RELATIVISTIC EFFECTS IN EXTRASOLAR PLANETARY SYSTEMS

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ABSTRACT

This paper considers general relativistic (GR) effects in currently observed extrasolar planetary systems. Although GR corrections are small, they can compete with secular interactions in these systems and thereby play an important role. Specifically, some of the observed multiple planet systems are close to secular resonance, where the dynamics is extremely sensitive to GR corrections, and these systems can be used as laboratories to test general relativity. For the three-planet solar system Upsilon Andromedae, secular interaction theory implies an 80% probability of finding the system with its observed orbital elements if GR is correct, compared with only a 2% probability in the absence of GR. In the future, tighter constraints can be obtained with increased temporal coverage.

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1. Introduction

The perihelion advance of the planet Mercury is one of the classic tests of general relativity [1]. The recent discoveries of extrasolar planets [2] provide us with a new ensemble of solar systems to study periastron advance and other relativistic effects. The current sample of extrasolar planets [3] includes many orbits that are surprisingly close to the stars, with semimajor axes \( a \approx 0.05 \text{ AU} \) and hence periods \( P \approx 4 \text{ days} \). Such small semimajor axes imply that these planets experience larger GR effects than the planets in our solar system. One way to quantify the efficacy of GR is through the dimensionless parameter \( \mu \equiv GM_*/(c^2a) \). A planet in a 4 day orbit has a \( \mu \) parameter nearly 10 times larger than that of Mercury, but direct measurement of periastron advance remains difficult because of the large number of observations required and because close planets tend to have nearly circular orbits. Fortunately, however, planetary systems with multiple planets and particular architectures allow for GR to exhibit much more pronounced effects. In these systems, secular interactions between the planets enforce time variations in the orbital elements (e.g., eccentricity \( e \)). These interactions are sensitive to the exact tuning of the system into secular resonance, and such tuning is affected by the relativistic corrections to the classical theory. As a result, these systems provide a new test of general relativity. In other systems, we find that this sensitivity to GR allows us to place new constraints on the system parameters.

2. Theory of Secular Interactions

We first outline the basic theory of secular interactions in multi-planet solar systems. In the absence of relativistic corrections, this topic has been widely discussed previously [4]. The basic result of these interactions is to force the orbits of participating planets to exchange angular momentum and thereby display time varying eccentricities (and other orbital elements). The amplitudes and time scales of these variations depend on the solar system configuration. Here we briefly outline the formalism and add the leading order relativistic correction. To the second order in eccentricity and inclination angle, the equations of motion for eccentricity \( e_j \) and argument of periastron \( \varpi_j \) decouple from those of inclination angle and the ascending node. Following standard convention [4], we work in terms of the variables defined by

\[
h_j \equiv e_j \sin \varpi_j \quad \text{and} \quad k_j \equiv e_j \cos \varpi_j,
\]

where the subscript refers to the \( j \)th planet in a solar system with \( N \) planets. The equations of
motion take the form
\[
\frac{dh_j}{dt} = \frac{1}{n_j a_j^2} \partial V_j \quad \text{and} \quad \frac{dk_j}{dt} = -\frac{1}{n_j a_j^2} \partial V_j,
\]
(2)
where \( V_j \) the secular part of the disturbing function, \( n_j \) is the mean motion, and \( a_j \) is the semi-major axis (for the \( j \)th planet). To consistent order in this approximation, the disturbing function can be written
\[
V_j = n_j a_j^2 \left[ \frac{1}{2} A_{jj}^2 e_j^2 + \sum_{k \neq j} A_{jk} e_j e_k \cos(\varpi_j - \varpi_k) \right].
\]
(3)
The physics of secular interactions is thus encapsulated in the \( N \times N \) matrix \( A_{ij} \), where the matrix elements take the form
\[
A_{jj} = n_j \left[ \frac{1}{4} \sum_{k \neq j} \frac{m_k}{M_* + m_j} \alpha_{jk} \tilde{\alpha}_{jk} b_{3/2}^{(1)}(\alpha_{jk}) + \frac{3 G M_*}{c^2 a_j^3} \right],
\]
(4)
\[
A_{jk} = -n_j \frac{1}{4} \frac{m_k}{M_* + m_j} \alpha_{jk} \tilde{\alpha}_{jk} b_{3/2}^{(2)}(\alpha_{jk}).
\]
(5)
The quantities \( \alpha_{jk} \) are defined such that \( \alpha_{jk} = a_j / a_k \) (\( a_k / a_j \)) if \( a_j < a_k \) (\( a_k < a_j \)). The complementary quantities \( \tilde{\alpha}_{jk} \) are defined so that \( \tilde{\alpha}_{jk} = a_j / a_k = \alpha_{jk} \) if \( a_j < a_k \), but \( \tilde{\alpha}_{jk} = 1 \) for \( a_k < a_j \). Finally, \( b_{3/2}^{(1)}(\alpha_{jk}) \) and \( b_{3/2}^{(2)}(\alpha_{jk}) \) are the Laplace coefficients [4]. The diagonal matrix elements include the leading order correction for general relativity [5]. These corrections are small in an absolute sense, with \( \mu \sim 4 \times 10^{-6}(a_j / 0.05\text{AU})^{-1} \), but they compete with the other terms and affect the eigenfrequencies when the system is near resonance. With the above definitions, the time variations of the eccentricity and argument of periastron are given by
\[
h_j(t) = \sum_i \Lambda_{ji} \sin(\lambda_i t + \beta_i), \quad k_j(t) = \sum_i \Lambda_{ji} \cos(\lambda_i t + \beta_i),
\]
(6)
where the \( \lambda_i \) are eigenvalues of the matrix \( A_{ij} \) and the \( \Lambda_{ji} \) are the corresponding eigenvectors. The phases \( \beta_i \) and the normalization of the eigenvectors are determined by the initial conditions, i.e., the values of eccentricity \( e_j \) and argument of periastron \( \varpi_j \) for each planet at \( t = 0 \) (taken to be the time when the orbital elements of the extrasolar planets are measured).

