Achievable Rate of Near-Field Communications Based on Physically Consistent Models

Mohamed Akrout\textsuperscript{1}, Volodymyr Shyianov\textsuperscript{1}, Faouzi Bellili\textsuperscript{1}, Amine Mezghani\textsuperscript{1}, Robert W. Heath\textsuperscript{2}

\textsuperscript{1}ECE Department, University of Manitoba, Winnipeg, Canada,
\textsuperscript{2}ECE Department, North Carolina State University, Raleigh, USA

\{akroutm,shyianov\}@myumanitoba.ca, \{Faouzi.Bellili,Amine.Mezghani\}@umanitoba.ca, rwheathjr@ncsu.edu

Abstract—This paper introduces a novel information-theoretic approach for studying the effects of mutual coupling (MC), between the transmit and receive antennas, on the overall performance of single-input-single-output (SISO) near-field communications. By incorporating the finite antenna size constraint using Chu’s theory and under the assumption of canonical-minimum scattering, we derive the MC between two radiating volumes of fixed sizes. Expressions for the self and mutual impedances are obtained by the use of the reciprocity theorem. Based on a circuit-theoretic two-port model for SISO radio communication systems, we establish the achievable rate for a given pair of transmit and receive antenna sizes, thereby providing an upper bound on the system performance under physical size constraints. Through the lens of these findings, we shed new light on the influence of MC on the information-theoretic limits of near-field communications using compact antennas.

Index Terms—Circuit theory for communications, near-field wireless communications, SISO, mutual coupling, induced EMF method, canonical minimum scattering antennas.

I. INTRODUCTION

With the increasing need for packing a massive number of antennas into 4G/5G platforms and towers, mutual coupling (MC) is inevitable between any two closely-spaced antennas whether being mounted \(i\) on two different transceivers in near-field (NF) SISO communications, or \(ii\) on the same transceiver in far-field (FF) multiple-input-multiple-output (MIMO) communications. Therefore, the conventional system design for FF communications is no longer appropriate and the strong NF interactions together with the significant MC effects must be meticulously studied to ultimately characterize the physical limitations of NF wireless systems. This is because MC can be a foe and a friend at the same time owing to the many advantages it has (e.g., effective current sheet) and the various challenges it raises (e.g., beam width/gain decrease, reflection resonances, etc.) \[1\], \[2\]. This calls for a principled approach of incorporating the MC in a physically-consistent model \[3\] starting at the Maxwell’s equations level \[4\], which many engineers continue to avoid.

To date, the topic of MC between a finite number of antennas has been extensively investigated in the literature using three different approaches \[1\]:

- Direct solvers which refer to the variety of direct numerical methods such as finite element/difference techniques and the method of moments;
- Element-by-element methods which involve the construction of an impedance or admittance matrix by considering each pair of antenna elements separately, with the remaining elements being electromagnetically removed;
- Minimum scattering methods which are a special class of the element-by-element methods for canonical minimum scattering (CMS) antennas \[5\] whose electromagnetic properties are expressed explicitly in terms of their radiation patterns when they are open-circuited only.

From another perspective, the footprint of today’s wireless technology is dominated by the antenna size, which cannot be miniaturized beyond the Chu limit \[6\]. Thus, the actual performance of wireless communication systems can only be gauged if the antenna size is considered as an integral part of their analysis. However, very few of studies \[7\], \[8\] have so far considered the antenna size as a physical constraint in their respective designs to characterize the achievable rate of communication systems. Moreover, their analysis ignored the mutual coupling effect due to the FF assumption.

To account for the antenna size limitations using Chu’s theory \[6\] together with the MC effects, this paper departs from the minimum scattering method to derive the mutual impedance between two small Chu’s CMS antennas by relying on the so-called “induced electromotive forces (EMF)” method \[9\]. The latter is exact for antennas with known current distributions which is indeed the case for electrically small antennas (including CMS antennas) examined in Chu’s theory \[6\].

Based on these considerations, the Chu’s CMS antenna becomes a useful model to incorporate both the size constraint and the mutual coupling independently of the antenna type. Indeed, the mutual impedance in antenna textbooks \[9\]–\[12\] is almost always computed for a specific antenna type. Our work, however, does not consider any specific antenna design to analyze the information-theoretic performance of SISO communications. This is key to studying the fundamental limits of communication systems based on a generic approach that is oblivious to their physical implementations.

