\textbf{F}_1\text{-schemes and toric varieties}

Anton Deitmar

\textbf{MCC classification:} 14A15, 11G25, 14L32, 14M32

\textbf{Keywords:} F1-schemes, toric varieties, field of one element

\textbf{Author’s address:} Math. Inst., Auf der Morgenstelle 10, 72076 Tübingen, Germany

\textbf{Abstract:} In this paper it is shown that integral \( \text{F}_1 \)-schemes of finite type are essentially the same as toric varieties. A description of the \( \text{F}_1 \)-zeta function in terms of toric geometry is given. \( \text{Etale morphisms and universal coverings are introduced.} \)

\section*{Introduction}

There are by now several attempts to make the theory of the field of one element \( \text{F}_1 \) rigorous. In [10] the authors formalize the transition from rings to schemes on a categorial level and apply this machinery to the category of sets to obtain the category of \( \text{F}_1 \)-schemes as in [1]. In [3] and [5] the authors extend the definition of rings in order to capture a structure that deserves to be called \( \text{F}_1 \). In [1] the author tried instead to fix the minimum properties any of these theories must share. The current paper extends this line of thought. We use terminology of [1] and [2].

In this paper, a ring will always be commutative with unit and a monoid will always be commutative. An \textit{ideal} \( \mathfrak{a} \) of a monoid \( A \) is a subset with \( A\mathfrak{a} \subseteq \mathfrak{a} \). A \textit{prime ideal} is an ideal \( \mathfrak{p} \) such that \( S\mathfrak{p} = A \setminus \mathfrak{p} \) is a submonoid of \( A \). For a prime ideal \( \mathfrak{p} \) let \( A_\mathfrak{p} = S^{-1}A \) be the \textit{localization} at \( \mathfrak{p} \). The \textit{spectrum} of
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a monoid $A$ is the set of all prime ideals with the obvious Zariski-topology (see [1]). Similar to the theory of rings, one defines a structure sheaf $\mathcal{O}_X$ on $X = \text{spec} (A)$, and one defines a scheme over $F_1$ to be a topological space together with a sheaf of monoids, locally isomorphic to spectra of monoids.

A $F_1$-scheme $X$ is of finite type, if it has a finite covering by affine schemes $U_i = \text{spec} (A_i)$ such that each $A_i$ is a finitely generated monoid. For a ring $R$, we write $X_R$ for the $R$-base-change of $X$, so $X_R = X_Z \otimes_Z R$.

For a monoid $A$ we let $A \otimes Z$ be the monoidal ring $Z[A]$. This defines a functor from monoids to rings which is left adjoint to the forgetful functor that sends a ring $R$ to the multiplicative monoid $(R, \times)$. This construction is compatible with gluing, so one gets a functor $X \mapsto X_Z$ from $F_1$-schemes to $Z$-schemes. In [2] we have shown that $X$ is of finite type if and only if $X_Z$ is a $Z$-scheme of finite type.

We say that the monoid $A$ is integral, if it has the cancellation property, i.e., if $ab = ac$ implies $b = c$ in $A$. This is equivalent to saying that $A$ injects into its quotient group or $A$ is a submonoid of a group.

By a module of a monoid $A$ we mean a set $M$ together with a map $A \times M \to M; (a, m) \mapsto am$ with $1m = m$ and $(ab)m = a(bm)$. A stationary point of a module is a point $m \in M$ with $am = m$ for every $a \in A$. A pointed module is a pair $(M, m_0)$ consisting of an $A$-module $M$ and a stationary point $m_0 \in M$.

1 Flatness

Recall the tensor product of two modules $M, N$ of $A$:

$$M \otimes N = M \otimes_A N = M \times N / \sim,$$

where $\sim$ is the equivalence relation generated by $(am, n) \sim (m, an)$ for every $a \in A, m \in M, n \in N$. The class of $(m, n)$ is written as $m \otimes n$. The tensor product $M \otimes N$ becomes a module via $a(m \otimes n) = (am) \otimes n$. For example, the module $A \otimes M$ is isomorphic to $M$.

Let now $(M, m_0)$ and $(N, n_0)$ be two pointed modules of $A$, then $(M \otimes N, m_0 \otimes n_0)$ is a pointed module, called the pointed tensor product.

