Fractal Dimension of Julia Set for Non-analytic Maps

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The Hausdorff dimensions of the Julia sets for non-analytic maps: \( f(z) = z^2 + \epsilon z^* \) and \( f(z) = z^2 + \epsilon \) are calculated perturbatively for small \( \epsilon \). It is shown that Ruelle’s formula for Hausdorff dimensions of analytic maps cannot be generalized to non-analytic maps.

1. INTRODUCTION

The Julia set \( J \) of a map is the closure of the unstable periodic points [1–4]. It is an invariant set of the map and is usually a “repeller”, that is, points close to \( J \) will be repelled away by successive iterations of the map. A simple example is the map on the complex plane:

\[
f(z) = z^2,
\]

for which \( J \) is the unit circle. Points close to \( J \) will flow to one of the two stable fixed points: 0 and \( \infty \). Thus \( J \) is the boundary or separator of basins of attraction. A much more complicated geometry appears for the Julia set of the map:

\[
f(z) = z^2 + c,
\]

where \( c \) is a non-zero constant (see Fig. 1(a) for an example and Ref. [4] for many other examples). In this case, \( J \) is a fractal and its topology undergoes drastic changes as \( c \) varies.

Before we proceed further, let us define a few notations. We denote \( f^n \) to be \( n \) successive iterations of the map. That is \( f^n(z) = f(f^{n-1}(z)) \). The set of all unstable cycles of length \( n \) is denoted by \( \text{Fix} f^n \). \( Df \) is the derivative matrix of \( f \). If \( f \) is an analytic map, i.e. \( \partial u/\partial x = \partial v/\partial y \) and \( \partial v/\partial x = -\partial u/\partial y \) with \( f(z = x + iy) = u + iv \), then \( \det Df = |df/dz|^2 \).

For analytic maps, the Hausdorff dimension \( D_H \) of the Julia set \( J \) can be calculated with a formula due to a theorem of Ruelle [5]:

\[
\lim_{n \to \infty} A_n(D_H) = 1,
\]

where

\[
A_n(D) = \sum_{z \in \text{Fix} f^n} \left| \frac{df^n}{dz} \right|^{-D}.
\]

Using the formula, Ruelle [5] and Widom et al. [6] calculated \( D_H \) for the map [1] in powers of \( c \) for small \( |c| \). It was not clear then whether the formulas [4] and [3] can be generalized to non-analytic maps. The natural generalization of [3] to non-analytic maps would be
\[ A_n(D) = \sum_{z \in \text{Fix} f^n} |\det Df^n|^{-D/2}. \]  

The calculations I present below show that the combination of (2) and (3) does not give the correct \( D_H \) for non-analytic maps in general and \( D_H \) can be calculated directly with the perturbation theory developed in Ref. [6].

II. THE MAP \( f(z) = z^2 + \epsilon z^* \)

Let us first consider the non-analytic map

\[ f(z) = z^2 + \epsilon z^*, \]  

where \( * \) denotes the complex conjugate. When \( \epsilon = 0 \) the Julia set \( J \) is the unit circle and can be parametrized as \( z(t) = e^{2\pi it} \). The map on \( J \) is

\[ f(z(t)) = z(2t). \]  

When \( \epsilon \neq 0 \) but small enough so that \( J \) is topologically equivalent to a circle we can still parametrize \( J \) so that Eq. (3) is satisfied [2,3]. If a map \( f_\epsilon \) with a parameter \( \epsilon \) satisfies

\[ f_\epsilon(z) = f_\epsilon(z^*), \]  

then

\[ z \in \text{Fix} f_\epsilon^n \iff z^* \in \text{Fix} f_\epsilon^n, \]  

which implies that

\[ J(f_\epsilon) = [J(f_\epsilon^*])^*, \]  

where \( J(f) \) is the Julia set of \( f \). In particular, if \( J \) can be parametrized as \( z(t) \) then

\[ z_\epsilon(t) = z_\epsilon^*(t). \]  

It is easy to see that the map (3) satisfies Eq. (7).

