Abstract

The smallness is proved of étale fundamental groups for arithmetic schemes. This is a higher dimensional analogue of the Hermite-Minkowski theorem. We also refer to the case of varieties over finite fields. As an application, we prove certain finiteness results of representations of the fundamental groups over algebraically closed fields.

1 Introduction

The Hermite-Minkowski theorem is a remarkable result in algebraic number theory. It says that for a number field $F$ and a finite set $S$ of $F$, there exist only finitely many extensions of $F$ unramified outside $S$ with given degree. By Galois theory we can interpret the theorem as the smallness of the Galois group $G_{F,S}$ of the maximal Galois extension of $F$ unramified outside $S$. In general, a profinite group is said to be small if there exist only finitely many open subgroups of the group of given index. One of our main results is the smallness of étale fundamental groups for arithmetic schemes as follows.

**Theorem** (Th. 2.8). Let $X$ be a connected scheme of finite type and dominant over the ring of integers $\mathbb{Z}$. Then the étale fundamental group $\pi_1(X)$ is small. Equivalently, there exist only finitely many étale coverings of $X$ with given degree.

On the contrary, non-complete varieties over a finite field, even function fields over a finite field in one variable with characteristic $p > 0$ have so many finite extensions that the Hermite-Minkowski theorem no longer holds.

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In fact, the Artin-Schreier equations produce infinitely many extensions of degree $p$ which ramify only at a place. Thus one can hope for the finiteness only for extensions of bounded degree with extra conditions, for example, fixing discriminants (see [6], Sect. 8.23). In Section 3 we use the fundamental groups with restricted ramification instead of discriminants. More precisely, we introduce a certain quotient $\pi_1(X, m)$ of the étale fundamental group $\pi_1(X)$. It classifies étale coverings of $X$ which allow ramification along the boundary according to a given modulus $m$. Here, the modulus $m$ is a collection of moduli associated with curves on $X$. These coverings have the advantage that they are stable under base change, while coverings with restricted ramification dealt in [9] are not. We examine the smallness of the fundamental group with restricted ramification for varieties over a finite field as follows.

**Theorem (Th. 3.6).** Let $X$ be a variety, proper over a curve defined over a finite field. Then $\pi_1(X, m)$ is small for any modulus $m$ on $X$.

As an application, in Section 4 we prove certain finiteness results of representations of the fundamental groups with restricted ramification over an algebraically closed field. In Appendix A we recall the class field theory of G. Wiesend [18] with slight modification by making use of the fundamental groups with restricted ramification.

After writing up this paper, it was pointed out by Professor Y. Taguchi that one of our main theorems (Th. 2.8) has been proved by G. Faltings ([4], Chap. VI, Sect. 2) assuming the scheme $X$ is affine and smooth over $\mathbb{Z}$. However there is only a sketch of the proof, and our proof is more systematic and simpler than his by virtue of Lemma 2.6. Thus we think that it is indispensable to leave Theorem 2.8 be in this paper.

Throughout this paper, a number field is a finite extension of the rational number field $\mathbb{Q}$. We denote by $\mathcal{O}_F$ the ring of integers of a number field $F$. For any field $K$, we denote by $\overline{K}$ a separable closure of $K$.

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2 Smallness of fundamental groups

The aim of this section is to prove the smallness of the étale fundamental group for an arithmetic scheme (Th. 2.8). First, we interpret the Hermite-Minkowski theorem as the smallness of a Galois group as follows.

**Definition 2.1** (cf. [5], Sect. 16.10, or of type (F) in Sect. 4.1 of [16]). A profinite group $G$ is said to be small if there exist only finitely many open normal subgroups $H$ with $(G : H) \leq n$ for any positive integer $n$.

For a number field $F$, the Galois group $G_{F,S}$ of the maximal Galois extension of $F$ unramified outside $S$ is small, where $S$ is a finite set of primes of $F$. This fact is nothing other than the Hermite-Minkowski theorem.

**Remark 2.2.** The notion of a small profinite group is used in another meaning in [13]. These two notions are quite different. For example, a direct product of countably many $\mathbb{Z}/n\mathbb{Z}$ is small in the sense of op. cit., and it is not small in our meaning. In general, any countably-based profinite group is small in the sense of op. cit.

**Proposition 2.3** ([16], Chap. III, Sect. 4.1, Prop. 8). Let $G$ be a profinite group. The following conditions are equivalent:

(i) $G$ is small.

(ii) There exist only finitely many open normal subgroups $H$ with $(G : H) \leq n$ for any positive integer $n$.

(ii') For every finite $G$-group $A$, $H^1(G, A)$ is a finite set.

(iii) For every finite group $A$, the set of continuous homomorphisms $\text{Hom}(G, A)$ is finite.

Secondly, we list basic properties of small profinite groups.

**Proposition 2.4** ([5], Sect. 16.10). Let $G$ be a profinite group.

(i) If $N$ is an open subgroup of $G$, then $G$ is small if and only if $N$ is small.

(ii) If $G$ is small and $N$ is a closed normal subgroup of $G$, then the quotient group $G/N$ is small.

(iii) If $G$ is topologically finitely generated, then $G$ is small.

**Lemma 2.5.** If $G$ and $G'$ are small profinite groups, then their free product $G * G'$ is also small.
Proof. For any finite group $A$, there is a bijection

$$\text{Hom}(G \ast G', A) \rightarrow \text{Hom}(G, A) \times \text{Hom}(G', A)$$

which maps $f : G \ast G' \rightarrow A$ to the pair $(f \mid_{G}, f \mid_{G'})$, where $f \mid_{G}$ and $f \mid_{G'}$ are the restrictions of $f$ to $G$ and $G'$ respectively. Hence we have the assertion.

The following lemma is a profinite group version of Lemma 5 in [21]. We can prove it by the same method, namely by employing the theory of “variety of groups”. Here we prove this by purely profinite group theoretic argument. This proof is due to Professor M. Matsumoto.

Lemma 2.6. Let $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$ be an exact sequence of profinite groups. If $G'$ and $G''$ are small, then so is $G$.

