(Zα)^4 order of the polarization operator in Coulomb field at low energy

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Abstract

We derive the low-energy expansion of (Zα)^2 and (Zα)^4 terms of the polarization operator in the Coulomb field. Physical applications such as the low-energy Delbrück scattering and "magnetic loop" contribution to the g factor of the bound electron are considered.

Key words: polarization operator, multiloop calculations, quantum electrodynamics, g factor

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1. Introduction

One of the predictions of the quantum field theory is a vacuum polarization by an external field. An important case thoroughly studied both experimentally and theoretically is the vacuum polarization effects in atomic field. Methods used for the study of this effect essentially depend on the nuclear charge |e|. At low |e|, the perturbation theory with respect to Zα is applicable (α = e^2 = 1/137 is the fine structure constant, \hbar = c = 1). At high |e|, the interaction with the external field should be taken into account exactly, which can be done with the help of the electron Green function in this field. This approach often requires quite involved numerical calculations, which usually fail to give the results for low |e|. Thus, the two approaches tend to be complementary. Usually, the perturbative calculations of vacuum polarization effect are limited by the leading order, since the first nonvanishing correction involves two more loops. Nowadays, the modern methods of calculation of the multiloop integrals are sufficiently powerful for the calculation of higher orders in Zα. It provides a possibility to compare the results of these approaches.

One of the basic nonlinear QED processes in the atomic field is the Delbrück scattering \cite{1}, the scattering of the photon in the Coulomb field due to the vacuum polarization. The amplitude of this process in the Born approximation has been obtained long ago for arbitrary energies in Ref. \cite{2}. At high energies and small scattering angles, when the quasiclassical approximation is valid, the amplitude is known exactly in Zα, see Refs. \cite{3,4,5,6,7,8}. Recently in Ref. \cite{9}, the Delbrück amplitude has been calculated numerically exactly in the parameter Zα at low energies. It was shown that the contribution of the higher orders (Coulomb corrections) to the
amplitude can be well fitted by the polynomial \( C_4(Z\alpha)^4 + C_6(Z\alpha)^6 \). The calculation of \((Z\alpha)^4\) term in perturbation theory would provide the independent check of the result of Ref. \[9\].

In present paper, we consider the polarization operator \( \Pi^{\mu\nu}(\omega, \mathbf{k}, \mathbf{q}) \) in the Coulomb field for small external momenta

\[
\omega \sim |\mathbf{k}| \sim |\mathbf{q}| \sim \lambda m,
\]

where \( \lambda \) is a dimensionless small parameter,

\[
\lambda \ll 1.
\]

We calculate the expansion of the polarization operator in the external Coulomb field in \( \lambda \) and \( Z\alpha \) up to the order \( \lambda^4 (Z\alpha)^4 \). The low-energy Delbrück scattering amplitude is readily expressed in terms of this operator. This polarization operator is also an essential ingredient of calculations of different physical observables in atoms, like Lamb shift and magnetic moment of the bound particle.

The polarization operator in the external Coulomb field is determined as follows:

\[
\Pi^{\mu\nu}(\omega, \mathbf{k}, \mathbf{q}) = 4\pi i e^2 \int d\mathbf{x} d\mathbf{y} d\mathbf{t} e^{-i\omega t + i\mathbf{k}\mathbf{x} - i\mathbf{q}\mathbf{y}} \langle \text{vac} | T J^\mu (t, \mathbf{x}) J^\nu (0, \mathbf{y}) | \text{vac} \rangle,
\]

where \( J^\mu = \bar{\psi} \gamma^\mu \psi \) is the electron current and the state \( |\text{vac}\rangle \) corresponds to the vacuum state in the presence of the Coulomb potential \( Z|e|/r \). In \( e^2 \) order, we have

\[
\Pi^{\mu\nu}(\omega, \mathbf{k}, \mathbf{q}) = 4\pi i e^2 \int \frac{d\omega}{2\pi} d\mathbf{x} d\mathbf{y} e^{i\mathbf{k}\mathbf{x} - i\mathbf{q}\mathbf{y}} \text{Tr} \left[ \gamma^\mu G (\mathbf{x}, \mathbf{y} | \omega - \omega) \gamma^\nu G (\mathbf{y}, \mathbf{x} | \omega) \right],
\]

where \( G (\mathbf{y}, \mathbf{x} | \omega) \) is the Green function of the electron in the Coulomb field. Due to the gauge invariance, the polarization operator obeys the constraints

\[
k_\mu \Pi^{\mu\nu}(\omega, \mathbf{k}, \mathbf{q}) = q_\nu \Pi^{\mu\nu}(\omega, \mathbf{k}, \mathbf{q}) = 0,
\]

where \( k^0 = q^0 = \omega \). Using these constraints, we can express \( \Pi^{\mu\nu} \) via five independent tensor structures, which we choose as follows

\[
m^3 \Pi^{\mu\nu} = f_1 (\mathbf{q}^\mu k^\nu - \mathbf{q}^\nu k^\mu) - f_3 e^{\mu\nu\rho\sigma} n_\rho e^{\nu\tau\rho\sigma} n_\sigma \frac{k_\rho (k - q)_\gamma (k - q)_\sigma q_\tau}{(k - q)^2} \\
+ (n^\mu k^\nu - \omega q^\mu)(\mathbf{q}^\mu n^\nu - \omega q^\nu) \left[ f_2 \delta_{\alpha\beta} + f_4 \frac{k_\alpha q_\beta}{\omega^2} - f_5 (k - q)_\alpha (k - q)_\beta \right],
\]

where \( n = (1, \mathbf{0}) \) and \( f_i \) are some scalar functions of \( \omega, \mathbf{k} \) and \( \mathbf{q} \).

