On connectivity of Julia sets of transcendental meromorphic maps and weakly repelling fixed points I

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Abstract

It is known that the Julia set of the Newton method of a non-constant polynomial is connected (Mitsuhiro Shishikura, Preprint, 1990, M/90/37, Inst. Hautes Études Sci.). This is, in fact, a consequence of a much more general result that establishes the relationship between simple connectivity of Fatou components of rational maps and fixed points which are repelling or parabolic with multiplier 1. In this paper we study Fatou components of transcendental meromorphic functions; that is, we show the existence of such fixed points, provided that immediate attractive basins or preperiodic components are multiply connected.

1. Introduction

The so-called Newton method is, in all likelihood, the most common of the root-finding algorithms, mainly because of its simplicity, high-efficiency index and quadratic order of convergence. The Newton method associated to a complex holomorphic function \( f \) is defined by the dynamical system

\[
N_f(z) := z - \frac{f(z)}{f'(z)}.
\]

As such, a natural question is what properties we might be interested in or, put more generally, what kind of study we want to make of it. From the dynamical point of view, and given the purpose of any root-finding algorithm, a fundamental issue is to understand the dynamics of \( N_f \) about its fixed points, as they correspond to the roots of the function \( f \); in other words, we would like to understand the fixed basins of attraction of \( N_f \), the sets of points that converge to a root of \( f \) under the iteration of \( N_f \).

Basins of attraction are actually just one type of stable component or component of the Fatou set \( \mathcal{F}(f) \), the set of points \( z \in \hat{\mathbb{C}} \) for which \( \{f^n\}_{n \geq 1} \) is defined and normal in a neighbourhood of \( z \) (recall that \( \hat{\mathbb{C}} \) stands for the Riemann sphere, the compact Riemann surface \( \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \)). The Julia set or set of chaos is its complement, \( J(f) := \hat{\mathbb{C}} \setminus \mathcal{F}(f) \).

At first, one could think that if the fixed points of \( N_f \) are exactly the roots of \( f \), then the Newton method is a neat algorithm in the sense that it will always converge to one of the roots. However, notice that not every stable component is a basin of attraction; even not every attracting behaviour is suitable for our purposes: Basic examples like the Newton method applied to cubic polynomials of the form \( f_a(z) = z(z-1)(z-a) \), for certain values of \( a \in \mathbb{C} \), lead to open sets of initial values converging to attracting periodic cycles. Actually, also the set of such parameters \( a \in \mathbb{C} \), for this family of functions, is an open set of the corresponding parameter space (see [6] or [8]).
A lot of literature concerning the Newton method’s Julia and Fatou sets has been written, above all when applied to algebraic functions. Przytycki [15] showed that every root of a polynomial $P$ has a simply connected immediate basin of attraction for $N_P$. Meier [13] proved the connectivity of the Julia set of $N_P$ when $\deg P = 3$, and later Tan [20] generalised this result to higher degrees of $P$. In 1990, Shishikura [18] proved the result that actually sets the basis of our work: For any non-constant polynomial $P$, the Julia set of $N_P$ is connected (or, equivalently, all its Fatou components are simply connected). In fact, he obtained this result as a corollary of a much more general theorem for rational functions, namely, the connectedness of the Julia set of rational functions with exactly one weakly repelling fixed point, that is, a fixed point which is either repelling or parabolic of multiplier 1 (see Section 3).

The present work, however, deals with the Newton method applied to transcendental maps. In the same direction, in 2002, Mayer and Schleicher [12] extended Przytycki’s theorem, showing that every root of a transcendental entire function $f$ has a simply connected immediate basin of attraction for $N_f$, and this work has been recently continued by Rückert and Schleicher [16], where they study Newton maps in the complement of such Fatou components. Our aim is to prove the natural transcendental versions of Shishikura’s results — although this paper covers just part of it — which can be conjectured as follows.

**Conjecture 1.1.** If the Julia set of a transcendental meromorphic function $f$ is disconnected, then there exists at least one weakly repelling fixed point of $f$.

We may assume that our transcendental meromorphic functions are defined on the plane $\mathbb{C}$, so infinity is an essential singularity.

**Remark.** Notice that essential singularities are always in the Julia set of a transcendental meromorphic function $f$, and therefore infinity can connect two unbounded connected components of $\mathcal{J}(f) \cap \mathbb{C}$ otherwise disconnected.

Now, transcendental meromorphic functions that come from applying the Newton method to transcendental entire functions happen to have no weakly repelling fixed points at all, so the next result is obtained forthwith.

**Conjecture 1.2 (Corollary).** The Julia set of the Newton method of a transcendental entire function is connected.

As it turns out, a possible proof of Conjecture 1.1 splits into several cases, according to different Fatou components, since the connectedness of the Julia set is equivalent to the simple connectedness of the connected components of its complement. In this paper we will see two of such cases, which, together, give raise to the following result.

**Theorem 1.3.** Let $f$ be a transcendental meromorphic function with either a multiply connected attractive basin or a multiply connected Fatou component with simply connected image. Then, there exists at least one weakly repelling fixed point of $f$.

Notice how this theorem actually connects with the result of Mayer and Schleicher mentioned above.

In order to prove this theorem, we use the method of quasi-conformal surgery and a theorem of Buff on virtually repelling fixed points. On the one hand, quasi-conformal surgery (see Section 2.1) is a powerful tool that allows to create holomorphic maps with some desired
behaviour. One usually starts glueing together — or cutting and sewing, this is why this procedure is called ‘surgery’ — several functions having the required dynamics; in general, the map $f$ obtained is not holomorphic. However, if we can create an appropriate almost-complex structure on $\hat{\mathbb{C}}$, the Measurable Riemann mapping theorem can be applied to find a holomorphic map $g$, plus some quasi-conformal homeomorphism that conjugates the functions $f$ and $g$. On the other hand, the property of being virtually repelling is only slightly stronger than that of weakly repelling, and in some cases it might just be easier to prove the existence of a virtually repelling fixed point.

The paper is structured as follows. The next section gives some basic definitions and properties of complex dynamics and related topics; in particular, it puts stress upon quasi-conformal surgery and virtually repelling fixed points. Some of the cases that result from the proof of Theorem 1.3 use a surgery process quite similar to that of Shishikura’s; thus, in Section 3 we recall his results and give part of his proof so as to show how surgery is used in our scenario. Later on, in our proof, we will focus on the differences between the two cases. Such proof, as well as the details on how Conjecture 1.1 splits, can be found in Section 4, dedicated to transcendental functions.

2. Preliminaries and tools

This section provides some general background on holomorphic dynamics, to be used later on. After a few initial basic definitions and results, the settings on quasi-conformal surgery and virtually repelling fixed points are also presented.

We consider $f$ to be a rational, transcendental entire or transcendental meromorphic function, and use the term complex function to denote either case. We write $f^n$ for the $n$th iteration of $f$; that is, $f^0(z) := z$ and $f^n(z) := f(f^{n-1}(z))$, when $n \geq 1$; as usual, $f^{-n}$ represents $(f^n)^{-1}$, the set of all inverse branches of $f^n$.

We say that $z_0 \in \hat{\mathbb{C}}$ is a periodic point of $f$ of (minimal) period $n \in \mathbb{N}$ if $f^n(z_0) = z_0$ and $f^k(z_0) \neq z_0$, for all $0 < k < n$; the multiplier of a periodic point $z_0$ of period $n$ is the value $\rho(z_0) := (f^n)'(z_0) \in \mathbb{C}$. A periodic point $z_0$ is called attracting if $|\rho(z_0)| < 1$, repelling if $|\rho(z_0)| > 1$ and parabolic if $\rho(z_0) = e^{2\pi i \theta}$, with $\theta \in \mathbb{Q}$. Also, $z_0$ is said to be weakly repelling if it is either repelling or parabolic of multiplier 1.

