On holonomy of Weyl connections in Lorentzian signature

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Abstract

Connected holonomy groups of Weyl connections in Lorentzian signature are classified.

1 Introduction

The holonomy group of a connection is an important invariant. This motivates the classification problem for holonomy groups. There are classification results for some cases of linear connections. There is a classification of reducible connected holonomy groups of affine torsion-free connections [25]. Important result is a classification of connected holonomy groups of Riemannian manifolds [6, 7, 10, 22]. Lorentzian holonomy groups are classified [4, 24, 13, 3, 19, 16]. There are partial results for holonomy groups of pseudo-Riemannian manifolds of other signatures [8, 9, 5, 11, 18, 17, 16, 15, 21, 28, 29].

Of certain interests are Weyl manifolds \((M, c, \nabla)\), where \(c\) is a conformal class of pseudo-Riemannian metrics and \(\nabla\) is a torsion-free linear connection preserving \(c\). In the Riemannian signature the connected holonomy groups of such connection are classified [2, 20].

The result of this paper is a complete classification of connected holonomy groups of Weyl connections in the Lorentzian signature.

2 Preliminaries

Denote by \((M, c)\) a conformal manifold, where \(M\) is a smooth manifold, and \(c\) is a conformal class of pseudo-Riemannian metrics on \(M\). Recall that two metrics \(g\) and \(h\) are conformally equivalent if and only if \(h = e^f g\), for some \(f \in \mathcal{C}^\infty(M)\).

Definition 1. A Weyl connection \(\nabla\) on a conformal manifold \((M, c)\) is a torsion-free linear connection that preserves the conformal class \(c\). The triple \((M, c, \nabla)\) is called a Weyl manifold.

By preserving a conformal class, we understand that if \(g \in c\), then there exists a 1-form \(\theta_g\) such that

\[ \nabla g = \theta_g \otimes g. \]
This formula is conformally invariant in the following sense:

\[ \text{if } h = e^f g, \ f \in C^\infty(M), \ \text{then } \nabla h = \theta_h \otimes h, \ \text{where } \theta_h = \theta_g - df. \]

For the holonomy algebra of Weyl connection of signature \((r, s)\) we have \(\mathfrak{hol}(\nabla) \subset \mathfrak{co}(r, s) = \mathbb{R} \text{id} \oplus \mathfrak{so}(r, s)\). If for a metric \(g \in c\) it holds \(\nabla g = 0\), then \(\mathfrak{hol}(\nabla) \subset \mathfrak{so}(r, s)\). Then the conformal structure is called closed and we are not interested in this case. Thus we assume that \(\mathfrak{hol}(\nabla) \subset \mathfrak{co}(r, s)\) and \(\mathfrak{hol}(\nabla) \not\subset \mathfrak{so}(r, s)\). The classification of the holonomy algebras of Weyl connections of positive signature is known \([2]\). Namely, let \(n = \text{dim } M\), then only the following holonomy algebras of non-closed Weyl structures are possible:

- \(\mathfrak{co}(n)\);
- \(\mathbb{R} \text{id} \oplus \mathfrak{so}(k) \oplus \mathfrak{so}(n - k)\), where \(1 \leq k \leq n - 1\);
- \(\mathbb{R} \text{id} \oplus \left\{ \begin{pmatrix} 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & -a \\ 0 & 0 & a & 0 \end{pmatrix} \bigg| a \in \mathbb{R} \right\} (n = 4)\).

**Definition 2.** Let \(\mathfrak{g}\) be a subalgebra of \(\mathfrak{gl}(n, \mathbb{R})\). A linear map \(R : \mathbb{R}^n \wedge \mathbb{R}^n \to \mathfrak{g}\) satisfying the condition

\[ R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 \quad \forall X, Y, Z \in \mathbb{R}^n \ (\text{Bianchi identity}) \]

is called an algebraic curvature tensor of type \(\mathfrak{g}\).

Let \(\mathcal{R}(\mathfrak{g})\) be the vector space of all algebraic curvature tensors of type \(\mathfrak{g}\). Let

\[ L(\mathcal{R}(\mathfrak{g})) := \text{span}\{R(X, Y) \mid R \in \mathcal{R}(\mathfrak{g}), \ X, Y \in \mathbb{R}^n\} \subset \mathfrak{g}. \]

**Definition 3.** A subalgebra \(\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})\) is called a Berger algebra if \(L(\mathcal{R}(\mathfrak{g})) = \mathfrak{g}\).

The next theorem follows from the Ambrose-Singer Theorem \([1]\).

**Theorem 1.** If \(\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})\) is the holonomy algebra of a torsion-free linear connection, then \(\mathfrak{g}\) is a Berger algebra.

We will denote the Minkowski space by \(\mathbb{R}^{1,n+1}\), and by the \((\cdot, \cdot)\) the metric on it. A basis \(p, e_1, \ldots, e_n, q\) of the space \(\mathbb{R}^{1,n+1}\) is called a Witt basis if \(p\) and \(q\) are isotropic vectors such that \((p, q) = 1\), and \(e_1, \ldots, e_n\) is an orthonormal basis of a subspace \(\mathbb{R}^n\) which is orthogonal to the vectors \(p\) and \(q\).

Denote by \(\mathfrak{co}(1, n + 1)_{\mathbb{R}p}\) the subalgebra of \(\mathfrak{co}(1, n + 1)\) preserving an isotropic line \(\mathbb{R}p\). It is clear that

\[ \mathfrak{co}(1, n + 1)_{\mathbb{R}p} = \mathbb{R} \text{id} \oplus \mathfrak{so}(1, n + 1)_{\mathbb{R}p}, \]
where \( \mathfrak{so}(1, n + 1)_{\mathbb{R}p} \) is the subalgebra of \( \mathfrak{so}(1, n + 1) \) which preserves \( \mathbb{R}p \). We can identify the Lie algebra \( \mathfrak{so}(1, n + 1)_{\mathbb{R}p} \) with the following matrix Lie algebra:

\[
\mathfrak{so}(1, n + 1)_{\mathbb{R}p} = \left\{ (a, A, X) \in \mathbb{R}^3 \mid a \in \mathbb{R}, A \in \mathfrak{so}(n), X \in \mathbb{R}^n \right\}.
\]

The non-zero brackets in \( \mathfrak{so}(1, n + 1)_{\mathbb{R}p} \) are:

\[
[(a, 0, 0), (0, 0, X)] = (0, 0, aX), \quad [(0, A, 0), (0, 0, X)] = (0, 0, AX),
\]

\[
[(0, A, 0), (0, B, 0)] = (0, [A, B], 0).
\]

We get the decomposition

\[
\mathfrak{so}(1, n + 1)_{\mathbb{R}p} = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n.
\]

An element of \( \mathfrak{co}(1, n + 1)_{\mathbb{R}p} \) will be denoted by \( (b, a, A, X) \), where \( b \in \mathbb{R} \) and \( (a, A, X) \in \mathfrak{so}(1, n + 1)_{\mathbb{R}p} \).

Recall that each subalgebra \( \mathfrak{h} \subset \mathfrak{so}(n) \) is compact and there exists the decomposition

\[
\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{z}(\mathfrak{h}),
\]

where \( \mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}] \) is the commutant of \( \mathfrak{h} \), and \( \mathfrak{z}(\mathfrak{h}) \) is the center of \( \mathfrak{h} \) [26].

**Definition 4.** A Lie subalgebra \( \mathfrak{g} \subset \mathfrak{so}(r, s) \) (or \( \mathfrak{g} \subset \mathfrak{co}(r, s) \)) is called weakly irreducible if it does not preserve any proper non-degenerate vector subspace of \( \mathbb{R}^{r,s} \).

The following theorem describes weakly irreducible subalgebras of \( \mathfrak{so}(1, n + 1)_{\mathbb{R}p} \) and belongs to Bérand-Bergery and Ikemakhen [4].

**Theorem 2.** A subalgebra \( \mathfrak{g} \subset \mathfrak{so}(1, n + 1)_{\mathbb{R}p} \) is weakly irreducible if and only if \( \mathfrak{g} \) is a Lie algebra of one of the following types.

**Type 1:**

\[
\mathfrak{g}^1_{\mathfrak{h}} = (\mathbb{R} \oplus \mathfrak{h}) \ltimes \mathbb{R}^n = \left\{ \begin{pmatrix} a & X^t & 0 \\ 0 & A & -X \\ 0 & 0 & -a \end{pmatrix} \mid a \in \mathbb{R}, X \in \mathbb{R}^n, A \in \mathfrak{h} \right\},
\]

where \( \mathfrak{h} \subset \mathfrak{so}(n) \) is a subalgebra.

**Type 2:**

\[
\mathfrak{g}^2_{\mathfrak{h}} = \mathfrak{h} \ltimes \mathbb{R}^n = \left\{ \begin{pmatrix} 0 & X^t & 0 \\ 0 & A & -X \\ 0 & 0 & 0 \end{pmatrix} \mid X \in \mathbb{R}^n, A \in \mathfrak{h} \right\},
\]

where \( \mathfrak{h} \subset \mathfrak{so}(n) \) is a subalgebra.
Type 3:

\[
g^{3,h,\varphi} = \{(\varphi(A), A, 0) \mid A \in \mathfrak{h}\} \times \mathbb{R}^n = \left\{ \begin{pmatrix} \varphi(A) & X^t & 0 \\ 0 & A & -X \\ 0 & 0 & -\varphi(A) \end{pmatrix} \mid X \in \mathbb{R}^n, A \in \mathfrak{h} \right\},
\]

where \( h \subset \mathfrak{so}(n) \) is a subalgebra satisfying the condition \( \mathfrak{j}(h) \neq \{0\} \), and \( \varphi : h \to \mathbb{R} \) is a non-zero linear map with the property \( \varphi|_{[h, h]} = 0 \).