It is important to note that the low-energy expansion of \( \Pi^{\mu\nu} \) itself and the functions \( f_{1-5} \) is not reduced to the multiple Taylor expansion in \( k^\mu, q^\mu, \omega \), i.e., \( f_{1-5} \) are nonanalytic functions of the external momenta. Nevertheless, the expansion in powers of the parameter \( \lambda \) from Eq. \( \text{(2)} \) is still possible. Different terms of the expansion come from one of two different regions of integration. We separate the contributions of these regions, using the dimensional regularization, in spirit of Refs. \[10, 11\]. The methods of separation and calculation of the contributions of these two regions are given in Sections \[2\] and \[3\] respectively. The results and conclusions are presented in Section \[4\].
2. Separation of the contributions of soft and hard regions

In order to demonstrate our method, let us consider the behaviour of the integral

$$J^{(D)}(q^2) = \int \frac{d^{D} \Delta d^{D} p}{\pi^{D}} f(p, \Lambda, q)$$

$$= \int \frac{\pi^{-D} d^{D} \Delta d^{D} p}{\Delta^2 (q + \Delta)^2 (p^2 + 1) [(p + q)^2 + 1][(p - \Delta)^2 + 1]}, \quad (7)$$

at $q^2 \ll 1$. The corresponding diagram is depicted in Fig. 1. The small-$q^2$ expansion of this integral, obtained in Ref. [12], has the form:

$$J^{(D)}(q^2) = \sum_{n=0}^{\infty} C_n(D) (-q^2)^n + (q^2)^{\frac{D}{2}-2} \sum_{n=0}^{\infty} D_n(D) (-q^2)^n, \quad (8)$$

$$C_n(D) = \frac{\Gamma(n + 3 - D/2)}{(n + 3 - D/2) (D - 3)} \left[ \frac{\Gamma(D/2 - 1)}{\Gamma(n + 5 - D) \Gamma(n + 3 - D/2)} \frac{\Gamma(n + 1) \Gamma(n + 3 - D/2)}{\Gamma(D/2 - 1) \Gamma(2n + 3)} \right]. \quad (9)$$

$$D_n(D) = \Gamma(2 - D/2) \Gamma(\frac{D}{2} - 1)^2 \frac{\Gamma(n + 1) \Gamma(n + 3 - D/2)}{\Gamma(D/2 - 2) \Gamma(2n + 3)}. \quad (10)$$

The representation (8) determines $J^{(D)}(q^2)$ at $0 < q^2 < 4$. The limit $q^2 \to 0$ essentially depends on $D$. For $D > 4$, this limit is equal to $C_0(D)$ and can be considered as the value of the function $J^{(D)}$ at $q^2 = 0$. For $D < 4$, there is no finite $q^2 \to 0$ limit of the representation (8). However, the value $J^{(D)}(0)$ defined via the analytic continuation with respect to $D$ from the region $D > 4$ is still $C_0(D)$. In other words, the limit $q^2 \to 0$ is not commuting with the analytic continuation with respect to $D$. The same claim is valid for the derivatives of $J^{(D)}(q^2)$ with respect to $q^2$.

As a consequence, the integrand expanded in $q$ gives only the terms $\propto (q^2)^n$ after the integration within the dimensional regularization. These terms correspond to the first sum in the right-hand side of Eq. (8):

$$\int \frac{d^{D} \Delta d^{D} p}{\pi^{D}} \left[ \sum_{n=0}^{\infty} \frac{q^i \cdots q^s}{n!} \left. \frac{\partial^n f(p, \Lambda, q)}{\partial q^{i_1} \cdots \partial q^{i_n}} \right|_{q=0} \right] = \sum_{n=0}^{\infty} C_n(D) \left( -\frac{q^2}{4} \right)^n. \quad (11)$$
Note that the expansion of the massless propagators is valid only in the region $\Delta \gg q$ (hard region). In order to obtain the rest terms of the expansion (8) one has to determine the contribution of the region $\Delta \sim q$ (soft region). To separate these contributions, we use the following trick. In the soft region the massive propagators can be expanded in both $q$ and $\Lambda$. Let us truncate this expansion at some fixed order $N$ in $q$ and $\Lambda$:

$$j_{\text{soft}}(N, p, \Lambda, q) = \sum_{n=0}^{N} j_{\text{soft}}^{(n)}(p, \Lambda, q),$$

$$j_{\text{soft}}^{(n)}(p, \Lambda, q) = \left. \frac{\partial^n p^4 j(p, \tau, \tau q)}{n! \partial \tau^n} \right|_{\tau=0}, \quad \text{so that } j_{\text{soft}}^{(n)} = O\left(q^{n\Delta}\right).$$

Now we can identically rewrite $J^{(D)}(q^2)$ as

$$J^{(D)}(q^2) = J_{\text{hard}}^{(D)}(q^2) + J_{\text{soft}}^{(D)}(q^2)$$

$$= \int \frac{d^D \Delta d^D p}{\pi^D} \left[ j(p, \Lambda, q) - j_{\text{soft}}(N, p, \Lambda, q) \right] + \int \frac{d^D \Delta d^D p}{\pi^D} j_{\text{soft}}(N, p, \Lambda, q).$$

The contribution of the soft region in the first term in Eq. (13) is suppressed as $q^{D+N-3}$, thus the integral of the difference $j(p, \Lambda, q) - j_{\text{soft}}(N, p, \Lambda, q)$ is determined by the region $\Delta \sim 1 \gg q$ up to $O\left(q^{2(D+N-3)/2}\right)$ term. In fact, the expansion of $j_{\text{soft}}(N, p, \Lambda, q)$ in $q$ gives scaleless functions of $\Lambda$, which vanish after the integration in the dimensional regularization. Finally, we have

$$J^{(D)}(q^2) = J_{\text{hard}}^{(D)}(q^2) + J_{\text{soft}}^{(D)}(q^2)$$

$$= \sum_{n=0}^{\infty} \int \frac{d^D \Delta d^D p}{\pi^D} j_{\text{hard}}^{(n)}(p, \Lambda, q) + \sum_{n=0}^{\infty} \int \frac{d^D \Delta d^D p}{\pi^D} j_{\text{soft}}^{(n)}(p, \Lambda, q),$$

$$j_{\text{hard}}^{(n)}(p, \Lambda, q) = \left. \frac{\partial^n p^4 j(p, \tau, \tau q)}{n! \partial \tau^n} \right|_{\tau=0}, \quad j_{\text{soft}}^{(n)}(p, \Lambda, q) = \left. \frac{\partial^n p^4 j(p, \tau, \tau q)}{n! \partial \tau^n} \right|_{\tau=0}.$$

