Approximating singularities by a cuspidal-edge on a maxface

PRADIP KUMAR and SAI RASMI RANJAN MOHANTY

Abstract. We give necessary and sufficient conditions on the singular Björling data such that the singular Björling problem’s solution has a prescribed nature of singularity. As an application, for a given maxface with a particular type of singularity, we find a sequence of maxfaces, all having a cuspidal-edge.

Mathematics Subject Classification. 53A35.

Keywords. Maxface singularities, Cuspidal-edge, Björling problem.

1. Introduction. Maximal immersions are zero mean curvature immersions in the Lorentz–Minkowski space $\mathbb{E}_4^3$. These are very similar to the minimal surface in $\mathbb{R}^3$, but if we allow some singularities (where maps are not immersions), the theory of these two differs. Maximal surfaces with singularity are called generalized maximal surfaces. A singularity on the generalized maximal surface is either branched or non-branched. Non-branched singular points are those points where the limiting tangent space does not collapse, and it contains a light-like vector. Various aspects of non-branched singularities have been studied in [3, 4, 6–8, 11, 13], etc.

Umehara and Yamada [13] proved that every non-branched maximal immersion as a map in $\mathbb{R}^3$ turns out to be frontal. They called a non-branched maximal immersion a maxface and discussed when this becomes the front near a singular point. Cuspidal-edge, swallowtails, cuspidal crosscaps, cuspidal butterflies, and cuspidal $S_{1}^-$ are few singularities that appear on a maxface $X$ as front or frontal.

On cuspidal-edges of the front, Saji, Umehara, and Yamada [12] introduced the singular curvature function which is closely related to the behavior of the Gaussian curvature of a surface near cuspidal-edges. Further, Martins and Saji [10] study differential geometric properties of cuspidal-edges with boundary
and give several differential geometric invariants. Toshizumi Fukui [5] also studied the local differential geometry of cuspidal-edges.

Not all maxfaces are front but if we approximate singularities on maxfaces with a cuspidal-edge (that is the first kind of singularities of fronts), then it may help us to understand the existence of some invariants related to other types of singularities as we have for the cuspidal-edge [5,10,12]. This idea motivates us to approximate singularities by a cuspidal-edge.

In this article, in Sect. 4, we construct a sequence (there may be many others) of maxfaces with a cuspidal-edge that “converges” to other singularities like shrinking or folded or as in Table 1. To prove this, we shall give necessary and sufficient conditions on the singular Björling data \( \{ \gamma, L \} \) such that it has a cuspidal-edge at some point.

Along with the cuspidal-edge, in this article, we find necessary and sufficient conditions on the singular Björling data \( \{ \gamma, L \} \) such that its corresponding maxface has swallowtails, cuspidal crosscaps, cuspidal butterflies, and cuspidal \( S^{-1} \). We summarize the conditions (given in the Propositions 3.1, 3.2, and 3.3) here in Table 1.

We believe Table 1 is very useful and apart from finding required convergent sequences, it may be a starting point of studying a suitable interpolation problem (finding a maxface containing two disjoint curves with prescribed nature of singularities along the curve). In [9], López discussed a kind of interpolation problem where he proves the existence of a maximal immersion (not the generalized) spanning two disjoint circular contours. Some discussion about finding maxfaces with two interpolating singular curves can be found in [2]. But we believe a general discussion requires an initial setup like the conditions as in Table 1. In article [1], David Brander has discussed similar conditions for the case of the non-maximal CMC surfaces with the special data.

The discussion of this article is close to [1,7,13].

2. Preliminary. This section reviews the definition of maxface, Weierstrass-Enneper representation, and the singular Björling problem.

