Research paper

Free surface intake vortices: scale effects due to surface tension and viscosity

FRANK SUERICH-GULICK, PhD Graduate, Department of Civil Engineering and Applied Mechanics, McGill University, Montréal, Canada
Email: frank.suerichgulick@mail.mcgill.ca

SUSAN J. GASKIN (IAHR Member), Associate Professor, Department of Civil Engineering and Applied Mechanics, McGill University, MacDonald Engineering Bldg, Room 492, Montréal, QC, Canada H3A 2K6
Email: susan.gaskin@mcgill.ca (author for correspondence)

MARC VILLENEUVE (IAHR Member), President, Lasalle Consulting Group, Lasalle, Canada
Email: mvilleneuve@gcl.qc.ca

ÉTIENNE PARKINSON, Head of Research & Development Vevey, Andritz Hydro, Vevey, Switzerland
Email: Etienne.Parkinson@andritz.com

ABSTRACT

Engineers frequently use physical scale models of hydropower intakes to assess and minimize the occurrence of harmful free surface vortices. The impact of surface tension, viscosity and turbulence on the scaling behaviour of the vortices is examined here using an analytical free surface vortex model developed from measurements in a laboratory-scale hydropower intake. First, the effect of surface tension on the free surface depression is computed using a finite-difference model over a wide range of depression scales and shapes. The impact and scaling behaviour of surface tension are found to be qualitatively different depending on whether the depression is dimple- or funnel-shaped. The influence of viscosity on scaling predicted by the analytical vortex model contradicts trends recorded by previous authors, which suggests that additional processes such as turbulent diffusion may play a significant role at larger scales. Scale effects due to the interplay of viscosity and turbulence require further investigation, whereas those due to surface tension are fairly easily quantified and predicted.

Keywords: Hydraulic models; hydropower intake; similarity (scaling) theory; surface tension; turbulence in rotating flow; turbulent diffusion; vortex dynamics

1 Introduction

Free surface vortices upstream from hydropower intakes occasionally cause serious problems in plant operation. It is common practice for engineers to use laboratory-scale physical models to assess and optimize proposed intake designs to maximize flow uniformity and steadiness and minimize the occurrence and intensity of vortices. Detailed measurements in a physical model of a simplified intake (see Fig. 1) were used to develop an analytical vortex model based on Burgers’s (1948) vortex model that predicts the vortex’s characteristic radius $r_0$, bulk circulation $\Gamma_\infty$ and tip depth $h_0$ (maximum depth of the free surface depression produced by the vortex) in terms of the intake velocity $U_i$ and relative intake submergence $s/d$ (Suerich-Gulick et al. 2014b). $s$ is the submergence of the intake pipe and $d$ is its inner diameter. This analytical model is used here to examine and try to quantify how scale effects due to surface tension, viscosity and turbulence affect the translation of vortex characteristics from a laboratory-scale model to the full-scale prototype.

It is widely recognized that the dominant parameter influencing vortex intensity is the Froude number (Quick 1962, Jain et al. 1978, Anwar 1983, Chang and Prosser 1987), defined as $F_d = U_i/(gd)^{1/2}$ or $F_s = U_i/(gs)^{1/2}$, where $g$ is the gravitational acceleration. Since the exact dependence on Froude number varies with the intake configuration and geometry (Knauss 1987), it is common practice to assess and optimize proposed intakes using laboratory-scale models operated at Froude similitude such that $F_M = F_P$, where $F_M$ and $F_P$ are the laboratory model and prototype values, respectively (Quick 1962, Chang and Prosser 1987). Since water is used in the laboratory model, it is
impossible to match the Weber (W), Reynolds (R) and Froude numbers simultaneously, leading to uncertainty about scale effects that has yet to be fully resolved (Tastan and Yıldırım 2010). The intake definitions for W and R are used here, using the submergence s as the characteristic length: \( W_d = \rho U_d^2 s / \sigma \) and \( R_s = U_s s / \nu \), where \( \rho, \nu \) and \( \sigma \) are the water density, kinematic viscosity and the air–water surface tension coefficients, respectively. Some authors define \( W_d \) and \( R_s \) using the intake diameter \( d \) instead of the intake submergence \( s \), indicated by the subscript.

Scale effects due to viscosity and surface tension have often been studied by documenting how the critical condition in physical models varies with increasing Reynolds and Weber numbers. Most authors define the critical condition for air entrainment as the operating condition at which the tip of the free surface depression just reaches the intake pipe (Daggett and Keulegan 1974, Jain et al. 1978, Anwar 1983, Odgaard 1986, Gulliver 1988, Hite and Mih 1994, Möller et al. 2012). A common approach is to try to identify a minimum \( W \) above which surface tension effects can be neglected (Daggett and Keulegan 1974, Jain et al. 1978, Anwar and Amphlett 1980, Anwar 1983), or a minimum \( R \) above which viscous effects can be neglected (Daggett and Keulegan 1974, Jain et al. 1978, Anwar 1983, Padmanabhan and Hecker 1984, Chang and Prosser 1987, Tastan and Yıldırım 2010).

Independence from surface tension effects in experiments has been reported at \( W_d \) values above 120 (Jain et al. 1978), 600 (Padmanabhan and Hecker 1984) and 748 (Möller et al. 2012), while Anwar (1983) reported persisting surface tension effects for \( W_d \) values up to \( 1.5 \times 10^4 \) for dimple depressions and up to \( 4 \times 10^4 \) for air core vortices, which have deep narrow funnels that reach far below the free surface. Tastan and Yıldırım (2010) observed that limiting values for \( R \) and \( W \) depend on both flow and geometrical conditions in experiments. Using theoretical analysis, Odgaard (1986) concluded that surface tension effects should be negligible for \( W_d > 720 \). Other authors have examined surface tension effects by numerically computing the profile of the free surface depression by the finite-difference method or by analytical or series approximations (Yıldırım and Jain 1981, Andersen et al. 2006, Stepanyants and Yeoh 2008a, Ito et al. 2010).

