DISTINGUISHING SIMPLE GROUPS

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ABSTRACT. The distinguishing number $D(\Gamma)$ of a graph $\Gamma$ is the least size of a partition of the vertices of $\Gamma$ such that no non-trivial automorphism of $\Gamma$ preserves this partition. We show that if the automorphism group of a graph $\Gamma$ is simple, then $D(\Gamma) = 2$. This is obtained by establishing the distinguishing number for all possible actions of simple groups.

1. INTRODUCTION

The concept of distinguishing number of a graph has been introduced by Albertson and Collins [1] in 1996. Since then an extensive literature on this number and related topics has been developed and the area is still flourishing.

In 2004, Tymoczko [24] has extended the concept to the distinguishing number of an arbitrary group action (cf. [4, 5]). This turned out to be very fruitful once, in 2011 [3], it was realized that in permutation group theory the problem had been investigated for many years as a part of the study of regular sets.

Given a group $G$ acting on a set $X$, by the distinguishing number of this action, denoted $D_X(G)$ or simply $D(G)$, we mean the least size of a partition of $X$ such that no element of $G$ preserves this partition, unless it fixes all $x \in X$. Then, the distinguishing number of a graph $\Gamma$ is the distinguishing number of the automorphism group of $\Gamma$.

Often, rather than about partition, authors speak about labeling (coloring) points (vertices) to destroy or break down the symmetry. Similarly, using the automorphism group, one defines the distinguishing number for other structures.

If the action of $G$ on $X$ if faithful we speak of the distinguishing number of the (corresponding) permutation group $(G, X)$. To have $D(G) = 2$ for such a group is equivalent to have a regular set (that is, a subset of $X$ whose set-stabilizer in $G$ is trivial [7, 21]).

The oldest remarkable result in the area is that by Gluck [12] who proved in 1983 that if the order of the group $G$ is odd, then for each action $G$ has a regular set, that is, $D(G) = 2$. This is also true for all primitive solvable permutation groups, but a few exceptions. Cameron et al. [7, 20] showed that all but finitely many primitive permutation groups $G$, not containing $A_n$, have $D(G) = 2$. Recently, similar results have been obtained for quasiprimitive and semiprimitive permutation groups in [10].

The Motion Lemma of Russell and Sundaram [19] states that if the minimal degree of a permutation group $G$ (the motion of the action) is at least $2\log_2 |G|$, then its
distinguishing number is two. Using this, Conder and Tucker [9] proved in 2011 that if we consider a vector space, group, or map, then in all but finitely many cases, the action of the automorphism group has distinguishing number two. They also suggested that, in general, having the distinguishing number two is a generic property.

Similar results, confirming this belief, have been proved for general linear groups in [4, 17]. Further, if one takes a Cartesian product of enough copies of the same graph, then the distinguishing number is two (see [2, 15]). Also, for all maps with more than 10 vertices, the action of the automorphism group on the vertices has distinguishing number two (see [22, 23], cf. [18]).

Now, it is important to realize that even if two graphs have the same automorphism group their distinguishing numbers may be different. This is so, because this number depends on the action of a group, not merely on its abstract structure. Many exceptions to the generic case for group actions of 2-distinguishability are connected with intransitive actions. The study of distinguishing number involves necessarily intransitive graphs. Therefore, as we demonstrate in this paper, the details of the construction of intransitive permutation groups may be especially useful in this study.

The distinguishing number is a local property of graphs in the sense that it is enough to replace one vertex by a clique to increase this number. This leads to a Brooks type inequality \( D(\Gamma) \leq Y(\Gamma) + 1 \) for connected graphs (see [8, 17]). We believe however that a real obstacle here are the maximal sizes \( \omega(\Gamma) \) of a clique and \( \omega(\overline{\Gamma}) \) of an anticlique, and in contrast with the chromatic number of a graph this provides an actual upper bound for the distinguishing number.

**Conjecture.** Let \( \Gamma \) be an arbitrary graph. Then

\[
D(\Gamma) \leq 1 + \max\{\omega(\Gamma), \omega(\overline{\Gamma})\}.
\]

The complete bipartite graph \( K_{n,n} \) shows that this inequality cannot be sharpened.

On the other hand, for intransitive graphs, it is easy to see that \( D(\Gamma) \) is bounded from above by the maximum of distinguishing numbers of the action of the automorphism group on the orbits. Yet, this bound is very rough. We show that in case of simple group the maximum may be replaced by the minimum. Constructions of intransitive groups, corresponding to the edges joining vertices in different orbits of graphs, usually decrease the distinguishing number in a radical way.

