A regularity criterion at one scale without pressure for suitable weak solutions to the Navier-Stokes equations

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Abstract

In this paper, we continue our work in [15] to derive \( \varepsilon \)-regularity criteria at one scale without pressure for suitable weak solutions to the Navier-Stokes equations. We establish an \( \varepsilon \)-regularity criterion below of suitable weak solutions, for any \( \delta > 0 \),

\[
\int_{Q(1)} |u|^{\frac{2}{2} + \delta} dx dt \leq \varepsilon.
\]

As an application, we extend the previous corresponding results concerning the improvement of the classical Caffarelli–Kohn–Nirenberg theorem by a logarithmic factor.

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1 Introduction

We focus on the following incompressible Navier-Stokes equations in three-dimensional space

\[
\begin{cases}
    u_t - \Delta u + u \cdot \nabla u + \nabla \Pi = 0, \\
    \text{div } u = 0, \\
    u|_{t=0} = u_0,
\end{cases}
\]

where \( u \) stands for the flow velocity field, the scalar function \( \Pi \) represents the pressure. The initial velocity \( u_0 \) is divergence free.

One important regularity criterion of suitable weak solutions of (1.1) is the following one due to [20, 21]: there exists an positive universal constant \( \varepsilon \) such that \( u \in L^\infty(Q(1/2)) \) provided the following conditions is satisfied

\[
\int_{Q(1)} |u|^3 + |\Pi|^\frac{3}{2} dx dt < \varepsilon.
\]

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This $\varepsilon$-regularity criterion plays an important role in the study of the Navier-Stokes equations (see, e.g., [7, 14, 16, 22, 25, 29] and the references therein). In [26, page 8], the authors gave a comment on regular criterion (1.2): “the bootstrapping enables to lower the exponent in the smallness condition from 3 to $\frac{5}{2} + \delta$ (at the cost of having to use smallness at all scales).” Indeed, Gustafson, Kang and Tsai in [12] established the following $\varepsilon$-regularity criterion at all scales
\[
\sup_{0< r \leq 1} \int_{Q(r)} |u|^2 dx \leq \varepsilon. \tag{1.3}
\]

We refer the reader to [12, 28] for other kind of $\varepsilon$-regularity criteria in terms of the velocity, the vorticity, the gradient of the vorticity at all scale. Kukavica [18] proposed three questions regarding regularity criterion (1.2). In particular, the second issue is that whether regularity criterion (1.2) holds for the exponent less than 3. Recently, Guevara and Phuc [11] first answered this question via establishing the regularity criteria below
\[
\|u\|_{L^{2p,2q}(Q(1))} + \|\Pi\|_{L^{p,q}(Q(1))} < \varepsilon, \quad \frac{3}{q} + \frac{2}{p} = \frac{7}{2} \quad \text{with } 1 \leq q \leq 2. \tag{1.4}
\]

Subsequently, in the spirit of [11], the authors in [13] further generalized Guevara and Phuc’s results by proving
\[
\|u\|_{L^{p,q}(Q(1))} + \|\Pi\|_{L^{1}(Q(1))} < \varepsilon, \quad 1 \leq \frac{2}{p} + \frac{3}{q} < 2, 1 \leq p, q \leq \infty. \tag{1.5}
\]

The third question posed by Kukavica is that the pressure can be removed in (1.2). In this direction, Wolf [31] successfully proved the following regularity criterion via introducing local pressure projection
\[
\int_{Q(1)} |u|^3 dx < \varepsilon.
\]

Furthermore, in [3], Chae and Wolf studied Liouville type theorems for self-similar solutions to the Navier-Stokes equations by proving $\varepsilon$-regularity criteria
\[
\sup_{-1 \leq t \leq 0} \int_{B(1)} |u|^q dx < \varepsilon, \quad \frac{3}{2} < q \leq 3. \tag{1.6}
\]

Developing the technique as in [3, 31], the authors in [15] obtained $\varepsilon$-regularity condition
\[
\int_{Q(1)} |u|^{20/7} dx < \varepsilon. \tag{1.7}
\]

Inspired by the comment to (1.2) mentioned above, Kukavica’s questions and recent progress (1.4)-(1.7), we try to prove the following result

**Theorem 1.1.** Let the pair $(u, \Pi)$ be a suitable weak solution to the 3D Navier-Stokes system (1.1) in $Q(1)$. For any $\delta > 0$, there exists an absolute positive constant $\varepsilon$ such that if $u$ satisfies
\[
\int_{Q(1)} |u|^{2+\delta} dx \leq \varepsilon, \tag{1.8}
\]
then, $u \in L^\infty(Q(1/16))$.

**Remark 1.1.** This theorem is an improvement of corresponding results in (1.4)-(1.6).
An outline of the proof for Theorem 1.1 is as follows. In the first step, following the path of [3, 15, 31], one establishes the Caccioppoli type inequalities just in terms of $u$ via the local energy inequality (2.2) (see Section 2 for more notations and details). As mentioned in [3, 31, 32], the advantage of local energy inequality (2.2) removed the non-local effect of the pressure term. However, as stated in [15], the cost of local energy inequality (2.2) without non-local pressure is that the velocity field $u$ is lack of the kinetic energy $\|u\|_{L^\infty}$. As observed in [15], $v = u + \nabla \Pi_h$ enjoys all the energy, namely, $\|v\|_{L^\infty} L^2$ and $\|v\|_{L^2 L^2}$, where $\Pi_h$ is a harmonic function. However, since $v$ depends on the radius $r$, one can not construct iteration in terms of $v$. This leads to the main difficulty in construction of Caccioppoli type inequality. The key point is the full application of $u = v - \nabla \Pi_h$ and absorption of $v$ in the right hand side by the left hand in local energy inequality (2.2). Compared with the proof in [15], we need to absorb more than one $v$ in the right hand side by the left hand by local energy inequality (2.2). After carefully choosing the suitable text function in (2.2) together with the interpolation inequality (2.9), we can treat the term $\int \int |v|^3 \phi^{2^{\beta+1}}$ (see Section 3 for more notations and details). Since $\Pi_2$ meets $\Delta \Pi_2 = - \text{divdiv}(u \otimes u)$, a natural method is using the representation of $\Pi_2$ to bound $\int \int |\Pi_2 \phi^{2^{\beta+1}}| |v| |v|^{\beta}$. Indeed, to the knowledge of the authors, it is worth remarking that we will utilize the representation of $\Pi_2$ with the test function rather than the pressure only used in all previous work. This guarantees that the representation of $\Pi_2$ and the integration domain in this term is consistent. This is of independent interest. Then we obtain the desired Caccioppoli type inequalities (1.12).

In the second step, we use Caccioppoli type inequalities (1.12) and induction arguments developed in [1, 3, 15, 23, 27] to prove Theorem 1.1. In contrast with the previous argument, it should be noted that there exist two difficult terms to bound in local energy inequality (2.2) since the integration of time in (1.8) is just $\frac{5+2\delta}{2}$ (see (4.3) for more details). The first one is $\int_{r_0}^t \int_{B_1} \Gamma \phi v \otimes \nabla \Pi_h : \nabla^2 \Pi_h$, which will be bounded by the introduction of new quantity in (4.24) in induction arguments. To control the second one $\int_{r_0}^t \int_{B_1} \Pi_2 v \cdot \nabla (\Gamma \phi)$, we decompose $\Pi_2$ into two parts (see (4.11)-(4.12)) and make use of Lemma 2.3 to obtain the desired estimates.

Next we turn attention to the Caccioppoli type inequality. For the reader’s convenience, before we formulate our proposition, we recall the known results proved in [3, 15, 32], respectively,

$$\|u\|_{L^2(\frac{3}{2}, \infty, Q(\frac{5}{2}))}^2 + \|\nabla u\|_{L^2(Q(\frac{5}{2}))}^2 \leq C\|u\|_{L^2(Q(1))}^2 + C\|u\|_{L^2(Q(1))}^3, \quad (1.9)$$

$$\|u\|_{L^2(\frac{3}{2}, \infty, Q(\frac{5}{2}))}^2 + \|\nabla u\|_{L^2(Q(\frac{5}{2}))}^2 \leq C\|u\|_{L^2(Q(\frac{5}{2}))}^{\frac{5}{2}} + C\|u\|_{L^2(Q(\frac{3}{2}, \infty, Q(1)))}^{\frac{5}{2}}, \quad 3 \leq q < 3, \quad (1.10)$$

$$\|u\|_{L^2(\frac{3}{2}, \infty, Q(\frac{5}{2}))}^2 + \|\nabla u\|_{L^2(Q(\frac{5}{2}))}^2 \leq C\|u\|_{L^2(Q(\frac{5}{2}))}^2 + C\|u\|_{L^2(Q(1))}^4, \quad (1.11)$$

Proposition 1.2. Assume that $u$ is a suitable weak solution to the Navier-Stokes equations, $3/2 \leq 2/p + 3/q < 2$ with $p \geq 2, q \geq 12/5$. Then we have, for any $R > 0$,

$$\|u\|_{L^2(\frac{3}{2}, \infty, Q(\frac{5}{2}))}^2 + \|\nabla u\|_{L^2(Q(\frac{5}{2}))}^2 \leq CR^{\frac{3\alpha - 5}{\alpha}} \|u\|_{L^p,q(Q(R))}^2 + CR^{\frac{3\alpha - 5}{\alpha}} \|u\|_{L^p,q(Q(R))}^4 + CR^{\frac{3\alpha - 5}{\alpha}} \|u\|_{L^p,q(Q(R))}^4, \quad (1.12)$$

where $\alpha = \frac{2}{p + \frac{3}{q}}$. 

We present two application of new $\varepsilon$-regularity criterion (1.8) without pressure at one scale. This is in part motivated by recent works [24, 30], where the authors found that at one scale it is useful to establish new $\varepsilon$-regularity criterion to obtain better box dimension and the improvement of the classical Caffarelli-Kohn-Nirenberg theorem by a logarithmic factor. Box dimension (Minkowski dimension) is a widely used fractal dimension (see Section 5 for the definition). The relationship between Hausdorff dimension and box dimension is that the former is less than the latter (see e.g. [8]). More information on box dimension can be found in [8]. Making use of Theorem 1.1 and following the path of [16, 29], one can derive that the (upper) box dimension of the singular points set $S$ is at most $37/30(\approx 1.23)$. This improves the previous box dimension of $S$ obtained in [16, 18, 25, 29]. We leave the proof to the interested author. Indeed, the proof is simpler than that of [13, 16, 23, 29] owing to $\varepsilon$-regularity criterion (1.8) without pressure holds at one scale. Finally, we are concerned with the improvement of the Caffarelli-Kohn-Nirenberg theorem by a logarithmic factor. In [4], Choe and Lewis introduced the generalized Hausdorff measure $\Lambda(S, r(\log(e/r))^\sigma)$ (for the detail, see Section 6) and proved that

$$\Lambda(S, r(\log(e/r))^\sigma) = 0(0 \leq \sigma < 3/44). \quad (1.13)$$

(1.13) with $\sigma = 0$ reduces to the celebrated Caffarelli-Kohn-Nirenberg theorem for the three-dimensional time-dependent Navier-Stokes system. Recently, there are some efforts to improve the bound of $\sigma$ in (1.13). $\sigma$ is bounded by $1/6$ by Choe and Yang in [5]. Later, Ren, Wang and Wu [24] improved the bound of $\sigma$ to $27/113$. Inspired by the new $\varepsilon$-regularity criterion (1.8), we have the following result

**Theorem 1.3.** Let $S$ stand for the set of all the potential interior singular points of suitable weak solutions to (1.1) and $0 \leq \sigma < 4/11$. There holds

$$\Lambda(S, r(\log(e/r))^\sigma) = 0.$$

**Remark 1.2.** Theorem 1.3 is an improvement of the known corresponding results in [4, 5, 24].

The remainder of this paper is structured as follows. In Section 2, we start with the details of Wolf’s the local pressure projection $\mathcal{W}_{\rho, \Omega}$ and recall the definition of local suitable weak solutions due to [31, 32]. Then, we establish some auxiliary lemmas. The Caccioppoli type inequality (1.12) is derived in Section 3. Section 4 is devoted to the proof of Theorem 1.1. Finally, we consider the improvements on Caffarelli–Kohn–Nirenberg theorem by a logarithmic factor in Section 5.

