Final state interaction on $B^+ \to \pi^- \pi^+ \pi^+$

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(Dated: January 1, 2016)

Abstract

The large localized CP asymmetries observed by LHCb in three-body charmless decays of $B$ mesons brings new challenge for experimentalists and their traditional models to fit data. With higher statistics, rescattering and three-body effects, previously ignored on data analysis, become more visible. The time is ripe to use better theoretical assumptions as an ingredient in order to improve the analysis. In this paper we address the issue of rescattering effects in the charmless three-body decays of $B$ mesons. In particular, we study the case of the $B^+ \to \pi^- \pi^+ \pi^+$ decay and show that the presence of hadronic loops shifts the P-wave phase near threshold to below zero, and modify the position of the $\rho$-meson peak and it width, in the Dalitz plot.

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Three-body B and D-meson decays are known experimentally to be dominated by low energy resonances on $\pi\pi$, $K\pi$ and $KK$ channels on Dalitz plot, usually analysed by means of isobar model. Although the isobar model allow the resonances to interfere, it does not include coupled channels and three-body effects. The simplicity of this model is not a problem for low statistics samples of charm and beauty decays, producing many interesting results\cite{1–4}. However, these effects previously ignored on isobar model, may arise and become visible in the large LHCb data\cite{5–7} and one will need to include them in the fit to data. Therefore, is urgent to improve isobar model including final state interactions (FSI) effects where resonances also play a role.

Besides the interest on light scalar resonances, the study of three-body decays of heavy meson is related to the understanding of Charge Parity (CP) violation effects. CP violation arises due to the interference between at least two different amplitudes with different weak and strong phases. In charmless charged B decays, LHCb observed a large CP asymmetries \cite{5–7} across the Dalitz plot, with regions having negative and positive values in the same decays. Studies of $B^{\pm} \rightarrow \pi^{\pm}\pi^{\mp}\pi^{\mp}$, subject of this work, showed that CP violation is affected by the strong coupling of the $\pi\pi$ and $KK$ channels on intermediate states of the decay from different final states\cite{7–9}, and by the interference between overlapping resonant amplitudes, like the S and P-wave interference around the $\rho(770)$ and $f_0(980)$ resonances\cite{7–10}. These studies strongly suggest that FSI may play an important role as a source of CP violation.

One can think of heavy meson three-body hadronic decays as a sequence of two processes. The first stage consists of the heavy quark weak transition, along with local quark-gluon interactions, compounding a short-distance process that we refer to as the weak vertex. This short-distance transition is followed by the hadronization and the long-distance process of FSI, where the three light mesons interact in all possible ways before reaching the detector. These two processes involve different scales and in general are treated separately with different theoretical frameworks.

\textbf{weak vertex}

The most common way to treat B and D weak decays is using the factorization technique, as in Refs.\cite{10, 11}, where the decay amplitude is represented by a product of quark currents that do not interact strongly with each other. An effective Hamiltonian describes
the transitions between the quark currents with the correct CKM matrix elements and Wilson coefficients. The latter describe perturbatively the quark-gluon interaction and are the direct link to short-distance physics. The factorization assumption works better when the decaying particle is heavy\cite{12}, what is not the case for the D meson. An alternative way to describe the weak vertex is the heavy mesons ChPT, developed independently by Burdman and Donoghue\cite{13} and Wise\cite{14}. In this approach, the charm and bottom mesons are coupled to the light SU(3) ChPT by means of SU(3) operators. In the effective Lagrangian, the quark-gluon interactions are hidden in coupling constants that need to be determined by experiments.

**FSI**

It is important to emphasize that in three-body decays there are two distinct final state interactions mechanisms. The first one is related to the two-body scattering and includes all possible interactions between two particles, exchanging resonances and coupled channels. This pure two-body interaction have been extensively studied with models based on ChPT\cite{15–18} and dispersion relations \cite{19–21}, applying the constraints of analyticity and unitarity of the S-matrix. Approaches that consider only this kind of FSI in three-body decays are called (2+1), or quasi-two-body, because interactions involving the third particle, usually referred to as “bachelor”, are ignored.

