A proposal to obtain a finite contribution of second derivative order to the gravitational field equations in $D = 4$ dimensions from a renormalized Gauss-Bonnet term in the action has recently received a wave of attention, and triggered a discussion whether the employed renormalization procedure yields a well-defined theory. One of the main criticisms is based on the fact that the resulting field equations cannot be obtained as the Euler-Lagrange equations from a diffeomorphism-invariant action.

In this work we use techniques from the inverse calculus of variation as an independent confirmation that the suggested truncated Gauss-Bonnet field equations cannot be variational, in any dimension. For this purpose, we employ canonical variational completion, based on the notion of Vainberg-Tonti Lagrangian, which consists in adding a canonically defined correction term to a given system of equations, so as to make them derivable from an action. We find that in $D > 4$ the suggested field equations can be variationally completed, which yields a theory with fourth order field equations. In $D = 4$ the variationally completed theory diverges.

Our findings are in line with Lovelock’s theorem, which states that, in 4 dimensions, the unique second-order Euler-Lagrange equations arising from a scalar density depending on the metric tensors and its derivatives, are the Einstein equations with a cosmological constant.

I. INTRODUCTION

The study of alternative and extended theories of gravity besides general relativity is motivated by observations in cosmology, such as the accelerating expansion of the universe, and by its tension with quantum theory. The latter has stipulated to consider quantum corrections to the Einstein-Hilbert action in form of higher curvature invariants. One such invariant, which is purely topological in four spacetime dimensions, is the Gauss-Bonnet invariant. Even though it does not contribute to the gravitational field equations in four dimensions, it has been shown that the contributions arising in higher dimensions can be renormalized in such a way as to yield a non-trivial contribution also in the limit of four dimensions [1, 2]. Considering these higher curvature terms as new terms in a classical theory of gravity instead of quantum corrections has led to the proposal of a new “renormalized 4D Gauss-Bonnet” theory gravity, which claims to yield a finite second derivative order contribution to the Einstein equations from a Gauss-Bonnet term in the gravitational action [3]. It is thus based on a similar concept as a dimensional regularization of the Einstein-Hilbert action, which is topological in $D = 2$ dimensions [4]. Also a generalization to further Lovelock curvature terms appearing at higher dimensions has been considered [5].

However attractive, the proposed model has also received criticism, and the correctness of the procedure to obtain field equations, or even symmetric solutions, from renormalizing the Gauss-Bonnet term in the gravitational action to obtain a finite, non-vanishing contribution in the limit of $D \to 4$ dimensions, has been challenged. Various works have shown inconsistencies in the proposed approach. The most obvious contradiction, which also concerns the aforementioned regularized theories in other dimensions, arises from the fact that the existence of a theory with the claimed properties would violate Lovelock’s result [6] that the only generally covariant Lagrangian field theory of the metric tensor alone, giving second-order field equations in four dimensions, is given by general relativity, possibly with a cosmological constant [6]. One manifestation of this contradiction is the “index problem”, which states that certain terms in the field equations vanish due to the number of possible index combinations in a given dimension, which is a discrete number, and therefore does not allow for a continuous limiting procedure. This has been pointed out already for the case of $D = 2$ regularized Einstein gravity [7], and applies also to the proposed 4D Gauss-Bonnet theory [8, 9]. Further, it has been shown that the proposed model yields consistent solutions only for highly symmetric...
spacetimes \cite{10}, and that the obtained regularized field equations cannot be obtained from a regular, diffeomorphism-invariant action \cite{11}.

In order to circumvent the aforementioned shortcomings and to obtain a consistent 4-dimensional theory preserving certain features of the proposed Gauss-Bonnet theory, various approaches have been studied. One possible approach is to consider the 4D Gauss-Bonnet theory as arising from the Kaluza-Klein reduction of a higher-dimensional theory. This approach leads to the appearance of a scalar Kaluza-Klein mode, which introduces additional contributions to the gravitational field equations also in 4 dimensions, which then reproduce the proposed contribution from the Gauss-Bonnet term \cite{12,13}. Another approach that yields a supplementary scalar mode is by introducing a counter-term in the gravitational action, which is constructed from a conformally rescaled metric \cite{14–16}, in analogy to a similar procedure for $D = 2$ Einstein gravity \cite{17}. These extensions are in line with Lovelock's theorem, as they introduce another dynamical field besides the metric, and so it is possible to obtain second-order, local, Lagrangian field equations in 4 dimensions. Hence, they fall into the Horndeski class of gravity theories \cite{18}. The resulting scalar-tensor field equations, however, are not equivalent to those originally proposed \cite{19}. Other possibilities include to explicitly break the invariance of the theory under diffeomorphisms, such as by deriving the field equations from a Hamiltonian approach \cite{20}, or to employ holography as a means to obtain non-trivial contributions from boundary terms \cite{21}.

