Resonant pumping in nonlinear Klein-Gordon equation and solitary packets of waves

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Abstract

Solution of the nonlinear Klein-Gordon equation perturbed by small external force is investigated. The frequency of perturbation varies slowly and passes through a resonance. The resonance generates a solitary packets of waves. Full asymptotic description of this process is presented.

Introduction

This work is devoted to the problem on a generation of solitary packets of waves by a small external driving force. We propose a new approach for generation of solitary packets of waves. We demonstrate that for perturbed nonlinear Klein-Gordon equation.

In our approach the wave packets appear due to passing of external driving force through resonance. After the resonance the envelope function of the wave packet is determined by nonlinear Schrödinger equation (NLSE). In the most important cases the envelope function is a sequence of solitary waves which are called solitons. The wave packets with the solitons as the envelope function are propagated without a deformation. The parameters of the solitons are obviously defined by the value of the driving force on a resonance curve.

Here we give the mathematical basis for the proposed approach. This basis allows to derive explicit formulas which define parameters for the solitary packets of waves with respect to the external driving force. Generation of the solitary packets of waves by a small driving force is described in detail. The formulas for the asymptotic solution before, after and in the neighborhood of the resonance curve are obtained.

Proposed approach is based on a local resonance phenomenon. The local resonance in linear ordinary differential equations was investigated in papers [2, 3]. Later this phenomenon was investigated in partial differential equations in linear case [4] and in weak nonlinear case [5, 6]. As was shown in these papers the amplitude of the wave which crosses the local resonance increases by linear

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The increase of the amplitude is proportional to the width of the local resonance layer.

After resonance a special proportion between the order of solution and scales of independent variables appears. This magic proportion gives NLSE for envelope function. The deriving of NLSE in such case is well known \cite{7, 8, 9} and justified \cite{10}.

Important kind of solution of NLSE is solitons. It is known the phenomenon of solitary waves generation for some nonlinear equations due to the modulation instability \cite{11}. For example, there exist the detailed analytical description of disappearance (generation) of soliton due to the modulation instability in the case of Kadomtsev-Petviashvili equation \cite{12}. Some results on soliton appearance in nonlinear Schrödinger equation due to instability are presented in \cite{13}. It is known the structural instability of the solitons for Davey-Stewartson equation \cite{14}. Perturbations used the different kinds of instability of the solution do not allow to obtain the solitons with given parameters.

The resonance generation of solitary waves by small external force is known due to computer simulation \cite{1}. Although computer simulations justify the possibility of soliton generation by an external driving force but also do not allow to connect the the parameters of the soliton and perturbation. Therefore the problem on soliton generation with the given parameters was still open.

The goal of this paper is the following: to demonstrate that the process of solitary waves generation due to local resonance is universal. This process allows to control parameters of generated waves. Earlier this phenomenon in the case of nonlinear Schrödinger equation was asymptotically investigated in \cite{15}. In this work we consider the similar phenomenon in the nonlinear Klein-Gordon equation. Our approach demonstrates that solitary waves with the given parameters can be obtained for nonlinear wave systems.

This paper has the following structure. The first section contains the main result and example. The second section contains the asymptotic construction in the pre-resonance domain. In the third section we construct the asymptotic solution in the neighborhood of the resonance curve. The fourth section of the paper is devoted to construction of the post-resonance asymptotics. All asymptotics are matched.

1 Main result

Let us consider the Klein-Gordon equation with cubic nonlinearity
\[
\partial_t^2 U - \partial_x^2 U + U + \gamma U^3 = \varepsilon^2 f(\varepsilon x) \exp \left\{ i \frac{S(\varepsilon^2 t, \varepsilon^2 x)}{\varepsilon^2} \right\} + \text{c.c.}, \quad 0 < \varepsilon \ll 1. \tag{1}
\]
Here $\gamma = \text{const}$; $f(y)$ is smooth and rapidly vanish as $y \to \pm \infty$. The phase function $S(y, z)$ of the driving force and all derivates with respect to $y, z$ are bounded.

We construct the formal asymptotic solution of the WKB-type using the combination of method of multiple scales \cite{16} and matching method \cite{17}. Below
we use scaled variables

\[ x_j = \varepsilon^j x, \quad t_j = \varepsilon^j t, \quad j = 1, 2. \]

The leading-order term of the asymptotic solution has an order \( \varepsilon^2 \) and oscillates with frequency of the driving force. The resonance curve is determined by

\[ l(t_2, x_2) \equiv (\partial_{t_2} S)^2 - (\partial_{x_2} S)^2 - 1 = 0. \]

We assume that the curve \( S = \text{const} \) does not touch the resonance curve:

\[ \partial_{x_2} l \partial_{x_2} S - \partial_{t_2} l \partial_{t_2} S \neq 0. \]

The frequency of the forced oscillations and frequency of the eigen oscillations of the linearized Klein-Gordon equation are equal on the resonance curve. It yields the local resonance layer in the neighborhood of the curve \( l(x_2, t_2) = 0 \). The asymptotics of the WKB-type is not valid into this layer. The leading-order term of the asymptotics is defined by Fresnel integral. After resonance the solution has an order \( \varepsilon \) and oscillates. The envelope function satisfies NLSE.

The accurate formulation of the result for this paper is following

**Theorem 1** In the domain \( -l \gg \varepsilon \) the asymptotic solution of (1) modulo \( O(\varepsilon^{N+1}) \) has the form

\[ U = \sum_{n \geq 2} \varepsilon^n U(t, x, \varepsilon), \]

where the leading-order term is

\[ \dot{U} = -\frac{f}{1} \exp(iS(t_2, x_2)/\varepsilon^2) + \text{c.c.} \]

The higher-order terms are determined from the recurrent system of algebraic equations (3), (4), (5), (7).

In the domain \( |l| \ll 1 \) the asymptotic solution of (1) modulo \( O(\varepsilon^{N+1}) \) has the form

\[ U = \sum_{n \geq 1} \varepsilon^n W(t_1, x_1, t_2, x_2, \varepsilon). \]

The leading-order term is

\[ \dot{W}(t_1, x_1, t_2, x_2, \varepsilon) = \dot{W}_1 \exp(iS(t_2, x_2)/\varepsilon^2) + \text{c.c.} \]

The function \( \dot{W}_1 \) is solution of the equation

\[ 2i\partial_{t_2} S \partial_{t_1} \dot{W}_1 - 2i\partial_{x_2} S \partial_{x_1} \dot{W}_1 - \lambda \dot{W}_1 = f \]

with the zero condition as \( \lambda \to -\infty \), where \( \lambda(x_1, t_1, \varepsilon) = l(t_2, x_2)/\varepsilon \). The higher-order terms are either solutions of the problems for equations (17) with zero
conditions as $\lambda \to -\infty$ or solutions of the algebraic equations (14).