Type 4:

\[
g^{4,h,m,\psi} = \{(0, A, X + \psi(A)) \mid A \in \mathfrak{h}, X \in \mathbb{R}^m \} = \left\{ \begin{pmatrix} 0 & X^t & \psi(A)^t & 0 \\ 0 & A & 0 & -X \\ 0 & 0 & 0 & -\psi(A) \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid X \in \mathbb{R}^m, A \in \mathfrak{h} \right\},
\]

where an orthogonal decomposition \( \mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^{n-m} \) is fixed, \( h \subset \mathfrak{so}(m) \), \( \dim \mathfrak{j}(h) \geq n - m \), and \( \psi : h \to \mathbb{R}^{n-m} \) is a surjective linear map with the property \( \psi|_{[h, h]} = 0 \).

3 Main results

Here we classify holonomy algebras \( g \subset \mathfrak{co}(1, n+1) \) such that \( g \not\subset \mathfrak{so}(1, n+1), n \geq 0 \).

First suppose that \( g \subset \mathfrak{co}(1, n+1) \) is irreducible. Then it is obvious that \( \text{pr}_{\mathfrak{so}(1,n+1)} g \subset \mathfrak{so}(1, n+1) \) is irreducible as well. Therefore, \( \text{pr}_{\mathfrak{so}(1,n+1)} g = \mathfrak{so}(1, n+1) \), since \( \mathfrak{so}(1, n+1) \) does not have any proper irreducible algebra [27]. Thus, \( g = \text{id} \oplus \mathfrak{so}(1, n+1) \).

Next, let us suppose that \( g \) preserves a non-degenerate subspace of \( \mathbb{R}^{1,n+1} \).

**Theorem 3.** Let \( g \subset \text{id} \oplus \mathfrak{so}(1, n+1) \) be a Berger algebra which is not weakly irreducible. Then \( g \) preserves an orthogonal decomposition \( \mathbb{R}^{1,n+1} = \mathbb{R}^{1,k+1} \oplus \mathbb{R}^{n-k}, 1 \leq k \leq n-1 \) and \( g \) is conjugated to one of the following subalgebras:

- \( g = \mathfrak{so}(1, k+1) \oplus \mathfrak{so}(n-k) \oplus \text{id}, 1 \leq k \leq n-1; \)
- \( g = (\mathbb{R}(1,-1,0,0) \oplus \mathfrak{k} \times \mathbb{R}^k) \oplus \mathfrak{so}(n-k), \) where \( \mathfrak{k} \subset \mathfrak{so}(k) \) is the holonomy algebra of a Riemannian manifold.

Now, we may assume that \( g \) does not preserve any non-degenerate subspace of \( \mathbb{R}^{1,n+1} \) and it is not irreducible, i.e., it is weakly irreducible and not irreducible. Let \( g \) preserve a degenerate subspace \( W \subset \mathbb{R}^{1,n+1} \). That means that \( g \) preserves the isotropic line \( W \cap W^\perp \). We fix a Witt basis \( p, e_1, \ldots, e_n, q \) in such a way that \( W \cap W^\perp = \mathbb{R}p \).
Theorem 4. A subalgebra $g \subset \mathfrak{co}(1, n+1)$, $g \not\subset \mathfrak{so}(1, n+1)$, is a weakly irreducible, not irreducible Berger algebra if and only if $g$ is conjugated to one of the following subalgebras of the Lie algebra $\mathfrak{co}(1, n+1)_{\mathbb{R}p}$:

- $\mathbb{R} \text{id} \oplus g^{1,h}$, $\mathbb{R} \text{id} \oplus g^{2,h}$, $\mathbb{R} \text{id} \oplus g^{3,h,\varphi}$, where $g^{1,h}$, $g^{2,h}$, $g^{3,h,\varphi}$ are from Theorem 2, and $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a Riemannian manifold;

- $g^{6,1,h} = \{(\theta(a, A), a, A, 0) \mid a \in \mathbb{R}, A \in \mathfrak{h}\} \times \mathbb{R}^n$, where $\theta : \mathbb{R} \oplus \mathfrak{h} \to \mathbb{R}$ is a non-zero map such that $\theta|_{[\h,\mathfrak{h}]} = 0$, and $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a Riemannian manifold;

- $g^{6,2,h} = \{(\theta(A), A, 0) \mid A \in \mathfrak{h}\} \times \mathbb{R}^n$, where $\theta : \mathfrak{h} \to \mathbb{R}$ is a non-zero map such that $\theta|_{[\h,\mathfrak{h}]} = 0$, and $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a Riemannian manifold;

- $g^{6,3,h,\varphi} = \{(\theta(A), \varphi(A), A, 0) \mid A \in \mathfrak{h}\} \times \mathbb{R}^n$, where $\theta : \mathfrak{h} \to \mathbb{R}$ and $\varphi : \mathfrak{h} \to \mathbb{R}$ are non-zero maps such that $\theta|_{[\h,\mathfrak{h}]} = 0$ and $\varphi|_{[\h,\mathfrak{h}]} = 0$, and $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a Riemannian manifold.

Below, in Section 9, for each Berger algebra $g$, we construct a Weyl connection with the holonomy algebra isomorphic to $g$.

Theorem 5. Each algebra $g \subset \mathfrak{co}(1, n+1)$ from Theorems 3 and 4 is the holonomy algebra of a Weyl connection.

Thus, for non-closed conformal structures of Lorentzian signature we obtained a complete classification of holonomy algebras. These algebras are exhausted by the Lie algebra $\mathfrak{co}(1, n+1)$ and subalgebras $g \subset \mathfrak{co}(1, n+1)$ from Theorems 3 and 4.

4 Weakly irreducible subalgebras of $\mathfrak{co}(1, n+1)_{\mathbb{R}p}$

Let $g \subset \mathfrak{co}(1, n+1)_{\mathbb{R}p}$ be a subalgebra. It is obvious that $g$ is weakly irreducible if and only if the projection of $g$ to $\mathfrak{so}(1, n+1)_{\mathbb{R}p}$ is weakly irreducible. Let $\overline{g} = \text{pr}_{\mathfrak{so}(1, n+1)_{\mathbb{R}p}} g$. It is clear that if $g$ is weakly irreducible and is not contained in $\mathfrak{so}(1, n+1)_{\mathbb{R}p}$, then only the following two situations are possible:

(a) $g = \mathbb{R} \text{id} \oplus \overline{g}$,

(b) $g = \{\theta(B) \text{id} + B \mid B \in \overline{g}\}$, where $\theta : \overline{g} \to \mathbb{R}$ is a non-zero linear map. Since $g$ is a Lie algebra, it holds $\theta|_{\overline{g} \overline{g}} = 0$.

Consider the case (b) in more details. Now we describe the map $\theta$ for each Lie algebra from Theorem 2.

Case b.1:
Let $\mathfrak{g} = \mathfrak{g}^{1,h} = (\mathbb{R} \oplus \mathfrak{h}) \ltimes \mathbb{R}^n$ be a subalgebra, where $\mathfrak{h} \subset \mathfrak{so}(n)$. From (1) it follows that $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{h}, \mathfrak{h}] \ltimes \mathbb{R}^n$. Thus we obtain the Lie algebra

$$\mathfrak{g} = \{(\theta(a, A), a, A, 0) \mid a \in \mathbb{R}, A \in \mathfrak{h}\} \ltimes \mathbb{R}^n,$$

where $\theta : \mathbb{R} \oplus \mathfrak{h} \to \mathbb{R}$ is a non-zero linear map such that $\theta\big|_{[\mathfrak{h}, \mathfrak{h}]} = 0$.

**Case b.2:**

Let $\mathfrak{g} = \mathfrak{h} \ltimes \mathbb{R}^n$. Consider the $\mathfrak{h}$-invariant orthogonal decomposition

$$\mathbb{R}^n = \mathbb{R}^{n_0} \oplus \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_r}$$

such that $\mathfrak{h}$ annihilates $\mathbb{R}^{n_0}$, and $\mathbb{R}^{n_\alpha}$, $\alpha = 1, \ldots, r$, are $\mathfrak{h}$-invariant and the induced representations in them are irreducible. Thus, $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{h}, \mathfrak{h}] \ltimes (\mathbb{R}^{n_0})^\perp$, where $(\mathbb{R}^{n_0})^\perp = \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_r}$. We get

$$\mathfrak{g} = \{(\theta(X, A), 0, A, X) \mid X \in \mathbb{R}^{n_0}, A \in \mathfrak{h}\} \ltimes (\mathbb{R}^{n_0})^\perp,$$

where $\theta : \mathbb{R}^{n_0} \oplus \mathfrak{h} \to \mathbb{R}$ is a non-zero linear map such that $\theta\big|_{[\mathfrak{h}, \mathfrak{h}]} = 0$.

**Case b.3:**

As in the case b.1, we get

$$\mathfrak{g} = \{(\theta(A), \varphi(A), A, 0) \mid A \in \mathfrak{h}\} \ltimes \mathbb{R}^n,$$

where $\theta : \mathfrak{h} \to \mathbb{R}$ is a non-zero linear map such that $\theta\big|_{[\mathfrak{h}, \mathfrak{h}]} = 0$.

**Case b.4:**

We will not use this Lie algebra, by that reason we will not describe it.

