Yukawa Textures from Family Symmetry and Unification

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Abstract

In this letter, we show how the combination of both a gauged $U(1)_X$ family symmetry and an extended vertical gauge symmetry in a single model, allows for the presence of additional Clebsch texture zeroes in the fermion mass matrices. This, leads to new structures for the textures, with increased predictivity, as compared to schemes with enhanced family symmetries only. We illustrate these ideas in the context of the Pati-Salam gauge group $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ supplemented by a $U(1)_X$ gauged family symmetry. In the case of symmetric mass matrices, two of the solutions of Ramond, Roberts and Ross that may not be obtained by family symmetries only, are accurately reproduced. For non-symmetric textures, new structures arise in models of this type. To distinguish between the solutions in this latter case, we performed a numerical fit to the charged fermion mass and mixing data. The best solution we found allows a fit with a total $\chi^2$ of 0.39, for three degrees of freedom.
In the recent years, there has been a lot of effort in trying to understand the pattern of quark and lepton masses and mixing angles. In order to explain the observed hierarchies in the most predictive way, it has been proposed that zero textures in the Yukawa matrices exist, like in the Fritzsch ansatz \[^1\] and the Georgi-Jarlskog (GJ) texture \[^2\]. Ramond, Roberts and Ross (RRR) \[^3\] have made a survey of possible symmetric textures which are both consistent with data and involve the maximum number of texture zeroes. Such textures can arise due to a family symmetry group \(G\) and some new heavy matter of mass \(M\) which transforms under \(G\), resulting in effective non-renormalisable operators that generate fermion masses \[^4\]. Such family symmetries arise in most of the superstring models. A realisation of this picture has been provided by Ibáñez and Ross (IR) \[^5\], based on the MSSM extended by a gauged family \(U(1)_X\) symmetry with \(\theta\) and \(\bar{\theta}\) singlet fields with opposite \(X\) charges, plus new heavy Higgs fields in vector representations. Anomaly cancellation occurs via a Green-Schwarz-Witten (GSW) mechanism, and the \(U(1)_X\) symmetry is broken not far below the string scale, generating Yukawa matrices of the form

\[
\lambda^U = \begin{pmatrix}
\varepsilon^8 & \varepsilon^3 & \varepsilon^4 \\
\varepsilon^3 & \varepsilon^2 & \varepsilon \\
\varepsilon^4 & \varepsilon & 1
\end{pmatrix}, \quad 
\lambda^D = \begin{pmatrix}
\varepsilon^8 & \varepsilon^3 & \bar{\varepsilon}^4 \\
\bar{\varepsilon}^3 & \bar{\varepsilon}^2 & \bar{\varepsilon} \\
\bar{\varepsilon}^4 & \bar{\varepsilon} & 1
\end{pmatrix}, \quad 
\lambda^E = \begin{pmatrix}
\bar{\varepsilon}^5 & \bar{\varepsilon}^3 & 0 \\
\bar{\varepsilon}^3 & \bar{\varepsilon}^2 & 0 \\
0 & 0 & 1
\end{pmatrix} \tag{1}
\]

These matrices resemble RRR solution 2 \[^3\] and for \(\varepsilon \equiv \bar{\varepsilon}^2\), \(\bar{\varepsilon} = 0.22\), they reproduce the known fermion mass hierarchy. However, the \(23 = 32\) element in the down mass matrix tends to be large, predicting a value for \(|V_{cb}|\) that has to be lowered by the insertion of coefficients or by a cancellation due to phases. On the other hand, one can imagine a different fit, that would naturally lead to a smaller 23 element. This can be achieved by introducing a small parameter \(\delta\) which originates from some flavour independent physics and appears as a factor in all non-renormalisable elements, so that e.g. the down quark mass matrix is modified to

\[
\lambda^D = \begin{pmatrix}
\delta \varepsilon^8 & \delta \varepsilon^3 & \delta \varepsilon^4 \\
\delta \bar{\varepsilon}^3 & \delta \bar{\varepsilon}^2 & \delta \bar{\varepsilon} \\
\delta \bar{\varepsilon}^4 & \delta \bar{\varepsilon} & 1
\end{pmatrix}. \tag{2}
\]

For \(\varepsilon \approx 0.22\) and \(\delta \approx 0.2\), in the down quark mass matrix, one finds that all entries have naturally the correct magnitude to reproduce the desired phenomenology.

