ISOMONODROMIC DEFORMATIONS AND
SU\(_2\)–INvariant Instantons on \(S^4\)

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Abstract. Anti-self-dual (ASD) solutions to the Yang-Mills equation (or instantons) over an anti-self-dual four manifold, which are invariant under an appropriate action of a three dimensional Lie group, give rise, via twistor construction, to isomonodromic deformations of connections on \(\mathbb{C}P^3\) having four simple singularities. As is well known this kind of deformations is governed by the sixth Painlevé equation \(P_{VI}(\alpha, \beta, \gamma, \delta)\). We work out the particular case of the SU\(_2\)-action on \(S^4\), obtained from the irreducible representation on \(\mathbb{R}^5\). In particular, we express the parameters \((\alpha, \beta, \gamma, \delta)\) in terms of the instanton number. The present paper contains the proof of the result announced in [16].

1. Introduction

Instantons are particular solutions to the Yang-Mills equation, in dimension four, and if they admit a sufficiently large symmetry group (a three dimensional one acting with cohomogeneity one) it is possible to make a one dimensional reduction of the self-duality equation. We will focus on the fact that these instantons, for an appropriate action of a Lie group, are related to an ordinary differential equation on the complex domain, which is known as the sixth Painlevé equation. We denote it by \(P_{VI}(\alpha, \beta, \gamma, \delta)\) and it is given by

\[
\frac{d^2y}{dx^2} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) \left( \frac{dy}{dx} \right)^2 - \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) \left( \frac{dy}{dx} \right) + \left( \alpha + \beta \frac{x}{y^2} + \gamma \frac{x-1}{(y-1)^2} + \delta \frac{x(x-1)}{(y-x)^2} \right) \frac{y(y-1)(y-x)}{x^2(x-1)^2},
\]

where \(\alpha, \beta, \gamma\) and \(\delta\) are complex parameters. This equation was found by Richard Fuchs in 1907 and is very important in the ODE’s theory over the complex domain. Its general solutions are transcendental functions and its key property is that it does not have movable critical points, what is usually called Painlevé’s property [7]. The method used by Fuchs, which is the way \(P_{VI}\) appears here, is that of isomonodromic deformations.

Our research is inspired by the work of Nigel Hitchin [8], where \(P_{VI}\) is related to some new examples of Einstein metrics. The basic idea is to use the symmetry group to produce a connection on the twistor space with logarithmic singularity along an anticanonical divisor. Since the twistor space is a fibre
bundle with fibre $\mathbb{C}P^1$ over the real 4-manifold, and the divisor will intersect a fibre generically in four distinct points, the connection gives rise to an isomonodromic deformation of connections on $\mathbb{C}P^1$ having four simple poles. This kind of deformations is governed by the sixth Painlevé equation, which was proved in [11], and this gives us the relation with invariant instantons mentioned before.

We apply the construction to the example of $SU_2$ acting on $S^4$ via the irreducible representation on $\mathbb{R}^5$; this action has two exceptional orbits of dimension two and all the others are three dimensional. The $SU_2$-invariant instantons are classified by the integer numbers of the form $n \equiv 1 \mod 4$. For each $n$ the Chern number of the instanton is $c_2 = \frac{n^2 - 1}{8}$ (see [4] or [5]). We will prove that the parameters of PV1 related to these instantons are given by

\[
\begin{align*}
\alpha^\pm &= \frac{1}{8}(n \pm 2)^2 \\
\gamma &= \frac{1}{8}n^2 \\
\beta &= -\frac{1}{8}n^2 \\
\delta &= -\frac{1}{8}(n^2 - 4).
\end{align*}
\]

It is known that for these parameters the equations are algebraically related ([17]). These values for the parameters were conjectured by Jan Segert in [18], but there has been no proof in the literature until now. Our point of view here is quite different from that of [18], since we use a more geometric approach.

We also make some explicit calculations by means of which it is possible to express the solution to Painlevé’s equation in terms of the corresponding instanton.

Finally, we would like to mention here that it is possible to look at instantons which are singular along the exceptional orbits. For instance instantons having a holonomic singularity as those considered by Kronheimer and Mrowka in [12]. It is not difficult to show what the corresponding parameters would be for these instantons in terms of the holonomy parameter. Unfortunately we are yet unable to prove that such instantons exist with the property of being invariant under the action, even in the case of $S^4$.

2. Preliminaries

In this section we remember the definition of the singular connection on the Penrose-Ward transform of an anti-self-dual (ASD) vector bundle, induced by the action of a three dimensional Lie group $G$. We review some of its properties, in particular that the singularity is logarithmic along an anticanonical divisor of the twistor space. This connection gives rise to an isomonodromic deformation of connections on $\mathbb{C}P^1$, and this leads us to the sixth Painlevé equation.

2.1. Cohomogeneity one ASD manifolds and their associated connection. In what follows $M$ will denote an ASD (in particular four dimensional) manifold, so we have its twistor space $Z$, and $G$ will be a compact three dimensional Lie group (we can think for example in $SU_2$). We suppose
that $G$ acts on $M$ having three dimensional principal orbit and preserving the conformal structure. In particular, the principal stabilizer $\Gamma \subset G$ is discrete and $M/G$ is diffeomorphic to $\mathbb{R}$, $S^1$, $[0,1)$ or $[0,1]$. We can always take a section $M/G \to M$ which intersects all the orbits perpendicularly. Such an action induces an action on the twistor space by biholomorphisms, that preserves the real structure and the twistor lines. At the infinitesimal level we can write the action as a map $\alpha: Z \times \mathfrak{g} \to TZ$, which when complexified is a homomorphism

$$(2.1) \quad \alpha: Z \times \mathfrak{g}_\mathbb{C} \to TZ.$$  

(We denote by $TZ$ the holomorphic tangent bundle of $Z$.) For the following definitions to make sense we will assume that $\alpha$ has rank three.

Set $Y = \{z \in Z : rk(\alpha_z) \leq 2\} \subset Z$. By looking at $\alpha$ as a section of $TZ \otimes \mathfrak{g}_\mathbb{C}$ we have that $s := \Lambda^3 \alpha \in H^0(Z, \Lambda^3 TZ \otimes \Lambda^3 \mathfrak{g}_\mathbb{C}) \cong H^0(Z, K_Z^{-1})$, is a section of the anticanonical bundle of $Z$. The set $Y$ is then exactly the zero locus of $s$, and it is nonempty because the anticanonical divisor has degree four restricted to each real twistor line. Then $Y$ is a codimension one analytic subvariety of $Z$; we denote by $Y^o$ the smooth part of $Y$.

From the fact that the action preserves the real structure we conclude that $\alpha$ is compatible with this structure, in the sense that the following identity is satisfied

$$d\tau \circ \alpha_z = \alpha_{\tau(z)}.$$  

This implies that $s$ restricted to a line vanishes identically, has two double zeroes in two antipodal points, or vanishes non degenerately in four points forming antipodal pairs. For a more detailed explanation see [8].