The Lorentz–Minkowski space \( \mathbb{E}^3_1 \) is a vector space \( \mathbb{R}^3 \) with metric \( \langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \) defined by \( \langle (a_1, b_1, c_1), (a_2, b_2, c_2) \rangle := a_1a_2 + b_1b_2 - c_1c_2 \) and the generalised maximal immersion is an immersion of a Riemann surface (with boundary) \( M \) to \( \mathbb{E}^3_1 \) such that the pullback metric on \( M \) does not vanish identically and it is positive definite wherever the metric does not vanish. Moreover at non-degenerate points the mean curvature is zero. Maxfaces are those generalized maximal immersions where singularities are only those points of \( M \) where the limiting tangent plane has a light-like vector. We have the following representation of the maxface.

2.1. Weierstrass–Enneper representation [13]. For a maxface \( X : M \to \mathbb{E}^3_1 \), there is a pair \( (g, \omega) \) of a meromorphic function and a holomorphic 1-form on \( M \) such that \( |g| \) is not identically equal to 1 and for \( \Phi := (1 + g^2, i(1 - g^2), -2g) \omega \), the map \( X \) is given by \( X(p) := \text{Re} \int_0^p \Phi \).

For a maxface, with the help of the Weierstrass data \( (g, \omega) \), below we define functions \( \alpha, \beta, \) and \( \eta \) as in [4,11], and [13].
Table 1. Criterion in terms of the Björling data

| Nature of singularities at $p$ | Function’s value at $p$ | $\gamma'$ | $L$ | $\gamma''$ | $\gamma'''$ | $L'$ | $L''$ | $\gamma'_1 \gamma'_2 - \gamma'_1 \gamma'_2$ | $L_1 L_2' - L'_1 L_2$ |
|-------------------------------|-------------------------|-----------|-----|-----------|-----------|-----|------|-------------------------------|-------------------|
| Cuspidal-edge                 | = 0                     | = 0       | = 0 |             | = 0       | = 0 | = 0  | $\neq 0$                      | $\neq 0$          |
| Swallowtails                  | = 0                     | $\neq 0$  | = 0 | $\neq 0$   | = 0       | = 0 | $\neq 0$ | $\neq 0$                      | $\neq 0$          |
| Cuspidal butterflies          | = 0                     | $\neq 0$  | = 0 | $\neq 0$   | = 0       | = 0 | = 0  | $\neq 0$                      | $\neq 0$          |
| Cuspidal $S_1^-$              | $\neq 0$                | = 0       | = 0 |             | = 0       | $\neq 0$ | $\neq 0$ | $\neq 0$                      | $\neq 0$          |
| Cuspidal-Crosscaps            | $\neq 0$                | = 0       | = 0 |             | $\neq 0$  | $\neq 0$ | $\neq 0$ | $\neq 0$                      | $\neq 0$          |
Table 2. Criterion in terms of the Weierstrass data

| Condition                                                                 | p is a singularity |
|---------------------------------------------------------------------------|--------------------|
| \( \text{Re}(\alpha) \neq 0, \text{Im}(\alpha) \neq 0 \)                   | cuspidal-edge     |
| \( \text{Re}(\alpha) \neq 0, \text{Im}(\alpha) = 0, \text{Re}(\beta) \neq 0 \) | swallow-tails      |
| \( \text{Re}(\alpha) \neq 0, \text{Im}(\alpha) = 0, \text{Re}(\beta) = 0, \text{Im}(\eta) \neq 0 \) | cuspidal butterflies |
| \( \text{Re}(\alpha) = 0, \text{Im}(\alpha) \neq 0, \text{Re}(\beta) = 0, \text{Re}(\eta) \neq 0 \) | cuspidal butterflies |
| \( \text{Re}(\alpha) = 0, \text{Im}(\alpha) \neq 0, \text{Re}(\beta) \neq 0 \) | cuspidal butterflies |

**Definition 2.1.** At \( p \in M \), let \((U, z)\) be a coordinate chart and \( \omega = f dz \), we define

\[
\alpha(z) = \frac{g'(z)}{g^2(z)f(z)}, \quad \beta(z) = \frac{g(z)}{g'(z)} \alpha'(z), \quad \text{and} \quad \eta(z) = \frac{g(z)}{g'(z)} \beta'(z).
\]

These functions help us to check the nature of a singularity.

