On Open String $\sigma$-Model and Noncommutative Gauge Fields

Oleg Andreev$^{*\dagger}$ and Harald Dorn$^{\dagger}$

Humboldt–Universität zu Berlin, Institut für Physik
Invalidenstraße 110, D-10115 Berlin, Germany

Abstract

We consider the ordinary and noncommutative Dirac-Born-Infeld theories within the open string $\sigma$-model. First, we propose a renormalization scheme, hybrid point splitting regularization, that leads directly to the Seiberg-Witten description including the two-form $\Phi$. We also show how such a form appears within the standard renormalization scheme just by some freedom in changing variables. Second, we propose a Wilson factor which has the noncommutative gauge invariance on the classical level and then compute the $\sigma$-model partition function within one of the known renormalization scheme that preserves the noncommutative gauge invariance. As a result, we find the noncommutative Yang-Mills action.

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1 Introduction

The open string $\sigma$-model was a hot topic in the eighties as it provided a basic tool to derive the low energy effective action (Dirac-Born-Infeld action) that is nonperturbative in $\alpha'$ [1, 2, 3]. Later it was also realized that it is useful for D-brane physics [4]. Recently, it has been pointed out by Seiberg and Witten that a special renormalization scheme, a point splitting regularization, results in a rather peculiar situation where the space-time (brane) coordinates do not commute (see [6] and refs. therein). The purpose of this paper is to further develop ideas about the appearance of noncommutative geometry by the open string $\sigma$-models.

Our conventions and some features of the quantization of open strings ending on D-branes that are relevant to our discussion are the following:

1. The $\sigma$-model action is given by

$$S = \frac{1}{4\pi\alpha'} \int_D d^2z \ g_{ij} \partial_a X^i \partial^a X^j - i \int_{\partial D} \ d\tau \ \left( \frac{1}{2} B_{ji} \dot{X}^i X^j + A_i(X) \dot{X}^i \right) + \phi ,$$

(1.1)

where $D$ means the string world-sheet (disk or half plane) whose boundary is $\partial D$. $g_{ij}$, $B_{ij}$, $\phi$ are the constant metric, antisymmetric tensor and dilaton fields, respectively. We also included in (1.1) the Abelian background gauge field $A_i(X)$. $X^i$ map the world-sheet to the brane and $i, j = 1, \ldots, p + 1$. The world-sheet indices are denoted by $a, b$.

$^{*}$e-mail: andreev@physik.hu-berlin.de
$^{\dagger}$Permanent address: Landau Institute, Moscow, Russia
$^{\dagger}$e-mail: dorn@physik.hu-berlin.de

For a recent review of this issue see [6] and references therein.
At this point, let us make a couple of remarks. First, we keep the explicit dependence on the constant dilaton field because it plays an important role in what follows. Second, for a constant \( B \) field \( B_{ij}e^{ab}\partial_aX^i\partial_bX^j \) is a total derivative, so we write this term as a boundary interaction.

2. A natural object to compute within the path integral is the \( \sigma \)-model partition function

\[
Z[\varphi, g, B, A, \alpha'] = \int \mathcal{D}X \, e^{-S} ,
\]

that results in the open string low energy effective action (Born-Infeld action) as well as the D-brane action (Dirac-Born-Infeld action) (for a review and refs., see, [4, 5]).

To compute the partition function (1.2) one first integrates over the internal points of the disc to reduce the integral to the boundary and next splits the integration variable \( X^i \) in the constant and the non-constant part as \( X^i(\tau) = x^i + \xi^i(\tau) \). As a result, one gets the infinite set of the vertices \( F, \partial F, \partial^2 F \) etc \( \Re \). Next the perturbation theory in \( \alpha' \) is used to compute the leading terms of \( Z \). The derivative-independent part of the partition function proves to be the Dirac-Born-Infeld action namely,

\[
Z[\varphi, g, B, A, \alpha'] = S_{\text{DBI}} + O(\partial F) = e^{-\varphi} \int [dx] \sqrt{\det(g + 2\pi\alpha'(F + B))} + O(\partial F) ,
\]

where \( F_{ij} = \partial_i A_j - \partial_j A_i \) and \([dx] = \frac{d^{d+1}x}{(2\pi\alpha')^{d+1}}\). The above result is based on the use of one of the standard renormalization schemes like Pauli-Villars or \( \zeta \)-function which was originally used by Fradkin and Tseytlin \( \Re \).

