ON DERIVED CATEGORIES OF ARITHMETIC TORIC VARIETIES

MATTHEW R BALLARD, ALEXANDER DUNCAN, AND PATRICK K. MCFADDIN

Abstract. We begin a systematic investigation of derived categories of smooth projective toric varieties defined over an arbitrary base field. We show that, in many cases, toric varieties admit full exceptional collections. Examples include all toric surfaces, all toric Fano 3-folds, some toric Fano 4-folds, the generalized del Pezzo varieties of VoskresenskiĂ and Klyachko, and toric varieties associated to Weyl fans of type $A$. Our main technical tool is a completely general Galois descent result for exceptional collections of objects on (possibly non-toric) varieties over non-closed fields.

1. Introduction

Recently, several intriguing threads relating derived categories and arithmetic geometry have emerged and motivated general structure questions for $k$-linear triangulated categories for arbitrary fields $k$. Such exploration has yielded many nice applications as well as further enticing problems, see as a sampling [AKW17][AAGZ13][ADPZ15][HT14][Hon15][LMS14]. Meanwhile over $\mathbb{C}$, structural results for derived categories seem to have deep implications for the underlying birational geometry, e.g. [AT14][ABB14][BB13][BMMS12][Kuz10][Via17]. Taking these together, derived categories become an important invariant for studying birational geometry over a general field [AB15]. A further benefit of this noncommutative approach is direct utility for solving problems in algebraic $K$-theory, for example [MP97].

With such tantalizing ties, one would like a fertile testing ground for questions. In this paper, we begin a systematic study of one such area: derived categories of arithmetic toric varieties. This area has the following nice features:

- rationality issues are deep in general but tractable in examples,
- robust tools already exist to investigate derived categories over the separable closure,
- and specific questions are often amenable to computational experimentation.

One of the best tools for understanding a derived category is an exceptional collection consisting of exceptional objects. As originally conceived in [Bej78], an exceptional object of a $k$-linear derived category is one whose endomorphism algebra is isomorphic to the base field $k$. When $k$ is not algebraically closed, this definition is too restrictive and instead we use the existing notion: an object of $D^b(X)$ is exceptional if its endomorphism algebra is a division algebra. Details are discussed in Section 2 below.

We illustrate this more general notion for two arithmetic toric varieties. The real conic $X = \{x^2 + y^2 + z^2 = 0\} \subset \mathbb{P}^2_\mathbb{R}$ has an exceptional collection $\{O_X, F\}$, where $\text{End}(F)$ is isomorphic to the quaternion algebra $\mathbb{H}$. Over $\mathbb{C}$, we have $X_\mathbb{C} \cong \mathbb{P}_\mathbb{C}^1$ and $F \otimes_{\mathbb{R}} \mathbb{C} \cong O(1)^{\otimes 2}$. As another example, consider the Weil restriction $Y$ of $\mathbb{P}_\mathbb{C}^1$ over $\mathbb{R}$ (“$\mathbb{P}_\mathbb{C}^1(\mathbb{C})$ viewed as an $\mathbb{R}$-variety”). Here $Y$ has an exceptional collection $\{O_Y, G, H\}$ where $\text{End}(G) \cong \mathbb{C}$ and $\text{End}(H) \cong \mathbb{R}$. Over $\mathbb{C}$, we have $Y \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{P}^1 \times \mathbb{P}^1$ with $G \otimes_{\mathbb{R}} \mathbb{C} \cong O(1,0) \oplus O(0,1)$ and $H \otimes_{\mathbb{R}} \mathbb{C} \cong O(1,1)$, where $O(i,j) = \pi_1^*O(i) \otimes \pi_2^*O(j)$.
A central question for derived categories of arithmetic toric varieties is the following:

**Question 1.1.** Let $X$ be a smooth projective toric variety over an arbitrary field. Does $X$ admit a full exceptional collection? If so, does $X$ possess a full exceptional collection of sheaves?

Over an algebraically-closed field of characteristic zero, there is always a full exceptional collection of objects [Kaw06, Kaw13] while the question of a full exceptional collection of sheaves is due to Orlov. Making allowances for different language, the question is also known to have a positive answer for Severi-Brauer varieties [AB15, Ber09], minimal toric surfaces [BSS11], and smooth projective toric varieties with absolute Picard rank at most 2 [Yan14].

In this article, we provide further evidence for a positive answer to Question 1.1, treating cases with low dimension or a high degree of symmetry.

**Theorem 1.2.** The following possess full exceptional collections of sheaves:

- smooth toric surfaces (Proposition 4.7),
- smooth toric Fano 3-folds (Proposition 4.11),
- all forms of 43 of the 124 smooth split toric Fano 4-folds (Section 4.3),
- all forms of centrally symmetric toric Fano varieties (Corollary 4.13), and
- all forms in characteristic zero of toric varieties corresponding to Weyl fans of root systems of type $A$ (Proposition 4.21).

Our results leverage extant work in the algebraically closed case such as [Ueh14] for 3-folds and [PN17] for 4-folds. We use Castravet and Tevelev’s recently discovered exceptional collection for $X(A_n)$ [CT17]. For the centrally symmetric toric Fano varieties (which are products of “generalized del Pezzo varieties” and projective lines [VK84]), we use an explicit exceptional collection closely related to the one found in [CT17]. Up to a twist by a line bundle, the authors had independently discovered the exact same collection! This suggests that symmetry imposes strong conditions on the possible exceptional collections, which paradoxically makes them easier to find.

To study arithmetic exceptional collections, we establish an effective Galois descent result for general exceptional collections. This applies to general varieties, not just toric ones.

**Theorem 1.3** (Theorem 2.17, Lemma 2.19). Let $X$ be a $k$-scheme and $L/k$ a $G$-Galois extension. Then $X_L$ admits a full (resp. strong) $G$-stable exceptional collection of objects of $\mathcal{D}^b(X_L)$ (resp. sheaves, resp. vector bundles) if and only if $X$ admits a full (resp. strong) exceptional collection of objects of $\mathcal{D}^b(X)$ (resp. sheaves, resp. vector bundles).

We highlight one corollary of a positive answer to Question 1.1. Arithmetic toric varieties are also studied in [MP97], which focused on computing their algebraic $K$-groups via decompositions in a certain noncommutative motivic category of $K_0$-correspondences. They showed that for an arithmetic toric $k$-variety $X$ with $G = \text{Gal}(k^s/k)$, the group $K_0(X_{k^s})$ is a direct summand of a permutation $G$-module (there exists a $\mathbb{Z}$-basis permuted by $G$).

**Question 1.4** (Merkurjev-Panin [MP97]). Let $X$ be an arithmetic toric variety over $k$ and $G = \text{Gal}(k^s/k)$. Is $K_0(X_{k^s})$ always a permutation $G$-module?

Question 1.1 can be viewed as a categorification of Question 1.4 as any such exceptional collection over $k$ immediately gives a permutation basis.
In order to show that every toric variety has a full exceptional collection over $\mathbb{C}$, the main tool used in [Kaw06, Kaw13] was the minimal model program (MMP) in birational geometry. The basic building blocks are toric stacks with Picard rank one, which always have full strong exceptional collections of line bundles. Indeed, runs of the MMP can be leveraged to effectively produce exceptional collections [BFKL17].

Over a non-closed field, one hopes to use the Galois-equivariant MMP, but the situation is more complicated. The most basic building blocks in this framework are those varieties $X$ which have $\rho^G = \text{rank}(\text{Pic}(X)^G) = 1$. Based on the results above and the hope of using the MMP in the arithmetic situation, we ask the following question in the vein of [Kin97, BH09, CMR10]:

**Question 1.5.** Let $X$ be a smooth toric $k$-variety and $L/k$ a $G$-Galois splitting field. If $\text{Pic}(X_L)^G$ is of rank 1, does $X_L$ possess a full strong $G$-stable exceptional collection consisting of line bundles?

**Acknowledgements.** The first author was partially supported by NSF DMS-1501813. He would also like to thank the Institute for Advanced Study for providing a wonderful research environment. These ideas were developed during his membership. The first author benefitted from discussions with Alicia Lamarche. The second author was partially supported by NSA grant H98230-16-1-0309. The third author would like to thank the Hausdorff Institute for their hospitality and lively research environment during the *K-theory and related fields* trimester program. A large portion of this manuscript was drafted during his time in Bonn. All authors also thank Fei Xie for pointing out that, due to an editing error, in a previous version of this paper, Proposition 4.7 stated that all smooth toric surfaces have strong collections of vector bundles instead of a not-necessarily strong collection of sheaves as was proven.

**Organization.** Section 2 treats Galois descent of exceptional collections consisting of objects on (possibly non-toric) varieties. In Section 3 we recall appropriate definitions of arithmetic toric varieties and establish additional descent results which are specific to toric varieties. In Section 4 we consider a range of examples. We begin by treating toric surfaces, followed by toric Fano 3-folds. For toric Fano 4-folds, we give partial results. We conclude by investigating the class of centrally symmetric toric Fano varieties, including the generalized del Pezzo varieties, and handling toric varieties associated to root systems of type $A$.

**Notation.** Throughout, $k$ denotes an arbitrary field and $k^s$ a separable closure. A *variety* is a geometrically integral separated scheme of finite type over $k$. All our schemes will be quasi-compact and quasi-separated. For a $k$-scheme $X$ and field extension $L/k$, we write $X_L := X \times_{\text{Spec } k} \text{Spec } L$. If $A$ is a $k$-algebra, we write $A_L = A \otimes_k L$. We use $\mathcal{D}^b(X)$ to denote the bounded derived category $\mathcal{D}^b(\text{Coh}(X))$ and we use $\mathcal{D}(X)$ to denote the derived category $\mathcal{D}(\text{Qcoh}(X))$. For an $\mathcal{O}_X$-algebra $A$, we write $\mathcal{D}^b(A)$ for the bounded derived category of complexes of $A$-modules which are coherent $\mathcal{O}_X$-modules.

2. Galois descent and exceptional collections

In this section we develop Galois descent for exceptional collections (in a generalized sense). We begin by recalling some definitions and conventions concerning structure theory of derived categories of schemes. We then give our main descent results for $G$-stable
exceptional collections (Theorem 2.17). We complete the section by collecting a few useful consequences to be used in the sequel.

2.1. Exceptional collections. We give some conventions for semiorthogonal decompositions of derived categories and in particular exceptional collections. Such collections have been widely studied over algebraically closed fields but have recently been treated in more generality \[ \text{[AAGZ13, AB15, ABB14, Ber09, BSS11, Ela09, Xie17, Yan14]} \]. We refer the reader to Remarks 2.15 and 2.18 for added details on some of these results.

For a triangulated category \( T \), we use the standard notation \( \text{Ext}^n_T(A, B) = \text{Hom}_T(A, B[n]) \). For objects \( A, B \) of \( \mathcal{D}^b(X) \), we use \( \text{End}_X(A) \) and \( \text{Ext}_X^n(A, B) \) to denote \( \text{End}_{\mathcal{D}^b(X)}(A) \) and \( \text{Ext}^n_{\mathcal{D}^b(X)}(A, B) \), respectively.

**Definition 2.1** (see \[ \text{[BK89]} \]). Let \( T \) be a triangulated category. A full triangulated subcategory of \( T \) is admissible if its inclusion functor admits left and right adjoints. A semiorthogonal decomposition of \( T \) is a sequence of admissible subcategories \( C_1, \ldots, C_s \) such that

1. \( \text{Hom}_T(A_i, A_j) = 0 \) for all \( A_i \in \text{Ob}(C_i), A_j \in \text{Ob}(C_j) \) whenever \( i > j \).
2. For each object \( T \) of \( T \), there is a sequence of morphisms \( 0 = T_s \to \cdots \to T_0 = T \) such that the cone of \( T_i \to T_{i-1} \) is an object of \( C_i \) for all \( i = 1, \ldots, s \).

