POLARIZATION TYPES OF ISOGENOUS PRYM-TYURIN VARIETIES

V. KANEV AND H. LANGE

Abstract. Let \( p : C \overset{\pi}{\longrightarrow} C' \overset{g}{\longrightarrow} Y \) be a covering of smooth, projective curves with \( \deg(\pi) = 2 \) and \( \deg(g) = n \). Let \( f : X \to Y \) be the covering of degree \( 2^{n} \), where the curve \( X \) parametrizes the liftings in \( C(\mathbb{Z}) \) of the fibers of \( g : C' \to Y \). Let \( P(X,\delta) \) be the associated Prym-Tyurin variety, known to be isogenous to the Prym variety \( P(C,C') \). Most of the results in the paper focus on calculating the polarization type of the restriction of the canonical polarization \( \Theta_{JX} \) on \( P(X,\delta) \). We obtain the polarization type when \( n = 3 \).

When \( Y \sim \mathbb{P}^{1} \) we conjecture that \( P(X,\delta) \) is isomorphic to the dual of the Prym variety \( P(C,C') \). This was known when \( n = 2 \), we prove it when \( n = 3 \), and for arbitrary \( n \) if \( \pi : C \to C' \) is étale. Similar results are obtained for some other types of coverings.

Introduction

Let \( W \) be a finite group which acts on a lattice \( L \). Assume the representation of \( W \) on \( L_{\mathbb{C}} = L \otimes_{\mathbb{Z}} \mathbb{C} \) is irreducible. Let \( \lambda \in L_{\mathbb{Q}} \) be a nonzero weight, i.e. \( w\lambda - \lambda \in L \) for every \( w \in W \). Suppose \( f : X \to Y \) is a covering of smooth, projective curves, with discriminant locus \( \mathcal{D} \), whose monodromy map can be decomposed as

\[
\pi_{1}(Y \setminus \mathcal{D}, y_{0}) \overset{m}{\longrightarrow} W \longrightarrow S(W\lambda)
\]

As shown in [K2] one can construct a correspondence in \( \text{Div}(X \times X) \) which induces on \( A = \text{Ker}(Nm_{f} : JX \to JY)^{0} \) an endomorphism \( \delta \) satisfying a quadratic equation \((\delta - 1)(\delta + q - 1) = 0\). The construction in [K2] is done in the case \( Y = \mathbb{P}^{1} \) and its extention to the general case \( g(Y) \geq 0 \) is immediate. We outline the necessary changes in Section 1. We let

\[
P(X,\delta) = (1 - \delta)A
\]

and call this Abelian variety the Przym-Tyurin variety associated with the covering \( f : X \to Y \).

Choosing another weight \( \lambda' \), one has another permutation representation \( W \to S(W\lambda') \). Composing with \( m : \pi_{1}(Y \setminus \mathcal{D}, y_{0}) \to W \) and applying Riemann’s Existence Theorem one obtains another covering \( p : C \to Y \), a correspondence in \( \text{Div}(C \times C) \), an induced endomorphism \( i \) and a Prym-Tyurin variety \( P(C,i) \). It is

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known that \( P(X, \delta) \) and \( P(C, i) \) are isogenous. This can be proved by the same arguments as in the case \( Y = \mathbb{P}^1 \) (see [K2], Section 6), we give an outline in Section II or using generalized Prym varieties associated to Galois coverings (see [D2]).

Let \( \Theta_{JX} \) and \( \Theta_{JC} \) be the restrictions of the canonical polarizations of \( JX \) and \( JC \) to \( P(X, \delta) \) and \( P(C, i) \) respectively. A natural question is: what is the connection between the polarization types of \( \Theta_{JX} \) and \( \Theta_{JC} \). Are there simple formulas by which, if one knows the polarization type associated with a given weight, one can calculate the polarization type associated with every other weight?

In all cases studied in the paper \( W \) is a Weyl group of an irreducible root system \( R \). Here the weights are the usual weights associated with irreducible representations of the corresponding complex simple Lie algebras.

The relation between \( P(C, i) \) and \( P(X, \delta) \) has been so far studied mainly when the restricted polarizations \( \Theta_{JX} \) and \( \Theta_{JC} \) are multiples of principal polarizations. In this case usually \( P(C, i) \) and \( P(X, \delta) \) are isomorphic as principally polarized Abelian varieties. For instance Recillas’ construction [K2] corresponds to \( Y = \mathbb{P}^1 \), \( W = W(R) \) where \( R \) is a root system of type \( A_3 \), \( \lambda' = \omega_1, \lambda = \omega_2 \). Here \( \omega_1, \omega_2 \) are the fundamental weights of the root system \( R \) according to the enumeration in [B]. Similarly Donagi’s tetragonal construction corresponds to \( Y = \mathbb{P}^1 \), \( W = W(D_4) \), \( \lambda' = \omega_1, \lambda = \omega_3 \text{ or } \lambda = \omega_4 \). These two constructions were generalized in [K2], when \( Y = \mathbb{P}^1 \), to \( W = W(A_n) \), \( \lambda' = \omega_1, \lambda = \omega_k, 2 \leq k \leq n - 1 \) and \( W = W(D_n), \lambda' = \omega_1, \lambda = \omega_{n - 1} \text{ or } \lambda = \omega_n \). In the first case one obtains Prym-Tyurin varieties isomorphic to Jacobians, in the second one Prym-Tyurin varieties isomorphic to Prym varieties.

The paper [La] deals with non-principally polarized Prym varieties. Starting from a double ramified covering \( \pi : C \to C' \) of a hyperelliptic curve \( C' \), Pantazis constructs another double ramified covering \( \pi' : X \to X' \) of a hyperelliptic curve \( X' \) and proves that the Prym variety \( P(X, X') \) is isomorphic to the dual \( P(C, C') \).

In our set-up this corresponds to \( Y = \mathbb{P}^1 \), \( W = W(B_2) \), \( \lambda' = \omega_1, \lambda = \omega_2 \).

Most of our results focus on the case \( W = W(B_n) \), \( \lambda' = \omega_1, \lambda = \omega_n \). The fundamental weight \( \omega_n \) is the dominant weight of the spinor representation of the Lie algebra so(2n + 1), so we call it the spinor weight. Here the relation between the coverings \( p : C \to Y \) and \( f : X \to Y \) may be described geometrically by the \( n \)-gonal construction [D1]. Start with a covering \( p : C \to C' \) and \( g : Y \to Y' \), where \( \deg(p) = 2 \), \( \deg(g) = n \). Consider the embedding \( g^* : Y \to C'^{(n)} \). Let \( Z \) be the curve defined by the cartesian diagram

\[
\begin{array}{ccc}
Z & \longrightarrow & C^{(n)} \\
\downarrow & & \downarrow \pi^{(n)} \\
Y & \longrightarrow & C'^{(n)} \\
\end{array}
\]

The points of \( Z \) parametrize the liftings of points of \( g^*(Y) \) in \( C^{(n)} \). Then \( X \) is the desingularization of \( Z \) and \( f : X \to Y \) is the associated covering of degree \( 2^n \). The two isogenous Prym-Tyurin varieties here are the Prym variety \( P(C, C') \) and \( P(X, \delta) \), where \( \delta : JX \to JX \) satisfies the equation \( (\delta|_A - 1)(\delta|_A + 2^{n-1} - 1) = 0 \) when restricted on \( A = \text{Ker}(Nm_f : JX \to JY)^0 \). We prove the following theorem in Section 4.
Theorem 1. Let $p : C \to C'$ be a covering with $\deg(\pi) = 2, \deg(g) = n$. Let $P = P(X, \delta), P' = P(C, C')$ and let $E_P, E_{P'}$ be the Riemann forms of the polarizations $\Theta_{JC}|_P, \Theta_{JC}|_{P'}$ respectively. Let $(d_1, \ldots, d_p)$ be the polarization type of $\Theta_{JC}|_{P'}$. There exists a canonical isogeny $\mu : \hat{P}' \to P$ such that

$$\mu^* E_P = \frac{2^{n-1} - 1}{d_1 d_p} \hat{E}_{P'}$$

where $\hat{P}', \hat{E}_{P'}$ are respectively the dual of $P(C, C')$ and the Riemann form of the dual polarization.

Every covering $p : C \to C'$ is ramified of degrees 2 and 3 respectively, be a simply ramified covering of type $B_n$. Let $f : X \to Y$ be the covering of degree 8 obtained by the 3-gonal construction and let $P(X, \delta)$ be the Prym-Tyurin variety. Then the polarization types of $\Theta_{JC}|_{P(C, C')}$ and $\Theta_{JX}|_{P(X, \delta)}$ are respectively

$$\left( \frac{1}{\mathcal{D}_{s, \perp 1}}, \frac{2, \ldots, 2}{\mathcal{D}_{t, \perp 2}}, \frac{2, \ldots, 2}{\mathcal{D}_{t, \perp 2}} \right)$$

$$\left( \frac{2, \ldots, 2}{\mathcal{D}_{t, \perp 2}}, \frac{4, \ldots, 4, 8, \ldots, 8}{\mathcal{D}_{t, \perp 2}} \right)$$

If $Y \cong \mathbb{P}^1$, then $P(X, \delta)$ is isomorphic to the dual of $P(C, C')$.

Assuming $C' \cong \mathbb{P}^1$ in Theorem 2 we obtain a new family of principally polarized Prym-Tyurin varieties. They have exponent 4 and are isomorphic to hyperelliptic Jacobians (Corollary 5.9).

The contents of the paper are as follows. Section 1 contains preliminary material. In Section 2 we study coverings $p : C \to C'$ with $\deg(\pi) = 2, \deg(g) = n$ with the emphasis on their connection with root systems of type $B_n$. We calculate the possible monodromy groups of the simply ramified coverings of type $B_n$ under some restrictions. Applying the general constructions of Section 1 we associate to a given covering $p : C \to C'$ of type $B_n$ and the spinor weight a
covering \( f : X \to Y \) of degree \( 2^n \), define and study canonical correspondences in \( \text{Div}(X \times C) \) and identify \( X \) with the \( n \)-gonal construction [D1]. Section 3 contains some results about \( P(X, \delta) \) valid for every \( n \). In Section 4 we recall the definition of dual Abelian variety and dual polarization, prove a criterion which relates the Riemann forms of isogenous Prym-Tyurin varieties, give a proof of Theorem 1 and verify the isomorphism \( \mu : \hat{P}(C, C') \sim P(X, \delta) \) when \( \pi : C \to C' \) is étale. In Section 5 we study coverings associated with Weyl groups of root systems of rank \( \leq 3 \). We give a new and simpler proof of Pantazis’ result \((W = W(B_2))\). We briefly discuss the simple case \( W = W(A_3) \), the Recillas’ construction. We give a proof of Theorem 2, and finally make explicit calculation of the Prym-Tyurin varieties and the polarization types when the monodromy group is contained in \( W(D_3) \). In Section 6 we study the cases \( W = W(B_4) \) and \( W = W(D_4) \). We cannot say much about the polarization types, however we include some relation between the Abelian varieties involved.

**Notation.** Let \( X = V/\Lambda \) and \( X' = V'/\Lambda' \) be complex tori. If \( f : X \to X' \) is a homomorphism, we denote by \( \tilde{f} : V \to V' \) the unique \( \mathbb{C} \)-linear map with \( \tilde{f}(\Lambda) \subset \Lambda' \) inducing \( f \). Identifying \( V \) and \( V' \) with the tangent spaces \( T_0X \) and \( T_0X' \) we have that \( \tilde{f} \) equals the differential \( df \) at \( 0 \). Furthermore, identifying \( \Lambda \) and \( \Lambda' \) with the homology groups \( H_1(X, \mathbb{Z}) \) and \( H_1(X', \mathbb{Z}) \) respectively, one has that \( \tilde{f}|_{\Lambda} \) equals \( f_* \), the induced homomorphism on homology. Similarly \( \tilde{f} : V \to V' \) and \( f_* : H_1(X, \mathbb{R}) \to H_1(X', \mathbb{R}) \) are the same maps if we make the identification of \( V \) and \( V' \) with \( H_1(X, \mathbb{R}) \) and \( H_1(X', \mathbb{R}) \) respectively as real vector spaces.

1. **Prym-Tyurin varieties**

1.1. Let \( \mathbb{R}^d = \oplus_{i=1}^d \mathbb{R}e_i, \mathbb{R}^e = \oplus_{j=1}^e \mathbb{R}f_j \). Let \( W \) be a finite group and let \( W \to \text{GL}(\mathbb{R}^d) \) and \( W \to \text{GL}(\mathbb{R}^e) \) be linear permutation representations, such that \( W \) acts transitively on both \( \{e_1, \ldots, e_d\} \) and \( \{f_1, \ldots, f_e\} \). Let \( A = (a_{ij}) \) be a \( d \times e \) matrix with integer entries and let \( S : \mathbb{R}^d \to \mathbb{R}^e \) be the linear map given by

\[
S(e_i) = \sum_{j=1}^e a_{ij} f_j.
\]

Let \( T : \mathbb{R}^d \to \mathbb{R}^e, T_1 : \mathbb{R}^d \to \mathbb{R}^d \), and \( T_2 : \mathbb{R}^e \to \mathbb{R}^e \) be linear maps which in the bases \( \{e_i\}, \{f_j\} \) have matrices with all entries equal to 1. We denote by \( tS : \mathbb{R}^e \to \mathbb{R}^d \) and \( tT : \mathbb{R}^e \to \mathbb{R}^d \) the linear maps with transposed matrices.

**Lemma 1.2.** Suppose \( S \) is \( W \)-equivariant. Then there exist integers \( a, b \in \mathbb{Z} \) such that

\[
\begin{align*}
(i) \quad tT \cdot S &= tS \cdot T = aT_1, \\
(ii) \quad T \cdot tS &= S \cdot tT = bT_2.
\end{align*}
\]

**Proof.**

\[
S(\sum_{i=1}^d e_i) = \sum_{j=1}^e \left( \sum_{i=1}^d a_{ij} \right) f_j := \sum_{j=1}^e b_j f_j
\]

Let \( w \in W \). One has

\[
S(\sum_{i=1}^d e_i) = S(\sum_{i=1}^d w(e_i)) = wS(\sum_{i=1}^d e_i) = \sum_{j=1}^e b_j (w(f_j))
\]
Since $W$ acts transitively on $\{f_j\}$ we conclude that $b_1 = \cdots = b_e = b$. So, $S \cdot i^*T = bT_2$. Transposing we obtain $T \cdot i^*S = bT_2$. This proves (ii). The proof of (i) is similar considering $i^*S(f_j) = \sum_{i=1}^d a_{ij}e_i$. One has $a = \sum_{j=1}^e a_{ij} f_i$ for every $i$.

