Reconstruction of Gray-scale Images

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Abstract

We present an algorithm to reconstruct gray scale images corrupted by noise. We use a Bayesian approach. The unknown original image is assumed to be a realization of a Markov random field on a finite two dimensional region $\Lambda \subset \mathbb{Z}^2$. This image is degraded by some noise, which is assumed to act independently in each site of $\Lambda$ and to have the same distribution on all sites. For the estimator we use the mode of the posterior distribution: the so called maximum a posteriori (MAP) estimator. The algorithm, that can be used for both gray-scale and multicolor images, uses the binary decomposition of the intensity of each color and recovers each level of this decomposition using the identification of the problem of finding the two color MAP estimator with the min-cut max-flow problem in a binary graph, discovered by Greig, Porteous and Seheult (1989). Experimental results and a detailed example are given in the text. We also provide a web page where additional information and examples can be found.

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1 Introduction

We consider here the problem of image reconstruction. Suppose a multicolor or gray-scale picture is subjected to noise and an observer has access only to this corrupted version. How can she estimate the original picture?

The analysis of this kind of problem has attracted a lot of interest and many approaches have been considered ([G]).

One of the methods proposed is the so-called MAP estimator. In this method one assumes that the original image is a random realization of a Markov random field that has been corrupted by some site-independent noise. One assumes that the distribution of the field (a priori distribution) is known as well as the distribution of the noise, that is, the conditional distribution of the observed image given the original one. The MAP estimator is the image that has the largest probability of having produced the observed one. This is the mode of the posterior distribution.

In multicolor images the literature proposes to use the so-called Potts model as a priori distribution. Roughly speaking, this model is a measure on the set of images that gives more weight to images that have neighboring pixels of the same color. This choice has a number of advantages and disadvantages. An important disadvantage is the fact that the algorithms used to compute the MAP estimator operate in exponential time in the number of pixels ([GJ]). Our main point in this paper is to propose an alternative a priori measure with the property that the computation time of the exact MAP estimator is polynomial in the number of pixels. Using this approach we have produced an algorithm and a program that reconstruct dirty images in polynomial time in the number of pixels.

In the remaining of this section we explain these ideas in some detail. In the next section we present some experimental results obtained from the implementation of our method to some images. The next section is more technical in nature. It explains why our MAP estimator can be computed efficiently (polynomial time) and provides a proof for the theoretical result presented below. The question of how to obtain estimators for some important parameters associated to the observed image is discussed in the appendix. We close this paper with some final remarks.

In order to motivate the discussion we have to introduce some notation.
Assume the image is a point of \( \{1, \cdots, c\}^\Lambda \), where \( \Lambda \) is a finite subset of \( \mathbb{Z}^2 \) (e.g. a square) with \( N \) sites. The image associates to each site or pixel in \( \Lambda \) one of \( c \) possible colors. Typically values for \( c \) are \( c = 2^8 \) or \( c = 2^{24} \). Assume \( 1 \leq K = \log_2 c \in \mathbb{Z} \) and let \( \underline{X} = (X_i)_{i=1,\ldots,N} \) for \( X_i = (X_1^i, X_2^i, \ldots, X^K_i) \in \{0,1\}^K \), \( \forall \ i \in \Lambda, \ K \geq 1 \), represent the true unknown (random) image. Denote the observed image by \( \underline{Y} = (Y_i)_{i=1,\ldots,N} \). We denote the space of pictures \( \{\{0,1\}^K\}^\Lambda \) by \( \Sigma_K \). In this notation we have \( c = 2^K \) possible colors in each pixel. Each \( X_i \) may correspond to a single binary number which gives the intensity of black in pixel \( i \) as \( \sum_{k=1}^K X_k^i 2^{-k} \) (in the case of grayscale picture) or it may correspond to three binary numbers, each giving the intensity of one of the three basic colors. In this second case, for instance, we could have \( K = 24 \) and \( \sum_{k=1}^{(1+c)8} X_k^i 2^{-k} \), for \( c = 0 \), 1 and 2 giving the intensity of, respectively, Red, Green and Blue at site \( i \). To simplify the discussion we assume, without loss, the first case since our approach in the second one is to reconstruct each color separately in order to reconstruct the whole picture.

In the Bayesian setting we assume (a) that the original picture \( \underline{X} \) is random with a known distribution which is called the a priori measure and (b) that we know how to model the noise.

