ON THE HAMILTONIAN STRUCTURE OF HIROT A-KIMURA
DISCRETIZATION OF THE EULER TOP

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Abstract. This paper deals with a remarkable integrable discretization of the $so(3)$
Euler top introduced by Hirota and Kimura. Such a discretization leads to an explicit
map, whose integrability has been understood by finding two independent integrals of
motion and a solution in terms of elliptic functions. Our goal is the construction of its
Hamiltonian formulation. After giving a simplified and streamlined presentation of their
results, we provide a bi-Hamiltonian structure for this discretization, thus proving its
integrability in the standard Liouville-Arnold sense.

1. Introduction

This paper deals with a remarkable integrable discretization for one of the basic inte-
grable systems, the three-dimensional Euler top, which describes the motion of the free
rigid body with a fixed point. Equations of motion of the Euler top in the body frame
read
\[ \dot{x}_1 = \alpha_1 x_2 x_3, \quad \dot{x}_2 = \alpha_2 x_3 x_1, \quad \dot{x}_3 = \alpha_3 x_1 x_2, \]
where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, and the real coefficients $\alpha_i$ are parameters of the system.
We will denote the vector of parameters by $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$. Throughout this
paper we will use an abbreviated notation, according to which $(ijk)$ stands for any cyclic
permutation of $(123)$. Thus, system (1) takes with this notation the form
\[ \dot{x}_i = \alpha_i x_j x_k. \]
The coordinates $x_i$ stand either for the angular velocities $\Omega_i$, in which case the coefficients
$\alpha_i$ are given by
\[ \alpha_i = \frac{I_j - I_k}{I_i}, \]
or otherwise for the angular momenta $M_i$, in which case the coefficients $\alpha_i$ are given by
\[ \alpha_i = \frac{1}{I_k} - \frac{1}{I_j}. \]
Here $I_i$ are the principal moments of inertia of the body. The relation between the
two formulations is given by $M_i = I_i \Omega_i$. Integrability features of the Euler top include
[1, 10, 11]: a bi-Hamiltonian structure, i.e. the existence of two compatible invariant
Poisson structures on the phase space; two independent integrals of motion, which are
in involution with respect to any of the invariant Poisson brackets; a Lax representation;

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explicit solutions in terms of elliptic functions. For the reader’s convenience, some of these features are briefly exposed in Sect. 2.

The general problem of integrable discretization of integrable systems is dealt with in the monograph [11]. One finds there also a detailed exposition of an integrable discretization of the Euler top, due to Veselov and Moser [8, 12]. The basic feature of this discretization is that it comes from a discrete Lagrangian formulation on the Lie group $SO(3)$. Upon a reduction to $so(3)^*$, it produces a correspondence, i.e. a multi-valued map, each branch of which is Poisson with respect to the Lie-Poisson bracket on $so(3)^*$, like the original phase flow. Moreover, it shares the integrals of motion and the Lax representation with the original continuous time flow. This Lax representation is related to matrix factorizations.

A class of discretizations of the Euler top sharing the integrals of motion with the continuous system has been introduced and studied in [2]. These discretizations are characterized by the equations of motion

$$\ddot{x}_i - x_i = \gamma \alpha_i (\ddot{x}_j + x_j)(\ddot{x}_k + x_k).$$

(5)

Here and below tilde denotes the shift $t \mapsto t + \epsilon$ in the discrete time $\epsilon \mathbb{Z}$, where $\epsilon$ is a (small) time step. In other words, in Eq. (5) (and in similar situation throughout the paper) we consider $x_i$ as functions on $\epsilon \mathbb{Z}$, and we write $x_i$ for $x_i(n\epsilon)$ and $\ddot{x}_i$ for $x_i(n\epsilon + \epsilon)$, $n \in \mathbb{Z}$. In Eq. (5), it is assumed that $\gamma \sim \epsilon/4$ is some real-valued function on the phase space. Then the map (5) approximates, for small $\epsilon$, the time $\epsilon$ shift along the trajectories of the continuous flow (1). In [2] the functions $\gamma$ have been characterized for which the map $(x_1, x_2, x_3) \mapsto (\ddot{x}_1, \ddot{x}_2, \ddot{x}_3)$ defined by Eq. (5) shares the invariant Poisson structure with the continuous system. In particular, the function $\gamma$ for the Veselov-Moser discretization has been determined. A further integrable discretization of the Euler top belonging to the family (5) was proposed in [4]. Interestingly, the simplest choice $\gamma = \epsilon/4$ leads to a a map which does not preserve the original Poisson structure. Discretizations (5) share the Lax matrix with the continuous time Euler top. They are implicit, since these formulas represent a system of algebraic (nonlinear) equations for $(\ddot{x}_1, \ddot{x}_2, \ddot{x}_3)$ which does not possess a simple closed-form solution.