The following theorem of Fatou [10] will be a key tool in the cases where the surgery technique is used. Its proof can be found in [14].

**Theorem 2.1 (Fatou).** Any rational map of degree greater than one has, at least, one weakly repelling fixed point.

The Fatou set is open by definition, and its connected components are commonly referred to as Fatou components. The following is a first classification of such components.

**Definition.** Let $f$ be a complex function, and let $U$ be a (connected) component of $\mathcal{F}(f)$; then $U$ is said to be preperiodic if there exist integers $n > m \geq 0$ such that $f^n(U) = f^m(U)$. We say that $U$ is periodic if $m = 0$, and fixed if $n = 1$. A Fatou component is called a wandering domain if it fails to be preperiodic.

The next classification of periodic Fatou components is essentially due to Cremer and Fatou, and was first stated in this form in [2].
Theorem 2.2 Classification. Let $U$ be a $p$-periodic Fatou component of a complex function $f$. Then $U$ is one of the following.

- **Immediate attractive basin:** $U$ contains an attracting $p$-periodic point $z_0$ and $f^{np}(z) \to z_0$, as $n \to \infty$, for all $z \in U$.
- **Parabolic basin or Leau domain:** $\partial U$ contains a unique $p$-periodic point $z_0$, and $f^{np}(z) \to z_0$, as $n \to \infty$, for all $z \in U$. Moreover $(f^p)'(z_0) = 1$.
- **Siegel disc:** There exists a holomorphic homeomorphism $\phi : U \to \mathbb{D}$ such that $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i \theta} z$, for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$.
- **Herman ring:** There exist $r > 1$ and a holomorphic homeomorphism $\phi : U \to \{1 < |z| < r\}$ such that $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i \theta} z$, for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$.
- **Baker domain:** $\partial U$ contains a point $z_0$ such that $f^{np}(z) \to z_0$, as $n \to \infty$, for all $z \in U$, but $f(z_0)$ is not defined. In our context, $z_0$ is an essential singularity.

Rational functions and transcendental entire functions of finite type (that is to say, with a finite number of singularities of the inverse function) have neither wandering domains nor Baker domains. The absence of wandering domains was proved by Sullivan [19] for rational functions and by Eremenko and Lyubich [9] and Goldberg and Keen [11] for such entire maps. As for Baker domains, while such Fatou components make no sense for rational functions because infinity is but a regular point, their absence for transcendental entire functions of finite type is, in fact, a consequence of a much stronger result of Eremenko and Lyubich [9], generalised to meromorphic maps by Bergweiler [3] using some of their ideas.

2.1. Quasi-conformal surgery

What is known today in holomorphic dynamics literature as quasi-conformal surgery is a technique to construct holomorphic maps with some prescribed dynamics. As mentioned, the term ‘surgery’ suggests that certain spaces and maps will be cut and sewed in order to construct the desired behaviour. This is usually the first step of the process and is known as topological surgery. On the other hand, the adjective ‘quasi-conformal’ indicates that the map one constructs in this first step is not holomorphic, but only quasi-regular, and it needs to be made holomorphic by means of the Measurable Riemann mapping theorem. This second step is called holomorphic smoothing.

Quasi-conformal mappings were first introduced in complex dynamics in 1981 by Sullivan, in a seminar at the IHES, and applied to the study of polynomial-like mappings by Douady and Hubbard [8]. In 1985 Sullivan published his study in [19], and two years later Shishikura gave a great impulse to the technique in its application to rational functions (see [17]).

We now introduce some basic concepts in order to understand the main results.

**Definition.** Let $U \subset \mathbb{C}$ be an open set; a measurable function $\mu : U \to \mathbb{C}$ is called a $k$-Beltrami coefficient of $U$ if $\|\mu\|_\infty = k < 1$.

Equivalently, one can associate to every $k$-Beltrami coefficient of $U$, $\mu$, an almost-complex structure $\sigma$, that is, a measurable field of (infinitesimal) ellipses in $TU$, defined up to multiplication by a positive real constant. More precisely, the argument of the minor axis of such ellipses at a point $z \in U$ is $\arg(\mu(z))/2$, and its ellipticity, that is, the ratio between its axes, equals $(1 - |\mu(z)|)/(1 + |\mu(z)|)$. Notice that this value is bounded between $(1 - \|\mu\|_\infty)/(1 + \|\mu\|_\infty) > 0$ and $1$ almost everywhere.
Definition. Let $U$ and $V$ be open sets in $\mathbb{C}$; a map $\phi : U \to V$ is said to be $k$-quasi-regular if it has locally square integrable weak derivatives and the function

$$\mu_{\phi}(z) := \frac{\partial \phi/\partial \bar{z}}{\partial \phi/\partial z}(z)$$

is a $k$-Beltrami coefficient. A $k$-quasi-conformal map is a quasi-regular homeomorphism.

It is easy to check that a quasi-regular map is locally the composition of a holomorphic function and a quasi-conformal map.

Definition. Let $U$ and $V$ be open sets in $\mathbb{C}$; a quasi-regular map $\phi : U \to V$ induces a contravariant functor $\phi^* : L^\infty(V) \to L^\infty(U)$ defined by

$$\phi^* \mu := \frac{\partial \phi/\partial \bar{z} + (\mu \circ \phi)(\partial \phi/\partial z)}{\partial \phi/\partial z + (\mu \circ \phi)(\partial \phi/\partial \bar{z})}.$$  

Notice that if $\mu : V \to \mathbb{C}$ is a Beltrami coefficient, then so is its pull-back $\phi^* \mu : U \to \mathbb{C}$. Moreover, if $\phi$ is a holomorphic map, then $\|\phi^* \mu\|_\infty = \|\mu\|_\infty$.

When the Beltrami coefficient $\mu$ is defined in terms of a quasi-regular map $\psi$ as above ($\mu \equiv \mu_{\psi}$), one can check that $\phi^* \mu_{\psi} = \mu_{\psi \circ \phi}$.

Definition. We call standard complex structure the constant Beltrami coefficient $\mu_0 := 0$ or, equivalently, the associated field of circles $\sigma_0$.

By Weyl’s lemma, we see that a quasi-regular map $\phi$ is holomorphic if and only if $\phi^* \mu_0 = \mu_0$. Now, it is clear that a quasi-conformal map $\phi$ defines a Beltrami coefficient $\mu_{\phi}$. Conversely, given a Beltrami coefficient $\mu$ and the so-called Beltrami equation

$$\frac{\partial \phi}{\partial \bar{z}} = \mu \cdot \frac{\partial \phi}{\partial z},$$

can we find an actual quasi-conformal map $\phi$ such that $\mu_{\phi} \equiv \mu^2$? The celebrated measurable Riemann mapping theorem answers this question positively; the following is a $U = V = \mathbb{C}$ version of the statement (see also [1] or [7]).

**Theorem 2.3** (Morrey, Bojarski, Ahlfors, Bers). Let $\mu$ be a Beltrami coefficient of $\mathbb{C}$; then, there exists a unique quasi-conformal map $\phi : \mathbb{C} \to \mathbb{C}$ such that $\phi(0) = 0$, $\phi(1) = 1$ and $\mu_{\phi} = \mu$.

The application of this result to complex dynamics is the following. Suppose that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasi-regular map with dynamics that we would like to see realised by a holomorphic map of $\hat{\mathbb{C}}$. Then, Theorem 2.3 guarantees the existence of such a map as long as we can construct an appropriate $f$-invariant almost-complex structure. The precise statement reads as follows.

