TOWARDS A THEORY OF LOGARITHMIC GLSM MODULI SPACES

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Abstract. In this article, we establish foundations for a logarithmic compactification of general GLSM moduli spaces via the theory of stable log maps \cite{15, 2, 25}. We then illustrate our method via the key example of Witten’s \(r\)-spin class. In the subsequent articles \cite{17, 16}, we will push the technique to the general situation. One novelty of our theory is that such a compactification admits two virtual cycles, a usual virtual cycle and a “reduced virtual cycle”. A key result of this article is that the reduced virtual cycle in the \(r\)-spin case equals to the \(r\)-spin virtual cycle as defined using cosection localization by Chang–Li–Li \cite{13}. The reduced virtual cycle has the advantage of being \(\mathbb{C}^*\)-equivariant for a non-trivial \(\mathbb{C}^*\)-action. The localization formula has a variety of applications such as computing higher genus Gromov–Witten invariants of quintic threefolds \cite{27} and the class of the locus of holomorphic differentials \cite{18}.

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1. Introduction

1.1. Gauged Linear Sigma models. One of the major advances in the subject of Gromov–Witten theory is the development of the so-called FJRW-theory by the third author and his collaborators. The Gromov–Witten theory of a Calabi–Yau hypersurface of a weighted projective space is conjectured to be equivalent to its FJRW-dual via the LG/CY correspondence, a famous duality from physics. In physics, the Gromov–Witten theory corresponds to a nonlinear sigma model.
while FJRW-theory corresponds to a Landau–Ginzburg model. Back in 1993, Witten gave a physical derivation of the LG/CY correspondence by constructing a family of theories which was known as gauged linear sigma model or GLSM \cite{Witten1993}. By varying the parameters of GLSM, Witten argued that GLSM converges to a nonlinear sigma model at a certain limit of parameters and a Landau-Ginzburg orbifold at a different limit. Hence, they are related by analytic continuation.

Several years ago, GLSM was put in a firm mathematical footing by Fan, Jarvis and the third author \cite{FanJarvisRuan2010}. Let us briefly describe the construction. The input data of a GLSM is an LG-space

\[ W : V / / G \rightarrow \mathbb{C} \]

for a GIT quotient \( V / / G \) with a \( \mathbb{C}^* \)-action \( \mathbb{C}^*_G \curvearrowright V \) (called the R-charge) such that \( W \) is homogenous of degree one. Moreover, we assume that the critical locus \( \text{Crit}_W = \{ dW = 0 \} \subset V / / G \) is compact. The most famous example is

\[ W = p(x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5) : \mathbb{C}^5 \times \mathbb{C} \rightarrow \mathbb{C} \]

with \( \mathbb{C}^* \)-action of weight \( (1, 1, 1, 1, -5) \). Here \( (x_1, x_2, x_3, x_4, x_5) \) are the coordinates of \( \mathbb{C}^5 \) and \( p \) is the coordinate of \( \mathbb{C} \). Furthermore, the R-charge has the weight \( (0, 0, 0, 0, 1) \). The GIT-quotient \( (\mathbb{C}^5 \times \mathbb{C}) / / \mathbb{C}^* \) has two chambers or phases depending on the character

\[ \theta(z) = z^n : \mathbb{C}^* \rightarrow \mathbb{C}^*. \]

If \( \theta > 0 \) (i.e., \( n > 0 \)), then the unstable locus is \( (0, 0, 0, 0, 0) \times \mathbb{C} \) and we have the GIT quotient \( ((\mathbb{C}^5 - \{(0, 0, 0, 0, 0)\}) \times \mathbb{C}) / / \mathbb{C}^* \cong \mathcal{O}_{\mathbb{P}^4}(-5) \). When \( \theta < 0 \), the unstable locus is \( \mathbb{C}^5 \times \{0\} \) and the GIT quotient is \( (\mathbb{C}^5 \times \mathbb{C}^*) / / \mathbb{C}^* \cong [\mathbb{C}^5 / \mathbb{Z}_5] \). This GLSM is supposed to be equivalent to the Gromov–Witten theory of the quintic 3-fold \( X_5 = \{ x_1^3 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0 \} \) in the chamber \( \theta > 0 \) and FJRW-theory of the LG orbifold

\[ F = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 : [\mathbb{C}^5 / \mathbb{Z}_5] \rightarrow \mathbb{C} \]

in the chamber \( \theta < 0 \). Let us use this example to illustrate Fan–Jarvis–Ruan’s algebraic GLSM theory. The geometric data for the above GLSM is

\[ \mathcal{M}^\theta = \{(C, \mathcal{L}, (s_1, s_2, s_3, s_4, s_5) \in H^0(\mathcal{L}^{\leq 5}), p \in H^0(\mathcal{L}^{-5} \otimes \omega_{\log})) : \ldots \} \]

satisfying certain stability condition where \( C \) is a pre-stable curve and \( \mathcal{L} \) is a line bundle over \( C \). For \( \theta > 0 \), the stability condition implies that \( (s_1, s_2, s_3, s_4, s_5) \) define a stable quasimap into \( \mathbb{P}^4 \) and we obtain a variant of Chang–Li’s p-field moduli space \cite{ChangLi2010}. For \( \theta < 0 \), the stability condition implies that the zeros of \( p \) form an effective divisor \( D \), and that \( p \) defines a weighted 5-spin structure \( \mathcal{L}^5 \cong \omega_{\log,C}(-D) \). In both cases, \( \mathcal{M}^\theta \) is a DM-stack with two-term perfect obstruction theory and has a virtual cycle in the Chow group. However, it is not proper.
TOWARDS LOGARITHMIC GLSM

To obtain a virtual cycle which we can integrate, we use $dW$ to define a cosection

$$\sigma: \text{Obs}_{\mathcal{M}^\theta} \to \mathcal{O}_{\mathcal{M}^\theta}$$

and apply Kiem—Li’s cosection localization technique [34] to define a localized virtual cycle $[\mathcal{M}]_{\sigma}^{\text{vir}}$ with support on the compact sub-locus $\mathcal{M}^\theta(\sigma) \subset \mathcal{M}^\theta$ satisfying the condition $(s_1, s_2, s_3, s_4, s_5, p) \in \text{Crit}_W$.

The above construction is beautiful. However, it is not directly useful for computational purposes. In many ways, we would like to have an alternative construction which is more friendly towards effective computation. To that end, we would like to avoid using a cosection.

In the same paper, Kiem–Li showed that if $\mathcal{M}$ is a compact moduli space with a two-term perfect obstruction theory and a cosection $\sigma$, then

$$\deg([\mathcal{M}]_{\sigma}^{\text{vir}}) = \deg([\mathcal{M}]^{\text{vir}})$$

This suggests that one should try to compactify the GLSM moduli space in a way such that its cosection extends without additional degeneracy loci. The main purpose of this and its subsequent articles is to construct such a compactification.

1.2. The logarithmic approach.

1.2.1. Stable maps relative to boundary divisors. The theory of stable maps relative to a smooth boundary divisor was first introduced in symplectic geometry in the 90’s by Li–Ruan [36] and Ionel–Parker [29, 30]. Since then, it has become one of main tools in the subject of Gromov–Witten theory.

During the last twenty years, its algebraic geometric version using expansions was first developed by Jun Li [37, 38]. A combination of expansions with logarithmic geometry was introduced by Kim [35], and one with orbifold structure was introduced by Abramovich–Fantechi [5]. In the general logarithmic setting relative to toroidal boundary divisors, the theory of stable log maps was developed by Abramovich–Chen–Gross–Siebert [2, 15, 25] without using expansions. A different approach using exploded manifolds was introduced by Brett Parker [45, 46, 47].

In this and the subsequent articles, we will apply the techniques of stable log maps to compactify the gauged linear sigma model (GLSM) of Fan-Jarvis-Ruan [23], and study their virtual cycles.

1.2.2. Log maps. A stable log map to a separated log Deligne–Mumford stack $Y$ is a morphism of log stacks $f: \mathcal{C} \to Y$ over a log scheme $S$ where $\mathcal{C} \to S$ is a twisted log curve and the underlying twisted map $f$ obtained by removing log structures is stable in the usual sense. For our purpose, we will only consider the case that $\mathcal{M}_Y$ is of Deligne–Faltings type of rank one. This amounts to saying that the logarithmic boundary of $Y$ is a Cartier divisor, see Section 2.1.8.
The central object of log maps is the stack $\mathcal{M}(Y, \beta)$ parameterizing stable log maps to $Y$ with a given collection of discrete data $\beta$ (Section 2.4.1). The case where $Y$ is a log scheme has been developed in [2, 15, 25]. The same method applies to the case of log Deligne–Mumford targets. Due to a lack of references, in Section 2 we record a proof of algebraicity of $\mathcal{M}(Y, \beta)$ together with many useful properties needed in our construction.

1.2.3. Modular principalization of the boundary. A stable log map is degenerate if it maps a component of the source curve to the boundary of $Y$. Denote by $\Delta \subset \mathcal{M}(Y, \beta)$ the locus consisting of degenerate fibers. In general, it is virtually a toroidal divisor, which becomes a major difficulty for the construction of a reduced perfect obstruction theory of the compactified GLSM. The key to overcome this difficulty is the following modular principalization of $\Delta$.

Let $f: C \to Y$ be a stable log map over a geometric log point $S$. For each irreducible component $Z \subset C$ we may associate an unique element $e_Z \in \overline{M}_S := \mathcal{M}_S/O_S^*$ called the degeneracy of $Z$ (Section 2.2.3). As elements of the monoid $\overline{M}_S$, they carry a natural partial ordering such that $e_{Z_1} \leq e_{Z_2}$ if $(e_{Z_2} - e_{Z_1}) \in \overline{M}_S$. Intuitively $e_Z$ measures the “speed” of degeneracy of $Z$ into the boundary of $Y$, and $e_{Z_1} \leq e_{Z_2}$ means that $Z_2$ degenerates “faster” than $e_{Z_1}$. The stable log map $f$ is said to have uniform maximal degeneracy if the set of degeneracies has a unique maximal element. It turns out that having uniform maximal degeneracy is an open condition and is stable under base change. Let $\mathcal{U}(Y, \beta) \subset \mathcal{M}(Y, \beta)$ be the sub-category fibered over log schemes consisting of objects with the uniform maximal degeneracy. In Section 3, we establish the following theorem.

**Theorem 1.1** (Theorem 3.15). The canonical morphism $\mathcal{U}(Y, \beta) \to \mathcal{M}(Y, \beta)$ is a proper, representable and log étale morphism of log Deligne–Mumford stacks.

The maximal degeneracy defines naturally a virtual Cartier divisor $\Delta_{\text{max}} \subset \mathcal{U}(Y, \beta)$ whose support is precisely the locus of degenerate log maps, see Section 3.4.1.

**Remark 1.2.** The category $\mathcal{U}(Y, \beta)$ is indeed the largest sub-category of $\mathcal{M}(Y, \beta)$ to which our construction of reduced perfect obstruction theory of compactified GLSM applies. Consequently, our construction applies to subcategories of $\mathcal{U}(Y, \beta)$ including aligned logarithmic structures of [1, 8.1]. The general construction of this paper allows us to work with various subcategories of $\mathcal{U}(Y, \beta)$ to carry out the computation of the GLSM virtual cycle. This will be a task of [16].

1.3. The $r$-spin case. Since the technique is quite involved, for the reader’s benefit it makes sense to work it out in full detail a first non-trivial simple example. This is another main purpose of the current
Our example of choice is the r-spin theory which corresponds to the GLSM of

\[ W = x^r p : [(\mathbb{C} \times \mathbb{C})/\mathbb{C}^*] \to \mathbb{C}, \]

where the coordinates on \( \mathbb{C} \times \mathbb{C} \) are \((x, p)\), the weight of action is \((1, -r)\) and \(R\)-charge is \((0, 1)\). Similarly to the case of quintic 3-folds, this model has two chambers as well. The relevant chamber for \(r\)-spin curve theory is the Landau–Ginzburg chamber \( \theta < 0 \), where the stable locus is \( \mathbb{C} \times \mathbb{C}^* \). Furthermore, we choose a stability condition such that \( p \) has no zero. By the previous discussion, \( p \) can be interpreted as defining an isomorphism \( L_r \cong \omega \log C \) and the GLSM moduli space is

\[ U_{g,k} = \{ (\mathcal{C}, \mathcal{L}, s \in H^0(\mathcal{L}), \mathcal{L}_r \cong \omega \log C) \}. \]

Let \((C/S, \mathcal{L})\) be an \(r\)-spin curve consisting of a log curve \( C \to S \) and an \(r\)-spin bundle \( \mathcal{L} \) over \( C/S \). Denote by \( 0_P \) and \( \infty_P \) the zero and infinity sections of \( P := \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_C) \) respectively, and \( M_P \) the log structure on \( P \) associated to \( \infty_P \). Consider the log stack \( \mathcal{P} = (P, M_{C|P} \oplus \mathcal{O} - M_{P}) \) with the projection \( \mathcal{P} \to C \). A log field is a section \( f : C \to P \). It is stable if \( \omega_{C/S} \otimes O(f^*0_P)^k \) is positive for \( k \gg 0 \).

Denote by \( \mathcal{S}_r^{1/r} \) the stack of stable log fields with discrete data \( \beta = (g, \gamma, c) \) consisting of the genus \( g \), the monodromy \( \gamma \) of the spin bundle along markings, and the contact order \( c \) along each markings with \( \infty_P \). We first achieve the compactification:

**Theorem 1.3** (Theorem 4.12). \( \mathcal{S}_r^{1/r} \) is represented by a proper log Deligne–Mumford stack.

**Remark 1.4.** The compactification of the moduli of abelian and meromorphic differentials using log stable maps has been studied previously in [14, 26]. The compactification considered in this paper (in the case \( r = 1 \)) is different from loc. cit. in that we do not put the log structure on \( P \) induced by the zero section.

**Remark 1.5.** It is worth emphasizing that the properness of \( \mathcal{S}_r^{1/r} \) is interestingly a non-trivial fact. As shown in Section 4.4.6, limit(s) of a one-parameter family of meromorphic sections of spin bundles may not exist regardless of the stability conditions. Log structures play an important role in the existence of the underlying limiting section!

Note that a log fields \( f : C \to P \) is equivalent to a log map \( f' : C \to (P, \mathcal{M}_P) \) whose underlying is a section of \( P \to C \). Since \( \mathcal{M}_P \) is Deligne–Faltings type of rank one, we may consider the stack \( \mathcal{S}^{1/r}_\beta \) of stable log fields with uniform maximal degeneracy with respect to \( \mathcal{M}_P \). Theorem 1.1 implies that \( \mathcal{S}^{1/r}_\beta \) is a proper log Deligne–Mumford stacks as well.

Next, we consider its virtual cycle. The stack \( \mathcal{S}_\beta^{1/r} \) admits a natural two term perfect obstruction theory and hence a virtual cycle \( [\mathcal{S}_\beta^{1/r}]^{vir} \).
But this virtual cycle is different from cosection localized virtual cycle. The main result of the paper is the following:

**Theorem 1.6** (Proposition 5.23 and 5.30). Under the condition that all markings are narrow and of trivial contact order, the space $\mathcal{U}^{1/r}_β$ carries an alternative “reduced” two term perfect obstruction theory together with a cosection $σ^{\text{red}}_{\mathcal{U}^{1/r}}$ on $\mathcal{U}^{1/r}_β$ that has no additional degeneracy loci. Furthermore, suppose $[\mathcal{U}^{1/r}_β]^{\text{red}}$ is the virtual cycle of the reduced perfect obstruction theory, then

$$i^*[\mathcal{U}^{1/r}_β]^{\text{vir}}_{\text{loc}} = [\mathcal{U}^{1/r}_β]^{\text{red}}$$

where $i: \mathcal{M}^{1/r}_{g,γ} → \mathcal{U}^{1/r}_β$ is the inclusion of the zero section, and $\mathcal{U}^{1/r}_β = \mathcal{U}^{1/r}_β \setminus Δ_{\text{max}}$.

**Remark 1.7.** We remark that the reduced perfect obstruction theory has the same virtual dimension as the original one. Therefore, it is not a traditional reduced virtual cycle, which changes the virtual dimension. Instead, the perfect obstruction theory is only “reduced” along the boundary of the moduli space.

1.4. **History of the $r$-spin virtual cycle.** There was a long line of works constructing both the moduli space of $r$-spin structures and its virtual cycle. Spin curves were proposed by Witten [52] in an effort to generalize his famous conjecture that the intersection theory of the moduli space of stable curves is governed by the KdV-hierarchy. The compactification was first constructed by Jarvis [31] using torsion-free sheaves and later by Abramovich–Jarvis [6] using line bundles on twisted curves.

The first construction of the virtual cycle is due to Polishchuck–Vaintrob [49]. From the modern point of view, their construction is better viewed as a quantum K-theoretic construction from which one can obtain a virtual cycle by taking some kind of Chern character (see [19]).

The picture was clarified significantly by Fan–Jarvis–Ruan with a vast generalization (FJRW-theory) of $r$-spin theory. The input data of FJRW theory is a nondegenerated quasi-homogeneous polynomial $W$ together with a so called *admissible* finite automorphism group $G$ of $W$. The $r$-spin theories are simply the case of $W = z^r$ and $G = \mathbb{Z}/r\mathbb{Z}$. The state space of the $r$-spin theory corresponds to the monodromy at the marked point, and is indexed by an integer $0 ≤ m < r$. The insertion $m > 0$ corresponds to the so called narrow sector in FJRW-theory and the corresponding virtual cycle was constructed as a localized topological Euler class. The role of $m = 0$ was clarified in general FJRW-theory as a new type of insertions called broad. They showed that the broad insertion is irrelevant in $r$-spin theory but a source of difficulty in general case. Fan–Jarvis–Ruan’s construction is analytic.
in nature although there is an algebraic construction of Polishchuk and Vaintrob using matrix factorizations [50]. However, it is not clear that these two are equivalent in the most general case.

The last piece of the puzzle before the present work was provided by Chang–Li–Li in [13], where they gave yet another algebraic geometric construction of FJRW virtual cycle for narrow sectors. This is the construction that we use in this article. Furthermore, they proved that all constructions of Polishchuk–Vaintrob, Chiodo, Fan–Jarvis–Ruan and Chang–Li–Li are equivalent.

Finally, the $A_r$-generalization of Witten’s integrable hierarchies conjecture was proved by Faber–Shadrin–Zvonkine [21] while the $D_n, E_6,7,8$-generalization was proved by Fan–Jarvis–Ruan [22].

1.5. Effective $r$-spin structures. A key input that led us to propose this new construction of the $r$-spin virtual cycle is a conjectural formula of $\overline{\mathcal{M}}_{g,n}^{1/r}_r$ by the second author. This formula was motivated by the recent study of the cycle of the locus of holomorphic differentials and of double ramification cycles. We outline here this train of thought.

We consider the open sub-stack $\mathcal{M}_{g,γ}^{1/r} \subset \overline{\mathcal{M}}_{g,γ}^{1/r}$ of $r$-spin structures on smooth orbifold curves. An $r$-spin structure $(\mathcal{C}/S, \mathcal{L}) \in \mathcal{M}_{g,γ}^{1/r}$ is called effective if $h^0(\mathcal{L}) > 0$. We denote by $S_0 \subset \mathcal{M}_{g,γ}^{1/r}$ the locus of effective $r$-spin structures and by $\overline{S}_0 \subset \overline{\mathcal{M}}_{g,γ}^{1/r}$ its closure. A. Polishchuk studied the geometry of effective $r$-spin structures (see [48]) and asked the following question: Can we express the $r$-spin virtual cycle to $\mathcal{M}_{g,γ}^{1/r}$ in terms of the cycle $\overline{S}_0$ and other natural cycles?

This problem was left aside until a precise conjecture was recently stated (see [44, Conjecture A.1]). This conjecture can be re-stated as follows: for large values of $r$, we have

$$\epsilon_* \left( \frac{1}{r} \overline{\mathcal{M}}_{g,γ}^{1/r}_r \right) \overline{\mathcal{M}}_{g,γ}^{1/r} + [\overline{S}_0] = \alpha(r) \in A^*(\overline{\mathcal{M}}_{g,n})$$

where $\alpha(r)$ is a polynomial in $r$ (here $\epsilon \colon \overline{\mathcal{M}}_{g,γ}^{1/r}_r \rightarrow \overline{\mathcal{M}}_{g,n}$ stands for the forgetful map of the spin structure).

Remark 1.8. Note that the conventions for the value of $\overline{\mathcal{M}}_{g,γ}^{1/r}_r \overline{\mathcal{M}}_{g,γ}^{1/r}$ are different in [13], [48], and [44].

This conjecture is very similar to a conjectural expression by Pixton for the so-called double-ramification (DR) cycles that was proved by Pandharipande, Pixton, and the second and fifth authors (see [44]). The main tool of their proof is the virtual localization formula of Graber and Pandharipande (see [24]).

In order to prove the new conjecture of [44], the second author built a localization formula by analogy with the proof of the expression of DR cycles. In this conjectural localization formula, the role of DR cycles
is replaced by cycles of effective $r$-spin structures. The second author checked the consistency of this formula by various computations in low genera.

From this point, our main problem was to construct the space where the conjectural localization formula should hold. The effort to pin down the geometry underlining this formula led to use the machinery of log geometry in this article. In work in progress \cite{18}, we prove the localization formula and show that it implies \cite[Conjecture A.1]{44}.

1.6. Plan. The paper is organized as follows. In Section 2, we discuss the general set-up of log stable maps in the orbifold setting. In Section 3, we introduce the new notion of log structures of “uniform maximal degeneracy”, which is crucial for the construction of the reduced virtual cycle. This is applied in Section 4, to construct the compactification of the moduli space of $r$-spin curves with a field. Finally, in Section 5, we construct the reduced perfect obstruction theories and cosections, and we prove Theorem 1.6.

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2. Moduli of twisted stable log maps

In this section, we introduce the set up of stable log maps needed for compactifying GLSM. It was defined with prestable source curves in \cite{2, 15, 25}. We take the opportunity to extend it to the orbifold setting.
2.1. Twisted log maps.

2.1.1. Twisted curves. Recall from [7] that a twisted $n$-pointed curve over $\mathcal{S}$ consists of the following data

$$(\mathcal{C} \rightarrow \mathcal{C} \rightarrow \mathcal{S}, \{\sigma_i\}_{i=1}^n)$$

where

1. $\mathcal{C}$ is a proper Deligne–Mumford stack, and is étale locally a nodal curve over $\mathcal{S}$;
2. $\sigma_i \subset \mathcal{C}$ are disjoint closed substacks in the smooth locus of $\mathcal{C} \rightarrow \mathcal{S}$;
3. $\sigma_i \rightarrow \mathcal{S}$ are étale gerbes banded by the multiplicative group $\mu_{r_i}$ for some non-negative integer $r_i$;
4. the morphism $\mathcal{C} \rightarrow \mathcal{C}$ is the coarse moduli morphism;
5. along each stacky critical locus of $\mathcal{C} \rightarrow \mathcal{S}$, the group action of $\mu_{r_i}$ is balanced;
6. $\mathcal{C} \rightarrow \mathcal{C}$ is an isomorphism over $\mathcal{C}_{\text{gen}}$, where $\mathcal{C}_{\text{gen}}$ is the complement of the markings $\sigma_i$ and the stacky critical locus of $\mathcal{C} \rightarrow \mathcal{S}$.

Given a twisted curve as above, by [7, 4.11] the coarse space $\mathcal{C} \rightarrow \mathcal{S}$ is a family of $n$-pointed usual pre-stable curves over $\mathcal{S}$ with the markings determined by the images of $\{\sigma_i\}$. We define the genus of the twisted curve as the genus of the corresponding coarse pre-stable curve. When there is no danger of confusion, we will simply write $\mathcal{C} \rightarrow \mathcal{S}$ for a family of twisted curves.

Twisted curves only have possible stacky structures along markings and nodes. We recall the local stacky structure below.

2.1.2. Stacky structure along nodes. Let $\mathcal{C} \rightarrow \mathcal{S}$ be a family of twisted curves with the coarse moduli $\mathcal{C} \rightarrow \mathcal{C}$. Let $\bar{q} \rightarrow \mathcal{C}$ be a geometric point of a node. Shrinking $\mathcal{S}$ if necessary, there exists an étale neighborhood $\mathcal{U} \rightarrow \mathcal{C}$ of $\bar{q}$ with an étale morphism

$$\mathcal{U} \rightarrow \text{Spec} \left( \mathcal{O}_{\mathcal{S}}[x,y]/(xy = t) \right)$$

for some $t \in \mathcal{O}_{\mathcal{S}}$. Then the pull-back $\mathcal{C} \times_{\mathcal{C}} \mathcal{U}$ is given by the stack quotient

$$(1) \quad \left[ \text{Spec} \left( \mathcal{O}_U[\tilde{x}, \tilde{y}]/(\tilde{x}\tilde{y} = t', \tilde{x}^r = x, \tilde{y}^r = y) \right) \right]/\mu_r$$

for some $t' \in \mathcal{O}_{\mathcal{S}}$. Here for a generator $\gamma \in \mu_r$, the $\mu_r$-action is given by $\gamma(\tilde{x}) = \zeta \tilde{x}$ and $\gamma(\tilde{y}) = \zeta' \tilde{y}$ for some primitive $r$-th roots of unity $\zeta$ and $\zeta'$. The balanced condition implies that $\zeta' = \zeta^{-1}$.

2.1.3. Stacky structure along markings. Let $\bar{p} \rightarrow \mathcal{C}$ be a geometric point of a marking corresponding to $\sigma_i$. Shrinking $\mathcal{S}$ if necessary, there exists an étale neighborhood $\mathcal{V} \rightarrow \mathcal{C}$ of $\bar{p}$ with an étale morphism

$$\mathcal{V} \rightarrow \text{Spec} \mathcal{O}_{\mathcal{S}}[z].$$
The pull-back $\mathcal{C} \times_{\mathcal{C}} \mathcal{V}$ is given by the stack quotient
\[(2) \quad \left[ \text{Spec} \left( \mathcal{O}_{\mathcal{U}}[\tilde{z}] / (\tilde{z}^{r_i} = \tilde{z}) \right) / \mu_{r_i} \right] \]
and for each $\zeta \in \mu_{r_i}$, the action is given by $\tilde{z} \mapsto \zeta \tilde{z}$.

2.1.4. Logarithmic twisted curves. A log twisted $n$-pointed curve over a fine and saturated log scheme $S$ in the sense of [43] consists of
\[(\pi : \mathcal{C} \rightarrow S, \{\sigma_i\}_{i=1}^n) \]
such that
1. The underlying data $(\mathcal{C} \rightarrow \mathcal{C} \rightarrow S, \{\sigma_i\}_{i=1}^n)$ is a twisted $n$-pointed curve over $S$, where $\mathcal{C} \rightarrow \mathcal{C}$ is the coarse moduli morphism.
2. $\pi$ is proper, logarithmically smooth, and integral morphism of fine and saturated logarithmic stacks.
3. If $U \subset \mathcal{C}$ is the non-critical locus of $\pi$, then $M_{\mathcal{C}}|_U \cong \pi^* M_S \oplus \bigoplus_{i=1}^n N_{\sigma_i}$, where $N_{\sigma_i}$ denotes the constant sheaf over $\sigma_i$ with fiber $\mathbb{N}$.

For simplicity, we may refer to $\pi : \mathcal{C} \rightarrow S$ as a log twisted curve. The pull-back of a log twisted curve $\pi : \mathcal{C} \rightarrow S$ along an arbitrary morphism of fine and saturated log schemes $T \rightarrow S$ is the log twisted curve $\pi_T : \mathcal{C}_T := \mathcal{C} \times_S T \rightarrow T$.

2.1.5. The combinatorial structure of log twisted curves. Given a log twisted curve $(\pi : \mathcal{C} \rightarrow S, \{\sigma_i\}_{i=1}^n)$ over $S$, we have an induced morphism of sheaves of monoids $\bar{\pi}^\flat : \pi^* \mathcal{M}_S \rightarrow \mathcal{M}_\mathcal{C}$ on $\mathcal{C}$. The structure of $\bar{\pi}^\flat$ is similar to the case of log curves without twists which we described below.

The morphism $\bar{\pi}^\flat$ is an isomorphism away from nodes and marked points.

At a marked point $p \rightarrow \mathcal{C}$ with $s = \pi(p)$, we have the stalk $\mathcal{M}_{\mathcal{C},p} \cong \pi^* \mathcal{M}_{S,s} \oplus \mathbb{N}$, and the morphism $\bar{\pi}^\flat_p : \pi^* \mathcal{M}_{S,s} \rightarrow \pi^* \mathcal{M}_{S,s} \oplus \mathbb{N}$ is the inclusion to the first factor.

At a node $q \rightarrow \mathcal{C}$ with the geometric point $\bar{q} \rightarrow q$ and image $\bar{s} = \pi(\bar{q})$, we have the stalk $\mathcal{M}_{\mathcal{C},\bar{q}} \cong \pi^* \mathcal{M}_{S,\bar{s}} \oplus \mathbb{N} \mathbb{N}^2$, where the direct sum is determined by a map
\[(3) \quad \mathbb{N} \rightarrow \mathcal{M}_{S,\bar{s}}, \quad 1 \mapsto \rho_{\bar{q}} \]
and the diagonal map $\mathbb{N} \rightarrow \mathbb{N}^2$. Indeed, the diagonal map is induced by the relation $t' = \tilde{x} \tilde{y}$ in the local chart (1). The two generators $(1,0), (0,1) \in \mathbb{N}^2$ correspond to the local coordinates $\tilde{x}, \tilde{y}$ of the two branches meeting at the node. The element $\rho_{\bar{q}}$ corresponds to the local section $t'$. The morphism $\bar{\pi}^\flat_{\bar{q}} : \pi^* \mathcal{M}_{S,\bar{s}} \rightarrow \pi^* \mathcal{M}_{S,\bar{s}} \oplus \mathbb{N} \mathbb{N}^2$ in this case is again the inclusion to the first factor.
2.1.6. The stack of log twisted curves. Denoted by $\mathcal{M}_{g,n}^{\text{tw}}$ the category of genus $g$ log twisted curves with $n$ marked points over the category of log schemes. By [43], the fibered category $\mathcal{M}_{g,n}^{\text{tw}}$ is represented by a log algebraic stack. Indeed, the underlying stack $\mathcal{M}_{g,n}^{\text{tw}}$ is the stack parameterizing twisted curves with the same discrete data. The boundary of $\mathcal{M}_{g,n}^{\text{tw}}$ parameterizing singular curves is a normal crossings divisor whose associated divisorial log structure defines the log structure of $\mathcal{M}_{g,n}^{\text{tw}}$.

2.1.7. Log stable maps with twisted source curves. We fix a log algebraic stack $Y$ as the target.

**Definition 2.1.** A log map to $Y$ over a fine and saturated log scheme $S$ consists of the following data $(\pi : C \to S, f : C \to Y)$ where $C \to S$ is a log twisted curve over $S$, and $f$ is a morphism of log stacks. The pull-back of a log map along an arbitrary morphism of log schemes is defined via the pull-back of log twisted curves as usual. Note that we do not require a log map to be representable.

When $Y$ is a separated log Deligne–Mumford stack, a log map is stable if the underlying twisted map is stable in the usual sense. In particular, a stable log map is representable.

For simplicity, we may write $f : C \to Y$ for a log map.

2.1.8. Deligne–Faltings targets of rank one. Throughout this paper, we will mainly focus on the following type of targets.

**Definition 2.2.** A log algebraic stack $Y$ is Deligne–Faltings type of rank one if there is a morphism of sheaves of monoids $N_Y \to \mathcal{M}_Y$ which locally lifts to a chart of $\mathcal{M}_Y$. Here $N_Y$ denotes the constant sheaf over $Y$ with fiber $\mathbb{N}$.

For a fine and saturated monoid $P$, denote by $\mathcal{A}_P$ the log algebraic stack with the underlying stack $[\text{Spec}(k[P])\!/\text{Spec}(k[P^{gp}])]$ and the log structure induced by the affine toric variety $\text{Spec}(k[P])$.

For simplicity, denote by $\mathcal{A} := \mathcal{A}_\mathbb{N}$. Let $\infty_\mathcal{A} \subset \mathcal{A}$ be the boundary divisor associated to the log structure $\mathcal{M}_{\mathcal{A}}$. The log stack $\mathcal{A}$ has the following universal property that if $Y$ is Deligne–Faltings type of rank one, then there is a strict morphism $Y \to \mathcal{A}$.