## 5 Auxiliary results

Let $\mathbb{R}^{r,s}$ be a pseudo-Euclidean space. We will identify the space of bivectors $\wedge^2\mathbb{R}^{r,s}$ with the Lie algebra $\mathfrak{so}(r, s)$ in such a way that

$$(X \wedge Y)Z = (X, Z)Y - (Y, Z)X.$$

**Definition 5.** For a subalgebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ its first prolongation is defined as:

$$\mathfrak{g}^{(1)} := \{\varphi \in \text{Hom}(\mathbb{R}^n, \mathfrak{g}) \mid \varphi(X)Y = \varphi(Y)X\}.$$

It is well-known that $(\mathfrak{so}(r, s))^{(1)} = 0$ and

$$(\mathfrak{so}(r, s) \oplus \mathbb{R} \text{id})^{(1)} = \{Z \wedge \cdot + (Z, \cdot) \text{id} \mid Z \in \mathbb{R}^{r,s}\} \cong \mathbb{R}^{r,s}. \quad (2)$$

**Lemma 1.** Let $\mathfrak{h}$ be a proper irreducible subalgebra of $\mathfrak{so}(r, s)$, then $(\mathfrak{h} \oplus \mathbb{R} \text{id})^{(1)} = 0$. If $\mathfrak{h}$ is a proper subalgebra of $\mathfrak{so}(n)$, then $(\mathfrak{h} \oplus \mathbb{R} \text{id})^{(1)} = 0.$
Proof. Let \( \mathfrak{h} \subset \mathfrak{so}(r, s) \) be an irreducible subalgebra. Suppose that \((\mathfrak{h} \oplus \mathbb{R} \text{id})^{(1)} \neq 0. \) Since

\[
(\mathfrak{h} \oplus \mathbb{R} \text{id})^{(1)} \subset (\mathfrak{so}(r, s) \oplus \mathbb{R} \text{id})^{(1)} \cong \mathbb{R}^{r,s}
\]

is an \( \mathfrak{h} \)-submodule, \((\mathfrak{h} \oplus \mathbb{R} \text{id})^{(1)} = \mathbb{R}^{r,s}. \) From (2) it follows that \( \mathfrak{h} = \mathfrak{so}(r, s). \)

Now suppose that \( \mathfrak{h} \subset \mathfrak{so}(n) \) is not irreducible. Then \( \mathfrak{h} \) preserves an orthogonal decomposition \( \mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}, \) for all \( k \leq n-1. \) Let \( \varphi \in (\mathfrak{h} \oplus \mathbb{R} \text{id})^{(1)}. \) From (2) it follows that there exists an element \( Z \in \mathbb{R}^n \) such that

\[
\varphi(X) = Z \wedge X + (Z, X) \text{id}
\]

for all \( X \in \mathbb{R}^n. \) Let \( X \in \mathbb{R}^k \) be non-zero. Since \( \varphi(X) \in \mathfrak{h} \oplus \mathbb{R} \text{id}, \) we get \( Z \in \mathbb{R}^k. \) Similarly, taking non-zero \( X \in \mathbb{R}^{n-k}, \) we get \( Z \in \mathbb{R}^{n-k}. \) This shows that \( Z = 0. \)

Lemma 2. It holds

\[
(\mathfrak{so}(1, n+1)_{\mathbb{R}p} \oplus \mathbb{R} \text{id})^{(1)} = \mathbb{R}(p \wedge \cdot + (p, \cdot) \text{id}) \cong \mathbb{R}p.
\]

Proof. As we have seen,

\[
(\mathfrak{so}(1, n+1) \oplus \mathbb{R} \text{id})^{(1)} = \{\varphi_Z = Z \wedge \cdot + (Z, \cdot) \text{id} \mid Z \in \mathbb{R}^{1,n+1}\}.
\]

The Lie algebra \( \mathfrak{so}(1, n+1)_{\mathbb{R}p} \) is spanned by the elements \( p \wedge q, p \wedge X \) and \( X \wedge Y, \) where \( X, Y \in \mathbb{R}^n. \) Let \( \varphi_Z \in (\mathfrak{so}(1, n+1) \oplus \mathbb{R} \text{id})^{(1)} \) and \( X \in \mathbb{R}^n. \) Then

\[
\varphi_Z(X) = Z \wedge X + (Z, X) \text{id},
\]

\[
\varphi_Z(q) = Z \wedge q + (Z, q) \text{id}.
\]

From the first equation it follows that \( Z \in \langle p, e_1, \ldots, e_n \rangle. \) From this and the second equation it follows that \( Z \in \mathbb{R}p. \)

Corollary 1. Let \( \mathfrak{f} \subset \mathfrak{so}(1, n+1)_{\mathbb{R}p} \oplus \mathbb{R} \text{id} \) be a subalgebra, then \( \mathfrak{f}^{(1)} \neq 0 \) if and only if \( \mathbb{R}(p \wedge q + \text{id}) \oplus \mathbb{R}^n \subset \mathfrak{f}. \)

Theorem 6. Let \( V \) be a pseudo-Euclidean space with an orthogonal decomposition \( V = V_1 \oplus V_2. \) Then the following relation is true:

\[
\mathcal{R}(\mathfrak{so}(V_1) \oplus \mathfrak{so}(V_2) \oplus \mathbb{R} \text{id}) \cong \mathcal{R}(\mathfrak{so}(V_1)) \oplus \mathcal{R}(\mathfrak{so}(V_2)) \oplus V_1 \otimes V_2.
\]

Proof. Take \( X_1, Y_1 \in V_1, X_2 \in V_2 \) and consider the Bianchi identity for \( R: \wedge^2 V \to \mathfrak{so}(V_1) \oplus \mathfrak{so}(V_2) \oplus \mathbb{R} \text{id}: \)

\[
R(X_1, Y_1)X_2 + R(Y_1, X_2)X_1 + R(X_2, X_1)Y_1 = 0
\]

(3)

It follows that \( R(X_1, Y_1)X_2 = 0 \) and \( R(X_1, Y_1) \in \mathfrak{so}(V_1). \) Since the Bianchi identity holds for vectors from \( V_1, R \mid_{\wedge^2 V_1} \in \mathcal{R}(\mathfrak{so}(V_1)). \) Similarly, \( R \mid_{\wedge^2 V_2} \in \mathcal{R}(\mathfrak{so}(V_2)). \)
Consider the equality

\[ R(X_2, Y_1)X_1 = R(X_2, X_1)Y_1. \]

Fix \( X_2 \), then it is easy to see, that

\[ \varphi_{X_2}(\cdot) := R(X_2, \cdot) \in (\mathfrak{so}(V_1) \oplus \mathbb{R} \text{id})^{(1)}. \]

Let \( R_3 = R|_{V_1 \otimes V_2} \), then

\[ R_3 : V_2 \rightarrow (\mathfrak{so}(V_1) \oplus \mathbb{R} \text{id})^{(1)} \cong V_1 \]

and \( R_3 \in V_1 \otimes V_2 \) (\( V_1 \otimes V_2 \) is irreducible \( (\mathfrak{so}(V_1) \oplus \mathfrak{so}(V_2)) \)-representation).

Thus, \( R = R_1 + R_2 + R_3 \), where \( R_1 = R|_{\wedge^2 V_1} \), \( R_2 = R|_{\wedge^2 V_2} \), and \( R_3 = R|_{V_1 \otimes V_2} \). Considering the tensor \( R_3 : V_1 \otimes V_2 \rightarrow \mathfrak{so}(V_1) \oplus \mathfrak{so}(V_2) \oplus \mathbb{R} \text{id} \) and fixing \( X_2 \in V_2 \), we obtain:

\[ R_3(\cdot, X_2)|_{V_1} \in (\mathfrak{so}(V_1) \oplus \mathbb{R} \text{id})^{(1)}, \quad R_3(\cdot, X_2)|_{V_1} = Z(X_2) \wedge |_{V_1} + (Z(X_2), \cdot) \text{id}|_{V_1}, \]

where \( Z : V_2 \rightarrow V_1 \) is a linear map. Similarly, by fixing \( X_1 \in V_1 \), we obtain:

\[ R_3(X_1, \cdot)|_{V_2} \in (\mathfrak{so}(V_2) \oplus \mathbb{R} \text{id})^{(1)}, \quad R_3(X_1, \cdot)|_{V_2} = W(X_1) \wedge |_{V_2} + (W(X_1), \cdot) \text{id}|_{V_2}, \]

where \( W : V_1 \rightarrow V_2 \) is a linear map.

For arbitrary \( X_1, Y_1 \in V_1 \), \( X_2, Y_2 \in V_2 \), we get following:

\[ R_3(X_1, X_2)Y_1 = (Z(X_2) \wedge X_1)Y_1 + (Z(X_2), X_1)Y_1, \]
\[ R_3(X_1, X_2)Y_2 = (W(X_1) \wedge X_2)Y_2 + (W(X_1), X_2)Y_2. \]

From the last two equations we conclude that \( (Z(X_2), X_1) = (W(X_1), X_2) \), i.e., \( W = Z^* \). The map \( Z : V_2 \rightarrow V_1 \) defines \( R_3 \) by the formula:

\[ R_3(X_1, X_2) = Z(X_2) \wedge X_1 + Z^*(X_1) \wedge X_2 + (Z(X_2), X_1) \text{id}, \quad R_3|_{\wedge^2 V_1} = 0, \quad R_3|_{\wedge^2 V_2} = 0. \]

\[ \square \]

**Corollary 2.** Let \( \mathfrak{h}_1 \) be a proper irreducible subalgebra of \( \mathfrak{so}(V_1) \), and \( \mathfrak{h}_2 \subset \mathfrak{so}(V_2) \), then \( \mathcal{A}(\mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathbb{R} \text{id}) = \mathcal{A}(\mathfrak{h}_1 \oplus \mathfrak{h}_2) \).

**Proof.** Acting in the same way as in Theorem 6, we get \( R = R_1 + R_2 + R_3 \), where \( R_1 = R|_{V_1 \times V_1} \in \mathcal{A}(\mathfrak{h}_1) \), \( R_2 = R|_{V_2 \times V_2} \in \mathcal{A}(\mathfrak{h}_2) \) and \( R_3 = R|_{V_1 \otimes V_2} \). According to Lemma 1 it holds \( R_3(\cdot, X_2) \in (\mathfrak{h}_1 \oplus \mathbb{R} \text{id})^{(1)} = 0. \)

\[ \square \]

**Corollary 3.** Let \( \mathbb{R}^{r,s} = V_1 \oplus V_2 \oplus V_3 \), then

\[ \mathcal{A}(\mathfrak{so}(V_1) \oplus \mathfrak{so}(V_2) \oplus \mathfrak{so}(V_3) \oplus \mathbb{R} \text{id}) = \mathcal{A}(\mathfrak{so}(V_1)) \oplus \mathcal{A}(\mathfrak{so}(V_2)) \oplus \mathcal{A}(\mathfrak{so}(V_3)). \]
6 Algebraic curvature tensors

For subalgebra \( \mathfrak{h} \subset \mathfrak{so}(n) \) consider the following space:

\[
P(\mathfrak{h}) = \{ P \in \text{Hom}(\mathbb{R}^n, \mathfrak{h}) \mid g(P(X)Y, Z) + g(P(Y)Z, X) + g(P(Z)X, Y) = 0, \; X, Y, Z \in \mathbb{R}^n \}
\]

defined in [24], see also [14]. The space \( P(\mathfrak{h}) \) is called the space of weak curvature tensors of type \( \mathfrak{h} \).