However, rescattering between the bachelor and the other mesons can also happen and be relevant. In these three-body FSI, involving hadronic loops, also known as triangle loops, there is a momentum sharing between the three particles in the final state. The exhaustive consideration of those effects in the amplitude is possible only by means of three-body calculations such as the solution of Faddeev equation\cite{22}. The importance of three-body FSI in heavy-meson decays have been investigated recently by many different groups\cite{23–31}, where the $D^+ \to K^-\pi^+\pi^+$ decay was chosen as a golden channel. This decay was study in a simple model that implemented three-body unitarity solving Faddeev and a compatible perturbative expansion of three-body effects on FSI\cite{22, 23, 25}, which was recently improved in Ref.\cite{24}. In particular, Refs.\cite{23, 24} showed that hadron loops introduce new complex structures to the $D^+ \to K^-\pi^+\pi^+$ amplitude, modifying both the S- and P-wave phase. That approach also succeeded to explain the observed discrepancy between the $K^-\pi^+$ S-
wave phase shift from scattering data [32, 33] and that extracted from $D^+ \rightarrow K^-\pi^+\pi^+$ decay [3, 34].

A more involved model to $D^+ \rightarrow K^-\pi^+\pi^+$ decay was performed in Ref. [30], where, inspired in $\pi\pi N$ system with the so called Z-diagrams [29], the FSI was considered as successive rescatterings between the bachelor $\pi$ with the $K\pi$ system forming another resonance. In this approach all the interactions are mediated by resonances which include the $K\pi$ coupled channels. Recently, another study for the $D^+ \rightarrow K^-\pi^+\pi^+$ decay was performed [31] using dispersion relation theory on three-body systems by means of Khuri-Treiman approach [35]. They consider the coupling between the $D^+ \rightarrow K^-\pi^+\pi^+$ and $D^+ \rightarrow \bar{K}^0\pi^0\pi^+$ and the two-body $K\pi$ couple channels. Although the work developed in Refs. [30] and [31] is more complex than those of [23, 24], they all share the same features: the importance of hadronic loops as a three-body rescattering effect on the final amplitude.

In spite of the differences between the phase space and quark contents, B and D meson decays share the same final state process. This allows us to extend the techniques developed for D decays to B charmless three-body decays. In this work, inspired on the importance of FSI to D meson decays, we study the relevance of a rescattering effects on the P-wave of $B^+ \rightarrow \pi^-\pi^+\pi^+$ decay. If one consider that $\pi^+$ and $\pi^-$ could scatter in the final state of this decay, we are including the presence of $\rho$ resonance family, well known on the $\pi\pi$ P-wave scattering [36]. Therefore, when fitting the data one needs to account for the overlap of resonances coming from different sources: the one produced by the rescattering and the one produced directly by the B vertex. This interference affects the values of the mass and width of the resulting resonances creating a practical problem to fit data using isobar model.

**Rescattering effects on $B \rightarrow 3\pi$**

Chau’s [37] description of heavy meson weak decays define the main topologies contributing to the process $B^+ \rightarrow \pi^-\pi^+\pi^+$, illustrated in the diagrams of Fig.1. The diagram $a.1$ contains an axial current, whereas $v.1$ and $v.2$ contain vector currents. One can see that the $v.1$ diagram does not contribute at the tree level, since it leads to the $\pi^+\pi^-\pi^0$ final state, whereas in the $v.2$ diagram, the internal $W$ emission implies a color suppression factor. For these reasons we choose to start our study investigating the contribution of diagram $a.1$.

If one consider the evaluation of the $a.1$ tree diagram through the $\pi^+\pi^-$ rescattering,
FIG. 1: From quarks to hadronized Feynman diagrams contribution to $B^+ \to \pi^- \pi^+ \pi^+$ decay. The $\rho \to \pi\pi$ decay were omitted above.

where by rescattering we mean the interaction between the $\pi^+$ produced from the W-boson with the $\pi^-$ from the $\rho$-meson, the amplitude for $B^+ \to \pi^- \pi^+ \pi^+$ decay can be represented by the series of diagrams in Fig. 2.

FIG. 2: Diagrams contributing to $B^+ \to \pi^- \pi^+ \pi^+$ decay amplitude.

