Immobilization of convex bodies in $\mathbb{R}^n$

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Abstract
We extend to arbitrary finite $n$ the notion of immobilization of a convex body $O$ in $\mathbb{R}^n$ by a finite set of points $P$ in the boundary of $O$. Because of its importance for this problem, necessary and sufficient conditions are found for the immobilization of an $n$-simplex. A fairly complete geometric description of these conditions is given: as $n$ increases from $n=2$, some qualitative difference in the nature of the sets $P$ emerges.

1 Introduction

Immobilization problems were introduced by [5] and were motivated by the need to understand the best position a machine (robot hand) can grasp an object, [6]. There is now an extensive literature in Robotics journals on immobilization, for example [4], [2], [7], [1], [10], [11], [9]. In [4] it was proved that four points suffice to immobilize any plane body and $2d$ points to immobilize a $d$–dimensional polytope. [2] answered Kuperberg’s conjecture in the affirmative proving that apart from a circular disk, any plane convex object with smooth boundary could be immobilized with three points. It was [1] who first brought out both the geometrical and algebraic aspects of immobilization by treating the case of a tetrahedron. Their results have recently been used in [3] to prove a necessary condition on rotors in tetrahedra. An overview of the classical results on immobilization focusing on analysis, existence and synthesis is given in [12].

For obvious practical reasons, much of the literature focuses on the problem in $\mathbb{R}^3$, and for the actual grasping of objects, robots take full advantage of the effects of friction. The aim of this article is quite different. Following [1], we study the purely geometric problem of grasping a smooth convex body $O$ at a finite set of points $P$ in its boundary and seek conditions so that $O$ is completely immobilized. [1] examined this problem in $\mathbb{R}^3$ and found a set of necessary and sufficient conditions for immobilizing a 3-simplex, i.e a tetrahedron; for reasons explained below, simplices play a particularly important role in the analysis. In this article, we reproduce their results, but generalise the problem to $\mathbb{R}^n$. We follow the approach of [1] by recasting the immobilization problem of the simplex $\Delta$, now in $\mathbb{R}^n$, as an extremal problem, but thereafter our techniques
are very different. Solution of the extremal problem leads to necessary and sufficient conditions for immobilization of $\Delta$ in terms of a matrix $A$ which encodes the geometry of $\Delta$ and the points $P$; these conditions, which are algebraic in character, agree with the results of [4] and [1] for $n = 2, 3$ respectively and reveal an interesting difference in behaviour between $n \leq 3$ and $n > 3$. We conclude the article by interpreting the conditions on $A$ geometrically, and thus show that there is a particular set $P$ which is optimal for immobilization in a sense described below. Some of the detailed proofs in the article have been relegated to the appendices.

2 Immobilization of Convex Bodies and Simplices

Let $O$ be a convex body in $\mathbb{R}^n$. We shall call a point $p$ in the boundary of $O$ a contact point, and a set $P$ of points in the boundary of $O$ will be called a contact set. A contact set $P$ is said to immobilize $O$ if, whenever $O$ is held fixed, any rigid motion of the points of $P$ causes one or more of its points to penetrate into the interior of $O$; equivalently, penetration by at least one point $p \in P$ occurs whenever $O$ is moved and the set $P$ is held fixed; in that case, we will call $P$ an immobilizing contact set.

For a smooth convex body in $\mathbb{R}^n$, for each point of contact $p_i \in P$, let $k_i$ be an outward pointing normal to $O$ at $p_i$ and $\pi_i$ the tangent hyperplane $k_i \cdot (x - p_i) = 0$; then $O$ lies in the intersection of the half spaces $H_i : k_i \cdot (x - p_i) \leq 0$.

For immobilization of $O$ it is necessary that $P$ contains at least $n + 1$ points and that the intersection of all half-spaces $H_i$, is a bounded polytope, completely enclosing $O$; we assume this throughout below. Suppose that $P$ contains precisely $n$ points and the corresponding normals are $k_1, \ldots, k_n$. A displacement $u$ of $O$ away from $p_i$ satisfies $k_i \cdot u < 0$. If the $n$ normals are linearly independent, then the system $k_i \cdot u = b_i, i = 1, \ldots, n$ has a unique solution for all choices of the $\{b_i\}$, in particular for all $b_i < 0$. The displacement $u$ moves $O$ away from all the $p_i$, so no penetration occurs. If the $\{k_i\}$ are linearly dependent, then they span $W$, a strict subspace of $\mathbb{R}^n$ and we simply choose $u$ orthogonal to $W$. Then $k_i \cdot u = 0$ for all $i$, so $O$ ‘slides’ along each hyperplane $\pi_i$ and is again not immobilized by $P$. Finally if $P$ has less than $n$ points, $O$ is even less constrained than when $P$ has $n$ points.

If the polytope $\cap H_i$ is unbounded, there is at least one direction $u$ such that if any of the points of $P$ is displaced by an arbitrary positive multiple of $u$, it remains within the polytope; then if $O$ is translated in this direction, it does not cut through any $\pi_i$, and so no penetration occurs.

Observe that if $O$ is a smooth convex body and the set $P$ contains precisely $n + 1$ points $p_0, \ldots, p_n$, the bounded polytope is a simplex $\Delta$. Thus a necessary condition for immobilization of $O$ is immobilization of the bounding simplex $\Delta$; the converse is false, since owing to the curvature of the boundary of $O$ into the interior of $\Delta$, a displacement of $P$ causing at least one or more $p_i$
to penetrate $\Delta$ may not cause any of these $p_i$ to penetrate $O$. In view of this necessity, understanding the immobilization of simplices is important for the study of immobilization of smooth convex bodies in general.

Let $\mathcal{K}$ denote the set $\{k_0, \ldots, k_n\}$ of outward normals to $O$ at the points $p_i$. In order to achieve the bounded simplex $\Delta$, it is necessary and sufficient that

1. each subset $\mathcal{K} - \{k_i\}$ of $n$ outward normals is linearly independent;
2. in the unique (up to an overall scalar multiple) dependency relation $\sum_{i=0}^n \lambda_i k_i = 0$ among the outward normals $k_i$, all the coefficients $\lambda_i$ are non-zero and of the same sign.

These conditions ensure that there is no direction $u$ in $\mathbb{R}^n$ such that $u \cdot k_i \leq 0$ for $i = 0, 1, \ldots, n$; then there is no translation $u$ of $O$ which does not cause $O$ to cut into at least one of the $\pi_i$. By rescaling, so that each outward normal $k_i$ is replaced by the outward normal $|\lambda_i| k_i$, the conditions 1 and 2 above may equivalently be replaced by

1. $\mathcal{K} - \{k_0\}$ is linearly independent
2. $\sum_{i=0}^n k_i = 0$.

It then follows that apart from an overall positive scaling factor (positive to ensure that normals remain outward pointing), $\sum_{i=0}^n k_i = 0$ is the unique dependency among the outward normals $k_i$, and that apart from this overall scaling factor, all the $k_i$ are now fixed.

We assume 1 and 2 below and in the next section construct a particular set of outward normals to $\Delta$ for which 1 and 2 are satisfied, and furthermore, the overall scaling factor is fixed. To do this, in §3, we initially change our view point: we start with the simplex $\Delta$, defined by its vertices, and in Proposition 1 derive a neat algebraic relationship between the set of vertices and a set of outward normals which satisfy 1 and 2 and which have a particular overall scaling. Then, for consistency with the approach taken in §1, in Proposition 2 we show how starting with the convex body $O$ and the contact set $P$ with its associated outward normals, we can construct $\Delta$ and rescale the normals to obtain the same relationship between vertices and normals as was found in Proposition 1.

### 3 Matrix Description of the Simplex $\Delta$ and Contact set $P$

Let $\Delta$ be a simplex in $\mathbb{R}^n$ having vertex set $V = \{v_0, v_1, \ldots, v_n\}$. Suppose the vertices are oriented so that $\text{vol}(\Delta)$, the $n-$ dimensional volume of $\Delta$, is positive. Let $[a_1, \ldots, a_n]$ denote the $n \times n$ matrix with columns $a_1, \ldots, a_n$. Then $\text{vol}(\Delta) = \frac{1}{n!} \det [v_1 - v_0, \ldots, v_n - v_0]$. To maintain symmetry, however, we regard
each vertex \( u \) with coordinates \((u_1, \ldots, u_n) = (1, u)\) in the hyperplane \( x_0 = 1 \) in \( \mathbb{R}^{n+1} \). Then letting \( V \) be the \((n+1) \times (n+1)\) matrix \([\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_n]\) it follows that \( \text{vol}(\Delta) = \frac{1}{n!} \det(V) \).

Let the face \( F_i, i = 0, 1, \ldots, n \) of \( \Delta \) be the \((n-1)\) dimensional simplex with vertex set \( V - \{v_i\} \), given a positive orientation. Let \( \text{vol}(F_i) \) denote the \((n-1)\) dimensional volume of \( F_i \). Then if \( h_i \) is the length of the altitude dropped from vertex \( v_i \) to face \( F_i \), regarding \( \Delta \) as a cone with base \( F_i \), its volume is given by

\[
\text{vol}(\Delta) = \int_0^{h_i} \left( \frac{x}{h_i} \right)^{n-1} \text{vol}(F_i) \, dx = \frac{1}{n} h_i \text{vol}(F_i). \tag{1}
\]

**Proposition 1** Let \( \Delta \) be a simplex in \( \mathbb{R}^n \) having vertex set \( V = \{v_0, \ldots, v_n\} \) with an orientation such that the \( n \)-dimensional volume of \( \Delta \) is positive. There exists a set of outward pointing normals \( k_i \) to the faces \( F_i \) of \( \Delta \) satisfying:

1. \( \sum_{i=0}^n k_i = 0 \),
2. \( |k_i| = \text{vol}(F_i), \ i = 0, 1, \ldots, n \).