Proof. Let $S$ be the set of open normal subgroups $N$ of $G$ (we write it as $N \triangleleft G$) with $(G : N) \leq n$. For each $N \in S$, we have $G' \cap N \triangleleft G'$ and $(G' : G' \cap N) \leq n$. Note that $G' \cap N$ is also normal in $G$. By taking quotient groups, we have an exact sequence $1 \rightarrow G'/G' \cap N \rightarrow G/G' \cap N \rightarrow G'' \rightarrow 1$. By the smallness of $G'$, there are only finitely many possibilities of open normal subgroups of $G'$ with the form $G' \cap N$ when $N$ runs through all the open normal subgroups of $G$ with $(G : N) \leq n$. Let $N'_1, \ldots, N'_r$ be such open normal subgroups of $G'$. Clearly we have

$$S = \bigcup_{i=1}^{r} \{N \triangleleft G \mid (G : N) \leq n, N'_i \subset N\}$$

$$\simeq \bigcup_{i=1}^{r} \{N \triangleleft G/N'_i \mid (G/N'_i : N) \leq n\}$$

Hence it is sufficient to prove the smallness of $G/N'_i$ and we may assume $G'$ is finite. Let $N_0$ be an open normal subgroup of $G$ with $G' \cap N_0 = 1$ (such $N_0$ exists since $G$ is Hausdorff and $G'$ is finite). When $N$ runs through all the open normal subgroups of $G$ with $(G : N) \leq n$, there exist only finitely many possibilities of open normal subgroups of $G$ with the form $N \cap N_0$ by Claim 2.7. Since $N \cap N_0$ is open in $G$, this implies that there are only finitely many possibilities of such $N$. Hence $G$ is small.

Claim 2.7. Let $1 \rightarrow G' \rightarrow G \xrightarrow{\varphi} G'' \rightarrow 1$ be an exact sequence of profinite groups, where $G'$ is a finite group and $G''$ is a small profinite group. For any positive integer $n$ there exist only finitely many $N \triangleleft G$ with $(G : N) \leq n$ and $G' \cap N = 1$. 

4
Proof. For \( N \in S \) with \( N \cap G' = 1 \), the restriction \( \varphi|_N : N \to \varphi(N) \) of \( \varphi \) is an isomorphism and we have \( (G'' : \varphi(N)) \leq n \). Let \( N'' \) be an open normal subgroup of \( G'' \) with \( (G'' : N'') \leq n \). By the smallness of \( G'' \), it is sufficient to prove the finiteness of the set \( S'' \) of \( N \in S \) with \( N \cap G' = 1 \) and \( \varphi(N) = N'' \). For any \( N \in S' \), by composing the isomorphism \( \varphi^{-1} : N'' \to N \) and the inclusion \( N \to \varphi^{-1}(N'') \), we have a section \( N'' \to \varphi^{-1}(N'') \) to \( \varphi^{-1}(N'') \to N'' \). This correspondence induces an injection \( S' \to S'' \), where \( S'' \) is the set of sections \( N'' \to \varphi^{-1}(N'') \) to \( \varphi^{-1}(N'') \to N'' \). Now we prove that the set \( S'' \) is finite. Suppose that \( S'' \neq \emptyset \). Take a section \( s_0 : N'' \to \varphi^{-1}(N'') \) in \( S'' \). Then we know that any section \( s : N'' \to \varphi^{-1}(N'') \) in \( S'' \) factors through the semi-direct product \( G' \rtimes s_0(N'') \) which is a subgroup of \( \varphi^{-1}(N'') \). Let \( K \) be the kernel of the conjugate action \( N'' \to s_0(N'') \to \text{Aut}(G') \), which is an open normal subgroup of \( N'' \). Then the restriction to \( K \) of any section \( s : N'' \to G' \rtimes s_0(N'') \) in \( S'' \) factors through \( G' \times s_0(K) \). This is completely determined by \( s_0 \) and the projection of \( s \mid_K \) to \( G' \). Since \( K \) is small, there are only finitely many possibilities of the restriction of sections \( s : N'' \to G' \rtimes s_0(N'') \) to \( K \). Since \( K \) is open in \( N'' \), we conclude that \( S'' \) is a finite set. \( \square \)

Finally, we prove the following theorem.

**Theorem 2.8.** Let \( X \) be a connected scheme of finite type and dominant over \( \mathbb{Z} \). Then the étale fundamental group \( \pi_1(X) \) is small.

To prove the theorem, we recall a homotopy exact sequence of étale fundamental groups.

**Lemma 2.9 ([10], Lem. 2).** Let \( S \) be a connected normal and locally Noetherian scheme with generic point \( \eta \). Let \( X \) be a scheme which is smooth and surjective over \( S \) and assume that its geometric generic fiber \( X_\eta \) is connected. Then the sequence of étale fundamental groups

\[
\pi_1(X_\eta) \to \pi_1(X) \to \pi_1(S) \to 1
\]

is exact.

**Proof of Theorem 2.8.** We shall first reduce to the case in which \( X \) is a normal scheme smooth over \( \mathcal{O}_F \) of a number field \( F \) and the geometric generic fiber \( X_\eta := X \otimes_{\mathcal{O}_F} \bar{F} \) is connected.