It is natural to ask, whether different renormalization schemes used to compute the path integral lead to the same answer. It was understood in the eighties that the ambiguity in the structure of \( Z \) related to the choice of a renormalization scheme should be a particular case of the field redefinition ambiguity present in the effective action reconstructed from the S-matrix. In other words, only structures in \( Z \) that are invariant under the field redefinition are relevant.

3. Seiberg and Witten pointed out that in the framework of the point splitting regularization scheme the leading terms of the perturbation theory are summed into the noncommutative version of the Dirac-Born-Infeld action namely,

\[
\hat{Z}[\hat{\varphi}, G, \theta, \hat{A}, \alpha'] = \hat{S}_{\text{DBI}} + \cdots = e^{-\hat{\varphi}} \int [dx] \sqrt{\det(G + 2\pi\alpha'\hat{F}) + \cdots} ,
\]

where \( \hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i - i\hat{A}_i \ast \hat{A}_j + i\hat{A}_j \ast \hat{A}_i \). Here the \( \ast \)-product is defined by

\[
f(x) \ast g(x) = e^{2(\theta x y)} \frac{\partial}{\partial y} \frac{\partial}{\partial z} f(x + y)g(x + z)|_{y=z=0} .
\]

It is well-known that such a product is noncommutative but associative.

As we mentioned above, the different renormalization schemes are equivalent, so using a change of variables one has to get

\[
\hat{S}_{\text{DBI}} = S_{\text{DBI}} + \text{total derivative terms} + O(\partial F) .
\]

In the case of interest the change of variables found by Seiberg and Witten is given by \( \Re \)

\[
G = (g - 2\pi\alpha'B)g^{-1}(g + 2\pi\alpha'B) , \quad \theta = -(2\pi\alpha')^2(g + 2\pi\alpha'B)^{-1}B(g - 2\pi\alpha'B)^{-1} ,
\]

\[
\hat{\varphi} = \varphi + \frac{1}{2} \ln \det \left( G(g + 2\pi\alpha'B)^{-1} \right) , \quad \hat{F} = F + F\theta F - (A\theta\partial)F + O(\theta^2) .
\]

\( ^2\)For the sake of simplicity, we use the matrix notations here and below.
Note that the last relation is simplified in the case of a constant field $\hat{F}$. Explicitly,

$$\hat{F} = (1 + F\theta)^{-1}F \ .$$

(1.8)

Moreover, it was also conjectured in [3] that the use of a suitable regularization that interpolates between the both mentioned above results in

$$Z[\varphi, G, \theta, A, \alpha'] = S_{\text{DBI}} + \cdots = e^{-\varphi} \int [dx] \sqrt{\det(G + 2\pi\alpha'(F + \Phi))} + \cdots ,$$

(1.9)

where $\Phi$ is some two-form. In this case the relations (1.7) are modified to

$$(G + 2\pi\alpha'\Phi)^{-1} = -\frac{1}{2\pi\alpha'}\theta + (g + 2\pi\alpha'B)^{-1} \ , \ \varphi = \varphi + \frac{1}{2} \ln \det \left(\frac{G + 2\pi\alpha'\Phi}{g + 2\pi\alpha'B}\right) .$$

(1.10)

2 Seiberg-Witten conjecture via $\sigma$-model

The aim of this section is to show how the Seiberg-Witten conjecture (1.9) may be simply derived within the $\sigma$-model approach. First, we start from the point splitting renormalization scheme. Then we propose how to modify it to get the desired result. Second, we start with the $\zeta$-function renormalization scheme and use the change of variables ($\sigma$-model couplings) to get (1.9). In the both cases $\Phi$ naturally appears due to freedom in choosing new variables.

2.1 Hybrid point splitting regularization

Following the ideas sketched in the introduction, we split the integration variable $X^i$ and moreover include the $B\hat{\xi}\xi$ term into the kinetic term that is usual for the problem at hand [2, 6]. So we have for the partition function (1.2)

$$Z[\varphi, g, B, A, \alpha'] = e^{-\varphi} \int [dx] \langle e^{-S_{\text{int}}(\varphi)} \rangle , \ \langle \cdots \rangle = \int D\xi e^{-S_0} \cdots ,$$

(2.1)

where $S_0 = \frac{1}{2} \int_{\partial D} d\tau d\tau' \xi^\dagger \hat{N}^{-1} \xi$ , $S_{\text{int}} = -i \int_{\partial D} d\tau A(x + \xi) \hat{\xi}$ ,

$$\hat{N} = -\alpha'G^{-1} \ln(\tau - \tau')^2 + \frac{i}{2} \theta\epsilon(\tau - \tau').$$

The matrices $G, \theta$ are given by Eq. (1.7).

