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Martingale defocusing and transience of a self-interacting random walk

Yuval Peres∗ Bruno Schapira† Perla Sousi‡

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Abstract

Suppose that \((X, Y, Z)\) is a random walk in \(\mathbb{Z}^3\) that moves in the following way: on the first visit to a vertex only \(Z\) changes by \(\pm 1\) equally likely, while on later visits to the same vertex \((X, Y)\) performs a two-dimensional random walk step. We show that this walk is transient thus answering a question of Benjamini, Kozma and Schapira. One important ingredient of the proof is a dispersion result for martingales.

Keywords and phrases. Transience, martingale, self-interacting random walk, excited random walk.

MSC 2010 subject classifications. Primary 60K35.

1 Introduction

In this paper we study the following self-interacting random walk \((X, Y, Z)\) in \(\mathbb{Z}^3\). On the first visit to a vertex only \(Z\) changes by \(\pm 1\) equally likely, while on later visits to the same vertex \((X, Y)\) performs a two dimensional random walk step, i.e. it changes by \((\pm 1, 0)\) or \((0, \pm 1)\) all with equal probability. This walk was conjectured in [7] to be transient.

Another process of this flavour was analysed in [15]: suppose that \(\mu_1, \mu_2\) are two zero-mean measures in \(\mathbb{R}^3\) and consider any adapted rule for choosing between \(\mu_1\) and \(\mu_2\). By adapted rule, we mean that the next choice every time depends on the history of the process up to this time. In [15] it was proved that if each measure is supported on the whole space, then for any adapted rule, the resulting walk in \(\mathbb{R}^3\) is transient. In [16] transience and recurrence properties and weak laws of large numbers were also proved for specific choices of one-dimensional measures; for instance when \(\mu_1\) is the distribution of simple random walk step and \(\mu_2\) the symmetric discrete Cauchy law.

A larger class of such processes are the so-called self-interacting random walks, which are not Markovian, since the next step depends on the whole history of the process up to the present time.

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For instance the edge or vertex reinforced random walks have attracted a lot of attention, see e.g. [2] [9] [13] [14] [17] [18] [19].

**Theorem 1.1.** Let $W_t = (X_t, Y_t, Z_t)$ be a random walk in $\mathbb{Z}^3$ such that on the first visit to a vertex only $Z_t$ changes to $Z_t \pm 1$ equally likely, while on later visits to a vertex $(X_t, Y_t)$ makes a two dimensional simple random walk step. Then $W$ is transient, i.e. $\|W_t\| \to \infty$ as $t \to \infty$ almost surely.

![Figure 1: The rectangles $[t_k, n] \times [-h_k, h_k]$ and the graph of $M$](image)

We now give a quick overview of the proof of Theorem 1.1. By conditioning on all the jumps of the two dimensional process $(Y, Z)$ and looking at the process $X$ only at the times when $(Y, Z)$ moves, we obtain a martingale $M$. Then we need to obtain estimates for the probability that $M$ is at 0 at time $n$ so that when multiplied by $1/n$ it should be summable. In Section 2 we state and prove a proposition that gives estimates for a martingale to be at 0 at time $n$ when it satisfies certain assumptions. We now state a simpler form of this proposition.

**Corollary 1.2.** Let $M$ be a martingale satisfying almost surely

$$\mathbb{E}[|M_{k+1} - M_k|^2 \mid \mathcal{F}_k] \geq 1 \quad \text{and} \quad |M_{k+1} - M_k| \leq (\log n)^a,$$

for all $k \leq n$ and some $a < 1$. Then there exists a positive constant $c$, such that

$$\mathbb{P}(M_n = 0) \leq \exp \left(-c(\log n)^{1-a}\right).$$

We remark that related results were recently proved by Alexander in [1] and by Armstrong and Zeitouni in [3].

In order to prove Corollary 1.2 we use the same approach as in [10]. More specifically, we consider the rectangles as in Figure 1 where the widths decay exponentially and $t_k = n - n/2^k$ for $k < \log_2(n)$. It is clear that $\{M_n = 0\}$ only if the graph of $M$ hits all the rectangles. Note that it suffices to show that for most rectangles conditionally on hitting them, the probability that the graph of $M$ does not hit the next one is lower bounded by $c/(\log n)^a$. This is the content of Proposition 2.1.
In order to control the probabilities mentioned above, we also have to make sure that the two-dimensional process visits enough new vertices in most intervals $[t_k, t_{k+1}]$. This is the content of Proposition 3.4 that we state and prove in Section 3.

In Section 4 we prove the following lemma, which shows that there is no dispersion result in the case $a = 1$, with general hypotheses like in Corollary 1.2.

**Lemma 1.3.** There exists a positive constant $c$, such that for any $n$, there exists a martingale $(M_k)_{k \leq n}$ satisfying almost surely

$$\mathbb{E}[(M_{k+1} - M_k)^2 \mid \mathcal{F}_k] \geq 1 \quad \text{and} \quad |M_{k+1} - M_k| \leq \log n,$$

for all $k \leq n$, yet

$$\mathbb{P}(M_n = 0) \geq c.$$

**Notation:** For functions $f, g$ we write $f(n) \lesssim g(n)$ if there exists a universal constant $C > 0$ such that $f(n) \leq Cg(n)$ for all $n$. We write $f(n) \gtrsim g(n)$ if both $f(n) \lesssim g(n)$ and $f(n) \gtrsim g(n)$. We write $B(x, r)$ to denote the ball in the $L^1$-metric centered at $x$ of radius $r$. Note also that in the rest of the paper we use $c$ for a constant whose value may change from line to line.

## 2 Martingale defocusing

In this section we state and prove a dispersion result for martingales. Then in Section 3 we use it to prove our main result, Theorem 1.1 when $a = 1/2$.