We use \( T = \langle C_1, \ldots, C_s \rangle \) to denote such a decomposition.

Particularly nice examples of semiorthogonal decompositions are given by exceptional collections, the study of which goes back to Beilinson \[ \text{[Bel78]} \].

**Definition 2.2.** Let \( T \) be a \( k \)-linear triangulated category. An object \( E \) in \( T \) is exceptional if the following conditions hold:

1. \( \text{End}_T(E) \) is a division \( k \)-algebra.
2. \( \text{Ext}_T^n(E, E) = 0 \) for \( n \neq 0 \).

A totally ordered set \( E = \{E_1, \ldots, E_s\} \) of exceptional objects is an exceptional collection if \( \text{Ext}_T^n(E_i, E_j) = 0 \) for all integers \( n \) whenever \( i > j \). An exceptional collection is full if it generates \( T \), i.e., the smallest thick subcategory of \( T \) containing \( E \) is all of \( \mathcal{D}^b(X) \). An exceptional collection is strong if \( \text{Ext}_T^n(E_i, E_j) = 0 \) whenever \( n \neq 0 \). An exceptional block is an exceptional collection \( E = \{E_1, \ldots, E_s\} \) such that \( \text{Ext}_T^n(E_i, E_j) = 0 \) for every \( n \) whenever \( i \neq j \). Given an exceptional collection \( E = \{E_1, \ldots, E_s\} \), we denote by \( \langle E \rangle \) the category generated by the objects \( E_i \).

**Remark 2.3.** Our notion of exceptional object generalizes the classical one, where item (1) of Definition 2.2 is replaced by: \( \text{End}_T(E) = k \) \[ \text{[Bon89], §2} \]. Over algebraically or separably closed fields, these definitions agree. Over non-closed fields, the classical definition is too restrictive to allow for the use of interesting arithmetic invariants in the study of exceptional collections on twisted forms, e.g., Brauer classes.

**Proposition 2.4** (Thm. 3.2 \[ \text{[Bon89]} \]). Let \( X \) be a \( k \)-scheme with exceptional collection \( \{E_1, \ldots, E_s\} \). If \( \mathcal{E}_i \) is the category generated by \( E_i \), there is a semiorthogonal decomposition \( \mathcal{D}^b(X) = \langle \mathcal{E}_1, \ldots, \mathcal{E}_s, A \rangle \), where \( A \) is the full subcategory with objects \( A \) such that \( \text{Hom}_{\mathcal{X}}(A, E_i) = 0 \) for all \( i \).

**Remark 2.5.** The reference assumes smoothness and projectivity but the conclusion is independent of this. Note further that if \( X \) admits a full exceptional collection then it is automatically smooth and proper by \[ \text{[Orl16], Propositions 3.30 and 3.31} \].
The existence of an exceptional collection on a scheme $X$ provides a means of studying derived geometry of $X$ in purely algebraic terms. Indeed, in such a situation, one may identify an “underlying” $k$-algebra which is derived equivalent to $X$. For exceptional blocks, one obtains a similar but slightly stronger fact.

**Proposition 2.6** (Thm. 6.2 [Bon89]). Let $X$ be a smooth projective $k$-scheme and let $\{E_1, \ldots, E_s\}$ be a full strong exceptional collection on $D^b(X)$. Then $R\text{Hom}_{D^b(X)}(E, -) : D^b(X) \to D^b(A)$ is a $k$-linear equivalence.

**Proposition 2.7.** If $E = \{E_1, \ldots, E_s\}$ is an exceptional block with $\text{End}(E_i) = D_i$, there is a $k$-algebra isomorphism $\text{End}(\oplus E_i) \simeq D_1 \times \cdots \times D_s$, and hence a $k$-linear equivalence $\langle E \rangle \simeq D^b(D_1 \times \cdots \times D_n)$.

The object $E = \oplus E_i$ of Proposition 2.6 is usually called a tilting object. If each $E_i$ is a sheaf (resp. vector bundle), then $E$ is called a tilting sheaf (resp. tilting bundle). Until recently, the theory of tilting objects has served as the main tool for extending the study of exceptional collections to non-closed fields. The results above show that any exceptional collection gives rise to both a tilting object and a semiorthogonal decomposition, and thus the admission of such a collection is a particularly special property of a given triangulated category. Our aim in the following subsection is to extend descent results for semiorthogonal decompositions and tilting objects to (our more general notion of) exceptional collections. We give a formal definition of tilting object for completeness.

**Definition 2.8.** A tilting object for a $k$-scheme $X$ is an object $E$ of $D^b(X)$ which satisfies the following conditions:

1. $\text{Ext}^n_X(E, E) = 0$ for $n > 0$.
2. $E$ generates $D^b(X)$.

**Remark 2.9** ($K$-theory and motivic decompositions). Exceptional collections have a particularly interesting manifestation in the realm of noncommutative motives. Indeed, an exceptional collection $\{E_1, \ldots, E_s\}$ on a smooth projective variety $X$ yields a decomposition $U(X) \simeq \bigoplus_i U(D_i)$ of its corresponding universal additive invariant [Tab15, §2.3], where $D_i = \text{End}(E_i)$. This defines a motivic decomposition by viewing $X$ as an object in the Merkurjev-Panin category of $K$-motives [MP97] or Kontsevich’s category of noncommutative Chow motives [Tab14, Thm. 6.10] via its associated dg-category of perfect complexes.

One nice consequence is that this decomposition is detected by algebraic $K$-groups [AB15, Prop. 1.10] in addition to a slew of other additive invariants in the sense of Tabuada [Tab15, §2.2]. Such invariants include algebraic $K$-theory with coefficients, homotopy $K$-theory, étale $K$-theory, (topological) Hochschild homology, and (topological) cyclic homology.

2.2. Galois descent. We develop Galois descent for exceptional collections consisting of objects in the derived category $D^b(X)$ of a (smooth projective) variety $X$. Throughout this section, pushforward and pullback functors are understood to be derived. For a $k$-scheme $X$ and finite Galois extension $L/k$, any element $g \in \text{Gal}(L/k)$ defines a morphism of $k$-schemes $g : X_L \to X_L$ which in turn defines the functor $g^* : D^b(X_L) \to D^b(X_L)$.

**Definition 2.10.** Let $X$ be a scheme with an action of a group $G$. A $G$-stable exceptional collection on $X$ is an exceptional collection $E = \{E_1, \ldots, E_s\}$ of objects in $D^b(X)$ such that for all $g \in G$ and $1 \leq i \leq s$ there exists $E \in E$ such that $g^*E_i \simeq E$. We say a $G$-stable...
exceptional collection $E$ is a $G$-orbit if, for every pair of objects $E, E' \in E$, there exists a $g \in G$ such that $g^*E \simeq E'$.

**Remark 2.11.** A simple example of a $G$-stable exceptional collection is a $G$-invariant exceptional collection, i.e., an exceptional collection $\{E_1, \ldots, E_s\}$ such that $g^*E_i \simeq E_i$ for all $1 \leq i \leq s$. It is often the case that toric varieties admit exceptional collections consisting of line bundles. If it is also the case that a group $G$ acts trivially on $\text{Pic}(X)$, such a collection is automatically $G$-invariant, and hence $G$-stable (see Lemma 2.20).

**Lemma 2.12.** Any $G$-stable exceptional collection may be written as a collection of $G$-stable exceptional blocks (after possibly reordering).

**Proof.** The decomposition of a $G$-stable exceptional collection into its $G$-orbits gives the desired exceptional blocks. Let $E$ be a $G$-stable exceptional collection and for elements $E, E' \in E$, we write $E \sim E'$ if $\text{Ext}^n(E, E') \neq 0$ for some $n$.

Let $A \subset E$ be a $G$-orbit. To see that $A$ is an exceptional block, suppose that $E \sim E'$ for $E, E' \in A$. Since $A$ is a $G$-orbit, $E' \simeq g^*E$ for some $g \in G$. Thus, $E \sim g^*E$, and acting again by $g$, we have $g^*E \sim (g^2)^*E$. Since $A$ is finite, we have $E \sim g^*E \sim \cdots \sim (g^s)^*E \sim E$ for some positive integer $s$. Thus, there is no ordering of the elements of $A$ such that they form a subset of an exceptional collection — a contradiction.

If $B$ is another $G$-orbit (distinct from $A$), we would like to see that these blocks can be ordered to form an exceptional collection. We claim that for any $E \in A$ and $F \in B$, one has $E \sim F$ only if $A$ precedes $B$ in the collection $E$ (i.e., $\text{Ext}^n(B, A) = 0$ for all $n$ and all $A \subset B$). To see this, assume that $E \sim F$ and $E \sim E'$ for some $E' \in A$. As $A$ is a $G$-orbit, $E' \simeq g^*E$ for some $g \in G$. Hence, just as above, we have a sequence $E \sim g^*E \sim \cdots \sim (g^s)^*E \sim E$. Thus, there is no ordering of the elements of $A$ and $B$ which forms an exceptional collection, contradicting the exceptionality of $E$. □

**Lemma 2.13.** Let $X$ be a $k$-scheme, $L/k$ a finite Galois extension with group $G$, and $\pi : X_L \to X$ the natural projection map. For any object $M$ in $\mathcal{D}(X_L)$ there is a natural isomorphism $\pi^*\pi_*(M) \simeq \bigoplus_{g \in G} g^*M$.

**Proof.** For any object $M$ of $\mathcal{D}(X_L)$, we have $\pi_*M \simeq \pi_*g^*M$, and adjunction yields a natural transformation $\pi^*\pi_* \to g^*$. Summing over all $g \in G$ provides the transformation $\alpha : \pi^*\pi_* \to \oplus_{g \in G} g^*$ and we show this is an isomorphism. Let $\mathcal{A}$ be the full subcategory of $\mathcal{D}(X_L)$ consisting of objects $A$ so that $\alpha_A$ is an isomorphism. Then $\mathcal{A}$ is triangulated and closed under coproducts. By [Nee96, Prop. 2.5], it suffices to show that $\mathcal{A}$ contains all perfect complexes (i.e., those complexes which are locally quasi-isomorphic to a bounded complex of finitely generated locally free $O_{X_L}$-modules). Since checking $\alpha$ is a quasi-isomorphism can be done Zariski-locally, we reduce the question to showing that $\mathcal{A}$ contains all bounded complexes of locally-free sheaves. We may further reduce to the case of bounded complexes of free sheaves of finite rank. A map of such complexes which is a component-wise isomorphism is in particular a quasi-isomorphism. The map $\alpha$ is clearly an isomorphism for free sheaves, and we conclude that $\mathcal{A} = \mathcal{D}(X_L)$. □

**Proposition 2.14** (Descent for orbits). Let $X$ be a $k$-scheme, $L/k$ a finite $G$-Galois extension, and $\pi : X_L \to X$ the natural projection map. If $E = \{E_1, \ldots, E_s\}$ is a $G$-orbit forming an exceptional collection on $X_L$, and if $E$ is any element of $E$, then there is an exceptional object $F$ in $\mathcal{D}^b(X)$ such that $\pi_*E \simeq F^{\oplus m}$ and $\pi^*F$ generates the category $\langle E \rangle$. 
Proof. By Lemma 2.12, exceptional \( G \)-orbits are completely orthogonal (and by definition carry a transitive action of \( G \)), which will be used throughout the proof. Fix an element \( E \in E \), so that \( E = E_i \) for some \( i \). Lemma 2.13 gives
\[
\pi^* \pi_* E \cong \bigoplus_{g \in G} g^* E
\]
We claim that \( \text{End}(\pi_* E) \) is a matrix algebra over a division algebra, and prove this by first showing that it is semisimple. Indeed, using \( \text{End}_X(M) \otimes_k L \cong \text{End}_{L^*}(\pi^* M) \) for any \( M \in D^b(X) \) [AB15, Rem. 2.1], we have
\[
\text{End}_X(\pi_* E) \otimes_k L \cong \text{End}_{L^*}(\pi^* \pi_* E) \cong \text{End}_{L^*} \left( \bigoplus_{g \in G} g^* E \right).
\]
Each \( g^* E \) is exceptional so that \( \text{End}_{L^*}(g^* E) =: D_g \) is a division algebra for each element \( g \in G \). Let \( H \leq G \) be the subgroup consisting of elements \( h \) satisfying \( h^* E \cong E \). For any system of coset representatives \( g \in G/H \), we have \( \text{End}_X(\pi_* E)_L \cong \bigoplus_{g \in G/H} M_m(D_g) \), where \( m = |H| \). This direct sum of matrix algebras over division algebras is semisimple, i.e., the Jacobson radical \( \text{rad}(\text{End}_X(\pi_* E)_L) = 0 \). We then have \( 0 = \text{rad}(\text{End}_X(\pi_* E)_L) = \text{rad}(\text{End}_X(\pi_* E))_L \) by [Ami58, Thm. 1], and hence \( \text{rad}(\text{End}_X(\pi_* E)) = 0 \). Thus, \( \text{End}_X(\pi_* E) \) is semisimple and so must also be a direct sum of matrix algebras over division algebras by the Artin-Wedderburn Theorem.