However, in order to naturally obtain such a contribution \(\delta\) in addition to the expansion parameter \(\varepsilon\) in the mass matrix entries, one has to go beyond the MSSM. The basic new idea of our approach is to combine the ideas of gauged \(U(1)_X\) family symmetry with unification in the form of an extended vertical gauge symmetry. As a concrete realisation of this idea we consider a specific supersymmetric unified theory \[^6\] that can be derived from a superstring model \[^7\], based on the Pati-Salam gauge group \(SU(4) \otimes SU(2)_L \otimes SU(2)_R\) \[^8\]. We emphasise that the results can be generalised to different groups; the important point is that the quarks and leptons are unified into a common representation, leading to new Clebsch relations. In this particular model, the left-handed quarks and leptons are accommodated in the following representations (see \[^6\] for details)

\[
F_i^{\alpha} = (4, 2, 1), \quad \bar{F}_i^{\dot{\alpha}} = (\bar{4}, 1, 2) \tag{3}
\]
where \( \alpha = 1, \ldots, 4 \) is an SU(4) index, \( a, x = 1, 2 \) are SU(2)_{L,R} indices, and \( i = 1, 2, 3 \) is a family index. The Higgs fields are contained in \( h_a^x = (1, \bar{2}, 2) \) while the two heavy Higgs representations are

\[
H^{ab} = (4, 1, 2), \quad \tilde{H}_{\alpha x} = (\bar{4}, 1, \bar{2})
\]

(4)

The Higgs fields are assumed to develop VEVs \(< H > = < \nu_H > \sim M_{GUT}, < \tilde{H} > = < \bar{\nu}_H > \sim M_{GUT} \), leading to the symmetry breaking at \( M_{GUT} \), \( SU(4) \otimes SU(2)_L \otimes SU(2)_R \rightarrow SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \) in the usual notation. Under this symmetry breaking, the bidoublet Higgs field \( h \) splits into two Higgs doublets \( h_1, h_2 \) whose neutral components subsequently develop weak scale VEVs, \(< h_1^0 > = v_1, < h_2^0 > = v_2 \) with \( \tan \beta \equiv v_2/v_1 \). The relevant part of the superpotential involving matter superfields is

\[
W = \lambda^{ij}_{1} \bar{F}_i h + \lambda^{jk}_{2} \bar{F}_j H \theta_k + \mu h h + \cdots
\]

(5)

where \( \theta_k \) are the superfields associated with the singlets. From Eq.5 we find that the Yukawa couplings satisfy the boundary conditions

\[
\lambda^{ij}_{1}(M_{GUT}) \equiv \lambda^{ij}_{U}(M_{GUT}) = \lambda^{ij}_{D}(M_{GUT}) = \lambda^{ij}_{E}(M_{GUT}) = \lambda^{ij}_{\nu}(M_{GUT}),
\]

(6)

When Eq.(6) is applied to the third family, successful predictions for top, bottom and tau masses are obtained. However these relations do not hold for the lighter families. The standard way forward is to suppose that there is some family symmetry which prohibits the renormalisable terms for the lighter families, but permits them for the third family. The lighter fermion masses are then accounted for by non-renormalisable operators whose order is controlled by the family symmetry.

The non-renormalisable operators which were studied in ref. [1], are formed from different group theoretical contractions of the indices in

\[
O^{apqw}_{\beta \gamma xz} \equiv F^{\alpha a}_{\beta x} h^w_{\gamma z} \bar{H} H^{ow}.
\]

(7)

and are of the form:

\[
O_{ij} \sim (F_i \bar{F}_j) h \left( \frac{H \bar{H}}{M^2} \right)^n + h.c.
\]

(8)

where \( M > M_{GUT} \) (in string models it can be associated with the string scale). When \( H, \bar{H} \) develop their VEVs, such operators will become effective Yukawa couplings of the form \( \bar{F} F h \) with a small coefficient of order \( M_{GUT}^2/M^2 \). These operators were previously written down independently of any particular family symmetry. Now we extend this scheme by introducing a \( U(1)_X \) family symmetry which is broken at a scale \( M_X > M_{GUT} \) by the VEVs of the singlet fields \( \theta \) and \( \bar{\theta} \). To get an idea of the implications, for simplicity we assume that (i) the singlet fields do not couple directly to the matter fields \( F \) and (ii) the the combination \( H \bar{H} \) has a zero quantum number under the symmetry. The new operators are then modified from those in Eq.8:

\[
O_{ij} \sim (F_i \bar{F}_j) h \left( \frac{H \bar{H}}{M^2} \right)^n \left( \frac{\theta^n \bar{\theta}^m}{M^{m+n}} \right) + h.c.
\]

