Incoherent Quantum Control

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Abstract

Conventional approaches for controlling open quantum systems use coherent control which affects the system’s evolution through the Hamiltonian part of the dynamics. Such control, although being extremely efficient for a large variety of problems, has limited capabilities, e.g., if the initial and desired target states have density matrices with different spectra or if a control field needs to be designed to optimally transfer different initial states to the same target state. Recent research works suggest extending coherent control by including active manipulation of the non-unitary (i.e., incoherent) part of the evolution. This paper summarizes recent results specifically for incoherent control by the environment (e.g., incoherent radiation or a gaseous medium) with a kinematic description of controllability and landscape analysis.

1 Introduction

The manipulation of atomic or molecular quantum dynamics commonly uses coherent quantum control, which may be extremely useful for a large variety of problems [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. The dynamical evolution of a closed quantum system under the action of a collection of coherent controls \( u = \{u_l(t)\} \) (e.g., Rabi frequencies of the applied laser field) is described by the equation

\[
\frac{d\rho_t}{dt} = -i \left[ H_0 + \sum_l Q_l u_l(t), \rho_t \right], \quad \rho|_{t=0} = \rho_0
\]

Here \( \rho_t \) is the system density matrix at time \( t \) (for an \( n \)-level quantum system, the set of all density matrices is \( D_n = \{\rho \in M_n | \rho \geq 0, \text{Tr} \rho = 1\} \), where \( M_n = \mathbb{C}^{n \times n} \) is the set of \( n \times n \) complex matrices), \( H_0 \) is the free system Hamiltonian describing evolution of the system in the absence of control fields and each \( Q_l \) is an operator describing the coupling of the system to the control field \( u_l(t) \).

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Coherent control of a closed system induces a unitary transformation of the system density matrix \( \rho_t = U_t \rho_0 U_t^\dagger \) and may have some limitations. The first limitation is due to the fact that unitary transformations of an operator preserve its spectrum; thus the spectrum of \( \rho_t \) is the same at any \( t \) and, for example, a mixed state \( \rho_0 \) will always remain mixed [13]. A second limitation is that a control \( u_{\text{opt}} \) which is optimal for some initial state \( \rho_0 \) may be not optimal for another initial state \( \tilde{\rho}_0 \) even if \( \rho_0 \) and \( \tilde{\rho}_0 \) have the same spectrum. This limitation originates from the reversibility of unitary evolution and is due to the fact that \( U_t \rho_0 U_t^\dagger \neq U_t \tilde{\rho}_0 U_t^\dagger \) if \( \rho_0 \neq \tilde{\rho}_0 \). To overcome these limitations at least to some degree, control by measurements [14, 15, 16, 17, 18] or incoherent control [19, 20, 21, 22] may be used, and in this work incoherent control by the environment [19] (ICE) is discussed. Some general mathematical notions for the controlled quantum Markov dynamics are formulated in Ref. [23].

The necessity to consider incoherent control relies also on the fact that coherent control of quantum systems (e.g., of chemical reactions) in the laboratory is often realized in a medium (solvent) which interacts with the controlled system and plays the role of the environment. Furthermore, then environment may be also affected to some degree by the coherent laser field, thus effectively realizing incoherent control of the system. Moreover, laser sources of coherent radiation at the present time have practical limitations, and some frequencies are very expensive to generate compared to the respective sources of incoherent control (e.g., incoherent radiation as considered in Sec. 2.1 of this work). Thus the latter incoherent control can be used in some cases to reduce the total cost of quantum control.

This paper summarizes recent results specifically for incoherent control by the environment [19] (ICE). A general theoretical formulation for incoherent control is provided in Sec. 2, followed by the examples of control by incoherent radiation (Sec. 2.1) and control through collisions with particles of a medium (e.g., solvent, gas, etc., Sec. 2.2). Relevant known results about controllability and the structure of control landscapes for open quantum systems in the kinematic picture are briefly outlined in Sec. 3.

