Poisson and diffusion approximation of stochastic Schrödinger equation with control

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Abstract

"Quantum trajectories" are solutions of stochastic differential equations of non-usual type. Such equations are called Belavkin or stochastic Schrödinger equations and describe random phenomena in continuous measurement theory of open quantum system. Recent investigations deal with the control theory in such model. Heuristic rules or heavy analytic theory is used to derive the stochastic model describing this framework.

In this article the Belavkin equations in presence of control are obtained as a limit of a discrete concrete procedure. Thus it gives a rigorous justification of the Poisson and diffusion approximation in quantum measurement theory with control. Furthermore we investigate some example using control in quantum mechanics.

Introduction

Many recent developments in quantum mechanics deal with “stochastic Schrödinger equations” (also called Belavkin equation) [10]. These equations are classical stochastic differential equations which describe random phenomena in continuous measurement theory. The solutions of these equations are called “quantum trajectories”, they give account of the time evolution of reference state of open quantum system undergoing a continuous measurement.

A classical physical model ([10]) used in quantum optics is the one of interaction: a two-level atom is in contact with an environment (a continuous field) and we want to know
the evolution of the little system (the atom system) by performing a measurement. Because of the "wave packet reduction", an indirect continuous measurement is then performed on the field in order to not destroy the information of the atom. This procedure means that we consider an observable of the field, we get then a partial information of the atom.

The first mathematical result of the evolution of the atom system are due to Davies in [9]. He gives namely a description of the time evolution of the state of the atom system from which we study the detection of photon emission. Thanks to this description, heuristic rules can be used to derive stochastic Schrödinger equations. In order to obtain rigorous result, a heavy background in operator theory is necessary. The theory of Von Neumann algebra and Fock space is classically used to describe the field and the interaction is given by a time unitary evolution described by quantum Langevin equations. Using quantum filtering theory, the stochastic model can be obtained in a rigorous way. This method needs a high analytic technology which uses fine result in non-commutative probability ([4],[5]).

These technical difficulties are multiplied with the introduction of control theory in this framework. Many applications, in quantum optics or engineering, needs namely an exterior intervention. For example, deterministic control as the intensity of laser is used to modify the evolution of a system, such control is called "open loop control". One can imagine more complicated procedure which adapt the control at each time depending on the past of the experience. This last kind of control is called “closed loop control” or “feedback control”. It depends on the past of the quantum trajectory, so it is a stochastic control.

The mathematical framework rendering the control effect is the modification of the unitary evolution at each time. The stochastic model of classical measurement theory must be adapted to describe this additional element. This an upper step in quantum filtering and the obtaining of rigorous result is more tedious ([21],[6],[5]).

In order to get around all the heavy background, our approach of such problems is to use an approximation by a concrete discrete model. Indeed we can consider a discrete version of the previous continuous model of interaction called quantum repeated interaction. The setup is as follow. The field is represented as a chain of identical little quantum system (with finite degree of liberty). Each pieces of environment interacts one after the other with the atom during a time \( h \). Such model is shown to converge when \( h \) goes to zero to classical continuous model in [2]. With this framework, we can perform an indirect measurement at each interaction and the different possible results give rise to a random sequence of state. In [17] and [18] it is shown that this sequence, also called discrete quantum trajectory, converges in distribution to the solution of the Belavkin equations. The continuous model is then justify by a concrete discrete procedure using classical probability theory.

The equivalent of control in the discrete model is obtained by the choice of a control after each measurement, this procedure can depend or not of the different observations. The main goal of this article is to prove that stochastic models with control can be obtained as a limit of discrete controlled model.

This article is structured as follow.

The first section is devoted to the discrete model of quantum repeated interaction with control. The principle of indirect measurement is described in this framework. A
probability space is namely defined to give account of the random property of the discrete quantum trajectory. It is proved that this sequence of state is a classical controlled Markov chain. Besides we obtain a stochastic finite difference equation satisfied by this Markov chain and we focus on a particular case of a two-level atom.

In the second section we investigate the approximation question in the case of the two-level atom. We give the different asymptotic tools to come into the convergence problem. The aim is to show that the stochastic difference equations of section (1) can appear as a discrete stochastic Schrödinger equations.

The third section is then devoted to the continuous case. The Belavkin equation are obtained as limit of the discrete model. The stochastic continuous model is justify by the passage to the limit of the discrete quantum trajectory. Hence we prove the Poisson and diffusion approximation of the theory continuous quantum measurement with control.

In the last section we investigate some applications of the previous model. On the one hand we study a concrete example of an atom driving by a laser, we establish the suitable discrete model. The convergence theorem of section (3) allow to obtain the continuous model in such situation. On the other hand we come into the problem of "optimal control" which uses general stochastic control result. This is applied to our subject in the diffusive case.

1 Discrete control model for quantum trajectory

The notion of discrete quantum trajectory is attached with the notion of discrete quantum measurement theory as it is explained in the introduction. In this section we study the evolution of a quantum system undergoing a sequence of quantum measurement. The wave packet reduction phenomenon resulting of direct measurement impose to use another procedure in order to observe a significant dynamic. A general setup is the one of interaction. The little system is namely in contact with an environment and the measurement is performed on the interacting system. We get then a partial information of the little system.

In this section we study a discrete model where the environment is considered as a chain of identical quantum system. Each system interacts one after the other during a time $h$. This is the classical setup of repeated quantum interaction and a measurement is performed at each interaction on the piece of environment. The random character of quantum mechanics gives then rise to a sequence of random state which render the effect of the successive result of the observation. This sequence is called discrete quantum trajectory and this procedure is called repeated quantum measurement.

Furthermore at each interaction we introduce a control strategy which depends on the evolution of the sequence of the state. It means that following the result of one interaction, we modify the next interaction and so on. The next subsection is devoted to the mathematical description of the discrete quantum trajectory with the effect of the control. We define namely a probability space rendering the controlled repeated measurement.
1.1 Repeated quantum measurement with control

The main goal of this section is then to describe the mathematical model of repeated quantum measurement by introducing a control. It is an upper step in order to generalize the theory of indirect measurement. We focus on probabilistic properties of the sequence representing the discrete quantum trajectory, in particular the Markov character.

As it was announced, the principle of indirect measurement is used in order to study the evolution of a quantum system. A small system (from which we want information) interacts with a field (a photon stream for example) on which we perform a measurement. In parallel, we add a control strategy.

Let us describe the discrete interaction setup. In order to fix the idea, we begin the description without control. The interacting field is represented as a chain of independent copies of pieces of environment. Each copy is a little quantum system and is represented by a Hilbert space $\mathcal{H}$; the small system is described by $\mathcal{H}_0$. The copies interact with the small system one after the other. The mathematical description of one interaction is as follow.

The compound system resulting of the coupling is described by the tensor product $\mathcal{H}_0 \otimes \mathcal{H}$ and the interaction is characterized by a unitary evolution $U$ acting on this Hilbert space. In the Schrödinger picture, if $\rho$ denotes any state on the tensor product the evolution is given by:

$$\rho \rightarrow U \rho U^\dagger.$$

Here the time of interaction is not specified. It will be important when considering approximations, but we keep in mind that the unitary operator $U$ depends on the interaction length time. After the above interaction, we consider a second copy of $\mathcal{H}$ which interacts with $\mathcal{H}_0$ in the same fashion and so on.

The sequence of interaction is described by the state space:

$$\Gamma = \mathcal{H}_0 \otimes \bigotimes_{k \geq 1} \mathcal{H}_k$$

The countable tensor product $\bigotimes_{k \geq 1} \mathcal{H}_k$ means the following. We consider that $\mathcal{H}_0$ and $\mathcal{H}$ are finite dimensional Hilbert spaces. Let $\{X_0, X_1, \ldots, X_n\}$ be a fixed orthonormal basis of $\mathcal{H}$, the projector on $X_0$: $|X_0\rangle\langle X_0|$ being the ground state (or vacuum state) of $\mathcal{H}$ (this is the bra-ket notation in mathematical physics see the remark below). The tensor product is taken with respect to $X_0$ (for all details concerning countable tensor product see [2]).

**Remark:** A vector $Y$ in a Hilbert space $\mathcal{H}$ is represented by the application $|Y\rangle$ from $\mathbb{C}$ to $\mathcal{H}$ which acts with the following way $|Y\rangle(\lambda) = |\lambda Y\rangle$. The linear form on $\mathcal{H}$ are represented by the operators $\langle Z|$ which acts on the vector $|Y\rangle$ by $\langle Z||Y\rangle = \langle Z, Y \rangle$ where $\langle , \rangle$ denotes the scalar product of $\mathcal{H}$.

The unitary evolution describing the $k$-th interaction is given by $U_k$ which acts non trivially like $U$ on $\mathcal{H}_0 \otimes \mathcal{H}_k$ whereas it acts like the identity operator on the other copies. If $\rho$ is a state on $\Gamma$, the effect of the $k$-th interaction is:

$$\rho \rightarrow U_k \rho U_k^*$$
Hence the result of the $k$ first interactions is described by the operator $V_k$ on $\mathcal{B}(\Gamma)$ defined by the recursive formula:

$$\begin{cases}
V_{k+1} &= U_{k+1}V_k \\
V_0 &= I
\end{cases}$$

and its effect on a state is given by:

$$\rho \rightarrow V_k \rho V_k^*.$$ 

The above equation gives us the description of quantum repeated interaction without control. The theory of repeated quantum measurement in this setup is detailed in [2]. Our purpose in this article is to introduce a control strategy.

The principle of control strategy is to modify the interaction at each step. This is the modification of the unitary operator which gives account of the control strategy ([6]). We can think about a controller which modifies the interaction following what he has observed (modification of the intensity of a laser for example).

From now, we introduce the time interaction $h$. Each pieces of environment interact ones after the others with the small system during a time $h$, a control procedure and a quantum measurement is performed at each interaction.

As a consequence at the $k$-th interaction the unitary operator $U_k$ depends on a parameter $u_{k-1}$ where $u_k \in \mathbb{R}^N$ for all $k \geq 0$, it depends also on the time interaction $h$. The sequence $u = (u_k(h))$ represents the control strategy and depends also on $h$. The $k$-th unitary operator is then denoted by $U_k(h, u_{k-1}(h))$. Here the operator $U_k(h, u_{k-1}(h))$ acts on $\mathcal{H}_0 \otimes \mathcal{H}_k$ and as identity elsewhere. So we define the sequence of operator on $\Gamma$:

$$V^u_k = U_k(h, u_{k-1}(h))U_{k-1}(h, u_{k-2}(h))\ldots U_1(h, u_0(h)).$$

It represents the unitary operator which describes the $k$ first interactions with control, the indice $u$ concerns the strategy ($u_k(h)$).

This is the mathematical model of quantum repeated interaction with control. Let us move on to the principle of quantum repeated indirect measurement. The idea is to perform a measurement with respect of an observable of the field at each interaction.

Let $A = \sum_{j=1}^{p} \lambda_j P_j$ be any observable on $\mathcal{H}$, then we consider its natural ampliation which defines an observable on $\Gamma$ by:

$$A^k := \bigotimes_{j=0}^{k} I \otimes \sum_{j=1}^{p} \lambda_j P_j \otimes \bigotimes_{j \geq k+1} I$$

The accessible data are the eigenvalues of $A^k$ and the result is inherently random. If $\rho$ is any state on $\Gamma$ the observation of $\lambda_j$ obeys to the probability law:

$$P[\text{to observe } \lambda_j] = Tr[\rho P^k_j], \quad j \in \{1, \ldots, p\}$$

If we have observed the eigenvalue $\lambda_j$ the “projection” postulate called “wave packet reduction” imposes the new state to be

$$\rho_j = \frac{P^k_j \rho P^k_j}{Tr[\rho P^k_j]}.$$
**Remark:** This new state is then the reference state of our system. If we want to perform another measurement of the same observable $A^k$ with the new state, it is straightforward that $P[\text{to observe } \lambda_j] = 1$.