Similarly, Reynolds number dependence has been observed to decrease asymptotically with increasing \( R \) (Daggett and Keulegan 1974, Jain et al. 1978, Anwar 1983, Chang and Prosser 1987), suggesting viscous effects may be negligible beyond a threshold \( R \) value. Suggested minimum values for \( R \) range from \( 4 \times 10^4 \) to \( 1.4 \times 10^5 \), depending on the geometry and the Froude number, using various definitions of \( R \) (Daggett and Keulegan 1974, Jain et al. 1978, Anwar 1983, Padmanabhan and Hecker 1984, Chang and Prosser 1987, Tastan and Yıldırım 2010). Detailed data that might help explain this trend are lacking.

Greater understanding of the processes driving scale effects would help engineers to interpret vortex observations more effectively or to estimate correction factors when the laboratory-scale model does not meet recommended \( W \) and \( R \) values. This latter scenario is more common for very large prototype intakes and/or when the model must include a significant stretch of the upstream river reach to capture approach flow conditions.

Odgaard (1986) proposes that the asymptotic \( R \) trend is due to turbulent mixing that enhances effective diffusivity in the vortex core. He models the intake vortex using Burgers’ (1948) vortex model in combination with a simple eddy diffusivity model. Burgers’ model assumes that the radial profiles of the azimuthal \( V_\theta(r) \) and radial \( V_r(r) \) velocities in the vortex are constant along the vortex axis \( z \) and that the axial velocity \( V_z(z) \) is independent of \( r \) and varies linearly with \( z \)

\[
V_\theta(r) = \frac{\Gamma_\infty}{2\pi r} \left[ 1 - \exp \left\{ - \left( \frac{r}{r_o} \right)^2 \right\} \right]
\]

\[
V_z(z) = \alpha z, \quad V_r(r) = -\frac{ar}{2}
\]

\[
r_o = 2 \left( \frac{V_\theta}{\mu} \right)^{1/2}, \quad a = \frac{\partial V_z}{\partial z}
\]

where \( r, \theta \) and \( z \) are the radial, azimuthal and axial cylindrical coordinates, with \( z \) aligned with the vortex axis pointing down from the free surface; \( V_r, V_\theta \) and \( V_z \) are the corresponding velocities. The gradient \( a \) is a constant with units \( s^{-1} \), \( \Gamma_\infty \) (units \( m^2 s^{-1} \)) is the bulk circulation of the vortex and \( v \) (units \( m^3 s^{-1} \)) is the kinematic viscosity of the fluid. Burgers’ model is based on the hypothesis that a stable vortex with a constant vorticity and \( V_\theta(r) \) profile along the vortex axis is produced by an equilibrium of axial stretching \( \partial V_z/\partial z \) and radial viscous diffusion. Detailed measurements of the velocity field of a free surface intake vortex suggest that the model captures the basic vortex structure quite well even if some subtle discrepancies exist (Suerich-Gulick et al. 2014b).

Odgaard (1986) suggests that momentum mixing in the vortex caused by turbulence increases the effective viscosity \( \nu_{\text{eff}} \) at larger scales. He replaces \( v \) in Burgers’ expression for \( r_o \) (Eq. 3) by \( \nu_{\text{eff}} = v + \nu_T \), and assumes that the eddy diffusivity \( \nu_T \) scales as \( \nu_T = \chi \Gamma_\infty \), following Squire (1965), with the non-dimensional constant \( \chi = 6 \times 10^{-5} \). The resulting model predicts Jain et al.’s (1978) critical submergence measurements fairly well (Gulliver 1988). Hite and Mih (1994) follow the same approach.

The proposal that \( \nu_{\text{eff}} \) increases with \( \Gamma_\infty \) would appear to contradict past results indicating that radial turbulent fluctuations are suppressed by flow rotation (Bradshaw 1973, Spalart 1998, Jacquin and Pantano 2002, Suerich-Gulick et al. 2014b) and that the spreading rate in the case of wing tip vortices is governed by viscous diffusion rather than by turbulent mixing (Zeman 1995). However, it is possible that radial turbulent diffusion is suppressed to a lesser degree at larger scales, as both \( \Gamma_\infty/v \) and \( R \) increase. Increasing eddy diffusivity would produce a gradual decrease in the relative contribution of molecular viscosity \( v \) to the effective diffusivity, until “viscous effects” become negligible at \( R_d \) values on the order of \( 10^5 \) (Odgaard’s 1986).
Although our measurements indicate that the contribution of turbulent mixing to radial diffusion of the vortices is negligible under the operating conditions examined (Suerich-Gulick et al. 2014b), in this case the maximum $v_\theta$ values from Odgaard’s (1986) model would only reach half the molecular viscosity, producing a 22% increase in the core radius $r_o$ (Eq. 3). This possible increase is comparable to the variation in $r_o$ observed experimentally in Suerich-Gulick et al. (2014b).