2 Incoherent control by the environment

The dynamical evolution of an open quantum system under the action of coherent controls in the Markovian regime is described by a master equation

\[
\frac{d\rho_t}{dt} = -i \left[ H_0 + H_{\text{eff}} + \sum_i Q_i u_i(t), \rho_t \right] + \mathcal{L}\rho_t
\]

(2)

The interaction with the environment modifies the Hamiltonian part of the dynamics by adding an effective Hamiltonian term \( H_{\text{eff}} \) to the free Hamiltonian \( H_0 \). Another important effect of the environment is the appearance of the term \( \mathcal{L} \) which describes non-unitary aspects of the evolution and is responsible for decoherence. This term in the Markovian regime has the general Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) form [24, 25]

\[
\mathcal{L}\rho = \sum_i \left( 2L_i \rho L_i^\dagger - L_i^\dagger L_i \rho - \rho L_i^\dagger L_i \right)
\]
where $L_i$ are some operators acting in the system Hilbert space. The explicit form of the GKSL term depends on the particular type of the environment, on the details of the microscopic interaction between the system and the environment, and on the state of the environment.

The coherent portion of the control in (2) addresses only the Hamiltonian part of the evolution while the GKSL part $L$ remains fixed (for the analysis of controllability properties for Markovian master equations under coherent controls see for example, Ref. [26]). However, the generator $L$ can also be controlled to some degree. For a fixed system-environmental interaction, the generator $L$ depends on the state of the environment, which can be either a thermal state at some temperature (including the zero temperature vacuum state) or an arbitrary non-equilibrium state. Such a state is characterized by a (possibly, time dependent) distribution of particles of the environment over their degrees of freedom, which are typically the momentum $k \in \mathbb{R}^3$ and the internal energy levels parameterized by some discrete index $\alpha \in A$ (e.g., for photons $\alpha = 1,2$ denotes polarization, for a gas of $N$-level particles $\alpha = 1, \ldots, N$ denotes the internal energy levels). Denoting the density at time $t$ of the environmental particles with momentum $k$ and occupying an internal level $\alpha$ by $n_{k,\alpha}(t)$, and the corresponding GKSL generator as $L = L[n_{k,\alpha}(t)]$, the equation (2) becomes

$$\frac{d\rho_t}{dt} = -i\left[H_0 + H_{\text{eff}} + \sum_i Q_i u_i(t), \rho_t \right] + L[n_{k,\alpha}(t)]\rho_t$$

(3)

Here both $u_i(t)$ and $n_{k,\alpha}(t)$ are used as the controls, and for $n_{k,\alpha}(t)$ the optimization is done over $k, \alpha$ in a time dependent fashion to obtain a desired outcome.

The solution of (3) with the initial condition $\rho|_{t=0} = \rho_0$ for each choice of controls $\{u_i(t)\}$ and $n_{k,\alpha}(t)$ can be represented by a family $P_t(\{u_i\}, n_{k,\alpha})$, $t \geq 0$ of completely positive (CP), trace preserving maps (see Sec. 3 for the explicit definitions) as

$$\rho_t = P_t(\{u_i\}, n_{k,\alpha})\rho_0$$

(4)

In general, for time dependent controls this family forms not a semigroup but a self-consistent two-parameter family of CP, trace preserving maps $P_{t,\tau}(\{u_i\}, n_{k,\alpha})$, $t \geq \tau \geq 0$, where each $P_{t,\tau}(\{u_i\}, n_{k,\alpha})$ represents the evolution from $\tau$ to $t$.

The target functional, also called the performance index, describes a property of the controlled system which should be minimized during the control and commonly consists of the two terms:

$$J(\{u_i\}, n_{k,\alpha}) = J_1(\{u_i\}, n_{k,\alpha}) + J_2(\{u_i\}, n_{k,\alpha})$$

The term $J_1(\{u_i\}, n_{k,\alpha})$, called the objective functional, represents the physical system’s property which we want to minimize. The term $J_2(\{u_i\}, n_{k,\alpha})$, called the cost functional, represents the penalty for the control fields.