Let us now consider a Hermitian vector bundle $E$ over $M$ endowed with an ASD connection $\mathcal{D}$, and structure group $S = SU_2$, together with a lift of the $G$-action by homomorphisms leaving the connection invariant. Denote by $\tilde{E}$ the holomorphic vector bundle over $Z$ induced by the pair $(E, \mathcal{D})$; usually known as Penrose-Ward transform [3]. It is possible to define a connection $\nabla$ in $\tilde{E}|_{Z \setminus Y}$, in the following way (as is done in [15]) :

$$\nabla_{\alpha(X)} := \mathcal{L}X \quad \forall X \in \mathfrak{g}_\mathbb{C}.$$  

Where $\mathcal{L}X$ is Lie derivative of sections. It is not difficult to verify the integrability of this connection.

**Proposition 2.1.** The connection $\nabla$ defined above in $\tilde{E}|_{Z \setminus Y}$ is flat.

**Proof.** The statement has a simple geometric interpretation. The Lie algebra $\mathfrak{g}_\mathbb{C}$ acts on $\tilde{E}$, which means that we have a Lie algebra homomorphism $\tilde{\alpha}: \mathfrak{g}_\mathbb{C} \to \mathfrak{X}(\tilde{E})$. The horizontal distribution of the connection $\nabla$ is simply the image of $\tilde{\alpha}$, from which we conclude its integrability since the Frobenius condition is necessarily satisfied. When the $G$ action can be extended to a $G^\mathbb{C}$ action the horizontal foliation is given by its orbits and the connection is $G^\mathbb{C}$-invariant. $\square$
By its definition $\nabla$ satisfies a compatibility condition with the real structure. These means that if $\sigma: \tau^*\tilde{E} \to \tilde{E}^*$, is the isomorphism given by \cite{3} theorem ?, which can be expressed as $\sigma = h \circ \hat{\tau}$, where $\hat{\tau}$ is the obvious lifting of $\tau$ and $h: \tilde{E} \to \tilde{E}^*$ is the isomorphism induced by the Hermitian structure, then

\begin{equation}
\sigma(\tau^*\nabla) = \nabla^*.
\end{equation}

Or in coordinates,

\begin{equation}
\tau^*A = -A^*;
\end{equation}

where $A$ is the connection 1-form.

**Example 2.1.** Let $M$ be a $G$-invariant conformal ASD manifold, such that $\alpha$ has rank three. We can think about the examples 2.1 and 2.2, or in \cite{8} more examples for $G = SU_2$ can be found.

Let $(V, \rho)$ be a two dimensional representation of $G$, and let us take the trivial vector bundle over $M$, $E = M \times V$, with the trivial flat connection.

The $G$ action on $E$ is simply the diagonal one

$$g \cdot (x, v) = (gx, \rho(g)v).$$

Then $\tilde{E} = Z \times V$, and the connection 1-form of $\nabla$ is given by

$$-\dot{\rho} \circ \alpha^{-1}: T(Z \setminus Y) \to \text{End}(V).$$

This can be seen by using the definition; more precisely, if $g(t)$ is a curve in $G$ such that $g(0) = I$, $\hat{g}(0) = X$, then it follows that

$$\nabla_X s(z) = \mathcal{L}_{\alpha^{-1}}(X)s(z) = \left. \frac{d}{dt} \right|_{t=0} \rho(g(t)^{-1})s(g(t)z) = (X - \dot{\rho}(\alpha^{-1}(X)))s(z).$$

The above is the connection considered by Hitchin in \cite{8}, for $G = SU_2$ and $(V, \rho)$ the canonical representation.

### 2.2. Logarithmic singularities

What is important about the connection $\nabla$ is its behaviour near the singularity. It has what is called logarithmic singularity along $Y^0$, which means that $\nabla$ has a simple pole along $Y^0$ and that it induces a flat connection on $\tilde{E} |_{Y^0}$ without singularities. We essentially follow Malgrange’s point of view in \cite{14} about this subject.

**Definition 2.1.** (1) A differential form $A$ in a complex manifold $X$ is said to have a simple pole along a smooth hypersurface $Y$, if it can be extended to a holomorphic section of $\Lambda^k T^* X \otimes [Y]$; here $[Y]$ is the line bundle associated to $Y$.

(2) Let $i: Y \hookrightarrow X$ be the inclusion map. We say that a 1-form $A$ has logarithmic singularity along $Y$ if it has a simple pole there and $i^* A = 0$. We are viewing $A$ as an element in $H^0(X, T^* X \otimes [Y])$ with the obvious extension of the map $i^*$.

The above definition can be expressed locally as follows. Let $A$ be a 1-form with a simple pole along $Y$, and let us take a coordinate system $(z_1, \ldots, z_n)$ in such a way that $Y$ is determined by $\{z_1 = 0\}$. In these coordinates
A = \sum j a_j dz_j , and as it has simple pole on Y a_j = \frac{b_j}{z_1}, with b_j holomorphic for all j. Therefore \( i^* A = \sum_j b_j|_Y i^* dz_j = \sum_{j \geq 2} b_j|_Y d(z_j|_Y) \), from which \( a_j \) is holomorphic for \( j \geq 2 \). Namely,

\[
A = \frac{b_1}{z_1} dz_1 + \sum_{j \geq 2} a_j dz_j,
\]

with \( \sum_{j \geq 2} a_j dz_j \) holomorphic. This is equivalent to saying that \( A = \frac{B}{z_1} \), \( B \) being holomorphic and \( B|_{TY} = 0 \). It can also be easily seen that \( A \) has logarithmic singularity along \( Y \) if and only if \( A \) and \( dA \) both have simple poles there.

Let us suppose now that \( E \) is a vector bundle over \( X \) endowed with an integrable connection defined on \( X \setminus Y \) (with \( X \) and \( Y \) as in definition 3.1). Let \( A \) be the connection 1-form associated to a local frame of \( E \).

**Definition 2.2.** We say that \( \nabla \) has a simple pole or a logarithmic pole along \( Y \) if \( A \) has a simple pole or a logarithmic pole along \( Y \) respectively.

**Remark 2.1.** If \( \nabla \) is a connection in \( E \to X \) with a logarithmic singularity along \( Y \), and \( f: Z \to X \) is a holomorphic function which is transversal to \( Y \), then \( f^* \nabla \) is a connection with logarithmic singularity along \( f^{-1}(Y) \).

**Theorem 2.2** (Hitchin). If \( s \) vanishes non degenerately, then the 1-form \( \alpha^{-1}: TZ \to g_C \) has logarithmic singularity along the smooth part of \( Y \).

From the above proposition together with proposition 2.2 one can easily deduce the following result:

**Corollary 2.3.** If \( s \) vanishes non degenerately, the connection associated to the action of \( G \) has a logarithmic singularity along the smooth part of \( Y \).