Since the scheme \( X \) is Noetherian, we have finite number of irreducible components \( X_1, \ldots, X_n \) of \( X \). The natural morphisms \( X_i \to X \) induce a morphism \( \sqcup_i X_i \to X \) which is an effective descent morphism. By using the descent theory ([8], Exp. IX, Th. 5.1), the group \( \pi_1(X) \) is a quotient of the
group generated by $\pi_1(X_i)$ for all $i$ and finitely many generators. Thus, we may reduce to the case $X$ is irreducible by Lemma 2.5. Since we have an isomorphism $\pi_1(X) \simeq \pi_1(X_{\text{red}})$ for the reduced closed subscheme $X_{\text{red}}$ of $X$ (op. cit., Exp. IX, Prop. 1.7), we may assume that $X$ is integral. Let $X' \to X$ be the normalization morphism of $X$. Since $X$ is an excellent scheme, it is a finite morphism. By the descent theory (op. cit., Exp. IX, Th. 5.1) again, we know that the group $\pi_1(X)$ is a quotient of the group generated by $\pi_1(X')$ and finitely many generators. By Lemma 2.5, we may assume that $X$ is a connected normal scheme. Let $S$ be the image of $X$ by the structure morphism $X \to \text{Spec}(\mathcal{O}_F)$. Since $X$ is flat over $\mathcal{O}_F$, the set $S$ is open in $\text{Spec}(\mathcal{O}_F)$. Note that the complement of $S$ is the finite set since $\mathcal{O}_F$ is a Dedekind domain. By Lemma 2.9 we have the following exact sequence:

$$\pi_1(X_{\mathbb{F}}) \to \pi_1(X) \to \pi_1(S) \to 1.$$ 

The fundamental group $\pi_1(S)$ is small by the Hermite-Minkowski theorem and $\pi_1(X_{\mathbb{F}})$ is topologically finitely generated ([7], Exp. II, Th. 2.3.1). Hence $\pi_1(X)$ is small by Lemma 2.6.

**Remark 2.10.** The same argument works for a connected scheme $X$ of finite type over a local field $k$ with characteristic 0. Since the group $\pi_1(\text{Spec}(k))$ is known to be topologically finitely generated, so is $\pi_1(X)$.

### 3 Fundamental groups with modulus

The notion of coverings with restricted ramification defined in [9] is not stable under base change even if our attention restricts to tame covers (cf. [14], Exam. 1.3). In this section, we shall introduce the notion of a covering with modulus which is a slight modification of G. Wiesend’s tame coverings ([19]). They are stable under base change and form a Galois category.

An arithmetic scheme is an integral separated scheme which is flat and of finite type over $\mathbb{Z}$. A variety $X$ over a finite field $k$ is an integral separated scheme of finite type over $k$. A curve on $X$ is an integral closed subscheme of dimension 1. Following op. cit., we assume that $X$ is an arithmetic scheme, or a variety over $k$. For any curve $C$ on $X$, let $k(C)$ be the function field of $C$, $\tilde{C}$ the normalization of $C$ and $\overline{C}$ the regular compactification of $\tilde{C}$, that
is, the proper curve which contains $\tilde{C}$ as an open subscheme and is regular outside $\tilde{C}$.

**Definition 3.1.** For every curve $C$ on $X$, a modulus $m_C$ on $C$ is an effective Weil divisor on $\tilde{C}$ supported in the boundary $\tilde{C} \setminus C$ of $\tilde{C}$, that is a formal sum $m_C = \sum_{P \in \tilde{C} \setminus C} m_P P$, whose coefficients $m_P$ are non-negative integers. A modulus $m = (m_C)_{C \subseteq X}$ on $X$ is a collection of moduli $m_C$ associated with curves $C$ on $X$.

Let $m$ be a modulus on $X$ and let $\pi_1(X, m)$ denote the quotient of $\pi_1(X)$ by the closed normal subgroup generated by the images of the $m_P$-th ramification subgroups $G_P^m$ of $G_P := \text{Gal}(\overline{k(C)}_P/k(C)_P)$ in the upper numbering for each curve $C$ on $X$ and $P \in \tilde{C} \setminus \tilde{C}$, where $k(C)_P$ is the completion of $k(C)$ at $P \in \tilde{C} \setminus \tilde{C}$. The group $\pi_1(X, m)$ is the fundamental group associated with the Galois category of the coverings with modulus $m$ defined as follows:

**Definition 3.2.** (i) Let $K$ be a complete discrete valuation field, $G_K$ the absolute Galois group of $K$ and $L$ a separable extension field of $K$. For any rational number $m > -1$, we say that the ramification of $L/K$ is bounded by $m$ if $G_K^m \subset G_L$, where $G_K^m$ is the $m$-th ramification subgroup of $G_K$ in the upper numbering.

(ii) For each curve $C$ on $X$, a finite étale morphism $C' \to \tilde{C}$ is called a covering of $C$ with modulus $m_C = \sum_{P} m_P P$ if the extension of complete discrete valuation fields $k(C')_P/k(C)_P$ is ramification bounded by $m_P$ for each $P \in \tilde{C} \setminus \tilde{C}$ and for each prime $P'$ of $k(C')$ over $P$.

(iii) A finite étale morphism $Y \to X$ is called a covering of $X$ with modulus $m$ if for every curve $C$ on $X$ and for each irreducible component $C'$ of $\tilde{C} \times_X Y$, the induced morphism $C' \to \tilde{C}$ is a covering of $C$ with modulus $m_C$.

**Remark 3.3.** Because of the theorem of Zariski-Nagata on the purity of the branch locus (cf. [8], Exp. X, Th. 3.1), our modulus $m = (m_C)_{C \subseteq X}$ might has infinitely many non-zero Weil divisors $m_C$.

Choose a point $P \in X$ and take a geometric point $\xi : \text{Spec}(\Omega) \to P$, where $\Omega$ is a separably closed extension of the residue field at $P$. We define a fiber functor $F$ by $F(Y) = \text{Hom}_X(\text{Spec}(\Omega), Y)$ for any covering $Y \to X$ of $X$ with modulus $m$.

**Lemma 3.4.** The category of coverings of $X$ with modulus $m$ together with the fiber functor $F$ is a Galois category. The associated fundamental group is isomorphic to $\pi_1(X, m)$.  

7
Proof. Let $Y_1, Y_2 \to X$ be coverings with modulus $m$, and $C$ a curve on $X$. The covering $\tilde{C} \times_X (Y_1 \times_X Y_2) \to \tilde{C}$ has modulus $m_C$ by Lemma 2.2 of [9]. Thus, the category of coverings of $X$ with modulus $m$ closed under fiber products. Let $Y_1 \to Y_2$ and $Y_2 \to X$ be two finite étale morphisms such that the composite $Y_1 \to X$ is a covering with modulus $m$. Then the cover $Y_2 \to X$ has modulus $m$ by Lemmas 2.2 and 2.4 of op. cit. The last assertion follows from the definition of a covering with modulus $m$.