**Theorem 7.** Every algebraic curvature tensor \( R \in \mathfrak{R}(\mathfrak{co}(1, n + 1)_{\mathbb{R}^p}) \) is uniquely determined by the elements:

\[
\begin{align*}
\mu, \lambda \in \mathbb{R}, \quad X_0, Z_0 \in \mathbb{R}^n, \quad \gamma \in \text{Hom}(\mathbb{R}^n, \mathbb{R}), \quad P \in P(\mathfrak{so}(n)), \quad K \in \odot^2 \mathbb{R}^n, \\
S + \tau \text{id} \in \mathfrak{R}(\mathfrak{co}(n)), \quad \text{where} \quad S \in \text{Hom}(\wedge^2 \mathbb{R}^n, \mathfrak{so}(n)), \quad \tau \in \text{Hom}(\wedge^2 \mathbb{R}^n, \mathbb{R})
\end{align*}
\]

by the equalities

\[
\begin{align*}
R(p, q) &= (\mu, \lambda, A_0, X_0), \\
R(p, V) &= ((Z_0, V), (Z_0, V), \gamma(V)) = -(A_0 + \mu E_n)V, \\
R(U, V) &= ((A_0U, V), (A_0U, V), S(U, V), L(U, V)), \\
R(U, q) &= (\gamma(U), (U, X_0) - \gamma(U), P(U), K(U)),
\end{align*}
\]

where \( A_0 \in \mathfrak{so}(n) \) is defined from the condition \( \tau(U, V) = (A_0U, V) \), and

\[
L(U, V) = P(V)U + \gamma(V)U - P(U)V - \gamma(U)V.
\]

The idea of the proof is the same as for the proof of a similar theorem from [12]. The decomposition \( \mathbb{R}^{1,n+1} = \mathbb{R}p \oplus \mathbb{R}^n \oplus \mathbb{R}q \) determines the decomposition of the space \( \wedge^2 \mathbb{R}^{1,n+1} \).

For \( R : \wedge^2 \mathbb{R}^{1,n+1} \to \mathfrak{co}(1, n + 1)_{\mathbb{R}^p} = \mathbb{R} \text{id} \oplus (\mathbb{R} \oplus \mathfrak{so}(n) \ltimes \mathbb{R}^n) \) one can consider the restrictions and projections and get various linear operators. It remains to rewrite the Bianchi identity in terms of these operators and solve the problem from linear algebra.

**Theorem 8.** Let \( \mathfrak{g} = \mathbb{R} \text{id} \oplus (\mathbb{R} \oplus \mathfrak{h}) \ltimes \mathbb{R}^n \), where \( \mathfrak{h} \subset \mathfrak{so}(n) \) is a proper subalgebra, and \( R \in \mathfrak{R}(\mathfrak{g}) \), then \( R \) satisfies Theorem 7 with the following additional constraints:

\[
Z_0 = 0, \quad \tau = 0, \quad A_0 = 0, \quad S \in \mathfrak{R}(\mathfrak{h}), \quad P \in P(\mathfrak{h}).
\]

**Proof.** Suppose that \( \mathfrak{g} = \mathbb{R} \text{id} \oplus (\mathbb{R} \oplus \mathfrak{h}) \ltimes \mathbb{R}^n \) and take \( R \) as in Theorem 7. The condition \( R \in \mathfrak{R}(\mathfrak{g}) \) is equivalent to the conditions \( R \in \mathfrak{R}(\mathfrak{co}(1, n + 1)_{\mathbb{R}^p}) \) and \( R(X, Y) \in \mathfrak{g} \) for all \( X, Y \in \mathbb{R}^{1,n+1} \). It follows that

\[
A_0 \in \mathfrak{h}, \quad Z_0 \wedge V \in \mathfrak{h}, \quad S(U, V) \in \mathfrak{h}, \quad P \in P(\mathfrak{h}) \quad \text{for all} \; U, V \in \mathbb{R}^n.
\]

**Case 1.** Suppose that \( \mathfrak{h} \subset \mathfrak{so}(n) \) is a proper irreducible subalgebra. Note that \( Z_0 \wedge V + (Z_0, V) \text{id} \in (\mathfrak{h} \oplus \mathbb{R} \text{id})^{(1)} \). According to Lemma 1, \( (\mathfrak{h} \oplus \mathbb{R} \text{id})^{(1)} = 0 \), therefore \( Z_0 = 0 \). Since
$S(U, V) \in \mathfrak{h}$ we get, that $S + \tau E_n \in \mathcal{R}(\mathfrak{h} \oplus \mathbb{R} E_n)$. From the classification from [25] it follows that $\mathcal{R}(\mathfrak{h} \oplus \mathbb{R} E_n) = \mathcal{R}(\mathfrak{h})$. Hence $\tau = 0$ and $A_0 = 0$.

**Case 2.** Suppose that $\mathfrak{h}$ is not irreducible. Then there exists $\mathfrak{h}$-invariant decomposition $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$. Let $\mathfrak{h} = \mathfrak{so}(k) \oplus \mathfrak{so}(n-k)$. Then, according to Theorem 6:

$$ (S + \tau \text{id})(X_1, X_2) = Z(X_2) \wedge X_1 + Z^*(X_1) \wedge X_2 + (Z(X_2), X_1) \text{id}. $$

Now, 

$$ \tau(X_1, X_2) = (Z(X_2), X_1) = (A_0 X_1, X_2) $$

and, hence, $A_0 X_1 = Z^*(X_1)$. Also, from the equality

$$ (A_0 X_2, X_1) = -(A_0 X_1, X_2) = -(Z(X_2), X_1), $$

we obtain $A_0 X_2 = -Z(X_2)$. But $A_0 \in \mathfrak{h}$, consequently $Z = 0$ and so $A_0 = 0$ and $\tau = 0$. The same result is true for an arbitrary subalgebra of $\mathfrak{so}(k) \oplus \mathfrak{so}(n-k)$.

**Theorem 9.** Let $\mathfrak{g} \subset \mathfrak{so}(1, n + 1)_{\mathbb{R}}$ be a subalgebra and let $\mathfrak{h} = \text{pr}_{\mathfrak{so}(n)} \mathfrak{g}$. If $\mathfrak{g}$ is a Berger subalgebra then $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a Riemannian manifold.

**Proof.** If $\mathfrak{h} = \mathfrak{so}(n)$, then $\mathfrak{h}$ is the holonomy algebra of a Riemannian manifold. Now suppose that $\mathfrak{h} \not\subset \mathfrak{so}(n)$, in this case, from Theorem 8 follows that $A_0 = 0$ and $Z_0 = 0$. Thus $\mathfrak{h}$ is generated by the images of the elements $S \in \mathcal{R}(\mathfrak{h})$, $P \in \mathcal{P}(\mathfrak{h})$. Therefore $\mathfrak{h}$ is the holonomy algebra of a Riemannian manifold according to [24].

### 7 Proof of Theorem 3

By the assumption of the theorem we have $\mathfrak{g}$-invariant decomposition

$$ \mathbb{R}^{1,n+1} = \mathbb{R}^{1,k+1} \oplus \mathbb{R}^{n-k}, $$

i.e.,

$$ \mathfrak{g} \subset \mathfrak{so}(1, k + 1) \oplus \mathfrak{so}(n - k) \oplus \mathbb{R} \text{id}. $$

First suppose, that $\text{pr}_{\mathfrak{so}(1,k+1)} \mathfrak{g}$ is irreducible. Then, $\text{pr}_{\mathfrak{so}(1,k+1)} \mathfrak{g} = \mathfrak{so}(1, k + 1)$. Consider the ideal $\mathfrak{a} = \mathfrak{g} \cap \mathfrak{so}(n-k) \subset \text{pr}_{\mathfrak{so}(n-k)} \mathfrak{g}$. Since $\mathfrak{so}(n-k)$ is a reductive Lie algebra, there exists an ideal $\mathfrak{b} \subset \text{pr}_{\mathfrak{so}(n-k)} \mathfrak{g}$ and a map $\varphi : \mathfrak{so}(1, n+1) \to \mathfrak{b} = \text{im} \varphi$ such that

$$ \text{pr}_{\mathfrak{so}(1,n+1)} = \{ A + \varphi(A) \mid A \in \mathfrak{so}(1, k + 1) \} \oplus \mathfrak{a}. $$

The algebra $\mathfrak{so}(1, k + 1)$ is simple for $k \geq 1$, so either $\text{ker} \varphi = 0$ or $\varphi = 0$. If $\text{ker} \varphi = 0$, then $\varphi$ is an isomorphism, which is impossible, since $\mathfrak{b} \subset \mathfrak{so}(n-k)$ is a compact Lie algebra, while $\mathfrak{so}(1, k + 1)$ is not compact. Thus, $\varphi = 0$ and, hence, $\text{pr}_{\mathfrak{so}(1,n+1)} = \mathfrak{so}(1, k + 1) \oplus \mathfrak{a}$. Applying Theorem 6 and Lemma 1 to $\mathfrak{a} \subset \mathfrak{so}(n-k)$, we obtain, that $\mathfrak{a} = \mathfrak{so}(n-k)$, therefore, $\mathfrak{g} = \mathfrak{so}(1, k + 1) \oplus \mathfrak{so}(n-k) \oplus \mathbb{R} \text{id}$. 