Both the originally proposed 4D Gauss-Bonnet theory and its scalar-tensor regularizations have received remarkable attention; in particular, highly symmetric solutions to the proposed field equations have been studied, such as black holes \cite{22–73}, wormholes \cite{74–76}, other compact objects \cite{77–80} and cosmology \cite{81–86}. Bounds on the theory have been obtained from its weak field limit \cite{87} as well as cosmological perturbations \cite{88,89} and the speed of gravitational waves \cite{85,90}. Also the asymptotic structure \cite{91} as well as aspects of quantum gravity \cite{92,93} and quantum cosmology \cite{94} have been studied.

Having reached the conclusion that the originally proposed 4D Gauss-Bonnet field equations cannot be variational, one may seek for alternative approaches to finding a set of regular, variational field equations, without the explicit introduction of a scalar degree of freedom. In particular, one may pose the question which set of variational field equations for the metric tensor alone would be as close as possible to the proposed equations. A constructive approach to answer this question is the method of canonical variational completion \cite{95,96}. Starting from an arbitrary set of differential equations defined on a specific coordinate chart, it yields a Lagrangian on the respective coordinate chart, whose Euler-Lagrange equations coincide with the original set of differential equations if and only if these are variational. In case they are not, it gives a canonical, in a sense minimal, correction term to be added to the original equations such that they become variational. A standard example is provided by the Ricci tensor, whose canonical variational completion is the full Einstein tensor. Another example of successful variational completion is the canonical variational field equation for Finsler gravity \cite{97}.

The aim of this article is to study the possibility of extending the proposed field equations for 4D Gauss-Bonnet gravity using the method of canonical variational completion. In the case we study here, where the original equations are not variational, one still obtains a Lagrangian whose Euler-Lagrange equations are a canonical extension of the original equations. However, it turns out that these canonically extended equations cannot be of second order in any dimension and do not make sense in $D = 4$.

For this purpose, we split the contribution to the gravitational field equations arising from the Gauss-Bonnet term, into two parts: one part which vanishes identically in $D = 4$ dimensions for combinatorial reasons, and does not allow for a limit $D \to 4$, and a part which is proportional to $D - 4$, and can hence be renormalized to yield a finite contribution also in $D = 4$ dimensions. We then apply the method of variational completion to each of these terms separately, and demonstrate that due to their degree of homogeneity in the dynamical variables of the theory, the obtained canonical correction still diverges in dimension $D = 4$.

The outline of this article is as follows. In Section II, we briefly review the proposed field equations of 4D Gauss-Bonnet gravity. A brief review of the method of canonical variational completion is provided in section III. We then apply the method to 4D Gauss-Bonnet gravity in Section IV. We end with a conclusion in Section V. Technical details on the derivative order of the variational completion of the truncated field equations are presented in Appendix A. Appendix B contains technical details on the mathematical nature of Euler-Lagrange expressions in the variational completion algorithm.

II. 4D GAUSS-BONNET GRAVITY

The proposed 4-dimensional extension of Gauss-Bonnet gravity is based on the $D$-dimensional action \cite{3}

\[ S = \int d^D x \sqrt{-g} \left[ \frac{M_p^2}{2} R - \Lambda_0 + \frac{\alpha}{D - 4} \mathcal{G} \right] + S_m, \]