In the domain $l \gg \varepsilon$ the asymptotic solution of (1) modulo $O(\varepsilon^{N+1})$ has the form

$$U(x, t, \varepsilon) = \sum_{n=1}^{N} \varepsilon^{n} \sum_{k=0}^{n-2} \ln^k(\varepsilon) \left( \sum_{\pm} \exp\{\pm i\Phi(x_2, t_2)/\varepsilon^2\} n_{\pm}(x_1, t_1, t_2) + \right. $$

$$\left. + \sum_{\chi \in K'_{n,k}} \exp\{i\chi(x_2, t_2)/\varepsilon^2\} n_{\chi}(x_1, t_1, t_2) \right).$$

The phase function $\Phi$ satisfies eikonal equation

$$(\partial_{t_2} \Phi)^2 - (\partial_{x_2} \Phi)^2 - 1 = 0$$

with conditions

$$\Phi|_{l=0} = S|_{t=0}, \quad \partial_{t_2} \Phi|_{l=0} = \partial_{t_2} S|_{l=0}.$$ 

The envelope function of the leading-order term is a solution of the nonlinear Schrodinger equation

$$2i \partial_{t_2} \Phi \partial_{x_2} \Psi_0 \Phi + \partial^2_{\xi} \Psi_0 \Phi + i[\partial^2_{t_2} \Phi - \partial^2_{x_2} \Phi] \Psi_0 \Phi + \gamma |\Psi_0 \Phi|^2 \Psi_0 \Phi = 0,$$

where the $\xi$ is defined from

$$\frac{dx_1}{d\xi} = \partial_{t_2} \Phi, \quad \frac{dt_1}{d\xi} = \partial_{x_2} \Phi.$$ 

The initial condition for $\Psi_0 \Phi$ is

$$\Psi_0 \Phi|_{l=0} = \int_{-\infty}^{\infty} d\sigma f(x_1) \exp(i \int_0^\sigma d\mu \lambda(x_1, t_1, \varepsilon)).$$

Integration in this integral is realized in the line of characteristic direction connected with (21), (22). The coefficient $n_{\pm} \Psi \Phi$ is determined from Cauchy problems for linearized Schrodinger equation (38). The coefficients $n_{\chi} \Psi \chi, \chi \in K'_{n,k}$ are determined from algebraic equations (39). The set $K_{n,k}$ is defined by

$$K_{1,0} = \pm \Phi; \quad K_{2,0} = \pm \Phi, \pm S, \quad K_{3,1} = \pm \Phi, \quad K_{n,k} = \cup \alpha + \beta + \delta,$$

where

$$\alpha \in K_{j_1,l_1}, \quad \beta \in K_{j_2,l_2}, \quad \delta \in K_{j_3,l_3}, \quad j_1 + j_2 + j_3 = n, \quad l_1 + l_2 + l_3 = k$$

The set $K'_{n,k} = K_{n,k} \backslash \{\pm \Phi\}$. 


This theorem is a direct consequence of theorems 2, 3 and 4 which are proved in the next parts of the paper.

To illustrate the theorem we consider equation (1) with the simplest of driving force. Let the phase function of the driven be \( S = (\varepsilon^2 t)^2 / 2 \). In this case the curve of the local resonance is the line \( t_2 = 1 \). In the domain \( t_2 > 1 \) the leading-order term of the asymptotics satisfies the Cauchy problem

\[
2i \partial_{t_2}^{1,0} \Psi + \partial^2_{\xi \xi}^{1,0} \Psi + \gamma | \Psi |^{1,0} \Psi = 0, \\
\Psi|_{t_2=1} = f(\xi)(1 + i)\sqrt{\pi}.
\]

The solution of this Cauchy problem contains solitary waves if the initial data is sufficiently large [12].

**Remark on WKB asymptotics.** Theorem 1 describes the special asymptotic solution of equation (1). It is defined by the driving force. One can add any solution of the WKB-type [16] of the order \( \varepsilon^2 \) to this constructed solution. It leads to an asymptotic solution for equation (1) of the form

\[
\tilde{U} = U(t, x, \varepsilon) + \sum_{n \geq 2} \varepsilon^n \hat{U}(t, x, \varepsilon).
\]

The coefficients \( \hat{U}(t, x, \varepsilon) \) of the asymptotics are calculated by standard methods of WKB-theory. This additional term leads to ponderous formulas and does not change the leading-order term of the asymptotics constructed in theorem 1.

## 2 Pre-resonance expansion

In this section the formal asymptotic solution is constructed in the domain before resonance. The asymptotic expansion has the form of the WKB-type. The leading-order term of the asymptotics has the order of the driving force and oscillates with its frequency. The constructed asymptotics is valid as \(-l \gg \varepsilon\). The result of this section is formulated below.

Let us construct the formal asymptotic solution for equation (1) in the form

\[
U = \sum_{n \geq 2} \varepsilon^n \hat{U}(t, x, \varepsilon), 
\]

where

\[
\hat{U} = \sum_{k \in \Omega_n} \hat{U}_k(t_2, x_2, \varepsilon x) \exp \left\{ ik \frac{S(t_2, x_2)}{\varepsilon^2} \right\}.
\]

Set \( \Omega_n \) for the higher-order term with the number \( n \) is described by the formula

\[
\Omega_n = \begin{cases} 
\{ \pm 1 \}, & n \leq 5; \\
\{ \pm 1, \pm 3, \ldots, \pm (2l + 3) \}, & l = \lfloor (n - 6)/4 \rfloor, \quad n \geq 6.
\end{cases}
\]
The functions $n_U^k$ and $n_{-k}$ are complex conjugated.

Let us substitute (2) in equation (1) and collect the terms of the same order of $\varepsilon$. As a result we obtain a recurrent sequence of algebraic equations.

$$
\frac{2}{U_1} = -\frac{f}{l},
$$

(3)

$$
\frac{3}{U_1} = 2i \frac{\partial t_1 f \partial x_2 S}{l^2},
$$

(4)

$$
\frac{4}{U_1} = \frac{2i f [\partial t_2 S \partial t_1 l - \partial x_2 S \partial x_2 l] - 4(\partial x_2 S)^2 \partial^2 f}{l^3} + \frac{2i \partial t_2 f \partial t_2 S + \partial^2 t_2 S f}{l^2},
$$

(5)

where

$$
l = (\partial t_2 S)^2 - (\partial x_2 S)^2 - 1.
$$

The curve where the phase function $S$ satisfies eikonal equation is called the resonance curve

$$
l[S] = (\partial t_2 S)^2 - (\partial x_2 S)^2 - 1 = 0.
$$

(6)

The amplitude $n_U^1$ has a singularity on this curve.

The formula for the $n$-th order term has the form

$$
n_U^k = \frac{1}{l} \left[ \partial^2_{t_2 t_2} U_k - 2ik \partial t_2 S \partial t_2 U_k + ik S t_2 t_2 \right] U_k - 2ik \partial x_2 S \partial x_2 U_k - ik \partial^2 x_2 S U_k - \partial^2_{x_1 x_1} U_k - 2\partial^2_{x_1 x_2} U_k - \partial^2_{x_2 x_2} U_k - 2ik \partial x_2 S \partial x_1 U_k + \gamma \sum_{n_1 + n_2 + n_3 = n, k_1 + k_2 + k_3 = k, k \in \Omega_n} \frac{n_1}{U_{k_1}} \frac{n_2}{U_{k_2}} \frac{n_3}{U_{k_3}}.
$$

(7)

**Lemma 1** The coefficient $n_U^k$ has the following behaviour

$$
n_U^k = O(l^{-(n-k)}), \quad k > 0, \quad l \to -0.
$$

(8)

**Proof.** Let us prove this lemma as $k = 1$. The validity of formula (8) for $n = 2, 3, 4$ directly obtains from (3), (4), (5). Suppose now that this formula is valid for the term $n_{-1}^1 U_1$. The increase of the order of the singularity as $l \to 0$ takes place due to differentiation with respect to $x_2, t_2$ and the nonlinear term in formula (7). Differentiation of the terms in formula (7) leads to formula (8).

