MULTI-PARAMETER EXTENSIONS OF A THEOREM OF PICHORIDES

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Abstract. Extending work of Pichorides and Zygmund to the d-dimensional setting, we show that the supremum of $L^p$-norms of the Littlewood-Paley square function over the unit ball of the analytic Hardy spaces $H^p_A(T^d)$ blows up like $(p-1)^{-d}$ as $p \to 1^+$. Furthermore, we obtain an $L \log d$-estimate for square functions on $H^1_A(T^d)$. Euclidean variants of Pichorides’s theorem are also obtained.

1. Introduction

Given a trigonometric polynomial $f$ on $\mathbb{T}$, we define the classical Littlewood-Paley square function $S_{\mathbb{T}}(f)$ of $f$ by

$$S_{\mathbb{T}}(f)(x) = \left( \sum_{k \in \mathbb{Z}} |\Delta_k(f)(x)|^2 \right)^{1/2},$$

where for $k \in \mathbb{N}$, we set

$$\Delta_k(f)(x) = \sum_{n = 2^k-1}^{2^k-1} \hat{f}(n)e^{i2\pi nx} \quad \text{and} \quad \Delta_{-k}(f)(x) = \sum_{n = -2^k+1}^{-2^k-1} \hat{f}(n)e^{i2\pi nx},$$

and for $k = 0$ we take $\Delta_0(f)(x) = \hat{f}(0)$.

A classical theorem of J.E. Littlewood and R.E.A.C. Paley asserts that for every $1 < p < \infty$ there exists a constant $B_p > 0$ such that

$$\|S_{\mathbb{T}}(f)\|_{L^p(\mathbb{T})} \leq B_p \|f\|_{L^p(\mathbb{T})} \quad (1)$$

for every trigonometric polynomial $f$ on $\mathbb{T}$, see, e.g., [5] or [19].

The operator $S_{\mathbb{T}}$ is not bounded on $L^1(\mathbb{T})$, and hence, the constant $B_p$ in (1) blows up as $p \to 1^+$. In [2], J. Bourgain obtained the sharp estimate

$$B_p \sim (p-1)^{-3/2} \quad \text{as} \quad p \to 1^+. \quad (2)$$

For certain subspaces of $L^p(\mathbb{T})$, however, one might hope for better bounds. In [12], S. Pichorides showed that for the analytic Hardy spaces $H^p_A(\mathbb{T})$ $(1 < p \leq 2)$, we have

$$\sup_{\|f\|_{L^p(\mathbb{T})} \leq 1} |S_{\mathbb{T}}(f)(x)| \sim (p-1)^{-1} \quad \text{as} \quad p \to 1^+. \quad (3)$$

Higher-dimensional extensions of Bourgain’s result (2) were obtained by the first author in [1]. In particular, given a dimension $d \in \mathbb{N}$, if $f$ is a trigonometric polynomial on $\mathbb{T}^d$, we define its $d$-parameter Littlewood-Paley square function by

$$S_{\mathbb{T}^d}(f)(x) = \left( \sum_{k_1, \ldots, k_d \in \mathbb{Z}} |\Delta_{k_1, \ldots, k_d}(f)(x)|^2 \right)^{1/2},$$
where for $k_1, \cdots, k_d \in \mathbb{Z}$ we use the notation $\Delta_{k_1, \cdots, k_d}(f) = \Delta_{k_1} \otimes \cdots \otimes \Delta_{k_d}(f)$, where the direct product notation indicates the operator $\Delta_{k_j}$ in the $j$-th position acting on the $j$-th variable. As in the one-dimensional case, for every $1 < p < \infty$, there is a positive constant $B_p(d)$ such that
\[
|S_{\mathbb{T}^d}(f)|_{L^p(\mathbb{T}^d)} \leq B_p(d) \|f\|_{L^p(\mathbb{T}^d)}
\]
for each trigonometric polynomial $f$ on $\mathbb{T}^d$. It is shown in [1] that
\[
B_p(d) \sim_d (p - 1)^{-3d/2} \quad \text{as} \quad p \to 1^+.
\] (4)

A natural question in this context is whether one has an improvement on the limiting behaviour of $B_p(d)$ as $p \to 1^+$ when restricting to the analytic Hardy spaces $H_A^p(\mathbb{T}^d)$. In other words, one is led to ask whether the aforementioned theorem of Pichorides can be extended to the polydisc. However, the proof given in [12] relies on factorisation of Hardy spaces, and it is known, see for instance [14, Chapter 5] and [15], that canonical factorisation fails in higher dimensions.

In this note we obtain an extension of (3) to the polydisc as a consequence of a theorem of Tao and Wright [16] on the endpoint mapping properties of Marcinkiewicz multiplier operators on the line, transferred to the periodic setting, with a variant of Marcinkiewicz interpolation to higher dimensions. In the last section we obtain a Euclidean version of Theorem 1 by using the aforementioned theorem of Tao and Wright [16], combined with a theorem of Peter Jones [7] on a Marcinkiewicz-type decomposition for analytic Hardy spaces over the real line.

Our main result in this paper is the following theorem.

