Sharp isoperimetric and Sobolev inequalities in spaces with nonnegative Ricci curvature

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Abstract

By using optimal mass transport theory we prove a sharp isoperimetric inequality in $\text{CD}(0, N)$ metric measure spaces assuming an asymptotic volume growth at infinity. Our result extends recently proven isoperimetric inequalities for normed spaces and Riemannian manifolds to a nonsmooth framework. In the case of $n$-dimensional Riemannian manifolds with nonnegative Ricci curvature, we outline an alternative proof of the rigidity result of Brendle (Comm Pure Appl Math 2021:13717, 2021). As applications of the isoperimetric inequality, we establish Sobolev and Rayleigh-Faber-Krahn inequalities with explicit sharp constants in Riemannian manifolds with nonnegative Ricci curvature; here we use appropriate symmetrization techniques and optimal volume non-collapsing properties. The equality cases in the latter inequalities are also characterized by stating that sufficiently smooth, nonzero extremal functions exist if and only if the Riemannian manifold is isometric to the Euclidean space.

Mathematics Subject Classification

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1 Introduction

The classical method in proving isoperimetric and related Sobolev inequalities in the Euclidean space is by symmetrization arguments, see e.g. Talenti [58, 59] and Lieb and Loss [39]. Recent developments in this area showed two new approaches. The first one is highlighted by the works of Cabré and Ros-Oton [9] and Cabré, Ros-Oton and Serra [10] which is based on PDE techniques, notably on Aleksandrov-Bakelman-Pucci estimates; this is the so called ABP-method. The second approach is based on the theory of optimal mass transportation (OMT, for short) and has been initiated in the seminal paper by Cordero-Erausquin, Nazaret and Villani [19] and the monograph of Villani [60].

Since the OMT-theory has been worked out in a rather general setting of metric measure spaces due to Lott and Villani [42] and Sturm [56, 57], it is a natural question to ask if this method is applicable to prove sharp geometric inequalities in general non-Euclidean settings.

A pioneering result in this direction is due to Cordero-Erausquin, McCann and Schmuckenschläger [18] who extended the Borell-Brascamp-Lieb inequalities to Riemannian manifolds by using the OMT-theory; in fact, this work stood at the basis of the papers of [42, 56, 57].

In the case of Riemannian and weighted Riemannian manifolds satisfying a strictly positive lower bound on the Ricci curvature, sharp Lévy-Gromov-type isoperimetric inequalities have been obtained by Milman [45, 46]. An extension of Milman’s results is due to Cavalletti and Mondino [13], who used the OMT-approach in combination to the measure disintegration method of Klartag [33] to prove a sharp Lévy-Gromov-type isoperimetric inequality on metric measure spaces satisfying the \( \text{CD}(\kappa, N) \) condition with \( \kappa > 0 \) and \( N \in [1, \infty) \). Another extension of [13, 45] is given by Ohta [50, 51] to the setting of (not necessarily reversible) Finsler manifolds.

Our goal is to show that the OMT-theory can also be successfully applied to prove sharp isoperimetric inequalities on spaces with nonnegative Ricci curvature in a synthetic or classical sense. It is well known that without an additional condition no isoperimetric inequality holds true in general spaces with nonnegative Ricci curvature.

For our results, the additional assumption is a volume growth control at infinity. To be more precise, let \( N > 1 \) be a real number, and \((M, d, m)\) be a metric measure space satisfying the \( \text{CD}(0, N) \) condition, see [42, 56, 57]. Let \( B_x(r) = \{ y \in M : d(x, y) < r \} \) be the metric ball with center \( x \in M \) and radius \( r > 0 \). By the generalized Bishop-Gromov volume growth inequality, see Sturm [57, Theorem 2.3], it follows that

\[
 r \mapsto \frac{m(B_x(r))}{r^N} \quad \text{is nonincreasing on } (0, \infty) \text{ for every } x \in M.
\]

Moreover, the \textit{asymptotic volume ratio}

\[
 \text{AVR}_{M, d, m} = \lim_{r \to \infty} \frac{m(B_x(r))}{\omega_N r^N}
\]

is independent of the choice of \( x \in M \). (Here, the constant \( \omega_N = \pi^N/\Gamma(1 + N/2) \) plays a normalization role and it is the volume of the Euclidean unit ball in \( \mathbb{R}^N \) whenever \( N \in \mathbb{N} \).) Our standing assumption is

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and we say that \((M, d, m)\) has Euclidean volume growth. In order to state our first result, we recall that the Minkowski content of \(\Omega \subset M\) is given by

\[
m^+(\Omega) = \liminf_{\varepsilon \to 0^+} \frac{m(\Omega \setminus \Omega_{\varepsilon})}{\varepsilon},
\]

where \(\Omega_{\varepsilon} = \{ x \in M : \exists y \in \Omega \text{ such that } d(x, y) < \varepsilon \}\) is the \(\varepsilon\)-neighborhood of \(\Omega\) with respect to the metric \(d\).

**Theorem 1.1** Let \((M, d, m)\) be a metric measure space satisfying the \(\text{CD}(0, N)\) condition for some \(N > 1\), and having Euclidean volume growth. Then for every bounded Borel subset \(\Omega \subset M\) it holds

\[
m^+(\Omega) \geq N \omega_n^\frac{1}{N} \text{AVR}_{M, d, m}^\frac{1}{N} m(\Omega)^N \text{Vol}_g^{N-1}. \tag{1.1}
\]

Moreover, inequality (1.1) is sharp.

Relying on the assumption of the Euclidean volume growth, our proof of (1.1) uses careful limiting procedure in an OMT-based distorted Brunn-Minkowski inequality given by Sturm [57]. Concerning the equality in (1.1), we could expect certain rigidity statement in the spirit of Cavalletti and Mondino [13] who proved that if equality occurs in the isoperimetric inequality for some space \(M\) with a strictly positive lower bound on its Ricci curvature, then \(M\) is isometric to a spherical suspension. At present time no such characterization is available for the equality in (1.1) in generic \(\text{CD}(0, N)\) spaces; the reason is that the argument based on the distorted Brunn-Minkowski inequality - in spite of the fact that it provides the sharp inequality (1.1) - seems to be too robust to identify the equality cases.

Let us note that Theorem 1.1 is in fact a generalization of the sharp isoperimetric inequalities on weighted normed cones/spaces, see Cabrè, Ros-Oton and Serra [10], on Riemannian manifolds, see Brendle [7] and weighted Riemannian manifolds, see Johne [31].

Considering a noncompact, complete \(n\)-dimensional Riemannian manifold \((M, g)\) with nonnegative Ricci curvature (\(\text{Ric} \geq 0\), for short), this can be seen as a metric measure space \((M, d_g, \text{Vol}_g)\), where \(d_g\) and \(\text{Vol}_g\) denote the natural metric and canonical measure on \((M, g)\), respectively. The asymptotic volume ratio of \((M, g)\) is a global geometric invariant given by \(\text{AVR}_g := \text{AVR}_{M, d_g, \text{Vol}_g}\). By the Bishop-Gromov volume comparison principle one has that \(\text{AVR}_g \leq 1\); moreover, \(\text{AVR}_g = 1\) if and only if \((M, g)\) is isometric to the usual Euclidean space \((\mathbb{R}^n, g_0)\). Under the assumption of Euclidean volume growth, i.e., \(0 < \text{AVR}_g \leq 1\), Theorem 1.1 implies that for every bounded and open subset \(\Omega \subset M\) with \(C^1\) smooth boundary \(\partial\Omega\), one has

\[
\mathcal{P}_g(\partial\Omega) \geq n \omega_n^\frac{1}{n} \text{AVR}_g^\frac{1}{n} \text{Vol}_g(\Omega)^\frac{n-1}{n}, \tag{1.2}
\]

where \(\mathcal{P}_g(\partial\Omega)\) stands for the perimeter of \(\partial\Omega\).
We notice that (1.2) has been recently obtained by Brendle [7] using the ABP-method, who also proved that equality holds in (1.2) for some \( \Omega \subset M \) with \( C^1 \)-regular boundary if and only if \( \text{AVR}_g = 1 \) and \( \Omega \) is isometric to a ball \( B \subset \mathbb{R}^n \). In our paper we also sketch an alternative proof of the characterization of the equality case based on the OMT-method.

Having inequality (1.2) at hand - which is equivalent to a sharp \( L^1 \)-Sobolev inequality on Riemannian manifolds with \( \text{Ric} \geq 0 \), together with the characterization of the equality case, it is a natural question to consider the validity of sharp \( L^p \)-Sobolev inequalities in the same geometric setting whenever \( p > 1 \). This problem has its genesis in the work of Aubin [4] who initiated in the early seventies the determination of sharp constants in Sobolev inequalities in curved settings. We notice that Aubin’s program is rather well-understood in the counterpart setting of negative curvature. Indeed, sharp \( L^p \)-Sobolev inequalities hold on Hadamard manifolds (i.e., simply connected, complete Riemannian manifold with nonpositive sectional curvature), see e.g. Mura- tori and Soave [49], whenever the Cartan-Hadamard conjecture holds, the latter being precisely the sharp Euclidean-type isoperimetric inequality on \((M, g)\). The validity of this conjecture is confirmed in low dimensions; see Weil [62] in 2-dimension, Kleiner [34] in 3-dimension and Croke [21] in 4-dimension, respectively.

For an \( n \)-dimensional complete Riemannian manifold \((M, g)\) with \( n \geq 2 \) and \( \text{Ric} \geq 0 \), endowed by its canonical measure \( \text{Vol}_g \), the simplest \( L^p \)-Sobolev inequality on \((M, g)\) reads as

\[
\left( \int_M |u|^{p^*} \, dv_g \right)^{1/p^*} \leq C \left( \int_M |\nabla_g u|^p \, dv_g \right)^{1/p}, \quad \forall u \in C^\infty_0(M), \tag{S}
\]

where \( C = C(n, p) > 0 \) is a universal constant, \( 1 < p < n \) and \( p^* = \frac{pn}{n-p} \) is the critical Sobolev exponent. A local chart analysis shows that the validity of (S) necessarily implies \( C \geq \text{AT}(n, p) \), where

\[
\text{AT}(n, p) = \pi^{-\frac{1}{2}} n^{-\frac{1}{2}} \left( \frac{p-1}{n-p} \right)^{1-1/p} \left( \frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)} \right)^{1/n}
\]

is the best Sobolev constant in (S) for the Euclidean space \((\mathbb{R}^n, g_0)\), see Aubin [4] and Talenti [58]. Moreover, in the Euclidean case, the unique family of extremals is also identified.