2.2. The combinatorial structure of twisted log maps. The combinatorial structure of log maps with twisted source curves is similar to the case without twists as in [25, 2, 15]. We introduce it following [3]. For our purpose, we assume $Y$ is Deligne–Faltings type of rank one.
2.2.1. The induced morphism of sheaves of monoids. Let \((\pi : C \to S, f : C \to Y)\) be a log map over \(S\). First consider the case where \(S\) is a geometric point with \(\mathcal{M}_S = Q\). Denote by \(\mathcal{M} := f^*\mathcal{M}_Y\). Thus, \(\mathcal{M}\) is a Deligne–Faltings log structure on \(C\) of rank one. This leads to a pair of morphisms of sheaves of monoids
\[(\bar{\pi}^\flat : Q \to \mathcal{M}_C, \bar{f}^\flat : \mathcal{M} \to \mathcal{M}_C)\]
where we view \(Q\) as the constant sheaf of monoids on \(C\). The morphism \(\bar{\pi}^\flat\) is described in Section 2.1.5. We describe the behavior of \(\bar{f}^\flat\) at generic points, marked points, and nodes of \(C\) as follows.

2.2.2. The stalks of \(\mathcal{M}\). Since \(\mathcal{M}\) is Deligne–Faltings type of rank one, for any point \(s \to C\) the sheaf \(\mathcal{M}_s\) is a constant sheaf of monoids with fiber either \(N\) or the trivial one \(\{0\}\).

2.2.3. The structure of \(\bar{f}^\flat\) at generic points. If \(s = \eta\) is a generic point, then we have a local morphism of monoids \(\bar{f}^\flat_{\eta} : \mathcal{M}_\eta \to Q\).

If \(\mathcal{M}_\eta = N\), we call the irreducible component \(Z \subset C\) containing \(\eta\) a degenerate component, and we call the element \(e_Z := \bar{f}^\flat_{\eta}(1) \in Q\) the degeneracy of \(f\) along \(Z\).

If \(\mathcal{M}_\eta = \{0\}\), we call the irreducible component \(Z \subset C\) containing \(\eta\) a non-degenerate component. The degeneracy of a non-degenerate component \(Z\) is defined to be \(e_Z = 0 \in Q\).

2.2.4. The structure of \(\bar{f}^\flat\) at marked points. If \(s = p \to \sigma_i\) is a marked point, then we have a local morphism of monoids \(\bar{f}^\flat_p : \mathcal{M}_p \to Q \oplus N\). Consider the composition
\[c_p : \mathcal{M}_p \xrightarrow{\bar{f}^\flat_p} Q \oplus N \xrightarrow{pr_2} N\]
If \(\mathcal{M}_p = N\), the morphism \(c_p\) is determined by \(c_p(1) \in N\). We call \(c_p\) or equivalently \(c_p(1)\) the contact order at \(p\). The marked point \(p\) has the trivial contact order if \(c_p(1) = 0\).

Let \(\eta\) be the generic point of the irreducible component \(Z\) containing \(p\), and assume that \(Z\) is degenerate. Since the generalization morphism \(\chi_{\eta,p} : Q \oplus N \to Q\) is just the projection to the first factor, we obtain
\[\bar{f}^\flat : N \to Q \oplus N, \quad 1 \mapsto e_Z + c_p(1) \cdot (0, 1)\]

2.2.5. The structure of \(\bar{f}^\flat\) at nodal points. Suppose \(s = q \to C\) is a nodal point contained in the closures of two generic points \(\eta_1, \eta_2\) of the two branches meeting at \(q\). Using the description of nodes in Section 2.1.5, we have a local morphism
\[\bar{f}^\flat_q : \mathcal{M}_q \to Q \oplus N^2\]
Let \((1, 0), (0, 1) \in N^2\) correspond to the two local coordinates around \(q\) of the two branches of \(\eta_1\) and \(\eta_2\) respectively.

\[\text{A morphism of monoids } h : P \to Q \text{ is local if } h^{-1}(Q^\times) = P^\times.\]
If $\mathcal{M}_q = \mathbb{N}$, choosing labeling carefully, we may assume that

$$f^\natural_q(1) = e + c_q \cdot (1, 0)$$

for some $c_q \in \mathbb{N}$ and $e \in \mathbb{Q}$. We call $c_q$ the contact order of the node $q$. Observe the following commutative diagram

$$\begin{array}{c}
\mathcal{M}_q \\
\downarrow \\
\mathcal{M}_{\eta_i}
\end{array} \xrightarrow{p_{\eta_i}} \begin{array}{c}
\mathcal{M}_{c_q} \\
\downarrow \\
\mathcal{M}_{c,\eta_i}
\end{array}$$

where the vertical arrows are the generalization morphism. Applying the commutativity of the above diagram with $i = 1$ to (4), we obtain that

$$f^\natural_q(1) = e_{Z_1} + c_q \cdot (1, 0)$$

where $e_{Z_1}$ is the degeneracy of the irreducible component $Z_1$ containing $\eta_1$. Using $i = 2$, we have

$$e_{Z_1} + c_q \rho_q = e_{Z_2}$$

where $e_{Z_2}$ is the degeneracy of the irreducible component $Z_2$ containing $\eta_1$. This is the nodal equation as in [15, (3.3.2)].

If $\mathcal{M}_q = \{0\}$, then $f^\natural_q$ is necessarily trivial, and $c_q = 0$. Since the commutativity of (5) holds in this case as well, taking generalization, we obtain $e_{Z_1} = e_{Z_2} = 0$. In particular, Equation (7) holds for all nodes.

2.2.6. The natural partial ordering. For a twisted curve $C$ over a geometric point, recall that its dual intersection graph $\mathcal{G}$ consisting of the set of vertices $V(\mathcal{G})$ corresponding to irreducible components, the set of edges $E(\mathcal{G})$ corresponding to nodes, and the set of half-edges $L(\mathcal{G})$ corresponding to marked points.

Let $q \to C$ map to a node joining two irreducible components $Z_1, Z_2$ with Equation (7). We introduce the partial ordering $\preceq$ as follows:

1. If $c_q > 0$, we write $v_1 \preceq v_2$
2. If $c_q = 0$, we write $v_1 \preceq v_2$ and $v_2 \preceq v_1$, or equivalently $v_1 \sim v_2$.

Then $\preceq$ extends to a partial order on the set $V(\mathcal{G})$, called the minimal partial order.

The minimal partial order yields an orientation of $\mathcal{G}$ as follows. Let $l \in E(\mathcal{G})$ be the corresponding edge joining vertices $v_1, v_2$ associated to $Z_1, Z_2$ respectively. The edge $l$ is oriented from $v_1$ to $v_2$ if $v_1 \preceq v_2$, and is oriented both ways if $v_1 \sim v_2$. 

2.2.7. The logarithmic combinatorial type. We introduce the log combinatorial type of the log map \((C \to S, f : C \to Y)\) over a geometric point \(S\) following [15, 3.4] and [2, 4.1.1]:

\[
G = (G, V(G) = V^a(G) \cup V^d(G), \preceq, (c_i)_{i \in L(G)}, (c_l)_{l \in E(G)})
\]

where
(a) \(G\) is the dual intersection graph of the underlying curve \(\overline{C}\).
(b) \(V^a(G) \cup V^d(G)\) is a partition of \(V(G)\) where \(V^d(G)\) consists of vertices of degenerate components.
(c) \(\preceq\) is the minimal partial order defined in Section 2.2.6.
(d) Associate to a leg \(i \in L(G)\) the contact order \(c_i \in \mathbb{N}\) of the corresponding marking \(\sigma_i\).
(e) Associate to an edge \(l \in E(G)\) the contact order \(c_l \in \mathbb{N}\) of the corresponding node.

Remark 2.3. Our definition of log combinatorial types is similar to the definition of types in [25, 1.10] and [3, 2.3.7]. Since we work with Deligne–Faltings type targets, we are able to include more combinatorial information such as the partition and partial order on \(G\).

These combinatorial data behave well under generalization:

Proposition 2.4. Let \(f : C \to Y\) be a log map over an arbitrary log scheme \(S\). Then

1. The contact order \(c_i\) along the \(i\)th marking \(\sigma_i\) is a constant over each connected components of \(S\).
2. Let \(W \subset C\) be a connected locus of nodes in \(C\). Then the contact order of the nodes is constant along \(W\).

Proof. The proof is identical to the case of [15, Lemma 3.2.4, 3.2.9]. □

2.3. Minimality.

2.3.1. The monoid. We recall the construction of minimal monoids in [15, 2, 25]. Consider a log map \((C \to S, f : C \to Y)\) over a geometric point \(S\) with the log combinatorial type \(G\). We introduce a variable \(\rho_l\) for each edge \(l \in E(G)\), and a variable \(e_v\) for each vertex \(v \in V(G)\). Denote by \(h_l\) the relation \(e_{v'} = e_v + c_l \cdot \rho_l\) for each edge \(l\) with the two ends \(v \preceq v'\) and contact order \(c_l\). Denote by \(h_v\) the following relation \(e_v = 0\) for each \(v \in V^a(G)\). Consider the following abelian group

\[
G = \bigoplus_{v \in V(G)} \mathbb{Z}e_v \bigoplus_{l \in E(G)} \mathbb{Z}\rho_l) / \langle h_v, h_l \mid v \in V^d(G), l \in E(G) \rangle
\]

Let \(G' \subset G\) the torsion subgroup. Consider the following composition

\[
(\bigoplus_{v \in V(G)} \mathbb{N}e_v \bigoplus_{l \in E(G)} \mathbb{N}\rho_l) \to G \to G/G'
\]
Let $\overline{M}(G)$ be the smallest submonoid that is saturated in $G/G^t$, and contains the image of the above composition. We call $\overline{M}(G)$ the minimal monoid associated to $G$, or associated to the log map.\footnote{The monoid $\overline{M}(G)$ is called the basic monoid in \cite{15}.}

**Proposition 2.5.** There is a canonical map of monoids $\phi: \overline{M}(G) \rightarrow \overline{M}(G)$ induced by sending $e_v$ to the degeneracy of the component associated to $v$, and sending $\rho_l$ to the element $\rho_q$ as in Equation (3) associated to $l$. In particular, the monoid $\overline{M}(G)$ is fine, saturated, and sharp.

**Proof.** This follows from the same proof of \cite{15}, Proposition 3.4.2. \hfill \Box

For later use, we observe the following.

**Corollary 2.6.** There is a canonical splitting $\overline{M}(G) = \overline{M}(G)' \oplus \mathbb{N}^d$ where $d$ is the number of edges in $E(G)$ whose contact orders are trivial. In particular, the image of the element $e_v$ is contained in $\overline{M}(G)'$ for all $v \in V(G)$.

**Proof.** Note that when $c_l = 0$, the element $\rho_l$ is not involved in the relation $h_l$. The collection of such $\rho_l$ generates the factor $\mathbb{N}^d$. \hfill \Box

2.3.2. **Minimal objects.** Same as in \cite{25, 15, 2}, we define the minimal objects using the canonical morphism $\phi$.

**Definition 2.7.** A log map $(\mathcal{C} \rightarrow S, f: \mathcal{C} \rightarrow Y)$ over $S$ is called minimal\footnote{The terminology used in \cite{25} is basic.} if for each of its geometric fibers, the induced canonical morphism as in Proposition 2.5 is an isomorphism.

The definition is justified by the openness of minimality.

**Proposition 2.8.** For any family of log maps $(\mathcal{C} \rightarrow S, f: \mathcal{C} \rightarrow Y)$ over a log scheme $S$, if the fiber $f_{\bar{s}}: \mathcal{C}_{\bar{s}} \rightarrow Y$ over a geometric point $\bar{s} \rightarrow S$ is minimal, then there is an étale neighborhood $U \rightarrow S$ of $\bar{s}$ such that the fiber $f_U: \mathcal{C}_U \rightarrow Y$ is minimal.

**Proof.** This follows from the same proof of \cite{15}, Proposition 3.5.2 and \cite{25}, Proposition 1.22. \hfill \Box

Minimal objects have the following universal property which is the key to the construction of the moduli stack.

**Proposition 2.9.** For any log map $f: \mathcal{C} \rightarrow Y$ over a log scheme $S$, there exists a minimal log map $f_m: \mathcal{C}_m \rightarrow Y$ over $S_m$ and a morphism of log schemes $\Phi: S \rightarrow S_m$ such that

1. The underlying morphism $\Phi$ is an isomorphism.
2. $f: \mathcal{C} \rightarrow Y$ is the pull-back of $f_m: \mathcal{C}_m \rightarrow Y$ along $\Phi$.

Furthermore, the pair $(f_m, \Phi)$ is unique up to a unique isomorphism.

**Proof.** The proof is identical to the situation of log maps with no orbifold twists on the source curves. We refer to \cite{25}, Proposition 1.24 and \cite{15}, Proposition 4.1.1 for the details. \hfill \Box
2.3.3. Finiteness of automorphisms. Let \( f: \mathcal{C} \to Y \) be a log map over \( S \) with \( S \) a geometric point. An automorphism of a stable log map is a pair \( (\psi: \mathcal{C} \to \mathcal{C}, \theta: S \to S) \) of compatible automorphisms of log schemes such that \( \psi \circ f = f \). Denote by \( \text{Aut}(f) \) the automorphism group of the log map \( f \), and \( \text{Aut}(f) \) the automorphism group of the corresponding underlying map. We have the following property:

**Proposition 2.10.** Suppose the log map \( f: \mathcal{C} \to Y \) over \( S \) is stable and minimal. Then the natural group morphism \( \text{Aut}(f) \to \text{Aut}(f\upharpoonright) \) is injective. In particular, the group \( \text{Aut}(f) \) is finite.

**Proof.** The proof is identical to the case of \([25, \text{Proposition 1.25}]\) and \([15, \text{Lemma 3.8.3}]\). \( \Box \)

2.4. The algebraicity of the stacks of twisted log maps.

2.4.1. The set-up and statement. Fix a separated log Deligne–Mumford stack \( Y \) as the target with \( \mathcal{M}_Y \) of Deligne–Faltings type of rank one.

Consider the discrete data

\[
\beta = (g, n, c = \{c_i\}_{i=1}^n, A)
\]

for twisted log maps in \( Y \) where \( g \) is the genus, \( n \) is the number of markings, \( c_i \) is the contact order of the \( i \)-th marking, and \( A \in H_2(Y) \) is a curve class.

Let \( \beta' = (g, n, c) \) be the reduced discrete data obtained by removing the curve class, and \( \beta = (g, n, A) \) the underlying discrete data by removing the contact orders.

Denote by \( \mathcal{M}(Y, \beta) \) the category of stable log maps to \( Y \) with the discrete data \( \beta \) fibered over the category of log schemes, and \( \mathcal{M}(Y, \beta) \) the stack of usual twisted stable maps to \( Y \). For our purposes, we view \( \mathcal{M}(Y, \beta) \) as a log stack equipped with the canonical log structure given by its universal curves. Composing with the forgetful morphism \( Y \to Y \), we obtain a canonical morphism

\[
\mathcal{M}(Y, \beta) \to \mathcal{M}(Y, \beta).
\]

We first establish the algebraicity following the method of \([8, 51]\).

**Theorem 2.11.** The morphism (10) is representable by log Deligne–Mumford stacks locally of finite type.

2.4.2. Reduction to the universal stack. Following the universal target strategy of Abramovich–Wise \([8]\), consider the canonical strict morphism \( Y \to \mathcal{A} \). For any log map \( f: \mathcal{C} \to Y \) over \( W \), the composition \( \mathcal{C} \to Y \to \mathcal{A} \) is a log map to \( \mathcal{A} \) over \( W \).

Denote by \( \mathfrak{M}(\mathcal{A}, \beta') \) the category of log maps to \( \mathcal{A} \) with the reduced discrete data \( \beta' \). The above composition defines a canonical morphism

\[
\mathfrak{M}(\mathcal{Y}, \beta) \to \mathfrak{M}(\mathcal{A}, \beta').
\]
On the other hand, consider the stack $\mathcal{M}_{g,n}(A)$ parameterizing (not necessarily representable) usual maps to $A$ from genus $g$, $n$-marked log twisted curves. It is an algebraic stack locally of finite type by [28, Theorem 1.2]. We further view $\mathcal{M}_{g,n}(A)$ as a log stack equipped with the canonical log structure induced by its universal twisted curve. We have:

**Proposition 2.12.** The canonical morphism

$$\mathcal{M}(A,\beta') \to \mathcal{M}_{g,n}(A),$$

induced by the forgetful morphism $A \to \overline{A}$ is representable by log Deligne–Mumford stacks locally of finite type. In particular, the fibered category $\mathcal{M}(A,\beta')$ is representable by log algebraic stacks locally of finite type.

**Proof of Theorem 2.11.** The underlying map $\underline{Y} \to A$ of $Y \to A$ induces a strict morphism of log stacks

$$\mathcal{M}(\underline{Y},\beta) \to \mathcal{M}_{g,n}(A),$$

where both stacks are equipped with the canonical log structures from their universal curves. The two morphisms (10) and (11) induce

$$\mathcal{M}(Y,\beta) \to \mathcal{M}(\underline{Y},\beta) \times_{\mathcal{M}_{g,n}(A)} \mathcal{M}(A,\beta'),$$

where the fiber product is in the fine and saturated category. The above morphism is an isomorphism. Indeed, the datum of a log map to $Y$ is equivalent to the datum of an underlying map to $\underline{Y}$ and a log map to $A$ with compatible compositions to $\overline{A}$. Thus, the algebraicity of Theorem 2.11 follows from Proposition 2.12. The Deligne–Mumford property is a consequence of Proposition 2.10. □

2.4.3. **Proof of Proposition 2.12.** The proof is essentially the one in [51, Corollary 1.1.1]. Here we record the details for completeness.

Let $U \to \mathcal{M}_{g,n}(A)$ be a strict morphism from a log scheme $U$. We will show that the product

$$\mathcal{U} := \mathcal{M}(A,\beta') \times_{\mathcal{M}_{g,n}(A)} U$$

is a log algebraic stack locally of finite type.

Denote by $V \subset \text{Log}_{U}$ the open substack of Olsson’s log stack that parameterizes morphisms of fine and saturated log structures $\mathcal{M}_{U} \to \mathcal{M}'$, see [41]. Thus $V$ is a log algebraic stack locally of finite type with the tautological morphism $V \to U$.

The composition $V \to U \to \mathcal{M}_{g,n}(A)$ defines a family of underlying twisted maps over $\underline{V}$, denoted by $f: \underline{C} \to \underline{A}$. Since $\mathcal{M}_{g,n}(A)$ carries the canonical log structure from its universal curve, pulling back the universal curve with its canonical log structure, we obtain a log curve $\pi: C \to V$. 
Write $M := f^*M_A$. Observe that to define a log map $f: C \to A$ compatible with the underlying morphism $f$ is equivalent to defining a morphism of log structures $f^*: M \to M_C$. Consider the stack $\text{Hom}_{\text{Sch}/C}(M, M_C)$ over $C$ which to each $C$-scheme $T$, associates the category of morphisms $M|_T \to M|_C|_T$, where $*|_T$ denotes the pull-back of $*$ to $T$. By [25, Proposition 2.9] and [51, Theorem 2.2], the morphism $\text{Hom}_{\text{Sch}/C}(M, M_C) \to C$ is representable by algebraic spaces, and is quasi-compact, quasi-separated, and locally of finite presentation.

Consider the push-forward $V := \pi_*\text{Hom}_{\text{Sch}/C}(M, M_C)$ along the underlying morphism of $\pi: C \to V$. By [28, Theorem 1.2], the stack $V$ is an algebraic stack over $V$ locally of finite type. Denote by $V \to V$ the strict morphism with underlying map $V \to V$. For each strict morphism $T \to V$, by construction we obtain a log curve $C_T \to T$ by pulling back $C \to V$, and a morphism of log structures $M|_{C_T} \to M|_{C_T}$, hence a log map $f_T: C_T \to A$. By Proposition 2.8 and 2.9, $U \subset V$ is the open substack parameterizing minimal objects. This completes the proof of Proposition 2.12.

### 2.4.4. Log smoothness of the universal stack.

The following log smoothness result of [8, Proposition 3.2] will be useful in our later construction.

**Proposition 2.13.** The tautological morphism $M(A, \beta') \to M_{g,n}$ by taking the source log curves, is log étale. In particular, the stack $M(A, \beta')$ is log étale and equi-dimensional.

**Proof.** Observe that $A$ is log étale over $\text{Spec} \mathbb{C}$. Furthermore, we may view $M_{g,n}^{tw}$ as the stack of pre-stable log maps to $\text{Spec} \mathbb{C}$. The result follows from the same proof of [8, Proposition 3.2], since the orbifold structures on source curves play no role. □

### 2.5. Relative boundedness of twisted log maps.

The boundedness of stable log maps without orbifold structures has been proved in [15, 2, 25] under certain assumptions. The general situation is proved in [4] by reducing to the case of [2]. For our purposes, we will only consider the Deligne–Faltings case of rank one in the orbifold situation.

Consider the forgetful morphism of log algebraic stacks $F: M(Y, \beta) \to \mathcal{M}(Y, \beta)$ where $\mathcal{M}(Y, \beta)$ has the canonical log structure from its universal curve. For each strict morphism $W \to M(Y, \beta)$, consider the projection $F_W: M(Y, \beta)_W := M(Y, \beta) \times_{\mathcal{M}(Y, \beta)} W \to W$

**Definition 2.14.** For a strict morphism $W \to \mathcal{M}(Y, \beta)$, the discrete data $\beta$ is called **combinatorially finite over** $W$ if the collection of log combinatorial types of log maps over geometric points of $M(Y, \beta)_W$ is finite.
Remark 2.15. Observe that if $W = \mathcal{M}(\underline{Y}, \beta)$ then $\mathcal{M}(Y, \beta)_W = \mathcal{M}(Y, \beta)$. Thus the above definition is compatible with combinatorially finiteness in [25, Definition 3.3].

We provide a relative boundedness result for later use.

**Proposition 2.16.** Suppose $\beta$ is combinatorially finite over $W$ for a strict morphism $W \to \mathcal{M}(\underline{Y}, \beta)$. Then $F_W$ is of finite type.

*Proof.* The proof of the statement is similar to the case of [15, Section 5.4] and [25, Section 3.2]. For completeness, we give the details with necessary modifications to fit our situation.

First observe that the statement is local on $W$. Thus we may assume $W$ is a scheme of finite type, and we are allowed to shrink $W$ if needed. Since the morphism $F$ is locally of finite type, it suffices to show that the underlying topological space of $\mathcal{M}(Y, \beta)_W$ is quasi-compact.

**Step 1: Reduction to combinatorially constant strata.**

As $\beta$ is combinatorially finite over $W$, $\mathcal{M}(Y, \beta)_W$ has finitely many strata such that the geometric fibers of each stratum have the same log combinatorial type. It remains to show that the underlying topological space of each such stratum is quasi-compact.

**Step 2: Construct combinatorially constant underlying strata.**

We shrink and stratify $W$ such that over each stratum:

1. The dual graphs of the underlying curves are constant, say $G$.
2. The partition $V(G) = V^n(G) \cup V^d(G)$ where $V^d(G)$ consists of components with images contained in the locus of $Y$ with non-trivial log structure, is constant.

Replace $W$ by a such stratum with the reduced scheme structure. Denote by $G$ and $V(G) = V^n(G) \cup V^d(G)$ the corresponding dual graph and partition over $W$. Let $G$ be a log combinatorial type of the fibers over $\mathcal{M}(Y, \beta)_W$.

**Step 3: Construct source log curve.**

Let $Q_1 = \mathbb{N}^n$ where $n$ is the number of edges in $E(G)$. Let $Q_2 = \overline{\mathcal{M}}(G)$ and consider the canonical map $\psi: Q_1 \to Q_2$ induced by the edge element $\rho_i$ as in Section 2.3.1. It induces a morphism of log stacks $A_{Q_2} \to A_{Q_1}$. Consider the fiber product $S = A_{Q_2} \times_{A_{Q_1}} W$ where $W \to A_{Q_1}$ is the canonical strict morphism. Pulling back the families over $W$, we obtain a log curve $\pi: C \to S$ and a usual pre-stable map $f: C \to \underline{Y}$.

**Step 4: The morphism of characteristic sheaves.**

Denote by $\mathcal{M}' := f^* \mathcal{M}_Y$. The log combinatorial type $G$ determines a unique morphism of characteristic sheaves $\overline{f^*}: \overline{\mathcal{M}}' \to \overline{\mathcal{M}}_c$ by the descriptions in Section 2.2.3, 2.2.4, and 2.2.5. Lifting $\overline{f^*}$ to a log map with the combinatorial type $G$ is equivalent to constructing a dashed
arrow making the following diagram commutative

\[
\begin{array}{ccc}
\mathcal{M}' & \xrightarrow{f^0} & \mathcal{M}_C \\
p_{\mathcal{M}'} & & p_{\mathcal{M}_C} \\
\mathcal{M} & \xrightarrow{f} & \mathcal{M}_C
\end{array}
\]

where the two vertical arrows are quotients by \(O_L^*\).

**Step 5: Parameterize morphisms of \(O^*\)-torsors.**

We first parameterize lifting \(f^0\) of \(\tilde{f}^0\) as a morphism of sheaves of monoids compatible with the \(O^*\)-action. Since \(Y\) is of Deligne–Faltings type of rank one, there is a global morphism \(h: \mathbb{N} \to \mathcal{M}\) that locally lifts to a chart. Consider the two \(O^*\)-torsors \(T_Y = q_{\mathcal{M}'}^{-1}(h(1))\) and \(T_C = q_{\mathcal{M}_C}^{-1}(\tilde{f}^0 \circ h(1))\). Observe that for any lifting \(f^0\), its restriction \(f^0|_{T_Y}\) factors through \(T_C\), and uniquely determines \(f^0\).

Consider the sheaf of morphisms of \(O^*\)-torsors \(I := \text{Isom}_C(T_Y, T_C)\) which is again an \(O^*\)-torsor. Let \(L\) be the corresponding line bundle. Using Grothendieck duality for Deligne–Mumford stacks (see for example [40]), the proof of [12, Proposition 2.2] shows that the push-forward \(\pi_* I\) parameterizes sections of \(L\) avoiding the zero section of \(L\). Thus it is an open sub-scheme of \(\pi_* L\) over \(S\). In particular, it is of finite type.

**Step 6: Construct logarithmic lifting.**

Note that \(\pi_* I\) associates, to each strict morphism \(T \to S\), the category of isomorphisms \(T_Y|_{C_T} \to T_C|_{C_T}\) where \(C_T = C \times_S T\). Recall from the above discussion that such isomorphisms is equivalent to a morphism of sheaves of monoids \(f_T^0: \mathcal{M}'_T \to \mathcal{M}_C|_{C_T}\) compatible with the \(O^*\)-action where \(\mathcal{M}'_T\) is the pullback of \(\mathcal{M}'\) along \(T \to S\). Such \(f_T^0\) is a morphism of log structures if it is compatible with the structure morphisms. This same proof as in [15, Lemma 5.4.3] implies that the locus of \(\pi_* I\) with such compatibility forms a closed subset \(V\). For our purposes, we may take \(V\) with the reduced scheme structure, hence it is of finite type.

Finally observe that the universal morphism \(f_V^0: \mathcal{M}'_V \to \mathcal{M}_C\) over \(V\) defines a family of minimal log maps \(f_V: \mathcal{C}_V \to Y\) over \(V\) with the constant log combinatorial type \(G\). The tautological morphism \(V \to \mathfrak{M}(Y, \beta)_W\) surjects onto the locus with the log combinatorial type \(G\). The boundedness follows since \(V\) is of finite type.

2.6. **The weak valuative criterion.** We fix a discrete valuation ring \(R\) with the maximal ideal \(m\) and the residue field \(R/m\). Let \(K\) be the quotient field of \(R\). We have the following version of valuative criterion necessary for properness.

\[\square\]
Proposition 2.17. Consider a commutative diagram of solid arrows of underlying stacks
\[
\begin{array}{ccc}
\text{Spec } K & \longrightarrow & \mathcal{M}(Y, \beta) \\
\downarrow & & \downarrow \Phi \\
\text{Spec } R & \longrightarrow & \mathcal{M}(Y, \beta)
\end{array}
\]

Possibly after replacing $R$ by a finite extension of DVRs, and $K$ by the induced extension of the quotient field, there exists a dashed arrow making the above diagram commutative. Furthermore, such a dashed arrow is unique up to a unique isomorphism.

Proof. The statement above can be proved identically as in [25, 15]. As the proof is quite long, we will only sketch the idea, mainly following [25], and will refer to the corresponding sections of [25, 15] for details.

Denote by $f_K: \mathcal{C}_K \to Y$ over $\eta = \text{Spec } K$. Similarly, let $f_R: \mathcal{C}_R \to Y$ over $S = \text{Spec } R$ be the usual pre-stable map induced by the bottom arrow. The commutativity of the square of the solid arrows is equivalent to the condition that the restriction $f_R|_{\eta}$ is the underlying map $f_K$. To construct the dashed arrow, we need to extend the minimal log map $f_K$ to the whole $S$ with the underlying map given by $f_R$. We need to show that possibly after a finite extension, such an extension exists and is unique.

The first step is to extend the log combinatorial type to the closed fiber over $\text{Spec } R/m$. Following Section 2.2.7, the dual graph $G$ and the partition $V^n(G) \cup V^d(G)$ is determined by the closed fiber of the underlying map $f_R$. The contact orders at the markings are determined by the discrete data $\beta$. It suffices to determine the contact order at each node of the closed fiber which will also determine the partial order $\preceq$ by the discussion in Section 2.2.6. Now the contact orders nodes can be determined étale locally around each node by the same argument as in [15, Section 6.2] or [25, Section 4.1]. This defines a unique log combinatorial type $G$.

The second step is to construct a unique log scheme $S$ with the underlying $\text{Spec } R$ extending the log scheme $\eta$ such that the fiber $\mathcal{M}_S|_{\text{Spec } R/m} = \mathcal{M}(G)$. This step can be carried out identically as in [25, Section 4.2], since it only uses the complement of markings and nodes, and orbifold structures play no role. We then obtain a unique morphism $S_R \to S$ where $C_R \to S_R$ is the canonical log curve associated to the underlying curve $\mathcal{C}_R \to S$. Thus pulling back the canonical log curve, we obtain a unique log curve $C \to S$.

Finally, to extend $f_K$ to the closed fiber, one needs to construct a morphism $f_R: C \to Y$ lifting $f_R$. This can be done using the same argument as in [25, Section 4.3] or a similar argument as in [15, Section...
6.3] by first constructing the log map étale locally on \( C \), then gluing them using the canonicity of the construction.

\[ \square \]

3. Stable log maps with uniform maximal degeneracy

In this section, we introduce a configuration of log structures which is the key to the construction of the reduced perfect obstruction theory, and subsequently Witten’s r-spin class.

We again fix the target \( Y \) with the log structure \( \mathcal{M}_Y \) of rank one Deligne–Faltings type.

3.1. Uniformed maximal degeneracy.

3.1.1. Maximal degeneracies. Consider a log map \( f : C \to Y \) over \( S \) with \( S \) a geometric point. Denote by \( G \) the corresponding log combinatorial type of \( f \), and \( \overline{\mathcal{M}(G)} \) the minimal monoid. Let \( \phi : \overline{\mathcal{M}(G)} \to \mathcal{M}_S \) be the canonical morphism as in Proposition 2.5.

Consider the natural partial order \( \preceq_{\mathcal{M}_S} \) on \( \mathcal{M}_S \) such that \( e_1 \preceq e_2 \) iff \( (e_2 - e_1) \in \mathcal{M}_S \). The partial order \( \preceq_{\overline{\mathcal{M}_S}} \) is a refinement of \( \preceq \) of \( G \) in the sense that \( v_1 \preceq v_2 \) in \( V(G) \) implies \( e_{v_1} \preceq e_{v_2} \) in \( \mathcal{M}_S \).

**Definition 3.1.** A degeneracy \( \phi(e_v) \in \overline{\mathcal{M}_S} \) is called maximal if \( \phi(e_v) \) is maximal in the set of all degeneracies under \( \preceq_{\overline{\mathcal{M}_S}} \). The corresponding vertex \( v \in V(G) \) is called a maximally degenerate vertex of \( f \).

As \( \preceq_{\overline{\mathcal{M}_S}} \) is a partial order, there could be more than one maximal degeneracy in \( \overline{\mathcal{M}_S} \). On the other hand, different vertices are allowed to have the same degeneracy in \( \overline{\mathcal{M}_S} \).

**Definition 3.2.** The log map \( f : C \to Y \) over \( S \) is said to have uniform maximal degeneracy if the set of degeneracies has a supremum under \( \preceq_{\overline{\mathcal{M}_S}} \). A family of log maps is said to have uniform maximal degeneracy if each geometric fiber has uniform maximal degeneracy.

The above definition for families is justified by the following.