Let \( Z \) be the center of \( \text{End}_X(\pi_* E) \) and \( Z_L \) the center of \( \text{End}_X(\pi_* E)_L \). Note that \( Z \) is an étale \( k \)-algebra, and to show that \( \text{End}(\pi_* E) \) is a matrix algebra, it suffices to show that \( Z \) has no zero divisors, and is thus a field. There is an embedding \( Z \hookrightarrow Z_L = \bigoplus_{g \in G/H} L_g \), where \( L_g \) is the center of the division algebra \( D_g \). The transitive action of \( G \) on \( \{E_1, \ldots, E_s\} \) implies that \( G \) acts transitively on a basis of \( Z_L \), so that \( Z = (Z_L)^G \) has no zero divisors.

We produce the object \( F \) using the identification \( \text{End}_X(\pi_* E) \cong M_n(D) \), where \( D \) is a division algebra. Let \( e_i = e_{ij} \) denote the usual idempotent matrices, so that \( \{e_i\} \) is a complete set of primitive orthogonal idempotents. Notice that \( F_i := \text{Im}(e_i) \) is a simple \( \text{End}_X(\pi_* E) \)-submodule of \( \pi_* E \) for each \( i \), and hence \( F_i \simeq F_j \) for each \( i, j \), and \( \text{End}_X(F_i) \simeq D \). Define \( F = \text{Im}(e_1) \subset \pi_* E \), included as a direct summand. We note that \( \pi_* E \cong \bigoplus F_i \simeq F^\oplus n \).

We now show that \( F \) is an exceptional object on \( X \). As stated above, \( \text{End}_X(F) \) is a division algebra, so it suffices to show that \( \text{Ext}_X^n(F, F) = 0 \) for \( n \neq 0 \). Using Lemma 2.13 and \((\pi^*, \pi_* )\)-adjunction, we have
\[
\text{Ext}_X^n(\pi_* E, \pi_* E) = \bigoplus_{g \in G} \text{Ext}_X^n(g^* E, E).
\]
For \( n \neq 0 \), each summand of the right-hand side is 0, which follows from the mutual orthogonality of the exceptional block \( E \) (when \( g^* E \not\cong E \)) and from exceptionality of \( E \) (when \( g^* E \cong E \)). Since \( F \) is a direct summand of \( \pi_* E \), it follows that \( \text{Ext}_X^n(F, F) \) is a summand of \( \text{Ext}_X^n(\pi_* E, \pi_* E) = 0 \).

Lastly, we show that \( \pi^* F \) generates the category \( \langle E \rangle \). Since \( F^\oplus m \simeq \pi_* E \), extending scalars to \( L \) gives \( (\pi^* F)^\oplus m = \pi^* (F^\oplus m) \simeq \pi^* \pi_* E \simeq \bigoplus g^* E \). Thus,
\[
\langle \pi^* F \rangle = \langle (\pi^* F)^\oplus m \rangle = \bigoplus g^* E = \langle g^* E \rangle_{g \in G} = \langle E \rangle.
\]
Remark 2.15. Proposition [2.14] provides a very specific case of descent for triangulated categories, the main advantage of which is that it allows one to identify a specific exceptional object that base extends to the given orbit. Moreover, a $G$-orbit which forms an exceptional collection consisting of vector bundles (resp. sheaves) descends to an exceptional collection consisting of vector bundles (resp. sheaves). Compare to the following descent result for semiorthogonal decompositions, which generalizes [Loc12 Cor. 2.15]. Although this result is useful for descending semiorthogonal decompositions, it does not identify exceptional objects.

Proposition 2.16 (Prop. 2.12, [AB15]). Let $T$ be a $k$-linear triangulated category such that $T_{k^s}$ is $k^s$-equivalent to $D^b(k^s, (k^s)^n)$. Then there exists an étale algebra $K$ of degree $n$ over $k$, an Azumaya algebra $A$ over $K$, and a $k$-linear equivalence $T \simeq D^b(K/k, A)$.

Let $X$, $F$, and $E$ be as in Proposition 2.14 and note that taking $T = \langle F \rangle$, we have $T_{k^s} = (\pi^*F)_{k^s} = \langle E \rangle_{k^s}$. Since $E = \{g^*E\}_{g \in G}$ is a full exceptional collection for $\langle E \rangle$, the bundle $E := \oplus (g^*E)_{k^s}$ is a tilting object for $\langle E \rangle_{k^s}$. This defines an equivalence $T_{k^s} \simeq \langle E \rangle_{k^s} \simeq D^b(K^s, \text{End}(E)) = D^b(k^s, (k^s)^n)$.

Proposition 2.16 yields an étale extension $K/k$, an Azumaya $K$-algebra $A$, and an equivalence $T \simeq D^b(K/k, A)$. In this case, since $T = \langle F \rangle$, we see that $A = \text{End}_X(F)$ is an Azumaya algebra over its center $Z$ (using the notation found in the proof of Proposition 2.14), which is simply a field extension of $k$.

Theorem 2.17 (Descent for stable collections). Let $X$ be a $k$-scheme, $L/k$ a finite $G$-Galois extension, and $\pi : X_L \to X$ the natural projection map. If $X_L$ admits a full $G$-stable exceptional collection $E$ of objects of $D^b(X_L)$, then $X$ admits a full exceptional collection $F$ of objects of $D^b(X)$. If $E$ is strong, so is $F$. If the elements of $E$ are vector bundles (resp. sheaves), the elements of $F$ are vector bundles (resp. sheaves).

Proof. By Lemma 2.12, we may write $E = \{E^1, \ldots, E^s\}$ as a collection of $G$-stable blocks, where each block is given by a $G$-orbit. Proposition 2.14 then associates to each block $E^i$ an exceptional object $F_i$ on $X$, and we show that $F = \{F_1, \ldots, F_s\}$ is a full exceptional collection on $X$. We first show that $\text{Ext}^n(F_i, F_j) = 0$ for all $n$ whenever $i > j$. Let $E^i$ and $E^j$ be elements of the collections $E^i$ and $E^j$, respectively. We then have

$$\text{Ext}^n_X(\pi_*E^i, \pi_*E^j) \simeq \bigoplus_{g \in G} \text{Ext}^n_{X_L}(g^*E^i, E^j).$$

Since $E^i$ and $E^j$ are elements of the exceptional collection $E$ and $i < j$, each summand is 0 for all $n$, so that $\text{Ext}^n_X(\pi_*E^i, \pi_*E^j) = 0$ for all $n$. The objects $F_i$ and $F_j$ are direct summands of $\pi_*E^i$ and $\pi_*E^j$, respectively, and it follows that $\text{Ext}^n_X(F_i, F_j) = 0$ for all $n$.

By Proposition 2.14, the exceptional collection $\{F_1, \ldots, F_s\}$ yields a semiorthogonal decomposition $D^b(X) = \langle \mathcal{F}_1, \ldots, \mathcal{F}_s, A \rangle$, where $\mathcal{F}_i = \langle F_i \rangle$ and $A$ is the full subcategory of objects $A$ with $\text{Hom}_{D^b(X)}(A, F_i) = 0$ for all $i$. In particular, the subcategories $\mathcal{F}_i$ are admissible. Extending scalars to $L$, we have
\((\mathcal{F}_L) = \langle \mathcal{E} \rangle\), as both categories are generated by \(\pi^*F\) by Proposition 2.14. The exceptional collection \(E = \{E^1, \ldots, E^s\}\) is full, hence we have a semiorthogonal decomposition

\[D^b(X_L) = \langle (\mathcal{F}_L), \ldots, (\mathcal{F}_s) \rangle.\]

Since our admissible subcategories \(\mathcal{F}_i\) base extend to a semiorthogonal decomposition, \([ABB14]\) Lem. 2.9 gives a semiorthogonal decomposition \(D^b(X) = \langle \mathcal{F}_1, \ldots, \mathcal{F}_s \rangle\). In particular, the collection \(\{F_1, \ldots, F_s\}\) generates \(D^b(X)\), so this collection is full.

If \(E\) is strong, the right side of (2.1) vanishes for \(i \neq j\) (and any \(n\)). It follows exactly as above that \(\text{Ext}^n_X(F_i, F_j) = 0\) for all \(n\) when \(i \neq j\), so that \(F\) is strong. \(\square\)

**Remark 2.18.** Similar descent results for collections of sheaves are given by Elagin in the algebraically closed case (i.e., \(k = \overline{k}\)) using the framework of equivariant exceptional collections in equivariant derived categories \([Elal09]\). Indeed, for a variety \(X\) with an action of a finite group \(G\) and a \(G\)-invariant exceptional collection (see Remark 2.11) consisting of sheaves, this descent result is given in terms of \(\alpha\)-twisted representations of \(G\) (see Theorem 2.2 of loc. cit.). For a \(G\)-stable exceptional collection consisting of sheaves, results are in terms of coinduced twisted representations of \(G\) (see Theorem 2.3 of loc. cit.).

**Lemma 2.19.** Let \(X\) be a \(k\)-scheme and \(L/k\) a finite \(G\)-Galois extension. If \(X\) admits an exceptional collection, then \(X_L\) admits a \(G\)-stable exceptional collection.

**Proof.** Let \(E_1, \ldots, E_s\) be the exceptional collection on \(X\) and consider \(\pi^*E_1, \ldots, \pi^*E_s\) on \(X_L\). To compute morphisms, we note that

\[\text{Hom}_{X_L}(\pi^*E_i, \pi^*E_j) = \text{Hom}_X(E_i, \pi_*\pi^*E_j) = \text{Hom}_X(E_i, E_j \otimes_k L) = \text{Hom}_X(E_i, E_j) \otimes_k L.\]

This vanishes if \(j > i\). Let \(A_i = \text{Hom}_X(E_i, E_i)\). We can split \(A_i \otimes_k L\) as a product of matrix algebras over division algebras \(A_{i,j} = M_{N_{i,j}}(D_{i,j})\) and correspondingly decompose

\[\pi^*E_i = \bigoplus F_{i,j}^{N_{i,j}}\]

with

\[\text{Hom}_{X_L}(F_{i,j}, F_{i,j'}) = D_{i,j}.\]

Note that \(F_{i,j}\) and \(F_{i,j'}\) are orthogonal for \(j \neq j'\). Thus, we have an exceptional collection. \(\square\)

**Lemma 2.20.** Let \(X\) be a \(k\)-scheme and \(L/k\) a finite extension with Galois group \(G\). If \(G\) acts trivially on \(\text{Pic}(X_L)\) and \(X_L\) admits an exceptional collection of line bundles, then \(X\) admits an exceptional collection of vector bundles.

