A SPACE OF MULTIPLIERS ON L

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Abstract. Conditions for a function (number sequence) to be a multiplier on the space of integrable functions on \( \mathbb{R} (T) \) are given. This generalizes recent results of Giang and Moricz.

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1. Introduction

In their recent paper [GM], Dăng Vũ Giang and F. Móricz gave a family of spaces, so that each element of such space is a multiplier on

\[ L = \left\{ f : \| f \|_L = \int_{\mathbb{R}} |f(x)| \, dx < \infty \right\}, \]

where \( \mathbb{R} = (-\infty, \infty) \). These results are obtained both in periodic and non-periodic cases. Proofs are strongly based on some sufficient conditions for a function to have an integrable Fourier transform, or for a trigonometric series to be a Fourier series of an integrable function, respectively.

More general conditions of such type were given in our recent paper [L]. Thus we can give less restrictive multiplier conditions, that is the results of [GM] follow from ours immediately. Some of our notation is the same as in [GM]. It allows us to compare easily our results.

Let \( f \) be an integrable function on \( \mathbb{R} \), and

\[ \hat{f}(x) = \int_{\mathbb{R}} f(t)e^{-ixt} \, dt \]

be its Fourier transform.

We say that a measurable bounded function \( \lambda \) is an \( M \)-multiplier if for every \( f \in L \) there exists a function \( g \in L \) such that

\[ \lambda(t) \hat{f}(t) = \hat{g}(t). \]

The norm of the corresponding operator \( \Lambda : L \to L \) which assigns to each \( f \in L \) the function \( \Lambda f = g \) accordingly to (1) may be calculated as usually:

\[ \| \Lambda \|_M = \sup_{\| f \|_L \leq 1} \frac{\| \Lambda f \|_L}{\| f \|_L}. \]

One may consider the space \( B \) of absolutely continuous functions on \( \mathbb{R} \), bounded over \( \mathbb{R} \), and endowed with the norm

\[ \| \lambda \|_B = \| \lambda \|_B + \int_{\mathbb{R}} |\lambda'(t)| \, dt, \]

where \( \| \lambda \|_B = \sup_{t \in \mathbb{R}} |\lambda(t)| \). There exist functions in \( B \) which are not multipliers (see, e.g., [T], p.170-172). Thus it is interesting to study some subspaces of \( B \) which are the spaces of multipliers.

2. Description of the space of multipliers

We introduce a subspace of \( B \) denoted by \( \mathcal{H} \) and defined as follows. Let

\[ S_f = \int_{0}^{\infty} \left| \int_{|t| \leq \frac{u}{2}} \frac{f(u-t) - f(u+t)}{t} \, dt \right| \, du. \]

Then, denoting

\[ A = \int_{0}^{\infty} \frac{|\lambda(t) - \lambda(-t)|}{t} \, dt, \]

we set

\[ \mathcal{H} = \{ \lambda : \| \lambda \|_{\mathcal{H}} = \| \lambda \|_B + S_\lambda + A < \infty \}. \]

Our main result is the following...
Theorem 1. If $\lambda \in \mathcal{H}$ then $\lambda$ is an $M$-multiplier, and

$$(2) \quad ||\Lambda||_M \leq C||\lambda||_{\mathcal{H}}.$$ 

Here and in what follows $C$ will mean absolute constants, and $C$ with indices, say $C_p$, will denote some constants depending only on the indices mentioned. The same letter may denote constants different in different places.

Let us make some remarks on the space $\mathcal{H}$ and compare Theorem 1 with earlier results.

Let $f_+$ be the odd continuation of the part of a function $f$ supported on $[0, \infty)$, and $f_-$ be the odd continuation of the part of $f$ supported on $(-\infty, 0]$.

Let, further, $ReH$ be the Hardy space with the norm $||f|| = ||f||_L + ||\tilde{f}||_L < \infty$, where

$$\tilde{f}(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)}{x-t} \, dt$$

is the Hilbert transform of $f$.

We say that $f \in H_s$ if $f_+ \in ReH$ and $f_- \in ReH$. It was proved in [L] that $||f||_1 + S_f < \infty$ is equivalent to the fact that $f \in H_s$.