To start we consider the noise. We fix an \( \epsilon > 0 \). Conditional on \( \underline{X} \), at each bit \( k \) of each pixel \( i \), independently, the observed value in bite \( k \) of pixel \( i \), called \( Y_i^k \), is equal to the true value \( X_i^k \) with probability \( 1 - \epsilon \) or, with probability \( \epsilon \), it corresponds to the switched value:

\[
P_i(Y_i^k | X_i^k) = \begin{cases} 
1 - \epsilon & \text{if } X_i^k = Y_i^k \\
\epsilon & \text{otherwise.}
\end{cases}
\]  

Hence

\[
P(\underline{Y} | \underline{X}) = \prod_{i \in \Lambda} \prod_{k \in \{1, \ldots, K\}} P_i(Y_i^k | X_i^k) \]  

\[
= \frac{1}{Z_h} \exp \left\{ h \sum_{i \in \Lambda} \sum_{k \in \{1, \ldots, K\}} 1_{\{X_i^k = Y_i^k\}}(Y_i) \right\}
\]  

where

\[
h = \log \left( \frac{1 - \epsilon}{\epsilon} \right).
\]  

and \( Z_h \) is the normalization constant.
Note that this hypothesis is not the same as the one in [FFG] where each pixel was either observed correctly with probability $1 - \epsilon$ or chosen uniformly among the other $c - 1$ colors.

Now we define the MAP (maximum a posteriori) estimator.

Denote by $\mu$ the a priori measure, that is, the distribution of $\mathbf{X}$. Then the MAP estimator after observing the image $\mathbf{Y}$, $\hat{\mathbf{X}} = \hat{\mathbf{X}}(\mathbf{Y}) \in \Sigma_K$ is any image which maximizes the posterior distribution $P(\mathbf{X} | \mathbf{Y}) \propto P(\mathbf{Y} | \mathbf{X}) \mu(\mathbf{X})$, that is

$$P(\hat{\mathbf{X}} | \mathbf{Y}) = \max_{\mathbf{X} \in \Sigma_K} P(\mathbf{X} | \mathbf{Y}) \quad (5)$$

Now we discuss the a priori measure. We want to consider here the situation on which very little is known about the original picture before the observation besides the information that it is some kind of real photo, perhaps taken by a satellite, as opposed to being a photo of something like a geometrical drawing, a cubist oil painting or a cell of a cartoon picture. We suppose that the only prior information available is that the original picture is locally smooth. By this we mean that the measure should be such that neighbor pixels on the picture are more likely to have colors which are near in some sense. More precisely we assume that there exists a real valued function $H$ which indicates how smooth is an image and take the a priori measure to be:

$$\mu(\mathbf{X}) = \frac{e^{-\beta H(\mathbf{X})}}{Z} \quad (6)$$

where $Z = \sum_{\mathbf{X}} e^{-\beta H(\mathbf{X})}$, with the sum taken over all possible images, is a normalization constant and $\beta \geq 0$ is a real parameter. This parameter measures the tendency to be smooth since if $\beta$ is large $\mu$ is concentrated on images with small values of $H$ while if $\beta$ is small $\mu$ is close to the uniform distribution on $\Sigma_K$, the set of all images. The motivation for this formula, in particular the minus sign in front of $\beta$, comes from statistical mechanics where it would be called Gibbs measure, $\beta$ would be the inverse temperature and $H$ would be the Hamiltonian (or Energy) function.

Plugging (3) into (5), and taking logarithms, we get that the images which
maximize the posterior distribution (5) are those which maximize

$$\beta H(X) + h \sum_{i \in \Lambda} \sum_{k \in \{1, \ldots, K\}} 1_{\{X_k^i = Y_k^i\}}(Y)$$  \hspace{1cm} (7)$$

The image that maximizes this expression makes the best compromise between being globally smooth, regulated by $\beta H$, and agreeing as much as possible with the observed image which is regulated by $h \sum_{i \in \Lambda} \sum_{k \in \{1, \ldots, K\}} 1_{\{X_k^i = Y_k^i\}}(Y)$. Of course there exists only one relevant parameter in the maximization problem, say $h/\beta$.

Although any strictly positive measure $\mu$ on $\Sigma_K$ can be written as above for some $H$, and thus (5) can be safely assumed for a generic a priori measure, we are interested only in the case on which $H$ is both simple and reasonable as a measurement of smoothness. This simplicity requirement will basically mean the assumption that $H$ has only local dependence as follows

$$H(X) = \sum_{<i,j>} d(X_i, X_j)$$  \hspace{1cm} (8)$$

where $d(X_i, X_j)$ is a measure for the distance between the colors at pixel $i$ and $j$ and the sum is taken over all pairs of neighbor pixels in the picture. That is, $<i,j> = \{(i, j) : |i - j| = 1\}$.

The choice of $d(X_i, X_j)$ depends on what kind of local properties one expects in the original picture.

