Spin Networks for Non-Compact Groups

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Abstract

Spin networks are a natural generalization of Wilson loop functionals. They have been extensively studied in the case where the gauge group is compact and it has been shown that they naturally form a basis of gauge invariant observables. Physically the restriction to compact gauge groups is enough for the study of Yang-mills theories, however it is well known that non-compact groups naturally arise as internal gauge groups for Lorentzian gravity models. In this context, a proper construction of gauge invariant observables is needed. The purpose of the present work is to define the notion of spin network states for non-compact groups. We first build, by a careful gauge fixing procedure, a natural measure and a Hilbert space structure on the space of gauge invariant graph connections. Spin networks are then defined as generalized eigenvectors of a complete set of hermitic commuting operators. We show how the delicate issue of taking the quotient of a space by non compact groups can be address in term of algebraic geometry. We finally construct the full Hilbert space containing all spin network states. Having in mind applications to gravity, we illustrate our results for the groups SL(2, R) and SL(2, C).

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1. INTRODUCTION

The purpose of this paper is to generalize the construction of an Hilbert space of Spin Network states to the case in which the gauge group is non-compact. Spin network states arise naturally in many fields of physics since they form a basis of gauge invariant functionals in Yang-Mills like theories [1]. In the context of gravity they were introduced by Rovelli and Smolin [2,3] and they were promoted as a basis of an Hilbert space of gauge and diffeomorphism invariant functionals in works by Ashtekar and Lewandowski [4,5] and by Baez [6]. This series of works have been focused on the case where the gauge group is compact. Compact gauge groups are natural as symmetry groups of gauge theory and Euclidean gravity, however non-compact gauge groups arise as symmetry groups of Lorentzian gravity.

For instance, $SL(2,\mathbb{C})$ arises in the original Ashtekar formulation of $3+1$ gravity in terms of self-dual variables. However, in this context, the lack of properly well defined spin network states has forced the community to work with the real $SU(2)$ Barbero connection at the price of introducing a new constant (Immirzi parameter), a more complicated dynamics [9] and the loss of a natural 4-dimensional geometrical interpretation of the phase space variables [10]. In the case of $2+1$ Lorentzian gravity, the partition function and the transition amplitudes have been computed in terms of spin networks (recoupling coefficients) of $SL(2,\mathbb{R})$ [11]. In this context, geometrical interpretation of the representation labels has led to the conclusion that space is continuous whereas time is discrete [11,12], in agreement with former results from 't Hooft [13].

In order to put these results on a firmer basis and construct geometrical operators in the Lorentzian context, one needs to understand better the nature of non-compact spin networks and how they form a natural Hilbert space related to the Hilbert space of gauge and diffeomorphism invariant connections. This is the purpose of this work.

Several issues concerning non-compact spin network have already been raised in the literature. First, Ashtekar and Lewandowski have already addressed the issue of completeness of the spin network functionals [14] versus the separability of the space of gauge invariant connections. We shall come back to these issues in section 3.3.2. There also has been some attempts to define non compact spin networks. Marolf [15] was the first one to show, in the context of $2+1$ gravity on the torus, that the loop transform using finite dimensional representation is ill defined when the group is non compact. He and, Ashtekar and Loll [16], have then studied the possibility to overcome this difficulty at the price of introducing additional and non natural structures.

Our approach shed new lights on this problem, it leads to a different point of view since we don’t insist on having spin networks labelled by finite dimensional representations. We show that one should work instead with the infinite dimensional unitary representations of the group. The emergence of spin networks labelled by infinite dimensional representation is not new. They already appeared in the context of spin foams models for Lorentzian 3d gravity [11,17] and 4-d gravity [18,19].

Moreover our formalism is more general since it sets a framework for all non-compact groups. Indeed, our purpose is to give a general account of the construction of non-compact spin networks and the structure of the stratified space of gauge invariant connections. Our presentation is valid for any semi-simple reductive group. The general exposition is therefore
quite mathematical. Having in mind further application to gravity we will illustrate the main problems and results in the context of \( \text{SL}(2, \mathbb{C}) \) and \( \text{SL}(2, \mathbb{R}) \).

1.1. Connection space and Cylindrical functions

We choose once for all a manifold \( \Sigma \) and \( P \) a locally trivial smooth principal \( G \)-bundle over \( \Sigma \), with \( G \) a semi-simple reductive group. We denote by \( \mathcal{A} \) the space of gauge \( G \)-connections and by \( \mathcal{G} \) the gauge group acting on connection by \( A^k = k^{-1}A_k + k^{-1}dk \). The theories we are interested in are Yang-Mills like in the sense that the phase space conjugate variables are given by a \( G \)-connection \( A \) (a magnetic potential) and a \( \text{ad}P \) valued densitized vector field \( E \), both are anti-hermitian. This phase space is the cotangent bundle to the space of connection \( T^*(\mathcal{A}) \). On such a phase space we want to impose the Gauss laws (gauge invariance) and eventually the diffeomorphism constraint. The representation of the operator algebra is done in the polarization where the wave functionals depend on the connection and therefore the Hilbert space structure is formally \( L^2(\mathcal{A}/\mathcal{G}, d\mu) \). The purpose of this work is to study the structure of this space. In fact we will first restrict our intention to special gauge invariant functionals of the connections called cylindrical functionals and we will study the possibility to give the space of cylindrical functions an Hilbert space structure. In the gauge invariant context, cylindrical functions are associated with graphs. Given a smooth oriented graph \( \Gamma \) composed of \( E \) oriented edges and \( V \) vertices we have the holonomy map:

\[
\Gamma : \mathcal{A} \to G^{\otimes E}
\]

\[
A \to (g_{e_1}, \ldots, g_{e_E}),
\]

(1.1)

where \( g_{e}(A) \in G \) denotes the holonomy of the connection along the edge \( e \) of the graph \( \Gamma \), which associates to any connection the holonomy of this connection along the \( E \) edges of the graph \( \Gamma \). The space of cylindrical functionals associated with \( \Gamma \) is the pullback by \( \Gamma \) of \( C^\infty(G^{\otimes E}) \) defined by:

\[
\Gamma^* \phi(A) = \phi(g_e(A)).
\]

(1.2)

The action of the gauge group on \( \mathcal{A} \) translates into an action at the vertices of the graph \( \Gamma \), if we denote by \( s(e) \) and \( t(e) \) the source and target of the edge \( e \) the gauge group action is given by

\[
g_e(A^k) = k_{s(e)}^{-1} g_e(A) k_{t(e)}
\]

(1.3)

The space of gauge invariant graph connection is denoted \( A_\Gamma = G^{\otimes E}/G^{\otimes V} \) and cylindrical gauge invariant functionals are functionals on \( A_\Gamma \). We want to construct a measure \( d\mu_\Gamma \) on \( A_\Gamma \) which provides an Hilbert space structure to the space of gauge invariant cylindrical functional \( \mathcal{H}_\Gamma = L^2(\Gamma, d\mu_\Gamma) \). This Hilbert space structure should give a representation of the Yang-Mills operator algebra restricted to \( \Gamma \). This operator algebra denoted \( \mathcal{O}_\Gamma \) is the quantization of the cotangent structure \( T^*(G^{\otimes E}/G^{\otimes V}) \) generated by the multiplication by gauge invariant function on \( G^{\otimes E} \) and by the gauge invariant derivation operators. We will see that this algebra is the operator algebra of a system of gauged particles (see section
The main constraint which determines (almost but not totally) the measure $d\mu_\Gamma$ is the fact that real classical quantities should be quantized as hermitian operators. In the case of a compact group, there is a unique solution up to an overall factor: the product of normalized Haar measures over each edge group element.

$$d\mu_\Gamma = \prod_{e \in E_\Gamma} dg_e.$$  \hfill (1.4)

This measure is then consistently extended to the space of all cylindrical functions and defines a measure on the space of generalized connection modulo gauge invariance \[20\]. In the non-compact case, it is no longer possible to integrate gauge invariant functional with (1.4) since the volume of the group is infinite. In order to construct the correct measure, we need to divide by the infinite volume of the gauge group, hence to gauge fix the gauge group action. We will do so in the following by showing that $A_\Gamma$ is isomorphic to $G^{h_\Gamma-1}$ where $h_\Gamma$ is the genus -handle number- of the surface obtained by blowing up the graph $\Gamma$. To be precise the isomorphism is only between dense subspaces. The measure we are looking for is obtained as the pushforward of the Haar measure on $G^{h_\Gamma-1}$.

The isomorphism is constructed by a gauge fixing procedure. This gauge fixing procedure is done in two steps. First in section 2. we choose a maximal tree in $\Gamma$ and we show that $A_\Gamma \sim G^{h_\Gamma}/Ad(G)$ where $Ad(G)$ denotes the adjoint diagonal action. In section 3. we introduce general useful facts about non compact groups and presents some important results of algebraic geometry allowing us to understand the geometry of non compact quotient spaces. Then in section 4. we continue the gauge fixing by showing that there exists an isomorphism between (dense subsets of) $G^{h_\Gamma}/Ad(G)$ and $G^{h_\Gamma-1}$. The isomorphism constructed being far from obvious. We finally show that the pull back measure is independent from all gauge choices leading to a well defined canonical measure $d\mu_\Gamma$ on $A_\Gamma$. In section 5. we show that $\text{Ad} G$-invariant and naively hermitic differential operators are indeed hermitic operator for the measure $d\mu_\Gamma$, we can define spin networks states as eigenvectors of a complete basis of such operators. In sections 6. and 7. we present explicit results for the rank one groups $\text{SU}(2), \text{SL}(2, \mathbb{C}), \text{SL}(2, \mathbb{R})$. Section 6. is devoted to the case where $h_\Gamma$, the genus of the graph, is one. This case is very different from the generic case treated in section 7. Finally in section 8. we discuss the construction of the full Hilbert space of all spin networks and show that it exhibits some interesting Fock substructure.

### 2. GAUGE FIXING CYLINDRICAL FUNCTIONS

#### 2.1. Constructing Flowers

$A_\Gamma = G^{\otimes E}/G^{\otimes V}$ is the space of graph invariant gauge connections. If $\Gamma = \bigcup_i \Gamma_i$ is a non connected graph then $A_\Gamma$ decomposes as the cross product $\otimes_i A_{\Gamma_i}$. It is therefore enough to understand the construction for the case of connected graphs and we will restrict in the following, unless specified otherwise, to connected graphs only.

$\Gamma$ is composed of $E$ oriented edges and $V$ vertices. Each oriented edge $e$ starts at the source vertex $s(e)$ and ends at the target vertex $t(e)$. A function on $A_\Gamma$ is a function on $G^{\otimes E}$
which satisfies gauge invariance at each vertex. More precisely, given group elements $k_v$ at each vertex $v$, $\phi$ satisfies:

$$\phi(g_{e_i}) = \phi(k_{s(e_i)}^{-1}g_{e_i}k_{t(e_i)})$$  \hspace{1cm} (2.1)

Our first goal is to define a measure to integrate such a function. For this purpose, we would like to identify the “true” degrees of freedom of $\phi$: we are going to gauge fix the gauge invariance (2.1).

There is a very simple and natural gauge fixing for graph connections which consists in eliminating as many variables $g_e$ as possible by fixing them to, say, the identity $1$. More precisely, we choose a maximal tree $T$ on our graph $\Gamma$. $T$ is a subset of edges which touches every vertex without ever forming a loop. The characteristic property of a maximal tree is that there exists a unique path along the tree $T$ which connects any two given vertices of $\Gamma$. In particular, $T$ is made up of $V - 1$ edges. Given two vertices $A$ and $B$, we can define the oriented product of group elements $h_{AB}^T$ along the path in $T$ connecting $A$ and $B$. Now, using the gauge invariance (2.1), we can fix all the group elements on the edges of $T$ to $1$. To achieve this, we first need to choose a vertex $A$ from which we are going to write our gauge fixing procedure. And we use (2.1) with

$$k_v = h_{vA}^T$$  \hspace{1cm} (2.2)

For an arbitrary edge $e$, the transformation reads

$$G_e^{(T)} = h_{As(e)}^Tg_eh_{t(e)A}^T$$  \hspace{1cm} (2.3)

Let’s consider an edge $e \in T$. There exists an unique path in $T$ linking it to $A$, else there would be a loop in the tree $T$. There is two situations: either the path connects $A$ to $s(e)$ or it connects $A$ and $t(e)$. Reversing the orientation of $e$, we can choose for example that the path connects $A$ to $t(e)$. Then, $h_{s(e)A}^T = g_eh_{t(e)A}^T$ and $h_{As(e)}^T = (h_{s(e)A}^T)^{-1}$ so that (2.3) reads $G_e^{(T)} = 1$. So (2.2) fixes all the group elements on the edges of the tree $T$ to $1$. This defines a function $\phi_T$ depending on the $g_T = E - V + 1$ group elements living on the edges not in $T$:

$$\phi_T(\{G_e^{(T)}, e \notin T\}) = \phi(g_e = G_e^{(T)} \text{ if } e \notin T \text{ or } = 1 \text{ else})$$  \hspace{1cm} (2.4)

This new function has a simple residual gauge invariance:

$$\forall k \in G, \phi_T(G_{f_i}^{(T)}) = \phi_T(k^{-1}G_{f_i}^{(T)}k), i = 1 \ldots g_T$$  \hspace{1cm} (2.5)

In other word, this gauge fixing procedure is an isomorphism

$$T : G^{\text{ad}}/\text{Ad}(G) \to A_\Gamma$$  \hspace{1cm} (2.6)

and $\phi_T$ is the pull back of $\phi$ by this isomorphism.

The residual gauge invariance corresponds to a graph with a single vertex which we call a flower. What happens is that we have contracted the whole tree $T$ to the single $A$, which is the remaining vertex. A priori, this construction and therefore the function $\phi_T$ depends
on the choice of the point $A$. In fact, the whole construction is independent of this choice. Shifting from the vertex $A$ to another vertex $B$, we can define the product of the group elements along the path in $T$ going from $A$ to $B$; let’s note it $h = h_{AB}^T$. The gauge fixing procedure carried from $B$ will create variables:

$$
\tilde{G}_e^{(T)} = h_{Bs(e)}^T g_e h_{t(e)}^T_B = h^{-1} h_{As(e)}^T g_e h_{t(e)}^T_A h = h^{-1} G_e^{(T)} h
$$

We will define a new function $\tilde{\phi}_T$ based on these new variables, but it will be equal to $\phi_T$ due to the gauge invariance (2.5) for $k = h$.