**Theorem 1.** Let $d \in \mathbb{N}$ be a given dimension. If $T_{m_j}$ is a Marcinkiewicz multiplier operator on $\mathbb{T}$ ($j = 1, \cdots, d$), then for every $f \in H_A^p(\mathbb{T}^d)$ one has
\[
|(T_{m_1} \otimes \cdots \otimes T_{m_d})(f)|_{L^p(\mathbb{T}^d)} \lesssim_{C_{m_1}, \cdots, C_{m_d}} (p - 1)^{-d} \|f\|_{L^p(\mathbb{T}^d)}
\]
as $p \to 1^+$, where $C_{m_j} = \|m_j\|_{\ell^\infty(\mathbb{Z})} + B_{m_j}$, $B_{m_j}$ being as in (5), $j = 1, \cdots, d$.

To prove Theorem 1, we use a theorem of T. Tao and J. Wright [16] on the endpoint mapping properties of Marcinkiewicz multiplier operators on the line, transferred to the periodic setting, with a variant of Marcinkiewicz interpolation for Hardy spaces which is due to S. Kislyakov and Q. Xu [8]. Since for every choice of signs the randomised version $\sum_{k \in \mathbb{Z}} \pm \Delta_k$ of $S_{\mathbb{T}}$ is a Marcinkiewicz multiplier operator on the torus with corresponding constant $B_m \leq 2$, Theorem 1 and Khintchine’s inequality yield the following $d$-parameter extension of Pichorides’s theorem [3].

**Corollary 2.** Given $d \in \mathbb{N}$, if $S_{\mathbb{T}^d}$ denotes the $d$-parameter Littlewood-Paley square function, then one has
\[
\sup_{\|f\|_{L^p(\mathbb{T}^d)} \leq 1} \|S_{\mathbb{T}^d}(f)\|_{L^p(\mathbb{T}^d)} \sim_d (p - 1)^{-d} \quad \text{as} \quad p \to 1^+.
\]

The present paper is organised as follows: In the next section we set down notation and provide some background, and in Section 3 we prove our main results. Using the methods of Section 2, in Section 4 we extend a well-known inequality due to A. Zygmund [18, Theorem 8] to higher dimensions. In the last section we obtain a Euclidean version of Theorem 1 by using the aforementioned theorem of Tao and Wright [16], combined with a theorem of Peter Jones [7] on a Marcinkiewicz-type decomposition for analytic Hardy spaces over the real line.
2. Preliminaries

2.1. Notation. We denote the set of natural numbers by \( \mathbb{N} \), by \( \mathbb{N}_0 \) the set of non-negative integers and by \( \mathbb{Z} \) the set of integers.

Let \( f \) be a function of \( d \)-variables. Fixing the first \( d-1 \) variables \((x_1, \ldots, x_{d-1})\), we write \( f(x_1, \ldots, x_d) = f_{x_1, \ldots, x_{d-1}}(x_d) \). The sequence of the Fourier coefficients of a function \( f \in L^1(\mathbb{T}^d) \) will be denoted by \( \hat{f} \).

Given a function \( m \in L^\infty(\mathbb{R}^d) \), we denote by \( T_m \) the multiplier operator corresponding to \( m \), initially defined on \( L^2(\mathbb{R}^d) \), by \( (T_m(f))(\xi) = m(\xi)\hat{f}(\xi), \xi \in \mathbb{R}^d \). Given \( \mu \in L^\infty(\mathbb{Z}^d) \), one defines (initially on \( L^2(\mathbb{T}^d) \)) the corresponding periodic multiplier operator \( T_\mu \) in an analogous way.

If \( \lambda \) is a continuous and bounded function on the real line and \( T_\lambda \) is as above, \( T_\lambda |_{\mathbb{T}} \) denotes the periodic multiplier operator such that \( T_\lambda(f)(x) = \sum_{n \in \mathbb{Z}} \lambda(n)\hat{f}(n)e^{2\pi inx} \) \((x \in \mathbb{T})\) for every trigonometric polynomial \( f \) on \( \mathbb{T} \).

Given two positive quantities \( X \) and \( Y \) and a parameter \( \alpha \), we write \( X \lessapprox Y \) (or simply \( X \lessapprox Y \)) whenever there exists a constant \( C_\alpha > 0 \) depending on \( \alpha \) so that \( X \leq C_\alpha Y \). If \( X \lessapprox_\alpha Y \) and \( Y \lessapprox_\alpha X \), we write \( X \sim_\alpha Y \) (or simply \( X \sim Y \)).

2.2. Hardy spaces and Orlicz spaces. Let \( d \in \mathbb{N} \). For \( 0 < p < \infty \), let \( H_p^A(\mathbb{D}^d) \) denote the space of holomorphic functions \( F \) on \( \mathbb{D}^d \), \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \), such that

\[
\|F\|_{H_p^A(\mathbb{D}^d)} = \sup_{0 \leq r_1, \ldots, r_d < 1} \int_{\mathbb{T}^d} |F(r_1 e^{i2\pi x_1}, \ldots, r_d e^{i2\pi x_d})|^p dx_1 \cdots dx_d < \infty.
\]

For \( p = \infty \), \( H_\infty^A(\mathbb{D}^d) \) denotes the class of bounded holomorphic functions on \( \mathbb{D}^d \). It is well-known that for \( 1 \leq p < \infty \), the limit \( f \) of \( F \in H_p^A(\mathbb{D}^d) \) as we approach the distinguished boundary \( \mathbb{T}^d \) of \( \mathbb{D}^d \), namely

\[
f(x_1, \ldots, x_d) = \lim_{r_1, \ldots, r_d \to 1^-} F(r_1 e^{i2\pi x_1}, \ldots, r_d e^{i2\pi x_d})
\]
exists a.e. in \( \mathbb{T}^d \) and \( \|F\|_{H_p^A(\mathbb{D}^d)} = \|f\|_{L^p(\mathbb{T}^d)} \). For \( 1 \leq p \leq \infty \), we define the analytic Hardy space \( H_p^A(\mathbb{T}^d) \) on the \( d \)-torus as the space of all functions in \( L^p(\mathbb{T}^d) \) that are boundary values of functions in \( H_p^A(\mathbb{D}^d) \). Moreover, it is a standard fact that \( H_p^A(\mathbb{T}^d) = \{ f \in L^p(\mathbb{T}^d) : \operatorname{supp}(\hat{f}) \subset \mathbb{N}_0^d \} \). Hardy spaces are discussed in Chapter 7 in [4], where the case \( d = 1 \) is treated, and in Chapter 3 of [14].

