Derivation in Strong Topology and Global Well-Posedness of Solutions to the Gross-Pitaevskii Hierarchy

THOMAS CHEN AND KENNETH TALIAFERRO

Department of Mathematics, University of Texas at Austin, Austin, Texas, USA

We derive the cubic defocusing GP hierarchy in \( \mathbb{R}^3 \) from a bosonic \( N \)-particle Schrödinger equation as \( N \to \infty \), in the strong topology corresponding to the space \( \mathcal{H}_1^{1/2} \) of sequences of marginal density matrices. In particular, we thereby eliminate the requirement of regularity \( \mathcal{H}_1^{1/2} \) for the initial data used in previous work. Moreover, the marginal density matrices obtained in this strong limit are allowed to be of infinite rank. This contrasts previous results where weak* limits were derived, and subsequently enhanced to strong limits based on the condition that the limiting density matrices have finite rank. Furthermore, we prove that positive semidefiniteness of marginal density matrices is preserved over time, which we combine with previous results Chen and Pavlović, to obtain the global well-posedness of solutions.

Keywords BBGKY hierarchy; Derivation of NLS; Global well-posedness; Gross-Pitaevskii hierarchy; Infinite rank limit; Quantum de Finetti.

Mathematics Subject Classification 35Q55; 81V70.

1. Introduction

The Gross-Pitaevskii (GP) hierarchy emerges, in the limit as \( N \to \infty \), from an \( N \)-body Schrödinger equation describing an interacting Bose gas under Gross-Pitaevskii scaling, via the associated BBGKY (Bogoliubov-Born-Green-Kirkwood-Yvon) hierarchy. Factorized solutions to the GP hierarchy are determined by a nonlinear Schrödinger equation (NLS). Through this procedure, one obtains a rigorous derivation of the NLS as a mean field description of the dynamics of a Bose-Einstein condensate.

In this paper, we extend previous results proven in [14, 15] concerning the derivation and global well-posedness of the cubic GP hierarchy in \( \mathbb{R}^3 \). We derive...
the GP hierarchy from the BBGKY hierarchy as $N \to \infty$ in the strong topology relative to the space $\mathcal{H}^1$ defined in (1.21), below, and we remove the requirement of regularity $\mathcal{H}^{1+\delta}$, with an arbitrary $\delta > 0$, for the initial data used in [15]. Moreover, we prove that solutions to the cubic defocusing GP hierarchy remain positive semidefinite over time, which we use, in combination with the higher order energy functionals and related results from [14], to prove global well-posedness.

The first derivation of the nonlinear Hartree equation (NLH) as a mean field description of a quantum manybody theory was given by Hepp in [37] using the Fock space formalism and coherent states. Subsequently, Spohn provided a derivation of the NLH using the BBGKY hierarchy in [50]. More recently, this topic was revisited by Fröhlich, Tsai, and Yau in [32]. In a series of very important works, Erdös, Schlein, and Yau gave the derivation of the NLS and NLH for a wide range of situations [24–27]; we will outline the main steps of their construction below, and will also mention related works of other authors.

The problems studied here are closely related to the study of Bose-Einstein condensation, where fundamental progress was made in recent years, see [2, 42–44] and the references therein.

1.1. Definition of the Model and Background

We consider a system of $N$ bosons in $\mathbb{R}^3$ described by a wave function $\Phi_N \in L^2_{\text{sym}}(\mathbb{R}^{3N})$ that satisfies the $N$-body Schrödinger equation

$$i\partial_t \Phi_N = H_N \Phi_N,$$  (1.1)

where the Hamiltonian $H_N$ is the self-adjoint operator on $L^2(\mathbb{R}^{3N})$ given by

$$H_N = \sum_{j=1}^{N} (-\Delta_{x_j}) + \frac{1}{N} \sum_{1 \leq i < j \leq N} V_N(x_i - x_j).$$  (1.2)

Here, $L^2_{\text{sym}}(\mathbb{R}^{3N})$ is the subspace of $L^2(\mathbb{R}^{3N})$ that is invariant under permutations (1.3) of the $N$ particle variables,

$$\Phi_N(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(N)}) = \Phi_N(x_1, x_2, \ldots, x_N) \quad \forall \pi \in S_N,$$  (1.3)

where $S_N$ is the $N$-th symmetric group. We note that permutation symmetry (1.3) is preserved by the $N$-body Schrödinger equation (1.1). The potential $V_N$ satisfies

$$V_N(x) = N^{d\beta} V(N^{\beta} x),$$  (1.4)

where $V \in \mathcal{S}(\mathbb{R}^3) \setminus \{0\}$ is spherically symmetric and nonnegative. The parameter $\beta$ typically has values in $[0, 1]$ (see the discussion below equation (1.15)); in this work, we will have $0 < \beta < \frac{1}{2}$.

Since the $N$-body Schrödinger equation (1.1) is linear and $H_N$ is self-adjoint, the global well-posedness of solutions in $L^2_{\text{sym}}(\mathbb{R}^{3N})$ is clear.

1.1.1. BBGKY Hierarchy. To begin with, one considers the density matrix

$$\gamma^{(N)}_{\Phi_N} = |\Phi_N \rangle \langle \Phi_N|,$$  (1.5)
and its marginals,

\[ \gamma_{\Phi_N}^{(k)}(t, \xi_k, \xi_k') := \int \Phi_N(t, \xi_k, \xi_{N-k}) \Phi_N(t, \xi_k', \xi_{N-k}) \, d\xi_{N-k}, \quad 1 \leq k \leq N, \] (1.6)

where \((\xi_k, \xi_{N-k}) \in \mathbb{R}^3 \times \mathbb{R}^{3(N-k)}\). The marginal density matrices satisfy the property of \textit{admissibility},

\[ \gamma_{\Phi_N}^{(k)} = \Tr_{k+1}(\gamma_{\Phi_N}^{(k+1)}), \quad k = 1, \ldots, N - 1, \] (1.7)

they define positive semidefinite operators on \(L^2(\mathbb{R}^3)\), and \(\Tr \gamma_{\Phi_N}^{(k)} = \| \Phi_N \|_{L^2(\mathbb{R}^3)}^2 = 1\), for an arbitrary \(N \in \mathbb{N}\).

Moreover, each \(\gamma_{\Phi_N}^{(k)}\) is completely symmetric under permutation of particle variables, and hermitean. That is,

\[ \gamma_{\Phi_N}^{(k)}(x_{\pi(1)}, \ldots, x_{\pi(k)}; x'_{\pi'(1)}, \ldots, x'_{\pi'(k)}) = \gamma_{\Phi_N}^{(k)}(x_1, \ldots, x_k; x'_1, \ldots, x'_k) \quad \text{and} \quad (1.8) \]

\[ \gamma_{\Phi_N}^{(k)}(\xi_k; \xi_k') = \gamma_{\Phi_N}^{(k)}(\xi'_k; \xi_k) \]

for all permutations \(\pi, \pi' \in S_k\).

It follows from the \(N\)-body Schrödinger equation (1.1) that

\[ i\hbar \frac{\partial}{\partial t} \gamma_{\Phi_N}^{(N)}(t) = [H_N, \gamma_{\Phi_N}^{(N)}(t)]. \] (1.9)

Accordingly, the \(k\)-particle marginals satisfy the BBGKY hierarchy

\[ i\hbar \frac{\partial}{\partial t} \gamma_{\Phi_N}^{(k)}(t, \xi_k; \xi_k') = -(\Delta_{\leq k} - \Delta_{\geq k}) \gamma_{\Phi_N}^{(k)}(t, \xi_k; \xi_k') + \frac{1}{N} \sum_{1 \leq i < j \leq k} [V_N(x_i - x_j) - V_N(x'_i - x'_j)] \gamma_{\Phi_N}^{(k)}(t, \xi_k; \xi_k') + \frac{N - k}{N} \sum_{i=1}^{k} \int_{x_{k+1}} d\xi_{k+1} [V_N(x_i - x_{k+1}) - V_N(x'_i - x_{k+1})] \] (1.10)

\[ \gamma_{\Phi_N}^{(k+1)}(t, \xi_k, x_{k+1}; \xi_k', x_{k+1}) \] (1.11)

for \(1 \leq k < N\), where \(\Delta_{\leq k} := \sum_{j=1}^{k} \Delta_{x_j}\), and similarly for \(\Delta_{\geq k}\).

1.1.2. Derivation of the GP Hierarchy. In \([24, 25, 28]\), the authors consider factorizing initial data, i.e.,

\[ \gamma_{\Phi_N}^{(k)}(0, \xi_k; \xi_k') = \gamma_{\Phi_0}^{(k)}(\xi_k; \xi_k') = \prod_{j=1}^{k} \phi_0(x_j) \phi_0(x'_j), \] (1.12)

as \(N \to \infty\), where \(\phi_0 \in H^1(\mathbb{R}^3)\). In particular, they prove that in the limit \(N \to \infty\), solutions to the BBGKY hierarchy converge in the weak-* topology, \(\gamma_{\Phi_N}^{(k)} \rightharpoonup \gamma^{(k)}\) for \(k \in \mathbb{N}\), on the space of trace class marginal density matrices. Moreover, it is proven
in [24, 25, 28] that the marginal density matrices obtained in the weak-\(\ast\) limit satisfy the infinite hierarchy

\[
\iota \gamma^{(k)}(t, x; x') = -(\Delta_{x_j} - \Delta_{x'_j}) \gamma^{(k)}(t, x_j; x'_j)
\]

\[
+ \kappa_0 \sum_{j=1}^{k} (B_{j,k+1} \gamma^{(k+1)}(t, x_j; x'_j)), \quad k \in \mathbb{N},
\]

which is referred to as the GP hierarchy. The interaction operator is defined by

\[
(B_{j,k+1} \gamma^{(k+1)})(t, x_j; x'_j) \ni \int dx_{k+1} dx'_{k+1} [\delta(x_j - x_{k+1}) - \delta(x'_j - x_{k+1})] \delta(x_j - x'_{k+1}) \delta(x'_j - x'_{k+1}).
\]

In the case \(\beta < 1\), one obtains a coupling constant \(\kappa_0 = \int V(x)dx\) (corresponding to the Born approximation of the scattering length). In the case \(\beta = 1\), the coefficient \(\kappa_0\) is the scattering length; the derivation of the GP hierarchy in this case is much more difficult, [24, 25]. We will have \(0 < \beta < 4\) in this paper, and set \(\kappa_0 = 1\).

In [24, 25, 28], solutions of the GP hierarchy are studied in spaces of \(k\)-particle marginals

\[
\mathcal{S}^1 := \{ (\gamma^{(k)})_{k \geq 1} \in \mathcal{S}^1 | \text{Tr}(|S^{(1)}, \gamma^{(k)})| < M^k \text{ for some constant } 0 < M < \infty \}
\]

where for \(\alpha > 0\),

\[
S^{(k,\alpha)} := \prod_{j=1}^{k} (1 - \Delta_{x_j})^{\alpha} (1 - \Delta_{x'_j})^{\alpha}.
\]

The solutions to the GP hierarchy obtained from the weak-\(\ast\) limit as described above exist globally in \(t\) and are positive semidefinite.

### 1.1.3. NLS from Factorized Solutions of GP

Given factorized initial data (1.12), one can easily verify that

\[
\gamma^{(k)}(t, x; x') = \prod_{j=1}^{k} \phi(t, x_j) \bar{\phi}(t, x'_j)
\]

is a solution (referred to as a factorized solution) of the GP hierarchy (1.13) if \(\phi(t) \in H^1(\mathbb{R}^d)\) solves the defocusing cubic NLS,

\[
i \phi_t = -\Delta_x \phi + |\phi|^2 \phi,
\]

for \(t \in I \subseteq \mathbb{R}\), and \(\phi(0) = \phi_0 \in H^1(\mathbb{R}^d)\). In this precise sense, the NLS emerges as a mean field description of the dynamics of Bose-Einstein condensates.

### 1.1.4. Uniqueness of Solutions of GP Hierarchies

The most involved part in this analysis is the proof of uniqueness of solutions to the GP hierarchy, and was
achieved in Erdős-Schlein-Yau in [24, 25, 28], in the space $\mathcal{S}^1$ using high dimensional singular integral estimates organized with Feynman graph expansions.

Subsequently, Klainerman and Machedon [40] presented an alternative method for proving uniqueness in a space of density matrices defined by the Hilbert-Schmidt type Sobolev norms

$$\|\gamma^{(k)}\|_{H^1_k} := \left\| S^{(k,1)} \gamma^{(k)} \right\|_{L^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k})}.$$  

While this is a different (strictly larger) space of marginal density matrices than the one considered by Erdős, Schlein, and Yau, [24, 25], the authors of [40] impose an additional a priori condition on space-time norms of the form

$$\left\| B_{j,k+1} \gamma^{(k+1)} \right\|_{L^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k})} < C^k,$$  

for some arbitrary but finite $C$ independent of $k$. The method of Klainerman-Machedon in [40] combines techniques from dispersive nonlinear PDEs with a reformulation of the combinatorial method introduced in [24–27], which is referred to as a “board game” argument.

The Klainerman-Machedon framework was used by Kirkpatrick, Schlein, and Staffilani in [39], to give a derivation of the cubic defocusing NLS in dimensions $d = 1, 2$, and by Chen and Pavlović in [11], to derive the quintic NLS for $d = 1, 2$.

As another line of research in this area, the study of the well-posedness theory of the GP hierarchy was initiated in [12–14, 16]. The authors introduced Banach spaces of sequences of marginal density matrices

$$\mathcal{H}_{\xi}^2 := \{ \text{symmetric } \Gamma = (\gamma^{(k)})_{k \in \mathbb{N}} \mid \|\Gamma\|_{\mathcal{H}_{\xi}^2} < \infty \}$$

with

$$\|\Gamma\|_{\mathcal{H}_{\xi}^2} := \sum_{k \in \mathbb{N}} \xi^k \left\| \gamma^{(k)} \right\|_{H^\xi}, \quad \xi > 0,$$

where

$$\left\| \gamma^{(k)} \right\|_{H^\xi} := \left\| S^{(k,\xi)} \gamma^{(k)} \right\|_{L^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k})}$$

is of Hilbert-Schmidt type, as in (1.19). Those spaces are equivalent to those considered by Klainerman and Machedon in [40]. Here, we call $\Gamma = (\gamma^{(k)})_{k \in \mathbb{N}}$ symmetric if each $\gamma^{(k)}(x_j, x'_j)$ satisfies (1.8). Moreover, we call $\Gamma$ positive semidefinite if $\gamma^{(k)}$ defines a positive semidefinite integral operator on $L^2(\mathbb{R}^{3k})$ for all $k$.

