MEROMORPHIC $L^2$ FUNCTIONS ON FLAT SURFACES

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Abstract. We estimate spectral gaps for the Hodge norm on quadratic differentials. To each tangent direction at any point $(X, q)$ in the principal stratum of quadratic differentials, we associate a Hodge norm, and control the logarithmic derivative of vectors perpendicular to the principal directions in terms of the $q$-areas of the components corresponding to thick–thin decompositions and the lengths of short curves in the $q$-metric. In the worst case scenario, one gets a spectral gap of size $C_{g,n} \text{sys}(X, q)^2$.

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1 Introduction

In [Fra], we defined a metric \( d_E \) on the space of unit area quadratic differentials on Riemann surfaces in terms of flat geometry, and wrote \( d^{ss} \) for the restriction of this metric to strongly stable leaves of the Teichmüller geodesic flow. We showed this flow exhibits hyperbolic behavior near quantitatively recurrent trajectories. More precisely,

**Theorem 1.1.** Let \( K \) be a compact subset of the unit tangent space to the moduli space \( \mathcal{M}_{g,n} \), and let \( \lambda \) denote Lebesgue measure on \( \mathbb{R} \). Let \( \theta \in (0,1) \). There are positive real constants \( C(K), r_0(K), \alpha(K, \theta) \) such that the following holds: if \( T > 0 \), and \( X, g_T(X) \in K \) and moreover,

\[
\lambda\{t \in [0, T] : g_t(X) \in K\} > \theta T, \quad \text{and}
\]

\[
d^{ss}(X_1, X), d^{ss}(X_2, X) < r_0(K), \quad \text{then}
\]

\[
\frac{d^{ss}(g_T X_1, g_T X_2)}{d^{ss}(X_1, X_2)} < C(K)e^{-\alpha(K, \theta)T}.
\]

A weaker version of this, using a construction called the *modified Hodge norm*, was used in [ABEM12], [EM11], and [Ham13] to provide Teichmüller space analogues to the results in the thesis of Margulis [Mar70] on the geodesic flow of compact negatively curved manifolds.

A substantial difficulty in the implementation of this technique came from the fact that the Hodge norm arguments depended on a compactness argument, and estimates from this argument became degenerate as quadratic differentials approached the boundary of the principal stratum. The purpose of Theorem 1.1 was to remove this technical difficulty by giving an alternative compactness argument that did not degenerate at the multiple zero locus.

The purpose of this paper is to give a more accurate description of the hyperbolic behavior of the Hodge norm that does not depend solely on compactness arguments. In addition, the estimates persist as one approaches the boundary of a stratum, and we estimate the rate at which hyperbolicity is lost as one approaches the boundary of moduli space; in some instances we even see long stretches in the cusps with no loss of hyperbolicity. In particular, the Hodge norm exhibits hyperbolic behavior over stretches when the surface does not make progress in more than one component of a Minsky product region. (See [Min96] or [Raf14] for precise descriptions of product regions.)

In particular, we define a number called the *Hourglass Ratio* \( H(X, q) \) of a half-translation surface \((X, q)\), which is equal to 1 if \( X \) is in the thick part of \( \mathcal{M}_{g,n} \), and in the cusps is less than or equal to \( \ell/\sqrt{A} \) if the surface can be cut into two essential or cylinder pieces with area at least \( A \) by a system of curves of length \( \ell \) in the \( q \)-metric.

Our main theorem is the following:
Theorem 1.2. Let \((X, q)\) be any quadratic differential in the principal stratum of half translation surfaces of genus \(g\) with \(n\) marked points. Let \((\tilde{X}, \omega)\) be the orienting double cover for \((X, q)\). Let \(g_t\) denote the Teichmüller flow, with \(g_t(X, \omega) = (X_t, \omega_t)\). Let \(\|\alpha\|_t\) denote the Hodge norm of \(\alpha \in H^1_{\text{odd}}(\tilde{X})\), where \(\alpha\) is the cohomology class of a (-1)-eigenform of the involution of \(X\) respecting the double cover, which is orthogonal to \(\text{Re}(\omega)\) and \(\text{Im}(\omega)\) with respect to the Hodge norm on \(H^1(\tilde{X}_t, \mathbb{R})\). Then
\[
\frac{d}{dt} |\log(\|\alpha\|_t)| < 1 - C_{g,n} H(X, q)^2,
\]
where \(C_{g,n} > 0\) is a constant depending on only \(g\) and \(n\).

Remark: This provides an infinitesimal spectral gap result for a norm on the tangent space at almost every point. The estimate does not degenerate near the multiple zero locus, even though the norm does. Thus we can give a qualitative estimate of how the gap decays in the cusp of the moduli space.

To make the analogy to negatively curved Riemannian manifolds, parallel transport of geodesic flow contracts perpendicular tangent directions at an exponential rate, where the exponent is the proportional to the square root of the sectional curvature. Although the Teichmüller metric is not Riemannian, this gives an estimate akin to “the sectional curvature is at most \(-C_{g,n} H(X, q)^4\) in all 2-planes containing the direction of \(q\),” at least for the purposes of estimate expansion/contraction of the unstable/stable manifolds for the flow, with the caveats that must be applied whenever attempting to use the Hodge norm as a norm on the tangent space to \(M_{g,n}\).

As an immediate corollary, we have

Theorem 1.3. Let all notation be as in theorem 1. Let \((X, q)\) be a unit area quadratic differential in the principal stratum, whose shortest essential simple closed curve has length \(\text{Sys}(X, q)\) in the \(q\)-metric. Then
\[
\frac{d}{dt} |\log(\|\alpha\|_t)| < 1 - C_{g,n} \text{Sys}(X, q)^2.
\]

Proof. This is immediate since the systole is less than or equal to the hourglass ratio (up to a constant multiple). \(\square\)

2 Quadratic Differentials and the Hodge Norm

2.1 Quadratic differentials. Let \(X\) be a Riemann surface of genus \(g\) with \(n\) punctures. Assume \(X\) admits a conformal metric of constant curvature \(-1\) with finite area, i.e. \(3g - 3 + n > 0\). A quadratic differential \(q\) on \(X\) is a holomorphic section of the tensor square of the cotangent bundle of \(X\) that extends meromorphically to the compact Riemann surface \(\overline{X}\) obtained by adding in a point at each cusp, such that there are no poles of order 2 or higher. The space of quadratic differentials on \(X\) has \(3g - 3 + n\) complex dimensions.
Such \((X, q)\) admits a system of holomorphic charts covering all points of \(X\) at which \(q\) is holomorphic and non-vanishing, such that if \(z\) is the local coordinate then \(q = dz^2 := dz \otimes dz\). Call these coordinate charts \(q\)-charts. The change of charts preserve a metric as well as foliations by vertical and horizontal lines. The metric and foliations have singular extensions to all of \(\overline{X}\) with cone-type singularities angle \((2 + k)\pi\), and \((2 + k)\)-pronged singularities, at each point where \(q\) vanishes to order \(k\), where \(k \geq -1\). We will mostly concern ourselves with the generic case, in which \(q\) has \(n\) simple poles and \(4g - 4 + n\) simple zeros. Such \((X, q)\) is said to belong to the principal stratum. We will also deal with quadratic differentials that are squares of holomorphic 1-forms.

2.2 Teichmüller deformations. For \(t \in \mathbb{R}\) let \(g_t\) be the matrix \(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\), acting on \(C = \mathbb{R}^2\) with the usual identification \(\begin{pmatrix} x \\ y \end{pmatrix} = x + iy\). Then \((X, q)\) is given by a system of charts with transition maps of derivative \(\pm 1\), and \(g_t\) simultaneously acts on these charts. Let \((X_t, q_t)\) denote the 1-parameter family of Riemann surfaces of genus \(g\) with \(n\) marked points whose complex structures are given by images of the \(q\)-charts under \(g_t\) in \(\mathbb{R}^2\). (This respects the identifications). It is well known that \(X_t\) is a geodesic parametrized by arc length with respect to the Teichmüller metric on the moduli space of Riemann surfaces.

2.3 Orienting double cover. Let \((X, q)\) be in the principal stratum, with completion \(\overline{X}\). The orienting double cover of \((X, q)\) is the Riemann surface and differential \((\tilde{X}, \tilde{q})\) obtained by taking the branched cover of \(X\) which is branched of order 2 over all zeros and poles of \(q\), and such that the preimages of the nonsingular points of the \(q\)-metric are given by vertical unit tangent vectors with respect to the \(q\)-metric. On \(\tilde{X}\), the pullbacks of the vertical and horizontal foliations of \(q\) are orientable, and induced by the square of a holomorphic 1-form \(\omega\) on \(X\). The induced double cover therefore admits a collection of charts whose transition maps are translations, i.e. the structure of a Riemann surface with a holomorphic 1-form. This 1-form is a \(-1\)-eigenvector for the holomorphic involution \(\iota: \tilde{X} \to \tilde{X}\) that permutes the sheets of the double cover. The 1-form \(\omega\) has \(4g - 4 + n\) double zeros, which are all fixed by \(\iota\). As is well known, the locus of holomorphic 1-forms realized by these double covers is locally described by the \(-1\)-eigenspace the action \(\iota\) on the first de Rham cohomology of \(\tilde{X}\), which we will call \(H^1_{\text{odd}}(\tilde{X}, \mathbb{C})\).

