Fast matrix decomposition in $\mathbb{F}_2$

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In this work an efficient algorithm to perform a block decomposition for large dense rectangular matrices with entries in $\mathbb{F}_2$ is presented. Matrices are stored as column blocks of row major matrices in order to facilitate rows operation and matrix multiplications with block of columns. One of the major bottlenecks of matrix decomposition is the pivoting involving both rows and column exchanges. Since row swaps are cheap and column swaps are order of magnitude slower, the number of column swaps should be reduced as much as possible. Here is presented an algorithm that completely avoids the column permutations. An asymptotically fast algorithm is obtained by combining the four Russian algorithm and the recursion with Strassen algorithm for matrix-matrix multiplication. Moreover optimal parameters for the tuning of the algorithm are theoretically estimated and then experimentally verified. A comparison with the state of the art public domain software SAGE shows that the proposed algorithm is generally faster.

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1. INTRODUCTION

An important tool in linear algebra is the matrix decomposition, which expresses a (rectangular) matrix as a product of two or more simpler matrices. Such decompositions are used for easy computation of rank, null space, and solving linear system and related problem.

There are many well-known algorithms for matrix decomposition defined in any field, finite or not. A common approach consists in reducing a matrix to the row-echelon form by row operations [Meyer 2000]. Once the row-echelon form is obtained, the rank will be equal to the number of non-zero rows and null space can be easily computed. Gauss $LU$ decomposition [Golub and Van Loan 1996] can be also used to solve linear systems (when the matrix is square and full rank) or compute the rank. Applied to rectangular or rank deficient matrices, it is costly as the computation of the row-echelon form. In fact, Gauss decomposition of a matrix $A$ produces an $LU$ factorization, i.e. $PAQ = LU$ where $P$ and $Q$ are permutation matrices, $L$ is the lower triangular matrix and $U$ is the row-echelon form of $A$ (up to column permutations). The asymptotic cost of a naive implementation of $LU$ decomposition for a dense $n \times n$ matrix is $O(n^3)$. However such a cost can be reduced using a combination of recursion and matrix-matrix multiplication. For example, using matrix-matrix multiplication (MMM) in the construction of the factorization, the asymptotic cost may be reduced by using fast MMM algorithms. The problem of fast matrix-matrix multiplication is still under development.
The naive MMM algorithm is based on the classical definition of the multiplication of two matrices; its cost is \(n^3\) multiplications and \(n^2(n-1)\) additions and so we classify the naive algorithm as an \(O(n^3)\) algorithm.

Strassen matrix-matrix multiplication algorithm [Strassen 1969] – which asymptotic cost is \(O(n^\log_2 7)\) in any field – uses only seven scalar multiplications (instead of the usual eight) to multiply \(2 \times 2\) matrices. In fact, as proved in [Winograd 1971], Strassen’s algorithm is optimal for \(2 \times 2\) matrices. Further asymptotic improvements [Coppersmith and Winograd 1990] can be obtained to perform multiplication of larger matrices. Hybrid algorithms incorporate Strassen and Winograd variants recursively to achieve high performance on large matrices [Huss-Lederman et al. 1996; Higham 1990; Douglas et al. 1994; Kaporin 1999]. The asymptotic cost \(O(n^\log_2 7)\) means that for a large enough \(n\), Strassen’s algorithm should theoretically perform multiplication significantly faster than the naive algorithm. However, asymptotic cost means that the actual cost of standard \(LU\) decomposition is about \(C_1 n^3\) while the actual cost of Strassen multiplication is about \(C_2 n^\log_2 7\) where \(C_2 \gg C_1\). Therefore, the use of Strassen algorithm is convenient only for large \(n\).

Strassen algorithm is recursive so that normally the recursion is terminated when the cost of recursion is larger than the classical matrix-matrix multiplication. That happens when \(C_2 n^\log_2 7 \approx C_1 n^3\), i.e. \(n \approx \exp(\ln(C_2/C_1)/(3-\log_2 7))\). For instance, if \(C_2/C_1 \approx 10\) we have \(n \approx 150000\) while in case \(C_2/C_1 \approx 5\) we have \(n \approx 4000\).

In practice, the computation of the switching point must consider additional costs and it is implementation dependent. For a detailed analysis see for example [Huss-Lederman et al. 1996; Higham 1990].

The efficient computation of MMM can be further improved in case of finite fields. In particular, in case of \(\mathbb{F}_2\), the Method of four Russian for Multiplication (M4RM) is a fast MMM algorithm, which cost is \(O(n^3/\log n)\) [Arlazarov et al. 1970; Aho et al. 1974; Albrecht et al. 2010]. Its asymptotic cost is better than classical matrix-matrix multiplication; but it is worse than recursive Strassen’s algorithm. However, if the actual cost of M4RM is about \(C_3 n^3/\log n\), we have that \(C_3 \ll C_1\) so that is competitive for not too big matrices. A combination of Strassen and M4RM is a good compromise for faster matrix-matrix multiplication [Albrecht et al. 2010]. The fast decomposition of an \(n \times m\) matrix with entries in a finite field \(\mathbb{F}_2\) is an important issue in algorithmic number theory and cryptanalysis [Shoup 2009; Bach and Shallit 1996]. In fact, some problems in cryptanalysis and number theory can be transformed in one involving a linear system with entries in \(\mathbb{F}_2\). The existence of solutions of a linear system can be deduced by analyzing the rank of the corresponding matrix.

In this paper we propose a new efficient algorithm to perform the matrix factorization for large dense rectangular matrices with entries in \(\mathbb{F}_2\). It uses an efficient implementation of M4RM algorithm and uses only row permutations. In Section 2 our notation is given and an appropriate data structure to store the matrix is described. In Section 3 the non-recursive block decomposition algorithm is presented and in Section 4 the corresponding recursive version is described. Moreover, in Section 5 some details about the choice of the algorithm parameters are given. Tests comparing our algorithm with Sage packages [Stein et al. 2012] are presented in Section 6.
2. THE USED MATRIX DATA STRUCTURE

Let $A$ be an $n \times m$ matrix having entries in $\mathbb{F}_2$, i.e. $A \in \text{Mat}(n, m, \mathbb{F}_2)$, denoted as

$$A = \begin{pmatrix}
a_{0,0} & \cdots & a_{0,m-1} \\
\vdots & \ddots & \vdots \\
a_{n-1,0} & \cdots & a_{n-1,m-1}
\end{pmatrix}.$$ 

We adopt the following convention for intervals of indices $a..b = \{a, a+1, \ldots, b-1\}$ and so, for instance, the sub-matrix $A_{a..b,c..d} \in \text{Mat}(b-a, d-c, \mathbb{F}_2)$ of $A$ represents the intersection of rows of index from $a$ to $b-1$ and columns of index from $c$ to $d-1$. The sub-matrix $A_{a..b,\cdot} \in \text{Mat}(b-a, m, \mathbb{F}_2)$ of $A$ is composed by the rows of index from $a$ to $b-1$. Furthermore, we denote by $\text{rank}(A)$ the rank of $A$, i.e. the maximum number of linearly independent row (or column) vectors of $A$.

