ON TAMENESS OF ALMOST AUTOMORPHIC DYNAMICAL SYSTEMS FOR GENERAL GROUPS

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ABSTRACT. Let $(X, G)$ be a minimal equicontinuous dynamical system, where $X$ is a compact metric space and $G$ some topological group acting on $X$. Under very mild assumptions, we show that the class of regular almost automorphic extensions of $(X, G)$ contains examples of tame but non-null systems as well as non-tame ones. To do that, we first study the representation of almost automorphic systems by means of semicocycles for general groups. Based on this representation, we obtain examples of the above kind in well-studied families of group actions. These include Toeplitz flows over $G$-odometers where $G$ is countable and residually finite as well as symbolic extensions of irrational rotations.

The probabilistic concept of independence is at the heart of several fundamental notions in ergodic theory like ergodicity, mixing, or positive entropy. Carried over to topological dynamics, independence gains a more combinatorial flavour and provides a basis for the local analysis of topological entropy (initiated by Blanchard in [1]) and related mixing properties, see [2]. For a comprehensive account of the combinatorial perspective on independence in topological dynamics with emphasis on entropy, see e.g., [3, 4]. We study the absence of independence due to tameness.

To gain some intuition, let us briefly discuss tameness for a binary subshift $(X, \sigma)$. In this case, a set $J \subseteq \mathbb{Z}$ is an independence set for $X$ if for each $z \in \{0, 1\}^J$ there is $x \in X$ with $x_j = z_j$ for every $j \in J$. The study of independence sets is of fundamental importance as the existence or absence of large independence sets implies strong dynamical consequences [3]. For example, the subshift $(X, \sigma)$ has positive topological entropy (as introduced by Adler, Konheim, and McAndrew [5]) if and only if $X$ has an independence set of positive asymptotic density. At the opposite end, $X$ has zero topological sequence entropy (as introduced by Goodman [6]) if and only if $X$ is a null system, that is, there is a finite upper bound on the size of independence sets. The lack of infinite independence sets is equivalent to tameness, a notion introduced to topological dynamics by Köhler [7].

The last decade saw an increased interest in tame systems (see e.g. [3, 8–12]; see also [13] for an up to date account) revealing their connections to other areas of mathematics like Banach spaces [14], circularly ordered systems [15], substitutions and tilings, quasicrystals, cut and project schemes and even model theory and logic [16–19]. A major breakthrough in the general understanding of tameness was achieved by Glasner’s recent structural result for tame minimal systems [20]. One of its consequences is that a tame minimal dynamical system which has an invariant measure is almost automorphic, uniquely ergodic and measure-theoretically isomorphic to its maximal equicontinuous factor [20, Corollary 5.4] (see also [3, 8, 9, 11] for previous results in this direction). In fact, a recent result shows that such systems are actually regularly almost automorphic, see [21, Theorem 1.2].

In view of these results, it is natural to ask whether there are non-tame regular almost automorphic extensions of equicontinuous systems. Further, as asked in [11]: if a regular extension is tame, can it be non-null? So far, few regular non-tame extensions are known (see [21, Corollary 3.7]) and the only positive answer to the second question is provided by specific Toeplitz shifts constructed in [3, Chapter 11]. We show that the answer to both questions is emphatically yes. In fact, under very mild assumptions any metric equicontinuous dynamical system $(T, G)$ has almost one-to-one extensions which are non-tame as well as extensions which are tame but not null, see Theorem 3.6 and Theorem 3.11. The basis for our construction are so-called semicocycle extensions which provide straightforward and flexible tools to obtain a variety of examples of almost automorphic systems.
For $\mathbb{Z}$-actions, it is known that a dynamical system is a semicocycle extension of a group rotation if and only if it is an almost automorphic extension of the same rotation [23] (provided the semicocycle is invariant under no rotation, see Section 2 for further details). As a matter of fact, this observation and its proof immediately carry over to actions of abelian groups. Our first goal is to extend this characterisation to general, non-abelian groups in Section 2. With this generalised notion of semicocycle extensions, we can rather directly construct a plethora of examples of almost automorphic systems. As an application, we obtain non-tame as well as tame but non-null symbolic systems such as Toeplitz flows over $G$-odometers with countable residually finite $G$ as well as symbolic extensions of irrational rotations. En passant, we obtain a generalisation (by completely different means) of the well-known fact that every minimal $\mathbb{Z}$-rotation on a compact metrizable monothetic group allows for an almost automorphic symbolic extension [23 Theorem 3.1], see Corollary 3.13.

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1. Background in topological dynamics

The statements of this section as well as their proofs can be found in standard references on topological dynamics and ergodic theory such as [22,25]. We say that a triple $(X, G, \Phi)$ is a topological dynamical system if $G$ is a topological group, $X$ is a compact Hausdorff topological space and $\Phi: G \times X \to X$ is a jointly continuous left action of $G$ on $X$. Most of the time, we keep the action $\Phi$ implicit. That is, we simply refer to $(X, G)$ as a topological dynamical system and write $g x$ for the image $\Phi(g, x)$. Given $g \in G$, we refer to the homeomorphism $x \mapsto g x$ also by $g$-translation and may identify an element $g \in G$ with that homeomorphism. We call the set $G X = \{ g x : g \in G \}$ the $G$-orbit of $x$ or simply orbit of $x \in X$ (under the action of $G$). The system $(X, G)$ is said to be minimal if for each $x \in X$ the orbit of $x$ is dense in $X$, that is, we have $\overline{G x} = X$.

A topological dynamical system $(X, G)$ is effective if distinct elements $g$ and $g'$ of $G$ define different homeomorphisms, that is, if for every $g, g' \in G$ with $g \neq g'$ there is $x \in X$ satisfying $g x \neq g' x$. We may always assume a system to be effective by identifying each element $g \in G$ with its $g$-translation as mentioned above. We say that $G$ acts freely on $x \in X$ if $g x \neq x$ for all $g \in G$ different from the neutral element $e_G \in G$. The dynamical system $(X, G)$ is free if $G$ acts freely on every $x \in X$. It is well known and straightforward to see that if $G$ is abelian and acts minimally on $X$, then $(X, G)$ is free if and only if $(X, G)$ is effective.

A topological dynamical system $(X, G)$ is equicontinuous if the collection of $g$-translations \( \{ x \mapsto g x : g \in G \} \subseteq X^X \) is a family of maps from $X$ to $X$ which is equicontinuous (with respect to the unique uniformity $\mathcal{U}_X$ that generates the topology on $X$). In this case, we have that for every $\alpha \in \mathcal{U}_X$ there is $\beta \in \mathcal{U}_X$ such that whenever $(x, x') \in \beta$ and $g \in G$ we have $(g x, g x') \in \alpha$. If $(X, G)$ is an equicontinuous topological dynamical system and if $X$ is metrizable, we can choose a compatible metric on $X$ such that each $g$-translation $x \mapsto g x$ is an isometry with respect to this metric. For that reason, whenever $X$ is metrizable, we will use the terms equicontinuous and isometric synonymously.

Recall that an invariant measure of a topological dynamical system $(X, G)$ (or: a $G$-invariant measure on $X$) is a Radon probability measure $\mu$ on $X$ such that $\mu(g A) = \mu(A)$ for all $g \in G$ and all $A \in \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the collection of all Borel sets of $X$. Given an invariant measure $\mu$, a set $A \in \mathcal{B}(X)$ is called invariant (with respect to $\mu$) if for all $g \in G$ we have $\mu(g A \triangle A) = 0$. If $\mu$ is an invariant measure such that for every invariant set $A \in \mathcal{B}(X)$ we have that either $\mu(A) = 0$ or $\mu(A) = 1$, then $\mu$ is referred to as being ergodic. It is well-known that every minimal equicontinuous system is uniquely ergodic, that is, it allows for a unique invariant measure which is necessarily ergodic (see also Theorem 1.3).
Let $(X, G)$ and $(Y, G)$ be two topological dynamical systems (with the same acting group $G$). A homomorphism from $(X, G)$ to $(Y, G)$ is a continuous map $\pi: X \to Y$ such that for every $x \in X$ and $g \in G$ we have $g\pi(x) = \pi(gx)$. If there is a homomorphism $\pi: X \to Y$ which is an onto map, then we say that $(Y, G)$ is a factor of $(X, G)$, $(X, G)$ is an extension of $(Y, G)$, and that $\pi$ is an epimorphism or a factor map. In the above situation, the terms isomorphism, automorphism, and endomorphism and accordingly, the notion of two systems being isomorphic have their standard meaning. A minimal topological dynamical system $(X, G)$ is coalescent if all its endomorphisms are automorphisms. Minimal equicontinuous systems are always coalescent (see, [24, page 81]). Further, factors of minimal systems are minimal and factors of equicontinuous systems are equicontinuous ([24, Corollary 2.6]).

Note that if $\pi: X \to Y$ is a factor map, then $R(\pi) = R = \{(x, x') \in X \times X : \pi(x) = \pi(x')\}$ is an invariant, closed equivalence relation (icer) on $X$. That is to say, the equivalence relation $R$ is a closed subset of $X \times X$ and if $(x, x') \in R$ and $g \in G$, then $(gx, gx') \in R$. Conversely, if $(X, G)$ is a topological dynamical system and $R$ is an icer on $X$, then the quotient space $X/R$ is a compact Hausdorff space. Furthermore, if $\pi: X \to X/R$ is the corresponding quotient map, then $\pi$ is an epimorphism from $(X, G)$ to $(X/R, G)$, where for all $g \in G$ the $g$-translation on $X/R$ is given by $\pi(x) \mapsto \pi(gx)$. Hence, factor maps and icers are just two ways of talking about the same thing and we will use them interchangeably.

If $(X, G)$ is a topological dynamical system, there is a smallest icer $S_{eq}$ (known as the equicontinuous structure relation) such that the factor system $(X/S_{eq}, G)$ is equicontinuous (see, for example, [24, Theorem 9.1]). We refer to $(X/S_{eq}, G)$ as well as to every system isomorphic to $(X/S_{eq}, G)$ as the maximal equicontinuous factor of $(X, G)$.