Following Widom et al. [3], we formally expand \( z(t) \) in powers of \( \epsilon \)

\[ z(t) = e^{2\pi it}[1 + \epsilon U_1(t) + \epsilon^* \tilde{U}_1(t) + \epsilon^2 U_2(t) + \epsilon^* \tilde{U}_2(t) + \epsilon^3 U_3(t) + \epsilon^* \tilde{U}_3(t) + \cdots], \]  

where the functions \( U_1(t), \tilde{U}_1(t), U_2(t), \tilde{U}_2(t), \tilde{U}_2(t), \ldots \) are all periodic with period 1. Eq. (10) implies that all the functions \( U(t) \) satisfies \( U(t) = U^*(t) \). Substituting (11) into (3) and equating terms with the same power of \( \epsilon \), we get

\[ U_1(2t) - 2U_1(t) = e^{-6\pi it}, \]  

\[ \tilde{U}_1(2t) - 2\tilde{U}_1(t) = 0, \]  

\[ U_2(2t) - 2U_2(t) = U_1^*(t) + e^{-6\pi it} \tilde{U}_1^*(t), \]  

\[ \tilde{U}_2(2t) - 2\tilde{U}_2(t) = \tilde{U}_1^* - \tilde{U}_1(t), \]  

\[ U_2(2t) - 2U_2(t) = 2U_1(t) \tilde{U}_1(t) + e^{-6\pi it} U_1^*(t). \]  

The solutions are

\[ U_1(t) = - \sum_{k=1}^{\infty} \frac{e^{-3\pi i2k}}{2^k}, \]  

\[ \tilde{U}_1(t) = 0, \]  

\[ U_2(t) = - \sum_{j,k,l=1}^{\infty} \frac{e^{3\pi i2j(2^{k-1} + 2^{j-1})t}}{2^{j+k+l}}, \]  

\[ \tilde{U}_2(t) = 0, \]  

\[ \tilde{U}_2(t) = \sum_{j,k=1}^{\infty} \frac{e^{3\pi i2j(2^{k-1} - 1)t}}{2^{j+k}}. \]
It is easy to see from Eq. (6) that unstable cycles of length \( n \) are

\[
\text{Fix } f^n = \{ z(t_j) : t_j = \frac{j}{2^n - 1}, j = 0, 1, \ldots, 2^n - 2 \}.
\]  

(22)

We now evaluate \( A_n(D) \) as defined in (4). Note that

\[
\det Df^n = \prod_{i=0}^{n-1} \det \begin{pmatrix} 2x_i + \text{Re}(\epsilon) & -2y_i + \text{Im}(\epsilon) \\ 2y_i + \text{Im}(\epsilon) & 2x_i - \text{Re}(\epsilon) \end{pmatrix}
\]

\[
= \prod_{i=0}^{n-1} (4x_i^2 - |\epsilon|^2)
\]

\[
= 4^n (1 - \frac{|\epsilon|^2}{4})^n \prod_{m=0}^{n-1} |z(2^mt_j)|^2,
\]

where the last equality holds to the second order in \( \epsilon \). Denote

\[
< G(t) >_n = \frac{1}{2^n - 1} \sum_{j=0}^{2^n-2} G(t_j),
\]

(23)

where \( t_j \)'s are given by Eq. (22).

\[
A_n(D) = \sum_{z \in \text{Fix } f^n} |\det Df^n|^{-D/2}
\]

\[
= 2^{-Dn} (2^n - 1)(1 - \frac{|\epsilon|^2}{4})^{-Dn/2} \prod_{m=0}^{n-1} |z(2^mt_j)|^{-D} >_n.
\]

(24)

Substituting Eqs. (17)-(21) into (11) and using the identity

\[
< e^{2\pi imt} >_n = \begin{cases} 1, & m = 0 \mod 2^n - 1 \\ 0, & m \neq 0 \mod 2^n - 1 \end{cases}
\]

(25)

it can be shown, after some algebra, that

\[
< z(2^mt_j)|^{-D} >_n = 1 + |\epsilon|^2 \left( \frac{D^2n}{4} - \frac{Dn}{2} - \frac{D^2}{2} - \frac{Dn}{2n+1} + \frac{D^2n}{2n+3} \right), \quad (n > 2).
\]

