CONVERGENCE OF RICCI FLOW ON $R^2$ TO PLANE

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Abstract. In this paper, we give a sufficient condition such that the Ricci flow in $R^2$ exists globally and the flow converges at $t = \infty$ to the flat metric on $R^2$.

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1. Introduction

In this short note, we are interested in the long-term behavior on $R^2$ of conformally flat solutions to the Ricci flow equation on $R^2$. Recall here that the Ricci flow equation for the one-parameter family of metric $g(t)$ on $R^2$ is

$$\partial_t g = -Rg, \text{ in } R^2.$$  

(1)

For these metrics $g(t)$, we take their forms as $g(x, t) = e^{u(x, t)}g_E$, where $g_E$ is the standard Euclidean metric on $R^2$. Then the Ricci flow equation becomes

$$\partial_t e^u = \Delta u, \text{ in } R^2,$$

where $\Delta$ is the standard Laplacian operator of the flat metric $g_E$ in $R^2$. The long-term existence of solutions of (1) or (6) has been studied in [3], where it is shown that

Theorem 1. The solutions to (1) with initial metric $g(0) = e^{u_0}g_E$ exist for all $t \geq 0$ if and only if

$$\int_{R^2} e^{u_0}dx = \infty.$$  

(3)

The global behavior of the Ricci flow has been studied in [14]. To state one of her result, we recall two concepts of the metric $g(t)$. One is below.

Definition 2. The aperture of the metric $g$ on $R^2$ is defined as

$$A(g) = \frac{1}{2\pi} \lim_{r \to \infty} \frac{L(\partial B_r)}{r}.$$  

Here $B_r$ denotes the geodesic ball (or disc) of radius $r$ and $L(\partial B_r)$ is the length of the boundary of $\partial B_r$.

The other is the Cheeger-Gromov convergence of the Ricci flow.

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Definition 3. The Ricci flow $g(t)$ is said to have modified subsequence convergence, if there exists a 1-parameter family of diffeomorphisms $\{\phi(t)\}_{t \geq 0}$ such that for any sequence $t_j \to \infty$, there exists a subsequence (denoted again by $t_j$ ) such that the sequence $\phi(t_j)^* g(t_j)$ converges uniformly on every compact set as $t_j \to \infty$.

Then we have the following result of L.F. Wu [14].

Theorem 4. Let $g(t) = e^{nu(t)} g_E$ be a solution to (1.1) such that $g(0) = e^{nu_0} g_E$ is a complete metric with bounded curvature and $\nabla u_0$ is uniformly bounded on $\mathbb{R}^2$. Then the Ricci flow has modified subsequence convergence as $t_j \to \infty$ with the limiting metric $g_\infty$ being complete metric on $\mathbb{R}^2$; furthermore, the limiting metric is flat if $A(g(0)) > 0$.

We point out that the diffeomorphisms $\phi(t_j)$ used in Theorem 4 are of the special form

$$\phi(t)(a, b) = (e^{-u(x_0, t)} a, e^{-u(x_0, t)} b) = (x_1, x_2) = x,$$

where $x_0 = (0, 0)$. The important fact for these diffeomorphisms is that

$$|\nabla g(t)f(x, t)| = |\nabla \phi(t)^* g(t)f((a, b), t)|,$$

for any smooth function $f$ and $x = \phi(t)(a, b)$.

In the interesting paper [8], which motivates our work here, the authors have proved the following.

Theorem 5. Suppose $g_0 = e^{nu_0} g_E$ has bounded curvature and $u_0$ is a bounded smooth function on $\mathbb{R}^2$. Then the Ricci flow $\partial_t g = -Rg$ exists for all $t \geq 0$ and has modified subsequence convergence to the flat metric in the $C^k$ topology of metrics on compact domains in $\mathbb{R}^2$ for each $k \geq 2$.