From now on we will consider only actions such that \( s \) vanishes non degenerately, since we are interested in connections having a logarithmic singularity. In a generic twistor line \( P \) the connection 1-form can be written as

\[
A_P(z)dz = \left( \frac{A_1(P)}{z - z_1} + \frac{A_2(P)}{z - z_2} + \frac{A_3(P)}{z - z_3} + \frac{A_4(P)}{z - z_4} \right) dz,
\]

where the \( A_i(P) \)'s \( \in g_C \) satisfy \( \sum_i A_i = 0 \), and \( (z_1, z_2) \) and \( (z_3, z_4) \) are antipodal pairs. (We are implicitly using an identification of \( P \) with \( \mathbb{C}P^1 \).) In view of the compatibility with the real structure, the following relations between the residues hold

\[
A_2 = -A_1^*, \quad A_4 = -A_3^*.
\]

**2.3. Isomonodromic deformations and the sixth Painlevé equation.**

By a monodromy representation on a space \( B \) we mean a representation of the fundamental group of \( B \), with values in some group \( \mathcal{G} \), which is compatible with change of base point isomorphisms. An isomonodromic deformation is just a family of monodromy representations which is constant modulo conjugation. More precisely, by an isomonodromic deformation on \( F \) we mean
a fiber bundle with fiber $F$, base space $X$ and total space $B$, together with a monodromy representation $(\rho_x, G)$ for each $x \in X$ such that the translation homomorphisms [19] are given by conjugation on $G$.

The family of connections $\nabla|_p$ parametrized by twistor lines defines an isomonodromic deformation (see [20]). As is shown in [8] the four points of intersection of a generic line with $Y$ have nonconstant cross ratio (as we change the line). Then, we can assume the intersection points to be $\{0, 1, x, \infty\}$ being $x$ the cross ratio, after an appropriate Möebius transformation. Doing so, the connection is given by

$$A(x, \zeta)d\zeta = \left(\frac{A_1(x)}{\zeta} + \frac{A_2(x)}{\zeta - 1} + \frac{A_3(x)}{\zeta - x}\right) d\zeta.$$ 

By the isomonodromic property the residues have to satisfy the Schlesinger’s equations

$$\frac{dA_1}{dx} = \frac{[A_1, A_3]}{x}, \quad \frac{dA_2}{dx} = \frac{[A_2, A_3]}{x - 1}, \quad \frac{dA_3}{dx} = \frac{[A_1, A_3]}{x} - \frac{[A_1, A_3]}{x - 1};$$

the last one simply says that $A_1 + A_2 + A_3 = -A_\infty = \text{constant}$. Note that the functions $\text{tr}(A_i^2)$ are constant in the deformation. We refer to [14] for the details, see also [8, 9, 10].

The following fact is central for our work and can be found in [11]. The proof is as well done by Mahoux in [13].

**Proposition 2.4 (Jimbo-Miwa).** Let $y(x) \in \mathbb{C}P^1 \setminus \{0, 1, x, \infty\}$ be the point at which $A(x, y(x))$ and $A_\infty$ have a common eigenvector, corresponding to the eigenvalue $\lambda$ of $A_\infty$. If the $A_i$ satisfy Schlesinger’s equations, then $y(x)$ satisfies the Painlevé equation $P_{VI}(\alpha, \beta, \gamma, \delta)$

$$\frac{d^2y}{dx^2} = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - x}\right) \left(\frac{dy}{dx}\right)^2 - \left(\frac{1}{x} + \frac{1}{x - 1} + \frac{1}{y - x}\right) \left(\frac{dy}{dx}\right) + \left(\alpha + \beta \frac{x}{y^2} + \gamma \frac{x - 1}{(y - 1)^2} + \delta \frac{x(x - 1)}{(y - x)^2}\right) \frac{y(y - 1)(y - x)}{x^2(x - 1)^2},$$

with parameters

$$\alpha = \frac{1}{2}(2\lambda - 1)^2, \quad \beta = 2\text{det}(A_0), \quad \gamma = -2\text{det}(A_1), \quad \delta = \frac{1}{2}(1 + 4\text{det}(A_2)).$$

3. **Relations between $\nabla$ and the Yang-Mills connection**

In this section we make some local calculations in the general set up, mostly to settle the notation. Many expressions will be essential in section 4.4.
3.1. The connection 1-form in a real twistor line. Take $x \in M$ with principal orbit type, and let $p^{-1}(x)$ be the real twistor line determined by it. Fix $z \in p^{-1}(x) \setminus Y$ and take a parallel section of $\tilde{E}$ with respect to the flat connection in a neighbourhood of $z$. Denote this section by $\tilde{u}$. We can take a “locally equivariant” trivialization (in the sense that it is equivariant for a neighbourhood of the identity) of $\tilde{E}$ in a neighbourhood of $x$, $\{s_1, s_2\}$. Let $\Phi$ be the Yang-Mills connection 1-form associated to this frame.

When one considers the frame $\{p^*s_1|_{p^{-1}(x)}, p^*s_2|_{p^{-1}(x)}\}$ one has a holomorphic trivialization of $\tilde{E}|_{p^{-1}(x)}$. With respect to it the identity

$$T\tilde{u}(z) = -A(T)\tilde{u}(z)$$

is satisfied, $T$ being a tangent vector to the twistor line at $z$.

Let $\{X_i\}$ be a basis of $\mathfrak{g}$; then there exist 1-forms $\alpha_i$, $i = 1, 2, 3$ such that $T = \alpha(\sum_i \alpha_i(T)X_i) = \sum_i \alpha_i(T)\alpha(X_i)$. Let $u$ be the local section of $E$ defined by $u(p(gz)) = \tilde{u}(gz)$ for $g$ in a suitable neighbourhood of the identity of $G$. Hence, by its definition $u$ verifies the equation

$$X_i \cdot u = 0.$$

Using the fact that $T$ is tangent to the fibre, that $\tilde{u}$ is holomorphic, and that by the definition of the holomorphic structure on $\tilde{E}$ one has $(p^*\mathcal{D})^{0,1} = \tilde{\mathcal{D}}$ (remember that $\mathcal{D}$ denotes the anti-self-dual connection), one obtains that

$$T\tilde{u}(z) = (p^*\nabla)T\tilde{u}(z) = (p^*\nabla)\sum_i \alpha_i(T)\alpha(X_i)\tilde{u}(z) = \sum_i \alpha_i(T)(p^*\nabla)\alpha(X_i)\tilde{u}(z)$$

$$= \sum_i \alpha_i(T)p^*(\nabla X, u)(z) = \sum_i \alpha_i(T)p^*\Phi(\alpha(X_i))\tilde{u}(z).$$

Then we finally arrive at the following relation between the two connections we are working with

**Proposition 3.1.** Let $x \in M$ be a point with principal orbit type, $\Phi$ be the Yang-Mills connection 1-form with respect to a locally equivariant frame, and $\alpha_i$ the 1-forms on $P = p^{-1}(x)$ such that $\alpha^{-1} = \sum_{i=1}^3 \alpha_iX_i$. Then, the connection 1-form of $\nabla|_P$ is given by

$$A_P = -\sum_{i=1}^3 \alpha_i\Phi(X_i(x)).$$

3.2. The anti-self-duality equation. The invariance of the anti-self-dual connection imposes restrictions on the degrees of freedom of $\Phi$. For example, because the orbit space $M/G$ is one dimensional, the anti-self-dual equation is reduced to an ordinary differential equation in one variable. In what follows we will use this fact to describe our connections in a more explicit way.

