Abstract. In this paper we provide a formulation for sweeping processes with arbitrary locally bounded retraction, not necessarily left or right continuous. Moreover we provide a proof of the existence and uniqueness of solutions for this formulation which relies on the reduction to the 1-Lipschitz continuous case by using a suitable family of geodesics for the asymmetric Hausdorff-like distance called excess.

1. Introduction

Evolution problems with unilateral constraints play a crucial role in many important mechanical applications. A well known problem of this kind is the so called sweeping process introduced by J.J. Moreau in the first seventies in the articles [46, 47, 48, 49, 53]. The most simple formulation of the sweeping processes is the following. Let $H$ be a real Hilbert space and let $C(t)$ be a given nonempty, closed and convex subset of $H$ depending on time such that the mapping $t \mapsto C(t)$ is locally Lipschitz continuous on $[0, \infty]$ when the family of closed subsets of $H$ is endowed with the Hausdorff metric. One has to find a locally Lipschitz continuous function $y : [0, \infty] \rightarrow H$ such that

\begin{align}
&y(t) \in C(t) \quad \forall t \in [0, \infty], \\
&- y'(t) \in N_{C(t)}(y(t)) \quad \text{for } L^1\text{-a.e. } t \in [0, \infty], \\
&y(0) = y_0,
\end{align}

$y_0$ being a prescribed point in $C(0)$. Here $L^1$ is the Lebesgue measure, and $N_{C(t)}$ is the exterior normal cone to $C(t)$ at $y(t)$ (all the definitions will be recalled in Section 2). Moreau was originally motivated by plasticity and friction dynamics (cf. [49, 52, 54]), but now sweeping processes have found applications to nonsmooth mechanics (see, e.g., [45, 37, 55]), to economics (cf., e.g., [30, 23, 28]), to electrical circuits (see, e.g., [2, 9, 1, 3]), to crowd motion modeling (cf., e.g., [42, 39, 43, 41, 40, 24]), and to other fields (see, e.g., the references in the recent paper [65]). The theoretical analysis of problem (1.4)-(1.7) has been expanded in various directions: the case of $C$ continuous was first dealt in [45], and in [44] the application of external forces is also considered; the nonconvex case has been studied in several papers, e.g., [66, 67, 68, 13, 15, 18, 4, 21, 63, 7, 26, 64, 29, 5, 62]; for stochastic versions see, e.g., [11, 12, 6, 16], while periodic solutions can be found in [14]. The continuous dependence properties of various sweeping problems are investigated, e.g., in [53, 10, 36, 56, 35, 57, 58, 59, 31, 32], and the control problems are studied, e.g., in [19, 20, 22].

2010 Mathematics Subject Classification. 34G25, 34A60, 47J20, 74C05.

Key words and phrases. Evolution variational inequalities, Functions of bounded variation, Sweeping processes, Convex sets, Retraction, Geodesics with respect to the excess.

The author is a member of GNAMPA-INdAM.
In [53] the formulation (1.1)-(1.3) is extended to the case when the mapping \( t \mapsto C(t) \) is of locally bounded variation and right continuous in the following natural way: it is proved that there is a unique \( y \in BV^r_{\text{loc}}([0, \infty[ ; \mathcal{H}) \), the space of right continuous \( \mathcal{H} \)-valued functions of locally bounded variation, such that there exist a positive measure \( \mu \) and a \( \mu \)-integrable function \( v : [0, \infty[ \rightarrow \mathcal{H} \) satisfying

\[
\begin{align*}
y(t) & \in C(t) \quad \forall t \in [0, \infty[, \\
Dy & = v\mu, \\
- v(t) & \in N_{C(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in [0, \infty[,
\end{align*}
\]

(1.4) (1.5) (1.6) (1.7)

where \( Dy \) denotes distributional derivative of \( y \), which is a measure since \( y \) is of bounded variation.

When \( t \) is a jump point for \( C \), the normality conditions (1.5)-(1.6) means that \( y(t) = y(t+) = \text{Proj}_{C(t)}(y(t-)) \), where \( \text{Proj} \) is the classical projection operator, thus \( y \) jumps from \( C(t-) \) to \( C(t+) \) along the shortest path which allows to satisfy the constraint (1.4).

The existence and uniqueness proof for problem (1.4)--(1.7) relies upon an implicit approximation scheme that is usually called "catching up algorithm" in view of the geometric meaning of the projection on \( C(t+) \). Actually in [53] the moving set \( C \) is assumed to be only with locally bounded right continuous retraction, rather than with locally bounded variation (the notion of retraction is recalled in (2.7)), i.e. the Hausdorff distance is replaced by the asymmetric distance \( e(A, B) := \sup_{x \in A} d(x, B) \), called \( \text{excess of } A \text{ over } B \), with \( A, B \) nonempty closed convex sets in \( \mathcal{H} \). When \( C \) does not enjoy the right continuity property with respect to the excess \( e \), a formulation involving measures like (1.4)--(1.7) is not possible since at a jump point \( t \) one has \( y(t) = \text{Proj}_{C(t)}(y(t-)) \) and \( y(t+) = \text{Proj}_{C(t+)}(y(t)) \), thus in [53] a weak solution is defined as the uniform limit of the approximate solutions defined by the catching up algorithm.

The purpose of the present paper is twofold. First we want to introduce a sort of "differential measure" formulation similar to (1.4)--(1.7) also in the case when \( C \) is with locally bounded retraction but not necessarily left or right continuous. On the other hand we provide an existence proof for our formulation which reduces the problem to the case when \( C \) has locally Lipschitz retraction. In order to perform this proof we reparametrize the convex valued curve \( \tilde{C} \) by the "arc length" \( \ell_{\tilde{C}}(t) \) with respect to the excess \( e \) (for simplicity let us assume now that there are no intervals where \( \ell_{\tilde{C}} \) is constant) so that we can write \( \tilde{C}(t) = \tilde{C}(\ell_{\tilde{C}}(t)) \) for every \( t \) and for a suitable \( e \)-Lipschitz reparametrization \( \tilde{C}(\sigma) \). The problem of reducing to the Lipschitz case is that \( \tilde{C} \) is only defined in the image of \( \ell_{\tilde{C}} \), therefore we have to fill in the jumps of \( \tilde{C} \) in an appropriate way in order to have \( \tilde{C} \) defined on the whole \([0, \infty[\). In this way we can take the Lipschitz solution of the sweeping process associated to \( \tilde{C} \), and then throw away the jumps to get the discontinuous solution related to \( C \). In the past papers [57, 33, 34, 60] we have shown that the choice of the paths connecting these jumps is nontrivial and an arbitrary connection can produce the wrong solution. In the paper [61] we found the proper family of geodesic connecting the jumps in the case when \( C \) is of bounded variation with respect to the Hausdorff metric. In the present paper we exploit a family of convex valued curves which are geodesics with respect to the excess \( e \). In a certain sense this procedure allows us to say that every discontinuous sweeping process with locally bounded retraction "is indeed" a Lipschitz continuous one. In this way we also generalize a result of [61] where a kind of "differential measure" formulation is given for sweeping processes with bounded variation with respect to the Hausdorff metric. In that paper we also allowed the behaviour on jumps to have a more general behaviour, therefore it is natural to wonder if any of these general behaviors can be obtained by a reparametrization of \( C \) using suitable curves connecting the jumps.
The paper is organized as follows. In the next section we present some preliminaries and in Sections 3 we state the main results of the paper. In Sections 4 and 5 we review the proof of the Lipschitz continuous case and we provide an integral formulation of the Lipschitz continuous sweeping processes. In Section 6 we define and study the class of convex valued geodesics needed in the proof of our main result which is presented in the last Section 7.

2. Preliminaries

Let us now recall the main notations and definitions used throughout in the paper. The set of integers greater than or equal to 1 will be denoted by \( \mathbb{N} \). Given an interval \( I \) of the real line \( \mathbb{R} \) and the family \( \mathcal{B}(I) \) of Borel subsets of \( I \), if \( \mu : \mathcal{B}(I) \to [0, \infty] \) is a measure, \( p \in [1, \infty] \), and if \( E \) is a Banach space, then the space of \( E \)-valued functions which are \( p \)-integrable with respect to \( \mu \) will be denoted by \( L^p(I, \mu; E) \) or by \( L^p(\mu; E) \). We do not identify two functions which are equal \( \mu \)-almost everywhere (\( \mu \)-a.e.).

2.1. Functions of bounded variation. In this subsection we assume that

\[
(X, d) \text{ is an extended complete metric space,}
\]

i.e. \( X \) is a set and \( d : X \times X \to [0, \infty] \) satisfies the usual axioms of a distance, but may take on the value \( \infty \). The notion of completeness remains unchanged. The general topological notions of interior, closure and boundary of a subset \( Y \subseteq X \) will be respectively denoted by \( \text{int}(Y) \), \( \text{cl}(Y) \) and \( \partial Y \). We also set \( d(x, A) := \inf_{a \in A} d(x, a) \). If \( (Y, d_Y) \) is a metric space then the continuity set of a function \( f : Y \to X \) is denoted by \( \text{Cont}(f) \), while \( \text{Discont}(f) := Y \setminus \text{Cont}(f) \).

Let \( S \subseteq Y \) we define \( \text{Lip}(f, S) := \sup \{ d(f(s), f(t)) / d_Y(t, s) : s, t \in S, s \neq t \} \). The Lipschitz constant of \( f \), and \( \text{Lip}(Y; X) := \{ f : Y \to X : \text{Lip}(f) < \infty \} \), the set of \( X \)-valued Lipschitz continuous functions on \( Y \). As usual \( \text{Lip}_{\text{loc}}(Y; X) := \{ f \in X^I : \text{Lip}(f, S) < \infty \forall S \text{ compact in } Y \} \).

