Campbell *et al.* demonstrated the existence of axion “hair” for Kerr black holes due to the non-trivial Lorentz Chern-Simons term and calculated it explicitly for the case of slow rotation. Here we consider the dilaton coupling to the axion field strength, consistent with low energy string theory and calculate the dilaton “hair” arising from this specific axion source.
In low energy string theory, the classical gravitational sector is modified by the inclusion of dilaton and axion couplings. There has been much recent interest in the new structures which are found to emerge in black hole solutions as a direct result of the non-minimal coupling of fields to Einstein gravity within string gravity.

The sparseness of black hole solutions is attributed to the “no hair” conjecture which limits the exterior field solutions to those required by a local gauge invariance. Hence the known static Schwarzschild holes characterized by the mass $M$ and Kerr-Newman and Reissner-Nordstrom, characterized additionally by angular momentum $A$ and/or charged gauge fields, $Q$. Attempts to enlarge the space of known solutions by explicit dependence on hair, lead to non-trivial solutions which are however, unstable against radial perturbations.

For scalar fields in particular, it has been shown by many authors that no solutions are available for minimally coupled fields even in the presence of non-trivial potentials. A slightly different result is obtained if non-minimal couplings are considered. This problem has been studied in the context of supergravity and Kaluza-Klein theories and it has been shown that even when non-trivial scalar “hair” is present, the scalar charge is not an independent parameter but is a function of $M$, $A$ and $Q$. The holes can therefore still be classified in terms of these three parameters and so the scalar “hair” does not violate the “no hair” conjecture. Similar remarks apply to string theories.

If the mass parameter of a black hole is large enough compared to the Planck mass, then the higher order curvature invariants predicted in the string effective action, may be neglected outside the event horizon. In particular, for small curvature, the non-minimal
coupling of the dilaton to the Gauss-Bonnet invariant of order $\alpha'$ in the string tension, can be neglected; moreover, the dilaton gauge field strength coupling may be retained despite the fact that it is also $O(\alpha')$. This is the case discussed by Garfinkle et al. [1] and Shapere et al. [2], where magnetic and dyonic black hole solutions are respectively obtained.

At the lowest order in the string tension, $(\alpha')^0$, the four dimensional theory consists of Einstein gravity, coupled to a free dilaton field and a dilaton-axion term:

$$S = \int d^4x \sqrt{-g} \left( R - 2(\nabla \phi)^2 - \frac{1}{3} e^{-4\phi} H^2 + \ldots \right).$$

(1)

where we assume natural units, $h = c = G = 1$. It is known [3] that for the minimally coupled case, axionic black holes correspond to static, spherically symmetric solutions of the Schwarzschild type with mass $M$ and a purely topological axion charge $q$,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega,$$

$$B_{\mu\nu} = q \frac{\epsilon_{\mu\nu}}{4\pi r^2}, \quad H_{\mu\nu\lambda} = 0,$$

(2)

where $d\Omega$ is the line element on the surface of a 2-sphere. For a pure Kalb-Ramond field ($H = dB$), its equation of motion, $d^* H = 0$ implies that locally, the dual of $H$ is given by $^* H = d\sigma$, where $\sigma$ is a free massless scalar. For any Einstein-scalar field system, the only (static and/or stationary) black hole solutions which exist are for constant $\sigma$ and other scalars. The non-minimal dilaton coupling of the string-induced action (1) does not alter this uniqueness theorem so long as the Kalb-Ramond field is itself minimally coupled, as the exterior derivative of a 2-form field. However, one of the novel features of
string theory is the fact that the three form field $H_{\mu\nu\lambda}$ is not minimally coupled: gauge
and gravitational anomalies \[9\] are removed from the theory through the introduction of
Lorentz and Yang-Mills Chern-Simons forms thus,

$$H = dB + \omega_L - \omega_Y$$

(3)