This above description gives the principle of measurement on the $k$-th copy. The quantum repeated measurement principle is the combination of the measurement principle and the repeated quantum interactions. Physically it means that each photons interacts with the atom and we perform a measurement after each interaction. After each procedure we have a new state given by the projection postulate: this is our discrete quantum trajectory.

The initial state on $\Gamma$ is chosen to be

$$\mu = \rho \otimes \bigotimes_{j \geq 1} \beta_j$$

where $\rho$ is any state on $\mathcal{H}_0$ and each $\beta_i = \beta$ is the reference state on $\mathcal{H}$. We denote by $\mu^u_k$ the state representing the new state after the $k$ first interactions, that is:

$$\mu^u_k = V^u_k \mu V^u_k \Star_k.$$

Let us now define the probabilistic framework and describe the effect of the successive measurements. We put $\Sigma = \{1, \ldots, p\}$ and on $\Sigma^N$ we define the cylinders of size $k$:

$$\Lambda_{i_1, \ldots, i_k} = \{\omega \in \Omega^N/\omega_1 = i_1, \ldots, \omega_k = i_k\}.$$  

We endow $\Sigma^N$ with the $\sigma$-algebra generated by all these sets. This is the cylinder $\sigma$-algebra. Remark that for all $j$, the unitary operator $U_j$ commutes with all $P^k$ for all $k < j$, for $\{i_1, \ldots, i_k\}$ (corresponding to the index of eigenvalues). Thus we can define the following non normalized state:

$$\tilde{\mu}^u_k(i_1, \ldots, i_k) = I \otimes P_{i_1} \otimes \ldots \otimes P_{i_k} \otimes I \ldots \mu^u_k I \otimes P_{i_1} \otimes \ldots \otimes P_{i_k} \otimes I \ldots$$

$$= P^k_{i_k} \ldots P^1_{i_1} \mu^u_k I P^1_{i_1} \ldots P^k_{i_k}.$$  

So we define a probability law on $\Sigma^N$, defined on the cylinders:

$$P[\Lambda_{i_1, \ldots, i_k}] = \text{Tr}[\tilde{\mu}^u(i_1, \ldots, i_k)].$$

This probability law satisfies the Kolmogorov consistency criterion, it defines then a probability on $\Sigma^N$. Hence we define the following random sequence of states:

$$\tilde{\rho}^u_k : \Sigma^N \longrightarrow \mathcal{B}(\Gamma)$$

$$\omega \longrightarrow \tilde{\rho}^u_k(\omega_1 \ldots \omega_k) = \frac{\tilde{\mu}^u_k(\omega_1 \ldots \omega_k)}{\text{Tr}[\tilde{\mu}^u_k(\omega_1 \ldots \omega_k)]}.$$  

This random sequence of states is our discrete quantum trajectory and the operator $\tilde{\rho}^u_k(i_1, \ldots, i_k)$ represents the state, if we have observed the results $(i_1, \ldots, i_k)$ during the $k$ first measurement with the control strategy $u$. This fact is precise in the following proposition.
Proposition 1 Let \( u \) be any strategy and \( (\tilde{\rho}_k^u) \) be the above random sequence of states we have for all \( \omega \in \Sigma^N \):

\[
\tilde{\rho}_{k+1}^u(\omega) = \frac{P_{\omega_{k+1}}^{k+1} U_{k+1}(h, u_k(h)) \tilde{\rho}_k^u(\omega) U_k^*(h, u_k(h)) P_{\omega_{k+1}}^{k+1}}{\text{Tr} \left[ \tilde{\rho}_k^u(\omega) U_k^*(h, u_k(h)) P_{\omega_{k+1}}^{k+1} U_{k+1}(h, u_k(h)) \right]}.
\]

This proposition is obvious but summarizes the quantum repeated measurement principle. The sequence \( \tilde{\rho}_k^u \) is the quantum trajectory rendering the effect of the successive measurements on \( \mathcal{G} \) with control.

In this article we consider three kind of control.

Definition 1 Let \( u \) be a control strategy which takes value in \( \mathbb{R}^n \).

1. If there exists some function \( u \) from \( \mathbb{R} \) to \( \mathbb{R}^n \) such that for all \( k \) \( u_k(h) = u(kh) \), the control strategy is called deterministic. It is also called “open loop control”.

2. If there exists some function \( u \) from \( \mathbb{R} \times \mathcal{B}(\mathcal{H}_0) \) to \( \mathbb{R}^n \) such that for all \( k \) \( u_k(h) = u(kh, \rho_k) \), the control strategy is called Markovian. It is also called “closed loop control” or “feedback control”. If for all \( k \) \( u_k(h) = u(\rho_k) \), this is an homogeneous Markovian strategy.

The following theorem is an easy consequence of the proposition (1) and the previous definition.

Theorem 1 For all control strategy \( u \), the sequence \( (\tilde{\rho}_n^u) \) is a non homogeneous Markov chain valued on the set of states of \( \mathcal{H}_0 \otimes_{i \geq 1} \mathcal{H}_i \) It is described as follows:

\[
P \left[ \tilde{\rho}_{n+1}^u = \mu / \tilde{\rho}_n^u = \theta_n, \ldots, \tilde{\rho}_0^u = \theta_0 \right] = P \left[ \tilde{\rho}_{n+1}^u = \mu / \tilde{\rho}_n^u = \theta_n \right]
\]

If \( \tilde{\rho}_n^u = \theta_n \) then \( \tilde{\rho}_{n+1}^u \) takes one of the values:

\[
\mathcal{H}_{i,n+1}^u(\theta_n) = \frac{P_{i,n+1}^n (U_{n+1}(h, u_n(h)) \theta_n U_{n+1}^*(h, u_n(h)) P_{i,n+1}^n)}{\text{Tr} \left[ (U_{n+1}(h, u_n(h)) \theta_n U_{n+1}^*(h, u_n(h)) P_{i,n+1}^n) P_{i,n+1}^n \right]} \quad i = 1, \ldots, p
\]

with probability \( \text{Tr} \left[ (U_{n+1}(h, u_n(h)) \theta_n U_{n+1}^*(h, u_n(h)) P_{i,n+1}^n) P_{i,n+1}^n \right] \).

Proof: This theorem is obvious thanks to the description given by the proposition (1) and the fact that the control strategy depends only on the past of the quantum trajectory.

In quantum theory it was assumed that we do not have access to the field (because it is more complicated, or impossible), we just have access to the small system. So the mathematical tool which gives account of this phenomenon is the partial trace operation given by the following theorem.
**Definition-Theorem 1** If we have a state \( \alpha \) on a tensor product \( \mathcal{H} \otimes \mathcal{K} \). There exists a unique state \( \eta \) on \( \mathcal{H} \) which is characterized by the property:

\[
\forall X \in \mathcal{B}(\mathcal{H}) \quad Tr_{\mathcal{H}}[\eta X] = Tr_{\mathcal{H} \otimes \mathcal{K}}[\alpha(X \otimes I)].
\]

Hence to obtain the trajectory concerning the small system we have to take the partial trace on \( \mathcal{H}_0 \). Let \( E_0 \) denotes the partial trace on \( \mathcal{H}_0 \) with respect to \( \otimes_{k \geq 1} \mathcal{H}_k \). We then define a random sequence of states on \( \mathcal{H}_0 \). For all \( \omega \) in \( \Sigma^N \) we put:

\[
\rho_n^u(\omega) = E_0[\rho_n^u(\omega)].
\]

This defines a sequence of state on \( \mathcal{H}_0 \) which contains the "partial" information given by the measurement and we have the following theorem which is a consequence of theorem (1).

**Theorem 2** For all control strategy \( u \), the random sequence defined by formula (5) is a non homogeneous Markov chain which takes value in the set of states on \( \mathcal{H}_0 \). If \( \rho_n^u = \chi_n \) then \( \rho_{n+1}^u \) takes one of the values:

\[
E_0 \left[ \frac{I \otimes P_i \hat{U}_{n+1}(h, u_n(h))(\chi_n \otimes \beta)\hat{U}^*_{n+1}(h, u_n(h)) I \otimes P_i}{Tr[\hat{U}_{n+1}(h, u_n(h))(\chi_n \otimes \beta)\hat{U}^*_{n+1}(h, u_n(h)) I \otimes P_i]} \right] \quad i = 1 \ldots p
\]

with probability \( Tr[\hat{U}_{n+1}(h, u_n(h))(\chi_n \otimes \beta)\hat{U}^*_{n+1}(h, u_n(h)) P_i] \).

**Remark:** Let us stress that \( \frac{I \otimes P_i \hat{U}_{n+1}(h, u_{n+1}(h))(\chi_n \otimes \beta)\hat{U}^*_{n+1}(h, u_{n+1}(h)) I \otimes P_i}{Tr[\hat{U}_{n+1}(h, u_{n+1}(h))(\chi_n \otimes \beta)\hat{U}^*_{n+1}(h, u_{n+1}(h)) I \otimes P_i]} \) is a state on \( \mathcal{H}_0 \otimes \mathcal{H} \), we have kept the notation \( E_0 \) to denote the partial trace on \( \mathcal{H}_0 \). Furthermore the operator \( \hat{U}_{n+1}(h, u_{n}(h)) \) is an operator on \( \mathcal{H}_0 \otimes \mathcal{H} \), it is defined through \( U_{n+1}(h, u_{n+1}(h)) \). Indeed it acts on \( \mathcal{H}_0 \otimes \mathcal{H} \) as \( U_{n+1}(h, u_{n}(h)) \) acts on \( \mathcal{H}_0 \otimes \mathcal{H}_{n+1} \).

Thanks to the above description, we can express a discrete evolution equation. Let denote:

\[
\mathcal{L}_i^u(\rho) = E_0 \left[ I \otimes P_i \hat{U}_k(h, u_{k-1}(h))((\rho \otimes \beta)\hat{U}_k^*(h, u_{k-1}(h)) I \otimes P_i) \right] \quad i = 1 \ldots p,
\]

we then have for all \( \omega \in \Sigma^N \) and all \( k > 0 \):

\[
\rho_{k+1}^u(\omega) = \sum_{i=0}^{p} \frac{\mathcal{L}_i^{k+1}(\rho_k^u(\omega))}{Tr[\mathcal{L}_i^{k+1}(\rho_k^u(\omega))]} \chi_{i+1}(\omega)
\]

(6)

where \( \chi_{i+1}(\omega) = \chi_i(\omega_k) \).

The above description, namely the equation (6) is going to be used to obtain an approximation of a continuous model of quantum measurement with control strategy. We come into the particular case of a two level atom. This case is often the central case in physical considerations, it is the subject of the following section.
1.2 A two-level atom

The case of a two-level atom is characterized by $\mathcal{H}_0 = \mathcal{H} = \mathbb{C}^2$. It is referred to the model of an atom in contact with a chain of spin.

In this section it is shown that the discrete controlled process $(\rho_n^u)$ is the solution of a finite difference stochastic equation which appears later as a discrete equivalent of a stochastic differential equation. Usually such type of equation is obtained thanks to the projection postulate we choose a suitable basis. Let $(\mathcal{X})$ be an orthonormal basis of $\mathcal{C}^L_k$. This theory, which refers to the classical, needs a high analytic technology in this context (Von Neumann algebra, non commutative probability, filtration and conditional expectation in Von Neumann algebra,...). The main goal of this section is to get around all this high technology and give a nice but rigorous presentation of the discrete quantum filtering equations for a two-level atom model (it is easy to generalize to higher dimension).

