Spectral Curve of Periodic Fisher Graphs

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Abstract

We study the spectral curves of dimer models on periodic Fisher graphs, obtained from a ferromagnetic Ising model on $\mathbb{Z}^2$. The spectral curve is defined by the zero locus of the determinant of a modified weighted adjacency matrix. We prove that either they are disjoint from the unit torus ($T^2 = \{(z, w) : |z| = 1, |w| = 1\}$) or they intersect $T^2$ at a single real point.

1 Introduction

In this paper we study the spectral curve of periodic, 2-dimensional Fisher graphs, either finite in one direction, and periodic in the other direction (cylindrical graph); or bi-periodic and obtained from a bi-periodic, ferromagnetic Ising model on $\mathbb{Z}^2$ (toroidal graph). To an edge-weighted, cylindrical (resp. toroidal) Fisher graph, one associates its spectral curve $P(z) = 0$ (resp. $P(z, w) = 0$). The real polynomial $P(z)(\text{resp.} P(z, w))$ defining the spectral curve arises as the characteristic polynomial of the Kasteleyn operator in the dimer model.

The study of spectral curve for periodic Fisher graphs (which is non-bipartite), is inspired by the work of Kenyon, Okounkov and Sheffield [KO06, KOS06]. They prove that the spectral curve of bipartite dimer models with positive edge weights is always a real curve of a special type, namely it is a Harnack curve. This implies many qualitative and quantitative results about the behavior of bipartite dimer models related to the phase transition.

The Fisher graph we consider in this paper is a graph with each vertex of the honeycomb lattice replaced by a triangle, see Figure 1.

A planar graph is one which can be embedded into the plane such that edges can intersect only at vertices. Fix an embedding of a planar graph $G$. A clockwise-odd orientation of $G$ is an orientation of the edges such that for each face (except the outer face) an odd number of edges pointing along it when traversed clockwise. For a planar graph, such an orientation always exists [Kas67]. The Kasteleyn matrix corresponding to such a graph is a $|V(G)| \times |V(G)|$ skew-symmetric matrix $K$ defined by

$$K_{u,v} = \begin{cases} W(uv) & \text{if } u \sim v \text{ and } u \rightarrow v \\ -W(uv) & \text{if } u \sim v \text{ and } u \leftarrow v \\ 0 & \text{else.} \end{cases}$$

where $W(uv) > 0$ is the weight associated to the edge $uv$.

Now let $G$ be a $\mathbb{Z}^2$-periodic planar graph. By this we mean $G$ is embedded in the plane so that translations in $\mathbb{Z}^2$ act by weight-preserving isomorphisms of $G$ which map each...
edge to an edge with the same weight. Let $G_n$ be the quotient graph $G/(n\mathbb{Z}\times n\mathbb{Z})$. It is a finite graph on a torus. Let $\gamma_{x,n}$ (resp. $\gamma_{y,n}$) be a path in the dual graph of $G_n$ winding once around the torus horizontally (resp. vertically). Let $E_H$ (resp. $E_V$) be the set of edges crossed by $\gamma_x$ (resp. $\gamma_y$). We give a crossing orientation for the toroidal graph $G_n$ as follows. We orient all the edges of $G_n$ except for those in $E_H \cup E_V$. This is possible since no other edges are crossing. Then we orient the edges of $E_H$ as if $E_V$ did not exist. Again this is possible since $G - E_V$ is planar. To complete the orientation, we also orient the edges of $E_V$ as if $E_H$ did not exist.

Let $K_1$ be a Kasteleyn matrix for the graph $G_1$. Given any parameters $z, w$, we construct a matrix $K(z, w)$ as follows. Let $\gamma_{x,1}$ and $\gamma_{y,1}$ be the paths introduced as above. Multiply $K_{u,v}$ by $z$ if the orientation on that edge is from $u$ to $v$, and multiply $K_{u,v}$ by $\frac{1}{z}$ if the orientation is from $v$ to $u$, and similarly for $w$ on $\gamma_y$. Define the characteristic polynomial $P(z, w) = \det K(z, w)$. The spectral curve is defined to be the locus $P(z, w) = 0$.

We also discuss cylindrical graphs in this paper. Assume we have a Fisher graph, periodic in the $z$ direction, finite in the $w$ direction, by removing all $w$-edges and $\frac{1}{w}$ edges.

An example of such a graph is shown in Figure 2.

Assume it is $m \times n$, meaning that it has width $m$ with respect to $z$, but period $n$ with respect to $w$. We embed this graph into a cylinder. Let $K(z)$ be the corresponding weighted adjacency matrix with an orientation shown in Figure 2. $K(z)$ is obtained from $K(z, w)$ by giving all edges crossed by $\gamma_y$ weight 0. Let $P(z) = \det K(z)$.

Kenyon and Okounkov ([KO06]) proved that the spectral curves of any periodic bipartite graph with positive edge weights are Harnack curves [Mik, MR01], whose intersection with $T_{x,y} = \{(z, w) : |z| = e^x, |w| = e^y\}$ can only be: i. no intersection; ii. a pair of conjugate points, each of which is of multiplicity 1; iii. a single real zero of multiplicity 2 (real node). Since the Fisher graph we consider in this paper is non-bipartite, no previous result is known. We prove the following theorems

**Theorem 1.1.** Consider a positively-weighted cylindrical Fisher graph, which is periodic in one direction, and finite on the other direction. Its quotient graph under the translation can be embedded into a cylinder, as illustrated in Figure 2. The intersection of $P(z) = 0$ and $T$ is either empty or a single real point.
Figure 2: Fisher graph on a cylinder

In Figure 1, each edge of the graph is either a side of the triangle (triangular edge), or an edge connecting different triangles (non-triangle edge). We call all the horizontal non-triangle edges in Figure 1, $a$-edges.

Theorem 1.2. Consider a bi-periodic Fisher Graph, given edge-orientations as illustrated in Figure 1. Assume all the triangular edges and $a$-edges have weights 1, and all the other edges have weights in $(0,1)$, then

1. the spectral curve $P(z, w) = 0$ is a Harnack curve;

2. the only possible intersection of $P(z, w) = 0$ with the unit torus $T^2 = \{(z, w) : |z| = 1, |w| = 1\}$ is a single real point of multiplicity 2.

Our result is very promising. Firstly, it leads to important properties of the dimer model on cylindrical graphs, such as weak convergence of Boltzmann measures and convergence rate of correlations for an infinite, periodic graph with finite width along one direction, see Proposition 3.4. Secondly, since the dimer model on the Fisher graph are closely related to the Ising model [MW] and the vertex model [Li], our result leads to a quantitative characterization of the critical temperature of the arbitrary periodic ferromagnetic, two-dimensional Ising model, as the solution of an algebraic equation. [Li2].

The outline of the paper is as follows. In Section 2, we explain the connection between the Kasteleyn operator, characteristic polynomial, spectral curve with the dimer model. In section 3, prove Theorem 1.1, as well as a phase transition characterized by the decay rate of edge-edge correlation for the dimer model on the cylindrical Fisher graph, resulting from Theorem 1.1. In section 4, we prove Theorem 1.2. The proof consists of 2 critical components, one is that $P(z, w) \geq 0$ for any $(z, w) \in T^2$, the other is an explicit correspondence between the Kasteleyn operator on the Fisher graph, and the Kasteleyn operator in the square-octagon lattice, introduced in [Dub12].

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2 Background

A perfect matching, or a dimer cover, of a graph is a collection of edges with the property that each vertex is incident to exactly one edge. A graph is bipartite if the vertices can be 2-colored, that is, colored black and white so that black vertices are adjacent only to white vertices and vice versa.

To a weighted finite graph $G = (V,E,W)$, the weight $W: E \to \mathbb{R}^+$ is a function from the set of edges to positive real numbers. We define a probability measure, called the Boltzmann measure $\mu$ with sample space the set of dimer covers. Namely, for a dimer cover $D$

$$\mu(D) = \frac{1}{Z} \prod_{e \in D} W(e)$$

where the product is over all edges present in $D$, and $Z$ is a normalizing constant called the partition function, defined to be

$$Z = \sum_{D} \prod_{e \in D} W(e),$$

the sum over all dimer configurations of $G$.

If we change the weight function $W$ by multiplying the edge weights of all edges incident to a single vertex $v$ by the same constant, the probability measure defined above does not change. So we define two weight functions $W, W'$ to be gauge equivalent if one can be obtained from the other by a sequence of such multiplications.

The key objects used to obtain explicit expressions for the dimer model are Kasteleyn matrices. They are weighted, oriented adjacency matrices of the graph $G$ defined as in Page 1.