In this paper we deal with the case when the automorphism group \( Aut(\Gamma) \) of a graph \( \Gamma \) is simple. Although, by [13], we know that for each simple group \( G \) there exists a graph \( \Gamma \) with \( Aut(\Gamma) = G \), in general, we know very little about such graphs (we say more on this topic after Corollary [4.2]). Therefore, our approach, rather than considering graphs, is to consider all possible actions of simple groups, and exclude those that are certainly not the automorphism groups of graphs.

Looking for simple groups is motivated by the fact that they may be viewed as the other extreme of solvable groups, and we wish to check if having distinguishing number two is also typical in this special case. Of course, we make use of that we have extensive and deep knowledge of simple groups. From the results established for primitive and quasiprimitive permutation groups we can infer that, with known
exceptions, transitive actions of simple groups have the distinguishing number two. So, in this paper, we focus on intransitive actions. Using a general result on the structure of intransitive permutation groups we show that in case of simple groups these actions have a special and very nice structure of parallel sums. This allows to check effectively all possible exceptions, using known classification results and computation machinery.

As we shall see, the case of simple groups illustrates perfectly all the discussed phenomena and may be a good starting point in further study. A counterpart of a special role of cliques and anticliques for graphs is the role of the alternating groups among simple groups. Our paper shades some light on this.

For computer calculations we have used system GAP 4.10.1. In cases of applying GAP, we give only those computational details that seem sufficient for the reader to verify our results using any computational system for permutation groups. Since in the case of simple groups all actions are faithful, we use the terminology of permutation groups rather than those of actions of groups.

2. INTRANSITIVE SIMPLE GROUPS

To describe the structure of intransitive simple groups we need to recall the concept of the subdirect sum of permutation groups and the general result on the structure of intransitive permutation groups. We give also some comments that allow to apply the recalled results in a proper and easy way. All considered permutation groups are finite and considered up to permutation isomorphism [11, p. 17] (i.e., two groups that differ only in labeling of points are treated as the same).

Given two permutation groups \( G \leq \text{Sym}(X) \) and \( H \leq \text{Sym}(Y) \), the direct sum \( G \oplus H \) is the permutation group on the disjoint union \( X \cup Y \) defined as the set of all permutations \((g,h)\), \( g \in G, h \in H \) such that (writing permutations on the right)

\[
x(g,h) = \begin{cases} 
  xg, & \text{if } x \in X \\
  xh, & \text{if } x \in Y
\end{cases}
\]

Thus, in \( G \oplus H \), permutations of \( G \) and \( H \) act independently in a natural way on the disjoint union of the underlying sets.

We define the notion of the subdirect sum following [14] (and the notion of intransitive product in [16]). Let \( H_1 \triangleleft G_1 \leq S_n \) and \( H_2 \triangleleft G_2 \leq S_m \) be permutation groups such that \( H_1 \) and \( H_2 \) are normal subgroups of \( G_1 \) and \( G_2 \), respectively. Suppose, in addition, that factor groups \( G_1/H_1 \) and \( G_2/H_2 \) are (abstractly) isomorphic and \( \phi : G_1/H_1 \rightarrow G_2/H_2 \) is the isomorphism mapping. Then, by

\[
G = G_1[H_1] \oplus_{\phi} G_2[H_2]
\]

we denote the subgroup of \( G_1 \oplus G_2 \) consisting of all permutations \((g,h)\), \( g \in G_1, h \in G_2 \), such that \( \phi(H_1 g) = H_2 h \). Each such group will be called the subdirect sum of \( G_1 \) and \( G_2 \).

If \( H_1 = G_1 \) and \( H_2 = G_2 \), then \( G = G_1 \oplus G_2 \) is the usual direct sum of \( G_1 \) and \( G_2 \). If \( H_1 \) and \( H_2 \) are trivial one-element subgroups, \( \phi \) is the isomorphism of \( G_1 \) onto \( G_2 \), and the sum is called, in such a case, the parallel sum of \( G_1 \) and \( G_2 \). Then the elements of \( G \) are of the form \((g, \phi(g))\), \( g \in G_1 \), and both the groups act in a parallel manner on their sets via isomorphism \( \phi \). In this case we use the notation \( G = G_1 \parallel_{\phi} G_2 \),
where \( \phi \) is an (abstract) isomorphism between \( G_1 \) and \( G_2 \), or simply \( G = G_1 \wr G_2 \) if there is no need to refer to \( \phi \). Note that \( G_1 \) and \( G_2 \) need to be abstractly isomorphic, but not necessarily permutation isomorphic, and they may act on sets of different cardinalities.