One important issue for later is the way the function $\phi_T$ changes when we modify the maximal tree $T$ on which it is based. Let’s therefore choose another maximal tree $U$. We can follow the same gauge fixing procedure based on the vertex $A$ to define variables $G_e^{(U)}$ for each edge $e$ not belonging to $U$ and define a function on the flower $\phi_U$. To relate $\phi_T$ and $\phi_U$, we would like to decompose the variables $G_e^{(U)}$ onto the variables $G_e^{(T)}$. Let’s more generally consider any oriented loop $L$ starting at the point $A$ and coming back to the point $A$ and try to express the oriented product of the group elements along it - say $H$ - in terms of the $G_e^{(T)}$. Such a loop must contain at least an edge not belonging to $T$; else the tree $T$ would contain a loop, which is impossible. Then, it is easy to realize that $H$ is the oriented product -following the orientation of the loop $L$- of the variables $G_e^{(T)}$ for $e \in L$ and not belonging to $T$. For an edge $e \notin U$, the group element $G_e^{(U)}$ can be expressed as the holonomy around the loop $L^{(U)}[e]$ following $U$ from $A$ to $s(e)$ and back from $t(e)$ to $A$. We can therefore decomposes $G_e^{(U)}$ into an oriented product of $G_f^{(T)}$. Coming back to the function $\phi_T$ and $\phi_U$, this implies that:

$$
\phi_T(G_e^{(T)}) = \phi_U(G_e^{(U)}) = \prod_{f \in L[e] \setminus T} G_f^{(T)}
$$

2.2. Examples

In the following we give some illustration of all these procedures for the following simple graph:

The function $\phi$ satisfies the following relation for every variables $k \in G$

$$
\phi(g_1, \ldots, g_9) = \\
\phi(k_A^{-1} g_1 k_C, k_A^{-1} g_2 k_B, k_A^{-1} g_3 k_B, k_C^{-1} g_4 k_B, k_C^{-1} g_5 k_D, k_E^{-1} g_7 k_F, k_E^{-1} g_8 k_F, k_D^{-1} g_9 k_F)
$$
We choose the tree $T$ as indicated on the above graph and we write down the gauge fixed variables $G^{(T)}$ based on the vertex $C$:

\begin{align*}
G_1^{(T)} &= g_4 g_2^{-1} g_1 \quad G_3^{(T)} = g_4 g_2^{-1} g_3 g_4^{-1} \\
G_7^{(T)} &= g_5 g_6^{-1} g_7 g_8^{-1} g_6 g_5^{-1} \quad G_9^{(T)} = g_5 g_9 g_8^{-1} g_6 g_5^{-1}
\end{align*}

(2.10)

Then one defines the function on the four petal flower by the following relation:

\[ \phi_T(G_1^{(T)}, G_3^{(T)}, G_7^{(T)}, G_9^{(T)}) = \phi(G_1^{(T)}, 1, G_3^{(T)}, 1, 1, 1, G_7^{(T)}, 1, G_9^{(T)}) \]  

(2.11)

One can do the same for another tree $U$:

\begin{align*}
G_2^{(U)} &= g_1^{-1} g_2 g_3^{-1} g_1 \quad G_4^{(U)} = g_4 g_3^{-1} g_1 \\
G_6^{(U)} &= g_5 g_9 g_7^{-1} g_6 g_5^{-1} \quad G_8^{(U)} = g_5 g_9 g_7^{-1} g_8 g_9^{-1} g_5^{-1}
\end{align*}

(2.13)

Then, we can also define the function $\phi_U$ as done above for the tree $T$. Now, following the procedure (2.8) of change of tree, we find the decomposition of the $G^{(U)}$ variables in terms of the $G^{(T)}$ variables:

\begin{align*}
G_2^{(U)} &= (G_1^{(T)})^{-1} (G_3^{(T)})^{-1} G_1^{(T)} \\
G_6^{(U)} &= G_9^{(T)} (G_7^{(T)})^{-1} (G_9^{(T)})^{-1} G_6^{(U)} = G_9^{(T)} (G_7^{(T)})^{-1}
\end{align*}

(2.14)

If one is skeptical, one can check these relations directly using the initial $g$ variables. Finally, we get the relation between the two functions on the flower:

\[ \phi_T(G_1^{(T)}, G_3^{(T)}, G_7^{(T)}, G_9^{(T)}) = \phi_U((G_1^{(T)})^{-1} (G_3^{(T)})^{-1} G_1^{(T)}, (G_3^{(T)})^{-1} G_1^{(T)}, G_9^{(T)} (G_7^{(T)})^{-1}, G_9^{(T)} (G_7^{(T)})^{-1} (G_9^{(T)})^{-1}) \]  

(2.15)
3. REDUCTIVE GROUPS, QUOTIENT SPACE AND ALGEBRAIC GEOMETRY

We have reduced so far the problem of constructing a measure on $A_{\Gamma}$ to the problem of constructing invariant measures on the spaces $A_h = G \times \cdots \times G/Ad(G)$, where $Ad(G)$ denotes the diagonal adjoint action of $G$.

\[ g \cdot (g_1, \cdots, g_h) \rightarrow (gg_1g^{-1}, \cdots, gg_hg^{-1}). \]  

(3.1)

The measure we are seeking for should be symmetric and invariant under right and left multiplication;

\[ d\mu(g_{\sigma_1}, \cdots, g_{\sigma_h}) = d\mu(g_1, \cdots, g_h), \]  

(3.2)

\[ d\mu(kg_1h, \cdots, g_h) = d\mu(g_1, \cdots, g_h), \]  

(3.3)

where $\sigma$ is a permutation. It should also satisfy reality conditions.

More precisely, suppose $P(X_1, \cdots, X_h)$ is a real ($P^\dagger = P$) $Ad(G)$-invariant element of $U(G^h)$ - $U(G)$ being the universal enveloping algebra of $G$. $P$ can be realized as a differential operator on $A_h$ using the correspondence between a Lie algebra element $X$ and left invariant derivative operator:

\[ \partial_X \phi(g_1, \cdots, g_i, \cdots, g_h) \equiv \phi(g_1, \cdots, g_iX, \cdots, g_h). \]  

(3.4)

This differential operator $P$ should be hermitian with respect to the measure $d\mu$ we are seeking for. In the case of a compact group, there is only one measure satisfying this conditions; $d\mu$ is just the product of normalized Haar measures for each group factor. The symmetry property is obvious and the implementation of the reality conditions are equivalent to the right and left invariance of the Haar measure. The integral of a gauge invariant function $\phi$ over $G^{\otimes 2}$ factorizes

\[ \int_{G^{\otimes 2}} d\phi(g_1, g_2) = vol(G) \int_{A^2} d\mu(g_1, g_2) \phi(g_1, g_2), \]  

(3.5)

since the volume of a compact group is finite, and we can normalize the Haar measure such that it is one. In the case of a non compact group, this is no longer possible since one would have to divide by the volume of the gauge group which is obviously infinite. We therefore have to be more cautious in the construction.

Before constructing explicitly the measure on $A_h$, we need to introduce some results concerning Lie algebras and Lie groups (all these facts can be studied more thoroughly in e.g [21]).

3.1. Reductive Lie group and Cartan subalgebra

The theory we are developing is valid for all, so-called, linear connected reductive semi-simple groups i-e algebraic matrix subgroups which are connected, invariant under conjugate transpose and of finite center. This contains all compact Lie groups but also non-compact ones which will be our main interest. Among these, we distinguish complex
groups $(SL(N, \mathbb{C}), SO(N, \mathbb{C}),$ and $Sp(N, \mathbb{C}))$ from real non compact groups (e.g $SL(2, \mathbb{R}), SO(N, 1),$ or $SL(N, \mathbb{R})$ . . .).

Given a Lie group $G$ and its Lie algebra $\mathcal{G}$, a Cartan Lie subalgebra $\mathcal{H}$ is a maximal abelian subalgebra of $\mathcal{G}$ stable under the conjugate transpose. A Cartan subgroup $H$ is the centralizer of a Cartan subalgebra $\mathcal{H}$ (i.e the subgroup of element of $G$ commuting with all the elements of $\mathcal{H}$). For each Cartan subgroup, we define the Weyl group as $W(H) = N(H)/H$, where $N(H)$ denotes the normalizer of $H$. In the case of compact groups, there is only one Cartan subalgebra, moreover any group element can be conjugated to the Cartan subgroup. In the non compact case, this is no longer true. First, in general, there is a finite number of non conjugate Cartan subalgebras, which all have the same rank (e.g $2$ for $SL(2, \mathbb{R})$, $1$ for $SO(2N + 1, 1)$, $N$ for $SL(N, \mathbb{R})$). Note that for complex groups (e.g $SL(N, \mathbb{C})$) there is only one Cartan subgroup. Second, not all elements of $G$ can be conjugated to a Cartan subgroup. The elements which can are called regular and the corresponding set is denoted $G_1$. $G_1$ consists of elements $x$ such that $Adx$ is diagonalizable. It is an open set in $G$ and its complement is of Haar measure 0. Moreover the action of $Ad(G)$ on $G_1$ is regular, $G_1/Ad(G)$ is equal to the disjoint union $\sqcup_i H_i/W(H_i)$ of the Cartan subgroups modulo their Weyl group.

Given a Cartan subalgebra $\mathcal{H}$ we have a Cartan decomposition of $\mathcal{G}$, $\mathcal{G} = \mathcal{H} \oplus \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Cartan subalgebra and $\mathcal{B}(\mathcal{H})$ an associated Borel subalgebra. $\mathcal{B}(\mathcal{H})$ uniquely decomposes in terms of eigenspaces of $ad\mathcal{H}$, i.e $\mathcal{B} = \bigoplus_{\alpha \in \Delta(H)} \mathcal{B}_\alpha$ where $\Delta(H)$ is the root space $\Delta \subset \mathcal{H}^*$. The root space decomposes into the union of positive roots $\Delta^+$ and negative roots $\Delta^-$. The generators of $\mathcal{H}$ are either compact $H^* = -H$ or non compact $H^* = H$.

Given a Cartan subgroup $H$ of $G$, we define a measure on $G/H$ as follows

$$\int_G f(g)dg = \int_{G/H} \left[ \int_H f(xh)dh \right] dx, \quad (3.6)$$

where $dg, dh$ are the invariant Haar measures on $G, H$ and $f$ is a compactly supported function on $G$. Note that $dx$ is still invariant under left multiplication, therefore, we have the identity

$$\int_G f(g)dg = \int_H \left[ \int_{G/H} f(h^{-1}xh)dx \right] dh. \quad (3.7)$$

Note that we cannot in the RHS of (3.7) innocently interchange the $x$ and $h$ integration as in (3.6). $G_1$ can be decomposed in an union of conjugacy classes $G_1 = \sqcup_i G_i^H$, where $G_i^H = \{ ghg^{-1}, h \in H_i, g \in G \}$, each conjugacy class covers $w(H_i) = \#W(H_i)$ times a connected component of $G_1$. The integral over $G$ can therefore be reexpressed as an integral over conjugacy classes, this is the Weyl integration formula:

$$\int_G f(g)dg = \sum_i \frac{1}{w(H_i)} \int_{H_i} \left[ \int_{G/H_i} f(xhx^{-1})dx \right] |\Delta_i(h)|^2 dh, \quad (3.8)$$

where

$$\Delta(e^H) = \prod_{\alpha \in \Delta^+(H_i)} \sinh \frac{\alpha(H_i)}{2}, \quad (3.9)$$

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$$\Delta(e^H) = \prod_{\alpha \in \Delta^+(H_i)} \sinh \frac{\alpha(H_i)}{2}, \quad (3.9)$$
for $H \in \mathcal{H}_i^C$, $e^H \in H_i$. Contrary to the case of compact groups, not all group elements can be obtain as an exponential $e^X$ with $X \in \mathcal{G}$. However we can realize any group element as $e^X$ with $X$ in the complexified Lie algebra $\mathcal{G}^C$.

3.2. Orbit space invariant theory and algebraic geometry

$A_h$ is defined as a quotient space by the action of a group $G$. We know that in general we do not get as a result a nice Hausdorff manifold. Several types of singularity can arise when we consider orbit spaces. Let’s look for instance at the case of $A_2 = G \times G / Ad(G)$. If $(g_1, g_2)$ are generic (non-commuting) elements of $G$, the isotropy group of these points is the center of $G$ hence a finite subgroup. But, if $g_1$ and $g_2$ are commuting elements of $G$, the isotropy group is non trivial. It is the intersection of the centralizer of $g_1$ and $g_2$. If say $g_1$ is regular, it is a Cartan subgroup and in general the dimension of the isotropy subgroup is, at least, the rank of the group. These non-generic points can act like attractors for the action of $Ad(G)$.

Suppose, for instance, that $G = SL(2, \mathbb{R})$ and $(g_1, g_2) = (1, e^{\sigma_3})$ where $\sigma_3 = diag(+1, -1)$. The isotropy group of this point is the group $e^{t\sigma_3}$. This point is non Haussdorff and is an attractor for some neighboring orbits. To see that, lets consider $g_u = (e^{u\sigma_3}, e^{\sigma_3})$ where $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\lim_{t \to \infty} e^{-t\sigma_3} g_u e^{t\sigma_3} = (1, e^{\sigma_3})$. The orbit $Ad(G).(e^{u\sigma_+}, e^{\sigma_3})$ is not closed since it contains the fixed point $(1, g_2)$. Therefore any neighborhood of the orbit $Ad(G).(e^{u\sigma_+}, e^{\sigma_3})$ contains $(1, e^{\sigma_3})$. So two different orbits associated with $u > 0$ and $u' < 0$ have non disjoint neighborhoods which means that the quotient space is not Haussdorff. One way to cure this problem is to exclude from the beginning the set of commuting elements, so that all orbits are closed. However this is not enough. Suppose, for instance, that $G = SL(2, \mathbb{R})$ and denote by $(x, y) \in \mathbb{R}^2$ the group element $(e^{\sigma_3}e^{x\sigma_+}, e^{\sigma_3}e^{y\sigma_-})$. The action of $e^{t\sigma_3}$ translates into the action $(x, y) \to (e^t x, e^{-t}y)$. The exclusion of the commuting elements translates into the condition $(x, y) \neq (0, 0)$. In this space, all orbits are closed. But one can see that any neighborhood of the orbit of $(x, 0)$ will intersect a neighborhood of the orbit of $(0, y)$ and the quotient is once again not Haussdorff. The solution is to exclude the points $(x, 0)$ or $(0, y)$, then we obtain a nice quotient.

This example illustrates the general problematic in defining quotient spaces. In fact, since $G$ is an algebraic group (being a subgroup of matrices) and since the $Ad(G)$ action is also algebraic, this problem has received a lot of attention in the mathematical literature when the group is complex [22, 23] under the name of invariant theory.

First, one needs to recall general facts from algebraic geometry and then give the definition of a regular or geometric quotient space. An affine algebraic variety $X$ over $\mathbb{C}$ is defined as being the set of zeros of a collection of polynomials of $\mathbb{C}^N$, $X = \cap_i V(P_i)$, where $V(P) = \{ x \in \mathbb{C}^N \mid P(x) = 0 \}$. The topology which is useful in this context is the Zariski topology where the closed sets are the algebraic subvariety of $X$, generated by $X(P) = X \cap V(P)$. The open sets are finite union of standard open sets $X_P = \{ x \in X \mid P(x) \neq 0 \}$. It is important to note that the open sets in Zariski topology are much bigger than in the usual topology. For instance any open set of $X$ is a dense subset of $X$ and any finite intersection of open sets is also dense. Given an affine algebraic variety $X$, one defines the algebra of regular functions, denoted $\mathbb{C}[X]$, as the algebra of
polynomials on $\mathbb{C}^N$ restricted to $X$, it is clear that $\mathbb{C}[X] = \mathbb{C}[\mathbb{C}^N]/I(X)$, where $I(X)$ is the ideal of polynomials which are zero on $X$.

One of the basic theorems in this context is that any subalgebra $A$ of $\mathbb{C}[\mathbb{C}^N]$ (or any commutative algebra) which is finitely generated and does not contain nilpotent elements is the algebra of regular functions on an affine variety $X$. Such an algebra is called affine. $X$ is called the spectrum of $A$ and defined as the set of homomorphism $A \to \mathbb{C}$. This theorem translates geometry into algebra.

Given an affine variety $X$, we say that it is irreducible if it cannot be decomposed as the union of two subvarieties. In the algebraic language, this translates into the condition that $\mathbb{C}[X]$ is integral. Given $\mathbb{C}[X]$, we can define the field of rational functions and we denote it $\mathbb{C}(X)$. We say that an application between two affine varieties $\phi : X \to Y$ is a morphism iff $\phi^*$ maps regular functions of $Y$ onto regular functions of $X$.