If \( f \in L^1(\mathbb{T}^d) \) is such that \( \operatorname{supp}(\hat{f}) \) is finite, then \( f \) is said to be a trigonometric polynomial on \( \mathbb{T}^d \), and if moreover \( \operatorname{supp}(\hat{f}) \subset \mathbb{N}_0^d \), then \( f \) is said to be analytic. It is well-known [4, 4.4] that for \( 1 \leq p < \infty \), the class of trigonometric polynomials on \( \mathbb{T}^d \) is a dense subspace of \( L^p(\mathbb{T}^d) \) and analytic trigonometric polynomials on \( \mathbb{T}^d \) are dense in \( H_p^A(\mathbb{T}^d) \).

We define the real Hardy space \( H^1(\mathbb{T}) \) to be the space of all integrable functions \( f \in L^1(\mathbb{T}) \) such that \( H_\mathbb{T}(f) \in L^1(\mathbb{T}) \), where \( H_\mathbb{T}(f) \) denotes the periodic Hilbert transform of \( f \). One sets \( \|f\|_{H^1(\mathbb{T})} = \|f\|_{L^1(\mathbb{T})} + \|H_\mathbb{T}(f)\|_{L^1(\mathbb{T})} \). Note that \( H^1(\mathbb{T}) \) can be regarded as a proper subspace of \( H^1(\mathbb{T}) \) and moreover, \( \|f\|_{H^1(\mathbb{T})} = 2\|f\|_{L^1(\mathbb{T})} \) when \( f \in H^1(\mathbb{T}) \).

Given \( d \in \mathbb{N} \), for \( 0 < p < \infty \), let \( H_p^A((\mathbb{R}_+^d)^d) \) denote the space of holomorphic functions \( F \) on \((\mathbb{R}_+^d)^d \), where \( \mathbb{R}_+^d = \{ x + iy : x > 0 \} \), such that

\[
\|F\|_{H_p^A((\mathbb{R}_+^d)^d)} = \sup_{y_1, \ldots, y_d \geq 0} \int_{\mathbb{R}^d} |F(x_1 + iy_1, \ldots, x_d + iy_d)|^p dx_1 \cdots dx_d < \infty.
\]

For \( p = \infty \), \( H_\infty^A((\mathbb{R}_+^d)^d) \) is defined as the space of bounded holomorphic functions in \((\mathbb{R}_+^d)^d \). For \( 1 \leq p \leq \infty \), for every \( F \in H_p^A((\mathbb{R}_+^d)^d) \) its limit \( f \) as we approach the
boundary $\mathbb{R}^d$, namely
\[
f(x_1, \ldots, x_d) = \lim_{y_1, \ldots, y_d \to 0^+} F(x_1 + iy_1, \ldots, x_d + iy_d),
\]
exists for $a.e.$ $(x_1, \ldots, x_d) \in \mathbb{R}^d$ and, moreover, $\|F\|_{H^p_A(\mathbb{R}^d_+)} = \|f\|_{L^p(\mathbb{R}^d)}$. Hence, as in the periodic setting, for $1 \leq p \leq \infty$ we may define the $d$-parameter analytic Hardy space $H^p_A(\mathbb{R}^d)$ to be the space of all functions in $L^p(\mathbb{R}^d)$ that are boundary values of functions in $H^p_A(\mathbb{R}^d_+)$.

The real Hardy space $H^1(\mathbb{R})$ on the real line is defined as the space of all integrable functions $f$ on $\mathbb{R}$ such that $H(f) \in L^1(\mathbb{R})$, where $H(f)$ is the Hilbert transform of $f$. Moreover, we set $\|f\|_{H^1(\mathbb{R})} = \|f\|_{L^1(\mathbb{R})} + \|H(f)\|_{L^1(\mathbb{R})}$.

We shall also consider the standard Orlicz spaces $L^{\log^+} L(\mathbb{T}^d)$. For $r > 0$, one may define $L^{\log^+} L(\mathbb{T}^d)$ as the space of measurable functions $f$ on $\mathbb{T}^d$ such that $\int_{\mathbb{T}^d} |f(x)| \log^+(1 + |f(x)|) \, dx < \infty$. For $r \geq 1$, we may equip $L^{\log^+} L(\mathbb{T}^d)$ with a norm given by
\[
\|f\|_{L^{\log^+} L(\mathbb{T}^d)} = \inf \{ \lambda > 0 : \int_{\mathbb{T}^d} \frac{|f(x)|}{\lambda} \log^+ \left(1 + \frac{|f(x)|}{\lambda}\right) \, dx \leq 1 \}.
\]
For more details on Orlicz spaces, we refer the reader to the books [3] and [19].