**Notations:** Throughout this paper, we denote

$$B(x, \mu) := \{y \in \mathbb{R}^3 | |x - y| \leq \mu\}, \quad B(\mu) := B(0, \mu), \quad \tilde{B}(\mu) := B(x_0, \mu),$$

$$Q(x, t, \mu) := B(x, \mu) \times (t - \mu^2, t), \quad Q(\mu) := Q(0, 0, \mu), \quad \tilde{Q}(\mu) := Q(x_0, t_0, \mu),$$

$$r_k = 2^{-k}, \quad \tilde{B}_k := \tilde{B}(r_k), \quad \tilde{Q}_k := \tilde{Q}(r_k).$$

For $p \in [1, \infty]$, the notation $L^p((0, T); X)$ stands for the set of measurable functions on the interval $(0, T)$ with values in $X$ and $\|f(t, \cdot)\|_X$ belongs to $L^p(0, T)$. For simplicity, we write

$$\|f\|_{L^p(Q(r))] := \|f\|_{L^p(-r^2, 0; L^q(B(r)))} \quad \text{and} \quad \|f\|_{L^p(Q(r))] := \|f\|_{L^p(Q(r))}.$$ 

We also denote

$$E(\mu) = \mu^{-1} \|u\|_{L^\infty(Q(\mu))}^2, \quad E_+(\mu) = \mu^{-1} \|
abla u\|_{L^2(Q(\mu))}^2,$$

$$E_p(\mu) = \mu^{p-5} \|u\|_{L^p(Q(\mu))}^p, \quad J_p(\mu) = \mu^{2(p-5)} \|
abla u\|_{L^p(Q(\mu))}^p.$$
In addition, denote the average of $f$ on the set $\Omega$ by $\overline{f}_\Omega$. For convenience, $\overline{f}_r$ represents $\overline{f}_{B(r)}$ and $\overline{\Pi}_r$ is denoted by $\overline{\Pi}_r$. $|\Omega|$ represents the Lebesgue measure of the set $\Omega$. For exponent $p \in [1, \infty)$, we define the Hölder conjugate $p^*$ through the relation $1/p^* = 1 - 1/p$.

We will use the summation convention on repeated indices. $C$ is an absolute constant which may be different from line to line unless otherwise stated in this paper.

## 2 Preliminaries

We begin with the Wolf’s local pressure projection $W_{p,\Omega} : W^{-1,p}(\Omega) \to W^{-1,p}(\Omega)$ ($1 < p < \infty$). More precisely, for any $f \in W^{-1,p}(\Omega)$, we define $W_{p,\Omega}(f) = \nabla \Pi$, where $\Pi$ satisfies (2.1). Let $\Omega$ be a bounded domain with $\partial \Omega \in C^1$. According to the $L^p$ theorem of Stokes system in [9, Theorem 2.1, p149], there exists a unique pair $(u, \Pi) \in W^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$-\Delta u + \nabla \Pi = f, \quad \text{div} u = 0, \quad u|_{\partial \Omega} = 0, \quad \int_\Omega \Pi dx = 0. \quad (2.1)$$

Moreover, this pair is subject to the inequality

$$\|u\|_{W^{1,p}(\Omega)} + \|\Pi\|_{L^p(\Omega)} \leq C\|f\|_{W^{-1,p}(\Omega)}.$$

Let $\nabla \Pi = W_{p,\Omega}(f)$ ($f \in L^p(\Omega)$), then $\|\Pi\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}$, where we used the fact that $L^p(\Omega) \hookrightarrow W^{-1,p}(\Omega)$. Moreover, from $\Delta \Pi = \text{div} f$, we see that $\|\nabla \Pi\|_{L^p(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|\Pi\|_{L^p(\Omega)}) \leq C\|f\|_{L^p(\Omega)}$. Now, we present the definition of suitable weak solutions of Navier-Stokes equations (1.1).

**Definition 2.1.** A pair $(u, \Pi)$ is called a suitable weak solution to the Navier-Stokes equations (1.1) provided the following conditions are satisfied,

1. $u \in L^\infty(-T, 0; L^2(\mathbb{R}^3)) \cap L^2(-T, 0; \dot{H}^1(\mathbb{R}^3)), \Pi \in L^{3/2}(-T, 0; L^{3/2}(\mathbb{R}^3));$

2. $(u, \Pi)$ solves (1.1) in $\mathbb{R}^3 \times (-T, 0)$ in the sense of distributions;

3. The local energy inequality reads, for a.e. $t \in [-T,0]$ and non-negative function $\phi(x, s) \in C_0^\infty(\mathbb{R}^3 \times (-T, 0))$,

$$\int_{B(r)} |v|^2 \phi(x,t) dx + \int_{-T}^{t} \int_{B(r)} |\nabla v|^2 \phi(x,s) ds dx \leq \int_{-T}^{t} \int_{B(r)} |v|^2 (\Delta \phi + \partial_t \phi) dx ds + \int_{-T}^{t} \int_{B(r)} |v|^2 u \cdot \nabla \phi dx ds$$

$$+ \int_{-T}^{t} \int_{B(r)} \phi(u \otimes v : \nabla^2 \Pi_h) dx ds + \int_{-T}^{t} \int_{B(r)} \phi \Pi_1 v \cdot \nabla \phi dx ds + \int_{-T}^{t} \int_{B(r)} \phi \Pi_2 v \cdot \nabla \phi dx ds. \quad (2.2)$$

Here, $\nabla \Pi_h = -W_{p,B(R)}(u), \nabla \Pi_1 = W_{p,B(R)}(\Delta u), \nabla \Pi_2 = -W_{p,B(R)}(u \cdot \nabla u), v = u + \nabla \Pi_h$. In addition, $\nabla \Pi_h, \nabla \Pi_1$, and $\nabla \Pi_2$ meet the following facts

$$\|\nabla \Pi_h\|_{L^p(B(R))} \leq C\|u\|_{L^p(B(R))}, \quad (2.3)$$

$$\|\Pi_1\|_{L^2(B(R))} \leq C\|\nabla u\|_{L^2(B(R))}, \quad (2.4)$$

$$\|\Pi_2\|_{L^{p/2}(B(R))} \leq C\|u^2\|_{L^{p/2}(B(R))}. \quad (2.5)$$
We list some interior estimates of harmonic functions $\Delta h = 0$, which will be frequently utilized later. Let $1 \leq p, q \leq \infty$ and $0 < r < \rho$, then it holds

$$\|\nabla^k h\|_{L^q(B(r))} \leq \frac{C r^{\frac{k}{q}}}{(\rho - r)^{\frac{k}{p} + 1}} \|h\|_{L^p(B(\rho))}, \tag{2.6}$$

$$\|h - \overline{h}_r\|_{L^q(B(\rho))} \leq \frac{C r^{\frac{q}{q} + 1}}{(\rho - r)^{q + 1}} \|h - \overline{h}_\rho\|_{L^q(B(\rho))}. \tag{2.7}$$

The proof of (2.6) rests on the mean value property of harmonic functions. This together with mean value theorem leads to (2.7). We leave the details to the reader.

For reader’s convenience, we recall an interpolation inequality. For each $2 \leq l \leq \infty$ and $2 \leq k \leq 6$ satisfying $\frac{2}{l} + \frac{3}{k} = \frac{3}{2}$, according to the Hölder inequality and the Young inequality, we know that

$$\|u\|_{L^{2,k}(Q(\mu))} \leq C \|u\|_{L^{\infty,2}(Q(\mu))}^{\frac{1}{2}} \|u\|_{L^2,\rho(Q(\mu))}^{\frac{3}{2}}$$

$$\leq C \|u\|_{L^{\infty,2}(Q(\mu))} \left(\|u\|_{L^2,\rho(Q(\mu))} + \|\nabla u\|_{L^2(Q(\mu))}\right)^{\frac{3}{4}}$$

$$\leq C \left(\|u\|_{L^{\infty,2}(Q(\mu))} + \|\nabla u\|_{L^2(Q(\mu))}\right). \tag{2.8}$$

**Lemma 2.1.** Let $1 \leq 2/p + 3/q < 2, 1 \leq p, q \leq \infty$ and $\alpha = \frac{2}{3} + \frac{1}{3}$. There is an absolute constant $C$ such that

$$\|u\|_{L^3(Q(\rho))}^3 \leq C \rho^{3(\alpha - 1)/2} \|u\|_{L^{p,q}(Q(\rho))} \left(\|u\|_{L^{\infty,2}(Q(\rho))} + \|\nabla u\|_{L^2(Q(\rho))}\right)^{(3 - \alpha)/2}. \tag{2.9}$$

**Remark 2.1.** Lemma 2.1 is obtained in [13]. Here we present a different proof from that in [13]. New proof allows one to apply it to more general case.

**Proof.** We denote

$$m = (3 - \alpha)\left(\frac{p}{\alpha}\right)^*, n = (3 - \alpha)\left(\frac{q}{\alpha}\right)^*.$$

Thanks to the Hölder inequality, we find that

$$\|u^\alpha u^{2 - \alpha}\|_{L^{m^*,n^*}} \leq \|u^\alpha\|_{L^{\frac{p}{\alpha},\frac{q}{\alpha}}} \|u^{2 - \alpha}\|_{L^{\frac{m}{\alpha},\frac{n}{\alpha}}}$$

$$\leq \|u\|_{L^{p,q}} \|u\|_{L^{m,n}}^{2 - \alpha}.$$

Using the latter inequality, the Hölder inequality, and (2.8), we infer that

$$\|u\|_{L^3}^3 \leq \|u\|_{L^{m,n}(Q(\rho))} \|u^\alpha u^{2 - \alpha}\|_{L^{m^*,n^*}}$$

$$\leq \|u\|_{L^{m,n}(Q(\rho))}^{3 - \alpha} \|u\|_{L^{p,q}}$$

$$\leq C \rho^{3(\alpha - 1)/2} \|u\|_{L^{p,q}} \|u\|_{L^{\alpha,\frac{2}{\alpha}}(Q(\rho))}$$

$$\leq C \rho^{3(\alpha - 1)/2} \|u\|_{L^{p,q}} \left(\|u\|_{L^{\infty,2}(Q(\rho))}^2 + \|\nabla u\|_{L^2(Q(\rho))}^2\right)^{(3 - \alpha)/2}. \tag{2.10}$$

This completes the proof.

In addition, we recall two well-known iteration lemmas.
Lemma 2.2. [10, Lemma V.3.1, p.161] Let $I(s)$ be a bounded nonnegative function in the interval $[r, R]$. Assume that for every $\sigma, \rho \in [r, R]$ and $\sigma < \rho$ we have

$$I(\sigma) \leq A_1(\rho - \sigma)^{-\alpha_1} + A_2(\rho - \sigma)^{-\alpha_2} + A_3 + \ell I(\rho)$$

for some non-negative constants $A_1, A_2, A_3$, non-negative exponents $\alpha_1 \geq \alpha_2$ and a parameter $\ell \in [0, 1)$. Then there holds

$$I(r) \leq c(\alpha_1, \ell)[A_1(R - r)^{-\alpha_1} + A_2(R - r)^{-\alpha_2} + A_3].$$

The following lemma is a generalization of corresponding result in [3] (see [3, Lemma 2.9, p.558]).