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Now suppose that \( \text{pr}_{\mathfrak{so}(1,k+1)} \mathfrak{g} \) is not irreducible. By Corollary 3, \( \text{pr}_{\mathfrak{so}(1,k+1)} \mathfrak{g} \) cannot preserve any non-degenerate subspace in \( \mathbb{R}^{1,k+1} \). Thus, \( \text{pr}_{\mathfrak{so}(1,k+1)} \mathfrak{g} \) preserves an isotropic line in \( \mathbb{R}^{1,k+1} \) and it is weakly irreducible. Choose a Witt basis \( p,e_1,\ldots,e_n,q \) in such a way, that \( \text{pr}_{\mathfrak{so}(1,k+1)} \mathfrak{g} \subset \mathfrak{so}(1,n+1)_{\text{pr}} \), then \( \mathfrak{g} \subset \mathfrak{so}(1,k+1)_{\text{pr}} \oplus \mathfrak{so}(n-k) \oplus \mathbb{R} \text{id.} \) We assume that \( \mathbb{R}^{1,k+1} \) and \( \mathbb{R}^{n-k} \) have bases \( p,e_1,\ldots,e_k,q \) and \( e_{k+1},\ldots,e_n \), correspondingly. Consider \( R \in \mathfrak{A}(\mathfrak{g}) \) as in Theorem 6. Since \( \mathfrak{g} \subset \mathfrak{so}(1,k+1) \oplus \mathfrak{so}(n-k) \oplus \mathbb{R} \text{id.} \) it holds \( R = R_1 + R_2 + R_3 \). Note that \( R_1 \) takes values in \( \mathfrak{g} \cap \mathfrak{so}(1,k+1) \), and \( R_2 \) takes values in \( \mathfrak{g} \cap \mathfrak{so}(n-k) \). By Lemma 2 and the proof of Theorem 6, the image of the map \( Z : \mathbb{R}^{n-k} \to \mathbb{R}^{1,k+1} \) in the formula for \( R_3 \) is contained in \( \mathbb{R}p \), i.e., there exists a map \( \alpha : \mathbb{R}^{n-k} \to \mathbb{R} \) such that \( Z(X_2) = \alpha(X_2)p \), where \( X_2 \in \mathbb{R}^{n-k} \). From the relation

\[
(Z^*(X_1),X_2) = (X_1,Z(X_2)) = (X_1,\alpha(X_2)p) = \alpha(X_2)(X_1,p)
\]

it follows that \( Z^*(p) = 0 \), \( Z^*(e_i) = 0 \), \( i = 1,\ldots,k \). And from the formula

\[
R_3(X_1,X_2) = \alpha(X_2)p \wedge X_1 + Z^*(X_1) \wedge X_2 + \alpha(X_2)(X_1,p) \text{id}
\]

we get

\[
\begin{align*}
R_3(p,X_2) &= 0, \\
R_3(e_i,X_2) &= \alpha(X_2)p \wedge e_i, \\
R_3(q,X_2) &= \alpha(X_2)p \wedge q + Z^*(q) \wedge X_2 + \alpha(X_2)\text{id}.
\end{align*}
\]

Since we assume that \( \mathfrak{g} \not\subset \mathfrak{so}(1,n+1) \) and that \( \mathfrak{g} \) is a Berger algebra, there exists \( R_3 \in \mathfrak{A}(\mathfrak{g}) \) such that \( \alpha \neq 0 \), and, hence, \( p \wedge \mathbb{R}^k \subset \mathfrak{g} \).

Assume that \( n-k \geq 2 \), and \( \alpha : \mathbb{R}^{n-k} \to \mathbb{R} \). For convenience, suppose that \( \alpha(e_{k+1}) = a \neq 0 \), \( \alpha(e_{k+2}) = \ldots = \alpha(e_n) = 0 \). In this case, \( Z^*(q) = ae_{k+1} \) and we obtain the following relations:

\[
R_3(q,e_{k+1}) = a(p \wedge q + \text{id}) \in \mathfrak{g},
\]

\[
R_3(q,e_j) = e_{k+1} \wedge e_j \in \mathfrak{g}, \quad k+2 \leq j \leq n.
\]

If \( j, i \neq k+1 \), then \( [e_{k+1},e_j,e_{k+1} \wedge e_i] = e_j \wedge e_i \in \mathfrak{g} \), and, hence \( \mathfrak{g} \) contains \( \mathfrak{so}(n-k) \). Note that if \( n-k = 1 \), then \( \mathfrak{so}(n-k) = 0 \).

Thus, \( \mathfrak{g} \subset \mathbb{R} \text{id} \oplus \mathbb{R} \oplus \mathfrak{so}(k) \oplus \mathfrak{so}(n-k) \rtimes \mathbb{R}^k \) contains \( p \wedge \mathbb{R}^k \), \( \mathfrak{so}(n-k) \) and \( \mathbb{R}(p \wedge q + \text{id}) \). Consider the projection \( \text{pr}_{\mathfrak{so}(1,k+1)} \mathfrak{g} \subset \mathfrak{so}(1,k+1)_{\text{pr}} \), it is weakly irreducible, contains \( p \wedge \mathbb{R}^k \) and its projection to \( \mathbb{R}p \wedge q \) is \( \mathbb{R}p \wedge q \). Therefore, \( \text{pr}_{\mathfrak{so}(1,k+1)} \mathfrak{g} \) is of type 1, i.e., \( \text{pr}_{\mathfrak{so}(1,k+1)} \mathfrak{g} = \mathbb{R} \oplus \mathfrak{e} \rtimes \mathbb{R}^k \). Thus, \( \mathfrak{g} = (\mathbb{R}(1,-1,0,0) \oplus \mathfrak{e} \rtimes \mathbb{R}^k) \oplus \mathfrak{so}(n-k) \). According to Theorem 9, \( \mathfrak{e} \subset \mathfrak{so}(k) \) is the holonomy algebra of a Riemannian manifold.

### 8 Proof of Theorem 4

For convenience, we will use the notation of Theorem 7 for the components of an algebraic curvature tensor \( R \).
Let \( g = \mathbb{R} \text{id} \oplus \mathfrak{g} \), where \( \mathfrak{g} \) of the type 1, 2 or 3 from Theorem 2. If \( g \) is a Berger algebra, then according to Theorem 9, \( \mathfrak{h} \) is the holonomy algebra of a Riemannian manifold. Consider the algebraic curvature tensor \( R \), which is defined by the condition \( \mu = 1 \) and all other elements are zeros. Since \( R \in \mathcal{R}(g) \), it holds \( \mathbb{R} \text{id} \subset L(\mathcal{R}(g)) \). Since \( \mathfrak{g} \) is a Berger algebra and \( g = \mathbb{R} \text{id} \oplus \mathfrak{g} \), \( g \) is a Berger algebra.

Assume that \( \mathfrak{g} = \mathbb{R} \text{id} \oplus \mathfrak{g} \), where \( \mathfrak{g} \) is of type 4, and \( R \in \mathcal{R}(g) \). Since \( \mathfrak{h} \subset \mathfrak{so}(m) \), according to Theorem 8, \( A_0 = 0 \), and \( Z_0 = 0 \). Choose an arbitrary non-zero element \( V \in \mathbb{R}^{n-m} \) and substitute to \( R(p, V) = (0, 0, 0, -\mu V) \). Since \( -\mu V = \psi(0) = 0 \), it holds \( \mu = 0 \).

We are going to show that \( \gamma = 0 \). Since \( X_0 \in \mathbb{R}^m \), then \( \gamma(U) = (U, X_0) = 0 \) for every \( U \in \mathbb{R}^{n-m} \), i.e., \( \gamma|_{\mathbb{R}^{n-m}} = 0 \). Now we take \( U \in \mathbb{R}^m \), \( V \in \mathbb{R}^{n-m} \) and arbitrary \( X, Y \in \mathbb{R}^n \), then for \( S \in \mathcal{R}(h) \), the formula
\[
(S(U, V)X, Y) = (S(X, Y)U, V)
\]
holds. Since \((S(X, Y)U, V) = 0\), it holds \( S(U, V) = 0 \). It follows that
\[
\text{pr}_{\mathbb{R}^{n-m}}(L(U, V)) = \psi(S(U, V)) = 0,
\]
but
\[
L(U, V) = P(V)U + \gamma(V)U - P(U)V - \gamma(U)V,
\]
therefore \( \text{pr}_{\mathbb{R}^{n-m}}(L(U, V)) = -\gamma(U)V = 0 \). From the last relation it follows that \( \gamma|_{\mathbb{R}^m} = 0 \).

Thus, \( \gamma = 0 \). Hence, \( g \) is not a Berger algebra, because \( L(\mathcal{R}(g)) \subset \mathfrak{so}(1, n + 1) \).

**Case b.1:**

Rewrite \( \theta : \mathbb{R} \oplus \mathfrak{h} \rightarrow \mathbb{R} \) in the form \( \theta = \theta_1 \oplus \theta_2 \), where \( \theta_1 = \theta|_{\mathbb{R}} \) and \( \theta_2 = \theta|_{\mathfrak{h}} \). Then,
\[
g = \{(\theta_1(a), a, 0, 0) \mid a \in \mathbb{R}\} \oplus \{(\theta_2(A), 0, A, 0) \mid A \in \mathfrak{h}\} \times \mathbb{R}^n.
\]
Choose the algebraic tensor \( R \) defined by the condition \( \lambda = 1, \mu = \theta_1(1) \) and all other elements are zeros. Since \( R \in \mathcal{R}(g) \), then \( \{(\theta_1(a), a, 0, 0) \mid a \in \mathbb{R}\} \subset L(\mathcal{R}(g)) \).

We will choose \( R \) as follows. Take \( P \in \mathcal{P}(h) \) and define \( \gamma(U) := \theta_2(P(U)) \). Choose \( X_0 \) such that \( (U, X_0) = \gamma(U) \). We suppose all other elements which define \( R \) to be zero. According to Theorem 9, \( \mathfrak{h} \) is the holonomy algebra of a Riemannian manifold, and, hence, it is a weak Berger algebra [24]. Therefore, elements \( P(U) \) generate \( \mathfrak{h} \), at the same time
\[
R(U, q) = (\theta_2(C(U)), 0, C(U), 0).
\]

Going through all \( P \in \mathcal{P}(h) \) and \( U \in \mathbb{R}^n \) we get \( \{(\theta_2(A), 0, A, 0) \mid A \in \mathfrak{h}\} \). Hence \( g \subset L(\mathcal{R}(g)) \), i.e., \( g \) is a Berger algebra.

**Case b.2:**

Assume that \( R \in \mathcal{R}(g) \). One can represent \( \theta : \mathfrak{h} \oplus \mathbb{R}^{n_0} \rightarrow \mathbb{R} \) in the form \( \theta = \theta_1 \oplus \theta_2 \), where \( \theta_1 = \theta|_{\mathfrak{h}} \) and \( \theta_2 = \theta|_{\mathbb{R}^{n_0}} \). Then,
\[
g = \{(\theta_1(A), 0, A, 0) \mid A \in \mathfrak{h}\} \oplus \{(\theta_2(X), 0, 0, X) \mid X \in \mathbb{R}^{n_0}\} \times (\mathbb{R}^{n_0})^\perp.
\]
We will show, that \( g \) is a Berger algebra only if \( \theta_2 = 0 \).