3. Proof of Theorem 1

Recall that a function $m \in L^\infty(\mathbb{R})$ is said to be a Marcinkiewicz multiplier on $\mathbb{R}$ if it is differentiable in every dyadic interval $[2^k, 2^{k+1})$, $k \in \mathbb{Z}$ and
\[
A_m = \sup_{k \in \mathbb{Z}} \left( \int_{[2^k, 2^{k+1})} |m'(\xi)| \, d\xi + \int_{(-2^{k+1}, -2^k]} |m'(\xi)| \, d\xi \right) < \infty
\]
If $m \in L^\infty(\mathbb{R})$ satisfies (4), then thanks to a classical result of J. Marcinkiewicz, see e.g. [3], the corresponding multiplier operator $T_m$ is bounded on $L^p(\mathbb{R})$ for all $1 < p < \infty$. In [14], Tao and Wright showed that every Marcinkiewicz multiplier operator $T_m$ is bounded from the real Hardy space $H^1(\mathbb{R})$ to $L^{1, \infty}(\mathbb{R})$, namely
\[
\|T_m(f)\|_{L^{1, \infty}(\mathbb{R})} \leq C_m \|f\|_{H^1(\mathbb{R})},
\]
where the constant $C_m$ depends only on $\|m\|_{L^\infty(\mathbb{R})} + A_m$, with $A_m$ as in (6). For the sake of completeness, let us recall that $L^{1, \infty}(\mathcal{M})$ stands for the quasi-Banach space of measurable functions on a measure space $\mathcal{M}$ endowed with the quasi-norm
\[
\|f\|_{L^{1, \infty}(\mathcal{M})} := \sup_{t > 0} \int_{\{x \in \mathcal{M} : |f(x)| > t\}} \, dx.
\]

Either by adapting the proof of Tao and Wright to the periodic setting or by using a transference argument, see Subsection 3.1, one deduces that every periodic Marcinkiewicz multiplier operator $T_m$ satisfies
\[
\|T_m(f)\|_{L^{1, \infty}(\mathbb{T})} \leq D_m \|f\|_{H^1(\mathbb{T})},
\]
where $D_m$ depends on $\|m\|_{L^\infty(\mathbb{Z})} + B_m$, $B_m$ being as in (8). Therefore, it follows that for every $f \in H^1_A(\mathbb{T})$ one has
\[
\|T_m(f)\|_{L^{1, \infty}(\mathbb{T})} \leq D'_m \|f\|_{L^1(\mathbb{T})},
\]
where one may take $D'_m = 2D_m$. We shall prove that for every Marcinkiewicz multiplier operator $T_m$ on the torus one has
\[
\sup_{\|f\|_{L^p(\mathbb{T})} \leq 1} \|T_m(f)\|_{L^p(\mathbb{T})} \leq B_m (p - 1)^{-1}
\]
\[\text{Marcinkiewicz originally proved the theorem in the periodic setting, see [10].}\]
as $p \to 1^+$. To do this, we shall make use of the following lemma due to Kislyakov and Xu [8].

**Lemma 3** (Kislyakov and Xu, [8] and [11]). If $f \in H^p_T(\mathbb{T})$ ($0 < p_0 < \infty$) and $\lambda > 0$, then there exist functions $h_\lambda \in H^p_T(\mathbb{T})$, $g_\lambda \in H^p_T(\mathbb{T})$ and a constant $C_{p_0} > 0$ depending only on $p_0$ such that

- $|h_\lambda(x)| \leq C_{p_0, \lambda} \min\{\lambda^{-1}|f(x)|, |f(x)|^{-1}\}$ for all $x \in \mathbb{T}$,
- $\|g_\lambda\|_{L^2(\mathbb{T})} \leq C_{p_0, \lambda} \int_{\mathbb{T}} |f(x)|^{p_0} dx$,
- $f = h_\lambda + g_\lambda$.

We remark that by examining the proof of Lemma 3 one deduces that when $1 \leq p_0 \leq 2$ the constant $C_{p_0}$ in the statement of the lemma can be chosen independent of $p_0$. To prove the desired inequality (10) and hence Theorem 1 in the one-dimensional case, we argue as in [11, Theorem 7.4.1]. More precisely, given a $1 < p < 2$, if $f$ is a fixed analytic trigonometric polynomial on $\mathbb{T}$, we first write $\|T_m(f)\|_{L^p(\mathbb{T})} = p \int_0^\infty \lambda^{p-1} \int_{\mathbb{T}} |f(x)|^p dx d\lambda$ and then by using Lemma 3 for $p_0 = 1$ we obtain $\|T_m(f)\|_{L^p(\mathbb{T})} \leq I_1 + I_2$, where

$$I_1 = p \int_0^\infty \lambda^{p-1} \int_{\mathbb{T}} |T_m(g_\lambda)(x)| > \lambda/2 |d\lambda$$

and

$$I_2 = p \int_0^\infty \lambda^{p-1} \int_{\mathbb{T}} |T_m(h_\lambda)(x)| > \lambda/2 |d\lambda.$$

To handle $I_1$, we use the boundedness of $T_m$ from $H^1_T(\mathbb{T})$ to $L^{1,\infty}(\mathbb{T})$ and Fubini’s theorem to deduce that

$$I_1 \leq (p - 1)^{-1} \int_\mathbb{T} |f(x)|^p dx.$$

To obtain appropriate bounds for $I_2$, we use the boundedness of $T_m$ from $H^2_T(\mathbb{T})$ to $L^2(\mathbb{T})$ and get

$$I_2 \leq \int_0^\infty \lambda^{p-3} \int_{\mathbb{T}} |f(x)|^2 dx d\lambda + \int_0^\infty \lambda^{p+1} \int_{\mathbb{T}} |f(x)|^{-2} dx d\lambda.$$

Hence, by applying Fubini’s theorem to each term, we obtain

$$I_2 \leq (2 - p)^{-1} \int_\mathbb{T} |f(x)|^p dx + (p + 2)^{-1} \int_\mathbb{T} |f(x)|^p dx.$$

Combining the estimates for $I_1$ and $I_2$ and using the density of analytic trigonometric polynomials in $(H^p_T(\mathbb{T}), \cdot \|L^p(\mathbb{T})\), (10)$ follows.