We also define the spaces

$$\mathcal{S}_x^2 := \left\{ \text{symmetric } \Gamma \in \bigoplus_{k=1}^\infty L^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k}) \mid \|\Gamma\|_{\mathcal{S}_x^2} < \infty \right\}$$

where for $x > 0$, 

$$\|\Gamma\|_{\mathcal{S}_x^2} = \sum_{k=1}^\infty \xi^k \text{Tr}(\left| S^{(k,\xi)} \gamma^{(k)} \right|).$$
is of trace-norm type. Clearly, $H_{\xi}^1 = \cup_{\xi > 0} H_{\xi}^1$ corresponding to (1.16). Moreover, $H_{\xi}^2 \subset H_{\xi}^{1+\delta}$ holds for any $\delta > 0$ and $0 < \xi < 1$.

In [15], Chen and Pavlović proved that solutions of the $N$-body Schrödinger equation converge to the solution of the GP hierarchy in $H_{\xi}^1$, for values $\beta \in (0, 1/4)$, provided that the initial data satisfies $\Gamma_0 \in H_{\xi}^{1+\delta}$ for an arbitrary $\delta > 0$, and for $\xi$ sufficiently small (depending on $0 < \xi < 1$). No factorization of solutions was assumed.

More recently, a derivation of the GP hierarchy in Klainerman-Machedon type spaces was given by Chen and Holmer in [21], for values $\beta \in (0, 2/3)$. Assuming a regularity requirement that follows from the condition (2.1) in Theorem 2.1 of our paper, they prove that solutions to the BBGKY hierarchy converge to solutions of the GP hierarchy satisfying the Klainerman-Machedon condition (1.20). This convergence is shown in the weak-* topology on the space of trace class marginal density matrices. See also [20].

Rodnianski and Schlein in [48] investigated the rate of convergence to the NLH, based on the approach of Hepp [37], which led to many further developments, including works of Grillakis-Machedon and G-M-Margetis [33–36], Chen [18], and Lee-Li-Schlein [9]. Many authors have contributed to this very active research field, and introduced a variety of different approaches; see for instance [1, 5, 23, 29–31, 45].

1.2. Outline of Main Results

The main results of this paper can be summarized as follows:

- **Derivation in $H_{\xi}^1$**: We show that solutions to the $N$-BBGKY hierarchy with initial data $\Gamma_{0,N}$ converge to those of the GP hierarchy strongly in $C([0, T], H_{\xi}^1)$ as $N \to \infty$ when the initial data is in $H_{\xi}^1$, see (3.3). In [15], this convergence is obtained with initial data in $H_{\xi}^{1+\delta}$, for an arbitrary, small $\delta > 0$ extra regularity. In this paper, we eliminate this condition, and provide the derivation of the GP hierarchy in the energy space. The detailed discussion is given in Section 4.

- **Strong convergence for limits of infinite rank**: The convergence proven in the work at hand is established in the strong topology on $H_{\xi}^1$. We note that in previous works following the BBGKY approach (except for [15]), including [21, 24–27], convergence along a subsequence is shown in the weak-* topology on the space of trace class marginal density matrices, using a compactness argument. Subsequently, this weak-* convergence is enhanced to strong convergence in trace norm, due to the special case of the limiting density matrices being of finite rank (since factorized initial data are considered). In the finite rank case (i.e., the rank of $\gamma(k)$ is bounded uniformly in $k$), the trace norm, corresponding to $H_{\xi}^1$, is equivalent to the Hilbert-Schmidt norm, corresponding to $H_{\xi}^1$. In contrast, we obtain a strong limit in $H_{\xi}^1$ without any finite rank requirement.

- **Global well-posedness**: Combining the higher order energy functional introduced in [14], combined with the quantum de Finetti theorem as formulated by Lewin-Nam-Rougerie in [41], we prove that solutions to the cubic defocusing GP hierarchy are globally well-posed. To this end, we prove that those solutions remain positive semidefinite over time if the initial data are positive semidefinite; this allows us to invoke a key result in [14] to arrive at global well-posedness. This is carried out in Section 5.
Global in time derivation of GP hierarchy: By combining our derivation of the GP hierarchy locally in time, and global well-posedness of the GP hierarchy, we arrive at a derivation of the GP hierarchy on arbitrarily large time intervals $[0, T]$; the details are presented in Section 6.

2. Statement of Main Theorems

In this section, we present the main theorems proven in this paper. Our first main result provides the derivation of the GP hierarchy from a bosonic $N$-body Schrödinger system via the associated BBGKY hierarchy as $N \to \infty$, in the strong topology with respect to the energy space $L^\infty_{\text{energy}}$, for a suitable $0 < \xi < 1$. In particular, this strong limit yields solutions $\Gamma(t) = (\gamma^{(k)}(t))_k$ to the GP hierarchy where $\gamma^{(k)}(t)$ does not need to be of finite rank. As noted above, a strong limit was obtained in the previous works [21, 24–27] only for the special case where the density matrices $(\gamma^{(k)}(t))_k$ have a finite rank.

Moreover, our result removes an extra regularity condition on the initial data which was assumed in [15]. In [15], solutions to the GP hierarchy were derived from the BBGKY hierarchy under the requirement that $\Gamma_0 \in \mathcal{S}^{1+\delta}_\xi$ for an arbitrarily small, but positive $\delta > 0$. Here, we assume that $\Gamma_0 \in \mathcal{S}^1_\xi$.

Theorem 2.1. Let $(\Phi_N)_N$ be a sequence of solutions to the $N$-body Schrödinger equation (1.1) with the corresponding marginal density matrices $\gamma^{(k)}_{\Phi_N}(t)$ given by (1.6).

Suppose that

$$\langle \Phi_N(0), H_N^k \Phi_N(0) \rangle < C^k N^k$$

(2.1)

and $\|\Phi_N(0)\|_2 = 1$ for all $N \in \mathbb{N}$ and $k \leq N$, where $C$ does not depend on $k$ or $N$. Moreover, assume that for some $0 < \xi' < 1$, and every $N \in \mathbb{N}$, we have

$$\Gamma^{(k)}(0) = (\gamma^{(1)}_{\Phi_N}(0), \ldots, \gamma^{(N)}_{\Phi_N}(0), 0, \ldots) \in \mathcal{H}_\xi^1$$

and that

$$\Gamma_0 := \lim_{N \to \infty} \Gamma^{(k)}(0)$$

exists in $\mathcal{H}_\xi^1$. Define the truncation operator $P_{\leq K}$ by

$$P_{\leq K}(\gamma) = (\gamma^{(1)}, \ldots, \gamma^{(K)}, 0, \ldots),$$

where $\frac{1}{2} b_1 \log N \leq K(N) \leq b_1 \log N$ for some $b_1 > 0$. Then, for sufficiently small $b_1 > 0$ (depending only on $\beta$ (see (1.4))) and sufficiently small $\xi > 0$ (depending on only $\xi'$) and sufficiently small $T > 0$ (depending only on $\xi$), the limit

$$\Gamma := \lim_{N \to \infty} P_{\leq K(N)} \Gamma^{(k)}$$

exists in $L^\infty_{\text{energy}} \mathcal{H}_\xi^1$ and satisfies the GP hierarchy with initial data $\Gamma_0 \in \mathcal{S}^1_\xi$. Moreover,

$$B\Gamma = \lim_{N \to \infty} B_N P_{\leq K(N)} \Gamma^{(k)}$$
Derivation and GWP of the GP Hierarchy

holds in $L^2_{\{0, T\}}(\mathcal{H})$. The abbreviated notations $B$ and $B_N$ for the interaction operators are defined in (3.8) and (3.22), below.

Our second main result establishes global well-posedness for solutions to the GP hierarchy, where our proof uses the quantum de Finetti theorem in the formulation presented in a recent paper by Lewin, Nam, and Rougerie [41], which we quote here:

**Theorem 2.2** (Quantum de Finetti Theorem). Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{H}^k = \bigotimes_{\text{sym}}^k \mathcal{H}$ denote the corresponding bosonic $k$-particle space. Let $\Gamma$ denote a collection of bosonic density matrices on $\mathcal{H}$, i.e.,

$$\Gamma = (\gamma^{(1)}, \gamma^{(2)}, \ldots)$$

with $\gamma^{(k)}$ a nonnegative trace class operator on $\mathcal{H}^k$. Then, the following hold:

- **(Strong Quantum de Finetti theorem, [38, 41, 51])** Assume that $\Gamma$ is admissible, i.e., $\gamma^{(k)} = \text{Tr}_{k+1} \gamma^{(k+1)}$, where $\text{Tr}_{k+1}$ denotes the partial trace over the $(k+1)$-th factor, $\forall k \in \mathbb{N}$. Then, there exists a unique Borel probability measure $\mu$, supported on the unit sphere in $\mathcal{H}$, and invariant under multiplication of $\phi \in \mathcal{H}$ by complex numbers of modulus one, such that

$$\gamma^{(k)} = \int d\mu(\phi) (|\phi\rangle \langle \phi|)^{\otimes k}, \ \forall k \in \mathbb{N}. \quad (2.2)$$

- **(Weak Quantum de Finetti theorem, [3, 4, 41])** Assume that $\gamma^{(N)}_N$ is an arbitrary sequence of mixed states on $\mathcal{H}^N$, $N \in \mathbb{N}$, satisfying $\gamma^{(N)}_N \geq 0$ and $\text{Tr}_{\mathcal{H}^N}(\gamma^{(N)}_N) = 1$, and assume that its $k$-particle marginals have weak-∗ limits

$$\gamma^{(k)}_N := \text{Tr}_{k+1, \ldots, N} \gamma^{(N)}_N \rightharpoonup^* \gamma^{(k)} \quad (N \to \infty), \quad (2.3)$$

in the trace class on $\mathcal{H}^k$ for all $k \geq 1$ (here, $\text{Tr}_{k+1, \ldots, N} \gamma^{(N)}_N$ denotes the partial trace in the $(k+1)$-st up to $N$-th component). Then, there exists a unique Borel probability measure $\mu$ on the unit ball in $\mathcal{H}$, and invariant under multiplication of $\phi \in \mathcal{H}$ by complex numbers of modulus one, such that (2.2) holds for all $k \geq 0$.

In the context of Theorem 2.1 proven here, strong convergence in Hilbert Schmidt norm implies weak-∗ convergence in the trace norm topology, provided that the limit point is trace class, see Proposition A.1 in the Appendix. Therefore, the solutions to the GP hierarchy obtained in Theorem 2.1 from the BBGKY hierarchy (1.11), in the limit as $N \to \infty$, satisfy the conditions of the quantum de Finetti theorem, either in its strong or its weak form.

Moreover, due to the bound (2.1), it follows that the initial data for the GP hierarchy satisfies $\Gamma_0 \in S^1_{\text{vis}}$ for some $0 < \xi < 1$. This implies that the higher order energy functionals introduced in [14], which correspond to $\Gamma_t$, are well-defined; see Section 5. We are therefore able to combine an application of the quantum de Finetti theorem with the higher order energy functionals for the cubic GP hierarchy that were introduced in [14], to prove the following result on the global well-posedness of solutions to the defocusing cubic GP hierarchy.
Theorem 2.3. Assume that
\[ \gamma_0^{(k)} = \int d\mu(\phi)(\langle \phi \rangle \langle \phi \rangle)^{\otimes k}, \quad k \in \mathbb{N} \]  
(2.4)
satisfies \( \Gamma_0 = (\gamma_0^{(k)})_{k=1}^\infty \in \mathcal{S}_{\xi}^1 \) for some \( 0 < \xi' < 1 \), where \( d\mu \) is a probability measure supported either on the unit sphere, or on the unit ball in \( L^2(\mathbb{R}^3) \). For \( I \subseteq \mathbb{R} \), we denote by
\[ \mathcal{W}^2_{\xi}(I) := \{ \Gamma \in C(I, \mathcal{H}^2_{\xi}) \mid B^+ \Gamma, B^- \Gamma \in L^1_{\text{loc}}(I, \mathcal{H}^2_{\xi}) \}, \]  
(2.5)
the space of local in time solutions of the GP hierarchy, with \( \xi' > 0 \) (depending only on \( \xi' \)), there is a unique global solution \( \Gamma \in \mathcal{W}^2_{\xi}(\mathbb{R}) \) to the cubic defocusing GP hierarchy (3.4) in \( \mathbb{R}^3 \) with initial data \( \Gamma_0 \). Moreover, \( \Gamma(t) \) is positive semidefinite and satisfies
\[ \| \Gamma(t) \|_{\mathcal{W}^1_{\xi}} \leq \| \Gamma_0 \|_{\mathcal{W}^1_{\xi'}} \]
for all \( t \in \mathbb{R} \). The dependence of \( \xi_1 \) on \( \xi' \) is detailed in Section 4.2.

2.1. Remarks

- We note that, by combining Theorem 2.1 and 2.3, one can show that Theorem 2.1 actually holds for \( T \) arbitrarily large, provided that \( \Gamma_0 \in \mathcal{S}_{\xi'}^1 \), and that \( \xi \) is sufficiently small. This is addressed in detail in Section 6.
- Although we only address the cubic defocusing GP hierarchy in \( \mathbb{R}^d \) for \( d = 3 \), we note that Theorem 2.3 can be proved in the same way for the more general cases considered in Theorem 7.2 of [14]. Let \( \kappa_0 \) be the constant in (1.14), and let \( p = 2, 4 \) correspond to the cubic and quintic GP hierarchies, respectively. Then, we have global well-posedness for the following cases:
  - Energy subcritical, defocusing GP hierarchy with \( p < \frac{4}{d-2} \) and \( \kappa_0 = +1 \).
  - \( L^2 \) subcritical, focusing GP hierarchy with \( p < \frac{4}{d} \) and \( \kappa_0 < 0 \) with \( |\kappa_0| \) sufficiently small (see Theorem 7.2 in [14] for an explicit bound on \( |\kappa_0| \)).
- A solution to the GP hierarchy obtained in a weak-* limit does not necessarily satisfy admissibility, even if the system at finite \( N \) does. However, we note that solutions to the GP hierarchy preserve admissibility (1.7), provided that it holds at the initial time \( t = 0 \); see Proposition B.1 in the Appendix.