**Proof.** Since \( \text{vol}(\Delta) > 0 \), the matrix \( V \) is invertible, so there exists the \((n+1) \times (n+1)\) matrix \( K = [\bar{k}_0, \bar{k}_1, \ldots, \bar{k}_n] \) such that

\[
K^T V = VK^T = -n\text{vol}(\Delta)I. \tag{2}
\]

Analogous to the decomposition \( \bar{v} = (1, v) \), we write each \( \bar{k}_i = (\kappa_i, k_i) \); the \( n \)-vectors \( k_i, \ i = 0, 1, \ldots, n \) are the vectors we seek as we now show. Then from \( K^T V = -n\text{vol}(\Delta)I \) there follows

\[
\kappa_i + k_i \cdot v_i = -n\text{vol}(\Delta), \quad 0 \leq i \leq n, \tag{3}
\]

\[
\kappa_i + k_i \cdot v_j = 0, \quad 0 \leq j \neq i \leq n, \tag{4}
\]

while from the first row of \( V K^T = -n\text{vol}(\Delta)I \), there follows

\[
\sum_{i=0}^n \kappa_i = -n\text{vol}(\Delta), \tag{5}
\]

\[
\sum_{i=0}^n k_i = 0. \tag{6}
\]

From (4) for each \( i \) and each \( j, l \neq i \)

\[
k_i \cdot (v_j - v_l) = 0. \tag{7}
\]

Thus \( k_i \) is perpendicular to each edge of \( F_i \) and is thus normal to \( F_i \).

For any point \( p_i \) of \( F_i \), since \( v_j \in F_i \), for all \( j \neq i \), it follows that \( p_i - v_j \) is parallel to \( F_i \) and hence

\[
k_i \cdot (p_i - v_j) = 0 \tag{8}
\]
which, using (4), gives for each 0 ≤ i ≤ n and for all j ≠ i
\[ \kappa_i = -\mathbf{k}_i \cdot p_i = -\mathbf{k}_i \cdot v_j. \] (9)

From (3) and (4), for each i and each j ≠ i,
\[ \mathbf{k}_i \cdot (v_j - v_i) = n \text{vol}(\Delta). \] (10)

Now since \( \mathbf{k}_i \) is perpendicular to \( F_i \), the projection of \( (v_j - v_i) \) along \( \mathbf{k}_i \) has the length \( h_i \) of the altitude from vertex \( v_i \) to the face \( F_i \). Thus (10) implies \( |\mathbf{k}_i| h_i = n \text{vol}(\Delta) \) and comparison with (1) shows that \( |\mathbf{k}_i| = \text{vol}(F_i) \).

Finally by (10) since \( n \text{vol}(\Delta) > 0 \), \( \mathbf{k}_i \) is outward pointing.

In this section, we started with \( \Delta \), specified by \( \mathcal{V} \); this led to the faces \( F_i \) and then, via equation (2), to the particular outward normals \( \mathbf{k}_i \) of magnitude \( \text{vol}(F_i) \) encoded by the matrix \( K \) in which the first row comprised elements \( \kappa_i = -\mathbf{k}_i \cdot p_i \) with \( p_i \) any point in \( F_i \). We now show that we reach the same endpoint if we start, as in §2, with the points \( p_i \) and the outward normals \( \mathbf{k}_i \) which satisfy \( \mathbf{k}_0 + \mathbf{k}_1 + \cdots + \mathbf{k}_n = 0 \), \( \mathbf{k}_1, \ldots, \mathbf{k}_n \) linearly independent and with a normalisation to be specified below.

**Proposition 2** Let \( O \) be a convex body in \( \mathbb{R}^n \) and \( \mathcal{P} = \{p_0, p_1, \ldots, p_n\} \) be a contact set of \( O \). Let \( \mathbf{k}_i \) be the outward pointing normals at \( p_i \) and let these normals satisfy conditions 1 and 2 of §2, namely \( \mathbf{k}_1, \ldots, \mathbf{k}_n \) are linearly independent and \( \sum_{i=0}^n \mathbf{k}_i = 0 \). Then if for \( i = 0, \ldots, n \), \( \pi_i \) denotes the hyperplane \( \mathbf{k}_i \cdot x = \mathbf{k}_i \cdot p_i \), there is a unique \( n \)-simplex \( \Delta \) with vertices \( v_j, j = 0, \ldots, n \) where \( v_j \) is the intersection point of all the \( \pi_i \) with \( i \neq j \); by orienting the set \( \mathcal{P} \) suitably we can ensure that \( \text{vol}(\Delta) > 0 \). Furthermore, if we write \( \kappa_j = -\mathbf{k}_j \cdot p_j \) and denote, as before, \( \mathcal{V} = [\bar{v}_0, \ldots, \bar{v}_n] \), \( K = [\overline{\mathbf{k}}_0, \overline{\mathbf{k}}_1, \ldots, \overline{\mathbf{k}}_n] \) where \( \bar{v}_j = (1, v_j), \bar{k}_j = (\kappa_j, \mathbf{k}_j) \), then there is an overall positive rescaling of the outward normals \( \{\mathbf{k}_i\} \) so that \( K \) and \( \mathcal{V} \) satisfy \( K^T \mathcal{V} = \mathcal{V} K^T = -n \text{vol}(\Delta) \mathcal{I} \), from which follow all the relations between normals and vertices established in Proposition [4].

**Proof.** The vertex \( v_j \) is the intersection point of the \( n \) hyperplanes \( \pi_i : \mathbf{k}_i \cdot x = \mathbf{k}_i \cdot p_i, \ 0 \leq i \neq j \leq n \). Since the \( n \) normals \( \mathbf{k}_i, i \neq j \) are linearly independent this defines \( v_j \) uniquely and hence the simplex \( \Delta \) with vertex set \( \mathcal{V} = \{v_0, \ldots, v_n\} \). Note that \( \Delta \) and \( \mathcal{V} \) are independent of the overall scaling of the normals since \( \mathbf{k}_i \) appears linearly in both sides of the equation for \( \pi_i \). Thus for \( j = 0, 1, \ldots, n, v_j \) satisfies
\[ \mathbf{k}_i \cdot v_j = \mathbf{k}_i \cdot p_i, \ 0 \leq i \neq j \leq n \] (11)
and writing \( \kappa_i = -\mathbf{k}_i \cdot p_i \), there follows
\[ \kappa_i + \mathbf{k}_i \cdot v_j = 0, \ 0 \leq i \neq j \leq n. \] (12)
Now summing (12) over all \( i \neq j \) and using \( \mathbf{k}_j = -\sum_{i \neq j} \mathbf{k}_i \), we have
\[ \kappa_i + \mathbf{k}_i \cdot v_i = \sum_{r=0}^n \kappa_r; \ 0 \leq i \leq n \] (13)
and we note that (12), (13) also hold independently of the overall scaling of the normals.

Now \( \sum_{r=0}^{n} \kappa_r < 0 \) : to see this, let \( X \) be strictly inside \( O \). Then since \( O \) is convex and the outward normal at each contact point \( p_r \) is \( k_r \), we must have \( (p_r - X) \cdot k_r > 0, r = 0, \ldots, n \). Hence \( \sum_{r=0}^{n} \kappa_r = - \sum_{r=0}^{n} k_r \cdot p_r = - \sum_{r=0}^{n} k_r \cdot (p_r - X) < 0 \). Defining vectors \( \bar{v}_j, \bar{k}_i \) and matrices \( V, K \) as in the statement of the Proposition, equations (12) and (13) then give \( K^T V = (\sum_{r=0}^{n} \kappa_r) I \). Since \( \sum_{r=0}^{n} \kappa_r < 0 \), \( K \) and \( V \) are non-singular, so that \( \text{vol}(\Delta) = \frac{1}{n!} \det V \neq 0 \).

Now by re-orienting the \( p_i \) if necessary (and thus, by (11), re-orienting the \( v_i \)) we can take \( \text{vol}(\Delta) > 0 \). Noting that \( \sum_{r=0}^{n} \kappa_r < 0 \), we now do an overall positive scaling of the \( k_r \) (and hence all the \( \kappa_r = -k_r \cdot p_r \)) so that

\[
\sum_{r=0}^{n} \kappa_r = -n \text{vol}(\Delta).
\]

We have thus now recovered (2)

\[
K^T V = VK^T = -n \text{vol}(\Delta) I.
\]

Hence with this orientation of contact points and overall scaling of the outward normals, all the relations between normals and vertices established in Proposition 1 follow.

**Remark 3** In practice to use (2) to obtain \( V \) from \( K \), it is first necessary to compute \(-n \text{vol}(\Delta) = -n \det V / n! \) in terms of \( K \) rather than \( V \). Taking determinants in (2) leads to \(-n \text{vol}(\Delta) = (-n - 1)! \det K \) and thus

\[
V = (-n - 1)! \det K \left( K^T \right)^{-1}, \text{ so giving the vertices.}
\]

We observe that

1) If all the faces are projected orthogonally onto a single face \( F_j \), the sum of the projected volumes is zero (as is easily visualised in the cases \( n = 2, 3 \)) which gives \( \sum_{i=0}^{n} k_i \cdot k_j = 0 \). Thus by projecting onto faces \( F_1, F_2, \ldots, F_n \) and using the linear independence of the corresponding normals, there follows \( \sum_{i=0}^{n} k_i = 0 \), thereby giving a geometric interpretation of (11).

2) From (4) and (10), for \( j \neq i \),

\[
\kappa_i = -k_i \cdot v_j = -k_i \cdot (v_j - 0)
\]

is \(-n \text{vol}(\Delta_i)\) where \( \Delta_i \) is the simplex with vertices of \( F_i \) together with the origin. As the origin is not in general a point of \( \Delta \), the quantities \( \kappa_i \) do not play a significant role later.

