Geometrical theory of balance systems and the entropy principle

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Abstract. In this work we present the theory of balance equations of Continuum Thermodynamics (balance systems) in a geometrical form using the Poincare-Cartan formalism of Multisymplectic Field Theory. A constitutive relation $C$ of a balance system $B_C$ is realized as a mapping between a (partial) 1-jet bundle of the configurational bundle $\pi: Y \to X$ and the dual bundle similar to the Legendre mapping of the Lagrangian Field Theory. Invariant (variational) form of the balance system $B_C$ is presented in three different forms and the space of admissible variations is defined and studied. Action of automorphisms of the bundle $\pi$ on the constitutive mappings $C$ is studied and it is shown that the symmetry group $\text{Sym}(C)$ of the constitutive relation $C$ acts on the space of solutions of balance system $B_C$. A version of the Noether Theorem for an action of a symmetry group of vertical automorphisms is presented with the usage of conventional multimomentum mapping. The “entropy principle” of Irreducible Thermodynamics (the requirement that the entropy balance should be the consequence of the balance system) is studied for general balance systems. The structure of corresponding (secondary) balance laws of a balance system $B_C$ is studied for the cases of different (partial) 1-jet bundles as domains of constitutive laws. Corresponding results may be considered as a generalization of the transition to the dual, Lagrange-Liu picture of Rational Extended Thermodynamics (RET).

1. Introduction.
This paper is the first part of a work where we present the theory of balance equations of Continuum Thermodynamics (balance systems) in a geometrical form using the Poincare-Cartan formalism of Multisymplectic Field Theory. A constitutive relation $C$ of a balance system $B_C$ is realized as a mapping between a (partial) 1-jet bundle of the configurational bundle $\pi: Y \to X$ and the dual bundle similar to the Legendre mapping of the Lagrangian Field Theory. Invariant (variational) form of the balance system $B_C$ is presented in three different forms and the space of admissible variations is defined and studied. Action of automorphisms of the bundle $\pi$ on the constitutive mappings $C$ is studied and it is shown that the symmetry group $\text{Sym}(C)$ of the constitutive relation $C$ acts on the space of solutions of balance system $B_C$. A version of the Noether Theorem for an action of a symmetry group of vertical automorphisms is presented with the usage of conventional multimomentum mapping. The “entropy principle” of Irreducible Thermodynamics (the requirement that the entropy balance should be the consequence of the balance system) is studied for general balance systems. The structure of corresponding (secondary) balance laws of a balance system $B_C$ is studied for the cases of different (partial) 1-jet bundles as domains of constitutive laws. Corresponding results may be considered as a generalization of the transition to the dual, Lagrange-Liu picture of Rational Extended Thermodynamics (RET).
case. On the other end there is the Rational Extended Thermodynamics where all the necessary derivatives of the basic fields are incorporated into the state space and the constitutive relations depend on the fields but not on their derivatives ([20, 21]). In between these two extreme positions there is a variety of situations where some derivatives of basic fields are included in the space state and some are not [10]. Quite often the choice of the derivatives (gradients of basic fields or their time derivatives) is related to the symmetry group of the described physical situation or to the covariance group required for the system of balance equations.

This work began at the Thermoconn 2005 conference in Messina after the lecture of Professor T. Ruggeri on Rational Extended Thermodynamics and the discussion I had there with Professor W. Muschik about the Entropy Principle.

After introducing some notations and settings in Section 2, in Section 3 we recall the basic structures of Multisymplectic Field Theory following [5, 14]. The only new material here is subsection 3.5 on the vertical contact structure in the space structures of Multisymplectic Field Theory following [5, 14]. The only new material here is subsection 3.5 on the vertical contact structure in the space $W_0$ of the united multisymplectic scheme and the characterization of Legendre mappings generated by the Lagrangians in terms of this structure.

In section 4 we define the partial 1-jet bundles $Z_p = J^1_p(\pi)$ for a configurational bundle $\pi : Y^{(n+1)+m} \to X^{n+1}$ of $m$ basic fields $y^i \in U$ over the physical or material space-time $X^{n+1}$. We discuss two examples of such jet bundles. One, $J^1_K(\pi)$, defined by a distribution $K \subset T(X)$ (or, with more details, by an almost product structure $T(X) = K \oplus K'$) on the space $X$, another, $J^1_K(\pi)$, for a case where $K \oplus K' = \langle \partial_t > \ominus T(B)$, $B$ being the material or physical space and the space $U$ of the basic fields splits as the product corresponding to the type of (first order) derivatives that enters the constitutive relations. In particular we define and study the partial Cartan structure on the 1-jet bundles $J^1_p(\pi)$ of these two types. A study of more general types of partial jet-bundles including the jet bundles of higher order will be pursued in the second part of this work.

In section 5 we study the prolongation of vector fields to the partial 1-jet bundles following similar prolongation procedures for conventional 1-jet bundles ([11, 14, 30]).

In Section 6 we define a general constitutive relation $C$ as a smooth mapping between the partial 1-jet bundle $J^1(\pi)$ and the (total) dual space $\tilde{Z} = \Lambda_2^{(n+1)+(n+2)}Y / \Lambda_1^{(n+1)+(n+2)}Y$

\[ C(x^\mu, y^i, z^i) = (x^\mu, y^i, F^\mu_i(x^\mu, y^i, z^i); \Pi_i(x^\mu, y^i, z^i)), \]

containing the current part $F^\mu_i dy^i \wedge \eta_\mu$ and the source part $\Pi_i dy^i \wedge \eta$. We introduce the covering constitutive relation $\tilde{C}$ defined by $C$, extending the Legendre transformations defined by a Lagrangian form $L \eta$. We define the Poincare-Cartan form $\Theta_{C,\nu} = F^\mu_i dy^i \wedge \eta_\mu + \Pi_i dy^i \wedge \eta$ of a constitutive relation $C$, the Poincare-Cartan form $\Theta_{\tilde{C}}$ of a covering relation $\tilde{C}$ and give several examples of types of constitutive relations: Lagrange Type $CL$, mixed type with a Lagrangian current part and the source term given by a dissipative potential $(L + D$ type), and vector-potential type.

In Section 7 we discuss three variational ways to get to the balance system $B_C$ corresponding to a constitutive relation $C$ (i.e. using vertical variations $\xi$ and the differential of the Poincare-Cartan form $\Theta_{C,\nu}$). In doing this a traditional way, i.e. requiring that $j^1(s)^* d \xi^i \Theta_C = 0$ or $j^1(s)^* i_{\xi^i} d \Theta_{\tilde{C}} = 0$ we have to put the condition $F^\mu_i \xi^i = 0$ on the vertical variations $\xi$ of the Poincare-Cartan form. Locally there are always enough of such $C$-admissible variations $\xi$ to separate balance equations (but globally this may not be true). That is why we present the third way, using the restricted horizontal differential $\hat{d}$ (see Appendix C) instead of conventional de-Rham differential $d$ for the invariant formulation of a balance system. In this case there are no restriction to the variations $\xi$. For a Lagrangian constitutive relation $CL$ the balance system coincides with the Euler-Lagrange system of equations defined by the Lagrangian $L$.
In Section 8 we discuss the action of extended geometrical (lifted from Y) transformations on the constitutive relations C and on the corresponding Poincare-Cartan form ΘC. We define the symmetry group Sym(C) of a constitutive relation C and prove that this symmetry group acts on the space of solutions Sol(BC) of the balance system BC. Using a connection ν in the configurational bundle π : Y → X we define the ν-homogeneous constitutive relations corresponding to the situation where C depends on the fields and their derivatives but not on the points of space-time X explicitly.

In Section 9 we prove the Noether Theorem for a balance system BC under an action of a (vertical) symmetry Lie group G ⊂ Aut(π) using the multimomentum mapping of a multisymplectic field theory [15, 19]. The Noether Theorem leads to the family of the balance equations which reduces to the conservation laws for special (or absent) source terms.

In Sections 10-12 we study the “entropy principle” for a balance system BC - the conditions on the constitutive relation C that allows the “secondary balance laws” - balance laws satisfied by any solution of the balance system BC. We characterize these conditions in terms of Lagrange-Liu multipliers λi generalizing the scheme of RET theory [20]. We consider the RET balance systems in the case of functionally independent LL-multipliers. Then we study the case of the full 1-jet bundle J1(π). We prove that the secondary balance laws correspond to the total or partial vector-potential type of the representation of the balance system BC. Other cases and an example of a balance system with two fields yi in a (1+1) space-time is studied in [29]. In the general situation, the (entropy) density h0 defining the secondary balance laws in the first case is a solution of the cyclic D-module of a very special form (Theorem 9).

In Section 13 we present the dual bundle picture of the RET balance systems in terms of LL-multipliers λi(x). We prove that the fulfillment of the entropy principle here reduces to the holonomy of the total constitutive section of the 1-jet bundle J1(Λ, Ω3(X)) of the Lagrange-Liu fields λi with values in the space of semi-basic 3-forms. In Appendices A-C we collect the information on the properties of partial volume forms ηµ, used in the text, the definition of the Iglesias differential [9] and definitions and principal properties of the (extended) horizontal differential d.

2. Settings.
In this section we present the bundle formulation of classical field theory used in the rest of this article. For deeper exposition we refer to the monographs [1, 6].

2.1. Space-time base manifold.
A state of a material body will be described by the collection of the fields {yi, i = 1,..., m} defined in a domain X = B × Rl ⊂ Rn+l of the physical or material space-time Rn+l with the boundary ∂X (or without a boundary). Here B ⊂ Rn is a domain in the n-dim physical or material (reference) space. We assume that the pseudo-Riemannian metric G is defined in X up to the boundary. An example of such a metric is the Euclidian metric G = dt2 + h, h being the canonical Euclidian metric in the physical space Rn+1, but in order to apply our scheme to a material manifold or relativistic systems we prefer to keep G more general. We will use local coordinates xµ, µ = 1, 2,..., n and the time variable t = x0 in X. We will be using Greek indices for the space-time variables and large Latin indices for space variables only.

Denote by η the volume n-form η = √|G|dx0 ∧ dx2 ... ∧ dxn corresponding to the metric G. We will be using the n-forms ηµ = i∂µη, µ = 0, 1, 2,..., n, for instance η0 = √|G|dx1 ∧ dx2 ... ∧ dxn.

2.2. Configurational (state) bundle.
The basic fields of a continuum thermodynamical theory yi (except for the entropy to be included later) take values in the space U ⊂ Rm which we will call, a basic state space of the system.
the exterior forms: (n+1)-form of the flows $F$

As is customary in the classical field theory (see [1, 6]) we organize these fields in the bundle

$$\pi_U : Y \to X, \quad X = \mathbb{R}_t \times \bar{B}, \quad Y = X \times U$$

with the base $X$ being the cylinder $\mathbb{R} \times \bar{B}$ and the fiber $U$.

To formulate balance equations in terms of exterior forms we will use the space of $n+(n+1)$-exterior forms in $X = \mathbb{R}^{n+1} : \Lambda^{n+(n+1)}(X) = \Lambda^n(X) \oplus \Lambda^{n+1}(X)$. This space has as its basis elements $\eta_\mu, \mu = 0, 1, 2, \ldots, n; \eta$ and is the space of smooth sections of the bundle of exterior forms of orders $n$ and $n+1$ over $X$.

2.3. Balance Equations.
Fields $y^i$ are to be determined as solutions of the field equations having the form of balance equations for the currents $F^\mu_{i,\mu}$, (where often $F^0_i = y^i$)

$$F^\mu_{i,\mu} = F^0_{i,x} + F^A_{i,x,A} = \Pi_i, \quad i = 1, \ldots, m.$$ (1)

Here $\Pi_i(x^\mu, y^i, y^i_{x\nu})$ is called the production and source of the component $y^i$, $F^0_i(x^\mu, y^i, y^i_{x\nu})$ is the density of the $i$-th balance law and $\sum_{A=1}^{n} F^A_i(x^\mu, y^i, y^i_{x\nu}) \frac{\partial}{\partial x^A} - \text{the flow}$ of the component $y^i$. These quantities are assumed to be functions of the of the point $x^\mu \in X$, fields $y^i$ and, possibly, their first derivatives $y^i_{x\nu}$ (recall that we consider here only first order balance laws).

In Rational Extended Thermodynamics all the derivatives of the basic fields are included in the space $U$ of variables $y^i$ and, thus, coefficients $F^\mu_{i,\mu}$ are functions of $x^\mu$ and $y^i$ only.

**Remark 1** In the conventional RET [20, 21] it is assumed that the coefficients of the balance laws (2) and (5) depend on the fields $y^i$ only. Here we use the more broad definition of RET balance systems allowing dependence on the point $x^\mu$.

As is customary in the classical field theory, the balance laws could be rewritten by introducing the exterior forms: $(n+1)$-form of the flows $F_i = F^\mu_i \eta_\mu, \quad i = 1, \ldots, m$, and the $(n+2)$-form of the production and source $\Pi_i = \Pi_i \eta$. Then the balance laws (1) take the form

$$dF_i = \Pi_i, \quad i = 1, \ldots, m,$$ (2)

**Example 1** As an example consider a system with 5 basic fields: mass density $\rho$, linear momentum density $\rho v^i$ and the total energy density $\epsilon = e + \frac{1}{2}v^2$ (sum of internal and kinetic energy per unit of mass). To each of them there corresponds the form

$$\begin{align*}
F_\rho &= \rho \eta_0 + \rho v^\mu \eta_\mu, \\
F_{\rho v^\nu} &= \rho v^\nu \eta_0 + (\rho v^\nu v^\mu - \epsilon^\mu v^\nu) \eta_\mu, \\
F_\epsilon &= (e + \frac{1}{2}v^2) \eta_0 + [(e + \frac{1}{2}v^2)v^\mu - \epsilon^\nu v^\mu + q^\nu] \eta_\mu.
\end{align*}$$ (3)

The production term is zero for $F_\rho$, equal to the density of body forces for $F_{\rho v^\nu}$ and is zero for the energy balance law. One can introduce internal energy $\epsilon$ as the basic variable. In this case

$$F_\epsilon = e \eta_0 + (ev^\mu + q^\mu) \eta_\mu, \quad \Pi_\epsilon = \eta_\mu \frac{\partial v^\nu}{\partial x^\mu} \eta,$$

see [20], Sec.5.3.
2.4. Entropy condition.
Entropy density $h^0$, entropy flux $h^A$, $A = 1, 2, \ldots, n$ and the entropy production $\Sigma$ are assumed to be a functions of the the same variables $x^\mu, y^i, y^i_{,\mu}$ as the coefficients of the balance laws (2). Entropy satisfies to the balance law
\[
d(h^\mu \eta_\mu) = \Sigma, \tag{4}
\]
with the positive production 4-form
\[
\Sigma = \Sigma(x^\mu, y^i, y^i_{,\mu}) \eta, \sigma \geq 0, \tag{5}
\]

The entropy principle requires that any solution of the balance equations (2) would also satisfy equation (5) and that the production $\sigma$ of entropy (in the system) should be non-negative.

To close system of equations (2) (or (2+5)) for $y^i$ one has to choose the constitutive relation $C$ of the thermodynamical system, i.e. to choose the flows and production forms as functions of $x^\mu, y^i$ and the appropriate derivatives of fields $y^i$. In particular, one have to choose the domain of the constitutive relation which is typically the full or partial jet-bundle of the configurational bundle $\pi$ of dynamical variables. By definition, the Rational Extended Thermodynamics is the zero order theory in that the domain of its constitutive relation is the space $Y$. In this article we consider the cases of constitutive relations of zero and first order only. Constitutive relations depend also on the background fields (metric $G$ in $X$ or a connection $\nu$ in the bundle $\pi: Y \to X$ in this paper). As we will see below, utilizing the entropy condition allows to reduce this process to the choice of entropy flow (n+1)-form and to the choice of production (n+2)-forms subject to the positivity condition.

3. Multisymplectic Field Theory.
3.1. The 1-jet bundle
Given a frame bundle $\pi: Y \to X$ we say that two sections $s, s' : U \to Y$ defined in a neighborhood $U$ of a point $x \in X$ define the same 1-jet $j^1s(x)$ if $s(x) = s'(x)$, $s_{sx} = s_{sx} : T_x(X) \to T_{s(x)}(Y)$. This defines an equivalence relation on the ... Space of equivalence classes of such local sections is defined $J^1_{\pi}$. The total space $J^1(\pi) = \bigcup_{x \in X} J^1_{\pi}(x)$ can be endowed with a smooth structure such that the mappings $J^1(\pi) \to Y \to X$ are fibrations. The fibration $\pi_{10} : J^1(\pi) \to Y$ is the affine bundle modeled in the vector bundle $\pi^*(T^*(X)) \otimes V(\pi)$, where $V(\pi) \subset T(Y)$ is the vertical subbundle of the bundle $\pi$.

Let $(x^\mu, y^i; \mu = 1, \ldots, n + 1 = \text{dim}(X); i = 1, \ldots, m)$ be an adopted local coordinate system in $Y$. Then the local coordinate system $(x^\mu, y^i, z^i_{,\mu}; \mu = 1, \ldots, n; i = 1, \ldots, m)$ can be defined in $J^1(\pi)$ by the condition
\[
z^i_{,\mu}(j^1s) = \frac{\partial y^i}{\partial x^\mu}(x).
\]

3.2. Lagrangian picture: Poincaré-Cartan Form.
Here we recall briefly (following [14, 5]), the Poincare-Cartan formalism of the multisymplectic field theory.

The volume form $\eta$ permits us to construct the vertical endomorphism
\[
S_\eta = (dy^i - z^i_{,\mu} dx^\mu) \wedge \eta_\nu \otimes \frac{\partial}{\partial z^\mu}\n
\]
which is a tensor field of type $(1, n+1)$ on the 1-jet bundle space $Z = J^1(\pi)$ of the configurational bundle $\pi: Y \to X$. 

For a Lagrangian \((n+1)\)-form \(L\eta\), \(L\) being a (smooth) function on the manifold \(Z = J^1(\pi)\) the Poincaré-Cartan \((n + 1)\) and \((n + 2)\)-forms are defined as follows:

\[
\Theta_L = L\eta + S^n_\eta(dL), \quad \Omega_L = -d\Theta_L,
\]

where \(S^n_\eta\) is the adjoint operator of \(S_\eta\). In coordinates we have

\[
\Theta_L = (L - z^i_\mu \frac{\partial L}{\partial z^i_\mu})\eta + \frac{\partial L}{\partial y^i} \eta_i \wedge \eta, \quad (7)
\]

Recall ([14]) that the couple \((Z, \Omega_L)\) is a multisymplectic manifold provided the Lagrangian \(L\) is regular, i.e. the matrix \(L_{z^i_\mu z^j_\nu}\) is nondegenerate.

An extremal of \(L\) is a section of \(\pi_{XY}\) such that for any vector field \(\xi_Z\) on \(Z\),

\[
(j^1\phi)^*(i_{\xi_Z} d\Theta_L) = 0,
\]

where \(j^1\phi\) is the first jet prolongation of \(\phi\).

A section \(\phi\) is an extremal of \(L\) if and only if it satisfies the Euler-Lagrange Equation (see, for instance, [1, 6])

\[
(j^1\phi)^* \left( \frac{\partial (L\sqrt{|G|})}{\partial y^i} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial (L\sqrt{|G|})}{\partial z^i_\mu} \right) \right) = 0, \quad 1 \leq i \leq m.
\]

3.3. Hamiltonian picture of the MS Field Theory.

In this section we present the dual, Hamiltonian Formulation of Elasticity Theory.

3.3.1. Canonical multisymplectic bundles \(\Lambda^k Y\). Denote by \(V(Y) \to Y\) the subbundle of vertical tangent vectors of the tangent bundle \(T(Y)\). Following [14] let \(\Lambda^k_0 Y\) denote the subbundle of the vector bundle \(\Lambda^k Y\) of exterior \(k\)-forms on \(Y\) consisting of those forms that vanish when \(r\) of their arguments are vertical (with respect to the fibration \(\pi_{XY}: Y \to X\))

\[
\Lambda^k_0 Y = \{ \sigma \in \Lambda^k Y \mid i_{\xi_1} \ldots i_{\xi_k} \sigma = 0, \forall \xi_i \in V(\pi). \}
\]

Later we will use it for \(k = n, n + 1; r = 1, 2\), or, more specifically, for \(k = 4, 5\).

Elements of the space \(\Lambda^n_0 Y\) are semibasic \(n\)-forms locally expressed as \(p(x, y)\eta\). Elements of the space \(\Lambda^n_0 Y\) have, in local adapted coordinates \((x^\mu, y^i)\) the form \(p(x, y)\eta + p_i^\mu dy^i \wedge \eta_\mu\).

This introduces coordinates \((x^\mu, y^i, p)\) on the manifold \(\Lambda^n_0 Y\) and \((x^\mu, y^i, p, p_i^\mu)\) on the manifold \(\Lambda^n_2 Y\).

The manifold \(\Lambda^k Y\) carries a canonical \(k\)-form \(\Theta_0^k\) define as follows:

\[
\Theta_0(\omega)(\xi_1, \ldots \xi_k) = \omega(\pi_{\Lambda^k}(\omega))((\pi_{\Lambda^k} \ast_1 (\xi_1), \ldots \pi_{\Lambda^k} \ast_1 (\xi_k)), \quad (9)
\]

where \(\omega \in \Lambda^k Y\), \(\xi_i \in T_\omega(\Lambda^k Y)\), and \(\pi_{\Lambda^k} : \Lambda^k Y \to Y\) is the canonical bundle projection.

By restriction, this form induces a \(k\)-form \(\Theta_0^k\) on the manifold \(\Lambda^n_0 Y\).

The local expressions in the case \(k = n + 1, r = 2\) are

\[
\Theta_0^{n+1} = p\eta + p_i^\mu dy^i \wedge \eta_\mu, \quad (10)
\]
3.3.2. Dual bundles: Hamiltonian systems. Basic for the Hamiltonian form of multisymplectic field theory are the following bundles: $\Lambda_2^{n+1}Y$ and its factor bundle over $Y$

$$Z^* = \Lambda_2^{n+1}Y / \Lambda_1^{n+1}Y, \; q : \Lambda_2^{n+1}Y \to Z^*.$$ 

The bundle $\Lambda_2^{n+1}(Y) \to Y$ can be identified with the affine dual to the bundle $\pi_1^1 : J^1(\pi) \to Y$ (see [19]).

A Hamiltonian is, in this approach, a section $h$ of the projection $q$. Having it available, we define $\Theta_h = h^*\Theta_2^{n+1}$.

A section $\sigma : X \to Z^*$ is said to satisfy the Hamilton equation (for a given Hamiltonian $h$) if

$$\sigma^*(\iota_\xi d\Theta_h) = 0,$$

for all vector fields $\xi$ on $Z^*$. In local coordinates $(x^\mu, y^i, z^j_\mu)$ a Hamiltonian $h$ is represented by a local function $H$:

$$p = -H(x^\mu, y^i, z^j_\mu)$$

and the Hamilton equations for a section $\sigma = (x^\mu, \sigma^i(x), \sigma^j_\mu(x))$ take the form:

$$\begin{align*}
\frac{\partial \sigma^i}{\partial x^\mu} &= \frac{\partial H}{\partial \sigma^j_\mu}, \\
\text{div}_G(p^\mu_i) &= \sum_\mu \left| \frac{\partial \sigma^\mu_i}{\partial x^\mu} + \sigma^\mu_i \frac{\partial \ln(\sqrt{|G|})}{\partial x^\mu} \right| = -\frac{\partial H}{\partial y^i}.
\end{align*}$$

In difference to the bundle $\Lambda^{n+1}(Y)$ the bundle $Z^*$ does not have a canonically defined form of the Poincare-Cartan type (see for instance, Sec. below where it is shown that under the transformation induced by an automorphism of the bundle $\pi$ the (locally defined) form $\Theta^*_{\pi}$ is not defined canonically, but its class $mod(\Lambda_1^{n+1}(Z^*))$ is. Taking $mod(\Lambda_1^{n+1}(Z^*))$ we get canonically defined element of the bundle $\Lambda_2^{n+1}Z^*/\Lambda_1^{n+1}Z^*$ on $Z^*$.

One may consider this class as defining the canonical $V^*(\pi)$-valued semi-basic n-form on $Y$.

Below we will see that such interpretation is sufficient for the separating components of a balance system to the individual balance laws with the help of independent vertical variations.