Suppose that \( g \) is a Berger algebra and \( \theta_2 \neq 0 \). Then there exists a \( V \in \mathbb{R}^{n_0} \) such that \( \theta_2(V) \neq 0 \). From the equality \( \theta_2(-\mu V) = 0 \) it follows that \( \mu = 0 \). From the expressions for \( R(p,q), R(U,V) \) and \( R(U,q) \) we obtain the following relations:

\[
\begin{align*}
\theta_2(X_0) &= 0, \quad \text{(5)} \\
\gamma(U) &= (U, X_0), \quad \text{(6)} \\
\theta_2(L(U,V)) &= 0. \quad \text{(7)}
\end{align*}
\]

If \( \theta_2 \neq 0 \), then \( \dim \ker \theta_2 = n_0 - 1 \). Choose a basis \( e_1, \ldots, e_{n_0}, \ldots, e_n \) such that \( \theta_2(e_1) \neq 0 \) and \( \theta_2(e_i) = 0 \) for \( i \geq 2 \). Due to the formulas (5) and (6), we have \( X_0 \in \text{span}\{e_2, \ldots, e_{n_0}\} \) and \( \gamma(e_1) = (e_1, X_0) = 0 \). Substituting to the formula (7) \( U = e_1 \) and \( V = e_i \) for \( i \geq 2 \), we obtain

\[
\theta_2(L(e_1, e_i)) = \theta_2(\gamma(e_i)e_1) = \gamma(e_i)\theta_2(e_1) = 0.
\]

Since \( \theta_2(e_1) \neq 0 \), then \( \gamma(e_i) = 0 \). So, \( \gamma = 0 \) and, hence, \( g \) is not a Berger algebra whenever \( \theta_2 \neq 0 \). When \( \theta_2 = 0 \), the algebra \( g \) is a Berger algebra, since \( (\theta(A), 0, A, 0) \subset L(\mathcal{R}(g)) \).

Case b.3:

The proof is similar as for the case b.1.

Case b.4:

It is not a Berger algebra, this fact follows from the case a.4.

9 Realization of Berger algebras

In this section we prove Theorem 5. To do this we show that for every Berger algebra \( g \subset \mathfrak{so}(1, n + 1) \) obtained above there exists a Weyl connection \( \nabla \) such that the holonomy algebra of \( \nabla \) is isomorphic to \( g \).

First we consider the algebras from Theorem 4. Let \( \mathfrak{h} \subset \mathfrak{so}(n) \) be the corresponding subalgebra. We will need the following lemma from [16].

**Lemma 3.** For an arbitrary holonomy algebra \( \mathfrak{h} \subset \mathfrak{so}(n) \) of a Riemannian manifold there exists a \( P \in \mathcal{P}(\mathfrak{h}) \) such that the vector space \( P(\mathbb{R}^n) \subset \mathfrak{h} \) generates the Lie algebra \( \mathfrak{h} \).

For each \( P \in \mathcal{P}(\mathfrak{h}) \) define matrices \( P_i = (P_{jk}^i)_{j,k=1,\ldots,n} \) such that \( P(e_i)e_j = P_{jk}^i e_k \).

Let \( \mathfrak{h} \subset \mathfrak{so}(n) \) be the holonomy algebra of a Riemannian manifold. Then according to the Borel–Lichnerowicz theorem there exists an orthogonal decomposition

\[
\mathbb{R}^n = \mathbb{R}^{n_0} \oplus \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_r}
\]

and the corresponding decomposition

\[
\mathfrak{h} = \{0\} \oplus \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_r
\]
into the direct sum of ideals such that \( h \) annihilates \( R_n \), \( h_i(R_n^j) = 0 \) for \( i \neq j \), and \( h_i \subset \mathfrak{so}(n_i) \) is an irreducible subalgebra for \( 1 \leq i \leq r \). Moreover, each Lie algebra \( h_i \) is the holonomy algebra of a Riemannian manifold. It is known [12] that it holds

\[
\mathcal{P}(h) = \mathcal{P}(h_1) \oplus \ldots \oplus \mathcal{P}(h_r).
\]

We will assume that the basis of \( \mathbb{R}^n \) is compatible with this decomposition of \( \mathbb{R}^n \).

Let \( v, x^1, \ldots, x^n, u \) be the standard coordinates on \( \mathbb{R}^{n+2} \). Consider the metric \( g \) given by the formula

\[
g = 2dvdu + \sum_{i=1}^{n} (dx^i)^2 + 2 \sum_{i=1}^{n} A_i dx^i du + H \cdot (du)^2,
\]

where

\[
A_i = \frac{1}{3} (P^i_{jk} + P^i_{kj}) x^j x^k
\]

and \( H \) is a function that will depend on the type of the holonomy algebra that we wish to obtain. Let \( \nabla \) be the Levi-Civita connection corresponding to \( g \). Let \( p, e_1, \ldots, e_n, q \) be the field of frames

\[
p = \partial_v, \quad e_i = \partial_i - A_i \partial_v, \quad q = \partial_u - \frac{1}{2} H \partial_v
\]

such that at every point it defines a Witt basis.

Consider the Weyl manifold \((M, c, \nabla)\), where

\[
M = \mathbb{R}^{n+2}, \quad c = [g], \quad \nabla = \nabla + K,
\]

and the tensor field \( K : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \) will be defined now.

The condition \( \text{Tor}(\nabla) = 0 \) implies

\[
K_XY = K_YX
\]

for all vector fields \( X, Y \) on \( M \). By the definition of a Weyl connection there exists 1-form \( \theta \) such that \( \nabla_X g = \theta(X)g \). Due to the fact that \( \nabla g = 0 \) it holds

\[
(\nabla_X g)(Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = -g(K_X Y, Z) - g(Y, K_X Z).
\]

On the other hand,

\[
(\nabla_X g)(Y, Z) = \theta(X)g(Y, Z) = g \left( \frac{1}{2} \theta(X) \text{id} Y, Z \right) + g \left( Y, \frac{1}{2} \theta(X) \text{id} Z \right).
\]

This implies

\[
g \left( K_X Y + \frac{1}{2} \theta(X) \text{id} Y, Z \right) + g \left( Y, K_X Z + \frac{1}{2} \theta(X) \text{id} Z \right) = 0.
\]

Therefore, for every \( X \in T_x M \) we get that \( K_X + \frac{1}{2} \theta(X) \text{id} \in \mathfrak{so}(T_x M, g_x) = \mathfrak{so}(1, n+1) \). Hence, \( K_X \in \mathfrak{co}(1, n+1) \). Since \( \nabla_X \partial_v \in \langle \partial_v \rangle \) [13], then \( \nabla_X \partial_v \in \langle \partial_v \rangle \) if and only if \( K_X \partial_v \in \langle \partial_v \rangle \), i.e., at every point, \( K_X \in \mathfrak{co}(1, n+1)_{\mathbb{R}^p} \). Moreover, \( K \in (\mathfrak{co}(1, n+1)_{\mathbb{R}^p})^{(1)} \), because \( K_X Y = K_Y X \). By
Lemma 2, \( K \in \mathbb{R}(p \wedge + g(p, \cdot) \text{id}) \) at every point. Hence, there exists a function \( f : \mathbb{R}^{n+2} \to \mathbb{R} \) such that
\[
K = f \cdot (\partial_e \wedge + g(\partial_v, \cdot) \text{id}) \quad \text{i.e.} \quad K_{\partial_e} \partial_b = f \cdot ((\partial_e \wedge + g(\partial_v, \partial_e)) \partial_b).
\]

(9)

For the Lie algebras \( \mathbb{R} \text{id} \oplus g^{3,h,\varphi} \) and \( g^{3,h,\varphi} \) define the numbers \( \varphi_i = \varphi(P(e_i)) \). For the Lie algebras \( g^{2,2h} \) and \( g^{2,h,\varphi} \) define the numbers \( \theta_i = \theta(P(e_i)) \). For the Lie algebra \( g^{\theta,1,h} \) let
\[
\alpha = \theta(1, 0), \quad \theta_i = \theta(0, P(e_i)).
\]

The following theorem shows that each algebra from Theorem 4 is the holonomy algebra.

**Theorem 10.** The holonomy algebra \( \mathfrak{hol}_0 \) of the Weyl connection \( \nabla = \nabla + K \) depends on the functions \( H \) and \( f \) in the following way

| \( H \) | \( f \) | \( \mathfrak{hol}_0(\nabla) \) |
|---|---|---|
| \( \frac{4}{3}v^3 + \sum_{i=1}^{n_0} (x^i)^2 \) | \( v \) | \( \mathbb{R} \text{id} \oplus g^{1,h} \) |
| \( v^2 + \sum_{i=1}^{n_0} (x^i)^2 \) | \( v \) | \( \mathbb{R} \text{id} \oplus g^{2,h} \) |
| \( v^2 + 2 \sum_{i=n_0+1}^{n} \varphi_i x^i v + \sum_{i=1}^{n_0} (x^i)^2 \) | \( v \) | \( \mathbb{R} \text{id} \oplus g^{3,h,\varphi} \) |
| \( (1 + \alpha)v^2 + 2 \sum_{i=n_0+1}^{n} \theta_i x^i v + \sum_{i=1}^{n_0} (x^i)^2 \) | \( \alpha v + \sum_{i=n_0+1}^{n} \theta_i x^i \) | \( g^{\theta,1,h} \) |
| \( 2 \sum_{i=n_0+1}^{n} \theta_i x^i v + \sum_{i=1}^{n_0} (x^i)^2 \) | \( \sum_{i=n_0+1}^{n} \theta_i x^i \) | \( g^{\theta,2,h} \) |
| \( 2 \sum_{i=n_0+1}^{n} (\theta_i + \varphi_i) x^i v + \sum_{i=1}^{n_0} (x^i)^2 \) | \( \sum_{i=n_0+1}^{n} \theta_i x^i \) | \( g^{\theta,3,h,\varphi} \) |

**Remark.** Note that the projection of \( \mathfrak{hol}_0(\nabla) \) on \( \mathfrak{so}(1, n + 1) \) is not necessary coincide with \( \mathfrak{hol}_0(\nabla) \).