We are now ready to precise the good notion of a quotient. Roughly a good quotient space is one for which orbits are separated by rational invariant functions. Let $G$ be an algebraic group acting on an affine irreducible variety $X$. A geometrical quotient of $X$ by the action of $G$ is an affine variety $Y$ together with a surjective morphism $\pi : X \to Y$ such that:

- (i) $\pi$ induces an isomorphism between $\mathbb{C}(Y)$ and $\mathbb{C}(X)^G$.
- (ii) The fibers of $\pi$ are the orbits of $G$ in $X$.

The condition (ii) tells us that $Y$ is a quotient space since it is a orbit space. The condition (i) tells us that this quotient space is a algebraic variety where points are separated by rational functions. We saw in the previous examples that such a good quotient space does not exist in general. Hopefully there is a fundamental theorem of Rosenlich which states that given a variety $X$ and an algebraic action of $G$ on $X$, it is always possible to choose an open dense set $X_0$ stable under $G$ such that $X_0/G$ is a good quotient. The proof of this theorem goes as follows. We first restrict $Y$ such that hypothesis (i) is satisfied. Then, hypothesis (i) implies that the orbit of $x$ is dense in $\pi^{-1}(x)$. However, in general, hypothesis (ii) is not true. What we do next is to restrict ourself to a subset $X_0$ of $X$ which contains only orbits of maximal dimension. This implies (ii). Then the geometric quotient $X_0/G$ exists as an algebraic variety.

In the case the group $G$ is reductive, there exists a fundamental theorem due to Hilbert and Nagata which states that if $X$ is irreducible and $G$ reductive then $\mathbb{C}[X]^G$ is finitely generated. Since $\mathbb{C}[X]^G$ does not contain nilpotent elements, this means that it is an affine algebra and therefore that it is the algebra of regular functions over its spectrum, which is denoted $X//G \equiv \text{spec}(\mathbb{C}[X]^G)$. It is equipped with a surjective morphism $\pi : X \to X//G$ and called the quotient of $X$ by $G$. The quotient of $X$ by $G$ is universal in the sense that any $G$-invariant morphism $p : X \to Y$ can be factorized over $X//G$, i.e there exists a morphism $q : X//G \to Y$ such that $p = q \circ \pi$. It is then possible to show that any fiber of $\pi$ contains a unique closed orbit. Geometrically this means that $X//G$ is the space of closed orbits of $G$ in $X$. This is a little bit disappointing since this means that $X//G$ could be a very rough description of the space of orbits. For instance $X//G$ is generally not a geometric quotient. For instance Let $X = \mathbb{C}^2$ and $G = \mathbb{C}^*$ acts by multiplication $(x,y) \to (tx,ty)$. The only
invariant polynomials $P(x, y)$ are the constant polynomials, so that $X//G$ is reduced to a point. In other words, there is one unique closed orbit, the one of $(0, 0)$. Fortunately, the following property is true if $G$ is linear reductive connected and semisimple. In this case, the algebra of fractions of $\mathbb{C}[X]^G$ (i.e. $\mathbb{C}(X//G)$) is equal to $\mathbb{C}(X)^G$. Equivalently this means that $X//G$ is the space of dense orbits, i.e. any fiber $\pi : X \to X//G$ contains a dense orbit. Therefore this theorem imply that in the case we consider, e.g. linear reductive group we can define the geometrical quotient space $X/G$ as the algebraic dual of the space of invariant polynomial.

Coming back to our specific problem we have $G$ a linear reductive connected semi-simple Lie group acting on $X = G^n$ by the adjoint action. By the general theory just exposed, we know that, in the case $G$ is complex, the universal orbit space $G^n//AdG$ consisting of dense orbits separated by invariant polynomials on $G^n$ is well defined. Moreover by the Rosenlich theorem, we know that it is always possible to exclude from $G^n$ a closed set such that the universal orbit space is well defined as a geometric quotient. However the drawbacks of these general methods, despite their beauty and generality, are that they say nothing about the real case in which we are interested and that they are not constructive. It is therefore still interesting to understand better and explicitly which closed set we have to exclude from $G^n$ in order to get a geometric quotient space (which is then a well defined affine variety). Moreover we are interested in the measure theoretical property of the quotient and we would like to show that the difference between the geometrical quotient and the quotient space $G^n//G$ is of measure zero.

First we know from Rosenlich theorem that it is possible to obtain a geometrical quotient by excluding a closed (in the Zariski topology) set from $G^n$. In the case of $G/Ad(G)$, this problem is well known. The solution is to exclude from $G$ the points at which the $Ad(G)$ action is not regular. The set of regular elements of $G$ is denoted by $G'$ and this is the set of $g \in G$ for which $Ad(G)$ is diagonalizable. The quotient space $G'/Ad(G)$ is equal to the union of the Cartan subgroups modulo their Weyl group $\sqcup H_i/W(H_i)$ (the intersection of $H_i$ with $H_j$ only contains the identity). We need to introduce the set of strictly regular elements of $G$ denoted $G_1$ and which is the set of regular elements of $G$ not equal to the identity, $G' = G_1 \cup \{Id\}$.

In the case of $G \times G$, the same strategy is working and we need to exclude from it non regular points. We therefore consider the action of $Ad(G)$ not on $G \times G$ but on a subspace. For a general group even if we know the existence of such a maximal subspace we don’t have an explicit description and this deserve a full mathematical study. We can however give an explicit description in the case of rank 1 groups, in that case we define

$$G_2 \equiv \{(g_1, g_2) \in G \times G; g_1 \in G_1 \text{ or } g_2 \in G_1, \text{ and } g_1 g_2 g_1^{-1} g_2^{-1} \in G_1\}, \quad (3.10)$$

We then have the following proposition

**Proposition 1** $G_2$ is a dense subset of $G \times G$, its complement is of Haar measure zero and $G_2/Ad(G)$ is a geometric quotient when $G$ is of rank one. Therefore, $A_2 = G_2/Ad(G)$ is an Hausdorff manifold of dimension $\dim G$ which separates rational functions and it is the base manifold of a homogeneous fiber bundle whose fiber is $G$ and total space $G_2 \approx A_2 \times G$. 

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This proposal is proved in the section 7 where we construct the dual spectrum space to the space of invariant polynomials and show that it is isomorphic to $G_2/\text{Ad}(G)$. The main point is that the condition of being in $G_1$ can be implemented as an algebraic inequality.

Note that, in the definition 3.10, $G_2$ is such that the centralizer of any element of $G_2$ is trivial. Suppose $g$ commutes with $(g_1, g_2) \in G_2$, since we can take e.g. $g_1$ to be regular hence $g$ can be diagonalized in the same basis as $g_1$ (the regularity assumption is essential here). $g$ cannot be regular since it commutes with $g_2$ and therefore it would mean that $g_2$ is also diagonal in the same basis, hence commutes with $g_1$, which is impossible. So the definition of $G_2$ implies that $g$ is diagonal and non-regular. If the rank of the group is 1, this means that $g$ is the identity. Whereas if the rank is higher, we clearly need additional conditions to get the same conclusion. It is important to understand that the definition of $G_2$ is not the naive definition where one just excludes from $G \times G$ the group elements which have a non trivial centralizer: in order to get a nice quotient, we need to take away more points and this is dictated by the fact that we want the quotient space to be the spectrum of the algebra of invariant functionals.

In the general case of higher rank group we define $G_2$ to be the following space

$$G_2 \equiv \{(g_1, g_2) \in G \times G; g_1 \in G_1 \text{ or } g_2 \in G_1, \text{ and } C(g_1, g_2) = Z_G\}$$

(3.11)

where, $C(g_1, g_2)$ denotes the centralizer of $g_1$ and $g_2$ and $Z_G$ denotes the center of the group. We will see in the next section that $G_2$ admits a quotient by $\text{Ad}(G)$ and we denote by $A_2$ the quotient. However, contrary to the definition 3.10 valid for rank 1 group, the definition 3.11 is not equivalent to a definition of $G_2$ as an algebraic dual.

4. CONSTRUCTION OF THE MEASURE ON $A_H$

This section is central to our paper. In this section we go back to our problem which is the construction of a measure on $A_h$. We show, by a non trivial gauge fixing procedure, that a dense subset of $G^h/\text{Ad}(G)$ can be identify with a dense subset of $G^{h-1}$ when $h > 1$. We first consider $A_2$, the case of one Cartan subgroup, then the case of several Cartan subgroups, we then consider the case of $A_h$. Finally, putting everything together we show that the measure on $A_{\Gamma}$ does not depend on all the gauge fixing choices, leading to the definition of a canonical measure.

4.1. Construction of the measure on $A_2$

1. Case of a unique Cartan subgroup $H$

Let’s consider the following embedding of $G_1$ into $A_2$. Given $g \in G_1$, we can conjugate it to the Cartan subgroup $H$ of $G$ i-e there exists $h \in H$, $x \in G/H$ such that $g = xhx^{-1}$. We choose a section $s : G/H \to G$ and we define the map:

$$j_s : G_1 \rightarrow A_2, \quad g = xhx^{-1} \rightarrow \text{Ad}(G).(h, s(x))$$

(4.1)
where $Ad(G).(h,s(x))$ is the orbit of $(h,s(x))$ under the conjugation by $G$. This is a gauge fixing since given $(g_1,g_2) \in A_2$ we can conjugate $g_1$ to the Cartan subgroup $H$ of $G$ i.e there exists $h \in H$, $y \in G/H$ such that $g_1 = yhy^{-1}$. Then $Ad(G).(g_1,g_2) = Ad(G).(h,y^{-1}g_2y)$. This fixes only partially the gauge since $H$ can act on $y$ by $y \to yk$ which means that we can still conjugate $\tilde{g}_2 = y^{-1}g_2y$ by a Cartan group element. Nevertheless, since we are in $A_2$, the centralizer of $g_1$ and $g_2$ is trivial, this means that the centralizer of $h \in H$ and $\tilde{g}_2$ is trivial. Since the centralizer of $h \in H \cap G_1$ is $H$, this implies that the conjugate action of $H$ on $\tilde{g}_2$ has no other fixed point than the center elements. Let’s suppose for the following that the center of $G$ is trivial. Then, given an arbitrary section $s : G/H \to G$, we can use the residual symmetry coming from the conjugation by $H$ to impose that $\tilde{g}_2$ belongs to the image of $s$. Finally, the gauge fixing we impose is $(g_1,g_2) \to (h,s(x)) \in H \times G/H$. We just have argued that every element of $G_2$ can be brought to this form. Moreover the condition that the centralizer of $(g_1,g_2)$ is implemented if we ask $g = xhx^{-1} \notin H$. Indeed, $xhx^{-1} \in H$ would mean that either $x \in H$ or that $x$ is a Weyl transformation. The first possibility, $x \in H$ is impossible since $x$ and $h$ don’t commute. So that $j_s$ gives a map from $G_1 \setminus H$ to $A_2$. The second possibility is related to the Gribov ambiguity, which makes the definition of $j_s$ still ambiguous.

This can be traced back to the fact that a given group element $g$ can be conjugated to different Cartan elements related by the action of the Weyl group $W(H)$, which is the residual conjugation action on the Cartan subgroup. There is two ways to solve this problem. First, we can require good transformations of the section $s$ under the action of the Weyl group

$$\forall x \in G/H, \forall w \in W(H), s(xw) = w^{-1} s(x) w$$

(4.2)

This renders $j_s$ well-defined and this is the hypothesis we will suppose in the following. Or we can impose $h$ to be in a fixed Weyl chamber. In this case, one must remove all the $1/w(H)$ factors from the following proofs.

With this map, we can pullback functions on $A_2$, or equivalently invariant functions on $G_2$, to functions on $G_1$ by $j_s^* F(xhx^{-1}) = F(h,s(x))$.

**Definition 1** Let $\mu$ be a measure on $A_2$ defined by

$$\int_{A_2} F(g_1,g_2) d\mu(g_1,g_2) \equiv \int_G j_s^* F(g) dg.$$  

(4.3)

**Proposition 2** Let $F$ be a $L^1$ function on $G_2$ with respect to the Haar measure. We also require that its gauge invariant version $^G F$ is well-defined:

$$^G F(g_1,g_2) = \int_G F(1g_1g^{-1},gg_2g^{-1})dg$$  

(4.4)

Then we have

$$\int_{G \times G} F(g_1,g_2)dg_1dg_2 = \int_{A_2} ^G F(g_1,g_2)d\mu(g_1,g_2),$$  

(4.5)
Let \( F(g_1, g_2) \) be a \( L^1 \) function on \( G \times G \).

\[
\int_{G \times G} F(g_1, g_2) dg_1 dg_2 = \frac{1}{w(H)} \int_{G/H \times H \times G} F(xhx^{-1}, g_2) \Delta(h) dxhdg_2
\]  

(4.6)

\[
= \frac{1}{w(H)} \int_{G/H \times H \times G} F(xhx^{-1}, xg_2x^{-1}) \Delta(h) dxhdg_2
\]  

(4.7)

\[
= \frac{1}{w(H)} \int_{G/H \times H \times G} F(xhx^{-1}, g_2) \Delta(h) dxhdg_2
\]  

(4.8)

where we have used the Weyl integration formula (3.8) in the first equality and the invariance under right and left translation of the Haar measure \( dg_2 \) in the second. Using the identity (3.7) for the integration on \( G_2 \) the integral can be expanded as

\[
\frac{1}{w(H)} \int_{G/H \times H \times H \times H \times H \times G} F(xhx^{-1}, xkyk^{-1}x^{-1}) \Delta(h) dxhdkdy
\]

\[
= \frac{1}{w(H)} \int_{H \times H \times H \times H \times H} \left[ \int_{G/H \times H} F(xhx^{-1}, xky(xk)^{-1}) dxdk \right] \Delta(h) dhdy
\]

\[
= \frac{1}{w(H)} \int_{H \times H \times H \times H} \left[ \int_{G/H \times H} F(xhxk^{-1}, xky(xk)^{-1}) dxdk \right] \Delta(h) dhdy
\]  

(4.9)

where we have used the fact that \( H \) is abelian to derive the last equality. Then using the definition of the \( G/H \) measure (3.6), we finally get

\[
\int_{G \times G} F(g_1, g_2) dg_1 dg_2 = \frac{1}{w(H)} \int_{H \times H \times G} G F(h, y) \Delta(h) dhdy
\]

(4.10)

where \( G \) is the gauge invariant version of \( F \):

\[
G F(g_1, g_2) = \int_{G} F(gg_1g^{-1}, gg_2g^{-1}) dg.
\]  

(4.11)

**Theorem 1** \( \mu \) is independent of \( s \), symmetric, invariant under right and left multiplication and invariant under taking the inverse:

\[
d\mu(g_1, g_2) = d\mu(g_2, g_1)
\]  

(4.12)

\[
d\mu(kg_1, g_2) = d\mu(g_1, g_2)
\]  

(4.13)

\[
d\mu(g_1, g_2) = d\mu(g_1^{-1}, g_2)
\]  

(4.14)

Let’s prove the above theorem for \( d\mu(kg_1, g_2) \) (left multiplication). One will be able to prove the other properties following the same line of thoughts. The easiest way to prove the theorem is to use a Faddeev-Popov gauge fixing procedure using the proposition 2. Let’s choose an invariant function \( F \) on \( G_2 \). We can choose any function \( \varphi \) on \( G_2 \) such that \( G \varphi = 1 \) and create the (gauge fixed) function \( \tilde{F} = F \varphi \). In the usual Faddeev-Popov procedure one would choose \( \varphi \) to be proportional to a \( \delta \) function of a gauge fixing condition, but this is not necessary. Applying proposition 3.
\[ \int_{A_2} F(g_1, g_2) d\mu(g_1, g_2) = \int_{G \times G} \tilde{F}(g_1, g_2) dg_1 dg_2, \quad (4.15) \]

Using the freedom in the choice of the function \( \varphi \) in proposition 2 and the fact that if \( \varphi(g_1, g_2) \) is a gauge fixing so is \( \varphi_k(g_1, g_2) = \varphi(k^{-1}g_1, g_2) \), we get

\[ \int_{A_2} F(g_1, g_2) d\mu(kg_1, g_2) = \int_{G \times G} F(k^{-1}g_1, g_2) \varphi(k^{-1}g_1, g_2) dg_1 dg_2 \]
\[ \quad = \int_{G \times G} F(g_1, g_2) \varphi(g_1, g_2) dg_1 dg_2 \]
\[ \quad = \int_{A_2} F(g_1, g_2) d\mu(g_1, g_2) \quad (4.16) \]

So we conclude to the left invariance of the measure \( d\mu \) defined on \( A_2 \).