Since \(\iota\) is holomorphic it commutes with the Hodge star operator, and the real and imaginary parts of a holomorphic 1-form \(\alpha + i\beta\), \(\alpha, \beta\) real, are related by Hodge star: \(\beta = *\alpha\), \(\alpha = -*\beta\). Thus \(H^1_{\text{odd}}(\tilde{X}, \mathbb{R})\) consists of those classes representable as real parts of holomorphic 1-forms in \(H^1_{\text{odd}}(X; \mathbb{C})\). Let \(\eta\) be an odd holomorphic 1-form. Then \((\iota/\eta)^2\) is \(\iota\)-invariant and therefore descends to a meromorphic function \(f\) on \(X\). A dimension count in fact shows that \(f\) is of the form \(q_2/q\), where \(q_2\) is another quadratic differential on \(X\). (For any \(q_2\) one gets such a function.)
2.4 Hodge norm and its first variation. The Hodge Norm of $c \in H^1(M, \mathbb{R})$ on a compact Riemann surface $M$ is given by $\|c\|_X^2 = \int_X \text{Re}(h_c) \wedge \text{Im}(h_c)$ where $h_c$ is the holomorphic 1-form whose real part is in the class $c$. The Hodge norm comes from a Hilbert space inner product:

$$\langle c_1, c_2 \rangle_{M} = -\frac{1}{2} \text{Im} \left( \int_X h_{c_1} \wedge \bar{h}_{c_2} \right).$$

As the complex structure of $M$ varies continuously, one can think of this group as a fixed vector space with a varying norm.

Let $(M_t, q_t)$ is a Teichmüller geodesic given by a 1-parameter family from a quadratic differential $q_0 = \omega_0^2$ defined on $M = M_0$, where $\omega_0$ us a holomorphic 1-form, and assume that the $q$-metric has unit area. There is a unit area holomorphic 1-form $\omega_t$ on $M_t$, whose cohomology class is $e^t \text{Re}(\omega_0) + i \cdot e^{-t} \text{Im}(\omega_0)$. The area form $|q|$ is given by $\text{Re}(\omega) \wedge \text{Im}(\omega)$. If $c \in H^1(M, \mathbb{R})$, Forni’s variational formula ([For01] lemma 2.1’, see also [FMZ12], section 2.6) says that

$$\frac{d}{dt} \left( \|c\|_M^2 \right)_{|t=0} = -2\text{Re} \left( \int_M \left( \frac{h_c}{\omega} \right)^2 |q| \right).$$

One sees that in the simplest case, $h_c = \pm \omega$, the derivative of the Hodge norm squared is $-2$, since one may apply the matrices $\begin{pmatrix} 1 & 0 \\ 0 & e^{-2t} \end{pmatrix}$ to obtain the family $M_t$, and the translation surface structures on $M_t$ whose holomorphic 1-forms have cohomologous real parts. It follows that the logarithmic derivative of the Hodge norm of $c$ is $-1$. Similarly, if $h_c = \pm i \omega$ one gets that the logarithmic derivative of the Hodge norm is $1$.

The variational formula tells us that these are the two extreme behaviors. A simple application of the Cauchy–Schwarz inequality implies that the logarithmic derivative of the Hodge norm is always in $[-1, 1]$, and the lower and upper extreme cases are attained by real and imaginary multiples of $\omega$ respectively. To attain statements about uniform hyperbolicity on certain regions of the moduli space, we would like to attain strict bounds for the growth rate of cohomology classes on the subspace

$$\langle [\text{Re}(\omega_t)], [\text{Im}(\omega_t)] \rangle \cap H^1_{\text{odd}}(X; \mathbb{R}),$$

i.e. to bound the logarithmic growth rate away from $\pm 1$. Here, the symbol $\perp$ is the symplectic complement with respect to the cup product. As it happens, it is also the Hodge orthogonal complement. This subspace is $g_t$-invariant.

The way we will proceed is as follows: Suppose that $\tilde{f}$ is the pullback of $f = q_2/q$ to the orienting double cover of $(X, q)$, where $q, q_2$ are quadratic differentials on $X$ and $\|\tilde{f}\|_2 = 1$. Let $(M_t, \omega_t) = g_t(\tilde{X}, \omega)$ be the orienting double covers corresponding to the family $g_t(X, q)$. Write $|\omega^2|$ for the area form obtained by pulling back $|q|$. Let $c \in H^1(\tilde{X}; \mathbb{R})$ be such $h_c/\omega$ is the pullback of $q_2/q$. Then if $\|\text{Im}(\tilde{f})\|_2 = \delta$, we have
\[
\frac{d}{dt} \left( \|c\|_{M_t}^2 \right)_{t=0} = -2 \text{Re} \left[ \int_M (\tilde{f})^2 \right] |\omega^2| \\
= -2 \int_M \text{Re}(\tilde{f})^2 - \text{Im}(\tilde{f})^2 |\omega^2| \\
= -2(1 - \delta^2 - \delta^2).
\]

It follows that

\[
- \frac{d}{dt} \log(\|c\|_{M_t}) \bigg|_{t=0} = 1 - 2\delta^2.
\]

We have a similar analysis when \(\|\text{Re}(\tilde{f})\| = \delta\). In particular, if \(\|\text{Re}(\tilde{f})\|, \|\text{Im}(\tilde{f})\| \geq \delta\),

\[
\left| \frac{d}{dt} \log(\|c\|_{M_t}) \right|_{t=0} \leq 1 - 2\delta^2.
\]

In the interest of proving spectral gap results, it is enough to prove

\[
\|\text{Im}(\tilde{f})\|_2 / \|\tilde{f}\|_2 = \|\text{Im}(f)\|_2 / \|f\|_2 \geq \delta
\]

whenever \((X, q)\) satisfies some geometric conditions on \((X, q)\) depending on \(\delta\). The equality is automatic. From now on, we will assume the following: If \(|q|\) is the area measure of the \(q\)-metric, then with respect to this area \(\|f\|_2 = 1\), \(f\) is meromorphic, and \(f\) orthogonal to the constant functions in \(L^2(X, |q|)\). These will be all we need to assume about \(f\), and in the case when \(q\) is in the principal stratum, such \(f\) consist of exactly the functions \(q_2/q\) with \(\int_X (q_2/q)|q| = 0\), up to scale.

## 3 Preliminary Notation and Estimates

### 3.1 Some fixed notation

In order to avoid a preponderance of named constants, \(C, C', C''\) will be used to denote topological constants, whose values will not be consistent throughout the paper. In each appearance, the important part is that we may choose their values to depend only on \(g\) and \(n\). Throughout, we will use the symbol \(u \asymp v\) to mean that there is some constant \(C \in (1, \infty)\), depending only on the particular moduli space we are considering, such that

\[
C^{-1} < u/v < C,
\]

where \(u, v\) are two positive quantities that depend on a point in the moduli space; however \(C\) will depend only on \(g\) and \(n\), or on some constant that may be chosen according to the hypotheses of a theorem. Crucially, however, the constants will never depend on a choice of flat surface.

Similarly, we may write \(u \lesssim v\) or \(v \lesssim u\) to mean that there exists some constant \(C\) for which

\[
u/v < C,\]
where \( u, v \) are related but variable positive quantities and where \( C \) is some topological constant. However, the implicit constant \( C \) may be different for each use of \( \prec \).

With these conventions, we may apply classical inequalities like Cauchy-Schwarz or add or multiply inequalities \( \preceq \) or \( \succ \).

1 denotes the constant function everywhere equal to 1; let \( 1_V \) denote the characteristic function of a set \( V \).

### 3.2 A gradient estimate.

Let \((X, q)\) be a point in quadratic differential space. For a measurable function \( f : X \to \mathbb{C} \), let \( \|f\|_p \) denote the \( L^p \) norm of \( f \) with respect to the area measure \( |q| \) of the \( q \)-metric. Let \( \langle f, g \rangle = \int_X f \bar{g}|q| \).

We have the following estimate by Treviño for the gradient of \( f \) at \( x \) with respect to the \( q \)-metric. What we state here is a trivial generalization of equation (26) in [Tre14], which requires only that \( f \) is holomorphic on a round disk centered at \( x \).

**Proposition 3.1.** Let \( \Sigma \) denote the set of singular points of \((X, q)\). Let \( B_R(x) \) denote the \( R \)-neighborhood of \( x \in X \) with respect to the \( q \)-metric, and assume \( f \) is analytic on \( X \setminus \Sigma \). Let \( r(x) \) be the largest \( R \) such that \( B_R(x) \) is an embedded disk not meeting \( \Sigma \). For any fixed \( t \in (0, 1] \), there is a constant \( c_t \) such that

\[
|\nabla f(x)| \leq c_t \|\text{Im}(f)\cdot 1_{B_t(r(x))}(x)\|_2.
\]

### 3.3 Holomorphic functions on expanding annuli.

This section will be dedicated to proving the Small Norm on Expanding Annuli Lemma, Lemma 3.3. It applies to expanding annuli, which we will define below:

**Definition 3.2.** An expanding annulus \( A \) is an open annulus in a surface with a flat metric, following the following additional conditions:

- The metric completion \( \bar{A} \) of \( A \) as a path metric space has two boundary components \( \gamma_1 \) and \( \gamma_2 \), which are rectifiable curves, at most one of which is degenerate (length 0).
- There is a number \( W \), called the width of \( A \) such that every point on \( p \in \bar{A} \setminus A \) is distance \( W \) from the boundary component not containing \( p \).
- \( A \) is foliated by level circles which are differentiable, piecewise smooth, and monotonically curved, and each level circle has nonzero geodesic curvature at some point.

Here, a level circle is a topological circle in \( A \), all of whose points are some distance \( d \) and \( w - d \) from the two boundary components, with \( 0 < d < w \). Monotonically curved means that for some orientation, the geodesic curvature is always non-negative. In the event that a level passes through a cone point, we require that the geodesic curvature condition is met at all non-cone points, and that the two arcs of the expanding annulus do not form an angle less than \( \pi \).

