From Fuchsian differential equations to integrable QFT

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Received 10 September 2014, revised 15 October 2014
Accepted for publication 17 October 2014
Published 4 November 2014

Abstract
We establish an intriguing correspondence between a special set of classical solutions of the modified sinh-Gordon equation (i.e., Hitchin’s ‘self-duality’ equations) on a punctured Riemann sphere and a set of stationary states in the finite-volume Hilbert space of the integrable 2D quantum field theory introduced by VA Fateev. An application of this correspondence to the problem of non-perturbative quantization of classically integrable nonlinear sigma models is briefly discussed. A detailed account of the results announced in this communication is contained in separate publications (Bazhanov and Lukyanov 2014 arXiv:1310.4390 [hep-th] and Bazhanov et al 2014 J. High Energy Phys. JHEP09(2014)147).

Keywords: integrable systems, quantum field theory, classical integrable equations, Fuchsian differential equations
PACS numbers: 02.30.Ik, 03.70.+k, 11.10.-z, 02.30.rz, 02.30.Hq

(Some figures may appear in colour only in the online journal)
finite-size 2D CFT (with the spatial coordinate compactified on a circle of the circumference \( R \)), a mathematically satisfactory construction of an infinite set of mutually commuting local integrals of motion (IM) can be given and the simultaneous diagonalization of these operators turns out to be a well-defined problem within the representation theory of the associated conformal algebra.

Different conformal algebras, as well as different sets of mutually commuting local IM yield a variety of integrable structures in CFT. The series of works [2] was dedicated to the simplest of these structures, associated with the diagonalization of the local IM from the quantum KdV hierarchy [3]. Subsequent studies of this problem culminated in a rather surprising link between the integrable structures of CFT and spectral theory of ordinary differential equations (ODEs) [4, 5]. In particular, in [5] a one-to-one correspondence was conjectured between the joint eigenbasis of the IM from the quantum KdV hierarchy and a certain class of differential operators of the second order \(-\partial_z^2 + V_L(z)\), with singular potentials \(V_L(z)\) (‘monster’ potentials in terminology of [5]). Apart from a regular singularity at \( z = 0 \) and an irregular singular point at \( z = \infty \), the monster potentials possess \( L \) regular singular points \( \{ x_a \}^L_{a=1} \). These potentials are not of much immediate interest in quantum mechanics, but arise rather naturally in the context of the theory of isomonodromic deformations. Solutions of the corresponding Schrödinger equations are single valued (monodromy-free) at \( z = x_a \) and their monodromy properties turn out to be similar to those of the radial wave functions for the three-dimensional isotropic anharmonic oscillator. The monodromy-free condition was formulated in a form of a system of \( L \) algebraic equations imposed on the set \( \{ x_a \}^L_{a=1} \). The correspondence proposed in [5] precisely relates the set of monster potentials \( V_L(z) \) and the joint eigenbasis for all quantum KdV IM in the level \( L \) subspace of the highest weight representation of the Virasoro algebra. In particular, this implies that a number of the potentials \( V_L(z) \) with a given value of \( L \) exactly coincides with a number of partitions \( p_L(L) \) of the integer \( L \) into parts of one kind.

Since 1998, the link to the spectral theory of ODE have been extended to a large variety of integrable CFT structures (for a review, see [6]), so that a natural question has emerged on whether a similar relation exist for massive integrable QFT. This question remained more or less dormant until the work [7], after which the so-called thermodynamic Bethe ansatz equations have started to appear in different contexts of SUSY gauge theories [8]. These remarkable developments have led to the work [9], which established a link between the eigenvalues of IM in the vacuum sector of the massive sine/sinh-Gordon model and some new spectral problem generalizing the one from [4].

This work is aimed to extend the results of [5, 9] and provide an explicit example of the correspondence between an infinite set of stationary states of massive integrable QFT in a finite volume and a set of singular differential operators of a certain type. At first glance, the best candidate for such study should be the sine-Gordon model, which always served as a basis for the development of integrable QFT. However, in spite of some technical complexity, a more general model introduced by Fateev [10] (which contains the sine-Gordon model as a particular case) turned out to be more appropriate for this task. The situation here is analogous to that in the Painlevé theory. Even though the Painlevé VI is the most complicated and general equation in the Painlevé classification, geometric structures behind this equation are much more transparent than those related to its degenerations. From this point of view, the fact that the sine-Gordon model is a certain degeneration of the Fateev model, could be understood as a QFT version of the relationship between the Painlevé VI and a particular case of Painlevé III.

