A REMARK ON Pin(2)-EQUIVARIANT FLOER HOMOLOGY

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Abstract. In this remark, we show how the monopole Frøyshov invariant, as well as the analogues of the Involutive Heegaard Floer correction terms \( d, \bar{d} \), are related to the Pin(2)-equivariant Floer homology \( \text{SWFH}\_G^{\mathbb{Z}} \). We show that the only interesting correction terms of a Pin(2)-space are those coming from the subgroups \( \mathbb{Z}/4, S^1, \) and Pin(2) itself.

1. Introduction

In [6], Manolescu resolved the triangulation conjecture, establishing that there exist non-triangulable manifolds in all dimensions at least 5. The proof relies on the construction of Pin(2)-equivariant Seiberg-Witten Floer homology, where Pin(2) is the group consisting of two copies of the complex unit circle, with a map \( j \) interchanging the two copies and so that \( ij = j_i \) and \( j^2 = -1 \).

Let \( F \) denote the field with two elements. As \( S^1 \)-equivariant monopole Floer homology associates to a three-manifold with spin\(^c\) structure a \( H^*(BS^1) = F[U] \)-module, Pin(2)-equivariant Floer homology associates to a rational homology three-sphere with spin structure a \( H^*(B\text{Pin}(2)) = F[q, v]/(q^3) \)-module, where cohomology is taken with \( F \)-coefficients. From the module structure of Pin(2)-equivariant Floer homology one obtains three invariants of homology cobordism \( \alpha, \beta, \gamma : \theta^3_H \to \mathbb{Z} \), where \( \theta^3_H \) is the integral homology cobordism group of integral homology three-spheres. These invariants satisfy:

\[
\alpha(Y) \equiv \beta(Y) \equiv \gamma(Y) \equiv \mu(Y) \mod 2
\]

and

\[
\alpha(-Y) = -\gamma(Y), \quad \beta(-Y) = -\beta(Y).
\]

In particular, these properties for \( \beta \) show that there is no element \( Y \in \theta^3_H \) of order 2 with \( \mu(Y) = 1 \). Galewski-Stern and Matumoto [1] and [8] showed that there exist nontriangulable manifolds in all dimensions at least 5 if and only if there exists an element \( Y \in \theta^3_H \) of order 2 with \( \mu(Y) = 1 \), from which Manolescu’s disproof of the triangulation conjecture follows.

Let \( G = \text{Pin}(2) \). Since the introduction of Manolescu’s \( G \)-equivariant Floer homology, denoted \( \text{SWFH}\_G^{\mathbb{Z}}(Y, s) \), other versions of Floer homologies with symmetries beyond the \( S^1 \)-symmetry have become available. Lin [4] constructed a Pin(2)-equivariant refinement of monopole Floer homology in the setting of Kronheimer-Mrowka [3]. In the setting of Heegaard Floer homology introduced by Ozsváth-Szabó in [11], [10], Hendricks and Manolescu [2] point out that naturality questions make it difficult to define a \( G \)-equivariant version of Heegaard Floer homology. However, they proceed by considering the subgroup \( \mathbb{Z}/4 = \langle j \rangle \subset G \) and define a Heegaard Floer analogue of \( \text{SWFH}\_G^{\mathbb{Z}} \) with respect to this smaller group, denoted \( \text{HFI}(Y, s) \). As for the above-mentioned theories, \( \text{HFI}(Y, s) \) is a module over \( H^*(B\mathbb{Z}/4) = F[U, Q]/(Q^2) \). Using \( \text{HFI}(Y, s) \), they associate two homology cobordism invariants \( \bar{d}(Y, s), \bar{d}(Y, s) \), from the module structure. However, \( \bar{d} \) and \( \bar{d} \) do not generally reduce to the Rokhlin invariant mod 2.

The purpose of this note is to relate the homology cobordism invariants obtained using theories equivariant with respect to different groups (especially the groups \( S^1, \mathbb{Z}/4 \) and \( G \) itself). In particular, we will see that, roughly speaking, all homology cobordism invariants that are constructed
from Manolescu’s homotopy type using the Borel homology of a subgroup of $\text{Pin}(2)$ are determined by the invariants defined using $\mathbb{Z}/4$, $S^1$ and $G$. For a precise statement, see Theorem 1.5.

We will work in the context of Manolescu’s construction of $G$-equivariant Floer homology, that is, $\text{SWFH}^G$.

We recall that in order to define $\text{SWFH}^G$, Manolescu first associates to a rational homology sphere with spin structure $(Y, \mathfrak{s})$ a $G$-equivariant stable homotopy type, denoted $\text{SWF}(Y, \mathfrak{s})$. Then $\text{SWFH}^G_*(Y, \mathfrak{s})$ is constructed from $\text{SWF}(Y, \mathfrak{s})$ by taking the $G$-equivariant Borel homology

$$\text{SWFH}^G_*(Y, \mathfrak{s}) = \tilde{H}^G_*(\text{SWF}(Y, \mathfrak{s})).$$

Using $\mathbb{Z}/4 = \langle j \rangle \subset G$, we may also consider the $\mathbb{Z}/4$-Borel homology of $\text{SWF}(Y, \mathfrak{s})$. We define

$$\text{SWFH}^{\mathbb{Z}/4}_*(Y, \mathfrak{s}) = \tilde{H}^{\mathbb{Z}/4}_*(\text{SWF}(Y, \mathfrak{s})).$$

Then $\text{SWFH}^{\mathbb{Z}/4}_*(Y, \mathfrak{s})$ has a $H^*(B\mathbb{Z}/4) = \mathbb{F}[U, Q]/(Q^2)$-module structure, from which we will define below homology cobordism invariants $\delta(Y, \mathfrak{s})$, $\hat{\delta}(Y, \mathfrak{s})$, and $\tilde{\delta}(Y, \mathfrak{s})$, which should correspond, respectively, to the invariants $d(Y, \mathfrak{s})/2$ and $\tilde{d}(Y, \mathfrak{s})/2$ of [2]. It is natural to ask to what extent these $\mathbb{Z}/4$ invariants are determined by $\alpha$, $\beta$, and $\gamma$, and, more generally, by the $\mathbb{F}[q, v]/(q^3)$-module structure of $\text{SWFH}^G_*(Y, \mathfrak{s})$. We show in Theorem 1.3 how to partially determine $\delta(Y, \mathfrak{s})$ and $\tilde{\delta}(Y, \mathfrak{s})$ from $\text{SWFH}^G_*(Y, \mathfrak{s})$, but that in general $\hat{\delta}(Y, \mathfrak{s})$ and $\tilde{\delta}(Y, \mathfrak{s})$ are not determined.

First, we show that although the $S^1$-Frøyshov invariant $\delta$ is not determined by $\hat{\delta}$ and $\tilde{\delta}$, it is determined by $\text{SWFH}^{\mathbb{Z}/4}$.