To prove the $d$-dimensional case, take $f$ to be an analytic trigonometric polynomial on $\mathbb{T}^d$ and note that if $T_m$ are periodic Marcinkiewicz multiplier operators ($j = 1, \ldots, d$), then for fixed $(x_1, \ldots, x_{d-1}) \in \mathbb{T}^{d-1}$ one can write

$$T_m(g(x_1, \ldots, x_{d-1}))(x_d) = T_{m_1} \otimes \cdots \otimes T_{m_d}(f)(x_1, \ldots, x_d),$$

where

$$g(x_1, \ldots, x_{d-1})(x_d) = T_{m_1} \otimes \cdots \otimes T_{m_{d-1}}(f)(x_1, \ldots, x_{d-1})(x_d).$$

Hence, by using (10) in the $d$-th variable, one deduces that

$$\|T_m(g(x_1, \ldots, x_{d-1}))\|_{L^p(\mathbb{T}^d)} \leq C_{m_\lambda} (p - 1)^{-d} \|g(x_1, \ldots, x_d)\|_{L^p(\mathbb{T}^d)}$$

where $C_{m_d} > 0$ is the implied constant in (10) corresponding to $T_{m_d}$. By iteratively applying this argument $d - 1$ times, one obtains

$$\|T_{m_1} \otimes \cdots \otimes T_{m_d}(f)\|_{L^p(\mathbb{T}^d)} \leq [C_{m_1} \cdots C_{m_d}]^p (p - 1)^{-dp} \|f\|_{L^p(\mathbb{T}^d)}$$

and this completes the proof of Theorem 1.
3.1. Proof of Corollary 2. We shall use the multi-dimensional version of Khintchine’s inequality: if \((r_k)_{k \in \mathbb{N}_0}\) denotes the set of Rademacher functions indexed by \(\mathbb{N}_0\) over a probability space \((\Omega, \mathcal{A}, P)\), then for every finite collection of complex numbers \((a_{k_1, \ldots, k_d})_{k_1, \ldots, k_d \in \mathbb{N}_0}\) one has

\[
\left\| \sum_{k_1, \ldots, k_d \in \mathbb{N}_0} a_{k_1, \ldots, k_d} r_{k_1} \otimes \cdots \otimes r_{k_d} \right\|_{L^p(\Omega^d)} \approx_p \left( \sum_{k_1, \ldots, k_d \in \mathbb{N}_0} |a_{k_1, \ldots, k_d}|^2 \right)^{1/2}
\]

for all \(0 < p < \infty\). The implied constants do not depend on \((a_{k_1, \ldots, k_d})_{k_1, \ldots, k_d \in \mathbb{N}_0}\), see e.g. Appendix D in [15], and do not blow up as we have the improved estimate

\[
\left\| \sum_{k_1, \ldots, k_d \in \mathbb{N}_0} a_{k_1, \ldots, k_d} r_{k_1} \otimes \cdots \otimes r_{k_d} \right\|_{L^p(\Omega^d)} \lesssim_p \left( \sum_{k_1, \ldots, k_d \in \mathbb{N}_0} |a_{k_1, \ldots, k_d}|^2 \right)^{1/2}
\]

Combining Theorem applied to \(d\)-fold tensor products of periodic Marcinkiewicz multipliers operators of the form \(\sum_{k \in \mathbb{Z}} \pm \Delta_k\), with the multi-dimensional Khintchine’s inequality as in [1]. Section 3) shows that the desired bound holds for analytic polynomials. Since analytic trigonometric polynomials on \(\mathbb{T}^d\) are dense in \((H^p_A(\mathbb{T}^d), \| \cdot \|_{L^p(\mathbb{T}^d)})\), we deduce that

\[
\sup_{\| f \|_{L^p(\mathbb{T}^d)} \leq 1} \left\| S_{T^d}(f) \right\|_{L^p(\mathbb{T}^d)} \lesssim_d (p - 1)^{-d} \quad \text{as } p \to 1^+.
\]

It remains to prove the reverse inequality. To do this, for fixed \(1 < p \leq 2\), choose an \(f \in H^p_A(\mathbb{T})\) such that

\[
\left\| S_T(f) \right\|_{L^p(\mathbb{T})} \geq C(p - 1)^{-1} \| f \|_{L^p(\mathbb{T})},
\]

where \(C > 0\) is an absolute constant. The existence of such functions is shown in [12]. Hence, if we define \(g \in H^p_A(\mathbb{T}^d)\) by

\[
g(x_1, \ldots, x_d) = f(x_1) \cdots f(x_d)
\]

for \((x_1, \ldots, x_d) \in \mathbb{T}^d\), then

\[
\left\| S_{T^d}(g) \right\|_{L^p(\mathbb{T}^d)} = \left\| S_T(f) \right\|_{L^p(\mathbb{T})} \cdot \left\| S_T(f) \right\|_{L^p(\mathbb{T})} \geq C^d(p - 1)^{-d} \| f \|_{L^p(\mathbb{T})}^d = C^d(p - 1)^{-d} \| g \|_{L^p(\mathbb{T}^d)}
\]

and this proves the sharpness of Corollary 2.