The notion of the coverings above is stable under base change as follows: For a morphism of finite type $f : X' \to X$ and any modulus $m = (m_C)_{C \subset X}$ on $X$, we put $f^* m := (m_{C'})_{C' \subset X'}$, where $m_{C'} = \sum_{P'} m_{P'} P'$ is the modulus on $C'$ defined as follows: If $f(C')$ is a curve on $X$, then $m_{C'} := f^* m_C$, namely $m_{P'} = e_{P'/P} m_P$ for $f(P') = P$, where $e_{P'/P}$ is the ramification index of the extension $k(C')_{P'}/k(C)_P$. On the other hand, if $f(C')$ is a closed point of $X$ then put $m_{C'} := 0$.

Lemma 3.5. We have a homomorphism $\pi_1(X', f^* m) \to \pi_1(X, m)$.

Proof. Let $Y \to X$ be a covering of $X$ with modulus $m$ and $C'$ a curve on $X'$. It is enough to show the induced covering $Y' \times_X X' \to X'$ of $X'$ has the modulus $f^* m = (m_{C'})_{C' \subset X'}$. If $f(C')$ is a closed point on $X$, it is separable base field extension and unramified. If $f(C')$ is a curve $C$ on $X$, it is a base change of $\tilde{C} \times_X Y \to \tilde{C}$ which has modulus $m_C$. Thus, the induced covering $\tilde{C'} \times_X Y \to \tilde{C'}$ is bounded by the modulus $m_{C'}$.

Now, we examine the smallness of fundamental groups with modulus. The fundamental group $\pi_1(X, m)$ is a quotient of $\pi_1(X)$. Thereby it is small for an arithmetic scheme $X$ by Lemma 2.2 (ii) and Theorem 2.8. When $X$ is a variety over $k$, we need the assumption on $X$ as in [18], Theorem 1, (c) (we recall it in the Appendix, cf. Th. A.4).

Theorem 3.6. Let $X$ be a variety, proper over a curve defined over $k$. Then $\pi_1(X, m)$ is small for any modulus $m$ on $X$.

To prove the theorem, we shall use the following notation and lemmas: For two moduli $m = (m_C)_C$ and $m' = (m'_C)_C$ on $X$, we write $m' \geq m$ if all the coefficients $m_P$ and $m'_P$ of moduli $m_C$ and $m'_C$ on $C$ satisfy $m'_P \geq m_P$ for each curve $C$ on $X$. Thereby we have a surjection $\pi_1(X, m') \to \pi_1(X, m)$.

Lemma 3.7. Let $f : X \to S$ be a smooth and of finite type morphism over $k$ from $X$ to a normal variety $S$ with generic point $\eta$. Assume that the geometric generic fiber $X_\eta$ is connected and the map $f$ has a section
Then, for any modulus $m$ on $X$ there is a modulus $m' \geq m$ on $X$ such that the sequence of fundamental groups

$$\pi_1(X_\eta) \to \pi_1(X, m') \to \pi_1(S, s^*m) \to 1$$

is exact.

**Proof.** For each curve $C'$ on $S$, its image $s(C')$ of the section $s : S \to X$ is a curve on $X$. Therefore we have $s^*m = (m_{s(C')})_{C' \subset S}$. By Lemma 3.5, the section $s : S \to X$ induces a homomorphism $\pi_1(S, s^*m) \to \pi_1(X, m)$. Take a modulus $m'$ on $X$ with $m' \geq m, m' \geq f^*s^*m$ and satisfying $m'_{s(C')} = m_{s(C')}$ for each curve $C'$ on $S$. Thus we obtain $\beta : \pi_1(X, m') \to \pi_1(S, n)$. By Lemma 2.9 we have the following commutative diagram

\[
\begin{array}{cccccc}
\pi_1(X_\eta) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(S) & \longrightarrow & 1 \\
\| & & \downarrow & & \downarrow & & \\
\pi_1(X_\eta) & \overset{\alpha}{\longrightarrow} & \pi_1(X, m') & \overset{\beta}{\longrightarrow} & \pi_1(S, s^*m) & \longrightarrow & 1,
\end{array}
\]

whose top row is exact. Therefore $\beta$ is surjective, and $\beta \circ \alpha = 0$. To prove the exactness of the bottom row, we must show that for any connected covering $X' \to X$ with modulus $m'$ which admits a section over $X_\eta$, there exists a connected covering $S' \to S$ with modulus $s^*m$ such that $X' = X \times_S S'$. By the exactness of the top row of the above diagram, we obtain a finite étale cover $S' \to S$ with $X' = X \times_S S'$. The definition of the modulus $s^*m$ on $S$ implies the covering $S' \to S$ has modulus $s^*m$ and the assertion follows from it.

**Lemma 3.8 ([20], Lem. 15).** Let $X$ be a variety over $k$ with dimension $> 0$. Assume that $X$ is proper over a curve which is defined over $k$. Then étale locally (in fact, finite étale and Zariski locally), there is a projective smooth morphism from $X$ to a regular variety $W$ over $k$ with geometrically irreducible fibers of dimension $1$ which possesses a section $s : W \to X$.