3. Notations for the GP and BBGKY Hierarchy

For convenience, we introduce additional notations for the GP and the BBGKY hierarchy in this section, mostly adopted from [12], which allow us to discuss them both on the same setting.
Let \( 0 < \xi < 1 \). We recall that
\[
\mathcal{H}_\xi^2 = \left\{ \text{symmetric } \Gamma \in \bigoplus_{k=1}^{\infty} L^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k}) \left| \| \Gamma \|_{\mathcal{H}_\xi^2} < \infty \right. \right\} \tag{3.1}
\]
where
\[
\| \Gamma \|_{\mathcal{H}_\xi^2} = \sum_{k=1}^{\infty} \xi^k \| \gamma^{(k)} \|_{H^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k})},
\]
with
\[
\| \gamma^{(k)} \|_{H^2} := \left\| S^{(k,x)} \gamma^{(k)} \right\|_{L^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k})} \tag{3.2}
\]
where \( S^{(k,x)} = \prod_{j=1}^{k} (1 - \Delta_{x_j})^{y/2}(1 - \Delta_{y_j})^{y/2} \).

We also recall the spaces
\[
\mathcal{D}_\xi^2 = \left\{ \text{symmetric } \Gamma \in \bigoplus_{k=1}^{\infty} L^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k}) \left| \| \Gamma \|_{\mathcal{D}_\xi^2} < \infty \right. \right\} \tag{3.3}
\]
where
\[
\| \Gamma \|_{\mathcal{D}_\xi^2} = \sum_{k=1}^{\infty} \xi^k \text{Tr}(\mathcal{S}^{(k,x)} \gamma^{(k)}).
\]
is of trace-norm type.

### 3.1. The GP Hierarchy

The cubic defocusing GP hierarchy is given by
\[
i \tilde{e}_{ij} \gamma^{(k)} = \sum_{j=1}^{k} \left[ -\Delta_{x_j} \gamma^{(k)} \right] + B_{k+1} \gamma^{(k+1)} \tag{3.4}
\]
for \( k \in \mathbb{N} \), where
\[
B_{k+1} \gamma^{(k+1)} = B_{k+1}^+ \gamma^{(k+1)} - B_{k+1}^- \gamma^{(k+1)}, \tag{3.5}
\]
where
\[
B_{k+1}^+ \gamma^{(k+1)} = \sum_{j=1}^{k} B_{j,k+1}^+ \gamma^{(k+1)}, \tag{3.6}
\]
and
\[
B_{k+1}^- \gamma^{(k+1)} = \sum_{j=1}^{k} B_{j,k+1}^- \gamma^{(k+1)}, \tag{3.7}
\]
with
\[
(B_{j,k+1}^{+})_{j}(t, x_1, \ldots, x_k, x'_1, \ldots, x'_k) \\
= \int \, dx_{k+1} \, dx_{k+1}' \, \delta(x_j - x_{k+1}) \delta(x_j - x'_{k+1}) \gamma_{j}^{(k+1)}(t, x_1, \ldots, x_{k+1}; x'_1, \ldots, x'_{k+1}),
\]
and
\[
(B_{j,k+1}^{-})_{j}(t, x_1, \ldots, x_k, x'_1, \ldots, x'_k) \\
= \int \, dx_{k+1} \, dx_{k+1}' \, \delta(x'_j - x_{k+1}) \delta(x'_j - x'_{k+1}) \gamma_{j}^{(k+1)}(t, x_1, \ldots, x_{k+1}; x'_1, \ldots, x'_{k+1}).
\]

The GP hierarchy can be rewritten in the following compact manner:
\[
i \hat{\partial}_t \Gamma + \hat{\Delta}_{\pm} \Gamma = B \Gamma
\]
\[
\Gamma(0) = \Gamma_0.
\]
(3.8)

where
\[
\hat{\Delta}_{\pm} \Gamma := (\Delta_{\pm} \gamma_{j}^{(k)})_{k \in \mathbb{N}}, \quad \text{with} \quad \Delta_{\pm}^{(k)} = \sum_{j=1}^{k} (\Delta_{x_j} - \Delta'_{x_j}),
\]
and
\[
B \Gamma := (B_{j,k+1}^{+} \gamma_{j}^{(k+1)})_{k \in \mathbb{N}}.
\]
(3.9)

We will also use the notation
\[
B^{+} \Gamma := (B_{j,k+1}^{+} \gamma_{j}^{(k+1)})_{k \in \mathbb{N}}, \quad B^{-} \Gamma := (B_{j,k+1}^{-} \gamma_{j}^{(k+1)})_{k \in \mathbb{N}}.
\]

Moreover, we define the free evolution operator \( U(t) \) by
\[
(U(t) \Gamma)^{(k)} = U^{(k)}(t) \gamma_{j}^{(k)},
\]
where
\[
(U^{(k)}(t) \gamma_{j}^{(k)})(\Delta_{x_j}, \Delta'_{x_j}) = e^{i\Delta_{x_j} t} e^{-i\Delta'_{x_j} t} (\Delta_{x_j}, \Delta'_{x_j})
\]
corresponds to the \( k \)-th component.

3.2. The BBGKY Hierarchy

The cubic defocusing BBGKY hierarchy in \( \mathbb{R}^3 \) is given by
\[
i \hat{\partial}_t \gamma_{N}^{(k)}(t) = \sum_{j=1}^{k} [-\Delta_{x_j}, \gamma_{N}^{(k)}(t)] + \frac{1}{N} \sum_{1 \leq j < k} [V_N(x_j - x_k), \gamma_{N}^{(k)}(t)]
\]
\[
+ \frac{(N - k)}{N} \sum_{1 \leq j \leq k} \text{Tr}_{k+1} [V_N(x_j - x_{k+1}), \gamma_{N}^{(k+1)}(t)],
\]
(3.10)
for \( k = 1, \ldots, N. \)
Derivation and GWP of the GP Hierarchy 1669

We extend this finite hierarchy trivially to an infinite hierarchy by adding the terms \( \gamma^{(k)}_N = 0 \) for \( k > N \), and write

\[
i \tilde{\epsilon}_{\gamma}^{(k)} = \sum_{j=1}^{k} \left[ -\Delta_{ij} \gamma^{(k)}_N \right] + (B_N \Gamma_N)^{(k)}
\]  

(3.11)

for \( k \in \mathbb{N} \). Here, we have \( \gamma^{(k)}_N = 0 \) for \( k > N \), and we define

\[
(B_N \Gamma_N)^{(k)} := \begin{cases} 
B_{N,k+1}^{(k+1)} + B_{N,k}^{error(k)} & \text{if } k \leq N \\
0 & \text{if } k > N.
\end{cases}
\]  

(3.12)

The interaction terms on the right hand side are defined by

\[
B_{N,k+1}^{(k+1)} = B_{N,k+1}^{+,main(k+1)} - B_{N,k}^{-,main(k+1)},
\]  

(3.13)

and

\[
B_{N,k}^{error(k)} = B_{N,k}^{+,error(k)} - B_{N,k}^{-,error(k)},
\]  

(3.14)

where

\[
B_{N,k}^{+,main(k+1)} := \frac{N-k}{N} \sum_{j=1}^{k} B_{N,j+1}^{+,main(k+1)}
\]  

(3.15)

and

\[
B_{N,k}^{+,error(k)} := \frac{1}{N} \sum_{i<j} B_{N,i,j,k}^{+,error(k)}
\]  

(3.16)

with

\[
(B_{N,i,j,k}^{+,main(k+1)})(t, x_1, \ldots, x_k; x'_1, \ldots, x'_k)
= \int dx_{k+1} V_N(x_j - x_{k+1}) \gamma^{(k+1)}_N(t, x_1, \ldots, x_k, x_{k+1}; x'_1, \ldots, x'_{k+1}, x_{k+1})
\]  

(3.17)

and

\[
(B_{N,i,j,k}^{+,error(k)})(t, x_1, \ldots, x_k; x'_1, \ldots, x'_k)
= V_N(x_j - x_j') \gamma^{(k)}_N(t, x_1, \ldots, x_k, x'_1, \ldots, x'_k).
\]  

(3.18)

Moreover,

\[
(B_{N,i,j,k}^{-,main(k+1)})(t, x_1, \ldots, x_k; x'_1, \ldots, x'_k)
= \int dx_{k+1} V_N(x'_j - x_{k+1}) \gamma^{(k+1)}_N(t, x_1, \ldots, x_k, x_{k+1}; x'_1, \ldots, x'_{k+1}, x_{k+1})
\]  

(3.19)

and

\[
(B_{N,i,j,k}^{-,error(k)})(t, x_1, \ldots, x_k; x'_1, \ldots, x'_k)
= V_N(x'_j - x'_j') \gamma^{(k)}_N(t, x_1, \ldots, x_k, x'_1, \ldots, x'_k).
\]  

(3.20)
We remark that in all of the above definitions, we have that $B_{N,k}^{\pm, \text{main}}$, $B_{N,k}^{\pm, \text{error}}$, etc. are defined to be given by multiplication with zero for $k > N$.

Then, we can write the BBGKY hierarchy compactly in the form

$$i\partial_t \Gamma_N + \hat{\Delta}_{\pm} \Gamma_N = B_N \Gamma_N$$

where

$$\hat{\Delta}_{\pm} \Gamma_N := (\Delta^{(k)}_{\pm} \Gamma_N)_{k \in N},$$

with $\Delta^{(k)}_{\pm} = \sum_{j=1}^{k} (\Delta_{x_j} - \Delta_{x_j'})$,

and

$$B_N \Gamma_N := (B_{N,k+1}^{(k+1)} \Gamma_N)_{k \in N}.$$ (3.22)

In addition, we introduce the notation

$$B_N^g \Gamma_N := (B_{N,k+1}^{(k+1)} \Gamma_N)_{k \in N}$$

adapted to (3.18) and (3.20).

4. Derivation of GP from BBGKY Hierarchy

In this section, we prove Theorem 2.1.

4.1. Local Well-Posedness of the BBGKY Hierarchy in $\mathcal{H}_\xi^1$

Taking $\delta = 0$ in Lemma 4.1 of [15] gives us local well-posedness of the $K$-truncated $N$-BBGKY hierarchy:

**Lemma 4.1.** Let $K < b_1 \log N$, for some constant $b_1 > 0$. Let

$$\Gamma^g_{0,N} := P_{\leq K} \Gamma_{0,N} = (\gamma_{0,N}^{(k)})_{k=1}^{K} \in \mathcal{H}_\xi^1,$$ (4.1)

and

$$T_0(\xi) := \xi^2 / c_0, \quad \xi \in \mathbb{R}_+.$$ (4.2)

where the constant $c_0 > 0$ is defined in Lemma D.1. Then, for $0 < \xi' < 1$ and $\xi$ satisfying (4.5), the following holds:

For $0 < T < T_0(\xi)$, and for $b_1 > 0$ sufficiently small (see (4.7) below), there exists a unique solution $\Gamma^g_N \in L^\infty_t \mathcal{H}_\xi^1$ of the BBGKY hierarchy (3.10) for $I = [0, T]$ such that $B_N \Gamma^g_N \in L^\infty_t \mathcal{H}_\xi^1$. Moreover,

$$\|\Gamma^g_N\|_{L^\infty_t \mathcal{H}_\xi^1} \leq C_0(T, \xi, \xi') \|\Gamma^g_{0,N}\|_{\mathcal{H}_\xi^1}$$ (4.3)
Proposition 4.2. Suppose that those of the results of both [15] and [14] hold. Furthermore, $(\Gamma^K_N)^{(i)} = 0$ for all $K < k \leq N$, and $t \in I$.

4.2. Constants

For convenience, we collect here the interdependence of various constants that appear in the formulation of the main theorems above. Throughout this paper, we will require that, given $\xi > 0$, the real, positive constants $\xi$ and $\xi_1$ satisfy

$$\begin{align*}
\xi < \eta \min \left\{ b, e^{-2\beta/b_1}, e^{-24\beta/b_1} \right\} \quad \text{and} \\
0 < \xi_1 < \theta^3 \xi < \theta^6 \xi',
\end{align*}$$

(4.5)

where $\theta := \min \{ \eta, (1 + \frac{1}{2} C_{\text{Sob}})^{-1/2} \}$; the constant $\eta > 0$ is defined in Lemma D.2, and $C_{\text{Sob}} > 0$ is the constant in the trace Sobolev inequality

$$\left( \int dx |f(x, x)|^2 \right)^{\frac{1}{2}} \leq C_{\text{Sob}} \left( \int dx_1 dx_2 |\nabla_x f(x_1, x_2)|^2 \right)^{\frac{1}{2}},$$

(4.6)

for $x_{1,2} \in \mathbb{R}^3$, see [14]. This will ensure that $\xi_1$ and $\xi$ are small enough so that the results of both [15] and [14] hold.

In (4.5), $b_1 > 0$ is a constant chosen sufficiently small that Lemma D.2 holds for all $K, N$ satisfying

$$K \leq b_1 \log N.$$  

(4.7)

To satisfy this requirement, $b_1$ only depends on $\beta$ (see (1.4)).

4.3. From $(K, N)$-BBGKY to $K$-Truncated GP Hierarchy

In this section, we show that solutions to the $(K, N)$-BBGKY hierarchy approach those of the $K$-Truncated GP hierarchy as $N \to \infty$.

Proposition 4.2. Suppose that $\Gamma_0 = (\zeta^{(k)}_0)_{k=1}^{\infty} \in S^1_{\zeta}$. Moreover, let $\Gamma^K \in \{ \Gamma \in L^\infty_{([0, \infty], \mathbb{R}^3)} \left| B\Gamma \in L^2_{([0, \infty], \mathbb{R}^3)} \right. \}$ be the solution of the GP hierarchy (3.4) with truncated initial data $\Gamma^K_0 = P_{\leq K} \Gamma_0$ constructed in [13], where $0 < \xi' < 1$ and $\xi$ satisfy (4.5), and $0 < T < T_0(\xi')$ (see (4.2)). Let $\Gamma^K_N$ solve the $(K, N)$-BBGKY hierarchy (3.10) with the same initial data $\Gamma^K_{0,N} := P_{\leq K} \Gamma_0$. Let

$$K(N) \leq b_1 \log N,$$  

(4.8)

as in Lemma 4.1. Then,

$$\lim_{N \to \infty} \| \Gamma^{(N)}_N - \Gamma^{(N)}_\infty \|_{L^\infty_{([0, \infty], \mathbb{R}^3)}} = 0,$$

(4.9)

and

$$\lim_{N \to \infty} \| B_N \Gamma^{(N)}_N - B \Gamma^{(N)}_\infty \|_{L^2_{([0, \infty], \mathbb{R}^3)}} = 0.$$  

(4.10)
Proof. In [13], the authors constructed a solution $\Gamma^K$ of the full GP hierarchy with truncated initial data, $\Gamma(0) = \Gamma^K_0 \in \mathcal{W}_\xi^{1}$, such that for an arbitrary fixed $K$, $\Gamma^K$ satisfies the GP hierarchy in integral representation,

$$\Gamma^K(t) = U(t)\Gamma^K_0 + i \int_0^t U(t-s) B\Gamma^K(s) \, ds. \quad (4.11)$$

and, in particular, $(\Gamma^K)^{(k)}(t) = 0$ for all $k > K$.