(26)

Substituting (26) into (24) yields

\[
A_n(D) = 2^{(1-D)} (1 + |\epsilon|^2 \left( \frac{D^2n}{4} - \frac{3Dn}{8} \right)), \quad (n > 1).
\]

(27)

If we were to use Eqs. (27) and (3) to obtain a Hausdorff dimension, we would get \( D_H = 1 - |\epsilon|^2 / (8 \ln 2) \), a value smaller than 1 for small but nonzero \( \epsilon \). We show in the following that this value of \( D_H \) is incorrect.

Let \[
\chi_n(D) = \sum_{j=0}^{2^n-2} \frac{|z(t_{j+1}) - z(t_j)|^D}{(2\pi)^D},
\]

(28)

where \( z(t_j) \in \text{Fix } f^n \) (Eq. (22)). The Hausdorff dimension \( D_H \) of the set \( \text{Fix } f^n \) in the limit \( n \to \infty \) is such that

\[
\lim_{n \to \infty} \chi_n(D_H) = 1.
\]

(29)

This \( D_H \) should also be the \( D_H \) of the Julia set \( J \). We now evaluate \( \chi_n(D) \) to the second order in \( \epsilon \). Putting Eqs. (17)-(21) into Eq. (13), we write
\[ z(t_{j+1}) - z(t_j) = C_0 + C_1 |\epsilon| + C_2 |\epsilon|^2. \] (30)

Then to the second order in \( \epsilon \),

\[ \chi_n(D) = \frac{(C_0, D)}{(2\pi)^n} (2^n - 1) \left[ 1 + \frac{D|\epsilon|}{|C_0|^2} < \text{Re}(C_0^* C_1) >_n \right. 
\quad + \left. \frac{D|\epsilon|^2}{|C_0|^2} \left( \frac{1}{2} < |C_1|^2 >_n + < \text{Re}(C_0^* C_2) >_n + \frac{D - 2}{2|C_0|^2} < (\text{Re}(C_0^* C_1))^2 >_n \right) \right]; \] (31)

where Eq. (23) is used. With the help of the identity (25) we get

\[ < |C_0|^2 > = 2(1 - \cos \frac{2\pi}{2^n - 1}), \] (32)
\[ < \text{Re}(C_0^* C_1) > = 0, \quad (n > 2) \] (33)
\[ < |C_1|^2 > = F(n), \] (34)
\[ < \text{Re}(C_0^* C_2) > = \frac{|C_0|^2}{2} (1 + \frac{1}{2^n}), \] (35)
\[ < (\text{Re}(C_0^* C_1))^2 > = \frac{1}{2} |C_0|^2 < |C_1|^2 >, \] (36)

where

\[ F(n) = \frac{2}{3} - 2 \sum_{k=1}^{\infty} \frac{1}{4^k} \cos 2\pi \frac{3 \cdot 2^{k-1} - 1}{2^n - 1}. \] (37)

The function \( F(n) \) can easily be solved for large \( n \) in the following way. Note that for \( n >> 1 \)

\[ F(n + 1) = \frac{1}{2} \left( 1 - \cos \frac{3 \cdot 2\pi}{2^n + 1} \right) + \frac{1}{4} F(n) \]
\[ = \frac{9\pi^2}{4n+1} + \frac{1}{4} F(n). \] (38)

Substituting \( F(n) = H(n)/4^n \) into Eq. (38), we have

\[ H(n + 1) = 9\pi^2 + H(n), \] (39)

which has the solution

\[ H(n) = 9\pi^2 n + a, \] (40)

where \( a \) is a constant independent of \( n \). From Eqs. (31), (32) - (36), and (40),

\[ \chi_n(D) = 2^{n(1-D)} [1 + |\epsilon|^2 \left( \frac{D}{2} + \frac{D - 2}{4|C_0|^2} \right)] \]
\[ = 2^{n(1-D)} (1 + \frac{9}{16} n D^2 |\epsilon|^2), \quad (n >> 1). \] (41)