Expanding the interaction part in $A$ and doing the Fourier transform, the partition function is

$$Z[\varphi, g, B, A, \alpha'] = e^{-\varphi} \int [dx] \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \cdots \int d\tau_n \int \cdots \int d^{p+1} k_n \ e^{ikx} A(k_1) \cdots A(k_n) \langle \prod_{j=1}^{n} \xi e^{ikj} \xi(\tau_j) \rangle_{G, \theta} ,$$

(2.2)

where $k = k_1 + \cdots + k_n$.

From now let us stick to the point splitting regularization. We are aware of the problem to define what is meant by the point splitted path integral for the partition function. Instead of giving a general answer, we define for our purpose a hybrid point splitting regularization. As used in [6] the point splitting regularization is not defined per se, but only in connection with a partial summation of perturbation theory that includes $B\hat{\xi}\xi$ term into the kinetic term. This results in an additive part $\frac{i}{2} \theta\epsilon(\tau - \tau')$ in the corresponding propagator. Everywhere the propagator ends on vertices which contain a $\tau$-derivative,

3It should be noted that though analogous formulae first appeared in the context of NCSYM [7] their appearance within the $\sigma$-model approach is due to Seiberg and Witten.
part of the regularization is to drop the $\partial \epsilon$ term. We use this rule throughout, also for possible operator expressions inside functional determinants. But we insist that the determinants for the remaining expressions are treated within the standard renormalization scheme like Pauli-Villars or $\zeta$-function. As a consequence we get a generalization of the corresponding formula in [6]

$$\langle \prod_{j=1}^{n} \xi_k \rangle_{G, \theta} = J e^{-\frac{i}{2} \sum_{i>j} k_i \theta_k \epsilon(\tau_i - \tau_j)} \langle \prod_{j=1}^{n} \hat{\xi} \rangle_{G, \theta_0},$$

(2.3)

where $\theta = \theta - \theta_0$. As to $J$, it is the quotient of functional determinants. We have to care of it because it is relevant for what follows. It is evident that there exists freedom in choosing the new parameter $\theta_0$. It turns out that exactly such freedom is responsible for the two-form $\Phi$. Let us go on to see that this is indeed the case. Undoing the Fourier transform we arrive at

$$Z[\varphi, G, \theta, A, \alpha'] = J e^{-\frac{i}{2} \sum_{i>j} \partial \theta \theta' \epsilon(\tau_i - \tau_j)} \langle \prod_{j=1}^{n} A(x_j + \xi_j) \rangle_{x_1=\ldots=x_n=x},$$

(2.4)

where $\partial = \frac{\partial}{\partial x_i}$.

An important point we should stress here is that the above redefinition of $\theta$ automatically leads to a proper redefinition of the kinetic term. This time we have

$$S_0 = \frac{1}{2} \int_0^\infty \int d\tau d\tau' \xi^\tau G N^{-1} \xi - \frac{i}{2} \int_0^\infty \int d\tau \xi^\tau \Phi \xi , \quad N = -\alpha' \ln(\tau - \tau')^2,$$

(2.5)

with the new matrices for the metric and antisymmetric field, given by the corresponding inversion of the relations (1.7)

$$G = \left( G^{-1} - \frac{1}{(2\pi \alpha')^2} \theta_0 G \theta_0 \right)^{-1}, \quad \Phi = - \frac{1}{(2\pi \alpha')^2} G \theta_0 G.$$

(2.6)

A simple algebra shows that the such defined $\theta, G, \Phi$ obey the Seiberg-Witten relation (1.10).

Since our regularization prescription allows us to treat the functional determinants along the lines of [1] we find for $J$

$$J = \left( \det \hat{N}_{G, \theta} \right)^{\frac{1}{2}} = \left( \frac{\det(g + 2\pi \alpha' B)}{\det(G + 2\pi \alpha' \Phi)} \right)^{\frac{1}{2}.}$$

(2.7)

The latter may be absorbed by the corresponding redefinition of the dilatonic field (string coupling constant). Thus we get the last relation in (1.10).