Accordingly, we have

$$B_N \Gamma^K_N - B\Gamma^K = B_N U(t)\Gamma^K_{0,N} - BU(t)\Gamma^K_0$$

$$+ i \int_0^t (B_N U(t-s) B_N \Gamma^K_N - BU(t-s) B\Gamma^K)(s) \, ds$$

$$= (B_N - B)U(t)\Gamma^K_{0,N} + BU(t)(\Gamma^K_{0,N} - \Gamma^K_0)$$

$$+ i \int_0^t (B_N - B)U(t-s) B\Gamma^K(s) \, ds$$

$$+ i \int_0^t B_N U(t-s)(B_N \Gamma^K_N - B\Gamma^K)(s) \, ds. \quad (4.12)$$

Here, we observe that we can apply Lemma D.2 with

$$\tilde{\Theta}^K_N := B_N \Gamma^K_N - B\Gamma^K$$

and

$$\Xi^K_N := (B_N - B)U(t)\Gamma^K_{0,N} + BU(t)(\Gamma^K_{0,N} - \Gamma^K_0)$$

$$+ i \int_0^t (B_N - B)U(t-s) B\Gamma^K(s) \, ds. \quad (4.14)$$

Given $\xi'$, we introduce parameters $\xi, \xi^n, \tilde{\zeta}''$ satisfying

$$\xi < \theta \xi'' < \theta^2 \xi'' < \theta^3 \xi'' \quad (4.15)$$

where the constant $\theta$ is defined as in (4.5), so that $0 < \theta \leq \eta$, where $\eta$ is defined as in Lemma D.2. Accordingly, Lemma D.2 implies that

$$\|B_N \Gamma^K_N - B\Gamma^K\|_{L^2_{N,0,1}\tilde{\zeta}''}$$

$$\leq C_0(T, \xi, \xi'', \|BU(t)(\Gamma^K_{0,N} - \Gamma^K_0)\|_{L^2_{N,0,1}\tilde{\zeta}''} + R_1(N) + R_2(N))$$

$$\leq C_1(T, \xi, \xi^n, \xi'', \|\Gamma^K_{0,N} - \Gamma^K_0\|_{L^2_{N,0,1}\tilde{\zeta}''} + R_1(N) + R_2(N)), \quad (4.16)$$

where we used Lemma A.1 in [15] to pass to the last line. Here,

$$R_1(N) := \|B_N - B)U(t)\Gamma^K_{0,N}\|_{L^2_{N,0,1}\tilde{\zeta}''}$$

and

$$R_2(N) := \left\| \int_0^t (B_N - B)U(t-s) B\Gamma^K(s) \, ds \right\|_{L^2_{N,0,1}\tilde{\zeta}''}. \quad (4.17)$$

$$R_2(N) := \left\| \int_0^t (B_N - B)U(t-s) B\Gamma^K(s) \, ds \right\|_{L^2_{N,0,1}\tilde{\zeta}''}. \quad (4.18)$$
Next, we consider the limit $N \to \infty$ with $K(N)$ as given in (4.8).

We have

\[
\lim_{N \to \infty} \| \Gamma_{0,N}^{K(N)} - \Gamma_0^{K(N)} \|_{\mathcal{H}_1^2} = \lim_{N \to \infty} \| P_{\leq K(N)}( \Gamma_{0,N} - \Gamma_0 ) \|_{\mathcal{H}_1^2} \\
\leq \lim_{N \to \infty} \| \Gamma_{0,N} - \Gamma_0 \|_{\mathcal{H}_1^2} = 0.
\]

(4.19)

By Lemmas 4.3 and 4.4 below, we have that

\[
\lim_{N \to \infty} R_1(N) = 0
\]

and

\[
\lim_{N \to \infty} R_2(N) = 0.
\]

Thus (4.16) $\to 0$ as $N \to \infty$, and hence the limit (4.10) holds. To prove (4.9), we observe that

\[
\Gamma_N^{K(N)}(t) - \Gamma^{K(N)}(t) = U(t)(\Gamma_N^{K(N)}(0) - \Gamma^{K(N)}(0)) \\
+ i \int_0^t U(t-s)(B_N \Gamma_N^{K(N)}(s) - B \Gamma^{K(N)}(s)) ds,
\]

and hence, for $0 < t < T$,

\[
\| \Gamma_N^{K(N)}(t) - \Gamma^{K(N)}(t) \|_{\mathcal{H}_1^2} \\
\leq \| U(t)(\Gamma_N^{K(N)}(0) - \Gamma^{K(N)}(0)) \|_{\mathcal{H}_1^2} \\
+ t^{1/2} \| U(t-s)(B_N \Gamma_N^{K(N)}(s) - B \Gamma^{K(N)}(s)) \|_{L_{\mathcal{H}_1^2}^2} \\
\leq \| \Gamma_N^{K(N)}(0) + \Gamma^{K(N)}(0) \|_{\mathcal{H}_1^2} + t^{1/2} \| B_N \Gamma_N^{K(N)} - B \Gamma^{K(N)} \|_{L_{\mathcal{H}_1^2}^2} \\
\leq \| \Gamma_N^{K(N)}(0) + \Gamma^{K(N)}(0) \|_{\mathcal{H}_1^2} + T^{1/2} \| B_N \Gamma_N^{K(N)} - B \Gamma^{K(N)} \|_{L_{\mathcal{H}_1^2}^2} \\
\to 0 \\text{as } N \to \infty \text{ by (4.10)}.
\]

(4.20)

Since the last line (4.20) is independent of $t$, the result (4.9) follows. \qed

**Lemma 4.3.** Under the same assumptions as in Proposition 4.2,

\[
\lim_{N \to \infty} \| (B_N - B) U(t) \Gamma_{0,N}^{K(N)} \|_{L_{\mathcal{H}_1^2}^2} = 0.
\]

(4.21)

**Proof.** We recall that for $g : \mathbb{R}^n \to \mathbb{C}$ of the form $g(x) = f(x,x)$ for some Schwartz class function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, one has

\[
\hat{g}(\xi) = \int \hat{f}(\xi - \eta, \eta) d\eta.
\]

(4.22)
We note that the Fourier transform of \( (U^{(k+1)}(t))_{0}^{(k+1)}(\bar{x}_{k+1}', \bar{y}_{k+1}') \) with respect to the variables \( (t, \bar{x}_{k+1}', \bar{y}_{k+1}') \) is given by

\[
\delta(\tau + |\bar{x}_{k+1}'|^2 - |\bar{y}_{k+1}'|^2)_{0}^{(k+1)}(\bar{x}_{k+1}', \bar{y}_{k+1}').
\] (4.23)

Recall that \( B_{N,1}^{+} U(t)_{0}^{(k+1)} \) is given by

\[
\int V_N(x_1 - x_{k+1})_{0}^{(k+1)}(t, x_1, \ldots, x_{k}, x_{k+1}; x'_1, \ldots, x'_{k}, x_{k+1}) dx_{k+1}
\]

and hence its Fourier transform with respect to the variables \( (t, \bar{x}_k, \bar{y}_k) \) is given by

\[
\int e^{-i\epsilon_k + i\eta_k} \hat{V}_N(u_1) * u_1 (F_{0}^{(k+1)})(\tau, u_1, \ldots, u_k, x_{k+1}; u'_1, \ldots, u'_{k}, x_{k+1}) dx_{k+1}
\]

where we substituted \( \eta \rightarrow \eta + v \). Thus, the above equals

\[
= \int \hat{V}_N(u_{k+1} + u'_{k+1}) F_{0}^{(k+1)}(\tau, u_1 - u_{k+1} - u'_{k+1}, u_2, \ldots, u_k, x_{k+1}; u'_1, \ldots, u'_{k}, x_{k+1}) du_{k+1} du'_{k+1}
\]

where the operator \( F \) is the Fourier transform with respect to the variables \( (t, \bar{x}_k, \bar{y}_k) \) and

\[
\delta(\ldots) := \delta(\tau + |u_1 - u_{k+1} - u'_{k+1}|^2 + |u_{k+1}|^2 - |u_1|^2 - |u'_{k+1}|^2).
\]

Equation (4.23) was used to pass to the last line (4.24). Similarly, the Fourier transform of \( B_{N,1}^{+} U(t)_{0}^{(k+1)} \) with respect to the variables \( (t, \bar{x}_k, \bar{y}_k) \) is given by

\[
\int \delta(\ldots) F_{0}^{(k+1)}(u_1 - u_{k+1} - u'_{k+1}, u_2, \ldots, u_k, u'_{k+1}) du_{k+1} du'_{k+1}.
\]

Thus,

\[
\| (B_{N,1}^{+} - B_{1}^{+}) U(t)_{0}^{(k+1)} \|_{L_{t}^{2} H}\]

\[
= \int \int \cdots \int \prod_{j=1}^{k} (u_j)_{0}^{2} \prod_{j=1}^{k} (u'_j)_{0}^{2} \left( \int (1 - \hat{V}_N(u_{k+1} + u'_{k+1})) \delta(\ldots) \right)
\]

\[
= \gamma_{0,N}^{(k+1)} (u_1 - u_{k+1} - u'_{k+1}, u_2, \ldots, u_k, u_{k+1}; u'_{k+1}) du_{k+1} du'_{k+1}.
\]
\[
\leq \int \int \int J(\tau, \mathbf{u}_k, \mathbf{u}'_k) \int \delta(\ldots)(u_1 - u_{k+1} - u'_{k+1})^2 (u_{k+1})^2 (u'_{k+1})^2 \\
\sum_{j=2}^{k} \prod_{i=1}^{k} (u_i)^2 \prod_{j=1}^{k} (u'_j)^2 |1 - \widehat{V}_\lambda(u_{k+1} + u'_{k+1})|^2 \\
\left[ \frac{1}{\lambda} (\tau) \left( u_1 - u_{k+1} - u'_{k+1}, u_2, \ldots, u_k, u_{k+1}, u'_{k+1} \right) \right]^2 \\
du'_{k+1} \, du_{k+1} \, du \, d\tau
\] (4.25)

where

\[
J(\tau, \mathbf{u}_k, \mathbf{u}'_k) := \int \int \int \frac{\delta(\ldots)(u_1)^2}{(u_1 - u_{k+1} - u'_{k+1})^2 (u_{k+1})^2 (u'_{k+1})^2} \, du_{k+1} \, du'_{k+1}
\]

and \( J(\tau, \mathbf{u}_k, \mathbf{u}'_k) \) is bounded uniformly in \( \tau, \mathbf{u}_k, \mathbf{u}'_k \), see Proposition 2.1 of [40].

Let \( \delta \) satisfy \( 0 < \delta < \beta \). Recall that \( \widehat{V}_\lambda(u) = \widehat{V}(N^{-\beta}u) \). The integral (4.25) can now be separated into the regions \( \{|u_{k+1} + u'_{k+1}| < N^\delta\} \) and \( \{|u_{k+1} + u'_{k+1}| \geq N^\delta\} \).

The portion of the integral (4.25) over \( \{|u_{k+1} + u'_{k+1}| < N^\delta\} \) is bounded by

\[
C_V N^{4(\delta-\beta)} \|v^{(k+1)}\|_{L^2}\|\mu^0\| \tag{4.26}
\]

because \( \nabla \widehat{V}(0) = 0 \) and \( \widehat{V} \in C^2 \), so by bounding the Taylor remainder term,

\[
\sup_{|u_{k+1} + u'_{k+1}| < N^\delta} |1 - \widehat{V}_\lambda(u_{k+1} + u'_{k+1})|^2 = \sup_{|u_{k+1} + u'_{k+1}| < N^\delta} |1 - \widehat{V}(N^{-\beta}(u_{k+1} + u'_{k+1}))|^2 \\
\leq \sup_{|u_{k+1} + u'_{k+1}| < N^\delta} C_V (N^{-\beta}(u_{k+1} + u'_{k+1}))^4 \\
\leq C_V N^{4(\delta-\beta)},
\]

where \( C_V \) is the \( L^\infty \) norm of the second derivative of \( V \).

The portion of the integral (4.25) over \( \{|u_{k+1} + u'_{k+1}| \geq N^\delta\} \) is bounded by

\[
a^2_{\lambda, N} := \int \int \int J(\tau, \mathbf{u}_k, \mathbf{u}'_k) \int_{|u_{k+1}| \geq N^\delta} \int \delta(\ldots)(u_1 - u_{k+1} - u'_{k+1})^2 \\
\sum_{j=2}^{k} \prod_{i=1}^{k} (u_i)^2 \prod_{j=1}^{k} (u'_j)^2 (1 + \|\nabla \widehat{V}\|_{L^\infty})^2 \\
\left[ \frac{1}{\lambda} (\tau) \left( u_1 - u_{k+1} - u'_{k+1}, u_2, \ldots, u_k, u_{k+1}, u'_{k+1} \right) \right]^2 \\
du'_{k+1} \, du_{k+1} \, du \, d\tau
\] (4.27)

\[
\leq C \|v^{(k+1)}\|_{L^2}^2 \|\mu^0\| \tag{4.28}
\]

We are now ready to bound the desired quantity (4.21) in the statement of the lemma.
Let $\Omega_{k,N} = \{|u_{k+1} + u'_{k+1}| < N^\delta\}$. Then,

$$\| (B^+_N - B^+) U(t) \Gamma_{0,N}^k \|_{L^2_{\mathbb{R}}\times\mathbb{R}^+}^2 - \| B^+_{\text{error}} U(t) \Gamma_{0,N}^k \|_{L^2_{\mathbb{R}}\times\mathbb{R}^+}^2 \to 0 \quad \text{as } N \to \infty \quad \text{by Proposition A.2 in [15]}$$

$$\leq \sum_{k=1}^K \sum_{j=1}^k (\xi')^k \| (B_{N,1:k+1}^+ - B_{k+1}^+) U^{(k+1)}(t) \gamma_{0,N}^{(k+1)} \|_{L^2_{\mathbb{R}}H^1}^2$$

$$\leq \sum_{k=1}^K k(\xi')^k \| (B_{N,1:k+1}^+ - B_{k+1}^+) U^{(k+1)}(t) \gamma_{0,N}^{(k+1)} \|_{L^2_{\mathbb{R}}H^1}^2$$

$$\leq \left( \sup_k k(\xi'/\xi')^k \right) \left( \sum_{k=1}^K (\xi')^k \| (B_{N,1:k+1}^+ - B_{k+1}^+) U^{(k+1)}(t) \gamma_{0,N}^{(k+1)} \|_{L^2_{\mathbb{R}}H^1}^2 \right)$$

$$\leq \left( \sup_k k(\xi'/\xi')^k \right) \left( C_0 (1 + \| \hat{V} \|_{\infty}) \sum_{k=1}^K (\xi')^k N^{2(\delta - \beta)} \| \gamma_{0,N}^{(k+1)} \|_{H^1} + \sum_{k=1}^K (\xi')^k a_{k,N} \right).$$

(4.29)

where (4.26) and (4.28) were used to pass to the last line (4.29).