One choice which is common in the literature (see, for instance [FFG] and its quotations) is

$$H_P(X) = \sum_{<i,j>} 1_{\{X_i \neq X_j\}}$$  \hspace{1cm} (9)$$

where the sum is taken over all pairs of neighbor sites in the lattice.

This choice is related to the Potts model in statistical mechanics (see for instance [M]) and explains the subscript. It is a model with very interesting properties that corresponds, in the two-color case, to the Ising model [MW]. To find a solution to (5) we have to find what is called in the physics literature a ground state for the Ising model with random magnetic field. This field is induced by the observed image.
Most of the interesting properties of those statistical mechanics models, like *phase transition*, appear in the so called *thermodynamic limit*, the limit on which the size of the system grows to the whole lattice ([R], [MW]). Even though in our discussion the lattice size is kept fixed we will need in section 4 some exact results on the thermodynamic limit in the two-color case (Ising model). Also some finite size considerations like the effect of boundary conditions (in our case we chose *free* boundary conditions) may be important.

The measure defined by (6), with (9) plugged in, would describe images on which neighbor pixels tend to be equal but if two pixels have different colors the cost of this *interface* does not depend on how different they are. If two pixels have different colors the most likely is that each one belongs to a single-color region separated by a *sharp* interface.

We could also try to represent the situation on which this is not necessarily the case choosing a finer notion of distance between $X_i$ and $X_j$. Two somewhat natural choices would be

$$H_1(\mathbf{X}) = -\sum_{<i,j>} \left| \sum_{k=1}^{K} (X^k_i - X^k_j) \frac{1}{2^k} \right|$$

(10)

or

$$H_2(\mathbf{X}) = -\sum_{<i,j>} \left( \sum_{k=1}^{K} (X^k_i - X^k_j) \frac{1}{2^k} \right)^2.$$  

(11)

Note that, for any choice of $H$, the MAP problem is well posed since we could, at least in principle, check the finitely many values in the right hand side of (7) and choose an image that maximizes it. But since $|\Sigma_K| = 2^{KN}$ with something like $K = 24$ and $N = 400 \times 600$ this approach is not computationally feasible. The problem with all the above choices is that it is not known how to do better than this time-consuming maximization by inspection in the multicolor case ($K > 1$) and thus the problem is, in practice, not solvable.

We propose another choice for $H$, intermediate between the Potts and the other two mentioned above, which respects nicely the notion of smoothness which led to the choice (10) but nevertheless induces to a polynomial time maximization problem. This choice is:
\[ H(\mathbf{X}) = \sum_{k=1}^{K} \sum_{\langle i,j \rangle} |X^k_i - X^k_j| \frac{1}{2^k} = \sum_{k=1}^{K} \sum_{\langle i,j \rangle} I_{\{X^k_i \neq X^k_j\}} \frac{1}{2^k} \quad (12) \]

With this, the problem of finding the \(2^K\)-color image which maximizes the posterior distribution \(P(\mathbf{X}|\mathbf{Y}) \propto P(\mathbf{Y}|\mathbf{X})\mu(\mathbf{X})\) is decomposed into \(K\) binary color maximization problems. Namely one has to solve

\[ P(\hat{X}_k|\mathbf{Y}_k) = \max_{X \in \Sigma_2} P(\mathbf{Y}_k|X)\mu_k(X) \quad (13) \]

for each \(k, 1 \leq k \leq K\) where \(\mathbf{Y}_k = \{Y^k_i\}_{i \in \Lambda}\) is the \(k\)-th component of the observed image and \(\mu_k\) is the Gibbs measure defined on the space of binary color images, \(\Sigma_2\), by (6) with

\[ H^k(X^k) = \sum_{\langle i,j \rangle} I_{\{X^k_i \neq X^k_j\}} \frac{1}{2^k}. \quad (14) \]

The \(K\) color image given by \(\mathbf{Y}\) is a solution of (5). Each one of these \(K\) binary images is called a layer. The weighted sum of the MAP for each layer gives our (gray-scale) image estimator:

\[ \hat{X}_i = \sum_{k=1}^{K} 2^{-k} \hat{X}^k_i. \]

On the multicolor case one solves a gray-scale problem for each basic color.

Each binary problem can be solved in polynomial time using the results by Greig, Porteous and Seheult who reformulated it as one involving finding a minimum cut on a capacitated network [FF] for which there exist fast algorithms. These ideas are presented briefly in the next section. From the statistical point of view the approach is different whether the parameters \(\beta\) and \(\epsilon\) are known or not. A truly Bayesian approach would associate an a priori measure to each one of the parameters \(\beta\) and \(\epsilon\). We leave this alternative to future work. Another possibility is to estimate these parameters from the observed image using classical frequentist analysis. This is possible under the hypothesis on the noise and on the original image being a sample of a Gibbs measure for the Potts Hamiltonian (9) (which is always
the case after the decomposition in binary colors) using exact results for the
two-dimensional Ising model obtained by Frigessi and Piccioni [FP]. We have
produced an algorithm based on [FP] for estimating the parameters $\epsilon$ and $\beta$.
This is explained in section 4 below.