**2.2. Singular Björling problem [7].** We explain the singular Björling problem in the following.

**Definition 2.2** (Singular Björling data [7]). Let \( \gamma : I \to \mathbb{E}^3 \) be a real analytic null curve and \( L : I \to \mathbb{E}^3 \) be a real analytic null vector field such that for all \( u \in I \), \( \gamma'(u) \) and \( L(u) \) are proportional, and \( \gamma'(u) \) and \( L(u) \) do not vanish simultaneously. Such \( \{\gamma, L\} \) is said to be a singular Björling data.

If the analytic extension of the function \( g : I \to \mathbb{C} \),

\[
g(u) := \begin{cases} 
L_1 + iL_2 & \text{if } \gamma' \text{ vanishes identically}, \\
L_3 & \text{if } L \text{ vanishes identically}
\end{cases}
\]

satisfies \(|g(z)| \neq 1\) on some simply connected domain \( \Omega \subset \mathbb{C} \), where \( z = u + iv \in \Omega \) and \( I \subset \Omega \). Then there is a unique generalized maximal immersion \( X : \Omega \to \mathbb{E}^3 \) which is given by (for \( u_0 \in I \) fixed) \( X(z) = \gamma(u_0) + \text{Re} \left( \int_{u_0}^z (\gamma'(w) - iL(w))dw \right) \) such that \( X(u, 0) = \gamma(u) \) and \( X_v(u, 0) = L(u) \). Moreover it has singularity set at least \( I \). After a translation, we can assume that \( \gamma(u_0) = 0 \), so we consider the solution as

\[
X_{\gamma, L}(z) = \text{Re} \left( \int_{u_0}^z (\gamma'(w) - iL(w))dw \right).
\]

The way singular Björling data is taken, the singularity set contains an interval \( I \) and for all \( u \), \( \gamma'(u) \) and \( L(u) \) do not vanish simultaneously. Therefore
$X_{\gamma,L}$ turns out to be a maxface in a neighborhood of singular points. In fact, the Weierstrass data for the maxface as in equation 2.2 is given by the analytic extension of $f(u) = (\gamma_1 - iL_1) - i(\gamma_2 - iL_2)$ and $g$ as in equation 2.1.

3. Necessary and sufficient conditions on the singular Björling data for prescribed type of singularity. In this section, we will calculate $\alpha, \beta,$ and $\eta$ as in Definition 2.1 for the maxface $X_{\gamma,L}$ at some singularity $t_0 \in I$. We get necessary and sufficient conditions on $\{\gamma, L\}$ so that $X_{\gamma,L}$ is a cuspidal-edge, swallowtails, cuspidal cross caps, cuspidal butterflies, and cuspidal $S_1^-$ singularities.

3.1. For cuspidal-edge at $u \in I$. Let $\{\gamma, L\}$ be the singular Björling data as in Definition 2.2. With the Gauss map as in equation 2.1, we calculate $\alpha$ as in Definition 2.1.

At $u \in I$, if $\gamma'(u) \neq 0$, then there exist a real number $c$ such that $L(u) = c \gamma'(u)$. So that $f(u) = (1 - ic)(\gamma_1(u) - i\gamma_2(u))$ and $g(u) = \frac{\gamma_1'(u) + i\gamma_2'(u)}{\gamma_3'(u)}$. In this case, we have $\alpha(u) = \frac{g'}{g^2f}(u) = \frac{g'}{g(1 - ic)\gamma_3'(u)}$. Here replacing the value of $f$ and $g$, we get

$$\alpha(u) = \frac{\gamma_3'(\gamma_1'' + i\gamma_2'') - (\gamma_1' + i\gamma_2')\gamma_3'' \gamma_3' \gamma_1' + i\gamma_2'}{\gamma_3'^2(1 - ic)\gamma_3'}\frac{\gamma_3'}{\gamma_3^2(1 - ic)} + \frac{(\gamma_1'' + i\gamma_2')(\gamma_1' - i\gamma_2')}{\gamma_3(1 - ic)(\gamma_1'' + \gamma_2''}(u).$$