Let us note first that the existing literature already contains rigidity results concerning inequality (S) on Riemannian manifolds with \( \text{Ric} \geq 0 \). Indeed, Ledoux [38] proved that if (S) holds with \( C = \text{AT}(n, p) \), then \((M, g)\) is isometric to the Euclidean space \((\mathbb{R}^n, g_0)\). Moreover, a quantitative form of Ledoux’s result were given by do Carmo and Xia [24] who proved that \((M, g)\) is topologically close to \((\mathbb{R}^n, g_0)\) whenever the constant \( C > \text{AT}(n, p) \) in (S) is sufficiently close to \( \text{AT}(n, p) \). In fact, a byproduct of do Carmo and Xia’s approach is that the validity of (S) with a constant \( C > 0 \) implies the volume non-collapsing property

\[
\text{AVR}_g \geq \left( \frac{\text{AT}(n, p)}{C} \right)^n. \tag{1.3}
\]
In particular, inequality (1.3) implies that for any complete Riemannian manifold \((M, g)\) with \(\text{Ric} \geq 0\) supporting the Sobolev inequality \((S)\), one necessarily has that \(\text{AVR}_g > 0\), i.e., \((M, g)\) has Euclidean volume growth. The converse is also true that follows by a general result of Coulhon and Saloff-Coste [20] where the constant \(C > 0\) in \((S)\) is generically determined.

Keeping our geometric setting, i.e., \((M, g)\) is a noncompact, complete Riemannian manifold with \(\text{Ric} \geq 0\), our second purpose is to provide sharp Sobolev inequalities on \((M, g)\) by the sharp isoperimetric inequality from relation (1.2). As expected, the asymptotic volume ratio \(\text{AVR}_g \in (0, 1]\) is explicitly encapsulated in these Sobolev inequalities and more spectacularly, they do provide sharp Sobolev constants.

To formulate this result let us denote by \(\tilde{W}^{1,p}(M) = \{u \in L^{p^*}(M) : |\nabla_g u| \in L^p(M)\}\). Using this notation we can state a sharp \(L^p\)-Sobolev inequality in the spirit of \((S)\):

**Theorem 1.2** Let \((M, g)\) be a noncompact, complete \(n\)-dimensional Riemannian manifold \((n \geq 2)\) with \(\text{Ric} \geq 0\) having Euclidean volume growth, i.e., \(0 < \text{AVR}_g \leq 1\). If \(p \in (1, n)\), then for every \(u \in \tilde{W}^{1,p}(M)\) one has

\[
\left( \int_M |u|^{p^*} \, dv_g \right)^{1/p^*} \leq S_g \left( \int_M |\nabla_g u|^p \, dv_g \right)^{1/p},
\]

where the constant \(S_g = \alpha T(n, p) \text{AVR}_g^{\frac{1}{n}}\) is sharp. Moreover, equality holds in (1.4) for some nonzero and nonnegative function \(u \in C^0(M) \cap \tilde{W}^{1,p}(M)\) if and only if \(\text{AVR}_g = 1\), and thus \((M, g)\) is isometric to the Euclidean space \((\mathbb{R}^n, g_0)\).

As a byproduct, Theorem 1.2 immediately implies the result of do Carmo and Xia [24]. Indeed, if \((S)\) holds for some \(C > 0\), then by the sharpness of \(S_g\) we have \(C \geq S_g\), which is equivalent to (1.3). Moreover, if \(C = \alpha T(n, p)\) then we obtain \(\text{AVR}_g \geq 1\), thus \(\text{AVR}_g = 1\), recovering Ledoux’s rigidity result [38] as well.

Note that inequality (1.4) belongs to the larger class of \(L^p\)-Gagliardo-Nirenberg inequalities \((1 < p < n)\); we are going to treat such an inequality in Theorem 3.1 whose proof indicates the way to obtain further sharp Sobolev inequalities (i.e., \(L^p\)-log-Sobolev and \(L^p\)-Faber-Krahn inequalities). In order to prove Theorem 1.2, we combine the sharp isoperimetric inequality (1.2) with a symmetrization argument from \((M, g)\) to \((\mathbb{R}^n, g_0)\) in the spirit of Aubin [4]. In this way we establish a Pólya-Szegő-type inequality involving the number \(\text{AVR}_g\), see Proposition 3.1. Another challenge is to prove the sharpness of the aforementioned inequalities (see e.g. the constant \(S_g\) in (1.4)). It turns out that the optimal volume non-collapsing properties of \((M, g)\) established by Ledoux [38] (and by one of us [36]) provide precisely the required tool.

Another class of problems concerns the Rayleigh-Faber-Krahn inequality where the sharp isoperimetric inequality (1.2) provides again the strongest possible rigidity statement. To formulate this result, recall that given a complete \(n\)-dimensional Riemannian manifold \((M, g)\) (with no curvature restriction for the moment), it is well known that the first eigenvalue of the Beltrami-Laplace operator \(-\Delta_g\) for the Dirichlet problem on a smooth bounded open set \(\Omega \subset M\) has the variational characterization
\( \lambda_{1,g}(\Omega) = \inf_{u \in C^\infty_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla_g u|^2 dv_g}{\int_{\Omega} u^2 dv_g}. \) (1.5)

According to Carron [12] (see also Hebey [30, Proposition 8.1]), if \( n \geq 3 \) and \( \text{Vol}_g(M) = +\infty \), the validity of the general Sobolev inequality (S) is equivalent to the validity of a generic Rayleigh-Faber-Krahn inequality on \((M, g)\), i.e., there exists \( \Lambda > 0 \) such that for any smooth bounded open set \( \Omega \subset M \) one has

\[
\lambda_{1,g}(\Omega) \geq \Lambda \text{Vol}_g(\Omega)^{-\frac{2}{n}}. \tag{1.6}
\]

In particular, Theorem 1.2 implies that inequality (1.6) holds whenever \((M, g)\) is a Riemannian manifold with \( \text{Ric} \geq 0 \) having Euclidean volume growth \( 0 < \text{AVR}_g \leq 1 \). In fact, in the latter geometric setting we can establish the sharp form of (1.6) (hereafter, \( j_\nu \) stands for the first positive root of the Bessel function \( J_\nu \) of the first kind with degree \( \nu \in \mathbb{R} \)):

**Theorem 1.3** Let \((M, g)\) be an \( n \)-dimensional Riemannian manifold as in Theorem 1.2. Then for every bounded and open subset \( \Omega \subset M \) with smooth boundary, we have

\[
\lambda_{1,g}(\Omega) \geq \Lambda_g \text{Vol}_g(\Omega)^{-\frac{2}{n}}, \tag{1.7}
\]

where the constant \( \Lambda_g = j_2^{2n-1}(\omega_n \text{AVR}_g)^{\frac{2}{n}} \) is sharp. Furthermore, equality holds in (1.7) for some bounded open subset \( \Omega \subset M \) with smooth boundary if and only if \( \text{AVR}_g = 1 \) and \( \Omega \) is isometric to a ball \( B \subset \mathbb{R}^n \).

The proof of Theorem 1.3 is based on inequality (1.2) and fine properties of Bessel functions. We note that, while a similar result is recently established by Fogagnolo and Mazzieri [25, Theorem 5.5] for \( n \in \{3, \ldots, 7\} \), Theorem 1.2 is valid in any dimension \( n \geq 2 \).

The paper is organized as follows. Section 2 is devoted to sharp isoperimetric inequalities on \( \text{CD}(0,N) \) spaces. First, in Sect. 2.1 we provide the proof for Theorem 1.1 and present some examples of \( \text{CD}(0,N) \) where our result applies. In Sect. 2.2, by using OMT-arguments, we outline a short, alternative proof to Brendle’s rigidity result concerning the equality in (1.2) in the context of Riemannian manifolds. In Sect. 3 we establish a sharp Gagliardo-Nirenberg inequality on Riemannian manifolds with \( \text{Ric} \geq 0 \) having Euclidean volume growth, see Theorem 3.1, whose particular case is precisely Theorem 1.2. To do this, we first provide a symmetrization argument, by establishing via the sharp isoperimetric inequality (1.2) an \( \text{AVR}_g \)-dependent Pólya-Szego inequality. Theorem 1.3 is proved in Sect. 3.2 by using the \( \text{AVR}_g \)-dependent Pólya-Szego inequality and fine features of Bessel functions.

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2 Sharp isoperimetric inequalities in CD(0, N) spaces

2.1 Proof of the sharp isoperimetric inequality

In this subsection we are going to prove inequality (1.1), its sharpness and provide some relevant examples and consequences. We first briefly recall the synthetic notion of nonnegative Ricci curvature introduced by Lott and Villani [42] and Sturm [56, 57].

Let \((M, d, m)\) be a metric measure space, i.e., \((M, d)\) is a complete separable metric space, \(m\) is a locally finite measure on \(M\) endowed with its Borel \(\sigma\)-algebra, and geodesic (for every two points \(x, y \in M\) there exists a minimizing geodesic \(\gamma : [0, 1] \to M\) parametrized proportional to arclength and \(\gamma(0) = x\) and \(\gamma(1) = y\)). In particular, we can define the \(s\)-interpolant set \(Z_s(\cdot, \cdot)\), i.e., for every \((x, y) \in M \times M\),

\[
Z_s(x, y) = \{ z \in M : d(x, z) = s d(x, y), \quad d(z, y) = (1-s)d(x, y) \}, \quad (2.1)
\]

and for any nonempty sets \(A, B \subset M\),

\[
Z_s(A, B) = \bigcup_{(x,y) \in A \times B} Z_s(x, y).
\]

We assume that the measure \(m\) on \(M\) is strictly positive, i.e., \(\text{supp}[m] = M\). As usual, \(P_2(M, d)\) is the \(L^2\)-Wasserstein space of probability measures on \(M\), while \(P_2(M, d, m)\) will denote the subspace of \(m\)-absolutely continuous measures.

For \(N \geq 1\), the \textit{Rényi entropy functional} \(\text{Ent}_N(\cdot|m) : P_2(M, d) \to \mathbb{R}\) with respect to the measure \(m\) is defined by

\[
\text{Ent}_N(\mu|m) = -\int_M \rho^{-\frac{1}{N}} \, d\mu = -\int_M \rho^{1-\frac{1}{N}} \, dm, \quad (2.2)
\]

the function \(\rho\) being the density of \(\mu^{ac}\) in \(\mu = \mu^{ac} + \mu^s = \rho m + \mu^s\), where \(\mu^{ac}\) and \(\mu^s\) represent the absolutely continuous and singular parts of \(\mu \in P_2(M, d)\), respectively.