**Corollary 2.4.** Let $\mu$ be a Beltrami coefficient of $\mathbb{C}$, and let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a quasi-regular map such that $f^* \mu = \mu$; then, $f$ is quasi-conformally conjugate to a holomorphic map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

**Proof.** Applying the measurable Riemann mapping theorem to $\mu$, there exists a quasi-conformal map $\phi$ with $\mu = \phi^* \mu_0$. Now, let us define $g := \phi \circ f \circ \phi^{-1}$; we just need to see that
g is indeed holomorphic:
\[ g^* \mu_0 = (\phi f \phi^{-1})^* \mu_0 = (\phi^{-1})^* f^* \phi^* \mu_0 = (\phi^{-1})^* f^* \mu = (\phi^{-1})^* \mu = \mu_0. \]

Remark. Notice that the dynamical condition of \( f \)-invariancy is represented by the expression \( f^* \mu = \mu \); that is, the function \( \mu \) (and therefore the associated almost complex structure) is preserved under the dynamics of \( f \).

2.2. On virtually repelling fixed points

We now introduce the concept of virtually repelling fixed point, which goes back to Epstein. It is slightly stronger than that of weakly repelling fixed point, and its definition is based on the holomorphic index (see also [5] or [14]).

Definition. The holomorphic index of a complex function \( f \) at a fixed point \( z \) is the residue
\[ \iota(f, z) := \frac{1}{2\pi i} \oint_z \frac{dw}{w - f(w)}. \]
In the case of a simple fixed point \( (\rho(z) \neq 1) \), the index is also given by
\[ \iota(f, z) = \frac{1}{1 - \rho(z)}. \]

If we have \( \text{Re}(\iota(f, z)) < m/2 \), where \( m \geq 1 \) denotes the multiplicity, then the fixed point \( z \) is called virtually repelling.

Remarks. • Virtually repelling fixed points are in particular weakly repelling, as in the multiple case the multiplier is \( \rho(z) = 1 \), and in the simple case we have
\[ \text{Re} \left( \frac{1}{1 - \rho(z)} \right) < \frac{1}{2} \iff |\rho(z)| > 1. \]

• Virtual repellency, unlike weak repellency, is not preserved under topological conjugacy, since the residue index is kept only under analytic conjugacy (see [14]). See also [18] for a proof of this property in weakly repelling fixed points.

Theorem 2.5 (Buff). Let \( U \subset \mathbb{D} \) be an open set, and let \( f : U \to \mathbb{D} \) be a proper holomorphic map of degree \( d \geq 2 \). If \( |f(z) - z| \) is bounded away from zero as \( z \in U \) tends to \( \partial U \), then \( f \) has at least one virtually repelling fixed point.

Remark. Observe that if we require \( U \) to be compactly contained in \( \mathbb{D} \), then \( f \) is a polynomial-like mapping (see [8]). By the Straightening theorem, \( f \) is hybrid equivalent — in particular, quasi-conformally conjugate — to a polynomial \( P \) in \( U \). It follows from Fatou’s theorem, Theorem 2.1, applied to \( P \) that \( f \) must have a weakly repelling fixed point in \( U \).

Of course, in our context we are not dealing with holomorphic maps, and so we shall adapt Buff’s result to our situation with the following version.

Corollary 2.6. Let \( f : V \to D \) be a proper transcendental holomorphic function with \( V \subset D \) and \( D \subset \mathbb{C} \) an open, simply connected set. If \( |f(z) - z| \) is bounded away from zero
as \( z \in V \) tends to either \( \partial V \) or \( \infty \), then there exists at least one virtually repelling fixed point of \( f \).

**Proof.** Since the set \( D \) is open and simply connected, we see that there exists a conformal Riemann mapping \( \phi : D \to \mathbb{D} \). This map takes the subset \( V \) to some \( \phi(V) = U \subset \mathbb{D} \), as \( V \) is contained in \( D \) (see Figure 1).

Let us now define the map \( g := \phi \circ f \circ \phi^{-1} \), which is clearly conjugate to \( f \) by the conformal conjugation \( \phi \). Observe that \( g \) is proper and \( |g(z) - z| \) is bounded away from zero as \( z \in U \) tends to \( \partial U \), for, so is \( |f(z) - z| \) as \( z \in V \) tends to either \( \partial V \) or the essential singularity. In this situation, \( g \) has at least one virtually repelling fixed point \( z_0 \) due to Theorem 2.5. Since conformal conjugacies preserve this property of fixed points, we see that there exists a virtually repelling fixed point \( \phi^{-1}(z_0) \) of \( f \) (in \( V \)).

**Remark.** In particular, Corollary 2.6 gives the existence of a weakly repelling fixed point of \( f \), which is the property we shall use in our arguments.

### 3. Shishikura’s rational case

Our work on connectivity of Julia sets of transcendental meromorphic functions is based on that of Shishikura’s for rational maps. In this section we would like to show the main results in his paper, as well as part of their proofs, since they also cover some very specific situations of our transcendental result. The case chosen is that concerning immediate attractive basins, and it has been rearranged such that the general structure matches the discourse on transcendental functions in Section 4.

The following theorem and corollary, along with all the other results and proofs in this section, are due to Shishikura and are extracted from [18].
Theorem 3.1. If the Julia set of a rational map \( f \) is disconnected, then there exist two weakly repelling fixed points of \( f \).

Corollary 3.2. The Julia set of a rational map with only one weakly repelling fixed point is connected; in other words, all its Fatou components are simply connected. In particular, the Julia set of the Newton method of a non-constant polynomial is connected.

Corollary 3.2 is an immediate consequence of the previous theorem, for the Newton method of a non-constant polynomial has all its fixed points attracting except for the one fixed point at infinity, which is (weakly) repelling.

In order to prove Theorem 3.1, Shishikura uses a case-by-case approach, according to different types of Fatou component — for a general complex function, these are wandering domains, preperiodic components and periodic components, the latter ones described in the Classification theorem, Theorem 2.2. For the Julia set of a rational map to be disconnected, there must exist at least one multiply connected Fatou component; namely, an immediate attractive basin, Leau domain, Herman ring or preperiodic component, since Siegel discs cannot be multiply connected and rational maps have neither wandering domains nor Baker domains. Furthermore, the preperiodic case may be treated in a slightly special way, since preperiodic components eventually landing on multiply connected periodic components can clearly be omitted, and so the image of a preperiodic Fatou component may be assumed simply connected.

The strategy that we have only just outlined can be shaped into the following theorem.

Theorem 3.3. Let \( f \) be a rational map of degree greater than one. Then we have the following assertions.

- If \( f \) has a multiply connected immediate attractive or parabolic basin, then there exist two weakly repelling fixed points.
- If \( f \) has a Herman ring, then there exist two weakly repelling fixed points.
- If \( f \) has a multiply connected Fatou component \( U \) such that \( f(U) \) is simply connected, then every component of \( \hat{\mathbb{C}} \setminus U \) contains a weakly repelling fixed point.

The next sections contain a two-step version of part of Shishikura’s proof for this result, namely, the attractive case. Thus, Section 3.1 deals but with fixed immediate attractive basins, while strictly periodic immediate attractive basins are explained Section 3.2. We refer to [18] for a complete proof of Theorem 3.3.

3.1. Fixed basin

Let us first sketch the process that forces the existence of at least two weakly repelling fixed points, provided that the rational map \( f \) has a multiply connected fixed immediate attractive basin. Since the basin is multiply connected, there exist at least two components of its complement; we want to show that two of them contain a weakly repelling fixed point each. Using quasi-conformal surgery, we can construct a rational map \( g \), conjugate to \( f \) where needed, with a weakly repelling fixed point in some suitable subset of the sphere so as for \( f \) to have such a point in one of the components of the complement of the basin.