Let us now use this approach for the calculation of the low-energy expansion of the polarization operator in the Coulomb field. The $(2\alpha)^N$ contribution ($N$ is even) is determined by the $N$-loop diagrams depicted in Fig. 2. In the dimensional regularization, it can be represented in the form

$$\Pi^{\mu\nu}_{(2\alpha)^N}(\omega, k, q) = \int \frac{d\epsilon d^D p}{(2\pi)^{D+1}} \frac{\prod_{i=1}^{N-1} d^D \Delta_i}{(2\pi)^{(N-1)D}} \Phi^{\mu\nu}(\epsilon, p, \Lambda_1, \ldots, \Lambda_{N-1}, \omega, k, q).$$

Similar to the previous example, there are two different region of integration

$$\text{hard region, when } \epsilon \sim p^i \sim \Delta_{1, N-1} \sim m,$$

$$\text{soft region, when } \left\{ \begin{array}{l} \epsilon \sim p^i \sim m, \\ \Delta_{1, N-1} \sim \lambda m. \end{array} \right.$$

Again, the expansion of the polarization operator is the sum of the integrals of the expansion of the integrand in hard and soft regions. In the coordinate representation, these regions have a simple physical meaning. The characteristic size of the electron field fluctuations (the size of
the electron loop) is of the order $1/m$. Hard region corresponds to the configurations where the distance between the Coulomb source and the electron loop is of the order of $1/m$. Soft region corresponds to the creation of the virtual electron-positron pair far from the Coulomb source. Obviously, there is no contribution from the region where only some of the momenta $\Delta_n$ are hard while the rest are soft. In the momentum representation, these regions correspond to massless tadpole diagrams which are zero in the dimensional regularization. Thus, the expansion of the polarization operator (15) has the form

$$\Pi_{\mu\nu}^{(Z\alpha)^N}(\omega, k, q) = \Pi_{\mu\nu}^{(Z\alpha)^N,\text{hard}}(\omega, k, q) + \Pi_{\mu\nu}^{(Z\alpha)^N,\text{soft}}(\omega, k, q)$$

$$= \sum_n \int \frac{d\varepsilon}{(2\pi)^{D+1}} \frac{d^D p}{(2\pi)^{D+1}} \prod_{i=1}^{N-1} d^D \Delta_i \Pi_{\mu\nu}^{(n)} + \sum_n \int \frac{d\varepsilon}{(2\pi)^{D+1}} \frac{d^D p}{(2\pi)^{D+1}} \prod_{i=1}^{N-1} d^D \Delta_i \Pi_{\mu\nu}^{(n)}.$$  

(18)

$$\Pi_{\mu\nu}^{(n)} = \frac{\partial^n \Pi_{\mu\nu}^{(\varepsilon, p, \Delta_1, \ldots, \Delta_{N-1}, \tau\omega, \tau k, \tau q)}}{(n!)^2 \partial^{n^2}} \bigg|_{\tau=0},$$

(19)

$$\Pi_{\mu\nu}^{(n)} = \frac{\partial^{n^2} \Pi_{\mu\nu}^{(\varepsilon, p, \Delta_1, \ldots, \Delta_{N-1}, \tau\omega, \tau k, \tau q)}}{(n!)^{n^2}} \bigg|_{\tau=0}. $$

(20)

Note that the simple power counting allows one to estimate the leading terms of the hard and soft contribution as

$$\Pi_{\mu\nu}^{(Z\alpha)^N,\text{hard}}(\omega, k, q) \sim \lambda^2,$$

$$\Pi_{\mu\nu}^{(Z\alpha)^N,\text{soft}}(\omega, k, q) \sim \lambda^{N-1}(D-1)+1.$$  

(21)

Using Eq. (18), one can calculate the contributions of the hard and soft regions separately.

3. Method of calculation

The contribution of the soft region can be graphically represented as the tree diagram shown in Fig. 3. The local multiphoton vertex depicted as a thick dot corresponds to the expansion of
the fermionic loop with respect to the soft momenta $\Lambda_i, k, q$. The expansion is expressed via the
integrals of the following form

$$
\int \frac{d\epsilon dDp}{(2\pi)^{2D+1}} \left( \epsilon^2 - p^2 - m^2 + i0 \right)^{-n} = i(-1)^n \Gamma(n - D/2 - 1/2) \frac{\Gamma(n) m^{2n-D-1}}{(4\pi)^{2D+1/2}}.
$$

The remaining integrals over $\Lambda_i$ can be easily evaluated in the coordinate representation. Nat-
urally, the contribution of the soft region can be calculated also with the help of the derivative
expansion of the one-loop effective QED action.

The contribution of the hard region determined by Eq. (18) isexpressed in terms of the $N$-
loop tadpoles. In particular, in $(Z\alpha)^2$ order, the basic integral has the following form

\begin{equation}
I = \int \frac{d\epsilon dDp dD\Delta}{(2\pi)^{2D+1}} \left( \epsilon^2 - p^2 - m^2 + i0 \right)^{n_1} \left( \epsilon^2 - (p-\Delta)^2 - m^2 + i0 \right)^{n_2} \left( \Delta^2 \right)^{n_3}.
\end{equation}

After the Wick rotation $\epsilon \rightarrow i\epsilon$ and rescaling $p \rightarrow \sqrt{\epsilon^2 + m^2} p$, $\Lambda \rightarrow \sqrt{\epsilon^2 + m^2} \Lambda$, we integrate
over $\epsilon$ and obtain

\begin{equation}
I = \frac{m^{1-2\gamma} \Gamma(\gamma - 1/2)}{2 \sqrt{\pi} \Gamma(\gamma)} \int \frac{dDp dD\Delta}{(2\pi)^{2D}} \left( p^2 + 1 \right)^{n_1} \left( (p-\Delta)^2 + 1 \right)^{n_2} \left( \Delta^2 \right)^{n_3},
\end{equation}

$$
\gamma = n_1 + n_2 + n_3 - D.
$$

The remaining integral is the two-loop tadpole in $D = 3 + \epsilon$ dimensions which can be easily
expressed in terms of $\Gamma$-functions.

Performing the similar integration over $\epsilon$ in $(Z\alpha)^3$ order, we express the contribution of the
hard region in terms of the integrals of the topology (and its subttopologies) depicted in Fig. 4.