Remember that it is possible to take a section $c: (0, 1) \to M$ intersecting perpendicularly each three dimensional orbit. Take an open set $U$ of the class of the identity in $G/\Gamma$ such that $G \to G/\Gamma$ is trivial as a principal
bundle. We can choose a frame \( \{s_1, s_2\} \) of \( E \) on \((0,1) \times U \) in such a way that the following is satisfied:

\[
D_c s_i = 0, \quad g \cdot s_i(c(t)) = s_i(g \cdot c(t)).
\]

Let now \( \Phi \) be the connection 1-form of \( D \) with respect to such a frame; since the chosen frame is equivariant equation (3.1) is satisfied. Because of the invariance we have that \( D \) is determined (in the complement of the exceptional orbits) by the map

\[
\phi : (0,1) \longrightarrow s \otimes g^*
\]
given by \( \phi_t(X) = \Phi(c(t))(X(c(t))) \); \( s \) is the Lie algebra of the structure group of \( E \).

For each \( \gamma \in \Gamma \) we define a new frame \( \{s_1^\gamma, s_2^\gamma\} \) of \( E \) by the expression

\[
s_1^\gamma(x) = \gamma \cdot s_1(\gamma^{-1}x).
\]

The connection 1-form of \( \gamma \cdot D \) with respect to this new frame is given by \( \Phi^\gamma = \gamma^* \Phi \).

Now, if \( \lambda_\gamma : (0,1) \times U \rightarrow S \) is the change of basis matrix from \( \{s_i\} \) to \( \{s_i^\gamma\} \), using the invariance of \( D \) we have that

\[
\gamma^* \Phi = \Phi^\gamma = \lambda_\gamma(x) \Phi \lambda_\gamma(x)^{-1} + \lambda_\gamma(x) d \lambda_\gamma(x)^{-1}.
\]

It is easy to see that \( \lambda_\gamma \) is constant. First let us see that it does not depend on \( t \):

\[
D_c s_i^\gamma = D_c (\gamma s_i)^{\gamma^{-1}} = (\gamma \cdot D)_c s_i = D_c s_i = 0,
\]

and on the other hand

\[
D_c s_i^\gamma = \sum_j \left( \frac{\partial \lambda_j}{\partial t} \right)_{ij} s_j;
\]

puting both expressions together we have \( \frac{\partial \lambda_j}{\partial t} = 0 \). It is straightforward to see that for fixed \( t \) the equation \( \lambda_\gamma(x) = \lambda_\gamma(c(t)) \) is satisfied for all \( x \in \{t\} \times U \). Then, \( \lambda_\gamma \) is constant and

\[
\gamma^* \Phi = \lambda_\gamma \Phi \lambda_\gamma^{-1}.
\]

Applying the above result at the point \( c(t) \) we obtain

(3.2) \[ \text{Ad}_{\lambda_\gamma}(\phi_t(X)) = \phi_t(\text{Ad}_{\gamma}(X)) \quad \forall \gamma \in \Gamma. \]

To find the anti-self-dual equation in terms of \( \phi_t \) let us write

\[
\phi_t = \phi_1 \otimes \sigma_1 + \phi_2 \otimes \sigma_2 + \phi_3 \otimes \sigma_3,
\]

where \( \phi_i : (0,1) \rightarrow s \) and \( \{\sigma_1, \sigma_2, \sigma_3\} \) is the dual basis to the one chosen for \( g \). The curvature of \( D \) evaluated at \( c(t) \) is then given by

\[
F^D_t = (d\Phi)_c(t) - \Phi_c(t) \wedge \Phi_c(t)
\]

\[
= \sum_i \dot{\phi}_i dt \wedge \sigma_i + \sum_{i<j} \left( \sum_k C_{ij}^k \dot{\phi}_k - [\phi_i, \phi_j] \right) \sigma_i \wedge \sigma_j,
\]
where $C_{ij}^k$ are the structure constants of $G$. Taking $c$ such that $\{c, X_1, X_2, X_3\}$ is a positively oriented basis for $TM$ we will have that the anti-self-dual equation $\ast F^D_t = -F^D_t$ can be written as

$$K_1 \dot{\phi}_1 = -\sum_{\ell=1}^{3} C_{23}^\ell \phi_\ell + [\phi_2, \phi_3], \ldots$$

(3.3)

$K_1 = \frac{||X_2|| ||X_3|| dt}{||X_1||}$ being the coefficient coming from the fact that the chosen basis is not orthonormal, plus other two equations that are obtained by cyclic permutations of the indexes $1,2,3$.

Remark 3.1. As we had seen before $\text{tr}(A^2_\infty)$ is constant in the deformation. Using this fact together with equation (3.1) we obtain that the function

$$\sum_{ij} \alpha_{i,\infty} \alpha_{j,\infty} \text{tr}(\phi_i \phi_j) = \text{tr}(A^2_\infty) = 2 \lambda^2$$

is an integration constant for equation (3.3); we denote by $\alpha_{i,\infty}(t)$ the residue at $\infty$ of the 1-form $\alpha_i$ on the line $P_i$. The same can be done with the other residues, but because of relations (2.5) between them only two independent constants of movement remain (in principle).

4. Painlevé’s equation and SU$_2$-invariant instantons on $S^4$

4.1. The SU$_2$ action on $S^4$. The SU$_2$ action on $S^4$ that we will consider is that coming from the irreducible representation in $\mathbb{R}^5$, which is obtained naturally by taking the SO$_3$ conjugacy action on trace free symmetric $3 \times 3$ matrices with real coefficients. The generic orbit is three dimensional and there are two exceptional orbits of dimension two, then $S^4/\text{SU}_2 \cong [0,1]$; for this description consult for example [4]. For our purposes it is more convenient to describe it from another point of view, starting with the complex four dimensional irreducible representation of SU$_2$. We identify $\mathbb{C}^4$ with the space of homogeneous polynomials of degree three, in two variables, and with complex coefficients

$$\mathbb{C}^4 \cong \left\{ p(x, y) = z_1x^3 + \sqrt{3}z_4x^2y + \sqrt{3}z_3xy^2 + z_2y^3 : z_i \in \mathbb{C}, i = 1, 2, 3, 4 \right\}.$$  