**Proof of Theorem 3.6.** By the descent theory for étale fundamental groups, we may assume that the variety $X$ is normal. Now, we shall prove the Theorem by induction on the dimension $d$ of $X$. The case of $d = 1$ is well-known (cf. [6], Th. 8.23.5), so we assume $d > 1$. By Lemma 3.8 there exist a finite étale morphism $X' \to X$, regular variety $W$ over $k$, and a proper smooth morphism $X' \to W$ with geometrically irreducible fibers of 1-dimension which admits a section $s : W \to X'$. Take the connected component $U'$ of $X'$ which contains the image of the section $s$ and denote the induced morphism by $i : U' \to X$. Since the function field of $U'$ is a finite extension field of that of
the natural homomorphisms \( \pi_1(U') \to \pi_1(X) \) and \( \pi_1(U', i^*m) \to \pi_1(X, \mathfrak{m}) \) have finite cokernel. Thus, we may assume \( X = X' \). Let \( \eta \) be the generic point of \( W \). Since the geometric generic fiber \( X_{\overline{\eta}} \) is irreducible, we have the following exact sequence by Lemma 3.7 for some moduli \( \mathfrak{m}' \geq \mathfrak{m} \) on \( X \):

\[
\pi_1(X_{\overline{\eta}}) \to \pi_1(X, \mathfrak{m}') \to \pi_1(W, s^*\mathfrak{m}) \to 1.
\]

Because \( X_{\overline{\eta}} \) is proper, \( \pi_1(X_{\overline{\eta}}) \) is topologically finitely generated (\[8\], Exp. X, Th. 2.9). The induction hypothesis says that \( \pi_1(W, s^*\mathfrak{m}) \) is small. Thus, the assertion follows from Lemma 2.6 and the surjection \( \pi_1(X, \mathfrak{m}') \to \pi_1(X, \mathfrak{m}) \).

**Remark 3.9.** It is known that the tame fundamental groups for curves over a finite field are topologically finitely generated. By the same manner as in the proof of the above theorem, for a variety \( X \) over a finite field which is proper over a curve, we can prove that the tame fundamental group \( \pi_1^{\text{tame}}(X) \) is also topologically finitely generated.

## 4 Application to representations of fundamental groups

Throughout this section, let \( k \) be an algebraically closed field. We always consider the general linear group \( \text{GL}_d(k) \) as a topological group with the discrete topology. First, we assume that its characteristic is 0. As an application of the Hermite-Minkowski theorem, the following finiteness of Galois representations over \( k \) is obtained (for the geometric version of this result, see \[12\], Th. 4 (i)):

**Theorem 4.1** (\[1\], see also \[12\], Th. 1). Let \( F \) be a number field. For any positive integer \( d \), there exist only finitely many isomorphism classes of continuous semisimple representations \( \rho : G_F \to \text{GL}_d(k) \) with bounded Artin conductor.

By using the fundamental groups with modulus defined in Section 3 instead of the Artin conductor, the theorem above is equivalent to the finiteness of representations of the fundamental group \( \pi_1(\text{Spec}(\mathcal{O}_F), \mathfrak{m}) \), where \( \mathfrak{m} \) is a modulus on the spectrum \( \text{Spec}(\mathcal{O}_F) \) of \( \mathcal{O}_F \). Generalizing this, we have the following theorem. Here, we consider the modified group \( \tilde{\pi}_1(X, \mathfrak{m}) \), the quotient of \( \pi_1(X, \mathfrak{m}) \) which classifies coverings with modulus in which, in addition, every \( \mathbb{R} \)-valued point of \( X \) splits completely.
Theorem 4.2. Let \( k \) be an algebraically closed field with characteristic 0.

(i) Let \( X \) be a regular arithmetic scheme. For any positive integer \( d \) and any modulus \( m \) on \( X \) there exist only finitely many isomorphism classes of semisimple continuous representations \( \rho : \tilde{\pi}_1(X, m) \to \text{GL}_d(k) \).

(ii) Let \( X \) be a regular variety, proper over a curve defined over a finite field \( k \). For any positive integer \( d \) and any modulus \( m \) on \( X \), there exist only finitely many isomorphism classes of semisimple continuous geometric representations \( \rho : \pi_1(X, m) \to \text{GL}_d(k) \).

Proof. It is sufficient for the proof of (i) (resp. (ii)) to show that there are only finitely many possibilities of open normal subgroups of \( \tilde{\pi}_1(X, m) \) which appear as the kernels of semisimple continuous (resp. continuous geometric) representations \( \rho : \tilde{\pi}_1(X, m) \to \text{GL}_d(k) \) since there exist only finitely many isomorphism classes of irreducible representations of a finite group over \( k \) (cf. [3], (21.25)).

Let \( \rho : \tilde{\pi}_1(X, m) \to \text{GL}_d(k) \) be a continuous (geometric) semisimple representation. By Jordan’s theorem (cf. [17], Chap. VI, Sect. 24, Th. 3) there exists an open normal subgroup \( N \) of \( \tilde{\pi}_1(X, m) \) containing the kernel \( \text{Ker}(\rho) \) of \( \rho \) such that \( N/\text{Ker}(\rho) \) is abelian and that \( (\tilde{\pi}_1(X, m) : N) \leq J(d) \), where \( J(d) \) is a positive integer depending only on \( d \). Since \( \tilde{\pi}_1(X, m) \) is small, there exist only finitely many such \( N \). Let \( f : X' \to X \) be the Galois covering corresponding to \( N \). Then there exists a surjective homomorphism \( \pi_1(X', m') \to N \) for the modulus \( m' := f^*m \) on \( X' \) and it induces a surjection \( \tilde{\pi}_1(X', m')^{ab} \to N^{ab} \).

In the case (i), \( N^{ab} \) is finite by Corollary A.3. Since \( N/\text{Ker}(\rho) \) is abelian, the commutator subgroup of \( N \) is contained in \( \text{Ker}(\rho) \). This proves the assertion (i).

In the case (ii), \( N^{ab} \) is not finite in general. Now we identify \( \text{Ker}(\rho) \) with the image of \( \text{Ker}(\rho) \) in \( N^{ab} \). Let \( k' \) be the algebraic closure of \( k \) in the function field of \( X \). Since \( \rho \) is geometric, the restriction of the canonical surjection \( \pi_1(X, m) \to \pi_1(\text{Spec}(k')) = G_{k'} \) to \( \text{Ker}(\rho) \) is also surjective. Then we have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \pi_1(X', m')^{ab, \text{geo}} & \longrightarrow & \pi_1(X', m')^{ab} & \longrightarrow & G_{k'} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N^{ab, \text{geo}} & \longrightarrow & N^{ab} & \longrightarrow & G_{k'} & \longrightarrow & 0.
\end{array}
\]

