Brans-Dicke wormholes in nonvacuum spacetime

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Abstract

Analytical wormhole solutions in Brans-Dicke theory in the presence of matter are presented. It is shown that the wormhole throat must not be necessarily threaded with exotic matter.

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The field equations of general relativity, being local in character, admit solutions with nontrivial topology. Among these, wormholes have been extensively studied [1]. Their most salient feature is that an embedding of one of their spacelike sections in Euclidean space displays two asymptotically flat regions joined by a throat.

The interest on wormholes is twofold. From the point of view of the Euclidean path integral formulation of quantum gravity, Coleman [2] and Giddings and Strominger [3], among others, have shown that the effect of wormholes is to modify low energy coupling constants and to provide probability distributions for them. In particular, Coleman [4] showed that, in the dilute wormhole approximation, the probability distribution for universes is infinitely peaked at $\Lambda = 0$, rendering all other values of the cosmological constant improbable.

On the purely gravitational side, the interest has been recently focused on traversable wormhole [1,5–8]. Most of the efforts are directed to study static configurations [9] that must have a number of specific properties in order to be traversable. The most striking of these properties is the violation of the energy conditions [10]. It implies that the matter that generates the wormhole is exotic [1], viz. its energy density is negative, as seen by static observers. Geometrically, this is a direct consequence of the singularity theorems of Hawking and Penrose [11]. Although we do not know of any such exotic material to date, quantum field theory might come to the rescue [12].

Finally, we should mention yet another proposal related to wormholes. It has been shown [5,13] that a nonstatic wormhole’s throat can be transformed into a time tunnel. Physical effects in this type of spacetimes have been studied in [14].

Wormhole solutions have also been discussed in alternative theories of gravity, such as $R + R^2$ theories [15], Moffat’s nonsymmetric theory [16], Einstein-Gauss-Bonnet theory [17], and Brans-Dicke (BD) theory [18]. In the last case, static wormhole solutions were found in vacuum, the source of gravity being the scalar field. Dynamical solutions are discussed in [19]. The aim of this paper is to look for static wormhole solutions of Brans-Dicke theory in a general setting, i.e. in the presence of matter that obeys a generic equation of state [20]. We shall also discuss whether the BD scalar can be the “carrier” of exoticty, as was shown
in [18] for the vacuum case.

Following the conventions of [21], the field equations of Brans-Dicke theory are

\[
R_{\mu\nu} = \frac{8\pi}{\Phi} \left( T_{\mu\nu} - \frac{\omega + 1}{2\omega + 3} T g_{\mu\nu} \right) + \omega \frac{\Phi_{,\mu} \Phi_{,\nu}}{\Phi^2} + \frac{\Phi_{,\mu\nu}}{\Phi} \tag{1}
\]

\[
\Phi_{,\mu} = \frac{8\pi}{2\omega + 3} T \tag{2}
\]

The assumption of a static spacetime entails that it is possible to choose a metric and a scalar field such that

\[
g_{\mu\nu,t} = 0 \quad \Phi_{,t} = 0 \quad g_{\iota\iota} = 0 \tag{3}
\]

\((i = r, \theta, \phi)\). We further require spherical symmetry, so that the line element can be written in Schwarzschild form:

\[
ds^2 = -e^{2\psi} dt^2 + e^{2\lambda} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{4}
\]

For the stress-energy tensor of matter we choose

\[
T^t_t = -\rho(r) \quad T^r_r = -\tau(r) \quad T^\theta_\theta = T^\phi_\phi = p(r) \tag{5}
\]

and zero otherwise. Finally, we adopt the following equation of state for matter:

\[-\tau + 2p = \epsilon \rho \tag{6}\]

where \(\epsilon\) is a constant. Now, the trace of the stress-energy tensor can be written as \(T = -\tau + 2p - \rho = \rho(\epsilon - 1)\). The field equations take the form

\[-\psi'' - (\psi')^2 + \lambda' \psi' + 2 \frac{\lambda'}{r} = -\frac{8\pi}{\Phi} \left[ \tau + \frac{\omega + 1}{2\omega + 3} T \right] e^{2\lambda} + (\omega + 1)(\ln \Phi)^2 + (\ln \Phi)'' - \lambda'(\ln \Phi)' \tag{7a}\]

\[1 - re^{-2\lambda} \left[ \psi' - \lambda' + \frac{1}{r} \right] = \frac{8\pi}{\Phi} \left[ \rho - \frac{\omega + 1}{2\omega + 3} T \right] r^2 + re^{-2\lambda}(\ln \Phi)' \tag{7b}\]

\[e^{2(\psi-\lambda)} \left[ \psi'' + (\psi')^2 - \lambda' \psi' + 2 \frac{\psi'}{r} \right] = \frac{8\pi}{\Phi} \left[ \rho + \frac{\omega + 1}{2\omega + 3} T \right] e^{2\psi} - \psi' e^{2(\psi-\lambda)}(\ln \Phi)' \tag{7c}\]
\[ \Phi'' - \Phi' \left( \lambda' - \psi' - \frac{2}{r} \right) = \frac{8\pi}{2\omega + 3} T e^{2\lambda} \]  