Our starting point is a special class of Fuchsian differential operators of the second order \( D = -\partial_z^2 + T_L(z) \) with \( 3 + L \) regular singular points at \( z = z_1, z_2, z_3 \) and \( z = x_1, \ldots, x_L \). The
variable $z$ can be regarded as a complex coordinate on the Riemann sphere with $3 + L$ punctures. Projective transformations of $z$ allows one to send three points $z_i$ to any designated positions. At the same time other parameters of $T_L(z)$ are chosen in such a way that the remaining $L$ regular singular points satisfy the monodromy-free condition. Therefore, monodromy properties of the differential operator $D$ with $L > 0$ turn out to be similar to those for $L = 0$ (i.e. the ordinary hypergeometric differential operator of the second order).

Next, we consider more general differential operators of the form

$$D(\lambda) = -\partial_z^2 + T_L(z) + \lambda^2 P(z),$$

where

$$P(z) = \frac{(z_\lambda - z_2)^{(a_1)}(z_1 - z_3)^{(a_2)}(z_2 - z_1)^{(a_3)}}{(z - z_1)^{2-a_1}(z - z_2)^{2-a_2}(z - z_3)^{2-a_1}},$$

and parameters $0 < a_i < 2$ obey the constraint $a_1 + a_2 + a_3 = 2$. Due to the last relation, $P(z)(dz)^2$ transforms as a quadratic differential under $\Sigma(G)$ transformations and the punctures $z_i$, $i = 1, 2, 3$ on the Riemann sphere can still be sent to any desirable positions. The monodromy properties of $D(\lambda)$ for $\lambda \neq 0$ change dramatically in comparison with the case $\lambda = 0$. However, one can still find positions of the punctures $x_i$, $i = 1, 2, 3$ so that they remain monodromy-free singular points for any values of $\lambda$. In this case the coordinates $x_i$, $i = 1, 2, 3$ obey a the system of $L$ algebraic equations similar to that from [5], and the moduli space of the operators $D(\lambda)$ constitute a finite discrete subset $\mathcal{A}^{(L)}$ of the moduli space of $D(0) = -\partial_z^2 + T_L(z)$. It appears that, for a given $L$, the cardinality of $\mathcal{A}^{(L)}$ coincides with the number of partitions $p(L)$ of the integer $L$ into parts of three kinds. We interpret this fact in the spirit of [5], and present arguments in support of the existence of a one-to-one correspondence between the elements of $\mathcal{A}^{(L)}$ and the level-$L$ common eigenbasis of local IM of the integrable hierarchy introduced by Fateev in [10] (see [18] for details). The arguments closely follow the line of [2] adapted to the algebra of extended conformal symmetry, which can be regarded as a quantum Hamiltonian reduction of the exceptional affine superalgebra $\tilde{D}(2, 1; a)$ [11] (the ‘corner-brane’ W-algebra, in terminology of [12]).

The above structure can be generalized to the case of massive QFT. The construction is based on the idea from [9], which was inspired by the works [7, 8]. As far as our attention has been restricted to the case of CFT, there was no need to separately consider the anti-holomorphic differential operator $D(\lambda) = -\partial_z^2 + \bar{T}_L(\bar{z}) + \lambda^2 \bar{P}(\bar{z})$, since there is only a nomenclature difference between the holomorphic and antiholomorphic cases. In massive QFT one should substitute $(D(\lambda), \bar{D}(\lambda))$ by a pair of $(2 \times 2)$-matrix valued differential operators

$$(D(\lambda), \bar{D}(\lambda)) = (\partial_z - A_z, \partial_{\bar{z}} - A_{\bar{z}})$$

with

$$A_z = -\frac{1}{2} \partial_z \sigma_3 + \sigma_+ e^{\eta} + \sigma_- \lambda^2 P(z) e^{-\eta},$$

$$A_{\bar{z}} = +\frac{1}{2} \partial_{\bar{z}} \sigma_3 + \sigma_- e^{\bar{\eta}} + \sigma_+ \bar{\lambda}^2 \bar{P}(\bar{z}) e^{-\bar{\eta}},$$

where $\sigma_3, \sigma_2 = (\sigma_1 \pm i \sigma_2)/2$ are the standard Pauli matrices. The pair $(A_z, A_{\bar{z}})$ forms an $\mathfrak{sl}(2)$ connection whose flatness is a necessary condition for the existence of solution of the

