technical details to Section 3. Informally, a polynomial $f(x) \in \mathbb{Z}[x]$ is said to be Higman-solvable if $f(0) \cdot f(1) \neq 0$ and if $f(x)$ admits a decomposition of the form $f(x) = g_1(x^{n_1} \cdots n_v) \cdots g_{c-1}(x^{n_{c-1}} n_v) \cdot g_c(x^n)$ with some additional conditions the polynomials $g_i(x)$. The following proposition shows that these conditions are satisfied with high probability, in the sense of probabilistic Galois theory.

We recall that the height of a polynomial is the maximal modulus of its coefficients.

**Proposition 1.6.** Fix $c, d_1, n_1, \ldots, d_c, n_c \in \mathbb{Z}_{\geq 1}$. For each $h \in \mathbb{Z}_{\geq 1}$, we consider the c-tuples $(g_1(x), \ldots, g_c(x))$ of monic polynomials in $\mathbb{Z}[x]$ with prescribed degrees $\deg(g_1(x)) = d_1, \ldots, \deg(g_c(x)) = d_c$ and heights at most $h$. Let $P(h)$ be the probability that the product

$$f(x) = g_1(x^{n_1} \cdots n_v) \cdots g_{c-1}(x^{n_{c-1}} n_v) \cdot g_c(x^n)$$

is Higman-solvable. Then $\lim_{h \to +\infty} P(h) = 1$.

We now turn to our main result. For a given Higman-solvable polynomial $f(x) \in \mathbb{Z}[x]$, we let $\text{inv}(f(x))$ be the non-zero, integer invariant of Definition 3.22. This $\text{inv}(f(x))$ will play the role of the invariant $k(f(x))$ of Conjecture 1.5. We also let $\text{irr}(f(x))$ be the number of irreducible factors of $f(x)$ of positive degree.

**Theorem 1.7.** Let $G$ be a finite (solvable) group admitting a fixed-point-free automorphism $\alpha$ and let $f(x)$ be any ordered identity of $\alpha$ that is Higman-solvable. If $\gcd(|G|, \text{inv}(f(x))) = 1$, then the Fitting height of $G$ is at most $\text{irr}(f(x))$.

Let us briefly discuss the assumptions. Every monic, Higman-solvable polynomial of positive degree is an ordered identity of a fixed-point-free automorphism of a finite, non-trivial group (Remark 5.4). Moreover, every fixed-point-free automorphism of a finite, non-trivial group has a monic, ordered identity that is Higman-solvable (Remark 5.5).

The strategy to prove Theorem 1.7 is straight-forward. First, we replace the ordered identities with the weaker notion of abelian identities. We then show that a natural power of the automorphism induced on the first lower Fitting subgroup $F_1(G)$ of $G$ satisfies an abelian identity with strictly fewer irreducible factors of positive degree. Our assumption on the prime divisors of $|G|$ will guarantee that the induced automorphism is still fixed-point-free. After at most $\text{irr}(f(x))$ iterations of this argument, we arrive at an abelian identity that is constant, say $c$. Since $|G|$ is assumed to be coprime to this constant $c$, we may conclude that the $\text{irr}(f(x))$'th lower Fitting subgroup of $G$ is the trivial group. We note that this strategy works if and only if the polynomial $f(x)$ is Higman-solvable.

In order to make the strategy precise, we will introduce a number of rather technical invariants of $f(x)$ in $\mathbb{Z}$ and $\mathbb{Z}[x]$. We list some of these in Table 1 for the convenience of the reader. The last column illustrates the special case $f(x) := \Phi_p(x)$, where $p$ is a prime and where $\Phi_p(x) = (x^p - 1)/(x - 1) = 1 + x + x^2 + \cdots + x^{p-1}$ is the cyclotomic polynomial that vanishes on the primitive $p$'th roots of unity. As we had observed in the examples above, this polynomial corresponds with the Frobenius conjecture.
Table 1: Invariants of a Higman-solvable polynomial $f(x) \in \mathbb{Z}[x]$.

| Auxiliary Invariant | Range | Definition | $f(x) := \Phi_p(x)$ |
|---------------------|-------|------------|-------------------|
| $f(x); f_*(x)$      | $\mathbb{Z}[x] \setminus \{0\}$ | 3.2 | $\Phi_p(x); 1$ |
| $\Delta(f(x))$     | $\mathbb{Z}[x] \setminus \{0\}$ | 3.3 | 1 |
| $f_1(x)$           | $\mathbb{Z}[x] \setminus \{0\}$ | 3.13 | 1 |
| $f_2^2(x)$         | $\mathbb{Z}[x] \setminus \{0\}$ | 3.18 | 1 |
| $||f(x)||$         | $\mathbb{Z}_{>0}$ | 3.2 | 1 |
| $\rho_1(f(x))$     | $\mathbb{Z} \setminus \{0\}$ | 3.15 | 1 |
| $\rho_2(f(x))$     | $\mathbb{Z} \setminus \{0\}$ | 3.16 | 1 |
| $\rho_3(f(x))$     | $\mathbb{Z} \setminus \{0\}$ | 3.20 | 1 |
| $\text{Discr}_+(f(x)); \text{Prod}(f(x))$ | $\mathbb{Z} \setminus \{0\}$ | 2.1.4 in [19] | $p^{p-2}; p^{p-1}$ |

| Main Invariant | Range | Definition | $f(x) := \Phi_p(x)$ |
|---------------|-------|------------|-------------------|
| len($f(x)$)   | $\mathbb{Z}_{>0}$ | 3.6 | 1 |
| irr($f(x)$)   | $\mathbb{Z}_{>0}$ | 3.7 | 1 |
| inv($f(x)$)   | $\mathbb{Z} \setminus \{0\}$ | 3.22 | $p^2(p-1)$ |

Overview. In Section 2, we collect preliminary results from the literature. In Section 3, we define our auxiliary invariants and Higman-solvable polynomials. We also prove a more precise version of Proposition 1.6. In Section 4 we work out the technical aspects of our strategy to prove the main theorem. In Section 5, we prove Theorem 1.7 with weaker assumptions and with stronger conclusions. In Section 6, we illustrate our techniques with three well-chosen examples. This paper extends techniques in [11] and [19], but we note that it can be read independently of those works.

2 Preliminaries

Theorem 2.1 (Rowley [21]). If a finite group $G$ admits a fixed-point-free automorphism, then $G$ is solvable.

Lemma 2.2 ([19, Proposition 3.1.3]). Let $G$ be a group with an automorphism $\alpha$. Then the identities of $\alpha$ form an ideal of $\mathbb{Z}[x]$.

Lemma 2.3 ([19, Lemma 4.1.1]). Let $G$ be a finite (solvable) group and let $\alpha : G \rightarrow G$ be a fixed-point-free automorphism. (i.) The map $-1 + \alpha : G \rightarrow G : g \mapsto g^{-1} \cdot \alpha(g)$ is bijective. (ii.) If $H$ is a Hall-subgroup of $G$, then some conjugate $K$ of $H$ satisfies $\alpha(K) = K$. (iii.) If $N$ is a normal subgroup of $G$ with $\alpha(N) = N$, then the induced automorphism $\overline{\alpha} : G/N \rightarrow G/N$ is also fixed-point-free.

Definition 2.4 ([20, Definition 2.1]). Let $X$ be a subset of the group $G$. We say that $X$ is arithmetically-free if $X$ contains no subset of the form $\{b, a, a \cdot b, a \cdot b^2, a \cdot b^3, \ldots\}$. We say that $X$ is product-free if it does not contain a subset of the form $\{b, a, a \cdot b\}$.

Theorem 2.5 (Moens [20, Theorem 3.14]). Let $A$ be a group and let $X$ be a finite, arithmetically-free subset of $A$. Then there exists a minimal $H(X, A) \in \mathbb{Z}_{>0}$ with the following property. If $L$ is a Lie ring that is graded by $A$ and supported by $X$, then $L$ is nilpotent of class at most $H(X, A)$.