Let us consider $n_U^k$ for $k > 1$. The validity of formula (8) for small values of $n$ and $k$ obtains by direct calculations. Consider the $n$-th order term. It
contain the terms with different values of \( k \). The higher-order term with \( k = 3 \) have the greatest order of singularity.

\[
\n^3U_3 = O(l^{-(n-3)}), \quad l \to -0.
\]

(9)

It takes place because the right hand side of (7) contains the term \( nU_\pm k U_{\pm 1} \). The calculation of the order of singularity for this term leads to formula (9).

The terms of the type of \( n^3 U_\pm n^j U^{\mp 1}_k U^{\pm 1} \), \( n_1 + n_2 + n_3 = n \) lead to weak singularities, for example for \( k_1 = 3 \) we obtain the order of singularity is equal to \( n - 9 \).

Consider nonlinear term \( n^j U_{k_1} U_{k_2} U_{k_3} \) from right hand side of (7) when the number of the higher-order term is equal to \( n \). Calculate the order of singularity for this term using the \((n-1)\)-th step of induction. Indexes of the amplitudes are connected by formulas

\[
\begin{align*}
\quad n_1 + n_2 + n_3 &= n, \\
\quad k_1 + k_2 + k_3 &= k.
\end{align*}
\]

Using (8) for \( n_1, n_2, n_3 < n \) we obtain that the order of the singularity for this term is equal to \( (n-k) \).

The right hand side of (7) contains derivatives of previous terms with respect to \( x_2, t_2 \). It leads to increase of the order of the singularity but the leading order nevertheless we obtain from nonlinear terms. The lemma is proved.

The domain of validity as \( l \to -0 \) for formal asymptotic solution in the form (2) is defined by

\[
\begin{align*}
\varepsilon^{n+1} U_{n+1} &< 1, \\
\varepsilon^n U_n &< 1.
\end{align*}
\]

It yields

\[-l \gg \varepsilon.\]

Using these lemmas we obtain the asymptotic representation for (2) as \( l \to -0 \)

\[
U = \sum_{n=2}^{N} \varepsilon^n \sum_{k \in \Omega_n} \exp\{ikS/\varepsilon^2\} \sum_{j=-(n-k)}^{\infty} \n^j U_k^j, \quad l \to -0.
\]

(10)

The following theorem is proved.

**Theorem 2** In the domain \(-l \gg \varepsilon\) the formal asymptotic solution of equation (1) modulo \( O(\varepsilon^{N+1}) \) has the form (2). The coefficients of the asymptotics \( U_k \) are defined from algebraic equations (3), (4), (5), (7).

3 Internal asymptotics

This part of the paper contains the asymptotic construction of the solution for equation (1) in the neighborhood of the curve \( l = 0 \). The domain of validity of this asymptotics intersects with domain of validity of expansion (2). These expansions are matched.
Theorem 3 In the domain $|l| \ll 1$ the formal asymptotic solution for equation (1) modulo $O(\varepsilon^{N+1})$ has the form

$$U = \sum_{n \geq 1} \varepsilon^n W(t_1, x_1, t_2, x_2, \varepsilon),$$  \hspace{1cm} (11)$$

where

$$W = \sum_{k \in \Omega_n} \tilde{W}_k(x_2, t_2, x_1, t_1) \exp \left\{ ik \frac{S(t_2, x_2)}{\varepsilon^2} \right\},$$  \hspace{1cm} (12)$$

The function $\tilde{W}_k$, $k = 1$ is solution of the problem for equation (17) with zero condition as $\lambda \to -\infty$ and solutions of algebraic equations (19) in the case $k \neq 1$. The functions $\tilde{W}_k$ and $\tilde{W}_{-k}$ are complex conjugated.

There is an essential difference between asymptotics (11) and external pre-resonance asymptotics (2). First the leading order term in (11) has an order $\varepsilon$ in contrast the leading order term in (2) has an order $\varepsilon^2$. Second the coefficients of asymptotics (11) depend on fast variables $x_1 = x_2/\varepsilon$ and $t_1 = t_2/\varepsilon$.

The proof of theorem 3 consists in three steps. First we derive equations for coefficients of the asymptotics. Second we solve the problems for asymptotic coefficients. And third we determine the domain of the validity for expansion (11).

3.1 The equations for coefficients

Let us construct the internal asymptotic expansion in the domain $|l| \ll 1$. Denote

$$\lambda(x_1, t_1, \varepsilon) = \frac{1}{\varepsilon^l}(\varepsilon x_1, \varepsilon t_1).$$  \hspace{1cm} (13)$$

In the domain $1 \ll \lambda \ll \varepsilon^{-1}$ both asymptotics (2) and (11) are valid. This fact allows us to obtain the asymptotic representation for coefficients of the internal asymptotics. Substitute $l = \varepsilon \lambda$ in formula (11) and expand the obtained expression with respect to powers of small parameter $\varepsilon$. It yields

$$\tilde{W}_k = \sum_{j=(n-k+1)}^{\infty} \lambda^{-j} U_k^{n+1}(x_2, t_2, x_1), \hspace{0.5cm} k \in \Omega_n, \hspace{0.5cm} \lambda \to -\infty.$$  \hspace{1cm} (14)$$

Let us obtain the differential equations for the coefficients of asymptotics (11). Substitute (11), (12) in equation (1) and collect the terms with equal powers of small parameter and exponents. It yields the equations for coefficients $\tilde{W}_k$. In particularly, the terms of the order $\varepsilon^2$ give us the equations for the leading-order terms of the asymptotics

$$2i\partial_{t_2} S \partial_{t_1} \frac{1}{W_1} - 2i\partial_{x_2} S \partial_{x_1} \frac{1}{W_1} - \lambda \frac{1}{W_1} = f,$$  \hspace{1cm} (15)$$
and complex conjugated equation for $\hat{W}_{-1}$.

The relation of the order $\varepsilon^3$ in equation (1) gives four equations. Two of them are complex conjugated differential equations for $\hat{W}_1$ and $\hat{W}_{-1}$:

$$
2i\partial_{t_2}S\partial_{t_1}\hat{W}_1 - 2i\partial_{x_2}S\partial_{x_1}\hat{W}_1 - \lambda \hat{W}_1 = \partial_{t_2}^2 \hat{W}_1 - \partial_{x_2}^2 \hat{W}_1 -
$$

$$
- i[\partial_{t_2}^2 S - \partial_{x_2}^2 S] \hat{W}_1 - 2i\partial_{t_2}S\partial_{t_1}\hat{W}_1 + 2i\partial_{x_2}S\partial_{x_1}\hat{W}_1 - 3\gamma |\hat{W}_1|^2 \hat{W}_1,
$$

(16)
two another equations are algebraic. These last equations allow us to determine the functions $\hat{W}_3$ and $\hat{W}_{-3}$

$$
\hat{W}_3 = \frac{\gamma}{8} (\hat{W}_1)^3.
$$

The higher-order terms are calculated by the same way. In particular, the terms in the case lower index is equal to 1 are determined by differential equations.

$$
2i\partial_{t_2}S\partial_{t_1} \hat{W}_1 - 2i\partial_{x_2}S\partial_{x_1} \hat{W}_1 - \lambda \hat{W}_1 = \hat{F}_1.
$$