In particular, the Lorentz Chern-Simons term, $\omega_L$, may be written in terms of the spin
connection on the manifold,

$$\omega_L = Tr (\omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega),$$

(4)

with a similar structure for $\omega_Y$ in terms of the gauge field $A_\mu$. In this paper we shall focus
on the significant effect of $\omega_L$ in producing black hole "hair"; thus we shall ignore the
gauge field sector in the following. It has been shown \[10\] that for all four dimensional
spacetimes conformal to a spacetime with a maximally symmetric 2-dim. subspace, the
Lorentz Chern-Simons form is exact (i.e., $\omega_L = d\beta$) so that it does not contribute to
the equations of motion. Thus for a Schwarzschild background, solution (2) holds with
constant dilaton field. However for (stationary) rotating black holes described by the Kerr
metric, the Lorentz Chern-Simons term is non-trivial and acts as a source for axion hair
by means of the Bianchi identity,

$$dH = Tr R \wedge R \neq 0.$$ 

(5)

This mechanism was used by Campbell, Duncan, Kaloper and Olive \[11\] to generate axion
field strength "hair" in the specific case of a slowly rotating black hole. In the limit of
small rotation where the angular momentum $A \ll M$, the Kerr metric becomes to $O(A)$:

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dt^2 + \left(1 - \frac{2M}{r} \right)^{-1} dr^2 - r^2 d\Omega - \frac{4MA \sin^2 \theta}{r} dtd\phi .$$  \hspace{1cm} (6)

The dual of the Hirzebruch signature density, $^* \text{Tr} (R \wedge R)$, then gives an $O(A)$ term which acts as a source for the pseudoscalar field $^* H = da$, the equation of motion for $a$ being,

$$\Box a = \frac{1}{4! \sqrt{-g}} R_{\alpha\beta\mu\nu} R_{\rho\sigma}^{\beta\alpha} .$$  \hspace{1cm} (7)

Campbell et al. [11] solve eq. (7) by using Green’s function techniques: for a solution to $O(A)$ it is sufficient to use the Green’s function for the Schwarzschild metric. Moreover, the back reaction of the axion on the metric (through the energy-momentum tensor of $a$) can be ignored in this approximation. The result [11] is regular and finite ($r > 2M$):

$$a(r, \theta) \equiv f(r) \cos \theta ; \quad f(r) = \frac{5A}{48M^3} \left( \frac{4M^2}{r^2} + \frac{8M^3}{r^3} + \frac{72M^4}{5r^4} \right) .$$  \hspace{1cm} (8)

In this calculation the dilaton field was neglected and effectively set to a constant. As a result of the dilaton-axion in (1), the axion hair (8) will in turn act as a source for dilaton hair. Here we shall explicitly derive this effect.

From the action (1) the equations of motion are:

$$R_{\mu\nu} = 2\nabla_\mu \phi \nabla_\nu \phi + e^{-4\phi} H_{\mu\lambda\rho} H^{\lambda\rho} - \frac{1}{3} g_{\mu\nu} e^{-4\phi} H^2$$

$$- \frac{1}{3} g_{\mu\nu} \nabla_\sigma (e^{-4\phi} H^{\lambda\beta\alpha} R_{\beta\alpha\lambda}) + \frac{2}{3} \nabla_\sigma (e^{-4\phi} H_\mu^\lambda R^\sigma_{\nu\sigma\lambda}) ,$$  \hspace{1cm} (9)

$$\nabla_\lambda (e^{-4\phi} H^{\mu\nu\lambda}) = 0 ,$$  \hspace{1cm} (10)

$$\Box \phi + \frac{1}{3} e^{-4\phi} H^2 = 0 .$$  \hspace{1cm} (11)
If we define,
\[
\frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\lambda\rho} Y_\rho = e^{-4\phi} H^{\mu\nu\lambda}
\] (12)
then eq. (10) is satisfied \((dY = 0)\) at least locally, for \(Y = db\). The Bianchi identity \(\ast (dH) = \ast Tr (R \wedge R)\) now becomes equivalent to eq. (7) after the rescaling,
\[
\partial_\mu b = -\frac{e^{-4\phi}}{6} \partial_\mu a .
\] (13)