The category $\text{Mod}_0(A)$ of pointed modules and pointed morphisms has a
terminal and initial object 0, so it makes sense to speak of kernels and cokernels. It is easy to see that every morphism \( f \) in \( \text{Mod}_0(A) \) possesses both. One defines the image of \( f \) as \( \text{im}(f) = \ker(\text{coker}(f)) \) and the coimage as \( \text{coim}(f) = \text{coker}(\ker(f)) \).

A morphism is called strong, if the natural map from \( \text{coim}(f) \) to \( \text{im}(f) \) is an isomorphism. Kernels and cokernels are strong. If \( A \xrightarrow{f} B \xrightarrow{g} C \) is given with \( g \) being strong and \( gf = 0 \), then the induced map \( \text{coker}(f) \to C \) is strong. Likewise, if \( f \) is strong and \( gf = 0 \), then the induced map \( A \to \ker g \) is strong. A map is strong if and only if it can be written as a cokernel followed by a kernel.

The usual notion of exact sequences applies, and we say that a sequence of morphisms is strong exact if it is exact and all morphisms in the sequence are strong.

A module \( F \in \text{Mod}_0(A) \) is called flat, if the functor \( X \mapsto F \otimes X \) is strong-exact, i.e., if for every strong exact sequence

\[
0 \to M \to N \to P \to 0
\]

the induced sequence

\[
0 \to F \otimes M \to F \otimes N \to F \otimes P \to 0
\]

is strong exact as well.

It is easy to see that a pointed module \( F \) is flat if and only if for every injection \( M \hookrightarrow N \) of pointed modules the map \( F \otimes M \to F \otimes N \) is an injection.

Examples. If \( A \) is a group, then every module is flat. Let \( S \) be a submonoid of \( A \). Then the localization \( S^{-1} A \) is a flat \( A \)-module. The direct sum \( G \oplus F \) of two flat modules is flat. Finally, consider the free monoid in one generator \( C_+ = \{1, \tau, \tau^2, \ldots\} \), then an \( A \)-module \( M \) is flat if and only if \( \tau m = \tau m' \) implies \( m = m' \) for all \( m, m' \in M \). This is equivalent to saying that \( M \) is a \( C_+ \)-submodule of a module of the quotient group \( C_\infty = \tau \mathbb{Z} \) of \( C_+ \). The same characterization holds for every integral monoid.

A morphism \( \varphi : A \to B \) of monoids is called flat if \( B \) is flat as an \( A \)-module. A morphism of \( \mathbb{F}_1 \)-schemes \( f : X \to Y \) is called flat if for every \( x \in X \) the morphism of monoids \( f^\# : O_{Y,f(x)} \to O_{X,x} \) is flat.
The following is straightforward.

- A morphism of monoids \( \varphi : A \to B \) is flat if and only if the induced morphism of \( \mathbb{F}_1 \)-schemes \( \text{spec } B \to \text{spec } A \) is flat.
- The composition of flat morphisms is flat.
- The base change of a flat morphism by an arbitrary morphism is flat.

**Remark.** It is easy to see that if \( \mathbb{Z}[F] \) is flat as \( \mathbb{Z}[A] \)-module, then \( F \) is flat as \( A \)-module. The converse is already false if \( A \) is a group. As an example let \( k \) be a field and let \( A \) be the group of all matrices of the form \( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \) where \( x \in k \). Let \( A \) act on \( k^2 \) in the usual way and trivially on \( k \). Consider the exact sequence of \( \mathbb{Z}[A] \)-modules,

\[
0 \longrightarrow k \xrightarrow{\alpha} k^2 \xrightarrow{\beta} k \longrightarrow 0,
\]

where \( \alpha(x) = \begin{pmatrix} x \\ 0 \end{pmatrix} \), \( \beta \begin{pmatrix} x \\ y \end{pmatrix} = y \). Let \( F = \{1\} \) the trivial \( A \)-module, then for every \( \mathbb{Z}[A] \)-module \( M \) one has \( M \otimes_{\mathbb{Z}[A]} \mathbb{Z}[F] = H_0(A, M) \). Note that \( H_0(A, k) = k \) and that \( H_0(\alpha) = 0 \), so it is not injective, hence \( \mathbb{Z}[F] \) is not flat.