**Proof.** The collection on \(X_L\) is automatically \(G\)-stable pointwise. Hence we can apply Theorem 2.17. \(\square\)

**Remark 2.21.** Note that while we may start with a collection of line bundles, the descended collection may not consist only of line bundles. An example of this is the real conic discussed in the introduction.

**Lemma 2.22.** Let \(X\) be a smooth \(k\)-variety and \(L/k\) a \(G\)-Galois extension. Let \(Y_1, \ldots, Y_s\) be a \(G\)-orbit of smooth transversal subvarieties of \(X_L\). Let \(Y_{\overline{1}} = \cap_{i \in I} Y_i\) and let \(H_{\overline{1}}\) be the normalizer of \(Y_{\overline{1}}\). If each \(Y_{\overline{1}}\) admits a full \(H_{\overline{1}}\)-stable exceptional collection, then \(X\) admits an exceptional collection, where \(X_L\) is the iterated blow up of \(X_L\) at the \(Y_i\) (in any order).

**Proof.** This is an iterated application of Orlov’s Theorem, see \([CT17]\) Lemma 7.2. \(\square\)
3. Arithmetic toric varieties

We introduce toric varieties over arbitrary fields. Such varieties, also known as arithmetic toric varieties, have been treated in \[\text{Dun16} \], \[\text{ELFST14} \], \[\text{MP97} \], \[\text{VKS1} \].

**Definition 3.1.** A torus (over \(k\)) is an algebraic group \(T\) (over \(k\)) such that \(T_{k^s} \cong \mathbb{G}_m^n\). A torus is split if \(T \cong \mathbb{G}_m^n\). A field extension \(L/k\) satisfying \(T_L \cong \mathbb{G}_m^n\) is called a splitting field of the torus \(T\). Any torus admits a finite Galois splitting field.

**Definition 3.2.** Given a torus \(T\), a toric \(T\)-variety is a normal variety with a faithful \(T\)-action and a dense open \(T\)-orbit. A toric \(T\)-variety is split if \(T\) is a split torus. A splitting field of a toric \(T\)-variety is a splitting field of \(T\). A variety is a toric variety if it is a toric \(T\)-variety for some torus \(T\).

**Definition 3.3.** Let \(X\) be a toric \(T\)-variety whose dense open \(T\)-orbit contains a \(k\)-rational point. Then we say \(X\) is neutral \[\text{Dun16} \] (or a toric \(T\)-model \[\text{MP97} \]). An orbit of a split torus always has a \(k\)-point, so a split toric variety is neutral; but the converse is not true in general.

**Remark 3.4.** In what follows, we will use the term toric variety to mean toric \(T\)-variety for some fixed torus \(T\), even though such a variety may have a toric structure for various tori. In fact, the choice of torus does not affect our analysis of toric varieties given below, and we refer interested readers to \[\text{Dun16} \] for such considerations.

Recall that a \(k\)-form of a \(k\)-variety \(X\) is a \(k\)-variety \(X'\) such that \(X_L \cong X'_L\) for some field extension \(L/k\). Any \(k\)-form of a toric variety is a toric variety \[\text{Dun16} \].

### 3.1. The split case

Let us begin by recalling some facts concerning toric varieties with \(T \cong \mathbb{G}_m^n\) (e.g., when \(k = \mathbb{C}\) or \(k = k^s\)), which are studied in terms of combinatorial data, e.g., lattices, cones, fans. Good references for toric varieties over \(\mathbb{C}\) include \[\text{Ful93} \], \[\text{CLS11} \], and many results hold generally in the split case.

Let \(N\) be a finitely generated free abelian group of rank \(n\) and \(M = \text{Hom}(N, \mathbb{Z})\). A subsemigroup \(\sigma \subset N_\mathbb{R}\) is a cone if \((\sigma^\vee)^\vee = \sigma\), where \(\sigma^\vee = \{u \in M \mid u(v) \geq 0\ \text{for all} \ v \in \sigma\}\).

A subsemigroup \(\tau\) is a face of \(\sigma\) if it is of the form \(\tau = \{v \in \sigma \mid u(v) = 0\ \text{for all} \ u \in S\}\) for some \(S \subseteq \sigma^\vee\). A cone \(\sigma\) is pointed if \(0\) is a face of \(\sigma\), and in this case \(\sigma^\vee\) generates \(M_\mathbb{R}\).

A fan \(\Sigma \subset N_\mathbb{R}\) is a finite collection of pointed cones such that (1) any face of a cone in \(\Sigma\) is a cone in \(\Sigma\) and (2) the intersection of any two cones in \(\Sigma\) is a face of each. To any fan \(\Sigma\) we associate a \(k\)-scheme \(U_\Sigma = \text{Spec } k[\sigma^\vee]\), and for any face \(\tau \subset \sigma\) the induced map \(U_\tau \rightarrow U_\sigma\) is an open embedding.

A fan \(\Sigma \subset N_\mathbb{R}\) is a finite collection of pointed cones such that (1) any face of a cone in \(\Sigma\) is a cone in \(\Sigma\) and (2) the intersection of any two cones in \(\Sigma\) is a face of each. To any fan \(\Sigma\) we associate a \(k\)-scheme \(U_\Sigma\) which is obtained by gluing the affine schemes \(U_\sigma\) along common subschemes \(U_\tau\) corresponding to faces.

On the other hand, beginning with a split torus \(T \cong \mathbb{G}_m^n\) and toric \(T\)-variety \(X\) with fixed embedding \(T \hookrightarrow X\), we recover \(M\) as the character lattice \(\text{Hom}(T, \mathbb{G}_m)\) of \(T\) and \(N\) as the cocharacter lattice \(\text{Hom}(\mathbb{G}_m, T)\). The association \(\Sigma \mapsto X_\Sigma\) defines a bijective correspondence between fans \(\Sigma \subset N_\mathbb{R}\) and toric \(T\)-varieties \(X\) (we remind the reader that here we assume \(T\) is a split torus; in general, fans \(\Sigma\) admitting an action by \(\text{Gal}(k^s/k)\) are in bijection with neutral toric \(T\)-varieties).

Let \(\Sigma(\ell)\) denote the collection of cones in \(\Sigma\) of dimension \(\ell\). Let \(\text{Div}_T(X)\) denote the free abelian group generated by the rays of \(\Sigma\), i.e., elements of \(\Sigma(1)\). By the Orbit-Cone Correspondence \[\text{CLS11} \] Thm. 3.2.6, \(\text{Div}_T(X)\) is isomorphic to the group of \(T\)-invariant Weil
divisors of $X$. For $X$ a (split) smooth projective toric variety, we have natural identifications $\text{Pic}(X) = \text{Pic}(X_{k^s}) = \text{Cl}(X_{k^s}) = \text{Cl}(X)$ which yield an exact sequence

$$0 \to M \to \text{Div}_T(X) \to \text{Pic}(X) \to 0.$$ 

In particular, if $X$ is of dimension $n$ and $m$ is the number of rays in $\Sigma$, the Picard rank of $X$ is $\rho = m - n$.

**Definition 3.5.** A variety $X$ is *Fano* (resp. *weak Fano*) if its anticanonical class $-K_X$ is ample (resp. nef and big). If $X$ is a normal variety, a Cartier $D$ divisor on $X$ is *nef* (“numerically effective” or “numerically eventually free”) if $D \cdot C \geq 0$ for every irreducible curve $C \subset X$. A divisor $D$ is *very ample* if $D$ is base point free and $\varphi_D : X \to \mathbb{P}(\Gamma(X, \mathcal{O}_X(D))^\vee)$ is an embedding. A divisor $D$ is *ample* if $\ell D$ is very ample for some $\ell \in \mathbb{Z}^+$. A line bundle $\mathcal{O}_X(D)$ is nef or (very) ample if the corresponding divisor $D$ is nef or (very) ample. A Cartier divisor is *numerically trivial* if $D \cdot C = 0$ for every irreducible complete curve $C \subset X$. Let $N^1(X)$ be the quotient group of Cartier divisors by the subgroup of numerically trivial divisors. The *nef cone* $\text{Nef}(X)$ is the cone in $N^1(X)$ generated by the nef divisors, and the *anti-nef cone* is the cone $-\text{Nef}(X) \subset N^1(X)$. A line bundle $\mathcal{O}_X(D)$ is nef (ample) if $D$ is nef (ample).

**Proposition 3.6.** A Cartier divisor $D$ on a split proper toric variety $X$ is nef (resp. ample) if and only if $D \cdot C \geq 0$ (resp. $D \cdot C > 0$) for all torus-invariant integral curves $C \subset X$.

**Proof.** When $k$ is algebraically closed, these are Theorems 3.1 and 3.2 of [Mus02]. One can see that the arguments remain valid in the split case more generally. \qed

### 3.2. The not necessarily split case

Here we provide a “black box” for producing exceptional collections on arbitrary forms of toric varieties by identifying certain special exceptional collections on a split toric variety. This reduces an arithmetic question to a completely geometric question.

Suppose $X$ is a toric $T$-variety. We say that an object $E \in \text{D}^b(X)$ is *$T$-equivariant* if $E$ is in the image of the forgetful functor from $\text{D}^b(\text{Coh}_T(X))$ (see §2 of [BFK14]). In particular, this implies that $t^*E \cong E$ for all $t \in T(k)$.

**Proposition 3.7.** Let $X$ be a split toric $T$-variety over a field $k$ and let $\Sigma$ be the associated fan. Suppose that $X$ admits an $\text{Aut}(\Sigma)$-stable full exceptional collection $E$ such that each object is $T$-equivariant. Then any $k$-form $X'$ of $X$ admits a full exceptional collection $E'$. Moreover, $E'$ is strong (resp. consists of vector bundles, consists of sheaves) as soon as $E$ is strong (resp. consists of vector bundles, consists of sheaves).

**Proof.** By Lemma [2.10] there exists a $G$-stable exceptional collection $F$ on $X_L$. From the proof of that lemma, the objects $F$ of $F$ are direct summands of $\pi^*E$ for each object $E \in E$, where each isomorphism class of simple direct summand is represented by exactly one $F$. Since $E$ is $\text{Aut}(\Sigma)$-stable and each object is $T$-equivariant, we may conclude that $F$ is $(T(L) \rtimes \text{Aut}(\Sigma))$-stable.

Let $X'$ be a $k$-form of $X$; there exists a finite Galois extension $L/k$ with Galois group $G$ such that $X'_L \cong X_L$. From Theorem 5.1 of [Dun10], the natural map

$$H^1(L/k, T(L) \rtimes \text{Aut}(\Sigma)) \to H^1(L/k, \text{Aut}(X)(L))$$


in Galois cohomology is surjective. Thus, we may assume that $X' = \sigma X$ is the twist by a cocycle $c : G \to T(L) \rtimes \text{Aut}(\Sigma)$. Recall that the cocycle condition is that $c(gh) = c(g)c(h)$ for all $g, h \in G$ where $\sigma(h)$ denotes the Galois action of $g$ on $T(L) \rtimes \text{Aut}(\Sigma)$.