In the above diagram, \( \pi_1(X', m')^{ab, \text{geo}} \) (resp. \( N^{ab, \text{geo}} \)) is the kernel of the surjective homomorphism \( \pi_1(X', m')^{ab} \to G_{k'} \) (resp. \( N^{ab} \to G_{k'} \)). Since \( \pi_1(X', m')^{ab} \to N^{ab} \) is surjective, we see that \( \pi_1(X', m')^{ab, \text{geo}} \to N^{ab, \text{geo}} \) is
surjective. Hence $N_{ab,\text{geo}}$ is finite by Corollary A.5. Thus there are only finitely many possibilities of $(\text{Ker}(\rho))_{\text{geo}}$ for each $N$, where $\text{Ker}(\rho)_{\text{geo}} := \text{Ker}(\rho) \cap N_{ab,\text{geo}}$. Note that $\text{Ker}(\rho)$ is the direct summand of $(\text{Ker}(\rho))_{\text{geo}}$ and $G_{k'}$. Hence there are only finitely many possibilities of open normal subgroups of $\pi_1(X, m)$ appearing as $\text{Ker}(\rho)$ for each $N$. This proves the assertion (ii). 

Next, assume the characteristic of $k$ is $p > 0$. In this case, H. Moon and Y. Taguchi proved the finiteness of mod $p$ Galois representations with solvable images over $k$ both for the number field case and for the function field case ([12], Th. 2, Th. 4 (ii)). For the function field case, the finiteness has been obtained in almost all the cases by G. Böckle and C. Khare (see [2]).

**Theorem 4.3.** Let $k$ be an algebraically closed field with characteristic $p > 0$.

(i) Let $X$ be a regular arithmetic scheme. For any positive integer $d$ and a modulus $m$ on $X$ there exist only finitely many isomorphism classes of semisimple continuous representations $\rho : \tilde{\pi}_1(X, m) \to \text{GL}_d(k)$ with solvable images.

(ii) Let $X$ be a regular variety, proper over a curve defined over a finite field $k$. For any positive integer $d$ and any modulus $m$ on $X$, there exist only finitely many isomorphism classes of semisimple continuous geometric representations $\rho : \pi_1(X, m) \to \text{GL}_d(k)$ with solvable images.

We use the finiteness of $\tilde{\pi}_1(X, m)/\tilde{\pi}_1(X, m)^{(r)}$ for an arithmetic scheme $X$, where $\tilde{\pi}_1(X, m)^{(r)}$ is the $r$-th commutator subgroup of $\tilde{\pi}_1(X, m)$ defined below.

**Definition 4.4.** For any topological group $G$ and non-negative integer $r$, define the $r$-th commutator subgroup $G^{(r)}$ of $G$ by the topological closure of the commutator subgroup $[G^{(r-1)}, G^{(r-1)}]$ in $G^{(r-1)}$ for $r \geq 1$, and $G^{(0)} := G$.

**Lemma 4.5.** Let $X$ be a regular arithmetic scheme. For any integer $r \geq 1$, $\tilde{\pi}_1(X, m)/\tilde{\pi}_1(X, m)^{(r)}$ is finite.

**Proof.** In the case of $r = 1$, the assertion follows from Corollary A.3. By induction on $r$, we may assume that $\tilde{\pi}_1(X, m)/\tilde{\pi}_1(X, m)^{(r-1)}$ is finite. Take a Galois cover $f : X' \to X$ corresponding to $\tilde{\pi}_1(X, m)^{(1)}$. We have a surjection $\tilde{\pi}_1(X', m') \to \tilde{\pi}_1(X, m)^{(1)}$ for the modulus $m' := f^*m$ on $X'$, and it induces a surjection

$$\tilde{\pi}_1(X', m')/\tilde{\pi}_1(X', m')^{(r-1)} \to \tilde{\pi}_1(X, m)^{(1)}/\tilde{\pi}_1(X, m)^{(r)}.$$ 

Thus the assertion follows from the hypothesis and the finiteness of $\tilde{\pi}_1(X, m)_{ab} = \tilde{\pi}_1(X, m)/\tilde{\pi}_1(X, m)^{(1)}$. 

\[12\]
Proof of Theorem 4.3. (i) For any group $G$, the solvability class of $G$ is the minimal integer $i \geq 0$ such that $G^{(i)} = 1$. Note that every solvable subgroup $G$ of $\text{GL}_d(k)$ has solvability class $\leq s = s(d)$, where $s := s(d)$ is a positive integer depending only on $d$ ([17], Chap. V, Sect. 20, Th. 8). Thus every continuous representation $\rho : \tilde{\pi}_1(X, m) \to \text{GL}_d(k)$ with solvable image factors through the quotient group of $\tilde{\pi}_1(X, m)$ by $\tilde{\pi}_1(X, m)^{(s)}$. Note that this quotient group is finite by Lemma 4.5. Thus the assertion (i) follows.

(ii) Let $\rho : \pi_1(X, m) \to \text{GL}_d(k)$ be a continuous geometric representation. By a theorem of Mal’cev-Kolchin (cf. [17], Chap. V, Sect. 19, Th. 7) and the structure of $\text{GL}_d(k)$ (cf. [11], Sect. 3), there exists a series of open normal subgroups $N_1 \supset N_2 \supset N_3 \supset \cdots \supset N_r = \ker(\rho)$ of $\pi_1(X, m)$ such that $(\pi_1(X, m) : N_i) \leq J'(d)$ and $N_i/N_{i+1}$ is abelian for each $1 \leq i \leq r - 1$. Here, $J'(d)$ is a positive integer depending only on $d$ and $r$ is bounded in terms of $d$. There are only finitely many possibilities of $N_1$ by the smallness of $\pi_1(X, m)$. Since $N_i/N_{i+1}$ is abelian for each $1 \leq i \leq r - 1$, we can prove by the same argument as in Theorem 4.2 (ii) that there are only finitely many possibilities of $N_{i+1}$ for each $N_i$. By induction on $i$, there are only finitely many possibilities of open normal subgroups of $\pi_1(X, m)$ appearing as $\ker(\rho)$. 

