ALMOST-EVERYWHERE CONVERGENCE OF FOURIER SERIES
FOR FUNCTIONS IN SOBOLEV SPACES

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Abstract. Let $S_\lambda F(x)$ be the spherical partial sums of the multiple Fourier series of function $F \in L_2(\mathbb{T}^N)$. We prove almost-everywhere convergence $S_\lambda F(x) \to F(x)$ for functions in Sobolev spaces $H^a_p(\mathbb{T}^N)$ provided $1 < p \leq 2$ and $a > (N - 1)(\frac{1}{p} - \frac{1}{2})$. For multiple Fourier integrals this is well known result of Carbery and Soria (1988). To prove our result, we first extend the transplantation technique of Kenig and Tomas (1980) from $L^p$ spaces to $H^a_p$ spaces, then apply it to the Carbery and Soria result.

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1. Main result.

Let $\{F_n\}, n \in \mathbb{Z}^N$, be the Fourier coefficients of a function $F \in L_2(\mathbb{T}^N), N \geq 2$, i.e.

$$F_n = (2\pi)^{-N} \int_{\mathbb{T}^N} F(y)e^{-iny}dy,$$

where $\mathbb{T}^N$ is $N$-dimensional torus: $\mathbb{T}^N = (\pi, \pi]^N$. Consider the spherical partial sums of the multiple Fourier series:

$$S_\lambda F(x) = \sum_{|n| < \lambda} F_n e^{inx},$$

where $nx = n_1x_1 + n_2x_2 + \ldots + n_Nx_N$ and $|n| = \sqrt{n_1^2 + n_2^2 + \ldots + n_N^2}$.

The aim of this paper is to investigate convergence almost-everywhere (a.e.) of these partial sums for functions in Sobolev spaces. To formulate the main result, we need to remind the definition of Sobolev spaces: the class of functions $L^p(\mathbb{T}^N)$ which for a given fixed number $a > 0$ make the norm

$$||F||_{H^a_p(\mathbb{T}^N)} = \sum_{n \in \mathbb{Z}^N} (1 + |n|^2)^{\frac{a}{2}}|F_n e^{inx}| L^p(\mathbb{T}^N)$$

finite is termed the Sobolev class $H^a_p(\mathbb{T}^N), p \geq 1$.

Theorem 1.1. Let $1 < p \leq 2$ and $a > (N - 1)(\frac{1}{p} - \frac{1}{2})$. Then for any function $F \in H^a_p(\mathbb{T}^N)$ the partial sums $S_\lambda F(x)$ converge to $F(x)$ a.e. in $\mathbb{T}^N$. Moreover, the maximal operator $S_* F(x) = \sup_{\lambda > 0} |S_\lambda F(x)|$ has the estimate

$$||S_* F||_{L^p(\mathbb{T}^N)} \leq C_{p,a} ||F||_{H^a_p(\mathbb{T}^N)}.$$
2. Theorem of C. Kenig and P. Tomas.

An $L_\infty(\mathbb{R}^N)$ function $M$ is regulated if every point of $\mathbb{R}^N$ is a Lebesgue point of $M$ (see [4]). Define for each real number $\lambda > 0$ an operator $E_{M,\lambda}$ on $L_2(\mathbb{R}^N)$ by $\hat{E}_{M,\lambda}f(\xi) = M(\xi/\lambda)\hat{f}(\xi)$ and $S_{M,\lambda}$ on $L_2(\mathbb{T}^N)$ by $\hat{S}_{M,\lambda}F(n) = M(n/\lambda)^2\hat{F}(n)$. The symbols $E_{M,\lambda}f(x)$ and $S_{M,\lambda}F(x)$ stand for the corresponding maximal operators, i.e., $E_{M,\lambda}f(x) = \sup_{\lambda > 0} |S_{M,\lambda}F(x)|$. A function $M$ is called $p$-maximal on $\mathbb{R}^N$ (or weak $p$-maximal on $\mathbb{R}^N$) if the operator $E_{M,\lambda}^p$ is bounded (or weakly bounded) on $L_p(\mathbb{R}^N)$; similar for $S_{M,\lambda}^p$ on $L_p(\mathbb{T}^N)$.