**Proposition 3.3.** For any family of log maps \( f : C \to Y \) over a log scheme \( S \), if the fiber \( f_s : C_s \to Y \) over a geometric point \( s \to S \) has uniform maximal degeneracy, then there is an étale neighborhood \( U \to S \) of \( s \) such that the pull-back family \( f_U : C_U \to Y \) over \( U \) has uniform maximal degeneracy.

Proposition 3.3 follows immediately from Lemma 3.4 and 3.6 below.

3.1.2. Generalization of degeneracies and partial orders. Consider a pre-stable log map \( f : C \to Y \) over a log scheme \( S \) together with a chart \( h : \overline{\mathcal{M}_{S,s}} \to \mathcal{M}_S \) where \( s \to S \) is a geometric point. Here \( f \) does not necessarily have uniform maximal degeneracy. Using the composition \( \overline{\mathcal{M}_{S,s}} \to \mathcal{M}_S \to \overline{\mathcal{M}_S} \), any \( e \in \overline{\mathcal{M}_{S,s}} \) is viewed as a section of \( \overline{\mathcal{M}_S} \). Let \( e_t \in \overline{\mathcal{M}_{S,t}} \) be the fiber of \( e \) over \( t \in S \). Denote by \( G \) the log combinatorial type of \( f_s \).
Lemma 3.4. Notations as above, suppose $e \in \overline{M}_{S,s}$ is the degeneracy of $v \in V(G)$. Then there is an étale neighborhood $U \to S$ of $s$ such that for any geometric point $t \in U$, the fiber $e_t \in \overline{M}_{S,t}$ is a degeneracy.

Proof. Shrinking $S$ if necessary, we may choose a section $\sigma : S \to C$ such that $\sigma(S)$ is contained in the smooth non-marked locus of $C \to S$, and intersects the component of $C_s$ corresponding to $v$. Consider the pull-back morphism

$$\sigma^*(\bar{f}) : (\sigma \circ f)^*\overline{M}_Y \to \sigma^*\overline{M}_C = \overline{M}_S.$$ 

The equality on the right hand side follows from the assumption that $\sigma(S)$ avoids all nodes and markings.

Since $\overline{M}_Y$ is of Deligne–Faltings type of rank one, we may choose a morphism $N \to \overline{M}_Y$ which locally lifts to a chart. Denote again by $1 \in \overline{M}_Y$ the image of $1 \in \mathbb{N}$ via this morphism. By the discussion in Section 2.2.3, the fiber of the image $\sigma^*(\bar{f})(1)_t \in \overline{M}_{S,t}$ over each geometric point $t \in S$ is the degeneracy of the component of $C_t$ intersecting $\sigma(S)$. In particular, we have $\sigma^*(\bar{f})(1)_s = e$. □

Conversely, every degeneracy of a nearby fiber is the generalization of some degeneracy from the central fiber:

Corollary 3.5. Notations as above, there is an étale neighborhood $U \to S$ of $s$ such that for any geometric point $t \in U$ and any degeneracy $e' \in \overline{M}_{S,t}$, there is a degeneracy $e \in \overline{M}_{S,s}$ such that $e_t = e'$.

Proof. Notations as in the proof of Lemma 3.4, we may further shrink $S$ and choose a finite set of extra markings $\{\sigma_i \to C\}$ avoiding nodes and the original markings, whose union $\bigcup\sigma_i(S)$ intersects each irreducible component of each geometric fiber of $C \to S$. □

The partial order $\preceq_{\overline{M}_{S,t}}$ is well-behaved under generalization:

Lemma 3.6. Notations as above, consider a pair of elements $e_1, e_2 \in \overline{M}_{S,s}$ with $e_1 \preceq_{\overline{M}_{S,s}} e_2$. Then we have $e_{1,t} \preceq_{\overline{M}_{S,t}} e_{2,t}$ in $\overline{M}_{S,t}$ for any geometric point $t \to S$.

Proof. By assumption, we have $(e_2 - e_1) \in \overline{M}_{S,s}$, hence

$$(e_2 - e_1)_t = (e_{2,t} - e_{1,t}) \in \overline{M}_{S,t}.$$ 

□

3.1.3. The category of log maps with uniform maximal degeneracies. Let $Y$ be a log stack of rank one Deligne–Faltings type. We introduce the fibered category $\mathcal{U}(Y, \beta')$ of pre-stable log maps to $Y$ with uniform maximal degeneracy and reduced discrete data $\beta'$ over the category of fine and saturated log schemes. If furthermore $Y$ is a separated log Deligne–Mumford stack, denote by $\mathcal{U}(Y, \beta) \subset \mathcal{U}(Y, \beta')$ the sub-category of stable log maps with discrete data $\beta$ as in (9).
By the universality as in Proposition 2.9, there are tautological morphisms of fibered categories as inclusions of subcategories:

\[ \mathcal{U}(Y, \beta) \to \mathcal{M}(Y, \beta) \quad \text{and} \quad \mathfrak{U}(Y, \beta') \to \mathfrak{M}(Y, \beta'). \]

We next introduce the minimality of the subcategory \( \mathfrak{U}(Y, \beta'). \)

### 3.2. Minimality with uniform maximal degeneracy.

#### 3.2.1. Log combinatorial type with uniform maximal degeneracy.

Let \( f: C \to Y \) be a pre-stable log map over \( S \) with uniform maximal degeneracy. First assume that \( S \) is a geometric point.

Let \( G \) be the log combinatorial type of \( f \), and \( \phi: \mathcal{M}(G) \to \mathcal{M}_S \) be the canonical morphism. Denote by \( V_{\text{max}} \subset V(G) \) be the subset of vertices having the maximal degeneracy in \( \mathcal{M}_S \). We call \( (G, V_{\text{max}}) \) the log combinatorial type with uniform maximal degeneracy.

#### 3.2.2. Minimal monoids with uniform maximal degeneracy.

Consider the torsion-free abelian group \( (\mathcal{M}(G)^{gp}/\sim)^{tf} \) where \( \sim \) is given by the relations \( (e_{v_1} - e_{v_2}) = 0 \) for any \( v_1, v_2 \in V_{\text{max}} \).

By abuse of notation, we may use \( e_v \) for the image of the degeneracy of the vertex \( v \) in \( (\mathcal{M}(G)^{gp}/\sim)^{tf} \). Thus, for any \( v \in V_{\text{max}} \) their degeneracies in \( (\mathcal{M}(G)^{gp}/\sim)^{tf} \) are identical, denoted by \( e_{\text{max}} \).

Let \( \mathcal{M}(G, V_{\text{max}}) \) be the saturated submonoid in \( (\mathcal{M}(G)^{gp}/\sim)^{tf} \) generated by

1. the image of \( \mathcal{M}(G) \to (\mathcal{M}(G)^{gp}/\sim)^{tf} \), and
2. the elements \( (e_{\text{max}} - e_v) \) for any \( v \in V(G) \).

By the above construction, we obtain a natural morphism of monoids \( \mathcal{M}(G) \to \mathcal{M}(G, V_{\text{max}}) \). On the other hand, we have a canonical morphism of monoids \( \phi: \mathcal{M}(G) \to \mathcal{M}_S \) by Proposition 2.5. Putting these together, we observe the following canonical factorization:

**Proposition 3.7.** There is a canonical morphism of monoids

\[ \phi_{\text{max}}: \mathcal{M}(G, V_{\text{max}}) \to \mathcal{M}_S. \]

such that the morphism \( \phi: \mathcal{M}(G) \to \mathcal{M}_S \) factors through \( \phi_{\text{max}} \).

**Corollary 3.8.** There is a canonical splitting

\[ \mathcal{M}(G, V_{\text{max}}) = \mathcal{M}(G, V_{\text{max}})' \oplus \mathbb{N}^d \]

where \( d \) is the number of edges in \( E(G) \) whose contact order is zero. Furthermore, the image of the element \( e_v \) is contained in \( \mathcal{M}(G)' \) for all \( v \in V(G) \).

**Proof.** This follows directly from Corollary 2.6 and the construction of \( \mathcal{M}(G, V_{\text{max}}) \). \( \square \)
Definition 3.9. We call $\overline{M}(G, V_{\text{max}})$ the minimal monoid with uniform maximal degeneracy associated to $(G, V_{\text{max}})$, or simply the minimal monoid associated to $(G, V_{\text{max}})$.

Definition 3.10. A stable log map $f : C \to Y$ over $S$ with $S$ a geometric point is called minimal with uniform maximal degeneracy if (14) is an isomorphism. A family of log maps is called minimal with uniform maximal degeneracy if each of its geometric fibers is so.

3.2.3. Openness of minimality with uniform maximal degeneracy. The definition of minimal objects in families with uniform maximal degeneracy is justified by the following analogue of Proposition 2.8:

Proposition 3.11. For any family of log maps $f : C \to Y$ over a log scheme $S$, if the fiber $f_s : C_s \to Y$ over a geometric point $s \to S$ is minimal with uniform maximal degeneracy, then there is an étale neighborhood $U \to S$ of $s$ such that the family $f_U : C_U \to Y$ is minimal with uniform maximal degeneracy.

Proof. By Proposition 3.3, replacing $S$ by an étale neighborhood of $s$, we may assume that $f : C \to Y$ over $S$ has uniform maximal degeneracy. For each geometric point $t \in S$, denote by $(G_t, V_{\text{max},t})$ the log combinatorial type of the fiber $f_t : C_t \to Y$ over $t$, see Section 3.2.1.

Let $f_m : C_m \to Y$ over $S_m$ be the associated minimal objects as in Proposition 2.9 such that $f$ is the pull-back of $f_m$ along a morphism $S \to S_m$. Shrinking $S$ if necessary, we choose two charts $\overline{M}_{S,s} \to \overline{M}_S$ and $\overline{M}_{S_m,s} \to \overline{M}_{S_m}$. We view elements of $\overline{M}_{S,s}$ and $\overline{M}_{S_m,s}$ as global sections of $\overline{M}_S$ and $\overline{M}_{S_m}$ via the following compositions respectively:

\[
\overline{M}_{S,s} \to \overline{M}_S \to \overline{M}_S \quad \text{and} \quad \overline{M}_{S_m,s} \to \overline{M}_{S_m} \to \overline{M}_{S_m}.
\]

For each geometric point $t \in S$ we have a commutative diagram of solid arrows

\[
\begin{array}{ccc}
\overline{M}_{S_m,s} & \xrightarrow{\chi} & \overline{M}_{S_m,t} \\
\downarrow & & \downarrow \\
\overline{M}(G_s, V_{\text{max},s}) & \xrightarrow{\chi_{s,t}} & \overline{M}(G_t, V_{\text{max},t}) \\
\overline{M}_{S,s} & \xrightarrow{\chi_{s,t}} & \overline{M}_{S,t} \\
\end{array}
\]

where the top and bottom horizontal arrows are the generalization morphisms given by the two charts above, the compositions of the vertical arrows are given by the morphism $S \to S_m$, and the factorization through $\overline{M}(G_t, V_{\text{max},t})$ follows from Proposition 3.7. By the construction in Section 3.2.2, the arrow on the top induces the dashed arrow $\chi$ making the above diagram commutative.

First observe that the lower commutative square in the above diagram implies that $\phi_{\text{max},t}$ is surjective. Indeed, the groupification of the
generalization morphism $\overline{\mathcal{M}}_{S, s}^{\text{gp}} \to \overline{\mathcal{M}}_{S, t}^{\text{gp}}$ is surjective. Since it factors through $\overline{\mathcal{M}}(G_t, V_{\max, t})^{\text{gp}}$, the morphism $\phi_{\max, t}^{\text{gp}}$ is also surjective. Furthermore, $\overline{\mathcal{M}}_{S, t}$ is the saturation of the submonoid in $\overline{\mathcal{M}}_{S, t}^{\text{gp}}$ generated by the image of $\overline{\mathcal{M}}_{S, s}$ which is precisely the image $\phi_{\max, t}(\overline{\mathcal{M}}(G_t, V_{\max, t}))$.

To see that $\phi_{\max, t}$ is injective, it remains to prove the injectivity of $\phi_{\max, t}^{\text{gp}}$. Consider the set
\begin{equation}
F = \{ e \in \overline{\mathcal{M}}_{S, s} \mid \chi_{s, t}(e) = 0 \}.
\end{equation}

By [41, Lemma 3.5], the group $F^{\text{gp}}$ is the kernel of the morphism $\overline{\mathcal{M}}_{S, s}^{\text{gp}} \to \overline{\mathcal{M}}_{S, t}^{\text{gp}}$. Let $K$ be the kernel of $\overline{\mathcal{M}}(G_s, V_{\max, s})^{\text{gp}} \to \overline{\mathcal{M}}(G_t, V_{\max, t})^{\text{gp}}$, hence $K \subset F^{\text{gp}}$. We will prove $F^{\text{gp}} = K$ by showing that the composition $F \hookrightarrow \overline{\mathcal{M}}(G_s, V_{\max, s}) \xrightarrow{\phi_{\max, s}} \overline{\mathcal{M}}(G_t, V_{\max, t})$ is trivial.

Indeed, consider the fine submonoid $\mathcal{N} \subset \overline{\mathcal{M}}(G_s, V_{\max, s})^{\text{gp}}$ generated by the degeneracy $e_v$ for each $v \in V(G_s)$, the element $p_l$ for each $l \in E(G)$, and the element $(e_{\max} - e_v)$ for each $v \in V(G)$. Let $e \in \overline{\mathcal{M}}(G_s, V_{\max, s})$ be one of the above three types. Observe that $\chi(e) = 0$ if $\chi_{s, t}(e) = 0$ by the construction in Section 3.2.2, hence $\chi(\mathcal{N} \cap F) = 0$. Since $\overline{\mathcal{M}}(G_s, V_{\max, s})$ is the saturation of $\mathcal{N}$ in $\overline{\mathcal{M}}(G_s, V_{\max, s})^{\text{gp}}$, $F$ is the saturation of $\mathcal{N} \cap F$. We conclude that $\chi(F) = 0$. \hfill $\square$

3.2.4. The universality. The minimal objects in $\mathcal{U}(Y, \beta')$ have a universal property similar to the case of Proposition 2.9:

**Proposition 3.12.** For any log map $f : C \to Y$ over a log scheme $S$ with uniform maximal degeneracy, there exists a log map $f_{\text{mu}} : C_{\text{mu}} \to Y$ over $S_{\text{mu}}$ which is minimal with uniform maximal degeneracy, and a morphism of log schemes $\Phi_u : S \to S_{\text{mu}}$ such that

1. The underlying morphism $\Phi_u$ is an isomorphism.
2. $f : C \to Y$ is the pull-back of $f_{\text{mu}} : C_{\text{mu}} \to Y$ along $\Phi_u$.

Furthermore, the pair $(f_{\text{mu}}, \Phi_u)$ is unique up to a unique isomorphism.

**Proof.** Let $f_m : C_m \to Y$ over $S_m$ be the associated minimal object as in Proposition 2.9, so that $f$ is the pull-back of $f_m$ along $\Phi : S \to S_m$ with $\Phi$ the identity of $S$.

Since the statement is local on $S$, we are free to shrink $S$ if needed. Thus, we may assume there are charts
\[ h_{S_m} : \mathcal{M}_{S_m, s} \to \mathcal{M}_{s, s} \quad \text{and} \quad h_S : \overline{\mathcal{M}}_{S, s} \to \mathcal{M}_S \]
for some geometric point $s \to S$. Denote by $(G, V_{\max})$ the log combinatorial type of the fiber $f_s$ over $s$. By Proposition 3.7, the morphism $\phi : \overline{\mathcal{M}}(G) = \overline{\mathcal{M}}_{S_m, s} \to \overline{\mathcal{M}}_{S, s}$ factors through $\phi_{\max} : Q := \overline{\mathcal{M}}(G, V_{\max}) \to \overline{\mathcal{M}}_{S, s}$. Write $\tilde{\phi} : \overline{\mathcal{M}}(G) \to Q$ for the canonical morphism.

Denote by $\mathcal{M}_{\text{mu}}$ the log structure on $S$ associated to the pre-log structure defined by the composition $h : Q \to \overline{\mathcal{M}}_{S, s} \xrightarrow{h_S} \mathcal{M}_S$. Thus, there is a morphism of log structures $\mathcal{M}_{\text{mu}} = Q \oplus h^{-1}O^*_S \to \mathcal{M}_S$. Therefore, $\mathcal{U}(Y, \beta')$ has a universal property similar to Proposition 2.9.
Then the following assignments on the right define a unique dashed arrow on the left which makes the diagram of log structures commutative:

\[
\begin{array}{ccc}
\mathcal{M}_{S_m} & \xrightarrow{\phi} & \mathcal{M}_S \\
\downarrow & & \downarrow \\
h_{S_m}(e) & \xrightarrow{h \circ \phi + v} & h_S \circ \phi(e) + u
\end{array}
\]

where \( u \in \mathcal{O}^* \) and \( v \in \mathcal{O}^* \) are the unique, invertible sections making the diagram commutative. This defines a morphism of log schemes \( S_{\mu} := (\sum \mathcal{M}_{S_{\mu}}) \to S_m \) through which \( S \to S_m \) factors. Further observe that such a morphism depends on the choice of charts \( h_S \) and \( h_{S_m} \). However, different choices of charts induce a unique isomorphism of \( S_{\mu} \) compatible with the arrows to and from \( S_m \) and \( S \) respectively.

Pulling back the log map over \( S_m \), we obtain a log map \( f_{\mu} : C \to Y \) over \( S_{\mu} \) which further pulls back to \( f \) over \( S \). Note that the geometric fiber \( f_{\mu,s} \) is minimal with uniform maximal degeneracy over \( s \). Further shrinking \( S \) and using Proposition 3.11, we obtain a family of log maps over \( S_{\mu} \) minimal with uniform maximal degeneracy as needed.

\[3.2.5. \text{Finiteness of automorphisms.} \text{ Consider a log map } f : C \to Y \text{ over } S \text{ with } S \text{ a geometric point. Suppose } f \text{ is minimal with uniform maximal degeneracy. Let } f_m : C \to Y \text{ over } S_m \text{ be the minimal log map given by Proposition 2.10 such that } f \text{ is the pull-back of } f_m \text{ along a morphism } \Phi : S \to S_m. \text{ Let } \text{Aut}(f) \text{ and } \text{Aut}(f_m) \text{ be the automorphism groups introduced in Section 2.3.3. They are related as follows.}
\]

**Proposition 3.13.** Notations as above, there is an injective homomorphism of groups \( \text{Aut}(f) \to \text{Aut}(f_m) \). In particular, \( \text{Aut}(f) \) is finite if \( f \) is stable.

**Proof.** We first construct this group homomorphism. Consider an element \( (\psi : C \to C, \theta : S \to S) \) in \( \text{Aut}(f) \). Note that \( f \) can be obtained as the pull-back of \( f_m \) via either \( S \xrightarrow{\Phi} S_m \) or the composition \( S \xrightarrow{\theta} S \xrightarrow{\Phi} S_m \). By the canonicity in Proposition 2.9, there is a unique isomorphism \( (\psi_m : C_m \to C_m, \theta_m : S_m \to S_m) \) in \( \text{Aut}(f_m) \) fits in the following commutative diagram:

\[
\begin{array}{ccc}
S & \xrightarrow{\theta} & S \\
\downarrow \Phi & & \downarrow \Phi \\
S_m & \xrightarrow{\theta_m} & S_m
\end{array}
\]

The arrow \( \text{Aut}(f) \to \text{Aut}(f_m) \) is then defined by \( (\psi, \theta) \mapsto (\psi_m, \theta_m) \).

To see the injectivity, observe that the morphism \( \mathcal{M}_{S_m}^{gp} \to \mathcal{M}_S^{gp} \) is surjective by the construction of Section 3.2.2. Thus \( \theta_m \) being the identity implies that \( \theta \) is also the identity. \[ \Box \]
3.3. The stack.

3.3.1. The statements. Consider the fibered categories of log maps with uniform maximal degeneracies as in Section 3.1.3. We now establish their algebraicity and properness. By Proposition 2.12, 2.16 and 2.17, it suffices to build these properties upon the stack of log maps. We first consider the case of the universal target.

**Theorem 3.14.** The tautological morphism as in (13)

\[ \mathcal{U}(A, \beta') \to \mathcal{M}(A, \beta') \]

is representable by log algebraic spaces of finite type. Furthermore, it is proper and log étale. In particular, the fibered category \( \mathcal{U}(A, \beta') \) is represented by a log smooth log algebraic stack locally of finite type.

Then consider the following cartesian diagram

\[
\begin{array}{ccc}
\mathcal{U}(Y, \beta) & \longrightarrow & \mathcal{U}(Y, \beta') \\
\downarrow & & \downarrow \\
\mathcal{M}(Y, \beta) & \longrightarrow & \mathcal{M}(Y, \beta')
\end{array}
\]

where the vertical arrows are given by (13), and the two horizontal arrows of the right square are induced by the canonical strict morphism \( Y \to A \). Note that imposing a curve class and requiring underlying maps being stable are both representable by open embeddings. The followig is an immediate consequence of the above theorem.

**Theorem 3.15.** The canonical morphism \( \mathcal{U}(Y, \beta) \to \mathcal{M}(Y, \beta) \) is a proper, representable and log étale morphism of log Deligne–Mumford stacks. In particular, \( \mathcal{U}(Y, \beta) \) is of finite type if \( \mathcal{M}(Y, \beta) \) is so.

We now give the proof of Theorem 3.14, which splits to two parts.

3.3.2. Representability, boundedness and log étaleness. For simplicity, write \( \mathcal{M} := \mathcal{M}(A, \beta') \) and \( \mathcal{U} := \mathcal{U}(A, \beta') \).

Consider Olsson’s log stack \( \text{Log}_{\mathcal{M}} \), which associates to each strict morphism \( T \to \mathcal{M} \) the category of morphisms of fine log structures \( \mathcal{M}_T \to \mathcal{M} \) over \( T \). By Proposition 3.12, we may view \( \mathcal{U} \) as the category fibered over the category of schemes parameterizing log maps minimal with uniform maximal degeneracy. By Proposition 3.11, the tautological morphism \( \mathcal{U} \to \text{Log}_{\mathcal{M}} \) is an open embedding. Since \( \text{Log}_{\mathcal{M}} \) is algebraic, \( \mathcal{U} \) is a log algebraic stack equipped with the universal minimal log structure. By Proposition 3.13, the morphism \( \mathcal{U} \to \mathcal{M} \) is representable. The log étaleness of \( \mathcal{U} \to \mathcal{M} \) follows from [41, Theorem 4.6 (ii), (iii)]. By Proposition 2.13, the stack \( \mathcal{U} \) is log étale.

To prove that \( \mathcal{U} \to \mathcal{M} \) is of finite type, consider a strict morphism \( T \to \mathcal{M} \) from a log scheme \( T \) of finite type, and write \( U := T \times_{\mathcal{M}} \mathcal{U} \). Since being of finite type is a property local on the target, it suffices to show that \( U \) is of finite type.
Denote by $\Lambda$ the collection of log combinatorial types of log maps over $T$. Since $T$ is of finite type, the set $\Lambda$ is finite. Let $\Lambda_{um} = \{(G,V_{\text{max}}) \mid G \in \Lambda\}$ be the collection of log combinatorial types of log maps over $U$ as in Section 3.2.1. The set $\Lambda_{um}$ is again finite as the number of choices of $V_{\text{max}} \subset V(G)$ for a fixed $G \in \Lambda$ is finite.

For each $(G,V_{\text{max}}) \in \Lambda_{um}$, the canonical morphism (14) induces a morphism of log stacks $\overline{\mathcal{A}}((G,V_{\text{max}})) \to \overline{\mathcal{A}}(G)$. Consider

$$\overline{\mathcal{A}}((G,V_{\text{max}})),T = T \times \text{Log} \overline{\mathcal{A}}((G,V_{\text{max}}))$$

where $T \to \text{Log}$ is the canonical strict morphism, and the morphism on the right is the composition $\overline{\mathcal{A}}((G,V_{\text{max}})) \to \overline{\mathcal{A}}(G) \to \text{Log}$. By [41, Corollary 5.25], there is an étale morphism

$$\overline{\mathcal{A}}((G,V_{\text{max}})),T \to \text{Log}T.$$

By the construction of $\mathcal{U}$, $U$ is an open sub-stack of $\text{Log}T$. By Definition 3.10 and Proposition 3.11, $U$ is covered by the image of the finite union:

$$\bigcup_{(G,V_{\text{max}}) \in \Lambda_{um}} \overline{\mathcal{A}}((G,V_{\text{max}})),T \to \text{Log}T.$$

Thus $U$ is of finite type.

3.3.3. Properness. Since $\mathcal{U} \to \mathcal{M}$ is representable and of finite type, for properness it suffices to prove the weak valuative criterion.

**Step 1: The set-up of the weak valuative criterion.**

Let $R$ be a discrete valuation ring, $m \subset R$ be its maximal ideal, and $K$ be its quotient field. Consider a commutative diagram of solid arrows of the underlying stacks

$$\begin{array}{ccc}
\text{Spec } K & \to & \mathcal{U} \\
\downarrow & & \downarrow \\
\text{Spec } R & \to & \mathcal{M}
\end{array}$$

It suffices to show that possibly after replacing $R$ by a finite extension of discrete valuation rings, and $K$ by the corresponding finite extension of quotient fields, there exists a unique dashed arrow making the above diagram commutative.

Let $f$ be a minimal log map over $S = (\text{Spec } R, \mathcal{M}_S)$ given by the bottom arrow of the above diagram. Denote by $s, \eta \in S$ the closed and generic points with the log structure pulled back from $S$ respectively. Let $f_{\eta}$ be the log map over $\eta = (s, \mathcal{M}_{\eta})$ minimal with uniform maximal degeneracy given by the top arrow. There is a canonical morphism $\eta \to \eta$ such that $f_{\eta}$ is the pull-back of $f_{\eta}$. We will construct the dashed arrow by extending $f_{\eta}$ to a log map over $\text{Spec } R$ which is the pull-back of $f$, and is minimal with uniform maximal degeneracy.

**Step 2: Determine the combinatorial type of the closed fiber.**
Passing to a finite extension of $R$ and $K$, denote by $G$ the log combinatorial type of the closed fiber $f_s$ of $f$, and by $(G, V_{\text{max}}^u)$ the log combinatorial type of $f_{\eta_0}$. We next determine the log combinatorial type $(G, V_{\text{max}}^u)$ of possible extensions of $f_{\eta_0}$.

We may assume that there exists a chart $h: \overline{\mathcal{M}}(G) \to \mathcal{M}_S$ after taking a further base change. For each $v \in V(G)$, denote by $e_v \in \overline{\mathcal{M}}(G)$ the corresponding degeneracy. Denote by $\text{gd}$ the following composition

$$\overline{\mathcal{M}}(G) \xrightarrow{h} \mathcal{M}_S \xrightarrow{\psi} \mathcal{M}_u \xrightarrow{\eta} \mathcal{M}_{\eta_0}.$$  

By Lemma 3.4, the general fiber of $\text{gd}(e_v)$ corresponds to a degeneracy of some vertex $v_\eta \in V(G_\eta)$. Consider the subset $V' \subset V(G)$ consisting of vertices $v$ such that $\text{gd}(e_v)_{\eta}$ corresponds to the degeneracy of vertices in $V_{\text{max}, \eta_0}$. We define a partial order on $V'$ as follows.

For any $v_1, v_2 \in V'$, observe $\text{gd}(e_{v_2}) - \text{gd}(e_{v_1}) \in K^\times$ as it is a difference of maximal degeneracies over $\eta$. We define

$$v_1 \preceq u v_2 \text{ if } (\text{gd}(e_{v_2}) - \text{gd}(e_{v_1})) \in R.$$  

Denote by $V_{\text{max}} \subset V'$ be the collection of maximal elements under this partial order $\preceq_u$.

We show that $(G, V_{\text{max}})$ is necessarily the log combinatorial type of any possible extension $f_{S_u}$ of $f_{\eta_0}$ over $S_u = (\text{Spec } R, \mathcal{M}_{S_u})$ with uniform maximal degeneracy. Given such an extension, let $V'_{\text{max}}$ be the collection of maximally degenerated vertices of the closed fiber of $f_{S_u}$. By Lemma 3.4 and 3.6, we have the inclusion $V'_{\text{max}} \subset V'$.

Consider the canonical morphism $\psi: S_u \to S$ along which $f$ pulls back to $f_{S_u}$. Thus $\text{gd}$ can be also given by the following composition

$$\overline{\mathcal{M}}(G) \xrightarrow{h} \mathcal{M}_S \xrightarrow{\psi} \mathcal{M}_{S_u} \xrightarrow{\eta} \mathcal{M}_{\eta_0}.$$  

Suppose $v_2 \in V'_{\text{max}}$. Then since $(\psi^\flat \circ h(e_{v_2}) - \psi^\flat \circ h(e_{v_1})) \in \mathcal{M}_{S_u}$ for any $v_1 \in V'$, we have $(\text{gd}(e_{v_2}) - \text{gd}(e_{v_1})) \in R$. This implies $V'_{\text{max}} \subset V_{\text{max}}$. The other direction $V_{\text{max}} \subset V'_{\text{max}}$ is similar.

**Step 3: Principalize degeneracies of elements in $V_{\text{max}}$.**

Let $\mathcal{K}_0 \subset \mathcal{M}_S$ be the log ideal generated by $\{h(e_v) \mid v \in V_{\text{max}}\}$. Let $S_0 \to S$ be the log blow-up along $\mathcal{K}_0$, and $f_{S_0}$ be the pull-back of $f$. We show that $\eta_u \to S$ factors through $S_0 \to S$ uniquely.

Indeed, let $(G, V_{\text{max}}^u, \eta_0)$ be the log combinatorial type of $f_{\eta_0}$. By Lemma 3.4 and 3.6, $\text{gd}(e_v)$ corresponds to the maximal degeneracy of $f_{\eta_0}$ for any $v \in V_{\text{max}}$. Thus $\mathcal{K}_0$ pulls back to a locally principal log ideal over $\eta_u$ via $\eta_u \to S$. It follows from the universal property of log blow-ups that there is a unique morphism $\eta_u \to S_0$ lifting $\eta_u \to S$.

Since the underlying of $S_0 \to S$ is projective, the underlying morphism of $\eta_u \to S_0$ extends to a strict morphism $S_0 \to S_0$ with the underlying $S_0 = \text{Spec } R$. In particular, we obtain a morphism $\psi_0: \eta_u \to S_0$. 

Denote by $f_{S_0}$ the pull-back of $f_{S_0}$ over $S_0$. Consider the composition

$$
\text{gd}_0 : \overline{\mathcal{M}}(G) \rightarrow \mathcal{M}_S \rightarrow \mathcal{M}_{S_0}
$$

We show that the elements in $V_{\max}$ have the same degeneracy associated to the closed fiber of $f_{S_0}$ by showing that

$$(17) \quad (\text{gd}_0(e_{v_2}) - \text{gd}_0(e_{v_1})) \in R^\times, \text{ for any } v_1, v_2 \in V_{\max}.
$$

Indeed, observe

$$(18) \quad (\text{gd}(e_{v_2}) - \text{gd}(e_{v_1})) \in R^\times, \text{ for any } v_1, v_2 \in V_{\max}.
$$

Since $S_0 \rightarrow S$ factors through $\hat{S}_0 \rightarrow S$, we have $(\text{gd}_0(e_{v_2}) - \text{gd}_0(e_{v_1})) \in \mathcal{M}_{S_0}$. Since $\text{gd} = \psi_0 \circ \text{gd}_0$, the claim follows from the fact that

$$
\psi_0 \circ (\text{gd}_0(e_{v_2}) - \text{gd}_0(e_{v_1})) = \text{gd}(e_{v_2}) - \text{gd}(e_{v_1}) \in R^\times.
$$

Step 4: Maximize the the degeneracy of elements in $V_{\max}$.

Fix $v_0 \in V_{\max}$. Consider the finite set $V(G) \setminus V_{\max} = \{v_1, \ldots, v_k\}$. Define $K_i \subset \mathcal{M}_{S_0}$ to be the log ideal generated by $\{\text{gd}_0(e_{v_1}), \text{gd}_0(e_{v_0})\}$ for $i = 1, 2, \ldots, k$. By (18) the log ideal $K_i$ is independent of the choice of $v_0 \in V_{\max}$. Consider the following diagram

$$
\begin{array}{c}
\hat{S}_0 \\
\downarrow \psi \downarrow \\
S_0
\end{array}
$$

where $\hat{S}_{i+1} \rightarrow \hat{S}_i$ is the log blow-up of the pull-back of $K_i$ via $\hat{S}_i \rightarrow S_0$.