The condition that geodesic curvature is not everywhere zero means that the circumference of a level circle is strictly monotonic in \( d \); the annulus is said to be
expanding in the direction in which the circumferences increase. For the purpose of the following lemma we will also want to assume that our expanding annuli are free of singularities. We will choose specific expanding annuli later to tackle the general case, but we can already state the lemma in enough generality here. The rough idea of the proof is to divide primitive expanding annuli $W$ into smaller annuli $W_j$ which are between $2^{-j} - 1$ and $2^{-j}$ away from the geodesic representatives of core curves. On each of the $W_j$ we can then make coarse estimates of various geometric bounds.

**Lemma 3.3.** Let $\delta_0 > 0$. Let $f$ be holomorphic on a region $W$ on a half-translation surface, and let $W$ be the union of regions $W_j, j_0 \leq j \leq N$, and assume there are additional regions $W_{j_0-1}$ and $W_{N+1}$, such that for all regions $W_j$ the following geometric conditions hold:

- For some implicit constants $C$ and $C'$, $C \cdot 2^{-j} \leq \text{diam}_q(W_j) \leq C' \cdot 2^{-j}$, i.e. $\text{diam}_q(W_j) \approx 2^{-j}$.
- For some choice of implicit constants (not necessarily the same as in the previous condition), $\text{Area}(W_j) \approx 2^{-2j}$.
- The $2^{-j-2}$-neighborhood of $W_j$ is contained in $W_{j-1} \cup W_j \cup W_{j+1}$ for $j_0 \leq j \leq N$.
- If $R(p)$ is the radius of the maximum flat disk in $W$ centered at $p$ on which $f$ is holomorphic then $R(p) \approx 2^{-j}$ if $p \in W_j$.

In each hypothesis we require that the implicit constants in $\approx$ do not depend on $j_0$, $j$, or $N$.

Suppose, moreover that we have the following bounds on $f$:

- $\|f\|_2 \leq 1$.
- $\|1_{W_{j_0}} \cdot f\|_\infty < \delta_0 2^{j_0}$, and
- $\|\text{Im}(f)\|_2 < \delta_0$.

Then we have the following three coarse inequalities, whose implicit constants depend only on the choices of implicit constants in the geometric conditions:

(a) $\|1_{W_j} f(x)\|_\infty \approx \delta_0 2^j$,

(b) $\|1_{W_j} f\|_1 \approx \delta_0 2^{j_0}$, and

(c) $\|1_{W_j} f\|_2 \approx \delta_0$.

**Proof.** First, write $I_j$ for $\|1_{W_j} \cdot \text{Im}(f)\|_2$. By Proposition 3.1 estimates, and the fact that the area of $W_j$ is comparable to $2^{-2j}$ we have

$$\|1_{W_j} \nabla f(x)\|_\infty \leq C 2^j (I_{j-1} + I_j + I_{j+1}).$$

Also write $M_j = \|1_{W_j} \cdot f\|_\infty$. We have $M_j - M_{j+1} \leq C 2^j (I_{j-1} + I_j + I_{j+1})$. Let $j_0$ be the smallest $j$ for whichever annulus we are estimating. Since $\sum_j I_j^2 \leq \delta_0^2$, the Cauchy–Schwarz inequality gives (a) by

$$|M_j - M_{j_0}| \leq \sum_{k=j_0}^j C 2^k (I_{k-1} + I_j + I_{k+1}) \leq C' 2^j \delta_0.$$
For convenience we will write $I_k^*$ for $(I_{k-1} + I_k + I_{k+1})$; we have $\sum_k I_{k*}^2 < 9\delta_0^2$. There might be infinitely many regions $W_j$ in $W$, but we may assume there are finitely many and apply the monotone convergence theorem to reduce to the case in which there are finitely many. That is, we show that there is a constant $C$, which depends only on the choices of constants in the assumptions of the theorem, $\|1_{W_N^*} \cdot f \|_2 < C\delta_0^2$, where $W_N^*$ is any finite union $W^* = \bigcup_{j=j_0}^N W_j$, so long as the constant $C$ in the bound does not depend on $N$.

Part (b) comes from summing part (a) times the area of $W_j$ over all $j$.

Proof of (c):

$$\|1_{W} \cdot f\|_2^2 \preceq \sum_{j=j_0}^N \text{Area}(W_j) M_j^2$$

$$\preceq \sum_{j=j_0}^N 2^{-2j} M_j^2$$

$$\preceq \sum_{j=j_0}^N 2^{-2j} \left[ M_{j_0} + \sum_{k=j_0}^j 2^k I_k \right]^2$$

$$\preceq \sum_{j=j_0}^N 2^{-2j} M_{j_0}^2 + \sum_{j=j_0}^N \left[ \sum_{k=j_0}^j 2^{-j} 2^k I_k \right]^2$$

$$\preceq (2^{-2j_0} M_{j_0}^2) + \sum_{j=j_0}^N \left[ \sum_{k=j_0}^j 2^{-j} 2^k I_k \right]^2$$

$$\preceq \delta_0^2 + \sum_{j=j_0}^N \sum_{k=j_0}^j \sum_{r=j_0}^j 2^{-j} 2^r I_k I_r$$

$$= \delta_0^2 + \sum_{j=j_0}^N \sum_{k=j_0}^j 2^{-j} \sum_{r=j_0}^j 2^r (I_k^2 + I_r^2)$$

$$\preceq \delta_0^2 + \sum_{j=j_0}^N \sum_{k=j_0}^N 2^{-j} 2^r I_k^2$$

$$\preceq \delta_0^2 + \sum_{k=j_0}^N I_{k*}^2 \sum_{j=k}^N 2^{k-j}$$

$$\preceq \delta_0^2 + \sum_{k=j_0}^N I_{k*}^2$$

$$\preceq \delta_0^2$$

Now take square roots.

We would like to point out a simple intuitive reason to expect a lemma like this. Let $F(z)$ be analytic on the annular region $\epsilon < z \leq 1/\epsilon$ in the complex plane, extending continuously to the boundary with small values of $F$ on the outer boundary. It will help to think of the real and imaginary parts of $F$ on circles $|z| = R$ all having their own Fourier decompositions; the Laurent series $\sum a_n z^n$ corresponds to $\sum a_n R^n e^{in\theta}$.
In the Laurent series for $F$ on this annulus, the constant term and the positive powers of $z$ must have small coefficients, since the outer boundary circle has a Fourier series with small $L^2$ norm. For negative $n$, $a_n z^n$ contributes equally to the $L^2$ norms of the real and imaginary parts of $F$ on any circle $|z| = R$, so this constrains $a_n$ for negative $n$ as well.

3.4 Holomorphic functions on cylinders. In this section we examine the case in which our surface has a cylinder of large modulus. We define cylinder and modulus:

DEFINITION 3.4. Let $(X, q)$ be a Riemann surface with quadratic differential. A cylinder in the $q$-metric is a non-empty open annulus $A$ in $X$ which is the disjoint union of all singularity-free $q$-geodesic representatives of a non-trivial free isotopy class $\alpha$ of simple closed curves in $X$. The curve $\alpha$ is said to be a cylinder curve for $(X, q)$.

DEFINITION 3.5. The modulus of a cylinder with core curve $\alpha$ on a surface $X$ with quadratic differential $q$ is given by $w_q(\alpha) / \ell_q(\alpha)$, where $\ell_q(\alpha)$ is the length of a $q$-geodesic representative of $\alpha$, and $w_q(\alpha)$ is the length of the shortest arc that passes through every $q$-geodesic representative of $\alpha$.

If a Riemann surface $Y$ is homeomorphic to an annulus, it admits a singularity-free quadratic differential metric for which the whole of $Y$ is a cylinder, and this quadratic differential is unique up to scalars, so the metric is unique up to homothety. The modulus of the annulus is the modulus of this cylinder. If the modulus is infinite, then there are two conformal classes: quotient of the upper half-plane by $(\mathbb{Z}, +)$, which we will call semi-infinite and the other by the quotient of $\mathbb{C}$ by $(\mathbb{Z}, +)$, which we call bi-infinite. The only connected Riemann surfaces properly containing bi-infinite annuli are $\mathbb{C}$ and $\hat{\mathbb{C}}$ so bi-infinite annuli will not appear embedded in any hyperbolizable Riemann surface. See e.g. [Hub06] for detail on the uniformization of annuli.

Fix $b < 1$. Write $B$ for $b^{-1}$. Pick a local isometric coordinate system on a cylinder $(-L, L) \times [0, 2\pi s]$ with the identification $(x, 0) \sim (x, 2\pi s)$ for all $x \in [-L, L]$. We will assume $L > 2b$.

The restriction of $f$ to an annulus $A^*$ can be viewed as a function on the annulus $|\log(|z|)| < L$ via the map $(x, y) \mapsto z = e^{(x+iy)/s}$. Since $f$ is holomorphic on this annulus, it is given by a convergent Laurent series $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$. For any $\epsilon \in (0, L/s)$, the sum converges absolutely and uniformly on the annulus $|\log(|z|)| < L/s - \epsilon$.

We would now like to make the following important estimate, which is true in general for any cylinder with coordinates $(-L, L) \times [0, 2\pi s] / (x, 0) \sim (x, 2\pi s)$, and any $f$ which is holomorphic on the cylinder and whose imaginary part has small $L^2$ norm.
Lemma 3.6. Fix a small number $b \in (0, 1/2)$, and set $B = b^{-1}$. $f(x, y) = \hat{f}(0)$. Assume $f$ is meromorphic on a cylinder whose core curve has length $2\pi s$ and whose width is $2L$, and set $\|\text{Im}(f)\|_2 = \delta$. (The norm here is with respect to the Lebesgue measure on the cylinder.) There is a constant $C$, depending only on $b$, such that the following hold whenever $L/s$ is sufficiently large:

(a) $|x| < L - Bs \Rightarrow |\hat{f}(0) - f(x, y)| \leq C\delta/s$.