Remark 4. For the subspace \(H^p_{A, \text{diag}}(\mathbb{T}^d)\) of \(L^p(\mathbb{T}^d)\) consisting of functions of the form \(f(x_1, \ldots, x_d) = F(x_1 + \cdots + x_d)\) for some one-variable function \(F \in H^p_A(\mathbb{T})\), we have the improved estimate

\[
\sup_{\| f \|_{L^p(\mathbb{T}^d)} \leq 1} \left\| S_{T^d}(f) \right\|_p \sim_p (p - 1)^{-1}, \quad p \to 1^+.
\]

This follows from invariance of the \(L^p\)-norm and Fubini’s theorem which allow us to reduce to the one-dimensional case. On the other hand, the natural inclusion of \(H^p_A(\mathbb{T}^k)\) in \(H^p_A(\mathbb{T}^d)\) yields examples of subspaces with sharp blowup of order \((p - 1)^{-k}\) for any \(k = 1, \ldots, d - 1\).

Both the original proof of Pichorides’s theorem and the extension in this paper rely on complex-analytic techniques, via canonical factorisation in [12] and conjugate functions in [11]. However, a complex-analytic structure is not necessary in order for an estimate of the form in Corollary 2 to hold. For instance, the same conclusion remains valid for \(g \in H^p(\mathbb{T}^d)\), the subset of \(L^p(\mathbb{T}^d)\) consisting of functions with \(\text{supp}(f) \subset (-\mathbb{N})^d\). Moreover, for functions of the form \(f + g\), where \(f \in H^p_A(\mathbb{T}^d)\) and \(g \in H^p(\mathbb{T}^d)\) with \(\| f \|_p = \| g \|_p \leq 1/2\), we then have \(\| f + g \|_p \leq 1\) and \(\| S_{T^d}(f + g) \|_p \leq \| S_{T^d}(f) \|_p + \| S_{T^d}(g) \|_p \lesssim (p - 1)^{-d}\) as \(p \to 1^+\).
The exponent $r$.

**Proposition 6.** Given $d \in \mathbb{N}$, there exists a constant $C_d > 0$ such that for every analytic trigonometric polynomial $g$ on $\mathbb{T}^d$ one has

$$
\|S_g(f)\|_{L^1(\mathbb{T}^d)} \leq C_d \|g\|_{L^{r}(\mathbb{T}^d)}.
$$

The exponent $r = d$ in the Orlicz space $L^{d}(\mathbb{T}^d)$ cannot be improved.
Proof. By using Lemma 3 and a Marcinkiewicz-type interpolation argument analogous to the one presented in Section 3 one shows that if \( T \) is a sublinear operator that is bounded from \( H^p_A(\mathbb{T}) \) to \( L^{1,p}(\mathbb{T}) \) and bounded from \( H^p_A(\mathbb{T}) \) to \( L^2(\mathbb{T}) \), then for every \( r \geq 0 \) one has

\[
\int_{\mathbb{T}} |T(f)(x)| \log^r (1 + |T(f)(x)|) \, dx \leq C_r \left[ 1 + \int_{\mathbb{T}} |f(x)| \log^{r+1} (1 + |f(x)|) \, dx \right]
\]

for every analytic trigonometric polynomial \( f \) on \( \mathbb{T} \), where \( C_r > 0 \) is a constant depending only on \( r \).

If \( T_{\omega_j} = \sum_{k \in \mathbb{Z}} r_k(\omega_j) \Delta_j \) denotes a randomised version of \( S_{\omega_j} \), \( j = 1, \ldots, d \), then \( T_{\omega_j} \) maps \( H^p_A(\mathbb{T}) \) to \( L^{1,p}(\mathbb{T}) \) and \( H^p_A(\mathbb{T}) \) to \( L^2(\mathbb{T}) \) and so, by using (14) and iteration, one deduces that

\[
\| (T_{\omega_1} \otimes \cdots \otimes T_{\omega_d})(f) \|_{L^1(\mathbb{T}^d)} \leq A_d \| f \|_{L^{\log^d} L(\mathbb{T}^d)}
\]

for every analytic polynomial \( f \) on \( \mathbb{T}^d \). Hence, the proof of (13) is obtained by using (15) and (11).