In terms of the field \(a(r, \theta)\), the dilaton equation of motion can be rewritten,
\[
\Box \phi + \frac{1}{N} e^{-4\phi} g^{\mu\nu} \partial_\mu a \partial_\nu a = 0 ,
\] (14)
where the normalization \(N \equiv 3 \times 6\). From eqs. (8) and (14), we see that the dilaton hair is an \(O(A^2)\) effect. Consider the perturbative expansion,
\[
\phi = \phi_o + A^2 \omega(x) + \ldots
\] (15)
where \(\phi_o \equiv \text{constant} (= 0)\) is the solution up to \(O(A^1)\). The scalar field \(\omega(x)\) represents the non-trivial dilaton hair to \(O(A^2)\); inserting this expansion into eq. (14) and expanding the exponential gives,
\[
\Box \omega \simeq -\frac{1}{N A^2} \partial_\mu a \partial^\mu a .
\] (16)

We may now use the same Green’s function technique to solve for \(\omega(x)\) in eq. (16). From the result of eq. (8), the source term appearing on the right hand side of eq. (16) is,
\[
\mathcal{F}(r, \theta) = -\frac{1}{N A^2} \left[ \frac{(r - 2M)}{r} f'(r)^2 \cos^2 \theta + \left( \frac{f(r)}{r} \right)^2 \sin^2 \theta \right] .
\] (17)

From the static Green’s function:
\[
\Box G(x, y) = \frac{\delta^3(x - y)}{\sqrt{-g}} ,
\] (18)
we have

\[ \omega = \int d^3y \sqrt{-g(y)} G(x, y) \mathcal{J}(y) . \]  

(19)

Eq. (18) is solved to \((A^0)\) in a Schwarzschild background for a point source at \(r_0 = b, \theta_0 = 0 = \phi_0\):

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \left( 1 - \frac{2M}{r} \right) \frac{\partial G}{\partial r} \right] + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial G}{\partial \theta} \right) = \frac{\delta(r-b) \delta(\cos \theta - 1)}{r^2} . \]  

(20)

The Green’s function \(G(r, \theta)\) can be expressed in terms of the Legendre functions \(P_l\) and \(Q_l\); the details of its derivation are given in ref. [3]. We then have,

\[ \omega = -\int_{2M}^{\infty} db \int_0^\pi d\theta_0 \int_0^{2\pi} d\phi_0 b^2 \sin \theta_0 G(r, \theta, b, \theta_0, \phi_0) \mathcal{J}(b, \theta_0, \phi_0) \]  

(21)

\[ = -2\pi \int_{2M}^{\infty} db \int_0^\pi d\theta_0 b^2 \sin \theta_0 G(r, \theta, b, \theta_0) \mathcal{J}(b, \theta_0) ; \]  

(22)

where in eq. (22), we have made use of the addition theorem of spherical harmonics; the \(\phi_0\) integration may be performed immediately since the source \(\mathcal{J}(r, \theta)\) does not depend on the variable \(\phi\). Substituting for \(G(r, \theta, b, \theta_0)\) [3] we obtain,

\[ \omega = -\sum_{l=0}^{\infty} \left( \frac{2l+1}{2M} \right) \int_{2M}^{r} db \int_0^\pi d\theta_0 P_l(b/M - 1)Q_l(r/M - 1)P_l(\cos \theta)P_l(\cos \theta_0) \]  

\[ \times \mathcal{J}(b, \theta_0)b^2 \sin \theta_0 \]  

\[ -\sum_{l=0}^{\infty} \left( \frac{2l+1}{2M} \right) \int_r^{\infty} db \int_0^\pi d\theta_0 Q_l(b/M - 1)P_l(r/M - 1)P_l(\cos \theta)P_l(\cos \theta_0) \]  

\[ \times \mathcal{J}(b, \theta_0)b^2 \sin \theta_0 \]  

(23)

