Rotating azimuthons in dissipative Kerr media excited by superpositions of Bessel beams

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We report the existence of persistently rotating azimuthons in media with self-focusing Kerr and absorption nonlinearities. The nonlinear loss is balanced by power influx from the peripheral reservoir stored in slowly decaying tails of the azimuthons. These modes are excited by a superposition of two Bessel beams with opposite vorticities, \( \pm s \), and slightly different conicities. The excited azimuthon exhibits opposite vorticity in its center to that of the input Bessel-beam superposition due to spontaneous inversion of the topological charge in the course of the azimuthon formation. Unlike azimuthons in loss-free media, number \( N \) of rotating intensity maxima and \( s \) are not mutually independent, being related by \( N = 2s \). The robustness of the rotating azimuthons is enhanced in comparison to similar static dissipative patterns. They can be excited in almost any transparent material, in the range of intensities for which the nonlinear absorption, induced by multiphoton absorption, is relevant. Close to the ionization threshold, the rotating azimuthons reproduce recently observed helical filaments of light in air and CS\(_2\).

I. INTRODUCTION

Solitons, solitary vortices \cite{1,2}, soliton clusters and necklace-shaped structures \cite{3,4}, and azimuthons \cite{5,6} are increasingly sophisticated, self-trapped light modes in nonlinear optical media, which have been predicted and experimentally realized in the last decades \cite{7}. Soliton clusters and azimuthons feature propagation-invariant or nearly-invariant intensity patterns that rotate uniformly as they propagate in transparent media, and require a stabilizing mechanism to arrest the collapse instability brought by the self-focusing Kerr nonlinearity. Such a mechanism may be provided by saturation of the Kerr nonlinearity \cite{3,5,8}. Many of these solitary structures have dissipative-soliton counterparts, which are supported by the balance between gain and losses, which occurs in laser cavities, in addition to the balance between the diffraction and self-focusing \cite{9,10}.

Relaxing the condition of strong localization, one can consider weakly localized states similar to Bessel beams \cite{11,12}, whose total norm (integral power, in terms of optics) diverges at \( r \to \infty \), where \( r \) is the radial coordinate in the two-dimensional plane, perpendicular to the propagation axis, \( z \). Unlike dissipative solitons, stationarity of such states is supported not through the balance of loss and gain, but rather due to the compensation of nonlinear dissipation, induced by multiphoton absorption in the optical material (which may generate weak plasma) and influx of power stored, in an indefinitely large (diverging) amount, in the weakly decaying tail of the quasi-Gaussian beam center that creates a weak plasma \( \text{[11][14]} \).

Similarly to the above-mentioned conservative systems, nonlinear dissipative Bessel beams with embedded vorticity have been predicted \cite{15,16}, and experimentally observed to induce tubular filamentation \cite{17}. A remarkable fact is that the dissipative nonlinear Bessel vortex beams can be stable in self-focusing media with the pure-cubic (Kerr) nonlinearity due to the stabilizing action of nonlinear absorption \cite{18}. More recently, launching arbitrary superpositions of Bessel beams with the same cone angle but different embedded vorticities has been shown to excite propagation-invariant (stationary) dissipation patterns of rather arbitrary shapes, the so-called “dissipatons” \cite{19}.

In this work, motivated by recent experiments exhibiting helical filamentation of light beams \cite{20,21}, we report the existence of what we call rotating dissipative azimuthons. These modes propagate steadily, with a constant rotation velocity, in nonlinearly absorbing Kerr media, being excited by superpositions of two Bessel vortex beams with opposite vorticities and slightly different conicities. Such coherent superpositions of Bessel beams can be readily generated in the experiment. In the linear approximation, their propagation was theoretically studied in Refs. \cite{22,23}.

Unlike soliton clusters and azimuthons in conservative systems, number \( N \) of rotating high-intensity (“hot”) spots and magnitude \( s \) of the vorticity at the center of the dissipative azimuthon are not independent integers, but are related by \( N = 2s \), which can be explained analytically. The rotating azimuthons exhibit a mixed linear-nonlinear behavior, resembling in some aspects the linear propagation of superpositions of Bessel beams with opposite vorticities and different cone angles, while in other respects they are similar to solitons clusters. Namely, their quasi-linear peripheral field imposes the same angular velocity of rotation as that induced by the input superposition of the Bessel beams [see Eq. (6) below]. However, the vorticity at the center of the azimuthon acquires a topological charge opposite to that of the input superposition of the Bessel beams, as a result of the...
topological-charge inversion in the course of the formation of the azimuthon, which can be explained by a trend to minimization of the Hamiltonian of the model’s conservative part. As a result, the dissipative azimuthon rotates in the direction opposite to the azimuthal gradient of the phase associated with the vorticity at the center, in the same way as soliton clusters do.