We are interested in the computation of the factorization of large dense matrices that do not fit into the cache. Thus, a good arrangement of the elements of the matrix in memory is important for an efficient data retrieval. Moreover, bits are naturally grouped in words whose size is a power of 2, typically 32, 64 or 128 for larger architectures. An element of $\mathbb{F}_2$ is naturally represented as one bit, so that elements in $\mathbb{F}_2$ are naturally grouped in words of 32, 64 or 128 bits. In particular, a string of elements in $\mathbb{F}_2$ is packed in an (unsigned) integer. The advantage of storing multiple elements of $\mathbb{F}_2$ as an integer is that it guarantees a natural parallelism of some operations.

From now on, we denote with $b$ the number of bits of the computer architecture, i.e. the number of bits in one machine word. Given two integers $x$ and $y$ whose bits represent elements in $\mathbb{F}_2$, the operation of exclusive or, denoted by $x \oplus y$, is the sum $\oplus$ in $\mathbb{F}_2$ applied to $x$ and $y$ bitwise; the and operation, denoted by $x \odot y$, is the multiplications $\odot$ in $\mathbb{F}_2$ applied to $x$ and $y$ bitwise. Infact, we have the following formulas:

$$x = \sum_{i=0}^{b} x_i 2^i, \quad y = \sum_{i=0}^{b} y_i 2^i, \quad x \oplus y = \sum_{i=0}^{b} (x_i \oplus y_i)2^i, \quad x \odot y = \sum_{i=0}^{b} (x_i \odot y_i)2^i.$$ 

The entries of a matrix are packed into integers that represent groups of elements of the matrix itself. In particular, $b$ consecutive entries in one row are packed into one integer. The way the integers are arranged changes how one accesses the elements of the matrix and you implement the elementary operations.

A matrix $A \in \text{Mat}(n, m, \mathbb{F}_2)$ is stored into a matrix of non-negative integers $\mathcal{A} \in \text{Mat}(n, \mu, \mathbb{N})$ with $\mu b - b < m \leq \mu b$. To access to the element $a_{ij}$ of $A$, we have to determine the corresponding column-block $q$ and then the right bit. Precisely, using integer division with remainder $j = qb + r$, $(0 \leq r < b)$, the element $a_{ij}$ corresponds to the $r$-th bit of the integer $\mathcal{A}_{iq}$.

**Remark 2.1.** According to the previous notation, the least significant bit is on the left respect to the most significant bit, i.e. we are using big endian bit order.
We consider a trivial example. Let $b = 3$ and $A$ be the following $(4 \times 5)$-matrix:

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Since $m = 5$ is not a multiple of $b = 3$, we directly memorize the matrix

$$A' = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

in a $4 \times 2$ matrix of 3 bit integer as follows

$$\mathcal{A} = \begin{pmatrix} 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 & 1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 \\ 1 \cdot 2^0 + 0 \cdot 2^1 + 0 \cdot 2^2 & 0 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 \\ 1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 & 1 \cdot 2^0 + 0 \cdot 2^1 + 0 \cdot 2^2 \\ 0 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 & 1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 1 & 2 \\ 3 & 1 \\ 4 & 3 \end{pmatrix}. \quad (1)$$

The elements of $\mathcal{A}$ can be organized using row-major order or column-major order. Our algorithm works better using column-major order. For example matrix $\mathcal{A}$ in (1) is stored as

$$\begin{pmatrix} 5 & 1 & 3 & 4 & | & 3 & 2 & 1 & 3 \end{pmatrix}.$$
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3. BLOCK DECOMPOSITION

In this section we give a (non-recursive) block decomposition that holds for matrices with entries in any field. For simplicity of notation, we are going to describe our strategy for matrices in $\mathbb{F}_2$. Let consider $A \in \text{Mat}(n, m, \mathbb{F}_2)$ such that it can be split in four blocks

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix},$$

where the block $B \in \text{Mat}(r, b, \mathbb{F}_2)$, with $r \leq b$, has full rank and the rows of $D$ are linearly dependent on those of $B$. In other words, there exists a matrix $Y$ such that $D = Y B$. The factorization is based on the identity:

$$\begin{bmatrix} B & C \\ D & E \end{bmatrix} = \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & E + YC \end{bmatrix} \begin{bmatrix} B & C \\ 0 & I \end{bmatrix}.$$

(2)

Due to the previous identity, we have that

$$\begin{bmatrix} I & 0 \\ 0 & E + YC \end{bmatrix} \begin{bmatrix} B & C \\ 0 & I \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & E + YC \end{bmatrix}$$

and notice that $\text{rank}(A) = \text{rank}(E + YC) + \text{rank}(B) = \text{rank}(E + YC) + r$. The sub-matrix $E + YC \in \text{Mat}(n - r, m - b, \mathbb{F}_2)$ is the Schur complement of $A$ (see [Haynsworth 1968; Zhang 2005]). Thus, we have reduced the decomposition to a smaller problem. Applying the same idea as before we can reduce the decomposition to smaller and smaller problems with a reduction steps that can be described as follows.