The general form of such integral is

\begin{equation}
J_{n_1 \ldots n_{10}} = \int \frac{dDk_1 dDk_2 dDk_3 dDk_4 \epsilon D_{0}^{-n_9} D_{10}^{-n_{10}}}{\pi^{2D} D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}},
\end{equation}

where

\begin{align*}
D_1 & = k_2^2 + 1, \\
D_5 & = (k_1 - k_2)^2, \\
D_6 & = (k_2 - k_3)^2, \\
D_7 & = (k_3 - k_4)^2, \\
D_9 & = (k_1 - k_3)^2, \\
D_{10} & = (k_2 - k_4)^2.
\end{align*}
Figure 4: Topology of the integrals required for the calculation of the low-energy expansion of the polarization operator in $(Z\alpha)^4$ order.

The IBP reduction procedure [13, 14] allows one to express any vacuum integral of the considered topology via the five master integrals shown in Fig. 5. On these diagrams the solid and dashed lines denote the massive $(k^2 + 1)^{-1}$ and massless $(k^2)^{-1}$ propagators. For each loop momentum the integration measure is taken as

$$\frac{d^{D}k}{\pi^{D/2}}.$$

Four of these integrals are trivially expressed in terms of $\Gamma$-functions. Their explicit forms are presented in Appendix. The only nontrivial master integral is $J_{\text{Cake}}$. After the IBP reduction, the master integral $J_{\text{Cake}}$ enters the polarization operator with the coefficient, having the first-order pole in the point $D = 3$. Therefore, we need to determine the $O(\epsilon^0)$ and $O(\epsilon^1)$ terms of the expansion of $J_{\text{Cake}}$.

We find it convenient to use the recurrence relation with respect to space-time dimension, see Ref. [15]. First, we use the Feynman parameterization to obtain the relation

$$J_{\text{Cake}}^{(D-2)} \overset{\text{def}}{=} J_{111111100}^{(D-2)} = 8J_{1112221200}^{(D)} + 8J_{1112221200}^{(D)} + 8J_{1122211200}^{(D)} + 8J_{1122211200}^{(D)} + 8J_{2222111100}^{(D)}.$$  \quad \quad (27)

Then, using the IBP identities, we express the integrals in the right-hand side of this relation via the five master integrals from Fig. 5. In particular, the last term in Eq. (27) can be expressed via
the master integrals as follows
\[
J_2^{(5+) \text{ def} } = J_{22211100}^{(5+)} = a_{\text{Cake}} f_2^{(5+) \text{ Cake}} + \frac{a_{\text{Clover}}}{\epsilon^2} f_2^{(5+) \text{ Clover}} + a_{\text{Infinity}} f_2^{(5+) \text{ Infinity}} + \frac{a_{\text{Tumbler}}}{\epsilon^2} f_2^{(5+) \text{ Tumbler}} + \frac{a_{\text{Melon}}}{\epsilon^2} f_2^{(5+) \text{ Melon}}.
\] (28)

The coefficients \(a_i\) are presented in the Appendix. They are chosen to be finite in the limit \(\epsilon \to 0\).

After the reduction, we have the following recurrence relation:
\[
J_2^{(3+\epsilon) \text{ Cake} } = \epsilon b_{\text{Cake}} f_2^{(5+) \text{ Cake} } + \frac{b_{\text{Clover}}}{\epsilon^2} f_2^{(5+) \text{ Clover} } + b_{\text{Infinity}} f_2^{(5+) \text{ Infinity} } + \frac{b_{\text{Tumbler}}}{\epsilon} f_2^{(5+) \text{ Tumbler} } + \frac{b_{\text{Melon}}}{\epsilon} f_2^{(5+) \text{ Melon} }.
\] (29)

Again, the coefficients \(b_i\) are chosen to be finite in the limit \(\epsilon \to 0\) and are presented in the Appendix. Now we use the following trick. Let us express \(J_2^{(5+) \text{ Cake} }\) from Eq. (28) and substitute into Eq. (27). We obtain
\[
J_2^{(3+\epsilon) \text{ Cake} } = \epsilon b_{\text{Cake}} f_2^{(5+) \text{ Cake} } + \frac{b_{\text{Clover}}}{\epsilon^2} f_2^{(5+) \text{ Clover} } + b_{\text{Infinity}} f_2^{(5+) \text{ Infinity} } + \frac{b_{\text{Tumbler}}}{\epsilon} f_2^{(5+) \text{ Tumbler} } + \frac{b_{\text{Melon}}}{\epsilon} f_2^{(5+) \text{ Melon} }.
\] (30)

Since the integral \(J_2^{(5)}\) is finite in \(D = 5\) and the coefficient in front of this integral in Eq. (30) contains \(\epsilon\) factor, the first term in the right-hand side of Eq. (30) does not contribute in \(\epsilon^0\) order. Expanding the coefficient \(a_i, b_i\) and the four simple master integrals, we obtain
\[
J_2^{(3+\epsilon) \text{ Cake} } = \frac{\pi^4}{6} + \epsilon \left[ \frac{\pi^4}{3} \left( C - \ln 2 - \frac{11}{8} \right) - \pi^2 - \frac{3}{4} \right] + O(\epsilon^2),
\] (31)
where \(C = 0.577...\) is the Euler constant. Note, that using this trick we have obtained the \(O(\epsilon^0)\) term of \(J_2^{(3+\epsilon) \text{ Cake} }\) “for free”.

In order to calculate the \(O(\epsilon)\) term, let us consider the general solution of the recurrence (27).

