$D = 3$ unification of curious supergravities

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ABSTRACT: We consider the dimensional reduction to $D = 3$ of four maximal-rank supergravities which preserve minimal supersymmetry in $D = 11, 7, 5$ and 4. Such “curious” theories were investigated some time ago, and the four-dimensional one corresponds to an $\mathcal{N} = 1$ supergravity with 7 chiral multiplets spanning the seven-disk manifold. Recently, this latter theory provided cosmological models for $\alpha$-attractors, which are based on the disk geometry with possible restrictions on the parameter $\alpha$. A unified picture emerges in $D = 3$, where the Ehlers group of General Relativity merges with the $S$-, $T$- and $U$-dualities of the $D = 4$ parent theories.

KEYWORDS: Supergravity Models, Extended Supersymmetry, M-Theory, Differential and Algebraic Geometry

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1 Introduction

Among compactifications of $D = 11$ supergravity on a $7$-manifold to $D = 4$, an interesting $\mathcal{N} = 1$ theory emerges, whose spectrum consists of seven chiral (Wess-Zumino) multiplets living in the seven-disk manifold

\[
\left[ {\text{SL}(2,\mathbb{R}) \over \text{U}(1)} \right]^\otimes 7.
\] (1.1)

This theory, proposed in [1] has some peculiar properties. It is the smallest member of a family of four “left curious” supergravities, defined in $D = (11, 7, 5, 4)$ dimensions, having a scalar manifold of (maximal) rank $(0, 4, 6, 7)$, respectively, and endowed with a minimal number $\nu$ of supersymmetries in the corresponding dimensions, $\nu = (32, 16, 8, 4)$, respectively. Such theories couple naturally to supermembranes and admit these membranes as solutions. In [7] the seven-disk manifold (1.1) was considered as providing possible restrictions on the parameter $\alpha$ of the cosmological $\alpha$-attractors models for inflation, depending on the embeddings of the single one-disk into (1.1).

When compactified on a $7$-manifold $X^7$ with independent Betti numbers $(b_0, b_1, b_2, b_3) = (b_7, b_6, b_5, b_4)$, the number of fields of spin $s = (2, 3/2, 1, 1/2, 0)$ in the resulting $D = 4$ supergravity is given by $n_s = (b_0, b_0 + b_1, b_1 + b_2, b_2 + b_3, 2b_3)$, and we may loosely associate Betti numbers with any supergravity with $n_s$ fields of spin $s$, whether or not manifolds with these Betti numbers actually exist. We may then define a generalized mirror transformation [1]

\[
(b_0, b_1, b_2, b_3) \rightarrow (b_0, b_1, b_2 - \rho/2, b_3 + \rho/2),
\] (1.2)

under which

\[
\rho(X^7) := \sum_{k=0}^{7} (-1)^{k+1} (k + 1) b_k = 7b_0 - 5b_1 + 3b_2 - b_3,
\] (1.3)

changes sign:

\[
\rho \rightarrow -\rho
\] (1.4)
(In the special case $b_1 = 0$, $\rho$ reversal reduces to the reflection symmetry of $G_2$ manifolds defined by Joyce [2, 3]). Generalised self-mirror theories are here defined to be those for which $\rho$ vanishes. Under further toroidal compactification to $D = 4$, the four curious supergravities have $\mathcal{N} = 8, 4, 2, 1$ supersymmetries and Betti numbers $(b_0, b_1, b_2, b_3) = (1, \mathcal{N} - 1, n, 3n - 5\mathcal{N} + 12)$ and thus are all self-mirror. (The $\mathcal{N} = 2$ theory is just the self-mirror $stu$ model [4, 5].) Similarly, we may define a generalized mirror transformation for 6-manifolds $X^6$ [1] with independent Betti numbers $(c_0, c_1, c_2, c_3) = (c_0, c_1, c_2, c_3)$

$$ (c_0, c_1, c_2, c_3) \rightarrow (c_0, c_1, c_2 - \chi/2, c_3 + \chi) \quad (1.5) $$

under which

$$ c( X^6 ) := \sum_{k=0}^{6} (-1)^k c_k = 2c_0 - 2c_1 + 2c_2 - c_3 \quad (1.6) $$

changes sign:

$$ \chi \rightarrow -\chi \quad (1.7) $$

(In the special case $c_1 = 0$, $\chi$ reversal reduces to ordinary mirror symmetry of Calabi-Yau [6]). Generalised self-mirror theories are here defined to be those for which $\chi$ vanishes. In the special case $X^7 = X^6 \times S^1$, $\rho = \chi$ and the two symmetries coincide.