It is known [Kas61, Kas67, Tes00, KOS06] that for a planar graph with a clockwise odd orientation, the partition function of dimers satisfies

$$Z_n = \sqrt{\det K}.$$

Given a Fisher graph with an orientation as illustrated in Figure 1, the quotient graph can be embedded into an $n \times n$ torus. When $n$ is even, if we reverse the orientations of all the edges crossed by $\gamma_x$ and all the edges crossed by $\gamma_y$, the resulting orientation is a crossing orientation. For $\theta, \tau \in \{0, 1\}$, given the orientation as in Figure 1, let $K_n^{\theta,\tau}$ be the Kasteleyn matrix $K_n$ in which the weights of edges in $E_H$ are multiplied by $(-1)^{\theta}$, and those in $E_V$ are multiplied by $(-1)^{\tau}$. Using the result proved in [Tes00], we can derive that when $n$ is even, the partition function $Z_n$ of the graph $G_n$ is

$$Z_n = \frac{1}{2} \text{Pf}(K_n^{00}) + \text{Pf}(K_n^{10}) + \text{Pf}(K_n^{01}) + \text{Pf}(K_n^{11}).$$

Let $E_m = \{e_1 = u_1v_1, ..., e_m = u_mv_m\}$ be a subset of edges of $G_n$. Kenyon [Ken97] proved that the probability of these edges occurring in a dimer configuration of $G_n$ with respect to the Boltzmann measure $P_n$ is

$$P_n(e_1, ..., e_m) = \frac{\prod_{i=1}^{m} W(u_i v_i)}{2Z_n^n} \left| \text{Pf}(K_n^{00})_{E_m^c} + \text{Pf}(K_n^{10})_{E_m^c} + \text{Pf}(K_n^{01})_{E_m^c} + \text{Pf}(K_n^{11})_{E_m^c} \right|$$

where $E_m^c = V(G_n) \setminus \{u_1, v_1, ..., u_m, v_m\}$, and $(K_n^{\theta,\tau})_{E_m^c}$ is the submatrix of $K_n^{\theta,\tau}$ whose lines and columns are indexed by $E_m^c$. 


The asymptotic behavior of $Z_n$ when $n$ is large is an interesting subject. One important concept is the partition function per fundamental domain, which is defined to be

$$\lim_{n \to \infty} \frac{1}{n^2} \log Z_n.$$  

Gauge equivalent dimer weights give the same spectral curve. That is because after Gauge transformation, the determinant is scaled by a nonzero constant, thus not changing the locus of $P(z, w)$.

A formula for enlarging the fundamental domain is proved in [CP01, KOS06]. Let $P_n(z, w)$ be the characteristic polynomial of $G_n$, and $P_1(z, w)$ be the characteristic polynomial of $G_1$, then

$$P_n(z, w) = \prod_{u^0 = z} \prod_{v^0 = w} P_1(u, v)$$

3 Graph on a Cylinder

3.1 Spectral Curve

Proof of Theorem 1.1 Without loss of generality, assume the period $n$, or the circumference of the cylinder, is even. Assume $P(z) = 0$ has a non-real zeros $z_0 \in \mathbb{T}^2$. Assume $z_0 = e^{i\alpha_0 \pi}$, $\alpha_0 \in (0, 1) \cup (1, 2)$

We classify all the real numbers in $(0, 1) \cup (1, 2)$ into 3 types

1. $\alpha_0 = \frac{p}{q}$ where $p, q$ are positive integers with no common factors, $p$ is odd
2. $\alpha_0$ is irrational
3. $\alpha_0 = \frac{p}{q}$ where $p, q$ are positive integers with no common factors, $p$ is even

First let us consider Case 1 and Case 2. There exists a sequence $\ell_k \in \mathbb{N}$, such that

$$\lim_{k \to \infty} z_0^{\ell_k} = -1$$

In other words, if we assume $z_0^{\ell_k} = e^{\sqrt{-1} \alpha_k \pi}$ where $\alpha_k \in [0, 2)$, then

$$\lim_{k \to \infty} \alpha_k = 1$$

According to the formula of enlarging the fundamental domain,

$$P(z_0) = 0$$

implies

$$P_k(z_0^{\ell_k}) = 0 \quad \forall k$$

Since the cylindrical graph is actually planar, if we reverse the orientation of all the edges crossed by $\gamma_2$, we get a clockwise-odd orientation, given that $n$ is even. Hence

$$P_k(-1) = Z_{2k}^2$$
where $Z_{\ell_k}$ is the partition function of dimer configurations of the cylinder with circumference $n\ell_k$ and height $m$. Therefore we have

\[
1 = \lim_{k \to \infty} \left| \frac{P_{\ell_k}(z_0^k) - P_{\ell_k}(-1)}{P_{\ell_k}(-1)} \right| 
\]

\[
= \lim_{k \to \infty} \frac{1}{Z_{\ell_k}^2} \left| P_{\ell_k}(e^{i\alpha_k \pi}) - P_{\ell_k}(e^{i\pi}) \right| 
\]

\[
= \lim_{k \to \infty} \frac{1}{Z_{\ell_k}^2} \left| \sum_{t=1}^{\infty} \left[ \frac{\pi(\alpha_k - 1)^{2t}}{(2t)!} \right] \frac{\partial^{2t} P_{\ell_k}(e^{i\theta})}{\partial \theta^{2t}} \right|_{\theta = \pi} 
\]

Since

\[
P_{\ell_k}(z) = \sum_{0 \leq j \leq m} P_j^{(\ell_k)}(z_j + \frac{1}{z_j})
\]

where $P_j^{(\ell_k)}$ is the signed sum of loop configurations winding exactly $j$ times around the cylinder (see Lemma 2.1), we have

\[
\frac{\partial^{2t} P_{\ell_k}(e^{i\theta})}{\partial \theta^{2t}} \big|_{\theta = \pi} = \sum_{1 \leq j \leq m} 2j^{2t}(-1)^t P_j^{(\ell_k)},
\]

and

\[
\lim_{k \to \infty} \frac{1}{Z_{\ell_k}^2} \left| \sum_{t=1}^{\infty} \left[ \frac{\pi m(\alpha_k - 1)^{2t}}{(2t)!} \right] \frac{\partial^{2t} P_{\ell_k}(e^{i\theta})}{\partial \theta^{2t}} \right|_{\theta = \pi} \leq \lim_{k \to \infty} \frac{1}{Z_{\ell_k}^2} \left| \sum_{t=1}^{\infty} \sum_{1 \leq j \leq m} |P_j^{(\ell_k)}| \right|
\]

To estimate $|P_j^{(\ell_k)}|$, let us divide the $m \times \ell_k n$ cylinder into $\ell_k n \times m \times 1$ layers. Connecting edges between two layers may be occupied once, unoccupied, or occupied twice (appear as doubled edges). Choose one layer $L_0$, we construct an equivalent class of loop configurations. Two loop configurations are equivalent if they differ from each other only in $L_0$, and coincide on all the other layers and boundary edges of $L_0$.

We claim that for any given equivalent class, there is at least one configuration including only even loops. To see that we choose an arbitrary configuration in that equivalent class including odd loops. Let $\mathcal{T}$ be the set of all triangles in $L_0$ belonging to some loop crossing $L_0$. We choose an odd loop $s_1$. Choose an triangle $\Delta_1 \in s_1 \cap L_0$. Moving along $L_0$ until we find a triangle $\Delta_2$ belonging to a different non-planar odd loop $s_2$. This is always possible because by Lemma 2.2, any loop configuration including odd loops always has an even number of odd loops, and all the odd loops are winding once along the cylinder. Change path through $\Delta_1$. Starting from $\Delta_1$, we change the doubled edge configuration to alternating edges along a path in $L_0$, and change the path through all triangles, belonging to even loops, or $s_1$, between $\Delta_1$ and $\Delta_2$, then we change the path through $\Delta_2$. For any even loop between $\Delta_2$ and $\Delta_2$, we change paths through an even number of triangles of that even loop, hence it is still an even loop. However, for $s_1$ and $s_2$, we change paths through an odd number of triangles, then both of them become even. We can continue this process until we eliminate all the odd loops, because there are always an even number of
Figure 3: before path change

Figure 4: after path change
odd loops in the configuration. An example of such an path change process is illustrated in the following figures.

Since we have at most $2^m$ different configurations in each equivalent class, and each equivalent class has at least one configuration including only even loops, we have

$$\frac{\text{# of loop configurations including odd loops}}{\text{# of even loop configurations}} < 2^m$$

We have a finite number of different edge weights, each of which is positive, hence the quotient of any two weights is bounded by a constant $C_1$, then

$$\sum_{1 \leq j \leq m} |P_j^{(l)}| \leq \text{Partition of configurations including nonplanar odd loops}$$

$$+ \text{Partition of configurations with only even loops}$$

$$\leq (2^m C_1^m + 1) \text{Partition of even loop configurations} \leq C_2^m Z_{l_k}^2$$

As a result

$$\lim_{k \to \infty} \frac{1}{Z_{l_k}^2} \sum_{t=1}^{\infty} \frac{|\pi(\alpha_k - 1)|^{2t} \partial \partial P_k(e^{i\theta})}{(2t)!} \partial \partial \theta^{2t}$$

$$\leq \lim_{k \to \infty} \frac{1}{Z_{l_k}^2} \sum_{t=1}^{\infty} \frac{|m\pi(\alpha_k - 1)|^{2t} C_2^m Z_{l_k}^2}{(2t)!} C_2^m Z_{l_k}^2$$

Since $m$ is a constant, and $\lim_{k \to \infty} \alpha_k = 1$, we have

$$\lim_{k \to \infty} \sum_{t=1}^{\infty} \frac{|m\pi(\alpha_k - 1)|^{2t} C_2^m Z_{l_k}^2}{(2t)!} C_2^m = 0$$

which is a contradiction to (3). Hence for all $\alpha_0$ in Case 1 and Case 2, $P(e^{i\alpha_0 \pi}) \neq 0$.