In [GM], the main theorem is similar to our Theorem 1, but with one of the spaces of a family $\{B_p : 1 \leq p < \infty\}$ instead of $\mathcal{H}$. This family was defined as follows. For $1 < p < \infty$ set

$$A_qf = \int_{\mathbb{R}} \left(\frac{1}{u} \int_{u \leq |t| \leq 2u} |f(t)|^q \, dt\right)^{\frac{1}{q}} \, du$$

where $\frac{1}{p} + \frac{1}{q} = 1$ here and in what follows, while for $p = 1$

$$A_\infty f = \int_{0}^{\infty} \, \text{ess sup} \, |f(t)| \, du.$$ 

Then for $1 \leq p < \infty$

$$B_p = \{\lambda : ||\lambda||_{B_p} = ||\lambda||_B + A_q\lambda' + A < \infty\}.$$ 

A theorem claimed as the main one in [GM] may be formulated like Theorem 1, with $B_p$ instead of $\mathcal{H}$. Its proof, after some simple computations, follows from the following

**Lemma.** (See Lemma 1 in [GM].) If a function $\lambda$ is locally absolutely continuous, satisfies the condition

$$(3) \quad \lim_{|t| \to \infty} \lambda(t) = 0,$$

Then for $1 \leq p < \infty$

$$B_p = \{\lambda : ||\lambda||_{B_p} = ||\lambda||_B + A_q\lambda' + A < \infty\}.$$ 

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**Lemma.** (See Lemma 1 in [GM].) If a function $\lambda$ is locally absolutely continuous, satisfies the condition

$$(3) \quad \lim_{|t| \to \infty} \lambda(t) = 0,$$
and for some $1 < q \leq \infty$ we have $A_q < \infty$, then $\hat{\lambda}$ belongs to $L$ if and only if $A < \infty$. Furthermore,

$$||\hat{\lambda}||_L \leq A + C_q A_q.$$ 

Indeed, the fact that $\lambda' \in L^1$ yields easily that $\lambda$ has finite limits $l_+$ and $l_-$ at $+\infty$ and $-\infty$, respectively. When they coincide, $l_+ = l_- = l$ (and this follows from $A < \infty$), one can take

$$\lambda_0(t) = \lambda(t) - l$$

and

$$\lambda_1(t) = \hat{\lambda}_0(-t).$$

If $\lambda_1 \in L$ then $\hat{\lambda}_1 = \lambda_0$, and taking

$$\Lambda f = lf + \lambda_1 * f$$

one gets

$$(\Lambda f)(t) = l\hat{f}(t) + \hat{\lambda}_1 \hat{f}(t)$$

$$= l\hat{f}(t) + \lambda_0(t)\hat{f}(t) = \lambda(t)\hat{f}(t),$$

and $\lambda$ is a multiplier on $L$.

Theorem 1 may be proved analogously, with application of one our result from [L] instead of Lemma.

**Theorem A.** (See [L], Theorem 2.) Let $\lambda$ be a locally absolutely continuous function, satisfying (3). Then for $|x| > 0$ we have

$$\hat{\lambda}(x) = \frac{i}{x} \left( \lambda\left(\frac{\pi}{2|x|}\right) - \lambda\left(-\frac{\pi}{2|x|}\right) \right) + \theta \gamma(x),$$

where $|\theta| \leq C$, and

$$\int \gamma(x) \, dx \leq ||\lambda'||_L + S \lambda'.$$

It is obvious that, in conditions of Theorem A, $\hat{\lambda} \in L$ iff $A < \infty$. As it was said above, in order to prove Theorem 1 it remains to repeat the proof of Theorem 1 from [GM] using Theorem A instead of Lemma 1.

Indeed, the following embeddings are almost obvious:

$$B_1 \subset B_{p_1} \subset B_{p_2} \subset B, \quad 1 < p_1 < p_2 < \infty,$$

while the following fact, proved in [L], is not so clear:

$$B_p \subset \mathcal{H}, \quad 1 \leq p < \infty.$$
Easy sufficient condition for an even function $\lambda$ defined on $[0, \infty)$ to be a Fourier multiplier, due to [BN], p.248, has the following relation (so-called quasiconvexity)

$$
\int_0^\infty t|d\lambda'(t)| < \infty
$$
as the main part. It is easy to verify that this condition is more restrictive than $S_{\lambda'} < \infty$. Indeed, we have

$$
S_{\lambda'} = \int_0^\infty \left| \int_0^{\frac{u}{2}} \frac{d x}{x} \int \frac{d \lambda'(t)}{u-x} \right| d u
\leq \int_0^\infty d u \int_0^{\frac{u}{2}} |d\lambda'(t)| \ln \frac{u}{2|u-t|} = \ln 3 \int_0^\infty t|d\lambda'(t)|.
$$

3. The case of Fourier series

Analogous results for the case of Fourier series were obtained in [GM] as well. We can generalize these results in the same manner as in the case of Fourier transforms.

Let now $L$ be the space of all complex-valued $2\pi$-periodic functions integrable over $\mathbb{T} = (-\pi, \pi]$, and

$$
||f||_L = \int_\mathbb{T} |f(x)| \, dx.
$$

Let

$$
\hat{f}(k) = \frac{1}{2\pi} \int_\mathbb{T} f(x)e^{-ikx} \, dx, \quad k = 0, \pm 1, \pm 2, \ldots
$$

be the Fourier coefficients of the function $f$.