**Theorem 1.1.** Let $(Y, \mathfrak{s})$ be a rational homology three-sphere with spin structure. Then

$$\delta(Y, \mathfrak{s}) = \frac{1}{2}(\min\{m \equiv 2\mu(Y, \mathfrak{s})+1 \mod 2 \mid \exists x \in \text{SWFH}^{\mathbb{Z}/4}_m(Y, \mathfrak{s}), \ x \in \text{Im } U^\ell \ \text{for all } \ell \geq 0, x \notin \text{Im } Q\} - 1)$$

We next relate the $S^1$ and $\mathbb{Z}/4$-invariants with those coming from $G$. Here, even given $\text{SWFH}^G_*(Y, \mathfrak{s})$, it is not possible to specify $\delta(Y, \mathfrak{s})$, $\hat{\delta}(Y, \mathfrak{s})$, or $\tilde{\delta}(Y, \mathfrak{s})$, although we have the following theorems.

**Theorem 1.2.** Let $(Y, \mathfrak{s})$ be a rational homology three-sphere with spin structure. Let

$$\delta_G(Y, \mathfrak{s}) = \frac{1}{2}(\min\{m \equiv 2\mu(Y, \mathfrak{s})+2 \mod 4 \mid \exists x \in \text{SWFH}^G_m(Y, \mathfrak{s}), \ x \in \text{Im } v^\ell \ \text{for all } \ell \geq 0, x \notin \text{Im } q^2\} - 2).$$

Then

$$\delta(Y, \mathfrak{s}) = \delta_G(Y, \mathfrak{s}) \text{ or } \delta_G(Y, \mathfrak{s}) + 1.$$

**Theorem 1.3.** Let $(Y, \mathfrak{s})$ be a rational homology three-sphere with spin structure. Let

$$\hat{\delta}_G(Y, \mathfrak{s}) = \frac{1}{2}(\min\{m \equiv 2\mu(Y, \mathfrak{s})+2 \mod 4 \mid \exists x \in \text{SWFH}^G_m(Y, \mathfrak{s}), \ x \in \text{Im } v^\ell \ \text{for all } \ell \geq 0, x \notin \text{Im } q^2\} - 2),$$

and

$$\tilde{\delta}_G(Y, \mathfrak{s}) = \frac{1}{2}(\min\{m \equiv 2\mu(Y, \mathfrak{s})+1 \mod 4 \mid \exists x \in \text{SWFH}^G_m(Y, \mathfrak{s}), \ x \in \text{Im } v^\ell \ \text{for all } \ell \geq 0, x \notin \text{Im } q^2\} - 1).$$

Then

$$\hat{\delta}(Y, \mathfrak{s}) = \hat{\delta}_G(Y, \mathfrak{s}) \text{ or } \hat{\delta}_G(Y, \mathfrak{s}) + 1$$

and

$$\tilde{\delta}(Y, \mathfrak{s}) = \tilde{\delta}_G(Y, \mathfrak{s}) \text{ or } \tilde{\delta}_G(Y, \mathfrak{s}) + 1.$$

To interpret $\delta_G(Y, \mathfrak{s})$, one may think of it just as the invariant $\gamma$, but with an adjustment coming from the $\mathbb{F}[v]$-torsion submodule of $\text{SWFH}^G_*(Y, \mathfrak{s})$. Similarly, $\hat{\delta}_G(Y, \mathfrak{s})$ is an adjustment of $\gamma$ as well, while $\tilde{\delta}_G(Y, \mathfrak{s})$ is an adjustment of $\beta$. We have the following as an immediate corollary of Theorem 1.3.
Corollary 1.4. Let \((Y, s)\) be a rational homology three-sphere with spin structure. Then
\[
\alpha(Y, s) \geq \delta(Y, s) \geq \beta(Y, s) \geq \delta(Y, s) \geq \gamma(Y, s).
\]

Given that the homology cobordism invariants from \(S^1\) and \(\mathbb{Z}/4 \subset \text{Pin}(2)\) cannot be determined from the \text{Pin}(2)-equivariant homology, it is natural to ask if there are other subgroups of \text{Pin}(2) which produce new homology cobordism invariants. We show that this is not the case. We call a homology cobordism invariant \(\delta_I\) a generalized Frøyshov invariant if it is constructed analogously to \(\delta\), but perhaps using a different subgroup \(H\) of \(G\); for the precise definition, see Section 3.3.

Theorem 1.5. Let \(\{\delta_I\}\) be the set of generalized Frøyshov invariants associated to a subgroup \(H \subset G\). Then
\[
\{\delta_I\} \subseteq \{\delta, \delta, \delta, \alpha, \beta, \gamma\},
\]
where the generalized Frøyshov invariants are viewed as maps \(\theta^3_H \to \mathbb{Z}\).

The proof of Theorem 1.5 also gives the next corollary. We note that the closed subgroups of \(G\) are precisely \(\mathbb{Z}/2 = \langle j^2 \rangle\), \(\mathbb{Z}/4 = \langle j \rangle\), \(S^1\), cyclic subgroups of \(S^1\), and generalized quaternion groups \(Q_4 = \langle e^{\pi i/m}, j \rangle\).

Corollary 1.6. Let \(H = Q_{4m} \subset G\) for \(m\) even and \((Y, s)\) any rational homology three-sphere with spin structure. Then the isomorphism type of \(\text{SWF}_H(Y, s)\) (as a \(H^*(BG)\)-module) is specified by the isomorphism type of \(\text{SWF}_H(Y, s)\) as a \(H^*(BH)\)-module.

Organization. In Section 2 we recall what we will need from equivariant topology. In Section 3 we define Gysin sequences and then use these to establish Propositions 3.6-3.8, which form the equivariant topology input for Theorem 1.1-1.3. In Subsection 3.2 we state the existence of Manolescu’s Seiberg-Witten Floer stable homotopy type, \(\text{SWF}(Y, s)\), and show Theorems 1.1-1.3 of the Introduction. In Subsection 3.3 we prove Theorem 3.12, which is the equivariant topology input for Theorem 1.5.

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2. Spaces of type SWF

2.1. \(G\)-CW Complexes. In this section we recall the definition of spaces of type SWF from [6], as well as briefly review the basics of equivariant topology, referring to Section 2 of [6] for further details. Spaces of type SWF are the output of the construction of the Seiberg-Witten Floer stable homotopy type of [6] and [7]; see Section 3.2. Throughout, all homology will be taken with \(\mathbb{F} = \mathbb{Z}/2\)-coefficients.

Let \(K\) be a compact Lie group. A (finite) \(K\)-CW decomposition of a space \((X, A)\) with \(K\)-action is a filtration \((X_n \ | \ n \in \mathbb{Z}_{\geq 0})\) of \(X\) such that

- \(A \subset X_0\) and \(X = X_n\) for \(n\) sufficiently large.
- The space \(X_n\) is obtained from \(X_{n-1}\) by attaching \(K\)-equivariant \(n\)-cells, copies of \((K/H) \times D^n\), for \(H\) a closed subgroup of \(K\).

