Considerations regarding the accuracy of fractional numerical computations

Octavian Postavaru · Flavius Dragoi · Antonela Toma

Received: 7 October 2021 / Revised: 20 May 2022 / Accepted: 30 June 2022 /
Published online: 8 August 2022
© The Author(s) 2022

Abstract
For solving numerically fractional differential equations, we have to take into account a rising flow of works as (Dehestani et al. in Appl Math Comput 336:433–453, 2018, https://doi.org/10.1016/j.amc.2018.05.017, Rahimkhani et al. Appl Math Model 40:8087–8107, 2016, https://doi.org/10.1016/j.apm.2016.04.026) that show the advantage of using the transformation $x \rightarrow x^\alpha$. In this paper, we aim to explain this transformation, and using the acquired knowledge, we are also discussing a method that is able to improve the accuracy of the numerical results for the delay fractional equations. We conclude the paper with two numerical examples to illustrate the analysis of this paper.

Keywords Fractional differential equations (primary) · Caputo’s derivative · Sharp bound · Polynomial coefficients

Mathematics Subject Classification 35R11 (primary) · 26A33 · 26D20 · 41A10

1 Preliminaries

The fractional differential equations (FDEs) are generalizations of the conventional differential equations to an arbitrary order. From the very beginning, FDEs have been regarded as an important tool in understanding the complex phenomena of nature.
However, it is remarkable to be mentioned that the power of fractional calculus allows description of complex phenomena in an analytical way. First time borrowed in classical mechanics and engineering, the tools of fractional calculus have found extensive applications in different branches of science, consequently, the significant increase in interest in developing numerical schemes for FDEs solutions. An overview of fractional order differential operators’ development can be found, for example, in [5, 8, 19].

In this paper we turn our attention to equations of the following type

\[ D^\gamma f(x) = h(x, f(x), g(x)), \quad 0 \leq x \leq 1, \quad (1.1) \]

with the Caputo fractional derivative of order \( \gamma \) defined as [14]

\[
(D^\gamma f)(x) = \frac{1}{\Gamma(n-\gamma)} \int_0^x \frac{f^{(n)}(s)}{(x-s)^{\gamma+1-n}} ds, \quad n-1 < \gamma \leq n, \quad n \in \mathbb{N},
\]

where \( \gamma > 0 \) is the order of the derivative, \( n \) is the smallest integer greater than \( \gamma \). It is well known that

\[
D^\gamma x^\alpha = \begin{cases} 
0, & \gamma \in \mathbb{N}_0, \quad \alpha < \gamma, \\
\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma+1)} x^{\alpha-\gamma}, & \text{otherwise}.
\end{cases} \quad (1.2)
\]

The involved Riemann-Liouville fractional integral operator is defined as

\[
I^q f(x) = \begin{cases} 
\frac{1}{\Gamma(q)} \int_0^x \frac{f(s)}{(x-s)^{1-q}} ds, & q > 0, \\
f(x), & q = 0.
\end{cases} \quad (1.3)
\]

To be able to treat numerically differential equation (1.1), we reduce the dynamical system to a system of algebraic equations, by using the approximation

\[
f_N(x) = \sum_{n=0}^{N-1} c_n x^n. \quad (1.4)
\]

In order to solve numerically some fractional differential equations, it has been shown that using the transformation \( x \to x^\alpha \) together with the Bernoulli wavelets [18] or with the Legendre-Laguerre functions [4], improves the accuracy of the numerical results. In this paper, we show that this transformation always improves the known results. Of course, the same method can be extended for calculating solutions of systems of fractional differential equation.
2 Error analysis

We start our discussion by considering the Legendre polynomials, when Eq. (1.4) becomes

$$f_N(x) = \sum_{n=0}^{N-1} c_n P_n(x),$$

and then, we show how this analysis can be extended to other types of polynomials. In the case of the Legendre polynomials $P_n(\cos \theta)$, it was shown that \[1, 10\]

$$\left( n + \frac{1}{2} \right)^{1/2} \sin^{1/2} \theta |P_n(\cos \theta)| \left( \frac{2}{\pi} \right)^{-1/2} < 1, \quad 0 \leq \theta \leq \pi,$$

an inequality which actually strengthened the Bernstein inequality, by replacing the factor $n^{1/2}$ by $(n + 1/2)^{1/2}$. It has been shown that for any $\epsilon > 0$, the factor $(n + 1/2)^{1/2}$ can not be replaced by $(n + 1/2 + \epsilon)^{1/2}$. Previously established by Stieltjes, the constant $\sqrt{2/\pi}$ instead of $A$ was introduced by Bernstein, and it turned out to be the best possible improvement [20].