**Definition 2.1.** Given an interval \( I \subseteq \mathbb{R} \), a function \( f : I \to X \), and a subinterval \( J \subseteq I \), the (pointwise) variation of \( f \) on \( J \) is defined by

\[
V(f, J) := \sup \left\{ \sum_{j=1}^{m} d(f(t_{j-1}), f(t_j)) : m \in \mathbb{N}, \, t_j \in J \forall j, \, t_0 < \cdots < t_m \right\}.
\]

If \( V(f, I) < \infty \) we say that \( f \) is of bounded variation on \( I \) and we set \( BV(I; X) := \{ f : I \to X : V(f, I) < \infty \} \) and \( BV_{\text{loc}}(I; X) := \{ f : I \to X : V(f, J) < \infty \forall J \text{ compact in } I \} \).

It is well known that the completeness of \( X \) implies that every \( f \in BV(I; X) \) admits one-sided limits \( f(t-) \), \( f(t+) \) at every point \( t \in I \), with the convention that \( f(\inf I-) := f(\inf I) \) if \( \inf I \in I \), and \( f(\sup I+) := f(\sup I) \) if \( \sup I \in I \). Moreover \( \text{Discont}(f) \) is at most countable. We set \( BV^I(I; X) := \{ f \in BV(I; X) : f(t-) = f(t) \forall t \in I \} \), \( BV^r(I; X) := \{ f \in BV(I; X) : f(t) = f(t+) \forall t \in I \} \). Accordingly we set \( BV_{\text{loc}}^I(I; X) := \{ u \in BV_{\text{loc}}(I; X) : u(t-) = u(t) \forall t \in I \} \) and \( BV_{\text{loc}}^r(I; X) := \{ u \in BV_{\text{loc}}(I; X) : u(t) = u(t+) \forall t \in I \} \) so that \( \text{Lip}_{\text{loc}}(I; X) \subseteq BV_{\text{loc}}^I(I; X) \).

2.2. Convex sets in Hilbert spaces. Throughout the remainder of the paper we assume that

\[
\begin{align*}
\mathcal{H} & \text{ is a real Hilbert space with inner product } (x, y) \mapsto \langle x, y \rangle, \\
\|x\| & := \langle x, x \rangle^{1/2},
\end{align*}
\]

and we endow \( \mathcal{H} \) with the natural metric defined by \( d(x, y) := \|x - y\| \), \( x, y \in \mathcal{H} \). We set

\[
D_r := \{ x \in \mathcal{H} : \|x\| \leq r \}, \quad r > 0
\]
monotone definition. which is also called the retraction of Exemple 2.8.2, p.46). We endow the set $C$ the function $\ell$ such that $K \in \mathcal{H}$ hold. If $K \in \mathcal{H}$ and $x \in K$, then $N_K(x)$ denotes the (exterior) normal cone of $K$ at $x$:

$$N_K(x) := \{u \in \mathcal{H} : (v - x, u) \leq 0 \quad \forall v \in K\} = \text{Proj}_{K}^{-1}(x) - x.$$  

(2.3)

From the previous definition in (2.3) it follows that the multivalued mapping $x \mapsto N_K(x)$ is monotone, i.e. $(u_1 - u_2, x_1 - x_2) \geq 0$ whenever $x_j \in K$, $u_j \in N_K(x_j)$, $j = 1, 2$ (see, e.g., [8, Exemple 2.8.2, p.46]). We endow the set $\mathcal{C}$ with the so called excess. Here we recall the definition.

**Definition 2.2.** The excess $e : \mathcal{C} \times \mathcal{C} \rightarrow [0, \infty]$ is defined by

$$e(A, B) := \sup_{a \in A} d(a, B), \quad A, B \in \mathcal{C},$$

which is also called the excess of $A$ over $B$.

The following facts are well known (see [50]):

$$e(A, A) = 0 \quad \forall A \in \mathcal{C},$$

(2.4)

$$e(A, B) \leq e(A, C) + e(C, B) \quad \forall A, B, C \in \mathcal{C}.$$  

(2.5)

Moreover we have that

$$e(A, B) = \inf\{\rho > 0 : A \subseteq B + D_\rho\} \quad \forall A, B \in \mathcal{C}.$$  

(2.6)

**Definition 2.3.** Given an interval $I \subseteq \mathbb{R}$, a function $C : I \rightarrow \mathcal{C}$, and a subinterval $J \subseteq I$, the retraction of $C$ on $J$ is defined by

$$R(C, J) := \sup \left\{ \sum_{j=1}^{m} e(C(t_{j-1}), C(t_j)) : m \in \mathbb{N}, \quad t_j \in J, \quad t_0 < \cdots < t_m \right\}.$$  

(2.7)

If $R(C, I) < \infty$ we say that $C$ is of bounded retraction on $I$ and we set $BR(I; \mathcal{C}) := \{C : I \rightarrow \mathcal{C} : R(C, I) < \infty\}$. We also set $BR_{loc}(I; \mathcal{C}) := \{C : I \rightarrow \mathcal{C} : R(C, J) < \infty \forall J \text{ compact in } I\}$.

For every $C \in BR_{loc}(I; \mathcal{C})$ we will define $\ell_C : [0, \infty[ \rightarrow [0, \infty[$ by

$$\ell_C(t) := R(C; [0, t]), \quad t \geq 0.$$  

(2.8)

The function $\ell_C$ is a sort of arc length with respect to the excess $e$ and is an increasing function such that $\ell_C(0) = 0$. It is well known (cf. [50]) that if $C \in BR_{loc}(I; \mathcal{C})$ then for every $t \in I$ we have

$$C(t+) := \lim_{s \rightarrow t+} \inf C(s) := \{x \in \mathcal{H} : \lim_{s \rightarrow t+} d(x, C(s)) = 0\} \in \mathcal{C},$$

(2.9)

$$C(t-) := \lim_{s \rightarrow t-} \inf C(s) := \{x \in \mathcal{H} : \lim_{s \rightarrow t-} d(x, C(s)) = 0\} \in \mathcal{C}.$$  

(2.10)

and

$$e(C(t), C(t+)) = \lim_{s \rightarrow t+} e(C(t), C(s)) = \ell_C(t+) - \ell_C(t),$$

(2.11)

$$e(C(t-), C(t)) = \lim_{s \rightarrow t-} e(C(s), C(t)) = \ell_C(t) - \ell_C(t-),$$  

(2.12)

therefore

$$\lim_{s \rightarrow t+} e(C(t), C(s)) = 0 \iff C(t) \subseteq C(t+).$$  

(2.13)
∀ \{C \in BR\} \text{ continuous when } \int \text{ and } {\mu}_{t} = |{\mu}_{j}|(I), \text{ i.e. } |{\mu}_{j}|(I) \text{ defines a norm on the space of measures with bounded variation. Finally we say that } \mu \text{ is with local bounded variation if } |{\mu}_{J}|(I) < \infty \text{ for every interval } I \text{ compact in } I \text{ (see, e.g. [25, Chapter I, Section 3]).}

If \nu : \mathcal{B}(I) \rightarrow [0, \infty] \text{ is a positive bounded Borel measure and if } g \in L^{1}(I, \nu; \mathcal{H}), \text{ then } g\nu \text{ will denote the vector measure defined by } g\nu(B) := \int_{B} g \, d\nu \text{ for every } B \in \mathcal{B}(I). \text{ In this case } |g\nu|(B) = \int_{B} |g(t)| \, d\nu \text{ for every } B \in \mathcal{B}(I) \text{ (see [25, Proposition 10, p. 174]).} \text{ Moreover a vector measure } \mu \text{ is called } \nu\text{-absolutely continuous if } \mu(B) = 0 \text{ whenever } B \in \mathcal{B}(I) \text{ and } \nu(B) = 0.

Assume that } \mu : \mathcal{B}(I) \rightarrow \mathcal{H} \text{ is a vector measure with bounded variation and let } f : I \rightarrow \mathcal{H} \text{ and } \phi : I \rightarrow \mathbb{R} \text{ be two step maps with respect to } \mu, \text{ i.e. there exist } f_{1}, \ldots, f_{m} \in \mathcal{H}, \phi_{1}, \ldots, \phi_{m} \in \mathcal{H} \text{ and } A_{1}, \ldots, A_{m} \in \mathcal{B}(I) \text{ mutually disjoint such that } |{\mu}_{j}|(A_{j}) < \infty \text{ for every } j \text{ and } f = \sum_{j=1}^{m} f_{j}, \phi = \sum_{j=1}^{m} \phi_{j}, \text{ where } 1_{S} \text{ is the characteristic function of a set } S, \text{ i.e. } 1_{S}(x) := 1 \text{ if } x \in S \text{ and } 1_{S}(x) := 0 \text{ if } x \notin S. \text{ For such step functions we define } \int_{I} f \, d{\mu}_{j} := \sum_{j=1}^{m} \int_{I} f_{j} \, d{\mu}_{j} \in \mathbb{R} \text{ and } \int_{I} \phi \, d\mu := \sum_{j=1}^{m} \phi_{j} \mu(A_{j}). \text{ If } St(|{\mu}_{j}|; \mathcal{H}) \text{ (resp. } St(|{\mu}_{j}|)) \text{ is the set of } \mathcal{H}\text{-valued (resp. real valued) step maps with respect to } \mu, \text{ then the maps } St(|{\mu}_{j}|; \mathcal{H}) \rightarrow \mathcal{H} : f \mapsto \int_{I} f \, d{\mu}_{j} \text{ and } St(|{\mu}_{j}|) : \phi \mapsto \int_{I} \phi \, d\mu \text{ are linear and continuous when } St(|{\mu}_{j}|; \mathcal{H}) \text{ and } St(|{\mu}_{j}|) \text{ are endowed with the } L^{1}\text{-seminorms } ||f||_{L^{1}(I; \mu; \mathcal{H})} := \int_{I} ||f||_{\mathcal{H}} \, d|\mu| \text{ and } ||\phi||_{L^{1}(I; \mu; \mathcal{H})} := \int_{I} ||\phi||_{\mathcal{H}} \, d|\mu|. \text{ Therefore they admit unique continuous extensions } I_{\mu} : L^{1}(I; \mu; \mathcal{H}) \rightarrow \mathbb{R} \text{ and } J_{\mu} : L^{1}(I; |\mu|) \rightarrow \mathcal{H}, \text{ and we set } \int_{I} (f, d{\mu}_{j}) := I_{\mu}(f), \int_{I} \phi \, d\mu := J_{\mu}(\phi), \text{ } \forall f \in L^{1}(I; \mu; \mathcal{H}), \phi \in L^{1}(I; |\mu|). 

