Some weaker forms of continuous functions in topological spaces

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Abstract

In this paper, the study of $g_{\omega\alpha}$-closed sets in topological spaces is continued, which are used to define and study some new spaces. Using $g_{\omega\alpha}$-closed sets and $g_{\omega\alpha}$-open sets, $g_{\omega\alpha}$-continuous and $g_{\omega\alpha}^*$-continuous functions are introduced and studied. It is found that these two classes of functions are weaker forms of continuous functions.

Keywords: Topological spaces, open sets, closed sets, $g_{\omega\alpha}$-continuous functions.

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1. Introduction

In 1970, Levine [15] introduced the concept of generalized closed sets (briefly g-closed sets). Since then many concepts related to g-closed sets were defined and investigated. The generalizations of generalized continuity were intensively studied in recent years by Balachandran, Devi, Maki and Sundaram [3, 16, 23]. Dunham [12], Battacharya and Lahiri [7], Dontchev [10] and Gnanambal [13], introduced $T_\frac{1}{2}$, semi-$T_\frac{1}{2}$, semi pre-$T_\frac{1}{2}$ and pre regular $T_\frac{1}{2}$-spaces respectively. Devi et.al [11, 9] introduced $T_{b\theta}$, $T_{d\alpha}$, a$T_b$ and a$T_d$-spaces. Veera kumar [25] introduced $T_\frac{1}{2}$, $T_\frac{1}{2}$, $T_p$ and $T_p$-spaces.

The class of continuous functions play an important role in general topological spaces. The stronger and weaker forms of continuity have been introduced and studied by several topologists. Recently, S. S. Benchalli et.al [6] introduced the notion of $g_{\omega\alpha}$-closed sets and studied some of their properties.

In this paper, some new spaces namely $T_{g_{\omega\alpha}}$-spaces, $g_{\omega\alpha}T$-spaces, $aT_{g_{\omega\alpha}}$-spaces and $\omega T_{g_{\omega\alpha}}$-spaces are introduced as applications of $g_{\omega\alpha}$-closed sets. And as a properties of $g_{\omega\alpha}$-closed sets $g_{\omega\alpha}$-continuous functions and $g_{\omega\alpha}^*$-continuous functions in topological spaces are introduced by using $g_{\omega\alpha}$-closure and $g_{\omega\alpha}$-interior under certain conditions and also study the relation between the newly defined concepts with already existing ones.

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2. Preliminaries

Throughout this paper \((X, \tau), (Y, \mu)\) and \((Z, \eta)\) (or simply \(X, Y\) and \(Z\)) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. For any subset \(A\) of a space \((X, \tau)\), \(cl(A)\) and \(int(A)\) denote the closure of \(A\) and interior of \(A\) respectively. For any subset \(A \subset X\), the closure of a subset \(A\) of a space \((X, \tau)\) is intersection of all closed sets those contain \(A\), and is denoted by \(cl(A)\).

**Definition 2.1.** A subset \(A\) of a topological space \((X, \tau)\) is called:

1. \(\alpha\)-open set \([19]\) if \(A \subseteq int(cl(int(A)))\) and \(\alpha\)-closed set if \(cl(int(cl(A))) \subseteq A\).
2. Semi-open set \([14]\) if \(A \subseteq cl(int(A))\) and Semi-closed set if \(int(cl(A)) \subseteq A\).

**Definition 2.2.** A subset \(A\) of a topological space \((X, \tau)\) is called a:

1. \(g\)-closed \([15]\) (resp. \(gs\)-closed \([2]\), \(ag\)-closed \([8]\), \(gp\)-closed \([17]\), \(gsp\)-closed \([10]\)) set if \(cl(A) \subseteq U\) (resp. \( scl(A) \subseteq U\), \( acl(A) \subseteq U\), \(pcl(A) \subseteq U\), \(spcl(A) \subseteq U\)) whenever \(A \subseteq U\) and \(U\) is open in \(X\).
2. \(gpr\)-closed \([13]\) (resp. \(rg\)-closed \([20]\), \(agr\)-closed \([26]\)) if \(pcl(A) \subseteq U\) (resp. \( cl(A) \subseteq U\), \( acl(A) \subseteq U\)) whenever \(A \subseteq U\) and \(U\) is regular-open in \(X\).
3. \(\omega\)-closed \([24]\) (resp. \(g'\)-closed \([21]\)) if \(cl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open (resp. \(g\)-open) in \(X\).
4. \(\omega\alpha\)-closed set \([4]\) if \(acl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\omega\)-open in \(X\).
5. \(g\omega\alpha\)-closed set \([6]\) if \(acl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\omega\alpha\)-open in \(X\).

The compliment of the above mentioned closed sets are their open sets respectively.

