Winding numbers and Fourier series

Jean–Pierre Kahane

This is an expository talk on a topic of classical analysis, arising from the $BMO$–theory of topological degree (Brézis–Nirenberg 1995) \[6\]. We sketch the history of the subject and some of its recent developments.

1 The starting point

In October 1995 Haïm Brézis visited Rutgers University and gave a lecture at the seminar of Israel Gelfand on his recent work with Louis Nirenberg on the extension of the notion of topological degree to a class of functions larger than the class of continuous functions, namely $VMO$, the class of functions with vanishing mean oscillation. Vanishing mean oscillation is expressed by the formula

$$
\lim_{|B| \to 0} \frac{1}{|B|^2} \int_{B \times B} |f(x) - f(y)| \, dx \, dy = 0
$$

when $f$ maps an open set $\Omega \subset \mathbb{R}^m$ into $\mathbb{R}^n$, $B$ denotes a ball contained in $\Omega$ and $|B|$ its volume, $dx = dx_1 \cdots dx_m$ and $|z| = |z_1^2 + \cdots + z_n^2|^{1/2}$. When $f$ maps a manifold into a manifold, this definition should be translated accordingly.

Gelfand was eager to have examples, and Brézis had plenty of them. First, $W^{s,p}(\Omega) \subset VMO(\Omega)$ when $0 < s < 1$, $1 < p < \infty$, $sp = n$, and $f \in W^{s,p}(\Omega)$ means

$$
\int_{\Omega \times \Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx \, dy < \infty.
$$

Then, the Sobolev classes

$$
H^{n/2}(\Omega) \subset VMO(\Omega)
$$
since $H^s = W^{s,2}$. In particular, going to mapping of the $n$–sphere into itself,

$$H^1(S^2, S^2) \subset VMO(S^2),$$

the original motivation of Brézis [5]. In the same way,

$$H^{1/2}(S^1, S^1) \subset VMO(S^1).$$

Gelfand felt satisfied with $S^1$, but not with the definition of $H^{1/2}$ by means of a double integral. Since $f$ can be expressed as $f(e^{it})$, $t \in \mathbb{R}$, and $|f(e^{it})| = 1$, how to write the definition in terms of the Fourier coefficients of $f$,

$$a_n = \int f(e^{it})e^{-int} \frac{dt}{2\pi}$$

(here and in the sequel we write $\int$ instead of $\int_0^{2\pi}$). Brézis had the answer immediately: the definition can be expressed as

$$\sum_{-\infty}^{\infty} |n| |a_n|^2 < \infty.$$

Then Gelfand asked another question. Can you express the topological degree by means of the $a_n$? Brézis could not answer on the spot. He went home, made a little computation, and got the simple and beautiful formula

$$\deg f = \sum_{-\infty}^{\infty} n|a_n|^2.$$

Here $\deg f$ denotes the usual topological degree if $f$ is continuous, and the VMO–degree in general, that is, $f \in H^{1/2}$.

This was integrated in the article of Brézis and Nirenberg [6] and initiated a series of other questions. The first is already in [6]. Most of them can be found in the “mise au point” by Brézis in 2006 [3]. We shall see some answers and new questions.

**Question 1 [6].**

What happens when $f$ is continuous (then $\deg f$ exists) and does not belong to $H^{1/2}(S^1, S^1)$, that is

$$\sum_{-\infty}^{\infty} |n| |a_n|^2 = \infty ?$$
Is there any summation process for the series
\[ \sum_{-\infty}^{\infty} n |a_n|^2 \]
such that \( \deg f \) can be computed in that way?

The first answers were given by Jacob Korevaar in 1999 [10]. Korevaar considers two summation processes, namely
\[ \lim_{n \to \infty} \sum_{-n}^{n} m |a_m|^2, \]
using symmetrical partial sums, and
\[ \lim_{r \to 1} \sum_{-\infty}^{\infty} r^{|m|} |a_m|^2, \]
the process of Abel–Poisson. The second is stronger than the first. Korevaar shows that they work when \( f \) has bounded variation, \( f \in C \cap BV \), but that none of them work under the mere assumption \( f \in C \). Actually they diverge for some \( f \), and they converge to a value different from \( \deg f \) for some other \( f \).

That led to another question.