Once one has a fast algorithm to reconstruct images it is natural to ask
what is the effect of iterating the whole process. More precisely, what hap-
pens if one takes the reconstructed image and applies the method again,
perhaps updating the value of $h/\beta$?

Note that to apply the method again for the (already) reconstructed
image is equivalent to assume that the reconstructed image could be thought
as obtained from some original picture which was chosen with respect to $\mu$
and then subjected to noise. Even though it is not difficult to verify that this
assumption is false the question is interesting both from the mathematical
and from the applied point of view. On the mathematical side one has a
mapping, $\mathbf{Y} \mapsto \hat{\mathbf{X}}(\mathbf{Y})$, from $\Sigma_K$ into itself, a discrete time dynamical system,
and it is natural to ask about iteration properties. On the practical minded
side one can ask about the effect of iteration on the quality of reconstruction
even if this can only be judged subjectively.

If we decide to update $h/\beta$ before each iteration the natural thing to do
would be to chose a larger value at each step since we assume the procedure
did a good job and removed some of the noise, therefore $\hat{\mathbf{X}}(\mathbf{Y})$ would be more
reliable than $\mathbf{Y}$. It would then be natural to increase $h$, without changing $\beta$.
A natural procedure to find each updated value is to use again the estimators
given by Frigessi and Piccioni ([FP]). In doing this we find, in fact, that this
parameter increases, after one iteration, as expected.

What is rather surprising is that once one tries this iteration procedure,
either keeping $h/\beta$ fixed or increasing it, at each step, one finds that it has no effect et all. The twice reconstructed image is exactly equal to the once
reconstructed one. In other words, $\hat{\mathbf{X}}$ as a function from $\Sigma_K$ into itself has a
fixed point in each once-reconstructed image.

More precisely write $F_{(h/\beta)}(\mathbf{X}) = \hat{\mathbf{X}}$ for the function from $\Sigma_K$ into itself
defined in (3). Then we have the following

**Proposition:** $F_s(\mathbf{X}) = F_t(F_s(\mathbf{X}))$ for all $t \geq s$.

We prove this proposition in the next section after reviewing some results
on networks and the connection with the maximization problem considered
Given this result a natural question arises. Is this property not true in general? Suppose one has a random function assuming values in some space \( S \) with some unknown parameter \( s \) which itself belongs to \( S \) chosen according to some \textit{a priori} measure (in our case, \( s \) corresponds to the original image). Define the MAP estimator as usual to obtain a function from \( S \) into itself. Does it always satisfy the this fixed point property? One could argue that since all the information about the original image which is contained in the observed one should be still present in the reconstructed image then this iteration should give no further results and the \textit{twice} reconstructed image should \textit{always} be equal to the once-reconstructed one. As it turns out this is not the case and it is not difficult to find counterexamples.

\section{Experimental Results}

We developed a program (\texttt{map}) in \textit{C} to reconstruct images according to the method proposed. This program reads an image in the \textit{portable bit map} format for true-color (.\texttt{ppm}) or gray-scale (.\texttt{pgm}) pictures. The intensity of each color in each pixel (or the single gray scale intensity there) is written as a (eight bits) binary number. The program then constructs a graph for each one of those layers (as defined after equation (14)), construct the corresponding graphs and finds the minimum cut for each one of them using a standard implementation of the Ford Fulkerson algorithm. The program uses integer arithmetic scaled by a factor of 10000. The values for \( \beta \) and \( H \) are supplied by command line switches. It is also possible to supply a bit mask to select which planes to reconstruct. The program is portable and it has been tested on \textit{SunOS}, \textit{Solaris} and \textit{Linux}.

The main problem in developing the program was the large amount of memory to hold the image, the graph and the auxiliary data structures. The solution was to use a compact representation of the graph, namely a matrix where each cell has a pixel and the values for the flow in each of 5 directions (4 neighbors plus \( s \) or \( t \)). This avoided the explicit representation of the edges and the extra space for the image itself.

A series of tests were made on a 700Mhz \textit{Pentium III} machine, with 128Mb of main memory, running Linux Debian potato 2.2, kernel version
2.4. The running times for restoration of six sample images are presented in table 1. These images are shown in tables 2, 3 and 4.

The images were obtained from a picture shot with a digital camera and converted to 8 bit gray-scale. The final result was saved in the file “fish.pgm”. Two additional files were obtained by clipping the image to a rectangle with half of the area (“fish2.pgm”) and to another one with a quarter of the total area (“fish3.pgm”).