In this study, using a similar approach developed in Ref.[24], we calculate the contribution of the diagrams in Fig. 2 to the $B^+ \to \pi^- \pi^+ \pi^+$ decay amplitude. Our goal is to identify the relative importance of the rescattering effects on the whole amplitude. The contribution from the weak vertex is not included, and will be studied in the near future. It is important to note that this will not affect the conclusions of this study, since both diagrams in Fig. 2 have the same weak vertex.

Tree Amplitude - $A_0$

The tree level contribution to the $B^+ \to \pi^- \pi^+ \pi^+$ decay amplitude, first diagram in Fig. 2, is given by:

$$iA_0 = iT_{B\rho} \, iT_{W\pi} \frac{i g_{\mu\nu}}{M_W^2} \, \frac{i}{D_\rho} \, iT_{\rho\pi\pi}^\nu; \quad (1)$$
where

\[ iT_{B\rho} = -i 2m_\rho \frac{e^+ \cdot P_B}{p_1^2} p_1 F_0^{B\rho}(p_1^2), \]

\[ F_0^{B\rho}(p_1^2) = \frac{F_{B\rho}(0)}{1 - M_\pi^2/m_{B^+}^2} \]

is the amplitude transition for \( B \rightarrow \rho \), driven by the form factor \( F_0^{B\rho}(p_1^2) \); and

\[ iT_{W\pi}^\mu = i f_\pi p_1^\mu, \]  
\[ iT_{\rho\pi\pi}^\nu = i C_W (-p_3 + p_2) \epsilon F^1_{\pi\pi}(s_{23}), \]

are, respectively, the pion production amplitude from the W-boson and the \( \rho \rightarrow \pi\pi \) production amplitude. Summing over the \( \rho \) polarisation yields

\[ \sum_{\lambda=0,\pm 1} e^\lambda \cdot (p_2 - p_3) e^{\lambda^*} \cdot p_B = -p_1 \cdot (p_2 - p_3), \]

and the tree amplitude is given by:

\[ A_0 = C_0 [-p_1 \cdot (p_2 - p_3)] F^1_{\pi\pi}(s_{23}); \]

\[ C_0 = 2m_\rho f_\pi C_W \frac{M_B}{M_W} F_{B\rho}(0), \]

where \( C_0 \) is a constant, and the product \(-p_1 \cdot (p_2 - p_3) = (s_{13} - s_{12})\) is the angular distribution of the P-wave. Therefore, the tree amplitude, \( A_0 \), is a function of \( s_{23} \), the invariant mass of the \( \rho \) decay channel.

**Production amplitude: \( \rho \rightarrow \pi\pi \)**

The \( \rho \) production amplitude is a crucial element in the description of \( B^+ \rightarrow \pi^- \pi^+ \pi^+ \) decay for both tree and rescattering amplitude. In a very general grounds, this process is given by the diagrams in Fig. 3, which take into account the point-like interaction and the vertex dressed by a loop of \( \pi\pi \).

The corresponding amplitude of this process is given by:

\[ F^1_{\pi\pi}(s) = \frac{G_\rho}{m_{B^+}^2 - s} \left[ 1 + (-\bar{\Omega}) T_{\pi\pi} \right] = \frac{G_\rho}{m_{B^+}^2 - s} \left[ 1 + (-\bar{\Omega}) \frac{\mathcal{K}}{1 + \bar{\Omega}} \right] = \frac{G_\rho}{m_R^2 - s} \left[ \frac{1}{1 + \bar{\Omega} \mathcal{K}} \right]. \]
where $G_{\rho}$ is the coupling $\rho \to \pi\pi$, $\mathcal{K}$ is the kernel of $\pi\pi$ scattering in P-wave and $\bar{\Omega}$ is $\pi\pi$ loop function. For convenience, we re-write Eq.(7) in order to include explicitly the $\pi\pi$ scattering phase, as done in Ref.[38], and the production amplitude becomes:

$$F_{\pi\pi}^1(s) = g \frac{\cos\delta(s)}{m_{\rho}^2 - s} e^{i\delta(s)} \quad (8)$$

where $\delta(s)$ is the $\pi\pi$ phase shift. In Eq. (8) we use the CERN-Munich parameterization [36] for the $\pi\pi$ phase shift, instead of using a model to the $\pi\pi$ scattering amplitude.