(b) If $Q \subset A^*$ is measurable and invariant under rotations in the $y$-coordinate then the average value of $f$ on $Q$ is $\hat{f}(0)$.

Let $A^*$ be the annulus on which $|x| < L - Bs$. Then

(c) $\|f \cdot 1_{A^*}\|_2^2 \leq C \left[ \|Ls\hat{f}(0)^2 + \text{Im}(f)\|_2^2 \right]$.

(How large is “sufficiently large” depends only on $b$.)

Proof. The functions $\{z^n | n \in \mathbb{Z}\}$ are orthogonal, but not orthonormal, for the $L^2$ norm coming from any finite rotation-invariant measure; in particular they are orthogonal with respect to $q$-area on the cylinder. Therefore, by uniform convergence there is a Parseval identity on the region $|\log(|z|)| < L/s - \epsilon$ for holomorphic functions and the area measure $|q|$. Passing to the limit as $\epsilon \to 0$, one gets the Parseval identity:

$$\|1_{A^*} \cdot f\|_2^2 = \sum_{n = -\infty}^{\infty} \int_{-L}^{L} \int_{\mathbb{R}} \left| \hat{f}(n) e^{n(x+iy)/s} \right|^2 dy \, dx = 2\pi s \left[ 2L|\hat{f}(0)|^2 + \sum_{n=1}^{\infty} \left( |\hat{f}(n)|^2 + |\hat{f}(-n)|^2 \right) \frac{\sinh(2nL/s)}{n/s} \right].$$

Unfortunately, the real/imaginary parts of $\hat{f}(n)z^n$ and $\hat{f}(m)z^m$ are only guaranteed to be orthogonal if $|m| \neq |n|$; however, if $L/s$ is sufficiently large, they are nearly orthogonal even if $m = -n$. More precisely, if $L > 0$, then there is a positive constant $c_{L/s}$ such that for all $n \in \mathbb{N}$ we have

$$\frac{\|\langle \text{Im}(\hat{f}(n)z^n), \text{Im}(\hat{f}(-n)z^{-n}) \rangle\|}{\|\text{Im}(\hat{f}(n)z^n)\|_2 \cdot \|\text{Im}(\hat{f}(-n)z^{-n})\|_2} < c_{L/s} < 1,$$

and $c_{L/s} \to 0$ as $L/s \to \infty$. The same bound holds with imaginary parts replacing real parts. This means that if we pretend that they are in fact orthogonal and estimate the norm of the sum of the components as the square root of the sums of these components’ norms squared, the error is by a multiplicative constant which depends only on $b$, and which tends to 1 exponentially quickly as $L/s \to \infty$. (The intuitive explanation of this is that most of the norm of concentrates at opposite ends of the cylinder $A^*$, so they essentially behave like functions with disjoint support; it is easy to estimate the contributions from regions where $|x| \geq 0$ and where $|x| \leq 0$.) This proves (c).

(b) follows from Fubini’s theorem, since the series converges uniformly on each circle and only the term for $n = 0$ has non-zero average.
In all estimates that follow, the more precisely of pretending that the real and imaginary components of the terms in the sum are actually orthogonal is only a small multiplicative constant. Similarly, for sufficiently large $L$ we may replace sinh with the exponential function, at the cost of a multiplicative factor close to 1. In particular, we have

$$\|1_A \cdot \text{Im}(f)\|_2^2 \leq sL|\hat{f}(0)|^2 + s^2 \sum_{n=1}^{\infty} \left(|(\hat{f}(n))|^2 + |\hat{f}(-n)|^2\right) \frac{e^{2nL/s}}{n}.$$  

Since the $L^2$ norm of Im$(f)$ is bounded above by $\delta$, this implies that for all $n > 0$, we get the following bound on Laurent coefficients:

$$|\hat{f}(\pm n)| \lesssim \sqrt{n}(\delta/s)e^{-nL/s},$$

where the implicit constant in the inequality above does not depend on $n$.

Now that we have coarse term-by-term bounds on the coefficients of the Laurent series, for $|x| < L - Bs$ we may estimate $f(x, y) - \hat{f}(0)$ by adding up its Laurent series:

$$|f(x, y) - \hat{f}(0)| \lesssim \sum_{n=1}^{\infty} \sqrt{n} \delta e^{-nL/s} e^{n(L/s - B)}$$

$$\lesssim \delta \sum_{n=1}^{\infty} \sqrt{n} e^{-nB}$$

$$\lesssim \delta/s.$$  

The last inequality, and therefore part (a), are due to convergence of the infinite series. \qed

## 4 Genus 2 Example

In this section we discuss a low-complexity example that contains the main analytic ideas, with simplified geometric input. The remaining sections will discuss a variant of Rafi’s thick–thin decomposition that generalizes the decomposition of the surface into pieces that appear in this example. This decomposition is strongly motivated by the product regions theorem of Minsky [Min96], and the concept of active interval for projections to the factors in Minsky’s theorem due to Rafi. The notion of active interval can be found in [Raf14], Theorem A. We postpone the careful discussion of these topics until we need them, but for those who already familiar with coarse geometry of Teichmüller space, we give a brief explanation of our choice of examples.

Let $(X, q)$ be formed as follows: let $T^*$ be a rectangular torus of side length $S$ with a geodesic slit of length $2\pi s$, and let $A^*$ be an $L \times 2\pi s$ rectangle. Form a translation surface by identifying the sides of length $2L$ with each other and the sides of $A^*$ of length $s$ with sides of the slit. The two endpoints of the slit are identified, forming a cone point of angle $6\pi$. We will assume $s/S$ and $s/L$ are smaller than $b/2$, where
$b \in (0, 1/2)$ is chosen as in Lemma 3.6 but allow $s/S$ and $s/L$ to be arbitrarily small (Fig. 1).

With respect to the cylinder coordinates in the previous section, the average value of $f$ on the region $|x| < L - Bs$ is $\hat{f}(0)$. We now break up $X$ into regions on which we can apply familiar estimates. The regions are as follows:

- $T$ will be the complement of the $bS$-neighborhood of the slit in the slit torus $T^*$.
- Let $U$ be the $(b \cdot s)$-neighborhood of the cone point.
- $A$ will be the annulus $|x| < L - Bs$ contained in $A^*$
- Let $E$ be the $Bs$-neighborhood of the slit with $U$ deleted.
- Let $M$ be the remaining expanding annulus consisting of points distance between $Bs$ and $bS$ away from the slit.

Now, let $d = \max(s/S, \sqrt{s/L})$. Let $\delta = \hat{\delta}d$. We will show that if $\hat{\delta} << 1$, then it is impossible to have a meromorphic $f$ on $X$ with $\|f\|_2 = 1$ and $\|\text{Im}(f)\|_2 < \delta$.

Pick a base point $p_T \in T$. We run the following steps:

**STEP 1:** If $p_1, p_2 \in T$ then $|f(p_1) - f(p_2)| < C\delta s^2$.

**Proof.** This follows from Proposition 3.1 estimates, since the diameter and injectivity radius are uniformly bounded below by a multiple of $S$ in $T$, and the diameter of $T$ is bounded above by a multiple of $S$, and the value of $\|f\|_2$ is bounded by 1.

**STEP 2:** $\hat{f}(0) < C/\sqrt{Ls}$.

**Proof.** By the Parseval identity $4\pi L s |\hat{f}(0)|^2 \leq \|f\|_2^2 = 1$.  

STEP 3: If \((x, y) \in A\) then \(|f(x, y) - \hat{f}(0)| < C\delta/\sqrt{Ls} \).

**Proof.** This is exactly part (a) from Lemma 3.6. \(\square\)

STEP 4: \(|1_T \cdot f|\infty < C/S\).

**Proof.** If not, then \(f \gg 1/S\) on all of \(T\), by step 1, and \(T\) has area \(S^2\), so \(\int_T |f|^2 \gg 1\). This contradicts \(\|f\|_2 = 1\). \(\square\)

STEP 5: If \(p_1, p_2\) are in the same expanding annular region (i.e. both in \(M\) or both in \(U\), then \(|f(p_1) - f(p_2)| \leq \frac{C\delta}{\min[d_q(z_0, p_1), d_q(z_0, p_2)]} \).

**Proof.** Follows from integrating the Proposition 3.1 estimate. \(\square\)

STEP 6: If \(p_1, p_2 \in E\), \(|f(p_1) - f(p_2)| < C\delta/s\).

**Proof.** This is essentially the same as step 1, except the geometry of \(E\) is controlled by \(s\) instead of \(S\). \(\square\)

STEP 7: If \(p_1, p_2 \in A \cup E \cup T \cup M\), then \(|f(p_1) - f(p_2)| < C\delta/s\).

**Proof.** Follows easily from steps 1–6 and the triangle inequality. \(\square\)

STEP 8: If \(\hat{\delta} \ll 1\), then we get a contradiction by showing that the sum of \(f\) and a constant function has small \(L^2\) norm, when by hypothesis the norm of such a function must be at least 1.

For this step we have two cases, depending on whether the torus or cylinder has greater area (up to a factor of \(4\pi\)). In each case, we will take the function \(f\) and subtract a constant function (coming from the larger piece) and show that the result has small \(L^2\) norm. This contradicts the fact that \(f\) has norm 1 and is orthogonal to the constant functions.