If } \nu \text{ is bounded positive measure and } g \in L^{1}(\nu; \mathcal{H}), \text{ arguing first on step functions, and then taking limits, it is easy to check that } \int_{I} (f, d(g\nu)) = \int_{I} (f, g) \, d\nu \text{ } \forall f \in L^{\infty}(\mu; \mathcal{H}). \text{ (2.15)}

The following results (cf., e.g., [25, Section III.17.2-3, pp. 358-362]) provide a connection between functions with bounded variation and vector measures which will be implicitly used in the paper.

**Theorem 2.1.** For every } f \in BV_{loc}(I; \mathcal{H}) \text{ there exists a unique vector measure of local bounded variation } \mu_{f} : \mathcal{B}(I) \rightarrow \mathcal{H} \text{ such that } \mu_{f}([c, d]) = f(d) - f(c), \mu_{f}([d, c]) = f(c) - f(d), \mu_{f}([c, d]) = f(d) - f(c), \mu_{f}([d, c]) = f(c) - f(d).
whenever $c < d$ and the left hand side of each equality makes sense. Conversely, if $\mu : \mathcal{B}(I) \rightarrow \mathcal{H}$ is a vector measure with local bounded variation, and if $f_\mu : I \rightarrow \mathcal{H}$ is defined by $f_\mu(t) := \mu([\inf I, t] \cap I)$, then $f_\mu \in BV_{loc}(I; \mathcal{H})$ and $\|f_\mu\| = \mu$.

**Proposition 2.1.** Let $f \in BV_{loc}(I; \mathcal{H})$, let $g : I \rightarrow \mathcal{H}$ be defined by $g(t) := f(t-) - f(t)$, for $t \in \text{int}(I)$, and by $g(t) := f(t)$, if $t \in \partial I$, and let $V_g : I \rightarrow \mathbb{R}$ be defined by $V_g(t) := V(g, [\inf I, t] \cap I)$. Then $\mu_g = f_\mu$ and $\|\mu_g\| = V(g, I)$.

The measure $\mu_f$ is called Lebesgue-Stieltjes measure or differential measure of $f$. Let us see the connection with the distributional derivative. If $f \in BV(I; \mathcal{H})$ and if $\mathcal{I} : \mathbb{R} \rightarrow \mathcal{H}$ is defined by

$$\mathcal{I}(t) := \begin{cases} f(t) & \text{if } t \in I \\ f(\inf I) & \text{if } \inf I \notin I, t \leq \inf I \\ f(\sup I) & \text{if } \sup I \notin I, t \geq \sup I \end{cases}$$

then, as in the scalar case, it turns out (cf. [57, Section 2]) that $\mu_f(B) = D\mathcal{I}(B)$ for every $B \in \mathcal{B}(\mathbb{R})$, where $D\mathcal{I}$ is the distributional derivative of $\mathcal{I}$, i.e.

$$-\int_{\mathbb{R}} \varphi'(t) \mathcal{I}(t) \, dt = \int_{\mathbb{R}} \varphi \, dD\mathcal{I} \quad \forall \varphi \in C^1_c(\mathbb{R}; \mathbb{R}),$$

$C^1_c(\mathbb{R}; \mathbb{R})$ being the space of real continuously differentiable functions on $\mathbb{R}$ with compact support. Observe that $D\mathcal{I}$ is concentrated on $I$: $D\mathcal{I}(B) = \mu_f(B \cap I)$ for every $B \in \mathcal{B}(I)$, hence in the remainder of the paper, if $f \in BV_{loc}(I; \mathcal{H})$ then we will simply write

$$Df := D\mathcal{I} = \mu_f, \quad f \in BV(I; \mathcal{H}),$$

and from the previous discussion it follows that

$$\|Df\| = |Df|(I) = \|\mu_f\| = V(f, I) \quad \forall f \in BV(I; \mathcal{H}).$$

If $I$ is bounded and $p \in [1, \infty]$, then the classical Sobolev space $W^{1,p}(I; \mathcal{H})$ consists of those functions $f \in C(I; \mathcal{H})$ such that $Df = gL^1$ for some $g \in L^p(I; \mathcal{H})$ and we have $W^{1,p}(I; \mathcal{H}) = AC^p(I; \mathcal{H})$. Let us also recall that if $f \in W^{1,1}(I; \mathcal{H})$ then the derivative $f'(t)$ exists for $L^1$-a.e. in $I$, $Df = f'L^1$, and $V(f,I) = \int_I \|f'(t)\| \, dt$ (cf., e.g. [8, Appendix]).

In [57, Lemma 6.4 and Theorem 6.1] it is proved that

**Proposition 2.2.** Assume that $I, J \subseteq \mathbb{R}$ are intervals and that $h : I \rightarrow J$ is nondecreasing.

(i) $Dh(h^{-1}(B)) = L^1(B)$ for every $B \in \mathcal{B}(h(\text{Cont}(h)))$.

(ii) If $f \in \text{Lip}(J; \mathcal{H})$ and $g : I \rightarrow \mathcal{H}$ is defined by

$$g(t) := \begin{cases} f'(h(t)) & \text{if } t \in \text{Cont}(h) \\ \frac{f(h(t+)) - f(h(t-))}{h(t+) - h(t-)} & \text{if } t \in \text{Discont}(h) \end{cases},$$

then $f \circ h \in BV(I; \mathcal{H})$ and $D(f \circ h) = g \, Dh$. This result holds even if $f'$ is replaced by any of its $L^1$-representatives.

3. MAIN RESULTS

In this section we state the main theorem of the present paper.
\textbf{Theorem 3.1.} Assume that $C \in BR_{\text{loc}}^r([0, \infty]; C_H)$ and $y_0 \in \mathcal{H}$. Then there exists a unique $y \in BV_{\text{loc}}^r([0, \infty]; \mathcal{H})$ such that there exists a Borel measure $\mu : \mathcal{B}([0, \infty]) \rightarrow [0, \infty]$ and a function $v \in L^1_{\text{loc}}(\mu; \mathcal{H})$ such that
\begin{align*}
y(t) &\in C(t), \\
Dy = v \mu, \\
- v(t) &\in N_{C(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in \text{Cont}(C), \\
y(t) &= \text{Proj}_{C(0)}(y(t)), \quad y(t^+) = \text{Proj}_{C(t^+)}(y(t)) \quad \forall t \in \text{Discont}(C) \setminus \{0\} \\
y(0) &= \text{Proj}_{C(0)}(y_0). \tag{3.5}
\end{align*}
Moreover $y$ is left continuous (respectively: right continuous) at $t \geq 0$ if and only if $u$ is left continuous (respectively: right continuous) at $t \geq 0$.

The following result shows that in the right continuous case the conditions (3.3)–(3.5) reduce to
\begin{align*}
- v(t) &\in N_{C(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in ]0, \infty[, \\
y(0) &= \text{Proj}_{C(0)}(y_0). \tag{3.7}
\end{align*}

\textbf{Theorem 3.2.} Assume that $C \in BR_{\text{loc}}^r([0, \infty]; C_H)$ and $y_0 \in \mathcal{H}$. Then there exists a unique $y \in BV_{\text{loc}}^r([0, \infty]; \mathcal{H})$ such that there exists a Borel measure $\mu : \mathcal{B}([0, \infty]) \rightarrow [0, \infty]$ and a function $v \in L^1_{\text{loc}}(\mu; \mathcal{H})$ such that
\begin{align*}
y(t) &\in C(t), \\
Dy = v \mu, \\
- v(t) &\in N_{C(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in [0, \infty[, \\
y(0) &= \text{Proj}_{C(0)}(y_0). \tag{3.11}
\end{align*}

An analogous result holds in the left continuous case:

\textbf{Theorem 3.3.} Assume that $C \in BR_{\text{loc}}^l([0, \infty]; C_H)$ and $y_0 \in \mathcal{H}$. Then there exists a unique $y \in BV_{\text{loc}}^l([0, \infty]; \mathcal{H})$ such that there exists a Borel measure $\mu : \mathcal{B}([0, \infty]) \rightarrow [0, \infty]$ and a function $v \in L^1_{\text{loc}}(\mu; \mathcal{H})$ such that
\begin{align*}
y(t) &\in C(t), \\
Dy = v \mu, \\
- v(t) &\in N_{C(t^+)}(y(t^+)) \quad \text{for } \mu\text{-a.e. } t \in [0, \infty[, \\
y(0) &= \text{Proj}_{C(0)}(y_0). \tag{3.15}
\end{align*}

4. Uniqueness and integral formulations

We start by proving the uniqueness of solutions to (3.1)–(3.5) in Theorem 3.1.

\textbf{Lemma 4.1.} Assume that $C \in BR_{\text{loc}}([0, \infty]; C_H)$ and $y_0 \in \mathcal{H}$. Then there exists a unique $y \in BV_{\text{loc}}([0, \infty]; \mathcal{H})$ such that there exists a Borel measure $\mu : \mathcal{B}([0, \infty]) \rightarrow [0, \infty]$ and a function $v \in L^1_{\text{loc}}(\mu; \mathcal{H})$ satisfying (3.1)–(3.5).

