Automorphism of complete $2,3,4$-ary trees

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Abstract. Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Automorphism of $G$ is an isomorphism from $G$ to itself. In other words, an automorphism of $G$ is a permutation $\varphi$ of $V(G)$, which has the property that $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(G)$, for every $u, v \in V(G)$ that is $\varphi$ preserves adjacency. The complete $n$-ary tree is a rooted-tree where its every internal vertex has exactly $n$ children. The height of a rooted tree is the number of edges on longest path connecting the root to a leaf (a vertex of degree one). In this paper, we determine the number of automorphism of complete $2,3,4$-ary trees with height $h$, for any positive integer $h$. We also conjecture the general formula to determine the number of automorphism of complete $n$-ary trees, for any $n \geq 5$.

1. Introduction

Graph theory is a branch of applied mathematics that is widely applied in everyday life. A graph $G$ consists of a set of vertices denoted by $V(G)$, and a set of edges of $G$ denoted by $E(G)$ where every edge connect exactly two vertices [1]. A tree is a connected graph with no cycle [6]. A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root. The height of a rooted tree is the number of edges on longest path connecting the root to a leaf (a vertex of degree one). An internal vertex is a vertex of degree at least two. An $n$-ary tree is a rooted tree in which every internal vertex has at most $n$ children. If every internal vertex in the tree has exactly $n$ children, it is called a complete $n$-ary tree. The level of a vertex $v$ in a rooted tree is the length of the unique path from the root to this vertex.

Gallian [3] states that a group is a non-empty set that is equipped with an operation and fulfills three properties namely associative, has an identity, and each element has an inverse. One of the relations between the two groups is mutually isomorphic. He also states that isomorphism $\varphi$ from group $G$ to group $\bar{G}$ is a bijection function and preserves operations in algebra such that for $a, b \in G$, $\varphi(ab) = \varphi(a)\varphi(b)$. If there is an
isomorphism from $G$ to $\bar{G}$, then it can be said that $G$ and $\bar{G}$ are isomorphic, denoted by $G \simeq \bar{G}$. In graph theory, an isomorphism of graphs $G$ and $H$ is a bijection $f$ between the vertex sets of $G$ and $H$, $f : V(G) \rightarrow V(H)$ such that any two vertices $u$ and $v$ of $G$ are adjacent in $G$ if and only if $f(u)$ and $f(v)$ are adjacent in $H$. An automorphism of a graph $G$ is an isomorphism from $G$ to itself which preserves adjacency as well as non-adjacency. Isomorphism and Automorphism are the key terms in Group theory as well as in graph theory. In 1984, B.D. McKay and N.C. Wormald [5] wrote on Automorphism of random graphs with specified vertices. Famous mathematician Lszl Babai [2] discussed on graph isomorphism and Automorphism property of two isomorphic graphs in the year 1994.

In [4], Majumder et al. studied automorphism of labeled simple connected graph from prescribed degrees. They determined the number of automorphisms in a simple connected graph labeled from a predetermined degree. In this paper, we will give the general formula to determine the number of automorphism of complete 2,3,4-ary trees. We also conjecture the general formula to determine the number of automorphism of complete $n$-ary trees, for any $n \geq 5$.

2. Main Results
In this section, we give the general formula to determine the number of automorphism of complete 2,3,4-ary trees of height $h$, for any positive integer $h$. First, consider the following definition.

![Tree $T_{n,h}$](image)

**Definition 2.1.** For $n \geq 2$ and $h \geq 1$, a graph $T_{n,h}$ is a complete $n$-ary tree of height $h$ having $\frac{n^h-1}{n-1}$ internal vertices and $n^h$ leaves.
We depict \( T_{n,h} \) as in Figure 1. Let \( A(T_{n,h}) \) be set of all automorphisms of \( T_{n,h} \). First, we give a formula to determine the number of automorphism of complete 2-ary tree of height \( h \).

**Example.** Consider all automorphisms of tree \( T_{2,1} \) as in Figure 2. Then, \( |A(T_{2,1})| = 2 \).

![Figure 2. All automorphisms of \( T_{2,1} \)](image)

Next, we derive the formula to determine \( |A(T_{2,h})| \) as in Table 1. From Table 1 we obtain that \( |A(T_{2,3})| = 2^4 \cdot [2^2, 2^1] \), where the number of vertices at \( 2^h \) level in \( T_{2,3} \) is 4 and the number of automorphism of \( T_{2,2} \) is \([2^2, 2^1]\). In addition, we obtain that

\[
|A(T_{2,h})| = 2^{2^{h-1}} [2^{2^{h-2}} \ldots 2^4 2^2 2^1] \\
= 2^1 2^2 2^4 \ldots 2^{2^{h-1}} \\
= 2^{2^0} 2^{2^1} 2^{2^2} \ldots 2^{2^{h-1}} \\
= 2^{d_0 + d_1 + \ldots + d_{h-1}} \\
= 2^h - 1
\]

Consequently, we have the following theorem.
Theorem 2.2. For any positive integer \( h \), the number of automorphism of any complete 2-ary tree of height \( h \) is \( 2^{2^h-1} \).