**Definition 2.3.** A subset \(A\) of a topological space \((X, \tau)\) is called:

1. \(T_1\)-space \([15]\) if every \(g\)-closed set is closed.
2. \(T^*\)-space \([26]\) if every \(g^*\)-closed set is closed.
3. \(T_0\)-space \([2]\) if every \(gs\)-closed set is \(g^*\)-closed.
4. \(\alpha\)-space \([19]\) if every \(\alpha\)-closed set is closed.
5. \(T_\omega\)-space \([22]\) if every \(\omega\)-closed set is closed.
6. \(\alpha T_\omega\)-space \([11]\) (resp. \(\alpha T_\omega\)-space \([27]\), \(\alpha T_\omega\)-space \([22]\), \(\alpha T_\omega\)-space \([11]\), \(\omega T_\alpha\)-space \([9]\)) if every \(ag\)-closed (resp. \(g^*\)-closed, \(\omega\)-closed, \(g\)-closed, \(\alpha\)-closed) set is closed.

**Definition 2.4.** A function \(f : X \to Y\) is called \(\alpha\)-continuous \([18]\) (resp. \(ag\)-continuous \([8]\), \(gp\)-continuous \([1]\), \(\omega\)-continuous \([24]\), \(\omega\alpha\)-continuous \([5]\)) if \(f^{-1}(V)\) is \(\alpha\)-closed (resp. \(ag\)-closed, \(gp\)-closed, \(\omega\)-closed, \(\omega\alpha\)-closed) in \(X\) for every closed set \(V\) in \(Y\).

3. New Spaces Using \(g\omega\alpha\)-Closed sets.

In this section the study of \(g\omega\alpha\)-closed sets in topological spaces is continued to define some new spaces, such as \(T_{g\omega\alpha}\)-spaces, \(g\omega\alpha\)-spaces, \(ag\omega_{g\omega\alpha}\)-spaces and \(\omega_{g\omega\alpha}\)-spaces, and study some of their properties.

**Definition 3.1.** A topological space \((X, \tau)\) is said to be \(T_{g\omega\alpha}\)-space if every \(g\omega\alpha\)-closed set is closed in \((X, \tau)\).

**Example 3.2.** Let \(X = \{a, b, c\}\) and \(\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}\) be a topology on \(X\), then the space \((X, \tau)\) is \(T_{g\omega\alpha}\)-space.

**Definition 3.3.** A topological space \((X, \tau)\) is said to be \(g\omega\alpha\)-space if every \(g\omega\alpha\)-closed set is \(\alpha\)-closed in \((X, \tau)\).

**Example 3.4.** In Example 3.2, the space \((X, \tau)\) is \(g\omega\alpha\)-space.

**Theorem 3.5.** A topological space \((X, \tau)\) is \(T_{g\omega\alpha}\)-space if and only if every \(g\omega\alpha\)-open set in \((X, \tau)\) is open in \((X, \tau)\).
Proof. Suppose the space \((X, \tau)\) is \(T_{\omega\alpha}\)-space. Let \(G\) be \(g\omega\alpha\)-open set in \((X, \tau)\). Then \(X - G\) is \(g\omega\alpha\)-closed set in \((X, \tau)\). Since \((X, \tau)\) is \(T_{\omega\alpha}\)-space, \(X - G\) is closed in \((X, \tau)\). Therefore \(G\) is open in \((X, \tau)\).

Conversely. Assume that every \(g\omega\alpha\)-open set is open in \((X, \tau)\). Let \(F\) be \(g\omega\alpha\)-closed set in \((X, \tau)\). By hypothesis \(X - F\) is open set in \((X, \tau)\). Therefore \(F\) is closed set in \((X, \tau)\). Hence \((X, \tau)\) is \(T_{\omega\alpha}\)-space.

**Theorem 3.6.** Every \(T_{\omega\alpha}\)-space is \(g\omega\alpha T\)-space.

**Proof.** Let \((X, \tau)\) be \(T_{\omega\alpha}\)-space. Let \(A\) be \(g\omega\alpha\)-closed set in \((X, \tau)\). Since \((X, \tau)\) is \(T_{\omega\alpha}\)-space, \(A\) is closed in \((X, \tau)\). But every closed set is \(\omega\alpha\)-closed. Therefore \(A\) is \(\omega\alpha\)-closed set in \((X, \tau)\). Hence \((X, \tau)\) is \(T_{\omega\alpha}\)-space.

The converse of the above theorem need not be true as seen from the following example.

**Example 3.7.** Let \(X = \{a, b, c\}\) and \(\tau = \{X, \phi, \{a\}\}\) be a topology on \(X\), then the space \((X, \tau)\) is \(g\omega\alpha T\)-space but not \(T_{\omega\alpha}\)-space. The set \(A = \{b\}\) is \(g\omega\alpha\)-closed and \(\omega\alpha\)-closed but not closed in \((X, \tau)\).

**Theorem 3.8.** If the space \((X, \tau)\) is \(T_{\omega\alpha}\)-space then every singleton of \((X, \tau)\) is either \(\omega\alpha\)-closed or open.

**Proof.** Suppose \(\{x\}\) is not \(\omega\alpha\)-closed set for some \(x \in X\). Then \(X - \{x\}\) is not \(\omega\alpha\)-open set and \(X\) is only \(\omega\alpha\)-open set containing \(X - \{x\}\). Therefore \(X - \{x\}\) is \(g\omega\alpha\)-closed set. Since \((X, \tau)\) is \(T_{\omega\alpha}\)-space, then \(X - \{x\}\) is closed. Hence \(\{x\}\) is open.