Eqs. (30) and (40) imply

\[ D = 1 + \frac{9}{16 \ln 2} |\epsilon|^2. \] (42)

III. THE MAP \( f(z) = z^2 + \epsilon \)

Next, we consider the non-analytic map

\[ f_\epsilon(z) = z^2 + \epsilon. \] (43)
The map \((43)\) has the property of Eq. \((7)\), so that
\[
J(\epsilon) = [J(\epsilon)]^*.
\]
Let us parametrize \(J\) in such a way so that
\[
f(z(t)) = z(-2t), \quad z(t) \in J.
\] (44)

The set of unstable cycles of length \(n\) is
\[
\text{Fix} f^n = \{z(t_j) : t_j = \frac{j}{(-2)^n - 1}, j = 0, \pm 1, \pm 2, \ldots\}. \tag{45}
\]
The number of elements in \(\text{Fix} f^n\) is \(|(-2)^n - 1|\). Following similar procedures as in the previous section, we have
\[
U_1(t) = -\sum_{k=1}^{\infty} e^{-\pi i 4^k t} \frac{t}{4^k}, \tag{46}
\]
\[
\tilde{U}_1(t) = -2 \sum_{k=1}^{\infty} e^{-\pi i 4^k t} \frac{t}{4^k}, \tag{47}
\]
\[
U_2(t) = -6 \sum_{j,k,l=1}^{\infty} e^{-\pi i (4^k - 1) j t} \frac{t}{4^j+k+l} \tag{48}
\]
\[
\tilde{U}_2(t) = -12 \sum_{j,k,l=1}^{\infty} e^{-\pi i (4^k + 4^l - 1) t} \frac{t}{4^j+k+l} + \sum_{k,l=1}^{\infty} e^{-\pi i (4^k + 4^l) t} \frac{t}{4^k+l}, \tag{49}
\]
\[
\hat{U}_2(t) = -4 \sum_{j,k,l=1}^{\infty} e^{-\pi i (4^k + 4^l/2) t} \frac{t}{2^j+2k+2l}. \tag{50}
\]

\(A_n(D)\) (Eq. \((4)\)) and \(\chi_n(D)\) ((Eq. \((28)\)) can be calculated to be
\[
A_n(D) = \chi_n(D) = 2^n(1-D)(1 + \frac{1}{4}nD^2|\epsilon|^2). \tag{51}
\]

In this case, \(A_n(D) = \chi_n(D)\) and it gives the correct Hausdorff dimension
\[
D_H = 1 + \frac{|\epsilon|^2}{4 \ln 2}. \tag{52}
\]
The reason for Ruelle’s formula to work in this case is that for the non-analytic map \((43)\) \(f^2(z)\) is analytic:
\[
f^2(z) = (z^2 + \epsilon^*)^2 + \epsilon, \tag{53}
\]
and that \((f^2) = J(f)\). Note that \((52)\) is the same as the \(D_H\) of the analytic map \((1)\) \(f(z) = z^2 + \epsilon\) \([3,4]\), to the second order in \(\epsilon\). Indeed, \(f^2(z)\) and thus \(J\) are identical for the two maps \((1)\) and \((43)\) for real \(\epsilon\). For complex \(\epsilon\), however, the two Julia sets look quite different (Fig. 1).

**IV. DISCUSSION**

Since Ruelle’s formula \((2)\) relies on the analyticity of the map, it is no surprise that it brakes down for non-analytic maps. When \(J\) is a closed curve, \(D_H\) can be calculated from \(\chi_n(D)\) (Eq. \((28)\)) for both analytic and non-analytic maps. When \(J\) is no longer topologically a circle, it can be difficult to utilize a formula based on distances between unstable cycle elements. In this case, it remains a challenge to formulate an efficient method for the calculation of \(D_H\) for non-analytic maps. Finally, the quantity \(A_n(D)\) (Eq. \((4)\)) can be very useful even for non-analytic maps. For example, it can be used to calculate the escape rate for points close to \(J\) \([5,6]\).

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