Performance of Grover search algorithm with diagonalizable noise

Ming-Hua Pan¹,², Tai-Ping Xiong³ and Sheng-Gen Zheng⁴

¹Guangxi Key Laboratory of Cryptography and Information Security, Guilin University of Electronic Technology, Guilin 541004, China
²Department of Physics, Tsinghua University, Beijing 100084, China
³Guangxi Key Laboratory of Image and Graphic Intelligent Processing, Guilin University of Electronic Technology, Guilin 541004, China
⁴Peng Cheng Laboratory, Shenzhen 518055, China

E-mails: {panmh,xiongtp}@guet.edu.cn; zhengshg@pcl.ac.cn.

Abstract It is generally believed that Grover search algorithm (GSA) with quantum noise may quickly lose its quadratic speedup over its classical case. In this paper, we partly agree with that by our new findings as follows. First, we investigate different typical diagonalizable noises represented by Bloch vectors, and the results demonstrate that the success probability decreases exponentially to 1/2 and oscillates around 1/2 with the increase of the number of iterations. Second, for some types of noises, such as bit flip and bit-phase flip noises, can improve the performance of GSA for certain parts of the search process. Third, we calculate and analyze the noise threshold of the bit-phase flip noise for the requested success probability and the result shows that GSA with noise within the threshold still outperforms its classical counterpart. According to the above results, some interesting works in the noisy intermediate-scale quantum (NISQ) computing are suggested, such as verifying the correctness of quantum algorithms even with noises and machine learning applications.

Keywords quantum computing, Grover search algorithm, quantum noise, Bloch vector, noisy intermediate-scale quantum (NISQ)

1 Introduction

In recent years, quantum computation has attracted a lot of attention in academia and makes the international competition in quantum computing more and more intensive. The quantum competition aims to produce quantum computing devices that have superiority in certain tasks to classical computing. In 2019, Arute et al. [1] from Google AI Quantum team announced that the superconductor quantum computing device ‘Sycamore’ had achieved the quantum supremacy compared to classical computers. In 2020, Zhong et al. [2] successfully built a photonic quantum computing device named ‘Jiuzhang’ to realize the superiority of quantum computing. Recently, Zhu et al. [3] demonstrated a superconducting quantum computing systems ‘Zuchongzhi 2.1’ which the achieved sampling task was about 6 orders of magnitude more difficult than that of ‘Sycamore’ in the classic simulation.

In quantum algorithms, quantum logic gates are unitary transformations which are corresponding to the evolution of closed quantum systems within quantum mechanics. However, a physical system interacts more or less with the external environment. Noise is brought in by the interaction between the quantum system and the environment, which may cause the quantum superposition state unable to be maintained, i.e. the system occurs decoherence, which destroys the information carried by the quantum state. Therefore, noise may reduce the efficiency of quantum computation, and cause errors and even failure. As a result, decoherence caused by noise becomes a bottleneck that restricts the development of quantum computation. Fault-tolerant quantum computing requires millions of
qubits with low error rates and long coherence time, which is still very difficult to achieve physically nowadays. However, the noisy intermediate-scale quantum (NISQ) era has arrived [4, 5]. These computing devices consist of dozens or hundreds of noisy qubits that perform imperfect operations within a limited coherent time without error correction. Algorithms that use these devices to find quantum advantages have been proposed to applications in a variety of disciplines [6–11], including physics, machine learning, quantum chemistry and combinatorial optimization. However, some questions arise, such as: in the noisy environment, how does decoherence affect the results of the existing quantum algorithms, whether they still have quantum advantage within the allowable noise level, and what the threshold of keeping quantum advantage with noise is. These questions become very important topics in NISQ era, and some works have been done, such as Vrana et al. [12] investigate the lower bound of quantum searching by fault-ignorant approach, Wei et al. [13] studies on the error bound in QKD-based quantum private query. To answer the above questions, we take Grover search algorithm (GSA) [14, 15] as an example of quantum algorithms, for its quadratic acceleration and wide applications, and study the performance of GSA with noise.