There is another formulation in dimension two. Since every complete Riemannian manifold of dimension two is a one dimension Kähler manifold, we can use the Kähler-Ricci flow formulation of the Ricci flow on $\mathbb{R}^2$. We shall consider the Ricci flow (1) as the Kähler-Ricci flow by setting

$$g_{ij} = g_{0ij} + \partial_i \partial_j \phi,$$

where $\phi = \phi(t)$ is the Kähler potential of the metric $g(t)$ relative to the metric $g_0$. Note that

$$g(0)_{ij} = g_{0ij} + \partial_i \partial_j \phi_0,$$

In this situation, the Ricci flow can be written as

$$\partial_t \phi = \log \frac{g_{011} + \phi_{11}}{g_{011}} - f_0, \quad \phi(0) = \phi_0,$$

where $f_0$ is the potential function of the metric $g_0$ in the sense that $R(g_0) = \Delta_{g_0} f_0$ in $\mathbb{R}^2$. Here $\Delta_{g_0} = g_0^{11} \partial_1 \partial_1$ in $\mathbb{R}^2$, which is the normalized Laplacian in Kähler geometry. Such a potential function has been introduced by
R. Hamilton in [7]. We remark that the initial data for the evolution equation (4) is \( \phi(0) \) which is non-trivial. For the equivalent of these two flows, one may see [1].

Our result is below.

**Theorem 6.** Suppose \( g_0 = e^{u_0}g_E \) has bounded curvature \( R_0 \) with (3) and \( f_0 \) is a bounded smooth function on \( \mathbb{R}^2 \) such that \( \Delta g_0 f_0 = R_0 \). Then the Ricci flow \( \partial_t g = -Rg \) with the initial metric \( g_0 \) exists for all \( t \geq 0 \) and has modified subsequence convergence to the flat metric in the \( C^k \) topology of metrics on compact domains in \( \mathbb{R}^2 \) for each \( k \geq 2 \).

We remark that because of the assumption about the potential function \( f_0 \), the initial metric \( g_0 \) is far from the cigar metric [9]. Here is the idea of the proof. We shall show that the limit \( f_\infty \) of \( f(t_j) \) is a constant function. Because of Theorem 4, we need only show that \( R(g_\infty) = \Delta g_\infty f_\infty = 0 \). The proof of Theorem 6 will be given in section 3.

2. **Maximum Principle and the Equivalence of Flows (4) and (1) in Dimension Two**

First we recall the maximum principle for the Ricci flow with bounded curvature. Given the Ricci flow \( g(t) \) on \( \mathbb{R}^2 \) with bounded curvature, we have the following well-known maximum principle.

**Lemma 7.** Fix any \( T > 0 \). If \( w(x,t) \) is a bounded smooth solution to the heat equation

\[
\partial_t w = \Delta_{g(t)} w, \quad R^2 \times (0,T]
\]

with the bounded initial data \( w(x,0) \), then \( |w(x,t)| \leq \sup_{R^2} |w(x,0)| \) for all \( t \in (0,T] \).

We now consider the equivalence of the flows (4) and (1) in dimension two. We use the argument in [1] (see Lemma 4.1 there). If \( g(t) \) is the Ricci flow in (1), we define

\[
u(x,t) = \int_0^t \log \frac{g_{11}(x,s)}{g_{11}(x,0)} ds - tf(0)
\]

and

\[
S_{11}(x,t) = g_{11}(x,t) - g_{11}(x,0) - u_{11}(x,t).
\]

Then by direct computation we have

\[
\frac{dS_{11}(x,t)}{dt} = 0, \quad S_{11}(x,0) = 0.
\]

Hence \( S_{11}(x,t) = 0 \) for all \( t > 0 \) and

\[
g_{11}(x,t) = g_{11}(x,0) + u_{11}(x,t).
\]

If \( u = u(x,t) \) is a solution to (1), then it is clear that

\[
g_{11}(x,t) = g_{11}(x,0) + u_{11}(x,t)
\]

satisfies (1).
3. proof of Theorem 6

The idea of the proof of Theorem 6 is similar to the argument in [2] and [8], see also [9].

