A new effective-field technique for the ferromagnetic spin-1 Blume-Capel model in a transverse crystal field

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A new approximating technique is developed so as to study the quantum ferromagnetic spin-1 Blume-Capel model in the presence of a transverse crystal field in the square lattice. Our proposal consists of approaching the spin system by considering islands of finite clusters whose frontiers are surrounded by non-interacting spins that are treated by the effective-field theory. The resulting phase diagram is qualitatively correct, in contrast to most effective-field treatments, in which the first-order line exhibits spurious behavior by not being perpendicular to the anisotropy axis at low temperatures. The effect of the transverse anisotropy is also verified by the presence of quantum phase transitions. The possibility of using larger sizes constitutes an advantage to other approaches where the implementation of larger sizes is costly computationally.

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I. INTRODUCTION

In general, many-body systems with interactions are very difficult to solve exactly. A way to overcome this difficulty is by approaching the many-body problem by a one-body problem, in which a mean-field replaces the interactions affecting the body. This idea is applied to the ferromagnetic Ising Model (see reference [1]). In the most simple mean-field approach, the nearest-neighbor interactions affecting each spin \( S_i \) are replaced in such a way that \( S_i \) now interacts with an effective field given by \( zJ\langle S_i \rangle \), where \( z \) is the coordination number, \( J \) the exchange constant, and \( \langle S_i \rangle \) is the thermal average of the spin \( i \). This is the so called "Weiss mean-field approach" [2]. Nevertheless, it neglects the spin correlations, and it leads the transition temperature \( T_c \) as well as the values of the critical exponents away from the exact values (\( T_c = zJ/k_B \), for all dimensions). However, for the one-dimensional case, the Ising model lacks of a phase transition at finite temperature, but Weiss’ approach wrongly predicts that \( T_c = 2J/k_B \). A further step for improving the solution of this problem, is to use the proposal of Hans Bethe, which consists in considering that a central spin should interact with all its nearest-neighbour spins forming a cluster [3]. Then, that cluster would interact to an effective field that approaches the next-nearest-neighbor spins surrounding the cluster. Thus, this improvement gives \( T_c = 2J/k_B \ln(z/(z-2)) \), which not only betters the approximation of the critical temperatures, but leads correctly to \( T_c = 0 \), for the one-dimensional case. In this way, the correlations between the spins has been included to some degree by considering a cluster of spins interacting with its nearest-neighbors.

A further step in approaching the many-body problem in spin systems is the effective-field approach. It is used in spin models with finite-size clusters based on the following Hamiltonian splitting:

\[
\mathcal{H} = \mathcal{H}_c + \mathcal{H}_v, \tag{1}
\]

where \( \mathcal{H}_c \) corresponds to the energy that is composed of spin variables of the finite cluster, whereas \( \mathcal{H}_v \), corresponds to the energy of the neighborhood, whose spins do not belong to the central sites of the finite cluster. In the canonical ensemble, the calculation of mean values of the spin variables \( G_c \) belongs to the subspace \( n_c \) of the finite cluster, and it is computed by the following procedure:

\[
\langle G_c \rangle = \frac{Tr G_c \exp(-\beta \mathcal{H})}{Tr \exp(-\beta \mathcal{H})} = \frac{\langle Tr_{n_c} G_c \exp(-\beta \mathcal{H}_c) \rangle}{\langle Tr_{n_c} \exp(-\beta \mathcal{H}_c) \rangle}, \tag{2}
\]

This equation is exact if \( [\mathcal{H}_c, \mathcal{H}_v] = 0 \). The great merit of Eq. (2), is that we can solve the model of the infinite system by using a finite system in the subspace \( n_c \). Various approximation methods use Eq. (2) as a starting point. One of them is the effective-field theory (EFT) proposed by Honmura and Kaneyoshi [4] for solving the spin-1/2 ferromagnetic system. Sousa et al. [5–8] applied EFT so as to treat different magnetic models with competing interactions. Recently,
Viana et al. developed a mean-field proposal for spin models, denominated effective correlated mean-field (ECMF), based on the following ansatz:

\[ \sigma_j = \lambda \langle S_c \rangle, \]

(3)

where \( \sigma_j \) are the neighbors of the central spins of the finite cluster, \( \langle S_c \rangle \) is the mean of the spin variable of the cluster, and \( \lambda \) is a term exhibiting the behavior of a molecular parameter.