Although this description applies to both fixed and periodic cases, in this section we just show the proof for the first one, that is to say: A rational map of degree greater than one with a multiply connected fixed immediate attractive basin has, at least, two weakly repelling fixed points.
Figure 2. The increasing sequence of open neighbourhoods of $\alpha$, where $U_{n_0-1}$ is simply connected and $U_{n_0}$ is multiply connected.

Let us call $\alpha$ the attracting fixed point of $f$ contained in the multiply connected fixed immediate attractive basin $A^*$. Take a small disc neighbourhood $U_0$ of $\alpha$ such that $f(U_0) \subset U_0$. For each $n \geq 0$, let $U_n$ be the connected component of $f^{-n}(U_0)$ that contains $\alpha$.

From the choice of $U_0$, we see that $A^* = \bigcup_{n \geq 0} U_n$. Therefore, there exists an $n > 0$ such that $U_n$ is multiply connected; otherwise, the union of the increasing simply connected open sets $U_n$ would be simply connected. More precisely, there exists an $n_0 > 0$ such that $U_{n_0}$ is multiply connected but $U_{n_0-1}$ is simply connected (see Figure 2). Rename $U := U_{n_0}$ for simplicity.

Since $U$ is multiply connected, there exist at least two connected components of $\hat{\mathbb{C}} \setminus U$; choose one of them and call it $E$. From the construction of $U$, notice that

$$f(U) = f(U_{n_0}) = U_{n_0-1} \subset U_{n_0} = U$$

and, therefore, $f(U) \subset U \subset A^*$.

Now that we have suitable sets to work with, the next step of this surgery process is the construction of some quasi-regular map, with certain desired dynamics, to which the Measurable Riemann mapping theorem (see Section 2.1) can be applied. The following lemma produces exactly such a function.

**Lemma 3.4 (Interpolation lemma).** Let $V_0$ and $V_1$ be simply connected open sets in $\hat{\mathbb{C}}$, with $\#(\hat{\mathbb{C}} \setminus V_0) \geq 1$, and let $f$ be a holomorphic map from a neighbourhood $N$ of $\partial V_0$ to $\hat{\mathbb{C}}$ such that $f(\partial V_0) = \partial V_1$ and $f(V_0 \cap N) \subset V_1$; choose a compact set $K$ in $V_0$ and two points $a \in V_0$ and $b \in V_1$. Then, there exists a quasi-regular mapping $f_1 : V_0 \to V_1$ such that:

- $f_1 = f$ in $V_0 \cap N_1$, where $N_1$ is a neighbourhood of $\partial V_0$ with $N_1 \subset N$;
- $f_1$ is holomorphic in a neighbourhood of $K$;
- $f_1(a) = b$.

Shishikura’s proof for the Interpolation lemma is somewhat technical and can be found in [18], although Figure 3 offers a sketch of it.

In our situation (see Figure 4), we write $V_0 := \hat{\mathbb{C}} \setminus E$ and $V_1 := f(U)$, call $K := \overline{f(U)}$ and choose $a = b \in f(U)$ arbitrarily. Thus, a quasi-regular mapping $f_1 : \hat{\mathbb{C}} \setminus E \to f(U)$ is obtained from Lemma 3.4.
We first construct two annuli $A_0 \subset V_0 \cap N$ and $A_1 \subset V_1$, with $\partial A_i = \partial V_i \cup \gamma_i$ and $K \cap A_0 = \emptyset$, $a \notin A_0$, $b \notin A_1$, in such a way that the restriction $f|_{A_0} : A_0 \to A_1$ be a covering map of degree $m$ and $A_0$ contains no critical points of $f$. Then we consider (conformal) Riemann mappings $\Psi_i : V_i \setminus A_i \to \mathbb{D}$ such that $\Psi_0(a) = \Psi_1(b) = 0$, and define $\tilde{f}$ on $V_0 \setminus A_0$ as $\tilde{f} := \Psi_1^{-1} \circ (z \mapsto z^m) \circ \Psi_0$. Thus both $f$ and $\tilde{f}$ are covering maps from $\gamma_0$ to $\gamma_1$ of the same degree without critical points, and hence homotopic. Take $\gamma'_1 \subset A_1$ and $\gamma'_0 := f^{-1}(\gamma'_1) \cap A_0$ as in the figure, and let $F$ be the natural linear interpolation map defined between $f$ on $\gamma'_0$ and $\tilde{f}$ on $\gamma_0$. Now the map $f_1 : V_0 \to V_1$, defined as $f$ between $\partial V_0$ and $\gamma'_0$, $F$ between $\gamma'_0$ and $\gamma_0$, and $\tilde{f}$ on $V_0 \setminus \overline{A_0}$, has the properties as required. The shaded regions indicate the dynamics of $F$.

Roughly speaking, the map $f_1$ simplifies $f$ outside $E$, where its behaviour cannot be controlled, although it still agrees with $f$ on the boundary of this set. We define yet another function $f_2 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by cutting and glueing $f$ and $f_1$ where needed:

$$
f_2 := \begin{cases} 
  f & \text{on } E, \\
  f_1 & \text{on } \hat{\mathbb{C}} \setminus E.
\end{cases}
$$
This function is quasi-regular, since \( f \) is rational and so holomorphic, \( f_1 \) is quasi-regular, and they coincide on an open annulus surrounding \( \partial E \). Furthermore, we see, just from its definition, that \( f_2 \) is holomorphic in \( E \) and in a neighbourhood of \( f(U) \), and it has a fixed point at \( a \), for \( f_2(a) = f_1(a) = b = a \). Notice that \( f_2(\mathbb{C} \setminus E) = f(U) \) and \( f(U) \subseteq \mathbb{C} \setminus E \); hence \( f(U) \) is invariant and the fixed point \( a \in f(U) \) is a global attractor of \( f_2 \) in \( \mathbb{C} \setminus E \). This concludes the topological step of the construction.

In order to apply the Measurable Riemann mapping theorem, it only remains to construct an appropriate \( f_2 \)-invariant almost-complex structure; so define

\[
\sigma := \begin{cases} 
\sigma_0 & \text{on } f(U) \\
(f_2^n)^* \sigma_0 & \text{on } f_2^{-n}(f(U)), \quad \text{for } n \in \mathbb{N} \\
\sigma_0 & \text{elsewhere}
\end{cases}
\]

(see Figure 5).

By construction, \( f_2^* \sigma = \sigma \) almost everywhere, since \( \sigma \) is defined based on the dynamics of \( f_2 \). Moreover, \( \sigma \) has bounded ellipticity; indeed, \( f_2 \) is holomorphic everywhere except in \( X := \mathbb{C} \setminus (E \cup f(U)) \), where it is quasi-regular. However, orbits pass through \( X \) at most once, since \( f_2(X) \subset f(U) \) and points never leave \( f(U) \) under iteration of \( f_2 \).

These are precisely the hypothesis of Corollary 2.4, so there exists a map \( g: \mathbb{C} \rightarrow \mathbb{C} \), holomorphic on the whole sphere, and hence rational, which is conjugate to \( f_2 \) by some quasi-conformal homeomorphism \( \phi \). Only for simplicity, let \( \psi \) be the inverse function of such homeomorphism, \( \psi := \phi^{-1} \).

Now Theorem 2.1 ensures the existence of a weakly repelling fixed point \( z_0 \) of \( g \), except when \( \deg g = 1 \) and \( g \) is an elliptic transformation. However, notice that

\[
g(\psi(\mathbb{C} \setminus E)) = \psi(f_2(\mathbb{C} \setminus E)) = \psi(f(U)) \subsetneq \psi(U) \subsetneq \psi(\mathbb{C} \setminus E),
\]

so \( g \) is a contraction and \( \psi(a) \) is an attracting fixed point of \( g \); in other words, \( g \) can never be an elliptic transformation. Also, observe that \( \psi(\mathbb{C} \setminus E) \) is contained in the basin of \( \psi(a) \).