Let $A^{(0)} = A$ and $P^{(0)}$ be a permutation matrix such that $P^{(0)} A^{(0)}$ can be partitioned as

$$P^{(0)} A^{(0)} = \begin{bmatrix} B^{(0)} \\ D^{(0)} \end{bmatrix} \begin{bmatrix} C^{(0)} \\ E^{(0)} \end{bmatrix}$$

where $B^{(0)} \in \text{Mat}(r_0, b, \mathbb{F}_2)$, $\text{rank}(B^{(0)}) = r_0 \leq b$

and the rows of $D^{(0)}$ are linearly dependent on those of $B^{(0)}$, i.e. $D^{(0)} = Y^{(0)} B^{(0)}$. Using the products in (2) we obtain the following decomposition

$$P^{(0)} A^{(0)} = \begin{bmatrix} I & 0 \\ Y^{(0)} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & A^{(1)} \end{bmatrix} \begin{bmatrix} B^{(0)} & C^{(0)} \\ 0 & I \end{bmatrix},$$

where $A^{(1)} = E^{(0)} + Y^{(0)} C^{(0)} \in \text{Mat}(n - r_0, m - b, \mathbb{F}_2)$. Now, we can apply the same idea to the Schur complement $A^{(1)}$. Let $P^{(1)}$ be a permutation matrix such
that

\[ P^{(1)}A^{(1)} = \begin{bmatrix} B^{(1)} & C^{(1)} \\ D^{(1)} & E^{(1)} \end{bmatrix} \]

where \( B^{(1)} \in \text{Mat}(r_1, b, \mathbb{F}_2) \), \( \text{rank}(B^{(1)}) = r_1 \leq b \)

and \( D^{(1)} = Y^{(1)}B^{(1)} \). Due to the decomposition (2), we reduce further the problem of the decomposition of \( A^{(1)} \) to the decomposition of \( A^{(2)} = E^{(1)} \oplus Y^{(1)}C^{(1)} \in \text{Mat}(n - r_0 - r_1, m - 2b, \mathbb{F}_2) \).

Going forward, at the \( j \)th stage, the original matrix \( A \) is transformed into the sub-matrix \( A^{(j)} \); notice that \( \text{rank}(A) = \text{rank}(A^{(j)}) + r_0 + r_1 + \cdots + r_{j-1} \). Next, we find a permutation \( P^{(j)} \) such that \( P^{(j)}A^{(j)} \) can be partitioned as follows

\[ P^{(j)}A^{(j)} = \begin{bmatrix} B^{(j)} & C^{(j)} \\ D^{(j)} & E^{(j)} \end{bmatrix} \]

with \( B^{(j)} \in \text{Mat}(r_j, b, \mathbb{F}_2) \), \( r_j \leq b \) and \( B^{(j)} \) full rank and \( D^{(j)} = Y^{(j)}B^{(j)} \). Using (2) again, we have

\[ \begin{bmatrix} I \\ 0 \\ P^{(j)}A^{(j)} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & Y^{(j)} & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & A^{(j+1)} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & B^{(j)} & C^{(j)} \end{bmatrix} \],

where

\[ A^{(j+1)} = E^{(j)} \oplus Y^{(j)}C^{(j)} \].

(4)

Note that \( A^{(j+1)} \) is the Schur complement of \( P^{(j)}A^{(j)} \) in \( \mathbb{F}_2 \) and, after \( \mu \) steps, we obtain

\[ PA = \begin{bmatrix} L_{11} & 0 \\ L_{12} & I \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} A^{(\mu-1)} \\ 0 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & I \end{bmatrix}, \]

(5)

where \( L \) is non-singular lower triangular matrix and \( U \) is the full rank upper block staircase. Let \( P^{(\mu-1)} \) be a permutation matrix such that \( P^{(\mu-1)}A^{(\mu-1)} \) can be partitioned as

\[ P^{(\mu-1)}A^{(\mu-1)} = \begin{bmatrix} B^{(\mu-1)} \\ D^{(\mu-1)} \end{bmatrix} \]

where \( B^{(\mu-1)} \in \text{Mat}(r_{\mu-1}, b', \mathbb{F}_2) \),

with \( b' = m - (\mu - 1)b \) satisfying \( r_{\mu-1} \leq b' \leq b \) and \( B^{(\mu-1)} \) is full rank.
Finally, \( D^{(\mu - 1)} = Y^{(\mu - 1)} B^{(\mu - 1)} \) and the decomposition (5) becomes

\[
\begin{bmatrix}
  I \\
  0
\end{bmatrix}
\begin{bmatrix}
  L_{11} \\
  P^{(\mu - 1)} L_{21} \\
  0
\end{bmatrix}
\begin{bmatrix}
  I \\
  0
\end{bmatrix}
\begin{bmatrix}
  I \\
  0
\end{bmatrix}
\begin{bmatrix}
  L_{11} \\
  P^{(\mu - 1)} L_{21} \\
  Y^{(\mu - 1)}
\end{bmatrix}
\begin{bmatrix}
  I \\
  0
\end{bmatrix}
\begin{bmatrix}
  U_{11} \\
  U_{12} \\
  0 \\
  B^{(\mu - 1)}
\end{bmatrix},
\]

where \( L \) is full rank lower trapezoidal matrix and \( U \) is the full rank upper block staircase. The previous steps can be resumed in the Lemma:

**Lemma 3.1.** Given any (rectangular) matrix \( A \), there exists a permutation \( P \) such that

\[
PA = LU,
\]

where \( U \) is full rank upper (block) triangular matrix and \( L \) is a full rank lower trapezoidal matrix.

Clearly, \( \text{rank}(A) \) is the rank of the matrix \( U \) which is the number of its rows: \( \text{rank}(A) = \sum_{j=0}^{\mu-1} r_j \).

Observe that the computation at the step \( j \) involves matrix-matrix multiplications to obtain the Schur complement (4) which are the most costly operations of the presented algorithm. The multiplication of a \( p \times b \)-matrix by a \( b \times q \)-matrix, in case \( p \gg b \), can be efficiently performed using the M4RM algorithm [Arlazarov et al. 1970; Aho et al. 1974; Albrecht et al. 2010]. Thus, in the computation of \( A^{(j+1)} \) it is convenient to use the M4RM algorithm, because this block operation is more efficient than the usual row operations.

This decomposition is based on the selection of \( B^{(j)} \) and the construction of \( Y^{(j)} \) at each step. An efficient algorithm for this will be discussed in the next sections. For this purpose the incremental construction of an inverse of a matrix and pseudo-inverse construction is a necessary tool.

**Remark 3.2.** The decomposition described in Lemma 3.1 when \( b = 1 \) is equivalent to the PLE factorization described in [Albrecht et al. 2011; Jeannerod et al. 2011]. However, computation with \( b = 1 \) is not convenient losing natural parallelism of integer operations.