1.3. Let $W, \{e_i\}, \{f_j\}$ and $S$ be as in \[1.1\]. Suppose $C, X$ and $Y$ are smooth, projective curves, $Y$ is irreducible (but $C$ and $X$ might be reducible). Let $f : X \to Y$ and $p : C \to Y$ be coverings of degrees $d$ and $e$ respectively which are not branched in $Y \setminus D$. Let $y_0 \in Y \setminus D$. Suppose the monodromy maps of $f$ and $p$ can be decomposed as

$$\pi_1(Y \setminus D, y_0) \xrightarrow{m} W \to S_d \quad \text{and} \quad \pi_1(Y \setminus D, y_0) \xrightarrow{n} W \to S_e,$$

where $W$ acts transitively on $\{e_1, \ldots, e_d\}$ and $\{f_1, \ldots, f_e\}$ as in \[1.1\].

A $W$-equivariant linear map $S$ as in \[1.1\] induces a correspondence, which abusing notation, we denote again by $S$. It is defined as follows. Fix bijections $f^{-1}(y_0) \to \{e_1, \ldots, e_d\}$ and $p^{-1}(y_0) \to \{f_1, \ldots, f_e\}$. If $y \in Y \setminus D$, choose a path $\gamma$ in $Y \setminus D$ which connects $y$ with $y_0$. Enumerate the points of the fibers over $y$ using covering homotopy along $\gamma$, so $f^{-1}(y) = \{x_1, \ldots, x_d\}$ and $p^{-1}(y) = \{z_1, \ldots, z_e\}$. Then

$$S(x_i) = \sum_{j=1}^e a_{ij}z_j, \quad i^*S(z_j) = \sum_{i=1}^d a_{ij}x_i.$$

Let $T \in \text{Div}(X \times C)$, $T_1 \in \text{Div}(X \times X)$ and $T_2 \in \text{Div}(C \times C)$ be the trace correspondences

$$T(x) = p^*(f(x)), \quad T_1(x) = f^*(f(x)), \quad T_2 = p^*(p(x)).$$

Lemma \[1.2\] yields

$$(1.1) \quad \begin{align*}
{i^*T} \circ S &= i^*S \circ T = (\deg S)T_1 \\
T \circ i^*S &= S \circ i^*T = (\deg i^*S)T_2
\end{align*}$$

Lemma 1.4. Let

$$A = \text{Ker}(Nm_f : JX \to JY)^0, \quad B = \text{Ker}(Nm_p : JC \to JY)^0.$$

Then the endomorphism $s : JX \to JC$ induced by the correspondence $S$ transforms $A$ into $B$ and $f^*(JY)$ into $p^*(JY)$.

Proof. Let $y \in Y$ and $z \in p^{-1}(y)$. Using \[1.1\] one has $S(f^*(y)) = S(i^*T(z)) = (\deg i^*S)T_2(z) = (\deg i^*S) p^*(y)$. So $s(f^*(JY)) \subset p^*(JY)$. If $x \in X$ and $y = f(x)$, then $f^*Nm_p(S(x)) = i^*T(S(x)) = (\deg S) f^*(f(x))$, so for every $u \in JX$ one has $f^*Nm_p(s(u)) = (\deg S) f^*Nm_f(u)$. This implies $s(A) \subset B$ since $f^* : JY \to JX$ has finite kernel and $A = \text{Ker}(Nm_f)^0$ is connected.

1.5. Let $W, L, \lambda \in L_Q, \lambda \neq 0, f : X \to Y$ be as in the introduction. We assume $Y$ is irreducible. Let $W\lambda = \{\lambda_1, \ldots, \lambda_d\}$. Let $(\mid \mid )$ be a symmetric, $W$-invariant, negative definite bilinear form such that $(w\lambda - \lambda | \lambda) \in \mathbb{Z}$ for $\forall w \in W$. The construction in [K2], Section 4 yields a lattice $N(R, \lambda) \cong L \oplus \mathbb{Z}$, an action of $W$ on $N(R, \lambda)$, a symmetric bilinear $W$-invariant form $(\mid \mid )$ on $N(R, \lambda)$, which extends $(\mid \mid )$, an orbit $W(\ell) = \{\ell_1, \ldots, \ell_d\} \subset N(R, \lambda)$ and a $W$-equivariant bijection $\ell_i \leftrightarrow \lambda_i$. One has

$$(1.2) \quad (\ell_i, \ell_j) = (\lambda_i | \lambda_j - \lambda_i) - 1.$$
and the following properties hold: \((\ell_i, \ell_j) \in \mathbb{Z}; (\ell_i, \ell_i) = -1 \) and \((\ell_i, \ell_j) \geq 0\) for \(i \neq j\). Let \(E \in GL_d(\mathbb{R})\) be the identity. One considers the \(W\)-equivariant linear map \(G = D - E : \mathbb{R}^d \to \mathbb{R}^d\) given by

\[
G(e_i) = \sum_{j=1}^{d} (\ell_i, \ell_j)e_j
\]

One defines a correspondence \(D - \Delta \in Div(X \times X)\) as in [13.3] Here \(\Delta\) is the diagonal. Let \(A = \text{Ker}(Nm_f : JX \to JY)^0\).

**Proposition 1.6.** The endomorphism \(\delta : JX \to JX\) induced by \(D\) leaves invariant \(A\) and \(f^*(JY)\). The restriction \(\delta|_A : A \to A\) satisfies the quadratic equation

\[
(\delta|_A - 1)(\delta|_A + q - 1) = 0,
\]

where \(q\), the exponent of the correspondence, is the integer

\[
q = -d(\lambda|\lambda)/\text{rk}(L).
\]

Furthermore \(q \geq 1\) and \(q = 1\) if and only if \(D = 0\).

**Proof.** This follows from Lemma 1.4, the equality \(G(G + qE) = mT\) proved in [K2], Proposition 5.3 and the argument of [K2], §5.4. □

Lemma 1.4 and Proposition 1.6 show that the differential of \(\delta\) at 0

\[
d\delta : T_0(JX) \to T_0(JX)
\]

can be diagonalized and has eigenvalues \(\text{deg} D, 1, 1 - q\) with eigenspaces which correspond to the Abelian subvarieties \(f^*(JY), (\delta + q - 1)A, (1 - \delta)A\) respectively.

**Definition 1.7.** We let \(P(X, \delta) = (1 - \delta)A\) and call this Abelian variety the Prym-Tyurin variety associated with the covering \(f : X \to Y\).

**Remark 1.8.** We notice that the choice of the \(W\)-invariant bilinear form \((\_ \mid \_)\) is irrelevant for the definition of the Prym-Tyurin variety. Indeed, multiplying \((\_ \mid \_)\) by an integer \(k\) one changes \(G\) to \(G' = kG + (k - 1)T\), so \(1 - \delta'|_A = k(1 - \delta|_A)\) and therefore the Prym-Tyurin variety remains the same.

**1.9.** Given the set-up of [1.5] namely a covering \(f : X \to Y\) whose monodromy map can be decomposed as \(\pi_1(Y \setminus \mathcal{D}, y_0) \xrightarrow{m} W \to S(W\lambda)\), let us choose another \(W\)-orbit of weights \(W\lambda' = \{\lambda'_1, \ldots, \lambda'_e\} \subset L_Q\). One associates with \(\pi_1(Y \setminus \mathcal{D}, y_0) \xrightarrow{m} W \to S(W\lambda')\) a covering \(f' : X' \to Y\). Let

\[
P \subset A \quad \text{and} \quad P' \subset A' = \text{Ker}(Nm_{f'} : JX' \to JY)^0
\]

be the two Prym-Tyurin varieties.
Proposition 1.10. The Abelian varieties \( P \) and \( P' \) are isogenous.

Proof. This is proved when \( Y = \mathbb{P}^1 \) in [K2], Section 6. The proof for arbitrary \( Y \) is essentially the same, so we only sketch it, referring for details to [K2]. One chooses an appropriate \( W \)-invariant bilinear form \( (\cdot | \cdot) \) on \( L \), constructs a lattice \( N \supset L \), an action of \( W \) on \( N \), a \( W \)-invariant, integer-valued, bilinear form \( B \) on \( N \), such that \( B|_L = (\cdot | \cdot) \), and two \( W \)-orbits \( \{\ell_1, \ldots, \ell_d\} \) and \( \{m_1, \ldots, m_c\} \) in \( N \) which are \( W \)-equivariantly bijective to \( W\lambda \) and \( W\lambda' \) respectively. One considers the linear map \( S : \mathbb{R}^d \to \mathbb{R}^e \) given by

\[
S(e_i) = \sum_{j=1}^e B(\ell_i, m_j) f_j
\]

This is a \( W \)-equivariant linear map and one employs it to construct a correspondence \( S \in \text{Div}(X \times X') \) as in [1.43]. By Lemma [1.4] it induces an endomorphism \( s : A \to A' \) and similarly \( \iota S \) induces \( \iota s : A' \to A \). By Remark [1.8] we may use the chosen common \( (\cdot | \cdot) \) in order to calculate \( P \) and \( P' \). Let \( G, G', \delta, \delta', q \) and \( q' \) be the corresponding data as in [1.3] and Proposition [1.6]. The following equalities of linear maps are verified in the course of the proof of Theorem 6.5 of [K2]:

(1.3) \( (G' + q' E) \cdot S = c_1 T, \quad S \cdot (G + q E) = c_2 T \)

for some \( c_1, c_2 \in \mathbb{Z} \). Transposing and passing to correspondences and homomorphisms this implies that

(1.4) \( s(A) \subset P', \quad s((\delta + q - 1)A) = 0 \)

\( \iota s(A') \subset P, \quad \iota s((\delta' + q' - 1)A') = 0 \)

One has by [K2], Lemma 6.5.1 that

(1.5) \( \iota S \cdot S = -q' G + d_1 T_1, \quad S \cdot \iota S = -q G' + d_2 T_2 \)

for some \( d_1, d_2 \in \mathbb{Z} \). Passing to correspondences and induces homomorphisms of \( A \) and \( A' \) this implies

(1.6) \( \iota S \circ s = q'(1 - \delta|A), \quad s \circ \iota s = q(1 - \delta'|A') \)

Restricting to \( P \) and \( P' \) one obtains

(1.7) \( \iota S \circ s|_{P} = qq' \cdot id_P, \quad s \circ \iota s|_{P'} = qq' \cdot id_{P'} \)

So \( s|_P : P \to P' \) and \( \iota s|_{P'} : P' \to P \) are isogenies.

\[\square\]

2. Coverings of type \( B_n \)

2.1. Let us consider a real vector space \( \mathbb{R}^n \) with basis \( \epsilon_1, \ldots, \epsilon_n \) and a cup product \( (\epsilon_j | \epsilon_k) = \delta_{jk} \). Denote by \( R \) the set

\[
R = \{ \pm \epsilon_j | j = 1, \ldots, n \} \cup \{ \pm \epsilon_j \pm \epsilon_k | 1 \leq j < k \leq n \} = R_s \cup R_t
\]

This is a root system of type \( B_n \) with \( R_s \) the set of short roots and \( R_t \) the set of long roots. For every \( \alpha \in R \) let \( s_\alpha : \mathbb{R}^n \to \mathbb{R}^n \) be the reflection \( s_\alpha(x) = x - \frac{2(x | \alpha)}{(\alpha | \alpha)} \alpha \). Then \( s_\alpha(R) = R \). The finite group \( W \) generated by \( s_\alpha, \alpha \in R \) is the Weyl group of type \( B_n \). The set of long roots \( R_t \) is a root system of type \( D_n \) and the reflections with respect to long roots generate a subgroup of index 2 in \( W \). We will usually denote the two Weyl groups by \( W(B_n) \) and \( W(D_n) \).
Consider the orbit $W\epsilon_1 = R_s = \{\epsilon_1, -\epsilon_1, \ldots, \epsilon_n, -\epsilon_n\}$. Acting on this set, the Weyl group $W(B_n)$ is identified with the permutation subgroup of $S(W\epsilon_1)$ consisting of permutations which commute with $-id$ (cf [13]). Each reflection $s_{\epsilon_j}$ acts as a transposition: $\epsilon_j \mapsto -\epsilon_j$ and each reflection $s_{\epsilon_j, \pm \epsilon_k}$ acts as a product of two independent transpositions.

Let $W = W(B_n)$. Let $Y$ be a smooth, projective, irreducible curve. Let $D \subset Y$ be a finite subset and let $y_0$ be a point in $Y \setminus D$. Let $m : \pi_1(Y \setminus D, y_0) \to W$ be a homomorphism. Composing with $W \hookrightarrow S(W\epsilon_1) = S_{2n}$ and applying Riemann’s existence theorem one obtains a covering $p : C \to Y$ of degree $2n$ whose monodromy map decomposes as

$$\pi_1(Y \setminus D, y_0) \overset{m}{\to} W \to S_{2n} \tag{2.1}$$

It is convenient to denote $\epsilon_{-j} = -\epsilon_j$ and consider $S_{2n}$ as the permutation group on the elements $\{1, -1, \ldots, n, -n\}$. Since the elements of $W$ commute with $-id$, the curve $C$ is equipped with an involution $i : C \to C$. Let $C' = C/i$ and let $\pi : C \to C'$ be the quotient map. One obtains the following decomposition of $p$

$$p : C \overset{\pi}{\to} C' \overset{g}{\to} Y. \tag{2.2}$$

Conversely, suppose a morphism $p : C \to Y$ of smooth, projective curves, can be decomposed as $C \overset{\pi}{\to} C' \overset{g}{\to} Y$ with $\deg \pi = 2$, $\deg g = n$. Assume $Y$ is irreducible. Let $i : C \to C$ be the involution with $\pi \circ i = \pi$. Let $D$ be the discriminant locus of $p$, let $y_0 \in Y \setminus D$ and let $p^{-1}(y_0) = \{x_1, x_1', \ldots, x_n, x_n'\}$ where $x_k' = i(x_k)$. Denoting $x_{-j} = x_j'$ and identifying $x_j$ with $\epsilon_j$, and $x_{-j}$ with $-\epsilon_j = \epsilon_{-j}$, $j = 1, \ldots, n$, we see that the monodromy map $\pi_1(Y \setminus D, y_0) \to S_{2n}$ can be decomposed as in (2.1).