(17)
The right hand side of equation (17) has the form

$$
\hat{F}_1 = -2i\partial_{t_2}S\partial_{t_1} \hat{W}_1 + 2i\partial_{x_2}S\partial_{x_1} \hat{W}_1 + (\partial_{t_2}S)^2 \hat{W}_1 - (\partial_{x_2}S)^2 \hat{W}_1 -
$$

$$
- \partial_{t_2}^2 \hat{W}_1 + \partial_{x_2}^2 \hat{W}_1 - \partial_{t_2}\partial_{t_1} \hat{W}_1 + \partial_{x_2}\partial_{x_1} \hat{W}_1
$$

$$
- \gamma \sum_{n_1 + n_2 + n_3 = n + 1, k_j \in \Omega_{n_j}, j = 1, 2, 3} \hat{W}_{k_1} \hat{W}_{k_2} \hat{W}_{k_3}.
$$

(18)
The higher-order terms in the case the lower index is not equal to 1 are determined by algebraic equations

$$
\hat{W}_k = \frac{\gamma}{k^2 - 1} \left( -2i\partial_{t_2}S\partial_{t_1} \hat{W}_k + 2i\partial_{x_2}S\partial_{x_1} \hat{W}_k + (\partial_{t_2}S)^2 \hat{W}_k - (\partial_{x_2}S)^2 \hat{W}_k -
$$

$$
- \partial_{t_2}^2 \hat{W}_k + \partial_{x_2}^2 \hat{W}_k - \partial_{t_2}\partial_{t_1} \hat{W}_k + \partial_{x_2}\partial_{x_1} \hat{W}_k
$$

$$
- \partial_{t_2}^{-4} \hat{W}_k + \partial_{x_2}^{-4} \hat{W}_k - \gamma \sum_{n_1 + n_2 + n_3 = n + 1, k_j \in \Omega_{n_j}, j = 1, 2, 3} \hat{W}_{k_1} \hat{W}_{k_2} \hat{W}_{k_3} \right).
$$

(19)

3.2 The solvability of equations for higher-order terms

In this section we present the explicit form for higher-order term $\hat{W}_1$ and investigate the asymptotic behaviour as $\lambda \to \pm\infty$. 

3.2.1 Characteristic variables

The function $W_1$ satisfies equation (17). The solution is constructed by characteristic method. Define the characteristic variables $\sigma, \xi$. We choose a point $(x_1^0, t_1^0)$ such that $\partial x_2 l(x_1^0, t_1^0) \neq 0$ as origin and denote by $\sigma$ the variable along the characteristic family for equation (17). We suppose $\sigma = 0$ on the curve $\lambda = 0$. The variable $\xi$ mensurates the distance along the curve $\lambda = 0$ from the point $(x_1^0, t_1^0)$. This point $(x_1^0, t_1^0)$ corresponds to $\xi = 0$. Let the positive direction for parameter $\xi$ coincide with positive direction of $x_2$ in the neighborhood of $(x_1^0, t_1^0)$.

The characteristic equations for (17) have a form

$$\frac{dt_1}{d\sigma} = 2\partial t_2 S(\varepsilon x_1, \varepsilon t_1), \quad \frac{dx_1}{d\sigma} = -2\partial x_2 S(\varepsilon x_1, \varepsilon t_1).$$  \hspace{1cm} (20)

The initial conditions for the equations are

$$x_1|_{\sigma=0} = x_1^0, \quad t_1|_{\sigma=0} = t_1^0.$$  \hspace{1cm} (21)

**Lemma 2** The Cauchy problem for characteristics has a solutions as $|\sigma| < c_1 \varepsilon^{-1}$, $c_1 = \text{const} > 0$.

**Proof.** The Cauchy problem (20), (21) is equivalent to the system of the integral equations

$$t_1 = t_1^0 + 2 \int_0^\sigma \partial t_2 S(\varepsilon x_1, \varepsilon t_1)d\zeta, \quad x_1 = x_1^0 - 2 \int_0^\sigma \partial x_2 S(\varepsilon x_1, \varepsilon t_1)d\zeta.$$  \hspace{1cm} (22)

Substitute $\tilde{t}_2 = (t_1 - t_1^0)\varepsilon$, $\tilde{x}_2 = (x_1 - x_1^0)\varepsilon$. It yields

$$\tilde{t}_2 = 2 \int_0^{\varepsilon\sigma} \partial t_2 S(\tilde{x}_2 - \varepsilon x_1^0, \tilde{t}_2 - \varepsilon t_1^0)d\zeta, \quad \tilde{x}_2 = -2 \int_0^{\varepsilon\sigma} \partial x_2 S(\tilde{x}_2 - \varepsilon x_1^0, \tilde{t}_2 - \varepsilon t_1^0)d\zeta.$$  \hspace{1cm}

The integrands are smooth and bounded functions on the plane $x_2, t_2$. There exists the constant $c_1 = \text{const} > 0$ such that the integral operator is contraction operator as $|\varepsilon| < c_1$. Lemma 2 is proved.

**Assumption.** We assume that the change of variables $(x_1, t_1) \rightarrow (\sigma, \xi)$ is unique in the neighborhood of the curve $\lambda = 0$. This assumption means that the characteristics for equation (17) do not touch the curve $\lambda = 0$. It means

$$\partial x_2 l \partial x_2 S - \partial t_2 l \partial t_2 S \neq 0.$$  \hspace{1cm}

It is convenient to use the following asymptotic formulas for change of variables $(x_1, t_1) \rightarrow (\sigma, \xi)$. 

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Lemma 3  In the domain $|\sigma| \ll \varepsilon^{-1}$ the asymptotics as $\varepsilon \to 0$ of the solutions for Cauchy problem (20), (21) have the form

$$x_1(\sigma, \xi, \varepsilon) - x_1^0(\xi) = -2\sigma \partial_{x_2} S + 2 \sum_{n=1}^{N} \varepsilon^n \sigma^{n+1} g_n(\varepsilon x_1, \varepsilon t_1) + O(\varepsilon^{N+1} \sigma^{N+2}),$$

(23)

$$t_1(\sigma, \xi, \varepsilon) - t_1^0(\xi) = 2\sigma \partial_{t_2} S + 2 \sum_{n=1}^{N} \varepsilon^n \sigma^{n+1} h_n(\varepsilon x_1, \varepsilon t_1) + O(\varepsilon^{N+1} \sigma^{N+2}),$$

(24)

where

$$g_n = \frac{d^n}{d\sigma^n}(\partial_{x_2} S) \bigg|_{\sigma=0}, \quad h_n = \frac{d^n}{d\sigma^n}(\partial_{t_2} S) \bigg|_{\sigma=0}. $$

The lemma proves by integration by parts of equations (22).

The next proposition gives us the asymptotic formula which connects variables $\sigma$ and $\lambda$ as $\sigma, \lambda \to \pm \infty$.

