BAVARD’S DUALITY THEOREM OF INVARIANT QUASIMORPHISMS

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Abstract. Let $H$ be a normal subgroup of a group $G$. A quasimorphism $f$ on $H$ is $G$-invariant if there is a non-negative number $D$ satisfying $|f(gxg^{-1}) - f(x)| \leq D$ for every $g \in G$ and every $x \in H$. The purpose in this paper is to prove Bavard’s duality theorem of $G$-invariant quasimorphisms, which was previously proved by Kawasaki and Kimura in the case $H = [G, H]$.

Our duality theorem gives a connection between $G$-invariant quasimorphisms and $(G, H)$-commutator lengths. Here for $x \in [G, H]$, the $(G, H)$-commutator length $\text{cl}_{G, H}(x)$ of $x$ is the minimum number $n$ such that $x$ is a product of $n$ commutators which are written by $[g, h]$ with $g \in G$ and $h \in H$. In the proof, we give a geometric interpretation of $(G, H)$-commutator lengths.

1. Introduction

1.1. $G$-invariant quasimorphisms. A real-valued function $f : G \to \mathbb{R}$ on a group $G$ is a quasimorphism if there is a non-negative number $D$ satisfying

$$|f(g_1 g_2) - f(g_1) - f(g_2)| \leq D$$

for every pair $g_1$ and $g_2$ of elements in $G$. A quasimorphism $f$ on $G$ is homogeneous if $f(x^n) = n \cdot f(x)$ for every $x \in G$ and every integer $n$. Quasimorphisms have been extensively studied in geometric group theory, and are closely related to the second bounded cohomology of groups. For an introduction to this subject, we refer to [3] and [7].

Let $H$ be a subgroup of $G$, and consider a quasimorphism $f$ on $H$. It is quite natural to ask when $f$ is extended to a quasimorphism on the whole group $G$. Such a problem has been actually studied in Kawasaki-Kimura [10] and Shtern [15].

Suppose that $H$ is normal. Then there is a condition that any extendable quasimorphism on $H$ clearly satisfies. Namely, if a quasimorphism $f$ on $H$ is extendable, then $f$ is invariant by the inner action of $G$ on $H$, up to bounded error, i.e. there is a non-negative number $D'$ such that

$$|f(gxg^{-1}) - f(x)| \leq D'$$

for every $g \in G$ and every $x \in H$. We call such a quasimorphism on $H G$-invariant. However, it is known that there is a $G$-invariant quasimorphism which is not extendable (see [10]). So the next problem is to ask when a $G$-invariant quasimorphism on $H$ is extendable. Dealing with this problem, Kawasaki and Kimura [10] considered to use a generalization

Key words and phrases. Quasimorphism, stable commutator length, Bavard’s duality, pseudo-character.
of Bavard’s duality theorem concerning $G$-invariant quasimorphisms, which is explained in the next subsection.

We also note that $G$-invariant quasimorphisms often appear in symplectic geometry. In fact, any symplectic manifold $(M, \omega)$ has the following two natural transformation groups. One is the group $\text{Symp}(M, \omega)$ of symplectomorphisms, and the other is the group $\text{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms. It is known that there are various $\text{Symp}(M, \omega)$-invariant quasimorphisms on $\text{Ham}(M, \omega)$ (see [6], [14], and [16] for example).

1.2. $(G,H)$-commutator length and its stabilization. The commutator length $\text{cl}_G(x)$ of an element $x$ in $[G,G]$ is the minimum $n$ such that there are $n$ commutators $c_1, \ldots, c_n$ of $G$ with $x = c_1 \cdots c_n$. Then it is known that the limit

$$\text{scl}_G(x) = \lim_{n \to \infty} \frac{\text{cl}_G(x)}{n}$$

exists and we call $\text{scl}_G(x)$ the stable commutator length of $x$. Bavard’s duality theorem gives a connection between quasimorphisms and commutator lengths in the following form:

**Theorem 1.1** (Bavard [1]). Let $G$ be a group and let $x \in [G,G]$. Then the following equality holds:

$$\text{scl}_G(x) = \frac{1}{2} \sup_{f \in Q^h(G) - H^1(G)} \frac{|f(x)|}{D(f)}$$

Here $Q^h(G)$ is the set of homogeneous quasimorphisms on $G$, and $H^1(G)$ is the set of homomorphisms from $G$ to $\mathbb{R}$. We consider that the right of the equality in Theorem 1.1 is 0 if every homogeneous quasimorphism on $G$ is a homomorphism.

Theorem 1.1 has several important applications. For example, Endo and Kotschik [5] used Theorem 1.1 to show the existence of homogeneous quasimorphisms which are not homomorphisms on the mapping class groups of surfaces. Other main applications of Theorem 1.1 are computations of stable commutator lengths. In fact, Theorem 1.1 allows us to compute the stable commutator lengths when $G$ has few homogeneous quasimorphisms which are not homomorphisms (see [11] and [17]). Other applications of Theorem 1.1 are found in [2], [4], and [13] for example.

The purpose in this paper is to prove Bavard’s duality theorem of $G$-invariant quasimorphisms. The notion associated to commutator lengths is the $(G,H)$-commutator lengths defined as follows: An element $x$ in $G$ is a $(G,H)$-commutator if there are $g \in G$ and $h \in H$ such that $x = [g,h]$. As is usual, we denote by $[G,H]$ the subgroup of $G$ generated by the $(G,H)$-commutators. The $(G,H)$-commutator length $\text{cl}_{G,H}(x)$ of an element $x$ in $[G,H]$ is the minimum number $n$ such that there are $(G,H)$-commutators $c_1, \ldots, c_n$ with $x = c_1 \cdots c_n$. Then it is clear that there exists a limit

$$\text{scl}_{G,H}(x) = \lim_{n \to \infty} \frac{\text{cl}_{G,H}(x^n)}{n},$$

and call $\text{scl}_{G,H}(x)$ the stable $(G,H)$-commutator length of $x$. 
To state our main result, we prepare some notation: Let $Q(H)^G$ be the set of $G$-invariant quasimorphisms on $H$, and $Q^h(H)^G$ the set of homogeneous $G$-invariant quasimorphisms on $H$. Let $H^1(H)^G$ be the set of homomorphisms from $H$ to $\mathbb{R}$, which are $G$-invariant. By using this notation, our main result is formulated as follows:

**Theorem 1.2 (Theorem 6.1).** Let $G$ be a group and $H$ a normal subgroup of $G$, and let $x \in [G,H]$. Then the following equality holds:

$$\text{scl}_{G,H}(x) = \frac{1}{2} \sup_{f \in Q^h(H)^G - H^1(H)^G} \frac{|f(x)|}{D(f)}$$

Note that Theorem 1.1 is the case $G = H$ of Theorem 1.2. So Theorem 1.2 is a generalization of Bavard’s duality. Kawasaki and Kimura [10] proved Theorem 1.2 when $[G,H] = H$.

Our proof of Theorem 1.2 is a generalization of the original proof of Bavard [1]. However, in the proof of Theorem 1.2 we introduce several notions which did not appear in the original proof. One of the important by-products is to give a geometric characterization of $(G,H)$-commutator lengths (Theorem 1.3).

1.3. **Geometric interpretation of $(G,H)$-commutator lengths.** Let $x$ be an element of the commutator subgroup $[G,G]$ of $G$. Then $x$ is identified with a homotopy class of loops in the classifying space $BG$ of $G$. Since $x$ vanishes in $H_1(G;\mathbb{Z})$, there is an oriented compact surface $S$ with connected boundary and a continuous map $f: S \to BG$ such that the loop $f|_{\partial S}: \partial S \to BG$ is $x$. Then the commutator length of $x$ coincides with the minimum genus of such a surface $S$ (see [3]).

In the proof of Theorem 1.2 we need a similar geometric characterization of $(G,H)$-commutator lengths. To obtain this, we introduce $(G,H)$-simplicial surfaces as follows.

Throughout the paper, every surface is assumed to be compact and oriented. We assume that our triangulation of a surface satisfies the following conditions:

- Every edge has an orientation. The endpoints of an edge may coincide.
- Every triangle (2-cell) $\sigma$ is surrounded by three edges $\partial_0 \sigma, \partial_1 \sigma$ and $\partial_2 \sigma$ as is depicted in Figure 1. We do not assume that $\partial_0 \sigma, \partial_1 \sigma$, and $\partial_2 \sigma$ are distinct.