The aim of this proposal is the improvement of the results obtained by other effective-field techniques, but we believe that the main advantage of our proposal is the simplicity in treating first-order phase transitions. Accordingly, in this work, we test our new technique in a quantum version of the spin-1 Blume-Capel model in the presence of a transverse crystal field in the square lattice.

We remark that the classical Blume-Capel model (BC) \([10, 11]\) is one of the most suitable models for studying magnetic systems from the point of view of the Statistical Mechanics. This model and its generalization, the Blume-Emery-Griffiths model, (BEG) was proposed to describe the \( \lambda \) transition in \(^4\text{He} - ^3\text{He} \) mixtures \([12]\) as well as ordering in a binary alloy \([13, 14]\). Furthermore, its applications also include the description of ternary fluids \([15, 16]\), solid-liquid-gas mixtures and binary fluids \([17, 18]\), microemulsions \([19, 20]\), ordering in semiconducting alloys \([21, 22]\) and electron conduction models \([23]\). Indeed, the BC model is found in many works using different lattices, spin degrees, including disorder and different Statistical Mechanic techniques \([24–32]\).

The Hamiltonian of the original BC model is as follows:

\[ H_N = -J \sum_{i \neq j} S_i^z S_j^z + D_z \sum_{i=1}^N (S_i^z)^2, \]

(4)

where \( J \) is the ferromagnetic coupling between the spins \( S_i^z = 0, \pm 1 \) of the lattice and \( D_z \) is the anisotropy parameter. At zero temperature, for \( 0 < D_z < D_c \), the energy of the Hamiltonian \( H_N \) is minimized when all spins are \( S_i^z = \pm 1 \), but for \( D_z > D_c \), all the spins take the value \( S_i^z = 0 \), so the system suffers a first-order phase transition at \( D = D_c \). The critical value \( D_c \) can be determined by equating the energy of the ordered and disordered states, i.e.,

\[ H_N(S_j^z = \pm 1) = H_N(S_j^z = 0). \]

(5)

When the temperature is taken into account, the BC model provides us a phase diagram with a tricritical point separating a second and a first-order frontier that divides the ferromagnetic order (F) and the paramagnetic region (PM). This rich critical behavior qualifies the BC model for representing different phase transitions. The phase diagram of the BC model with equivalent-neighbor interactions can be seen in Fig.2 of reference \([33]\), which is qualitatively the same for dimensions greater than one.

A variant of this model is the Biaxial Blume-Capel model that considers a transverse crystal or anisotropy field to the easy axis of magnetization:

\[ H_N = -J \sum_{i \neq j} S_i^z S_j^z + \sum_{j=1}^N \left[ -D_x (S_j^x)^2 + D_z (S_j^z)^2 \right]. \]

(6)

Thus, the anisotropies \( D_x \) and \( D_z \) are now relevant physical parameters that play an important role in the quantum phase transitions that emerge. So, the term containing \( D_z \) enriches the phase diagram of the BC model. In classical phase transitions the spins are oriented according to temperature fluctuations. On the other hand, the quantum phase transitions occur at very low temperatures, so the spins are oriented by quantum fluctuations associated to states of energy minima. Particularly, in this model there are critical values of \( D_x \) and \( D_z \) for which ferromagnetic-paramagnetic phase transitions exist for \( T \simeq 0 \).