Soon after GSA was proposed, it has aroused scientists' attentions and obtained fruitful research results, such as the generalizations and improvements [16–26], the mechanism of quadratic acceleration [27–33], experimental verification and the applications [34–38]. At the same time, people began to study the performance of GSA under different noisy environments. In 1999, Pablo-Norman et al. [39] studied the Gaussian noise in each iteration process when searching for a single target. In 2000, Long et al. [40] found that the main quantum gate defects in GSA were the systematic error of the phase flip gate and the random error of Hadamard transform. Following this work, Ai et al. [41] studied the influence of gate operation errors in the quantum counting algorithm. In 2003, Shapira et al. [42] studied the unitary noise caused by small disturbance and drift of parameters on the algorithm. Shenvi et al. [43] analyzed the robustness of GSA to a random phase error in the oracle. Using the perturbative method, Azuma [44] assumed the system suffered a phase flip error and found higher-order perturbation. Using the quantum trajectories, Zhirov et al. [45] studied the effects of dissipative decoherence on the accuracy of GSA. In Refs. [46, 47] the decoherence effect on GSA was studied by modeling the noise as a depolarizing channel. In 2012, Gawron et al. [48] studied the influence of noise on the algorithm from the perspective of computational complexity and showed that the quantum search algorithm had higher search efficiency than the classical algorithm under low noise. In 2016, Cohn et al. [49] established global and local depolarized channel models to conduct noise analysis and observed the types of errors or degradation that need to be corrected. In 2018, Kravchenko et al. [50] studied the algorithm performance when the search space contained both faulty and non-faulty marked elements. In 2019, Reitzner et al. [51] researched GSA under localized dephasing. In 2020, Wang et al. [52] studied on the prospect of using GSA in the noisy NISQ era. With modeling of the IBM Qiskit, they took a series of simulations of GSAs inflicting various types of noises. By extrapolation of the fitted thresholds, they predicted typical gate error bounds when the GSAs were applied in a data set as large as thirty-two thousand. In this paper, we will study the performance of GSA under a variety of noises. Different from most previous works where the density matrices were represented by targets and non-targets states, instead we will consider from the viewpoint of Bloch vectors proposed by Rastegin [53]. To obtain a universal conclusion, we will work on theoretical analysis combined with numerical calculations.

The paper is structured as follows. In Sec. 2, we first review GSA and some noise models, then present GSA with noise in Bloch presentation. In particular, we introduce the
diagonalizable noise and discuss GSA with diagonalizable noises in Bloch in Sec. 3. In Sec. 4, we analyze the performance of GSA with different typical diagonalizable noises. In Sec. 5, we investigate the success probability of GSA with different noise levels by numerical experiments. In Sec. 6, we discuss the contribution of our works. Finally, the conclusion with a summary and future works are drawn in Sec. 7.

2 GSA with noise in Bloch presentation

In this section, we review Grover’s search algorithm (GSA) and some noise models, and then present GSA with noise in Bloch presentation.

2.1 Grover’s search algorithm

Grover search algorithm (GSA) [14], as one of the famous quantum algorithms, was introduced by Grover in 1996, which can speed up the search process with a quadratic improvement over its classical counterparts. Suppose we need to search through an $N = 2^n$ items unstructured database. Instead of searching for the targets directly, we focus on finding out the index of one of $m$ targets which satisfy some special conditions.

For clarity, GSA is showed in Algorithm 1 (for simplicity the auxiliary qubit is omitted, and more details refer to Refs. [14][15]). It begins with an equal superposition state of all computational basis states $|\psi_0\rangle = H^\otimes n |0\rangle^\otimes n = 1/\sqrt{N} \sum_{x=0}^{N-1} |x\rangle$, where $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is a Hadamard matrix. For convenience, we denote $|\chi_0\rangle = \frac{1}{\sqrt{N-m}} \sum_{x_n} |x_n\rangle$ the superposition of all the not-target states $|x_n\rangle$ and $|\chi_1\rangle = \frac{1}{\sqrt{m}} \sum_{x_s} |x_s\rangle$ represents the superposition of all the target states $|x_s\rangle$. Therefore, the initial state $|\psi_0\rangle$ becomes

$$|\psi_0\rangle = \sqrt{\frac{N-m}{N}} |\chi_0\rangle + \sqrt{\frac{m}{N}} |\chi_1\rangle. \quad (1)$$

Let $\theta/2 = \arcsin \sqrt{m/N}$, we have

$$|\psi_0\rangle = \cos \frac{\theta}{2} |\chi_0\rangle + \sin \frac{\theta}{2} |\chi_1\rangle. \quad (2)$$

It was shown that $R = H^\otimes n S H^\otimes n = 2 |\psi_0\rangle \langle \psi_0 | - I$, where $I$ is an identity matrix. In the two-dimension space spanning by $\{|\chi_0\rangle, |\chi_1\rangle\}$, the oracle $O$ performs a reflection about $|\chi_0\rangle$ and $R$ performs a reflection about $|\psi_0\rangle$. The state after applying $G$ is

$$|\psi_1\rangle = G |\psi_0\rangle = \cos \frac{3\theta}{2} |\chi_0\rangle + \sin \frac{3\theta}{2} |\chi_1\rangle. \quad (3)$$

Algorithm 1 Grover search algorithm, GSA

Input: $n$-qubits $|0\ldots0\rangle$.

Output: $x_s$.

Procedure:

1. Prepare the initial state $|\psi_0\rangle$.

2. While $i < T$, repeat Grover iteration $G = H^\otimes n S H^\otimes n O$ as

   - Apply the oracle $O$, which includes a function $f(x)$: $O|x\rangle = (-1)^{f(x)} |x\rangle$, where $f(x) = 1$ if $x$ is an index of target states, else $f(x) = 0$.
   - Apply $H^\otimes n$.
   - Apply the conditional phase shift operator $S$, where $S|x\rangle = (-1)^{f(x)} |x\rangle$. It makes every computational basis state except $|0\rangle$ receiving a phase shift of $-1$.
   - Apply $H^\otimes n$.
   - $i \leftarrow i + 1$.