Now, for $(k, N) \in \mathbb{N} \times \mathbb{N}$, we define

$$\tilde{a}_{k,N}^2 := \iint J(\tau, \nu, \nu') \int_{|u_{k+1} + u'_{k+1}| \geq N^\delta} \int \delta(\ldots, u_1 - u_{k+1} - u'_{k+1})^2 (u_{k+1})^2 (u'_{k+1})^2 \prod_{j=2}^k (u_j)^2 \prod_{j=1}^k (u'_j)^2 (1 + \| \hat{V} \|_{\infty})^2 \left| \int_{l=0}^{(k+1)} (u_1 - u_{k+1} - u'_{k+1}, u_2, \ldots, u_k, u_{k+1}; u'_{k+1}) \right|^2 du_{k+1} du_{k+1} du \, d\tau,$$

(4.30)

and observe that $\tilde{a}_{k,N} = a_{k,N}$ (as defined in (4.27)), for $k \leq N$, because $\gamma_{0,N}^{(k)} = \gamma_0^{(k)}$ for $k \leq N$. Thus, we have that

$$\text{(4.29)} \leq \left( \sup_k k(\xi'/\xi')^k \right) \left( C_0 (1 + \| \hat{V} \|_{\infty}) \sum_{k=1}^\infty (\xi')^k N^{2(\delta - \beta)} \| \gamma_0^{(k+1)} \|_{H^1} + \sum_{k=1}^\infty (\xi')^k a_{k,N} \right).$$

(4.31)
It follows from the definition (4.30) of $a_{k,N}^2$ that $\sum_{k=1}^{\infty} (\xi^2)^k a_{k,N} \leq C \| \Gamma_0 \|_{L^2_{\xi}}^2$ and that, for fixed $k$, $a_{k,N} \searrow 0$ monotonically as $N \to \infty$. This is because $a_{k,N}^2$ is an integral where the integrand is independent of $N$ and the region of integration shrinks as $N$ grows. Thus, by the monotone convergence theorem, $\sum_{k=1}^{\infty} (\xi^2)^k a_{k,N} \searrow 0$ as $N \to \infty$. Therefore (4.31) $\to 0$ as $N \to \infty$. $\square$

**Lemma 4.4.** Under the same assumptions as in Proposition 4.2,  
\[
\lim_{N \to \infty} \| \int_0^t (B_N - B) U(t-s) \Gamma_k^N(s) \, ds \|_{L^2_{\xi}} = 0.
\]

**Proof.** We have that  
\[
\left\| \int_0^t (B_N - B) U(t-s) \Gamma_k^N(s) \, ds \right\|_{L^2_{\xi}} \leq \int_0^T \| (B_N - B) U(t-s) \Gamma_k^N(s) \|_{L^2_{\xi}} \, ds.
\]

By the same arguments as in the proof of Lemma 4.3 above, the integral above goes to zero as $N \to \infty$ provided that $\| U(t-s) \Gamma_k^N(s) \|_{L^2_{\xi}}$ is uniformly bounded in $N$. See [13] for a proof of the boundedness of $\| U(t-s) \Gamma_k^N(s) \|_{L^2_{\xi}}$.

**4.4. Control of $\Gamma_k^N$ and $\Gamma_k^N$ as $N \to \infty$**

We begin by stating an energy estimate used by Erdös, Schlein, and Yau in [25]. We define the notation $R_{\psi} := \prod_{j=1}^{\infty} (1 - \Delta_j)^{\alpha/2}$.

**Proposition 4.5.** Suppose that $\psi$ is symmetric with respect to permutations of its $N$ variables. Fix $k \in \mathbb{N}$ and $0 < C < 1$. Then there is $N_0 = N_0(k, C)$ such that  
\[
\langle \psi, (H_N + N)^N \psi \rangle \geq C^N N^k \langle \psi, R^{(k, 2)} \psi \rangle
\]
for all $N > N_0$.

**Proposition 4.6.** Suppose that $b_1 > 0$, $b_1 \log(N) \geq K(N) \geq \frac{1}{2} b_1 \log(N)$, and that $\zeta > 0$ satisfies  
\[
\zeta < \eta \min \left\{ \frac{1}{C} e^{-8\beta/b_1}, e^{-24\beta/b_1} \right\},
\]
where  
\[
\text{Tr} S^{(1, 1)}_{\Gamma^N_k}(0) < C^k.
\]

Then  
\[
\lim_{N \to \infty} \| B_N \Gamma^N_k - P_{\leq K(N) - 1} B_N \Gamma^b \|_{L^2_{\xi}} = 0.
\]
Proof. From Lemma 6.1 in [15], we have that
\[ \| B_N \Gamma^k_N - P_{0, K-1} B_N \Gamma^{\Phi_N} \|_{L^2_{\xi, t}} \leq C(T, \zeta)(\eta^{-1} \zeta)^k K \| (B_N \Gamma^{\Phi_N})^{(K)} \|_{L^2_{\xi, t}} \]  \hspace{1cm} (4.34)
holds for a finite constant $C(T, \zeta)$ independent of $K, N$.

It follows immediately from the definition of $V_N$ that
\[ \| \nabla V_N \|_{L^1} \leq CN^{4\beta}. \]

Thus, we have that
\[ \| (B_N \Gamma^{\Phi_N})^{(K)} \|_{L^2_{\xi, t}} \]
\[ = \int dt \int d\Sigma_{K} d\Sigma_{K'} \left[ \sum_{l=1}^{K} \left[ \prod_{m=1}^{l} \langle \nabla x_m \rangle \langle \nabla x'_m \rangle \right] V_N(x_l - x_{K+1}) \Phi_N(t, \Sigma_{K}, x_{K+1}, \ldots, x_{N}) dx_{K+1} \ldots dx_N \right]^2 \]
\[ \leq C(T \| V_N \|_{L^\infty} + \| \nabla V_N \|_{L^1}) K^2 \sup \left( \| R^{(K, 1)} \Phi_N \|_{L^2} \| R^{(K, 1)} \Phi_N \|_{L^2} \right)^2 \]
\[ = CNT^{4\beta} K^2 \sup \left( \text{Tr}(S^{(K, 1), \gamma_N}\langle \gamma_N \rangle(t)) \right). \hspace{1cm} (4.35) \]

Since $\langle \Phi_N(0), H^{\xi}_N, \Phi_N(0) \rangle < C^k N^K$, it follows from Proposition 4.5, that
\[ \text{Tr}(S^{(K, 1), \gamma_N}\langle \gamma_N \rangle(t)) = \langle \Phi_N(t), R^{(K, 2)} \Phi_N(t) \rangle \]
\[ \leq \frac{1}{N^k C^k} \langle \Phi_N(t), (H_N + N^k \Phi_N(t) \rangle \]
\[ = \frac{1}{N^k C^k} \langle \Phi_N(0), (H_N + N^k \Phi_N(0) \rangle \]
\[ \leq \frac{1}{N^k C^k} (2^k \langle \Phi_N(0), H^k \Phi_N(0) \rangle + 2N^k \langle \Phi_N(0), \Phi_N(0) \rangle) \]
\[ \leq C^k. \hspace{1cm} (4.36) \]

Combining (4.34), (4.35), and (4.36) yields
\[ \| B_N \Gamma^k_N - P_{0, K-1} B_N \Gamma^{\Phi_N} \|_{L^2_{\xi, t}} \]
\[ \leq C(T, \zeta)(\eta^{-1} \zeta)^k K \| (B_N \Gamma^{\Phi_N})^{(K)} \|_{L^2_{\xi, t}} \]  by (4.34)
\[ \leq C(T, \zeta)(\eta^{-1} \zeta)^k KCT^{1/2} N^{4\beta} K \sup \text{Tr}(S^{(K, 1), \gamma_N}\langle \gamma_N \rangle(t)) \]  by (4.35)
\[ \leq C(T, \zeta)(\eta^{-1} \zeta)^k KCT^{1/2} N^{4\beta} C^K \]  by (4.36)
\[ \leq C(T, \zeta)(\eta^{-1} \zeta)^k KN^{4\beta} C^K \]
\[ \rightarrow 0 \text{ as } N \rightarrow \infty \]

because $K(N) \geq \frac{1}{2} b_1 \log(N)$ and $\zeta$ satisfies (4.32). \qed
4.5. **Proof of Theorem 2.1**

We are now ready to conclude the proof of Theorem 2.1. To this end, we recall again the solution $\Gamma^K$ of the GP hierarchy with truncated initial data, $\Gamma^K(t = 0) = P_{\leq K} \Gamma_0 \in \mathcal{H}_1^1$. In [13], the authors proved the existence of a solution $\Gamma^K$ that satisfies the $K$-truncated GP hierarchy in integral form,

$$\Gamma^K(t) = U(t) \Gamma^K(0) + i \int_0^t U(t - s) BT\Gamma^K(s) \, ds$$

(4.37)

where $(\Gamma^K)^{(k)}(t) \equiv 0$ for all $k > K$. Moreover, it is shown in [13] that this solution satisfies $BT\Gamma^K \in L^2_{t \in I} \mathcal{H}_1^1$, where $I := [0, T]$.

Additionally, the following convergence was proved in [13]:

(a) The limit

$$\Gamma := \lim_{K \to \infty} \Gamma^K$$

exists in $L^\infty \mathcal{H}_1^1$.

(b) The limit

$$\Theta := \lim_{K \to \infty} BT\Gamma^K$$

exists in $L^2 \mathcal{H}_1^1$, and in particular,

$$\Theta = BT\Gamma.$$  

(4.40)

(c) The limit $\Gamma$ in equation (4.38) satisfies the full GP hierarchy with initial data $\Gamma_0$.

Clearly, we have that

$$\|BT - B_N P_{\leq K(N)} \Gamma^K\|_{L^2_{t \in I} \mathcal{H}_1^1}$$

$$\leq \|BT - BT \Gamma^K\|_{L^2_{t \in I} \mathcal{H}_1^1}$$

$$+ \|BT \Gamma^K - B_N \Gamma^K_N\|_{L^2_{t \in I} \mathcal{H}_1^1}$$

$$+ \|BT \Gamma^K_N - B_N P_{\leq K(N)} \Gamma^K\|_{L^2_{t \in I} \mathcal{H}_1^1}.$$  

(4.41)

(4.42)

(4.43)

In the limit $N \to \infty$, we have that (4.41) → 0 from (4.39) and (4.40). By Proposition 4.2, (4.42) → 0. (4.43) → 0 follows from Proposition 4.6. This is because $\Gamma_0 \in \mathcal{S}_1^1$, and hence (4.33) holds. Therefore,

$$\lim_{N \to \infty} \|BT - B_N \Gamma^K\|_{L^2_{t \in I} \mathcal{H}_1^1} = 0.$$  

Moreover, we have that

$$\|P_{\leq K(N)} \Gamma^K\|_{L^\infty_{t \in I} \mathcal{H}_1^1}$$

$$\leq \|P_{\leq K(N)} \Gamma^K\|_{L^\infty_{t \in I} \mathcal{H}_1^1}$$

$$+ \|\Gamma^K\|_{L^\infty_{t \in I} \mathcal{H}_1^1}$$

$$+ \|\Gamma^K_N - \Gamma^K\|_{L^\infty_{t \in I} \mathcal{H}_1^1}.$$  

(4.44)

(4.45)

(4.46)
By the Duhamel formula, and applying the Cauchy-Schwarz inequality in time, we have

$$
(4.44) = \| \int_0^t U(t-s) B_N(P_{\leq K(N)} \Gamma^{F_N} - \Gamma^N_N) (s) \, ds \|_{L^\infty_{t} L^1_r}^2
\leq T^{1/2} \| B_N \Gamma^N_N - B_N P_{\leq K(N)} \Gamma^{F_N} \|_{L^2_{t} L^1_r}^2
\to 0 \text{ as } N \to \infty \text{ by Proposition 4.6.}
$$

(4.45) \to 0 \text{ as } N \to \infty \text{ by (4.38). Finally, (4.46) \to 0 \text{ as } N \to \infty \text{ follows from Proposition 4.2. Thus}

$$
\lim_{N \to \infty} \| P_{\leq K(N)} \Gamma^{F_N} - \Gamma \|_{L^\infty_{t} L^1_r}^2 = 0.
$$

This completes the proof of Theorem 2.1. \qed

5. Global Well-Posedness

In this section, we prove Theorem 2.3. To this end, we first prove positive semidefiniteness of solutions to the GP hierarchy in Theorem 5.2, below, and subsequently global well posedness of the GP hierarchy in Theorem 5.4.

To prove positive semidefiniteness, we recall the quantum de Finetti theorem, Theorem 2.2, and we invoke the following lemma from [10].

**Lemma 5.1.** Let \( \mu \) be a Borel probability measure in \( L^2(\mathbb{R}^3) \), and assume that

$$
\int d\mu(\phi) \| \phi \|_{\|_{\mu}}^2 \leq M^{2k} \tag{5.1}
$$

holds for some finite constant \( M > 0 \), and all \( k \in \mathbb{N} \). Then,

$$
\mu(\{ \phi \in L^2(\mathbb{R}^3) \mid \| \phi \|_{\|_{\mu}} > M \}) = 0.
$$

**Proof.** From Chebyshev’s inequality, we have that

$$
\mu(\{ \phi \in L^2(\mathbb{R}^3) \mid \| \phi \|_{\|_{\mu}} > \lambda \}) \leq \frac{1}{\lambda^{2k}} \int d\mu(\phi) \| \phi \|_{\|_{\mu}}^{2k} \leq \frac{M^{2k}}{\lambda^{2k}}
$$

for any \( k > 0 \). For \( \lambda > M \), the right hand side tends to zero when \( k \to \infty \). \qed

We recall that, for \( I \subseteq \mathbb{R} \),

$$
\mathcal{W}^2_\xi(I) = \{ \Gamma \in C(I, \mathcal{H}^2_\xi) \mid B^* \Gamma, B^* \Gamma \in L^2_{loc}(I, \mathcal{H}^2_\xi) \}.
$$

We are now ready to prove positive semidefiniteness of solutions to the GP hierarchy.