We call the quadratic variation of a martingale $M$, the process $(V_t)_{t \geq 1}$ defined by

$$V_t = \sum_{\ell=1}^{t} \mathbb{E}[(M_\ell - M_{\ell-1})^2 \mid \mathcal{F}_{\ell-1}].$$

**Proposition 2.1.** Let $\rho > 0$ be given. There exists a positive constant $c$ and $n_0 \geq 1$ such that the following holds for any $a \in (0, 1)$. Suppose that $M$ is a martingale with quadratic variation $V$ and suppose that $(G_k)_k$ is an i.i.d. sequence of geometric random variables with mean 2 satisfying

$$|M_{k+1} - M_k| \leq G_k \quad \forall k. \quad (2.1)$$

For each $1 \leq k < \log_2(n)$ we let $t_k = n - n/2^k$ and

$$A_k = \{V_{t_{k+1}} - V_{t_k} \geq \rho(t_{k+1} - t_k)/\log n \}.$$

Suppose that for some $N \geq 1$ and $1 \leq k_1 < \ldots < k_N < \log_2(n)/2$, it holds

$$\mathbb{P}\left(\bigcap_{i=1}^{N} A_{k_i}\right) = 1. \quad (2.2)$$

Then we have for all $n \geq n_0$

$$\mathbb{P}(M_n = 0) \leq \exp\left(-cN/(\log n)^a\right).$$

**Remark 2.2.** We note that the choice of mean 2 for the geometric random variables in $(2.1)$ is arbitrary. Any other value would be fine as well.
Proof of Corollary 1.2. If we divide the martingale $M$ by $(\log n)^a$, then it satisfies the hypotheses of Proposition 2.1 with $N = \log_2(n)/2$, and hence the statement of the corollary follows. \qed

Before proving Proposition 2.1 we state and prove a preliminary result that will be used in the proof later.

Lemma 2.3. There exists $\rho > 0$ such that the following holds. Suppose that $M$ is a martingale satisfying assumption (2.1) of Proposition 2.1. Let $m < \ell$ and $h > \log(\ell - m + 1)$ be given and let $\tau = \min\{t \geq m : |M_t - M_m| \geq h\}$. Suppose that $P(V_{\ell} - V_m \geq h^2/\rho) = 1$. Then we have almost surely

$$P(\tau \leq \ell | F_m) \geq \frac{1}{2}.$$  

Proof. It is well known that the process $(M_t^2 - V_t)$ is a martingale. Since $\tau \wedge \ell \geq m$ is a stopping time, by the optional stopping theorem we get

$$E[M_{\tau \wedge \ell}^2 - V_{\tau \wedge \ell} | F_m] = M_m^2 - V_m. \quad (2.3)$$

Now we claim that

$$E[M_{\tau \wedge \ell}^2 - M_m^2 | F_m] \lesssim h^2. \quad (2.4)$$

Indeed we can write

$$E[M_{\tau \wedge \ell}^2 - M_m^2 | F_m] = E[(M_{\tau \wedge \ell} - M_{(\tau-1)\wedge \ell}) + M_{(\tau-1)\wedge \ell} - M_m]^2 \leq 2E[(M_{\tau \wedge \ell} - M_{(\tau-1)\wedge \ell})^2] + 2h^2,$$

where the last inequality follows from the definition of $\tau$. In order to bound the first term in the right hand side above, we use (2.1) and the fact that $\tau \geq m$. This way we get

$$|M_{\tau \wedge \ell} - M_{(\tau-1)\wedge \ell}| \leq \max_{m \leq t \leq \ell} |G_t|.$$

So we now obtain

$$E[(M_{\tau \wedge \ell} - M_{(\tau-1)\wedge \ell})^2 | F_m] \lesssim \log(\ell - m + 1)^2,$$

which proves our claim (2.4), using also the hypothesis $h > \log(\ell - m + 1)$. Since by assumption we have $P(V_{\ell} - V_m \geq h^2/\rho) = 1$, we obtain that almost surely

$$E[V_{\tau \wedge \ell} - V_m | F_m] \geq E[(V_{\ell} - V_m)1(\tau \geq \ell) | F_m] \geq \frac{h^2}{\rho}P(\tau \geq \ell | F_m).$$

This together with (2.3) and (2.4) and by taking $\rho$ sufficiently small proves the lemma. \qed

We are now ready to give the proof of Proposition 2.1

Proof of Proposition 2.1 We will argue as in [10], by saying that in order for $M_n$ to be at $0$, the graph of $M$, i.e. the process $((t, M_t))_{t \leq n}$, has to cross the space-time rectangles $H_k$, for all $k = 1, \ldots, \log_2(n)$, which are defined by

$$H_k := [t_k, n] \times [-h_k, h_k] \quad \text{with} \quad h_k := \rho \sqrt{\frac{t_{k+1} - t_k}{(\log n)^{2\alpha}}}. (2.5)$$
We now define
\[ \sigma_k = \inf \{ t \geq t_k : |M_t| \geq h_k \} \]
For each \( k \leq \log_2(n) \) such that \( \mathbb{P}(A_k) = 1 \) we can apply Lemma 2.3 with \( m = t_k, \ell = t_{k+1} \) and \( h = 2h_k \) if \( n \) is sufficiently large so that \( h > \log(\ell - m + 1) \). We thus deduce that for \( \rho \) sufficiently small and for all \( n > n_0 \) we have almost surely
\[ \mathbb{P}(\sigma_k \leq t_{k+1} | \mathcal{F}_{t_k}) \geq \frac{1}{2}. \] (2.6)