Physically-consistent models come into play as an effective tool to describe the underlying circuits of the NF communications models \[3\] while accounting for the MC effects and the antenna size constraint. In fact, to rigorously understand the limits of miniaturization, it is essential to use a quantitative measure not only of the MC itself but, most importantly, of the influence that it exerts on the overall system performance.

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As a matter of fact, in any physically-consistent model, the noise cannot be regarded from a statistical viewpoint only, e.g., treating it as an additive Gaussian-distributed random variable. Rather, there is a need for a precise description depending on the noise sources at both the transmitter and the receiver along with the self/mutual impedances of the Chu’s CMS antennas. From an information-theoretic perspective, the effects of MC cannot genuinely be comprehended unless aspects of the entire NF communication system are taken into account.

In this work, we derive the achievable rate of NF and FF SISO communications with a restriction on the size of the transmit and receive antennas. Using a physically consistent circuit model of radio communications, we find the maximum mutual information per unit of time between the input and output signals of the system under the antenna size constraint. For a given size pair \((a_{T}, a_{R})\) of the transmit and receive Chu’s CMS antennas, we determine the transmit-receiver mutual impedances in the NF using the induced EMF method.

To analyze the impact of MC on the performance of NF communications, this paper provides a general framework at the intersection of electromagnetic and communication theories by focusing on the SISO case only. This constitutes a stepping stone to the analysis of MIMO systems through the use of multi-port circuit models. The latter will be the subject of a standalone follow-up work. A full version of this paper is available in [13], which includes detailed derivations, more simulation results, and indepth discussion of the prior art.

We also mention the common notations used in this paper. Given any complex number \(z\), \(\Re\{z\}\), \(\Im\{z\}\), and \(\{z\}\) return its real part, imaginary part, and complex conjugate, respectively. Throughout the paper, \(c\) denotes the speed of light in vacuum (i.e., \(c \approx 3 \times 10^8\)) \(\lambda\) is the wavelength, and \(k_{0} = 1.38 \times 10^{-23} \text{m}^2\text{kg}^{-1}\text{s}^{-2}\text{K}^{-1}\) is the Boltzmann constant. Moreover, \(\mu = 1.25 \times 10^{-6} \text{m} \text{kg}^{-1}\text{s}^{-2}\text{A}^{-2}\) and \(\epsilon = 8.85 \times 10^{-12} \text{m}^{-3}\text{kg}^{-1}\text{s}^{-2}\text{A}^2\) are the permeability and permittivity of vacuum, respectively. Finally, \(k = \omega \sqrt{\frac{\mu}{\epsilon}} = \frac{2\pi}{\lambda}\) and \(\eta = \sqrt{\frac{\mu}{\epsilon}}\) are the wave number and the wave impedance of a plane wave in free space, respectively.

II. SISO COMMUNICATION SYSTEM MODEL

From circuit theory, transmitted/received signals are either voltages or currents that flow through the ports of the transmit/receive antennas. Finding the relationship between port variables at the transmitter(s) and receiver(s) is key to modelling both NF and FF communication channels in a physically consistent way.

\[
\begin{bmatrix}
V_{T} \\
I_{T}
\end{bmatrix}
= Z_{\text{SISO}}
\begin{bmatrix}
V_{R} \\
I_{R}
\end{bmatrix}
\]

As depicted in Fig. 1, a SISO communication channel can be viewed as a two-port network connecting the port currents, \((I_{T}(f), I_{R}(f))\), and the port voltages, \((V_{T}(f), V_{R}(f))\), through an impedance matrix \(Z_{\text{SISO}}\) as follows:

\[
\begin{bmatrix}
V_{T}(f) \\
V_{R}(f)
\end{bmatrix}
= \begin{bmatrix}
Z_{T} & Z_{TR} \\
Z_{RT} & Z_{R}
\end{bmatrix}
\begin{bmatrix}
I_{T}(f) \\
I_{R}(f)
\end{bmatrix}.
\]

The diagonal entries, \(Z_{T}\) and \(Z_{R}\), represent the self-impedances of transmit and receive antennas, respectively. The off-diagonal entries, \(Z_{RT}\) and \(Z_{TR}\), represent the mutual transmit-receive and receive-transmit impedances, respectively, between transmit and receive antennas.