We call, for easier reference, a covering $p : C \overset{\pi}{\to} C' \overset{g}{\to} Y$ with $\deg \pi = 2$ a covering of type $B_n$, or shortly a $B_n$-covering. It has simple ramification of type $B_n$ at a point $b \in D$, if the local monodromy at $b$ is a reflection in $W(B_n)$. If this is a reflection $s_\alpha \in R_\ell$ (a long root), then $g : C' \to Y$ is simply ramified at $b$, i.e. $|g^{-1}(b)| = n - 1$, and $\pi : C \to C'$ is unramified at $g^{-1}(b)$. If $\alpha \in R_\ell$ (a short root), then $g$ is unramified at $b$ and $\pi$ has one branch point among $g^{-1}(b)$. We say a ramified covering $p : C \to Y$, which can be decomposed as in (2.2), is a simply ramified $B_n$-covering if it has simple ramification of type $B_n$ at all discriminant points. In this case we denote by $D = D_\ell \cup D_t$ the corresponding splitting. It is clear that $p : C \overset{\pi}{\to} C' \overset{g}{\to} Y$ is a simply ramified $B_n$-covering if and only if the covering $g : C' \to Y$ is either unramified, or simply ramified, $g(\text{Discr}(C \to C')) \cap \text{Discr}(C' \to Y) = \emptyset$, and no two discriminant points of $\pi$ belong to the same fiber of $g$.

The ordinary simply ramified coverings $f : X \to Y$ of degree $n$, where the local monodromies are transpositions, may be considered as simply ramified coverings of type $A_{n-1}$, since $W(A_{n-1}) \cong S_n$. A simply ramified $B_n$-covering $p : C \overset{\pi}{\to} C' \overset{g}{\to} Y$ is an ordinary simply ramified covering if and only if $D_\ell = \emptyset$, i.e. $g : C' \to Y$ is étale.

2.2. Given $m : \pi_1(Y \setminus D, y_0) \to W$ and $p : C \to Y$ as above, one calculates, using (1.2), that the correspondence associated with $L = Q(R), W\epsilon_1 = \{\epsilon_{\pm j}|j = 1, \ldots, n\}$ and $(\epsilon_j, \epsilon_k) = -\delta_{jk}$ is obtained from the linear map $D(\ell_j) = \ell_{-j}, j \in [-n, n]$. Therefore this correspondence is the graph of the involution $i : C \to C$. Here
We want to calculate the monodromy group of ramified variety \( \subseteq \{ \text{imprimitive if there is a subset } \Phi \}
\)

Let us first recall some facts about ordinary coverings. Let \( f : X \to Y \) be a covering of smooth curves of degree \( n \). The curve \( Y \) is assumed irreducible. Let \( G \subseteq S_n \) be the monodromy group. The irreducibility of \( X \) is equivalent to the transitivity of \( G \). Let us recall (see [2.4]) that a transitive subgroup of \( S_n \) is called imprimitive if there is a subset \( \Phi \subseteq \{1, \ldots, n\} \), \( 1 < |\Phi| < n \), such that for every \( g \in G \) one has either \( g(\Phi) = \Phi \), or \( g(\Phi) \cap \Phi = \emptyset \). A transitive group is called primitive if it is not imprimitive. It is clear that the monodromy group of \( f : X \to Y \) is imprimitive if and only if there is a nontrivial decomposition \( f : X \to X_1 \to Y \).

We call a covering of smooth, irreducible curves \( f : X \to Y \) primitive if no such decomposition exists. If \( f : X \to Y \) has simple ramification, then for any decomposition \( X \to X_1 \to Y \) with \( \deg(X \to X_1) > 1 \) the covering \( X_1 \to Y \) must be étale. So, for a simply ramified covering of smooth irreducible curves \( f : X \to Y \), primitivity is equivalent to the surjectivity of \( f_* : \pi_1(X,*) \to \pi_1(Y,*) \).

It is shown in [BE], Lemma 2.4 that a primitive subgroup of \( S_n \) which contains a transposition equals \( S_n \). So, if \( f : X \to Y \) is a primitive covering of degree \( n \) with at least one simple branching, then the monodromy group is \( S_n \).

Using the notation of [2.1] let \( a_j = \{\epsilon_j, -\epsilon_j\} \) and denote \( \Sigma = \{a_1, \ldots, a_n\} \). Every element of \( W(B_n) \) induces a permutation of \( \Sigma \). One obtains the following exact sequence.

\[
0 \to G_2 \to W(B_n) \to S_n \to 0
\]

where \( G_2 \cong (\mathbb{Z}/2\mathbb{Z})^n \) is the subgroup generated by the reflections \( s_{\epsilon_j}, j = 1, \ldots, n \). Let \( G_1 \) be the subgroup generated by \( s_{\epsilon_i, -\epsilon_j}, 1 \leq i < j \leq n \). It maps isomorphically to \( S_n \), so \( W(B_n) \) is a semidirect product of \( G_1 \) and \( G_2 \). Furthermore, \( G_1 \) has two orbits when acting on \( \{\pm \epsilon_j | j = 1, \ldots, n\} \): \( \Sigma_1 = \{\epsilon_1, \ldots, \epsilon_n\} \) and \( \Sigma_2 = \{-\epsilon_1, \ldots, -\epsilon_n\} \). The conjugates of \( G_1 \) map surjectively to \( S_n \) in [2.3]. Two other subgroups of \( W(B_n) \) which have the same property are: \( W(B_n) \) itself, the group \( W(D_n) \) generated by the reflections with respect to long roots \( \{s_{\epsilon_j} \pm \epsilon_k | 1 \leq j < k \leq n\} \). The conjugates of the group of the next lemma is another example.

**Lemma 2.5.** Let \( \sigma = -id \in W(B_n) \). Let \( G = N_{W(B_n)}(G_1) \) be the normalizer of \( G_1 \). Then \( G = G_1 \cup G_1 \sigma \). The group \( G \) acts transitively on \( \{\pm \epsilon_j | j = 1, \ldots, n\} \) and if \( n \geq 3 \) the only reflections which belong to \( G \) are \( \{s_{\epsilon_j} \pm \epsilon_k | 1 \leq j < k \leq n\} \).

**Proof.** Let \( g \in N_{W(B_n)}(G_1) \). Then either \( g(\Sigma_1) = \Sigma_1, g(\Sigma_2) = \Sigma_2 \) or \( g(\Sigma_1) = \Sigma_2, g(\Sigma_2) = \Sigma_1 \). In the first case \( g \in G_1 \). In the second case \( g \in G_1 \sigma \). If \( n \geq 3 \) no reflection can satisfy \( s_\alpha(\Sigma_1) = \Sigma_2 \), hence the reflections which belong to \( G_1 \) are \( s_{\epsilon_j} \pm \epsilon_k, 1 \leq j < k \leq n \).

**Lemma 2.6.** Let \( G \subseteq W(B_n) \) be a subgroup, which contains a reflection with respect to a long root, and the image of \( G \) in \( S_n \) is a primitive group. Then one of the following alternatives holds.
(i) \( G = W(B_n) \),
(ii) \( G = W(D_n) \),
(iii) \( n \geq 3 \) and \( G = wN_{W(B_n)}(G_1)w^{-1} \) for some \( w \in W(B_n) \),
(iv) \( G = wG_1w^{-1} \) for some \( w \in W(B_n) \).

Furthermore \( G \) is transitive only in cases (i) – (iii) and in cases (iii) and (iv) the set of reflections in \( G \) equals \( w\{s_{e_j-e_k}|1 \leq j < k \leq n\}w^{-1} \).

**Proof.** The image \( \overline{\Gamma} \) of \( G \) in \( S_n \) is a primitive subgroup which contains a transposition. So, \( \overline{\Gamma} = S_n \) by [BE], Lemma 2.4. By hypothesis \( G \) contains some \( s_\alpha, \alpha \in R_\ell \). Let \( w_1 \in W(B_n), w_1(\alpha) = \epsilon_1 - \epsilon_2 \). Substituting \( G \) by \( w_1Gw_1^{-1} \) we may assume that \( s_{\epsilon_1-\epsilon_2} \in G \). Let \( h \in S_n \) be a permutation with \( h(1) = 2, h(2) = 3 \). Let \( g \in G \), \( \overline{g} = h \). Then \( gs_{\epsilon_1-\epsilon_2}g^{-1} \) is either \( s_{\epsilon_2-\epsilon_3} \) or \( s_{\epsilon_2+\epsilon_3} \). If the latter case occurs we replace \( G \) by \( s_{\epsilon_2}Gs_{\epsilon_3} \) and obtain \( s_{\epsilon_2-\epsilon_3} \in G \). Repeating this argument with (34), (45) etc. we obtain that replacing \( G \) by some \( wGw^{-1} \) we may assume that \( G \supset \langle s_{\epsilon_1-\epsilon_2}, \ldots, s_{\epsilon_{n-1}-\epsilon_n} \rangle = G_1 \). Suppose \( G \) contains the pair \( s_{\epsilon_i-\epsilon_j}, s_{\epsilon_i+\epsilon_j} \), for some \( i \neq j \). Let \( 1 \leq k < \ell \leq n \). Let \( g \in G \) be an element such that the permutation \( \overline{g} \in S_n \) transforms \( i \mapsto k, j \mapsto \ell \). Then \( g\{s_{\epsilon_i-\epsilon_j}, s_{\epsilon_i+\epsilon_j}\}g^{-1} = \{s_{\epsilon_k-\epsilon_\ell}, s_{\epsilon_k+\epsilon_\ell}\} \). Therefore \( G \) contains \( s_\beta \) for every \( \beta \in R_\ell \). Hence \( G = W(B_n) \) or \( G = W(D_n) \). If no pair \( \{s_{\epsilon_i-\epsilon_j}, s_{\epsilon_i+\epsilon_j}\} \) is contained in \( G \), then the only reflections contained in \( G \) are \( \{s_{\epsilon_i-\epsilon_j}|1 \leq i < j \leq n\} \). Hence if \( g \in G \), then for every \( \beta \) one has \( gs_{\epsilon_i-\epsilon_j}g^{-1} = s_{\epsilon_k-\epsilon_\ell} \), for some \( k \neq \ell \). Therefore \( G \subset N_{W(B_n)}(G_1) \). Since \( |N_{W(B_n)}(G_1):G_1| = 2 \) one has that either \( G = G_1 \) or \( G = N_{W(B_n)}(G_1) \). If \( n = 2 \), then \( N_{W(B_2)}(G_1) = W(D_2) \). If \( n \geq 3 \) we apply Lemma 2.6.

**Proposition 2.7.** Let \( p : C \rightarrow C', g : Y \) be a \( B_n \)-covering with irreducible \( C \). Suppose \( g : C' \rightarrow Y \) is primitive (cf. [2.3]). Furthermore suppose that \( g : C' \rightarrow Y \) is ramified, in one of the branch points \( b \in \mathcal{D} \subset Y \) it is simply ramified and \( \pi \) is unramified in \( g^{-1}(b) \). Then either \( G = W(B_n) \), or \( G = W(D_n) \), or \( G \) is conjugated to \( N_{W(B_n)}(G_1) \) in case \( n \geq 3 \). The latter case happens if and only if \( p : C \rightarrow Y \) fits into a commutative diagram

(2.4)

\[
\begin{array}{ccc}
C & \xrightarrow{\pi} & C' \\
\downarrow f \ & \ & \downarrow g \\
\overline{Y} \ & \xrightarrow{h} & Y
\end{array}
\]

where \( h : \overline{Y} \rightarrow Y \) is a covering of degree 2.

**Proof.** The possible alternatives for \( G \) follow from Lemma 2.6. In case (iii), renumbering the fiber \( p^{-1}(y_0) \), one may assume \( G = N_{W(B_n)}(G_1) \). Then one has a
commutative diagram of $G$-equivariant maps

\[
\begin{array}{ccc}
\{\pm \varepsilon_j | 1 \leq j \leq n\} & \{\alpha_j | 1 \leq j \leq n\} & \{\Sigma_1, \Sigma_2\} \\
\end{array}
\]

This yields (2.4). Viceversa, suppose (2.4) holds for some $f : C \to \tilde{Y}$ and $h : \tilde{Y} \to Y$. Let $\tau : Y \to \tilde{Y}$ be the involution such that $h \circ \tau = h$. For every $x \in C$ one has $h \circ f(x) = h \circ f(i \cdot x)$ since $h \circ f = g \circ \pi$. It is impossible that $f(i \cdot x) = f(x)$ for $\forall x \in C$, since this would imply a decomposition $C' \to \tilde{Y} \to Y$, while by hypothesis $C' \to Y$ is primitive. Therefore $f(i \cdot x) = \tau f(x)$. Now, one can number the points of $p^{-1}(y_0)$ so that a diagram (2.5), with maps commuting with the monodromy action, takes place. Hence $G$ is conjugated to $N_{W(B_n)}(G_1)$.

When $g : C' \to Y$ is simply ramified, primitivity is equivalent to the surjectivity of $g_* : \pi_1(C', *) \to \pi_1(Y, *)$. So, we obtain the following corollary.

**Corollary 2.8.** Let $p : C \overset{\pi}{\longrightarrow} C' \overset{\varrho}{\longrightarrow} Y$ be a simply ramified $B_n$-covering with irreducible $C$. Assume $g_* : \pi_1(C', *) \to \pi_1(Y, *)$ is surjective. Let $G$ with $G \subset W(B_n) \subset S_{2n}$ be the monodromy group of the covering $p : C \to Y$.

If $\mathfrak{D}_s \neq \emptyset$, then $G = W(B_n)$.

If $\mathfrak{D}_s = \emptyset$, then one of the following alternatives holds: either $G = W(B_n)$; or $G = W(D_n)$; or $G$ is conjugate to $N_{W(B_n)}(G_1)$ in case $n \geq 3$. The latter alternative holds if and only if $p : C \overset{\pi}{\longrightarrow} C' \overset{\varrho}{\longrightarrow} Y$ fits into a commutative diagram (2.4) with étale $\tilde{Y} \to Y$.

**Proof.** $\mathfrak{D}_t \neq \emptyset$ since otherwise $g_* : \pi_1(C', *) \to \pi_1(Y, *)$ would not be surjective. The subgroups $W(D_n)$ and $N_{W(B_n)}(G_1)$ do not contain reflections with respect to short roots $s_{\varepsilon_j}$, so only $G = W(B_n)$ is possible if $\mathfrak{D}_s \neq \emptyset$. Let $\mathfrak{D}_s = \emptyset$. By Lemma 2.3 the reflections which belong to $N_{W(B_n)}(G_1)$ do not interchange $\Sigma_1$ and $\Sigma_2$. Therefore, if $G$ is conjugated to $N_{W(B_n)}(G_1)$, the covering $\tilde{Y} \to Y$ is étale.