2. The case of many Cartan subgroups

In general, we have many Cartan subgroups and let’s note them \( H_1, H_2, \ldots, H_n \). \( G_1 \) can be decomposed in disconnected components \( G^{(i)} = Ad(G).H_i \), each conjugated to the Cartan subgroup \( H_i \). For each of these components, one can choose a section \( s_i : G/H_i \to G \) and define a map

\[ j_i : G^{(i)} \subset G_1 \to A_2 \]
\[ g = yhy^{-1} \to Ad(G).(h, s_i(y)) \quad (4.17) \]

From this map, one can define a measure \( d\mu_i \) on \( A_2 \) as in the definition (i) by

\[ \int_{A_2} F(g_1, g_2) d\mu_i(g_1, g_2) \equiv \int_{G_i} j_i^*F(g)dg \quad (4.18) \]

**Proposition 3**  Given any \( L^1 \) function \( F \) on \( G_2 \) we have

\[ \int_{G \times G} F(g_1, g_2) dg_1 dg_2 = \int_{A_2} G^F(g_1, g_2) d\mu(g_1, g_2), \quad (4.19) \]

where

\[ d\mu(g_1, g_2) = \sum_i \frac{1}{w(H_i)} d\mu_i(g_1, g_2) \quad (4.20) \]

and where \( G^F \) is the gauge invariant version of \( F \):

\[ G^F(g_1, g_2) = \int_{G} F(gg_1g^{-1}, gg_2g^{-1}) dg \quad (4.21) \]
Let $F(g_1, g_2)$ be a $L^1$ function on $G \times G$.

$$\int_{G \times G} F(g_1, g_2) dg_1 dg_2 = \sum_i \frac{1}{w(H_i)} \int_{G/H_i \times H_i \times G} F(xh x^{-1}, g_2) \Delta(h) dx dh dg_2$$  \hspace{1cm} (4.22)

$$= \sum_i \frac{1}{w(H_i)} \int_{G/H_i \times H_i \times G} F(xh x^{-1}, x g_2 x^{-1}) \Delta(h) dx dh dg_2$$  \hspace{1cm} (4.23)

where we have used the Weyl integration formula (3.8) in the first equality and the invariance under right and left translation of the Haar measure in the second. Using the identity (3.7) for the integration on $G_2$ the integral can be expanded as

$$\int_{G \times G} F(g_1, g_2) dg_1 dg_2 = \sum_i \frac{1}{w(H_i)} \int_{G/H_i \times H_i \times G} \left[ \int_{G/H_i \times H_i} F(xh x^{-1}, x g_2 x^{-1}) dx \right] \Delta(h) dh dy$$  \hspace{1cm} (4.24)

Using the fact that $H_i$ is abelian and the definition of the $G/H_i$ measure (3.6) we finally get

$$\int_{G \times G} F(g_1, g_2) dg_1 dg_2 = \sum_i \frac{1}{w(H_i)} \int_{H_i \times H_i \times G} G F(h, y) \Delta(h) dh dy$$  \hspace{1cm} (4.25)

$$= \sum_i \frac{1}{w(H_i)} \int_{H_i \times H_i} d \mu^{(i)}(g_1, g_2) G F(g_1, g_2)$$  \hspace{1cm} (4.26)

Using the above proposition, we can generalize theorem 1 to the multi-Cartan case using the same proof as done before.

### 4.2. The measure on $A_h$

We are now interested in generalizing the case of $A_2$ to $A_h$. Applying the Rosenlich theorem stated in (3.2), it is always possible to choose an open dense set $G_h \subset G^h$ such that the geometric quotient $A_h = G_h / Ad(G)$ is well-defined as in proposition 1. Following the choice made for the 2-petals case we get the following definition when $G$ is rank one.

$$G_h \equiv \{ (g_1, \cdots, g_h) \in G^h \mid \exists (i, j) \in [1, \cdots, h], (g_i, g_j) \in G_2 \}$$  \hspace{1cm} (4.27)

**Definition 2**

Note that $G_h$ is such that the centralizer of any element in $G_h$ is the identity.
Definition 3 We choose two edges $i, j$ on the $n$-petal flower. Then, we can define a measure on $A_h$

$$
\mu^{(ij)}[f(g_1, g_2, \ldots, g_n)] = \int_{G_2^{(ij)}} d\mu(g_i, g_j) \prod_{k \neq i, j} d g_k f(g_1, g_2, \ldots, g_n)
$$

(4.29)

where we have taken the measure gauge fixed measure $d\mu(g_i, g_j)$ on the two chosen edges and the Haar measure on the other edges.

This measure is well-defined since $\prod_{k \neq i, j} d g_k f(g_1, g_2, \ldots, g_n)$ is an invariant function on $G_2^{(ij)}$ (and therefore a function on $A_2$, which we can integrate using $d\mu(g_i, g_j)$).

Proposition 4 Given any $L^1$ function $F$ on $G^n$ we have

$$
\int_{G^n} F(g_1, \ldots, g_n) d g_1 \ldots d g_n = \int_{A_n} ^G F(g_1, \ldots, g_n) d\mu(g_1, \ldots, g_n),
$$

(4.30)

where $^G F$ is the gauge invariant version of $F$:

$$
^G F(g_1, \ldots, g_n) = \int_G F(g g_1 g^{-1}, \ldots, g g_n g^{-1}) d g
$$

(4.31)

This proposition is easily proved using the proposition 2 and the invariance of the Haar measure under right and left multiplication. And its leads to the following theorem:

Theorem 2 The measures $d\mu^{(ij)}$ don’t depend on the choice of edges $i, j$. And one defines a unique measure $d\mu(g_1, g_2, \ldots, g_n)$ on $A_h$. Moreover, this measure is symmetric under permutation of $g_1, \ldots, g_n$, under right and left multiplication and under taking the inverse of one of its argument:

$$
d\mu(g_{\sigma_1}, \ldots, g_{\sigma_n}) = d\mu(g_1, \ldots, g_n),
$$

$$
d\mu(k g_1 h, \ldots, g_n) = d\mu(g_1, \ldots, g_n),
$$

$$
d\mu(g_i^{-1}, \ldots, g_n) = d\mu(g_1, \ldots, g_n)
$$

(4.32)

4.3. Measure on an arbitrary graph

To construct the measure on an arbitrary graph $\Gamma$, we are going to choose a maximal tree $T$ and carry on the gauge fixing procedure described in the first section in order to reduce the graph $\Gamma$ to a flower. We then define the measure $d\mu_T$ such that for all gauge invariant functions $\phi$ on $\Gamma$, we have:

$$
\int d\mu_T(g_1, \ldots, g_E) \phi(g_1, \ldots, g_E) = \int \tilde{d}\mu(g_1, \ldots, g_F) \phi_T(g_1, \ldots, g_F)
$$

(4.33)

where $\tilde{d}\mu$ is the measure on the $F$ petal flower.
This definition a priori depends on the choice of the tree $T$. We are going to prove that this is not the case. So we choose two maximal trees $T$ and $U$. The gauge fixed function are related by (2.8):

$$\phi_T(G_e(T)) = \phi_U(G_e(U) = \prod_{f \in \mathcal{E} \setminus T} G_f(T)$$

and we want to prove that:

$$\int \tilde{d}\mu(G_1(T), \ldots, G_F(T)) \phi_T(G_1(T), \ldots, G_F(T)) = \int \tilde{d}\mu(G_1(U), \ldots, G_F(U)) \phi_U(G_1(U), \ldots, G_F(U))$$

or equivalently:

$$\int \tilde{d}\mu(G_1(U), \ldots, G_F(U)) \phi_U(G_1(U), \ldots, G_F(U)) = \int \tilde{d}\mu(G_1(T), \ldots, G_F(T)) \phi_U(G_e(U) = \prod_{f \in \mathcal{E} \setminus T} G_f(T)$$

(4.35)

We are going to show this equality by doing some elementary changes of variables which would correspond to elementary moves between the two maximal trees and we will show that the measure is invariant under each such move. Let’s first define what we mean by an elementary move.

**Definition 4** Given a graph $\Gamma$ and a maximal tree $T$ on it, let’s choose a vertex $v$ such that there is at least one edge linked to it which is not in the tree $T$. Let’s pick one and call it $f$. There exists an unique path in $T$ linking the other vertex of $f$ to $v$. This path goes along an unique edge $e \in T$ linked to $v$. Then an elementary change of tree, or elementary move, is exchanging the role of $e$ and $f$ and considering the maximal tree $U = T \cup f \setminus e$.

$$\begin{align*}
e \in T & \quad f \not\in T \quad \text{\rightarrow} \quad e \notin U \quad f \in U \\
\in T & \quad f \notin T \quad \text{\rightarrow} \quad e \not\in U \quad f \in T
\end{align*}$$

The interest in such a definition lies into the following proposition:

**Proposition 5** Having chosen two maximal trees $T$ and $U$ on a graph $\Gamma$, there exists a sequence of elementary moves going from $T$ to $U$.

Then, as we will see, the change of variable from $G(U)$ to $G(T)$ is very simple for such a move since it is implemented either by an inversion or by left multiplication, so that it will simplify the study of change of trees.

Let’s first prove the proposition. Given $\Gamma$ and two maximal trees $T, U$ in it we can distinguish four types of edges: edges belonging to both trees $T$ and $U$, edges in $V = T \setminus U$, edges in $W = U \setminus T$ and edges in neither trees. By elementary moves on either the tree
or the tree $U$, we would like to reduce the sets $V$ and $W$ down to nothing. Let’s take a
closer look at the set $V$. First, $V$ might not be connected. In this case, we would carry on
the following procedure on each of its connected parts. Let’s denote one of the connected
part $V_1$ and work on it. $V_1$ is a tree as part of the tree $T$. In particular, it is not closed
and has some open ends i.e. edges connected to $V_1$ only by one vertex. By doing elementary
moves, we are going to remove them from $V_1$, and then, by repeating the same operations,
one could erase completely the set $V_1$. And finally, by repeating the procedure on the other
connected parts of $V$, one could absorb completely the set $T \setminus U$.

So let’s choose an edge $e$ at an open end of $V_1$. It has two vertices: $v$ in the exterior of
$V_1$ and $w$ in the interior of $V_1$. There exists an unique path $P$ along $U$ which links these two
vertices and $e \notin U$ is not in it. In $P$, there exists at least one edge in $U$ but not in $T$ else
there would be a loop in the tree $T$.

Let’s suppose that such an edge $f \in U \setminus T$ touches directly the edge $e$ (at the vertex $v$).
Then, we can do an elementary move exchanging $e \leftrightarrow f$ and create a tree $	ilde{U} = U \cup e \setminus f$
closer to the tree $T$ than the tree $U$.

Let’s now come back to the general case in which we have to follow a sequence of edges
$f_1, \ldots, f_n \in T \cap U$ starting from the vertex $v$ along the path $P$ to an edge $f$ in $U \setminus T$. Then,
we do the allowed elementary moves on the tree $U$ exchanging $e \leftrightarrow f_1, \ldots, f_{n-1} \leftrightarrow f_n$, thus
creating the trees $U_1, \ldots U_n$. Starting from $v$, all the edges $e, f_1, \ldots, f_{n-1}$ are both in $T$ and
$U_n$. $f_n$ is in $T \setminus U_n$, $f$ is in $U_n \setminus T$, and all the other edges on the way back to $w$ are in $U_n$.
So that we are in the same simple case as above and we finally do the move $f_n \leftrightarrow f$ on the
tree $U_n$ creating a tree $	ilde{U}$ such that the whole loop $P$ from $v$ to $v$ is in both $T$ and $\tilde{U}$ except
the edge $f$ which is in neither. Practically, we had a loop with all the edges in $U$ but one
which is in $T$ (it’s $e$), and by elementary moves, we move it around until it meets an edge
which is not in $T$ and they “cancel” each other.

This ends the absorption of the edge $e$: the set $T \setminus \tilde{U}$ contains one edge less than $T \setminus U$.
And we now repeat the same procedure using the new tree $\tilde{U}$.

We now are able to prove that :

**Theorem 3** The Jacobian of the change of variables (4.35) is 1, so that the measure $d\mu_T$
is invariant under changes of tree.

This theorem will assure the existence of a measure $d\mu^{(r)} = d\mu_T$ independent from
the choice of the tree $T$ and therefore from the whole gauge fixing procedure.
which is the measure we will use to integrate our gauge invariant functions and define a space of $L^2$ gauge invariant functions. This space will in fact be the Hilbert space of spin networks defined on the graph $\Gamma$.

Proposition 5 means that we only have to prove the theorem 3 for elementary moves. So, let’s realize an elementary move on the tree $T$ around the vertex $v$ and define the new maximal tree $U = T \cup f \setminus e$. For every edge $a \notin U$ on the $U$-flower, we define the variables $G_a^{(U)}$. And we want to express them in terms of the variables $G_b^{(U)}$. For $a \notin U$ and $b \in T$, we want to relate $G_a^{(U)}$ to $G_b^{(T)}$. It can be easily seen that these two variables are equal up to a multiplication on the left or right or on both side by $G_f^{(T)}$ or its inverse. And for the only other case $a = e$, we will have $G_e^{(U)} = (G_f^{(T)})^{\pm1}$. Then using the invariance of the measure of the flower by multiplication or by taking the inverse of one of its argument (theorem 2), we can conclude that the above change of variables has a trivial Jacobian.

5. SPIN NETWORKS STATES

In this section, we are going to define the spin networks as eigenvectors of a set of commuting Laplacian operators, which will be shown to be hermitian.

5.1. Laplacian operators

Let’s consider a graph $\Gamma$ and a gauge invariant function defined on it. These functions depend on $E$ group elements $g_1, \ldots, g_E$. Let’s denote by $X$ an element of the Lie algebra and by $\partial_X^R$ (resp. $\partial_X^L$) the right (resp. left) invariant derivative acting on the $j$-th group element associated with the edge $e$:

\begin{align}
\partial_X^R f(g_1, \cdots, g_N) &= f(Xg_1, \cdots, g_N) \\
\partial_X^L f(g_1, \cdots, g_N) &= f(g_1(-X), \cdots, g_N)
\end{align}

The gauge group action acts on derivative operators by conjugation at each vertex:

$$\partial_X^R \rightarrow \partial_X^R \kappa_{s(e)} X_k^{-1(e)}.$$

We are interested in gauge invariant differential operators. The algebra of such operators is generated by Laplacian operators $\Delta_e^{(i)}$ where $e$ labels the edges of $\Gamma$ and $i$ runs from 1 to the rank $r$ of the group. For each edge $e$, the space of Laplacians is in one to one correspondence with the Casimir operators of the Lie algebra. Therefore this set of Laplacians gives a complete basis of commuting operators. Indeed, they are commuting since for a given $e$ two Casimirs commute and for different edges $e$’s the differential operators $\partial_X^R$ commute with each other.

