The theorem of implicit operator in the sectorial neighborhoods

R Yu Leontev
Mathematical Analysis and Differential Equations Department, Institute of Mathematics and Information Technologies, Irkutsk State University, 1 K. Marks St., 664003 Irkutsk, Russia
E-mail: romanisu@yandex.ru

Abstract. A nonlinear operator equation with parameter in a linear normed space is considered. The operator in the main part of the equation is not continuously invertible so, the standard theorem of implicit operator cannot be applied. We obtain sufficient conditions for the existence of a continuous solution. Method of constructing this solution is given.

Let $X, Y$ be the Banach spaces, $\Lambda$ be a linear normalized space. Consider the following nonlinear operator equation

$$F(x, \lambda) = 0,$$

where $F : X \times \Lambda \to Y$.

Operator $F$ is continuous in the neighbourhood of zero with respect to $x$ and $\lambda$. We have the equality $F(0, 0) = 0$. There arises the issue of existence of small continuous solutions such that $x(0) = 0$, and the issue of techniques of their constructing.

**Definition 1.** Let us call the open set $S \subset \Lambda$, such that $0 \in \partial S$, where $\partial S$ is the boundary of set $S$, the sectorial neighbourhood of point $0 \in \Lambda$.

As an example of the sectorial neighbourhood of zero for $\Lambda = \mathbb{R}$ we consider the deleted neighbourhood of zero, as well as the left or the right semineighbourhood of zero. If $\Lambda = \mathbb{R}^2$, then the sectorial neighbourhood of zero will be represented, for example, by an open sector, whose boundary contains point $\lambda = 0$.

Let operator $F(x, \lambda)$ have the Frechet derivative with respect to the first argument, which is continuous with $x$ and $\lambda$ in neighbourhood zero, and let the following assessment be fulfilled:

$$\|F_x^{-1}(0, \lambda)\| = O\left(\frac{1}{a(\lambda)}\right) \text{ for } \forall \lambda \in S,$$

where $a(\lambda)$ is a continuous functional, $a(\lambda) : S \to \mathbb{R}_+$; $a(0) = 0$ (i.e. $\lim_{S \ni \lambda \to 0} a(\lambda) = 0$); $S$ is a sectorial neighbourhood of zero.

Our purpose is to give sufficient conditions of existence of the small solution $x(\lambda)$ for equation (1) in the sectorial neighbourhood of zero $S$ and propose a technique of constructing this solution.

Operator $F_x(0, \lambda)$ has a bounded inverse one in domain $S$, but operator $F_x(0, 0)$ is not continuously invertible due to equality $a(0) = 0$. So, the standard theorem of implicit operator [1, 2, 3] cannot be applied in the given case. In paper [4], the author has identified the classes of
equations for the case, when equation (1) in case of assessment (2) is reduced to an equivalent equation, which has a unique small solution, whose existence may be proved, while applying the principle of contracting maps, and construct the asymptotic of the solution with the use of the method of sequential approximations.

We do not assume that operator \( F_x(0,0) \) is Fredholm or has a closed domain of values. Parameter \( \lambda \) is an element of an arbitrary linear normalized space.

Estimates of the form (2) have practically played the decisive role in the papers on the theory of branching solutions. On this basis, in papers [5, 6], introduced and investigated was the concept of simple solution.

In monograph [7] the relationship of the estimates of such a kind, with the Jordan structure of the Frechet derivative \( F_x(0,\lambda) \) was ascertained. In the same publication, the method of regulation of computing of simple solutions by definite shift with respect to parameter \( \lambda \) coordinated with the computation error was proposed.

Let \( r \) be a positive number, and \( S \) be a sectorial neighbourhood of zero. Introduce the set:

\[
\Omega = \{(x, \lambda) \in X \times \Lambda, \|x\| \leq a(\lambda)r, \lambda \in S\}.
\]

The following definition suggests the concept of the minimal continuous branch, investigation of the issues of existence and constructing of which for the equations of the form (1) is right the idea of the present paper.