For a triangulated surface $S$, let $E(S)$ be the set of edges of $S$ and $T(S)$ the set of triangles of $S$. A $G$-labelling of $S$ is a function $f: E(S) \to G$ satisfying $f(\partial_1 \sigma) = f(\partial_0 \sigma) \cdot f(\partial_2 \sigma)$ for every $\sigma \in T(S)$. We call a pair $(S,f)$ consisting of a triangulated surface $S$ together with a $G$-labelling $f$ a $G$-simplicial surface. A $G$-simplicial surface with boundary $x$ is a $G$-simplicial surface $(S,f)$ such that $\partial S$ has only one edge and $f$ sends it to $x$.

Note that if a $G$-labelling of $S$ is given, then there is a continuous map $f': S \to BG$ sending $e \in E(S)$ to the loop associated to $f'(e)$. On the other hand, if a continuous map $f': S \to BG$ is given, then there is a $G$-labelling $f$ of $S$ such that the associated continuous
map of \( f \) is homotopy equivalent to \( f' \). Therefore for \( x \in [G, G] \), the commutator length of \( x \) coincides with the minimum genus of a \( G \)-simplicial surface with boundary \( x \).

Now we are ready to define \( (G, H) \)-simplicial surfaces. A \( (G, H) \)-labelling of \( S \) is a \( G \)-labelling \( f : E(S) \to G \) such that either \( f(\partial_0 \sigma) \) or \( f(\partial_2 \sigma) \) belongs to \( H \) for every \( \sigma \in T(S) \). We call a pair \( (S, f) \) of triangulated surface \( S \) together with a \( (G, H) \)-labelling \( f \) a \( (G, H) \)-simplicial surface.

It turns out that there is a close relation between \( (G, H) \)-simplicial surfaces and \( (G, H) \)-commutators. In fact, we show that for an element \( x \) of \( G \) there is a \( (G, H) \)-simplicial surface with boundary \( x \) if and only if \( x \) is contained in \([G, H]\) (see Section 5). Moreover, \( (G, H) \)-simplicial surfaces give the following characterization of \( (G, H) \)-commutator lengths.

**Theorem 1.3** (Theorem 5.3). Let \( G \) be a group and \( H \) a normal subgroup. For an element \( x \) in \([G, H]\), the \( (G, H) \)-commutator length of \( x \) coincides with the minimum of the genus of a connected \( (G, H) \)-simplicial surface with boundary \( x \).

### 1.4. \( H \)-quasimorphisms.

Recall that the defect \( D(f) \) of a quasimorphism \( f \) on \( G \) is the minimum number \( D \geq 0 \) such that \( |f(xy) - f(x) - f(y)| \leq D \) for every pair \( x \) and \( y \) of elements in \( G \). The defect is a pseudo-norm on the space \( Q(G) \) of quasimorphisms, and the kernel of \( D \) is the space \( H^1(G) \) of homomorphisms from \( G \) to \( \mathbb{R} \). Thus the space \( Q(G)/H^1(G) \) is a normed space equipped with the norm induced by \( D \).

Let \( C_n(G) \) be the inhomogeneous complex of the group \( G \). Namely, \( C_n(G) \) is the free \( \mathbb{R} \)-module generated by \( G^n \), and the differential \( \partial : C_n(G) \to C_{n-1}(G) \) is defined by

\[
\partial(g_1, \cdots, g_n) = (g_2, \cdots, g_n) + \sum_{i=1}^{n-1} (-1)^i (g_1, \cdots, g_i, g_{i+1}, \cdots, g_n) + (-1)^n (g_1, \cdots, g_{n-1}).
\]

Regard \( C_n(G) \) as the normed space by the \( l^1 \)-norm. Then the space of cycles \( Z_n(G) \) is a closed subspace and hence \( C_n(G)/Z_n(G) \) is a normed space. An important observation in the proof of the original Bavard duality is to identify the normed space \( Q(G)/H^1(G) \) with the topological dual of \( C_2(G)/Z_2(G) \), and applied the Hahn-Banach theorem.

In the case of \( G \)-invariant quasimorphisms, we consider instead the space of \( H \)-quasimorphisms, defined as follows. A function \( f : G \to \mathbb{R} \) is an \( H \)-quasimorphism if there is a non-negative
number $D''$ such that $|f(gx) - f(g) - f(x)| \leq D''$ and $|f(xg) - f(x) - f(g)| \leq D''$ for every $g \in G$ and every $x \in H$. We call the minimum of $D''$ the defect of the $H$-quasimorphism $f$, and denote it by $D''(f)$. We call an $H$-quasimorphism with $D''(f) = 0$ an $H$-homomorphism.

It turns out that $H$-quasimorphisms are closely related to $G$-invariant quasimorphisms. In fact, a quasimorphism $f$ on $H$ is $G$-invariant if and only if $f$ extends to $G$ as an $H$-quasimorphism (see Section 3).

As is the case of the defect of usual quasimorphisms, the defect $D''$ of $H$-quasimorphisms is a pseudo-norm of the space $Q_H(G)$ of $H$-quasimorphisms, and the kernel of $D''$ is the space $H^1_H(G)$ of $H$-homomorphisms. Then the normed space $Q_H(G)/H^1_H(G)$ is isomorphically identified with the topological dual of $C'_2(G)/Z'_2(G)$. Here $C'_2(G)$ is the submodule of $C_2(G)$ generated by the set of elements $(g_1, g_2) \in G \times G$ such that either $g_1$ or $g_2$ belongs to $H$, and $Z'_2(G)$ is $C'_2(G) \cap Z_2(G)$.

Let $B'_1$ be the image of $\partial : C'_2 \to C_1$. We consider that $B'_1$ is a normed space whose norm is given by the isomorphism $C'_2/Z'_2 \overset{\cong}{\to} B'_1$. It turns out that $x \in [G, H]$ implies $x \in B'_1$, and the following limit exists:

$$\text{fill}_{G,H}(x) = \lim_{n \to \infty} \frac{\|x^n\|}{n}$$

We call $\text{fill}_{G,H}(x)$ the $(G, H)$-filling norm of $x$. Using the geometric characterization of $(G, H)$-commutator lengths, we show the following theorem:

**Theorem 1.4** (Theorem 5.1). For each $x \in [G, H]$,

$$\text{fill}_{G,H}(x) = 4 \cdot \text{scl}_{G,H}(x)$$

Applying Hahn-Banach theorem to $B'_1$, we deduce our duality theorem (Theorem 1.2) from this theorem. This is the outline of the proof of Theorem 1.2.

1.5. **Organization of this paper.** In Section 2, we give a few direct applications of Theorem 1.2. Section 3 is devoted to the study of algebraic properties of $G$-invariant quasimorphisms and $H$-quasimorphisms. In Section 4, we study the space of $H$-quasimorphisms and $(G, H)$-filling norms. In Section 5, we introduce $(G, H)$-simplicial surfaces and give a geometric characterization of $(G, H)$-commutator lengths (Theorem 1.3) and prove Theorem 1.4. In Section 6, we complete the proof of Theorem 1.2.

Throughout the paper, $G$ is a group and $H$ is a normal subgroup of $G$.

**Acknowledgement.** The first author is supported by JSPS KAKENHI 18J00765, and the third author is supported by JSPS KAKENHI 19K14536.

2. **Applications**

In this section, we give a few direct applications of our duality theorem (Theorem 1.2). We start with the following equivalence of $\text{scl}_2$ and $\text{scl}_{G,H}$ in some cases. Kawasaki and Kimura [10] showed the following theorem for the case (1) when $H = [G, H]$. 
Theorem 2.1. Let $H$ be a normal subgroup of a group $G$. Suppose that one of the following conditions is satisfied:

1. The group homomorphism $G \to G/H$ has a section.
2. $H$ is a finite index subgroup of $G$.

Then the following inequalities holds for every $x \in [G, H]$:

$$\text{scl}_{G}(x) \leq \text{scl}_{G,H}(x) \leq 2 \cdot \text{scl}_{G}(x)$$

Proof. The inequality $\text{scl}_{G}(x) \leq \text{scl}_{G,H}(x)$ is obvious. So we prove $\text{scl}_{G,H}(x) \leq 2 \cdot \text{scl}_{G}(x)$.