However the converse of the above theorem need not be true as seen from the following example.

**Example 3.9.** Let \(X = \{a, b, c, d\}\) and \(\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}\) be a topology on a space \((X, \tau)\), then the space \((X, \tau)\) satisfies the conclusion of the above theorem 3.3 but \((X, \tau)\) is not \(T_{\omega\alpha}\)-space.

**Theorem 3.10.** If \((X, \tau)\) is \(1/2 T_{\alpha}\)-space then it is \(g\omega\alpha T\)-space.

**Proof.** Let \(A\) be \(g\omega\alpha\)-closed set in \((X, \tau)\). Then \(A\) is \(ag\)-closed. Since \((X, \tau)\) is \(1/2 T_{\alpha}\)-space, \(A\) is \(\alpha\)-closed. Hence \((X, \tau)\) is \(g\omega\alpha T\)-space.

The converse of the above theorem need not be true as seen from the following example.

**Example 3.11.** In Example 3.7, the space \((X, \tau)\) is \(g\omega\alpha T\)-space but not \(1/2 T_{\alpha}\)-space. Since the set \(A = \{a, b\}\) is \(ag\)-closed but not \(\alpha\)-closed set in \((X, \tau)\).

**Theorem 3.12.** If a space is \(g\omega\alpha T\)-space then every singleton of \((X, \tau)\) is either \(\omega\alpha\)-closed or \(\alpha\)-open.

**Proof.** Suppose \(\{x\}\) is not \(\omega\alpha\)-closed set for some \(x \in X\). Then \(X - \{x\}\) is not \(\omega\alpha\)-open set and \(X\) is only \(\omega\alpha\)-open set containing \(X - \{x\}\). Therefore \(X - \{x\}\) is \(g\omega\alpha\)-closed set. Since \((X, \tau)\) is \(g\omega\alpha T\)-space, then \(X - \{x\}\) is \(\alpha\)-closed. Hence \(\{x\}\) is \(\alpha\)-open.

**Example 3.13.** Let \(X = \{a, b, c, d\}\) and \(\tau = \{X, \phi, \{b, c\}, \{b, c, d\}, \{a, b, c\}\}\) is a topological space, then the space \((X, \tau)\) satisfies the conclusion of the above theorem 3.5 But \((X, \tau)\) is not \(g\omega\alpha T\)-space.

**Theorem 3.14.** If \(A\) topological space \((X, \tau)\) is \(T_c\)-space and \(1/2 T_{\alpha}\)-space then \(X\) is \(T_{\omega\alpha}\)-space.

**Proof.** Let \(A\) be \(g\omega\alpha\)-closed set in \((X, \tau)\). Then \(A\) is \(gs\)-closed set in \((X, \tau)\). Since \(X\) is \(T_c\)-space, we have \(A\) is \(g^*\)-closed. Since \(X\) is \(1/2 T_{\alpha}\)-space, we have \(A\) is closed in \((X, \tau)\). Hence \(X\) is \(T_{\omega\alpha}\)-space.

**Definition 3.15.** A topological space \((X, \tau)\) is said to be \(ag T_{\omega\alpha}\)-space if every \(ag\)-closed set is \(g\omega\alpha\)-closed in \((X, \tau)\).

**Example 3.16.** In Example 3.2, the space \((X, \tau)\) is \(ag T_{\omega\alpha}\)-space.
Definition 3.17. A topological space \((X, \tau)\) is said to be \(\omega a T_{g\omega a}\)-space if every \(\omega a\)-closed set is \(g\omega a\)-closed in \((X, \tau)\).

Example 3.18. In Example 3.2, the space \((X, \tau)\) is \(\omega a T_{g\omega a}\)-space.

Theorem 3.19. A topological space \((X, \tau)\) is \(\omega a T_{g\omega a}\)-space if and only if every \(\omega a\)-open set in \((X, \tau)\) is \(g\omega a\)-open in \((X, \tau)\).

Proof. Suppose the space \((X, \tau)\) is \(\omega a T_{g\omega a}\)-space, let \(G\) be \(\omega a\)-open set in \((X, \tau)\). Then \(X - G\) is \(\omega a\)-closed set in \((X, \tau)\). Since \((X, \tau)\) is \(\omega a T_{g\omega a}\)-space, \(X - G\) is \(g\omega a\)-closed in \((X, \tau)\). Therefore \(G\) is \(g\omega a\)-open in \((X, \tau)\).

Conversely, assume that every \(\omega a\)-open set is \(g\omega a\)-open in \((X, \tau)\). Let \(F\) be \(\omega a\)-closed set in \((X, \tau)\), then \(X - F\) is \(\omega a\)-open set in \((X, \tau)\). By hypothesis, \(X - F\) is \(g\omega a\)-open set in \((X, \tau)\). Therefore \(F\) is \(g\omega a\)-closed set in \((X, \tau)\). Hence \((X, \tau)\) is \(\omega a T_{g\omega a}\)-space.

Theorem 3.20. Every \(\omega a T_b\)-space is \(\omega a T_{g\omega a}\)-space.