Given \(N \geq 1\), the \textit{curvature-dimension condition} \(\text{CD}(0, N)\) states that for all \(N' \geq N\) the functional \(\text{Ent}_{N'}(\cdot|m)\) is convex on the \(L^2\)-Wasserstein space \(P_2(M, d, m)\), i.e., for each \(\mu_0, \mu_1 \in P_2(M, d, m)\) there exists a geodesic \(\Gamma : [0, 1] \to P_2(M, d, m)\) joining \(\mu_0\) and \(\mu_1\) such that for every \(s \in [0, 1]\),

\[
\text{Ent}_{N'}(\Gamma(s)|m) \leq (1-s)\text{Ent}_{N'}(\mu_0|m) + s\text{Ent}_{N'}(\mu_1|m).
\]

An almost immediate consequence of the above definition is the (distorted) \textit{Brunn-Minkowski inequality} on the metric measure space \((M, d, m)\) satisfying the \(\text{CD}(0, N)\) condition; in particular, if \(A, B \subset M\) are two Borel sets such that \(m(A) \neq 0 \neq m(B)\), then for every \(s \in [0, 1]\) and \(N' \geq N\) one has

\[
m(Z_s(A, B))^{rac{1}{N'}} \geq (1-s)m(A)^{rac{1}{N'}} + sm(B)^{rac{1}{N'}}, \quad (2.3)
\]

see e.g. Sturm [57, Proposition 2.1].

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Proof of inequality (1.1). Let us recall our setting: \((M, d, m)\) is a metric measure space satisfying the CD\((0, N)\) condition for some real number \(N > 1\), having Euclidean volume growth, and let \(\Omega \subset M\) be a bounded Borel set.

Let \(x_0 \in \Omega\) and \(R > 0\) be arbitrarily fixed and let \(d_0 := \text{diam}(\Omega) < \infty\). If \(s \in [0, 1]\), we claim that

\[
Z_s(\Omega, B_{x_0}(R)) \subseteq \Omega_{s(d_0 + R)}, \tag{2.4}
\]

where \(\Omega_\varepsilon\) is the \(\varepsilon\)-neighborhood of \(\Omega\) with \(\varepsilon > 0\).

Let \(s > 0\) (for \(s = 0\), (2.4) is trivial). Indeed, if \(z \in Z_s(\Omega, B_{x_0}(R))\), then by definition, there exist \(x \in \Omega\) and \(y \in B_{x_0}(R)\) such that \(d(z, x) = sd(x, y)\) and \(d(z, y) = (1-s)d(x, y)\). In particular, we have that \(d(x, y) \leq d(x, x_0) + d(x_0, y) < d_0 + R\), thus, \(\text{dist}(z, \Omega) \leq d(z, x) < s(d_0 + R)\), which proves the claim (2.4).

On the other hand, the Brunn-Minkowski inequality (2.3) implies for every \(s \in [0, 1]\) that

\[
m(Z_s(\Omega, B_{x_0}(R))) \geq (1-s)m(\Omega) + sm(B_{x_0}(R)).
\]

Then, by (2.4) and using the definition of the Minkowski content, we have

\[
m^+(\Omega) = \lim_{\delta \to 0} \frac{m(\Omega \setminus \Omega_\delta)}{\delta} = \lim_{s \to 0} \frac{m(Z_s(\Omega, B_{x_0}(R))) - m(\Omega)}{s(d_0 + R)}
\geq \lim_{s \to 0} \frac{m(Z_s(\Omega, B_{x_0}(R))) - m(\Omega)}{s(d_0 + R)}
\geq \lim_{s \to 0} \frac{((1-s)m(\Omega) + sm(B_{x_0}(R)))^N - m(\Omega)}{s(d_0 + R)}
= Nm(\Omega) \frac{m(B_{x_0}(R))^N - m(\Omega)^N}{d_0 + R}.
\]

Since the latter estimate is valid for every \(R > 0\), we can take the limit \(R \to \infty\) at the right hand side, and by the definition of the asymptotic volume ratio we obtain that

\[
m^+(\Omega) \geq N\omega_N^{\frac{1}{N}} \text{AVR}_{M,d,m}^\frac{1}{N} m(\Omega)^{\frac{N-1}{N}},
\]

which is precisely relation (1.1).

Sharpness of (1.1). In order to show the sharpness of (1.1), we argue by contradiction. Assume that there is a constant

\[
C > N\omega_N^{\frac{1}{N}} \text{AVR}_{M,d,m}^\frac{1}{N}
\]

such that for all bounded Borel subsets \(\Omega \subset M\) with positive measure it holds

\[
m^+(\Omega) \geq Cm(\Omega)^{\frac{N-1}{N}}. \tag{2.5}
\]
To obtain the desired contradiction, we choose \( \Omega = B_{x_0}(r), \ r > 0 \), and observe that \( (B_{x_0}(r))_\delta \subseteq B_{x_0}(r + \delta) \) for every \( \delta > 0 \). By the monotonicity of the function \( r \mapsto \frac{m(B_{x_0}(r))}{r^N} \) one has that

\[
\frac{m(B_{x_0}(r + \delta))}{(r + \delta)^N} \leq \frac{m(B_{x_0}(r))}{r^N}.
\]

Consequently, we obtain that

\[
\frac{m((B_{x_0}(r))_\delta) - m(B_{x_0}(r))}{\delta} \leq \frac{m(B_{x_0}(r + \delta)) - m(B_{x_0}(r))}{\delta} \leq m(B_{x_0}(r)) \frac{1}{\delta} \left[ \left( \frac{r + \delta}{r} \right)^N - 1 \right].
\]

Letting \( \delta \to 0 \) in the above inequality, it follows that

\[
m^+(B_{x_0}(r)) \leq N \left( \frac{m(B_{x_0}(r))}{r^N} \right)^{\frac{1}{N}} m(B_{x_0}(r))^{\frac{N-1}{N}}. \tag{2.6}
\]

Using the definition of the asymptotic volume ratio, by (2.5) and (2.6) one has

\[
C \leq \lim_{r \to \infty} \frac{m^+(B_{x_0}(r))}{m(B_{x_0}(r))^{\frac{N-1}{N}}} \leq N \omega_N^{\frac{1}{N}} \text{AVR}_{M,d,m}^{\frac{1}{N}} < C,
\]

a contradiction, proving the sharpness of (1.1). \( \square \)

**Remark 2.1** Let us mention that the usage of Brunn-Minkowski inequalities is a powerful tool to prove isoperimetric and other related geometric inequalities; for recent results in this direction we refer to Kolesnikov and Milman [35], and Milman and Rotem [47]. However, the double limiting process in the above proof of (1.1) (by taking first \( s \to 0 \) and then \( R \to \infty \)) conceals those fine information that are crucial to characterize the equality cases. Therefore, the use of the Brunn-Minkowski inequality (2.3) in general seems to be too rough to establish rigidity statements.

In the sequel we present some classes of CD\((0, N)\) spaces where our results can be applied.

**Example 2.1 (Weighted cones)** Let \( \Sigma \subseteq \mathbb{R}^n \) be an open convex cone with vertex at the origin, and \( H : \mathbb{R}^n \to [0, \infty) \) be a gauge function (i.e., symmetric, convex and positively homogeneous of degree one). We endow the space \( \mathbb{R}^n \) with the induced metric \( d_H(x, y) = H(x - y) \). Let \( w \) be a continuous function in \( \Sigma \), positive in \( E \), and positively homogeneous of degree \( \alpha \geq 0 \) such that \( w^{\frac{1}{\alpha}} \) is concave in \( \Sigma \) whenever \( \alpha > 0 \). In particular, the concavity of \( w^{\frac{1}{\alpha}} \) is equivalent to the fact that the triplet \( (\Sigma, d_H, w \mathcal{L}^n) \) is a CD\((0, n + \alpha)\) space, see e.g. Villani [60], which also means that
\( \text{Ric}_w^\alpha \geq 0 \) on \( \Sigma \), see (2.8) below. Moreover, by the homogeneity properties of \( H \) and \( w \), it is easy to check that

\[
\text{AVR}_{\Sigma, dH, w} = \frac{\int_{B_{dH}^{(1)} \cap \Sigma}}{\omega_n} w > 0.
\]

In particular, Theorem 1.1 implies the main result of Cabré, Ros-Oton and Serra [10, Theorem 1.3], i.e., for every open set \( \Omega \subset \mathbb{R}^n \) with enough smooth boundary and \( \sum_w < \infty \) one has

\[
\frac{P_{w,H}(\Omega; \Sigma)}{\left( \int_{\Omega \cap \Sigma} w \right)^{n+\alpha-1}} \geq \frac{P_{w,H}(W; \Sigma)}{\left( \int_{W \cap \Sigma} w \right)^{n+\alpha-1}}, \tag{2.7}
\]

where \( P_{w,H} \) denotes the anisotropic weighted perimeter w.r.t. \( w \) and \( H \), and \( W \) is the Wulff set associated with the gauge \( H \), i.e., \( W = \{ x \in \mathbb{R}^n : x \cdot \nu \leq H(\nu), \nu \in S^{n-1} \} \), see Wulff [63].

We notice that the right hand side of the isoperimetric inequality (2.7) can be easily rewritten by means of \( \text{AVR}_{\Sigma, dH, w} \), coming from the homogeneity of \( w \) and \( H \). Particular forms of (2.7) can be found also in Cabré and Ros-Oton [9]. Let us note that the images of the Wulff set \( W \) under dilations are isoperimetric sets. Very recently, Cinti, Glaudo, Pratelli, Ros-Oton and Serra [16] proved that all isoperimetric sets in (2.7) are of this form (possibly up to some translations).

**Example 2.2 (Weighted Riemannian manifolds)** Let \( (M, g) \) be a noncompact, complete, \( n \)-dimensional Riemannian manifold and consider the Bakry-Émery Ricci curvature on the Riemannian metric measure space \( (M, g, w dv_g) \) given by

\[
\text{Ric}_w^\alpha := \text{Ric} - D^2(\log w) - \frac{1}{\alpha} D \log w \otimes D \log w, \tag{2.8}
\]

where \( \alpha > 0 \), and \( Dw \) and \( D^2w \) denote the differential and Hessian of a differentiable function \( w : M \rightarrow (0, \infty) \). It is known that if \( \text{Ric}_w^\alpha \geq 0 \) then \( (M, g, w dv_g) \) is a CD(0, \( N \)) space with \( N = n + \alpha \). In particular, if \( \Omega \subset M \) is an open set with smooth boundary, then \( m^+(\Omega) = \int_{\partial \Omega} w \) and \( m(\Omega) = \int_{\Omega} w \), where \( m = wdv_g \) is the weighted measure on \((M, g)\). Thus, assuming that

\[
\Lambda_\alpha = \lim_{r \to \infty} \frac{\int_{B_r(x)} w}{r^{n+\alpha}} > 0,
\]

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then Theorem 1.1 applies and we immediately obtain the sharp isoperimetric inequality

\[ \int_{\partial \Omega} w \geq (n + \alpha) \Lambda_{\alpha}^{\frac{1}{n+\alpha}} \left( \int_{\Omega} w \right)^{\frac{n+\alpha-1}{n+\alpha}}. \quad (2.9) \]

Inequality (2.9) has been recently obtained by Johne [31], extending the ABP-method used by Brendle [7] in the unweighted case \( w = 1 \). The equality case is characterized by Brendle [7] whenever \( w = 1 \); this fact will be discussed in more details in § 2.2.