\textbf{Proof.} Assume by contradiction that there are two solutions $y_1$ and $y_2$. If $B \in \mathcal{B}([0, \infty])$, then by [51, Proposition 2] and by the monotonicity of the normal cone we get
\begin{align*}
\int_{B \cap \text{Cont}(C)} \text{d}D\|y_1(\cdot) - y_2(\cdot)\|^2 &\leq 2 \int_{B \cap \text{Cont}(C)} \langle y_1 - y_2, \text{d}D(y_1 - y_2) \rangle \\
&\leq 2 \int_{\text{Cont}(C)} \langle y_1 - y_2, \text{d}D(y_1 - y_2) \rangle \leq 0,
\end{align*}
while if \( t \in \text{Discont}(C) \), from (3.4) we infer that
\[
D(\|y_1(\cdot) - y_2(\cdot)\|^2)(\{t\}) = \|y_1(t^+) - y_2(t^+)\|^2 - \|y_1(t^-) - y_2(t^-)\|^2 \\
= \|\text{Proj}_{C(t)}(y_1(t)) - \text{Proj}_{C(t)}(y_2(t))\|^2 - \|y_1(t^-) - y_2(t^-)\|^2 \\
\leq \|y_1(t) - y_2(t)\|^2 - \|y_1(t^-) - y_2(t^-)\|^2 \\
= \|\text{Proj}_{C(t)}(y_1(t^-)) - \text{Proj}_{C(t)}(y_2(t^-))\|^2 - \|y_1(t^-) - y_2(t^-)\|^2 \\
\leq \|y_1(t^-) - y_2(t^-)\|^2 - \|y_1(t^-) - y_2(t^-)\|^2 = 0.
\]
Therefore for every \( B \in \mathcal{B}([0, T]) \) we find
\[
D(\|y_1(\cdot) - y_2(\cdot)\|^2)(B) \\
= D(\|y_1(\cdot) - y_2(\cdot)\|^2)(B \cap \text{Cont}(u)) + D(\|y_1(\cdot) - y_2(\cdot)\|^2)(B \cap \text{Discont}(u)) \\
= \int_{B \cap \text{Cont}(u)} dD(\|y_1(\cdot) - y_2(\cdot)\|^2) + \sum_{t \in B \cap \text{Discont}(u)} D(\|y_1(\cdot) - y_2(\cdot)\|^2)(\{t\}) \leq 0
\]
which implies that \( t \mapsto \|y_1(t^-) - y_2(t^-)\|^2 \) is nonincreasing and leads to the uniqueness of the solution since \( y_i(t) = \text{Proj}_{C(t)}(y_i(t^-)) \) and \( y_i(t^+) = \text{Proj}_{C(t)}(y_i(t)) \) for \( i = 1, 2 \). \( \Box \)

5. The 1-Lipschitz case

Our proof of the main Theorem 3.1 is based on the fact that problem (3.1)–(3.5) is reduced to the case when \( C \) is 1-Lipschitz continuous with respect to \( e \), i.e.
\[
e(C(t), C(s)) \leq s - t \quad \forall t, s \in \mathbb{R} : 0 \leq t \leq s. \tag{5.1}
\]
To be more precise the following theorem holds:

**Theorem 5.1.** Assume that \( y_0 \in H \) and that \( C : [0, \infty[ \rightarrow \mathcal{C}_H \) is 1-Lipschitz continuous with respect to \( e \), i.e. (5.1) holds. Then exists a unique function \( y \in \text{Lip}_\text{loc}([0, \infty[ ; H) \) such that
\[
y(t) \in C(t), \tag{5.2}
y' \in N_{C(t)}(y(t)) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \geq 0, \tag{5.3}
y(0) = \text{Proj}_{C(0)}(y_0). \tag{5.4}
\]
We will call this unique solution \( S(C, y_0) \) so that we define \( S : \text{Lip}_\text{loc}([0, \infty[ ; H) \times H \rightarrow \text{Lip}_\text{loc}([0, \infty[ ; H), \) the solution operator associating with every pair \((C, y_0)\) the only \( y \) satisfying (3.8)–(3.11).

This result was proved in [53] as a consequence of the more general formulation (3.8)–(3.11). Since we want to follow the opposite direction, for the sake of completeness and in order to be independent of the proofs of [53], we provide here a direct proof. As far as we know this proof has an element of novelty based on the fact that we exploit an integral formulation of (5.3) in order to pass to the limit in the approximation procedure. Let us start by proving an integral formulation for the more general inclusion (3.3).

**Lemma 5.1.** Let us assume that \( \mu : \mathcal{B}([0, \infty[) \rightarrow [0, \infty[ \) is a Borel measure and that \( C \in \text{BR}_\text{loc}^r([0, \infty[ ; \mathcal{C}_H) \). If \( v \in \text{L}^1\text{loc}(\mu ; H) \), \( y \in \text{BV}_\text{loc}^r([0, \infty[ ; H) \) and \( y(t) \in C(t) \) for every \( t \in [0, \infty[ \), then the following two conditions are equivalent.

(i) \( v(t) \in N_{C(t)}(y(t)) \) for \( \mu \)-a.e. in \( [0, T[ \), for every \( T > 0 \).

(ii) \[
\int_{[0,T[} \langle y(t) - z(t), v(t) \rangle \, d\mu(t) \leq 0 \quad \text{for every } T > 0 \text{ and every bounded } \mu\text{-measurable function } z : [0, T[ \rightarrow H \text{ such that } z(t) \in C(t) \text{ for every } t \in [0, T[.}
\]
Proof. Let us start by assuming that (i) holds and let \( z : [0, T] \to \mathcal{H} \) be a bounded \( \mu \)-measurable function such that \( z(t) \in C(t) \) for every \( t \in [0, T] \). Then it follows that
\[
\langle y(t) - z(t), v(t) \rangle \leq 0 \quad \text{for } \mu\text{-a.e. } t \in [0, T],
\]
and integrating over \([0, T]\) we infer condition (ii). Now assume that (ii) is satisfied and recall that if \( f \in L^1(\mu, [0, T]; \mathcal{H}) \) then there exists a \( \mu \)-zero measure set \( Z \) such that \( f([0, T] \setminus Z) \) is separable (see, e.g., [38, Property M11, p. 124]), therefore from (the proof of) [27, Corollary 2.9.9., p. 156] it follows that
\[
\lim_{h \searrow 0} \frac{1}{\mu([t, t+h])} \int_{[t,t+h]} \|f(\tau) - f(t)\|_E \, d\mu(\tau) = 0 \quad \text{for } \mu\text{-a.e. } t \in [0, T]. \tag{5.5}
\]
The points \( s \) satisfying (5.5) are called right \( \mu \)-Lebesgue points of \( f \). Let \( L \) be the set of right \( \mu \)-Lebesgue points for \( \tau \mapsto v(\tau) \), fix \( t \in L \), and choose \( \zeta_t \in C(t) \) arbitrarily. Since \( y \in BV_{\text{loc}}([0, \infty[; \mathcal{H}) \) we have that \( t \) is a right \( \mu \)-Lebesgue point of \( y \), therefore
\[
\frac{1}{\mu([t, t+h])} \int_{[t,t+h]} \|y(\tau) - y(t)\|_E \, d\mu(\tau)
\leq \frac{1}{\mu([t, t+h])} \int_{[t,t+h]} (\|y\|_\infty \|v(\tau) - v(t)\| + \|v\|_\infty \|y(\tau) - y(t)\|) \, d\mu(\tau), \tag{5.6}
\]
thus
\[
\lim_{h \searrow 0} \frac{1}{\mu([t, t+h])} \int_{[t,t+h]} \langle y(\tau), v(\tau) \rangle \, d\mu(\tau) = \langle y(t), v(t) \rangle, \tag{5.7}
\]
i.e. \( t \) is a right \( \mu \)-Lebesgue point of \( \tau \mapsto \langle y(\tau), v(\tau) \rangle \). If \( \zeta : [0, T] \to \mathcal{H} \) is defined by \( \zeta(\tau) := \text{Proj}_{C(t)}(\zeta_t), \tau \in [0, T] \), we have that \( \zeta \) is right continuous (see [53, Proposition 2e, Remark 1]), therefore \( \zeta \) is bounded and measurable, \( \zeta(t) = \zeta_t \), and arguing as before we see that \( t \) is a right \( \mu \)-Lebesgue point of \( \tau \mapsto \langle \zeta(\tau), v(\tau) \rangle \). Since \( \zeta(\tau) \in C(t) \) for every \( \tau \in [0, T] \), the function \( z(\tau) := 1_{[0,T]\setminus[t,t+h]}(\tau)\zeta(\tau) + 1_{[0,T] \setminus [t,t+h]}(\tau) y(\tau), \tau \in [0, T] \), is well defined for every sufficiently small \( h > 0 \) and \( z(\tau) \in C(t) \) for every \( \tau \in [0, T] \), thus we can take \( z \) in condition (ii) and we get
\[
\int_{[t,t+h]} \langle y(\tau), v(\tau) \rangle \, d\mu(\tau) \leq \int_{[t,t+h]} \langle \zeta(\tau), v(\tau) \rangle \, d\mu(\tau).
\]
Dividing this inequality by \( \mu([t, t+h]) \) and taking the limit as \( h \searrow 0 \) we get \( \langle y(t) - \zeta_t, v(t) \rangle \leq 0 \). Therefore, as \( \mu(L) = 0 \), we have proved that
\[
\langle y(t) - \zeta, v(t) \rangle \leq 0 \quad \forall \zeta \in C(t), \quad \text{for } \mu\text{-a.e. } t \in [0, T],
\]
i.e. condition (i) holds. \qed

As a corollary we get the desired integral formulation for (5.3).