We say that a factor map $\pi: X \to Y$ is almost one-to-one if the set

$$X_0 = \{x \in X : \pi^{-1}(\{\pi(x)\}) = \{x\}\}$$

(1.1)

is dense in $X$. In this case, we call the system $(X, G)$ an almost one-to-one extension of $(Y, G)$. A topological dynamical system is called almost automorphic if its maximal equicontinuous factor is minimal and the corresponding factor map $\pi$ is almost one-to-one. In this case, we call points $x \in X$ with $\pi^{-1}(\{\pi(x)\}) = \{x\}$ almost automorphic. Observe that almost automorphic systems are necessarily minimal. If the projection $\pi(X_0)$ of the almost automorphic points to the maximal equicontinuous factor is measurable and of full measure (with respect to the unique invariant measure on $\pi(X)$), we say $(X, G)$ is regular. Clearly, every system isomorphic to an almost automorphic system is almost automorphic itself.

**Proposition 1.1** (cf. [23, Proposition 1.1], [25, V(6.1)5, page 480]). If $(X, G)$ is a minimal topological dynamical system, then the following statements are equivalent.

(a) $(X, G)$ is almost automorphic.

(b) $(X, G)$ is an almost one-to-one extension of an equicontinuous system.

Furthermore, if $(X, G)$ is an almost one-to-one extension of a minimal equicontinuous system $(\mathcal{T}, G)$, then $(\mathcal{T}, G)$ is the maximal equicontinuous factor of $(X, G)$.

1.1. **The Ellis semigroup and equicontinuous systems.** By $X^X$ we denote the collection of all (not necessarily continuous) maps from $X$ to itself. We endow $X^X$ with the product topology which coincides with the topology of pointwise convergence (a net $(\xi_n)$ in $X^X$ converges to $\xi \in X^X$ if and only if $\xi_n(x) \to \xi(x)$ for every $x \in X$). By Tychonoff’s theorem, $X^X$ is a compact Hausdorff space. Furthermore, $X^X$ has a semigroup structure defined by composition of maps.

Given a topological dynamical system $(X, G)$, the Ellis semigroup $E(X)$ associated to $(X, G)$ is defined as the closure of the set of $g$-translations $\{x \mapsto gx : g \in G\}$ in the space $X^X$. We may take the liberty to consider elements of $G$ as elements in $E(\mathcal{T})$. Note that, in general, there may be elements in $E(X)$ which are neither bijective nor continuous.

**Theorem 1.2** (cf. [24, Theorem 7, p. 54]). Let $\pi: X \to Y$ be a factor map between two topological dynamical systems $(X, G)$ and $(Y, G)$. Then there exists a unique continuous semigroup epimorphism $\Phi: E(X) \to E(Y)$ such that $\pi(\xi) = \Phi(\xi)\pi(x)$ holds for every $x \in X$ and $\xi \in E(X)$.

Note that there is a natural left action of $G$ on $E(X)$ given by $E(X) \ni \xi \mapsto g\xi \in E(X)$ for each $g \in G$. Clearly, $(E(X), G)$ is a topological dynamical system.
Theorem 1.3 (cf. [24, pp. 52–53]). Suppose \((T, G)\) is a minimal equicontinuous dynamical system. Then \(E(T)\) is a compact Hausdorff topological group, each \(\xi \in E(T)\) is a homeomorphism on \(T\), and \((E(T), G)\) is a minimal equicontinuous dynamical system, too. There is also a jointly continuous action of \(E(T)\) on \(T\) extending the action of \(G\) on \(T\) so that \((T, E(T))\) is a minimal equicontinuous dynamical system. If \(T\) is metrizable, then so is \(E(T)\). We further have:

(a) The system \((T, G)\) is a factor of \((E(T), G)\) and for every \(\theta \in T\) the map \(p_\theta : E(T) \to T\) given by

\[ p_\theta(\xi) = \xi \theta \]

is a factor map. Furthermore, let \(\text{Stab}(\theta) = \{\xi \in E(T) : \xi \theta = \theta\}\) be the stabiliser of \(\theta \in T\) with respect to the action of \(E(T)\) on \(T\). Then \(\text{Stab}(\theta)\) is a closed subgroup of \(E(T)\) and \((E(T)/\text{Stab}(\theta), G)\) is isomorphic to \((T, G)\). In particular, \(p_\theta\) is an open map and the push-forward of the Haar measure on \(E(T)\) through the projection onto \(E(T)/\text{Stab}(\theta)\) gives the unique invariant measure \(\mu_{\text{ET}}\) of \((T, G)\).

(b) If \(G\) is abelian, then \(E(T)\) is abelian as well and \((T, G)\) is isomorphic to \((E(T), G)\).

Remark 1.4. For later reference, let us briefly collect some properties of the action of \(E(T)\) on \(T\) provided by the above statement. First, note that an immediate consequence of the minimality of \((T, G)\) is that \(E(T)\) acts transitively on \(T\), that is, for each pair \(\theta_1, \theta_2 \in T\) there is \(\xi \in E(T)\) with \(\xi \theta_1 = \theta_2\).

Secondly, notice that the unique \(G\)-invariant measure \(\mu_{ET}\) on \(T\) necessarily coincides with the unique invariant measure of the minimal and equicontinuous dynamical system \((T, E(T))\).

Finally, observe that if \(T\) is metrizable and \(\rho\) is a metric on \(T\) with respect to which \(G\) acts isometrically on \(T\), then the action of \(E(T)\) on \(T\) is also isometric with respect to \(\rho\).

Remark 1.5. If we have that \(\text{Stab}(\theta)\) is trivial, then Theorem 1.3 (a) yields that \((T, G)\) is actually isomorphic to \((E(T), G)\) and hence free, provided \(G\) acts effectively on \(T\). The assumption of an action acting freely (in a weak sense) will enter our constructions of tame non-null systems in Section 3.

Corollary 1.6. Suppose \((T, G)\) is a minimal equicontinuous dynamical system and \((\theta_n)\) is a net in \(T\) with \(\theta_n \to \theta\) for some \(\theta \in T\). Then there is a subnet \((\theta_n')\) of \((\theta_n)\) and a net \((\xi_m)\) in \(E(T)\) with \(\xi_m\theta_n' = \theta\) and \(\xi_m \to e\), where \(e\) denotes the neutral element in the group \(E(T)\).

Proof. Let \(p_\theta : E(T) \to T\) be the factor map determined by \(\theta\) (see Theorem 1.3). For each \(n\), choose \(\theta_n' \in p_\theta^{-1}(\theta_n)\). By compactness of \(E(T)\), there is a subnet \((\theta_n')\) of \((\theta_n)\) with \(\theta_n' \to \hat{\theta}\) for some \(\hat{\theta}\) which lies in \(p_\theta^{-1}(\theta) \subseteq E(T)\), due to the continuity of \(p_\theta\). Let \(\xi_m = \hat{\theta}^{-1}\). Since \(E(T)\) is a topological group, we have \(\xi_m \to e\). At the same time, it holds that \(\xi_m\theta_n' = \xi_m\theta_n'\theta = \xi_m\theta_n \theta = \xi_m\theta_n \theta = \theta = p_\theta(\theta) = \theta\).

Corollary 1.7. Suppose \((T, G)\) is a minimal equicontinuous dynamical system and \(T\) is infinite. Let \(\theta, \theta' \in T\). Then in each neighbourhood of the neutral element \(e\in E(T)\) there is \(g \in G\) (considered as an element of \(E(T)\)) with \(g \notin \text{Stab}(\theta) \cup \text{Stab}(\theta')\), that is, \(g\theta \neq \theta\) and \(g\theta' \neq \theta'\).

Proof. Clearly, every subgroup of \(E(T)\) which has non-empty interior (in \(E(T)\)) is, in fact, open (in \(E(T)\)). Further, it is straightforward to see and well known that an open subgroup of a compact group is necessarily of finite index. As \(T\) is infinite and homeomorphic to \(E(T)/\text{Stab}(\theta)\) and \(E(T)/\text{Stab}(\theta')\) (due to Theorem 1.3), we clearly have that neither \(\text{Stab}(\theta)\) nor \(\text{Stab}(\theta')\) has finite index. Hence, \(\text{Stab}(\theta)\) and \(\text{Stab}(\theta')\) have empty interior.

As \(\text{Stab}(\theta)\) and \(\text{Stab}(\theta')\) are closed, we thus have that the complement of \(\text{Stab}(\theta) \cup \text{Stab}(\theta')\) is open and dense. As \(G\) embeds densely in \(E(T)\) (by definition), the statement follows.

2. Semicocycle extensions

In this section, we introduce a representation of almost automorphic systems by means of Sturmian-like subshifts with a compact alphabet. This construction builds upon so-called

\*The notion of transitivity should not be confused with the notions of topological or point transitivity (see [24, p.31]) which are commonly referred to by the abbreviated term transitive, too. We would like to stress that throughout this work, we refer by transitive solely to the above concept.
semicocycles. For $G = \mathbb{Z}$, these have already proved useful in the study of factors of Toeplitz shifts (see \cite{22} and \cite{27}) but also of almost one-to-one extensions of non-equicontinuous systems (see \cite{28} for a nice exposition). For symbolic $\mathbb{Z}$-shifts, similar techniques can be found in \cite{23, 24, 29}. Here, we extend the constructive idea of semicocycles to general groups, in particular, to non-abelian ones. We would like to mention that although the underlying ideas are close to those for semicocycle extensions of $\mathbb{Z}$-actions some care has to be taken in the course of this generalisation.

We will frequently deal with maps $f: \mathbb{T}_0 \subseteq \mathbb{T} \to K$, where $\mathbb{T}$ and $K$ are Hausdorff topological spaces and $\mathbb{T}_0$ is some subset of $\mathbb{T}$. Despite the fact that such $f$ may not be defined everywhere in $\mathbb{T}$, we will always consider the graph of $f$ as a subset of $\mathbb{T} \times K$. That is, we will refer by $\text{gr} f$ to the set $\{(\theta,k) \in \mathbb{T} \times K : \theta \in \mathbb{T}_0, k = f(\theta)\} \subseteq \mathbb{T} \times K$. Moreover, we denote by $F \subseteq \mathbb{T} \times K$ the closure of the graph of $f$ as a subset of $\mathbb{T} \times K$, that is, $F = \overline{\text{gr} f}$. Finally, for every $\theta \in \mathbb{T}$, we define the $\theta$-section of $F$ as the set $F(\theta) = \{k \in K : (\theta,k) \in F\}$.