We also report a gyroscopic effect in the spontaneous formation of dissipative azimuthons, viz., that the rotation accelerates the formation of the spinning steady-shape patterns, in comparison to similar non-rotating steady patterns, or “dissipatons”, which arise when cone angles of the two Bessel beams are equal in the input state. We also observe enhanced stability of the rotating dissipative azimuthons, as compared to dissipative Bessel vortex beams and static dissipatons. In the case of instability, the rotating dissipatons feature richer dynamics, including formation of persistently pulsating azimuthons.

Basic results, produced by systematic numerical simulations of the model, are reported in Section II. The gyroscopic effect and a detailed analysis of the stability of the rotating dissipative azimuthons are presented in Section III and IV, respectively. The paper is concluded in Section V.

II. DISSIPATIVE AZIMUTHONS EXCITED BY SUPERPOSITIONS OF BESSEL BEAMS WITH OPPOSITE TOPOLOGICAL CHARGES

We consider the propagation of monochromatic light beams along the $z$ axis, $E = A \exp(-i\omega t + ikz)$, with carrier frequency $\omega$ and propagation constant $k = n\omega/c$, where $c$ is the speed of light in vacuum, and $n$ is the linear refractive index. The nonlinear Schrödinger equation (NLSE) that governs the paraxial propagation of field envelope $A$, is [15]

$$\partial_z A = \frac{i}{2k} \nabla_z^2 A + \frac{ikn_2}{n} |A|^2 A - \frac{\beta^{(M)}}{2} |A|^{2M-2} A,$$  \hspace{1cm} (1)

where $\nabla_z^2 \equiv \partial_z^2 + (1/r)\partial_r + (1/r^2)\partial_\varphi^2$ is the transverse Laplacian written in polar coordinates $(r, \varphi)$ in the transverse plane, $n_2 > 0$ is the nonlinear refractive index, and $\beta^{(M)} > 0$ is the multiphoton absorption coefficient of order $M$.

In the absence of the nonlinear terms, Eq. (1) is satisfied by Bessel beams carrying vorticity with any integer topological charge $s$. In the form explicitly representing the paraxial approximation they are $A(r, \varphi) \propto J_s(k\theta_s r) \exp(is\varphi) \exp(i\delta_s z)$, where $J_s$ is the Bessel function of the first kind and order $s$, $\theta_s > 0$ is the cone angle, and

$$\delta_s = -k\theta_s^2/2 < 0$$ \hspace{1cm} (2)

is the contribution to the propagation constant associated with the conical geometry of the Bessel beam. In the linear regime, Eq. (1) is actually satisfied by any superposition of Bessel beams with arbitrary topological charges, amplitudes and cone angles. In particular, superpositions with different cone angles produce propagation-invariant intensity patterns which rotate in the course of the propagation, as demonstrated in [22, 23]. Rotatory polarization patterns in free space, emulating optical activity of the effective medium, have also been demonstrated in superpositions of orthogonally polarized Bessel beams with different cone angles [24].

In the nonlinear realm, starting from Refs. [25] for the pure-Kerr medium, and [11, 12] including more general nonlinearities and dissipative nonlinear terms, undis-
torted and unattenuated propagation in the form of non-
linear Bessel beams was demonstrated experimentally. A
linear Bessel beam launched in the nonlinear medium
spontaneously transforms into an appropriate nonlinear
beam. Subsequently, similar properties have been
demonstrated for vortical Bessel beams, which too were
shown to transform into propagation-invariant nonlinear
counterparts \[15\]-\[17\]. The existence of fully stable non-
linear Bessel vortex beams in media with pure-cubic Kerr
nonlinearity has been established in Ref. \[15\], where
the stabilizing mechanism is provided by nonlinear ab-
sorption. In a more recent study, arbitrary superposi-
tions of Bessel beams with different topological charges
and amplitudes but identical cone angles, launched into
the medium with nonlinear absorption \[14\] were shown
to excite propagation-invariant “dissipatons”, which are
rather arbitrary structures composed of vortices and
bright spots where power is continuously dissipated. In
those studies, stationarity propagation in media with
nonlinear absorption was enabled by a feeding mech-
nism provided by power influx from the reservoir with an
indefinitely large capacity, maintained by the slowly de-
caying quasi-linear tails of nonlinear Bessel vortex beams
dissipatons. This mechanism is not possible with
strongly localized dissipative solitons, which should be
maintained by intrinsic gain.