Proof. Let \((X, \tau)\) be \(\omega a T_b\)-space. Let \(A\) be \(\omega a\)-closed set in \((X, \tau)\). Since \((X, \tau)\) is \(\omega a T_b\)-space, \(A\) is closed and therefore it is \(g\omega a\)-closed set in \((X, \tau)\). Hence \((X, \tau)\) is \(\omega a T_{g\omega a}\)-space.

The converse of the above theorem need not be true as seen from the following example.

Example 3.21. In Example 3.6, the space \((X, \tau)\) is \(\omega a T_{g\omega a}\)-space but not \(\omega a T_b\). Since the set \(A = \{a, b, c\}\) is \(\omega a\)-closed but not closed in \((X, \tau)\).

Theorem 3.22. If \((X, \tau)\) is \(\omega a T_{g\omega a}\)-space and \(T_{g\omega a}\)-space, then \((X, \tau)\) is \(\omega a T_b\)-space (resp. \(\omega a T_c\)-space, \(\omega a T_{d\omega a}\)-space, \(\omega a T_{\omega a}\)-space, \(\omega a T_{\ast\omega a}\)-space).

Proof. Let \(A\) be \(\omega a\)-closed set in \((X, \tau)\). Since \((X, \tau)\) is \(\omega a T_{g\omega a}\)-space, \(A\) is \(g\omega a\)-closed set in \((X, \tau)\). Again since \((X, \tau)\) is \(T_{g\omega a}\)-space, implies \(A\) is closed. Hence \((X, \tau)\) is \(\omega a T_b\)-space (resp. from [6] \((X, \tau)\) is \(\omega a T_c\)-space, \(\omega a T_{d\omega a}\)-space, \(\omega a T_{\omega a}\)-space, \(\omega a T_{\ast\omega a}\)-space).

Theorem 3.23. If \((X, \tau)\) is an \(\omega a T_{g\omega a}\)-space then every singleton subset of \((X, \tau)\) is closed or \(\omega a\)-open.

Proof. Suppose \(\{x\}\) is not closed set for some \(x\in X\). Then \(X - \{x\}\) is not open set and \(X\) is only open set containing \(X - \{x\}\). Therefore \(X - \{x\}\) is \(\omega a\)-closed set. Then \(X - \{x\}\) is \(g\omega a\)-closed, because \((X, \tau)\) is \(\omega a T_{g\omega a}\)-space. Hence \(\{x\}\) is \(\omega a\)-open.

The converse of the above theorem need not be true as seen from the following example.

Example 3.24. In Example 3.3, the space \((X, \tau)\) satisfies the conclusion of above theorem 3.10 but \((X, \tau)\) is not \(\omega a T_{g\omega a}\)-space. The set \(A = \{a, b, c\}\) is \(\omega a\)-closed but not \(g\omega a\)-closed in \((X, \tau)\).

Remark 3.25. \(T_{g\omega a}\)-space is independent of \(T_{\omega a}\)-space.

Example 3.26. Let \(X = \{a, b, c\}\) and \(\tau = \{X, \phi, [a], [b, c]\}\) then the space \((X, \tau)\) is \(T_{\omega a}\)-space but not \(T_{g\omega a}\)-space. The set \(A = \{c\}\) is \(g\omega a\)-closed but not closed in \((X, \tau)\).

Example 3.27. Let \(X = \{a, b, c\}\) and \(\tau = \{X, \phi, [a], [a, b], [a, c]\}\) then the space \((X, \tau)\) is \(T_{g\omega a}\)-space but not \(T_{\omega a}\)-space. The set \(A = \{a, b\}\) is \(\omega a\)-closed but not closed in \((X, \tau)\).

Theorem 3.28. A topological space \((X, \tau)\) is \(\omega a T_{g\omega a}\)-space if and only if every \(\omega a\)-open set in \((X, \tau)\) is \(g\omega a\)-open in \((X, \tau)\).

Proof. Suppose the space \((X, \tau)\) is \(\omega a T_{g\omega a}\)-space, let \(G\) be \(\omega a\)-open set in \((X, \tau)\). Then \(X - G\) is \(\omega a\)-closed set in \((X, \tau)\). Since \((X, \tau)\) is \(\omega a T_{g\omega a}\)-space, \(X - G\) is \(g\omega a\)-closed in \((X, \tau)\). Therefore \(G\) is \(g\omega a\)-open in \((X, \tau)\).

Conversely, assume that every \(\omega a\)-open set is \(g\omega a\)-open in \((X, \tau)\). Let \(F\) be \(\omega a\)-closed set in \((X, \tau)\), then \(X - F\) is \(\omega a\)-open set in \((X, \tau)\). By hypothesis, \(X - F\) is \(g\omega a\)-open set in \((X, \tau)\). Therefore \(F\) is \(g\omega a\)-closed set in \((X, \tau)\). Hence \((X, \tau)\) is \(\omega a T_{g\omega a}\)-space.
4. $gωα$-Continuous Functions.

In this section $gωα$-continuous functions and $gωα^∗$-continuous functions in topological spaces are introduced and study some of their properties and characterizations.