Identifying $X_L = X_{L'}$ twisting gives $\sigma'(g) = c(g)\sigma(g)$ where $\sigma$ is the action of $G$ induced from $X$ and $\sigma'$ is induced from $X'$. The punchline is that the action $\sigma'$ factors through the image of $(T(L) \rtimes \text{Aut}(\Sigma)) \rtimes G$ described above. Thus the exceptional collection $F$ is $G$-stable for the $X'$ action as well. The proposition now follows by Theorem 2.17.

**Corollary 3.8.** Let $X$ be a split toric $T$-variety over a field $k$ and let $\Sigma$ be the associated fan. If $X$ admits an $\text{Aut}(\Sigma)$-stable full (strong) exceptional collection of line bundles, then every $k$-form of $X$ admits a full (strong) exceptional collection of vector bundles.

**Proof.** Recall that every line bundle is isomorphic to a $T$-equivariant line bundle by standard results on toric varieties. The corollary now follows by Proposition 3.7.

**Lemma 3.9.** Let $X$ and $Y$ be smooth projective toric varieties over $k$. Let $G = \text{Gal}(k^s/k)$. Assume we have a $K$-positive toric flip $X \dasharrow Y$ such that over $k^s$ the flipping loci $F_i$ are disjoint and permuted by $G$. Let $H_i$ be the normalizer of $F_i$. If $X_L$ admits a full $G$-stable exceptional collection and $Y_i$ admits a full $H_i$-stable exceptional collection, then $Y$ admits a full exceptional collection.

**Proof.** Passing to $k^s$ we are free to use [BFK17] giving semi-orthogonal decompositions for the flip over each $Y_i$. Since the $Y_i$ are disjoint, we can concatenate these collections to get a $G$-stable collection.

### 3.3. Products of toric varieties

Recall that, given groups $G, H$ along with a homomorphism $\rho : H \to S_n$, the wreath product $G \wr H$ is the group $G^n \rtimes H$ where $H$ acts on $G^n$ by permuting the copies of $G$. We say a toric variety $X$ is indecomposable if it cannot be written as a product $X_1 \times X_2$ where $X_1$ and $X_2$ are positive-dimensional toric varieties.

**Lemma 3.10.** Suppose $Z = X_1^{n_1} \times \cdots \times X_r^{n_r}$ is a product of proper split toric varieties $X_1, \ldots, X_r$, where $X_i \neq X_j$ for $i \neq j$ and each $X_i$ is indecomposable. Then

$$\text{Aut}(\Sigma) \simeq (\text{Aut}(\Sigma_1) \wr S_{n_1}) \times \cdots \times (\text{Aut}(\Sigma_r) \wr S_{n_r}),$$

where $\Sigma$ is the fan of $Z$ and $\Sigma_1, \ldots, \Sigma_r$ are the fans of $X_1, \ldots, X_r$.

**Proof.** First, consider $Z = X_1 \times X_2$ where $X_1, X_2$ are proper split toric varieties. Let $N$ (resp. $N_1, N_2$) be the cocharacter lattice and $\Sigma$ (resp. $\Sigma_1, \Sigma_2$) be the fan of $Z$ (resp. $X_1, X_2$). Here $N = N_1 \oplus N_2$ and $\Sigma$ is the set of cones of the form $\sigma_1 \times \sigma_2$ where $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$. The faces of a cone $\sigma_1 \times \sigma_2$ are precisely the cones of the form $\sigma'_1 \times \sigma'_2$ where $\sigma'_1$ is a face of $\sigma_1$ and $\sigma'_2$ is a face of $\sigma_2$. The fan $\Sigma_1$ can be canonically identified with the subfan of $\Sigma$ via the bijection $\sigma \mapsto \sigma \times \{0\}$.

Now, suppose also that $Z = Y \times W$ is a product of proper split toric varieties where $Y$ is indecomposable. Let $\Sigma_Y$ be the fan of $Y$, which we can canonically identify with a subfan of $\Sigma_Z$. Every cone of $Y$ is of the form $\sigma_1 \times \sigma_2$ where $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$. Since fans are closed under taking faces, $\sigma_1 \times \{0\}$ and $\{0\} \times \sigma_2$ are also cones in $\Sigma_Y$. Thus every cone in $\Sigma_Y$ is a product of cones in the intersections $\Sigma_Y \cap \Sigma_1$ and $\Sigma_Y \cap \Sigma_2$.

In particular, since $X$ is proper, we have that the space $N_Y \otimes \mathbb{R}$ is the direct sum of $(N_Y \otimes \mathbb{R}) \cap (N_1 \otimes \mathbb{R})$ and $(N_Y \otimes \mathbb{R}) \cap (N_2 \otimes \mathbb{R})$, and $\Sigma_Y$ is a product of the fans $\Sigma_Y \cap \Sigma_1$ and $\Sigma_Y \cap \Sigma_2$. 


and $\Sigma_Y \cap \Sigma_2$. Since $Y$ is indecomposable, one of these fans is indecomposable and $\Sigma_Y$ must be a subfan of either $\Sigma_1$ or $\Sigma_2$.

Returning to the general case, we conclude that the decomposition $\Sigma = \Sigma_1^{n_1} \times \cdots \times \Sigma_r^{n_r}$ is unique up to ordering. The description of the automorphism group is immediate. □

**Lemma 3.11.** Let $Z$ be a proper toric $k$-variety with splitting field $L/k$. Suppose $Z_L = \prod_{i=1}^n X_i$ where each $X_i$ is an indecomposable split proper $L$-variety admitting a full (strong) $\text{Aut}(\Sigma_i)$-stable exceptional collection of line bundles, where $\Sigma_i$ is the fan of $X_i$. Then $Z$ has a full (strong) exceptional collection of vector bundles.

**Proof.** It is a well known that the exterior product collection is an exceptional collection. For each isomorphism class among the $X_i$ fix a full (strong) $\text{Aut}(\Sigma_i)$-stable exceptional collection of line bundles. This ensures that the exterior product collection is stable under the action of $(\text{Aut}(\Sigma_{X_1}) \wr S_{\alpha_1}) \times \cdots \times (\text{Aut}(\Sigma_{X_r}) \wr S_{\alpha_r})$. Since this group is $\text{Aut}(\Sigma)$ by Lemma 3.10 the exterior product collection descends by Corollary 3.8 □

4. LOW DIMENSION OR HIGH SYMMETRY

We provide exceptional collections for smooth toric surfaces, Fano 3-folds, some Fano 4-folds, centrally-symmetric toric varieties, and toric varieties corresponding to root systems of type $A$.

4.1. Surfaces. Here we prove that every toric surface has a full exceptional collection. We begin by recalling the (classical) minimal model program for surfaces over non-closed fields.

Suppose $f : X \to X'$ is a birational morphism of smooth projective surfaces over a field $k$. If $k$ is separably closed, then by Proposition 5 of [Coo88] the morphism factors into a sequence $X = X_0 \to X_1 \to \cdots \to X_r = X'$ where each morphism $X_i \to X_{i+1}$ is the blowup of a point on $X_{i+1}$. Over a non-closed field $k$, we can factor $f : X \to X'$ into a sequence where each morphism $X_i \to X_{i+1}$ is defined over $k$ and is a blowup of a (necessarily finite) Galois orbit of $k^a$-points on $X_{i+1}$.

Blowing up a point produces an exceptional curve: a smooth rational curve with self-intersection $-1$. By Castenhuvo’s contractibility criterion, such a curve can always be obtained as the result of a blow-up. If one finds a skew Galois orbit of such curves on $X$, then there exists a birational morphism $f : X \to X'$ contracting these curves. Repetition of this procedure eventually terminates.

**Definition 4.1.** A minimal surface $X$ is a smooth projective surface over a field $k$ such that every birational morphism $X \to X'$ to a smooth projective surface $X'$ is an isomorphism.

Any smooth projective surface can be obtained by iteratively blowing up Galois orbits of separable points starting from a minimal model. A toric variety is geometrically rational. Minimal geometrically rational surfaces were classified by Manin [Man66] and Iskovskikh [Isk79]. One checks that the toric surfaces in their collection are the following (see also a direct proof in [Xie17]):

**Lemma 4.2.** A minimal smooth projective toric surface is a $k^a/k$-form of one of the following:

(1) $\mathbb{P}^2$, $\text{Aut}(\Sigma) = S_3$. 


Proof. A minimal geometrically rational surface is either a del Pezzo surface or has a conic bundle structure \cite{Man66,Isk79}. Over the separable closure, a del Pezzo surface is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a blow up of $\mathbb{P}^2$ at up to 8 points in general position. Blowing up only one or two points never results in a minimal surface, and no more than three points can be simultaneously torus invariant and in general position. Thus every del Pezzo surface is a $k^*/k$-form of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\text{dP}_6$. Over the separable closure, a conic bundle structure has at most 2 singular fibers since their images must be torus invariant points on the base $\mathbb{P}^1$. A minimal conic bundle with at most two singular fibers over the separable closure must be either a del Pezzo surface or a minimal ruled surface. □

Here we exhibit full strong exceptional collections consisting of $G$-stable blocks for each minimal toric surface exhibited above (none of these collections are original). The fans associated to the split forms of these surfaces are given in Figure 1. In each case, we fix a torus $T$ which gives $X$ the structure of a toric $T$-surface. As remarked above, this gives a homomorphism $G \to \text{Aut}(\Sigma)$ as well as an action of $G$ on $\text{Pic}(X_L)$, where $L$ is a splitting field of $T$, $G = \text{Gal}(L/k)$, and $\Sigma$ is the fan corresponding to the split toric surface $X_L$. We produce $G$-stable exceptional collections in each case by exhibiting $\text{Aut}(\Sigma)$-stable collections.
Example 4.3. Let \( X \) be a toric \( T \)-surface whose split form is \( \mathbb{P}^2 \) with \( \text{Aut}(\Sigma) = S_3 \). The \( S_3 \)-action on \( \text{Pic}(\mathbb{P}^2) = \mathbb{Z} \) is clearly trivial, so that the exceptional collection \( \{ \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \} \), given in [Bier85] yields a full strong \( \text{Aut}(\Sigma) \)-stable exceptional collection. By Corollary 3.8, \( X \) admits a full strong exceptional collection.

Example 4.4. Let \( X \) be a toric surface whose split form is \( \mathbb{P}^1 \times \mathbb{P}^1 \) with \( \text{Aut}(\Sigma) = D_8 \), and consider the natural projections \( p_1, p_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \). Let \( \mathcal{O}(p, q) = p_1^* \mathcal{O}(p) \otimes p_2^* \mathcal{O}(q) \). By [KN90], the collection \( \{ \mathcal{O}, \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,1) \} \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \) is exceptional since \( \{ \mathcal{O}, \mathcal{O}(1) \} \) is an exceptional collection for \( \mathbb{P}^1 \). The \( D_8 \)-action preserves this collection, with orbits given by the blocks \( E^0 = \{ \mathcal{O} \} \), \( E^1 = \{ \mathcal{O}(1,0), \mathcal{O}(0,1) \} \), and \( E^2 = \{ \mathcal{O}(1,1) \} \). In particular, this collection above is \( \text{Aut}(\Sigma) \)-stable, and Corollary 3.8 yields an exceptional collection on \( X \).