The noise was added “artificially” by a program which scan every bit in the image and inverts it with a given probability. The probabilities used were 10% and 15%. The modified files were named fish.10.pgm, fish2.10.pgm, fish3.10.pgm and fish.15.pgm, fish2.15.pgm, fish3.15.pgm, respectively.

Each image was restored in all bit planes using 3 values of $\beta$, namely 0.1, 0.3, 0.5. An additional restoration, just on the most significant bit plane were made using the estimated $\hat{\beta}$ as indicated in the appendix. The values are presented in the tables 2, 3 and 4.

We did not try to define a metric to measure the quality of the reconstruction, relying on a subjective analysis. Besides the removal of noise, it was observed an improvement in the shades and (consequently) in the third dimension perception, even if some blurring is introduced.

These and other examples are available directly from the authors or at http://www.ime.usp.br/~gubi/MAP/index.html

| File    | Beta 0.10  | Beta 0.30  | Beta 0.50 |
|---------|-----------|-----------|-----------|
| fish.10.pgm | 6.40      | 17.77     | 16.78     |
| fish.15.pgm | 6.05      | 14.38     | 19.54     |
| fish2.10.pgm | 2.81      | 4.45      | 6.35      |
| fish2.15.pgm | 2.65      | 4.38      | 5.86      |
| fish3.10.pgm | 1.38      | 2.43      | 3.20      |
| fish3.15.pgm | 1.34      | 2.37      | 3.02      |

Table 1: Restoration running times for three sample images with two noise levels (10% and 15% per bit)
| Original | 10% | 15% |
|----------|-----|-----|
| ![Original Image](image1.png) | ![10% Image](image2.png) | ![15% Image](image3.png) |
| ![10% Image](image4.png) | ![10% Image](image5.png) | ![15% Image](image6.png) |
| ![10% Image](image7.png) | ![10% Image](image8.png) | ![15% Image](image9.png) |
| ![10% Image](image10.png) | ![10% Image](image11.png) | ![15% Image](image12.png) |

Table 2: Sample image (fish.pbm)
| Original | 10%  | 15%  |
|----------|------|------|
| ![Image](original.png) | ![Image](10.png) | ![Image](15.png) |
| $\beta = 0.1$ | ![Image](beta01.png) | ![Image](beta01.png) |
| $\beta = 0.3$ | ![Image](beta03.png) | ![Image](beta03.png) |
| $\beta = 0.5$ | ![Image](beta05.png) | ![Image](beta05.png) |
| $\hat{\beta}_{10\%} = \hat{\beta}_{15\%} = 0.45$ | ![Image](beta10.png) | ![Image](beta15.png) |

Table 3: Same photo of table 2 but cut in half (fish2.pbm)
| Original | 10%            | 15%            |
|----------|----------------|----------------|
| ![Original Image](fish3.jpg) | ![Image 10%](fish3_10.jpg) | ![Image 15%](fish3_15.jpg) |
| \[\beta = 0.1\] | ![Image β = 0.1](fish3_0.1.jpg) | ![Image β = 0.1](fish3_0.1.jpg) |
| \[\beta = 0.3\] | ![Image β = 0.3](fish3_0.3.jpg) | ![Image β = 0.3](fish3_0.3.jpg) |
| \[\beta = 0.5\] | ![Image β = 0.5](fish3_0.5.jpg) | ![Image β = 0.5](fish3_0.5.jpg) |
| \[\hat{\beta}_{10\%} = 0.549\] \[\hat{\beta}_{15\%} = 0.590\] | ![Image β = 0.5](fish3_0.5.jpg) | ![Image β = 0.5](fish3_0.5.jpg) |

Table 4: 1\textsuperscript{th} of same image (fish3.jpg)
3 MAP estimator and networks

In this section we present some capacitated network ideas which are used to solve the two-color maximization problem described before as discovered by Greg, Porteous and Seheult [GPS] and use them to prove Proposition.

As mentioned before with the choice of $H$ given by (12) the maximization in (3) is decomposed into $K$ two-color problems. Therefore we assume in this section that we are in the binary color case $K = 1$.

The MAP estimator given $Y$ is the image $\hat{X} \in \{0,1\}^\Lambda$ which gives the maximum of

$$L_\alpha(X|Y) = \alpha \sum_{i \in \Lambda} \mathbf{1}_{\{X_i = Y_i\}}(X) + \sum_{<i,j>} \mathbf{1}_{\{X_i = X_j\}}(X)$$

(15)

where $\alpha = h/\beta$.

A network $N$ is a graph $G = (V,E)$, where $V$ is a finite set of vertices and $E$ is a set of couples of vertices, with a capacity $c(e) \geq 0$ associated with each edge $e \in E$.