Recall (see [3]) that given an Ehresmann connection $\nu : Y \to J^1(\pi)$ on the bundle $\pi$ with the vertical projector

$$P_v = \partial y^i \otimes (dy^i + \Gamma^i_\mu dx^\mu),$$

defines naturally the linear section $\delta_\nu : Z^* \to \Lambda_2^{n+1}Y$ given by

$$\delta_\nu(F^\mu_i dy^i \wedge \eta_\mu) = (F^\mu_i \Gamma^i_\mu)\eta + F^\mu_i dy^i \wedge \eta_\mu.$$  \hspace{1cm} (11)

Section $\delta_\nu$ defines the pullback of the form $\Theta_2^{n+1}$:

$$\delta_\nu^*\Theta_2^{n+1} = (F^\mu_i \Gamma^i_\mu)\eta + F^\mu_i dy^i \wedge \eta_\mu.$$ \hspace{1cm} (12)

Form $\Theta_\nu = \delta_\nu^*\Theta_2^{n+1}$ is defined correctly on the manifold $Z^*$. 

3.3.3. Bundle $\tilde{Z}$ for the balance systems. To present the system of balance laws in the multisymplectic form we will need to use the vector bundles $\Lambda_i^{(n+1)+(n+2)} Y = \Lambda_i^{n+1} Y \oplus \Lambda_i^{n+2} Y$, where $i = 1, 2$ and the vector bundle

$$\tilde{Z} = \Lambda_2^{(n+1)+(n+2)} Y / \Lambda_1^{(n+1)+(n+2)} Y = \Lambda_2^{n+1} / \Lambda_1^{n+1} \times Y \Lambda_2^{n+2} / \Lambda_1^{n+2}. \quad (13)$$

Locally, elements of the factor bundle $\tilde{Z}$ can be presented in the form $p^i dy^i \wedge \eta_i + q_i dy^i \wedge \eta$.

Exterior forms $\Theta^k_{i,\nu}$ for $k = n + 1, n + 2; i = 1, 2$ induce on the bundle $\mathcal{Z}$ the $(n+1) + (n+2)$ form

$$\tilde{\Theta}_\nu = p^i dy^i \wedge \eta_i + q_i dy^i \wedge \eta \quad (14)$$

where $n + 1$ and $n + 2$ components of this form are lifted from the canonical forms on the components $\Lambda_2^k Y / \Lambda_1^k Y$ for $k = n + 1, n + 2$.

3.4. Legendre Transformation.

Let $L$ be a Lagrangian function. We define the fiber mapping over $Y$ leg$_L : Z \to \Lambda_2^{n+1} Y$, as follows:

$$\text{leg}_L(j^1_{\phi}(X_1, \ldots, X_{n+1})) = (\Theta_L)_{j^1_{\phi}(X_1, \ldots, X_{n+1})},$$

where $j^1_{\phi} \in Z, X_i \in T_{\phi(x)} Y$ and $X_i \in T_{j^1_{\phi}(X)} Z$ are such that $\pi_{YZ} \circ \tilde{X}_i = X_i$.

Notice that addition of a constant to the Lagrangian leads to the constant shift (in $p$) of the image of Legendre mapping in $\Lambda_2^{n+1} Y$ (which is a submanifold of codimension 1 if the Lagrangian is regular). Thus, the space $\Lambda_2^{n+1} Y$ is foliated by these shifts.

In local coordinates, we have

$$\text{leg}_L(x^\mu, y^i, z^i_\mu) = (x^\mu, y^i, p = L - z^i_\mu \frac{\partial L}{\partial z^i_\mu}, p^i = \frac{\partial L}{\partial \dot{y}^i}).$$

The Legendre transformation $\text{Leg}_L : Z \to Z^*$ is defined as the composition $\text{Leg}_L = q \circ \text{leg}_L$. Locally

$$\text{Leg}_L(x^\mu, y^i, z^i_\mu) = (x^\mu, y^i, p^i = \frac{\partial L}{\partial \dot{y}^i}).$$

Recall [14] that the Legendre transformation $\text{Leg}_L : Z \to Z^*$ is a local diffeomorphism if and only if $L$ is regular.

If, in addition, the Lagrangian $L$ is hyperregular (i.e. if $\text{Leg}_L$ is a global diffeomorphism), one can define a Hamiltonian $h : Z^* \to \Lambda_2^{n+1} Y$ by setting $h = \text{leg}_L \circ \text{Leg}_L^{-1}$. Then

$$\text{Leg}_L^* \Theta_h = \Theta_L, \text{ Leg}_L^* \Omega_h = \Omega_L.$$**

In this case $\text{Leg}_L : (Z, \Omega_L) \to (Z^*, \Omega_h)$ is a multisymplectomorphism.

3.5. Unified formalism.

In this subsection we describe the unified geometrical setting of a classical field theory developed in [14, 4] as the generalization of Skinner-Rusk (Dirac) geometrical mechanics.

Introduce the fiber product $W_0 = Z \times_Y \Lambda_2^{n+1} Y$ with canonical projectors $p_i$ to the $i$-th factor. We consider canonical coordinates $(x^\mu, y^i, z^i_\mu, p, p^i_\mu)$ on $W_0$.

Define the $(n+1)$-forms $\Theta = p_i^* \Theta_{n+1}^i$ and $\Theta^L = p_i^* \Theta_L$, and the corresponding $(n+2)$-forms $\Omega = d\Theta, \Omega^L = d\Theta^L$. In addition, we introduce the differences $\Theta^* = \Theta^L - \Theta, \Omega^* = \Omega^L - \Omega$. In local coordinates we have

$$\Theta^* = \left[ L - z^i_\mu \frac{\partial L}{\partial z^i_\mu} - p \right] \eta + \left[ \frac{\partial L}{\partial \dot{y}^i} - p^i_\mu \right] dy^i \wedge \eta_i.$$**
Define the submanifold $W_1 \subset W_0$ as the graph of the Legendre mapping $\overline{L}: Z \to \Lambda_2^{n+1} Y$, see [4] and [14]. Locally it is given by equations $p_i^\mu = \frac{\partial L}{\partial z^i_\mu} = 0$. Then we have

$$\Theta^*|_{W_1} = \left[ L - z^i_\mu \frac{\partial L}{\partial z^i_\mu} - p \right] \eta.$$  \hfill (15)

Introduce the function $H_0$ on $W_0$ as follows ([14]): $H_0 = p + p^i_\mu z^i_\mu - pr^2 YL$.

Consider now the submanifold $W_2 \subset W_1$ defined by equation $H_0 = 0$, i.e.

$$p = -(p^i_\mu z^i_\mu - L),$$

which defines, for a given Lagrangian the hamiltonian section of $q : \Lambda_2^{n+1} Y \to Z^*$ (a “local energy”, see above).

Notice that the form $\Theta^*$ (and therefore, $\Omega^*$) vanish when they restrict to $W_2$. This allows to identify $W_2$ with the graph of the Legendre mapping $\overline{L} : Z \to Z^*$

### 3.6. Vertical contact structure.

Notice that the dimensions of the fiber of the bundle $\pi^1_0 : J^1(\pi) \to Y$ and that of the dual bundle $Z^* = (\Lambda_2^n Y/\Lambda_2^1 Y) \to Y$ are the same. In addition to the $(n+1)$ and $(n+2)$-forms introduced above, the bundle $\pi_Y W_0 : W_0 \to Y$ has one more geometrical structure, namely the contact structure in the fibers $W_0, y$ of the bundle $\pi_Y W_0 : W_0 \to Y$. Indeed, fibers of this bundle are endowed with the contact 1-form

$$\hat{\theta} = dp + z^i_\mu dp^i_\mu = d(p + z^i_\mu p^i_\mu) = p^i_\mu dz^i_\mu.$$ \hfill (16)

This “vertical contact structure” allows to distinguish the Legendre mappings defined by some lagrangian $L \in C^\infty(Z)$ from the general bundle mappings $\hat{C} : Z \to Z^*$, namely,

**Proposition 1**  
(i) Let $L\eta$ be a Lagrangian defined on the space $Z = J^1(\pi)$. Then the intersection of the graph $\Gamma_L$ of the Legendre transformation $\overline{L}: J^1(\pi) \to \Lambda_2^{n+1} Y$ with the fibers $W_0, y$ of the bundle $\pi_Y W_0 : W_0 \to Y$ are the **Legendre submanifolds** of the fibers $W_0, y$.

(ii) Let $\hat{C} : J^1(\pi) \to \Lambda_2^{n+1} Y$ be any smooth bundle morphism (over $Y$) given by

$$\hat{C}(x, y, z^i_\mu) = (x, y; p(x, y, z^i_\mu), p^i_\mu(x, y, z^i_\mu)).$$

Then the intersection of the graph $\Gamma_{\hat{C}} \subset W_0$ of this morphism with the fibers $W_0, y$ of the bundle $W_0 \to Y$ are the **Legendre submanifolds** of the (contact) fibers $W_0, y$ if and only there exists a (locally defined) function $L \in C^\infty(J^1(\pi))$ such that

$$p_j^\nu = L z^j_\nu, \quad p = L - z^i_\mu L z^i_\mu.$$  

**Proof:** To prove the first statement, notice that we have in local coordinates

$$\overline{L}(x^\mu, y^i, z^i_\mu) = (x^\mu, y^i, p = L - z^i_\mu \frac{\partial L}{\partial z^i_\mu}, p^i_\mu = \frac{\partial L}{\partial z^i_\mu}).$$

Thus, along $\Gamma_L$ we have

$$\hat{\theta}|_{\Gamma_L} = dv((L - z^i_\mu \frac{\partial L}{\partial z^i_\mu}) + z^i_\mu \frac{\partial L}{\partial z^i_\mu}) - \frac{\partial L}{\partial z^i_\mu} dz^i_\mu = 0.$$  

For the second part we notice that the restriction of the 1-form $\hat{\theta}$ on the fiber $W_0, y$ to the graph of $\hat{C}$ has the form

$$p_{_{z^i_\mu}} dz^i_\mu + z^i_\mu p^i_\mu dz^i_\mu = \partial_{z^i_\mu} (p + z^i_\mu p^i_\mu) = p_j^\nu.$$  

Introducing the function $L = p + z^i_\mu p^i_\mu$ we immediately get the necessary expressions for components of the mapping $\hat{C}$. 
4. Partial jet-bundles $Z_p = J^1_p(\pi)$ of the bundle $\pi : Y \to X$.

Here we present a construction of two types of partial jet bundles $Z_p = J^1_p(\pi)$ of a fiber bundle $\pi : Y \to X$. One, denoted as $J^1_K(\pi)$ is related to a subbundle $K$ of the tangent bundle $T(X)$ (or, more exact, with an almost product structure $T(X) = K \oplus K'$ (see [13])), the other, denoted as $J^1_S(\pi)$, is defined by the decomposition $S$ of the state space $U$ into the direct sum of subspaces with different sets of derivatives entering the constitutive relations.

**Definition 1** Let $\pi : Y \to X$ be a fiber bundle and let $K(X) \subset T(X)$ be a subbundle of the tangent bundle of manifold $X$.

(i) Let $\phi_1, \phi_2$ be a two sections of the bundle $\pi$ such that $\phi_1(x) = \phi_2(x)$. We say that these two sections are $K$-**equivalent of order 1 at a point** $x \in X$ and denote this as $\phi_1 \sim_K x \phi_2$ if the restrictions of the tangent mappings $\phi_{1x} : T(X) \to T_{\phi_1(x)}(Y)$ to the subspace $K_x(X)$ coincide.

(ii) The space of classes of $K_x$-equivalence of order 1 will be called a $K$-**partial 1-jet** (of a section) at a point $x$. The space of $K$-partial 1-jets at a point $x \in X$ will be denoted by $J^1_K x(\pi)$.

(iii) The union $\bigcup_{x \in X} J^1_K x(\pi)$ will be denoted by $J^1_K(\pi_{XY})$ and will be called the space of $K$-partial 1-jets of sections of the bundle $\pi$.

**Example 2** For $K = \emptyset$ (empty), the bundle $J^1_K(\pi) = \{0\}_Y$ - affine bundle with zero-dimensional fiber (this is the case of RET). For $K = T(X)$ the bundle $J^1_K(\pi) = J^1(\pi)$ is the conventional 1-jet bundle.

**Proposition 2** (i) The space $J^1_K(\pi)$ of $K$-partial 1-jets of sections of the bundle $\pi_{XY}$ has a natural structure of affine bundle $\pi_{10 K}$ over $Y$ and of the fiber bundle over $X$.

(ii) There is a canonical surjection of affine bundles

$$w_K : J^1(\pi) \to J^1_K(\pi),$$

associating with any class of equivalent sections of order 1 of the bundle $\pi$ containing a section $\phi$ the class of $K_x$-equivalent of order 1 sections of $\pi$ containing section $\phi$.

(iii) Let $T(X) = K(X) \oplus K'(X)$ be a decomposition of the tangent bundle of $X$ into the direct sub of vector subbundles (an almost product structure $(AP)$). Then the commutative diagram

$$\begin{array}{ccc}
J^1(\pi) & \xrightarrow{w_K} & J^1_K(\pi) \\
\downarrow w_{K'} & & \downarrow \pi_{10 K} \\
J^1_{K'}(\pi) & \xrightarrow{\pi_{10 K'}} & Y
\end{array}$$

is the diagram determining $J^1(\pi)$ as the fiber product of partial affine 1-jet bundles with respect to $K$ and $K'$ over $Y$.

(iv) (Functoriality) Let the lower square of the diagram

$$\begin{array}{ccc}
J^1_K(\pi_1) & \xrightarrow{j^1(f)_K} & J^1_K(\pi_2) \\
\downarrow \pi_1^1 & & \downarrow \pi_2^1 \\
Y_1 & \xrightarrow{f} & Y_2 \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
X & = & X
\end{array}$$
Proof: To prove the last statement of the proposition notice that for any section \( s \) of the bundle \( \pi_1 \) and any (local) section \( \xi \) of the subbundle \( K \subset T(X) \) we have for the section \( s' = f \circ s \) of the bundle \( \pi_2 \): \[
\xi \cdot s'(x) = \frac{\partial f}{\partial y^i}(\xi \cdot s^i)(x)
\]
and thus, the derivatives of components of a section \( s' \) in the directions of the subbundle \( K \) are defined by the linear mapping of the derivatives of components of the section \( s \) in the same direction. Therefore the mapping sending the point \( (x, y, z^j_{\xi_k}) \), \( \xi_k \) being a local basis of the bundle \( K \) to the point \( (x, y' = f(x, y), z^j_{\xi_k} = \frac{\partial f}{\partial y^i}(x, y)z^j_{\xi_k} \) is defined correctly (independent of the choice of section \( s \)) and determines the mapping \( \hat{f}_K \).

Remark 2 If a subbundle \( K \subset T(X) \) is chosen, the G-orthogonal complement to \( K : K' = K^\perp \) can be taken as the complemental subbundle \( K' \) in the AP structure \( T(X) = K \oplus K' \).

Let \( \nu \) be a section of the bundle \( (J^1(\pi), \pi_0, Y) \) over \( Y \) (a jet field or Ehresmann connection on \( \pi \) [11]). The section \( \nu \) determines a section of the bundle \( J^1_K(\pi) \). Thus, in the affine fibers \( J^1_K(\pi) \) of \( \pi^1 \) (respectively \( \pi^1_K \)) a point \( \nu(y) \) is chosen. This defines an identification of affine space \( J^1_K(\pi) \) with the vector space \( V_y \otimes T^*_\pi(y)(X) \), \[
I_\nu : J^1_y(\pi) \simeq V_y \otimes T^*_\pi(y)(X),
\]
and a similar identification of the partial 1-jet bundle \[
J^1_K y(\pi) \simeq V_y \otimes T^*_K \pi(y)(X),
\]
where \( T^*_K \pi(y)(X) = T^*_x(X)/K^\perp(X) \). Thus, a choice of a connection \( \nu \) identifies (noncanonically) 1-jet bundles \( J^1_K(\pi) \) with the vector bundles: \( J^1_K \nu(\pi) \simeq V_\pi \otimes T^*_K(\pi)(X) \), see [11], Sec.17.2.

4.1. Space-time splitting case.
As an example of a situation where the partial jet spaces \( J^1_K(\pi) \) with different \( K \) mixes to produce more complex partial jet bundle consider the situation where the fiber \( U \) of the configurational bundle \( Y \) splits into the subspaces of fields that enters the constitutive relations with only time, only space and space-time derivatives:
\[
U = U_0 \oplus U_t \oplus U_x \oplus U_{tx},
\]
corresponding to the splitting \( S \) of the set of indices:
\[
S = S_0 \cup S_t \cup S_x \cup S_{tx} : [0, m] = [0, m_0] \cup [m_0 + 1, m_1] \cup [m_1 + 1, m_2] \cup [m_2 + 1, m].
\]
Here \( U_0 \) includes the fields \( y^i, i \in S_0 \) whose first derivatives do not enter the CR; \( U_t \) includes the fields \( y^i, i \in S_t \) whose time derivative enters the CR but their spacial gradient does not; \( U_x \) is formed by the fields \( y^i, i \in S_x \) whose spacial gradient but not the time derivative enter the CR and \( U_{tx} \) is formed by the fields \( y^i, i \in S_{tx} \) all derivatives of which enter the CR. We assume that all the fields are tensor or tensor density fields in the space \( X \). This splitting is, therefore, \( Diff(T) \times Diff(B) \)-invariant.

Using the decomposition (19) together with the splitting \( T(X) = T(R_t) \oplus T(B) \) (see Sec.2) we can introduce the following
Table 1. 1- and 2-partial jet spaces.

| Y | $U_0$ | $U_t$ | $U_x$ | $U_{tx}$ |
|---|---|---|---|---|
| $J_1^1(\pi)$ | 0 | $u^i_t$ | $u^i_x$ | $u^i_{tx}$ |
| $J_2^2(\pi)$ | $u^i_t, u^i_x$ | $u^i_{tx}, u^i_{tt}, u^i_{tx}$ | $u^i_{ttx}, u^i_{txx}, u^i_{txxx}$ |

**Definition 2** Let $S$ be a diagram of a splitting of the fiber of the bundle $\pi$ as the sum of subbundles (19). We define the **partial 1-jet bundle** $J_1^1(\pi)$ which has as its fiber space the first derivatives of sections $s : X \to Y$ of the following type

$$u^i_t, i \in S_t = [m_0 + 1, m_1] ; \quad u^i_{x}, i \in S_x = [m_1 + 1, m_2] ; \quad u^i_{tx}, i \in S_{tx} = [m_2 + 1, m].$$

**Proposition 3**

(i) The bundle $J_1^1(\pi)$ is defined correctly with respect to the diffeomorphisms from $Diff(R_t) \times Diff(B)$ of the base manifold $X = R_t \times B^3$ containing arbitrary diffeomorphisms of $B^3$ and the independent time diffeomorphisms of $R_t$.

(ii) Corresponding to the decomposition (19) we have the decomposition of the bundle $Y \to X$ as the fiber product of vector bundles over $X$

$$Y = Y_0 \times_X Y_t \times_X Y_x \times_X Y_{tx}. \quad (20)$$

(iii) For the partial 1-jet bundle $J_1^1(\pi)$ we have the following decomposition into the fiber product of affine bundles

$$J_1^1(\pi) = 0(Y_0) \times_Y J_1^1(Y_t) \times_Y J_1^1(Y_x) \times_Y J_1^1(Y_{tx}) \quad (21)$$

over $X$.

(iv) (Functoriality) Let the lower square of the diagram

$$\begin{array}{ccc}
J_1^1(\pi_1) & \xrightarrow{j_1^1(f)} & J_1^1(\pi_2) \\
\pi_1 & \downarrow & \pi_2 \\
Y_1 & \xrightarrow{f} & Y_2 \\
\pi_1 & \downarrow & \pi_2 \\
X & \xrightarrow{} & X
\end{array}$$

represent a morphism of the bundles $f : Y_1 \to Y_2$ over the manifold $B$ such that $f(U_1 K_i) \subset U_2 K_i$ for the subbundles $K_i$ are the subbundles or the splitting $S$, i.e. $K_i = < 0, < \partial_t >, < \partial_x >, T(X)$. Then there exists the morphism of the bundles $j_1^1(f) : J_1^1(\pi_1) \to J_1^1(\pi_2)$ such that the diagram above is commutative.

Table 1 lists the derivatives of sections corresponding to such a decomposition of the state space.

The notion of geometric automorphisms of the bundle $\pi$ that can be lifted to the bundle $J_1^1(\pi)$ is now introduced.

**Definition 3**

(i) An automorphism $\phi \in Aut(\pi)$ of the bundle $\pi$ is called $S$-admissible if it is the automorphism of the fiber product bundle decomposition (20), i.e. there are automorphisms: $\phi_0$ of the bundle $Y_0 \to X$, $\phi_t$ of the bundle $Y_t \to X$ etc. such that

$$\phi = \phi_0 \times_Y \phi_t \times_Y \phi_x \times_Y \phi_{tx}.$$
(ii) A projectable vector field $\xi \in \mathcal{X}(Y)$ is called $S$-admissible if transformations of its local flow $\phi_t$ are $S$-admissible. Denote by $\mathcal{X}_S(\pi)$ the Lie algebra of all $S$-admissible projectable vector fields in $Y$.

Finally, we have the following

**Proposition 4** The affine bundle $J^1(J_p^1(\pi))$ over $J_p^1(\pi)$ has its fiber modeled on the vector space

$$T^*(X) \otimes U_0 \oplus T^*_{x,t,t}(X) \otimes U_t \oplus T^*_{t,x,xx}(X) \otimes U_{tx},$$

where the types of partial derivatives of fields from different components of $U$ included in the second partial jet bundle are marked.

### 4.2. Partial Cartan distribution.

The 1-jet space $Z = J^1(\pi)$ is endowed with the canonical Cartan distribution $Ca$ locally (in the adapted coordinates) defined by the 1-forms

$$\omega^i = dy^i - z^i_\mu dx^\mu.$$  

When the subbundle $K$ is not integrable we will have to use non-holonomic frames in $X$ and the corresponding coframes. The following simple Lemma gives the representation of the Cartan distribution in $Z$ in terms of such a non-holonomic frame.

**Lemma 1** Let $(x^i, y^i)$ be a local adapted coordinate chart in the bundle $\pi$. Let $\xi_\mu$ be a (local) nonholonomic frame of the tangent bundle $T(X)$. Denote by $\psi^\mu$ its dual coframe $(<\psi^\mu, \xi_\nu> = \delta^\mu_\nu)$. Let $\hat{\psi}^\mu$ be the pullback of the 1-form $\psi^\mu$ to $Z = J^1(\pi)$ by the projection $\pi^1$. Introduce the (local) coordinates in the fibers of the bundle $J^1(\pi) \rightarrow Y$ by

$$z^i_\mu(j^1(s))(x) = (<\xi_\mu, s^i>)(x)$$

for all sections $s : X \rightarrow Y$. Then the Cartan distribution $Ca$ in $J^1(\pi)$ is defined by the forms

$$\omega^i = dy^i - \sum_\mu z^i_\mu \hat{\psi}^\mu,$$  \hspace{1cm} (22)

and is generated by the vector fields

$$Ca(z) = <N_\mu = \xi_\mu + \sum_i z^i_\mu \partial_{y^i}, \partial_{z^i_\mu}, \mu = 1, \ldots, n + 1 >.$$  

In a contrast to the full 1-jet bundle in the maximal (RET) case the fiber of $J_p^1(\pi)$ is one point and has no local geometrical structure. We will show that in the intermediate case partial 1-jet bundles $J^1_K(\pi), J^1_S(\pi)$ have the “partial Cartan distribution”, corresponding to the structure of the fibers of $J_p^1(\pi) \rightarrow Y$. This distribution although depending not just on the subbundle $K$ but on the complementary distribution $K'$ as well (i.e. on the whole almost product structure $T(X) = K \oplus K'$, [13]) plays an important role for the partial jet bundles similar to that of the conventional Cartan distribution.

We start with the case of a decomposition $T(X) = K \oplus K'$ of the tangent bundle of the base $X$ and the corresponding decomposition $T^*(X) = K^* \oplus K'^*$ of the cotangent bundle into the direct sum of two integrable subbundles. Locally, one can choose a coordinate chart $x^\mu = <x^\nu; x^\sigma>$ such that (with respect to the index splitting $\mu = <\nu, \sigma>$) $K = <\partial_{x^\nu}>; K' = <\partial_{x^\sigma}>$.

Now we define the 1-forms on the partial 1-jet bundle $J^1_K(Y)$:

$$\omega^i_K = dy^i - \sum_\nu z^i_\nu dx^\nu.$$
A balance law

Partial Cartan distribution is the linear span of the vector fields

\[ \text{Definition 4} \]

(i) A balance equation with a domain \( D \) is an \( n \times (n + 1) \) semibasic form \( B \) defined in the domain \( D \):

\[ B = F^\mu \eta_\mu + \Pi \eta, \quad F^\mu, \Pi \in C^\infty(D). \]

(ii) A section \( s \in \Gamma(\pi) \) is a solution of the balance equation \( B \) if

\[ \tilde{d}j_p^1(s)^*(B) = dj_p^1(s)^*(F^\mu \eta_\mu) - j_p^1(s)^* \Pi \eta = 0. \]

Here \( \tilde{d} \) is the Iglesias differential, see Appendix B.