3. General Relativity in Observed Systems

To illustrate the action of GR in extrasolar planetary systems, we have calculated the secular interactions for two observed systems [3] with and without including the general relativistic terms.
We use the systems Upsilon Andromedae and HD160691 because they contain inner planets with $a \sim 0.05$ AU and outer planets with $a \sim 1$ AU. As shown below, this type of solar system architecture allows for GR to compete with secular interactions. Figure 1 shows the mean eccentricity $\langle e \rangle$ of the innermost planet, as driven by secular interactions and averaged over many cycles, plotted as a function of $\sin i$ for the two systems.

For Upsilon Andromedae (top panel), general relativity acts to damp the excitation of eccentricity by secular interactions. Since the viewing angle is not measured, the resulting mean eccentricity $\langle e \rangle$ is shown as a function of $\sin i$. For large $\sin i$ values, the inner planet would be driven to mean eccentricity values $\langle e \rangle \approx 0.4$ without GR, but only $\langle e \rangle \approx 0.016$ when relativistic corrections are included. The observed eccentricity for the inner planet $e_{\text{obs}} \approx 0.011$ is much closer to the relativistic mean value. Since secular theory uses the observed orbital elements in the boundary conditions, the system always has some chance of displaying the observed (low) eccentricity of the inner planet (even if $\langle e \rangle \sim 0.4$), so the implications for GR must be stated in terms of probabilities: If the observed eccentricity of the inner planet has a measurement error that is gaussian distributed with width $\sigma_{\text{obs}} = 0.015$, then the probability of observing the system in its current state would be only 0.023 in the absence of GR. The probability of finding the inner planet with its observed eccentricity is 0.78 if GR is included. The validity of general relativity is thus strongly favored.

For the HD160691 system (bottom panel), the inclusion of relativistic terms acts in the opposite direction, i.e., it leads to greater predicted values of $\langle e \rangle$. For values of $\sin i \sim 1$, the predicted mean driven eccentricities are small enough to be consistent with the observed low value $e_{\text{obs}} \sim 0$. As $\sin i$ decreases, however, the level of eccentricity forcing increases and the system reaches a resonance near $\sin i \approx 0.5$. The observed low value of eccentricity, in conjunction with these considerations, thus constrain the viewing angle of the HD160691 system to be nearly edge-on; if we require $\langle e \rangle \lesssim 0.05$, consistent with observational uncertainties, the viewing angle is confined to the range $\sin i \gtrsim 0.93 \ (i \gtrsim 70^\circ)$. Since inclination angles are notoriously difficult to measure in these systems, this constraint on $i$ is quite valuable. For example, this limit on the viewing angle has important implications for the possibility of observing the inner planet in transit [6].