### 3.1 The computation of \( Y^{(j)} \)

Assume that the permutation \( P \) applied to matrix \( A \) results in

\[
PA = \begin{bmatrix}
  B \\
  C \\
  D \\
  E
\end{bmatrix},
\]

where the rectangular matrix \( B \in \text{Mat}(r,b,\mathbb{F}_2) \), with \( r \leq b \), has full rank and satisfies \( D = YB \) for an opportune matrix \( Y \) which we have to determine.
In case \( r = b \) matrix \( B \) is non-singular and we easily deduce that \( Y = DB^{-1} \). Instead, when the matrix \( B \) is full rank with less rows than columns, a pseudo-inverse \( B^\dagger \) has to be computed. For example the pseudo-inverse of Moore-Penrose [Ben-Israel and Greville 2003] is given by

\[
B^\dagger = B^T (BB^T)^{-1}
\]

and satisfies

\[
BB^\dagger = BB^T (BB^T)^{-1} = I,
\]
\[
DB^\dagger = Y BB^\dagger = Y.
\]

Thus, \( Y \) can be computed by simple right multiplication by a pseudo-inverse of \( B \). Although the use of the Moore-Penrose's pseudo-inverse is correct, a more efficient pseudo-inverse can be constructed. Let \( J \in \text{Mat}(b, r, \mathbb{F}_2) \) be an insertion matrix whose effect is to insert \( b - r \) zero-rows into a matrix with \( r \) rows. Let \( R \in \text{Mat}(b, b, \mathbb{F}_2) \) be the matrix containing linearly independent rows which makes the square matrix \( JB + R \) non-singular. Notice that \( J^T \) is a projection matrix which satisfies

\[
J^T J = I, \quad J^T R = 0.
\]

(6)

If \( r = b \), i.e. if \( B \) is square and non-singular, insertions are not necessary and we have \( J = I, R = 0 \).

**Example 3.3.** In case \( b = 3 \) and \( r = 2 \) we can see a situation as

\[
J_j = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Let consider \( Z = (JB + R)^{-1} J \). Due to the properties in (6) and the relation \( D = YB \), we have

\[
BZ = B(JB + R)^{-1} J = J^T JB(JB + R)^{-1} J
\]
\[
= J^T (JB + R)(JB + R)^{-1} J = J^T J = I
\]
\[
DZ = YBZ = Y.
\]

Thus the matrix \( Z \) has the same effect of the pseudo-inverse (it is a pseudo-inverse different from the Moore-Penrose one) and it is used in the computation of \( Y \) during the factorization procedure. In order to build \( Z \), we need an algorithm to compute the inverse of a (small) square matrix in \( \mathbb{F}_2 \); it will be treated in the next section.

3.2 Incremental construction of the inverse of a square matrix in \( \mathbb{F}_2 \)

Let \( B \) non-singular with all principal minors non-singular, it is possible to incrementally build its inverse \( Z \). This requirement is not restrictive because every non-singular matrix by a row permutation satisfies it (due to Gauss \( LU \) decomposition,
see [Kincaid and Cheney 2002] page 156 Theorem 1). Let $B_k$ be the $k^{th}$ principal minor of $B$ and $Z_k$ its inverse. We can directly obtain $B_{k+1}^{-1}$ from $B_k^{-1}$ using the following factorization

$$B_{k+1} = \begin{bmatrix} B_k & c \\ r^T & \alpha \end{bmatrix} = \begin{bmatrix} I & 0 \\ r^T Z_k & 1 \end{bmatrix} \begin{bmatrix} B_k & c \\ 0 & 1 \end{bmatrix}$$ (7)

which holds when

$$\alpha \oplus r^T Z_k c = 1.$$ (8)

**Remark 3.4.** In case of $\mathbb{F}_q$, the previous condition becomes $\alpha - r^T Z_k c = \beta$ where $\alpha, \beta \in \mathbb{F}_q$ and $\beta \neq 0$. Moreover, the last block in (7) becomes

$$\begin{bmatrix} B_k & c \\ 0 & \beta \end{bmatrix}.$$

Inverting factorization (7) we get immediately:

$$B_{k+1}^{-1} = \begin{bmatrix} B_k & c \\ r^T & \alpha \end{bmatrix}^{-1} = \begin{bmatrix} Z_k & Z_k c \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ r^T Z_k & 1 \end{bmatrix}$$ (9)

(Notice that $B_1$ is a $1 \times 1$ matrix and thus $B_1 = Z_1 = [1]$). Therefore, due to (9) we can compute $Z_{k+1}$ from $Z_k$ as

$$Z_{k+1} = B_{k+1}^{-1} = \begin{bmatrix} Z_k \oplus (Z_k c)(r^T Z_k) Z_k c \\ r^T Z_k \end{bmatrix},$$ (10)

which needs two matrix-vector multiplication and a rank one update. Moreover, setting $\tilde{c} = Z_k c$, the last matrix in (10) can be written as a matrix-matrix product:

$$Z_{k+1} = \begin{bmatrix} Z_k \oplus \tilde{c}(r^T Z_k) \tilde{c} \\ r^T Z_k \end{bmatrix} = H_{k+1} \begin{bmatrix} Z_k & 0 \\ 0 & 1 \end{bmatrix}, \quad H_{k+1} = \begin{bmatrix} I \oplus \tilde{c} r^T \tilde{c} \\ r^T \end{bmatrix}.$$ (11)

Let $M_k$ be the matrix obtained multiplying by $Z_k$ the first $k$ rows of $B$:

$$M_k = \begin{bmatrix} I & Z_k c & Z_k C \\ r^T & \alpha & e^T \\ D & d & E \end{bmatrix} = \begin{bmatrix} I & \tilde{c} & \tilde{C} \\ r^T & \alpha & e^T \\ D & d & E \end{bmatrix}, \quad B = \begin{bmatrix} B_k & c & C \\ r^T & \alpha & e^T \\ D & d & E \end{bmatrix}.$$

Due to (11), the update of $M_{k+1}$ results in

$$M_{k+1} = \begin{bmatrix} Z_{k+1} & 0 \\ 0 & I \end{bmatrix} B = \begin{bmatrix} H_{k+1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Z_k & 0 \\ 0 & 1 \end{bmatrix} B = \begin{bmatrix} H_{k+1} & 0 \\ 0 & I \end{bmatrix} M_k.$$
and thus, $H_{k+1}$ is used in updating both $M_{k+1}$ and $Z_{k+1}$. Moreover, we obtain the following update formulas:

$$H_{k+1} \begin{bmatrix} \tilde{C} \\ e^T \end{bmatrix} = \begin{bmatrix} I \oplus \tilde{c}r^T & \tilde{c}r^T \\ r^T & 1 \end{bmatrix} \begin{bmatrix} \tilde{C} \\ e^T \end{bmatrix} = \begin{bmatrix} \tilde{C} \oplus \tilde{c}s^T \\ s^T \end{bmatrix} \quad \text{where} \quad s^T = r^T \tilde{C} \oplus e^T.$$

Note that it is possible to perform the incremental build when all principal minors are non-singular, which is equivalent to satisfy equation (8) for all $k$. Such a condition is used to dynamically select linear independent rows in the construction of $B^{(j)}$.