(iii) A balance law \( B \) is called trivial if any section of the bundle \( \pi \) is its solution.

Balance laws with a domain \( D \subset J_p^1(\pi) \) form a vector space - subspace \( BL(D) \subset \Lambda_{sb}^*(D) \) of the subalgebra of semi-basic forms \( \Lambda_{sb}^*(D) \) of the exterior algebra \( \Lambda^*(D) \) in the domain \( D \).

Denote by \( d_\mu \) the total derivative by \( x^\mu \):

\[ d_\mu = \partial_{x^\mu} + z^i_\mu \partial_{y^i} + z^i_\mu \sigma^j u \partial_{z^j}. \]

\[ \text{Lemma 2} \]

A balance law \( B = F^\mu \eta_\mu + \Pi \eta \) is trivial if and only if \( \Pi = \sum_\mu d_\mu F^\mu \), where \( d_\mu = \partial_{x^\mu} + z^i_\mu \partial_{y^i} \) is the total derivative by \( x^\mu \).

Proof is standard.

\[ \text{Definition 5} \]

A balance system defined in a domain \( D \subset J_p^1(\pi) \) is a subspace in the space \( BL(D) \) of the balance laws defined in \( D \).
Remark 3 As defined, the notion of balance system is very broad. To be more practical, one has to deal with systems large enough to specify all the components $s^i(x)$ of the basic fields $y^i$ but small enough to be determined. The second condition is usually achieved by requiring that the number of equations in a balance system is equal to the number $m$ of basic fields. The first condition requires the fulfillment of some regularity conditions (see [29], Section 13) that may depend on the problem studied with the balance system.

5. Lift of vector fields to $Z_p, Z^*, W$. In this section we discuss the prolongation (lift) of infinitesimal geometrical transformations (vector fields) from $Y$ to the partial 1-jet bundles $Z_p = J^1_p$ and to the dual space $Z$.

We start with the following simple

Lemma 3 Let $G \subset \text{Aut}(\pi)$ be a Lie group of automorphisms of the bundle $\pi$.

(i) The projection $G_0$ of the group $G$ to $X$ lifts to the natural bundle of exterior algebras $\Lambda^*(X)$ such that the pullback of the forms

$$\pi^* : \Lambda^*(X) \to \Lambda^*(Y)$$

is equivariant with respect to the projection $g \to g_0$.

(ii) Subbundles $\Lambda^k(Y)$ are invariant under the lifted action of $G$.

Let the action of $G$ on $X$ by $g \to g_0$ preserves the subbundle $K \subset T(X)$. Then one can naturally define the action of $G$ on the partial 1-jet bundle $J^1_K(\pi)$ in such a way that the projections $J^1_K(\pi) \to Y \to X$ become $G$-morphisms (see below).

Similarly, if an action of the group $G$ leaves the splitting (20) invariant and its projection $G_0$ leaves invariant the space-time decomposition $T(X) = T(\mathbb{R}_t) \oplus T(B)$ of the tangent bundle, one may lift its action to $J^1_K(\pi)$.

If an action of $G$ can be lifted to $J^1_K(\pi)$ and to $\Lambda^{n+(\alpha+1)}Y$, by taking the fiber product of these actions we may lift the action of $G$ to the space $W_0$.

Let us look in more detail at these prolongations of transformations (in global as well as in infinitesimal variants).

5.1. Lift of vector fields and transformations to $J^1_p(\pi)$.

5.1.1. Case $Z_p = Z = J^1(\pi)$. Recall ([11, 30]) that there exists the natural lift $\xi \to \xi^1$ of an arbitrary vector field $\xi \in \mathcal{X}(\pi)$ to the vector field $\xi^1$ in $J^1(\pi)$ defined by the conditions described in the following

Proposition 6 (see [11, 30]).

(i) For any vector field $\xi \in \mathcal{X}(Y)$ there is a unique vector field $\xi^1 \in \mathcal{X}(J^1(\pi))$ (1-jet prolongation of $\xi$) defined by the conditions:

(a) The vector field $\xi^1 \in \mathcal{X}(J^1(\pi))$ is projectable to $Y$ and

$$\pi_{10*}(\xi^1) = \xi,$$

(b) The local flow of the vector field $\xi^1$ preserves the Cartan distribution $Ca$ (such a vector field is called an infinitesimal contact transformation).

(ii) The lift $\xi^1$ of a vector field $\xi = \xi^\mu(x,y)\partial_{x^\mu} + \xi^i(x,y)\partial_{y^i}$ has in local adapted coordinates the form

$$\xi^1 = \xi^\mu(x)\partial_{x^\mu} + \xi^i(x,y)\partial_{y^i} + \left(d_\mu \xi^i - z^i_\nu \frac{\partial \xi^\nu}{\partial x^\mu}\right) \partial_{y^\nu}, \quad (24)$$

where $d_\mu \xi^i = \frac{\partial \xi^i}{\partial x^\mu} + z^i_\nu \frac{\partial \xi^\nu}{\partial y^\mu}$ is the total derivative of the function $\xi^i$ and similarly for $\xi^\mu$. 

(iii) The mapping $\xi \to \xi^1$ is the homomorphism of Lie algebras:

$$[\xi, \eta]^1 = [\xi^1, \eta^1]$$

for all $\xi, \eta \in \mathcal{X}(Y)$.

(iv) For a projectable vector field $\xi \in \mathcal{X}(\pi)$, $\xi = \xi^\mu(x) \partial_{x^\mu} + \xi^i(x, y) \partial_{y^i}$ the 1-jet prolongation $\xi^1$ coincides with the flow prolongation (see below).

The flow prolongation (lift) is defined by the local flow $\phi_t(y)$ of the vector field $\xi \in \mathcal{X}(\pi)$. Let $\phi_t(x)$ be the flow induced by $\phi_t$ in $X$ having the vector field $\xi^\mu(x) \partial_{x^\mu}$ as the generator. The flow $\phi_t$ acts on sections $y = s(x)$ of the bundle $\pi$ by the rule: $s \mapsto (\phi_s)(x) = \phi_t s(\phi_t^{-1}(x))$. Differentiating $t$ at $t = 0$ we get the generator of the action on the 1-jet part $s^1_{x^\nu}$ in the form (24) (see ([11, 30])). The flow lift of automorphisms from $\text{Aut}(\pi)$ and of corresponding vector fields is the homomorphism of groups (Lie algebras)

$$\text{Aut}(\pi) \to \text{Aut}(\pi^2_0), \; \mathcal{X}(\pi) \to \mathcal{X}(\pi^2_0) \quad (25)$$

5.1.2. Case of $Z_\nu = J^1_k(\pi)$. Let now $K$ be a subbundle of the tangent bundle $T(X)$ and let $T(X) = K \oplus K'$ be an AP-structure containing $K$. Let $\eta_\nu$ be a local basis of $K$ (in an integrable case $\eta_\nu = \partial_{x^\nu}$) and denote by $z^i_\nu$ the corresponding local coordinates in the fiber of the bundle $\pi_{10} : J^1_k(\pi) \to Y$ (see above).

**Definition 6**

(i) Denote by $\mathcal{X}_K(\pi)$ the Lie algebra of $\pi$-projectable vector fields $\xi$ in $Y$ such that the field $\bar{\xi}$ generated by $\xi$ in $X$ preserves the distribution $K \subset T(M)$: $\bar{\phi}_t K = K$ for the local flow $\phi_t$ of the vector field $\xi$.

(ii) Denote by $\mathcal{X}_{K \oplus K'}(\pi)$ the Lie algebra of $\pi$-projectable vector fields $\xi$ in $Y$ such that the field $\bar{\xi}$ generated by $\xi$ in $X$ preserves the distributions $K, K' \subset T(M)$: $\bar{\phi}_t K = K$ for the local flow $\phi_t$ of the vector field $\xi$ (and the same for $K'$).

**Lemma 4** Let the AP-structure $T(X) = K \oplus K'$ be integrable and let $(x^\nu, x^\sigma)$ be a (local) integrating chart. Then

(i) A $\pi$-projectable vector field $\xi = \xi^\mu(x) \partial_{x^\nu} + \xi^i(x, y) \partial_{y^i}$ belongs to $\mathcal{X}_K(\pi)$ if and only if

$$\bar{\xi} = \xi^\mu(x^\nu, x^\sigma) \partial_{x^\nu} = \xi^\nu(x) \partial_{x^\nu} + \xi^\sigma(x^\sigma) \partial_{x^\sigma},$$

i.e. if the components $\xi^\sigma(x)$ do not depend on the variables $x^\nu$.

(ii) A $\pi$-projectable vector field $\xi = \xi^\mu(x) \partial_{x^\nu} + \xi^i(x, y) \partial_{y^i}$ belongs to $\mathcal{X}_{K \oplus K'}(\pi)$ (preserves the almost product structure $T(X) = K \oplus K'$) if and only if

$$\bar{\xi} = \xi^\mu(x) \partial_{x^\nu} = \xi^\nu(x^1) \partial_{x^\nu} + \xi^\sigma(x^\sigma) \partial_{x^\sigma},$$

**Proof:**

$$[\bar{\xi}, \partial_{x^\nu}] = -(\partial_{x^\nu} \cdot \xi^\nu) \partial_{x^\nu} - (\partial_{x^\sigma} \cdot \xi^\sigma) \partial_{x^\sigma}.$$  

This vector field belongs to $K$ if and only if $\partial_{x^\nu} \cdot \xi^\nu = 0$ for all $\nu$ and $\sigma$. The second statement is proved in the same way.

**Proposition 7** Let the AP-structure $T(X) = K \oplus K'$ be integrable and let $(x^\nu, x^\sigma)$ be a (local) integrating chart.

(i) For a vector field $\xi = \xi^\mu(x) \partial_{x^\nu} + \xi^i(x, y) \partial_{y^i} \in X_K(\pi)$ the following properties are equivalent

(a) There exist a vector field $\xi^1 \in \mathcal{X}(J^1_k(\pi))$ such that
1. The local flow of the vector field $\xi^1$ preserves the partial Cartan distribution $Ca_K$.
2. $\pi_{10} \circ \xi^1 = \xi$.

(b) The vector field $\xi$ has, in a local integrating chart $(x^\nu, x^\sigma)$ the form

$$\xi = \xi^\nu(x^\nu)\partial_{x^\nu} + \xi^\sigma(x^\sigma)\partial_{x^\sigma} + \xi^i(x^\nu, y)\partial_{y^i}.$$ 

In particular the projection $\hat{\xi}$ of the vector field $\xi$ in $X$ preserves the almost product structure $K \oplus K'$.

(ii) In the case where these conditions are fulfilled the vector field $\nu^i\partial_{x^i}$ of the contact ideal of exterior forms,

$$\nu^i\partial_{x^i} =\xi^i(x^\nu)\partial_{x^\nu} + \xi^\sigma(x^\sigma)\partial_{x^\sigma} + \xi^i(x^\nu, y)\partial_{y^i} + \left(d_{\nu} \xi^i - z^i_{\nu, x^\nu}\right)\partial_{z^i}$$

is unique and is given by the formula

$$\xi^i = \xi^\nu(x^\nu)\partial_{x^\nu} + \xi^\sigma(x^\sigma)\partial_{x^\sigma} + \xi^i(x^\nu, y)\partial_{y^i}$$

or

$$\xi^i = \xi^\nu(x^\nu)\partial_{x^\nu} + \xi^\sigma(x^\sigma)\partial_{x^\sigma} + \xi^i(x^\nu, y)\partial_{y^i} + \left(d_{\nu} \xi^i - z^i_{\nu, x^\nu}\right)\partial_{z^i}$$

(26)

(iii) The mapping $\xi \rightarrow \xi^1$ is the homomorphism of Lie algebras:

$$[\xi, \eta]^1 = [\xi^1, \eta^1]$$

for all $\xi, \eta \in \mathcal{X}_{K,K'}(\pi)$.

**Proof:** Let $\hat{\xi} = \xi_1(x^\nu)\partial_{x^\nu} + \xi_1(x^\sigma)\partial_{x^\sigma} + \lambda_1\partial_{z_1}$ be a prolongation to the partial jet bundle $Z_K$ of the vector field $\xi$. Then, preservation of the partial Cartan structure is equivalent to the condition that for all the generators $\omega^i_K = dy^i - \sum_{\nu} z^i_{\nu, x^\nu}$ of the contact ideal of exterior forms,

$$\mathcal{L}_{\xi} \omega^i_K = \sum_{j} q_j^i \omega^i_K,$$

for some functions $q_j^i \in C^\infty(Z_\nu)$. We calculate

$$\mathcal{L}_{\xi} \omega^i_K = (di + i_\xi d)(dy^i - \sum_{\nu} z^i_{\nu, x^\nu}) = d[\xi^i - z^i_{\nu, x^\nu}] + i_\xi ( -dz^i_{\nu, x^\nu} \wedge dy^{\nu}) =$$

$$= d\xi^i - \xi^\nu dz^i_{\nu} - z^i_{\nu, x^\nu} \lambda^1 \partial_{x^\nu} + \xi^\nu dx^\nu = \xi^i_x, dx^\nu + \xi^i_{x^\nu} dy^j - z^i_{\nu, x^\nu} dx^\nu + \xi^i_{x^\sigma} dx^\sigma - \lambda^1 \partial_{x^\nu} =$$

$$= \sum_{j} q_j^i (dy^j - \sum_{\nu} z^i_{\nu, x^\nu}),$$

or

$$(\xi^i_{x^\nu} - z^i_{\nu, x^\nu}) dx^\sigma + \xi^i_{x^\sigma} dy^j + [\xi^i_{x^\nu} - \lambda^1 \partial_{x^\nu} - z^i_{\nu, x^\nu}] dx^\nu = \sum_{j} q_j^i (dy^j - \sum_{\nu} z^i_{\nu, x^\nu}).$$

This equality is fulfilled if and only if we have

$$\begin{cases}
\xi^i_{x^\sigma} - z^i_{\nu, x^\nu} = 0, \\
q_j^i = \xi^i_{x^\nu}, \\
\xi^i_{x^\nu} - \lambda^1 \partial_{x^\nu} - z^i_{\nu, x^\nu} = -q_j^i z^i_{\nu, x^\nu}.
\end{cases}$$

Since neither $\xi^i_1$ nor $\xi^\nu$ depend on $z^i_{\nu}$, the first system is equivalent to the requirement that both $\xi^i_1$ and $\xi^\nu$ are independent of $x^\sigma$. Then the second condition determines $q_j^i$ and second - $\lambda^1 = \xi^i_{x^\nu} + \xi^i_{x^\sigma} z^i_{\nu} - z^i_{\nu, x^\nu}$, and the prolongation $\hat{\xi}$ takes the form described in the Proposition.

Criteria for prolongation of a projectable vector field $\xi \in X(Y)$ to the partial 1-jet bundle $J_1^K(\pi)$ in a case of a general (non-integrable) AP structure $T(X) = K \oplus K'$ is given in [29].
5.1.3. Case of $Z_0 = Z_\mathcal{S} = J_0^1(\pi)$. Consider now the case of the partial 1-jet bundle $J_0^1(\pi)$ generated by the $(x,t)$-decomposition $S$ (see Definition 2 above). By Proposition 3 the bundle $Y$ is the fiber product of the bundles $Y = Y_0 \times_{Y_1} Y_3 \times_{Y_2} Y_4 \times_{Y_5} Y_{6}$ and the partial 1-jet has the form of the fiber product $J_0^1(\pi) = 0(Y_0) \times_{Y_1} J_0^1(Y_1) \times_{Y_2} J_0^1(Y_2) \times_{Y_3} J_0^1(Y_3) \times_{Y_4} J_0^1(Y_4) \times_{Y_5} J_0^1(Y_5) \times_{Y_6} J_0^1(Y_6)$.

A natural class of automorphisms of the bundle $\pi$ is the class of $S$-automorphisms of $\pi$ (see Definition 3) and the corresponding class of $S$-admissible vector fields $\xi \in \mathcal{X}_S(\pi)$ in $Y$. These are the geometrical or infinitesimal automorphisms of the bundle $\pi$ that preserve the $S$-type of fields under transformation.

In this case we have the canonical integrable AP-structure $T(\mathcal{X}) = T(B_\oplus) \subset \delta_t$ with a local chart $(x,t)$. Applying the arguments used for the study of prolongation $\xi \rightarrow \xi^1$ to the partial 1-jet bundles $J_0^1(\pi)$ we get the following analog of the previous Proposition:

**Proposition 8** (i) A vector field $\xi \in \mathcal{X}_S(\pi)$ preserves the AP-structure $T(B_\oplus) \subset \delta_t$ if and only if

$$\bar{\xi} = \ell^\mu(x,t)\partial_{x^\mu} = \xi^A(x)\partial_{x^A} + \xi^i(t)\partial_i,$$

(ii) For any $S$-admissible $\pi$-projectable vector field $\xi \in \mathcal{X}_S(\pi)$ the following statements are equivalent

(a) There is a vector field $\xi^1 \in \mathcal{X}(Z_S)$ such that

1. The vector field $\xi^1 \in \mathcal{X}(Z_S)$ is $\pi_{10}$-projectable and

$$\pi_{10*}(\xi^1) = \xi,$$

2. The local flow of the vector field $\xi^1$ preserves the partial Cartan distribution $\mathcal{C}_{\pi/\mathcal{S}}$ at $Z_S(\pi)$.

(b) The vector field $\xi$ has the following form

$$\xi = \xi^A(x)\partial_{x^A} + \xi^i(t)\partial_i + \sum_{i_0 \in S_0} \xi^{i_0}(x,t; y^{j_0}, j_0 \in S_0)\partial_{y^{j_0}} + \sum_{i_1 \in S_1} \xi^{i_1}(t; y^{j_1}, j_1 \in S_t)\partial_{y^{j_1}} +$$

$$+ \sum_{i_2 \in S_2} \xi^{i_2}(x; y^{j_2}, j_2 \in S_x)\partial_{y^{j_2}} + \sum_{i_3 \in S_{1x}} \xi^{i_3}(x,t; y^{j_3}, j_3 \in S_{1x})\partial_{y^{j_3}}.$$

where the dependence of the vertical components $\xi^1$ of the vector field $\xi$ on the variables $x^A, t$ is specified by the subset $S_\mathcal{A}$ containing index $i$.

(iii) In the case where these conditions are fulfilled the vector field $\xi^1$ is unique and is given by the formula (recall that $x^0 = t$)

$$\xi^1 = \xi + \sum_{i_1 \in S_1} \left( \frac{d\xi^{i_1}}{dx_0} - z^{i_1}_{x^0} \frac{\partial \xi^0}{\partial x^0} \right) \partial_{z^{i_1}_{x^0}} + \sum_{i_2 \in S_2} \left( \frac{d\xi^{i_2}}{dx^A} - z^{i_2}_{x^A} \frac{\partial \xi^B}{\partial x^A} \right) \partial_{z^{i_2}_{x^A}} +$$

$$+ \sum_{i_3 \in S_{1x}} \left( \frac{d\xi^{i_3}}{dx^A} - z^{i_3}_{x^A} \frac{\partial \xi^A}{\partial x^A} \right) \partial_{z^{i_3}_{x^A}}. \quad (27)$$

(iv) Mapping $\xi \rightarrow \xi^1$ is the homomorphism of Lie algebras: $[\xi, \eta]^1 = [\xi^1, \eta^1]$ for all $\xi, \eta \in \mathcal{X}_K(Y)$.

5.2. Prolongation of $\pi$-automorphisms to the dual bundles $\Lambda^1 X$ and $\mathcal{Z}$.

Automorphisms of the bundle $\pi$ (and, correspondingly, projectable vector fields $\xi \in \mathcal{X}(\pi)$) have a natural (flow) prolongation to the projective diffeomorphisms (and vector fields) of the double bundles $\Lambda^1 X \rightarrow Y \rightarrow X$ of semi-basic $k$-forms on $Y$ (see [11] or [14]). This lift $\xi \rightarrow \xi^1$ is defined by the pullback $\phi^*_{\xi^1}$ of the exterior forms on $Y$ by the local flow $\phi_t$ of a vector field $\xi \in \mathcal{X}(\pi)$.

Another way to lift a general vector field $\xi \in X(Y)$ is defined by the following construction studied in [14] for the case where the metric $G$ is Euclidean (put $\lambda = 0$ in the following definition).
(Definition-Proposition, [14]) Let \( \alpha \) be a pullback to \( \Lambda_n^{+1} Y \) of a \( \pi \)-semibasic form \( \alpha = \alpha^k(x,y)\eta_v \) on \( Y \). Let \( \xi \in \mathcal{X}(Y) \).

(i) There exists a unique vector field \( \xi^\alpha \) on \( \Lambda_n^{+1} \) satisfying the following conditions

(a) Vector field \( \xi^\alpha \) is \( \pi_{\Lambda_n^{+1} Y} \)-projectable and
\[
\pi_{\Lambda_n^{+1} Y} \cdot \xi^\alpha = \xi,
\]

(b)
\[
\mathcal{L}_{\xi^\alpha} \Theta^{n+1} = -da.
\]

(ii) Vector field \( \xi^\alpha \) has the local form
\[
\xi^\alpha = \xi + \xi^\alpha_1 p \partial_p + \xi^\alpha_2 \nu \partial_\nu,
\]
where
\[
\xi^\alpha_1 p = -p \left( \frac{\partial \xi^\alpha}{\partial x^\mu} + \xi^\alpha \frac{\partial \lambda}{\partial x^\mu} \right) - p^\mu_i \left( \frac{\partial \xi^\alpha}{\partial x^\mu} - \xi^\alpha_2 \lambda \right)
\]
\[
\xi^\alpha_2 \nu = -p \left( \frac{\partial \xi^\alpha}{\partial x^\nu} - \xi^\alpha \lambda \right) + p^\mu_j \frac{\partial \xi^\alpha}{\partial x^\mu} - p^\mu_i \left( \frac{\partial \xi^\alpha}{\partial x^\nu} - \xi^\alpha_2 \lambda \right) - \frac{\partial \alpha^\mu}{\partial u^\nu} - \frac{\partial \alpha^\mu}{\partial u^\nu}.
\]

(iii) Let a vector field \( \xi \in \mathcal{X}(Y) \) be \( \pi \)-projectable. Then the 0-lift \( \xi^0 \) of \( \xi \) coincides with the flow prolongation \( \xi^1 \) defined above.

Let now \( \phi \in Aut(\pi) \) be an automorphism of the bundle \( \pi \). Arguments in the beginning of this subsection shows that the flow lift \( \phi^1 \) of \( \phi \) to the bundle \( \Lambda^k Y \) leaves its subbundles \( \Lambda^k Y \) invariant. In particular, \( \phi^1 \) acts on the subbundles \( \Lambda_1^{n+1} Y, \Lambda_1^{n+2} Y \) leaving their subbundles \( \Lambda_1^{n+1} Y, \Lambda_1^{n+2} Y \) invariant and leaving canonical forms \( \Theta_2^{n+1} Y \) and \( \Theta_2^{n+2} Y \) invariant. Therefore, \( \phi^1 \) generates the automorphism \( \phi^* \) of the bundle \( \tilde{Z} = Z^{n+1} \oplus Z^{n+2} \) leaving both terms invariant.

Let \( H : Z^{n+1} \rightarrow \Lambda_2^{n+1} Y \) to be a section (see Sec.3.3) of the bundle \( \Lambda_2^{n+1} Y \rightarrow Z^{n+1} \). Then for the induced form \( \Theta_H = H^* \Theta_2^{n+2} = H(x, y, p^\mu)\eta + p^\mu dy^i \wedge \eta_i \) we have
\[
\bar{\phi}^* \Theta_H(x, y, p^\mu) = \bar{\phi}^* H^* \Theta_2^{n+2} = (H \circ \bar{\phi})^* \Theta_2^{n+2} = H \circ \bar{\phi}(x, y, p^\mu)\eta + p^\mu dy^i \wedge \eta_i.
\]

Thus, though the \((0, n+1)\)-term of the form \( \Theta_H \) is changed, its \((1, n)\)-term is invariant. For the infinitesimal action of vector field \( \xi^* \) - generator of the 1-parametrical group of diffeomorphisms \( \phi_1^* \) we get from the previous formula
\[
\mathcal{L}_{\xi^*} \Theta_H = \langle \xi^* \cdot H \rangle \eta.
\]

In particular, we will be using this formula for the sections defined by a connection \( \nu \) in the bundle \( \pi \) with \( H = p^\mu \bar{\Gamma}_\mu(x) \).