Now let us consider $\alpha_0$ in Case 3. From the argument above we derive that as long as

$$|\alpha - 1| < \delta,$$

where $\delta$ is a small positive number depending only on $m$,

$$P(e^{i\alpha \pi}) \neq 0.$$
hence

\[ P(e^{i\alpha_0\pi}) \neq 0. \]

Now we consider \( \frac{1}{q} \geq \frac{\delta}{2} \), only finitely many \( q \)'s satisfy this condition. Let \( \ell \) be a prime number satisfying \( \ell > [\frac{3}{\delta}] + 1 \), then after enlarging the fundamental domain to an \( m \times \ell n \) cylinder, the corresponding \( \delta \) will not change because it depends only on \( m \). For any

\[ 1 \geq \frac{1}{q} \geq \frac{\delta}{2}, \]

we have

\[ \frac{\delta}{2} > \frac{1}{\ell q}. \]

Let \( \ell \alpha_1 \pmod{2} = \alpha_0 \). Namely,

\[ \alpha_1 = \frac{p + 2sq}{\ell q} \quad \text{for } s = 1, 2, ..., \ell \]

in reduced form. By the previous argument \( P(e^{i\alpha_1\pi}) \neq 0 \), hence \( P(\ell)(e^{\alpha_0\pi\sqrt{-1}}) \neq 0 \). Hence when \( \alpha_0 \) is in Case 3, \( P_\ell(z_0) \neq 0 \) after enlarging the fundamental domain to \( m \times n \ell \), where \( \ell \) depends only on \( m \), and is independent of \( \alpha_0 \). If \( P(z_0) = 0 \), \( z_0 \) is not real, then we enlarge the fundamental domain to \( m \times n \ell \), where \( \ell \) is a big prime number depending only on \( m \), we derive that \( P_\ell(z_0^\ell) = 0 \), however this is impossible since \( P(\ell) = 0 \) can have only real root on \( \mathbb{T} \). \( \square \)

**Corollary 3.1.** For any non-real \( z \in \mathbb{T} \), all eigenvalues of \( K(z) \) are of the form \( i\lambda_j, j = 1, \ldots, 6mn \), where \( i \) is the imaginary unit. 3mn of the \( \lambda_j \)'s are positive, the other \( \lambda_j \)'s are negative.

**Proof.** From the definition of \( K(z) \), \( iK(z) \) is a Hermitian matrix. We claim that \( iK(z) \) has 3mn positive and 3mn negative eigenvalues for any non-real \( z \in \mathbb{T} \). In fact, for a planar graph with no \( z \) vertices, \( K_0 \) is a anti-symmetric real matrix with eigenvalues \( \pm i\lambda_j \), \( 1 \leq j \leq 3mn - m \), and det \( K_0 > 0 \), because \( | PfK_0 | \) is the partition function of dimer configurations with positive edge weights, and det \( K_0 = (PfK_0)^2 \).

By the previous theorem, every time we add a pair of boundary vertices connected by a \( z \) edge, we have a anti-Hermitian matrix \( K_{r+1} \) of order \( 6mn - 2m + 2(r+1) \) with \( K_r \) as a principal minor. By induction hypothesis \( iK_r \) has the same number of positive and negative eigenvalues, the interlacing theorem implies that \( iK_{r+1} \) has at least \( 3mn - m + r \) positive and \( 3mn - m + r \) negative eigenvalues. By previous theorem, det \( K_{r+1} > 0 \) for non-real \( z \), so the other two eigenvalues of \( iK_{r+1} \) can only be one positive and one negative. \( \square \)

**Lemma 3.2.** If \( P(1) = 0 \), then \( \frac{\partial P(z)}{\partial z} \Big|_{z=1} = 0 \). Let \( z = e^{\sqrt{-1}\theta} \), then \( \frac{\partial P(e^{\sqrt{-1}\theta})}{\partial \theta} \big|_{\theta=1} = 0 \). For generic choice of edge weights, \( \frac{\partial^2 P(z)}{\partial z^2} \big|_{z=1} \neq 0 \), \( \frac{\partial^2 P(e^{\sqrt{-1}\theta})}{\partial \theta^2} \big|_{\theta=1} \neq 0 \).

**Proof.** We prove the result for derivatives with respect to \( z \), the derivatives with respect to \( \theta \) are very similar. Since

\[ P(z) = \sum_j P_j(z^j + \frac{1}{z^j}) \]
we have

\[
\frac{\partial P}{\partial z} \bigg|_{z=1} = \sum_j j P_j (z^{j-1} - \frac{1}{z^{j+1}}) \bigg|_{z=1} = 0
\]

\[
\frac{\partial^2 P}{\partial z^2} \bigg|_{z=1} = \sum_j 2j^2 P_j
\]

Each monomial in \(P(1)\) corresponds to a loop configuration including only even loops. Consider a configuration including \(2^k\) even loops, each of which winding exactly once around the cylinder. Except those non-contractible even loops, the rest of the graph is covered by doubled-edge configuration. Consider \(P(1)\) as a polynomial of edge weights. The monomial corresponding to that configuration has a coefficient

\[
S_{0,k} = \pm 2^{2k}
\]

because each single loop can have two different orientations, corresponding to two terms in the expansion of the determinant. However, the coefficient of the monomial corresponding to the same configuration in \(\frac{\partial^2 P(z)}{\partial z^2} \bigg|_{z=1}\) is

\[
S_{2,k} = (2k)^2 + 2k(2k-2)^2 + \left(\frac{2k}{2}\right)(2k-4)^2 + \cdots + \left(\frac{2k}{2k}\right)(2k-4k)^2
\]

Obviously, \(\frac{S_{2,k}}{S_{0,k}}\) is a number depending on \(k\). Therefore \(P(1)\) and \(\frac{\partial^2 P(z)}{\partial z^2} \bigg|_{z=1}\) cannot divide each other. Let

\[
W_2 = \{ (w_e)_{e \in E} : \frac{\partial^2 P(z)}{\partial z^2} \bigg|_{z=1} = 0 \}
\]

\[
W_0 = \{ (w_e)_{e \in E} : P(1) = 0 \}
\]

The intersection of \(W_2\) and \(W_0\) forms a proper subvariety of \(W\). Hence if \(P(1) = 0\), and we choose the edge weights generically, \(\frac{\partial^2 P(z)}{\partial z^2} \bigg|_{z=1} \neq 0\). \(\square\)

**Remark.** Lemma 3.3 implies that if \(P(1) = 0\), for generic choice of edge weights, the intersection is of multiplicity 2.

**Proposition 3.3.** Let

\[
F(r) = \frac{1}{2\pi} \int_0^{2\pi} \log P(re^{i\theta}) d\theta
\]

If \(P(z) = 0\) does not intersect the unit torus \(\mathbb{T}\), \(F(r)\) is differentiable at \(r = 1\); If \(P(z) = 0\) has a real zero of multiplicity 2 at \(z = 1\),

\[
\lim_{r \to 1^+} \frac{\partial F(r)}{\partial r} - \lim_{r \to 1^-} \frac{\partial F(r)}{\partial r} = 2
\]

**Proof.** Since \(P(z)\) is a Laurent polynomial in \(z\), according to the Jensen’s formula

\[
\log |P(0)| = -\sum_{k=1}^n \log \left( \frac{r}{|a_k|} \right) + \frac{1}{2\pi} \int_0^{2\pi} \log |P(re^{i\theta})| d\theta
\]
where \(a_1, \cdots, a_n\) are the zeros of \(P\) in the interior of the disk \(\{z : |z| < r\}\). If \(P(z)\) has no zeros on the circle \(\{z : |z| = r\}\), then

\[
\frac{\partial F}{\partial r} = \lim_{\Delta r \to 0} \frac{F(r + \Delta r) - F(r)}{\Delta r} = \lim_{\Delta r \to 0} \frac{1}{\Delta r} \sum_{k=1}^{n} \log \left( \frac{r + \Delta r}{r} \right) = \frac{n}{r}
\]

The proposition follows from substituting \(r = 1\), and the fact that the intersection of \(P(z) = 0\) with \(\mathbb{T}\) can only be a single real point.

**Remark.** Jensen’s formula implies that the curve \(P(z) = 0\) is Harnack if and only if

\[
r_0 \left( \lim_{r \to r_0} - \lim_{r \to r_0^-} \frac{\partial F(r)}{\partial r} \right) \leq 2
\]

for any \(r_0 > 0\). In fact, if the height of the cylinder \(m \leq 3\), we can always derive that the corresponding spectral curve is Harnack. To see that, first of all, being Harnack is a closed condition, for generic choice of edge weights, the intersection of \(P(z) = 0\) with \(|z| = 1\) is at most two points (counting multiplicities). Without loss of generality, assume for some \(r(0 < r < 1)\), \(P(z) = 0\) intersects \(|z| = r\) at 3 different points \(z_1, z_2, z_3\), then \(\bar{z}_1, \bar{z}_2, \bar{z}_3\), lie also on the intersection of \(P(z) = 0\) with \(|z| = r\), given that \(P(z)\) is a real-coefficient polynomial. Hence the intersection of \(P(z) = 0\) with \(|z| = r\) is at least 4 points. Moreover, by symmetry, the reciprocal of those points are also roots of \(P(z) = 0\), then \(P(z) = 0\) has at least 8 roots, which is a contradiction to the fact \(m \leq 3\).