The first general class of objective functionals appears in the problem of minimizing the expectation value of some observable associated to the system at a target time $T > 0$. The system is assumed at the initial time $t = 0$ to be in the state $\rho_0$. Any observable characterizing the system (e.g., its energy, population of some level, etc.) is represented by some self-adjoint operator $O$ acting in the system Hilbert space, and the corresponding objective functional has the form

$$J_1(\{u_i\}, n_{k,\alpha}) = \text{Tr} [\rho_T(\{u_i\}, n_{k,\alpha})O] \equiv \text{Tr} [(P_T(\{u_i\}, n_{k,\alpha})\rho_0)O]$$

(5)
Here \( \rho_T(u_t, n_{k,\alpha}) \equiv P_T\{(u_t), n_{k,\alpha}\}\rho_0 \) is the final density matrix of the system evolving from the initial state \( \rho_0 \) under the action of \( u_t \) and \( n_{k,\alpha} \). The physical meaning of this objective functional is that it represents the average measured value of the observable \( O \) at the final time \( T \) when the system evolves from the initial state \( \rho_0 \) under the action of the controls \( (u_t), n_{k,\alpha} \).

The second general class of objective functionals appears in the problem of optimal state-to-state transfer. Suppose that initially the system is in a state \( \rho_0 \) and the control goal is to steer the system at some target time \( T \) into some desired target state \( \rho_{\text{target}} \). In this case one seeks controls \( (u_t) \) and \( n_{k,\alpha} \) which minimize the distance between the states \( \rho_T(u_t, n_{k,\alpha}) \) and \( \rho_{\text{target}} \). The corresponding objective functional has the form

\[
J_1[(u_t), n_{k,\alpha}] = \|\rho_T(u_t, n_{k,\alpha}) - \rho_{\text{target}}\| \equiv \|P_T\{(u_t), n_{k,\alpha}\}\rho_0 - \rho_{\text{target}}\| 
\]

(6)

where \( \| \cdot \| \) is a suitable matrix norm. Usually the Hilbert-Schmidt norm \( \|A\| = \sqrt{\text{Tr} A^\dagger A} \) can be used.

The third important class of objective functionals appears in the problem of producing a desired target CP, trace preserving map \( P_{\text{target}} \). In this case the objective functional has the form

\[
J_1[(u_t), n_{k,\alpha}] = \|P_T\{(u_t), n_{k,\alpha}\} - P_{\text{target}}\| 
\]

(7)

where \( \| \cdot \| \) is a suitable norm in the space of all CP, trace preserving maps. In particular, in conventional models of quantum computation the target transformation \( P_{\text{target}} \) is a unitary gate (e.g., a phase \( U_\phi \) or Hadamard \( U_H \) gate, and for these examples \( P_{\text{target}} = U_\phi \) or \( P_{\text{target}} = U_H \), respectively) [27, 28]. More general non-unitary target transformations can arise [e.g., in quantum computing with mixed states [29] or for generating controls robust to variations of the initial system’s state [30] (see also Sec. 3.1)].

The cost functional \( J_2 \) can be chosen to have the form

\[
J_2[(u_t), n_{k,i}] = \sum_l \int_0^T dt \alpha_l(t)|u_t(t)|^2 + \max_{0 \leq t \leq T} \sum_i \int d\mathbf{k} \beta_i(\mathbf{k})n_{k,i}(t) 
\]

Here each function \( \alpha_l(t) \geq 0 \) [resp., \( \beta_i(\mathbf{k}) \geq 0 \)] is a weight describing the cost for the control \( u_t \) at time \( t \) (resp., for the density of particles of the environment with momentum \( \mathbf{k} \) and occupying the internal level \( i \)). The first term minimizes the energy of the optimal coherent control. The second term minimizes the total density of the environment.

The control functions belong to some sets of admissible controls \( (u_t) \in \mathcal{E} \) and \( n_{k,\alpha} \in \mathcal{D} \). The following three important problems arise.

Optimal controls. Find, for a given initial state \( \rho_0 \) and a target time \( T \), some (or all) controls \( (u_t) \in \mathcal{E} \) and \( n_{k,\alpha} \in \mathcal{D} \) which minimize the performance index.