### 3.3 Construction of row permutation

The steps of reduction from $A^{(j)}$ to $A^{(j+1)}$ need the computation of permutation matrix $P^{(j)}$ that reorders the rows of $A^{(j)}$ in order to obtain $P^{(j)}A^{(j)}$ partitioned as in (3). The block $B^{(j)}$ must be non-singular with all principal minors non-singular. This permutation can be computed using only the first $b$ columns of $A^{(j)}$ and must satisfy

$$P^{(j)}A^{(j)}_{*,0..b} = \begin{bmatrix} B^{(j)} \\ D^{(j)} \end{bmatrix}.$$

The selection of the rows and the construction of the inverse of $B^{(j)}$ are done together using incremental update of Section 3.2, where the selected rows must satisfy condition (8). Observe that to check condition (8), the $\ell$-row of the submatrix $A^{(j)}_{*,0..b}$ is partitioned as $A^{(j)}_{\ell,0..b} = (r^T, \alpha, \beta^T)$ where the $\beta^T$ portion of the row is ignored in the computation.

### 3.4 Insertion of linearly independent rows

In the computation of row permutation it may happen that condition (8) is not satisfied by all the rows of the column block $A^{(j)}_{*,0..b}$. Obviously, such a situation does not arise if $n \geq m$ and when the matrix is full rank.\nnAt this stage, standard algorithms introduce column permutations to satisfy condition (8). If such a condition cannot be satisfied even using column permutations, it means that last rows are linearly dependent on the previous ones and then the algorithm must end. Column permutations are not executed in our algorithm, so that new linearly independent rows are inserted using the two matrices $J$ and $R$, introduced in Section 3.1. We observe that it is easy to build a row that satisfies condition (8) and that, in particular, the row $(r^T, \alpha, \beta^T) = (0^T, 1, 0^T)$ trivially satisfies it. In practice, the presented process mixes row permutations and row insertions; it can be respectively split in a row permutations followed by a rows insertions:

$$P^{(j)}A^{(j)}_{*,0..b} = \begin{bmatrix} B^{(j)} \\ D^{(j)} \end{bmatrix}, \quad J_jB^{(j)} + R_j.$$

Notice that the permutation $P^{(j)}$ and the insertion $J_j$ are chosen in such a way the square block $J_jB^{(j)} + R_j$ has all principal minors non-singular and satisfies (6).
The following procedure performs the operation described in Sections 3.2-3.3 and 3.4 to obtain an incremental construction of the pseudo-inverse $Z$.

**Procedure buildZ($A$, $r$)**

```plaintext
1 $i_0 \leftarrow 0$;
2 for $j = 0..b$ do $Z_j \leftarrow 2^j$; $M_j \leftarrow 2^j$; $P_j \leftarrow -1$;
3 for $k_{\text{bit}} = 0..b$ do
4   $c \leftarrow 2^{k_{\text{bit}}}$;
5   for $i = 0..b$ do
6     if $M_i \odot 2^{k_{\text{bit}}} \neq 0$ then $c \leftarrow c \oplus 2^i$;
7   end
8   for $i = i_{b..r}$ do
9     if popCount($c \odot A_i$) is odd then
10        $P_{k_{\text{bit}}} \leftarrow i$; $A_i \leftarrow A_i$; $M_{k_{\text{bit}}} \leftarrow A_i$; $i_b \leftarrow i_b + 1$; break;
11     end
12   end
13 $y \leftarrow M_{k_{\text{bit}}}$; // Update of $Z$ and $M$
14 for $j = 0..b$ do
15   if $y \odot 2^j \neq 0$ then $Z_{k_{\text{bit}}} \leftarrow Z_{k_{\text{bit}}} \oplus Z_j$; $M_{k_{\text{bit}}} \leftarrow M_{k_{\text{bit}}} \oplus M_j$;
16 end
17 for $j = 0..b$ do
18   if $c \odot 2^j \neq 0$ then $Z_j \leftarrow Z_{k_{\text{bit}}} \oplus Z_j$; $M_j \leftarrow M_{k_{\text{bit}}} \oplus M_j$;
19 end
20 $m \leftarrow 0$;
21 for $i = 0..b$ do
22   if $P_i = -1$ then
23     // $\overline{m}$ is the integer with bits complemented
24     for $j = 0..b$ do $Z_j \leftarrow (Z_j/2 \odot \overline{m}) \oplus (Z_j \odot m)$;
25   else
26     $m \leftarrow 2m + 1$;
27   end
28 end
```

## 4. BLOCK RECURSIVE ALGORITHM

In this section we describe a recursive version of the algorithm presented in Section 3. Let consider $A \in \text{Mat}(n, m, \mathbb{F}_2)$ split into two sub-matrices $A_L \in \text{Mat}(n, p, \mathbb{F}_2)$ and $A_R \in \text{Mat}(n, m - p, \mathbb{F}_2)$ with $p = \lceil m/2 \rceil$ such that

$$A = \begin{bmatrix} A_L & A_R \end{bmatrix}.$$

Applying the decomposition in Lemma 4.1 to the left part $A_L$, we have $PA_L = LU$ and so

$$PA = \begin{bmatrix} PA_L & PA_R \end{bmatrix} = \begin{bmatrix} LU & PA_R \end{bmatrix}.$$
Since \( PA_R = \begin{bmatrix} C \\ D \end{bmatrix} \), where \( C \in \text{Mat}(r, m - p, \mathbb{F}_2) \) and \( D \in \text{Mat}(n - r, m - p, \mathbb{F}_2) \), we obtain

\[
PA = \begin{pmatrix} L_{0..r,} & 0 \\ L_{r..n,} & I \end{pmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} L_{0..r,}^{-1} C \\ \begin{bmatrix} D \oplus L_{r..n,}L_{0..r,}^{-1} C \end{bmatrix} \end{bmatrix}.
\]