The production amplitude, Eq. (8), is also inside the hadronic loop in rescattering calculation. Therefore, this amplitude needs to be integrated with the loop. We parametrize the experimental data by a sum of poles with a complex constant in the numerator[24], and write

$$F_{\pi\pi}^1(s) = \sum_{i=1}^{3} \frac{N_{\rho_i}}{s - \Theta_i} ; \quad (9)$$

where the values of $N_{\rho_i}$ and $\Theta_i = a + ib$ were available in table I of Ref.[24]. The advantage of this form is to allow one to use Feynman rules to integrate the loop, which will become clear in the following.

**Rescattering Amplitude - A1**

The rescattering contribution is given by the second diagram in Fig. 2 and depends only on the $s_{12}$, the invariant mass of the rescattered pions. However, the $B^+ \to \pi^-\pi^+\pi^+$ decay is fully symmetric and one can consider the symmetric process, given in Fig.4, in order to compare the rescattering contribution, now as a function of $s_{23}$, to the tree amplitude, calculated above. Then, the rescattering contribution can be written in terms of the tree amplitude as

$$A_1 = -i \int \frac{d^4\ell}{(2\pi)^4} \frac{T_{\pi\pi} A_0}{\Delta_\pi^+ \Delta^-_{\pi}} , \quad (10)$$
where $T_{\pi\pi}$ is the $\pi\pi$ scattering amplitude represented by the green bubble at Fig. 4 and $\Delta_{\pi}^+$, $\Delta_{\pi}^-$ are the pions propagators inside the loop: $\Delta_{\pi}^+ = (P_B - l)^2 - M_{\pi}^2$ and $\Delta_{\pi}^- = (l - p_3)^2 - M_{\pi}^2$.

It is important to note that the $B \rightarrow \rho$ form factor is no longer a constant because the transferred momentum is integrated over in the loop. In this case, we use a single pole approximation and

$$F_{B\rho}^0(p_i^2) = -\frac{F_{B\rho}(0) m_{B^*}^2}{\Delta_{B^*}}; \quad \Delta_{B^*} = (P_B - l)^2 - m_{B^*}^2. \quad (11)$$

The rescattering amplitude is then given by:

$$A_1 = -i C_O (m_{B^*}^2) \int \frac{d^4l}{(2\pi)^4} \frac{T_{\pi\pi}(s_{23}) [-p_1' \cdot (p_2' - p_3)]}{\Delta_{\pi}^+ \Delta_{\pi}^- \Delta_{B^*}} F_{\pi\pi}^1(l^2), \quad (12)$$

where $p_i'$ are the momentum of the pions inside the loop and $F_{\pi\pi}^1$ is given by Eq.(9). The scalar product in Eq.(12) can be written in terms of the meson propagators and the loop integral momentum $l$ as

$$-p_1' \cdot (p_2' - p_3) = \frac{1}{2} [\Delta_{\pi}^+ + 2 \Delta_{\pi}^- - 2 s_{23} + 3 M_{\pi}^2 + M_B^2 - l^2] \quad (13)$$

and

$$A_1 = \frac{i C_O m_{B^*}^2 N_\rho}{2} \int \frac{d^4l}{(2\pi)^4} \frac{T_{\pi\pi}(s_{23}) \Delta_{\pi}^+ (\Delta_{\pi}^- + 2 \Delta_{\pi}^- - 2 s_{23} + 3 M_{\pi}^2 + M_B^2 - l^2)}{\Delta_{\pi}^+ \Delta_{\pi}^- \Delta_{B^*} [l^2 - \Theta_{\rho}]}.$$