Example 2.6. The finite subset $X$ of $A$ is product-free if and only if $H(X, A) \leq 1$. 

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Theorem 2.7 (Moens [19, Theorem 4.3.1]). Let $X$ be a finite, arithmetically-free subset of the multiplicative group $(\mathbb{F}^\times, \cdot)$ of a field $\mathbb{F}$. Then $H(X, \mathbb{F}^\times) \leq |X|^{2^{|X|}}$.

Theorem 2.8 (Moens [19, Theorem 4.2.2]). Let $L$ be a finite Lie ring with an automorphism $\beta$ with an identity $f(x) \in \mathbb{Z}[x]$ with root set $X$. Suppose that $\gcd(|L|, \text{Discr}_x(f(x))) \cdot \text{Proj}(f(x))) = 1$. Then there exists an embedding $\varepsilon : \text{Discr}_x(f(x)) \cdot L \hookrightarrow K$ of the Lie ring $\text{Discr}_x(f(x)) \cdot L$ into a Lie ring $K$ that is graded by $(\mathbb{F}^\times, \cdot)$ and supported by $X$.

Definition 2.9. For $d, h \in \mathbb{Z}_{\geq 1}$, we define $E_d(h)$ to be the number of monic polynomials in $\mathbb{Z}[x]$ of degree $d$ and height at most $h$ for which the Galois group is not the full symmetric group on $d$ letters.

Theorem 2.10 (van der Waerden [26]). For each $d \in \mathbb{Z}_{\geq 1}$, we have $E_d(h) = o(h^d)$.

So the number of monic polynomials of degree $d$ and height at most $h$ that are reducible or decomposable is $o(h^d)$. We refer to Bhargava’s recent solution [4] of the van der Waerden conjecture for an optimal estimate on the growth rate of $E_d(h)$.

3 Higman-solvable polynomials and their properties

3.1 Higman-solvable polynomials $f(x)$ and $\text{len}(f(x))$

Definition 3.1. Let $a(x), b(x) \in \mathbb{Z}[x]$. We say that $a(x)$ is powerful if it is in the set $\mathbb{Z}[x^2] \cup \mathbb{Z}[x^3] \cup \mathbb{Z}[x^4] \cdots$. And $a(x)$ is a closed divisor of $b(x)$ if $a(x)$ is a divisor of $b(x)$ and

(a) every powerful divisor of $b(x)$ divides $a(x)$,

(b) if $A$ is the root set of $a(x)$ and $B$ is the root set of $b(x)$, then $A \cdot A \cap B \subseteq A$, and

(c) $\gcd(a(x), b(x)/a(x)) = 1$.

It is clear that the gcd of all closed divisors of $a(x)$ is again a closed divisor of $a(x)$.

Definition 3.2 ($\|f(x)\|$, $\mathcal{T}(x), f_\ast(x)$). Let $f(x) \in \mathbb{Z}[x]$. If $f(x) \in \mathbb{Z}$, then we define $\|f(x)\| := 0$ and $f_\ast(x) := \mathcal{T}(x) := f(x)$. So we assume that $f(x) \notin \mathbb{Z}$. We then let $\|f(x)\|$ be the maximal $n \in \mathbb{Z}_{\geq 0}$ such that $f(x) \in \mathbb{Z}[x^n]$. Then there is a unique $\mathcal{T}(x) \in \mathbb{Z}[x]$ such that $f(x) = \mathcal{T}(x)\|f(x)\|$. We then define $f_\ast(x)$ to be the gcd of all closed divisors of $\mathcal{T}(x)$.

So $f_\ast(x)$ is a closed divisor of $\mathcal{T}(x)$ and $f(x) = \mathcal{T}(x)^\|f(x)\|/\|f(x)\|$.

Definition 3.3 ($\Delta(f(x))$). We define the map $\Delta : \mathbb{Z}[x] \to \mathbb{Z}[x] : f(x) \mapsto f_\ast(x)$.

Definition 3.4 (Higman-solvable). Let $f(x) \in \mathbb{Z}[x]$. We say that $f(x)$ is Higman-solvable if $f(0) \cdot f(1) \neq 0$ and $\Delta^l(f(x)) \in \mathbb{Z}$ for some $l \in \mathbb{Z}_{\geq 0}$.

Example 3.5. If $f(x) := (x^4 + 3x^2 + 1)(x^2 + 1)(x + 2)$, then $\Delta^2(f(x)) = 5$ and $\Delta^3(f(x)) = 1$. If $f(x) := (x^4 - 5)(x^2 - 2)(x + 1)$, then $\Delta^2(f(x)) = 5$ and $\Delta^3(f(x)) = 1$. If $f(x) := (x^4 - 2)(x^3 - 3)$, then $\Delta(f(x)) = f(x)$, so that $f(x)$ is not Higman-solvable. If $f(x) := (x^2 - 1)/(x - 1)$, for some composite number $n$, then $\Delta(f(x)) = f(x)$, so that $f(x)$ is not Higman-solvable.

Definition 3.6 ($\text{len}(f(x))$). Let $f(x) \in \mathbb{Z}[x]$ be Higman-solvable. We define the Higman-length $\text{len}(f(x))$ of $f(x)$ to be the minimal $l \in \mathbb{Z}_{\geq 0}$ such that $\Delta^l(f(x)) \in \mathbb{Z}$.

Definition 3.7 ($\text{irr}(f(x))$). For $f(x) \in \mathbb{Z}[x] \setminus \{0\}$, we define $\text{irr}(f(x))$ to be the number of irreducible factors of $f(x)$ of positive degree (counted with multiplicity).

Proposition 3.8. If $f(x) \in \mathbb{Z}[x]$ is Higman-solvable, then $\text{len}(f(x)) \leq \text{irr}(f(x))$. 

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Proof. Note that, for a Higman-solvable polynomial $f(x)$, we have: \( \text{irr}(f(x)) = \text{irr}(\Delta(f(x))) \iff f(x) \in \mathbb{Z} \). We now use induction on \( \text{irr}(f(x)) \). If \( \text{irr}(f(x)) = 0 \), then we indeed have \( \text{len}(f(x)) = 0 \leq 0 = \deg(f(x)) \). So we assume \( \text{irr}(f(x)) > 0 \). Since \( f(x) \notin \mathbb{Z} \), we have \( 1 + \text{irr}(\Delta(f(x))) \leq \text{irr}(f(x)) \). The induction hypothesis yields \( \text{len}(\Delta(f(x))) \leq \text{irr}(\Delta(f(x))) \). So also \( \text{len}(f(x)) = 1 + \text{len}(\Delta(f(x))) \leq 1 + \text{irr}(\Delta(f(x))) \leq \text{irr}(f(x)) \). \( \square \)

Lemma 3.9. Let \( m, n \in \mathbb{Z}_{\geq 1} \) and let \( a(x) \in \mathbb{Z}[x] \setminus \{x, -x, x-1, -x+1\} \).

- (a) If \( a(x) \) is irreducible, then \( a(x) \) is Higman-solvable and \( \text{len}(a(x)) \leq 1 \).
- (b) If \( a(x) \) is Higman-solvable, then \( a(x^n) \) is Higman-solvable and \( \text{len}(a(x^n)) = \text{len}(a(x)) \).
- (c) Suppose that \( a(x) \) is Higman-solvable and monic of degree \( m \) and let \( C \) be its companion operator. Let \( b(x) \in \mathbb{Z}[x] \) be any polynomial of degree at most \( n \) that is irreducible and indecomposable and that is coprime to the non-zero, integer polynomial \( \prod_{i=1}^{n} \chi_{C \circ C}(x) \cdot \chi_{C}(x^1) \cdot \chi_{C^2}(x^2) \cdots \chi_{C^{m+n}}(x^{m+n}) \). Then also the product \( a(x) \cdot b(x) \) is Higman-solvable.

Proof. (a) By construction, \( \Delta(a(x)) \) coincides with the content of \( a(x) \).