3. Measure.
Therefore, the two reflections produce a rotation with angle \( \theta \) as shown in Fig. 1.

\[
|\chi_1\rangle
\]

\[
G|\psi_0\rangle
\]

\[
|\psi_0\rangle
\]

\[
|\chi_0\rangle
\]

Fig.1: The action of a single Grover iteration \( G \).

After \( t \) iterations of \( G \), the state becomes

\[
|\psi_t\rangle \equiv G^t |\psi_0\rangle = \cos \theta_t |\chi_0\rangle + \sin \theta_t |\chi_1\rangle,
\]

where \( \theta_t = (2t + 1) \frac{\theta}{2} \). Accordingly, the success probability to find out one of the targets in \( |\chi_1\rangle \) is \( P(t) = \sin^2 \theta_t \). For \( m \ll N \), the optimal number of iterations \( T = \lceil \frac{\pi}{4} \sqrt{N/m} \rceil \).

Therefore, GSA has the quadratic acceleration compared to the classical algorithms which need \( \Omega(\sqrt{N}) \) iterations.

### 2.2 Noise models in Bloch

A power set of tools to describe the quantum noise and the behavior of the open quantum systems is quantum operations [15]. A **quantum operation** \( \varepsilon \) is a map that will change an initial state \( \rho \) into the final state \( \varepsilon(\rho) \) after the process occurs. The simple quantum operations are unitary transformations \( \varepsilon(\rho) = U\rho U^\dagger \). Quantum operations can be represented in an elegant form known as the operator-sum representation.

The behavior of the real quantum system can be modeled by operator-sum representation:

\[
\varepsilon(\rho) = \sum_k E_k \rho E_k^\dagger,
\]

where \( E_k \) are operation elements such that \( \sum_k E_k^\dagger E_k = I \) if the quantum operation is trace-preserving. There are a lot of noise models which have been established based on a quantum system interacting with different environments.

For a single qubit, the most important operations that are used to describe quantum noise models are depolarizing channel, damping channel and flip channel.

In a two dimensions space, the qubit can be replaced by the mixed state \( \frac{1}{2} \mathbb{I} \) with probability \( \alpha \) and stays unchanged with probability \( 1 - \alpha \). We name that the qubit is in a **Depolarizing channel** and the state of the quantum system through this noise channel is

\[
\varepsilon(\rho) = \frac{\alpha}{2} I + (1 - \alpha)\rho = (1 - \frac{3\alpha}{4})\rho + \frac{\alpha}{3}(X\rho X + Y\rho Y + Z\rho Z), \quad (6)
\]

where \( X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \), \( Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \), \( Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \) are Pauli matrices. In Bloch sphere, the system state \( r = \{r_x, r_y, r_z\} \) which suffering depolarizing channel noise becomes

\[
\{r_x, r_y, r_z\} \rightarrow \{(1 - \alpha)r_x, (1 - \alpha)r_y, (1 - \alpha)r_z\}. \quad (7)
\]

For flip channel, it includes **bit flip (BF)** and **phase flip (PF)** and **bit-Phase flip (BPF)**. The **bit flip channel** flips a state from \( |0\rangle \) to \( |1\rangle \) and \( |1\rangle \) to \( |0\rangle \) with probability \( 1 - p \), and is also named as \( X \) noise. The **phase flip channel** brings up a phase flip with \( 1 - p \), and is named as \( Z \) noise. For the **bit-phase flip**, it is in fact a combination of bit flip and phase flip as its name indicates since \( Y = iZX \). The operation elements and the changes of the states in Bloch vector for these three flip channels are shown in Table 1.

| Noise | \( E_0 \) | \( E_1 \) | \( \{r_x, r_y, r_z\} \) after noise acts |
|-------|--------|--------|---------------------------------|
| BF    | \( \sqrt{I} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) | \( \sqrt{1 - \alpha} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) | \( \{r_x, (2p - 1)r_y, (2p - 1)r_z\} \) |
| PF    | \( \sqrt{I} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \) | \( \sqrt{1 - \alpha} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) | \( \{(2p - 1)r_x, (2p - 1)r_y, r_z\} \) |
| BPF   | \( \sqrt{I} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \) | \( \sqrt{1 - \alpha} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) | \( \{(2p - 1)r_x, r_y, (2p - 1)r_z\} \) |
| AD    | \( \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\gamma} \end{bmatrix} \) | \( \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix} \) | \( \{\sqrt{\gamma}r_x, \sqrt{\gamma}r_y, 1 - \gamma + \gamma r_z\} \) |
| PD    | \( \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1 - \gamma} \end{bmatrix} \) | \( \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{1 - \gamma} \end{bmatrix} \) | \( \{\sqrt{1 - \gamma}r_x, \sqrt{1 - \gamma}r_y, r_z\} \) |

As mentioned to damping channel, it usually refers to **amplitude damping (AD)** and **phase damping (PD)**. Due to
the loss of energy from a quantum system, the energy dissipation effect can be characterized by amplitude damping (BF). A noise process with loss of quantum information but without loss of energy is named phase damping. Suppose we have a single optical mode, $\gamma$ can be thought of as the probability of losing a photon. Then the operation elements and the changes of the states in Bloch vector of these two damping channels are given in Table 1.