## 2 Algebraic extensions

Let \( A \) be a submonoid of \( B \). An element \( b \in B \) is called *algebraic over* \( A \), if there exists \( n \in \mathbb{N} \) with \( b^n \in A \). The extension \( B/A \) is called *algebraic*, if every \( b \in B \) is algebraic over \( A \). An algebraic extension \( B/A \) is called *strictly algebraic*, if for every \( a \in a \) the equation \( x^n = a \) has at most \( n \) solutions in \( B \).

If \( B/A \) is algebraic, then \( \mathbb{Z}[B]/\mathbb{Z}[A] \) is an algebraic ring extension, but the converse is wrong in general, as the following example shows: Let \( A = \mathbb{F}_1 \) and \( B \) be the set of two elements, 1 and \( b \) with \( b^2 = b \).

A monoid \( A \) is called *algebraically closed*, if every equation of the form \( x^n = a \) with \( a \in A \) has a solution in \( A \). Every monoid \( A \) can be embedded into
an algebraically closed one, and if \( A \) is a group, then there exists a smallest such embedding, called the algebraic closure of \( A \). For example, the algebraic closure \( \overline{\mathbb{F}_1} \) of \( \mathbb{F}_1 \) is the group \( \mu_\infty \) of all roots of unity, which is isomorphic to \( \mathbb{Q}/\mathbb{Z} \).

### 3 Etale morphisms

Recall that a homomorphism \( \varphi: A \to B \) of monoids is called a local homomorphism, if \( \varphi^{-1}(B^\times) = A^\times \) (every \( \varphi \) satisfies “⊃”). For a monoid \( A \) let \( m_A = A \smallsetminus A^\times \) be its maximal ideal. It is easy to see that a homomorphism \( \varphi: A \to B \) is local if and only if \( \varphi(m_A) \subseteq m_B \).

A local homomorphism \( \varphi: A \to B \) is called unramified if

- \( \varphi(m_A)B = m_B \) and
- \( \varphi \) injects \( A^\times \) into \( B^\times \) and \( B/\varphi(A) \) is a finite strictly algebraic extension.

Note that if \( \varphi \) is unramified, then so are all localizations \( \varphi_p: A_{\varphi^{-1}(p)} \to B_p \) for \( p \in \text{spec } B \).

A morphism \( f: X \to Y \) of \( \mathbb{F}_1 \)-schemes is called unramified, if for every \( x \in X \) the local morphism \( f^\#: \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x} \) is unramified.

A morphism \( f: X \to Y \) of \( \mathbb{F}_1 \)-schemes is called locally of finite type, if every point in \( Y \) has an open affine neighborhood \( V = \text{spec } A \) such that \( f^{-1}(V) \) is a union of open affines \( \text{spec } B_i \) with \( B_i \) finitely generated as a monoid over \( A \). The morphism is of finite type if for every point in \( Y \) the number of \( B_i \) can be chosen finite. The morphism is called finite, if every \( y \in Y \) has an open affine neighborhood \( V = \text{spec } A \) such that \( f^{-1}(V) \) is affine, equal to \( \text{spec } B \), where \( B \) is finitely generated as \( A \)-module.

A morphism \( f: X \to Y \) of finite type is called étale, if \( f \) is flat and unramified. It is called an étale covering, if it is also finite.

**Proposition 3.1** The étale coverings of \( \text{spec } \mathbb{F}_1 \) are the morphisms of the form \( \text{spec } A \to \text{spec } \mathbb{F}_1 \), where \( A \) is a finite cyclic group. The scheme \( \text{spec } \mathbb{F}_1 \) has no non-trivial étale coverings.
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\textbf{Proof:} Clear. \(\Box\)

A connected scheme over \(F_1\), which has only the trivial étale covering, is called \textit{simply connected}.

\textbf{Proposition 3.2} The schemes \(\text{spec } \overline{F_1}, \text{spec } C_+ \times \overline{F_1}\) and \(\mathbb{P}^1_{\overline{F_1}}\) are simply connected.

\textbf{Proof:} The first has been dealt with. For the second, let \(A = \mu_\infty \times C_+\). Then \(\text{spec } A = \text{spec } C_+ \times \overline{F_1}\). Let \(f : X \to \text{spec } A\) be an étale covering. As \(f\) is finite, \(X\) is affine, say \(X = \text{spec } B\). Let \(\varphi : A \to B\) denote the corresponding morphism of monoids. The space \(\text{spec } A\) consists of two points, the generic point \(\eta_A\) and the closed point \(c_A\). Likewise, let \(\eta_B, c_B\) denote the generic and closed points of \(\text{spec } B\). One has \(f(\eta_B) = \eta_A\). We will show that \(f(c_B) = c_A\).