The contact points \( p_i \) lie in the interior of each face \( F_i \). Thus each \( p_i \) is a linear combination of the vertices of \( F_i \) with coefficients that are non-negative and sum to unity. Define \( \bar{p}_i = (1, p_i) \) and \( P = [\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_n] \); we then have

\[
P = V \Lambda\]
where $\Lambda$ is an $(n+1) \times (n+1)$ matrix whose columns list the coefficients of $\bar{p}_j$ in terms of $v_0, v_1, \ldots, v_n$. The matrix $\Lambda = (\lambda_{ij})$ then has the following properties: $\lambda_{ii} = 0$ since $v_i \not\in F_i$; $\lambda_{ij} > 0$ for all $i \neq j$; $\sum_{i=0}^n \lambda_{ij} = 1$. $\Lambda$ is thus a stochastic matrix with the additional property that the diagonal entries are zero. These matrices enjoy certain properties which we exploit later. The matrix $\Lambda$ thus provides an efficient encoding of the set of contact points.

4 The Penetration Function

The conditions for immobilization of $\Delta$ are now recast in terms of an extremal problem. Let $E_n$ denote the group of all rigid motions of Euclidean space $\mathbb{R}^n$. Given the simplex described by the matrix $V$ (and thus its normals $K$) and the contact points by the matrix $P$ define the penetration function $\Phi : E_n \rightarrow \mathbb{R}$ by

$$\Phi(g) = \sum_{i=0}^{n} (g(p_i) - p_i) \cdot k_i. \quad (14)$$

Then $\Phi$ varies continuously with $g$ and $-\Phi(g)$ measures the total amount of normal penetration into $\Delta$ by the points $p_i \in P$ under the action of $g$ in $E_n$ (weighted by the volumes of the faces).

Now each $g \in E_n$ has a unique decomposition $g = tr$ where $t \in T_n$, the group of translations of $\mathbb{R}^n$ and $r \in SO(n)$, the group of orientation preserving rotations of $\mathbb{R}^n$ about the origin. For each $t \in T_n$ and $r \in SO(n)$, $r^{-1}tr$ is also a translation from which it follows that $T_n r = r T_n$ for all $r \in SO(n)$, so that $T_n$ is a normal subgroup of $E_n$ and $SO(n)$ is the quotient group.

An element $g = tr \in E_n$ has a convenient matrix representation. Let $t_{\bar{x}}$ denote the translation $x \mapsto x + \bar{a}$ and let $r$ have matrix $R$. Then $g(x) = Rx + \bar{a}$. As in §3 let $\bar{x} = (1, x)$ and $\bar{a} = (1, \bar{a})$. Then the $(n+1) \times (n+1)$ matrix $G$ with top row $(1, 0, 0, \ldots, 0)$ and lower $n \times (n+1)$ array $[\bar{a}, R]$ satisfies

$$G \bar{x} = \begin{bmatrix} 1 & 0^T & \bar{a} \\ 0 & R & \bar{a}^T \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ gx \end{bmatrix} = \bar{g}(x). \quad (15)$$

With this formulation of $g = t_{\bar{a}}r$, it follows that if $g$ varies continuously so do $t_{\bar{a}}$ and $r$, and conversely. Thus the map from $E_n$ to $T_n \times SO(n)$ given by $g = tr \mapsto (t, r)$ and its inverse are continuous.

Lemma 4 $\Phi$ is well defined on $E_n/T_n = SO(n)$.

Proof. Let $r \in SO(n)$; we show that $\Phi$ is constant on $T_n r$. For $g = t_{\bar{a}}r$

$$\Phi(t_{\bar{a}}r) = \sum_{i=0}^{n} (t_{\bar{a}}rp_i - p_i) \cdot k_i = \sum_{i=0}^{n} (rp_i + \bar{a} - p_i) \cdot k_i = \Phi(r)$$

since $\sum_{i=0}^{n} k_i = 0$.

We thus regard $\Phi$ as a continuous map from $SO(n)$ to $\mathbb{R}$. 

7
Lemma 5 For each \( r \in SO(n) \), there is a unique \( t(r) \in T_n \) and hence \( g(r) = t(r)r \in E_n \) so that \( \Phi(g(r)) = \Phi(r) \) and

\[
(g(r)p_i - p_i) \cdot k_i = \frac{\Phi(r)}{n+1}, \quad i = 0, 1, \ldots, n.
\]

In particular \( g(r) = I \in E_n \) if and only if \( r = I \in SO(n) \). Furthermore \( g(r) \) varies continuously with \( r \).

Remark 6 Lemma 5 states that for each rotation in \( SO(n) \) (and hence from the decomposition \( g = tr \), for every rigid motion), there is a unique translation \( t(r) \) so that the normal penetrations into \( \Delta \) at the \( p_i \) caused by \( t(r)r \) are all equal.

Proof. Equation \( (16) \) holds if and only if there is a unique \( a \) so that

\[
(rp_i + a - p_i) \cdot k_i = \frac{\Phi(r)}{n+1}, \quad i = 1, \ldots, n
\]

(17)

Since \( \{k_1, k_2, \ldots, k_n\} \) are linearly independent, \( a \) is uniquely specified by the \( n \) equations

\[
(rp_i + a - p_i) \cdot k_i = \frac{\Phi(r)}{n+1}, \quad i = 1, \ldots, n
\]

(18)

But then from Lemma 4

\[
(rp_0 + a - p_0) \cdot k_0 = \sum_{i=0}^n (rp_i + a - p_i) \cdot k_i = \Phi(r) - \frac{n}{n+1} \Phi(r) = \frac{\Phi(r)}{n+1}
\]

(19)

which gives \( (17) \).

When \( r = I \in SO(n) \), then by \( (14) \), \( \Phi(I) = 0 \) so that \( a = 0 \) by \( (17) \) and hence \( g(r) = I \in E_n \). Conversely if \( r \neq I \in SO(n) \), then by \( (15) \), \( t_{a} \neq I \in E_n \) for any \( a \), so in particular \( g(r) \neq I \). The system \( (17) \) for \( a \) can be written

\[
k_i \cdot a = b_i, \quad i = 0, 1, \ldots, n
\]

(19)

where \( b_i = \frac{\Phi(r)}{n+1} - (rp_i - p_i) \cdot k_i \). Using the notation of \( \S 3 \) and writing \( \tilde{a} = (0, a) \) and \( \tilde{b} = (b_0, b_1, \ldots, b_n) \), the system \( (19) \) is

\[
K T \tilde{a} = \tilde{b}
\]

which has solution

\[
\tilde{a} = -\frac{\tilde{b}}{n \text{vol}(\Delta)}
\]

from which, invoking the continuity of \( \Phi(r) \) with \( r \), it follows that \( a \) and hence \( t(r) \) and \( g(r) \) all depend continuously on \( r \in SO(n) \).

Lemma 7 For each \( t \in T_n \), \( \Phi(t) = 0 \). If \( t \neq I \in E_n \), then under the action of \( t \) at least one point \( p_i \) penetrates \( \Delta \).
Proof. Since $\sum_{i=0}^{n} k_i = 0$,

$$
\Phi(t_a) = \sum_{i=0}^{n} (p_i + a - p_i) \cdot k_i = a \cdot \sum_{i=0}^{n} k_i = 0.
$$

If $t_a \neq I$, then $a \neq 0$. Since $\{k_1, \ldots, k_n\}$ are linearly independent at least one $a \cdot k_i$, $i = 1, \ldots, n$ is non-zero and since $\sum_{i=0}^{n} k_i = 0$ at least one $a \cdot k_i$, $i = 0, \ldots, n$ is strictly negative. Hence $(t_a p_i - p_i) \cdot k_i = a \cdot k_i < 0$ for at least one $i$, so that $p_i$ penetrates $\Delta$ under the action of $t_a$.

**Proposition 8** The set $\mathcal{P}$ immobilizes $\Delta$ if and only if $\Phi : SO(n) \to \mathbb{R}$ has a strict local maximum at $I$.

**Proof.** Let $\mathcal{P}$ immobilize $\Delta$. Then each rigid motion $g \neq I$ in a sufficiently small neighbourhood $N$ of $I$ in $\mathbb{E}_n$ causes at least one $p_i$ to penetrate $\Delta$. By the continuity of $r \mapsto g(r)$ there is a neighbourhood $N'$ of $I \in SO(n)$ so that $r \in N'$ implies $g(r) \in N$. Now suppose that $\Phi$ does not have a strict local maximum at $I \in SO(n)$. Then each neighbourhood of $I \in SO(n)$, and in particular $N'$, contains a rotation $r \neq I$ such that $\Phi(r) \geq 0$. For this $r$, $g(r) \in N$, and by Lemma 7 $g(r) \neq I$ and $(g(r)p_i - p_i) \cdot k_i = \Phi(r) \geq 0$ for $i = 0, 1, \ldots, n$, so that $g(r)$ causes no $p_i$ to penetrate $\Delta$; this gives the required contradiction.

Conversely let $\Phi$ have a strict local maximum at $I \in SO(n)$. Then $\Phi(r) < 0$ for all $r \neq I$ in a sufficiently small neighbourhood $N'$ of $I$ in $SO(n)$. Now using the continuity of $tr \mapsto (t, r)$, consider a rigid motion $g = tr \neq I$ in a sufficiently small neighbourhood of $I$ in $\mathbb{E}_n$ so that $r \in N'$. If $r = I$, then $g = t \neq I$ and hence by Lemma 7 at least one $p_i$ penetrates $\Delta$; if $r \neq I$, then $\Phi(g) = \Phi(r) < 0$ so that at least one term $(g(p_i) - p_i) \cdot k_i < 0$ so that $p_i$ penetrates $\Delta$. Thus $\mathcal{P}$ immobilizes $\Delta$.