CASE 1: \(s/S \geq \sqrt{s/L}\), or equivalently \(1 \geq Ls \geq S^2\). We thus have \(\delta = \hat{\delta}s/S\). Let \(g(x) = f(x) - \hat{f}(0)\).

In this case we get the table of coarse upper bounds seen below. How to read this table: the entry in each box in row \(X\) column \(Y\) represents \(\ast\) in the top row of column \(Y\) when \(V\) takes the value in the first column of row \(X\). In case there are two boundary components we give two upper bounds, one for each component, above and below the box in the table where they would be expected to appear: All terms in the second column are \(O(\hat{\delta})\) but their sum is at least 1, a contradiction.

CASE 2: \(s/S \leq \sqrt{s/L}\), or equivalently \(Ls \leq S^2 \leq 1\). Now, \(\delta = \hat{\delta}\sqrt{s/L}\). Now, let \(g(x) = f(x) - f(p_T)\). We make a similar upper bound table as before, but we suggestively switch the positions of \(A\) and \(T\). This gives us a similar contradiction. Perhaps the final row of the table deserves some explanation. The entire goal is to control the value of the last entry using the Parseval identity from part (c) of Lemma 3.6. We actually already know that the contribution to the \(L^2\) norm of \(f - \hat{f}(0)\) coming from \(A\) is controlled by \(\delta\), from the proof of part (c) of Lemma 3.6.
UPPER BOUNDS FOR $g = f - f(0)$:

| $V$ | $\text{Area}(V) \asymp *$ | $| g \cdot \partial V | \asymp *$ | $\int_V g|q| \asymp *$ | $\int_V |g|^2|q| \asymp *$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| $A$ | 1 $\delta/s \asymp \delta$ | $\int_A g|q| = 0$ | $\delta^2$ | $\delta^2 \asymp \delta^2$ |
| $E$ | $s^2$ $\delta/s = \delta/S$ | $s\delta \asymp S\delta$ | $\delta^2 \asymp \delta^2$ | $\delta^2 \asymp \delta^2$ |
| $U$ | $s^2$ $\infty$ | $s\delta \asymp S\delta$ | $\delta^2$ | $\delta^2$ |
| $M$ | $S^2$ $\delta/s = \delta/S$ | $S\delta$ | $\delta^2$ | $\delta^2$ |
| $T$ | $S^2$ $\delta/s = \delta/S$ | $S\delta$ | $\delta^2$ | $\delta^2$ |

Note that the terms $\hat{g}(n)$ and $\hat{f}(n)$ agree for $n = 0$, so the contribution of this part is controlled by $C\delta^2$. Thus we only need to control $\hat{g}(0)$, which is done using the upper bound on its boundary values and the maximum modulus principle; this gives a pointwise bound for $g$ on all of $A$ which can be integrated.

As with the previous case, we deduce that the $L^2$ norm of $g$ is $O(\delta)$ and therefore $o(1)$ as $\delta \to 0^+$.

5 Thick–Thin Decompositions

5.1 Rafi’s thick–thin decomposition. In this section we recall some results of Minsky and Rafi regarding the geometry of quadratic differentials. Throughout, a curve will always mean a homotopy class of un-oriented simple closed curve in $X$ which is not homotopic to a puncture or a point.

**Definition 5.1.** Let $b \in \mathbb{R}^+$, $A_b = \{(x, y) \in \mathbb{R}^2 : 0 < y < b\}/(x, y) \sim (x + 1, y)$. If $A$ is an annulus conformal to $A_b$, we say that $b$ is the modulus of $A$, and write $\text{Mod}(A) = b$.

Every annulus is conformal to either $\mathbb{C}\setminus\{0\}$ or to $A_b$ for a unique $b$.

**Definition 5.2.** If $\alpha$ is a curve in a Riemann surface $X$, and $A$ is an annulus in $X$, write $A \simeq \alpha$ if $A$ has a deformation retract in the homotopy class $\alpha$. We say $\alpha$ is
the core curve of $A$. The *extremal length* of $\alpha$ is defined to be

$$\text{Ext}_\alpha(X) := \left( \sup_{A \simeq \alpha \text{Mod}(A)} \right)^{-1}.$$  

**Definition 5.3.** Let $(X, q)$ be a quadratic differential, and let $\alpha$ be a simple closed curve in $X$. We say $\alpha$ is a *cylinder curve* if there is a $q$-geodesic representative of $\alpha$ that does not pass through a singular point of the $q$-metric. If $\alpha$ is a cylinder curve, then there is a one-parameter family of geodesic representatives of $\alpha$ that do not pass through singular points, and the union of all such geodesics is an annulus in $X$, which we call a *cylinder*. Moreover, these curves, plus the boundary curves of the cylinder are all the $q$-geodesic representatives of $\alpha$. If $\alpha$ is not a cylinder curve, then in the completion $\overline{X}$ of $X$ (with respect to the $q$-metric), any length-infimizing sequence of constant speed parametrized loops in the class $\alpha$ converge (as parametrized geodesics, up to the action of the dihedral group) to a unique loop in $\overline{X}$, which we call the $q$-geodesic representative of $\alpha$.

**Definition 5.4.** If $\alpha$ is a curve, we say that an (open) annulus $A \subset X$ is the *expanding annulus for* $\alpha$ if the following hold:

- $A$ has core curve $\alpha$
- If $\alpha$ is a cylinder curve, $A$ does not contain any geodesic in the cylinder of $\alpha$
- One boundary component of $A$ is a $q$-geodesic representative for $A$
- In the (completion of the) annular cover of $X$ corresponding to $A$, the two boundaries of $A$ are a uniform distance apart. That is, there is a number $w$ such that if we take the metric completion of $A$ and $p$ is any point belonging to either boundary component, then $p$ is a distance $w$ from the other boundary component.
- $A$ is maximal with respect to the previous properties.

As a convention, we will also say that a puncture $p$ admits an expanding annulus, which is the maximal singularity-free annulus whose boundary is equidistant from $p$ that deformation retracts to a loop about $p$. (The double cover of such an annulus will be isometric to a finite cover of $\{z \in \mathbb{C} : \}$ in its intrinsic geometry inherited from the $q$-metric.)

As mentioned earlier, it is possible for an annulus to be expanding without being the expanding annulus of a curve. An expanding annulus may contain singularities in our interior, but we will be especially interested in expanding annuli which do not.

A curve can have at most two expanding annuli and at most one (maximal) cylinder. Since our surfaces are orientable, it makes sense to talk about which side of a curve an expanding annulus belongs to.

We will use the following approximation to the modulus that can be computed more easily from flat geometry:
\begin{quote}
Notation 5.5. If $A$ is a cylinder, let $\mu(A)$ be its modulus; if $A$ is an annulus not intersecting a cylinder whose boundaries are uniform distance apart, let $\mu(A)$ be the log of the ratio of the lengths of its boundary components (choose the ratio that is greater than 1 before taking the log). If $A$ only has one boundary curve and the other boundary is a puncture, set $\mu(A) = \infty$.

The following is a consequence of estimates from [Min92], section 4, reformulated in [CRS08], section 5:

\textbf{Theorem 5.6.} There is a number $\epsilon_0$, and positive constants depending only on $g$ and $n$, such that if $q$ is a quadratic differential on $X \in \mathcal{M}_{g,n}$, then if $\text{Ext}_\alpha(X) < \epsilon_0$, then $X$ contains an expanding annulus or cylinder $A$ with core $\alpha$ with

$$\mu(A) \gtrsim \frac{1}{\text{Ext}_\alpha(X)}.$$ 

Moreover, of the expanding annuli and cylinder that exist for $\alpha$, $A$ may be chosen to be the cylinder or expanding annulus maximizing $\mu$. If $A$ is an expanding annulus, then there is a singularity-free annulus of $A' \subset A$ whose boundaries are each constant distance from the boundaries of $A$, with

$$\mu(A') \gtrsim \mu(A).$$

\textbf{Definition 5.7.} A singularity-free annulus whose boundary components are uniform distance apart and have monotone geodesic curvature is said to be a primitive regular annulus.

The annulus $A'$ in Theorem 5.6 is a primitive regular annulus. The expanding annulus for a curve may contain multiple primitive regular annuli. For our analysis, primitive regular annuli will be more useful than the canonical choice of expanding annuli, because they are regions to which we can apply the estimates of section 3.

\textbf{Definition 5.8.} The \textit{Margulis constant} is a number $\epsilon$ such that for any point $p$ on any hyperbolic surface for which there is no embedded disk of radius $\epsilon/2$ centered at $p$, then $p$ is either part of a cusp or a collar neighborhood of a closed geodesic of radius less than $\epsilon$.

In particular, no two closed geodesics shorter than the Margulis constant can intersect.

Now, fix a number $\epsilon_0$ smaller than the Margulis constant and small enough to meet the hypotheses of Theorem 5.6, and consider the collection of subsurfaces obtained by designating representatives of curves $\alpha$ of extremal length $\epsilon_0$ or smaller by deleting their geodesic representatives in the hyperbolic metric in the conformal class $X$.

Now, for each such subsurface $Y$, take the cover $\hat{Y}$ of $X$ associated to $\pi_1(Y)$, pull back the $q$-metric to this cover and call it the $\hat{q}$-metric. There is an open annulus of infinite area corresponding to each component of $\partial Y$, with geodesic boundary. If $\alpha$
is not a cylinder curve, there is a unique such annulus $A_\alpha$. If $\alpha$ is a cylinder curve, let $A_\alpha$ be the unique such annulus containing the entire cylinder. Delete from $\hat{Y}$ the union of the open annuli $A_\alpha$; we then say that

$$\hat{Y} \setminus \bigcup_{\alpha \subset \partial Y} A_\alpha =: Y_q$$

is the $q$-geodesic representative of the subsurface $Y$.