In order to answer why the transformation $\cos \theta \rightarrow \cos^\alpha \theta$ improves the numerical computation, we ask how this transformation affects Eq. (2.1). First of all, this transformation restricts the interval $[0, \pi]$ to $[0, \pi/2]$ (the transformation it is not well defined on the interval $[\pi/2, \pi]$).

**Lemma 1** For $\theta \in [0, \pi/2], n \geq 0$ and $\alpha \geq 0$ we have

$$M_n^\alpha(\cos \theta) = \left( n + \frac{1}{2} \right)^{1/2} (1 - \cos^{2\alpha} \theta)^{1/4} |P_n(\cos^\alpha \theta)| \left( \frac{2}{\pi} \right)^{-1/2} < 1.$$

**Proof** The above inequality must also be fulfilled in the following situation

$$\left( n + \frac{1}{2} \right)^{1/2} \max_{\theta \in [0, \pi/2]} (1 - \cos^{2\alpha} \theta)^{1/4} |P_n(\cos^\alpha \theta)| \equiv \left( n + \frac{1}{2} \right)^{1/2} g_n < \left( \frac{2}{\pi} \right)^{1/2}.$$

In analogy to [20], we have

$$g_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

A direct calculation determines the relationship

$$\left( 2n + \frac{5}{2} \right)^{1/2} g_{n+1} > \left( 2n + \frac{1}{2} \right)^{1/2} g_n.$$
and from inequalities associated with Wallis’s product [7], we find

\[
\lim_{n \to \infty} \left(2n + \frac{1}{2}\right)^{1/2} g_n = \left(\frac{2}{\pi}\right)^{1/2}.
\]

Therefore, the inequality (2.1) is not improved by the transformation \(\cos \theta \rightarrow \cos^\alpha \theta\). The visual interpretation of this transformation is given in Fig. 1, for \(n = 4\) and \(\alpha = 0.6, 1, 1.4\). We see that by changing \(\alpha\), we modify the position of peaks and not their amplitude.

This answer shows that the advantage of replacing \(\cos \theta\) by \(\cos^\alpha \theta\) in the numerical computations comes from somewhere else, and the first thought is to see if this improvement is not somehow related to the value of the fractional derivative (or integration).

We want to see how the order of the polynomial affects our results, therefore we should introduce some helpful mathematical preliminaries. First, we define the weighted semi-norm

\[
\| f \| = \int_0^1 f'(x) \frac{1}{(1-x^2)^{1/4}} \, dx.
\]

**Theorem 1** Let

\[
f_N(x) = \sum_{n=0}^{N-1} c_n' Q_n(x),
\]

with \(Q_n(x)\) a polynomial of the type (1.4), and let \(f^{(n)}\), \(n = 1, m - 1\), be absolutely continuous functions with \(f^{(m)}\) of bounded variations. Then for \(m \in [1, N - 1]\), we have:

\[\square\]
i) when $m = 1$, we get
\[
\| f(x) - f_N(x) \| \leq \frac{4 \| f' \|}{\sqrt{\pi(2N - 5)}},
\]

ii) when $m \geq 2$, then
\[
\| f(x) - f_N(x) \| \leq \frac{2 \| f^{(m)} \|}{(m - 1) \sqrt{\pi(2N - 2m - 1)}} \prod_{k=2}^{m} \left( N - k + \frac{1}{2} \right)^{-1}.
\]

**Proof** If the polynomial $Q_n(x)$ is of the Legendre type, we find the proof of this statement in [10].

Let
\[
Y = \text{span}\{c'_0, c'_1, c'_2, \ldots, c'_{N-1}\}.
\]

Since $Y$ is a finite dimensional vector space, $f$ has the unique best approximation out of $Y$ such as $f_0 \in Y$, that is,
\[
\| f - f_0 \| \leq \| f - y \|.
\]

Because $f_0 \in Y$, there exist unique coefficients $c'_0, c'_1, \ldots, c'_{N-1}$ such that
\[
f \simeq f_0 = \sum_{n=0}^{N-1} c'_n Q_n(x).
\]

Since the best approximation is unique,
\[
\| f - \sum_{n=0}^{N-1} c_n P_n \| = \| f - \sum_{n=0}^{N-1} c'_n Q_n \|,
\]
using the results [10], we complete the proof. \qed

Basically, any polynomial can be put into the form (1.4). For example, for the Legendre polynomials we have the transformation
\[
P_n(x) = \Xi T(x),
\]
where
\[
T_n(x) = (1, x, x^2, \ldots, x^{N-1})^T.
\]
and

$$\Xi = \begin{pmatrix} \xi_{00} & 0 & 0 & \ldots & 0 \\ \xi_{10} & \xi_{11} & 0 & \ldots & 0 \\ \xi_{20} & \xi_{21} & \xi_{22} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_{m0} & \xi_{m1} & \xi_{m2} & \ldots & \xi_{mm} \end{pmatrix},$$

with $\xi_{ij} = 2 \left(\binom{i}{j} \frac{(i + j - 1)}{i}\right)$, and $i, j \in \overline{0, N - 1}$.