The action is as usual: SU$_2$ acts on the variables by matrix multiplication

$$\left( \begin{array}{cc} a & b \\ -b & a \end{array} \right) \cdot (x, y) = (x, y) \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right)^t = (ax + by, -bx + ay),$$

and on polynomials by precomposition $g \cdot p(x, y) = p(g^{-1} \cdot (x, y))$. The above representation is quaternionic if we identify $\mathbb{C}^4$ with $\mathbb{H}^2$ by the map

$$(z_1, z_2, z_3, z_4) \mapsto (z_1 + z_2j, z_3 + z_4j),$$

and preserves the canonical bilinear form of $\mathbb{H}^2$ (this is what the $\sqrt{3}$ factors are necessary for). We are considering $\mathbb{H}^2$ as a left $\mathbb{H}$-module.
When we consider the action on quaternionic lines we obtain the SU\(_2\) action on \(S^4\) (remember the identification \(\mathbb{H}P^1 \cong S^4\)), and looking at the action on complex lines we obtain the action on \(\mathbb{C}P^3\) (the twistor space of \(S^4\)). Denote by \([h_1, h_2]_{\mathbb{C}}\) a point in \(\mathbb{C}P^3\) and by \([h_1, h_2]_{\mathbb{H}}\) the corresponding point under twistor projection in \(\mathbb{H}P^1\), where \((h_1, h_2) \in \mathbb{H}^2\). The description of the twistor space of \(S^4\) can be found in [1].

In this example the divisor \(Y\) is the quartic in \(\mathbb{C}P^3\) of cubic polynomials having a repeated root, i.e. the zero locus of the discriminant. The SU\(_2\) action on \(\mathbb{C}P^3\) can be extended to an \(SL_2(\mathbb{C})\)-action. Writing a polynomial in the form \(p(x, y) = y^3q(\frac{x}{y})\), \(SL_2(\mathbb{C})\) acts in \(q\) by Möbius transformations in its roots. There are three different orbits, one is the dense one of dimension three \(\mathbb{C}P^3 \setminus Y \cong SL_2(\mathbb{C})/S_3\) (\(S_3\) is the permutation group of three elements), and the others are of dimensions two and one. From this point of view the set \(Y\) is the union of the lower dimensional orbits.

A line in \(\mathbb{C}P^3\) is determined by two linearly independent polynomials \(q_1, q_2\). The points of intersection with \(Y\) correspond to those polynomials of the form \(q_1 + \zeta q_2\) which have zero discriminant, namely, those for which \(z\) is such that the system

\[
q_1 + \zeta q_2 = 0
\]

\[
q_1' + \zeta q_2' = 0
\]

has a solution. Therefore \(\zeta = -\frac{q_1(\alpha)}{q_2(\alpha)}\), for a root \(\alpha\) of \(q_1'q_2 - q_1q_2'\). For a generic line this polynomial has degree four, with four distinct roots, each of which gives a point of intersection of the line with \(Y\). So, we have that the divisor intersects a generic line in four different points.

The two bidimensional orbits in \(S^4\) are those of the points \(x_0 = [1, 0]_{\mathbb{H}}\) and \(y_0 = [0, 1]_{\mathbb{H}}\), whose stabilizer is the subgroup

\[
\tilde{O}_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} : |a|^2 = 1 \right\} \cup \left\{ \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} : |b|^2 = 1 \right\} \subset SU_2
\]

(we denote the elements of this subgroup simply by \(a\) in the diagonal case, and by \(b\) in the other case), what can be easily verified. In fact we have

\[
\begin{align*}
a \cdot x^3 &= \bar{a}^3 x^3 & a \cdot xy^2 &= ax y^2 \\
b \cdot x^3 &= -b^3 y^3 & b \cdot xy^2 &= -bx^2 y
\end{align*}
\]

In view of this the two exceptional orbits are isomorphic to \(SU_2/\tilde{O}_2 \cong SO_3/O_2 \cong \mathbb{R}P^2\), and we will denote them by \(\mathbb{R}P^2\). The real twistor lines over the points \(x_0\) and \(y_0\) are given by

\[
p^{-1}(x_0) = \{ [h, 0]_{\mathbb{C}} : h \in \mathbb{H} \} \quad p^{-1}(y_0) = \{ [0, h]_{\mathbb{C}} : h \in \mathbb{H} \},
\]

and they have as stabilizer, under the \(SL_2(\mathbb{C})\) action, the subgroup

\[
\left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} : a \in \mathbb{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} : b \in \mathbb{C}^* \right\};
\]
what can be proved with a simple calculation. Moreover, in each of these lines there are only two points with non discrete stabilizer, which tells us that they intersect $Y$ in exactly two points. In particular none of them is transversal to $Y$. The intersection can be written explicitly in the following way:

$$p^{-1}(x_0) \cap Y = \{(1,0)_c; [j,0]_c\}$$

$$p^{-1}(y_0) \cap Y = \{(0,1)_c; [0,j]_c\}.$$

Observe that the stabilizer of the point $[1,0]_c$ is the subgroup of lower triangular matrices and the stabilizer of $[j,0]_c$ is the subgroup of upper triangular matrices, while that of the points $[0,1]_c$ and $[0,j]_c$ is the subgroup of diagonal matrices. Therefore, the first two are in the singular part of $Y$ and the other two in the smooth one (i.e. in the two dimensional orbit).

The subgroup $\Gamma = \{\pm 1, \pm i, \pm j, \pm k\} \subset SU_2$ has as fixed point set the maximal circle $\Sigma := \{[s,t]_H : s,t \in \mathbb{R}\}$ (which intersects perpendicularly all the orbits), and we can see that

$$\sigma(x^+) \cap \Sigma = \{[1,0]_H, [1,\sqrt{3}]_H, [-1,\sqrt{3}]_H\}$$

$$\sigma(x^-) \cap \Sigma = \{[\sqrt{3},1]_H, [0,1]_H, [-\sqrt{3},1]_H\},$$

where $x^+ = [1,0]_H = x_0$ and $x^- = [\sqrt{3},1]_H$.

We parametrize the segment on $\Sigma$ from $x^+$ to $x^-$ by $c : [0,1] \rightarrow S^4$, given by $c(t) = [\sqrt{3},t]_H$ (this parametrization is not geodesic but it simplifies some calculations). This segment intersects each orbit in exactly one point and its ends are on the exceptional orbits. For each $t$ we have a real twistor line that we will denote by $P_t$:

$$P_t = p^{-1}(c(t)) = \{[z(x^3 + tx^2y) + w(tx^2y + y^3)]_c : z, w \in \mathbb{C}\} \subset \mathbb{C}P^3,$$

or equivalently $P_t = \{(\sqrt{3}z : \sqrt{3}w : tz : tw) \in \mathbb{C}P^3 : z, w \in \mathbb{C}\}$.

In order to obtain the intersection $Y \cap P_t$ we do as above. The polynomials $q_1, q_2$ corresponding to the line $P_t$ are $q_1(x) = x^3 + tx$, $q_2(x) = tx^2 + 1$.