Searching for steady states in the rotating frame, we
look for solutions to Eq. (1) with input
\[ A(r, \varphi, z = 0) = a_s [J_s(k\theta_s r) \exp(i\varphi)
+ J_{-s}(k\theta_{-s} r) \exp(-i\varphi)] , \]
composed of two Bessel beams with equal amplitudes \(a_s\),
opposite vorticities \(\pm s\), \(s = 1, 2 \ldots\) and slightly different
cone angles, \(\theta_{\pm s} > 0\), hence the respective shifts of the
axial wave vectors,
\[ \delta_{\pm s} = -k\theta_{\pm s}^2/2, \]
are slightly different too. It is worthy to note that the
solution of the linearized version of Eq. (1) seeded by
input (3), namely,
\[ A(r, \varphi, z) = a_s [J_s(k\theta_s r) \exp(i\varphi + i\delta_s z)
+ J_{-s}(k\theta_{-s} r) \exp(-i\varphi + i\delta_{-s} z) ] , \]
features vorticity \(s\) at its center \((r \to 0)\) for \(\theta_s > \theta_{-s}\),
and \(-s\) for \(\theta_s < \theta_{-s}\), with the phase increasing by \(2\pi s\)
counterclockwise and clockwise, respectively, in a circle
of small radius \(r\) about the beam center. With increasing
radius, the vorticity keeps switching between \(s\) to \(-s\).
The angular velocity of the rotation of the intensity pat-
tern, \([A(r, \varphi, z)]^2\), corresponding to the linear solution in
Eq. (5) is \[22\] \[23\]
\[ \Omega = \frac{\delta_{-s} - \delta_s}{2s} = \frac{k}{4s}(\theta_s^2 - \theta_{-s}^2), \]
where Eq. (4) is used, which is counterclockwise (posi-
tive) for \(\theta_s > \theta_{-s}\) and clockwise (negative) for \(\theta_s < \theta_{-s}\).

Thus, the direction of rotation of the intensity pattern
coincides with the direction of the azimuthal gradient of
the phase about the beam center in these linear super-
positions of Bessel beams.

Figure 1(a) is an example of the nonlinear evolution
produced by initial condition (3) with \(s = 1\) and slightly
different cone angles. The simulation of Eq. (1) was
performed with parameters corresponding to the high-
intensity light propagation in air, in the regime for which
the Kerr self-focusing and nonlinear absorption are sig-
ificant (see the caption to the figure for details). The
numerical solution demonstrates that the propagating
beam fast enough attains a steady rotatory state, featur-
ing compressed lobes as a result of the Kerr self-focusing,
as seen in Fig. 1(b). The solid black curve in Fig. 1(c)
shows that, the peak intensity approaches a constant
value in the course of the propagation, confirming the
stationarity of the rotating pattern. Another manifesta-
tion of the steady propagation is the fact that the rate of
the nonlinear loss (NLL) of the power,
\[ \text{NLL} = 2\pi|\beta(M)| \int_0^\infty |A(r)|^{2M} r dr , \]
also attains a nearly constant value, as shown by the red
solid curve in Fig. 1(c). Thus, the rotating structure is
a genuine dissipative azimuthon.

As shown in Figs. 1-5 for different values of \(s\), the intensity pattern of dissipative azimuthons preserves the
2\(s\)-fold rotational symmetry of the input configuration
given by Eq. (3). Thus the number of hot spots in the
rotating azimuthons, \(N = 2s\), is only determined by mag-
nitude \(s\) of the topological charge in the two beams which
build the input. In this respect, dissipative azimuthons
differ from their conservative counterparts and soliton
clusters, for which the number of hot spots and the vor-
ticity at the center are independent integers \[3\] \[6\]. The
latter property of the conservative model is explained by
the fact that the number of intensity maxima in the circu-
lar pattern is determined by the modulational instability
of the axially uniform state.