(b) Since \( \|a(x^n)\| = n \cdot \|a(x)\| \), we have \( \Delta(a(x^n)) = \Delta(a(x)) \). So the polynomial \( a(x^n) \) is Higman-solvable of length \( \text{len}(a(x^n)) = \text{len}(a(n)) \).

(c) The eigenvalues of \( C \) are precisely the roots of \( a(x) \). So the roots of \( \chi_{C \circ C}(x) \) are the products \( \lambda \cdot \mu \) of the roots \( \lambda, \mu \) of \( g(x) \), and the roots of \( \chi_{C}(x^i) \) are the product \( \omega \cdot \lambda \) of the roots \( \omega \) of \( x^i - 1 \) with the roots \( \lambda \) of \( a(x) \). Since \( b(x) \) is coprime to \( x(x-1) \), we have \( b(0) \cdot b(1) \neq 0 \), so that also \( f(0) \cdot f(1) \neq 0 \). Since \( b(x) \) is coprime to \( C \), it suffices to show that every powerful divisor of \( f(x) \) of positive degree divides \( a(x) \). Let \( u(x) \) be a powerful divisor of \( f(x) \) of positive degree. Then \( \|u(x)\| \leq \deg(f(x)) = m + n \). If \( \gcd(b(x), u(x)) \neq 1 \), then there is a root \( \lambda \) of \( b(x) \) and a root \( \mu \) of \( a(x) \) such that \( (\lambda/\mu)^{\|u(x)\|} \). In this case, \( \gcd(b(x), \chi_{C}(x^i)) \neq 1 \), for \( i := \|u(x)\| \). This contradiction finishes the proof. \( \square \)

Remark 3.10. The proof of claim (c) also shows that \( \text{len}(a(x^n)) \leq \text{len}(f(x)) \leq \text{len}(a(x)) + 1 \). The upper bound is reached if and only if \( a(t) \) is powerful.

Let us now prove a more precise version of Proposition 1.6.

Proposition 3.11. Fix \( c, d_1, n_1, \ldots, d_c, n_c \in \mathbb{Z}_{\geq 1} \) and set \( k := \max\{|i| \leq c-1 \text{ and } n_i \geq 2\} \).

For each \( h \in \mathbb{Z}_{\geq 1} \), we consider the \( c \)-tuples \((g_1(x), \ldots, g_c(x))\) of monic polynomials in \( \mathbb{Z}[x] \) with prescribed degrees \( \deg(g_1(x)) = d_1, \ldots, \deg(g_c(x)) = d_c \) and heights at most \( h \).

Let \( P(h) \) be the probability that the product \( f(x) = g_1(x^{n_1}) \cdots g_{c-1}(x^{n_{c-1}})g_c(x^{n_c}) \) is Higman-solvable of Higman-length \( h + k \). Then \( \lim_{h \to +\infty} P(h) = 1 \).

Proof. Set \( d := d_1 + \cdots + d_c \) and let \( B_c(h) \) be the number of \( c \)-tuples \((g_i(x))_{1 \leq i \leq c} \) for which \( f(x) \) fails to satisfy the properties of the proposition. Since \( P(h) = 1 - B_c(h)/(2h + 1)^d \), we need only show that \( B_c(h) = o(h^d) \). We proceed by induction on \( c \). Suppose first that \( c = 1 \). Lemma 3.9 (a) and (b) imply that if \( f(x) = g_1(x^{n_1}) \) is not Higman-solvable, then \( g_1(x) \) is reducible. Theorem 2.10 implies that the number of reducible \( g_1(x) \) is \( o(h^d) \). Since none of the \( g_1(x) \) are constant, we conclude that \( B_1(h) = o(h^d) \). Now suppose that \( c > 1 \). We prove the claim for \( n_{c-1} > 1 \), the other case being a straightforward variation.

For a given \( c \)-tuple, we define \( a(x) := g_1(x^{n_1}) \cdots g_{c-1}(x^{n_{c-1}}) \) and \( b(x) := g_c(x) \). Lemma 3.9 (c) and Remark 3.10 imply that if \( f(x^{1/n_c}) \) is not Higman-solvable of Higman-length \( k \), then \( a(x) \) is not Higman-solvable of length \( k - 1 \), or \( b(x) \) is reducible or decomposable, or \( b(x) \) divides the polynomial defined in the proposition.

The induction hypothesis guarantees that there are at most \( o(h^d) \) tuples of the first kind, while Theorem 2.10 guarantees that there are at most \( o(h^d) \) of the second kind. For the
remaining tuples, it is clear that there are only \( o(h^d) \) of the third kind. So the number of \( c \)-tuples for which \( f(x^{1/n}) \) is not Higman-solvable of Higman-length \( k \) is \( o(h^d) \). Lemma 3.9 (b) now implies that \( B_c(h) = o(h^d) \).

**Remark 3.12.** By replacing Theorem 2.10 with Bhargava’s recent solution [4] of the van der Waerden conjecture, we can make the growth rates of \( B_c(h) \) and \( P(h) \) explicit.

### 3.2 More invariants and their properties

**Definition 3.13** \((\mathcal{T}_{n,i}(x), \mathcal{T}_n(x))\). Let \( \mathcal{T}(x) \) be given by \( b_0 + b_1 \cdot x + \cdots + b_m \cdot x^m \). For each \( n \in \mathbb{Z}_{\geq 2} \) and \( i \in \{0, \ldots, n-1\} \), we define the partial sums \( \mathcal{T}_{n,i}(x) := \sum_{j=1 \mod n} b_j \cdot x^j \).

We also define \( \mathcal{T}_n(x) := \gcd(\mathcal{T}_{n,0}(x), \ldots, \mathcal{T}_{n,n-1}(x)) \).

We note that \( \mathcal{T}_{n,i}(x) \in \mathbb{Z}[x] \) and \( \mathcal{T}_n(x) \in \mathbb{Z}[x] \).

**Lemma 3.14.** There exists some \( \rho \in \mathbb{Z}_{\geq 1} \) such that, for all \( n \in \mathbb{Z}_{\geq 2} \), we have \( \rho \cdot f_*(x) \in \mathcal{T}_{n,0}(t) \cdot \mathbb{Z}[x] + \cdots + \mathcal{T}_{n,n-1}(t) \cdot \mathbb{Z}[x] \).

**Proof.** Consider \( n \geq 2 \). Then the \( \mathcal{T}_{n,i}(x) / \mathcal{T}_n(x) \) are coprime over \( \mathbb{Q} \). So Bezout’s theorem gives some \( r_n \in \mathbb{Z} \setminus \{0\} \) such that \( r_n \cdot \mathcal{T}_n(x) \in \mathcal{T}_{n,0}(t) \cdot \mathbb{Z}[x] + \cdots + \mathcal{T}_{n,n-1}(t) \cdot \mathbb{Z}[x] \). Now define \( \rho := r_2 \cdot r_{n+1} \in \mathbb{Z} \setminus \{0\} \). Since each \( \mathcal{T}_n(x) \) is powerful, we have \( \mathcal{T}_n(x) | f_*(x) \). For all \( n \geq 2 \), we then have \( \rho \cdot f_*(x) \in r_n \cdot \mathcal{T}_n(x) \cdot \mathbb{Z}[x] \subseteq \mathcal{T}_{n,0}(t) \cdot \mathbb{Z}[x] + \cdots + \mathcal{T}_{n,n-1}(t) \cdot \mathbb{Z}[x] \). \( \square \)

The above existence result can be made effective by means of the Euclidean algorithm. The integers \( \rho \) form a non-zero principal ideal of \( \mathbb{Z} \). This justifies the following definition.

**Definition 3.15** \((p_1(f(x)))\). We define \( p_1(f(x)) \) to be the minimal \( \rho \in \mathbb{Z}_{\geq 1} \) for which Lemma 3.14 holds.