The present paper deals with the following beautiful explicit discretization of equations of motion (2), introduced by Hirota and Kimura [5]:

$$\ddot{x}_i - x_i = \delta_i (\ddot{x}_j x_k + x_j \ddot{x}_k).$$

(6)

Here one can take

$$\delta_i = \epsilon \alpha_i/2,$$

(7)

we will adopt this choice for the vector of parameters $\delta = (\delta_1, \delta_2, \delta_3) \in \mathbb{R}^3$ throughout the paper. This discretization is explicit, since the algebraic equations (6) are linear with respect to $(\ddot{x}_1, \ddot{x}_2, \ddot{x}_3)$, and thus they can be solved in a closed form (see Sect. 3 for further details). Hirota and Kimura presented some of the integrability attributes for their discretization: two independent integrals of motion and a solution in terms of elliptic functions. Other attributes, like the Hamiltonian formulation and the Lax representation, has not been mentioned by them. The main goal of the present paper is to fill the first of these two gaps by providing a bi-Hamiltonian structure for the Hirota-Kimura discretization, and thereby to prove its integrability in the standard Liouville-Arnold sense.
We found it worthwhile to give also a simplified and streamlined presentation of the results found in [5]. Indeed, the discretization of the Lagrange top given by Kimura and Hirota later in [7], as well as some preliminary results by Ratiu [9], indicate that the map (6) might be just a tip of an iceberg, a huge collection of discretizations of integrable systems of classical mechanics. We plan to develop this topic in a series of upcoming publications.

It is an established fact that many of the most important integrable systems can be found in the classical literature on differential geometry. Usually this refers to solitonic partial differential equations, like the sine-Gordon equation, but it turns out to be true also for the integrable map (6): a 1951 paper in “Mathematische Nachrichten” by H. Jonas is devoted to a birational map \((x, y, z) \mapsto (\tilde{x}, \tilde{y}, \tilde{z})\) given by

\[
\begin{align*}
x + \tilde{x} + y\tilde{z} + z\tilde{y} &= 0, \\
y + \tilde{y} + z\tilde{x} + x\tilde{z} &= 0, \\
z + \tilde{z} + x\tilde{y} + y\tilde{x} &= 0,
\end{align*}
\]

which differs only unessentially from (6). The map (8) has an origin in the spherical geometry, \((x, y, z)\) and \((\tilde{x}, \tilde{y}, \tilde{z})\) being the cosines of the side lengths of two spherical triangles with complementary angles. Jonas’ results include integrals of the map (8) and its solution in terms of elliptic functions. Thus, [6] seems to be one of the earliest precursors of the theory of integrable maps.

2. Euler Top

The aim of this Section is to recall some of the main features of the integrable continuous-time Hamiltonian flow (2).

Proposition 1. Let \(\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3\) be a constant vector. A quadratic function

\[
H^{(\beta)} = \frac{1}{2}(\beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_3^2)
\]

is an integral of motion for (2) if and only if \(\beta \perp \alpha\), i.e. if \(\beta_1 \alpha_1 + \beta_2 \alpha_2 + \beta_3 \alpha_3 = 0\).

Proof: An easy computation based on Eq. (2) shows that

\[
\frac{d}{dt} H^{(\beta)} = (\beta_1 \alpha_1 + \beta_2 \alpha_2 + \beta_3 \alpha_3) x_1 x_2 x_3.
\]

Since the orthogonal complement of the vector \(\alpha\) is two-dimensional, there are two independent integrals of motion. It is sometimes convenient to use a special basis of the orthogonal complement just mentioned, consisting of vectors with one vanishing component.

Corollary 1. The three quadratic functions

\[
G_i = \frac{1}{2}(\alpha_j x_k^2 - \alpha_k x_j^2)
\]

are integrals of motion for (2). Of course, only two of them are (linearly) independent since \(\alpha_1 G_1 + \alpha_2 G_2 + \alpha_3 G_3 = 0\).