That is, we have

$$\alpha(u) = \frac{1}{\gamma_3^2(1 - ic)}[-\gamma_3'' \gamma_1' + \gamma_1'' \gamma_2' + i(\gamma_1' \gamma_2' - \gamma_1'' \gamma_2')](u).$$

We denote:

$$D(\gamma_{12}, \gamma_{12}'') := \gamma_1' \gamma_2' - \gamma_1'' \gamma_2; \quad D(L_{12}, L_{12}') := L_1L_2 - L_2L_1.'$$

So that $\alpha(u) = \frac{D(\gamma_{12}, \gamma_{12}'')}{\gamma_3^2(1 - ic)}$. Moreover $\gamma$ is a null curve and $c = \frac{L_3}{\gamma_3}$, therefore at $u$, when $\gamma'(u) \neq 0$, we get

$$\alpha(u) = -\frac{L_3D(\gamma_{12}, \gamma_{12}''')}{(\gamma_3'' + \gamma_3')\gamma_3^2} + i\frac{D(\gamma_{12}', \gamma_{12}'')}{(\gamma_3'' + \gamma_3')\gamma_3'}.$$  

Similarly, for the case when $L(u) \neq 0$, we get the following:

$$\alpha(u) = -\frac{D(L_{12}, L_{12}')}{(\gamma_3'' + \gamma_3')L_3} + i\frac{\gamma_3D(L_{12}, L_{12}')}{(\gamma_3'' + \gamma_3')L_3}.$$  

We know $u$ is a cuspidal-edge for the maxface if and only if Re($\alpha$) and Im($\alpha$) at $u$ are non-zero.

Therefore at those points $u \in I$, where $\gamma' \neq 0$ and $u$ a is cuspidal-edge for the maxface (as in equation 2.2), we must have $\gamma' \neq 0, L \neq 0,$ and $D(\gamma_{12}, \gamma_{12}'') \neq 0$ at $u$ and this implies $D(L_{12}, L_{12}') \neq 0$ at $u$. 

On the other hand, at those points $u \in I$, where $L \neq 0$ and $u$ is a cuspidal-edge, we must have $\gamma' \neq 0, L \neq 0,$ and $D(L_{12}, L'_{12}) \neq 0$ at $u,$ and this implies $D(\gamma'_{12}, \gamma''_{12}) \neq 0$ at $u.$

This proves the following:

**Proposition 3.1.** Let $\{\gamma, L\}$ be the singular Björling data. Then the maxface $X_{\gamma, L}$ as in equation 2.2 has a cuspidal-edge at $u \in I$ if and only if at $u,$ $\gamma' \neq 0, L \neq 0,$ and $D(\gamma'_{12}, \gamma''_{12}) \neq 0$ or $D(L_{12}, L'_{12}) \neq 0.$

This proposition has many applications, in particular, we will use it to prove Theorem 4.1. Moreover, constructing examples having cuspidal-edge singularity turns out be handy.