The $\pi\pi$ scattering amplitude in this channel can receive contributions from both S and P-wave, but in this study we will consider only the P-wave contribution. In this case, the s-channel amplitude projection on P-wave result

$$T_{\pi\pi}(s_{23}) = \frac{3 (t - u)}{s_{23} - 4 M_{\pi}^2} T_{\pi\pi}^P(s_{23}); \quad (14)$$

$$t - u = 2 p_1 \cdot (p_2 - p_3) - 2 l \cdot (p_2 - p_3), \quad (15)$$

where we used the CERN-Munich parametrization[36] to describe the $T_{\pi\pi}^P$ scattering amplitude.
Finally, the rescattering amplitude for $B^+ \to \pi^- \pi^+ \pi^+$ is given by:

$$A_1 = i \frac{C_O m_B^2}{2} T_{\pi\pi}^P (s_{23}) N_\rho \left\{ I_1 - I_2 \right\},$$  \hspace{1cm} (16)

where $I_1$ and $I_2$ are, respectively, functions of scalar and tensorial loops integrals

$$I_1 = (s_{12} - s_{13}) \frac{i}{16 \pi^2} \left\{ \left( M_B^2 - 2 s_{23} + 3 M_\pi^2 + \Theta_i \right) \Pi_{\pi^+\pi^- B^* \rho_i}^I + \Pi_{\pi^+\pi^- B^* \rho_i}^{\pi^+\pi^- B^*} ight\},$$  \hspace{1cm} (17)

$$I_2 = 2 (p_2 - p_3)_\mu \frac{i}{16 \pi^2} \left\{ \left( M_B^2 - 2 s_{23} + 3 M_\pi^2 + \Theta_i \right) \Pi_{\pi^+\pi^- B^* \rho_i}^{\mu} + \Pi_{\pi^+\pi^- B^* \rho_i}^{\mu} \right\},$$  \hspace{1cm} (18)

and the indices on function $\Pi_{xyz}$ refer to the particles that are taking part in the loop. All these loop integrals can be solved using Feynman technique. A guide for all calculations can be found in Ref. [39].

**Results**

We compare numerically the contributions from tree, Eq.(5), and rescattering amplitudes, Eq.(16), to the total $B^+ \to \pi^- \pi^+ \pi^+$ decay amplitude.

In Fig.5, we show the results for the modulus what clarify the relative weight of each amplitude into the final one. In the plot on the left one can see that the tree and rescattering amplitudes interfere destructively with the dominance of the former. Is possible to note a presence of a smaller peak around 1.7 GeV, which is the contribution of the $\rho(1450)$ coming from the $T_{\pi\pi}$ scattering amplitude. In the plot on the right, the two contributions are superimposed, with the rescattering contribution rescaled, showing that the line shapes are very different. Not only the position of the peak is different, but also the two curves are asymmetric in opposite directions, with the rescattering curve being wider than the curve from the tree amplitude.

In the Fig.6 individual contributions to the P-wave phase are shown as a function of the $\pi\pi$ invariant mass. Although they look similar, at threshold the rescattering contribution, Eq.(16), starts bellow $-60^\circ$ and crosses $90^\circ$ in a different position than the tree contribution. Even if one shift up the rescattering contribution, they have a clear different energy dependence on the lower sector. This rescattering behave at low energy is due to the hadronic
FIG. 5: Magnitude square of $B^+ \to \pi^- \pi^+ \pi^+$ decay amplitude: (left) the tree Eq.(5), rescattering Eq.(16) and final amplitude, (right) only tree and rescattering rescaled.

loop and is in complete agreement with what was seen in the $D^+ \to K^- \pi^+ \pi^+$ study in Refs.[24, 39], the first work to show the effect of rescattering in three-body decay.

FIG. 6: P-wave phase for tree, Eq.(5), and rescattering, Eq.(16), contributions in $B^+ \to \pi^- \pi^+ \pi^+$ decay.

In order to analyse with more detail the effects of the rescattering when fitting the data, three samples of toy Monte Carlo with 10,000 events were generated using the software Laura++[40] to each of the amplitudes derived in the previous section: tree and rescattering, and to the full amplitude. In the sequence we tried to fit those amplitudes using isobar model tools, such as Relativistic Breit-Wigner[41] functions.

The analyses on rescattering amplitude, function $A_1$, did not produced a reasonable fit with Breit-Wigner functions. However, we were able to parametrized this function using one Landau and two Gaussian functions, the result is shown in Fig.7.
Fig. 7: Result of the fit performed on rescattering contribution, Eq.(16), with two Gauss and one Landau functions.