We want to define the spin networks as the basis of eigenstates vectors for this complete set of gauge invariant differential operators. In order to do that, we need to show that these operators are hermitian with respect to the measure $d\mu^{(T)}$ that we have just constructed. We are going to give the proof for the quadratic Laplacian operator $\Delta_e = \sum_i \partial_X^R \partial_X^{-1}$.
∂Re, ∂Rc for short), where Xi denotes an orthonormal basis of the Lie algebra. The general case is similar, it simply needs more cumbersome notations.

Because of the measure dμ(Γ) has being defined on the gauge fixed flower corresponding to Γ, we have to follow the gauge fixing procedure leading to group variables Gi, . . . , GF on the flower and express the operators ∆e in term of the derivatives ∂̃L,R with respect to these new variables.

To start with, let’s look at an example: the case of the two petal flower coming from either the Θ graph or the eyeglass graph. Let’s first gauge fix the Θ graph:

We gauge fix from the point A. We have G1 = g1g3−1 and G2 = g2g3−1. It is then easy to check that ∂1R = ∂̃1R, ∂2R = ∂̃2R and ∂3R = ∂L1 + ∂L2 so that ∆1 = L1, ∆2 = L2 and ∆3 = Λ1 + Λ2 + 2∆12, where Λ12 = ∂̃1R ∂̃2R.

In the case of the eyeglass graph:

We have G1 = g1 and G2 = g3g2g3−1. Thus ∆1 = Λ1, ∆2 = Λ2 as the Laplacian is invariant under Ad(G) and ∆3 = (∂̃2R − ∂̃2L)2.

In the generic case, for a given edge e, if there exists a maximal tree T which doesn’t go through e, then e will be on the gauge fixed flower and ∆e will simply be the Laplacian ∆e with derivatives with respect to the flower variable Ge.

What happens to edges which are in every possible trees, such as the middle edge in the eyeglass graph, is slightly more tricky. For such an edge e, we choose to gauge fix from its departure vertex v. Then ∂eR will be equal to the sum of ∂̃fR for all edges f whose loop starts with e and ∂̃fL for all edges whose loop finishes with e.

In any case, the initial differential operators ∆e can be written as a sum of ∆εRR = ∂̃εR ∂̃εR and ∆εLR = ∂̃εL ∂̃εR where i, j = 1, . . . , g. These operators are Ad(G) invariant operators on Gh. This is easily seen since the gauge invariance of a function φT is equivalent to

\[
(\sum_{e|s(e)=v} \partial^Re + \sum_{e|t(e)=v} \partial^Le)\phi = 0,
\]

for all vertices v. Choosing a tree amounts to use these equations to express all derivatives along edges belonging to the tree in terms of the other ones that we named ∂̃εR. We are then left with only one relation \[\sum_{i=1}^{h}(\partial^Re + \partial^Le)\phi = 0\]

5.2. Spin Networks as Laplacian eigenvectors

**Theorem 4** The Laplacian operators ∆εRR, ∆εLR are hermitian with respect to the measure dμh, h > 1.
We will give the proof for $\tilde{\Delta}_i \equiv \tilde{\Delta}^{RR}_i$. The proof for a general operator is similar.

Let's consider

$$\int_{A_h} (\varphi \tilde{\Delta}_i \psi - \psi \tilde{\Delta}_i \varphi) d\mu_h$$

(5.4)

where $\varphi, \psi$ are gauge invariant functions. And let's introduce a gauge fixing function $\phi$, which is such that

$$\int_G g^\phi dg = 1$$

(5.5)

where $g^\phi(g_1, \ldots, g_N) = \phi(g g_1 g^{-1}, \ldots, g g_N g^{-1})$. The integral (5.4) can be written as

$$\int_{G^h} (\varphi \tilde{\Delta}_i \psi - \psi \tilde{\Delta}_i \varphi) \phi d g_1 \cdots d g_h$$

(5.6)

Using the invariance of the Haar measure under left multiplication we can integrate by part the right invariant derivatives, this leads to (we take off the $\tilde{}$ for simplicity of the notations):

$$\int_{A_h} (\varphi \Delta_i \psi - \psi \Delta_i \varphi) d\mu = \int_{G^h} \psi \partial_{X_i}^R \phi \partial_{X_j}^R \phi - \varphi \partial_{X_j}^R \psi \partial_{X_i}^R \phi d g_1 \cdots d g_h$$

(5.7)

Let's look at the first term, we can write it as

$$\int_{G^h} d g_1 \cdots d g_h \psi \partial_{X_i}^R \phi \partial_{X_j}^R \phi = \int_{A_h} \psi \left[ \int_{G^h} g^\phi \partial_{X_i}^{R^h} \phi \partial_{X_j}^{R^h} \phi \right] d\mu_h$$

(5.8)

where we have used the definition of the invariant measure (4) and $g^\phi = \phi$. Using the following identity

$$g^\phi = \partial_{Ad(g)^{-1} \cdot X_i} g^\phi,$$

(5.9)

and the invariance of the quadratic differential operator

$$\sum_i \partial_{Ad(g) \cdot X_i}^R \otimes \partial_{Ad(g) \cdot X_i}^R = \sum_i \partial_{X_i}^R \otimes \partial_{X_i}^R$$

(5.10)

one gets

$$\int_{A_h} \psi \left[ \int_{G} d g \partial_{X_i}^R \varphi \partial_{X_i}^R \phi \right] d\mu_h.$$ 

(5.11)

Using that $g^\phi = \varphi$ since $\varphi$ is gauge invariant and that the condition (5.5) implies $\partial_{X_i} \int g^\phi dg = 0$, we conclude that the integral (5.4) is zero.

**Definition 5** Since the operators $\Delta_e$ form a set of commuting hermitian operators on the Hilbert space $L^2(d\mu(\Gamma))$, we can diagonalize them and their eigenvectors form an orthonormal basis of $L^2(d\mu(\Gamma))$. We call these eigenvectors **spin networks**.
It should be clear that these vectors should be considered as generalized $\delta$ normalizable vectors if one of the eigenvalues they carry is part of a continuous spectrum: one should consider them as invariant distributions and not invariant functions.

We obtain a basis of functions which are labelled by the eigenvalues of the Laplacians - the Casimirs of the group - on each edge of the graph $\Gamma$. In other words, if we call $d\rho(\lambda)$ the spectral measure of the Laplacian operator one gets

$$H_\Gamma \equiv L^2(d\mu^{(\Gamma)}) = \bigoplus_e \int d\rho(\lambda_e) \otimes v(I_v(\lambda)), \quad (5.12)$$

where $I_v(\lambda)$ is the space of intertwiners between the representations carried by the edges meeting at the vertex $v$.

In the case of $G = SU(2)$, there is a one-to-one correspondence between the eigenvalues of the Casimir and the irreducible unitary representations. And by the previous reasoning, we have totally reconstructed the usual structure of spin networks with representations labelling the edges of the spin networks if the graph is trivalent. In a more general context, one should be careful that the eigenvalues of the Laplacians do not always completely characterize the representations, there can be a degeneracy where several representations carry the same Laplacians. This is the case, for instance, of the series of discrete representations of $SL(2, \mathbb{R})$, where the degeneracy is 2. This is not the case however for the unitary representations of $SL(2, \mathbb{C})$ which are totally determined by the values of their two Casimirs. We have presented the spin networks in the particular case of rank 1 but as we said in the beginning, all the propositions work the same for a more general group.

### 5.3. Unfolding vertices and $SU(2)$ spin networks

Using the gauge fixing procedure, we can unfold all the vertices of a given graph $\Gamma$. By this, we mean replace each vertex by a (minimal) tree which has only 3-valent vertices. More explicitly, let’s consider a vertex $v$. We can match the edges meeting at $v$ two by two (if their number is odd, we leave one edge on its own) and create a 3-valent vertex for each of these pairs. Then we repeat this process until over.

Let’s call the unfolded graph $\Gamma_0$. As the flowers corresponding to $\Gamma$ and to $\Gamma_0$ are the same, we have $L^2(d\mu^{(\Gamma)}) = L^2(d\mu^{(\Gamma_0)})$. We can then construct spin networks on the graph $\Gamma_0$, by labelling all its edges - both the ones already in $\Gamma$ and the new ones which are inside the vertices of $\Gamma$ - with representations of the group $G$. These spin networks span $L^2(d\mu^{(\Gamma_0)})$ and therefore also $L^2(d\mu^{(\Gamma)})$. That way, we have unfolded the structure of the vertices of the spin networks based on the graph $\Gamma$. We have reduced the problem of characterizing nodes to the case of 3-valent ones. One thing which one should be aware of is that for high rank groups, it happens that the space of trivalent intertwiners is infinite dimensional e.g $sl(3, \mathbb{R})$.

In the case of $G = SU(2)$, the 3-valent nodes are 3-valent intertwiners intertwining between the representations labelling the three edges. These intertwiners are unique to a normalization. So we have fully characterized $SU(2)$ spin networks - both their edges and nodes - by the above unfolding procedure. In the general case, we can have many possible 3-valent intertwiners and their space has to be studied in order to fully characterize the nodes of the spin networks.
6. THE ONE LOOP CASE

In this section, we restrict ourself to rank one groups and we deal with the graph made of a single loop with a single bivalent vertex, which describes the quotient space $G/\text{Ad}(G)$. This case is essentially different from the cases of flowers with higher $h \geq 2$ number of petals for which the quotient space $G^h/\text{Ad}(G)$ can be mapped onto $G^{h-1}$ as described in the previous section. And both the techniques used and the problems encountered differ from the other cases. Nevertheless, this study is important since the characters of the unitary representations are supposed to be orthonormal vectors of $L^2(G/\text{Ad}(G))$, therefore it is interesting as the simplest part of the gauge invariant connection space and it illustrates the problem of the possible non-connectivity of the quotient space and the (super)selection rule issue which comes with it [25].

6.1. $SL(2, \mathbb{C})$

In this section, we consider the case where the group is $SL(2, \mathbb{C})$ and we are interested in describing the quotient $(SL(2, \mathbb{C}))/\text{Ad}(SL(2, \mathbb{C}))$ as defined in section (3.2). Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$, the algebra $\mathbb{C}[SL(2, \mathbb{C})]^{SL(2, \mathbb{C})}$ of polynomials invariant under the adjoint action is generated by $X(g) = (1/2)\text{tr}(g)$, since such polynomials are linear combination of $\text{tr}(g^n)$. $\text{tr}(g^n)$ can be expressed as a polynomial in $X$ due to the relation $g^2 - \text{tr}(g)g + 1 = 0$, more precisely $\text{tr}(g^n) = T_n(X)$ with $T_n$ the Chebichev’s polynomials of the first kind. Therefore the spectrum of the invariant polynomial affine algebra is just $\mathbb{C}$ and the quotient morphism is the trace. Moreover $\text{tr}^{-1}(x), x \neq \pm 1$ is exactly an orbit of a strictly regular element and $\text{tr}^{-1}(\pm 1) = \{ \pm \text{Id} \} \cup \{ \pm \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C} \} \cup \{ \pm \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \mid z \in \mathbb{C} \}$. One can therefore think of $SL(2, \mathbb{C})/\text{Ad}(SL(2, \mathbb{C}))$ as the geometric quotient of $G_1 \cup \pm \text{Id}$. The $G$-invariant measure (Weyl measure) induced by the $SL(2, \mathbb{C})$ Haar measure is given by

$$\mu(f) = \int_{\mathbb{C}} |X^2 - 1|f(X)dX$$

(6.1)

where the integration region is over the complex plane minus the interval $[-1, +1]$ with the usual Lebesgue measure on $\mathbb{C}$.

More explicitly, $SL(2, \mathbb{C})$ has only one Cartan subgroup $H$ which is the set of diagonal matrices $\text{diag}(\lambda, \lambda^{-1}), \lambda \in \mathbb{C}$. The Weyl group is $Z_2$ and $\text{diag}(\lambda, \lambda^{-1})$ is conjugate to $\text{diag}(\lambda^{-1}, \lambda)$. The Weyl integration formula reads:

$$\int_{SL(2, \mathbb{C})} f(g)dg = \int_H dh \left[ \int_{SL(2, \mathbb{C})/H} f(xhx^{-1})dx \right] |\Delta(h)|^2$$

(6.2)

The invariant measure is obtained by removing the redundant integration over $SL(2, \mathbb{C})/H$ and integration solely on $H$ and one finds back the measure (6.1).

The unitary principal series of $SL(2, \mathbb{C})$ is a family of unitary irreducible representations of $SL(2, \mathbb{C})$ indexed by pairs $(j, \rho)$ with $j \in \mathbb{Z}/2$ and $\rho \in \mathbb{R}$. There are realized in $L^2(\mathbb{C})$ and the action $R_{j,\rho}$ of $SL(2, \mathbb{C})$ is given by
\[ R_{j,\rho} \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = |bz + d|^{-2-2i\rho} \left( \frac{bz + d}{|bz + d|} \right)^{2j} f \left( \frac{az + c}{bz + d} \right) \]  

(6.3)

for \( z \in \mathbb{C} \) and \( f \in L^2(\mathbb{C}) \). The characters are

\[ \chi_{j,\rho} \begin{pmatrix} e^{x+i\theta} & 0 \\ 0 & e^{-x-i\theta} \end{pmatrix} = \frac{e^{i\rho x} e^{ij\theta} + e^{-i\rho x} e^{-ij\theta}}{|e^{x+i\theta} - e^{-x-i\theta}|^2}. \]  

(6.4)

Using the measure (6.1) on \( X = (e^{x+i\theta} + e^{-x-i\theta})/2 \) and making a change of variables to \( x, \theta \), it is straightforward to check that

\[ \mu(\chi_{j_1,\rho_1} \chi_{j_2,\rho_2}) = \delta_{j_1,j_2} \delta(\rho_1 - \rho_2) \]  

(6.5)

so that the above characters form an orthonormal basis of the Hilbert space \( L^2(A_1) = L^2(SL(2, \mathbb{C})//Ad(SL(2, \mathbb{C}))) \).

6.2. \( SU(2) \)

In the case of \( SU(2) \), there is again a unique Cartan subgroup \( H \), composed of the diagonal matrices \( h_\theta = \text{diag}(e^{i\theta}, e^{-i\theta}) \), \( \theta \in [-\pi, \pi] \). The Weyl group is \( Z_2 \): \( h_\theta \) and \( h_{-\theta} \) are conjugate. The \( SU(2) \)-invariant measure is

\[ \mu_{SU(2)}(f) = \frac{2}{\pi} \int_{-1}^{1} dX \sqrt{1 - X^2} f(X) = \frac{2}{\pi} \int_{0}^{\pi} d\theta \sin^2 \theta f(\theta) \]  

(6.6)

where \( X = 1/2tr(g) = \cos \theta \).

An orthonormal basis of \( L^2(SU(2)/Ad(SU(2))) \) is given by the characters of the irreducible (finite-dimensional) representation of \( SU(2) \):

\[ \chi_{j}(h_\theta) = \frac{\sin(j + 1)\theta}{\sin(\theta)} \]  

(6.7)

where \( j \) runs over the non-negative integers (twice the spin). For making a change of variable from \( X \) to \( \theta \), it is easy to check that

\[ \mu_{SU(2)}(\chi_j \chi_k) = \delta_{jk} \]  

(6.8)

6.3. \( SL(2, \mathbb{R}) \)

In the case of \( SL(2, \mathbb{R}) \), we have two Cartans subgroups; a compact one, which corresponds to space rotations

\[ H_0 = \left\{ k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \ 0 \leq \theta \leq 2\pi \right\} \]  

(6.9)

and a non-compact one, which corresponds to boosts
\[ H_1 = \pm \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \ t \in \mathbb{R} \right\}. \]

(6.10)

\( W(H_0) \) is trivial but \( W(H_1) = \mathbb{Z}_2 \) and \( a_t \) is conjugate to \( a_{-t} \). A regular element of \( SL(2, \mathbb{R}) \) can be conjugated to \( H_0 \) or to \( H_1 \) and a \( Ad(SL(2, \mathbb{R})) \) invariant function \( f \) will be described by its action on both Cartan subgroups i.e by two functions \( f_0(\theta) \) and \( f_1^\pm(t), t \geq 0 \).