To prove sharpness for \( d = 1 \), let \( N \) be a large positive integer to be chosen later and take \( V_N = 2K_{2N+1} - K_{2N} \) to be the de la Vallée Poussin kernel of order \( 2^N \), where \( K_n \) denotes the Fejér kernel of order \( n \). Then, for every analytic trigonometric polynomial \( f \), one has

\[
\| f_N \|_{L^{\log^d} L(\mathbb{T})} \lesssim N^r
\]

Then, one can easily check that \( f_N \in H^p_A(\mathbb{T}) \), \( \Delta_{N+1}(f_N)(x) = \sum_{k=2^N} 2^N e^{i2\pi kx} \) and \( \| f_N \|_{L^{\log} L(\mathbb{T})} \lesssim N^r \). Hence, if we assume that (12) holds for some \( L \log^r L(\mathbb{T}) \), then we see that we must have

\[
N \lesssim \| \Delta_{N+1}(f_N) \|_{L^1(\mathbb{T})} \lesssim \| S_{\omega}(f_N) \|_{L^1(\mathbb{T})} \lesssim \| f_N \|_{L^{\log} L(\mathbb{T})} \lesssim N^r
\]

and so, if \( N \) is large enough, it follows that \( r \geq 1 \), as desired. To prove sharpness in the \( d \)-dimensional case, take \( g_N(x_1, \cdots, x_d) = f_N(x_1) \cdots f_N(x_d) \), \( f_N \) being as above, and note that

\[
N^d \lesssim \| \Delta_{N+1}(f_N) \|_{L^1(\mathbb{T}^d)} \lesssim \| S_{\omega}(g_N) \|_{L^1(\mathbb{T}^d)} \lesssim \| g_N \|_{L^{\log} L(\mathbb{T}^d)} \lesssim N^r.
\]

Hence, by taking \( N \to \infty \), we deduce that \( r \geq d \).

Remark 7. Note that, by using (14) and (11), one can actually show that there exists a constant \( B_d > 0 \), depending only on \( d \), such that

\[
\| S_{\omega}(f) \|_{L^{\log} L(\mathbb{T}^d)} \leq B_d \| f \|_{L^{\log^{d-1}} L(\mathbb{T}^d)}
\]

for every analytic trigonometric polynomial \( f \) on \( \mathbb{T}^d \). Notice that if we remove the assumption that \( f \) is analytic, then the Orlicz space \( L^{\log^{d-1}} L(\mathbb{T}^d) \) in (16) must be replaced by \( L^{\log^{3d/2}} L(\mathbb{T}^d) \), see [1].

5. Euclidean variants of Theorem 1

In this section we obtain an extension of Pichorides’s theorem to the Euclidean setting. Our result will be a consequence of the following variant of Marcinkiewicz-type interpolation on Hardy spaces.

Proposition 8. Assume that \( T \) is a sublinear operator that satisfies:

- \( \| T(f) \|_{L^{1,p}(\mathbb{R})} \leq C \| f \|_{L^1(\mathbb{R})} \) for all \( f \in H^p_A(\mathbb{R}) \) and
- \( \| T(f) \|_{L^2(\mathbb{R})} \leq C \| f \|_{L^2(\mathbb{R})} \) for all \( f \in H^2_A(\mathbb{R}) \),

where \( C > 0 \) is an absolute constant. Then, for every \( 1 < p < 2 \), \( T \) maps \( H^p_A(\mathbb{R}) \) to \( L^p(\mathbb{R}) \) and moreover,

\[
\| T \|_{H^p_A(\mathbb{R}) \to L^p(\mathbb{R})} \leq [(p-1)^{-1} + (2-p)^{-1}]^{1/p}.
\]
Proof. Fix $1 < p < 2$ and take an $f \in H^p_\lambda(\mathbb{R})$. From a classical result due to Peter Jones [7, Theorem 2] it follows that for every $\lambda > 0$ one can write $f = F_\lambda + \lambda f$, where $F_\lambda \in H^2_\lambda(\mathbb{R})$, $\lambda f \in H^p_\lambda(\mathbb{R})$ and, moreover, there is an absolute constant $C_0 > 0$ such that

- $\int_{\mathbb{R}} |F_\lambda(x)| dx \leq C_0 \int_{\{x \in \mathbb{R} : N(f)(x) > \lambda\}} N(f)(x) dx$ and
- $\|f\|_{L^p(\mathbb{R})} \leq C_0 \lambda$.

Here, $N(f)$ denotes the non-tangential maximal function of $f \in H^p_\lambda(\mathbb{R})$ given by

$$N(f)(x) = \sup_{|x-x'| < t} |(f * P_t)(x')|,$$

where, for $t > 0$, $P_t(s) = t/(s^2 + t^2)$ denotes the Poisson kernel on the real line. Hence, by using the Peter Jones decomposition of $f$, we have

$$\|T(f)\|_{L^p(\mathbb{R})}^p = \int_0^\infty p \lambda^{p-1} |\{x \in \mathbb{R} : |T(f)(x)| > \lambda/2\}| d\lambda \leq I_1 + I_2,$$

where

$$I_1 = p \int_0^\infty \lambda^{p-1} |\{x \in \mathbb{R} : |T(F_\lambda)(x)| > \lambda/2\}| d\lambda$$

and

$$I_2 = p \int_0^\infty \lambda^{p-1} |\{x \in \mathbb{R} : |T(\lambda f)(x)| > \lambda/2\}| d\lambda.$$

We shall treat $I_1$ and $I_2$ separately. To bound $I_1$, using our assumption on the boundedness of $T$ from $H^2_\lambda(\mathbb{R})$ to $L^{1,q}(\mathbb{R})$ together with Fubini’s theorem, we deduce that there is an absolute constant $C_1 > 0$ such that

$$I_1 \leq C_1 (p - 1)^{-1} \int_{\mathbb{R}} |N(f)(x)|^p dx. \quad (17)$$

To bound the second term, we first use the boundedness of $T$ from $H^2_\lambda(\mathbb{R})$ to $L^2(\mathbb{R})$ as follows

$$I_2 \leq C \int_0^\infty p \lambda^{p-3} \left( \int_{\mathbb{R}} |f_\lambda(x)|^2 dx \right) d\lambda$$

and then we further decompose the right-hand side of the last inequality as $I_{2,\alpha} + I_{2,\beta}$, where

$$I_{2,\alpha} = C \int_0^\infty p \lambda^{p-3} \left( \int_{\{x \in \mathbb{R} : N(f)(x) > \lambda\}} |f_\lambda(x)|^2 dx \right) d\lambda$$

and

$$I_{2,\beta} = C \int_0^\infty p \lambda^{p-3} \left( \int_{\{x \in \mathbb{R} : N(f)(x) \leq \lambda\}} |f_\lambda(x)|^2 dx \right) d\lambda.$$