Let us show that we can obtain such a formula for $(\rho_n^u)$.

\[
\rho_{k+1}^u = f(\rho_k^u, X_{k+1}).
\]  

(7)

where $(X_k)_k$ is a sequence of random variables. In order to obtain such a formula we study how to obtain $\rho_{k+1}^u$ through the measurement after the $(k+1)$-th interaction when the initial state after $k$ procedures is $\rho_k^u$.

The state $\rho_k^u$ can be namely considered as an initial state (according to the Markov property of theorem (2)). Thus we consider a single interaction with a system $(\mathcal{H}, \beta)$ (actually this is the $k + 1$-th copy). Remember that each Hilbert space are $\mathbb{C}^2$. We consider an observable of the form $A = \lambda_0 P_0 + \lambda_1 P_1$ and the unitary operator describing the $k + 1$-th interaction is a unitary $4 \times 4$ matrix. We consider it as an operator on $\mathcal{H}_0$:

\[
U_{k+1}(h, u_k(h)) = \left( \begin{array}{cc} L_{00}(kh, u_k(h)) & L_{01}(kh, u_k(h)) \\ L_{10}(kh, u_k(h)) & L_{11}(kh, u_k(h)) \end{array} \right)
\]

where each $L_{ij}(kh, u_k(h))$ are operators on $\mathcal{H}_0$. In order to compute the state given by the projection postulate we choose a suitable basis. Let $(X_0 = \Omega, X_1 = X)$ be an orthonormal basis of $\mathbb{C}^2$. For $\mathcal{H}_0 \otimes \mathcal{H}$, we consider the following basis $\Omega \otimes \Omega, X \otimes \Omega, \Omega \otimes X, X \otimes X$. This basis allows us to consider the above way of writing for $U_{k+1}(h, u_k(h))$. For $\beta$ we choose:

\[
\beta = |\Omega\rangle \langle \Omega|.
\]

As a consequence, the state after the interaction is:

\[
\rho_{k+1}^u = U_{k+1}(h, u_k(h))(\rho_k^u \otimes \beta)U_{k+1}^*(h, u_k(h)) \\
= \left( \begin{array}{cc} L_{00}(kh, u_k(h)) \rho_k^u L_{00}^*(kh, u_k(h)) & L_{01}(kh, u_k(h)) \rho_k^u L_{01}^*(kh, u_k(h)) \\ L_{10}(kh, u_k(h)) \rho_k^u L_{10}^*(kh, u_k(h)) & L_{11}(kh, u_k(h)) \rho_k^u L_{11}^*(kh, u_k(h)) \end{array} \right).
\]

We apply the indirect quantum measurement principle. For the two possible results of the measurement remember that we have:

\[
\mathcal{L}_0^{u,k+1}(\rho_k) = E_0[I \otimes P_0 \mu_{k+1}^u I \otimes P_0],
\]

(8)

\[
\mathcal{L}_1^{u,k+1}(\rho_k) = E_0[I \otimes P_1 \mu_{k+1}^u I \otimes P_1].
\]

(9)
These are operators on $H_0$. We denote the two probability by $p_{k+1}^u = \text{Tr}[\mathcal{L}_0^{u,k+1}(\rho_k^u)]$ and $q_{k+1}^u = \text{Tr}[\mathcal{L}_1^{u,k+1}(\rho_k^u)]$. The non normalized state: $\mathcal{L}_0^{u,k+1}(\rho_k^u)$ appears with probability $p_{k+1}^u$ and $\mathcal{L}_1^{u,k+1}(\rho_k^u)$ with probability $q_{k+1}^u$.

Thanks to this two probabilities we can define a random variable $\nu_{k+1}$ on $\{0, 1\}$ by:

\[
\begin{cases} 
\nu_{k+1}(0) = 0 & \text{with probability } p_{k+1}^u \\
\nu_{k+1}(1) = 1 & \text{with probability } q_{k+1}^u 
\end{cases}
\]

As a consequence we can describe the state on $H_0$ with the following equation. We have for all $\omega \in \Sigma_N$:

\[
\rho_{k+1}^u(\omega) = \frac{\mathcal{L}_0^{u,k+1}(\rho_k^u(\omega))}{p_{k+1}^u(\omega)}(1 - \nu_{k+1}(\omega)) + \frac{\mathcal{L}_1^{u,k+1}(\rho_k^u(\omega))}{q_{k+1}^u(\omega)}\nu_{k+1}(\omega).
\] (10)

In order to obtain the final discrete quantum evolution equation we consider the centered and normalized random variable:

\[
X_{k+1} = \frac{\nu_{k+1} - q_{k+1}^u}{\sqrt{q_{k+1}^u p_{k+1}^u}}.
\]

We define the associated filtration on $\{0, 1\}^N$:

\[
\mathcal{F}_k = \sigma(X_i, i \leq k).
\]

So by construction we have $E[X_{k+1}/\mathcal{F}_k] = 0$ and $E[X_{k+1}^2/\mathcal{F}_k] = 1$. Using $(X_k)$ we can write the discrete evolution equation for our quantum trajectory:

\[
\rho_{k+1}^u = \mathcal{L}_0^{u,k+1}(\rho_k^u) + \mathcal{L}_1^{u,k+1}(\rho_k^u) + [-\sqrt{\frac{q_{k+1}^u}{p_{k+1}^u}}\mathcal{L}_0^{u,k+1}(\rho_k^u) + \sqrt{\frac{p_{k+1}^u}{q_{k+1}^u}}\mathcal{L}_1^{k+1}(\rho_k^u)]X_{k+1}.
\] (11)

The above equation can be considered in a general way and the unique solution starting from $\rho_0$ is our quantum trajectory. This is the discrete quantum filtering equation with control for a two-level atom undergoing repeated measurement. Remember that this equation depends on $h$ through the unitary operator $U$. In the following section, we introduce some suitable asymptotic for this operator in order to consider the limit $h$ goes to zero in the equation (11).

## 2 Description of asymptotic

The unitary operator describing the interaction at each step depends on $h$. We want to consider the limit $h$ goes to zero in order to obtain the stochastic model of continuous quantum measurement with control. Let us start with the discrete quantum trajectory describing by the filtering equation (11), this section is devoted to introduce asymptotic of
the operator appearing in this equation. In order to obtain the continuous model, the aim is to consider the equation (11) as a discrete stochastic differential equation. Indeed the discrete process \((p^u_k)\) satisfies:

\[
\begin{align*}
\rho^u_{k+1} - \rho^u_0 &= \sum_{i=0}^k [\rho^u_{i+1} - \rho^u_i] \\
&= \sum_{i=0}^k [\mathcal{L}^u_{0,i} + \mathcal{L}^u_{1,i}] + \sum_{i=0}^k \left[ -\sqrt{\frac{q_{i+1}^u}{p_{i+1}^u}} \mathcal{L}^u_{0,i} + \sqrt{\frac{p_{i+1}^u}{q_{i+1}^u}} \mathcal{L}^u_{1,i} \right] X_{t+1} \\
&= \sum_{i=0}^k \left[ -\sqrt{\frac{q_{i+1}^u}{p_{i+1}^u}} \mathcal{L}^u_{0,i} + \sqrt{\frac{p_{i+1}^u}{q_{i+1}^u}} \mathcal{L}^u_{1,i} \right] X_{t+1} \\
&= \sum_{i=0}^k \left[ -\sqrt{\frac{q_{i+1}^u}{p_{i+1}^u}} \mathcal{L}^u_{0,i} + \sqrt{\frac{p_{i+1}^u}{q_{i+1}^u}} \mathcal{L}^u_{1,i} \right] X_{t+1} \\
&= \sum_{i=0}^k \left[ -\sqrt{\frac{q_{i+1}^u}{p_{i+1}^u}} \mathcal{L}^u_{0,i} + \sqrt{\frac{p_{i+1}^u}{q_{i+1}^u}} \mathcal{L}^u_{1,i} \right] X_{t+1} \\
&= \sum_{i=0}^k \left[ -\sqrt{\frac{q_{i+1}^u}{p_{i+1}^u}} \mathcal{L}^u_{0,i} + \sqrt{\frac{p_{i+1}^u}{q_{i+1}^u}} \mathcal{L}^u_{1,i} \right] X_{t+1} \\
&= \sum_{i=0}^k \left[ -\sqrt{\frac{q_{i+1}^u}{p_{i+1}^u}} \mathcal{L}^u_{0,i} + \sqrt{\frac{p_{i+1}^u}{q_{i+1}^u}} \mathcal{L}^u_{1,i} \right] X_{t+1} \tag{12}
\end{align*}
\]

This equation appears then as a discrete equivalent of a stochastic differential equation, this is precise below. Let us give the asymptotic condition of the different operator to express the candidate to be the limit.

Consider a partition of \([0, T]\) in subintervals of equal size \(1/n\). The time of interaction is supposed now to be \(h = 1/n\), the dynamic laws concerning the evolution of an open quantum system imposed that the unitary operator of evolution depends on the time interaction. The unitary operator describing the \(k\)-th interaction with control is of the following form:

\[
U_{k+1}(n, u_k(n)) = \begin{pmatrix}
L_{00}(k/n, u_k(n)) & L_{01}(k/n, u_k(n)) \\
L_{10}(k/n, u_k(n)) & L_{11}(k/n, u_k(n))
\end{pmatrix},
\]

where we have introduced the dependance in \(n\).

Without control strategy, Attal-Pautrat [2] have shown that if the coefficients \(L_{ij}\) of \((U_n)\) follows well-defined asymptotic, the operator \(V_{nt} = U_{nt} \cdots U_1\) (defined on \(\Gamma\)) weakly converges to an operator \(V_t\). The operator process \((V_t)\) acts on a Fock space and satisfies a quantum Langevin equation.

In the control approach, we can not have a similar result because the control strategy can depends on the quantum trajectory \((p^u_k)\). In the deterministic control strategy case, a convergence result for \(V_{nt}^u\) can be obtained, but this is not our subject. However as no control is a particular control with \(u = 0\), similar asymptotic must be kept to describe the coefficients \(L_{ij}\).

In order to give the asymptotic we need to define some function which will appear in the definition of the different coefficients. Let \(\mathbb{H}_2(\mathbb{C})\) denote the \(2 \times 2\) self-adjoint matrix, we define two function:

\[
H : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{H}_2(\mathbb{C}) \\
(t, s) \mapsto H(t, s)
\]

\[
C : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{M}_2(\mathbb{C}) \\
(t, s) \mapsto C(t, s)
\]

The regularity of \(H\) and \(C\) will be imposed later. The function \(H\) represents an Hamiltonian depending on time and \(C\) is any operator function. The coefficient of \(U\) are defined thanks to this two function.
Thus we put for all $k$:

\[
L_{00}(k/n, u_k(n)) = I + \frac{1}{n} \left( -iH(k/n, u_k(n)) - \frac{1}{2} C(k/n, u_{k+1}(n))C(k/n, u_k(n))^* \right) \\
+ \circ \left( \frac{1}{n} \right) \tag{13}
\]

\[
L_{00}(k/n, u_k(n)) = \frac{1}{\sqrt{n}} C(k/n, u_k(n)) + \circ \left( \frac{1}{n} \right) \tag{14}
\]

Furthermore we suppose that all $\circ (1/n)$ are uniform in $k$. All the justification about the choice of asymptotic can be found in.

As the different probability $p_k$, $q_k$ and the operators $L^k, u_i$ depends on the observable, we are going to classify the observable in order to determine the asymptotic behavior of the equation (11).