It follows from Theorem 1.2 that for every $\epsilon > 0$ there is a $G$-invariant homogeneous quasimorphism $\phi$ such that

$$\text{scl}_{G,H}(x) - \epsilon \leq \frac{1}{2} \frac{\phi(x)}{D(\phi)}.$$

If one of the conditions (1) and (2) is satisfied, then there exists a quasimorphism $\psi$ on $G$ such that $\psi|_H = \phi$ and $D(\psi) \leq D(\phi)$ (see [10] for the case (1) and [9] for (2)). Let $\overline{\psi}$ denote the homogenization of $\psi$ (see Section 3). Then it is known that $D(\overline{\psi}) \leq 2D(\psi)$ (see [3]). Therefore, Theorem 1.2 implies

$$\text{scl}_{G,H}(x) - \epsilon \leq \frac{1}{2} \frac{\phi(x)}{D(\phi)} \leq \frac{\overline{\psi}(x)}{D(\overline{\psi})} \leq 2 \cdot \text{scl}_{G}(x).$$

Since $\epsilon$ is an arbitrary positive number, this completes the proof.

Example 2.2. Let $B_n$ and $P_n$ denote the braid group and the pure braid group on $n$ strands. Since $P_n$ is a finite index normal subgroup of $B_n$, Theorem 2.1 implies that $\text{scl}_{B_n}$ and $\text{scl}_{B_n,P_n}$ are equivalent. On the other hand, it is known that $\text{scl}_{P_n}$ and $\text{scl}_{B_n,P_n}$ are not equivalent when $n \geq 3$ (see [10]).

Example 2.3. If $G/H$ is a free group, then the quotient $G \to G/H$ has a section, and hence $\text{scl}_{G,H}$ and $\text{scl}_G$ are equivalent. For example, we have that $\text{scl}_{B_n}$ and $\text{scl}_{[B_n,B_n]}$ are equivalent since $B_n/[B_n,B_n] = \mathbb{Z}$. Note that this result was obtained in Kawasaki-Kimura [10] when $n \geq 5$.

Let $G'$ be a subgroup of $G$, and $H'$ a subgroup of $H$. Suppose that $H'$ is a normal subgroup of $G'$. We consider the comparison of $\text{scl}_{G,H}$ and $\text{scl}_{G',H'}$.

We say that $G'$ is $m$-displaceable in $G$ (see [2]) if there are $m$ elements $g_1, \ldots, g_m$ of $G$ such that the subgroups

$$G', g_1G'g_1^{-1}, \ldots, g_mG'g_m^{-1}$$

pair-wise commute.

Theorem 2.4. Suppose that $G'$ is an $m$-displaceable subgroup of $G$. For every $x \in [G', H']$, the following inequality holds:

$$\text{scl}_{G,H}(x) \leq \frac{1}{m + 1} \cdot \text{scl}_{G',H'}(x).$$
This result is a variant of [2] Theorem 2.8. We use the following proposition which is a variant of [2] Proposition 2.11.

**Proposition 2.5.** Let $G'_1, \cdots, G'_N$ be subgroups of $G$ such that $G'_i$ and $G'_j$ commute for $i \neq j$. For each $i$, let $H'_i$ be a subgroup of $G'_i$, and set $K_G = G'_1 \cdots G'_N$ and $K_H = H'_1 \cdots H'_N$. Then for every $\phi \in Q^h(K_H)^{K_G}$, the following equality holds:

$$D(\phi) = \sum_{i=1}^{N} D(\phi|_{H'_i}).$$

**Proof.** Let $x \in K_G$ and $y \in K_H$, and write

$$x = x_1 \cdots x_N, \ y = y_1 \cdots y_N,$$

where $x_i \in G'_i$ and $y_i \in H'_i$. Then we have

$$[x, y] = [x_1, y_1] \cdots [x_N, y_N].$$

Since $[x_i, y_1]$ and $[x_j, y_j]$ commute for $i \neq j$ and $\phi$ is homogeneous, we have

$$\phi([x, y]) = \sum_{i=1}^{N} \phi([x_i, y_i]).$$

Thus the desired equality follows from Lemma [3.6].

**Proof of Theorem 2.4.** Since $G'$ is $m$-displaceable, there are $g_1, \cdots, g_m \in G$ such that the subgroups $G', g_1G'g_1^{-1}, \cdots, g_mG'g_m^{-1}$ pair-wise commute. Put $G'_i = g_iG'g_i^{-1}$ and $H'_i = g_iH'g_i^{-1}$ for $i = 0, 1, \cdots, m$. Here we consider that $g_0$ is the identity of $G$. For every $\phi \in Q^h(H)^G$, we have $D(\phi|_{H'_i}) = D(\phi|_{H'})$. Set $K_G = G'_1 \cdots G'_m$ and $K_H = H'_0H'_1 \cdots H'_m$. Applying Proposition 2.5 we have

$$D(\phi) \geq D(\phi|_{K_H}) = (m + 1)D(\phi|_{H'}).$$

Applying Theorem 1.2 we have

$$\text{scl}_{G', H'}(x) = \frac{1}{2} \sup_{\phi' \in Q^h(H')^G} \frac{\phi'(x)}{D(\phi')} \geq (m + 1) \cdot \frac{1}{2} \sup_{\phi \in Q^h(H)^G} \frac{\phi(x)}{D(\phi)} = (m + 1) \cdot \text{scl}_{G, H}(x).$$

This completes the proof.

**Example 2.6.** Let $m, n,$ and $N$ be positive integers such that $3 \leq n \leq N$ and $mn \leq N$. Then the braid group $B_n$ is $m$-displaceable in $B_N$. Thus, for each $x \in [B_n, P_n]$, we have

$$\text{scl}_{B_N, P_N}(x) \leq \frac{1}{m + 1} \text{scl}_{B_n, P_n}(x).$$

**Example 2.7.** Let $g$ and $h$ be integers such that $2 \leq g \leq h$. Let $\Sigma_{g, 1}$ denotes the surface of genus $g$ with 1 boundary. If a positive integer $m$ satisfies $mg \leq h$, then the mapping class group $M_{g, 1}$ of $\Sigma_{g, 1}$ is $m$-displaceable in $M_{h, 1}$. Thus, for each $x \in [M_{g, 1}, I_{g, 1}]$,

$$\text{scl}_{M_{h, 1}, I_{h, 1}}(x) \leq \frac{1}{m + 1} \text{scl}_{M_{g, 1}, I_{g, 1}}(x).$$
Here $\mathcal{I}_{g,1}$ is the Torelli group of $\Sigma_{g,1}$.

Let $B_\infty = \bigcup_{n=2} B_n$ be the infinite braid group and $P_\infty = \bigcup_{n=2} P_n$ the infinite pure braid group. For $\alpha \in [B_\infty, P_\infty]$, there exists an integer $n$ so that $\alpha \in [B_n, P_n]$. Since $B_n$ is $m$-displaceable in $B_\infty$ for arbitrarily large $m$, we have $\text{scl}_{B_\infty, P_\infty}(\alpha) = 0$ by Theorem 2.4. Thus $\text{scl}_{B_\infty, P_\infty}$ is identically zero. On the other hand, note that $\text{scl}_{P_\infty}$ is not identically zero since there is a surjective homomorphism $P_\infty \to P_3$ (by forgetting strands) and $P_3$ has a homogeneous quasimorphism which is not a homomorphism.

3. $H$-QUASIMORPHISM

Here we study some algebraic properties of $G$-invariant quasimorphisms and $H$-quasimorphisms. First we introduce the following notation. For real numbers $a$ and $b$ and for a non-negative number $D$, we write $a \sim_D b$ to mean $|b - a| \leq D$.

Recall that a quasimorphism $f$ on $H$ is $G$-invariant if there is $D' \geq 0$ such that

$$f(gxg^{-1}) \sim_{D'} f(x)$$

holds for every $g \in G$ and every $x \in H$. For a $G$-invariant quasimorphism $f$ on $H$, we write $D'(f)$ to indicate the number

$$\sup\{||f(gxg^{-1}) - f(x)|| \mid g \in G, x \in H\}.$$

Let $Q(H)^G$ denote the set of $G$-invariant quasimorphisms on $H$. We call a real-valued function $f$ on $G$ strictly $G$-invariant if $f(gxg^{-1}) = f(x)$ for every $g \in G$ and every $x \in H$.