When \( D_x = 0 \) and \( D_z \neq 0 \) a quantum phase transition (at \( T = 0 \)) exists for the critical value of \( D_z \) corresponding to an energy minimum of zero value related to the state \( S_j^z = 0 \). However, for \( D_z = 0 \) and \( D_x \neq 0 \), the quantum phase transition occur for a critical value of \( D_x \) related to a energy minimum \(-D_x\) related to the state where \( S_j^z = 0 \).
II. IMPLEMENTATION OF THE TECHNIQUE

In the present proposal the system consists of finite-size clusters of interacting sites $S_j = S_j^x, S_j^y, S_j^z$ and non-interacting sites $\sigma_j$, which belong to a set of infinite $N$ particles. In this work we use the cluster scheme shown in Fig. 1, where we may observe islands of finite size $C_w$ that are composed by $N_c$ interacting sites $S_{w,j}$, in which different $C_w$ clusters do not interact, and spins $\sigma_w$ have only $z$-component.

The Hamiltonian of the model given in Eq. (6) is, regarding the scheme of Fig. 1, as follows:

$$H_N = H_\sigma + H_w,$$

where

$$H_\sigma = -J \sum_{j=1}^{n} \sum_{r=1}^{4} \sigma_j S_{j,r}^z + D_z \sum_{j=1}^{n} \sigma_j^2,$$

that corresponds to the portion of the energy of spins $\sigma_j$. Furthermore, we have

$$H_w = \sum_w \left\{ -J \sum_{i \neq j}^{N_w} S_{w,i}^z S_{w,j}^z + \sum_{j=1}^{N_w} \left[ -D_z (S_{w,j}^z)^2 + D_z (S_{w,j}^z)^2 \right] \right\},$$

which is the portion of the energy that excludes spins $\sigma_j$. Thus, we have the following equations associated to the term $-\beta H_N$:

$$-\beta H_\sigma = K_4 \sum_{j=1}^{4} \sigma_j S_{j,r}^z - Kd_z \sum_{j=1}^{n} \sigma_j^2$$

and also

$$-\beta H_w = \sum_w \left\{ K_4 \sum_{i \neq j}^{N_w} S_{w,i}^z S_{w,j}^z + \sum_{j=1}^{N_w} \left[ Kd_z (S_{w,j}^z)^2 - Kd_z (S_{w,j}^z)^2 \right] \right\},$$

where $\beta = 1/k_B T$, $d_x = D_x/J$ and $K = \beta J$.

Now we have to perform the mean in the $v$ space of the spins $\sigma_j$, given by:

$$\langle \langle \sigma_j \rangle \rangle_v = \frac{\langle Tr_v \sigma_j \exp (-\beta H_N) \rangle}{\langle Tr_v \exp (-\beta H_N) \rangle}$$

$$= \frac{\langle Tr_v \sigma_j \exp (-\beta H_\sigma) \rangle}{\langle Tr_v \exp (-\beta H_\sigma) \rangle}$$

$$= \langle \langle \frac{\partial}{\partial \phi_j} \ln (Z_j) \rangle \rangle$$

where

$$Z_j = Tr_v \exp (-\beta H_\sigma) = 2 \exp (-Kd_z) \cosh (\phi_j) + 1,$$

and also

$$\phi_j = K_4 \sum_{r=1}^{4} S_{j,r}^z.$$

In this way we have

$$\langle \langle \sigma_j \rangle \rangle_v = \frac{2 \exp (-Kd_z) \sinh (\phi_j)}{2 \exp (-Kd_z) \cosh (\phi_j) + 1} = \langle G(\phi_j) \rangle.$$
Now, by performing this trinomial operator we get the following result:

\[
\langle \langle \sigma_j \rangle \rangle_v = \left\langle \left\langle \exp \left( K \sum_{r=1}^{4} S^z_{j,r} \frac{\partial}{\partial x} \right) G_j(x) \right| x = 0 \right\rangle
\]

\[
= \left\langle \prod_{r=1}^{4} \exp \left( K S^z_{j,r} \frac{\partial}{\partial x} \right) G_j(x) \right| x = 0 \right\rangle
\]

\[
= \left\langle \prod_{r=1}^{4} \left[ \sum_{p=0}^{2} b_p (S^z_{j,r})^p \right] G_j(x) \right| x = 0 \right\rangle
\]