### 3 Discussion

It is a well-known fact that polynomials of high order are sensitive to perturbations in their coefficients.

**Problem 1** We want to discuss the following problem

$$D^\gamma f(x) = f(x) + g(x), \quad (3.1)$$

where $g(x)$ is a known function.

Using Eqs. (1.2) and (1.4), we may write

$$D^\gamma f(x^\alpha) = D^\gamma \sum_{n=0}^{N} c_n x^{\alpha n} = \sum_{n=1}^{N} c_n \Gamma(n\alpha + 1) \frac{\Gamma(n\alpha - \gamma + 1)}{\Gamma(n\alpha - \gamma + 1)} x^{\alpha n - \gamma},$$

and we get the following equation

$$\sum_{n=1}^{N} c_n \frac{\Gamma(n\alpha + 1)}{\Gamma(n\alpha - \gamma + 1)} x^{\alpha n - \gamma} = \sum_{n=0}^{N} c_n x^{\alpha n} + g(x),$$

which can be solved by the collocation method. If we denote by $x_i, i = \overline{0, N - 1}$, the Newton-Cotes nodes, we get the following system of equations

$$c_0 + c_1 x_1^{\alpha} \left(1 - \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma+1)} x_1^{1-\gamma/\alpha}\right) + c_2 x_2^{2\alpha} \left(1 - \frac{\Gamma(2\alpha+1)}{\Gamma(2\alpha-\gamma+1)} x_2^{1-\gamma/(2\alpha)}\right) + \cdots + c_{N-1} x_{N-1}^{(N-1)\alpha} \left(1 - \frac{\Gamma((N-1)\alpha+1)}{\Gamma((N-1)\alpha-\gamma+1)} x_{N-1}^{1-\gamma/((N-1)\alpha)}\right) = -g(x_i). \quad (3.2)$$

On the other hand, when $\alpha = \gamma$, equation (3.1) becomes

$$\sum_{n=1}^{N} c_n \frac{\Gamma(n\alpha + 1)}{\Gamma(n\alpha - \alpha + 1)} x^{\alpha (n-1)} = \sum_{n=0}^{N} c_n x^{n\alpha} + g(x).$$
Considerations regarding the accuracy of fractional... 1791

and we can determine the coefficients $c_i$ by identifying the coefficients of $x^{n\alpha}$, getting

$$c_0 = 0,$$
$$c_k = \frac{g(x)}{\Gamma(k\alpha + 1)}, \quad k \geq 1.$$  \hspace{1cm} (3.3)

The difference between Eqs. (3.2) and (3.3) is major: while for solving the first system of equations we need numerical substitution and consequently an approximation of the $c_i$ coefficients, the second result is exact (fractional). Hence, one can write

$$\| D^\gamma f(x^\alpha) - D^\gamma f_N(x^\alpha) \| \leq \| D^\gamma f(x^\gamma) - D^\gamma f_N(x^\gamma) \|, \quad \alpha \neq \gamma.$$ 

Therefore, according to Theorem 1, increasing the polynomial degree should increase the accuracy of the result, but with the increase in the polynomial degree, the coefficient determination is required to be very good as the polynomials become sensitive to the perturbation of the coefficients. In conclusion, the transformation $x \to x^\alpha$ is very indicative when dealing with the fractional derivative $D^\alpha$.

We may apply this result to delay problems.

**Problem 2** We consider the following differential equation

$$D^\gamma f(x) = f(x) + f(x - \tau) + g(x),$$

where $g(x)$ is a known function.

For Taylor polynomials, we have (3)

$$T_m(x - \tau) = \Lambda T_m(x),$$  \hspace{1cm} (3.4)

where $\Lambda$ is the following matrix

$$\Lambda = \begin{pmatrix}
\lambda_{00} & 0 & 0 & \ldots & 0 \\
\lambda_{10} & \lambda_{11} & 0 & \ldots & 0 \\
\lambda_{20} & \lambda_{21} & \lambda_{22} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda_{m0} & \lambda_{m1} & \lambda_{m2} & \ldots & \lambda_{mm}
\end{pmatrix},$$

with $\lambda_{ij} = \binom{i}{i-j} (-\tau)^{i-j}$.

The property (3.4) can be used in the expression of $f(x - \tau)$, and applying a similar analysis as the one applied in Problem 1, we conclude that the transformation $x \to x^\alpha$ gives the best result.

**Remark 1** Problem 1 can be considered as a particular case of Problem 2.
4 Examples

In this section, we consider two examples that we solve by two different methods: fractional-order hybrid of block-pulse functions and Bernoulli polynomials, and fractional-order Bernoulli wavelets. The numerical calculations are performed by using Mathematica 11.