A. A circuit-theoretic SISO communication model

We consider the circuit-based model, depicted in Fig. 2, to study a SISO communication system where the transmit and receive antennas are at either NF or FF separation distances. There, the wireless channel \(Z_{\text{SISO}}\) can model both the NF and FF communication channels, \(Z_{\text{SISO}}^{\text{NF}}\) and \(Z_{\text{SISO}}^{\text{FF}}\), respectively, depending on whether the MC effects are taken into account or not.

\[
\begin{bmatrix}
V_{T} \\
I_{T}
\end{bmatrix}
= Z_{\text{SISO}}
\begin{bmatrix}
V_{R} \\
I_{R}
\end{bmatrix}
\]

\[
\begin{bmatrix}
V_{T} \\
I_{T}
\end{bmatrix}
= \begin{bmatrix}
Z_{T} & Z_{TR} \\
Z_{RT} & Z_{R}
\end{bmatrix}
\begin{bmatrix}
I_{T}(f) \\
I_{R}(f)
\end{bmatrix}.
\]

Fig. 2: SISO communication model including the signal generator, the transmit and receive antennas, and the associated intrinsic noise. The output voltage \(V_{R}\) is connected to the load impedance of a receive device.

The noiseless communication channel between the transmit and receive antennas is represented by the frequency-domain impedance matrix \(Z_{\text{SISO}}\) given in (1). For more realistic scenarios, we hereafter consider a noisy communication channel of the noiseless model in (1).

1) Incorporating noise into the noiseless communication channel model: Accounting for the noise sources at the transmit and receive antennas boils down to injecting their background noise at both the input and output ports. This can be done when the two-port network is only composed of passive components which have the same absolute temperature \(T\) as the surrounding environment [3]. This situation is called the thermal equilibrium noise and corresponds to the case where the two-port network noise originates solely from the thermal agitation of electrons as they flow inside its all-passive components. To this end, we introduce two voltage sources \(V_{\text{NT}}(f)\) and \(V_{\text{NR}}(f)\) at the transmitter and receiver ports as depicted in Fig. 2. There, the terminals of the non-ideal generator voltage \(V_{T}(f)\) (i.e., with internal resistance \(R\)) are connected to the transmitting antenna through a noisy input port with the current-voltage pair \((V_{1}(f), I_{1}(f))\). Likewise, the noisy receive antenna terminals are connected to the outside world through the output port with current-voltage pair \((V_{2}(f), I_{2}(f))\).
When the mutual coupling is ignored (e.g., the FF scenario), the noise voltages $\overline{V}_{N,T}(f)$ and $\overline{V}_{N,R}(f)$ are uncorrelated from each other, i.e.:

$$E\left[\overline{V}_{N,T}(f) \overline{V}_{N,R}(f)\right] = 0. \quad (2)$$

Besides, their auto-correlations (a.k.a., second-order moments) are determined using the truncated Fourier transform [14]:

$$\lim_{T_0 \to \infty} \frac{1}{T_0} E\left[\overline{V}_{N,T}(f)\right] = 4 k_b T \Re\{Z_T\}, \quad (3a)$$

$$\lim_{T_0 \to \infty} \frac{1}{T_0} E\left[\overline{V}_{N,R}(f)\right] = 4 k_b T \Re\{Z_R\}. \quad (3b)$$

In NF communications, however, the noise voltages $\overline{V}_{N,T}(f)$ and $\overline{V}_{N,R}(f)$ are correlated and the assumption in (2) does not hold anymore. In other words, one must take the cross-correlation between $\overline{V}_{N,T}(f)$ and $\overline{V}_{N,R}(f)$ into account by considering the following covariance matrix of the noise-induced voltages:

$$\lim_{T_0 \to \infty} \frac{1}{T_0} \begin{bmatrix} E\left[\overline{V}_{N,T}(f)\right] & E\left[\overline{V}_{N,T}(f) \overline{V}_{N,R}(f)\right] \\ E\left[\overline{V}_{N,R}(f) \overline{V}_{N,T}(f)\right] & E\left[\overline{V}_{N,R}(f)\right] \end{bmatrix} = 4 k_b T \Re\{Z_{SISO}\}. \quad (4)$$

Together, the impedance matrix and the noise covariance matrix provide the circuit-theoretic description of the NF and FF communication channels modelled in Fig. 2.