Assume the contrary. Then \(\varphi^{-1}(m_B)\) is empty, hence \(\varphi\) maps \(A\) to the unit group \(B^\times\). The localization at the closed point \(c_B\) then maps \(\mu_\infty \times C_\infty\) to \(B^\times\) and is unramified, hence injective. But as \(C_+ \to C_\infty\) is not finite, neither can \(\varphi\) be finite, a contradiction. So we conclude \(f(c_B) = c_A\), and so the corresponding localization, which is \(\varphi\) itself, is unramified. Let \(s = \varphi(\tau)\), where \(\tau\) is the generator of \(C_+\). Then \(\varphi(m_A)B = m_B\) implies \(m_B = sB\), and so \(B = B^\times \cup sB\) (disjoint union). Also, \(B^\times\) is an algebraic extension of \(A^\times \cong \mu_\infty\), hence equals \(\varphi(A^\times)\). As \(B\) is finitely generated and flat as \(A\)-module, there are \(b_1, \ldots, b_r \in B\) with

\[sB = B^\times s^N \cup B^\times s^Nb_1 \cup \ldots \cup B^\times s^N b_r.\]

If we assume \(r > 0\), then \(b_1\) is algebraic over \(\varphi(A) = B^\times \cup B^\times s^N\), so let \(N\) be the smallest number in \(\mathbb{N}\) such that \(b_1^N \in \varphi(A)\). Then \(b_1^N \not\in B^\times \cong \mu_\infty\), because, as the extension is strictly algebraic, then \(b_1\) would be in \(B^\times\) already. So \(b_1^N \in B^\times s^N\). As the group \(B^\times\) is divisible, we can replace \(b_1\) with a \(B^\times\) multiple to get \(b_1^N = s^M\) for some \(M \in \mathbb{N}\). Then \(b_1^N \not\in B^\times s^N b_1\), as \(b_1 = b^s k b_1\) leads to \(s^M = b_1^N = (b^s)^N s^{kN+M}\) which contradicts the injectivity of \(\varphi\). But then \(b_1\) must be in one of the other \(B^\times s^N\)-orbits, which contradicts the disjointness of these orbits. We conclude \(r = 0\), i.e. \(B = B^\times \cup B^\times s^N \cong A\) as claimed. The assertion for \(\mathbb{P}^1_{\overline{F_1}}\) is an easy consequence. \(\Box\)
4 Toric varieties

Recall a toric variety is an irreducible variety $V$ over $\mathbb{C}$ together with an algebraic action of the $r$-dimensional torus $\text{GL}_r^1$, such that $V$ contains an open orbit.

As toric varieties can be constructed via lattices it follows that every toric variety is the lift $X_\mathbb{C}$ of an $\mathbb{F}_1$-scheme $X$. For integral schemes of finite type there is a converse direction given in the following theorem, which shows that integral $\mathbb{F}_1$-schemes of finite type are essentially the same as toric varieties.

**Theorem 4.1** Let $X$ be a connected integral $\mathbb{F}_1$-scheme of finite type. Then every irreducible component of $X_\mathbb{C}$ is a toric variety. The components of $X_\mathbb{C}$ are mutually isomorphic as toric varieties.

**Proof:** Let $U = \text{spec } A$ be an open affine subset of $X$. Let $\eta$ be the generic point of $X$, then the localization $G = A_\eta$ is the quotient group of $A$. At the same time, $G$ is the stalk $\mathcal{O}_{X, \eta}$, so $G$ does not depend on the choice of $U$ up to canonical isomorphism. Let $\varphi : A \to G$ be the quotient map, which is injective as $X$ is integral. The $\mathbb{C}$-algebra homomorphism,

$$
\mathbb{C}[A] \to \mathbb{C}[G] \otimes \mathbb{C}[A] \\
a \mapsto \varphi(a) \otimes a
$$