(1)
where the Gauss-Bonnet scalar is given by
\[ G = 6R^{\mu\nu|\rho\sigma}R_{\rho\sigma} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} . \] (2)
By variation with respect to the metric \( g_{\mu\nu} \) one obtains the field equations
\[ E_{\mu\nu} = M_5^2G_{\mu\nu} + \Lambda_0g_{\mu\nu} - \frac{2\alpha}{D-4}G_{\mu\nu} = T_{\mu\nu} , \] (3)
where
\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \] (4)
is the Einstein tensor, and the term originating from the Gauss-Bonnet scalar is given by
\[ G_{\mu\nu} = 15g_{\mu[\nu}R^{\rho\sigma}_{\rho\sigma}R_{\omega\sigma]\nu]} = \frac{1}{2}G_{\mu\nu} - 2R_{\mu\rho\sigma}\lambda^\rho_\sigma + 4R_{\mu\rho\sigma\sigma}R_{\rho^\sigma} + 4R_{\mu\rho}R_{\rho^\sigma} - 2RR_{\mu\nu} . \] (5)
It has been argued in [3] that since \( G_{\mu\nu} = 0 \) in \( D = 4 \) dimensions, this theory has a well-defined limit for \( D \rightarrow 4 \). However, this is a fallacy, since it can be shown that the latter term is given by [8, 11]
\[ - G_{\mu\nu} = (D-4)A_{\mu\nu} + W_{\mu\nu} , \] (6)
where we introduced the tensors
\[ A_{\mu\nu} = \frac{D}{D-2} \left[ \frac{2D}{D-1}RR_{\mu\nu} - \frac{4D-2}{D-3}R^\rho_\lambda C_{\mu\rho\lambda\nu} - 4R_{\mu\rho}R_{\nu\sigma} + 2R_{\rho\lambda}R^{\rho\lambda}g_{\mu\nu} - \frac{1}{2}D + 2 \right] R^2g_{\mu\nu} \] (7)
and
\[ W_{\mu\nu} = 2C_{\mu\rho\lambda\sigma}C_{\nu\rho\lambda\sigma} - \frac{1}{2}C_{\tau\rho\lambda\sigma}C_{\tau\rho\lambda\sigma}g_{\mu\nu} , \] (8)
using the Weyl tensor
\[ C_{\mu\rho\lambda\sigma} = R_{\mu\rho\lambda\sigma} + \frac{1}{D-2}(R_{\mu\rho}g_{\nu\sigma} - R_{\mu\rho}g_{\nu\sigma} + R_{\nu\rho}g_{\mu\sigma} - R_{\nu\rho}g_{\mu\sigma}) + \frac{1}{(D-1)(D-2)}R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) . \] (9)
and that one cannot extract a factor \( D - 4 \) from the latter term \( W_{\mu\nu} \), which vanishes in \( D = 4 \) dimensions for combinatorial reasons. Hence, the field equations (3), which now take the form
\[ E_{\mu\nu} = M_5^2G_{\mu\nu} + \Lambda_0g_{\mu\nu} + 2\alpha \left( A_{\mu\nu} + \frac{W_{\mu\nu}}{D-4} \right) = T_{\mu\nu} , \] (10)
do not have a smooth limit for \( D \rightarrow 4 \), due to the appearance of the last term. It has thus been argued that this term should be omitted in \( D = 4 \) dimensions, and only the truncated part \( A_{\mu\nu} \) be considered as the \( D \rightarrow 4 \) limit of the field equations. However, as shown in [11], this term cannot originate from the variation of a diffeomorphism-invariant action. In the following we will show that even dropping the diffeomorphism invariance request, this term cannot originate from any action at all and also the variational completion of these truncated field equations degenerates in \( D = 4 \).

III. VARIATIONAL COMPLETION

We now briefly review the method of canonical variational completion [95, 96], which we will employ in the next section. Given an arbitrary PDE system, the inverse variational problem consists in finding out whether there exists a Lagrangian function \( L \) having (11) as its Euler-Lagrange equations.
In the following, we will limit our attention to second order PDE systems:
\[ \mathcal{E}_A(x^\mu,y^B,y^B_{\mu},y^B_{\mu\nu}) = 0 , \] (11)
where \( x^\mu, \) \( (\sigma,\mu,\nu = 0,...,n-1) \) are coordinates on a smooth manifold \( M, y^A, \) \( (A,B = 1,...,m) \) are components of sections into fibre bundles \( (Y \xrightarrow{\pi} M,F) \) over \( M \) and \( y^A_{\mu} = \partial_\mu y^A, \) \( y^A_{\mu\nu} = \partial_\mu \partial_\nu y^A \) are their derivatives.
We will regard $\mathcal{E}_A$ as functions defined on a specific fibered coordinate chart $(V^2, \psi^2)$ on the jet bundle $J^2Y$; Lagrangians will be regarded as differential forms

$$\lambda = \mathcal{L} d^n x$$
on $V^2$.

The system (11) is called:

- **locally variational**, if, corresponding to each fibered chart $(V^2, \psi^2)$ on $J^2Y$, there exists a Lagrangian $\lambda_V$ on $V^2$ having (11) as its Euler-Lagrange equations;

- **globally variational**, if (11) admits a Lagrangian $\lambda = \mathcal{L} d^n x$ defined on the entire $J^2Y$, i.e., the various Lagrangians $\lambda_V$ can be smoothly glued together into a single Lagrangian $\lambda$.

In the following, by "variational", unless elsewhere specified, we will mean locally variational. It is important to note that, here, the term "local" means "defined over a specific coordinate chart", i.e., it has a different meaning than the one it commonly has in physics. It does neither imply that the Lagrangian is coordinate invariant, nor that it cannot contain, e.g., integrals.

There are basically two ways of checking whether a given PDE system is locally variational: checking the so-called Helmholtz conditions, [96], or explicitly finding a Lagrangian. These two methods are tightly related, as follows.

Given the PDE system (11), we introduce on the given coordinate neighborhood $V^2$ the so-called *Vainberg-Tonti Lagrangian* function

$$\mathcal{L}_\varepsilon = y^A \int_0^1 \mathcal{E}_A(x^\mu, t y^B, t y^B_\mu, t y_B^\nu) dt.$$  \hspace{1cm} (12)

If the above integral is well defined, then the Vainberg-Tonti Lagrangian gives rise to the Euler-Lagrange expressions

$$\hat{E}_A := \frac{\partial \mathcal{L}_\varepsilon}{\partial y^A} - d_\mu \frac{\partial \mathcal{L}_\varepsilon}{\partial y^A_\mu} + d_\mu d_\nu \frac{\partial \mathcal{L}_\varepsilon}{\partial y^A_{\mu\nu}},$$  \hspace{1cm} (13)

where $d_\mu = \frac{d}{dt} y^\mu$.