Let us now define the Wilson factor as

$$P \exp \left( i \int A(x + \xi) \right) = \sum_{n=0}^\infty \frac{i^n}{n!} \int \cdots \int d\tau_n e^{\frac{i}{2} \sum_{i>j} \partial \theta \theta' \epsilon(\tau_i - \tau_j)} \prod_{j=1}^{n} A(x_j + \xi_j) \rangle_{x_1=\ldots=x_n=x}.$$ 

(2.8)

It is clear that this factor is equal to the usual path ordering $P$ applied to the exponential in the sense of the $*$-product.

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4We change our notations due to specializing the renormalization scheme. So, $A$ denotes the gauge variable in this scheme etc.
Finally, the partition function becomes

$$Z[\phi, G, \theta, A, \alpha'] = e^{-\phi} \int [dx] \int D\xi e^{-S_0} P * \exp\left(i \int_{\partial D} d\tau \, A(x + \xi) \dot{\xi}\right). \quad (2.9)$$

Let us now specialise the gauge field to

$$A_i(X) = \frac{1}{2} f_{ji} X^j, \quad (2.10)$$

with a constant matrix $f$. Then the related noncommutative field strength is simply given by $F = f - f\theta f$.

For the gauge field (2.10) the Wilson factor (2.8) turns out to be

$$P * \exp\left(\frac{i}{2} \int_{\partial D} d\tau (x + \xi)^T f \dot{\xi}\right) = \exp\left(\frac{i}{2} \int_{\partial D} d\tau (x + \xi)^T F \dot{\xi}\right). \quad (2.11)$$

Now we just have the functional integral treated in [1] and find

$$Z[\phi, G, \theta, A, \alpha'] = S_{DBI} = e^{-\phi} \int [dx] \sqrt{\det(G + 2\pi \alpha' (F + \Phi))}. \quad (2.12)$$

With this formula we have finished an explicit calculation proving that the partition function for a constant field strength within the hybrid point splitting renormalization scheme is given by the Dirac-Born-Infeld action referring to the noncommutative gauge field. This has been argued for in [6] by referring to the analogy of the involved structures compared to the commutative case.

2.2 $\zeta$-function regularization

As we have mentioned in the introduction, the different renormalization schemes used to compute the path integral (1.2) lead to the same result after a proper change of the variables ($\sigma$-model couplings) is done. In fact, such a change of the couplings is nothing but the resummation of the perturbation theory in $\alpha'$. After this is understood, it immediately comes to mind to realize a resummation by changing the path integral variables. A possible way to do this is to take a new variable as a function of $\alpha'$. Let us now show how it works. Specialising to the $\zeta$-function renormalization scheme [1], we define the path integral measure in the same way as it was done in [8]. So we have

$$Z[\phi, g, B, A, \alpha'] = e^{-\phi} \int [dx] \sqrt{\det g} \langle e^{-S_{int}} \rangle, \quad \langle \ldots \rangle = \int D\xi e^{-S_0} \ldots, \quad (2.13)$$

Above we have rescaled $X^i$ as $X^i \to \sqrt{2\pi \alpha'} X^i$. $N$ is the boundary value of the Neumann function.

Now let us change the variable $\xi^i$ as $\xi^i = \left(g^{-1}A\xi^i\right)^i$. In fact, the measure is defined in such a way that the effect is only due to the action $S$. The latter becomes

$$S_0 = \frac{1}{2} \int_{\partial D} d\tau d\tau' \xi^i \dot{\xi}^i g N^{-1} \xi, \quad S_{int} = -i\pi \alpha' \int_{\partial D} d\tau \xi^T \left(B + F\right) \dot{\xi} + O(\partial F), \quad (2.14)$$

where $G = \Lambda^T g^{-1} \Lambda$, $\theta = -\left(2\pi \alpha'\right)^2 \Lambda^{-1} B \Lambda^{T-1}$.