Since $(\text{gd}(e_0) - \text{gd}(e_{v_1})) \in \mathcal{M}_{\eta_0}$, the log ideal $K_i$ pulls back to a locally principal log ideal over $\eta_0$ via $\psi_0$. Thus we obtain a sequence of dashed arrows $\psi_i : \eta_i \rightarrow \hat{S}_i$ lifting $\psi_0$ as in the above diagram.

Since log blow-ups are projective, we obtain a strict morphism $S_k \rightarrow \hat{S}_k$ with underlying $\overline{S}_k = \text{Spec } R$ extending the underlying morphism of $\psi_0$. Thus for each $i$ we have morphisms $\psi_i : \eta_i \rightarrow S_i$ and $S_i \rightarrow S_0$. Let $f_{S_k} : C_{S_k} \rightarrow A$ over $S_k$ be the pull-back of $f_{S_0}$.

Consider the composition $\text{gd}_k : \overline{\mathcal{M}}(G) \rightarrow \mathcal{M}_S \rightarrow \mathcal{M}_{S_k}$. Since the pull-back of $K_i$ is locally principal over $S_k$, we have that either $(\text{gd}_k(e_{v_0}) - \text{gd}_k(e_{v_1}))$ or $(\text{gd}_k(e_{v_1}) - \text{gd}_k(e_{v_0}))$ belongs to $\mathcal{M}_{S_k}$. We next show that the latter is not possible.

Indeed the construction in Step 2 implies that $(\text{gd}(e_{v_0}) - \text{gd}(e_{v_1})) \in \mathcal{M}_{\eta_0} \setminus R^\times$. Since $\text{gd} = \psi_0 \circ \text{gd}_k$, we necessarily have that $(\text{gd}_k(e_{v_0}) - \text{gd}_k(e_{v_1})) \in \mathcal{M}_{S_k} \setminus R^\times$ for any $i = 1, \ldots, k$. Thus $f_{S_k}$ over $S_k$ has uniform maximal degeneracy by Proposition 3.3.

Step 5: Verify the extension and uniqueness.

We show that $f_{S_k}$ is the unique extension of $f_{\eta_k}$ as needed. First observe that the pull-back of $f_{S_k}$ along $\psi_k$ is the log map $f_{\eta_k}$ minimal with uniform degeneracy. Thus the universality of Proposition 3.12
implies that $\psi_k$ induces an isomorphism between the generic fiber $f_{S_k,\eta}$ of $f_{S_k}$ and $f_{\eta_u}$. Using Proposition 3.12 again, we obtain a log map $f_{S_u}$ over $S_u$ which is minimal with uniform maximal degeneracy, and a morphism $S_k \to S_u$ with the identity underlying morphism, along which $f_{S_u}$ pulls back to $f_{S_k}$. This provides the desired extension of $f_{\eta_u}$.

To see the uniqueness, let $f_{S_u}$ over $S_u$ be any extension of $f_{\eta_u}$. Note that there is a canonical morphism $S_u \to S_k$ along which $f_{S_u}$ pull-back to $f_{S_k}$. Since the log combinatorial type $(G, V_{\text{max}})$ is unique as shown in Step 2, the log ideal $K_0$ as in Step 3 pulls back to a locally principal log ideal over $S_u$, hence there is a unique morphism $S_u \to S_0$ such that $f_{S_u}$ is the pull-back of $f_{S_0}$.

By Condition (2) of Section 3.2.2, the log ideal $K_i$ as in Step 4 pulls back to a locally principal log ideal over $S_u$, hence a unique morphism $S_u \to S_0$ such that $f_{S_u}$ is the pull-back of $f_{S_0}$. Applying the universality of Proposition 3.12 one more time, we obtain an isomorphism $S_u \to S_k$ compatible with pull-back of log maps.

This completes the proof of Theorem 3.14. □

3.4. The logarithmic twist. Here we introduce the log twist which is the key to extending the cosection across the boundary.

Consider the stack $\mathcal{U} := \mathcal{U}(\mathcal{A}, \beta')$ with its universal pre-stable log map $f_{\mathcal{U}} : \mathcal{C}_{\mathcal{U}} \to \mathcal{A}$ and the projection $\pi_{\mathcal{U}} : \mathcal{C}_{\mathcal{U}} \to \mathcal{U}$.

3.4.1. The boundary torsor of $\mathcal{U}$. Consider the global section $e_{\text{max}} \in \Gamma(\mathcal{U}, \mathcal{M}_{\mathcal{U}})$ which is the maximal degeneracy over each geometric point. Consider the $\mathcal{O}_{\mathcal{U}}^*$-torsor over $\mathcal{U}$

$$T_{\text{max}} := e_{\text{max}} \times_{\mathcal{M}_{\mathcal{U}}} \mathcal{M}_{\mathcal{U}}$$

and the corresponding line bundle $L_{\text{max}} \supset T_{\text{max}}$. The composition

$$T_{\text{max}} \to \mathcal{M}_{\mathcal{U}} \to \mathcal{O}_{\mathcal{U}}$$

induces a morphism of line bundles

$$(20) \quad L_{\text{max}} \to \mathcal{O}_{\mathcal{U}}.$$

Since $\mathcal{U}$ is log smooth by Theorem 3.14, the dual of the above defines a section of $L_{\text{max}}^\vee$ whose vanishing locus is a Cartier divisor $\Delta_{\text{max}} \subset \mathcal{U}$ such that $L_{\text{max}}^\vee \cong \mathcal{O}_{\mathcal{U}}(\Delta_{\text{max}})$.

3.4.2. The torsor from the target. By Section 2.1.8, the characteristic sheaf $\mathcal{M}_{\mathcal{A}}$ admits a global section $\delta_\infty \in \Gamma(\mathcal{A}, \mathcal{M}_{\mathcal{A}})$ whose image in $\mathcal{M}_{\mathcal{A}}$ is a local generator. Consider the $\mathcal{O}_{\mathcal{A}}^*$-torsor over $\mathcal{A}$:

$$T_{\infty} := \delta_\infty \times_{\mathcal{M}_{\mathcal{A}}} \mathcal{M}_{\mathcal{A}}$$

and the corresponding line bundle $\mathcal{O}_{\mathcal{A}}(-\infty, \mathcal{A}) \supset T_{\infty}$. The composition