Lemma 4  Let be $\sigma \ll \varepsilon^{-1}$, then:

$$\lambda = \phi(\xi) \sigma + O(\varepsilon \sigma^2), \quad \phi(\xi) = \frac{d\lambda}{d\sigma} \bigg|_{\sigma=0} \quad \sigma \to \infty.$$

Proof. From formula (13) we obtain the representation in the form

$$\lambda = \sum_{j=1}^{\infty} \lambda_j(x_1, t_1, \varepsilon) \sigma^j \varepsilon^{j-1},$$

where

$$\lambda_j(x_1, t_1, \varepsilon) = \frac{1}{j!} \frac{d^j}{d\sigma^j} \lambda(x_1, t_1, \varepsilon) \bigg|_{\sigma=0}. $$

It yields

$$\lambda = \frac{d\lambda}{d\sigma} \bigg|_{\sigma=0} \sigma + O(\varepsilon \sigma^2 \frac{d^2 \lambda}{d\sigma^2}).$$

Let be

$$\left| \frac{d^2 \lambda}{d\sigma^2} \right| \geq \text{const}, \quad \xi \in \mathbb{R}.$$

The function $d\lambda/d\sigma$ is not equal to zero

$$\frac{d\lambda}{d\sigma} = \frac{1}{2} \left( - \partial_{x_2} \lambda \partial_{x_2} S + \partial_{t_2} \lambda \partial_{t_2} S \right) \neq 0.$$ 

Let us suppose $d\lambda/d\sigma > 0$. It yields

$$\lambda = \phi(\xi) \sigma + O(\varepsilon \sigma^2), \quad \phi(\xi) = \frac{d\lambda}{d\sigma} \bigg|_{\sigma=0}$$

The lemma is proved.
3.2.2 Solutions of the equations for higher-order terms

The higher-order terms \( \hat{W}^n_{\pm 1} \) are solutions of equation (17) with the given asymptotic behaviour \( \lambda \to -\infty \). Equation (17) can be written in characteristic variables as

\[
i \frac{d}{d\sigma} \hat{W}^n_1 - \lambda \hat{W}^n_1 = \hat{F}^n_1.
\]  

Lemma 5 The solution of equation (17) with the asymptotic behaviour (14) as \( \lambda \to -\infty \) has a form

\[
\hat{W}^n_1 = \exp(-i \int_0^\sigma d\zeta \lambda(x_1, t_1, \varepsilon)) \int_{-\infty}^\zeta d\zeta \hat{F}^n_1(x_1, t_1, \varepsilon)) \exp(-i \int_0^\zeta d\chi \lambda(x_1, t_1, \varepsilon)).
\]  

Proof. By direct substitution we see that expression (26) is the solution of (25). The asymptotics of this solution as \( \lambda \to -\infty \) can be obtained by integration by parts and substitution

\[
\frac{d}{d\sigma} = 2 \partial_{x_2} S \partial_{t_1} - 2 \partial_{x_2} S \partial_{x_1}.
\]

It yields

\[
\hat{W}^n_1 = \sum_{j=0}^{\infty} \left( \frac{2 \partial_{x_2} S \partial_{t_1} - 2 \partial_{x_2} S \partial_{x_1}}{i \lambda} \right)^j \left[ \frac{\hat{F}^n_1}{i \lambda} \right], \quad \lambda \to -\infty.
\]  

From formula (18) we obtain that formulas (27) and (14) are equivalent. The lemma is proved.

3.3 Asymptotics as \( \lambda \to \infty \) and domain of validity of the internal asymptotics

The domain of validity of the internal expansion is determined by the asymptotics of higher-order terms. In this section we show that the \( n \)–th order term of the asymptotic solution increases as \( \lambda^{n-1} \) when \( \lambda \to \infty \). This increase of higher-order terms allows us to determine the domain of validity for internal asymptotics (11) as \( \lambda \to \infty \).

3.3.1 Asymptotics of higher-order terms

This section contains two propositions concerning asymptotic behaviour as \( \lambda \to \infty \) for higher-order terms in (11). The first lemma describes the asymptotic behaviour of higher-order terms as \( \lambda \to \infty \) and the second one contains a result about asymptotics of the phase function.
Lemma 6 The asymptotic behaviour of $\tilde{W}_1^n$ as $1 \ll \lambda \ll \varepsilon^{-1}$ has a form

\[
\tilde{W}_1^n = \sum_{j=0}^{n-1} \sum_{k=0}^{j-1} \left( \lambda^j \ln |\lambda| \tilde{W}_1^{(j,k)}(\xi) \right) \exp(-i \int_0^\sigma d\xi \lambda(x_1, t_1, \varepsilon)) + 
+ \sum_{j=0}^{\infty} \left( \frac{2 \partial_{t_1} S \partial_{t_1} - 2 \partial_{x_1} S \partial_{x_1}}{i\lambda} \right)^j \frac{n}{i\lambda} F_1^n, \quad (28)
\]

Proof. Let us calculate the asymptotics of the leading-order term

\[
\tilde{W}_1 = \exp(-i \int_0^\sigma d\xi \lambda(x_1, t_1, \varepsilon)) \int_{-\infty}^{\xi} d\xi f(x_1) \exp(i \int_0^\sigma d\chi \lambda(x_1, t_1, \varepsilon)) = 
\exp(-i \int_0^\sigma d\xi \lambda(x_1, t_1, \varepsilon)) \int_{-\infty}^{\infty} d\xi f(x_1) \exp(i \int_0^\sigma d\chi \lambda(x_1, t_1, \varepsilon)) - 
\exp(-i \int_0^\sigma d\xi \lambda(x_1, t_1, \varepsilon)) \int_{-\sigma}^{\infty} d\xi f(x_1) \exp(i \int_0^\sigma d\chi \lambda(x_1, t_1, \varepsilon)).
\]

Further by integration by parts of the last term we obtain formula (28) as $n = 1$, where

\[
\tilde{W}_1^{(0,0)}(\xi) = \int_{-\infty}^{\infty} d\sigma f(x_1) \exp(i \int_0^\sigma d\chi \lambda(x_1, t_1, \varepsilon)),
\]

$\tilde{F}_1 = f(x_1)$.

To calculate the asymptotics of $\tilde{W}_1^n$ in formula (28) we use the asymptotics with respect to $\sigma$ of the leading-order term. Integral (28) contains the term with linear increase with respect to $\sigma$ when $n = 2$. We eliminate this growing part from integral explicitly. The residuary integral converges as $\sigma \to \infty$. It can be calculated in the same manner as it was calculated for $\tilde{W}_1$. It yields formula (28) as $n = 2$, where

\[
\tilde{W}_1^{(1,0)}(\xi) = \tilde{W}_1^{(0,0)}(\xi).
\]

The same direct calculations are realized for the $n-$th order term. The lemma is proved.

To complete the proof of theorem 3 we need to obtain the domain of validity of asymptotics (11). The formal series (11) is asymptotic when

\[
\frac{\varepsilon^{n+1} W^{n+1}}{\varepsilon^n W^n} \ll 1, \quad \varepsilon \to 0.
\]

Lemma 8 gives $\lambda \ll \varepsilon^{-1}$. After substitution $\lambda = \varepsilon l$ we obtain $l \ll 1$. Theorem 3 is proved.
3.3.2 Asymptotics of the phase function as \( \lambda \to \infty \)

To obtain the asymptotics as \( \lambda \to \infty \) we need to derive the asymptotics of the phase function in formula (28).