An important result from variational calculus states that a system of partial differential equations for which the integral (12) makes sense is locally variational if and only if the obtained Euler-Lagrange equations $\hat{E}_A = 0$ coincide with the original equations (11). Actually, it can be shown that the correction terms

$$H_A = \hat{E}_A - \mathcal{E}_A$$  \hspace{1cm} (14)

are linear combinations of the coefficients of the so-called *Helmholtz form*; the Helmholtz conditions actually state that local variability of a given PDE system is equivalent to the vanishing of the associated Helmholtz form; the explicit coordinate expressions of the Helmholtz conditions can be found, e.g. in [96]. That is, if the system (11) is locally variational, then the Vainberg-Tonti Lagrangian, is a Lagrangian for (11), defined on the given coordinate neighborhood; moreover, any other Lagrangian for (11) will differ from the Vainberg-Tonti Lagrangian by a divergence expression - which will bring no contribution to the Euler-Lagrange equations.

In other words, the mapping attaching to a class of equivalent Lagrangians, their common Euler-Lagrange expressions (more technically, their common *Euler-Lagrange source form*), and the mapping attaching to a given variational source form (i.e., to a set of Euler-Lagrange expressions), its Vainberg-Tonti Lagrangian, are inverse to each other.

The system (13) of partial differential equations, which is locally variational by construction, is called the *canonical variational completion* of the original system (11). Thus, the canonical variational completion of (11) is obtained by adding the corresponding Helmholtz expressions; if (11) is not variational, then, these provide nontrivial correction terms to the original system. A standard example in this sense is the following.

**Example: the Einstein tensor as the canonical variational completion of the Ricci tensor** [95].

Historically, it is known that the first variant of gravitational field equations proposed by Einstein was

$$R_{\mu\nu} = \kappa T_{\mu\nu}.$$  \hspace{1cm} (15)

Later on, he noticed that this system was inconsistent, since the right hand side is covariantly divergence-free, while the Ricci tensor is not; based on a heuristic argument involving the contracted Bianchi identity, he then added the correction term $\frac{\kappa}{2} R_{\mu\nu}$ to the left hand side, thus obtaining the nowadays known form of the fundamental equations of general relativity.
IV. VARIATIONAL COMPLETION OF 4D GAUSS-BONNET GRAVITY

Let us start by a simple remark. It is already known, [96, p. 147], that, if a second order PDE system is variational, then it must be linear in the second order derivatives of the dependent variable. The truncated Gauss-Bonnet terms $A_{\mu\nu}$ are not linear in the second order derivatives of $g_{\mu\nu}$, see Appendix A, in any dimension. Hence, independently of the value of $D$, they cannot represent the Euler-Lagrange expressions of any Lagrangian.

In the following, we aim to determine a correction term to be added to $A_{\mu\nu}$, so as to make them variational.

More precisely, we will apply the above described canonical variational completion algorithm to the truncated Gauss-Bonnet gravity equations

$$\dot{E}_{\mu\nu} = M_{D}^{2} G_{\mu\nu} + \Lambda_{0} g_{\mu\nu} + 2 \alpha A_{\mu\nu} = T_{\mu\nu},$$

where the over-set circle denotes truncation, and note that their Vainberg-Tonti Lagrangian diverges in $D = 4$ dimensions.

Let us calculate the Vainberg-Tonti Lagrangian from the field equations (10). In order to use the components of the metric $g_{\mu\nu}$ as our dynamical variables $y^{A}$, we need to raise the indices and to restore the density factor in the field equations (3), such that they read

$$E^{\mu\nu} = \frac{1}{2} \sqrt{-g} E^{\mu\nu},$$

where the factor $-\frac{1}{2}$ arises from the definition of the energy-momentum tensor in the field equations (3). Hence, these are the correct Euler-Lagrange expressions obtained from variation of the action (1). Without this density factor, the equations cannot be variational, which can be proven using the Helmholtz conditions, see Appendix B for the mathematical details. Thus, omitting the density would break the relation between the Vainberg-Tonti Lagrangian of an already variational system, and the variation, which recovers these equations as its Euler-Lagrange equations.