Example 4.5. Let \( X \) be a toric surface whose split form is the Hirzebruch surface \( \mathbb{F}_a \); here \( \text{Aut}(\Sigma) = C_2 \). Let \( e_1, e_2 \) be the standard basis for \( \mathbb{Z}^2 \). As in [CLS11, Ex. 4.1.8], let \( u_1 = -e_1 + ae_2 \), \( u_2 = e_2 \), \( u_3 = e_1 \), and \( u_4 = -e_2 \) be the generators of \( \Sigma(1) \) with corresponding toric divisors \( D_i \). The Picard group of \( \mathbb{F}_a \) is freely generated by \( \{ D_1, D_2 \} \) and \( D_1 \) is linearly equivalent to \( D_2 \). The only nontrivial fan automorphism \( \sigma \) takes \( e_1 \mapsto -e_1 + ae_2 \) and \( e_2 \mapsto e_2 \). Thus \( \sigma \) leaves \( D_2, D_4 \) fixed and interchanges \( D_1 \) and \( D_3 \). We conclude the action of \( C_2 \) on \( \text{Pic}(\mathbb{F}_a) \) is trivial, and thus, any exceptional collection is necessarily \( G \)-stable (see Lemma 2.20). An exceptional collection for \( \mathbb{F}_a \) is given by \( \{ \mathcal{O}, \mathcal{O}(D_3), \mathcal{O}(D_4), \mathcal{O}(D_3 + D_4) \} \) [KN90]. Corollary 3.8 then gives an exceptional collection on \( X \).

Example 4.6. Let \( X \) be a toric surface whose split form is \( d\mathbb{P}_6 \); here \( \text{Aut}(\Sigma) = D_{12} \). Viewing \( d\mathbb{P}_6 \) as the blowup of \( \mathbb{P}^2 \) at 3 non-colinear points, let \( H \) be the pullback of the hyperplane divisor on \( \mathbb{P}^2 \) and \( E_i \) the exceptional divisors, \( i = 1, 2, 3 \). As shown in [Kin97, Prop. 6.2(ii)], the collection

\[
\{ \mathcal{O}, \mathcal{O}(H - E_1), \mathcal{O}(H - E_2), \mathcal{O}(H - E_3), \mathcal{O}(H), \mathcal{O}(2H - (E_1 + E_2 + E_3)) \}
\]

gives an exceptional collection for \( d\mathbb{P}_6 \), which is \( \text{Aut}(\Sigma) \)-stable.

Let us rephrase this in the notation of [BSS11]. There are two morphisms \( d\mathbb{P}_6 \to \mathbb{P}^2 \) realizing \( d_2 \) as a blowup of \( \mathbb{P}^2 \), and we denote the collection of all six exceptional divisors by \( L_i \) and \( M_i \), with \( i = 1, 2, 3 \). Let \( H \) and \( H' \) denote the pullbacks of the hyperplane divisors on \( \mathbb{P}^2 \) under the maps contracting \( M_i \) and \( L_i \), respectively, where we identify \( H \) with the divisor given in King’s collection above (and thus we also identify \( E_i \) with \( M_i \)). Then \( H = L_1 + M_2 + M_3 \), and it follows that

\[
2H - (E_1 + E_2 + E_3) = L_1 + L_2 + M_3 = H'
\]

using the relation \( L_i + M_j = L_j + M_i \). Furthermore, one checks that \( H - E_1 = L_2 + M_3 \), \( H - E_2 = L_1 + M_3 \), and \( H - E_3 = L_1 + M_2 \). As described in [BSS11, §2], the element \( \sigma \) in \( S_3 \times C_2 = D_{12} \) which cyclically permutes the six lines \( L_i, M_i \) also satisfies \( \sigma(H) = H' \) and \( \sigma^2(H) = H \). We arrange the exceptional collection above into blocks \( E^0 = \{ \mathcal{O} \} \), \( E^1 = \{ \mathcal{O}(H - E_1), \mathcal{O}(H - E_2), \mathcal{O}(H - E_3) \} \) and \( E^2 = \{ \mathcal{O}(H), \mathcal{O}(2H - (E_1 + E_2 + E_3)) \} \). In particular, the exceptional collection given above is \( \text{Aut}(\Sigma) \)-stable, and by Corollary 3.8, we have an exceptional collection on \( X \).

Proposition 4.7. Every toric surface admits a full exceptional collection of sheaves.

Proof. There is a sequence of blowups \( X = X_0 \to \cdots \to X_s = X' \) where \( X' \) is minimal, so must be one of the varieties given in Lemma 4.2. By Examples 4.3-4.6, \( X' \) admits a full
strong exceptional collection of vector bundles, and thus $X'_L$ admits a $G$-stable exceptional collection. By Lemma 2.22, $X_L$ admits a $G$-stable exceptional collection. □

**Remark 4.8.** The authors would like to thank F. Xie for pointing out a mistake in the statement of a previous version of Proposition 4.7. Xie also discusses exceptional collections of toric surfaces in [Xie17], although her definition of exceptional object is not the same as ours. In the second arXiv version of that paper, Xie sketched in Remark 8.8 how one might construct an exceptional collection for toric surfaces. After the authors posted a preliminary version of this paper to the arXiv, Xie updated her preprint with Corollary 8.8, which proves the analog of the above proposition for vector bundle but using her notion of exceptional collection.

### 4.2. The toric Frobenius and toric Fano 3-folds

In Table 1 we present the classification of smooth toric Fano 3-folds given in [Bat99, WW82], adopting Batyrev’s enumeration. For each $X = X_\Sigma$, we record the following invariants:

- $\sigma(1) = |\Sigma(1)|$ is the number of rays of $\Sigma$ [BT09].
- $k_0$ is the rank of the Grothendieck group $K_0(X)$, which coincides with the number of maximal cones in the fan $\Sigma$ [BT09].
- $\text{Aut}(\Sigma)$ is the automorphism group of the (lattice $N$ which preserves the) fan $\Sigma$ corresponding to $X$.
- $\rho$ is the Picard rank of $X$ [WW82].
- $\rho^G$ is the $\text{Aut}(\Sigma)$-invariant Picard rank of $X$, i.e., the rank of $\text{Pic}(X)^{\text{Aut}(\Sigma)}$.
- $\text{fr} = |\text{Frob}(X)|$ is the number of isomorphism classes of line bundles produced by the push forward of the structure sheaf under the Frobenius morphism [BT09, Ueh14].
- $\text{fr}^- = |\text{Frob}(X) \cap -\text{Nef}(X)|$ is the number of isomorphism classes of line bundles in $\text{Frob}(X)$ which lie in the anti-nef cone of $X$ [Ueh14].

#### 4.2.1. Toric Frobenius

Let $X$ be a split toric variety of dimension $n$ with fixed torus embedding $T \hookrightarrow X$ and take $\ell \in \mathbb{Z}^+$. Define the $\ell$th Frobenius map on $T = \mathbb{G}_m^m$ to be $(x_1, \ldots, x_n) \mapsto (x_1^\ell, \ldots, x_n^\ell)$. The unique extension to $X$ will be denoted $F_\ell$ and called the $\ell$th Frobenius morphism. Alternatively, if $\Sigma \subset N$ is the fan associated to $X$, define a lattice $N' = \frac{1}{\ell}N$. The inclusion $N \subset N'$, which sends a cone in $N_\mathbb{R}$ to the cone with the same support in $N'_\mathbb{R}$, induces a finite surjective morphism which is precisely the $\ell$th Frobenius morphism $F_\ell : X \to X$.

The sheaf $(F_\ell)_*(O_X)$ splits into line bundles and Thomsen provides an algorithm for computing its direct summands [Tho00]. We let $\text{Frob}(X)$ denote the union of all isomorphism classes of line bundles arising as direct summands of $(F_\ell)_*(O_X)$ as $\ell$ varies over $\mathbb{Z}^+$. Note that $\text{Frob}(X)$ is a finite set.

**Conjecture 4.9 (Bondal [Bon06]).** If $X$ is a smooth proper toric variety then the collection $\text{Frob}(X)$ generates $\mathbb{D^b}(X)$.

For a toric variety $X$ in which Bondal’s Conjecture is true, we will say that the Frobenius generates the derived category of $X$. In loc. cit., Bondal proves that one actually gets a full strong exceptional collection when all summands are nef. In particular, Conjecture 4.9 is true. He also notes his arguments work for all for all but two toric Fano threefolds. To cover all toric Fano threefolds, Uehara noticed that discarding line bundles which do not lie in the set $-\text{Nef}(X)$ do, in fact, form a full strong exceptional collection [Ueh14].
Toric Fano 3-fold $X$ & $\sigma(1)$ & $k_0$ & $\text{Aut}(\Sigma)$ & $\rho$ & $\rho^G$ & $\text{ft}$ & $\text{ft}^-$

1. $\mathbb{P}^3$ & 4 & 4 & $S_4$ & 1 & 1 & 4 & 4
2. $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(2))$ & 5 & 6 & $S_3$ & 2 & 2 & 7 & 6
3. $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$ & 5 & 6 & $S_3$ & 2 & 2 & 6 & 6
4. $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1))$ & 5 & 6 & $C_2 \times C_2$ & 2 & 2 & 6 & 6
5. $\mathbb{P}^2 \times \mathbb{P}^1$ & 5 & 6 & $D_{12}$ & 2 & 2 & 6 & 6
6. $\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1,1))$ & 6 & 8 & $D_8$ & 3 & 2 & 8 & 8
7. $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(l)), l^2 = 1$ on $\mathbb{P}^8$ & 6 & 8 & $D_8$ & 3 & 3 & 8 & 8
8. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ & 6 & 8 & $C_2 \times S_4$ & 3 & 1 & 8 & 8
9. $\mathbb{P}^8 \times \mathbb{P}^1$ & 6 & 8 & $C_2 \times C_2$ & 3 & 3 & 8 & 8
10. $\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1, -1))$ & 6 & 8 & $D_8$ & 3 & 2 & 8 & 8
11. $\mathbb{B}_{\mathbb{P}^2}(\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1)))$ & 6 & 8 & $C_2$ & 3 & 3 & 9 & 8
12. $\mathbb{B}_{\mathbb{P}^2}(\mathbb{P}^2 \times \mathbb{P}^1)$ & 6 & 8 & $C_2$ & 3 & 3 & 8 & 8
13. $\mathbb{dP}_7 \text{-bundle over } \mathbb{P}^1$ & 7 & 10 & $C_2$ & 4 & 4 & 10 & 10
14. $\mathbb{dP}_7 \text{-bundle over } \mathbb{P}^1$ & 7 & 10 & $C_2 \times C_2$ & 4 & 3 & 10 & 10
15. $\mathbb{dP}_7 \times \mathbb{P}^1$ & 7 & 10 & $C_2 \times C_2$ & 4 & 3 & 10 & 10
16. $\mathbb{dP}_7 \text{-bundle over } \mathbb{P}^1$ & 7 & 10 & $C_2$ & 4 & 4 & 10 & 10
17. $\mathbb{dP}_6 \times \mathbb{P}^1$ & 8 & 12 & $C_2 \times C_2 \times S_3$ & 5 & 2 & 12 & 12
18. $\mathbb{dP}_6 \text{-bundle over } \mathbb{P}^1$ & 8 & 12 & $C_2 \times C_2$ & 5 & 4 & 12 & 12

Table 1. Toric Fano 3-folds

**Lemma 4.10.** Let $X$ be a toric variety over $k$ with splitting field $L$. Suppose $E$ is a full (strong) exceptional collection for $\mathcal{D}^b(X_L)$ where either $E = \text{Frob}(X_L)$ or $E = \text{Frob}(X_L) \cap - \text{Nef}(X_L)$. Then there exists a full (strong) exceptional collection for $\mathcal{D}^b(X)$.