Let $f$ be a quasimorphism on $G$. For each $x \in G$, it is known that there exists a limit

$$\overline{f}(x) = \lim_{n \to \infty} \frac{f(x^n)}{n},$$

and call the function $\overline{f} : G \to \mathbb{R}$ the homogenization of $f$. It is known that $\overline{f}$ is a homogeneous quasimorphism. Before stating the next lemma, recall that a homogeneous quasimorphism $f$ on $G$ is (strictly) conjugation invariant, i.e. $f(gxg^{-1}) = f(x)$ for every pair $g$ and $x$ of elements in $G$ (see Section 2.2.3 in [3]).

**Lemma 3.1.** Let $f$ be a $G$-invariant quasimorphism on $H$. Then its homogenization $\overline{f}$ is strictly $G$-invariant. In particular, if $f$ is homogeneous, then $D'(f) = 0$.

**Proof.** It is known that $f(x) \sim_{D(f)} \overline{f}(x)$ for every $x \in G$ (see Lemma 2.21 of [3]). Let $g \in G$ and $x \in H$. For every positive integer $n$, we have

$$n\overline{f}(gxg^{-1}) = \overline{f}(gx^n g^{-1}) \sim_{D(f)} f(gx^n g^{-1}) \sim_{D'(f)} f(x^n) \sim_{D(f)} \overline{f}(x^n) = n\overline{f}(x).$$

Therefore we have the inequality

$$|\overline{f}(gxg^{-1}) - \overline{f}(x)| \leq \frac{2D(f) + D'(f)}{n}$$

for every positive integer $n$. This implies $\overline{f}(gxg^{-1}) = \overline{f}(x)$. \qed
Here we introduce the following notion relevant to quasimorphisms.

**Definition 3.2.** Let $G$ be a group and $H$ a normal subgroup of $G$. A function $f : G \to \mathbb{R}$ is an $H$-quasimorphism if there is a number $D''$ such that

$$|f(gx) - f(g) - f(x)| \leq D''$$

and

$$|f(xg) - f(x) - f(g)| \leq D''$$

for every $g \in G$ and every $x \in H$. We denote by $D''(f)$ the infimum of such a non-negative number $D''$, and call it the defect of the $H$-quasimorphism $f$.

**Lemma 3.3.** Let $f : G \to \mathbb{R}$ be an $H$-quasimorphism. Then the restriction $f|_H$ of $f$ to $H$ is a $G$-invariant quasimorphism.

**Proof.** It is clear that $f|_H$ is a quasimorphism on $H$ whose defect is not bigger than $D''(f)$. For $g \in G$ and $x \in H$, we have

$$f(g) \sim_{D''(f)} f(gxg^{-1}) + f(gx^{-1}) \sim_{D''(f)} f(gxg^{-1}) + f(g) + f(x^{-1}) \sim_{2D''(f)} f(gxg^{-1}) - f(x) + f(g).$$

This means

$$f(gxg^{-1}) - f(x) \sim_{4D''(f)} 0,$$

and hence $f$ is a $G$-invariant quasimorphism on $H$. \qed

**Proposition 3.4.** Let $f : H \to \mathbb{R}$ be a $G$-invariant quasimorphism on $H$. Then there is an $H$-quasimorphism $f' : G \to \mathbb{R}$ with $f'|_H = f$ and $D''(f) \leq D(f) + D'(f)$. Moreover, if $f$ is homogeneous, we can take $f'$ to satisfy $D''(f') = D(f)$.

**Proof.** Let $S$ be a subset of $G$ such that $e \in S$ and the map $S \times H \to G$, $(s, x) \mapsto sx$ is a bijection. Let $f'$ be a real-valued function on $G$ which satisfies the following properties:

1. $f'(e) = 0$. For an element $s \in S - \{e\}$, let $f'(s)$ be an arbitrary real number.
2. For $s \in S$ and $h \in H$, define $f'(sh) = f'(s) + f(h)$.

We show that $f'$ is an $H$-quasimorphism such that $D''(f) \leq D(f) + D'(f)$ and $f'|_H = f$. Let $g \in G$ and $x \in H$. Let $s \in S$ and $h \in H$ so that $sh = g$. Then we have

$$f'(gx) = f'(shx) = f'(s) + f'(hx) \sim_{D(f)} f'(s) + f'(h) + f'(x) = f'(sh) + f'(x) = f'(g) + f'(x),$$

and hence

$$|f'(gx) - f'(g) - f'(x)| \leq D(f).$$

Next put $y = g^{-1}xg$. Then we have

$$f'(xg) - f'(x) - f'(g) = f'(gy) - f'(ggy^{-1}) - f'(g) \sim_{D'(f)} f'(gy) - f'(y) - f'(g) \sim_{D(f)} 0,$$

and hence

$$|f'(xg) - f'(x) - f'(g)| \leq D(f) + D'(f).$$

Thus we have shown that $f'$ is an $H$-quasimorphism satisfying $D''(f) \leq D(f) + D'(f)$. 

Suppose that $f$ is homogeneous. Then Lemma 3.1 implies $D'(f) = 0$, and hence we have $D''(f) \leq D(f)$. On the other hand, it is clear that $D(f) \leq D''(f)$. This completes the proof. □

Combining Lemma 3.3 and Proposition 3.4, we have that a quasimorphism $f : H \to \mathbb{R}$ is $G$-invariant if and only if $f$ has an extension which is an $H$-quasimorphism.

An $H$-homomorphism is a function $f : G \to \mathbb{R}$ satisfying

$$f(gx) = f(g) + f(x) = f(x) + f(g)$$

for every $g \in G$ and every $x \in H$.

**Corollary 3.5.** A $G$-invariant homomorphism $f : H \to \mathbb{R}$ has an extension $f' : G \to \mathbb{R}$ which is an $H$-homomorphism.

*Proof.* Since a homomorphism is homogeneous, it follows from Proposition 3.4 that there is an $H$-quasimorphism $f' : G \to \mathbb{R}$ with $D''(f') = D(f) = 0$. □

We end this section with mentioning $(G,H)$-commutators. Recall that an element $x$ of $G$ is a $(G,H)$-commutator if there are $g \in G$ and $h \in H$ such that $x = [g,h] = ghg^{-1}h^{-1}$. Since

$$[g,h] = [ghg^{-1},g^{-1}], \quad [h,g] = [g^{-1},ghg^{-1}],$$

$x$ is a $(G,H)$-commutator if and only if there are $g \in G$ and $h \in H$ with $x = [h,g]$. As is usual, we write $[G,H]$ to mean the subgroup of $G$ generated by the $(G,H)$-commutators. Note that $[G,H]$ is a normal subgroup of $G$ and $[G,H] \subset H$.

For an element $x$ in $[G,H]$, define the $(G,H)$-commutator length $\text{cl}_{G,H}(x)$ of $x$ by

$$\text{cl}_{G,H}(x) = \{n \mid \text{There are } n \text{ $(G,H)$-commutators } c_1, \cdots, c_n \text{ such that } x = c_1 \cdots c_n\}.$$  

It easily follows from Fekete’s lemma that there is a limit

$$\text{scl}_{G,H}(x) = \lim_{n \to \infty} \frac{\text{cl}_{G,H}(x^n)}{n},$$

and we call $\text{scl}_{G,H}(x)$ the stable $(G,H)$-commutator length of $x$.

**Lemma 3.6.** For a $G$-invariant homogeneous $H$-quasimorphism $f : H \to \mathbb{R}$, the following equalities hold:

$$D(f) = \sup_{h_1,h_2 \in H} |f([h_1,h_2])| = \sup_{g \in G,h \in H} |f([g,h])|$$

In particular, $f([g,h]) \leq D(f)$ holds for every $g \in G$ and every $h \in H$.

*Proof.* The first equality

$$D(f) = \sup_{h_1,h_2 \in H} |f([h_1,h_2])|$$

is known (see Lemma 3.6 of [1] or Lemma 2.24 of [3]). The inequality

$$\sup_{h_1,h_2 \in H} |f([h_1,h_2])| \leq \sup_{g \in G,h \in H} |f([g,h])|$$
is obvious. Since \( f \) is a homogeneous \( G \)-invariant quasimorphism, \( f \) is strictly \( G \)-invariant (Lemma 3.1) and hence satisfies

\[
f([g, h]) = f(ghg^{-1}h^{-1}) \sim_D f(ghg^{-1}) + f(h^{-1}) = f(h) - f(h) = 0.
\]

Thus we have

\[
\sup_{g \in G, h \in H} |f([g, h])| \leq D(f),
\]

which completes the proof. \( \square \)

4. Filling norm

Let \( Q_H = Q_H(G) \) be the space of \( H \)-quasimorphisms on \( G \), and \( H^1_H = H^1_H(G) \) the space of \( H \)-homomorphisms on \( G \). Then the defect \( D' \) of \( H \)-quasimorphisms is a pseudo-norm on \( Q_H \) whose kernel is \( H^1_H \), and hence \( Q_H/H^1_H \) is a normed space. The purpose in this section is to identify \( Q_H/H^1_H \) with a topological dual of a certain normed space arising from the inhomogeneous complex of the group \( G \).