Note that under a rescaling $g_{\mu\nu} \rightarrow t g_{\mu\nu}$ the terms in the field equations transform as

$$g_{\mu\nu} \rightarrow t g_{\mu\nu}, \quad G_{\mu\nu} \rightarrow G_{\mu\nu}, \quad A_{\mu\nu} \rightarrow t^{-1} A_{\mu\nu}, \quad W_{\mu\nu} \rightarrow t^{-3} W_{\mu\nu}.$$  

so that after raising indices we have

$$g^{\mu\nu} \rightarrow t^{-1} g^{\mu\nu}, \quad G^{\mu\nu} \rightarrow t^{-2} G^{\mu\nu}, \quad A^{\mu\nu} \rightarrow t^{-3} A^{\mu\nu}, \quad W^{\mu\nu} \rightarrow t^{-3} W^{\mu\nu}.$$  

This gives us the Vainberg-Tonti Lagrangian density

$$\mathcal{L} = -\frac{1}{2} g_{\mu\nu} \int_{0}^{1} t^{D/2} \sqrt{-g} \left[ t^{-2} M_{D}^{2} G^{\mu\nu} + t^{-1} \Lambda_{0} g^{\mu\nu} + 2 \alpha A^{\mu\nu} + \left( \frac{W_{\mu\nu}}{D - 4} \right) \right] dt$$

$$= -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \left[ \frac{2 M_{D}^{2}}{D - 2} G^{\mu\nu} + \frac{2 \Lambda_{0}}{D} g^{\mu\nu} + \frac{4 \alpha}{D - 4} \left( A^{\mu\nu} + \frac{W_{\mu\nu}}{D - 4} \right) \right]$$

$$= \sqrt{-g} \left[ \frac{M_{D}^{2}}{2} \mathcal{R} - \Lambda_{0} - \frac{2 \alpha}{D - 4} \left( A^{\mu\nu} + \frac{W_{\mu\nu}}{D - 4} \right) \right],$$

where we used $g_{\mu\nu} g^{\mu\nu} = D$ as well as

$$g_{\mu\nu} G^{\mu\nu} = g_{\mu\nu} \left( \mathcal{R}^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \right) = R - \frac{D}{2} \mathcal{R},$$

and the appearing traces are given by

$$A_{\mu}^{\mu} = \frac{D - 3}{D - 2} \left( 2 R_{\mu\nu} R^{\mu\nu} - \frac{D}{2(D - 1)} R^{2} \right),$$

$$W_{\mu}^{\mu} = (D - 4) \left( \frac{2}{D - 2} R_{\mu\nu} R^{\mu\nu} - \frac{1}{(D - 1)(D - 2)} R^{2} - \frac{1}{2} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right).$$
We see that the first two terms give us the Einstein-Hilbert Lagrangian density, as one would have expected. For the last terms, however, we observe the following:

1. First, note that the procedure giving us the Vainberg-Tonti Lagrangian density is not well-defined for the last two terms in $D = 4$ dimensions, since in that case one would obtain a term proportional to $t^{-1}$ in the integral (20), which would diverge. Hence, we can perform the integration only in $D > 4$ dimensions, in which case one obtains a factor $(D - 4)^{-1}$ in the result.

2. The combination of the last two terms satisfies

$$A^\mu{}_{\mu} + \frac{W^\mu{}_{\mu}}{D - 4} = \frac{1}{2} g \, ,$$

and so the full Lagrangian (20) indeed recovers the gravitational part of the action (1), hence confirming the validity of the variational completion procedure for the full field equations (10).

3. The Vainberg-Tonti Lagrangian of the truncated field equations (16), in which the term $W^\mu{}_{\mu}$ has been omitted, yields the truncated Lagrangian

$$\tilde{\mathcal{L}} = \sqrt{-g} \left[ \frac{M^2}{2} R - \Lambda_0 - \frac{2\alpha}{D - 4} A^\mu{}_{\mu} \right]$$

in any dimension $D$ except $D = 4$. Its variation does not give back the original truncated field equations (16), but their canonical variationally completed field equations, which have been obtained using the Mathematica package xAct [98] and xTras [99],

$$\tilde{\mathcal{E}}_{\mu\nu} = M^2 G_{\mu\nu} + \Lambda_0 g_{\mu\nu} + \frac{4\alpha(D - 3)}{(D - 1)(D - 2)(D - 4)} \left[ g_{\mu\nu} \left( R - \frac{D}{4} R^2 + (D - 1) R^\alpha{}_{\rho\sigma} R^\rho{}_{\sigma} \right) - 2(D - 1) \Box R_{\mu\nu} + (D - 2) \nabla_\mu \nabla_\nu R + D R R_{\mu\nu} - 4(D - 1) R^\alpha{}_{\rho\sigma} R_{\mu\rho\nu\sigma} \right] = T_{\mu\nu} \, ,$$

which clearly contain fourth order derivatives acting on the metric, usually canceled from the contributions coming from the $W^\mu{}_{\mu}$ term, see Appendix A. Also, it becomes explicitly visible that these field equations diverge in $D = 4$.