**Proof.** Both $\text{Frob}(X_L)$ and $\text{Nef}(X_L)$ are canonical constructions and thus are $\text{Aut}(X_L)$-stable. In particular, $E$ is $\text{Aut}(\Sigma)$-stable and so Corollary 3.8 applies. □

**Proposition 4.11.** Let $X$ be a smooth projective toric Fano 3-fold over a field $k$. Then $X$ admits a full strong exceptional collection consisting of vector bundles.

**Proof.** Let $X_L$ be the associated split toric Fano 3-fold. The main result of [Ueh14] guarantees that the set $E = \text{Frob}(X_L) \cap - \text{Nef}(X_L)$ defines a full strong exceptional collection on $X$. Lemma 4.10 completes the proof. □

### 4.3. Toric Fano 4-folds

There are 124 split smooth toric Fano 4-folds, which were first classified in [Bat99] (a missing case was added in [Sat00]). In [PN17], Prabhu-Naik exhibits full strong exceptional collections for all 124 of these 4-folds. However, it is not clear that these collections are $\text{Aut}(\Sigma)$-stable, so they do not necessarily lead to full strong exceptional collections in the arithmetic case.

All collections obtained using Method 1 of loc. cit. produce $\text{Aut}(\Sigma)$-stable collections (note that this is precisely the method used in [Ueh14] for toric Fano 3-folds, and we will refer to this as the Bondal-Uehara Method). Together with Lemmas 3.11 and 4.10 this gives stable exceptional collections for 43 of the 124 smooth toric Fano 4-folds. However, there are examples when the Bondal-Uehara Method fails to produce an exceptional collection. In this case, all is not lost (see Section 4.4).
More precisely, the varieties (61), (62), (63), (64), (77), (105), (107), (108), (110), (122), and (123) of [PN17] are shown to have exceptional collections using the Bondal-Uehara Method. Hence, they admit exceptional collections which are Aut(Σ)-stable and thus provide exceptional collections for the arithmetic forms. Secondly, for the varieties (109), (114), and (115), the set \( \text{Frob}(X) \) is a full exceptional collection, which is \( G \)-stable by Lemma 4.10.

Lastly, Lemma 3.11 guarantees the existence of exceptional collections on products. Hence, the following varieties also admit stable exceptional collections: (0), (4), (9), (17), (24), (25), (26), (27), (45), (52), (53), (54), (55), (56), (58), (67), (73), (88), (90), (92), (93), (97), (103), (111), (112), (113), (118), (119), (120).

4.4. Centrally symmetric toric Fano varieties. Polytopes with the highest degree of symmetry are the centrally symmetric polytopes, i.e., \( -P = P \). The smooth split toric varieties \( X \) whose anti-canonical polytope is full-dimensional and centrally symmetric were classified in [VK84]. It was shown that any such variety (which we refer to as a centrally symmetric toric Fano varieties) is isomorphic to a product of projective lines and generalized del Pezzo varieties \( V_n \) of dimension \( n = 2m \). Note that \( V_2 = \text{dP}_6 \) and \( V_4 \) is the missing (116) from the list in Section 4.3 (this is (118) in the enumeration found in [Bat99]). The goal of this section is to exhibit full stable exceptional collections on \( V_n \), which in turn yields stable exceptional collections for any centrally symmetric toric Fano variety, in light of Lemma 3.11.

In [CT17, Theorem 6.6], Castravet and Tevelev found Aut(Σ)-stable full strong exceptional collections for the varieties \( V_n \). The authors of this paper had independently discovered the same exceptional collection (up to a twist by a line bundle). Nevertheless, the perspective here may be of independent interest, so we sketch the argument.

The variety \( V_n \) with \( n = 2m \) has rays given by

\[
\begin{align*}
e_1 &= (1, 0, \cdots, 0) & \tilde{e}_1 &= (-1, 0, \cdots, 0) \\
e_2 &= (0, 1, \cdots, 0) & \tilde{e}_2 &= (0, -1, \cdots, 0) \\
\vdots & & \vdots \\
e_n &= (0, 0, \cdots, 1) & \tilde{e}_n &= (0, 0, \cdots, -1) \\
e_{n+1} &= (-1, -1, \cdots, -1) & \tilde{e}_{n+1} &= (1, 1, \cdots, 1)
\end{align*}
\]

and whose maximal cones are given as follows. Among the rays \( e_1, \ldots, e_{n+1} \), omit a single \( e_i \). From the remaining \( n = 2m \) rays, choose \( \frac{n}{2} \) of them and take their antipodes [VK84, Proof of Thm. 5]. Note that \( V_2 = \text{dP}_6 \) (whose fan is given in Figure 1). The number of maximal cones \( c(n) \) of \( V_n \) is given by

\[
c(n) = \frac{(n + 1)!}{(\frac{n}{2})!^2} = \frac{(2m + 1)!}{m!^2}.
\]

There’s a natural action of \( S_{n+1} \times C_2 \), where \( S_{n+1} \) permutes \( e_1, \ldots, e_{n+1} \) and \( \tilde{e}_1 \ldots \tilde{e}_{n+1} \) in the obvious way. The \( C_2 \)-action is simply the antipodal map on the cocharacter lattice — we will refer to it as “the involution.” Clearly, the involution interchanges \( e_i \) and \( \tilde{e}_i \).

The variety \( V_n \) is of importance in birational geometry due its appearance in the factorization of the standard Cremona transformation of \( \mathbb{P}^n \). In fact, as is well-known, \( V_n \) can be explicitly obtained from \( \mathbb{P}^n \) as follows. First blow up the torus fixed points, then flip the (strict transforms) of the lines through these points, then flip the (strict transforms) of
planes through these points, . . . , up until, and not including, the half-dimensional linear subspaces. The resulting variety is $V_n$. For more, see [Cas03].

Since $V_n$ and the blow up of $\mathbb{P}^n$ at its torus fixed points are isomorphic in codimension 1, they have isomorphic Picard groups. We use a basis $H, E_1, \ldots, E_{n+1}$ for Pic($V_n$), which correspond to the hyperplane section and the exceptional divisors of the blown up $\mathbb{P}^n$. We have

$$[e_i] = E_i, \quad [\bar{e}_i] = (H - \sum_{j=1}^{n+1} E_j) + E_j$$

where $S_{n+1}$ permutes the $E_i$ leaving $H$ fixed, and the involution is represented by the following matrix

$$\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 - n & 0 & -1 & \cdots & -1 \\
1 - n & -1 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 - n & -1 & -1 & \cdots & 0
\end{pmatrix}.$$

For each $c \in \mathbb{Z}$ and $J \subset \{1, \ldots, n+1\}$, define

$$F_{c,J} := c \left( \sum_{i=1}^{n+1} E_i - H \right) - \sum_{j \in J} E_j.$$

Note that the involution takes $F_{c,J}$ to $F_{|J|-c,J}$. Then,

**Proposition 4.12.** The set of $F_{c,J}$ with

1. $|J| - \frac{n}{4} \leq c \leq \frac{n}{4}$, or
2. $\frac{n+2}{4} \leq c \leq |J| - \frac{n+2}{4}$.

form a full strong $(S_{n+1} \times C_2)$-stable exceptional collection on $V_n$ under any ordering of the blocks such that $|J|$ is (non-strictly) decreasing.

**Proof Sketch.** This collection is the same as that of [CT17, Theorem 6.6] up to a twist by a line bundle. Thus, we only sketch an argument here. One checks that the description of “forbidden cones” given by Borisov and Hua in [BH09] shows that relevant cohomology groups vanish — this shows that it is a strong exceptional collection. To prove generation, one considers the series of flips required to reach $\mathbb{P}^n$ blown up at $n+1$ points. Using the description of the semi-orthogonal decompositions in [BF17], the line bundles can be shown to generate the necessary admissible subcategories of each intermediate birational model. □

Since any centrally symmetric toric Fano variety is a product of projective lines and the varieties $V_n$, Lemma 3.11 yields the following:

**Corollary 4.13.** Any form of a centrally symmetric toric Fano variety admits a full strong exceptional collection consisting of vector bundles.
4.5. Toric varieties from the Weyl fans of type $A$. One method for identifying toric varieties with large symmetry groups is to start with root systems. Let $R$ be a root system in a Euclidean space $E$. The $\mathbb{Z}$-lattice generated by $R$ is denoted $M(R)$, while its dual in $E^\vee$ is denoted by $N(R)$. For every set $S$ of simple roots in $E$, we have the dual cone corresponding to a closed Weyl chamber

$$\sigma_S := \{ f \in E^\vee \mid \langle f, \alpha \rangle \geq 0, \forall \alpha \in S \}.$$ 

The cones $\sigma_S$ are the maximal cones for a fan $\Sigma_R$ in $E^\vee$. We denote the associated toric variety by $X(R)$. Recall that an automorphism of $R$ is an element of $\text{GL}(E)$ preserving $R$.

Let $W(R)$ be the Weyl group and $\Gamma(R)$ the symmetry group of the Dynkin diagram of $R$. It is well-known that $\text{Aut}(R) \simeq W(R) \rtimes \Gamma(R)$.

Any automorphism of $R$ induces an action on the fan $\Sigma(R)$, giving an homomorphism $\phi : \text{Aut}(R) \to \text{Aut}(\Sigma(R))$.

**Lemma 4.14.** The map $\phi : \text{Aut}(R) \to \text{Aut}(\Sigma(R))$ is an isomorphism.

*Proof.* First note that the set $R$ can be reconstructed from $\Sigma(R)$ by taking the union of the extremal rays generating the dual cones $\sigma_S^\vee$ for all $\sigma_S$. Thus any symmetry of the fan induces a symmetry of $R$. This gives the inverse map to $\phi$. □

Here we focus on the case $R = A_n$. In [LM00], the authors showed that $X(A_n)$ is a moduli space of rational curves with $(n + 1)$ marked points and 2 poles. Another useful proof appeared in [BB11].

Using this perspective, Castravet and Tevelev exhibited an exceptional collection on $X(A_n)$ that is stable under the action of permuting the marked points and flipping the poles, i.e., an $(S_{n+1} \rtimes C_2)$-stable collection. Here we demonstrate that Castravet and Tevelev’s exceptional collection satisfies the conditions of Proposition 3.7 and hence descends to an exceptional collection on any form of $X(A_n)$ (in characteristic 0).

To do this requires a bit of translating divisors and actions from the moduli-theoretic language to the toric language. We recall the moduli-theoretic language.

**Definition 4.15.** Let $N$ be a set of order $n$. A chain of polar $\mathbb{P}^1$’s is a $(\{0, \infty\} \cup N)$-marked linear nodal chain of $\mathbb{P}^1$’s with 0 on the left tail and $\infty$ on the right tail. A chain of polar $\mathbb{P}^1$’s is stable if

1. marked points do not coincide with nodes,
2. only $N$-marked points are allowed to coincide,
3. each component of the chain has at least three special points (nodes or marked points).

We write $LM_N$ for the corresponding moduli space. We also use $LM_n$ depending on the context. Note that the universal curve over $LM_n$ is isomorphic to $LM_{n+1}$.

**Theorem 4.16.** The toric variety $X(A_{n-1})$ is isomorphic to $LM_n$. Moreover, if we fix an embedding $A_{n-1} \to A_n$, the corresponding map $X(A_n) \to X(A_{n-1})$ is the universal curve. Moreover, $X(A_n) \to X(A_{n-1})$ is a toric morphism.