2 Method

2.1 Experiment

A laboratory-scale model of a simplified low-head hydropower intake is constructed with two tall pier-like plates mounted perpendicular to the downstream wall of the channel, one on each side of the intake opening, as shown in Fig. 1. Each pier produces two vortices in its wake: a submerged vortex with one end connected to the channel bed, and a free surface vortex with one end connected to the free surface. The other end of each vortex is drawn into the intake pipe. A range of vortex intensities is produced by different combinations of $\Gamma_\infty$ and $r_o$. The evolution of the relative difference $\delta=\Delta h/h_{\theta,0}$ between the profile tip depths is examined, where $\Delta h=h_{\theta,0} - h_{\theta,0}$ (Yıldırım and Jain 1981).

The free surface profile is controlled by the equilibrium of the forces exerted by gravity, centripetal acceleration and surface tension. Following Andersen et al. (2006), Stepanyants and Yeoh (2008a) and Ito et al. (2010), we use Laplace’s (1880) model stating that surface tension reduces the pressure across the air–water interface by $\rho_\theta \kappa(r)$, where $\kappa(r)$ is the local mean curvature of the air–water interface and $\rho_\theta = \sigma/\rho g$ is the squared characteristic length of the air–water interface. The resulting radial profile of the depression $h_{\theta}(r)$ is given by

$$h_{\theta}(r) = \int_\infty^{\hat{r}} \left( \frac{V_\theta(\hat{r})^2}{g\hat{r}} - \rho_\theta \kappa(\hat{r}) \right) d\hat{r}$$

(Andersen et al. 2006). A constant value for $l_o$ of 2.73 mm is used here, which corresponds to a clean air–water interface at 15°C. The variations in $l_o$ associated with the range of experimental temperatures (13–15°C) are negligible compared to those that might be caused by impurities in the water or at the air–water interface. The kinematic energy associated with $V_r$ and $V_\theta$ would...
slightly increase the depth of the depression, but this contribution is negligible compared to that of $V_o$ (Odgaard 1986). We assume that the curved path of the vortex axis entering the intake pipe has a negligible impact on the free surface depression. However, it might be more significant for shallower submergence values or a different geometry.

The mean local curvature $\kappa(r)$ is given by

$$\kappa(r) = -\frac{1}{2} \left\{ \frac{h_r}{r[1 + (h_r)^2]^{3/2}} + \frac{h_{rr}}{[1 + (h_r)^2]^{3/2}} \right\}$$  \hspace{1cm} (5)

where $h_r$ and $h_{rr}$ are the first and second derivatives of $h$ with respect to $r$, respectively (Andersen et al. 2006). The first term on the right is the curvature about the horizontal axis (perpendicular to the page in a two-dimensional section of the profile such as Fig. 4a) and the second term is the curvature about the vortex’s (vertical) axis of rotation. At the vortex tip, the free surface profile has a positive (concave) horizontal axis curvature. Then, at values of $r$ beyond $r > r_0$, it passes through an inflection point and the horizontal axis curvature becomes negative (convex). The surface tension force thus pushes the interface upward in the core portion of the vortex and pulls it down very slightly just outside the core.

Since our primary goal is to get a larger view of trends in surface tension effects over a range of shapes and scales rather than to obtain the exact shape of the depression, the free surface profile is computed by directly substituting Burgers’s relation for $V_o(r)$ from Eq. (1) into Eq. (4)

$$h_o(r) = \int_0^r \frac{\Gamma_{\infty}^2}{4\pi^2 g^3} \left[ 1 - \exp \left( \frac{r^2}{r_0^2} \right) \right]^2 - \hat{\xi}_o \kappa(\hat{\xi}) \, d\hat{\xi}$$  \hspace{1cm} (6)

This approximation neglects the effect of the free surface depression on the velocity field and hence indirectly on the depression itself as well. Stepanyants and Yeoh’s (2008b) results suggest that this approximation produces a negligible error in $\delta$ in the case of a mild dimple depression and an error of 26% for an extremely deep, funnel-type depression with a nominal free surface depression slope $\zeta = h_o/r_o = 110$. This is judged to be an acceptable level of error for the purpose of this study.

Equation (6) is discretized along $r$ by central differences and an equilibrium profile $h_o(r)$ is computed numerically for the given $r_0$ and $\Gamma_{\infty}$ by gradually decreasing $r_0$ from a large value (which produces a very shallow depression) to the desired $r_o$ (Suerich-Gulick 2013). Funnel vortices are characterized by a large peak in surface curvature at the tip ($r = 0$), requiring a large relaxation factor and smoothing of the computed curvature profile $h_o(r)$ between each iteration to suppress oscillations there.

3 Analysis and discussion

3.1 Surface tension effects

The free surface profile code is first tested using the free surface and velocity profiles measured and computed by Andersen et al. (2006) for a moderate funnel of nominal depression slope $\zeta = 15$; the results compare very well. The code is then tested by computing the free surface depression from the measured velocity profiles in our experiment, where surface tension effects are significant, and comparing the computed tip depth $h_o$ to $h_o$ recorded in the film segments.

Figure 2a shows each measured tip depth $h_o$ compared to the tip depth $h_o$ computed using Eq. (6) with $\Gamma_{\infty}$ and $r_o$ obtained by fitting Burgers’s profile (Eq. 1) to the measured $V_o(r)$ profiles. The shape of the symbols on the graph indicates the corresponding intak velocity $U_l$ and their shade indicates the relative submergence $s/d$. The horizontal error bars show the spread between the two values of $h_o$ computed from $r_o$ and $\Gamma_{\infty}$ obtained from the two fitting methods, and the symbols show the mean. Although the spread is somewhat large for some points, the agreement is close enough to indicate that both the method used to measure the azimuthal velocities and Burgers’s model used to describe the measured profiles are sufficiently

---

Figure 2. (a) Correspondence between the measured and computed tip depths. (b) Relative surface tension effect as a function of the depression scale and nominal slope, adapted from Suerich-Gulick et al. (2014a) with permission from ASCE
accurate to predict the free surface depression from the velocity measurements.