Then, the intersection points are those for which $\zeta = -\frac{q_1(\alpha)}{q_2(\alpha)}$, where $\alpha$ is a root of

$$q_1'q_2 - q_1q_2' = tx^4 + (3-t^2)x^2 + t.$$

The last polynomial has four roots if $t^4 - 10t^2 + 9 = (t^2 - 1)(t^2 - 9) \neq 0$ and $t \neq 0$, then $|P_t \cap Y| = 4$ when $t \in (0,1)$ and $|P_0 \cap Y| = |P_1 \cap Y| = 2$.

4.2. Description of the invariant instantons over $S^4$. If we want to apply the description described in section 2, and so relate solutions to the Painlevé sixth equation with anti-self-dual solutions to the Yang-Mills equation, we need to have rank two vector bundles over $S^4$ which are $SU_2$-equivariant with respect to the action described in 4.1. On the other hand we require that they admit an ASD invariant connection. For the construction of such bundles we refer to [4] and [5]. In the present section we will give a description of the results. Remember that the orientation of $S^4$ considered
here is the standard one of $\mathbb{H}P^1$, with respect to which the Hopf bundle is self-dual ($c_2 = -1$).

Suppose that $E \rightarrow S^4$ is a rank two Hermitian vector bundle on which $SU_2$ acts covering the action on $S^4$ described in 4.1. Our action on $S^4$ has two exceptional orbits, of dimension two, corresponding to the points $x^+$ and $x^-$. The fibres of $E$ over these points have to be invariant under their respective stabilizers, then they are two dimensional representations of $\tilde{O}_2 \subset SU_2$. By means of a simple exercise it is possible to find all the representations of this kind, which are classified by the integers congruent with 1 modulo 4 and zero (the trivial representation). Restricted to the diagonal subgroup $S^1 \subset \tilde{O}_2$ each non trivial representation splits as $\mathbb{C}_{(r)} \oplus \mathbb{C}_{(-r)}$, for a positive odd integer or zero $r$, $\mathbb{C}_{(r)}$ being the one dimensional representation of $S^1$ with weight $r$. According to the previous considerations, each lifting of the action on $S^4$ to a rank two vector bundle defines a pair of integers $(n_+, n_-)$, where $n_+ \equiv 1 \mod 4$, 0.

Conversely, it is proved that such a pair of integers uniquely determines (modulo $SU_2$-isomorphisms) a vector bundle $E \rightarrow S^4$ together with an $SU_2$ action (by morphisms) covering the given action on $S^4$. The vector bundles are obtained from the trivial one over the open sets $U^\pm := S^4 \setminus \mathbb{R}P^2_\pm$ by means of a “clutching construction”. We summarize the previous discussion in the following proposition. For a detailed proof consult [4].

**Proposition 4.1** (Bor). For each pair of integers $(n_+, n_-)$ as above, there exists a complex vector bundle $E_{(n_+, n_-)} \rightarrow S^4$ with $SU_2$-action, lifted from $S^4$, such that $n_+$ and $n_-$ are the weights of the representations of the singular stabilizers on the corresponding fibres $E_{x^+}$ and $E_{x^-}$. Moreover, the Chern number of the vector bundle corresponding to $(n_+, n_-)$ is given by

$$c_2 = \frac{n_+^2 - n_-^2}{8}.$$ 

It is possible to prove that those vector bundles for which $n_+ > 1$ and $n_- > 1$ do not admit any self-dual or anti-self-dual $SU_2$-invariant connection ([6]). But using an equivariant version of the ADHM construction one can obtain the following statement [6]:

**Proposition 4.2.** For each $n = 1 \mod 4$ there exists a unique $SU_2$-invariant anti-self-dual connection on $E_n := E_{(1,n)}$ (i.e., for $n_+ = 1$).

And these vector bundles are just what we needed in order to apply the general construction mentioned in section 2, since as we have seen, the lines in the family $P_t$ intersect the divisor in four different points for $t \in (0,1)$, and then the hypotheses of proposition 2.5 are satisfied.

4.3. **The parameters of $P_{V1}$.** Using the $SL_2(\mathbb{C})$ action it is possible to calculate the parameters of $P_{V1}$ corresponding to the invariant instantons described in the previous section in a geometric way, though they can be found also by means of the expression (3.4) as we will see later. Let $z \in Y^\circ$, $U$
be a trivializing neighbourhood of $E_n$ and $(z_1, z_2, z_3)$ be coordinates on $\mathbb{C}P^3$ such that $Y$ is defined by $z_1 = 0$. Consider a rational curve $\xi: \mathbb{C}P^1 \to \mathbb{C}P^3$ transversal to $Y$ at $z$. The 1-form $\xi^* A$, defined on $\xi^{-1}(U \setminus Y)$, has logarithmic pole at $\xi^{-1}(z)$ with residue $B(z)$, i.e. equal to the residue of $A$ evaluated at the point $z$. Thus, to find the residue of $A$ at a given point it is sufficient to know $A$ restricted to a rational line, transversal to $Y$ at that point. To find the parameters of Painlevé’s equation we have to calculate the residues, or equivalently the trace of its squares. Observe that if $z \in Y^\circ$ and $g \in \text{SL}_2(\mathbb{C})$, then the residue of the connection at the point $gz$ is conjugate to the residue at $z$ since as the connection is $\text{SL}_2(\mathbb{C})$-invariant we can take a trivialization of $E_n$ around $gz$ for the connection 1-form to be $g^* A$; thus, the trace of the squares of the residues does not depend on the point of $Y^\circ$ at which we calculate it. From these two observations we conclude that it is sufficient to take a point in $Y^\circ$, a twistor line transversal to $Y^\circ$ at that point, and calculate the residue of the connection restricted to the line at such point. For that we will fix $z_0 = [0, 1]_C \in Y^\circ$; the stabilizer of $z_0$, as was mentioned before, is \( \left\{ \left( \begin{array}{cc} a & 0 \\ 0 & \frac{1}{a} \end{array} \right) : a \in \mathbb{C}^* \right\} \cong \mathbb{C}^* \). The group $\mathbb{C}^*$ acts on $\mathbb{C}^4$ in the following way

\[
\begin{align*}
  a \cdot x^3 &= a^{-3} x \\
  a \cdot xy^2 &= axy^2 \\
  a \cdot y^3 &= a^3 y^3.
\end{align*}
\]

When we projectivize we have three lines, invariant under the $\mathbb{C}^*$ action, that pass through $z_0$; these lines are $\zeta \mapsto [xy^2 + \zeta x^2 y]$, $\zeta \mapsto [\sqrt{3} xy^2 + \zeta y^3]$ and $\zeta \mapsto [\sqrt{3} xy^2 + \zeta x^3]$ (with $\zeta \in \mathbb{C}P^1 \setminus \{\infty\} \cong \mathbb{C}$). The first two have weights -2 and 2 respectively, so they correspond to tangent lines to $Y^\circ$, and the third one, which is then transversal, has weight -4.