Lemma 2.3. Let $f \in L^q(Q(1))$ with $\frac{3}{7} > q > 1$ and $0 < r_0 < 1/2$. Suppose that for all $(x_0, t_0) \in Q(1/2)$ and $r_0 \leq r \leq \frac{1}{2}$,

$$\|f - \bar{f}_{B(r)}\|_{L^p(\tilde{Q}(r))} \leq C r^\gamma. \tag{2.11}$$

Let $\nabla \Pi = \mathcal{W}_{q, B(1)}(\nabla \cdot f)$. Then for all $(x_0, t_0) \in Q(1/2)$ and $r_0 \leq r \leq \frac{1}{2}$, it holds

$$\|\Pi - \bar{\Pi}_{B(r)}\|_{L^p(\tilde{Q}(r))} \leq C r^\gamma.$$

Proof. In view of the definition of pressure projection $\mathcal{W}_{q, B(1)}$, we know that

$$\|\Pi\|_{L^q(B(1))} \leq C\|f - \bar{f}_{B(1)}\|_{L^q(B(1))}. \tag{2.12}$$

We introduce a cut-off function $\phi(x)$ such that $\phi(x) = 1, x \in \bar{B}(\frac{3r}{4}), \phi(x) = 0, x \in \bar{B}(r)$. Note that

$$\Delta \Pi = \nabla \mathcal{W}_{q, B(1)}(\nabla \cdot f).$$

We split $\Pi$ into two part $\Pi = \Pi(1) + \Pi(2)$, where

$$\Delta \Pi(1) = -\nabla \mathcal{W}_{q, B(1)}(\nabla \cdot (\phi(f - \bar{f}_{B(r)}))),$$

which follows from that

$$\Delta \Pi(2) = 0, x \in \bar{B}(3r/4).$$

Thanks classical Calderón-Zygmund theorem, we have

$$\|\Pi(1) - \bar{\Pi}(1)_{B(r)}\|_{L^q(\tilde{B}(r))} \leq C\|f - \bar{f}_{B(r)}\|_{L^q(\tilde{B}(r))}.$$

This and hypothesis (2.11) yield

$$\|\Pi(1) - \bar{\Pi}(1)_{B(r)}\|_{L^p(\tilde{Q}(r))} \leq C r^\gamma. \tag{2.13}$$

The interior estimates of harmonic functions (2.7) and the triangle inequality guarantee that, for $\theta < 1/2$, we have

$$\int_{\bar{B}(\theta r)} |\Pi(2) - \bar{\Pi}(2)_{B(\theta r)}|^q dx \leq C(r\theta)^{3+q} \int_{\bar{B}(r/2)} |\Pi(2) - \bar{\Pi}(2)_{B(r/2)}|^q dx$$

$$\leq C\theta^{3+q} \int_{\bar{B}(r/2)} |\Pi - \bar{\Pi}(r/2)|^q dx + \int_{\bar{B}(r/2)} |\Pi(1) - \bar{\Pi}(1)_{B(r/2)}|^q dx.$$
We derive from the latter inequality and (2.13) that

\[ \|\Pi(2) - \Pi_{B(\theta r)}\|_{L^p_q(Q(\theta r))} \leq C\theta^{3/4+1}\|\Pi - \Pi_{B(r/2)}\|_{L^p_q(Q(r/2))} + Cr^\tau. \]

With the help of the triangle inequality again, (2.13) and the last inequality, we infer that

\[ \|\Pi - \Pi_{B(\theta r)}\|_{L^p_q(Q(\theta r))} \leq \|\Pi(1) - \Pi_{B(\theta r)}\|_{L^p_q(Q(\theta r))} + \|\Pi(2) - \Pi_{B(\theta r)}\|_{L^p_q(Q(\theta r))} \]

\[ \leq Cr^\tau + C\theta^{3/4+1}\|\Pi - \Pi_{B(r/2)}\|_{L^p_q(Q(r/2))} + Cr^\tau \]

\[ \leq C\theta^{3/4+1}\|\Pi - \Pi_{\tilde{B}(r)}\|_{L^p_q(Q(r))} + Cr^\tau, \]

where we used the fact that \( \|g - \tilde{g}_{B(r)}\|_{L^l(B(r))} \leq C\|g - c\|_{L^l(B(r))} \) with \( l \geq 1 \).

Now, thanks to \( \frac{3}{q} + 1 > \tau \), invoking iteration Lemma 2.2 and (2.12), we see that

\[ \|\Pi - \Pi_{\tilde{B}(r)}\|_{L^p_q(Q(r))} \leq C\tau^\tau\|\Pi - \Pi_{\tilde{B}(1/2)}\|_{L^p_q(Q(1/2))} + Cr^\tau \]

\[ \leq C\tau^\tau\|f - \tilde{f}_{B(1)}\|_{L^p_q(Q(1))} + Cr^\tau \]

\[ \leq C\tau^\tau. \]

This completes the proof of this lemma. \( \square \)

To prove Theorems 1.3, we need the following result.

**Proposition 2.4.** [24] Let \( S \) stand for the set of all the potential interior singular points of suitable weak solutions to (1.1) and \( \tau \) be defined in Lemma 6.2. Then, there holds, for \( 0 \leq \sigma < \frac{1}{\tau + 1} \),

\[ \Lambda(S, r(\log(e/r))^{\tau}) = 0. \]

3 Caccioppoli estimate

This section contains the proof of Proposition 1.2. Proposition 1.2 turns out to be a corollary of the following proposition.

**Proof of Proposition 1.2.** Consider \( 0 < R/2 \leq r < \frac{3\rho + \rho}{4} < \frac{\rho + \rho}{2} < \rho \leq R \). Let \( \phi(x, t) \) be non-negative smooth function supported in \( Q(\frac{\rho + \rho}{2}) \) such that \( \phi(x, t) \equiv 1 \) on \( Q(\frac{3\rho + \rho}{4}) \), \(|\nabla \phi| \leq C/(\rho - r)\) and \(|\nabla^2 \phi| + |\partial_t \phi| \leq C/(\rho - r)^2\).

Let \( \nabla \Pi_h = \mathcal{W}_{q,B(\rho)}(u) \), then, there holds

\[ \|\nabla \Pi_h\|_{L^p_q(Q(\rho))} \leq C\|u\|_{L^p_q(Q(\rho))}, \quad \|\Pi_1\|_{L^2(Q(\rho))} \leq C\|\nabla u\|_{L^2(Q(\rho))}, \quad \|\Pi_2\|_{L^\frac{p}{p-1}(Q(\rho))} \leq C\|u\|_{L^\frac{p}{p-1}(Q(\rho))}. \]
By virtue of interior estimate of harmonic function (2.6) and (3.1), we conclude that
\[
\|\nabla \Pi_h\|_{L^2,\infty(Q(\frac{r}{2}))} \leq C \frac{\rho^{\frac{5n-4}{2\alpha}}}{(\rho - r)^{\frac{\alpha}{2}}} \|\nabla \Pi_h\|_{L^2,2(Q(\rho))} \leq \frac{C\rho^{\frac{5n-4}{2\alpha}}}{(\rho - r)^{\frac{\alpha}{2}}} \|u\|_{L^p,q(Q(\rho))},
\]
(3.4)
\[
\|\nabla^2 \Pi_h\|_{L^2,\infty(Q(\frac{r}{2}))} \leq C \frac{\rho^{\frac{5n-4}{2\alpha}}}{(\rho - r)^{\frac{\alpha}{2} + 1}} \|\nabla \Pi_h\|_{L^2,2(Q(\rho))} \leq \frac{C\rho^{\frac{5n-4}{2\alpha}}}{(\rho - r)^{\frac{\alpha}{2}}} \|u\|_{L^p,q(Q(\rho))},
\]
(3.5)
\[
\|\nabla \Pi_h\|_{L^2,4(Q(\frac{r}{2}))} \leq C \frac{\rho^{\frac{13n-8}{4\alpha}}}{(\rho - r)^{\frac{\alpha}{2}}} \|\nabla \Pi_h\|_{L^2,2(Q(\rho))} \leq \frac{C\rho^{\frac{13n-8}{4\alpha}}}{(\rho - r)^{\frac{\alpha}{2}}} \|u\|_{L^p,q(Q(\rho))}.
\]
(3.6)

We define \( \beta = \frac{1}{\alpha - 1} \) and choose \( \phi^{2\beta} \) as the non-negative function in the local energy inequality to get
\[
\int_{B(\frac{r}{2})} |v|^2 \phi^{2\beta}(x, t) dx + \int_{Q(\frac{r}{2})} |\nabla v|^2 \phi^{2\beta}(x, s) dx ds
\leq \frac{C}{(\rho - r)^2} \int_{Q(\frac{r}{2})} |v|^2 + \frac{C}{(\rho - r)} \int_{Q(\frac{r}{2})} |v|^2 |u| \phi^{2\beta - 1}
+ \int_{Q(\frac{r}{2})} \phi^{2\beta} |u \otimes v : \nabla^2 \Pi_h| + \frac{C}{(\rho - r)} \int_{Q(\frac{r}{2})} |\nabla \Pi_h| \phi^{2\beta - 1}
+ \frac{C}{(\rho - r)} \int_{Q(\frac{r}{2})} |\Pi_2 \phi^{\beta - 1}||v\phi^{\beta}|
= : I + II + III + IV + V.
\]
(3.7)

By means of the triangle inequality, (3.1) and the Hölder inequality, we see that
\[
\int_{Q(\frac{r}{2})} |v|^2 \leq \int_{Q(\rho)} |u|^2 + |\nabla \Pi_h|^2 \leq C \rho^{\frac{5n-4}{\alpha}} \|u\|_{L^p,q(Q(\rho))}^2,
\]
(3.8)
which implies that
\[
I \leq \frac{C\rho^{\frac{5n-4}{\alpha}}}{(\rho - r)^2} \|u\|_{L^p,q(Q(\rho))}^2.
\]

It is obvious that
\[
II \leq \frac{C}{(\rho - r)} \int_{Q(\rho)} |v|^2 \phi^{2\beta - 1} |v - \nabla \Pi_h|
\leq \frac{C}{(\rho - r)} \int_{Q(\rho)} |v|^3 \phi^{2\beta - 1} + \phi^{2\beta - 1} |v|^2 |\nabla \Pi_h|
= \frac{C}{(\rho - r)} \int_{Q(\rho)} |v|^3 \phi^{3 - \alpha} + \frac{C}{(\rho - r)} \int_{Q(\rho)} \phi^{2\beta - 1} |v|^2 |\nabla \Pi_h|
=: II_1 + II_2,
\]
(3.9)
where the fact $2\beta - 1 - \beta(3 - \alpha) = 0$ is used.

Utilizing similar argument as in proof of (2.9) in Lemma 2.1 and the Young inequality, we write
\[
H_1 \leq \frac{C}{(\rho - r)^{\beta}} \int_{L^2(\mathbb{R}^3)} (|v|^{2} + \|\nabla v\|^{2})\, dr \leq \frac{C}{(\rho - r)^{\beta}} \int_{L^2(\mathbb{R}^3)} (|v|^{2} + \|\nabla v\|^{2})\, dr 
\]
where the fact $2\beta - 1 - \beta(3 - \alpha) = 0$ is used.

Invoking the Hölder inequality, Young’s inequality and (3.11), we obtain
\[
H_2 \leq \frac{C}{(\rho - r)^{\beta}} \int_{L^2(\mathbb{R}^3)} (|v|^{2} + \|\nabla v\|^{2})\, dr 
\]
where the fact $2\beta - 1 - \beta(3 - \alpha) = 0$ is used.