**Example 3.1.** Let $\gamma(u) = (\sin u, -\cos u, u)$ and $L(u) = u(\cos u, \sin u, 1)$ on $I = (0, 1).$ Then we have $L(u) = u\gamma'(u)$ and $D(\gamma'_{12}, \gamma''_{12}) = 1.$

It is clear that $\gamma' \neq 0, L \neq 0,$ and $D(\gamma'_{12}, \gamma''_{12}) \neq 0$ on $(0, 1).$ Hence all points are cuspidal-edges on $(0, 1).$

**Example 3.2.** Let $\gamma(u) = (u - u^3, u^2, u + u^3)$ and $L(u) = u^2(1 - u^2, 2u, 1 + u^2)$ on $I = (0, 1).$ Then $L(u) = u^2\gamma'(u)$ and $D(\gamma'_{12}, \gamma''_{12}) = 2(u^2 + 1).$ It is clear that $\gamma' \neq 0, L \neq 0,$ and $D(\gamma'_{12}, \gamma''_{12}) \neq 0$ on $(0, 1).$ Hence all points are cuspidal-edges on $(0, 1).$

In the following, we will find necessary and sufficient conditions on the singular Björling data such that the maxface (as in equation 2.2) has swallowtails, cuspidal cross-caps, etc., at $u \in I.$

### 3.2. For swallowtails and cuspidal butterflies at $u.$ If $L \neq 0$ at $u,$ then $\gamma' = dL,$ where $d$ is a function in a neighborhood of $u.$ In this case, from equation 3.3, we have $\alpha = iD(L_{12}, L'_{12})/(d - i)L_3^3$ therefore $\alpha' = i(D(L_{12}, L'_{12})(d - i)L_3^3 - D(L_{12}, L'_{12})(d'L_3^3 + 3(d - i)L_3L_4^3))/(d - i)^2L_3^3.$

This gives

$$\beta = \frac{\alpha'}{(d - i)L_3^3\alpha},$$

$$\beta' = \frac{D(L_{12}, L'_{12})(d - i)L_3^3 - D(L_{12}, L'_{12})(d'L_3^3 + 3(d - i)L_3L_4^3)}{(d - i)^2L_3^3D(L_{12}, L'_{12})},$$

$$\eta = \frac{g'}{g'} = \frac{\beta'}{(d - i)L_3^3\alpha} = -i \frac{\beta'L_3^2}{D(L_{12}, L'_{12})}. \tag{3.6}$$
We know $X_{\gamma,L}$ has a swallowtail at $u \in I$ if and only if at $u$, Re $\alpha \neq 0$, Im $\alpha = 0$, and Re $\beta \neq 0$.

The first two conditions Re $\alpha \neq 0$ and Im $\alpha = 0$ are satisfied at $u$ if and only if at $u$, $D(L_{12}, L'_{12}) \neq 0$, $L \neq 0$, and $\gamma' = 0$. Since at $u$, $d = \frac{\gamma'}{L} = 0$, $\beta = \frac{D(L_{12}, L'_{12})d''L_3^2 + i(D(L_{12}, L''_{12})L_3^3 - 3D(L_{12}, L'_{12})L_3^2L_3')}{L_3^2D(L_{12}, L'_{12})}$.

Therefore at $u$, Re $\beta \neq 0$ if and only if at $u$, $D(L_{12}, L'_{12}) \neq 0$, $L \neq 0$, $\gamma' = 0$, and $d' \neq 0$. Since at $u$, $d = 0$ and Re $\beta = 0$ if and only if $d'' = 0$, we get from equations 3.5 and 3.6

$$\eta = \frac{D(L_{12}, L'_{12})(D(L'_{12}, L''_{12}) + D(L_{12}, L''_{12}))}{D^2(L_{12}, L'_{12})} + \frac{D(L_{12}, L''_{12})(L_3D(L'_{12}, L'_{12}) + L_3D(L_{12}, L''_{12}))}{D^2(L'_{12}, L'_{12})} - \frac{i}{L_3}(\delta''L_3^2 - 3iL_3') + 6iL_3^2 \frac{L_3D(L'_{12}, L'_{12})}{L_3D(L_{12}, L'_{12})}.$$  

Therefore at $u$, Im $\eta \neq 0$ if and only if at $u$, $D(L_{12}, L'_{12}) \neq 0$, $L \neq 0$, $\gamma' = 0$, $\eta'' = 0$, $\gamma''' \neq 0$, and $L \neq 0$. So we have the following:

**Proposition 3.2.** $X_{\gamma,L}$ has a swallowtail at $u$ if and only if at $u$, $\gamma' = 0$, $\gamma'' \neq 0$, $L \neq 0$, and $D(L_{12}, L'_{12}) \neq 0$. On the other hand, $X_{\gamma,L}$ has a cuspidal butterfly at $u$ if and only if at $u$, $\gamma' = 0$, $\gamma'' = 0$, $\gamma''' \neq 0$, $L \neq 0$, and $D(L_{12}, L'_{12}) \neq 0$.

Similar calculation gives the following.

**Proposition 3.3.** $X_{\gamma,L}$ has a cuspidal cross cap at $u$ if and only if at $u$, $\gamma' \neq 0$, $L = 0$, $L' \neq 0$, and $D(\gamma_{12}, \gamma'_{12}) \neq 0$. And $X_{\gamma,L}$ has a cuspidal $S^-_{11}$ at $u$ if and only if at $u$, $\gamma' \neq 0$, $L = 0$, $L' = 0$, $L'' \neq 0$, and $D(\gamma_{12}, \gamma''_{12}) \neq 0$.

In Table 1, we summarized all conditions of Propositions 3.1, 3.2, and 3.3. Using the conditions, it is easy to find a maxface with the the followly singularities: cuspidal-edges, cuspidal cross caps, and cuspidal $S^-_{11}$ as in the following:

**Example 3.3.** Let $\delta$ be a null real analytic curve and $\mu$ be a null vector field defined on the interval $I$ such that $\mu = \delta'$, $D(\delta'_{12}, \delta''_{12}) \neq 0$, and both $\delta$ and $\mu$ are never zero on $I$. Let $a, b,$ and $c$ be three different real numbers on $I$. Now we construct $\gamma(u) = \delta(u)$ and $L(u) = (u - b)(u - c)^2 \mu(u)$. Then $\gamma$ and $L$ are Björing data for the maxface $X_{\gamma,L}$ such that $L(u) = (u - b)(u - c)^2 \gamma'(u)$. We see that

- at $a$, $L \neq 0$, $\gamma' \neq 0$, and $D(\gamma'_{12}, \gamma''_{12}) \neq 0$,
- at $b$, $L = 0$, $L' \neq 0$, $\gamma' \neq 0$, and $D(\gamma_{12}, \gamma''_{12}) \neq 0$, and
• at \( c, L = 0, L' = 0, L'' \neq 0, \gamma' \neq 0, \) and \( D(\gamma''_1, \gamma''_2) \neq 0. \)

Therefore \( a, b, \) and \( c \) are a cuspidal-edge, a cuspidal crosscap and a cuspidal \( S_1^- \) resp. for the maxface \( X_{\gamma,L}. \)

Similarly, for three different real numbers \( m, n, \) and \( p, \) if we take \( \gamma'(u) = (u - n)(u - p)^2\delta'(u) \) and \( L(u) = \mu(u), \) where \( \delta \) and \( \mu \) the are same as above, then \( m, n, \) and \( p \) are a cuspidal-edge, a swallowtail, and a cuspidal butterfly resp.

**Example 3.4.** Let \( \gamma(u) = (\sin u, -\cos u, u) \) and \( L(u) = u(u - 1)^2(\cos u, \sin u, 1) \) be the Björling data, then \(-1, 0\) and \( 1 \) are a cuspidal-edge, a cuspidal cross cap, and a cuspidal \( S_1^- \) resp.

We can construct many such examples. Moreover as we mentioned in the introduction, another direct application of Table 1 is to find a sequence of maxfaces converging to other types of singularities. We discuss this in the next section.

**4. Approximating various singularities by a cuspidal-edge.** We start with an example that will explain the essence of the main Theorem 4.1 of this section.