Notice that any function \(H^{(\beta)}\) is a linear combination of the \(G_i\)’s:

\[
\alpha_i H^{(\beta)} = \beta_j G_k - \beta_k G_j.
\]
In the angular velocities formulation, a basis of the orthogonal complement \( \alpha^\perp \) can be chosen consisting of \( \beta^{(1)} = (I_1, I_2, I_3) \) and \( \beta^{(2)} = (I_1^2, I_2^2, I_3^2) \). In the angular momenta formulation, a basis of \( \alpha^\perp \) consists of \( \beta^{(1)} = (1/I_1, 1/I_2, 1/I_3) \) and \( \beta^{(2)} = (1, 1, 1) \).

**Proposition 2.** Let \( \beta \perp \alpha \), and let \( \gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3 \) satisfy

\[
\alpha_i = \beta_j \gamma_k - \beta_k \gamma_j,
\]

so that \( \gamma \perp \alpha \). Then the system (2) is Hamiltonian with the Hamilton function \( H^{(\beta)} \) with respect to the Poisson bracket

\[
\{x_i, x_j\}^{(\gamma)} = \gamma_k x_k.
\]

**Proof:** A direct verification:

\[
\{x_i, H^{(\beta)}\}^{(\gamma)} = \beta_j x_j \{x_i, x_j\}^{(\gamma)} + \beta_k x_k \{x_i, x_k\}^{(\gamma)} = \left( \beta_j \gamma_k - \beta_k \gamma_j \right) x_j x_k = \alpha_i x_j x_k.
\]

Propositions 1 and 2 show the bi-Hamiltonian property of the Euler top. Referring to the angular velocities the system has two Hamiltonian formulations:

\[
H = \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2) \quad \text{with} \quad \{\Omega_i, \Omega_j\} = \frac{I_k}{I_i I_j} \Omega_k,
\]

and

\[
H = \frac{1}{2} (I_1^2 \Omega_1^2 + I_2^2 \Omega_2^2 + I_3^2 \Omega_3^2) \quad \text{with} \quad \{\Omega_i, \Omega_j\} = \frac{1}{I_i I_j} \Omega_k.
\]

Referring to the angular momenta, the system also has two Hamiltonian formulations:

\[
H = \frac{1}{2} \left( \frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3} \right) \quad \text{with} \quad \{M_i, M_j\} = M_k,
\]

and

\[
H = \frac{1}{2} (M_1^2 + M_2^2 + M_3^2) \quad \text{with} \quad \{M_i, M_j\} = \frac{1}{I_k} M_k.
\]

3. Hirota-Kimura discretization of the Euler top

We now turn to the study of the map (6). Though the vector of parameters \( \delta \) is arbitrary, we will think of it as related to \( \alpha \) as in Eq. (7).

3.1. Integrals of motion. An explicit form of this map can be easily obtained. Considering Eq. (6) as a system of linear equations for the updated variables \( \tilde{x}_i \), one finds immediately its solution:

\[
\begin{pmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3
\end{pmatrix} = \begin{pmatrix}
1 & -\delta_1 x_3 & -\delta_1 x_2 \\
-\delta_2 x_3 & 1 & -\delta_2 x_1 \\
-\delta_3 x_2 & -\delta_3 x_1 & 1
\end{pmatrix}^{-1} \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}.
\]

Note also that, considering Eq. (6) as a system of linear equations for \( \tilde{x}_i \), one finds the alternative formula

\[
\begin{pmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3
\end{pmatrix} = \begin{pmatrix}
1 & \delta_1 \tilde{x}_3 & \delta_1 \tilde{x}_2 \\
\delta_2 \tilde{x}_3 & 1 & \delta_2 \tilde{x}_1 \\
\delta_3 \tilde{x}_2 & \delta_3 \tilde{x}_1 & 1
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}.
\]
We will use the notation

\[
A(x, \delta) = \begin{pmatrix}
1 & -\delta_1 x_3 & -\delta_1 x_2 \\
-\delta_2 x_3 & 1 & -\delta_2 x_1 \\
-\delta_3 x_2 & -\delta_3 x_1 & 1
\end{pmatrix},
\]

so that the equations of the map can be written as

\[
\tilde{x} = A^{-1}(x, \delta)x = A(x, -\delta)x.
\]

**Proposition 3.** The quantities

\[
F_i = 1 - \frac{\delta_i \delta_j x^2_j}{1 - \delta_i \delta_j x^2_k},
\]

are integrals of motion for the map (\(\text{(6)}\)). Of course, there are only two independent integrals since \(F_1 F_2 F_3 = 1\).