Given the unusual properties and possible cosmological applications of these curious supergravities, in the present note we give a $D = 3$ three-way unified picture in terms of

1) compactifications of $M$-theory in terms of toroidal moduli;

2) dimensional reduction of the four curious supergravities $D = (11, 7, 5, 4)$ to $D = 3$;

3) dimensional reduction of 4 curious supergravities in $D = 4$ to $D = 3$. In particular, the resulting $\mathcal{N} = 2$, $D = 3$ supergravity has the scalar manifold given by the eight-disk manifold

$$ \left[ \frac{\text{SL}(2, \mathbb{R})}{\text{U}(1)} \right]^{\otimes 8} \quad (1.8) $$

which can be regarded as the unification of $S$-, $T$- and $U$- dualities of the $\mathcal{N} = 1$, $D = 4$ corresponding theory mentioned above, augmented by the disk manifold $\frac{\text{SL}(2, \mathbb{R})_{\text{Ehlers}}}{\text{U}(1)}$ pertaining to the $D = 4$ Ehlers group $\text{SL}(2, \mathbb{R})_{\text{Ehlers}}$.

The paper is organized as follows.

In section 2 we recall the embedding of $[\text{SL}(2, \mathbb{R})]^{\otimes 8}$ into $E_{8(8)}$. In section 3 we give an interpretation of the four curious supergravities in terms of sequential reductions of $M$-theory on an eight-manifold with only toroidal moduli of $T^8$, $T^4 \times T^4$, and $T^2 \times T^2 \times T^2 \times T^2$ (“$M$-theoretical path”). Then, in section 4 we consider the so-called “Ehlers path”, by compactifying these theories from $D = 4$ to $D = 3$. Finally, section 5 contains some concluding remarks.
2 \( E_{8(8)} \) and the eight-disk manifold

Almost all exceptional Lie algebras \( \mathcal{E} \) enjoy a rank-preserving (generally non-maximal nor symmetric) embedding of the type

\[
\mathcal{E} \supset [\mathfrak{sl}(2)]^{\oplus r}, \quad r := \text{rank}(\mathcal{E}).
\]

(2.1)

This holds for \( \mathcal{E} = \mathfrak{e}_8, \mathfrak{e}_7, \mathfrak{f}_4, \mathfrak{g}_2 \), with \( r = 8, 7, 4, 2 \), respectively. The unique exception\(^1\) is provided by the rank-6 exceptional algebra \( \mathfrak{e}_6 \), which embeds only \([\mathfrak{sl}(2)]^{\oplus 6}\), and not \([\mathfrak{sl}(2)]^{\oplus 4}\).

In the following treatment, we will focus on the maximally non-compact (i.e., split) real form \( E_{8(8)} \) of \( e_8 \), considering it at the Lie group level \((E_{8(8)} \supset [\text{SL}(2, \mathbb{R})]^{\oplus 8})\), in the context of \( D = 3 \) supergravity theories.

More specifically, starting from \( E_{8(8)} \) we will analyze two paths yielding the same \( \mathcal{N} = 2, D = 3 \) supergravity theory,\(^2\) coupled to 8 matter multiplets, whose scalars coordinatize the completely factorized rank-8 Hodge-Kähler symmetric, eight-disk manifold \((1.8)\).