Remember that we have:

\[
X_k(n)(i) = \begin{cases} 
-\sqrt{\frac{q_{k+1}(n)}{p_{k+1}(n)}} & \text{with probability } p_{k+1}(n) \text{ if } i = 0 \\
\sqrt{\frac{p_{k+1}(n)}{q_{k+1}(n)}} & \text{with probability } q_{k+1}(n) \text{ if } i = 1
\end{cases} \tag{15}
\]

Let define for all state $\rho$ and for all $(k, n) \in \mathbb{N}^2$:

\[
\mathcal{J}(t, s)(\rho) = C(t, s) \rho C^*(t, s).
\]

If the observable is of the form $A = \lambda_0 \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) + \lambda_1 \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right)$, we obtain the asymptotic for the probabilities:

\[
p^u_{k+1}(n) = 1 - \frac{1}{n} Tr \left[ \mathcal{J}(k/n, u_k(n))(\rho^u_k(n)) \right] + \circ \left( \frac{1}{n} \right)
\]
\[
q^u_{k+1}(n) = \frac{1}{n} Tr \left[ \mathcal{J}(k/n, u_k(n))(\rho^u_k(n)) \right] + \circ \left( \frac{1}{n} \right)
\]

The discrete equation becomes:

\[
\rho^u_{k+1}(n) - \rho^u_k(n) = \\
\frac{1}{n} L(k/n, u_k(n))(\rho^u_k(n)) + \circ \left( \frac{1}{n} \right) \\
+ \left[ \frac{\mathcal{J}(k/n, u_k(n))(\rho^u_k(n))}{Tr \left[ \mathcal{J}(k/n, u_k(n))(\rho^u_k(n)) \right]} - \rho^u_k(n) + \circ(1) \right] \sqrt{q^u_{k+1}(n)p^u_{k+1}(n)} X_{k+1}(n) \tag{17}
\]

where for all state $\rho$ we have:

\[
L(t, s)(\rho) = -i[H(t, s), \rho] - 1/2 \{C(t, s)C^*(t, s), \rho\} + \mathcal{J}(t, s)(\rho)
\]

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with for all matrix $P, Q \ [P, Q] = PQ - QP$ and $\{P, Q\} = PQ + QP$.

In the same fashion, if the observable is non diagonal in the basis $(\Omega, X)$, we consider $P_0 = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$ and $P_1 = \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix}$ we have:

\[
p_{k+1} = p_{00} + \frac{1}{\sqrt{n}} Tr [\rho_k^u (p_{01} C(k/n, u_{k+1}(n)) + p_{10} C^*(k/n, u_k(n)))] \\
+ \frac{1}{n} Tr [\rho_k^u p_{00} (C(k/n, u_k(n)) + C^*(k/n, u_k(n)))] + o \left(\frac{1}{n}\right)
\]

\[
q_{k+1}^u = q_{00} + \frac{1}{\sqrt{n}} Tr [\rho_k^u (q_{01} C(k/n, u_k(n)) + q_{10} C^*(k/n, u_k(n)))] \\
+ \frac{1}{n} Tr [\rho_k^u q_{00} (C(k/n, u_k(n)) + C^*(k/n, u_k(n)))] + o \left(\frac{1}{n}\right)
\]

The discrete equation becomes:

\[
\rho_{k+1}^u - \rho_k^u = \\
\frac{1}{n} L(k/n, u_k(n)) (\rho_k^u) + o \left(\frac{1}{n}\right) + [e^{i\theta} C(k/n, u_k(n)) \rho_k^u e^{-i\theta} \rho_k^u C^*(k/n, u_k(n))] \\
- Tr[\rho_k^u (e^{i\theta} C(k/n, u_k(n)) + e^{-i\theta} C^*(k/n, u_k(n)))] \rho_k^u + o(1)] \frac{1}{\sqrt{n}} X_{k+1}(n) \quad (18)
\]

where $\theta$ is a parameter which appears during the computation.

These both equations (16, 18) show that there are two different evolution for the discrete quantum trajectory. The following section is devoted to the continuous limit. In the first case we show that the discrete process $(\rho_k^u)$ converges to the solution of a stochastic differential equation driven by a counting process whereas the second case has a diffusive evolution.

### 3 Convergence to the continuous model

Even without control, as it was announced, the setup of the stochastic model describing continuous measurement needs a heavy background in quantum filtering theory. Thanks to the work of Davies, stochastic differential equation can be derived with heuristic rules. These equations are called Belavkin or stochastic Schrodinger equations and the solutions are called continuous quantum trajectory. The difficulties necessary to obtain such model in a rigorous way contrast with the intuition and the description of the discrete procedure which gives rise to the discrete quantum trajectories.

In this article, the continuous model with control is obtained as a weak limit of the discrete quantum trajectory $(\rho_k^u)$. Two kind of stochastic differential equations of non-usual type appear following the observable which is considered. One is driven by a Brownian motion and the other type is driven by a counting process.
3.1 The diffusive case

We consider the general case of a Markovian strategy. Thus for all \((k, n) \in \mathbb{N}^2\) we have \(u_k(n) = u(k/n, \rho^n_k)\) where \(u\) is a function from \(\mathbb{R} \times M_2(\mathbb{C})\) to \(\mathbb{R}^n\). The regularity of this function will be specified later. Such strategy means that we perform a control which depends at each step of the evolution of \((\rho^n_k)\). A deterministic strategy is a particular case where \(u(k/n, \rho_k) = u(k/n)\).

As it was announced the diffusive behavior appears when a non diagonal observable is considered. Let \(A\) be any such observable and \(\rho_0\) be any initial state. The discrete quantum trajectory satisfies:

\[
\rho^n_{[nt]} = \rho_0 + \sum_{k=1}^{[nt]-1} \rho^n_{k+1} - \rho^n_k = \sum_{k=1}^{[nt]-1} \left( \frac{1}{n} L(k/n, u_k(n))(\rho^n_k) + \circ \left( \frac{1}{n} \right) \right) + \sum_{k=1}^{[nt]-1} \left[ C(k/n, u_k(n))\rho^n_k + \rho^n_k C^*(k/n, u_k(n)) \right]
\]

\[ - Tr[\rho^n_k(C(k/n, u_k(n)) + C^*(k/n, u_k(n)))] \rho^n_k + o(1) \frac{1}{\sqrt{n}} X_{k+1}(n), \]

where we have suppressed the parameter \(\theta\). Let us stress that this process depends on \(n\).

Thanks to this equation we can define the processes for all \(t\):

\[
W_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} X_k(n) \\
V_n(t) = \frac{[nt]}{n} \\
\rho^n_n(t) = \rho^n_{[nt]}(n) \\
u_n(t, W) = u([nt]/n, W) \ \forall W \in M_2(\mathbb{C}) \\
C_n(t, s) = C([nt]/n, s) \ \forall s \\
H_n(t, s) = H([nt]/n, s) \ \forall s
\]

Hence the discrete quantum trajectory satisfies the stochastic differential equation:

\[
\rho^n_n(t) = \rho_0 + \int_0^t L_n(s-, \rho^n_n(s-), u_n(s-, \rho^n_n(s-)))dV_n(s) + \int_0^t \Theta_n(s-, \rho^n_n(s-), u_n(s-, \rho^n_n(s-)))dW_n(s)
\]

\[ (20) \]
where for all \(n,s\) and state \(\rho\) we have:

\[
L_n(s, \rho, a) = -i[H_n(s, a), \rho] - \frac{1}{2}\{C_n(s, a)C_n^*(s, a), \rho\} + C_n(s, a) \rho C_n^*(s, a) + o(1) \\
\Theta_n(s, \rho, a) = C_n(s, a)\rho + \rho C_n^*(s, a) - Tr[\rho (C_n(s, a) + C_n^*(s, a))]\rho + o(1)
\]

(21)

(22)

The aim of this section is to prove that this sequence (20) converges in distribution for the Skorohod topology to the solution of the following stochastic differential equation:

\[
d\rho^u_t = L(t, \rho^u_t, u(t, \rho^u_t))dt + \Theta(t, \rho_t, u(t, \rho^u_t))dW_t,
\]

(23)

where \((W_t)\) is a standard Brownian motion. For \(L\) and \(\Theta\), we have the following definition.

\[
L(s, \rho, a) = -i[H(s, a), \rho] - \frac{1}{2}\{C(s, a)C^*(s, a), \rho\} + C(s, a) \rho C^*(s, a) \\
\Theta(s, \rho, a) = C(s, a)\rho + \rho C^*(s, a) - Tr[\rho (C(s, a) + C^*(s, a))]\rho.
\]

(24)

(25)

This stochastic differential equation is called diffusive Stochastic Schrödinger equation with control or diffusive-Belavkin equation with control. The solution of this equation are naturally called continuous controlled quantum trajectory. So in order to consider the existence and the uniqueness of solution for such equation we must impose regularity of the different function.

**Remark:** Concerning the regularity, we suppose that for all \(T > 0\) there exists a constant \(M(T)\) and \(K(T)\) such that the function \(L\) and \(\Theta\) satisfy for all \(t \leq T\) and \((\mu, \rho) \in M_2(\mathbb{C})^2:\)

\[
\sup \{\|L(t, \mu, a) - L(t, \rho, a)\|, \|\Theta(t, \mu, a) - \Theta(t, \rho, a)\|\} \leq K(T)\|\mu - \rho\|
\]

\[
\sup \{\|L(t, \rho, a)\|, \sup \{\|\Theta(t, \rho, a)\|\} \leq M(T)(1 + \|\rho\| + \|a\|)
\]

(26)

About the first variable, we just impose continuity for \(H, C\) and \(u\) which implies the same property for all the other function. Regarding (26), this is the Lipschitz property uniformly on \([0, T]\). This is classical conditions in stochastic differential equations theory.

If the global Lipschitz condition is not realized (in concrete application for example) one can use a truncation method for the different function (this impose local Lipschitz condition). In this case, if the truncation is large enough, the equation (20) is not modified because the process \((\rho_n(t))\) takes value in the set of state and so is bounded.

As all the process with which we deal are bounded, so we can suppose that all the function satisfy (26).

**Theorem 3** Let \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) be a probability space of a standard Brownian motion \((W_t)\). Assume that \(L\) and \(\Theta\) satisfy (26). Let \(\rho\) be any \(2 \times 2\) matrix. The stochastic differential equation:

\[
d\rho_t^u = L(t, \rho_t^u, u(t, \rho_t^u))dt + \Theta(t, \rho_t^u, u(t, \rho_t^u))dW_t,
\]

(27)

admits a unique solution \((\rho_t^u)\) with \(\rho_0 = \rho\).
This theorem is just a consequence of the Lipschitz condition (26) (see for a proof). If we consider such equation in a general way with the initial condition being a state, it is not easy to see that the solution takes value in the set of state. This physical interest will be a consequence of the convergence theorem.

For all $T > 0$ we define $D[0, T]$ the space of càdlàg process of $\mathbb{M}_2 C$ endowed with the Skorohod topology. Before to express the theorem of convergence we need some technical hypothesis about $L_n, \Theta_n, L$ and $\Theta$.

Let $T_1[0, \infty)$ denote the set of nondecreasing mapping $\lambda$ from $[0, \infty)$ to $[0, \infty)$ with $\lambda(0) = 0$ such that $\lambda(t + h) - \lambda(t) \leq h$ for all $t, h \geq 0$. Starting with the definition of $L_n, \Theta_n, L$ and $\Theta$ we define for $L_n$ for example:

$$\tilde{L}_n : D[0, \infty) \times T_1[0, \infty) \longrightarrow D[0, \infty)$$

such that for all $t \geq 0$ $L_n(X) \circ \lambda(t) = L_n(\lambda(t), X_{\lambda(t)}, u_n(\lambda(t), X_{\lambda(t)}))$. We introduce the two following condition concerning a function $\tilde{G}$ and a sequence $\tilde{G}_n$ as above.