## 5 Conditions for a Maximum of the Penetration Function

Given $\Delta$ and $\mathcal{P}$, we now examine conditions under which $\Phi$ has a strict local maximum at $I \in SO(n)$. Regarding $g \in SO(n)$ as an $n \times n$ matrix and noting that $p_i \cdot k_i$ can be identified as the trace of the $n \times n$ matrix $k_ip_i^T$, it follows that

$$
\sum_{i=0}^{n} g(p_i) \cdot k_i = \sum_{i=0}^{n} \text{tr}(k_ip_i^T g^T) = \sum_{i=0}^{n} \text{tr}(g^T k_ip_i^T) = \text{tr}(g^T A) = \text{tr}(A^T g),
$$

(20)

where $A$ is the matrix

$$
A = \sum_{i=0}^{n} k_ip_i^T
$$

(21)
which depends solely on the geometry of $\Delta$ and the set of contact points $\mathcal{P}$. Hence
\[
\Phi(g) = \text{tr}(A^T(g - I))
\] (22)
and we now consider conditions on $A$ so that $\Phi$ has a strict local maximum at $I$.

Now each $g \in SO(n)$ may be written as
\[
g = \exp S = \sum_{k \geq 0} \frac{S^k}{k!}
\]
where $S$ is a skew symmetric matrix, and where $g = I$ corresponds to $S = 0$. Hence $\Phi(g) = \Psi(S)$ where
\[
\Psi(S) = \text{tr}(A^T S) + \frac{1}{2!} \text{tr}(A^T S^2) + \cdots
\] (23)

We regard (23) as a power series about $S = 0$ and examine it for a strict local maximum at $S = 0$.

Lemma 9 For $\Psi$ to have a local maximum at $S = 0$, it is necessary that for all skew symmetric matrices $S$, $\text{tr}(A^T S) = 0$.

Proof. This is just the usual condition to make $S = 0$ a stationary point of $\Psi$, which is necessary for an extreme value at $S = 0$.

We now examine further conditions to ensure sufficiency of a strict local maximum at $S = 0$.

Lemma 10 For $\Psi$ to have a strict local maximum at $S = 0$, it is sufficient that for all skew-symmetric matrices $S \neq 0$:
(i) $\text{tr}(A^T S) = 0$;
(ii) $\text{tr}(A^T S^2) < 0$.

Proof. Assuming (i) and (ii), it follows from (i) that
\[
\Psi(S) = \frac{1}{2!} \text{tr}(A^T S^2) + \frac{1}{3!} \text{tr}(A^T S^3) + \cdots
\] (24)

For all $S \neq 0$ in a neighbourhood $N$, the sign of $\Psi(S)$ is the same as that of $\text{tr}(A^T S^2)$. Since by (ii), $\text{tr}(A^T S^2) < 0$ for all $S \neq 0$ in $N$, it follows that $\Psi(S) < 0$ for all $S \neq 0$ in $N$ and hence $\Psi$ has a strict local maximum at $S = 0$.

We now examine the implications for $A$ to ensure that conditions (i) and (ii) hold.
Lemma 11 \( \text{tr}(A^T S) = 0 \) for all skew-symmetric matrices \( S \) if and only if \( A \) is symmetric.

**Proof.** Let \( S^{(ij)} \) be the skew symmetric matrix with a 1 in the position \((i, j), -1 \) in position \((j, i)\) and 0 everywhere else. Then \( \text{tr}(A^T S^{(ij)}) = a_{ji} - a_{ij} = 0 \). The converse follows as each skew symmetric \( S \) can be expressed as \( S = \sum_{i<j} c_{ij} S^{(ij)} \).

**Definition 12** A real symmetric matrix \( A \) is almost positive definite if the sum of every pair of eigenvalues is positive. Here, an eigenvalue can only be added to itself if it is a repeated eigenvalue. This condition is weaker than positive definiteness.

Lemma 13 For a real symmetric matrix \( A \), \( \text{tr}(A^T S^2) < 0 \) for all skew-symmetric matrices \( S \neq 0 \) if and only if \( A \) is almost positive definite.

**Proof.** We have \( \text{tr}(A^T S^2) = \text{tr}(S A S) = -\text{tr}(S^T A S) \). \( A \) can be diagonalised as \( P^T A P = D \) where \( P \) is orthogonal. Hence
\[
\text{tr}(S^T A S) = \text{tr}(S^T P D P^T S) = \text{tr}(P^T S^T P D P^T S) .
\]
Now \( S \) is skew-symmetric if and only if \( P^T S P \) is skew-symmetric, hence \( \text{tr}(S^T A S) > 0 \) for all skew \( S \neq 0 \) if and only if \( \text{tr}(S^T D S) > 0 \) for all skew \( S \neq 0 \). Choosing now \( S = S^{(ij)} \), then \( S^{(ij)} S^{(ij)^T} \) is diagonal with a 1 in position \( i \) and \( j \) and 0 elsewhere. Hence
\[
\text{tr}(S^{(ij)^T D S^{(ij)}}) = \text{tr}(S^{(ij)} S^{(ij)^T} D) = \lambda_i + \lambda_j
\]
so that if \( S = \sum_{i<j} c_{ij} S^{(ij)} \), then
\[
\text{tr}(S^T D S) = \sum_{i<j} c_{ij}^2 (\lambda_i + \lambda_j) > 0
\]
if and only if each pair \( \lambda_i + \lambda_j > 0 \), which gives the result.

Combining Lemmas 10, 11 and 13 we thus have:

**Corollary 14** \( \Psi \) has a strict local maximum at \( S = 0 \) if (i) \( A \) is symmetric and (ii) \( A \) is almost positive definite.

We want to examine the extent to which the conditions (i) and (ii) are necessary. Given that (i) holds, from (24) considering what happens with a real function of a real variable, one could envisage \( \Psi \) having a strict quartic maximum at \( S = 0 \), given by
\[
\Psi(S) = \frac{1}{4!} \text{tr}(A^T S^4) + \cdots
\]
in which \( \text{tr}(A^T S^2) = 0, \text{tr}(A^T S^3) = 0 \) and \( \text{tr}(A^T S^4) < 0 \). This however does not happen; it transpires that Corollary 14 has a converse. Thus we have
Proposition 15 $\Psi$ has a strict local maximum at $S = 0$ if and only if:

(i) $A$ is symmetric; and

(ii) $A$ is almost positive definite.

Proof. The if part is given by Corollary 14.

Now (i) is necessary by Lemma 9 and Lemma 11. To establish the necessity of
(ii) we now assume that $\Psi$ has a strict local maximum at $S = 0$ and that $A$ is symmetric, in which case $\Psi(S)$ is given by (21).

We complete the proof by contradiction, so we assume that $A$ is not almost positive definite. Then by Lemma 13 there is a skew $S_0 \neq 0$ so that $\text{tr}(A^TS_0^2) \geq 0$. We now consider two cases:

(a) If $\text{tr}(A^TS_0^2) > 0$, then replacing $S_0$ by $\epsilon S_0$ where $\epsilon > 0$ is sufficiently small that $\epsilon S_0 \in N$, the neighbourhood of $S = 0$ defined after (23), it follows that

$\Psi(\epsilon S_0) > 0$, giving the required contradiction;

(b) If there is no skew $S \neq 0$ such that $\text{tr}(A^TS^2) > 0$, then $\text{tr}(A^TS_0^2) = 0$. Using the same arguments as those in Lemma 14 we may take $A$ to be diagonal; also, since there is no $S \neq 0$ such that $\text{tr}(A^TS^2) > 0$, there is no pair of eigenvalues $\lambda_i, \lambda_j$ with $\lambda_i + \lambda_j < 0$; hence $\lambda_i + \lambda_j \geq 0$ for all pairs of eigenvalues. Next if

$S_0 = \sum_{i<j} c_{ij} S^{(ij)}$ then from $\text{tr}(A^TS_0^2) = 0$ there follows $\sum_{i<j} c_{ij}^2 \lambda_i + \lambda_j = 0$

so that $\lambda_i + \lambda_j = 0$ for all pairs $i, j$ for which $c_{ij} \neq 0$. We can thus assume $\lambda_1 + \lambda_2 = 0$. Then for $S = S^{(12)}$, we have that $S^2$ is diagonal with entries $-1, 1, 0, 0, \ldots, 0$ from which follows $\text{tr}(A^TS^{2r}) = (-1)^r(\lambda_1 + \lambda_2) = 0$ for all $r \geq 1$ and since $S^{2r+1}$ is skew $\text{tr}(A^TS^{2r+1}) = 0$ for all $r \geq 0$. Hence for $S = S^{(12)} \neq 0$, we have $\Psi(S) = 0$, again contradicting the existence of a strict local maximum at $S = 0$.

Proposition 14 gives necessary and sufficient conditions for the immobilization of $\Delta$. These are couched as algebraic conditions on the matrix $A$: we recall that $A$ depends solely on the geometry of $\Delta$ and the contact set $P$.

We now establish a relation between the matrix $A$ and the matrix $A$, introduced in § 3, and which like $A$ also only depends on the geometry of $\Delta$ and $P$; we then use the special properties of $A$ to show that symmetry of $A$ alone suffices for immobilization of $\Delta$ in the cases $n = 2, 3$, but not for $n \geq 4$.

Lemma 16 The eigenvalues of the $(n + 1) \times (n + 1)$ matrix $-n\nu(\Delta)A$ are precisely those of the $n \times n$ matrix $A$ together with the value $-n\nu(\Delta)$.