**Example 2.3 (Euclidean cones)** Following [5] and [32], we present here a general class of examples such that we have equality in (1.1); these are the so-called *Euclidean cones* and are defined as follows. Let \( (M, d, m) \) be a complete metric measure space such that \( \text{diam} M \leq \pi \). We define \( \text{Con}(M) \) as the quotient of \( M \times [0, \infty) \) by identifying all points of \( M \times \{ 0 \} \) with a single point \( O \) - the origin of \( \text{Con}(M) \). The metric on \( \text{Con}(M) \) is defined by

\[ d_c((x, s), (y, t)) = \sqrt{s^2 + t^2 - 2st \cos d(x, y)}, \]

and the measure by \( d\mathcal{m}_c(x, s) = d\mathcal{m}(x) \otimes s^n ds \). Let us assume that the space \( (\text{Con}(M), d_c, \mathcal{m}_c) \) satisfies the \( \text{CD}(0, n + 1) \) condition. According to Bacher and Sturm [5], this holds true when \( M \) is an \( n \)-dimensional Riemannian manifold with \( \text{Ric} \geq n - 1 \), or more generally, if \( M \) is a weighted Riemannian manifold satisfying the \( \text{CD}(n - 1, n) \) condition (even when \( n \) is not necessarily an integer). Moreover, according to Ketterer [32] the same holds if we consider \( M \) to be a \( \text{CD}^*(n - 1, n) \) space. A direct calculation gives that

\[ \text{AVR}_{\text{Con}(M), d_c, \mathcal{m}_c} = \frac{m(M)}{(n + 1) \omega_{n+1}} > 0. \]

In conclusion the statement of Theorem 1.1 applies.

We claim that balls \( B_O(R) \) centered at the origin of the cone are isoperimetric sets for all \( R > 0 \). To see this, first note that \( \mathcal{m}_c(B_O(R)) = \frac{m(M) R^{n+1}}{n+1} \), and furthermore we claim that \( B_O(R + \epsilon) = (B_O(R))_\epsilon \). To see this last equality we note first that the inclusion \( B_O(R + \epsilon) \subseteq (B_O(R))_\epsilon \) is trivial. To check the opposite inclusion, pick a point \( (y, t) \in (B_O(R))_\epsilon \); then there exists a point \( (x, s) \in B_O(R) \) such that \( \epsilon > d_c((x, s), (y, t)) = \sqrt{s^2 + t^2 - 2st \cos d(x, y)} \geq |s - t| \). Since \( (x, s) \in B_O(R) \) we have that \( s < R \) and thus \( t < R + \epsilon \). From here we obtain that \( \mathcal{m}_c(B_O(R)) = m(M) R^n \) and we conclude that equality holds in (1.1) for \( B_O(R) \). We do not know if all isoperimetric sets are of this form.

**Remark 2.2** The characterization of the equality case in (1.1) is a challenging problem even in the particular settings of the above examples. In the case of Example 2.1 (weighted cones), a careful stability argument is carried out in [16] in order to characterize the isoperimetric sets in (2.7), while in the case of Example 2.2 we expect a strong rigidity of the manifold (and presumably of the weight) akin to the one stated.
by Brendle [7] (and discussed below) in the unweighted setting. The question of characterization of the equality case in the isoperimetric inequality in Theorem 1.1 for general metric measure spaces requires further investigations that will be considered in a forthcoming work.

2.2 Rigid isoperimetric inequalities in Riemannian manifolds with $\text{Ric} \geq 0$ : the canonical case

In this subsection we focus to the following result:

**Theorem 2.1** (Brendle [7]) Let $(M, g)$ be an $n$-dimensional Riemannian manifold as in Theorem 1.2. Equality holds in (1.2) for some $\Omega \subset M$ with $C^1$ regular boundary if and only if $\text{AVR}_g = 1$ and $\Omega$ is isometric to a ball $B \subset \mathbb{R}^n$.

Let us note first that besides Brendle’s ABP-based proof (which is valid in any dimension), Theorem 2.1 has been proven in the 3-dimensional case by Agostiniani, Fogagnolo and Mazzieri [1] by using Huisken’s mean curvature flows; moreover, Fogagnolo and Mazzieri [25] extended their arguments to manifolds up to 7 dimensions. We shall also outline a short, alternative proof of Theorem 2.1 by using tools from the OMT-theory. Our primordial motivation by doing so is that we present an approach that might be useful also to wider classes of possibly nonsmooth settings. At this stage however, certain technical issues prevent us to carry out the proof in more general structures.

**Outline of the proof of Theorem 2.1.** Let $\Omega \subset M$ be a bounded, connected and open set with smooth boundary such that equality holds in (1.2). We divide the proof in two steps.

**Step 1:** we show that at the points of the isoperimetric set $\Omega$ the manifold $M$ is locally isometric to the Euclidean space. Let us consider the probability measures

$$
\mu = \frac{1}{\text{Vol}_g(\Omega)} dv_g \quad \text{and} \quad v = \frac{1}{\text{Vol}_g(\Omega_r)} dv_g
$$

and the associated optimal transport map $T_r(x) = \exp_x (-\nabla_g u_r(x))$ for a.e. $x \in \overline{\Omega}$, where $1_A$ denotes the indicator function of the set $A \subset M$, and $u_r : \overline{\Omega} \to \mathbb{R}$ is a $c = d_g^2/2$-concave function, see Cordero-Erausquin, McCann and Schmuckenschläger [18] and McCann [43]. Then $T_r : \overline{\Omega} \to \Omega_r$ is injective except of a null set and the change of variables formula holds; in particular, we have the Monge-Ampère equation

$$
\frac{1}{\text{Vol}_g(\Omega)} = \frac{1}{\text{Vol}_g(\Omega_r)} \det DT_r(x) \quad \text{for a.e.} \ x \in \Omega.
$$

By the construction of the optimal transport map $T_r$, it turns out that $|\nabla_g u_r(x)| \leq r + 2d_0$ for a.e. $x \in \overline{\Omega}$, where $d_0 = \text{diam}(\Omega)$. Let $t > 0$ be arbitrarily fixed. For $r \geq t$, we introduce the family of scaling functions $w_{t,r} = t u_{t,r}$. The latter estimate implies that there exists $C_0 > 0$ (not depending on $r$) such that for every $r \geq t$ and a.e. $x \in \overline{\Omega}$, $|\nabla_g w_{t,r}(x)| \leq tC_0$, i.e., the family $\{w_{t,r}\}_{r \geq t}$ is equicontinuous. By Arzelà-Ascoli’s
theorem we obtain a sequence \( \{ w_{t, r_k} \}_k \) that converges uniformly to some function \( w_t : \overline{\Omega} \to \mathbb{R} \) as \( k \to \infty \) (and \( r_k \to \infty \)); moreover, \( w_t \) is \( c \)-concave, which follows by the general theory of \( c \)-concave functions (see Villani [60]).

Up to a smoothing argument à la Greene and Wu [28, 29], we assume that \( w_{t, r} \) is enough regular, thus the divergence theorem, Schwarz inequality and the equality in (1.2) yield

\[
\int_{\Omega} \left( 1 - \frac{\Delta_g w_{t, r}(x)}{n} \right) dv_g = \text{Vol}_g(\Omega) - \frac{1}{n} \int_{\partial \Omega} \langle \nabla_g w_{t, r}, n \rangle_g d\sigma_g
\]

\[
\leq \text{Vol}_g(\Omega) + \frac{1}{n} \frac{t}{r}(r + 2d_0)P_g(\partial \Omega)
\]

\[
= \text{Vol}_g(\Omega) + \frac{t}{r}(r + 2d_0)\omega_n^\frac{1}{n} \text{AVR}_g \text{Vol}_g(\Omega)^{\frac{n-1}{n}} \text{ (2.11)}
\]

where \( n(x) \) stands for the unit outward normal vector at \( x \in \partial \Omega \). Since \( \text{Ric} \geq 0 \), the volume distortions in \((M, g)\) verify

\[
v_s(x, y) = \lim_{r \to 0} \frac{\text{Vol}_g(Z_s(x, B_r(y)))}{\text{Vol}_g(B_s(r))} \geq 1 \text{ (2.12)}
\]

for every \( s \in (0, 1) \) and \( x, y \in M \) with \( y \notin \text{cut}(x) \), see [18, Corollary 2.2], where \( \text{cut}(x) \subset M \) is the cut-locus of \( x \). Thus, the Jacobian determinant inequality from [18, Lemma 6.1] becomes

\[
\left( \det DT_{t, r}(x) \right)^\frac{1}{n} \geq \left( 1 - \frac{t}{r} \right) v_{1-t/r}(T_r(x), x) + \frac{r}{t} v_{t/r}(x, T_r(x)) \left( \det DT_r(x) \right)^\frac{1}{n}
\]

\[
\geq 1 - \frac{t}{r} + \frac{r}{t} \left( \det DT_r(x) \right)^\frac{1}{n} \text{ for a.e. } x \in \Omega \text{ (2.13)}
\]

Combining (2.10), (2.11) and (2.13), and letting \( r \to \infty \), it follows by Fatou’s lemma that

\[
\int_{\Omega} \left( 1 - \frac{\Delta_g w_t}{n} - \left( \det D F_t \right)^\frac{1}{n} \right) dv_g \leq 0,
\]

where \( F_t : \overline{\Omega} \to F_t(\overline{\Omega}) \) is the optimal transport map \( F_t(x) = \text{exp}_x(-\nabla_g w_t(x)) \). On the other hand, since \( w_t : \overline{\Omega} \to \mathbb{R} \) is \( c \)-concave, one has the pointwise estimate

\[
0 \leq 1 - \frac{\Delta_g w_t(x)}{n} - \left( \det D F_t(x) \right)^\frac{1}{n} \text{ for a.e. } x \in \Omega \text{ (2.14)}
\]

see e.g. Wang and Zhang [61]. By the latter two inequalities we obtain the second order PDE

\[
1 - \frac{\Delta_g w_t(x)}{n} = \left( \det D F_t(x) \right)^\frac{1}{n} \text{ for a.e. } x \in \Omega \text{ (2.15)}
\]
Having (2.15) for every $t > 0$, and by using that $w_{st} = sw_t$, the Jacobian determinant inequality for the map $F_{st}$ and relation (2.12) imply for a.e. $x \in \Omega$ that

$$1 - s \frac{\Delta_g w_t(x)}{n} = 1 - \frac{\Delta_g w_{st}(x)}{n} = (\det DF_{st}(x))^{\frac{1}{n}}$$
$$\geq (1 - s) (v_{1-s}(F_t(x), x))^{\frac{1}{n}} + s (v_s(x, F_t(x)))^{\frac{1}{n}} (\det DF_t(x))^{\frac{1}{n}}$$
$$\geq 1 - s + s \left( 1 - s \frac{\Delta_g w_t(x)}{n} \right) = 1 - s \frac{\Delta_g w_t(x)}{n}.$$