3 The \( M \)-theory path

The first path starts from \( M \)-theory (or, more appropriately, \( \mathcal{N} = 1, D = 11 \) supergravity), and performs iterated compactifications on tori \( T^8, T^4 \times T^4 \), and on \( T^2 \times T^2 \times T^2 \times T^2 \); this corresponds to the following chain of maximal and symmetric embeddings:

\[
E_{8(8)} \supset \text{SO}(8, 8)
\]
\[
\supset \text{SO}(4, 4) \times \text{SO}(4, 4)
\]
\[
\supset [\text{SO}(2, 2)]^{\oplus 4} \cong [\text{SL}(2, \mathbb{R})]^{\oplus 8}.
\]

(3.1)

(3.2)

(3.3)

Each step of this chain has an interpretation in terms of truncations of the massless spectrum of \( M \)-theory dimensionally reduced to \( D = 3 \), such as to preserve \( \mathcal{N} = 16, 8, 4, 2 \) local supersymmetries. As we discuss below, the last three are obtained keeping only the geometric moduli of the tori \( T^8, T^4 \times T^4 \) and \( T^2 \times T^2 \times T^2 \times T^2 \), respectively. It is worth here recalling that the classical moduli space of a \( d \)-dimensional torus is \(\mathbb{R}^+ \times \frac{\text{SL}(d, \mathbb{R})}{\text{SO}(d)}\), spanned by \( g_{IJ} = g(IJ)\).

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\(^1\)It should be here pointed that \( \mathfrak{e}_6 \) stands on its own among exceptional Lie algebras for at least another reason: it is the unique exceptional Lie algebra which does not embed maximally its principal (Kostant’s) \( \mathfrak{sl}(2) \) algebra. Indeed, while all Lie algebras maximally embed \( \mathfrak{sl}(2) \) (\( \mathfrak{e}_8 \) and \( \mathfrak{e}_7 \) actually maximally embed three and two \( \mathfrak{sl}(2) \)’s, respectively), \( \mathfrak{e}_6 \) embeds its \( \mathfrak{sl}(2) \) through the chain of maximal embeddings \( \mathfrak{e}_6 \supset \mathfrak{f}_4 \supset \mathfrak{sl}(2) \) (in other words, \( \mathfrak{e}_6 \) "inherits" the \( \mathfrak{sl}(2) \) of \( \mathfrak{f}_4 \)).

\(^2\)\( E_{8(8)} \) belongs to the so-called exceptional \( E_n(n) \)-sequence [9–11] of symmetries of maximal supergravities in \( 11 - n \) dimensions.

\(^3\)For a thorough analysis of the geometric structure of scalar manifolds of \( D = 3 \) supergravity theories, see [12].

\(^4\)The rank of a manifold is defined as the maximal dimension (in \( \mathbb{R} \)) of a flat (i.e., with vanishing Riemann tensor), totally geodesic submanifold (see e.g. §6, page 209 of [13]).
whereas the quantum one (in a stringy sense) reads
\[
\mathcal{M}_d := \frac{\text{SO}(d,d)}{\text{SO}(d) \times \text{SO}(d)}, \text{ spanned by } g_{IJ} = g_{(IJ)} \text{ and } B_{IJ} = B_{[IJ]}.
\] (3.5)

The first, starting step of the \textit{M-theoretical path} (3.1)–(3.3) corresponds to:\(^{5}\)
\[
\text{M-theory } T^8 \xrightarrow{\text{geom+non-geom}} \mathcal{N} = 16, D = 3 : \frac{\text{E}_8(8)}{\text{SO}(16)}, \quad (B,F) = (128,128)
\] (3.6)

namely a compactification retaining \textit{both} geometric ($g_{IJ}, A_{\mu IJ}$) and non-geometric ($g_{\mu I}, A_{IJK}$) moduli of $T^8$, down to maximal supergravity in $D = 3$ [15] ($I, J, K = 1, \ldots, 8$, and $\mu = 0,1,2$); note that the 128 bosonic massless degrees of freedom can be organized in \text{SO}(8) irreprs. as follows:
\[
g_{IJ}, A_{\mu IJ}, g_{\mu I}, A_{IJK},
\] (3.7)

where the 1-form $A_{\mu IJ} = A_{\mu[IJ]}$ (playing the role of the “\textit{M-theoretical B-field}”) gets then dualized to scalar fields $A_{IJ}$ in $D = 3$.