Corollary 5.1. Let \( C : [0, \infty[ \to C_\mathcal{H} \) be such that (5.1) holds. If \( y \in \text{Lip}_{\text{loc}}([0, \infty[; \mathcal{H}) \) and \( y(t) \in C(t) \) for every \( t \in [0, \infty[ \), then the following two conditions are equivalent.
(i) \(-y'(t) \in N_{C(t)}(y(t)) \) for \( \mathcal{L}^1\)-a.e. in \( [0, T] \), for every \( T > 0 \).
(ii) \( \int_0^T \langle y(t) - z(t), y'(t) \rangle \, dt \leq 0 \) for every \( T > 0 \) and every bounded \( \mathcal{L}^1\)-measurable function \( z : [0, T] \to \mathcal{H} \) such that \( z(t) \in C(t) \) for every \( t \in [0, T] \).

Proof. We can apply Lemma (5.1) with \( \mu = \mathcal{L}^1 \) and \( v = -y' \). \qed
Now we provide the proof of Theorem 5.1 following an implicit discretization scheme. The proof is obviously inspired by the paper [53] and a crucial fact for getting proper a priori estimates is the following inequality holding for \(A, B \in C(H)\) and \(x, y \in H\) (see [53, Lemma 1(2a)]):

\[
\|\text{Proj}_A(x) - \text{Proj}_B(y)\|^2 - \|x - y\|^2 \leq 2d(x, A)e(B, A) + 2d(y, B)e(A, B) \tag{5.8}
\]

**Proof of Theorem 5.1.** First of all observe that the uniqueness of solutions can be inferred exactly as in Lemma 4.1, since (3.1)–(3.5) hold with \(\mu = L^1\) and \(v = v'\). Concerning existence, for every \(n \in \mathbb{N} \cup \{0\}\), set

\[
t^n_j := \frac{j}{2^n}, \quad j \in \mathbb{N} \cup \{0\}
\]

and we define recursively the sequence \((y^n_j)_{j=0}^{\infty}\) by the three following conditions:

\[
y^n_0 := \text{Proj}_{C(0)}(y_0),
\]

\[
y^n_j \in C(t^n_j), \quad j \in \mathbb{N}. \tag{5.10}
\]

\[
y^n_j - y^{n-1}_j \in -N_{C(t^n_j)}(y^n_j), \quad j \in \mathbb{N}. \tag{5.11}
\]

Observe that (5.10)-(5.12) is an implicit time discretization scheme for (5.3)–(5.4). From the fact that \(N_{C(t^n_j)}(y^n_j)\) is a cone we infer that (5.12) is equivalent to \(y^n_j - y^n_{j-1} \in N_{C(t^n_j)}(y^n_j)\), which can be rewritten as

\[
y^n_j = \text{Proj}_{C(t^n_j)}(y^n_{j-1}) = \text{Proj}_{C(\frac{t^n_j}{2^n})}(y^n_{j-1}) \quad \forall j \in \mathbb{N}. \tag{5.13}
\]

Therefore using the 1-Lipschitzianity of \(C\) with respect to \(e\), we get

\[
\|y^{n+1}_1 - y^n_0\|^2 = \|\text{Proj}_{C(t^{n+1}_1)}(y^n_0) - y^n_0\|^2 = d(y^n_0, C(t^{n+1}_1))^2 \\
\leq e(C(0), C(t^{n+1}_1))^2 \leq (t^{n+1}_1)^2 = \frac{1}{2^{2n+2}} \tag{5.14}
\]

and more generally, using also (5.8), for every \(j \geq 1\) we have

\[
\|y^{n+1}_{2j+1} - y^n_j\|^2 = \|\text{Proj}_{C(t^{n+1}_{2j+1})}(y^{n+1}_{2j+1}) - y^n_j\|^2 \\
\leq \|y^{n+1}_{2j+1} - y^n_j\|^2 + 2d(y^{n+1}_{2j+1}, C(t^{n+1}_{2j+1}))d(y^n_j, C(t^{n+1}_{2j+1})) \\
\leq \|y^{n+1}_{2j} - y^n_j\|^2 + 2e(C(t^{n+1}_{2j+1}), C(t^{n+1}_{2j+1}))e(C(t^n_{2j+1}), C(t^n_{2j+1})) \\
= \|y^{n+1}_{2j} - y^n_j\|^2 + 2e(C \left( \frac{2j}{2^{n+1}} \right), C \left( \frac{2j + 1}{2^{n+1}} \right))e(C \left( \frac{2j + 1}{2^{n+1}} \right), C \left( \frac{2j + 1}{2^{n+1}} \right)) \\
\leq \|y^{n+1}_{2j+1} - y^n_j\|^2 + 2 \left( \frac{2j + 1}{2^{n+1}} - \frac{2j}{2^{n+1}} \right) \left( \frac{2j + 1}{2^{n+1}} - \frac{2j + 1}{2^{n+1}} \right) \\
= \|y^{n+1}_{2j} - y^n_j\|^2 + 2 \frac{1}{2^{2n+1}} \frac{1}{2^{n+1}} = \|y^{n+1}_{2j} - y^n_j\|^2 + \frac{2}{2^{2n+2}}. \tag{5.15}
\]

Moreover for every \(j \in \mathbb{N}\) the nonexpansivity of the projection yields

\[
\|y^{n+1}_{2j+1} - y^n_j\|^2 = \|\text{Proj}_{C(\frac{t^{n+1}}{2^n})}(y^{n+1}_{2j+1}) - \text{Proj}_{C(\frac{t^{n}}{2^n})}(y^n_j)\|^2 \\
\leq \|y^{n+1}_{2j+1} - y^n_{j-1}\|^2 \\
= \|y^{n+1}_{2(j-1)+1} - y^n_{j-1}\|^2, \tag{5.16}
\]
thus, (5.14), (5.15), (5.17), and a recursive argument yield for \( j \geq 1 \)
\[
\|y_{2j+1}^{n+1} - y_j^n\|^2 \leq \|y_1^{n+1} - y_0^n\|^2 + \frac{2j}{2^{2n+2}} \leq \frac{1}{2^{2n+2}} + \frac{2j}{2^{2n+2}} = \frac{1 + 2j}{2^{2n+2}}
\]
and for \( j \geq 1 \)
\[
\|y_{2j+1}^{n+1} - y_j^n\|^2 \leq \|y_{2(j-1)+1}^{n+1} - y_{j-1}^n\|^2 \leq \frac{1 + 2(j-1)}{2^{2n+2}} = \frac{2j - 1}{2^{2n+2}}.
\]

We now define the step function \( y_n : [0, \infty[ \longrightarrow \mathcal{H} \) by
\[
y_n(t) := y_j^n \quad \text{if} \quad t \in \left[t_j^n, t_{j+1}^n\right], \quad j \in \mathbb{N} \cup \{0\},
\]
and we are going to prove that the sequence \( \langle y_n \rangle \) is locally uniformly Cauchy. To this aim we fix \( T > 0, t \in [0, T] \), and \( n \in \mathbb{N} \). If \( k, K \in \mathbb{N} \cup \{0\} \) are the unique integers such that
\[
\frac{k}{2^n} = t_k^n \leq t < t_{k+1}^n = \frac{k + 1}{2^n}, \quad \frac{K}{2^n} \leq T < \frac{k + 1}{2^n},
\]
we have two possibilities: either
\[
t_k^n = \frac{k}{2^n} = \frac{2k}{2n+1} = t_{2k}^{n+1} \leq t < \frac{2k + 1}{2n+1} = t_{2k+1}^{n+1}
\]
or
\[
\frac{2k + 1}{2n+1} = t_{2k+1}^{n+1} \leq t < \frac{2k + 2}{2n+1} = \frac{k + 1}{2^n}.
\]
In the first case (5.21), from (5.18) we get that
\[
\|y_{n+1}(t) - y_n(t)\|^2 = \|y_{2k}^{n+1} - y_k^n\|^2 \leq \frac{2k - 1}{2^{2n+2}},
\]
in the second case (5.22), from (5.17) we infer that
\[
\|y_{n+1}(t) - y_n(t)\|^2 = \|y_{2k+1}^{n+1} - y_k^n\|^2 \leq \frac{1 + 2k}{2^{2n+2}}.
\]
Therefore in every case thanks to (5.20) we find that
\[
\|y_{n+1}(t) - y_n(t)\|^2 \leq \frac{1 + 2k}{2^{2n+2}} \leq \frac{1 + 2K}{2^{2n+2}} \leq \frac{1 + 2^{n+1}T}{2^{2n+2}}
\]
thus
\[
\sum_{n=0}^{\infty} \|y_{n+1}(t) - y_n(t)\| \leq \sum_{n=0}^{\infty} \frac{(1 + 2^{n+1}T)^{1/2}}{2^{n+1}} < \infty,
\]
and we have that \( \langle y_n \rangle \) is uniformly Cauchy on \( [0, T] \). It follows that there exists a right continuous function \( y : [0, \infty[ \longrightarrow \mathcal{H} \) such that
\[
y_n \rightarrow y \quad \text{uniformly on} \quad [0, T], \quad \forall T > 0.
\]
In particular it follows, by the closedness of \( \mathcal{C}(t) \), that
\[
y(t) \in \mathcal{C}(t) \quad \forall t \geq 0.
\]
Now take \( t, s \in \mathbb{R} \) with \( 0 \leq t < s \). Then there exist \( j_n, k_n \in \mathbb{N} \) such that \( j_n < k_n \) and
\[
\frac{j_n}{2^n} \leq t < \frac{j_n + 1}{2^n}, \quad \frac{k_n}{2^n} \leq s < \frac{k_n + 1}{2^n},
\]
thus
\[
\|y_{j_n}(t) - y_{j_n}(s)\|^2 \leq \|y_{j_n}(t) - y_{j_n}(s)\|^2 \leq \frac{1 + 2^{j_n+1}T}{2^{2j_n+2}} \leq \frac{1 + 2^{k_n+1}T}{2^{2k_n+2}} \leq \frac{1 + 2^{n+1}T}{2^{2n+2}}.
\]
Definition 6.1. Assume that to reduce the affine interpolant of $y$ and we have found that $y \in t$, hence thanks to the integral characterization of Corollary 5.1 we have proved the theorem.