2) The receive LNA model: The LNA is modeled as a noisy frequency-flat device with gain $\beta$, i.e., with the input-output voltage relationship:

$$V_{LNA}(f) = \beta V_{Rin}(f) \quad [V]. \quad (5)$$

For an amplifier with an input impedance $R_{in}$ and a noise figure $N_f$, we determine the second-order statistics of the intrinsic noise voltage, $\overline{V}_{LNA,N}(f)$, generated inside the LNA using the truncated Fourier transform as follows:

$$\lim_{T_0 \to \infty} \frac{1}{T_0} E\left[\overline{V}_{LNA,N}(f)\right] = 4 k_b T R_{in} (N_f - 1). \quad (6)$$

The amplifier noise voltage, $\overline{V}_{LNA,N}(f)$, is uncorrelated with the transmit noise voltage, $\overline{V}_{N,T}(f)$, and the receive noise voltage, $\overline{V}_{N,R}(f)$, i.e.:

$$\lim_{T_0 \to \infty} \frac{1}{T_0} E\left[\overline{V}_{LNA,N}(f) \overline{V}_{N,T}(f)\right] = 0, \quad (7a)$$

$$\lim_{T_0 \to \infty} \frac{1}{T_0} E\left[\overline{V}_{LNA,N}(f) \overline{V}_{N,R}(f)\right] = 0. \quad (7b)$$

3) The input-output relationship of the channel model:

Adding the noise to both the input and output ports of the SISO communication channel model transforms its linear input-output relationship (1) into an affine one. By applying Kirchhoff’s voltage law (KVL) in Fig. 2, we obtain an affine noisy two-port communication channel model:

$$\begin{bmatrix} V_1'(f) \\ V_2'(f) \end{bmatrix} = Z_{SISO} \begin{bmatrix} I_1(f) \\ I_2(f) \end{bmatrix} + \begin{bmatrix} \overline{V}_{N,T}(f) \\ \overline{V}_{N,R}(f) \end{bmatrix}. \quad (8)$$

Moreover, using basic circuit analysis, the relationship between the output voltage $V_R(f)$ and the input voltage $V_T(f)$ is obtained as follows:

$$V_R(f) = V_{LNA}(f) + \beta R_{in} \times \frac{Z_{TR}(f) (V_T(f) + \overline{V}_{N,T}(f)) + (R + Z_T) \overline{V}_{N,R}(f)}{(R_{in} + Z_R(f)) (R + Z_T(f)) - Z_{TR}^2(f)}, \quad (9)$$

which can be rewritten in the same form as the conventional model for wireless communication channels:

$$V_R(f) = H(f) V_T(f) + W(f). \quad (10)$$

In (10), $H(f)$ represents the transfer function of the channel and $W(f)$ is the Fourier transform of the noise $w(t)$ given by:

$$H(f) = \frac{\beta R_{in} Z_{TR}(f)}{(R_{in} + Z_R(f)) (R + Z_T(f)) - Z_{TR}^2(f)}, \quad (11a)$$

$$W(f) = \overline{V}_{LNA,N}(f) + \beta R_{in} \times \frac{Z_{TR}(f) \overline{V}_{N,T}(f) + (R + Z_T(f)) \overline{V}_{N,R}(f)}{(R_{in} + Z_R(f)) (R + Z_T(f)) - Z_{TR}^2(f)}, \quad (11b)$$

wherein the self and mutual (NF and FF) impedances, i.e., diagonal and off-diagonal entries of $Z_{SISO}$, can be derived using the electromagnetic coupling model as will be shown in Section III. One can also notice that the noise in (11b) depends on self/mutual impedances, unlike the conventional assumption of independent additive Gaussian noise. The derived SISO model in (10) and (11) requires the expressions of the NF and FF mutual impedances, namely, $Z_{SISO}^{NF}$ and $Z_{SISO}^{NF}$, to fully characterize the SISO communication circuit model given in Fig. 2. The analytical expressions of these two impedance matrices will be established in Section III.