**Definition 4.1.** A function $f : X \to Y$ is called $gωα$-continuous if the inverse image of every closed set in $Y$ is $gωα$-closed in $X$.

**Example 4.2.** Let $X = Y = [a, b, c]$ and $\tau = [X, \phi, [a], [b], [c]]$ and $\mu = \{Y, \phi, [b], [a, c]\}$. Define a function $f : X \to Y$ by $f(a) = c$, $f(b) = a$ and $f(c) = b$, then $f$ is $gωα$-continuous.

**Theorem 4.3.** A function $f : X \to Y$ is $gωα$-continuous if and only if $f^{-1}(V)$ is $gωα$-open in $X$ for every open set $V$ in $Y$.

**Proof.** Let $f : X \to Y$ be a $gωα$-continuous function and $V$ be an open set in $Y$. Then $X - V$ is closed in $Y$. Since $f$ is $gωα$-continuous, $f^{-1}(X - V) = Y - f^{-1}(V)$ is $gωα$-closed in $X$. Therefore $f^{-1}(V)$ is $gωα$-open in $X$.

Conversely, assume $f^{-1}(V)$ is a $gωα$-open in $X$ for every open set $V$ in $Y$. Let $F$ be a closed set in $Y$. Then $Y - F$ is an open set in $Y$. By assumption $f^{-1}(Y - F) = X - f^{-1}(F)$ is $gωα$-open set in $X$. This implies $f^{-1}(F)$ is $gωα$-closed in $X$. Therefore $f$ is $gωα$-continuous.

**Remark 4.4.** If a function $f : X \to Y$ is continuous (resp $α$-continuous, $ω$-continuous, $gα$-continuous) then function is $gωα$-continuous, but converse is not true in general, as seen from the following example.

**Example 4.5.** Let $X = Y = [a, b, c]$, $\tau = [X, \phi, [a], [b], [c]]$ and $\mu = \{Y, \phi, [b], [a, c]\}$ be topologies on $X$ and $Y$ respectively. Define a function $f : X \to Y$ by $f(a) = a$, $f(b) = b$ and $f(c) = c$. Then $f$ is $gωα$-continuous function but not continuous, $α$-continuous, $ω$-continuous and $gα$-continuous. Since for a closed set $\{a, c\}$ in $Y$, $f^{-1}[\{a, c\}] = [a, c]$ is not closed, $α$-closed, $ω$-closed and $gα$-closed in $X$.

**Remark 4.6.** If $f : X \to Y$ is $gωα$-continuous, then $f$ is $ag$-continuous (resp $gsp$-continuous, $gps$-continuous, $gpr$-continuous, $rg$-continuous, $agr$-continuous) in $X$, but converse is not true in general.

**Example 4.7.** Let $X = Y = [a, b, c]$ and $\tau = [X, \phi, [a]]$ and $\mu = \{Y, \phi, [a], [b], [a, b]\}$ be topologies on $X$ and $Y$ respectively. Define a function $f : X \to Y$ by $f(a) = c$, $f(b) = a$ and $f(c) = b$. Then $f$ is $ag$-continuous, $gs$-continuous, $gps$-continuous, $gpr$-continuous, $rg$-continuous, $agr$-continuous but not $gωα$-continuous. Since for the closed set $A = [b, c]$ in $Y$, $f^{-1}(\{b, c\}) = [a, c]$ is not $gωα$-closed in $X$.

**Remark 4.8.** The concept of $gωα$-continuous function is independent with $ωα$-continuous function as seen from the following examples.

**Example 4.9.** Let $X = Y = [a, b, c]$ and $\tau = [X, \phi, [a]]$ and $\mu = \{Y, \phi, [a], [b], [a, b]\}$ be topologies on $X$ and $Y$ respectively. Define a function $f : X \to Y$ by $f(a) = a$, $f(b) = b$ and $f(c) = c$. Then $f$ is $ωα$-continuous but not $gωα$-continuous. Since for the closed set $A = [a, c]$ in $Y$, $f^{-1}(\{a, c\}) = [a, c]$ is not $gωα$-closed but it is $ωα$-closed in $X$.

**Example 4.10.** Let $X = Y = [a, b, c]$ and $\tau = [X, \phi, [a], [b], [a, b]]$ and $\mu = \{Y, \phi, [a], [b], [a, b]\}$ be topologies on $X$ and $Y$ respectively. Define a function $f : X \to Y$ by $f(a) = a$, $f(b) = b$ and $f(c) = c$. Then $f$ is $gωα$-continuous but not $ωα$-continuous. Since for the closed set $A = [b]$ in $Y$, $f^{-1}(\{b\}) = [b]$ is not $ωα$-closed but it is $gωα$-closed in $X$.

**Remark 4.11.** The concept of $gωα$-continuous function is independent with $gα^∗$-continuous and $gα$-continuous function as seen from the following examples.