(9)
where again $M'$ represents an energy scale which can be associated with the string scale or the $U(1)_X$ breaking scale. The single power of $(H\bar{H})$ is present in every entry of the matrix and plays the role of the factor $\delta$ in Eq.2, while higher powers of $(H\bar{H})$ are suppressed. In contrast to the MSSM case, here the factor $(H\bar{H})$ carries important group theoretical Clebsch information. In fact Eq.3 amounts to assuming a sort of factorisation of the operators with the family hierarchies being completely controlled by the $\theta, \bar{\theta}$ fields as in IR. Here, $m$ and $n$ are dependent on $i, j$, while the splittings between different charge sectors of the same family are controlled by the Clebsch factors in $(H\bar{H})$. The Clebsch factors have of course a family dependence, i.e. they depend on $i, j$. The relevant $n = 1$ operators, appear in Table 1.

We should comment on the origin of the operators in Eq.4. There are two possible sources of these operators which we can envisage: (1) the operators may emerge directly from the superstring construction; (2) they may be generated from the effective field theory below the string scale. In case (1) the operators will be suppressed by powers of the string scale, while in (2) the operators will be suppressed by powers of the mass of some new heavy states in vector-like representations which mix with the quark and lepton representations. The first possibility will be addressed in [10]. Here, we would only like to remark that, in this case, the larger contributions naturally arise by contracting fields belonging to the same sector of the superstring (Neveu-Schwarz, or Ramond sector), and that the operators that arise in this way, lead to correct phenomenological predictions. As for the second possibility, this would require heavy vector-like representations of matter fields $\psi(x) + \bar{\psi}(-x)$ where the $X$ charges are shown in parentheses. Flavour mixing occurs via insertions of $\theta$ and $\bar{\theta}$ fields leading to the so called spaghetti diagrams recently discussed in the context of the MSSM [11]. In the case of the Pati-Salam model, additional heavy Higgs fields in adjoint representations may generate the operators in Eq.4. An example of a spaghetti diagram which can yield a 23 entry in the Yukawa matrix is shown in Fig.1. Note that the $H\bar{H}$ pair will be in the Pati-Salam representation of the heavy $\Sigma$ field which does not change flavour since it has zero $X$ charge (in parentheses). Flavour changing occurs only in the heavy $\psi, \bar{\psi}$ sector (where these fields have the same Pati-Salam representations as the chiral matter fields $F_i, \bar{F}_j$) via insertions of the $\bar{\theta}$ fields.

Note that Table 1 includes cases of zero Clebsch coefficients, where the contribution to the up-type matrix, for example, is precisely zero. Similarly there are zero Clebsch coefficients for the down-type quarks (and charged leptons). The existence of such zero Clebsch coefficients enables us to obtain textures with different places for zeroes in the up and down mass matrices, something that is not easy in the case that only a flavour symmetry is included in the model. Apart from the zero Clebsch coefficients another advantage of combining quark-lepton unification with a family symmetry is that one can also account for the mass splitting within a particular family, since now the same expansion parameter appears in both the up and down quark mass matrices.

To see how all this occurs, let us take a specific example. Consider the symmetric $n = 1$
Figure 1: A Pati-Salam spaghetti diagram.

Table 1: The $n = 1$ operators with Clebsch coefficients as shown.
operator texture,
\[
\lambda = \begin{pmatrix}
0 & O^M & O^N \\
O^M & O^W + \text{s.d.} & O^N \\
O^N & O^N & O_{33}
\end{pmatrix}
\] (10)

where \(O_{33}\) is the renormalisable operator and \(\text{s.d.}\) stands for a sub-dominant operator with a suppression factor compared to the other dominant operator in the same entry. Substituting the Clebsch coefficients from Table 1 we arrive at the following \(\lambda^{U,D,E}\) Yukawa matrices, at the GUT scale,