When \(A\) is a point, we call \((X, A)\) a pointed \(K\)-CW complex.

Let \(X\) and \(Y\) pointed \(K\)-CW complexes, at least one of which is a finite complex. We define the smash product \(X \wedge Y\) as a \(K\)-space by letting \(K\) act diagonally. In the case that \(X = V^+\) is the one-point compactification of a finite-dimensional \(K\)-representation \(V\), we call \(\Sigma V^+ Y = V^+ \wedge Y\) the suspension of \(Y\) by \(V\). Define also
\[
X \wedge_K Y = (X \wedge Y)/K.
\]
Let $EK$ denote a contractible space with free $K$-action, and let $EK_+$ denote $EK$ with a disjoint base point added. The reduced Borel homology and cohomology of $X$ are defined by:

\[ \tilde{H}_K^*(X) = \tilde{H}_*(EK_+ \wedge_K X), \]
\[ \tilde{H}_K^*(X) = \tilde{H}^*(EK_+ \wedge_K X). \]

Borel homology and cohomology are invariants of $K$-equivariant homotopy equivalence.

Furthermore, there is a projection:

\[ EK_+ \wedge_K X \xrightarrow{\pi_1} EK_+ / K = BK_+, \]

which induces a map $\pi_1^* : H^*(BK) \to \tilde{H}^*_K(X)$. Via $\pi_1^*$, $\tilde{H}_K^*(X)$ and $\tilde{H}^*_K(X)$ inherit the structure of $H^*(BK)$-modules. On $\tilde{H}_K^*(X)$, $H^*(BK)$ increases grading, while on $\tilde{H}^*_K(X)$, $H^*(BK)$ decreases grading.

For a subgroup $L \subseteq K$ and a $K$-CW complex $X$, we may relate the $L$-Borel cohomology and $K$-Borel cohomology of $X$. Indeed, there is a quotient map

\[ EK_+ \wedge L X \xrightarrow{\pi} EK_+ \wedge_K X \]

which induces a map in cohomology:

\[ \tilde{H}_L^*(X) \xrightarrow{\pi^*} \tilde{H}_L^*(X). \]

We call the map $\pi^*$ the restriction map from $K$ to $L$ and write $\pi^* = \text{res}^K_L$.

Setting $X$ a point in $[2]$, we have a restriction map (of algebras) $H^*(BK) \to H^*(BL)$. Then $\tilde{H}_L^*(X)$ inherits an $H^*(BK)$-module structure by restriction. We denote $\tilde{H}_L^*(X)$, viewed as a $H^*(BK)$-module, by $\text{res}^K_L \tilde{H}_L^*(X)$. Then the map

\[ \tilde{H}_L^*(X) \xrightarrow{\text{res}^K_L} \text{res}^K_L \tilde{H}_L^*(X) \]

is a map of $H^*(BK)$-modules.

Let $G = \text{Pin}(2)$ and $BG$ its classifying space. In addition to the definition of $G$ from the Introduction, one may think of $G$ as the set $S^1 \cup jS^1 \subset \mathbb{H}$, where $S^1$ is the unit circle in the $\langle 1, i \rangle$ plane, with group action on $G$ induced from the group action of the unit quaternions. Thus $S^\infty = S(\mathbb{H}^\infty)$ with its quaternion action is a free $G$-space. Since $S^\infty$ is contractible, we identify $EG = S^\infty$.

Manolescu shows in [6] that

\[ H^*(BG) = \mathbb{F}[q,v]/(q^3), \]

where $\deg q = 1$ and $\deg v = 4$.

For convenience, we also record

\[ H^*(BZ/2) = \mathbb{F}[W], \]
\[ H^*(BZ/4) = \mathbb{F}[U, Q]/(Q^2 = 0), \]
\[ H^*(BS^1) = H^*(\mathbb{C}P^\infty) = \mathbb{F}[U], \]

where $\deg U = 2$ and $\deg W = \deg Q = 1$.

For a subset $S$ of a group $K$, let $\langle S \rangle$ denote the subgroup generated by $S$. There are inclusions $\mathbb{Z}/2 = \langle j^2 \rangle \subset S^1 \subset G$, and $\mathbb{Z}/4 \cong \langle j \rangle \subset G$. We will describe the corresponding restriction maps in Proposition [3.1].

We will also need to use that Borel cohomology behaves well with respect to suspension.
Proposition 2.1 ([6] Proposition 2.2). Let $V$ a finite-dimensional representation of a compact Lie group $K$. Then, as $H^*(BK)$-modules:

$$
\tilde{H}_K^n(\Sigma^V X) \cong \tilde{H}_{n-\dim V}^K(X)
$$

We mention three irreducible representations of $G$:

- The trivial one-dimensional representation $\mathbb{R}$.
- The one-dimensional real vector space on which $j \in G$ acts by $-1$, and on which $S^1$ acts trivially, denoted $\mathbb{R}$.
- The quaternionic representation $\mathbb{H}$, where $G$ acts by left multiplication.

Definition 2.2. Let $s \in \mathbb{Z}$. A space of type SWF at level $s$ is a pointed, finite $G$-CW complex $X$ with

- The $S^1$-fixed-point set $X^{S^1}$ is $G$-homotopy equivalent to $(\mathbb{R}^s)^+$, the one-point compactification of $\mathbb{R}^s$.
- The action of $G$ on $X - X^{S^1}$ is free.

We define $\mu(X) \in \mathbb{Q}/2\mathbb{Z}$ by $\mu(X) = \frac{s}{2} \mod 2$.

We will often have occasion later to work with $2\mu(X)$, which we view as an element of $\mathbb{Q}/4\mathbb{Z}$.

We note that for a space $X$ of type SWF,

$$
\tilde{H}_G^s(X^{S^1}) = \tilde{H}_G^s((\mathbb{R}^s)^+) = \tilde{H}_G^s(S^0) = H_{s-\cdot}(BG),
$$

and

$$
\tilde{H}_G^s(X^{S^1}) = H^{*-s}(BG),
$$

using Proposition 2.1.