Another manifestation of the robustness of the dissi-
pative azimuthons is that they preserve the angular ve-
locity imposed by the superposition of the two vortex
Bessel beams in input (3), viz., \(\Omega = (\delta_{-s} - \delta_s)/2s\); hence
the rotation period is \(\tau_{\text{rotation}} = 4\pi s/|\delta_{-s} - \delta_s|\), and,
given the 2\(s\)-fold rotational symmetry of the rotating pat-
tern, it periodically repeats itself after passing distance
\(2\pi/|\delta_{-s} - \delta_s|\), independent of the topological charge, \(s\).
The preservation of the angular velocity is explained by
the fact that the dissipative azimuthon is surrounded by
the quasi-linear asymptotic field (supporting the above-
mentioned power reservoir) that continues to rotate as
the input linear superposition does, i.e., with angular ve-
locity \(\xi\). Since the whole structure is stationary in the
rotating frame, the inner nonlinear region rotates syn-
chronously with the small-amplitude periphery.

A general feature of dissipative azimuthons that makes
them substantially different from the linear Bessel super-
position in Eq. (3) is that the direction of rotation of the intensity pattern is opposite to the direction of the azimuthal phase gradient close to the central vortex. The reason is that the topological charge of the vortex at the center reverses its sign in the course of the azimuthon formation from the initial Bessel beam superposition, as shown in Fig. 2. The intensity pattern of the input Bessel beam superposition in Fig. 2(a) has $\theta_s > \theta_{-s}$, hence the input vorticity at the center is $s = 1$, corresponding to the counterclockwise phase increase by $2\pi s$ around the origin, as shown by the azimuthal phase profile in Fig. 2(c). Accordingly, the intensity pattern in the azimuthon established by the evolution of intensity pattern, which is displayed in Fig. 2(b), rotates counterclockwise too. However, the vorticity at the center inverts in the course of the evolution to $-s = -1$, corresponding to the clockwise phase increase by $2\pi s$ around the origin seen in Fig. 2(d). Inversion of the vorticity affects the whole central zone of the azimuthon including the hot spots. At larger radii the vorticity oscillates, and in the most peripheral zone the periodic alternations coincide with those of the input Bessel beam superposition.

It is relevant to note that a possibility of dynamical inversion of the sign of the topological charge of an optical vortex in a conservative medium, which interacts with a material lattice structure, was previously predicted theoretically [26] and demonstrated experimentally [27]. In the present setting, the rotating intensity pattern emerging in the azimuthon mode may play a role of such a lattice. Indeed, the rotation of a layer with radius $R$ at angular velocity $\Omega$ tends to add the term generated by the Galilean transform, $kR^2\Omega (\varphi - \Omega z/2)$, to the phase of the wave field (subject to the periodicity constraint, $kR^2\Omega = m$, with integer $m$). If, on the other hand, vorticity phase $\pm s \varphi$ dominates at $r$ small enough, the minimization of the gradient term in the Hamiltonian of the conservative part of Eq. (1), with density $\rho_s \equiv \rho, \varphi$, suggests to choose mutual signs of $\Omega$ and $\pm s \varphi$, which help to cancel different contributions to the phase.

Thus, dissipative azimuthons clearly exhibit a mixed linear-nonlinear structure: The stationary intensity pattern, including the quasi-linear periphery, rotating with angular velocity given by Eq. (4), which is imposed by input (3), and a restructured nonlinear center including the circular chain of hot spots, whose vorticity is inverted with respect to the rotation direction, as in soliton clusters [3, 5].

For a more comprehensive study of the properties of the dissipative azimuthons [in particular, their (in)stability], we introduce

$$\delta \equiv \frac{1}{2} (\delta_s + \delta_{-s}) , \quad \Delta \delta \equiv \frac{1}{2} (\delta_s - \delta_{-s}) , \quad (8)$$

where $\delta_{\pm s}$ is defined as per Eq. (2), and then scaled radius and propagation distance

$$\rho \equiv \sqrt{2k|\delta|} r , \quad \zeta \equiv |\delta| z , \quad \tilde{A} \equiv \left( \frac{\beta (M)}{2|\delta|} \right)^{1/(M-1)} A , \quad (9)$$

which transform NLSE (1) into

$$\partial_z \tilde{A} = i\nabla_\perp^2 \tilde{A} + i\alpha |\tilde{A}|^2 \tilde{A} - |\tilde{A}|^{2M-2} \tilde{A} , \quad (10)$$

where now $\nabla_\perp = \partial_{\rho}^2 + (1/\rho) \partial_{\rho} + (1/\rho^2) \partial_{\varphi}^2$, and

$$\alpha \equiv \left( \frac{2|\delta|}{\beta (M)} \right)^{1/(M-1)} \frac{k n_2}{n |\delta|} . \quad (11)$$