$$T_{\infty} \to \mathcal{M}_{\mathcal{A}} \to \mathcal{O}_{\mathcal{A}}$$

corresponds to the canonical embedding $\mathcal{O}_{\mathcal{A}}(-\infty, \mathcal{A}) \to \mathcal{O}_{\mathcal{A}}$. 

```
3.4.3. *The universal twist.* We construct the *log twist* as follows.

**Lemma 3.16.** Suppose all contact orders in \( \beta' \) are trivial. Then \( f_U^\circ \) induces a morphism compatible with the \( O_{\Cal C_U}^* \)-action

\[
\tilde{f}_U^\circ: (\pi_U^*\mathcal{T}_{\text{max}}) \otimes (f_U^*\mathcal{T}_\infty^\vee) \to \mathcal{M}_{\text{cl}}, \quad a \otimes (\mathcal{T}_\infty) \mapsto a - f_U^\circ(b)
\]

where \( \mathcal{T}_\infty^\vee \) is the dual torsor of \( \mathcal{T}_\infty \).

**Proof.** Consider the sequence of inclusions

\[
\pi_U^*\mathcal{T}_{\text{max}} \subset \pi_U^*\mathcal{M}_U \subset \mathcal{M}_{\text{cl}},
\]

and the composition

\[
f_U^*\mathcal{T}^\vee_\infty \subset f_U^*\mathcal{M}^{gp}_A \to \mathcal{M}^{gp}_{\text{cl}},
\]

where the last arrow induced by \( f_U^\circ \). Putting these together, we obtain

\[
(\pi_U^*\mathcal{T}_{\text{max}}) \otimes (f_U^*\mathcal{T}^\vee_\infty) \to \mathcal{M}^{gp}_{\text{cl}}, \quad a \otimes (\mathcal{T}_\infty) \mapsto a - f_U^\circ(b).
\]

To see this morphism factors through \( \mathcal{M}_{\text{cl}} \), it suffices to show the image of the composition

\[
(\pi_U^*\mathcal{T}_{\text{max}}) \otimes (f_U^*\mathcal{T}^\vee_\infty) \to \mathcal{M}^{gp}_{\text{cl}} \to \overline{\mathcal{M}}^{gp}_{\text{cl}}
\]

is contained in \( \overline{\mathcal{M}}^{gp}_{\text{cl}} \). Note the image is of the form \( e_{\text{max}} - \bar{f}_U^\circ(\delta_\infty) \). Since \( e_{\text{max}} \) is the maximal degeneracy and the contact orders are all trivial, we have \( e_{\text{max}} - \bar{f}_U^\circ(\delta_\infty) \in \overline{\mathcal{M}}^{gp}_{\text{cl}} \) by the description in Section 2.2. \( \square \)

**Proposition 3.17.** Suppose the contact orders in \( \beta' \) are all trivial. Then there is a natural morphism of line bundles over \( \mathcal{C}_U \)

\[
\hat{f}_U: \pi_U^*\mathcal{L}_{\text{max}} \otimes f_U^*\mathcal{O}(\infty_A) \to \mathcal{O}_{\mathcal{C}_U}
\]

such that \( \hat{f}_U \) vanishes along non-maximally degenerate components, and is surjective everywhere else.

**Proof.** The morphism \( \hat{f}_U \) is obtained by composing \( \tilde{f}_U^\circ \) as in Lemma 3.16 with the structural morphism \( \mathcal{M}_{\text{cl}} \to \mathcal{O}_{\mathcal{C}_U} \), and using the corresponding line bundles \( \mathcal{T}_\infty \subset \mathcal{O}_A(-\infty) \) and \( \mathcal{T}_{\text{max}} \subset \mathcal{L}_{\text{max}} \).

Consider a non-maximally degenerate component \( Z \) with degeneracy \( e_Z \). Then over the generic point of \( Z \) we have

\[
e_{\text{max}} - \hat{f}_U^\circ(\delta_\infty) = e_{\text{max}} - e_Z \in \mathcal{M}_U \setminus \{0\}
\]

as \( e_{\text{max}} \) is the maximal degeneracy. Since the target of \( \hat{f}_U \) is the trivial line bundle, we conclude that \( \hat{f}_U \) vanishes over the non-maximally degenerate components.

Then observe that \( e_{\text{max}} - \hat{f}_U^\circ(\delta_\infty) = 0 \) in \( \overline{\mathcal{M}}^{gp}_{\text{cl}} \) over the the maximally degenerate components except those nodes joining maximally degenerate components with non-maximally degenerate components. \( \square \)
3.5. **A partial expansion.** Denote by $\mathcal{N}_{\text{max}} \subset \mathcal{M}_U$ the sub-log structure generated by $\mathcal{T}_{\text{max}} \subset \mathcal{M}_U$. Then $e_{\text{max}} \subset \Gamma(U, \overline{\mathcal{M}}_U)$ is a global section whose image in $\overline{\mathcal{N}}_{\text{max}}$ is a local generator.

Denote by $A_{\text{max}} := A$ the log stack with the boundary divisor $\Delta$ given by the origin. The inclusion $\mathcal{N}_{\text{max}} \hookrightarrow \mathcal{M}_U$ defines a morphism of log stacks $m: U \rightarrow A_{\text{max}}$ with $m^{-1}(\Delta) = \Delta_{\text{max}}$.

Let $E_b \subset A_e$ be the exceptional divisor of $b$ and $\infty_{A_{\text{max}}} \subset A_{\text{max}}$ be the proper transform of $\infty_{A} \times A_{\text{max}} \subset A \times A_{\text{max}}$.

**Lemma 3.18.** Suppose the contact orders in $\beta'$ are all trivial. Then there is a commutative diagram of log stacks

\[
\begin{array}{ccc}
C_U & \xrightarrow{f_e^*} & A^e \\
\downarrow f_{U \times m} & & \downarrow b \\
A \times A_{\text{max}} & & \\
\end{array}
\]

such that

1. The inverse image $(f_e^*)^{-1}(\infty_{A^e})$ is empty.
2. For any geometric point $w \rightarrow \Delta_{\text{max}}$, an irreducible component $Z \subset C_w$ over $w$ dominates $E_b$ via $f_e^*$ if and only if $Z$ is maximally degenerate with non-trivial degeneracy.

**Proof.** We first construct the morphism $f_e^*$. Denote by

$K \subset \mathcal{M}_{A \times A_{\text{max}}} := \mathcal{M}_A \oplus \mathcal{O}^* \mathcal{M}_{A_{\text{max}}}$

the log ideal generated by $\mathcal{T}_{\text{max}}$ and $\mathcal{T}_{\infty}$. The pull-back $(f_{U \times m})^* K \subset \mathcal{M}_{C_U}$ is the log ideal generated by $\mathcal{T}_{\text{max}}$ and the image $f_{U \times m}^*(\mathcal{T}_{\infty})$. Since $b$ is the log blow-up of $K$, to show that $(f_{U \times m})^* K$ is locally principal, it suffices to show that $(f_{U \times m})^* K$ is locally principal, which follows from Lemma 3.16.

Now consider geometric points $w \rightarrow \Delta_{\text{max}}$ and $x \rightarrow C_w$. Denote by $\epsilon_{\text{max}}'$ and $\delta'$ the corresponding local generators of $\mathcal{T}_{\text{max}}$ and $f_{U \times m}^*(\mathcal{T}_{\infty})$ in a neighborhood $W \subset C_w$ of $x$ respectively. Then by Lemma 3.16 we have $\epsilon_{\text{max}}' - \delta' \in \mathcal{O}_{C_w}$. Let $\alpha(\epsilon_{\text{max}}' - \delta') \in \mathcal{O}_W$ be the corresponding image.

By construction of $b$, locally in the smooth topology we can choose a coordinate of $\mathcal{E}_b \setminus \infty_{A^e}$ mapping to $\alpha(\epsilon_{\text{max}}' - \delta')$ via $(f_{U \times m})^*$. Thus $f_{U \times m}^*(W)$ dominates $\mathcal{E}_b \setminus \infty_{A^e}$ if and only if $\alpha(\epsilon_{\text{max}}' - \delta') \neq 0$ on $W$. Statement (2) follows from the fact that $\alpha(\epsilon_{\text{max}}' - \delta')$ vanishes only along non-maximally degenerate components of $C_w$. 

To see (1), observe that \((f^e_U,w)_{-1}(\infty_{\mathcal{A}^e})\) is supported on the poles of the section \(\alpha(e'_\text{max} - \delta')\) over the maximally degenerate components of \(\mathcal{C}_w\). But \(\alpha(e'_\text{max} - \delta')\) has no poles by Lemma 3.16. □

We give another description of (22). Since \(E^c := \pi^*_U \Delta_{\text{max}} - \mathcal{E}_b\) is effective, there is a natural inclusion \((f^e_U)^*\mathcal{O}(\mathcal{E}_b) \to \pi^*_U \mathcal{O}(\Delta_{\text{max}})\), hence

\[
\pi^*_U \mathcal{L}_{\text{max}} \otimes (f^e_U)^*\mathcal{O}(\mathcal{E}_b) \cong \pi^*_U \mathcal{O}(-\Delta_{\text{max}}) \otimes (f^e_U)^*\mathcal{O}(\mathcal{E}_b) \to \mathcal{O}_{\text{cl}}.
\]

Lemma 3.19. The two morphisms (23) and (22) are identical.

Proof. Since \(b^* [\infty_{\mathcal{A}} \times \mathcal{A}_\text{max}] = [\mathcal{E}_b] + [\infty_{\mathcal{A}^e}]\), pulling back (22) via \(b\), we have

\[
\pi^*_U \mathcal{L}_{\text{max}} \otimes (f^e_U)^*\mathcal{O}(\mathcal{E}_b + \infty_{\mathcal{A}^e}) \to \mathcal{O}_{\text{cl}}.
\]

By Lemma 3.18, the above morphism becomes

\[
\pi^*_U \mathcal{L}_{\text{max}} \otimes (f^e_U)^*\mathcal{O}(\mathcal{E}_b) \to \mathcal{O}_{\text{cl}},
\]

which is (23). □

4. LOGARITHMIC FIELDS

4.1. \(r\)-spin curves and their moduli. The case of stable \(r\)-spin curves has been studied in [31, 32, 6, 19]. Following the strategy of [6], we extend \(r\)-spin structures to twisted pre-stable curves.

4.1.1. \(r\)-spin structures.

Definition 4.1. An \(n\)-marked, genus \(g\), \(r\)-spin curve over a scheme \(S\) consists of the following data

\[(\mathcal{C} \to S, \mathcal{L}, \mathcal{L}^r \cong \omega_{\mathcal{C}/S}^{\log})\]

where

(1) \(\mathcal{C} \to S\) is a family of genus \(g\), \(n\)-marked twisted pre-stable curves.

(2) \(\mathcal{L}\) is a representable line bundle over \(\mathcal{C}\) with a given isomorphism \(\mathcal{L}^r \cong \omega_{\mathcal{C}/S}^{\log}\) where \(\omega_{\mathcal{C}/S}^{\log}\) is the log cotangent bundle of the log smooth morphism \(\mathcal{C} \to S\).

The pull-back of \(r\)-spin curves is defined in the usual sense. For simplicity, we may write \((\mathcal{C} \to S, \mathcal{L})\) for an \(r\)-spin curve over \(S\).

Notation 4.2. For the purposes of this paper, we would like to view the family of curves \(\mathcal{C} \to S\) as a family of log curves equipped with the canonical log structure pulled-back from the stack of log curves as in Section 2.1.6. This avoids adding extra underlines to both \(\mathcal{C}\) and \(S\).

Notation 4.3. Unlike the usual notations in logarithmic geometry, the log cotangent bundle of \(\mathcal{C} \to S\) in this paper is denoted by \(\omega_{\mathcal{C}/S}^{\log}\) rather than \(\omega_{\mathcal{C}/S}\). We reserve the notation \(\omega_{\mathcal{C}/S}\) for the dualizing line bundle of the family \(\mathcal{C} \to S\). This choice of notations is compatible with the commonly used notations in FJRW theory.
4.1.2. Monodromy representation along markings and nodes. Consider an $r$-spin curve $(C \to S, \mathcal{L})$ and its $i$-th marking $\sigma_i \subset C$ with the cyclic group $\mu_r$. As the line bundle $\mathcal{L}$ is representable, the action of $\mu_r$ on $\mathcal{L}|_{\sigma_i}$ factors through a group homomorphism

$$\gamma_i : \mu_r \hookrightarrow \mathbb{G}_m$$

which is called the monodromy representation along $\sigma_i$.

In this paper, we use $\gamma = (\gamma_i)_{i=1}^n$ to denote the collection of monodromy representations along the $n$ marked points. This is a discrete invariant of $r$-spin curves.

Consider a geometric point $q \to C$ which is a node. Étale locally around $q$, we have the model (1). Denote by $C_{q+}$ and $C_{q-}$ the two components intersecting at $q$ with respect to the two coordinates $x$ and $y$ respectively. We obtain two monodromy representations $\gamma_{q\pm} : \mu_r \to \mathbb{G}_m$ of $\mathcal{L}|_q$ at $q \in C_{q\pm}$ respectively. The representability of $\mathcal{L}$ implies that both $\gamma_+$ and $\gamma_-$ are injective. The balanced condition of $C$ at the node $q$ implies that the composition

$$\mu_r \xrightarrow{\gamma_+ \times \gamma_-} \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$$

is trivial, where the second arrow is the multiplication morphism.

4.1.3. $r$-spin structure as twisted stable maps. Given an $r$-spin curve $(C \to S, \mathcal{L})$ we obtain a unique commutative diagram as follows:

$$\begin{array}{ccc}
(\mathcal{C},\omega_{\mathcal{C}/S}^{\log})^{1/r} & \to & (C,\omega_{C/S}^{\log})^{1/r} \\
\downarrow & & \downarrow \\
C & \to & C \\
\downarrow & & \downarrow \\
S_1 & \to & S_2.
\end{array}$$

where

(1) $C \to C$ is the coarsification. Here we equip both $C \to S_1$ and $C \to S_2$ with their canonical log structures as a family of log curves. This is a log étale morphism. Furthermore, the bottom morphism $S_1 \to S_2$ induces an identity of the underlying structure $S_1 = S_2 = S$. see [43, Theorem 1.9].

(2) $(C,\omega_{C/S}^{\log})^{1/r} \to C$ is strict and étale with the underlying morphism given by taking the $r$-th root stack of $\omega_{C/S}^{\log}$ over $C$.

(3) $C \to (C,\omega_{C/S}^{\log})^{1/r}$ is induced by the $r$-spin structure $\mathcal{L}' \cong \omega_{C/S}^{\log}$. 


Our description of the $r$-spin structure is similar to the case of [6, Section 1.5] except that we equip the two families of curves with their canonical log structure for later use.

Conversely, by pulling back the universal $r$-th root along $\mathcal{C} \to (C, \omega_{C/S}^{\log})^{1/r}$ we obtain an $r$-spin bundle over $\mathcal{C}$. To summarize, we have

**Lemma 4.4.** The data of an $r$-spin curve $(\mathcal{C} \to S, \mathcal{L})$ is equivalent to the diagram (24).

### 4.1.4. The stack of $r$-spin structures.

Denote by $\mathcal{M}^{1/r}_{g, \gamma}$ the stack of genus $g$, $n$-marked, $r$-spin curves with monodromy data $\gamma$ along markings. It can be viewed a fibered category over the category of usual schemes as the log structures on the curves are the canonical ones.

**Proposition 4.5.** The stack $\mathcal{M}^{1/r}_{g, \gamma}$ is a smooth, log smooth algebraic stack locally of finite presentation. Furthermore, the tautological morphism removing the $r$-spin structures $\mathcal{M}^{1/r}_{g, \gamma} \to \mathcal{M}^{tw}_{g,n}$ is locally of finite type, quasi-separated, strict, and log étale.

**Proof.** Denote by $\pi : \mathcal{C} \to \mathcal{M}^{tw}_{g,n}$ the universal curve, and $\mathcal{C} \to C$ the universal coarse moduli morphism. Also, denote by $(\mathcal{C}, \omega_{\mathcal{C}/\mathcal{M}^{tw}_{g,n}}^{\log})^{1/r}$ the root stack over $\mathcal{C}$ parameterizing $r$-th roots of $\omega_{\mathcal{M}^{tw}_{g,n}}^{\log}$. As $\omega_{\mathcal{C}/\mathcal{M}^{tw}_{g,n}}^{\log} \cong \omega_{\mathcal{C}/\mathcal{M}^{tw}_{g,n}}^{\log}$, we observe that $\tilde{\mathcal{C}} := (C, \omega_{C/\mathcal{M}^{tw}_{g,n}}^{\log})^{1/r} \times_{C} \mathcal{C} \cong (\mathcal{C}, \omega_{\mathcal{C}/\mathcal{M}^{tw}_{g,n}}^{\log})^{1/r}$ with an étale projection $\tilde{\mathcal{C}} \to \mathcal{C}$.

Consider $S \to \mathcal{M}^{tw}_{g,n}$ with the pull-back family $\tilde{\mathcal{C}} \to \mathcal{C} \to S$. By the description of (24), giving an $r$-spin bundle $\mathcal{L}_S$ over $\mathcal{C}_S$ is equivalent to giving a section $s$ of the projection $\tilde{\mathcal{C}} \to \mathcal{C}$ such that the composition $\mathcal{C}_S \to \tilde{\mathcal{C}} \to \mathcal{C}$ is representable. Thus the stack $\mathcal{M}^{1/r}_{g, \gamma}$ is an open substack of the stack $\pi_* \tilde{\mathcal{C}}$ parameterizing sections of the morphism $\tilde{\mathcal{C}} \to \mathcal{C}$ over $\mathcal{M}^{tw}_{g,n}$ with discrete data $\gamma$. By [28, Theorem 1.3], the stack $\mathcal{M}^{1/r}_{g, \gamma}$ is algebraic, and the tautological morphism $\mathcal{M}^{1/r}_{g, \gamma} \to \mathcal{M}^{tw}_{g,n}$ is locally of finite type and quasi-separated.

As $\mathcal{M}^{tw}_{g,n}$ carries the canonical locally free log structure, it remains to show that the morphism $\mathcal{M}^{1/r}_{g, \gamma} \to \mathcal{M}^{tw}_{g,n}$ is étale in the usual sense. We check it using the infinitesimal lifting property.

Let $A \to B$ be a small extension of Artin rings, and consider the commutative diagram of solid arrows

\[
\begin{array}{ccc}
\text{Spec } B & \longrightarrow & \mathcal{M}^{1/r}_{g, \gamma} \\
\downarrow & & \downarrow \\
\text{Spec } A & \longrightarrow & \mathcal{M}^{tw}_{g,n}
\end{array}
\]
It suffices to show that there is a unique dashed arrow marking the above diagram commutative. Pulling back the universal families, it remains to construct the section given by the dashed arrows fitting in the commutative diagram of solid arrows

\[
\begin{array}{ccc}
\tilde{C}_{\text{Spec } B} & \xrightarrow{\bullet} & \tilde{C}_{\text{Spec } A} \\
\downarrow & & \downarrow \\
C_{\text{Spec } B} & \xrightarrow{\bullet} & C_{\text{Spec } A}
\end{array}
\]

But since the vertical arrows are étale, by the infinitesimal lifting of étale morphisms, such dashed arrow exist and is unique. □

The following is an analogue of [6, Corollary 2.2.2]

**Corollary 4.6.** The tautological morphism \(M_{1/r}^{g,n} \to M_{g,n}\) is proper and quasi-finite.

**Proof.** By viewing \(r\)-spin curves as twisted stable maps, the properness follows from [7, Theorem 1.4.1]. Since the morphism \(M_{1/r}^{g,n} \to M_{g,n}^{\text{tw}}\) is étale and \(M_{g,n}^{\text{tw}} \to M_{g,n}\) has zero dimensional fibers, we conclude that the composition \(M_{1/r}^{g,n} \to M_{g,n}^{\text{tw}} \to M_{g,n}\) is quasi-finite. □

4.1.5. Log \(r\)-spin curves and their stacks.

**Definition 4.7.** A log \(r\)-spin curve over a log scheme \(S\) consists of

\[(C \to S, L)\]

where \(C \to S\) is a log curve (not necessarily equipped with the canonical log structure), and \(L\) is an \(r\)-spin structure over the canonical log curve of \(C \to S\). The pull-back of the log \(r\)-spin curve is defined as usual using fiber products in the fine and saturated category.

As every log curve is obtained by the unique pull-back from the associated canonical log curve, we have that

**Corollary 4.8.** The log stack \(M_{1/r}^{g,n}\) with its canonical log structure given by its universal curve represents the category of log \(r\)-spin curves fibered over the category of log schemes.

4.2. Log \(r\)-spin fields and their moduli.

4.2.1. Log \(r\)-spin fields. Given a log \(r\)-spin curve \((C \to S, L)\), consider the \(\mathbb{P}^1\)-bundle

\[\mathcal{P} := \mathbb{P}(L \oplus \mathcal{O}_C) \to L.\]

Denote by \(0_\mathcal{P}\) and \(\infty_\mathcal{P}\) the zero and infinity section of the above \(\mathbb{P}^1\)-bundle with normal bundles \(L\) and \(L^\vee\) respectively. Let \(\mathcal{M}_{\infty_\mathcal{P}}\) be the log structure over \(\mathcal{P}\) associated to the Cartier divisor \(\infty_\mathcal{P}\). It is Deligne–Faltings type of rank one, see Section 2.1.8.
Denote by $\mathcal{P}' = (\mathcal{P}, \mathcal{M}_\infty)$ and $\mathcal{P} = (\mathcal{P}, \mathcal{M}_C|_\mathcal{P} \oplus \mathcal{O} \cdot \mathcal{M}_\infty)$ the corresponding log stacks where $\mathcal{M}_C|_\mathcal{P}$ is the pull-back of $\mathcal{M}_C$. There is a natural projection

$$\mathcal{P} \to \mathcal{C}. \quad (25)$$

**Definition 4.9.** A log $r$-spin field over a log scheme $S$ consists of

$$(\mathcal{C} \to S, \mathcal{L}, f: \mathcal{C} \to \mathcal{P})$$

where

1. $(\mathcal{C} \to S, \mathcal{L})$ is a log $r$-spin curve over $S$.
2. $f$ is a section of $\mathcal{P} \to \mathcal{C}$.

It is called stable if $\omega_{C/S}^{\log} \otimes f^* \mathcal{O}(0_P)^k$ is positive for $k \gg 0$. The pull-back of a log $r$-spin field is defined as usual via pull-back of log curves.

4.2.2. Associated log map of log $r$-spin fields. Note that giving a log $r$-spin field $f: \mathcal{C} \to \mathcal{P}$ is equivalent to giving an associated log map

$$\mathcal{C} \to \mathcal{P}'. \quad (26)$$

In fact, the inclusion $\mathcal{M}_\infty \to \mathcal{M}_C|_\mathcal{P} \oplus \mathcal{O} \cdot \mathcal{M}_\infty$ defines a natural morphism $\mathcal{P} \to \mathcal{P}'$. Thus (26) is given by the composition $\mathcal{C} \to \mathcal{P} \to \mathcal{P}'$.

On the other hand, given a morphism (26) we recover the $r$-spin field $f$ via $\mathcal{C} \to \mathcal{P}' \times_\mathcal{C} \mathcal{C} =: \mathcal{P}$. For convenience, we may use $f$ for the corresponding log map (26) when there is no danger of confusion.

**Definition 4.10.** A log $r$-spin field has uniform maximal degeneracy if its associated log map has uniform maximal degeneracy.

It is called minimal (with uniform maximal degeneracy) if the associated log map (26) is minimal (with uniform maximal degeneracy).

4.2.3. The discrete data of log $r$-spin fields. The discrete data of log $r$-spin fields is given by

$$\beta := (g, \gamma_i)_{i=1}^n, c = (c_i)_{i=1}^n \quad (27)$$

where

1. $g$ is the genus.
2. $\gamma_i$ is the monodromy representation at the $i$-th marking.
3. $c_i$ is the contact order of the associated log map at the $i$-th marking.

Comparing to the discrete data in (9), the above (27) does not specify the curve class. However, since we only allow sections, the curve class is uniquely determined by the collection of contact orders $c$:

$$A = [0_P] + \sum_{i=1}^n c_i \cdot [\mathcal{P}_{\sigma_i}] \quad (28)$$

where $0_P$ is the zero section of the projection $\mathcal{P} \to \mathcal{C}$, and $\mathcal{P}_{\sigma_i}$ is the fiber over the $i$-th marking $\sigma_i$. 
4.2.4. Automorphisms of minimal stable log $r$-spin fields. An automorphism of a log $r$-spin field can be defined similarly as in Section 2.3.3 by taking into account the automorphisms on the target $\mathcal{P}$ induced by the automorphisms of the curve.

**Proposition 4.11.** Consider a log $r$-spin field $f: \mathcal{C} \to \mathcal{P}$ over $S$ with $S$ a geometric point. Suppose it is minimal (with uniform maximal degeneracy). Then its automorphism group is finite.

**Proof.** By Proposition 2.10 and 3.13, it suffices to show that the underlying structure $(\mathcal{C}, \mathcal{L}, f: \mathcal{C} \to \mathcal{P})$ has finite automorphisms. To simplify notation, we will abuse notation and write $(\mathcal{C}, \mathcal{L}, f: \mathcal{C} \to \mathcal{P})$ instead of $(\mathcal{C}_i, \mathcal{L}_i, f_i: \mathcal{C}_i \to \mathcal{P})$ in this proof.

For this, it suffices to prove that $f_i: \mathcal{C}_i \to \mathcal{P}$ has finitely many automorphisms for any irreducible component $\mathcal{C}_i \subseteq \mathcal{C}$ with all special points of $\mathcal{C}_i$ marked. Since $\mathcal{P} \to \mathcal{C}$ is representable, $f_i$ has finitely many automorphisms when $\mathcal{C}_i$ is a stable curve.

It remains to consider the situation when $\mathcal{C}_i$ is of genus zero and has at most two special points. In these cases $\omega_{\mathcal{C}_i}^\log$ and hence $\mathcal{L}$ have non-positive degree. Furthermore $f_i$ cannot be the zero or infinity section since $\mathcal{O}(f_i^*0_\mathcal{P})$ would then have non-positive degree. Suppose $\mathcal{C}_i$ has at most one special point. Then $\mathcal{L}$ has negative degree along $\mathcal{C}_i$. Since $f(\mathcal{C}_i)$ intersects the zero section properly, it must intersect the infinity section properly at at least one special point. We note that for stability $\mathcal{O}(f_i^*0_\mathcal{P})$ must have positive degree, so that $f_i$ intersects properly with the zero section. If $\mathcal{C}_i$ has only one special point, we may view the intersection point(s) as additional marking(s) since they are preserved by automorphisms. It therefore suffices to consider the case that $\mathcal{C}_i$ has exactly two special points, in which case $\mathcal{L}$ is trivial, and we may view $f_i$ as a stable map to $\mathbb{P}^1$. But since $f_i$ has non-trivial intersection with the zero section, this stable map and hence $f_i$ have finitely many automorphisms. \[\square\]

4.2.5. The stacks of log $r$-spin fields. Let $\mathcal{S}_r^{-1/r}$ be the category of stable $r$-spin fields over the category of log schemes with the discrete data $\beta$. Let $\mathcal{U}_r^{-1/r} \subset \mathcal{S}_r^{-1/r}$ be the subcategory consisting of objects with uniform maximal degeneracy. Next we show that

**Theorem 4.12.** The two categories $\mathcal{U}_r^{-1/r}$ and $\mathcal{S}_r^{-1/r}$ are represented by proper log Deligne–Mumford stacks.

For later use, we introduce $\mathcal{S}$ the stack over $\mathcal{M}_{g, \gamma}^{1/r}$, which associates to each strict morphism $T \to \mathcal{M}_{g, \gamma}^{1/r}$, the category of sections $f$ of the underlying projective bundle $\mathcal{P}_T := \mathbb{P}(\mathcal{L}_T \oplus \mathcal{O}_{\mathcal{C}_T}) \to \mathcal{C}_T$ with the curve class given by (28). Here $(\mathcal{C}_T \to \mathcal{T}, \mathcal{L}_T)$ is the spin structure given by $T \to \mathcal{M}_{g, \gamma}^{1/r}$. We may view $\mathcal{S}$ as a log stack with the strict morphism to $\mathcal{M}_{g, \gamma}^{1/r}$. 
Note that $S$ is an open substack of the stack parameterizing twisted stable maps with the family of targets $\mathcal{P}_{\mathcal{M}_{g,n}} \to \mathcal{M}_{g,n}$, as requiring $f$ to be a section of $\mathcal{P} \to \mathcal{C}$ is an open condition. The following is a consequence of [7, Theorem 1.4.1]:

**Lemma 4.13.** The stack $S$ is algebraic locally of finite type. Furthermore, the tautological morphism $S \to \mathcal{M}_{g,n}$ is proper and of Deligne–Mumford type.

**Proof of Theorem 4.12.** By Theorem 3.14, the tautological morphism $U_{1/\beta} \to \mathcal{X}_{1/\beta}$ is proper, log étale, and representable by algebraic spaces of finite type. Thus to prove Theorem 4.12, it remains to prove the statements for $\mathcal{X}_{1/\beta}$ only. We first verify the representability.

Consider the tautological morphism by removing log structures $\mathcal{X}_{1/\beta} \to S$.

By Proposition 2.12, this morphism is represented by algebraic stack locally of finite type. The algebraicity of $S$ in Lemma 4.13 implies that the stack $\mathcal{X}_{1/\beta}$ is also algebraic and locally of finite type. Proposition 4.11 further implies that $\mathcal{X}_{1/\beta}$ is a Deligne–Mumford stack.

It remains to prove the properness. We will divide this into two parts: the boundedness part will be proved in Section 4.3, and the valuative criterion will be checked in Section 4.4.

### 4.3. Boundedness

We next prove the following result:

**Proposition 4.14.** The stack $\mathcal{X}_{1/\beta}$ is of finite type.

Consider the tautological morphism

$$\mathcal{X}_{1/\beta} \to \mathcal{M}_{g,n}$$

by taking the corresponding coarse curves. Using the above morphism, the proof of Proposition 4.14 splits into the following two lemmas.

**Lemma 4.15.** The tautological morphism (29) is of finite type.

**Proof.** Note that the morphism (29) is given the composition $\mathcal{X}_{1/\beta} \to S \to \mathcal{M}_{1/\beta} \to \mathcal{M}_{g,n}$ where the middle arrow is of finite type by Lemma 4.13, and the right arrow is of finite type by Corollary 4.6. It remains to show that the morphism $\mathcal{X}_{1/\beta} \to S$ is of finite type.

Let $T \to S$ be any strict morphism from a log scheme $T$ of finite type, and write $\mathcal{X}_T := \mathcal{X}_{1/\beta} \times_S T$. It suffices to show that $\mathcal{X}_T$ is of finite type. By Proposition 2.16, it suffices to show that the discrete
data $\beta$ is combinatorially finite over $T$, see Definition 2.14. We prove this by applying the strategy similar to [15, Proposition 5.3.1].

Denote by $f_T$ the universal section of $P_T \to \mathcal{C}_T$ over $T$. As $T$ is of finite type, there are finitely many dual graphs for geometric fibers of the source curve $\mathcal{C}_T \to T$. Let $\mathcal{G}$ be any such dual graph of $\mathcal{C}_t$ for some geometric point $t \to T$. It remains to show that the choices of log combinatorial types as in (8) with the given dual graph $\mathcal{G}$ is finite.

Note that the partition $V(\mathcal{G}) = V^v(\mathcal{G}) \sqcup V^d(\mathcal{G})$ as in (8) is uniquely determined by the underlying section $f_t$ of $P_t$. Indeed, $V^d(\mathcal{G})$ consists of irreducible components whose images via $f_t$ are contained in the infinity section of $P_t$. The contact orders along the marked points are determined by $\beta$. Since $\mathcal{G}$ is a finite graph, the choices of partial orderings $\preccurlyeq$ on $V(\mathcal{G})$ is also finite. We fix such a choice, denoted again by $\preccurlyeq$. It remains to show that the choices of the contact orders at each nodes are finite.

Let $Z \subset \mathcal{C}_t$ be an irreducible component. Let $q \in \mathcal{C}_t$ be a nodal point joining $Z$ with another irreducible component $Z'$. We call $q$ an incoming node if $Z' \preccurlyeq Z$, and an outgoing node if $Z \preccurlyeq Z'$. The same discussion as in [15, Proposition 5.2.4] implies that

\[(30) \quad \deg f^*(\infty_P)|_Z = \sum_{\text{q incoming node}} c_q - \sum_{\text{q outgoing node}} c_q,
\]

where $c_q$ is the contact order at the node $q$. Note that if a node $q$ is both incoming and outgoing, then necessarily $c_q = 0$.

To bound the choices of contact orders at the nodes, we construct a partition:

$V(\mathcal{G}) = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$

inductively as follows. First, we choose $V_1$ to be the collection of largest elements in $V(\mathcal{G})$ with respect to $\preccurlyeq$. Supposing that $V_1, \ldots, V_i$ are chosen, we choose $V_{i+1} \subset V(\mathcal{G}) \setminus (\bigcup_{j=1}^{i} V_j)$ to be the collection of largest elements with respect to $\preccurlyeq$.

By construction, a node $q$ joining component(s) in the same $V_i$ must have $c_q = 0$. Let $Z_1$ be any component corresponding to an element in $V_1$. Then $Z_1$ has only incoming node(s). By (30), the choices of contact orders at these nodes are finite, as contact orders are non-negative integers. In particular, there are finitely many choices for the contact orders of the outgoing nodes attached to components of $V_2$.

Now suppose the number of choices of contact orders of the outgoing nodes attached to components of $V_i$ is finite. Using (30) and the condition that contact orders are non-negative integers again, we conclude that the incoming nodes of components of $V_i$, hence the outgoing nodes of components of $V_{i+1}$ have finitely many choices of contact orders. By induction, the number of choices of contact orders at each nodes is finite. This finishes the proof. \qed
Lemma 4.16. The image of the morphism $\mathcal{S}_\beta^{1/r} \to \mathcal{M}_{g,n}$ is contained in an open substack of finite type.

Proof. To bound the image of $\mathcal{S}_\beta^{1/r} \to \mathcal{M}_{g,n}$, it suffices to show that the number of rational components of the fibers over $\mathcal{S}_\beta^{1/r}$ is bounded. For this, it suffices to show that the numbers of unstable components of the source curves are bounded.

Consider any geometric point $t \to \mathcal{S}_\beta^{1/r}$ with the fiber $f_t : C_t \to \mathcal{P}_t$. By the stability as in Definition 4.9, the line bundle $f_t^*(\mathcal{O}(0_{\mathcal{P}_t}))$ has non-negative degree along each component of $C_t$, and positive degree along each unstable component of $C_t$. Furthermore, since the spin bundle $\mathcal{L}_t$ over $t$ is representable, the degree of $f_t^*(\mathcal{O}(0_{\mathcal{P}_t}))$ along each unstable component is at least $\frac{1}{r}$. Since $\deg f_t^*(\mathcal{O}(0_{\mathcal{P}_t})) = (2g - 2 + n)/r$, the number of unstable components of $C_t$ is at most $(2g - 2 + n)$. □