**Theorem 5.2.** Assume that

$$
\gamma^{(k)}_{10} = \int d\mu(\phi) (|\phi \rangle \langle \phi|)^{\otimes k}, \quad k \in \mathbb{N} \tag{5.2}
$$
Proposition 5.3  

In [14], it is shown that these higher order energy functionals are conserved: 

where $d\mu$ is a probability measure supported either on the unit sphere, or on the unit ball in $L^2(\mathbb{R}^3)$. Then, for $0 < \xi' < 1$ and $\xi > 0$ satisfying (4.5), and for $0 < T < \min\{T_0(\xi), T_1(\xi)\}$ (see (4.2) and (5.13)), there is a unique solution $\Gamma \in \mathcal{W}^\ell_t([0, T])$ to the cubic defocusing GP hierarchy (3.4) in $\mathbb{R}^3$ with initial data $\Gamma_0$. Moreover, $\Gamma(t)$ is positive semidefinite for $t \in [0, T]$.

Proof. By [12] and Proposition C.3, there exists a unique solution $\Gamma$ to the GP hierarchy in $\mathcal{W}^\ell_t([0, T])$ with initial data $\Gamma_0$.

By the quantum de Finetti theorem (Theorem 2.2) and Lemma 5.1, there exists a positive semidefinite Borel probability measure $\mu$ on the unit sphere in $L^2(\mathbb{R}^3)$ such that

$$\tilde{\gamma}^{(k)}_0 = \int d\mu(\phi) (|\phi\rangle \langle \phi|)^{\otimes k}$$

and $\|\phi\|_{\mu}^2 \leq (\xi')^{-1} \|\Gamma_0\|_{\mu}$ $\mu$-almost everywhere. Let $S_t$ be the flow map of the cubic defocusing NLS. Since the NLS is well-posed in $H^1$,

$$\tilde{\gamma}^{(k)}(t) := \int d\mu(\phi) (|S_t \phi\rangle \langle S_t \phi|)^{\otimes k}$$

is well-defined, positive semidefinite, and $\tilde{\Gamma} := \{\tilde{\gamma}^{(k)}\}_{k=1}^\infty$ satisfies the cubic defocusing GP hierarchy.

Moreover, we claim that $\tilde{\Gamma} \in \mathcal{W}^\ell_t([0, T])$. To prove this fact, let $\langle \mathcal{E}^{(m)}(\Gamma(t)) \rangle_{\Gamma(t)}$, $m \in \mathbb{N}$, denote the higher order energy functionals for the cubic GP hierarchy introduced in [14]. They are given by

$$\langle K^{(m)}(\Gamma(t)) \rangle_{\Gamma(t)} := \text{Tr}_{1,3,5,\ldots,2m+1}(K^{(m)}(\Gamma(t)(2m))(t))$$

for $m \in \mathbb{N}$, where

$$K_\ell := \frac{1}{2}(1 - \Delta_v)\text{Tr}_{\ell+1}^\perp + \frac{1}{4}B_{\ell,\ell+1}^+, \quad \ell \in \mathbb{N},$$

$$K^{(m)} := K_1 K_3 \cdots K_{2m-1}.$$ 

In [14], it is shown that these higher order energy functionals are conserved:

Proposition 5.3 (C-Pavlović [14]). Suppose that $\Gamma \in \mathcal{S}^\ell_t$ is symmetric, admissible, and solves the GP hierarchy. Then, for all $m \in \mathbb{N}$, the higher order energy functionals (5.5) are bounded and conserved, $\langle K^{(m)}(\Gamma(t)) \rangle_{\Gamma(t)} = \langle K^{(m)}(\Gamma(0)) \rangle_{\Gamma(0)}$.

Using the de Finetti theorem, we can eliminate the requirement of admissibility. We write

$$E[\phi] := \frac{1}{2}\|\phi\|_{\mu}^2, \quad \|\phi\|_{L^2}^2 = \frac{1}{4}\|\phi\|_{L^4}^4$$

for the conserved energy of the solution of the NLS. Then, it can be easily checked that

$$\langle \mathcal{E}^{(m)}(\tilde{\Gamma}(t)) \rangle_{\tilde{\Gamma}(t)} = \int d\mu(\phi) \left( \frac{1}{2} + E[S_t \phi] \right)^m.$$
Moreover, with $T > 0$ as in (5.13), below; see for instance [40] or [8] for details. Moreover,

$$
\| \langle \nabla \rangle S_\phi \|_{L^1_{[0,T]}(R^3)} \leq \sup_{t \in [0,\bar{T}]} \sqrt{1 + 2E[S_\phi]} = \sqrt{1 + 2E[\phi]}
$$

with $T > 0$ as in (5.13), below; see for instance [40] or [8] for details. Moreover,

$$
\langle \nabla \rangle S_\phi \|_{L^1_{[0,T]}(R^3)} \leq \sup_{t \in [0,\bar{T}]} \sqrt{1 + 2E[S_\phi]} = \sqrt{1 + 2E[\phi]} \tag{5.11}
$$

We then obtain that

$$
(5.9) \leq C \sum_{k=1}^{\infty} (2\tilde{\zeta})^k \int d\mu(\phi) \left( \frac{1}{2} + E[\phi] \right)^{k+1}
$$

$$
= C \sum_{k=2}^{\infty} (2\tilde{\zeta})^k \langle \mathcal{H}^{(k)} \rangle \chi_0
$$

We have that the sequence of higher energy functionals $\langle \mathcal{H}^{(m)} \rangle$, for $m \in \mathbb{N}$, satisfies

$$
\| \Gamma(t) \|_{\delta^2} \leq \sum_{m \in \mathbb{N}} (2\tilde{\zeta})^m \langle \mathcal{H}^{(m)} \rangle \chi_0
$$

$$
= \sum_{m \in \mathbb{N}} (2\tilde{\zeta})^m \langle \mathcal{H}^{(m)} \rangle \chi_0
$$

$$
\leq \| \Gamma(0) \|_{\delta^2},
$$

by Theorem 6.2 in [14]. As a consequence, we find that

$$
\| \tilde{\Gamma}(t) \|_{\delta^2} \leq \| \tilde{\Gamma}(0) \|_{\delta^2}
$$

$$
= \sum_{m \in \mathbb{N}} (2\tilde{\zeta})^m \langle \mathcal{H}^{(m)} \rangle \chi_0
$$

$$
\leq \| \Gamma(0) \|_{\delta^2}
$$

$$
< \infty.
$$

Moreover,

$$
\| B \tilde{\Gamma} \|_{L^1_{[0,T]}H^1}
$$

$$
\leq \sum_{k=1}^{\infty} (\tilde{\zeta})^k \int d\mu(\phi) \| \langle \nabla \rangle S_\phi \|^2_{L^1_{[0,T]}L^2(R^3)} \| \langle \nabla \rangle S_\phi \|_{L^1_{[0,T]}L^2(R^3)}
$$

$$
\leq \sum_{k=1}^{\infty} (\tilde{\zeta})^k \int d\mu(\phi) \| \langle \nabla \rangle S_\phi \|^2_{L^1_{[0,T]}L^2(R^3)} \| \langle \nabla \rangle S_\phi \|_{L^1_{[0,T]}L^2(R^3)}
$$

$$
\leq \sum_{k=1}^{\infty} (\tilde{\zeta})^k \int d\mu(\phi) \| \langle \nabla \rangle S_\phi \|^2_{L^1_{[0,T]}L^2(R^3)} \| \langle \nabla \rangle S_\phi \|_{L^1_{[0,T]}L^2(R^3)}
$$

Here, we use the bound

$$
\| \langle \nabla \rangle S_\phi \|_{L^1_{[0,T]}L^2(R^3)} \leq C(T) \| \langle \nabla \rangle \phi \|_{L^1} \leq C(T) \sqrt{1 + 2E[\phi]}, \tag{5.10}
$$

with $T > 0$ as in (5.13), below; see for instance [40] or [8] for details. Moreover,

$$
\| \langle \nabla \rangle S_\phi \|_{L^1_{[0,T]}L^2(R^3)} \leq \sup_{t \in [0,\bar{T}]} \sqrt{1 + 2E[S_\phi]} = \sqrt{1 + 2E[\phi]} \tag{5.11}
$$

We then obtain that

$$
(5.9) \leq C \sum_{k=1}^{\infty} (2\tilde{\zeta})^k \int d\mu(\phi) \left( \frac{1}{2} + E[\phi] \right)^{k+1}
$$

$$
= C \sum_{k=2}^{\infty} (2\tilde{\zeta})^k \langle \mathcal{H}^{(k)} \rangle \chi_0
$$
Derivation and GWP of the GP Hierarchy

\[ \leq C\xi^{-1} \| \Gamma_0 \|_{\mathcal{H}_1^1} \]

\[ < \infty. \]

(5.12)

Finally, we pick \( T_1(\xi) > 0 \) sufficiently small that (5.10) above holds for

\[ 0 < T < T_1(\xi), \]

noting that the constant \( C(T) \) in (5.10) depends on \( \| \phi \|_{H^1} < (\xi')^{-1/2} \| \Gamma_0 \|_{\mathcal{H}_1^1} \) and thus on \( \xi \), where \( \xi \) and \( \xi' \) are related as in (4.5).

Thus, we have shown that \( \Gamma \in \mathcal{W}_1^1([0, T]). \) By uniqueness of solutions to the GP hierarchy in \( \mathcal{W}_1^1([0, T]), \) we conclude that \( \Gamma = \tilde{\Gamma}. \)

In particular, we note that \( \Gamma(t) \) is positive semidefinite for \( t \in [0, T]. \)

Now that we have positive semidefiniteness of solutions to the GP hierarchy, we are able to to global well-posedness of solutions to the GP hierarchy, using an induction argument as in [14] below.

**Theorem 5.4.** Suppose that \( \Gamma_0 = (\xi)_{k=1}^{\infty} \in \mathfrak{S}_1^1 \) is as in Theorem 5.2. Then, for \( 0 < \xi' < 1 \) and \( \xi_1 \) satisfying (4.5), there is a unique global solution \( \Gamma \in \mathcal{W}_1^1(\mathbb{R}) \) to the cubic defocusing GP hierarchy (3.4) in \( \mathbb{R}^3 \) with initial data \( \Gamma_0. \) Moreover, \( \Gamma(t) \) is positive semidefinite and satisfies

\[ \| \Gamma(t) \|_{\mathcal{H}_1^1} \leq \| \Gamma_0 \|_{\mathcal{H}_1^1} \]

(5.14)

for all \( t \in \mathbb{R}. \)

**Proof.** Let \( I_0 \) be the time interval \([jT, (j + 1)T]\), where \( 0 < T < \min\{T_0(\xi_1), T_1(\xi_1)\} \) (see (4.2) and (5.13)) and \( \xi, \xi_1 \) satisfy (4.5). By [12] and Proposition C.3, we have that there is a unique solution \( \Gamma \) to the GP hierarchy in \( \mathcal{W}_1^1(I_0). \) Moreover, by Theorem 5.2, \( \Gamma \) is positive semidefinite on \( I_0. \) It follows as in the proof of Theorem 7.2 in [14] that the higher order energy functionals \( \langle K^{(m)} \rangle_{\Gamma(t)} \), which are defined in equation (5.5), are conserved on \( I_0. \) Thus, as in inequality (7.18) in [14], we have that on \( I_0, \)

\[ \| \Gamma(t) \|_{\mathcal{H}_1^1} \leq \| \Gamma_0 \|_{\mathcal{H}_1^1} \]

(5.15)

\[ \leq \sum_{m \in \mathbb{N}} (2\xi)^m \langle K^{(m)} \rangle_{\Gamma(t)} \]

(5.16)

\[ \leq \sum_{m \in \mathbb{N}} (2\xi)^m \langle K^{(m)} \rangle_{I_0} \]

(5.17)

\[ \leq \| \Gamma_0 \|_{\mathcal{H}_1^1}. \]

(5.18)

Note that positive semidefiniteness of \( \Gamma \) is needed to pass from (5.15) to (5.16) because the definition of \( \| \Gamma(t) \|_{\mathcal{H}_1^1} \) involves taking absolute values, but the definition of \( \langle K^{(m)} \rangle_{\Gamma(t)} \) does not.

Therefore \( \Gamma(T) \in \mathfrak{S}_1^1, \) and so by [12] and Proposition C.3, there is a unique solution \( \Gamma \in \mathcal{W}_1^1(I_1) \) of the GP hierarchy with initial data \( \Gamma(T). \) By another application of Theorem 5.2 and energy conservation (5.15) \( \sim \) (5.18), \( \Gamma \) is positive
semidefinite on \(I\) and \(\Gamma(2T) \in \mathcal{S}_1^1\). Thus, we can repeat the argument and find that we have a unique solution \(\Gamma \in \mathcal{W}^{1,1}_0(\mathbb{R})\). Moreover,
\[
\|\Gamma(t)\|_{\mathcal{X}_1^1} \leq \|\Gamma(t)\|_{\mathcal{X}_1^1} \leq \|\Gamma_0\|_{\mathcal{S}_1^1}
\]
for all \(t \in \mathbb{R}\). \(\square\)

6. Global Derivation of the GP Hierarchy

In this section, we show that the validity of Theorem 2.1 can be extended to arbitrarily large values of \(T\), provided that \(\Gamma_0 \in \mathcal{S}_1^1\) has the form (2.4), and that \(\xi\) is sufficiently small. This is obtained from combining Theorem 2.1 and Theorem 2.3 in a recursive manner.

We begin by observing that, in the statement of Theorem 2.1, instead of assuming that
\[
\Gamma_0 := \lim_{N \to \infty} \Gamma^{\Phi_N}(0)
\]
holds in \(\mathcal{X}_1^1\), we may assume that
\[
\Gamma_0 := \lim_{N \to \infty} P_{\leq k(N)} \Gamma^{\Phi_N}(0)
\]
holds. Indeed, the proof of Theorem 2.1 is unaffected by this replacement.

We also note that initial condition \(\langle \Phi_N(0), H_N^2 \Phi_N(0) \rangle\) implies that \(\Gamma^{\Phi_N}(t) \in \mathcal{X}_1^1\) for any \(t \in \mathbb{R}\), provided that \(\zeta < (4(C + 1))^{-1}\). This follows from (4.36). In fact, given \(\zeta\), we have a bound \(\bar{C}\), uniform in \(N\) and \(t\), such that
\[
\|\Gamma^{\Phi_N}(t)\|_{\mu^1} < \bar{C}^k. \quad (6.1)
\]

We also note that, by Theorem 2.3, the solution to the GP hierarchy \(\Gamma(t) \in \mathcal{X}_1^1\), for all \(t \in \mathbb{R}\), provided that \(\xi_1\) is sufficiently small.