Arguing in the same manner as in (3.9), we infer that
\[
III = \int_{L^2(\mathbb{R}^3)} (|v|^{2} + \|\nabla v\|^{2})\, dr 
\]
where the fact $2\beta - 1 - \beta(3 - \alpha) = 0$ is used.

From the Hölder inequality, Young’s inequality and (3.11), (3.12), we see that
\[
III \leq \frac{C}{(\rho - r)^{\beta}} \int_{L^2(\mathbb{R}^3)} (|v|^{2} + \|\nabla v\|^{2})\, dr 
\]
where the fact $2\beta - 1 - \beta(3 - \alpha) = 0$ is used.

Similarly, we have
\[
III \leq \frac{C}{(\rho - r)^{\beta}} \int_{L^2(\mathbb{R}^3)} (|v|^{2} + \|\nabla v\|^{2})\, dr 
\]
where the fact $2\beta - 1 - \beta(3 - \alpha) = 0$ is used.

As a consequence, we have
\[
III \leq \frac{2}{32} \frac{C}{(\rho - r)^{\beta}} \int_{L^2(\mathbb{R}^3)} (|v|^{2} + \|\nabla v\|^{2})\, dr 
\]
where the fact $2\beta - 1 - \beta(3 - \alpha) = 0$ is used.
In light of Hölder’s inequality, (3.2), (3.8) and Young’s inequality, we deduce that

\[
IV \leq \frac{C}{(p-r)} \|v\|_{L^2(Q(\frac{\rho}{\tau}))} \|\Pi_1\|_{L^2(Q(\frac{\rho}{\tau}))} \\
\leq \frac{C}{(p-r)^2} \|v\|_{L^2(Q(\frac{\rho}{\tau}))}^2 + \frac{1}{16} \|\Pi_1\|_{L^2(Q(\rho))}^2 \\
\leq \frac{C}{(p-r)^2} \|v\|_{L^2(Q(\frac{\rho}{\tau}))}^2 + \frac{1}{16} \|\nabla u\|_{L^2(Q(\rho))}^2 \\
\leq \frac{C\rho^{\frac{9a-4}{2\sigma}}}{(p-r)^2} \|u\|_{L^{p,q}(Q(\rho))}^2 + \frac{1}{16} \|\nabla u\|_{L^2(Q(\rho))}^2. \tag{3.14}
\]

To proceed further, we denote \(\eta = \phi^{\beta-1}\). The fact \(\partial_t \partial_j \Pi_2 = -\partial_i \partial_j(u_i u_j)\) and Leibniz’s formula allow us to get

\[
\partial_t \partial_j(\Pi_2 \eta) = -\eta \partial_t \partial_j(u_j u_i) + 2\partial_i \eta \partial_j \Pi + \Pi \partial_i \partial_j \eta.
\]

Integrating by parts, we have

\[
\eta \Pi_2(x) = \Gamma * (-\eta \partial_t \partial_j(u_j u_i) + 2\partial_i \eta \partial_j \Pi_2 + \Pi_2 \partial_i \partial_j \eta) \\
= -\partial_t \partial_j \Gamma * (\eta(u_j u_i)) \\
+ 2\partial_i \Gamma * (\partial_j \eta(u_j u_i)) + 2\partial_i \Gamma * (\partial_i \eta \Pi_2) \\
- \Gamma * (\partial_i \partial_j \eta u_j u_i) - \Gamma * (\partial_i \partial_i \eta \Pi_2) =: \Pi_{21}(x) + \Pi_{22}(x) + \Pi_{23}(x), \tag{3.15}
\]

By Young’s convolution inequality, setting \(\tau = \frac{5q-12}{2q} > 0\), we arrive at

\[
\|\Pi_{22}(x)\|_{L^{1,2}} \leq \frac{C\rho^\tau}{\rho - r} \left( \|\Pi_2\|_{L^{1, \frac{2q}{2q-2}}} + \|u\|_{L^{1, \frac{2q}{2q-2}}}^2 \right) \\
\leq \frac{C\rho^\tau}{\rho - r} \|u\|_{L^{2,q}}^2 \\
\leq \frac{C\rho^{\frac{\tau}{2} + \frac{2(p-2)}{p}}}{\rho - r} \|u\|_{L^{p,q}}^2 \\
\leq \frac{C\rho^{\frac{9a-8}{2\sigma}}}{\rho - r} \|u\|_{L^{p,q}}^2. \tag{3.16}
\]

Likewise,

\[
\|\Pi_{23}(x)\|_{L^{1,2}} \leq \frac{C\rho}{(\rho - r)^2} \left( \|\Pi_2\|_{L^{1, \frac{12}{12}}} + \|u\|_{L^{1, \frac{12}{12}}}^2 \right) \\
\leq \frac{C\rho}{(\rho - r)^2} \|u\|_{L^2}^2 \\
\leq \frac{C\rho^{\frac{11a-8}{2\sigma}}}{(\rho - r)^2} \|u\|_{L^{p,q}}^2. \tag{3.17}
\]

Note that

\[
u_i u_j = v_i v_j - (\nabla \Pi_h)_i(v)_j + (\nabla \Pi_h)_i(\nabla \Pi_h)_j - (v)_i(\nabla \Pi_h)_j.
\]
Hence, some integrations by parts ensure that
\[
\Pi_{21} = -\partial_i \partial_j \Gamma \left[ \eta(v_i v_j - (\nabla \Pi_h)_i(v)_j + (\nabla \Pi_h)_i((\nabla \Pi_h)_j - (v)_i(\nabla \Pi_h)_j) \right]
= -\partial_i \partial_j \Gamma \left[ \eta(v_i v_j) + 2\partial_i \partial_j \Gamma \left[ \eta((\nabla \Pi_h)_i(v)_j) - \partial_i \partial_j \Gamma \left[ \eta((\nabla \Pi_h)_i((\nabla \Pi_h)_j) \right) \right] \right]
= : \Pi_{211} + \Pi_{212} + \Pi_{213}.
\]

It is clear that
\[
V \leq \frac{C}{\rho - r} \int Q(\rho) |v^{\phi^{\beta}}| (|\Pi_{211}| + |\Pi_{212}| + |\Pi_{213}| + |\Pi_{22}| + |\Pi_{23}|)
= : V_1 + V_2 + V_3 + V_4 + V_5.
\]

To bound $IV_1$, the classical Calderón-Zygmund theorem allows us to argue as the deduction of (2.9) to obtain that
\[
V_1 \leq \frac{C}{(\rho - r)^2} \|v^{\phi^{\beta}}\|_{L^{m,n}(Q(\rho))} \|\Pi_{211}\|_{L^{m^*,n^*}}
\leq \frac{C}{(\rho - r)^2} \|v^{\phi^{\beta}}\|_{L^{m,n}(Q(\rho))} \|v^{\phi^{\beta}}\|_{L^{m^*,n^*}}
\leq \frac{C}{(\rho - r)^2} \|v^{\phi^{\beta}}\|_{L^{m,n}(Q(\rho))} \|v\|_{L^{p,q}}
\leq \frac{C}{(\rho - r)^2} \|v\|_{L^{p,q}}^2 (\|v\|_{L^{p,q}}^2 + \|\nabla(v^{\phi^{\beta}})\|_{L^2(Q(\rho))}^2),
\]

where we used the Young inequality and the fact that $\beta - 1 - \beta(2 - \alpha) = 0$.

Likewise, we get
\[
V_2 \leq \frac{C}{\rho - r} \|v^{\phi^{\beta}}\|_{L^{\infty,2}} \|\Pi_{212}\|_{L^{1,2}}
\leq \frac{C}{\rho - r} \|v^{\phi^{\beta}}\|_{L^{\infty,2}} \|\eta((\nabla \Pi_h)_i(v)_j)\|_{L^{1,2}}
\leq \frac{C}{\rho - r} \|v^{\phi^{\beta}}\|_{L^{\infty,2}} \|\nabla \Pi_h\|_{L^{2,\infty}} \|\eta(v)_j\|_{L^{2,2}}
\leq \frac{1}{128} \|v^{\phi^{\beta}}\|_{L^{\infty,2}}^2 + \frac{C}{(\rho - r)^2} \|v\|_{L^2}^2 \|\nabla \Pi_h\|_{L^{2,\infty}}^2
\leq \frac{1}{128} \|v^{\phi^{\beta}}\|_{L^{\infty,2}}^2 + \frac{C}{(\rho - r)^5} \|u\|_{L^{p,q}(Q(\rho))}^4.
\]

Similarly, we get
\[
V_3 \leq \frac{C}{\rho - r} \|v^{\phi^{\beta}}\|_{L^{\infty,2}} \|\Pi_{213}\|_{L^{1,2}}
\leq \frac{C}{\rho - r} \|v^{\phi^{\beta}}\|_{L^{\infty,2}} \|\nabla \Pi_h\|_{L^{2,4}}^2
\leq \frac{1}{128} \|v^{\phi^{\beta}}\|_{L^{\infty,2}}^2 + \frac{C}{(\rho - r)^2} \|\nabla \Pi_h\|_{L^{2,4}}^4
\leq \frac{1}{128} \|v^{\phi^{\beta}}\|_{L^{\infty,2}}^2 + \frac{C}{(\rho - r)^8} \|u\|_{L^{p,q}(Q(\rho))}^4.
\]
Combining the Hölder inequality and (3.16) ensures that

\[ V_4 \leq \frac{C}{\rho - r} \| \nu \phi^\beta \|_{L^{\infty,2}} \| \Pi_{22} \|_{L^{1,2}} \]
\[ \leq \| \nu \phi^\beta \|_{L^{\infty,2}} \frac{C \rho^{\frac{9\alpha - 8}{2\alpha - 1}}}{(\rho - r)^2} \| u \|_{L^{p,q}}^2 \]
\[ \leq \frac{1}{128} \| \phi^\beta v \|_{L^{\infty,2}}^2 + C \rho^{\frac{9\alpha - 8}{2\alpha - 1}} (\rho - r)^4 \| u \|_{L^{p,q}}^4. \] (3.21)

Arguing as in the proof of the last inequality, we deduce that

\[ V_5 \leq \frac{C}{(\rho - r)} \| \nu \phi^\beta \|_{L^{\infty,2}} \| \Pi_{25} \|_{L^{1,2}} \]
\[ \leq \| \nu \phi^\beta \|_{L^{\infty,2}} \frac{C \rho^{\frac{11\alpha - 8}{2\alpha - 1}}}{(\rho - r)^3} \| u \|_{L^{p,q}}^2 \]
\[ \leq \frac{1}{128} \| \phi^\beta v \|_{L^{\infty,2}}^2 + C \rho^{\frac{11\alpha - 8}{2\alpha - 1}} (\rho - r)^6 \| u \|_{L^{p,q}}^4. \]

We derive from the Cauchy-Schwarz inequality and (3.8) that

\[ \iint_{Q(\rho)} |\nabla (\nu \phi^\beta)|^2 dx ds \leq 2 \left( \iint_{Q(\rho)} |\nabla \nu|^2 |\phi^\beta|^2 dx ds + \frac{1}{2} \iint_{Q(\rho)} |\nabla \phi^\beta|^2 |\phi^\beta|^{-2} dx ds \right) \]
\[ \leq 2 \iint_{Q(\rho)} |\nabla \nu|^2 |\phi^\beta|^2 dx ds + \frac{C \rho^{\frac{5\alpha - 4}{\alpha}}}{(\rho - r)^2} \| u \|^2_{L^{p,q}(Q(\rho))}. \] (3.22)