B. Circuit-theoretic models for near- and far-field SISO communications

The SISO channel model established in (10) and (11) is not tailored to CMS antennas. In fact, their equivalent circuits must be specified to fully characterize their electromagnetic radiation properties, and hence their respective mutual impedances. Using Chu’s CMS antenna modelled in Fig. 2 from [13] at both the transmitter and the receiver, the SISO model in Fig. 2 yields the NF circuit-theoretic model depicted in Fig. 3. There, the two current-dependent sources, $I_1$ and $I_2$, account for the transmitter-receiver and receiver-transmitter reaction currents when the transmitter and the receiver are in close proximity (i.e., in the NF).

Unlike the NF circuit model in Fig. 3, the far-field circuit model does not involve the receiver-transmitter reaction current since the electrical properties of the receiver do not influence those of the transmitter, i.e., $Z_{TR} = 0$. As will be shown in Section III, this difference affects the computation of the mutual impedances of Chu’s CMS antennas only. The self-impedances, $Z_T$ and $Z_R$, remain unchanged since they do not depend on the separating distance between the transmitter and the receiver. Furthermore, both $Z_T$ and $Z_R$ are obtained from Fig. (3) as the self-impedances of the transmit and receive.
of two Chu’s CMS antennas, two Hertz dipoles, and establish a relationship between the mutual impedances of
overcome this limitation, we resort to the equivalence theorem ([13]) are known outside their encompassing spheres only. To
whose radiated EM fields (described in Appendix I-1 from
the antenna, making it incompatible with Chu’s CMS antennas
under a known current distribution. The induced EMF method tromotive forces that are induced on the antenna structure
mutual impedance is obtained based on the Friis’ equation.

$$Z_{\text{Hertz}}(f) = \frac{c^2 R_1 + j \frac{2\pi f c a_T R_1}{2\pi f c a_T} - (2\pi f a_T)^2 R_1}{c^2 R_2 + j \frac{2\pi f c a_R R_2}{2\pi f c a_R} - (2\pi f a_R)^2 R_2} \quad \text{[Ω]},$$

$$Z_{\text{Chu}}(f) = \frac{\sqrt{\mathcal{R}\{Z_{\text{Hertz}}^{\text{RT}}\}} \mathcal{R}\{Z_{\text{Hertz}}^{\text{TR}}\}}{\sqrt{\mathcal{R}\{Z_{\text{Chu}}^{\text{RT}}\}} \mathcal{R}\{Z_{\text{Chu}}^{\text{TR}}\}} \quad \text{[Ω]},$$

III. COMPUTATION OF THE NEAR- AND FAR-FIELD IMPEDANCE MATRICES

We now derive the mutual impedances involved in the NF and FF SISO channel impedance matrices, $Z_{\text{SISO}}^{\text{NF}}$ and $Z_{\text{SISO}}^{\text{FF}}$, respectively. The NF mutual impedances are derived using the induced EMF method owing to the so-called Chu-Hertz equivalence established in Appendix II-B from [13]. The FF mutual impedance is obtained based on the Friis’ equation.

A. Computation of the near-field mutual impedances

1) Induced EMF analysis and the Chu-Hertz equivalence: The induced EMF method consists in studying the electromotive forces that are induced on the antenna structure under a known current distribution. The induced EMF method requires the explicit knowledge of the current distribution of the antenna, making it incompatible with Chu’s CMS antennas whose radiated EM fields (described in Appendix I-1 from [13]) are known outside their encompassing spheres only. To overcome this limitation, we resort to the equivalence theorem and establish a relationship between the mutual impedances of two Hertz dipoles, $Z_{\text{Hertz}}^{\text{RT}}$, $Z_{\text{Hertz}}^{\text{TR}}$, and the mutual impedances of two Chu’s CMS antennas, $Z_{\text{Chu}}^{\text{RT}}$, $Z_{\text{Chu}}^{\text{TR}}$. We are now ready to state the radiation resistance equivalence,

\textbf{Result 1.} The radiation of a Hertz dipole $\mathcal{R}\{Z_{\text{Hertz}}^{\text{SISO}}\}$ having a uniform current distribution, $I$, and a Chu’s antenna $\mathcal{R}\{Z_{\text{Chu}}^{\text{SISO}}\}$ can be made equal by adequately choosing the mode coefficient, $A_1$, of the TM$_1$ mode pertaining to the Chu’s electric antenna as follows:

$$A_1 = j \frac{Ik^2 c}{4\pi f} \sqrt{3\mathcal{R}\{Z_{\text{Hertz}}^{\text{SISO}}\}/2\pi f},$$

$$\mathcal{R}\{Z_{\text{Chu}}^{\text{SISO}}\} \approx \mathcal{R}\{Z_{\text{Hertz}}^{\text{SISO}}\}.$$

\textbf{Proof.} See Appendix I from [13].