Theorem (C. Kenig and P. Tomas [4]). Suppose that $M$ is a regulated $L_\infty(\mathbb{R}^N)$ function and let $1 < p < \infty$. Then $M$ is $p$-maximal or weak $p$-maximal on $\mathbb{R}^N$ if and only if $M$ is $p$-maximal or weak $p$-maximal on $\mathbb{T}^N$.

3. Theorem of Carbery and Soria.

Let the symbol $E_\lambda f(x)$ stands for the spherical partial integrals of multiple Fourier integrals of a function $f \in L_2(\mathbb{R}^N)$:

$$E_\lambda f(x) = (2\pi)^{-N} \int_{|\xi| < \lambda} \hat{f}(\xi)e^{ix\xi}d\xi.$$  

Here a Fourier transform of function $f$ is defined as

$$\hat{f}(\xi) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} f(x)e^{-ix\xi}dx.$$  

Carbery and Soria in their paper [11] considered, among the other problems, almost-everywhere convergence of Fourier integrals $E_\lambda f(x)$ for functions $f$ in Sobolev spaces $H^a_p(\mathbb{R}^N)$. Note, the norm in this space has the form

$$||f||_{H^a_p(\mathbb{R}^N)} = \left\| \int_{\mathbb{R}^N} (1 + |\xi|^2)^{-\frac{a}{2}} \hat{f}(\xi)e^{ix\xi}d\xi \right\|_{L_p(\mathbb{R}^N)}.$$  

Theorem (Carbery and Soria, [11]). Let $N \geq 2$. If $f \in H^a_p(\mathbb{R}^N)$ with

$$1 < p \leq 2, \quad a > (N - 1)(\frac{1}{p} - \frac{1}{2}),$$  

then $E_\lambda f \rightarrow f$ a.e.

Our aim is to obtain this result for Fourier series $S_\lambda F(x)$ using the idea of the proof of the theorem of C. Kenig and P. Tomas. To do this we should first reduce things to $L_p$ spaces.

Let $\Delta$ be the Laplace operator. Then

$$E_\lambda f = (1 - \Delta)^{-\frac{\lambda}{2}} E_\lambda (1 - \Delta)^{\frac{\lambda}{2}} f = (1 - \Delta)^{-\frac{\lambda}{2}} E_\lambda g, \quad g = (1 - \Delta)^{\frac{\lambda}{2}} f \in L_p(\mathbb{R}^N).$$  

Define

$$E_{(\lambda,a)}g = (1 - \Delta)^{-\frac{\lambda}{2}} E_\lambda g = (2\pi)^{-\frac{N}{2}} \int_{|\xi|^2 < \lambda} (1 + |\xi|^2)^{-\frac{a}{2}} \hat{g}(\xi)e^{ix\xi}d\xi, \quad g \in L_p(\mathbb{R}^N).$$  

The operator $S_{(\lambda,a)}G, G \in L_p(\mathbb{T}^N)$ on the torus $\mathbb{T}^N$ is defined similarly:

$$S_{(\lambda,a)}G = \sum_{|n|^2 < \lambda} (1 + |n|^2)^{-\frac{a}{2}} G_n e^{inx}, \quad G \in L_p(\mathbb{T}^N).$$  

Let $E_{(\lambda,a)}$ and $S_{(\lambda,a)}$ be the corresponding maximal operators.

It is not hard to see, that there is no function $M$ so that we could write the operators $E_{(\lambda,a)}$ and $S_{(\lambda,a)}$ in the form of operators $E_{M,\lambda}$ and $S_{M,\lambda}$. Therefore one can not use here the theorem of C. Kenig and P. Tomas. Nevertheless, we have the following statement.