(12)

where \( \frac{\partial}{\partial x} \) is the differential operator, and also

\[
b_0 = 1, \ b_1 = \sinh \left( K \frac{\partial}{\partial x} \right) \ e \ b_2 = \cosh \left( K \frac{\partial}{\partial x} \right) - 1.
\]

(13)

It is important to note that in this calculation process we have many correlation means of sites \( S^z_{j,r} \), however, sites \( S^z_{j,r} \) belong to different clusters \( C_w \), thus sites \( S^z_{j,r} \) are independent. Accordingly, the following first-order relations of approach can be used by regarding the definition of the magnetization \( m \):

\[
\langle S^z_{j,r} \rangle = m
\]

(14)

\[
\langle S^z_{j,r_1} S^z_{j,r_2} \rangle = \langle S^z_{j,r_1} \rangle \langle S^z_{j,r_2} \rangle = m^2
\]

(15)

\[
\langle S^z_{j,r_1} S^z_{j,r_2} S^z_{j,r_3} \rangle = \langle S^z_{j,r_1} \rangle \langle S^z_{j,r_2} \rangle \langle S^z_{j,r_3} \rangle = m^3
\]

(16)

\[
\langle S^z_{j,r_1} S^z_{j,r_2} S^z_{j,r_3} S^z_{j,r_4} \rangle = \langle S^z_{j,r_1} \rangle \langle S^z_{j,r_2} \rangle \langle S^z_{j,r_3} \rangle \langle S^z_{j,r_4} \rangle = m^4.
\]

(17)

This treatment also applies for the terms \( e \left\langle (S^z_{j,r})^2 \right\rangle = q \). So, Eq. (12) can be rewritten in the following form:

\[
\langle \langle \sigma_j \rangle \rangle_v = \left| b_0 + b_1 m + b_2 q \right|^4 G_j(x) \left| x = 0 \right.
\]

(18)

Now, by performing this trinomial operator we get the following result:

\[
\langle \langle \sigma_j \rangle \rangle_v = \sum_{p_1=0}^{4-p_1} \sum_{p_2=0}^{4-p_1} a_1^{4-p_1-p_2} a_2^{p_2} a_3^{p_1} \exp \left( (4 - p_1 - 2p_2) K \frac{\partial}{\partial x} \right) G_j(x) \left| x = 0 \right.
\]

(19)

where

\[
a_1 = \frac{1}{2} (q + m), \ a_2 = \frac{1}{2} (q - m) \ e \ a_3 = 1 - q.
\]

(20)

Now we apply the following relation

\[
\exp \left( \omega \frac{\partial}{\partial x} \right) G_j(x) \left| x = 0 \right. = G_j(x = \omega),
\]

(21)

so:

\[
\langle \langle \sigma_j \rangle \rangle_v = \sum_{p_1=0}^{4-p_1} \sum_{p_2=0}^{4-p_1} a_1^{4-p_1-p_2} a_2^{p_2} a_3^{p_1} G_j(x = 4 - p_1 - 2p_2),
\]

(22)

and we have also that

\[
a_1^{e_1} a_2^{e_2} a_3^{e_3} = \left( \frac{1}{2} \right)^{e_1+e_2} \sum_{t_1=0}^{e_1} \sum_{t_2=0}^{e_2} \sum_{t_3=0}^{e_3} (-1)^{t_2+t_3} q^{t_1+e_2-t_1-t_2+t_3} m^{t_1+t_2}.
\]

(23)
In this way we have the following simplified expression:

$$\langle \langle \sigma_j \rangle \rangle_v = \sum_{k=0}^{4} A_k m^k.$$  

(24)

where $A_k = A_k (K, d_z, q)$. We verified that $A_k$ is zero for even values of $k$.