Using the equivalent radiated power, which is a direct consequence of the radiation resistance equivalence in (13), we derive the following relationship between the mutual impedances of two Chu’s antennas and two Hertz dipoles:

\begin{align}
Z_{\text{Hertz}}^{\text{RT}}(f) &= \sqrt{\mathcal{R}\{Z_{\text{Hertz}}^{\text{RT}}\}} \mathcal{R}\{Z_{\text{Hertz}}^{\text{TR}}\} \\
Z_{\text{Hertz}}^{\text{TR}}(f) &= \sqrt{\mathcal{R}\{Z_{\text{Chu}}^{\text{RT}}\}} \mathcal{R}\{Z_{\text{Chu}}^{\text{TR}}\}.
\end{align}

Proof. See Appendix II-B from [13].

It is therefore enough to apply the induced EMF method on two Hertz dipoles and then deduce the mutual impedance between two Chu’s antennas using Result 2.

2) Near-and far-field mutual impedance calculation: By resorting to basic EM field theory, we consider a pair of transmit and receive Hertz dipoles. The dipoles are separated by a distance $d$, aligned with respect to (w.r.t.) their axes $w$ and $z$, respectively, and arbitrarily rotated with angles $\beta$ and $\gamma$ w.r.t. their connecting axis $r$ as depicted in Fig. 4. Using Result 2, it is easy to show that the mutual impedances between two Chu’s CMS antenna are given by (15).

Unlike the computation of the NF mutual impedances where a minimum of EM theory was invoked to model the MC effect, the FF channel model is governed by the Friis’ equation. Using basic circuit analysis, the FF mutual impedance, $Z_{\text{Chu}}^{\text{RT}}(f)$, is expressed as (see Appendix III from [13]):

$$Z_{\text{Chu}}^{\text{RT}}(f) = \frac{\sqrt{\mathcal{R}\{Z_{\text{Hertz}}^{\text{RT}}\}} \mathcal{R}\{Z_{\text{Hertz}}^{\text{TR}}\}}{\sqrt{\mathcal{R}\{Z_{\text{Chu}}^{\text{RT}}\}} \mathcal{R}\{Z_{\text{Chu}}^{\text{TR}}\}} = \frac{1}{\sqrt{\mathcal{R}\{Z_{\text{Hertz}}^{\text{RT}}\}} \mathcal{R}\{Z_{\text{Hertz}}^{\text{TR}}\}} \frac{1}{\sqrt{\mathcal{R}\{Z_{\text{Chu}}^{\text{RT}}\}} \mathcal{R}\{Z_{\text{Chu}}^{\text{TR}}\}} \frac{1}{\sqrt{\mathcal{R}\{Z_{\text{Hertz}}^{\text{RT}}\}} \mathcal{R}\{Z_{\text{Hertz}}^{\text{TR}}\}} \frac{1}{\sqrt{\mathcal{R}\{Z_{\text{Chu}}^{\text{RT}}\}} \mathcal{R}\{Z_{\text{Chu}}^{\text{TR}}\}}.$$

In (18), $d$ is the distance between the transmit and receive antennas which have gains $G_T$ and $G_R$, respectively. Since antennas are reciprocal, we have $Z_{\text{RT}}(f) = Z_{\text{TR}}^{-1}(f)$. Moreover, the signal attenuation in the FF region between the transmitter and the receiver is very large, i.e.: $|Z_{\text{Chu}}^{\text{RT}}(f)| = |Z_{\text{Chu}}^{\text{TR}}(f)| \approx |Z_{\text{Hertz}}^{\text{RT}}(f)| \approx 0$.