### 3.2 Limit Measure of Cylindrical Approximation

From the proof we know that all terms in \(P_{m \times n}(-1)\) are positive, if \(n\) is even. We can always enlarging the fundamental domain in \(z\) direction without changing edge weights to get

\[
P_{m \times 2n}(-1) = P_{m \times n}(i)P_{m \times n}(-i) = P_{m \times n}^2(i) = |Pf_{m \times 2n}K(-1)|^2
\]

Therefore \(P_{m \times n}(i)\) is the partition function of dimer configurations of the \(m \times 2n\) cylinder graph. According to the formula of enlarging the fundamental domain,

\[
P_{m \times 2ln}(-1) = \prod_{z^{2l} = -1} P_{m \times n}(z),
\]

we have

\[
\lim_{l \to \infty} \frac{1}{4l} \log P_{m \times 2ln}(-1) = \lim_{l \to \infty} \frac{1}{4\pi} \frac{2\pi}{2l} \sum_{z^{2l} = -1} \log P_{m \times n}(z) = \frac{1}{4\pi} \int_{\mathbb{T}} \log P_{m \times n}(z) \frac{dz}{iz}
\]

The convergence of the Riemann sums to the integral follows from the fact that the only possible zeros of \(P(z)\) on \(\mathbb{T}\) is a single real node. \(\Box\) is defined to be the **partition function** per fundamental domain.
Any probability measure on the infinite banded graph, with depth \( m \) on one direction, and period \( n \) on the other direction, is determined by the probability of cylindrical sets. Namely, we choose a finite number of edges \( e_1, e_2, \cdots, e_k \) arbitrarily, and the probabilities

\[
Pr(e_1 \& e_2 \& \cdots \& e_k)
\]

that \( e_1, e_2, \cdots, e_k \) occur in the dimer configuration simultaneously for all finite edge sets determines the probability measure. We consider the measures on the infinite graph as weak limits of measures on cylindrical graphs. First of all, we prove a lemma about the entries of the inverse Kasteleyn matrix using the cylindrical approximation.

**Lemma 3.4.**

\[
\lim_{l \to \infty} K_{m \times 2ln}^{-1}(-1)(k_v, s_v), (k_w, s_w) = \frac{1}{2\pi} \int_T z^{k_v - k_w} \frac{\text{cofactor} K_{m \times n}(s_v, s_w)(z)}{P_{m \times n}(z)} dz
\]

**Proof.** To that end, we construct a transition matrix \( S \) to make \( S^{-1} K_{m \times 2ln} \) block diagonal, with each block corresponding to a \( m \times n \) quotient graph. Define

\[
S = (e_0, \cdots, e_0^{6mn}, e_1, \cdots, e_1^{6mn}, \cdots, e_{2l-1}, \cdots, e_{2l-1}^{6mn})
\]

where

\[
e_k^s(j, t) = \begin{cases} 
  e^{i\pi(j+1)(2k+1)/4l} & \text{if } s = t \\
  0 & \text{if } s \neq t
\end{cases}
\]

then

\[
S^{-1} K_{m \times 2ln} S = \begin{pmatrix}
K_{m \times n}(e^{\frac{i\pi}{l}}) & 0 \\
0 & K_{m \times n}(e^{\frac{3i\pi}{l}}) \\
& \ddots \\
0 & & & K_{m \times n}(e^{\frac{i(4l-1)\pi}{2l}})
\end{pmatrix}
\]

Since \( S^{-1} = \frac{1}{2l} S^t \), we have

\[
K_{m \times 2ln}^{-1}(-1)(k_v, s_v), (k_w, s_w) = \frac{1}{2l} \sum_{j=0}^{2l-1} \frac{\text{cofactor} K_{m \times n}(s_v, s_w)}{P_{m \times n}(e^{\frac{(2j+1)\pi(k_v-k_w)}{4l}})} e^{i(2j+1)\pi(k_v-k_w)/2l}
\]

where \( k_v \) is the index of the fundamental domain for vertex \( v \) and \( s_v \) is the index of vertex in the fundamental domain.

If \( P_{m \times n}(z) \) has no zero on \( T \), we have

\[
\lim_{l \to \infty} K_{m \times 2ln}^{-1}(-1)(k_v, s_v), (k_w, s_w) = \frac{1}{2\pi} \int_T z^{k_v - k_w} \frac{\text{cofactor} K_{m \times n}(s_v, s_w)(z)}{P_{m \times n}(z)} dz
\]

If 1 is an order-2 zero of \( P_{m \times n}(z) \), let

\[
Q(z) = z^{k_v - k_w} \frac{\text{cofactor} K_{m \times n}(s_v, s_w)(z)}{P_{m \times n}(z)}
\]
Since \( \det K(1) \) is an anti-symmetric real matrix of even order, and non-invertible, the dimension of its null space is non-zero and even. Hence \( \text{Adj} K(1) \) is a zero matrix, and \( 1 \) is at least a zero of order \( 1 \) for \( \text{cofactor} K \). Then in a neighborhood of \( 1 \),

\[
Q(z) = \frac{R_{es_{z=1}}Q(z)}{z-1} + R(z),
\]

where \( R(z) \) is analytic at \( 1 \).

\[
\lim_{l \to \infty} K^{-1}_{m \times 2ln}(-1)_{(K_v,s_v),(K_w,s_w)} = \lim_{\delta \to 0+} \lim_{l \to \infty} \frac{1}{2l} \left( \sum_{0 \leq j < \frac{16}{\pi} - \frac{1}{2}} + \sum_{\frac{16}{\pi} - \frac{1}{2} \leq j \leq 2l - \frac{16}{\pi} - \frac{1}{2}} + \sum_{2l - \frac{16}{\pi} - \frac{1}{2} < j \leq 2l - 1} \right) Q(e^{(2j+1)i\pi \frac{2l}{\pi}}) \quad (7)
\]

For the second term,

\[
\lim_{\delta \to 0+} \lim_{l \to \infty} \frac{1}{2l} \sum_{0 \leq j < \frac{16}{\pi} - \frac{1}{2}} Q(e^{(2j+1)i\pi \frac{2l}{\pi}}) = \lim_{\delta \to 0+} \frac{1}{2\pi} \int_{\delta}^{2\pi - \delta} Q(e^{i\theta})d\theta = p.v. \lim_{\delta \to 0+} \frac{1}{2\pi} \int_{\delta}^{2\pi} Q(z) \frac{dz}{iz} = \frac{1}{2} R_{es_{z=1}}Q(z) + \sum R_{es_{|z|<1}} \frac{Q(z)}{z} \quad (8)
\]

For the first and the third term

\[
\lim_{\delta \to 0+} \lim_{l \to \infty} \frac{1}{2l} \left( \sum_{0 \leq j < \frac{16}{\pi} - \frac{1}{2}} + \sum_{\frac{16}{\pi} - \frac{1}{2} \leq j \leq 2l - \frac{16}{\pi} - \frac{1}{2}} + \sum_{2l - \frac{16}{\pi} - \frac{1}{2} < j \leq 2l - 1} \right) Q(e^{(2j+1)i\pi \frac{2l}{\pi}}) \left( \frac{R_{es_{z=1}}Q(z)}{e^{(2j+1)i\pi \frac{2l}{\pi}} - 1} + R(e^{(2j+1)i\pi \frac{2l}{\pi}}) \right)
\]

Since \( R(z) \) is analytic in a neighborhood of \( 1 \),

\[
\lim_{\delta \to 0+} \lim_{l \to \infty} \frac{1}{2l} \left( \sum_{0 \leq j < \frac{16}{\pi} - \frac{1}{2}} + \sum_{\frac{16}{\pi} - \frac{1}{2} \leq j \leq 2l - \frac{16}{\pi} - \frac{1}{2}} + \sum_{2l - \frac{16}{\pi} - \frac{1}{2} < j \leq 2l - 1} \right) R(e^{(2j+1)i\pi \frac{2l}{\pi}}) = 0 \quad (12)
\]

Moreover,

\[
\lim_{\delta \to 0+} \lim_{l \to \infty} \frac{1}{2l} R_{es_{z=1}}Q(z) \left( \sum_{0 \leq j < \frac{16}{\pi} - \frac{1}{2}} + \sum_{\frac{16}{\pi} - \frac{1}{2} \leq j \leq 2l - \frac{16}{\pi} - \frac{1}{2}} + \sum_{2l - \frac{16}{\pi} - \frac{1}{2} < j \leq 2l - 1} \right) \frac{1}{e^{(2j+1)i\pi \frac{2l}{\pi}} - 1} = 0 \quad (13)
\]

\[
\lim_{\delta \to 0+} \lim_{l \to \infty} \frac{1}{2l} R_{es_{z=1}}Q(z) \sum_{0 \leq j < \frac{16}{\pi} - \frac{1}{2}} \left( \frac{1}{e^{(2j+1)i\pi \frac{2l}{\pi}} - 1} + \frac{1}{e^{-(2j+1)i\pi \frac{2l}{\pi}} - 1} \right) = 0 \quad (14)
\]

\[
= - \lim_{\delta \to 0+} \frac{1}{2l} R_{es_{z=1}}Q(z) \left[ \frac{16}{\pi} + \frac{1}{2} \right] = 0 \quad (15)
\]

And the theorem follows from (6), (7), (11), (12), (15). \( \square \)
\textbf{Proposition 3.5.} Using a large cylinder to approximate the infinite periodic banded graph, we derive that the weak limit of probability measures of the dimer model exists. The probability of a cylindrical set under this limit measure is

\[
Pr(e_1&e_2&\cdots&e_k) = \lim_{l \to \infty} \prod_{j=1}^{k} w_{e_j} \sqrt{\det K_{m \times 2l}^{-1}} \begin{pmatrix} u_1 & v_1 & \cdots & u_k & v_k \\ u_1 & v_1 & \cdots & u_k & v_k \end{pmatrix}
\]

where \( w_{e_j} \) is the weight of \( e_j \), and \( u_j, v_j \) are the two vertices of \( e_j \).