The proof of the following statement is straightforward and left to the reader.

**Proposition 2.1.** Assume that $\mathbb{T}$ and $K$ are Hausdorff topological spaces and $K$ is compact. Let $\mathbb{T}_0 \subseteq \mathbb{T}$ be dense, $f: \mathbb{T}_0 \subseteq \mathbb{T} \to K$ be a mapping, and $F = \overline{\text{gr} f} \subseteq \mathbb{T} \times K$.

1. Let $\theta \in \mathbb{T}_0$. The function $f$ is continuous at $\theta$ (with respect to the subspace topology on $\mathbb{T}_0$ inherited from $\mathbb{T}$) if and only if $F(\theta) = \{f(\theta)\}$.

2. If $f$ is continuous at every $\theta \in \mathbb{T}_0$, then for every dense set $\mathbb{T}_1 \subseteq \mathbb{T}$ and every function $g: \mathbb{T}_1 \to K$ with $g(\theta') \in F(\theta')$ for all $\theta' \in \mathbb{T}_1$, the set $\text{gr} g$ is dense in $F$.

For the rest of this work, $K$ is always assumed to be a compact Hausdorff space. We say that $\theta \in \mathbb{T}$ is a *discontinuity point* of $f$ if $F(\theta)$ has more than one element. We write

$$D_f = \{\theta \in \mathbb{T} : \#F(\theta) > 1\} \subseteq \mathbb{T}$$

for the set of all discontinuity points of $f$. If $f: \mathbb{T}_0 \to \mathbb{T}$ is continuous, we clearly have $\mathbb{T}_0 \cap D_f = \emptyset$. In this case, if $\mathbb{T}_0$ is further dense in $\mathbb{T}$, the mapping

$$\mathbb{T} \setminus D_f \to K, \quad \theta \mapsto k_\theta,$$

where $k_\theta$ is such that $F(\theta) = \{k_\theta\}$, is a well-defined continuous extension of $f$. For simplicity, we may refer to this mapping by $f$ as well.

Recall that a triple $(\mathbb{T}, G, \theta_0)$ is a pointed dynamical system if $(\mathbb{T}, G)$ is a topological dynamical system and $\theta_0$ is an element in $\mathbb{T}$ with $G\theta_0 = \mathbb{T}$. A $(K\text{-valued})$ semicocycle over a pointed dynamical system $(\mathbb{T}, G, \theta_0)$ is a map $f: G\theta_0 \to K$ which is continuous with respect to the subspace topology on $G\theta_0 \subseteq \mathbb{T}$. If $f$ is a semicocycle, then for every $\theta \in \mathbb{T}$ the section $F(\theta)$ is nonempty and $F(g\theta_0) = \{f(g\theta_0)\}$ for every $g \in G$ (see Proposition 2.1).

Given a $K\text{-valued}$ semicocycle $f$ over a pointed system $(\mathbb{T}, G, \theta_0)$, we now define a topological dynamical system $(X_f, G)$ associated to $f$. To that end, observe that $G$ acts from the left on the product topological space $K^G$ by means of the shift action

$$G \times K^G \ni (h, (x_g)_{g \in G}) \mapsto \sigma^h((x_g)_{g \in G}) = (x_{gh})_{g \in G} \in K^G.$$

By slightly abusing notation, we may identify $f$ with the mapping $g \mapsto f(g\theta_0)$ on $G$ and hence consider $f$ an element of $K^G$. We define $(X_f, G)$ as the orbit closure of $f$ under the above shift action. Throughout this work, given $x \in K^G$, we synonymously refer by $x_g$ and $x(g)$ to the image of $g$ under $x$.

It is natural to ask whether $(X_f, G)$ is an extension of $(\mathbb{T}, G)$. Towards an answer to this question, we introduce an equivalence relation $\sim$ on $\mathbb{T}$ by putting

$$\theta_1 \sim \theta_2 \text{ if and only if for every } \xi \in E(\mathbb{T}) \text{ we have } F(\xi \theta_1) = F(\xi \theta_2),$$

where $E(\mathbb{T})$ denotes the Ellis semigroup of $(\mathbb{T}, G)$. We denote the equivalence class of $\theta \in \mathbb{T}$ by $[\theta]$ and say the semicocycle $f$ is *invariant under no rotation* if $\sim$ is the identity relation, that is, if $[\theta] = 1$ for every $\theta \in \mathbb{T}$.

**Lemma 2.2.** If $(\mathbb{T}, G)$ is a minimal equicontinuous system, then the relation $\sim$ is an icer.

**Proof.** Since $G$ embeds in $E(\mathbb{T})$, we easily see that $\sim$ is $G$-invariant. It remains to show that $\sim$ is closed in $\mathbb{T} \times \mathbb{T}$. Let $\theta_1, \theta_2 \in \mathbb{T}^2$ be given and suppose there is a net $(\theta_1^n, \theta_2^n) \to (\theta_1, \theta_2)$ with $\theta_1^n \sim \theta_2^n$. Due to Corollary \ref{1} we may assume without loss of generality that there is a net $(\xi_n)$ in $E(\mathbb{T})$ with $\xi_n \theta_1^n = \xi_n$ and $\xi_n \to e$. Then $F(\xi_1^n) = F(\xi_n \theta_1^n) = F(\xi_n \theta_2^n)$ for every $n \in \mathbb{N}$. The icer follows.

\[ \text{Proof.} \]
for every $\xi \in E(T)$. Since $\xi_0^{\theta_0} \to \xi \theta_2$ and $F$ is closed, this yields $F(\xi \theta_1) \subseteq F(\xi \theta_2)$ for each $\xi \in E(T)$. Interchanging the roles of $\theta_1$ and $\theta_2$, we obtain $F(\xi \theta_2) \subseteq F(\xi \theta_1)$ for every $\xi \in E(T)$ and hence $\theta_1 \sim \theta_2$. \hfill $\square$

If not stated otherwise, we throughout assume the system $(\mathbb{T}, G)$ to be minimal and equicontinuous. In this case, given some $\theta_0 \in \mathbb{T}$ and a semicocycle $f$ over $(\mathbb{T}, G, \theta_0)$ which is invariant under no rotation, we refer to the system $(X_f, G)$ as a *semicocycle extension of $(\mathbb{T}, G)$*. As we will see in Theorem 2.5, a semicocycle extension of $(\mathbb{T}, G)$ is indeed an extension of $(\mathbb{T}, G)$. The following statement provides a simple but useful sufficient criterion for invariance under no rotation.

**Proposition 2.3.** Let $f$ be a semicocycle over $(\mathbb{T}, G, \theta_0)$. If for each $\theta_1, \theta_2 \in \mathbb{T}$ there is $\xi \in E(\mathbb{T})$ such that $\xi \theta_1 \in D_f$ and $\xi \theta_2 \notin D_f$, then $f$ is invariant under no rotation.

**Proof.** This immediately follows from the fact that, by definition of $D_f$, we have

$$\#F(\xi \theta_1) > 1 = \#F(\xi \theta_2).$$

\hfill $\square$

The next statement suggests that we may realise invariance under no rotation by possibly changing the pointed dynamical system (see also [27, Theorem 6.5] for a similar statement for semicocycles over $\mathbb{Z}$-odometers).

**Lemma 2.4.** Suppose $(\mathbb{T}, G)$ is a minimal equicontinuous system, $\theta_0 \in \mathbb{T}$ and $f$ is a semicocycle over $(\mathbb{T}, G, \theta_0)$. Then $(X_f, G)$ is a semicocycle extension of $(\mathbb{T}/\sim, G)$.

**Proof.** Since $\sim$ is an icer on $\mathbb{T}$, we have that $(\mathbb{T}/\sim, G)$ is a (necessarily minimal and equicontinuous) factor of $(\mathbb{T}, G)$ so that $(\mathbb{T}/\sim, G, [\theta_0])$ is clearly a pointed dynamical system. Further, note that $f : G[\theta_0] \subseteq \mathbb{T}/\sim \to K$ given by $g[\theta_0] \mapsto f(g \theta_0)$ is well-defined and that

$$F([\theta]) = F(\theta)$$

for all $\theta \in \mathbb{T}$. It follows that $f$ is, in fact, continuous, by Proposition 2.1. With the epimorphism $\Phi$ from Theorem 1.2, we obtain $F(\Phi(\xi)[\theta]) = F([\xi \theta]) = F(\xi \theta)$ for all $\xi \in E(T)$ which implies that $f$ is invariant under no rotation. \hfill $\square$

The next result is a generalisation of the respective statement for $\mathbb{Z}$-actions (cf. [27, Theorem 6.4] and [22, Theorem 5.2]). The main idea of its proof is as in these references. However, as the group $G$ is not necessarily abelian, the homogeneous space $\mathbb{T}$ may not possess a compatible group structure (see Theorem 1.3). This fact requires an extra passage through the Ellis semigroup $E(\mathbb{T})$ of $\mathbb{T}$ (see, in particular, the proof of equation (2.5)).

**Theorem 2.5.** Let $(X, G)$ and $(\mathbb{T}, G)$ be topological dynamical systems and assume $(\mathbb{T}, G)$ is minimal and equicontinuous. The following statements are equivalent.

1. $(X, G)$ is an almost automorphic extension of $(\mathbb{T}, G)$.
2. $(X, G)$ is topologically isomorphic to a semicocycle extension $(X_f, G)$ of $(\mathbb{T}, G)$.

**Remark 2.6.** Note that due to Proposition 1.1 the above statement yields that $(\mathbb{T}, G)$ is the maximal equicontinuous factor of any of its semicocycle extensions.

**Proof of Theorem 2.5.** We first show that (2) implies (1). Let $f$ be a semicocycle over the pointed minimal equicontinuous system $(\mathbb{T}, G, \theta_0)$ (where $\theta_0 \in \mathbb{T}$) and assume $f$ is invariant under no rotation. Note that by Proposition 1.1 it suffices to show that $(X_f, G)$ is an almost one-to-one extension of $(\mathbb{T}, G)$.