Reachable sets. Find, for a given final time \( T > 0 \) and an initial state \( \rho_0 \), the set of all states reachable from \( \rho_0 \) up to the time \( T \), i.e., the set

\( \mathcal{R}_T(\rho_0) = \{P_T\{(u_t), n_{k,\alpha}\}\rho_0 | t \leq T, (u_t) \in \mathcal{E}, n_{k,\alpha} \in \mathcal{D}\} \)

Landscape analysis. Find, for a given \( T > 0 \), an initial state \( \rho_0 \) and a self-adjoint operator \( O \), all extrema (global and local, and saddles, if any) of the objective functional \( J_1[u_t(t), n_{k,\alpha}(t)] = \text{Tr}\{P_T\{(u_t), n\}\rho_0\}O \) defined by (5) [and similarly for the objective functionals defined by (6) and (7)] or of the corresponding performance index.
2.1 Incoherent control by radiation

Non-equilibrium radiation is characterized by its distribution in photon momenta and polarization. For control with distribution of incoherent radiation the magnitude of the photon momentum \( |k| \) can be exploited along with the polarization and the propagation direction in cases where polarization dependence or spatial anisotropy is important (e.g., for controlling a system consisting of oriented molecules bound to a surface).

A thermal equilibrium distribution for photons at temperature \( T \) is characterized by Planck’s distribution

\[
n_k = \frac{1}{\exp\left(\frac{c|k|}{k_B T}\right) - 1}
\]

where \( c \) is the speed of light, \( \hbar \) and \( k_B \) are the Planck and the Boltzmann constants which we set to one below. Non-equilibrium incoherent radiation may have a distribution given as an arbitrary non-negative function \( n_k(t) \). Some practical means to produce non-equilibrium distributions in the laboratory may be based either on filtering thermal radiation or on the use of independent monochromatic sources.

The master equation for an atom or a molecule interacting with a coherent electromagnetic field \( E_c(t) \) and with incoherent radiation with a distribution \( n_k(t) \) in the Markovian regime has the form:

\[
\frac{d\rho}{dt} = -i[H_0 + H_{\text{eff}} - \mu E_c(t), \rho] + L_{\text{Rad}}[n_k(t)]\rho_t \tag{8}
\]

The coherent part of the dynamics is generated by the free system’s Hamiltonian \( H_0 = \sum_n \varepsilon_n P_n \) with eigenvalues \( \varepsilon_n \), forming the spectrum \( \text{spec}(H_0) \), and the corresponding projectors \( P_n \), the effective Hamiltonian \( H_{\text{eff}} \) resulting from the interaction between the system and the incoherent radiation, dipole moment \( \mu \), and electromagnetic field \( E_c(t) \).

The GKSL generator \( L = L_{\text{Rad}} \) induced by the incoherent radiation with distribution function \( n_k(t) \) has the form (e.g., see Ref. [31])

\[
L_{\text{Rad}}[n_k(t)]\rho = \sum_{\omega \in \Omega} \left[ \gamma^{+}_{\omega}(t) + \gamma^{-}_{\omega}(t) \right] \left( 2\mu_{\omega} \rho \mu_{\omega}^\dagger - \mu_{\omega}^\dagger \mu_{\omega} \rho - \rho \mu_{\omega}^\dagger \mu_{\omega} \right) \tag{9}
\]

Here the sum is taken over the set of all system transition frequencies \( \Omega = \{ \varepsilon_n - \varepsilon_m | \varepsilon_n, \varepsilon_m \in \text{spec}H_0 \} \), \( \mu_{\omega} = \sum_{\varepsilon_n - \varepsilon_m = \omega} P_m \mu P_n \), and the coefficients

\[
\gamma^{\pm}_{\omega}(t) = \pi \int d|k| \delta(|k| - \omega) |g_k|^2 [n_k(t) + (1 \pm 1)/2]
\]

determine the transition rates between energy levels with transition frequency \( \omega \). The transition rates depend on the photon density \( n_k(t) \). The form-factor \( g_k \) determines the coupling of the system to the \( k \)-th mode of the radiation. Equation (8) together with the explicit structure (9) of the GKSL generator provides the theoretical formulation for analysis of control by incoherent radiation.