\textit{Proof.} On a finite cylindrical graph of \( m \times 2l \)

\[
Pr(e_1&e_2&\cdots&e_k) = \frac{Z_{e_1,\ldots,e_k}}{Z}
\]

where \( Z \) is the partition function of dimer configurations on that graph, and \( Z_{e_1,\ldots,e_k} \) is the partition function of dimer configurations for which \( e_1,\ldots,e_k \) appear simultaneously. Since

\[
Z^2 = \det K_{m \times 2l}(-1)
\]

\[
Z^2_{e_1,\ldots,e_k} = \prod_{j=1}^{k} w_{e_j} \text{cofactor}K_{m \times 2l, (u_1,v_1,\ldots,u_k,v_k)}(-1)
\]

\textbf{(17)} follows from the fact that if originally we have a clockwise-odd orientation, we still have a clockwise-odd orientation when removing edges and ending vertices, while keeping the orientation on the rest of the graph. The proposition follows from Jacobi’s formula for the determinant of minor matrices. \hfill \square

We consider two edges \( e_1 \) and \( e_2 \) with weight \( x_1 \) and \( x_2 \) on \( m \times 2l \) cylinder, and compute the covariance.

\[
Pr_{m \times 2l}(e_1\&e_2) - Pr_{m \times 2l}(e_1)Pr_{m \times 2l}(e_2)
\]

\[
= x_1x_2 \sqrt{\det K_{m \times 2l}^{-1}} \begin{pmatrix} v_1 & w_1 & v_2 & w_2 \\ v_1 & v_1 & v_2 & w_2 \end{pmatrix}(-1)
\]

\[
- x_1x_2 \sqrt{\det K_{m \times 2l}^{-1}} \begin{pmatrix} v_1 & w_1 & v_2 & w_2 \\ v_1 & w_1 & v_2 & w_2 \end{pmatrix}(-1) \cdot \det K_{m \times 2l}^{-1} \begin{pmatrix} v_2 & w_2 \\ v_2 & w_2 \end{pmatrix}(-1)
\]

\[
= x_1x_2(K_{m \times 2l}(v_1,w_1)K_{m \times 2l}(v_2,w_2) - K_{m \times 2l}(v_1,w_2)K_{m \times 2l}(v_2,w_1)) - K_{m \times 2l}(v_1,v_2)K_{m \times 2l}(w_1,w_2)
\]

\textbf{(21)}

In order to compute the covariance as \( l \searrow \infty \), we only need to compute the entries of \( K_{m \times 2l}^{-1}(-1) \) as \( l \searrow \infty \).

Then we have the following proposition

\textbf{Proposition 3.6.} Consider the dimer model on a Fisher graph, embedded into an \( m \times ln \) cylinder, as illustrated in Figure 4. Let the circumference of the cylinder go to infinity, i.e. \( l \to \infty \), and keep the height of the cylinder unchanged. Consider two edges \( e_1 \) and \( e_2 \). As \( |e_1 - e_2| \searrow \infty \): if \( P(z) = 0 \) does not intersect \( T \), the edge-edge correlation decays exponentially; if \( P(z) = 0 \) has a node at 1, the edge-edge correlation tends to a constant.

\textit{Proof.} The theorem follows from formula \textbf{(22)}, and the estimates of the entries of inverse Kasteleyn matrix. By \textbf{(11)}, the second term goes to 0 as \( |e_1 - e_2| \to \infty \), the first term is 0, if no zero exists on \( T \), the first term is a nonzero constant if the spectral curve has a real node at 1. \hfill \square
Example 3.7. (1 × 2 cylindrical graph) Assume we have a dimer model on a Fisher graph embedded into an infinite cylinder of height 1 and period 2. One period of the graph is illustrated in Fig 5.

The characteristic polynomial of the model is

\[ P(z) = (b_1b_2z - a_1a_2)(\frac{b_1b_2}{z} - a_1a_2). \]

The probability that an \(a_1\)-edge occurs is

\[
Pr(a_1) = \frac{a_1a_2}{4\pi} \left( p.v. \int_{T} \frac{1}{a_1a_2 - b_1b_2z} \frac{dz}{\sqrt{-1z}} + p.v. \int_{T} \frac{1}{a_1a_2 - b_1b_2z} \frac{dz}{\sqrt{-1z}} \right)
\]

\[
= \begin{cases} 
1 & \text{if } a_1a_2 > b_1b_2 \\
0 & \text{if } a_1a_2 < b_1b_2 \\
\frac{1}{2} & \text{if } a_1a_2 = b_1b_2 
\end{cases}
\]

\( P(z) = 0 \) has a real node at 1 if and only if \( a_1a_2 = b_1b_2 \). At the critical case, the covariance of an \( a_1 \) edge and a \( b_2 \) edge, as their distance goes to infinity, is

\[
Pr(a_1 \& b_2) - Pr(a_1)Pr(b_2) = 0 - \frac{1}{4} = \frac{1}{4}
\]

4 Graph on a Torus

4.1 Combinatorial and Analytic Properties

To compute the characteristic polynomial \( P(z, w) \) of the Fisher graph, we give an orientation to edges as illustrated in Figure 1.

Lemma 4.1. The characteristic polynomial \( P(z, w) \) for a periodic Fisher graph with period \((m, 0)\) and \((0, n)\) is a Laurent polynomial of the following form:

\[
P(z, w) = \sum_{i,j} P_{ij}(z^i w^j + \frac{1}{w^j z^i})
\]

where \((i, j)\) are integral points of the Newton polygon with vertices \((\pm m, 0), (0, \pm n), (\pm m, \mp n)\):

Proof. Let \( p = 6mn \). By definition,

\[
P(z, w) = \det K(z, w) = \sum_{i_1, \ldots, i_p} (-1)^{\sigma(i_1, \ldots, i_p)} k_{1,i_1} \times \cdots \times k_{p,i_p}.
\]
Here $\sigma$ is the number of even cycles of the permutation $(i_1, \ldots, i_p)$. The sum is over all possible permutations of $p$ elements. Each term of $P(z, w)$ corresponds to an oriented loop configuration occupying each vertex exactly twice. For the graph $G_1$ with $m$ $z$-edges and $n$ $w$-edges, $P(z, w)$ is a Laurent polynomial with leading terms $z^m; \frac{1}{z^m}, w^n; \frac{1}{w^n}, z^m w^n; \frac{1}{z^m w^n}$. $z^i w^j$ corresponds to loops of homology class $(i, j)$. $P(z, w)$ is symmetric with respect to $z^i w^j$ and $\frac{1}{w^j z^i}$. That is because for each term of $z^i w^j$, if we reverse the orientation of all loops, we get a term of $\frac{1}{w^j z^i}$ with coefficients of the same absolute value, corresponding to the product of weights of edges included in the configuration. The sign of the term is multiplied by $(-1)^p = 1$.

To show that all the powers $(i, j)$ lie in the polygon, we multiply all the $b$-edges by $z$ (or $\frac{1}{z}$), and all the $c$-edges by $w$ (or $\frac{1}{w}$), according to their orientation. This way the corresponding characteristic polynomial becomes $P(z^n, w^m)$. Let $(\tilde{i}, \tilde{j})$ be a power of monomial in $P(z^n, w^m)$. At each triangle, all the possible contributions of local configurations to the power of the monomial can only be $(0, 0), (0, \pm 1), (\pm 1, 0), (\pm 1, \mp 1)$. Examples are illustrated in the following Figure 4. The left graph has two doubled edges, and the contribution to the power of the monomial is $(0, 0)$, the right graph has a loop winding from the $z$-edge to the $w$-edge, and the contribution to the power of the monomial is $(1, -1)$.

Consider all the $2mn$ triangles. For each edge, we considered its contribution twice, since we counted it from both triangles it connected. Hence we have

$$-2mn \leq 2\tilde{i} \leq 2mn$$
$$-2mn \leq 2\tilde{j} \leq 2mn$$
$$-2mn \leq 2(\tilde{i} + \tilde{j}) \leq 2mn$$

Since $\tilde{i} = ni$, $\tilde{j} = mj$, and the Newton polygon $N(P)$ is defined to be

$$N(P) = \text{convex hull}\{(i, j) \in \mathbb{Z}^2 | z^i w^j \text{ is a monomial in } P(z, w)\},$$

the lemma follows.

