Extended Wronskian formula for solutions to the Korteweg-deVries equation

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Abstract. A matrix extension is presented for constructing a much broader class of exact solutions to the KdV equation through the Wronskian formulation. The present method can be applied to other soliton equations.

1. Introduction

It is well known that Hirota first proposed the bilinear methods [1]. Equivalently at the same time a more compact form of obtaining N-soliton solution is through Wronskian technique. Satsuma gave Wronskian representation of the multisoliton solution of the KdV equation [2]. Then the Wronskian technique was developed by Freeman and Nimmo [3,4]. And this technique together with the Hirota method is considered as one of efficient and direct approaches to deriving soliton solutions for nonlinear evolution equations possessing bilinear forms. This technique admits direct and simple verifications of the solutions, and there have been generalizations surrounding it. For example, it can be generalized to find rational solution and mixed soliton-rational solutions in Wronskian form for few soliton equations by following the idea of long wave limitation[5]. Another meaningful generalization came from Siriaunpiboon and co-workers [6], who altered the conditions satisfied by Wronskian entries and derived a more general Wronskian solution to the KdV equation. Their result enables us to write more sorts of solutions into Wronskian form. They could not only include soliton solutions, rational solution [7,8], but also obtain the positons which Matveev derived from the Darboux transformations [9,10]. Very recently, Ma and You, based on their previous discussions, gave a systematic analyze to the solution structures in detail and constructed diverse exact solutions to the KdV equation, such as rational solutions, solitons, positons, negatons, breathers, complexitons, and more generally their interaction solutions[11,12]. It is noted that He proposed a straight forward and concise method, called Exp-function method to search for explicit exact solution of various evolution equations[16]. In this letter, following the substantial extensions by Siriaunpiboon et al. and Ma, respectively, we propose a matrix form to the KdV equation in using the Wronskian method. It provided us a method to present a novel class of exact solutions to the KdV equation under the help of its bilinear form.

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This paper is organized as follows. In Section 2, a general sufficient conditions will be presented for guaranteeing Wronskian to be solutions of the KdV equation and this yields an approach to a broad class of Wronskian solutions of the KdV equation. In Section 3, solitons, rational solutions and limiting solutions arising from the sufficient conditions in the Wronskian form for the KdV equation will be analyzed. In Section 4, we will construct real complexiton solutions to the KdV equation. A few concluding remarks will be given in Section 5.

2. Matrix generalization

In this section, we generalize the Ma’s procedure to the matrix form. Let us consider the KdV equation in the following form

\[ u_t - 6uu_x + u_{xxx} = 0. \]  

Employing the transformation \( u = 2(\ln f)_x \), the bilinear form of the KdV equation is then \[ (D_x D_t + D_t^4)f \cdot \bar{f} = 0, \]  

where \( D_x \) and \( D_t \) are Hirota derivatives defined by

\[ D_x^n D_t^m f \cdot g = (\partial_x - \partial_{x'})^m(\partial_t - \partial_{t'})^n f(x,t)g(x',t')|_{x'=x,t'=t}. \]

In the following, we adopt the compact Freeman and Nimmo’s notation \[ 3
\]

\[ f = W(\varphi_1, \varphi_2, \ldots, \varphi_N) = \begin{vmatrix}
\varphi_1 & \varphi_2 & \cdots & \varphi_N \\
\varphi_1' & \varphi_2' & \cdots & \varphi_N' \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_1^{N-1} & \varphi_2^{N-1} & \cdots & \varphi_N^{N-1}
\end{vmatrix} = |0, 1, \ldots, N-1| = |N-1|. \]  

First we recall some concerned results about the Wronskian technique to the KdV equation. In using the Wronskian technique to solve the KdV equation, one usually starts from

\[ -\varphi_{i,xx} = \lambda_{i} \varphi_{i}, \quad \varphi_{i,t} = -4\varphi_{i,xxx}, \quad 1 \leq i \leq N \]  

where \( \lambda_{i} \) are arbitrary real constants.