(C1) For each compact subset $K \in D[0, \infty) \times T_1[0, \infty)$ and $t > 0,$

$$\sup_{(X, \lambda) \in K} \sup_{s \leq t} \| \tilde{G}_n(X, \lambda)(s) - \tilde{G}(X, \lambda)(s) \| \rightarrow 0$$

(C2) For $(X_n, \lambda_n) \in D[0, \infty) \times T_1[0, \infty)$ / $\sup_{s \leq T} \| X_n(s) - X(s) \| \rightarrow 0$

and

$$\sup_{s \leq t} | \lambda_n(s) - \lambda(s) | \rightarrow 0 \text{ for each } t > 0 \text{ implies}$$

$$\sup_{s \leq t} \| \tilde{G}(X_n, \lambda_n)(s) - \tilde{G}(X, \lambda)(s) \| \rightarrow 0$$

This condition are often satisfied, for example if $H$, $L$ and $u$ are continuous.

**Theorem 4** Let $\rho$ be any state on $\mathcal{H}_0$. Let $(\rho_n(t))$ be the discrete quantum trajectory satisfying:

$$\rho_n(t) = \rho_0 + \int_0^t L_n(s-, \rho_n(s-), u_n(s-, \rho_n(s-)) dV_n(s)$$

$$+ \int_0^t \Theta_n(s-, \rho_n(s-), u_n(s-, \rho_n(s-)) dW_n(s).$$

Let $(\rho^u_t)$ be the solution of:

$$\rho^u_t = \rho + \int_0^t L(s, \rho^u_s, u(s, \rho^u_s)) ds + \int_0^t \Theta(s, \rho^u_s, u(s, \rho^u_s)) dW_s,$$  

starting with $\rho_0 = \rho$.

Assume that $L_n$, $\Theta_n$, $\tilde{L}$ and $\Theta$ satisfy (C1) and (C2).

Then for all $T > 0$ the process $(\rho_n(t))$ converges in distribution in $D[0, T]$ to the process $(\rho_t)$. 

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If the initial condition is a state, a consequence of this theorem is that the solution of (30) is a stochastic process which takes value in the set of state, because all the conditions to be a state are closed with respect to the weak convergence. The boundness character of such process implies that a truncation method is in fact unnecessary and confirms that we can suppose that (26) is always satisfied. The theorem (3) and (4) implies then that the Stochastic Schrödinger equation with control (30) admits a unique solution which takes value in the set of state.

As a consequence the discrete procedure which gives rise to the discrete quantum trajectory is a mean to establish and justify the continuous stochastic model of diffusive-Belavkin equation with control. A totally different approach for such model is given by in [5].

The convergence theorem (4) is a consequence of the following version of theorem of Kurtz and Protter [13],[14].

**Theorem 5** Let \((F^n_t)\) be a filtration for all \(n\) such that \((W^n_n(t))\) is a \((F^n_t)\)-martingale. Let \((W^n_n, V^n_n)\) be defined by. Suppose that \((W^n_n, V^n_n)\) converges in distribution in the Skorohod topology to \((W, V)\) where \((W)\) is a standard Brownian motion and \(V = t\) for all \(t \geq 0\).

Suppose that:

\[
\sup_n \{E^n[[W^n_n, W^n_n]_t]\} < \infty, \tag{31}
\]

and suppose that \(\tilde{L}_n, \tilde{L}, \tilde{\Theta}_n\) and \(\tilde{\Theta}\) satisfy the condition (C1) and C(2).

Suppose that the process \((\rho^n_n(t))\) is \((F^n_t)\)-adapted. Hence the process \((\rho^n_n(t))\) solution of the equation (20) converges in distribution in \(D[0, T]\) for all \(T > 0\) to the unique solution \((\rho^n_t)\) of (30).

The condition (31) and the weak convergence of \((W^n)\) is attached with the notion of "goodness" for a sequence of stochastic process. It is an important property to consider weak convergence of stochastic integral with respect from \((W^n_n(t))\) (see [13]).

On the one hand we verify the property for \((W^n_n(t))\) and on the other hand we deal with the functions \(\tilde{L}_n, \tilde{L}, \tilde{\Theta}_n\) and \(\tilde{\Theta}\).

Let \(t > 0\) and \(n\) be fixed we define:

\[F^n_t = \sigma(X_i, i \leq [nt]).\]

It is obvious that \((W^n(t))\) and \((\rho^n_n(t))\) are \((F^n_t)\)-adapted because the strategy is supposed to be Markovian. We have the following property:

**Proposition 2** The process \((W^n(\cdot), F^n_t)\) is a martingale and converges to a standard Brownian motion \(W_t\) when \(n\) goes to infinity.

Moreover we have: \(\sup_n E[[W^n_n, W^n_n]_t] < \infty.\)

As a consequence, we have the convergence in distribution for the process \((W_n, V_n)\) to \((W, V)\) when \(n\) goes to infinity.
Proof: The definition of the random variable $X_k$ is made to have $E[X_{i+1}/\mathcal{F}_s^n] = 0$ which implies $E[\frac{1}{n} \sum_{i=[ns]+1}^{[nt]} X_i/\mathcal{F}_s^n] = 0$ for $t > s$. Thus if $t > s$ we have the martingale property:

$$E[W_n(t)/\mathcal{F}_s^n] = W_n(s) + E\left[\frac{1}{\sqrt{n}} \sum_{i=[ns]+1}^{[nt]} X_i/\mathcal{F}_s^n\right] = W_n(s).$$

By definition of $[Y, Y]$ for a stochastic process we have

$$[W_n, W_n]_t = W_n(t)^2 - 2 \int_0^t W_n(s-)dW_n(s) = \frac{1}{n} \sum_{i=1}^{[nt]} X_i^2$$

Thus we have

$$E[[W_n, W_n]_t] = \frac{1}{n} \sum_{i=1}^{[nt]} E[X_i^2] = \frac{1}{n} \sum_{i=1}^{[nt]} E[E[X_i^2/\sigma\{X_i, l < i\}]]$$

$$= \frac{1}{n} \sum_{i=1}^{[nt]} 1 = \frac{[nt]}{n}.$$

Hence we have $sup_n E[[W_n, W_n]_t] \leq t < \infty$.

Let us prove the weak convergence. It is enough to prove that (see):

$$\lim_{n \to \infty} E[((W_n, W_n)_t - t)^2] = 0.$$ 

We have $E[X_i^2] = E[E[X_i^2/\sigma\{X_i, l < i\}]] = 1$ and if $i < j$ $E[(X_i^2 - 1)(X_j^2 - 1)] = E[(X_i^2 - 1)(X_j^2 - 1)/\sigma\{X_i, l < j\}] = E[(X_i^2 - 1)E[(X_j^2 - 1) = 0$. Thus we have:

$$E[((W_n, W_n)_t - \frac{[nt]}{n})^2] = \frac{1}{n^2} \sum_{i=1}^{[nt]} E[(X_i^2 - 1)^2] + \frac{1}{n^2} \sum_{i<j} E[(X_i^2 - 1)(X_j^2 - 1)]$$

$$= \frac{1}{n^2} \sum_{i=1}^{[nt]} E[(X_i^2 - 1)^2]$$

Thanks to the fact that $p_{00}$ and $q_{00}$ are not equal to zero (because the observable is not diagonal) each terms $E[(X_i^2 - 1)^2]$ is bounded uniformly in $i$ so we have:

$$\lim_{n \to \infty} E[((W_n, W_n)_t - \frac{[nt]}{n})^2] = 0$$

Like $\frac{[nt]}{n} \to t$ in $L_2$ we have the desired convergence. The weak convergence for Skorohod topology for $(W_n, V_n)$ is straightforward. \qed

As the condition $(C1)$ and $(C2)$ are assumed to be satisfied, thanks to the property (2) and the theorem (5) of Kurtz and Protter, we have proved the convergence theorem (4) in the diffusive case.
All the technical difficulties of Von Neumann algebra or non-commutative probability is gotten around. The physical model is then rigorously justify as a limit of a discrete procedure and we have obtained all the model including control theory. This section was devoted to the diffusive-Belavkin equation with control, this is the non-diagonal character of the observable (in the fixed basis) which implies the weak convergence of \((W_n(t))\) to \((W_t)\). The case of a diagonal observable gives rise to a counting noise, it is investigated in the next section.

3.2 Poisson approximation of control quantum measurement

In this section we investigate the convergence of the discrete quantum trajectory when the measurement is performed with respect to a diagonal observable in the basis \((\Omega, X)\). In this situation the discrete quantum trajectory satisfy:

\[
\rho^u_{n[t]}(t) = \rho_0 + \sum_{k=0}^{\lfloor nt \rfloor - 1} \frac{1}{n} \left[ L(k/n, u_k(n)) (\rho^u_k) - J(k/n, u_k(n)) (\rho^w_k) + Tr[J(k/n, u_k(n)) (\rho^w_k)] \rho_k \right] + \sum_{k=0}^{\lfloor nt \rfloor - 1} \frac{J(k/n, u_k(n)) (\rho^w_k)}{Tr[J(k/n, u_k(n)) (\rho^w_k)]} \nu_{k+1} \nu_k + 1
\]

(32)

Let define some function:

\[
\rho^u_n(t) = \rho^u_{n[t]}, \\
N_n(t) = \sum_{k=1}^{[nt]} \nu_k, \quad V_n(t) = \frac{[nt]}{n}, \\
R_n(t, \rho, a) = R([nt]/n, \rho, a), \quad Q_n(t, \rho, a) = Q([nt]/n, \rho, a)
\]

With suitable definition for \(R\) and \(Q\) as in the diffusive case, we can write then:

\[
\rho^u_n(t) = \int_0^t R_n(s-, \rho^u_n(s-), u_n(s-, \rho^u_n(s-))) dV_n(s) + \int_0^t Q_n(s-, \rho^u_n(s-), u_n(s-, \rho^u_n(s-))) dN_n(s)
\]

Remember that \(\nu_k\) satisfy:

\[
\nu_{k+1}(0) = 0 \quad \text{with probability} \quad p_{k+1}(n) = 1 - \frac{1}{n} Tr[J(k/n, u_{k+1}(n)) (\rho^w_k)] + o \left( \frac{1}{n} \right) \\
\nu_{k+1}(1) = 1 \quad \text{with probability} \quad q_{k+1}(n) = \frac{1}{n} Tr[J(k/n, u_{k+1}(n)) (\rho^w_k)] + o \left( \frac{1}{n} \right)
\]

It appears clearly that the evolution is not diffusive. The driving process of the stochastic differential equation in this situation is actually a counting process. In order to introduce this fact let us consider for all \(t > 0\) the term \(E[\sum_{k=0}^{[nt]-1} \nu_{k+1}]\). By conditioning with
\( G_k = \sigma \{ \nu_i, i \leq k \} \) one can show:

\[
E \left[ \sum_{k=0}^{\lfloor nt \rfloor - 1} \nu_{k+1} \right] = \int_0^t E \left[ Tr \left[ J_n(s-, u_n(s-, \rho_n^u(s-)))(\rho_n^u(s-)) \right] \right] dV_n(s) + o(1)
\]

So if we assume that there is a continuous time process \((\rho_t)\) which is the limit of \((\rho_n^u(t))\) when \(n\) goes to infinity, we must have a convergence for the process \((N_n(t))\) to a continuous time counting process.

In order to rigorously obtain it, we must use random Poisson measure theory (see for the general theory [12]). As often in Poisson approximation, random coupling theory is used to obtain convergence theorem. The solution of the stochastic differential equation and the discrete quantum trajectory is going to be realized in the same probability space.

Before to give the associated stochastic differential equation in this framework, we introduce a suitable probability space to use random coupling theory.