**Lemma 4.2.** For configurations with odd loops corresponding to a non-vanishing term in $P(z, w)$, all odd loops have non-trivial homology, and the total number of odd loops is even.
Proof. For any loop configuration on the Fisher graph embedded on a torus, the number of odd loops is always even. That is because the total number of vertices are even, while odd loops always involve odd number of vertices. Any term in $P_{ij}$ including odd loops can appear only when odd loops have non-trivial homology. That is because for a contractible odd loop, we can reverse the orientation of that loop to negate the sign of that term. The term with reversed orientation on the odd loop cancels with the original term, because the homology class $(0, 0)$ of the configurations is not changed given the odd loop is contractible. 

4.2 Generalized Fisher Correspondence

Consider the Fisher graph obtained by replacing each vertex of the honeycomb lattice by a triangle. Assume all the triangle edges have weight 1, and all the non-triangle edges have positive weights not equal to 1. Furthermore, we assume that at each triangle, there is an even number of adjacent edges with weight less than 1. We introduce a generalized Fisher correspondence between the Ising model on the triangular lattice and the dimer model on the Fisher lattice as follows:

1. If two adjacent spins have the same sign, and the dual edge of the Fisher lattice has weight strictly greater than 1, then the dual edge separating the two spins is present in the dimer configuration.

2. If two adjacent spins have the same sign, and the dual edge of the Fisher lattice has weight strictly less than 1, then the dual edge separating the two spins is not present in the dimer configuration.

3. If two adjacent spins have the opposite sign, and the dual edge of the Fisher lattice has weight strictly greater than 1, then the dual edge separating the two spins is not present in the dimer configuration.

4. If two adjacent spins have the opposite sign, and the dual edge of the Fisher lattice has weight strictly less than 1, then the dual edge separating the two spins is present in the dimer configuration. If at each triangle, an even number of incident edges have weight less than 1, we change the configuration on an even number of incident edges. As a result, the number of present edges incident to each triangle is still odd, which is a dimer configuration. This correspondence is 2-to-1 since negating the spins at all vertices corresponds to the same dimer configuration.

Around each triangle of the triangular lattice, we always have an even number of sign changes. If all the non-triangular edges have weight strictly greater than 1, then the number of present edges incident to each triangle of the Fisher graph is equal to the number of edges separating the same spins, which is odd. Whenever we have an edge with weight strictly less than 1, we change the configuration of that edge, according to the principle described above. Figure 9 is an example of the generalized Fisher correspondence given $a > 1, b > 1, c < 1, d > 1, e < 1$.

Choose the interaction $J_e$ associated to a bond as follows:

$$J_e = \frac{1}{2} \log w_e$$

where $w_e$ is the weight of the dual edge. $J > 0$ corresponds to the ferromagnetic interaction.
4.3 Duality Transformation

We call a geometric figure built with a certain number of bonds of a lattice a closed polygon if at every lattice point only an even number of bonds occurs. It is clear that every configuration of “+” and “−” spins on a lattice can be associated a closed polygon of the dual lattice in the following way. A dual bond belongs to the polygon if it separates different spins and does not belong to the polygon if it separates equal spins. The same closed polygon is associated to two symmetric configurations in which the “+” and “−” spins are interchanged.

Let $T_{mn}$ be the quotient graph of the triangular lattice on the plane, as defined on Page 4. Let $H_{mn}$ be the dual graph of $T_{mn}$, $H_{mn}$ is a honeycomb lattice which can be embedded into an $m \times n$ torus. Without loss of generality, assume both $m$ and $n$ are even.

Define an Ising model on $T_{mn}$ with interactions $\{J_e\} e \in E(T_{mn})$. Assume the Ising model on $T_{mn}$ has partition function $Z_{T_{mn},I}$. Then $Z_{T_{mn},I}$ can be written as,

$$Z_{T_{mn},I} = 2 \prod_{e \in E(T_{mn})} \exp(J_e) \sum_{C^* \in S_{00}^*} \prod_{e \in C^*} \exp(-2J_e) := 2 \prod_{e \in E(T_{mn})} \exp(-J_e)Z_{F_{mn},D_{00}},$$

where $S_{00}^*$ is the set of closed polygon configurations of $H_{mn}$, with an even number of occupied bonds crossed by both $\gamma_x$ and $\gamma_y$. The sum is over all configurations in $S_{00}^*$. Similarly, we can define $S_{01}^*, (S_{10}^*, S_{11}^*)$ to be the set of closed polygon configurations of $H_{mn}$, with an even(odd,odd) number of occupied bonds crossed by $\gamma_x$, and an odd(even,odd) number of occupied bonds crossed by $\gamma_y$.

$F_{mn}$ is the Fisher graph obtained by replacing each vertex of $H_{mn}$ by a triangle, with weights on all the non-triangle edges given by

$$w_e = e^{2J_e}$$

Let $Z_{F_{mn},D}$ be the partition function of dimer configurations on $F_{mn}$, a Fisher graph embedded into an $m \times n$ torus, with weights $e^{2J_e}$ on edges of $H_{mn}$, and weight 1 on all the other edges. Then

$$Z_{F_{mn},D} = Z_{F_{mn},D_{00}} + Z_{F_{mn},D_{01}} + Z_{F_{mn},D_{10}} + Z_{F_{mn},D_{11}},$$

where

$$Z_{F_{mn},D_{\theta,\tau}} = \prod_{e \in E(T_{mn})} \exp(2J_e) \sum_{C^* \in S_{\theta,\tau}^*} \prod_{e \in C^*} \exp(-2J_e).$$

For example, $Z_{F_{mn},D_{01}}$ is the dimer partition function on $F_{mn}$ with an even number of occupied edges crossed by $\gamma_x$, and an odd number of occupied edges crossed by $\gamma_y$. It also
corresponds to an Ising model which has the same configuration on the two boundaries parallel to $\gamma_y$, and the opposite configurations on the two boundaries parallel to $\gamma_x$. Similar results hold for all the $Z_{F_{mn}, D_{\theta, \tau}}, \theta, \tau \in \{0, 1\}$.

On the other hand, if we consider the high temperature expansion of the Ising model on $T_{mn}$, we have

$$Z_{T_{mn}, I} = \sum_{\sigma} \prod_{e=uv \in E(T_{mn})} \exp(J_e \sigma_u \sigma_v)$$

$$= \prod_{e=uv \in E(T_{mn})} \cosh J_e \sum_\sigma \prod_{e=uv \in E(T_{mn})} (1 + \sigma_u \sigma_v \tanh J_e)$$

$$= \prod_{e=uv \in E(T_{mn})} \cosh J_e \prod_{C \in S} \prod_{e \in C} 2^{mn} \tanh J_e,$$

where $S$ is the set of all closed polygon configurations of $T_{mn}$. Let $\tilde{F}_{mn}$ be a Fisher graph embedded into an $m \times n$ torus, with weights $\tanh J_e$ on edges of $G_n$, with each vertex of the triangular lattice replaced by a gadget, as illustrated in the following Figure.

![Figure 9: Fisher Correspondence](image)

There is an 1-to-2 correspondence between closed polygon configurations on the left graph and the dimer configurations on the right graph. An edge is present in the closed polygon configuration of the left graph if and only if it is present in the dimer configuration of the right graph. Assume $\tilde{F}_{mn}$ has weight $\tanh J_e$ on edges of $T_{mn}$, and weight 1 on all the other edges. In other words, the edge with weight $\tanh J_e$ of $\tilde{F}_{mn}$ and the edge with weight $e^{2J_e}$ of $F_{mn}$ are dual edges. Then we have

$$Z_{T_{mn}, I} = 2 \prod_{e \in E(T_{mn})} \exp(-J_e) Z_{T_{mn}, D_{00}} = \prod_{e \in E(T_{mn})} \cosh J_e Z_{\tilde{F}_{mn}, D}$$

where $Z_{\tilde{F}_{mn}, D}$ is the partition function of dimer configurations on $\tilde{F}_{mn}$. Hence we have

$$Z_{T_{mn}, D_{00}} = \frac{1}{2^{mn+1}} \prod_{e \in E_{T_{mn}}} (1 + \exp(2J_e)) Z_{\tilde{F}_{mn}, D}.$$

More generally, we can expand all the $Z_{F_{mn}, D_{\theta, \tau}}$ as follows:

$$Z_{F_{mn}, D_{\theta, \tau}} = \frac{1}{2^{mn+1}} \prod_{e \in E_{T_{mn}}} (1 + \exp(2J_e)) Z_{\tilde{F}_{mn}, D}((-1)^\tau, (-1)^\theta).$$

$Z_{\tilde{F}_{mn}, D}(-1, 1)$ is the dimer partition function of $\tilde{F}_{mn}$ with weights of edges crossed by $\gamma_x$ multiplied by $-1$. Similarly for $Z_{\tilde{F}_{mn}, D}(1, -1)$ and $Z_{\tilde{F}_{mn}, D}(-1, -1)$. Therefore

$$Z_{F_{mn}, D_{00}} = \max_{\theta, \tau \in \{0, 1\}} Z_{F_{mn}, D_{\theta, \tau}}$$
Now we consider a Fisher graph \( \hat{F}_{mn} \). \( \hat{F}_{mn} \) is the same graph as \( F_{mn} \) except edge weights. Namely, \( \hat{F}_{mn} \) has weight 1 on all the triangle edges. Around each triangle, we have an even number of connecting edges satisfying

\[
\hat{w}_e = \frac{1}{w_e},
\]

where \( w_e \) (resp. \( \hat{w}_e \)) is the weight of edge \( e \) for the graph \( F_{mn} \) (resp. \( \hat{F}_{mn} \)). All the other edge weights satisfy

\[
\hat{w}_e = w_e.
\]