In this notation, input (3) takes the form of

$$A (\rho , \varphi , 0) = b_s \left[ J_s \left( \sqrt{1 + \frac{\Delta \delta}{\delta} \rho} \right) e^{is \varphi} + J_{-s} \left( \sqrt{1 - \frac{\Delta \delta}{\delta} \rho} e^{-is \varphi} \right) \right] , \quad (12)$$

which, given the smallness of $|\Delta \delta/\delta|$ for close values of the cone angles, may be approximated by

$$A (\rho , \varphi , 0) = b_s \left[ J_s \left( (1 + \eta \rho) e^{is \varphi} + J_{-s} \left( (1 - \eta \rho) e^{-is \varphi} \right) \right] , \quad (13)$$

FIG. 2. Results for the setting with the same parameters as in Fig. 1 (a) The transverse intensity profile, measured in TW/cm², of the superposition of Bessel beams in the input given by Eq. (3), with $s = 1$, $\theta_1 = 0.26^\circ$, $\theta_{-1} = 0.24^\circ$, and $a_1^2 = 20$ TW/cm². The corresponding scaled parameters in equations (10) and (11) are $M = 8$, $\alpha = 0.88$, and $\eta = 0.04$, $b_1 = 0.87$, respectively. (b) The intensity profile of the azimuthon produced by the propagation over distance $z = 100$ cm. The counterclockwise angular velocity is $\Omega = 3.40 \pm 0.08$ deg/cm from the numerical simulation and $\Omega = 3.447$ deg/cm from Eq. (5). Panels (c) and (d) display the azimuthal phase profiles along a circle of small radius $r$ for the input Bessel beam superposition (3), and for the azimuthon established by the propagation, respectively.
where we set $\sqrt{1 + \Delta \delta / \delta} \simeq 1 \pm \eta$, and

$$\eta \equiv \frac{\Delta \delta}{2\delta}, \quad b_s \equiv \left( \frac{\Delta \delta}{2|\delta|} \right)^{\frac{1}{2}}, \quad a_s. \quad (14)$$

The properties of dissipative azimuthons, as previously reported for static dissipations and nonlinear Bessel vortex beams, are essentially the same, regardless of the multiphoton absorption order $M$ in Eq. 3, therefore we henceforth fix $M = 4$ (note, in particular, that $M = 4$ in water at 527 nm). The dissipative azimuthon is then determined by vorticity $s$, the strength of the Kerr nonlinearity relative to the nonlinear absorption, $\alpha$, the amplitude parameter, $b_s$, and the radial mismatch, $\eta$ of the input Bessel beams, see Eqs. 11 and 14. Positive and negative $\eta$ correspond, respectively, to $\theta_s > \theta_{s-s}$ and $\theta_s < \theta_{s-s}$, i.e., positive (counterclockwise) and negative (clockwise) angular velocity. In the scaled notation, the angular velocity is

$$\omega = \frac{2\eta}{s}, \quad (15)$$

and the rotation period is $\zeta_{\text{rotation}} = \pi s / |\eta|$. The abovementioned inversion of the vorticity implies that the respective topological charges are $-s$ and $s$ for $\eta > 0$ and $\eta < 0$.

### III. THE GYROSCOPIC EFFECT IN DISSIPATIVE AZIMUTHONS

Compared to nonlinear Bessel beams and static dissipations [15, 19], rotation is seen in Figs. 1 and 2 to accelerate the formation of the dissipative patterns. The dashed curves in Fig. (c) show the peak intensity and nonlinear power-loss rate as functions of the propagation distance, and Fig. (d) displays the static intensity pattern established, after some propagation distance, in the nonlinear medium, when the cone angles of the input Bessel beams are made equal to the mean value of the two slightly different angles of the original input. It is seen that the profile with equal conicities is not yet formed because continues to attenuate, in comparison with the original one that propagates without attenuation. The non-rotating pattern requires a much longer propagation distance to form than its steady counterpart, whose formation is promoted by what may be called a gyroscopic effect. Namely, if the static pattern is created by input [13] with $\eta = 0$, the slow relaxation towards the static state is characterized by a certain relaxation length, $\zeta_{\text{rel}}$. On the other hand, if slight difference of conicities of the two Bessel beams in the input promotes the formation of a steadily rotating state, characterized by a rotation period $\zeta_{\text{rotation}} = \pi s / |\eta|$, the rotation is expected to eclipse the slow relaxation provided that $\zeta_{\text{rotation}} \lesssim \zeta_{\text{rel}}$, i.e., for