\[
\lambda^U = \begin{pmatrix}
0 & 0 & 2\lambda^U_{13} \\
0 & \lambda^U_{22} & 2\lambda^U_{23} \\
2\lambda^U_{13} & 2\lambda^U_{23} & 1
\end{pmatrix},
\lambda^D = \begin{pmatrix}
0 & \sqrt{2}\lambda^D_{12} & 0 \\
\sqrt{2}\lambda^D_{12} & \frac{\sqrt{3}}{2}\lambda^D_{22} & 0 \\
0 & 0 & 1
\end{pmatrix},
\lambda^E = \begin{pmatrix}
0 & \sqrt{2}\lambda^D_{12} & 0 \\
\sqrt{2}\lambda^D_{12} & 3\frac{\sqrt{3}}{5}\lambda^D_{22} & 0 \\
0 & 0 & 1
\end{pmatrix}
\] (11)

where \(\lambda^D_{22}\) and \(\lambda^E_{22}\) arise from the dominant \(O^W_{22}\) operator and \(\lambda^U_{22}\) comes from a sub-dominant operator that is relevant because of the texture zero Clebsch in the up sector of \(O^W_{22}\). The zeroes in the matrices correspond to those of the RRR solution 5, but of course in our case they arise from the Clebsch zeroes rather than from a family symmetry. The numerical values corresponding to RRR solution 5 are,

\[
\lambda^U = \begin{pmatrix}
0 & 0 & 2 \times 10^{-3} \\
0 & 3 \times 10^{-3} & 3 \times 10^{-2} \\
2 \times 10^{-3} & 3 \times 10^{-2} & 1
\end{pmatrix},
\lambda^D = \begin{pmatrix}
0 & 5 \times 10^{-3} & 0 \\
5 \times 10^{-3} & 2 \times 10^{-2} & 0 \\
0 & 0 & 1
\end{pmatrix}
\] (12)

In our model the hierarchy \(\lambda^U_{22} \ll \lambda^D_{22}\) is explained by a Clebsch zero and a suppression factor of the sub-dominant operator. Using Eq.(12) we can read off the values of the couplings which roughly correspond to a unified matrix of effective (and dominant) couplings:

\[
\lambda_{ij} = \begin{pmatrix}
0 & 3 \times 10^{-3} & 1 \times 10^{-3} \\
3 \times 10^{-3} & 2 \times 10^{-2} & 2 \times 10^{-2} \\
1 \times 10^{-3} & 2 \times 10^{-2} & 1
\end{pmatrix}
\] (13)

where we have extracted the Clebsch factors. Thus, the Clebsch factors reproduce the values in Eq.(12) required by phenomenology for the dominant effective couplings displayed in Eq.(13).

The case we are examining is different from the IR analysis in two aspects: (a) the fermion mass matrices of different charge sectors have the same origin, and thus the same expansion parameter and (b) all variations between these sectors arise from Clebsch factors. The structure of the mass matrices is again determined by a family symmetry \(U(1)_X\), but now the charge assignment of the various states are as in Table 2.

| \(U(1)_X\) | \(Q_i\) | \(u^c_i\) | \(d^c_i\) | \(L_i\) | \(e^c_i\) | \(\nu^c_i\) | \(h_1\) | \(h_2\) | \(H\) | \(H\) |
|---|---|---|---|---|---|---|---|---|---|---|
| \(U(1)_X\) | \(\alpha_i\) | \(\alpha_i\) | \(\alpha_i\) | \(\alpha_i\) | \(\alpha_i\) | \(\alpha_i\) | \(-\alpha_3 - \alpha_3\) | \(-\alpha_3 - \alpha_3\) | \(x\) | \(-x\) |

Table 2: \(U(1)_X\) charges for symmetric textures.
The need to preserve $SU(2)_L$ invariance requires left-handed up and down quarks (leptons) to have the same charge. This, plus the additional requirement of symmetric matrices, indicates that all quarks (leptons) of the same $i$-th generation transform with the same charge $\alpha_i$. Finally, lepton-quark unification under $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ indicates that quarks and leptons of the same family have the same charge. The full anomaly free Abelian group involves an additional family independent component, $U(1)_{FI}$, and with this freedom $U(1)_X$ is made traceless without any loss of generality, giving $\alpha_1 = -(\alpha_2 + \alpha_3)$.

If the light Higgs $h_2$, $h_1$, responsible for the up and down quark masses respectively, arise from the same bidoublet $h = (1, 2, 2)$, then they have the same $U(1)_X$ charge so that only the $33$ renormalisable Yukawa coupling to $h_2$, $h_1$ is allowed, and only the $33$ element of the associated mass matrix will be non-zero. The remaining entries are generated when the $U(1)_X$ symmetry is broken, via Standard Model singlet fields, which can be either chiral or vector ones.