Siriaunpiboon et al. extend the Eq.(2.4) to the following:

\[ -\varphi_{i,xx} = \sum_{j=1}^{i} \lambda_{j} j \varphi_{j}, \quad \varphi_{i,t} = -4\varphi_{i,xxx}, \quad 1 \leq i \leq N \]  

where \( \lambda_{ij} \) are arbitrary real constants.

Ma further generalized the condition above as follows

\[ -\varphi_{i,xx} = \sum_{j=1}^{N} \lambda_{j} j \varphi_{j}, \quad \varphi_{i,t} = -4\varphi_{i,xxx}, \quad 1 \leq i \leq N \]  

Where \( \lambda_{ij} \) are arbitrary real constants, and constructed a very broad class of exact solutions to the KdV equation, among which solitons, positons and complexiton solutions.

To generalize general sufficient condition, we give the following lemma which can be proved by using determinant definition.

Lemma 2.1. Suppose that \( |A| \) is an \( N \times N \) determinant, \( P \) is some operator, \( P_{r}(j) \) \( |A| \)

Means \( P \) only acts on every entry in the j-th column of \( |A| \) and \( P_{r}(j) \) means \( P \) only acts on entry in the j-th row of \( |A| \). Then we have
\[ \sum_{j=1}^{N} P_{c}(j)|A| = \sum_{j=1}^{N} P_{r}(j)|A|. \] (2.7)

In what follows, we assume that \( A = (a_{ij}) \) is a \( N \times N \) matrix, and the vectors of function \( \Phi \) are defined by \( \Phi = (\phi_{1}, \phi_{2}, \ldots, \phi_{N})^{T} \). By virtue of above lemma, we can prove the following matrix generalization about the Wronskian solutions of the KdV equation.

Theorem 2.1 Assume that the column vector \( \Phi = (\phi_{1}, \phi_{2}, \ldots, \phi_{N})^{T} \) satisfies the two sets of conditions

\[ -\Phi_{xx} = A^{2}\Phi, \] (2.8)
\[ \Phi_{tt} = -4\Phi_{xxx}, \] (2.9)

where \( A^{2} \) to be \( N \times N \) square matrix. Then \( f = |N - 1| \) defined by Eq.(2.3) gives a solution to the bilinear KdV equation (2.1).

Obviously, the conditions (2.8) and (2.9) are an extension of the conditions of (2.4) for the \( N = 1 \) case. The proof of the above theorem is similar to Ref. [3], and we omit it.

The conditions (2.8) and (2.9) are two matrix linear partial differential equations. We are going to solve it explicitly. It is not difficult that the solution formula of (2.8) and (2.9) can be expressed as

\[ \Phi = \cosh(As - 4A^{2}t)\mu + \sinh(As - 4A^{2}t)\nu \] (2.10)

or

\[ \Phi = \cos(As + 4A^{2}t)\mu + \sin(As + 4A^{2}t)\nu, \] (2.11)

where \( \mu, \nu \) are arbitrary initial vectors.

3. Applications of the solution formulas

We have presented in the preceding section a solution formula to solve the representative system of the matrix differential equations defined by (2.8) and (2.9). In this section, we would like to consider the construction of solutions, which contains soliton, rational solutions, positon and mixed solutions.

Note that a constant similar transformation for the coefficient matrix \( \Lambda = A^{2} \) does not change the resulting Wronskian solution to the KdV equation [11]. Therefore, by linear algebra, we only need to consider some types of the coefficient matrix. If \( \Lambda \) is selected appropriately, we can obtain a very broad class of exact solution to the KdV equation, including soliton, positon and rational solutions. In what follows, we will focus on the different cases of \( \Lambda \) to present solutions mentioned above.

3.1. Solitons

First, let us expand \( \exp(As - 4A^{3}t) \) as

\[ \exp(As - 4A^{3}t) = \sum_{j=0}^{\infty} \frac{1}{j!} x^{j} A^{j} \sum_{l=0}^{\infty} \frac{(-1)^{l}}{l!} 4^{l} t^{l} A^{3l} = \sum_{h=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-4)^{l}}{(h-3l)!} x^{h-3l} t^{l} A^{h}. \] (3.1)

Therefore, if we take two specific choice of \( \mu = \nu = 1 \) in Eq.(2.10), then we have

\[ \exp(-As + 4A^{3}t) = \sum_{h=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-4)^{l}}{(h-3l)!} x^{h-3l} t^{l} (-A)^{h}. \] (3.2)
Where $\mu$ is arbitrary real vector.