defines an action of the algebraic group $G = \text{spec } \mathbb{C}[G]$ on $\text{spec } \mathbb{C}[A]$. Since this is compatible with the restriction maps of the structure sheaf, we get an algebraic action of the group scheme $G$ on $X_\mathbb{C}$. As $X$ is integral, $G = \text{spec } \mathbb{C}[G] = \text{spec } \mathbb{C}[A_\eta]$ also is an open subset $V_\mathbb{C}$ of $X_\mathbb{C}$, and for $U_\mathbb{C} = \text{spec } \mathbb{C}[A]$ the map

$$
\mathcal{O}(U_\mathbb{C}) = \mathbb{C}[A] \xrightarrow{\varphi} \mathbb{C}[G] = \mathcal{O}(V_\mathbb{C})
$$

is the restriction map of the structure sheaf $\mathcal{O}$ of $X_\mathbb{C}$. The map $\mathbb{C}[A] \to \mathbb{C}[G]$ is injective and $\mathbb{C}[G]$ has zero Jacobson radical, so it follows that $V_\mathbb{C}$ is dense in $X_\mathbb{C}$, so in particular it meets every irreducible component. The group $G$ is a finitely generated abelian group, so $G \cong \mathbb{Z}^r \times F$ for a finite abelian group...
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\( F \). Hence \( G \equiv \text{GL}_1^* \times F \) as a group-scheme. As \( G \) meets every component of \( X_C \), the latter are permuted by \( F \). Whence the claim.

To formulate the next result, we will briefly recall the standard construction of toric varieties, see [4]. Let \( N \) be a lattice, i.e., a group isomorphic to \( \mathbb{Z}^n \) for some \( n \). A fan \( \Delta \) in \( N \) is a finite collection of \emph{proper convex rational polyhedral cones} \( \sigma \) in the real vector space \( N \otimes \mathbb{R} = N \otimes \mathbb{R} \), such that every face of a cone in \( \Delta \) is in \( \Delta \) and the intersection of two cones in \( \Delta \) is a face of each. (Here zero is considered a face of every cone.) We explain the notation further: A convex cone is a convex subset \( \sigma \) of \( N \otimes \mathbb{R} = N \otimes \mathbb{R} \), it is \emph{polyhedral}, if it is finitely generated and \emph{rational}, if the generators lie in the lattice \( N \). Finally, a cone is called \emph{proper} if it does not contain a non-zero sub vector space of \( N \).

Let a fan \( \Delta \) be given. Let \( M = \text{Hom}(N, \mathbb{Z}) \) be the dual lattice. for a cone \( \sigma \in \Delta \) the dual cone \( \tilde{\sigma} \) is the cone in the dual space \( M \otimes \mathbb{R} = M \otimes \mathbb{R} \) consisting of all \( \alpha \in M \otimes \mathbb{R} \) such that \( \alpha(\sigma) \geq 0 \). This defines a monoid \( A_\sigma = \tilde{\sigma} \cap M \). Set \( U_\sigma = \text{spec}(\mathbb{C}[[A_\sigma]]) \). If \( \tau \) is a face of \( \sigma \), then \( A_\tau \supseteq A_\sigma \), and this inclusion gives rise to an open embedding \( U_\tau \hookrightarrow U_\sigma \). Along these embeddings we glue the affine varieties \( U_\sigma \) to obtain a variety \( X_\Delta \) over \( \mathbb{C} \), which has a given \( \mathbb{F}_1 \)-structure. Then \( X_\Delta \) is a toric variety, the torus being \( U_0 \cong \text{GL}_1^* \). Every toric variety is given in this way.

**Lemma 4.2** Let \( B \) be a submonoid of the monoid \( A \) of finite index. Then the map \( \psi : \text{spec} A \to \text{spec} B \) defined by \( \psi(p) = p \cap B \) is a bijection.

**Proof:** Let \( N \in \mathbb{N} \) be such that \( a^N \in B \) for every \( a \in A \). To see injectivity, let \( \psi(p) = \psi(q) \) and let \( a \in p \). Then \( a^N \in q \) and so \( a \in q \) as \( q \) is a prime ideal. This shows \( p \subseteq q \) and by symmetry we get equality. For surjectivity, let \( p_B \in \text{spec} B \) and let \( p = \{ a \in A : a^N \in p_B \} \). Then \( \psi(p) = p_B \). □