Let \( A' = D \oplus L_{r..n,}L_{0..r,}^{-1} C \), we can recursively apply the factorization \( P'A' = L'U' \) and we obtain

\[
PA = \begin{pmatrix} L_{0..r,} & 0 \\ L_{r..n,} & I \end{pmatrix} \begin{bmatrix} L_{0..r,}^{-1} C \\ \begin{bmatrix} D \oplus L_{r..n,}L_{0..r,}^{-1} C \end{bmatrix} \end{bmatrix} = \begin{pmatrix} U & 0 \\ \begin{bmatrix} 0 \\ L_{r..n,}^{-1} \end{bmatrix} \end{pmatrix} \begin{bmatrix} L_{0..r,}^{-1} C \\ \begin{bmatrix} D \oplus L_{r..n,}L_{0..r,}^{-1} C \end{bmatrix} \end{pmatrix}.
\]

and then

\[
\begin{bmatrix} I & 0 \\ 0 & P' \end{bmatrix} PA = \begin{pmatrix} U & 0 \\ \begin{bmatrix} 0 \\ L' \end{bmatrix} \end{pmatrix} \begin{bmatrix} L_{0..r,}^{-1} C \\ \begin{bmatrix} D \oplus L_{r..n,}L_{0..r,}^{-1} C \end{bmatrix} \end{pmatrix}.
\]

Remark 4.1. Notice that here we perform the matrix-matrix multiplications using our implementation of the Strassen’s algorithm.

5. PERFORMANCE TUNING

The algorithm presented in Section 3 completely avoids column permutations and the cost of the decomposition is given by the cost of the matrix-matrix multiplication. The cost to multiply \( A \in \text{Mat}(n, b, \mathbb{F}_2) \) and \( B \in \text{Mat}(b, b, \mathbb{F}_2) \), using the M4RM algorithm, is approximately given by the costs of the Xor operations. In particular, if we neglect the cost of the memory access and other minor costs, we obtain that

\[
\text{cost of M4RM} = T^b_c + nR^b_c, \quad T^b_c = \sum_{j=1}^{K} (2^{\ell_j} - 1)Xor_b, \quad R^b_c = K Xor_b \tag{12}
\]

where

1. \( \ell_j \) are positive integers such that \( \ell_1 + \ell_2 + \cdots + \ell_K = b; \)
2. \( 2^{\ell_j} \) is the size of the \( j^{th} \) of the \( K \) tables used to perform M4RM algorithm;
3. \( T^b_c \) is the cost of the tables construction;
4. \( R^b_c \) is the cost of the rows operations;
5. \( Xor_b \) is the cost of the Xor operation using integer of \( b \) bit size.

When \( K \) divides \( b \), we assume \( \ell_j = c = b/K \) and the expression in (12) becomes

\[
\text{cost of M4RM} = T^b_c + nR^b_c, \quad T^b_c = \left( \frac{b}{c} \right) (2^c - 1)Xor_b, \quad R^b_c = \left( \frac{b}{c} \right) Xor_b \tag{13}
\]
Fast matrix decomposition in $\mathbb{F}_2$

In order to simplify the cost estimation, from now on, we assume that $K$ divides $b$ and we perform the cost analysis starting by (13). In Subsection 5.1 we are going to discuss about the choices of the parameters $b$ and $c$. Moreover, in Subsection 5.2 we will give an idea about the switching point from the M4RM multiplication and the Strassen multiplication. Finally, in Subsection 5.3 a complexity estimation of our algorithm is given.

5.1 Integer word-size and performance of M4RM

Let us consider two matrices $A \in \text{Mat}(n, 2b, \mathbb{F}_2)$ and $B \in \text{Mat}(2b, 2b, \mathbb{F}_2)$. Due to (13) and choosing $2b$ bits as word-size, the cost to perform the M4RM algorithm to multiply $A$ and $B$ is approximately $T_{2b}^{2b} + nR_{2b}^{2b}$. On the other hand, using a word-size of $b$ bits, the cost of M4RM becomes $4T_b^b + n4R_b^b$.

Considering the cost of the rows contribution and the respective ratio, we have that

$$\frac{\text{Cost of } b\text{-bits row}}{\text{Cost of } 2b\text{-bits row}} = \frac{4nR_b^b}{nR_{2b}^{2b}} = 2\left(\frac{\text{Xor}_b}{\text{Xor}_{2b}}\right)$$

and so, in case of $\text{Xor}_{2b} < 2\text{Xor}_b$, it is convenient to use $2b$ bits.

The cost of the tables construction is respectively given by $4T_b^b$ and $T_{2b}^{2b}$ and the corresponding ratio is then

$$\frac{\text{Cost per } b\text{-bits tables}}{\text{Cost per } 2b\text{-bits tables}} = \frac{4T_b^b}{T_{2b}^{2b}} = 2\left(\frac{\text{Xor}_b}{\text{Xor}_{2b}}\right).$$

Since the cost of the $\text{Xor}_{32} \approx \text{Xor}_{64}$ in 64 bit architecture, it is convenient to choose $b = 64$. For hardware that supports SSE 128 bit instructions, the following relation $2\text{Xor}_{64} > \text{Xor}_{128} \geq \text{Xor}_{64}$ holds and thus it is convenient to use $b = 128$.

Once the size $b$ has been chosen, it should be more convenient to choose the size $c$ of the tables used for the M4RM algorithm. Since the cost to perform the product of the two matrices $A \in \text{Mat}(n, b, \mathbb{F}_2)$ and $B \in \text{Mat}(b, b, \mathbb{F}_2)$ is given by (13), the optimal table size $c$ (when $b$ and $n$ are known) can be estimated minimizing (13), which is the minimum of the following function

$$C(b, c, n) = \left(\frac{b}{c}\right)(2^c - 1 + n).$$

**Remark 5.1.** Actually, to determine the minimum of the function $C$ is quite complicated. However, it can be observed that $C$, as a function of $n$, is a straight line with a slope that decreases when $c$ increases. So, when $n$ exceeds the value $C(b, c, n) = C(b, c + 1, n)$, it is convenient to use a table of size $c + 1$ instead of size $c$. In the case of the product of square matrices, the minimum cost is obtained minimizing $C(b, c, b)$ (with respect to $c$) and we obtain the following values:

$$\arg \min_c \{C(32, c, 32)\} \approx 4.08 \quad \arg \min_c \{C(64, c, 64)\} \approx 4.77 \quad \arg \min_c \{C(128, c, 128)\} \approx 5.5.$$
Table I. Cost of tables construction and cost of the rows operation for M4RM.