Recall that \( C_n(G) \) is the free \( \mathbb{R} \)-module generated by the \( n \)-tuple direct product \( G^n \), and the differential \( \partial: C_n(G) \rightarrow C_{n-1}(G) \) is given by

\[
\partial(g_1, \ldots, g_n) = (g_2, \ldots, g_n) + \sum_{i=1}^{n-1} (-1)^i (g_1, \ldots, g_i, g_{i+1}, \ldots, g_n) + (-1)^n (g_1, \ldots, g_{n-1}).
\]

Note that in the case \( n = 2 \), the differential is described by

\[
\partial(g_1, g_2) = g_2 - g_1g_2 + g_1.
\]

Let \( C'_2 = C'_2(G) \) be the \( \mathbb{R} \)-submodule of \( C_2(G) \) generated by the set

\[
\{(g_1, g_2) \in G \times G \mid H \text{ contains either } g_1 \text{ or } g_2\}.
\]

Put \( B'_1 = \partial C'_2 \) and \( Z_2' = Z_2(G; \mathbb{R}) \cap C'_2 \). Consider \( C'_2 \) as a normed space by the \( l^1 \)-norm. Then \( Z_2' \) is a closed subspace of \( C'_2 \), and hence \( C'_2/Z_2' \) is a normed space. We consider \( B'_1 \) as a normed space by the isomorphism \( C'_2/Z_2' \cong B'_1 \), and we write \( \|x\|' \) to indicate the norm on \( B'_1 \).

Lemma 4.1. If \( g \in G \) and \( h \in H \), then \([g, h] \in B'_1 \) and \( \|[g, h]\|' \leq 3 \).

Proof. This is deduced from the following equality:

\[
\partial([g, h], hg) - \partial(g, h) + \partial(h, g) = [g, h]
\]

\( \square \)

Lemma 4.2. If \( x, y \in [G, H] \), then \( \|xy\|' \leq \|x\|' + \|y\|' + 1 \).
Proof. Since \(x, y \in [G, H]\), we have \(xy \in [G, H]\). Therefore Lemma 4.1 implies \(x, y, xy \in B'_1\). Since \(\partial(x, y) = y - xy + x\), we have

\[\|xy - x - y\|' \leq 1.\]

Therefore

\[\|xy\|' = \|(x + y) + (xy - x - y)\|' \leq \|x + y\|' + \|xy - x - y\|' \leq \|x\|' + \|y\|' + 1.\]

\[\square\]

Let \(x \in [G, H]\). Lemma 4.2 implies

\[\|x^{m+n}\|' + 1 \leq (\|x^m\|' + 1) + (\|x^n\|' + 1)\]

for every pair \(m\) and \(n\) of positive integers. Therefore Fekete’s lemma implies that there is a limit

\[\text{fill}_{G,H}(x) = \lim_{n \to \infty} \frac{\|x^n\|'}{n}.\]

We call \(\text{fill}_{G,H}(x)\) the \((G, H)\)-filling norm of \(x\).

**Proposition 4.3.** Let \(x \in [G, H]\). Then there is an inequality \(\|x\|' \leq 4 \cdot \text{cl}_{G,H}(x) - 1\).

Proof. Suppose \(\text{cl}_{G,H}(x) = n\), and let \(c_1, \ldots, c_n\) be \((G, H)\)-commutators satisfying \(x = c_1 \cdots c_n\). Then Lemma 4.1 and Lemma 4.2 imply

\[\|x\|' \leq \|c_1\| + \cdots + \|c_n\| + (n - 1) \leq 4n - 1.\]

\[\square\]

**Corollary 4.4.** If \(x \in [G, H]\), then \(\text{fill}_{G,H}(x) \leq 4 \cdot \text{scl}_{G,H}(x)\).

Proof. By Proposition 4.3, we have an inequality

\[\frac{\|x^n\|'}{n} \leq 4 \cdot \frac{\text{cl}_{G,H}(x^n)}{n} - \frac{1}{n}\]

By taking the limits, we have \(\text{fill}_{G,H}(x) \leq 4 \cdot \text{scl}_{G,H}(x)\).

\[\square\]

Thus we have one side of the inequalities in Theorem 1.4. We prove the other side in the next section, using a geometric characterization of \((G, H)\)-commutator lengths.

For a normed space \(V\), let \(V^*\) denote the topological dual of \(V\).

**Proposition 4.5.** The normed spaces \(Q_H/H^1_H\) and \((C'_2/Z'_2)^*\) are isometric.

Proof. Let \(f : G \to \mathbb{R}\) be an \(H\)-quasimorphism on \(G\). Then \(f\) is identified with an \(\mathbb{R}\)-linear map \(f : C_1(G) \to \mathbb{R}\). Since \(f\) is an \(H\)-quasimorphism, we have

\[|f \circ \partial(g, x)| = |f(x) - f(gx) + f(g)| \leq D''(f),\]

and

\[|f \circ \partial(x, g)| = |f(g) - f(xy) + f(x)| \leq D''(f)\]
for every \( g \in G \) and every \( x \in H \). This means that \( f \circ \partial : C'_2 \to \mathbb{R} \) is continuous. Since \( f \circ \partial \) vanishes on \( Z'_2 \), we have that \( f \circ \partial \) induces a continuous linear map \( \tilde{f} : C'_2/Z'_2 \to \mathbb{R} \), whose operator norm is \( D''(f) \). Thus we have constructed the correspondence from \( Q_H(G) \) to \((C'_2/Z'_2)^*\). The kernel of this correspondence is the space of \( H \)-homomorphisms. Thus we have constructed a norm preserving map \( Q_H/H^1_H \to (C'_2/Z'_2)^* \).

Next we construct the correspondence \((C'_2/Z'_2)^* \to Q_H/H^1_H \). Let \( f : C'_2/Z'_2 \to \mathbb{R} \) be bounded. Then \( f \) is identified with a linear map \( \overline{f} : B'_1 \to \mathbb{R} \). Since our coefficient is \( \mathbb{R} \), there is a linear map \( f' : C_1(G) \to \mathbb{R} \) such that \( f'|_{B_1'} = \overline{f} \), which is not necessarily continuous. The function \( f' : C_1(G) \to \mathbb{R} \) is identified with a real-valued function \( f' : G \to \mathbb{R} \) on \( G \). Let \( D'' \) be the operator norm of \( f : C'_2 \to \mathbb{R} \). Then we have

\[
|f'(gx) - f'(g) - f'(x)| = |f'(\partial(g, x))| = |f(g, x)| \leq D''.
\]

Similarly, we can show \(|f'(xg) - f'(x) - f'(g)| \leq D''\). This implies that \( f' \) is an \( H \)-quasimorphism.