Thus, if $y(t) - y(s) = \lim_{n \to \infty} \|y_n(t) - y_n(s)\| \leq \lim_{n \to \infty} \frac{1}{n} + s - t = |t - s|$ and we have found that $y$ is a 1-Lipschitz continuous function. Now let $x_n$ be the piecewise affine interpolant of $y_n$:

$$x_n(t) := y_{j-1}^n + \frac{t - t_{j-1}^n}{t_j^n - t_{j-1}^n} (y_j^n - y_{j-1}^n), \quad \text{if } t \in [t_{j-1}^n, t_j^n].$$

It is immediately seen that $x_n \to y$ uniformly on $[0, T]$, (5.31) moreover we have

$$x_n'(t) = \frac{y_j^n - y_{j-1}^n}{t_j^n - t_{j-1}^n} \in - N_C(t_j^n) (y_j^n) \quad \forall t \in ]t_{j-1}^n, t_j^n[ \quad (5.32)$$

and

$$\|x_n'(t)\| = \frac{\|y_j^n - y_{j-1}^n\|}{t_j^n - t_{j-1}^n} \leq 1 \quad \forall t \neq t_j^n, \quad (5.33)$$

therefore $\|x_n'\|_{L^2([0, T]; H)}$ is bounded and, at least for a subsequence which we do not relabel, we have that $x_n' \to y'$ weakly in $L^2([0, T]; H)$.

Thus, if $z : [0, T] \to H$ is a bounded measurable function such that $z(t) \in C(t)$ for every $t \in [0, T]$, we have that

$$\int_0^T \langle y(t) - z(t), y'(t) \rangle \, dt = \lim_{n \to \infty} \int_0^T \langle y_n(t) - z(t), x_n'(t) \rangle \, dt \leq 0, \quad (5.35)$$

hence thanks to the integral characterization of Corollary 5.1 we have proved the theorem. \hfill \Box

6. Geodesics for the retraction

In this section we introduce the class of geodesics with respect to the excess $e$ which allow us to reduce the BR sweeping processes to the Lipschitz continuous case.

**Definition 6.1.** Assume that $A, B \in C_H$ and set $\rho := e(A, B)$. We define the curve $F_{A,B} : [0, 1] \to C_H$ by

$$F_{A,B}(t) := \begin{cases} A & \text{if } t = 0 \\ B + (1-t)D_\rho = B + D(1-t)\rho & \text{if } 0 < t \leq 1 \end{cases} \quad (6.1)$$
Proposition 6.1. If \( A, B \in \mathcal{G}_H \) and \( F_{(A,B)} : [0,1] \to \mathcal{G}_H \) is defined as in Definition 6.1 we have
\[
e(F(s), F(t)) = (t-s)e(A,B) \quad \forall s, t \in [0,1], \ s < t,
\]
in particular \( \text{Lip}(F) = (t-s) \) and we can call \( F \) a \( e \)-geodesic connecting \( A \) to \( B \).

Proof. For every \( t > 0 \) we have \( F(0) = A \subseteq B + D_\rho = A + B_{(1-t)\rho} + B_{\rho} \) thus \( e(F(0), F(t)) \leq t\rho \). If \( 0 < s \leq t \) we have
\[
F(s) = B + D_{(1-s)\rho} = B + B_{(1-t)\rho} + D_{(t-s)\rho} = F(t) + D_{(t-s)\rho}.
\]
Therefore \( e(F(s), F(t)) \leq (t-s)\rho = (t-s)e(A,B) \). On the other hand we have, for instance, \( e(A,B) \leq e(A, F(s)) + e(F(s), F(t)) + e(F(t), B) \leq se(A,B) + e(F(s), F(t)) + (1-t)e(A,B) \), hence \( (t-s)e(A,B) \leq e(F(s), F(t)) \) and (6.2) is proved.

The next Lemma shows that the solutions of the sweeping processes driven by \( F_{(A,B)} \) are always straight line segments connecting the initial datum \( y_0 \) to its projection to \( B \).

Lemma 6.1. Let \( A, B \in \mathcal{G}_H \) be such that \( \rho := e_{\mathcal{H}}(A, B) < \infty \) and let \( F_{(A,B)} = F : [0,1] \to \mathcal{G}_H \) be defined by (6.1). If \( y_0 \in A \), then let \( t_0 \in [0,1] \) be the unique number such that
\[
\|y_0 - \text{Proj}_B(y_0)\| = (1-t_0)\rho
\]
and define \( y \in \text{Lip}([0,1]; \mathcal{H}) \) by
\[
y(t) := \begin{cases} y_0 & \text{if } t \in [0,t_0[ \\ y_0 + \frac{t-t_0}{1-t_0}(\text{Proj}_B(y_0) - y_0) & \text{if } t \neq t_0, \ t \in [t_0,1[. \\
\text{Proj}_B(y_0) & \text{if } t = 1 
\end{cases}
\]
Then
\[
y(t) \in F_{(A,B)}(t) \quad \forall t \in [0,1],
\]
\[
y'(t) + N_{F_{(A,B)}(t)}(y(t)) \geq 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,1],
\]
\[
y(0) = \text{Proj}_{F_{(A,B)}(0)}(y_0) = y_0,
\]
i.e. \( y \) is the unique solution of the sweeping process driven by \( F_{(A,B)} \) with initial condition \( y_0 \in A \).

Proof. We use the notation \( K_\rho := K + D_\rho \) for \( K \in \mathcal{G}_H \) and \( \rho \geq 0 \). If \( y_0 \in B \) we have that \( t_0 = 1, y(t) = y_0 = \text{Proj}_B(y_0) \in \mathcal{G}(t) \) and \( y'(t) = 0 \) for every \( t \in [0,1] \), and we are done. Therefore we assume that \( y_0 \notin B \), i.e. \( t_0 < 1 \), thus from [60, Lemma 4.1-(ii)] and formula (6.4) we deduce that
\[
y_0 \in \partial B_{(1-t_0)\rho} \iff t = t_0.
\]
that is \( t_0 \) is the first time when \( \partial B_{(1-t)\rho} \) meets \( y_0 \). If \( t \in [t_0,1] \) we have that
\[
\|y(t) - \text{Proj}_B(y_0)\| = \frac{1-t}{1-t_0} \|y_0 - \text{Proj}_B(y_0)\| = \frac{1-t}{1-t_0}(1-t_0)\rho = (1-t)\rho,
\]
therefore
\[
y(t) \in \partial B_{(1-t)\rho} \quad \forall t \in [t_0,1].
\]
Therefore, since
\[
y'(t) = \begin{cases} 0 & \text{if } t \in ]0,t_0[ \\
\frac{1}{1-t_0}(\text{Proj}_B(y_0) - y_0) & \text{if } t \in ]t_0,1[. 
\end{cases}
\]
we infer from [60, Lemma 4.1-(iii)] that \( y'(t) \in N_{\mathcal{F}(t)}(y(t)) \) for every \( t \in [0, 1] \setminus \{ t_0 \} \). \( \square \)

7. Sweeping processes with arbitrary locally bounded retraction

In this section we provide the proofs of the main theorems.

Proof of Theorem 3.1. Now we assume that \( C \) has bounded local retraction and we recall that \( \ell_C : [0, \infty[ \to [0, \infty[ \) is defined by

\[
\ell_C(t) := R(C; [0, t]), \quad t \geq 0.
\]

The function \( \ell_C \) is increasing, therefore \( \ell_C^{-1}(\tau) \) is always a (possibly degenerate) interval for every \( \tau \in \ell_C([0, \infty[) \) and we can define \( \mathcal{\bar{C}} : \ell_C([0, \infty[) \to \mathcal{\mathcal{H}} \) in the following way:

\[
\mathcal{\bar{C}}(\tau) := \begin{cases} 
C(t) & \text{if } \ell_C^{-1}(\tau) \text{ is a singleton and } \ell_C^{-1}(\tau) = \{ t \} \\
C(t+) & \text{if } \ell_C^{-1}(\tau) \text{ is not a singleton and } \inf \ell_C^{-1}(\tau) = t.
\end{cases}
\]

Let us observe that if \( \ell_C \) is not constant on any nondegenerate interval, then one has \( C(t) = \mathcal{\bar{C}}(\ell_C(t)) \) for every \( t \geq 0 \), so that \( \mathcal{\bar{C}} \) can be considered as a reparametrization of \( C \) by the “arc length” \( \ell_C \). Let us also observe that \( C \) is set-theoretically increasing in time on the intervals where \( \ell_C \) is constant, therefore the solution of the sweeping process driven by \( C \) is expected to be constant on these intervals. We now claim that \( \mathcal{\bar{C}} \) is 1-Lipschitz continuous with respect to \( e \), and to this aim we take \( \sigma, \tau \in \ell_C([0, \infty[) \) with \( \sigma < \tau \) and assume that \( s = \inf \ell_C^{-1}(\sigma) \leq \sup \ell_C^{-1}(\tau) = s^* \), \( t = \inf \ell_C^{-1}(\tau) \leq \sup \ell_C^{-1}(\tau) = t^* \) for some \( s, s^*, t, t^* \geq 0 \). If \( \ell_C^{-1}(\sigma) \) and \( \ell_C^{-1}(\tau) \) are both singletons, then \( e(\mathcal{\bar{C}}(\sigma), \mathcal{\bar{C}}(\tau)) = e(C(s), C(t)) = R(C; [s, t]) = \tau - \sigma. \) If \( \ell_C^{-1}(\sigma) \) is not a singleton and \( \ell_C^{-1}(\tau) \) is a singleton, then \( e(\mathcal{\bar{C}}(\sigma), \mathcal{\bar{C}}(\tau)) = e(C(s), C(t+)) = e(C(s), C((t + t^*)/2)) \leq R(C; [s, (s + t^*)/2]) = \tau - \sigma. \) Finally if both \( \ell_C^{-1}(\sigma) \) and \( \ell_C^{-1}(\tau) \) are not singletons, then \( e(\mathcal{\bar{C}}(\sigma), \mathcal{\bar{C}}(\tau)) = e(C(s), C(t+)) = e(C((s + s^*)/2), C((t + t^*)/2)) \leq R(C; [(s + s^*)/2, (t + t^*)/2]) = \tau - \sigma. \) and we have proved that \( \mathcal{\bar{C}} \) is 1-Lipschitz continuous with respect to \( e \). Now we extend the definition of \( \mathcal{\bar{C}} \) over the whole \([0, \infty[ \) by setting