The numerical simulations illustrating the capabilities of learning control by incoherent radiation to prepare prespecified mixed states from a pure state is available [19] along with a
theoretical analysis of the set of stationary states for the generator $L_{\text{Rad}}$ for some models [21]. Incoherent control by radiation can extend the capabilities of coherent control by exciting transitions between the system’s energy levels for which laser sources are either unavailable at the present time or very expensive compared with the corresponding sources of incoherent radiation. Ref. [22] provides a simple experimental realization of the combined coherent (by a laser) and incoherent (by incoherent radiation emitted by a gas-discharge lamp) control of certain excitations in Kr atoms.

2.2 Incoherent control by a gaseous medium

This section considers incoherent control of quantum systems through collisions with particles of a surrounding medium (e.g., a gas or solvent of electrons, atoms or molecules, etc.). This case also includes coherent control of chemical reactions in solvents if the coherent field addresses not only the controlled system but the solvent as well. The particles of the medium in this treatment serve as the control and the explicit characteristic of the medium exploited to minimize the performance index is in general a time dependent distribution of the medium particles over their momenta $k$ and internal energy levels $\alpha \in A$. This distribution is formally described by a non-negative function $n : \mathbb{R}^3 \times A \times \mathbb{R} \to \mathbb{R}_+$, whose value $n_{k,\alpha}(t)$ (where $k \in \mathbb{R}^3$, $\alpha \in A$, and $t \in \mathbb{R}_+$) has the physical meaning of the density at time $t$ of particles of the surrounding medium with momentum $k$ and in internal energy level $\alpha$. In this scheme one prepares a suitable, in general non-equilibrium, distribution of the particles in the medium such that the medium drives the system evolution through collisions in a desired way.

It may be difficult to practically create a desired non-equilibrium distribution of medium particles over their momenta. In contrast, a non-equilibrium distribution in the internal energy levels can be relatively easily created, e.g., by lasers capable of exciting the internal levels of the medium particles or through an electric discharge. Then the medium particles can affect the controlled system through collisions and this influence will typically depend on their distribution. A well known example of such control is the preparation of population inversion in a He–Ne gas-discharge laser. In this system an electric discharge passes through the He–Ne gas and brings the He atoms into a non-equilibrium state of their internal degrees of freedom. Then He–Ne collisions transfer the energy of the non-equilibrium state of the He atoms into the high energy levels of the Ne atoms. This process creates a population inversion in the Ne atoms and subsequent lasing. A steady electric discharge can be used to keep the gas of helium atoms in a non-equilibrium state to produce a CW He–Ne laser. This process can serve as an example of incoherent control through collisions by considering the gas of He atoms as the control environment (medium) and the Ne atoms as the system which we want to steer to a desired (excited) state.

Quantum systems controlled through collisions with gas or medium particles in certain regimes can be described by master equations with GKSL generators whose explicit structure is different from the generator $L_{\text{Rad}}$ describing control by incoherent radiation. If the medium is sufficiently dilute, such that the probability of simultaneous interaction of the control system with two or more particles of the medium is negligible, then the reduced dynamics of the system will be Markovian [32, 33] and will be determined by two body scattering events between the system and one particle of the medium. Below we provide a formulation for
control of quantum systems by a dilute medium, although the assumption of diluteness is not a restriction for ICE, and dense mediums might be used for control as well.

The master equation for a system interacting with coherent fields $u_l(t)$ and with a dilute medium of particles with mass $m$ has the form (3) with the generator $\mathcal{L}[n_{k,\alpha}(t)] = \mathcal{L}_{\text{Medium}}[n_{k,\alpha}(t)]$ specified by the distribution function of the medium $n_{k,\alpha}(t)$ and by the $T$-operator (transition matrix) for the scattering of the system and a medium particle. Below we assume that the particles of the medium are characterized only by their momenta and do not have internal degrees of freedom; otherwise, the state of one particle of the medium should have the form $|k, \alpha\rangle$, where $\alpha$ specifies the internal degrees of freedom. A transition matrix element is $T_{n,n'}(k,k') = \langle n, k | T | n', k' \rangle$, where $|n, k\rangle \equiv |n\rangle|k\rangle$ denotes the product state of the system discrete eigenstate $|n\rangle$ (an eigenstate of the system’s free Hamiltonian $H_0$ with eigenvalue $\varepsilon_n$) and a translational state of the system and a medium particle with relative momentum $k$. If the system is fixed in space (we consider this case below corresponding to the system particle being much more massive than the particles of the surrounding medium) then $|k\rangle$ is a translation state of a medium particle. The general case of relative system medium particle motion can be considered as well using suitable master equations.