$^5$In fact, it is nothing but $GL(p + 1)$ transforms whose matrices depend on $\alpha'$. 
It is evident that as far as \( \Lambda \) depends on \( \alpha' \) we automatically get a resummation of the perturbation theory because the vertices (\( \sigma \)-model couplings) are redefined. It is also clear which role the dilaton field \( \varphi \) has to play. It is responsible for the cancellation of the poles in \( \alpha' \) within the perturbation theory i.e., the \( \alpha' \)-dependence of \( Z \) always looks like \( Z \sim (\alpha')^{-p+1}\left(1 + O(\alpha')\right) \). Note that \( \Lambda \) is an arbitrary function in \( g, B, \alpha' \). Moreover \( B \) plays a key role as it is responsible for \( \alpha' \)-dependence of \( \Lambda \) because \( g \) is dimensionless. One can also consider \( \Lambda \) as a function in \( G, \theta, \alpha' \). In this case the change of the variables is given by 

\[
\xi_i = (\Lambda^T - 1 G) \xi_i.
\]

In general, it is not clear how to exactly compute the partition function. The best what we can do is to find the leading terms within the perturbation expansion as it was done in \([1, 9, 10]\). For our purposes the one-loop approximation is also sufficient. So the problem is reduced to finding the corresponding determinant. It is straightforward to write down a solution of the problem \([1]\)

\[
S_{\text{DBI}} = e^{-\varphi} \int [dx] \sqrt{\det g} \sqrt{\det(1 - \frac{1}{2\pi \alpha'} \theta G + 2\pi \alpha' \Lambda^{-1} F \Lambda^T - 1 G)}.
\]

(2.15)

In doing transformations with the determinant the important thing one should keep in mind is that the resummation of the perturbation theory assumes that the partition function depends on \( \alpha' \) as \( Z \sim (\alpha')^{-p+1}\left(1 + O(\alpha')\right) \). As to the determinant, we postulate that it is given by \( \sqrt{\det(A + 2\pi \alpha' B)} \) with some matrices \( A \) and \( B \). So it is clear that we should get rid of the \( \frac{1}{\alpha'} \)-term. The latter argument assumes

\[
S_{\text{DBI}} = e^{-\varphi} \int [dx] \sqrt{\det(g + 2\pi \alpha' B)} \sqrt{\det(1 + (1 - \frac{1}{2\pi \alpha'} \theta G)^{-1} 2\pi \alpha' \Lambda^{-1} F \Lambda^T - 1 G)}.
\]

(2.16)

Above we have used that \( \det(g - \frac{1}{2\pi \alpha'} g \theta G) = \det(g + 2\pi \alpha' B) \).

Moreover we bring \( \Lambda^{-1} \) to a form

\[
\Lambda^{-1} = (G^{-1} - \frac{1}{2\pi \alpha'} \theta) \Delta,
\]

(2.17)

where \( \Delta \) is a new matrix that depends on \( G, \theta, \alpha' \). This results in

\[
S_{\text{DBI}} = e^{-\varphi} \int [dx] \sqrt{\det(g + 2\pi \alpha' B)} \sqrt{\det(1 + 2\pi \alpha' F \Delta^T (G^{-1} + \frac{1}{2\pi \alpha'} \theta) \Delta)}.
\]

(2.18)

Our postulate for the determinant yields the following \( \alpha' \)-dependence of \( \Delta^T (G^{-1} + \frac{1}{2\pi \alpha'} \theta) \Delta \)

\[
\Delta^T (G^{-1} + \frac{1}{2\pi \alpha'} \theta) \Delta = \frac{1}{2\pi \alpha'} \Gamma + \Sigma(\alpha')
\]

(2.19)

where \( \Sigma(\alpha') \) is regular in the limit \( \alpha' \to 0 \).

Substituting (2.19) into (2.18) we find

\[
S_{\text{DBI}} = e^{-\varphi} \int [dx] \sqrt{\det(1 + F \Gamma)} \sqrt{\det(\Sigma^{-1} + 2\pi \alpha'(1 + F \Gamma)^{-1} F)}.
\]

(2.20)

Above we have also defined a new dilatonic field as

\[
\varphi = \varphi + \frac{1}{2} \ln \frac{\det \Sigma^{-1}}{\det(g + 2\pi \alpha' B)}
\]