There are many mean-field proposals that have been done so as to approach $\sigma_j$. In this paper we use the following relation:

$$\sigma_j = \lambda m,$$  

(25)

where $\lambda$ is a parameter to be determined. Then we apply Eq. (25) in Eq. (24), which leads to the following result:

$$\langle \langle \lambda m \rangle \rangle_v = \sum_{k=0}^{4} A_k m^k \lambda = \sum_{k=0}^{4} A_k m^{k-1}.$$  

(26)

Note that from this equation we have that $\lambda = \lambda (K, d_z, q, m)$.

A. The Interacting Cluster

In Fig. 2 we may see that a $C_w$ cluster contain spins $S_j$, each of them interacting between next-nearest neighbors in the respective finite-size square lattice. We can also observe that $\sigma_k$ represents the neighbors of the central sites $S_j^z$, which compose the finite cluster of $N_c$ sites. Thus, the Hamiltonian is conveniently written in the following form:

$$- \beta H_{N_c} = K \sum_{i \neq j} S_i^z S_j^z + K \sum_j \left[ -d_z (S_j^z)^2 + d_x (S_j^x)^2 \right] + \sum_{j=1}^{N_c} C_j S_j^z.$$  

(27)

where

$$C_j = K \sum_{k=1}^{n_j} \sigma_k.$$  

(28)

In the present work the ansatz given in Eq. (9) is our fundamental assumption, so

$$C_j = n_j \lambda k \langle S_j^z \rangle.$$  

(29)

In what follows the size of the finite clusters is considered to be $N_c = 1, 2, 4, 9, 25, 36, 49$ with 64 central sites. Thus, we have the following relations:

$$N_c = 1: C_j = 4 \lambda k \langle S_j^z \rangle.$$  

(30)

$$N_c = 2: C_j = 3 \lambda k \langle S_j^z \rangle.$$  

(31)

$$N_c = 4: C_j = 2 \lambda k \langle S_j^z \rangle.$$  

(32)

$$N_c \geq 9: C_j = 2 \lambda k \langle S_j^z \rangle \text{ or } C_j = 0.$$  

(33)
The magnetic properties per particle such as \( m = \langle S^z_c \rangle \) and \( q = \langle (S^z_c)^2 \rangle \) are given by the following statistical definitions:

\[
\langle S^z_c \rangle = \frac{1}{N_c} \text{Tr} \left( \sum_{j=1}^{N_c} S^z_j \right) \exp(-\beta \mathcal{H}_{N_c})
\]

\[
\langle (S^z_c)^2 \rangle = \frac{1}{N_c} \text{Tr} \left( \sum_{j=1}^{N_c} (S^z_j)^2 \right) \exp(-\beta \mathcal{H}_{N_c})
\]

where the partition function in the space of sites \( S^z_j \) is given by

\[
Z_{N_c} = \text{Tr} \exp(-\beta \mathcal{H}_{N_c}) = \sum_{j=1}^{3N_c} E_j
\]

where \( E_j \) are the eigenvalues of the Hamiltonian. Particularly, when \( N_c = 1 \) the eigenstates \( |s_z\rangle \) of the orthogonal basis are given by:

\[
|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e^{-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

The hamiltonian matrix \(-\beta \mathcal{H}_1\) is given by

\[
-\beta \mathcal{H}_1 = \begin{pmatrix}
C_1 + Kd_x/2 - Kdz & 0 & Kd_x/2 \\
0 & Kd_x & 0 \\
Kd_x/2 & 0 & -C_1 + Kd_x/2 - Kdz
\end{pmatrix}.
\]