**Corollary 2.9.** Let $p : C \overset{\pi}{\longrightarrow} C' \overset{\varrho}{\longrightarrow} \mathbb{P}^1$ be a simply ramified $B_n$-covering with irreducible $C$. Let $G \subset W(B_n) \subset S_{2n}$ be the monodromy group of $p : C \to \mathbb{P}^1$.

If $\mathfrak{D}_s \neq \emptyset$, then $G = W(B_n)$. If $\mathfrak{D}_s = \emptyset$, then $G = W(D_n)$.

**Remark 2.10.** We notice that when $g(Y) \geq 1$ the monodromy group of $p : C \overset{\pi}{\longrightarrow} C' \overset{\varrho}{\longrightarrow} Y$ might very well be $W(B_n)$, even when $\pi : C \to C'$ is étale. The fundamental group $\pi_1(Y \setminus \mathfrak{D}, y_0)$ is generated by $\gamma_1, \ldots, \gamma_n, \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$, where $\gamma_1, \ldots, \gamma_n$ are homotopy classes of loops encircling the branch points of $p : C \to Y$, with the only relation

\[
\gamma_1 \cdots \gamma_n = [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g].
\]

When $\pi : C \to C'$ is étale the monodromy map $m : \pi_1(Y \setminus \mathfrak{D}, y_0) \to W(B_n)$ has the property that $m(\gamma_i) \in W(D_n)$ for $\forall i = 1, \ldots, n$ (cf. [K2], pp.179,180), but the only restriction for $m(\alpha_j), m(\beta_j) \in W(B_n)$ comes from the relation (2.6). Reversing
and applying Riemann’s existence theorem it is easy to construct, when \( g(Y) \geq 1 \), coverings of type \( B_n \) with étale \( \pi : C \to C' \) and full monodromy group \( W(B_n) \).

**Definition 2.11.** Let \( p : C \xrightarrow{\pi} C' \xrightarrow{g} Y \) be a covering with \( \deg(\pi) = 2 \) and irreducible \( C \). We say it is a simple \( B_n \)-covering if:

(i) \( p \) is simply ramified \( B_n \)-covering and both \( \pi \) and \( g \) are ramified;

(ii) \( g : C' \to Y \) is primitive, or equivalently \( g_* : \pi_1(C',*) \to \pi_1(Y,*) \) is surjective;

**Remark 2.12.** Notice that if \( \deg(g) \) is prime, then \( g : C' \to Y \) is primitive. So in this case the simply ramified \( B_n \)-coverings with \( \mathcal{D}_s \neq \emptyset \) and \( \mathcal{D}_t \neq \emptyset \) are simple \( B_n \)-coverings. The same statement holds when \( \deg(g) \) is arbitrary and \( Y \cong \mathbb{P}^1 \). By Corollary 2.8 every simple \( B_n \)-covering has full monodromy group \( W(B_n) \).

2.13. The short root \( \epsilon_1 \) is the fundamental weight \( \omega_1 \) of the root system \( R \) of type \( B_n \) (cf. [B]). Let us consider the fundamental weight \( \omega_n = \frac{1}{n}(\epsilon_1 + \cdots + \epsilon_n) \). This is the dominant weight of the spinor representation of the Lie algebra \( so(2n+1) \).

We call \( \omega_n \) the spinor weight. One has

\[
W\omega_n = \{ \lambda_A = \frac{1}{2} \sum_{j \notin A} \epsilon_j - \sum_{j \in A} \epsilon_j \mid A \subset \{1, \ldots, n\} \}
\]

Let \( m : \pi_1(Y \setminus \mathcal{D}, y_0) \to W \subset S_{2n} \) be the monodromy map of the covering \( p : C \xrightarrow{\pi} C' \xrightarrow{g} Y \) as in 2.1. Composing \( m \) with the permutation representation \( W \to S(W\omega_n) \) and applying Riemann’s existence theorem one obtains a smooth, projective curve \( X \) and a covering \( f : X \to Y \) of degree \( 2^n \). Let us fix a bijection \( \gamma \) and \( y \) in \( Y \). One calculates the points of \( f^{-1}(y) \) using covering homotopy along \( \gamma \) and the fixed bijection. Thus \( f^{-1}(y) = \{ x_A \mid A \subset \{1, \ldots, n\} \} \) in correspondence with \( 2^n \).

Calculating the action of \( s_{\epsilon_j} \) and \( s_{\epsilon_j \pm \epsilon_k} \) on \( W\omega_n \) one obtains that if \( p : C \xrightarrow{\pi} C' \xrightarrow{g} Y \) is a simply ramified \( B_n \)-covering, then the local monodromy of \( f : X \to Y \) is a product of \( 2^{n-1} \) independent transpositions at a point \( b \in \mathcal{D}_s \) and a product of \( 2^{n-2} \) independent transpositions at a point \( b \in \mathcal{D}_t \).

One calculates a correspondence \( D \) as in 1.3 letting \( L = Q(R) = \oplus_{j=1}^n \mathbb{Z} \epsilon_j \), \( (\epsilon_j | \epsilon_k) = -2\delta_{jk} \) and \( \lambda = \omega_n \). It is shown in [K2], Section 8.8 that one has for \( D \) the formula

\[
D(x_A) = \sum_{B \neq A} (|A| + |B| - 2|A \cap B| - 1)x_B
\]

and furthermore \( q = 2^{n-1} \) and \( \deg D = 2^{n-1}(n-2) + 1 \). Let \( \delta : JX \to JX \) be the endomorphism induced by \( D \), let

\[
A = \text{Ker}(Nm : JX \to JY)^0 \quad \text{and} \quad P(X, \delta) = (1 - \delta)A.
\]

Applying Proposition 1.10 to \( W = W(B_n), \lambda = \omega_n \) and \( \lambda' = \omega_1 \), one obtains that \( P(X, \delta) \) and \( P(C, i) = P(C, C') \) are isogenous. In the next paragraph we want to describe this isogeny more explicitly.

2.14. The correspondence \( S \in \text{Div}(X \times C) \) which establishes the isogeny between \( P(X, \delta) \) and \( P(C, C') \) is constructed in [K2], Section 8.8. We recall this briefly. One considers the bilinear form \( (\epsilon_j | \epsilon_k) = -2\delta_{jk} \). Here

\[
\mathbb{R}^{d} = \oplus_{A \subset \{1, \ldots, n\}} \mathbb{R}e_A, \quad \mathbb{R}^{c} = \oplus_{j=1}^n (\mathbb{R}f_j \oplus \mathbb{R}f_{-j})
\]
One has

\[ S(e_A) = 2S_0(e_A) + nT(e_A) \]

where

\[ S_0(e_A) = \sum_{j \notin A} f_{-j} + \sum_{j \in A} f_j, \quad T(e_A) = \sum_{j=1}^{n} (f_j + f_{-j}) \]

Since \( S \) and \( T \) are \( W \)-equivariant, so is \( S_0 \). The construction of §1.3 yields correspondences in \( \text{Div}(X \times C) \), which abusing notation we denote again by \( S, S_0 \) and 
\( T \). One has for every \( y \in Y \setminus \mathfrak{D} \) that 
\( f^{-1}(y) = \{ z_A | A \subset \{ 1, \ldots, n \} \} \) \( , p^{-1}(y) = \{ x_j, x'_j | j = 1, \ldots, n \} \), with 
\( x'_j = i(x_j) := x_{-j} \), and \( S(z_A) = 2S_0(z_A) + nT(z_A) \), where

\[ S_0(z_A) = \sum_{j \notin A} x'_j + \sum_{j \in A} x_j, \quad T(z_A) = p^*(f(z_A)) \]

Replacing \( S \) by \( -S \) one has \( -S(z_A) = 2S_1(z_A) - (n + 1)T(z_A) \), where

\[ S_1(z_A) = \sum_{j \notin A} x'_j + \sum_{j \in A} x_j \]

We may use the correspondence \( S_1 \) in order to fit the covering \( f : X \to Y \) into the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{S_1} & C^{(n)} \\
| f \downarrow & & \downarrow \pi^{(n)} \\
Y & \xrightarrow{g^*} & C'^{(n)}
\end{array}
\]

Here \( g^* \) denotes the map associating to a point \( y \in Y \) the whole fibre \( g^{-1}(y) \) considered as a point of the \( n \)-fold symmetric product \( C'^{(n)} \) of \( C' \). It is clear that \( S_1 : X \to C^{(n)} \) is generically injective. We will later need some properties of \( S_0 \) which we now prove.

**Lemma 2.15.** Let \( s_0 : JX \to JC \) and \( t^{s_0} : JC \to JX \) be the homomorphisms induced by \( S_0 \) and \( tS_0 \) respectively. Let \( T_1, T \in \text{Div}(X \times C) \) and \( T_2 \in \text{Div}(C \times C) \) be the trace correspondences (cf. §1.3). Then

(i) \( s_0 \) and \( t^{-s_0} \) induce homomorphisms \( s_0 : \text{Ker}(Nm) \to P(C, C') \), \( t^{s_0} : \text{Ker}(Nm) \to P(C, C') \) and isogenies \( S_0 : P(X, \delta) \to P(C, C') \), \( t^{s_0} : P(C, C') \to P(X, \delta) \), such that \( t^{s_0}s_0|_{P(X, \delta)} = 2^{-1}id_{P(X, \delta)} \), \( s_0^{t}s_0|_{P(C, C')} = 2^{-1}id_{P(C, C')} \)

(ii) \( tS_0(S_0(z_A)) = z_A - D(z_A) + (n - 1)T_1(z_A) \) for \( \forall A \subset \{ 1, \ldots, n \} \)

(iii) \( t^{s_0}(S_0(x_k)) = 2n^{-2}(x_k - i(x_k)) + 2n^{-2}T_2(x_k) \) for \( \forall k \in [-n, n] \)

**Proof.** Part (i) follows from Proposition 1.10 and its proof, observing that \( s = 2s_0, t^s = 2t^{s_0} \). All the correspondences are obtained from linear maps as in §1.3 so (abusing notation) we need to verify the following equalities of linear maps:

\[ tS_0 \cdot S_0 = E - D + (n - 1)T_1, \quad S_0 \cdot t^{s_0} = 2n^{-2}(E - I) + 2n^{-2}T_2 \]

where \( I \) is the involution \( I(f_k) = f_{-k} \) and \( E \) is the identity map. We use [K2], Lemma 6.5.1 (we notice that the identity map is denoted by \( I \) in that paper). The data \( W(B_n), L = \oplus_{j=1}^{n} Z\epsilon_j, \lambda = \omega_n \) and \( (\epsilon_j | \epsilon_k) = -2\delta_{jk} \) yields \( G = D - E \) and \( q = 2n^{-1} \). The data \( W(B_n), L = \oplus_{j=1}^{n} Z\epsilon_j, \lambda' = \omega_1 \) and \( (\epsilon_j | \epsilon_k) = -2\delta_{jk} \) yields
\( G' = 2(I - E) + T_2 \) and \( q' = 4(\text{cf. [K2], §8.8.1 and §4.7}) \). According to [K2], Lemma 6.5.1 one has
\[ ^tS \cdot S = -q'G + d_1T_1 = 4(E - D) + d_1T_1 \]
Replacing \( S \) by \( 2S_0 + nT \) and using Lemma 1.2 one obtains
\[ ^tS_0 \cdot S_0 = E - D + f_1T_1 \]
The degrees of the correspondences \( S_0, ^tS_0, D - E \) and \( T_1 \) are \( n, 2^{n-1} \), \( 2^{n-1}(n - 2) \) and \( 2^n \) respectively. Therefore \( f_1 = n - 1 \) and Equality (ii) is verified. Equality (iii) is proved similarly using \( S \cdot ^tS = -q'G' + d_2T_2 \).

3. SOME RESULTS FOR ARBITRARY \( n \)

3.1. Let the situation be as in (2.1). So \( p : C \to Y \) with decomposition (2.2) denotes a covering of type \( B_n \). Assume moreover that \( p \) is a simple \( B_n \)-covering. In particular \( C \) is an irreducible curve. According to (2.4, Hurwitz formula gives
\[ g(C') = \frac{|D_1|}{2} + ng(Y) - n + 1, \quad g(C) = \frac{|D_s|}{2} + |D_1| + 2ng(Y) - 2n + 1. \]
Hence the Prym variety \( P(C, C') \) of the covering \( \pi : C \to C' \) is of dimension
\[ \dim P(C, C') = g(C') + \frac{|D_s|}{2} - 1 = \frac{|D_s|}{2} + |D_1| + ng(Y) - n. \]
Consider the covering \( f : X \to Y \) defined in (2.13). The curve \( X \) is smooth by construction. It is connected, and therefore irreducible, because by Corollary 2.8 the monodromy group of \( p : C \to Y \) is \( W(B_n) \subset S_{2n} \) and therefore the monodromy group of \( f : X \to Y \) is a transitive subgroup of \( S_{2n} \). Since the covering \( f \) is of degree \( 2^n \), we obtain
\[ g(X) = 2^{n-2}|D_s| + 2^{n-3}|D_1| + 2^n g(Y) - 2^n + 1. \]
Counting the number of points over branch points in \( Y \) in the diagram (2.18), it is easily seen that \( S_1 : X \to C(n) \) is injective, taking into account that \( g^* \) is injective. Hence we may consider the points of \( X \) as points of the symmetric product \( C(n) \). In other words, we denote the points of \( X \) by \( x = x_1 + \cdots + x_n \) with \( x_i \in C \).