Let
\[ f = -\partial_t \phi. \]

Then, taking the time derivative of (4), we have
\[ \partial_t f = \Delta_g f, \quad f(0) = -\partial_t \phi(0) = f_0. \]

By Lemma 7 we know that \( f \) is uniformly bounded in \( \mathbb{R}^2 \). The important fact for us is that
\[ \Delta_g f = R. \]

See [9] for a proof of this. If \( f_0 \) has some decay at space infinity, one can can the same decay by the argument of Dai-Ma [4].

It is well-known that \( R \) is uniformly bounded in any finite interval and \( |f_t| \) and \( |\nabla f|^2 \) are bounded for each \( t \geq 0 \) (via the use of \( f(x, t) = f(x, 0) + \int_0^t R(x, s) ds \)).

Our next task is to obtain a better control on \( |\nabla f| \) as \( t \to \infty \). To get this, we let
\[ F(x, t) = t|\nabla f|^2 + f^2. \]

Then we have
\[ \partial_t F \leq \Delta_g F, \quad \text{in } \mathbb{R}^2. \]

Using the maximum principle (Lemma 7), we know that
\[ \sup_{x \in \mathbb{R}^2} |\nabla f(x, t)|^2 \leq \frac{C}{1 + t} \]

for some uniform constant \( C > 0 \). Once we have this bound, we can follow the argument in Lemmata 8,9, and 10 in [8] to conclude that the curvature bounds that there are uniform constants \( C_k \), for any \( k \geq 1 \), such that
\[ \sup_{\mathbb{R}^2} |\nabla^k R(x, t)|^2 \leq \frac{C_k}{(1 + t)^{k+2}}, \quad t > 0. \]

We are now ready to complete the proof of Theorem 6. Proof of Theorem 6 We shall use the modified convergence sequence \( g(t_j) \) in Theorem 4. We need only show that the limiting metric has flat curvature and this will be obtained by showing that the limiting function \( f_\infty \) of \( f(x, t_j) \) is constant. Since \( f(x, t) \) is uniformly bounded by a constant \( K > 0 \) on \( \mathbb{R}^2 \times [0, \infty) \), for the fixed \( x_0 = (0, 0) \in \mathbb{R}^2 \) and for any sequence \( t_j \to \infty \), there exists a subsequence, still denoted by \( t_j \), such that \( c = \lim f(x_0, t_j) \) exists. By the construction of the metrics \( g(t_j) \), for any compact subset \( K \subset \mathbb{R}^2 \), the limiting metric \( g_\infty \) is equivalent to any \( \phi(t_j)^*g(t_j) \) for every large \( t \); that is, there is a uniform constant \( C = C(K) > 0 \) such that
\[ d_t(x, x_0) \leq C d_{g_\infty}(x, x_0) \]
for every $x \in K$, where $d_t(x, x_0)$ is the distance between $x$ and $x_0$ in $\phi(t)^*g(t)$ and $d_{g_\infty}(x, x_0)$ is the distance of the limiting metric. For $x \in K$, we can establish (for all $t > 1$),

$$|f(x, t) - f(x_0, t)| \leq d_t(x, x_0) \sup_{x \in K} |\nabla f(x, t)| \leq \frac{C d_{g_\infty}(x, x_0)}{1 + t},$$

where we have used the fact that for $x = \phi(t)(a, b)$,

$$|\nabla g(t)f(x, t)| = |\nabla \phi(t)^*g(t)f((a, b), t)|,$$

which is uniformly bounded. It follows that $f(x, t_j)$ is also convergent to $c$, which is $f_\infty(x) = c$ as $t_j \to \infty$, and then $\partial_1 \partial_\bar{1} f(x, t_j) \to 0$. Then we have $\Delta_{g_\infty} f_\infty = 0 = R_\infty$ and then $\phi(t_j)^*g(t_j) \to g_\infty$ locally in $C^2$ with $g_\infty$ of flat curvature. The $C^k$-convergence of $\phi(t_j)^*g(t_j)$ to this flat limit then follows from the previous curvature estimates obtained in [7] (see Lemmata 8, 9, and 10 in [8]). This completes the proof of Theorem 6.

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