A Class Field Theory

We recall the class field theory of G. Wiesend [18] and restate it by using the fundamental groups $\pi_1(X, m)^{ab}$ defined in Section 3. We keep the notation of Section 3. Let $X$ be an arithmetic scheme, or a variety over a finite field $k$. We denote by $X_0$ the set of closed points of $X$. For any curve $C$ on $X$, let $C_\infty$ be the set of closed points $P$ in $\overline{C} \setminus \tilde{C}$ and the infinite places of $k(C)$ if $k(C)$ is a number field (cf. op. cit., Sect. 2).

For each curve $C$ on $X$, there is a natural map

$$k(C)^\times \to I_{\tilde{C}} := \bigoplus_{P \in \tilde{C}_0} \mathbb{Z} \oplus \bigoplus_{P \in C_\infty} k(C)_P^\times,$$

defined by the normalized valuations corresponding to $P \in \tilde{C}$ and the inclusions of $k(C)$ into its completions $k(C)_P$ for $P \in C_\infty$. Define the class group $\mathcal{C}_X$ associated with $X$ by the cokernel of the induced homomorphism

$$\bigoplus_{C \in X} k(C)^\times \to \bigoplus_{C \in X} I_{\tilde{C}} \to \bigoplus_{P \in X_0} \mathbb{Z} \oplus \bigoplus_{P \in C_\infty} k(C)_P^\times.$$
The topology of the direct sum and the quotient topology make this an Abelian topological group. A canonical continuous homomorphism
\[ \rho : C_X \rightarrow \pi_1(X)^{ab} \]
defined as follows (op. cit., Lem. 2): For each closed point \( P \in X_0 \), class field theory of finite fields gives \( \mathbb{Z} \rightarrow \hat{\mathbb{Z}} \simeq \pi_1(P) \rightarrow \pi_1(X)^{ab} \). For any curve \( C \) on \( X \) and \( P \in C_\infty \), we have a continuous homomorphism \( k(C)_P^\times \rightarrow \pi_1(\text{Spec}(k(C)_P))^{ab} \rightarrow \pi_1(X)^{ab} \) by local class field theory. Then global class field theory shows the homomorphism defined above factors through \( C_X \). Thus we have \( \rho : C_X \rightarrow \pi_1(X)^{ab} \).

Let \( m \) be a modulus on \( X \) (Def. 3.1). For each curve \( C \) on \( X \) and \( P \in \overline{C} \setminus \tilde{C} \), we denote by \( U^m_P \) the \( m_P \)-th higher unit subgroup \( 1 + m_P^{m_P} \) of \( k(C)_P \), where \( m_P \) is the maximal ideal corresponding to \( P \). We make the notational convention \( U^1_P := k(C)_P^\times \) and \( m := 1 \) for the archimedean local fields \( k(C)_P = \mathbb{R}, \mathbb{C} \).

**Definition A.1.** For a modulus \( m \) on \( X \), the ray class group \( C_X(m) \) with modulus \( m \) is the quotient of \( C_X \) by the images of
\[ U_X(m) := \bigoplus_{C \subseteq X} \bigoplus_{P \in C_\infty} U^m_P. \]
The group \( C_X(m) \) carries the quotient topology.

For each curve \( C \) on \( X \) and \( P \in \overline{C} \setminus \tilde{C} \), we have the following commutative diagram by local class field theory:
\[
\begin{array}{ccc}
U^m_P & \longrightarrow & (G_P^{ab})^{m_P} \\
\downarrow & & \downarrow \\
C_X & \longrightarrow & \pi_1(X)^{ab} \longrightarrow \pi_1(X,m)^{ab},
\end{array}
\]
where \( G_P := \text{Gal}(\overline{k(C)}_P/k(C)_P) \). We also consider the modified group \( \widetilde{\pi}_1(X,m)^{ab} \), the quotient of \( \pi_1(X,m)^{ab} \) which classifies those coverings in which, in addition, every \( \mathbb{R} \)-valued point of \( X \) splits completely. Note that, we have \( \widetilde{\pi}_1(X,m)^{ab} = \pi_1(X,m)^{ab} \) for a variety \( X \) over \( k \). By the very definition of \( \widetilde{\pi}_1(X,m) \), the composite \( C_X \xrightarrow{\rho} \pi_1(X)^{ab} \rightarrow \pi_1(X,m)^{ab} \) factors through \( C_X(m) \). Thus the map \( \rho : C_X \rightarrow \pi_1(X)^{ab} \) induces a natural homomorphism \( \rho_m : C_X(m) \rightarrow \pi_1(X,m)^{ab} \).

One of the main theorems in [18] is the following:
Theorem A.2 \(\text{(op. cit., Th. 1 (a))}\). Let \(X\) be a regular arithmetic scheme.

(i) The map \(\rho : \mathcal{C}_X \to \pi_1(X)_{\text{ab}}\) is surjective.

(ii) The kernel of \(\rho\) is the connected component \(\mathcal{C}_X^0\) of 0 in \(\mathcal{C}_X\).

(iii) Any closed subgroup of \(\mathcal{C}_X\) which contains \(\mathcal{C}_X^0\) is the pre-image of a closed subgroup of \(\pi_1(X)_{\text{ab}}\).

(iv) The map \(\rho\) induces a bijection between open subgroups of \(\pi_1(X)_{\text{ab}}\) and those of \(\mathcal{C}_X\).

Corollary A.3. The map \(\rho_m : \mathcal{C}_X(m) \to \tilde{\pi}_1(X, m)_{\text{ab}}\) is an isomorphism of finite abelian groups.