(2.21)

\[\text{Note that } \Lambda = g + 2\pi \alpha' B \text{ recovers the Seiberg-Witten relations }[1, 7].\]
Furthermore due to our postulate for the determinant we can represent $\Sigma^{-1}(\alpha') = \Pi + 2\pi\alpha'\Omega$ with some $\alpha'$-independent matrices $\Pi$ and $\Omega$. So the Eq. (2.20) is rewritten in the following form

$$S_{\text{DBI}} = e^{-\varphi} \int [dx] \sqrt{\det(1 + F\Theta)} \sqrt{\det(\Pi + 2\pi\alpha'(\Omega + (1 + F\Gamma)^{-1}F))}.$$  \hspace{1cm} (2.22)

Let us now see what the matrices $\Sigma, \Pi, \Omega$ are. To do so, it is worth to remark that $\Lambda_{ij} = g_{ij}$ is a trivial transform in a sense that it does not lead to any resummation of the perturbation theory. So we can consider it as a unity in a space of all transforms. It is reasonable to normalize transforms with respect of this unity by requiring that $\Delta$ has the following asymptotic behaviour $\Delta^j_i = \delta^j_i + O(\alpha')$ as $\alpha' \to 0$. On the other hand, it also seems reasonable because $\delta^j_i$ is the only $\alpha'$-independent tensor with such an index structure we have. Then it immediately follows from (2.19) that $\Gamma = \theta$. This allows to define a new field strength $F$ as $F = (1 + F\theta)^{-1}F$ that of course coincides with the Seiberg-Witten definition (1.8). However, the point is that we have not used the fact that two theories should be in the same class of equivalence as gauge theories. As a result, the Dirac-Born-Infeld action is rewritten as

$$S_{\text{DBI}} = e^{-\varphi} \int [dx] \sqrt{\det(1 + F\Theta)} \sqrt{\det(G + 2\pi\alpha'(\Phi + F))}.$$  \hspace{1cm} (2.23)

Furthermore it is clear from the general covariance of the partition function that $\Pi$ must be treated as a metric $G$. As to $\Omega$, it coincides with $\Phi$ from Eq.(1.9). Of course, (2.17), (2.19), (2.21) are equivalent to the relations (1.10).

Finally, we get

$$S_{\text{DBI}} = e^{-\varphi} \int [dx] \sqrt{\det(1 + F\Theta)} \sqrt{\det(G + 2\pi\alpha'(\Phi + F - (A\theta\partial)F + O(\theta^2))}.$$  \hspace{1cm} (2.24)

What we actually need is only the last factor of the integrand to get the equivalence (1.10). To fix this problem, one subtle point we should remind is that different formulations coincide up to derivative terms. It also assumes that we can integrate by parts that of course makes no sense for constant field strengths. In fact, we have to consider slowly varying field strengths. As a consequence, the relation between $F$ and $\Phi$ becomes more involved [6]. In the leading order of $\theta$ it is given by

$$F = F - F\Theta F - (A\theta\partial)F + O(\theta^2) = F_0 - (A\theta\partial)F + O(\theta^2).$$  \hspace{1cm} (2.25)

It is easy to see that in this order the actions (1.9) and (2.24) coincide namely,

$$\int [dx] \left( \sqrt{\det(1 + F\Theta)} \sqrt{\det(G + 2\pi\alpha'(\Phi + F_0))} - \sqrt{\det(G + 2\pi\alpha'(\Phi + F_0 - (A\theta\partial)F))} \right) = 0.$$  \hspace{1cm} .

Unfortunately calculations become more and more involved as far as we consider higher order terms.

To conclude, let us comment on the background independence of the action $S_{\text{DBI}}$. Since the action $S_{\text{DBI}}$ is explicitly invariant the expression (2.24) we find in above is also invariant. Next what we drop is total derivative terms, so the action $S_{\text{DBI}}$ is invariant at least in the leading order in $\theta$ modulo total derivatives.

3 Alternative way

In fact, there are two ways in getting the noncommutative Yang-Mills theory with the $*$-product structure within the $\sigma$-model. The first one $a \ la$ Seiberg-Witten is to start from the Wilson factor that has...
the ordinary gauge invariance on the classical level and then get the noncommutative gauge invariance on the quantum level just by using a proper renormalization scheme. The second way we propose, based on our experience with the hybrid renormalization scheme, is to start from the Wilson factor that has the noncommutative gauge invariance on the classical level and use a regularization that maintains it on the quantum level. So our aim in this section is to show how to realize this proposal.