Taking into account the explicit form of the coefficient \(b_{\text{Cake}}\), we obtain
\[
J_2^{(D)} = J_0(D) \left[ P(D) + \sum_{i=1}^{4} \sum_{n=1}^{\infty} \frac{\epsilon_i^{(D+2n)}}{J_0(D+2n)} \right],
\] (32)

where
\[
J_0(D) = \frac{\Gamma(1-D/2) \Gamma(3-D/2 - 11/2)}{\Gamma(D-2) \Gamma(D-3)^2},
\] (33)
\[
c_{\text{Clover}} = \frac{b_{\text{Clover}}}{\epsilon^3 b_{\text{Cake}}}, \quad c_{\text{Infinity}} = \frac{b_{\text{Infinity}}}{\epsilon b_{\text{Cake}}},
\]
\[
c_{\text{Tumbler}} = \frac{b_{\text{Tumbler}}}{\epsilon^2 b_{\text{Cake}}}, \quad c_{\text{Melon}} = \frac{b_{\text{Melon}}}{\epsilon^2 b_{\text{Cake}}},
\]
\[
c_{\text{Tumbler}} = \frac{b_{\text{Tumbler}}}{\epsilon^2 b_{\text{Cake}}},
\]
\[
c_{\text{Melon}} = \frac{b_{\text{Melon}}}{\epsilon^2 b_{\text{Cake}}},
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c_{\text{Tumbler}} = \frac{b_{\text{Tumbler}}}{\epsilon^2 b_{\text{Cake}}},
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c_{\text{Tumbler}} = \frac{b_{\text{Tumbler}}}{\epsilon^2 b_{\text{Cake}}},
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c_{\text{Melon}} = \frac{b_{\text{Melon}}}{\epsilon^2 b_{\text{Cake}}},
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\[
c_{\text{Tumbler}} = \frac{b_{\text{Tumbler}}}{\epsilon^2 b_{\text{Cake}}},
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c_{\text{Melon}} = \frac{b_{\text{Melon}}}{\epsilon^2 b_{\text{Cake}}},
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c_{\text{Tumbler}} = \frac{b_{\text{Tumbler}}}{\epsilon^2 b_{\text{Cake}}},
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c_{\text{Melon}} = \frac{b_{\text{Melon}}}{\epsilon^2 b_{\text{Cake}}},
\]
\[
c_{\text{Tumbler}} = \frac{b_{\text{Tumbler}}}{\epsilon^2 b_{\text{Cake}}},
\]
\[
c_{\text{Melon}} = \frac{b_{\text{Melon}}}{\epsilon^2 b_{\text{Cake}}},
\]
and \( P(\mathcal{D}) = P(\mathcal{D} + 2) \) is a periodic function of \( \mathcal{D} \). Note that, using the explicit form of the simple master integrals, Eq. (61), the sums in Eq. (32) can be checked to converge rapidly. In order to fix the function \( P(\mathcal{D}) \), we have to calculate the leading asymptotic of \( J^{(3)}_{\text{Cake}} \) at \( \mathcal{D} \to +\infty \). However, the calculation of this asymptotic is not a simple problem. Instead, we may proceed in the alternative way by applying the method of difference equations described in Ref. [16]. According to this method, we derive the recurrence relation in \( x \) for \( J^{(3)}_{x_{1111111000}} \):

\[
J^{(3)}_{x_{1111111000}} = C(x) J^{(3)}_{x_{1111111000}} + F(x),
\]

\[
C(x) = \frac{x(x + 11 - 3\mathcal{D})}{(x + 5 - 3\mathcal{D}/2)(x + 3 - \mathcal{D})},
\]

where \( F(x) \) is expressed in terms of finite sums of \( \Gamma \)-functions. The solution of this relation is

\[
J^{(3)}_{x_{1111111000}} = G^{(3)}(x) \left[ R^{(3)}(x) + \sum_{y=1}^{\infty} \frac{F(y)}{G(y)} \right],
\]

\[
G^{(3)}(x) = \frac{\Gamma(x + 5 - 3\mathcal{D}/2)\Gamma(x + 3 - \mathcal{D})}{\Gamma(x)\Gamma(x + 11 - 3\mathcal{D})},
\]

where \( R^{(3)}(x) \) is a periodic function of \( x \), which can be determined through the \( J^{(3)}_{x_{1111111000}} \) asymptotic behaviour in \( x \to \infty \) limit. At large \( x \), the \( x \)-dependence of \( J^{(3)}_{x_{1111111000}} \) factorizes into \( \int d^3k_1 \left( k_1^2 + 1 \right)^x \). Using the large-\( x \) behaviour of the function \( G^{(3)}(x) \), we find

\[
\left. \frac{J^{(3)}_{x_{1111111000}}}{G^{(3)}(x)} \right|_{x \to \infty} \sim x^{3-D}.
\]

Therefore, for \( \mathcal{D} > 3 \) we have \( R(x) = 0 \) and we can use Eq. (36) to estimate numerically the function \( P(\mathcal{D}) \) in Eq. (32). It should be noted that the recurrence relation (34) can be hardly applied near \( \mathcal{D} = 3 \) due to the slow convergence of the sum in the right-hand side of Eq. (35). Performing the estimation of \( P(\mathcal{D}) \) for several non-integer values of \( \mathcal{D} \), we find that in all cases the value of the function \( P(\mathcal{D}) \) is compatible with zero up to \( 10^{-10} \). Thus, our ansatz is \( P(\mathcal{D}) = 0 \), and we have

\[
J^{(3+\epsilon)}_{\text{Cake}} = \frac{8^{\epsilon+3}\Gamma\left(-\frac{1}{2} - \frac{\epsilon}{2}\right)\Gamma\left(\frac{3}{2} - 1\right)}{\epsilon\Gamma(\epsilon)^3} \sum_{n=0}^{\infty} T(2n + \epsilon),
\]

\[
T(\nu) = \frac{8^{-\nu-5}(\nu^3 + 40\nu^2 + 35\nu + 10)\Gamma\left(-\frac{\nu}{2} - 1\right)^2 \Gamma(\nu)^3}{\Gamma\left(\frac{\nu}{2} + 2\right)}
\]

\[
- \pi^{7/2} \left( 6417\nu^5 + 13266\nu^4 + 83689\nu^3 + 980\nu^2 - 585\nu - 126 \right)\Gamma(\nu)
\]

\[
+ 6144\nu(3\nu - 1)(3\nu + 1)\Gamma(-2\nu)\Gamma(3\nu + 4)\cos\left(\frac{\nu\pi}{2}\right)\sin^2(\nu\pi)\cos\left(\frac{3\nu\pi}{2}\right)
\]

\[
+ \pi^{5/2} \left( 585\nu^3 + 372\nu^2 + 7\nu - 12 \right)\Gamma(\nu)\Gamma\left(\frac{3\nu}{2}\right)^3 \tan\left(\frac{3\nu\pi}{2}\right)
\]