Associated to a space $X$ of type SWF at level $s$, we take the Borel cohomology $\tilde{H}_G^s(X)$, from which Manolescu [6] defines $a(X), b(X),$ and $c(X)$:

$$
\begin{align*}
    a(X) &= \min \{ r \equiv 2\mu(X) \mod 4 \mid \exists x \in \tilde{H}_G^r(X), v^l x \neq 0 \text{ for all } l \geq 0 \}, \\
    b(X) &= \min \{ r \equiv 2\mu(X) + 1 \mod 4 \mid \exists x \in \tilde{H}_G^r(X), v^l x \neq 0 \text{ for all } l \geq 0 \} - 1, \\
    c(X) &= \min \{ r \equiv 2\mu(X) + 2 \mod 4 \mid \exists x \in \tilde{H}_G^r(X), v^l x \neq 0 \text{ for all } l \geq 0 \} - 2.
\end{align*}
$$

Using $S^1$-Borel cohomology, Manolescu [6] also defines

$$
\begin{align*}
    d(X) &= \min \{ r \mid \exists x \in \tilde{H}_G^r(X^s), U^l x \neq 0 \text{ for all } l \geq 0 \}.
\end{align*}
$$

The well-definedness of $a, b, c, d$ follows from the Equivariant Localization Theorem. We list a version of this theorem for spaces of type SWF:

Theorem 2.3 ([13] III (3.8)). Let $X$ be a space of type SWF. Then the inclusion $X^{S^1} \to X$, after inverting $v$, induces an isomorphism of $\mathbb{F}[q, v, v^{-1}]/(q^3)$-modules:

$$
v^{-1} \tilde{H}_G^s(X^{S^1}) \cong v^{-1} \tilde{H}_G^s(X).
$$

Furthermore,

$$
\begin{align*}
    U^{-1} \tilde{H}_G^{s_{3}}(X^{S^1}) &\cong U^{-1} \tilde{H}_G^{s_{3}}(X), \\
    U^{-1} \tilde{H}_G^{s_{4}}(X^{S^1}) &\cong U^{-1} \tilde{H}_G^{s_{4}}(X), \\
    W^{-1} \tilde{H}_G^{s_{2}}(X^{S^1}) &\cong W^{-1} \tilde{H}_G^{s_{2}}(X).
\end{align*}
$$
For $X$ a space of type SWF, $X$ is a finite $G$-complex and so we have that $\tilde{H}_{G}^{*}(X)$ is finitely generated as an $F[v]$-module. In particular, the $F[v]$-torsion part of $\tilde{H}_{G}^{*}(X)$ is bounded above in grading. Similarly, the $F[U]$-torsion parts of $\tilde{H}_{G}^{*}(X)$ and $\tilde{H}_{Z/4}^{*}(X)$, as well as the $F[W]$-torsion of $\tilde{H}_{Z/2}^{*}(X)$, are bounded above in grading.

Following [7], we define analogues of $a, b$ and $c$ for $\mathbb{Z}/4$.

**Definition 2.4.** For $X$ a space of type SWF, we define $\bar{d}(X)$ and $d(X)$ by

\begin{align*}
\bar{d}(X) &= \min\{r \equiv 2\mu(X) \text{ mod } 2 \mid \exists x \in \tilde{H}_{Z/2}^{r}(X), U^{l}x \neq 0 \text{ for all } l \geq 0\}, \\
\bar{d}(X) &= \min\{r \equiv 2\mu(X) + 1 \text{ mod } 2 \mid \exists x \in \tilde{H}_{Z/2}^{r}(X), U^{l}x \neq 0 \text{ for all } l \geq 0\} - 1,
\end{align*}

The well-definedness of $\bar{d}(X)$ and $\bar{d}(X)$ follows from Theorem 2.3.

**2.2. Stable $G$-Equivariant Topology.** Here we define stable equivalence for $G$-spaces and define the Manolescu invariants $\alpha, \beta$ and $\gamma$, as well as their $\mathbb{Z}/4$-analogues.

**Definition 2.5** (see [7]). Let $X$ and $X'$ be spaces of type SWF, $m, m' \in \mathbb{Z}$, and $n, n' \in \mathbb{Q}$. We say that the triples $(X, m, n)$ and $(X', m', n')$ are stably equivalent if $n - n' \in \mathbb{Z}$ and there exists a $G$-equivariant homotopy equivalence, for some $r \gg 0$ and some nonnegative $M \in \mathbb{Z}$ and $N \in \mathbb{Q}$:

$$
\Sigma^{r\mathbb{R}}\Sigma^{(M-m)\mathbb{R}}\Sigma^{(N-n)\mathbb{H}}X \rightarrow \Sigma^{r\mathbb{R}}\Sigma^{(M-m')\mathbb{R}}\Sigma^{(N-n')\mathbb{H}}X'.
$$

Let $\mathcal{E}$ be the set of equivalence classes of triples $(X, m, n)$ for $X$ a space of type SWF, $m \in \mathbb{Z}$, $n \in \mathbb{Q}$, under the equivalence relation of stable $G$-equivalence. The set $\mathcal{E}$ may be considered as a subcategory of the $G$-equivariant Spanier-Whitehead category $\mathcal{S}$, by viewing $(X, m, n)$ as the formal desuspension of $X$ by $m$ copies of $\mathbb{R}$ and $n$ copies of $\mathbb{H}$. We define Borel cohomology for $(X, m, n) \in \mathcal{E}$, as an isomorphism class of $H^*(BK)$-modules, by

$$
\tilde{H}_{K}^{*}((X, m, n)) = \tilde{H}_{K}^{*}(X)[m + 4n],
$$

for $K$ any closed subgroup of $G$. The well-definedness of (11) follows from Proposition 2.1.

Finally, we define the invariants $\alpha, \beta, \gamma, \delta, \bar{\delta}$ and $\delta$ associated to an element of $\mathcal{E}$.

**Definition 2.6.** For $[(X, m, n)] \in \mathcal{E}$, we set

$$
\alpha((X, m, n)) = \frac{a(X)}{2} - \frac{m}{2} - 2n, \quad \beta((X, m, n)) = \frac{b(X)}{2} - \frac{m}{2} - 2n, \quad \gamma((X, m, n)) = \frac{c(X)}{2} - \frac{m}{2} - 2n,$$

$$
\delta((X, m, n)) = \frac{d(X)}{2} - \frac{m}{2} - 2n, \quad \bar{\delta}((X, m, n)) = \frac{\bar{d}(X)}{2} - \frac{m}{2} - 2n, \quad \delta((X, m, n)) = \frac{\bar{d}(X)}{2} - \frac{m}{2} - 2n.
$$

The invariants above do not depend on the choice of representative of the class $[(X, m, n)] \in \mathcal{E}$.

For notational convenience later, we also define

$$
\mu((X, m, n)) = \mu(X) - \frac{m}{2} - 2n \text{ mod } 2.
$$

\footnote{This convention is slightly different from that of [7]. The object $(X, m, n)$ in the set of stable equivalence classes $\mathcal{E}$, as defined above, corresponds to $(X, \frac{m}{2}, n)$ in the conventions of [7].}
3. Gysin Sequences

3.1. Gysin Sequences for Change-of-Groups. Let $K$ and $L$ be compact Lie groups so that there is a fiber sequence of Lie groups (for $n = 0, 1, \text{or } 3$),

$$L \subset K \rightarrow S^n. \tag{12}$$

Then there is a Gysin sequence, as in \cite{13}[§III.2], given by:

$$\ldots \rightarrow H^*(BK) \xrightarrow{e(K, L)} H^{*+n+1}(BK) \xrightarrow{p^*} H^{*+n+1}(BL) \rightarrow H^{*+1}(BK) \rightarrow \ldots \tag{13}$$

where $e(K, L)$ is the Euler class of the sphere bundle $S^n \rightarrow BL = EK/L \rightarrow BK$ and $p : BL \rightarrow BK$ is the projection (note that $p^* = \text{res}_K^L$).