**Proposition 4.3** Suppose that \( \Delta \) is a fan in a lattice of dimension \( n \). For \( j = 0, \ldots, n \) let \( f_j \) be the number of cones in \( \Delta \) of dimension \( j \). Set

\[
c_j = \sum_{k=j}^n f_{n-k}(-1)^{k+j} \binom{k}{j}.
\]
Let $X$ be the corresponding toric variety, then the $\mathbb{F}_1$-zeta function of $X$ equals

$$\zeta_X(s) = s^{c_0}(s-1)^{c_1} \cdots (s-n)^{c_n}. $$

**Proof:** Let $\sigma \in \Delta$ be a cone of dimension $k$. Let $F$ be a face of $\tilde{\sigma}$. Let $p_F = A_\sigma \setminus F$. Then $p_F$ is a non-empty prime ideal in $A_\sigma$. The map $F \mapsto p_F$ is a bijection between the set of all faces of $\tilde{\sigma}$ and the set of non-empty prime ideals of $A_\sigma$. The set $S_p = A \setminus p$ equals $M \cap F$. The quotient group $\text{Quot}(S_p)$ is isomorphic to $\mathbb{Z}^f$, where $f$ is the dimension of $F$. There is a bijection between the set of faces of $\sigma$ and the set of faces of $\tilde{\sigma}$ mapping a face $\tau$ to the face $F$ of all $\alpha \in \tilde{\sigma}$ with $\alpha(\tau) = 0$. The dimension of $F$ then equals $n - \dim(\tau)$. So let $f_\sigma^j$ denote the number of faces of $\sigma$ of dimension $j$. Then the zeta polynomial of $X_\sigma$ equals

$$N_\sigma(x) = \sum_{k=0}^{n} f_\sigma^k (x-1)^{n-k}. $$

Let $N_\Delta$ be the zeta polynomial of $X_\Delta$. We get

$$N_\Delta(x) = \sum_{k=0}^{n} f_k (x-1)^{n-k}$$

$$= \sum_{k=0}^{n} f_k \sum_{j=0}^{n-k} \binom{n-k}{j} x^j (-1)^{n-k-j}$$

$$= \sum_{k=0}^{n} f_{n-k} \sum_{j=0}^{k} \binom{k}{j} x^j (-1)^{k-j}$$

$$= \sum_{j=0}^{n} x^j \sum_{k=j}^{n} f_{n-k} \binom{k}{j} (-1)^{k-j}. $$

This implies the claim. \(\square\)

## 5 Valuations

On the infinite cyclic monoid $C_+ = \{1, \tau, \tau^2, \ldots\}$ we have a natural linear order given by $\tau^k \leq \tau^l \iff k \leq l$. Let $\varphi, \psi$ be two monoid morphisms from a
monoid $A$ to $C_+$. Then define $\varphi \leq \psi \iff \varphi(a) \leq \psi(a) \ \forall a \in A$. A valuation on $A$ is a non-trivial homomorphism $v : A \to C_+$ which is minimal with respect to the order $\leq$ among all non-trivial homomorphisms from $A$ to $C_+$. Let $V(A)$ denote the set of valuations on $A$.

**Lemma 5.1** Let
\[
1 \longrightarrow A \longrightarrow B \xrightarrow{\varphi} F \longrightarrow 1
\]
be an exact sequence of monoids, where $F$ is a finite abelian group. Then for every valuation $v \in V(A)$ there exists a unique valuation $w$ on $B$ and $k \in \mathbb{N}$ such that
\[
w|_A = v^k.
\]
Mapping $v$ to $w$ sets up a bijection from $V(A)$ to $V(B)$.

**Proof:** Let $F'$ be a subgroup of $F$ and let $B'$ be the preimage of $F'$ under $\varphi$. We get two exact sequences
\[
1 \longrightarrow A \longrightarrow B' \longrightarrow F' \longrightarrow 1,
\]
and
\[
1 \longrightarrow B' \longrightarrow B \longrightarrow F/F' \longrightarrow 1.
\]
Assume we have proven the lemma for each of these two sequences, then it follows for the original one. In this way we reduce the proof to the case when $F$ is a finite cyclic group. We first show existence of $w$ for given $v$. For this let $f_0$ be a generator of $F$ and let $l$ be its order. Choose a $b_0$ in the preimage $\varphi^{-1}(f_0)$. Then $b_0^l \in A$, and $v(b_0^l) = \tau^n$ for some $n \geq 0$. If $n = 0$, then set $k = 1$ and define $w : B \to C_+$ by $w(b_0^l a) = v(a)$ for $a \in A$ and $j \geq 0$. If $n > 0$, then set $k = l/{\gcd}(l,n)$ and let $w : B \to C_+$ be defined by $w(b_0^l a) = \tau^j v(a)^k$. This shows existence of the extension $w$. \hfill $\square$