The code is then used to compute the free surface depression produced by Burgers’s vortices, with and without surface tension, for a range of $\Gamma_\infty$ and $r_o$ values. As shown in Fig. 2b, the results reveal that the relative surface tension effect $\delta$ scales very differently depending on the shape of the depression, which is quantified here using the nominal depression slope $\zeta = h_{n0}/r_o$. The transition between the dimple and funnel modes occurs around $1 < \zeta < 10$, depending on the scale. For dimple-shaped depressions corresponding to $\zeta \gtrsim 1$ – 10, $\delta$ becomes independent of $\zeta$, while for funnel-shaped depressions ($\zeta \gtrsim 1$ – 10), $\delta$ varies with both the scale and shape $\zeta$. Furthermore, the results show that for a given scale $r_o/l_o$, the relative surface tension effect $\delta$ is much more significant in a dimple than in a funnel vortex, in agreement with the results of Yildirim and Jain (1981).

Figure 3a shows that once the limiting dimple shape is reached (towards the upper right of the graph), $\delta$ converges to a unique function $f_\delta$ that depends only on the scale $r_o/l_o$ and is independent of $\Gamma_\infty$.

$$f_\delta \left( \frac{r_o}{l_o} \right) = \left[ \exp \left( -0.44 \left( \frac{r_o}{l_o} \right)^2 \right) + 1.9 \left( \frac{r_o}{l_o} \right)^{1.6} \right]^{-1}$$

At large scales ($r_o/l_o \gtrsim 3$), $f_\delta$ tends towards a straight line so that $\delta \sim (r_o/l_o)^{2(-0.9)} \sim (r_o/l_o)^{-1.8}$. Likewise, when $\delta$ is plotted in Fig. 3b as a function of the product $(r_o/l_o)^2 \zeta = r_o h_{n0}/l_o^2$, the curves collapse to a straight line of slope $\delta \sim [(r_o/l_o)^2 \zeta]^{-0.6}$ at large values of $(r_o/l_o)^2 \zeta$ corresponding to the funnel shape. The dashed lines in Fig. 3a and 3b indicate lines of constant bulk circulation $\Gamma_\infty$, while the solid lines in Fig. 3b indicate lines of constant scale $r_o/l_o$.

These trends can be compared to the scaling behaviour of the local curvature $k_{n0}$ of the free surface at $r = 0$ in the absence of surface tension (Eq. 5) in the limits $\zeta \ll 1$ (dimple) and $\zeta \gg 1$ (funnel). If $r$ is assumed to scale with $r_o$, $h_r$ with $\zeta$ and $h_{n0}$ with $h_{n0}/r_o^2$, then Eq. (5) produces $k_{n0} \sim h_{n0}/r_o^2$ for the dimple and $k_{n0} \sim r_o^{-1}$ for the funnel. If it is further estimated that $\delta \sim k_{n0}(r_o/h_{n0})$, this produces the scaling behaviour $\delta \sim (r_o/h_{n0})^{-2}$ for the dimple and $\delta \sim [(r_o/h_{n0})^2 \zeta]^{-1}$ for the funnel (equivalent to the relation used by Odgaard 1986 in his analysis). So the same essential scaling behaviour is produced by the computations and the theoretical analysis, except that the slopes of the computed trends ($-1.8$ and $-0.6$ for the dimple and funnel, respectively) are weaker than those ($-2$ and $-1$) produced by the rough theoretical analysis. Stepanyants and Yeoh (2008a) also obtain $\delta \sim r_o^{-2}$ for the dimple. The difference between our numerical result and that of Stepanyants and Yeoh (2008b) might be due to simplifications in the velocity profile model used here, or to the different solution methods, since Stepanyants and Yeoh use a series solution with analytical functions to approximate the shape of the tip instead of computing a discretized profile. However, it seems quite possible that the difference in slopes between the present computed trend and theoretical analysis is due to physics rather than numerical error, since the theoretical analysis does not capture the non-linearity of the process by which surface tension changes the shape of the free surface depression and in turn affects the magnitude of the surface tension and so forth.

To understand these trends, we examine how the computed shape of the free surface depression is modified by surface tension and how that effect depends on the initial shape and scale of the depression. Comparison of the profiles in Fig. 4a computed with surface tension (dashed and dotted lines) and without (solid line) shows that surface tension acts differently on different regions of the profile depending on the shape of the depression. Within a given scale ($r_o/l_o = 0.3$) there is a much stronger relative reduction of the depression $h_r/h_{n0}$ for the dimple vortices ($\zeta \lesssim 20$) than for the funnel vortices ($\zeta \gtrsim 20$). Surface tension acts strongly over a much larger radius in the dimple vortex (up to $r/r_o \approx 2$) than in the funnel vortex where the effect is restricted to an inner region $r/r_o \lesssim 0.5$ that shrinks as the funnel gets deeper. In the funnel vortices, surface tension appears to essentially clip off the tip of the depression, thereby significantly diminishing the spike in curvature at the tip that characterizes funnel vortices.
This can be seen in Fig. 4b, which shows the curvature profiles \( \kappa_{\sigma} (r/r_o) \) that correspond to the free surface profiles in Fig. 4a. In the absence of surface tension (solid line), \( r_o \kappa \) for \( \zeta = 350 \) reaches a peak of 510 at \( r = 0 \). Figure 4c shows that surface tension has much less effect at a larger scale (\( r_o/l_{\sigma} = 1.5 \)) in both the dimple (\( \zeta \leq 5 \) here) and funnel vortices. The profiles for the two lowest values of \( \zeta \) in Fig. 4a and 4c coincide, indicating that the dimple shape limit has been reached, where \( \delta \) becomes independent of \( \zeta \).