According to these estimates, we must have equalities everywhere in the above chain of inequalities; thus, we necessarily have that the volume distortions should verify

$$v_{1-s}(F_t(x), x) = v_s(x, F_t(x)) = 1 \text{ for a.e. } x \in \overline{\Omega}.$$ (2.16)

By our earlier result [6, Theorem 4.1] (see also Chavel [14, Theorem III.4.3]), we can conclude that the sectional curvatures along the geodesic segments $s \mapsto \exp_x(-s\nabla_g w_t(x)) = F_{ts}(x)$, $s \in [0, 1]$, connecting $x$ to $F_t(x)$ are constantly equal to 0; this shows in particular that in all points of $\overline{\Omega}$ the sectional curvatures identically vanish. This fact implies that $\overline{\Omega} \subset M$ is locally isometric to the Euclidean space, see e.g. Petersen [54, Theorem 5.5.8]. This concludes the first step of the proof.

**Step 2:** we upgrade the local isometries to a global one. In order to carry out the second step we use first a covering argument combined with the Bishop-Gromov comparison and (2.15) to conclude that for some $\alpha > 0$,

$$\text{Hess}_g(-w_t)(x) = t\alpha \text{Id} \text{ for every } t > 0, x \in \Omega.$$ (2.17)

In particular, we have that $\Delta_g(-w_t) = n\alpha t$, and by (2.15), it yields

$$\det DF_t(x) = (1 + \alpha t)^n \text{ for every } t > 0, x \in \Omega.$$ (2.18)

The Jacobian determinant inequality, the equality in (1.2) and the Monge-Ampère equation (2.10) yield

$$\int_{\Omega} (\det DF_t(x))^{\frac{1}{n}} d\nu_g \geq \text{Vol}_g(\Omega) + \frac{f}{n} \mathcal{P}_g(\partial \Omega).$$

By (2.15) and the homogeneity property $w_t = tw$, the latter estimate becomes equivalent to

$$\mathcal{P}_g(\partial \Omega) \leq \int_{\Omega} \Delta_g(-w)d\nu_g.$$ 

Let us observe that $\Delta_g w = -n\alpha =$constant, thus $w$ is smooth up to the boundary $\partial \Omega$. Since $|\nabla_g w| \leq 1$ on $\Omega$, by the divergence theorem and Schwarz inequality we obtain that
\[ \mathcal{P}_g(\partial \Omega) \leq \int_{\Omega} \Delta_g(-w) \, dv_g = \int_{\partial \Omega} \langle \nabla_g(-w)(x), n(x) \rangle_g \, d\sigma_g(x) \leq \int_{\partial \Omega} d\sigma_g = \mathcal{P}_g(\partial \Omega). \]  

(2.19)

In particular, we have equality in the Schwarz inequality, which implies that

\[ \nabla_g(-w)(x) = n(x) \quad \text{for every } x \in \partial \Omega. \]  

(2.20)

If \( x_0 \in \bar{\Omega} \) is the global minimum of \(-w\) over \( \bar{\Omega} \), by (2.20) we clearly have that \( x_0 \) cannot belong to \( \partial \Omega \); thus \( x_0 \in \Omega \) and subsequently, \( \nabla_g w(x_0) = 0 \). Thus, \( x_0 \in \Omega \) is a fixed point of \( x \mapsto F_t(x) = \exp_x(-t \nabla_g w(x)) \) for every \( t > 0 \).

We now show that the optimal transport map \( F_t \) passes information from the infinitesimal volume distortion at the critical point \( x_0 \) of \( w \), to infinity. To do that, let \( B_{x_0}(r) \subset \Omega \) be a ball with enough small radius \( r > 0 \). Since \( B_{x_0}(r) \subset \Omega \) is isometric to a ball in \( \mathbb{R}^n \) with the same radius \( r > 0 \), for every \( r > 0 \) it follows by (2.18) that

\[ \text{Vol}_g(F_t(B_{x_0}(r))) = \int_{B_{x_0}(r)} \det D F_t(x) \, dv_g = (1 + \alpha t)^n \text{Vol}_g(B_{x_0}(r)) = (1 + \alpha t)^n \omega_n r^n. \]

(2.21)

Due to (2.17) and \( \nabla_g w(x_0) = 0 \), a simple estimate shows that there exists \( C > 0 \) such that for every \( t > 0 \), one has \( F_t(B_{x_0}(r)) \subseteq B_{x_0}(t(\alpha r + Cr^2) + r) \). Combining (2.21) and the latter inclusion, we obtain \( \omega_n r^n (1 + \alpha t)^n \leq \text{Vol}_g(B_{x_0}(t(\alpha r + Cr^2) + r)) \). For a fixed \( r > 0 \), dividing by \( t^n \) and taking \( t \to \infty \), the definition of AVR\(_g\) implies that \( r^n \alpha^n \leq \text{AVR}_g(\alpha r + Cr^2)^n \). Dividing by \( r^n \) this inequality and taking \( r \to 0 \), it follows that \( 1 \leq \text{AVR}_g \). Thus, we have \( \text{AVR}_g = 1 \), i.e., \((M, g) \) is isometric to the Euclidean space \((\mathbb{R}^n, g_0)\). In particular, it follows that \( \mathcal{P}_g(\partial \Omega) = n \omega_n^{\frac{1}{n}} \text{Vol}_g(\Omega)^{\frac{n-1}{n}} \); being in the Euclidean setting (up to an isometry), the latter equality implies that \( \Omega \subset M \) is isometric to a ball \( B \subset \mathbb{R}^n \).

**Remark 2.3** We notice the 'duality' of our OMT-argument with respect to Brendle's proof. On one hand, the ABP-method applied by Brendle [7] begins with a specific PDE akin to (2.15) with a Neumann boundary value condition and uses an estimate of the type (2.14) to arrive via a Ricci flow to the rigidity result. On the other hand, our OMT-argument begins with optimal transport rays and the Monge-Ampère equation (2.10) by using (2.14) to conclude the PDE (2.15) together with the Neumann boundary value condition (2.20), whose solution gives the required information about the whole manifold, i.e., \( \text{AVR}_g = 1 \).

**Remark 2.4** The smoothness of the boundary of the isoperimetric set is an essential requirement not only in our argument, but also in Agostiniani, Fogagnolo and Mazzieri [1, 25] and Brendle [7]. However, we expect that the smoothness assumption on the boundary might be replaced by a more general condition. Indeed, a careful inspection of our proof shows that the same argument can be extended to cover the case of domains \( \Omega \) with Lipschitz regular boundaries. On the other hand, isoperimetric sets \( \Omega \subset M \) (i.e., satisfying equality in (1.1) or (1.2)) are sets of finite perimeter. In the Euclidean case, the structure of sets with finite perimeter is well-understood, see Ambrosio,
Fusco and Pallara [2]; this should give useful information on the regularity of ∂Ω also in our case. Therefore, we believe that the rigidity statement should hold true with no additional apriori boundary regularity assumption as such a property should already be encoded into the initial fact that Ω is a set of finite perimeter. However, the proof of such a general statement is far from trivial, even in the Euclidean setting, see e.g. the survey notes by Fusco [26] on the early works of De Giorgi.

Riemannian manifolds with Ric ≥ 0 have been widely studied in the literature, stating various classifications and topological rigidities, see e.g. Anderson [3], Cheeger and Colding [15], Colding [17], Li [41], Liu [40], Menguy [44], Perelman [53], Reiris [55], Zhu [65]. To conclude this section, we present two Riemannian manifolds with Ric ≥ 0 that satisfy in addition also the Euclidean volume growth condition, providing as well their explicit asymptotic volume ratios.

Example 2.4 (Rotationally invariant metric on \( \mathbb{R}^n \)) Let \( n \geq 3 \) and \( f : [0, \infty) \to [0, 1] \) be a smooth nonincreasing function such that \( f(0) = 1 \) and \( \lim_{s \to \infty} f(s) = a \in (0, 1] \). We consider the rotationally invariant metric on \( \mathbb{R}^n \) defined by the warped product metric

\[
g = dr^2 + F(r)^2 d\theta^2,
\]

where \( F(r) = \int_0^r f(s) ds \) and \( d\theta^2 \) is the standard metric on the sphere \( S^{n-1} \). If \( x = (x_1, \theta_1) \) and \( \hat{x} = (x_2, \theta_2) \) are two points in \( \mathbb{R}^n \), it turns out that \( d_g(x, \hat{x}) \geq |x_1 - x_2| \), which implies that \((M, g)\) is complete. Furthermore, it is well known that the sectional (thus, the Ricci) curvature of \((\mathbb{R}^n, g)\) is nonnegative, see Carron [11].

For \( R \gg 1 \), one has that \( \text{Vol}_g(B_0(R)) = \int_{B_0(R)} dv_g \sim n \omega_n \int_0^R F(r)^{n-1} dr \). The latter estimate and L’Hôspital’s rule give that

\[
\text{AVR}_g = \lim_{R \to \infty} \frac{\text{Vol}_g(B_0(R))}{\omega_n R^n} = \lim_{R \to \infty} \frac{n \int_0^R F(r)^{n-1} dr}{R^n} = \frac{F(R)^{n-1}}{R^{n-1}} = a^{n-1} \in (0, 1].
\]

When \( a = 1 \), i.e., \( \text{AVR}_g = 1 \), by our monotonicity assumption it turns out that \( f \equiv 1 \) on \([0, \infty)\); thus \( F(r) = r \) and the metric \( g = g_0 = dr^2 + r^2 d\theta^2 \) becomes Euclidean.