**Proposition 3.1.** We specify the Euler classes associated to the groups $\mathbb{Z}/2, \mathbb{Z}/4, S^1$, and $G$, as follows.

1. Associated to $S^1 \rightarrow G \rightarrow S^0$, we have $e(G, S^1) = q$. Further, $p^*v = U^2$.
2. Associated to $\mathbb{Z}/4 \rightarrow G \rightarrow S^1$, we have $e(G, \mathbb{Z}/4) = q^2$. Further, $p^*q = Q$ and $p^*v = U^2$.
3. Associated to $\mathbb{Z}/2 = \langle j^2 \rangle \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 = \langle j \rangle$, we have $e(\mathbb{Z}/4, \mathbb{Z}/2) = q$. Further, $p^*Q = 0$, $p^*U = W^2$.
4. Associated to $\mathbb{Z}/2 = \langle j^2 \rangle \rightarrow S^1 \rightarrow S^1/\langle j^2 \rangle = S^1$, we have $e(S^1, \mathbb{Z}/2) = 0$. Further, $p^*U = W^2$.

We call the Gysin sequences above Types [1]-[4], respectively.

**Proof.** In each case [1]-[4] it is straightforward to see that the Euler class is specified by the algebraic structure of the entries of the exact sequence (13). For example, we prove (1). Since $H^1(BS^1) = 0$, we have that $p^*q = 0$. By exactness of (13), $q$ is in the image of the Euler class, and since the Euler class $e(G, S^1)$ is of degree 1, we have $e(G, S^1) = q$. The other cases are similar. □

More generally, for $X$ a $K$-CW complex, we have a sphere bundle:

$$S^n \rightarrow EK \times_L X \rightarrow EK \times_K X \tag{14}$$

and a Gysin sequence:

$$H^*_K(X) \xrightarrow{e(X)} H^{*+n+1}_K(X) \xrightarrow{p^*} H^{*+n+1}_L(X) \rightarrow \ldots, \tag{15}$$

where $e(X)$ is the Euler class of the bundle (14). By construction, we have a map of bundles:

$$EK \times_L X \xrightarrow{\pi_L} EK \times_K X \xrightarrow{\pi_K} BL \rightarrow BK. \tag{16}$$

and by functoriality of the Euler class, we have

$$e(X) = \pi_K^*(e(K, L)). \tag{16}$$

**Fact 3.2.** By (16), $e(X) = q, q^2, Q, 0$ for types [1]-[4], respectively, for any $K$-CW complex $X$.

We can now relate, in the case of spaces of type SWF, the $\mathbb{Z}/2, \mathbb{Z}/4, S^1$, and $G$-cohomology theories.

We adapt a definition of [5] to our setting.

**Definition 3.3.** Let $S = (L \rightarrow K \rightarrow S^n)$ be one of the sequences of groups in Proposition 3.1. An abstract $S$-Gysin sequence $\mathcal{G}$ consists of the following:
(1) A $H^*(BK)$-module $M^K$, a $H^*(BL)$-module $M^L$, both graded by a $\mathbb{Z}$-coset of $\mathbb{Q}$ and bounded below.

(2) An exact triangle of $H^*(BK)$-modules.

\[ \begin{array}{ccc}
M^K & \xrightarrow{e(K,L)} & M^K \\
\downarrow\wr & & \downarrow\wr \\
\wr & & \wr \\
\end{array} \]

where $e(K,L)$ is the Euler class of $H^*(BK)$ as in Proposition 3.1 acting on the $H^*(BK)$-module $M^K$. Further, $\wr$ has degree $-n$ and $\wr$ has degree 0.

(3) In sufficiently high degrees, the triangle (17) is isomorphic to the exact triangle corresponding to $S$ from Proposition 3.1, perhaps with grading shifted.

**Proposition 3.4.** For every $X \in \mathcal{E}$, and $S = (L \to K \to S^n)$ the Gysin sequence

\[ \begin{array}{ccc}
\tilde{H}_*^*(X) & \xrightarrow{e(K,L)} & \tilde{H}_*^*(X) \\
\downarrow\wr & & \downarrow\wr \\
\wr & & \wr \\
\end{array} \]

is an abstract $S$-Gysin sequence, for $S$ of type [1]-[4]. The grading shift in (4) is $2\mu(X)$, upward.

**Proof.** Properties (1) and (2) of Definition 3.3 are automatically satisfied for (18); we prove Property (3). In sufficiently high degrees $d \geq N$ for some $N$, $\tilde{H}_*^*(X)$ must be isomorphic to $H^{d+2\mu(X)}(BK)$ and $\tilde{H}_*^*(X)$ must be isomorphic to $H^{d+2\mu(X)}(BL)$, using that $X$ is of type SWF. Recall from the proof of Proposition 3.1 that there is only one choice of maps $p^*$ and $\wr$ that make the triples $H^*(BK), H^*(BK)$ and $H^*(BL)$ into exact triangles. Since $\tilde{H}_*^*(X)$ and $\tilde{H}_*^*(X)$ are isomorphic to $H^{\ast+2\mu(X)}(BK)$ and $H^{\ast+2\mu(X)}(BL)$, the same reasoning as in the proof of Proposition 3.1 shows there is only one choice of maps $p^*$ and $\wr$ that make (18) exact in high degrees (Namely, the $p^*$ and $\wr$ listed). This establishes property (3) of Definition 3.3.

In order to prove Proposition 3.6, the precursor to Theorem 1.1, we will need to compare $S^1$ and $\mathbb{Z}/4$-invariants despite there being no Gysin sequence relating $S^1$ and $\mathbb{Z}/4$ homology. As an intermediate step, we define

\[ \delta_{\mathbb{Z}/2}(X) = \frac{1}{2}(\min\{m \mid \exists x \in H^m_{\mathbb{Z}/2}(X), W_\ell x \neq 0 \text{ for all } \ell \geq 0\}) \]

for $X \in \mathcal{E}$.

A priori, $\delta_{\mathbb{Z}/2}(X) - \mu(X)$ may be a half-integer (the next lemma implies it is, in fact, an integer).

**Lemma 3.5.** Let $X \in \mathcal{E}$. Then $\delta_{\mathbb{Z}/2}(X) = \delta(X)$.

**Proof.** We will use the Gysin sequence of type [4] associated to $X$. For now, fix $p^*$ and $\wr$ to refer to the Gysin sequence maps of that type.