Let \( X_{\gamma,L} \) be the maxface with the singular Björling data given by

\[
\gamma(t) = (0, 0, 0), \quad L(t) = (1 - t^2, 2t, 1 + t^2).
\]

The maxface \( X_{\gamma,L} \) has a shrinking singularity on \( I = (-1, 1) \) (in fact on \( \mathbb{R} \)). Moreover this is not a front, but below we will give a sequence of maxfaces (front) converging to \( X_{\gamma,L} \) and having a cuspidal-edge.

For \( n > 1, \) we define

\[
\gamma_n'(t) = \frac{1}{n}L_n(t),
\]

\[
L_n(t) = \left(1 - \frac{1}{n}\right)(1 - t^2, 2t, 1 + t^2).
\]

Then for each \( n, \) the data \( \{\gamma_n, L_n\} \) turns out to be a singular Björling data. Let \( X_{\gamma_n,L_n} \) be the corresponding maxface. Moreover, for each \( t \in \mathbb{R}, \) we have \( \gamma_n'(t) \neq 0, \) \( L_n(t) \neq 0, \) and \( D(\gamma_{12}''', \gamma_{12}''') \neq 0. \) Therefore every point on \( \mathbb{R} \) is a cuspidal-edge. In Fig. 1, we have shown the maxfaces \( X_{\gamma_n,L_n} \) for \( n = 3, 5, 15, \) and \( 50. \)

In this example, we start with a maxface having a shrinking singularity and we give a sequence of maxfaces having a cuspidal-edge “converging” to the shrinking singularity. Below we will give a general discussion towards this. First we will define the norm in which we talk about the convergence.

Let \( \Omega \subset \mathbb{C} \) be a bounded simply connected domain, \( \overline{\Omega} \) be its closure. Let \( X \in C(\overline{\Omega}, \mathbb{R}^3), \) the space of continuous maps. For each \( z \in \overline{\Omega}, \) we denote

\[
\|X(z)\| := \max\{X_1(z), X_2(z), X_3(z)\} \quad \text{and} \quad \|X\|_{\Omega} := \sup_{z \in \overline{\Omega}} \|X(z)\|.
\]

Here \( C(\overline{\Omega}, \mathbb{R}^3) \) becomes a Banach space under the norm \( \|\cdot\|_{\Omega}. \)

In the proposition below, we will give a sequence of maxfaces for general \( \{\gamma, L\}. \)
Figure 1. Sequence of maxfaces having cuspidal-edges that bend to a shrinking singularity

Proposition 4.1. Let $X_{\gamma,L}$ be a maxface and for $t_0 \in I, \gamma' \neq 0, D(\gamma'_1, \gamma'_2) \neq 0$. Then there is a sequence of maxfaces $X_n$ defined in a neighborhood $\Omega$ of $t_0$ such that each maxface $X_n$ has a cuspidal-edge at $t_0$ and $X_n \to X_{\gamma,L}$ in the norm $\|\cdot\|_{\Omega}$.

Proof. As $\gamma'(t_0) \neq 0$, there is an interval $I_1$ containing $t_0$ such that for all $t \in I_1, \gamma'_2(t) \neq 0$. Without loss of generality, we can assume for all $t \in I_1, \gamma'_2(t) > 0$. On $I_1$, we define $c(t) = \frac{L_3(t)}{\gamma'_3(t)}$.

For each $n$, we define $\delta'_n$ and $\mu_n$ such that

$$\delta'_n = \gamma' + \left(\frac{1}{n}, \frac{1}{n}, h_n\right),$$

$$\mu_n = \left(c(t) + \frac{1}{n}\right) \delta'_n.$$ 

Here $h_n = -\gamma'_3 + \sqrt{\gamma'_3^2 + 2 \left(\frac{1}{n^2} + \frac{\gamma'_3 + \gamma'_2}{n}\right)}$.