**Proof:** Equation \(\tilde{F}_i = F_i\) can be re-written as

\[
(1 - \delta_k \delta_j \tilde{x}_j^2)(1 - \delta_i \delta_j x^2_k) = (1 - \delta_i \delta_j \tilde{x}_j^2)(1 - \delta_k \delta_i x^2_j),
\]

which is equivalent to

\[
\delta_j (\tilde{x}_k^2 - x_k^2) - \delta_k (\tilde{x}_j^2 - x_j^2) = \delta_i \delta_j \delta_k (\tilde{x}_j^2 x_k^2 - x_j^2 x_k^2),
\]

that is, to

\[
\delta_j (\tilde{x}_k + x_k)(\tilde{x}_k - x_k) - \delta_k (\tilde{x}_j + x_j)(\tilde{x}_j - x_j) = \delta_i \delta_j \delta_k (\tilde{x}_j x_k + x_j \tilde{x}_k)(\tilde{x}_k x_j - x_k \tilde{x}_j).
\]

Using the equations of motion (\(\text{(6)}\)) on both sides of the latter formula, we arrive at

\[
(\tilde{x}_k + x_k)(\tilde{x}_i x_j + x_i \tilde{x}_j) - (\tilde{x}_j + x_j)(\tilde{x}_k x_i + x_k \tilde{x}_i) = (\tilde{x}_i - x_i)(\tilde{x}_k x_j - x_k \tilde{x}_j),
\]

which is an algebraic identity.

\[\square\]

The relation between \(F_i\)’s and the integrals of the continuous time Euler top is straightforward:

\[
F_i = 1 + \frac{\epsilon^2 \alpha_i}{4} G_i + O(\epsilon^4).
\]

**Corollary 2.** Let \(\beta \perp \delta\). Then the following three functions are integrals of motion for the map (\(\text{(6)}\)):

\[
H_i^{(\beta)} = \frac{H^{(\beta)}}{1 - \delta_j \delta_k x^2_i},
\]

where the common numerator \(H^{(\beta)}\) is an integral of the continuous time Euler top given in Eq. (\(\text{(9)}\)).
Lemma 1. For the map (6) the following holds:

\[
\delta_i H_i^{(\beta)} = \frac{-(\beta_j \delta_j + \beta_k \delta_k)x_i^2 + \beta_j \delta_i x_j^2 + \beta_k \delta_i x_k^2}{1 - \delta_j \delta_k x_i^2} = \frac{\beta_j (\delta_i x_j^2 - \delta_j x_j^2) + \beta_k (\delta_i x_k^2 - \delta_k x_k^2)}{1 - \delta_j \delta_k x_i^2} = \frac{\beta_j}{\delta_k} \left(1 - \frac{1 - \delta_k \delta_j x_i^2}{1 - \delta_j \delta_k x_i^2}\right) + \frac{\beta_k}{\delta_j} \left(1 - \frac{1 - \delta_j \delta_k x_i^2}{1 - \delta_j \delta_k x_i^2}\right) = \frac{\beta_j}{\delta_k} \left(1 - \frac{1}{F_k}\right) + \frac{\beta_k}{\delta_j} (1 - F_j).
\]

3.2. Invariant volume form. Next, we establish the existence of an invariant measure for the map (6). Let us first give the following useful Lemma.