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5 To the best of our knowledge, the differential operator $D(\lambda)$ for $L = 0$ was originally introduced (up to change of variables) in the unpublished work (2001) of A B Zamolodchikov and the second author (see also [12]). For $L > 0$, it appeared in [13].
auxiliary linear problem $D(\lambda) \Psi = D(\tilde{\lambda}) \Psi = 0$. The zero-curvature condition yields the modified sinh-Gordon equation (MShG)

$$ \partial_z \partial_{\bar{z}} \eta - e^{2\eta} + \rho^4 \mathcal{P}(z) \mathcal{P}(\bar{z}) e^{-2\eta} = 0 \quad (\rho^2 = \lambda \bar{\lambda}). $$

We consider a particular class of singular solutions of this equation, defined of the following requirements:

(i) $e^{-\eta}$ should be a smooth, single valued complex function without zeroes on the Riemann sphere with $3 + L + \tilde{L}$ punctures. Since $z = \infty$ is assumed to be a regular point on the sphere, then $e^{-\eta} \sim |z|^2$ as $|z| \to \infty$.

(ii) $e^{-\eta}$ develops a singular behavior at $z = z_i$: $e^{-\eta} \sim |z - z_i|^{-2m_i}$, as $|z - z_i| \to 0$ ($i = 1, 2, 3$), where parameters $m_i$ are restricted within the domains $\frac{1}{2} \leq m_i \leq \frac{1}{2} (2 - a_i)$.\(^6\)

(iii) $e^{-\eta}$ also develops a singular behavior at $z = x_a$ ($a = 1, \ldots, L$) and $z = \bar{y}_b$ ($b = 1, \ldots, \tilde{L}$):

$$ e^{-\eta} \sim \frac{z - x_a}{\bar{z} - x_a}, \quad e^{-\eta} \sim \frac{z - y_b}{\bar{z} - \bar{y}_b}. $$

The positions of these punctures are constrained by the requirement that $e^{z \pm i\eta \sigma_3} \Psi$ is single-valued in the neighborhood of the punctures $z = x_a$ and $z = \bar{y}_b$ (where $\Psi$ is a general solution of the MShG auxiliary linear problem).

Following [13], the above monodromy-free condition (iii) can be transformed into a set of $L + \tilde{L}$ constraints imposed on the regular part of local expansions of $(\partial_z \eta, \partial_{\bar{z}} \eta)$ at the monodromy-free punctures

$$ \partial_z \eta = \frac{1}{z - x_a} + \frac{1}{2} \gamma_a + o(1), \quad \partial_{\bar{z}} \eta = -\frac{1}{\bar{z} - \bar{z}_a} + o(1), $$

$$ \partial_z \eta = \frac{1}{\bar{z} - \bar{y}_b} + \frac{1}{2} \bar{\gamma}_b + o(1), \quad \partial_{\bar{z}} \eta = -\frac{1}{z - y_b} + o(1), $$

where $\gamma_a = \partial_z \log \mathcal{P}(z)|_{z=x_a}$, $\bar{\gamma}_b = \partial_{\bar{z}} \log \mathcal{P}(z)|_{z=\bar{y}_b}$.

It is expected that as far as positions of the punctures $z_i$ are fixed, the triple $m = (m_1, m_2, m_3)$ and the pair $(L, \tilde{L})$ are chosen, the MShG equation has a finite set $A_{m}^{(L, \tilde{L})}$ of solutions satisfying all of the above requirements. Then we can define the moduli space $A_m$ which is a union of these finite sets: $A_m = \bigcup_{L, \tilde{L}} A_{m}^{(L, \tilde{L})}$. Notice that, to a certain extent, $A_m$ can be regarded as a Hitchin moduli space [14].

An essential ingredient of the formal theory of the MShG equation is an existence of an infinite hierarchy of one-forms, which are closed by virtue of the MShG equation itself. With these forms, on can define an infinite set of conserved charges $\{q_{2n-1}, \tilde{q}_{2n-1}\}_{n=1}^{\infty}$, which can be used to characterize the elements of the moduli space $A_m$. Indeed, in spite of the fact that the flat connection $A = A_z dz + A_{\bar{z}} d\bar{z}$ associated with an element of $A_m$ is not single-valued on the punctured sphere, it does return to the original value after a continuation along the Pochhammer loop—the contour $\gamma_p$ depicted in figure 1.