Take $x = (x(g))_{g \in G} \in X_f$. By definition, there is a net $(h_n)$ in $G$ with $\lim_n f(g h_n \theta_0) = x(g)$ for each $g \in G$. It is natural to take some accumulation point $\theta_x \in \mathbb{T}$ of the net $(h_n \theta_0)$ and define

$$\pi : X_f \to \mathbb{T}, \quad x \mapsto \theta_x.$$ \hfill (2.3)

Observe that for every $g \in G$ we necessarily have

$$x(g) \in F(g \theta_x) = F(g \pi(x)),$$ \hfill (2.4)

where $F = \bigcup_{g \in G} f(\theta_x) \subseteq \mathbb{T} \times K$ as above. Our goal is to prove that $\pi$ is, in fact, an almost one-to-one factor map from $X_f$ to $\mathbb{T}$.

\[\text{We would like to remark that if } \mathbb{T} \text{ is metrizable, Stone’s Metrization Theorem yields that } \mathbb{T}/\sim \text{ is also metrizable.} \]
To that end, we first show that $\pi$ is uniquely defined. Note that this will yield that $\pi$ is continuous. Take two accumulation points $\theta^1_2, \theta^2_2 \in \mathcal{T}$ of the net $(h_n \theta_0)$. We will prove that
\[
F(\pi \theta^1_2) = F(\pi \theta^2_2) \quad \text{for every } \xi \in E(\mathbb{T}).
\] (2.5)
Since $f$ is invariant under no rotation, this yields that, in fact, $\theta^1_2 = \theta^2_2$.

To show (2.5), we introduce some notation. Let $p: E(\mathbb{T}) \rightarrow \mathcal{T}$ be a factor map as given by Theorem 1.3. Fix $\hat{\theta}^1_2, \hat{\theta}^2_2 \in \mathcal{T}$ with $p(\hat{\theta}^1_2) = \hat{\theta}^1_2$ and $p(\hat{\theta}^2_2) = \hat{\theta}^2_2$. Further, let
\[
\hat{F} = (p \times id_K)^{-1}(F) = \{(\xi, y) \in E(\mathbb{T}) \times K : (p(\xi), y) \in F\}.
\]
Clearly, if $p(\xi) = p(\xi')$ for $\xi, \xi' \in E(\mathbb{T})$, then $\hat{F}(\xi) = \hat{F}(\xi')$.

Since $p$ is an open map, the set $E_0 = p^{-1}(G \theta_0)$ is dense in $E(\mathbb{T})$. Similarly, as $\hat{f} = f \circ p: E_0 \rightarrow K$ verifies $gr \hat{f} = (p \times id_K)^{-1}(gr f)$, we have that $gr \hat{f}$ is dense in $\hat{F}$. Observe that $\hat{f}$ is continuous. Now, Proposition 2.1 (2) and equation (2.4) yield that $\Theta^1 = \{(g \theta^1_2, x(g)) \in \mathbb{T} \times K : g \in G\}$ is dense in $\hat{F}$ and that $\Theta^1 = \{(g \theta^2_2, x(g)) \in E(\mathbb{T}) \times K : g \in G\}$ is dense in $\hat{F}$.

We are ready to prove (2.5).

Fix $\xi_0 \in E(\mathbb{T})$ and take $y \in F(\xi_0 \theta^1_2)$. Clearly, $p(\xi_0 \theta^1_2) = \xi_0 p(\theta^1_2) = \xi_0 \theta^1_2$ so that $(\xi_0 \theta^1_2, y) \in \hat{F}$. Due to the denseness of $\hat{\Theta}^1$ in $\hat{F}$, there is a net $(g_n)$ such that $lim_n (g_n \theta^1_2, x(g_n)) = (\xi_0 \theta^1_2, y)$, in particular, $lim_n g_n \theta^1_2 = \xi_0 \theta^1_2$ so that $lim_n g_n = \xi_0$ since $E(\mathbb{T})$ is a group. It follows that $lim_n (g_n \theta^1_2, x(g_n)) = (\xi_0 \theta^1_2, y)$. Therefore, $y \in \hat{F}(\xi_0 \theta^1_2) = \hat{F}(\xi_0 \theta^1_2)$.

This proves that $\hat{F}(\xi_0 \theta^1_2) \subseteq \hat{F}(\xi_0 \theta^2_2)$. Interchanging the roles of $\theta^1_2$ and $\theta^2_2$, we obtain the reverse inclusion and hence (2.5).

The map $\pi$ in (2.3) is thus uniquely defined and continuous. Further, we clearly have
\[
\pi(\sigma^s(f(g_0 h_0)))_{g_0 \in G} = \pi((f(g_0 h_0))_{g_0 \in G}) = \eta g h_0 = s \pi((f(g_0 h_0))_{g_0 \in G})
\]
for all $s, h \in G$. Hence, by continuity of $\pi$ and denseness of $\{\sigma^h f(\theta_0) : h \in G\}$ in $X_f$, we have $\pi(\sigma^s x) = \pi(x)$ for all $x \in X_f$. Thus, $(\mathbb{T}, G)$ is a factor of $(X_f, G)$. Further, note that (2.4) implies that $\pi$ is almost one-to-one which hence proves (2).

Finally, we prove that (1) implies (2). To this end, suppose we have an almost automorphic system $(K, G)$ with the maximal equicontinuous factor $(\mathbb{T}, G)$ and let $\pi: K \rightarrow \mathbb{T}$ be the associated factor map. Take an almost automorphic point $k \in K$ and define $f$ on $G \pi(k) \subseteq \mathbb{T}$ by $f(gr(k)) = gk \in K$. Since $k$ is an almost automorphic point, it is not hard to see that $f$ is a continuous map. Thus, $f$ is a $K$-valued semicocycle over $(\mathbb{T}, G, \pi(k))$.

Let $X_f = \{f(g_0 \pi(k))_{g_0 \in G} : h \in G\} \subseteq K^G$. It remains to show that $(X_f, G)$ is isomorphic to $(X, G)$ and that $f$ is invariant under no rotation, that is $\sim$ is a trivial equivalence relation on $\mathbb{T}$. We begin by noting that the map $\psi: X_f \ni x \mapsto x(e) \in K$, where $e = \varepsilon_0$ is the neutral element of $G$, is onto and continuous. To show that $\psi$ is injective, take $g, h \in G$ and observe that $f(g h \pi(k)) = g h k = g f(h \pi(k))$. This implies $x_g = x_h$ for each $(x_g)_{g \in G} \in X_f$.

Thus, if $x_e = y_e$, then $(x_g)_{g \in G} = (y_g)_{g \in G}$ proving injectivity. Moreover, it is easy to see that $\psi(\sigma^s x) = g \psi(x)$ for every $g \in G$. Hence, $(X_f, G)$ is isomorphic to $(K, G)$ so that $(\mathbb{T}, G)$ is a maximal equicontinuous factor of $(X_f, G)$. At the same time, Lemma 2.4 and the fact that (2) implies (1) yield that $(X_f, G)$ is an almost automorphic extension of $(\mathbb{T}/\sim, G)$ which is -due to Proposition 1.1- a maximal equicontinuous factor of $(X_f, G)$, too.

Therefore, we may consider the factor map $\mathbb{T} \ni x \mapsto [\xi] \in \mathbb{T}/\sim$ as an endomorphism of $(\mathbb{T}, G)$. Due to the coalescence of equicontinuous systems, the factor map $\mathbb{T} \ni x \mapsto [\xi] \in \mathbb{T}/\sim$ must be a homeomorphism, so that $\sim$ is the identity relation. Thus, $(X_f, G)$ is a semicocycle extension of $(\mathbb{T}, G)$ isomorphic to $(K, G)$.

\[ \square \]

Remark 2.7. Observe that a semicocycle extension $(X_f, G)$ of $(\mathbb{T}, G)$ is regular if and only if $GD_f$ is measurable and $\pi(\mathbb{T})(GD_f) = 0$.

2.1. A brief discussion of a weak form of freeness. In order to obtain conditions which ensure that a semicocycle extension $(X_f, G)$ is tame, we will have to assume a certain form of freeness of its maximal equicontinuous factor $(\mathbb{T}, G)$ (see Lemma 3.2). This section provides a simple auxiliary statement which will prove useful in this context.

Recall that $G$ is said to act almost freely on $\theta \in \mathbb{T}$ if there are only finitely many $g \in G$ with $g \theta = \theta$. In other words, $G$ acts almost freely on $\theta$ if $\text{Stab}_G(\theta) = \{g \in G : g \theta = \theta\}$ is
finite. To weaken the notion of freeness even further, let us introduce the following relation. Given \( \theta \in \mathbb{T} \) and \( g, g' \in \text{Stab}_G(\theta) \), we write

\[ g \sim g' \iff \text{there is a neighbourhood } U \text{ of } \theta \text{ such that for all } \omega \in U \text{ we have } g\omega = g'\omega. \]

Clearly, \( \sim \) defines an equivalence relation on \( \text{Stab}_G(\theta) \).

**Definition 2.8.** We say that \( G \) acts locally almost freely on \( \theta \in \mathbb{T} \) if the quotient of \( \text{Stab}_G(\theta) \) with respect to \( \sim \) is finite.

It is immediate that \( G \) acts locally almost freely on \( \theta \) if it acts almost freely on \( \theta \). An example of an effective minimal equicontinuous system \((T, G)\) where \( G \) acts locally almost freely on every \( \theta \) but not almost freely on any \( \theta \) is given by an action of the isometric subgroup of the topological full group of a \( \mathbb{Z} \)- odometer (see [31, Example 7.3]).

The following observation will be applied in the proof of Lemma 3.2.

**Proposition 2.9.** Let \((X_f, G)\) be a semicocycle extension of the minimal equicontinuous system \((T, G)\) and let \( \pi \) be the corresponding factor map from \((X_f, G)\) to \((T, G)\). Let \( \theta \in \mathbb{T} \), \( x \in \pi^{-1}(\theta) \), and \( g, g_1, g_2 \in G \) with \( g_1 \sim g_2 \). Then

\[ x(gg_1) = x(gg_2). \]

**Proof.** We keep the notation as in the proof of Theorem 2.5. Let \((h_n)\) be a net in \( T \) such that \( x(g) = \lim_n f(gh_n\theta_0) \) for each \( g \in G \). Since \( g_1 \) and \( g_2 \) coincide on a neighbourhood of \( \theta \), we obtain

\[ x(gg_1) = \lim_n f(gg_1h_n\theta_0) = \lim_n f(gg_2h_n\theta_0) = x(gg_2), \]

where we used \( \theta = \lim_n h_n\theta_0 \) (see equation (2.3)) in the second equality. \( \square \)

3. **Regular non-tame and tame non-null examples**

Throughout this section, we assume \( T \) to be an infinite compact metric space equipped with a metric \( \rho \) with respect to which the group \( G \) acts isometrically on \( T \). As before, we assume \((T, G)\) to be minimal.