4. The Magnitude of Relativistic Effects

Next we want to find an analytic criterion that characterizes the size of relativistic effects. In
these systems, general relativity is significant when the final term in equation (4) competes with the others. In practice, only the inner planet has significant relativistic corrections. We can also assume that the system is sufficiently hierarchical so that \( b_{3/2}^{(1)}(\alpha_{j1}) = 3\alpha_{j1} + \mathcal{O}(\alpha_{j1}^2) \), and that only one outer planet competes with the relativistic correction. In this limit, we obtain the requirement

\[
\frac{m_j}{M_*} \alpha_{j1}^3 \approx \frac{4GM_*}{c^2a_1},
\]

where the subscript ‘1’ denotes the inner planet and ‘j’ the outer planet. Both sides of this equation represent small dimensionless quantities. When their ratio is of order unity, however, relativistic effects compete with secular interactions, i.e., this constraint is equivalent to the requirement that one of the dimensionless fields \( \Pi \sim 1 \), where

\[
\Pi \equiv \frac{4GM_*^2a_j^3}{c^2m_ja_1^4} \approx 6.3\left(\frac{m_j}{M_*}\right)^{-1}\left(\frac{M_*}{M_\odot}\right)^2\left(\frac{a_j}{1\text{ AU}}\right)^3\left(\frac{a_1}{0.05\text{ AU}}\right)^{-4}.
\]

The second equality indicates that relativistic effects compete with secular interactions for a Jovian planet in a \( \sim 1 \text{ AU} \) orbit perturbing an inner planet in a \( \sim 0.05 \text{ AU} \) orbit. The relative size of the relativistic effect grows with increasing semi-major axis \( a_j \) of the perturber, but the absolute size of the secular effect decreases. For closer perturbing planets \( (a_j \ll 1 \text{ AU}) \), interactions are stronger but relativity plays little role; for more distant planets \( (a_j \gg 1 \text{ AU}) \), the interactions are weak but are dominated by relativity, which only makes the inner planet precess forward in its orbit. The condition for GR to compete with secular interactions can also be written in terms of time scales. GR itself defines a characteristic time scale \( \tau_{gr} \),

\[
\tau_{gr} \equiv \frac{c^2a_1^{3/2}}{3(GM_\odot)^{3/2}} \approx 3011 \text{ yr} \left(\frac{a_1}{0.05 \text{ AU}}\right)^{5/2}\left(\frac{M_*}{1.0M_\odot}\right)^{-3/2},
\]

the time required for the periastron to precess one radian forward in its orbit. When this time scale is comparable to the secular interaction time scale, \( \tau_{gr} \sim \tau_{sec} \), then GR plays a significant role in the dynamics. For planets with \( \sim 4 \text{ day} \) orbits, \( \tau_{gr} \) is comparable to the secular interaction time scales for many of the observed extrasolar planetary systems, where \( \tau_{sec} = 10^2 - 10^5 \text{ yr} \) [6].

5. The Sign of Relativistic Effects

Figure 1 shows that GR can lead to either larger (HD160691) or smaller (Ups And) eccentricity forcing, compared with classical theory. However, most previous discussions of relativistic effects in planetary systems emphasize its stabilizing influence – the tendency for GR to reduce the amplitude
of eccentricity forcing. It is thus useful to explore when the two types of behavior occur, and what system parameters are required.

For a two planet system, we consider an idealized case in which the inner planet has a nearly circular orbit, or at least cycles through the \( e = 0 \) state. This condition is common in that close planets in observed systems tend to have nearly circular orbits [3]. In this limit, the eccentricity amplitude \( \eta \) of the inner planet (forced by the outer planet) can be written in the form

\[
\eta^2 = \frac{A_{22}^2 e_{2(0)}^2}{(\lambda_1 - \lambda_2)^2},
\]

where \( e_{2(0)} \) is the initial eccentricity of the outer planet. For a two planet system, the difference in eigenvalues is positive definite and is given by the expression

\[
\Delta \lambda = \lambda_1 - \lambda_2 = [(A_{11} - A_{22})^2 + 4A_{12}A_{21}]^{1/2}.
\]

Since the eigenvalues cannot be degenerate for a two planet system, secular interactions do not lead to resonance. Notice that only the diagonal matrix elements contain GR corrections and only the inner planet is close enough to the star for such corrections to matter. The first matrix element can thus be written \( A_{11} = A_0 + 3n_1 \mu \), where \( A_0 \) is the matrix element in the absence of GR and \( \mu = GM_\ast/(c^2a_1) \) is the relativistic correction factor. With this construction, the question of whether GR acts to increase or decrease the amplitude of eccentricity oscillation depends on the sign of the derivative \( d\eta/d\mu \), i.e.,

\[
\text{sign}\left(\frac{d\eta}{d\mu}\right) = -\text{sign}(A_0 + 3n_1 \mu - A_{22}) = -\text{sign}\left[1 + \Pi - \frac{m_1}{m_2} \left(\frac{a_1}{a_2}\right)^{1/2}\right].
\]