Let us define the Wilson factor as

\[
W[C] = P \times \exp \left( i \int_C d\tau A\dot{X} \right) = \sum_{n=0}^{\infty} i^n \int \cdots \int d\tau_n H(\tau_{12}) \cdots H(\tau_{n-1n}) A\dot{X}(\tau_1) \cdots \star A\dot{X}(\tau_n),
\]

(3.1)

where the \(\star\)-product is defined with respect to a translational mode of \(X\). Such a factor coincides with the one defined in (2.8). This is clear just by substituting the expansion of unity \(1 = H(\tau_{12}) \cdots H(\tau_{n-1n}) + (\text{all perms.})\) and changing the variables \(\tau\) in such a way to get the ordering \(\tau_1 > \tau_2 > \cdots > \tau_n\). Note that the definition is nothing but a slightly modified version of the non-Abelian Wilson factor. Of course, it is simply to fit the non-Abelian case into the above definition just by allowing the gauge field to be a \(N \times N\) hermitian matrix and taking a trace. Explicitly,

\[
W[C] = TrP \times \exp \left( i \int_C d\tau A\dot{X} \right).
\]

(3.2)

An important point we should stress here is that the Wilson factor as it is defined in (3.1) or (3.2) is almost invariant under the gauge transformation of the noncommutative Yang-Mills theory namely, \(\delta_X A\dot{i} = \partial_i \lambda - i A_i \* \lambda + i \lambda \* A_i\). The latter is clear from an analogy with the non-Abelian Yang-Mills theory where the gauge invariant expression is given by a trace of \(P \times \exp \int A\dot{X}\). Of course, this is easy to see by directly doing the gauge transforms. Fortunately for us, what saves the day is that we are interested in the partition function. Indeed, as we saw splitting the integration variable \(X^i\) automatically provides the integral over the zero mode (translational mode) \(x^i\). This is exactly what we need because in the noncommutative case the integral does the same job as the trace in the non-Abelian case, i.e. it provides the cyclic property \(\int d^{D+1} x f(x) \* g(x) = \int d^{D+1} x g(x) \* f(x)\) that is crucial for the gauge invariance. Thus the partition function is gauge invariant.

To be more precise, the partition function is given by

\[
Z[A] = \int DX \exp \left( -S_0 + i \int_C d\tau A_i \dot{X}^i \right),
\]

(3.3)

\[
S_0 = \frac{1}{4\pi\alpha'} \int d^D z G_{ij} \partial_a X^i \partial^a X^j.
\]

To see that this partition function indeed leads to the noncommutative Yang-Mills theory it is instructive to compute its asymptotic behaviour as \(\alpha' \to 0\). The main point is that we have to be careful to preserve the noncommutative gauge invariance. It turns out that the renormalization scheme based on the \(\zeta\)-function or some of its modification does this job. The computations are analogous to the ordinary non-Abelian case. As a result, we find

\[
Z[A] = \int [dx] \sqrt{\det G} \left( 1 + \frac{1}{4} (2\pi\alpha')^2 F_{ij} \* F_{kl} G^{ik} G^{jl} + O(\alpha'^4) \right),
\]

(3.4)

with \(F_{ij} = \partial_i A_j - \partial_j A_i - i A_i \* A_j + i A_j \* A_i\).

7 Some similar motivations are provided by the quantization of open strings in a constant \(B\) field background where the deformation parameter \(\theta\) is explicitly related with zero modes (see, e.g., \[1\] and refs. therein).

8 \(H\) means the Heaviside step function.

9 It becomes gauge invariant after the integration over the translational mode of \(X\).
It is straightforward to generalize these computations for the non-Abelian case. The only new thing that appears is $Tr$.

Finally, let us briefly show how to incorporate SUSY within the above formalism. For simplicity, let us consider the NSR formalism. In other words, we add a set of the fermionic fields $\psi^i$ whose metric also is $G$. Following [9], it is simply to suggest what the Wilson factor should be. Moreover, the formalism developed in this paper allows to use the super-space notations. Thus, the Wilson factor is given by

$$W[C] = \mathcal{P} \star \exp\left(i \int_C d\tau ADX \right) = \sum_{n=0}^{\infty} i^n \int \ldots \int d\tau_n H(\tau_{12}) \ldots H(\tau_{n-1n}) ADX(\tau_1) \star \cdots \star ADX(\tau_n),$$

(3.5)

where we use the super-space notations namely, $d\tau = d\tau d\vartheta$, $H(\tau_{nm}) = H(\tau_{nm}) + \vartheta_n \delta_m \delta(\tau_{nm})$, $X^i = X^i + \vartheta \psi^i$, $D = \partial_\tau - \partial_\vartheta$. As in the bosonic case the noncommutative multiplication law is defined in terms of the translational modes of $X^i$.