**Proof of Theorem 10.** Since the coefficients of the metric \( g \) are polynomial functions, the connection \( \nabla \) is analytic and from the proof of Theorem 3.9.2 in [23] it follows that \( \mathfrak{hol}_0(\nabla) \) is generated by the elements of the form
\[
\nabla_{Z_a} \cdots \nabla_{Z_1} R(X, Y)_0 \in \mathfrak{so}(1, n + 1), \quad X, Y, Z_1, \ldots, Z_a \in T_0M; \quad \alpha = 0, 1, 2, \ldots.
\]

Denote by \( \Gamma_a \) the matrix \((\Gamma^c_{ba})_{b,c=1,...,n,u}\) of the Christoffel symbols. We have the following recursion formula
\[
\nabla_{a_a} \cdots \nabla_{a_1} R(\partial_a, \partial_b) = \partial_{a_a} \nabla_{a_{a-1}} \cdots \nabla_{a_1} R(\partial_a, \partial_b) + [\Gamma_{a_a}, \nabla_{a_{a-1}} \cdots \nabla_{a_1} R(\partial_a, \partial_b)].
\]

(10)

For the Christoffel symbols of the connection \( \nabla \) it holds
\[
\Gamma_a = \Gamma_a + K_a, \quad \Gamma_a = (\Gamma^c_{ba})_{b,c=1,...,n,u}; \quad K_a = (K^c_{ba})_{b,c=1,...,n,u}; \quad K_{\partial_a} \partial_b = K^c_{ba} \partial_c.
\]

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where $\Gamma_a$ are the Christoffel symbols for the Levi-Civita connection $\nabla$ of the metric $(8)$ and $K_a$ is given by the formula $(9)$. Thus,

$$
\Gamma_v = \begin{pmatrix}
0 & 0 & \frac{1}{2} \partial_v H \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \Gamma_i = \begin{pmatrix}
0 & * & * \\
0 & \frac{1}{2} F_i + (\delta_{ik} f)_{k=1}^n \\
0 & 0 & 0
\end{pmatrix},
$$

$$
\Gamma_u = \begin{pmatrix}
\frac{1}{2} \partial_v H & * & * \\
0 & \frac{1}{2} F + f E_n & -\frac{1}{2} \text{grad}_n H + \frac{1}{2} (\partial_v H) A \\
0 & 0 & -\frac{1}{2} \partial_v H + 2 f
\end{pmatrix},
$$

where

$$
F = (F_{ij})_{i,j=1,...,n}, \quad F_{ij} = \partial_j A_i - \partial_i A_j = 2 P_{jk}^i x^k, \quad F_i = (F_{1i}, \ldots, F_{ni})^t, \quad \text{grad}_n H = (\partial_1 H, \ldots, \partial_n H)^t, \quad A = (A_1, \ldots, A_n)^t.
$$

We mark by $*$ the elements that are not important for us. Now, using the formula

$$
R(\partial_a, \partial_b) = \partial_a \Gamma_b - \partial_b \Gamma_a + [\Gamma_a, \Gamma_b],
$$

one can compute the components of the curvature tensor

$$
R(\partial_v, \partial_i) = \begin{pmatrix}
0 & * & * \\
0 & 0 & ((\delta_{ik} \partial_v f)_{k=1}^n)^t \\
0 & 0 & 0
\end{pmatrix},
$$

(11)

$$
R(\partial_i, \partial_j) = \begin{pmatrix}
0 & * & * \\
0 & 0 & ((-P_{ik}^j + \delta_{jk} \partial_i f - \delta_{ik} \partial_j f)_{k=1}^n)^t \\
0 & 0 & 0
\end{pmatrix},
$$

(12)

$$
R(\partial_v, \partial_n) = \begin{pmatrix}
\frac{1}{2} \partial_v^2 H & * & * \\
0 & (\partial_v f) E_n & -\frac{1}{2} \partial_v \text{grad}_n H + \frac{1}{2} (\partial_v^3 H) A \\
0 & 0 & -\frac{1}{2} \partial_v^2 H + 2 \partial_v f
\end{pmatrix},
$$

(13)

$$
R(\partial_1, \partial_n) = \begin{pmatrix}
\frac{1}{2} \partial_1 \partial_v H & * & * \\
0 & P_1 + (\partial_1 f) E_n & Z_i \\
0 & 0 & -\frac{1}{2} \partial_1 \partial_v H + 2 \partial_v f
\end{pmatrix},
$$

(14)

where $Z_i = ((Z_{ik})_{i=1}^n)^t$ is a vector with the coordinates

$$
Z_{ik} = -\frac{1}{2} \partial_i \partial_k H - \delta_{ik} \partial_a f + \left(\frac{1}{2} \partial_i \partial_v H) A_k + \frac{1}{2} \partial_v \partial_k (\partial_i A_k - \frac{1}{2} F_{ik} - \delta_{ik} f) - \frac{1}{4} \sum_{m=1}^n F_{km} F_{im} + \delta_{ik} f^2.
$$

Also we need the following covariant derivative

$$
\nabla_v R(\partial_v, \partial_n) = \begin{pmatrix}
\frac{1}{2} \partial_v^2 H & * & * \\
0 & (\partial_v^3 f) E_n & -\frac{1}{2} \partial_v^2 \text{grad}_n H + \frac{1}{2} (\partial_v^3 H) A \\
0 & 0 & -\frac{1}{2} \partial_v^2 H + 2 \partial_v^2 f
\end{pmatrix},
$$

(15)

Proof of the inclusion $\mathfrak{g} \subset \mathfrak{ho}_0$. 

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Lemma 4. It holds $\mathbb{R}^n \subset \mathfrak{hol}_0(\nabla)$.

Proof. If $1 \leq i \leq n_0$, then $R(\partial_i, \partial_a) = (0, 0, 0, \frac{1}{2}e_i)$, hence, $\mathbb{R}^{n_0} \subset \mathfrak{hol}_0(\nabla)$. We will show that $\mathbb{R}^{n_0} \subset \mathfrak{hol}_0(\nabla)$ for $1 \leq \alpha \leq r$. Assume that $e_i, e_j, e_k \in \mathbb{R}^{n_0}$. Then

$$R(\partial_i, \partial_j)_0 = \left(0, 0, 0, \sum_{e_k \in \mathbb{R}^{n_0}} P_{ik}^j e_k - s_i e_j + s_j e_i \right),$$

where $s_i = (\partial_i f)(0)$. We claim that there exist $i, j$ such that

$$\sum_{e_k \in \mathbb{R}^{n_0}} P_{ik}^j e_k - s_i e_j + s_j e_i \neq 0.$$

For the first three connections from the statement of the theorem it holds $s_i = 0$ and there is nothing to prove. In the rest three cases it holds $s_i = \theta_i$. Suppose that $P_{ik}^j = s_i \delta_{jk} - s_j \delta_{ik}$ and, hence,

$$P_k = \sum_{e_i \in \mathbb{R}^{n_0}} s_i e_i \wedge e_k.$$

Since $P|_{\mathbb{R}^{n_0}} \neq 0$, then the only possible option is that $\mathfrak{h}_\alpha = \mathfrak{so}(n_\alpha)$. For $n_\alpha \geq 3$ it holds $\theta|_{\mathfrak{h}_\alpha} = 0$ and we have a contradiction. For $n_\alpha = 2$ we will assume for convenience that $e_1, e_2$ is a basis of $\mathbb{R}^{n_0}$. In this case

$$s_1 = \theta(P_1) = -s_2 \xi, \quad s_2 = \theta(P_2) = s_1 \xi, \quad \text{where} \quad \xi = \theta \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right).$$

Therefore $s_1 = -s_1 \xi^2$ and hence $s_1 = s_2 = 0$.

Thus there exists non-zero $Y \in \mathbb{R}^{n_0}$, such that $(0, 0, 0, Y) \in \mathfrak{hol}_0(\nabla)$. Since the image of $P|_{\mathbb{R}^{n_0}}$ generates the algebra $\mathfrak{h}_\alpha$ and $\mathfrak{hol}_0(\nabla)$ is a Lie algebra, then from (14) it follows that for every $A \in \mathfrak{h}_\alpha$ there exist $\gamma, \beta$ such that

$$\zeta := \begin{pmatrix} \gamma & * & * \\ 0 & A & * \\ 0 & 0 & -\gamma \end{pmatrix} + \beta \text{id} \in \mathfrak{hol}_0(\nabla).$$

Then $[\zeta, (0, 0, 0, Y)] = (0, 0, 0, \gamma Y + AY) \in \mathfrak{hol}_0(\nabla)$ and hence $(0, 0, 0, AY) \in \mathfrak{hol}_0(\nabla)$. Similarly, for every $A_1, \ldots, A_s \in \mathfrak{h}_\alpha$ it holds $(0, 0, 0, A_s \cdots A_1 Y) \in \mathfrak{hol}_0(\nabla)$. Since $\mathfrak{h}_\alpha \subset \mathfrak{so}(n_\alpha)$ is irreducible, then $\mathbb{R}^{n_0} \subset \mathfrak{hol}_0(\nabla)$. The lemma is proved.

To complete the proof of the inclusion $\mathfrak{g} \subset \mathfrak{hol}_0(\nabla)$ it is necessary to consider each algebra $\mathfrak{g}$ separately. Take, for instance, $\mathfrak{g} = \mathfrak{g}^{1,0}$ and let $\nabla$ be the corresponding connection. Then

$$R(\partial_x, \partial_a)_0 = (1, -1, 0, *),$$

$$\nabla \circ R(\partial_x, \partial_a)_0 = (0, 1, 0, *).$$

From this and Lemma 4 we obtain $\text{id} \oplus \mathbb{R}(0, 1, 0, 0) \subset \mathfrak{hol}_0(\nabla)$. Next, from (14) it holds $(0, 0, P_1, 0) \in \mathfrak{hol}_0(\nabla)$. Finally $\mathfrak{h} \subset \mathfrak{hol}_0(\nabla)$ since $\mathfrak{h}$ is generated by the elements $P_1$.  