For our study we need to lift a projectable vector field \( \xi \) to the bundle \( \tilde{X} = \Lambda_2^{(n+1)+2} (\pi)/\Lambda_1^{(n+1)+2} = \Lambda_2^{(n+1)+2} / \Lambda_1^{(n+1)+2} \), where \( \lambda = \ln(|G|) \).

**Proposition 9** For any projectable vector field \( \xi \in \mathcal{X}(\pi) \) there exists a unique projectable vector field \( \xi^{(n+2)} \) on the bundle \( \Lambda_2^{(n+2)} \) that leaves the canonical form \( p_i dy^i \wedge \eta \) invariant. That vector is given by the relation
\[
\xi^{(n+2)} = \xi + \xi_k \partial_{p_k}, \xi_k = p_k (\xi^\alpha \frac{\partial \lambda}{\partial x^\mu} - \frac{\partial \xi^\alpha}{\partial x^\mu} - p^\mu_j \frac{\partial \xi^\alpha}{\partial \lambda^k}),
\]
where \( \lambda = \ln(|G|) \).
Definition 8

(i) A (general) constitutive relation (CR) \( C \) of a field theory with the configurational bundle \( \pi : Y \rightarrow X \) and the partial 1-jet space \( Z_p = J^1_p(\pi) \) is a smooth morphism of bundles over \( Y \)

\[
C : J^1_p(\pi) \rightarrow \tilde{Z} = \Lambda^{(n+1)+(n+2)}_2 Y / \Lambda^{(n+1)+(n+2)}_1 Y.
\]

In local coordinates \((x^\mu, y^i, z^i_\mu)\) on \( J^1_p(\pi) \) and \((p^\mu_i, q_i)\) on \( \tilde{Z} \) a CR-mapping \( C \) has the form

\[\text{Proof:} \quad \text{We have, for a vector field of the form } \xi^{(n+2)} = \xi + \xi_k \partial_{p_k} \]

\[
L_{\xi^{(n+2)}}(p_i dy^i \wedge \eta) = (d_{\xi^{(n+2)}} + i_{\xi^{(n+2)}} d)(p_i dy^i \wedge \eta)
\]

\[
= d[p_i \xi^\mu \eta - p_i \xi^\mu dy^i \wedge \eta] + i_{\xi^{(n+1)}}(dp_i \wedge dy^i \wedge \eta)
\]

\[
= [d(p_i \xi^\mu) \eta - d(p_i \xi^\mu) \wedge \eta] - [p_i \xi^\mu dy^i \wedge \eta] + \xi_i dy^i \wedge \eta - \xi_i dp_i \wedge \eta
\]

\[
+ \xi^\mu dp_i \wedge dy^i \wedge \eta
\]

\[
= [\xi_i dp_i \wedge \eta + p_i \frac{\partial \xi_i}{\partial y^j} du^j \wedge \eta - \xi^\mu dp_i \wedge dy^i \wedge \eta + \xi^\mu dp_i \wedge \eta]
\]

\[
= \left( \frac{\partial \xi^\mu}{\partial x^\mu} p_i + p_i \frac{\partial \xi^\mu}{\partial y^j} - (p_i \xi^\mu \frac{\partial \lambda}{\partial x^\mu}) + \xi_i \right) dy^i \wedge \eta.
\]

Here we have used the relation \( d\eta_\mu = \lambda_{x^\mu} \eta \).

Equating the obtained expression to zero we get the expression for \( \xi_i \) as in the Proposition.

Since the commutator of a projectable vector field \( \xi^\mu(x) \partial_{x^\mu} + \xi_i(x^\nu, y^j) \partial_{y^j} \) is vertical, lift local automorphisms of \( \pi \) (and the corresponding infinitesimal transformations - Lie derivatives with respect to the lifted vector fields) preserve the subbundles \( \Lambda^k_0 Y \subset \Lambda^k Y \) and, therefore, define the lifts of automorphism transformations (global, local or infinitesimal) to the corresponding automorphisms of the (double) bundles \( \Lambda^k_0 Y \rightarrow Y \rightarrow X \).

Combining the last result with the prolongation \( \xi^{0} \) from Definition-Proposition 7 and the prolongation \( \xi^{(n+2)} \) from Proposition 9 and using factorization by \( \Lambda_2^{(n+1)+(n+2)} \) we get the following

Corollary 1

For any projectable vector field \( \xi \in X(\pi) \) there exists the unique projectable vector field \( \xi^1 \) in the space \( \tilde{Z} = \Lambda^2_2^{(n+1)+(n+2)} / \Lambda_1^{(n+1)+(n+2)} \) - prolongation of \( \xi \), preserving the \((n+1)+(n+2)\) form \( p_i^\mu du^i \wedge \eta_\mu + p_k du^k \wedge \eta \).

5.3. Transformations of \( W_0 \) and \( W_1 \).

Taking the fiber product of the action of \( A = X(\pi^1) \) (global, local or infinitesimal) in \( J^1_p(\pi) \) and its action by the (global, local or infinitesimal) automorphisms of the bundle \( \Lambda^k_0 Y \rightarrow Y \rightarrow X \) induced first by the projection to \( Y \) and then by the lift to \( \Lambda^k Y \) described in the last subsection, we define the (global, local or infinitesimal) action of the group \( Aut(\pi^1) \) on the bundles \( W^{co} = J^1_p(\pi) \times Y \Lambda^2_2^{(n+1)+(n+2)} \) and \( W_p = J^1_p(\pi) \times Y (\Lambda^2_2^{(n+1)+(n+2)} / \Lambda_1^{(n+1)+(n+2)}) = Z_p \times \tilde{Z} \).

Combining this action with the homomorphism \( \hat{X}(\pi) \rightarrow X(\pi^1) \) induced by the lift prolongation we get the action of \( \hat{X}(\pi) \) by the projected diffeomorphisms of \( W^{co} \) and \( W_p \).

6. General Constitutive Relations (CR).

In this section we define general constitutive relations and the Poincare-Cartan forms defined by these relations. We also give examples of several types of constitutive relations.
(ii) A general constitutive relation $\mathcal{C}$ is called **regular** if the mapping $\mathcal{C}$ is the diffeomorphism of $Z_p$ onto the submanifold of $Z$.

(iii) A covering constitutive relation $\hat{\mathcal{C}}$ of the field theory with the configurational bundle $\pi : Y \to X$ and the partial 1-jet space $Z_\pi = J^1_p(\pi)$ is a smooth mapping of bundles

$$\hat{\mathcal{C}} : J^1_p(\pi) \to \Lambda^{(n+1)+(n+2)}_2 Y$$

or, what is the same, a section $\sigma$ of the bundle $\pi^* Y_{(n+1)+(n+2)}$ above. In local coordinates $(x^\mu, y^i, z^i_\mu)$ on $J^1_p(\pi)$ and $(p, p^\mu_i, q_i)$ on $\Lambda^{n+1}_2 Y \oplus \Lambda^{n+2}_2 Y$ a CCR-mapping $\hat{\mathcal{C}}$ has the form

$$\hat{\mathcal{C}}(x^\mu, y^i, z^i_\mu) = (x^\mu, y^i; p(x^\mu, y^i, z^i_\mu); F^\mu_i(x^\mu, y^i, z^i_\mu); \Pi_i(x^\mu, y^i, z^i_\mu))$$

The physical case corresponds to the choice $n = 3$.

(iv) A constitutive relation $\mathcal{C}$ (respectively a covering CR $\hat{\mathcal{C}}$) is called a conservative relation (CR) (resp. a covering CCR) if $\Pi_i = 0, i = 1, \ldots, m$.

(v) For a given constitutive relation $\mathcal{C}$ denote by $\mathcal{C}_-$ the constitutive relation obtained from $\mathcal{C}$ by changing the sign of the production $(n + 2)$ part:

$$\mathcal{C}_-(x^\mu, y^i, z^i_\mu) = (x^\mu, y^i; F^\mu_i(x^\mu, y^i, z^i_\mu); -\Pi_i(x^\mu, y^i, z^i_\mu))$$

**Remark 4** The definition given here is very broad, including, in particular, a zero mapping. Thus, to get a useful class of constitutive relations one has to put some nondegeneracy conditions to this mapping including, but not reducing to, the regularity of a CR defined above.

We have the following simple

**Proposition 10** (i) Constitutive relations (and the covering constitutive relations) form the $C^\infty(J^1_p(Y))$-module $\mathcal{C}R$ and $\hat{\mathcal{C}}R$ respectively.

(ii) Let $\hat{\mathcal{C}} \in \hat{\mathcal{C}}R$ be a CCR, then combining the defining mapping $\hat{\mathcal{C}} : J^1_p(\pi) \to \Lambda^{(n+1)+(n+2)}_2 Y$ with the projection by $\Lambda^{(n+1)+(n+2)}_1 Y$ we associate with a CCR $\hat{\mathcal{C}}$ the constitutive relation $C \in \mathcal{C}R$.

(iii) A canonical linear mapping $\mathcal{C}R \to \hat{\mathcal{C}}R$ (section of the projection above) is defined by the formula

$$(\hat{\mathcal{C}})(z) = (x^\mu, y^i; -z^i_\mu F^i_\mu(x^\mu, y^i, z^i_\mu); F^\mu_i(x^\mu, y^i, z^i_\mu); \Pi_i(x^\mu, y^i, z^i_\mu)).$$

$\mathcal{C}R \hat{\mathcal{C}}$ will be called the lifted CCR of the constitutive relation $C$.

**Example 3** In the maximal (RET) case $U = U_0$, the partial 1-jet space $J^1_p(\pi) = Y$ coincides with $Y$, (its fiber is a point $R^q$) and the constitutive relation is just the section of the bundle $\hat{Z} \to Y$.

Let $\mathcal{C}$ be a general constitutive relation. Taking the pullback of the canonical form $\hat{\Theta}_\nu$ on the bundle $\hat{Z}$ we get the **Poincare-Cartan form of the constitutive relation $\mathcal{C}$**

$$\Theta_{\mathcal{C}, \nu} = C^* (\hat{\Theta}_\nu) = (F^\mu_i \Gamma^i_\mu) \eta + F^\mu_i dy^i \wedge \eta_\mu + \Pi_i dy^i \wedge \eta.$$

Using the canonical forms on the bundle $\Lambda^{(n+1)+(n+2)}_2 Y$ we get the **Poincare-Cartan form of the covering constitutive relation $\hat{\mathcal{C}}$**

$$\Theta_{\hat{\mathcal{C}}, \nu} = \hat{\Theta}_\nu = (F^\mu_i \Gamma^i_\mu) \eta + F^\mu_i dy^i \wedge \eta_\mu + \Pi_i dy^i \wedge \eta.$$
\[ \Theta = C(\Theta^n + \Theta^{n+2}) = p\eta + F^\mu_i dy^i \wedge \eta. \]  

(35)

For the lifted CCR \( \tilde{C} \) of a CR \( C \) we have

\[ \Theta \tilde{C} = -z^i \mu F^\mu_i \eta + F^\mu_i dy^i \wedge \eta. \]  

(36)

Below we list several types of constitutive relations that are widely used in physics and continuum mechanics.

**Example 4** A *Lagrange constitutive relation* defined by a smooth (Lagrangian) function \( L \in C^\infty(Z_p) \) is given by the mapping:

\[ C_L(x^\mu, y^i, z^i_\mu) = (p^\mu_i = \frac{\partial L}{\partial z^i_\mu}; \Pi_i = \frac{\partial L}{\partial y^i}). \]  

(37)

Correspondingly, a *covering Lagrange constitutive relation* is defined by a smooth function \( L \in C^\infty(Z_p) \) giving the mapping

\[ \hat{C}_L(x^\mu, y^i, z^i_\mu) = (p = -z^i \mu \frac{\partial L}{\partial z^i_\mu}, p^\mu_i = \frac{\partial L}{\partial z^i_\mu}; \Pi_i = \frac{\partial L}{\partial y^i}). \]  

(38)

Notice that the covering Lagrange relation defined here does not coincide with the covering CR defined by the Legendre transformation of the Lagrangian \( L_\eta \).

**Example 5** A *semi-Lagrangian CR* is defined by a smooth function \( L \in C^\infty(Z_p) \) and arbitrary functions \( Q_i \in C^\infty(Z_p), i = 1, \ldots, m \):

\[ C_{L,Q_i}(x^\mu, y^i, z^i_\mu) = (p^\mu_i = \frac{\partial L}{\partial z^i_\mu}; \Pi_i = Q_i(x^\mu, y^i, z^i_\mu)). \]  

(39)

**Remark 5** When the domain of \( C \) is a partial 1-jet bundle \( J^1_1(p) \) the components \( F^\mu_i \) of the dissipative potential \( D \) are absent from the fibers of \( J^1_1(p) \). In the case of RET the semi-Lagrangian CR is trivially zero.

A very important example of a semi-Lagrangian CR is the following

**Example 6** \( L + D \)-system. Let \( L \) be a smooth function \( L \in C^\infty(Z_p) \). Let the time derivatives \( z^i_0 \) of all basic fields belong to \( Z_p \) and let \( D \in C^\infty(Z_p) \) be one more function (a dissipative potential). Define the constitutive relation \( C_{L,D} \) that differs from Lagrangian CR \( C_L \) by the condition \( \Pi_i = L, y^i + D, z^i_0 \). Thus, the corresponding Poincare-Cartan form is

\[ \Theta_{L,D} = \Theta_L + D(z^i_0) dy^i \wedge \eta. \]  

(40)

**Example 7** Vector-potential CR. Consider a RET case. Let \( h = h^\mu(x, y) \eta_\mu \) be a semi-basic \( n \)-form on \( Y \). Define a constitutive relation by the formula

\[ C_h(x^\mu, y^i) = (p^\mu_i = \frac{\partial h^\mu}{\partial y^i}; \Pi_i = \Pi_i(x, y)). \]  

(41)

This is the case of the dual formulation in terms of Lagrange-Liu variables (\( y^i \) replaces the \( \lambda^i \) here), see Section 13 below and [29].
7. Balance System defined by a Constitutive Relation $\mathcal{C}$.
In the Lagrangian Field Theory the Euler-Lagrange system of equations is obtained by (9) using arbitrary variations $\xi \in \mathcal{X}(Z)$:

$$j^1_p(s)di_\xi \Theta_L(z) = 0. \quad (42)$$

For a general constitutive relation (31) and the corresponding Poincare-Cartan form $\Theta_{\mathcal{C},\nu}$ we have an analog of the variational formula given by the Cartan formula

$$L_\xi \Theta_{\mathcal{C},\nu} = i_\xi d\Theta_{\mathcal{C},\nu} + di_\xi \Theta_{\mathcal{C},\nu}. \quad$$

Thus, we can try to formulate the balance laws corresponding to the CR mapping $\mathcal{C}$ by taking the pullback via $j^1_p(s)$ of the first or second term in the Cartan formula and request it to be zero for a set of variation vector fields $\xi$, large enough to separate the balance equations corresponding to the CR $\mathcal{C}$. Yet, as we see below, both of these cases leads to some restrictions to the set of admissible variations $\xi$. We should have $m$ linearly independent vector fields $\xi$ in order to extract all $m$ balance equations from the invariant formulation of the type (42). Locally this is always possible but still leads to some restrictions on the type of variations. We will see that there is a way around this difficulty if one uses the reduced horizontal differential $\tilde{d}$ (see Appendix B, comp. [7]) instead of the conventional De-Rham differential $d$ in the CR version of (42).

Below we present three different ways of obtaining the conventional form of balance system from an invariant formulation.

7.1. Poincare-Cartan formulation of a balance system.
We start with the Poincare-Cartan way of obtaining the balance equations and for this we take an arbitrary vector field $\xi \in \mathcal{X}(Z_p)$ locally having form $\xi = \xi^\mu \partial_{x^\mu} + \xi^i \partial_{z^i}$, and plug it into the the Poincare-Cartan form $\Theta_{\mathcal{C},\nu}$

$$i_\xi \Theta_{\mathcal{C},\nu} = F^\mu_i \xi^i \eta_\mu - F^\mu_i \xi^i dy^j \wedge \eta_{\mu\nu} - \Pi_i \xi^\mu dy^j \wedge \eta_\mu + \eta_\mu. \quad (43)$$

Then we apply the I-differential $\tilde{d}$

$$\tilde{d}i_\xi \Theta_{\mathcal{C},\nu} = d(F^\mu_i \xi^i) \wedge \eta_\mu + (F^\mu_i \xi^i (\partial_{x^\nu} \lambda_G) \eta) - d(F^\mu_i \xi^i dy^j \wedge \eta_{\mu\nu} +$$

$$+ \sum_{\mu<\nu} F^\mu_i \xi^i dy^j \wedge ((\partial_{x^\nu} \lambda_G) \eta_\mu - (\partial_{x^\nu} \lambda_G) \eta_\mu) - (\Pi_i \xi^i) \eta + \Pi_i \xi^\mu dy^j \wedge \eta_\mu.$$}

Requesting the vector field $\xi$ to be vertical (i.e. taking $\xi^\mu = 0$) we get for the Poincare-Cartan form of a CR $\mathcal{C}$

$$\tilde{d}i_\xi \Theta_{\mathcal{C},\nu} = d(F^\mu_i \xi^i) \wedge \eta_\mu + (F^\mu_i \xi^i (\partial_{x^\nu} \lambda_G) \eta) - (\Pi_i \xi^i) \eta =$$

$$= \xi^i F^\mu_{i,x^\nu} \eta + F^\mu_{i,y^j} dy^j \wedge \eta_\mu + F^\mu_{i,z^j} dz^j \wedge \eta_\mu) + (F^\mu_i \xi^i (\partial_{x^\nu} \lambda_G) \eta) - \Pi_i \xi^i \eta + F^\mu_i d\xi^i \wedge \eta_\mu.$$}

Applying now the pullback by the 1-jet $j^1_p(s)$ of a section $s \in \Gamma(\pi)$ and using

$$j^1_p(s)[F^\mu_{i,y^j} dy^j \wedge \eta_\mu + F^\mu_{i,z^j} dz^j \wedge \eta_\mu] = F^\mu_{i,y^j} s^j_i \eta + F^\mu_{i,z^j} s^j_i \eta,$$

we get

$$j^1_p(s)\tilde{d}i_\xi \Theta_{\mathcal{C}} = \xi^i [(F^\mu_i \circ j^1_p(s))_{,x^\nu} + F^\mu_i (\partial_{x^\nu} \lambda_G) - \Pi_i \circ j^1_p(s)] \eta +$$
A constitutive relation chart

The parentheses on the right-hand side contain the pullback by $j^1(s)$ of the total derivative $d_\mu \xi^i = \xi^i_{,x^\mu} + z^j_\mu \xi^i_{,y^j} + z^j_\mu \xi^j_{,z^j_\mu}$ by $x^\mu$ of the component $\xi^i$ of the vector field $\xi$ along section $s$ contracted with the form $F^1_\mu$.

In order to extract the balance equations from the invariant form of the variational principle obtained by equating to zero the obtained expression, we require the second term in (49) vanishes.

Thus, the vector field $\xi$ should satisfy the additional condition explicitly formulated in the next definition.

**Definition 9** (i) For a constitutive relation $C$ (or, more precisely, for the current form $F = F^\mu dy^\mu \wedge \eta_\mu \in \Lambda^n Y/\Lambda^n Y$) denote by $X(C)$ the sheaf associated to the pre-sheaf of vector fields over $Y$ that for an open set $U \subset Y$ consists of vertical vector fields $\xi \in \Gamma(U, V(\pi))$ whose flow prolongation $\xi^i = \xi^i_{,y^j} + (d_\mu \xi^i)_{,z^j_\mu}$ satisfies the condition $\xi^i_{,y^j} = \xi^i$ and

$$FDiv(\xi^i) = F^\mu d_\mu \xi^i = 0$$

in $U$. Vector fields - sections of the sheaf $X(C)$ will be called $C$-admissible.

(ii) A constitutive relation $C$ with the current form $F$ is called locally separable if each point $y \in Y$ has a neighborhood $U_y$ such that there are $m$ vector fields $\xi_k$ in the space of sections $\Gamma(U_y, X(C))$ linearly independent at each point $y_1 \in U_y$.

(iii) A constitutive relation $C$ is called separable in an an open subset $W \subset Y$ if there are $m$ vector fields $\Gamma(W, X(C))$ linearly independent at each point of $W$.

Having introduced these notions we can now formulate the Poincare-Cartan version of the variational principle for the balance system with the domain $J^1_p(\pi)$ and the constitutive relation $C$.

**Definition 10** Let $C$ be a constitutive relation with the domain $J^1_p(\pi)$. We say that a section $s : D_s \to Y$ of the bundle $\pi : Y \to X$, $D_s$ being an open subset in $X$, satisfies the balance system (system of balance laws) defined by $CR\ C$ if for all $C$-admissible vector fields $\xi \in X(C)|_{\pi^{-1}(D_s)}$ (i.e. over $D_s$)

$$\tilde{d}(j^1_p(s)^*(i_\xi \Theta C)) = j^1_p(s)^*(\tilde{d}(i_\xi \Theta C)) = 0.$$  

Here $\tilde{d}$ is the Iglesias differential of a $(n+(n+1))$-form on $X$ (see Appendix B). In simple terms, for a $(n) + (n+1)$-form $\omega^n + \omega^{n+1}$

$$\tilde{d}(\omega^n + \omega^{n+1}) = d\omega^n - \omega^{n+1}.$$  

Let $z \in J^1_p(\pi)$ and let $V \subset X$ be a neighborhood of the projection $x = \pi_1(z) \in X$ over which the bundle $Y$ is trivial $Y|_V \approx V \times U$ and $W \subset U$ be an open set such that $V \times W$ is the neighborhood of the point $y = \pi_{10}(z)$. We may assume that $V$ and $W$ are domains of the adapted chart $(x^\mu, y^i)$. Vector fields $\xi_j = \partial_{y^j} \in V(\pi)|_{V \times W}$ have the property that in this local chart $\frac{\partial \xi_j}{\partial x^\mu} = 0$ in $\pi_{10}(V \times W)$. Therefore these $m$ vector fields in $X(V \times W)$ are $C$-admissible for any constitutive relation $C$. As a result we get

**Proposition 11** Any constitutive relation $C$ is locally separable.

Globally defined $C$-admissible vector fields have an important meaning for the balance system $B_C$ (see Section 10 below). There exists a geometrical situation where, at each point, there is a natural class of $m$ linearly independent globally defined vector fields $\xi$ admissible for all $CR$.
Proposition 12 Let \( \pi : Y \to X \) be a trivial principal bundle of a connected abelian \( n \)-dimensional Lie group \( A \). Then the (globally defined) fundamental vector fields on \( Y \) (generated by the right action of \( A \) on \( Y \)) satisfy the condition \((45)\).

Proof is trivial.

Corollary 2 Let \( \pi : Y \to X \) be a trivial principal bundle of a connected abelian \( m \)-dimensional Lie group \( A \) (in particular, a trivial vector bundle over \( X \)). Let \( C \) be an arbitrary constitutive relation. Then the equivalence statement of Theorem 1 below is valid for any constitutive relation \( C \).

Now we formulate the main result of this section in the Poincaré-Cartan formulation.

Theorem 1 If a constitutive relation \( C \) is locally separable, then the following statements for a section \( s \in \Gamma(\pi)(D_s) \), \( D_s \subset X \) are equivalent:

(i) 
\[
dot j^1_{ap}(s) \xi_0 \Theta_C \equiv j^1_{ap}(s) \dot{\xi}_C = 0, \text{ for all } \xi \in \mathcal{X}(\mathcal{C})|_{D_s},
\]

(ii) Section \( s \) is the solution of the following system of balance laws - balance system:
\[
(F^\mu \circ j^1_p(s))_{,x^\mu} + F^\mu_i(\partial_{x^\nu} \lambda_G) = \Pi_i(j^1_p(s)), \quad i = 1, \ldots, m.
\]

Henceforth we will refer to the formula \((48)\) as \((\spadesuit)\).

Proof: It is clear that (i) follows from (ii). If (i) true, choose \( m \) (local) linearly independent vector fields \( \xi_k \in \mathcal{X}(G) \) in a neighborhood of a point \( x \in X \). Then, for such (locally defined) vector fields \( \xi \) the last term in \((44)\) is zero and we get the system of linear equations
\[
\xi_k(s(x))((F^\mu_i \circ j^1_p(s))_{,x^\mu} + F^\mu_i(\partial_{x^\nu} \lambda_G) - \Pi_i j^1_p(s)) = 0
\]
at each point \( y \) in a neighborhood of the point \( x \) for \( i \) unknowns \((F^\mu_i \circ j^1_p(s))_{,x^\mu} + F^\mu_i(\partial_{x^\nu} \lambda_G) - \Pi_i j^1_p(s)\). The matrix \( \xi_k(s(x)) \) of this system is nondegenerate, so the solution to the system is zero. Being true in a neighborhood of any point \( x \in X \), (ii) is true on all \( X \).