Besides, the family \( \mathcal{G} = \{ g^n \}_{n \geq 1} \) omits the open set \( \psi(X) \); therefore \( \mathcal{G} \) is normal in \( \psi(\mathbb{C} \setminus E) \) by Montel’s theorem, that is, \( \psi(\mathbb{C} \setminus E) \subset \mathcal{F}(g) \). However, weakly repelling fixed points belong to the Julia set, and so \( z_0 \in \psi(E) \). Because such points are preserved under conjugacy, also \( f_2 \) has a weakly repelling fixed point \( \phi(z_0) \), in \( E \); and so does \( f \), since both functions coincide precisely on this set (see Figure 6).
Figure 6. The properties of \( g \) (including the existence of a weakly repelling fixed point) are transferred to \( f^2 \) due to the conjugacy \( \phi \) (recall that \( V_0 = \hat{C} \setminus E \)).

The set \( E \) was arbitrarily chosen from at least two components of \( \hat{C} \setminus U \), which means that \( f \) has at least two weakly repelling fixed points. This concludes the proof of Theorem 3.3 for fixed immediate attractive basins.

3.2. Periodic basin

In this section, we focus our attention on the case of periodic immediate attractive basins of period greater than one. The surgery process involved here is quite similar to that for fixed immediate attractive basins (see Section 3.1), and so we will give the differences in detail and try to abridge the arguments when identical.

Analogously to the fixed case, let \( \langle \alpha \rangle \) be the attracting cycle of \( f \) contained in the multiply connected \( p \)-periodic immediate attractive basin, \( A^* \), and let \( A^*(\alpha) \) be the connected component of \( A^* \) containing \( \alpha \). Take a small disc neighbourhood \( U_0 \) of \( \alpha \) such that \( f^p(U_0) \subset U_0 \) and, for each \( n \geq 0 \), define \( U_n \) as the connected component of \( f^{-n}(U_0) \) such that \( U_n \cap \langle \alpha \rangle \neq \emptyset \).

As before, we can put \( A^*(\alpha) = \bigcup_{n \geq 0} U_{np} \), so, in the sequence \( \{U_k\}_k \), there is a multiply connected set \( U \) with simply connected image. Shishikura formalises this statement with the following lemma.

**Lemma 3.5.** Let \( f \) be a rational map of degree greater than one with a multiply-connected \( p \)-periodic immediate attractive basin. Then, there exists a connected open set \( U \), contained in the basin, such that:

- \( U \) is multiply connected and \( f(U) \) is simply connected;
- \( U \) is a connected component of \( f^{-1}(f(U)) \);
- \( f^p(U) \subset U \).

Next, let \( E \) be one of the connected components of the complement of \( U \). Since \( U \subset A^* \) and \( p > 1 \), its image \( f(U) \) must lie in either \( E \) or some other component of \( \hat{C} \setminus U \). Then, let us assume that \( k - 1 \) iterations of \( U \) under \( f \) belong to \( E \) and precisely the \( k \)th iteration lands outside it, with \( k \in \mathbb{N} \); that is to say, \( f^i(U) \subset E \), for all \( 0 < i < k \), and \( f^k(U) \subset \hat{C} \setminus E \). (Notice that this assumption is not restrictive: Since \( f^p(U) \subset U \), necessarily \( k \) must range \( 0 < k \leq p \).) See Figure 7 for an overview of all possible cases.

In analogy to the fixed case, we will define a quasi-regular map \( f_2 : \hat{C} \to \hat{C} \) that will map \( \hat{C} \setminus E \) strictly inside itself, this time after \( k \) iterations. More precisely, set \( V_0 := \hat{C} \setminus E \) and \( V_1 := f(U) \), which lies in either \( E \) (when \( k > 1 \)) or \( \hat{C} \setminus E \) (when \( k = 1 \)). Set also \( K := f^k(U) \) and choose \( b \in f(U) \) and \( a = f^{k-1}(b) \in K \). By the Interpolation lemma, Lemma 3.4, there
exists a quasi-regular map $f_1 : \hat{\mathbb{C}} \setminus E \to f(U)$ which agrees with $f$ on $\partial E$, is holomorphic in a neighbourhood of $K$ and satisfies $f_1(a) = b$.

Observe that if $k = 1$, then the situation is completely equal to the fixed case (see Figure 8).

From here on we proceed as in Section 3.1, setting $f_2 = f$ on $E$ and $f_2 = f_1$ on $\hat{\mathbb{C}} \setminus E$. This makes $f_2$ a quasi-regular map of $\hat{\mathbb{C}}$, holomorphic in both $E$ and a neighbourhood of $f^k(U)$, with a $k$-periodic point $f_2^k(a) = f^{k-1}(f_1(a)) = f^{k-1}(b) = a$. Observe also that $f_2^k(\hat{\mathbb{C}} \setminus E) = f^k(U)$ and $\overline{f^k(U)} \subseteq \hat{\mathbb{C}} \setminus E$; it follows that $f_2^k$ is a contraction and $a$ a global attractor in $\hat{\mathbb{C}} \setminus E$.

As before, we may define an almost-complex structure $\sigma$ by

$$
\sigma := \begin{cases} 
\sigma_0 & \text{on } f(U), \\
(f_2^n)^*\sigma & \text{on } f_2^{-n}(f(U)) \text{ for } n \in \mathbb{N} \\
\sigma_0 & \text{elsewhere.}
\end{cases}
$$

Observe that $\sigma = \sigma_0$ on $\bigcup_{i=1}^k f^i(U)$ (see Figure 9).

Furthermore, $\sigma$ is $f_2$-invariant by construction and has bounded distortion, since orbits pass through $\hat{\mathbb{C}} \setminus (E \cup \overline{f^k(U)})$ (the set where $f_2$ is not holomorphic) at most once.

With this setting, and following the fixed case, Theorem 2.1 and Corollary 2.4 guarantee the existence of a weakly repelling fixed point of $f$ in $E$, which is exactly what we wanted to prove.
4. The transcendental case

Shishikura’s theorem, Theorem 3.1, inspires the analogous result in the transcendental world, that is, our Conjecture 1.1 on connectedness of Julia sets of transcendental meromorphic functions and its relationship to the existence of weakly repelling fixed points.

Following Shishikura, we can use the Classification theorem, Theorem 2.2, to individualise the main statement according to Fatou components.

**Conjecture 4.1.** Let \( f \) be a transcendental meromorphic function. Then,

- if \( f \) has a multiply connected immediate attractive or parabolic basin, Baker domain or wandering domain, or
- if \( f \) has a Herman ring, or
- if \( f \) has a multiply connected Fatou component \( U \) such that \( f(U) \) is simply connected,

there exists at least one weakly repelling fixed point of \( f \).

**Remark.** The case of the multiply connected wandering domain was already proved by Bergweiler and Terglane [4] in a different context, namely, in the search of solutions of certain differential equations with no wandering domains.

Now Theorem 1.3 clearly follows from the cases of the immediate attractive basin and the preperiodic Fatou component, which we shall prove in this section. The first two sections contain the proof of the first statement, rewritten as the following theorem, while the preperiodic case can be found in Section 4.3.

**Theorem 4.2.** Let \( f \) be a transcendental meromorphic function with a multiply connected \( p \)-periodic immediate attractive basin \( A^* \). Then, there exists at least one weakly repelling fixed point of \( f \).