4.1 Example 1

In this example, we consider the equation [11, 18]

\[
D^q f(x) = \left( \frac{x^{q+1}}{\Gamma(q+2)} \right)^2 + x - f^2(x), \quad 0 < q \leq 1, \quad 0 \leq x \leq 1,
\] (4.1)

with

\[ f(0) = 0. \]

The solution of this equation is

\[ f(x) = \frac{x^{q+1}}{\Gamma(q+2)}. \]

4.1.1 Fractional-order hybrid of block-pulse functions and Bernoulli polynomials

As the title suggests, in this subsection we solve the proposed problem using fractional-order hybrid of block-pulse functions and Bernoulli polynomials (FOHBPB). Details on the theoretical method and error calculation can be found in [15].

To solve this problem, we use the notation

\[
D^q f(x) = C^T B^\alpha(x),
\] (4.2)

where

\[ C = [c_{10}, \ldots, c_{N0}, c_{11}, \ldots, c_{N1}, \ldots, c_{1M}, \ldots, c_{NM}]^T, \]

and

\[ B^\alpha(x) = [b_{10}^\alpha(x), \ldots, b_{N0}^\alpha(x), b_{11}^\alpha(x), \ldots, b_{N1}^\alpha(x), \ldots, b_{1M}^\alpha(x), \ldots, b_{NM}^\alpha(x)]^T. \]

The function \( b_{nm}^\alpha \), with \( n = 0, 1, \ldots, N \) and \( m = 0, 1, \ldots, M \), is presented in Appendix A. Applying the Riemann-Liouville operator (1.3) to the equation (4.2), we obtain

\[
f(x) = C^T B^\alpha(x, q) + f(0) + f'(0)x,
\] (4.3)
Considerations regarding the accuracy of fractional... 1793

Table 1  In the first two columns, we have presented the FOHBPB method for $N = 1$ and $M = 4$. In the last two columns, we have presented the FOBW method for $k = 2$ and $M = 7$, for Example 1

| $x$  | FOHBPB $q = \alpha = 0.8$ | FOHBPB $q = 0.8, \alpha = 1$ | FOBW $q = \alpha = 0.5$ | FOBW $q = 0.5, \alpha = 1$ |
|------|-------------------|-------------------------------|-------------------|-------------------|
| 0    | 0                 | $3.3 \times 10^{-17}$         | $8.4 \times 10^{-10}$ | $3.0 \times 10^{-4}$ |
| 0.1  | 0                 | $2.8 \times 10^{-17}$         | $1.4 \times 10^{-9}$  | $3.5 \times 10^{-5}$ |
| 0.2  | 0                 | $4.2 \times 10^{-17}$         | $1.7 \times 10^{-9}$  | $2.2 \times 10^{-5}$ |
| 0.3  | 0                 | $4.2 \times 10^{-17}$         | $1.9 \times 10^{-8}$  | $2.1 \times 10^{-5}$ |
| 0.4  | 0                 | $5.6 \times 10^{-17}$         | $1.4 \times 10^{-8}$  | $1.8 \times 10^{-5}$ |
| 0.5  | 0                 | $5.6 \times 10^{-17}$         | $1.1 \times 10^{-8}$  | $1.6 \times 10^{-5}$ |
| 0.6  | 0                 | $5.6 \times 10^{-17}$         | $8.9 \times 10^{-9}$  | $1.0 \times 10^{-5}$ |
| 0.7  | 0                 | $7.4 \times 10^{-9}$          | $1.9 \times 10^{-5}$  | $1.0 \times 10^{-4}$ |
| 0.8  | 0                 | $6.9 \times 10^{-9}$          | $1.6 \times 10^{-4}$  | $1.0 \times 10^{-4}$ |
| 0.9  | 0                 | $7.7 \times 10^{-9}$          | $7.0 \times 10^{-4}$  | $7.0 \times 10^{-4}$ |
| 1    | 0                 | $1.1 \times 10^{-16}$         | $1.0 \times 10^{-8}$  | $2.2 \times 10^{-3}$ |

where

$$\overline{B}^\alpha (x, q) = [I^q b_{10}^\alpha (x), \ldots, I^q b_{N0}^\alpha (x), I^q b_{11}^\alpha (x), \ldots, I^q b_{N1}^\alpha (x), \ldots, I^q b_{NM}^\alpha (x)]^T,$$

and the quantities $I^q b_{nm}^\alpha (x)$, with $n = 0, 1, \ldots, N$ and $m = 0, 1, \ldots, M$, are presented in Appendix A.