We would like to divide by the volume of \( G/H_0 \) and of \( G/H_1 \). These two volumes are infinite and the ratio of these two volumes is also infinite. So this leads to an ambiguity and we are left with a one-parameter family (up to normalization) of possible \( Ad(SL(2, \mathbb{R})) \) invariant measures

\[ \mu_{SL(2, \mathbb{R})}(f) = \alpha_0 \int_0^{2\pi} d\theta \sin^2 \theta f_0(\theta) + \alpha_1 \int_0^{+\infty} dt \sinh^2 t f_1^\pm(t) \]  

(6.11)

The formal property that we want our measure to satisfy is:

\[ \mu(Gf) = \int_G dg f(g) \]  

(6.12)

where \( Gf \) denotes the averaging over the gauge group \( G \) of a compact supported non invariant function \( f \). The subtle point is that the centralizer of a generic group element under the adjoint action is (conjugated to) either \( H_0 \) or \( H_1 \) depending on the group element. So the averaging should take this into account. Therefore if \( f_0 \) (resp \( f_1 \)) is supported on the space \( G_{(0)} \) (resp \( G_{(1)} \)) of group elements which can be conjugated to \( H_0 \) (resp \( H_1 \)), we define

\[ G_{i}(g = xh_i x_i^{-1}) = \int_{G/H_i} dx_i f(x_i h_i x_i^{-1}) \]  

(6.13)

In this case, we easily prove using (3.8) that (6.12) is satisfied for the unique choice \( \alpha_i = 1 \).

However, this is not the whole story. The natural way to get an invariant measure is to choose a cutoff \( \lambda \) and \( G_\lambda \) a compact subset of \( G \), with \( G_\lambda \to G \) when \( \lambda \) grows to infinity. Then we take the invariant measure to be the limit

\[ \mu(f) = \lim_{\lambda \to \infty} \frac{\int_{G_\lambda} f(g) dg}{\int_{G_\lambda} dg}, \]  

(6.14)

for a \( G \) invariant function \( f \). The resulting measure that it leads to is \( (\alpha_0 = 1, \alpha_1 = 0) \). And this measure gives a zero weight to function with support on \( G_{(1)} \).

The way to reconcile these points of view is the following. One needs to define two Hilbert spaces, one (denoted \( H_0 \)) for the functions with support on \( G_{(0)} \) which is given by the measure \( (\alpha_0 = 1, \alpha_1 = 0) \) and one (denoted \( H_1 \)) for the functions with support on \( G_{(1)} \) which is given by the measure \( (\alpha_0 = 0, \alpha_1 = 1) \). This would take into account the fact that the space of \( Ad(G) \) invariants is disconnected with incommensurable volume of centralizer. Physically, this means that the two sectors cannot communicate i.e we can not find physical operators mapping physical states between the two sectors. This was rigorously shown by Gomberoff and Marolf \[25\] in a similar context but in the language of group averaging and rigging maps.
SL(2, \mathbb{R}) has three series of principal unitary representations: the continuous series $C_s$ labelled by a positive real number $s$ and two discrete series $D_n^\pm$ both labelled by an integer $n \geq 1$ and a sign. The characters of the continuous series are

$$\chi_s(k_\theta) = 0$$

(6.15)

$$\chi_s(\pm a_t) = \frac{\cos st}{\sinh t}$$

(6.16)

and the characters of the discrete series $D_n^\pm$ are

$$\chi_n^\pm(k_\theta) = \pm \frac{e^{\pm i(n-1)\theta}}{2i \sin \theta}$$

(6.17)

$$\chi_n^\pm(a_t) = \frac{e^{-(n-1)|t|}}{2|\sinh t|} \text{ with a factor } (-1)^n \text{ for } -a_t$$

(6.18)

It is clear that the characters of the discrete series (resp. continuous series) are orthonormal with respect to the Hilbert space structure $\mathcal{H}_0$ (resp. $\mathcal{H}_1$). Moreover both characters are eigenvalues of the Laplacian. More explicitly, the Laplacian reads:

$$\Delta = \frac{1}{\sin \theta} \frac{\partial^2}{\partial \theta^2} \sin \theta + \frac{1}{4} \text{ on } \mathcal{H}_0 \quad \Delta = -\frac{1}{\sinh t} \frac{\partial^2}{\partial t^2} \sinh t + \frac{1}{4} \text{ on } \mathcal{H}_1 \text{ for } t \geq 0$$

(6.19)

so that the eigenvalue of $\chi_s$ is $s^2 + 1/4$ and the one corresponding to $\chi_n^\pm$ is $m(1-m)$ with $m = n - 1/2$. Moreover, one could notice that, for a generic measure (6.11) with arbitrary $(\alpha_0, \alpha_1)$, the discrete characters $\chi_n^\pm$ are not orthonormal, which would be in contradiction with the fact that the Laplacian is Hermitian, unless we restrict ourself to the choice $\mathcal{H}_0$ i.e $(\alpha_0 = 1, \alpha_1 = 0)$ which appear to be the only self-consistent choice of measure when taking into account the discrete series.

The characters of the continuous series are fully characterized as distributions which are eigenvectors of the Laplacian, invariant under the Weyl group (residual gauge symmetry) and with full support only on $G(1)$. However, this is not the case for the discrete series. There are several distributions which are both eigenvector of the Laplacian and invariant with support in $G(0)$. The solution of this puzzle lies in the definition of the Laplacian and more particularly in the space of functions on which it is defined. Indeed, among all the invariant distributions, one should choose the ones which are not only a distribution on $G(0) \coprod G(1)$ but on the full group $G$: one asks $\chi_n^\pm$ to be an eigenvalue of the Laplacian as a distribution on $G$. More precisely, in order to satisfy the eigenvalue equation $\int_G \chi(g)\Delta f(g)dg = \lambda \int_G \chi(g)f(g)dg$, one needs to integrate by parts. If $f$ is a compact supported function on $G(0) \coprod G(1)$, all the boundary terms vanishes trivially. However, if $f$ is a compact supported function on $G$, the vanishing of the boundary terms leads to some boundary conditions on $\chi$. The eigen-distributions that can be extended to distributions on $G$ are called regular. Now it is easy to check that such distributions (normalizable with
respect to the scalar product on $\mathcal{H}_0$ are in one-to-one correspondence with unitary representations. This was first shown by Harish-Chandra and was the foundation of his works on harmonic analysis on non-compact group [26].

To sum up, the issue is about the domain of definition of the Laplacian $\Delta$. We make it act on the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ for consistency of the group averaging, but the eigenvalue problems is well-defined for distributions on the whole group $G$ (taking into account the null elements, which are not regular). Nevertheless, we can conclude that the discrete characters $\chi_n$ restricted to $G_{(0)}$ form a basis of $\mathcal{H}_0$ and the continuous characters $\chi_s$ form a basis of $\mathcal{H}_1$. And the tail of $\chi_n$ on $G_{(1)}$ is due to non-trivial boundary conditions in the eigenvalue problem.

The case of one petal graph is quite complicated, this is essentially due to the fact that the space $A_1$ is not connected, since we have excluded all null rotations and that taking them into account is not straightforward. Fortunately, as we shall now see, the situation for higher loop graphs is simpler since the generic centralizer of a point of $G^h$ is $G$ for all elements.

7. THE TWO PETAL FLOWER: EXAMPLES

7.1. $SL(2, \mathbb{C})$

In this section, we consider the case where the group is $SL(2, \mathbb{C})$ and we are interested in describing the quotient $(SL(2, \mathbb{C}))^2//Ad(SL(2, \mathbb{C}))$ as defined in section (3.3.2). Let $(g_1, g_2) \in SL(2, \mathbb{C})^2$ and denote $X_1(g_1, g_2) = (1/2)tr(g_1)$, $X_2(g_1, g_2) = (1/2)tr(g_2)$ and $X_3(g_1, g_2^{-1}) = (1/2)tr(g_1g_2)$. This defines an $Ad(SL(2, \mathbb{C}))$-invariant morphism $\pi : SL(2, \mathbb{C})^2 \to \mathbb{C}^3$. We have the following property

**Proposition 6** $\pi$ gives an isomorphism between the algebra of invariant polynomials $\mathbb{C}[SL(2, \mathbb{C})^2]|SL(2, \mathbb{C})$ and $\mathbb{C}[X_1, X_2, X_3]$.

**Proof:** Let

$$G_2(SL(2, \mathbb{C})) = \{(g_1, g_2) \in SL(2, \mathbb{C})|tr(g_1)^2 \neq 4 \text{ or } tr(g_2)^2 \neq 4, \text{ and } tr([g_1, g_2]_G) \neq 2\}.$$ 

The image of this set by $\pi$ is the complement in $\mathbb{C}^3$ of $\Delta$, where $\Delta$ is the closed subset of $\mathbb{C}^3$ such that the polynomials $X_1^2 - 1$ or $X_2^2 - 1$, and $\Theta(X_1, X_2, X_3) \equiv (X_3 - X_1X_2)^2 - (X_1^2 - 1)(X_2^2 - 1)$ are equal to zero ($[\_\_\_]_G$ denotes the group commutator). This is clear since $tr([g_1, g_2]_G) - 2 = 4\Theta(X_1, X_2, X_3)$. The key point is that this gives an isomorphism between $G_2(SL(2, \mathbb{C}))/SL(2, \mathbb{C})$ and $\mathbb{C}^3 \setminus \Delta$. We can construct explicitly the inverse map, Let $s(X) = (s_1, s_2)$ be defined by:

$$s_1(X) = \begin{pmatrix} X_1 + \sqrt{X_1^2 - 1} & 0 \\ 0 & X_1 - \sqrt{X_1^2 - 1} \end{pmatrix},$$

$$s_2(X) = \begin{pmatrix} X_2 - \frac{x_1x_2-x_3}{\sqrt{X_1^2-1}} & 1 \\ \frac{\Theta(X)}{x_1^2-1} & X_2 + \frac{x_1x_2-x_3}{\sqrt{X_1^2-1}} \end{pmatrix}. \quad (7.1)$$
One should be careful since $s$ needs the definition of a square root, this means that this is a multivalued function on $\mathbb{C}^3 \setminus \Delta$, however any change in the choice of the square root determination is implemented by a gauge transformation, therefore $s$ is well defined as a function valued into $G_2(\text{SL}(2,\mathbb{C}))/\text{SL}(2,\mathbb{C})$. Suppose we define $\tilde{s}(\vec{X})$ the map corresponding to another determination of the square root, i.e the one obtained from $\sqrt{X_1^2 - 1} \to -\sqrt{X_1^2 - 1}$ we have that

$$\tilde{s}(\vec{X}) = \text{Ad} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \cdot s(\vec{X}).$$  \hspace{1cm} (7.3)

It is easy to see that $\pi \circ s$ is the identity mapping on $\mathbb{C}^3 \setminus \Delta$. It is also true that $s \circ \pi$ is the identity mapping on $G_2(\text{SL}(2,\mathbb{C}))/\text{SL}(2,\mathbb{C})$. First, given $(g_1, g_2) \in G_2(\text{SL}(2,\mathbb{C}))$ we can diagonalize $g_1$ since it is regular. This does not fix completely the action of the gauge group since one can still act by a diagonal gauge transformation and a Weyl transformation (i.e $g_1 \to g_1^{-1}$). Any diagonal transformation $\text{diag}(\lambda, \lambda^{-1})$ is acting on $g_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & \lambda^2b \\ \lambda^{-2}c & d \end{pmatrix}$. Now, $\text{tr}(g_1g_2g_1^{-1}g_2^{-1}) - 2 = (\lambda - \lambda^{-1})^2bc$. The condition $\Theta \neq 0$ translates into $bc \neq 0$, so that one can fix the residual action by asking $b = 1$.

\textbf{Proposition 7} \hspace{1cm} The invariant measure $\mu$ defined in definition (\ref{def:inv-measure}) is simply the Lebesgue measure in $\mathbb{C}^3$ when translated in terms of $X_1, X_2, X_3$. More precisely, let $F$ be a function on $\mathbb{C}^3$, $\pi^* F$ is an invariant function and

$$\int_{(\text{SL}(2,\mathbb{C}))^2/\text{SL}(2,\mathbb{C})} \pi^* F(g_1, g_2) d\mu(g_1, g_2) = \int_{\mathbb{C}^3} F(\vec{X}) d^2X_1 d^2X_2 d^2X_3.$$ \hspace{1cm} (7.4)

\textbf{Proof:} Let us recall that the Haar measure for $\text{SL}(2,\mathbb{C})$ is defined as $dg = d^2ad^2bd^2cd^2d$ $\delta^2(ad - bc - 1)$ if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let

$$y = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}, \quad h = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad g = yhy^{-1}.$$ \hspace{1cm} (7.5)

The measure on $A_2$ is defined by $d\mu = dg$ it is easy to see that

$$dg = |\lambda - \lambda^{-1}|^2 d^2(\lambda + \lambda^{-1}) d^2ad^2d.$$ \hspace{1cm} (7.6)

Moreover $X_1 = \lambda + \lambda^{-1}$, $X_2 = a + d$ and $X_3 = \lambda a + \lambda^{-1}d$ thus $dg = d^2X_1 d^2X_2 d^2X_3$.

Let’s define the invariant functionals

$$\Phi_{j,\rho}(g_1, g_2) = \chi_{j_1,\rho_1}(g_1)\chi_{j_2,\rho_2}(g_2)\chi_{j_3,\rho_3}(g_1g_2)$$ \hspace{1cm} (7.7)

An explicit computation gives

$$\int \Phi_{j,\rho} d\mu(g_1, g_2) = \prod_{i=1}^3 \delta(\rho_i) \delta_{j_i}$$ \hspace{1cm} (7.8)
7.2. \(SU(2)\)

We can deduce the \(SU(2)\) case from the previous formalism. We have the constraint
\[ I_3 \equiv \{ X_i \in [1, 1], \Theta(X) = (X_3 - X_1 X_2)^2 - (X_1^2 - 1)(X_2^2 - 1) < 0 \} \]
and the invariant measure is
\[ \int_{I_3} d\vec{X}. \] (7.10)

Writing \(X = \cos \theta\), we can re-express the above constraint in terms of \(\theta_1, \theta_2, \theta_3\) ∈ \([0, \pi]\):
\[ \cos(\theta_1 + \theta_2) \leq \cos \theta_3 \leq \cos(\theta_1 - \theta_2) \] (7.11)

which is simply the constraint arising when multiplying two elements of \(SU(2)\) or equivalently summing two vectors in a spherical space. Indeed two group elements \(g_1, g_2\) in \(SU(2) \sim S^3\) determine a triangle in \(S^3\) with vertices \(1, g_1, g_2\). The invariant geometry of this triangle is determined by the three lengths which are \(\theta_1, \theta_2, \theta_3\). In these variables, the measure is \(\sin \theta_1 \sin \theta_2 \sin \theta_3 d\theta_1 d\theta_2 d\theta_3\) and the domain of integration is
\[ \theta_1 + \theta_2 \leq \theta_3 \text{ and cyclic perm.} \] (7.12)
\[ \theta_1 + \theta_2 + \theta_3 \leq 2\pi. \] (7.13)

One can also express the invariant geometry of the triangle in terms of two edges: their lengths \(\theta_1, \theta_2\) and the angle \(\theta_3\) they form. This angle is determined by the edge lengths by \(\cos \theta_3 = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \cos \theta_3\). In these geometric variables, the condition on the variables reads \(\Theta = -\sin^2 \theta_1 \sin^2 \theta_2 \sin^2 \theta_3 \neq 0\) which means that we exclude degenerate triangles. In these new variables, the measure is \(\sin^2 \theta_1 \sin^2 \theta_2 d\theta_1 d\theta_2 \sin \theta_3 d\theta_3\). Now one can easily check that
\[ \int_{I_3} \chi_{j_1}(X_1)\chi_{j_2}(X_2)\chi_{j_3}(X_3)d\vec{X} = \delta_{j_1,j_2,j_3} \frac{1}{d_{j_3}}. \] (7.14)
as expected.