Next we claim that a.s. conditionally on \( \mathcal{F}_{\sigma_k} \) the martingale has probability of order \( 1/(\log n)^a \), to reach level \( \pm h_k (\log n)^a/\rho^2 \) before returning below level \( \pm h_k/\sqrt{2} \) (if at least one of these events occurs before time \( n \)). Indeed assume for instance that \( M_{\sigma_k} \geq h_k \). Then the optional stopping theorem shows that on the event \( E_k = \{ M_{\sigma_k} \geq h_k \} \cap \{ \sigma_k \leq n \} \) we have
\[ h_k \leq M_{\sigma_k} = \mathbb{E}[M_{T_1 \land T_2 \land n} | \mathcal{F}_{\sigma_k}], \] (2.7)
where
\[ T_1 := \inf \{ t \geq \sigma_k : |M_t| \geq h_k (\log n)^a/\rho^2 \}, \]
and
\[ T_2 := \inf \{ t \geq \sigma_k : |M_t| \leq h_k/\sqrt{2} \}. \]
We deduce from (2.7) that on \( E_k \)
\[ h_k \leq \mathbb{E}[M_{T_1 \land T_2 \land n} | \mathcal{F}_{\sigma_k}] + \frac{h_k (\log n)^a}{\rho^2} \mathbb{P}(n < T_1 \land T_2 | \mathcal{F}_{\sigma_k}) + \frac{h_k}{\sqrt{2}}. \]
Then by using again the bound (2.1) we get that on \( E_k \)
\[ \mathbb{E}[M_{T_1 \land T_2 \land n} | \mathcal{F}_{\sigma_k}] \leq \frac{h_k (\log n)^a}{\rho^2} \mathbb{P}(T_1 < T_2 \land n | \mathcal{F}_{\sigma_k}) + \mathbb{E} \left[ \max_{t_k \leq t \leq n} G_t \right] \]
\[ \leq \frac{h_k (\log n)^a}{\rho^2} \mathbb{P}(T_1 < T_2 \land n | \mathcal{F}_{\sigma_k}) + c_1 \log(n - t_k + 1), \]
where \( c_1 \) is a positive constant. It follows that if \( n \) is large enough, then on \( E_k \)
\[ \mathbb{P}(T_1 \land n < T_2 | \mathcal{F}_{\sigma_k}) \gtrsim \frac{1}{(\log n)^a}. \]
Similarly we get the same inequality with the event \( \{ M_{\sigma_k} \geq h_k \} \) replaced by \( \{ M_{\sigma_k} \leq -h_k \} \), and hence we get that almost surely
\[ \mathbb{P}(T_1 \land n < T_2 | \mathcal{F}_{\sigma_k}) \mathbb{1}(\sigma_k \leq n) \gtrsim \frac{1}{(\log n)^a} \mathbb{1}(\sigma_k \leq n), \] (2.8)
which proves our claim. We now notice that on the event \( \{ t_k \leq T_1 \leq n \} \) we have by Doob’s maximal inequality
\[ \mathbb{P} \left( \sup_{i \leq n - t_k} |M_{i + T_1} - M_{T_1}| \geq h_k (\log n)^a / (2\rho^2) \Big| \mathcal{F}_{T_1} \right) \lesssim \frac{n - t_k}{(h_k (\log n)^a / \rho^2)^2} < c_1, \] (2.9)
where \( c_1 \) is a constant that we can take smaller than 1 by choosing \( \rho \) small enough. Note that we used again (2.1) in order to bound the \( \mathcal{L}^2 \) norm of the increments of the martingale \( M \).
Next we define a sequence of stopping times, which are the hitting times of the space-time rectangles \((H_k)\) defined in \((2.5)\). More precisely, we let \(s_0 = 0\) and for \(i \geq 1\) we let

\[ s_i = \min \{ t > s_{i-1} : (t, M_t) \in H_i \}. \]

Thus for each \(k < \log_2(n)\) such that \(\mathbb{P}(A_k) = 1\) by using \((2.6)\), \((2.8)\) and \((2.9)\), we get that on the event \(\{s_{k-1} \leq n\}\)

\[ \mathbb{P}(s_k > n \mid \mathcal{F}_{s_{k-1}}) \geq \frac{1}{(\log n)^a}. \]

Using the assumption that the event \(\bigcap_{i=1}^N A_i\) happens almost surely we obtain

\[ \mathbb{P}(M_n = 0) \leq \mathbb{P}(s_{k_1}, \ldots, s_{k_N} \leq n) \leq \left(1 - \frac{c}{(\log n)^a}\right)^N \leq \exp\left(-cN/(\log n)^a\right), \]

for a positive constant \(c\), and this concludes the proof.

\[ \square \]

### 3 Proof of transience

In this section we prove Theorem 1.1. We first give an equivalent way of viewing the random walk \(W\). Let \(\xi_1, \xi_2, \ldots\) be i.i.d. random variables taking values \((0, 0, \pm 1)\) equally likely. Let \(\zeta_1, \zeta_2, \ldots\) be i.i.d. random variables taking values \((\pm 1, 0, 0), (0, \pm 1, 0)\) all with equal probability, and independent of the \((\xi_i)\). Assume that \((W_0, \ldots, W_t)\) have been defined, and set

\[ r_W(t) = \# \{W_0, \ldots, W_t\}. \]

Then

\[ W_{t+1} = \begin{cases} W_t + \xi_{r_W(t)} & \text{if } r_W(t) = r_W(t-1) + 1, \\ W_t + \zeta_{r_W(t)} & \text{otherwise} \end{cases}. \]

To prove Theorem 1.1 it will be easier to look at the process at the times when the two dimensional process moves. So we define a clock process \((\tau_k)_{k \geq 0}\) by \(\tau_0 = 0\) and for \(k \geq 0\),

\[ \tau_{k+1} = \inf \{ t > \tau_k : (X_t, Y_t) \neq (X_{\tau_k}, Y_{\tau_k}) \} = \inf \{ t > \tau_k : t - r_W(t) = k \}. \]

Note that \(r_W(0) = 1\) and \(\tau_k < \infty\) a.s. for all \(k\). Observe that by definition the process \(U_t := (X_{\tau_t}, Y_{\tau_t})\) is a \(2d\)-simple random walk, and that \(r_W(\tau_t) = \tau_t - t + 1\). Note that

\[ Z_t = \sum_{i=1}^{r_W(t)-1} \langle \xi_i, (0, 0, 1) \rangle. \]

We set \(\mathcal{F}_t = \sigma(\xi_1, \ldots, \xi_{\tau_t-1})\), so that \(Z_{\tau_t}\) is \(\mathcal{F}_t\)-measurable for all \(t\).