The curve \( X \) admits an involution \( \sigma \), induced by the involution \( ^t \) defined by the double covering \( \pi \), namely
\[ \sigma(x_1 + \cdots + x_n) = x_1' + \cdots + x_n'. \]
Note that for \( n \geq 3 \) the involution \( \sigma \) is fixed-point free. Hence, denoting \( X' = X/\sigma \) and by \( P(X, X') \) the corresponding Prym variety, we have
\[ \dim P(X, X') = \frac{g(X) - 1}{2} = 2^{n-3}|D_s| + 2^{n-4}|D_1| + 2^{n-1}(g(Y) - 1). \]

3.2. For any \( x = x_1 + \cdots + x_n \in X \) and any subset \( A \subset \{1, \ldots, n\} \) we denoted \( x_A = \sum_{i \in A} x_i + \sum_{i \in \bar{A}} x_i' \). In particular, \( x = x_{\bar{A}} \). In (2.18) we defined a correspondence \( D \) on \( X \) and we saw that it is given by equation (2.8). In particular we have for \( x = x_{\bar{A}} \),
\[ D(x) = \sum_{|B| \geq 2}(|B| - 1)x_B \]
\[ = \sum_{|B| = 2} x_B + 2 \sum_{|B| = 3} x_B + \cdots + (n - 1)x_{\{1, \ldots, n\}} \]
This implies \( \deg D = 2^{n-1}(n - 2) + 1 \). Moreover \( D \) is of exponent \( q = 2^{n-1} \) (see (2.13).
Lemma 3.3. The correspondence $D$ commutes with $\sigma$: $\sigma D = D \sigma$.

Proof. Note first that for all subsets $B \subset \{1, \ldots, n\}$ we have $\sigma(x_B) = x_{\overline{B}}$, where $\overline{B} = \{1, \ldots, n\} \setminus B$. Hence

$$D\sigma(x_A) = D(x_{\overline{A}}) = \sum_{C \neq \overline{A}} (|C| - |A|) x_C$$

Setting $C = \overline{B}$ and noting that $|A| - |B| = |A| + |B| - 2|A \cap B|$, we have

$$D\sigma(x_A) = \sum_{B \neq A} (|A| - |B| - 2|A \cap B| - 1)x_{\overline{B}} = \sigma D(x_A).$$

\[\square\]

Lemma 3.4. For any $x = x_0 \in X$,

$$(D - 1)(x + \sigma(x)) = (n - 2)f^*f(x).$$

Proof. Writing $\Sigma = \{1, \ldots, n\}$ we have using Lemma 3.3

$$(D - 1)(x + \sigma(x)) = \sigma(Dx_0 - x_0) + (Dx_0 - x_0)$$

$$= (n - 2)\sum_{B \neq A} (|B| - 1)x_B - n x_0$$

$$= (n - 2)(|A| - 1)x_0 + \sum_{B \neq A} (|B| - 1)x_B - n x_0$$

$$= (n - 2)\sum_{B \neq A} (|B| - 1)x_B - n x_0$$

$$= (n - 2)\sum_{B \neq A} (|B| - 1)x_B - n x_0$$

$$= (n - 2)f^*f(x).$$

\[\square\]

As in [2.13] consider $A = \text{Ker}(Nm_f : JX \to JY)^0$, the connected component of the kernel of the norm map of $f$. If $\delta \in \text{End}(JX)$ denotes the endomorphism induced by the correspondence $D$, we defined the Prym-Tyurin variety associated to $D$ by

$$P(X, \delta) = (1 - \delta)A.$$ 

Lemma 3.3 implies that $D$ induces an endomorphism of the Prym variety $P(X, X')$, also denoted by $\delta$. Using this notation we have

Proposition 3.5. $P(X, \delta)$ and $P(X, X')$ are related as follows:

(i) $P(X, \delta) \subset P(X, X')$,
(ii) $P(X, \delta) = (\delta - 1)P(X, X').$

Proof. (i) follows immediately from Lemma 3.4. For the proof of (ii) note first that $P(X, X') \subset A$, since $Nm_f(x - \sigma(x)) = 0$ for every $x \in X$. Moreover, $JX = P(X, X') + \pi^*JX'$. Intersecting with $A$ gives

$$A = P(X, X') + (\pi^*JX' \cap A)^0.$$ 

On the other hand, $P(X, \delta) = (\delta - 1)A$ by definition. Hence it suffices to check that $(\delta - 1)(\pi^*JX' \cap A^0) = 0$.

But $\pi^*JX' \cap A = \{a' + \sigma(a') \mid a' \in JX', \ Nm_f(a' + \sigma(a')) = 0\}$ and we have $Nm_f(a' + \sigma(a')) = 2Nm_f(a')$. This implies, using Lemma 3.3 and the proof of Lemma 3.4

$$2(D - 1)(a' + \sigma(a')) = 2(n - 2)f^*Nm_f(a') = 0.$$

\[\square\]
Here $(\delta - 1)(\pi^*JX' \cap A)$ consists of torsion points, which implies $(\delta - 1)((\pi^*JX' \cap A)^0) = 0.$

**3.6.** We need a construction due to Donagi (cf. [1]). Consider again the spinor weight $\omega_n$. The subgroup $W(D_n) \subset W(B_n) = W$ is of index 2 and one has a splitting (cf. (2.7))

$$W \omega_n = \{ \lambda_A \mid |A| \text{ is even} \} \cup \{ \lambda_A \mid |A| \text{ is odd} \}.$$  

The subgroup $W(D_n)$ of type $\tau$ yields a decomposition of type $D_n$, namely the dominant weights associated to the two semispinor representations of the Lie algebra $so(2n)$. In the situation of (2.13) the monodromy group of the covering $f : X \to Y$ is $\pi_1(Y \setminus \mathcal{D}, y_0) \to W(B_n) \to S(W \omega_n)$. Let $G = m(\pi_1(Y \setminus \mathcal{D}, y_0))$ be the monodromy group. The splitting (3.3) yields a decomposition

$$f : X \xrightarrow{g} \tilde{Y} \xrightarrow{h} Y.$$ 

Here $h$ is of degree 2 and its monodromy map is the composition of $m$ with $W(B_n) \to S_2$. The latter is obtained from $W(B_n)$ acting on the two subsets of $\mathcal{D}$.

If $G \subset W(D_n)$, then $\tilde{Y}$ is a disjoint union $\tilde{Y} = Y_1 \sqcup Y_2$ of two copies of $Y$ and respectively $X = X_1 \sqcup X_2$. If $G \not\subset W(D_n)$, then $\tilde{Y}$ is irreducible and the monodromy group of $g : X \to \tilde{Y}$ equals $G \cap W(D_n)$.

If $p : C \to C' \to Y$ is a simply ramified $B_n$-covering then, calculating the action of the reflections on $W \omega_n$, we see that $h : \tilde{Y} \to Y$ is ramified in $\mathcal{D}$, and $g : X \to \tilde{Y}$ is ramified in $h^{-1}(\mathcal{D})$. The Prym variety $P(\tilde{Y}, Y)$ is of dimension

$$\dim P(\tilde{Y}, Y) = \frac{|\mathcal{D}|}{2} + g(Y) - 1.$$ 

Using the correspondence $S_1 \in \text{Div}(X \times C)$ of (2.14) and the commutative diagram (2.9) we may give another interpretation of the map $g : X \to \tilde{Y}$. Two points $x = x_1 + \cdots + x_n$ and $\hat{x} = \hat{x}_1 + \cdots + \hat{x}_n$ of $X$ are called equivalent, denoted $x \sim \hat{x}$, if and only if $f(x) = f(\hat{x})$ and the points $x_i$ and $\hat{x}_i$ of $C$ differ by an even number of changes. Then $\tilde{Y}$ is the quotient of $X$ modulo this equivalence relation and $g : X \to \tilde{Y}$ is the map associating to any $x = x_0 \in X$ the equivalence class $\{x_A \mid |A| \text{ even} \}$.

For $n$ even, the involution $\sigma$ respects the equivalence relation and $f$ factorizes as follows

$$X \to X' \to \tilde{Y} \to Y.$$ 

For $n$ odd, the involution $\sigma$ exchanges the equivalence classes and we have instead a commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{g} & \tilde{Y} \\
\downarrow & & \downarrow h \\
X' & \xrightarrow{h} & Y \\
\end{array}$$ 

Let $\tau : \tilde{Y} \to \tilde{Y}$ denote the involution associated to the covering $h$. Then $g(\sigma(x)) = \tau(g(x))$ implies that $g$ induces a homomorphism of Prym varieties $Nm_g : P(X, X') \to P(\tilde{Y}, Y)$. 
Lemma 3.7. \( Nm_g : P(X, X') \rightarrow P(\hat{Y}, Y) \) is the zero map for \( n \) even and surjective for \( n \) odd.

Proof. For even \( n \) we have \( g(x_0) = g(\sigma(x_0)) \) for any \( x = x_0 \in X \), which implies the assertion. The surjectivity for odd \( n \) is obvious. \( \square \)

Proposition 3.8. For any odd integer \( n \geq 3 \),

(i) \( P(X, \delta) \subset K := Ker(P(X, X') \xrightarrow{Nm_g} P(\hat{Y}, Y))^0 \).

(ii) \( g^*P(\hat{Y}, Y) \subset (\delta + 2^n - 1)P(X, X') \)

Proof. (i) For any \( x = x_0 \in X \) we have \( D(x_0) = \sum_{k=2}^{n} \sum_{|A|=k} (k-1)x_A \), which implies

\[
g(D(x_0) - x_0) = \left( \sum_{k=0}^{n} \right) (k-1)g(x_0) + \left( \sum_{k=0}^{n} \right) (k-1)\tau(g(x_0))
\]

Define

\[
M := \sum_{k=0}^{n} \sum_{|A|=k} (k-1) = \sum_{k=0}^{n} \sum_{|A|=k} (k-1)
\]

and note that the equality is an immediate consequence of the well known binomial identities \( \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \) and \( \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \). We get

\[
g(D(x_0) - x_0) = M(g(x_0) + \tau g(x_0))
\]

and thus for all \( a \in JX \)

\[
g(\delta(a) - a) = M[Nm_g(a) + \tau Nm_g(a)] = Mh^*Nm_f(a).
\]

Now recall that \( P(X, \delta) = (\delta - 1)A \). So any \( b \in P(X, \delta) \) is of the form \( b = (\delta - 1)(a) \) with \( Nm_f(a) = 0 \). This implies \( Nm_g(b) = Mh^*Nm_f(a) = 0 \) and thus the assertion.

(ii) If \( z \in \hat{Y} \), then \( g^*(z - \tau(z)) = g^*(z) - \sigma(g^*(z)) \). This shows that \( g^*(P(\hat{Y}, Y)) \subset P(X, X') \).

Let \( JX = V/A, J\hat{Y} = W/\Gamma \), and let \( V^-, W^- \) be the anti-invariant subspaces of \( \sigma \) and \( \tau \) respectively. Since \( E_{JX} = -(\cdot,\cdot)_X \) and \( E_{J\hat{Y}} = -(\cdot,\cdot)_{\hat{Y}} \), we have by the projection formula, \( E_{J\hat{Y}}(g_v, v, w) = E_{JX}(v, g^*w) \) for any \( v \in V \) and \( w \in W \). So, \( g^*(W) \) is orthogonal to \( Ker(g_* : V \rightarrow W) \). We saw that \( g^*(W^-) \subset V^- \). By Proposition 8.3 (i) and Part (i) one has \((1-\delta)\) \( V^- \subset V^- \cap Ker(g_*) \). The subspace \((\delta + q - 1)\) \( V^- \), with \( q = 2^{n-1} \), is the orthogonal complement of \((1-\delta)\) \( V^- \) with respect to \( E_\Xi = \frac{1}{2}(E_{JX}|_{V^-}) \). Therefore \( g^*(W^-) \subset (\delta + 2^{n-1} - 1)\) \( V^- \), which proves (ii). \( \square \)

Remark 3.9. We notice that the assumption \( p : C \rightarrow Y \) is a simple \( B_n \)-covering, which we made for simplicity at the beginning of this section, was essentially used only in [3.1] for calculating various dimensions using the Hurwitz formula. The constructions and the proofs of the statements in §3.2, 3.8 hold under the more general assumption that \( p : C \rightarrow Y \) is an arbitrary \( B_n \)-covering, in particular the nonsingular curves \( C \) and \( X \) might be reducible. Indeed the proofs of the statements are based on identities between various correspondences. One verifies these identities on the generic fibers of \( p : C \rightarrow Y \) and \( f : X \rightarrow Y \), as done in the text above, then taking closures obtains the identities on the whole curves.
4. Dual Abelian varieties and the spinor weight

4.1. We first recall some material from [BL1] (see also [BL2], Section 14.4 and [K3], §3.4). Let \((P, L)\) be a polarized Abelian variety, \(P = V/\Lambda\), let \(E : \Lambda \times \Lambda \to \mathbb{Z}\) be the Riemann form and let \((d_1, \ldots, d_p)\) be the polarization type of \(L\). One has \(V \cong \Lambda_R := \Lambda \otimes \mathbb{R}\) as \(\mathbb{R}\)-vector spaces. Let \(I : \Lambda_R \to \Lambda_R, \ I^2 = -id\) be the operator defining the complex structure of \(V\). The dual Abelian variety \(\hat{P}\) is isomorphic to \(\text{Hom}_R(\Lambda, \mathbb{R})/\text{Hom}_\mathbb{Z}(\Lambda, \mathbb{Z})\) where the complex structure on \(\text{Hom}_R(\Lambda, \mathbb{R})\) is defined by the unique operator \(J\) such that \(\langle J\omega, Iv \rangle = \langle \omega, v \rangle\). The canonical homomorphism \(\varphi_L : P \to \hat{P}\) is given by the \(\mathbb{C}\)-linear map \(\varphi : \Lambda_R \to \text{Hom}_R(\Lambda, \mathbb{R})\), where \(\varphi(v) = E(v, -)\). Let \(\Lambda^*\) be the dual lattice of \(\Lambda\), i.e.

\[ \Lambda^* = \{v \in V|E(v, \lambda) \in \mathbb{Z} \text{ for } \forall \lambda \in \Lambda\}. \]

Then \(\varphi\) induces an isomorphism \(V/\Lambda^* \sim \hat{P}\). The dual polarization \(L_{\delta}\) on \(\hat{P}\) is given by the unique Riemann form \(\hat{E}\) such that \(\varphi^*\hat{E} = d_1d_pE\). So, the pair \((\hat{P}, L_{\delta})\) is isomorphic to \((V/\Lambda^*, d_1d_pE)\).