The first term $I_{2,\alpha}$ can easily be dealt with by using the fact that $\|f_\lambda\|_{L^p(\mathbb{R})} \leq C_0 \lambda$,

$$I_{2,\alpha} \leq C' \int_0^\infty p \lambda^{p-1} |\{x \in \mathbb{R} : N(f)(x) > \lambda\}| d\lambda = C' \int_{\mathbb{R}} |N(f)(x)|^p dx,$$

where $C' = C_0 C$. To obtain appropriate bounds for $I_{2,\beta}$, note that since $|f_\lambda|^2 = |f - F_\lambda|^2 \leq 2|f|^2 + 2|F_\lambda|^2$ one has $I_{2,\beta} \leq I'_{2,\beta} + I''_{2,\beta}$, where

$$I'_{2,\beta} = 2C \int_0^\infty p \lambda^{p-3} \left( \int_{\{x \in \mathbb{R} : N(f)(x) > \lambda\}} |f(x)|^2 dx \right) d\lambda$$

and

$$I''_{2,\beta} = 2C \int_0^\infty p \lambda^{p-3} \left( \int_{\{x \in \mathbb{R} : N(f)(x) \leq \lambda\}} |F_\lambda(x)|^2 dx \right) d\lambda.$$
To handle $I_{2,\beta}'$, note that since $f \in H^p_\theta(\mathbb{R})$ (1 < $p < 2$) one has $|f(x)| \leq N(f)(x)$ for a.e. $x \in \mathbb{R}$ and hence, by using this fact together with Fubini’s theorem, one obtains

$$I_{2,\beta}' \leq C(2 - p)^{-1} \int_{\mathbb{R}} [N(f)(x)]^p dx.$$ 

Finally, for the last term $I_{2,\beta}'$, we note that for a.e. $x \in \{N(f) \leq \lambda\}$ one has

$$|F_{\lambda}(x)| \leq |f(x)| + |f_{\lambda}(x)| \leq N(f)(x) + |f_{\lambda}(x)| \leq (1 + C_0)\lambda$$

and hence,

$$I_{2,\beta}' \leq C' \int_0^\infty \lambda^{p-2} \left( \int_{\mathbb{R}} |F_{\lambda}(x)| dx \right) d\lambda$$

$$\leq C' \int_0^\infty \lambda^{p-2} \left( \int_{\{x \in \mathbb{R} : N(f)(x) > \lambda\}} N(f)(x) dx \right) d\lambda \leq C''(p - 1)^{-1} \int_{\mathbb{R}} [N(f)(x)]^p dx,$$

where $C'' = 4(1 + C_0)$ and in the last step we used Fubini’s theorem. Since $I_2 \leq I_{2,\alpha} + I_{2,\beta}' + I_{2,\beta}'$, we conclude that there is a $C_2 > 0$ such that

$$I_2 \leq C_2 [(p - 1)^{-1} + (2 - p)^{-1}] \int_{\mathbb{R}} [N(f)(x)]^p dx. \quad (18)$$

It thus follows from (17) and (18) that

$$\|T(f)\|_{L^p(\mathbb{R})} \leq [(p - 1)^{-1} + (2 - p)^{-1}]^{1/p} \|N(f)\|_{L^p(\mathbb{R})}.$$ 

To complete the proof of the proposition note that one has

$$\|N(f)\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})} \quad (f \in H^p_\theta(\mathbb{R})), \quad (19)$$

where one can take $C_p = A_0^{1/p}$, $A_0 \geq 1$ being an absolute constant, see e.g. p.278-279 in vol.I in [19], where the periodic case is presented. The Euclidean version is completely analogous. Hence, if 1 < $p < 2$, one deduces that the constant $C_p$ in (19) satisfies $C_p \leq A_0$ and so, we get the desired result.
Corollary 10. For \( d \in \mathbb{N} \), one has
\[
\|S_{\mathbb{R}^d}\|_{H^p_{\mathbb{R}^d} \rightarrow L^p_{\mathbb{R}^d}} \sim_d (p - 1)^{-d}
\]
as \( p \to 1^+ \).

Remark 11. The multiplier operators covered in Theorem \( \mathbb{9} \) are properly contained in the class of general multi-parameter Marcinkiewicz multiplier operators treated in Theorem 6’ in Chapter IV of \( \mathbb{12} \). For a class of smooth multi-parameter Marcinkiewicz multipliers M. Wojciechowski \( \mathbb{17} \) proves that their \( L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d) \) operator norm is of order \( (p - 1)^{-d} \) and that they are bounded on the \( d \)-parameter Hardy space \( H^p(\mathbb{R} \times \cdots \times \mathbb{R}) \) for all \( 1 \leq p \leq 2 \). Note that the multi-parameter Littlewood-Paley square function is not covered by this result; see also \( \mathbb{1} \) for more refined negative statements.

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