The next step corresponds to the first, maximal and symmetric embedding (3.1), which amounts to retaining only the geometric moduli of $T^8$ (i.e., to setting $g_{\mu I} = 0 = A_{IJK}$ in the bosonic sector), thus giving rise upon compactification to half-maximal supergravity coupled to $n = 8$ matter multiplets in $D = 3$:
\[
\text{M-theory } T^8 \xrightarrow{\text{geom}} \mathcal{N} = 8, D = 3, n = 8 : \frac{\text{SO}(8,8)}{\text{SO}(8) \times \text{SO}(8)}, \quad (B,F) = (64,64)
\] (3.8)

The subsequent maximal and symmetric embedding (3.2) corresponds to a compactification on $T^4 \times T^4$ retaining only the corresponding geometric moduli ($i,j = 1, \ldots, 4$, and $i',j' = 5, \ldots, 8$):
\[
g_{ij}, A_{\mu ij}, g_{i'j'}, A_{\mu i'j'},
\] (3.9)

thus giving rise to the following $\mathcal{N} = 4, D = 3$ supergravity model:
\[
\text{M-theory } T^4 \times T^4 \xrightarrow{\text{geom}} \mathcal{N} = 4, D = 3, n = 8 : \frac{\text{SO}(4,4)}{\text{SO}(4) \times \text{SO}(4)} \times \frac{\text{SO}(4,4)}{\text{SO}(4) \times \text{SO}(4)}, \quad (B,F) = (32,32)
\] (3.10)

The last step is given by the maximal and symmetric embedding (3.3), corresponding to a compactification on $T^2 \times T^2 \times T^2 \times T^2$ retaining only the related geometric moduli
\[
g_{11}, g_{12}, g_{22}, A_{\mu 12}, g_{33}, g_{34}, g_{44}, A_{\mu 34}, g_{55}, g_{56}, g_{66}, A_{\mu 56}, g_{77}, g_{78}, g_{88}, A_{\mu 78},
\] (3.11)

thus giving rise to the $\mathcal{N} = 2, D = 3$ supergravity model whose scalar manifold is given by the eight-disk manifold (1.8):
\[
\text{M-theory } T^2 \times T^2 \times T^2 \times T^2 \xrightarrow{\text{geom}} \mathcal{N} = 2, D = 3 : \left[ \text{SL}(2,\mathbb{R}) \right]^{\otimes 8}, \quad (B,F) = (16,16)
\] (3.12)

Some comments are in order.

\(^{5}\) “$B$” and “$F$” denote the number of bosonic and fermionic massless degrees of freedom throughout.
1. All symmetric scalar manifolds in (3.6), (3.8), (3.10) and (3.12) have rank 8, as a consequence of the fact that all embeddings of the chain (3.1)–(3.3) are rank-preserving.

2. The theories (3.6), (3.8), (3.10) and (3.12) are nothing but the $D = 3$ reduction of the four curious supergravities, studied in [1] and mentioned in section 1. These latter are defined in $D = q + 3 = 11, 7, 5, 4$ Lorentzian space-time dimensions (with $q := \dim_{\mathbb{R}} \mathbb{A} = 8, 4, 2, 1$, where $\mathbb{A} = \mathbb{O}$ (octonions), $\mathbb{H}$ (quaternions), $\mathbb{C}$ (complex numbers), $\mathbb{R}$ (reals) denote the four Hurwitz division algebras), with scalar manifolds of rank 0, 4, 6, 7 respectively. As observed in [1], such $\mathcal{N} = 8, 4, 2, 1$ curious supergravities respectively correspond to $\mathcal{N} - 1 = 7, 3, 1, 0$ lines of the Fano plane, and hence they admit a division algebraic interpretation consistent with the so-called “black-hole/qubit” correspondence (cfr. e.g. [16] for an introduction and a list of refs.). By further compactifying them respectively on $T^8$, $T^4$, $T^2$, $T^1 = S^1$ down to $D = 3$, the rank of the corresponding scalar manifold (after dualization) increase by 8, 4, 2, 1, so that all the resulting $D = 3$ theories have rank-8 scalar manifolds, as given by (3.6), (3.8), (3.10) and (3.12). They have $\mathcal{N} = 2^4, 2^3, 2^2, 2$ local supersymmetry in $D = 3$, with $2^8$, $2^7$, $2^6$ and $2^5$ total number of massless states, respectively. In this perspective, the dimensional reduction to $D = 3$ provides a unified view of the curious supergravities.