In order to avoid the divergent result in the Vainberg-Tonti Lagrangian density (20), one may attempt to consider not the metric $g_{\mu\nu}$ as dependent variable $y^A$ in the variational completion algorithm shown in section III, but rather its inverse $g^{\mu\nu}$. To derive the Vainberg-Tonti Lagrangian density, one must then consider the rescaling $g^{\mu\nu} \rightarrow t g^{\mu\nu}$, under which the terms in the field equations (10) behave as

$$g_{\mu\nu} \rightarrow t^{-1} g_{\mu\nu} \, , \quad G_{\mu\nu} \rightarrow G_{\mu\nu} \, , \quad A_{\mu\nu} \rightarrow t A_{\mu\nu} \, , \quad W_{\mu\nu} \rightarrow t W_{\mu\nu} \, .$$

The field equations to be used, together with a proper density factor, would then take the form

$$\mathcal{E}_{\mu\nu} = -\frac{1}{2} \sqrt{-g} E_{\mu\nu} \, ,$$

where $E_{\mu\nu}$ is given by the field equations (10). For the Vainberg-Tonti Lagrangian density one then finds the expression

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \int_0^1 t^{-D/2} \sqrt{-g} \left[ M^2 G_{\mu\nu} + t^{-1} \Lambda_0 g_{\mu\nu} + 2 t \alpha \left( A_{\mu\nu} + \frac{W_{\mu\nu}}{D - 4} \right) \right] \, dt \, .$$

However, note that the appearance of powers $t^k$ with $k \leq -1$ in this integral, which diverge when integrated over the domain $[0, 1]$. Hence, this integral is not well-defined, and therefore cannot be used in order to obtain the Vainberg-Tonti Lagrangian density. This confirms our conclusions enumerated above.

V. CONCLUSION

The variational completion algorithm is a powerful tool to answer the question if a certain set of field equations are the Euler-Lagrange equations of an action principle and, in the negative case, it determines a correction term to
the original equations, which makes them variational. We used this tool to analyse the field equations of the recently proposed “renormalized 4D Gauss-Bonnet” theory of gravity [3].

With this article, we have confirmed from the standpoint of the inverse problem of the calculus of variations that the field equations (16) cannot be variational in any dimension. Moreover, we have proven that they do not possess a variational completion in $D = 4$ dimensions, but only in dimension $D > 4$. These completed field equations then necessarily contain higher than second order derivatives acting on the metric. The reason for this is that the separate terms $A_{\mu \nu}$ and $W_{\mu \nu}$ do not reproduce their individual contribution to the field equations, but additional terms of fourth derivative order, which would otherwise cancel when they are summed to form the Gauss-Bonnet term.

ACKNOWLEDGMENTS

MH and CP were supported by the Estonian Ministry for Education and Science through the Personal Research Funding Grants PRG356 and PSG489, as well as the European Regional Development Fund through the Center of Excellence TK133 “The Dark Side of the Universe”. The authors would like to acknowledge networking support by the COST Actions CANTATA (CA15117) and QGMM (CA18108), supported by COST (European Cooperation in Science and Technology).

Appendix A: Properties of the field equations from the $A^\mu \nu$ term

We claimed in item 3 of Section IV that the variationally completed field equations of the truncated Einstein-Gauss Bonnet field equations (16) contain higher than second order derivatives in any dimension. This is true due to the following line of argument.

The trace of $A_{\mu \nu}$ contains non-trivial terms which are quadratic in the second derivatives of the metric, which do not factor in a way that Bianchi identities cancel these terms. Hence also $A_{\mu \nu}$ itself contains such terms. This can be explicitly be realized by introducing a counting parameter $\epsilon$ and replacing every term $\partial_\mu \partial_\nu g_{\rho \sigma}$ by $\epsilon \partial_\mu \partial_\nu g_{\rho \sigma}$. Doing so we can express $A^\mu \nu$ as power series in $\epsilon$ and find, with help of the computer algebra program xAct for Mathematica [98],

$$A^\mu \nu = \epsilon^2 \frac{D - 2}{2(D - 1)} g^{\mu \sigma} \xi^\rho g^{\nu \omega} g^{\tau \upsilon}$$

$$\left( D(-\partial_\xi g_{\mu \lambda} + 2 \partial_\xi \partial_\upsilon g_{\mu \sigma})\partial_\xi \partial_\upsilon g_{\rho \tau} + (D - 1)\partial_\xi \partial_\upsilon g_{\mu \lambda}\partial_\xi g_{\rho \tau} + \partial_\xi \partial_\upsilon g_{\mu \lambda}\partial_\xi g_{\rho \tau} g_{\sigma \zeta} - D\partial_\xi \partial_\lambda g_{\mu \sigma}\partial_\xi g_{\rho \omega} + (D - 1)\partial_\mu \partial_\nu g_{\rho \sigma} - 4 \partial_\mu \partial_\nu g_{\xi \tau} + 2 \partial_\mu \partial_\nu g_{\xi \omega} \right)$$

$$+ 2(D - 1)\partial_\mu \partial_\nu g_{\mu \lambda}(\partial_\xi \partial_\upsilon g_{\xi \tau} - 2 \partial_\xi \partial_\upsilon g_{\xi \omega} + \partial_\xi \partial_\upsilon g_{\xi \tau})$$

+ lower order terms in $\epsilon$.