\[
+ \frac{12288(3\nu - 1)(3\nu + 1)\Gamma(2\nu + 2)\cos\left(\frac{\nu\pi}{2}\right)\sin(2\nu\pi)}{\Gamma\left(\frac{\nu}{2} + 2\right) \sin(2\nu\pi)}
\]

\[
\frac{\pi^{3/2} \left( 3897\nu^4 + 3870\nu^3 + 797\nu^2 - 198\nu - 54 \right)\Gamma(\nu)\Gamma\left(\frac{3\nu}{2}\right)^3 \tan\left(\frac{3\nu\pi}{2}\right)}{36864(3\nu - 1)(3\nu + 1)\Gamma(2\nu + 2)\Gamma\left(\frac{3\nu}{2} + \frac{1}{2}\right) \sin(2\nu\pi) \sin\left(\frac{\nu\pi}{2}\right)}.
\]
The series in this representation are converging rapidly. The first term in $T(\nu)$, the most slowly decreasing one, behaves as $3^{-3n}$. Thus, roughly speaking, each two consecutive terms in the sum give three more decimal digits of precision. We claim this expression to be valid for arbitrary $\epsilon$. For example, one can easily reproduce terms of the expansion of $J_{\text{Cake}}$ near $D = 4$ obtained in [17].

Using Eqs. (28), (39), we obtain

$$J^{(5)}_2 = -\frac{1}{81} \pi^2 \left( 63\pi^2 - 488 - 72\zeta_3 \right) + \sum_{n=1}^{\infty} T_2(n),$$

$$T_2(n) = \frac{(1-n)^2 \pi^2 (56n^3 + 80n^2 + 35n + 5) \pi^2 ((n-1)!)^3}{18(2n+1)^2(3n+1)!} - \frac{(1-n)^2 \pi^2 ((2n)!)^3}{9n (36n^2 - 1)(n!)^2(4n+1)!} - \frac{16\pi^2 (2n-1)!(4n-1)!}{27 (36n^2 - 1)^2 (6n + 3)!} \times \left( 11088576n^6 + 7641216n^5 + 691632n^4 - 424512n^3 - 79356n^2 + 616n + 585 \right) + \frac{2(1-n)^2 \pi^2 ((2n)!)^3}{27n^2 (36n^2 - 1)^2 (n!)^2(4n+2)(6n+1)!} \times \left( 4489344n^6 + 2794176n^5 + 206352n^4 - 146880n^3 - 26032n^2 - 28n + 153 \right) - \frac{4\pi^2 (2n)!(4n)! (H_{2n-1} + 2H_{4n-1} - 3H_{6n-1})}{27n^2 (36n^2 - 1)(6n + 3)!} \times \left( 102672n^5 + 106128n^4 + 33472n^3 + 1960n^2 - 585n - 63 \right) - \frac{(1-n)^2 \pi^2 ((2n)!)^3}{54n^2 (36n^2 - 1)(n!)^3(4n+1)!(6n+1)!} \times \left( 31176n^4 + 15480n^3 + 1594n^2 - 198n - 27 \right),$$

$$J^{(5)}_2 = 0.0516516357945 \ldots$$

Here $H_n = \sum_{k=1}^{n} k^{-1}$ is a harmonic number. Unfortunately, we have not been able to express the sums in Eq. (41) in terms of $\zeta$-functions and alike. However, the numerical convergence of the above series is perfect and we obtain from Eqs. (31), (41)

$$J^{(3+\epsilon)}_{\text{Cake}} = \frac{\pi^4}{6} - \epsilon \times 58.3184377060 \ldots + O(e^2).$$

Methods of calculation and values of multiloop vacuum integrals in arbitrary space-time dimension $D$ are of independent interest, because these integrals appear as parts of the amplitudes for various physical processes: from QCD and QED radiative corrections [18] to the thermodynamics of finite temperature QCD-like theories [19]. In particular, the master integrals in Fig 5 enter the basis intensively used in modern QCD calculations [20, 21, 22, 23].

10
4. Results and Conclusion

The perturbative expansion of the form factors in Eq. (6) has the form

\[ f_i = \sum_{n=2,4,...} \alpha (Z\alpha)^n f_i^{(n)}. \]  

(44)

We have calculated the low-energy expansion of \( f_i^{(2)} \) and \( f_i^{(4)} \) up to \( O(\lambda^2) \), which corresponds to the expansion of the polarization operator up to \( O(\lambda^4) \). Using Eq. (18), we obtain in \((Z\alpha)^2\) order

\[ f_1^{(2)} = \frac{7}{2(4\pi)^2} + \frac{1}{6!} \frac{|k - q|}{m} + \frac{1}{2^2(5!)^2m^2} \left[ \frac{117}{4} \omega^2 + 49 k \cdot q - \frac{57}{2} \left( k^2 + q^2 \right) \right]. \]

(45)

\[ f_2^{(2)} = \frac{73}{2^2(4\pi)^2} - \frac{11}{6!} \frac{|k - q|}{m} \]
\[ + \frac{1}{2^3(5!)^2m^2} \left[ \frac{10923}{2} \omega^2 + 3095 k \cdot q + 1111 \left( k^2 + q^2 \right) \right]. \]

(46)

\[ f_3^{(2)} = \frac{7}{6!} \frac{|k - q|}{m} + \frac{587}{2^3(5!)^2} m^2. \]

(47)

\[ f_4^{(2)} = \frac{573}{2^2(5!)^2} m^2. \]

(48)

\[ f_5^{(2)} = \frac{4}{6!} \frac{|k - q|}{m} + \frac{1369}{2^3(5!)^2} \left( k - q \right)^2. \]

(49)

The \( O(\lambda^0) \) terms of \( f_{1,2}^{(2)} \) have been obtained in Ref. [2]. When \( k = \omega = 0 \), the expansion of \( f_1^{(2)}(q) \) agrees with the exact result obtained in Ref. [24]. The next-to-leading terms, proportional to \( |k - q| \), come from the soft region and exhibit the above-mentioned nonanalytic behavior.