Proof. We prove by induction on \( h \). Let \( P(h) = |A(T_{2,h})| = 2^{2^h-1} \), for any \( h \in \mathbb{N} \).

- Basic Step
  For \( h = 1 \), then we have \( P(1) = |A(T_{2,1})| = 2^{2^1-1} = 2^1 = 2 \). From Figure 2 this is true that \( P(1) = 2 \).

- Induction Step.
  Assume that the formula \( P(h) \) is true for \( h = k \), so we have
  \[
  |A(T_{2,k})| = 2^{2^k-1} \text{ for } k \geq 1.
  \]

  Now, we prove that the formula \( P(k + 1) \) is true, that is we must show that \( |A(T_{2,k+1})| = 2^{2^{k+1}-1} \). Since the number of vertices at \( k \)th level in \( T_{2,k+1} \) is \( 2^{2^k} \) then from Table 2 we have
  \[
  |A(T_{2,k+1})| = |A(T_{2,k})| \cdot 2^{2^k} = 2^{2^{k+1}-1} \cdot 2^{2^k} = 2^{2^{k+1}-1+2^k} = 2^{2^k+1-1}.
  \]

  Since \( P(k + 1) : |A(T_{2,k+1})| = 2^{2^{k+1}-1} \) then the formula \( P(h) : |A(T_{2,h})| = 2^{2^h-1} \) is true for any \( h \in \mathbb{N} \).

Example. Consider all automorphisms of \( T_{3,1} \) as in Figure 3

Thus, \( |A(T_{3,1})| = 6 \).

Using the similar way as in \( T_{3,h} \), we give a formula to determine the number of automorphism of \( T_{3,h} \).

Consider the construction formula \( |A(T_{3,h})| \) as in Table 2 From Table 2 \( |A(T_{3,3})| = \)

| \( h \) | the number of leaves | \( |A(T_{3,h})| \) | Formula |
|---|---|---|---|
| 1 | 3 | 6 | \((3!)^1\) |
| 2 | 9 | \(6^4\) | \((3!)^2 \cdot (3!)^1\) |
| 3 | 27 | \(6^3 \cdot 3\) | \((3!)^3 \cdot (3!)^3 \cdot (3!)^1\) |
| \vdots | \vdots | \vdots | \vdots |
| \( h \) | \(3^h\) | \( ? \) | \((3!)^{3^{h-1}} \cdot (3!)^{3^{h-2}} \cdot \ldots \cdot (3!)^9 \cdot (3!)^3 \cdot (3!)^1\) |
Figure 3. All automorphisms of $T_{3,1}$

$(3!)^9.([3!]^3.(3!)^1)$, where the number of vertices at $2^h$ level in $T_{3,3}$ is 9 and the number of automorphism of $T_{3,2}$ is $[(3!)^3.(3!)^1]$. In addition, we obtain that

$$|A(T_{3,h})| = (3!)^{3h-1}((3!)^{3h-2} \ldots (3!)^9.(3!)^3.(3!)^1$$

$$= (3!)^{1}.(3!)^{3}.(3!)^9 \ldots 3^{3^h-1}$$

$$= (3!)^{3^0}.(3!)^{3^1}.(3!)^{3^2} \ldots (3!)^{3^{h-1}}$$

$$= (3!)^{3^0+3^1+3^2+\ldots+3^{h-1}}$$

$$= (3!)^{\frac{3^h-1}{2}}$$

Consequently, we have the following theorem.

**Theorem 2.3.** For any positive integer $h$, the number of automorphism of any complete 3-ary tree of height $h$ is $3^{\frac{3^h-1}{2}}$.

**Proof.** We prove by induction on $h$. Let $P(h) = |A(T_{3,h})| = (3!)^{\frac{3^h-1}{2}}$, for any $h \in \mathbb{N}$.

- **Basic Step**
  For $h = 1$, then we have $P(1) = |A(T_{3,1})| = (3!)^{\frac{3^1-1}{2}} = (3!)^1 = (3!) = 6$. From Figure 2 this is true that $P(1) = 6$.

- **Induction Step.**
  Assume that the formula $P(h)$ is true for $h = k$, so we have

  $$|A(T_{3,k})| = (3!)^{\frac{3^k-1}{2}} for k \geq 1.$$
Now, we prove that the formula $P(k + 1)$ is true, that is we must show that $|A(T_{3,k+1})| = (3!)^{\frac{3k+1-1}{2}}$. Since the number of vertices at $k^{th}$ level in $T_{2,k+1}$ is $(3!)^{3k}$ then from Table 2 we have

$$|A(T_{3,k+1})| = |A(T_{3,k})|(3!)^{3k} = (3!)^{\frac{3k-1}{2}}(3!)^{3k} = (3!)^{\frac{3k-1}{2} + 3k} = (3!)^{\frac{(3k-1)}{2}} = (3!)^{\frac{(3k+1-1)}{2}}$$

Since $P(k + 1) : |A(T_{3,k+1})| = (3!)^{\frac{3k+1-1}{2}}$ then the formula $P(h) : |A(T_{3,h})| = (3!)^{\frac{3h-1}{2}}$ is true for any $h \in \mathbb{N}$.