Substituting Eqs. (4.2) and (4.3) into Eq. (4.1), we obtain

$$C^T \overline{B}^\alpha (x) = \left( \frac{x^{q+1}}{\Gamma (q+2)} \right)^2 + x - \left( C^T \overline{B}^\alpha (x, q) \right)^2,$$

and by collocating this equation in the Newton-Cotes nodes $x_i$ given by

$$x_i = \left( \frac{i + 1}{2N(M+1)} \right)^{1/\alpha}, \quad i = 0, 1, \ldots, 2N(M+1) - 2,$$

we get $N(M+1)$ algebraic equations which can be solved by Newton’s iterative method.

In order to solve this example, we consider $N = 1$ and $M = 4$. In Table 1, we compared $q = \alpha = 0.8$ with $q = 0.8, \alpha = 1$. As you can see, for the case $q = \alpha = 0.8$, we got the exact result.

4.1.2 Fractional-order Bernoulli wavelets

In this section, we analyze the method of numerical solving of differential equations using the fractional-order Bernoulli wavelets (FOBW) method, described in [18].
We start from the notation
\[ D^q f(x) = G^T \Psi^\alpha(x), \] (4.4)

where \( G \) and \( \Psi^\alpha(x) \) are \( 2^{k-1}M \times 1 \) vectors, given by
\[ G = [g_{1,0}, g_{1,1}, \ldots, g_{1,M-1}, g_{2,0}, g_{2,1}, \ldots, g_{2,M-1}, \ldots, g_{2k-1,M-1}]^T, \]

and
\[ \Psi^\alpha(x) = [\psi^\alpha_{1,0}(x), \psi^\alpha_{1,1}(x), \ldots, \psi^\alpha_{1,M-1}(x), \psi^\alpha_{2,0}(x), \psi^\alpha_{2,1}(x), \ldots, \psi^\alpha_{2,M-1}(x), \psi^\alpha_{2k-1,M-1}(x)]^T, \]

where the functions \( \psi^\alpha_{n,m}(x) \), \( n = 1, 2, \ldots, 2^{k-1} \) and \( m = 0, 1, \ldots, M - 1 \), are defined in Appendix B.

We define
\[ f(x) = G^T P^{(q,\alpha)} \Psi^\alpha(x), \] (4.5)

where the fractional integration operational matrix \( P^{(q,\alpha)} \) is given by
\[ P^{(q,\alpha)} = \Theta F^{(q,\alpha)} \Theta^{-1}, \] (4.6)

and the operators \( \Theta \) and \( F^{(q,\alpha)} \) are defined in Appendix B. Eq. (4.5) together with Eq. (4.6) allows us to write
\[ f(x) = G^T P^{(q,\alpha)} \Psi^\alpha(x). \] (4.7)

If we use the following notation:
\[ E^T \Psi^\alpha(x) = \left( \frac{x^{q+1}}{\Gamma(q+2)} \right)^2 + x, \]
this quantity together with the notations (4.4) and (4.7)) allows us to write the equation
\[ G^T \Psi^\alpha(x) + \left( G^T P^{(q,\alpha)} \Psi^\alpha(x) \right)^2 = E^T \Psi^\alpha(x). \]

By collocating this equation at the zero shifted Legendre polynomials and using Newton’s iterative method, we get the unknown vector \( G \).

In case \( k = 2 \) and \( M = 7 \), in Table 1, we compared \( q = \alpha = 0.5 \) with \( q = 0.5 \), \( \alpha = 1 \). As can be seen, for the case \( q = \alpha = 0.5 \), the results are at least three orders of magnitude better.
4.2 Example 2

Consider the linear fractional differential equation

\[ D^2 y(x) + 3Dy(x) + 2D^{v_1}y(x) + D^{v_2}y(x) + 5y(x) = f(x), \]
where

\[ 0 < v_2 < v_1 < 1, \quad 0 \leq x \leq 1, \]

and

\[ f(x) = 1 + 3x + \frac{2}{\Gamma(3 - v_1)}x^{2 - v_1} + \frac{1}{\Gamma(3 - v_2)}x^{2 - v_2} + 5\left(1 + \frac{1}{2}x^2\right), \]

subject to the initial conditions

\[ y(0) = 1, \quad y'(0) = 0. \]

The exact solution of this problem for \( v_2 = 0.0159 \) and \( v_1 = 0.1379 \) is

\[ y(x) = 1 + \frac{1}{2}x^2. \]

4.2.1 Fractional-order hybrid of block-pulse functions and Bernoulli polynomials

Using the same technique as in Example 1, the following equation is obtained

\[ C^T B^\alpha(x) + 3C^T \bar{B}^\alpha(x, 1) + 2C^T \bar{B}^\alpha(x, 2 - v_1) + C^T \bar{B}^\alpha(x, 2 - v_2) + 5(C^T \bar{B}^\alpha(x, 2) + 1) = g(x), \]

which by collocating in Newton-Cotes nodes, allows us to calculate the unknown \( C \). This problem is solved for \( N = 2 \) and \( M = 1 \). The first two derivatives are multiples of 1 and that is why we expect the best result to be for \( \alpha = 1 \). As can be seen in the first two columns of Table 2, the case \( \alpha = 1 \) is better than the case \( \alpha = 1.5 \).