**Question 2 [3].**
Does \( \deg f \) depend on the absolute value of the Fourier coefficients \( a_n \) only? Since the energy of \( f \) is defined by the \( |a_n|^2 \); and the topological degree of a mapping from \( S^1 \) to \( S^1 \) is nothing but the winding number, the question can be asked in a pleasant form: can we hear the winding number?

The answer is negative. Jean Bourgain and Gaby Kozma were able to construct two functions belonging to \( C(S^1, S^1) \) with the same \( |a_n| \) and different degrees [2]. It is a very difficult construction.

**Question 3 [3, 9].**
Let us return to the summation processes. Let us begin by the Abel–Poisson process, since it is stronger than most usual processes. If we assume that \( f \) satisfies a Hölder condition of order \( \alpha > 0 \), that is, in Zygmund’s notation, \( f \in \Lambda_{\alpha}(S^1, S^1) \) (Zygmund, as many authors, says “Lipschitz condition of order \( \alpha \)” [12]), is it true that
\[ \deg f = \lim_{r \to 1} \sum_{-\infty}^{\infty} n r^{|n|} |a_n|^2? \]
The answer is positive for \( \alpha > \frac{1}{3} \) and negative for \( \alpha \leq \frac{1}{3} \). This comes from a more precise statement, which involves the classes \( \lambda^p_\alpha \) of Zygmund ([12], p. 45), defined as

\[
\lambda^p_\alpha = \{ g : \int |g(t + s) - g(s)|^p ds = o(t^\alpha), \ t \downarrow 0 \},
\]

and a non–classical summation process, namely

\[
\lim_{t \to 0} \sum_{n=-\infty}^{\infty} |a_n|^2 \frac{\sin nt}{t}.
\]

This limit exists and is equal to \( \text{deg} f \) when \( f \in C \cap \lambda^3_{1/3}(S^1, S^1) \) but there exists \( f \in \Lambda_{1/3}(S_1, S_1) \) such that it doesn’t exist, or it exists and is different from \( \text{deg} f \) [8].

The positive part of this statement is valid when \( C \cap \lambda^3_{1/3}(S^1, S^1) \) is replaced by \( W^3_{1/3}(S^1, S^1) \) [3]. We don’t know if it is still valid under the assumption \( f \in VMO \cap \lambda^3_{1/3}(S^1, S^1) \); that would provide a common generalization to both statements.

**Question 4** [9].

We introduced three summation processes, and there are many others. It was a popular subject in the 1920’s, and the best reference is Hardy’s book of 1949, Divergent Series [7]. I returned to this topic in [9].

Hardy considers series of terms indexed by positive integers, say

\[
\sum_{1}^{\infty} u_m.
\]

In our situation

\[
u_m = m(|a_m|^2 - |a_{-m}|^2).
\]

Ordinary convergence to \( s \) means

\[(C) \quad s = \lim_{n \to \infty} \sum_{1}^{n} u_m\]

Cesàro summability of order \( k \) \((k > -1)\) to \( s \) means

\[(C,k) \quad s = \lim_{n \to \infty} \left( \frac{n + k}{k} \right)^{-1} \sum_{m} \left( \frac{n + k - m}{k} \right) u_m ;\]
\((C, O)\) is the same as \(C\), and \((C, 1)\) deals with the arithmetic means of partial sums; it is the process used by Fejér in Fourier series. The processes

\[(R, k) \quad s = \lim_{r \downarrow 0} \sum_{1}^{\infty} u_m \left( \frac{\sin mt}{mt} \right)^k\]

\((k\) being a positive integer\) are called Riemann summation processes of order \(k\). The original Riemann process is \((R, 2)\) and it was used in order to study everywhere convergent trigonometric series. The process we just used is \((R, 1)\). Whenever we consider \((R, k)\) we assume \(\sum_{1}^{\infty} |u_m| m^{-k} < \infty\). The process

\[(A) \quad s = \lim_{n \uparrow 1} \sum_{1}^{\infty} r^m u_m\]

is the Abel, or Abel–Poisson, summation process.