Let \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) be a filtered probability space which supports a Poisson point process \(N\) on \(\mathbb{R} \times \mathbb{R}\). We have the random Poisson measure denoted by:

\[
N(\omega, ds, dx)\text{ whose the compensator is } dx \otimes dt.
\]

Indeed for all Borel subset \(B\) we have:

\[
P[N(B) = k] = \frac{\Lambda(B)^k}{k!} \exp(-\Lambda(B)),
\]

where \(\Lambda\) denotes the Lebesgue measure on \(\mathbb{R} \times \mathbb{R}\). Hence we have \(E[N(B)] = \Lambda(B)\). Moreover for all sequence \((A_i)\) of disjoints Borel subset the random variable \((N(A_i))\) are mutually independent. Besides the random measure is supposed to be \(\mathcal{F}_t\) adapted (see [12] for more details).

We define the following sequence of random variable which are defined on the set of states:

\[
\tilde{\nu}_{k+1}(\eta, \omega) = 1_{N(\omega, G_k(\eta)) > 0}
\]

where \(G_k(\eta) = \{(t, u)/\frac{k}{n} \leq t < \frac{k+1}{n}, 0 \leq u \leq -n\ln(Tr[L_0^{k+1}(n)(\eta)])\}\). Let \(\rho_0 = \rho\) be any state and \(T > 0\), we define the process \((\tilde{\rho}_k)\) for \(k < \lfloor nT \rfloor\) by the recursive formula:

\[
\tilde{\rho}_{k+1} = L_0^{k+1}(\tilde{\rho}_k) + L_1^{k+1}(\tilde{\rho}_k) + \left[ -\frac{L_0^{k+1}(\tilde{\rho}_k)}{Tr[L_0^{k+1}(\tilde{\rho}_k)]} + \frac{L_1^{k+1}(\tilde{\rho}_k)}{Tr[L_1^{k+1}(\tilde{\rho}_k)]} \right] (\tilde{\nu}_{k+1}(\tilde{\rho}_k, \cdot) - Tr[L_0^{k+1}(\tilde{\rho}_k)])
\]

Thanks to a property of the definition of Poisson measure, the following proposition is obvious:

**Proposition 3** Let \(T\) be fixed. The discrete process \((\tilde{\rho}_k^u)_{k<\lfloor nT \rfloor}\) defined by (34) have the same distribution of the discrete quantum trajectory \((\rho_k^u)_{k<\lfloor nT \rfloor}\) defined by the quantum repeated measurement.

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We have then defined the discrete quantum trajectory in the probability space of a Poisson point process \( N \). We can express the continuous time stochastic differential equation:

\[
\mu_t^u = \mu_0 + \int_0^t R(s-, \mu_{s-}^u, u(s-, \mu_{s-}^u))ds
+ \int_0^t \int_{\mathbb{R}} Q(s-, \mu_{s-}^u, u(s-, \mu_{s-}^u))1_{0 < x < \text{Tr}[\mathcal{J}(t, u(s-, \mu_{s-}^u)) \mu_{s-}^u]} N(., ds, dx)
\]

The solution is here denoted by \((\mu)\) in order to not mistake with the discrete processes. The existence and uniqueness of a solution is given by the following theorem due to Jacod and Protter in [11]:

**Theorem 6** Let \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) be a probability space which supports a Poisson point process \( N \). Assume that \( Q \) and \( R \) satisfy the condition (26) and are continuous in the first variable. Hence for all \( X_0 \in \mathbb{M}_2(\mathbb{C}) \), the following stochastic equation:

\[
X_t = X_0 + \int_0^t R(s-, X_{s-}, u(s-, X_{s-}))ds
+ \int_0^t \int_{\mathbb{R}} Q(s-, X_{s-}, u(s-, X_{s-}))1_{0 < x < \text{sup}\{0, \text{Re}(\text{Tr}[\mathcal{J}(s-, u(s-, X_{s-}))])\}} N(., ds, dx)
\] (35)

admits a unique solution \((X_t)\).

Furthermore for all càdlàg process \((Y_t)\) which takes value in \( \mathbb{M}_2(\mathbb{C}) \), the process \((\tilde{N}_t^Y)\) defined by:

\[
\tilde{N}_t^Y = \int_0^t \int_{\mathbb{R}} 1_{0 < x < \text{sup}\{0, \text{Re}(\text{Tr}[\mathcal{J}(s-, u(s-, Y_{s-}))])\}} N(., ds, dx)
\] (36)

is a counting process whose the stochastic intensity (predictable compensator) is:

\[
t \mapsto \int_0^t \text{sup}\{0, \text{Re}(\text{Tr}[\mathcal{J}(t, u(s-, Y_{s-}))])\}
\]

We do not give the proof of this theorem. The existence and uniqueness of \((X_t)\) is provided by the fact that the ordinary differential equation \(dX_t = R(t, X_t)ds\) is solving and the fact that for all \( t \) and for all càdlàg matricial process \((Y_t)\):

\[
\int_0^t \text{sup}\{0, \text{Re}(\text{Tr}[\mathcal{J}(s-, u(s-, Y_{s-}))])\} < \infty.
\]

This last property ensures namely that for all process \((Y_t)\), the process \((\tilde{N}_t^Y)\) is a counting process without time explosion. The following remark give an idea of the general proof of the theorem (6).

**Remark:** Suppose that for there exists a constant \( K \) such that:

\[
\forall (t, X) \in \mathbb{R}_+ \times \mathbb{M}_2(\mathbb{C}), \text{ sup}\{0, \text{Re}(\mathcal{J}(t, u(t, X)))(X))\} < K,
\]

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the random function \( t \to N(., [0, K] \times [0, t]) = \mathcal{N}_t \) defines a standard Poisson process with intensity \( K \). Thus for the filtration \( \mathcal{F}_t \) we can choose the natural filtration of this process. The Poisson random measure and the previous process generate on \([0, T]\) (for a fixed \( T \)) a sequence \( \{(\tau_i, \xi_i), i \in \{1, \ldots, N_i\}\} \). Each \( \tau_i \) represents the jump time of \( \mathcal{N} \). Moreover the random variables \( \xi_i \) are random uniform variables on \([0, K]\). Consequently we can write the solution in the following way:

\[
X_t = X_0 + \int_0^t R(s-, X_{s-}, u(s-, X_{s-}))ds + \sum_{i=1}^{N_t} Q(\tau_i-, X_{\tau_i-}, u(\tau_i-, X_{\tau_i-}))1_{0 \leq \xi_i \leq \text{Re}(\text{Tr}[\mathcal{J}(\tau_i-, u(\tau_i-, X_{\tau_i-}))(X_{\tau_i-})])},
\]

For our purpose we can remark that for all process \((\mu_t)\) which takes value in the set of state of \( \mathbb{C}^2 \) we have:

\[
\text{Re}(\text{Tr}[\mathcal{J}(s-, u(s-, \mu_{s-}))](\mu_{s-})) = \text{Tr}[\mathcal{J}(s-, u(s-, \mu_{s-}))(\mu_{s-})] > 0.
\]

Thus we have the following theorem which expresses the convergence result in this situation and gives the jump-Belavkin equation model with control.

**Theorem 7** Let \((\tilde{\rho}_{[nt]}^u)_{0 \leq t \leq T}\) be the discrete quantum trajectory defined by:

\[
\tilde{\rho}_{[nt]} = \rho_0 + \sum_{k=0}^{[nt]-1} R(k/n, p_k^u, u_n(k/n, p_k^u)) + \sum_{k=0}^{[nt]-1} Q(k/n, p_k^u, u_n(k/n, p_k^u))\tilde{v}_{k+1}(p_k^u, \cdot).
\]

Hence if \( \mu_0 = \rho_0 \), for all \( T > 0 \) we have the weak convergence of \((\tilde{\rho}_{[nt]}^u)_{0 \leq t \leq T}\) in \( \mathcal{D}[0, T]\) for the Skorohod topology to the process \((\mu_t)\) solution of the stochastic differential equation:

\[
\mu_t^u = \mu_0 + \int_0^t R(s-, \mu_{s-}^u, u(s-, \mu_{s-}^u))ds + \int_0^t \int_\mathbb{R} Q(s-, \mu_{s-}^u, u(s-, \mu_{s-}^u))1_{0 < x < \text{Tr}[\mathcal{J}(s-, u(s-, \mu_{s-}^u))(\mu_{s-}^u)]}N(., ds, dx).
\]

Thus we have the following corollary which concerns physical considerations about the fact that the process takes value in the set of state.

**Corollary 1** The solution \((\mu_t)\) of (38) is a stochastic process which takes value in the set of state. Moreover the stochastic process \((\tilde{N}_t)\) defined by:

\[
\tilde{N}_t = \int_0^t \int_\mathbb{R} 1_{0 < x < \text{Tr}[\mathcal{J}(s, u(s-, \mu_{s-}))(\mu_{s-})]}N(., ds, dx),
\]

is a counting process with stochastic intensity:

\[
t \mapsto \int_0^t \text{Tr}[\mathcal{J}(s-, u(s-, \mu_{s-}))(\mu_{s-})]ds.
\]

For all \( T > 0 \) the process \((N_u(t))\) converges in \( \mathcal{D}[0, T]\) to \((\tilde{N}_t)\).
We do not give the proof of the theorem (7). This proof is very technical and all the important and interesting details are in [18]. The idea is to compare the discrete quantum trajectory with an Euler scheme of (30) with Poisson approximation technics. The Euler scheme approximation theorem for equation with counting process with intensity can be also find in.

Thus the theorem (7) express the Poisson approximation model for continuous jump-Belavkin equation with control.

4 Example and application

Starting with the description of quantum trajectories with control, this section is devoted to some applications of this model.

Firstly a continuous model of a two-level atom controlled by a laser is derived from a discrete description. We shall focus on the counting process, this model is called Resonance fluorescence in quantum optics. Secondly we deal with the problem of “optimal control” and we applied the classical result of stochastic control in our situation.

4.1 Laser monitoring atom

This section is devoted to the description of an example of quantum measurement using control: Resonance Fluorescence in presence of a laser. A two level atom is namely in contact with an electromagnetic field (the laser). This coupling system is in contact with a photon counter which detects the photon emission. The discrete setup is as follow.

Let us describe one interaction. The small system is representing by $\mathcal{H}_0$ and any initial state $\rho$. The laser is representing by $(\mathcal{H}_l, \mu^l)$ and the photon counter by $(\mathcal{H}_c, \beta^c)$. Each Hilbert space are $\mathbb{C}^2$ endowed with the orthonormal basis $(\Omega, X)$ and the unitary operator is denoted by $U$. The compound system after interaction is:

$$\mathcal{H}_0 \otimes \mathcal{H}_l \otimes \mathcal{H}_c,$$

and the state after interaction is:

$$\alpha = U(\rho \otimes \mu^l \otimes \beta^c)U^*$$

Let $(\Omega \otimes \Omega \otimes \Omega, X \otimes X \otimes X, \Omega \otimes \Omega \otimes X, \Omega \otimes X \otimes \Omega, \Omega \otimes X \otimes X, X \otimes X \otimes X)$ be an orthonormal basis of $\mathcal{H}_0 \otimes \mathcal{H}_l \otimes \mathcal{H}_c$. In this basis, as in the previous section, the unitary operator is considered as a $4 \times 4$ matrix: $(L_{ij}(n))_{0 \leq i,j \leq 3}$ where each $L_{ij}(n)$ are operator on $\mathcal{H}_0$ and depends on the time interaction.

For the different reference state we choose:

$$\mu^l = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \beta^c = |X_0\rangle \langle X_0|$$

Hence for the state $\alpha = (\alpha_{uv})_{0 \leq u,v \leq 3}$ after interaction we have:

$$\alpha_{uv} = (aL_{u0}(n)\rho + bL_{u1}(n)\rho)L^*_{v0}(n) + (cL_{v0}(n)\rho + dL_{v1}(n)\rho)L^*_{v1}(n)$$

(40)
The measurement is performed on the counter side. Let \( A \) denote any observable of \( \mathcal{H}_c \) then \( I \otimes I \otimes A \) denotes the corresponding observable on \( \mathcal{H}_0 \otimes \mathcal{H}_l \otimes \mathcal{H}_c \). We perform a measurement and by partial trace operation with respect to \( \mathcal{H}_l \otimes \mathcal{H}_c \) we obtain a new state on \( \mathcal{H}_0 \).