In the special case when, in addition, \( G_1 = G_2 = G \) and \( \phi \) is the identity, we write \( G^{(2)} \) for \( G \wr G \). More generally, for \( r \geq 2 \), by \( G^{(r)} \) we denote the permutation group in which the group \( G \) acts in the parallel way (via the identity isomorphisms) on \( r \) disjoint copies of a set \( X \). This group is called the parallel multiple of \( G \), and its element are denoted \( g^{(r)} \) with \( g \in G \). In particular, we admit \( r = 1 \) and put \( G^{(1)} = G \). For example, the cyclic group generated by the permutation \( g = (1, 2, 3)(4, 5, 6)(7, 8, 9) \) is permutation isomorphic to the parallel multiple \( C_3^{(3)} \), where \( C_3 \) denotes the permutation group on \( \{1, 2, 3\} \) generated by the cycle \((1, 2, 3)\).

The main fact established in [16] is that every intransitive group has the form of a subdirect sum, and its components can be easily described. Let \( G \) be an intransitive group acting on a set \( X = X_1 \cup X_2 \) in such a way that \( X_1 \) and \( X_2 \) are disjoint fixed blocks of \( G \). Let \( G_1 \) and \( G_2 \) be restrictions of \( G \) to the sets \( X_1 \) and \( X_2 \), respectively (they are called also constituents). Let \( H_1 \leq G_1 \) and \( H_2 \leq G_2 \) be the subgroups fixing pointwise \( X_2 \) and \( X_1 \), respectively. Then we have

**Theorem 2.1.** [16] Theorem 4.1] If \( G \) is a permutation group as described above, then \( H_1 \) and \( H_2 \) are normal subgroups of \( G_1 \) and \( G_2 \), respectively, the factor groups \( G_1/H_1 \) and \( G_2/H_2 \) are abstractly isomorphic, and

\[
G = G_1[H_1] \oplus \phi \ G_2[H_2],
\]

where \( \phi \) is an isomorphism of the factor groups.

There is some subtlety regarding the parallel powers we have to be aware. A well-known fact is that the automorphism group \( \text{Aut}(G) \) of a group \( G \) may have outer automorphisms that are not given by the conjugation action of an element of \( G \). In the case of permutation groups \( G \leq \text{Sym}(X) \) some outer automorphism may still be given by the conjugation action of an element in \( \text{Sym}(X) \). This corresponds generally to permuting elements of \( X \), and such automorphisms are called permutation automorphisms. They form a subgroup of \( \text{Aut}(G) \), which we denote by \( \text{PAut}(G) \). Often \( \text{PAut}(G) = \text{Aut}(G) \), but some permutation groups have also other automorphisms, which we will call nonpermutation automorphisms.

Now, if \( G = H \oplus \psi H \), for some permutation group \( H \), where \( \psi \in \text{PAut}(G) \), then \( G \) is permutation isomorphic to \( H^{(2)} \) (i.e., \( G = H^{(2)} \), according to our convention). If \( \psi \) is a nonpermutation automorphism, then \( G \neq H^{(2)} \). (More precisely, we should speak here about isomorphisms induced by automorphisms and make distinction between base sets of components, but we assume that this is contained in the notion of the disjoint union, and we will make it explicit only when the need arises). An example is the alternating group \( A_6 \) that has a nonpermutation automorphism \( \psi \) (cf. [6]). Then, \( A_6^{(2)} \) and \( A_6|\psi|A_6 \) are not permutation isomorphic. As we shall see, this example is exceptional. The distinguishing number \( D(A_6|\psi|A_6) = 3 \).

With this result we may turn to describing the structure of intransitive simple permutation groups. Without loss of generality we may assume that groups we consider
have no fixed points, as fixed points do not affect the distinguishing number. More precisely, if a permutation group $G$ has precisely $m > 0$ fixed points, then $G = G' \oplus I_m$, where $G'$ has no fixed points, and $I_m$ is the trivial group acting on $m$ points, that is, containing only the identity permutation. Obviously, $D(G) = D(G')$, and $G$ and $G'$ are abstractly isomorphic. We have the following.

**Theorem 2.2.** Let $G$ be a simple intransitive group with no fixed points. Then $A$ has the form of a parallel sum $G = H \parallel \phi K$, where $H$ and $K$ are permutation groups abstractly isomorphic to $G$.