**Example 4.12.** Let $X = Y = [a, b, c]$ and $\tau = [X, \phi, [a], [b], [a, b]]$ and $\mu = \{Y, \phi, [a], [b], [a, b]\}$ be topologies on $X$ and $Y$ respectively. Define a function $f : X \to Y$ by $f(a) = a$, $f(b) = b$ and $f(c) = a$. Then $f$ is $gωα$-continuous but not $gα^∗$-continuous and $gα$-continuous. Since for the closed set $A = [b, c]$ in $Y$, $f^{-1}(\{b, c\}) = [a, b]$ is not $gα^∗$-closed and $gα$-closed, but it is $gωα$-closed in $X$. 
Example 4.13. Let $X = Y = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\mu = \{Y, \emptyset, \{b\}, \{b, c\}\}$ be topologies on $X$ and $Y$ respectively. Define a function $f : X \to Y$ by $f(a) = a$, $f(b) = b$ and $f(c) = c$. Then $f$ is $g^*$-continuous but not $g\omega a$-continuous. Since for the closed set $A = \{a\}$ in $Y$, $f^{-1}(\{a\}) = \{a\}$ is not $g\omega a$-closed but it is $g^*$-closed set in $X$.

Example 4.14. Let $X = Y = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{a, c\}\}$ and $\mu = \{Y, \emptyset, \{b\}, \{b, c\}\}$ be topologies on $X$ and $Y$ respectively. Define a function $f : X \to Y$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then $f$ is $g^*$-continuous but not $g\omega a$-continuous. Since for the closed set $A = \{a, b\}$ in $Y$, $f^{-1}(\{a, b\}) = \{a, b\}$ is not $g\omega a$-closed but it is $g^*$-closed set in $X$.

Remark 4.15. The following diagram summarizes the above discussions.

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**Theorem 4.16.** Let $f : X \to Y$ be a function from $X$ into $Y$. Then followings are equivalent:

1. The function $f$ is $g\omega a$-continuous.
2. For each $x \in X$ and each open set $V$ in $Y$ with $f(x) \in V$, there is a $g\omega a$-open set $U_x$ in $X$ such that $x \in U_x$ and $f(U) \subset V$.
3. For each $x \in X$, the inverse of every neighbourhood of $f(x)$ is $g\omega a$-nhd of $x$.
4. For every subset $A$ of $X$, $f(\text{g\omega cl}(A)) \subset \text{cl}(f(A))$.
5. For every subset $B$ of $Y$, $\text{g\omega cl}(f^{-1}(B)) \subset f^{-1}(\text{cl}(B))$.
6. For every subset $B$ of $Y$, $f^{-1}(\text{int}(B)) \subset g\omega a-\text{int}(f^{-1}(B))$.

**Proof.** (1) $\Rightarrow$ (2): Suppose (1) holds. Let $x \in X$ and $V$ be an open set in $Y$ with $f(x) \in V$, then $x \in f^{-1}(V)$. Since $f$ is $g\omega a$-continuous, $f^{-1}(V)$ is $g\omega a$-open set in $X$. Put $U = f^{-1}(V)$, then $x \in U$ and $f(U) = f(f^{-1}(V)) \subset V$. Therefore (ii) holds.

(2) $\Rightarrow$ (1): Suppose (2) holds. Let $x \in X$ and $V$ be an open set in $Y$ containing $f(x)$. By hypothesis, there exists a $g\omega a$-open set $U_x$ in $X$ such that $x \in U_x$ and $f(U_x) \subset V$. This implies $x \in f^{-1}(V)$, which implies $f^{-1}(V)$ is $g\omega a$-nhd of $x$. Since $x$ is arbitrary, $f^{-1}(V)$ is $g\omega a$-nhd of each of its points. From Theorem 5.3 in [6], $f^{-1}(V)$ is a $g\omega a$-open set in $X$. Therefore $f$ is $g\omega a$-continuous.

(1) $\Rightarrow$ (3) Suppose (1) holds and let $x \in X$ and $V$ be an open nhd of $f(x)$, therefore there exists an open set $W$ in $Y$ such that $f(x) \in W \subset V$ and hence $x \in f^{-1}(W) \subset f^{-1}(V)$. Since $f$ is $g\omega a$-continuous and $f^{-1}(W)$ is a $g\omega a$-open set in $X$. Therefore $f^{-1}(V)$ is $g\omega a$-nhd of $x$. Consequently, for each $x \in X$, the inverse of every nhd of $f(x)$ is $g\omega a$-nhd of $x$.

(3) $\Rightarrow$ (1) Suppose (3) holds. Let $x \in X$ and $H$ be an open set in $Y$ containing $f(x)$. This implies $H$ is nhd of $f(x)$. By (iii), $f^{-1}(H)$ is $g\omega a$-nhd of $x$. Since $x$ is arbitrary, $f^{-1}(H)$ is $g\omega a$-nhd of each of its points. From Theorem 5.3 in [6], $f^{-1}(H)$ is a $g\omega a$-open set in $X$. Therefore $f$ is $g\omega a$-continuous.
This implies $\alpha$. Suppose as seen from the following example. Let $\omega$ be a function $f : X \to Y$ be a function:

(1) If $X$ is $T_{\omega\alpha}$-space then $f$ is $\omega\alpha$-continuous if and only if $f$ is continuous.