4 The Ehlers path

The second path yielding the $\mathcal{N} = 2, D = 3$ supergravity theory with scalar manifold (1.8) starts with the so-called Ehlers embedding (cfr. e.g. [17], and refs. therein) for maximal supergravity in $D = 4 \rightarrow D = 3$, and then proceeds with a chain of maximal, symmetric and rank-preserving embeddings which has already been considered in [7, 14, 18]:

$$E_{8(8)} \supset E_{7(7)} \times \text{SL}(2, \mathbb{R})_{\text{Ehlers}}$$

$$\supset \text{SO}(6, 6) \times \text{SL}(2, \mathbb{R})_{\text{Ehlers}} \times \text{SL}(2, \mathbb{R})$$

$$\supset \text{SO}(4, 4) \times [\text{SL}(2, \mathbb{R})]^2 \times \text{SL}(2, \mathbb{R})_{\text{Ehlers}} \times \text{SL}(2, \mathbb{R})$$

$$\supset [\text{SL}(2, \mathbb{R})]^8$$

Since this path, which we name Ehlers path, starts with a $D = 4 \rightarrow D = 3$ dimensional reduction, it is immediate to realize that the $D = 3$ scalar manifolds given in (3.6), (3.8), (3.10) and (3.12) are nothing but the dimensional reduction of the $D = 4$ cosets of $\mathcal{N} = 8, 4, 2, 1$ curious supergravities with rank-7 scalar manifolds (after dualization; cfr. table XVIII of [1]).

While for $\mathcal{N} = 8, 4, 2$ the dimensional reduction $D = 4 \rightarrow D = 3$ is well-known from the study of Maxwell-Einstein systems coupled to non-linear sigma models ([19], thereby including the c-map [20, 21] relating projective special Kähler manifolds to quaternionic
manifolds), for $\mathcal{N} = 1$ the dimensional reduction reads

$$ (B, F) = (16, 16) : \begin{array}{cccc}
\left[ \frac{\text{SL}(2, \mathbb{R})}{\text{U}(1)} \right]^{\otimes 7} & \rightarrow & \left[ \frac{\text{SL}(2, \mathbb{R})}{\text{U}(1)} \right]^{\otimes 8} \\
\mathcal{N} = 1, D = 4, n_c = 7, n_v = 0 & & \mathcal{N} = 2, D = 3, n_c = 8, n_v = 0
\end{array} \tag{4.5} $$

and it stands on a different footing. Indeed, the $\mathcal{N} = 1$, $D = 4$ supergravity theory is coupled only to 7 chiral multiplets, with no vectors at all. Therefore, under (spacelike) dimensional reduction $D = 4 \rightarrow D = 3$, the chiral multiplets’ scalar manifold (1.1) gets enlarged only by a further factor manifold $\frac{\text{SL}(2, \mathbb{R})}{\text{U}(1)} \otimes \text{U}(1)$, spanned by the axio-dilaton given by the $S^1$-radius of compactification and by the dualization of the corresponding Kaluza-Klein vector. In other words, the added $\frac{\text{SL}(2, \mathbb{R})}{\text{U}(1)}$ manifold pertains to the two degrees of freedom of the $D = 4$ massless graviton (since in $D = 3$ the graviton does not propagate any degree of freedom): as mentioned in section 1, the \textit{seven-disk manifold} (1.1) $[1, 7]$ gets enlarged to the \textit{eight-disk manifold} (1.8) by including the $D = 4$ Ehlers group $\text{SL}(2, \mathbb{R})_{\text{Ehlers}}$.