\[
\mathcal{\bar{C}}(\sigma) := \mathcal{F}(C(t)-, C(t))(\sigma - \ell_C(t-)) \ell_C(t- - \ell_C(t-)) \quad \text{if } \sigma \in ]\ell_C(t-), \ell_C(t)[, \text{ if } \ell_C(t-) \neq \ell_C(t), \quad (7.2)
\]

\[
\mathcal{\bar{C}}(\ell_C(t-)) := C(t-) \quad \text{if } \ell_C(t-) \notin \ell_C([0, \infty[), \text{ if } \ell_C(t-) \neq \ell_C(t), \quad (7.3)
\]

and

\[
\mathcal{\bar{C}}(\sigma) := \mathcal{F}(C(t), C(t+))(\sigma - \ell_C(t)) \ell_C(t+ - \ell_C(t)) \quad \text{if } \sigma \in ]\ell_C(t+), \ell_C(t+)[, \text{ if } \ell_C(t) \neq \ell_C(t+), \quad (7.4)
\]

thus the resulting curve \( \mathcal{\bar{C}} : [0, \infty[ \to \mathcal{H} \) is a 1-Lipschitz continuous function with respect to \( e \) because \( \mathcal{\bar{F}}(C(t-), C(t)) \) and \( \mathcal{\bar{F}}(C(t), C(t+)) \) are geodesics connecting respectively \( C(t-) \) to \( C(t) \) and \( C(t) \) to \( C(t+) \), and because \( C(\inf \ell_C^{-1}(\ell_C(t-))+) \subseteq C(t-) \) if \( \ell_C(t-) \in \ell_C([0, \infty[). \) Therefore we have that \( S(\mathcal{\bar{C}}) \in \text{Lip}_{\text{loc}}([0, \infty[ ; \mathcal{H}) \) where \( S \) is the solution operator of the Lipschitz sweeping process defined in Theorem 5.1. Let us set

\[
y := S(\mathcal{\bar{C}}, y_0), \quad y := S(\mathcal{\bar{C}}, y_0) \circ \ell_C,
\]

and let us prove that \( y \) solves (3.1)-(3.5). It is obvious that \( y(t) \in C(t) \) when \( \ell_C^{-1}(\ell_C(t)) = \{ t \} \). If instead \( \ell_C^{-1}(\ell_C(t)) \) is not a singleton, we have that \( y(t) = \hat{y}(\ell_C(t)) \in \mathcal{\bar{C}}(\ell_C(t)) = C(\inf \ell_C^{-1}(\ell_C(t)+)) \subseteq C(t) \), thus condition (3.1) is satisfied. It is very easy to check that (3.5) holds true. Since \( \hat{y} \) is Lipschitz continuous and \( \ell_C \) is increasing, it is clear that \( y \in BV([0, T] ; \mathcal{H}) \), and
that $y$ is left continuous (respectively: right continuous) if and only if $\ell_c$ is left continuous (respectively: right continuous), so that $\text{Discont}(y) = \text{Discont}(\ell_c) = \text{Discont}(C)$. If $v : [0, \infty[ \to H$ is defined by

$$v(t) := \begin{cases} \hat{y}'(\ell_c(t)) & \text{if } t \in \text{Cont}(\ell_c) \\ \frac{\hat{y}(\ell_c(t+)) - \hat{y}(\ell_c(t-))}{\ell_c(t+) - \ell_c(t-)} & \text{if } t \in \text{Discont}(\ell_c) \end{cases},$$

(7.5)

then, since $\ell_c$ is increasing, from Proposition 2.2-(ii) we infer that $y \in BV_{\text{loc}}([0, \infty[; H)$ and $Dy = vD\ell_c$, i.e. (3.2) holds with $\mu = D\ell_c$. Let us set

$$Z := \{ t \in [0, \infty[ : -\hat{y}'(t) \notin N_{\tilde{C}(t)}(\hat{y}(t)) \}.$$

(7.6)

From formula (5.3) we deduce that

$$L^1(Z) = 0,$$

therefore, thanks to Proposition 2.2-(i), we have that

$$D\ell_c(\{ t \in \text{Cont}(\ell_c) : -v(t) \notin N_{C(t)}(y(t)) \})$$

$$= D\ell_c(\{ t \in \text{Cont}(\ell_c) : -\hat{y}'(t) \notin N_{\tilde{C}(\ell_c(t))}(\hat{y}(\ell_c(t))) \})$$

$$= D\ell_c(\{ t \in \text{Cont}(\ell_c) : \ell_c(t) \in Z \}) = L^1(Z) = 0,$$

(7.7)

hence (3.3) also holds with $\mu = D\ell_c$. Now let us fix $t \in \text{Discont}(\ell_c)$ and observe that if $\sigma \in ]\ell_c(t-), \ell_c(t)[$ then $\tilde{C}(\sigma) = F_{C(t-),C(t)}(\sigma - \ell_c(t-)/(\ell_c(t) - \ell_c(t-)))$, thus by the semigroup property of (5.2)-(5.4) and by Lemma 6.1 we have

$$v(t) = \hat{y}(\ell_c(t))$$

$$= S(\tilde{C}, y_0)(\ell_c(t))$$

$$= S(\tilde{C}(\cdot + \ell_c(t-)), S(\tilde{C}, y_0)(\ell_c(t-)))(\ell_c(t) - \ell_c(t-))$$

$$= S(F_{C(t-),C(t)} \cdot, \hat{y}(\ell_c(t-)))(1)$$

$$= S(F_{C(t-),C(t)}, y(t-))(1)$$

$$= \text{Proj}_{C(t)}(y(t-)).$$

(7.8)

On the other hand if $\sigma \in ]\ell_c(t), \ell_c(t+)]$ then $\tilde{C}(\sigma) = F_{C(t),C(t+)}(\sigma - \ell_c(t)/(\ell_c(t) + \ell_c(t)))$, therefore

$$v(t+) = \hat{y}(\ell_c(t+))$$

$$= S(\tilde{C}, y_0)(\ell_c(t))$$

$$= S(\tilde{C}(\cdot + \ell_c(t), S(\tilde{C}, y_0)(\ell_c(t)))(\ell_c(t+) - \ell_c(t))$$

$$= S(F_{C(t),C(t+)} \cdot, \hat{y}(\ell_c(t)))(1)$$

$$= S(F_{C(t),C(t+)} \cdot, y(t))(1)$$

$$= \text{Proj}_{C(t+)}(y(t)).$$

(7.9)

thus (3.4) is satisfied and we are done since uniqueness was proved in Section 4. \hfill \Box

Proof of Theorem 3.2. We know from Theorem 3.1 that $y$ is right continuous, thus we only have to prove formula (3.10). If $t \in \text{Discont}(u)$ then the first condition of (3.4) reads

$$\langle y(t) - \zeta, y(t) - y(t-) \rangle \leq 0 \quad \forall \zeta \in C(t+).$$

(7.10)
Hence, since $\text{Cont}(y) = \text{Cont}(\ell_C)$ and $\ell_C$ is right continuous, for every $z \in L^\infty([0,T];\mathcal{H})$ with $z([0,T]) \subseteq \mathcal{Z}$ we have

\[
\int_{[0,T]} \langle y(t) - z(t), dDy(t) \rangle = \int_{\text{Cont}(u) \cap [0,T]} \langle y(t) - z(t), dDy(t) \rangle + \sum_{t \in \text{Discont}(u) \cap [0,T]} \langle y(t) - z(t), y(t) - y(t-) \rangle \leq 0
\]

and we can conclude invoking Lemma 5.1. □

The proof of Theorem 3.3 is analogous.