To complete the construction of the correspondence \((C'_2/Z'_2)^* \to Q_H/H^1_H \), we let \( f'' \) be another linear extension of \( \overline{f} : B'_1 \to \mathbb{R} \) and show that \( f' - f'' \) is an \( H \)-homomorphism. Since \( f' \) and \( f'' \) coincide on \( B'_1 \), we have

\[
f'(gx) - f'(g) - f'(x) = f'(gx - g - x) = f''(gx - g - x) = f''(gx) - f''(g) - f''(x).
\]

This means

\[
(f' - f'')(gx) = (f' - f'')(g) + (f' - f'')(x).
\]

Similarly, we can show

\[
(f' - f'')(xg) = (f' - f'')(x) + (f' - f'')(g),
\]

and hence \( f' - f'' \) is an \( H \)-homomorphism. Thus we have completed the construction of the correspondence \((C'_2/Z'_2)^* \to Q_H/H^1_H \). Since these correspondences are mutually inverses and the correspondence \( Q_H/H^1_H \to (C'_2/Z'_2)^* \) is an isometry, we complete the proof. \( \square \)

**Corollary 4.6.** The normed space \( Q_H/H^1_H \) is a Banach space.

Applying the Hahn-Banach theorem, we have the following corollary:

**Corollary 4.7.** For \( x \in [G, H] \), the following holds:

\[
||x||' = \sup_{f \in Q_H/H^1_H} \frac{|f(x)|}{D''(f)}
\]

5. Geometric characterization of \( \text{cl}_{G,H} \)

Let \( x \) be an element in \([G, G]\). An element of \( G \) is identified with a homotopy class of loops in \( BG \). Since \( x \) is zero in the integral homology group of \( G \), there is a compact orientable surface \( S \) with connected boundary and a map \( f : S \to BG \) which sends the boundary of \( S \) to \( x \). The commutator length is the minimum genus of such a surface \( S \). In
this section, we give a similar geometric interpretation of \((G, H)\)-commutator lengths, and show the following theorem:

**Theorem 5.1.** For \(x \in [G, H]\), the equality \(\text{fill}_{G,H}(x) = 4 \cdot \text{scl}_{G,H}(x)\) holds.

Recall that we have shown the inequality \(\text{fill}_{G,H}(x) \leq 4 \cdot \text{scl}_{G,H}(x)\) (Corollary 4.4). So it suffices to show \(\text{fill}_{G,H}(x) \geq 4 \cdot \text{scl}_{G,H}(x)\).

5.1. \((G, H)\)-labellings of simplicial surfaces. We first recall our terminology from Section 1. Every surface is assumed to be compact and oriented. In this paper, a triangulation of \(S\) is a CW-structure which satisfies the following conditions: Every 1-cell is oriented. Every 2-cell \(\sigma\) is surrounded by three 1-cells \(\partial_0 \sigma\), \(\partial_1 \sigma\), and \(\partial_2 \sigma\) as is depicted in Figure 1 in Section 1. We do not assume that \(\partial_0 \sigma, \partial_1 \sigma,\) and \(\partial_2 \sigma\) are distinct 1-cells. Similarly, we do not assume that the endpoints of 1-cells are different.

We call a surface \(S\) equipped with its triangulation simplicial surface. Let \(E(S)\) be the set of 1-cells in \(S\) and \(T(S)\) the set of 2-cells in \(S\). A \(G\)-labelling of \(S\) is a function \(f: E(S) \to G\) which satisfies

\[
f(\partial_0 \sigma) \cdot f(\partial_2 \sigma) = f(\partial_1 \sigma)
\]

for every \(\sigma \in T(S)\). A \((G, H)\)-triangle of a \(G\)-labelling \(f\) is a 2-cell \(\sigma\) in \(S\) such that either \(f(\partial_0 \sigma)\) or \(f(\partial_2 \sigma)\) belongs to \(H\). A \((G, H)\)-labelling of \(S\) is a \(G\)-labelling of \(S\) such that every 2-cell of \(S\) is a \((G, H)\)-triangle. A \((G, H)\)-simplicial surface is a pair \((S, f)\) of a triangulated surface \(S\) together with a \((G, H)\)-labelling \(f\) of \(S\). A \((G, H)\)-simplicial surface \(S\) with boundary \(x\) is a \((G, H)\)-simplicial surface \((S, f)\) such that the boundary of \(S\) has only one 1-cell and \(f\) sends it to \(x\).

Clearly, our simplicial surface is a geometric realization of a 2-dimensional simplicial set with certain conditions. From now on we write \(S\) when we consider the simplicial surface as a simplicial set, and \(|S|\) when we consider the simplicial surface as a topological space. Using simplicial set, we can describe \(G\)-labellings simpler. Regard \(G\) as a small category in the usual way, and let \(NG\) denote its nerve. Then a \(G\)-labelling of \(S\) is identified with a simplicial map \(f: S \to NG\). Since the geometric realization \(BG = |NG|\) of \(NG\) is the classifying space of \(G\), a \(G\)-labelling \(f\) of \(S\) induces a continuous map \(|f|: |S| \to BG\). On the other hand, for every continuous map \(f': |S| \to BG\), there is a \(G\)-labelling \(f: S \to NG\) such that \(|f| \simeq f'\). For an introduction to simplicial sets, we refer to [8] and [12].

For simplicity, we call a non-degenerate simplex of a simplicial set a cell. In fact, for a simplicial set \(K\), there is a bijective correspondence between the non-degenerate simplices in \(K\) and cells of the geometric realization \(|K|\) of \(K\).

We call an element \(x \in B'_1\) an integral \((G, H)\)-boundary if there is a chain \(c \in C'_2\) with integral coefficients such that \(\partial c = x\). For an integral \((G, H)\)-boundary \(x\), we write \(\|x\|'_{Z}\) to mean the infimum of \(\|c\|_1\) such that \(c \in C'_2\) is integral and \(\partial c = x\).
Lemma 5.2. Let \( x \in [G,H] \). Then there is a \((G,H)\)-simplicial surface \( S \) with boundary \( x \). If the number of 2-cells of \( S \) coincides with \( \| x \|'_2 \), then \( S \) is connected.

Proof. This proof is similar to a proof of the fact that every loop homologous to 0 is represented as a boundary of a compact connected surface.

By the proof of Lemma 4.1, \( x \in [G,H] \) implies that \( x \) is an integral \((G,H)\)-boundary. Thus there is an integral chain \( c \in C'_2 \) satisfying \( \partial c = x \). Then there is a 2-dimensional simplicial set \( K \) and a simplicial map \( f: K \to NG \) satisfying the following conditions:

- Every face of a cell in \( K \) is a cell.
- The number of 2-cells of \( K \) coincides with \( \| c \|_1 \).
- \( H_2(K;Z) \cong Z \) and \( K \) has a 2-cycle \( c \) such that \( f(c_K) = c \), \( \| c_K \|_1 = \| c \|_1 \), and \( c_K \) represents the generator of \( H_2(K;Z) \).
- If a pair \( e_1 \) and \( e_2 \) of 1-cells in \( K \) satisfies \( f(e_1) = f(e_2) \) in \( G \), then \( e_1 = e_2 \).
- \( K \) has only one 0-cell.

Adding 1-cells and 0-cells appropriately to \( K \), we have a simplicial surface \( S \) with a simplicial map \( p: S \to K \) such that \( p \) sends 2-cells of \( S \) to 2-cells of \( K \) bijectively, and the boundary of \( S \) has only one 1-cell. Then the composition \( S \to K \to NG \) is a \((G,H)\)-simplicial surface with boundary \( x \).

Suppose that \( \| c \|_1 = \| x \|'_2 \) and \( S \) is not connected. Let \( S' \) be the connected component of \( S \) containing the boundary of \( S \). The restriction \( f|_{S'}: S' \to NG \) is a \((G,H)\)-simplicial surface with boundary \( x \) whose number of 2-cells is smaller than \( \| c \|_1 = \| x \|' \). This is a contradiction. \( \square \)

On the other hand, we will show that if there is a \((G,H)\)-simplicial surface with boundary \( x \) then \( x \) is contained in \([G,H]\) (see Proposition 5.6).

We are now ready to state the geometric characterization of \((G,H)\)-commutator lengths.

Theorem 5.3. Let \( x \) be an element in \([G,H]\). Then the \((G,H)\)-commutator length \( cl_{G,H}(x) \) of \( x \) is the minimum of the genus of a \((G,H)\)-simplicial surface with boundary \( x \).

The proof of Theorem 5.3 is postponed to the next subsection. In the rest of this subsection, we deduce Theorem 5.1 from Theorem 5.3.

The following lemma can be shown in a standard way, so we omit the proof.