After a short discussion of (weak forms of) topological independence, we will turn to the construction of tame non-null and non-tame semicocycle extensions of \((T, G)\). In the last section, we provide several symbolic examples.

3.1. **Tameness and nullness.** In the following, we briefly discuss the concepts of tameness and nullness. For the sake of a concise presentation and later applications, this discussion is held in the framework of semicocycle extensions \((X_f, G)\).

Given subsets \( A_0, A_1 \subseteq X_f \), we say that \( J \subseteq G \) is an independence set for \((A_0, A_1)\) if for each finite subset \( I \subseteq J \) and every \( a \in \{0, 1\}^I \) we have \( \bigcap_{i \in I} \sigma^{-a_i} A_i \neq \emptyset \). A pair of points \( x_0, x_1 \in X_f \) is an IT-pair if for each pair of neighbourhoods \( U_0 \) and \( U_1 \) of \( x_0 \) and \( x_1 \), respectively, there is an infinite independence set. Instead of providing the original definition of tameness, we make use of the following alternative characterisation (see [3, Proposition 6.4]). We say \((X_f, G)\) is non-tame if there is an IT-pair \((x_0, x_1)\) with \( x_0 \neq x_1 \).

Note that the existence of such an IT-pair implies the existence of distinct \( k_0, k_1 \in K \) such that for all compact neighbourhoods \( V_0 \subseteq K \) and \( V_1 \subseteq K \) of \( k_0 \) and \( k_1 \), respectively, there is an infinite independence set \( J \) for \((\mathcal{A}(V_0), \mathcal{A}(V_1))\), where

\[ A(V_j) = \{ x \in X_f : x_j \in V_j \} \quad (j = 0, 1). \]

Vice versa, the existence of disjoint compact subsets \( A_0, A_1 \subseteq X_f \) with an infinite independence set implies non-tameness [3, Proposition 6.4]. This immediately yields the following

**Proposition 3.1.** The system \((X_f, G)\) is non-tame if and only if there are disjoint compact sets \( V_0, V_1 \subseteq K \) and a sequence \((g_i)_{i \in \mathbb{N}} \) in \( G \) such that for every \( \ell \in \mathbb{N} \) and each \( a \in \{0, 1\}^\ell \) there is \( x \in X_f \) with \( x_{g_i} \in V_{a_i} \) for \( i = 1, \ldots, \ell \) or, equivalently, such that for each \( a \in \{0, 1\}^\ell \) there is \( x \in X_f \) with \( x_{g_{\ell}} \in V_{a_{\ell}} \).

The following observation will be key in the construction of tame examples.
Lemma 3.2. Let \( (X_f, G) \) be a semicycle extension of the minimal equicontinuous metric dynamical system \((T, G)\). Suppose \( D_f \) is countable and suppose that for each \( \theta \in T \) we have that \( G \theta \cap D_f \) is finite and that \( G \) acts locally almost freely on every \( \theta \in D_f \). Then \( (X_f, G) \) is tame.

Proof. For a contradiction, we are given disjoint compact sets \( V_0, V_1 \subseteq K \) and a sequence \( \{g_i\}_{i \in \mathbb{N}} \) in \( G \) as in Proposition 3.1. Without loss of generality, we may assume that there is \( \xi \in E(T) \) with \( g_i \to \xi \) as \( i \to \infty \) (where the \( g_i \) are considered as elements in \( E(T) \)).

Given \( x \in X_f \), recall that \( x_{g_i} \in F(g_i, \pi(x)) \) (see equation (2.4)), where \( \pi \) denotes the factor map from \((X_f, G)\) to \((T, G)\). Let us assume first that \( \lim_{i \to \infty} g_i \pi(x) = \pi(x) \notin D_f \).

Note that in this case there is \( i_0 \in \mathbb{N} \) such that at least one of the following conditions holds

- \( \forall i \geq i_0 : x_{g_i} \notin V_0 \) (which is the case if \( F(\xi) = \{ f(\xi) \} \subseteq V_0 \)) or
- \( \forall i \geq i_0 : x_{g_i} \notin V_1 \) (which is the case if \( F(\pi(x)) = \{ f(\pi(x)) \} \subseteq V_1 \)),

where we understand \( f \) to be defined on \( T \setminus D_f \) in the sense of (2.1). Hence, if there is \( a \in \{0, 1\}^N \) with \( x_{g_i} \in V_{ai} \) for every \( i \in \mathbb{N} \), then \( a \) is necessarily eventually constant. Thus, the set

\[
\{ a \in \{0, 1\}^N : \text{there is } x \in X_f \text{ with } \pi(x) \notin D_f \text{ and } x_{g_i} \in V_{ai} \ (i \in \mathbb{N}) \}
\]

consists of eventually constant sequences and is therefore at most countable.

Since \( D_f \) is countable, we clearly have that \( \{ \theta \in T : \pi(\theta) \in D_f \} \) is countable, too. As \( \{0, 1\}^N \) is uncountable, there must hence be \( \theta \in T \) (with \( \pi(\theta) \in D_f \)) such that

\[
\{ a \in \{0, 1\}^N : \text{there is } x \in \pi^{-1}(\theta) \text{ with } x_{g_i} \in V_{ai} \ (i \in \mathbb{N}) \}
\]

is uncountable. Pick such \( \theta \in T \).

Suppose we are given \( i_0 \in \mathbb{N} \) with \( g_{i_0} \theta \notin D_f \). Then for all \( x \in \pi^{-1}(\theta) \) we have that \( x_{g_{i_0}} \in F(g_{i_0}, \theta) \) (due to (2.4)). In other words, there is only one (if any) admissible value for the \( i_0 \)-th entry of any sequence \( a \in \{0, 1\}^N \) verifying \( x_{g_i} \in V_{ai} \ (i \in \mathbb{N}) \) for some \( x \in \pi^{-1}(\theta) \).

We may hence assume without loss of generality that \( g_i \theta \in D_f \) for all \( i \in \mathbb{N} \). By our assumptions, there are finitely many \( \theta_1, \ldots, \theta_n \) such that \( G \theta \cap D_f = \{ \theta_1, \ldots, \theta_n \} \). We may assume that \( g_i \theta = \theta_{\ell} \) for \( \ell = 1, \ldots, n \).

Clearly, \( \text{Stab}_G(\theta) = g_1^{-1} \text{Stab}_G(\theta_1) g_1 \) so that \( G \) acts locally almost freely on \( \theta_{\ell} \), too. Let \( a_1, \ldots, a_m \) be representatives of the equivalence classes of \( \theta_{\ell} \). For \( i \in \mathbb{N} \), we set \( \ell(i) \in \{1, \ldots, n\} \) such that \( g_{\ell(i)} \theta = g_i \theta \) and set \( j(i) \in \{1, \ldots, m\} \) such that \( g_{\ell(i)}^{-1} g_i \sim a_{j(i)} \). Now, for all \( i \in \mathbb{N} \) and all \( x \in \pi^{-1}(\theta) \), we obtain \( x_{g_i} = x_{g_{\ell(i)} g_{\ell(i)}^{-1} g_i} = x_{g_{\ell(i)} a_{j(i)}} \) where we used Proposition 2.9 in the last step. Let \( N \in \mathbb{N} \) be such that for all \( i \in \mathbb{N} \) there is \( k_i \leq N \) with \( \ell(k_i) = \ell(i) \) and \( j(k_i) = j(i) \). Observe that such \( N \) clearly exists. However, this implies that every sequence \( a \in \{0, 1\}^N \) which satisfies for some \( x \in \pi^{-1}(\theta) \) and all \( i \in \mathbb{N} \) that \( x_{g_i} \in V_{a_i} \), is completely determined by its first \( N \) entries. That is, there are only finitely many such sequences \( a \). This contradicts the assumptions on \( \theta \) and finishes the proof. \( \square \)

Naturally related to the idea of tameness is the concept of nullness. We refrain from rephrasing the original definition. Instead, we provide the following characterisation for systems of the form \((X_f, G)\) which is obtained by similar arguments as Proposition 3.1 (cf. Proposition 5.4).

Proposition 3.3. The system \((X_f, G)\) is non-null if and only if there are disjoint compact sets \( V_0, V_1 \subseteq K \) such that for each \( \ell \in \mathbb{N} \) there is a finite sequence \( \{g_i\}_{i=1}^{r} \) in \( G \) such that for each \( a \in \{0, 1\}^f \) there exists \( x \in X_f \) with \( x_{g_i} \in V_{a_i} \).

3.2. Technical preparations. In this part, we provide some tools which will be successively used in the next sections. We start by defining properties of a family of sequences \( \{a_n\}_{n=1}^{\infty} \) where \( n \in \mathbb{N} \) of real numbers which will serve as radii of certain balls in a later step of our construction. In particular, we ask that the following holds for every \( n \in \mathbb{N} \) (if applicable)

1. \((r_n^+)_{n \in \mathbb{N}}\) is a strictly decreasing null-sequence.
2. \((r_n^+)_{n \in \mathbb{N}}\) is a strictly decreasing null-sequence.
3. There exists \( m \in \mathbb{N} + 1 \) such that \( r_{m+1} = r_m^+ \).
(R4) Suppose \( r_{i+1}^n = r_i^n \) for some \( i,j \in \mathbb{N} \). If \( j = 1 \mod 4 \), then \( r_{j+1}^{n+1} = r_{j+1}^n \). If \( j = 2 \mod 4 \), then \( r_{j+3}^{n+1} = r_{j+3}^n \).

(R5) For all \( \theta \in \mathbb{T} \) and every \( i \in \mathbb{N} \), we have \( B_{r_i^n}(\theta) \setminus B_{r_{i+1}^n}(\theta) \neq \emptyset \).