It is classical, and easy to see, that \((C, k')\) is stronger than \((C, k)\) and \((R, k')\) is stronger than \((R, k)\) if \(k' > k\), and that \((A)\) is stronger than all \((C, k)\) and stronger than \((R, 2)\). We write

\[k' > k \implies (C, k) \implies (C, k')\]
\[k' > k \implies (R, k) \implies (R, k')\]

\[(C, k) \implies (A)\]
\[(R, 2) \implies (A)\]

Hardy mentioned some more subtle results:

\[(R, 2) \implies (C, 2 + \delta) \quad (\delta > 0)\]
\[(R, 1) \implies (C, 1 + \delta) \quad (\delta > 0)\]

The first is due to Kuttner (1935) \[11\] and the second to Zygmund (1928) \[13\]. Moreover, Kuttner gave an example showing that

\[(R, 3) \nrightarrow (A).\]

It is easy to see that

\[(R, 1) \nleftrightarrow (C).\]

More interesting is the fact that

\[(R, 1) \nleftrightarrow (C, 1).\]
In the opposite direction, $(C) \not\rightarrow (R, 1)$.

These last results can be found at the end of [9].

**Question 5 [4, 1].**

In September 2008, Brézis made a report on topological degree and asked the following question. Let $f \in C(S^1, S^1)$, with Fourier coefficients $a_n (n \in \mathbb{Z})$. Is it true that

$$
\sum_{-\infty}^{\infty} |n| |a_n|^2 \leq |\deg f| + 2 \sum_{1}^{\infty} n |a_n|^2 ?
$$

Since it is true when the first member is finite, the question can be written as

$$
\sum_{1}^{\infty} n |a_n|^2 < \infty \implies \sum_{-\infty}^{\infty} |n| |a_n|^2 < \infty ?
$$

I could answer in a particular case ($f \in \Lambda_\alpha$, $\alpha > 0$) and Bourgain in the general case.

The answer is positive. Moreover

$$
\sum_{-\infty}^{\infty} |n| |a_n|^2 \leq 32 \sum_{1}^{\infty} n |a_n|^2 .
$$

The implication is valid in a more general situation. Let $s > 0$. Then

$$
\sum_{1}^{\infty} n^{2s} |a_n|^2 < \infty \implies \sum_{-\infty}^{\infty} |n|^{2s} |a_n|^2 < \infty .
$$

But, except when $s = \frac{1}{2}$, there is no constant $C = C(s)$ such that

$$
\sum_{-\infty}^{\infty} |n|^{2s} |a_n|^2 < C \sum_{1}^{\infty} n^{2s} |a_n|^2 .
$$

These results are valid when $f$ is supposed to be $VMO$ instead of continuous, but not when $f$ is supposed to be bounded (counterexample: a Blaschke product).

The study of Fourier series of continuous or $VMO$ unimodular functions is an interesting byproduct of the study of winding numbers.
References

[1] J. Bourgain et J.–P. Kahane. *Sur les séries de Fourier des fonctions continues unimodulaires (2009)*, submitted to Annales de l’Institut Fourier.

[2] J. Bourgain and G. Kozma. *One cannot hear the winding number*, J. Europ. Math. soc. 9 (2007), 637–658.

[3] H. Brézis. *New questions related to the topological degree*, The unity of mathematics, in honour of I.M. Gelfand, Progress in Mathematics 244 (2006), 137–154.

[4] H. Brézis. *Oral communication*, Conference NODE in honour of J. Mawhin and J. Habetz, Bruxelles, September 2008.

[5] H. Brézis and J.–M. Coron. *Large solutions for harmonic maps in two dimensions*, Comm. Math. Phys. 92 (1983), 203–225.

[6] H. Brézis and J. Nirenberg. *Degree theory and BMO*, Part I, Selecta mathematica 1 (1995), 197–263.

[7] G.H. Hardy. *Divergent series*, Oxford 1949, New–York 1991.

[8] J.–P. Kahane. *Sur l’équation fonctionnelle \( \int_\pi (\psi(t+s) - \psi(s))^3 ds = sint \)*, C.R. Math. Acad. Sc. Paris 341 (2005), 141–145.

[9] J.–P. Kahane. *Winding numbers and summation processes*, Complex Variables and Elliptic Equations, à paraître.

[10] J. Korevaar. *On a question of Brezis and Nirenberg concerning the degree of circle maps*, Selecta mathematica 5 (1999), 107–122.

[11] B. Kuttner. *The relation between Riemann and Cesàro summability*, Proc. London Math. Soc. (2) 38 (1935), 273–283.

[12] A. Zygmund. *Trigonometric series I*, Cambridge University Press 1959.

[13] A. Zygmund. *Sur la dérivation des séries de Fourier*, Bull. Acad. Polon. 1924, 243–249.