Often, $Y_q$ is a surface with boundary, but there are degenerate cases in which it is not. In addition, some points on the boundary components of $Y_q$ may be identified in $X$. However, for any sufficiently small $\epsilon > 0$, the $\epsilon$-neighborhood of $Y_q$ in $\hat{Y}$ is homeomorphic to the corresponding hyperbolic surface with boundary $Y$. An example in section 5 of [Raf07] shows that $Y_q$ can be a spine of $Y$. However, we have the following definition and theorem from [Raf07] controlling the geometry of $Y_q$:

**Definition 5.9.** Let $(X, q) \in QD(T_g, n)$ and let $Y_q$ be the $q$-geodesic representative of a thick component $Y$ of the thick–thin decomposition of $(X, q)$. If $Y$ is a pair of pants, let $\text{size}_q(Y)$ be the maximum of the $q$-lengths of the boundary components of $Y$. Otherwise, let $\text{size}_q(Y)$ be the minimum of the $q$-lengths of essential simple closed curves in $Y$. We say that $\text{size}_q(Y)$ is the size of the subsurface $Y$ in the $q$-metric.

**Theorem 5.10.** Let $X, q, Y$ be as above. Let $\beta$ be any essential curve in $Y$, and let $\ell_q(\beta)$ denote its length in the $q$-metric and $\ell_\sigma(\beta)$ its length in the hyperbolic metric on $X$. Let $\text{diam}_q$ and $\text{Area}_q$ denote diameter and area (respectively) in the $q$-metric. Then we have the following coarse estimates of the geometry of $Y_q$, in which all constants depend only on $g$ and $n$:

- $\ell_q(\beta) \asymp \text{size}_q(Y) \ell_\sigma(\beta)$
- $\text{diam}_q(Y_q) \asymp \text{size}_q(Y)$
- $\text{Area}_q(Y) := \int_{Y_q} |q| \leq C \cdot \text{size}_q(Y)^2$.

We have the following lemma ([EMR19], Lemma 3.5) relating size to expanding annulus:

**Lemma 5.11.** There exist positive constants $C_1, C_2$, depending only on topology ($g$ and $n$), such that every thick component $Y$ of $(X, q)$ has the following property: every puncture or boundary component of $Y$ with $q$-geodesic length less than $C_1 \cdot \text{size}_q(Y)$ admits an expanding annulus of circumference at least $C_2 \cdot \text{size}_q(Y)$ in the direction of the subsurface $Y$.

**Corollary 5.12.** There is a constant $c > 0$, depending only on $g, n, c$, such that the following holds. If $Y$ is a thick component of $(X, q)$ and $Y$ is not a pair of pants, and every boundary curve $\alpha \subset \partial Y$ satisfies $\ell_q(\alpha) < c \cdot \text{size}_q(Y)$, then $\text{Area}_q(Y) \asymp \text{size}_q(Y)^2$. (If $Y$ is a pair of pants, then the same statement is vacuously true.)
For non-pants, one direction is due to Rafi, and the other follows immediately if we can show any of the expanding annuli are contained in $Y$. Now, the $q$-diameter of the union of any two boundary curves is bounded below by a constant multiple of $\text{size}_q(Y)$, since the union of two such curves and any arc connecting them contains a curve that is essential in $Y$. In particular, the $q$-geodesic representatives of the boundary components must be disjoint if $c$ is sufficiently small. This implies the expanding annuli in the direction of $Y$ do not leave $Y$. □

5.2 The primitive annuli decomposition. We would like to use a modified thick–thin decomposition of a surface $(X, q)$, which consists of some annuli of large modulus and the complement of their union. We will not use maximal cylinders and expanding annuli exactly, however; we remove a bounded amount of modulus near each boundary circle, so that each annulus satisfies the hypothesis of Lemma 3.3 or Lemma 3.6. This will also allow us to choose our primitive annuli to not intersect. The decomposition of our surface $(X, q)$ that we get as a result will be essentially a refinement of the thick–thin decomposition of our original surface. Every sufficiently short curve with respect to the hyperbolic metric on $X$ is guaranteed to have some primitive annulus of large modulus in the $q$-metric, but we will also introduce a few more annuli whose core curves bound disks that enclose the singularities of $q$-metric, either individually or as clusters of nearby singularities, of which at most one was one of the original punctures of $X$. In addition, some expanding annuli from the original surface’s thick–thin decomposition may have multiple primitive annuli of large modulus. However, these are the only types of curves that will occur. In particular, all components will be either be in the homotopy class of original components of the thick–thin decomposition (as subsurfaces or annuli) or they will be original components with some disks cut out.

Lemma 5.13. There exist constants $\mu_0 > B > 2$ depending only on the topology of $X$, such that the following hold:
Suppose $(X, q)$ is taken from the principal stratum and then punctured at all singularities, to create a surface $(X', q)$ with the same metric but more punctures. For every expanding annulus $A$ in $(X', q)$ associated to a curve or puncture satisfying $\mu_0 < \mu(A) \leq \infty$, we pick two level circles of the expanding annulus: let $o_A$ be the (topological) circle $\gamma_A$ consisting of points $p$ such that the distance from $p$ to the puncture or geodesic boundary component of $A$ is $1/B$ times the distance from $p$ to the other (non-geodesic) boundary of $A$ in the completion $\overline{X}$ of $X$ (with respect to the $q$-metric). If $A$ is homotopic to a puncture let $i_A$ be the puncture, and otherwise let $i_A$ be the simple closed curve contained in $A$ that consists of points $B$ times the $q$-length of the $q$-geodesic boundary component of $A$. For each cylinder $A$ with $\mu(A) > \mu_0$, let $i_A$ and $o_A$ be geodesics homotopic to the core curve of $A$, each distance $B$ times the $q$-length of the core curve away from a boundary component. Then, the curves $i_A$ and $o_A$ are pairwise disjoint for each annulus $A$, and $i_A$ is shorter unless they are both core curves of a cylinder. Moreover, the corresponding
annuli are disjoint from each other; that is, for \( A_1, A_2 \) distinct, the annulus bounded by \( i_{A_1} \) and \( o_{A_1} \) is disjoint from the annulus bounded by \( i_{A_2} \) and \( o_{A_2} \). The constant \( B \) can be taken arbitrarily large, although larger values of \( B \) will necessitate a larger value of \( \mu_0 \).

Before we begin the proof, we remark that the point of the constant \( B \) that we can take a primitive annulus \( A \) and replace it with one that loses a little bit of modulus by shedding a protective outer layer whose is roughly \( B \) times the perimeter of the outer boundary; in so doing we avoid intersections. This allows the annuli to shrink at a bounded additive cost to their \( \mu \)-values.

**Proof.** To make use of non-positively curved geometry we pass to the completion of the orienting double cover of \((X, q)\). The universal cover of this completion will be denoted \((\hat{X}, \hat{q})\). It is complete, homeomorphic to a disk, and non-positively curved in the sense of Alexandrov.

If \( S_q \) is the set of singularities of \( q \), note that in the orienting double cover, the inverse images of any expanding annulus \( A \) corresponding to an element of \( S_q \) or a curve, must satisfy one of the following three possibilities:

1. It consists of a single annulus which is a double cover of \( A \), and it is the expanding annulus associated to the inverse image of the corresponding puncture or curve, which is a single curve.
2. It consists of a pair of disjoint annuli, which are the expanding annuli for a pair of disjoint curves in the orienting double cover.
3. It consists of a pair of disjoint annuli \( A_1 \) and \( A_2 \), each of which consists of a disjoint union of level circles for distance to a geodesic lift of the geodesic boundary of \( A \), but which are proper subsets of disjoint expanding annuli for two curves that are the preimage of the core curve of \( A \), and the outer boundaries of \( A_1 \) and \( A_2 \) intersect some boundary \( A_2 \) and \( A_1 \), respectively. (This is necessary for \( A \) to be a maximal expanding annulus.)

In case 3, we claim that the width of \( \mu_i \) is more than a third of the width of the canonical expanding annulus containing \( A_i \), as long as \( \mu_0 \) is large enough to guarantee that the width of \( A \) exceeds the length of the geodesic boundary of \( A \). (By theorem 4.5 of [Min92] such a \( \mu_0 \) exists.) Indeed, such an annulus about \( A_1 \) would intersect both boundary components of \( A_2 \), and then expand an additional distance of at least the length of the lifts of the \( q \)-geodesic of \( A \), which would imply that it contained a singularity.

Clearly, it is enough for us to prove that the preimages of these annuli to the orienting double cover are all disjoint, or more generally, that the connected components of the preimages in \((\hat{X}, \hat{q})\) are all disjoint.

The annulus bounded by \( i_A \) and \( o_A \) is covered by a disjoint union of connected components, which are covers of degree 1, 2, or \( \infty \) (with deck group isomorphic to \( \mathbb{Z} \)) in \((\hat{X}, \hat{q})\). Now, any such connected component consists of a pair \( i_A \) and \( o_A \). Any connected component \( \gamma \) of the inverse image of a level circle of \( A \) in \((\hat{X}, \hat{q})\), when
deleted, divides \((\hat{X}, \hat{q})\) into two components, since such a component is either a topological circle, or fellow travels a bi-infinite geodesic. (For a complete, Alexandrov non-positively curved metric on the disk, there is a boundary at \(\infty\) homeomorphic to a circle, and we can just collapse this boundary and invoke the Jordan separation theorem.) Denote by \(I_\gamma\) and \(O_\gamma\) the two parts of the induced partition of the singularities of \((\hat{X}, \hat{q})\), by which we mean the collections of inverse images of singularities in \((X, q)\) lying in the two connected components, including those which have cone angle \(2\pi\). Distinguish between the two sets in the following way: the singularities on \(I_\gamma\) are those that are closer the corresponding connected component of the preimage of \(\hat{i}_A\) of \(i_A\) than to the component \(\hat{o}_A\) that maps to \(o_A\), and the singularities in \(O_\gamma\) are closer to \(\hat{o}_A\) than to \(\hat{i}_A\).