4.4. Valuative criterion. Let $R$ be a discrete valuation ring, $m_R \subset R$ be its maximal ideal, and $K$ be its quotient field. Let $(C_\eta \to \eta, \mathcal{L}_\eta; C_\eta \to \mathcal{P}_\eta)$ be a minimal stable object over $\eta = (\text{Spec} K, \mathcal{M}_\eta)$. Possibly after a finite extension of $R$, we wish to uniquely extend $f_\eta$ to a family $f : \mathcal{C} \to \mathcal{P}$ over $S = (\text{Spec} R, \mathcal{M}_S)$.

4.4.1. Reduce to the case of nondegenerate irreducible generic fiber. By Proposition 2.17, it suffices to extend the underlying section $\underline{f}_\eta$ to a family of sections $f$ over Spec $R$. Taking partial normalization along the splitting nodes of $\underline{C}_\eta$ and labeling them, it suffices to extend $\underline{f}_\eta$ over each irreducible components. Thus we may assume that $\underline{C}_\eta$ is irreducible.

We may further assume that the image of $\underline{f}_\eta$ is not entirely contained in $0_{\mathcal{P}_\eta}$ or $\infty_{\mathcal{P}_\eta}$, as otherwise we may simply extend $\underline{f}_\eta$ as $0_{\mathcal{P}_\eta}$ or $\infty_{\mathcal{P}_\eta}$ respectively. Passing to a finite extension if necessary, we may assume that $\underline{f}_\eta$ intersects $0_{\mathcal{P}_\eta}$ and $\infty_{\mathcal{P}_\eta}$ properly along $\eta$-points of $\underline{C}_\eta$.

As the log structures are irrelevant for extending the underlying structure, we will drop the underline in this section for simplicity, and all stacks are assumed to be underlying stacks unless otherwise specified. It remains to prove the following result.

Proposition 4.17. Let $(C_\eta, \mathcal{L}_\eta)$ be an irreducible $r$-spin curve, and $f_\eta$ be a section of $\mathcal{P}_\eta := \mathbb{P}(\mathcal{L}_\eta \oplus \mathcal{O}_{C_\eta}) \to C_\eta$. Denote by $0_{\mathcal{P}_\eta}$ and $\infty_{\mathcal{P}_\eta}$ the zero and infinity sections of $\mathcal{P}_\eta$. Suppose that

1. $f_\eta$ is neither the zero nor the infinity section.
2. $f_\eta$ intersects the infinity section only along marked points.
3. $\omega_{C_\eta}^{\log} \otimes f_\eta^*(\mathcal{O}(0_{\mathcal{P}_\eta}))^k$ is positive for $k \gg 0$.

 Possibly after a finite extension, there is a unique $r$-spin curve $(\mathcal{C}, \mathcal{L})$ over Spec $R$ and a section $f$ of $\mathcal{P} := \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_C) \to \mathcal{C}$ extending the triple $(C_\eta, \mathcal{L}_\eta, f_\eta)$ such that $\omega_{\mathcal{C}}^{\log} \otimes f^*(\mathcal{O}(0_{\mathcal{P}}))^k$ is positive for $k \gg 0$. 

Remark 4.18. In the above proposition, marked points are allowed to be broad, namely the inertia group along the marking can be trivial.

Notation 4.19. In this section, we call \((C_i, L_i, f_i)\) an \(r\)-spin triple if \((C_i, L_i)\) is an \(r\)-spin curve over Spec \(R\), and \(f_i\) is a section of \(P_i := \mathbb{P}(L_i \oplus O) \rightarrow C_i\). Their generic fibers over \(\eta\) are decorated by subscripts \(\eta\).

We state a useful tool:

Lemma 4.20. Consider an \(r\)-spin curve \((C_\eta, L_\eta)\) with its coarse moduli \(C_\eta \rightarrow C_\eta\). Let \(C \rightarrow \text{Spec } R\) be a pre-stable curve extending \(C_\eta\). Possibly after a finite base change, there is a unique \(r\)-spin curve \((C, L)\) over Spec \(R\) extending \((C_\eta, L_\eta)\), and a unique twisted stable map \(f : C_1 \rightarrow P := \mathbb{P}(L \oplus O_C)\) over Spec \(R\) extending \(f_\eta\).

Proof. To prove the statement, we apply properness of twisted stable maps twice, see [7]. First, we extend the \(r\)-spin structure using the twisted stable map point of view as in (24). We then extend \(f_\eta\) to \(f\) as twisted stable maps. □

4.4.2. Construct an extension with auxiliary markings. Denote by \(\Lambda\) the set of markings of \(C_\eta\). Taking a finite base change if necessary, we may assume that \(f_\eta\) intersects \(0_P\) properly along \(\eta\)-points of \(C_\eta\).

Denote by \(\Lambda_0\) the set of these intersection points which are non-marked in \(C_\eta\). Let \(C'_\eta\) be the marked curve given by \(C_\eta\) together with the set of markings \(\Lambda_0 \cup \Lambda\).

Let \(C'_\eta \rightarrow C'_\eta\) be the coarse moduli morphism. Possibly after a finite base change, let

\[
C'_1 \rightarrow \text{Spec } R
\]

be any family of pre-stable curves with the set of markings \(\Lambda \cup \Lambda_0\) extending \(C'_\eta\). Let \(C_1 \rightarrow \text{Spec } R\) be the family of pre-stable curves obtained by removing the set of markings \(\Lambda_0\) from \(C'_1\). By Lemma 4.20, we obtain an \(r\)-spin curve \((C_1, L_1) \rightarrow \text{Spec } R\) extending \((C_\eta, L_\eta)\) with the coarse moduli \(C_1 \rightarrow C_1\).

Let \(\hat{C}_1 \rightarrow C'_1\) be the \(r\)-th root stack along the markings in \(\Lambda_0\). Then \(\hat{C}_1\) has the set of markings \(\Lambda \cup \Lambda_0\). Consider the line bundle over \(\hat{C}_1\):

\[
\hat{L}_1 = L_1|_{\hat{C}_1} \otimes O_{\hat{C}_1}(\sum_{x \in \Lambda_0} x).
\]

One verifies directly that \((\hat{L}_1)^{\otimes r} \cong \omega_{\hat{C}_1/\text{Spec } R}^{\log}\), hence \((\hat{C}_1, \hat{L}_1)\) is an \(r\)-spin curve over Spec \(R\).

Denote by \(\hat{P}_1 := \mathbb{P}(\hat{L}_1 \oplus O)\). As \(\hat{C}_1\) is irreducible, the section \(f_\eta\) induces a section \(\hat{f}_{1, \eta}\) of \(\hat{P}_{1, \eta} \rightarrow \hat{C}_{1, \eta}\) which is neither the zero nor the infinity sections by assumption.

Lemma 4.21. Notations as above, possibly after a finite extension, there is an \(r\)-spin triple \((\hat{C}_2, \hat{L}_2, \hat{f}_2)\) extending \((\hat{C}_{1, \eta}, \hat{L}_{1, \eta}, \hat{f}_{1, \eta})\).
Proof. Observe that \( \tilde{f}_{1,\eta} \) intersects the zero and infinity section of \( \tilde{\mathcal{P}}_{1,\eta} \) only along markings in \( \Lambda \cup \Lambda_0 \). Possibly after a further finite base change, we obtain a stable map \( \tilde{f}_2: \tilde{\mathcal{C}}_2 \to \tilde{\mathcal{P}}_1 \) extending \( \tilde{f}_{1,\eta} \).

We claim that the composition \( \tilde{\mathcal{C}}_2 \to \tilde{\mathcal{P}}_1 \to \tilde{\mathcal{C}}_1 \) contracts only rational components with precisely two special points. Let \( Z \subset \tilde{\mathcal{C}}_2 \) be a contracted component. Then \( Z \) cannot be a rational tail: Otherwise, \( \tilde{f}_2|_Z \) surjects onto a fiber of \( \tilde{\mathcal{P}}_1 \to \tilde{\mathcal{C}}_1 \). As over the generic point all intersections with zero and infinity sections are marked, \( Z \) contains at least two special points.

Suppose \( Z \) has at least three special points. Then two of the special points are either a marked point or a node joining \( Z \) with a tree of rational components contracting to a point of \( \tilde{\mathcal{C}}_1 \). The above discussion implies that such a tree contains at least one marked point. This is impossible since \( \tilde{\mathcal{C}}_2 \to \tilde{\mathcal{C}}_1 \) preserves marked points.

Since \( \tilde{\mathcal{C}}_2 \to \tilde{\mathcal{C}}_1 \) contracts only rational bridges and is compatible with markings, we have \( \omega_{\tilde{C}_2/\text{Spec} \, R}^{\log} = \omega_{\tilde{C}_1/\text{Spec} \, R}^{\log}|_{\tilde{C}_2} \). We check that \( (\tilde{\mathcal{C}}_2, \tilde{\mathcal{C}}_2 := \tilde{\mathcal{L}}_1|_{\tilde{C}_2}) \) is an \( r \)-spin curve over \( \text{Spec} \, R \), hence \( \tilde{\mathcal{P}}_1|_{\tilde{\mathcal{C}}_2} = \tilde{\mathcal{P}}_2 := \mathbb{P}(\tilde{\mathcal{L}}_2 \oplus \mathcal{O}) \). Thus \( \tilde{f}_1 \) pulls back to a section \( \tilde{f}_2 \) of \( \tilde{\mathcal{P}}_2 \to \tilde{\mathcal{C}}_2 \) as needed. \( \square \)

4.4.3. Remove auxiliary markings.

Lemma 4.22. Let \( (\tilde{\mathcal{C}}_2, \tilde{\mathcal{L}}_2, \tilde{f}_2) \) be as in Lemma 4.21. Let \( \tilde{\mathcal{C}}_2 \to \mathcal{C}_2 \) be obtained by first rigidifying along markings in \( \Lambda_0 \), then removing \( \Lambda_0 \) from the set of markings. Then there is an \( r \)-spin triple \( (\mathcal{C}_2, \mathcal{L}_2, f_2) \) extending \( (\mathcal{C}_2, \mathcal{L}_2, f_2) \) such that

1. \( \tilde{\mathcal{L}}_2 = \mathcal{L}_2|_{\mathcal{C}_2} \otimes \mathcal{O}_{\mathcal{C}_2}(\sum_{x \in \Lambda_0} x) \),
2. \( f_2 \) and \( \tilde{f}_2 \) are isomorphic away from the sections in \( \Lambda_0 \),
3. \( f_2 \) sends sections in \( \Lambda_0 \) to the zero section of \( \mathcal{P}_2 := \mathbb{P}(\mathcal{L}_2 \oplus \mathcal{O}) \).

Proof. We first construct the spin bundle \( \mathcal{L}_2 \). Let \( \mathcal{C}_2 \to \mathcal{C}_2 \) be the coarse moduli morphism. Then \( \mathcal{C}_2 \) over \( \text{Spec} \, R \) extends \( \mathcal{C}_2 \) as a family of pre-stable curves with the set of markings \( \Lambda \). By Lemma 4.20, we obtain an \( r \)-spin curve \( (\mathcal{C}_3, \mathcal{L}_3) \) over \( \text{Spec} \, R \) extending \( (\mathcal{C}_2, \mathcal{L}_2) \).

Let \( \tilde{\mathcal{C}}_3 \to \mathcal{C}_3 \) be the \( r \)-th root construction along sections in \( \Lambda_0 \), and view \( \tilde{\mathcal{C}}_3 \) as a family of pre-stable curves with markings \( \Lambda \cup \Lambda_0 \). Consider the line bundle \( \mathcal{L}_3 := \mathcal{L}_3|_{\tilde{\mathcal{C}}_3} \otimes \mathcal{O}_{\tilde{\mathcal{C}}_3}(\sum_{x \in \Lambda_0} x) \) over \( \tilde{\mathcal{C}}_3 \). We check that \( \tilde{\mathcal{L}}_3 \) is an \( r \)-spin bundle over \( \tilde{\mathcal{C}}_3 \). Since \( \tilde{\mathcal{C}}_{3,\eta} = \tilde{\mathcal{C}}_{1,\eta} = \tilde{\mathcal{C}}_{2,\eta} \), the \( r \)-spin structure \( (\tilde{\mathcal{C}}_3, \tilde{\mathcal{L}}_3) \) over \( \text{Spec} \, R \) extends \( (\tilde{\mathcal{C}}_{2,\eta}, \tilde{\mathcal{L}}_{2,\eta} = \tilde{\mathcal{L}}_{1,\eta}) \). As both \( \tilde{\mathcal{C}}_2 \) and \( \tilde{\mathcal{C}}_3 \) have the same coarse curve \( \mathcal{C}_2 \), by the uniqueness of Lemma 4.20, we conclude that \( (\tilde{\mathcal{C}}_3, \tilde{\mathcal{L}}_3) \cong (\tilde{\mathcal{C}}_2, \tilde{\mathcal{L}}_2) \), hence \( \tilde{\mathcal{C}}_3 = \mathcal{C}_2 \) and \( \tilde{\mathcal{L}}_2 := \mathcal{L}_3 \). This proves (1).

We now construct the section \( f_2 \). Possibly after a finite extension, \( f_\eta \) extends to a twisted stable map \( \tilde{f}_2: \tilde{\mathcal{C}}_2 \to \mathcal{P}_2 \). Consider the following
where the dashed arrow is a rational map which is well-defined and isomorphic away from the fibers over sections in $\Lambda_0$. Thus away from the preimages of $\hat{C}_2 \to C_2$ over $\Lambda_0$, the morphism $\hat{f}_2$ is isomorphic to $\tilde{f}_2$. The morphism $\hat{C}_2 \to C_2$ is a contraction of rational components over $\Lambda_0$ of the closed fiber.

Let $Z_x \subset \hat{C}_2$ be the preimage of a point $x \in C_2, \text{Spec } R/m_R$ in $\Lambda_0$. Suppose $Z_x$ is not a point. Then $\hat{f}_2|_{Z_x}$ surjects onto a fiber of $P_2 \to C_2$, hence intersects the infinity section non-trivially. This no possible, as such intersection point(s) must be marked in $\Lambda$, and is disjoint from sections in $\Lambda_0$. Thus $\hat{C}_2 \to C_2$ is an isomorphism, and $f_2 := \hat{f}_2$ is a section of $P_2 \to C_2$. This proves (2).

The third statement follows from that $f_{2,\eta} = f_\eta$ sends sections in $\Lambda_0$ to the zero section of $P_2 \to C_2$.

4.4.4. **Contract unstable components.** Let $C_2 \to C_2$ be the coarse moduli morphism where $C_2$ is a family of pre-stable curves over $\text{Spec } R$ with the set of markings $\Lambda$. An irreducible component $Z \subset C_2$ is unstable if $\omega_\log^{C_2} \otimes f^*(O(0_{P_2}))^k$ fails to be positive on $Z$ for $k \gg 0$. Let $Z \subset C_2$ be the image of $\tilde{Z}$. Then $Z$ is unstable if $\tilde{Z}$ is so. Note that all unstable components are over the closed point $\text{Spec } R/m_R$, and are rational components with at most two markings.

**Lemma 4.23.** Let $C_2 \to C_3$ be a contraction of an unstable component $Z$ with two special points. Possibly after a further finite base change, we obtain an $r$-spin triple $(C_3, L_3, f_3)$ extending $(C_\eta, L_\eta, f_\eta)$ such that

1. $C_3 \to C_3$ is the coarse moduli morphism.
2. $C_2 \to C_3$ contracts $Z \subset C_2$ to a point.
3. $P_2 = P_3 \times_{C_n} C_2$ and $f_2$ is the pull-back of $f_3$.

Here we do not require that $f_2$ is of the form in Lemma 4.22.

**Proof.** By Lemma 4.20, we obtain an $r$-spin curve $(C_3, L_3)$ over $\text{Spec } R$ extending $(C_\eta, L_\eta)$ with the coarse moduli morphism $C_3 \to C_3$. Consider the cartesian diagram of solid arrows

\[
\begin{array}{ccc}
C_2 & \rightarrow & \hat{C}_2 \\
\downarrow & & \downarrow \\
C_3 & \rightarrow & \hat{C}_3 \\
\end{array}
\quad
\begin{array}{ccc}
(C_2, \omega_{C_2/\text{Spec } R}^{\log})^{1/r} & \rightarrow & C_2 \\
\downarrow & & \downarrow \\
(C_3, \omega_{C_3/\text{Spec } R}^{\log})^{1/r} & \rightarrow & C_3 \\
\end{array}
\]

The square on the right is cartesian as $\omega_{C_3/\text{Spec } R}^{\log} = \omega_{C_2/\text{Spec } R}^{\log}|_{C_3}$. 

Let \( \mathcal{C}_2'' \to \hat{\mathcal{C}}_2 \) be the twisted stable map extending the isomorphism \( \mathcal{C}_{2,n} \to \hat{\mathcal{C}}_{3,n} \). By pulling back the universal \( r \)-th root via \( \mathcal{C}_2'' \to \hat{\mathcal{C}}_2 \to (C_2, \omega^{\log}_{C_2/\text{Spec } R})^{1/r} \), we obtain an \( r \)-spin bundle \( \mathcal{L}_2'' \) over \( \mathcal{C}_2'' \) extending \( \mathcal{L}_n'' \). By uniqueness of Lemma 4.20, we obtain \( (\mathcal{C}_2'', \mathcal{L}_2'') = (\mathcal{C}_2, \mathcal{L}_2) \) hence the dashed horizontal arrow as above. The skewed dashed arrow is then the composition \( \mathcal{C}_2 \to \hat{\mathcal{C}}_2 \to \mathcal{C}_3 \). Thus (2) follows.

It follows from the above construction that \( \mathcal{L}_2 = \mathcal{L}_2|_{\mathcal{C}_2} \). Hence \( \mathcal{P}_2 \) is the pull-back of \( \mathcal{P}_3 \) via \( \mathcal{C}_2 \to \mathcal{C}_3 \), and \( f_2 \) is the pull-back of \( f_2 \). This proves (3).

We next remove unstable rational tails. We first prove

**Lemma 4.24.** Choosing the extension (31) appropriately, we may assume that \( \mathcal{C}_2 \) in Lemma 4.22 has unstable rational tails contained in the zero section of \( \mathcal{P}_2 \) via \( f_2 \).

**Proof.** Let \( \mathcal{Z} \subset \mathcal{C}_2 \) be an unstable rational tail not contained in the zero section of \( \mathcal{P}_2 \). Then \( \mathcal{Z} \) contains no section from \( \Lambda_0 \). Since \( \mathcal{Z} \) is from a component \( \hat{\mathcal{Z}} \subset \hat{\mathcal{C}}_2 \), and \( \hat{f}_2 \) is a twisted stable map, \( \mathcal{Z} \) maps to a rational tail \( Z \subset \mathcal{C}_1 \). Blowing down \( Z \), we obtain another extension (31) together with the same set of sections \( \Lambda \cup \Lambda_0 \). □

We then contract the unstable rational tails inductively as follows.

**Lemma 4.25.** Let \( (\mathcal{C}_2, \mathcal{L}_2, f_2) \) be an \( r \)-spin triple extending \( (\mathcal{C}_n, \mathcal{L}_n, f_n) \). Suppose all unstable rational tails of \( \mathcal{C}_2 \) are contained in the zero section of \( \mathcal{P}_2 \to \mathcal{C}_2 \) via \( f_2 \). Let \( \mathcal{C}_2 \to \mathcal{C}_2 \) be the coarse moduli morphism, and \( \mathcal{C}_2 \to \mathcal{C}_3 \) be the contraction of an unstable rational tail \( Z \). Then there is a triple \( (\mathcal{C}_3, \mathcal{L}_3, f_3) \) extending \( (\mathcal{C}_n, \mathcal{L}_n, f_n) \) with coarse moduli morphism \( \mathcal{C}_3 \to \mathcal{C}_3 \) such that all unstable rational tails of \( \mathcal{C}_3 \) are contained in the zero section of \( \mathcal{P}_3 = \mathbb{P}(\mathcal{L}_3 \oplus \mathcal{O}) \) via \( f_3 \).

**Proof.** By Lemma 4.20, we obtain the \( r \)-spin curve \( (\mathcal{C}_3, \mathcal{L}_3) \) extending \( (\mathcal{C}_n, \mathcal{L}_n) \) with the coarse moduli morphism \( \mathcal{C}_3 \to \mathcal{C}_3 \), and a twisted stable map \( f_4 : \mathcal{C}_4 \to \mathcal{P}_3 \) over \( \text{Spec } R \) extending \( f_n \). Let \( x \) be the image of \( Z \to \mathcal{C}_3 \). As the pairs \( (\mathcal{C}_2, \mathcal{L}_2) \) and \( (\mathcal{C}_3, \mathcal{L}_3) \) are isomorphic away from the preimage of \( Z \) and \( x \), \( f_4 \) and \( f_2 \) are isomorphic away from the preimage of \( Z \) and \( x \). Thus the composition \( \mathcal{C}_4 \to \mathcal{P}_3 \to \mathcal{C}_3 \) is a contraction of rational components \( \mathcal{Z}_x \) over \( x \in \mathcal{C}_3 \). Suppose \( \mathcal{Z}_x \) is not a point. Since \( f_4 \) is a twisted stable map, \( f_4|_{\mathcal{Z}_x} \) must intersect the infinity section along a marking in \( \Lambda \). This is a contradiction. □

We start with an \( r \)-spin triple as in Lemma 4.22 and 4.24, and inductively apply Lemma 4.23 and 4.25 by contracting unstable components. After finitely many steps, we obtain \( (\mathcal{C}, \mathcal{L}, f) \) as in Proposition 4.17.

4.4.5. **Separatedness.** Consider stable extensions \( (\mathcal{C}_i, \mathcal{L}_i, f_i) \) of \( (\mathcal{C}_n, \mathcal{L}_n, f_n) \) for \( i = 1, 2 \). Let \( \mathcal{C}_i \to \mathcal{C}_i \) be the coarse moduli for \( i = 1, 2 \). By Lemma
4.20, it suffices to show that there is an isomorphism $C'_1 \cong C'_2$ extending the one over $\eta$.

Let $C'_3$ be a family of prestable curves over $\text{Spec } R$ extending $C_\eta$ with dominant morphisms $C'_3 \to C_i$ for $i = 1, 2$. We may assume $C'_3$ has no rational components with at most two special points contracted in both $C_1$ and $C_2$ by further contracting these components.

Let $C'_3 \to C_1 \times C_2 \times C'_3$ be the family of twisted stable maps over $\text{Spec } R$ extending the obvious one $C_\eta \to C_1 \times C_2 \times C_3$. Observe that the composition $C'_3 \to C_1 \times C_2 \times C_3 \to C_3$ is the coarse moduli morphism. Indeed, if there is a component of $C'_3$ contracted in $C_3$, then it will be contracted in both $C_1$ and $C_2$ as well.

Let $C'_3 \to (C'_3, \omega_{C'_3/\text{Spec } R}^{\log})^{1/2}$ be the twisted stable map extending the spin structure over $\eta$. Then we obtain a (not necessarily representable) spin bundle $\mathcal{L}_3$ over $C_3$. We next compare $C_3$ and $C_i$ for $i = 1, 2$.

Set $U_i = C_3$ for $i = 1, 2$. Let $U_i^{(k)}$ be obtained by removing from $U_i^{(k)}$ the rational components with precisely one special point in $U_i^{(k)}$ and that are contracted in $C_i$. Note that these removed rational components need not be proper, and their closure may have more than one special points in $C_3$. We observe that this process must stop after finitely many steps. Denote by $U_i \subset C_3$ the resulting open subset.

**Lemma 4.26.**

1. $U_1 \cup U_2 = C_3$.
2. $C_3 \setminus U_1 \subset U_2$ and $C_3 \setminus U_2 \subset U_1$.

**Proof.** Suppose $z \in C_3 \setminus (U_1 \cup U_2) \neq \emptyset$. Then there is a tree of rational curves in $C_3$ attached to $z$ and contracted in both $C_1$ and $C_2$. This contradicts the assumption on $C_3$. Statement (2) follows from (1). □

Consider the coarse moduli morphism $C_3 \to C_3$. Denote by $U_i := C_3 \times C_i U_i$ for $i = 1, 2$. Since $U_i \to C_i$ contracts only rational components with two special points in $U_i$, we have $\omega_{C_i/\text{Spec } R|U_i}^{\log} = \omega_{U_i/\text{Spec } R}^{\log}$ which further pulls back to $\omega_{C_i/\text{Spec } R|U_i}^{\log} = \omega_{U_i/\text{Spec } R}^{\log}$. Thus the pull-back $\mathcal{L}_i|_{U_i}$ is an $r$-th root of $\omega_{U_i/\text{Spec } R}^{\log}$. Recall the $r$-spin bundle $\mathcal{L}_3|_{U_i}$. Note that $\mathcal{U}_{i, \eta} = \mathcal{C}_\eta$ and $\mathcal{L}_3|_{U_i, \eta} \cong \mathcal{L}_i, \eta$. This isomorphism extends to $\mathcal{L}_3|_{U_i} = \mathcal{L}_i|_{U_i}$ uniquely for $i = 1, 2$. This allows us to glue the pull-backs $f_i|_{U_i}$ to a field $f_3 : C_3 \to \mathcal{P}_3$.

**Lemma 4.27.** $\deg f_{3,s}^* \mathcal{O}(0_{P_3})|_{\mathcal{U}_{i,s}} \geq \deg f_{i,s}^* \mathcal{O}(0_{P_i})$, for $i = 1, 2$.

**Proof.** Note that $\omega_{C_i/\text{Spec } R|U_i}^{\log} = \omega_{C_i/\text{Spec } R|U_i}^{\log}(D')$ for some effective divisor $D'$ supported on the special points $U_i \setminus U_i$ of $C_3$. Further note that $\mathcal{L}_3$ and $\mathcal{L}_i$ are the $r$-th roots of $\omega_{C_i/\text{Spec } R}^{\log}$ and $\omega_{C_i/\text{Spec } R}^{\log}$ respectively and $\mathcal{L}_3|_{U_i} = \mathcal{L}_i|_{U_i}$. Thus there exists an effective divisor $D$ supported on $U_i \setminus U_i$ such that $\mathcal{L}_3|_{U_i,s} \cong \mathcal{L}_i|_{U_i,s}(D)$.

If $D = 0$, then we are done. If $D \neq 0$, then it is not possible that one, or equivalently, both $f_{3,s}|_{U_i,s}$ and $f_{i,s}|_{U_i,s}$ map into the infinity section.
because in that case both of $L_3|_{D_{i,s}}$ and $L_i|_{D_{i,s}}$ are trivial. Suppose $f_{3,s}|_{D_{i,s}}$ and $f_{i,s}|_{D_{i,s}}$ are not the infinity section. By comparing the degree component-wise, we may assume $U_{i,s}$ is irreducible, hence $f_{3,s}|_{D_{i,s}}$ and $f_{i,s}|_{D_{i,s}}$ are viewed as rational sections of $\text{Vb}(L_3|_{D_{i,s}})$ and $\text{Vb}(L_i|_{D_{i,s}})$, respectively. Since they agree on $U_{i,s}$ and $L_3|_{D_{i,s}} \cong L_i|_{D_{i,s}}(D)$ for an effective $D$, we have

$$\deg f_{3,s}^* \mathcal{O}(0_{P_{3}})|_{D_{i,s}} - \deg f_{i,s}^* \mathcal{O}(0_{P_{i}})|_{D_{i,s}} \geq 0.$$  

\[ \square \]

Suppose $C_1 \neq C_2$. Then we have $U_i \neq C_i$ for some $i$, say $i = 1$. By construction each connected component of $C_3 \setminus U_1$ is a tree of proper rational curves in $U_2$ with no marked point, hence $\mathcal{T} := (C_3 \setminus U_1) \subset U_2$.

By construction, the composition $\mathcal{T} \to C_3 \to C_2$ is a closed immersion and $f_3|_{\mathcal{T}} = f_2|_{\mathcal{T}}$. Since $\deg \omega_{\mathcal{T}/\text{Spec} R}|_{\mathcal{T}} < 0$ (unless $\mathcal{T} = \emptyset$), the stability of $f_2$ implies

$$\deg f_3^* \mathcal{O}(0_{P_{3}})|_{\mathcal{T}} = \deg f_2^* \mathcal{O}(0_{P_{2}})|_{\mathcal{T}} > 0.$$  

Using Lemma 4.27, We calculate

$$\deg f_{3,s}^* \mathcal{O}(0_{P_{3}}) = \deg f_{3,s}^* \mathcal{O}(0_{P_{3}})|_{D_{i,s}} + \deg f_{3,s}^* \mathcal{O}(0_{P_{3}})|_{\mathcal{T}} \geq \deg f_{1,s}^* \mathcal{O}(0_{P_{1}}) + \deg f_{3,s}^* \mathcal{O}(0_{P_{3}})|_{\mathcal{T}}.$$  

Since $\deg f_{3,s}^* \mathcal{O}(0_{P_{3}}) = \deg f_{1,s}^* \mathcal{O}(0_{P_{1}})$, we conclude that $\mathcal{T} = C_3 \setminus U_1 = \emptyset$.

Observe that $C_3 = U_1 \to C_1$ contracts proper rational components with precisely two special points. Let $Z \subset C_3$ be such a component, and let $\mathcal{Z} = Z \times_{C_3} C_3$. Since $f_3|_{\mathcal{Z}} = U_1$ is the pull-back of $f_1$, we have

$$\deg f_{3}^* \mathcal{O}(0_{P_{3}})|_{\mathcal{Z}} = 0.$$  

On the other hand, since $Z$ has two special points in $C_3$ and is contracted in $C_1$, it is not contracted in $C_2$. Denote by $Z' \subset C_2$ the component dominating $Z \subset C_2$. Then $Z'$ has precisely two special points. Furthermore $f_2|_{Z'}$ and $f_3|_{Z}$ coincide away from the two special points. Using (33), we observe that $\deg f_{2}^* \mathcal{O}(0_{P_{2}})|_{Z'} = 0$, which contradicts the stability of $f_2$. Thus $C_3 \to C_1$ is an isomorphism.

This completes the proof of Proposition 4.17. \[ \square \]

4.4.6. *Failure of properness without log structure along $\infty_p$. As our target has the non-trivial log structure $\mathcal{M}_{\infty_p}$ along the infinity section (see Section 4.2.1), a non-degenerate component can only intersect $\infty_p$ along nodes or markings. Hence we have condition (2) in Proposition 4.17. It turns out that this condition is necessary for proving the weak valuative criterion for the moduli of meromorphic sections of the spin bundle. We exhibit this necessity using the following example.*

Consider the case that $r = 1$. Let $C = \mathbb{P}^1$ with three marked points $z = 1, 2, \infty$ where $z = u/v$ for a fixed homogeneous coordinates $[u : v]$
of \( \mathbb{P}^1 \). Consider a family of meromorphic differentials \( f_t = t \frac{dz}{z} \) over \( C \) where \( t \) is the parameter over a punctured disc \( \Delta \). Observe that \( f_t \) intersects the infinity section transversally at a single non-marked point \( z = 0 \). We claim that the limit as \( t \to 0 \) does not exist as a section of \( \mathbb{P}(\omega^{\log} \oplus \mathcal{O}) \) with finite automorphisms.

Suppose possibly after a finite base change, the family \( f_t \) extends to a family of sections of \( \mathbb{P}(\omega^{\log}_{C_{\Delta}/\Delta} \oplus \mathcal{O}) \to C_{\Delta} \) over the family of prestable curves \( C_{\Delta} \to \Delta \).

Consider the closed fiber \( f_0 \) over \( C_0 \). As our main focus will be the limiting behavior around \( z = 0 \in C \), and as the three marked points do not collide, we may assume that there is a component \( M \subset C_0 \) containing all the three markings. Observe that the restriction \( f_0|_M \) is the zero section. Furthermore, there is a connected tree of rational curves \( T \subset C_0 \) glued to the point \( z = 0 \in M \), hence \( C_0 = M \cup T \).

First consider a rational tail \( Z \subset C_0 \) contained in \( T \). If \( f_0 \) has finite automorphisms, then \( f_0|_Z \) cannot be the infinity section. Suppose \( Z \) does not intersect the infinity section. Then for degree reasons \( f_0|_Z \) is the zero section, in which case \( f_0 \) again has infinite automorphisms. Thus \( C_0 \) has a unique rational tail \( Z \subset T \) intersects the infinity section transversally at a single point. In particular, \( T \) is a chain of rational curves.

Consider the component \( Z' \subset T \) glued to \( z = 0 \in M \). Suppose \( Z' \neq Z \). The restriction \( f_0|_{Z'} \) cannot be the zero section, as otherwise \( f_0 \) can have infinite automorphisms by scaling \( Z' \). Thus \( f_0|_{Z'} \) must intersect the infinity section. Then both \( f_0|_Z \) and \( f_0|_{Z'} \) intersects the infinity section. This contradicts the fact that the generic fiber \( f_t \) only intersects the infinity section transversally at \( z = 0 \). Consequently we have \( T = Z \), and \( f_0|_Z \) intersects the zero section as well.

Finally, observe that \( \mathbb{P}(\omega^{\log}_{C_{\Delta}/\Delta} \oplus \mathcal{O})|_Z \) is the Hirzebruch surface \( H_1 \) with zero section given by the \((-1)\)-curve. An straightforward calculation shows that there is no section of \( H_1 \to Z \) intersects infinity section transversally at a single point and the zero section at another point. Alternatively, such a section would correspond to a meromorphic differential on \( Z \) holomorphic outside a single point and with a simple pole at this point. By the residue theorem such a differential does not exist.

5. Cosections and the reduced virtual cycle

5.1. The logarithmic perfect obstruction theory. The perfect obstruction theory of stable log maps has been formulated in [25, 8] in different but equivalent ways using the log cotangent complexes of [42]. Here we will follow the method of [8].

5.1.1. The canonical perfect relative obstruction theory. Let \( \mathcal{S} := \mathcal{S}^{1/r}_\beta \) be the stack of stable log \( r \)-spin fields with the discrete data \( \beta \) as in
(27). Let $\beta'$ be the reduced discrete data as in Section 2.4.1. Consider the universal stack $\mathcal{M}(A, \beta')$ as in Section 2.4.2. Consider the fiber product in the fine and saturated category

$$\mathcal{M} := \mathcal{M}(A, \beta') \times_{\mathcal{M}_{A,n}^{1/r}} \mathcal{M}_{g,n}^{1/r}$$

where the two arrows to $\mathcal{M}_{g,n}^{tw}$ are the tautological morphisms. By Propositions 2.13 and 4.5, $\mathcal{M}$ is log smooth and equi-dimensional.

By (11), we have the tautological morphism

$$\mathcal{J} \to \mathcal{M}$$

induced by the associated log maps (26) and the spin structures. This leads to the following commutative diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f_{\mathcal{J}}} & \mathcal{J} \\
\downarrow h & & \downarrow \pi_{\mathcal{J}} \\
\mathcal{P} & \xrightarrow{\pi_{\mathcal{J}}} & \mathcal{M}
\end{array}$$

where $f_{\mathcal{J}}: C_{\mathcal{J}} \to \mathcal{P}_{\mathcal{J}}$ is the universal log $r$-spin field, $\pi_{\mathcal{J}}: C_{\mathcal{J}} \to \mathcal{J}$ and $\pi_{\mathcal{M}}: C_{\mathcal{M}} \to \mathcal{M}$ are the universal log curves. Note that the two squares are both cartesian with vertical strict arrows.

We reserve the letter $L$ for the log cotangent complexes of [42], and the letter $T$ for its dual. For what follows, without further decoration all functors are automatically in the derived sense.

Observe that the left and right Cartesian squares imply

$$f_{\mathcal{J}}^* L_{\mathcal{P}_{\mathcal{J}}/C_{\mathcal{J}}} \cong h^* L_{\mathcal{P}_{\mathcal{M}}/C_{\mathcal{M}}}, \quad \text{and} \quad \pi_{\mathcal{J}}^* L_{\mathcal{J}/C_{\mathcal{M}}} \cong L_{C_{\mathcal{J}}/C_{\mathcal{M}}}$$

respectively. By the commutativity of arrows to $C_{\mathcal{M}}$, we have

$$h^* L_{\mathcal{P}_{\mathcal{M}}/C_{\mathcal{M}}} \to L_{C_{\mathcal{J}}/C_{\mathcal{M}}}$$

hence the morphism

$$f_{\mathcal{J}}^* L_{\mathcal{P}_{\mathcal{J}}/C_{\mathcal{J}}} \to \pi_{\mathcal{J}}^* L_{\mathcal{J}/C_{\mathcal{M}}}.$$

Since $\mathcal{P}_{\mathcal{J}} \to C_{\mathcal{J}}$ is log smooth and integral, we have $L_{\mathcal{P}_{\mathcal{J}}/C_{\mathcal{J}}} \cong \Omega_{\mathcal{P}_{\mathcal{J}}/C_{\mathcal{J}}}$ the log cotangent bundle. Pushing forward along $\pi_{\mathcal{J}}$, and using the adjunction morphism $\pi_{\mathcal{J}}^* \pi_{\mathcal{J}}^! L_{\mathcal{J}/C_{\mathcal{M}}} \to L_{\mathcal{J}/C_{\mathcal{M}}}$, we obtain