(7d)

To solve the system made up of Eqs. (7) we shall follow the philosophy sketched in [22]. We shall look for a differential equation relating \( \psi \) and \( \lambda \), starting from the equations of motion and the equation of state. The equation we shall obtain is second order and nonlinear in \( \psi \) but, after a change of variables, first order and linear in \( \lambda \). We shall then make a specific choice for \( \psi \) consistent with asymptotic flatness and nonexistence of horizons and singularities. We shall finally substitute this \( \psi \) into the linear equation and solve for \( \lambda \).

As explained in [21], from Eqs. (6), (7c), and (7d), it can be shown that \( \Phi = \Phi_0 e^{c\psi} \) where \( c = (\epsilon - 1)/(2\omega + 3 + (\omega + 1)(\epsilon - 1)) \), and \( \Phi_0 \) is related to the value of the gravitational coupling constant when \( r \to \infty \). In the case \( \omega \to \infty \) or \( \epsilon \to 1 \), we get general relativity back (although in the latter case, other solutions different from \( \Phi = \text{const} \) might exist).

After a bit of algebra, we get the equation:

\[ A \psi'' + B (\psi')^2 + 2A \psi' - A \lambda' \psi' + \frac{2}{r^2} (e^{2\lambda} - 1) = 0 \]  

(8)

where

\[ A = -2 \frac{2 + \epsilon + 2\omega}{2 + \epsilon + \omega(1 + \epsilon)} \]

\[ B = -\frac{8 + \epsilon^2(\omega + 2) + 4\omega^2(1 + \epsilon) + 8\epsilon + 11\omega + 12\omega\epsilon}{[2 + \epsilon + \omega(\epsilon + 1)]^2} \]

In the spirit of [22], we make the ansatz \( \psi = -\alpha/r \), where \( \alpha \) is a positive constant. With this election, which guarantees that the gravitational constant takes the correct value at \( r \to \infty \), Eq. (8) takes the form

\[ h(r) + f(r) e^{2\lambda} + g(r) \lambda' = 0 \]  

(9)

where

\[ h(r) = B \left( \frac{\alpha}{r^2} \right)^2 - \frac{2}{r^2} \quad f(r) = \frac{2}{r^2} \quad g(r) = -\frac{A\alpha}{r^2} + \frac{4}{r} \]
A suitable change of variables transforms Eq. (9) into a Bernoulli equation, and afterwards into a linear equation. Its general solution is given by

\[ e^{-2\lambda} = \frac{e^{2s/\varphi}}{\varphi} \left( 1 + \frac{R}{\varphi} \right)^{-(s+1)} \{ I + K \} \tag{10} \]

where

\[ \varphi = \frac{r}{\alpha}, \quad s = \frac{B}{A}, \quad R = -\frac{A}{4}, \quad l = -\frac{B}{A^2} \]

\[ I \equiv \int e^{-2s/\varphi} \left( 1 + \frac{R}{\varphi} \right) ^{8l} d\varphi \]

and \( K \) is a constant. It is not valid when \( A \to 0 \), i.e. for \( \omega = -1 - \epsilon/2 \). The binomial \((1 + R/\varphi)^{8l}\) is related to the hypergeometric function \(_2F_1[23]\). Using the relation \([23]\)

\[ e^{t}pF_q(\alpha_1, \ldots \alpha_p; \beta_1 \ldots \beta_q; -xt) = \sum_{n=0}^{\infty} p+1F_q(-n, \alpha_1, \ldots \alpha_p; \beta_1, \ldots \beta_q; x) \frac{t^n}{n!}, \tag{11} \]

the integral \( I \) can be written

\[ I = 2s \sum_{n=0}^{\infty} \int _3F_1(-n, -8l, b; b; R/2s) \left( \frac{-2s}{\varphi} \right)^n \tag{12} \]

Integrating out the terms corresponding to \( n = 0 \) and \( n = 1 \), we finally get

\[ I = \varphi - 8l R \ln \varphi + \varphi \sum_{n=2}^{\infty} 3F_1(-n, 8l, b; b; R/2s)(-1)^n \left( \frac{2s}{\varphi} \right)^n \frac{1}{n!(n-1)} \tag{13} \]

It is easily seen that \( e^{2\lambda} \to 1 \) when \( \varphi \to \infty \).

