RELAXATION SCHEMES FOR THE JOINT LINEAR CHANCE CONSTRAINT BASED ON PROBABILITY INEQUALITIES

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(Communicated by Jie Sun)

Abstract. This paper is concerned with the joint chance constraint for a system of linear inequalities. We discuss computationally tractable relaxations of this constraint based on various probability inequalities, including Chebyshev inequality, Petrov exponential inequalities, and others. Under the linear decision rule and additional assumptions about first and second order moments of the random vector, we establish several upper bounds for a single chance constraint. This approach is then extended to handle the joint linear constraint. It is shown that the relaxed constraints are second-order cone representable. Numerical test results are presented and the problem of how to choose proper probability inequalities is discussed.

1. Introduction. Stochastic programming is a crucial tool for solving optimization problems under uncertainty. An important topic in stochastic programming is how to deal with problems with chance constraints, initially studied in Charnes et al. (1958) and Soyster (1973). Various elegant methods have been reported, see [10, 27, 28, 15, 18, 14, 25, 3]. Systems with chance constraints have good stability properties, but it is difficult to transform the chance constraints into equivalent and computationally-tractable constraints [5, 17]. This paper, therefore, focuses on relaxation schemes for linear joint chance constraints, namely

\[ \mathbb{P}(A(\xi)x \leq b(\xi)) \geq 1 - \epsilon, \quad x \in X \]  

(1)

\( A(\xi)x \leq b(\xi) \) is an \( m \times n \) linear constraint system that depends on a random vector \( \xi \), \( \epsilon \in (0,1) \) is a tolerance variable and \( X \) is a deterministic convex polyhedron.

A variety of methods have been presented to handle chance constraint. Below we outline five typical methods.

1. (Chen et al., 2010 [7]) The CVaR Approach. Use the relationships

\[ \mathbb{P}(A_i(\xi)x \leq b_i(\xi)) \geq 1 - \epsilon \iff \text{VaR}_\epsilon(b_i(\xi) - A_i(\xi)x) \leq 0 \iff \text{CVaR}_\epsilon(b_i(\xi) - A_i(\xi)x) \leq 0 \]

2020 Mathematics Subject Classification. 90C15, 90C25, 90C59.
Key words and phrases. Probability inequalities, chance constraints, linear inequalities, stochastic optimization.

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to relax the \(i\)th constraint of the system \(A(\xi)x \leq b(\xi)\) into the ‘easy’ constraint \(\text{CVaR}_\epsilon(b_i(\xi) - A_i(\xi)x) \leq 0\), combined with Bonferroni inequality or other ideas to decouple the probability system \(\mathbb{P}(A(\xi)x \leq b(\xi)) \geq 1 - \epsilon\) into individual chance constraints.

2. (Cheung et al. 2012 [10]) With restrictions on the random variables’ moments, the primal problem can be transformed into a solvable problem by inequality scaling. For example, the probability term in the chance constraints can be scaled by using Markov inequality.

\[
\mathbb{P}(F(x,\xi) \geq \lambda) \leq \frac{\mathbb{E}(F(x,\xi))}{\lambda},
\]

where \(F(x,\xi), \lambda \in \mathbb{R}^+\).

After the inequality scaling, the constraint can be solved by Sample Average Approximation (SAA) method [2, 19].

3. (Jiang and Guan, 2016 [15], Peyman and Daniel, 2016 [17], Gao and Kleywegt 2018 [12]) They considered a probability density function (pdf) of a random variable without any assumption about r.v.s’ moments. They imposed a restriction on the unknown pdf such that the difference between the pdf of the r.v. and a known pdf was within a reasonable range. They use \(\phi\) divergence or Wasserstein distance to measure the difference between two pdfs. With this approach, the stochastic programming problem can be relaxed to a two-stage programming problem that is computationally-tractable.

4. (Bai et al. 2019 [3]) It is well-known that, assuming discrete distribution of \(\xi\), the joint chance constraint can be converted to a system of 0-1 integer constraints. When certain variables are fixed, the integer constraint is reduced to a knapsack constraint. This idea is combined with an augmented Lagrangian scheme to develop into a solution procedure that is particularly suitable for nonconvex objective functions. Since we target on linear programs with joint chance constraint in this paper, we do not go further in this direction.