$$|\eta| \gtrsim |\eta|_{\text{min}} = \pi s / \zeta_{\text{rel}}. \quad (16)$$

In the example displayed Fig. 3 with $s = 1$, $b_1 = 0.75$ and $M = 4$, $\alpha = 1$, the characteristic relaxation distance, $\zeta_{\text{rel}} \sim 70$, of the non-rotating pattern predicts $|\eta|_{\text{min}} \simeq 0.04$. As seen in Fig. 3(a), the long relaxation stage, following the short initial stage of compression under the combined action of the self-focusing and absorption, is indeed eliminated at $\eta \gtrsim |\eta|_{\text{min}}$. Eventually, the snapshot of the intensity pattern in the rotating state is quite similar to that of the non-rotating one, cf. Figs. 3(b) and (c), but the rotating pattern forms much faster.

### IV. STABILITY OF THE DISSIPATIVE AZIMUTHONS

#### A. Stability limits

Similar to nonlinear Bessel vortex beams [18] and dissipatons [19], dissipative azimuthons remain stable for sufficiently low values of the normalized Kerr coefficient $\alpha$ [see Eq. 11], and they become unstable above a threshold value, $\alpha > \alpha_{\text{th}}$, which is significantly higher than the instability threshold for non-rotating states [18, 19], implying that the rotation enhances the stability. By means...
of systematic simulations, we have identified the threshold as a function of \( \alpha \) for several values of vorticity \( s \), using input (13).

Given the large number of parameters, in the numerical simulations we fixed values \( |b_s| = 0.6 \) and \( |b_b| = 0.8 \) [see Eq. (14)], which represent typical intensities at which the Kerr nonlinearity and multiphoton absorption are substantial in fused silica, and \( \eta = 0.04 \), which implies relative variations of the cone angle \( \simeq 10\% \).

For \( s = 1 \), the threshold is found to be \( \langle \alpha_{th} \rangle_s = 2.25 \pm 0.05 \) for \( |b_s| = 0.6 \) and \( |b_b| = 0.8 \). The dissipative azimuthons with a higher vorticity are somewhat less stable, featuring \( \langle \alpha_{th} \rangle_s = 2.15 \pm 0.05 \) for \( |b_s| = 0.6 \), and \( \langle \alpha_{th} \rangle_s = 2.15 \pm 0.05 \) for \( |b_s| = 0.8 \). Further, for \( s = 3 \), it was found that \( \langle \alpha_{th} \rangle_s = 2.25 \pm 0.05 \) for \( |b_s| = 0.6 \), and \( \langle \alpha_{th} \rangle_s = 2.15 \pm 0.05 \) for \( |b_s| = 0.8 \). The difference between the cases of \( s = 2 \) and \( 3 \) plausibly originates from their differing symmetries. These instability thresholds are about twice as large as typical values \( \alpha_{th} \simeq 1 \) for nonlinear Bessel beams and non-rotating dissipatons found in Refs. [18] and [19], respectively, which indicates the enhanced robustness provided by the rotation. Figures 4 and 5 show intensity and phase patterns of the rotating dissipative azimuthons with \( s = 2 \) and \( 3 \), and \( \alpha = 1 \), which places them well within the stability region.

### B. Instability development scenarios

Direct simulations make it possible to identify two basic scenarios of the development of unstable patterns. Close to the stability boundary, i.e., for the lowest values of \( \alpha \) above \( \alpha_{th} \), unstable rotating patterns spontaneously turn into oscillating ones, which keep rotating and may remain robust by themselves. For higher values of \( \alpha \), an unstable azimuthon decays into random patterns, by progressively loosing its symmetry. Three examples of this dynamics, for \( s = 1, 2 \), and \( 3 \) are shown in Figs. 6 and 7 respectively. The instability develops, at first, through a long sequence of regular patterns featuring the...
same 2\(\pi\)-fold symmetry as the input, which follow each other in a random way. Then the pattern become less and less regular, through developing intrinsic oscillations, until developing complete randomness.