Example 8 For a Lagrangian constitutive relation (See Example 3) the balance system \((\spadesuit)\) takes the form
\[
\frac{\partial L}{\partial \dot{z}^\mu} (j^1_p(s))_{,x^\mu} + \frac{\partial L}{\partial \dot{z}^\mu} j^1_p(s)(\partial_{x^\nu} \lambda_G) = \frac{\partial L}{\partial y^i} (j^1_p(s)) = 0
\]
or
\[
\frac{\partial}{\partial x^\mu} \frac{\partial L}{\partial \dot{z}^\mu} (j^1_p(s))_{,x^\mu} + \frac{\partial L}{\partial \dot{z}^\mu} j^1_p(s)(\partial_{x^\nu} \lambda_G) - \frac{\partial L}{\partial y^i} (j^1_p(s)) = 0,
\]
which is the system of Euler-Lagrange equations for the Lagrangian form \( L \eta \). Here \( \lambda_G = \ln(|G|) \).

Example 9 \( L + D \)-system. For a \( L + D \)-system where the Poincaré-Cartan form is
\[
\Theta_{L,D} = \Theta_L + D_{\dot{z}^i} dy^i \wedge \eta,
\]
the corresponding balance system \((\spadesuit)\) has the form
\[
\mathcal{E}_{L}(s)_i = \frac{\partial}{\partial x^\mu} \frac{\partial L}{\partial \dot{z}^\mu} (j^1_p(s))_{,x^\mu} + \frac{\partial L}{\partial \dot{z}^\mu} j^1_p(s)(\partial_{x^\nu} \lambda_G) - \frac{\partial L}{\partial y^i} (j^1_p(s)) = \frac{\partial D}{\partial y^i} (j^1(s)).
\]
These are the Euler-Lagrange equations with a dissipative Rayleigh potential \( D \) (see [17]).

Example 10 System of conservation laws. If we take \( \Pi_i = 0 \) in the constitutive relation \( C \) then the balance system \((\spadesuit)\) takes the form of the system of conservation laws
\[
(F^\mu \circ j^1_p(s))_{,\mu} = 0.
\]
7.2. Euler-Lagrange formulation of the balance system.

Now we will see what happens if we apply the standard order \( i \mathcal{D} \) of operations that is used in the Lagrangian Theory (9.2) to the covering Poincare-Cartan form \( \Theta_{\mathcal{C}_-} \) of the constitutive relation \( \mathcal{C} \)

\[
\Theta_{\mathcal{C}_-} = (-z^i_\mu F^\mu_i)\eta + F^1_i dy^i \wedge \eta_\mu - \Pi_1 dy^i \wedge \eta.
\]

We reversed the sign of the source term in the Poincare-Cartan form to compensate for the different order of operation of contraction and applying the differential.

For a given section \( s \in \Gamma_V(\pi) \), \( V \subset X \) we request the fulfilment of the equation

\[
j^1_p(s)^* (i_\xi \tilde{d} \Theta_{\mathcal{C}_-}) = 0
\]

for large enough family of (locally defined) vector fields \( \xi \in \mathcal{V}(\pi^1) \) vertical with respect to the projection \( \pi^1: Z_p \to X \) guarantying the sections \( s \) to be a solution of the balance system of \( m \) independent balance equations. Remark that vector fields \( \xi^1 \) for \( \xi \in \mathcal{V}(\pi) \) is a special case of considered vector fields. We take the \( (n+1)+ (n+2) \)-form \( \Theta_{\mathcal{C}_-} \) of the type (36) and apply first the Iglesias differential \( \tilde{d} \) and then \( i_\xi \) for a vertical vector field \( \xi = \xi^i \partial_{y^i} + \xi_\mu \partial_{\eta^\mu} \). Recall that for \( \xi \in \mathcal{V}(\pi) \)

\[
\xi^1 = \xi^i \partial_{y^i} + d_\mu \xi_\mu \partial_{\eta^\mu}.
\]

We get

\[
i_\xi \tilde{d} \Theta_{\mathcal{C}_-} = i_\xi \left[ -d(z^i_\mu F^\mu_i) \wedge \eta + dF^\mu_i \wedge dy^i \wedge \eta_\mu - F^1_i dy^i \wedge \eta_\mu + \Pi_1 dy^i \wedge \eta \right]
\]

\[
= -\xi^1 \cdot (z^i_\mu F^\mu_i) \eta + \xi^1 \cdot F^\mu_i dy^i \wedge \eta_\mu - \xi^1 dF^\mu_i \wedge \eta_\mu - \lambda_{G,x^\nu} F^\mu_i \xi^i \eta + \xi^i \Pi_1 \eta
\]

\[
= -[D_\mu \xi^1 F^\mu_i + (z^i_\mu \xi^1 \cdot F^\mu_i)]\eta + \xi^1 \cdot F^\mu_i (\omega^i + z^i_\mu d\eta^\nu) \wedge \eta_\mu - \xi^1 \cdot F^\mu_i (\omega^i + z^i_\mu d\eta^\nu) \wedge \eta_\mu
\]

\[
= -[D_\mu \xi^1 F^\mu_i + (z^i_\mu \xi^1 \cdot F^\mu_i)]\eta + \xi^1 \cdot F^\mu_i (\omega^i + z^i_\mu d\eta^\nu) \wedge \eta_\mu
\]

Equating this to zero we see that the \( \xi \)-weighted balance equation will be true for a section

\[
j^1_p(s)^* \xi^1 [d_\mu F^\mu_i + \lambda_{G,x^\nu} F^\mu_i - \Pi_1] = 0
\]

if and only if \( j^1_p(s)^* \xi^1 F^\mu_i = 0 \) for this vertical vector field \( \xi \) (notice that \( \mathcal{X}(\mathcal{C}) = \mathcal{X}(\mathcal{C}_-) \)). This brings us back to the extension of the condition \( FDiv(\xi) = 0 \) to the space of all \( \pi^1 \)-vertical vector fields (see above)

**Definition 11**

(i) A vector field \( \xi \in X(U) \), \( U \subset Z_p \) is called \( \mathcal{C} \)-admissible if \( \xi^1 F^\mu_i = 0 \) in \( U \).

(ii) A constitutive relation \( \hat{\mathcal{C}} \) is called separable in \( U \) if the space of projections to \( Y \) of the space of \( \mathcal{C} \)-admissible vector fields in \( U \) has, at each point \( \pi_{10}(U) \) dimension \( m \).
The arguments above prove the following

**Theorem 2** If a constitutive relation $C$ is locally separable, then the following statements for a section $s \in \Gamma(p)(U)$, $U \subset X$ are equivalent:

(i) \[ j_p^1(s)^*(i_{\xi} \tilde{d}\Theta_{C_\mu}) = 0 \quad \text{for all } \xi \in \mathcal{X}(\mathcal{C}|_U). \quad (53) \]

(ii) Section $s$ is the solution of the following system of balance laws - **balance system**:

\[
(F_i^\mu \circ j_p^1(s))_{,x^\nu} + F_i^\mu(\partial_{x^\nu}\lambda_G) = \Pi_i(j_p^1(s)), \quad i = 1, \ldots, m. \quad \star
\]

**Remark 6** It is instructing to compare the calculation above in the proof of the last Theorem with the similar one for the Poincare-Cartan form corresponding to a Lagrangian $L \in C^\infty(Z)$. In this last case, the first term in $\Theta_L$ is $(L - z^\nu L_{z^\nu})\eta$. It such a case $d(L - z^\nu L_{z^\nu})\eta = dL - L_{z^\nu}dz^\nu - z^\nu dL_{z^\nu}$ and the second term in this expression is cancelled by the similar term coming from the first one. As a result the term $D_\mu \xi^i F_i^\mu$ does not appear and there are no restrictions on the variation $\xi$.

### 7.3. Horizontal (reduced) differential formulation of the balance system.

Here we use (see Appendix C) the reduced horizontal differential $\tilde{d}$ acting from $J^k(\pi)$ to $J^{k+1}(\pi)$ for all $k$ by the formulas

\[
\begin{aligned}
\tilde{d}f(x,y,z^\mu_1, \ldots, z^\mu_{m-1}) &= (f_x + z^\mu_1 f_y + z^\mu_2 f_{x,y} + \ldots + z^\mu_{m-1} f_{x,y, \ldots, z^\mu_{m-1}}) dx^\nu, \\
\tilde{d}dy^i &= 0, \quad \tilde{d}z^\mu_{m-1} = 0.
\end{aligned}
\]

Now, let us postulate the balance system corresponding to the CR $\mathcal{C}_{\xi,\nu}$ in the form

\[ j_p^1(s)^*(i_{\xi} \tilde{d}\Theta_{\mathcal{C}_{\xi,\nu}}) = 0, \quad (55) \]

for all variations $\xi \in \mathcal{X}(Z_p)$.

Notice that the additional term in $\Theta^{n+1}_{\pi}$ of the form $h(z)\eta$ produced by an arbitrary transformation by an adopted transformation $\phi \in \text{Aut}(\pi)$ will be eliminated by applying the reduced horizontal differential $\tilde{d}$ since $\tilde{d}(\eta) = 0)$, so that this equation is independent on a choice of a connection $\nu$ in the Poincare-Cartan form $\tilde{\Theta}_{\mathcal{C}_{\xi,\nu}}$.

We have, for any vector field $\xi \in \mathcal{X}(Z_p)$

\[
\begin{aligned}
i_{\xi} \tilde{d}\Theta_{\mathcal{C}_{\mu}} &= i_{\xi}[\tilde{d}(F_i^\mu dy^i \wedge \eta_\mu) + \Pi_i dy^i \wedge \eta] \\
&= i_{\xi}[-(D_{x^\rho}F_i^\mu)du^i \wedge \eta - F_i^\mu dy^i \wedge \tilde{d}\eta_\mu) + \Pi_i dy^i \wedge \eta] \\
&= i_{\xi}[-(D_{x^\rho}F_i^\mu - F_i^\mu \lambda_G) + \Pi_i]dy^i \wedge \eta \\
&= \xi[D_{x^\rho}F_i^\mu - F_i^\mu \lambda_G + \Pi_i]dy^i \wedge \eta + \xi^\mu [-D_{x^\rho}F_i^\mu - F_i^\mu \lambda_G + \Pi_i]dy^i \wedge \eta,
\end{aligned}
\]

since $\tilde{d}\eta_\mu = \lambda_G + \Pi_i dy^i \wedge \eta$. Now we take the pullback by the section $j_p^1(s)$. Since the forms $y^i \wedge \eta_\mu$ and $\eta$ are linearly independent, the condition (55) will be fulfilled for all vector fields $\xi \in \mathcal{X}(Z_p)$ if and only if the balance system of equations $\star$

\[
j_p^1(s)^*[D_{x^\rho}F_i^\mu + F_i^\mu \lambda_G] = \Pi_i(j_p^1(s)), \quad i = 1, \ldots, m
\]

is satisfied by the section $s$. Thus, we get
Theorem 3 Let \( C \) be a constitutive relation. For a section \( s \in \Gamma(\pi) \) the following statements are equivalent

(i) For all vector fields \( \xi \in \mathcal{X}(Z_p) \),
\[
j^1_p(s)i_{\xi^1}d\Theta_{C_\pi} = 0.
\]

(ii) Section \( s \) is the solution of the balance system (★)
\[
j^1_p(s)[D_{x^\mu}F_i^\mu - F_i^\mu\lambda_{G,x^\mu}] = \Pi_i(j^1_p(s)), \ i = 1, \ldots, m.
\]

Remark 7 If we would like to use the conventional horizontal differential \( d_H \) instead of augmented one in the formulated above we would still remove the term of the form \( q(z) \eta \) of the Poincare-Cartan form of \( CR \ C \), but in the calculation above we would get, for a vertical vector field \( \xi \) on \( Y \) an extra term
\[
i_{\xi^1}[F_i^\mu d_Hdy^i \wedge \eta]\]
and, after taking the pullback by a section \( s : X \to Y \) we would get an extra term \( (F_i^\mu d_\mu \xi^i)\eta \). This brings us back to the requirement that vector field \( \xi \) of variation is \( C \)-admissible.

8. Action of geometrical transformations on the constitutive relations.

In this section we study the action on the constitutive relations of the natural prolongations of \( Y \) studied in Section 5.

Let \( \xi \in \mathcal{X}(\pi) \) be an infinitesimal automorphism (vector field) of the bundle \( \pi \), i.e. a projectable vector field in \( Y \) \( \xi = \xi^\mu(x)\partial_{x^\mu} + \xi^i(x,y)\partial_{y^i} \). Let \( \xi^1 \) be its prolongation to the projectable contact vector field in \( J^1_p(\pi) \) (see Propositions 6-8 Sec.5). Thus, we have
\[
\xi^1 = \xi^\mu(x)\partial_{x^\mu} + \xi^i(x,y)\partial_{y^i} + \left( D_{\mu} \xi^i - z^i_\mu \frac{\partial \xi^\mu}{\partial x^\mu} \right) \partial_{z^i_\mu},
\]
where the summation in the last term is taken over the \( z^i_\mu \) that are present in the partial 1-jet bundle. In the RET case we do not need to introduce any prolongation.

Let \( \xi^{1*} \) be the prolongation of \( \xi \) to the projectable vector field in \( \tilde{Z} = \Lambda^*_{2,n+(n+1)}/\Lambda^*_{1,n+(n+1)} \):
\[
\xi^{1*} = \xi^\mu(x)\partial_{x^\mu} + \xi^i(x,y)\partial_{y^i} + \left( -p^\nu \left( \frac{\partial \xi^\mu}{\partial x^\nu} - \xi^\mu \lambda_{x^\nu} \right) - p^\nu \frac{\partial \xi^i}{\partial u^\nu} - p^i \left( \frac{\partial \xi^\nu}{\partial x^\nu} - \xi^\nu \lambda_{x^\nu} \right) - p^i \xi^\mu \lambda_{x^\nu} \right) \partial_{p^\mu} + \left( p_k \left( \xi^\mu \frac{\partial \lambda_{x^\nu}}{\partial x^\nu} - \frac{\partial \xi^\mu}{\partial x^\nu} \right) - p^i \frac{\partial \xi^i}{\partial u^\nu} \right) \partial_{p^k}.
\]

Let now \( \phi^1 \) be a local flow in \( J^1_p(\pi) \) of the vector field \( \xi^1 \) and \( \psi_l \) be a local flow in \( \tilde{Z} \) of the vector field \( \xi^{1*} \).

Let \( C : Z_p \to \tilde{Z} \) be a constitutive relation and \( \Theta_{C,\nu} \) - corresponding \( \nu \)-lifted Poincare-Cartan form of \( C \). We have
\[
\phi_{-1}^{1*}\Theta_{C,\nu} = \phi_{-1}^{1*}(C^*\tilde{\nu}) = \phi_{-1}^{1*}C^*\phi_1^*\psi_l \tilde{\nu} = \left( \phi_{-1}^{1*}C^*\psi_1 \right) \tilde{\nu} + (F^\mu_{\nu}\Gamma^I_\mu) \circ \psi_l = (\psi_l \circ C \circ \phi_{-1}^*) \tilde{\nu} + (F^\mu_{\nu}\Gamma^I_\mu) \circ \psi_l = C^{0*} \Theta_{C,\nu} + (F^\mu_{\nu}\Gamma^I_\mu) \circ \psi_l \eta = \Theta_{C^{0*},\nu} + C^{0*} \circ ((F^\mu_{\nu}\Gamma^I_\mu) \circ \psi_l) \eta,
\]
where we have used the fact that $\psi_t$ acts on the form $\Theta_{C,\mu}$ leaving its $dy^i \wedge \eta_{\mu}$ part invariant. Here we introduced the notation $C^{\phi_1 t} = \psi_t \circ C \circ \phi_{-1}^t$.

Taking the derivative by $t$ at $t = 0$ we get the generalized Lie derivative of mapping $C$ with respect to the vector fields $(\xi^1, \xi^4)$ (see [11], Chapter 11) - the vector field over the mapping $C : Z_p \to \hat{Z}$:

$$L_{(\xi^1, \xi^4)}C = C_+ (\xi^1) - \xi^4 \circ C.$$  \hfill (56)

In local adapted coordinates we have

$$L_{(\xi^1, \xi^4)}C = \left[ \left( \xi^\nu \partial_{\nu} + \xi^i \partial_i \right) + \left( d_\sigma \xi^i - z^i \frac{\partial \xi^\nu}{\partial x^\sigma} \right) \partial_{z^i} \right] F^\mu_i + \left( F^\nu_i \left( \frac{\partial \xi^\mu}{\partial x^\nu} - \xi^\mu \lambda_{x^\nu} \right) + F^\mu_i \left( \frac{\partial \xi^\nu}{\partial x^\mu} - \xi^\nu \lambda_{x^\mu} \right) + F^\nu_i \xi^\mu \lambda_{x^\nu} \right) \partial_{\xi^i} + \left( \xi^\nu(x) \partial_{\nu} + \xi^i(x, y) \partial_i \right) d_\mu \xi^i - z^i \frac{\partial \xi^\nu}{\partial x^\mu} \right) \partial_{z^i} \] \Pi_k - \left( \Pi_k \left( \frac{\partial \lambda_{x^\mu}}{\partial x^\nu} - \frac{\partial \xi^\nu}{\partial x^\mu} \right) - \Pi_j \frac{\partial \xi^i}{\partial x^j} \right) \partial_{p^k} \] \hfill (57)

For partial 1-jet bundles we assume restrictions to the automorphisms that were introduced in Section 5. For instance, for $J^1_3(\pi)$ we assume that the automorphisms of $\pi$ preserve the structure of fiber product (20).

**Definition 12**  
(i) A diffeomorphism $\Phi$ of $W_p = Z_p \times \hat{Z}$ is called a **generalized symmetry transformation** of constitutive relation $C$ if $\Phi(\Gamma_C) = \Gamma_{C'}$ for the graph $\Gamma_C$ of the mapping $C$. A generalized symmetry $\Phi$ of $C$ is called a **trivial symmetry** of $C$ if restriction of $\Phi$ to $\Gamma_C$ is identity.

(ii) A couple of diffeomorphisms $\phi^1 \in Diff(Z_p), \psi \in Diff(\hat{Z})$ is said to generate the **symmetry transformation** of $C$ if the diffeomorphism $\Psi = \psi \times \phi^{-1}$ of $W_p$ is the generalized symmetry of $C$. This is equivalent to the condition

$$\psi \circ C(z) = C \circ \phi(z) \text{ for all } z \in Z_p.$$  

A symmetry is, of course, a special case of a generalized symmetry.

(iii) An automorphism $\phi \in Aut_p(\pi)$ is called a **geometrical symmetry transformation** of a constitutive relation $C$ if the diffeomorphism $\Psi = \tilde{\phi}^* \times \phi^{-1}$ of $W_p$ is the symmetry of $C$, i.e. if $C^{\phi_1 t} = C$.

(iv) An automorphism $\phi \in Aut_p(\pi)$ is called a **geometrical symmetry transformation** of a covering constitutive relation $\hat{C}$ if the diffeomorphism $\Psi = \tilde{\phi}^* \times \phi^{-1}$ of $W_p$ is the symmetry of $\hat{C}$, i.e. if $\hat{C}^{\phi_1 t} = \hat{C}$.

(v) Let $\xi \in X_p(\pi)$ be a projectable vector field. We say that $\xi$ is a **geometrical infinitesimal symmetry** of the constitutive relation $C$ if $L_{(\xi^1, \xi^4)}C = 0$.

Properties presented in the next Proposition follows directly from the given definitions.

**Proposition 13**  
(i) A vector field $\xi \in X_p(\pi)$ is an infinitesimal symmetry of $C$ if (and only if) the (local) diffeomorphisms $\Phi^\xi_t = \psi_t \times \phi^{-1}_t$ of $W_p$ defined by the prolongation of $\xi$ map $\Gamma_C$ into itself: $\Phi^\xi_t(\Gamma_C) = \Gamma_C$, i.e. $\Phi^\xi_t$ is the symmetry of $C$. 
(iii) Generalized symmetries of $C$ form the group $GSym(C) \subset Diff(W_p)$.

(iii) Trivial symmetries of $C$ form the normal subgroup $TSym(C)$ of $GSym(C)$.

(iv) Geometrical symmetries $\Phi$ form the subgroup $Sym(C) \subset Aut_p(\pi)$.

(v) Infinitesimal symmetries of $C$ form Lie algebra $\mathfrak{g}(C) \subset \mathcal{X}_p(\pi)$ with the bracket of vector fields in $Y$ as the Lie algebra operation.

(vi) A vector field $X \in \mathcal{X}(W_p)$ is the generator of the 1-parametrical group of generalized symmetries of $C$ if and only if it is tangent to the graph $\Gamma_C$.

Condition that the generalized Lie bracket (57) is zero has the form of a system of first order differential equations for the coefficients of the form $\Theta_C$:

$$
\begin{cases}
(F^\mu_i \cdot \sigma + F^\mu_j \frac{\partial}{\partial y^i}) \xi^j + F^\mu_i \frac{\partial}{\partial y^j} \xi^j = 0, \\
(\Pi_k : z^j, y^k) \xi^j + \Pi_j \frac{\partial}{\partial y^k} \xi^j = 0, \\
(\Pi_k : z^j, y^k) \xi^j + \Pi_j \frac{\partial}{\partial y^k} \xi^j = 0, \quad k = 1, \ldots, m.
\end{cases}
$$

The differential (vector field) part in these equations for a fixed $\mu$ represents the same first order differential operator acting on the components of the vector function with values in the space dual to the vertical tangent vector of the bundle $\pi$, i.e. in $V(\pi^*)$ lifted to the space $Z_p$. It is the same vector field that stays in the left side of equation for $\Pi_i$.

For the vertical vector fields $\xi = \xi^i \partial_{y^i}$ the system (58) takes the form

$$
\begin{cases}
(F^\mu_i \cdot \sigma + F^\mu_j \frac{\partial}{\partial y^i}) \xi^j + F^\mu_i \frac{\partial}{\partial y^j} \xi^j = 0, \\
(\Pi_k : z^j, y^k) \xi^j + \Pi_j \frac{\partial}{\partial y^k} \xi^j = 0.
\end{cases}
$$

We can rewrite the last system as conditions on the vertical vector field $\xi = \xi^i \partial_{y^i}$:

$$
\begin{cases}
(F^\mu_i \cdot \sigma + F^\mu_j \frac{\partial}{\partial y^i}) \xi^j + F^\mu_i \frac{\partial}{\partial y^j} \xi^j = 0, \quad \mu = 1, \ldots, n; i = 1, \ldots, m, \\
(\Pi_k : z^j, y^k) \xi^j + \Pi_j \frac{\partial}{\partial y^k} \xi^j = 0, \quad k = 1, \ldots, m.
\end{cases}
$$

Recall that here $d_\sigma \xi^i = \frac{\partial \xi^i}{\partial z^\sigma} + z_\sigma \frac{\partial \xi^i}{\partial y^\sigma}$.

Let $\xi = \xi^i \partial_{y^i} \in \mathcal{X}(C)$ and let $\phi \in Aut_p(\pi)$ be as above.

**Lemma 5** For a transformed constitutive relation $C^\phi = \psi \circ C \circ \phi^{-1}$ we have $F\text{div}(C^\phi) = \phi^* F\text{div}(C)$.

Proof of this Lemma see [29].

Let $\phi \in Aut_p(\pi)$ be a geometrical symmetry of a CR $C$. For a solution $s \in \Gamma(\pi)$ of the balance system (52), i.e. for a section $s \in \Gamma(\pi)$ such that

$$
(j^1_p(s))^{\ast} i_{\xi} \tilde{\Theta}_{C} = 0
$$

for all $\xi \in \mathcal{X}(C)$ we have

$$
(j^1_p((\phi^1 \ast s))^{\ast} i_{\xi} \tilde{\Theta}_{C} = [\phi^1 \ast (j^1_p(s))]^{\ast} i_{\xi} \tilde{\Theta}_{C} = (j^1_p(s))^{\ast} \circ \phi^1 \ast i_{\xi} \tilde{\Theta}_{C} =
$$
\( \mathcal{C}^\phi = \mathcal{C} \). Last expression is equal zero if (vertical) vector field \( \phi_1 \in \mathcal{X}(\mathcal{C}) \). But, by Lemma 5 above \( \phi_1 \mathcal{X}(\mathcal{C}) = \mathcal{X}(\mathcal{C}^\phi) \). Since for a geometrical symmetry transformation \( \mathcal{C}^\phi = \mathcal{C} \) we have proved the following

**Theorem 4** Let \( \phi \in \text{Aut}_p(\pi) \) be a geometrical symmetry of the CR \( \mathcal{C} \). Then the mapping \( s \to \phi^*s \) maps the set \( \text{Sol}(\mathcal{C}) \) of solutions of the balance system \( \star \) into itself.