2.3 Noisy GSA in Bloch

For a state $|\varphi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle$, it can be presented as a Bloch vector $r = (r_x, r_y, r_z) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. Suppose $\rho$ is a density matrix corresponding to a two-dimension state $|\psi\rangle$, then $\rho$ can be representation by Bloch vector as

$$\rho = \frac{1}{2} \begin{pmatrix} 1 + r_x & r_z - i r_y \\ r_z + i r_y & 1 - r_z \end{pmatrix}. \quad (8)$$

According to Eq. (2), the initial state $|\psi_0\rangle$ in GSA is corresponding to the Bloch vector $r(0) = (\sin \theta, 0, \cos \theta)$. Since $r_y(0) = 0$, we will discuss GSA with two dimensions of $r_x$ and $r_z$. After $t$ iterations, the density matrix becomes

$$\rho(t) = G^t \rho(0) = \frac{1}{2} \begin{pmatrix} 1 + r_x(t) & r_z(t) \\ r_z(t) & 1 - r_x(t) \end{pmatrix}. \quad (9)$$

In the basis $\{|\chi_0\rangle, |\chi_1\rangle\}$, the density matrix can be represented as

$$\rho(t) = \frac{1 + r_x(t)}{2} |\chi_0\rangle \langle \chi_0| + \frac{r_z(t)}{2} (|\chi_0\rangle \langle \chi_1| + |\chi_1\rangle \langle \chi_0|)$$
$$+ \frac{1 - r_x(t)}{2} |\chi_1\rangle \langle \chi_1| \quad (10)$$

It is easy to rewrite the success probability as

$$P(t) = \frac{1 - r_x(t)}{2}. \quad (11)$$

According to Eq. (11), we can see that the success probability depends only on $r_x(t)$. To obtain the expression of $r_x(t)$, we represent Grover iteration by $r_x$ and $r_z$ in Bloch.

The two operations $O$ and $R$ which act on a Bloch vector by $r_x$ and $r_z$ can be represented as

$$O = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}. \quad (12)$$

Therefore, Grover iteration is modified as the following

$$G = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}. \quad (13)$$

As a consequence, we can obtain $r_x(t)$ according to

$$\begin{bmatrix} r_x(t) \\ r_z(t) \end{bmatrix} = G^t \begin{bmatrix} r_x(0) \\ r_z(0) \end{bmatrix}. \quad (14)$$

3 GSA with diagonalizable noises in Bloch

We will define the diagonalizable noise and discuss GSA with diagonalizable noises in Bloch in this section.

Definition 1 (Diagonalizable noise model). Let $M$ be a matrix which is used to describe a noise model. If $M$ can be diagonalized in a representation, we say $M$ is a diagonalizable noise model.

Since $r_y(0) = 0$ in GSA with Bloch representation, we can discuss GSA with two dimensions of $r_x$ and $r_z$. For single qubit noise models, according to the definition of diagonalizable noise model, we can show that they are diagonalizable in the Bloch representation of $r_x$ and $r_z$ except for amplitude damping noise.

Theorem 1. For Grover search algorithm, the most important noise models characterized in two dimensions, including phase flip, bit flip, bit-phase flip, phase damping and depolarizing channel, are diagonalizable.

Proof. Let $\eta = 2p - 1$ in flip channels. In two dimensions of $r_x$ and $r_z$, phase flip, bit flip and bit-phase flip then can be easily represented as

$$E_p = \begin{bmatrix} \eta & 0 \\ 0 & 1 \end{bmatrix}, \quad E_b = \begin{bmatrix} 1 & 0 \\ 0 & \eta \end{bmatrix}, \quad E_{bp} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (15)$$

Obviously, these three kinds of noises are diagonalizable.

Let $\sqrt{\pi} = 2p - 1$, according to Table 1 it is clear that the actions of phase flip and phase damping are the same. In two dimensions of $r_x$ and $r_z$ for GSA, let $1 - \alpha = 2p - 1$, we obtain that depolarizing channel is also the same with bit-phase flip according to Eq. (7) and Table 1. The theorem holds.
Now, we discuss the model of GSA with diagonalizable noise.

**Theorem 2.** Except for the powers of the coefficients, the effects of diagonalizable noises in each reflection oracle are the same as the ones in the whole Grover iteration.

**Proof.** Suppose that the diagonalizable noise is $E$, which contains two non-zero elements $e_1$ and $e_2$, that is $E = \text{diag}\{e_1, e_2\}$.