Inserting all these estimates into (2.2) and using (3.22), we conclude that

\[ \sup_{-\rho^2 \leq t \leq 0} \int_{B(\rho)} |\nu \phi^\beta|^2 dx + \iint_{Q(\rho)} |\nabla (\nu \phi^\beta)|^2 dx d\tau \]
\[ \leq \frac{1}{4} \left( \| \nu \phi^\beta \|^2_{L^{2,\infty}(Q(\rho))} + \| \nabla (\nu \phi^\beta) \|^2_{L^2(Q(\rho))} \right) + \frac{C \rho^{\frac{5\alpha - 4}{\alpha}}}{(\rho - r)^2} \| u \|^2_{L^{p,q}(Q(\rho))} \]
\[ + \frac{C \rho^{\frac{13\alpha - 8}{\alpha}}}{(\rho - r)^8} \| u \|^4_{L^{p,q}(Q(\rho))} + \frac{C \rho^{\frac{9\alpha - 8}{\alpha}}}{(\rho - r)^4} \| u \|^4_{L^{p,q}(Q(\rho))} + \frac{C \rho^{\frac{11\alpha - 8}{\alpha}}}{(\rho - r)^6} \| u \|^4_{L^{p,q}}. \]

This in turn implies

\[ \sup_{-\rho^2 \leq t \leq 0} \int_{B(\rho)} |\nu \phi^\beta|^2 dx + \iint_{Q(\rho)} |\nabla (\nu \phi^\beta)|^2 dx d\tau \]
\[ \leq C \rho^{\frac{5\alpha - 4}{\alpha}} \| u \|^2_{L^{p,q}(Q(\rho))} + \frac{C \rho^{\frac{10\alpha - 8}{\alpha}}}{(\rho - r)^2} \| u \|^2_{L^{p,q}(Q(\rho))} \]
\[ + \frac{C \rho^{\frac{13\alpha - 8}{\alpha}}}{(\rho - r)^8} \| u \|^4_{L^{p,q}(Q(\rho))} + \frac{1}{16} \| \nabla u \|^2_{L^2(Q(\rho))} \]
\[ + \frac{C \rho^{\frac{13\alpha - 8}{\alpha}}}{(\rho - r)^8} \| u \|^4_{L^{p,q}(Q(\rho))} + \frac{C \rho^{\frac{9\alpha - 8}{\alpha}}}{(\rho - r)^4} \| u \|^4_{L^{p,q}(Q(\rho))} + \frac{C \rho^{\frac{11\alpha - 8}{\alpha}}}{(\rho - r)^6} \| u \|^4_{L^{p,q}(Q(\rho))}. \] (3.23)
The interior estimate of harmonic function \( (2.6) \) and \( (3.1) \) implies that
\[
\|\nabla \Pi_h\|_{L^2, \sigma Q(r)}^2 \leq \frac{C r^3 \rho^{-3} 2}{(\rho - r)^{2.3}} \|\nabla \Pi_h\|_{L^2 Q(\rho)}^2
\]
\[
\leq C \rho \frac{\alpha - 1}{(\rho - r)^3} \|u\|_{L^p, q}(Q(\rho)).
\]

With the help of the triangle inequality, interpolation inequality \( (2.8) \) and the last inequality, we get
\[
\|u\|_{L^2, \sigma Q(r)}^2 \leq \|v\|_{L^2, \sigma Q(r)}^2 + \|\nabla \Pi_h\|_{L^2, \sigma Q(r)}^2
\]
\[
\leq C \{\|v\|_{L^2, \infty Q(r)}^2 + \|\nabla v\|_{L^2 Q(r)}^2\} + C \rho \frac{\alpha - 1}{(\rho - r)^3} \|u\|_{L^p, q}(Q(\rho)). \tag{3.24}
\]

Employing \( (2.6) \) and \( (3.4) \) once again, we have the estimate
\[
\|\nabla^2 \Pi_h\|_{L^2 Q(r)}^2 \leq \frac{C r^3}{(\rho - r)^{3+2}} \|\nabla \Pi_h\|_{L^2 Q(\rho)}^2 \leq C \rho \frac{\alpha - 1}{(\rho - r)^3} \|u\|_{L^p, q}(Q(\rho)).
\]

This together with the triangle inequality and \( (3.23) \) leads to
\[
\|\nabla u\|_{L^2 Q(r)}^2 \leq \|\nabla v\|_{L^2 Q(r)}^2 + \|\nabla^2 \Pi_h\|_{L^2 Q(r)}^2
\]
\[
\leq \left\{1 + \frac{\rho^3}{(\rho - r)^3}\right\} C \rho \frac{\alpha - 1}{(\rho - r)^2} \|u\|_{L^p, q}(Q(\rho)) + C \rho \frac{\alpha - 1}{(\rho - r)^2} \rho^3 \|u\|_{L^p, q}(Q(\rho))
\]
\[
+ C \rho \frac{\alpha - 8}{(\rho - r)^3} \|u\|_{L^p, q}(Q(\rho))^4 + \frac{1}{16} \|\nabla u\|_{L^2 Q(r)}^2
\]
\[
+ C \rho \frac{\alpha - 8}{(\rho - r)^3} \|u\|_{L^p, q}(Q(\rho))^4 + C \rho \frac{\alpha - 8}{(\rho - r)^2} \|u\|_{L^p, q}^4 + C \rho \frac{11 \alpha - 8}{(\rho - r)^6} \|u\|_{L^p, q}^4. \tag{3.25}
\]

As an immediate consequence of \( (3.24) \) and \( (3.25) \), we get
\[
\|u\|_{L^2, \sigma Q(r)}^2 + \|\nabla u\|_{L^2 Q(r)}^2
\]
\[
\leq \left\{1 + \frac{\rho}{(\rho - r)^3}\right\} C \rho \frac{\alpha - 1}{(\rho - r)^2} \|u\|_{L^p, q}(Q(\rho)) + C \rho \frac{\alpha - 1}{(\rho - r)^2} \rho^3 \|u\|_{L^p, q}(Q(\rho))
\]
\[
+ C \rho \frac{\alpha - 8}{(\rho - r)^3} \|u\|_{L^p, q}(Q(\rho)) + \frac{1}{16} \|\nabla u\|_{L^2 Q(r)}^2
\]
\[
+ C \rho \frac{\alpha - 8}{(\rho - r)^3} \|u\|_{L^p, q}(Q(\rho))^4 + C \rho \frac{11 \alpha - 8}{(\rho - r)^6} \|u\|_{L^p, q}^4.
\]

Now, we are in a position to apply \([10, Lemma V.3.1, p.161]\) to the latter to find that
\[
\|u\|_{L^2, \sigma Q(\frac{r}{2})}^2 + \|\nabla u\|_{L^2 Q(\frac{r}{2})}^2
\]
\[
\leq CR \rho \frac{\alpha - 1}{(\rho - r)^2} \|u\|_{L^p, q}(Q(\rho)) + CR \rho \frac{\alpha - 1}{(\rho - r)^2} \|u\|_{L^p, q}(Q(\rho)) + CR \rho \frac{11 \alpha - 8}{(\rho - r)^6} \|u\|_{L^p, q}(Q(\rho)). \tag{3.26}
\]

This achieves the proof of this proposition. \(\square\)
4 Induction arguments and proof of Theorem 1.1

In this section, we utilize an especial case of (1.12) with \( p = q = (5 + 2\delta)/2 \) and induction arguments to finish the proof of Theorem 1.1. To this end, we begin with a critical proposition, which can be seen as the bridge between the previous step and the next step for the given statement in the induction arguments.

**Proposition 4.1.** Assume that \( \int_{Q(r)} |v|^{\frac{10}{3}} \leq r^5N \), with \( r_k \leq r \leq r_{k_0} \). There is a constant \( C \) such that the following result holds. For any given \( (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^- \) and \( k_0 \in \mathbb{N} \), we have for any \( k > k_0 \),

\[
\sup_{-r_k^2 \leq t-t_0 \leq 0} \int_{B_k} |v|^2 + r_k^{-3} \int_{Q_k} |\nabla v|^2 \leq C \sup_{-r_k^2 \leq t-t_0 \leq 0} \int_{B_{k_0}} |v|^2 + C \sum_{l=k_0}^k r_l \left( \int_{Q_l} |v|^{\frac{10}{3}} \right)^{\frac{3}{5}} \left( \int_{Q_{k_0}} |u|^{\frac{5+2\delta}{2}} \right)^{\frac{2}{5+2\delta}} + C \sum_{l=k_0}^k r_l^{\frac{5+2\delta}{5+2\delta}} \left( \int_{Q_l} |v|^{\frac{10}{3}} \right)^{\frac{3}{5}} \left( \int_{Q_{k_0}} |u|^{\frac{5+2\delta}{2}} \right)^{\frac{2}{5+2\delta}} (4.1)
\]

\[
+ C \sum_{l=k_0}^k r_l^{\frac{2+4\delta}{5+2\delta}} r_l^{-\frac{3}{5}} \left\| v \right\|_{L^\infty(Q_l)}^{\frac{5+2\delta}{5+2\delta}} \left( \int_{Q_{k_0}} |\nabla u|^2 \right)^{\frac{2}{5+2\delta}} (Q(1))
\]

\[
+ C \sum_{l=k_0}^k r_l \left( \int_{Q_l} |v|^{\frac{10}{3}} \right)^{\frac{3}{5}} \left( \int_{Q_{k_0}} |\nabla u|^2 \right)^{\frac{2}{5+2\delta}} (Q(1))
\]

\[
+ C \sum_{l=k_0}^k r_l^{\frac{1+2\delta}{5+2\delta}} \left( \int_{Q_l} |v|^{\frac{10}{3}} \right)^{\frac{3}{5}} \left\{ N^{3/5} + N \left\| u \right\|_{L^\infty(Q_l)}^{\frac{5+2\delta}{2}} \right\} (Q(1))
\]

\[
+ C \sum_{l=k_0}^k r_l^{\frac{4+4\delta}{5+2\delta}} r_l^{-\frac{3}{5}} \left\| v \right\|_{L^\infty(Q_l)}^{\frac{5+2\delta}{5+2\delta}} \left( \int_{Q_{k_0}} |\nabla u|^2 \right)^{\frac{2}{5+2\delta}} (Q(1)).
\] (4.2)

**Proof.** In order to simplify the presentation, we suppose \((x_0, t_0) = (0, 0)\). Let us introduce the backward heat kernel

\[
\Gamma(x, t) = \frac{1}{4\pi(r_k^2 - t)^{3/2}} e^{-\frac{|x|^2}{4(r_k^2 - t)}}.
\]

Furthermore, we choose the smooth cut-off function below

\[
\phi(x, t) = \begin{cases} 
1, & (x, t) \in Q(r_{k_0}+1), \\
0, & (x, t) \in Q(\frac{3}{2}r_{k_0}+1);
\end{cases}
\]

satisfying

\[
0 \leq \phi \leq 1 \quad \text{and} \quad r_{k_0}^2 |\partial_t \phi(x, t)| + r_{k_0}^4 |\partial_x^2 \phi(x, t)| \leq C.
\]

Easy calculations lead to the following fact:
(i) There is a constant $c > 0$ independent of $r_k$ such that, for any $(x, t) \in Q(r_k)$,
$$\Gamma(x, t) \geq cr_k^{-3}.$$

(ii) For any $(x, t) \in Q(r_k)$, we have
$$|\Gamma(x, t)\phi(x, t)| \leq Cr_k^{-3}, \quad |\nabla\phi(x, t)\Gamma(x, t)| \leq Cr_k^{-4}, \quad |\phi(x, t)\nabla\Gamma(x, t)| \leq Cr_k^{-4}.$$

(iii) For any $(x, t) \in Q(3r_{k_0}/4) \setminus Q(r_{k_0}/2)$, one can deduce that
$$\Gamma(x, t) \leq Cr_{k_0}^{-3}, \quad \partial_t\Gamma(x, t) \leq Cr_{k_0}^{-4},$$
from which it follows that
$$|\Gamma(x, t)\partial_t\phi(x, t)| + |\Gamma(x, t)\Delta\phi(x, t)| + |\nabla\phi(x, t)\nabla\Gamma(x, t)| \leq Cr_{k_0}^{-5}.$$