For the calculation we need to know the $\mathbb{C}^*$-vector bundles over $\mathbb{C}$, i.e. the holomorphic vector bundles over $\mathbb{C}$ that come with a $\mathbb{C}^*$ action. The classification of these objects is simple and is given by the following lemma that can be found in [2].

**Lemma 4.3** (Atiyah). Let $E$ be a $\mathbb{C}^*$-holomorphic vector bundle over $\mathbb{C}$. Then $E \cong \mathbb{C} \times E_0$ with the diagonal action on $\mathbb{C} \times E_0$: $\lambda \cdot (w, v) = (\lambda^k w, \lambda v)$.

We are now in position to establish the following theorem that says which are the parameters of PV1 related to the family of vector bundles described in 4.2.

**Theorem 4.4.** Let $\{E_n : n \text{ odd}\}$ be the family of $\text{SU}_2$-invariant instantons over $S^4$ described in 4.2. Then, the Painlevé equation corresponding to the isomonodromic deformation obtained from the Penrose-Ward transform of $E_n$, with the connection defined by the action, has the following parameters:

\[
\alpha^k = \frac{1}{8} (n \pm 2)^2, \quad \beta = -\frac{1}{8} n^2, \quad \gamma = \frac{1}{8} n^2, \quad \delta = -\frac{1}{8} (n^2 - 1).
\]

**Proof.** Let $\xi: \mathbb{C}P^1 \to Z$ be the rational curve given by the expression $\xi(t) = [\zeta, 1]_C$; we know that it is transversal to $Y$ at the point $z_0 = [0, 1]_C$. Restricting the vector bundle to the line we obtain a $\mathbb{C}^*$-holomorphic vector bundle
over \( \mathbb{C}P^1 \), and in view of the previous lemma we have that

\[
(\xi \big|_{\mathbb{C}P^1 \setminus \{\infty\}})^* E \cong \mathbb{C} \times E_{n, z_0}
\]

with the \( \mathbb{C}^* \) action given by \( \lambda \cdot (\zeta, v) = (\lambda^{-4} \zeta, \lambda \cdot v) \). Moreover, by the construction, \( E_{n, z_0} \) is a representation of \( \mathbb{C}^* \) with weights \(-n, n\); therefore, taking a base of eigenvectors, we have a \( \mathbb{C}^* \)-isomorphism between \( \mathbb{C} \times E_{n, z_0} \) and \( \mathbb{C} \times \mathbb{C}^2 \) with the action \( \lambda \cdot (\zeta, \lambda_1, \lambda_2) = (\lambda^{-4} \zeta, \lambda^{-n} \lambda_1, \lambda^n \lambda_2) \). Then the holomorphic field generated by the action can be expressed as follows

\[
X = -4\zeta \frac{\partial}{\partial \zeta} - n\lambda_1 \frac{\partial}{\partial \lambda_1} + n\lambda_2 \frac{\partial}{\partial \lambda_2}.
\]

Recall that this field has to be horizontal because of the definition of the connection. Now, the 1-forms that annihilate the horizontal subspace are

\[
\theta_\alpha = d\lambda_\alpha + \sum_\beta \lambda_\beta A_{\alpha \beta}, \quad \alpha = 1, 2;
\]

(see proof of theorem 2.1 in [3]). Thus we have that

\[
0 = \theta_\alpha (X) = (-1)^\alpha \lambda_\alpha - 4\zeta \sum_\beta \lambda_\beta A_{\alpha \beta}(\zeta),
\]

from what we obtain \( A_{12} = A_{21} = 0 \) and \( A_{11} = -A_{22} = \frac{n}{4\zeta} \), and we conclude that the invariant we wanted to find is

\[
\text{tr}(\text{Res } A)^2 = \frac{1}{8} n^2.
\]

This, together with proposition 2.5 finishes the proof.

In the next section we will see another proof of the previous theorem, based on more explicit calculations; see remark 4.1.

4.4. Some explicit calculations. In this section we will calculate explicitly the connection form of the flat connection, restricted to the family of lines we are considering, in terms of the Yang-Mills connection. This will give us a better understanding of what is happening and how to obtain the solution to Painlevé’s equation from the solution to the anti-self-dual equation. Remember that for any real twistor line \( P \) we can write the flat connection in the following way (see equation (3.1))

\[
A(T) = -\sum_{i=1}^{3} \alpha_i(T) p^* \Phi(\alpha(X_i)),
\]

where \( \{X_1, X_2, X_3\} \) is a base for \( \mathfrak{su}_2 \), \( T \) is a tangent vector to \( P \), \( \Phi \) is the Yang-Mills connection 1-form on \( S^4 \), and the 1-forms \( \alpha_i \) are defined by

\[
\alpha^{-1}(T) = \sum_{i=1}^{3} \alpha_i(T) X_i.
\]
The map \( \phi: (0, 1) \to \mathfrak{su}_2 \otimes \mathfrak{su}_2 \) defined in section 3.2, which determines the flat connection, has to be of the form

\[
\phi_t = a_1(t)X_1 \otimes \sigma_1 + a_2(t)X_2 \otimes \sigma_2 + a_3(t)X_3 \otimes \sigma_3
\]

because of its invariance under the \( \Gamma \) action (see equation (3.2)), with \( a_i: (0, 1) \to \mathbb{R} \).

The (anti-)self-dual equation can be written in terms of the \( a_i \)'s in the form

\[
(-)^{\frac{1}{2}}K_1\dot{a}_1 = a_2a_3 - a_1,
\]

plus other two equations obtained by cyclic permutation of the indexes 1,2,3. The coefficients \( K_i \), coming from the fact that the basis formed by the generating vectors of the action and \( \dot{c} \) is not orthonormal, are given by the expressions

\[
K_1(t) = \frac{(t^2 - 1)(t^2 - 9)}{4t}, \quad K_2(t) = 4\frac{(t - 3)(t + 1)}{(t + 3)(t - 1)}, \quad K_3(t) = 4\frac{(t + 3)(t - 1)}{(t - 3)(t + 1)}.
\]

In fact, the functions \( a_i \) can be extended to all of \( \Sigma \) (as in \([5]\)) and they satisfy the boundary conditions: \( a_1(0) = 1, a_2(1) = -n, a_1(1) = a_3(1) = 0 \) \([5]\).

**Example 4.1.** For the trivial bundle \( E_1 \) the anti-self-dual connection is determined by the functions \( a_1(t) = a_2(t) = a_3(t) = 1 \) (of course, it is also self-dual).