If $\Lambda$ is chosen by

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ \lambda_2 & \ddots \\ 0 & \cdots & \lambda_N \end{pmatrix}$$

(3.4)

Therefore, from Eq.(2.10), we get

$$\phi_j = \exp(\lambda_j x - 4\lambda_j^2 t), 1 \leq j \leq N,$$

(3.5)

and the classical soliton solutions can be generated.

3.2. Rational solutions

In the following, we would like to show that the method of matrix presents a broad class of rational solutions to the KdV equation.

If square matrix $\Lambda$ has the following particular form

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ \vdots & \ddots \\ 0 & \cdots & 1 \end{pmatrix},$$

(3.6)

then noting that

$$(\Lambda)^N = 0,$$

(3.7)

we find that

$$\Phi = \sum_{h=0}^{N-1} \sum_{l=0}^h \frac{(-4)^l}{(2h-3l)!l!} x^{2h-3l} \Gamma^l \mu,$$

(3.8)

where $\mu$ is arbitrary real vector.

Hence, we can immediately obtain

$$\phi_j = \mu_j + \frac{1}{2!} x^2 \mu_{j-1} + \frac{1}{4!} (-x^4 - 4x^2 t) \mu_{j-2} + \cdots + \frac{(-4)^j}{(2j-3l)!l!} x^{2j-3l} \Gamma^l, 1 \leq j \leq N,$$

(3.9)

where $\mu_j$ is arbitrary real constants.

Taking a set of specific choice for $\mu_0 = 1, \mu_j = 0(1 \leq j \leq N)$ yields

$$\phi_0 = 1, \phi_1 = \frac{1}{2!} x^2, \phi_2 = \frac{1}{4!} x^4 - 4x^2 t,$$

$$\phi_3 = \frac{1}{6!} x^6 - \frac{4}{3!} x^3 t + 8t^2, \cdots \cdots .$$

(3.10)

Then the associated Wronskian solution to the KdV equation is given by

$$u = -2\partial_x^2 \ln W(\phi_0, \phi_1 \cdots \phi_{k-1}).$$

(3.11)
The first three rational solutions of lower order are
\[ u_1 = \frac{2}{x^2}, \quad u_2 = \frac{6x^4 - 144xt}{(x^3 + 12t)^2}, \]
\[ u_3 = \frac{12x(x^9 - 43200t^3 + 5400x^3t^2)}{(x^6 - 60x^3t - 700t^2)^2}. \]  
(3.12)

More general rational solutions can be generated from which evaluating other value of \( I^\mu \).

3.3. Limit solutions

Now we consider the following type of Jordan blocks of coefficient matrix \( \Lambda \)
\[
J(\lambda_i) = \begin{pmatrix}
\lambda_i & 0 \\
1 & \lambda_i \\
\vdots & \ddots & \ddots \\
0 & \cdots & 1 & \lambda_i
\end{pmatrix}_{l \times l} = \lambda_i I_l + E_l, \tag{3.13}
\]
where \( \lambda_i \neq 0 \), \( l \) is positive integers, \( I_l \) represents \( \times l \) matrix,
\[
E_{k_i} = \begin{pmatrix}
0 & 0 \\
1 & 0 \\
\vdots & \ddots & \ddots \\
0 & \cdots & 1 & 0
\end{pmatrix}_{k_i \times k_i}. \tag{3.14}
\]

Obviously, the Jordan block has the real eigenvalue \( \lambda_i \) with algebraic multiplicity \( l \).