Now, we give a formula to determine the number of automorphism of $T_{4,h}$.

Consider the construction formula $|A(T_{4,h})|$ as in Table 3. From Table 3 we obtain

| $h$ | the number of leaves | $|A(T_{4,h})|$ | Formula |
|-----|---------------------|----------------|---------|
| 1   | 4                   | 24             | $(4!)^1$ |
| 2   | 14                  | $24^2$         | $(4!)^4,[(4!)^1]$ |
| 3   | 64                  | $24^21$        | $(4!)^4,[(4!)^4, (4!)^5]$ |
| ... | ...                 | ...            | ...     |
| $h$ | $4^h$               | ?              | $(4!)^{4^{h-1}}[(4!)^{4^{h-2}} \ldots (4!)^1.6.(4!)^4, (4!)^1]$ |

$$|A(T_{4,h})| = (4!)^{4^{h-1}}[(4!)^{4^{h-2}} \ldots (4!)^1.6.(4!)^4, (4!)^1]$$

$$= (4!)^{1}.(4!)^4. (4!)^1 6 \ldots 4^{4^{h-1}}$$

$$= (4!)^{3}.(4!)^4. (4!)^2 \ldots (4!)^{4^{h-1}}$$

$$= (4!)^{4^0+4^1+4^2+\ldots+4^{h-1}}$$

$$= (4!)^{4^{h-1}}$$

Consequently, we have the following theorem.

**Theorem 2.4.** For any positive integer $h$, the number of automorphism of any complete 4-ary tree of height $h$ is $(4!)^{\frac{4^{h-1}}{3}}$. 
Proof. We prove by induction on $h$. Let $P(h) = |A(T_{4,h})| = (4!)^{\frac{h-1}{3}}$, for any $h \in \mathbb{N}$.

- **Basic Step**
  For $h = 1$, then we have $P(1) = |A(T_{4,1})| = (4!)^{\frac{1-1}{3}} = (4!)^1 = (4!) = 24$. It is easy to check that $|A(T_{4,1})| = 24$.

- **Induction Step.**
  Assume that the formula $P(h)$ is true for $h = k$, so we have
  
  \[ |A(T_{4,k})| = (4!)^{\frac{k-1}{3}} \]

  for any $k \geq 1$.

  Now, we prove that the formula $P(k + 1)$ is true, that is we must show that
  \[ |A(T_{4,k+1})| = (4!)^{\frac{k+1-1}{3}}. \]

  Since the number of vertices at $k$th level in $T_{4,k+1}$ is $(4!)^{k}$, then from Table 3 we have
  
  \[
  |A(T_{4,k+1})| = |A(T_{4,k}).(4!)^{k}|
  = (4!)^{\frac{k-1}{3}}.(4!)^{4k}
  = (4!)^{\frac{k-1}{3}+4k}
  = (4!)^{\frac{4k-1}{3}}
  = (4!)^{\frac{4k+1-1}{3}}
  \]

  Since $P(k + 1) : |A(T_{4,k+1})| = (4!)^{\frac{k+1-1}{3}}$ then the formula $P(h) : |A(T_{4,h})| = (4!)^{\frac{h-1}{3}}$ is true for any $h \in \mathbb{N}$.

From Theorem 2.2, Theorem 2.3 and Theorem 2.4, we propose a conjecture as follows.

**Conjecture 2.5.** The number of automorphism of any complete $n$-ary tree of height $h$ is $(n!)^{\frac{n^h-1}{n-1}}$, for any $n \geq 2$ and $h \geq 1$.

3. Conclusion

Graph automorphism is an interesting problem. From the main result, we obtain the general formulas to determine the number of automorphism of complete 2,3,4-ary trees of height $h$, for any positive integer $h$. First, we prove that the number of automorphism of complete 2-ary tree $A(T_{2,h})$ is $(2!)^{(2^h-1)}$, the number of automorphism of complete 3-ary tree $A(T_{3,h})$ is $(3!)^{\frac{3^h-1}{2}}$, and the number of automorphism of complete 4-ary tree $A(T_{4,h})$ is $(4!)^{\frac{4^h-1}{4}}$. In particular, we conjecture that the number of automorphism of complete 4-ary tree $A(T_{n,h})$ is $(n!)^{\frac{n^h-1}{n-1}}$, for any $n \geq 2$ and $h \geq 1.$
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