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For the remaining algebras one can show in the same way that $R(\partial_a, \partial_b)$ generate $\mathfrak{g}$.

**Proof of the inclusion** $\mathfrak{hol}_0 \subset \mathfrak{g}$.

Since $\nabla$ preserves $\langle \partial_v \rangle$, then $\mathfrak{hol}_0(\nabla) \subset \mathfrak{co}(1, n+1)_{\mathbb{R}^p}$. Next, the elements $\Gamma_a$ and $R(\partial_a, \partial_b)$ have the form

$$
\begin{pmatrix}
\alpha + \beta & * & *
\\
0 & \alpha \text{id}_{\mathbb{R}^n} + B & *
\\
0 & 0 & \alpha - \beta
\end{pmatrix},
$$

where $\alpha, \beta$ are functions and $B$ is a function with values in $\mathfrak{h}$. Moreover, from (10) it follows that every $\nabla_{a_1} \cdots \nabla_{a_l} R(\partial_a, \partial_b)$ also has the form (16). Then for each algebra $\mathfrak{g}$ one can check that $R(\partial_a, \partial_b)_0 \in \mathfrak{g}$ and $\nabla_{a_1} \cdots \nabla_{a_l} R(\partial_a, \partial_b)_0 \in \mathfrak{g}$. The theorem is proved.

Next, we will show that the algebras from Theorem 3 are the holonomy algebras. The first algebra can be realized in the same way as in [20]. Now, suppose that $\mathfrak{g} = (\mathbb{R}(1, -1, 0, 0) \oplus \mathfrak{k} \ltimes \mathbb{R}^k) \oplus \mathfrak{so}(n-k)$, where $\mathfrak{k} \subset \mathfrak{so}(k)$ is the holonomy algebra of a Riemannian manifold.

First of all consider the metric

$$
g = 2dvdu + h, \quad h = \sum_{i,j=1}^{n} h_{ij} dx^i dx^j,
$$

where $h_{ij} = h_{ij}(x^1, \ldots, x^n, u)$. Let $\nabla$ be the connection of $g$, $K$ be as in the proof of the previous theorem, and $\nabla = \nabla + K$.

The Christoffel symbols for the connection $\nabla$ are as follows

$$
\Gamma_v = (0),
$$

$$
\Gamma_i = \begin{pmatrix}
0 & \left( \frac{1}{2} \frac{\partial h_{i1}}{\partial u}, \ldots, -\frac{1}{2} \frac{\partial h_{in}}{\partial u} \right) & 0 \\
0 & \tilde{\Gamma}_i & \left( \left( \frac{1}{2} \frac{\partial h_{im}}{\partial u} \right)_{k=1}^{n} \right)^{\mathfrak{e}_i} \\
0 & 0 & 0
\end{pmatrix}
$$

$$
\Gamma_u = \begin{pmatrix}
0 & 0 & 0 \\
0 \left( \frac{1}{2} \frac{\partial h_{km}}{\partial u} \right)_{k,l=1,\ldots,n} & 0 & 0
\end{pmatrix} + f \begin{pmatrix}
0 & 0 & 0 \\
0 & E_n & 0 \\
0 & 0 & 0
\end{pmatrix},
$$

where $\tilde{\Gamma}_i$ are the Christoffel symbols of $h$. The vector field $\partial_v$ is a parallel isotropic vector field.

Let $(N, h)$ be a Riemannian manifold of dimension $k$ with the holonomy algebra $\mathfrak{k} \subset \mathfrak{so}(k)$. Consider the manifold $M = \mathbb{R} \times N \times \mathbb{R}^{n-k} \times \mathbb{R}$. 

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We will consider coordinates \( v, x^1, ..., x^k, x^{k+1}, ..., x^n, u \), where \( x^1, ..., x^k \) are local coordinates on \( N \). We choose the function \( f \) that defines \( K \) as
\[
f = x^{k+1}.
\]
Let
\[
h = h^1 + h^2,
\]
where
\[
h^2 = \sum_{i,j=k+1}^n h^2_{ij} dx^i dx^j, \quad h^2_{ij} = e^{-2fu} \delta_{ij}.
\]
We obtain the Weyl manifold \((M, [g], \nabla)\). The above formulas for the Christoffel symbols show that the distribution generated by the vector fields \( \partial_{x^{k+1}}, ..., \partial_{x^n} \). Hence the holonomy algebra preserves a vector subspace \( \mathbb{R}^{n-k} \) is parallel. Since
\[
R(\partial_{k+1}, \partial_u)_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & 2 \end{pmatrix},
\]
then the Weyl structure is non-closed. From Theorem 3 it follows that the only posible holonomy algebra is \( \mathfrak{g} \).

References

[1] W. Ambrose, I. M. Singer, “A theorem on holonomy”, Trans. Amer. Math. Soc., 75 (1953), 428–443.
[2] F. Belgun, A. Moroianu, “Weyl-Parallel Forms, Conformal Products and Einstein-Weyl Manifolds”, Asian Journal of Mathematics, 15:4 (2011), 499–520.
[3] H. Baum, “Holonomy groups of Lorentzian manifolds: a status report”, Global differential geometry, Springer Proc. Math., 17, Springer-Verlag, Berlin–Heidelberg, 2012, 163–200.
[4] L. Bérard-Bergery, A. Ikemakhen, “On the holonomy of Lorentzian manifolds”, Differential geometry: geometry in mathematical physics and related topics (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., 54, Part 2, Amer. Math. Soc., Providence, RI, 1993, 27–40.
[5] L. Bérard-Bergery, A. Ikemakhen, “Sur l’holonomie des variétés pseudo-riemanniennes de signature (n,n)”, Bull. Soc. Math. France, 125:1 (1997), 93–114.
[6] M. Berger, “Sur les groupers d’holonomie des variétés à connexion affine et des variétés riemanniennes”, Bull. Soc. Math. France, 83 (1955), 279–330.
[7] A. L. Besse, Einstein manifolds, Ergeb. Math. Grenzgeb. (3), 10, Springer-Verlag, Berlin, 1987, xii+510 pp.
[8] N.I Bezvitnaya, “Holonomy algebras of pseudo-hyper-Kählerian manifolds of index 4”, Differential Geom. Appl., 31:2 (2013), 284–299.
[9] N.I Bezvitnaya, “Holonomy groups of pseudo-quaternionic-Kählerian manifolds of non-zero scalar curvature”, Ann. Global Anal. Geom., 39:1 (2011), 99–105.
[10] R.L Bryant, “Metrics with exceptional holonomy”, Ann. Math., 126 (1987), 525–576.
[11] A. Fino, I. Kath, “Holonomy groups of \( G_2 \)-manifolds”, Trans. Amer. Math. Soc., 371:11 (2019), 7725–7755.
[12] A.S. Galaev, “The spaces of curvature tensors for holonomy algebras of Lorentzian manifolds”, Differential Geom. Appl., 22:1 (2005), 1–18.
[13] A. S. Galaev, “Metrics that realize all Lorentzian holonomy algebras”, *Int. J. Geom. Methods Mod. Phys.*, 3:5-6 (2006), 1025–1045

[14] A. S. Galaev, “One component of the curvature tensor of a Lorentzian manifold”, *J. Geom. Phys.*, 60:6-8 (2010), 962–971

[15] A. S. Galaev, “Note on the holonomy groups of pseudo-Riemannian manifolds”, *Mathematical Notes*, 93:5-6 (2013), 810–815

[16] A. S. Galaev, “Holonomy groups of Lorentzian manifolds”, *Russian Math. Surveys*, 70:2 (2015), 249–298

[17] A. S. Galaev, “Holonomy algebras of Einstein pseudo-Riemannian manifolds”, *Journal of the London Mathematical Society-Second Series*, 98, Part 2 (2018), 393–415

[18] A. S. Galaev, “Holonomy Classification of Lorentz-Kahler Manifolds”, *Journal of Geometric Analysis*, 29:2 (2019), 1075–1108

[19] A. S. Galaev, Th. Leistner, “Recent developments in pseudo-Riemannian holonomy theory”, *Handbook of pseudo-Riemannian geometry and supersymmetry*, IRMA Lect. Math. Theor. Phys., 16, Eur. Math. Soc, 2010, 581–627

[20] J. Grabbe, “On the holonomy groups of Weyl manifolds”, 2014, 14 pp., arXiv:1410.4253.

[21] A. Ikemakhen, “Sur l’holonomie des variétés pseudo-riemanniennes de signature (2,2+n)”, *Publ. Mat.*, 43:1 (1999), 55–84.

[22] D.D. Joyce, *Riemannian holonomy groups and calibrated geometry*, Oxford Graduate Texts in Mathematics, 12, Oxford University Press, Oxford, 2007, x+303 pp. pp.

[23] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*. V. I, II, Interscience Tracts in Pure and Applied Mathematics, Interscience Publishers John Wiley & Sons, Inc., New York–London–Sydney, 1963, 1969, xi+329 pp., xv+470 pp.

[24] Th. Leistner, “On the classification of Lorentzian holonomy groups”, *J. Differential Geom.*, 76:3 (2007), 423–484

[25] S. Merkulov, L. Schwachhöfer, “Classification of irreducible holonomies of torsion-free affine connections”, *Ann. of Math. (2)*, 150:1 (1999), 77–149

[26] A. L. Onishchik, É. B. Vinberg, *Lie groups and algebraic groups*, Springer Ser. Soviet Math., Springer-Verlag, Berlin, 1990, xx+328 pp.

[27] A. J. Di Scala, C. Olmos, “The geometry of homogeneous submanifolds of hyperbolic space”, *Math. Z.*, 237:1 (2001), 199–209

[28] Volkhausen, Christian, “Local Type II metrics with holonomy in \( G_2^* \)”, *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 89:2 (2019), 179–201

[29] Volkhausen, Christian, “Local Type III metrics with holonomy in \( G_2^* \)”, *Annals of Global Analysis and Geometry*, 56:1 (2019), 113–136

[30] H. Wu, “On the de Rham decomposition theorem”, *Illinois J. Math.*, 8:2 (1964), 291–311