First, we establish $\delta(X) \geq \delta_{\mathbb{Z}/2}(X)$. Let $x \in \tilde{H}^m_{S^1}(X)$ be $U$-nontorsion. For sufficiently large $\ell$, by Property (3) of Definition 3.3, $p^*(U^\ell x) \neq 0$. By the equivariance property (2) of the same definition, $p^*(U^\ell x) = W_{2\ell}p^*x$, and so $p^*x$ is a $W$-nontorsion element of $\tilde{H}^m_{\mathbb{Z}/2}(X)$. For notational convenience, define

\[ d_{\mathbb{Z}/2}(X) = \min\{m \equiv 2\mu(X) \mod 2 \mid \exists x \in H^m_{\mathbb{Z}/2}(X), W_\ell x \neq 0 \text{ for all } \ell \geq 0\} \]

We have then

\[ \min\{m \equiv 2\mu(X) \mod 2 \mid \exists x \in H^m_{S^1}(X), U^\ell x \neq 0 \text{ for all } \ell \geq 0\} \geq d_{\mathbb{Z}/2}(X). \]
We note that the only $U$-nontorsion elements of $\tilde{H}^*_X(X)$ are in degree $d \equiv 2\mu(X) \mod 2$, by Proposition 3.4. So the left-hand side of (20) is $2\delta(X)$. We also define
\[ \bar{d}_{Z/2}(X) = \min\{m \equiv 2\mu(X) + 1 \mod 2 \mid \exists x \in H^m_{Z/2}(X), W^\ell x \neq 0 \text{ for all } \ell \geq 0\}. \]
By definition, $\delta_{Z/2}(X) = \frac{\min\{d_{Z/2}(X), \bar{d}_{Z/2}(X)\}}{2}$.

We next show the inequality opposite to (20).

Let $x \in \tilde{H}^m_{Z/2}(X)$ be a $W$-nontorsion element, with $m \equiv 2\mu(X) + 1 \mod 2$. Then, by Property (3) of Definition 3.3, $i^*(W^{2\ell}x) = U^\ell x$ must be nonzero. In particular, $i^*x \in \tilde{H}^{m-1}_X(X)$ is $U$-nontorsion. Then we obtain
\[ 2\delta(X) \leq \bar{d}_{Z/2}(X) - 1. \]
Furthermore,
\[ \bar{d}_{Z/2}(X) = d_{Z/2}(X) \pm 1, \]
since for $x$ in one of the sets corresponding to $d_{Z/2}$ or $\bar{d}_{Z/2}$, $Wx$ is in the other.

But, combining (20) and (21), we have $d_{Z/2}(X) \leq \bar{d}_{Z/2}(X) - 1$. So, from (22), we obtain
\[ d_{Z/2}(X) = \bar{d}_{Z/2}(X) - 1. \]
It follows that $2\delta(X) = d_{Z/2}(X)$.

Using the definition of $\delta_{Z/2}(X)$, the proof is complete. □

The following statement corresponds to Theorem 1.1 of the Introduction.

**Proposition 3.6.** Let $X \in \mathcal{E}$. Then:
\[ \delta(X) = \frac{1}{2}\left(\min\{m \equiv 2\mu(X) + 1 \mod 2 \mid \exists x \in H^m_{Z/4}(X), U^\ell x \neq 0 \text{ for all } \ell \geq 0, Qx = 0\} - 1\right). \]

**Proof.** We denote the right-hand side of (23) by $\delta_{Z/4}(X)$. We will consider the abstract Gysin sequence of type (3) associated to $X$; fix $p^*$ and $i^*$ to refer to the maps in this type of Gysin sequence. Using Lemma 3.5, we need only show
\[ \delta_{Z/4}(X) = \delta_{Z/2}(X). \]
We start by showing $\delta_{Z/4}(X) \leq \delta_{Z/2}(X)$. We note that any $W$-nontorsion element $x$ of $\tilde{H}^*_X(X)$ with deg $x \equiv 2\mu(X) + 1 \mod 2$ must have $U^\ell i^*x = i^*W^{2\ell}x \neq 0$ for $\ell$ sufficiently large. However, $Q^\ell i^*x = 0$ by exactness of (17). Thus, if $x \in \tilde{H}^m_{Z/2}(X)$ with $m \equiv 2\mu(X) + 1 \mod 2$ is $W$-nontorsion, then there exists an element $i^*x \in \tilde{H}^m_{Z/4}(X)$ which is $U$-nontorsion, and which is annihilated by $Q$. Thus
\[ \min\{m \equiv 2\mu(X) + 1 \mod 2 \mid \exists x \in \tilde{H}^m_{Z/4}(X), U^\ell x \neq 0 \text{ for all } \ell \geq 0, Qx = 0\} - 1 \leq \bar{d}_{Z/2}(X) - 1. \]
By the proof of Lemma 3.5, $\frac{\bar{d}_{Z/2}(X) - 1}{2} = \delta_{Z/2}(X)$. Then
\[ \delta_{Z/4}(X) \leq \delta_{Z/2}(X). \]
Next we show
\[ \delta_{Z/2}(X) \leq \delta_{Z/4}(X). \]
Indeed, fix $m \equiv 2\mu(X) + 1 \mod 2$ and $x \in \tilde{H}^m_{Z/4}(X)$ so that $x$ is $U$-nontorsion and satisfies $Qx = 0$. Then $x \in \text{Im } i$, say $x = iy$, by exactness of (17). However, since $x$ is $U$-nontorsion, $y$ is nontorsion as well, so we see:
\[ \min\{m \mid \exists x \in \tilde{H}^m_{Z/2}(X), W^\ell x \neq 0 \text{ for all } \ell \geq 0\} \leq 2\delta_{Z/4}(X). \]
Recalling the definition of $\delta_{\mathbb{Z}/2}(X)$, the above is precisely $\delta_{\mathbb{Z}/2}(X) \leq \delta_{\mathbb{Z}/4}(X)$, completing the proof. □

The following proposition corresponds to Theorem 1.2 from the Introduction.

**Proposition 3.7.** Let $X \in \mathfrak{C}$. Fix $N \in \mathbb{Z}$. Then $\delta(X) \leq 2N + \mu(X) + 1$ if and only if

$$\min \{ m \equiv 2\mu(X) + 2 \mod 4 \mid \exists x \in \tilde{H}_G^m(X), v^\ell x \neq 0 \text{ for all } \ell \geq 0, qx = 0 \} - 2 \leq 2N + \mu(X).$$

**Proof.** Denote the left-hand side of (26) by $\delta(X)$. First, we show that $\delta(X) \leq 2N + \mu(X)$ implies $\delta(X) \leq 2N + \mu(X) + 1$.

Say $\delta_{G}(X) \leq 2N + \mu(X)$, that is, there exists a $v$-nontorsion element $x \in \tilde{H}_G^{4N+2\mu(X)+2}(X)$ so that $qx = 0$. Then, by exactness of (17), $x \in \text{Im } v^\ell$, say $x = v^\ell y$. Since $x$ is $v$-nontorsion, $y$ must be $U$-nontorsion. Thus, $\delta(X) \leq \frac{\deg y}{2} = 2N + \mu(X) + 1$.