The instability-development scenario, demonstrated by the numerical solutions, is essentially the same for \(s = 1, 2\) and \(3\), but the symmetry with respect to the rotation by \(\Delta \varphi = \pi\) for \(s = 1\) is much more robust than the symmetry with \(\Delta \varphi = \pi/2\) for \(s = 2\) or \(\Delta \varphi = \pi/3\) for \(s = 3\). The patterns for \(s = 1\) are much simpler, and, typically, the resulting random pattern still keep the symmetry with respect to the rotation by \(\Delta \varphi = \pi\) [see, e.g., Fig. 6(b)].

The phase pattern globally reflects the structure of the intensity pattern. However, if one considers the phase as a function of azimuthal angle \(\varphi\) along circumferences with different radii \(\rho\), conspicuous evolution with \(\zeta\) can be identified. Indeed, while the phase circulation is \(2\pi s\) in the center of the input configuration (3), and \(-2\pi s\) just after the topological charge has reversed its sign, which occurs immediately, a central domain of small radius in which the phase circulation is zero quickly emerges in the course of the evolution of the unstable azimuthon, and the radius of this domain grows until it fills the entire central area of the pattern, except for a few remaining phase dislocations, whilst the pattern becomes random. Thus, the evolution leads to a conclusion that the center of the pattern does not carry any vorticity anymore. Alternations of domains with the phase circulations \(2\pi s\) and \(-2\pi s\) still occur at different increasing values of the radius and, as expected, in the peripheral area (at large radii) the alternations become identical to that of the input Bessel-beam superposition.

![FIG. 6. An example of the instability development, starting from the two-lobe dissipative azimuthon, with \(s = 1\), and leading to establishment of a random set of isolated hot spots. In this case, the input is similar to that displayed in Fig. 3(c). Examples of a transient regular amplitude pattern, obtained at \(\zeta = 20.625\) (a), and of a random pattern, obtained at \(\zeta = 93.75\) (b). The developing pattern is eventually fully randomized (not shown here). Parameters are \(M = 4, \alpha = 4\) for the NLSE, and \(s = 1, |b_s| = 0.8, \eta = 0.04\) for the Bessel-beam input (3).](image)

![FIG. 7. An example of the instability development, starting from the four-lobe dissipative azimuthon, with \(s = 2\). In this case, the input is similar to that displayed in Fig. 4(a). Examples of a transient regular amplitude pattern, obtained at \(\zeta = 16.875\) (a), and of a randomized pattern, obtained at \(\zeta = 105\) (b). Parameters are \(M = 4, \alpha = 4\) for the NLSE, and \(s = 2, |b_s| = 0.8, \eta = 0.04\) for the Bessel-beam input (3).](image)

C. **Pulsating and breathing dissipative azimuthons**

In the case of vorticity \(s = 2\), a peculiar instability scenario exhibits spontaneous transition a the four-lobe pattern into a two-lobe one. The development of this scenario is quite slow, allowing the initial four-lobe pattern to propagate considerable distances, keeping an apparently stable shape. The transition commences with oscillations of the amplitude of two pairs of hot spots which constitute the cross (four-lobe) structure of the input. Such oscillations may lead directly to the destruction of the structure, or proceed in an apparently persistent way. Due to a limited propagation distance in the simulations

![FIG. 8. Examples of the instability development of the six-lobe dissipative azimuthon, proceeding through a sequence of regular patterns, the input pattern being close to that shown in Fig. 5(a), while the final pattern (not shown here) is a random set of isolated hot spots. Examples of transient regular patterns at (a) \(\zeta = 7.5\) and (b) \(\zeta = 13.125\). Parameters are \(M = 4, \alpha = 4\) for the NLSE, and \(s = 3, |b_s| = 0.8, \eta = 0.04\) for the Bessel-beam input.](image)
and very low instability growth rate, it is not always possible to distinguish these outcomes. An example is produced in Fig. 9 for \( s = 2, |b_s| = 0.8, \) and \( \alpha = 2.6, \) in which the emerging oscillatory state appears to be stable. In this case, input (3) quickly forms a dissipative azimuthon, that propagates without any visible instability up to \( \zeta \approx 75. \) Then, amplitudes of the lobes start to oscillate until the pattern reshapes into a fully robust two-lobe structure.