**Proof.** By Theorem 2.1, $G = H[H'] \oplus K[K']$, where each of $H$ and $K$ acts nontrivially on at least two points. Now, the group $B' \oplus I_m$ is a normal subgroup of $G$, as $B' < B$. Since $C$ is nontrivial, and $G$ is simple, we infer that $H'$ is trivial. Similarly, we observe that $K'$ is trivial. It follows that $A = H \parallel K$, $\phi$ is an isomorphism between $H$ and $K$, and $G$ is isomorphic to both $H$ and $K$, as required. $\square$

The proposition above means that intransitive simple permutation groups with nontrivial orbits have always the form of parallel sums. Some remarks are needed to make a proper use of this result.

Note that (using the inverse isomorphism $\phi^{-1}$) we see easily that $G_1 \parallel G_2$ and $G_2 \parallel G_1$ are permutation isomorphic, and since we treat permutation isomorphic groups as identical, the operation of the parallel sum may be considered to be commutative. Further, decomposing each summand step by step we can get a decomposition into transitive components. In particular, in $G = (G_1 \parallel G_2) \parallel G_3$ all the involved groups must be abstractly isomorphic, and $G$ is permutation isomorphic with $G_1 \parallel (G_2 \parallel G_3)$, where isomorphisms $\phi'$ and $\psi'$ are suitably determined by $\phi$ and $\psi$. Thus we may also consider this operation to be associative (up to permutation isomorphism).

So, generally, a simple intransitive permutation group (with no fixed points) is a parallel sum of two or more transitive components that are all abstractly isomorphic, and the action on the union of orbits is given by a system of suitable isomorphisms between components.

Note however, that in case, when a group has two different actions on the set of the same cardinality, various systems of isomorphism may lead to different permutation groups; so the pointing out only transitive components may not define the parallel sum uniquely. (For example, projective symplectic group $PSp(4,3)$ has two nonequivalent actions on the set of cardinality $n = 40$). A similar remark applies when there are two identical components with a nonpermutation automorphism.

### 3. Distinguishing number and regular sets

Consider distinguishing labelings for parallel sums. It is easy to see that in this case, if we have a distinguishing $k$-labeling (i.e., one with $k$ labels) for one of the components, then it is easy to construct a distinguishing $k$-labeling for the whole sum.

**Lemma 3.1.** Let $G = H \parallel K$ be a parallel sum of two groups. Then,

$$D(G) \leq \min\{D(H), D(K)\}.$$
Proof. Suppose that in the definition of \( G \) above, \( H \) acts on \( X \) and \( K \) act on \( Y \). Note that, by the definition of the parallel sum, the only permutation in \( G \) fixing all the points in \( X \) is the identity. Therefore, given is a having a distinguishing \( k \)-labeling \( \ell_1 \) of \( X \) for the action of \( H \), one can obtain a distinguishing \( k \)-labeling \( \ell \) of \( X \cup Y \) for the action of \( G \) just by assigning an arbitrary label to all points in \( Y \). A symmetrical arguments proves the claim. \( \Box \)

Now, let us consider the parallel multiple \( G = A_n^{(k)} \) of the alternating group \( A_n \). As we will see below, if \( k \) is large enough with regard to \( n \), then \( D(G) = 2 \), which is one more confirmation of the phenomenon discussed in the introduction.

**Lemma 3.2.** If \( G = A_n^{(k)} \) is the parallel multiple of the alternating group \( A_n \), \( n \geq 3 \) and \( k \geq 1 \), then \( D(G) \) is the smallest integer \( d \) such that \( d^k \geq n - 1 \).

**Proof.** We need to show that for \( d^k \geq n - 1 \), \( G \) has a distinguishing \( d \)-labeling, and if \( d^k < n - 1 \), then no such labeling exists for \( G \).

We may assume that \( A_n^{(k)} \) acts on \( X = X_1 \cup \ldots \cup X_r \), where each \( X_i \) is a copy of \( \{1, 2, \ldots, n\} \), and the action of \( g^{(r)} \) is the same on all copies as the action of \( g \in A_n \) on \( \{1, 2, \ldots, n\} \). Let \( \ell \) denote a \( d \)-labeling of \( X \) with \( \ell(i, j) \) denoting the label of \( i \)-th element in \( X_j \). We define a labeling \( \ell' \) of \( \{1, 2, \ldots, n\} \) by \( \ell'(i) = (\ell(i, 1), \ell(i, 2), \ldots, \ell(i, k)) \).