In order to fix the constant \( K \), we must select a value for the dimensionless radius \( (\varphi_{th}) \) such that the “flaring out” condition

\[ \lim_{\varphi \to \varphi_{th}^+} e^{-2\lambda} = 0^+ \tag{14} \]

is satisfied. In the case \( R \leq 0 \), \( \varphi_{th} \) must necessarily be greater than \( |R| \), so that the flaring out condition holds for all values of \( \omega \) and \( \epsilon \) except, obviously, those where \( R \) diverges, which are given by \( \omega = -(2+\epsilon)/(1+\epsilon) \). Nevertheless, the absolute size of the throat also depends
on $\alpha$. The aforementioned properties of $\lambda$, together with the definition of $\psi$, bear out that the metric tensor describes two asymptotically flat spacetimes joined by a throat.

Let us now study the issue of weak energy condition (WEC) violation. Using the field equations and the expression for the trace, we easily obtain

$$\frac{2e^{2\lambda}}{r^2} - \frac{4\psi'}{r} - \frac{2}{r^2} = \frac{16\pi}{\Phi} e^{2\lambda} + \frac{4\Phi'}{r\Phi} - \omega \left( \frac{\Phi'}{\Phi} \right)^2 + 2 \frac{\Phi'}{\Phi} \psi'$$

At the throat, $e^{2\lambda} \to \infty$, and then

$$\tau_\text{th} \approx \frac{\Phi_\text{th}}{8\pi r_\text{th}^2}$$

To calculate $\rho_\text{th}$, we use the nontrivial component of the equation $T^\mu_{\nu;\mu} = 0$:

$$\tau' = \psi'(\rho - \tau) - \frac{2\tau}{r} - \frac{\epsilon \rho + \tau}{r}$$

Using Eqs. (16) and (17), and the derivative of Eq. (15),

$$\rho_\text{th} \approx \tau_\text{th} \frac{c + 1 + \varphi_\text{th}}{1 - \epsilon \varphi_\text{th}}$$

And finally, from Eq. (13),

$$p_\text{th} \approx \frac{\tau_\text{th}}{2} \frac{\epsilon (c + 1) + 1}{1 - \epsilon \varphi_\text{th}}$$

We shall show now that WEC may be violated (at least near the throat) with nonexotic matter. This means that we shall present the parameters for which a wormhole solution exists whenever the matter content of the theory satisfying the inequalities

$$\rho_\text{th} \geq 0 \quad \rho_\text{th} - \tau_\text{th} \geq 0 \quad \rho_\text{th} + p_\text{th} \geq 0$$

or equivalently,

\[1\] This situation is analogous to what Kar and Sahdev have found for wormholes in general relativity.\textsuperscript{22}
\[
\frac{c + 1 + \varphi_{th}}{1 - \epsilon \varphi_{th}} \geq 1
\]  
(21)

\[
\frac{\epsilon(c + 1) + 3 + 2(c + \varphi_{th})}{1 - \epsilon \varphi_{th}} \geq 0
\]  
(22)

In addition, a necessary condition for the violation of the weak energy condition for matter plus Brans-Dicke field at the throat is given by

\[
\frac{2(\omega + 1) + \epsilon}{2\omega + 3} \rho_{th} \leq 0
\]  
(23)

As an example, let us study the case \(\epsilon = 2\). From Eqs. (16), (18), and (19), the inequalities (20) will be satisfied if

\[
\left( \varphi_{th} \geq -\frac{1}{9\omega + 12} \text{ and } \varphi_{th} < \frac{1}{2} \right) \text{ or } \left( \varphi_{th} \leq -\frac{1}{9\omega + 12} \text{ and } \varphi_{th} > \frac{1}{2} \right)
\]  
(24)

Inequality (23) will be satisfied for \(\omega \in (-2, -3/2)\). Finally, we have to impose that \(\varphi_{th} \geq |A/4|\), which implies that

\[
\varphi_{th} \geq \left| \frac{2 + \omega}{4 + 3\omega} \right|
\]  
(25)

These inequalities constrain \(\varphi_{th}\) to an interval in which a nonexotic wormhole can be constructed, for instance, in the case \(\omega = -1.75\). We should recall that a definite interval for \(\varphi_{th}\) does not determine the radius of the throat, because of the dependence of \(\varphi\) on \(\alpha\).

Summing up, we showed that Brans-Dicke theory in the presence of matter with a fairly general equation of state admits analytical wormhole solutions. They generalize the vacuum ones presented by Agnese and La Camera [18]. It should be noted that there exists some regions of the parameter space in which the Brans-Dicke field may play the role of exotic matter, implying that it might be possible to build a wormholelike spacetime with the presence of ordinary matter at the throat.

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