**Theorem 3.1.** Let $1 < p < \infty$. If the operator $E_{(\lambda,a)}^*$ is bounded (or weakly bounded) on $L_p(\mathbb{R}^N)$, then the operator $S_{(\lambda,a)}^*$ is also bounded (or weakly bounded) on $L_p(\mathbb{T}^N)$.

Note, unlike the theorem of C. Kenig and P. Tomas, this theorem is not "if and only if" type, but it is just enough for our purpose.
4. The Lorentz spaces.

To investigate the weak boundedness it is convenient to introduce the Lorentz space \( L_{p, \infty}(\Omega) \), \( \Omega \subseteq \mathbb{R}^N \), \( 1 < p < \infty \), (or the weak \( L_p(\Omega) \) space) consisting of all measurable functions \( f \) which make the norm

\[
||f||_{L_{p, \infty}(\Omega)} = \sup_{t > 0} \{t(d_f(t))^{1/p}\}
\]

finite. In this definition the symbol \( d_f(t) \) stands for the distribution function

\[
d_f(t) = \mu\{x \in \Omega; |f(x)| > t\},
\]

where \( \mu(E) \) is the Lebesque measure of the set \( E \).

It is not hard to verify, that

\[
t^p \mu\{x \in \Omega; |f(x)| > t\} \leq \||f||_{L_{p, \infty}(\Omega)}^p \leq \||f||_{L_p(\Omega)}^p.
\]

We also need the Lorentz space \( L_{p, 1}(\Omega) \) with the norm

\[
||f||_{L_{p, 1}(\Omega)} = \int_0^\infty (d_f(t))^{1/p} dt.
\]

These both Lorentz spaces are Banach spaces and \( (L_{p, 1}(\Omega))^* = L_{p', \infty}(\Omega) \) (see [3], p. 52).

Let \( 1 < p < \infty \) and \( q = \frac{p}{p-1} \). Then by virtue of the Hahn-Banach theorem one has

\[
||f||_{L_{p, 1}(\Omega)} = \sup_{||g||_{L_{q, \infty}(\Omega)} \leq 1} \int_\Omega fg dx
\]

and

\[
||f||_{L_{p, \infty}(\Omega)} = \sup_{||g||_{L_{q, 1}(\Omega)} \leq 1} \int_\Omega fg dx.
\]

5. A linearization.

Define the Banach space \( L_p(\Omega, l^\infty(\mathbb{Z}^+)) \) as the collection of all sequences of \( L_p(\Omega) \) functions \( \{f_k\} \) such that the norm \( \sup_k ||f_k||_{L_p(\Omega)} \) is finite. The Banach space \( L_p(\Omega, l^1(\mathbb{Z}^+)) \) is defined similarly. Then \( E^*_s(\alpha) \) may be viewed as an operator defined on \( L_p(\mathbb{R}^N) \) and taking values in \( L_p(\mathbb{R}^N, l^\infty(\mathbb{Z}^+)) \) and \( S^*_s(\alpha) \) defined on \( L_p(\mathbb{T}^N) \) with values in \( L_p(\mathbb{T}^N, l^\infty(\mathbb{Z}^+)) \). Using this reduction and duality, the following linearization of maximal operators can be proved exactly by the same way as in [3].

Lemma 5.1. Let \( 1 < p < \infty \) and \( q = \frac{p}{p-1} \). The operator \( E^*_s(\alpha) \) is bounded in \( L_p(\mathbb{R}^N) \) if and only if

\[
||\sum_k E_{(\lambda_k, \alpha)} f_k||_{L_q}(\mathbb{R}^N) \leq C \||f||_{L_p(\mathbb{R}^N)}, \quad \{f_k\} \in L_q(\mathbb{R}^N, l^1(\mathbb{Z}^+)),
\]

uniformly in all sequences of positive reals \( \{\lambda_k\} \).