Obviously, \( \ell \) is a distinguishing \( d \)-labeling for \( A_n^{(k)} \) if and only if \( \ell' \) is distinguishing for \( A_n \) on \( \{1, 2, \ldots, n\} \). The later holds if and only if no more than two points have the same label (otherwise there is a permutation in \( A_n \) preserving the labeling). Since the labels \( \ell'(i) \) are \( k \)-tuples of \( d \) possible values, there are \( d^k \) of them. So, if \( d^k < n - 1 \), then either more than two points must have the same label or there are two pairs with the same label, and therefore the labeling is not distinguishing. Otherwise, One may choose labels \( \ell(i, j) \) so that \( \ell \) is distinguishing. This proves the result. \( \Box \)

In general, if \( G \) is a parallel sum of permutation isomorphic components, it needs not to be a parallel multiple. As we have already explained, this is so, since permutation groups may have nonpermutation automorphisms. For alternating groups we have an interesting exception mentioned in the previous section.

**Lemma 3.3.** Let \( G \) be the parallel sum of components permutation isomorphic to a fixed alternating group \( A_n \), such that \( G \neq A_n^{(r)} \) for any \( r \geq 1 \). Then, \( n = 6 \), \( G = A_6^{(r)}|_{\psi}A_6^{(s)} \) for some \( r, s \geq 1 \) and \( D(G) = 2 \) with the exception of \( G = A_6|_{\psi}A_6 \neq A_6^{(2)} \), in which case \( D(A_6|_{\psi}A_6) = 3 \).

**Proof.** It is well known \([6]\) that the only alternating group \( A_n \) having a nonpermutation automorphism is \( A_6 \) and that the index \([\text{Aut}(A_6) : \text{PAut}(A_6)] = 2 \), which means that up to permutation automorphisms there is only one nonpermutation automorphism in \( A_6 \). Hence, if \( G \neq A_6^{(k)} \), then \( G = A_6^{(r)}|_{\psi}A_6^{(s)} \).

First, consider the case of two components \( G = A_6|_{\psi}A_6 \). Using GAP, we find a nonpermutation automorphisms \( \psi \) of \( A_6 \) given by the images of generators of \( A_6 \):

\[
(2, 3)(4, 5)|_{\psi} = (2, 5)(3, 4), \quad \text{and} \quad ((1, 2, 3, 4)(5, 6))|_{\psi} = (1, 2, 3, 4)(5, 6).
\]
out a regular set for $H$ mutations of computations. Let $n$ permutation groups. To provide easy verification for the reader we give some details

Proof. This can be easily checked using GAP, that $A_6 \triangleright A_6$ has no regular set, and $D(A_6 \triangleright A_6) = 3$. In turn, the group $G' = A_6^{(2)} \triangleright A_6$, obtained by applying the same nonpermutation automorphism, has regular sets of all sizes from 4 to 14. It follows that $D(G') = 2$, and using Lemma 3.1 $D(A_6^{(1)} \triangleright A_6^{(s)}) = 2$ for all $r > 2$.

Now, we will combine the above lemmas with what is known on primitive groups. In [20], Seress lists all primitive groups that have no regular set (cf. [21, Theorem 2.2]). Using this list one can distinguish all simple groups in primitive action that have no regular set. In the list below the first entry denotes the degree of the action, and the second—the group itself. If the group is abstractly isomorphic to one of $A_n$ or another group in the list with a different name, then this is indicated. There are no other isomorphisms between the groups in the list except those indicated. (This may be inferred from the classification of the finite simple groups, using, e.g., [11, Appendix A]). Note that all groups in the list appears in their natural actions with the exception of $L_2(11)$ which acts here on 11 points (rather than 12).

$$
\mathcal{L} = \{(6, L_2(5) \cong A_5), (7, L_3(2) \cong L_2(7)), (8, L_2(7) \cong L_3(2)), (9, L_2(8)),
(10, L_2(9) \cong A_6), (11, L_2(11)), (13, M_{11}), (12, M_{12}),
(13, L_3(3)), (15, L_4(2) \cong A_8), (22, M_{22}), (23, M_{23}), (24, M_{24})\}.
$$

Performing suitable computation one may check that although these groups have no regular sets, their sums have. First we establish the following.

**Lemma 3.4.** For each group $H$ in the list $\mathcal{L}$, $H^{(2)}$ has a regular set.

**Proof.** This can be easily checked using GAP or other systems for computation in permutation groups. To provide easy verification for the reader we give some details of computations. Let $n$ be the degree of $H$, $X = \{1, 2, \ldots, n\}$, and $Y = \{1', 2', \ldots, n'\}$.