(iv) For any $(x, t) \in Q_l \setminus Q_{l+1}$,
$$\Gamma \leq Cr_{l+1}^{-3}, \quad \nabla\Gamma \leq Cr_{l+1}^{-4}.$$

Now, plugging $\varphi_1 = \phi\Gamma$ into the local energy inequality (2.2) and utilizing the fact that $\Gamma_t + \Delta\Gamma = 0$, we arrive at that
$$\int_{B_1} |v|^2\phi(x, t)\Gamma + \int_t^{t-r_{k_0}^2} \int_{B_1} |\nabla v|^2\phi(x, s)\Gamma$$
$$\leq \int_{-r_{k_0}^2}^t \int_{B_1} |v|^2(\Gamma\Delta\phi + \Gamma\partial_t\phi + 2\nabla\Gamma\nabla\phi)$$
$$+ \int_{-r_{k_0}^2}^t \int_{B_1} |v|^2\nabla(\phi\Gamma) - |v|^2\nabla \Pi_2 \cdot \nabla(\phi\Gamma)$$
$$+ \int_{-r_{k_0}^2}^t \int_{B_1} \Gamma\phi(v \otimes v - \nabla \Pi_2 \otimes v : \nabla^2\Pi_2)$$
$$+ \int_{-r_{k_0}^2}^t \int_{B_1} \Pi_1 v \cdot \nabla(\Gamma\phi) + \int_{-r_{k_0}^2}^t \int_{B_1} \Pi_2 v \cdot \nabla(\Gamma\phi),$$
(4.3)
where
$$\nabla \Pi_1 = \mathcal{W}_{2, B_1}(\Delta u), \quad \nabla \Pi_2 = -\mathcal{W}_{2+2\delta, B_1}(\nabla \cdot (u \otimes u)).$$

First, we give the low bound estimates of the terms on the left hand side of inequality (4.3).
Indeed, by virtue of (i), we know that
$$\int_{B_1} |v|^2\phi\Gamma \geq c\int_{B_1} |v|^2,$$
and
$$\int_{-r_{k_0}^2}^t \int_{B_1} \phi\Gamma|\nabla v|^2 \geq cr_k^{-3}\int_{Q_k} |\nabla v|^2.$$

Second, we focus on the estimation of the right hand side of (4.3). As the support of $\partial_t\phi$ is included in $Q(3r_{k_0}/4)/Q(r_{k_0}/2)$, we deduce
$$\int_{-r_{k_0}^2}^t \int_{B_1} |v|^2|\Gamma\Delta\phi + \Gamma\partial_t\phi + 2\nabla\Gamma\nabla\phi| \leq C \sup_{-r_{k_0}^2 \leq t \leq 0} \int_{B_{k_0}} |v|^2.$$
(4.4)
The Hölder inequality and (iv) entails that

\[
\iint_{Q_{k_0}} |v|^2 v \cdot \nabla (\phi \Gamma) d\tau \\
\leq \sum_{l=k_0}^{k-1} \iint_{Q_l/Q_{l+1}} |v|^3 |\nabla (\phi \Gamma)| + \iint_{Q_k} |v|^3 |\nabla (\phi \Gamma)| \\
\leq C \sum_{l=k_0}^{k} r_l^{-4} \iint_{Q_l} |v|^3 \\
\leq C \sum_{l=k_0}^{k} r_l^{-4} \left( \iint_{Q_l} |v|^{\frac{10}{3}} \right)^{\frac{2}{9}} r_l^{\frac{1}{2}} \\
\leq C \sum_{l=k_0}^{k} r_l \left( \iint_{Q_l} |v|^{\frac{10}{3}} \right)^{\frac{9}{10}}.
\]

Similar arguments lead to

\[
\iint_{Q_{k_0}} |v|^2 \nabla \Pi_h \cdot \nabla (\phi \Gamma) \\
\leq \sum_{l=k_0}^{k-1} \iint_{Q_l/Q_{l+1}} |v|^2 |\nabla \Pi_h||\nabla (\phi \Gamma)| + \iint_{Q_k} |v|^2 |\nabla \Pi_h||\nabla (\phi \Gamma)| \\
\leq C \sum_{l=k_0}^{k} r_l^{-4} \iint_{Q_l} |v|^2 |\nabla \Pi_h| \\
\leq C \sum_{l=k_0}^{k} r_l^{-4} \left( \iint_{Q_l} |v|^{\frac{10}{3}} \right)^{\frac{2}{9}} \left( \iint_{Q_l} |\nabla \Pi_h|^\frac{5+2\delta}{3} \right)^\frac{2}{5+2\delta} r_l^{\frac{4\delta}{5+2\delta}} \\
\leq C \sum_{l=k_0}^{k} r_l^{-4+3+\frac{4\delta}{5+2\delta}} \left( \iint_{Q_l} |v|^{\frac{10}{3}} \right)^{\frac{2}{9}} r_l^{\frac{2}{5+2\delta}} \left( \iint_{Q_l} |\nabla \Pi_h|^\frac{5+2\delta}{3} \right)^\frac{2}{5+2\delta} \\
\leq C \sum_{l=k_0}^{k} r_l^{\frac{1+\delta}{5+2\delta}} \left( \iint_{Q_l} |v|^{\frac{10}{3}} \right)^{\frac{3}{5}} \left( \iint_{Q_l} |u|^{\frac{5+2\delta}{3}} \right)^\frac{2}{5+2\delta} \left( \iint_{Q_l} |\nabla \Pi_h|^\frac{5+2\delta}{3} \right)^\frac{2}{5+2\delta} r_l^{\frac{4\delta}{5+2\delta}}. \tag{4.5}
\]

Similar, by Hölder’s inequality and (2.6), we get

\[
\iint_{Q_{k_0}} |v|^2 |\nabla^2 \Pi_h|| (\phi \Gamma) \\
\leq \sum_{l=k_0}^{k-1} \iint_{Q_l/Q_{l+1}} |v|^2 |\nabla^2 \Pi_h|| (\phi \Gamma) + \iint_{Q_k} |v|^2 |\nabla^2 \Pi_h|| (\phi \Gamma) \\
\leq C \sum_{l=k_0}^{k} r_l^{-2} \iint_{Q_l} |v|^2 |\nabla^2 \Pi_h| \\
\leq C \sum_{l=k_0}^{k} r_l^{-2} \left( \iint_{Q_l} |v|^{\frac{10}{3}} \right)^{\frac{2}{5}} \left( \iint_{Q_l} |\nabla^2 \Pi_h|^\frac{5+2\delta}{3} \right)^\frac{2}{5+2\delta} r_l^{\frac{4\delta}{5+2\delta}}.
\]
\[
\begin{align*}
\leq C \sum_{l=k_0}^{k} r_l^{-3+3+\frac{44}{5+25}} \left( \iint_{Q_l} |v|^{10/3} \right)^{\frac{4}{10}} r_l^{-\frac{3}{5+25}} \left( \iint_{Q_l} |\nabla \Pi_h|^{\frac{5+25}{2}} \right)^{\frac{2}{5+25}} \\
\leq C \sum_{l=k_0}^{k} r_l^{6+45/4+25} \left( \iint_{Q_l} |v|^{10/3} \right)^{\frac{3}{7}} \left( \iint_{Q_l} |u|^{\frac{5+25}{2}} \right)^{\frac{2}{5+25}}.
\end{align*}
\tag{4.6}
\]

From the Hölder inequality, (iv), (2.6) and (2.3), we deduce that
\[
\begin{align*}
\iint_{Q_{k_0}} \phi \Gamma |v| |\nabla \Pi_h| |\nabla^2 \Pi_h| \\
\leq C \sum_{l=k_0}^{k} r_l^{-3} \left\| v \right\|_{L^{\frac{5+25}{5+25}}(Q_l)} \left\| \nabla \Pi_h \right\|_{L^{\frac{12(5+25)}{5+25}}(Q_l)} \left\| \nabla^2 \Pi_h \right\|_{L^{\frac{12(5+25)}{5+25}}(Q_l)} \\
\leq C \sum_{l=k_0}^{k} r_l^{-3} \left\| v \right\|_{L^{\frac{5+25}{5+25}}(Q_l)} \left\| \nabla \Pi_h \right\|_{L^{\frac{12(5+25)}{11+10}}(Q_l)} \left\| \nabla^2 \Pi_h \right\|_{L^{\frac{12(5+25)}{11+10}}(Q_l)} \\
\leq C \sum_{l=k_0}^{k} r_l^{\frac{2+45}{5+25} - \frac{3}{5+25}} \left\| v \right\|_{L^{\frac{5+25}{5+25}}(Q_l)} \left\| u \right\|_{L^{\frac{12(5+25)}{11+10}}(Q_l)}.
\end{align*}
\tag{4.7}
\]

We introduce \( \chi_l = 1 \) on \( |x| \leq 7/8 r_l \) and \( \chi_l = 0 \) if \( |x| \geq r_l \). It is obvious that \( \chi_{k_0} \phi \Gamma = \phi \Gamma \) on \( Q_{k_0} \). By taking advantage of the support of \( (\chi_l - \chi_{l+1}) \), we derive from (iv) that \( |\nabla((\chi_l - \chi_{l+1}) \phi \Gamma)| \leq C r_{l+1}^{-4} \). Applying (ii) yields that \( |\nabla(\chi_k \phi \Gamma)| \leq C r_k^{-4} \). Therefore, we write
\[
\begin{align*}
\iint_{Q_{k_0}} v \cdot \nabla(\phi \Gamma) & = \sum_{l=k_0}^{k-1} \iint_{Q_l} v \cdot \nabla((\chi_l - \chi_{l+1}) \phi \Gamma) \Pi_1 + \iint_{Q_k} v \cdot \nabla((\chi_k \phi \Gamma) \Pi_1 \\
& = \sum_{l=k_0}^{k-1} \iint_{Q_l} v \cdot \nabla((\chi_l - \chi_{l+1}) \phi \Gamma)(\Pi_1 - \Pi_{l+1}) + \iint_{Q_k} u \cdot \nabla((\chi_k \phi \Gamma)(\Pi_1 - \Pi_{k+1}) \\
& \leq C \sum_{l=k_0}^{k-1} r_{l+1}^{-4} \iint_{Q_l} |v| |\Pi_1 - \Pi_{l+1}| + r_k^{-4} \iint_{Q_k} |v| |\Pi_1 - \Pi_{k+1}| \\
& = : I + II.
\end{align*}
\tag{4.8}
\]

Combining the Hölder inequality, (2.7) and (2.4) yield
\[
\begin{align*}
I \leq C \sum_{l=k_0}^{k-1} r_{l+1}^{-4} \left( \iint_{Q_l} |v|^{\frac{10}{3}} \right)^{\frac{3}{10}} \left( \iint_{Q_l} |\Pi_1 - \Pi_{l+1}|^2 \right)^{\frac{1}{2}} r_l \\
\leq C \sum_{l=k_0}^{k-1} r_{l+1}^{-4+\frac{2}{3}} \left( \iint_{Q_l} |v|^{\frac{10}{3}} \right)^{\frac{3}{10}} r_l^{3+\frac{4}{3}} \left( \iint_{Q_{k_0}} |\Pi_1 - \Pi_{k+1}|^2 \right)^{\frac{1}{2}} r_l \\
\leq C \sum_{l=k_0}^{k-1} r_{l+1} \left( \iint_{Q_l} |v|^{\frac{10}{3}} \right)^{\frac{3}{10}} \left( \iint_{Q_{k_0}} |\Pi_1|^2 \right)^{\frac{1}{2}} \\
\leq C \sum_{l=k_0}^{k-1} r_{l+1} \left( \iint_{Q_l} |v|^{\frac{10}{3}} \right)^{\frac{3}{10}} \left( \iint_{Q_{k_0}} |\nabla u|^2 \right)^{\frac{1}{2}}.
\end{align*}
\tag{4.9}
\]