Notice that (R3) and (R4) imply that the sequence \( r_i^{n+1} \) is obtained from \( (r_i^n)_{m_{\infty}} \) by adding two extra entries \( r_{i+1}^{n+1} \) and \( r_{i+3}^{n+1} \) between \( r_i^n = r_j^{n+1} \) and \( r_{i+1}^n = r_{j+1}^{n+1} \) for each even index \( i \geq m_{\infty} \) and appropriately chosen \( j \in \mathbb{N} \).

Observe that it is always possible to find a family with the above properties: First, choose a null-sequence \( (r_i) \) such that for some \( \theta \in \mathbb{T} \) we have \( B_{r_i}(\theta) \setminus B_{r_{i+1}}(\theta) \neq \emptyset \). This is possible since under the present assumptions \( \mathbb{T} \) cannot have isolated points. Now, as pointed out in Remark 1.4, \( E(\mathbb{T}) \) acts transitively and isometrically on \( \mathbb{T} \) which clearly gives that \( B_{r_i}(\theta) \setminus B_{r_{i+1}}(\theta) \neq \emptyset \) actually holds for all \( \theta \in \mathbb{T} \). An appropriate re-labelling of \( (r_i) \) (in an obviously non-injective fashion) provides us with a family \( (r_i^n) \) which satisfies the above. It is worth mentioning and will be used frequently that we can choose the radii \( r_i^n \) arbitrarily small.

By means of the above radii \( (r_i^n) \), we now construct real-valued functions \( f_n \), which will be the building blocks of the semicocycles in the next section. Given \( n \in \mathbb{N} \), let \( f_n : (0, \infty) \to [0, 1] \) be a continuous function which vanishes outside \( (0, r_1^n) \) and verifies \( f_n(x) = 1 \) for all \( x \in [r_i^n, r_{i+1}^n] \) when \( i = 2 \mod 4 \) and \( f_n(x) = 0 \) for all \( x \in [r_i^n, r_{i+1}^n] \) when \( i = 0 \mod 4 \). Obviously, \( f_n \) cannot be extended to a continuous function on \( [0, \infty) \). The next statement follows from the fact that for each \( n \geq 1 \) the function \( f_{n+1} \) assumes both the values 0 and 1 on all but finitely many of those intervals of the form \( [r_i^{n+1}, r_i^n] \) on which \( f_n \) is constant (specifically: on all such intervals contained in \( (0, r_1^{n+1}) \), see Figure 3.1).

**Lemma 3.4.** For each \( \alpha \in \mathbb{N}_0 \), each \( s \in \mathbb{N} \) and all \( a \in \{0, 1\}^s \), there is \( j \in \mathbb{N} \) such that with \( I_a = [r_{j+1}^{n+1}, r_j^{n+1}] \) we have \( (f_n(x))_{a = \alpha_1 + \ldots + \alpha_s = a} = 1 \) for all \( x \in I_a \).

**Remark 3.5.** Notice that (R5) yields that for each \( \theta \in \mathbb{T} \) there is \( \theta' \in \mathbb{T} \) with \( \rho(\theta, \theta') \in I_a \).

### 3.3. Tame non-null extensions

We now turn to the construction of tame but non-null semicocycle extensions. Note that such extensions ask for at least two orbits in \( \mathbb{T} \): one containing \( \theta_0 \) (along which \( f \) is continuous) and another one which hits a non-empty set of discontinuity points \( D_f \) of \( f \).

**Theorem 3.6.** Let \( \mathbb{T} \) be an infinite compact metric space on which \( G \) acts minimally by isometries. Suppose there are at least two distinct \( G \)-orbits in \( \mathbb{T} \) and assume there is a point \( \theta \in \mathbb{T} \) on which \( G \) acts locally almost freely. Then there exists a tame and non-null almost automorphic extension \( (X_f, G) \) of \( (\mathbb{T}, G) \).

If, additionally, for one (and hence every) point \( \theta \in \mathbb{T} \) the orbit \( G\theta \subseteq \mathbb{T} \) is measurable, then \( (X_f, G) \) can be chosen to be regular.

**Remark 3.7.** Observe that a sufficient condition for orbits in \( \mathbb{T} \) to be measurable is to assume that \( G \) is \( \sigma \)-compact: Then, each \( G \)-orbit is \( \sigma \)-compact as well and hence a countable union of compact sets and therefore measurable.
Proof of Theorem 3.6. Pick \( \theta \in \mathbb{T} \) such that \( G \) acts locally almost freely on \( \theta \), let \( g_0 \) coincide with the neutral element \( e_G \) in \( G \) and choose a sequence \( (g_n)_{n=1}^{\infty} \) in \( G \) such that \( (g_n\theta)_{n=0}^{\infty} \) has pairwise distinct elements and \( g_n\theta \to \theta \) as \( n \to \infty \).

Consider a collection of radii \( \{(r^n_i)_{i=1}^{\infty} : n = 1, 2, \ldots \} \) which satisfies (R1)-(R5) Without loss of generality, we may assume that for each \( n \geq 1 \) the radius \( r_1^n \) is sufficiently small to guarantee that

\[
\{B^n_{1^{-1}}(g_n\theta) : n = 1, 2, \ldots \} 
\]

is a collection of pairwise disjoint balls.

For \( n \in \mathbb{N} \), let \( f_n : (0, \infty) \to [0, 1] \) denote the function associated to the sequence \( (r^n_i)_{i=1}^{\infty} \) as described in Section 3.2. Given \( s \in \mathbb{N} \), set \( \alpha(s) = \sum_{j=0}^{s-1} j \) and let \( J_s \subseteq (0, \infty) \) be a closed interval such that \( \bigcup_{n \in \{0,1\}} I_n^{(s)} \subseteq J_s \) and \( f_{\alpha(s)+1}(x) = f_{\alpha(s)+2}(x) = \ldots = f_{\alpha(s+1)}(x) = 0 \) if \( x \) is a boundary point of \( J_s \). Here, the intervals \( I_n^{(s)} \) are provided by Lemma 3.4.

Given \( n \in \mathbb{N} \) with \( \alpha(s) < n \leq \alpha(s+1) \), define \( f_n : [0, \infty) \to [0, 1] \) by

\[
f_n(x) = \begin{cases} f(x) & \text{if } x \in J_s, \\ 0 & \text{otherwise.} \end{cases}
\]

Note that all \( f_n \) are continuous. Further, for every \( s \in \mathbb{N} \), all \( \ell = 1, \ldots, s \), every \( a \in \{0, 1\}^s \), and every \( x \in I_n^{(s)} \subseteq J_s \) we have

\[
f_{\alpha(s)+\ell}(x) = f_{\alpha(s)+\ell}(x) = a \ell,
\]

due to Lemma 3.4.

Now, for \( \omega \in \mathbb{T} \) define

\[
f(\omega) = \begin{cases} 0 & \text{if } \omega = \theta, \\ \sum_{n=1}^{\infty} f_n(\rho(\omega, g_n\theta)) & \text{otherwise.} \end{cases}
\]

Due to the disjointness of the collection of balls in (3.1), this defines a function \( f : \mathbb{T} \to [0, 1] \) which is further continuous on \( \mathbb{T} \). Hence, if \( \theta_0 \in \mathbb{T} \) is such that \( G\theta_0 \cap \{\theta\} = \emptyset \), then \( f \) is a semicycle over \((\mathbb{T}, G, \theta_0)\).

We construct \( F = \bigcup F \subseteq \mathbb{T} \times [0, 1] \) and \( X_f \) as described in Section 3.2. Clearly, \( D_f = \{\theta\} \). To see that \( f \) is invariant under no rotation, pick any distinct \( \theta_1, \theta_2 \in \mathbb{T} \) and choose \( \xi \in E(\mathbb{T}) \) such that \( \xi\theta_1 = \theta \). Then, \( \xi\theta_1 \in D_f \) and \( \xi\theta_2 \in \mathbb{T} \setminus D_f \) so that Proposition 2.3 yields that \( f \) is invariant under no rotation. Hence, by Theorem 2.5 \((X_f, G)\) is an almost automorphic extension of \((\mathbb{T}, G)\). Further, by Lemma 3.2, we immediately obtain that \((X_f, G)\) is tame.

Now, suppose we are given \( a \in \{0, 1\}^s \) for some \( s \in \mathbb{N} \). Since \( G \) acts minimally on \( \mathbb{T} \), we can choose \( h_a \in G \) such that \( \rho(h_a\theta_0, \theta) \) is in the interval \( I_n^{(s)} \) (see also Remark 3.5). As \( G \) acts by isometries, we further have \( \rho(g_n h_a \theta_0, g_n \theta) = \rho(h_a \theta_0, \theta) \) for every \( n \geq 1 \). In particular, this gives \( f_n(g_n h_a \theta_0) = f_n(\rho(g_n h_a \theta_0, g_n \theta)) \) for \( n = \alpha(s)+1, \ldots, \alpha(s+1) \), due to the definition of \( f \) and the disjointness of the balls in (3.1).

Hence, with \( (x_g)_{g \in G} = (f(gh_a \theta_0))_{g \in G} \) we have

\[
x_{g_n} = f_n(\rho(g_n h_a \theta_0, g_n \theta)) = f_n(\rho(h_a \theta_0, \theta)) = a_n \in V_{a_n} \quad \text{for } n = \alpha(s)+1, \ldots, \alpha(s+1),
\]

where \( V_0 = \{0\} \) and \( V_1 = \{1\} \) and where we used (3.2). Since \( s \in \mathbb{N} \) and \( a \in \{0, 1\}^s \) were arbitrary, we obtain that \((X_f, G)\) is null-non by means of Proposition 3.3.

In order to see the second part, first note that the assumption of a measurable orbit implies that every orbit of \((\mathbb{T}, G)\) is measurable and further, that every orbit is necessarily of the same \( m_\gamma \)-measure, due to Remark 1.4. Since \( m_\gamma \) is further ergodic and orbits are clearly invariant sets, this implies that orbits are of measure zero as we assume that there is more than one orbit in \( \mathbb{T} \). Now, let \( \pi \) denote the factor map from \((X_f, G)\) to \((\mathbb{T}, G)\). Clearly, the projection of the almost automorphic points \( \pi(X_0) \) (see (1.1)) coincides with the complement of \( G\theta_0 \) and is hence of full measure.