4.2.2 Fractional-order Bernoulli wavelets

Using the FOBW technique, we obtain

\[ G^T (I + 3P^{(1, \alpha)} + 2P^{(2-v_1, \alpha)} + P^{(2-v_2, \alpha)} + 5P^{(2, \alpha)} + 5E^T) = A^T, \]

where the notations \( 1 = E^T \Psi^\alpha(x) \) and \( g(x) = A^T \Psi^\alpha(x) \) are used.
In the first two columns, we have presented the FOHBPB method for \( N = 2 \) and \( M = 1 \). In the last two columns, we have presented the FOBW method for \( k = 1 \) and \( M = 3 \), for Example 2.

| \( x \) | FOHBPB \( \alpha = 1.5 \) | FOHBPB \( \alpha = 1 \) | FOBW \( \alpha = 1.5 \) | FOBW \( \alpha = 1 \) |
|-------|-----------------|-----------------|-----------------|-----------------|
| 0     | 0               | 0               | 9.8\((-3)\)  | 5.5\((-17)\)  |
| 0.1   | 4.3\((-18)\)   | 0               | 4.6\((-3)\)  | 5.5\((-17)\)  |
| 0.2   | 1.6\((-17)\)   | 0               | 5.4\((-7)\)  | 6.9\((-17)\)  |
| 0.3   | 3.3\((-17)\)   | 0               | 2.4\((-3)\)  | 6.9\((-17)\)  |
| 0.4   | 5.4\((-17)\)   | 0               | 2.8\((-3)\)  | 8.3\((-17)\)  |
| 0.5   | 7.4\((-17)\)   | 0               | 1.7\((-3)\)  | 5.5\((-17)\)  |
| 0.6   | 9.1\((-17)\)   | 0               | 7.6\((-5)\)  | 5.5\((-17)\)  |
| 0.7   | 9.9\((-17)\)   | 0               | 1.6\((-3)\)  | 5.5\((-17)\)  |
| 0.8   | 9.6\((-17)\)   | 0               | 2.1\((-3)\)  | 5.5\((-17)\)  |
| 0.9   | 7.5\((-17)\)   | 0               | 4.7\((-4)\)  | 5.5\((-17)\)  |
| 1     | 3.3\((-17)\)   | 0               | 4.3\((-3)\)  | 5.52\((-17)\) |

In case \( k = 1 \) and \( M = 3 \), in Table 2, we compared \( \alpha = 1 \) with \( \alpha = 1.5 \). As can be seen, for the case \( \alpha = 1 \), the results are at least ten orders of magnitude better.

## 5 Conclusions

In this paper we explain why the transformation \( x \rightarrow x^\alpha \) gives us smaller errors in solving fractional differential equations. We also consider two numerical examples, in which we illustrate the results in this paper.

Beyond the fundamental aspect of a strictly mathematical nature, the accuracy given by the transformation \( x \rightarrow x^\alpha \) should bring improvement for obtaining the numerical supports of some equations that are the basis for understanding the nature. In recent decades, an increasing number of works confirmed that the tools of fractional calculus can help in capable of delivering providing more realistic results in a large number of practical applications compared to their integer-order counterparts. Most of fractional-order models in practical applications are nonlinear or involve some time delays, [9]. The employment of the \( x \rightarrow x^\alpha \) transformation can be extended to multi-dimensional models, such as two-dimensional linear autonomous incommensurate fractional-order system [2] or three-dimensional fractional-order Hindmarsh-Rose type models [6]. Also, in recent years, there has been an increased interest in the theory and applications of fractional differential equations. We can address this transformation further to Grünwald-Letnikov-type linear fractional variable order discrete-time systems [13] and also to systems with sequential fractional difference [12].

## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.
6 Appendix A

The FOHBPB functions are given on $[0, 1)$, by $[15–17]$

$$b_{nm}^\alpha(x) = \begin{cases} \beta_m(Nx^\alpha - n + 1), & x \in \left[\left(\frac{n-1}{N}\right)^{\frac{1}{\alpha}}, \left(\frac{n}{N}\right)^{\frac{1}{\alpha}}\right), \\ 0, & \text{otherwise}, \end{cases}$$

with $\beta_m(x^\alpha)$ the fractional-order Bernoulli functions

$$\beta_m(x^\alpha) = \sum_{k=0}^{m} \binom{m}{k} \alpha_{m-k} x^{k\alpha}, \quad 0 \leq x \leq 1,$$

where $\alpha_k, k = 1, 2, \ldots, m$ are the Bernoulli numbers.