In our case, all charge and mass matrices have identical structure under the $U(1)_X$ symmetry, since all known fermions are accommodated in the same multiplets of the gauge group. The charge matrix is of the form

$$\begin{pmatrix}
-2\alpha_2 - 4\alpha_3 & -3\alpha_3 & -\alpha_2 - 2\alpha_3 \\
-3\alpha_3 & 2(\alpha_2 - \alpha_3) & \alpha_2 - \alpha_3 \\
-\alpha_2 - 2\alpha_3 & \alpha_2 - \alpha_3 & 0
\end{pmatrix}$$

Then, including the common factor $\delta$ which multiplies all non-renormalisable entries, the following hierarchy of dominant effective Yukawa couplings is obtained

$$\lambda \approx \begin{pmatrix}
\delta \epsilon^{[2+6a]} & \delta \epsilon^{[3a]} & \delta \epsilon^{[1+3a]} \\
\delta \epsilon^{[3a]} & \delta \epsilon^2 & \delta \epsilon^{1+3a} \\
\delta \epsilon^{1+3a} & \delta \epsilon & 1
\end{pmatrix},$$

where we have used a vector-like pair of $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ singlets to Higgs the family symmetry. Here $\epsilon = (< \theta > /M)^{[\alpha_2 - \alpha_3]}$ where $M$ is the unification mass scale which governs the higher dimension operators, while a unique charge combination $a = \alpha_3/(\alpha_2 - \alpha_3)$ appears in the exponents of all matrices, as a result of quark-lepton unification. In Eq.[13] the flavour independent suppression factor $\delta = \langle H \bar{H} \rangle /M^2$ results from the operators in Eq.[10]. The factor of $\delta$ actually helps the numerical fit and setting $a = 1$, $\epsilon = 0.22$ and $\delta \approx 0.2$ successfully reproduces the desired values of the elements of the Yukawa matrix in Eq.[13]. Thus, starting with all of the “fundamental” Yukawa couplings\(^1\) $\sim 1$, we can reproduce the correct phenomenology of charged fermion masses and mixings.

Let us now look at the case of non-symmetric textures\(^2\), with an additional zero in the $23$ entry. It turns out\([9, 10]\) that it is optimal to consider a scheme in which the dominant operators in the Yukawa matrix are $O_{33}$, $O_{32}^C$, $O_{22}^W$, $O_{21}$, $\tilde{O}_{21}$ and $O_{12}$, where the last three operators are left general and will be specified later. We are also aware from the analysis in ref.\([9]\) that $O_{12}$ must have a zero Clebsch coefficient in the up sector. A combination of two operators

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\(^1\)i.e. dimensionless couplings in the full $SU(4) \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_X$ gauge theory.

\(^2\)Asymmetric textures have been discussed in a different framework in\([12]\).
must then provide a non-zero $O_{21}$ entry to generate a big enough $V_{ub}$, while an additional much more suppressed operator elsewhere in the Yukawa matrix gives the up quark a small mass. At $M_{GUT}$, the Yukawa matrices that we can form from the operators listed above, are of the form

$$\lambda' = \begin{bmatrix} 0 & H_{12} e^{i\phi_{21}} x_{12}^I & 0 \\ H_{21} x_{21}^I e^{i\phi_{21}} + \tilde{H}_{21} \bar{x}_{21}^I e^{i\phi_{21}} & 0 & H_{12} e^{i\phi_{12}} x_{12}^I \\ 0 & H_{22} x_{22}^I e^{i\phi_{22}} & 0 \\ H_{32} x_{32}^I e^{i\phi_{32}} & H_{33} e^{i\phi_{33}} & 0 \end{bmatrix},$$  

\hspace{1em} (16)

where only the dominant operators are listed and complex phases have been taken into account. The $I$ superscript labels the charge sector and $x_{ij}^I$ refers to the Clebsch coefficient relevant to the charge sector $I$ in the $ij$th position. $\phi_{ij}$ are unknown phases and $H_{ij}$ is the magnitude of the effective dimensionless Yukawa coupling in the $ij$th position. Any subdominant operators that we introduce will be denoted below by a prime and it should be borne in mind that these will only affect the up matrix. So far, the known Clebsch coefficients are

$$x_{12}^U = 0, \quad x_{22}^U = 0, \quad x_{22}^D = 1, \quad x_{22}^E = -3, \quad x_{32}^U = -1, \quad x_{32}^E = -3.$$  