**Lemma 7** As \( \lambda \to \infty \):

\[
\int_0^\sigma d\xi = S \varepsilon^2 + \frac{1}{\varepsilon}(\partial_{x_2}^2 S(x_1 - x_1^0) + \partial_{t_2}^2 S(t_1 - t_1^0)) + O(\varepsilon^3). \quad (29)
\]

**Proof.** Substitute the asymptotics of \( \lambda \) from lemma 6. Calculate the asymptotics of the integral in formula (29)

\[
\int_0^\sigma d\xi(x_1, t_1, \varepsilon) = \int_0^\sigma \frac{d\zeta}{2} \left[ (-\partial_{x_2} \partial_{x_2}^2 S + \partial_{t_2} \partial_{t_2}^2 S) \zeta + O(\varepsilon^2) \right] =
\]

\[
(-\partial_{x_2} \partial_{x_2}^2 S + \partial_{t_2} \partial_{t_2}^2 S) \frac{\sigma^2}{4} + O(\varepsilon^3).
\]

The asymptotics of the phase function \( S(x_2, t_2) \) in the neighborhood of the curve \( l = 0 \) is represented by a segment of the Taylor series. It yields

\[
\frac{1}{\varepsilon}(\partial_{x_2}^2 S(x_1 - x_1^0) + \partial_{t_2}^2 S(t_1 - t_1^0)) +
\]

\[
\frac{1}{2} \left[ S_{x_2x_2}^2 (x_1 - x_1^0)^2 + 2 S_{x_2t_2} (x_1 - x_1^0) (t_1 - t_1^0) + S_{t_2t_2} (t_1 - t_1^0)^2 \right]
\]

\[
O(\varepsilon^2 |t_1 - t_1^0| + |t_1 - t_1^0|^3).
\]

Substitute instead of \((x_1 - x_1^0)\) and \((t_1 - t_1^0)\) their asymptotic behaviour with respect to \(\varepsilon\) from lemma 3. This substitution and result of lemma 4 complete the proof of lemma 7.

The asymptotics as \( \lambda \to -\infty \) contains fast oscillating terms with phase functions \( kS, k \in \mathbb{Z} \). The leading-order term of the asymptotics as \( \lambda \to \infty \) contains the oscillations with an additional phase function. We obtain this result from lemma 6. Denote this new phase function by \( \Phi(x_2, t_2)/\varepsilon^2 \). The asymptotics of this function is obtained in lemma 7. The nonlinearity and additional phase function lead to more complicated structure of the phase set for higher-order terms of the asymptotics as \( \lambda \to \infty \).

**Lemma 8** The phase set \( K_n \) for the \( n \)-th order term of the asymptotics as \( \lambda \to \infty \) is determined by formula

\[
K_1 = \pm \Phi; \quad K_2 = \pm \Phi, \pm S, \quad K_n = \cup_{j_1+j_2+j_3=n} \chi_{j_1} + \chi_{j_2} + \chi_{j_3}, \quad \chi_{jk} \in K_j.
\]

The proof of this lemma follows from the asymptotic formula for \( n \)-th order term. Representation (11), formula (28) and lemma 4 allow us to construct the
asymptotics as $\lambda \to \infty$ of the internal expansion in an explicit form

$$U = \sum_{n=1}^{N} \varepsilon^n \left( \sum_{j=0}^{n-2} \sum_{k=0}^{n-2} \lambda^j \ln^k |\lambda| W_{1,j,k}(\xi) \right) \times \exp \left[ -i \left( \frac{1}{\varepsilon} (\partial_2^2 S(x_1 - x_0^0) + \partial_2^2 S(t_1 - t_0^0)) + O(\varepsilon^3) \right) \right] + \sum_{n=1}^{N} \varepsilon^n \left( \sum_{j=0}^{\infty} \left( \frac{2\partial_2^2 S \partial_4 - 2\partial_2^2 S \partial_1}{i\lambda} \right)^j \left( \frac{P_1}{i\lambda} \right) \right) \exp \left\{ i \frac{S(t_2, x_2)}{\varepsilon^2} \right\} + \sum_{n=2}^{N} \varepsilon^n \left( \sum_{k \in \Omega, k \neq \pm 1} \right) W_k \exp \left\{ i k \frac{S(t_2, x_2)}{\varepsilon^2} \right\} + c.c.. \quad (30)$$

This representation and formula (30) complete the proof of the lemma.

4 Post-resonance expansion

This section contains the construction of the asymptotics of the solution for (1) after passage through resonance. The constructed solution has the order $\varepsilon$ and oscillates. The envelope function of these oscillations satisfies nonlinear Schrödinger equation. This section consists in two parts. The first part contains the construction of the formal asymptotic solution. We obtain the equations for higher-order terms of the asymptotics. Asymptotic behaviour for higher-order terms as $l \to -0$ follows from section 3.3.2. In the second part of this section we determine the domain of validity for this external asymptotics near resonance curve $l(x_2, t_2) = 0$. The matching method gives us the initial conditions for higher-order terms of the asymptotics.

The main result of this section is formulated in the following theorem.

**Theorem 4** In the domain $l \gg \varepsilon$ the formal asymptotic solution of equation (1) modulo $O(\varepsilon^{N+1})$ has a form

$$U(x, t, \varepsilon) = \sum_{n=1}^{N} \varepsilon^n \left( \sum_{k=0}^{n-2} \ln^k (\varepsilon) \left( \sum_{\Phi} \exp \left\{ \pm i \Phi(x_2, t_2)/\varepsilon^2 \right\} \Psi_{\pm \Phi}(x_1, t_1, t_2) + \sum_{\chi \in \mathcal{K}_{n,k}} \exp \left\{ i \chi(x_2, t_2)/\varepsilon^2 \right\} \Psi_{\chi}(x_1, t_1, t_2) \right) \right). \quad (31)$$

Here the function $\Phi(x_2, t_2)$ satisfies eikonal equation

$$(\partial_2^2 \Phi)^2 - (\partial_2^2 \Phi)^2 - 1 = 0 \quad (32)$$

and initial condition on the curve $l = 0$:

$$\Phi_{l=0} = S_{l=0}, \quad \partial_2 \Phi_{l=0} = \partial_2 S_{l=0}.$$
The leading-order term of the asymptotics is a solution of the Cauchy problem for nonlinear Schrödinger equation

\[ 2i\partial_{t_2}\Phi\partial_{t_1}^{1,0}\Psi\Phi + \partial_{\xi}^{1,0}\Psi\Phi + \gamma|\Psi\Phi|^{2}\Psi\Phi = 0, \]

where the \( \xi \) is defined from

\[ \frac{dx_1}{d\xi} = \partial_{t_2}\Phi, \quad \frac{dt_1}{d\xi} = \partial_{x_2}\Phi. \]

The coefficients \( n_k, n_{k'} \Psi \pm \Phi \) are determined from Cauchy problems for linearized Schrödinger equation. The coefficients \( n_k, n_{k'} \Psi, \chi \in K_{n,k} \) are determined from algebraic equations. The set \( K_{n,k} = K_{n,k} \backslash \{ \pm \Phi \} \).

Theorem 1 follows from theorems 2, 3 and 4.

### 4.1 Structure of the second external asymptotics

Let us construct the formal asymptotic solution from theorem 4. Substitute (31) in original equation and collect the terms of the same order with respect to \( \varepsilon \). It yields \( N + 1 \) equations and residual of the order \( \varepsilon^{N+1} \). After collecting the terms with the same phase functions we obtain the recurrent system of equation for coefficients of (31).