As an example we take the group $H = L_2(5)$ acting on $n = 6$ elements. To point out a regular set for $H^{(2)}$ we need to define the underlying set $X$ and generating permutations for $H^{(2)}$. In each case for $X$ we take $X = \{1, 2, \ldots, n\} \cup \{1', 2', \ldots, n'\}$. Then, we find a set of generators for $H$ (one may use, for example, the GAP listing of primitive groups of degree $n$). In the considered example we get $g = (1, 2, 5)(3, 4, 6)$ and $h = (3, 5)(4, 6)$. We form permutations $g^{(2)}$ and $h^{(2)}$ obtaining

$$
g^{(2)} = (1, 2, 5)(3, 4, 6)(1', 2', 5')(3', 4', 6'),
$$

$$
h^{(2)} = (3, 5)(4, 6)(3', 5')(4', 6').
$$

Clearly, the above permutations generate $H^{(2)}$. Now the reader can check that, for instance, the set $S = \{1, 2, 3, 2', 4', 6'\}$ has the trivial stabilizer.

The proof in other cases is the same. The only change is in the choice of permutations $g$ and $h$ generating $H$, and the regular set $y$. These details are given in Table [1].
We also need to consider subdirect sums of elements from $\mathcal{L}$ that arise by using a nonpermutation automorphism. It is easy to check that the groups in $\mathcal{L}$ having nonpermutation automorphisms are the following:

$$\mathcal{L}^* = \{L_3(2), L_2(11), M_{12}, L_3(3), L_4(2)\}.$$  

Moreover, one may also check that the index $[\text{Aut}(G) : P\text{Aut}(G)] = 2$ in each of these cases. We use this to prove the following.
Lemma 3.5. For each group $H$ in the list $\mathcal{L}$, and each automorphism $\psi$ of $H$, if the parallel sum $G = H \big|_{\psi} H$ is different from $H^{(2)}$, then $G$ has a regular set.

Proof. In view of the remarks above we may restrict to groups $H \in \mathcal{L}^*$ and to one nonpermutation automorphism $\psi$ in each case.

As in the proof of Lemma 3.3, first we find a nonpermutation automorphism of $H$.

For $H = L_3(2)$, using GAP, one can find a nonpermutation automorphism $\psi$ given by the images of generators

$$(1, 2)(5, 7)\psi = (1, 2)(3, 6), \quad \text{and} \quad (2, 3, 4, 7)(5, 6)\psi = (2, 3, 4, 7)(5, 6).$$

Hence, the group $G = H \big|_{\psi} H$ on the set $X = \{1, \ldots, 7\} \cup \{1', \ldots, 7'\}$ is generated by permutations

$$g = (1, 2)(5, 7)(1', 2')(3', 6') \quad \text{and} \quad h = (2, 3, 4, 7)(5, 6)(2', 3', 4', 7')(5', 6').$$

It is easy to check that $H$ has regular sets of all sizes from 4 to 10, and one of them is, for example, $\{1, 5, 2', 3'\}$.

For other groups in $\mathcal{L}^*$ the constructions are the same, and the only difference is the nonpermutation automorphism used. In Table 2 generators of $H \big|_{\psi} H$ and examples of regular sets are given. Generators are given in the form as above, allowing to decode nonpermutation automorphisms used.

$$\qed$$

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$n$ & $H$ & generators $g$ and $h$ of $H \big|_{\psi} H$ & regular set \\
\hline
7 & $L_3(2)$ & $(1, 2)(5, 7)(1', 2')(3', 6')$, & $\{1, 5, 2', 3'\}$ \\
 & & $(2, 3, 4, 7)(5, 6)(2', 3', 4', 7')(5', 6')$ & \\
\hline
11 & $L_2(11)$ & $(1, 3)(2, 7)(5, 9)(6, 11)(1', 3')(5', 4', 6', 7', 8', 10', 9', 12')$, & $\{1, 2, 3, 3', 5'\}$ \\
 & & $(3, 5, 11)(4, 9, 7)(6, 10, 8)(3', 5', 11')(4', 9', 7')(6', 10', 8')$ & \\
\hline
12 & $M_{12}$ & $(1, 3, 5, 7, 2, 4, 8, 9)(6, 10, 11, 12)(1', 3')(5', 4', 6', 7', 8', 10', 9', 12')$, & $\{1, 2, 3, 7, 9, 1', 3'$, \\
 & & $(1, 7, 8, 3, 9, 6, 4, 11, 10, 12, 2)(3', 2', 1', 7', 11', 9', 12', 5'$, \\
 & & $6', 4', 10')$ & $5', 9', 12')$ \\
\hline
13 & $L_3(3)$ & $(3, 5, 11)(6, 7, 9)(8, 12, 13)(3', 8', 7')(5', 12', 9')(6', 11', 13')$, & $\{1, 2, 3, 1', 2', 3'\}$ \\
 & & $(1, 13, 7)(2, 10, 6)(3, 5, 12)(4, 11, 9)(1', 13', 7')(2', 10', 6')$ & \\
 & & $(3', 5', 12')(4', 11', 9')$ & \\
\hline
15 & $L_4(2)$ & $(1, 9, 5, 14, 13, 2, 6)(3, 15, 4, 7, 8, 12, 11)(1', 4', 2', 14', 13', 7', 8')(3', 10', 15', 9', 5', 6', 12')$, & $\{1, 3, 5, 3', 9', 13'\}$ \\
 & & $(1, 3, 2)(4, 8, 12)(5, 11, 14)(6, 9, 15)(7, 10, 13)(1', 2', 3'$ \\
 & & $(4', 14', 10')(5', 12', 9')(6', 13', 11')(7', 15', 8')$ & \\
\hline
\end{tabular}
\end{table}