**Example 4.2.** The Hopf bundle, which is self-dual, has the \( a_i \)'s given by the expressions

\[
a_1(t) = \frac{t^2 - 9}{t^2 + 3}, \quad a_2(t) = -2t\frac{t + 3}{t^2 + 3}, \quad a_3(t) = -2t\frac{t - 3}{t^2 + 3}.
\]

Applying the antipodal map one sees that the anti-self-dual form on \( E_{-3} \) is thus given by

\[
a_1(t) = 3\frac{(1 - t^2)}{t^2 + 3}, \quad a_2(t) = -6\frac{(t + 1)}{t^2 + 3}, \quad a_3(t) = -6\frac{(t - 1)}{t^2 + 3}.
\]

Our next step is to find \( \alpha^{-1} \). In order to do so we take homogeneous coordinates on \( \mathbb{CP}^3 \) \( (\lambda, \mu, \zeta) = (\frac{z_1}{z_2}, \frac{z_3}{z_2}, \frac{z_4}{z_2}) \), in the open set \( \{z_2 \neq 0\} \), associated with which we have a tangent basis on \( \mathbb{CP}^3 \) that we denote by \( \{\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \mu}, \frac{\partial}{\partial \zeta}\} \).

As a basis for \( \mathfrak{su}_2 \) we take

\[
X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
\]

As an intermediate step it is convenient to take a complex basis for \( \mathfrak{sl}_2(\mathbb{C}) \) defined by \( \{-iX_1, \frac{1}{2}(X_2 + iX_3), -\frac{1}{2}(X_2 - iX_3)\} \), with respect to which the matrix associated to \( \alpha \) at the point \( (\lambda, \mu, \zeta) \) is

\[
\begin{pmatrix}
-6\lambda & -\sqrt{3}\lambda\mu & \sqrt{3}\zeta \\
-2\mu & \sqrt{3} & 2\zeta - \sqrt{3}\mu^2 \\
-4\zeta & 2\mu & \sqrt{3}\lambda - \sqrt{3}\mu\zeta
\end{pmatrix}.
\]
In homogeneous coordinates our family of lines is parametrized by \((\lambda, \frac{1}{\sqrt{3}} t \lambda, \frac{1}{\sqrt{3}} t)\), and so the vector \(T = \frac{\partial}{\partial \lambda} + \frac{t}{\sqrt{3}} \frac{\partial}{\partial \mu}\) is tangent to \(P_t\), and consequently, using (4.2), we can easily obtain that

\[
(4.3) \quad \alpha^{-1}(T) = -\frac{1}{3 \Delta(t, \lambda)} \left[ i(t^2 - 1)(t^2 - 9) \frac{\lambda}{2} X_1 - t(t + 1)(t - 3)(\lambda^2 + 1)X_2 \right.
+ \left. it(t - 1)(t + 3)(1 - \lambda^2)X_3 \right],
\]

where \(\Delta(t, \lambda) = \frac{1}{2} (8t^3 \lambda^4 - 2 (t^4 + 18t^2 - 27) \lambda^2 + 8t^3) = - \det(\alpha)(\lambda, \lambda t/\sqrt{3}, t/\sqrt{3})\). From the above expression we can find without difficulty the intersection points of the divisor with each line, which are the poles of the flat connection. They are given by the equation \(\Delta(t, \lambda) = 0\) for each fixed \(t\); from which it follows that

\[
P_t \cap Y = \{ \pm \sqrt{\mu_{\pm}} \}, \quad \text{where} \quad \mu_{\pm} = \frac{t^4 + 18t^2 - 27 \pm \sqrt{(t^2 - 1)(t^2 - 9)^3}}{8t^3}.
\]

We put \(z_4 = -z_1 = \sqrt{\mu_-}, z_3 = -z_2 = \sqrt{\mu_+}\). Note that \(\mu_{\pm} \in \mathbb{R}^+\) and \(\mu_+ \mu_- = 1\). We make the Möbius transformation such that \(z_1 \to 0, z_2 \to 1, \) and \(z_4 \to \infty\). Let \(\mu = \mu_+ - \mu_-\), then the residues of the forms \(\alpha_i\) are given by

\[
(4.4) \quad \alpha_{1,0}(t) = \frac{i(t^2 - 1)(t^2 - 9)}{16t^4 \mu} = \alpha_{1,\infty}(t) = \alpha_{1,1}(t)
\]

\[
\alpha_{2,0}(t) = \frac{(\mu_+ - 1)(t(t + 1)(t - 3))}{8t^2 \mu z_1} = -\alpha_{2,\infty}(t) = \alpha_{2,1}(t) = -\alpha_{2,1}(t)
\]

\[
\alpha_{3,0}(t) = \frac{i(\mu_+ - 1)(t(t - 1)(t + 3))}{8t^2 \mu z_1} = -\alpha_{3,\infty}(t) = \alpha_{3,1}(t) = -\alpha_{3,1}(t).
\]

The cross ratio \(x\) of the four singular points, which is the variable in the Painlevé equation, is related with \(t\) by the expression

\[
x = \frac{(t + 1)(t - 3)^3}{(t - 1)(t + 3)^3}.
\]

The connection 1-form of \(\nabla\) will be given in each line by

\[
(4.5) \quad A(t, \lambda) = -\sum_{i=1}^3 a_i(t) \alpha_i(t, \lambda) X_i
= \frac{1}{3 \Delta(t, \lambda)} \left( i(t^2 - 1)(t^2 - 9) \frac{\lambda}{2} a_1(t) X_1 - t(t + 1)(t - 3)(\lambda^2 + 1) a_2(t) X_2 \right.
+ \left. it(t - 1)(t + 3)(1 - \lambda^2) a_3(t) X_3 \right),
\]
and the residues will then be

\[
A_0(t) = -\sum_{i=1}^{3} a_i(t)\alpha_{i,0}(t)X_i = -A_x^i(t)
\]

\[
A_\infty(t) = -\sum_{i=1}^{3} a_i(t)\alpha_{i,\infty}(t)X_i = -A_x^1(t);
\]

the identities between the residues can be verified from (4.6). Remember that the solution to the sixth Painlevé equation is given by the function \(y(x)\), where \(y(x)\) is the only point in \(\mathbb{CP}^1 \setminus \{0, 1, x, \infty\}\) such that \(A(t, y(x))\) and \(A_\infty(t)\) have a common eigenvector corresponding to one of the eigenvalues. Then, with the above expressions it is possible to find \(y(x)\) in terms of the instanton.

**Remark 4.1.** Another proof of theorem 4.4. We have from equation 3.4

\[
\text{tr}(A_\infty^2) = \sum_{i=1}^{3} a_i(t)^2\alpha_{i,\infty}(t)^2 \text{tr}(X_i^2) = -2 \sum_{i=1}^{3} a_i(t)^2\alpha_{i,\infty}(t)^2.
\]

From expressions (4.4) it is easy to see that \(\lim_{t \to 1} \alpha_{1,\infty}(t) = \lim_{t \to 1} \alpha_{3,\infty}(t) = 0\) and \(\lim_{t \to 1} \alpha_{2,\infty}(t) = \frac{1}{4}\). Therefore, using the boundary conditions on the \(a_i\)'s, we have

\[
\text{tr}(A_\infty^2) = -2 \lim_{t \to 1} \sum_{i=1}^{3} a_i(t)^2\alpha_{i,\infty}(t)^2 = \frac{n^2}{8},
\]

which is the same value obtained in the proof of theorem 4.4.

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