\(^6\) At $m_i = \frac{1}{2} (2 - a_i)$ the leading asymptotic of $e^{-\eta}$ as $z \to z_i$ involves logarithms. Here we ignore such subtleties.
Therefore one can consider the Wilson loop

\[ W = \text{Tr} \left[ P \exp \left( \oint_{\gamma} A \right) \right], \]

whose significant advantage is that it does not depend on the precise shape of the integration contour. In particular, it is not sensitive to deformations of \( \gamma_p \) which sweep through the monodromy-free punctures. The Wilson loop is an entire periodic function of the spectral parameter \( \theta = \frac{1}{2} \log (\lambda/\bar{\lambda}) \), which possesses an asymptotic expansion:

\[
\log W \approx \begin{cases} 
C e^\theta + \sum_{n=1}^{\infty} q_{2n-1} e^{-(2n-1)\theta} & (\theta \to +\infty), \\
C e^{-\theta} + \sum_{n=1}^{\infty} \bar{q}_{2n-1} e^{(2n-1)\theta} & (\theta \to -\infty) .
\end{cases}
\]

Here \( C \) is some constant, whereas \( \{q_{2n-1}, \bar{q}_{2n-1}\}_{n=1}^{\infty} \) denotes the infinite set of conserved charges.

The main objective of this work is to propose the correspondence between elements of the moduli space \( A_m \) and a subset \( H_k^{(0)} \) of stationary states of the Fateev model in a finite volume. To describe \( H_k^{(0)} \) explicitly, let us recall some basic facts about this model.

The Fateev model is governed by the following Lagrangian in \( 1 + 1 \) Minkowski space

\[
\mathcal{L} = \frac{1}{16\pi} \sum_{i=1}^{3} \left( \left( \partial_\tau \varphi_i \right)^2 - \left( \partial_x \varphi_i \right)^2 \right) + 2\mu \left( e^{i\varphi_1} \cos \left( \alpha_1 \varphi_1 + \alpha_2 \varphi_2 \right) \right) + e^{-i\varphi_1} \cos \left( \alpha_1 \varphi_1 - \alpha_2 \varphi_2 \right) .
\]

For the three scalar fields \( \varphi_i(x, t) \). Here \( \alpha_i \) are real coupling constants obeying a single constraint \( \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \frac{1}{2} \). The parameter \( \mu \) in the Lagrangian sets the mass scale, \( \mu \sim [\text{mass}] \). We assume a finite-size geometry (with the spatial coordinate \( x \) compactified on a circle of circumference \( R \)) and impose the periodic boundary conditions \( \varphi_i(x + R, t) = \varphi_i(x, t) \). Due to the periodicity of the Lagrangian in \( \varphi_i \), the space of states \( \mathcal{H} \) in the model splits into orthogonal subspaces \( \mathcal{H}_k \) characterized by the three

\[ \end{equation}
‘quasimomenta’ \( k = (k_1, k_2, k_3) \). Similarly to the quantum mechanical problem of a particle in a periodic potential, the subspaces \( H_k \) possess the band structure, where the subspace \( H_k^{(0)} \), corresponding to the first Brillouin zone, is of primary interest. The Fateev model is integrable, in particular it has an infinite set of commuting local Hamiltonians of the form:

\[
\mathcal{H}_k = \sum_{\alpha} \mathcal{H}_\alpha(k),
\]

where \( \mathcal{H}_\alpha(k) \) are the local Hamiltonians associated with the Lorentz spins of the associated local densities. For \( \alpha \equiv \ldots, 2, 4, 6, \ldots \) being the Lorentz spins of the associated local densities. For \( 0 \leq k_i \leq \frac{1}{2} \), the sets of eigenvalues \( \{ \mathcal{H}_k^{(+)}(n), \mathcal{H}_k^{(-)}(n) \}_{n=1}^{\infty} \) fully specify the common eigenbasis of the local IM in \( H_k^{(0)} \).