Lemma 5.4. For \( x \in [G,H] \),

\[
\lim_{n \to \infty} \frac{\| nx \|'_2}{n} = \| x \|'
\]

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Let \( n \) be a positive integer. Since \( x \in [G,H] \), \( nx \) is an integral \((G,H)\)-boundary and hence there is \( c \in C'_2 \) with integral coefficient such that \( nx = \partial c \) and...
\[ \|nx\|_Z = \|c\|'. \] Define the chain \( c' \in C'_2 \) by
\[
c' = c - \sum_{i=1}^{n-1} (x,x^i).
\]
Then we have \( \|c'\|' \leq \|nx\|_Z + n - 1 \) and \( \partial c' = x^n \). Thus it follows from Lemma 5.2 that there is a \((G,H)\) simplicial surface \((S,f)\) with boundary \(x^n\) such that the number of 2-cells of \( S \) is \( \|c'\|' \). Let \( S' \) be the connected component of \( S \) including the boundary of \( S \), and \( f' \) the restriction of \( f \) to \( S' \). Let \( g \) be the genus of \( S' \), and let \( p, e, s \) be the numbers of 0-cells, 1-cells, and 2-cells in \( S' \), respectively. Then the following hold:

1. By observing the Euler characteristic of \( S \), we have
\[
s - e + p = 1 - 2g.
\]
2. Every 1-cell not contained in the boundary of \( S' \) appears two times as faces of 2-cells. Since the boundary of \( S' \) has one 1-cell, we have
\[
1 + 2(e - 1) = 3s.
\]
Thus we have
\[
s = 4g + 2p - 3 \geq 4g - 1 \geq \text{cl}(x^n) - 1.
\]
Here we use Theorem 5.3 to deduce the last inequality. Since \( \|nx\|_Z + n - 1 \geq s \), we have
\[
\|nx\|_Z \geq 4 \cdot \text{cl}(x^n) - n.
\]
Multiplying \( n^{-1} \) and taking their limits, we have
\[
\|x\|' \geq 4 \cdot \text{scl}_{G,H}(x) - 1
\]
for every \( x \in [G,H] \). Thus we have
\[
\|x^m\|' \geq 4 \cdot \text{scl}_{G,H}(x^m) - 1 = 4m \cdot \text{scl}_{G,H}(x) - 1.
\]
Multiplying \( m^{-1} \) and taking the limits, we have
\[
\text{fill}_{G,H}(x) \geq 4 \cdot \text{scl}_{G,H}(x).
\]
This completes the proof of Theorem 5.1. \( \square \)

5.2. **Proof of Theorem 5.3** We start the proof of Theorem 5.3. Theorem 5.3 is deduced from Proposition 5.5 and Proposition 5.6. Throughout this section, we write \( N \) to indicate the minimum of the genus of a \((G,H)\) simplicial surface with boundary \( x \). The following proposition implies the inequality \( N \leq \text{cl}_{G,H}(x) \).

**Proposition 5.5.** Let \( x \in [G,H] \) and put \( m = \text{cl}_{G,H}(x) \). Then there is a connected \((G,H)\) simplicial surface with boundary \( x \) whose genus is \( m \).
Proof. Let $x \in [G, H]$. Let $c_1, \cdots, c_m$ be $(G, H)$-commutators satisfying $x = c_1 \cdots c_m$. We want to show that there is a $(G, H)$-simplicial surface with boundary $x$ whose genus is $m$. Thus it suffices to find a $(G, H)$-triangulation of the $(4m + 1)$-gon depicted in Figure 2. To construct this, we first embed three 2-simplices to the part $g_1 \xrightarrow{h_1} g_1 \xrightarrow{h_1}$ as is depicted in the left of Figure 3, and after that, we embed $m - 2$ triangles as is depicted in the right of Figure 5. Since $[G, H] \subset H$, we have that these triangles are $(G, H)$-triangles. This completes the proof.

On the other hand, the inequality $N \geq cl_{G, H}(x)$ follows from the following proposition:

**Proposition 5.6.** Let $x \in G$ and let $n$ be a positive integer. Suppose that there is a connected $(G, H)$-simplicial $(G, H)$ with boundary $x$ whose genus is $n$. Then there are $n$ $(G, H)$-commutators $c_1, \cdots, c_n$ such that $x = c_1 \cdots c_n$. In particular, if there is a $(G, H)$-simplicial surface with boundary $x$, then $x$ is contained in $[G, H]$.

Proof. Let $(S, f)$ be a connected $(G, H)$-simplicial surface with boundary $x$ such that the genus of $S$ is $n$. Let $S'$ be the subcomplex of $S$ consisting of simplices of $S$ mapped to $H$ by $f$. In other words, $S'$ consists of 2-cells whose boundary consists of 1-cells labelled by $H$. Let $\sigma$ be a 2-cell not contained in $S'$. Since $\sigma$ is a $(G, H)$-triangle, $\partial_1 \sigma$ is not contained in $S'$, one of $\partial_0 \sigma$ and $\partial_2 \sigma$ is not contained in $S'$, and the other is contained in $S'$. In particular, $\sigma$ has exactly two 1-faces not contained in $S'$. Consider the line segment in the geometric realization $|S|$ of $S$ which connects the two central points of the two faces of $\sigma$ not contained in $S'$, and let $C$ be the union of these line segments. Then $C$ is a 1-dimensional CW-complex contained in $|S|$, and satisfies the following properties:

1. Every connected component of $C$ is homeomorphic to $S^1$.
2. For each connected component $\gamma$ of $C$, the inclusion $\gamma \hookrightarrow |S|$ is homotopic to a map factored through $|S'|$.
3. $|S'|$ is a deformation retract of $|S| - C$. 

![Figure 2.](image-url)
We first show (1). Since every 1-cell of $S$ is contained in at most two 2-faces, there are no vertices in $C$ with degree greater than 2. Thus $C$ is a compact 1-dimensional manifold which may have a non-empty boundary. Suppose that $C$ has a non-empty boundary and $v$ is a point in $\partial C$. If $v$ is not contained in the boundary of $S$, then $v$ is contained in exactly two 2-cells $\sigma$ and $\tau$. Since $\sigma$ and $\tau$ are not mapped to $NH$, $v$ is contained in the line segments of $\sigma$ and $\tau$. Thus $v$ is not a boundary point in $C$. On the other hand, if $v$ is contained in the boundary of $S$, then $v$ is clearly a boundary. However, in that case, $C$ is a one-dimensional compact manifold having only one boundary point. This is a contradiction and hence $C$ has no boundary and $x$ is contained in $H$. This completes the proof of (1).

Next we show (2). Let $\gamma$ be a connected component of $C$. There are two ways to verify (2). Since $\gamma$ is a simple closed curve in the orientable surface $|S|$, the normal bundle of $\gamma$ is trivial and hence we can slide $\gamma$ to $|S'|$. The second one is more concrete. Let $\sigma$ be a 2-cell not contained in $S'$. Since $\sigma$ is a $(G,H)$-triangle, we can slide the line segment to $S'$ along the directions of each 1-simplex containing an endpoint (see Figure 3). Combining these deformations for all the line segments forming $C$, we have a homotopy from $\gamma \mapsto |S|$ to a map factored through $|S'|$. Considering similar deformations, we have the condition (3).

We consider each connected component $\gamma$ of $C$ is a simple closed curve whose basepoint is mapped to a vertex in $|S'|$ by the homotopy considering in (2).

We define the simple closed curve $\gamma_1, \ldots, \gamma_n$ as follows: Recall that a simple closed curve in a topological surface $|S|$ is non-separable if the number of connected components of $|S| - \gamma$ is not bigger than the number of connected components of $|S|$. First, if $C$ has a connected component $\gamma$ which is non-separable in $|S|$, then put $\gamma_1 = \gamma$ and put $C_1 = C - \gamma$. Next if $C_1$ has a connected component which is non-separable in $|S| - \gamma_1$, then let $\gamma_2$ denote the connected component and put $C_2 = C_1 - \gamma_2$. Iterating this, we have simple closed curves $\gamma_1, \ldots, \gamma_k$ such that each $\gamma_i$ is a non-separable simple closed curve in $C - (\gamma_1 \cup \cdots \cup \gamma_{i-1})$, and every connected component of $C - (\gamma_1 \cup \cdots \cup \gamma_k)$ is separable in $|S| - (\gamma_1 \cup \cdots \cup \gamma_k)$. Since the genus of $S$ is $n$, we have that $k \leq n$, and hence $|S| - C$ has $n - k$ handles. If
$k < n$, there is a non-separable simple closed curve $\gamma_{k+1}$ in $|S| - C$. If $k + 1 \neq n$, we can further take a non-separable simple closed curve $\gamma_{k+2}$ in $|S| - (C \cup \gamma_{k+1})$. Iterating this, we can take simple closed curves $\gamma_{k+1}, \ldots, \gamma_n$ so that $|S| - (\gamma_1 \cup \cdots \cup \gamma_n)$ has no non-separable simple closed curves. For each $i$ we take a basepoint of $\gamma_i$ so that it is mapped to a vertex in $S'$ by the deformation retract considered in (3).