Some observations are:

1. All symmetric scalar manifolds in (4.6), (4.7) and (4.8) have rank 7, as a consequence of the fact that all embeddings of the chain (4.1)–(4.4) are rank-preserving.

2. The chain of embeddings (4.1)–(4.4) has been used in [18] (also cfr. [14]) to study the tripartite entanglement of seven qubits inside $E_7$. Moreover, it was recently exploited in [7] in order to obtain the $\mathcal{N} = 1$, $D = 4$ theory with 7 WZ multiplet given in the fourth line of (4.5).

3. The maximal and symmetric embedding (4.2) corresponds to the truncation of maximal $D = 4$ supergravity to half-maximal supergravity coupled to 6 matter (vector) multiplets:

$$ \frac{E_{7(7)}}{\text{SU}(8)} \rightarrow \frac{\text{SL}(2, \mathbb{R})}{\text{U}(1)} \times \frac{\text{SO}(6, 6)}{\text{SO}(6) \times \text{SO}(6)} \quad \mathcal{N} = 8, D = 4, (B, F) = (128, 128) \rightarrow \mathcal{N} = 4, D = 4, n_c = 6, (B, F) = (64, 64) \tag{4.6} $$

4. The subsequent step (4.3) corresponds to the truncation of half-maximal $D = 4$ supergravity coupled to 6 vector multiplets to the $\mathcal{N} = 2, D = 4$ \textit{stu} model coupled to 4 hypermultiplets, whose quaternionic scalars coordinatize the symmetric scalar manifold $\frac{\text{SO}(4, 4)}{\text{SO}(3) \times \text{SO}(3)}$; since this latter is the c-map [20] of the corresponding vector-multiplets’ projective special Kähler manifold $\left[ \frac{\text{SL}(2, \mathbb{R})}{\text{U}(1)} \right]^{\otimes 3}$, this model is \textit{self-mirror} (also cfr. e.g. [22]):

$$ \left[ \frac{\text{SL}(2, \mathbb{R})}{\text{U}(1)} \right] \times \frac{\text{SO}(6, 6)}{\text{SO}(6) \times \text{SO}(6)} \rightarrow \left[ \frac{\text{SL}(2, \mathbb{R})}{\text{U}(1)} \right]^{\otimes 3} \times \frac{\text{SO}(4, 4)}{\text{SO}(4) \times \text{SO}(4)} \quad \mathcal{N} = 4, D = 4, n_c = 6, (B, F) = (64, 64) \rightarrow \mathcal{N} = 2, D = 4, n_v = 3, n_H = 4, \text{self-mirror \textit{stu} model, } (B, F) = (32, 32) \tag{4.7} $$

5. The last step (4.3) corresponds to the truncation of the self-mirror $D = 4$ \textit{stu} model to an $\mathcal{N} = 1$, $D = 4$ theory with 7 WZ multiplets, whose scalars span the \textit{seven-disk}
manifold (1.1) [1, 7]:

\[
\left[ \frac{\text{SL}(2, R)}{U(1)} \right]^3 \times \frac{\text{SO}(4, 4)}{\text{SO}(4) \times \text{SO}(4)} \quad \rightarrow \quad \left[ \frac{\text{SL}(2, R)}{U(1)} \right]^7.
\]

This step is non-trivial for what concerns the retaining of a $N=1$ local supersymmetry in the gravity theory with non-linear sigma model given by (1.8). Besides the necessary truncation of the $N=1$ gravitino multiplet coming from the supersymmetric $\mathcal{N}=2 \rightarrow \mathcal{N}=1$ reduction of the $\mathcal{N}=2$ gravity multiplet, one has to truncate all $\mathcal{N}=1$ vector multiplets coming from the supersymmetry reduction of the three $\mathcal{N}=2$ vector multiplets; furthermore, a truncation of half of the $\mathcal{N}=1$ chiral multiplets stemming from the supersymmetry reduction of the four $\mathcal{N}=2$ hypermultiplets must be performed. This last step is particularly challenging for the consistency with local $\mathcal{N}=1$ supersymmetry, which is however granted by the results in [23] (also cfr. [24]); see, in particular, the discussion around eq. (6.145) therein.