In \((Z\alpha)^4\) order we obtain

\[ f_1^{(4)} = \frac{1}{27} \left\{ \frac{2}{\pi^2} J_2^{(5)} - \frac{4}{3} \zeta_3 + \frac{5}{9} \zeta_2 + \frac{13}{18} \right\} \]
\[ + \left( \frac{1}{2^{105} m^2} \right) \left[ \left( -\frac{109}{2\pi^2} J_2^{(5)} + \frac{475}{3} \zeta_3 - \frac{4111}{30} \zeta_2 + \frac{2575}{72} \right) \omega^2 \right] \]
\[ + \left( k^2 + q^2 \right) \left[ \frac{6}{\pi^2} J_2^{(5)} - \frac{25}{18} \zeta_2 \zeta_3 + \frac{29}{8} \right] + k \cdot q \left( \frac{51}{\pi^2} J_2^{(5)} + \frac{481}{30} \zeta_2 - \frac{3053}{108} \right). \]

(50)

\[ f_2^{(4)} = \frac{1}{27} \left( -\frac{10}{3} \zeta_3 + \frac{1267}{72} \zeta_2 - \frac{3661}{144} \right) \]
\[ + \left( \frac{1}{2^{105} m^2} \right) \left[ \left( \frac{473}{4\pi^2} J_2^{(5)} - \frac{145}{2} \zeta_3 + \frac{279829}{720} \zeta_2 - \frac{1233655}{2592} \right) \omega^2 \right] \]
\[ + \left( k^2 + q^2 \right) \left[ \frac{95}{2} \zeta_3 - \frac{359}{360} \zeta_2 + \frac{18769}{288} \right] + k \cdot q \left( -\frac{235}{2\pi^2} J_2^{(5)} - \frac{32999}{180} \zeta_2 + \frac{610}{3} \zeta_3 + \frac{7571}{144} \right). \]

(51)
Figure 6: The "magnetic loop" contribution to the bound electron $g$ factor.

\[ f^{(4)}_3 = \frac{1}{2^{10} 5^3} \frac{(k - q)^2}{m^2} \left( \frac{87}{2\pi^2} f^{(5)}_2 - \frac{12641}{180} \xi_2 + \frac{24551}{216} \right), \]

\[ f^{(4)}_4 = \frac{1}{2^{10} 5^3} \frac{\omega^2}{m^2} \left( \frac{-71}{2\pi^2} f^{(5)}_2 + \frac{25}{3} \zeta_3 + \frac{2857}{60} \xi_2 - \frac{111685}{1296} \right), \]

\[ f^{(4)}_5 = \frac{1}{2^{10} 5^3} \frac{(k - q)^2}{m^2} \left( \frac{31}{2\pi^2} f^{(5)}_2 - 30\zeta_3 + \frac{11639}{180} \xi_2 - \frac{31319}{432} \right). \]

In $(Z\alpha)^4$ order, the nonanalytic contribution to the form factors is suppressed as $O(\lambda^5)$ and thus is far beyond the accuracy chosen. Note, that the technique used in this paper can be applied without modification to the calculation of the higher terms of the low-energy expansion in $(Z\alpha)^2$ and $(Z\alpha)^4$ orders.

The Coulomb corrections to the form factors $f_1$ and $f_2$ in order $\lambda^0$ were calculated in Ref. [9] numerically. Although the interaction with the Coulomb field was taken into account exactly, it turned out that the results can be well fitted by the polynomial function of $Z\alpha$:

\[ f_1 = \frac{7}{2(4!)} (Z\alpha)^2 + 3.35 \cdot 10^{-4} (Z\alpha)^4 + 1.6 \cdot 10^{-4} (Z\alpha)^6, \]

\[ f_2 = -\frac{73}{2^5(4!)} (Z\alpha)^2 - 3.55 \cdot 10^{-3} (Z\alpha)^4 - 2.1 \cdot 10^{-3} (Z\alpha)^6. \]

In $O(\lambda^1)$ order, our results (50), (51) for the $(Z\alpha)^4$-order corrections numerically coincide with those of Eqs. (55), (56) with an accuracy of a few percent.

As the demonstration of possible applications of our result, let us calculate the $O((Z\alpha)^7)$ contribution of the "magnetic loop" to the $g$ factor of the bound electron, see Fig. 6. The corresponding correction to the $g$ factor of the electron in $nL_J$ state has the form [25]
where \( a \) and \( b \) are determined by the form of the bound electron wave function

\[
\psi(r) = \left( \frac{a(r) \Omega_{JLM}(n)}{ib(r) \Omega_{JLM}(n)} \right).
\]

The characteristic scale of the function \( G(q) \) is \( mZ\alpha/n \ll m \), so we have two regions of integration: \( q \sim mZ\alpha/n \) and \( q \sim m \). In the leading order, only the first region is essential, the contribution of this region is of the order \( O((Z\alpha)^5) \). The leading correction \( \sim O((Z\alpha)^0) \) for \( L \neq 0 \) states also comes from the region \( q \sim mZ\alpha/n \), while for \( L = 0 \) the whole interval \( mZ\alpha/n \lesssim q \lesssim m \) is essential. This correction has been found in Ref. [25]. Note that the integrals in Eq. (18) of Ref. [25] can be taken analytically, and the correction to \( g \) factor up to the order \( O((Z\alpha)^0) \) can be represented as

\[
\left( \frac{\Delta g}{g_0} \right)_{(Z\alpha)^5+(Z\alpha)^0} = \frac{7\alpha (Z\alpha)^5}{288n^3 J(J+1)(2J+1)}
\]

\[
+ \delta_{L=0} \left( \frac{4\alpha (Z\alpha)^6}{135n} \right) \left[ \log \frac{n}{2Z\alpha} - \frac{641}{240} - \frac{(n+1)(4n-1)}{6n^2} \right]
\]

\[
+ \delta_{L=0} \left( \frac{2\alpha (Z\alpha)^6}{45\pi n^3(2L+1)(2k-1)^2} \right) \left( \frac{3}{L(L+1)} - \frac{1}{n^2} \right). \tag{57}
\]