## 6 Cohomology

Cohomology is not defined over $\mathbb{F}_1$. I am grateful to Ofer Gabber for bringing the following example to my attention. Let $X$ be the topological space consisting of three points $\eta, X_+, x_-$. The open sets besides the trivial ones are
Let $A$ be a subgroup of the abelian group $B$ and let $C = B/A$. Let $\mathcal{F}$ be the sheaf of abelian groups on $X$ with $\mathcal{F}(U_\pm) = A$ and $\mathcal{F}(U) = B$ and the restriction being the inclusion. Let $\mathcal{G}$ be the constant sheaf $B$ and let $\mathcal{H}$ be the quotient sheaf $\mathcal{G}/\mathcal{F}$. As $\mathcal{G}$ is flabby, the long cohomology sequence terminates and looks like this:

$$0 \to H^0(\mathcal{F}) \to H^0(\mathcal{G}) \to H^0(\mathcal{H}) \to H^1(\mathcal{F}) \to 0$$

In concrete terms this is

$$0 \to A \to B \to C \times C \to (C \times C)/\Delta \to 0,$$

where $\Delta$ means the diagonal in $C \times C$. Let $f : X \to X$ be the homeomorphism with $f(x_+) = x_-$, $f(x_-) = x_+$, and $f(\eta) = \eta$. There is a natural isomorphism $f_* \mathcal{F} \cong \mathcal{F}$ and for the other sheaves as well. On the global sections of $\mathcal{F}$ and $\mathcal{G}$ this induces the trivial map, whereas on $H^0(\mathcal{H})$ it induces the flip $(a,b) \mapsto (b,a)$, which on $H^1(\mathcal{F})$ amounts to the same as the inversion $a \mapsto -a$. The naturality of these isomorphisms means that if the sheaves and the cohomology groups are defined over $\mathbb{F}_1$, then so must be the flip. This, however, is not the case, as for a set $S$ the inversion on the abelian group $\mathbb{Z}[S]$ is not induced by a self-map of $S$.

Even more convincing is the fact that in this example there are different injective resolutions which produce different cohomology groups.

References

[1] Deitmar, A.: Schemes over $\mathbb{F}_1$. in: Number Fields and Function Fields - Two Parallel Worlds. Progress in Mathematics, Vol. 239 Geer, Gerard van der; Moonen, Ben J.J.; Schoof, Ren (Eds.) 2005.

[2] Deitmar, A.: Remarks on zeta functions and $K$-theory over $F1$. Proc. Japan Acad. Ser. A Math. Sci. (2007).

[3] Durov, N.: New Approach to Arakelov Geometry. http://arxiv.org/abs/0704.2030

[4] Fulton, W.: Introduction to toric varieties. Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993
[5] Haran, S.: Non-additive geometry. Comp. Math. 143, 618-688 (2007).

[6] Kurokawa, B.; Ochiai, H.; Wakayama, A.: Absolute Derivations and Zeta Functions. Documenta Math. Extra Volume: Kazuya Kato’s Fiftieth Birthday (2003) 565-584.

[7] Kurokawa, N.: Zeta functions over $F_1$. Proc. Japan Acad. Ser. A Math. Sci. 81 (2005), no. 10, 180-184 (2006).

[8] Soulé, C.: Les variétés sur le corps à un élément. Mosc. Math. J. 4, no. 1, 217–244, 312 (2004).

[9] Tits, J.: Sur les analogues algébriques des groupes semi-simples complexes. 1957 Colloque d’algèbre supérieure, Bruxelles du 19 au 22 décembre 1956 pp. 261–289 Centre Belge de Recherches Aathématiques Établissements Ceuterick, Louvain; Librairie Gauthier-Villars, Paris.

[10] Toen, B.; Vaquie, M: Under Spec Z. http://arxiv.org/abs/math/0509684

Mathematisches Institut
Auf der Morgenstelle 10
72076 Tübingen
Germany
deitmar@uni-tuebingen.de