The second equality uses the approximations \( m_1, m_2 \ll M_\ast, a_1 \ll a_2 \), and \( b_{3/2}^{(1)} \approx 3a_1/a_2 \); the dimensionless field \( \Pi \) is defined in equation (8). The third term must dominate in order for the sign to be positive, and hence for relativity to lead to greater eccentricity amplitudes. Since \( a_1 < a_2 \) by definition, relativity will lead to greater stability (smaller amplitudes) unless \( m_1 > m_2 \). As a result, relativity can amplify eccentricity forcing only for sufficiently massive inner planets, where the requirement for amplification can be written in the form

\[
m_1 > m_2 \left(\frac{a_2}{a_1}\right)^{1/2} + 4\mu M_\ast \left(\frac{a_2}{a_1}\right)^{7/2}.
\]

This constraint thus implies that the inner planet mass \( m_1 \) must be large compared to the outer planet mass \( m_2 \). The two planet systems observed to date [3] show an interesting trend: None
of these solar systems have planetary masses that satisfy this inequality, and hence GR leads to
greater stability in all of these (two planet) cases.

Now we consider a three planet system in which the inner planet has little effect on the outer
two planets due to its smaller mass and/or inner position. These characteristics apply to both
HD160691 and UpsAnd. In this hierarchical limit, the $3 \times 3$ matrix $A_{ij}$ can be approximated by
taking $A_{12} = A_{13} = A_{21} = A_{31} = 0$. The eigenvalues of this reduced matrix then take the form

$$\lambda_1 = A_{11}, \lambda_{2,3} = \frac{1}{2} \left\{ (A_{22} + A_{33} \pm [(A_{33} - A_{22})^2 + 4A_{23}A_{32}]^{1/2}) \right\}. \quad (14)$$

With this level of complexity, the system can have degenerate eigenvalues, e.g., when $\lambda_1 = A_{11}$ is
equal to either $\lambda_2$ or $\lambda_3$. Without loss of generality, suppose that $\lambda_1$ and $\lambda_2$ are nearly degenerate.
The effect of GR is to add a small positive contribution to $A_{11}$ and hence $\lambda_1$ (we again take $A_{11} = A_{0} + 3n_1\mu$), where the added term describes the relativistic precession of the inner planet’s orbit.
If $\lambda_1 \lesssim \lambda_2$, then GR brings the eigenvalues closer together and thus acts to increase eccentricity
forcing. If $\lambda_1 \gtrsim \lambda_2$, then GR makes the eigenvalues more unequal and acts to decrease eccentricity
forcing. The full cubic equation for $\text{det}[A - \lambda I]$ is more complicated, but allows for similar behavior.

6. Conclusion

For multiple planet systems with favorable architectures, the effects of general relativity can be
significant (Fig. 1). When solar systems have two relatively massive outer planets and a third inner
planet of smaller mass near secular resonance, GR effects are large enough to move the system in
or out of a resonant condition. Since the resonance condition depends on planetary masses, which
in turn depend on $\sin i$, and since small planets cannot survive in exact resonance, this effect can
be used to constrain the allowed range of $\sin i$ in observed extrasolar planetary systems. For the
HD160691 system, this constraint implies that $\sin i > \sim 0.93$. This GR effect thus provides another
means of probing the properties of these solar systems. For other systems, general relativity causes
the inner planet to precess forward in its orbit fast enough to compromise eccentricity excitation
from a second planet, i.e., the mean eccentricity values are much smaller than they would be in flat
space. This latter effect can be used as a new test of general relativity. For example, the Upsilon
Andromedae system has an $\sim 80\%$ chance of being observed with its measured parameters if GR is
correct, but only a $\sim 2\%$ chance if the GR corrections were absent. Although the weak-field limit
of GR (including the perihelion advance of Mercury) is well established in our own solar system.
[1], the results of this paper add to our understanding by: [A] Developing a new relativistic effect – with amplified sensitivity – due to GR competing with secular interactions, and [B] Providing an independent confirmation of GR in other solar systems. In the future, additional extrasolar planetary systems and/or higher precision observational data can provide more stringent tests of general relativity.

References

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Fig. 1.— The effects of general relativistic corrections on two extrasolar planetary systems. The mean eccentricity $\langle e \rangle$ of the innermost planet, as driven by secular interactions and averaged over many cycles, is plotted as a function of $\sin i$ for the Upsilon Andromedae system (top panel) and the HD160691 system (bottom panel). Both systems have three planets detected to date. The predictions of secular theory are shown with relativistic corrections as the solid curves, and without relativistic corrections as the dashed curves. Notice that the relativistic terms act in opposite ways in the two systems: Inclusion of relativity acts to damp eccentricity excitation by the secular interactions in the Upsilon Andromedae system. In HD160691, however, they allow the system to approach a resonance for $\sin i \approx 0.5$. 