The Riemann-Liouville fractional integral operator for hybrid of block-pulse functions and Bernoulli polynomials is given by the following expression, $[15, 17]$

$$I^\beta b_{nm}^\alpha(x) = \begin{cases} 0, & x \in \left(-\infty, \left(\frac{n-1}{N}\right)^{\frac{1}{\alpha}}\right), \\ U(x), & x \in \left[\left(\frac{n-1}{N}\right)^{\frac{1}{\alpha}}, \left(\frac{n}{N}\right)^{\frac{1}{\alpha}}\right), \\ U(x) - V(x), & x \in \left[\left(\frac{n}{N}\right)^{\frac{1}{\alpha}}, \infty\right), \end{cases}$$

with

$$U(x) = \sum_{k=0}^{m} \sum_{r=0}^{k} \binom{m}{k} \binom{k}{r} \alpha_{m-k} N^r (1-n)^{k-r} x^{\alpha r + \beta}$$

$$\times \frac{\Gamma(\alpha r + 1)}{\Gamma(\beta + \alpha r + 1)} \left[1 - I^{1 - \frac{1}{\alpha}} \left(\frac{n - 1}{N} \frac{1}{x} ; r\alpha + 1, \beta\right)\right],$$

and

$$V(x) = \sum_{k=0}^{m} \sum_{r=0}^{k} \binom{m}{k} \binom{k}{r} \alpha_{m-k} N^r (1-n)^{k-r} x^{\alpha r + \beta}$$
\[
\frac{\Gamma(\alpha r + 1)}{\Gamma(\beta + \alpha r + 1)} \left[ 1 - I \left( \frac{n}{N}^{1/\alpha} \frac{1}{x}; r\alpha + 1, \beta \right) \right].
\]

7 Appendix B

The fractional-order Bernoulli wavelets \( \psi_{n,m}^\alpha(x) \) are defined [18]

\[
\psi_{n,m}^\alpha(x) = \begin{cases} 
2^{k-1} \tilde{\beta}(2^{k-1} x^\alpha - \hat{n}), & \frac{\hat{n}}{2^{k-1}} \leq x^\alpha < \frac{\hat{n} + 1}{2^{k-1}}, \\
0, & \text{otherwise},
\end{cases}
\]

with

\[
\tilde{\beta}_m(2^{k-1} x^\alpha - \hat{n}) = \begin{cases} 
1, & m = 0, \\
\frac{1}{\sqrt{(-1)^{m-1}(m)!^2}} \beta_m(2^{k-1} x^\alpha - \hat{n}), & m > 0,
\end{cases}
\]

where \( \hat{n} = n - 1, m = 0, 1, \ldots, M - 1, n = 1, 2, \ldots, 2^{k-1} \) and \( \beta_m(t) \) are the well-known Bernoulli polynomials of order \( m \).

For \( k = 2 \), we have [18]

\[
\Theta = \begin{cases} 
\Phi = [a_{i,j}]_{2^{k-1}M \times M}, & 0 \leq x < \left( \frac{1}{2} \right)^{1/2}, \\
\Phi' = [a'_{i,j}]_{2^{k-1}M \times M}, & \left( \frac{1}{2} \right)^{1/2} \leq x < 1,
\end{cases}
\]

with

\[
a_{i,j} = \begin{cases} 
\frac{1}{2^{k-1}} \left( \begin{array}{c} 2^i \frac{1}{\lambda_{i-1}} \\ \end{array} \right) \frac{1}{2^{j-1}} \left( \begin{array}{c} i - 1 \\ \end{array} \right) \frac{1}{\lambda_{i-1}} & i = j, \\
2^{j-1} \left( \begin{array}{c} i - 1 \\ \end{array} \right) \frac{1}{\lambda_{i-1}} & j < i < M, \\
0, & \text{otherwise},
\end{cases}
\]

and

\[
a'_{i,j} = \begin{cases} 
0, & 1 \leq i \leq M, \\
(-1)^{j-M} a_{i-M,j}, & M + 1 \leq i \leq 2^{k-1} M,
\end{cases}
\]

where \( \lambda_0 = 1 \) and \( \lambda_i = \sqrt{\frac{(-1)^{i-1}(i)!^2}{(2i)!}} \beta_{2i}, i = 1, 2, \ldots, M - 1. \)
The fractional integration operational matrix of FOBW has the expression [18]

\[
F^{(v,\alpha)} = \begin{bmatrix}
\omega^{(v,\alpha)}_{0,0,0} & \omega^{(v,\alpha)}_{0,1,0} & \cdots & \omega^{(v,\alpha)}_{0,M-1,0} \\
\sum_{r=0}^{1} \omega^{(v,\alpha)}_{1,0,r} & \sum_{r=0}^{1} \omega^{(v,\alpha)}_{1,1,r} & \cdots & \sum_{r=0}^{1} \omega^{(v,\alpha)}_{1,M-1,r} \\
\cdots & \cdots & \cdots & \cdots \\
\sum_{r=0}^{M-1} \omega^{(v,\alpha)}_{M-1,0,r} & \sum_{r=0}^{M-1} \omega^{(v,\alpha)}_{M-1,1,r} & \cdots & \sum_{r=0}^{M-1} \omega^{(v,\alpha)}_{M-1,M-1,r}
\end{bmatrix}
\]