**Definition 3.16** \((p_2(f(x)))\). If \( f_*(x) \) is constant, then we define \( p_2(f(x)) := f_*(x) \).

If \( f_*(x) \) is not a constant, but \( \mathcal{T}(x) / f_*(x) \) is a constant, then we define \( p_2(f(x)) := \mathcal{T}(x) / f_*(x) \). Else, we define \( p_2(f(x)) := \operatorname{Res}(f_*(x), \mathcal{T}(x) / f_*(x)) \).

**Lemma 3.17.** We have \( p_2(f(x)) \in \mathbb{Z} \setminus \{0\} \), and for all \( k \in \mathbb{Z}_{\geq 1} \), we have \( p_2(f(x))^k \cdot f_*(x) \in f_*(x)^{k} \cdot \mathbb{Z}[x] + \mathcal{T}(x) \cdot \mathbb{Z}[x] \).

**Proof.** We may assume that \( f_*(x) \) and \( \mathcal{T}(x) / f_*(x) \) are not constant. Since \( f_*(x) \) is a closed divisor of \( \mathcal{T}(x) \), we have \( p_2(f(x)) \in \mathbb{Z} \setminus \{0\} \). Next observe that \( p_2(f(x))^{k-1} \cdot f_*(x) \in f_*(x) \cdot \mathcal{T}(x) / f_*(x) \cdot \mathbb{Z}[x] = f_*(x) \cdot \mathbb{Z}[x] + \mathcal{T}(x) \cdot \mathbb{Z}[x] \) for all \( k \geq 1 \). Then also \( p_2(f(x))^k \cdot f_*(x) \in f_*(x)^k \cdot \mathbb{Z}[x] + \mathcal{T}(x) \cdot \mathbb{Z}[x] \) for all \( k \geq 1 \). \( \square \)

We use the standard notation \( \text{lc}(f_*(x)) \) for the leading coefficient of \( f_*(x) \in \mathbb{Z}[x] \setminus \{0\} \).

**Definition 3.18** \((f_2^*(x))\). If \( f_*(x) \in \mathbb{Z} \), then we define \( f_2^*(x) := f_*(x) \). Else, we let \( C \) be the companion operator of the monic polynomial \( \text{lc}(f_*(x))^{-1} \cdot f_*(x) \). Then we define the polynomial \( f_2^*(x) \) as the multiple \( \text{lc}(f_*(x))^{k_2} \cdot \chi_{C \otimes C}(x) \) of the characteristic polynomial of the Kronecker square \( C \otimes C \) of \( C \).

We note that \( f_2^*(x) \in \mathbb{Z}[x] \setminus \{0\} \).

**Lemma 3.19.** Let \( Z \) be the (possibly empty) root set of \( f_*(x) \) in an algebraically-closed field \( \mathbb{F} \). If \( \text{lc}(f_*(x)) \neq 0 \) mod \( \text{char}(\mathbb{F}) \), then \( Z \cdot Z \) is the root set of \( f_2^*(x) \) in \( \mathbb{F} \).

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Proof. We assume that $f_\ast(x) \not\in \mathbb{Z}$, since otherwise there is nothing to prove. We see that $Z$ is the root set of $\text{lc}(f_\ast(x))^{-1} \cdot f_\ast(x)$. So $Z$ is the set of eigenvalues of the corresponding companion operator $C$. So $Z \cdot Z$ is the set of eigenvalues of the Kronecker square $C \otimes C$. So $Z \cdot Z$ is the set of roots of $\chi_{C \otimes C}(t)$ and therefore of $f_\ast^2(x)$. \hfill \Box

**Definition 3.20** ($u(x), \rho_3(f(x))$). We define $u(x) := \gcd(f_\ast^2(x), \overline{f(x)})$. If $\overline{f(x)}/u(x)$ is constant, then we define $\rho_3(f(x)) := \overline{f(x)}/u(x)$. If $f_\ast^2(x)/u(x)$ is a constant, but $\overline{f(x)}/u(x)$ is not a constant, then we define $\rho_3(f(x)) := f_\ast^2(x)/u(x)$. Else, we define $\rho_3(f(x)) := \text{Res}(f_\ast^2(x)/u(x), \overline{f(x)}/u(x))$.

**Lemma 3.21.** We have $\rho_3(f(x)) \in \mathbb{Z} \setminus \{0\}$, and for all $k \in \mathbb{Z}_{\geq 1}$, we have $\rho_3(f(x))^k \cdot f_\ast(x)^k \in f_\ast^2(x)^k \cdot \mathbb{Z}[x] + \overline{f(x)} \cdot \mathbb{Z}[x]$.

**Proof.** We may again assume that $f_\ast^2(x)/u(x)$ and $\overline{f(x)}/u(x)$ are not constant. By construction, $f_\ast^2(x)/u(x)$ and $\overline{f(x)}/u(x)$ are coprime. So $\rho_3(f(x)) \in \mathbb{Z} \setminus \{0\}$. For all $k \geq 1$, we get $\rho_3(f(x))^k \cdot u(x)^k \in u(x)^k \cdot (f_\ast^2(x)/u(x))^k \cdot \mathbb{Z}[x] + u(x)^k \cdot \overline{f(x)} \cdot \mathbb{Z}[x] \subseteq f_\ast^2(x)^k \cdot \mathbb{Z}[x] + \overline{f(x)} \cdot \mathbb{Z}[x]$. Since $f_\ast(x)$ is a closed divisor of $\overline{f(x)}$, we see that $u(x)/f_\ast(x)$. So also $\rho_3(f(x))^k \cdot f_\ast(x)^k \in f_\ast^2(x)^k \cdot \mathbb{Z}[x] + \overline{f(x)} \cdot \mathbb{Z}[x]$ for all $k \geq 1$. \hfill \Box

### 3.3 The invariant $\text{inv}(f(x))$

**Definition 3.22** ($\text{inv}(f(x))$). Let $f(x) \in \mathbb{Z}[x]$ be Higman-solvable. If $\text{len}(f(x)) = 0$, then we define $\text{inv}(f(x)) := f(x)$. Else, we recursively define

$$\text{inv}(f(x)) := f(1) \cdot \text{lc}(f(x)) \cdot \rho_1(f(x)) \cdot \rho_2(f(x)) \cdot \rho_3(f(x)) \cdot \text{Discr}_\ast(f(x)) \cdot \text{Prod}(f(x)) \cdot \text{inv}(\Delta(f(x))) .$$

**Proposition 3.23.** If $f(x) \in \mathbb{Z}[x]$ is Higman-solvable, then $\text{inv}(f(x)) \in \mathbb{Z} \setminus \{0\}$.

**Proof.** We use induction on $\text{len}(f(x))$. If $\text{len}(f(x)) = 0$, then there is nothing to prove. So we assume $\text{len}(f(x)) > 0$. Since $\Delta(f(x))$ is again Higman-solvable, the induction hypothesis states that $\text{inv}(\Delta(f(x))) \in \mathbb{Z} \setminus \{0\}$. It now remains to observe that: $f(1) \in \mathbb{Z} \setminus \{0\}$ by assumption, $\text{lc}(f(x)) \in \mathbb{Z} \setminus \{0\}$ by construction, $\rho_1(f(x)) \in \mathbb{Z} \setminus \{0\}$ by definition, $\rho_2(f(x)) \in \mathbb{Z} \setminus \{0\}$ by Lemma 3.17, $\rho_3(f(x)) \in \mathbb{Z} \setminus \{0\}$ by Lemma 3.21, and $\text{Discr}_\ast(f(x)), \text{Prod}(f(x)) \in \mathbb{Z} \setminus \{0\}$ by Proposition 2.1.5 of [19]. \hfill \Box

### 4 A reduction from $(G, \alpha, f(x))$ to $(F_1(G), \alpha \|f(x)\|, f_\ast(x))$

**Definition 4.1.** Let $G$ be a group and let $\varphi : G \rightarrow G$ be a map. We say that $\varphi$ is nilpotent, if there exists some $n \in \mathbb{N}$ such that $\varphi^n(x) = 1$ for all $x \in G$. We say that $G$ is a $\varphi$-group if $\varphi$ is nilpotent.