Without losing generality, we only need consider one of Jordan block above. It is easy to show that Eq.(2.10) has following solution
\[
\Phi(\lambda_i) = J^s(\lambda_i) \exp(Ax - 4A^2t)\mu, \tag{3.15}
\]
where \( \mu \) is arbitrary real vector. Noticing that
\[
J^s(\lambda_i) = (\lambda_i I_l + E_l)^s = \lambda_i^s I_l + s \lambda_i^{s-1} E_l + \cdots + C^s_{j} \lambda_i^{s-j} E^j_l + \cdots + E^s_l \tag{3.16}
\]
Then it follows that
\[
\phi_j(\lambda_i) = (\mu_1 \frac{1}{(j - 1)!} \lambda_i^{j-1} + \cdots + \mu_{j-1} \partial_{\lambda_i}^{j-1}) \exp(Ax - 4A^2t), \tag{3.17}
\]
where \( \mu \) is arbitrary real vector constant, and \( 1 \leq j \leq l \).

Therefore, according to theorem 2.1, we obtain a Wronskian solution to the KdV equation:
\[
f = W(\phi_1(\lambda_i), \phi_2(\lambda_i), \cdots, \phi_{k_i}(\lambda_i)). \tag{3.18}
\]
Since the solutions what we discussed can also be derived by taking an appropriate limit of the parameters under the help of the bilinear form, we call the resulting solutions as limit solutions. It is noted that such a kind of solutions can also be obtained through the IST as multi-pole solutions [15], or through a limit procedure is Darboux transformation [13,14].
We remark that the matrix form defined by (2.10) is more general than (2.4), and thus they have broader applications. We note here that we take the Jordan block form (3.6) or (3.13), so that we can obtain rational solutions and limit solutions corresponding to the eigenvalue $\lambda_j = 0$ or $\lambda_j \neq 0$, respectively. In the following section, we will see another interesting application in the case of complex eigenvalues.

4. Complexiton solution

In this section, we will focus on the case of complex eigenvalues to present complexiton solutions. In order to construct complexiton solutions, let us consider the another Jordan block

$$
\Lambda = \begin{pmatrix}
    J_1 & 0 \\
    J_2 & \ddots \\
    0 & \cdots & J_l
\end{pmatrix},
$$

where

$$
J_i = \begin{pmatrix}
    A_i & 0 \\
    I_2 & A_i \\
    \vdots & \ddots & \ddots \\
    0 & \cdots & I_2 & A_i
\end{pmatrix},
$$

$$
A_i = \begin{pmatrix}
    \alpha_i & -\beta_i \\
    \beta_i & \alpha_i
\end{pmatrix},
$$

where $\alpha_i, \beta_i$ are real constants.

In this case, the Jordan block in (4.2) has a pair of complexiton eigenvalues $\alpha_i \pm \beta_i \sqrt{-1}$. We start from two order matrix

$$
\Lambda = \begin{pmatrix}
    \alpha & -\beta \\
    \beta & \alpha
\end{pmatrix} = \alpha I_2 + \beta \sigma_2,
$$

where

$$
I_2 = \begin{pmatrix}
    1 & 0 \\
    0 & 1
\end{pmatrix},
$$

$$
\sigma_2 = \begin{pmatrix}
    0 & -1 \\
    1 & 0
\end{pmatrix}.
$$

Thus, the general solution to the KdV equation is given by

$$
\Phi = \exp[\alpha(x - 4\alpha^2 t + 12\beta^2 t)I_2 + \beta(x + 4\beta^2 t - 12\alpha^2 t)\sigma_2] C,
$$

where $C$ is arbitrary vector constant.

Note that

$$
\sigma_2^2 = -I_2,
$$

we have

$$
\Phi = \exp(\alpha(x - 4\alpha^2 t + 12\beta^2 t))[\cos(\beta x + 4\beta^2 t - 12\alpha^2 t)I_2 + \sin(\beta x + 4\beta^2 t - 12\alpha^2 t)\sigma_2] C. \quad (4.7)
$$
For general case

\[ J_i = A' + E', \quad (4.8a) \]

where

\[ A' = I_4 \bigotimes A_i = \begin{pmatrix} A_i \\ \vdots \\ A_i \end{pmatrix}_{i \times i}, \quad E' = E_4 \bigotimes I_i = \begin{pmatrix} 0 \\ I_2 \\ 0 \\ I_2 \end{pmatrix}_{i \times i}, \quad (4.8) \]

where symbol \( \bigotimes \) indicates tensor product.

