ON THE GROWTH OF THE ENERGY OF ENTIRE SOLUTIONS TO
THE VECTOR ALLEN-CAHN EQUATION

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Abstract. We prove that the energy over balls of entire, nonconstant, bounded solutions to the vector Allen-Cahn equation grows faster than \((\ln R)^k R^{n-2}\) for any \(k > 0\), as the volume \(R^n\) of the ball tends to infinity. This improves the growth rate of order \(R^{n-2}\) that follows from the general weak monotonicity formula. Moreover, our estimate can be considered as an approximation to the corresponding rate of order \(R^{n-1}\) that is known to hold in the scalar case.

1. Introduction

Consider the semilinear elliptic system

\[
\Delta u = W(u) \text{ in } \mathbb{R}^n, \quad n \geq 2, \quad (1.1)
\]

where \(W : \mathbb{R}^m \to \mathbb{R}, \ m \geq 2,\) is sufficiently smooth and nonnegative. This system has variational structure, as solutions (in a smooth, bounded domain \(\Omega \subset \mathbb{R}^n\)) are critical points of the energy

\[
E(v; \Omega) = \int_{\Omega} \left\{ \frac{1}{2} |\nabla v|^2 + W(v) \right\} dy
\]

(subject to their own boundary conditions).

In the scalar case, namely \(m = 1,\) Modica [14] used the \(P\)-function technique [20] and intrinsically scalar arguments to show that every entire, bounded solution to (1.1) satisfies the pointwise gradient bound

\[
\frac{1}{2} |\nabla u|^2 \leq W(u) \text{ in } \mathbb{R}^n, \quad (1.2)
\]

(see also [7] and [10]). Using this, together with Pohozaev identities, it was shown in [15] that such solutions satisfy the following strong monotonicity formula:

\[
\frac{d}{dr} \left( \frac{1}{r^{n-1}} \int_{B(x,r)} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dy \right) \geq 0, \quad r > 0, \ x \in \mathbb{R}^n, \quad (1.3)
\]

where \(B(x,r)\) stands for the \(n\)-dimensional ball of radius \(r\) that is centered at \(x.\) In particular, it follows that each entire, bounded and nonconstant solution to the scalar problem satisfies

\[
\int_{B(x,r)} \left\{ \frac{1}{2} |\nabla v|^2 + W(v) \right\} dy \geq cr^{n-1}, \quad r > 0, \quad \text{for some } c > 0. \quad (1.4)
\]

In the vector case, that is \(m \geq 2,\) the analog of the gradient bound (1.2) does not hold in general. In passing, let us mention that if a solution \(u\) satisfied the analog of the gradient bound (1.2), then it would also satisfy the strong monotonicity formula (1.3) (see [1], [19]).
All is not lost, however, as using the fact that every solution to (1.1) satisfies the weak monotonicity formula

$$\frac{d}{dr} \left( \frac{1}{r^{n-2}} \int_{B(x,r)} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dy \right) \geq 0, \quad r > 0, \quad x \in \mathbb{R}^n,$$

(1.5)

and with some more work in the case $n = 2$, it follows readily that, given $x \in \mathbb{R}^n$, each nonconstant solution to the system (1.1) satisfies:

$$\int_{B(x,r)} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dy \geq \begin{cases} cr^{n-2} & \text{if } n \geq 3, \\ c \ln r & \text{if } n = 2, \end{cases}$$

(1.6)

for all $r > 1$ and some $c > 0$.

Let us mention that the above lower bound is sharp in the case $n = 2$. Indeed, for the Ginzburg-Landau system

$$\Delta u = (|u|^2 - 1) u, \quad u : \mathbb{R}^n \to \mathbb{R}^m, \quad \left( \text{here } W(u) = \frac{(1-|u|^2)^2}{4} \text{ vanishes on } \mathbb{S}^{m-1} \right),$$

(1.7)

arising in superconductivity, there are entire, bounded solutions with energy over $B(0, r)$ of order $\ln r$, as $r \to \infty$, if $n = m = 2$ (see [5] and the references therein).

In this note, we will specialize to the class of potentials that satisfy the following properties: $W \in C^2$ and there exist $N \in \mathbb{N}$ points $a_i \in \mathbb{R}^m$ such that

$$W(a_i) = 0, \quad W > 0 \quad \text{in } \mathbb{R}^m \setminus \{a_i\} \quad \text{and} \quad W_{uu}(a_i)\nu \cdot \nu > 0 \quad \forall \nu \in \mathbb{S}^{m-1}, \quad i = 1, \cdots, N,$$

(1.8)

where $\cdot$ stands for the Euclidean inner product in $\mathbb{R}^m$. For this class of potentials, the system (1.1) is known as the vector Allen-Cahn equation and models multiphase transitions (see [3] [6]).