We use two quite different strategies in order to prove this theorem. The first one is based on Shishikura’s surgery construction and applies when either \( A^* \) is bounded or preimages of a sufficiently small neighbourhood of the attractive point in \( A^* \) do not behave too wildly. The second technique, used in the rest of the cases, involves Buff’s theorem, Theorem 2.5, on virtually repelling fixed points.
Let us first assume that $A^*$ is bounded. In this very particular case we can also assume the existence of a connected open set $U \subset A^*$ such as Lemma 3.5 gives — that is to say, multiply connected and such that $f(U)$ is simply connected, $U$ is a connected component of $f^{-1}(f(U))$ and $f^p(U) \subset U$ — since the basin has no accesses to infinity and therefore preimages of compact sets (in the construction of $U$) keep compact.

We have $U \subset A^* \subset \mathcal{F}(f)$, so the essential singularity must be contained in the complement $\hat{\mathbb{C}} \setminus U$. Moreover, since $U$ is multiply connected, there exists at least one connected component $E$ of $\hat{\mathbb{C}} \setminus U$ which does not contain the singularity. As in the rational (periodic) case (see Section 3.2), we assume that the iterations of $U$ under $f$ do not jump outside $E$ until the $k$th one, and proceed analogously to find a function $f_2$ that preserves $f$ on $E$ but has attracting dynamics (interpolation function $f_1$) on $\hat{\mathbb{C}} \setminus E$.

Notice that $f_2$ is indeed quasi-regular: On $\hat{\mathbb{C}} \setminus E$, the map $f_1$ is quasi-regular and infinity is no longer an essential singularity; on $E$, now $f$ sends the poles to the (non-special) point at infinity — as $f$ is meromorphic, $f_2$ is holomorphic on $E$ as a map defined on the Riemann sphere. By definition of $f_1$, the functions $f$ and $f_1$ agree on the neighbourhood $V_0 \cap N_1$, so the glueing is continuous.

At this point, the topological step of the surgery process is done. The further holomorphic smoothing and end of the proof goes on exactly as in Section 3.2; therefore $f$ has a weakly repelling fixed point in $E$.

As for the unbounded case, we cannot apply the previous surgery construction in general, since the existence of asymptotic values and Fatou components with the essential singularity on their boundary can lead to unbounded preimages of bounded sets, while trying to construct $U$. Instead, we will use this very property to force the situation described in Buff’s theorem, (Theorem 2.5) and, in particular, Corollary 2.6.

Therefore let us assume from now on that $A^*$ is unbounded. The cases of the fixed basin ($p = 1$) and the (strictly) periodic basin ($p > 1$) are next treated separately.

4.1. Fixed basin

In this case, the immediate attractive basin $A^*$ consists of a single (fixed) Fatou component. Let $\alpha \in A^*$ be its one attracting fixed point. We first construct a nested sequence of open sets containing $\alpha$ as follows. Let $U_0$ be a neighbourhood of $\alpha$ such that $\overline{f(U_0)} \subset U_0$; that is, put $U_0 := \varphi^{-1}(\Delta)$, where $\varphi$ is the linearisation map of the fixed point $\alpha$ and $\Delta$ is a disc in its linearisation coordinates; and define $U_n$ as the connected component of $f^{-n}(U_0)$ that contains $\alpha$, for all $n \in \mathbb{N}$. Notice that $U_0 \subset U_1 \subset \ldots$ because of the choice of the initial neighbourhood $U_0$.

Since $A^*$ is multiply connected, there exists an $n_0 \in \mathbb{N}$ such that $U_0,\ldots,U_{n_0-1}$ are simply connected and $U_{n_0}$ is multiply connected. This implies that the complement of $U_{n_0}$ has at least one bounded connected component, since its fundamental group is $\pi_1(U_{n_0}) \neq \{0\}$. In view of this, let $E$ be one of the bounded connected components of $\hat{\mathbb{C}} \setminus U_{n_0}$ (see Figure 10).

As Figure 10 suggests, at some point the sets $\{U_k\}_k$ might become unbounded, and so further preimages of such sets could have poles and prepoles on their boundaries. The actual condition for this fact to happen can be written in terms of the intersection set $\partial E \cap \mathcal{J}(f)$ and is specified in the following lemma.

**Lemma 4.3.** Let $f$ be a transcendental meromorphic function with an unbounded multiply connected fixed immediate attractive basin $A^*$, and let $\{U_k\}_{k=0}^{n_0}$ and $E$ be as above. Then, the following are equivalent:

1. $U_0,\ldots,U_{n_0-1}$ are all bounded;
2. $\partial E \cap \mathcal{J}(f) = \emptyset$;
3. $\partial E$ contains no poles.
The sequence \( \{U_k\}_k \) and the bounded set \( E \) (in grey, the multiply connected set \( U_{n_0} \)).

Proof. Let us first see how (1) implies (2). The boundaries of \( U_0, \ldots, U_{n_0-1} \) belong to the Fatou set and are bounded. Since \( \partial E \) is mapped onto \( \partial U_{n_0-1} \), it follows that \( \partial E \cap \mathcal{J}(f) = \emptyset \). Statement (2) trivially gives (3). For (3) implies (1), suppose that there exists a \( k \in \mathbb{N} \), with \( 0 < k < n_0 \), such that \( U_k \) is unbounded. Since this is an increasing sequence, \( U_k, U_{k+1}, \ldots \) are all unbounded and, in particular, so is \( U_{n_0-1} \). However, \( \partial U_{n_0-1} \) is simply connected, and the set \( E \) is bounded. Then \( \partial E \) must contain at least one pole, which contradicts (3).

Therefore, in the case where \( \partial E \) never meets \( \mathcal{J}(f) \), the set \( U_{n_0} \) can be renamed \( U \), and we have the following situation: \( U \) is multiply connected and \( f(U) = f(U_{n_0}) = U_{n_0-1} \) is simply connected; \( U \) is a connected component of \( f^{-1}(f(U)) = f^{-1}(U_{n_0-1}) \), by definition;

\[
f(U) \subset U_{n_0-1} \subset U_{n_0} = U,
\]

since \( U_{n_0-1} \) is bounded and \( U \) open. Now this situation is but the setting we had in the case of \( \mathcal{A}^* \) bounded, with \( p = 1 \) (see Figure 11). Surgery can thus be applied in the same fashion (see Section 3.1) to obtain a quasi-regular map that sends \( \hat{C} \setminus E \) to \( U_{n_0-1} \) and equals \( f \) on \( E \). Observe that the essential singularity is no longer there and, therefore, the holomorphic map that we obtain from the surgery procedure is a rational map. This gives the desired weakly repelling fixed point in \( E \).

A different case is the situation where \( \partial E \) does intersect \( \mathcal{J}(f) \). Lemma 4.3 asserts the existence of at least one pole \( P \) in \( \partial E \). From now on, this is the situation we deal with.

As mentioned, in this case we no longer use quasi-conformal surgery, but Buff’s theorem Theorem 2.5; in other words, we want to find an open subset of \( \hat{C} \) that contains a preimage of itself and with boundary that does not share fixed points with the boundary of such preimage. (We shall see it suffices that infinity not be on the preimage’s boundary.)

Let us first construct a (shrinking) nested sequence of sets, in the complement of the open sets \( \{U_k\}_k \), by defining \( V_n \) to be the connected component of \( \hat{C} \setminus U_n \) that contains \( E \), for all \( 0 \leq n \leq n_0 \). Notice that the closed sets \( V_0, \ldots, V_{n_0-1} \) are all unbounded, for \( U_{n_0} \) is the first multiply connected set of its sequence, and \( V_{n_0} = E \) is bounded by definition. Notice also that this component containing \( E \) is simply connected (since \( U_n \) is connected) and indeed unique,
Figure 11. Sketch of the case where $\partial E$ never meets the Julia set, on the Riemann sphere (The shaded set represents $U$; surgery can be applied as in the case where $A^*$ is bounded and $p = 1$; compare with Figure 4).