Proof. The matrix $A = \sum_{i=0}^n k_i p_i^T$ is the product of the $n \times (n + 1)$ matrix $[k_0, k_1, \ldots, k_n]$ with the transpose of the $n \times (n + 1)$ matrix $[p_0, p_1, \ldots, p_n]$. Reintroducing the $(n + 1) \times (n + 1)$ matrices $K = [k_0, k_1, \ldots, k_n]$ and $P = [p_0, p_1, \ldots, p_n]$ we have

\[ K P^T = \begin{bmatrix} k_0 & \cdots & k_n \\ k_0 & \cdots & k_n \end{bmatrix} \begin{bmatrix} 1 & p_0^T \\ 1 & p_1^T \\ \vdots & \vdots \\ 1 & p_n^T \end{bmatrix} = \begin{bmatrix} -\nu(\Delta) & k^T \\ 0 & A \end{bmatrix}, \quad (25) \]
where we have used $\sum k_j = -n\text{vol}(\Delta)$, $\sum k_j = 0$ and where $b = \sum_j k_j p_j$ plays no further role in the discussion. Thus the eigenvalues of $KP^T$ are precisely those of $A$ together with a further eigenvalue $-n\text{vol}(\Delta)$.

But from §3 $P = V \Lambda$ and $KV^T = -n\text{vol}(\Delta)I$ so that

$$KP^T = K\Lambda^T V^T = -n\text{vol}(\Delta) K\Lambda^T K^{-1}.$$ 

Hence $KP^T$ is similar to $-n\text{vol}(\Delta)\Lambda^T$ which gives the result.

Now from §3, $\Lambda$ is a stochastic matrix with diagonal entries all zero and all off-diagonal entries positive. The following proposition lists the possibilities for the eigenvalues of such a matrix (the proof is actually for $\Lambda^T$ which has the same eigenvalues as $\Lambda$).

**Proposition 17** Let $\Lambda = (\lambda_{ij})$ be a $m \times m$ matrix such that:

(i) $\sum_{j=1}^{m} \lambda_{ij} = 1$ for $1 \leq i \leq m$;
(ii) $\lambda_{ii} = 0$ for all $1 \leq i \leq m$;
(iii) $\lambda_{ij} > 0$ for all $1 \leq i \neq j \leq m$.

Then for all $m \geq 3$:

- 1 is an eigenvalue of $\Lambda$ with eigenvector $(1, 1, \ldots, 1)^T$
- all other eigenvalues lie strictly within the unit circle.

**Proof.** We consider vectors in $\mathbb{C}^m$ with the supremum norm $||z|| = \max\{|z_k| : 1 \leq k \leq m\}$. Then for $k = 1, 2, \ldots, m$

$$||(\Lambda z)_k|| = \sum_{j=1}^{m} |\lambda_{kj}| |z_j| \leq \sum_{j=1}^{m} |\lambda_{kj}| ||z|| = ||z||,$$

Hence if $\lambda$ is an eigenvalue of $\Lambda$ with eigenvector $z$

$$|\lambda|||z|| = ||(\Lambda z)|| = ||\Lambda z|| = \max\{|(\Lambda z)_k| : 1 \leq k \leq m\} \leq ||z||,$$

so that $|\lambda| \leq 1$.

If $z = (1, 1, \ldots, 1)^T$, then $(\Lambda z)_k = \sum_{j=1}^{m} \lambda_{kj} = 1$ so that $\Lambda z = (1, 1, \ldots, 1)^T$ and hence $z = (1, 1, \ldots, 1)^T$ is an eigenvector corresponding to $\lambda = 1$.

Now let $\Lambda z = \mu z$ where $|\mu| = 1$ and let $|z_k|$ be maximal among $|z_k|$, $1 \leq k \leq m$. Then

$$|z_k| = |\mu z_k| = |(\Lambda z)_k| = \sum_{j=1}^{m} |\lambda_{kj} z_j| \leq \sum_{j=1}^{m} |\lambda_{kj}| |z_j|$$

and using $\sum_j \lambda_{kj} = 1$, it follows that

$$0 \leq \sum_{j=1}^{m} \lambda_{kj} (|z_j| - |z_k|).$$
But since \( \lambda_{k^*j} > 0 \) and \(|z_j| - |z_{k^*}| \leq 0\) for all \( j \neq k^* \), it follows \(|z_j| = |z_{k^*}|\) for all \( j \neq k^* \). Thus the components \( z_j \) of eigenvector \( z^* \) all lie on a circle of radius \(|z_{k^*}|\) in \( \mathbb{C} \), and for \( m \geq 3 \), unless all the \( z_k \) are the same, the point \( \sum_{j=1}^{m} \lambda_{k^*j} z_j \) lies strictly inside the convex hull of the \{\( z_j \}\} and so has modulus strictly less than \(|z_{k^*}|\) contradicting the equality \(|z_{k^*}| = |\sum_{j=1}^{m} \lambda_{k^*j} z_j|\) in (20) above. Hence the eigenvector \( z^* \) is a multiple of \((1, 1, \ldots, 1)^T\) and thus corresponds to \( \lambda = 1 \); thus all other eigenvalues \( \lambda \) satisfy \(|\lambda| < 1\).

**Remark 18** The final argument relies on the fact that for \( m \geq 3 \), \( \Lambda \) has elements satisfying \( 0 < \lambda_{ij} < 1 \); this is not true when \( m = 2 \) in which case

\[
\Lambda = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

and then \( \Lambda \) has an eigenvalue of \(-1\) corresponding to eigenvector \((-1, 1)^T\).

This proposition leads to a difference in the sufficiency conditions for immobilization between the cases \( n \leq 3 \) and \( n > 3 \).

**Proposition 19** When \( n \leq 3 \) a sufficient condition for immobilization is simply the symmetry of \( A \).

**Proof.** We show that for \( n \leq 3 \), the fact that the points \( p_i \) are internal to the faces \( F_i \) together with symmetry of \( A \) implies that \( A \) is almost positive definite. The result then follows from Corollary 14.

If \( A \) is symmetric its eigenvalues are real and following Lemma 16, we write them as \(-\text{vol}(\Delta)\lambda_1, \ldots, -\text{vol}(\Delta)\lambda_n\), where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( \Lambda \) other than unity. Since \( \Lambda \) is tracefree, \( \lambda_1 + \lambda_2 + \cdots + \lambda_n + 1 = 0 \). For \( n = 2 \), \( \lambda_1 + \lambda_2 = -1 \) so that \(-\text{vol}(\Delta)\lambda_1 - \text{vol}(\Delta)\lambda_2 = \text{vol}(\Delta)\) and so \( A \) is almost positive definite. For \( n = 3 \), \( \lambda_1 + \lambda_2 + \lambda_3 + 1 = 0 \), so, for example \( \lambda_1 + \lambda_2 = -(1 + \lambda_3) < 0 \) since \(|\lambda_3| < 1\) by Proposition 17 and \( \lambda_3 \) is real, similarly for each pair of eigenvalues and hence \( A \) is almost positive definite.

This method clearly fails for \( n \geq 4 \) and indeed the reduced sufficiency condition does not hold for \( n \geq 4 \) as the example in Appendix A shows.

For the case \( n = 3 \) Proposition 19 was implied by the work of [1] and for the case \( n = 2 \), it was implied by the work of [4]. In both cases different methods were used. The use here of stochastic matrices which are widely studied seems advantageous.

### 6 Geometric Interpretation of Conditions for Immobilization

We finally attempt to give a geometric description of the conditions for immobilization of \( \Delta \). The ideal result would be to give a complete geometric description
of the elements of \( S \), the set of all contact sets \( P \) which immobilize \( \Delta \). This is possible when \( n = 2 \), (see [4]) but already becomes awkward for \( n = 3 \). Instead, we settle for something less. We are able to give a good geometric description of some of the elements of \( S \) and one which we call \( \mathcal{G} \) stands out as being in some sense optimal. We will also describe a geometric process, which starting with \( \mathcal{P} = \mathcal{G} \) displaces the point \( p_i \), each within its face \( F_i \), to reach other elements of \( S \). In this process, symmetry of \( A \) is maintained, but it is harder to keep track of almost positive definiteness.

Our procedure is as follows: we initially construct some special \( n \times (n + 1) \) matrices \([p_0, p_1, \ldots, p_n]\) of contact points which ensure that \( A \) is symmetric, but where we temporarily relax the condition \( p_i \in F_i \). From the linearity of \( A \) with respect to the \( p_i \) in \( A = \sum \lambda_i p^T_i \), it follows that any linear combination of these special matrices also yields a matrix \( A \) which is symmetric, and we choose the linear combinations to ensure that \( p_i \in F_i \); we will also examine almost positive definiteness of such matrices \( A \).

The following special sets \([p_0, p_1, \ldots, p_n]\) guarantee symmetry of \( A \):

(i) \([t_0k_0, t_1k_1, \ldots, tnk_n]\) where \( t_i, i = 0, 1, \ldots, n \) are arbitrary. Then \( A = \sum_{i=0}^n t_i k_i k_i^T \) is clearly symmetric and positive definite if all \( t_i > 0 \).

(ii) \([\bar{z}, \bar{z}, \ldots, \bar{z}]\) where \( \bar{z} \in \mathbb{R}^n \) is arbitrary. Then \( A = \sum k_i \bar{z}^T \) which is the zero matrix.

(iii) \([v_1, \ldots, v_n]\). This corresponds to the \( (n + 1) \times (n + 1) \) matrix \( V = [\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_n] \) for which \( KV^T = -n \text{vol}(\Delta)I_{n+1} \) and then by (25), \( A = -n \text{vol}(\Delta)I_{n+1} \).

(iv) \([k_1, k_0, 0, \ldots, 0] \) gives \( A = k_0k_1^T + k_1k_0^T \) which is clearly symmetric: similarly any interchange of \( k_i \) and \( k_j, i \neq j \) with all other entries 0.