Without loss of generality, we assume that an even number of edges with weight strictly less than 1 are crossed by \( \gamma_x \), and an even number of edges with weight greater than 1 are crossed by \( \gamma_y \). Then there is a 1-to-1 correspondence between configurations in \( Z_{F_{mn},D_\theta\tau} \) and \( Z_{\hat{F}_{mn},D_\theta\tau} \) by changing the configurations on all the edges with weight strictly less than 1. Hence we have

\[
Z_{F_{mn},D_\theta\tau} = \prod_{\{e: w_e < 1\}} w_e Z_{\hat{F}_{mn},D_\theta\tau}.
\]

As a result

\[
Z_{\hat{F}_{mn},D_{00}} = \max_{\theta, \tau \in \{0, 1\}} Z_{\hat{F}_{mn},D_{\theta\tau}} \tag{23}
\]

**Proposition 4.3.** Assume all the triangle edges have weight 1, and all the non-triangle edges have weight not equal to 1. Assume around each triangle, an even number of edges have weight strictly less than 1. Assume the size of the graph \( m \) and \( n \) are even, and the number of edges crossed by \( \gamma_x \) and \( \gamma_y \) with weight strictly less than 1 are both even. Then

\( P(z, -1) = 0 \) have no zeros on the unit circle \( \mathbb{T} \).

**Proof.** When both \( m \) and \( n \) are even, we have

\[
PfK(1, 1) = Z_{00} - Z_{01} - Z_{10} - Z_{11}
\]

\[
PfK(1, -1) = Z_{00} + Z_{01} - Z_{10} + Z_{11}
\]

\[
PfK(-1, 1) = Z_{00} - Z_{01} + Z_{10} + Z_{11}
\]

\[
PfK(-1, -1) = Z_{00} + Z_{01} + Z_{10} - Z_{11}
\]

By (23), \( PfK(1, -1) > 0, PfK(-1, 1) > 0, PfK(-1, -1) > 0 \), given all the edge weights are strictly positive. Hence \( P(z, -1) \) have no real roots on \( \mathbb{T} \).

Assume \( P(z) = 0 \) has a non-real zeros \( z_0 \in \mathbb{T}^2 \). Assume

\[
z_0 = e^{\sqrt{-1} \alpha_0 \pi}, \quad \alpha_0 \in (0, 1) \cup (1, 2)
\]

If \( \alpha_0 \) is rational, namely \( \alpha_0 = \frac{p}{q} \), then after enlarging the fundamental domain to \( m \times qn \), \( P_{\theta}(z^q, -1) = 0 \), while \( z^q \) is real, which is a impossible.

Now let us consider the case of irrational \( \alpha_0 \). There exists a sequence \( \ell_k \in \mathbb{N} \), such that

\[
\lim_{k \to \infty} z_0^{\ell_k} = 1
\]

In other words, if we assume \( z_0^{\ell_k} = e^{\sqrt{-1} \alpha_k \pi} \) where \( \alpha_k \in (-1, 1) \), then

\[
\lim_{k \to \infty} \alpha_k = 0.
\]
According to the formula of enlarging the fundamental domain,

\[ P_{\ell k}(z_0^k, -1) = 0 \quad \forall k \]

By (23),

\[ P(1, -1) = (Z_{00} + Z_{01} - Z_{10} + Z_{11})^2 \geq (Z_{01} + Z_{11})^2 \]

Therefore we have

\[ 1 \leq \lim_{k \to \infty} \left| \frac{P_{\ell k}(z_0^k, -1) - P_{\ell k}(1, -1)}{(Z_{\ell k,01} + Z_{\ell k,11})^2} \right| \quad (24) \]

On the other hand

\[ \lim_{k \to \infty} \left| \frac{P_{\ell k}(z_0^k, -1) - P_{\ell k}(1, -1)}{(Z_{\ell k,01} + Z_{\ell k,11})^2} \right| = \lim_{k \to \infty} \frac{1}{(Z_{\ell k,01} + Z_{\ell k,11})^2} \sum_{t=1}^{\infty} \frac{[\pi \alpha_k]^2 t^2 \partial^{2t} P_{\ell k}(e^{\sqrt{-1} \theta})}{(2t)!} \big|_{\theta = 0} \quad (25) \]

\[ = \lim_{k \to \infty} \frac{1}{(Z_{\ell k,01} + Z_{\ell k,11})^2} \sum_{t=1}^{\infty} \frac{[\pi \alpha_k]^2 t^2 \partial^{2t} P_{\ell k}(e^{\sqrt{-1} \theta})}{(2t)!} \frac{\partial^{2t} P_{\ell k}(e^{\sqrt{-1} \theta})}{\partial \theta^{2t}} \big|_{\theta = 0} \quad (26) \]

\[ = \lim_{k \to \infty} \frac{1}{(Z_{\ell k,01} + Z_{\ell k,11})^2} \sum_{t=1}^{\infty} \frac{[\pi \alpha_k]^2 t^2 \partial^{2t} P_{\ell k}(e^{\sqrt{-1} \theta})}{(2t)!} \big|_{\theta = 0} \quad (27) \]

Since

\[ P_{\ell k}(z, -1) = \sum_{0 \leq j \leq m} P_j^{(\ell k)}(z + \frac{1}{z^j}) \]

where \( P_j^{(\ell k)} \) is the signed sum of loop configurations winding exactly \( j \) times around the cylinder (see Lemma 2.1), we have

\[ \frac{\partial^{2t} P_{\ell k}(e^{\sqrt{-1} \theta})}{\partial \theta^{2t}} \big|_{\theta = 0} = \sum_{1 \leq j \leq m} 2j^{2t}(-1)^t P_j^{(\ell k)}, \]

and

\[ \lim_{k \to \infty} \frac{1}{(Z_{\ell k,01} + Z_{\ell k,11})^2} \sum_{t=1}^{\infty} \frac{[\pi \alpha_k]^2 t^2 \partial^{2t} P_{\ell k}(e^{\sqrt{-1} \theta})}{(2t)!} \frac{\partial^{2t} P_{\ell k}(e^{\sqrt{-1} \theta})}{\partial \theta^{2t}} \big|_{\theta = 0} \]

\[ \leq \lim_{k \to \infty} \frac{1}{(Z_{\ell k,01} + Z_{\ell k,11})^2} \sum_{t=1}^{\infty} \frac{[\pi \alpha_k]^2 t^2 \partial^{2t} P_{\ell k}(e^{\sqrt{-1} \theta})}{(2t)!} \sum_{1 \leq j \leq m} |P_j^{(\ell k)}| \]

Using the same technique as described in the proof of Theorem 1.2, we have

\[ \sum_{1 \leq j \leq m} |P_j^{(\ell k)}| \leq \text{Partition of configurations including nonplanar odd loops} \]

+ \text{Partition of configurations with only even loops} \leq (2^m C_1^{6m} + 1) \text{Partition of even loop configurations} \leq C_2^m Z_{\ell k}^2 \]

Moreover, there is a one-to-one correspondence between dimer configurations in \( Z_{00} \) and \( Z_{01} \), similarly between dimer configurations in \( Z_{10} \) and \( Z_{11} \). We divide the \( m \times \ell_k n \) torus into \( \ell_k n \) circles, each circle has circumference \( m \). Fix one circle \( C_0 \), and fix configurations...
out side the circle and on the boundary of the circle. For each dimer configuration in $Z_{00}$, if we rotate the configuration to alternating edges along $C$, we get a dimer configuration in $Z_{01}$. An example of such an transformation is illustrated in the Figures 11 and 12.

As a result,

$$(Z_{t_k,00} + Z_{t_k,01} - Z_{t_k,10} + Z_{t_k,11})^2 \leq C^m(Z_{t_k,01} + Z_{t_k,11})^2$$

where $C$ is a constant independent of $k$. Hence

$$\lim_{k \to \infty} \frac{1}{(Z_{t_k,01} + Z_{t_k,11})^2} \sum_{t=1}^{\infty} (\pi \alpha_k)^{2t} \left( \frac{\partial^{2t} P_k(e^{\sqrt{-1} \theta})}{\partial \theta^{2t}} \right)_{\theta=0} \leq \frac{1}{(Z_{t_k,01} + Z_{t_k,11})^2} \sum_{t=1}^{\infty} (m \pi \alpha_k)^{2t} \frac{2^t C_m}{(2t)!} Z_{t_k}^2$$

$$\leq \lim_{k \to \infty} C_m^m \sum_{t=1}^{\infty} \frac{(m \pi \alpha_k)^{2t}}{(2t)!} C_m^m = 0$$

which is a contradiction to (27). Therefore the proposition follows.

Lemma 4.4. $P(z, w) \geq 0$, $\forall (z, w) \in T^2$.