It then follows that

\[ A^s = (A' + E')^s = A'^s + sA'^{s-1}E' + \cdots + C^s A'^{s-i}E'^i \]

\[ = (I_4 + E' \partial_{\alpha_i} + \cdots + \frac{1}{s!}E'^s \partial_{\alpha_i}^s + \cdots + \frac{1}{s!}E'^s \partial_{\alpha_i}^s)A'^s, \quad (4.9) \]

where we have employed the following formula

\[ \partial_{\alpha_i} A_i h = \partial_{\alpha_i} (\alpha_i I_2 + \beta_i \sigma_2)^b = h(\alpha_i I_2 + \beta_i \sigma_2)^{h-1}, \quad h = 1, 2 \ldots. \quad (4.10) \]

Therefore,

\[ \Lambda^s = \begin{pmatrix} I_2 \\ I_2 \partial_{\alpha_i} \\ \frac{1}{2!} I_2 \partial_{\alpha_i}^2 \\ \vdots \\ \frac{1}{(s-1)!} I_2 \partial_{\alpha_i}^{s-1} \end{pmatrix} \begin{pmatrix} I_2 \\ I_2 \partial_{\alpha_i} \\ \frac{1}{2!} I_2 \partial_{\alpha_i}^2 \\ \vdots \\ \frac{1}{(s-1)!} I_2 \partial_{\alpha_i}^{s-1} \end{pmatrix} A'^s = T(\alpha_i)A'^s. \quad (4.11) \]

Then by Theorem 2.1, the general solution of the KdV equation is given by

\[ \phi_2(\alpha_i) = T(\alpha_i) \exp(Ax - 4A^3t)C = T(\partial_{\alpha_i})(I_4 \bigotimes \exp(Ax - 4A^3t)C) \]

\[ = T(\partial_{\alpha_i}) \begin{pmatrix} \exp(Ax - 4A^3t) \\ 0 \\ \vdots \\ 0 \\ \exp(Ax - 4A^3t) \end{pmatrix} C, \quad (4.12) \]

where \( C \) is arbitrary vector constant.

Hence, the complexiton solution to the KdV equation is given by

\[ f = W(\phi_1(\alpha_1)^T, \phi_2(\alpha_1)^T, \cdots, \phi_{l_4}(\alpha_1)^T). \quad (4.13) \]

The change of \( \alpha \) into \( \beta \) will lead to the another representation of the complexiton solutions to the KdV equation.
On the other hand, Eqs.(2.8) and (2.9) have another trigonometric function solutions (2.11). Similarly, the trigonometric function solutions can also be denoted by

\[ \Phi = \exp(\alpha x + 4\alpha^2 t - 12\alpha\beta^2 t)\left[\cos(\beta x - 4\beta^3 t + 12\beta\alpha^2 t)I_2 + \sin(\beta x - 4\beta^3 t + 12\beta\alpha^2 t)\sigma_2\right]C, \]  

where \( C \) is arbitrary vector constant.

5. Concluding remarks
In this paper, we have constructed an extension of Wronskian technique to solve matrix equation. Through the extension, we have shown that a broad class of explicit exact solutions to the KdV equation can be generated by the Wronskian formulation. It is noted that the resulting solution formulas provide a direct and comprehensive approach to construct diverse exact solutions to the KdV equation, such as solitons, rational solutions, positons, negatons, complexitons solutions. The key technique is to apply the variation of parameters in solving the matrix equation which involved partial differential equations of second and three order and to analyze solution structures in detail.

Acknowledgements
The authors would like to thank Dr. W.X. Ma for his stimulating discussions. This project is supported by the Special Funds for Educational Committee of Zhejiang Province of China (No 20050280) and Natural Science Foundation of Zhejiang Province of China (No Y605044).

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