We will use the notation $T_{\omega}(k,k') := \sum_{m,n: \varepsilon_m - \varepsilon_n = \omega} T_{m,n}(k,k')|m\rangle\langle n|$. The density of particles of the medium at momentum $k$ is denoted as $n_k(t)$, and the set of all transition frequencies $\omega$ of the system among the energy levels of $H_0$ is denoted as $\Omega$. In this notation the GKSL generator is

$$L_{\text{Medium}}[n_k(t)]\rho = 2\pi \sum_{\omega \in \Omega} \int d\omega n_k(t) \int d\omega' \delta \left( \frac{|\omega'|^2}{2m} - \frac{|\omega|^2}{2m} + \omega \right) \times \left[ T_{\omega}(k',k)\rho T_{\omega}^\dagger(k',k) - \frac{1}{2} \left\{ T_{\omega}^\dagger(k',k) T_{\omega}(k',k), \rho \right\} \right]$$

(10)

where $\{,\}$ denotes the anti-commutator. If the medium is at equilibrium with inverse temperature $\beta$, then the density has the stationary Boltzmann form $n_k(t) \equiv n_k = C(\beta, n) \exp(-\beta|k|^2/2m)$. Here the normalization constant $C(\beta, n)$ is determined by the condition $\int d\omega n_k = n$, where $n$ is the total density of the medium. The structure of Eq. (10) has been discussed previously for equilibrium media [32, 33] and for non-equilibrium stationary media [34]. Non-equilibrium media may be characterized by generally time dependent distributions. Equation (3) with $\mathcal{L}[n_k(t)] = \mathcal{L}_{\text{Medium}}[n_k(t)]$ provides the general formulation for theoretical analysis of control by a coherent field $u_l(t)$ and by a non-equilibrium medium with density $n_k(t)$.

As a simple illustration of such incoherent control, Fig. 1 reproduces the numerical results from Ref. [19] for optimally controlled transfer of a pure initial state of a four-level system into three different mixed target states [i.e., the objective function (6) is chosen]. The control is modelled by collisions with a medium prepared in a static non-equilibrium distribution $n_l(t)$ whose form is optimized by learning control using a genetic algorithm (GA) [35] based on the mutation and crossover operations. Since the initial and target states have different spectra, they can not be connected by a unitary evolution induced by coherent control. However, Fig. 1 shows that ICE through collisions can work perfectly for such situations.
Figure 1: (From Ref. [19]. Copyright (2006) by the American Physical Society.) Results of ICE simulations with a surrounding non-equilibrium medium as the control for target states (a) $\rho_{\text{target}} = \text{diag}(0.3; 0.3; 0.2; 0.2)$, (b) $\rho_{\text{target}} = \text{diag}(0.3; 0.2; 0.3; 0.2)$, and (c) $\rho_{\text{target}} = \text{diag}(0.4; 0.1; 0.4; 0.1)$. Each case shows: the objective function vs GA generation, the optimal distribution vs momentum, and the evolution of the diagonal elements of the density matrix for the optimal distribution. In the plots for the objective function the upper curve is the average value for the objective function and the lower one is the best value in each generation.