**Definition 2.** If there are numbers \( r_0 \in (0, r], \varepsilon > 0 \) such that only one solution \( x^*(\lambda) \) of all the small solutions of equation (1) defined in domain \( \Omega \) occurs in domain

\[
\Omega_0 = \{(x, \lambda) \in X \times \Lambda, \|x\| \leq a(\lambda)r_0, \lambda \in S, 0 < \|\lambda\| < \varepsilon\},
\]

then solution \( x^*(\lambda) \) will be called the minimal solution of equation (1) in domain \( S \), continuous at point \( \lambda = 0 \) (from now on, briefly “the minimal continuous branch”).

Note, if equation (1) for \( \forall \lambda \in S \) has the solution \( x(\lambda) = 0 \), then the minimal continuous branch will be represented by zero solution. This definition implies either uniqueness of the solution, which we call the minimal continuous branch, or absence of such a solution in principle.

Next, the following theorem is given for the equation (1), whose conditions are sufficient for the existence of the minimal continuous branch in the sectorial neighbourhood of zero \( S_0 \subseteq S \subseteq \Lambda \).

**Theorem 1.** Let the following conditions be satisfied in domain \( \Omega \):
1) operator \( F(x, \lambda) \) is continuous with respect to \( x \) and \( \lambda \) and has the partial Frechet derivative \( F_x(x, \lambda) \), continuous with respect to \( x \) and \( \lambda \);
2) \( F(0,0) = 0 \); operator \( F_x(0,\lambda) \) is continuously invertible for \( \forall \lambda \in S \) on the total space \( Y \), furthermore, for \( S \ni \lambda \to 0 \) we have the estimate (2);
3) one can find some constant \( L > 0 \), such that for each \( \lambda \in S \) the following inequality is valid:

\[
\|F_x(x, \lambda) - F_x(0, \lambda)\| \leq L\|x\|;
\]

4) we have the following estimate \( \|F(0, \lambda)\| = o(a^2(\lambda)) \), when \( S \ni \lambda \to 0 \).

Hence there is a number \( r_0 \in (0, r] \) and a sectorial neighbourhood of zero \( S_0 \subseteq S \) such that for each \( \lambda \in S_0 \) equation (1) has a minimal continuous branch \( x(\lambda) \to 0 \), when \( S_0 \ni \lambda \to 0 \), in the sphere \( \|x\| \leq a(\lambda)r_0 \), which may be constructed by the method of sequential approximation.

The proof of this theorem is based on the substitution \( x = a(\lambda)V \) and application of the principle of contracting maps to the obtained equation.
Example 1. Let us demonstrate that equation

\[ F(x, \lambda) \equiv \int_0^1 tsx(s) \, ds + \lambda x(t) - \int_0^1 x^3(s) \, ds - f(t, \lambda) = 0, \]

where \( x(t) \in C_{[0,1]}, \) \( f(t, \lambda) = m(t)\lambda^n, \) \( m(t) \in C_{[0,1]}, \) \( n > 2, \) \( S \) is a deleted neighbourhood of zero, has a continuous for \( t \in [0, 1] \) solution \( x_\lambda(t) \to 0 \) for \( S \ni \lambda \to 0. \)

Solution. First of all, let us find the Frechet derivative with respect to the first argument for operator \( F(x, \lambda). \) To this end, consider the difference

\[ F(x + h) = F(x) = \int_0^1 tsx(s + h(s)) \, ds + \lambda x(t) + \lambda h(t) - \int_0^1 x^3(s + h(s)) \, ds - f(t, \lambda) = \]

\[ f(t, \lambda) - \int_0^1 tsx(s) \, ds - \lambda x(t) + \int_0^1 x^3(s) \, ds + f(t, \lambda) = \]

\[ \int_0^1 tsh(s) \, ds + \lambda h(t) - \int_0^1 3x^2(s)h(s) \, ds - \int_0^1 3x(s)h^2(s) \, ds - \int_0^1 h^3(s) \, ds. \]

Here, the Frechet differential will consist of all the addends, which enter with respect to \( h \) linearly into expression \( F(x + h) - F(x), \) i.e. the Frechet differential writes:

\[ F_x(x, \lambda)h = \int_0^1 tsh(s) \, ds + \lambda h(t) - 3 \int_0^1 x^2(s)h(s) \, ds. \]