In the next lemma we consider the last case needed to prove our main result.

Lemma 3.6. For each pair of groups $H \cong K$ in the list $\mathcal{L}$ that are different as permutation groups, the group $H \big|_{\phi} K$ has a regular set.

Proof. Looking at the list $\mathcal{L}$ we see that there are exactly five possibilities for $H$ and $K$:

...
(i) \( n = 5, 6 \) with \( A_5 \cong L_2(5) \),
(ii) \( n = 7, 8 \) with \( L_3(2) \cong L_2(7) \),
(iii) \( n = 6, 10 \) with \( A_6 \cong L_2(9) \),
(iv) \( n = 11, 12 \) with two actions of \( M_{11} \),
(v) \( n = 8, 15 \) with \( A_8 \cong L_4(2) \).

Again, for each pair calculations are similar, so we comment only one example.

As the generators of \( G = L_2(5) \|_5 A_5 \) (formed similarly as the proof of the previous lemma) we take \((1,3,4)(2,5,6)(8,9,11),(1,2)(3,4)(7,8)(9,10)\).

We need to note that the group \( G = L_2(5) \|_5 A_5 \) is unique up to permutation isomorphism. This is because \( A_5 \) has only permutation automorphism (even if \( L_2(5) \) has a nonpermutation automorphism). The situation is similar in the remaining cases (one of the groups has only permutation automorphisms), so we have only one permutation group of the form \( H \|_5 K \) in each case.

Using GAP, it is not difficult to find a regular set in \( L_2(5) \|_5 A_5 \) (in this case we found that \( \{1,3,7,9\} \) is regular).

In Table 3, we list the degree \( n \) in the form the sum of the degrees of transitive components \( H \) and \( K \), the generators of the sum \( H \|_5 K \), and an example of a regular set in \( H \|_5 K \). This allows to decode the isomorphism \( \psi \) and to check easily that the set pointed out is regular.

\[ \square \]

### Table 3.

| \( n \) | group | generators of \( H \|_5 K \) | regular set |
|--------|-------|-----------------------------|-------------|
| 6 + 5  | \( L_2(5) \|_5 A_5 \) | \((1,3,4)(2,5,6)(8,9,11),(1,2)(3,4)(7,8)(9,10)\) | \( \{1,3,7,9\} \) |
| 8 + 7  | \( L_2(7) \|_5 L_3(2) \) | \((1,6,5)(2,3,7)(9,11,10)(12,15,13),(1,4)(2,7)(3,5)(6,8)(9,12)(14,15)\) | \( \{1,3,5,10,12,14\} \) |
| 10 + 6 | \( L_2(9) \|_5 A_6 \) | \((1,5,3,9,6)(2,7,8,4,10)(11,12,13,14,15),(1,5,3,9,6)(2,7,8,4,10)(11,13,14,15,16)\) | \( \{1,3,13,15\} \) |
| 12 + 11| \( M_{11}(12) \|_5 M_{11} \) | \((1,12)(2,10,5,7)(3,8)(4,6,11,9)(13,21,17,19)(16,20,23,18),(1,3)(2,7)(8,11)(9,10)(14,16)(15,18)(19,22)(21,23)\) | \( \{1,3,7,9,11,14,18\} \) |
| 15 + 8 | \( L_4(2) \|_5 A_8 \) | \((1,9,5,14,13,2,6)(3,15,4,7,8,12,11)(16,17,18,19,20,21,22),(1,3,2)(4,8,12)(5,11,14)(6,9,15)(7,10,13)(21,22,23)\) | \( \{1,4,7,8,18,20,23\} \) |

### 4. Results

Now, we are ready to prove our main result. Below the well-known notation for abstract groups is used for permutation groups obtained from the corresponding abstract group in its natural action. An exception is \((L_2(11),11)\) meaning the projective special linear group \( L_2(11) \) acting on 11 points. \( A_6 \|_5 A_6 \) denotes the exceptional intransitive group described in Lemma 3.3.
Theorem 4.1. Let $G$ be a simple permutation group with no fixed points. Then, $D(G) = 2$ except for the following cases:

1. If $G = A_n^{(k)}$, then $D(G)$ is the smallest integer $d$ such that $dk \geq n - 1$;
2. If $G \in \{L_3(2), M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, A_6 \wr A_6\}$, then $D(G) = 3$.