Remark 3.8. In view of the above theorem, we may ask if there is a minimal isometric dynamical system with more than one orbit such that its almost automorphic extensions are null if and only if they are tame. Note that the assumption of a point \( \theta \in \mathbb{T} \) on which \( G \) acts

\footnote{Note that this immediately yields that there are actually uncountably many orbits in \( \mathbb{T} \).}
Clearly, the canonical action of the special orthogonal group $SO(3)$ on the 2-sphere $S^2$ is minimal, isometric and not locally almost free on any $\theta \in S^2$. While $(S^2, SO(3))$ only allows for exactly one orbit, we may consider the natural action of the product $G = SO(3) \times H$ on $T = S^2 \times T_1$, where $(T_1, H)$ is some minimal isometric dynamical system which allows for at least two distinct orbits. It is straightforward to see that $(T, G)$ is still minimal and isometric and that $(T, G)$ allows for more than one orbit. Furthermore, $G$ does not act locally almost freely on any $\theta \in T$. However, we clearly have that if there is some non-null and tame almost automorphic extension $(X, H)$ of $(T, G)$, then $(S^2 \times X, G)$ also is a non-null and tame almost automorphic extension of $(T, G)$.

**Remark 3.9.** According to [20, Corollary 5.4] and [21, Theorem 1.2], a minimal tame dynamical system on a metric space which allows for an invariant measure is necessarily regular already almost automorphic. In fact, a close inspection of the proof in [21] shows that every tame almost automorphic system with the property that its maximal equicontinuous subgroup is metrizable and such that orbits in $T$ are measurable is automatically regular. Against this background, we could also reformulate the second part of Theorem 3.6 as follows: If, additionally, for one (and hence every) point $\theta \in T$ the orbit $G\theta \subseteq T$ is measurable, then $(X_f, G)$ is regular.

**Remark 3.10.** It is worth remarking that free minimal equicontinuous systems $(T, G)$, where $T$ is assumed to be metrizable and infinite, may, in fact, allow for only two orbits: Let $T$ be a compact topological group which allows for a dense subgroup $G \subseteq T$ of index 2. Then the natural action of $G$ on $T$ has exactly two orbits and is clearly free and minimal.

Note that a well-known example of such $T$ is given by the product $T = \{0, 1\}^N$ of countably many copies of the finite field $\{0, 1\}$. We may consider $T$ as a vector space (and hence a group) over the field $\{0, 1\}$. Pick a base $B$ of $T$ which contains in particular those elements which have exactly one entry equal to 1. Observe that $B$ necessarily contains an element $b$ with infinitely many entries equal to 0 and infinitely many entries equal to 1. Now, it is straightforward to see that the linear span of $B \setminus \{b\}$ is a subgroup of index 2 in $\{0, 1\}^N$ while it is clearly dense in $T$.

### 3.4. Non-tame extensions

We now turn to the construction of non-tame examples. As Lemma 3.2 suggests, such examples may in general need at least countably many discontinuity points of the associated semicocycle. Therefore, the invariance under no rotation has to be obtained in a more elaborate fashion than in the previous section.

**Theorem 3.11.** Let $T$ be an infinite compact metric space on which $G$ acts minimally by isometries. Suppose there are at least two distinct orbits in $T$ under $G$. Then there exists a non-tame almost automorphic extension $(X_f, G)$ of $(T, G)$.

If, additionally, for one (and hence every) point $\theta \in T$ the orbit $G\theta \subseteq T$ is measurable, then $(X_f, G)$ can be chosen to be regular.

**Proof.** Pick distinct $\theta, \theta' \in T$ from one and the same orbit, that is, $G\theta = G\theta'$. Let $r > 0$ be such that $\rho(\theta, \theta') > 2r$. Due to Corollary 1.7 there is $g_1 \in G$ with $B_r(\theta) \ni g_1\theta \neq \theta$ and $g_1\theta' \neq \theta'$. Let $g_0$ coincide with the neutral element $e_G$ of $G$ and choose a sequence $(g_n)_{n=2}^\infty$ in $G$ such that $(g_n\theta)_{n=0}^\infty$ consists of pairwise distinct elements with $g_n\theta \to \theta$ as $n \to \infty$ and $g_n\theta \in B_r(\theta) \cap B_r(g_1\theta)$ for all $n \in \mathbb{N}$.

Consider a collection of radii $\{(r_n^1)\}_{n=1}^\infty : n = 1, 2, \ldots\}$ which satisfies (R1)-(R5). By choosing the radii $r_n^1$ sufficiently small, we may assume that first,

$$\{B_{2r_1}(g_1\theta)\} \cup \{B_{2r_{n-1}^1}(g_n\theta) : n = 2, 3, \ldots\}$$

is a family of pairwise disjoint balls, secondly,

$$B_r(\theta) \cap B_r(g_1\theta) \supseteq \cl\left(\bigcup_{n=1}^\infty B_{r_n^1}(g_n\theta)\right)$$

and thirdly, that

$$\rho(\theta', g_1\theta') > 2 \cdot r_1^1.$$
Let $\Theta = \{g_n\theta: n = 0, 1, \ldots\} \cup \{\theta\}$. Define $f: \mathbb{T} \to [0,1]$ by

$$f(\omega) = \begin{cases} 0 & \text{if } \omega \in \Theta, \\ f_1(\rho(\omega, \theta')) + \sum_{n=1}^{\infty} f_n(\rho(\omega, g_n\theta)) & \text{otherwise.} \end{cases}$$

(3.6)

Observe that $f$ is continuous outside the set $\Theta \subseteq G\theta$. Further, by the assumptions, there is $\theta_0 \in \mathbb{T} \setminus G\theta$ so that the restriction of $f$ to $G\theta_0$ is, in fact, continuous. We may hence consider $f$ to be a semicocycle over $(\mathbb{T}, G, \theta_0)$. We construct $F$ and $X_f$ as described in Section 2.

Clearly, $D_f = \Theta$.

We next show that $f$ is invariant under no rotation. For the sake of the construction of symbolic examples in the next section, we are going to prove strictly more than we actually need for the present purpose. To that end, let us define

$$U_{g\theta} = \overline{\bigcup_{n=1}^{\infty} B_{r_1}(g_n\theta) \cup \beta_{r_1}(\theta')} = \overline{\bigcup_{n=1}^{\infty} B_{r_1}(g_n\theta) \cup \beta_{r_1}(\theta')}.$$  

(3.7)

Claim 3.12. For distinct $\theta_1$ and $\theta_2$ in $\mathbb{T}$, there is $\xi \in E(\mathbb{T})$ such that $\xi \theta_1 \in \Theta$ and $\xi \theta_2 \notin U_{g\theta}$.

Observe that $\Theta \subseteq U_{g\theta}$. Hence, taken the above claim for granted, we immediately obtain the invariance under no rotation from Proposition 2.3 so that $(X_f, G)$ is indeed a semicocycle extension—and thus, an almost automorphic extension—of $(\mathbb{T}, G)$.

Proof of Claim 3.12. Fix $\theta_1, \theta_2 \in \mathbb{T}$ with $\theta_1 \neq \theta_2$. We have to distinguish between the following cases where we repeatedly used that $E(\mathbb{T})$ acts transitively and isometrically on $\mathbb{T}$ (see Remark 1.4).

Case 1 ($0 < \rho(\theta_1, \theta_2) \leq r_1$): In this case, since $(r^n_1)_{n \in \mathbb{N}}$ is a strictly decreasing null sequence (due to (R1)), there is $n_0$ such that $r^{n_0}_1 \geq \rho(\theta_1, \theta_2) > r^{n_0+1}_1$. Choose $\xi \in E(\mathbb{T})$ such that $\xi \theta_1 = g_{n_0+1}\theta \in \Theta$. Then, we have $\xi \theta_2 \in B_{r_1}(g_{n_0+1}\theta) \setminus B_{r_1}(g_{n_0+1}\theta)$ so that the disjointness of the family of balls in (3.3) gives that $\xi \theta_2 \notin U_{g\theta}$.

Case 2 ($r_1 < \rho(\theta_1, \theta_2) \leq r$): In this case, choose $\xi \in E(\mathbb{T})$ such that $\xi \theta_1 = \theta' \in \Theta$. Then $\xi \theta_2 \in B_r(\theta') \setminus B_{r_1}(\theta')$ which is clearly in the complement of $U_{g\theta}$ due to (3.4) and the fact that $\rho(\theta', \theta) > 2r$.

Case 3 ($r < \rho(\theta_1, \theta_2)$): Choose $\xi \in E(\mathbb{T})$ such that $\xi \theta_1 = \theta \in \Theta$. If $\xi \theta_2 \notin U_{g\theta}$, we are done. Hence, it remains to consider $\xi \theta_2 \in U_{g\theta}$. In this case, we necessarily have $\xi \theta_2 \in B_{r_1}(\theta')$, due to (3.4). Now, observe that $g_1 \xi \theta_1 = g_1 \theta \in \Theta$. However, by the reverse triangle inequality,

$$\rho(\theta', g_1 \xi \theta_2) \geq \rho(\theta', g_1 \theta) - \rho(g_1 \theta', g_1 \xi \theta_2) = \rho(\theta', g_1 \theta) - \rho(\theta', \xi \theta_2) > 2r_1 - r_1 \geq r_1,$$

where we used (3.5) in the second to the last step. Hence, $g_1 \xi \theta_2 \notin B_{r_1}(\theta')$. At the same time, $\rho(g_1 \xi \theta_2, g_1 \xi \theta_1) = \rho(\theta_2, \theta_1) > r$, so that by (3.4) we indeed obtain $g_1 \xi \theta_2 \notin U_{g\theta}$. This proves the claim.

In order to finish the proof of the first part, it remains to show that $(X_f, G)$ is non-tame. To that end, suppose we are given $a \in \{0,1\}^s$ for some $s \in \mathbb{N}$. Choose $h_a \in G$ such that $\rho(h_a\theta_0, \theta)$ is in the interval $I_a^{(s)}$ from Lemma 3.4 which is possible due to Remark 3.5 and since $(\mathbb{T}, G)$ is minimal. Since $G$ acts by isometries, we have $\rho(g_n h_a\theta_0, g_n \theta) = \rho(h_a\theta_0, \theta)$ for every $n \geq 1$ so that $f(g_n h_a\theta_0, g_n \theta) = f_n(\rho(h_a\theta_0, \theta))$ for all $n = 1, \ldots, s$ (due to the definition of $f$ and due the disjointness of the family of balls in (3.3)). Hence, with $(x_g)_{g \in G} = (f(g h_a\theta_0))_{g \in G}$, Lemma 3.4 gives

$$x_{g_a} = f_n(\rho(h_a \theta_0, \theta)) = a_n \in V_{a_n} \text{ for each } n = 1, \ldots, s,$$

where $V_0 = \{0\}$ and $V_1 = \{1\}$. Since $s \in \mathbb{N}$ and $a \in \{0,1\}^s$ were arbitrary, we obtain that $(X_f, G)$ is non-tame by means of Proposition 3.1. This finishes the proof of the first part.