We define networks on the set of vertices $V$, given by the sites in $\Lambda$ plus two extra ones denoted by $s$ (source) and $t$ (sink), that is

$$V = \Lambda \cup \{s\} \cup \{t\}.$$ 

For the set of arcs we take

$$E = (\cup_i \{(s,i)\}) \cup (\cup_i \{(i,t)\}) \cup (\cup_{<i,j>} \{(i,j)\})$$

where the first two unions are taken on $i \in \Lambda$ and the last one is taken over pairs of nearest neighbor sites in $\Lambda$.

Given the observed image $Y$ and a real number $\alpha$ we define the capacities of the network $N_\alpha(Y)$ as follows.

If $Y(i) = 1$ we set $c_\alpha(Y,i) = \alpha$ as its capacity, otherwise set $c_\alpha(Y,i,t) = \alpha$; to each arc $e = (i,j)$ of neighbor sites in $\Lambda$ we associate $c_\alpha(Y,e) = 1$. All other arcs have capacity zero.

For each image $X$ let

$$A(X) = \{s\} \cup \{i \in \Lambda : X(i) = 1\}$$

14
\[ B(\underline{X}) = \{t\} \cup \{i \in \Lambda : X(i) = 0\}. \]

These two sets define a cut of the network
\[ C(\underline{X}) = \{(i, j) \in E : i \in A(\underline{X}), j \in B(\underline{X})\}. \]

Notice that the cut \( C(\underline{X}) \) consists of a set of arcs whose removal (cut) makes it impossible to find a path going from \( s \) to \( t \) through arcs with non-zero capacity. If we now define the capacity of the cut \( C(\underline{X}) \) by the quantity
\[
C_{\underline{Y}, \alpha}(\underline{X}) = \sum_{(i,j) \in C(\underline{X})} c_{\underline{Y}, \alpha}(i, j)
\]

it is very simple to check that
\[
L_{\alpha}(\underline{X}|\underline{Y}) = a - C_{\underline{Y}, \alpha}(\underline{X}),
\]

where \( a \) is a constant which does not depend on \( \underline{X} \). Therefore to find the MAP estimator one has to find the cut which minimizes (16): the so called minimum cut [FF].

Ford and Fulkerson showed that the value of the capacity of the minimum cut is equal to the maximum flow through the network from source to sink. Recall that a flow \( f \) in a network \( N \) (on \( G(V, E) \) with capacity \( \{c(e)\}_{e \in E} \)) from \( s \) to \( t \) is a collection of real numbers \( \{f(e)\}_{e \in E} \), where \( f(e) \) can be thought as the amount of fluid per unit time going through the pipeline \( e \), such that the flow on each arch does not exceed its capacity, \( 0 \leq f(e) \leq c(e) \), for all \( e \in E \), and such that the flow is conserved
\[
\sum_{j \in V} f(i, j) - \sum_{j \in V} f(j, i) = 0
\]

for all \( i \neq s \).

Well known fast algorithms to find this maximum flow exist and therefore the problem is solved in practice. The algorithm presented in this paper uses this method to reconstruct each one of the (8 or 24) layers.

**Proof of the Proposition.** To simplify the notation write \( \hat{\underline{X}} = F_{\alpha}(\underline{Y}) \). Let \( \alpha' \geq \alpha \). The networks defined by the pair \( (\underline{Y}, \alpha) \), denoted by \( N_{\alpha}(\underline{Y}) \), and by \( (\hat{\underline{X}}, \alpha') \), \( N_{\alpha'}(\hat{\underline{X}}) \), can differ only in the capacities assigned to each arc.
The proposition asserts that the cut defined by \( \hat{X} \) also has minimum capacity in \( N_{\alpha'}(\hat{X}) \), that is we want to prove that
\[
C_{\hat{X},\alpha'}(\hat{X}) \leq C_{\hat{X},\alpha'}(Z),
\]
for any image \( Z \in \Sigma_1 \).