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and
\[
II \leq C r_k \left( \iint_{Q_l} |v|^\frac{10}{7} \right)^{\frac{3}{10}} \left( \iint_{Q_{k_0}} |\nabla u|^2 \right)^{\frac{1}{2}}. \tag{4.10}
\]

It follows from (4.9) and (4.10) that
\[
\iint_{Q_{k_0}} v \cdot \nabla (\phi \Gamma) \Pi_1 \leq C \sum_{l=k_0}^{k} r_l \left( \iint_{Q_l} |v|^\frac{10}{7} \right)^{\frac{3}{10}} \left( \iint_{Q_{k_0}} |\nabla u|^2 \right)^{\frac{1}{2}}.
\]
To bound the term involving \( \Pi_2 \), noting that
\[
\Delta \Pi_2 = -\text{div div} (u \otimes u) = -\text{div div} (v \otimes v - v \otimes \nabla \Pi_h - \nabla \Pi_h \otimes v + \nabla \Pi_h \otimes \nabla \Pi_h),
\]
we decompose \( \Pi_2 \) into two parts
\[
\Pi_2 = \Pi_{21} + \Pi_{22}
\]
with
\[
\Delta \Pi_{21} = -\text{div div} (v \otimes v - v \otimes \nabla \Pi_h - \nabla \Pi_h \otimes v), \quad \tag{4.11}
\]
\[
\Delta \Pi_{22} = -\text{div div} (\nabla \Pi_h \otimes \nabla \Pi_h). \quad \tag{4.12}
\]

For \( r_k \leq r \leq r_{k_0} \), we compute directly that
\[
\iint_{Q(r)} |v \otimes v - v \otimes v_r|^{10/7} \leq \iint_{Q(r)} |v|^{20/7} \leq C r^5 \left( \iint_{Q(r)} |v|^\frac{10}{7} \right)^{\frac{6}{7}} \leq C r^5 N^{6/7}. \tag{4.13}
\]
Making use of Hölder’s inequality and (2.6), we see that
\[
\|v \otimes \nabla \Pi_h - v \otimes \nabla \Pi_{hrr}\|_{L^{10/7}(Q(r))} \leq C \|v\|_{L^\frac{10}{7}(Q(r))} \|\nabla \Pi_h\|_{L^{5+25\delta}(Q(r))} r^{\frac{5}{2} - \frac{45}{7(5+25\delta)}}
\]
\[
\leq C r^{\frac{3}{2} - \frac{45}{7(5+25\delta)}} \left( \iint_{Q(r)} |v|^\frac{10}{7} \right)^{\frac{3}{10}} \|\nabla \Pi_h\|_{L^{5+25\delta}(Q(r))}
\]
\[
\leq C r^{\frac{27+14\delta}{10(5+25\delta)}} N^\frac{3}{10} \|u\|_{L^{5+25\delta}(Q(1))}. \tag{4.14}
\]

Since (4.13) and (4.14) are valid, one can invoke Lemma 2.3 to obtain
\[
\|\Pi_{21} - \Pi_{21r}\|_{L^\frac{10}{7}(Q(r))} \leq C r^{7/2} \left( \iint_{Q(r)} |v|^\frac{10}{7} \right)^{3/5} + C r^{\frac{27+14\delta}{10(5+25\delta)}} N^\frac{3}{10} \|u\|_{L^{5+25\delta}(Q(1))}
\]
\[
\leq C r^{\frac{27+14\delta}{10(5+25\delta)}} \left\{ \left( \iint_{Q(r)} |v|^\frac{10}{7} \right)^{3/5} + N \|u\|_{L^{5+25\delta}(Q(1))} \right\}. \tag{4.15}
\]
For \( k \leq l \leq k_0 \), concatenating the Hölder inequality and (4.15) yield
\[
r_l^{-4} \iint_{Q_l} |v| |\Pi_{21} - \Pi_{21B(r)}| \leq C r_l^{-4} \left( \iint_{Q_l} |v|^\frac{10}{7} \right)^{\frac{3}{10}} \left( \iint_{Q_l} |\Pi_2 - (\Pi_2)_B|\right)^{\frac{7}{10}}
\]
\[
\leq C r_l^{-4} \left( \iint_{Q_l} |v|^\frac{10}{7} \right)^{\frac{3}{10}} \left( \iint_{Q_l} |\Pi_2 - (\Pi_2)_B|\right)^{\frac{7}{10}}.
\]
\[
\leq Cr^{-4} \left( \int_{Q_1} |v|^{10} \right)^{1/10} \left\{ N^{3/5} + N^3 \|u\|_{L^{5+2/5}(Q(1))} \right\}
\]

\[
\leq Cr^{1+2} \left( \int_{Q_1} |v|^{10} \right)^{1/10} \left\{ N^{3/5} + N^3 \|u\|_{L^{5+2/5}(Q(1))} \right\}.
\] (4.16)

According to the Poincaré inequality for a ball, Hölder’s inequality, (2.6) and (2.3), we get

\[
\|\nabla \Pi_h \otimes \nabla \Pi_h - \nabla \Pi_h \otimes \nabla \Pi_{h,r}\|_{L^{6(5+24)}(B(\bar{r}))} \\
\leq Cr\|\nabla \Pi_h \|_{L^{19+1445}(B(\bar{r}))} \|\nabla^2 \Pi_h\|_{L^{19+1445}(B(\bar{r}))} \\
\leq Cr^{-1+6} \|\nabla \Pi_h\|_{L^{5+24}(Q(1))} \|\nabla \Pi_h\|_{L^{5+24}(Q(1))} \\
\leq Cr \|u\|_{L^{5+24}(B(1))}^2 \\
\leq Cr \|u\|_{L^{5+24}(B(1))}^2.
\] (4.17)

which implies that

\[
\|\nabla \Pi_h \otimes \nabla \Pi_h - \nabla \Pi_h \otimes \nabla \Pi_{h,r}\|_{L^{5+24}(Q(1))} \leq Cr^{27+1445} \|u\|_{L^{5+24}(Q(1))}^2.
\] (4.18)

In view of Lemma 2.3 and (4.18), we have

\[
\|\Pi_{22} - \bar{\Pi}_{22}\|_{L^{5+24}(Q(1))} \leq Cr^{27+1445} \|u\|_{L^{5+24}(B(1))}^2.
\] (4.19)

By combining Hölder’s inequality and (4.19), we deduce that

\[
r_{1}^{-4} \int_{Q_1} |v| \|\Pi_{22} - \bar{\Pi}_{22}\| \\
\leq r_{1}^{-4} \|v\|_{L^{5+24}(Q(1))} \|\Pi_{22} - \bar{\Pi}_{22}\|_{L^{5+24}(Q(1))} \\
\leq Cr^{-1+2+28} r_{1}^{-2} \|v\|_{L^{5+24}(Q(1))} \|u\|_{L^{5+24}(Q(1))}^2.
\] (4.20)

According to (4.20) and (4.16), we know that

\[
\int_{Q_{k_0}} v \cdot \nabla (\phi \Gamma) \Pi_2 \\
\leq C \sum_{l=k_0}^{k-1} r_{l+1}^{-4} \int_{Q_l} |v| \|\Pi_{21} - \bar{\Pi}_{21}\| + r_{k}^{-4} \int_{Q_k} |v| \|\Pi_{21} - \bar{\Pi}_{21}\| \\
+ C \sum_{l=k_0}^{k-1} r_{l+1}^{-4} \int_{Q_l} |v| \|\Pi_{22} - \bar{\Pi}_{22}\| + r_{k}^{-4} \int_{Q_k} |v| \|\Pi_{22} - \bar{\Pi}_{22}\| \\
\leq C \sum_{l=k_0}^{k} \|v\|_{L^{5+24}(Q(1))} \left\{ \left( \int_{Q(l)} |v|^{10} \right)^{3/5} + \left( \int_{Q(l)} |v|^{10} \right)^{3/5} \|u\|_{L^{5+24}(Q(1))} \right\}.
\]
Finally, gathering the above inequalities yields the desired inequality. \(\square\)

With Proposition 4.1 at our disposal, we can now give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** By Hölder’s inequality, it suffices to consider the case with \(\delta\) sufficient small. By the interior estimate (2.6) of harmonic function and (3.1), we have

\[
\|\nabla \Pi_h\|_{L^\infty(\mathring{B}(1/8))} \leq C\|\nabla \Pi_h\|_{L^{\frac{5+2\delta}{4}}(B(1))} \leq C\|u\|_{L^{\frac{5+2\delta}{4}}(B(1))}. \tag{4.22}
\]

We temporarily assume that, for any Lebesgue point \((x_0, t_0)\) in \(Q(1/8)\),

\[
|v(x_0, t_0)| \leq C\varepsilon^{\frac{2(1+25)}{3(5+2\delta)}}. \tag{4.23}
\]

It follows from (4.22) and (4.23) that

\[
\|u\|_{L^{\frac{5+2\delta}{4}}(\mathring{Q}(1/8))} \leq \|\nabla \Pi_h\|_{L^{\frac{5+2\delta}{4}}(\mathring{Q}(1/8))} + \|v\|_{L^{\frac{5+2\delta}{4}}(\mathring{Q}(1/8))} \leq C\|u\|_{L^{\frac{5+2\delta}{4}}(Q(1))} + C\varepsilon^{\frac{2(1+25)}{3(5+2\delta)}}.
\]

The well-known Serrin regularity criterion implies that that \((0, 0)\) is a regular point. Thus, it is enough to show (4.23) to complete the proof of Theorem 1.1. In what follows, let \((x_0, t_0)\) in \(Q(1/8)\) and \(r_k = 2^{-k}\). According to the Lebesgue differentiation theorem, it suffices to show

\[
\int_{Q_k} \int_{Q_k} |v|^{\frac{10}{3}} + r_k^{\frac{3(5+2\delta)}{2(1+25)}} |v|^{\frac{5+2\delta}{5+2\delta}} \leq \varepsilon^{2/3}, \quad k \geq 3. \tag{4.24}
\]

First, we prove that (4.24) holds true for \(k = 3\). Indeed, from (3.23) with \(\alpha = \frac{5+2\delta}{9}\) in Section 3, Proposition 1.2 and hypothesis (1.8), we get

\[
\sup_{-\frac{1}{2} \leq t \leq 0} \int_{B(\frac{3}{8})} |v|^2 dx + \int_{Q(\frac{4}{3})} |\nabla v|^2 dx d\tau \\
\leq C\|u\|^2_{L^{\frac{5+2\delta}{2}}(Q(1/2))} + C\|u\|^\frac{5+2\delta}{4}_{L^{\frac{5+2\delta}{2}}(Q(1/2))} \tag{4.25}
\]

\[
+ C\|u\|^4_{L^{\frac{5+2\delta}{2}}(Q(1/2))} + \frac{1}{16}\|\nabla u\|^2_{L^2(Q(1/2))} \\
\leq C\|u\|^2_{L^{\frac{5+2\delta}{2}}(Q(1))} + C\|u\|^\frac{5+2\delta}{3}_{L^{\frac{5+2\delta}{2}}(Q(1))} + C\|u\|^4_{L^{\frac{5+2\delta}{2}}(Q(1))} \tag{4.26}
\]

and

\[
\int_{Q(\frac{4}{3})} |v|^{\frac{10}{3}} \leq C\left( \sup_{-\frac{1}{2} \leq t \leq 0} \int_{B(\frac{3}{8})} |v|^2 dx \right)^{1/2} + C\left( \int_{Q(\frac{4}{3})} |\nabla v|^2 \right)^{1/2},
\]

and

\[
\|v\|^\frac{5+2\delta}{4}_{L^{\frac{5+2\delta}{2}}} \leq C\left( \frac{\sup_{-\frac{1}{2} \leq t \leq 0} \int_{B(\frac{3}{8})} |v|^2 dx}{\int_{Q(\frac{4}{3})} |\nabla v|^2} \right)^{1/2} + C\left( \int_{Q(\frac{4}{3})} |\nabla v|^2 \right)^{1/2}.
\]
These inequalities together with \( (4.25) \) means
\[
\iint_{\tilde{Q}_3} |v|^{10/3} + (\frac{1}{8})^{\frac{3(5+2\alpha)}{2(1+2\alpha)} \| v \|_{L^{\frac{5+2\alpha}{1+2\alpha}}(\tilde{Q}(R))}} \leq C \varepsilon + C \varepsilon^{2/5}.
\]