**Lemma 1.** For the map (6) the following holds:

\[
\frac{\ddot{x}_i - \delta_i \ddot{x}_j \ddot{x}_k}{1 - \delta_j \delta_k \ddot{x}_i^2} = \frac{x_i + \delta_i x_j x_k}{1 - \delta_j \delta_k x_i^2},
\]

(14)

and, as a corollary,

\[
\frac{(\ddot{x}_i - \delta_i \ddot{x}_j \ddot{x}_k)^2}{(1 - \delta_i \delta_k \ddot{x}_j^2)(1 - \delta_j \delta_k \ddot{x}_i^2)} = \frac{(x_i + \delta_i x_j x_k)^2}{(1 - \delta_i \delta_k x_j^2)(1 - \delta_j \delta_k x_i^2)}.
\]

(15)

**Proof:** We prove, for instance, Eq. (14). It is equivalent to

\[
(\ddot{x}_i - \delta_i \ddot{x}_j \ddot{x}_k)(1 - \delta_j \delta_k x_i^2) = (x_i + \delta_i x_j x_k)(1 - \delta_j \delta_k x_i^2),
\]

or to

\[
\ddot{x}_i - x_i - \delta_i \ddot{x}_j \ddot{x}_k - \delta_i x_j x_k = -\delta_j \delta_k x_i \ddot{x}_i (\ddot{x}_i - x_i) - \delta_i \delta_x (x_i^2 \ddot{x}_j \ddot{x}_k + \ddot{x}_i x_j x_k).
\]

Upon using equations of motion (6) on both sides of the latter formula, we find that it is equivalent to

\[
(\ddot{x}_j - x_j)(\ddot{x}_k - x_k) = \delta_j \delta_k (x_i \ddot{x}_j + \ddot{x}_i x_j)(x_i \ddot{x}_k + \ddot{x}_i x_k),
\]

which is a direct consequence of Eq. (6).

□

Now we are in the position to prove the following claim.

**Proposition 4.** There holds:

\[
\det \frac{\partial \ddot{x}}{\partial x} = \frac{\phi(\ddot{x})}{\phi(x)},
\]

where \(\phi(x)\) is any of the functions

\[
\phi(x) = (1 - \delta_i \delta_j x_i^2)(1 - \delta_j \delta_k x_i^2),
\]

(16)

\[
\phi(x) = (1 - \delta_i \delta_j x_i^2)^2.
\]

(17)

*The ratio of any two different functions \(\phi(x)\) is an integral of motion for (6) due to Proposition [3]. Equivalently, the three-form

\[
\Omega = \frac{1}{\phi(x)} dx_1 \wedge dx_2 \wedge dx_3
\]

(18)
is invariant under the map (6).

**Proof:** First of all, we derive the following formula for the Jacobian of the map (6):

\[
\det \frac{\partial \tilde{x}}{\partial x} = \frac{\det A(\tilde{x}, -\delta)}{\det A(x, \delta)}.
\]  

(19)

Indeed, differentiating Eq. (6) with respect to \(x_1, x_2, x_3\), one obtains the columns of the matrix equation

\[
\begin{pmatrix}
1 & -\delta_1 x_3 & -\delta_1 x_2 \\
-\delta_2 x_3 & 1 & -\delta_2 x_1 \\
-\delta_3 x_2 & -\delta_3 x_1 & 1
\end{pmatrix}
\frac{\partial \tilde{x}}{\partial x} = \begin{pmatrix}
1 & \delta_1 \tilde{x}_3 & \delta_1 \tilde{x}_2 \\
\delta_2 \tilde{x}_3 & 1 & \delta_2 \tilde{x}_1 \\
\delta_3 \tilde{x}_2 & \delta_3 \tilde{x}_1 & 1
\end{pmatrix}.
\]

Computing determinants leads to Eq. (19), which can be written in length as

\[
\det \frac{\partial \tilde{x}}{\partial x} = \frac{1 - \delta_j \delta_k \tilde{x}_i^2 - \delta_i \delta_k \tilde{x}_j^2 - \delta_i \delta_j \tilde{x}_k^2 + 2 \delta_i \delta_j \delta_k \tilde{x}_i \tilde{x}_j \tilde{x}_k}{1 - \delta_j \delta_k \tilde{x}_i^2 - \delta_i \delta_k \tilde{x}_j^2 - \delta_i \delta_j \tilde{x}_k^2 - 2 \delta_i \delta_j \delta_k \tilde{x}_i \tilde{x}_j \tilde{x}_k} = \frac{(1 - \delta_i \delta_k \tilde{x}_j^2)(1 - \delta_j \delta_k \tilde{x}_i^2) - \delta_j \delta_i \tilde{x}_i \tilde{x}_j \tilde{x}_k}{(1 - \delta_i \delta_k \tilde{x}_j^2)(1 - \delta_j \delta_k \tilde{x}_i^2) - \delta_j \delta_i \tilde{x}_i \tilde{x}_j \tilde{x}_k}^2.
\]

Now the claim of Proposition with \(\phi\) as in Eq. (16), say, follows from Eq. (15).