References

1. Antonov, V.A., Holsevnikov, K.V.: An estimate of the remainder in the expansion of the generating function for the Legendre polynomials (generalization and improvement of Bernstein’s inequality). Vestnik Leningrad Univ. Math. 13, 163–166 (1981)
2. Brandibur, O., Kaslik, E.: Exact stability and instability regions for two-dimensional linear autonomous multi-order systems of fractional-order differential equations. Fract. Calc. Appl. Anal. 24(1), 225–253 (2021). https://doi.org/10.1515/fca-2021-0010
3. Davaeifar, S., Rashidinia, J.: Solution of a system of delay differential equations of multi pantograph type. J. Taibah Univ. Sci. 11, 1141–1157 (2017). https://doi.org/10.1016/j.jtusci.2017.03.005
4. Dehestani, H., Ordokhani, Y., Razzaghi, M.: Fractional-order Legendre-Laguerre functions and their applications in fractional partial differential equations. Appl. Math. Comput. 336, 433–453 (2018). https://doi.org/10.1016/j.amc.2018.05.017
5. Diethelm, K.: The Analysis of Fractional Differential Equations. Springer-Verlag, Berlin, Heidelberg (2010)
6. Kaslik, E.: Analysis of two- and three-dimensional fractional-order Hindmarsh-Rose type neuronal models. Fract. Calc. Appl. Anal. 20(3), 623–645 (2017). https://doi.org/10.1515/fca-2017-0033
7. Kazarinoff, N.D.: Analytic Inequalities. Holt, Rinehart and Winston, New York (1961)
8. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies (2006). https://doi.org/10.1016/S0304-0208(06)80001-0
9. Liu, H., Wang, D., Xiao, A.: Dissipativity and stability analysis for fractional functional differential equations. Fract. Calc. Appl. Anal. 18(6), 1399–1422 (2015). https://doi.org/10.1515/fca-2015-0081
10. Lorch, L.: Alternative proof of a sharpened form of Bernstein’s inequality for Legendre polynomials. Appl. Anal. 14, 237–240 (1983). https://doi.org/10.1080/00036818308839426
11. Mohammadi, F., Cattani, C.: A generalized fractional-order Legendre wavelet Tau method for solving fractional differential equations. Fractal Frac. 6(5), Art. 275 (2022). https://doi.org/10.3390/fractalfract6050275
12. Mozyrska, D., Girejko, E., Wyrwas, M.: Fractional nonlinear systems with sequential operators. Centr. Eur. J. Phys. 11(10), 1295–1303 (2013). https://doi.org/10.2478/s11534-013-0223-3
13. Mozyrska, D., Oziablo, P., Wyrwas, M.: Stability of fractional variable order difference systems. Fract. Calc. Appl. Anal. 22(3), 807–824 (2019). https://doi.org/10.1515/fca-2019-0044
14. Podlubny, I.: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Academic Press, New York (1998)
15. Postavaru, O., Dragoi, F., Toma, A.: Enhancing the accuracy of solving Riccati fractional differential equations. Fract. Frac. 6(5), Art. 275 (2022). https://doi.org/10.3390/fractalfract6050275
16. Postavaru, O., Toma, A.: A numerical approach based on fractional-order hybrid functions of block-pulse and Bernoulli polynomials for fractional numerical solutions of fractional optimal control problems. Math. Comput. Simul. 194, 269–284 (2022). https://doi.org/10.1016/j.matcom.2021.12.001
17. Postavaru, O., Toma, A.: Numerical solution of two-dimensional fractional-order partial differential equations using hybrid functions. Partial Differ. Equ. Appl. Math. 4, 100099 (2021). https://doi.org/10.1016/j.padiff.2021.100099
18. Rahimkhani, P., Ordokhani, Y., Babolian, E.: Fractional-order Bernoulli wavelets and their application. Appl. Math. Model. 40, 8087–8107 (2016). https://doi.org/10.1016/j.apm.2016.04.026
19. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach Science Publishers, Switzerland (1993)
20. Szegö, G.: Orthogonal Polynomials. Amer. Math. Soc. Colloq. Publ., Vol. 23, 4th Ed., Providence, R.I. (1975)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.