Proof. Let \(N\) be a closed subgroup of \(\pi_1(X)_{\text{ab}}\) corresponding to the image \(H\) of \(\mathcal{U}_X(m)\) in \(\mathcal{C}_X/\mathcal{C}_X^0\) by Theorem \(\text{A.2}\) (iii). We have the following commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & H & \longrightarrow & \mathcal{C}_X/\mathcal{C}_X^0 & \longrightarrow & \mathcal{C}_X(m) & \longrightarrow & 0 \\
\downarrow & & \downarrow \bar{\rho} & & \downarrow \rho_m & & \\
0 & \longrightarrow & N & \longrightarrow & \pi_1(X)_{\text{ab}} & \longrightarrow & \tilde{\pi}_1(X, m)_{\text{ab}} & \longrightarrow & 0,
\end{array}
\]

where \(\bar{\rho}\) is induced by \(\rho\). Theorem \(\text{A.2}\) says that \(\bar{\rho}\) is bijective. Thus, the induced map \(\rho_m\) is also bijective. To prove the finiteness of these groups, we may take a sufficiently large modulus \(m\). Then \(\mathcal{C}_X(m)\) carries the discrete topology. Since the compact group \(\tilde{\pi}_1(X, m)_{\text{ab}}\) equals the discrete group \(\mathcal{C}_X(m)\), both are finite.

For the geometric case, we need to assume that the variety is proper over a curve which is defined over a finite field. However, the classical theory of curves over a finite field is naturally included.

Theorem A.4 \(\text{(op. cit., Th. 1, (c))}\). Let \(X\) be a regular variety, proper over a curve which is defined over \(k\).

(i) The image of the map \(\rho : \mathcal{C}_X \to \pi_1(X)_{\text{ab}}\) consists of those elements of \(\pi_1(X)_{\text{ab}}\) whose images in \(\pi_1(\text{Spec}(k)) = G_k\) are integral powers of the Frobenius.

(ii) The kernel of \(\rho\) is the connected component \(\mathcal{C}_X^0\) of 0 in \(\mathcal{C}_X\).

(iii) Any closed subgroup of \(\mathcal{C}_X\) which contains \(\mathcal{C}_X^0\) is the pre-image of a closed subgroup of \(\pi_1(X)_{\text{ab}}\).

(iv) The map \(\rho\) induces a bijection between open subgroups of \(\pi_1(X)_{\text{ab}}\) and those of \(\mathcal{C}_X\) which have nontrivial image in \(\mathcal{C}_{\text{Spec}(k)} = \mathbb{Z}\).
Corollary A.5. (i) The map $\rho_m : C_X(m) \to \pi_1(X, m)^{ab}$ is injective and has dense image.
(ii) The kernel $\pi_1(X, m)^{ab, \text{geo}}$ of $\pi_1(X, m)^{ab} \to \pi_1(\text{Spec}(k))^{ab} \simeq \hat{\mathbb{Z}}$ is finite.

Proof. (i) We can prove the assertion by the same manner as in the proof of Corollary A.3.
(ii) By the proof of Theorem A.4, we have the following commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & C_X^{\text{geo}} & \longrightarrow & C_X & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
& & \rho^{\text{geo}} & \downarrow & \rho & & & \rho_m & \downarrow & \rho_m & & \rho_m & \longrightarrow & \hat{\mathbb{Z}} & \longrightarrow & 0,
\end{array}
$$

where $C_X^{\text{geo}}$ and $\pi_1(X)^{ab, \text{geo}}$ are defined by the exactness of the corresponding rows and $\rho^{\text{geo}}$ is surjective. Thus, we have

$$
\begin{array}{cccccc}
0 & \longrightarrow & C_X(m)^{\text{geo}} & \longrightarrow & C_X(m) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
& & \rho_m^{\text{geo}} & \downarrow & \rho_m & & & \rho_m & & \rho_m & & \rho_m & & \hat{\mathbb{Z}} & \longrightarrow & 0
\end{array}
$$

with a bijection $\rho_m^{\text{geo}}$. Since the compact subgroup $\pi_1(X, m)^{ab, \text{geo}}$ equals the discrete subgroup $C_X(m)^{\text{geo}}$, both are finite.

Remark A.6. We can show that the finiteness of $\pi_1(X, m)^{ab, \text{geo}}$ for a normal variety $X$ over a finite field without the assumption “$X$ is proper over a curve”. In fact, it is sufficient to show the surjectivity of $\rho_m^{\text{geo}} : C_X(m)^{\text{geo}} \to \pi_1(X, m)^{ab, \text{geo}}$. This fact is reduced to the density theorem of S. Lang and the finiteness of $\pi_1(X)^{ab, \text{geo}}$ as follows. We have the following commutative diagram

$$
\begin{array}{cccccc}
\bigoplus_{P \in X_0} \mathbb{Z} & \longrightarrow & C_X(m) & \longrightarrow & C_X(m) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
& & \rho & & \rho & & \rho_m & & \rho_m & & \rho_m & & \hat{\mathbb{Z}} & \longrightarrow & 0
\end{array}
$$

where the left vertical map is defined by mapping 1 to the Frobenius element at $P \in X_0$. By approximation, the top horizontal homomorphism is surjective and $\rho$ has dense image. Thus, $\rho_m$ also has dense image. Consider the
following commutative diagram:

\[
\bigoplus_{C \subseteq X \ P \in C} U_0^P / U_m^P \longrightarrow \mathcal{C}_X(m)^\text{geo} \longrightarrow \mathcal{C}_X(0)^\text{geo} \longrightarrow 0 \\
\bigoplus_{C \subseteq X \ P \in C} (G_P^\text{ab})^0 / (G_P^\text{ab})_m^P \longrightarrow \pi_1(X, m)^{ab, \text{geo}} \longrightarrow \pi_1(X, 0)^{ab, \text{geo}} \longrightarrow 0.
\]

Since \(\pi_1(X, 0)^{ab, \text{geo}}\) is finite ([15], Th. 7.4) the map \(\rho_0^\text{geo}\) is surjective and \(\rho_P\) is bijective by local class field theory. Thus, \(\rho_m^\text{geo}\) is surjective.

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Shinya Harada
Graduate School of Mathematics
Kyushu University
6-10-1, Hakozaki, Higashiku, Fukuoka-city, 812-8581 Japan
JSPS Research Fellow,
s.harada@math.kyushu-u.ac.jp

Toshiro Hiranouchi
Graduate School of Mathematics
Kyushu University
6-10-1, Hakozaki, Higashiku, Fukuoka-city, 812-8581 Japan
JSPS Research Fellow,
hiranouchi@math.kyushu-u.ac.jp