If we consider the angular integrals first, we observe that the only non-zero contributions in \(G(r, \theta)\) come from the \(l = 0\) and \(l = 2\) terms of the Legendre series, corresponding to pointlike and quadrupole sources respectively. Thus we require the following functions,

\[ P_0(\cos \theta) = 1 ; \quad P_2(b/M - 1) = \frac{1}{2} \left( \frac{3b^2}{M^2} - \frac{6b}{M} + 2 \right) \]
\[ Q_0(b/M - 1) = \frac{1}{2} \ln \left( \frac{b}{b - 2M} \right) ; \]
\[ Q_2(b/M - 1) = \frac{1}{4} \left( \frac{3}{M^2} (b - M)^2 - 1 \right) \ln \left( \frac{b}{b - 2M} \right) - \frac{3}{2} \left( \frac{b}{M} - 1 \right) \]  

Having substituted for these we are left with eight radial integrals to evaluate over the domain, \( r \in [2M, \infty) \). The integrations were carried out using a Mathematica program. After further algebraic manipulation, we find that the solution to eq. (16) is,

\[ \omega(r, \theta) = \frac{1}{169344 N M^5} \left[ \frac{14889}{2} \left( \frac{1}{r} + \frac{M}{r^2} \right) + \frac{9926M^2}{r^3} + \frac{9989M^3}{r^4} + \frac{21112M^4}{5r^5} \right. \]
\[ - \frac{15176M^5}{r^6} - \frac{29376M^6}{r^7} - \frac{31752M^7}{r^8} \]
\[ + 2P_2(\cos \theta) \left\{ \frac{2471M^2}{r^3} + \frac{2513M^3}{r^4} - \frac{2656M^4}{r^5} - \frac{19580M^5}{r^6} \right. \]
\[ - \frac{31320M^6}{r^7} - \frac{31752M^7}{r^8} \right\} \] \( ; (r > 2M) \)  

where \( P_2(\cos \theta) = \frac{1}{2}(3\cos^2 \theta - 1) \).

Thus \( \omega(r, \theta) \) is the \( O(A^2) \) dilaton “hair” around a slowly rotating Kerr black hole. It is important to stress that the axion and dilaton “hair” arise without the presence of a net axion/ dilaton source; the rotation of the hole itself is the source through the Lorentz Chern-Simons coupling. The shape of the dilaton field \( \omega \) is shown in fig. (1): \( \omega(r, \theta) \) is a monotonic decreasing function of \( r \) and is finite at the horizon. The quadrupole term gives rise to a small angular dependence which is more appreciable in the vicinity of the horizon.
Unlike the axion, the dilaton “hair” is associated with a charge $D$ where,

$$D = \frac{1}{4\pi} \int d^2\Sigma \nabla_\mu \phi$$

(26)

and the integral is taken over a two-sphere at spatial infinity. Evaluating $D$ using $\omega$ in eq. (25) gives,

$$D = -\frac{1}{N} \left( \frac{709}{8064} \right) \frac{A^2}{2M^5}$$

(27)

The negative sign implies that the dilaton $\omega$ corresponds to a long range attractive force between weakly rotating black holes. This charge is not however a new free parameter since it is determined by the mass $M$ and angular momentum $A$ of the black hole. The integral of the source depends on $A^2$ and so this charge vanishes as $A \to 0$ in the Schwarzschild limit.

We have shown that the effective string theory can give rise to non-trivial dilaton hair through the coupling due to the Lorentz Chern-Simons term. As in the other cases of non-minimal coupling of a scalar field with gravitation, the dilaton charge is not an independent parameter and so the “no-hair” conjecture still holds.

In this paper, we have not considered the back reaction on the gravitational field due to the matter fields. This would be important in order to see how the causal structure of spacetime is modified near the horizon. A rigorous discussion of this topic would however require an exact solution of the field equations, which seems at the moment a highly non-trivial task.

We notice however that some exact solutions have been found for a rotating dilaton black hole in special cases of the effective string theory [12], but where the Lorentz Chern-
Simons term is neglected.

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Fig.1 Dilaton hair, $\omega N$ (in polar co-ordinates) exterior to horizon; $M = 1$. 