|       | 32 bits     | 64 bits     | 128 bits    |
|-------|-------------|-------------|-------------|
|       | $T_c$ 1000 $R_c$ | $T_c$ 1000 $R_c$ | $T_c$ 1000 $R_c$ |
| 2     | 0.028  10.22 | 0.048  15.80 | 0.086  49.56 |
| 3     | 0.044  6.25  | 0.066  10.77 | 0.151  34.19 |
| 4     | 0.069  5.18  | 0.099  9.71  | 0.223  21.89 |
| 5     | 0.140  3.78  | 0.151  7.30  | 0.349  19.89 |
| 6     | 0.222  2.91  | 0.254  6.07  | 0.688  17.25 |
| 7     | 0.252  2.82  | 0.465  5.06  | 2.322  15.72 |
| 8     | 0.484  2.44  | 0.721  4.73  | 3.734  14.52 |
| 9     | 0.772  2.40  | 1.709  4.37  | 7.446  16.12 |
| 10    | 4.227  1.83  | 3.465  4.98  | 11.364 16.87 |

Normalized cost

|       | $16 \times T_c$ | $16000 \times R_c$ | $4 \times T_c$ | $4000 \times R_c$ | $T_c$ | $1000 \times R_c$ |
|-------|-----------------|-------------------|---------------|------------------|-------|-----------------|
| 2     | 0.444           | 164.80            | 0.192         | 62.90            | 0.086 | 49.40           |
| 3     | 0.700           | 101.20            | 0.264         | 43.10            | 0.151 | 34.06           |
| 4     | 1.112           | 83.00             | 0.398         | 38.82            | 0.223 | 23.08           |
| 5     | 2.248           | 60.36             | 0.606         | 29.20            | 0.349 | 19.89           |
| 6     | 3.560           | 45.80             | 1.016         | 24.26            | 0.688 | 17.28           |
| 7     | 4.028           | 46.64             | 1.862         | 20.30            | 2.322 | 15.73           |
| 8     | 7.748           | 38.20             | 2.886         | 19.20            | 3.734 | 14.49           |
| 9     | 12.348          | 37.64             | 6.838         | 17.20            | 7.446 | 16.16           |
| 10    | 67.640          | 29.32             | 13.862        | 20.04            | 11.364| 16.83           |

Time measured in microseconds

In Table I the costs to perform rows operations and table constructions are given; in Table II we report the costs to perform product of square matrices of size 32, 64 and 128.

5.2 How to choose switching point for Strassen Matrix-Matrix multiplication

Starting by the formula (13), we can easily obtain the cost to multiply (using the M4RM algorithm) two matrices of size $n$. Let $N = \lceil n/b \rceil$ be the number of blocks in which we divide the matrix. We have to apply $N^2$ times the M4RM algorithm and we obtain the following cost

$$ M(n) = N^2(\tau_c^b + nR_c^b) = \frac{Xor_b}{b_c}(n^2(2c - 1) + n^3). $$

To perform Strassen matrix-matrix multiplication algorithm, our implementation needs essentially 22 additions and 7 multiplications of matrices having size $\frac{N}{2}$. Since the cost to compute one addition is given by $(\frac{nN}{4}) Xor_b$, the whole cost for Strassen’s
Table II. Cost of the product of 2 square matrices having size respectively 32, 64, 128. The normalized costs are also reported.

\[
\begin{array}{cccccc}
\text{bit} & 32 \times 32 & 64 \times 64 & 128 \times 128 \\
2 & 0.362 & 22.88 & 1.060 & 8.48 & 6.43 \\
3 & 0.255 & 16.16 & 0.755 & 6.04 & 4.51 \\
4 & 0.240 & 15.20* & 0.725 & 5.80 & 2.88 \\
5 & 0.262 & 16.64 & 0.625 & 5.00* & 2.90 \\
6 & 0.312 & 20.00 & 0.645 & 5.12 & 2.89* \\
7 & 0.345 & 22.08 & 0.790 & 6.32 & 3.91 \\
8 & 0.570 & 36.64 & 1.025 & 8.16 & 5.52 \\
9 & 0.845 & 54.40 & 1.955 & 15.60 & 9.57 \\
10 & 4.355 & 279.80 & 3.635 & 29.08 & 13.56 \\
\end{array}
\]

Then, it is convenient to use Strassen’s algorithm as long as

\[
S(n) = \frac{11n^2}{2b} \cdot \text{Xor}_b + 7S\left(\frac{n}{2}\right).
\]

In other words, it is convenient to use Strassen’s algorithm when

\[
n \geq 44c + 6(2^c - 1).
\]
Table III. Cost of the BuildZ procedure considering blocks of $b$ bits

| Line | Cost               |
|------|--------------------|
| 2    | $2b$               |
| 4    | $b$                |
| 6    | $b(b-1)$          |
| 9    | $5b + rb\text{ popC}$ |
| 13   | $b$                |
| 15   | $3b(b-1)/2$       |
| 18   | $3b(b-1)/2$       |
| 24   | $4b^2$            |

Using Table III the total cost to build all the matrices $Z$ along the whole decomposition is given by the following formula

$$M(5b + 8b^2) + b\text{ popC}\sum_{k=1}^{M}Nb = m(5 + 8b) + mn\text{ popC} = O(n^2).$$

(14)

As seen in Section 5, the cost to perform the product, using M4RM algorithm, of a matrix in $\text{Mat}(n, b, \mathbb{F}_2)$ by a matrix in $\text{Mat}(b, b, \mathbb{F}_2)$ is given by (13). The cost to perform the M4RM algorithm to a matrix in $\text{Mat}(n, b, \mathbb{F}_2)$ by a matrix in $\text{Mat}(b, m, \mathbb{F}_2)$ becomes

$$\left(\frac{m}{c}\right)(2^c - 1 + n),$$

where the parameter $c$ is the size of the used tables. Noticing that $m = Mb$ and $n = Nb$, we analyze the cost in term of operations on $b$ bits in the following two extreme situations.

a) When $A$ is full rank the cost for the M4RM algorithm applied into the factorization:

$$\text{cost}_{\text{full}} = \sum_{k=1}^{M} \left(\frac{b}{c}\right)(M - k + 1)(2^c - 1 + (N - k)b) = \frac{n^3}{3bc} + O(n^2).$$

b) When $\text{rank}(A) = 1$ the cost is given by

$$\text{cost}_{1} = \sum_{k=1}^{M} \left(\frac{b}{c}\right)(M - k + 1)(2^c - 1 + Nb) = \frac{n^3}{2bc} + O(n^2).$$