Proof. If $G$ is primitive, then by [20], we get the list $L$ of fourteen exceptional simple permutation groups that have no regular set. The distinguishing numbers for these groups are computed in [10]. If $G$ is transitive imprimitive, then as a simple group is quasiprimitive, and by [10, Theorems 2], $D(G) = 2$.

Thus, we may assume that $G$ is intransitive, and by Theorem [22] (and the following remarks), $G$ is a parallel sum of at least two transitive components, all abstractly isomorphic to $G$.

If at least one of the components $K$ is different from $A_n$ and is not on the list $L$, then $D(K) = 2$, and the result follows by Lemma [3.1]. So, we may assume that all the components are isomorphic to some $A_n$ or one of the groups in the list $L$.

Suppose first that $G = H^{(k)}$ is a parallel multiple. Then, in the case when $H = A_n$, the result follows by Lemma [3.2] and in case when $H \in L$, the result follows by Lemma [3.3] combined with Lemma [3.1].

Next, suppose that $G$ is a parallel sum of the same components $H$ different from $H^{(k)}$. Then, if $H = A_n$, the result follows by Lemma [3.3]. The fact that $D(A_6 \wr A_6) = 3$ may be easily checked using GAP. If $H \in L$, then the result is by Lemma [3.5] and Lemma [3.1].

It remains to consider the situation when $G$ has two transitive components $H$ and $K$ that are not permutation isomorphic, but are abstractly isomorphic. In this case the result follows by Lemma [3.6] combined with Lemma [3.1].

Corollary 4.2. If the automorphism group of a graph $\Gamma$ is simple, then $D(\Gamma) = 2$.

Proof. First, if $\Gamma$ is transitive, then the theorem above shows that all transitive exceptions are double-transitive, which means that none of them is the automorphism group of a graph.

For $G = A_6 \wr A_6$ consider the orbitals of $G$ (orbits in the action of $G$ on pairs). There are three orbitals: two corresponding to each orbit consisting of pairs of points in the orbit, and one consisting of pairs with a point in each orbit. The latter, that there is only one orbital consisting of such pairs, may be easily seen using GAP. This means that the automorphism group of a graph admitting all permutations in $G$ as automorphisms is $S_6 \oplus S_6$, not $G$.

In case of $A_n^{(k)}$, $k \geq 2$, the situation is similar. The difference is that in this case there are two orbitals corresponding to every two orbits of $A_n^{(k)}$ consisting of pairs with a point in each orbit. It can be easily seen, without GAP, that the orbitals of $A_n^{(k)}$ are the same as those of $S_n^{(k)}$, and the former is not the automorphism group of any graph.

How little we know about graphs whose automorphism group is simple shows the following question, which seems open and nontrivial:
Problem. Does there exist a graph whose automorphism group is simple and primitive?

In fact, the only examples of graphs with simple automorphism group we knew so far involved construction of Cayley graphs [13]. So, we may ask more: Does there exist a transitive graph whose automorphism group is isomorphic to a simple group \( G \) and whose order is less than the order of \( G \)?

We restricted the question to transitive graphs, since the results of this paper allow us to construct examples that are intransitive. Consider the orbitals of the group \((A_5, 15)\), where \( A_5 \) acts on 15 points. Using GAP one may check that there exists a 4-coloring of the orbitals of \((A_5, 15)\) such that the automorphism group of the resulting colored graph is precisely \((A_5, 15)\). Consider now the parallel sum \( G = (A_5, 15) || (A_5, 15) \). Based on the 4-coloring above one may construct a 2-coloring of the orbitals of \( G \), forming a graph \( \Gamma \), such that \( Aut(\Gamma) = A_5 \).

Simple groups have many transitive actions determined by the action of the group on its cosets for any subgroup. In view of the example above, it seems plausible that many of them form automorphism groups of colored graphs, and suitable parallel sums provide examples of (uncolored) graphs whose automorphism group is simple. Also, we have not excluded the possibility that there are intransitive graphs whose automorphism groups are of the form given in Lemmas 3.5 and 3.6.

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