$$\pi_{\mathcal{J}}^* f_{\mathcal{J}}^* \Omega_{\mathcal{P}_{\mathcal{J}}/C_{\mathcal{J}}} \to L_{\mathcal{J}/C_{\mathcal{M}}}.$$

Since the morphism $\mathcal{J} \to \mathcal{M}$ is strict, we have that

$$L_{\mathcal{J}/C_{\mathcal{M}}} \cong L_{\mathcal{M}/C_{\mathcal{M}}}$$

where $L_{\mathcal{M}/C_{\mathcal{M}}}$ is the cotangent complex in the usual sense.

**Proposition 5.1.** The morphism (35) defines a perfect obstruction theory for $\mathcal{J} \to \mathcal{M}$ in the sense of Behrend-Fantechi [9, Section 7].
Proof. Note that \( \pi_{S_0} f^* f^* \Omega_{\mathcal{P}/\mathcal{C}} \) is a two-term complex perfect in \([-1, 0]\). It suffices to show that (35) is an obstruction theory.

Let \( S_0 \to S \) be a strict closed embedding induced by a square-zero ideal. Given a commutative diagram of solid arrows

\[
\begin{array}{c}
S_0 \ar[r] & \mathcal{I} \\
\downarrow & \\
S_1 \ar[r] & \mathcal{M}
\end{array}
\]

we want to study the dashed arrow lifting the bottom arrow. Using the associated log map (26), the above diagram of solid arrows translates to the following commutative diagram of solid arrows, and the dashed arrow translates to the dashed arrow below:

\[
\begin{array}{c}
C_{S_0} \ar[r] & \mathcal{P}_{C_S} \\
\downarrow & \\
C_S \ar[r] & A \times C_S
\end{array}
\]

Note that the vertical arrow on the right hand side is strict and smooth with the tangent bundle \( T_{\mathcal{P}_{C_S}/A \times C_S} \cong T_{\mathcal{P}_{C S}/C_S} \). Now the statement follows from the deformation theory of log morphisms in [42, Theorem 5.9].

Observe that the above construction of perfect obstruction theory is compatible with arbitrary base changes.

Lemma 5.2. For any morphism \( S \to \mathcal{M} \), consider \( \mathcal{I} := S \times_K \mathcal{I} \) with the pull-back \( f_{\mathcal{I}} : C_{\mathcal{I}} \to \mathcal{P}_{\mathcal{I}} \). Then the perfect obstruction theory (35) of \( \mathcal{I} \to \mathcal{M} \) pulls back to a perfect obstruction theory

\[
\pi_{\mathcal{I}_S} f_{\mathcal{I}}^* \Omega_{\mathcal{P}/\mathcal{C}} \to L_{\mathcal{I}_S/\mathcal{S}}.
\]

of the strict morphism \( \mathcal{I}_S \to S \).

5.1.2. The case of maps with uniform maximal degeneracy. Replacing \( \mathcal{M}(A, \beta') \) by \( \mathcal{U}(A, \beta') \) in (34), we obtain

\[
(36) \quad \mathcal{U} := \mathcal{U}(A, \beta') \times_{\mathcal{M}^{1/r}_{y, y}} \mathcal{M}^{1/r}_{y, y}
\]

By Theorem 2.13, the natural projection \( \mathcal{U} \to \mathcal{M} \) is log étale. Thus \( \mathcal{U} \) is log smooth and equi-dimensional.

Now consider \( \mathcal{U} := \mathcal{U}_\beta^{1/r} \) and the universal log \( r \)-spin field \( f_\mathcal{U} : C_\mathcal{U} \to \mathcal{P}_{\mathcal{U}} \) over \( \mathcal{U} \). Since \( \mathcal{U} = \mathcal{U} \times_{\mathcal{I}} \mathcal{I} \), applying Lemma 5.2, we obtain a relative perfect obstruction theory

\[
\pi_{\mathcal{U}_S} f_{\mathcal{U}}^* \Omega_{\mathcal{P}/\mathcal{C}} \to L_{\mathcal{U}_{\mathcal{S}}/\mathcal{U}}.
\]

For later use, denote by \( T_{\mathcal{P}_{\mathcal{U}}/\mathcal{C}_{\mathcal{U}}} = \Omega_{\mathcal{P}_{\mathcal{U}}/\mathcal{C}_{\mathcal{U}}}^* \) the log cotangent bundle of the projection \( \mathcal{P}_{\mathcal{U}} \to C_{\mathcal{U}} \). Taking dual of the above morphism, we
obtain
\[ T_{\mathscr{U}/\mathfrak{U}} \to \pi_{\mathscr{U},*} f_{\mathscr{U}}^* T_{\mathcal{P}_{\mathscr{U}}/\mathcal{C}_{\mathscr{U}}} =: E_{\mathscr{U}/\mathfrak{U}}. \]

The following result will be useful for later calculation.

**Lemma 5.3.** Let \( f: \mathcal{C} \to \mathcal{P} \) be a log \( r \)-spin fields over a log scheme \( S \), and \( \mathcal{L} \) the corresponding spin bundle over \( \mathcal{C} \). Then we have
\[ f^* T_{\mathcal{P}/\mathcal{C}} \cong f^*(\mathcal{O}_{\mathcal{P}}(0_{\mathcal{P}})) \cong \mathcal{L} \otimes f^*(\mathcal{O}_{\mathcal{P}}(\infty_{\mathcal{P}})). \]

**Proof.** Note that the usual tangent bundle is \( T_{\mathcal{P}/\mathcal{C}} = \mathcal{O}_{\mathcal{P}}(0_{\mathcal{P}} + \infty_{\mathcal{P}}) \).

The log tangent bundle \( T_{\mathcal{P}/\mathcal{C}} \subset T_{\mathcal{P}/\mathcal{C}} \) is the subsheaf consisting of vector fields vanishing along \( \infty_{\mathcal{P}} \). Thus we have \( T_{\mathcal{P}/\mathcal{C}} = \mathcal{O}_{\mathcal{P}}(0_{\mathcal{P}}) \) which proves the first equality.

The second equality follows from the observation that \( \mathcal{O}_{\mathcal{P}}(0_{\mathcal{P}} - \infty_{\mathcal{P}}) = \mathcal{L}|_{\mathcal{P}} \), where \( \mathcal{L}|_{\mathcal{P}} \) is the pull-back of \( \mathcal{L} \) along the morphism \( \mathcal{P} \to \mathcal{C} \). \( \square \)

**5.2. The relative cosection.** Consider the universal log \( r \)-spin fields \( f_{\mathscr{U}}: \mathcal{C}_{\mathscr{U}} \to \mathcal{P}_{\mathscr{U}} \) over \( \mathfrak{U} := \mathfrak{U}_{1/r}^{1/r} \). Denote by \( \pi_{\mathscr{U}}: \mathcal{C}_{\mathscr{U}} \to \mathfrak{U} \) the projection, and \( \mathcal{L}_{\mathfrak{U}} \) the universal \( r \)-spin bundle over \( \mathcal{C}_{\mathfrak{U}} \).

Throughout the rest of this section, we impose the following condition which is necessary for the cosection construction.

**Assumption 5.4.** All marked points are narrow with the zero contact order in \( \beta \).

**Notation 5.5.** For a vector bundle \( V \) over a log stack \( X \), write \( \text{Vb}(V) \) to be the geometric vector bundle associated to \( V \) with the strict morphism \( \text{Vb}(V) \to X \). For any morphism \( Y \to X \), denote by \( V|_Y \) and \( \text{Vb}(V)|_Y \) the pull-back of \( V \) and \( \text{Vb}(V) \) respectively.

**5.2.1. The twisted spin section.** Consider the canonical inclusion
\[ \iota: \mathcal{O}_{\mathfrak{U}} \to \mathcal{O}_{\mathfrak{U}}(0_{\mathfrak{U}}). \]

Using the isomorphisms
\[ \mathcal{O}_{\mathfrak{U}}(0_{\mathfrak{U}}) \cong \mathcal{L}|_{\mathfrak{U}} \otimes \mathcal{O}_{\mathfrak{U}}(1) \cong \mathcal{L}|_{\mathfrak{U}} \otimes \mathcal{O}_{\mathfrak{U}}(\infty_{\mathfrak{U}}), \]
we obtain
\[ f_{\mathfrak{U}}^* \iota: \mathcal{O}_{\mathfrak{U}} \to \mathcal{O}_{\mathfrak{U}}(0_{\mathfrak{U}}) \cong f_{\mathfrak{U}}^* \mathcal{O}_{\mathcal{P}}(0_{\mathfrak{P}}) \otimes \mathcal{O}_{\mathfrak{U}}(\infty_{\mathfrak{U}}). \]

By Assumption 5.4, we may pull back (22) via \( \mathfrak{U} \to \mathfrak{U} \) and obtain
\[ f_{\mathfrak{U}}^*: \pi_{\mathfrak{U}}^* \mathbb{L}_{\mathbb{L}} \otimes f_{\mathfrak{U}}^* \mathcal{O}(\infty_{\mathfrak{P}}) \to \mathcal{O}_{\mathfrak{U}}. \]

By abuse of notation, \( \mathbb{L}_{\mathbb{L}} \) denotes the pull-back of the corresponding line bundle over \( \mathfrak{U} \). Then we obtain a morphism
\[ f_{\mathfrak{U}}^* T_{\mathfrak{U}/\mathfrak{P}_{\mathfrak{U}}} \to \mathcal{O}(\infty_{\mathfrak{P}}) \otimes f_{\mathfrak{U}}^* \mathcal{L}_{\mathfrak{U}} \otimes \pi_{\mathfrak{U}}^* \mathbb{L}_{\mathbb{L}}. \]
Composing with (39), we obtain the twisted spin section
\[ s^{\gamma} := (\otimes \hat{f}^\gamma) \circ (f^\gamma)^* : \mathcal{O}_{C^\gamma} \to \mathcal{L}^\gamma \otimes \pi^*_{\mathcal{L}} L^{\gamma}_{\text{max}}, \]
or equivalently a morphism \( s^{\gamma} : C^\gamma \to \text{Vb}(\mathcal{L}^\gamma \otimes \pi^*_{\mathcal{L}} L^{\gamma}_{\text{max}}) \).

5.2.2. The twisted superpotential and its differentiation. Write for simplicity
\[ \omega_{\log, C^\gamma} := \omega^\log_{\mathcal{L}^\gamma / \mathcal{C}^\gamma} \quad \text{and} \quad \omega_{\mathcal{L}^\gamma} := \omega_{\mathcal{C}^\gamma / \mathcal{L}^\gamma}. \]

The \( r \)-spin structure \( \mathcal{L}^\gamma_r \cong \omega_{\log, C^\gamma} \) defines an isomorphism
\[ (\mathcal{L}^\gamma \otimes \pi^*_{\mathcal{L}} L^{\gamma}_{\text{max}})^{\otimes r} \cong \omega_{\log, C^\gamma} \otimes \pi^*_{\mathcal{L}} L^{\gamma}_{\text{max}}^{\otimes r} \]
hence a non-linear morphism
\[ W : \text{Vb}(\mathcal{L}^\gamma \otimes \pi^*_{\mathcal{L}} L^{\gamma}_{\text{max}}) \to \text{Vb}(\omega_{\log, C^\gamma} \otimes \pi^*_{\mathcal{L}} L^{\gamma}_{\text{max}}^{\otimes r}) \]
called the twisted superpotential. For convenience, we may equip both the source and the target of \( W \) with the log structures pulled back from \( C^\gamma \). In particular, \( W \) is a strict morphism. Differentiating \( W \), we have
\[ dW : T_{\text{Vb}(\mathcal{L}^\gamma \otimes \pi^*_{\mathcal{L}} L^{\gamma}_{\text{max}})/\mathcal{C}^\gamma} \to W^* T_{\text{Vb}(\omega_{\log, C^\gamma} \otimes \pi^*_{\mathcal{L}} L^{\gamma}_{\text{max}}^{\otimes r})/\mathcal{C}^\gamma}. \]

Pulling back \( dW \) via the twisted spin section (42) gives
\[ s^{\gamma}_* (dW) : \mathcal{L}^\gamma \otimes \pi^*_{\mathcal{L}} L^{\gamma}_{\text{max}} \to \omega_{\log, C^\gamma} \otimes \pi^*_{\mathcal{L}} L^{\gamma}_{\text{max}}^{\otimes r}. \]

Pushing forward along \( \pi^\gamma \) and applying the projection formula, we have
\[ (44) \quad \pi^\gamma_*, s^{\gamma}_* (dW) : \pi^\gamma_*(\mathcal{L}^\gamma) \otimes L^{\gamma}_{\text{max}} \to \pi^\gamma_*(\omega_{\log, C^\gamma}) \otimes L^{\gamma}_{\text{max}}^{\otimes r}. \]

Denote by \( \Sigma := \sum_i \sigma_i \), the sum of marked points of \( C^\gamma \). Since all the markings are narrow, recall from [13, Lemma 3.2] that the push-forward of the natural inclusion \( \mathcal{L}^\gamma (-\Sigma) \hookrightarrow \mathcal{L}^\gamma \) is an isomorphism
\[ (45) \quad \pi^\gamma_*, \mathcal{L}^\gamma (-\Sigma) \xrightarrow{\cong} \pi^\gamma_*, \mathcal{L}^\gamma. \]

Twisting down (43) by the markings and pushing forward, we have
\[ (46) \quad \pi^\gamma_*, s^{\gamma}_* (dW)(-\Sigma) : \pi^\gamma_*(\mathcal{L}^\gamma (-\Sigma)) \otimes L^{\gamma}_{\text{max}} \to \pi^\gamma_*(\omega_{\log, C^\gamma}) \otimes L^{\gamma}_{\text{max}}^{\otimes r}. \]

The two morphisms (44) and (46) fit in a commutative diagram
\[ (47) \quad \pi^\gamma_*, (\mathcal{L}^\gamma (-\Sigma)) \otimes L^{\gamma}_{\text{max}} \xrightarrow{\pi^\gamma_*, s^\gamma_*(dW)(-\Sigma)} \pi^\gamma_*, \omega_{\log, C^\gamma} \otimes L^{\gamma}_{\text{max}}^{\otimes r} \]
\[ \cong \]
\[ \pi^\gamma_*, (\mathcal{L}^\gamma) \otimes L^{\gamma}_{\text{max}} \xrightarrow{\pi^\gamma_*, s^\gamma_*(dW)} \pi^\gamma_*, (\omega_{\log, C^\gamma}) \otimes L^{\gamma}_{\text{max}}^{\otimes r} \]
where the vertical arrows are induced by twisting down by \( \Sigma \), and the left vertical arrow is the isomorphism (45).
We give a point-wise description of $R^1\pi_{\mathcal{U},*}\mathcal{S}_{\mathcal{U}}^*(dW)(-\Sigma)$. Consider a geometric point $w \to \mathcal{U}$ with the pullback $(C_w/w, \mathcal{L}_w, f_w)$. Using Serre duality and the spin structure $\mathcal{L}_w \cong \omega_{\log,w}$, we have

$$H^1(\mathcal{L}_w(-\Sigma)) \otimes \mathcal{L}_{\text{max},w}^\vee \cong H^0(\mathcal{L}_w^\vee(\Sigma) \otimes \omega_w) \otimes \mathcal{L}_{\text{max},w}^\vee$$

$$\cong H^0(\mathcal{L}_w^\vee \otimes \omega_{\log,w}) \otimes \mathcal{L}_{\text{max},w}^\vee$$

$$\cong H^0(\mathcal{L}_w^{-1}) \otimes \mathcal{L}_{\text{max},w}^\vee.$$ 

The fiber $R^1\pi_{\mathcal{U},*}\mathcal{S}_{\mathcal{U}}^*(dW)(-\Sigma)_w$ is then given by

$$(48) \quad H^0(\mathcal{L}_w^{-1}) \otimes \mathcal{L}_{\text{max},w}^\vee \to \mathcal{L}_{\text{max},w}^{\otimes-r}, \quad s \mapsto rs_{w}^{-1} \cdot \tilde{s}.$$ 

where $s_w$ is the fiber of (42) at $w$, hence $s_w^{-1} \in H^0(\mathcal{L}_w^{-1}) \otimes \mathcal{L}_{\text{max},w}^{(1-r)}$.

5.2.3. The relative cosection. Pushing forward (41), we obtain

$$(49) \quad \mathcal{E}_{\mathcal{U}/\mathcal{X}} \to \pi_{\mathcal{U},*}(\mathcal{L}_{\mathcal{U}}) \otimes \mathcal{L}_{\text{max}}^\vee.$$ 

Composing with the left and top arrows of (47), we obtain

$$(50) \quad \mathcal{E}_{\mathcal{U}/\mathcal{X}} \to \pi_{\mathcal{U},*}\omega_{\mathcal{U}} \otimes \mathcal{L}_{\text{max}}^{\otimes-r},$$

whose $H^1$ defines the relative cosection

$$(51) \quad \sigma_{\mathcal{U}/\mathcal{X}} : \text{Obs}_{\mathcal{U}/\mathcal{X}} := H^1(\mathcal{E}_{\mathcal{U}/\mathcal{X}}) \to R^1\pi_{\mathcal{U},*}\omega_{\mathcal{U}} \otimes \mathcal{L}_{\text{max}}^{\otimes-r} \cong \mathcal{L}_{\text{max}}^{\otimes-r}.$$ 

By abuse of notation, denote by $\Delta_{\text{max}} \subset \mathcal{U}$ the pre-image of $\Delta_{\text{max}} \subset \mathcal{X}$. Let $\mathcal{X}^{\circ} := \mathcal{U} \setminus \Delta_{\text{max}}$. Then $\mathcal{X}^{\circ}$ is the stack parameterizing sections of the $r$-spin bundle. Note that $\mathcal{X}^{\circ}$ is the stack $X$ as in [13, Section 3].

**Lemma 5.6.** In the $r$-spin case, the restriction of (37) to $\mathcal{X}^{\circ}$ is the perfect obstruction theory in [13, (3.2)], and $\sigma_{\mathcal{U}/\mathcal{X}|\mathcal{X}^{\circ}}$ is the relative cosection in [13, (3.5)].

**Proof.** By assumption, we have that

$$f_{\mathcal{U},*}\mathcal{O}(\infty_p) = \mathcal{O}_{\mathcal{X}^{\circ}} \quad \text{and} \quad \mathcal{L}_{\text{max}}|_{\mathcal{X}^{\circ}} = \mathcal{O}_{\mathcal{X}^{\circ}}.$$ 

Then the statement follows from the construction of $\sigma_{\mathcal{U}/\mathcal{X}}$.

5.2.4. Surjectivity of $\sigma_{\mathcal{U}/\mathcal{X}}$ along the boundary.

**Lemma 5.7.** Suppose a narrow marking has the trivial contact order. Then its image via $f_{\mathcal{U}}$ is contained in the zero section $0_{\mathcal{P}} \subset \mathcal{P}_{\mathcal{U}}$.

**Proof.** We first show that the image of such a narrow marking avoids the infinity section. Since this is an open condition, it suffices to check this over a geometric fiber $w \to \mathcal{U}$ with the log twisted field $f_w : C_w \to \mathcal{P}_w$.

Suppose $f_w(\sigma_i) \in \infty_{\mathcal{P}}$ for a narrow marking $\sigma_i \in C_w$. Then there is an irreducible Zariski neighborhood $V \subset C_w$ of $p$ such that $\mathcal{M}_{C_w} \cong \pi_{\mathcal{U}}\mathcal{M}_{C_w} \oplus \pi_{\mathcal{U}}N_{\mathcal{U}}$, see Section 2.1.4. Since the contact order of $\sigma_i$ is trivial, $f_w|_V$ induces a morphism of $\mathcal{O}^*$-torsors over $V$ of the form $f_w|_V : f_w\mathcal{T}_{\infty_p}|_V \to \mathcal{T}_{e_V}$ where $e_V \in \pi_{\mathcal{U}}\mathcal{M}_{C_w}$ is the degeneracy of $f_w$.
along $V$, $T_e = eV \times_{\mathcal{M}_{C_w}} \mathcal{M}_{C_w}$, and $T_{\infty, P} \subset \mathcal{M}_{P_w}$ is the preimage of the torsor $T_{\infty}$ as in (21) via $P_w \to A$. Taking the corresponding line bundles, we obtain a morphism $f^*\mathcal{O}(\infty_P)|_V \to \mathcal{O}_V$ whose dual $\mathcal{O}_V \to f^*\mathcal{O}(\infty_P)|_V$ is non-vanishing at $p$ since the contact order at $p$ is trivial. Note that $f^*\mathcal{O}(\infty_P)|_V \cong \mathcal{L}_V^\vee|_V$, we thus obtain a local section of $\mathcal{L}_V^\vee$ non-vanishing at a narrow marking. But by [13, Lemma 3.2] and [6, Proposition 3.0.3] such a local section vanishes at $\sigma_i$. This is a contradiction.

Since $f_\mathcal{V}(\sigma_i)$ avoids $\infty_P$, locally $f_\mathcal{V}$ is a section of $\mathcal{L}_\mathcal{V}$ around $\sigma_i$, hence vanishes along $\sigma_i$. This completes the proof. □

We next prove the surjectivity of $\sigma_{\mathcal{V}/\mathfrak{U}}$ along $\Delta_{\max}$.

**Proposition 5.8.** The vanishing locus $(\sigma_{\mathcal{V}/\mathfrak{U}} = 0) \subset \mathcal{V}$ is given by the locus along which $f_\mathcal{V}$ is the zero section.

**Proof.** By Lemma 5.6 and [13, Lemma 3.6], $\sigma_{\mathcal{V}/\mathfrak{U}}|_{\mathcal{V}}$ vanishes along the locus where $f_\mathcal{V}$ is the zero section. It remains to show that $\sigma_{\mathcal{V}/\mathfrak{U}}$ is surjective along $\Delta_{\max}$. Since $\mathcal{L}_{\max}^{\otimes -r}$ is a line bundle, the image of $\sigma_{\mathcal{V}/\mathfrak{U}}$ is a torsion-free sub-sheaf of $\mathcal{L}_{\max}^{\otimes -r}$. Thus it suffices to show that $\sigma_{\mathcal{V}/\mathfrak{U}}$ is surjective at each geometric point of $\Delta_{\max}$.

Let $w \in \Delta_{\max}$ be a geometric point with $(\mathcal{C}_w/w, \mathcal{L}_w, f_w)$. Taking $H^1$ of (41) over $w$, we have

$$H^1(\mathcal{L}_w \otimes f_w^*\mathcal{O}(\infty_P)) \to H^1(\mathcal{L}_w \otimes \mathcal{L}_{\max,w}^\vee).$$

By construction $\sigma_{\mathcal{V}/\mathfrak{U},w}$ is the following composition

$$H^1(\mathcal{L}_w \otimes f_w^*\mathcal{O}(\infty_P)) \longrightarrow H^1(\mathcal{L}_w) \otimes \mathcal{L}_{\max,w}^\vee$$

(by (45))

$$\cong$$

(by (46))

$$\longrightarrow \mathcal{L}_{\max,w}^{\otimes -r}$$

where the first arrow is (52). Applying Serre duality and taking the dual, we have $\sigma_{\mathcal{V}/\mathfrak{U},w}$:

$$H^0(\mathcal{L}_w^\vee \otimes f_w^*\mathcal{O}(\infty_P) \otimes \omega_w) \longrightarrow H^0(\mathcal{L}_w^\vee \otimes \omega_w) \otimes \mathcal{L}_{\max,w}$$

(by (52))

$$\cong$$

$$H^0(\mathcal{L}_w^{r-1} \otimes \mathcal{L}_{\max,w}^\vee)$$

$$\cong$$

$$\mathcal{L}_{\max,w}^{\otimes r}$$

where the first and last arrow is given by the dual of (52) and (48) respectively. We describe $\sigma_{\mathcal{V}/\mathfrak{U},w}$ via the above composition as follows.

Suppose $v_0 \in \mathcal{L}_{\max,w}^{\otimes r}$ is a non-zero vector. Applying dual of (48), we obtain a vector $v_1 := (rs^r_{w} \otimes \omega_w)|_Z \in H^0(\mathcal{L}_w^{r-1} \otimes \mathcal{L}_{\max,w})$. By Proposition 3.17, the section $v_1$ is non-trivial along the sub-curve $Z \subset \mathcal{C}_w$ consisting of maximally degenerate components, and vanishes along $\mathcal{C}_w \setminus Z$.

By Lemma 5.7, since $Z$ contains no markings, we have $(\mathcal{L}_w|_Z)^r \cong \omega_w|_Z$. Thus $v_1$ is a section of $\mathcal{L}_w^\vee \otimes \omega_w \otimes \mathcal{L}_{\max,w}^\vee$ non-trivial along $Z$, and vanishes along $\mathcal{C}_w \setminus Z$. 
Finally observe that the dual of (52) is given by
\[ L_\omega \otimes \omega \otimes l_{\text{max},w} \otimes f'_w \to L'_\omega \otimes f'_w \mathcal{O}(-\infty_{\mathcal{P}_w}) \]

hence \( \sigma'_w|_{\mathcal{U}w}(v_0) = v_1 \otimes f'_w \). By Proposition 3.17 again, \( f'_w \) hence \( v_1 \otimes f'_w \) is non-trivial along \( Z \). In particular, \( \sigma'_w|_{\mathcal{U}w}(v_0) \neq 0 \).

The above analysis implies that \( \sigma'_w|_{\mathcal{U}w} \) is injective, hence \( \sigma_{\mathcal{U}w} \) is surjective. This completes the proof. □

5.3. Factorization of the relative obstruction.

5.3.1. An auxiliary twist. Denote by \( \mathcal{L}_{\mathcal{U},-} := \mathcal{L}_{\mathcal{U}}(-\Sigma) \) for simplicity. Similar to the construction of \( \mathcal{P}_{\mathcal{U}} \) in Section 4.2.1, we formulate the stack \( \mathcal{P}_{\mathcal{U},-} \) with \( \mathcal{L}_{\mathcal{U}} \) replaced by \( \mathcal{L}_{\mathcal{U},-} \). The log structure of \( \mathcal{P}_{\mathcal{U},-} \) is defined to be
\[ \mathcal{M}_{\mathcal{P}_{\mathcal{U},-}} := \mathcal{M}_{\mathcal{C}_\mathcal{U}}|_{\mathcal{P}} \oplus \mathcal{M}_{\mathcal{\infty}_{\mathcal{P},-}} \]

where \( \mathcal{\infty}_{\mathcal{P},-} \subseteq \mathcal{P}_{\mathcal{U},-} \) is the corresponding infinity section. The natural morphism \( \text{Vb}(\mathcal{L}_{\mathcal{U},-}) \to \text{Vb}(\mathcal{L}_{\mathcal{U}}) \) induces a birational map of log stacks \( \mathcal{P}_{\mathcal{U},-} \to \mathcal{P}_{\mathcal{U}} \) which is isomorphic away from fibers over marked points. Denote by \( \mathcal{P}_{\mathcal{U},\text{reg}} \subseteq \mathcal{P}_{\mathcal{U},-} \) the open sub-stack where the above rational map is well-defined. Let \( \mathbf{t}: \mathcal{P}_{\mathcal{U},\text{reg}} \to \mathcal{P}_{\mathcal{U}} \) be the corresponding morphism. Denote by \( \mathcal{P}_{\mathcal{U},\text{reg}} \) the pull-back of \( \mathcal{P}_{\mathcal{U},\text{reg}} \) with the corresponding morphism \( \mathbf{t}: \mathcal{P}_{\mathcal{U},\text{reg}} \to \mathcal{P}_{\mathcal{U},\text{reg}} \).

**Lemma 5.9.** There is a canonical factorization
\[ C_{\mathcal{U}} \xrightarrow{f_{\mathcal{U}}} \mathcal{P}_{\mathcal{U}} \xrightarrow{\mathbf{t}} \mathcal{P}_{\mathcal{U},\text{reg}} \]

**Proof.** Note that \( \mathcal{P}_{\mathcal{U},\text{reg}} \subseteq \mathcal{P}_{\mathcal{U}} \) is obtained by removing the fiber of \( \mathcal{\infty}_{\mathcal{P},-} \) over marked points. The statement follows from Lemma 5.7. □

Denote by \( \mathcal{P}_{\mathcal{U},-} \to \mathcal{A} \) the morphism of log stacks such that \( \mathcal{M}_{\mathcal{\infty}_{\mathcal{P},-}} \) is the pull-back of \( \mathcal{M}_{\mathcal{A}} \). Consider the natural morphism \( \mathcal{U} \to \mathcal{U} \) induced by the composition of \( f_{\mathcal{U},-} \) with \( \mathcal{P}_{\mathcal{U},-} \to \mathcal{A} \). The above lemma implies that \( \mathcal{U} \) can be viewed as the log stack parameterizing log twisted sections \( f_{\mathcal{T},-}: \mathcal{C}_\mathcal{T} \to \mathcal{P}_{\mathcal{T},-} \) for any \( \mathcal{T} \to \mathcal{U} \). The same construction in Section 5.1.1 provides a perfect obstruction theory of \( \mathcal{U} \to \mathcal{U} \):

\[ T_{\mathcal{U}/\mathcal{U}} \to \pi_{\mathcal{U}} \circ f_{\mathcal{U},-} T_{\mathcal{P}_{\mathcal{U},\text{reg}}/\mathcal{C}_{\mathcal{U}}} \cong \pi_{\mathcal{U}} \circ f_{\mathcal{U},-} T_{\mathcal{P}_{\mathcal{U},-}/\mathcal{C}_{\mathcal{U}}} =: E_{\mathcal{U}/\mathcal{U},-}. \]

On the other hand since \( f_{\mathcal{U}}^* \mathcal{O}(\mathcal{\infty}_{\mathcal{P}}) \cong f_{\mathcal{U},-}^* \mathcal{O}(\mathcal{\infty}_{\mathcal{P},-}) \), we calculate
\[ f_{\mathcal{U},-}^* T_{\mathcal{P}_{\mathcal{U},-}/\mathcal{C}_{\mathcal{U}}} \cong \mathcal{L}_{\mathcal{\infty}_{\mathcal{P},-}} \otimes f_{\mathcal{U},-}^* \mathcal{O}(\mathcal{\infty}_{\mathcal{P},-}) \]
\[ \cong \mathcal{L}_{\mathcal{U}}(-\Sigma) \otimes f_{\mathcal{U}}^* \mathcal{O}(\mathcal{\infty}_{\mathcal{P}}) \]
\[ \cong f_{\mathcal{U}}^* T_{\mathcal{P}_{\mathcal{U}}/\mathcal{C}_{\mathcal{U}}}(-\Sigma). \]
Using (45) and Lemma 5.7, we have
\[ \pi_{\mathcal{U}, *} \mathcal{T}_{P_{\mathcal{U}, \reg}/C_{\mathcal{U}}} \cong \pi_{\mathcal{U}, \ast} \mathcal{T}_{P_{\mathcal{U}}/C_{\mathcal{U}}}. \]

To summarize:

**Lemma 5.10.** The two perfect obstruction theories (37) and (53) are identical.

We now view $\mathcal{U}$ with the universal family $f_{\mathcal{U}, -}: C_{\mathcal{U}} \to P_{\mathcal{U}, -}$.

5.3.2. Partial expansion and contraction. The morphism $m: (\mathcal{A}, \beta') \to A_{\max}$ from Section 3.5 induces a morphism $\mathcal{U} \to A_{\max}$ which will again be denoted by $m$ by abuse of notation. Consider the following cartesian diagram of fine log stacks
\[
\begin{array}{ccc}
\mathcal{P}_{\mathcal{U}, -} & \to & \mathcal{A} \times A_{\max} \\
\downarrow b & & \downarrow b \\
\mathcal{P}_{\mathcal{U}, -} & \to & A_{\max}
\end{array}
\]
where the bottom is the product of $m$ and $\mathcal{P}_{\mathcal{U}, -} \to A$.

By construction, one checks that the bottom arrow satisfies the flatness conditions in [33, Proposition (4.1)], hence is integral in the sense of [33, Definition (4.3)]. In particular, the underlying structure of the above cartesian diagram is a cartesian diagram of the underlying algebraic stacks. We remark that the above diagram is indeed cartesian in the fine and saturated category. Since the saturation plays no role in the following discussion, we omit the details here.

In the above diagram, since the right vertical arrow is log étale, the left vertical arrow is again log étale. By abuse of notation, we denote both vertical arrows by $b$. Let $\infty_{\mathcal{P}_{-}} \subset \mathcal{P}_{\mathcal{U}, -}$ be the pre-image of $\infty_{\mathcal{A}_{\mathcal{U}}} \subset \mathcal{A}_{\mathcal{U}}$, and write $\mathcal{P}^{\circ}_{\mathcal{U}, -} := \mathcal{P}_{\mathcal{U}, -} \setminus \infty_{\mathcal{P}_{-}}$. Denote by $\mathcal{E}_{b} \subset \mathcal{P}^{\circ}_{\mathcal{U}, -}$ the exceptional divisor contracted by $b$. In the following, we view the (relative) normal bundle $\mathcal{N}_{\infty_{\mathcal{P}_{-}}} \cong \mathcal{A}_{\mathcal{U}}$ as a line bundle over $C_{\mathcal{U}}$.

**Lemma 5.11.** $\mathcal{N}_{\infty_{\mathcal{P}_{-}}}^{\vee} \cong \mathcal{L}_{\mathcal{U}, -} \otimes \pi_{\mathcal{U}}^{\ast} \mathcal{L}_{\max}^{\vee}$.

**Proof.** Observe that $b^{\ast} [\infty_{\mathcal{P}_{-}}] = [\infty_{\mathcal{P}_{-}}] + [\mathcal{E}_{b}]$ where $[\ast]$ denotes the corresponding divisor class. Pulling back to $C_{\mathcal{U}}$ via the identification $C_{\mathcal{U}} \cong \infty_{\mathcal{P}_{-}} \cong \infty_{\mathcal{P}_{-}}$, we obtain
\[ \mathcal{O}(\infty_{\mathcal{P}_{-}})|_{\infty_{\mathcal{P}_{-}}} \cong (\mathcal{O}(\infty_{\mathcal{P}_{-}}) \otimes \mathcal{O}(\mathcal{E}_{b}))|_{\infty_{\mathcal{P}_{-}}}. \]
Using $\mathcal{O}(\infty_{\mathcal{P}_{-}})|_{\infty_{\mathcal{P}_{-}}} \cong \mathcal{N}_{\infty_{\mathcal{P}_{-}}} \cong \mathcal{L}_{\mathcal{U}, -}^{\vee}$ and $\mathcal{O}(\infty_{\mathcal{P}_{-}})|_{\infty_{\mathcal{P}_{-}}} \cong \mathcal{N}_{\infty_{\mathcal{P}_{-}}}$, we obtain
\[ \mathcal{L}_{\mathcal{U}, -}^{\vee} \cong \mathcal{N}_{\infty_{\mathcal{P}_{-}}} \otimes \mathcal{O}(\mathcal{E}_{b})|_{\infty_{\mathcal{P}_{-}}}. \]

Finally, observe that $\mathcal{O}(\mathcal{E}_{b})|_{\infty_{\mathcal{P}_{-}}} \cong \pi_{\mathcal{U}}^{\ast} \mathcal{L}_{\max}^{\vee}$, which leads to the desired isomorphism. \qed
Lemma 5.12. There is a commutative diagram of log stacks

\[
\begin{array}{ccc}
\mathcal{P}_{\tilde{u},-}^c & \xrightarrow{c} & Vb(\mathcal{L}_{u,-} \otimes \pi_{u}^* \mathcal{L}_{\max}^\vee) \\
\downarrow & & \downarrow \\
\mathcal{C}_{\tilde{u}} & \xrightarrow{\tilde{c}} & \\
\end{array}
\]

where \(c\) is a birational morphism contracting \(\mathcal{E}_c\), the proper transform of \(\mathcal{P}_{\tilde{u},-} \times_{\mathcal{C}_{\tilde{u}}} \Delta_{\max}\), to the zero section of \(Vb(\mathcal{L}_{u,-} \otimes \pi_{u}^* \mathcal{L}_{\max}^\vee)\).

Proof. Note that once the underlying morphism of \(c\) is defined, the morphism on the level of log structures is automatically obtained since the right skew arrow is strict. We may assume for simplicity that all stacks in the rest of this proof have the trivial log structure.

Note that \(3[\infty_{\mathcal{P}_{-}}]\) is a relative nef divisor of the family of nodal rational curves \(\mathcal{P}_{\tilde{u},-}^c \rightarrow \mathcal{C}_{\tilde{u}}\). Let \(\tilde{c}: \mathcal{P}_{\tilde{u},-}^c \rightarrow \mathcal{P}_{\tilde{u},-}\) be the induced contraction, and \(\mathcal{E}_c \subset \mathcal{P}_{\tilde{u},-}^c\) the exceptional locus contracted by \(\tilde{c}\). Then \(\mathcal{E}_c\) is the proper transform of \(\mathcal{P}_{\tilde{u},-} \times_{\mathcal{C}_{\tilde{u}}} \Delta_{\max}\). Observe that the resulting projection \(\mathcal{P}_{\tilde{u},-}^c \rightarrow \mathcal{C}_{\tilde{u}}\) is again a smooth \(\mathbb{P}^1\)-fibration since the contracted locus consists of a family of \((-1)\)-curves over \(\mathcal{C}_{\tilde{u}}\).

Furthermore, note that \(\tilde{c}\) induces an embedding \(\mathcal{N}_{\infty_{\mathcal{P}_{-}}} \rightarrow \mathcal{P}_{\tilde{u},-}^c\) over \(\mathcal{C}_{\tilde{u}}\) with complement \(0_{\mathcal{P}_{-}} \subset \mathcal{P}_{\tilde{u},-}^c \setminus \mathcal{N}_{\infty_{\mathcal{P}_{-}}}\) given by the image of the zero section \(0_{\mathcal{P}_{-}} \subset \mathcal{P}_{\tilde{u},-}^c\). We thus obtain

\[
\mathcal{P}_{\tilde{u},-}^c \cong \mathbb{P}(\mathcal{N}_{\infty_{\mathcal{P}_{-}}} \oplus \mathcal{O}) \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{N}_{\infty_{\mathcal{P}_{-}}}^\vee).
\]

Thus \(c\) is obtained from \(\tilde{c}\) by removing \(\infty_{\mathcal{P}_{-}}\) and its image in \(\mathcal{P}_{\tilde{u},-}^c\). □

Consider the canonical morphism induced by the divisor \(\mathcal{E}_c\)

\[
(56) \quad \iota_{\mathcal{E}_c}: \mathcal{O}_{\mathcal{P}_{\tilde{u},-}} \rightarrow \mathcal{O}_{\mathcal{P}_{\tilde{u},-}^c}(\mathcal{E}_c) \cong \mathcal{L}_{\max}^\vee(-\mathcal{E}_b),
\]

and the morphism of log tangent bundles

\[
dc: T_{\mathcal{P}_{\tilde{u},-}^c/\mathcal{C}_{\tilde{u}}} \rightarrow c^*TVb(\mathcal{L}_{u,-} \otimes \pi_{u}^* \mathcal{L}_{\max}^\vee)/\mathcal{C}_{\tilde{u}}.
\]

Lemma 5.13. \(dc = \otimes \iota_{\mathcal{E}_c}\).

Proof. Consider the morphism of log cotangent bundles

\[
c^*: \Omega_{Vb(\mathcal{L}_{u,-} \otimes \pi_{u}^* \mathcal{L}_{\max}^\vee)/\mathcal{C}_{\tilde{u}}} \rightarrow c^*\Omega_{\mathcal{P}_{\tilde{u},-}^c/\mathcal{C}_{\tilde{u}}}.
\]

Note that \(c\) is an isomorphism away from the divisor \(\mathcal{E}_c\). Furthermore, the contraction \(c\) is the blow-up of the zero section of \(Vb(\mathcal{L}_{u,-} \otimes \pi_{u}^* \mathcal{L}_{\max}^\vee)\). A local coordinate calculation shows that \(c^* = \otimes \iota_{\mathcal{E}_c}^\vee\). Taking dual, we obtain the desired equality. □
5.3.3. Twisted spin section via partial expansion. Consider the commutative diagram of solid arrows

\[
\begin{array}{c}
\mathcal{C}^- \\
\downarrow f_{\mathcal{U},-} \quad \downarrow \quad \downarrow A_{\mathcal{C},-} \\
\mathcal{P}^-_{\mathcal{U},-} & \longrightarrow & \mathcal{A}^e_{\mathcal{C},-} \\
\end{array}
\]

where the square is the pull-back of (55) via \( \mathcal{U} \to \mathcal{U} \). We then obtain the dashed arrow \( f^e_{\mathcal{U},-} \). Consider the following composition

\[
s_{\mathcal{U},-} : C_{\mathcal{U}} \xrightarrow{f_{\mathcal{U},-}} \mathcal{P}^-_{\mathcal{U},-} \xrightarrow{c_{\mathcal{U}}} \text{Vb}(L_{\mathcal{U},-} \otimes \pi_{\mathcal{U}}^* L_{\max}^\vee)
\]

where \( c_{\mathcal{U}} \) is the pull-back of the contraction \( c \) as in Lemma 5.12.

**Lemma 5.14.** The section \( s_{\mathcal{U}} \) in (42) is given by the composition

\[
C_{\mathcal{U}} \xrightarrow{s_{\mathcal{U}}} \text{Vb}(L_{\mathcal{U},-} \otimes \pi_{\mathcal{U}}^* L_{\max}^\vee) \longrightarrow \text{Vb}(L_{\mathcal{U}} \otimes \pi_{\mathcal{U}}^* L_{\max}^\vee).