These results clearly demonstrate that surface tension effects do not scale in the same way for dimple vortices and funnel vortices. This qualitative difference in behaviour might partly explain the variability in recommendations found in the literature for the minimum laboratory model size required to avoid surface tension effects. The results also strongly suggest that empirical surface tension scaling laws derived by studying the onset of air entrainment (i.e. deep funnel vortices) must not be directly employed to interpret vortex observations in reduced scale models of hydropower plants, where spatial constraints are such that only dimple vortices are commonly observed.

### 3.2 Scale effects associated with viscosity and turbulence

In Eq. (9) of Suerich-Gulick et al. (2014b), the range of expected tip depths \( h_{a,0,\text{est}} \) for a specific geometry was estimated as a function of the intake velocity \( U_i \) and relative submergence \( s/d \). We obtain \( h_{a,0,\text{est}} \) (including surface tension) by adding the surface tension correction function \( f_{\sigma} \) (Eq. 7)

\[
\frac{h_{a,0,\text{est}}}{d} = c_0 \left( 1 - \frac{f_{\sigma}}{\beta} \right) \frac{R_s F^2_{s}}{k} \left( \frac{l_p}{s} \right)^2 \left( \frac{d}{c_s} - c_2 \right)^2
\]

with the non-dimensional coefficients \( c_0 = 3.7 \times 10^{-5} \) and \( c_2 = 0.28 \). \( c_4 \) has two values, 1.0 and 0.6, corresponding to low and high values of \( \Gamma_{\infty} \), respectively. \( l_p \) is the length of each pier, \( k \) is the distance between the piers and \( R_s F^2_{s} = R_d F^2_{d} = U_i^2/(\nu g) \). This expression is valid for all the conditions examined, regardless of vortex shape, except that \( f_{\sigma} \) (Eq. 7) is valid only for dimple-type depressions and will overestimate the surface tension effect for funnel- and transition-type depressions. For a more accurate correction for these latter shapes, \( \delta \) can be read off Fig. 2b for a given nominal depression slope \( \zeta \) and scale \( r_o/l_{\sigma} \) and substituted for \( f_{\sigma} \) in Eq. (8). The solid lines indicate curves of constant \( r_o/l_{\sigma} \).

Figure 5a shows the measured values compared with the range of values \( h_{a,0,\text{est}} \) estimated from Eq. (8), with \( r_o \) in \( f_{\sigma} \) (Eq. 7) estimated using \( \beta = 0 \). Surface tension has a minimal impact relative to the variability associated with variations in \( r_o \) and \( \Gamma_{\infty} \). The predicted range of \( h_{a,0,\text{est}} \) values is shown by the boxes: the upper limit is obtained from \( c_4 = 0.6 \) and \( \beta = 0.15 \).
and the lower limit is obtained from $c_4 = 1.0$ and $\beta = 0.85$. Possible causes of $\Gamma_{\infty}$ variability include as-yet poorly understood strengthening processes within the vortex, and turbulent fluctuations in the surrounding flow. Variability in $r_o$ would appear to be mainly due here to variations in the shape of the $V(z)$ profile in the upper portion of the flow, ranging from a non-linear profile $V(z) \sim (h - z)^{-1}$ to a more linear one $V(z) \sim z$. This effect is indicated in Eq. (8) by the parameter $\beta$. As discussed in Suerich-Gulick et al. (2014b), $\beta$ indicates the vertical extent of the linear profile of $V(z)$, where $0 \leq \beta \leq 1 - 0.5d/s$ is the proportion of the distance $s$ from the free surface to the top of the inlet over which the linear profile occurs. $\beta = 0$ indicates that $V(z)$ inside the vortex follows the non-linear profile of the flow outside the vortex, while larger values of $\beta$ indicate that $V(z)$ follows a linear profile in $z$ from the free surface down to $z = \beta s$, producing a steeper slope in $V(z)$ at the free surface. In the experiment, $\beta$ is observed to fall in the range $0.15 \leq \beta \leq 0.85$ (Suerich-Gulick et al. 2014b). Turbulence in the vortex core might also produce variations in $r_o$ by enhancing the effective radial diffusivity in Eq. (3).

To compare Eq. (8) with other results, it is reformulated in terms of the critical relative submergence $s'_c$ for air entrainment, where the tip of the vortex depression reaches the top of the outlet so that $h_{o,0,\text{crit}} = s$. Substituting $s'_c = (s/d)_{\text{crit}}$ for $h_{o,0}/d$ in Eq. (8) and isolating terms in $s'_c$ produces

$$s'_c^5 = \frac{c_0(1-f_o)\Gamma}{(1-\beta)\left(d^2\ell/P\right)F_d^{5/3}}$$

As shown in Fig. 5b, this relation predicts that $s'_c$ scales roughly as $s'_c \sim A^{1/3}F_d^{1/3}$ at smaller values of $R_d^{1/2}F_d$, with $A^2 = c_0(1-f_o)(1-\beta)^{-1}(d^2\ell/P\ell^3)$. It then flattens out at larger values, becoming less sensitive to $R_d^{1/2}F_d$ at deeper submergences. The two curves are produced by the two values of $c_4 = 0.6$ and 1 and they have a singularity at $s'_c = 0.6$ and 1.7, respectively. The shift to a milder slope at larger $F_d$ reproduces trends observed in physical models of vertical intakes by Tastan and Yıldırım (2010) and of horizontal intakes by Jiming et al. (2000).