Let us consider equations under \( \exp(i\Phi/\varepsilon^2) \). The terms of the order \( \varepsilon^1 \) give us the equation (32) for the phase function of eigen oscillations. The initial data is determined by matching condition and represented by value of driven phase \( S \) on the resonance curve \( l = 0 \)

\[ \Phi|_{t=0} = S|_{t=0}, \quad \partial_{t_2}\Phi|_{t=0} = \partial_{t_2}S|_{t=0}. \]

The terms of the order \( \varepsilon^2 \)

\[ 2i\left( \partial_{t_2}\Phi\partial_{t_1}^{1,0}\Psi\Phi - \partial_{x_2}\Phi\partial_{x_1}^{1,0}\Psi\Phi \right) = 0 \]

give us the homogeneous transport equation

\[ \partial_{t_2}\Phi\partial_{t_1}^{1,0}\Psi\Phi - \partial_{x_2}\Phi\partial_{x_1}^{1,0}\Psi\Phi = 0. \]  

This equation allows us to determine the dependence of the leading-order term on characteristic variable \( \zeta \). Equation (33) along the characteristics

\[ \frac{dx_1}{d\zeta} = -\partial_{x_2}\Phi, \quad \frac{dt_1}{d\zeta} = \partial_{t_2}\Phi \]  

(34)
can be written in the form of ordinary differential equation

$$\frac{d^{1.0}}{d\zeta} \Psi \phi = 0.$$  \hspace{1cm} (35)

It yields $\Psi \phi$ depends on $\xi$, where the $\xi$ is defined by

$$\frac{dx_1}{d\xi} = \partial_{t_2} \Phi, \quad \frac{dt_1}{d\xi} = \partial_{x_2} \Phi.$$

The terms of the order $\varepsilon^3$ which oscillates as $\exp(i\Phi/\varepsilon^2)$ are

$$2i \left( \partial_{t_2} \Phi \partial_{t_1} \Psi \phi - \partial_{x_2} \Phi \partial_{x_1} \Psi \phi \right) + \frac{2i\partial_{t_2} \Phi \partial_{t_2} \Psi \phi + [(\partial_{t_1} \xi)^2 - (\partial_{x_1} \xi)^2] \partial_{\xi \xi} \Psi \phi + i[\partial_{t_2}^2 \Phi - \partial_{x_2}^2 \Phi] \Psi \phi + \gamma] \Psi \phi \right|^{1.0} \Psi \phi = 0.$$

It is convenient to write this equation in the form of ordinary differential equation in terms of characteristic variables

$$\frac{d^{2.0}}{d\xi} \Psi \phi = -2i \partial_{t_2} \Phi \partial_{t_2} \Psi \phi - [(\partial_{t_1} \xi)^2 - (\partial_{x_1} \xi)^2] \partial_{\xi \xi} \Psi \phi + i[\partial_{t_2}^2 \Phi - \partial_{x_2}^2 \Phi] \Psi \phi + \gamma] \Psi \phi \right|^{1.0} \Psi \phi = 0. \hspace{1cm} (36)$$

Equation \ref{36} shows that the right hand side of equation \ref{36} does not depend on $\zeta$. To avoid secularities in the asymptotics we demand the right hand side of equation is equal to zero. It allows to determine the dependence of the leading-order term on slow variable $t_2$

$$2i \partial_{t_2} \Phi \partial_{t_2} \Psi \phi + [(\partial_{t_1} \xi)^2 - (\partial_{x_1} \xi)^2] \partial_{\xi \xi} \Psi \phi + i[\partial_{t_2}^2 \Phi - \partial_{x_2}^2 \Phi] \Psi \phi + \gamma] \Psi \phi \right|^{1.0} \Psi \phi = 0. \hspace{1cm} (37)$$

The equations for the higher-order terms are obtained by the same manner

$$2i \left( \partial_{t_2} \Phi \partial_{t_1} \Psi \phi - \partial_{x_2} \Phi \partial_{x_1} \Psi \phi \right) = 2i \partial_{t_2} \Phi \partial_{t_2} \Psi \phi - \partial_{\xi \xi} \Psi \phi - \gamma] \Psi \phi \right|^{1.0} \Psi \phi \right|^{1.0} \Psi \phi = 0,$$

where $k_1 + l_1 + m_1 = n + 2$, $k_2 + l_2 + m_2 = k$, $\alpha + \beta + \delta = \Phi$, $\alpha \in K_{k_1,k_2}$, $\beta \in K_{l_1,l_2}$, $\delta \in K_{m_1,m_2}$. 17
To construct the uniform asymptotic expansion with respect to $\zeta$ we obtain the linearized Schrödinger equation for higher-order term

$$2i\partial_{t_1}\Phi\partial_{t_2} n_{k}^{n} \Phi + \partial_{\xi}^{2} n_{k}^{n} \Phi + i[\partial_{t_2}^{2} \Phi - \partial_{x_2}^{2} \Phi] n_{k}^{n} \Phi =$$

$$- \partial_{t_1} \xi \partial_{\xi}^{2} n_{1,k}^{n-1} \Phi + \gamma \sum_{k_1,k_2,l_1,l_2,m_1,m_2,\alpha,\beta,\delta} k_1,k_2,l_1,l_2,m_1,m_2,\alpha,\beta,\delta \Psi_{\alpha}^{\beta} \Psi_{\delta},$$

(38)

where $k_1 + l_1 + m_1 = n + 2$, $k_2 + l_2 + m_2 = k$, $\alpha + \beta + \delta = \Phi$, $\alpha \in K_{k_1,k_2}$, $\beta \in K_{l_1,l_2}$, $\delta \in K_{m_1,m_2}$.

The amplitudes $\Psi_{\alpha}$ as $\chi \neq \pm \Phi$ are determined by algebraic equations

$$\left[-(\chi t_2)^2 + (\chi x_2)^2 + 1\right] n_{k}^{n} \Psi_{\chi} = F_{\chi}, \quad \chi \neq \pm \Phi.$$ 

(39)

Here the right hand side of the equation depends on previous terms and their derivatives

$$F_{\chi} = -2i\chi t_2 \partial_{t_1} n_{-3,k}^{n} \Psi_{\chi} + 2i\chi x_2 \partial_{x_1} n_{-4,k}^{n} \Psi_{\chi} - 2i\chi t_2 \partial_{t_1} n_{-2,k}^{n} \Psi_{\chi} - i[\chi t_2 - \chi x_2] n_{-2,k}^{n} \Psi_{\chi} -$$

$$\partial_{t_2}^{2} n_{-3,k}^{n} \Psi_{\chi} - \partial_{t_2}^{2} n_{-4,k}^{n} \Psi_{\chi} - \gamma \sum_{k_1,k_2,l_1,l_2,m_1,m_2,\alpha,\beta,\delta} k_1,k_2,l_1,l_2,m_1,m_2,\alpha,\beta,\delta \Psi_{\alpha}^{\beta} \Psi_{\delta},$$

where $k_1 + l_1 + m_1 = n - 4$, $k_2 + l_2 + m_2 = k$, $\alpha + \beta + \delta = \chi$, $\alpha \in K_{k_1,k_2}$, $\beta \in K_{l_1,l_2}$, $\delta \in K_{m_1,m_2}$.

These equations are similar to equations for amplitudes from pre-resonance section.

The obtained result is formulated below

**Lemma 9** The coefficients of formal asymptotic solution satisfy recurrent system of equations.

The right hand side of equation (38) has a singularity as $l \to 0$. The singularity appears due to $\Psi_{\chi}$ as $\chi \neq \pm \Phi$. The analysis of the right hand side of the equation allows us to calculate the order of singularity as $l \to 0$. It is equal to $O(l^{-(n-1)})$. Below we prove the solvability of equation (38) with the given asymptotics as $l \to 0$.