In the recent paper [15] it was conjectured that the set of vacuum eigenvalues \( \{ \mathcal{H}_k^{(+)}(n), \mathcal{H}_k^{(-)}(n) \}_{n=1}^{\infty} \) (i.e. corresponding to the unique state in \( H_k^{(0)} \) with the lowest value of the energy \( E = \mathcal{H}_k^{(+)}(n) + \mathcal{H}_k^{(-)}(n) \)) essentially coincides with the set of conserved charges \( \{ q_{2n-1}, \tilde{q}_{2n-1} \}_{n=1}^{\infty} \) associated with the unique element \( \mathcal{A}_m^{(0)}(0) \) of the moduli space \( \mathcal{A}_m \), provided that

\[
\alpha_i^2 = \frac{a_i}{4}, \quad k_i = \frac{1}{a_i} \left( 2m_i + 1 \right), \quad \mu R = 2\rho.
\]

In this work we extend the conjecture of [15] to the whole spectrum of the local IM in the subspace \( H_k^{(0)} \). Namely, we conjecture that the corresponding eigenvalues coincide with the values of the classical conserved charges associated with the elements of the moduli space \( \mathcal{A}_m \). Thus, for the values of \( k_i \), restricted to the segment \( [0, \frac{1}{2}] \), there is a remarkable correspondence between the joint eigenbasis of the local IM in \( H_k^{(0)} \) space and the solutions of MShG equation specified by the elements of \( \mathcal{A}_m \) (see [18] for further details).

Among various applications, the above correspondence between the classical and quantum integrable systems provides a powerful tool for deriving functional and integral equations which determine the spectrum of local IM in massive QFT. We believe that this correspondence is, in fact, a general phenomenon which open a new way of approaching integrable QFT and, most importantly, the problem of non-perturbative quantization of classically integrable nonlinear sigma models. Here, we are motivated by the following consideration.

The above discussion has been focused on the ‘symmetric’ regime of the Fateev model where all the couplings \( \alpha_i \) are real, so that the Lagrangian is completely symmetric under simultaneous permutations of the real fields \( \varphi_i \) and the real couplings \( \alpha_i \). The theory is apparently non-unitary in this case. In the most interesting unitary regime one of the couplings, say \( \varphi_3 \), is pure imaginary: \( \alpha_3^2 > 0, \alpha_2^2 > 0, \alpha_3^2 := -b^2 < 0 \), and the Lagrangian becomes real

\[
\mathcal{L} = \frac{1}{16\pi} \sum_{i=1}^{\bar{3}} \left( \left( \frac{\partial \varphi_i}{\partial \tau} \right)^2 - \left( \frac{\partial \varphi_i}{\partial \sigma} \right)^2 \right) - 2\mu \left( e^{ib\sigma} \cos \left( \alpha_1 \varphi_1 + \alpha_2 \varphi_2 \right) \right.
\]

\[
\left. + e^{-ib\sigma} \cos \left( \alpha_1 \varphi_1 - \alpha_2 \varphi_2 \right) \right).
\]

The physical content in the unitary regime is different from the symmetric one. However, assuming the same periodic boundary conditions for each field \( \varphi_i \) (\( i = 1, 2, 3 \)), we can use the same symbols \( H \) and \( H_k \) to denote the spaces of states and their certain linear subspaces in the both cases (except that \( \mathbf{k} \) in the unitary regime should be regarded as a pair of quasimomenta, \( \mathbf{k} = (k_1, k_2) \), because of lack of periodicity in \( \varphi_3 \)-direction). As before, we focus on the component \( H_k^{(0)} \) corresponding to the first Brillouin zone. An important property of the local IM is that their existence and their form are not sensitive to the choice of
parameter values. Thus the eigenstates in $\mathcal{H}_k^{(0)}$ are again specified by the joint spectra of local IM.

Having in mind relations between $\mathcal{H}_k^{(0)}$ and $A_m$ in the symmetric regime, let us consider the MShG equation in the regime $a_1, a_2 > 0, a_3 < 0$ (the constraint $a_1 + a_2 + a_3 = 2$ is still assumed). A brief inspection shows that the set of requirements imposed on the MShG field remains quite meaningful in this case. Only the asymptotic condition which describes the behavior of the solution in the vicinity of the third puncture $z_3$, requires a special attention. For $a_i > 0$ we had the freedom to control the asymptotic behavior of $\eta$ as $z \to z_i$, with the arbitrary parameters $m_i$. If $a_3 < 0$, the situation is different: the leading asymptotic behavior of the solution at $z = z_3$ is determined by the MShG equation itself \cite{9}:

\[ e^{-\eta|\mathcal{P}(z)|^2} \propto |z - z_3|^{a_3}. \]

Taking this into account one can still define the moduli space $A_m$, which now will be labeled by the pair $m = (m_1, m_2)$. At the same time the definition of the set of the conserved charges \( q_i, \bar{q}_i \) remains unchanged. We expect that the relation between the subspace $\mathcal{H}_k^{(0)}$ and the moduli space $A_m$ will continue to holds for the unitary regime as well.