We call \(\mathbb{Q}\) the law of the process \(U\). We denote by \(\mathbb{P}_U()\) the law of the process \(W\) conditionally on the whole process \(U\), or in other words on the whole sequence \((\zeta_t)_{t \geq 1}\). We write \(\mathbb{P} = \mathbb{Q} \times \mathbb{P}_U\) for the law of the overall process \(W\).

In the following claim we show that the process \(Z\) observed only at the times when the two-dimensional process moves, is a martingale.

**Claim 3.1.** Let \(M_t = Z_{\tau_t}\). Then \(\mathbb{Q}\)-a.s. we have that \((M_t)\) is an \((\mathcal{F}_t)\)-martingale under \(\mathbb{P}_U\).
Proof. We already noticed that $M_t$ is adapted to $\mathcal{F}_t$ for all $t$. Now since the $\xi_i$’s are i.i.d. and have mean 0 it follows from Wald’s identity that

$$\mathbb{E}_U [Z_{\tau_{t+1}} | \mathcal{F}_t] = \mathbb{E}_U [Z_{\tau_t} + \sum_{i=\tau_{t-1}}^{\tau_{t+1}-t-1} \langle \xi_i, (0,0,1) \rangle | \mathcal{F}_t] = Z_{\tau_t},$$

since $\tau_{t+1} - t - 1$ is the first time after $\tau_t - t + 1$ when we visit an already visited site, and is thus a stopping time.

Remark 3.2. We note that the jumps of the martingale are stochastically dominated by geometric random variables. More precisely, we can couple the process $M$ (or $W$) with a sequence $(G_t)_{t \geq 0}$ of i.i.d. geometric random variables with parameter $1/2$, such that

$$|M_{t+1} - M_t| \leq G_t \quad \text{for all } t \geq 0. \quad (3.1)$$

Before proceeding, we give some more definitions. For $t \geq 0$, set

$$r_U(t) = \#\{U_0, \ldots, U_t\},$$

i.e. $r_U(t)$ is the cardinality of the range of the two-dimensional process up to time $t$. We also set for $t \geq 0$

$$V_t := \sum_{\ell=1}^{t} \mathbb{E}_U [(M_{\ell} - M_{\ell-1})^2 | \mathcal{F}_{\ell-1}].$$

Claim 3.3. Suppose that $U_\ell$ is a fresh site, i.e. $U_\ell \notin \{U_0, U_1, \ldots, U_{\ell-1}\}$. Then

$$\mathbb{E}_U [(M_{\ell+1} - M_\ell)^2 | \mathcal{F}_\ell] \geq 2.$$

Proof. Notice that when $U_\ell$ is a fresh site, then $M_{\ell+1} - M_\ell$ can be written as

$$M_{\ell+1} - M_\ell = \sum_{i=1}^{\tau} \lambda_i,$$

where $(\lambda_i)_i$ are i.i.d. random variables taking values $\pm 1$ equally likely and

$$\tau = \inf\{k \geq 2 : (\lambda_{k-1}, \lambda_k) \in \{(-1,+1), (+1,-1)\}\}.$$

Then by the optional stopping theorem we deduce

$$\mathbb{E}_U [(M_{\ell+1} - M_\ell)^2 | \mathcal{F}_\ell] = \mathbb{E} \left[ \left( \sum_{i=1}^{\tau} \lambda_i \right)^2 \right] = \mathbb{E}[\tau] \geq 2,$$

since $\tau \geq 2$ by definition.

Before proving Theorem 1.1 we state a proposition that we prove later, which combined with the above claim guarantees that the quadratic variation $V$ of the martingale $M$ satisfies assumption (2.2) of Proposition 2.1. The following proposition only concerns the 2d-simple random walk.
Proposition 3.4. For $k \geq 1$ we let $t_k = n - n/2^k$ and for $\rho > 0$ define

$$\mathcal{K} = \left\{ 1 \leq k \leq (\log n)^{3/4} : r_U(t_{k+1}) - r_U(t_k) \geq \rho (t_{k+1} - t_k)/\log n \right\}.$$

Then there exist positive constants $\alpha, c$ and $\rho_*$ such that for all $\rho < \rho_*$

$$\mathbb{P}\left( \#\mathcal{K} \leq \rho(\log n)^{3/4} \bigg| U_n = 0 \right) \lesssim \exp(-c(\log n)^{\alpha}).$$

Proof of Theorem 1.1. Let $\mathcal{K}$ and $\rho$ be as in Proposition 3.4. Note that $\mathcal{K}$ is completely determined by the 2d-walk. Setting $A = \{ \#\mathcal{K} \geq \rho(\log n)^{3/4} \}$ we then have

$$\mathbb{P}(U_n = M_n = 0) = \mathbb{E}[1(U_n = 0)\mathbb{P}_U(M_n = 0) 1(A)] + \mathbb{E}[1(U_n = 0)\mathbb{P}_U(M_n = 0) 1(A^c)].$$