We now use these special sets to construct some immobilizing sets.

### 6.1 Centroids of the Faces

Let \( g = \frac{1}{n} \sum_{i=0}^n v_i \). Then \( g_i = g - \frac{v_i}{n} \) is the centroid of the face \( F_i \). Let \( G = [\bar{g}_0, \bar{g}_1, \ldots, \bar{g}_n] \) be the \( (n + 1) \times (n + 1) \) matrix of the contact set of centroids. Then for this set

\[
KG^T = K[\bar{g}, \bar{g}, \ldots, \bar{g}]^T - \frac{1}{n}KV^T = 0 + \text{vol}(\Delta)I_{n+1}
\]

so that by (25) \( A = \text{vol}(\Delta)I_n \) which is symmetric and positive definite and hence the set of centroids is immobilizing. We can also see this by noting that for \( G \) the corresponding matrix \( \Lambda \) is \( \frac{1}{n}(J-I) \) where \( J \) is the \( (n + 1) \times (n + 1) \) matrix all of whose entries are 1. \( J \) has one eigenvalue of \( n + 1 \) and \( n \) eigenvalues of 0, so that \( \Lambda \) has one eigenvalue of 1 and \( n \) eigenvalues of \( -\frac{1}{n} \), whence by Lemma [10] \( A \) has \( n \) eigenvalues of \( \text{vol}(\Delta) \).
6.2 Centred contact sets

A set of contact points $\mathcal{P} = \{p_0, p_1, \ldots, p_n\}$ where $p_i \in F_i$ is centred at $z \in \mathbb{R}^n$ if the normal lines $l_i : x = p_i + t k_i$, $t \in \mathbb{R}$ are concurrent, meeting in $z$. Given $\Delta$, the set $Z$ of such points $z \in \mathbb{R}^n$ may be constructed as the intersection of the $n + 1$ cylinders $C_i$ where $C_i = \{l_i : p_i \in F_i\}$. This set is a polytope in $\mathbb{R}^n$ which we show in Appendix B always includes an open set of points interior to $\Delta$, but not necessarily the whole of the interior points of $\Delta$ (for $n = 2$, $\Delta$ a triangle, if $\Delta$ is acute angled, then the set is a hexagon completely enclosing $\Delta$, of twice the area of $\Delta$; if $\Delta$ is obtuse angled, the set is a parallelogram only including part of $\Delta$ and which becomes unbounded as the obtuse angle approaches $\pi$).

Let $z$ be a point of concurrency interior to $\Delta$. Then there exists $t_i > 0$ so that $p_i = z + t_i k_i \in F_i$ and for the contact set $\mathcal{P} = \{p_i\}$

$$A = \sum_{i=0}^{n} k_i z^T + \sum_{i=0}^{n} t_i k_i k_i^T = \sum_{i=0}^{n} t_i k_i k_i^T$$

which is symmetric and positive definite, so that $\mathcal{P}$ immobilizes $\Delta$.

**Remark 20** For $n = 2$, $\Delta$ a triangle, the collection of contact sets centred at $z$ whether or not $z$ lies within $\Delta$, is precisely the collection of all immobilizing contact sets [3]. This is not true for $n \geq 3$. From the results of the appendices, the space of centred immobilizing contact sets has dimension $n$ whereas the space of all immobilizing contact sets has dimension $\frac{1}{2}n(n + 1) - 1$. These are equal when $n = 2$ but not otherwise.

6.3 Displacements

A comparatively complete description of immobilizing contact sets of $\Delta$ can be given by examining, instead of the contact set itself, displacements $\Delta p_i$ of the points $p_i$ within the faces $F_i$ from one immobilizing contact set to another. Leaving aside the condition of almost positive definiteness for the time being, the conditions $p_i$ lying in the hyperplane containing $F_i$ and $A$ symmetric are:

$$k_i \cdot p_i = k_i \cdot v_j, \ j \neq i, \ i = 0, 1, \ldots, n; \ \sum_{i=0}^{n} k_i \wedge p_i = 0,$$

the latter merely stating that the antisymmetric part of $A$ vanishes. Then a displacement $\Delta p_i$ within $F_i$ maintains the symmetry of $A$ only if $\Delta p_i$ satisfy

$$k_i \cdot \Delta p_i = 0, \ i = 0, 1, \ldots, n; \ \sum_{i=0}^{n} k_i \wedge \Delta p_i = 0. \quad (27)$$

While the system (27) is necessary to keep each displaced point $p_i$ in the face $F_i$ and to maintain symmetry of $A$, it is not sufficient, because it places no constraint on the size of the displacements $\Delta p_i$: the system (27) achieves the
more modest objective of keeping each point \( p_i \) in the hyperplane containing the face \( F_i \) (equivalently, the displacement \( \Delta p_i \) is parallel to \( F_i \)) together with maintaining symmetry of \( A \).

We now exhibit a basis for the space \( \Omega \) of displacements \(\Delta P = [\Delta p_0, \Delta p_1, \ldots, \Delta p_n]\) satisfying the system (27). There is no such displacement set where all but one of the \( \Delta p_i \) are zero, since, if for example, \( \Delta p_0 \) were the only non-zero displacement, we would have \( k_0 \cdot \Delta p_0 = 0 \) and \( k_0 \wedge p_0 = 0 \) which requires \( \Delta p_0 \) to be both perpendicular and parallel to \( k_0 \).

There are, however, displacements where all but two of the \( \Delta p_i \) are zero. If the non-zero displacements are \( \Delta p_i \) and \( \Delta p_j \), then (27) reduces to

\[
 k_i \cdot \Delta p_i = 0, \quad k_j \cdot \Delta p_j = 0, \quad (28)
\]

\[
 k_i \wedge \Delta p_i + k_j \wedge \Delta p_j = 0. \quad (29)
\]

Wedging (29) with \( k_i \) gives \( \Delta p_j \wedge (k_i \wedge k_j) = 0 \) so that \( \Delta p_j \) is a linear combination of \( k_i \) and \( k_j \); likewise \( \Delta p_i \). Then using (28), it follows that

\[
 \Delta p_i = t k_{ij} \quad \text{and} \quad \Delta p_j = t k_{ji}, \quad (30)
\]

where

\[
 k_{ij} = k_j - \frac{\langle k_i \cdot k_j \rangle}{|k_i|^2} k_i \quad (31)
\]

is the projection of \( k_j \) perpendicular to \( k_i \) and \( t \) is a parameter. We define a collection of displacement sets \( \Delta P_{ij}, 0 \leq i < j \leq n \) by

\[
 \Delta P_{ij} = [\Delta p_0, \Delta p_1, \ldots, \Delta p_n]; \quad \Delta p_i = k_{ij}; \quad \Delta p_j = k_{ji}; \quad \Delta p_m = 0 \quad \text{for all} \quad m \neq i, j. \quad (32)
\]

We have thus shown that the displacements \( \Delta P = [\Delta p_0, \Delta p_1, \ldots, \Delta p_n] \) with each \( \Delta p_i \) parallel to \( F_i \), which also maintains symmetry of \( A \), and for which only \( \Delta p_i \) and \( \Delta p_j \), \( i \neq j \), are non-zero is necessarily given by \( \Delta P = t \Delta P_{ij} \) for some \( t \). In the displacement \( \Delta P_{ij} \), the points \( p_i \) and \( p_j \) move in a dependent way in directions parallel to the orthogonal projection of \( k_j \) onto \( F_i \) and the orthogonal projection of \( k_i \) onto \( F_j \) respectively; since these directions are both linear combinations of \( k_i \) and \( k_j \), they are both orthogonal to \( F_{ij} \), the \( n-2 \) dimensional face in which \( F_i \) meets \( F_j \).

In Appendix C, we show that the collection \( C \) of \( \Delta P_{ij}, 0 \leq i < j \leq n \) spans the space \( \Omega \) of displacements which satisfy (27), but they are not independent (see Appendix D).

We can see the effect of the displacement set \( t \Delta P_{01} \) on the immobilizing set of centroids \( G \). Since each \( g_i \) is strictly interior to \( F_i \), for sufficiently small \( |t| \), the displaced points remain in their faces and, by construction, \( A \) remains...
symmetric, so we examine almost positive definiteness. For the displaced set we have

\[[p_0, p_1, \ldots, p_n] = [g_0, g_1, \ldots, g_n] + t[k_{01}, k_{10}, 0, \ldots, 0]\]

so that

\[A = \text{vol}(\Delta)I_n + tB_{01}\]

where \(B_{01}\) is the \(n \times n\)-matrix

\[B_{01} = k_0k_{01}^T + k_1k_{10}^T\]

Now take a basis of \(\mathbb{R}^n\) given by \(\{k_0, k_1, u_2, \ldots, u_{n-1}\}\) where each \(u_i\) is orthogonal to span \(\{k_0, k_1\}\). Then

\[
\begin{align*}
B_{01}k_0 &= k_1|k_{10}|^2 \\
B_{01}k_1 &= k_0|k_{01}|^2 \\
B_{01}u_j &= 0, \quad j = 2, \ldots, n
\end{align*}
\]

so that \(B_{01}\) is similar to a matrix \((b_{ij})\) where \(b_{10} = |k_{10}|^2, b_{01} = |k_{01}|^2\) and all other entries are zero. Thus \(B_{01}\) has eigenvalues \(\pm |k_{01}|, |k_{10}|\) and all other eigenvalues are 0, so that \(A\) has eigenvalues \(\text{vol}(\Delta) \pm t|k_{01}|, |k_{10}|\) and the remaining \(n-2\) eigenvalues are all \(\text{vol}(\Delta)\), hence for the displaced contact set, the smallest sum of any pair of eigenvalues is

\[2\text{vol}(\Delta) - |t||k_{01}|, |k_{10}|\]

so that any displacement \(t\Delta P_{01}, t \neq 0\), reduces the minimal sum of a pair of eigenvalues and hence for \(|t|\) sufficiently large the almost positive definite condition fails.