As an illustration of the above correspondence consider the problem of calculating the vacuum energy $E_k^{(\text{vac})}$ in the unitary regime of the Fateev model for non-zero values of the quasimomenta $k = (k_1, k_2)$. This problem lies beyond the scope of all traditional methods of integrable quantum field theory including the thermodynamic Bethe ansatz and nonlinear integral equations. In the unitary regime the Fateev model admits a dual sigma-model description \cite{10} as a two-parameter deformation of the $SU(2)$ principal chiral field theory defined by the action

\[ A = \int d^2x \left( a_1 \mathrm{Tr} \left( \partial_\mu g^2 \partial_\mu g^{-1} \right) + 2 I \left( L_\mu^3 \right)^2 + 2 r \left( R_\mu^3 \right)^2 \right) / \left( 4(u+r)(u+l) - r l \left( \mathrm{Tr} \left( g \sigma_3 g^{-1} \sigma_3 \right) \right) \right)^2, \]

where $g \in SU(2)$ and $L_\mu^3$ and $R_\mu^3$ stand for the left and right currents, $L_\mu^3 := \frac{1}{2I} \mathrm{Tr} \left( \partial_\mu g g^{-1} \sigma_3 \right)$, $R_\mu^3 := \frac{1}{2r} \mathrm{Tr} \left( g^{-1} \partial_\mu g \sigma_3 \right)$. The coupling constants $(u, r, l)$ are related to the parameters $a_1, a_2$,

\[ a_1 = +\frac{\pi}{2\sqrt{l(r+u)}}, \quad a_2 = +\frac{\pi}{2\sqrt{l(r+u)}}. \]

The quasimomenta $(k_1, k_2)$ determine the twisted boundary conditions for the matrix valued field $g$,

\[ g(t, x + R) = e^{i\pi k_2 \sigma_3} g(t, x) e^{i\pi k_1 \sigma_1}. \]

According to the correspondence stated above the value of the vacuum energy $E_k^{(\text{vac})} = L^{+ \times +} + L^{- \times -}$ is determined by the values of the conserved charges $(q_i, \bar{q}_i)$ for an appropriate solution $\eta^{(\text{vac})}(z)$ of the MShG equation, having no monodromy-free punctures, $L = \mathcal{L} = 0$. Then, extending the result of \cite{15}, one obtains,

\[ R E_k^{(\text{vac})} = -\frac{8}{\pi} \int d^2z \left| \mathcal{P}(z) \right| \sinh^2 \hat{\eta}(z) \]

\[ + \pi \sum_{i=1}^{2} \frac{4 m_i + 2 - a_i}{4a_i} \left( \frac{1}{2} \log \rho^2 \right) \left( \frac{a_i}{2} \right) \Gamma \left( \frac{a_i}{2} \right) \left( 1 - \frac{a_i}{2} \right) \]

where $\hat{\eta}(z) = \eta^{(\text{vac})}(z) - \frac{1}{2} \log \rho^2 \left| \mathcal{P}(z) \right|$. To verify this relation the MShG equation has been numerically solved for various set of the parameters $a_i$ and $k_i$ \cite{17}. The resulting values of
$E_{k}^{(\text{vac})}$ are in good agreement with the ultra-violet and infra-red asymptotics formulae, which were analytically derived directly from the definition of the Fateev model.

Finally, note that the sigma-model description is especially useful in the strong coupling limit ($a_1, a_2 \to \infty$ with $a_1/a_2$ kept fixed), which can be regarded as the classical limit, whereas the classical integrability of the theory was only recently established in [16]. A detailed account of the results announced in this communication is contained in separate publications [17, 18].

Acknowledgments

We are deeply indebted to AB Zamolodchikov for teaching us QFT and previous collaborations. The research of VB is partially supported by the Australian Research Council.

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