(2) If $X$ is $\omega\alpha T$-space, then $f$ is $\omega\alpha$-continuous if and only if $f$ is $\alpha$-continuous.

Proof. (1) Suppose $X$ is $T_{\omega\alpha}$-space and $f$ is $\omega\alpha$-continuous. Let $V$ be an open set in $Y$, by hypothesis $f^{-1}(V)$ is $\omega\alpha$-open set in $X$ and hence $f^{-1}(V)$ is open. Therefore $f$ is continuous. Converse is obvious, because every open set is $\omega\alpha$-open.

(2) Suppose $X$ is $\omega\alpha T$-space and $f$ is $\omega\alpha$-continuous. Let $V$ be an open set in $Y$, by hypothesis $f^{-1}(V)$ is $\omega\alpha$-open set in $X$ and hence $f^{-1}(V)$ is $\alpha$-open. Therefore $f$ is $\alpha$-continuous. Converse is obvious, because every $\alpha$-open set is $\omega\alpha$-open.

Theorem 4.18. If $A$ is $\omega\alpha$-closed set in $X$ where $X$ is $\omega\alpha T_{\omega\alpha}$-space, and if $f : X \to Y$ is $\omega\alpha$-irresolute and pre-$\alpha$-closed then $f(A)$ is $\omega\alpha$-closed set in $Y$.

Proof. Let $U$ be any $\omega\alpha$-open set in $Y$, such that $f(A) \subset U$, where $A$ is $\omega\alpha$-closed set in $X$ and $X$ is $\omega\alpha T_{\omega\alpha}$-space, then $A \subset f^{-1}(U)$. Since $f$ is $\omega\alpha$-irresolute, $f^{-1}(U)$ is $\omega\alpha$-open in $X$ containing a $\omega\alpha$-closed set A. Therefore $acl(A) \subset f^{-1}(U)$. This implies $f(acl(A)) \subset U$. Now $f$ is pre-$\alpha$-closed and $acl(A)$ is $\alpha$-closed, implies $f(acl(A))$ is $\alpha$-closed in $Y$. Therefore $acl(f(acl(A))) = f(acl(A))$. Moreover, $acl(f(A)) \subset acl(f(acl(A))) \subset U$. This implies $acl(f(A)) \subset U$. Therefore $f(A)$ is $\omega\alpha$-closed set in $Y$.

Theorem 4.19. Suppose $f : X \to Y$ is pre-$\alpha$-closed and $\omega\alpha$-open bijective and $f$ is $\omega\alpha T$-space then $Y$ is also $\omega\alpha T$-space.

Proof. Suppose $y \in Y$, then $y = f(x)$ for some $x \in X$ as $f$ is bijective. Since $X$ is $\omega\alpha T$-space, implies $\{x\}$ is either $\omega\alpha$-closed or $\alpha$-open, by theorem 3.12. If $\{x\}$ is $\alpha$-closed, then $f(\{x\}) = \{y\}$ is $\alpha$-closed, because $f$ is pre-$\alpha$-closed. Also, if $\{x\}$ is $\omega\alpha$-open, $f(\{x\}) = \{y\}$ is $\omega\alpha$-open as $f$ is $\omega\alpha$-open. Therefore $Y$ is $\omega\alpha T$-space.

Definition 4.20. A function $f : X \to Y$ is $\omega\alpha^*$-continuous if $f^{-1}(V)$ is $\omega\alpha$-closed in $X$ for every $\alpha$-closed set $V$ in $Y$.

Theorem 4.21. A function $f : X \to Y$ is $\omega\alpha^*$-continuous, if and only if $f^{-1}(V)$ is $\omega\alpha$-open in $X$ for every $\alpha$-open set $V$ in $Y$.

Proof. Similar to proof of Theorem 4.3.

Remark 4.22. Every $\omega\alpha^*$-continuous function is $\omega\alpha$-continuous. But converse need not be true in general as seen from the following example.
Example 4.23. Let \( X = Y = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \mu = \{Y, \phi, \{a\}\} \) be topologies on \( X \) and \( Y \) respectively. Define a function \( f : X \to Y \) by \( f(a) = a, f(b) = b \) and \( f(c) = c \). Then \( f \) is \( g_{\omega\alpha} \)-continuous but not \( g_{\omega\alpha}^* \)-continuous as \( A = \{b\} \) is \( \alpha \)-closed in \( Y \), but \( f^{-1}(\{b\}) = \{b\} \) is not \( g_{\omega\alpha} \)-closed in \( X \).

Remark 4.24. From above discussion we have the following implications.

\[
\text{Continuous} \xleftarrow{g_{\omega\alpha}} \text{g}_{\omega\alpha} \text{-Continuous} \xrightarrow{g_{\omega\alpha}^*} \text{g}_{\omega\alpha}^* \text{-Continuous}
\]

Theorem 4.25. If \( f : X \to Y \) is \( g_{\omega\alpha}^* \)-continuous and \( \omega\alpha \)-open and \( A \) is \( g_{\omega\alpha} \)-closed in \( Y \) where \( Y \) is \( g_{\omega\alpha} \)-space, then \( f^{-1}(A) \) is \( g_{\omega\alpha} \)-closed subset in \( X \).