Figure 12. The increasing sequence of open sets $\{U_k\}_k$ and the decreasing sequence $\{V_k\}_k$. In this example, $U_{n_0 - 1}$ is the first unbounded set in the sequence and, consequently, $V_n = \mathbb{C} \setminus U_n$ for all $n < n_0 - 1$. The shaded set corresponds to $V_{n_0 - 1}$, while $V_{n_0} = E$. The same situation has been drawn on the plane and on the Riemann sphere.

and that $V_0 \supset V_1 \supset \ldots \supset V_{n_0} = E$, since $U_0 \subset U_1 \subset \ldots$ and all the $\{V_k\}_k$ must contain $E$ (see Figure 12).

From Lemma 4.3 and from the fact that $U_0$ is bounded, there exists an $n_1 \in \mathbb{N}$, with $0 < n_1 < n_0$, such that $U_0, \ldots, U_{n_1 - 1}$ are bounded and $U_{n_1}, U_{n_1 + 1}, \ldots$ are unbounded. Moreover, since the preimage of an unbounded set may contain poles on its boundary, we can assume that there exists an $n_2 \in \mathbb{N}$, with $0 < n_1 < n_2 \leq n_0$, such that $P \notin \partial V_0, \ldots, \partial V_{n_2 - 1}$ and $P \in \partial V_{n_2}$. The following lemma shows that, in this case, $P \in \partial V_n$ for all $n_2 \leq n \leq n_0$.

**Lemma 4.4.** Suppose that there exists a $k < n_0$ such that $P \in \partial V_k$. Then $P \in \partial V_j$, for all $k \leq j \leq n_0$. 

Figure 13. The situation where $n_2 = n_0$; that is, the first set $V_k$ that contains the pole $P$ on its boundary is $V_{n_0} = E$ itself. Then, a preimage $X$ of $V_{n_0-1}$ must exist in $E$.

Proof. It is clear that $P \in \partial V_{n_0}$, given that $E = V_{n_0}$. Now, suppose that there exists a $k < j < n_0$ such that $P \notin \partial V_j$.

By definition, $E \subset V_j$ and therefore $P \in \hat{V}_j$. However, on the other hand, since $V_j \subset \hat{C} \setminus U_j$, we see that $U_k \subset U_j \subset \hat{C} \setminus V_j$. It follows that $P \in \partial U_k$. However, we assumed that $P \notin \partial V_j$, so we deduce that $P \in \text{int}(\hat{C} \setminus V_j)$. This contradicts the fact that $P \in \hat{V}_j$.

If $n_2 = n_0$, then the first set $V_k$ which contains $P$ on its boundary is $E$ itself (see Figure 13). As $V_{n_0-1}$ is unbounded, there exists some connected component $X$ of $f^{-1}(V_{n_0-1})$ such that $P \in \partial X$. Furthermore, the preimage $X$ must be contained in $E$, since points immediately outside $E$ belong to $U_{n_0}$ (with image under $f$ being $U_{n_0-1}$), and hence cannot be a preimage of points in $V_{n_0-1} \subset \hat{C} \setminus U_{n_0-1}$. Of course the boundaries $\partial V_{n_0-1}$ and $\partial X$ do not have any common fixed point because $|f(z) - z|$ is bounded away from zero as $z \in X$ tends to $\partial X$, so the map $f : X \to V_{n_0-1}$ satisfies the hypothesis of Corollary 2.6, and therefore $f$ has at least one weakly repelling fixed point.

The most general case is that where $0 < n_1 < n_2 < n_0$. One example of this situation is given by Figure 14, namely when $n_2 = n_1 + 1$ and $n_0 = n_2 + 2$.

Observe that, in this case, the interior of the sets $\{V_k\}_k$ with $k \geq n_2$ might have more than one connected component (as shown in the example of Figure 14). In order to avoid this, in our setting we define yet another sequence $\{W_k\}_k$, where each $W_n$ is the unbounded connected component of $V_n$, for all $n_2 \leq n < n_0$. Notice that such an unbounded component must be indeed unique, since the sets $\{V_k\}_k$ are all simply connected (see Figure 15).

With these tools, our proof will continue as follows. For every $n_2 \leq n < n_0$, we will first consider the preimage sets of $W_n$ attached to $P$. If any connected component of $f^{-1}(W_n)$ happens to be bounded, then Buff’s theorem can be applied and the proof will finish, as we will show in Lemma 4.5. However, if all of them are unbounded, then it is clear that both $W_n$ and each of its preimages would have infinity as a fixed point (of the restricted map) on their boundaries, contradicting the hypotheses of Corollary 2.6. In this case we will jump to the next step and repeat the procedure with $W_{n+1}$. We will now make this argument precise.
Figure 14. A possible distribution of the sets $U_1, \ldots, U_n$, with $0 < n_1 < n_2 < n_0$, and more precisely $n_2 = n_1 + 1$ and $n_0 = n_2 + 2$. To simplify, the sets $\{U_k\}_k$ have been drawn only with one access to infinity. Observe that $\text{V}_{n_2}$ and $\text{V}_{n_2+1}$ have two and three connected components, respectively. The shaded area represents $\text{V}_{n_2+1}$.

Figure 15. The open set $W_{n_2}$ is the unbounded component of the interior of the (shaded) set $\text{V}_{n_2}$.

As boundedness of preimages plays quite an important role, for the sake of clarity we define for $n_2 \leq n < n_0$ the families of sets

$$X_n := \{X \subset \hat{\mathbb{C}} \text{ bounded connected component of } f^{-1}(W_n) : P \in \partial X \}.$$ 

In other words, $X_n$ is the set of bounded connected components of $f^{-1}(W_n)$ with $P$ on their boundary. Now the following lemma proves the key point of our iterative process.

**Lemma 4.5.** Fix $n^* \in \mathbb{N}$ such that $n_2 \leq n^* < n_0$ and suppose that $\mathcal{X}_n = \emptyset$, for all $n_2 \leq n < n^*$, but $\mathcal{X}_{n^*} \neq \emptyset$. Then, there exists at least one weakly repelling fixed point of $f$. 

Figure 16. A bounded preimage $X$ of $W_{n* - 1}$ containing $P$ on its boundary must be always in $W_{n*}$ and hence in $W_{n* - 1}$. Buff’s theorem gives then a weakly repelling fixed point. Here the dashed lines represent $V_{n* - 1}$, while the continuous ones correspond to $V_{n*}$.

Figure 17. In the situation where $X$ lies in one of the bounded components $B$ of $\hat{V}_{n*}$, there exists a preimage $Y$ of $W_{n* - 1}$ such that $X \subset Y \subset B$.

Proof. Let $X \in \mathcal{X}_{n*}$. It is clear that $X \subset V_{n* + 1} \subset V_{n*} \subset V_{n* - 1}$, where the first inclusion follows from the fact that $V_{n*} \setminus V_{n* + 1} \subset U_{n* + 1}$ and its points never fall in $W_{n*}$ under iteration of $f$ (see Figure 16). If $X \subset W_{n*}$, then the map $f : X \to W_{n*}$ satisfies the hypothesis of Corollary 2.6, which provides a weakly repelling fixed point of $f$. Otherwise, $X$ is contained in one of the bounded components $B$ of $V_{n*}$ (see Figure 17). Consider preimages of $W_{n* - 1}$, that is, to say, connected components of $f^{-1}(W_{n* - 1})$; since $W_{n*} \subset W_{n* - 1}$, there exists a preimage $Y$ of $W_{n* - 1}$ such that $X \subset Y$. However, also $Y \subset V_{n*}$ (for the same reason that $X \subset V_{n* + 1}$), which means that $Y \subset B$ by continuity. This makes $Y$ bounded, since so is $B$, therefore $Y \in \mathcal{X}_{n* - 1}$ and $\mathcal{X}_{n* - 1} \neq \emptyset$, contradicting our initial assumption.