Let us note that one get the measure (7.10) directly by a gauge fixing procedure without having to appeal to the general theorem 1. Let \(g_i = \left(\begin{array}{cc} a_i & b_i \\ -\bar{b}_i & \bar{a}_i \end{array}\right), i = 1, 2\). The gauge conditions we want to impose are \(b_1(g_1) = 0\) and \(Im(b_2)(g_2) = 0\), \(Im\) denotes the imaginary part. The Faddeev-Popov determinant is the determinant of the \(3 \times 3\) matrix 
\[ (Re(b_1)([X_i, g_1]); Im(b_1)([X_i, g_1]); Im(b_2)([X_i, g_2])); \]
where \(X_i, i = 1, \cdots, 3\) is a basis of \(su(2)\). This determinant is proportional to \((a_1 - (a_1)^{-1})^2 b_2\), it should multiply the gauge fixed measure \(dg_1 dg_2 \delta^2(b_1) \delta(Im(b_2))\), a direct computation leads to the results 7.10.
7.3. \(SL(2, \mathbb{R})\)

In the case of \(SL(2, \mathbb{R})\), the constraint reads \(\vec{X}(g_1, g_2) \in J_3\) where
\[
J_3 \equiv \{X_i \in \mathbb{R}, \Theta(\vec{X}) \neq 0, (X_1^2 \neq 1 \text{ or } X_2^2 \neq 1)\} \text{ and } \vec{X} \neq I_3\}.
\tag{7.15}
\]
The invariant measure is given by
\[
\int_{J_3} d\vec{X}.
\tag{7.16}
\]
This result can be obtained using both previous methods, gauge fixing or application of the general formulas in the context of \(SL(2, \mathbb{R})\). It is interesting to note that \(J_3\) corresponds to a real section of \(\mathbb{C}^3\) which is complementary to \(I_3\) describing \(SU(2)\). As in the case of \(SU(2)\), one can give a geometrical interpretation of the configuration space \(J_3\) and the non-degeneracy condition. This will lead to a nice understanding of the singularity properties.

It is well known that \(SL(2, \mathbb{R})\) is isomorphic to \(AdS_3\), the Anti-de-Sitter space in three dimensions, which can be described as an hyperboloid in flat four dimension space, \(AdS_3 = \{-(X_0)^2 + (X_1)^2 + (X_2)^2 - (X_3)^2 = -1\}\), the isomorphism being
\[
g(X) = \begin{pmatrix}
X_0 + X_1 & X_2 + X_3 \\
X_2 - X_3 & X_0 - X_1
\end{pmatrix}.
\tag{7.17}
\]
\(AdS_3\) is a Lorentzian space and \(SO(2, 2)\) is its isometry group. Then the space of couples of group elements \((g_1, g_2)\) corresponds to the space of geodesic triangles in \(AdS_3\) with one vertex fixed to be the identity. The adjoint action of \(SL(2, \mathbb{R})\) on \((g_1, g_2)\) translates into the action of the subgroup of \(SO(2, 2)\) which fixes the identity, hence into the action of the Lorentz group \(SO(2, 1)\) which rotates the triangles. So the space of orbits is the space which describes the intrinsic geometry of Anti-de-Sitter triangles. Such triangles can be of four types: they can be space-like, time-like, null or degenerate (meaning that the three vertices of the triangle belong to the same geodesic), depending on whether they lay in a space-like, time-like or null plane. The edges of the triangles can also be of four types: they can be time-like, space-like, null or degenerate (meaning that the two vertices of the edge coincide). Unlike the \(SU(2)\) case, the invariant geometry cannot be fully characterized by the edge lengths since the length (more precisely the square length) is zero for both a null edge and a degenerate edge. However, the following proposition shows that if we restrict the space of triangles to triangles which satisfy the condition \(\Theta \neq 0\) then the geometry of the triangle is uniquely determined by the lengths of its edges. The geometrical meaning of this condition is the following:

**Proposition 8** The condition \(\Theta(g_1, g_2) = 0\) is equivalent to the condition that the \(AdS\) triangle \((1, g_1, g_2)\) is either null or degenerate. Moreover \(\Theta(g_1, g_2) < 0\) (resp. \(\Theta(g_1, g_2) > 0\)) iff the \(AdS\) triangle \((1, g_1, g_2)\) is spacelike (resp. timelike).

In order to prove this proposition, we need to do some \(AdS\) geometry. It is convenient to consider \(AdS\) space as embedded in the projective space \(\mathbb{R}^+P^3\), which is the space of half
lines in $\mathbb{R}^4$: $AdS_3 = \{(X_0 : X_1 : X_2 : X_3) \in \mathbb{R}^+P^3\} = -(X_0)^2+(X_1)^2+(X_2)^2-(X_3)^2 < 0\}$. The advantages of such a representation of the $AdS$ space is to simplify the geodesic geometry of $AdS$. First the geodesics of $AdS$ are the straight lines of $\mathbb{R}^+P^3$. Moreover the geodesic planes of $AdS_3$ are the intersection of $\mathbb{R}^+P^3$ planes with $AdS$, they are therefore given by linear equations $P_{(y_0,y_1,y_2,y_3)} = \{(X_0 : X_1 : X_2 : X_3) \in \mathbb{R}^+P^3\} = 0$ (the indices are raised using a lorentzian $(-,+,+,−)$ metric). Thus the geodesic hyperplanes of $AdS$ are in one to one correspondence with points of $\mathbb{R}^+P^3$. Geometrically this means that all the geodesics orthogonal to a given plane meet in one point. If the plane $P_Y$ is spacelike then $Y \cdot Y < 0$ and the refocusing point is in $AdS$. This corresponds to the attractive nature of negative cosmological constant where all time-like geodesics re-focus in a finite proper time. If the plane $P_Y$ is time-like then $Y \cdot Y > 0$. And if the plane $P_Y$ is null then $Y \cdot Y = 0$. Moreover, in this latter case, $P_Y$ is tangent to the quadric $Y \cdot Y = 0$.

Next, we identify $\mathbb{R}^+P^3$ with the space of $2 \times 2$ matrices modulo multiplication by a positive scalar using $Y \in \mathbb{R}^+P^3 \rightarrow g(Y)$ as in (7.17). Now let’s consider the triangle $(1,g_1,g_2)$ and suppose that it is non-degenerate i-e $[g_1,g_2] = g_1g_2 - g_2g_1 \neq 0$. Let’s denote by $Y(g_1,g_2)$ the $\mathbb{R}^+P^3$ element satisfying $g(Y(g_1,g_2)) = [g_1,g_2]$. It is clear that $P_{Y(g_1,g_2)}$ is the plane of the triangle $(1,g_1,g_2)$ since $tr([g_1,g_2]) = 0$ if $g = 1, g_1$ or $g_2$. Then a straightforward computation shows that

$$Y(g_1,g_2) \cdot Y(g_1,g_2) \equiv det([g_1,g_2]) = 8\Theta(g_1,g_2),$$

(7.18)

and leads to the conclusion of the proposition [8].

This proposition tells us that the space $J_3$ is the space of non-degenerate and non-null triangles. This space is disconnected, and has two disconnected regions depending on whether the normal to the triangle is timelike or spacelike. There is no natural distinction between past and future in $AdS_3$ since it is periodic in time and a timelike geodesic will come back to the initial normal surface. Therefore we can define two Hilbert space structures, one for the functions with support on spacelike triangles and one for the functions with support on timelike triangles, with the scalar product defined by the measure $\mathcal{L}$. The situation is however drastically different from the one-loop case where the invariant space was also disconnected but in that case the centralizer group was drastically different in both regions. In the present two petals case, we see that the centralizer i-e the group which fixes a given triangle, is trivial in both sectors. This means that there is no superselection rule avoiding to construct invariant operators mapping one sector to another. Indeed one realizes that, even when we extend the space $J_3$ to the space $\tilde{J}_3$ of all non-degenerate triangles (allowing null cases), the centralizer of any triangle of $\tilde{J}_3$ is still trivial. Therefore we can extend the definition of the measure to $\tilde{J}_3$ which is connected and there exists one unique invariant measure on this space. In other words, we see that if $\phi(g_1,g_2)$ is a function with compact support on $\tilde{J}_3$ then $G\phi(g_1,g_2) = \int \phi(gg_1g^{-1},gg_2g^{-1})dg$ is well defined for all $(g_1,g_2) \in \tilde{J}_3$. This means that the invariant distributions on $J_3$ obtained by group averaging can be extended to invariant distributions on $\tilde{J}_3$. We expect the spin network functionals to be of this type and therefore the Hilbert space structure to be uniquely fixed in that case.

8. THE HILBERT SPACE OF SPIN NETWORKS
8.1. The Compact Group Case: The Ashtekar-Lewandowski construction

The Ashtekar-Lewandowski approach consists in the use projective techniques in the compact group case to define the space of generalized connections, a space of continuous functions upon it (cylindrical functions) and a measure called the Ashtekar-Lewandowski (AL) measure \[20\] which endow this space with a Natural and diffeomorphism invariant Hilbert space structure. For a recent and complete review of this approach, one can look at \[9\].

First, we define a space of gauge invariant “connections” for each graph \(\Gamma\), embedded in a spacelike manifold.

\[
A_{\Gamma} = G^{\otimes E} / G^{\otimes V} = \{ [(g_{e_1}, \ldots, g_{e_E})]_{G^{\otimes V}} \} = \{ (k_{s(e_i)}^{-1} g_{e_i} k_{t(e_i)}, i = 1 \ldots E), k_v \in G \} \quad (8.1)
\]

We define a partial order \(\prec\) over the set of graphs: \(\Gamma_1 \prec \Gamma_2\) iff \(\Gamma_1\) can be obtained from \(\Gamma_2\) by removing edges and bivalent vertices. We then define projections \(p_{\Gamma_2\Gamma_1} : A_{\Gamma_2} \rightarrow A_{\Gamma_1}\) for \(\Gamma_1 \prec \Gamma_2\), by removing the extra edges and contracting the extra bivalents vertices:

\[
\begin{align*}
\text{removing the edge } i & \quad (g_1, \ldots, g_i, \ldots, g_E) \rightarrow (g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_E) \\
\text{bivalent vertex between 1 and 2} & \quad (g_1, g_2, \ldots, g_E) \rightarrow (g_1 g_2^\epsilon, \ldots, g_E) \\
\end{align*}
\]

with \(\epsilon = \pm 1\) depending on the relative orientation on \(g_1\) and \(g_2\).

Let’s illustrate these rules with the example of the reduction of the \(\Theta\) graph to a single loop:

\[
\begin{align*}
(1,2,3) & \sim (h^{-1}g_1k, h^{-1}g_2k, h^{-1}g_3k) \rightarrow (1,3) & \sim (h^{-1}g_1k, h^{-1}g_3k) \\
(1,3) & \sim (h^{-1}g_1k, h^{-1}g_3k) \rightarrow G_1 = g_1g_3^{-1} \sim h^{-1}G_1h
\end{align*}
\]

Then, we can define the projective limit \(\tilde{A}\) as the set of families of elements of \(A_{\Gamma}\) consistent with the projections:

\[
\tilde{A} = \left\{ (a_{\Gamma})_{\Gamma\text{graph}} \in \times_{\Gamma} A_{\Gamma} / \forall \Gamma_{1,2}, \Gamma_1 \prec \Gamma_2 \Rightarrow p_{\Gamma_2\Gamma_1} a_{\Gamma_2} = a_{\Gamma_1} \right\} \quad (8.4)
\]

In the case of a compact group \(G\), the spaces \(A_{\Gamma}\) are topological, compact and Hausdorff and the projections are continuous, therefore \(\tilde{A}\) with the Tychonov topology (product topology) is compact and Haussdorff. We can now construct continuous function on \(\tilde{A}\). We start by defining the spaces:

\[
C^0(A_{\Gamma}) = \{ f \in \mathcal{F}(A_{\Gamma}, \mathbb{C}), f \text{ continuous} \} \quad (8.5)
\]
The projections $p$ induce some injections between the spaces of functions $C^0(A_{\Gamma_1})$ and $C^0(A_{\Gamma_2})$ for $\Gamma_1 \prec \Gamma_2$:

\[ i_{\Gamma_1 \Gamma_2} : C^0(A_{\Gamma_1}) \rightarrow C^0(A_{\Gamma_2}) \]  

(8.6) 

\[ \phi(\{g_e\}_e \in \Gamma_1) \rightarrow \tilde{\phi}(\{g_e\}_e \in \Gamma_2) = \phi(p_{\Gamma_2 \Gamma_1}\{g_e\}_e \in \Gamma_2) \]  

(8.7) 

We define the following equivalence relation:

\[ f_{\Gamma_1} \in C^0(A_{\Gamma_1}) \sim f_{\Gamma_2} \in C^0(A_{\Gamma_2}) \iff \exists \Gamma_3 \succ \Gamma_1, \Gamma_2, i_{\Gamma_1 \Gamma_3} f_{\Gamma_1} = i_{\Gamma_2 \Gamma_3} f_{\Gamma_2} \iff \forall \Gamma_3 \succ \Gamma_1, \Gamma_2, i_{\Gamma_1 \Gamma_3} f_{\Gamma_1} = i_{\Gamma_2 \Gamma_3} f_{\Gamma_2} \]  

(8.8) 

This allows us to define the space of Cylindrical functions:

\[ Cyl(\overline{A}) = \bigcup_{\Gamma} C^0(A_{\Gamma}) / \sim \]  

(8.9) 

We divide by the previous equivalence relation in order to remove the redundancies due to the existence of the injections. On $Cyl(\overline{A})$, we can define a norm

\[ \| [f_{\Gamma}] \| = \sup_{x_{\Gamma} \in A_{\Gamma}} |f_{\Gamma}(x_{\Gamma})| \]  

(8.10) 

Then the completed space is an abelian C$^*$ algebra, to which we can apply the Gelfand-Naimark theorem. It states that it is the algebra of continuous functions on a certain compact Haussdorff space called the Gelfand spectrum of the C$^*$ algebra. In [20], Ashtekar and Lewandowski prove that its Gelfand spectrum is simply $\overline{A}$, i.e., that we have the following isomorphism:

\[ Cyl(\overline{A}) \approx C^0(\overline{A}) \]  

(8.11) 

Choosing measures $d\mu(\Gamma)$ -the Haar measure- on the spaces of discrete connections $A_{\Gamma}$ and checking that they are consistent with the injections

\[ \forall \Gamma_1 \prec \Gamma_2, i_{\Gamma_1 \Gamma_2} d\mu(\Gamma_2) = d\mu(\Gamma_1) \]  

(8.12) 

we can define a measure $\overline{d\mu}$ -the Ashtekar-Lewandowski measure- on $\overline{A}$ by considering their projective limit. And our final Hilbert space will be $\mathcal{H}_{cyl} = L^2(\overline{A}, \overline{d\mu})$.