3 Kinematic description of incoherent control

Physically admissible evolutions of an $n$-level quantum system can be represented by CP, trace preserving maps (Kraus maps) [36]. A map $\Phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$ is positive if for any $\rho \in \mathcal{M}_n$ such that $\rho \geq 0$: $\Phi(\rho) \geq 0$. A linear map $\Phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$ is CP if for any $l \in \mathbb{N}$ the map $\Phi \otimes I_l : \mathcal{M}_n \otimes \mathcal{M}_l \rightarrow \mathcal{M}_n \otimes \mathcal{M}_l$ is positive (here $I_l$ denotes the identity map in $\mathcal{M}_l$). A CP map $\Phi$ is called trace preserving if for any $\rho \in \mathcal{M}_n : \text{Tr} \Phi(\rho) = \text{Tr} \rho$. The conditions of trace preservation and positivity for physically admissible evolutions are necessary to guarantee that $\Phi$ maps states into states. The condition of complete positivity has the following meaning. Consider the elements of $\mathcal{M}_l$ as operators of some $l$-level ancilla system which does not evolve, i.e., its evolution is represented by the identity mapping $I_l$. Suppose that the $n$-level system does not interact with the ancilla. Then the combined evolution of the total system will be represented by the map $\Phi \otimes I_l$ and the condition of complete positivity requires that for any $l$ this map should transform all states of the combined system into
states, i.e. to be positive.

Any CP, trace preserving map $\Phi$ can be expressed using the Kraus operator-sum representation as

$$\Phi(\rho) = \sum_{i=1}^{\lambda} K_i \rho K_i^\dagger$$

where $K_i \in \mathbb{C}^{n \times n}$ are the Kraus operators subject to the constraint $\sum_{i=1}^{\lambda} K_i^\dagger K_i = \mathbb{I}_n$ to guarantee trace preservation. This constraint determines a complex Stiefel manifold $V_n(\mathbb{C}^{\lambda n})$ whose points are $n \times (\lambda n)$ matrices $V = (K_1; K_2; \ldots; K_\lambda)$ (i.e., each $V$ is a column matrix of $K_1, \ldots K_\lambda$) satisfying the orthogonality condition $V^\dagger V = \mathbb{I}_n$.

The explicit evolution $P_t\{(u_{t}), n_{k,\alpha}\}$ in (4) is unlikely to be known for realistic systems. However, since this evolution is always a CP, trace preserving map, it can be represented in the Kraus form

$$P_t\{(u_{t}), n_{k,\alpha}\} \rho = \sum_{i=1}^{\lambda} K_i(t, (u_{t}), n_{k,\alpha}) \rho K_i^\dagger(t, (u_{t}), n_{k,\alpha}),$$

where $\sum_{i=1}^{\lambda} K_i^\dagger(t, (u_{t}), n_{k,\alpha}) K_i(t, (u_{t}), n_{k,\alpha}) = \mathbb{I}_n$.

Assume that any Kraus map can be generated in this way using the available coherent and incoherent controls $\{u_{t}(t)\}$ and $n_{k,\alpha}(t)$. Then effectively the Kraus operators can be considered as the controls [instead of $\{u_{t}(t)\}$ and $n_{k,\alpha}(t)$] which can be optimized to drive the evolution of the system in a desired direction. This picture is called the kinematic picture in contrast with the dynamical picture of Sec. 2. In the next two subsections we briefly outline the controllability and landscape properties in the kinematic picture.

### 3.1 Controllability

Any classical or quantum system at a given time is completely characterized by its state. The related notion of state controllability refers to the ability to steer the system from any initial state to any final state, either at a given time or asymptotically as time goes to infinity, and the important problem in control analysis is to establish the degree of state controllability for a given control system. Assuming for some finite-level system that the set of admissible dynamical controls generates arbitrary Kraus type evolution, the following theorem implies then that the system is completely state controllable.

**Theorem 1** For any state $\rho_f \in \mathcal{D}_n$ of an $n$-level quantum system there exists a Kraus map $\Phi_{\rho_f}$ such that $\Phi_{\rho_f}(\rho) = \rho_f$ for all states $\rho \in \mathcal{D}_n$.