Thus, under the assumptions of Theorem 2.1, at time \(T\), we have
\[
\Gamma^{\Phi_N}(T) \in \mathcal{X}_1^1 \quad \text{and} \quad \Gamma(T) = \lim_{N \to \infty} P_{\leq k(N)} \Gamma^{\Phi_N}(T) \quad \text{in} \quad \mathcal{X}_1^1,
\]
provided that \(\xi_1\) is sufficiently small (note that we also require \(\xi_1 < (4(C + 1))^{-1}\)). By another application of Theorem 2.1, we have that at time \(2T\),
\[
\Gamma^{\Phi_N}(2T) \in \mathcal{X}_1^1 \quad \text{and} \quad \Gamma(2T) = \lim_{N \to \infty} P_{\leq k(N)} \Gamma^{\Phi_N}(2T) \quad \text{in} \quad \mathcal{X}_1^1,
\]
provided that \(\xi_2 < \xi_1\) is sufficiently small. (6.3) says that
\[
\sum_{k=1}^\infty \|\Gamma(2T)^{(k)} - P_{\leq k(N)} \Gamma^{\Phi_N}(2T)^{(k)}\|_{\mu^1} \to 0 \quad \text{as} \quad N \to \infty.
\]
However, by (6.1) and the dominated convergence theorem for sequences, we actually have the stronger statement
\[
\sum_{k=1}^{\infty} \xi_k \|\Gamma(2T)^{(k)} - P_{\leq K(N)} \Gamma^{\psi_N}(2T)^{(k)}\|_{H^1} \to 0 \quad \text{as } N \to \infty,
\]
where we have \(\xi_1\) instead of \(\xi_2\). Thus, at time 2T we actually have
\[
\begin{align*}
\Gamma^{\psi_N}(2T) & \in \mathcal{H}_1 \quad \text{and} \\
\Gamma(2T) & = \lim_{N \to \infty} P_{\leq K(N)} \Gamma^{\psi_N}(2T) \quad \text{in } \mathcal{H}_1.
\end{align*}
\] (6.4)

Note that (6.4) is the same as (6.2), but with \(T\) replaced by \(2T\). Thus, we may iterate the argument again, and conclude that Theorem 2.1 holds for \(T\) arbitrarily large, provided that \(\Gamma_0 \in \mathcal{S}_1\) has the form (2.4), and that \(\xi\) is sufficiently small.

Appendix A. Strong vs. Weak-* Convergence

**Proposition A.1.** Suppose that \(\gamma_N^{(k)}\) is a sequence of operators on \(L^2(\mathbb{R}^k)\) such that \(\gamma_N^{(k)} \to \gamma_\infty^{(k)}\) strongly in Hilbert Schmidt norm. Suppose also that \(\gamma_N^{(k)}\) and \(\gamma_\infty^{(k)}\) are trace class operators such that \(\text{Tr} |\gamma_N^{(k)}| \leq 1\) for all \(N\). Then \(\gamma_N^{(k)} \to \gamma_\infty^{(k)}\) in the weak-* topology induced by the trace norm.

**Proof.** We follow the usual construction of a metric for the weak-* topology induced by the trace norm, as presented in [25], for example. Let \(\mathcal{K}_k\) be the space of compact operators on \(L^2(\mathbb{R}^k)\) equipped with the operator norm topology. Let \(\mathcal{K}_1\) be the space of trace class operators on \(L^2(\mathbb{R}^k)\). By [46], we have that \(\mathcal{K}_1 = \mathcal{K}_1^*\). Since \(\mathcal{K}_k\) is separable, there exists a sequence \(\{J_i^{(k)}\}_{i=1}^{\infty} \in \mathcal{K}_k\) of Hilbert Schmidt operators, dense in the unit ball of \(\mathcal{K}_k\). Note that Hilbert Schmidt operators are dense in the space of compact operators, because, by [46], every compact operator on a Hilbert space is of the form \(\lim_{N \to \infty} \sum_{n=1}^{N} \lambda_n \langle \psi_n, \cdot \rangle \phi_n\), with \(\{\psi_n\}_{n=1}^{\infty}\) and \(\{\phi_n\}_{n=1}^{\infty}\) orthonormal sets, and \(\{\lambda_n\}_{n=1}^{\infty}\) positive real numbers such that \(\lambda_n \to 0\).

On \(\mathcal{K}_1\), we define the metric \(\eta_k\) by
\[
\eta_k(\gamma^{(k)}, \tilde{\gamma}^{(k)}) := \sum_{i=1}^{\infty} 2^{-i} |\text{Tr} J_i^{(k)} (\gamma^{(k)} - \tilde{\gamma}^{(k)})|.
\]

By [47], the topology induced by the metric \(\eta_k\) is equivalent to the weak-* topology on \(\mathcal{K}_1\).

Now, since \(\{J_i^{(k)}\}_{i=1}^{\infty} \in \mathcal{K}_k\) are Hilbert Schmidt, we have
\[
\text{Tr} |J_i^{(k)} (\gamma_N^{(k)} - \gamma_\infty^{(k)})| \leq (\text{Tr} (J_i^{(k)})^2)^{1/2} (\text{Tr} |\gamma_N^{(k)} - \gamma_\infty^{(k)}|^2)^{1/2} \quad \text{(A.1)}
\]
\[
\to 0 \quad \text{as } N \to \infty.
\] (A.2)

Moreover,
\[
\text{Tr} |J_i^{(k)} (\gamma_N^{(k)} - \gamma_\infty^{(k)})| \leq \|J_i^{(k)}\|_{L^2 \to L^2} \text{Tr} |\gamma_N^{(k)} - \gamma_\infty^{(k)}| \leq 1 + \text{Tr} |\gamma_\infty^{(k)}| \quad \text{(A.3)}
\]
Thus, by the dominated convergence theorem for sequences, \( \eta_k(\gamma^{(k)}_N, \gamma^{(k)}_\infty) \to 0 \) as \( N \to \infty \), and so \( \gamma^{(k)}_N \to \gamma^{(k)}_\infty \) in the weak-* topology on \( L^1 \).

\[ \square \]

Appendix B. Conservation of Admissibility for the GP Hierarchy

In this part of the appendix, we prove that the GP hierarchy conserves admissibility. This result has been used in many papers, but we have not found an explicit proof. For the convenience of the reader, we present it here.

**Proposition B.1.** Suppose that \( \Gamma_0 = (\gamma^{(k)}_0)_{k=1}^\infty \in \mathcal{H}_\xi^1 \) is admissible and satisfies \( \text{Tr} \gamma^{(k)}_0 = 1 \) for all \( k \in \mathbb{N} \). Then, for \( 0 < \xi < 1 \) and \( \xi \) satisfying (4.5), the unique solution \( \Gamma \in \mathcal{H}_\xi^1(I) \) to the GP hierarchy obtained in [12] is admissible for all \( t \in I \), provided that \( A := \{A^{(k)}\}_{k=1}^\infty \in \mathcal{H}_\xi^1(I) \), where

\[
A^{(k)}(t, x_k; x_k') := -\gamma^{(k)}(t, x_k; x_k') + \int \gamma^{(k+1)}(t, x_{k+1}; x_{k+1}'; x_{k+1}) dx_{k+1}. \tag{B.1}
\]

**Proof.** We first note that for \( f \in \mathcal{F}(\mathbb{R}^n \times \mathbb{R}^n) \), we have

\[
\int ((\Delta_{x_1} - \Delta_{x_1}) f)(x, x) dx = 0. \tag{B.2}
\]

Indeed, this follows from

\[
\int ((\Delta_{x_1} - \Delta_{x_1}) f)(x, x) dx =
= \int \int \delta(x_1 - x_2) (\Delta_{x_1} - \Delta_{x_2}) f(x_1, x_2) dx_1 dx_2
= \int \int \int \delta(x_1 - x_2) e^{iu_1 x_1 + iu_2 x_2} ((u_2^2 \hat{f}(u_1, u_2) - u_1^2 \hat{f}(u_1, u_2)) dx_1 dx_2
= \int \int \int \delta(x_1 - x_2) e^{iu_1 x_1 + iu_2 x_2} ((u_2^2 \hat{f}(u_1, u_2) - u_1^2 \hat{f}(u_1, u_2)) dx_1 dx_2
= \int \int \delta(u_1 + u_2) ((u_2^2 \hat{f}(u_1, u_2) - u_1^2 \hat{f}(u_1, u_2)) du_1 du_2
= \int (u_2^2 - u_1^2) \hat{f}(u_1, -u_1) du_1
= 0,
\]

which implies (B.2).

Next, we note that the definition of admissibility implies that \( \gamma^{(k)} \) is admissible at time \( t \) if and only if

\[
A^{(k)}(t, x_k, x_k') = 0.
\]

Since \( \Gamma \) satisfies the GP hierarchy, we have that

\[
i \hat{c}_k A^{(k)}(x_k; x_k') = (\Delta_{x_k} - \Delta_{x_k'}) \gamma^{(k)}(x_k; x_k')
- \kappa_0 \left[ \gamma^{(k)}(x_k, x_k'; x_k') - \gamma^{(k+1)}(x_k, x_k'; x_k') \right]. \tag{B.3}
\]

\[
- \kappa_0 \left[ \gamma^{(k)}(x_k, x_k'; x_k') - \gamma^{(k+1)}(x_k, x_k'; x_k') \right]. \tag{B.4}
\]
\[ + \int \left[ \left( (-\Delta_{x_2} + \Delta_{x_1}) \gamma^{(2)} \right) (x_1, x_2; x'_1, x'_2) \right. \]
\[ + k_0 \gamma^{(3)}(x_1, x_2, x'_1, x'_2) \]
\[ - k_0 \gamma^{(3)}(x_1, x_2, x'_1, x'_2) \]
\[ + k_0 \gamma^{(3)}(x_1, x_2, x'_1, x'_2) \]
\[ - k_0 \gamma^{(3)}(x_1, x_2, x'_1, x'_2) \right\} dx_2 \]
\[ = \int (\Delta_{x_1} - \Delta_{x'_1}) \gamma^{(2)}(x_1, x_2; x'_1, x'_2) \, dx_2 \]
\[ - (\Delta_{x_1} - \Delta_{x'_1}) A^{(1)}(x_1; x'_1) \]
\[ - \int \gamma^{(3)}(x_1, x_1, x'_1, x'_1) \right\} dx_2 \]
\[ + k_0 A^{(2)}(x_1, x_1; x'_1, x'_1) - k_0 A^{(2)}(x_1, x'_1, x'_1) \]
\[ + \int \left[ \left( (-\Delta_{x_1} + \Delta_{x'_1}) \gamma^{(2)} \right) (x_1, x_2; x'_1, x'_2) \right. \]
\[ + k_0 \gamma^{(3)}(x_1, x_2, x'_1, x'_2) \]
\[ - k_0 \gamma^{(3)}(x_1, x_2, x'_1, x'_2) \right\} dx_2 \]
\[ = - (\Delta_{x_1} - \Delta_{x'_1}) A^{(1)}(x_1; x'_1), \]
\[ + k_0 A^{(2)}(x_1, x_1; x'_1, x'_1) - k_0 A^{(2)}(x_1, x'_1, x'_1) \]

where (B.1) was used to pass from (B.3) to (B.6) and from (B.4) to (B.7). Moreover, (B.2) and density of \( \mathcal{S} \) in \( H^1 \) was used to pass from (B.5) to (B.8). Symmetry of \( \gamma^{(k)} \) was used to pass to (B.9).

Observe that (B.9) is precisely the right hand side of the first equation in the GP hierarchy. Thus \( A^{(1)} \), and similarly \( A^{(k)} \) for \( k > 1 \), satisfies the GP hierarchy. \( A(0) = 0 \), so by uniqueness of solutions to the GP hierarchy [12], \( A = 0 \).

**Appendix C. Continuity of Solutions to the GP Hierarchy**

In [12], it is shown that there is a unique solution \( \Gamma \) to the GP hierarchy (3.4) in \( \{ \Gamma \in L^\infty_{t\in[0,T]} H^2_x \mid B^+ \Gamma, B^- \Gamma \in L^2_{t\in[0,T]} H^2_x \} \). In this part of the appendix, we show that this solution \( \Gamma \) is an element of \( C([0, T], H^1_x) \).

**Lemma C.1.** If \( \gamma^{(k)} \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \), then
\[
\lim_{t \to 0} \| (U^{(k)}(t) - U^{(k)}(0)) \gamma^{(k)} \|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} = 0.
\]

**Proof.** We recall that \( U^{(k)}(t) = e^{-it(-\Delta_{x_2} + \Delta_{x_1})} \). Since \(-\Delta_{x_2} + \Delta_{x_1}\) is a self-adjoint operator, we have from theorem VIII.7 in [46] that \( U^{(k)}(t) \) is a strongly continuous one-parameter unitary group, and the lemma follows. \( \square \)
Lemma C.2. If $\Gamma \in \mathcal{H}^2(\mathbb{R}^d \times \mathbb{R}^d)$, then
\[
\lim_{t \to 0} \|(U(t) - U(0))\Gamma\|_{\mathcal{H}^1} = 0.
\]

Proof. \[
\|(U(t) - U(0))\Gamma\|_{\mathcal{H}^1} = \sum_{k=1}^{\infty} 2^k \|U^{(k)}(t) - U^{(k)}(0)\|_{L^2} \to 0 \quad \text{as } t \to 0
\]
by Lemma C.1, the fact that $\|U^{(k)}(t)\|_{L^2} \leq 1$, and the dominated convergence theorem for series. \qed

Proposition C.3. The solution $\Gamma$ to the GP hierarchy constructed in [12] lies in $C([0, T], \mathcal{H}^1)$.