Proof. Assume there exists $(z_0, w_0) \in T^2$, such that $P(z_0, w_0) < 0$, since $P(z, -1) > 0$ and $P(-1, w) > 0$, if we consider $z = e^{i\theta}, w = e^{i\phi}, z_0 = e^{i\theta_0}, w_0 = e^{i\phi_0}$, and $(\theta, \phi) \in [-\pi, \pi]^2$, on the boundary of the domain $[-\pi, \pi]^2$, $P(e^{i\theta}, e^{i\phi})$ are strictly positive, while $P(e^{i\theta_0}, e^{i\phi_0}) < 0$, by continuity there exists a neighborhood $O_h = (\phi_0 - h, \phi_0 + h)$, such that for any $\gamma$, such that for any $\phi \in O_h$, $P(e^{i\theta_0}, e^{i\phi}) < 0$. Consider straight lines in $[-\pi, \pi]^2$, connecting $(\theta_0, \phi)$ to $(\pi, \phi)$, namely

$$\gamma_\phi(t) = ((\pi - \theta_0)t + \theta_0, \phi)$$

for $\phi \in (\phi_0 - h, \phi_0 + h)$, there exists $t_\phi$, such that

$$P(e^{i((\pi - \theta_0)t_\phi + \theta_0)}, e^{i\phi}) = 0$$

By continuity, there exists a $\phi$ in the open interval $(\phi_0 - h, \phi_0 + h)$ such that $\frac{\phi}{q}$ a of -1. For instance $\phi = \frac{2}{p} \pi$ in reduced form where $q$ is odd. Then after enlarging the fundamental domain to pm x n, we have $P(e^{i((\pi - \theta_0)t_\phi + \theta_0)}, -1) = 0$, which is a contradiction.

4.4 Proof of Theorem 1.2

In this section, we prove that the spectral curve of Fisher graphs associated to any periodic, ferromagnetic Ising model is a Harnack curve, and its intersection with the unit torus $T^2 = \{(z, w) : |z| = 1, |w| = 1\}$ is either empty or a single real intersection of multiplicity 2.
First of all, it is proved in [Dub12] that the Ising characteristic polynomial $P(z, w)$ has the same zero locus as the characteristic polynomial of the square-octagon lattice, denoted by $P_C(z, w)$, with suitable clock-wise odd orientation. $P_C(z, w)$ is the characteristic polynomial of a bipartite graph with positive edge weights and clockwise-odd orientation, it is proved in [KO06] that $P_C(z, w) = 0$ is a Harnack curve, i.e., for any fixed $x, y > 0$ the number of zeros of $P_C(z, w)$ on the torus $\mathbb{T}_{x,y} = \{(z, w) : |z| = x, |w| = y\}$ is at most 2 (counting multiplicities). Hence the number of zeros of $P_F(z, w)$ on the unit torus is at most 2. Since $P_F(z, w) \geq 0$, $\forall (z, w) \in \mathbb{T}^2$, each zero $P_F(z, w)$ on $\mathbb{T}^2$ is at least of multiplicity 2. If $P_F(z, w)$ have a non-real zero $(z_0, w_0)$ on $\mathbb{T}^2$, then $(\bar{z}_0, \bar{w}_0) \neq (z_0, w_0)$ is also a zero of $P_F(z, w)$, and each of them is at least of multiplicity 2. Then $P_F(z, w)$ has at least 4 zeros on $\mathbb{T}^2$, which is a contradiction to the fact that $P_F(z, w) = 0$ is a Harnack curve. Hence the intersection of $P_F(z, w) = 0$ with $\mathbb{T}^2$ is either empty or a single real intersection of multiplicity 2.

**Example 4.5.** (1 × n fundamental domain) Assume the Fisher graph has period 1 × n, and all edge weights are strictly positive. The intersection of $P(z, w) = 0$ with $\mathbb{T}^2$ is either empty or a single real node. That is, they intersect at one of the point $(\pm 1, \pm 1)$ and the intersection is of multiplicity 2.

**Proof.** Without loss of generality, assume $P(z, w)$ is linear with respect to $z$ and $\frac{1}{z}$, then all terms in $P(z, w)$ fall into two categories

- **i)** occupy $z$-edge exactly once
- **ii)** does not occupy $z$-edge or occupy $z$-edge twice

1 × n fundamental domain is comprised of n 1 × 1 blocks, for each block, we give weights $a_1, b_1, c_1, a_2, b_2, c_2$ for triangle edges, assume $a_i = a_1a_2$, $b_i = b_1b_2$, $c_i = c_1c_2$, $1 \leq i \leq n$. See Figure 6. Obviously such an arrangement of edge weights is gauge equivalent to the arrangement giving all triangle edges weight 1 and non-triangle edges positive edge weights.

![Figure 12: Fisher Graph on a Cylinder](image)

Each configuration in case **i)** consists of a single even loop and some double edges, the partition of all configurations in **i** is

$$P_i = -\prod_{i=1}^{n}[(a_i - b_i c_i)w + (c_i - a_i b_i)]\frac{1}{z} - \prod_{i=1}^{n}[(a_i - b_i c_i)\frac{1}{w} + (c_i - a_i b_i)]z$$

Configurations in case **ii)** depend on configurations of each 1×1 block, which are determined by configurations of boundary edges. Let $V^{0,0}_i$ denote the partition at $i$-th block when both edges in $z$ direction are unoccupied, $V^{2,0}_i$ denote the partition at $i$-th block when left edge in $z$ direction is occupied twice, while right edge is unoccupied, similarly for $V^{0,2}_i$.
and $V_i^{2,2}$. Then we have

$$
V_{i}^{0,0} = a_i^2 + c_i^2 + a_i c_i (w + \frac{1}{w}) \\
V_{i}^{0,2} = a_i c_i b_i (\frac{1}{w} - w) \\
V_{i}^{2,0} = b_i a_i c_i (w - \frac{1}{w}) \\
V_{i}^{2,2} = b_i^2 - b_i (w + \frac{1}{w}) + 1
$$

Let $k_i \in \{0, 2\}$, $1 \leq i \leq n$, then partition of all configurations in $ii$ is

$$P_{0,2} = \sum_{k_1, \ldots, k_n} \prod_{i=1}^{n} V_{i}^{k_i, k_{i+1}}$$

Assume $w = e^{\sqrt{-1} \phi}$, then $V_{i}^{0,0} \geq 0$, $V_{i}^{2,2} \geq 0$. Periodicity implies that $V_{i}^{0,2}$ and $V_{i}^{2,0}$ always appear in pairs, then all terms in $P_{0,2}$ where $\sin \phi$ appears depend only on $\sin^2 \phi$, moreover, given all edge weights are positive, each term with $\sin \phi$ is nonnegative, therefore

$$
P(z, w) = P_1 + P_{0,2} = \prod_{i=1}^{n} [(a_i - b_i c_i)w + (c_i - a_i b_i)] \frac{1}{z} - \prod_{i=1}^{n} [(a_i - b_i c_i) \frac{1}{w} + (c_i - a_i b_i)]z + \prod_{i=1}^{n} [a_i^2 + c_i^2 + a_i c_i (w + \frac{1}{w})] + \prod_{i=1}^{n} [b_i^2 - b_i (w + \frac{1}{w}) + 1] + F(w)
$$

where $F(w) \geq 0$. Assume $G(z, w) = z[P(z, w) - F(w)]$. Fix $w$, $G(z, w)$ is a quadratic polynomial with respect to $z$. Assume

$$
C_0 = \prod_{i=1}^{n} [(a_i - b_i c_i) \frac{1}{w} + (c_i - a_i b_i)] \\
C_1 = \prod_{i=1}^{n} [a_i^2 + c_i^2 + a_i c_i (w + \frac{1}{w})] + \prod_{i=1}^{n} [b_i^2 - b_i (w + \frac{1}{w}) + 1]
$$

then the root for $G(z, w) = 0$ will be

$$
z_{1,2} = \frac{-C_1 \pm (C_1^2 - 4|C_0|^2)^\frac{1}{2}}{2C_0}
$$

On the other hand,

$$
C_1^2 - 4|C_0|^2 \geq 4 \prod_{i=1}^{n} (a_i^2 + c_i^2 + 2a_i c_i \cos \phi)(b_i^2 - 1 - 2b_i \cos \phi) - \prod_{i=1}^{n} [(a_i - b_i c_i)^2 + (c_i - a_i b_i)^2 + 2(a_i - b_i c_i)(c_i - a_i b_i) \cos \phi)] \\
(a_i^2 + c_i^2 + 2a_i c_i \cos \phi)(b_i^2 + 1 - 2b_i \cos \phi) - [(a_i - b_i c_i)^2 + (c_i - a_i b_i)^2 + 2(a_i - b_i c_i)(c_i - a_i b_i) \cos \phi)]
$$

$$
= 4a_i b_i c_i \sin^2 \phi \geq 0
$$
Therefore, $C_1^2 - 4|C_0|^2 \geq 0$. Given $a_i, b_i, c_i > 0$, equality holds only if $w$ is real. If $C_1^2 - 4|C_0| > 0$, then $|z_{1,2}| \neq 1$, $G(z, w)$ has no zeros on $\mathbb{T}^2$. Since $G(1, 1) = \det K(1, 1) > 0$, $P(z, w) - F(w) > 0$ on $\mathbb{T}^2$, so $P(z, w) > 0$ on $\mathbb{T}^2$. If $C_1^2 - 4|C_0|^2 = 0$, then $w$ is real, $z_1 = z_2$ are real, $P(z, w)$ has real node on $\mathbb{T}^2$. □

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