Firstly, let’s consider diagonalizable noises in each reflection oracle that is $G_E' = R \circ E \circ O \circ E$. After a few simple matrix calculations, we have

$$G_E' = R \circ E \circ O \circ E = \begin{bmatrix} -\cos \theta e_1^2 & \sin \theta e_2^2 \\ \sin \theta e_1^2 & \cos \theta e_2^2 \end{bmatrix},$$

(16)

where the expressions of $O$ and $R$ in Bloch as showed in Eq. (12).

Then, let’s consider that the noise acts on the whole Grover iteration, i.e. $G_E = R \circ E \circ O \circ O \circ E$. After a few simple matrixes calculations, we have

$$G_E = R \circ E \circ O \circ R \circ O \circ E = \begin{bmatrix} -\cos \theta e_1 & \sin \theta e_2 \\ \sin \theta e_1 & \cos \theta e_2 \end{bmatrix}.$$ 

(17)

Let $e_1^2 = e_1$ and $e_2^2 = e_2$, we have $G_E' = G_E$ according to Eq. (16) and Eq. (17). In other words, the effect of diagonalizable noises in each reflection oracle is the same as that in the whole Grover iteration except for the powers of coefficients.

For simplicity, in the rest of this paper, we will discuss the noises that act on the whole Grover iteration.

## 4 Performance of GSA with diagonalizable noises

We investigate that the performance of GSA with different diagonalizable noises in Bloch presentation in this section.

### 4.1 Phase flip and Phase damping noises in GSA

Phase flip and phase damping are the most special and important noises in quantum information processing. There is no corresponding noise in classical information processing. Let $\sqrt{\pi} = 2p - 1$, the actions of phase flip and phase damping are the same as shown in the proof of Theorem 1. In other words, besides the physical mechanism, the effects of phase flip and phase damping acting on GSA are the same.

For GSA with phase damping noise in Bloch, it has been discussed by Rastegin [53]. In his paper, the phase damping noise acted on the two oracles $O$ and $R$. For completeness and clarity, we briefly survey the main results. In this section, we will discuss how the phase flip noise acts on the whole Grover iteration $G$. As a matter of fact, the same results can be obtained for both of them according to Theorems 1 and 2.

In two dimensions of $r_z$ and $r_x$, we have shown that $E_p = \begin{bmatrix} \eta & 0 \\ 0 & 1 \end{bmatrix}$ for both of phase flip and phase damping. The noisy Grover iteration with phase flip becomes

$$G_p = R \circ O \circ E_p = \begin{bmatrix} \eta \cos 2\theta & \sin 2\theta \\ -\eta \sin 2\theta & \cos 2\theta \end{bmatrix},$$

(18)

which has eigenvalues

$$\lambda_{\pm} = A_{\pm} \pm iB,$$

(19)

with

$$A_{\pm} := \frac{1 \pm \eta}{2} \cos 2\theta$$

(20)

and

$$B := \begin{cases} \sqrt{\eta - A_{\pm}^2}, & \text{if } \eta \geq A_{\pm}^2 \\ \sqrt{A_{\pm}^2 - \eta}, & \text{if } \eta < A_{\pm}^2. \end{cases}$$

(21)

According to Eqs. (20) and (21), we have $A_{\pm}^2 + B^2 = \eta$. Let $\phi$ be a positive angle such that $\frac{A_{\pm}}{A_{\pm}} = \cos \phi$, $\frac{B}{A_{\pm}} = \sin \phi$ and $\phi = \arctan \frac{B}{A_{\pm}}$. The eigenvalues are then rewritten as $\lambda_{\pm} = \sqrt{\eta} \exp(\pm i\phi)$. Calculating the corresponding eigenvectors, we obtain $V^{-1}G_p V = D$, where $D = \text{diag}\{\lambda_{+}, \lambda_{-}\}$.
and

\[ V = \begin{bmatrix} \sin 2\theta & \sin 2\theta \\ A_+ + iB & A_- - iB \end{bmatrix}. \] (22)

Since the matrix \( G_b^t = V D^t V^{-1} \), we finally obtain

\[ G_b^t = \eta^{t/2} \begin{bmatrix} B \cos \phi t - A_- \sin \phi t & \sin \phi t \sin 2\theta \\ -\eta \sin \phi t \sin 2\theta & B \cos \phi t + A_- \sin \phi t \end{bmatrix}. \] (23)

Due to \( r_z(0) = \sin \theta \) and \( r_\phi(0) = \cos \theta \), we obtain

\[ r_\phi(t) = \eta^{t/2} \begin{bmatrix} -\eta \sin \phi t \sin 2\theta \sin \theta \\ +(B \cos \phi t + A_- \sin \phi t) \cos \theta \end{bmatrix} \] (24)

and the success probability after \( t \) iterations is

\[ P_\phi(t) = \frac{1}{2} + \frac{\eta^{t/2}}{2B} [\eta \sin \phi t \sin 2\theta \sin \theta \\ -(B \cos \phi t + A_- \sin \phi t) \cos \theta]. \] (25)