\hspace{1em} (17)

We have just enough freedom in rotating the phases of $F_{1,2,3}$ and $\bar{F}_{1,2,3}$ to get rid of all but one of the phases in Eq.\hspace{1em}16. When the subdominant operator is added, the Yukawa matrices are

$$\lambda^U = \begin{bmatrix} 0 & H_{12} e^{i\phi_{21}} & 0 \\ H_{21} x_{21}^I e^{i\phi_{21}} & 0 & H_{12} e^{i\phi_{12}} \\ 0 & H_{32} x_{32}^I & H_{33} \end{bmatrix},$$

$$\lambda^D = \begin{bmatrix} 0 & H_{12} x_{12}^D & 0 \\ H_{21} x_{22}^D & 0 & H_{12} x_{12}^D \\ 0 & H_{32} x_{32}^D & H_{33} \end{bmatrix},$$

$$\lambda^E = \begin{bmatrix} 0 & H_{12} x_{12}^E \hspace{1em} 0 \\ H_{21} x_{22}^E \hspace{1em} 0 & H_{12} x_{12}^E \\ 0 & H_{32} x_{32}^E & H_{33} \end{bmatrix}. $$ \hspace{1em} (18)

where we have defined

$$H_{21}^{U} e^{i\phi_{21}} \equiv H_{21} x_{21}^U e^{i\phi_{21}} + \tilde{H}_{21} \bar{x}_{21}^U e^{i\phi_{21}}$$

$$H_{21}^{D,E} \equiv H_{21} x_{21}^{D,E} e^{i\phi_{21}} + \tilde{H}_{21} \bar{x}_{21}^{D,E} e^{i\phi_{21}}$$  \hspace{1em} (19)

We may now remove $\phi_{22}'$ by phase transformations upon $\bar{F}_{1,2,3}$ but $\phi_{21}^U$ may only be removed by a phase redefinition of $F_{1,2,3}$, which would alter the prediction of the CKM matrix $V_{CKM}$. Thus, $\phi_{21}^U$ is a physical phase, that is it cannot be completely removed by phase rotations upon the fields. Once the operators $O_{21}, \bar{O}_{21}, O_{12}$ have been chosen, the Yukawa matrices at $M_{GUT}$ including the phase in the CKM matrix are therefore identified with $H_{ij}, H_{22}', \phi_{21}^U$.
The MS scheme. The relevant renormalisation group equations (RGEs) are listed in ref. \[9\].

We obtain the best fit. The result appears in Table 3.

This type of non-symmetric texture can be described by a structure of the kind,

\[
\lambda = \begin{pmatrix}
\delta \epsilon_{big} & \delta \epsilon_3 & \delta \epsilon_{big} \\
\delta \epsilon_3 & \delta \epsilon^{lor2} & \delta \epsilon_{big} \\
\delta \epsilon_{big} & \delta \epsilon & 1
\end{pmatrix}
\]

(20)

where we identify $\epsilon \equiv \lambda = 0.22$ and set $\delta \approx 0.1$.

Let us numerically analyse the non-symmetric case we have been discussing, since it goes beyond solutions that have been discussed so far in the literature. This is done by performing a global fit of possible textures, arising from different operator assignments, to $m_e, m_\mu, m_u, m_c, m_t, m_d, m_s, m_b, \alpha_S(M_Z), |V_{ub}|, |V_{cb}|$ and $|V_{us}|$ using $m_\tau$ as a constraint. The values of the 8 parameters introduced in Eq.\[18\] ($\phi_{21}^o \equiv \phi$, $H_{21}^0 \equiv H_{21}'$, $H_{21}^D \equiv H_{21}$, $H_{22}'$, $H_{22}$, $H_{12}$, $H_{32}$, $H_{33}$), plus $\alpha_S$ at the GUT scale are determined by the fit.