Fix \( Z \) and let \( E_I = \mathcal{C}(\hat{X}) \setminus \mathcal{C}(Z) \), \( E_{II} = \mathcal{C}(\hat{X}) \cap \mathcal{C}(Z) \) and \( E_{III} = \mathcal{C}(Z) \setminus \mathcal{C}(\hat{X}) \). By definition of MAP estimator we have
\[
C_{Y,\alpha}(\hat{X}) \leq C_{Y,\alpha}(Z),
\]
which implies
\[
D_{Y,\alpha}(E_I) \leq D_{Y,\alpha}(E_{III}).
\]
where, for a set of edges \( E \),
\[
D_{Y,\alpha}(E) = \sum_{e \in E} c_{Y,\alpha}(e)
\]

Therefore to check (18) it is enough to verify
\[
D_{\hat{X},\alpha'}(E_I) \leq D_{\hat{X},\alpha'}(E_{I})
\]
\[
D_{\hat{X},\alpha'}(E_{III}) \geq D_{\hat{X},\alpha'}(E_{III}).
\]

We start with inequality (21). Assume \( e \in E_I \). If \( e \) is an external edge (connecting some site \( i \) with either the source \( s \) or the sink \( t \)), then, since \( e \in \mathcal{C}(\hat{X}) \), it must be \( c_{\hat{X},\alpha'}(e) = 0 \) which is less or equal than \( c_{Y,\alpha}(e) \). On the other hand, if \( e \) is an internal edge (i.e. \( e = (i, j) \) for \( i \) and \( j \) nearest neighbors in \( \Lambda \)) then its capacity equals 1 in both cases. This concludes the proof of (21).

Suppose now \( e \in E_{III} \). Inequality (22) follows from the observation that \( c_{\hat{X},\alpha'}(e) \) can not be zero.

\[\Box\]

### 4 Final Remarks

The different approaches used in image reconstruction are based in quite different set of theoretical ideas and it is not clear how to compare their results.
One possible measure for the quality of the reconstruction, used in [FP] to compare 9 algorithms, is to evaluate the proportion of pixels classified correctly. Since our main goal here was to present a method for the reconstruction of multicolor images we leave the comparison with other methods for future work. In any case, since we are working with exact MAP’s for the chosen Hamiltonian, our method will be as good (and as bad) as the usual two colors MAP estimators, regarding the proportion of bits reconstructed.

For the usual MAP reconstruction problem in the multicolor case no fast algorithm is known ([FFG]). For a probabilistic approach via Simulated Annealing in order to get the exact estimator one needs to decrease very slowly some parameter while the computation goes on and thus needs a prohibitively large amount of time [Gi].

One possibility is to accept approximated MAP estimators which can be obtained fast enough. One can do this with simulated annealing by updating the parameter fast enough but then we lose control on how close the approximation is to the exact one. An approximate of the MAP estimator with a probabilistic analysis of the error in the three color case was developed in [FFG]. Another approach can be found in [J].

Our approach is not completely Bayesian as we also consider a situation on which the a priori measure has some unknown parameters but do not assume any prior knowledge about them. In the Appendix we describe a method to estimate these parameters from the picture itself using classical statistical methods, as proposed by Frigessi and Piccioni [FP]. In those cases we verify that plugging those estimated values into the formulas used to get the MAP estimator provides a reconstruction which appears to be the best. As mentioned before we do not try to quantify this.

Ricardo Maronna proposed that instead of estimating $\beta$ and $h$ one could look for the $\alpha = \beta/h$ that maximizes (7). The estimator for the true image will then be the $X$ which realizes this maximum with the best $\alpha$. It is not clear yet how to justify this theoretically. Another alternative would be to choose the uniform distribution on an appropriate range as a priori distribution for $\epsilon$ and $\beta$. However the computation of the MAP estimator in this case seems prohibitive.

As an experimental observation we remark that reconstructing only the first layer of a dirty image (and leaving the others as they are) gives a quite good visual result. A possible explanation of this fact is that, as a conse-
sequence of the binary decomposition, each layer is “half” as important as the previous one.

Looking for algorithms which give smooth solutions we propose the following hierarchical procedure. First consider layer 1 and find $\hat{Z}^1$, as before for $\hat{X}^1$. Then given layer $\hat{Z}^\ell$, for $\ell = 1, \ldots, k - 1$, define $\hat{Z}^k$ as the (binary) image that maximizes

$$L_\alpha(X_k|Y) = \alpha \sum_{i \in A} 1_{\{X^k_i = Y^k_i\}}(X) + \sum_{\ell=1}^{k} \prod_{\ell=1}^{k} 1_{\{X^\ell_i = X^\ell_j\}}(X).$$

(23)

In words, this algorithm will try to get the same value for two neighbors in layer $k$ if these neighbors have the same value for all previous layers. If not they are not coupled. Visual realizations of this algorithm give also good results. Although there is no Hamiltonian for this model, the algorithm is well defined. Moreover each layer corresponds to the so called diluted Ising model.

Assume that we get $N$ independent samples of the same image $Y_1, \ldots, Y_N$. This means that the original realization $X$ of the image is the same but the noises are independent. A generalization of our approach deals with this problem in the following way. We modify the capacities associated to the graph $G$. To each arc $(s, i)$ connected to the source we associate the capacity

$$\alpha \left( \sum_{n=1}^{N} (2Y_{n,i}^k - 1) \right)^+. $$

and then to each arc $(i, t)$ connected to the sink we associate the capacity

$$\alpha \left( \sum_{n=1}^{N} (2Y_{n,i}^k - 1) \right)^-. $$

In the same vein one could use the remaining two colors to get information about the color being reconstructed.