*Proof.* This is [LM00, Theorem 2.6.3]. See also [BB11, Theorem 3.19]. The map is consequently toric by [BB11, Proposition 1.4]. □
Under this isomorphism, the closures of the torus orbits on \(X(A_n)\) have the following moduli-theoretic description. Fix a partition \(N_1 \sqcup N_2 = N\) and let \(\delta_{N_i}\) denote the divisor parametrizing polar chains of length exactly 2 having the first marked by \(N_1\) and the last marked by \(N_2\). For a partition with more parts, \(N_1 \sqcup N_2 \sqcup \cdots \sqcup N_t = N\), one has the locus \(Z_{N_1,\ldots,N_t}\) parametrizing polar chains of length exactly \(t\), where the \(i\)-th \(\mathbb{P}^1\) is marked by \(N_i\). These loci are precisely the proper torus orbit closures on \(X(A_n)\).

Note that each loci is a complete intersection
\[
Z_{N_1,\ldots,N_t} := \delta_{N_1} \cap \delta_{N_1 \cup N_2} \cap \cdots \cap \delta_{N_1 \cup \cdots \cup N_{t-1}}.
\]
Moreover, we have an isomorphism
\[
Z_{N_1,\ldots,N_t} \simeq LM_{N_1} \times LM_{N_2} \times \cdots \times LM_{N_t}
\]
where the left node of each \(\mathbb{P}^1\) is marked with 0 and the right node is marked with \(\infty\). Thus, we have toric morphisms
\[
i_{N_1,\ldots,N_t} : LM_{N_1} \times LM_{N_2} \times \cdots \times LM_{N_t} \rightarrow LM_N.
\]
Also, for each subset \(K \subset N\), we get a forgetful map \(\pi_K : LM_N \rightarrow LM_K\), which is a toric morphism since it is a composition of maps from Theorem 4.16.

Recall there is a set of line bundles \(\mathbb{G}_N\) on \(LM_N\) \([\text{CT}17\text{, Definition 1.5}]\), and one generates a larger set \(H_N\) of sheaves via
\[
H_N := \{(i_{N_1,\ldots,N_t})_*(G_{l_1} \boxtimes \cdots \boxtimes G_{l_t}) \mid \forall N_1 \cup \cdots \cup N_t = N, \ G_{l_j} \in \mathbb{G}_{N_j}\},
\]
where \(i_{N_1,\ldots,N_t} : Z_{N_1,\ldots,N_t} \hookrightarrow LM_N\) is the inclusion.

**Theorem 4.17.** There is an ordering on the set
\[
CT_N := H_N \cup \left( \bigcup_{K \subseteq N} \{ \pi_K^*E \mid E \in H_K \} \right) \cup \{O\}
\]
making it into an \((S_N \times C_2)\)-stable exceptional collection under permutations of the two sets of markings.

**Proof.** This is \([\text{CT}17\text{, Proposition 1.5}]\). \(\square\)

**Proposition 4.18.** The action of \(S_{n+1} \times C_2\) given by permuting the two sets of marked points corresponds to the action of \(\text{Aut}(A_n)\) on \(X(A_n)\).

**Proof.** We use the standard presentation of the root system for \(A_n\) as \(e_i - e_j\) for \(1 \leq i < j \leq n + 1\) and follow \([\text{BB11}\text{, Construction 3.6}]\). The embedding \(A_n \hookrightarrow A_{n+1}\) gives the universal curve \(X(A_{n+1}) \rightarrow X(A_n)\). For \(i \in \{1,\ldots,n\}\), we take the \((n+1)\) projections \(A_{n+1} \rightarrow A_n\), whose kernels are generated by \(e_i - e_{n+1}\) for \(1 \leq i \leq n + 1\). These give sections \(s_i : X(A_n) \rightarrow X(A_{n+1})\). Finally, for the polar sections, we have the dual vector \(v_{n+2}\). The vectors \(v_{n+2}\) and \(-v_{n+2}\) give toric invariant divisors which are isomorphic to \(X(A_n)\) \([\text{BB11}\text{, Proposition 1.9}]\). The isomorphisms give the other sections \(s_0\) and \(s_\infty\).

The Weyl group is the permutation group of the \(e_i\), and hence of the \(e_i - e_{n+2}\). In particular, it permutes the \(s_i\). The outer involution acts on the fan by negation and thus exchanges the cone corresponding to \(v_{n+2}\) with the cone corresponding to \(v_{n+2}\). \(\square\)

**Corollary 4.19.** The set \(CT_N\) is \(\text{Aut}(\Sigma(A_n))\)-stable.
Proof. This is an immediate corollary of Lemma 4.14 and Proposition 4.18. □

Proposition 4.20. Each object in the collection $\mathbf{CT}_N$ is torus-equivariant.

Proof. Line bundles are always isomorphic to torus-equivariant line bundles, so all objects in $\mathbb{G}_N$ are torus-equivariant. There is a canonical equivariant structure on tensor products and on pullbacks by equivariant morphisms (see §2 of [BFK14]); thus each object $G_1 \boxtimes \ldots \boxtimes G_n$ is torus-equivariant for $G_{i_j} \in \mathbb{G}_{N_j}$. Let $i : Z \to X$ be shorthand for some map $i_{N_1, \ldots, N_t}$. There is a splitting of tori $T = S \times S'$ where $Z$ is an $S$-toric variety and $S'$ acts trivially on $i(Z)$. Let $\psi : T \to S$ denote the projection. We have a composition of functors

$$D^b(\text{Coh}_S Z) \to D^b(\text{Coh}_T Z) \to D^b(\text{Coh}_T X)$$

where the first map is the functor $\text{Res}_\psi$ (§2.9 of [BFK14]) and the second map is the $T$-equivariant pushforward (§2.5 of [BFK14]). This composition reduces to the ordinary pushforward $i_* : D^b(Z) \to D^b(X)$ when the equivariant structure is forgotten. We conclude that each object of $H_K$ is torus-equivariant and the result follows. □

We now prove the main result of this section:

Proposition 4.21. Let $k$ be a field of characteristic zero and $X$ a form of $X(A_n)$ over $k$. Then $X$ admits a full exceptional collection of sheaves.

Proof. Combining Theorem 4.17, Corollary 4.19, and Proposition 4.20 allows us to appeal to Proposition 5.7 and conclude that $\mathbf{CT}_N$ descends to an exceptional collection of sheaves on $X$. □

Remark 4.22. To remove the characteristic zero assumption one needs to extend generation results of [CT17] to nonzero characteristic. This could conceivably be done by reversing the flow of reasoning in [CT17], using the fact we know the collections for $V_n$ in any characteristic. We do not pursue this.

References

[AAGZ13] Alexey Ananyevskiy, Asher Auel, Skip Garibaldi, and Kirill Zainoulline. Exceptional collections of line bundles on projective homogeneous varieties. Adv. Math., 236:111–130, 2013.

[AB15] Asher Auel and Marcello Bernardara. Semiorthogonal decompositions and birational geometry of del pezzo surfaces over arbitrary fields. https://arxiv.org/abs/1511.07576, 11 2015.

[ABB14] Asher Auel, Marcello Bernardara, and Michele Bolognesi. Fibrations in complete intersections of quadrics, Clifford algebras, derived categories, and rationality problems. J. Math. Pures Appl. (9), 102(1):249–291, 2014.

[ADPZ15] Kenneth Ascher, Krishna Dasaratha, Alexander Perry, and Rong Zhou. Rational points on twisted K3 surfaces and derived equivalences. https://arxiv.org/abs/1506.01374, 06 2015.

[AKW17] Benjamin Antieau, Daniel Krashen, and Matthew Ward. Derived categories of torsors for abelian schemes. Adv. Math., 306:1–23, 2017.

[Ami58] S. A. Amitsur. The radical of field extensions. Bull. Res. Council Israel. Sect. F, 7F:1–10, 1957/1958.

[AT14] Nicolas Addington and Richard Thomas. Hodge theory and derived categories of cubic fourfolds. Duke Math. J., 163(10):1885–1927, 2014.

[Bat99] V. V. Batyrev. On the classification of toric Fano 4-folds. J. Math. Sci. (New York), 94(1):1021–1050, 1999. Algebraic geometry, 9.

[BB11] Victor Batyrev and Mark Blume. The functor of toric varieties associated with Weyl chambers and Losev-Manin moduli spaces. Tohoku Math. J. (2), 63(4):581–604, 2011.
[KN90] A. B. Kvichansky and D. Yu. Nogin. Exceptional collections on ruled surfaces. In Helices and vector bundles, volume 148 of London Math. Soc. Lecture Note Ser., pages 97–103. Cambridge Univ. Press, Cambridge, 1990.

[Kuz10] Alexander Kuznetsov. Derived categories of cubic fourfolds. In Cohomological and geometric approaches to rationality problems, volume 282 of Progr. Math., pages 219–243. Birkhäuser Boston, Inc., Boston, MA, 2010.

[LM00] A. Losev and Y. Manin. New moduli spaces of pointed curves and pencils of flat connections. Michigan Math. J., 48:443–472, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[LM14] Max Lieblich, Davesh Maulik, and Andrew Snowden. Finiteness of K3 surfaces and the Tate conjecture. Ann. Sci. Éc. Norm. Supé. (4), 47(2):285–308, 2014.

[Man66] Ju. I. Manin. Rational surfaces over perfect fields. Inst. Hautes Études Sci. Publ. Math., (30):55–113, 1966.

[MP97] A. S. Merkurjev and I. A. Panin. K-theory of algebraic tori and toric varieties. K-Theory, 12(2):101–143, 1997.

[Mus02] Mircea Mustaţă. Vanishing theorems on toric varieties. Tohoku Math. J. (2), 54(3):451–470, 2002.

[Nee96] Amnon Neeman. The Grothendieck duality theorem via Bousfield’s techniques and Brown representability. J. Amer. Math. Soc., 9(1):205–236, 1996.

[Orl16] Dmitri Orlov. Smooth and proper noncommutative schemes and gluing of DG categories. Adv. Math., 302:59–105, 2016.

[PN17] Nathan Prabhu-Naik. Tilting bundles on toric Fano fourfolds. J. Algebra, 471:348–398, 2017.

[Sat00] Hiroshi Sato. Toward the classification of higher-dimensional toric Fano varieties. Tohoku Math. J. (2), 52(3):383–413, 2000.

[Tab14] Gonçalo Tabuada. Additive invariants of toric and twisted projective homogeneous varieties via noncommutative motives. J. Algebra, 417:15–38, 2014.

[Tab15] Gonçalo Tabuada. Noncommutative motives, volume 63 of University Lecture Series. American Mathematical Society, Providence, RI, 2015. With a preface by Yuri I. Manin.

[Tho00] Jesper Funch Thomsen. Frobenius direct images of line bundles on toric varieties. J. Algebra, 226(2):865–874, 2000.

[Tœi12] Bertrand Toën. Derived Azumaya algebras and generators for twisted derived categories. J. Algebra, 326(2):658–684, 2010.

[Ueh14] Hokuto Uehara. Exceptional collections on toric Fano threefolds and birational geometry. Internat. J. Math., 25(7):1450072, 32, 2014.

[Via17] Charles Vial. Exceptional collections, and the Néron-Severi lattice for surfaces. Adv. Math., 305:895–934, 2017.

[VK84] V. E. Voskresenskiĭ and A. A. Klyachko. Toric Fano varieties and systems of roots. Izv. Akad. Nauk SSSR Ser. Mat., 48(2):237–263, 1984.

[WW82] Keiichi Watanabe and Masayuki Watanabe. The classification of Fano 3-folds with torus embeddings. Tokyo J. Math., 5(1):37–48, 1982.

[Xie17] Fei Xie. Toric surfaces over an arbitrary field. https://arxiv.org/abs/1610.06612, 05 2017.

[Yan14] Youlong Yan. Tilting sheaves on Brauer-Severi schemes and arithmetic toric varieties. Electronic Thesis and Dissertation Repository, 2014.