There are $N$ and $I_2 \subset I_1$ containing $t_0$ such that for all $t \in I_2$ and $n > N$, $\gamma'_3^2 + 2 \left(\frac{1}{n^2} + \frac{\gamma'_3 + \gamma'_2}{n}\right) \neq 0$, and $\frac{1}{n} \neq -c(t_0)$.

For $n > N$, $\{\delta_n, \mu_n\}$ turns out to be a singular Björling data on $I_2$. These can be extended analytically on some domain $\mathcal{U}$ that contains $I_2$. We take a bounded simply connected domain $\Omega$ containing $t_0$ such that $\overline{\Omega} \subset \mathcal{U}$.

Moreover we see that at $t_0$ and $n > N_1 > N, \delta'_n \neq 0, \mu_n \neq 0$, and $D(\delta'_n, \delta''_n) \neq 0$.

Therefore for $n > N_1$, the maxfaces $X_{\delta_n, \mu_n}$ for the singular Björling data $\{\delta_n, \mu_n\}$ have a cuspidal-edge at $t_0$ and hence cuspidal-edge in an interval containing $t_0$.

Let $z \in \Omega$, we have $X_{\delta_n, \mu_n}(z) - X_{\gamma,L}(z) = \Re \int_{w_0}^{z} (\delta'_n(w) - \gamma'(w))(1 - i(c(w) + \frac{1}{n})) dw$. It is direct to see that $\|X_{\delta_n, \mu_n} - X_{\gamma,L}\|_{\Omega} \to 0$. □

Remark 4.1. If we have a maxface $X_{\gamma,L}$ with $L \neq 0$ and $D(L_{12}, L'_{12}) \neq 0$ at 0, then with a little change, we have a sequence of functions $g_n$ (similar to $h_n$ in the above proposition) and we can take
\[ \mu_n = L + \left( \frac{1}{n}, \frac{1}{n}, g_n \right) \text{ and } \delta_n' = \left( d(t) + \frac{1}{n} \right) \mu_n. \]

With a similar argument as above, we find a sequence of maxfaces \( X_{\delta_n, \mu_n} \) with singular Björling data \( \{ \delta_n, \mu_n \} \) having a cuspidal-edge at 0 and \( X_{\delta_n, \mu_n} \to X_{\gamma, L} \) in the norm \( \| . \|_\Omega \).

**Remark 4.2.** For a constant null curve \( \gamma \) and a null vector field \( L \) such that for all \( t, \) \( D(L_{12}, L'_{12}) \neq 0, \) the maxface \( X_{\gamma, L} \) has a shrinking singularity. For this case, we can choose \( \gamma_n \) and \( L_n \) similar to the example at the beginning of the section. Moreover little variation will hold for the folded singularity as well.

We conclude the article with the following theorem which is a direct consequence of Proposition 4.1 and Remarks 4.1, 4.2.

**Theorem 4.1.** Let \( X_{\gamma, L} \) be the maxface with singular Björling data \( \{ \gamma, L \} \) such that at \( t_0 \in I, \) it has a shrinking or a folded singularity or any of the singular point as in Table 1. Then there is a sequence of maxfaces \( X_n \) defined on a domain \( \Omega \) containing \( t_0 \) such that each \( X_n \) has cuspidal-edge at \( t_0. \) Moreover \( X_n \to X_{\gamma, L} \) in the norm \( \| . \|_\Omega. \)

**Acknowledgements.** The authors are very thankful to the anonymous referees for their valuable comments which helped a lot to improve the article.

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Pradip Kumar
Department of Mathematics
Shiv Nadar University
Dadri
Uttar Pradesh 201314
India
e-mail: pradip.kumar@snu.edu.in

Sai Rasmi Ranjan Mohanty
Department of Mathematics
Shiv Nadar University
Dadri
Uttar Pradesh 201314
India
e-mail: sm743@snu.edu.in

Received: 12 February 2021
Revised: 4 April 2022
Accepted: 13 April 2022.