In order to find the next-to-leading correction, we separate the contributions of the two regions similar to what has been described above. The details of this calculation will be presented elsewhere. It turns out that the complete \( O((Z\alpha)^7) \) result for the correction to \( g \) factor can be expressed via several first terms of expansion of the function \( f_1 \) near \( q = 0 \), namely

\[
\left( \frac{\Delta g}{g_0} \right)_{(Z\alpha)^7} = \frac{4\alpha (Z\alpha)^7}{n^3 J(J+1)(2J+1)} \times
\]

\[
\left[ \log \frac{n}{2Z\alpha} - \frac{641}{240} - \frac{(n+1)(4n-1)}{6n^2} \right] f_1^{(2)}(0)
\]

\[- \delta_{L=0} \frac{8\alpha (Z\alpha)^7}{3n^5} \left( 1 + 5n^2 \right) \left( 1 + 5n^2 \right) f_1^{(2)''}(0) + \frac{4\alpha (Z\alpha)^7}{n^3 J(J+1)(2J+1)} f_1^{(4)}(0). \tag{58}
\]

Now, owing to Eq. (59), we have the last essential ingredient to obtain the correction. Using Eqs. (45), (50), we obtain

\[
\left( \frac{\Delta g}{g_0} \right)_{(Z\alpha)^7} = \frac{7\alpha (Z\alpha)^7}{288n^3 J(J+1)(2J+1)} \times
\]

\[
\left[ \log \frac{n}{2Z\alpha} - \frac{641}{240} - \frac{(n+1)(4n-1)}{6n^2} \right] + \delta_{L=0} \frac{19\alpha (Z\alpha)^7}{7200n^5} \left( 1 + 5n^2 \right)
\]

\[
+ \frac{\alpha (Z\alpha)^7}{32n^3 J(J+1)(2J+1)} \left( \frac{2}{\pi^2} f_2^{(5)} - \frac{4}{3} \zeta_3 + \frac{5}{9} \zeta_2 + \frac{13}{18} \right). \tag{59}
\]
In particular, for $1S_{1/2}$ and $2P_{1/2}$ states we have

$$
\begin{align*}
\left( \frac{\Delta g}{g_0} \right)_{1S} & = 1.62 \times 10^{-2} \alpha (Z \alpha)^5 + 9.431 \times 10^{-6} \alpha (Z \alpha)^6 \left( \ln \frac{1}{2Z\alpha} - 2.67 \right) \\
& + 4.1 \times 10^{-2} \alpha (Z \alpha)^7, \\
8 \left( \frac{\Delta g}{g_0} \right)_{2P} & = 1.62 \times 10^{-2} \alpha (Z \alpha)^5 + 5.8946 \times 10^{-3} \alpha (Z \alpha)^6 \\
& + 3.26 \times 10^{-2} \alpha (Z \alpha)^7.
\end{align*}
$$

The contribution of the $O((Z \alpha)^7)$ term is rather essential, e.g., for $Z = 6$ (carbon) the ratio of this term to $O((Z \alpha)^6)$ term for $1S_{1/2}$ state is −0.81. The last term in Eq. (59) corresponds to the contribution of the electron loop with four Coulomb exchanges. It is interesting to compare the magnitude of this term with that of the first two terms. As it was claimed in Ref. [25] this term appears to be numerically small. E.g., for the ground state, the contribution of the last term is only 2.2 percent.

### Appendix

The explicit form of the four simple master integrals from Fig. [5] reads:

$$
\begin{align*}
\alpha_{\text{Cak}} & = \frac{e^3}{48(3 \epsilon + 2)} (9 e^2 + 3 \epsilon - 4), \\
\alpha_{\text{Cak}} & = \frac{e^3}{384 \epsilon^2} (9 e^2 + 3 \epsilon - 4), \\
\alpha_{\text{Inf}} & = - \frac{e^3}{16} \left( \frac{3 e^2}{8} - \frac{51 e}{16} - \frac{5}{3} (\epsilon + 1) \right) (2 \epsilon + 2)^3, \\
\alpha_{\text{Tumb}} & = - \frac{e^3}{16} \left( \frac{73 e^5}{6} + 16 e^4 + \frac{967 e^3}{54} + \frac{397 e^2}{27} + \frac{17 e}{3} + \frac{2}{3} \right) (\epsilon + 1) (3 \epsilon / 2 + 3 / 2) \epsilon (2 \epsilon + 2) (3 \epsilon) \epsilon, \\
\alpha_{\text{Melon}} & = \frac{e^3}{16} \left( \frac{6 e^5}{4} - \frac{117 e^3}{4} - \frac{51 e^2}{8} - \frac{3}{4} (3 \epsilon / 2 + 5 / 2) (2 \epsilon + 2) \epsilon (\epsilon + 2) (3 \epsilon) \right).
\end{align*}
$$
Here \( x_n = \Gamma (x+n) / \Gamma (x) = x(x+1) \ldots (x+n-1) \). The coefficients in Eq. (27) are

\[
\begin{align*}
\epsilon b_{\text{Cake}} &= -\frac{\epsilon (\epsilon + 1)(\epsilon + 2)(\epsilon + 3)}{48(3\epsilon - 2)(3\epsilon + 2)}, \\
\epsilon^2 b_{\text{Clover}} &= \frac{(\epsilon + 1)(\epsilon + 3)^4 \left(14\epsilon^3 + 40\epsilon^2 + 35\epsilon + 10\right)}{384\epsilon^4(3\epsilon - 2)(3\epsilon + 2)}, \\
\epsilon^{-1} b_{\text{Infinity}} &= -\left(3 - \frac{7\epsilon}{4} - 93\epsilon^2 - \frac{585\epsilon^3}{4}\right) \frac{(\epsilon + 2)\epsilon_4(2\epsilon + 2)}{4(3\epsilon - 2)^2}, \\
\epsilon^{-1} b_{\text{Tumbler}} &= \left(7 + \frac{65\epsilon}{2} - \frac{490\epsilon^2}{9} - \frac{4184\epsilon^3}{9} - 737\epsilon^4 - \frac{713\epsilon^5}{2}\right) \frac{16(\epsilon + 1/2)_2(3\epsilon - 2)^5}{(\epsilon + 3/2 + 5/2)_1}, \\
\epsilon^{-1} b_{\text{Melon}} &= \left(3 + 11\epsilon - \frac{797\epsilon^2}{18} - 215\epsilon^3 - \frac{433\epsilon^4}{2}\right) \frac{3(3\epsilon/2 + 5/2)_2(2\epsilon + 2)_5}{16(3\epsilon - 2)^7}.
\end{align*}
\]

(63)

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