IV. ACHIEVABLE RATE OPTIMIZATION

The maximum achievable rate is the largest mutual information between the input and the output of the communication channel. It is seen from Fig. 3 that the transmit voltage waveform $v_T(t)$ is under full control of the system...
\[ Z_{\text{Chu}}^{\text{TR}} = Z_{\text{Chu}}^{\text{TR}} = -\frac{3k_0^2c^2}{4\pi^2f^2} \sqrt{\mathbb{R}\{Z_{\text{Chu}}^{\text{TR}}\} \mathbb{R}\{Z_{\text{Chu}}^{\text{TR}}\}} \times \left[ \frac{1}{2} \sin(\beta) \sin(\gamma) \left( \frac{1}{j k_0 d} + \frac{1}{(j k_0 d)^2} \right) \cos(\gamma) \cos(\beta) \left( \frac{1}{j k_0 d} + \frac{1}{(j k_0 d)^2} \right) \right] e^{-j k_0 d}. \] (15)

\[ X(f) = |Z_{\text{Chu}}^{\text{TR}}(f)|^2 \mathbb{R}\{Z_{\text{Chu}}^{\text{TR}}(f)\} + (R + \frac{Z_{\text{Chu}}^{\text{TR}}(f)}{2^1}) Z_{\text{Chu}}^{\text{TR}}(f) \mathbb{R}\{Z_{\text{Chu}}^{\text{TR}}(f)\} + (R + Z_{\text{Chu}}^{\text{TR}}(f)) \mathbb{R}\{Z_{\text{Chu}}^{\text{TR}}(f)\} + (R + Z_{\text{Chu}}^{\text{TR}}(f))^2 \mathbb{R}\{Z_{\text{Chu}}^{\text{TR}}(f)\}. \] (16)

\[ Y(f) = |(R_{\text{in}} + Z_{\text{Chu}}^{\text{TR}}(f)) (R + Z_{\text{Chu}}^{\text{TR}}(f) - (Z_{\text{Chu}}^{\text{TR}}(f))^2 |^2. \] (17)

\[ C = \max_{\mathbb{P}\{\mathbf{v}\}} I(\mathbf{v}_T(t); \mathbf{v}_R(t)) \text{ [bits/s]}, \] (20)

where \( I(\mathbf{v}_T(t); \mathbf{v}_R(t)) \) is the mutual information per unit time between the two random processes representing the input and output voltages/signals of the communication system [15], [16]. Moreover, \( \mathbb{P}\{\mathbf{v}\} \) is the probability measure on the space of input/generator voltages, \( \mathbf{v}_T(t) \), which for any finite set of time instants \( \{t_1, t_2, \ldots, t_n\} \), specifies the following joint cumulative distribution function \( \forall (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n \):

\[ \mathbb{P}\{\mathbf{v}(t_1) \leq v_1, \mathbf{v}(t_2) \leq v_2, \ldots, \mathbf{v}(t_n) \leq v_n \}. \] (21)

In designing the probability law of the generator, we suppose that the expected per-frequency power, \( P_1(f) \), satisfies:

\[ P_1(f) \leq P_{\text{max}}, \forall f, \] (22)

where \( P_{\text{max}} \) is the maximum power that the generator can supply due to regulatory restrictions or hardware constraints. In the sequel, we study the impact of the physical size constraints at both the transmitter and the receiver by restricting the volumes encompassing their antennas to be of finite radii, namely \( a_T \) and \( a_R \), respectively.

To evaluate the maximum achievable rate given in (20) using the channel response \( H(f) \) established in (11a) and the Fourier transform of the noise \( W(f) \) obtained in (11b), we introduce the two quantities \( X(f) \) and \( Y(f) \) given in (16) and (17), respectively. By employing the noise auto/cross-correlation properties in (2)-(7) as well as (16) and (17), we show that:

i) The square magnitude of the channel response \( H(f) \) is expressed as:

\[ |H(f)|^2 = \beta^2 \frac{P_{\text{in}}^2 |Z_{\text{Chu}}^{\text{TR}}(f)|^2}{Y(f)}. \] (23)

ii) The PSD of the noise is given by:

\[ N(f) = \frac{1}{4R} \lim_{T_0 \to \infty} \frac{1}{T_0} \mathbb{E}[|W(f)|^2] = k_b T \frac{P_{\text{in}}}{R} \left[ (N_f - 1) + \beta^2 P_{\text{in}} X(f) Y(f) \right]. \] (24)

Using the established expression of the achievable rate under antenna size constraints for both NF and FF scenarios, we now examine the effect of the transmit and receive antenna sizes on the overall performance of SISO communications.