If \((\gamma_1, \ldots, \gamma_p, \gamma_{p+1}, \ldots, \gamma_{2p})\) is a symplectic basis of \(\Lambda\), \(E(\gamma_i, \gamma_{p+j}) = \delta_{ij}d_i\), then

\[ \left( -\frac{1}{d_p}\gamma_2p, \ldots, -\frac{1}{d_1}\gamma_{p+1}, \frac{1}{d_p}\gamma_p, \ldots, \frac{1}{d_1}\gamma_{2p} \right) \]

is a symplectic basis of \(\Lambda^*\) with respect to \(d_1d_pE\). So, the polarization type of \(\hat{E}\) is

\[ (d_1, \frac{d_1d_p}{d_{p-1}}, \ldots, \frac{d_1d_p}{d_{p-1}}, \ldots, d_p). \]

Let \((A = V/\Lambda, \Theta)\) be a principally polarized Abelian variety. Let \(\delta : A \to A\) be an endomorphism, which is symmetric with respect to the Rosati involution of \((A, \Theta)\) and satisfies the equation \((\delta - 1)(\delta - q + 1) = 0\) for some \(q \geq 2\). Let \(P = (1 - \delta)A\). Let \(E\) be the Riemann form of \(\Theta\) and let \(E_P\) be the Riemann form of the restricted polarization \(\Theta|_P\). The next proposition is a generalization to arbitrary polarizations of some material well known in the case of principal polarizations (see [BM], Section 7 and [K2], Proposition 2.4).

Proposition 4.2. Let \((Z = V_Z/\Lambda_Z, \theta)\) be a polarized Abelian variety with polarization type \((d_1, \ldots, d_p)\) and let \(E_\theta\) be its Riemann form. Let \(f : A \to Z\) be a surjective homomorphism which satisfies

\[ E_\theta(f(\alpha), f(\beta)) = E((1 - \delta)(\alpha), \beta) \]

for \(\forall \alpha, \beta \in \Lambda\).

Then \(\mu = \varphi_{\Theta}^{-1} \circ f : \hat{Z} \to A\) transforms \(Z\) onto \(P\), \(f|_P : P \to Z\) and \(\mu : \hat{Z} \to P\) are isogenies, and

\[ \mu^*E_P = \frac{q}{d_1d_p}E_\theta. \]

Furthermore \(\hat{Z} \to P\) is an isomorphism if and only if \(\tilde{f} : \Lambda \to \Lambda_Z\) is surjective.

Proof. One has \(P = V^\perp/\Lambda \cap V^\perp\), where \(V^\perp\) is the eigenspace of \(\tilde{\delta} : V \to V\) with eigenvalue \(1 - q\). Let \(\phi = \tilde{f} : V \to V_Z\). Define \(\phi : V_Z \to V\) by

\[ E(\phi(x), w) = E_Z(x, \phi(w)) \]

for all \(x \in V_Z\) and \(w \in V\). This is a \(\mathbb{C}\)-linear map which induces \(\varphi_{\Theta}^{-1} \circ f \circ \varphi_{\Theta} : Z \to A\). Claim. The following properties hold:
(a) \( {}^t\phi : V \to V_Z \) is injective,
(b) \( {}^t\phi \circ \phi = 1 - \delta \),
(c) \( \phi \circ \delta = (1 - q)\phi \),
(d) \( \phi \circ {}^t\phi = q \cdot \text{id} \).

Indeed, \( {}^t\phi(x) = 0 \) iff \( E(\phi(x), w) = 0 \) for all \( w \in V \). Since \( E(\phi(x), w) = E_Z(x, \phi(w)) \) and \( \phi : V \to V_Z \) is epimorphic by hypothesis, one obtains \( x = 0 \). One has

\[
E(\phi(x), w) = E_\theta(\phi(v), \phi(w)) = E((1 - \delta)v, w)
\]

for all \( v, w \in V \). Therefore \( {}^t\phi \circ \phi = 1 - \delta \). In order to prove (c) it suffices to verify that

\[
{}^t\phi \circ \phi \circ \delta = (1 - q)\phi \circ \phi \circ \phi.
\]

This follows from (b) and the equation \((\delta - 1)(\delta + q - 1) = 0\). In order to prove (d) it suffices to verify that \( \phi \circ {}^t\phi \circ \phi = q\phi \). This follows from (b) and (c). The claim is proved.

Using (b) and (a) one has that \( {}^t\phi : V_Z \to V^* \) is an isogeny, so \( \mu : \hat{Z} \to P \) is an isogeny. Furthermore \( \text{Ker} \phi = \text{Ker}(\phi \circ \phi) = \text{Ker}(1 - \delta) = (\delta + q - 1)V \).

Therefore \( f|_P : P \to Z \) is an isogeny. One has

\[
E(\phi(x), {}^t\phi(y)) = E_\theta(x, \phi \circ {}^t\phi(y)) = q E_\theta(x, y)
\]

Since \( \phi \) induces \( \nu^{-1}_\Theta \circ \tilde{\phi} \circ \varphi = \mu \circ \varphi \) we conclude that \( \nu_\Theta(\mu \circ E_P) = q E_\theta \). This implies \( \mu \circ E_P = \frac{-q}{\delta} E_\theta \) (cf. [BL2], Proposition 2.4.3). The last statement follows from the isomorphism (cf. [BL2], Proposition 2.4.3)

\[
\text{Ker}(\tilde{f} : \hat{Z} \to A) \cong \text{Hom}(\Lambda_Z/\tilde{f}(\Lambda), C^*)
\]

\( \square \)

4.3. Recall that if \( \hat{Y} \to Y \) is a double covering of smooth, projective, irreducible curves then \( \dim P(\hat{Y}, Y) = g(\hat{Y}) - g(Y) \) and the restriction of the canonical polarization \( \Theta_{P} \) on \( P(\hat{Y}, Y) \) has type \( (2, \ldots, 2) \) if \( \hat{Y} \to Y \) is unramified and type \( (1, \ldots, 1, 2, \ldots, 2) \) if \( \hat{Y} \to Y \) is ramified, where 2 appears \( g(\hat{Y}) \) times and 1 appears \( g(Y) \) times (cf. [Fay], [Mum]). Suppose \( p : C \xrightarrow{\pi} C' \xrightarrow{g} \mathbb{P}^1 \) is a simply ramified \( B_n \)-covering with irreducible \( C \). Let \( \mathcal{D} = \mathcal{D}_s \cup \mathcal{D}_r \) be the splitting of the discriminant locus as in [2.1].

Let \( P' = P(C, C') \) and let \( E_{P'} \) be the Riemann form of the restriction \( \Theta_{JC|P'} \). Calculating the genera of \( C \) and \( C' \) by the Hurwitz formula one obtains that the polarization type \( (d_1, \ldots, d_p) \) of \( \Theta_{JC|P'} \) is \( (2, \ldots, 2) \) if \( |\mathcal{D}_s| = 0 \) or \( |\mathcal{D}_s| = 2 \) and \( (1, \ldots, 1, 2, \ldots, 2) \) where 1 appears \( \frac{1}{2} |\mathcal{D}_s| - 1 \) times and 2 appears \( \frac{1}{2} |\mathcal{D}_r| + 1 - n \) times.

4.4. Let \( P' \) and \( E_{P'} \) be the dual Abelian variety and the dual polarization. Consider the covering \( f : X \to \mathbb{P}^1 \) of degree \( 2^n \) associated with the spinor weight \( \omega_n \) as in [2.1].

Let \( P = P(X, \delta) \) be the Prym-Tyurin variety. Here \( A = JX, P = (1 - \delta)JX \) and \( q = 2^{n-1} \). The correspondence \( S_0 \in \text{Div}(X \times C) \) constructed in [2.1] induces a homomorphism \( s_0 : JX \to P' = P(C, C') \).

We claim the hypothesis of Proposition 4.2 is satisfied for \( s_0 : JX \to P' \), where \((A, \Theta) = (JX, \Theta_{JX}), (Z, \theta) = (P', \Theta_{JC|P'}) \). Abusing notation we will write \( s_0 \) instead of \( S_0 \), or instead of the induced homomorphism on homology \( s_0 \), when things are clear from the context.

By Lemma 2.15 \( s_0 : JX \to JC \) is surjective. Furthermore a standard fact is that the homomorphisms \( s_0 : JX \to JC \) and \( {}^t s_0 : JC \to JX \) induced by \( S_0 \in \mathcal{O} \).
Proposition 4.6. Let $\Theta$ forms of $P$ and $C$ and $C$ is irreducible. Let $\Theta$ forms of $\Theta_JX|_{d, \Theta_JX|_{d}}$ respectively and let $(d_1, \ldots, d_p)$ be the polarization type of $\Theta_JX|_{d}$. Then the homomorphism $\varphi^{-1} \circ \delta_0 : \hat{P} \rightarrow JX$ yields an isogeny $\mu : \hat{P} \rightarrow P$ such that

$$\mu^{*}E_p = \frac{2^{n-1}}{d_{1}d_{p}} \hat{E}_{p}.$$ 

Conjecture 4.5. Assume $p : C \xrightarrow{\pi} C' \xrightarrow{\varrho} \mathbb{P}^1$ is a simply ramified $B_n$-covering and $C$ is irreducible. Let $\hat{P}'$ be the dual of the Prym variety $P(C, C')$. Then the homomorphism $\mu : \hat{P}' \rightarrow P(X, \delta)$ is an isomorphism.

Equivalently, the polarization type of $\Theta_JX|_{d(X, \delta)}$ is $(2^{n-2}, \ldots, 2^{n-2})$ if $|D_s| = 0$ or $|D_s| = 2$ and $(2^{n-2}, \ldots, 2^{n-2}, 2^{n-1}, \ldots, 2^{n-1})$ where $2^{n-2}$ appears $\frac{1}{2}|D_s| + 1 - n$ times and $2^{n-1}$ appears $\frac{1}{2}|D_s| - 1$ times, if $|D_s| > 2$.

The case $n = 2$ is known to be true and is due to Mumford [Mum] (the case $|D_s| = 0$), Dalalyan [Da] (the case $|D_s| = 2$) and Pantazis [Pa] (the case $|D_s| > 2$).

We include a simple proof in Proposition 5.2. We give a proof in the case $n = 3, |D_s| > 0$ in Theorem 5.1 below. The case of étale $\pi : C \rightarrow C'$ is verified in the next proposition. Conjecture 4.5 remains open for simple $B_n$-coverings of $\mathbb{P}^1$ with $n \geq 4$ (cf. Remark 2.12).

Proposition 4.6. Let $p : C \xrightarrow{\pi} C' \xrightarrow{\varrho} \mathbb{P}^1$ be a simply ramified $B_n$-covering with irreducible $C$. Assume $\pi : C \rightarrow C'$ is étale. Let $f : X \rightarrow \mathbb{P}^1$ be the covering of degree $2^n$ associated with the spinor weight. Then $\Theta_JX|_{d(X, \delta)}$ has polarization type $(2^{n-2}, \ldots, 2^{n-2})$.

Proof. By Corollary 2.12 the monodromy group of $p : C \rightarrow \mathbb{P}^1$ is $W(D_n)$. Hence $X = X_0 \cup X_1$, where $X_0 \rightarrow \mathbb{P}^1$ and $X_1 \rightarrow \mathbb{P}^1$ are coverings of degree $2^n$ which correspond to the two semispinor weights of the root system of type $D_n$ (cf. [K2]). One has $JX = JX_0 \times JX_1$. Let $P = P(X, \delta)$. The restriction of $s_0 : (JX_0 \times JX_1) \rightarrow P(C, C')$ on $JX_0$ and $JX_1$ is studied in Section 8.6 of [K2], where correspondences on $X_i$ of exponent $2^{n-3}$ and associated endomorphisms $\delta_i : JX_i \rightarrow JX_i$ are defined. Let $P' = P(C, C')$. From the polarized isomorphism of $P'$ with the Prym-Tyurin varieties $P(X_0, s_0)$ and $P(X_1, s_1)$ proved there (see [K2], Proposition 6.1.4 and Theorem 6.4(iii)), it follows that $(s_0|_{X_0})_s : H_1(JX_0, \mathbb{Z}) \rightarrow H_1(P', \mathbb{Z})$ is surjective for $i = 0$ or $i = 1$. Applying Theorem 1.1 and Proposition 1.2 we conclude that $\mu : \hat{P}' \rightarrow P$ is an isomorphism and $\mu^{*}E_p = \frac{2^{n-1}}{d_{1}d_{p}} \hat{E}_{p}$. Therefore $E_p$ has type $(2^{n-2}, \ldots, 2^{n-2})$. \qed
5. Coverings with monodromy group contained in \(W(R)\), with \(\text{rank}(R) \leq 3\)

### 5.1. Bigonal construction, \(W = W(B_2)\)

Let \(W = W(R)\) where \(R\) is of type \(B_2\). Consider the two fundamental weights \(\omega_1\) and \(\omega_2\). We are in the situation of \([\text{4.1}]\) and \([\text{4.13}]\) Let \(p : C \to C' \to Y\) be a simple \(B_2\)-covering with irreducible \(C\). Equivalently, \(\deg \pi = \deg g = 2\), \(\mathcal{D}_s = g(Discr(C \to C'))\) and \(\mathcal{D}_t = Discr(C' \to Y)\) are nonempty, and \(\mathcal{D}_s \cap \mathcal{D}_t = \emptyset\).

Let \(f : X \to Y\) be the covering of degree 4 associated with \(\omega_2\). The monodromy group of \(p : C \to Y\) is \(W(B_2)\), so \(X\) is irreducible. Let \(\sigma : X \to X\) be the involution defined in \([\text{4.1}]\) and let \(f : X \to X' \to Y\) be the corresponding decomposition of \(f\). Calculating the action of the reflections on \(\omega_2\) one verifies that \(Discr(X' \to Y) = \mathcal{D}_s\) and \(g'(Discr(X \to X')) = \mathcal{D}_t\). Here the isogenous Prym-Tyurin varieties are the ordinary Prym varieties \(P(C, C')\) and \(P(X, X')\).

#### Proposition 5.2.

(i) The polarization types of \(\Theta_{JC}|_{P(C, C')}\) and \(\Theta_{JX}|_{P(X, X')}\) are respectively

- \((1, 1, 2, 2, 2, 2, 2, 2, 2, 2),\)
- \(\frac{1}{2}|\mathcal{D}_s|-1 \frac{1}{2}|\mathcal{D}_t|-1 2g(Y)\)

(ii) If \(Y = \mathbb{P}^1\), then \(\mu : \hat{P}(C, C') \to P(X, X')\) defined in Theorem 4.1 is an isomorphism.

**Proof.** (i) This follows from the Hurwitz formula (cf. \([\text{4.13}]\)).

(ii) Let \(P = P(X, X')\) and \(P' = P(C, C')\). Let us consider the isogeny \(\mu : \hat{P}' \to P\) from Theorem 4.1. Let \(E_P\) and \(E_{P'}\) be the Riemann forms of \(\Theta_{JX}|_P\) and \(\Theta_{JC}|_{P'}\) respectively. Suppose first that \(|\mathcal{D}_s| > 2\) and \(|\mathcal{D}_t| > 2\). By (i) the polarization type of \(E_P\) is the same as that of \(E_{P'}\). We have by Theorem 4.1 that \(\mu^*E_P = E_{P'}\). Therefore \(\mu : P' \to P\) is a polarized isomorphism.