Proof. Let \( A \) be \( g_{\omega\alpha} \)-closed set in \( Y \) where \( Y \) is \( g_{\omega\alpha} \)-space and \( F \) be a \( \omega\alpha \)-open set in \( X \) such that \( f^{-1}(A) \subseteq F \), then \( A \subseteq f(F) \). Since \( f \) is \( \omega\alpha \)-continuous, implies \( f(F) \) is \( \omega\alpha \)-open in \( Y \). Since \( A \) is a \( g_{\omega\alpha} \)-closed set in \( Y \), implies \( acl(A) \subseteq f(F) \). This implies \( f^{-1}(acl(A)) \subseteq F \). Since \( f \) is \( g_{\omega\alpha}^* \)-continuous and \( acl(A) \) is \( \alpha \)-closed in \( Y \), implies \( f^{-1}(acl(A)) \subseteq g_{\omega\alpha} \)-closed set in \( X \). Therefore \( acl(f^{-1}(acl(A))) \subseteq F \). This implies, \( acl(f^{-1}(A)) \subseteq acl(f^{-1}(acl(A))) \subseteq F \). That is \( acl(f^{-1}(A)) \subseteq F \). This shows that \( f^{-1}(A) \) is \( g_{\omega\alpha} \)-closed set in \( X \).

Theorem 4.26. If \( f : X \to Y \) is \( g_{\omega\alpha}^* \)-continuous and \( \omega\alpha \)-closed function and \( g : Y \to Z \) is \( g_{\omega\alpha}^* \)-continuous, then \( g \circ f \) is \( g_{\omega\alpha}^* \)-continuous.

Proof. Suppose \( F \) is \( \alpha \)-closed set in \( Z \). As \( g \) is \( g_{\omega\alpha}^* \)-continuous \( g^{-1}(F) \) is \( g_{\omega\alpha} \)-closed set in \( Y \). Now \( f \) is \( g_{\omega\alpha}^* \)-continuous and \( \omega\alpha \)-closed function, implies \( f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F) \) is \( g_{\omega\alpha} \)-closed in \( X \). Therefore \( g \circ f \) is \( g_{\omega\alpha} \)-continuous.

Theorem 4.27. Let \( f : X \to Y \) and \( g : Y \to Z \) be two functions. Then:

1. Let \( Y \) be \( g_{\omega\alpha} \)-space. If \( f \) is \( g_{\omega\alpha} \)-continuous and \( g \) is \( g_{\omega\alpha}^* \)-continuous, then \( g \circ f \) is \( g_{\omega\alpha}^* \)-continuous.
2. Let \( Y \) be \( g_{\omega\alpha} \)-space. If \( f \) and \( g \) are \( g_{\omega\alpha}^* \)-continuous, then \( g \circ f \) is \( g_{\omega\alpha}^* \)-continuous.
3. If \( f \) is \( g_{\omega\alpha} \)-continuous and \( g \) is \( \alpha \)-continuous, then \( g \circ f \) is \( g_{\omega\alpha} \)-continuous.

Proof. (1) Let \( F \) be any \( \alpha \)-closed set in \( Z \). Since \( g \) is \( g_{\omega\alpha}^* \)-continuous function and \( Y \) is \( g_{\omega\alpha} \)-space, implies \( g^{-1}(F) \) is closed in \( Y \). Since \( f \) is \( g_{\omega\alpha} \)-continuous, implies \( f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F) \) is \( g_{\omega\alpha} \)-closed set in \( X \). Therefore \( g \circ f \) is \( g_{\omega\alpha}^* \)-continuous function.

(2) Let \( F \) be any \( \alpha \)-closed set in \( Z \). As \( g \) is \( g_{\omega\alpha}^* \)-continuous function and \( Y \) is \( g_{\omega\alpha} \)-space, \( g^{-1}(F) \) is \( \alpha \)-closed in \( Y \). Since \( f \) is \( g_{\omega\alpha}^* \)-continuous, implies \( f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F) \) is \( g_{\omega\alpha} \)-closed set in \( X \). Therefore \( g \circ f \) is \( g_{\omega\alpha}^* \)-continuous function.

(3) Let \( F \) be a closed set in \( Z \). As \( g \) is \( \alpha \)-continuous, \( g^{-1}(F) \) is \( \alpha \)-closed in \( Y \). Since \( f \) is \( g_{\omega\alpha}^* \)-continuous, implies \( f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F) \) is \( g_{\omega\alpha} \)-closed set in \( X \). Therefore \( g \circ f \) is \( g_{\omega\alpha} \)-continuous function.

5. Conclusion

In this paper, the study of \( g_{\omega\alpha} \)-closed sets is continued. Further \( g_{\omega\alpha} \)-continuous and \( g_{\omega\alpha}^* \)-continuous functions in topological spaces are introduced and investigated. The notions \( g_{\omega\alpha} \)-closed sets and \( g_{\omega\alpha}^* \)-continuous functions can be used to study compact spaces, connected spaces and bitopological spaces.
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