Example 2.5 (Asymptotically locally Euclidean manifolds) Following Agostiniani, Fogagnolo and Mazzieri [1, Definition 4.13], a complete, noncompact Riemannian manifold \((M, g)\) is asymptotically locally Euclidean manifold if there exist a compact set \( K \subset M \), a ball \( B \subset \mathbb{R}^n \), a diffeomorphism \( \Psi : M \subset K \to \mathbb{R}^n \setminus B \), a number \( \tau > 0 \) and a finite subgroup \( G \) of \( SO(n) \) acting freely on \( \mathbb{R}^n \setminus B \) such that
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\[(\Psi^{-1} \circ \pi)^* g(z) = g_0 + O(|z|)^{-\tau}; \quad (2.22)\]

\[\left| \partial_i ((\Psi^{-1} \circ \pi)^* g) \right|(z) = O(|z|)^{-\tau - 1}; \quad (2.23)\]

\[\left| \partial_i \partial_j ((\Psi^{-1} \circ \pi)^* g) \right|(z) = O(|z|)^{-\tau - 2}, \quad (2.24)\]

where \(\pi : \mathbb{R}^n \to \mathbb{R}^n / G\) stands for the natural projection, \(z \in \mathbb{R}^n \setminus B\) and \(i, j \in \{1, \ldots, n\}\).

Due to assumptions (2.22)-(2.24), it turns out that \((M, g)\) has Euclidean volume growth; furthermore, one has that

\[\text{AVR}_g = \frac{1}{\text{Card}(G)}, \quad (2.25)\]

see [1, rel. (4.31)]. In particular, \((M, g)\) is isometric to \((\mathbb{R}^n, g_0)\) if and only if \(G = \{\text{Id}\} \subset SO(n)\); otherwise, \(0 < \text{AVR}_g < 1\).

When \(n = 3\), the finite subgroups of \(SO(3)\) are isomorphic to either a cyclic group \(\mathbb{Z}/m = \mathbb{Z}_m\) \((m \in \mathbb{N} \setminus \{0, 1\})\), a dihedral group \(D_m\), or the rotational symmetry group of a regular solid, i.e., (a) the symmetry group of the tetrahedron \(A_4\), (b) the symmetry group of the cube \(S_4\) (or octahedron), (c) the symmetry group of the dodecahedron \(A_5\) (or icosahedron). These subgroups of \(SO(3)\) together with (2.25) can be efficiently applied to produce sharp isoperimetric inequalities on 3-dimensional asymptotically locally Euclidean manifolds.

### 3 Sharp and rigid Sobolev inequalities on Riemannian manifolds with \(\text{Ric} \geq 0\)

Let \(u : M \to \mathbb{R}\) be a fast decaying function, i.e., \(\text{Vol}_g(\{x \in M : |u(x)| > t\}) < +\infty\) for every \(t > 0\). For such a function, let inspired by Aubin [4] and Druet, Hebey and Vaugon [23], we associate its Euclidean rearrangement function \(u^* : \mathbb{R}^n \to [0, \infty)\) which is radially symmetric, nonincreasing in \(|x|\), and for every \(t > 0\) is defined by

\[\text{Vol}_{g_0} (\{x \in \mathbb{R}^n : u^*(x) > t\}) = \text{Vol}_g (\{x \in M : |u(x)| > t\}). \quad (3.1)\]

By (3.1) and the layer cake representation, see Lieb and Loss [39], it turns out that

\[\text{Vol}_g(\text{supp}(u)) = \text{Vol}_{g_0}(\text{supp}(u^*)), \quad (3.2)\]

and if \(u \in L^q(M)\) for some \(q \in (0, \infty)\), the Cavalieri principle reads as

\[\|u\|_{L^q(M)} = \|u^*\|_{L^q(\mathbb{R}^n)}. \quad (3.3)\]

For every \(0 < t < \|u\|_{L^\infty(M)}\), let us consider the sets

\[\Omega_t = \{x \in M : |u(x)| > t\} \quad \text{and} \quad \Omega^*_t = \{x \in \mathbb{R}^n : u^*(x) > t\}, \quad (3.4)\]
respectively. The key ingredient in our arguments is the following Pólya-Szegő inequality.

**Proposition 3.1** Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold with \(\text{Ric} \geq 0\) and having Euclidean volume growth, and \(u : M \to \mathbb{R}\) be a fast decaying function such that \(|\nabla_g u| \in L^p(M)\), \(p > 1\). Then one has

\[
\|\nabla_g u\|_{L^p(M)} \geq \text{AVR}_g^{1/p} \|u^*\|_{L^p(\mathbb{R}^n)}.
\]  

(3.5)

In addition, if equality holds in (3.5) for some nonnegative \(u \in C^n(M) \setminus \{0\}\), then \(\text{AVR}_g = 1\), i.e., \((M, g)\) is isometric to the Euclidean space \((\mathbb{R}^n, g_0)\), and \(\Omega_t \subset M\) is isometric to the ball \(\Omega^*_t \subset \mathbb{R}^n\) for a.e. \(0 < t < \|u\|_{L^\infty(M)}\).

**Proof** The proof is inspired by Aubin [4], and Brothers and Ziemer [8]. Without loss of generality, we may assume that \(u \geq 0\) since \(\|\nabla_g u\|_{L^p(M)} = \|\nabla_g |u|\|_{L^p(M)}\), and by density, it is enough to consider functions belonging to \(C^n(M)\). For every \(0 < t < \|u\|_{L^\infty(M)}\) = \(L\), let

\[\Pi_t := u^{-1}(t) \subset M\] and \(\Pi_t^* := (u^*)^{-1}(t) \subset \mathbb{R}^n\),

and due to (3.4),

\[
\mathcal{V}(t) := \text{Vol}_g(\Omega_t) = \text{Vol}_{g_0}(\Omega^*_t).
\]

Consider the set of critical points \(C = \{x \in M : \nabla_g u(x) = 0\}\) and \(C^* = \{x \in \mathbb{R}^n : \nabla u^*(x) = 0\}\), respectively. Since \(u \in C^n(M)\), we have that the set of critical values \(u(C)\) is a null measure set in \(\mathbb{R}\) by Sard’s theorem. The set \(\Pi_t\) for \(t \not\in u(C)\) is a smooth, regular surface of class \(C^n\). Similarly we have that \(u^*(C^*)\) is also of null measure.

The co-area formula implies that

\[
\mathcal{V}(t) = \text{Vol}_g(C \cap u^{-1}(t, L)) + \int_t^L \left( \int_{\Pi_s} \frac{1}{\nabla_g u} \, d\mathcal{H}^{n-1} \right) \, ds
\]

(3.6)

\[
= \text{Vol}_{g_0}(C^* \cap (u^*)^{-1}(t, L)) + \int_t^L \left( \int_{\Pi_t^*} \frac{1}{\nabla u^*} \, d\mathcal{H}^{n-1} \right) \, ds.
\]

(3.7)

By the monotonicity of the function \(\mathcal{V}\), we conclude that it is differentiable almost everywhere. Furthermore notice that the functions \(t \mapsto \text{Vol}_g(C \cap u^{-1}(t, L))\), \(t \mapsto \text{Vol}_{g_0}(C^* \cap (u^*)^{-1}(t, L))\) have the same properties. Moreover, these latter functions have vanishing derivatives almost everywhere. Thus (3.6) and (3.7) imply that

\[- \mathcal{V}'(t) = \int_{\Pi_t} \frac{1}{\nabla_g u} \, d\mathcal{H}^{n-1} = - \int_{\Pi_t^*} \frac{1}{\nabla u^*} \, d\mathcal{H}^{n-1} \text{ for a.e. } 0 < t < L.
\]

(3.8)

In the sequel, we consider only those values of \(t > 0\) for which \(\mathcal{V}'(t)\) is well-defined and the above formula holds.
Since \( u^* \) is radially symmetric, the set \( \Pi^*_t \) is an \((n - 1)\)-dimensional sphere. Furthermore, \(|\nabla u^*|_t := |\nabla u^*|\) is constant on the \((n - 1)\)-dimensional sphere \( \Pi^*_t \) and by (3.8) it follows that

\[
\gamma'(t) = -\frac{\mathcal{H}^{n-1}(\Pi^*_t)}{|\nabla u^*|_t} \quad \text{for a.e. } 0 < t < L. \tag{3.9}
\]

By (3.8) and Hölder’s inequality we infer that

\[
\mathcal{H}^{n-1}(\Pi_t) = \int_{\Pi_t} d\mathcal{H}^{n-1} = \int_{\Pi_t} \frac{1}{|\nabla_g u|_t} \, d\mathcal{H}^{n-1} \\
\leq \left( \int_{\Pi_t} \frac{1}{|\nabla_g u|} \, d\mathcal{H}^{n-1} \right)^{\frac{p-1}{p}} \left( \int_{\Pi_t} |\nabla_g u|^{p-1} \, d\mathcal{H}^{n-1} \right)^{\frac{1}{p}} \\
\leq (-\gamma'(t))^{\frac{p-1}{p}} \left( \int_{\Pi_t} |\nabla_g u|^{p-1} \, d\mathcal{H}^{n-1} \right)^{\frac{1}{p}}.
\]

Since \( \text{Vol}_g(\Omega_t) = \text{Vol}_{g_0}(\Omega^*_t) \) and \( \Pi_t \) is a \( C^1 \) smooth regular surface for a.e. \( 0 < t < L \) (and for such surfaces the \((n - 1)\)-dimensional Hausdorff measure and the perimeter coincide), by the isoperimetric inequality (1.2) we have

\[
\mathcal{H}^{n-1}(\Pi_t) = \mathcal{P}_g(\Pi_t) \geq \text{AVR}^\frac{1}{n} \, \mathcal{P}_{g_0}(\Pi^*_t) = \text{AVR}^\frac{1}{n} \, \mathcal{H}^{n-1}(\Pi^*_t) \quad \text{for a.e. } 0 < t < L. \tag{3.10}
\]

Thus, by (3.10) and relation (3.9), the previous estimate implies that

\[
\int_{\Pi_t} |\nabla_g u|^{p-1} \, d\mathcal{H}^{n-1} \geq \left( \mathcal{H}^{n-1}(\Pi_t) \right)^{\frac{p}{p-1}} (-\gamma'(t))^{\frac{p}{p-1}} \\
\geq \text{AVR}_g^\frac{p}{n} \left( \mathcal{H}^{n-1}(\Pi^*_t) \right)^{\frac{p}{p-1}} \left( \frac{\mathcal{H}^{n-1}(\Pi^*_t)}{|\nabla u^*|_t} \right)^{\frac{p}{p-1}} \\
= \text{AVR}_g^\frac{p}{n} \int_{\Pi^*_t} |\nabla u^*|^{p-1} \, d\mathcal{H}^{n-1}. \tag{3.11}
\]

By combining again the co-area formula with this estimate, it follows that

\[
\int_M |\nabla_g u|^p \, dv_g = \int_0^\infty \int_{\Pi_t} |\nabla_g u|^{p-1} \, d\mathcal{H}^{n-1} \, dt \\
\geq \text{AVR}_g^\frac{p}{n} \int_0^\infty \int_{\Pi^*_t} |\nabla u^*|^{p-1} \, d\mathcal{H}^{n-1} \, dt = \text{AVR}_g^\frac{p}{n} \int_{\mathbb{R}^n} |\nabla u^*|^p \, dx,
\]

which concludes the proof of inequality (3.5).