Similarly, the operator \( S^*_s(\alpha) \) is bounded in \( L_p(\mathbb{T}^N) \) if and only if

\[
||\sum_k S_{(\lambda_k, \alpha)} F_k||_{L_q}(\mathbb{T}^N) \leq C \||F||_{L_p(\mathbb{T}^N)}, \quad \{F_k\} \in L_q(\mathbb{T}^N, l^1(\mathbb{Z}^+)),
\]

uniformly in all sequences of positive reals \( \{\lambda_k\} \).

Similar results hold for weak boundedness of operators \( E^*_s(\alpha) \) and \( S^*_s(\alpha) \). Namely, using the equalities (4.2) and (4.3) one can prove that the pair of inequalities

\[
||E^*_s(\alpha)f||_{L_{p, \infty}(\mathbb{R}^N)} \leq ||f||_{L_p(\mathbb{R}^N)}, \quad f \in L_p(\mathbb{R}^N),
\]

(5.3)

\[
||S^*_s(\alpha)F||_{L_{p, \infty}(\mathbb{T}^N)} \leq ||F||_{L_p(\mathbb{T}^N)}, \quad F \in L_p(\mathbb{T}^N),
\]

is equivalent to the pair of inequalities

\[
||\sum_k E_{(\lambda_k, \alpha)} f_k||_{L_{q, 1}(\mathbb{R}^N)} \leq C \||f||_{L_{q, 1}(\mathbb{R}^N)}, \quad \{f_k\} \in L_{q, 1}(\mathbb{R}^N, l^1(\mathbb{Z}^+)),
\]

(5.5)
In [5] this identity was proved for the operators \( E \).

Therefore, \( \sum_k S(\lambda_k, a)F_k \in L_q(T^N) \leq C \sum_k |F_k| \in L_{q, 1}(T^N) \), \( \{ F_k \} \in L_{q, 1}(T^N, l^1(Z^+)) \).

6. Proof of Theorem 5.1

We first prove that \( S^*_\lambda \) is bounded in \( L_p(T^N) \) if \( E^*_\lambda \) is bounded in \( L_p(\mathbb{R}^N) \).

According to the linearization Lemma 5.1 it suffices to show that from the inequality (6.1) uniformly in all sequences \( \lambda_k \) it follows the inequality (5.2) uniformly in all sequences \( \lambda_k \).

Let \( G(x) \) be any continuous periodic function on \( T^N \). Then

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathbb{T}^N} G(x) e^{-\varepsilon |x|^2} dx = \int_{\mathbb{T}^N} G(x) dx,
\]
as in Stein and Weiss [5], p. 261.

Now suppose that the inequality (5.1) holds true. It suffices to prove (5.2) for an arbitrary trigonometric polynomials \( F_k(x) = P_k(x) \). Let \( Q(x) \) be a trigonometric polynomial on \( T^N \) and \( L_\varepsilon(x) = e^{-\varepsilon |x|^2} \). The following identity can be proved by virtue of (6.1) (see Stein and Weiss [5], p. 261):

\[
\int_{\mathbb{T}^N} \sum_k S(\lambda_k, a)P_k(x)Q(x) dx = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathbb{T}^N} \sum_k E(\lambda_k, a)(P_k L_\varepsilon/P)Q(x)L_{\varepsilon/p}(x) dx.
\]

In [5] this identity was proved for the operators \( S_{M, \lambda} \) and \( E_{M, \lambda} \). But this proof remains valid if the multiplier is only bounded and continuous.

Applying the inequality (5.1), one has by virtue of the last equality

\[
\left| \int_{\mathbb{T}^N} \sum_k S(\lambda_k, a)P_k(x)Q(x) dx \right| \leq C_p \lim \sup_{\varepsilon \to 0} \left[ \varepsilon^\frac{Q_1}{Q_2} \left( \sum_k |P_k| L_{\varepsilon/q}(R^N) \right) \right] |
\](\( Q \)) \( L_{\varepsilon/p}(R^N) \) \( L_p(R^N) \).

where we used (6.1) in the last equality. Taking the supremum over all trigonometric polynomials \( Q \) with \( L_p \) norm 1, we obtain (5.2), and this completes the proof of the theorem concerning the boundedness of the operators.