Let \(|\mathcal{D}_s| = 2\). Then \(X' \cong \mathbb{P}^1, P(X, X') = JX\) and \(E_P = E_{JX}\) is a principal polarization. Here \(E_{P'} = 2E_{\Xi}\) for a principal polarization \(\Xi\) on \(P' = P(C, C')\), \(E_{P'}\) has type \((2, \ldots, 2)\) and \(\psi\) of the decomposition

\[
\varphi_{\Xi} : P' \xrightarrow{2\mu} P' \xrightarrow{\psi} \hat{P}'
\]
determines a polarized isomorphism between \((P', E_{\Xi})\) and \((\hat{P}', \frac{1}{2}E_{P'})\). By Theorem 4.1 one has \(\mu^*E_{JX} = \frac{1}{2}\hat{E}_{P'}\). Therefore \(P' \cong \hat{P}' \cong JX\).

Let \(|\mathcal{D}_t| = 2\). Then \(C' \cong \mathbb{P}^1, P' = JC\) and \(E_P = 2E_{\Xi}\) for a principal polarization \(\Xi\). Identifying \(JC\) with its dual \(\hat{P}'\) we have for \(\mu : JC \to P\) the formula \(\mu^*E_P = 2E_{JC}\). Therefore \(\mu\) is a polarized isomorphism of \(JC\) with \((P(X, X'), \Xi)\).

**Remark 5.3.** The isomorphism \(\hat{P}(C, C') \cong P(X, X')\) is due to Pantasis [Pa] and the isomorphism \(P(C, C') \cong JX\), when \(|\mathcal{D}_s| = 2\), is due to Dalalyan [Da]. Our proof, based on Theorem 4.1, is new and simpler than the original ones.

#### 5.4. Recillas’ construction, \(W = W(A_3) = W(D_3)\)

Let \(p : C \to Y\) be a simply ramified covering of degree 4 with irreducible \(C\). Suppose it cannot be decomposed through an étale covering \(C' \to Y\) of degree 2. Then the covering is
primitive, so its monodromy group is $S_4$ (cf. \[23\]). Furthermore $p^* : JY \to JC$ is injective, $A = Ker(Nm_\rho)$ is connected, and the polarization type of $\Theta_{JC\mid A}$ is $(1,\ldots,1,4,\ldots,4)$ where 4 appear $g(Y)$ times (cf. [BL2], Corollary 12.1.5). In the set-up of the introduction we have $W = S_4 = W(R)$, where $R$ is of type $A_3$ and $\lambda = \omega_1$. Replacing $\lambda$ by the second fundamental weight $\lambda' = \omega_2$ one obtains a covering $f : X \xrightarrow{\pi} X' \xrightarrow{g} Y$, where $X$ is irreducible, $p$ is étale of degree 2 and $g$ is a simply ramified covering of degree 3. Constructing an appropriate correspondence (cf. [K2], §8.5.3) one easily verifies that $X$ is isomorphic to a curve in $C(2)$, namely the closure of $\{x + y \in C(2) \mid x \neq y, p(x) = p(y)\}$. This construction, when $Y \cong \mathbb{P}^1$, is due to Recillas \[Re\]. Here we have $i = 0, P(C,i) = A$ and $\Theta_{JC\mid P(C,i)}$ has polarization type as above, while $P(X,\delta)$ is the ordinary Prym variety $P(X,X')$, so $\Theta_{JC\mid P(X,X')}$ has polarization type $(2,\ldots,2)$. Summarizing one obtains the following statement, due to Recillas when $Y \cong \mathbb{P}^1$ \[Re\].

**Proposition 5.5.** Let $p : C \to Y$ be a simply ramified covering of degree 4 with irreducible $C$, which cannot be decomposed through an étale covering of degree 2. Let $f : X \xrightarrow{\pi} X' \xrightarrow{g} Y$ be the associated covering of degree 6 obtained by the Recillas construction. Then $A = Ker(Nm_\rho : JC \to JY)$ and $P(X,X')$ are isogenous Abelian varieties and the polarization types of $\Theta_{JC\mid A}$ and $\Theta_{JC\mid P(X,X')}$ are respectively

$$(1,\ldots,1,4,\ldots,4),$$

$$(2,\ldots,2,\ldots,2)$$

If $Y \cong \mathbb{P}^1$, then $J(C)$ and $P(X,X')$ are isomorphic as principally polarized Abelian varieties.

**5.6.** Replacing $\lambda = \omega_1$ by $\lambda' = \omega_3$ one obtains a covering $f : X \to Y$ which is equivalent to $p : C \to Y$, since $-id$ is $W(A_3)$-equivariant and transforms $\omega_1$ to $\omega_3$. Hence there is a polarized isomorphism between $(A,\Theta_{JC\mid A})$ and $(A',\Theta_{JC\mid A'})$, where $A' = Ker(Nm_\rho : JX \to JY)$.

Inverting Recillas’ construction one covers the case $W = W(D_3), \lambda = \omega_1, \lambda' = \omega_2$ or $\lambda' = \omega_3$ where $\omega_1, \omega_2, \omega_3$ are the fundamental weights of a root system of type $D_3$.

**5.7.** The next case we consider is $W = W(R)$, where $R$ is of type $B_3$, $\lambda = \omega_1$ and $\lambda' = \omega_3$ (cf. Section \[2\]). Let $p : C \xrightarrow{\pi} C' \xrightarrow{p'} Y$ be a simple $B_3$-covering. Since $\deg p' = 3$, so $p' : C' \to Y$ is primitive, the simplicity of the covering $p$ is equivalent to the condition that $p : C \to Y$ is a simply ramified $B_3$-covering and both $\pi : C \to C'$ and $p' : C' \to Y$ are ramified. Let $f : X \to Y$ be the covering of degree 8 associated with $\omega_3$. The curve $X$ is irreducible, since the monodromy group of $p : C \to Y$ is $W(B_3)$ by Corollary \[2.8\]. Let $f : X \xrightarrow{g} \tilde{Y} \xrightarrow{h} Y$ be the decomposition defined in \[3.6\].

**Lemma 5.8.** The following equalities hold

(i) $P(X,\delta) = Ker(Nm_\rho : P(X,X') \to P(\tilde{Y},Y))^0$

(ii) $(\delta + 3)P(X,X') = g^*P(\tilde{Y},Y)$.

**Proof.** In Proposition \[2.8\] we proved inclusions, so it only remains to compare the dimensions. We have $\dim P(X,\delta) = \dim P(C,C')$ since these are isogenous Abelian
The restriction $\Theta_{\ref{5.3}}$ for $n = 3$ we conclude that $\dim P(X, \delta) = \dim P(X, X') - P(Y, Y)$, so (i) holds.

One has $P(X, \delta) = (1 - \delta)P(X, X') + (\delta + 3)P(X, X')$, and the two Abelian varieties on the right intersect in a finite number of 4-torsion points. Therefore $\dim(\delta + 3)P(X, X') = \dim P(Y, Y)$. Hence (ii) holds as well.

**Theorem 5.1.** Let $p : C \to C' \to Y$ be a simple $B_3$-covering. Let $f : X \to Y$ be the covering of degree 8 associated with the spinor weight and let $P(X, \delta)$ be the Prym-Tyurin variety. Then the polarization types of $\Theta_{\ref{JC}}|_{P(C, C')}$ and $\Theta_{\ref{JX}}|_{P(X, \delta)}$ are respectively

\begin{align}
(5.1) \quad & (1, \ldots, 1, 2, \ldots, 2, 2, \ldots, 2) \\
& [\mathfrak{D}_8, -1] \quad [\mathfrak{D}_8 + 2g(Y) - 2] \quad g(Y) \\
\end{align}

\begin{align}
(5.2) \quad & (2, \ldots, 2, 4, \ldots, 4, 8, \ldots, 8) \\
& [\mathfrak{D}_8 + 2g(Y) - 2] \quad [\mathfrak{D}_8, -1] \quad g(Y) \\
\end{align}

If $Y \cong \mathbb{P}^1$, then Conjecture $\ref{J\delta}$ holds: $P(X, \delta)$ is isomorphic to the dual of $P(C, C')$.

Proof. By $\ref{3.1}$ one has $g(C') = \frac{1}{4}\mathfrak{D}_8 + 3g(Y) - 2$, so (5.1) follows from the well known fact about ordinary Prym varieties (cf. $\ref{3.3}$. By $\ref{3.0}$ the covering $g : X \to \tilde{Y}$ of degree 4 is simply branched in $h^{-1}(\mathfrak{D}_8)$ and has monodromy group $W(D_3) \cong S_4$. Therefore $g$ cannot be decomposed through an étale covering of $\tilde{Y}$, so $g^* : J\tilde{Y} \to JX$ is injective. We have a commutative diagram

\begin{align}
& \xymatrix{ JX \ar[r]^{g^*} \ar[d] & J\tilde{Y} \ar[d] \\
P(X, \sigma) \ar[r]^{g^*} & P(\tilde{Y}, Y) }
\end{align}

The restriction $\Theta_{\ref{J\tilde{Y}}}|_{P(\tilde{Y}, Y)}$ has type $(1, \ldots, 1, 2, \ldots, 2)$ where 1 appears $\frac{1}{2}\mathfrak{D}_8 - 1$ times and 2 appears $g(Y)$ times. The pull-back of $\Theta_{\ref{JX}}$ by $g^*$ is $4\Theta_{\ref{J\tilde{Y}}}$. The involution $\sigma : X \to X$ is fixed point free, so $\Theta_{\ref{JX}}|_{P(X, X')} = 2\Xi$ for a principal polarization $\Xi$. Using $\ref{5.3}$, the fact that $g^*$ is injective and the equality $(\delta + 3)P(X, X') = g^*P(\tilde{Y}, Y)$ of Lemma $\ref{5.8}$ we conclude that $\Xi|_{\mathfrak{D}_8 + 3g(Y)}$ has polarization type $(2, \ldots, 2, 4, \ldots, 4)$ where 2 appears $\frac{1}{2}\mathfrak{D}_8 - 1$ times and 4 appears $g(Y)$ times.

Recall that a polarization $L$ on an Abelian variety $B$ determines $\varphi_L : B \to \hat{B}$ and $K(L) := Ker(\varphi_L)$. If $(A, \Theta)$ is a principally polarized Abelian variety with a Prym-Tyurin endomorphism $\delta$ and if $B = (\delta + g - 1)A, P = (1 - \delta)A$, then $B$ and $P$ are a pair of complementary Abelian subvarieties of $A$. Hence the finite groups $K(\Theta|_{\ref{J}})$ and $K(\Theta|_{\ref{P}})$ are isomorphic according to $\ref{11.2}$, Corollary 12.1.5. Applying this to $(A, \Theta) = (P(X, X'), \Xi)$ we conclude that the polarization of $\Xi|_{\mathfrak{P}(X, \delta)}$ is $(1, \ldots, 1, 2, \ldots, 2, 4, \ldots, 4)$ where the lengths of the sequences $(2, \ldots, 2), (4, \ldots, 4)$ and $(1, \ldots, 1)$ are respectively $\frac{1}{2}\mathfrak{D}_8 - 1, g(Y)$ and $\dim P(X, \delta) - \dim(\delta + 3)P(X, X')$. The latter number equals $\dim P(X, \delta) - P(Y, Y) = \frac{1}{2}\mathfrak{D}_8 + 2g(Y) - 2$. Since $\Theta_{\ref{JX}}|_{P(X, X')} = 2\Xi$ we obtain, restricting on $P(X, \delta)$ the polarization type of $\ref{3.2}$.
Let $Y = \mathbb{P}^1$. Let $P = P(X, \delta)$ and $P' = P(C, C')$. One has $|\mathcal{D}_t| \geq 4$ since $p' : C' \to \mathbb{P}^1$ is a triple covering, simply ramified in $\mathcal{D}_t$. By hypothesis $\pi : C \to C'$ is ramified, so $|\mathcal{D}_t| \geq 2$.

Suppose first $|\mathcal{D}_s| > 2$ and $|\mathcal{D}_t| > 4$. By Theorem 4.1 we have $\mu^* E_P = 2E_{P'}$. By (5.1) and (5.2), applied with $g(Y) = 0$, the polarization types of $E_P$ and $2E_{P'}$ are the same. Therefore $\mu : P' \to P$ is an isomorphism.

Let $|\mathcal{D}_s| = 2$. Then $Y \cong \mathbb{P}^1$, $P = P(X, X')$, and $E_P = 2E_\Xi$ for some principal polarization $\Xi$ on $P$. Since $\pi : C \to C'$ is ramified in two points we have $E_{P'} = 2E_{\Xi'}$ for some principal polarization $\Xi'$ of $P' = P(C, C')$. By Theorem 4.1 we have $\mu^* E_P = E_{P'}$. Therefore $\mu^* E_\Xi = E_\Xi'$ and we have polarized isomorphisms $(P(C, C'), E_{\Xi'}) \cong (P(C, C'), E_{\Xi'}) \cong (P(X, X'), E_\Xi)$.

We notice that in this example the correspondence $(\mu, X, \delta)$ transforms isomorphically $\mathcal{P}_\Xi$ for some principal polarization $\Xi$, according to (5.2). By Theorem 4.1 we have $\mu^* E_P = 4E_{JC}$. Identifying $JC$ with its dual we obtain a polarized isomorphism $(JC, \Theta_{JC}) \cong (P(X, \delta), \Xi)$. □

The polarized isomorphism $P(C, C') \cong P(X, X')$ is a particular case of the tetragonal construction of Donagi. The limit case $|\mathcal{D}_t| = 4$ yields a new class of principally polarized Prym-Tyurin varieties.

**Corollary 5.9.** Let $\pi : C \to \mathbb{P}^1$ be a hyperelliptic curve. Let $p' : C' \to \mathbb{P}^1$ be a simple triple covering, so that $p : C \xrightarrow{\pi} C' \xrightarrow{p'} \mathbb{P}^1$ is a simply ramified $B_3$-covering (cf. §2.7). Let $f : X \to \mathbb{P}^1$ be the associated covering of degree $8$ (cf. §2.7), and let $\delta : JX \to JX$ be the associated Prym-Tyurin endomorphism, satisfying $(\delta - 1)(\delta + 3) = 0$. Then the Prym-Tyurin variety $P(X, \delta) = (1 - \delta)JX$ has a principal polarization $\Xi$ with the property that $\Theta_{JX|P(X, \delta)} \equiv 4\Xi$. Furthermore $(P(X, \delta), \Xi) \cong (JC, \Theta_{JC})$.