Proof. As proven in [12], the solution $\Gamma$ satisfies
\[
\Gamma \in L^1_{[0, T] \mathcal{H}^1}, \quad \tag{C.1}
B\Gamma \in L^2_{[0, T] \mathcal{H}^1}, \quad \text{and} \tag{C.2}
\Gamma(t) = U(t)\Gamma_0 + i\kappa_0 \int_0^t U(t - s)B\Gamma(s) \, ds. \tag{C.3}
\]
Thus, in $\mathcal{H}^1$, we have that
\[
\lim_{h \to 0} [\Gamma(t + h) - \Gamma(t)] = \lim_{h \to 0} \left[ U(t + h)\Gamma_0 + i\kappa_0 \int_0^{t+h} U(t + h - s)B\Gamma(s) \, ds \right.
\]
\[
- U(t)\Gamma_0 - i\kappa_0 \int_0^t U(t - s)B\Gamma(s) \, ds \left. \right]
= \lim_{h \to 0} U(h)\Gamma_0 + \lim_{h \to 0} (U(h) - U(0)) \left[ i\kappa_0 \int_0^t U(t - s)B\Gamma(s) \, ds \right] \quad \text{[\ast]} \tag{C.4}
+ \lim_{h \to 0} i\kappa_0 \int_t^{t+h} U(t + h - s)B\Gamma(s) \, ds. \tag{C.5}
\]
By Lemma C.2, (C.4) = 0. By (C.1) and (C.3), $[\ast] \in \mathcal{H}^1$, so it follows from Lemma C.2 that (C.5) = 0. Now
\[
\left\| \int_t^{t+h} U(t + h - s)B\Gamma(s) \, ds \right\|_{\mathcal{H}^1}
\leq \int_t^{t+h} \|U(t + h - s)B\Gamma(s)\|_{\mathcal{H}^1} \, ds
\]
Derivation and GWP of the GP Hierarchy

\[ \leq \sqrt{h} \| U(t + h - s)B\Gamma(s) \|_{L^2_{t, h}\| \cdot \|^{3/2}} \]

\[ = \sqrt{h} \| B\Gamma(s) \|_{L^2_{t, h}\| \cdot \|^{3/2}} \]

\[ \rightarrow 0 \quad \text{as} \quad h \rightarrow 0 \]

by (C.2), so (C.6) = 0. \hfill \square

Appendix D. Iterated Duhamel Formula and Board Game Argument

In this part of the appendix, we recall a technical result from [15] that is used in parts of this paper. It corresponds to Lemma B.3 in [15].

Let \( \Xi = (\Xi^{(k)})_{k \in \mathbb{N}} \) denote a sequence of functions \( \Xi^{(k)} \in L^2_{t \in [0, T]} H^1(\mathbb{R}^3 \times \mathbb{R}^3) \), for \( T > 0 \). Then, we define the associated sequence \( \text{Duh}_j(\Xi) \) of \( j \)-th level iterated Duhamel terms based on \( B^{(k)}_{\text{main}} \) (see Section 3.2 for notations), with components given by

\[
\text{Duh}_j(\Xi^{(k)}(t)) := i \int_0^t dt_1 \cdots \int_0^{t_{j-1}} dt_j \cdot \sum \text{Duh}_{\text{main}}^{(k+1)} \cdot \text{Duh}_{\text{main}}^{(k+2)} \cdots (\Xi^{(k)})(t),
\]

with the conventions \( t_0 := t \), and

\[
\text{Duh}_0(\Xi^{(k)}(t)) := (\Xi^{(k)}(t))
\]

for \( j = 0 \). Using the board game estimates of [24, 25, 40], one obtains:

Lemma D.1. For \( \Xi = (\Xi^{(k)})_{k \in \mathbb{N}} \) as above,

\[
\| \text{Duh}_j(\Xi^{(k)}(t)) \|_{L^2_{t, h}\| \cdot \|^{3/2}} \leq k c_0^2 (c_0 T)^j \| \Xi^{(k+1)} \|_{L^2_{t, h}\| \cdot \|^{3/2}},
\]

where the constants \( c_0, C_0 \) depend only on \( d, p \). For this work, the dimension is given by \( d = 3 \) and the nonlinearity is given by \( p = 2 \) (cubic GP hierarchy).

Lemma D.1 is used for the proof of the next result (by suitably exploiting the splitting \( B_N = B^{\text{main}}_N + B^{\text{error}}_N \)), which corresponds to Lemma B.3 in [15].

Lemma D.2. Let \( \delta' > 0 \) be defined by

\[
\beta = \frac{1 - \delta'}{4}.
\]

Assume that \( N \) is sufficiently large that the condition

\[
K < \frac{\delta'}{\log C_0} \log N,
\]

holds, where the constant \( C_0 \) is as in Lemma D.1.
Assume that $\Xi^K_N \in L^2_{\text{loc}}(\mathbb{R}^1_\xi)$ for some $0 < \tilde{\xi} < 1$, and that $\tilde{\xi}$ is small enough that $0 < \tilde{\xi} < \eta_{\xi}^{-1}$, with

$$\eta < (\max\{1, C_0\})^{-1}.$$  \hfill (D.6)

Let $\Theta^K_N$ and $\Xi^K_N$ satisfy the integral equation

$$\Theta^K_N(t) = \Xi^K_N(t) + i \int_0^t B_N U(t-s) \Theta^K_N(s) \, ds$$  \hfill (D.7)

The superscript "K" in $\Theta^K_N$ and $\Xi^K_N$ means that only the first $K$ components are nonzero, and $B_N = B_N^{\text{main}} + B_N^{\text{error}}$.

Then, the estimate

$$\|\Theta^K_N\|_{L^2_{\text{loc}}(\mathbb{R}^1_\xi)} \leq C_1(T, \tilde{\xi}, \eta_{\xi}^{-1}) \|\Xi^K_N\|_{L^2_{\text{loc}}(\mathbb{R}^1_\xi)}$$  \hfill (D.8)

holds for a finite constant $C_1(T, \tilde{\xi}, \eta_{\xi}^{-1}) > 0$ independent of $K, N$.

**Funding**

The work of T.C. was supported by NSF grants DMS-1009448 and DMS-1151414 (CAREER).

**References**

[1] Adami, R., Golse, G., Teta, A. (2007). Rigorous derivation of the cubic NLS in dimension one. *J. Stat. Phys.* 127:1194–1220.

[2] Aizenman, M., Lieb, E.H., Seiringer, R., Solovej, J.P., Yngvason, J. (2004). Bose-Einstein quantum phase transition in an optical lattice model. *Phys. Rev. A* 70:023612.

[3] Ammari, Z., Nier, F. (2008). Mean field limit for bosons and infinite dimensional phase-space analysis. *Ann. H. Poincaré* 9:1503–1574.

[4] Ammari, Z., Nier, F. (2011). Mean field propagation of Wigner measures and BBGKY hierarchies for general bosonic states. *J. Math. Pures Appl.* 95:585–626.

[5] Anapolitanos, I. (2011). Rate of convergence towards the Hartree-von Neumann limit in the mean-field regime. *Lett. Math. Phys.* 98(1):1–31.

[6] Beckner, W. Convolution estimates and the Gross-Pitaevskii hierarchy. Preprint, http://arxiv.org/abs/1111.3857.

[7] Cazenave, T. (2003). *Semilinear Schrödinger Equations*. Courant Lecture Notes, Vol. 10. Providence, RI: Amer. Math. Soc.

[8] Cazenave, T., Weissler, F. B. (1988). The Cauchy problem for the nonlinear Schrödinger equation in $H^1$. *Manuscripta Math.* 61:477–494.

[9] Chen, L., Lee, J.O., Schlein, B. (2011). Rate of convergence towards Hartree dynamics. *J. Stat. Phys.* 144:872–903.

[10] Chen, T., Hainzl, C., Pavlovic, N., Seiringer, R. Unconditional uniqueness for the cubic Gross-Pitaevskii hierarchy via Quantum De Finetti. *Commun. Pure Appl. Math.* Preprint, http://arxiv.org/abs/1307.3168.
[11] Chen, T., Pavlović, N. (2011). The quintic NLS as the mean field limit of a Boson gas with three-body interactions. *J. Funct. Anal.* 260:959–997.

[12] Chen, T., Pavlović, N. (2010). On the Cauchy problem for focusing and defocusing Gross-Pitaevskii hierarchies. *Discr. Contin. Dyn. Syst.* 27:715–739.

[13] Chen, T., Pavlović, N. (2013). A new proof of existence of solutions for focusing and defocusing Gross-Pitaevskii hierarchies. *Proc. Amer. Math. Soc.* 141:279–293.

[14] Chen, T., Pavlović, N. Higher order energy conservation and global well-posedness of solutions for the Gross-Pitaevskii hierarchies. *Commun. PDE.* Preprint, http://arxiv.org/abs/0906.2984.

[15] Chen, T., Pavlović, N. (2014). Derivation of the cubic NLS and Gross-Pitaevskii hierarchy from manybody dynamics in $d = 2, 3$ based on spacetime norms. *Ann. H. Poincaré* 15:543–588.

[16] Chen, T., Pavlović, N., Tzirakis, N. (2010). Energy conservation and blowup of solutions for focusing GP hierarchies. *Ann. Inst. H. Poincare (C) Anal. Non-Lin.* 27:1271–1290.

[17] Chen, T., Pavlović, N., Tzirakis, N. (2012). Multilinear Morawetz identities for the Gross-Pitaevskii hierarchy. *Contemp. Math.* 581:39–62.

[18] Chen, X. (2012). Second order corrections to mean field evolution for weakly interacting bosons in the case of 3-body interactions. *Arch. Ration. Mech. Anal.* 203:455–497.

[19] Chen, X. (2012). Collapsing estimates and the rigorous derivation of the 2D cubic nonlinear Schrödinger equation with anisotropic switchable quadratic traps. *J. Math. Pures Appl.* (9) 98:450–478.

[20] Chen, X. (2013). On the rigorous derivation of the 3D cubic nonlinear Schrödinger equation with a quadratic trap. *Arch. Ration. Mech. Anal.* 210:365–408.

[21] Chen, X., Holmer, J. On the Klainerman-Machedon conjecture of the quantum BBGKY hierarchy with self-interaction. Preprint, arXiv:1303.5385.

[22] Chen, X., Holmer, J. (2013). On the rigorous derivation of the 2D cubic nonlinear Schrödinger equation from 3D quantum many-body dynamics. *Arch. Rational Mech. Anal.* 210:909–954.

[23] Elgart, A., Erdös, L., Schlein, B., Yau, H.-T. (2006). Gross-Pitaevskii equation as the mean field limit of weakly coupled bosons. *Arch. Rat. Mech. Anal.* 179:265–283.

[24] Erdös, L., Schlein, B., Yau, H.-T. (2006). Derivation of the Gross-Pitaevskii hierarchy for the dynamics of Bose-Einstein condensate. *Comm. Pure Appl. Math.* 59:1659–1741.

[25] Erdös, L., Schlein, B., Yau, H.-T. (2007). Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems. *Invent. Math.* 167:515–614.

[26] Erdös, L., Schlein, B., Yau, H.-T. (2009). Rigorous derivation of the Gross-Pitaevskii equation with a large interaction potential. *J. Amer. Math. Soc.* 22:1099–1156.

[27] Erdös, L., Schlein, B., Yau, H.-T. (2010). Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensates. *Ann. of Math.* (2) 172:291–370.

[28] Erdös, L., Yau, H.-T. (2001). Derivation of the nonlinear Schrödinger equation from a many body Coulomb system. *Adv. Theor. Math. Phys.* 5:1169–1205.
[29] Fröhlich, J., Graffi, S., Schwarz, S. (2007). Mean-field- and classical limit of many-body Schrödinger dynamics for bosons. *Comm. Math. Phys.* 271:681–697.

[30] Fröhlich, J., Knowles, A., Pizzo, A. (2007). Atomism and quantization. *J. Phys. A* 40:3033–3045.

[31] Fröhlich, J., Knowles, A., Schwarz, S. (2009). On the mean-field limit of Bosons with Coulomb two-body interaction. *Comm. Math. Phys.* 288:1023–1059.

[32] Fröhlich, J., Tsai, T.-P., Yau, H.-T. (2000). On a classical limit of quantum theory and the non-linear Hartree equation. GAFA 2000 (Tel Aviv, 1999). *Geom. Funct. Anal.* Special Volume, Part I:57–78.

[33] Grillakis, M., Machedon, M. (2013). Pair excitations and the mean field approximation of interacting bosons. I. *Comm. Math. Phys.* 324:601–636.

[34] Grillakis, M., Machedon, M., Margetis, A. (2011). Second-order corrections to mean field evolution of weakly interacting bosons II. *Adv. Math.* 228:1788–1815.

[35] Grillakis, M., Machedon, M., Margetis, A. (2010). Second-order corrections to mean field evolution for weakly interacting bosons. I. *Comm. Math. Phys.* 294:273–301.

[36] Grillakis, M., Margetis, A. (2008). A priori estimates for many-body Hamiltonian evolution of interacting boson system. *J. Hyperbolic Differ. Equ.* 5:857–883.

[37] Hepp, K. (1974). The classical limit for quantum mechanical correlation functions. *Comm. Math. Phys.* 35:265–277.

[38] Hudson, R. L., Moody, G. R. (1975/76). Locally normal symmetric states and an analogue of de Finetti's theorem. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 33:343–351.

[39] Kirkpatrick, K., Schlein, B., Staffilani, G. (2011). Derivation of the two dimensional nonlinear Schrödinger equation from many body quantum dynamics. *Amer. J. Math.* 133(1):91–130.

[40] Klainerman, S., Machedon, M. (2008). On the uniqueness of solutions to the Gross-Pitaevskii hierarchy. *Commun. Math. Phys.* 279(1):169–185.

[41] Lewin, M., Nam, P.T., Rougerie, N. Derivation of Hartree’s theory for generic mean-field Bose systems. Preprint, arXiv:1303:0981.

[42] Lieb, E.H., Seiringer, R., Solovej, J.P., Yngvason, J. (2005). *The Mathematics of the Bose Gas and Its Condensation*. Basel: Birkhäuser.

[43] Lieb, E.H., Seiringer, R. (2002). Proof of Bose-Einstein condensation for dilute trapped gases. *Phys. Rev. Lett.* 88:170409.

[44] Lieb, E.H., Seiringer, R., Yngvason, J. (2001). A rigorous derivation of the Gross-Pitaevskii energy functional for a two-dimensional Bose gas. *Commun. Math. Phys.* 224:17–31.

[45] Pickl, P. (2011). A simple derivation of mean field limits for quantum systems. *Lett. Math. Phys.* 97(2):151–164.

[46] Reed, M., Simon, B. (1980). *Methods of Modern Mathematical Physics*. Vol. I. New York: Academic Press Inc.

[47] Rudin, W. (1973). *Functional Analysis*. McGraw-Hill Series in Higher Mathematics. New York, NY: McGraw-Hill Book Co.

[48] Rodnianski, I., Schlein, B. (2009). Quantum fluctuations and rate of convergence towards mean field dynamics. *Comm. Math. Phys.* 291(1):31–61.

[49] Schlein, B. (2008). *Derivation of Effective Evolution Equations from Microscopic Quantum Dynamics*. Lecture Notes for the Minicourse Held at the 2008 CMI Summer School in Zurich.
[50] Spohn, H. (1980). Kinetic equations from Hamiltonian dynamics. *Rev. Mod. Phys.* 52:569–615.

[51] Stormer, E. (1969). Symmetric states of infinite tensor products of C*-algebras. *J. Functional Analysis* 3:48–68.

[52] Tao, T. (2006). *Nonlinear Dispersive Equations. Local and Global Analysis. CBMS 106.* Providence, RI: AMS.