\(\square\)

3.3. **Invariant Poisson structure.** In the construction of an invariant Poisson structure for the map (6) we shall use the following results from [3] (Proposition 15 and Corollary 16 there).

Let \(f : M \rightarrow M\) be a smooth mapping of an \(n\)-dimensional manifold \(M\), and let \(\Omega\) be a volume form invariant under \(f\), i.e., \(f^* \Omega = \Omega\). Define \(\omega\) to be the dual \(n\)-vector field to \(\Omega\) such that \(\omega \wedge \Omega = 1\). Here the symbol \(\wedge\) denotes the contraction between multi-vector fields and forms. If \(I_1, \ldots, I_{n-2}\) are integrals of \(f\) with \(dI_1 \wedge \cdots \wedge dI_{n-2} \neq 0\), then the bi-vector field \(\sigma = \omega \wedge dI_1 \wedge \cdots \wedge dI_{n-2}\) is an invariant Poisson structure for \(f\). If \(J_1, \ldots, J_{n-2}\) is another set of independent integrals and \(\tau = \omega \wedge dJ_1 \wedge \cdots \wedge dJ_{n-2}\) is the corresponding Poisson structure, then \(\sigma\) and \(\tau\) are compatible, i.e., for any constants \(a, b\), the bi-vector field \(a\sigma + b\tau\) is a Poisson structure, again.

In particular, for \(n = 3\), if a three-form (18) is invariant under \(f\), so that the dual tri-vector field is given by

\[
\omega = \phi(x) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3},
\]

then for any integral \(I\) of \(f\) the bi-vector field

\[
\sigma = \omega \wedge dI = \phi(x) \left( \frac{\partial I}{\partial x_3} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + \frac{\partial I}{\partial x_1} \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + \frac{\partial I}{\partial x_2} \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} \right)
\]

(20)

is an invariant Poisson structure for \(f\), as well as any linear combination of such bi-vector fields. The Poisson brackets of coordinate functions are given by

\[
\{x_i, x_j\} = \phi(x) \frac{\partial I}{\partial x_k}.
\]

(21)

Applying this result to the integrals \(\log F_1, \log F_2, \log F_3\), with the three volume densities (16), we arrive at the following statement.
\textbf{Proposition 5.} The following brackets give compatible invariant polynomial Poisson structures for the map (\ref{eq:x}):

\[
\{x_i, x_j\} = C_i \delta_j x_k (1 - \delta_k \delta_i x^2_j) - C_j \delta_i x_k (1 - \delta_k \delta_j x^2_i) ,
\]

where \(C_1, C_2, C_3\) are arbitrary constants.

Notice that the Poisson brackets \((\ref{eq:x})\) yield three compatible polynomial Poisson structures. Indeed, setting \(C_2 = C_3 = 0\) and \(C_1 = 1\), we get

\[
\{x_1, x_2\}_1 = \delta_2 x_3 (1 - \delta_3 \delta_1 x^2_2), \quad \{x_2, x_3\}_1 = 0, \quad \{x_3, x_1\}_1 = -\delta_3 x_2 (1 - \delta_1 \delta_2 x^2_3),
\]

setting \(C_1 = C_3 = 0\) and \(C_2 = 1\), we get

\[
\{x_1, x_2\}_2 = -\delta_1 x_3 (1 - \delta_2 \delta_3 x^2_1), \quad \{x_2, x_3\}_2 = \delta_3 x_1 (1 - \delta_1 \delta_2 x^2_3), \quad \{x_3, x_1\}_2 = 0,
\]

while setting \(C_1 = C_2 = 0\) and \(C_3 = 0\), we get

\[
\{x_1, x_2\}_3 = 0, \quad \{x_2, x_3\}_3 = -\delta_2 x_1 (1 - \delta_3 \delta_1 x^2_2), \quad \{x_3, x_1\}_3 = \delta_1 x_2 (1 - \delta_2 \delta_3 x^2_1).
\]

It is easy to verify that the brackets \((\ref{eq:x}), \ (\ref{eq:y}), \ (\ref{eq:z})\) admit as Casimir functions the integrals \(F_1, F_2, F_3\), respectively.