By the classification theorem of compact surfaces, there is a homeomorphism from $|K|$ to the surface depicted in Figure 5 which maps $\gamma_1, \ldots, \gamma_n$ to the simple closed curves as are depicted by red curves in Figure 5. Define the simple closed curves $\delta_1, \ldots, \delta_n$ and paths $\alpha_1, \ldots, \alpha_n$ as are depicted in Figure 5. Here we assume that the basepoint of $\delta_i$ coincides with the basepoint of $\gamma_i$. The basepoint of $|S|$ is the vertex contained in the boundary of $|S|$, and $\alpha_i$ connects the basepoint of $|S|$ with the basepoint of $\gamma_i$ and $\delta_i$.

Let $\overline{\alpha}_i$ be the reverse of the path $\alpha_i$. Then $f(\overline{\alpha}_i \cdot \gamma_i \cdot \alpha_i)$ is a loop of $BG$, and we let $h_i$ be the element of $G$ associated to $f(\overline{\alpha}_i \cdot \gamma_i \cdot \alpha_i)$ of $G$. Similarly, we let $g_i$ be the element of $G$ associated to $f(\overline{\alpha}_i \cdot \delta_i \cdot \alpha_i)$. Then we have

$$x = [g_1, h_1] \cdots [g_n, h_n].$$

Thus it suffices to prove that $h_i$ is an element of $H$.

For each $i$, there is a map $\varphi_i : |S| \to |S|$ which satisfies the following:

(a) There is a based homotopy from the identity of $|S|$ to $\varphi_i$.
(b) $\varphi_i(\gamma_i) \subset |S'|$
(c) $\varphi_i$ sends the basepoint of $\gamma_i$ to a vertex in $|S'|$.

To see this, for $i \leq k$, since the inclusion $\gamma_i \cup \{\ast\} \hookrightarrow |S|$ is a cofibration, the homotopy of $\gamma_i \hookrightarrow |S|$ sending $\gamma_i$ to $|S'|$ discussed in (2) in the former in this proof extends to the homotopy $H_i$ of $|S|$ from the identity of $|S|$. Let $\varphi_i$ be the end of this homotopy $H_i$. For $i > k$, we can similarly prove the existence of $\varphi_i$ by using deformation retracts from $|S| - C$ to $|S'|$ considered in (3).

By (a), we have

$$h_i = |f|_*(\overline{\alpha}_i \cdot \gamma_i \cdot \alpha_i) = |f|_*(\varphi_i(\overline{\alpha}_i \cdot \gamma_i \cdot \alpha_i)) = [f \circ \varphi_i \circ \alpha_i]^{-1} \cdot [f \circ \varphi_i \circ \gamma_i] \cdot [f \circ \varphi_i \circ \alpha_i].$$
It follows from (c) that $f \circ \varphi_i \circ \alpha_i$ is a loop of $BG$, and hence $[f \circ \varphi_i \circ \alpha_i]$ is an element in $G$. It follows from (b) that $f \circ \varphi_i \circ \gamma_i$ is a loop in $BH$ and hence it is contained in $H$. Since $H$ is normal, we have that $h_i$ is an element in $H$. This completes the proof.

**Remark 5.7.** In the definition of $(G,H)$-simplicial surfaces, we do not admit a triangle $\sigma$ such that $\partial_1 \sigma$ is contained in $H$ but neither $\partial_0 \sigma$ nor $\partial_1 \sigma$ is contained in $H$. However, a similar geometric characterization of $(G,H)$-commutator lengths holds if we admit such a triangle. To state it explicitly, we prepare some terminology. We call a triangle $\sigma$ in a simplicial surface with $G$-labelling $f$ a **pseudo-($G,H$)-triangle** if one of $f(\partial_0 \sigma)$, $f(\partial_1 \sigma)$, and $f(\partial_2 \sigma)$ belongs to $H$, and define a **pseudo-($G,H$)-surface** to be a simplicial surface with a $G$-labelling such that every 2-cell of it is a pseudo-($G,H$)-triangle. Then by almost the same proof of Theorem 5.3, it can be shown that the $(G,H)$-commutator length of $x \in [G,H]$ coincides with the minimum genus of a connected pseudo-($G,H$)-surface with boundary $x$.

**6. Proof of the main theorem**

The purpose of this section is to complete the proof of our generalization of Bavard’s duality. This part is a straightforward generalization of the corresponding part of Bavard’s original proof.

Let $Q^h(H)^G$ denote the space of homogeneous $G$-invariant quasimorphisms on $H$, and $H^1(H)^G$ the space of $G$-invariant homomorphisms from $H$ to $\mathbb{R}$. Recall that $Q_H = Q_H(G)$ is the space of $H$-quasimorphisms on $G$ and $H^1_H = H^1_H(G)$ is the space of $H$-homomorphisms on $G$ (see Section 3).

**Theorem 6.1.** For $a \in [G,H]$, the following equality holds:

$$\text{scl}_{G,H}(a) = \frac{1}{2} \sup_{f \in Q^h(H)^G \cap H^1(G)^G} \frac{|f(a)|}{D(f)}.$$ 

**Proof.** Let $f \in Q^h(H)^G$. Suppose $\text{cl}_{G,H}(a) = m$ and let $c_1, \ldots, c_m$ be $(G,H)$-commutators such that $a = c_1 \cdots c_m$. Then we have

$$f(a) \sim_{(m-1)D(f)} f(c_1) + \cdots + f(c_m) \sim_{mD(f)} 0.$$
Thus we have $|f(a)| \leq (2m - 1)D(f) = (2 \cdot \text{cl}_{G,H}(a) - 1)D(f)$. Therefore

$$|f(a)| = \frac{|f(a^n)|}{n} \leq \frac{\text{cl}(a^n)}{n} \cdot 2D(f) - \frac{D(f)}{n}.$$ 

By taking the limit, we have an inequality

$$|f(a)| \leq 2D(f) \cdot \text{cl}_{G,H}(a).$$

Thus we have

$$\text{scl}_{G,H}(a) \geq \sup_{f \in Q^H \cap H^1(H)^G} \frac{|f(a)|}{D(f)}.$$

Next we show the converse of the inequality. By Theorem 5.1 and Corollary 4.7, we have

$$4 \cdot \text{scl}_{G,H}(a) = \text{fill}_{G,H}(a) = \lim_{n \to \infty} \frac{\|a^n\|}{n} = \lim_{n \to \infty} \left( \sup_{f \in Q^H \cap H^1_H} \frac{|f(a^n)|}{nD''(f)} \right)$$

For each $n$, take $f_{n,m} \in Q_H$ to satisfy

$$\sup_{f \in Q^H \cap H^1_H} \frac{|f(a^n)|}{nD''(f)} \sim \frac{1}{m} \frac{|f_{n,m}(a^n)|}{nD''(f_{n,m})}.$$ 

Then we have

$$\|f_{n,m}|_H - f_{n,m}|_H\| \leq D(f_{n,m}|_H) \leq D''(f_{n,m}).$$

Here $f_{n,m}|_H$ is the homogenization of the quasimorphism $f_{n,m}|_H$ on $H$. Then

$$\frac{|f_{n,m}|_H(a)|}{D''(f_{n,m})} = \frac{|f_{n,m}|_H(a^n)|}{nD''(f_{n,m})} \sim \frac{1}{m} \frac{f_{n,m}(a^n)}{nD''(f_{n,m})} \sim \frac{1}{m} \sup_{f \in Q^H \cap H^1_H} \frac{|f(a^n)|}{nD''(f)}$$

Thus we have that

$$\lim_{n \to \infty} \frac{|f_{n,m}|_H(a)|}{D''(f_{n,m})} = \lim_{n \to \infty} \left( \sup_{f \in Q^H \cap H^1_H} \frac{|f(a^n)|}{nD''(f)} \right) = 4 \cdot \text{scl}_{G,H}(a)$$

Therefore we have

$$\text{scl}_{G,H}(a) \leq \frac{1}{4} \sup_{f \in Q^H \cap H^1_H} \frac{|f|_H(a)|}{D''(f)} \leq \frac{1}{2} \sup_{f \in Q^H \cap H^1_H} \frac{|f|_H(a)|}{D(f)} \leq \frac{1}{2} \sup_{f \in Q^H \cap H^1_H} \frac{|f(a)|}{D(f)}.$$ 

Here we use $D(f|_H) \leq 2D(f|_H) \leq D''(f)$ and the fact that the restriction to $H$ of an $H$-quasimorphism is $G$-invariant (Lemma 3.3). 

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