According to the conditions of the theorem, the estimate of the norm of operator \( F_x^{-1}(0, \lambda) \) shall satisfy the condition (2). Let us find \( F_x^{-1}(0, \lambda). \) To this end, let us resolve the equation

\[ \int_0^1 tsh(s) \, ds + \lambda h(t) = f(t) \quad (3) \]

with respect to \( h(t), \) \( \int_0^1 ts[\cdot] \, ds + \lambda[\cdot] = F_x(0, \lambda). \) Let us withdraw the multiplier \( t \) from the sign of the integral and introduce the denotation:

\[ \int_0^1 s \cdot h(s) \, ds = C_1. \quad (4) \]

Hence from (3) we obtain:

\[ h(t) = \frac{1}{\lambda} (f(t) - tC_1). \]

Having substituted the obtained equation for \( h(t) \) into equality (4), we obtain an algebraic equation with respect to \( C_1, \) after resolving of which we obtain:

\[ C_1 = \frac{3}{3\lambda + 1} \int_0^1 sf(s) \, ds. \]
Hence operator $F_x^{-1}(0, \lambda)$ has the form:

$$F_x^{-1}(0, \lambda)f = f(t) - \frac{3t}{(3\lambda + 1)\lambda} \int_0^1 sf(s) \, ds.$$  

Consequently, the estimate (2) is satisfied:

$$\|F_x^{-1}(0, \lambda)\| = O\left(\frac{1}{|\lambda|}\right) \lambda \to 0, \lambda \neq 0.$$  

Now let us verify whether other conditions of theorem 1 are satisfied or not:  
1) operator $F(x, \lambda)$ is continuous with respect to $x$ and $\lambda$ in some neighbourhood of zero $\|x\| \leq r, |\lambda| < \rho$. The Frechet derivative $F_x(x, \lambda)$ is also continuous with respect to $x$ and $\lambda$ in some neighbourhood of zero $\|x\| \leq r, |\lambda| < \rho$;  
2) operator $F_x(0, \lambda)$ is continuously invertible in some deleted neighbourhood of zero, and the estimate (2) is valid;  
3) from the estimate

$$\|F_x(x, \lambda)h - F_x(0, \lambda)h\| = \left\|3 \int_0^1 x^2(s)h(s) \, ds\right\| \leq 3r\|x\|\|h\|,$$

where $\|x\| \leq r$, it follows that

$$\|F_x(x, \lambda) - F_x(0, \lambda)\| \leq 3r\|x\|;$$

4) $\|F(0, \lambda)\| = \|f(t, \lambda)\| = \|m(t)\|\lambda^n\| = \|m(t)\|\|\lambda\|^n$, where $n > 2.$  

Consequently, all the conditions of theorem 1 are satisfied, what allows one to state that the given equation has a solution $x(\lambda) \to 0$ for $\lambda \to 0$ in some deleted neighbourhood of zero $0 < |\lambda| < r_0$. This solution has the form $x = \lambda V$, where $V(\lambda)$ is constructed by the method of sequential approximations with the initial approximation $V_0 = 0$.

References

[1] Veinburg M M 1969 The Theory of Branching Solutions of Nonlinear Equations (Moscow: Nauka)  
[2] Lusternik L A and Sobolev V I 1965 Elements of Functional Analysis (Moscow: Nauka)  
[3] Trenogin V A 2002 Functional Analysis (Moscow: Fizmatlit)  
[4] Sidorov N A 2004 Minimal branches of solutions of nonlinear equations and asymptotic regulators Nonlinear Boundary-Value Problems 14 (Donetsk: Institute of Applied Mathematics and Mechanics) 161–4  
[5] Gelman A E 1963 On simple solutions of operator equations in the case of branching Russian Math. Doklady 152(5) 1042–4.  
[6] Krasnoselsky M A, Vainikko G M, Zabreiko P P, Rutitsky Ya B and Stetsenko V Ya 1969 An approximate Solution of Operator Equations (Moscow: Nauka)  
[7] Sidorov N A 1982 General Issues of Regulation in Problems of the Branching Theory (Irkutsk: Irkutsk State University)