The control strategy is realized by the use of the laser. If we choose the time interaction to be \( 1/n \) and the state of the laser is:

\[
\mu_k = \left( \begin{array}{cc}
    a_k & b_k \\
    c_k & d_k 
\end{array} \right),
\]

where for \( a_k \) we have \( a_k = a(k/n) \) and a similar definition for the other coefficient.

If \( \rho_k \) denotes the state on \( \mathcal{H}_0 \) after \( k \) first measurement, for \( \alpha_{uv}^{k+1} = U_{k+1}(n)(\rho_k \otimes \mu_k \otimes \beta)U_{k+1}^*(n) \) we have:

\[
\alpha_{uv}^{k+1}(n) = (a(k/n)L_{u0}(n)\rho_k + b(k/n)L_{u1}(n)\rho_k)L_{v0}^*(n) \\
+ (c(k/n)L_{v0}(n)\rho_k + d(k/n)L_{v1}(n)\rho_k)L_{v1}^*(n)
\]

We are not exactly in the same situation of the previous section where the effect of the control was rendered by the unitary operator. Moreover the state of the interacting system was \( |X_0\rangle\langle X_0| \) where \( X_0 \) is the first vector of a fixed orthonormal basis. However we can replace this model in this setup if we consider a G.N.S representation of \((\mathcal{H}_l \otimes \mathcal{H}_c, \mu_k \otimes \beta)\) ([1]). The G.N.S representation of a finite dimensional Hilbert space is described as follow.

Let \( \mathcal{H} = \mathbb{C}^n \) be a finite dimensional Hilbert space endowed with a reference state \( \rho \).

We consider \( B(\mathcal{H}) \) the space of endomorphisms of \( \mathcal{H} \) equipped with the scalar product:

\[
\langle A, B \rangle = \text{Tr}[\rho A^* B].
\]

Let \( I \) be the identity operator, we choose it as a first vector of an orthonormal basis of \( B(\mathcal{H}) \). Then we have the G.N.S representation \( \pi \) of \( B(\mathcal{H}) \) into \( B(B(\mathcal{H})) \) given by:

\[
\pi(A)B = AB.
\]

Hence we have: \( \langle I, \pi(A)I \rangle = \text{Tr}[\rho A] \). With this representation we have that \( \rho = |I\rangle\langle I| \).

This remark concerning G.N.S representation is to justify that we can use the previous result to obtain the convergence. In this case such way is not necessary and make heavy the computation. Just with the description of the stochastic difference equation we can obtain the continuous model.

The most interesting case is the counting case where \( A = |\Omega\rangle\langle \Omega| \). The counting process renders namely the number of photon which is detected in the photon counter. The asymptotic for the unitary operator follows the asymptotic of Attal-Pautrat in [2]. Let \( \delta_{ij} = 1 \) if \( i = j \) we denote:

\[
\epsilon_{ij} = \frac{1}{2}(\delta_{0i} + \delta_{0j})
\]

The coefficients must follow the convergence condition:

\[
\lim_{n \to \infty} n^{\epsilon_{ij}}(L_{ij}(n) - \delta_{ij}I) = L_{ij}
\]
where $L_{ij}$ are operator on $H_0$.

Let $P_0 = |\Omega\rangle\langle \Omega|$ and $P_1 = |X\rangle\langle X|$ be the eigenprojector of $A$. If $\rho_k$ denotes the random state after $k$ measurements we denote:

$$L_{00}^{k+1}(\rho_k) = E_0[I \otimes I \otimes P_0(U_{k+1}(n)(\rho_k \otimes \mu^I_k \otimes \beta)U^*_k(n))I \otimes I \otimes P_0]$$
$$= \alpha_{00}^{k+1}(n) + \alpha_{11}^{k+1}(n)$$

$$L_{11}^{k+1}(\rho_k) = E_0[I \otimes I \otimes P_1(U_{k+1}(n)(\rho_k \otimes \mu^I_k \otimes \beta)U^*_k(n))I \otimes I \otimes P_1]$$
$$= \alpha_{22}^{k+1}(n) + \alpha_{33}^{k+1}(n)$$

This is namely the two unnormalised state, the operator $L_{00}^{k+1}(\rho_k)$ appears with probability $p_{k+1} = Tr[L_{00}^{k+1}(\rho_k)]$ and $L_{11}^{k+1}(\rho_k)$ with probability $q_{k+1} = Tr[L_{11}^{k+1}(\rho_k)]$.

For the Laser state we choose the following way of writing:

$$\mu^I_k = \frac{1}{1 + |h(k/n)|^2} \begin{pmatrix} 1 & h(k/n) \\ \bar{h}(k/n) & |h(k/n)|^2 \end{pmatrix}$$

with the asymptotic:

$$h(k/n) = \frac{1}{\sqrt{n}} f(k/n) + o\left(\frac{1}{n}\right)$$

For a continuous equivalent model in the Von Neumann algebra setup, one can see [4] and the justification of the discrete model can be found in [2]. We assume that $f$ is continuous, it renders the effect of the intensity of the laser.

So we can defined our discrete quantum trajectory in the same fashion of the previous section and we have the discrete evolution equation:

$$\rho_k = \frac{L_{00}^{k+1}(\rho_k)}{p_{k+1}} + \left[ -\frac{L_{00}^{k+1}(\rho_k)}{p_{k+1}} + \frac{L_{11}^{k+1}(\rho_k)}{q_{k+1}} \right] 1_1$$

For our future application concerning resonance fluorescence we impose that $L_{01} = -L^*_{10}$, and $L_{11} = L_{21} = L_{31} = L_{30} = 0$.

With the previous condition and according to the fact that $U$ must be unitary one can show that there exists a self-adjoint operator $H$ such that:

$$L_{00} = -(iH + \frac{1}{2} \sum_{i=1}^2 L^*_i L_i)$$

It is the similar representation of the section (2). We can express the convergence result in this situation.

**Proposition 4** Let $(\Omega, F, F_t, P)$ be the probability space of a Poisson point process $N$ on $\mathbb{R}^2$. 

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The discrete quantum trajectory \( \rho_{[nt]} \) defined by the equation (43) weakly converges in \( D([0,T]) \) for all \( T \) to the solution of the following stochastic differential equation:

\[
\begin{align*}
\mathbf{\mu}_t &= \mathbf{\mu}_0 + \int_0^t \left[ -i[H, \mathbf{\mu}_{s-}] + \frac{1}{2} \{ \sum_{i=1}^{2} L^*_{i0} L_{i0}, \mathbf{\mu}_{s-} \} + L_{10} \mathbf{\mu}_{s-} - L^*_{10} \\
&\quad + \left( f(s-) L_{10} \mathbf{\mu}_{s-} - f(s-) L^*_{10}, \mathbf{\mu}_{s-} \right) - Tr[L_{20} \mathbf{\mu}_{s-} - L^*_{20}] \mathbf{\mu}_{s-} \right] ds \\
&\quad + \int_0^t \int_\mathbb{R} \left[ -\mathbf{\mu}_{s-} + \frac{L_{20} \mathbf{\mu}_{s-} - L^*_{20}}{Tr[L_{20} \mathbf{\mu}_{s-} - L^*_{20}]} \right] 1_{0 < x < Tr[L_{20} \mathbf{\mu}_{s-} - L^*_{20}]} N(dx, ds).
\end{align*}
\]

(45)

**Proof:** For example we have the following asymptotic for \( \mathcal{L}_0^{k+1}(\rho_k) \):

\[
\begin{align*}
\mathcal{L}_0(\rho_k) &= \rho_k + \frac{1}{n} \left[ L_{00} \rho + \rho L^*_{00} + L_{10} \rho L^*_{10} + f\left( \frac{k}{n} \right) \left[ L_{01} \rho + \rho L^*_{01} \right] + f\left( \frac{k}{n} \right) \left[ L_{10} \rho + \rho L^*_{01} \right] \right] \\
&\quad + \circ \left( \frac{1}{n} \right)
\end{align*}
\]

(46)

This above asymptotic, the condition about the operator \( L_{ij} \) and the theorem (7) prove the proposition. \( \square \)

Thus we have obtained the continuous stochastic equation of the evolution of a two level atom driving by a laser. It is an example of an application of a deterministic control. The effect of the control is characterized by the function \( f \). Let us now give a simple consequence of this control in a special case of such interaction.

Consider a model where \( H \) is assumed to be \( H = 0 \). We put:

\[
C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L_{10} = k_l C, \quad L_{20} = k_c C,
\]

with \( |k_l|^2 + |k_c|^2 = 1 \). The constant \( k_l \) and \( k_c \) are called decay rates.

Without control the stochastic model of a two level atom in presence of a photon counter is given by:

\[
\begin{align*}
\mathbf{\mu}_t &= \mathbf{\mu}_0 + \int_0^t \left[ + \frac{1}{2} \{ C, \mathbf{\mu}_{s-} \} + C \mathbf{\mu}_{s-} C^* - Tr[C \mathbf{\mu}_{s-} C^*] \mathbf{\mu}_{s-} \right] ds \\
&\quad + \int_0^t \int_\mathbb{R} \left[ -\mathbf{\mu}_{s-} + \frac{C \mathbf{\mu}_{s-} C^*}{Tr[C \mathbf{\mu}_{s-} C^*]} \right] 1_{0 < x < Tr[C \mathbf{\mu}_{s-} C^*]} N(dx, ds).
\end{align*}
\]

(47)

Let denote \( \tilde{N}_t = \int_0^t \int_\mathbb{R} 1_{0 < x < Tr[C \mathbf{\mu}_{s-} C^*]} N(dx, ds) \) and \( T = \inf \{ t > 0; \tilde{N}_t > 0 \} \). In [3] it was proved that:

\[ \forall t > T : \quad \mathbf{\mu}_t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |\Omega\rangle \langle \Omega| \]

(48)
It means that at most one photon appears on the photon counter. The mathematical reason is that if we write the equation (47) in the following way:

$$\mu_t = \int_0^t \Psi(\mu_{s-})ds + \int_0^t \Phi(\mu_{s-})d\tilde{N}_s,$$

we have for $$\mu = |\Omega\rangle \langle \Omega|$$: $$\Phi(\mu) = \Psi(\mu) = 0$$.

In the presence of laser the control $$f$$ gives rise to the term $$[fL_{10} - fL_{10}^*]$$, hence if $$\mu = |\Omega\rangle \langle \Omega|$$ we still have $$\Phi(\mu) = 0$$ but we do not have anymore $$\Psi(\mu) = 0$$ and the property (48) is not satisfied. As a consequence it is possible to observe more than one photon in the photon counter.

So we have seen a concrete model of deterministic control model. In the next section we deal with general strategy and the particular problem of optimal control. Consideration about optimal control is an interesting mean to point out the importance of Markovian strategy.

### 4.2 optimal control

This section is then devoted to what is called “optimal control” problem. Thanks to the stochastic model described by the discrete and the continuous model we can consider a general situation (in the both case) where we want to optimize a certain character of the quantum trajectory. The idea of optimal control is to find a control strategy which satisfy the optimization constraint. In this section we give the classical mathematical description of such problem and investigate the general result in the discrete and in the continuous model. Let us begin with the discrete model.