So eigenvalues obtained from the matrix \(-\beta \mathcal{H}_1\) correspond to the following expressions:

\[
E_1 = Kd_x
\]

\[
E_{2,3} = Kd_x/2 - Kdz + \frac{1}{2} \sqrt{K^2d_x^2 + 4C_1^2}
\]

while the eigenvectors are:

\[
|E_1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |0\rangle
\]

\[
|E_2\rangle = \begin{pmatrix} 1 \\ 0 \\ R_2 \end{pmatrix} = |1\rangle + R_2 | -1\rangle
\]

\[
|E_3\rangle = \begin{pmatrix} R_3 \\ 0 \\ 1 \end{pmatrix} = R_3 |1\rangle + | -1\rangle
\]

where

\[
R_2 = \frac{Kd_x}{2C_1 + \sqrt{K^2d_x^2 + 4C_1^2}}
\]

\[
R_3 = -\frac{Kd_x}{2C_1 + \sqrt{K^2d_x^2 + 4C_1^2}}
\]
For $N_c > 1$, eigenvectors and eigenvalues of $-\beta H_N$ are obtained by numerical methods.

An important issue is the thermodynamic treatment of the spin system, accordingly, we use the free energy given by the following equation:

$$\phi = -\frac{1}{N_c} t \ln (Z_{N_c}) + \gamma m^2,$$

where $t = k_B T / J$ is the reduced temperature and $\gamma$ is a parameter to be determined. At the equilibrium, the free energy is minimized, thus:

$$\frac{\partial \phi}{\partial m} = f_m = 0,$$

where the function $f_m$ stands for the equation of state given by

$$f_m = m - \langle S^z \rangle,$$

and we can determine the parameter $\gamma$ by using Eq. (47).

### III. RESULTS

We firstly obtained the phase diagram of the BC model in the plane $t - d_z$ in a square lattice, based on the considerations of the previous section. In Fig. 3 we show the phase diagram for clusters containing $N_c = 1$ (in (a)), $N_c = 4$ (in (b)) and $N_c = 16$ (in (c)) central sites. We faced the computational problem of solving the model for big clusters, inasmuch as the number of accessible states corresponds to $3^{N_c}$ states. For instance, for $N_c = 16$ and $N_c = 64$ sites, we have accessible states of order $10^7$ and $10^{30}$, respectively, which are huge numbers. Accordingly, for clusters of sizes $N_c > 16$, we prefer only to calculate the critical temperature $t_c$, for $d = 0$, and the coordinates of the tricritical point $P(d_c, t_c)$. In this figure we may observe that the critical value of the anisotropy corresponds to $d_c = 2.0$, for $t \to 0$, which agrees with exact results obtained when equating the energy of the ordered state ($S^z_j = \pm 1$), with the energy of the disordered one ($S^z_j = 0$), for a finite system of $N$ sites, i.e.,

$$d_c = \frac{z}{2},$$

where $z$ is the coordination number of the lattice. The black circle represents the tricritical point that separates the second-order and the first-order frontier. We observe that the critical temperature $t_c$ decreases as the size of the cluster $N_c$ increases. The first-order frontier correctly falls perpendicularly to the anisotropy axis, however, in general, effective-field results do not reproduce this feature of the first-order frontier (see the IEF curve in Figure 5 of reference [34]).

In Table 1 we present the values of $t_c$, obtained for each cluster size $N_c$, where we compare the results of this work using ECMF with the mean-field approximation that uses clusters (MFT), where $\lambda = 1$, in this case. We remark that Yüksel et al. [34] obtained $t_c \simeq 1.690$, by using a Metropolis Monte Carlo simulation (SMC), whereas Silva et al. [27] obtained $t_c \simeq 1.714$ using Wang-Landau sampling. In references [33] and [26] we have $t_c \simeq 1.695$ and $t_c \simeq 1.681$, respectively. We may observe that the results obtained by the ECMF approach are close to the SMC values when the cluster size is increased. Nevertheless, if compared with the MFT results, $t_c$ is better estimated by the ECMF method, regarding the SMC results as a reference.

The calculations of the tricritical points $P(d_c, t_c)$ through the ECMF technique are shown in Table 2. We see that when the cluster size $N_c$ is increased, the values of the critical anisotropy $d_c$ approach the values $d_c = 1.966(2)$ and $d_c = 1.974$, obtained by Monte Carlo simulations of references [27] and [34], respectively. In what the value $t_c$ concerns, the convergence is closer to that of reference [34], which gives $t_c \approx 0.56$, obtained by Monte Carlo simulations [34].