**Remark 5.10.** We notice that in this example the correspondence $D \in \text{Div}(X \times X)$ has 8 fixed points, two ones in each of the fibers $f^{-1}(y)$ where $y$ belongs to the branch locus of $p' : \mathbb{P}^1 \to \mathbb{P}^1$. The equality $\Theta_{JX|P(X, \delta)} \equiv 4\Xi$ cannot be deduced from Ortega’s criterion [Or], since this criterion requires the existence of 4 fixed points $p_1, \ldots, p_4$ of $D$ which satisfy $p_1, \ldots, p_4 \in D(p_k)$ and $p_k \notin D(p_k) - p_k$ for $k = 1, \ldots, 4$. Hence these 4 points have to belong to a single fiber of $f : X \to \mathbb{P}^1$, which is not the case in Corollary 5.9.

5.11. We studied in §§5.7, 5.10 simple $B_3$-coverings. Let us consider another class of simply ramified $B_3$-coverings. Namely, let $p : C \xrightarrow{\pi} C' \xrightarrow{p'} Y$ be with unramified $\pi$ and let $p$ have monodromy group $W(D_3)$ (cf. Corollary 2.8). Since $W(D_3)$ has two orbits when acting on $W_3$, namely $\{\lambda_A\}_{A \text{ even}}$ and $\{\lambda_A\}_{A \text{ odd}}$, the covering $f : X \to Y$ splits into disjoint union

$$f : X_1 \sqcup X_2 \to Y.$$ Let $f_i = f|_{X_i}$. The canonical involution $\sigma : X \to X$ associated with $-id : W_3 \to W_3$ transforms isomorphically $X_1$ and $X_2$ into each other. If $x \in X_i$ we have

$$D(x) = f_i^*(f_i(x)) - x + 2\sigma(x) \quad (5.4)$$

Let $\varphi : Z \to Y$ be a fixed covering of degree 4 equivalent to any of $f_i : X_i \to Y$, $i = 1, 2$. The covering $\varphi : Z \to Y$ is simply ramified with monodromy group
Proof. Let $E = P$. By Lemma 5.12 decomposition $\pi$ universal covering, we have so (1
\begin{align*}
\Delta^-(JY \times JY) := \{(\lambda, -\lambda) | \lambda \in JY\}. 
\end{align*}
One obtains an exact sequence
\begin{align*}
0 \longrightarrow B \times B \longrightarrow A \longrightarrow JY \longrightarrow 0
\end{align*}

Lemma 5.12. The Prym-Tyurin variety $P(X, \delta) = (1 - \delta)A$ equals the antidiagonal of $B \times B$, $P(X, \delta) = \Delta^-(B \times B)$.

Proof. If $(\lambda, \mu) \in B \times B$, then (5.4) yields $(1 - \delta)(\lambda, \mu) = 2(\lambda - \mu, \mu - \lambda)$, so
\begin{align*}
(1 - \delta)\Delta(B \times B) = 0, \quad (1 - \delta)\Delta^-(B \times B) = \Delta^-(B \times B)
\end{align*}
Let us apply the results of Section 3 to the type of coverings we consider (see Remark 3.3). Here $\tilde{Y} = Y \cup Y$, the covering $g : X \to \tilde{Y}$ is $\varphi \cup \varphi : Z \cup Z \to Y \cup Y$, $P(X, X') = \Delta^-(JZ \times JZ)$, $P(Y, Y) = \Delta^-(JY \times JY)$. One has the canonical decomposition $JZ = \varphi^*(JY) + B$, so
\begin{align*}
\Delta^-(JZ \times JZ) = (\varphi^* \times \varphi^*)\Delta^-(JY \times JY) + \Delta^-(B \times B)
\end{align*}
By Proposition 3.8 (ii) one has
\begin{align*}
(\varphi^* \times \varphi^*)\Delta^-(JY \times JY) \subset (\delta + 3)\Delta^-(JZ \times JZ)
\end{align*}
so $(1 - \delta)$ annihilates this Abelian subvariety. By Proposition 3.3 (ii) one has $P(X, \delta) = (1 - \delta)\Delta^-(JZ \times JZ)$, so using 5.3 we obtain $P(X, \delta) = \Delta^-(B \times B)$.

Proposition 5.13. Let $p : C \xrightarrow{\pi} C' \xrightarrow{\varphi} Y$ be a simply ramified $B_3$-covering with unramified $\pi$ and monodromy group $W(D_3)$. Let $f : X \to Y$ be the associated covering of degree 8. Then the polarization type of $\Theta_{JX}|_{P(X, \delta)}$ is $(2, \ldots, 2, 8, \ldots, 8)$ where 8 appears $g(Y)$ times.

Proof. Let $E_B$ be the Riemann form of $\Theta_{JX}|_B$. Since $\varphi : Z \to Y$ is primitive, the polarization type of $E_B$ is $(1, \ldots, 1, 4, \ldots, 4)$ where 4 appears $g(Y)$ times (cf. 5.4). By Lemma 5.12 $P(X, \delta) = P \subset B \times B$ is the antidiagonal, so denoting by $\tilde{\gamma}$ the universal covering, we have $E_P = (p_1^*E_B + p_2^*E_B)|_{\tilde{\gamma}}$ and for every $u = (x, -x) \in \tilde{B} \times \tilde{B}$ and every $v = (y, -y) \in \tilde{B} \times \tilde{B}$ we have
\begin{align*}
E_P(u, v) = (p_1^*E_B + p_2^*E_B)(x, -x; y, -y) = 2E_B(x, y)
\end{align*}
Therefore the polarization type of $E_P$ is $(2, \ldots, 2, 8, \ldots, 8)$.

6. Rank(R) = 4

6.1. The case $W = W(B_3)$. Let $p : C \xrightarrow{\varphi} C' \xrightarrow{\varrho} Y$ be a simple $B_3$-covering and let $f : X \to Y$ the covering of degree 16 associated with the spinor weight $\omega_4$. The curve $X$ is irreducible, since the monodromy group of the covering $p$ is $W(B_3)$ by Corollary 2.8. The involution $\iota$ on $C$ induces an involution $\sigma$ on the curve $X$ (see 3.1). Denote $X' := X/\sigma$. We cannot say much about the induced polarizations on Prym-Tyurin varieties involved. However we can say something about the relations between these varieties.
For $i = 0, \ldots, 3$ define a correspondence $D_i$ of $X$ by

$$D_i(x_0) = \sum_{|B| = i+1} x_B.$$ 

So $D = \sum_{i=0}^3 iD_i$.

**Proposition 6.2.** If $\delta_i$ denotes the endomorphism of $JX$ associated to $D_i$, we have

$$P(X, \delta) = (\delta_0 + 2)P(X, X').$$

**Proof.** For any $x = x_0 \in X$ we have

$$(D - 1)(x - \sigma(x)) = 2(D_2 - D_0)(x) + 4\sigma(x) - 4x.$$ 

Now $D_2(x) = D_0(\sigma(x))$ implies $(D - 1)(x - \sigma(x)) = 2(D_0 + 2)(\sigma(x) - x)$. According to Proposition 3.5(ii) this gives the assertion. $\square$

According to Proposition 3.5(i), $P(X, \delta)$ is an abelian subvariety of the Prym variety $P(X, X')$. Hence we can speak about its complementary variety with respect to the polarization.

**Proposition 6.3.** The complementary abelian subvariety of $P(X, \delta)$ in $P(X, X')$ is given by

$$(\delta + 7)P(X, X') = (\delta_0 - 2)P(X, X').$$

**Proof.** According to Propositions 3.5(ii) and 1.6 the complementary abelian subvariety is given by $(\delta + 7)P(X, X')$. From the proof of Proposition 6.2 we see that $(D + 7)(x - \sigma(x)) = 2(D_0 - 2)(\sigma(x) - x)$. This implies the last assertion. $\square$

Let $\rho : X \to X'$ denote the double covering associated to the involution $\sigma$. The next proposition describes the endomorphism $(\delta_0 + 2)(\delta_0 - 2)$ on $\rho^*JX'$.

**Proposition 6.4.** For any $x = x_0 \in X$,

$$(D_0 + 2)(D_0 - 2)(x + \sigma(x)) = 4D_1(x).$$

In particular $(\delta_0 + 2)(\delta_0 - 2)(\rho^*(JX')) = \delta_1(JX)$.

**Proof.** For any $x = x_0 \in X$, $(D_0 - 2)(x) = \sum_{|B|=1} x_B - 2x$ and hence

$$(D_0 + 2)(D_0 - 2)(x) = 4x_0 + 2 \sum_{|B|=2} x_B - 2 \sum_{|B|=1} x_B + 2 \sum_{|B|=1} x_B - 4x_0$$

$$= 2D_1(x).$$

Similarly we get $(D_0 + 2)(D_0 - 2)(\sigma(x)) = 2D_1(x)$. Adding both equations gives the assertion. $\square$

**6.5. The case $W = W(D_4)$.** Let $p : C \xrightarrow{\pi} C' \xrightarrow{\varphi} Y$ denote a simply ramified $B_4$-covering with $\pi$ unramified. Suppose moreover that $p$ has monodromy group $W(D_4)$ (cf. Corollary 2.8). We notice that this condition is automatically satisfied if $Y \cong \mathbb{P}^1$ by Corollary 2.9. When $g(Y) \geq 1$ the monodromy group might be $W(B_n)$ or conjugated to $N_{W(B_n)}(G_1)$ (see Remark 2.10), cases which we do not consider in the present paper. As in \cite[6.11]{KanevLange} the curve $X$, associated to the spinor weight $\omega_4$, is the disjoint union of two smooth irreducible curves $X = X_1 \sqcup X_2$. For $i = 1$ and 2 the map $f_i := f|X_i : X_i \to Y$ is an 8 : 1 covering. Using the notation of above, we have for any $x = x_0 \in X$, $f_1^{-1}(f(x)) = \{x_B\}_{|B| \text{ even}}$ and $f_2^{-1}(f(x)) = \{x_B\}_{|B| \text{ odd}}$.
Hence the involution $\sigma : X \to X$ induced by the involution $'$ on $C$ restricts to involutions $\sigma_i = \sigma|X_i : X_i \to X_i$ for $i = 1$ and 2. They induce factorizations of $f_i$, namely

$$f_i : X_i \to X'_{i} \to Y.$$  

The map $X_i \to X'_{i} = X_i/\sigma_i$ is an étale double covering, since $\sigma$ and thus the involution $\sigma_i$ are fixed-point free. The map $X'_{i} \to Y$ is a 4:1 covering, simply ramified exactly over the $|D_i|$ ramification points of $p : C \to Y$. This implies $g(X'_i) = \frac{|D_i|}{2} + 4g(Y) - 3 = g(C')$ and hence

$$\dim P(X_i, X'_i) = \frac{|D_i|}{2} + 4g(Y) - 4 = \dim P(C, C').$$

Consider the correspondence $D$ on $X$ defined in [2.13]

**Lemma 6.6.** For any $y \in Y$ and $i = 1$ and 2 we have

$$D(f_i^*(y)) = 8f_i^*(y) + f_i^{*}(y).$$

**Proof.** We prove this only for $i = 1$, the proof being the same for $i = 2$. Let $x = x_0 = p_1 + \cdots + p_4 \in X$ with $f_1(x) = y$. Moreover let $\Sigma = \{1, 2, 3, 4\}$ and all sets $B$ are subsets of $\Sigma$. Then $f_1^*(y) = x + \sum_{|B|=2} x_B + \Sigma x \Sigma$ and hence

$$Df_1^*(y) = D(x) + \sum_{|B|=2} D(x_B) + D(x_\Sigma)$$

$$= \sum_{|B|=2} x_B + 2\sum_{|B|=3} x_B + 3x_\Sigma + 6x_0 + 6x_\Sigma + 7\sum_{|B|=2} x_B + 2(3\sum_{|B|=1} x_B + \sum_{|B|=3} x_B)$$

$$+ \sum_{|B|=2} x_B + 2\sum_{|B|=1} x_B + 3x_0$$

$$= 9x_0 + 9x_\Sigma + 9\sum_{|B|=2} x_B + 8\sum_{|B|=1} x_B + 8\sum_{|B|=3} x_B$$

$$= 8f_1^*(y) + f_1^{*}(y).$$

Consider the abelian variety $A = \ker(Nm_f : JX \to JY)$ and for $i = 1$ and 2 define $B_i = \ker(Nm_{f_i} : JX_i \to JY_i)$. Then $B_1 \times B_2$ and $Q = \{(f_1^*(m), -f_2^*(m)) \mid m \in JY\}$ are abelian subvarieties of $A$ and it is easy to see that $A = B_1 \times B_2 + Q$.

**Proposition 6.7.** Denoting by $\delta$ also the restriction to $A$ of the endomorphism defined by $D$, we have $(\delta - 1)Q = 0$ and $Q \subset (\delta + 1)A$ in concordance with $q = 8$.

**Proof.** For any divisor $c$ of degree 0 on $Y$ we have

$$(\delta - 1)(f_1^*(c), -f_2^*(c)) = (D - 1)f_1^*(c) - (D - 1)f_2^*(c)$$

$$= 8f_1^*(c) + f_1^*(c) - f_1^*(c) - 8f_1^*(c) - f_2^*(c) + f_2^*(c)$$

$$= 0.$$  

This proves the first assertion. According to Lemma 6.6 $(D - 1)(f_1^*(y)) = 8f_1^*(y)$. This implies $(D + 7)(f_1^*(y)) = 8f_1^*(y) + 8f_1^*(y)$ and similarly $(D + 7)(-f_2^*(y)) = -8f_1^*(y) - 8f_2^*(y)$. Hence $(D + 7)(f_1^*(y), -f_2^*(y)) = 8(f_1^*(y), -f_2^*(y))$ which implies the second assertion.

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V. Kanev, Dipartamento di Matematica ed Applicazioni, Università di Palermo, Via Archirafi n.34, 90123 Palermo, Italy, and Institute of Mathematics of the Bulgarian Academy of Sciences

E-mail address: kanev@math.unipa.it

H. Lange, Mathematisches Institut, Universität Erlangen-Nürnberg, Bismarckstrasse 14, D-91054 Erlangen, Germany

E-mail address: lange@mi.uni-erlangen.de