If equality holds in (3.5) for some nonnegative \( u \in C^1(M) \setminus \{0\} \), by (3.10) we necessarily have for a.e. \( 0 < t < L \) that

\[
\mathcal{P}_g(\Pi_t) = \text{AVR}_g^\frac{1}{n} \, \mathcal{P}_{g_0}(\Pi^*_t) = n \omega_n^\frac{1}{n} \, \text{AVR}_g^\frac{1}{n} \, \text{Vol}_{g_0}(\Omega^*_t)^{\frac{n-1}{n}} = n \omega_n^\frac{1}{n} \, \text{AVR}_g^\frac{1}{n} \, \text{Vol}_g(\Omega_t)^{\frac{n-1}{n}}.
\]
Since by Sard’s theorem, for almost every $t$ the isoperimetric domains $\Omega_t$ have regular $C^1$ boundaries, Theorem 2.1 can be applied. Thus $\text{AVR}_g = 1$, i.e., $(M, g)$ is isometric to the Euclidean space $(\mathbb{R}^n, g_0)$, and $\Omega_t \subset M$ is isometric to the ball $\Omega_t^* \subset \mathbb{R}^n$ for a.e. $0 < t < L$. □

### 3.1 Gagliardo-Nirenberg interpolation inequality

Sharp Gagliardo-Nirenberg inequalities on $\mathbb{R}^n$ are known after Del Pino and Dolbeault [22] and Cordero-Erausquin, Nazaret and Villani [19]. In the sequel we establish a sharp Gagliardo-Nirenberg inequality on Riemannian manifolds with Ric $\geq 0$ whose particular form provides the statement of Theorem 1.2.

When $p \in (1, n)$ and $1 < \alpha \leq \frac{n}{n-p}$, the Gagliardo-Nirenberg inequality on $\mathbb{R}^n$ reads as

$$
\|u\|_{L^{\alpha,p}(\mathbb{R}^n)} \leq \mathcal{G}_{\alpha,p,n} \|\nabla u\|_{L^p(\mathbb{R}^n)}^\theta \|u\|_{L^{\alpha(p-1)+1}(\mathbb{R}^n)}^{1-\theta}, \forall u \in \dot{W}^{1,p}(\mathbb{R}^n),
$$

where

$$
\theta = \frac{p^*(\alpha - 1)}{\alpha p(p^* - \alpha p + \alpha - 1)},
$$

and the best constant

$$
\mathcal{G}_{\alpha,p,n} := \left(\frac{\alpha - 1}{p'}\right)^\theta \left(\frac{p'}{n}\right)^{\theta + \frac{\theta}{p}} \left(\frac{\alpha(p-1)+1}{\alpha-1}\right)^{\frac{1}{p\sigma} - \frac{1}{\sigma}} \left(\frac{\alpha(p-1)+1}{\alpha-1}\right)^{\frac{\theta}{\sigma}} \left(\frac{\alpha(p-1)+1}{\alpha-1}\right)^{\frac{\theta}{\sigma}} \frac{1}{\sigma} \left(\alpha_nB\left(\frac{\alpha(p-1)+1}{\alpha-1} - \frac{n}{p'}, \frac{n}{p}\right)\right)^{\frac{\theta}{n}}
$$

is achieved by the unique family of functions $h^{\lambda}_{\alpha,p}(x) = (\lambda + |x|^{\frac{p}{p-1}})^{-\frac{1}{\sigma}}$, $x \in \mathbb{R}^n$, $\lambda > 0$. Here, $p^* = \frac{np}{n-p}$, $p' = \frac{p}{p-1}$ and $\dot{W}^{1,p}(\mathbb{R}^n) = \{u \in L^{p^*}(\mathbb{R}^n) : |\nabla u| \in L^p(\mathbb{R}^n)\}$, while $B(\cdot, \cdot)$ is the Euler beta-function.

Let us recall from the Introduction the function space

$$
\dot{W}^{1,p}(M) = \{u \in L^{p^*}(M) : |\nabla u| \in L^p(M)\}.
$$

**Theorem 3.1** Let $(M, g)$ be an $n$-dimensional Riemannian manifold as in Theorem 1.2. Let $p \in (1, n)$, $1 < \alpha \leq \frac{n}{n-p}$ and the constants $\theta$ and $\mathcal{G}_{\alpha,p,n}$ given by (3.13) and (3.14), respectively. Then

$$
\|u\|_{L^{\alpha,p}(M)} \leq K_{G}^{\mathcal{G}_{\alpha,p,n}} \|\nabla u\|_{L^p(M)}^\theta \|u\|_{L^{\alpha(p-1)+1}(M)}^{1-\theta}, \forall u \in \dot{W}^{1,p}(M),
$$

where the constant $K_{G}^{\mathcal{G}_{\alpha,p,n}} = \mathcal{G}_{\alpha,p,n} \text{AVR}_g^{-\frac{\alpha}{p}}$ is sharp. Moreover, equality holds in (3.15) for some nonzero and nonnegative function $u \in C^n(M) \cap \dot{W}^{1,p}(M)$ if and only if $\text{AVR}_g = 1$ and $u = h^{\lambda}_{\alpha,p}$ a.e. for some $\lambda > 0$ (up to isometry).
Proof} Clearly, it is enough to prove (3.15) for nonnegative functions $u \in C_0^\infty(M)$; the inequality for general $u \in \dot{W}^{1,p}(M)$ will follow by approximation.

We recall that the Euclidean rearrangement function $u^*$ of $u$ satisfies the optimal Gagliardo-Nirenberg inequality (3.12), thus relations (3.3) and (3.5) imply that

$$
\|u\|_{L^p(M)} = \|u^*\|_{L^p(\mathbb{R}^n)} \\
\leq G_{\alpha,p,n} \|\nabla u^*\|_{L^p(\mathbb{R}^n)}^{\theta} \|u^*\|_{L^{p(\alpha-1)+1}(\mathbb{R}^n)}^{1-\theta} \\
\leq G_{\alpha,p,n} AVR_g^{-\frac{\theta}{n}} \|\nabla u\|_{L^p(M)}^{\theta} \|u\|_{L^{p(\alpha-1)+1}(M)}^{1-\theta},
$$

(3.16)

which proves inequality (3.15).

To prove the statement about sharpness, assume by contradiction, that the constant $K_{g}^{GN} = G_{\alpha,p,n} AVR_g^{-\frac{\theta}{n}}$ is not sharp in (3.15), i.e., there exists $0 < C < K_{g}^{GN}$ such that

$$
\|u\|_{L^p(M)} \leq C \|\nabla u\|_{L^p(M)} \|u\|_{L^{p(\alpha-1)+1}(M)}, \quad \forall u \in \dot{W}^{1,p}(M).
$$

Since $(M, g)$ is a complete Riemannian manifold with $\text{Ric} \geq 0$, the validity of the latter inequality implies the quantitative non-collapsing volume property

$$
\text{Vol}_g(B_x(r)) \geq \left( \frac{G_{\alpha,p,n}}{C} \right)^{\frac{n}{\theta}} \omega_n r^n \quad \text{for all } x \in M \text{ and } r \geq 0,
$$

see Ledoux [38] (for $\theta = 1$) and Xia [64], Kristály [36], and Kristály and Ohta [37] (for general $\theta$ from (3.13)). By the latter relation, one has that

$$
AVR_g = \lim_{r \to \infty} \frac{\text{Vol}_g(B_x(r))}{\omega_n r^n} \geq \left( \frac{G_{\alpha,p,n}}{C} \right)^{\frac{n}{\theta}}.
$$

This inequality is equivalent to $C \geq K_{g}^{GN}$ which contradicts our initial assumption $0 < C < K_{g}^{GN}$. In conclusion, the constant $K_{g}^{GN}$ is sharp in inequality (3.15).

Assume now that equality holds in (3.15) for some nonzero and nonnegative function $u \in C^\infty(M) \cap \dot{W}^{1,p}(M)$. Clearly, $u$ is fast decaying and by (3.3), (3.5) and (3.12) we have

$$
\|u\|_{L^p(M)} = K_{g}^{GN} \|\nabla u\|_{L^p(M)}^{\theta} \|u\|_{L^{p(\alpha-1)+1}(M)}^{1-\theta} \geq K_{g}^{GN} AVR_g^{\frac{\theta}{n}} \|\nabla u^*\|_{L^p(\mathbb{R}^n)}^{\theta} \|u^*\|_{L^{p(\alpha-1)+1}(\mathbb{R}^n)}^{1-\theta} \\
\geq \|u^*\|_{L^p(\mathbb{R}^n)} = \|u\|_{L^p(M)}.
$$

Consequently, we have equalities in the above chain of inequalities. In particular, by the first equality we have equality in the Pólya-Szegő inequality (3.5); thus Proposition 3.1 implies that $AVR_g = 1$, i.e., $(M, g)$ is isometric to the Euclidean space $(\mathbb{R}^n, g_0)$. Furthermore, the second equality implies that $u^*$ is an extremal in the Euclidean Gagliardo-Nirenberg inequality (3.12), which shows that $u^* = h_{\alpha,p}^\lambda$ for some $\lambda > 0$.

Since the set

\[\text{Vol}_g(B_x(r)) \geq \left( \frac{G_{\alpha,p,n}}{C} \right)^{\frac{n}{\theta}} \omega_n r^n \quad \text{for all } x \in M \text{ and } r \geq 0,\]

see Ledoux [38] (for $\theta = 1$) and Xia [64], Kristály [36], and Kristály and Ohta [37] (for general $\theta$ from (3.13)). By the latter relation, one has that

\[AVR_g = \lim_{r \to \infty} \frac{\text{Vol}_g(B_x(r))}{\omega_n r^n} \geq \left( \frac{G_{\alpha,p,n}}{C} \right)^{\frac{n}{\theta}}.\]

This inequality is equivalent to $C \geq K_{g}^{GN}$ which contradicts our initial assumption $0 < C < K_{g}^{GN}$. In conclusion, the constant $K_{g}^{GN}$ is sharp in inequality (3.15).