The following property will be used repeatedly (but not always explicitly).

**Lemma 4.2.** Suppose that $N$ is normal in $G$ with $\alpha(N) = N$. Let $\overline{\alpha} : G/N \rightarrow G/N$ be the natural automorphism of $G/N$. Let $f(x) \in \mathbb{Z}[x]$. If $G/N$ is an $f(\alpha)$-group and $N$ is an $f(\alpha)$-group, then also $G$ is an $f(\alpha)$-group.

**Proof.** By assumption, there exist $k_1, k_2 \in \mathbb{Z}_{\geq 1}$ such that $f(\alpha)^{k_1}(G) \subseteq H$ and $f(\alpha)^{k_2}(N) \subseteq \{1\}$. So $\varphi^{k_1+k_2}(G) = \{1\}$. \hfill \Box

In what follows, we will often write slightly inaccurately “$N$ is an $f(\alpha)$-group” instead of “$N$ is an $f(\overline{\alpha})$-group,” since no confusion can arise. We had already introduced ordered identities of automorphisms. We now consider two more types of identities.
Definition 4.3. Let $G$ be a group with an automorphism $\alpha$. The polynomial $f(x) \in \mathbb{Z}[x]$ is an identity of $\alpha$ if there exist $h_1(x), \ldots, h_k(x) \in \mathbb{Z}[x]$ with $f(x) = h_1(x) + \cdots + h_k(x)$ and the map $G \to G$ sending $g$ to $(h_1(\alpha(g)) \cdots (h_k(\alpha(g)))$ vanishes identically. We say that $f(x)$ is an abelian identity of $\alpha$ if $f(x)$ is an (ordered) identity of all the automorphisms $\pi : S \to S$ induced on the abelian, characteristic sections $S$ of $G$.

It is clear that every ordered identity of $\alpha$ is also an identity of $\alpha$, and that every identity of $\alpha$ is also an abelian identity of $\alpha$. We will need this weakest notion of identity to make our induction work lateron.

Lemma 4.4. Let $\alpha$ be an automorphism of a group $G$. Then the abelian identities of $\alpha$ form an ideal of $\mathbb{Z}[x]$.

Proof. Let $f(x), g(x) \in \mathbb{Z}[x]$. Suppose first that $f(x), g(x)$ are abelian identities of $\alpha$ and set $h(x) := f(x) + g(x)$. Then $f(\alpha)$ and $g(\alpha)$ vanish on every characteristic, abelian section $S$ of $G$. For every $s \in S$, we then have $h(\alpha)s = (f(\alpha)s) \cdot (g(\alpha)s) = 1 \cdot 1 = 1$. So also $f(x) + g(x)$ is an abelian identity of $\alpha$. We next suppose that $f(x)$ is an abelian identity of $\alpha$ but that $g(x)$ need not be one. We set $h(x) := f(x) \cdot g(x)$ and we choose a characteristic, abelian section $S$ of $G$. For every $s \in S$, we then have $h(\alpha)s = g(\alpha)f(\alpha)s = g(\alpha)(1) = 1$. So also $f(x) \cdot g(x)$ is an abelian identity of $\alpha$. This finishes the proof.

Lemma 4.5. Let $G$ be a finite, solvable group of derived length $k$ with automorphism $\alpha$. If $f(x) \in \mathbb{Z}[x]$ is an abelian identity of $\alpha$, then $G$ is an $(f(\alpha))$-group and $f(x)^k$ is an identity of $\alpha$.

Proof. We see that $f(\alpha)$ vanishes on each of the $k$ abelian factors of the derived series of $G$. So then $f(\alpha)^k(G) = \{1\}$ and $f(x)^k$ is an identity of $\alpha$.

Proposition 4.6. Let $f(x) \in \mathbb{Z}[x]$ satisfy $f(0) \neq 0$ and let $q$ be a prime satisfying $\gcd(q, \text{lcm}(f(x))) = 1$. Let $Q$ be a finite $q$-group and let $\beta$ be an automorphism. Suppose that $\overline{f}(x)$ is an abelian identity of $\beta$ and suppose that the Frattini-quotient $Q/\Phi(Q)$ is an $f_*(\beta)$-group. Then also $Q$ is an $f_*(\beta)$-group.

Proof. The lower central series $(\Gamma_i(Q))_{i \geq 0}$ of $Q$ naturally gives rise to a finite, graded Lie ring $L := L_1 \oplus L_2 \oplus \cdots \oplus L_c$ of class $c = c(L) = c(Q)$, where $L_i := \Gamma_i(Q)/\Gamma_{i+1}(Q)$. Let $\gamma : L \to L$ be the Lie ring automorphism that is naturally induced on $L$ by $\beta$. Then $\overline{f}(x)$ vanishes on $L$. Since $\Phi(Q) = Q^q \cdot [Q, Q]$, we also have $f_*(\gamma)^{k_0}(L_1) \subseteq q \cdot L_1$, for some sufficiently large $k_0 \in \mathbb{Z}_{\geq 1}$.

Let us prove that $(L_1, +)$ is an $f_*(\gamma)$-group. We first consider the special case: $q \cdot L = \{0_L\}$. Then $L$ is a Lie algebra over the prime field $\mathbb{F}_q$ and $f_*(\gamma)^{k_0}(L_1) = \{0_L\}$. After extending the scalars, we may further assume that the ground field $F$ of $L$ is algebraically-closed. We proceed by induction on the class $c$ of $L$. If $c = 1$, then $L = L_1$ is an $f_*(\gamma)$-group by assumption. So we assume $c > 1$. The induction hypothesis yields some $k_1 \in \mathbb{Z}_{\geq 1}$ such that $f_*(\gamma)^{k_1}(L_1 \oplus \cdots \oplus L_{c-1}) = \{0_L\}$. So the generalized eigenvalues of $\gamma$ on $L_1$ and $L_{c-1}$ are roots of $f_*(\gamma)$. The generalized eigenvalues of $\gamma$ on $L_1 = [L_1, L_{c-1}]$ will therefore be of the form $\lambda - \mu$ with $f_*(\lambda) = f_*(\mu) = 0$. Since $(q, \text{lcm}(f(x))) = 1$, we may apply Lemma 3.19 to obtain $f_2^2(\gamma)^{k_2}(L_1) \subseteq f_2^2(\gamma)^{k_2}([L_1, L_{c-1}]) = \{0_L\}$, for some sufficiently large $k_2 \in \mathbb{Z}_{\geq 1}$. But, by assumption, we also have $\overline{f}(\gamma)(L_1) = \{0_L\}$. So $f_2^2(\gamma)$ and $\overline{f}(x)$ are both identities of $\gamma$ on $L_1$. Lemma 3.21 shows that the ideal of $\mathbb{Z}[x]$ that is generated by $f_2^2(\gamma)$ and $\overline{f}(x)$ contains the polynomial $\rho_3(f_2^2(\gamma)) \cdot f_*(\gamma)^{k_2}$. Lemma 4.4 then implies that $\rho_3(f_2^2(\gamma)) \cdot f_*(\gamma)^{k_2}(L_1) = \{0_L\}$. Since $(q, \rho_3(f(x))) = 1$, also $f_*(\gamma)^{k_2}(L_1) = \{0_L\}$. Set $k_3 := \max\{k_1, k_2\}$. Then we conclude that $f_*(\gamma)^{k_3}(L_1 \oplus \cdots \oplus L_{c-1}) + f_*(\gamma)^{k_3}(L_{c-1}) = \{0_L\}$. We now consider the general case.
$q^e \cdot L = \{0_L\}$, where $e \in \mathbb{Z}_{\geq 1}$. Then $q \cdot L$ is a characteristic ideal of $L$ and the additive exponent of the quotient $L/(q \cdot L)$ is $q$. So we can apply the above argument to $L/(q \cdot L)$ in order to conclude that $f_*(\gamma^{k_3}(L)) \subseteq q \cdot L$, for some $k_3 \in \mathbb{Z}_{\geq 1}$. But then $f_*(\gamma^{e \cdot k_3}(L)) \subseteq q^e \cdot L = \{0_L\}$.