It differs from the trends with a constant slope on a log–log scale found by other researchers. Gulliver (1988) observed $s'_c \sim F_d^{2/3}$ in experiments with vertical intakes, and Rao et al. (1997) derived the same relation from Yıldırım and Kocabas’s (1995) analysis of lateral intakes in cross-flow or at the end of a channel. Jain et al. (1978) observed $s'_c \sim F_d^{2/3}N_{F,0}^{-2/3}K^{-1}$ and Odgaard (1986) analytically derived $s'_c \sim N_{F,0}^{-2/3}R_d^{1/2}F_d^{1/2}$, both for flow in a cylinder, where $N_{F,0} = \Gamma_{\infty}s/Q$ is the non-dimensional circulation and $K$ is a viscous correction factor.

The decreased sensitivity to $F_d$ predicted by Eq. (9) at greater $F_d$ values is due to the non-linear $V(z)$ profile that roughly follows $V(z)/U_i \sim d/(s - z)$, as opposed to the linear profile $V(z)/U_i = z/s$ assumed by Odgaard (1986). The non-linear profile has a much milder gradient $\partial V(z)/\partial z$ at the free surface at large submergences, which produces more diffuse and thus weaker vortices, requiring a greater relative increase in $F_d$ to produce an air core vortex than at lower submergences. The linear profile assumed by Odgaard (1986) might explain why his model tends to over-predict $s'_c$ compared with observations at larger $s'_c$ values in several configurations (Jain et al. 1978, Gulliver 1988). It should also be noted that $\Gamma_{\infty}$ depends on $U_i$, $s/d$ and the relative pier length $h_p/d$ in the current experiment, whereas it is imposed using adjustable guide vanes in the experiments considered by Odgaard (1986) and Jain et al. (1978), which should affect the corresponding scaling relations.

Equation (8) can also be used to evaluate how the characteristics of a free surface vortex in a laboratory-scale model operated at Froude similitude would translate to a much larger prototype intake with the same geometry at a geometric scaling ratio $\alpha = \ell_p/\ell_M$, where $\ell_M$ and $\ell_p$ are the characteristic lengths in the laboratory model and the prototype, respectively. Neglecting surface tension $(1-f_o)$ and assuming the model and prototype are geometrically identical so that $(d/k)$, $(d/s)$ and $(\ell_p/\ell_M)$ are identical in both the model and prototype, Eq. (8) becomes

$$h' = \frac{h_{o,0}}{d} \sim \frac{c_0}{(1-\beta)}\left(d \left(\frac{d}{c_4s} - c_2\right)\right)^2F_d^2\ell_p$$
Since \( F_M = F_P \), the ratio of the outflow velocities \( U_{i,P} / U_{i,M} = \alpha^{1/2} \). Because water is usually employed in the laboratory model, \( v_M = v_P \) and the prototype-to-model ratio of the Reynolds numbers is therefore \( R_{P} / R_{M} = \alpha^{3/2} \). If it is further assumed that the velocity profiles outside and inside the vortex follow the same shape in the model and prototype so that \([d / (C_s t)] = c_1 \) and \((1 - \beta)\) are identical in the model and prototype, we obtain

\[
\frac{h'_{P}}{h'_{M}} = \frac{R_{P}}{R_{M}} = \alpha^{3/2} \tag{11}
\]

This indicates that the vortex depression \( h'_{P} \) produced in the prototype would be significantly greater in relative terms than \( h'_{M} \) produced in the laboratory model if the scaling ratio \( \alpha \) is large. Scaling ratios of 20 are common and can reach as high as 200 in some cases (Hecker 1981). This predicted result is due to the fact that the viscosity of the fluid is identical in both model and prototype while circulation increases with intake size and is thus larger in the prototype.

This result is contradicted by observations of decreasing sensitivity to Reynolds number at large \( R \) values (Daggett and Keulegan 1974, Jain et al. (1978), Anwar, H.O. 1983, Chang and Prosser 1987), as well as the high success rate of laboratory-scale modelling for predicting prototype vortex activity in the past (Hecker 1981, Montilla et al. 2004). The discrepancy suggests that additional processes or changes in flow structure must come into play at larger scales. There are several possibilities. To begin with, the laboratory and prototype intakes could have different axial profiles \( V_z(z) \), producing \( \beta_M \neq \beta_P \), and/or slightly different flow structures outside the vortex, while they are assumed here to be identical. If turbulent diffusion enhances the effective viscosity controlling \( \nu \), at larger scales and/or higher flow rates as suggested by Einstein and Li (1951), Anwar (1969) and Odgaard (1986), this would reduce the value of \( h_{n_0} \) at larger scales. To properly assess this possibility, it would be necessary to make simultaneous measurements of the \( V_z(z) \) and \( V_o(r) \) profiles such as those made in Suerich-Gulick et al. (2014b), but at larger scales and \( R \) values.

Increased turbulence in the surrounding flow might also modify scaling behaviour at higher \( R \) values by preventing vortices from forming or intensifying, as observed by Padmanabhan and Hecker (1984) and Tastan and Yildirim (2010). Existing work on the interaction of external turbulence with the trailing vortices produced in the wake of airplane wings indicates that its impact depends on the length- and time-scale characteristics of both the background turbulence and of the central vortex of interest (Zeman 1995, Jacquin and Pantano 2002, Beninati and Marshall 2005).