Assume now that equality holds in (3.15) for some nonzero and nonnegative function $u \in C^\infty(M) \cap \dot{W}^{1,p}(M)$. Clearly, $u$ is fast decaying and by (3.3), (3.5) and (3.12) we have

\[\|u\|_{L^p(M)} = K_{g}^{GN} \|\nabla u\|_{L^p(M)}^{\theta} \|u\|_{L^{p(\alpha-1)+1}(M)}^{1-\theta} \geq K_{g}^{GN} AVR_g^{\frac{\theta}{n}} \|\nabla u^*\|_{L^p(\mathbb{R}^n)}^{\theta} \|u^*\|_{L^{p(\alpha-1)+1}(\mathbb{R}^n)}^{1-\theta} \geq \|u^*\|_{L^p(\mathbb{R}^n)} = \|u\|_{L^p(M)}.\]

Consequently, we have equalities in the above chain of inequalities. In particular, by the first equality we have equality in the Pólya-Szegő inequality (3.5); thus Proposition 3.1 implies that $AVR_g = 1$, i.e., $(M, g)$ is isometric to the Euclidean space $(\mathbb{R}^n, g_0)$. Furthermore, the second equality implies that $u^*$ is an extremal in the Euclidean Gagliardo-Nirenberg inequality (3.12), which shows that $u^* = h_{\alpha,p}^\lambda$ for some $\lambda > 0$. Since the set
\{x \in \mathbb{R}^n : u^*(x) > 0, \nabla u^*(x) = 0\}

is a singleton containing only 0 \in \mathbb{R}^n, it turns out by Brothers and Ziemer [8, Theorem 1.1] that (up to an isometry), \( u = u^* \) a.e. The converse is trivial. \( \square \)

**Remark 3.1**  
(i) Theorem 1.2 is a simple consequence of Theorem 3.1 by considering \( \theta = 1 \).
(ii) Further sharp Sobolev-type inequalities can be proved on Riemannian manifolds with with \( \text{Ric} \geq 0 \), similarly to Theorem 3.1, as the dual Gagliardo-Nirenberg inequality (\( \alpha < 1 \)), \( L^p \)-log-Sobolev inequality (\( \alpha \rightarrow 1 \)), and Faber-Krahn inequality (\( \alpha \rightarrow 0 \)), see [19]. In fact, these inequalities follow by Proposition 3.1, while their sharpness by Kristály [36].

### 3.2 Rayleigh-Faber-Krahn inequality: first eigenvalues in sharp form

In this subsection we prove Theorem 1.3; the following auxiliary result will be used whose proof is based on a simple application of the layer cake representation.

**Lemma 3.1** Let \((M, g)\) be a complete Riemannian manifold, \( R > 0 \) and \( x_0 \in M \) be arbitrarily fixed, and \( f : [0, R] \rightarrow \mathbb{R} \) be a \( C^1 \)-function on \([0, R]\). Then

\[
\int_{B_{x_0}(R)} f(d_g(x_0, x)) dv_g = f(R) Vol_g(B_{x_0}(R)) - \int_{0}^{R} f'(r) Vol_g(B_{x_0}(r)) dr.
\]

The precursor of Theorem 1.3 reads as follows.

**Theorem 3.2** Let \((M, g)\) be an \( n \)-dimensional Riemannian manifold as in Theorem 1.2. Then for every smooth bounded open set \( \Omega \subset M \) and \( u \in W^{1,2}_0(\Omega) \) we have

\[
\text{Vol}_g(\Omega)^{-\frac{2}{n}} \int_{\Omega} u^2 dv_g \leq R_g \int_{\Omega} |\nabla u|^2 dv_g,
\]

where the constant \( R_g = j_{n-1}^{-1}(\omega_n \text{AVR}_g)^{-\frac{2}{n}} \) is sharp. In addition, equality holds in (3.17) for some set \( \Omega \subset M \) and for some nonzero function \( u \in W^{1,2}_0(M) \) if and only if \((M, g)\) is isometric to \( (\mathbb{R}^n, g_0) \) and \( \Omega \) is isometric to a ball \( B \subset \mathbb{R}^n \).

**Proof** Let \( \Omega \subset M \) be any smooth bounded open set and \( B \subset \mathbb{R}^n \) be a ball with \( \text{Vol}_g(\Omega) = \text{Vol}_{g_0}(B) \). If \( u : \Omega \rightarrow \mathbb{R} \) is any nonzero function with the usual smoothness properties, its Euclidean rearrangement function \( u^* : B \rightarrow [0, \infty) \) satisfies the properties (3.2), (3.3) and (3.5); in particular, the latter relations combined with the Euclidean Rayleigh-Faber-Krahn inequality immediately gives

\[
\frac{\int_{\Omega} |\nabla u|^2 dv_g}{\text{Vol}_g(\Omega)^{-\frac{2}{n}} \int_{\Omega} u^2 dv_g} \geq \frac{\text{AVR}_g^{\frac{2}{n}} \int_{B} |\nabla u^*|^2 dx}{\text{Vol}_{g_0}(B)^{-\frac{2}{n}} \int_{B} (u^*)^2 dx} \geq j_{n-1}^{\frac{2}{n}}(\omega_n \text{AVR}_g)^{\frac{2}{n}} = R_g^{-1},
\]

(3.18)
which is exactly inequality (3.17).

The sharpness of the constant \( R_g = \frac{j_{n-1}}{2} (\omega_n \text{AVR}(g))^{-\frac{2}{n}} \) in (3.17) is more delicate, which requires fine properties of Bessel functions of the first kind \( J_v, v \in \mathbb{R} \), see e.g. Olver et al. [52]; for completeness, we outline its proof. By contraction, we assume there exists \( C < R_g \) such that for every smooth bounded open set \( \Omega \subset M \),

\[
\text{Vol}_g(\Omega)^{-\frac{2}{n}} \int_{\Omega} u^2 \text{d}v_g \leq C \int_{\Omega} |\nabla_g u|^2 \text{d}v_g, \quad \forall u \in W^{1,2}_0(\Omega). \tag{3.19}
\]

For convenience of notation, we choose \( \nu = \frac{n}{2} - 1 \geq 0 \). For every \( R > 0 \) and \( x_0 \in M \), we consider the function \( u_R : B(x_0, R) \to \mathbb{R} \) defined by

\[
u(x) = d_g(x_0, x)^{-\nu} J_v \left( j_v \frac{d_g(x_0, x)}{R} \right), \quad x \in B(x_0, R).
\]

It is clear that (3.19) can be applied to the function \( u_R \) and to the set \( \Omega = B(x_0, R) \), i.e.,

\[
\text{Vol}_g(B(x_0, R))^{-\frac{2}{n}} \int_{B(x_0, R)} u_R^2 \text{d}v_g \leq C \int_{B(x_0, R)} |\nabla_g u_R|^2 \text{d}v_g. \tag{3.20}
\]

Basic properties of Bessel functions combined with the eikonal equation

\[
|\nabla_g d_g(x_0, x)| = 1 \text{ for a.e. } x \in M,
\]

imply that

\[
\int_{B(x_0, R)} |\nabla_g u_R|^2 \text{d}v_g = \frac{j_v^2}{R^2} \int_{B(x_0, R)} d_g(x_0, x)^{-2v} J_{v+1}^2 \left( j_v \frac{d_g(x_0, x)}{R} \right) \text{d}v_g.
\]

Applying Lemma 3.1, a change of variables, the Lebesgue’s dominated convergence theorem and finally an integration by parts, one has

\[
\lim_{R \to \infty} \int_{B(x_0, R)} |\nabla_g u_R|^2 \text{d}v_g = j_v^2 \omega_n \text{AVR}_g \left( J_{v+1}^2(j_v) - \int_0^1 t^n \frac{\text{d}}{\text{d}t} (t^{-2v} J_{v+1}(j_v t)) \text{d}t \right)
\]

\[
= j_v^2 \omega_n \text{AVR}_g \int_0^1 t^2 J_{v+1}(j_v t) \text{d}t.
\]

Since \( J_v(j_v) = 0 \), a similar reasoning as above shows that
\[
\lim_{R \to \infty} \frac{\int_{B_{2R}(R)} u^2 \, dv_g}{R^2} = n \omega_n AVR_g \int_0^1 tJ_v^2(j_v(t)) \, dt.
\]

Letting \( R \to \infty \) in (3.20) and taking into account the latter relations, we obtain that

\[
(\omega_n AVR_g)^{-2} \leq Cj_v^2.
\]

This inequality clearly contradicts \( C < R_g = j_v^{-2}(\omega_n AVR(g))^{-2} \), concluding the sharpness of \( R_g \) in (3.17).

Assume now that equality holds in (3.17) for some open bounded set \( \Omega \subset M \) and some nonzero and nonnegative function \( u \in W^{1,2}_0(\Omega) \). In particular, equality in (3.17) (thus in (3.18)) implies equality in (3.5), where \( u^* : B \to \mathbb{R} \) is the Euclidean rearrangement function of \( u \), and \( B \subset \mathbb{R}^n \) is a ball with \( \text{Vol}_g(\Omega) = \text{Vol}_g(B) \). Furthermore, since \( u \) verifies a second order PDE involving the operator \( \Delta_g \) (obtained as the Euler-Lagrange equation associated with (1.5)), it turns out by standard regularity theory that \( u \in C^\infty(\Omega) \). Consequently, by Proposition 3.1 one has \( AVR_g = 1 \) and \( \Omega \) is isometric to \( B \subset \mathbb{R}^n \). The converse is trivial.

\section*{Proof of Theorem 1.3}

The proof rests upon Theorem 3.2. Indeed, the sharp estimate of the first eigenvalue \( \lambda_{1,g}^D(\Omega) \) in (1.7) immediately follows by (3.17), noticing that \( R_g = \Lambda_g^{-1} \). Standard compactness and variational argument show that the infimum in (1.5) is achieved. Thus, if equality holds in (1.7) for some \( \Omega \subset M \), then there exists some nonzero and sign-preserving function \( u \in W^{1,2}_0(\Omega) \) which produces equality in (3.17) as well, concluding the proof, cf. Theorem 3.2.

\section*{Remark 3.2}

Having sharp Sobolev inequalities on Riemannian manifolds with \( \text{Ric} \geq 0 \) (see § 3.1-3.2), a natural question would be to establish such inequalities in the framework of general \( \text{CD}(0, N) \) spaces. Based on the sharp isoperimetric inequality in Theorem 1.1, a Pólya-Szegő inequality is expected to hold on \( \text{CD}(0, N) \) spaces; a similar argument has been completed recently on \( \text{CD}(\kappa, N) \) spaces with \( \kappa > 0 \), see Mondino and Semola [48].

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\section*{Declarations}

\section*{Conflict of interest}

The authors state that there is no conflict of interest.

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