A bounded sequence $\{\lambda = \lambda(k)\}$, with $||\lambda||_m = \sup |\lambda(k)| < \infty$, is called an $M$-multiplier if for every $f \in L$ there exists a function $g \in L$ such that

$$
(4) \quad \lambda(k) \hat{f}(k) = \hat{g}(k), \quad k = 0, \pm 1, \pm 2, \ldots
$$

As above (4) assigns a bounded linear operator $\Lambda$, and it is worth studying spaces of multipliers which are subspaces of the space

$$
bv = \{\lambda : ||\lambda||_{bv} = ||\lambda||_m + ||\Delta \lambda||_1 < \infty\},
$$

where $|| \cdot ||_1$ is the norm in $l^1$, and

$$
\Delta \lambda(k) = \left\{ \begin{array}{ll}
\lambda(k) - \lambda(k+1) & \text{if } k \geq 0, \\
\lambda(k) - \lambda(k-1) & \text{if } k < 0.
\end{array} \right.
$$

Again a sequence $\lambda \in bv$ exists which is not a multiplier (see [Z], Vol.1, p.184).

We introduce a subspace of $bv$

$$
h = \{\lambda : ||\lambda||_{h} = ||\lambda||_m + \alpha + a < \infty\},
$$
where
\[ s_\lambda = \sum_{m=2}^{\infty} \left| \frac{\sum_{k=1}^{m} \Delta \lambda(m-k) - \Delta \lambda(m+k)}{k} \right| \]

and
\[ a = \sum_{k=1}^{\infty} \frac{|\lambda(k) - \lambda(-k)|}{k}. \]

Note that the condition \( s_\lambda < \infty \) is called the Boas-Telyakovskii condition (see, e.g., [T1]).

The analog of Theorem 1 for Fourier series may be formulated as follows.

**Theorem 2.** If \( \lambda \in h \) then \( \lambda \) is an \( M \)-multiplier, and
\[ ||\Lambda||_M \leq C ||\lambda||_h. \]

The proof again may be reduced to the proof of the multiplier theorem for Fourier series in [GM] with application of the following corollary to Theorem A instead of corresponding weaker result in [GM] (see Lemma 3).

Let \( \ell(x) = \lambda(k) + (k-x)\Delta \lambda(k) \) for \( x \in [k-1,k] \), with \( \lim_{|k|\to\infty} \lambda(k) = 0 \).

**Theorem B.** (see [L], Theorem 5). For every \( y, 0 < |y| \leq \pi \),
\[ \sum_{k=-\infty}^{\infty} \lambda(k)e^{iky} = \frac{i}{y} \left( \ell\left(\frac{\pi}{2\pi}\right) - \ell\left(-\frac{\pi}{2\pi}\right) \right) + \theta \gamma(y) \]
where \( \theta \leq C \), and
\[ \int_{\mathbb{T}} |\gamma(y)| dy \leq ||\Delta \lambda||_1 + s_\lambda. \]

This is a somewhat stronger form of Telyakovskii’s result in [T1].

It is obvious now that the function \( \ell \), having the sequence \( \lambda \) as its Fourier coefficients, is integrable over \( \mathbb{T} \) when \( \lambda \in h \). Thus it is enough to substitute this result for Lemma 3 in [GM], and so changed proof establishes Theorem 2.

Analogously to the case of Fourier transforms, the multiplier properties in the case of Fourier series were proved in [GM] for a family of sequences \( \{bv_p, 1 \leq p < \infty \} \). Each family is a subspace of \( bv \) and is defined as follows. Let \( I_n = \{2^n, 2^n+1, ..., 2^{n+1}-1\} \), and
\[ a_p = \sum_{n=0}^{\infty} 2^n \left( \sum_{|k| \in I_n} |\Delta \lambda(k)|^q \right)^{\frac{1}{q}} \]
for \( 1 < p < \infty \), while for \( p = 1 \)
\[ a_\infty = \sum_{n=0}^{\infty} \max_{|k| \in I_n} |\Delta \lambda(k)|. \]

Then for \( 1 \leq p < \infty \)
\[ \{bv_p, 1 \leq p < \infty \} = \{ \lambda : ||\lambda||_{bv} = ||\lambda||_{bv_1} + a_\infty + a_\infty < \infty \}. \]
It is well known that

\[ bv_1 \subset bv_{p_1} \subset bv_{p_2} \subset bv, \quad 1 < p_1 < p_2 < \infty \]

but for us more important is that for all \( p \)

\[ bv_p \subset h. \]

This was proved for \( p = 1 \) by Telyakovskii [T2], and for \( p > 1 \) by Fomin [F]. Therefore the result for Fourier series in [GM] is contained in Theorem 2 as the partial case.

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