The matrices $\lambda^I$ are diagonalised numerically and $|V_{ub}(M_{GUT})|, |V_{us}(M_{GUT})|$ are determined by $V_{CKM} = V_U V_D^\dagger$, where $V_U, V_D$ are the matrices that act upon the $(u, c, t)_L$ and $(d, s, b)_L$ column vectors respectively to transform from the weak eigenstates to the mass eigenstates of the quarks. We use the boundary conditions $\alpha_1(M_{GUT}) = \alpha_2(M_{GUT}) = 0.708$, while $\lambda_{u,c,t,d,s,b,e,\mu,\tau}$, $|V_{us}|$ and $|V_{ub}|$ are run\[1\] from $M_{GUT}$ to 170 GeV$\approx m_t$ using the RGEs for the MSSM. The $\lambda_i$ are evolved to their empirically derived running masses using 3 loop QCD$\otimes$1 loop QED \[3\]. $m_\tau^e$ and $\lambda^\mu_h(m_\tau)$ then\[3\] fix $\tan \beta$ through the relation \[13\] $\cos \beta = \frac{\sqrt{2} m_\tau^e}{\alpha_H^\mu(h)}$, where $v = 246.22$ GeV is the VEV of the Standard Model Higgs. There are twelve data points and nine parameters so we have three degrees of freedom (dof). The parameters are all varied until the global $\chi^2$/dof is minimised. The data used (with 1$\sigma$ errors quoted) is taken from \[13\]. For

\[
O_{12} \equiv O^R, \quad O_{21} + \tilde{O}_{21} \equiv O^M + O^A
\]

(21)

we obtain the best fit. The result appears in Table \[3\].
Out of 16 possible models that fit the texture required by Eqs.17, 11 models fit the data with $\chi^2/\text{dof} < 3$, 5 with $\chi^2/\text{dof} < 2$ and 3 with $\chi^2/\text{dof} < 1$. The operators listed as $O_{12}, O_{21}, \tilde{O}_{21}$ describe the structure of the models and the entries $H_{22}, H_{12}, H_{21}, \cos\phi, H_{33}, H_{22}', H_{21}'$ are the GUT scale input parameters of the best fit values of the model. The estimated 1σ deviation in $\alpha_S(M_Z)$ from the fits is $\pm 0.003$ and the other parameters are constrained to better than 1% apart from $\cos\phi$, whose 1σ fit errors often cover the whole possible range. We conclude that the $\chi^2$ test has some discriminatory power, since if all of the models were equally good, we would statistically expect to have 11 models with $\chi^2/\text{dof} < 1$, 3 models with $\chi^2/\text{dof} = 1 - 2$ and 2 models with $\chi^2 = 2 - 3$ out of the 16 tested.

Summarising, we have combined the idea of a gauged $U(1)_X$ family symmetry with that of quark-lepton unification within the framework of a string-inspired Pati-Salam model. The non-renormalisable operators are composed of a factor $(H\bar{H})$ and a factor involving the singlet fields $\theta, \bar{\theta}$ as in Eq.9. The singlet fields $\theta, \bar{\theta}$ break the $U(1)_X$ symmetry and provide the horizontal family hierarchies while the $H, \bar{H}$ fields break the $SU(4)\otimes SU(2)_L\otimes SU(2)_R$ symmetry and give the vertical splittings arising from group theoretic Clebsch relations between different charge sectors. We have studied both symmetric and asymmetric mass textures. In the first case, as an example, we have shown how one of the RRR textures can be reproduced by the structure in Eq.[15], where $\delta$ results from the factor $\langle H\bar{H}\rangle/M^2$ and $\epsilon$ results from the factor $(<\theta>/M)^{|\alpha_2 - \alpha_3|}$ in the operators of Eq.[10]. This results to an accurate reproduction of RRR solution 5 which cannot be reproduced by family symmetry alone. The Giudice ansatz [14] (RRR solution 3), which also cannot normally be achieved, can similarly be reproduced by our scheme. Our approach has also been extended to non-symmetric textures, which are motivated by particular superstring constructions. For these, we performed a numerical analysis since they are beyond the symmetric RRR regime and found that from the possible textures, one of them allows a low energy fit to the data, with a total $\chi^2$ of 0.39 for three degrees of freedom. All the textures that we present, rely on the existence of Clebsch zeroes, which are a feature of the operators in our model. Moreover, the combination of family symmetry and unification has the additional advantage that quark and lepton masses are related by Clebsch factors, as in the Georgi-Jarlskog scheme, which improves the overall predictive power of the approach. More detailed issues, including the superstring construction itself, are deferred to a longer publication [11].
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