(3.2)

On the event $A$, using Claim 3.3 we get that there exist $k_1, \ldots, k_{\rho(\log n)^{3/4}} \in \mathcal{K}$ such that

$$\mathbb{P}_U\left( \bigcap_{i=1}^{\rho(\log n)^{3/4}} A_{k_i} \right) = 1,$$

where the events $(A_i)$ are as defined in Proposition 2.1. We can now apply this proposition (with $a = 1/2$) to obtain

$$\mathbb{P}_U(M_n = 0) 1(\#\mathcal{K} \geq \rho(\log n)^{3/4}) \lesssim \exp(-c(\log n)^{1/4}).$$

Therefore from (3.2) we deduce

$$\mathbb{P}(U_n = M_n = 0) \lesssim \frac{1}{n} \exp(-c(\log n)^{1/4}) + \frac{1}{n} \exp(-c(\log n)^{\alpha}),$$

where $\alpha$ is as in Proposition 3.4. Since this last upper bound is summable in $n$, this proves that 0 is visited only finitely many times almost surely. Exactly the same argument would work for any other point of $\mathbb{Z}^3$, proving that $W$ is transient.

Before proving Proposition 3.4 we state and prove a standard preliminary lemma and a corollary that will be used in the proof.

Lemma 3.5. Let $U$ be a simple random walk in $\mathbb{Z}^2$ starting from 0. Then there exists a positive constant $c$, such that for all $t \leq n \log n$ satisfying $\log(n/t) \lesssim (\log n)^{3/4}$ we have

$$\mathbb{P}\left( \#\{U_0, \ldots, U_n\} \cap B(0, \sqrt{t}) \geq \frac{t}{(\log n)^{1/16}} \right) \lesssim \exp(-c(\log n)^{1/16}).$$

Proof. To prove this we first decompose the path into excursions that the random walk makes across $B(0, 2\sqrt{t}) \setminus B(0, \sqrt{t})$ before time $n$. More precisely define $\sigma_0 = 0$, and for $i \geq 0$,

$$\sigma_i' = \inf\{k \geq \sigma_i : U_k \notin B(0, 2\sqrt{t})\},$$

$$\sigma_{i+1} = \inf\{k \geq \sigma_i' : U_k \in B(0, \sqrt{t})\}.$$

Let

$$N := \max\{i : \sigma_i \leq n\},$$

where
be the total number of excursions before time \( n \), and for each \( i \leq N \), let
\[
R_i := \# \{ U_{\sigma_i}, \ldots, U_{\sigma_i^t} \},
\]
be the number of points visited during the \( i \)-th excursion. Of course we have
\[
\# \{ U_0, \ldots, U_n \} \cap B(0, \sqrt{t}) \leq \sum_{i=1}^{N} R_i. \tag{3.3}
\]

Note that every time the random walk is on the boundary of the ball \( B(0, 2\sqrt{t}) \), it has probability of order \( 1/(\log(n/t) + 4\log\log n) \) to hit the boundary of the ball \( B(0, \sqrt{n}\log n) \) before hitting \( B(0, \sqrt{t}) \) (see for instance [12]). If \( T \) is the first exit time from \( B(0, \sqrt{n}\log n) \), then
\[
\mathbb{P}(T \leq n) \lesssim e^{-c(\log n)^4}, \tag{3.4}
\]
where \( c \) is a positive constant. On the event \( \{ T \geq n \} \), it is easy to see that \( N \) is dominated by a geometric random variable with mean of order \( \log(n/t) \). We thus get
\[
\mathbb{P}
\left(
N \geq (\log(n/t) + 4\log\log n)(\log n)^{1/16}
\right)
\leq \mathbb{P}(T \leq n) + \mathbb{P}
\left(
N \geq (\log(n/t) + 4\log\log n)(\log n)^{1/16}, T \geq n
\right)
\lesssim \exp \left( -c(\log n)^4 \right) + \exp \left( -c(\log n)^{1/16} \right) \lesssim \exp \left( -c(\log n)^{1/16} \right). \tag{3.5}
\]

Since we have \( \mathbb{E}_x[\sigma_i' - \sigma_i] \lesssim t \) for all \( x \in B(0, 2\sqrt{t}) \), by using the Markov property we can deduce
\[
\mathbb{P}
\left(
\sigma_i' - \sigma_i \geq t(\log n)^{1/16}
\right) \leq \exp(-c(\log n)^{1/16}).
\]

Moreover, it follows from [3] Lemma 4.3] and the fact that \( \log n \asymp \log t \) that
\[
\mathbb{P}
\left(
\# \{ U_{\sigma_i}, \ldots, U_{\sigma_i+t(\log n)^{1/16}} \} \geq \frac{t}{(\log n)^{7/8}}
\right) \leq \exp\left(-c(\log n)^{1/16}\right).
\]

Combining the last two inequalities, we get that for any \( i \),
\[
\mathbb{P}
\left(
R_i \geq \frac{t}{(\log n)^{7/8}}
\right) \leq 2 \exp(-c(\log n)^{1/16}),
\]
where \( c \) is a positive constant. Using the assumption that \( \log(n/t) \lesssim (\log n)^{3/4} \) together with \( (3.3) \) and \( (3.5) \) concludes the proof of the lemma.