4 Conclusions

The free surface profile calculations reveal that both the shape and the scale of the free surface profile determine how surface tension will modify the shape and total depth of the depression. The results suggest that the magnitude of surface tension effects in a laboratory-scale vortex can be fairly easily estimated using the proposed correction factor. In comparison, scale effects linked to viscosity and turbulence appear to be much more difficult to explain and predict. By incorporating the non-linear velocity profile of the flow surrounding the vortex, the proposed analytical model successfully reproduces the decreasing sensitivity to Froude number observed experimentally by earlier authors at deep submergences. However the model in its current form fails to reproduce the independence from Reynolds number observed by previous authors at large Reynolds values, suggesting that additional processes must intervene or that the flow structure may change at larger scales.

Potential changes in the flow structure at larger scales, as well as perturbations and enhanced diffusion caused by turbulence will have to be examined more closely to better understand the observed scaling behaviour. The proposed vortex model linking vortex characteristics to intake geometry and approach flow is a useful tool to evaluate how different processes interact to control vortex characteristics at both the laboratory scale and at larger scales.

Acknowledgements

We would like to acknowledge the advice and contributions of David Morisette, Pierre Tadeo, Tristan Aubel, Maryse Page, Anne-Marie Giroux and Sébastien Houde.

Funding

This work was supported by the National Sciences and Engineering Research Council of Canada (Postgraduate Scholarship B), the Fonds Québécois pour les Sciences et les Technologies (Bourse de recherche au doctorat) and Hydro-Québec’s research centre.

| Notation | Description               |
|----------|---------------------------|
| \( a \)  | axial gradient of axial velocity \( (s^{-1}) \) |
| \( c_{1-6} \) | non-dimensional model coefficients \( (-) \) |
| \( d \)  | intake pipe inner diameter \( (m) \) |
| \( f_s \) | surface tension correction function \( (-) \) |
| \( F \)  | intake Froude number \( (-) \) |
| \( g \)  | gravitational acceleration \( (m s^{-2}) \) |
| \( h_0 \) | free surface depression tip depth \( (m) \) |
| \( h' \) | non-dimensional depression tip depth \( (-) \) |
| \( h_r \) | local derivative of \( h \) w.r.t. \( r \) \( (-) \) |
| \( h_s \) | local second derivative of \( h \) w.r.t. \( r \) \( (m^{-1}) \) |
| \( k \)  | distance between the piers \( (m) \) |
| \( K \)  | viscous correction factor \( (-) \) |
| \( l_p \) | pier length \( (m) \) |
| \( \ell \) | intake characteristic length \( (m) \) |
| \( l_s \) | air–water interface characteristic length \( (m) \) |
| \( N_\nu \) | non-dimensional viscosity parameter \( (-) \) |
| \( N_{fr*} \) | non-dimensional circulation \( (-) \) |
| \( Q \)  | intake flow rate \( (m^3 s^{-1}) \) |
\[ R = \text{Reynolds number (–)} \]
\[ r = \text{vortex radial coordinate (m)} \]
\[ r_o = \text{vortex characteristic radius (m)} \]
\[ s = \text{intake submergence (m)} \]
\[ s_c = \text{non-dimensional critical submergence (–)} \]
\[ U_i = \text{mean intake velocity (m s}^{-1}) \]
\[ W = \text{intake Weber number (–)} \]
\[ \alpha = \text{model-to-prototype scaling factor (–)} \]
\[ \beta = Y_s \text{ linearization ratio (–)} \]
\[ \Delta h = \text{tip depth difference due to surface tension (m)} \]
\[ \chi = \text{eddy diffusivity scaling coefficient (–)} \]
\[ \delta = \text{relative surface tension effect (–)} \]
\[ \Gamma_\infty = \text{vortex bulk circulation (m}^2 \text{ s}^{-1}) \]
\[ \kappa = \text{local mean free surface curvature (m}^{-1}) \]
\[ v = \text{water molecular viscosity (m}^2 \text{ s}^{-1}) \]
\[ v_{\text{eff}} = \text{effective viscosity (m}^2 \text{ s}^{-1}) \]
\[ v_T = \text{eddy diffusivity (m}^2 \text{ s}^{-1}) \]
\[ \pi = \text{trigonometric constant (–)} \]
\[ \theta = \text{vortex azimuthal coordinate (rad)} \]
\[ \rho = \text{water density (kg m}^{-3}) \]
\[ \eta = \text{value at (r = 0)} \]
\[ M = \text{experimental model value} \]
\[ P = \text{prototype value} \]
\[ \text{comp} = \text{computed value} \]
\[ \text{d} = \text{value based on intake diameter} \]
\[ \text{est} = \text{estimated value} \]
\[ \text{nt} = \text{without surface tension effect} \]
\[ s = \text{value based on intake submergence} \]
\[ \sigma = \text{with surface tension effect} \]

The underlying research materials for this article can be accessed at http://digitool.library.mcgill.ca/thesisfile119367.pdf (Ph.D. Thesis)

References

Andersen, A., Bohr, T., Stenum, B., Rasmussen, J.J., Lautrup, B. (2006). The bathtub vortex in a rotating container. J. Fluid Mech. 556, 121–146.

Anwar, H.O. (1969). Turbulent flow in a vortex. J. Hydraulic Res. 7(1), 1–29.

Anwar, H.O. (1983). The non-dimensional parameters of free-surface vortices measured for horizontal and vertically inverted intakes. Houille Blanche 1, 11–25.

Anwar, H.O., Amphlett, M.B. (1980). Vortices at vertically inverted intake. J. Hydraulic Res. 18(2), 123–134.

Beninati, M.L., Marshall, J.S. (2005). External turbulence interaction with a columnar vortex. J. Fluid Mech. 540, 221–245.

Bradshaw, P. (1973). Effects of streamline curvature on turbulent flow. AGARDograph No. 169.