Next, say that $\delta(X) \leq 2N + \mu(X) + 1$. Let $x \in \tilde{H}_G^{4N+2\mu(X)+2}(X)$ be $U$-nontorsion. Then by (1) of Proposition 3.1, $v^\ell x$ must be $v$-nontorsion. In particular, $v^\ell x$ is a $v$-nontorsion element of $\tilde{H}_G^{4N+2\mu(X)+2}(X)$ with $q(v^\ell x) = 0$. Thus $\delta_{G}(X) \leq 2N + \mu(X)$, as needed. □

The following proposition corresponds to Theorem 1.3 from the Introduction.

**Proposition 3.8.** Let $X \in \mathfrak{C}$. Then $\tilde{\delta}(X) \leq 2N + \mu(X) + 1$, for some integer $N$, if and only if

$$\min \{ m \equiv 2\mu(X) + 1 \mod 4 \mid \exists x \in \tilde{H}_G^m(X), v^\ell x \neq 0 \text{ for all } \ell \geq 0, q^2x = 0 \} - 1 \leq 2N + \mu(X).$$

Further, $\tilde{\delta}(X) \leq 2N + \mu(X) + 1$ if and only if

$$\min \{ m \equiv 2\mu(X) + 1 \mod 4 \mid \exists x \in \tilde{H}_G^m(X), v^\ell x \neq 0 \text{ for all } \ell \geq 0, q^2x = 0 \} - 1 \leq 2N + \mu(X).$$

**Proof.** We will consider Gysin sequences of Type 2. Denote the left-hand side of (27) by $\delta_G(X)$, and that of (28) by $\tilde{\delta}_G(X)$.

First, we show that $\delta_G(X) \leq 2N + \mu(X) + 1$ implies $\delta_G(X) \leq 2N + \mu(X)$. Indeed, say $\delta_G(X) \leq 2N + \mu(X) + 1$ and $x \in \tilde{H}_G^{4N+2\mu(X)+3}(X)$ so that $x$ is $U$-nontorsion. Then by (2) of Proposition 3.1, $w x$ is $v$-nontorsion. Further, $q^2 lx = 0$ by exactness of (17). Thus

$$\min \{ m \equiv 2\mu(X) + 2 \mod 4 \mid \exists x \in \tilde{H}_G^m(X), v^\ell x \neq 0 \text{ for all } \ell \geq 0, q^2x = 0 \} \leq 4N + 2\mu(X) + 2.$$

Then $\delta_G(X) \leq 2N + \mu(X)$, as needed.

Next, suppose $\delta_G(X) \leq 2N + \mu(X)$; we will show $\delta_G(X) \leq 2N + \mu(X) + 1$. Choose $x \in \tilde{H}_G^{4N+2\mu(X)+2}(X)$ so that $x$ is $v$-nontorsion and $q^2x = 0$. Then $x \in \text{Im } v$, say $x = v y$, and $y$ is $U$-nontorsion, in grading $4N + 2\mu(X) + 3$. We then obtain $\delta_G(X) \leq 2N + \mu(X) + 1$, as needed.

The proof for $\tilde{\delta}(X)$ is completely analogous. □

Propositions 3.7 and 3.8 cannot be improved, as we will see in Example 3.13, in that there exist spaces $X_1, X_2$ with isomorphic $\tilde{H}_G^*(X_i)$ but different $\delta, \tilde{\delta}$ and $\tilde{\delta}$ invariants.

### 3.2. Seiberg-Witten Floer Homology

In this section we convert the results of the previous section into statements for three-manifolds. First we recall the existence of the Seiberg-Witten Floer stable homotopy type.
Theorem 3.9 (Manolescu [3, 7]). There is an invariant \( SWF(Y, s) \), the Seiberg-Witten Floer spectrum class, of rational homology three-spheres with spin structure \((Y, s)\), taking values in \( \mathcal{E} \). A spin cobordism \((W, t)\), with \( b_2(W) = 0 \), from \( Y_1 \) to \( Y_2 \), induces a map \( SWF(Y_1, t|_{Y_1}) \rightarrow SWF(Y_2, t|_{Y_2}) \). The induced map is a homotopy-equivalence on \( S^1 \)-fixed-point sets.

Manolescu constructs \( SWF(Y, s) \) by using finite-dimensional approximation to the Seiberg-Witten equations on the Coulomb slice. From \( SWF(Y, s) \) one extracts homology cobordism invariants as in the following definition.

Definition 3.10. For \((Y, s)\) a spin rational homology three-sphere, the Manolescu invariants \( \alpha(Y, s), \beta(Y, s), \gamma(Y, s) \) and \( \delta(Y, s) \) are defined by \( \alpha(SWF(Y, s)), \beta(SWF(Y, s)), \gamma(SWF(Y, s)) \) and \( \delta(SWF(Y, s)) \), respectively. We further define

\[
\delta(Y, s) = \delta(SWF(Y, s)) \quad \text{and} \quad \bar{\delta}(Y, s) = \delta(SWF(Y, s)).
\]

All these quantities are invariant under homology cobordism.

Proof of Theorems 3.11,3.13. These Theorems follow by applying Propositions 3.6,3.8 to \( SWF(Y, s) \), and dualizing.

3.3. Equivariant Homology of subgroups of \( G \). Here we make precise and prove Theorem 3.15.

We first define generalized Frøyshov invariants.

Let \( H \subseteq G \) be a Lie subgroup of \( G \). Note that \( H^*(BG) \) is periodic; that is, cup product with \( v \in H^*(BG) \) defines an isomorphism of \( \mathbb{F} \)-modules

\[
H^n(BG) \to H^{n+4}(BG)
\]

for all \( n \geq 0 \). It turns out that \( H^*(BH) \) is also periodic; fix \( P \in H^*(BH) \) so that cup product with \( P \) induces an isomorphism \( H^*(BH) \to H^{*+|P|}(BH) \).

For \( X \) a space of type SWF at level \( s \), let \( t : X^{S^1} \to X \) denote the inclusion map of the \( S^1 \)-fixed point set, and let \( t^* \) denote the induced map in Borel cohomology

\[
t^* : \tilde{H}^*_t(X) \to \tilde{H}^*_t(X^{S^1}) = H^{*+s}(BH).
\]

Definition 3.11. For a homogeneous element \( e \) of \( H^*(BH)/P \) (with \( \mathbb{Z}/\deg P \)-grading) and \( X \in \mathcal{E} \) we define the generalized Frøyshov invariant \( \delta_{H, e}(X) \) by:

\[
\min\{m \equiv 2\mu(X) + \deg e \mod \deg P \mid \exists x \in \tilde{H}^m_t(X), t^*x = P^k e, \text{ for some } k \} - \deg e
\]

The well-definedness of the quantity \( \delta_{H, e}(X) \) is guaranteed by the Equivariant Localization Theorem. It is apparent that all of \( \alpha, \beta, \gamma, \delta \), and \( \tilde{\delta} \) and \( \bar{\delta} \) are special cases of generalized Frøyshov invariants.