**Lemma 10** The asymptotics as $l \to 0$ of the solution of equation (38) has the form

$$n_{k}^{n} \Psi_{\phi}(x_1,t_1,t_2) = \sum_{j=-(n-2)}^{1} \sum_{m=0}^{j-1} n_{k}^{n} j^{m}(x_1,t_1) \mathcal{U}(\ln l)^{m} + O(1), \quad l \to 0. \quad (40)$$

**Proof.** Determine the order of the singularity of the right hand side of the equation as $l \to 0$. First consider equation (38) for $n = 3, k = 0$. The solution of this equation gives us the coefficient $3_{0}$ $\Psi_{\phi}$. The nonlinearity contains the term
The function $2^0 \Psi_S$ has the singularity of the order $l^{-1}$ as $l \to 0$. It determines the order of singularity for right hand side $l^{-2}$. We construct the asymptotics of $\Psi \chi$ in the form

$$3^0 \Psi \chi = 3^0 \Psi_S l^{-1} + 3^0 \Psi \ln(l) + 3^0 \lim l \ln(l) + 3^0 \Psi \phi,$$

(41)

Substitute (41) in equation for $n = 3$. It leads to recurrent system of equations for coefficients $3^0 (j,k)$

$$-2i \partial_2 \Psi \partial_2 l \Psi (1, 0) = - \Psi (2, 0) |2l|^2,$$

$$2i \partial_2 \Psi \partial_2 l \Psi (0, 1) = L[3^0 (1, 0)],$$

$$2i \partial_2 \Psi \partial_2 l \Psi (1, 1) = L[3^0 (1, 0)].$$

Here we denote the linear operator by

$$L[\Psi] = 2i \partial_2 \Psi \partial_2 l \Psi + \partial_2^2 \Psi + i[\partial_2^2 \Psi - \partial_2^2 \Psi] |2l|^2 \Psi + \gamma (2^0 \Psi \phi |2l|^2 + (\Psi \phi)^2 \Psi^*).$$

The regular part $\hat{3}^0 \Psi \phi$ of the asymptotics satisfies the nonhomogeneous linear Schrödinger equation. The right hand side of the equation is smooth

$$L[\hat{3}^0 \Psi \phi] = -l \ln |l| L[3^0 (1, 1)] - 2i \partial_2 \Psi \partial_2 l \Psi (1, 1).$$

The initial condition for the regular part of the asymptotics is determined below by matching with the internal asymptotic expansion.

The structure of the terms $\Psi \pm \Psi$ for $n > 3$ has a similar form. The right hand side of equation (38) depends on junior terms. These singularities can be eliminate

$$n^k \hat{F}_\phi = \sum_{j=0}^{-(n-2)-j+1} \sum_{m=0}^{n^k (j,m)} \beta m \ln |l| n^k (j,m) + \hat{F}_\phi.$$

The coefficients $n^k (j,m)$ do not contain singularities as $l \to 0$. These coefficients are easy calculated.

The direct substitution of (40) in equation and collecting the terms with the same order of $l$ complete the proof of lemma 10.

### 4.2 The domain of validity of the second external asymptotics and matching procedure

The domain of validity of the second external asymptotics is determined by

$$\varepsilon \frac{n+1}{n} \ll 1.$$
Formulas \(31\) and \(40\) give the condition
\[ l \ll \varepsilon. \]

The domain \(|l| \ll 1\) of validity of the internal asymptotics and domain of validity of the second external asymptotics are intersected. This fact allows to complete the construction of the second external asymptotics by matching method \[17\]. The structure of singular parts of the internal asymptotics as \(\lambda \to +\infty\) and external asymptotics as \(l \to 0\) are equivalent. The coefficients are coincided due to our constructions. The matching of regular parts of these asymptotics takes place due to

\[ n_{\Psi \Phi} |_{l=0} = \frac{n}{W^{(0,0)}(\xi)}. \]

The function \(W^{(0,0)}(\xi)\) is determined in lemma \[6\].

In particular, the initial condition for the leading-order term has a form

\[ \Psi_{\Psi \Phi} |_{l=0} = \int_{-\infty}^\infty d\sigma f(x_1) \exp(i \int_0^\sigma d\chi \lambda(x_1, t_1, \varepsilon)). \]

The soliton theory for nonlinear Schrodinger equation leads us to the fact that the function \(\Psi_{\Psi \Phi}\) contains the solitary waves when \(f(x_1)\) is sufficiently large.

Theorem \[4\] is proved.

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References

[1] L.Friedland, A.G.Shagalov. Exitetion of solitons by adiabatic multyresonant forcing. Phys.rev.lett., v.8, 20, 1998, pp.4357-4360.
[2] J. Kevorkyan, Passage through resonance for a one-dimensional oscillator with slowly varying frequency. SIAM J. Appl.Math., 1971, v.20, pp.364-373.
[3] Rubenfeld L. The passage of weakly coupled nonlinear oscillators through internal resonance. Stud. Appl. Math., 1977, v.57, pp.77-92.
[4] Neu J.C. Resonantly interacting waves. SIAM J. Appl. Math., 1983, v.43, n1, pp.141-156.
[5] L.A. Kalyakin, Lokal’nyi rezonans v slabonelineinykh zadachakh. Matematicheskie zametki, 1988, t.44, n5, pp.697-699.
[6] S.G.Glebov, O slabonelineinoi zadache s lokal’nym rezonansom. Differentsial’nye uravneniya, 1995, t.31, n8, pp.1402-1408.
[7] P.L. Kelley, Self-focusing of optical beams. Phys. Rev. Lett., 1965, v.15, pp.1005-1008.

[8] V.I. Talanov, O samofokusirovke mal'ykh puchkov v nelineinykh sredakh, Pis'ma v ZhETF, 1965, 2, pp.218-222.

[9] V.E. Zakharov, Ustoichivost' periodicheskikh voln s konechnoi amplitudoi na poverkhnosti glubokoi zhidkosti. Zhurnal prikladnoi mekhaniki i tekhnicheskoi fiziki, 1968, 2, pp.86-94.

[10] L.A. Kalyakin. Dlinnovolnovye asymptotiki.Integriruemye uravneniya kak asymptoticheskii predel nelineinykh sistem. Uspekhi matematicheskikh nauk. 1989, v.44, n1, pp.5-34.

[11] B.B. Kadomtsev, V.I. Petviashvili. Ob ustoichivosti uedinennykh voln v slabodispersiruyushchikh sredakh. DAN SSSR, 1970, t.194, n4, pp.753-756.

[12] V.E. Zakharov, S.V. Manakov, S.P. Novikov, L.P. Pitaevskii. Teoriya solitonov: metod obratnoi zadachi. M.: Nauka, 1980.

[13] N.V. Alexeeva, I.V. Barashenkov, D.E. Pelinovsky. Dynamics of the parametrically driven NLS solitons beyond the onset of the oscillatory instability. Nonlinearity, v.12, 1999, pp.103-140.

[14] R.R. Gadylshin, O.M. Kiselev, On solitonless structure of the perturbed soliton solution for the Davey-Stewartson equation, Theor. Math. Phys., 1996, v.106, pp.167-173.

[15] S.G. Glebov, O.M. Kiselev, V.A. Lazarev. Soliton generation by local resonance. Proceedings of the Steklov Institute of Mathematics. Suppl., 2003, issue 1, S84-S90.

[16] A. Jeffrey and T. Kawahara. Asymptotic methods in nonlinear wave theory. Pitman Publishing INC, 1982. 256 pp.

[17] A. M. Il'in Matching of Asymptotic Expansions of Solutions of Boundary Value Problem, AMS, 1992.