We come back to the description of the discrete quantum trajectory as a Markov chain. Let $$n$$ be fixed, thanks to the theorem (2), the discrete quantum trajectory ($$\rho^u_k$$) is given with the following way. Let $$\rho$$ be any state, if $$\rho^u_k = \rho$$ then $$\rho^u_{k+1}$$ takes one of the value:

$$H_i^{u,k} (\rho) = \frac{L_{i0}(k/n, u_k(n)) (\rho) L_{i0}^*(k/n, u_k(n))}{\text{Tr}[ L_{i0}(k/n, u_k(n)) (\rho) L_{i0}^*(k/n, u_k(n)) ]},$$

with probability for $$i = 0$$: $$p^{u}_{k+1} (\rho) = \text{Tr}[ L_{00}(k/n, u_k(n)) (\rho) L_{00}^*(k/n, u_k(n))]$$ and for $$i = 1$$: $$q^{u}_{k+1} (\rho) = \text{Tr}[ L_{10}(k/n, u_k(n)) (\rho) L_{10}^*(k/n, u_k(n))]$$.

Remember that ($$u_k$$) designs the control strategy. More generally, for all $$k$$ we accept all strategy $$u_k$$ which depends on $$\rho_i$$ for all $$i \leq k$$. We define $$\mathcal{U}$$ the set of all admissible strategy which satisfies this condition. We are interested in the “optimal control” in finite horizon. Let $$c$$ and $$\phi$$ be two function.

Let $$N$$ be fixed, the optimal control problem in finite horizon is to consider the optimal cost:

$$\min_{u \in \mathcal{U}} E \left[ \sum_{k=0}^{N-1} c(k, \rho^u_k, u_k) + \phi(\rho^u_N) \right]$$

(49)
If there is some strategy which realizes the minimum, this strategy is called the optimal strategy. Let \( u \in U \) be a fixed strategy and let \( T \) be the set of stopping time, a variant of “optimal control” theory is to consider:

\[
\min_{\tau \in T : \tau < N} \mathbb{E} \left[ \sum_{k=0}^{\tau} c(k, \rho_k^u, u_k) + \phi(\rho_N^u) \right]
\]  

We do not develop this theory in this article. This is the theory of optimal stopping time problem and can be also considered without control. Let us investigate the classical result in stochastic control.

For this we define:

\[
V^k(\rho) = \min_{u \in U} \mathbb{E} \left[ \sum_{j=k}^{N-1} c(k, \rho_j^u, u_j) + \phi(\rho_N^u) / \rho_n = \rho \right]
\]

**Remark** The function \( c \) and \( \phi \) are determined by the optimization constraint. The equation which appears in the following theorem is called the cost equation and the function \( c \) and \( \phi \) are called cost function.

**Theorem 8** Let \( U \) be a compact set and suppose that \( c \) is a continuous function. The solution of:

\[
\begin{align*}
V^k(\rho) &= \min_{u \in U} \left\{ \rho_{k+1}^u(\rho)H_0^{u,k}(\rho) + q_{k+1}^u(\rho)H_1^{u,k}(\rho) + c(k, \rho, u_k) \right\} \\
V^N(\rho) &= \phi(\rho)
\end{align*}
\]

(51)

give the optimal cost:

\[
V^k(\rho) = \min_{u \in U} \mathbb{E} \left[ \sum_{j=k}^{N-1} c(k, \rho_j^u, u_j) + \phi(\rho_N^u) / \rho_n = \rho \right].
\]

The optimal strategy is given by:

\[
u^* : \rho \rightarrow u^*_k(\rho) \in \arg \min_{u \in U} \left\{ \rho_{k+1}^u(\rho)H_0^{u,k}(\rho) + q_{k+1}^u(\rho)H_1^{u,k}(\rho) + c(k, \rho, u_k) \right\}
\]

(52)

Furthermore this strategy is Markovian.

**Proof:** The proof is based of what is called dynamic programming in stochastic control theory. Let \( u \) be any strategy and let \( V \) we have

\[
\mathbb{E} \left[ (V^{k+1}(\rho_{k+1}^u) - V^k(\rho_k^u)) / \sigma \{ \rho_i^u, i \leq k \} \right] = \rho_{k+1}^u V^{k+1}(H_0^{u,k}(\rho_k^u)) + q_{k+1}^u V^{k+1}(H_1^{u,k}(\rho_k^u)) - V^k(\rho_k^u)
\]

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then we have:

\[
\mathbf{E}[V^N(\rho^u_N) - V^0(\rho)] = \sum_{k=0}^{N-1} \mathbf{E}[V^{k+1}(\rho^u_{k+1}) - V^k(\rho^u_k)]
\]

\[
= \sum_{k=0}^{N-1} \mathbf{E}[p^u_{k+1} V^{k+1}(\mathcal{H}_0^u(\rho^u_{k+1})) + q^u_{k+1} V^{k+1}(\mathcal{H}_1^u(\rho^u_{k+1})) - V^{k}(\rho^u_k)]
\]

\[
\geq -\sum_{k=0}^{N-1} \mathbf{E}[c(k, \rho^u_k, u_k)] \quad \text{(by definition of the min)}
\]

So we have \(V^0(\rho) \leq \mathbf{E} \left[ \sum_{k=0}^{N-1} c(k, \rho^u_k, u_k) + \phi(\rho^u_N) \right] \) for all strategy \(u\).

Moreover we have equality if we choose the strategy (52). This strategy is Markovian because the function \(c\) depends only on \(\rho_k\) at time \(k\). \(\square\)

The system (51) which describes the cost equation is called the discrete Hamilton-Jacobi Bellman equation.

The fact that the optimal strategy is Markovian is another justification of the choice of such model of control for the discrete quantum trajectory. This theorem claims that we need just Markovian strategy in order to solve the “optimal control” problem.

The next last section is devoted to the same investigation in the continuous time model of quantum trajectories.

### 4.3 optimal control for continuous quantum trajectory

In the third section we have proved the Poisson and the diffusion approximation in quantum measurement theory. Thus we have the diffusive evolution equation:

\[
\rho_t = \rho_0 + \int_0^t L(s, \rho^u_s, u(s, \rho^u_s))ds + \int_0^t \Theta(s, \rho^u_s, u(s, \rho^u_s))dW_s.
\]

The Poisson approximation is given by:

\[
\rho_t = \rho_0 + \int_0^t R(s, \rho^u_s, u(s, \rho^u_s))ds
\]

\[
+ \int_0^t \int_{\mathcal{R}} Q(s, \rho^u_s, u(s, \rho^u_s))\mathbf{1}_{0 < x < Tr(\mathcal{J}(s, u(s, \rho^u_s)))}N(dx, ds)
\]

with definition for \(L, \Theta, R\) and \(Q\) in section (3).

In this section we consider the same problem of ”optimal control” as in the discrete
case. Let \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) be a probability space where we can consider such equations:

\[
\rho_t = \rho_0 + \int_0^t L(s, \rho_s^u, u_s)ds + \int_0^t \Theta(s, \rho_s^u, u_s)dW_s.
\]

\[
\rho_t = \rho_0 + \int_0^t R(s-, \rho_{s-}^u, u_{s-})ds + \int_0^t \int_0 \int R(s, \rho_{s-}^u, u_{s-})1_{0<x<T}(J(s-,\rho_{s-}^u)N(dx,ds)
\]

where \(u = (u_t)\) designs a \(\mathcal{F}_t\) adapted strategy. With the condition (26) the previous equation admits a unique solution, the solution takes value in the set of state on \(\mathcal{H}_0\). As the discrete case we consider not only Markovian strategy and we denote by \(\mathcal{U}\) be the set of all admissible strategy which satisfy the condition of adaptation.

Let \(c\) and \(\phi\) be two cost function as in the discrete model. Let \(T > 0\), the optimal problem in finite horizon is given by:

\[
\min_{u \in \mathcal{U}} \mathbb{E}\left[\int_0^T c(s, \rho_s^u, u_s)ds + \phi(\rho_T^u)\right].
\]

We introduce the following function:

\[
V(t, \rho) = \min_{u \in \mathcal{U}} \mathbb{E}\left[\int_t^T c(s, \rho_s^u, u_s)ds + \phi(\rho_T^u)\right].
\]

This function represents the result of optimal control after \(t\) assuming \(\rho_t = \rho\). We have:

\[
V(0, \rho_0) = \min_{u \in \mathcal{U}} \mathbb{E}\left[\int_0^T c(s, \rho_s^u, u_s)ds + \phi(\rho_T^u)\right].
\]

In this article we just give the result for the optimal control problem for the diffusive case. A similar result for the Poisson case can be found in [5].

Before to give the theorem, we need the expression of the infinitesimal generator of the process \((\rho_t^u)\) in the diffusive case. Indeed it will appear in the continuous equivalent of Hamilton-Jacobi-Bellmann equation describing in the discrete case. For this we consider \((\rho_t^u)\) as a process which takes value in \(\mathbb{R}^3\) with the identification of the state and the Bloch sphere. The identification is the following:

\[
\mathbb{R}^3 \quad \mapsto \quad \mathcal{M}_2(\mathbb{C})
\]

\[
(x, y, z) \quad \mapsto \quad \frac{1}{2} \begin{pmatrix} 1 + x & y + iz \\ y - iz & 1 - x \end{pmatrix}
\]

Computing the expression of \(L\) and \(\Theta\), the stochastic differential equation concerning the diffusive case can be written as a system of stochastic differential equation of the form:

\[
\rho_t = \rho_0 + \int_0^t L(s, \rho_s^u, u_s)ds + \int_0^t \Theta(s, \rho_s^u, u_s)dW_s.
\]
where $L$ is a function from $\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}$ to $\mathbb{R}^3$ and the same holds for $\Theta$. For an example of such computation one can see. We introduce the $3 \times 3$ matrix $\Pi$ defined by $\Pi_{ij} = \Theta_i \Theta_j$ where $\Theta_i$ corresponds to the function coordinate of $\Theta$.

So if $f$ designs any bounded $C^2$ function from $\mathbb{R}^3$ to $\mathbb{R}$ we define:

$$A^{u,t} f(x) = \frac{1}{2} \sum_{i,j=1}^{3} \Pi_{ij}(t,x,u) \frac{\partial f(x)}{\partial x_i} \frac{\partial f(x)}{\partial x_j} + \sum_{i=1}^{3} L_i(t,x,u) \frac{\partial f(x)}{\partial x_i}. \quad (58)$$

this is the infinitesimal generator of $(\rho_t)$. In particular if $u$ is a fixed constant, we have that for all $f$ $C^2$ and bounded, the following process:

$$\mathcal{M}_t = f(\rho_t) - f(\rho_0) - \int_0^t A^{u,s} f(\rho_s) ds$$

is a martingale for the filtration generated by $(\rho_t)$.

Let us express now the theorem concerning the optimal control in this case:

**Theorem 9** Suppose there is a function $(t, \rho) \to V(t, \rho)$ which is $C^1$ in $t$ and $C^2$ in $\rho$ such that:

$$\left\{ \frac{\partial V(t,\rho)}{\partial t} + \min_{u \in U} \{ A^{u,t} V(t,\rho) + c(t, \rho, u) \} \right\} = 0$$

$$V(T, \rho) = \phi(\rho) \quad (59)$$

This function gives the solution of the optimal problem:

$$V(t, \rho) = \min_{u \in U} \mathbb{E} \left[ \int_t^T c(s, \rho_s^u, u_s) ds + \phi(\rho^u_T) \left/ \rho^u_t = \rho \right. \right].$$

and if:

$$u^*(t, \rho) \in \arg \min_{u \in U} \{ A^{u,t} V(t, \rho) + c(t, \rho, u) \} \quad (60)$$

is an admissible strategy then it defines an optimal strategy. Furthermore this strategy is Markovian.

The equation (59) is called the Hamilton-Jacobi-Bellmann equation.

So we do not give the proof of this theorem (see [15], [19]). One more time the optimal strategy is Markovian, this confirms the choice of such strategy in the model of quantum trajectories with control.

A similar result holds for the Poisson case. The infinitesimal generator for such process is given in [8].

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