This expansion holds when the table size $c$ and $b$ are fixed. In case of $c \equiv c(n)$ or when $b \equiv b(n)$ estimation holds if the growth is asymptotically bounded by $\lim_{n \to \infty} c(n)/\log n < \infty$ and $\lim_{n \to \infty} b(n)/n = 0$. Thus, from (14) the asymptotic cost of the factorization is dominated by the leading term $n^3/(3bc)$ for the full rank case and $n^3/(2bc)$ when $\text{rank}(A) = 1$. This is the cost in term of number of operations on blocks of $b$ bits. Andrén, Hellstrom and Markstrom in [Andrén et al. 2007] analyzed the cost of factorization in terms of number of rows operation.
Thus, in order to compare our cost with the one in this reference, we have to multiply our cost by \( b/n \) obtaining

\[
\text{row-cost}_{\text{full}} = \frac{n^2}{3c} + \mathcal{O}(n), \quad \text{row-cost}_1 = \frac{n^2}{2c} + \mathcal{O}(n).
\]

Assuming as in [Andrén et al. 2007] the table size \( c = \log_2 n \), we obtain

\[
\text{row-cost}_{\text{full}} = \frac{n^2}{3 \log_2 n} + \mathcal{O}(n), \quad \text{row-cost}_1 = \frac{n^2}{2 \log_2 n} + \mathcal{O}(n).
\]

Notice that the theoretical minimum for the complexity obtained in [Andrén et al. 2007] is \( n^2/(2 \log_2 n) + \mathcal{O}(n) \) and it apparently contradicts estimation \( \text{row-cost}_{\text{full}} \).

But this minimum is obtained considering row operations performed on the whole row while we use operations on strings of \( b \)-bits, so that there is no contradiction. Our algorithm in the worst case is at least asymptotically twice faster than the one described in [Andrén et al. 2007], although our algorithm needs to store the whole M4RM tables while Andrén et al. does not need additional memory.

**Remark 5.2.** In practice, the worst case is never reached because we have adopted an implementation strategy which gives us an advantage in the case of matrices having very low rank, and in particular when the rank is equal to one.

6. PERFORMANCE TESTS AND COMPARISON

To evaluate the performance of our algorithm, we compute the decomposition of \( n \times n \) matrices with entries in \( \mathbb{F}_2 \). Our block decomposition works well both for dense matrices and for relatively sparse matrices. Notice that in the former case the obtained matrices have rank most probably equal to \( n - 1 \) (or \( n \)); in the latter case we have low-rank matrices.

First, we have constructed a sample of random dense matrices of size \( n \) (where the size \( n \) ranges from 256 and 65536).

In Table IV we give the minimum value of ten observed running times for computing our block decomposition (recursive and non-recursive), in case \( b = 32, 64, 128 \).

In Table V, the minimum running time of ten trials to obtain a matrix decomposition is given. In particular, we compare the running times to obtain M4RI, PLUQ, PLE decompositions adopted into SAGE [Stein et al. 2012] with the ones to get our 128 block recursive decomposition.

Then, we constructed low-rank matrices as follows. We consider the samples

\[
S_n = \{ S_{n,i} \mid i = 2, \ldots, n \}
\]

of 39 relatively sparse matrices of size \( n \) having respectively \( i \) non-zero elements per rows.

In Figure 2 we have plotted the observed running times (in milliseconds) for M4RI (cross), PLUQ (square), PLE(circles) and our 128 bit block recursive (triangles) decomposition. This is done for every matrix in the sample \( S_n \) The size \( n \) ranges from 1024 to 65536.
Table IV. Compare running times with $b = 32, 64, 128$ recursive (rec) and non recursive versions of our algorithm. Execution obtained using LLVM 3.0 compiler (MAC OSX) on a 3.06GHz Intel(R) Xeon.

| $n$  | 32      | 32 rec  | 64      | 64 rec  | 128     | 128 rec  |
|------|---------|---------|---------|---------|---------|---------|
|      | milli-seconds |        |        |         |         |         |
| 256  | 0.106   | 0.108   | 0.131   | 0.143   | 0.247   | 0.250   |
| 512  | 0.347   | 0.350   | 0.327   | 0.330   | 0.553   | 0.557   |
| 1024 | 1.574   | 1.741   | 1.070   | 1.171   | 1.443   | 1.527   |
| 2048 | 9.260   | 10.862  | 5.076   | 5.927   | 5.172   | 5.795   |
| 4096 | 64.849  | 76.458  | 32.454  | 37.996  | 27.274  | 30.792  |

| $n$  | seconds |         |         |         |         |         |
|------|---------|---------|---------|---------|---------|---------|
| 8192 | 0.492   | 0.549   | 0.248   | 0.267   | 0.203   | 0.206   |
| 16384| 3.922   | 3.981   | 1.939   | 1.981   | 1.646   | 1.523   |
| 32768| 32.011  | 29.791  | 23.859  | 15.440  | 12.848  | 11.498  |
| 65536| 253.169 | 216.323 | 190.065 | 114.794 | 102.071 | 86.156  |

Table V. Compare running times with three algorithms (PLUQ, M4RI, PLE) adopted into SAGE and our recursive algorithm with $b = 128$ using LLVM 3.0 compiler (MAC OSX) on a 3.06GHz Intel(R) Xeon.

| $n$  | M4RI    | PLUQ    | PLE     | our     |
|------|---------|---------|---------|---------|
|      | milli-seconds |      |         |         |
| 256  | 0.248   | 0.239   | 0.239   | 0.252   |
| 512  | 0.847   | 0.836   | 0.821   | 0.562   |
| 1024 | 3.428   | 2.653   | 2.551   | 1.539   |
| 2048 | 12.810  | 10.590  | 10.003  | 5.850   |
| 4096 | 62.231  | 53.686  | 50.400  | 31.406  |

| $n$  | seconds |         |         |         |         |
|------|---------|---------|---------|---------|---------|
| 8192 | 0.349   | 0.340   | 0.334   | 0.207   |
| 16384| 2.462   | 2.425   | 2.392   | 1.532   |
| 32768| 18.674  | 18.558  | 18.341  | 11.529  |
| 65536| 135.906 | 128.567 | 125.783 | 86.538  |

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Fig. 2. Running times for computing a matrix decomposition of $n \times n$ low-rank matrices using LVM 3.0 compiler (MAC OSX) on a 2.53GHz Intel(R) Core 2 Duo.

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