As to the weak boundedness of the operators, the proof is similar: we again apply the linearization Lemma 5.1 and assuming (5.3), in order to prove (5.6) it will suffice to know that

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sum_k |P_k| L_{\varepsilon/q}(\mathbb{R}^N) \leq \sum_k |P_k| L_{q, 1}(R^N),
\]

which follows from the Poisson summation formula (see [5], p. 225, 231).

7. Proof of Theorem 5.1

Let \( 1 < p \leq 2 \), \( q = \frac{2N}{N+2/p} \) and \( a = \frac{N-1}{2} \). Application of the Marcinkiewicz interpolation theorem to the estimates, obtained in [1], paragraphs 3 and 4, gives

\[
||E^*_\lambda f||_{L_q(\mathbb{R}^N)} \leq C ||f||_{L_p(\mathbb{R}^N)}, \quad f \in L_p(\mathbb{R}^N).
\]

Therefore, \( E^*_\lambda f \) is finite a.e. on \( \mathbb{R}^N \) for each \( f \in L_p(\mathbb{R}^N) \), \( p > 1 \). Then by Stein’s theorem on a sequence of translation invariant linear operators: \( L_p \to L_p \), \( 1 \leq p \leq 2 \) (see, for example, [7], p. 73), one may conclude, that

\[
||E^*_\lambda f||_{L_{p, \infty}(\mathbb{R}^N)} \leq C ||f||_{L_p(\mathbb{R}^N)}, \quad f \in L_p(\mathbb{R}^N), \quad p > 1, \quad a = \frac{N - 1}{2},
\]

By virtue of Theorem 5.1 we have for the same values of the parameters \( p \) and \( a \) an estimate

\[
||S^*_\lambda f||_{L_{p, \infty}(\mathbb{R}^N)} \leq C ||f||_{L_p(\mathbb{T}^N)}, \quad F \in L_p(\mathbb{T}^N).
\]

On the other hand, from the estimates, obtained in [1], paragraph 3, it follows

\[
||E^*_\lambda f||_{L_2(\mathbb{R}^N)} \leq C_a ||f||_{L_2(\mathbb{R}^N)}, \quad f \in L_2(\mathbb{R}^N), \quad a > 0,
\]
or by virtue of Theorem 5.1

\[(7.2) \quad \|S^*_a F\|_{L^p(T^N)} \leq C_a \|F\|_{L^2(T^N)}, \quad F \in L^2(T^N), \quad a > 0.\]

Now applying first the Marcinkiewicz interpolation theorem to the estimates (7.1) and (7.2) (with \(a = \frac{N-1}{2}\)), we obtain

\[\|S^*_a F\|_{L^p(T^N)} \leq C \|F\|_{L^p(T^N)}, \quad F \in L^p(T^N), \quad p > 1, \quad a = \frac{N-1}{2},\]

then applying to this estimate and (7.2) (with \(a > 0\)) Stein’s interpolation theorem on an analytic family of linear operators (see, for example, \([7]\), p. 46) we have

\[\|S^*_a F\|_{L^p(T^N)} \leq C_{p,a} \|F\|_{L^p(\mathbb{R}^N)}, \quad F \in L^p(T^N), \quad \text{where } 1 < p \leq 2 \text{ and } a > (N-1)(\frac{1}{p} - \frac{1}{2}).\]

Turning back to Sobolev spaces, we rewrite this inequality as

\[\|S^*_a G\|_{L^p(T^N)} \leq C_{p,a} \|G\|_{H^p(T^N)}, \quad G \in H^p(T^N), \quad \text{where } p \text{ and } a \text{ satisfy the same conditions.}\]

This is the estimate (1.3). First part of the statement of Theorem 1.1 follows from this estimate by using the standard technic (see \([5]\)).

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