Hence, also the untraced tensor $A_{\mu \nu}$, which is part of the truncated field equations, must contain terms of the form $\partial_\mu \partial_\nu g_{\rho \sigma} \partial_\xi \partial_\upsilon g_{\xi \tau}$.

But, any variational PDE system which is of second order must be linear in the second order derivatives acting on the fundamental dynamical variable [96, p. 147]. Hence the truncated field equations cannot be variational and the variation of $A^\mu \nu$ cannot be of second order only, but must contain higher derivatives.

Appendix B: Necessity of densitising in variational completion

In Section IV we applied the variational completion algorithm to the original and to the truncated Einstein Gauss-Bonnet gravity field equations in any dimension.

An important first step in applying the algorithm was to define the densitized field equations in equation (17). In the following, we are going to prove that, given that the expressions $E^{\mu \nu} = -\frac{1}{4} E^{\mu \nu} \sqrt{-g}$ are the Euler-Lagrange expressions of a Lagrangian $\lambda = \tilde{L} d^n x$, then the expressions $E^{\mu \nu}$ cannot arise as the Euler-Lagrange expressions of any Lagrangian (either coordinate-invariant or not).

Mathematically more precise, variationality is generally discussed for certain differential forms $\epsilon$ on a jet bundle of a fibered manifold, rather than for PDE’s. These differential forms are called source forms and their local coefficients $\epsilon_A$ are the left hand sides of the given PDE’s. Multiplying a PDE system by a positive factor (such as $\sqrt{-g}$) will
Inevitably lead to a different source form; thus, this factor does not affect the set of solutions of them PDE system, but does affect its variationality.

To fix the notation, let \((Y \rightarrow M, F)\) be a fiber bundle over \(M\), with a local coordinate system \((x^\mu, y^A)\) adapted to the fibration. Sections (physically interpreted as fields) are maps \(\gamma : U \rightarrow Y\), where \(U \subset M\) is open, are locally described as \(\gamma : (x^\mu) \rightarrow (y^A(x^\mu))\). On the second order jet bundle \(J^2Y\), we denote the induced coordinates by \((x^\mu, y^A, y^A_{\mu}, y^A_{\mu\nu})\).

It is important to note that, on the jet bundle \(J^2Y\), the quantities \(x^\mu, y^A, y^A_{\mu}, y^A_{\mu\nu}\) are interpreted as coordinate functions (i.e., they are independent of one another); only when composed by (prolonged) sections, they provide the derivatives of the functions \((y^A(x^\mu))\). In other words, \(y^A_{\mu}\) are some "slots", into which, when we insert a section \(\gamma\), we obtain the partial derivatives \(\frac{\partial y^A}{\partial x^\mu}\).

In [96, p. 147], it was shown that, for a second order source form \(\varepsilon\) on \(J^2Y\), with local coefficients \(\varepsilon_A = \varepsilon_A(x^\mu, y^B, y^B_{\mu}, y^B_{\mu\nu})\), local variationality is equivalent to the following Helmholtz conditions being identically satisfied by \(\varepsilon_A\):

\[
\begin{align*}
H_{AB}(\varepsilon) &:= \frac{\partial \varepsilon_A}{\partial y^B_{\mu\nu}} - \frac{\partial \varepsilon_B}{\partial y^A_{\mu\nu}} = 0 & (B1) \\
H_{\mu\nu}^A(\varepsilon) &:= \frac{\partial \varepsilon_A}{\partial y^B_{\mu\nu}} + \frac{\partial \varepsilon_A}{\partial y^B_{\nu\mu}} - d_\mu \left( \frac{\partial \varepsilon_A}{\partial y^B_{\mu\nu}} + \frac{\partial \varepsilon_B}{\partial y^A_{\mu\nu}} \right) = 0 & (B2) \\
H_{AB}(\varepsilon) &:= \frac{\partial \varepsilon_A}{\partial y^B_{\mu\nu}} - \frac{\partial \varepsilon_B}{\partial y^A_{\mu\nu}} - \frac{1}{2} d_\nu \left( \frac{\partial \varepsilon_A}{\partial y^B_{\mu\nu}} - \frac{\partial \varepsilon_B}{\partial y^B_{\nu\mu}} \right) = 0 & (B3)
\end{align*}
\]

Here, \(d_\mu = \partial_\mu + y^A_{\nu\rho} \frac{\partial}{\partial y^A_{\nu\rho}} + y^A_{\mu\nu} \frac{\partial}{\partial y^A_{\mu\nu}} + y^A_{\mu\nu\rho} \frac{\partial}{\partial y^A_{\mu\nu\rho}}\) is the total derivative operator (of order three) acting on functions \(f : J^2Y \rightarrow \mathbb{R}\), \(f = f(x^\mu, y^B, y^B_{\mu}, y^B_{\mu\nu})\). In particular, \(d_\mu y^A = y^A_{\mu}\).