In this case, it was shown recently in [2], by extending the density estimates of [8] to this vector setting, that nonconstant bounded, minimal solutions satisfy

$$\liminf_{r \to \infty} \frac{1}{r^{n-1}} \int_{B(x,r)} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dy > 0, \quad \forall x \in \mathbb{R}^n.$$

(1.9)

For a simple proof of this result when $n = 2$, under weaker assumptions on $W$, we refer to [18]. In comparison, let us note that the solution mentioned in relation to (1.7) is minimal. On the other side, if $u$ is a nonconstant solution which is periodic in each variable, as in [4] for the vector Allen-Cahn equation or [13] for (1.7), it is easy to see that

$$\liminf_{r \to \infty} \frac{1}{r^n} \int_{B(x,r)} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dy > 0, \quad \forall x \in \mathbb{R}^n.$$

It was shown recently in [18] that the above relation also holds for nonconstant radial solutions, as the ones in [12] for the scalar Allen-Cahn equation $\Delta u = u^3 - u$.

It is therefore natural to ask what can be said, besides of the lower bound (1.6), about arbitrary bounded solutions to the vector Allen-Cahn equation. Our main result is the following improvement of the lower bound (1.6) for this class of systems.

**Theorem 1.1.** Assume that $W \in C^2(\mathbb{R}^m; \mathbb{R})$, $m \geq 2$, satisfies (1.8). If $u \in C^2(\mathbb{R}^n; \mathbb{R}^m)$, $n \geq 2$, is a nonconstant, entire and bounded solution to the elliptic system (1.1), for any
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$x \in \mathbb{R}^n$ and $k > 0$, it holds that

$$\frac{1}{(\ln r)^k} \frac{1}{r^{n-2}} \int_{B(x,r)} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dy \to \infty, \text{ as } r \to \infty.$$  \hfill (1.10)

Our result implies that, in contrast to the Ginzburg-Landau system (1.7), the growth in the latter estimate cannot be achieved for any nonconstant bounded solution of the vector Allen-Cahn equation for any $n, m \geq 2$. Moreover, it can be considered as an approximation to the corresponding lower bound (1.4) that holds in the scalar case and to (1.9) in the case of minimal solutions.

Our proof of Theorem 1.1 will be based on the monotonicity formula (1.5) and the following lemma from [17]:

**Lemma 1.2.** Assume that $W \in C^1(\mathbb{R}^m; \mathbb{R})$, $m \geq 2$, and that $u \in C^2(\mathbb{R}^n; \mathbb{R}^m)$ satisfies (1.1). Then, for every $x \in \mathbb{R}^n$ and any positive numbers $R_0 < R_1$, there exists $r(x) \in (R_0, R_1)$ such that

$$r(x) \int_{\partial B(x,r(x))} \left| \frac{\partial u}{\partial \nu} \right|^2 dS(y) + 2 \int_{B(x,r(x))} W(u) dy \leq \frac{1}{\ln(R_1/R_0)} \ln \left( \frac{\tilde{E}(u,x,R_1)}{\tilde{E}(u,x,R_0)} \right) \int_{B(x,r(x))} e(u) dy,$$  \hfill (1.11)

where

$$e(u) \equiv \frac{1}{2} |\nabla u|^2 + W(u)$$

and

$$\tilde{E}(u,x,r) \equiv \frac{1}{r^{n-2}} \int_{B(x,r)} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dy.$$  

This lemma was proven in [17] for the special case of the Ginzburg-Landau system (1.7) but the proof carries over verbatim to the general case. In particular, it is based on the identity

$$\frac{d}{dr} \left( \tilde{E}(u,x,r) \right) = \frac{1}{r^{n-2}} \int_{\partial B(x,r)} \left| \frac{\partial u}{\partial \nu} \right|^2 dS(y) + \frac{2}{r^{n-1}} \int_{B(x,r)} W(u) dy,$$  \hfill (1.12)

which implies at once the weak monotonicity formula (1.5) for nonnegative $W$ and is a direct consequence of Pohozaev identities for systems (see [21] for the case of general $W$). Note also that, in contrast to [17], we have included the boundary integral in the left-hand side of (1.11) (we have also kept the factor 2).

The rest of this article is devoted to the proof of Theorem 1.1.

## 2. Proof of the main result

**Proof of Theorem 1.1.** Since the problem is translation invariant, without loss of generality, we may carry out the proof for $x = 0$.

For future reference, we note that by standard elliptic regularity theory (see [11]), there exists a constant $C_0 > 0$ such that

$$|u| + |\nabla u| \leq C_0 \text{ in } \mathbb{R}^n.$$  \hfill (2.1)