In the continuous limit \(\epsilon \to 0\) these three brackets correspond to the invariant linear brackets \{\cdot, \cdot\}^{(\gamma)} of the Euler top, given in Proposition \[\ref{prop:1}\] with \(\gamma = (0, -\alpha_3, \alpha_2), \gamma = (\alpha_3, 0, -\alpha_1)\), and \(\gamma = (-\alpha_2, \alpha_1, 0)\), respectively. Clearly, these three linear brackets are linearly dependent. On the contrary, the three polynomial brackets \((\ref{eq:x}), \ (\ref{eq:y}), \ (\ref{eq:z})\) are linearly independent, if one considers linear combinations with scalar coefficients. However, they become linearly dependent, if one considers more general linear combinations. Indeed, the volume density \(\phi\) in Eq. \((\ref{eq:volume})\) can be multiplied by an arbitrary integral without violating the Poisson property. Thus, in formulating the compatibility property of such Poisson tensors it is natural to consider their linear combinations with coefficients being integrals of motion rather than just numbers. In particular, the linear combination of the brackets \((\ref{eq:x}), \ (\ref{eq:y}), \ (\ref{eq:z})\) with the coefficients

\[
(C_i, C_j, C_k) = \left(\frac{\delta_i}{F_j}, \delta_j F_i, \delta_k\right)
\]

vanishes, so that there are only two independent brackets among them.

\textbf{3.4. Explicit solutions.} Explicit solutions were given in \[5\], but it has not been explained there how to determine the parameters of the elliptic functions involved in their formulas, using the initial conditions. We would like to fill in this gap here. We use the following addition formulas for the Jacobi elliptic functions:

\[
\begin{align*}
\text{cn}(\xi + \eta) - \text{cn}(\xi - \eta) & = -\frac{2 \text{sn} \xi \text{dn} \xi \text{sn} \eta \text{dn} \eta}{1 - k^2 \text{sn}^2 \xi \text{sn}^2 \eta}, \\
\text{sn}(\xi + \eta) - \text{sn}(\xi - \eta) & = \frac{2 \text{cn} \xi \text{dn} \xi \text{sn} \eta}{1 - k^2 \text{sn}^2 \xi \text{sn}^2 \eta}, \\
\text{dn}(\xi + \eta) - \text{dn}(\xi - \eta) & = -\frac{2 k^2 \text{sn} \xi \text{cn} \xi \text{sn} \eta \text{cn} \eta}{1 - k^2 \text{sn}^2 \xi \text{sn}^2 \eta},
\end{align*}
\]
and the related formulas

\[
\text{sn}(\xi + \eta)\text{dn}(\xi - \eta) + \text{sn}(\xi - \eta)\text{dn}(\xi + \eta) = \frac{2\text{sn} \xi \text{dn} \xi \text{cn} \eta}{1 - k^2\text{sn}^2 \xi \text{sn}^2 \eta},
\]

\[
\text{cn}(\xi + \eta)\text{dn}(\xi - \eta) + \text{cn}(\xi - \eta)\text{dn}(\xi + \eta) = \frac{2\text{cn} \xi \text{dn} \xi \text{cn} \eta \text{dn} \eta}{1 - k^2\text{sn}^2 \xi \text{sn}^2 \eta},
\]

\[
\text{sn}(\xi + \eta)\text{cn}(\xi - \eta) + \text{sn}(\xi - \eta)\text{cn}(\xi + \eta) = \frac{2\text{sn} \xi \text{cn} \xi \text{dn} \eta}{1 - k^2\text{sn}^2 \xi \text{sn}^2 \eta}.
\]

Assume that the coefficients \( \delta_i \) are given by formulas (7) with \( \alpha_i \) coming from Eqs. (3) or (4) with \( I_1 < I_2 < I_3 \), so that \( \delta_1 < 0, \delta_2 > 0, \delta_3 < 0 \).