5 Conclusion

Summarizing, there exist (at least) three different ways to obtain the four $\mathcal{N}=16, 8, 4, 2$ curious supergravities (3.6), (3.8), (3.10) and (3.12) with symmetric scalar manifolds of (maximal) rank 8 in $D=3$:

1. Toroidal compactification of $M$-theory from $D=11$ to $D=3$, respectively retaining geometric and non-geometric moduli of $T^8$, and then geometric moduli of $T^8$, of $T^4 \times T^4$, and of $T^2 \times T^2 \times T^2 \times T^2$. This is given by the $M$-theoretical path (3.1)–(3.3) discussed in section 3.

2. Toroidal compactification of the four curious supergravities [1] (defined in 11, 7, 5, 4 dimensions) respectively on $T^8$, $T^4$, $T^2$, $T^1=S^1$ down to $D=3$; this is discussed at point 3 of section 3.

3. $S^1$-dimensional reduction $D=4 \rightarrow D=3$ of the $\mathcal{N}=8, 4, 2, 1$, $D=4$ curious supergravities with rank-7 scalar manifolds (after dualization; cfr. table XVIII of [1]). This is given by the Ehlers path (4.1)–(4.4) discussed in section 4.

By comparing the two paths (3.1)–(3.3) and (4.1)–(4.4), it is evident that they exhibit different and features.

The $M$-theoretical path (3.1)–(3.3) is deeply rooted in $M$-theory, and it makes “octality”, pertaining to the symmetry of the fully factorised rank-8 Hodge-Kähler symmetric coset (eight-disk manifold (1.8)) in $D=3$, completely manifest: the $\text{SL}(2, R)$’s of $T$-duality (from the $T^2$-factors of the 8-dimensional internal manifold), the $\text{SL}(2, R)$’s of $S$-duality

\footnote{In a different framework, more pertaining to the first path (3.1)–(3.3) to (1.8), $\mathcal{N}=1$ local supersymmetry for the $D=4$ theory with scalar manifold (1.8) was obtained in [1] by considering $M$-theory compactified on a suitable 7-dimensional manifold with $G_2$-structure.}
and $U$-duality, and the $D = 4$ Ehlers group $\text{SL}(2, \mathbb{R})_{\text{Ehlers}}$ (of gravitational origin) get unified, and they stand on the same footing.

On the other hand, the Ehlers path (4.1)–(4.4), makes only “septality”, pertaining to the full-fledged symmetry of the fully factorised rank-7 Hodge-Kähler symmetric coset in $D = 4$ (seven-disk manifold (1.1)), completely manifest: only the $\text{SL}(2, \mathbb{R})$’s of $S$-, $T$- and $U$- dualities get unified.

However, notwithstanding the first step (4.1) which seems to single out the $D = 4$ Ehlers group $\text{SL}(2, \mathbb{R})_{\text{Ehlers}}$, a complete equivalence between the two paths is reached at their final steps. It would be worth pursuing an $E_{11}$ interpretation [25] of these four maximal rank theories preserving minimal supersymmetry in $D = 11, 7, 5, 4$.

We also recall that in $D = 4$ the four curious supergravities with $N = 8, 4, 2, 1$ are associated with $7, 3, 1, 0$ vertices of the Fano plane [1]. Similarly, in $D = 3$ the $N = 16, 8, 4, 2$ theories are associated with the $7, 3, 1, 0$ quadrangles of the Fano plane and the dual Fano plane.\footnote{See the decompositions of the 56 under $E_7 \supset [\text{SL}(2)]^\otimes 7$ and the 248 under $E_8 \supset [\text{SL}(2)]^\otimes 8$ in [14].}

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