**Corollary 3.6.** Let \( U \) be a simple random walk in \( \mathbb{Z}^2 \), let \( t \leq n \) satisfying \( \log(n/(n-t)) \lesssim (\log n)^{3/4} \) and let \( \varepsilon < 1/32 \). Then there exists a positive constant \( c \) such that
\[
\mathbb{P}
\left(
\# \{ U_0, \ldots, U_t \} \cap B(0, (\log n)^{\varepsilon}\sqrt{n-t}) \geq \frac{n-t}{(\log n)^{16-2\varepsilon}} \mid U_n = 0
\right) \lesssim \exp \left( -c(\log n)^{1/16} \right).
\]

**Proof.** First we use the rough bound:
\[
\# \{ U_0, \ldots, U_t \} \cap B(0, (\log n)^{\varepsilon}\sqrt{n-t}) \leq \# \{ U_0, \ldots, U_{n/2} \} \cap B(0, (\log n)^{\varepsilon}\sqrt{n-t}) + \# \{ U_{n/2}, \ldots, U_n \} \cap B(0, (\log n)^{\varepsilon}\sqrt{n-t}).
\]
We now note that if $A$ is an event only depending on the first $n/2$ steps of the random walk, then we have
\[
\mathbb{P}(A \mid U_n = 0) = \frac{\mathbb{P}(U_n = 0 \mid A) \mathbb{P}(A)}{\mathbb{P}(U_n = 0)} \lesssim \mathbb{P}(A),
\] (3.7)
where the last inequality follows from the local central limit theorem. By reversibility we obtain
\[
\mathbb{P}\left(\#\{U_{n/2}, \ldots, U_n\} \cap B(0, (\log n)^{\varepsilon} \sqrt{n - t}) \geq \frac{n - t}{2(\log n)^{1/6} - 2\varepsilon} \mid U_n = 0\right) = \mathbb{P}\left(\#\{U_0, \ldots, U_{n/2}\} \cap B(0, (\log n)^{\varepsilon} \sqrt{n - t}) \geq \frac{n - t}{2(\log n)^{1/6} - 2\varepsilon} \mid U_n = 0\right). \tag{3.8}
\]
The statement now readily follows by combining Lemma 3.5 with (3.7) and (3.8).

**Proof of Proposition 3.4.** Let us consider the events
\[
A_k := \left\{ r_U(t_{k+1}) - r_U(t_k) \geq \rho \frac{t_{k+1} - t_k}{\log n} \right\},
\]
with $\rho > 0$ some constant to be fixed later. Let also $\varepsilon < 1/48$,
\[
B_k = \left\{ \#\{U_0, \ldots, U_{t_k}\} \cap B(0, (\log n)^{\varepsilon} \sqrt{n - t_k}) \leq \frac{t_{k+1} - t_k}{(\log n)^{1/6} - 2\varepsilon} \right\},
\]
and
\[
\tilde{B}_k := B_k \cap \left\{ U_{t_k} \in B(0, \sqrt{n - t_k}) \right\}.
\]
Set for $k = 1, \ldots, (\log n)^{3/4}$,
\[
\mathcal{G}_k = \sigma(U_0, \ldots, U_{t_k}),
\]
and note that $\tilde{B}_k \in \mathcal{G}_k$.

**Claim 3.7.** For any $k \leq (\log n)^{3/4}$ we have almost surely
\[
\mathbb{P}(A_k^c \mid \mathcal{G}_k) \mathbf{1}(\tilde{B}_k) \lesssim \frac{1}{(\log n)^{\varepsilon}} \mathbf{1}(\tilde{B}_k).
\]

**Proof.** To prove the claim we use two facts. On the one hand it follows from [5, Theorem 1.5] that if $\rho$ is small enough, then a.s.
\[
\mathbb{P}\left(\#\{U_{t_k+1}, \ldots, U_{t_k+1}\} \leq 2\rho \frac{t_{k+1} - t_k}{\log n} \mid \mathcal{G}_k\right) \leq \exp \left(-c(\log n)^{1/6}\right).
\]
Moreover, on the event $\{U_{t_k} \in B(0, \sqrt{n - t_k})\}$, with probability at most $\exp(-c(\log n)^{2\varepsilon})$ the random walk exits the ball $B(0, (\log n)^{\varepsilon} \sqrt{n - t_k})$ before time $t_{k+1}$. Therefore we obtain on the event $\{U_{t_k} \in B(0, \sqrt{n - t_k})\}$ that
\[
\mathbb{P}\left(\#\{U_{t_k+1}, \ldots, U_{t_k+1}\} \cap B(0, (\log n)^{\varepsilon} \sqrt{n - t_k}) \leq 2\rho \frac{t_{k+1} - t_k}{\log n} \mid \mathcal{G}_k\right) \lesssim \exp \left(-c(\log n)^{2\varepsilon}\right). \tag{3.9}
\]
Suppose now on the other hand that a point is at distance at least $r = O(\sqrt{t})$ from $U_{t_k}$. Then it is well known (see for instance [12]) that the probability that the walk hits it during the time interval
\([t_k, t_k + t]\) is \(O(\log(\sqrt{t}/r)/\log \sqrt{t})\). If we apply this with \(t = t_{k+1} - t_k\), \(r = \sqrt{n - t_k}/\log n\), and use that \(#B(0, r) \asymp r^2\), we deduce that

\[
\mathbb{E}\left[\frac{t_{k+1} - t_k}{(\log n)^2} + \frac{t_{k+1} - t_k \log \log n}{(\log n)^{17/16 - 2\varepsilon}}\right] 1(\tilde{B}_k) \\
\leq \frac{t_{k+1} - t_k}{(\log n)^{1 + \varepsilon}} 1(\tilde{B}_k).
\]  

(3.10)

We now have almost surely

\[
\mathbb{P}\left(A_k | G_k\right) 1(\tilde{B}_k) \geq \mathbb{P}\left(\#\{U_{t_{k+1}}, \ldots, U_{t_{k+1}}\} \cap \mathcal{B}(0, (\log n)^\varepsilon \sqrt{n - t_k}) > 2\rho \frac{t_{k+1} - t_k}{\log n} | G_k\right) 1(\tilde{B}_k) \\
- \mathbb{P}\left(\#\{U_0, \ldots, U_t\} \cap \{U_{t_{k+1}}, \ldots, U_{t_{k+1}}\} \cap \mathcal{B}(0, (\log n)^\varepsilon \sqrt{n - t_k}) \geq \rho \frac{t_{k+1} - t_k}{\log n} | G_k\right) 1(\tilde{B}_k) \\
\geq \left(1 - \exp\left(-c(\log n)^{1/6}\right) - \frac{c_1}{(\log n)^\varepsilon}\right) 1(\tilde{B}_k),
\]

where the last inequality follows from (3.9), (3.10) and Markov’s inequality.