\]

**Proof.** Since \( t : \mathcal{P}_{\mathcal{U},\text{reg}} \to \mathcal{P}_{\mathcal{U}} \) is well-defined along the zero section, pulling back (38) we have

\[
t^*t : \mathcal{O}_{\mathcal{P}_{\mathcal{U},\text{reg}}} \to \mathcal{O}_{\mathcal{P}_{\mathcal{U},\text{reg}}} (0_{\mathcal{P}_-}) \otimes \mathcal{O}_{C_{\mathcal{U}}}(\Sigma)|_{\mathcal{P}_{\mathcal{U},\text{reg}}}.
\]

Since \( \mathcal{O}_{\mathcal{P}_{\mathcal{U},\text{reg}}} (0_{\mathcal{P}_-}) \cong \mathcal{L}_{\mathcal{U},-} |_{\mathcal{P}_{\mathcal{U},\text{reg}}} \otimes \mathcal{O}_{\mathcal{P}_{\mathcal{U},\text{reg}}} (\infty_{\mathcal{P}_-}) \), further pulling back to \( \mathcal{P}_{\mathcal{U},\text{reg}} = b^{-1}(\mathcal{P}_{\mathcal{U},\text{reg}}) \), we have

\[
b^*t^*t : \mathcal{O}_{\mathcal{P}_{\mathcal{U},\text{reg}}} \to \mathcal{L}_{\mathcal{U},-} |_{\mathcal{P}_{\mathcal{U},\text{reg}}} \otimes \mathcal{O}_{\mathcal{P}_{\mathcal{U},\text{reg}}} (\infty_{\mathcal{P}_-} + \mathcal{E}_b) \otimes \mathcal{O}_{C_{\mathcal{U}}}(\Sigma)|_{\mathcal{P}_{\mathcal{U},\text{reg}}}
\]

which naturally restricts to

\[
b^*t^*t : \mathcal{O}_{\mathcal{P}^-_{\mathcal{U},-}} \to \mathcal{L}_{\mathcal{U},-} |_{\mathcal{P}^-_{\mathcal{U},-}} \otimes \mathcal{O}_{\mathcal{P}^-_{\mathcal{U},-}} (\infty_{\mathcal{P}_-} + \mathcal{E}_b) \otimes \mathcal{O}_{C_{\mathcal{U}}}(\Sigma)|_{\mathcal{P}^-_{\mathcal{U},-}}.
\]

Since \( f_{\mathcal{U}} \) factors through \( f^e_{\mathcal{U},-} \), we have \( (f_{\mathcal{U},-}^e)^* (b^*t^*t) = f^e_{\mathcal{U},-} \).

By Lemma 3.19, we have \( (f^e_{\mathcal{U},-})^* (\otimes \iota_{\mathcal{E}_i}) = (\otimes \tilde{f}_{\mathcal{U}}^e) \) in (41). Putting things together, we have

\[
s_{\mathcal{U}} = (\otimes \tilde{f}_{\mathcal{U}}^e) \circ f^e_{\mathcal{U},-} = (f^e_{\mathcal{U},-})^* (\otimes \iota_{\mathcal{E}_i}) \circ (f^e_{\mathcal{U},-})^* (b^*t^*t)
\]

\[
= (f^e_{\mathcal{U},-})^* ((\otimes \iota_{\mathcal{E}_i}) \circ (b^*t^*t)).
\]

Note that \( (\otimes \iota_{\mathcal{E}_i}) \circ (b^*t^*t) \) is the following morphism

\[
\mathcal{O}_{\mathcal{P}^-_{\mathcal{U},-}} \to \mathcal{L}_{\mathcal{U},-} |_{\mathcal{P}^-_{\mathcal{U},-}} \otimes \mathcal{O}_{\mathcal{P}^-_{\mathcal{U},-}} (\infty_{\mathcal{P}_-}) \otimes \mathcal{L}_{\mathcal{max}}^\vee \otimes \mathcal{O}_{C_{\mathcal{U}}}(\Sigma)|_{\mathcal{P}^-_{\mathcal{U},-}}
\]

which factors through the natural morphism

\[
\mathcal{O}_{\mathcal{P}^-_{\mathcal{U},-}} \to \mathcal{L}_{\mathcal{U},-} |_{\mathcal{P}^-_{\mathcal{U},-}} \otimes \mathcal{O}_{\mathcal{P}^-_{\mathcal{U},-}} (\infty_{\mathcal{P}_-}) \otimes \mathcal{L}_{\mathcal{max}}^\vee.
\]

Write \( V = \text{Vb}(L_{\mathcal{U},-} \otimes \pi_{\mathcal{U}}^* L_{\max}^\vee) \) for simplicity. The section \( s_{\mathcal{U},-} \) is the pull-back of the following canonical morphism via itself

\[
\iota_- : \mathcal{O}_V \to \mathcal{O}_V (0_V) \cong (L_{\mathcal{U},-} \otimes \pi_{\mathcal{U}}^* L_{\max}^\vee)|_V.
\]
This pulls back to
\[ \mathcal{C}_-^\ell : \mathcal{O}_{\mathcal{P}^e_{\mathcal{U}, -}} \to \mathcal{L}_{\mathcal{U}, -}|_{\mathcal{P}^e_{\mathcal{U}, -}} \otimes \mathcal{L}^r_{\max}, \]
which is the restriction of (59). Since \( s_{\mathcal{U}} \) factors through \( f^e_{\mathcal{U}, -} \), the section (59) pulls back to \( s_{\mathcal{U}, -} \) via \( f^e_{\mathcal{U}, -} \). This finishes the proof. \( \square \)

5.3.4. Relative cosection via partial expansion. For simplicity, write \( \mathcal{L}_{\mathcal{U}, -} := \mathcal{L}_{\mathcal{U}, -} \otimes \pi_{\mathcal{U}}^\times \mathcal{L}^r_{\max} \) and \( \tilde{\mathcal{H}}_{\mathcal{U}} := \omega_{\mathcal{U}} \otimes \pi_{\mathcal{U}}^\times \mathcal{L}^{-\otimes r}_{\max} \).

Consider the following composition
\[ \mathcal{P}^e_{\mathcal{U}, -} \xrightarrow{\epsilon} \text{Vb}(\tilde{\mathcal{H}}_{\mathcal{U}, -}) \xrightarrow{W_{\text{-}}} \text{Vb}(\tilde{\mathcal{H}}_{\mathcal{U}, -}((1 - r)\Sigma)) \xrightarrow{W_{\text{-}}} \text{Vb}(\tilde{\mathcal{H}}_{\mathcal{U}}). \]

Take the differentiation
\[ T_{\mathcal{P}^e_{\mathcal{U}, -}/\mathcal{C}_{\mathcal{U}}} \xrightarrow{d\epsilon} \mathcal{C}_{\text{Vb}(\tilde{\mathcal{H}}_{\mathcal{U}, -})/\mathcal{C}_{\mathcal{U}}} \xrightarrow{dW_{\text{-}}} (W_{\text{-}} \circ \mathcal{C})^* \text{Vb}(\tilde{\mathcal{H}}_{\mathcal{U}}). \]

Using (54) and pulling back to \( \mathcal{C}_{\mathcal{U}} \), we have
\[ \mathcal{L}_{\mathcal{U}, -} \otimes f^e_{\mathcal{U}} \mathcal{O}(\infty P) \xrightarrow{(f^e_{\mathcal{U}, -})^* d\epsilon} \mathcal{L}_{\mathcal{U}, -} \xrightarrow{s_{\mathcal{U}}^\times dW_{\text{-}}} \mathcal{H}_{\mathcal{U}}. \]

Further pushing forward, we obtain
\[ \mathbb{E}_{\mathcal{U}/\mathcal{H}} := \pi_{\mathcal{U}}^\times(\mathcal{L}_{\mathcal{U}, -} \otimes f^e_{\mathcal{U}} \mathcal{O}(\infty P)) \xrightarrow{\pi_{\mathcal{U}}^\times(s_{\mathcal{U}}^\times dW_{\text{-}})} \mathcal{H}_{\mathcal{U}} \]

**Proposition 5.15.** The composition (61) is (50). In particular, the relative cosection (51) is the \( \bar{H}^1 \) of (61).

**Proof.** By Lemma 3.19, we have \((f^e_{\mathcal{U}, -})^*(\otimes \mathbb{E}_{\mathcal{U}}) = (\otimes \tilde{\mathcal{H}}_{\mathcal{U}})\), where \( \mathbb{E}_{\mathcal{U}} \) is as in (56). By Lemma 5.13, \((f^e_{\mathcal{U}, -})^* d\epsilon \) is obtained by tensoring (41) by \( \mathcal{O}_{\mathcal{C}_{\mathcal{U}}}(-\Sigma) \). By (45), the arrow \( \pi_{\mathcal{U}}^\times(f^e_{\mathcal{U}, -})^* d\epsilon \) is (49). Further observe that the arrow \( \pi_{\mathcal{U}}^\times, s_{\mathcal{U}}^\times dW_{\text{-}} \) is (46). This proves the statement. \( \square \)

5.3.5. The twisted Hodge bundle. Denote by \( \tilde{\mathcal{H}}_{\mathcal{U}} := \omega_{\mathcal{U}} \otimes \pi_{\mathcal{U}}^\times \mathcal{L}^{-\otimes r}_{\max} \). Consider the direct image cone \( C(\pi_{\mathcal{U}}^\times, \tilde{\mathcal{H}}_{\mathcal{U}}) \) as in [12, Definition 2.1]. This is an algebraic stack over \( \mathcal{U} \) parameterizing sections of \( \tilde{\mathcal{H}}_{\mathcal{U}} \), see [12, Proposition 2.2]. We further equip it with the log structure pulled back from \( \mathcal{U} \). For simplicity, we write \( \mathcal{H} := C(\pi_{\mathcal{U}}^\times, \tilde{\mathcal{H}}_{\mathcal{U}}) \), and denote by \( s_{\mathcal{H}} : C_{\mathcal{H}} \to \text{Vb}(\tilde{\mathcal{H}}_{\mathcal{U}}) \) the universal section over \( \mathcal{H} \).

By [12, Proposition 2.5], the strict morphism \( \mathcal{H} \to \mathcal{U} \) has a perfect obstruction theory
\[ \mathbb{T}_{\mathcal{H}/\mathcal{U}} \to \mathbb{E}_{\mathcal{H}/\mathcal{U}} := \pi_{\mathcal{H}}^\times \tilde{\mathcal{H}}_{\mathcal{U}}. \]

By projection formula, we have
\[ R^1 \pi_{\mathcal{U}}^\times \tilde{\mathcal{H}}_{\mathcal{U}} = (R^1 \pi_{\mathcal{U}}^\times \omega_{\mathcal{U}} \otimes \mathcal{L}^{-\otimes r}_{\max}) \cong \mathcal{L}^{-\otimes r}_{\max}. \]
This implies that \( R^0 \pi_{\mathcal{U},*} \tilde{\omega}_\mathcal{U} \cong R^0 \pi_{\mathcal{U},*} \omega_\mathcal{U} \otimes \mathcal{L}_{\max}^{-\otimes r} \) is indeed a vector bundle whose associated geometric vector bundle is \( \mathcal{H} \). In particular, the morphism \( \mathcal{H} \to \mathcal{U} \) is strict and smooth. Thus \( T_{\mathcal{H}/\mathcal{U}} \) is a vector bundle over \( \mathcal{H} \) concentrated in degree zero, and the following morphism is trivial:

\[
0 = H^1(T_{\mathcal{H}/\mathcal{U}}) \to R^1 \pi_{\mathcal{H},*} \tilde{\omega}_\mathcal{H} \cong \mathcal{L}_{\max}^{-\otimes r}.
\]

The section \( s_\mathcal{U} \) as in (42) defines a section \( \tilde{s}_\mathcal{U} \colon \mathcal{C}_\mathcal{U} \to \text{Vb}(\mathcal{L}_{\mathcal{U}}^{\otimes r} \otimes \pi_{\mathcal{U}}^* \mathcal{L}_{\max}^{-\otimes r}) \cong \text{Vb}(\omega_{\log,\mathcal{U}} \otimes \pi_{\mathcal{U}}^* \mathcal{L}_{\max}^{-\otimes r}) \).

By Lemma 5.14, \( s_\mathcal{U} \) is a global section of \( \mathcal{L}_{\mathcal{U}}(-\Sigma) \otimes \pi_{\mathcal{U}}^* \mathcal{L}_{\max}^{-\otimes r} \). Thus \( \tilde{s}_\mathcal{U} \) factors through a section

\[
\mathcal{C}_\mathcal{U} \to \text{Vb}(\omega_{\mathcal{U}} \otimes \pi_{\mathcal{U}}^* \mathcal{L}_{\max}^{-\otimes r}),
\]

which is again denoted by \( \tilde{s}_\mathcal{U} \). This induces a morphism

\[
\mathcal{U} \to \mathcal{H}
\]

such that \( \tilde{s}_\mathcal{U} \) is the pull-back of \( s_\mathcal{H} \).

5.3.6. Obstruction factorization.

Lemma 5.16. There is a canonical commutative diagram

\[
\begin{array}{ccc}
\mathcal{T}_{\mathcal{U}/\mathcal{U}} & \to & \mathcal{T}_{\mathcal{H}/\mathcal{U}}|_{\mathcal{U}} \\
\downarrow & & \downarrow \\
\mathcal{E}_{\mathcal{U}/\mathcal{U}} & \to & \mathcal{E}_{\mathcal{H}/\mathcal{U}}|_{\mathcal{U}}
\end{array}
\]

where the bottom arrow is (61), and the left and right vertical arrows are the perfect obstruction theories (37) and (62) respectively.

Proof. Consider the following commutative diagram over \( \mathcal{C}_\mathcal{U} \):

\[
\begin{array}{ccc}
\mathcal{C}_{\mathcal{U}} & \to & \mathcal{C}_{\mathcal{H}} \\
\mathcal{P}_{\mathcal{C}_{\mathcal{U}}} & \to & \text{Vb}(\tilde{\omega}_{\mathcal{U}}) \\
\downarrow & & \downarrow \\
(W_-)^{oc} & \to & \text{Vb}(\tilde{\omega}_{\mathcal{U}})
\end{array}
\]

By abuse of notation, \( s_\mathcal{H} \) is the composition \( \mathcal{C}_{\mathcal{H}} \to \text{Vb}(\tilde{\omega}_{\mathcal{H}}) \to \text{Vb}(\tilde{\omega}_{\mathcal{U}}) \). We obtain a commutative diagram of log cotangent complexes

\[
\begin{array}{ccc}
\pi_{\mathcal{U}}^* \mathcal{T}_{\mathcal{U}/\mathcal{U}} & \to & \mathcal{T}_{\mathcal{C}_{\mathcal{U}}} \\
\downarrow & & \downarrow \\
(f_{\mathcal{U}}^-)^* \mathcal{T}_{\mathcal{C}_{\mathcal{U}}} & \to & \mathcal{T}_{\mathcal{Vb}(\tilde{\omega}_{\mathcal{U}})}
\end{array}
\]

By abuse of notation, \( s_\mathcal{H} \) is the composition \( \mathcal{C}_{\mathcal{H}} \to \text{Vb}(\tilde{\omega}_{\mathcal{H}}) \to \text{Vb}(\tilde{\omega}_{\mathcal{U}}) \). We obtain a commutative diagram of log cotangent complexes

\[
\begin{array}{ccc}
\pi_{\mathcal{U}}^* \mathcal{T}_{\mathcal{U}/\mathcal{U}} & \to & \mathcal{T}_{\mathcal{C}_{\mathcal{U}}}/\mathcal{C}_{\mathcal{U}} \\
\downarrow & & \downarrow \\
(f_{\mathcal{U}}^-)^* \mathcal{T}_{\mathcal{C}_{\mathcal{U}}} & \to & \mathcal{T}_{\mathcal{Vb}(\tilde{\omega}_{\mathcal{U}})}
\end{array}
\]
Since \( b \) in (57) is log étale, we have \( (f_{\mathfrak{w}, -})^* T_{P_{\mathfrak{u}, -}/\mathfrak{u}} \cong (f_{\mathfrak{w}, -})^* T_{P_{\mathfrak{u}, -}/\mathfrak{u}} \). Applying \( \pi_{\mathfrak{w}, *\mathfrak{u}} \) and using adjunction we obtain

\[
\begin{array}{c}
\pi_{\mathfrak{w}, *}(f_{\mathfrak{w}, -})^* T_{P_{\mathfrak{u}, -}/\mathfrak{u}} \pi_{\mathfrak{w}, *}(s_{\mathfrak{u}})^* T_{V_{\mathfrak{u}, -}/\mathfrak{u}} \cong (f_{\mathfrak{w}, -})^* T_{P_{\mathfrak{u}, -}/\mathfrak{u}} \pi_{\mathfrak{w}, *}(s_{\mathfrak{u}})^* T_{V_{\mathfrak{u}, -}/\mathfrak{u}}
\end{array}
\]

which is (64).

\[\square\]

**Proposition 5.17.** The injection \( H^1(T_{\mathfrak{w}/\mathfrak{u}}) \to \text{Ob}_\mathfrak{w}/\mathfrak{u} \) factors through the kernel of the relative cosection \( \sigma_{\mathfrak{w}/\mathfrak{u}} \) in (51).

**Proof.** By Lemma 5.16, taking \( H^1 \) of (64), we obtain a commutative diagram

\[
\begin{array}{ccc}
H^1(T_{\mathfrak{w}/\mathfrak{u}}) & \to & H^1(T_{\mathfrak{u}/\mathfrak{u}}) = 0 \\
\downarrow & & \downarrow \\
\text{Ob}_\mathfrak{w}/\mathfrak{u} & \to & L_{\max}^{-\otimes r}
\end{array}
\]

where \( H^1(T_{\mathfrak{u}/\mathfrak{u}}) = 0 \) follows from the smoothness of \( \mathfrak{u} \to \mathfrak{u} \).

\[\square\]

**5.4. The reduced relative perfect obstruction theory.** The dual of (20) induces a complex with amplitude \([0, 1]\) over \( \mathfrak{u} \):

\[
F := \mathcal{O}_{\mathfrak{u}} \xrightarrow{\epsilon} L_{\max}^{-\otimes r}.
\]

Since \( \mathfrak{u} \to \mathfrak{u} \) is log smooth, \( \epsilon \) is injective. Consider the cokernel \( \text{cok} \epsilon \).

Then \( F = \text{cok} \epsilon [-1] \) in the derived category. The composition

\[
\mathcal{E}_{\mathfrak{u}/\mathfrak{u}} \to H^1(E_{\mathfrak{u}/\mathfrak{u}})[-1] \cong L_{\max}^{-\otimes r}[-1] \to \text{cok} \epsilon [-1]
\]

defines a morphism of complexes \( \mathcal{E}_{\mathfrak{u}/\mathfrak{u}} \to F|_{\mathfrak{u}} \), and hence a triangle

\[
(65) \quad \mathcal{E}_{\mathfrak{u}/\mathfrak{u}}^{\text{red}} \to \mathcal{E}_{\mathfrak{u}/\mathfrak{u}} \to F|_{\mathfrak{u}} \to \mathcal{E}_{\mathfrak{u}/\mathfrak{u}}^{\text{red}} \]  

where the notation \( |_{\ast} \) stands for derived pull-back to \( \ast \).

**Lemma 5.18.** \( H^1(E_{\mathfrak{u}/\mathfrak{u}}^{\text{red}}) = \mathcal{O}_{\mathfrak{u}/\mathfrak{u}} \).

**Proof.** Taking the long exact sequence of (65) and using \( H^0(F) = 0 \), we have an exact sequence

\[
0 \to H^1(E_{\mathfrak{u}/\mathfrak{u}}^{\text{red}}) \to H^1(E_{\mathfrak{u}/\mathfrak{u}}) \to H^1(F|_{\mathfrak{u}}) \to 0.
\]

Since \( H^1(E_{\mathfrak{u}/\mathfrak{u}}) \to H^1(F|_{\mathfrak{u}}) \) is precisely the morphism \( L_{\max}^{-\otimes r} \to \text{cok} \epsilon \), it follows that \( H^1(E_{\mathfrak{u}/\mathfrak{u}}^{\text{red}}) = \mathcal{O}_{\mathfrak{u}/\mathfrak{u}} \).

\[\square\]

The composition

\[
(66) \quad \mathcal{E}_{\mathfrak{w}/\mathfrak{u}} \to \mathcal{E}_{\mathfrak{u}/\mathfrak{u}}|_{\mathfrak{w}} \to F|_{\mathfrak{w}},
\]

yields a triangle

\[
(67) \quad \mathcal{E}_{\mathfrak{u}/\mathfrak{u}}^{\text{red}} \to \mathcal{E}_{\mathfrak{w}/\mathfrak{u}} \to F|_{\mathfrak{w}} \to \mathcal{E}_{\mathfrak{u}/\mathfrak{u}}^{\text{red}} \]
Lemma 5.19. The perfect obstruction theories $T_{\mathcal{H}/U} \to E_{\mathcal{H}/U}$ and $T_{\mathcal{U}/U} \to E_{\mathcal{U}/U}$ factor through $T_{\mathcal{H}/U} \to E_{\mathcal{H}/U}^{\text{red}}$ and $T_{\mathcal{U}/U} \to E_{\mathcal{U}/U}^{\text{red}}$ respectively. Furthermore, they fit in a commutative diagram

\begin{equation}
\begin{array}{c}
T_{\mathcal{U}/U} \ar[d] & \ar[r] & T_{\mathcal{H}/U} \ar[d] \\
E_{\mathcal{U}/U}^{\text{red}} & \ar[r] & E_{\mathcal{H}/U}^{\text{red}} \\
\end{array}
\end{equation}

Proof. By Lemma 5.16, we have a commutative diagram of solid arrows:

\begin{equation}
\begin{array}{c}
T_{\mathcal{U}/U} \ar[d] & \ar[r] & T_{\mathcal{H}/U} \ar[d] \\
E_{\mathcal{U}/U}^{\text{red}} & \ar[r] & E_{\mathcal{H}/U}^{\text{red}} \\
\end{array}
\end{equation}

where the two horizontal lines are triangles (67) and (65), and the two curved arrows are the corresponding perfect obstruction theories.

Since $\mathcal{H} \to \mathcal{U}$ is representable and smooth, the complex $T_{\mathcal{H}/U}$ is represented by the relative tangent bundle $T_{\mathcal{H}/U}$. Thus the composition $T_{\mathcal{H}/U} \to E_{\mathcal{H}/U} \to F|_{\mathcal{H}}$ is the zero morphism. This yields the lower dashed arrow $T_{\mathcal{H}/U} \to E_{\mathcal{H}/U}^{\text{red}}$.

Now by the commutativity, the composition $T_{\mathcal{U}/U} \to E_{\mathcal{U}/U} \to F|_{\mathcal{U}}$ is the same as $T_{\mathcal{U}/U} \to T_{\mathcal{H}/U}|_{\mathcal{U}} \to F|_{\mathcal{U}}$, hence is trivial. Thus, we obtain the top dashed arrow $T_{\mathcal{U}/U} \to E_{\mathcal{U}/U}^{\text{red}}$. \qed

Lemma 5.20. The two complexes $E_{\mathcal{U}/\mathcal{U}}^{\text{red}}$ and $E_{\mathcal{H}/\mathcal{U}}^{\text{red}}$ are perfect with tor-amplitude in $[0,1]$.

Proof. Since $E_{\mathcal{H}/\mathcal{U}}$, $E_{\mathcal{U}/\mathcal{U}}$ and $F$ are perfect in $[0,1]$, the complexes $E_{\mathcal{H}/\mathcal{U}}^{\text{red}}$ and $E_{\mathcal{U}/\mathcal{U}}^{\text{red}}$ are at least perfect in $[0,2]$. It suffices to show that $H^2(E_{\mathcal{H}/\mathcal{U}}^{\text{red}}) = 0$ and $H^2(E_{\mathcal{U}/\mathcal{U}}^{\text{red}}) = 0$.

Taking the long exact sequence of (65), we have an exact sequence

$$H^1(E_{\mathcal{H}/\mathcal{U}}) \to H^1(F|_{\mathcal{H}}) \to H^2(E_{\mathcal{H}/\mathcal{U}}^{\text{red}}) \to 0$$

Since the left arrow is $L^{-\otimes r}_{\max} \to \text{cok } \epsilon$, we have $H^2(E_{\mathcal{H}/\mathcal{U}}^{\text{red}}) = 0$.

Similarly using (67), we have an exact sequence

$$H^1(E_{\mathcal{U}/\mathcal{U}}) \to H^1(F|_{\mathcal{U}}) \to H^2(E_{\mathcal{U}/\mathcal{U}}^{\text{red}}) \to 0.$$
By (66), the left arrow is the following composition
\[ H^1(\mathcal{E}_{\mathcal{U}/\mathcal{U}}) \to H^1(\mathcal{E}_{\mathcal{B}/\mathcal{U}|\mathcal{U}}) \to H^1(\mathcal{F}|_{\mathcal{U}}). \]

By construction, \( \mathcal{F}|_{\mathcal{U}\setminus \Delta_{\max}} = 0 \) is the zero complex. It suffices to show that the above composition is surjective along a neighborhood of \( \Delta_{\max} \), which follows from Proposition 5.8 and Lemma 5.16.

**Lemma 5.21.** The two arrows \( T_{\mathcal{D}/\mathcal{U}} \to \mathcal{E}_{\mathcal{B}/\mathcal{U}} \) and \( T_{\mathcal{U}/\mathcal{U}} \to \mathcal{E}_{\mathcal{B}/\mathcal{U}} \) define perfect obstruction theories of \( \mathcal{D} \to \mathcal{U} \) and \( \mathcal{U}/\mathcal{U} \) respectively.

**Proof.** We verify the case of \( T_{\mathcal{U}/\mathcal{U}} \to \mathcal{E}_{\mathcal{B}/\mathcal{U}} \). The case of \( T_{\mathcal{D}/\mathcal{U}} \to \mathcal{E}_{\mathcal{B}/\mathcal{U}} \) is similar. By the triangle (67) and the factorization of Lemma 5.19, we have a surjection \( H^0(T_{\mathcal{U}/\mathcal{U}}) \to H^0(\mathcal{E}_{\mathcal{B}/\mathcal{U}}) \) and an injection \( H^1(T_{\mathcal{U}/\mathcal{U}}) \to H^1(\mathcal{E}_{\mathcal{B}/\mathcal{U}}) \). Since \( \mathcal{F} \) is perfect in \([0, 1]\), (67) implies that \( H^0(T_{\mathcal{U}/\mathcal{U}}) \to H^0(\mathcal{E}_{\mathcal{B}/\mathcal{U}}) \) is also injective, and hence an isomorphism.

The proof of the above lemma leads to the following

**Corollary 5.22.**

1. \( H^0(\mathcal{E}_{\mathcal{B}/\mathcal{U}}) = H^0(\mathcal{E}_{\mathcal{U}/\mathcal{U}}) \).
2. Diagram (69) induces a morphism between long exact sequences

\[
\begin{array}{ccccccc}
0 & \longrightarrow & H^0(\mathcal{F}|_{\mathcal{U}}) & \longrightarrow & H^1(\mathcal{E}_{\mathcal{U}/\mathcal{U}}) & \longrightarrow & H^1(\mathcal{E}_{\mathcal{B}/\mathcal{U}}) & \longrightarrow & H^1(\mathcal{F}|_{\mathcal{U}}) & \longrightarrow & 0 \\
& & \downarrow{\cong} & & \downarrow{\sigma_{\mathcal{U}/\mathcal{U}}^{\text{red}}} & & \downarrow{\sigma_{\mathcal{B}/\mathcal{U}}} & & \downarrow{\cong} & & \\
0 & \longrightarrow & H^0(\mathcal{F}|_{\mathcal{U}}) & \longrightarrow & H^1(\mathcal{E}_{\mathcal{B}/\mathcal{U}}) & \longrightarrow & H^1(\mathcal{E}_{\mathcal{B}/\mathcal{U}}) & \longrightarrow & H^1(\mathcal{F}|_{\mathcal{U}}) & \longrightarrow & 0 \\
\end{array}
\]

where the morphism \( \sigma_{\mathcal{U}/\mathcal{U}}^{\text{red}} \) is surjective along \( \Delta_{\max} \).

**Proof.** It remains to verify the surjectivity of \( \sigma_{\mathcal{U}/\mathcal{U}}^{\text{red}} \) along \( \Delta_{\max} \). This follows from the surjectivity of \( \sigma_{\mathcal{B}/\mathcal{U}} \) along \( \Delta_{\max} \) by Proposition 5.8.

We summarize our construction below.

**Proposition 5.23.** The morphism \( \mathcal{U} \to \mathcal{U} \) admits a reduced perfect obstruction theory

\[
(70) \quad T_{\mathcal{U}/\mathcal{U}} \to \mathcal{E}_{\mathcal{B}/\mathcal{U}},
\]

and a reduced relative cosection

\[
(71) \quad \sigma_{\mathcal{U}/\mathcal{U}}^{\text{red}} : \text{Obs}_{\mathcal{U}/\mathcal{U}}^{\text{red}} := H^1(\mathcal{E}_{\mathcal{B}/\mathcal{U}}) \to \mathcal{O}_{\mathcal{U}}
\]

with the following properties

1. \( \mathcal{E}_{\mathcal{U}/\mathcal{U}}|_{\mathcal{U}\setminus \Delta_{\max}} = \mathcal{E}_{\mathcal{B}/\mathcal{U}}|_{\mathcal{U}\setminus \Delta_{\max}}. \)
2. \( \sigma_{\mathcal{U}/\mathcal{U}}^{\text{red}}|_{\mathcal{U}\setminus \Delta_{\max}} = \sigma_{\mathcal{B}/\mathcal{U}}|_{\mathcal{U}\setminus \Delta_{\max}}. \)
3. \( \sigma_{\mathcal{U}/\mathcal{U}}^{\text{red}} \) is surjective along \( \Delta_{\max} \).

In particular, \( \sigma_{\mathcal{U}/\mathcal{U}}^{\text{red}} \) and \( \sigma_{\mathcal{B}/\mathcal{U}} \) have the same degeneracy loci.
Proof. The perfect obstruction theory has been verified in Lemma 5.20 and 5.21. The formation of \( \sigma_{\mathcal{U}/\mathcal{U}}^{\text{red}} \) and its surjectivity along \( \Delta_{\text{max}} \) follows from Corollary 5.22.

Finally (1) follows from the observation \( F|_{\mathcal{U}\setminus\Delta_{\text{max}}} = 0 \). Statement (2) follows from (1) and (69). \( \square \)

Notation 5.24. Since \( \mathcal{U} \) is equi-dimensional, denote by \( [\mathcal{U}] \) the virtual fundamental class of \( \mathcal{U} \) defined by the relative perfect obstruction theory (70), see [9].

5.5. The reduced absolute perfect obstruction theory.

5.5.1. Resolution of the base.

Lemma 5.25. Let \( \mathcal{V} \subset \mathcal{U} \) be a finite type open substack, and write \( \Delta_{\text{max},\mathcal{V}} = \Delta_{\text{max}} \cap \mathcal{V} \). Then there exists a birational, log étale, projective morphism of log stacks \( \tilde{\phi}: \tilde{\mathcal{V}} \to \mathcal{V} \) such that

1. \( \tilde{\phi}|_{\Delta_{\text{max},\mathcal{V}}} \) is an isomorphism onto \( \mathcal{V} \setminus \Delta_{\text{max},\mathcal{V}} \).
2. The log structure of \( \tilde{\mathcal{V}} \) is locally free. In particular, the underlying stack of \( \tilde{\mathcal{V}} \) is smooth.

Proof. Recall from Corollary 3.8 that there is a canonical splitting \( \mathcal{M}_{\mathcal{V}} = \mathcal{M}'_{\mathcal{V}} \oplus \mathcal{O}^* \mathcal{M}_{\mathcal{V}}^\prime \) where \( \mathcal{M}'_{\mathcal{V},s} = \mathbb{N}^d \) is the factor corresponding to nodes with the trivial contact order for each geometric point \( s \to \mathcal{V} \). Indeed, given a node over \( s \), if it has the trivial contact order, then it can either be smoothed out or remain the trivial contact order in a neighborhood of \( s \). Observe that \( \mathcal{M}'_{\mathcal{V}} \) is trivial along \( \mathcal{V} \setminus \Delta_{\text{max},\mathcal{V}} \) as the curves have no degenerate components away from \( \Delta_{\text{max}} \).

Denote by \( \mathcal{A}'_{\mathcal{V}} \) and \( \mathcal{A}''_{\mathcal{V}} \) the Artin fans associated to the log structures \( \mathcal{M}'_{\mathcal{V}} \) and \( \mathcal{M}''_{\mathcal{V}} \) respectively, see [4, Proposition 3.1.1]. By Theorem 3.14, we have a strict, smooth morphism of log stacks \( \mathcal{V} \to \mathcal{A}'_{\mathcal{V}} \times \mathcal{A}''_{\mathcal{V}} \). Let \( \mathcal{Y} \to \mathcal{A}'_{\mathcal{V}} \) be the projective sub-division provided by [4, Theorem 4.4.2]. In particular, it is projective and log étale, and \( \mathcal{M}_{\mathcal{Y}} \) is locally free. Consider the induced projective, log étale morphism

\[
\tilde{\phi}: \tilde{\mathcal{V}} := \mathcal{V} \times \mathcal{A}'_{\mathcal{V}} \times \mathcal{A}''_{\mathcal{V}} (\mathcal{Y} \times \mathcal{A}''_{\mathcal{V}}) \to \mathcal{V}.
\]

Now (2) follows from the construction. Since \( \mathcal{M}'_{\mathcal{V}} \) is the trivial log structure on \( \mathcal{V} \setminus \Delta_{\text{max},\mathcal{V}} \), \( \tilde{\phi} \) is an isomorphism away from \( \Delta_{\text{max},\mathcal{V}} \). This proves (1). \( \square \)

Let \( \mathcal{W} \subset \mathcal{U} \) be a finite type open substack containing the image of \( \mathcal{V} \). We fix a resolution \( \tilde{\phi}: \tilde{\mathcal{V}} \to \mathcal{V} \) as in Lemma 5.25. Consider the fiber products

\[
\tilde{\mathcal{H}} := \mathcal{V} \times_{\mathcal{U}} \tilde{\mathcal{V}} \quad \text{and} \quad \tilde{\mathcal{W}} := \mathcal{U} \times_{\mathcal{U}} \tilde{\mathcal{V}}.
\]

The perfect obstruction theories \( T_{\mathcal{H}/\mathcal{U}} \to E_{\mathcal{H}/\mathcal{U}}^{\text{red}} \) and \( T_{\mathcal{W}/\mathcal{U}} \to E_{\mathcal{W}/\mathcal{U}}^{\text{red}} \) in Lemma 5.21 pull back to perfect obstruction theories

\[
T_{\tilde{\mathcal{H}}/\tilde{\mathcal{U}}} \to E_{\tilde{\mathcal{H}}/\tilde{\mathcal{U}}}^{\text{red}} \quad \text{and} \quad T_{\tilde{\mathcal{W}}/\tilde{\mathcal{U}}} \to E_{\tilde{\mathcal{W}}/\tilde{\mathcal{U}}}^{\text{red}}.
\]
Since \( \tilde{\mathfrak{G}} \) is equi-dimensional, let \( [\tilde{\mathfrak{U}}]^{\text{red}} \) be the virtual cycle of \( \tilde{\mathfrak{U}} \) defined by the above perfect obstruction theory as in [9]. By Lemma 5.25 and the virtual push-forward of [20, 39], we obtain:

**Lemma 5.26.** \( \tilde{\phi}_*[\tilde{\mathfrak{U}}]^{\text{red}} = [\mathfrak{U}]^{\text{red}} \)

### 5.5.2. The absolute reduced theory and cosection

Consider the morphism of triangles:

\[
\begin{array}{ccc}
T_{\tilde{\delta}/\tilde{\mathfrak{Q}}} & \longrightarrow & T_{\tilde{\delta}} \\
\downarrow & & \downarrow \\
E_{\tilde{\delta}/\tilde{\mathfrak{Q}}}^{\text{red}} & \longrightarrow & E_{\tilde{\delta}}^{\text{red}} \\
\end{array}
\]

**Lemma 5.27.** The induced morphism \( H^1(E_{\tilde{\delta}/\tilde{\mathfrak{Q}}}^{\text{red}}) \rightarrow H^1(E_{\tilde{\delta}}^{\text{red}}) \) is an isomorphism and \( H^1(E_{\tilde{\delta}}^{\text{red}}) \cong \mathcal{O}_{\tilde{\delta}} \).

**Proof.** Since \( \tilde{\mathfrak{G}} \) is smooth, we have \( H^1(T_{\tilde{\delta}}) = 0 \). Consider the induced morphism between long exact sequences

\[
\begin{array}{ccc}
H^0(T_{\tilde{\delta}}) & \longrightarrow & H^0(T_{\tilde{\mathfrak{Q}}}|_{\tilde{\delta}}) \\
\downarrow & & \downarrow \\
H^0(E_{\tilde{\delta}}^{\text{red}}) & \longrightarrow & H^0(E_{\tilde{\mathfrak{Q}}}|_{\tilde{\delta}}) \\
\end{array}
\]

Since \( \tilde{\delta} \rightarrow \tilde{\mathfrak{G}} \) is smooth, \( H^0(T_{\tilde{\delta}}) \rightarrow H^0(T_{\tilde{\mathfrak{Q}}}|_{\tilde{\delta}}) \) and \( H^0(E_{\tilde{\delta}}^{\text{red}}) \rightarrow H^0(E_{\tilde{\mathfrak{Q}}}|_{\tilde{\delta}}) \) are both surjective. Thus \( H^1(E_{\tilde{\delta}}^{\text{red}}) \rightarrow H^1(E_{\tilde{\mathfrak{Q}}}|_{\tilde{\delta}}) \) is an isomorphism. Lemma 5.18 implies that \( H^1(E_{\tilde{\delta}}^{\text{red}}) \cong \mathcal{O}_{\tilde{\delta}} \). \( \square \)

Now consider the morphism of triangles:

\[
\begin{array}{ccc}
T_{\tilde{\mathfrak{U}}/\mathfrak{Q}} & \longrightarrow & T_{\tilde{\mathfrak{U}}} \\
\downarrow & & \downarrow \\
E_{\tilde{\mathfrak{U}}/\mathfrak{Q}}^{\text{red}} & \longrightarrow & E_{\tilde{\mathfrak{U}}}^{\text{red}} \\
\end{array}
\]

By [10, Proposition A.1. (1)], we obtain a perfect obstruction theory \( T_{\tilde{\mathfrak{U}}} \rightarrow E_{\tilde{\mathfrak{U}}}^{\text{red}} \) of \( \tilde{\mathfrak{U}} \) with the corresponding virtual circle \( [\tilde{\mathfrak{U}}]^{\text{red}} \).

The bottom morphism in (68) induces a morphism of triangles

\[
\begin{array}{ccc}
E_{\tilde{\mathfrak{U}}/\mathfrak{Q}}^{\text{red}} & \longrightarrow & T_{\mathfrak{Q}}|_{\tilde{\mathfrak{U}}} \\
\downarrow & & \downarrow \\
E_{\tilde{\delta}/\mathfrak{Q}}^{\text{red}} & \longrightarrow & T_{\mathfrak{Q}}|_{\tilde{\delta}} \\
\end{array}
\]
Taking $H^1$ and applying Lemma 5.27, we have a commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
H^1(\mathcal{E}_{\tilde{U}/\tilde{V}}) \ar[r] & H^1(\mathcal{E}_{\tilde{U}/\tilde{V}}) \\
\sigma_{\tilde{U}/\tilde{V}} & \sigma_{\tilde{U}/\tilde{V}} \ar[u] \\
\mathcal{O} & \mathcal{O} \ar[u]
\end{array}
\end{array}
\end{array}
\]

Observe that $\sigma_{\tilde{U}/\tilde{V}}$ is the pull-back of $\sigma_{\tilde{U}/\tilde{V}}$ in (71). We call $\sigma_{\tilde{U}/\tilde{V}}$ the absolute reduced cosection.

5.5.3. **Proof of Theorem 1.6.** Denote by $\tilde{\Delta}_{\text{max}} := \tilde{U} \times \tilde{U}/\Delta_{\text{max}}$. By Lemma 5.25 (1), we have the identity $U^\circ := U \setminus \tilde{\Delta}_{\text{max}} = U \setminus \Delta_{\text{max}}$. Consider the open embedding $\iota: U^\circ \rightarrow \tilde{U}$ with the trivial perfect obstruction theory. Thus the virtual pull-back $\iota^!$ in the sense of [39] is just the flat pull-back.

Denote by $\sigma_{U^\circ} = \sigma_{\tilde{U}/\tilde{V}}|_{U^\circ}$. By Lemma 5.25, 5.6, and Proposition 5.23 (2), the morphism $\sigma_{U^\circ}$ is the absolute cosection in [13, Proposition 3.4] in the $r$-spin case. We then obtain:

**Lemma 5.28.** $[\mathcal{U}^\circ]_{\sigma_{U^\circ}}$ is the Witten’s top Chern class as in [13, Definition-Proposition 3.9].

On the other hand, let $\tilde{U}(\sigma_{\tilde{U}/\tilde{V}})$ (respectively $\mathcal{U}^\circ(\sigma_{U^\circ})$) be the degeneracy loci of $\sigma_{\tilde{U}/\tilde{V}}$ (respectively $\sigma_{U^\circ}$). Since $\sigma_{\tilde{U}/\tilde{V}}$ is the pull-back of $\sigma_{U^\circ}$, Proposition 5.23 (3) implies that $\sigma_{\tilde{U}/\tilde{V}}$ is surjective along $\tilde{\Delta}_{\text{max}}$, hence $\tilde{U}(\sigma_{\tilde{U}/\tilde{V}}) = \mathcal{U}^\circ(\sigma_{U^\circ})$.

Let $i^!_{\sigma_{\tilde{U}/\tilde{V}}}$ be the cosection localized virtual pull-back as in [11]. Since $i^!_{\sigma_{\tilde{U}/\tilde{V}}} = i^!$ and $\tilde{U}(\sigma_{\tilde{U}/\tilde{V}}) = \mathcal{U}^\circ(\sigma_{U^\circ})$, applying [11, Theorem 2.6] we have the following equalities in $A_*(\mathcal{U}^\circ(\sigma_{U^\circ}))$:

\[
[i^!_{\sigma_{\tilde{U}/\tilde{V}}}] = [\mathcal{U}^\circ(\sigma_{U^\circ})].
\]

Let $\tilde{i}: \mathcal{U}^\circ(\sigma_{U^\circ}) \rightarrow \tilde{U}$ be the closed embedding. By [34], we have:

**Lemma 5.29.** $\tilde{i}_*[\mathcal{U}^\circ]_{\sigma_{U^\circ}} = [\tilde{U}]_{\text{red}}$.

Finally, let $i = \tilde{\phi} \circ \tilde{i}: \mathcal{U}^\circ(\sigma_{U^\circ}) \rightarrow U$ be the closed embedding. Applying Lemma 5.26, we have:

**Proposition 5.30.** $i_*[\mathcal{U}^\circ]_{\sigma_{U^\circ}} = [U]_{\text{red}}$.

This completes the proof of Theorem 1.6.

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