### 8.1. Homogeneous constitutive relations.

If the state space of a theory contains enough fields to make the constitutive relations free from the explicit dependence on \( (t, x) \in X \) (general relativity or theory of uniform materials are two examples), then the corresponding balance system simplifies and while studying it one does not need to introduce assumptions on the character of the space-time dependence of the balance system. Definition given below is an invariant way to distinguish a class of such CR.

Any local chart \( x^\mu \) in \( X \) defined the local (translational) action of \( R^n \) in \( X \) associating with the basic vectors \( e_\mu \) the vector field \( \partial_{x^\mu} \). Vice versa, any \( n \)-dimensional commutative subalgebra \( \mathfrak{h} \) of the Lie algebra of vector fields \( \mathcal{X}(U) \), \( U \) being an open connected subset of \( X \), defines the locally transitive action of \( R^n \) in \( U \) and, therefore, a local chart in a neighborhood of any point in \( U \).

**Definition 13**

(i) Let \( \nu \) be a connection in the bundle \( \pi \) satisfying to the conditions of Propositions 13 or 14 with "partial" meaning \( K \oplus K' \) or \( S \) respectively. We will call a constitutive relation \( \mathcal{C} \) \textbf{\( \nu \)-homogeneous} if any point \( z \in Z_p \) there exists a local chart in a neighborhood \( U_x \), \( x = \pi^1(z) \) such that the Poincare-Cartan form \( \Theta_{\mathcal{C},\nu} \) of the CR \( \mathcal{C} \) is invariant under the local flows \( \phi^\nu_\xi \) of the flow lifts \( \hat{\xi} \) of \( \nu \)-horizontal vector fields \( \hat{\xi}, \xi \in \mathfrak{h} \) in the neighborhood of \( y = \pi^{10}(z) \):

\[
\mathcal{L}_{\xi^1} \Theta_{\mathcal{C},\nu} = 0 \mod q\eta.
\]

(ii) A constitutive relation \( \mathcal{C} \) is called a homogeneous if there is a connection \( \nu \) on the bundle \( \pi \) such that \( \mathcal{C} \) is \( \nu \)-homogeneous.

Next statement follows directly from the definitions.

**Proposition 14** Let \( \nu \) be a connection in the bundle \( \pi \). Then the following properties of a constitutive relation \( \mathcal{C} \) are equivalent:

(i) \( \mathcal{C} \) is \( \nu \)-homogeneous,

(ii) For all \( \xi \in \mathfrak{h} \), the \( \nu \)-horizontal lift \( \hat{\xi} \) is the infinitesimal symmetry of the constitutive mapping \( \mathcal{C} \) in sense of Definition 12.

(iii) The graph \( \Gamma_{\mathcal{C}} \subset Z_p \times \tilde{Z} \) of mapping \( \mathcal{C} \) is invariant under the flow generated by (flow) lifts of \( \nu \)-horizontal vector fields \( \hat{\xi}, \xi \in \mathfrak{h} \).

**Remark 8** In a case where connection \( \nu \) is flat, the association \( \xi \to \xi^1 \) is the Lie algebra endomorphism \( \mathfrak{h} \to \text{Aut}(\pi^1) \subset \mathcal{X}(Z_p) \).
9. Vertical Noether Theorem.

Let a Lie group $G \subset Sym(C) \subset Aut(\pi)$ be a subgroup of the symmetry group of a constitutive relation $C$. Let $\mathfrak{g}$ be the Lie algebra of the group $G$ and $\mathfrak{g}^*$ be its dual space. The Noether Theorem of Lagrangian Field Theory associates the group $G$ with the family of conserved currents. These currents are defined in terms of the (multi)-momentum mapping that in the case of a multisymplectic field theory was constructed in [19]. We recall this definition in our context and use it to construct Noether currents for a balance system defined by a CR $C$. Here we restrict ourselves to the case where group $G$ is formed by the pure gauge transformations, correspondingly the vector fields $\xi \in \mathfrak{g} \subset X(\pi)$ are vertical.

For an element $\xi \in \mathfrak{g}$ we denote by the same letter the corresponding vector field in $Y$, by $\xi'$ - the lifted vector field in $J^1_\pi(\pi)$ preserving the Cartan distribution, by $\xi^1$ - the vector field in $\tilde{Z}$ preserving the canonical multisymplectic forms (contact vector field): $L_{\xi^1}*(\Theta^2_n + \Theta^3_{n+2}) = 0$.

The condition that the vector field $\xi$ generates the symmetry of the CR $C$ discussed in the last Section has the form

$$L_{\xi^1} \Theta^{n+1}_n = \Theta^{n+2}_{n+1} = 0.$$  \hfill (59)

Recall the canonical multimomentum mapping (MM) following [19, 15].

**Definition 14** The multimomentum mapping $J: Z^* \to \Lambda^\pi(X) \otimes \mathfrak{g}^*$ is defined as

$$J(z^*)(\xi) = i_{\xi^1} \Theta^{n+1}_n, \text{ for all } z^* \in \Lambda^{n+1}, \xi \in \mathfrak{g}.$$

**Lemma 6** ([19, 15]) For the MM-mapping $J$

$$i_{\xi^1} \Theta^{n+1}_n = -dJ(z^*)(\xi).$$

**Proof:** We have $L_{\xi^1} \Theta^{n+1}_n = 0 = i_{\xi^1} \Theta^{n+1}_n + di_{\xi^1} \Theta^{n+1}_n = i_{\xi^1} \Theta^{n+1}_n + dJ(z^*)(\xi)$.

It seems natural to define the multimomentum mapping $J^C$ for an arbitrary constitutive relation $C$ in the same way as it was defined in [19] for the Legendre transformation corresponding to a Lagrangian $L_\eta$.

In the next definition we assume that the vector fields $\xi \in \mathcal{X}(Y)$ satisfy to the conditions that guarantee existence of the lift $\xi \to \xi^1$ from $Y$ to $J^1(\pi)$ (See Section 5).

**Definition 15** A multimomentum mapping of a constitutive relation $C$ is the mapping $J^C: J^1_\pi(\pi) \to \Lambda^\pi(X) \otimes \mathfrak{g}^*$

$$J^C(z)(\xi) = C^*_z J(C^{n+1}(\pi))(\xi^1) = i_{\xi^1} \Theta^{n+1}_n(z),$$

where $C^{n+1}$ is the $(n+1)$-component of the constitutive mapping $C$.

**Remark 9** Notice that acting on $\pi^1$-vertical vector fields $J^C(z)$ depends only on the current of the constitutive mapping and, therefore, it is independent on a choice of connection $\nu$ and $J^C(z) = J^{C_\nu}(z)$.

**Lemma 7** If the mapping $C$ is regular (i.e. if it is the diffeomorphism onto its image), then

$$i_{\xi^1} \Theta^{n+1}_n = -dJ^C(z)(\xi), \text{ for all } z \in J^1_\pi(\pi), \xi \in \mathfrak{g} \subset \mathcal{V}(Y).$$

**Proof:** Follows from the previous Lemma by using the $G$-equivariance of the constitutive relation $C$ giving $C_{\xi^1} = \xi^1 \circ C$. More specifically, we have

$$(dJ^C(z)(\xi) = (dC^*J)(z)(\xi^1) = C^*(dJ)(\xi^1) = dJ(C(z))(\xi^1) = dJ(C(z))(\xi^1) = (\xi^1) = i_{\xi^1} \Theta^{n+1}_n = i_{\xi^1} C^*_z \Theta^{n+1}_n = i_{\xi^1} \Theta^{n+1}_n.$$

Now we formulate the vertical version of the Noether Theorem for the balance system $\mathcal{B}_C$ corresponding to a regular constitutive relation.
Theorem 5 Let $C$ be a regular constitutive relation defined on a partial $1$-jet bundle $Z_p = J^1_p(\pi)$. Let a Lie group $G \subset \text{Sym}(C) \subset V(\pi)$ be a symmetry group of the flux part $\Theta_C^{n+1}$ of $C_{\text{R}}$ formed by the $\pi$-vertical $C$-admissible vector fields. Then for all $\xi \in \mathfrak{g}$ and for all solutions $s \in \Gamma(\pi)$ of the balance system $B_C$,

$$d[J^C(j^1(s)(x))(\xi)] = j^1(s)^*i_{\xi} \Theta_C^{n+2} = j^1(s)^*[\omega^i(\xi^j)\Pi_i]\eta,$$

(61)

where $\omega^i = dy^i - \sum_{\mu} z^i_{\mu} dx^\mu$ are the basic Cartan forms in $J^1_p(\pi)$.

Associating the secondary balance law (70) with each $\xi \in \mathfrak{g}$ defines the linear mapping

$$\mathfrak{g} \rightarrow \mathfrak{g} \mathfrak{L}_C.$$

Proof: We have

$$d[J^C(j^1(s))(\xi)] = j^1(s)^*d[J^{-1}_{C-}(\xi)] = -j^1(s)^*i_{\xi}d\Theta^{n+1}_{C-} = -j^1(s)^*i_{\xi} \Theta_C^{n+2}$$

(62)

$$= j^1(s)^*i_{\xi} \Theta_C^{n+2} = j^1(s)^*[\omega^i(\xi^j)\Pi_i]\eta.$$

where we have used the Euler-Lagrange form of the balance system (59).

Corollary 3 If, in addition to the conditions of Theorem 6 the balance system $B_C$ is the conservation system (i.e. if $\Pi_i = 0$, $i = 1, \ldots, m$), then for all $\xi \in \mathfrak{g}$ and for all solutions $s \in \Gamma(\pi)$ of the balance system $B_C$ the Noether conservation law holds:

$$d[J^C(j^1(s)(x))(\xi)] = 0,$$

(63)

Let condition (68) be fulfilled i.e. $G$ is a symmetry of both flux and source terms of the constitutive relation $C$. Then

$$d[i_{\xi} \Theta_C^{n+2}] = -i_{\xi} d\Theta_C^{n+2} = -i_{\xi} (d\Pi_i \wedge dy^i \wedge \eta)$$

$$= -(\xi^i \cdot \Pi_i)dy^i \wedge \eta - \xi^i d\Pi_i \wedge \eta + \xi^\mu d\Pi_i \wedge dy^i \wedge \eta_{\mu}$$

$$= -(\xi^i \cdot \Pi_i)(\omega^i + z^i_{\mu} dx^\mu) \wedge \eta - \xi^i(\Pi_i,x^\sigma dx^\sigma + \Pi_i,y^j(\omega^j + z^j_{\sigma} dx^\sigma))$$

$$+ \Pi_i,z^i(\omega^i - z^i_{\sigma} dx^\sigma)) \wedge \eta + \xi^\mu d\Pi_i \wedge (\omega^i + z^i_{\sigma} dx^\sigma) \wedge \eta_{\mu}$$

$$= Cont + \xi^\mu d\Pi_i \wedge z^i_{\mu} dx^\sigma \wedge \eta_{\mu}$$

$$= Cont + \xi^\mu [\Pi_i,x^\sigma dx^\sigma + \Pi_i,y^j(\omega^j + z^j_{\sigma} dx^\sigma)] \wedge \eta_{\mu} = Cont.$$

Here $Cont$ means a contact form. During this calculation we repeatedly used the equality $dx^\nu \wedge \eta = 0$. Applying the pullback by $j^1(s)$ we finish the proof of the following

Proposition 15 Let, in addition to the conditions of Theorem 6, $G$ be the symmetry of the source part of the constitutive relation, i.e. (68) is true. Then

$$d[i_{\xi} \Theta_C^{n+2}] = Cont \Rightarrow d[j^1(s)^*i_{\xi} \Theta_C^{n+2}] = 0$$

for all sections $s$. Therefore, locally (and in a top. trivial domain, globally)

$$j^1(s)^*i_{\xi} \Theta_C^{n+2} = d\Phi_C(s, \xi, z)$$

for some $(n+1)$ form $\Phi_C$ (g-potential of the source $C^{n+2}$) linearly depending on the vector field $\xi$.

Corollary 4 If $G$ is the Lie group of symmetries of a regular constitutive relation $C$ then (locally)

$$d[J^C(j^1(s)(x))(\xi) - \Phi_C(s, \xi, z)] = 0$$

for all solutions $s \in \Gamma(\pi)$ of the balance system $B_C$. 

10. Entropy condition and the secondary balance laws.

A natural question that leads directly to the “entropy condition” for a balance system

\[(F^\mu \circ j^1_p(s))_{,\nu} = \Pi_i(j^1_p(s)), \ i = 1, \ldots, m,\]

defined by a CR \( \mathcal{C} \) is - except for the linear combinations of balance equations in the system \( \star \), are there balance laws for the bundle \( Y^1_p(\pi) \) that follow from the balance system \( \star \) in the following sense:

**Definition 16** Fix a CR \( \mathcal{C} \) and consider the corresponding balance system \( (\star) \). We call a balance law

\[
(K^\mu \circ j^1_p(s))_{,\mu} = Q \circ j^1_p(s).
\]

of the same type (i.e. with the coefficients defining on \( J^1_p(\pi) \)) given by a \( (n+1)+(n+2) \)-form \( K^\mu \eta_\mu + Q \eta \) on the space \( Y^1_p(\pi) \) generated by the CR \( \mathcal{C} \) (or the secondary balance laws for the system \( \star \)) if any solution \( s : X \to Y \) of the balance system \( (\star) \) is at the same time solution of the balance law (64).

**Remark 10** Here we exclude from consideration all the higher order (in the jet hierarchy order sense) balance laws that are consequences of the balance system (see [12, 27]).

All of the balance laws that follow from the balance system (including the balance laws in the system \( (\star) \) themselves and their linear combinations) form the vector space \( \mathcal{B}L \mathcal{C} \). The simple class of secondary balance laws beyond the linear combinations of the balance laws of the system \( (\star) \) is determined by the following

**Proposition 16** Let a vertical vector field \( \xi = \xi^i \partial_{p^i} \in V(\pi) \) belongs to the \( X(\mathcal{C}) \), i.e. the condition \( F \text{Div}(\xi) = 0 \) is fulfilled. Then the balance law

\[
j^1_p(a)(\xi \circ j^1_p)(s) = \xi^i \Pi_i \Leftrightarrow ((\xi^i F^\mu_i \circ j^1_p(s))_{,\nu} = (\xi^i \Pi_i) \circ j^1_p(s)
\]

belongs to the space \( \mathcal{B}L \mathcal{C} \).

**Proof:** Follows from \( d(j^1_p(a)(\xi \circ j^1_p)(s) = \xi^i d(j^1_p(a)(F^\mu_i \circ j^1_p(s)) + j^1_p(a)(F \text{Div}(\xi)) \eta. \)

**Example 11** The **entropy principle** of Thermodynamics (see ([20, 21, 23])) requires that the entropy balance

\[
h_{\mu}^{\nu}= \Sigma,
\]

with the entropy density \( h^0 \), entropy flux \( h^A \), \( A = 1, 2, 3 \), and entropy production \( \Sigma \) belongs to the space \( \mathcal{B}L \mathcal{C} \) of the balance system of a given theory. This requirement place a serious restrictions on the form of constitutive relation \( \mathcal{C} \) and leads to the construction of a dual system in terms of Lagrange-Liu fields \( \lambda \) considered below (see Sec.3 or [20]).

Notice that the second law of thermodynamics requires, in addition to the existence of the entropy balance law, the fulfillment of the condition \( \Sigma > 0 \) of **positivity of the entropy production**.

Below we consider the case where \( Z_p = J^1_S(\pi) \). This case includes the RET case, the full 1-jet bundle case and several intermediate cases.

Consider the balance system \( (\star) \)

\[
(F^\mu_i \circ j^1_p(s))_{,\nu} = \Pi_i(j^1_p(s)), \ i = 1, \ldots, m,
\]

and calculate explicitly the derivatives in it. We get
\[ F^\mu_{i,x^\nu}(j^1_p(s)) + F^\mu_{i,y^j}(j^1_p(s))s^j_{,x^\nu} + \sum_{j \in U_1 \cup U_2} F^\mu_{i,z^j_x}(j^1_p(s))s^j_{,x^\nu} + \]
\[ + \sum_{j \in U_1 \cup U_2} F^\mu_{i,z^j_x A}(j^1_p(s))s^j_{,x^\nu,x^A} = \Pi_i(j^1_p(s)). \]

Here \( A = 1, \ldots, n; \mu = 0, \ldots, n \). Denote by \( \Phi \) the set of indices \( z_p^j \) such that the derivatives \( s^j_{,x^\nu} \) of sections \( s \in \Gamma(\pi) \) belong to \( J^1_p(\pi) \). For instance for the splitting \( S \) we will have the index set \( \Phi \) formed by \( z_0^j, j \in U_1 \cup U_2; z_A^j, j \in U_x \cup U_2 \). Split the second term in the left side leaving terms where derivatives \( z^j_p \) do not belong to \( \Phi \) on the left and moving all the other terms (such that for corresponding indices \( j, \mu \) one has \( z^j_p \in \Phi \)) together with the first term to the right. We get in \( J^2_p(\pi) = J^1(J^1_p(\pi) \cap J^2(\pi)) \),
\[ \sum_{z^j_p \notin \Phi} F^\mu_{i,y^j}(j^1_p(s))s^j_{,x^\nu} + \sum_{j \in U_1 \cup U_2} F^\mu_{i,z^j_x}(j^1_p(s))s^j_{,x^\nu} + \sum_{j \in U_1 \cup U_2} F^\mu_{i,z^j_x A}(j^1_p(s))s^j_{,x^\nu,x^A} = \]
\[ = \Pi_i(j^1_p(s)) - F^\mu_{i,x^\nu}(j^1_p(s)) - \sum_{z^j_p \in \Phi} F^\mu_{i,y^j}(j^1_p(s))s^j_{,x^\nu}, \]

or
\[ j^{1s}(s) \left[ \sum_{z^j_p \notin \Phi} F^\mu_{i,y^j}(z^j_p) + \sum_{j \in U_1 \cup U_2} F^\mu_{i,z^j_x}(z^j_p) + \sum_{j \in U_1 \cup U_2} F^\mu_{i,z^j_x A}(z^j_p) \right] = \]
\[ = j^{1s}(s) \left[ \Pi_i - F^\mu_{i,x^\nu} - \sum_{z^j_p \in \Phi} F^\mu_{i,y^j}(z^j_p) \right]. \]

We may consider these relations as defining, for each point \( z = (x^\mu, y^j, z^j_p) \in J^1_p(\pi) \) the system \((i = 1, \ldots, m)\) of affine planes \( A_i(z) \) in the fiber of \( J^2_p(\pi) \):
\[ \sum_{z^j_p \notin \Phi} F^\mu_{i,y^j}(z^j_p) + \sum_{j \in U_1 \cup U_2} F^\mu_{i,z^j_x}(z^j_p) + \sum_{j \in U_1 \cup U_2} F^\mu_{i,z^j_x A}(z^j_p) = \Pi_i - F^\mu_{i,x^\nu} - \sum_{z^j_p \in \Phi} F^\mu_{i,y^j}(z^j_p) \]
(66)

Notice that this is the affine plane in the fiber product of affine bundles \( J^1(J^2_p(\pi)) \) of first and second derivatives of different fields \( y^j \). For instance, a fiber of this bundle includes all the first derivatives of fields \( y^j \) from \( U_0 \), first derivatives by \( x^A \) and second derivatives by \( x^A \) of the fields in \( U_t \) etc.

For each \( i \) the affine planes \( A_i(z) \) form, upon varying the point \( z \), the affine subbundle \( A_i \subset J^2_p(\pi) \). The intersection \( W \) of these planes generates the affine subbundle \( W \subset J^2_p(\pi) \) (generically) of codimension \( m \).

The balance law (64) has a similar form to the balance laws of the system (1) and, as a result, it generated the affine subbundle of \( J^2_p(\pi) \) with the fiber given by the affine hyperplane
\[ \sum_{z^j_p \notin \Phi} K^\mu_{y^j}(z^j_p) + \sum_{j \in U_1 \cup U_2} K^\mu_{z^j_x}(z^j_p) + \sum_{j \in U_1 \cup U_2} K^\mu_{z^j_x A}(z^j_p) = Q - K^\mu_{x^\nu} - \sum_{z^j_p \in \Phi} K^\mu_{y^j}(z^j_p). \]
(67)

Since for any point of a fiber \( J^2_p x^\nu(\pi) \) over a point \( z \in J^1_p(\pi) \) there exists a section \( s : X \to Y \) such that \( j^1_p(s)(x) = z \) and its second jet passes through a given point, the following algebraic formulation of the “entropy principle” is true.
Proposition 17  The balance law (64) belongs to the space $\mathcal{BL}(C)$ if and only if for any point $z \in J^1_p(Y)$ the intersection $W$ of the affine hyperplanes (66) is contained in the affine hyperplane (67).

In such a situation the following simple result of affine algebra is useful

**Lemma 8**  Let $E^N$ be a vector space (over a field $k$) and let $W$ be an affine subspace in $E$ equal to the intersections of fibers of linear functionals $f^i : E \to k$: $W = \cap_i f_i^{-1}(c_i)$, $c_i \in k$. Let $h : E \to k$ be a linear functional on $E$ and let $c \in k$. Then the following conditions are equivalent

(i) $W \subset h^{-1}(c)$,
(ii) $h = \sum \lambda_i f^i$ for some $\lambda_i \in k$.

**Corollary 5**  The balance law (64) belongs to the space $\mathcal{BL}(C)$ if and only if for any point $z \in Z_S$ there exist functions $\lambda^i(z) \in C^\infty(Z_S)$ (Lagrange-Liu multipliers) such that

$$
\begin{align*}
\sum_i \lambda^i(z) \begin{pmatrix}
F^\mu_{i,y^j} & z^j_{y^j} \notin \Phi \\
F^\mu_{i,x_t} & j \in U_t \cup U_{tx} \\
F^\mu_{i,j} & j \in U_z \cup U_{tx}
\end{pmatrix} &= \begin{pmatrix}
K^\mu_{y^j} \\
K^\mu_{x_t} \\
K^\mu_{z^j}
\end{pmatrix}, \\
\sum_i \lambda^i(z) \left(\Pi_i - F_{i,x^n}^\mu - \sum_{z_{y^j} \in \Phi} F_{i,y^j}^\mu z^j_{y^j}\right) &= Q(z) - K_{i,x^n}^\mu - \sum_{z_{y^j} \in \Phi} K_{i,y^j}^\mu z^j_{y^j}.
\end{align*}
$$

(68)

The first three groups of equations here present equations for the flux component $K^\mu$. Thus, the solvability conditions (mixed derivatives conditions) should be satisfied by the left parts of equations in these groups. On the other hand, the last equation determines the source term $Q$ of the balance equation (64) provided $K$ was found.

Denote by $d^\mu_v : \Lambda^*(J^1_p(\pi)) \to \Lambda^*(J^1_p(\pi))$ the (“partial”) vertical differential of forms (including scalar functions) on the bundle $\pi^1 : J^1_p(\pi) \to X$. This differential includes derivatives by all the variables $z_{y^j} \in \Phi$ and by $y^j$, $j \notin \Phi$. Then the first group of relations in (68) can be presented in the form

$$
d^\mu_v K^\mu = \lambda^i(z) d^\mu_v F^\mu_i. 
$$

(69)

**Remark 11**  **Correspondence between the secondary balance laws (64)** and the sets $\{\lambda^i(z)\}$ of LL-multipliers is linear. This correspondence defines the mapping $\mathcal{BL}_C \to$ from the space of all balance laws generated by a constitutive relation $C$ and the space of vector fields in $J^1_p(\pi)$ of the type $\lambda = \lambda^i(z) \partial_{y^i}$.

Next we study the integrability conditions (82-83) for different cases.

11. Case of RET: functionally independent $\lambda^i$.

Consider a case of a constitutive relation $C$ for a maximal state space $U$, where the bundle $J^1_p(\pi) \to Y$ has zero fiber and the constitutive law $C$ is the section $Y \to \Lambda^u_{2}^{n+(n+1)}(Y)$ of the exterior form bundle $\Lambda^u_{2}^{n+(n+1)}(Y)$ on $Y$. In this case the calculations above simplify dramatically and the conditions of the previous Corollary take the form

$$
\begin{align*}
\sum_i \lambda^i(y) d_i F^\mu_i = d_i K^\mu, \iff \sum_i \lambda^i(y) F^\mu_{i,y^j} = K^\mu_{y^j}, \forall j = 1, \ldots, m, \\
\sum_i \lambda^i(y) \Pi_i(y) = Q(y) - K^\mu_{x^n},
\end{align*}
$$

(70)
with \(d_v\) being in this case the vertical differential in the bundle \(\pi\): \(d_v = d|_{Y(\pi)}\).