First we deal with the case in which one of the annuli is a cylinder. Clearly, if two cylinders intersect, then their core curves intersect essentially, which contradicts the fact that they both admit cylinders of modulus greater than \(\mu_0\). However, the width of a primitive expanding annulus that contains a point on the boundary of a cylinder is at most the length of the core curve of the cylinder, because it does not contain two distinct preimages of that point. Therefore it does not contain any point in the cylinder which is distance \(B\) times the length of the core curve.

Now, we observe that \(I_{A_1} = I_{A_2}\) if and only if they arise from the same lift of an annulus, since their core curves are homotopic if finite, and they lie on the same side(s) of the same infinite geodesic or cylinder of geodesics if \(A_1\) and \(A_2\) are infinite. Now, we break the analysis of pairs of expanding annuli into three cases.

CASE 1: \(I_{A_1} \cap I_{A_2} = \emptyset\). If \(B\) is large enough, every point in the annulus or strip bounded by \(\hat{i}_{A_1}\) and \(\hat{o}_{A_1}\) is closer to \(I_{A_1}\) than to \(I_{A_2}\). We have a similar statement if we reverse the indices 1 and 2, and combining these gives the desired disjointness.

CASE 2: \(I_{A_1}\) and \(I_{A_2}\) are neither disjoint nor nested. Let \(\ell_i\) be the lengths of the geodesic corresponding to \(A_i\) and let \(W_i\) be the widths. Then the curve with length \(\ell_1\) contains points in \(I_{A_2}\) and \(O_{A_2}\), so \(\ell_1 > W_2\). Similarly \(\ell_2 > W_1\). We thus get \(\ell_1 + \ell_2 > W_1 + W_2\). But for all sufficiently large \(\mu\) we have \(\ell_i < W_i\), so this is impossible.

CASE 3: They are nested; without loss of generality \(I_{A_1} \subset I_{A_2}\). Let \(\ell_i, W_i\) be as in the previous case. Then \(A_2\) cannot correspond to a single point, and is therefore covered by components of infinite diameter. Consider the bi-infinite geodesic corresponding to \(A_2\). Now, a component of the inverse image of \(A_1\) begins at a singularity in \(I_{A_2}\) or a geodesic joining a bi-infinite sequence of singularities inside of \(I_{A_2}\), and it can expand in the direction containing \(O_{A_2}\) for an additional width of at most \(\ell_2\) after it first meets the geodesic boundary of the lift of \(A_2\), just as in our analysis when one of \(A_1\) and \(A_2\) was a cylinder. Just as in that case, we conclude that the expanding annulus for \(A_1\) does not reach far enough past the geodesic boundary if \(B > 1\).

**Lemma 5.14.** Let \(\Sigma\) be the set of singularities of \((X, q)\). If \(\mu_0\) is chosen sufficiently large in Lemma 5.13, the cylinders and expanding annuli from Lemma 5.13 are deleted from \((X, q)\), and \(\Sigma\) is also deleted, then for each remaining connected com-
ponent $Y_i$, there is a corresponding component $Z_i$ of the thick–thin decomposition of $(X \setminus \Sigma, q)$ representing the homotopy class of the subsurface $Y_i$ up to deletion of additional punctures. Let $p \in Y_i$, and $r(p)$ denote the radius of the largest embedded disk in $X \setminus \Sigma$ centered at $p$. Let $\gamma$ be any boundary component of $Y_i$, with length $\ell(\gamma)$. We then have the following estimates: $\ell(\gamma), r(p), \sqrt{\text{Area}_q(Y_i)} \asymp \text{size}_q(Z_i)$.

All implied constants depend only on $g, n, B$, and $\mu_0$.

We note that it does not matter whether we measure these with respect to the intrinsic metric on $Y_i$ or on the ambient metric on $X$.

**Proof.** This follows easily from Rafi’s thick-thin decomposition and Lemma 5.13. □

**Definition 5.15.** Let $S$ vary over all systems $U$ of curves from Lemma 5.13 that separate $(X, q)$ into two disjoint components, neither of which is a disjoint union of expanding annuli. If $X_U$ denotes the component with smaller area, and $\ell(U)$ denotes the sum of the lengths of the curves in $U$, define the *hourglass ratio* of $(X, q)$ by

$$H(X, q) := \min \left\{ 1 \cup \left\{ \frac{\ell(U)}{\text{Area}_q(U)^{1/2}} : U \in \mathcal{PAD} \right\} \right\}. $$

Remark: We call this the hourglass ratio because it measures to what extent there is a small passage separating two much larger components. We also remark that we do not require the two components to be connected, but if we did this would only change the value of $H(X, q)$ up to a multiplicative constant.

**Definition 5.16.** We refer to the collection of components bounded by the collection of simple closed curves in Lemma 5.13 as the *Primitive Annuli Decomposition* of $(X, q)$ and denote it by $\mathcal{PAD}(X, q)$.

## 6 Meromorphic Functions and Efficient Paths

**Proposition 6.1.** Let $Y$ be a component of $\mathcal{PAD}(X, q)$, i.e. a connected component of the complement of the specific collection of simple closed curves on $(X, q)$ from Lemmas 5.13 and 5.14. Let $f$ be meromorphic and $L^2$ on $(X, |q|)$. If $Y$ is not an expanding annulus, then for any $p_1, p_2 \in Y$ we have

$$|f(p_1) - f(p_2)| \asymp \|\text{Im}(f)\|_2 / \|1_Y\|_2. $$

Moreover, if there is a path from $x_1 \in X$ to $x_2 \in X$ in $X$ which is distance at least $r$ away from every singularity, then

$$|f(x_1) - f(x_2)| \asymp \|\text{Im}(f)\|_2 / r. $$

**Proof.** For the first claim, our main tools are Proposition 3.1 and Lemma 3.6. We may multiply the gradient bound of Proposition 3.1 and combine it with the diameter and injectivity radius bounds of Lemma 5.14 for each non-cylinder component to
bound the change in the value of $f$ along that component. To control the change in the value of $f$ along a cylinder component, we can directly apply Lemma 3.6.

The second claim is not much different. We can take our original path and consider any components of $\mathcal{PAD}(X,q)$ it enters. Any two points in the same non-annular component $Y_i$ are joined by a path in $Y_i$ whose length is $O(\text{size}_q(Y_i))$, and on which the distance to the nearest singularity is bounded below by $C\text{size}_q(Y_i)$. Any two points on an expanding annulus can be joined by an arc which consists of an arc constant distance from both boundary components and an arc that is perpendicular to all such arcs, i.e., part of a level circle and part of a geodesic joining the two boundaries. We can assume that our path takes this form whenever it enters a non-cylinder component; since the distance to singularities is constant up to a bounded multiple on each such $Y_i$ this does not cause the closest approach to a singularity to decrease by more than a bounded factor.

Now, our new path enters each thick component and cylinder at most once. We integrate the estimate from Proposition 3.1 on all excursions into expanding annulus components following these types of paths, and apply the first claim for segments of paths that belong to components that are not expanding annuli. \hfill \Box

PROPOSITION 6.2. Let $\|\cdot\|_2$ denote the $L^2$ norm with respect to the $q$-area on a half-translation surface $(X,q)$. There is a constant $C_{g,n} > 0$ depending only on the genus and number of marked points of $X$, such that if $f$ is a nonzero $L^2$ meromorphic function on a unit area half-translation surface $(X,q)$ with $\int_X f|q| = 0$, then

$$\|\text{Im}(f)\|_2 > C_{g,n} H(X,q) \|f\|_2.$$ 

Proof. To simplify we may assume $(X,q)$ has unit area and $\|f\|_2 = 1$ and $\|\text{Im}(f)\|_2 = \hat{\delta} H(X,q)$. We get a contradiction if $\hat{\delta} < 0.1$.

As usual, we will use the letter $C, C', C''$ to denote various positive constants depending only on $g$ and $n$. Their values may change from step to step. \hfill \Box

STEP 1: Some component $X_0$ of $\mathcal{PAD}(X,q)$ has area at least $C$, and we may take $X_0$ to not be an expanding annulus.

Proof. The number of components of $\mathcal{PAD}(X,q)$ is bounded depending only on $g$ and $n$, and the area of an expanding annulus component is at most $C$ times the area of its neighboring component(s). So we may assume that the largest cylinder or largest non-annular component has area at least $C > 0$; we may take this component to be $X_0$. \hfill \Box

STEP 2: If $x_0 \in X_0$, and $X_1$ is any component of $\mathcal{PAD}$ which is not an expanding annulus, then $x_1 \in X_1$ then $|f(x_1) - f(x_0)| < \hat{\delta}\text{Area}_q(X_1)$.

Proof. Let $g(x) = f(x) - f(x_0)$. We will first prove this for some $x_1 \in X_1$, and then extend to all $x_1 \in X_1$. On each piece $X_1$ that is not an expanding annulus component, $f$ is constant up to an additive error of $C\frac{\hat{\delta}}{\text{size}_q(X_1)}$ by Proposition 6.1.
Moreover, if $X_1$ is a component of $\mathcal{PAD}(X, q)$ but not an expanding annulus, then there is a path $\gamma$ from $X_0$ to $X_1$ such that

$$\frac{d(\gamma, \Sigma)}{\sqrt{\text{Area}(X_1)}} > CH(X, q).$$

So for some $x_1 \in X_1$, an application of Proposition 6.1 gives us

$$H(X, q)|g(x_1)| \prec \frac{\|\text{Im}(f)\|_2}{\text{Area}_q(X_1)^{1/2}}.$$

Moreover, by Proposition 6.1, this is actually true for all $x_1 \in X_1$. The claim then follows by dividing through by $H(X, q)\text{Area}_q(X_1)^{1/2}$.