The second part follows similarly as in the proof of Theorem 3.6 since the complement of the projection of the almost automorphic points of $X_f$ coincides with $GD_f$ and is hence a countable union of orbits.

\footnote{In fact, in order to immediately obtain invariance under no rotation in a way as simple as in Theorem 3.6 we could construct $f$ in such a way that $\theta'$ is the unique point in $D_f$ with $\theta \in F(\theta)$ (by simply replacing the summand $f_1(\rho(\omega, \theta'))$ by $2f_1(\rho(\omega, \theta'))$ in (3.5)). However, as we also aim at symbolic examples, we won’t follow this path.}
3.5. Symbolic examples. Note that in the constructions above we obtained \( K \)-valued semicocycles with \( K = \{0, 1\} \). It is natural to ask whether we can find symbolic examples, that is, \( \{0, 1\} \)-valued semicocycles which yield tame non-null and non-tame extensions, respectively. In the proofs above, it was not only important to control the number of times a given orbit hits the set of discontinuity points \( D_f \) but also that there are orbits which don’t hit \( D_f \) at all. Due to the fact that we dealt with \( \{0, 1\} \)-valued examples, we could ensure that the set \( D_f \) was at most countable which simplified the related problems considerably.

Now, in order to restrict the set of values to \( \{0, 1\} \), we may change the functions \( f_n \) to obtain maps \( f'_n : (0, \infty) \to \{0, 1\} \). On the side of non-tame extensions, we obtain the following statement with these \( \{0, 1\} \)-valued functions as in the proofs of Theorem 3.6 and Theorem 3.11 we may create new discontinuity points. This may even imply \( GD_f = \mathbb{T} \) which is an obvious obstruction for the constructions.

Nonetheless, under suitable extra assumptions, we can still obtain regular symbolic semicocycle extensions. On the side of non-tame extensions, we obtain the following statement and en generalisation of [23, Theorem 3.1].

**Corollary 3.13.** Suppose \( G \) is countable and \( (\mathbb{T}, G) \) is a metric minimal isometric dynamical system (with \( \mathbb{T} \) infinite). Then there is a symbolic regular almost automorphic extension \( (X_f, G) \) which is non-tame.

**Proof.** The proof works almost literally as the proof of Theorem 3.11 if we construct \( f \) by means of the functions \( f'_n \) instead of \( f_n \) and if we assume that the family of radii \( \{r^n_i\} \) verifies not only (R1)–(R5) but also

\[(R6) \quad \text{For every } \theta \in \mathbb{T} \text{ and all } i, n \in \mathbb{N} \text{ we have } m_\mathbb{T}(B_{r^n_i}(\theta) \setminus B_{r^{n+1}_i}(\theta)) = 0.\]

Note that (R6) holds as soon as there is just one \( \theta \in \mathbb{T} \) with \( m_\mathbb{T}(B_{r^n_i}(\theta) \setminus B_{r^{n+1}_i}(\theta)) = 0 \) (see also Remark 1.4). This can clearly be realised since \( m_\mathbb{T} \) is finite and the family \( \{r^n_i\} \) is countable.

We leave the remaining details of the proof to the reader but would like to make the following comments:

- Now, the set of discontinuity points \( D_f \) is the union of the countable set \( \Theta \) and countably many sets of the form \( B_{r^n_i}(\theta) \setminus B_{r^{n+1}_i}(\theta) \) which are assumed to be of measure zero due to (R6). Hence, as \( G \) is countable, \( m_\mathbb{T}(GD_f) = 0 \) so that there is a point \( \theta_0 \) along whose orbit \( f \) is indeed continuous.
- Since \( \Theta \subseteq D_f \subseteq U_{\theta_0} \) (see equation (3.7)), we obtain that \( f \) is indeed invariant under no rotation due to Claim 3.12 and Proposition 2.3. Hence, \( (X_f, G) \) is an almost automorphic extension of \( (\mathbb{T}, G) \) which is further regular since \( m_\mathbb{T}(GD_f) = 0. \)

**Remark 3.14.** For the special case of irrational rotations on \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \), a similar result was announced in [20, Remark 5.8]. As a matter of fact, it has been proven in [21, Corollary 3.7] that a symbolic extension \( (X_f, \mathbb{Z}) \) of an irrational rotation on \( \mathbb{R}/\mathbb{Z} \) is non-tame if \( D_f \) is a Cantor set. In the light of the above statement and the fact that already a single discontinuity point can destroy nullness (according to Theorem 3.6), this seems to suggest that the question of whether an almost automorphic system is tame or null simply boils down to a question of the size of \( D_f \). However, the mechanism which establishes the non-tameness in [21, Corollary 3.7] can easily be destroyed by placing countably many points in the gaps of the respective Cantor set. That is, for every prescribed value \( \gamma \) between 0 and 1, there is a \( \{0, 1\} \)-valued semicocycle \( f \) such that \( D_f \) is of Hausdorff measure \( \gamma \) while \( (X_f, \mathbb{Z}) \) is a symbolic tame extension of an irrational rotation.

To obtain tame non-null examples, we have to introduce a stronger version of the assumption (R6) in order to still be able to apply Lemma 3.2. This boils down to restrictions on the space \( \mathbb{T} \) which, on the other, also allow for symbolic non-tame extensions even if \( G \) is uncountable.

**Corollary 3.15.** If \( \mathbb{T} \) is a Cantor set and \( (\mathbb{T}, G) \) is a minimal equicontinuous dynamical system with at least two distinct orbits, then there is a symbolic almost one-to-one extension \( (X_f, G) \) which is non-tame.
If, additionally, for one (and hence every) point $\theta \in \mathbb{T}$ the orbit $G\theta \subseteq \mathbb{T}$ is measurable, then $(X_f, G)$ can be chosen to be regular.

If $G$ acts locally almost freely on some point $\theta \in \mathbb{T}$, then all of the above holds true if we replace non-tame by tame non-null.

**Proof.** Let us assume to be given a family of radii $(r^n_i)$ which verifies not only (R1)–(R5) but also

(R6') For every $\theta \in \mathbb{T}$ and all $i, n \in \mathbb{N}$ the ball $B_{r^n_i}(\theta)$ is clopen.

Under the assumption of (R6'), functions of the form $\omega \mapsto f'_n(\rho(\omega, \theta))$ are continuous on $\mathbb{T} \setminus \{\theta\}$ so that the proofs of Theorem 3.6 and Theorem 3.11 respectively, translate literally to the present setting.

To see that (R6') can always be ensured under the above hypothesis, note that we may assume without loss of generality that $\mathbb{T}$ is equipped with the compatible $G$-invariant metric $\rho$ given by

\[ \rho(\theta, \theta') = \sup_{g \in G} d(h(g\theta), h(g\theta')), \]

where $h$ denotes a homeomorphism from $\mathbb{T}$ to $\{0, 1\}^N$ and $d$ denotes the Cantor metric $d(x, y) = 2^{-\min\{n: x_n \neq y_n\}}$ on $\{0, 1\}^\mathbb{N}$ which only assumes values in $\{0\} \cup \{1/2^\ell: \ell \in \mathbb{N}\}$. Hence, we can assume the metric $\rho$ to assume only countably many values, too. This certainly allows to guarantee that for some $\theta \in \mathbb{T}$ and all $i, n \in \mathbb{N}$, the balls $B_{r^n_i}(\theta)$ are clopen. As in the previous examples, Remark 1.4 yields that this carries over to all $\theta \in \mathbb{T}$ so that (R6') can be realised. \[ \square \]

If $\mathbb{T}$ is a Cantor set and $G$ a discrete countable group, then the family of free minimal equicontinuous systems $(\mathbb{T}, G)$ is well understood: the group $G$ is necessarily residually finite and $(\mathbb{T}, G)$ is isomorphic to a $G$-odometer (see [32, Theorem 2.7]). The next statement follows from [32, Theorem 2.7] and Corollary 3.13 and Corollary 3.15 combined with a characterisation of Toeplitz flows as symbolic almost one-to-one extensions of free minimal $G$-odometers. For a thorough discussion of $G$-odometers and Toeplitz flows over residually finite groups, we refer the reader to [32, 34].

**Corollary 3.16.** Let $G$ be a countable discrete group and assume $\mathbb{T}$ is a Cantor set. Then every free minimal equicontinuous $G$-action $(\mathbb{T}, G)$ is the maximal equicontinuous factor of regular Toeplitz flows $(X_1, G)$ and $(X_2, G)$ such that $(X_1, G)$ is non-tame and $(X_2, G)$ is tame but non-null.

The point in the proof of Corollary 3.15 is that despite the fact that $f$ only assumes values in $\{0, 1\}$, the set of discontinuity points is still only countable or even finite. While in general, such a straightforward argument is not available, we obtain

**Corollary 3.17.** Every irrational rotation on $\mathbb{R}/\mathbb{Z}$ allows for a symbolic almost one-to-one extension which is tame but non-null.

**Proof.** As in the previous examples, we leave the details to the reader and only briefly discuss the differences to the proofs in the previous section. Suppose we have a rotation by an irrational angle $\alpha$. We proceed similarly as in the proof of Theorem 3.6 where we replace the functions $f_n$ by $f'_n$. This time, we choose the family $(r^n_i)$ to satisfy (R1)–(R5) and further assume that for distinct $r^n_i \neq r^m_j$ we have that $\mathbb{Z}\alpha + r^n_i \cap \mathbb{Z}\alpha + r^m_j = \emptyset$. As $r^n_i \to 0$ (due to (R1)), this ensures that every orbit hits the countable set of the respective discontinuity points at most finitely many times which again allows the application of Lemma 3.2. \[ \square \]

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