We extend this to an arbitrary non-zero displacement

\[
\Delta P = \sum_{0 \leq i < j \leq n} t_{ij} \Delta P_{ij}
\]

as follows. Following from above, we now have

\[A = \text{vol}(\Delta)I_n + \sum_{0 \leq i < j \leq n} t_{ij}B_{ij}\]

where

\[B_{ij} = k_i k_{ij}^T + k_j k_{ji}^T.\]

Now from before \(B_{01}\) is similar to \((b_{ij})\) which is trace-free. Thus \(B_{01}\) itself and similarly all \(B_{ij}\) are trace-free. Hence the sum of the eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_n\) of \(A\) satisfies

\[\lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{tr}(A) = \text{tr}(\text{vol}(\Delta)I_n + \sum t_{ij}B_{ij}) = n\text{vol}(\Delta).\]

Now \(\lambda_1 = \lambda_2 = \cdots = \lambda_n = \text{vol}(\Delta)\) if and only if \(A = \text{vol}(\Delta)I_n\) (only if requires us to use symmetry of \(A\) so that \(A\) is diagonalizable). So for any non-zero perturbation \(\sum t_{ij}B_{ij}\) of \(A\), since the sum of eigenvalues is fixed, at least one
eigenvalue increases strictly and at least one eigenvalue decreases strictly, so that the smallest sum of any pair of eigenvalues of $A$ has to decrease for any non-zero displacement $\sum t_{ij} \Delta P_{ij}$ from $\mathcal{G}$. Because the eigenvalues of $A$ change continuously with $\Delta P$, using the dimensionality results in the appendices, we can summarise the results as follows:

1. in the $\frac{1}{2}n(n + 1) - 1$ dimensional space of contact sets $[p_0, p_1, \ldots, p_n]$ for which $A$ is symmetric and where each $p_i$ lies in the $(n - 1)$-dimensional hyperplane containing $F_i$, there is an open neighbourhood $N$ of $\mathcal{G}$, the centroids, which immobilizes $\Delta$;

2. the contact set $\mathcal{G}$ is optimal in that any displacement within the neighbourhood $N$ causes the smallest sum of a pair of eigenvalues of $A$ to decrease from $2\text{vol}(\Delta)$, and if such a displacement is large enough to cause this sum to vanish, then immobilization is lost.

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7 Appendix A

We give an example of a simplex $\Delta$ in $\mathbb{R}^4$ and points $p_0, p_1, \ldots, p_4$ on $\Delta$ satisfying the symmetry condition but failing the almost positive definiteness condition. Consider the 4-simplex having vertices $v_0 = (-\frac{3}{4}, -1, 0, -3)$, $v_1 = (-\frac{83}{36}, 0, 0, 1)$, $v_2 = (1, 1, 0, -3)$, $v_3 = (\frac{35}{18}, 0, -1, 1)$ and $v_4 = (\frac{35}{18}, 0, 1, 1)$. The standard outward normal vectors of the simplex are $k_0 = (0, 34, 0, 17)$, $k_1 = (16, -\frac{34}{3}, 0, -\frac{19}{18})$, $k_2 = (0, -34, 0, \frac{17}{2})$, $k_3 = (-8, \frac{17}{3}, 34, -\frac{187}{36})$ and $k_4 = (-8, \frac{17}{3}, -34, -\frac{187}{36})$. The points

\[
p_0 = \frac{3}{10} v_0 + \frac{2}{5} v_2 + \frac{3}{20} v_3 + \frac{3}{20} v_4 \tag{33}
\]
\[
p_1 = \frac{1}{10} v_0 + \frac{1}{10} v_2 + \frac{2}{5} v_3 + \frac{7}{5} v_4 \tag{34}
\]
\[
p_2 = \frac{8}{5} v_0 + \frac{2}{5} v_1 + \frac{1}{10} v_3 + \frac{1}{10} v_4 \tag{35}
\]
\[
p_3 = \frac{1}{10} v_0 + \frac{7}{10} v_1 + \frac{1}{10} v_2 + \frac{1}{10} v_4 \tag{36}
\]
\[
p_4 = \frac{1}{10} v_0 + \frac{7}{10} v_1 + \frac{1}{10} v_2 + \frac{1}{10} v_3 \tag{37}
\]

are interior to their faces and satisfy the symmetry condition since

\[
\sum_{i=0}^{4} k_i p_i^T = \begin{bmatrix}
\frac{238}{5} & 0 & 0 & 0 \\
0 & \frac{136}{5} & 0 & 0 \\
0 & 0 & \frac{34}{5} & 0 \\
0 & 0 & 0 & -\frac{68}{5}
\end{bmatrix}.
\]

However, a pair of eigenvalues of this matrix has a negative sum.
8 Appendix B

We demonstrate that the set \( Z \) of points of concurrency of centred contact sets \( P \) defined in §6 contains an open subset of \( \mathbb{R}^n \) which lies in the interior of \( \Delta \). Through any \( z \in \mathbb{R}^n \) pass \( n + 1 \) lines \( \ell_j, j = 0, 1, \ldots, n \), where \( \ell_j \) has direction \( k_j \), so meets orthogonally \( \pi_j \), the hyperplane containing \( F_j \). The line \( \ell_j \) meets \( \pi_j \) in a point \( z_j \) satisfying

\[
z_j = z + t_j k_j \quad (38)
\]

and

\[
k_j \cdot z_j = k_j \cdot v_i \quad \text{for any } i \neq j \quad (39)
\]

The requirements that each \( z_j \) lies in \( F_j \) and the aim to show that there exists such points \( z \) within \( \Delta \) are best encoded by expressing the points \( z_i \) and \( z \) as linear combinations of the vertices \( [v_0, v_1, \ldots, v_n] \). Thus we write

\[
z_j = \sum_{i=0}^{n} \lambda_{ij} v_i \quad (40)
\]

where \( (\lambda_{ij}) \) is a stochastic matrix just as in §3 and we seek

\[
z = \sum_{i=0}^{n} \mu_i v_i \quad (41)
\]

where \( \sum_{i=0}^{n} \mu_i = 1 \) and \( \mu_i > 0 \) to ensure that \( z \) lies within \( \Delta \). The combination of \( v_i \) and \( k_j \) in the system (38) - (41) suggest that it is advantageous to use the machinery developed in §3 so we again extend into \( \mathbb{R}^{n+1} \) by writing \( \bar{z} = (1, z) \) and \( \bar{z}_i = (1, z_i) \). Then (40) and (41) give

\[
[\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_n] = V \Lambda \quad (42)
\]

and

\[
\bar{z} = V \mu, \quad (43)
\]

where \( \mu \) is the \((n+1)\) column \((\mu_0, \mu_1, \ldots, \mu_n)\). Then from the equation \( K^T V = -n \text{vol}(\Delta) I \), we have

\[
-n \text{vol}(\Delta) \lambda_{ij} = \bar{k}_j^T \bar{z}_j \quad (44)
\]

and

\[
-n \text{vol}(\Delta) \mu_i = \bar{k}_j^T \bar{z}. \quad (45)
\]

From §3 equation (41), for all \( i \neq j \),

\[
k_j \cdot v_i = -\kappa_j, \quad (46)
\]

so by (38) dotted with \( k_j \), (39) and (46) we have

\[
k_j \cdot z + t_j |k_j|^2 = -\kappa_j
\]
from which it follows

\[ t_j = -\frac{k_j \cdot z + \kappa_j}{|k_j|^2} = -\frac{\bar{k}_j^T \bar{z}}{|k_j|^2} + \frac{n\text{vol}(\Delta)\mu_j}{|k_j|^2} \]  

(47)

by means of (45) and where we recall that \( \bar{k}_j = (\kappa_j, k_j) \). Hence (48) becomes

\[ z_j = z + \frac{n\text{vol}(\Delta)}{|k_j|^2} |k_j|^2 \mu_j k_j, \]  

(48)

which, given \( z \in \mathbb{R}^n \), locates the \( z_j \in \pi_j \).

We want to extend equation (48) into \( \mathbb{R}^{n+1} \) so that we can use (44) and (45) to find a relation between the coefficients \( \mu \) and \( \Lambda \). As the zero components of both \( \bar{z}_j \) and \( \bar{z} \) are both unity we cannot simply replace \( k_j \) by \( \bar{k}_j \) in (48); we will need to extend \( k_j \) to \( \mathbb{R}^{n+1} \) in such a way that the zero component vanishes. For this we use

\[ \sum_l k_l = 0 \]

and replace \( k_j \) by

\[ k_j + \rho_j \sum_l k_l \]

and then choose \( \rho_j \) so that the 0 component in the corresponding expression

\[ \bar{k}_j + \rho_j \sum_l \bar{k}_l \]

vanishes. Thus we require

\[ 0 = \kappa_j + \rho_j \sum_l \kappa_l = \kappa_j + \rho_j (-n\text{vol}(\Delta)), \]

using (5) and now (48) may be extended into \( \mathbb{R}^{n+1} \) as

\[ \bar{z}_j = z + \frac{n\text{vol}(\Delta)}{|k_j|^2} \mu_j \left( \bar{k}_j + \frac{\kappa_j}{n\text{vol}(\Delta)} \sum_l \bar{k}_l \right) \]  

(49)