We can finally derive that $Q$ is an $f_*(\gamma)$-group. A simple induction on $i \in \mathbb{Z}_{\geq 1}$ shows that $f_*(\beta^{i \cdot e \cdot k_3}(Q)) \subseteq \Gamma_{i+1}(Q)$. So $f_*(\beta^{i \cdot e \cdot k_3}(Q)) \subseteq \Gamma_{c+1}(Q) = \{1Q\}$. 

**Proposition 4.7.** Let $G$ be a finite, solvable group with an automorphism $\alpha$ and let $f(x)$ be an abelian identity of $\alpha$. Suppose that $f(0) \neq 0$ and $\gcd(|G|, f(1) \cdot \rho_1(f(x)) \cdot \rho_3(f(x))) = 1$. Then the automorphism $\alpha^{|f(x)|} : F_1(G) \to F_1(G)$ is fixed-point-free and $F_1(G)$ is an $f_*(\alpha^{|f(x)|})$-group.

The first statement is easy to prove. Let us use the abbreviation $\beta := \alpha^{|f(x)|}$. Then $f(x)$ is an abelian identity of $\beta$. Since $G$ is solvable, we may apply Lemma 4.5. So some natural power of $f(x)$, say $f(x)^{k_0}$, is an identity of $\beta$. Any fixed-point $x$ of $\beta$ therefore satisfies $x f(x)^{k_0}$. Since $\gcd(|G|, f(1)) = 1$, we conclude that $x = 1$. So $\beta$ is a fixed-point-free automorphism of $G$. The second statement is more difficult to prove. We proceed by induction on $|G|$. If $|G| = 1$, then $F_1(G) = \{1\}$, so that there is nothing to prove. So we assume that $|G| > 1$. We follow the strategy of Higman in [11].

**Claim 4.8.** We may assume that $G$ has Fitting height exactly 2.

**Proof.** If the Fitting height is at most 1, then there is nothing left to prove. So suppose $F_2(G) \neq \{1\}$. Then $F_1(G)$ is an proper, characteristic section of $G$. By the induction hypothesis, we have that $F_1(F_1(G)) = F_2(G)$ is a $f_*(\beta)$-group. Similarly, $G/F_2(G)$ is a proper, characteristic section of $G$, so that $F_1(G/F_2(G)) = F_1(G)/F_2(G)$ is an $f_*(\beta)$-group. But then all of $F_1(G)$ is a $f_*(\beta)$-subgroup.

**Claim 4.9.** We may assume that the first upper Fitting-subgroup $\overline{F_1}(G)$ of $G$ is a $q$-group, for some prime $q$.

**Proof.** Otherwise, the nilpotent group $\overline{F_1}(G)$ is the direct product of two proper, non-trivial, characteristic subgroups $A$ and $B$. Then the induction hypothesis implies that $F_1(G/A) = F_1(G/B)$ are $f_*(\beta)$-groups. By definition, there then exist $k_1, k_2 \in \mathbb{Z}_{\geq 1}$ such that $f_*(\beta^{k_1}(F_1(G)) \subseteq A$ and $f_*(\beta^{k_2}(F_1(G)) \subseteq B$. Since $A \cap B = \{1\}$, we conclude that also $\overline{F_1}(G)$ is a $f_*(\beta)$-group.

**Claim 4.10.** We may assume that the quotient $G/\overline{F_1}(G)$ is an elementary-abelian $p$-group, for some prime $p \neq q$.

**Proof.** Let $E/\overline{F_1}(G)$ be a proper, characteristic subgroup of $G/\overline{F_1}(G)$ and lift it to a proper characteristic subgroup $E$ of $G$ containing $\overline{F_1}(G)$. The induction hypothesis forces $F_1(E)$ to be an $f_*(\beta)$-group. If $F_1(E) = \{1\}$, then the induction hypothesis forces $F_1(E)/F_1(E) = F_1(G)/F_1(E)$ to be a $f_*(\beta)$-group. But then also $F_1(G)$ is a $f_*(\beta)$-group and our proof is done. So we may suppose that $F_1(E) = \{1\}$. This means that the normal subgroup $E$ is nilpotent and contained in the maximal nilpotent normal subgroup $\overline{F_1}(G)$ of $G$. So $E/\overline{F_1}(G)$ is the trivial group. We conclude that $G/\overline{F_1}(G)$ is characteristically-simple. Since $G$ is solvable, the quotient is an elementary-abelian $p$-group, for some prime $p$. Since $G$ has Fitting height exactly 2, it is not nilpotent, and $p \neq q$.

**Claim 4.11.** We may assume that $\overline{F_1}(G) = \overline{F_1}(G)$. 

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Proof. Otherwise, the Sylow-$p$ subgroup of the nilpotent group $G/F_1(G)$ is proper, non-trivial, and characteristic so that it lifts to a proper, characteristic subgroup $E$ of $G$ that is not a $q$-group. The induction hypothesis implies that $F_1(E)$ is an $f_*(\beta)$-group. If it is not non-trivial, then the induction hypothesis also claims that $F_1(G/F_1(E)) = F_1(G)/F_1(E)$ is a $f_*(\beta)$-group. But in this case all of $F_1(G)$ is a $f_*(\beta)$-group, and we are done. The other case cannot occur, since it would imply that $E \subseteq F_1(G)$ is a $q$-group.

Claim 4.12. We may assume that $F := \overline{F_1(G)} = \overline{F_1(G)}$ is an elementary-abelian $q$-group.

Proof. Otherwise, the Frattini-subgroup $\Phi(F)$ of $F$ is non-trivial. The induction hypothesis then forces $F_1(G/\Phi(F)) = F/\Phi(F)$ to be an $f_*(\beta)$-group. Proposition 4.6 then guarantees that all of $\overline{F}$ is an $f_*(\beta)$-group, and we are done.

Lemma 2.3 provides a Sylow-$p$ subgroup $P$ of $G$ satisfying $\beta(P) = P$. Then $G = F \rtimes P$ and $G \rtimes \langle \beta \rangle = F \rtimes (P \rtimes \langle \beta \rangle)$. So $P \rtimes \langle \beta \rangle$ acts on $F$ via conjugation within $G \rtimes \langle \beta \rangle$. We now change our perspective and notation. We identify the abelian section $F$ of $G$ with the additive group of a non-trivial, finite-dimensional vector space $V$ over the prime field $\mathbb{F}_p$. Then $P \rtimes \langle \beta \rangle$ naturally acts on $V$ via the homomorphism $\gamma : P \rtimes \langle \beta \rangle \to \text{GL}(V) : A \mapsto \overline{A}$.

Claim 4.13. $\overline{P}$ acts fixed-point-freely on $V$, $\overline{\beta}$ normalizes $\overline{P}$ and acts fixed-point-freely on $\overline{\beta}$, and $\overline{\beta}(\overline{V}) = \{0_V\}$.

Proof. Claim 1 follows from the fact that $F = \overline{F_1(G)}$ is self-centralizing. Consider claim 2. Since $P$ is normal in $P \rtimes \langle \beta \rangle$, its image $\overline{P}$ is normal in $\overline{P} \rtimes \langle \overline{\beta} \rangle$. Now suppose that $A \in P$ and that $\overline{\beta}^{-1} \circ \overline{A} \circ \overline{\beta}$ acts trivially on $V$. Then the element $\beta^{-1} \cdot A \cdot \beta$ of $P$ acts trivially on $F$. Since $F$ is self-centralizing, we conclude that $\beta^{-1} \cdot A \cdot \beta = 1_P$. So also $\beta^{-1} \circ \overline{A} \circ \overline{\beta} = 1_v$. Claim 3 is simply a change of notation.