In this and the next subsection we suppress the index associated with \(x^\mu\). No derivatives by \(x^\mu\) will be calculated in these subsections, so this will not lead to any ambiguities.

Let us fix an index \(\mu\) in the first of these relations and notice that locally in \(Y\) the exactness of the form in the left side is equivalent to its closeness, i.e. to the following integrability condition

\[
d_v \left( \sum_i \lambda^i(y) d_v F^\mu_i \right) = \sum_i d_v \lambda^i(y) \wedge d_v F^\mu_i(y) = 0. \tag{71}
\]

Restrict now to the case where differentials \(d_v F^\mu_i(y)\) form the coframe of the vertical tangent space \(V(\pi)\) at each point \(y \in Y\). This condition ensures that the densities in the balance laws are (locally) functionally independent as functions of \(y^j\). This means that the functions \(w^i = F^\mu_i(y)\) can be taken as local coordinates in the fibers of the bundle \(\pi\). Now we use the following variant of the Cartan lemma ([16])

**Lemma 9** Let \(W \subset E^k\) be a connected star-shaped domain in a real vector space \(E^k\), let \(w_i\) be a coordinate chart in the domain \(W\) and let the functions \(\lambda^i(w)\) satisfy the relation

\[
\sum_i d\lambda^i \wedge dw^i = 0, \quad \forall w \in W.
\]

Then there exist smooth functions \(a_{ij}(w)\) with \(a_{ij}(w) = a_{ji}(w)\) such that

\[
d\lambda^i = \sum_j a_{ij}(w) dw^j.
\]

**Corollary 6** In the conditions of the previous Lemma, there exists a function \(h^0 \in C^1(W)\) such that

\[
\lambda^i = \frac{\partial h^0}{\partial w^i}.
\]

Applying this lemma to the condition (71) we find that Lagrange-Liu multipliers \(\lambda^i(x, y)\) have the form

\[
\lambda^i = \frac{\partial h^0}{\partial w^i}|_{w^i=F^\mu_i(y)},
\]

for a smooth function \(h^0 \in C^1(W)\). Substituting these expressions into (70) we find that one should have

\[
\begin{aligned}
d_v K^A(y) &= \sum_i \frac{\partial h^0}{\partial w^i}|_{w^i=F^\mu_i(y)} d_v F^A_i, \quad \forall A = 1, \ldots, m, \\
Q(y) &= \sum_i \frac{\partial h^0}{\partial w^i}|_{w^i=F^\mu_i(y)} \Pi_i(y).
\end{aligned} \tag{72}
\]

These relations defines uniquely the production term \(Q(y)\) in (64). Flux terms \(K^A, A = 1, \ldots, n\) in the balance law (64) can be found up to the addition of an arbitrary function of \(x\) provided the 1-form on the right is closed with respect to the vertical differential \(d_v\). This closeness condition has the form

\[
d_v \frac{\partial h^0}{\partial w^j}(w^k = F^\mu_k) \wedge d_v F^A_i = 0 \iff \frac{\partial^2 h^0}{\partial w^i \partial w^j}(F^\mu_k) d_v F^\mu_i \wedge d_v F^A_i = 0. \tag{73}
\]

Here \(d_v F^\mu_i = dw^j\).

As a result we have proved the following
Theorem 6 Let \((\star)\) be a balance system with the functionally independent density functions \(F_i^0\). There is a bijection between the regular secondary balance laws \((64)\) of the balance system \((\star)\) defined in an open subset \(W \subset Y\) - elements of \(\mathcal{BL}(C)\) such that the multipliers \(\lambda^i = \frac{\partial K^0}{\partial w^i}\) are functionally independent and the smooth functions (potentials) \(h^0(w^1, \ldots, w^m) \in C^\infty(W)\), modulo addition of an arbitrary function \(\pi^*f(y), f \in C^\infty(\pi(W))\), with the nondegenerate vertical Hessian \(\text{Hess}(h^0) = \frac{\partial^2 h^0}{\partial w^i \partial w^j} dw^j \wedge d_e F_i^A(w) = 0, A = 1, \ldots, n\).

This bijection is given by the relations
\[
\begin{align*}
K^0(y) &= h^0(F_k^0)(y), \\
d_e K^A(y) &= \sum_i \lambda^i d_e F_i^A, \quad \forall A = 1, \ldots, n, \\
Q(y) &= \sum_i \lambda^i \Pi_i(y) + K^\mu_{x^\mu},
\end{align*}
\]

where Lagrange-Liu multipliers \(\lambda^i\) are defined by
\[
\lambda^i(y) = \left. \frac{\partial h^0}{\partial w^i} \right|_{w^k = F_k^0(y)}.
\]

Making the change of variables \(y^k \to w^i = F_i^0\) in the flux functions \(F_i^A\) i.e. using \(\tilde{F}_i^A = F_i^A(y^k = y^k(w))\) we rewrite the integrability conditions \((71)\) in the form
\[
\frac{\partial^2 h^0}{\partial w^i \partial w^j} dw^j \wedge d_e \tilde{F}_i^A(w) = 0, A = 1, \ldots, n.
\]

Example 12 It is clear that all the linear functions \(h(w)\) satisfy these conditions. They correspond to the linear combinations of balance laws of the system \((\star)\).

Let now \(h^0(w) = k_{ij} w^i w^j\) be a non-degenerate quadratic function of its arguments \((\det(k_{ij}) \neq 0)\). Then, condition \((76)\) takes the form \(k_{ij} dw^j \wedge d_e F_i^A(w) = 0\). Introducing the linear change of variables \(v^i = k_{ij} w^j\) and using Lemma 10 we see that the integrability condition is equivalent to the condition that the forms below are closed
\[
k_{ij} F_j^A dw^i = F_k^A(v) dv^k.
\]

Example 13 Let \(F_i^0 = y^i\). Consider a case where secondary balance laws are generated by a \(C\)-admissible vector field \(\xi = \xi^i \partial_{y^i}\) (see Proposition 16). In this case it follows from Theorem 7
\[
h^0(y) = \sum_k y^k \xi^k(x, y).
\]

Therefore, in this case LL-multipliers have the form \(\lambda^i = \xi^i + \sum_k y^k \partial_{y^j} \xi^k\).

Let now \(h^0\) be a concave function, i.e. let the Hessian of \(h^0\) be negative definite in its domain
\[
\frac{\partial^2 h^0}{\partial w^i \partial w^j} < 0.
\]
Then, the transformation \( y^i \to \lambda^j = \frac{\partial \theta_0}{\partial u^i}, \) is the globally defined diffeomorphism. Using \( \lambda^i \) instead of \( u^i = F^0_k(y) \) as the new variables along the fiber of the bundle \( \pi : Y \to X \) we write the integrability condition (85) in the form

\[
d\lambda^i \wedge d\nu \tilde{F}^A_i(\lambda) = 0, \tag{77}
\]

where \( \tilde{F}^A_i(\lambda) = \tilde{F}^A_i(w(\lambda)) \). Using Lemma 9 we see that this condition is (locally) equivalent to the existence of the functions (potentials) \( h^A(\lambda) \) defined uniquely, up to an addition of functions of \( x \) such that

\[
\tilde{F}^A_i(\lambda) = \frac{\partial h^A}{\partial \lambda^i}. \tag{78}
\]

**Definition 17** Let \( h^0(w) \) be a concave function of variables \( w^j \) such that the condition (71) is satisfied. Then the \( n \)-form

\[
h = h^\mu \eta_\mu = h^0 \eta_0 + h^A \eta_A
\]

for which condition (78) is fulfilled is called a (local) \( n \)-potential of the balance system (\( \star \)). The balance laws (64) corresponding to the function \( h^0 \) are said to be generated by the four-potential \( h \).

In a case where \( h^0 \) is a nondegenerate at a point \( (x, w_0 = F^0_k(x, y_0)) \) in the sense that its vertical Hessian

\[
Hess(h^0)(x, w_0) = \left( \frac{\partial^2 h^0}{\partial w^i \partial w^j} \right) |_{x, w = w_0}
\]

is a nondegenerate matrix, the change of variables \( \{w^k\} \to \{\lambda^i = \frac{\partial h^0}{\partial w^j}\} \) exists locally, in a neighborhood of the point \( (x, w_0) \) and the previous conclusions for convex \( h^0 \) will follow locally.

As a result we get the following

**Theorem 7** Let the differentials \( d_u \tilde{F}^0_i(y) \) in a neighborhood of a point \( y_0 \) form a coframe in the fibers \( Y_x \), i.e. the functions \( w^k(u) = F^0_k(x, u) \) are functionally independent. Then,

(i) Locally, in a neighborhood of the point \( y_0 \) there is a bijection between the secondary balance laws (64) in \( BL_C \) such that the multipliers \( \lambda^i = \frac{\partial K^0}{\partial u^i} \) are functionally independent and the functions \( h^0(w^k) \) of variables \( x, w^k = F^0_k(y) \) are nondegenerate (with the nondegenerate vertical Hessian) at the point \( w_0 = F^0_k(y_0) \) defined up to an addition of an arbitrary function of \( x \) and \textbf{such that} for the variables \( \lambda^i = \frac{\partial h^0}{\partial w^j}|_{w^k = F^0_k(y)} \) defined in a neighborhood of the point \( y_0 \) the integrability conditions

\[
d\lambda^i \wedge d\nu \tilde{F}^A_i(\lambda) = 0 \iff d_u(\tilde{F}^A_i d\lambda^i) = 0, \quad A = 1, 2, \ldots, n \tag{79}
\]

where \( \tilde{F}^A_i(\lambda) = F^A_i(x, u(w(\lambda))) \) are fulfilled.

(ii) For a given secondary balance law (64) the function \( h^0(x, w) \) is defined by the condition

\[
h^0(x, w) = K^0(x, y(w)), \quad \text{where } w^k(y) = F^0_k(x, y),
\]

the integrability conditions (70) are fulfilled, the local vertical coordinates \( \lambda^i = \frac{\partial h^0}{\partial w^j}|_{w^k = F^0_k(y)} \) defined in a neighborhood of the point \( y_0 \) satisfy the integrability condition and there exist functions \( h^A \) such that

\[
\tilde{F}^A_i(\lambda) = \tilde{F}^A_i(w(\lambda)) = \frac{\partial h^A}{\partial \lambda^i}.
\]
thus defining the \((n+1)\)-potential

\[ h(\lambda) = h^\mu(x, \lambda) \eta_\mu, \]

corresponding to the balance law (64).

(iii) Vice versa, for a given function \( h^0(w^k) \) of variables \( x, w^k = F^0_k(y) \) nondegenerate (see above) at the point \( w_0 = F^0_k(y_0) \), the balance law components are defined from the conditions:

\[
\begin{align*}
K^0(y) &= h(F^0_k(y)), \\
d_v K^A(y) &= \sum_i \lambda^i d_v F^A_i, \quad \forall A = 1, \ldots, n, \\
Q(y) &= \sum_i \lambda^i \Pi_i(y) + K^\mu_{x\mu},
\end{align*}
\]

(80)

where the Lagrange-Liu multipliers \( \lambda^i \) defined by

\[
\lambda^i(y) = \frac{\partial h^0}{\partial w^i}|_{w^k = F^0_k(y)}
\]

are functionally independent.

(iv) Equivalently, there exists a (local) bundle isomorphism \( \varphi : Y \to \Lambda \) and a section \( h = h^\mu(\lambda^i) \eta_\mu \) of the bundle \( \Omega^{n-1}_1(\Lambda) \) of semi-basic \( n \)-forms such that

\[
\tilde{\Theta}^{n+1} = \tilde{F}^\mu_j d\lambda^j \wedge \eta_\mu = d_v h.
\]

(81)

All statements except the last one were proved before the theorem. The last one follows directly from (78).

The integrability condition (71) represents an \textbf{overdetermined system of linear equations} for the function \( h^0 \). Therefore, except for solutions linear in \( y^i \), solutions might exist only under certain conditions on the current part of the constitutive relation \( C \).

Consider a special case where densities \( F^0_i(x, y) \) coincide with the basic fields \( y^i \). In this case \( w^i = y^i \) and condition (71)

\[
\frac{\partial^2 h^0}{\partial y^i \partial y^j} dy^j \wedge d_v F^A_i(y) = 0, \quad A = 1, \ldots, n
\]

takes the form

\[
F^{A}_{i,j} h^0_{,y^i y^j} dy^j \wedge dy^k = 0, \quad A = 1, 2, \ldots, n, \text{ or}
\]

\[
F^{A}_{i,j} \frac{\partial^2}{\partial y^j \partial y^k} h^0 = 0, \quad \text{for all } k \neq j, A = 1, 2, \ldots, n.
\]

(82)

This system has all the functions \( h^0 = \sum_k c_k(x) y^k \) linear in \( y^i \) as trivial solutions, corresponding to the linear combination of balance equations of the system (\( \star \)). Generically there are no other solutions. This has its reflection in the known fact that the entropy principle (existence of a nontrivial solution of this system with the negative definite Hessian) place very restrictive conditions to the constitutive law \( C \).

Introduce the vertical vector fields \( \eta^A_k = F^{A}_{i,y^i}(x, y) \partial_{y^i} \) and the system (82) takes the form

\[
(\eta^A_k \partial_{y^j} - \eta^A_j \partial_{y^k}) h^0 = 0, \quad k \neq j, A = 1, 2, \ldots, n.
\]

(83)

This system determines the cyclic \( D_\mathcal{E} \)-module \( \mathcal{M}_C \) over the ring \( \mathcal{E} \) of smooth functions containing the coefficients \( F^A_i ([2]) \). Choosing the admissible filtration of the algebra \( D_\mathcal{E} \) we
The characteristic variety \( \text{Char}(\mathcal{M}_C) \) of D-module \( \mathcal{M}_C \) (more precisely, the relative characteristic variety with respect to the projection \( Y \to X \) (see [2], Sec.1.6)) is defined as the support of the \( \mathcal{E}[\zeta]\)-module \( \text{gr}(\mathcal{M}_C) \) - i.e. as the coisotropic subset of the (vertical) cotangent bundle \( V^*(Y) \to Y \) where

\[
P^A_{ij}(\zeta) = 0, \text{ for all } i, j, A.
\]

The set of expressions on the right can be considered as the collection of coefficients of the 2-form \( (\eta^A(\zeta) dq^i) \wedge (\zeta dq^j) \) with the coefficients being functions in \( Y \). This wedge product is zero if and only if the 1-forms are proportional, i.e. if \( \eta^A(\zeta) = \mu^A(x,y)\zeta_i, A = 1, \ldots, n; i = 1, \ldots, m \) with some factors \( \mu^A(x,y) \). The left side of this equation can be considered as the result of the application of linear operator \( K^A \) with the matrix \( K^A \) and the set of lines generated by the common eigenvectors of \( K^A \) with components \( \zeta_j \). Then the condition for \( \zeta \) obtained above takes the form

\[
K^A \zeta = \mu^A(y) \zeta.
\]

Thus, a (vertical) covector \( (y, \zeta) \) belongs to the support of the module \( \text{gr}(\mathcal{M}_C) \) if and only if \( \zeta \) is the common (real) eigenvector of operators \( K^A(y) \) for all \( A = 1, \ldots, n \). Generically there are no nonzero vector with this property and \( \text{Char}(\mathcal{M}_C)(x) = U_x, \ x \in X \).

As a result we have proved the following

**Theorem 8** (i) Let \( C \) be a constitutive relation of the RET type with \( F^0_i = y^i \). Then a function \( h^0(y^i) \) generates a secondary balance law with the functionally independent Lagrange-Liu multipliers \( \lambda^i \) in a neighborhood of a point \( (x_0, y_0) \) if and only if it is a solution of the cyclic D-module \( \mathcal{M}_C \)

\[
(\eta^A_k \partial_{y^k} - \eta^A_j \partial_{y^j}) h^0 = 0, \ k \neq j, A = 1, 2, \ldots, n,
\]

where \( \eta^A_k = F^A_{i_k y^k}(x,y)\partial_{y^k} \) and such that the (vertical) Hessian

\[
\text{Hess}(h^0)(x_0, y_0) = \left( \frac{\partial^2 h^0}{\partial y^i \partial y^j} \right) \bigg|_{x=x_0, y=y_0}
\]

is nondegenerate in a neighborhood of a point \( (x_0, y_0) \).

(ii) The characteristic variety of the cyclic D\_C-module \( \mathcal{M}_C \) is the union of the zero section of the vertical cotangent bundle \( V^*(Y) \to Y \) and the set of lines generated by the common eigenvectors of the linear operators \( K^A(y) : V^*_y(Y) \to V^*_y(Y) \) defined by

\[
(K^A(y)\zeta)_j = F^A_{j_k y^k}\zeta_k.
\]

A deeper study of the D-module \( \mathcal{M}_C \) and the corresponding secondary balance laws for a RET system with a constitutive relation \( C \) will be presented elsewhere.

For a case of a RET balance system with functionally dependent LL-multipliers and for an example of a model balance system with 2 fields \( y^i \) and one space variable, see [29].

**12. Complete case -** \( J^2_0(\pi) = J^1(\pi) \).

In this case the first system of equations in (70) is absent while the next two combined have the form

\[
\sum_{i} \lambda^i F^\mu_{i,x^\nu} = K^\mu_{x^\nu}, \quad \mu, \nu = 0, 1, \ldots, n; i = 1, \ldots, m. \quad \Rightarrow d_{\nu} K^\mu = \lambda^i d_{x^\nu} F^\mu_1. \quad (84)
\]
The condition of local solvability of these equations for $K^\mu$ for a given set of functions $\lambda^i \in C^\infty(J^1(\pi))$ has the form

$$d_v \lambda^i \wedge d_v F_i^\mu = -d_v(F_i^\mu d_v \lambda^i) = 0. \quad (85)$$

Here $d_v$ is the vertical differential in the bundle $J^1(\pi) \to Y$, i.e. differential by all $z^i_\mu$. If this condition is fulfilled for a given set of LL-multipliers $\lambda^i$, then the equations (84) determine the functions $K^\mu$ up to an addition to an arbitrary function of $x, y$:

$$K^\mu = K^\mu + H^\mu(x, y). \quad (86)$$

Then the last equations in (82) determine the production term $Q(z)$. Thus, in this case with every solution $K^\mu$ there enters the whole family of solutions:

$$\begin{cases} K^\mu = K^\mu + H^\mu(x, y), \\ Q(z) = \sum_i \lambda^i(z) \left( \Pi_i - F_i^{\mu,x} - \sum_{z^j_\mu \in \Phi} F_i^{\mu,y} z^j_\mu \right) + K^\mu_{,x} + \sum_{z^j_\mu \in \Phi} K^\mu_{,y} z^j_\mu. \end{cases} \quad (87)$$

Returning to the condition (84) we notice that for a fixed $y \in Y$, this condition has a local (and in a good domain, global) solution

$$d_v \lambda^i \wedge d_v F_i^\mu = -d_v(F_i^\mu d_v \lambda^i), \quad (88)$$

with some functions $h^\mu(y, z^i_\mu)$ (functions $h^\mu$ are defined up to addition of arbitrary functions of $y$). The special form of the left side suggests using $\lambda^i$ as coordinates in the fibers of the bundle $Z = J^1(\pi)$ and interpreting the last relation as the complete vertical differential of $h^\mu$.

To do this assume that the functions $\lambda^i$ are functionally independent in a domain $W \subset Z$ as functions of variables in the fibers of the bundle $\pi_{\ast 10} : J^1(\pi) \to Y$ and, therefore, they can be used to replace some of the coordinates $z^j_\mu$. For instance we may assume that the condition

$$\text{Det} \left( \frac{\partial \lambda}{\partial z^j_0} \right) \neq 0 \text{ is fulfilled in } W \text{ (this condition is suggested as an example, it is not used in the formulation or in the proof of the next Theorem).}$$

Changing coordinates in the domain $W (y, z^i_\mu) \to (y, \lambda^i, z^j_A, A \neq 0)$ we get from (87) that function $h^\mu$ are functions of $y$ and $\lambda^i$ but not of $z^j_A$. Thus, we introduce in the domain $W$ the “vector potential form”

$$h = h^\mu(y, w^1, \ldots, w^m) \eta_\mu, \quad (89)$$

such that in variables $(y, \lambda^i, z^j_A, A \neq 0)$

$$F_i^\mu = \frac{\partial h^\mu}{\partial w^i} |_{w^i = \lambda^i}, \quad (90)$$

In particular, $F_i^\mu = \hat{F}_i^\mu(y, \lambda^i, y, z^j_A)$ are functions of $y$ and of LL-multipliers $\lambda^i$ (but not of $z^j_A$ directly).

**Theorem 9** Let $Z = J^1(\pi)$ be a complete 1-jet bundle of the bundle $\pi$ and let $C$ be an admissible constitutive relation with the balance system $(\star)$. Let a balance equation (64) be an element of $BL_C$ with the LL-multipliers $\lambda^i$ be functionally independent in a domain $W \subset Z$ such that $H^1(W) = 0$. Then there exists (unique
up to an addition of arbitrary functions of $y$) a “vector-potential” semi-basic $n$-form $h = h^\mu(y, w^1, \ldots w^m)\eta_\mu$ with functions $h^\mu$ of $x, y$ and $m$ variables $w^i$ such that

$$F^\mu_i(z) = \frac{\partial h^\mu}{\partial w^i}|_{w^j = \lambda^j(z)},$$

and

$$h^\mu(y, w)|_{w^j = \lambda^j(y, z)} = \lambda^j F^\mu_j - K^\mu.$$

Vice versa, for any smooth bundle epimorphism $\lambda : z \to (y, \lambda^j(z))$ from $Z \to Y$ to the trivial bundle $Y \times \Lambda^m_Y \equiv \mathbb{R}^m \to Y$ trivial on the base and for all smooth mappings of bundles $h : \Lambda^m_Y \to H^{n+1} = Y \times \mathbb{R}^{n+1} : (w^j) \to (h^\mu(y, w^i))$ trivial on the base

$$J^1(\pi) \xrightarrow{\lambda} \Lambda^m_Y \xrightarrow{\pi} H^{n+1}_Y$$

such that

$$F^\mu_i(z)\lambda^*_w (dw^i) = F^\mu_i d_v \lambda^i = d_v (h^\mu \circ \lambda) = \lambda^*(d_v h^\mu)$$

($\lambda^*_w$ being the restriction of tangent mapping $\lambda^*_w$ to the vertical subbundle $V(\pi^1_0) \subset T(Z)$) for all $\mu$ the balance law

$$K^\mu = \lambda^j F^\mu_j - h^\mu,$$

and where production term $Q$ is defined by the relation

$$Q(z) = \sum_i \lambda^i(z) \left( \Pi_i - F^\mu_{i,x^\mu} - \sum_{z^j_\mu \in \Phi} F^\mu_{i,y^j z^j_\mu} \right) + K^\mu_{x^\mu} + \sum_{z^j_\mu \in \Phi} K^\mu_{y^j z^j_\mu}.$$

belongs to $B\mathcal{L}_C$.

**Proof:** All statements of the first part but the last one follows from the discussion preceding the Proposition. The first statement follows by combining (84) with (88).

For the second part, condition (94) is equivalent to the solvability condition (99) while the definition (95) is the same as (92). Finally, (96) is the second relation in (87).

**Remark 12** The epimorphism condition in the second part of the Theorem guarantees the functional independence of the functions $\lambda^i$.

**Remark 13** It is instructive to compare relations (94) and (95) with the similar relations in the RET theory, see [20], Ch.1.

**Definition 18** Let $C$ be a CR with the Poincare-Cartan form $\Theta_C = F^\mu dy^\mu \wedge \eta_\mu + \Pi dy^i \wedge \eta$. We say that that CR admits a LLM-representation (Lagrange-Liu-Muller) $(\lambda, h)$ in a domain $W \subset Z$ with the smooth bundle mapping $\lambda$ trivial on the base $Y$ and of the vertical rank $m$ (i.e. restriction of this mapping to all fibers $\pi^1_0(1)$ is of rank $m$)
and the semi-basic $n$-form $h = h^\mu(y,w)\eta_\mu$ on $\Lambda$ ($n$-potential) such that

$$F^\mu_i(z)\lambda^*_z \omega(z) \wedge \eta_\mu = F^\mu_i\omega \lambda^*_i \wedge \eta_\mu = \omega((h^\mu \circ \lambda)\eta_\mu) = \lambda^* \omega \eta_\mu$$

is true in the domain $W$.

**Corollary 7** The following conditions are equivalent

(i) $C$ admits the $(\lambda, h)$-representation in the domain $W$.

(ii) The balance law (78) defined in the Theorem 7 belongs to $\mathcal{B} \mathcal{L}_C(W)$.

For the intermediate cases of $Z_p = J^1_k(\pi)$ and $Z_p = J^1_b(\pi)$, see [29].