Theorem 3.12. Let \( H \subseteq G \) a Lie subgroup, and let \( \{\delta_{H, e}\} \) be the set of generalized Frøyshov invariants associated to \( H \). Then

\[
\{\delta_{H, e}\} \subseteq \{\delta, \tilde{\delta}, \bar{\delta}, \alpha, \beta, \gamma\},
\]

where the generalized Frøyshov invariants are viewed as maps \( \mathcal{E} \to \mathbb{Z} \). Moreover, \( \tilde{\delta}(X) \) and \( \bar{\delta}(X) \) are not generally determined by \( \tilde{H}^*_t(X) \).

Proof. We refer to Example 3.13 for the last assertion, so we need only determine \( \delta_{H, e} \).

First, consider strict subgroups \( \mathbb{Z}/n = H \subset S^1 \).

If \( n \) is odd, then \( H^*(B\mathbb{Z}/n; \mathbb{Z}/2) \cong \mathbb{F} \), concentrated in degree 0, and so there are no generalized Frøyshov invariants. For \( n \) even, \( H^*(B\mathbb{Z}/n; \mathbb{Z}/2) \cong H^*(BS^1; \mathbb{Z}/2) \), and in particular the only associated generalized Frøyshov invariant is \( \delta_{\mathbb{Z}/n, 1} \). The same argument as in Lemma 3.5 shows
δ_{Z/n,1} = δ. Thus the generalized Frøyshov invariants associated to a subgroup of \( S^1 \) are determined by \( \delta \) and determine \( \delta \).

Next, consider a strict subgroup \( H \subset G \) not contained in \( S^1 \) and not equal to \( \mathbb{Z}/4 \). Then \( H \) is a generalized quaternion group \( Q_{4m} = (e^{\pi i/m}, j) \) with \( m \geq 2 \).

First, say \( m \) even. We note \( H^1(BQ_{4m}) = \text{Hom}(Q_{4m}, F) = F^2 \). Then, since \( H^1(BG) = F \), we see that the Gysin sequence associated to the sphere bundle

\[
S^1 \to BQ_{4m} \to BG
\]
splits:

\[
H^*(BQ_{4m}) = H^*(BG) \oplus H^*(BG)[-1]
\]
as an \( H^*(BG) \)-module. Recall \( H^*(BG) \) acts on \( H^*(BQ_{4m}) \) by the map \( p : H^*(BG) \to H^*(BQ_{4m}) \).

Let \( r \) generate \( H^1(BG)[-1] \) in the decomposition (30). Then the homogeneous elements of \( H^*(BQ_{4m})/(v) \) are

\[
1, q, q + r, r, q^2, q^2 + qr, qr, \text{ and } q^2 r
\]
Furthermore, we note from the definition of \( \delta_{H,e} \) that, for \( X \in \mathfrak{C} \),

\[
\delta_{H,f}(X) \geq \delta_{H,e}(X) \text{ if } f \text{ divides } e.
\]

Repeating the argument in the proof of Propositions 3.7 and 3.8, we see

\[
\delta_r(X) \geq \alpha(X), \quad \delta_{q+r}(X) \geq \alpha(X), \quad \delta_q(X) \geq \beta(X), \quad \delta_{q+r}(X) \geq \beta(X), \quad \delta_{q^2}(X) \geq \gamma(X), \quad \delta_{q^2}(X) \leq \gamma(X),
\]

However, \( \delta_r(X) \leq \delta_1(X) \) by (31), so \( \delta_r(X) \leq \delta_1(X) = \alpha(X) \). Similarly, one obtains that all the inequalities in (32) and (33) are equalities. Thus \( \delta_{H,e}(X) \) are in fact determined by \( \alpha(X), \beta(X) \), and \( \gamma(X) \). This completes the proof for the \( m \) even case.

If \( m \) is odd, we have \( H^*(BQ_{4m}) \cong H^*(B\mathbb{Z}/4) \). The argument of Lemma 3.5 then adapts to show that the generalized Frøyshov invariants of \( Q_{4m} \) and \( \mathbb{Z}/4 \) agree.

**Theorem 1.5** follows from Theorem 3.12 while Corollary 1.6 is a consequence of the proof of Theorem 3.12. We close with an example showing that \( SWFH^*_+(Y, s) \), as an \( H^*(BG) \)-module, does not determine \( \delta, \bar{\delta}, \) or \( \hat{\delta} \).

**Example 3.13.** There are pointed \( G \)-stable homotopy types \( X_1 \) and \( X_2 \) so that

\[
\bar{H}_G^*(X_1) = \bar{H}_G^*(X_2) = \mathcal{V}_8^+ \oplus \mathcal{V}_1^+ \oplus \mathcal{V}_2^+ \oplus \mathbb{F}_3^2 \oplus \mathbb{F}_4
\]
where \( \mathcal{V}_n^+ \) denotes the \( \mathbb{F}[v] \)-module \( \mathbb{F}[v] \), with grading shifted up by \( n \), and \( \mathbb{F}_n \) denotes a copy of \( \mathbb{F} \) concentrated in degree \( n \). Furthermore,

\[
\delta(X_1) = \delta(X_1) = 2, \quad \delta(X_1) = 0, \quad \text{and } \delta(X_2) = \delta(X_2) = 3.
\]

To specify the \( q \)-action, let \( t_8, t_{11}, \) and \( t_2 \) be \( \mathbb{F}[v] \)-generators of \( \mathcal{V}_8^+, \mathcal{V}_1^+, \text{ and } \mathcal{V}_2^+ \) respectively, while \( y_3, y'_3 \) generate \( \mathbb{F}_3^2 \) and \( y_4 \) generates \( \mathbb{F}_4 \). Then \( qt_8 = v^2 t_1, qt_1 = t_2, qt_2 = y_3 \) and \( qt_3 = y_4 \).

We give a description of the chain complexes of \( X_1 \) and \( X_2 \) over \( C^*_{CW}(G) = \mathbb{F}[s, j]/(sj = j^3 s, s^2 = j^4 + 1 = 0) \) where \( \deg s = 1, \deg j = 0 \). Indeed \( C^*_{CW}(X_1) \) is \( \mathbb{F}[f, x_1, x_3, x_4, x_5, y_3] \) with \( \partial(x_1) = f, \partial(x_3) = (1 + j)^3 x_1, \partial(x_4) = (1 + j)x_3, \partial(x_5) = (1 + j)x_4 + s x_3 \) and \( \partial(y_3) = (1 + j)^2 s x_1 \). We have \( C^*_{CW}(X_2) = \mathbb{F}[f, x_1, x_3, x_4, y_3, y'_3, y_5] \) where the differentials are as before, and \( \partial(y_5) = (1 + j)^2 s y_3 \). The calculation of the Manolescu invariants for both examples is an application of the techniques of [12], [11].
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