It now suffices to verify that $f_*(\overline{\beta})(V) = \{0_V\}$. Let us do this in the remainder of the proof. We may assume that the base field $F$ of $V$ is algebraically-closed. Since gcd($q, |\overline{P}|$) = 1, we can simultaneously diagonalize the elements of $\overline{P}$. So $V = \bigoplus_{\chi} V_\chi$, where $\chi$ runs over the characters of $\overline{P}$ with non-zero character space $V_\chi$. We recall that, for every $\overline{A} \in \overline{P}$ and every $v \in V_\chi$, we have $\overline{A}(v) = \chi(\overline{A}) \cdot v$. One can then verify that the group $(\overline{\beta})$ naturally acts on the set of character spaces by permutations.

Claim 4.14. For every character space $V_\chi$, we have $\overline{\beta}(V_\chi) \neq V_\chi$.

Proof. Suppose otherwise: $\overline{\beta}(V_\chi) = V_\chi$. For every $\overline{A} \in \overline{P}$ and $v \in V_\chi$, we then have $(\overline{\beta}^{-1} \circ \overline{A} \circ \overline{\beta})(v) = \overline{A}(v)$. So every element of $V_\chi$ is a common fix-point of $\overline{A}^{-1} \cdot (\overline{\beta}^{-1} \cdot A \cdot \beta)$. Since $\overline{\beta}$ acts fixed-point-freely on $\overline{P}$, Lemma 2.3 shows that every element of $\overline{P}$ is of the form $\overline{A}^{-1} \cdot (\overline{\beta}^{-1} \cdot \overline{A} \cdot \overline{\beta})$. So every element of $V_\chi$ is a common fix-point of $\overline{P}$. Claim 4.13 then forces $V_\chi = \{0_V\}$. But every character space is non-zero by definition.

Claim 4.15. We have $f_*(\overline{\beta})(V) = \{0_V\}$.

Proof. Let $V_\chi$ be an arbitrary character space and consider its orbit $V_\chi, \overline{\beta}(V_\chi) =: V_{\chi_1}, \ldots, \overline{\beta}^{n-1}(V_\chi) =: V_{\chi_1}, \ldots, \overline{\beta}^{n-1}(V_\chi) =: V_{\chi_n-1}$ under the action of $(\overline{\beta})$, where $n$ is the minimal element of $\mathbb{Z}_{\geq 1}$ such that $\overline{\beta}(V_\chi) = V_{\chi_n}$. Claim 4.14 shows that $n \geq 2$. For every $v \in V_\chi$, we then have $0_v = \overline{\beta}(v) = \overline{\beta}_n(v) + \cdots + \overline{\beta}_{n,n-1}(v)$. Since the $\overline{\beta}_{n,i}(v)$ belong to linearly independent spaces $V_{\chi_i}$, each term of this sum vanishes. By definition (cf. Lemma 3.14), there exist polynomials $a_0(x), \ldots, a_{n-1}(x) \in \mathbb{Z}[x]$ such that $\rho_1(f(x)) \cdot f_*(x) =$
Then states that $\beta_1 = k_1$ for some $\alpha_i$. Proof. We first apply Theorem 4.1. Let $G$ be a finite, solvable group with a fixed-point-free automorphism $\alpha : G \to G$ and let $f(x)$ be an abelian identity of $\alpha$. Suppose that $f(0) \neq 0$ and $\gcd(|G|, f(1) - \alpha f(x)) = 1$. Then $f_\alpha(0) = \beta_1$. Let $\beta : S \to G$ be an abelian identity of $\beta$. According to Lemma 3.1, we have $f_\beta(0) = \beta_2$. Trivially, we have $\beta \in S$. So $\beta S$ is fixed-point-free. Then $f_\beta(0) = 0$. By construction, $f_\beta(0) = 0$. This finishes the proof of the proposition. We can easily “upgrade” this result.

**Theorem 4.16.** Let $G$ be a finite, solvable group with a fixed-point-free automorphism $\alpha : G \to G$ and let $f(x)$ be an abelian identity of $\alpha$. Suppose that $f(0) \neq 0$ and $\gcd(|G|, f(1) \cdot \alpha f(x)) = 1$. Then $f_\alpha(0) = \beta_1$. Let $\beta : S \to G$ be an abelian identity of $\beta$. According to Lemma 3.1, we have $f_\beta(0) = \beta_2$. Trivially, we have $\beta \in S$. So $\beta S$ is fixed-point-free. Then $f_\beta(0) = 0$. By construction, $f_\beta(0) = 0$. This finishes the proof of the proposition. We can easily “upgrade” this result.

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**5 Proof of the main theorem**

**Theorem 5.1.** Let $G$ be a finite, solvable group with a fixed-point-free automorphism $\alpha : G \to G$ and let $f(x)$ be an abelian identity of $\alpha$. Suppose that $f(x)$ is Higman-solvable and $\gcd(|G|, \text{inv}(f(x))) = 1$. For each $i \in \mathbb{Z}_{\geq 0}$, $\Delta^{i+1}(f(x))$ is an abelian identity of the fixed-point-free automorphism $\alpha^{\|f(x)\|} : F_{i+1}(G) \to F_{i+1}(G)$.

**Proof.** We first apply Theorem 4.16. We then proceed by induction on $i \in \mathbb{Z}_{\geq 0}$. If $i = 0$, there is nothing left to prove. So we assume $i > 0$. By definition, $\text{inv}(\Delta(f(x)))$ divides $\text{inv}(f(x))$. So $\gcd(|F_i(G)|, \text{inv}(\Delta(f(x))))$ divides $\gcd(|G|, \text{inv}(f(x)))$ and is therefore 1. The induction hypothesis then states that $\Delta^{i}(\Delta(f(x))) = \Delta^{i+1}(f(x))$ is an abelian identity of the fixed-point-free automorphism $\alpha^{\|f(x)\|} : F_{i+1}(G) \to F_{i+1}(G)$. If $m \in \mathbb{Z} \setminus \{0\}$ is an abelian identity of $\alpha$ and $\gcd(|G|, m) = 1$, then $G = \{1\}$.

**Lemma 5.2.** Let $G$ be a finite, solvable group and let $\alpha : G \to G$ be an automorphism. If $m \in \mathbb{Z} \setminus \{0\}$ is an abelian identity of $\alpha$ and $\gcd(|G|, m) = 1$, then $G = \{1\}$.

**Proof.** Lemma 4.5 states that $G^m = \{1\}$, for some $k \in \mathbb{Z}_{\geq 0}$. So $G = \{1\}$.

We now prove a strong form of Theorem 1.7 by replacing ordered identities with identities, by removing the condition on the torsion of $|G|$, and by obtaining a sharper bound on the Fitting height of $G$.

**Theorem 5.3.** Let $G$ be a finite group with a fixed-point-free automorphism $\alpha$ and let $f(x)$ be an identity of $\alpha$ that is Higman-solvable.
(a) Then $G$ is solvable and the product $A \cdot B$ of two subgroups $A$ and $B$ of coprime order, such that the Fitting height $h(A)$ of $A$ satisfies

$$h(A) \leq \text{len}(f(x)) \leq \text{irr}(f(x)) \leq \text{deg}(f(x)),$$

and such that $|B|$ divides a natural power of the non-zero integer $\text{inv}(f(x))$.

(b) Suppose, moreover, that the roots of $f(x)$ form an arithmetically-free subset $X$ of the group $(\mathbb{Q}^\times, \cdot)$. Then the derived length $dl(A)$ of $A$ satisfies

$$dl(A) \leq \text{len}(f(x)) \cdot H(X, \mathbb{Q}^\times) \leq \text{deg}(f(x))^{2\text{deg}(f(x)) + 1}.$$

Note that if $\gcd(|G|, \text{inv}(f(x))) = 1$, then $B = \{1\}$ and $G = A$. So we do indeed recover Theorem 1.7.