Suppose, to the contrary, that (1.10) does not hold. Then, there would exist constants $k, C_1 > 1$ and a sequence $R_j > 1$ with $R_j \to \infty$ such that

$$\tilde{E}(u,0,R_j) \leq C_1 (\ln R_j)^k, \quad j \geq 1.$$  \hfill (2.2)
On the other side, thanks to (1.6), we have that

\[ \tilde{E}(u, 0, R_j^\frac{1}{2}) \geq C_2, \quad j \geq 1, \]

for some \( C_2 > 0 \) (\( C_2 < C_1 \) without loss of generality). In passing, we note that our motivation for the power 1/2 comes from the proof of the \( \eta \)-compactness lemma in [16]. By Lemma 1.2, we infer that there exist \( r_j \in (R_j^\frac{1}{2}, R_j) \) such that

\[
\begin{align*}
    r_j \int_{B(0,r_j)} |\frac{\partial u}{\partial y}|^2 dS(y) + 2 \int_{B(0,r_j)} W(u)dy &\leq \frac{2}{\ln R_j} \ln \left( \frac{C_1 (\ln R_j)^k}{C_2} \right) r_j^{n-2} \tilde{E}(u, 0, r_j) \\
    \text{using (1.5)} &\leq \frac{2}{\ln R_j} \ln \left( \frac{C_1 (\ln R_j)^k}{C_2} \right) C_1 r_j^{n-2} (\ln R_j)^k \\
    &\leq C_3 (\ln R_j)^{k-\frac{1}{2}} r_j^{n-2} \\
    &\leq C_4 (\ln r_j)^{k-\frac{1}{2}} r_j^{n-2},
\end{align*}
\]

for some \( C_3, C_4 > 0 \) and all \( j \geq 1 \).

Let \( F \in C^\infty(\mathbb{R}^m; \mathbb{R}^m) \) be such that

\[ F(v) = v - a_i \quad \text{if} \quad |v - a_i| \leq \delta, \quad i = 1, \ldots, N, \quad (2.4) \]

for some small \( \delta > 0 \) (so that the \( \delta \)-neighborhoods of the \( a_i \)'s are disjoint). We note that such a function can be constructed by first defining it in a \( \delta \)-neighborhood of each \( a_i \) and then extending it componentwise. For future reference, let us note at this point that, by virtue of (1.8) and (2.1), there exists a constant \( C_5 > 0 \) such that

\[ |F(u) \cdot W_u(u)| \leq C_5 W(u), \quad x \in \mathbb{R}^n. \quad (2.5) \]

Taking the inner product of (1.1) with \( F(u) \) and integrating by parts the resulting identity over \( B(0, r_j) \) yields that

\[
\begin{align*}
    \int_{B(0,r_j)} F_u(u) \nabla u \cdot \nabla u dy &= \int_{B(0,r_j)} \frac{\partial}{\partial y} F(u) dS(y) - \int_{B(0,r_j)} F(u) \cdot W_u(u) dy \\
    \text{using (2.1), (2.5)} &\leq C_6 r_j^{n-1} \left( \int_{B(0,r_j)} |\frac{\partial u}{\partial y}|^2 dS(y) \right)^{\frac{1}{2}} + C_5 \int_{B(0,r_j)} W(u)dy \\
    \text{using (2.3)} &\leq C_7 (\ln r_j)^{k-\frac{1}{2}} r_j^{n-2},
\end{align*}
\]

for some \( C_6, C_7 > 0 \) and all \( j \geq 1 \).

Let

\[ A_j = \{ x \in B(0, r_j) : |u(x) - a_i| > \delta, \quad i = 1, \ldots, N \}. \]

It follows from the first part of (1.8), (2.1) and (2.3) that the \( n \)-dimensional Lebesgue measure of \( A_j \) satisfies

\[ \mathcal{H}^n(A_j) \leq C_8 (\ln r_j)^{k-\frac{1}{2}} r_j^{n-2}, \]

for some \( C_8 > 0 \) and all \( j \geq 1 \). Therefore, using once more (2.1), and (2.4), we obtain that

\[
\begin{align*}
    \int_{B(0,r_j)} F_u(u) \nabla u \cdot \nabla u dy &\geq \int_{B(0,r_j) \backslash A_j} |\nabla u|^2 dy - C_9 (\ln r_j)^{k-\frac{1}{2}} r_j^{n-2} \\
    &\geq \int_{B(0,r_j)} |\nabla u|^2 dy - C_{10} (\ln r_j)^{k-\frac{1}{2}} r_j^{n-2},
\end{align*}
\]
for some $C_9, C_{10} > 0$ and all $j \geq 1$. Hence, it follows from (2.6) that
\[
\int_{B(0,r_j)} |\nabla u|^2 \, dy \leq C_11 (\ln r_j)^{k-\frac{1}{2}} r_j^{n-2},
\]
for some $C_{11} > 0$ and all $j \geq 1$.

By combining the above relation with (2.3), we arrive at
\[
\tilde{E}(u,0,r_j) \leq C_{12} (\ln r_j)^{k-\frac{1}{2}},
\]
for some $C_{12} > 0$ and all $j \geq 1$.

We have therefore reduced the exponent in (2.2) by $1/2$ (for a different sequence $r_j \to \infty$). Iterating this scheme a finite number of times, we arrive at
\[
\tilde{E}(u,0,s_j) \to 0 \quad \text{as} \quad j \to \infty,
\]
for some sequence $s_j \to \infty$.

On the other hand, the weak monotonicity formula (1.5) (recall also (1.12)) implies that $u \equiv a_i$ for some $i \in \{1, \ldots, N\}$, which contradicts the assumption that $u$ is nonconstant. □

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