5 Appendix: Estimation of $\beta$ and $\epsilon$

In this section we apply ideas from Frigessi and Piccioni ([FP]) which exploit well known (but highly nontrivial) results from statistical mechanics ([R], [MW]) to obtain estimators for the parameters $\beta$ and $\epsilon = (1 + e^h)^{-1}$. 

18
The result is a program called *estima* that has as input a multicolor image and returns one estimated $\beta$ and one estimated $\epsilon$ for each layer.

The measure defined in (6) with $H$ given by (9) in the two-color case (Ising model) in the thermodynamic limit gives rise to a translation invariant measure which may be ergodic or not according to the value of $\beta$.

This choice of $H$ does not favor zeros or ones. This symmetry is perhaps easier to see if we represent a configuration as an element $S$ in $\{−1, +1\}^Z$ with $−1$ replacing zeros. This is the usual Ising notation while the one using zeros and ones is known in statistical mechanics as *lattice gas* representation. In the situation considered here they are equivalent.

More precisely in if $Y \in \{0, 1\}^\Lambda$ let $S$ be the corresponding Ising configuration given by

$$S = \{S_i, i \in \Lambda : S_i = 2Y_i - 1\}$$

(24)

And instead of Hamiltonian (9) we use

$$H_P(S) = -\sum_{<i,j>} S_i S_j$$

(25)

The corresponding infinite volume limit measure is not ergodic if $\beta$ is smaller than some (known) critical value $\beta_c$. In this case the limit measure is a mixture with equal weights of two measures $\mu_+^{\beta}$ and $\mu_-^{\beta}$, the first favoring configurations with more $+1$’s than $-1$’s and the other favoring configurations with more $-1$’s than $+1$’s. If $\beta \geq \beta_c$ then $\mu_+^{\beta} = \mu_-^{\beta}$. If we denote by $E_+^{\beta}(f)$ ($E_-^{\beta}(f)$ ) the expected value of a function $f$ defined on $\{−1, +1\}^Z$ the symmetry between $\mu_+^{\beta}$ and $\mu_-^{\beta}$ imply

$$E_+^{\beta}(S_i S_j) = E_-^{\beta}(S_i S_j) = r_\beta(|i - j|)$$

(26)

$r_\beta(|i - j|)$ is called the two point correlation function for the infinite volume system at inverse temperature $\beta$. We will need two of those correlation functions in what follows: $r_\beta(1)$, for nearest neighbors, and $r_\beta(\sqrt{2})$ for neighbors along the diagonal on the lattice.

Suppose $S$ corresponds (in the Ising notation) to the original image and $R$ corresponds to the observed image after noise. Under the hypothesis on
the noise

$$E^{+}_{\beta}(R_iR_j) = E^{+}_{\beta}(S_iS_j)(1 - 2\epsilon)^2$$  \hspace{1cm} (27)$$

Therefore the ratio between $r_{\beta}(1)$ and $r_{\beta}(\sqrt{2})$ computed for the observed image depends only on $\beta$. Call this ratio $\phi(\beta)$.

If $R \in \{-1, 1\}^\Lambda$ corresponds to an observed two-color picture [FP] found a sequence $(\hat{\beta}, \hat{\epsilon})$ of consistent estimators for $(\beta, \epsilon)$ given by

$$\hat{\beta} = \phi^{-1} \left( \frac{G^1(R)}{G^2(R)} \right)$$  \hspace{1cm} (28)$$

and

$$\hat{\epsilon} = \frac{1}{2} \left\{ 1 - \left( \frac{G^1(R)}{r_{\beta}(1)} \right)^{\frac{1}{2}} \right\}$$  \hspace{1cm} (29)$$

with

$$G^a(R) = \frac{\sum_{(i,j) \in \Lambda^0} R_iR_j}{4|\Lambda^0|}$$  \hspace{1cm} (30)$$

where for $a = 1$ the sum is taken over pairs of nearest neighbor sites along the lattice directions in $\Lambda^0$, the interior of $\Lambda$, and for $a = 2$ the sum is over neighbor sites along the two lattice diagonals again in the interior of $\Lambda$.

Expressions for $r_{\beta}(1)$ and $r_{\beta}(\sqrt{2})$ are complicated involving elliptic integrals with different formulas for $\beta$ smaller, equal and larger than $\beta_c$ and the function $\phi^{-1}$ must be computed numerically. Our program estima finds these estimators from a observed image.

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