Next, if we write \(Q(\cdot) = \mathbb{P}(\cdot | U_n = 0)\) for the Doob transform of \(U\), then \((U_k)_{k \leq n}\) is a Markov chain under \(Q\). We let \(A_k = A_k \cap \{U_{t_{k+1}} \in \mathcal{B}(0, 2\sqrt{n - t_k})\}\). Then we have almost surely

\[
Q(A_k | G_k) 1(\tilde{B}_k) \geq Q(\tilde{A}_k | G_k) 1(\tilde{B}_k) \geq \mathbb{P}(\tilde{A}_k | G_k) 1(\tilde{B}_k) \geq p 1(\tilde{B}_k),
\]

(3.11)

where the penultimate inequality follows by the local central limit theorem as in (3.7) and the last inequality from Claim 3.8 and the fact that

\[
\mathbb{P}\left(U_{t_{k+1}} \in \mathcal{B}(0, 2\sqrt{n - t_k}) | U_{t_k} \in \mathcal{B}(0, \sqrt{n - t_k})\right) \geq c > 0.
\]

Then we introduce the process \((M_k)_{k \leq (\log n)^{3/4}}\), defined by \(M_1 = 0\) and for \(k \geq 2\),

\[
M_k := \sum_{\ell = 1}^{k-1} \left\{ 1(A_{\ell} \cap \tilde{B}_{\ell}) - Q(A_{\ell} | G_{\ell}) 1(\tilde{B}_{\ell}) \right\}.
\]

Note that by construction it is a \((G_k)\)-martingale, under the measure \(Q\). Since the increments of this martingale are bounded, it follows from Azuma-Hoeffding’s inequality that for any \(\kappa > 0\), there exists a positive constant \(c\) such that

\[
\mathbb{P}\left(|M_{(\log n)^{3/4}}| \geq \kappa (\log n)^{3/4} \mid U_n = 0\right) \leq \exp(-c(\log n)^{3/4}).
\]

(3.12)

As a consequence of Corollary 3.6, we get that

\[
1 - \mathbb{P}\left(\cap_{k \leq (\log n)^{3/4}} B_k | U_n = 0\right) \leq \exp(-c(\log n)^{1/16}).
\]

(3.13)

Claim 3.8. There exists a positive constant \(c\) such that

\[
\mathbb{P}\left(\sum_{k=1}^{(\log n)^{3/4}} 1(U_k \in \mathcal{B}(0, \sqrt{n - t_k})) \leq c(\log n)^{3/4} \mid U_n = 0\right) \leq \exp(-c(\log n)^{3/4}).
\]
Proof. By using reversibility and the local central limit theorem again, it suffices in fact to show the result without conditioning on $U_n = 0$, and replacing the times $t_k$ by $n - t_k$. In other words, it suffices to prove that

$$P \left( \sum_{k=1}^{(\log n)^{3/4}} 1(U_{2k} \in \mathcal{B}(0, 2^{k/2})) \leq c(\log n)^{3/4} \right) \leq \exp(-c(\log n)^{3/4}), \quad (3.14)$$

for some $c > 0$. This is standard, but for the sake of completeness we give a short proof now. We will prove in fact a stronger statement. Call $v_k := \inf \{ t \geq 0 : U_t \notin \mathcal{B}(0, 2^{k/2}) \}$. Obviously it is sufficient to prove (3.14) with the events $\{ v_k > 2^k \}$ in place of $\{ U_{2k} \in \mathcal{B}(0, 2^{k/2}) \}$. Set $\mathcal{H}_k = \sigma(U_0, \ldots, U_{v_k})$. Then it is well known that we can find a constant $\alpha > 0$, such that a.s. for any $k$,

$$P(v_{k+1} > 2^{k+1} | \mathcal{H}_k) \geq \alpha.$$ 

Then by considering the martingale

$$M_k' := \sum_{\ell=1}^{k} \left( 1(v_\ell > 2^\ell) - P(v_\ell > 2^\ell | \mathcal{H}_{\ell-1}) \right),$$

and using the Azuma-Hoeffding inequality the desired estimate follows. So the proof of the claim is complete.

By taking $\rho$ and $\kappa$ sufficiently small and using (3.11), (3.12), (3.13) and Claim 3.8 finishes the proof of the proposition.

4 Example

In this section we construct the martingale of Lemma 1.3. Before doing so, we recall a self-interacting random walk $(X, Y, Z)$ in $\mathbb{Z}^3$ which was mentioned in [7] and is closely related to the random walk of Theorem 1.1; on the first visit to a vertex only $(X, Y)$ performs a two-dimensional step, while on later visits to the same vertex only $Z$ changes by $\pm 1$ equally likely. Our proof in this case does not apply, or at least another argument is required. Indeed, by looking again at the process $Z$ at the times when $(X, Y)$ moves, we still obtain a martingale, but we do not have a good control on the jumps of this martingale. In particular, up to time $n$, they could be of size of order $\log n$, which might be a problem as Lemma 1.3 shows.