Burgers, J.M. (1948). A mathematical model illustrating the theory of turbulence. Adv. Appl. Mech. 1, 171–199.

Chang, E., Prosser, M.J. (1987). Basic results of theoretical and experimental work. In Swirling flow problems at intakes. J. Knauss, ed. A.A. Balkema, Rotterdam, NL, 39–55.

Daggett, L.L., Keulegan, G.H. (1974). Similitude in free-surface vortex formations. J. Hydraulic Div. 100(HY11), 1565–1581.

Einstein, H.A., Li, H. (1951). Steady vortex flow in a real fluid. Proc. Heat Trans. Fluid Mech. Inst. 4, 33–43.

Gulliver, J.S. (1988). Discussion of free-surface air core vortex. J. Hydraulic Div. 114(4), 447–449.

Hecker, G. (1981). Model-prototype comparison of free surface vortices. J. Hydraulic Div. 107(HY10), 1243–1259.

Hite, J.E., Mih, W.C. (1994). Velocity of air-core vortices at hydraulic intakes. J. Hydraulic Eng. 120(3), 284–297.

Ito, K., Sakai, T., Eguchi, Y., Monji, H., Ohshima, H., Uchibori, A., Xu, Y. (2010). Improvement of gas entrainment prediction method – Introduction of surface tension effect. J. Nucl. Sci. Technol. 47(9), 771–778.

Jacquin, L., Pantano, C. (2002). On the persistence of trailing vortices. J. Fluid Mech. 471, 159–168.

Jain, A.K., Raju, K.G.R., Garde, R.J. (1978). Vortex formation at vertical pipe intakes. J. Hydraulic Div. 104(HY10), 1429–1445.

Jiming, M., Yuanbo, L., Jitang, H. (2000). Minimum submergence before double-entrance pressure intakes. J. Hydraulic Eng. 126(8), 628–631.

Knauss, J. (1987). Prediction of critical submergence. In Swirling flow problems at intakes, J. Knauss, ed. Balkema, Rotterdam, NL, 57–76.

Laplace, P.S. (1800). Oeuvres complètes de Laplace, Vol. 4. Gauthier-Villars, Paris, 419–498.

Möller, G., Meyer, A., Detert, M., Boes, R. (2012). Luftteintragsrate durch Einlaufwirbel – Modellfamille nach Froude. Proc. Int. Conf. Internationales Wasserbausymposium, Graz, AT. G. Zenz, ed. verlag des Technischen Universität, 371–378.

Montilla, G., Marcano, A., Castro, C. (2004). Air entrainment at Guri Dam intake operating at low heads. Proc. Int. Conf. Hydraulics of dams and river structures, Tehran, F. Yazdandoost, J. Attari, eds. Balkema, Rotterdam, NL, 53–60.

Odgaard, A. (1986). Free-surface air core vortex. J. Hydraulic Eng. 112(7), 610–620.

Padmanabhan, M., Hecker, G.E. (1984). Scale effects in pump sump models. J. Hydraulic Eng. 110(HY11), 1540–1556.

Quick, M. (1962). Scale relationships between geometrically similar free spiral vortices, Pt. 2. Civil Eng. Public Works Rev. 57, 1319–1320.

Rao, S., Diwanji, V., Srivastava, Y. (1997). Discussion of critical submergence for intakes in open channel flow. J. Hydraulic Div. 120(3), 284–297.

Rao, S., Diwanji, V., Srivastava, Y. (1997). Discussion of critical submergence before double-entrance pressure intakes. J. Hydraulic Eng. 120(3), 1429–1445.

Rao, S., Diwanji, V., Srivastava, Y. (1997). Discussion of critical submergence for intakes in open channel flow. J. Hydraulic Div. 120(3), 284–297.

Rao, S., Diwanji, V., Srivastava, Y. (1997). Discussion of critical submergence before double-entrance pressure intakes. J. Hydraulic Eng. 120(3), 284–297.

Squires, S. (1965). The growth of a vortex in turbulent flow. Aeronaut. Quart. 16(Pt.3), 302–306.

Stepanyants, Y., Yeoh, G. (2008a). Burgers–Rott vortices with surface tension. Z. Angew. Math. Phys. 59, 1057–1068.
Stepanyants, Y., Yeoh, G. (2008b). Stationary bathtub vortices and a critical regime of liquid discharge. *J. Fluid Mech.* 604, 77–98.

Suerich-Gulick, F. (2013). Axial stretching, viscosity, surface tension and turbulence in free surface vortices at low-head hydropower intakes. *PhD Thesis*. McGill University, Montreal.

Suerich-Gulick, F., Gaskin, S.J., Parkinson, E., Villeneuve, M. (2014a). The characteristics of free surface vortices at low-head hydropower intakes. *J. Hydraulic Eng.* 140, 291–299.

Suerich-Gulick, F., Gaskin, S.J., Villeneuve, M., Parkinson, E. (2014b). Free surface intake vortices: Theoretical model and measurements. *J. Hydraulic Res.* doi:10.1080/00221686.2014.896425

Tastan, K., Yildirim, N. (2010). Effects of dimensionless parameters on air-entraining vortices. *J. Hydraulic Res.* 48(1), 57–64.

Yildirim, N., Jain, S. (1981). Surface tension effect on profile of a free vortex. *J. Hydraulic Div.* 107(HY1), 132–136.

Yildirim, N., Kocabas, F. (1995). Critical submergence for intakes in open channel flow. *J. Hydraulic Eng.* 121(12), 900–905.

Zeman, O. (1995). The persistence of trailing vortices: A modeling study. *Phys. Fluids* 7(1), 135–143.