The coordinates of the tricritical point were determined through a Landau expansion,
\[
\phi(d, t) = \sum_{p=0}^{\infty} A_p(d, t) m^p.
\]  

From this equation, we are interested in solving the following system of equations 

\[
A_2(d_t, t_t) = 0 \\
A_4(d_t, t_t) = 0,
\]

so as to obtain the tricritical point. Thus, we may note that 

\[
A_p = \frac{-t}{N_c^2 Z_{N_c}} \left( \frac{\partial^p Z_{N_c}}{\partial m^p} \right)_{m=0} + \left( \frac{\partial^p}{\partial m^p} (\gamma m^2) \right)_{m=0}
\]

corresponds to the equation that determines the coefficients \(A_p\).

In Fig. 4 we show the phase diagram in the \(t - d_x\) plane for \(N_c = 1\) (case (a)) and \(N_c = 2\) (case (b)). In Fig. 5 we show the detail of this diagram for low temperature region. Both cases exhibit phase transitions of first and second order, as well as the presence of two tricritical points, respectively. For \(N_c = 1\) were obtained \(P(d_t = 11.93, t_t = 2.10^{-4})\) and \(P(d_t = 6.67, t_t = 0.99)\), whereas for \(N_c = 2\), we have \(P(d_t = 10.73, t_t = 10^{-2})\) and \(P(d_t = 6.16, t_t = 0.94)\).

Another important aspect is shown in Fig. 6, for a cluster with \(N_c = 1\) within the ECMF approach and corresponds to Fig. 5. There it is shown a free energy minimum at the transition point located at \(P (t = 0.001, d_x = 11.842076)\). This is signaling a first-order phase transition point. Three minima at the same level can be observed, one for \(m = 0\), and two symmetrical ones at \(m = \pm m_0\), which clearly identify a first-order phase transition due to coexisting phases.

In this point the temperature is close to zero and the value of the free energy tends to \(\phi = -E_1/K = -d_x\), which corresponds to the eigenvalue of the disordered state \(|E_1\rangle\), where \(m = 0\). This is a quantum phase transition, without the influence of the temperature fluctuations. In Fig. 7 is shown the \(\phi\) behavior of the free energy minimum for the phase transition point \(P (t = 0.001, d_x = 10.85502)\) using the ECMF approach with \(N_c = 2\) sites. This, of course, is a signal of a second-order phase transition.

Recently, it has been studied the classical BC model using an effective-field technique (EFT) \[36\] with \(N_c = 1\) site on the square lattice. The same qualitative results can also be observed in reference \[37\]. Similarly, in Fig. 8 we exhibit the mean-field case of the present model (\(\lambda = 1\)), where we can see a tricritical point for \(N_c = 1\) (frontier line (a)), whereas, for \(N_c = 2\) (frontier line (b)), two tricritical points are present. Thus, a detailed criticality of these frontiers are shown in Fig.9 for the low temperature region, so as to observe that the lower tricritical point in line (b) is very close to the zero temperature.

**IV. CONCLUSIONS**

In this paper we study the ferromagnetic spin-1 Blume-Capel (BC) model with nearest-neighbor interactions in the presence of a transverse crystal-field, within a mean-field approach. We call this new approach as effective correlated mean-field (ECMF). For the bidimensional case, we implemented the model in the square lattice. The results show that when the size of the cluster \(N_c\) increases, the values of the critical temperature \(t_c\) (for null anisotropy) tend to 1.690, which is the Monte Carlo estimate of Yüksel et al. \[31\] (see Table 1). Another important result is related to the estimate of the tricritical point \(P(d_t, t_t)\) in comparison with the results of Silva \[27\] and Yüksel \[34\] (see Table 2). Our ECMF values of \(d_t\) reasonably agree with the results of the Monte Carlo simulations, as \(N_c\) increases, whereas \(t_t\) tends to the effective-field (EFT) result developed in reference \[34\].