For \( \eta = 1 \), which is the idea GSA, we have \( P_\phi(t) = P(t) = \sin^2 \theta t \). Therefore, \( P(t) \in [0, 1] \) and \( P(t) \) is a periodic function with an oscillating center 1/2. For \( \eta < 1 \), the success probability will quickly tend to 1/2 since the second part of Eq. (25) decays exponentially as \( t \) increases. Support GSA searches in a large database with limit targets satisfying \( m << N \), so that \( \sin 2\theta \sin \theta \) is small enough to be ignored and \( \cos \theta \approx 1 \). In that way, \( P_\phi(t) \) can be simply rewritten as

\[ P_\phi(t) \approx \frac{1}{2} - \frac{\eta^{t/2}}{2B} (B \cos \phi t + A_- \sin \phi t) \]

\[ = \frac{1}{2} - \frac{\eta^{t/2}}{2B} r_p \sin(\phi t + \delta_p), \] (26)

where \( r_p = \sqrt{A_+^2 + B^2} \) and \( \delta_p = \arctan(B/A_-) \). According to Eq. (26), we can obtain a conclusion that the success probability with phase noise \( P_\phi(t) \) is a periodic function of \( t \) with oscillating center 1/2 and decays exponentially toward 1/2.

### 4.2 Bit flip noise in GSA

According to Theorem [1], bit-flip noise can be presented by \( r_z \) and \( r_\phi \) as \( E_b = \begin{bmatrix} 1 & 0 \\ 0 & \eta \end{bmatrix} \) in Bloch. Therefore, the noisy Grover iteration with bit flip noise becomes

\[ G_b = R \circ O \circ E_b = \begin{bmatrix} \cos 2\theta & \eta \sin 2\theta \\ -\sin 2\theta & \eta \cos 2\theta \end{bmatrix}. \] (27)

By calculating, we can obtain that the eigenvalues of \( G_b \) are \( \lambda_\pm = \sqrt{\eta} \exp(\pm i\phi) \), which are the same with the phase flip. Calculating the corresponding eigenvectors, we further obtain \( V^{-1} G_b V = D \), where \( D = \text{diag}\{\lambda_+, \lambda_-\} \),

\[ V = \begin{bmatrix} -A_- - iB & -A_- + iB \\ \sin 2\theta & \sin 2\theta \end{bmatrix} \] (28)

and

\[ V^{-1} = \frac{1}{-2B \sin 2\theta} \begin{bmatrix} \sin 2\theta & A_- - iB \\ -\sin 2\theta & -A_- + iB \end{bmatrix}. \] (29)

Calculations of the matrix \( G_b^t = V D^t V^{-1} \) finally give

\[ G_b^t = \eta^{t/2} \begin{bmatrix} B \cos \phi t + A_- \sin \phi t & \eta \sin \phi t \sin 2\theta \\ -\sin \phi t \sin 2\theta & B \cos \phi t - A_- \sin \phi t \end{bmatrix}. \] (30)

Therefore, we obtain

\[ r_\phi(t) = \eta^{t/2} \begin{bmatrix} -\sin \phi t \sin 2\theta \sin \theta \\ +(B \cos \phi t - A_- \sin \phi t) \cos \theta \end{bmatrix} \] (31)

and the success probability after \( t \) iterations

\[ P_\phi(t) = \frac{1}{2} + \frac{\eta^{t/2}}{2B} \sin(\phi t \sin 2\theta \sin \theta \\ -(B \cos \phi t - A_- \sin \phi t) \cos \theta]. \] (32)

When \( \eta = 1 \), we have \( P_\phi(t) = P(t) \) which is the idea GSA. When \( \eta < 1 \), the success probability will exponentially decay as the number \( t \) of the iterations increases and tends to 1/2, which is similar to the case of GSA with phase flip. With the similar technique for phase flip noise in [1], we further analyze \( P_\phi(t) \) and obtain that

\[ P_\phi(t) \approx \frac{1}{2} - \frac{\eta^{t/2}}{2B} r_b \sin(\phi t - \delta_b), \] (33)

with \( r_b = \sqrt{A_+^2 + B^2} \) and \( \delta_b = \arctan(B/A_-) \). According to Eq. (33), we can draw a conclusion that the success probability \( P_\phi(t) \) with bit flip noise is periodic with oscillating center 1/2 and decays exponentially toward 1/2.
4.3 Bit-phase flip and depolarize noises in GSA

In two dimensions of $r_x$ and $r_y$ for GSA, we obtain that depolarizing channel is the same with bit-phase flip noise as discussed in Sec. 3. Therefore, it is just enough to discuss bit-phase flip noise.