On the other hand, with the sample generated with only the tree amplitude, a good fit was obtained with a Relativistic Breit-Wigner[41] parametrization for the $\rho(770)$ with no barrier factors and with floating mass and width. This result is consistent once the only complex structure in the amplitude is the production amplitude of Eq.(8). To this first fit, the results for the resonance parameters were: $m_{\rho} = 0.775 \pm 0.001\text{GeV}$, $\Gamma_{\rho} = 0.148 \pm 0.001\text{GeV}$, where the uncertainties are due to the sample size. Comparing to the PDG[42] values: $m_{\rho} = 0.77526 \pm 0.00025\text{GeV}$ and $\Gamma_{\rho} = 0.1491 \pm 0.0008\text{GeV}$, one can see that they nicely agree.

When we add the rescattering contribution to the tree one, defining the complete amplitude to $B^+ \rightarrow \pi^-\pi^+\pi^+\pi^+$ decay, there is an destructive interference around the $\rho(770)$ energy between an Breit-Wigner like function and one that is not. The effect of this interference is visible on resulting a sample that can only be fitted by a sum of two Breit-Wigner functions, with the following parameters: $m_{\rho} = 0.756 \pm 0.001\text{GeV}$, $\Gamma_{\rho} = 0.163 \pm 0.003\text{GeV}$ and $m_{\rho2} = 1.50 \pm 0.01\text{GeV}$, $\Gamma_{\rho2} = 0.194 \pm 0.033\text{GeV}$; where the uncertainties are statistical. The first $\rho$ contribution is clearly the result of the tree and rescattering interference, while the second $\rho$ comes only from the rescattering (Fig.5, left). Comparing with the results of the tree amplitude fit, one can see that the inclusion of the rescattering amplitude slightly changes the $\rho(770)$ mass and increase significantly it width. The calculation of rescattering amplitude did not include any free parameter to account for it relative weight to the tree amplitude, what could enlarge or decrease the observed influence on changing the resonance parameters.
Final Remarks

Our analyses on the relevance of rescattering effects in the $B^+ \to \pi^- \pi^+ \pi^+$ decay amplitude confirm the importance of hadronic loops as observed in Refs.[24, 30]. It shifts the phase at threshold to below zero and change the shape of the $\rho$ meson in the final amplitude. When we include rescattering effects we allow the interference of two kind of resonances: the one produced directly from the main vertex, i.e. the one covered by the isobar model, and the one coming from the two-body scattering, which is only possible by rescattering. The above analysis show that the interference between those resonances in the same energy region change the width and the line shape of the resonance in the resulting amplitude, what become visible when performing the fit using isobar model. This effect will also be observed on the Dalitz plot with the complexity of the other variable interference. In principle, if one consider the possibility of rescattering interference when performing fits to data, letting the mass and width of isobar model free, it will be possible to measure the amount of rescattering contribution in the final amplitude. It is important to point out that in this work we did not consider all the FSI effect nor all the dynamics in the $B^+ \to \pi^- \pi^+ \pi^+$. We focus on the main topologies and in the understanding of the rich FSI interference on the amplitude.

Acknowledgements

We would like to thank A.C. dos Reis, A. Gomes and M.R. Robilotta for the fruitful discussions and careful reading of this manuscript. The work of PCM was supported by CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico).

Appendix A: loop integrals

For a generic triangle loop we have

$$I_{xyz} = \frac{i}{(4\pi)^2} \Pi_{xyz} \quad (A1)$$

$$\Pi_{xyz} = - \int_0^1 da \int_0^1 db \frac{1}{D_{xyz}}, \quad (A2)$$

with

$$D_{xyz} = (1-a)m_x^2 + a(1-b)m_y^2 + abm_z^2 - i\epsilon$$
\[-a(1 - a)(1 - b)(p_x - p_y)^2 - a(1 - a)b(p_x - p_z)^2\]
\[-a^2b(1 - b)(p_y - p_z)^2;\]  \hspace{1cm} (A3)

The indices on Eqs.(17) and (18) refers to the following propagators:

\[\Delta^+ = (P_B - l)^2 - M^2;\]
\[\Delta^- = (l - p_3)^2 - M^2;\]
\[\Delta_{B^*} = (P_B - l)^2 - m^2_{B^*}\]
\[\Delta_\rho = l^2 - \Theta_i;\]

The integrals are solved numerically.

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