Proof of Lemma 1.3. Define $M_0 = 0$. Let $(S_k^1)_{k,i}$ be independent (over $i$) simple random walks on $\mathbb{Z}$ and let $(\tilde{S}_k^1)_{k,i}$ be independent (over $i$) random walks with jumps that take values $\pm [\log n]$ equally likely and start from 0. Let $k_*$ be the first integer such that $n/2^{k_*} \leq (\log n)^2$. We now let

$$M_k = S_k^1 \quad \text{for} \quad k \leq n/2.$$ 

We define $t_1$ by

$$n - t_1 = \frac{n}{2} + \inf \left\{ t \geq 0 : \left| M_{n/2} + \tilde{S}_{t_1}^1 \right| \leq \log n \right\}.$$
If $t_1 \geq 0$, then we let
\[ M_{k+n/2} = M_{n/2} + \tilde{S}_k \quad \text{for} \quad 0 \leq k \leq \frac{n}{2} - t_1. \]

If $t_1 < 0$, then we let
\[ M_{k+n/2} = M_{n/2} + \tilde{S}_k \quad \text{for} \quad 0 \leq k \leq \frac{n}{2}. \]
Suppose that we have defined $t_\ell > 0$, we now define $t_{\ell+1}$ inductively. We let
\[ M_{k+n-t_\ell} = M_{n-t_\ell} + S_{k}^{\ell+1} \quad \text{for} \quad 0 \leq k \leq \frac{t_\ell}{2}, \]
and we also define $t_{\ell+1}$ by
\[ n - t_{\ell+1} = n - \frac{t_\ell}{2} + \inf \left\{ t \geq 0 : \left| M_{n-t_\ell/2} + \tilde{S}_{k+1}^\ell \right| \leq \log n \right\}. \]
If $t_{\ell+1} \geq 0$, then we let
\[ M_{k+n-t_\ell/2} = M_{n-t_\ell/2} + \tilde{S}_k^{\ell+1} \quad \text{for} \quad 0 \leq k \leq \frac{t_\ell}{2} - t_{\ell+1}. \]
If $t_{\ell+1} < 0$, then we let
\[ M_{k+n-t_\ell/2} = M_{n-t_\ell/2} + \tilde{S}_k^{\ell+1} \quad \text{for} \quad 0 \leq k \leq \frac{t_\ell}{2}. \]

In this way we define the times $t_\ell$ for all $\ell \leq k^*$, unless there exists $\ell$ such that $t_\ell < 0$, in which case we set $t_m = 0$ for all $\ell + 1 \leq m \leq k^*$. If $k^* > 1$, then at time $n - t_{k^*} + 1$ if $d(M_{n-t_{k^*}},0) \neq 0$, then the martingale makes a jump of size $\pm d(M_{n-t_{k^*}},0)$ equally likely. If $d(M_{n-t_{k^*}},0) = 0$, then with probability $1/(\log n)^2$ it jumps to $\pm \log n$, while with probability $1 - 1/(\log n)^2$ it stays at 0. From time $n - t_{k^*} + 2$ until time $n$ at every step with probability $1/(\log n)^2$ it jumps to $\pm \log n$, while with probability $1 - 1/(\log n)^2$ it stays at its current location.

By the definition of the martingale it follows that it satisfies the assumptions of the lemma. It only remains to check that there exists a positive constant $c$ such that $\mathbb{P}(M_n = 0) > c$. We define the events
\[ E = \{ M_{n-t_{k^*}+1} = 0 \} \quad \text{and} \quad E' = \{ M_\ell = 0, \text{ for all } \ell \in \{ n - t_{k^*} + 2, \ldots, n \} \}. \]

We now have
\[
\mathbb{P}(M_n = 0) \geq \mathbb{P}(t_1 > 0, \ldots, t_{k^*} > 0, M_{n-t_{k^*}} = 0, E, E') \\
+ \mathbb{P}(t_1 > 0, \ldots, t_{k^*} > 0, M_{n-t_{k^*}} \neq 0, E, E').
\]

(4.1)

By the definition of the times $t_i$, it follows that $t_{i+1} \leq t_i/2$, and hence we deduce that $t_i \leq n/2^i$, which implies that $t_{k^*} \leq n/2^{k^*} \leq (\log n)^2$. We now obtain
\[
\mathbb{P}(E, E' \mid t_1 > 0, \ldots, t_{k^*} > 0, M_{n-t_{k^*}} \neq 0) \gtrsim \left( 1 - \frac{1}{(\log n)^2} \right)^{(\log n)^2},
\]
\[
\mathbb{P}(E, E' \mid t_1 > 0, \ldots, t_{k^*} > 0, M_{n-t_{k^*}} = 0) \gtrsim \left( 1 - \frac{1}{(\log n)^2} \right)^{(\log n)^2}.
\]

(4.2)

Using the estimate for a simple random walk that if $h > 0$, then
\[
\mathbb{P}_h(S_k > 0, \forall k \leq n) \lesssim \frac{h}{\sqrt{n}},
\]
we get for a positive constant $c_1$ that
\[ P(t_{\ell+1} > 0 \mid t_\ell > 0) = 1 - P \left( \inf \{ t \geq 0 : |M_{n-t_{\ell/2}} + S_{t_{\ell/2}}^{\ell+1} \} \leq \log n \} \leq \frac{t_\ell}{2} \mid t_\ell > 0 \right) \]
\[ \geq 1 - \frac{c_1}{\log n}. \]

Hence from (4.1) and (4.2) together with the above estimate and the fact that $k^* \asymp \log n$, we finally conclude
\[ P(M_n = 0) \gtrsim \left( 1 - \frac{c_1}{\log n} \right)^{c_2 \log n} \cdot \left( 1 - \frac{1}{(\log n)^2} \right)^{(\log n)^2} \geq c_3 > 0 \]

and this finishes the proof of the lemma.

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