Another noteworthy propagation regime triggered by the instability for \( s = 1 \) is the formation of a permanently breathing dissipative azimuthon. As seen in Fig. 10 in this case oscillations occur between a pattern in which the hot spots with the largest amplitude are closest to the center [Fig. 10(a)], and another one, in which the hot spots are located on a secondary ring [Fig. 10(b)]. This regime sets in as a persistent one, after a transient stage in which the intensity maxima of the farther separated spots are lower.

There is also a dynamical regime exhibiting spontaneous transformation of a transient oscillatory state, which keeps a high level of symmetry, into another one with a lower symmetry, as shown in Fig. 11. The initial six-lobe azimuthon is similar to the one displayed Fig. 5 but for a larger scaled strength of the Kerr nonlinearity, \( \alpha \) [see Eq. (11)]. At the first, relatively long, stage of the evolution (\( \zeta \lesssim 75 \)), the pattern remains unchanged in the rotating reference frame, before the instability commences. Then, the first oscillatory state emerges, see Figs. 11(a) and (b), which keeps essential symmetry: the initial invariance with respect to the rotation by \( \Delta \varphi = \pi / 3 \) rotation is lost, being reduced to the invariance for \( \Delta \varphi = \pi \), but the axial symmetry is conserved. The amplitude oscillates between two opposite spots on one hand, and the four other on the other hand. Then, the structure switches into a second oscillatory state, which keeps solely the symmetry with respect to the rotation by \( \Delta \varphi = \pi \), see Figs. 11(c) and (d). In this state, oscillations occur between opposite spots in two pairs, labeled (1, 4) and (2, 5) in the figure, while the spots belonging to the third pair, (3, 6), keep a low intensity. Note that the apparent rotation by 30° relating Figs. 11(c) and (d), is actually a consequence of the oscillations. Indeed, the rotation angle per se between the configurations in panels (c) and (d), separated by propagation distance \( \Delta \zeta = 3.125 \), is \( \Delta \varphi = \pi \Delta \zeta = (2\eta / s) \Delta \zeta \) [see Eq. (15)], which yields only \( \Delta \varphi = 4.77° \).

V. CONCLUSIONS

We have reported the existence of a novel type of azimuthons, which represent the propagating optical field with a stationary intensity pattern in a uniformly rotating reference frame in the Kerr medium with nonlinear loss, induced by multiphoton absorption in the material. Similar to nonlinear Kerr fundamental \( |1112 \) and vortical \( |1519 \) Bessel beams and “dissipatons” \( |19 \), the steady propagation of the rotating azimuthons in the lossy medium is maintained by the flux from the peripheral reservoir, which stored an indefinitely large amount of power in the slowly decaying tails of the beam. Unlike conservative azimuthons \( |34 |0 \), the number \( N \) of “hot spots” (intensity maxima) and the vorticity of the input are linked by \( N = 2s \), rather than being mutually independent.

The rotating dissipative azimuthons are excited by the coherent superposition of two Bessel beams with opposite topological charges and slightly different cone angles, cf. Refs. [22] [23]. In comparison to the non-rotating intensity patterns excited by the Bessel-beam pairs with identical conicities, the rotating azimuthons form faster and are more robust. The existence and stability of the
these modes in the self-focusing Kerr medium is provided by the nonlinear absorption. If the absorption is turned off, the input superposition of the Bessel beams does not result in formation of any stationary pattern. Unstable rotating azimuthons in this model are also interesting to consider, because the development the instability produces various dynamical regimes, including persistently pulsating azimuthons and breathing azimuthons, as well as the transition to “turbulent” patterns.

These results may help to understand the physics underlying the recently observed helical filaments, excited by superpositions of Bessel-Gauss beams with opposite vorticities in air and CS$_2$ [20, 21], where the interplay of the Kerr nonlinearity and multiphoton absorption, induced by the ionization of air, plays a key role in the propagation.

FIG. 11. An example of the unstable evolution of the six-lobe dissipative azimuthon. The respective input [3] is similar to that presented in Fig. [3]. At the first stage of evolution, the pattern oscillates between configurations shown in (a), at $\zeta = 125$, and (b), at $\zeta = 128.125$. At the second stage, the pattern oscillates between the configurations shown in (c), at $\zeta = 165.625$, and (d), at $\zeta = 168.750$. Parameters are $s = 3$, $M = 4$, $\alpha = 3$, $|b_s| = 0.6$, and $\eta = 0.04$.

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