Now multiplying by \( \bar{k}_i^T \) and using (44) and (45), there follows

\[ -n\text{vol}(\Delta)\lambda_{ij} = -n\text{vol}(\Delta)\mu_i + \frac{n\text{vol}(\Delta)}{|k_j|^2} \mu_j \left( \bar{k}_j^T k_j + \frac{\kappa_j}{n\text{vol}(\Delta)} \bar{k}_j^T \left( \sum_l \bar{k}_l \right) \right). \]  

(50)

Now for the final term above \( \bar{k}_j^T k_j = \kappa_i \kappa_j + k_i \cdot k_j \) while \( \bar{k}_j^T \sum_l \bar{k}_l = \bar{k}_j \cdot (\sum_l \kappa_l, 0) \) since \( \sum_l k_l = 0 \), and this equals \( \kappa_i \sum_l \kappa_l = -n\text{vol}(\Delta)\kappa_i \) so that the final bracket in (50) just reduces to \( k_i \cdot k_j \) and thus (50) becomes

\[ \lambda_{ij} = \mu_i - \frac{\mu_j}{|k_j|^2} k_i \cdot k_j. \]  

(51)

We note that \( \lambda_{jj} = 0 \) as required and for each \( j \), \( \sum_{i=0}^n \mu_i = 1 \) gives \( \sum_{i=0}^n \lambda_{ij} = 1 \) (where we use \( \sum_i k_i = 0 \)). For a point \( z \) interior to \( \Delta \) to yield the corresponding
point \( z_i \) interior to \( F_i \) we need to find \( \mu_i > 0, i = 0, 1, \ldots, n \) so that \( \lambda_{ij} > 0 \) for all \( 0 \leq i \neq j \leq n \). A particular solution to this is

\[
\mu_i = \frac{|k_i|}{\sum_{l=0}^{n} |k_l|}
\]

(52)

since then

\[
\lambda_{ij} = \frac{|k_j| (||k_j|||k_i| - k_i \cdot k_j)}{|k_j|^2 \sum_{l=0}^{n} |k_l|}
\]

which is positive for all \( i \neq j \). For the point \( z \) corresponding to the solution \( (52) \), by continuity, there is a full neighbourhood \( N \) of \( z \) interior to \( \Delta \) so that each \( x \in N \) projects along \( k_i \) to a point \( x_i \) interior to \( F_i \) and thus yields an immobilizing set of contact points.

We note that for \( n = 2 \), for the centroid immobilizing set where \( \lambda_{ij} = \frac{1}{2} \) for \( 0 \leq i \neq j \leq 2 \), equation \( (51) \) has a solution given by

\[
\mu_0 = \frac{-\frac{1}{2}k_0^2 k_1 \cdot k_2}{\Delta^2}
\]

and so on cyclically, where \( \Delta^2 \) is \( |k_i \times k_j|^2 \) for any pair \( i \neq j \). This shows that for the triangle, the centroid contact set is a centred contact set (which is just a complicated way of saying that the perpendicular bisectors of the sides of a triangle are concurrent). There is no corresponding result for \( n \geq 3 \) so that, in general, for \( n \geq 3 \) the centroid contact set is not a centred contact set.
We demonstrate that the collection $C$ of displacements $\Delta P_{ij}, 0 \leq i < j \leq n$ defined by (31) and (32) in §6 spans the space $\Omega$ of displacements $\Delta P = [\Delta p_0, \Delta p_1, \ldots, \Delta p_n]$ with each $\Delta p_i$ parallel to $F_i$ and which also maintains symmetry of $A$; that is those $\Delta P$ which satisfy the system (27).

We firstly observe that for each $i = 0, 1, \ldots, n$ the $k_{ij}, j \neq i$ span the space of directions parallel to $F_i$: a vector $v$ lies in this space if and only if $k_i \cdot v = 0$.

Now $v = \sum_j v_{ij} k_j$ since the $k_j$ span $\mathbb{R}^n$ so that $\sum_j v_{ij} (k_i \cdot k_j) = 0$ whence

$$v_{ii} = -\sum_{j \neq i} \frac{k_i \cdot k_j}{|k_j|^2} v_{ij}$$

giving $v = \sum_{j \neq i} v_{ij} k_j$. For each $i$, the set $\{k_{ij} : j \neq i\}$ is dependent: from $\sum_{j=0}^n k_j = 0$ there follows the unique dependency

$$\sum_{j \neq i} k_{ij} = 0. \quad (53)$$

Thus each $\Delta P$ having each $\Delta p_i$ parallel to $F_i$ may be written (non-uniquely) as $\Delta P = [\Delta p_0, \Delta p_1, \ldots, \Delta p_n]$ where

$$\Delta p_i = \sum_{j=0}^n c_{ij} k_{ij} \quad (54)$$

in which $c_{ii} = 0$, and this ensures that the parallel condition $k_i \cdot \Delta p_i = 0$ is satisfied.

We now exploit the lack of uniqueness in the representation (54). For $i = 0, 1, \ldots, n$, in view of (53) we have

$$\Delta p_i = \sum_{j=0}^n c_{ij} k_{ij} + \mu_i \sum_{j=0, j \neq i}^n k_{ij}$$

and we choose $\mu_i$ so that

$$c_{i0} + \mu_i = c_{0i} \quad \text{for} \quad i = 0, 1, \ldots, n,$$

(so that $\mu_0 = 0$) and now define $c'_{ij}, 0 \leq i, j \leq n$ by

$$c'_{0i} = c_{0i}, \quad i = 0, 1, \ldots, n; \quad (55)$$

$$c'_{i0} = c_{i0} + \mu_i, \quad i = 0, 1, \ldots, n; \quad (56)$$

$$c'_{ij} = c_{ij} + \mu_i, \quad 1 \leq i \neq j \leq n; \quad (57)$$

$$c'_{ii} = 0, \quad 0 \leq i \leq n. \quad (58)$$

(59)
Then for the given \( \Delta P \) we now have

\[
\Delta p_i = \sum_{j=0}^{n} c'_{ij} k_{ij}
\]  

(60)

where \( c'_{ii} = 0 \) and \( c'_{i0} = c'_{0i} \) for \( i = 0, 1, \ldots, n \). We now study the effect of the symmetry condition

\[
\sum_{i=0}^{n} k_i \wedge \Delta p_i = 0
\]  

(61)

on the representation (60) and observe from (31) that

\[
k_i \wedge k_{ij} = k_i \wedge k_j.
\]  

(62)

Hence substituting (60) into (61) and using (62) there follows

\[
0 = \sum_{i=0}^{n} k_i \wedge \sum_{j=0}^{n} c'_{ij} k_{ij} = \sum_{i,j=0}^{n} c'_{ij} k_i \wedge k_j
\]  

(63)

\[
= \sum_{j=0}^{n} c'_{0j} k_0 \wedge k_j + \sum_{i=0}^{n} c'_{i0} k_i \wedge k_0 + \sum_{i,j=1}^{n} c'_{ij} k_i \wedge k_j
\]  

(64)

\[
= \sum_{i,j=1}^{n} c'_{ij} k_i \wedge k_j,
\]  

(65)

the terms with \( i \) or \( j \) being zero cancelling because of the skew symmetry of \( k_i \wedge k_j \) and the normalisation \( c'_{0j} = c'_{j0} \). But since the wedge products \( k_i \wedge k_j \) are independent for \( 1 \leq i < j \leq n \) it follows that \( c'_{ij} = c'_{ji} \) for all \( 1 \leq i < j \leq n \), whence \( c'_{ij} = c'_{ji} \) for \( 0 \leq i < j \leq n \) together with \( c'_{ii} = 0, \ 0 \leq i \leq n \). Thus

\[
\Delta P = [\Delta p_0, \Delta p_1, \ldots, \Delta p_n] = [\sum c'_{0j} k_0, \sum c'_{1j} k_1, \ldots, \sum c'_{nj} k_n]
\]  

(66)

\[
= \sum_{0 \leq i < j \leq n} c'_{ij} \Delta P_{ij}
\]  

(67)

showing that the \( \Delta P_{ij}, \ 0 \leq i < j \leq n \) span \( \Omega \).
10 Appendix D

We show that the displacements $\Delta P_{ij}, 0 \leq i < j \leq n$ which satisfy equation (27) are not independent. If $\sum_{0 \leq i < j \leq n} c_{ij} \Delta P_{ij}$ vanishes, then each $\Delta p_m = 0$. Noting that $\Delta P_{ij}$ only has non-zero components in the $i$th and $j$th locations, for the linear combination $\sum_{0 \leq i < j \leq n} c_{ij} \Delta P_{ij}$, we have for $0 \leq m \leq n$

$$0 = \Delta p_m = \sum_{i=0}^{m-1} c_{im} k_{mi} + \sum_{j=m+1}^{n} c_{mj} k_{mj},$$

where for $m = 0$, the first sum in the right hand side vanishes and for $m = n$, the second sum vanishes. Hence by (31) for $0 \leq m \leq n$,

$$\sum_{i=0}^{m-1} c_{im} k_i + \sum_{j=m+1}^{n} c_{mj} k_j - \left( \sum_{i=0}^{m-1} c_{im} \frac{k_i \cdot k_m}{|k_m|^2} + \sum_{j=m+1}^{n} c_{mj} \frac{k_j \cdot k_m}{|k_m|^2} \right) k_m = 0.$$

But the only dependency between $k_0, k_1, \ldots, k_n$ is $\sum_{j=0}^{n} k_j = 0$ from which it follows that

$$c_{0m} = c_{1m} = \cdots = c_{(m-1)m} = c_{m(m+1)} = \cdots = c_{mn}.$$

Applying this for each $m = 0, 1, \ldots, n$ gives that all the $c_{ij}, 0 \leq i < j \leq n$ must be equal. Thus there is precisely one dependency between the $\Delta P_{ij}$ and the space of displacement sets maintaining symmetry of $A$ is $\frac{1}{2} n(n+1) - 1$ dimensional.