V. NUMERICAL RESULTS AND DISCUSSIONS

In this section, we present numerical results for the maximum achievable rate of SISO communications under antenna size constraints at both the transmitter and the receiver by inspecting the behavior of the following metrics:

- the SNR as a function of the frequency with \( \text{SNR}(f) = \frac{P_1(f) |H(f)|^2}{N(f)} \),
- the maximum achievable rate in (20) under transmit/receive antenna size constraints with uniform power allocation (PA),
- the maximum achievable rate under optimal PA, \( P^*_T(f) \).

We also examine the system performance under the following two configurations:

a) colinear configuration as depicted in Fig. 5a where the axes of transmit and receive antennas are colinear,

b) parallel configuration as depicted in Fig. 5b where the axes of transmit and receive antennas are parallel.

\[ \gamma = \pi \quad \beta = 0, \] (a)

\[ \gamma = \frac{\pi}{2} \quad \beta = \frac{\pi}{2}. \] (b)

In both scenarios, the Chu’s sphere encompassing the transmit (resp. receive) antenna has a radius \( a_T \) (resp. \( a_R \)) with \( a_T + a_R \leq d \). The latter condition ensures that the transmit
and receive spheres do not overlap in the near-field region so that the equivalence theorem can still be used to find the corresponding mutual impedances.

We first plot in Fig. 6 the SNR as a function of the electrical distance/separation, $d/\lambda$, for two antenna sizes, $a/\lambda_p \in \{0.2, 0.25\}$ with $(a_T, a_R) = (a, a)$, for both the colinear and parallel configurations. Here, $\lambda_p$ is the peak wavelength, i.e., the wavelength calculated at the highest SNR. In [24], we set the noise factor to $N_t = 5 \text{ dB}$ and the noise temperature to $T = 300 \text{ [K]}$. Fig. 6 reveals that the colinear antenna configuration yields higher SNR than the parallel configuration as long as $d/\lambda \gtrapprox 0.38$. Thereafter, the SNR decreases quickly and becomes much smaller than that of the parallel configuration. The threshold distance $d \approx 0.38\lambda$ can be interpreted as the limit between the NF and FF regions. In fact, when antennas are closely spaced, most of the communication happens through the radial/NF component of the electric field $E_r$ which is aligned along the colinear axis and vanishes in the FF region. It should be noticed that when $d \approx \lambda$ the degradation of the SNR due to finite antenna sizes almost vanishes. This can be attributed to the fact that the antenna size relative to the wavelength increases thereby making the antennas not electrically small anymore, i.e., they do not store a lot of NF reactive energy.

Next, we study the impact of the SNR, or equivalently the transmit power $P_t(f)$, on the performance of compact antennas under optimal power control. Fig. 7 depicts the achievable rate for two antenna sizes, $a$, with $\lambda/a \in \{0.2, 0.25\}$. The horizontal axis shows the ratio $d/\lambda$ for both uniform power allocation, i.e., $P_t(f) = P_{\text{max}}/W$, as well as the optimal power allocation $P_t(f) = \mathcal{P}^*(f)$. The bandwidth is fixed to $W = 0.2f_c$ with $f_c = 25 \text{ [GHz]}$ and the maximum transmit power is $P_{\text{max}} = 10 \text{ [mW]}$. The usual between the colinear and parallel configurations is again observed at $d \approx 0.38\lambda$ which separates the NF and FF regions. It can be noticed that the optimal power allocation provides a gain of up to 50% in terms of achievable rate in the NF region and over 200% in the FF region.

VI. CONCLUSION

This paper employs a circuit-theoretic model of wireless communication in which the channel is derived using CMS Chu’s antennas at both the transmitter and the receiver. After establishing an equivalence between Chu antennas and Hertz dipoles, we computed the mutual impedance using the induced EMF method. This led to a full characterization of the input/output relationship of the proposed circuit-equivalent model. We also examined the effect of the mutual coupling on the achievable rate for both colinear and parallel relative antenna orientations under uniform and optimal power allocation strategies (see [13] for more details). This analysis can be extended to colinear/parallel MIMO systems using multi-port circuit theory where the benefit of spatial multiplexing can be investigated under the mutual coupling effect.

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