13. RET balance systems. Lagrange-Liu dual formulation.

In this section we suggest a bundle picture of the Rational Extended Thermodynamics in terms of dual variables. We will be using terminology from [20], Ch.1. (see also [29], Sec.3). Recall that for the conventional RET case where $F^i_0 = y^i$ ([20]) whenever the entropy density $h^0(x,y)$ is convex by vertical variables $y^i$, the change of variables $\{y^i\} \to \lambda^i = \frac{\partial h^0}{\partial y^i}$ is a globally defined diffeomorphism $\varphi_x : U_x \to \Lambda_x$ of the fibers $U_x$ onto the space $\Lambda$ of variables $\lambda^i$. This allows us to introduce the dual bundle $\pi_{Y^*} : Y^* \to X$ with the fiber $\Lambda$ with the corresponding isomorphism of bundles

$$Y \xrightarrow{\varphi^*} Y^*$$

$$\begin{array}{ccc}
\pi_{Y^*} & \pi_Y \\
\downarrow & \downarrow \\
\Lambda & \Lambda
\end{array}$$

$$X \xrightarrow{\pi_{Y^*}} X$$

Since in this section we are repeatedly using $\Lambda$ for the space of dual variables, it will be convenient to change the notation for the space (or bundle) of exterior $k$-forms from $\Lambda^k$ to $\Omega^k$.

Taking the pullback of the bundle of $n + (n+1)$-forms on $X$ via the projection $\pi_{Y^*}$ or, what is the same, forming the fiber product of the bundle $\pi_{Y^*}$ with the $((n+1)+(n+2))$-bundle $\pi_{(n+1)+(n+2)}$ (see Sec.2) we get the following commutative diagram

$$\begin{array}{ccc}
\Lambda \times \Omega^{(n+1)+(n+2)} & \to & \Omega^{(n+1)+(n+2)}(X) \\
\downarrow \pi_{\Omega} & & \downarrow \pi_{\Omega^*} \\
\Lambda & \to & \Omega^*(X)
\end{array}$$

where the left column represent a typical fiber of the middle column bundle over a point $x \in X$.

A point of a fiber of the bundle $\pi_{\Omega^*}$ can be presented as

$$(\lambda, \sum_\mu q^\mu(x, \lambda) + p(x, \lambda)),$$

where $q^\mu(x, \lambda), p(x, \lambda)$ are functions defined on the space $Y^*$.

Introduce the 1-jet bundle $J^1(\pi_{Y^*})$ of the bundle $\pi_{\Omega^*}$. A point of the fiber of this 1-jet bundle $\pi_{Y^*} : J^1(\pi_{Y^*}) \to \Lambda = Y^*_x \times (\Omega^{(n+1)+(n+2)}(X)$ (over a fixed base point $x \in X$) can be presented as

$$q^\mu_i \eta_\mu \wedge d\lambda^i + p_i \eta \wedge d\lambda^i,$$

were we have used the standard isomorphism $J^1(E \to U) \approx E \otimes T^*(U)$ of bundles over $U$ induced by a connection in the bundle $E \to U$. In this case we are using a connection induced in the central column of the bundle (100) by the Levi-Civita connection $\Gamma^G$ of the metric $G$ in the bundle of $(n+1)+(n+2)$-forms over $X$. 

Organize the spaces introduced above into the following bundle picture, where on the right are the local coordinates in the fibers of the bundles

\[
\begin{align*}
J^1(\Lambda \times \Omega^{(n+1)+(n+2)}) & \longrightarrow J^1(Y^* \times X \Omega^{(n+1)+(n+2)}) \\
\pi_1^1 & \rightarrow \pi_1^1 \\
(\Lambda \times \Omega^{(n+1)+(n+2)}) & \longrightarrow Y^* \times X \Omega^{(n+1)+(n+2)} \\
\pi_{\Lambda \Omega} & \rightarrow \pi_{\Lambda \Omega} \\
\Lambda & \longrightarrow \Lambda \\
\pi & \rightarrow \pi_{Y^* X} \\
\end{align*}
\]

(102)

A choice of a section \( \hat{h} \) of bundle \( \pi_{\Lambda \Omega} \) determines the dual entropy density \( \hat{h}^0(\lambda) \), its flow \( \hat{h}^\nu(\lambda) \) and the entropy production \( \Sigma(\lambda) \) as the function of the dual variables \( \lambda^i \).

A choice of a section \( c = (q^\mu_i(\lambda), p_i(\lambda)) \) of the 1-jet bundle \( \pi_2^* = \pi_{\Lambda \Omega} \circ \pi_1^1 \) determines, in addition to the previous quantities, the quantities \( q^\mu_i \) and \( p_i \) as functions of the dual variables \( \lambda^i \).

If we identify

\[
\tilde{F}^\mu_i \equiv q^\mu_i(\lambda), \quad \tilde{\Pi}_i \equiv p_i(\lambda),
\]

we see that a choice of a section \( c \) of the jet bundle \( \pi_2^* \) is equivalent to the choice of all the constitutive relations of the theory simultaneously.

Recall that a section \( c \) of the bundle \( \pi_2^* \) is called holonomic if it is a 1-jet of a section \( \hat{h} \) of the bundle \( \pi_{Y^* \times \Omega^{(n+1)}} \):

\[
c(\lambda) = j^1(\hat{h})(\lambda).
\]

It follows that if the \( \Omega^{(n+1)} \)-component \( c^{(n+1)} \) of the section \( c \) is holonomic, fields \( \tilde{F}^\mu_i(\lambda), \hat{h}^\mu(\lambda) \) satisfy the relations

\[
d_\eta \hat{h}^\mu = \tilde{F}^\mu_i d\lambda^i \Leftrightarrow \tilde{F}^\mu_i = \frac{\partial \hat{h}^\mu}{\partial \lambda^i}
\]

(104)

and vice versa.

To see this we recall (see, for instance [11, 12]) that the 1-jet space \( J^1(\Lambda \times X \Omega^{(n+1)}) \) is endowed with the canonical contact structure defined by the forms \( \omega^\mu = dq^\mu - q^\mu_i d\lambda^i \).

Necessary and sufficient conditions for a section \( c = (\hat{h}^\mu(\lambda), q^\mu_i(\lambda)) \) to be holonomic are

\[
c^*(\omega^\mu) = d\hat{h}^\mu - q^\mu_i d\lambda^i = 0
\]

for all \( \mu = 0, 1, \ldots, n \) which is the other form of relations (104) with the identification (103) above.

Assume now that the dual space \( \Lambda \) of variables \( \lambda^i \) is the vector space and consider now the Liouville vector field \( \zeta = \lambda^i \frac{\partial}{\partial \lambda^i} \) in the (vector) space \( \Lambda \).

We require additionally that the section \( c \) satisfies the (residual entropy) condition

\[
i_\zeta(\Pi_i(\lambda) \eta \wedge d\lambda^i) = \Pi_i \lambda^i = \Sigma(\lambda) \geq 0
\]

(105)

In such a way we ensure the fulfilment of conditions (4-5) including the positivity of entropy production \( \Sigma \).

As a result we have proved the following
Proposition 18 The following statements are equivalent

(i) Constitutive relations defined by the section $c$ of the bundle $\pi_2^*$ satisfy the entropy principle.

(ii) The $\Omega^n$-component of section $c$ is holonomic and the $\Omega^{n+1}$-component $\Pi_i d\lambda^i \wedge \eta$ of section $c$ satisfies the positivity condition

$$i_\xi[i(\lambda)d\lambda^i \wedge \eta] = \Pi_i \lambda^i \eta = \Sigma(\lambda)\eta \geq 0.$$  \hspace{1cm} (106)

In the last inequality we use the nonnegativity defined by the mass form $dM = \rho \eta$.

Example 14 Let a function $\Psi(\lambda)$ be given such that the radial monotonicity condition

$$\zeta \cdot \Psi \geq 0$$  \hspace{1cm} (107)

is fulfilled. This condition is equivalent to the geometrical requirement that the sublevel domains $\Psi^{-1}(-\infty,c)$ of the function $\Psi$ are “star-shaped” with respect to the origin.

Consider a production vector $\Pi_i$ of the form

$$\Pi_i = \frac{\partial \Psi}{\partial \lambda^i} \Leftrightarrow \Pi_i d\lambda^i \wedge \eta = d\Psi \wedge \eta$$

with the function $\Psi(\lambda)$. Then the positivity condition $\lambda^i \Pi_i \geq 0$ is fulfilled due to the condition (107).

Now we would like to present the balance system in terms of dual fields $\lambda^i(x)$ instead of the original fields $y^i(x)$ in the way similar to the Euler-Lagrange Equations in the multisymplectic Poincare-Cartan formalism (see above):

$$\partial_x\left[(j^1 \ast (\lambda) \hat{F}_\mu)\right] = (j^1 \ast (\lambda) \Pi_i)(\lambda(x)).$$  \hspace{1cm} (108)

To do this we start with a section

$$c = j^1(\hat{h}) - \Pi = (\lambda^i; \hat{\mu}(\lambda) \eta_\mu + \Sigma(\lambda)\eta; \frac{\partial \hat{h}_\mu}{\partial \lambda^i} d\lambda^i \wedge \eta_\mu - \Pi_i(\lambda) d\lambda^i \wedge \eta)$$  \hspace{1cm} (109)

of the 1-jet bundle $\pi_2^*$ satisfying the conditions of the Proposition 18 above.

Taking the differential $\hat{d}$ of the vertical part of section $c$ - the $(n+1)+(n+2)$ form $c_v = \frac{\partial \hat{h}_\mu}{\partial \lambda^i} d\lambda^i \wedge \eta_\mu - \Pi_i(\lambda) d\lambda^i \wedge \eta$ we get

$$d(\frac{\partial \hat{h}_\mu}{\partial \lambda^i} \wedge d\lambda^i \wedge \eta_\mu + \Pi_i(\lambda) d\lambda^i \wedge \eta = -\partial_{x^\mu} \left( \frac{\partial \hat{h}_\mu}{\partial \lambda^i} \right) d\lambda^i \wedge \eta - \frac{\partial \hat{h}_\mu}{\partial \lambda^i} \lambda_{G,x^\mu} d\lambda^i \wedge \eta + \Pi_i(\lambda) d\lambda^i \wedge \eta.$$  

Now we take the interior derivative of this form in the direction of an arbitrary vertical vector field $\xi \in T(\Lambda)$ (corresponding, in Poincare-Cartan formalism, to the vertical variation of a section $(\hat{h}, \Sigma)$ in the direction of $\xi$) and get

$$i_\xi c_v = -\partial_{x^\mu} \left( \frac{\partial \hat{h}_\mu}{\partial \lambda^i} \right) \xi^i \wedge \eta - \frac{\partial \hat{h}_\mu}{\partial \lambda^i} \lambda_{G,x^\mu} \xi^i \wedge \eta + \Pi_i(\lambda) \xi^i \wedge \eta$$

Taking the pullback of this $(n+1)+(n+2)$ form with respect to a section $\lambda = \lambda(x)$ of the bundle $\pi_{Y^* X} : Y^* = X \times \Lambda \to X$ we get

$$\lambda^i(i_\xi c_v) = i_{\xi \circ \lambda(x)} \lambda^* d\epsilon_v = i_{\xi \circ \lambda(x)} d\lambda^* \epsilon_v =$$
\[\xi^i(x, \lambda(x)) \left[ - (D_\mu \left( \frac{\partial \hat{h}^\mu}{\partial \lambda^i}(x, \lambda(x)) \right) - \frac{\partial \hat{h}^\mu}{\partial \lambda^i}(x, \lambda(x)) \lambda G_{x, \mu} + \Pi_i(x, \lambda(x)) \right] \eta. \]

Equating this expression to zero and requiring that the last equation would be fulfilled for a section \(\lambda(x)\) for an arbitrary (vertical) vector field \(\xi\) in the space \(\Lambda\) we see that the condition \(\lambda^*(i_\xi c_v) = 0\) is equivalent to the fulfillment of the balance system of equations

\[\partial_{x^\nu} \left( \frac{\partial \hat{h}^\mu}{\partial \lambda^i}(\lambda(x)) \right) = \Pi_i(\lambda(x))\]

which is, with the identification \(F^\mu_i = \frac{\partial \hat{h}^\mu}{\partial \lambda^i}\) equivalent to the dual system of RET balance equations (see [20], Ch.1) for the Lagrange-Liu fields \(\lambda^i(x)\). Thus we have proved the following statement

**Theorem 10** Let

\[S = j^1(\hat{h}) - \Pi = (\lambda^i; \hat{h}^\mu(\lambda) \eta_\mu + \Sigma(\lambda) \eta; \frac{\partial \hat{h}^\mu}{\partial \lambda^i} d\lambda^i \wedge \eta_\mu - \Pi_i(\lambda) d\lambda^i \wedge \eta)\]

be a (constitutive) section of the 1-jet bundle \(\pi^*_2\) satisfying to conditions of Proposition 25 above. Then the following statements about a section \(\lambda(x)\) of the bundle \(\pi_{Y \times X} : Y^* = X \times \Lambda \to X\) are equivalent:

(i) For any vertical vector field \(\xi\) in the space \(\Lambda\)

\[\lambda^*(i_\xi c_v) = 0\]

(ii) With the identification \(\hat{F}^\mu_i(\lambda) = \frac{\partial \hat{h}^\mu}{\partial \lambda^i}\), the system of dual fields \(\lambda = \lambda(t, x^\nu)\) satisfies the balance system (108), the entropy principle and the second law of thermodynamics.

**14. Conclusion.**

The basic structures of a multisymplectic theory of systems of balance laws (balance systems) was developed in this paper. Constitutive relations of balance systems appear in this scheme as a generalized Legendre transformations \(\mathcal{C}\) between the (partial) 1-jet bundles of the configurational bundle \(\pi : Y \to X\) and the dual bundle of the semi-basic exterior \((n+1)+(n+2)\)-forms on \(Y\).

The action of geometrical (gauge) transformations on the constitutive laws \(\mathcal{C}\) and on the corresponding Poincare-Cartan forms was studied. It is shown, in particular, that the geometrical symmetry group of a constitutive relation acts on the space of solutions of the corresponding balance system. A version of the Noether Theorem was obtained for the symmetry groups of a constitutive law \(\mathcal{C}\).

The entropy principle was formulated for general balance systems and the restrictions it put on the constitutive laws were studied. These considerations were applied to Rational Extended Thermodynamics (RET) to construct the dual geometrical picture of RET, present the balance system of RET in an invariant form and to interpret the entropy principle as the holonomicy of the current component of the constitutive relations.

In the second part of this work we will study the partial jet bundles of higher order compatible with the covariance groups of a balance system (see [18, 32, 31]) and extend the scheme presented here to this situation. The action of the groups of point transformations and the gauge groups on the phase and dual jet-bundles of a field theory in producing, rearranging and ordering the systems of balance laws (“balance systems”) of mixed tensorial structure and of different differential order will be studied in the framework of the present scheme.
More detailed study of the structure of secondary balance laws of a balance system is the other direction of the future work. Applications to the continuum mechanics (uniform materials, nonlinear visco-elasticity and the electrodynamics of continua) will be considered.

Another direction fore future work would be to extend the constructed scheme to the case of the base manifolds with the boundary \((X, \partial X)\). Even in the case of homogeneous Thermodynamics the mathematical (geometrical) description of interaction of a thermodynamical system with the environment presents a challenge (see, for instance, the works [24, 25, 26]).

Acknowledgments
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Appendix A. Properties of forms \(\eta_\mu\).
Here we collect some properties of the forms \(\eta_\mu\) that are repeatedly used in the text.

We have
\[
\eta_\mu = i_{\partial_{x^\mu}} \eta = (-1)^\mu \sqrt{|G|} dx^0 \wedge \ldots \wedge x^{\mu-1} \wedge dx^{\mu+1} \ldots \wedge dx^n
\]
and
\[
dx^\mu \wedge \eta_\mu = \eta.
\]

The differential \(d\eta_\mu\) has the form
\[
d\eta_\mu = (-1)^\mu \sqrt{|G|} i_{\partial_{x^\nu}} dx^\mu \wedge dx^0 \wedge \ldots \wedge x^{\nu-1} \wedge dx^{\nu+1} \ldots \wedge dx^n = (\partial_{x^\nu} \lambda_G) \eta,
\]
where \(\lambda_G = \ln(\sqrt{|G|})\).

Introduce the (n-1)-forms
\[
\eta_{\mu\nu} = i_{\partial_{x^\nu}} i_{\partial_{x^\mu}} \eta.
\]

Then we have
\[
\eta_{\mu\nu} = \begin{cases} 
  i_{\partial_{x^\nu}} \eta_\mu = (-1)^{\mu+\nu} \sqrt{|G|} dx^0 \wedge x^{\nu-1} \wedge dx^{\nu+1} \wedge \ldots \wedge x^{\mu-1} \wedge dx^{\mu+1} \ldots \wedge dx^n, & \text{if } \nu < \mu; \\
  i_{\partial_{x^\nu}} \eta_\mu = (-1)^{\mu+\nu-1} \sqrt{|G|} dx^0 \wedge x^{\nu-1} \wedge dx^{\nu+1} \wedge \ldots \wedge x^{\mu-1} \wedge dx^{\mu+1} \ldots \wedge dx^n, & \text{if } \nu > \mu.
\end{cases}
\]
and, in particular, for all \(\mu, \nu\), \(\eta_{\mu\nu} = -\eta_{\nu\mu}\). We also have
\[
dx^\sigma \wedge \eta_{\mu\nu} = \begin{cases} 
  \eta_\mu & \text{if } \sigma = \nu, \\
  -\eta_\nu & \text{if } \sigma = \mu, \\
  0, & \text{otherwise},
\end{cases}
\]
for all \(\mu, \nu\). To see this we first check it explicitly for \(\nu < \mu\) and for \(\nu > \mu\) we use \(dx^\sigma \wedge \eta_{\mu\nu} = -dx^\sigma \wedge \eta_{\nu\mu}\) and use the proved result.

For the differentials of these forms we calculate for the case \(\mu < \nu\)
\[
d\eta_{\mu\nu} = ((\partial_{x^\nu} \lambda_G) \eta_\mu - (\partial_{x^\nu} \lambda_G) \eta_\nu).
\]
and then notice that using antisymmetry of \(\eta_{\mu\nu}\) we get the same result for the case \(\nu > \mu\).
Appendix B. Iglesias Differential.

The differential $\tilde{d}$ is a specific example of a type of operator introduced by D. Iglesias and used in [9].

\[
\begin{aligned}
\tilde{d} : & \Omega^{k+1}(X) \to \Omega^{k+2}(X) : \\
\tilde{d}(\alpha^k + \beta^{k+1}) &= ((-d\alpha + \beta) + d\beta).
\end{aligned}
\]

(B.1)

Lemma 10 $\tilde{d} \circ \tilde{d} = 0$.

Proof 1 We have

\[
\tilde{d} \tilde{d}(\alpha^k + \beta^{k+1}) = \tilde{d}((-d\alpha + \beta) + d\beta) = \\
(-d(-d\alpha + \beta) + d\beta) + d(d\beta) = -d\beta + d\beta + 0.
\]

The complex

\[
0 \to \Omega^1(X) \oplus \Omega^0(X) \to \ldots \to \Omega^k(X) \oplus \Omega^{k-1}(X) \to \ldots \Omega^n(X) \oplus \Omega^{n-1}(X) \to 0 \oplus \Omega^n(X) \to 0
\]

is generated by the de Rham complex of a manifold $X$ and corresponds to couples of forms $\alpha^k + \beta^{k+1}$. This complex can be considered as dual to the complex of chains generated by couples $(C^{k+1}, \partial C^k)$ of submanifolds $C^{k+1} \subset X^n$ of dimension $k$ with the boundary $\partial C^k$. Duality is defined by integration

\[
<\alpha^k + \beta^{k+1}, (C^{k+1}, \partial C^k)> = \int C \beta + \int_{\partial C} \alpha.
\]

We have, obviously,

\[
<\tilde{d}(\alpha^k + \beta^{k+1}), (C^{k+1}, \partial C^k)> = 0
\]

for all $(C, \partial C)$ iff $\tilde{d}(\alpha^k + \beta^{k+1}) = 0$.

Appendix C. Reduced horizontal complex.

We restrict the usual ([12, 7]) horizontal differential $d_H$ for a bundle $\pi : Y \to X$ to the subcomplex of $\pi_\infty$-semi-basic form of the De Rham complex $\Lambda^*(J_\infty(\pi))$. Horizontal differential $d_H$ acts from $J_k(\pi)$ to $J_{k+1}(\pi)$ for all $k$ by the formulas

\[
\begin{aligned}
\hat{d}f(x, y, z_\mu^{i_1 \ldots i_k}) &= d_\mu dx^\mu = (f_x^\nu + z_\nu^{i_1} f_{y^{i_1}} + z_\nu^{i_2} f_{x^{i_2}} + \ldots + z_\nu^{i_k} f_{x^{i_k}})dx^\nu, \\
\hat{d}dx^\mu &= 0.
\end{aligned}
\]

(C.1)

Here the total derivative $d_\mu$ is the lift of the partial derivative $\partial_{x^\nu}$ to the vector field in $J_\infty(\pi)$ in the sense of [12, 27]:

\[
d_\mu f(x, y, z) = \frac{\partial f}{\partial x^\mu} + z_\mu^{i_1} \frac{\partial f}{\partial y^i} + \sum_{i, \mu_1 \ldots \mu_k} z_\mu^{i_1 \ldots i_k} \frac{\partial f}{\partial z_\mu^{i_1 \ldots i_k}}.
\]

The following properties follow directly from the definition of the total derivative:

Lemma 11 Acting on the functions from $C^\infty(J_\infty(\pi))$,

(i) $[d_\mu, \partial_{x^\nu}] = 0$,

(ii) $[d_\mu, \partial_{y^i}] = 0$. 

We extend the differential $df$ restricted to the subcomplex of semi-basic forms to the operator $\Lambda^k(J^\infty(\pi)) \to \Lambda^{k+1}(J^\infty(\pi))$ by requiring that

$$\hat{d}dy^i = 0, \hat{dd}z_{\mu_1\ldots\mu_k} = 0.$$ 

Operator $\hat{d}$ extended this way preserves the subcomplex $\Lambda^*_\pi(J^\infty(\pi))$ of $\pi^1_0$-semibasic forms (with the generators $dy^i, dx^\mu$) and maps the subspaces of the forms annihilated by $\pi \pi$-vertical arguments into itself

$$\hat{d} : \Lambda^k(J^\infty(\pi)) \to \Lambda^{k+1}(J^\infty(\pi)).$$

**Lemma 12** Let $CA(J^\infty(\pi))$ be the ideal in $\Lambda(J^\infty(\pi))$ of the contact forms (forms that annul the Cartan distribution), then for any form $\nu \in \Lambda(J^\infty(\pi))$ we have

$$(\hat{d} - d)\nu \in CA(J^\infty(\pi)).$$

Proof: Both operators $d$ and $\hat{d}$ are derivations of the exterior algebra, therefore it is sufficient to prove the statement for generators of this algebra $f(x,u,z), dx^\mu, dy^i, dz^j_{\mu}, \mu = \mu_1, \ldots, \mu_k$. For differentials the result is obvious - both operators annul them. For the functions we have

$$\hat{df} = f_{,x^\mu}dx^\mu + f_{,y^i}dy^i + \sum_\mu f_{,\bar{z}^i_{\bar{\mu}}}dz^j_{\bar{\mu}} = f_{,x^\mu}dx^\mu + f_{,y^i}dz^j_{\bar{\mu}} + \sum_\mu f_{,\bar{z}^i_{\bar{\mu}}}dz^j_{\bar{\mu}}dx^\nu.$$ 

Subtracting from this expression the similar (but simpler) expression for $df$ we get

$$(\hat{d} - d)f = f_{,y^i}(z^j_{\bar{\mu}}dx^\mu - dy^i) + \sum_\mu f_{,\bar{z}^i_{\bar{\mu}}} \left( \bar{z}^j_{\bar{\mu}}dx^\mu - dz^j_{\bar{\mu}} \right) = -f_{,y^i}dy^i + \sum_\mu f_{,\bar{z}^i_{\bar{\mu}}} \omega^i_{\bar{\mu}},$$

that finishes the proof.

**Proposition 19** Let $\phi$ be an automorphism of the bundle $\pi$ and $\phi^\infty$ - its contact (=flow) prolongation to the $J^\infty(\pi)$. Then

$$\hat{d}\phi^\infty \ast \omega \equiv \phi^\infty \ast \hat{d}\omega \mod CA^*$$

for all $\pi^1_0$-semibasic forms $\omega$ on $J^\infty(\pi)$. Here $CA^*(J^\infty(\pi))$ is the ideal in generated by the Cartan forms (forms that annulate the Cartan distribution).

For the proof see [29].

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