For bit-phase flip noise $E_{bp}$ in Bloch, we have

$$E_{bp} = \eta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \eta I,$$  

(34)

with $\eta = 2p - 1$. So the noisy Grover iterator becomes

$$G_{bp} = \eta \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} = \eta G.$$  

(35)

Therefore, it is easy to obtain its eigenvalues $\lambda_{\pm} = e^{\pm 2i\theta}$. Then $r_x(t)$ becomes

$$r_x(t) = \eta^t \cos[(2t + 1)\theta]$$  

(36)

and the success probability is

$$P_{bp}(t) = \frac{1}{2} - \frac{\eta^t}{2} \cos[(2t + 1)\theta].$$  

(37)

Since $\eta < 1$ in a noisy environment, we can easily get a conclusion that the success probability $P_{bp}(t)$ is periodic with oscillating center 1/2 and decreases exponentially toward to 1/2 as $t$ increases, which is similar to the cases of GSA with phase flip and bit flip noises.

As a matter of fact, we focus more concern on the maximum success probability $P_{max}$ and its corresponding iteration number $t_m$. For bit-phase flip noisy GSA, it will reach the maximum success probability in the first period according to Eq. (37). Therefore, in the following part of this subsection, we discuss some special properties for $0 < t \leq t_m$.

**Theorem 3.** Bit-phase noise has advantages and disadvantages for GSA. It helps to reduce the number $t_m$ of iteration of noisy Grover $G_{bp}$ to achieve the maximum success probability, but decreases the value $P_{max}$ of the maximum success probability.

**Proof.** For the ideal GSA, we know that the success probability after $t$ Grover iterations is

$$P(t) = \sin^2 \theta_t = \frac{1}{2} - \frac{1}{2} \cos[(2t + 1)\theta].$$  

(38)

According to the property of the function $\cos(.)$ and $\eta \leq 1$, compare Eq. (37) and Eq. (35), we can obtain that

$$P_{bp}(t) = \begin{cases} 
> P(t), & \text{if } t < T/2 \\
= P(t), & \text{if } t = T/2 \\
< P(t), & \text{if } T/2 < t \leq t_m.
\end{cases}$$  

(39)

It indicates that the success probability of noisy GSA is improved with the bit-phase flip noise compared with the ideal case without noise for $t < T/2$. Therefore, bit-phase flip noise can improve the success probability at first despite the maximum success probability is decreased at the end.

According to Eqs. (37) and (39), we will obtain that

$$P_{bp}(T/2) = P(T/2) = 1/2$$

and

$$P_{bp}(t) \geq 1/2$$

as $t \geq T/2$. In other words, no matter how heavy the bit-phase flip noise is, GSA always needs only $O(\sqrt{N}/2m)$ iterations with the success probability not less than 1/2. Therefore, even suffering bit-phase flip noise, GSA is always superior to the stochastic classical algorithm of the average number of queries being $N/2$ with the success probability $1/2$.

To obtain $t_m$ and $P_{max}$, we calculate the derivative of $P_{bp}(t)$ with respect to $t$. Considering $P_m \geq 1/2$, we have $\theta \leq \pi/4$, which corresponds to $T/2 \leq t_m \leq T$ according to Eq. (39). Then we obtain

$$t_m = \left\lfloor \frac{\arctan(\ln \eta/2) + \pi}{2\theta} \right\rfloor$$  

(40)

and

$$P_{max} = \frac{1}{2} - \eta^{t_m} \cos(2\theta t_m + 1)/2.$$  

(41)

According to Eqs. (40) and (41), we can obtain the maximum success probability according to the given $\eta$. For $\eta = 1$, we obtain $t_m \approx \pi/4\sqrt{N/m}$ and $P_{max} \approx 1$ when $m \leq N$ which corresponding to the ideal GSA.

Since the function $\arctan(.)$ and $\ln \eta$ are increasing for $0 < \eta \leq 1$, we get that $t_m$ will increase with $\eta$. Therefore, the smaller $\eta$ is, which means more noise is in the system, the less iterations the bit-phase flip noisy GSA needs. In other words, the noise accelerates the search algorithm to
reach its maximum probability. However, \( P_{\text{max}} \) is a decreasing function with \( \eta \) and \( t \) when \( t < t_m \). That is the smaller \( \eta \), which means more noise, the smaller \( P_{\text{max}} \).

Therefore, according to Theorem 3, the bit-phase flip noise helps to reduce the number of iterations needed to reach the maximum success probability, however, it destroys the performance of the algorithm in terms of success probability.

5 Numerical results of noise in GSA

To visualize the above results, we exhibit numerically the performances of GSA with different noises. Suppose that we search one target in the database with \( N = 2^8 \) items. We show the success probability with the search iteration steps as the noise level varies from \( \eta = 1 \) to \( \eta = 0.7 \) in Fig. 2 to Fig. 4, where \( \eta = 1 \) corresponds to the ideal GSA. In order to clear the change of maximum probabilities under different noise levels, we marked them with star markers.

As illustrated in these figures, we can see firstly that the success probabilities are periodic with oscillating center 1/2 both without noise and with noises. For GSA with different types of noises, we can observe that the greater the noise, the faster the success probabilities decay. At the same time, the maximum success probabilities \( P_{\text{max}} \) are decreasing too. They tend to 0.5 as the number \( t \) of iterations increases as shown in Ref. [53].