It is a simple matter to check that the expression (3.5) is almost gauge invariant as well as to compute the partition function in the leading order in $\alpha'$. The result is again given by the noncommutative Yang-Mills action (see Eq.(3.4)). It is also straightforward to fit the non-Abelian case into the above definition just by doing in the same way as we did in the bosonic case.

4 Concluding Comments

Motivated by our experience with the $\sigma$-model, we would like to propose an exactly gauge invariant version of the Wilson factor within the noncommutative Yang-Mills theory,

$$W[C] = \frac{1}{V} \int d^{p+1}x \sqrt{\det G} \ Tr P * \exp\left(i \int_C d\tau A\dot{X} \right), \quad \text{with} \quad V = \int d^{p+1}x \sqrt{\det G} .$$

(4.1)

Some discussions of the Wilson factors and Dirac-Born-Infeld action that have some interference with what we described in above are due to [12].

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References

[1] E. S. Fradkin, and A. A. Tseytlin, Phys.Lett. B163 (1985) 123.
[2] A. Abouelsaood, C. G. Callan, C. R. Nappi, and S. A. Yost, Nucl.Phys. B280 (1987) 599; C. G. Callan, C. Lovelace, C. N. Nappi, and S. A. Yost, Nucl.Phys. B288 (1987) 525.
[3] J. Dai, R. G. Leigh, and J. Polchinski, Mod.Phys.Lett.A4 (1989) 2073.
[4] A. A. Tseytlin, “Born-Infeld action, supersymmetry and string theory”, Report No. Imperial/TP/98-99/67, hep-th/9908105, to appear in the Yuri Golfand memorial volume, ed. M. Shifman, World Scientific (2000).
[5] J. Polchinski, “TASI lectures on D-branes”, Report No. NSF-ITP-96-145, hep-th/9611050.
[6] N. Seiberg, and E. Witten, JHEP 9909 (1999) 032.
[7] B. Pioline, and A. Schwarz, JHEP 9908 (1999) 021.
[8] O. Andreev, R. Metsaev, and A. Tseytlin, Yad. Fiz. 51 (1990) 564; Sov. J. Nucl. Phys. 51 (1990) 359.

[9] O. Andreev, and A. Tseytlin, Nucl.Phys. B311 (1988) 205.

[10] A. A. Tseytlin, Phys.Lett. B202 (1988) 81; H. Dorn, and H.-J. Otto, Zeitschr. f. Phys. C32 (1986) 599.

[11] C. Chu and P. Ho, Nucl.Phys. B550 (1999) 151.
A. Fayyazuddin, and M. Zabine, “A note on bosonic strings in constant $B$ field”, Report No. USITP-99-8, hep-th/9911018.

[12] L. Cornalba, and R. Schiappa, “Matrix Theory Star Products from the Born-Infeld Action”, Report No. CTP2887, hep-th/9907211; L. Cornalba, “D-brane Physics and Noncommutative Yang-Mills Theory”, hep-th/9909081.
N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa, ”Wilson loops in Noncommutative Yang Mills”, Report No. KEK-TH-649, hep-th/9910004.
K. Okuyama, ”A Path Integral Representation of the Map between Commutative and Noncommutative Gauge Fields”, Report No. KEK-TH-655, hep-th/9910138.
P. Watts, ”Noncommutative String Theory, the R-Matrix, and Hopf Algebras”, Report No. DIAS-99-13, hep-th/9911026.
J. Ambjorn, Y.M. Makeenko, J. Nishimura, R.J. Szabo, ”Finite N Matrix Models of Noncommutative Gauge Theory”, Report No. NBI-HE-99-44, hep-th/9911041.
T. Lee, ”Noncommutative Dirac-Born-Infeld Action for D-brane”, Report No. KIAS-P99109, hep-th/9912038.