The assertion \( (4.24) \) with \( k = 3 \) is valid. Next, we assume that, for any \( 3 \leq l \leq k \),
\[
\iint_{\tilde{Q}_l} |v|^{10/3} + r_l^{\frac{3(5+2\alpha)}{2(1+2\alpha)} \| v \|_{L^{\frac{5+2\alpha}{1+2\alpha}}(\tilde{Q}(r))}} \leq \varepsilon^{2/3}.
\]

As a consequence, for any \( r_k \leq r \leq r_3 \), we see that
\[
\iint_{\tilde{Q}(r)} |v|^{10/3} + r^{\frac{3(5+2\alpha)}{2(1+2\alpha)} \| v \|_{L^{\frac{5+2\alpha}{1+2\alpha}}(\tilde{Q}(r))}} \leq C \varepsilon^{2/3}.
\] (4.27)

For any \( 3 \leq i \leq k \), we apply Proposition 4.1 to \( N = C \varepsilon^{2/3} \) and \( k_0 = 3 \) to obtain
\[
\sup_{-\varepsilon^2 \leq t - t_0 \leq 0} \left( \iint_{B_t} |v|^2 + r_i^{-3} \iint_{\tilde{Q}_i} |\nabla v|^2 \right) 
\leq C \left( \sup_{-\varepsilon^2 \leq t - t_0 \leq 0} \left( \iint_{B_{k_0}} |v|^2 + C \sum_{l=3}^i r_l \left( \iint_{\tilde{Q}_l} |v|^{10/3} \right)^{\frac{9}{10}} \right) \right.
\]
\[
+ C \sum_{l=3}^i r_l^{\frac{5+2\alpha}{5+25}} \left( \iint_{\tilde{Q}_l} |v|^{10/3} \right)^{\frac{2}{5+25}} \left( \iint_{Q_1} |u|^{\frac{5+2\alpha}{2}} \right)^{\frac{5}{5+25}}
\]
\[
+ C \sum_{l=3}^i r_l^{\frac{5+2\alpha}{5+25}} r_l^{-\frac{3}{2}} \| v \|_{L^{\frac{5+2\alpha}{1+2\alpha}}(\tilde{Q}(r))} \| u \|_{L^{\frac{5+2\alpha}{2}}(Q(1))}^{\frac{3}{5+25}}
\]
\[
+ C \sum_{l=3}^i r_l \left( \iint_{\tilde{Q}_l} |v|^{10/3} \right)^{\frac{3}{10}} \left( \iint_{Q_{k_0}} |\nabla u|^2 \right)^{\frac{1}{3}}
\]
\[
+ C \sum_{l=3}^i r_l^{\frac{5+2\alpha}{5+25}} \left( \iint_{\tilde{Q}_l} |v|^{10/3} \right)^{\frac{3}{10}} \left( \iint_{Q(r)} |v|^{10/3} \right)^3 + \left( \iint_{Q(r)} |v|^{10/3} \right)^{3/10} \| u \|_{L^{\frac{5+2\alpha}{2}}(Q(1))} \right)
\]
\[
+ \left( \iint_{\tilde{Q}_l} |v|^{10/3} \right)^{\frac{3}{10}} \left( \iint_{Q(r)} |v|^{10/3} \right)^{3/10} \| u \|_{L^{\frac{5+2\alpha}{2}}(Q(1))} \right)
\]
\[
\leq C \varepsilon^{\frac{3}{5+25}} + C \sum_{l=3}^i r_l \varepsilon^{\frac{3}{5} + \frac{9}{10}} + C \sum_{l=3}^i r_l^{\frac{5+2\alpha}{5+25}} \varepsilon^{\frac{3}{5} + \frac{2}{5+25}}
\]
\[
+ C \sum_{l=3}^i r_l^{\frac{5+2\alpha}{5+25}} \varepsilon^{\frac{3}{5} + \frac{2}{5+25}} + C \sum_{l=3}^i r_l^{\frac{5+2\alpha}{5+25}} \varepsilon^{\frac{3}{5} + \frac{4}{5+25}}
\]
\[
+ C \sum_{l=3}^i r_l^{\frac{5+2\alpha}{5+25}} \varepsilon^{\frac{3}{5} + \frac{4}{5+25}} + C \sum_{l=3}^i r_l^{\frac{5+2\alpha}{5+25}} \varepsilon^{\frac{3}{5} + \frac{4}{5+25}} \left\{ \varepsilon^{\frac{3}{5} + \frac{4}{5+25}} \right\}
\]
\[
+ C \sum_{l=3}^i r_l^{\frac{5+2\alpha}{5+25}} \varepsilon^{\frac{3}{5} + \frac{4}{5+25}}
\]

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where the hypothesis (1.8) and (4.27) are used.

We derive from (2.8) that
\[
\iint_{Q_{k+1}} |v|^{10} \leq C \left( \sup_{-r_k^2 \leq t < 0} \int_{B_k} |v|^2 \right)^{\frac{2}{5}} \left( \iint_{Q_k} |\nabla v|^2 \right)^{\frac{1}{5}} + C \left( \sup_{-r_k^2 \leq t < 0} \int_{B_k} |v|^2 \right)^{\frac{2}{5}},
\]
and
\[
r_{k+1}^{-\frac{3}{2}} \|v\|_{L^{1+25} \cap L^{6(5+25)}(Q_{k+1})} \leq C \left( \frac{1}{r_k} \sup_{-r_k^2 \leq t < 0} \int_{B_k} |v|^2 \right)^{\frac{3}{4}} + C \left( r_k^{-3} \iint_{Q_k} |\nabla v|^2 \right)^{\frac{1}{4}}.
\]
Hence, there holds
\[
\frac{1}{r_k^{3+1}} \iint_{Q_{k+1}} |v|^{10} \leq C \left( \frac{1}{r_k^3} \sup_{-r_k^2 \leq t < 0} \int_{B_k} |v|^2 \right)^{\frac{3}{4}} + C \left( \frac{1}{r_k^3} \sup_{-r_k^2 \leq t < 0} \int_{B_k} |v|^2 \right)^{\frac{3}{4}} \left( r_k^{-3} \iint_{Q_k} |\nabla v|^2 \right)^{\frac{1}{4}},
\]
and
\[
r_{k+1}^{-\frac{3(5+25)}{1 + 25}} \|v\|_{L^{1+25} \cap L^{6(5+25)}(Q_{k+1})} \leq C \left( \frac{1}{r_k} \sup_{-r_k^2 \leq t < 0} \int_{B_k} |v|^2 \right)^{\frac{3+25}{1+25}} + C \left( r_k^{-3} \iint_{Q_k} |\nabla v|^2 \right)^{\frac{1}{2}} \frac{5+25}{1+25},
\]
\[
\leq C \varepsilon^{\frac{3}{2}}.
\]
Putting together (4.29) and (4.30), we have
\[
\iint_{Q_{k+1}} |v|^{10} + r_{k+1}^{-\frac{3(5+25)}{1+25}} \|v\|_{L^{1+25} \cap L^{6(5+25)}(Q_{k+1})} \leq \varepsilon^{2/3}.
\]
This completes the proof of this theorem. \qed

5 Improvement on Caffarelli–Kohn–Nirenberg theorem by a logarithmic factor

The goal of this section is to prove Theorem 1.3. According to Proposition 2.4 in Section 2, the key point for the extension of \( \sigma \) in Theorem 1.3 is to develop [24, Lemma 4.2, p.818].
Therefore, it suffices to present an improvement of this lemma. To make the paper more readable, we will apply (1.8) for $\delta = 0$ to show Theorem 1.3.

First, we give the definition of generalized parabolic Hausdorff measure as follows.

**Definition 6.1.** Let $h$ be an increasing continuous function on $(0, 1]$ with $\lim_{r \to 0} h(r) = 0$ and $h(1) = 1$. For fixed parameter $\delta > 0$ and set $E \subset \mathbb{R}^3 \times \mathbb{R}$, we denote by $D(\delta)$ the family of all coverings $\{Q(x_i, t_i; r_i)\}$ of $E$ with $0 < r_i \leq \delta$. We denote

$$\Psi_\delta(E, h) = \inf_{D(\delta)} \sum_i h(r_i)$$

and define the generalized parabolic Hausdorff measure as

$$\Lambda(E, h) = \lim_{\delta \to 0} \Psi_\delta(E, h).$$

In what follows, we set $m(r) = (\Gamma(r))^{\sigma} = (\log(e/r))^{\sigma}$, where $\sigma \in (0, 1)$ will be determined later.

Before going further, we set

$$F(m) = \{(x, t) \mid \limsup_{r \to 0} E^*_E(r) m(r) \leq 1\}.$$ 

In addition, we need the following fact due to Choe and Lewis [4]

**Lemma 6.1.** [4] Assume that $(x, t) \in F(m) \cap S$. Then, there exists a positive constant $c_1$ independent of $(x, t)$ such that

$$\limsup_{r \to 0} \frac{E(r)}{m^2(r)} \leq c_1. \quad (6.1)$$

As said above, it suffices to prove the following lemma.

**Lemma 6.2.** Let $(x, t) \in F(m) \cap S$. Then, there exists a positive constant $c_2$ independent of $(x, t)$ such that

$$\liminf_{r \to 0} J_q(r) m(r)^{\tau} \geq c_2,$$

where $\tau = \frac{35-14q}{4}$.

**Proof.** Assume that the statement fails, then, for any $\eta > 0$, there exists a singular point $(x, t)$ and a sequence $r_n \to 0$ such that

$$J_q(r_n) m(r_n)^{\tau} < \eta. \quad (6.2)$$

It follows from Lemma 2.5 and (6.1) and that, for $\theta_n < 1/8$,

$$E_{5/2}(\theta_n r_n) \leq C \theta_n^{5/2} m^{5/2}(r_n) + C \theta_n^{\frac{25-10q}{4}} m(r_n)^{\frac{5-2q}{2}} J_q(r_n)$$

$$\leq C \left[ m(r_n)^{\tau} J_q(r_n) \right]^{\frac{2}{1-\tau}}$$

$$\leq C \eta^{\frac{2}{1-\tau}},$$
where $\theta_n = [m(r_n)^{-q}J_q(r_n)]^{\frac{4}{35} - \frac{10}{27}}$. Note that $\theta_n$ goes to 0 as $n \to \infty$ by (6.2). Let $\rho_n = \theta_n r_n$ and $\varepsilon_2 = C\eta^{-2}\varepsilon / 2$ such that $\varepsilon_2 < \min\{1, \varepsilon / 2\}$. For sufficiently large $n$, we see that

$$E_{5/2}(\theta_n r_n) \leq \varepsilon_2.$$ 

This together with (1.8) implies that $(x, t)$ is a regular point. Thus, we reach a contradiction and finish the proof. 

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