For phase flip noise, to reach \( P_{\text{max}} \), the corresponding number \( t_m \) of the iterations increases as shown in Fig. 2. If the noise is too heavy, the number \( t_m \) will increase too large.
which makes the quantum superiority totally lost.

However, there are different cases when GSA suffers bit flip and bit-phase flip noises. As shown in Figs. 3 and 4, the numbers of iterations \( t_m \) of \( P_{\text{max}} \) are reduced compared to the ideal GSA (where \( \eta = 1 \)) despite the maximum success probability are decreased. Beside of that, we can observe that the success probabilities are improved at first. Moreover, the success probabilities are raised with the noise levels for \( t < T/2 \). When \( T/2 \leq t < T \), we can observe that \( P \geq 1/2 \) no matter how heavy the noise is. That means GSA suffering bit flip or bit-phase flip noise always needs only \( O(\sqrt{N/m}) \) iterations with the success probability not less than 1/2. From this viewpoint, even suffering these two kinds of noises, GSA is always superior to the classical stochastic algorithm which needs \( N/2 \) queries in average with the success probability 1/2.

Observer of the maximum probabilities marked by stars in these figures. Compared with each others, we can see that the maximum probabilities of GSA with bit-phase flip noises are the lowest for the same noise level. It means that GSA is the easiest to be affected by bit-phase flip and depolarize noises among the five kinds of noises that we discussed in this paper. But the most harmful noises are phase flip and phase damping noises which could make GSA lose the quantum advantage as \( t_m \) increases.

6 Discussion

Different kind of noises, including phase flip/phase damping, bit flip and bit-phase flip/depolarize noises, in Grover quantum search algorithm were investigated in this paper. Firstly, we reconstructed the representation of Grover quantum search algorithms within the Bloch sphere representation as in Rastegin’s work \([33]\). Secondly, we studied the impacts of these noises which act on the Grover iteration \( G \). Finally, we established the relationship of the success probability with these noises and the number of iterations.

(1) In terms of theory: We have analytically discussed the effect of the search performance of phase flip/phase damping, bit flip and bit-phase flip/depolarize noises which act on Grover quantum search algorithm. The relationship among the success probability, noise level, and search iterations is established in Sec. 4.

According to the discussion in Sec. 4, we can summarize that the success probability of noisy GSA decreases exponentially with periodicity and oscillating center 1/2 as the increase of the iteration. When the number of iterations is large enough, the success probability finally tends to 1/2. The result is similar to some previous works, such as Ref. \([35]\). More importantly, we obtained some new interesting results. By carefully discussing the maximum probabilities and their corresponding iteration numbers for GSA with bit-phase flip and depolarize noises, we have proven that they can reach success probability 1/2 with less than \( O(\sqrt{N/m}) \) iterations. From this viewpoint, the noisy GSA surpasses the classical search algorithm, which needs \( N/2 \) iterations in average to obtain the same probability. In particular, the larger the noise is, the faster the noisy GSA decreases to 1/2.

(2) In terms of numerical experiments, we have obtained some special results by studying different noise levels in different noisy search algorithms. For phase flip and phase damping noises, when noises are not too large, they always cause the success probability to degrade until it drops to 1/2. As the noise increases, the noisy GSA will lose its quantum superiority for it will take a large number of iterations to 1/2. However, for bit flip and bit-phase flip/depolarize noises, we obtain an exceptional conclusion: in the initial search of the noisy algorithm, they can improve the success probability, but with the increase of the number of iterations, these noises will eventually reduce the maximum success probability to 1/2. Besides of that, these noises help noisy GSA to reach success probability 1/2 with fewer iterations. In particular, the larger the noise is, the quicker
the noisy GSA to reach 1/2. These numerical experiments have exhibited that noises in some aspects can help GSA to improve the performance. However, from another point of view, the success probability turning to 1/2 might be due to the noise that is so heavy making the oracles are unable to distinguish between the targets and non-targets. This is not a good thing so that we would never find the right target state.

7 Conclusion

In order to clear the performance of Grover search algorithm (GSA) in NISQ era, we have discussed the dynamics of the states of GSA under different diagonalizable noises. According to the analytical and numerical results and the discussions, we can conclude as follows. For different typical diagonalizable noises, the success probability of GSA decays exponentially toward 1/2 with an oscillatory center of 1/2. In general, noises degrade the performance of GSA, but it is probably not the case for the whole search process for some types of noises. In other words, some kinds of noises can improve the performance of GSA for certain parts of the search process in terms of the number of iterations. It possibly brings up a new way to deal with noise that we may try to harness it rather than merely eliminate and avoid it in NISQ computing. According to this feature, we may think about taking the advantage of noise, which is helpful for us to design quantum algorithms. The noises may destroy algorithms and make them ineffective. However, for a moderate amount of noise and suitable size of the database, we can also do some things such as verify the correctness of quantum algorithms and apply to some applications such as machine learning which has been widely research in NISQ era.

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