**STEP 3: Proof of the proposition.** We now assume $\hat{\delta} < < 1$. As in step 2, set $g(x) = f(x) - f(x_0)$. Then we must have $\|g\|_2 \geq 1$, since $\|f\|$ is orthogonal to constants and $f - g$ is constant. However, by Step 2 we clearly have $\int_{X_1} |g|^2 |q| \prec (\hat{\delta})^2$.

Step 2 also implies that

$$\|g(x)\| \prec \frac{\hat{\delta}}{d(x, \Sigma)}$$

on all components that are not expanding annuli. We may use this as the boundary condition needed to apply Lemma 3.3. Summing these we conclude $\|g\|_2 < C\hat{\delta}$, a contradiction. $\square$

*Proof of Theorem 1.2.* Follows immediately as as corollary given the discussion in section 2. $\square$

## 7 Contraction Along Axes of Pseudo-Anosov Homeomorphisms

A pseudo-Anosov homeomorphism induces a map on the space of measured foliations, with north-south dynamics on the space $\mathcal{PMF}$ of projective measured foliations. If the axis of the pseudo-Anosov diffeomorphism is contained in the principal stratum of quadratic differentials, we may use the Hodge norm as a norm on the tangent space to the space of quadratic differentials, which is locally $H^1_{\text{odd}}(\tilde{X}; \mathbb{C})$. If a class is of the form $\alpha + i\beta$ with $\alpha, \beta \in H^1_{\text{odd}}(\tilde{X})$, the flow in period coordinates is given by $g_t(\alpha + i\beta) = e^{t\alpha} + ie^{-t}\beta$. Let $\phi$ be pseudo-Anosov with translation length $T$. One can apply the flow and then the inverse of the pseudo-Anosov homeomorphism to get a self-map of $H^1_{\text{odd}}(X; \mathbb{R})$. As is well known, this map is symplectic, and the eigenvectors have norms in $[e^{-T}, e^T]$. The top and bottom eigenvalues are simple and positive real; they correspond to the classes of the horizontal and vertical foliations. This map is also symplectic; so the eigenvalues take the form

$$e^T = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_2 - 1 > \lambda_1^{-1} = e^{-T}.$$
Theorem 7.1. Let $\phi$ be pseudo-Anosov of translation distance $T$. Let $(X, q)$ be a half-translation surface on the axis of $\phi$ with $(X_t, q_t) = g_t(X, q)$. If $\phi$ belongs to the principal stratum, then

$$\log \left( \frac{\lambda_1}{\lambda_2} \right) \geq C_{g,n} \int_0^T H(X_t, q_t) dt. \square$$

We would like to have a similar theorem when $\phi$ does not have axis in the principal stratum. In this case the period coordinates do not make sense, but one may find a neighborhood of $X$ covered by a finite collection of cones in vector spaces of the form $H^1_{odd}(X', q') \times H^1_{odd}(X', q')$, where $(X', q')$ is a nearby quadratic differential in the principal stratum. Then, possibly after passing to a power of $\phi$, we may assume that our pseudo-Anosov does not non-trivially permute these cones. Then we have self-maps of these cones, and the eigenvalues of these maps satisfy

$$e^T = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_2^{-1} > 1 = e^{-T}.$$ 

We give a brief sketch of the proof here. Relevant definitions can be found in [Fra].

Step 1: We consider the coding of a geodesic in the principal stratum that fellow travels the axis of $\phi$, via a train track splitting sequence $T$, following [Fra]. Such an axis will contain a subsequence that is a splitting sequence for a power of $\phi$. However, every edge splits, so we fix some periodic splitting sequence $S$ in which every edge splits, which can be concatenated with $T$. We let $L_T$ and $L_S$ be the associated maps on period coordinates. We consider the pseudo-Anosov homeomorphisms $\phi_k$ associated to the splitting sequences $T^k S$ as $k \to \infty$.

We make the following claims:

Claim 1: The axes of the $\phi_n$ all belong to a fixed compact subset of $\mathcal{M}_{g,n}$ that depends only on $T$ and $S$.

Claim 2: For each $\epsilon > 0$, there is some $k_0$ such that for all $k > k_0$, all but a fraction $(1 - \epsilon)$ of the length of the axis of $\phi_k$ $\epsilon$-fellow travels the axis of $\phi_k$ in a parametrized fashion. (This follows easily from hyperbolic properties of the flow on compact invariant sets, see e.g. [Ham10]).

Claim 3: The average value of the hourglass ratio along the axis of $\phi_k$ converges to the average value of the hourglass ratio along the axis of $\phi$.

Claim 4: If $\lambda_2$ and $\lambda_1$ are the top two eigenvalues for $L_U$, for $U = T, S, T^N S$ etc. Then the spectral gap of of a linear map on the cone $X$ of equivalence classes tangential measures, is $\frac{\lambda_2}{\lambda_1} = \lim_{k \to \infty} \text{diam}_{\text{Hilb}}(U^k SX)$, where $X$ is the cone of equivalence classes of tangential measures, and $\text{diam}_{\text{Hilb}}$ is the diameter in the Hilbert metric on $X$. From [Fra] we know that some power of $L_S$ is a contraction on $X$ in the Hilbert metric, and if we linearize the Hilbert metric near the attracting fixed point on a hyperplane representing the cone $X$ up to scaling, then the derivative of the map $L_S$ on $X$ with respect to the Hilbert metric is $L_S/\lambda_1 \circ P$, where $P$ is a projection onto the hyperplane.

We then get the conclusion by taking square the $N^{th}$ root and the limit as $N \to \infty$. 
A construction of Bell and Schleimer in [BS15] gives examples of pseudo-Anosov homeomorphisms with $\lambda_2/\lambda_1$ arbitrarily close to 1, whenever the complex dimension of the moduli space $3g - 3 + n$ is at least 4. We remark that our construction implies that such homeomorphisms must live deeper and deeper in the cusps of moduli space or have quasiconformal dilatation tending to 1; however, it is well known that the quasiconformal dilatation is bounded below by a constant greater than 1 for each fixed $g, n$.

8 Invariant Transverse Measures and Unique Ergodicity

The following theorem is from [Sm], building on a theorem from [Tre14], which proved the result in the case when the $(X, q)$ has $q = \omega^2$ for some holomorphic 1-form $\omega$.

Theorem 8.1. Let $\kappa(t)$ denote the systole of $(X_t, q_t) = g_t(X, q)$ with respect to the $q_t$-metric. If

$$\int_0^{\infty} \kappa(t)^2 dt$$

diverges, then the vertical foliation on $(X, q)$ is uniquely ergodic.

We would like to replace $\kappa(t)^2$ with the square of the hourglass ratio, but such a theorem cannot be true. A counterexample can be given by Strebel differential. That is, a surface obtained from a rectangle by gluing the top and bottom by a translation, and the left and right sides by a piecewise-translation. However, in this case, the invariant ergodic measures are all topologically equivalent.

Following the methods of Smith and Treviño, one may prove

Theorem 8.2. Let $(X_t, q_t) = g_t(X, q)$. If

$$\int_0^{\infty} H(X_t, q_t)^2 dt$$

diverges, then the invariant transverse measures on vertical foliation represent only one point in $\mathcal{MF}$.

Proof. It is well known that $(X, q)$ decomposes into cylinders of closed leaves of the vertical foliation and minimal components for the foliation, see for example the survey of Masur and Tabachnikov [MT02].

If the integral diverges, we observe that the surface is either a Strebel differential (in which case the result is trivial) or the vertical foliation is minimal. Indeed, if not there are at least two cylinders or minimal components whose area does not tend to zero but which are separated by a system of curves whose length decays exponentially, so $H(X_t, q_t) = O(e^{-t})$. 

In the minimal case, we may further reduce to the case in which \((X, q)\) is in the minimal stratum by a small deformation along the strongly stable leaf into the principal stratum to some \((X', q')\); then \(H(X_t, q_t) - H(X'_t, q'_t)\) decays exponentially. The argument of Treviño and Smith now carries through with the following modification: our estimate for the \(L^2\) norm of the imaginary part of a meromorphic function orthogonal to the constant functions may be substituted into their argument. □

9 Further Questions

We expect that there should be similar geometric criteria that give bounds for the derivative of the Hodge norm on bigger larger subspaces. For example, we expect that mutually orthogonal \(L^2\) meromorphic functions must essentially live on disjoint subsurfaces and cylinders, i.e, disjoint Minsky product region factors. More precisely, for each \(n\), one may define a “level \(n\) hourglass constant” and use this to give upper and lower bounds the derivative of the Hodge norm on the \(n\)th exterior power of \(H^1_{\text{odd}}(\tilde{X}, \mathbb{R})\). This can be done by similar methods. If such an integral diverges, we would hope to be able to conclude that the vertical foliation of \((X, q)\) has ergodic measures forming a simplex with at most \((n - 1)\) vertices in \(\mathcal{PMF}\). This could be seen as a quantitative generalization of McMullen’s generalization of Masur’s criterion ([McM], Thm. 1.4).

Similarly, one should also be able to bound more Lyapunov exponents of pseudo-Anosov maps by similar integrals. Finally, we leave as a question the qualitative sharpness of our spectral gap estimate: that is, can it be improved in any way except by estimating the multiplicative constants?

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