The results that consider the transverse crystal \(D_x\) can be observed in the phase diagram in the plane \(t - d_x\). There we have the presence of first- and second-order phase transitions for \(D_x > 0\), together with two tricritical points, whereas for \(D_x < 0\), we only have a second-order criticality. The evidence of a quantum phase transition is shown in Fig. 6, where the energy minima is threefold degenerated with \(\phi = -E_1/K\), related to the eigenvalue of the disordered state \((m = 0)\), see Eq. (11).

The main merit of the ECMF approximation is the determination of the molecular parameter \(\lambda\), which is obtained from the effective-field theory. Furthermore, the values of the critical temperature determined by this approach (for a
TABLE I: Critical temperatures obtained for various cluster sizes using the ECMF and MFT techniques.

| Technique/Nc | 1   | 2   | 4   | 9   | 16  | 25  | 36  | 49  | 64  |
|--------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| ECMF         | 2.468 | 2.420 | 2.342 | 2.277 | 2.241 | 2.216 | 2.108 | 1.972 | 1.914 |
| MFT          | 2.666 | 2.552 | 2.406 | 2.309 | 2.259 | 2.236 | 2.224 | 2.145 | 2.098 |

given value of $N_c$) converge faster in comparison with other techniques like Monte Carlo [34], and when we compare them with the results obtained by the usual mean field approach (see table 1).

On the other hand, the possibility of working with larger sizes constitutes an advantage for analyzing finite-size effects. Finally, we hope that this new technique of approach can be applied satisfactorily in other models and lattices.

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TABLE II: Values of tricritical points obtained with the MFT-MC approximation for several sizes of the cluster.

| Ne  | 1   | 2   | 4   | 9   | 16  | 25  | 36  | 49  | 64  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $d_t$ | 1.848 | 1.857 | 1.872 | 1.874 | 1.876 | 1.896 | 1.916 | 1.921 | 1.935 |
| $t_t$ | 1.182 | 1.164 | 1.113 | 1.095 | 1.051 | 1.002 | 0.899 | 0.835 | 0.792 |

FIG. 1: The scheme showing the cluster of spins to be used in the ECMF approach.

FIG. 2: Scheme for sites located on a square lattice, where we have the central sites (filled circles) and neighboring sites (open circles).
FIG. 3: Phase Diagrams obtained for $d_z = 0$, using clusters with $N_c = 1, 4$ and 16 central sites, corresponding to lines (a), (b) and (c), respectively. The continuous lines represent second-order frontiers, whereas dashed lines are for the first-order ones. The black circles represent tricritical points.

FIG. 4: Phase diagrams obtained for $d_z = 0$, using clusters of $N_c = 1$ and 2 central sites that correspond to lines (a), (b), respectively.
FIG. 5: Detailed low temperature region on the right of Fig. 4. We remark that lines (a) and (b) correspond to clusters of $N_c = 1$ and 2 central sites, respectively.

FIG. 6: Energy minima obtained for the phase transition point $P(t = 0.0010(1), d_x = 11.8420(1))$, which corresponds to a first-order point in Fig. 5, for the frontier line corresponding to the size $N_c = 1$. 


FIG. 7: Energy minimum obtained for the phase transition point $P(t = 0.0010(1), d_x = 10.855(1))$, which corresponds to a second-order point in Fig.5, for the frontier corresponding to the size $N_c = 2$.

FIG. 8: Phase diagrams obtained for $d_z = 0$, using the traditional MFT ($\lambda = 1$), where we have in (a) $N_c = 1$ central sites and in (b) $N_c = 2$ central sites.
FIG. 9: Detailed low temperature region on the right of Fig.8. The frontier line (a) ends at zero temperature as a first-order line, whereas the frontier line (b) is of second-order at zero temperature with a tricritical point very close to it.