Then the above addition formulas suggest to look for the solution in one of two forms:

\[
x_1 = A_1 \text{cn}(\nu n + \varphi_0), \quad x_2 = A_2 \text{sn}(\nu n + \varphi_0), \quad x_3 = A_3 \text{dn}(\nu n + \varphi_0),
\]

(26)

or

\[
x_1 = A_1 \text{dn}(\nu n + \varphi_0), \quad x_2 = A_2 \text{sn}(\nu n + \varphi_0), \quad x_3 = A_3 \text{cn}(\nu n + \varphi_0),
\]

(27)

with \( \nu \) being a parameter to be determined and \( \varphi_0 \) an arbitrary phase. Both possibilities (26) and (27) are realized (in different regions of the phase space). Consider first the possibility (26). It is easy to see that equations of motion (6) are satisfied by functions (26), if and only if the following conditions hold [5]:

\[
A_1 = -\delta_1 A_2 A_3 \frac{\text{cn}(\nu/2)}{\text{sn}(\nu/2)\text{dn}(\nu/2)},
\]

(28)

\[
A_2 = \delta_2 A_1 A_3 \frac{\text{cn}(\nu/2)\text{dn}(\nu/2)}{\text{sn}(\nu/2)},
\]

(29)

\[
A_3 = -\delta_3 A_1 A_2 \frac{\text{dn}(\nu/2)}{k^2\text{sn}(\nu/2)\text{cn}(\nu/2)}.
\]

(30)

The amplitudes \( A_i \) should be determined from the values of the integrals of motion. Substitute the ansatz (26) into the integrals (13), then a direct computation based on the relations \( \text{cn}^2 \xi = 1 - \text{sn}^2 \xi \) and \( \text{dn}^2 \xi = 1 - k^2\text{sn}^2 \xi \) leads to

\[
A_1^2 = \frac{1 - F_3}{\delta_2 \delta_3}, \quad A_2^2 = \frac{1 - F_3^{-1}}{\delta_1 \delta_3}, \quad A_3^2 = \frac{1 - F_1^{-1}}{\delta_1 \delta_2},
\]

and

\[
k^2 = \frac{1 - F_3^{-1}}{1 - F_1^{-1}}.
\]

Thus, this ansatz holds, if and only if \( F_1 < F_3^{-1} < 1 \), that is, if \( F_2 > 1 \). With the values just found, relations (28)–(30) lead to

\[
\text{sn}^2(\nu/2) = 1 - F_1.
\]
Turning to the possibility (27) (omitted in [5]), we find that equations of motion (6) are satisfied by functions (27), if and only if the following conditions hold:

\[ A_1 = -\delta_1 A_2 A_3 \frac{dn(\nu/2)}{k^2 sn(\nu/2) cn(\nu/2)}, \]

\[ A_2 = \delta_2 A_1 A_3 \frac{cn(\nu/2) dn(\nu/2)}{sn(\nu/2)}, \]

\[ A_3 = -\delta_3 A_1 A_2 \frac{cn(\nu/2)}{sn(\nu/2) dn(\nu/2)}. \]

Substituting the ansatz (27) into the integrals (13), we find:

\[ A_2^1 = \frac{1 - F_3}{\delta_2 \delta_3}, \quad A_2^2 = \frac{1 - F_1}{\delta_1 \delta_3}, \quad A_3^2 = \frac{1 - F_1^{-1}}{\delta_1 \delta_2}, \]

and

\[ k^2 = \frac{1 - F_1}{1 - F_3^{-1}}. \]

Therefore, this ansatz holds, if and only if \( F_3^{-1} < F_1 < 1 \), that is, if \( F_2 < 1 \), and then relations (31)–(33) lead to

\[ sn^2(\nu/2) = 1 - F_3^{-1}. \]

Thus, in both cases all parameters of the solution are expressed in terms of the initial data (more precisely, in terms of the integrals of motion).

4. Concluding remarks

In this paper, we studied a remarkable birational map of \( \mathbb{R}^3 \), which serves as an integrable discretization of the Euler top, on one hand, and plays a role in the spherical geometry, on the other. Along with a streamlined presentation of results obtained previously in [6] and in [5], namely the conserved quantities and the solution in terms of elliptic functions, we found an invariant volume form and a family of compatible invariant Poisson tensors for this map. Thus, it becomes a well-established representative of integrable maps, with a standard definition of integrability in the Liouville-Arnold sense. One more standard attribute of integrable systems remains to be found for this map, namely the Lax representation. This would provide a key to understanding the nature of analogous discretizations proposed in [7], [9], which seem to belong to the most mysterious objects in the universe of integrable maps.

Acknowledgments

M.P. has been supported by the European Community through the FP6 Marie Curie RTN ENIGMA (Contract number MRTN-CT-2004-5652).

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