**Proof.** (a) The group is solvable by Rowley’s Theorem 2.1. According to Lemma 2.3, we can find a Hall-$\text{inv}(f(x))$ subgroup $B$ of $G$ and a Hall-$\text{inv}(f(x))'$-subgroup $A$ of $G$ satisfying $\alpha(A) = A$. Then $G = A \cdot B$ and we consider the automorphism $\alpha_A : A \rightarrow A$ obtained by restriction. Then $f(x)$ is an abelian identity of $\alpha_A$. Let $l := \text{len}(f(x))$. If $l = 0$, then $f(x) = \text{inv}(f(x))$ is a non-zero constant, so that we need only apply Lemma 5.2 to conclude that $A = \{1_A\}$. So we assume that $l > 0$. Theorem 5.1 then shows that the non-zero constant polynomial $\Delta'(f(x))$ is an abelian identity of some automorphism of $F_2(A)$. Since $\Delta'(f(x))$ divides $\text{inv}(f(x))$, we need only apply Lemma 5.2 to conclude that $\Gamma_2(A) = \{1\}$. Proposition 3.8 further states that $l \leq \text{irr}(f(x)) \leq \text{deg}(f(x))$.

(b) We assume that $l := \text{len}(f(x)) > 0$, since otherwise there is nothing left to prove. Let $S_i := F_i(A)/F_{i+1}(A)$ be the $i$th factor of the lower Fitting series of $A$. Then we have the coarse bound $dl(A) \leq dl(S_0) + \cdots + dl(S_{l-1}) \leq c(S_0) + \cdots + c(S_{l-1})$. Proposition 3.8 shows that $l \leq \text{deg}(f(x))$. So it suffices to show that $c(S_i) \leq H(X, \mathbb{Q}^\times) \leq |X|^{2|X|}$, for each $i \geq 0$. Let $L$ be the Lie ring that naturally corresponds with the lower central series of some such $S_i$ and let $\beta : L \rightarrow L$ be the induced Lie ring automorphism. Then $f(x)$ is an identity of $\beta$. Since $\gcd(|L|, \text{Discr}_x(f(x)) \cdot \text{Prod}(f(x))) = 1$, we may use Theorem 2.8 to embed $\text{Discr}_x(f(x)) \cdot L$ into a Lie ring $K$ that is graded by $\mathbb{Q}^\times$ and supported by $X$. Theorem 2.5 then states that $K$ is nilpotent of class $c(K) \leq H(X, \mathbb{Q}^\times)$. Theorem 2.7 further shows that $H(X, \mathbb{Q}^\times) \leq |X|^{2|X|}$. By combining these observations, we obtain $c(S_i) = c(L) = c(\text{Discr}_x(f(x)) \cdot L) \leq c(K) \leq H(X, \mathbb{Q}^\times) \leq |X|^{2|X|}$.

**Remark 5.4.** Every monic, Higman-solvable polynomial $f(x)$ of positive degree is an ordered identity of a fixed-point-free automorphism of a finite, non-trivial group.

Indeed, let $p$ be any prime not dividing the non-zero integer $f(0) \cdot f(1)$. Then the companion operator $\alpha$ of $f(x)$ defines a fixed-point-free automorphism of the elementary-abelian $p$-group of rank $\text{deg}(f(x))$ and $f(x)$ is an ordered identity of $\alpha$. We refer to [19, Section 3] for more interesting constructions.

**Remark 5.5.** Every fixed-point-free automorphism $\alpha$ of a finite, non-trivial group $G$ has a monic, ordered identity that is Higman-solvable.

Indeed, the polynomial $f(x) := -1 + |G| \cdot x^{|\alpha| - 1} + x^{|\alpha|}$ is an ordered identity of $\alpha$. By Perron’s criterion, this $f(x)$ is irreducible and therefore Higman-solvable. We refer to [19, Section 3] for more interesting constructions.

**Remark 5.6.** Suppose that $f(x) \in \mathbb{Z}[x]$ is not divisible by $x$ or by any cyclotomic polynomial. Then its root set is arithmetically-free.

Indeed, for every pair of roots $(\lambda, \mu)$, the arithmetic progression $\lambda, \lambda \cdot \mu, \lambda \cdot \mu^2, \ldots$ contains infinitely-many distinct elements and is therefore not contained in the root set.
6 Examples

Example 6.1. Let $G$ be a finite group with a f.p.f. automorphism $\alpha : G \longrightarrow G$, satisfying $\{g^2 \cdot \alpha(g) \cdot \alpha^2(g), \alpha^3(g), \alpha^4(g), \alpha^5(g)^4 \cdot \alpha^6(g)^2 \cdot \alpha^7(g)g \in G\} = \{1\}$. Then $G$ is the product of a metabelian subgroup $A$ and a Hall-$\langle 2 \cdot 3 \cdot 5 \rangle$ subgroup $B$.

Proof. We see that $f(x) := (x^4 + 3x^2 + 1)(x^2 + 1)(x + 2)$ is an identity of the automorphism. Example 3.5 shows that $f(x)$ is Higman-solvable with $\text{len}(f(x)) = 2$. One can verify that $\text{inv}(f(x))$ divides a natural power of $2 \cdot 3 \cdot 5$. Moreover, the root set $X$ of $f(x)$ is product-free, so that $H(X, \underline{Q}^{-}) = 1$. So we need only apply Theorem 5.3. □

Example 6.2. Let $G$ be a finite group with a f.p.f. automorphism $\alpha : G \longrightarrow G$, satisfying $\{g^{10} \cdot \alpha^3(g)^{-2} \cdot \alpha^2(g)^{-5} \cdot \alpha(g)^{10} \cdot \alpha^4(g)^{-2} \cdot \alpha^3(g)^{-5} \cdot \alpha^2(g)g \in G\} = \{1\}$. Then $G$ is the product of a metabelian-by-nilpotent subgroup $A$ and a Hall-$\langle 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \rangle$ subgroup $B$.

Proof. We see that $f(x) := (x^4 - 5)(x^2 - 2)(x + 1)$ is an identity of the automorphism. Example 3.5 shows that $f(x)$ is Higman-solvable with $\text{len}(f(x)) = 3$. One can verify that $\text{inv}(f(x))$ divides a natural power of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$. So we may apply Theorem 5.3 and conclude that $B$ is solvable with Fitting height at most 3. We next observe that the roots of $f(x)$ do not form an arithmetically-free subset of $\langle \underline{Q}^{-} \rangle$. But we know, from Theorem 5.1, that $\Delta^1(f(x)) = (x^4 - 5)(x^2 - 2)$ is an abelian identity of the f.p.f. automorphism $\alpha : F_1(B) \longrightarrow F_1(B)$. Since $\Delta(f(x))$ has a product-free set of roots, say $Y$, Theorem 5.3 states that $d(\Delta(f(x))) \cdot H(Y, \underline{Q}^{-}) \leq (3 - 1) \cdot 1 = 2$. A slightly more complicated argument allows us to show that $B$ is abelian-by-nilpotent. □

Example 6.3. Let $G$ be a finite group with f.p.f. automorphism $\alpha : G \longrightarrow G$, satisfying $\{\alpha^3(g)^{-1} \cdot g^3 \cdot \alpha^3(g)^{-1} \cdot g^3 \cdot \alpha^5(g)^{-3} \cdot \alpha^5(g)g \in G\} = \{1\}$. Then $G$ is the product of an abelian subgroup $A$ and a Hall-$\langle 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 73 \rangle$ subgroup $B$.

Proof. We note that $h(x) := (x^2 - 5)(x^3 - 3)$ is an identity of $\alpha$. Example 3.5 shows that it is not Higman-solvable. But, since $h(x)$ has no roots of finite order, it divides some Higman-solvable polynomial $f(x)$ with $\text{len}(f(x)) \leq 1$. This $f(x)$ is then an identity of $\alpha$, according to Lemma 2.2, and it satisfies the assumptions of Theorem 5.3. It now remains to observe that $f(x) := (x^6 - 2^3)(x^3 - 3^2)$ is such a polynomial with a product-free set of roots and that $\text{inv}(f(x))$ divides a natural power of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 73$. □

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