Using this result, the end of the proof becomes straightforward. For every $n \in \mathbb{N}$ such that $n_2 < n < n_0$, check whether $\mathcal{X}_n \neq \emptyset$. As it turns out, the last family of sets of the sequence
Figure 18. $U$ is a multiply connected subset of $A^*$ such that $f(U)$ is simply connected. If $U$ were unbounded, then the point at infinity would be in $\partial(\hat{C} \setminus (U \cup E))$ (see Figure 19).

$\{X_k\}_k$ always has this property, $\mathcal{X}_{n_0-1} \neq \emptyset$, since preimages of $W_{n_0-1}$ with $P$ on their boundary lie in $V_{n_0} = E$, which is bounded by definition. Therefore, take the smallest $n$ for which $\mathcal{X}_n \neq \emptyset$ holds, and Lemma 4.5 gives a weakly repelling fixed point of $f$.

4.2. Periodic basin

This case begins with the same setting as the fixed basin, although it soon becomes much simpler. Let $A^*$ be the multiply connected $p$-periodic immediate attractive basin of $f$, and let $\langle \alpha \rangle \subset A^*$ be its attracting $p$-periodic cycle. As before, we define $U_0$ to be a suitable neighbourhood of $\alpha$ such that $f^p(U_0) \subset U_0$, and $U_n$ as the connected component of $f^{-n}(U_0)$ that intersects $\langle \alpha \rangle$, for all $n \in \mathbb{N}$. Analogously to the fixed case, we have

$$U_l \subset U_{p+l} \subset U_{2p+l} \subset \ldots, \text{ for all } 0 \leq l < p.$$ 

Again, there exists an $n_0 \in \mathbb{N}$ such that $U_0, \ldots, U_{n_0-1}$ are simply connected and $U_{n_0}$ is multiply connected, for so is $A^*$. Call $U = U_{n_0}$, and let $E$ be one of the bounded connected components of $\hat{C} \setminus U$ (see Figure 18).

Remark. Notice the impossibility of using Lemma 4.3 to separate the different cases, as we did in the previous section. Indeed, in this periodic case the sequence $\{U_k\}_k$ is no longer nested, and so our proof cannot be extended beyond fixed basins.

When $\partial E$ has no poles, analogously to the previous case, we will apply the periodic-case surgery described in Section 3.2 to find a weakly repelling fixed point of $f$. First notice that the curve $f(\partial E)$ is bounded, since $\partial E$ is bounded by definition and has no poles by hypothesis. It follows that $f(\partial E) = \partial U_{n_0-1}$, because $f(\partial E)$ is at least one of its connected components and $U_{n_0-1}$ is simply connected. We conclude that $U_{n_0-1}$ must be bounded, since so is $f(\partial E)$.

Now this means we can use the Interpolation lemma, Lemma 3.4, to obtain a quasi-regular map $f_1 : \hat{C} \setminus E \to U_{n_0-1} = f(U)$, as in the previous cases, and the surgery process goes on and finishes as it did in the rational periodic case.

When $\partial E$ does contain a pole $P$, the image $f(U)$ must be unbounded and, therefore, contained in one of the unbounded connected components of $\hat{C} \setminus U$. Consider a simply
Figure 19. If there exists a pole $P$ on $\partial E$, then there exists a set $D \subset E$ such that $f(D) = V$, where $V$ is an unbounded simply connected set that contains $U$ but not $f(U)$ (the thick lines correspond to $\partial U$, while the sets $D$ and $V$ appear dark- and light-shaded, respectively).

connected, unbounded, closed set $V \subset \hat{\mathbb{C}}$, containing $U$ but not its image $f(U)$ (see Figure 19): this is always possible because we are in the case $p > 1$. Notice that also $E \subset V$ by construction of $V$ (which is simply connected) and boundedness of $E$. Now there exists a preimage $D$ of $V$, with $P \in \partial D$, and $D \subset E$ since points immediately outside $E$ are in $U$ and thus mapped to $f(U) \subset \hat{\mathbb{C}} \setminus V$. Moreover, we have $D \subset E \subset V$, so $\partial D \cap \partial V = \emptyset$ and Corollary 2.6 gives a weakly repelling fixed point of $f$.

This step concludes the periodic immediate attractive case and, with it, the proof of Theorem 4.2.

4.3. Preperiodic Fatou components

Recall that our main aim in this paper is to prove Theorem 1.3, as stated earlier in the introduction, and so far we have just closed one of its natural subcases, that is, the immediate attractive basin. Notice, though, that our proof became specially laborious in those situations where we were unable to apply quasi-conformal surgery techniques, in other words, when we could not find a multiply connected open set with simply connected image.

However, the case we will deal with in this section starts exactly with and is actually defined by this very hypothesis, so it is no surprise that the preperiodic case shall be proved using only surgery, in fact, using surgery in a fashion very similar to that of Shishikura’s for the rational (preperiodic) case. We want to prove the following.

**Theorem 4.6.** Let $f$ be a transcendental meromorphic function with a multiply connected (strictly preperiodic) Fatou component $U$ such that $f(U)$ is simply connected. Then, there exists at least one weakly repelling fixed point of $f$.

It is clear that $U$ is a connected component of $f^{-1}(f(U))$, since $U$ is a Fatou component itself. Let $E$ be one of the bounded components of $\hat{\mathbb{C}} \setminus U$ (one such component always exists because $U$ is multiply connected).
Figure 20. The two possible situations: in (a), the iterations of $U$ always stay in $E$, $f^k(U) \subset E$ for all $k \in \mathbb{N}$; whereas in (b), there exists a $k \in \mathbb{N}$ such that $f^i(U) \subset E$ for all $0 < i < k$ and $f^k(U) \subset \hat{\mathbb{C}} \setminus E$.

Figure 21. The new almost-complex structure $\sigma$.

In analogy to the rational case, let us focus our attention on the sequence of iterations $\{f^k(U)\}_{k \in \mathbb{N}}$. Notice that, in the preperiodic case, such iterations will not necessarily eventually abandon $E$ because they will never come back to $U$. This fact gives raise to two quite different situations, depicted in Figure 20.

Notice that case (b) is exactly the situation we already treated in the attractive case, so an analogous procedure gives a global quasi-regular map $f_2$, with its conjugate rational function $g$, plus the subsequent weakly repelling fixed point of $f$ in $E$.

For case (a) we define a quasi-regular map $f_2 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ in exactly the same way, that is, via $f_1 : \hat{\mathbb{C}} \setminus E \to f(U)$. However, in this case we define our $f_2$-invariant almost-complex structure as

$$
\sigma := \begin{cases} 
\sigma_0 & \text{on } f^n(U) \text{ for } n \in \mathbb{N} \\
\mathcal{F}_2^* \sigma_0 & \text{on } \hat{\mathbb{C}} \setminus E \\
(f_2^*)^n \sigma_0 & \text{on } f_2^{-n}(\hat{\mathbb{C}} \setminus E) \text{ for } n \in \mathbb{N} \\
\sigma_0 & \text{elsewhere}
\end{cases}
$$

(see Figure 21).
Therefore, we see that $f^2 \sigma = \sigma$ almost everywhere, by construction, and that $\sigma$ has bounded ellipticity, since $f_2$ is holomorphic everywhere except in $\hat{\mathbb{C}} \setminus E$, where it is quasi-regular but orbits clearly pass at most once through.

As usual, a rational map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ conjugate to $f_2$ is obtained from Corollary 2.4 and $f$ inherits from it a weakly repelling fixed point in $E$.

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