Now, let us assume that the source form \(\varepsilon\) satisfies the Helmholtz conditions. Multiplying \(\varepsilon\) by a factor \(f = f(x^\mu, y^B)\) depending on the dynamical variables \(y^B\) but not on their derivatives, we obtain new source form \(f\varepsilon\), with local coefficients \(f\varepsilon_A\).

The first Helmholtz condition (B1) is, indeed, not affected by the rescaling. But, substituting \(f\varepsilon_A\) instead of \(\varepsilon_A\) into (B2) gives:

\[
H_{\mu\nu}^A(f\varepsilon) := fH_{AB}^A(\varepsilon) - (d_\mu f) \left( \frac{\partial \varepsilon_A}{\partial y^B_{\mu\nu}} + \frac{\partial \varepsilon_B}{\partial y^A_{\mu\nu}} \right).
\]

The term \(fH_{AB}^A(\varepsilon)\) vanishes by the variationality assumption on \(\varepsilon\); using (B1) in the remaining term, we get \(\frac{\partial \varepsilon_A}{\partial y^B_{\mu\nu}} + \frac{\partial \varepsilon_B}{\partial y^A_{\mu\nu}} = 2 \frac{\partial \varepsilon_A}{\partial y^A_{\mu\nu}}\) and therefore:

\[
H_{AB}^\nu(f\varepsilon) = -2(d_\mu f) \frac{\partial \varepsilon_A}{\partial y^A_{\mu\nu}}. 
\]

Now, let us apply the above result for \(\varepsilon_A := \mathcal{E}^{\mu\nu}, y^A := g_{\mu\nu}, f := \frac{1}{\sqrt{-g}}\). These are functions on the jet bundle \(J^2\text{Met}(M)\), where \(\text{Met}(M)\) is the bundle of nondegenerate tensors of type \((0, 2)\) over the spacetime manifold \(M\). On this bundle, a system of fibered coordinate functions has the form \((x^\mu; g_{\mu\nu}; g_{\mu\nu,\rho}; g_{\mu\nu,\rho\tau})\).

A brief direct computation, using \(\frac{\partial g}{\partial g_{\nu\rho}} = g^{\nu\rho} g\), gives:

\[
d_\mu f = \frac{1}{2} (-g)^{-1/2} g^{\nu\rho} g_{\nu\rho,\mu} = \frac{1}{\sqrt{-g}} \Gamma^\nu_{\mu\nu},
\]

where the \(\Gamma^\nu_{\mu\nu}\) are formal Christoffel symbols, i.e., in their expressions, \(x^\mu, g_{\mu\nu}\) and \(g_{\mu\nu,\rho}\) are all regarded as independent variables (it is only along given sections that we can state that \(g_{\mu\nu} = g_{\mu\nu}(x^\mu)\)). In particular, we cannot tune the coordinates \(x^\mu\) in such a way as to have \(\Gamma^\nu_{\mu\nu} = 0\) even at a single point (let alone having this equality identically satisfied).
The second factor in $H_{AB}^{\nu}(f \xi) = H_{\mu}(\alpha \beta)(g^\delta)(f \xi)$ is:
\[
\frac{\partial \xi_x}{\partial y^\nu} = \frac{\partial \xi_{\alpha \beta}}{\partial y_{\gamma \delta \mu \nu}}.
\] (B5)

We can easily convince ourselves that $H_{AB}^{\nu}(f \xi)$ do not identically vanish, as follows. Instead of calculating their full expression, we check just the trace $g_{\alpha \beta} H_{\mu}(\alpha \beta)(g^\delta)(f \xi)$, in the simplest particular case, i.e., $D = 4$; in this case, $E_{\alpha \beta} = G_{\alpha \beta} \sqrt{-g}$ (and therefore $g_{\alpha \beta} E_{\alpha \beta} = -R \sqrt{-g}$), that is,
\[
2 \Gamma_{\mu \tau}^{\nu} \frac{\partial R}{\partial g_{\gamma \delta \mu \nu}} = 2(\Gamma_{\gamma \delta \mu \nu} - \Gamma_{\gamma \nu} g^{\delta \mu} - \Gamma_{\tau \nu} g^{\gamma \delta}) \neq 0,
\]
where we have used the identity: \(\frac{\partial R}{\partial g_{\gamma \delta \mu \nu}} = g^{\gamma \delta} g^{\nu \mu} - g^{\mu \nu} g^{\gamma \delta}\).

Therefore, there is no chance that the full expressions $H_{\mu}(\alpha \beta)(g^\delta)(f \xi)$ would identically vanish for arbitrary $D$, which means that the functions $E_{\mu \nu} = -\frac{2}{\sqrt{-g}} E_{\mu \nu}$ cannot be the Euler-Lagrange expressions of any Lagrangian (either coordinate invariant or not).

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