**Proof.** Consider the spectral decomposition of the final state $\rho_f = \sum_{i=1}^{n} p_i |\phi_i\rangle \langle \phi_i|$, where $p_i$ is the probability to find the system in the state $|\phi_i\rangle$ ($p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$). Choose an arbitrary orthonormal basis $\{|\chi_j\rangle\}$ in the system Hilbert space and define the operators

$$K_{ij} = \sqrt{p_i} |\phi_i\rangle \langle \chi_j|, \quad i, j = 1, \ldots, n.$$
The operators $K_{ij}$ satisfy the normalization condition $\sum_{i,j=1}^{n} K_{ij}^\dagger K_{ij} = I_n$ and thus determine the Kraus map $\Phi_{\rho_f}(\rho) = \sum_{i,j=1}^{n} K_{ij} \rho K_{ij}^\dagger$. The map $\Phi_{\rho_f}$ acts on any state $\rho \in D_n$ as

$$\Phi_{\rho_f}(\rho) = \sum_{i,j=1}^{n} p_i |\phi_i\rangle \langle \chi_j| \rho |\phi_j\rangle \langle \chi_j| = (\text{Tr} \rho) \sum_{i=1}^{n} p_i |\phi_i\rangle \langle \phi_i| = \rho_f$$

and thus satisfies the condition of the Theorem. □

The potential importance of this result is that it shows that there may exist a single incoherent evolution which is capable for transferring all initial states into a given target state, and moreover, the target state can be an arbitrary pure or a mixed state[30]. Thus this theorem shows that non-unitary evolution can break the two general limitations for coherent unitary control described in the second paragraph in the Introduction.

3.2 Control landscape structure

In the kinematic description, under the assumption that any Kraus map can be generated, the objective functional becomes a function on the Stiefel manifold $V_n(C^{\lambda n})$. In practice, various gradient methods may be used to minimize such an objective function. If the objective function has a local minimum then gradient based optimization methods can be trapped in this minimum and will not provide a true solution to the problem. For such an objective function, if the algorithm stops in some minimum one can not be sure that this minimum is global and therefore this solution may be not satisfactory. This difficulty does not exist if a priori information about absence of local minima for the objective function is available as provided by the following theorem for a general class of objective functions of the form $J_1[K_1, \ldots, K_\lambda] = \text{Tr} \left[ (\sum_{i=1}^{\lambda} K_i \rho K_i^\dagger)O \right]$ in the kinematic picture.

**Theorem 2** For any $n \in \mathbb{N}$, $\rho \in D_n$, and for any Hermitian $O \in \mathcal{M}_n$ the objective function $J_1[K_1, \ldots, K_\lambda] = \text{Tr} \left[ (\sum_{i=1}^{\lambda} K_i \rho K_i^\dagger)O \right]$ on the Stiefel manifold $V_n(C^{\lambda n})$ does not have local minima or maxima; it has global minimum manifold, global maximum manifold, and possibly saddles whose number and the explicit structure depend on the degeneracies of $\rho$ and $O$.

The case $n = 2$ has been considered in detail in Ref. [37], where the global minimum, maximum, and saddle manifolds are explicitly described for each type of initial state $\rho$. In particular, it is found that the objective function $J_1$ for a non-degenerate target operator $O$ and for a pure $\rho$ (i.e., such that $\rho^2 = \rho$) does not have saddle manifolds; for the completely mixed initial state $\rho = \frac{1}{2}I$, $J_1$ has one saddle manifold with the value of the objective function $J_{\text{saddle}} = 1/2$; and for any partially mixed initial state $J_1$ has two saddle manifolds corresponding to the values of the objective function $J_{\text{saddle}}^\pm = (1 \pm \|w\|)/2$, where $w = \text{Tr} [\rho \sigma]$ and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices (the vector $w$ is in the unit ball, $\|w\| \leq 1$ and this vector characterizes the initial state as $\rho = \frac{1}{2}[I + \langle w, \sigma \rangle]$). The case of arbitrary $n$ is considered in Ref. [38].

4 Conclusions

This paper outlines recent results for incoherent control of quantum systems through their interaction with an environment. A general formulation for incoherent control through GKSL...
dynamics is given, followed by examples of incoherent radiation and a gaseous medium serving as the incoherent control environments. The relevant known results on controllability of open quantum systems subject to arbitrary Kraus type dynamics, as well as properties of the corresponding control landscapes, are also discussed.

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