REFERENCES

[1] V. Acary, O. Bonnefon, B. Brogliato, “Nonsmooth Modeling and Simulation for Switched Circuits. Lecture notes in electrical engineering”. Springer (2011).
[2] K. Addy, S. Adly, B. Brogliato, D. Goeleven, A method using the approach of Moreau and Panagiotopoulos for the mathematical formulation of non-regular circuits in electronics, Nonlinear Anal. Hybrid Syst. 1 (2007), 30-43. Nonlinear Anal. Hybrid Syst. 1 (2007), 30-43.
[3] S. Adly, T. Haddad, L. Thibault, Convex sweeping process in the framework of measure differential inclusions and evolution variational inequalities, Math. Program. Ser. B 148 (2014), 5-47.
[4] H. Benabdellah, Existence of solutions to the nonconvex sweeping process, J. Differential Equations 164 (2000), 286-295.
[5] F. Bernicot, J. Venel, Differential inclusions with proximal normal cones in Banach spaces, J. Convex Anal. 17 (2010), 451-484.
[6] F. Bernicot, J. Venel, Stochastic perturbations of sweeping process, J. Differential Equations 251 (2011), 1195-1224.
[7] M. Bounkhel, L. Thibault, Nonconvex sweeping process and prox-regularity in Hilbert space, J. Nonlinear Convex Anal. 6 (2005), 359-374.
[8] H. Brezis, “Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert”, North-Holland Mathematical Studies, Vol. 5, North-Holland Publishing Company, Amsterdam, 1973.
[9] B. Brogliato, L. Thibault, Existence and uniqueness of solutions for non-autonomous complementarity dynamical systems, J. Convex Anal. 17 (2010), 961-990.
[10] M. Brokate, P. Krejčí, H. Schnabel, On uniqueness in evolution quasivariational inequalities, J. Convex Anal. 11 (2004), 111-130.
[11] C. Castaing, Version aléatoire de problème de rafle par un convexe variable, C.R. Acad. Sci. Paris, Sér. A 277 (1973), 1057-1059.
[12] C. Castaing, Equations différentielles. Rafle par un convexe aléatoire à variation continue raite, C. R. Acad. Sci. Paris, Sér. A 282 (1976), 515-518.
[13] C. Castaing, T.X. Duc Ha, M. Valadier, Evolution equations governed by the sweeping process, Set-Valued Anal. 1 (1993), 109–139.
[14] C. Castaing, M.D.P. Monteiro Marques, BV periodic solutions of an evolution problem associated with continuous moving convex sets, Set-Valued Anal. 3 (1995), 381–399.
[15] C. Castaing, M.D.P. Monteiro Marques, Evolution problems associated with non-convex closed moving sets with bounded variation, Portugaliae Mathematica 53 (1996), 73–87.
[16] C. Castaing, M.D.P. Monteiro Marques, P. Raynaud de Fitte, A Skorokhod problem governed by a closed convex moving set, J. Convex Anal. 23 (2016), 387-423
[17] C. Castaing, M. Valadier, “Convex Analysis and Measurable Multifunctions”, Springer, Berlin - Heidelberg - New York, 1977.
[18] G. Colombo, V.V. Goncharov, The sweeping process without convexity, Set-Valued Anal. 7 (1999), 357–374.
[19] G. Colombo, R. Henrion, N.D. Hoang, B.S. Mordukhovich, Discrete approximations of a controlled sweeping process, Set-Valued Var. Anal. 23 (2015), 69–86.
[20] G. Colombo, R. Henrion, Nguyen D. Hoang, B. S. Mordukhovich, Optimal control of the sweeping process: the polyhedral case, J. Differential Equations 260 (2016), 3397–3447.
[21] G. Colombo, M.D.P. Monteiro Marques, Sweeping by a continuous prox-regular set, J. Differential Equations 187 (2003), 46-62.
[22] G. Colombo, M. Palladino, The minimum time function for the controlled Moreau’s sweeping process, SIAM J. Control 54 (2016), 2036–2062.
[23] B. Cornet, Existence of slow solutions for a class of differential inclusions, J. Math. Anal. Appl. 96 (1983), 130-147.
[24] S. Di Marino, B. Maury, F. Santambrogio, Measure sweeping processes, J. Convex Anal. 23 (2016), 567–601.
[25] N. Dinculeanu, “Vector Measures”, International Series of Monographs in Pure and Applied Mathematics, Vol. 95, Pergamon Press, Berlin, 1967.
[26] J.F. Edmond and L. Thibault, BV solutions of nonconvex sweeping process differential inclusions with perturbation, J. Differential Equations 226 (2006), 135-179.
[27] H. Federer, “Geometric Measure Theory”, Springer-Verlag, Berlin-Heidelberg, 1969.
[28] S. Flam, J.-B. Hiriart-Urruty, A. Jourani, Feasibility in finite time, J. Dynamical Control Systems 15 (2009), 537-555.
[29] T. Haddad, A. Jourani, L. Thibault, Reduction of sweeping process to unconstrained differential inclusion, Pac. J. Optim. 4 (2008), 493-512.
[30] C. Henry, An existence theorem for a class of differential equations with multivalued right-hand side, J. Math. Anal. Appl. 41 (1973), 179-186.
[31] O. Klein, V. Recupero Hausdorff metric BV discontinuity of sweeping processes, J. Phys.: Conf. Ser., 727 (2016), 012006.
[32] J. Kopfová, V. Recupero, BV-norm continuity of sweeping processes driven by a set with constant shape, J. Differential Equations 261 (2016), 5875-5899.
[33] P. Krejčí, V. Recupero, Comparing BV solutions of rate independent processes, J. Convex Anal. 21 (2014), 121-146.
[34] P. Krejčí, V. Recupero, BV solutions of rate independent differential inclusions, Math. Bohem. 139 (2014), 607–619.
[35] P. Krejčí, T. Roche, Lipschitz continuous data dependence of sweeping processes in BV spaces, Discrete Contin. Dyn. Syst. Ser. B 15 (2011), 637-650.
[36] P. Krejčí, A. Vladimirov, Polyhedral sweeping processes with oblique reflection in the space of regulated functions, Set-Valued Anal. 11 (2003), 91-110.
[37] M. Kunze, M. D. P. Monteiro Marques, An introduction to Moreau’s sweeping processes, Impact in Mechanical Systems - Analysis and Modelling, B. Brogliato (Ed.) Lecture Notes in Physics 551, Springer (2000), 1-60.
[38] S. Lang, “Real and Functional Analysis - Third Edition”, Springer Verlag, New York, 1993.
[39] B. Maury, A. Roudneff-Chupin, F. Santambrogio, A macroscopic crowd motion model of gradient flow type, Math. Models Methods Appl., 20, 1787-1821, 2010.
[40] B. Maury, A. Roudneff-Chupin and F. Santambrogio, Congestion-driven dendritic growth, Discrete Contin. Dyn. Syst., 34 (4), 1575-1604, 2014.
[41] B. Maury, A. Roudneff-Chupin, F. Santambrogio, J. Venel, Handling Congestion in Crowd Motion Modeling, Netw. Heterog. Media, 6 (3), 485-519, 2011.
[42] B. Maury, J. Venel, A mathematical framework for a crowd motion model, C. R. Math. Acad. Sci. Paris 346 (2008), 1245-1250.
[43] B. Maury, J. Venel, A discrete contact model for crowd motion, ESAIM: M2AN 45 (2011), 145-168.
[44] M.D.P. Monteiro Marques, Perturbations convexes semi-continues supérieurement dans les espaces de Hilbert, Sem. Anal. Conexe Montpellier (1984), exposé 2.
[45] M.D.P. Monteiro Marques, “Differential Inclusions in Nonsmooth Mechanical Problems - Shocks and Dry Friction”, Birkhauser Verlag, Basel, 1993.
[46] J. J. Moreau, Rafle par un convexe variable, I, Sem. d’Anal. Conexe, Montpellier, Vol. 1 (1971), Exposé No. 15.
[47] J. J. Moreau, Rafle par un convexe variable, II, Sem. d’Anal. Conexe, Montpellier, Vol. 2 (1972), Exposé No. 3.
[48] J. J. Moreau, Problème d’évolution associé à un convexe mobile d’un espace hilbertien, C.R. Acad. Sci. Paris Sér. A-B 276 (1973), A791–A794.
[49] J. J. Moreau, On unilateral constraints, friction and plasticity, in “New Variational Techniques in Mathematical Physics” (G. Capriz and G. Stampacchia, Ed.) pp. 173-322, C.I.M.E. II Ciclo 1973, Ediz. Cremonese, Roma, 1974.
[50] J. J. Moreau, Multiapplications à retraction finie, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 1 (1974), 169–203.
[51] J.J. Moreau, Sur les mesures différentielles de fonctions vectorielles et certains problèmes d’évolution, C. R. Math. Acad. Sci. Paris Sér. A 282 (1976), 837-840.
[52] J. J. Moreau, Application of convex analysis to the treatment of elastoplastic systems, in “Application of Methods of Functional Analysis to Problems in Mechanics” (P. Germain, B. Nayroles, Eds.) Lecture Notes in Mathematics Vol. 503, Springer, Berlin-Heidelberg-New York (1976), 56–89.

[53] J.J. Moreau, Evolution problem associated with a moving convex set in a Hilbert space, J. Differential Equations 26 (1977), 347-374.

[54] J. J. Moreau, An introduction to unilateral dynamics, in “Novel Approaches in Civil Engineering (M. Frémond, F. Maceri Eds.)”, Springer, Berlin (2002).

[55] L. Paoli, Multibody Dynamics with Unilateral Constraints and Dry Friction: How the Contact Dynamics Approach May Handle Coulomb’s Law Indeterminacies? J. Convex Anal. 23 (2016), 849-876.

[56] V. Recupero, Sobolev and strict continuity of general hysteresis operators, Math. Methods Appl. Sci. 32 (2009), 2003-2018.

[57] V. Recupero, BV solutions of rate independent variational inequalities, Ann. Sc. Norm. Super. Pisa Cl. Sc. (5) 10 (2011), 269-315.

[58] V. Recupero, A continuity method for sweeping processes, J. Differential Equations 251 (2011) 2125–2142.

[59] V. Recupero, BV continuous sweeping processes, J. Differential Equations 259 (2015), 4253-4272.

[60] V. Recupero, Sweeping processes and rate independence, J. Convex Anal. 23 (2016), 921-946.

[61] V. Recupero, F. Santambrogio, Sweeping processes with prescribed behavior on jumps, Ann. Mat. Pura Appl. 197 (2018), 1311-1332.

[62] M. Sene, L. Thibault, Regularization of dynamical systems associated with prox-regular moving sets, J. Nonlin. Convex Anal. 15 (2014), 647-663.

[63] L. Thibault, Sweeping process with regular and nonregular sets, J. Differential Equations 193 (2003), 1-26.

[64] L. Thibault, Regularization of nonconvex sweeping process in Hilbert space, Set-Valued Anal. 16 (2008), 319-333.

[65] L. Thibault Moreau Sweeping Process with Bounded Truncated Retraction, J. Convex Anal. 23 (2016), 1051-1098.

[66] M. Valadier, Quelques problèmes d’entrainement unilatéral en dimension finie, Sem. Anal. Convexe Montpellier (1988), Exposé 8.

[67] M. Valadier, Lipschitz approximation of the sweeping (or Moreau) process, J. Differential Equations 88 (1990), 248-264.

[68] M. Valadier, Raflle et viabilité, Sem. Anal. Convexe Montpellier (1992), Exposé 17.

Vincenzo Recupero, Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, I-10129 Torino, Italy.
E-mail address: vincenzo.recupero@polito.it