### 8.2. An Alternative: the GNS construction

An elegant way of constructing the Hilbert space $\mathcal{H}_{cyl}$ is using the GNS (Gelfand-Naimark-Segal) construction [24,27]. One considers the algebra $\mathcal{A}$ of all cylindrical functions $f_\Gamma$ (on all graphs $\Gamma$) with the normal multiplication law between functions. One defines the norm sup on this space as in the previous paragraph:

\[ \| f_\Gamma \| = \sup_{A_{\Gamma}} |f_\Gamma| \]  

(8.13)
One can then complete $\mathcal{A}$ to a $C^*$ algebra $\bar{\mathcal{A}}$. On $\bar{\mathcal{A}}$, we define a state $\omega$ - a positive linear form - which is simply the integration:

$$\omega(f_{\Gamma}) = \int_{A_{\Gamma}} d\mu^{(\Gamma)} f_{\Gamma} = \int_{SU(2)^E} dg_1 \ldots dg_E f_{\Gamma}(g_1, \ldots, g_E)$$  \hspace{1cm} (8.14)

$\omega$ induced an inner product $\langle f_{\Gamma_1}| f_{\Gamma_2} \rangle = \omega(f_{\Gamma_1}^* f_{\Gamma_2})$ remembering that the product of the two cylindrical functions is a cylindrical function based on any graph bigger than both $\Gamma_1$ and $\Gamma_2$. We then define the Gelfand ideal

$$\mathcal{I} = \{a \in \bar{\mathcal{A}}| \omega(a^* a) = 0\}$$  \hspace{1cm} (8.15)

We get a positive definite scalar product on the space $\mathcal{H}_{\text{gns}} = \bar{\mathcal{A}}/\mathcal{I}$. And we get the physical Hilbert space by completing this space to $\overline{\mathcal{H}}_{\text{gns}}$. It is straightforward to check that the equivalence relation $\sim$ is the same as defined by $\mathcal{I}$ so that $\overline{\mathcal{H}}_{\text{gns}} = \mathcal{H}_{\text{cyl}}$.

8.3. The Non-Compact Group Case

Let's now assume the group $G$ is non-compact. The obstacle to applying the AL construction is the non-compactness of the $A_{\Gamma}$ spaces. There is no problem defining the projections and injections. However, the space $\bar{\mathcal{A}}$ is non-compact and therefore we can not obtain it as
Gelfand spectrum. Moreover, we can not define a norm \( \|f_\Gamma\| \) on the spaces of continuous functions \( C^0(A_\Gamma) \) so that \( Cyl(\Lambda) \) does not have any norm and can not be completed into a \( C^* \) algebra. And finally, the family of measures \( d\mu^{(\Gamma)} \) is not consistent with the partial order on the set of graphs. To save some results of the AL approach, one could try to compactify the spaces \( A_\Gamma \) or to impose some cut-off on the group. Nevertheless, this makes it hard to deal with the gauge invariance. One could also change the definition of \( C^0(A_\Gamma) \) by taking the bounded continuous functions in order to define a norm on these spaces. Nevertheless, it is not clear what would be its Gelfand spectrum.

One promising approach would be to use the fact that \( A_\Gamma \) is an algebraic space. It can therefore be recovered not as a Gelfand spectrum but as an algebraic spectrum of the affine algebra \( P(A_\Gamma) \) of polynomial function. The problem is that the union of all such affine algebras modulo \( \sim \) in no longer finitely generated so the usual theorems of algebraic geometry can not applied and it is not clear if one can define an algebraic dual of that space. But we still think that this road is worth pursuing.

Here, we choose to concentrate on defining a Hilbert space -the Hilbert space of spin networks- and we don’t tackle the problem of constructing it as a \( L^2 \) space. Our construction will be based on the results obtained from the GNS approach; in particular, we won’t need the projections/injections structure. The drawback of this approach is that we don’t construct the space of generalized connections \( \tilde{A} \). So we can not interpret our Hilbert space as a \( L^2 \) space: we lose some aspects of the “wave function” interpretation. But for all practical purpose the Hilbert space structure is all of what we need.

So, what is the structure we are left with? In the non-compact group case the trivial representation \( j = 0 \) is not a \( L^2 \) representation. Any function not depending on a group element is clearly not normalizable. In other words the trivial representation doesn’t appear in the decomposition of \( L^2(A_\Gamma) \). This mean that we have built directly the spaces \( \tilde{H} \) defined in the previous paragraph. And we build the configuration space as a direct sum of these spaces:

\[
H_{\text{config}} = \bigoplus_{\Gamma \in \tilde{G}} \tilde{H}_\Gamma
\]

(8.17)

There is a possible normalization ambiguity in the above summation. A priori, we are free to normalize the different \( H_\Gamma \) spaces as we wish. This relative normalization of the Hilbert spaces can be traced down to an ambiguity in the definition of the Haar measure used to define the measure \( d\mu^{(\Gamma)} \) of each Hilbert space. In the compact case, we fix these measures to be probability measures \( \mathbb{P} \) and we normalize the Haar measure such that the group gets a unit volume. This makes the measures consistent with the projection structure of the Ashtekar-Lewandowski construction. In the non-compact group case, it is impossible to define such a normalization. However, looking at the way the Haar measure comes into the definition of (3) the measures over different graphs, it is natural to require that the Haar measure be normalized the same way for all measures. More precisely, if we take an integrable function over \( G^{(n+1)}/Ad(G) \), we can integrate out one of its variable using the Haar measure, and we would get an integrable function over \( G^n/Ad(G) \). Then, it is natural to require that the integrals of the two functions be equal.
This argument fixes the Haar measure up to a constant. And if we rescale the Haar measure by a factor $\alpha$, then the measures $d\mu_n$ are to be scaled by $\alpha^{(n-1)}$. And we can think of the normalization of the Haar measure as the choice of a scale in our physical theory.

Now the space $H_{\text{config}}$ defined in (8.17) doesn’t seem to be a $L^2$ space. Nevertheless, it carries some Fock space structure. In that frame, the projection/injection structure of the AL approach would be replaced by creation and annihilation operators. These would act like isometries between the different Hilbert spaces $\tilde{H}_{\Gamma} = L^2(G^E/G^V) = L^2(G^{h\Gamma}/Ad(G))$ and could fix the normalization ambiguity. More precisely, let’s consider an infinite graph $\Gamma_\infty$ i.e a sequence of graphs $(\Gamma_i)_{i \in \mathbb{N}}, \Gamma_i \in \tilde{G}$ such that $\Gamma_i \prec \Gamma_{i+1}$ and the inclusion is strict. Then the space

$$F_{\Gamma_\infty} = \bigoplus_i \tilde{H}_{\Gamma_i}, \quad (8.18)$$

looks like a Fock space where the addition of a loop would be a creation operator.

The difficulty of endowing $F_{\Gamma_\infty}$ of a Fock space structure comes from the residual $Ad(G)$ non-compact gauge symmetry. There exists a natural gauge fixing through the possibility of erasing this remaining symmetry by considering cylindrical functions which are gauge invariant but at a single vertex of the graph. Indeed, following the gauge fixing procedure described in section 8.3 based at the point $A$, the space of such graph connections is simply $G^{h\Gamma}$ and the corresponding Hilbert space of states is $L^2(G^{h\Gamma}, dg \otimes h\Gamma)$. It is then possible to pile these spaces into a Fock space of states $F$ by carefully summing over graphs. The connection states can be seen as a set of loops whose base point is $A$ (flower around the vertex $A$). The creation and annihilation operators acting as usually to go from $L^2(G^n)$ to $L^2(G^{n \pm 1})$ then create or destroy a loop from $A$. Thus, from this point of view, $F$ represents the fluctuations of the connection around the point $A$. Then, what about the gauge invariance at the point $A$? Imposing it directly on $F$ leads to divergence problems. Nevertheless, instead of imposing gauge invariance, we could place ourself at the point $A$ and ignore the gauge invariance but instead impose that the considered states transform nicely under $G$ and belong to a given representation of the group $G$. However this means introducing by hand in the theory an observer at the point $A$, represented by the chosen representation. And in the present work, we prefer to tackle the issue of considering fully gauge invariant functionals and study the sum of the spaces $L^2(G^{h\Gamma}/Ad(G))$.

### 8.4. Towards a Fock space for the space of connections

We are interested by gluing together $L^2(G^n/Ad(G))$ spaces. An useful analogy is interpreting these spaces as state spaces of particles living on the group $G$. Indeed, $L^2(G^n, (dg)^n) - dg$ is the Haar measure on $G$ - is the space corresponding to a free particle living on the group $G^n$ or equivalently $n$ free particles living on the group $G$. Its action evaluated on a function $g(t) : \mathbb{R} \to G^n$ is

$$S_{\text{free}} = \frac{1}{2} \int dt \text{Tr} \left((g^{-1} \partial_t g)^2\right) \quad (8.19)$$

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One can check this action is invariant under (constant) left and right multiplication in \( G^n \). We can do the Hamiltonian analysis of this system and the phase space is the tangent bundle of the group \( G \). The equation of motion is:

\[
\partial_t (g^{-1} \partial_t g) = 0 \tag{8.20}
\]

We choose \( \pi^{(l)} = g^{-1} \partial_t g \) as momentum (instead of the canonical momentum), it is the Noether charge associated to the left invariance. The solutions are then parameterized as

\[
g(t, g_0) = g_0 \exp(\pi^{(l)} t) \tag{8.21}
\]

We could also choose the right momentum defined by \( \pi^{(r)} = -\partial_t g g^{-1} \) and then the solutions would be

\[
g(t, g_0) = \exp(-\pi^{(r)} t) g_0 \tag{8.22}
\]

which are the geodesics.

The Poisson bracket reads

\[
\{ g, \dot{g} \} = 0 \quad \{ g, \pi^{(l)} \} = X g \quad \{ \pi^{(l)}, \pi^{(l)} \} = X[Y, X] \tag{8.23}
\]

where \( g, \dot{g} \) are group elements, \( X, Y \) are in the Lie algebra, and \( \pi^{(l)} = \text{Tr}(X \pi^{(l)}) \) is the component of \( \pi^{(l)} \) in the \( X \) direction.

One can create a Fock space for the free particles states

\[
\mathcal{F} = \bigoplus_{n \geq 0} L^2(G^n) \tag{8.24}
\]

We can construct creation operators \( a^\dagger_\varphi \) and annihilation operators \( a_\varphi \) which are adding or removing a one particle state to a (symmetrized) \( n \) particle state - let’s call it \( \psi \):

\[
(a_\varphi \psi)(g_1, g_2, \ldots, g_{n-1}) = \int dg_n \psi(g_1, g_2, \ldots, g_n) \varphi(g_n) \tag{8.25}
\]

\[
(a^\dagger_\varphi \psi)(g_1, g_2, \ldots, g_{n+1}) = \sum_i \psi(g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n) \varphi(g_i) \tag{8.26}
\]

We can also write a (free) field theory corresponding to this Fock space. Indeed, Let us define a field operator \( \Phi(g) = a_\delta \), \( \Phi^\dagger(g) = a^\dagger_\delta \), where \( \delta \) denote the Dirac delta function supported at \( g \). Then, the action of the total impulsion operator on the Hilbert space of \( N \) particles can be written in terms of the field operators \( \sum_{i=1}^{N} \pi^{(l/r)}_{X} = \int dg \Phi^\dagger(g)(-i\nabla^{(l/r)}_{X})\Phi(g) \), where \( \nabla^{(l/r)}_{X} \) denote the left or right invariant derivative operator in the direction of \( X \). In the same way the Hamiltonian operator can be written as.
H = - \int_G dg \Phi^\dagger(g) \Delta \Phi(g), \quad (8.27)

and the action governing the quantization and the dynamic of the field is expressed in term of a space time field \( \Phi(t, g) \):

\[
S[\Phi(t, g)] = \int_{\mathbb{R} \times G} dtdg \Phi^\dagger(g) \left( i \frac{\partial}{\partial t} + \Delta \right) \Phi(g).
\]

We now wish to follow the same steps for gauged particles i.e in the case that we gauge the global \( Ad(G) \) symmetry. This can be achieved by introducing a gauge fields \( A \) living in the Lie algebra \( \mathcal{G} \), and the action reads (we have slightly modified the action given in [28]):

\[
S_{\text{gauged}}[g(t) \in G^n, A] = - \frac{1}{2} \int dt \text{Tr} \left( (g^{-1} \partial_t g)^2 \right) + \int dt \text{Tr} \left( (g \partial_t g^{-1}) A + A(g^{-1} \partial_t g) + gAg^{-1} A - A^2 \right) \tag{8.29}
\]

This action is invariant under the following \( Ad(G) \) gauge invariance for arbitrary \( G \)-valued \( h(t) \):

\[
\begin{cases}
  g \rightarrow hgh^{-1} \\
  A \rightarrow hAh^{-1} + h\partial_t h^{-1}
\end{cases} \tag{8.30}
\]

The space of states of our system will be \( L^2(G^n/Ad(G)) \). For these gauged particles, we would like to do the same thing as for the free particles i.e write down creation and annihilation operators and a corresponding field theory. The problem is the change of symmetry: the symmetry in \( L^2(G^n/Ad(G)) \) is a global symmetry \( Ad(G) \) on the system of \( n \) particles and it is hard to have a Fock space interpretation. An analogy would be to study a system of \( N \) particles in space-time which would be invariant under global Poincaré transformations.

The easiest way to write creation and annihilation operators would be through gauge fixing. Starting from a graph \( \Gamma \) and going through the gauge fixing procedure, we have seen that the space \( L^2(A\Gamma) \) is naturally isomorphic to \( L^2(G^n) \). Therefore the space associated with an infinite graph is similar Fock space. And in this context creation and annihilation operators are adding or removing a loop to the graph. We feel that it will be interesting to have a deeper understanding of this ideas and of the field theory behind this. Note that the action of the field theory behind the gauged particle is obtain by introducing a gauge field \( A(t) \) and a term to the action (8.28):

\[
\int dtdg A(t)(\nabla^{(l)} - \nabla^{(r)}) \Phi(g, t). \tag{8.31}
\]

**CONCLUSION**

In this paper, we have defined the notion of Spin network states for non-compact reductive groups. We have shown how to construct the quotient space of graph connections as
the algebraic dual of a polynomial algebra. We have also constructed, by a careful gauge fixing procedure, a canonical measure on this space which turned out to be independent of any gauge fixing choices. This measure defines a Hilbert space structure for each graph, and spin networks states are defined as generalized eigenvectors of invariant, hermitic differential operators. We have explicitly realized all these ideas in the context of SL(2, R) and SL(2, C) by a direct analysis of the quotient space and measure in the simplest cases. Finally we have discussed the nature of the full Hilbert space based on all graphs and we have shown that a natural Fock structure appears in this context.

The work we have done is the first step toward a full comprehension of non-compact spin networks, i-e identical to the one we have for the compact ones. As we have stressed in our paper, an understanding of the full Hilbert space as an $L^2$ space is still missing. We expect that this should come together with an interpretation of the full space of gauged connections as an algebraic dual. Also, a more detailed and explicit study of the space of spin networks would be interesting to pursue in order to reach a deeper understanding of their analytic properties, in the spirit of the work of Harish-Chandra on characters of non-compact groups. Finally, we feel that the Fock space structure which is emerging in our construction is something important that should be put on firmer basis. Nevertheless, this work opens the possibility to study the Hilbert spaces of non-compact spin networks that arises in Lorentzian formulations of gravity and allows us to discuss the spectra of geometrical operators in this context [12].

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