5. (Yang and Xu, 2016 [27]) The fifth method requires a special assumption that allows the probability term in the chance constraint to be eliminated,

\[
\sup_{\delta \in T} \mathbb{P}(\delta \in T) = \frac{1}{1 + \theta^2},
\]

where \(\delta \in \mathbb{R}^n\) is a random vector \((E(\delta) = 0, \text{and } E(\delta\delta^T) = \Sigma)\), \(T \subseteq \mathbb{R}^n\) is a closed convex set and \(\theta = \inf_{y \in T} \sqrt{y^T\Sigma^{-1}y}\). In this example, due to its special assumptions, the chance constraint is equivalent to a constraint without r.v..
part (section 5) is regarding numerical experiment, in which an example is shown to demonstrate the efficiency of methods.

2. Single chance-constrained linear inequalities. Our first assumption is the following, which is called the linear decision rule in the literature.

\[ A(\xi) = A^{(0)} + \Sigma_{i=1}^{N} \xi_i A^{(i)}, \]

\[ b(\xi) = b^{(0)} + \Sigma_{i=1}^{N} \xi_i b^{(i)}, \]

where \( A^{(j)} \in \mathbb{R}^{m \times n}, b^{(j)} \in \mathbb{R}^{m}, j = 0, 1, 2, \ldots, N, \) are given matrices and vectors, \( \xi_i : \Omega \rightarrow \mathbb{R}, i = 1, 2, \ldots, N \) are \( N \) independent r.v.s defined in probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Without loss of generalization, we assume \( \mathbb{E} \xi_i = 0 \) and \( \text{Var}(\xi_i) = \mathbb{E} \xi_i^2 = \sigma_i^2, i = 1, 2, \ldots, N \). For \( j = 0, 1, 2, \ldots, N, \) let \( A^{(i)} \) denote the \( j \)-th row of \( A^{(i)} \) and \( b^{(i)} \) denote the \( j \)-th component of \( b^{(i)} \), respectively. In addition, denote \( B_i = \Sigma_{j=1}^{N} (A^{(j)}_i x - b^{(j)}_i)^2 \sigma_j^2, i = 1, 2, \ldots, m \).

This section concentrates on the case of single chance constraint, i.e. \( m = 1 \), where the matrices become \( A(\xi) = A_{1 \times n} = (a_{11}^{(0)}, \ldots, a_{1n}^{(0)}) + \Sigma_{i=1}^{N} \xi_i (a_{11}^{(i)}, \ldots, a_{1n}^{(i)}) \) and \( b(\xi) = b_{1 \times 1} = b^{(0)} + \Sigma_{i=1}^{N} \xi_i b^{(i)} \).

2.1. Method based on generalization of the Chebyshev inequality. The Chebyshev inequality is commonly used in chance constraint problems, and it can transform the constraint into a conic constraint that can be solved directly. In this section, we review this method in detail, as it has many similarities to the method presented in the next section.

**Theorem 2.1.** (generalization of the Chebyshev inequality [16]) Suppose that \( X \in \mathbb{R} \) is a r.v., \( \text{Var}(X) = \sigma^2 \). For any \( x \in \mathbb{R} \) and \( a \in \mathbb{R} \),

\[ \mathbb{P}(X - EX \geq x) \leq \frac{\sigma^2 + a^2}{(x + a)^2}. \] (4)

**Theorem 2.2.** When \( m = 1 \), we can use generalization of the Chebyshev inequality to transform the single chance-constrained linear inequalities (1), \( \mathbb{P}(A_{1 \times n}(\xi)x \leq b(\xi)) \geq 1 - \varepsilon, \) into the following form:

\[ \frac{B_1 + a^2}{(b^{(0)}_1 - A^{(0)}_1 x + a)^2} \leq \varepsilon. \] (5)

Then, the optimal solution of optimization problem after transformation is a feasible point of primal problem and the corresponding optimal value is an upper bound of optimal value of primal problem.

**Proof.** Consider the left-hand-side of the primal chance constraint,

\[ \mathbb{P}(A_{1 \times n}(\xi)x \leq b(\xi)) = 1 - \mathbb{P}(A_{1 \times n}(\xi)x \geq b(\xi)) \]

\[ = 1 - \mathbb{P}(A^{(0)}_1 + \Sigma_{i=1}^{N} \xi_i A^{(i)}_1 x \geq b^{(0)}_1 + \Sigma_{i=1}^{N} \xi_i b^{(i)}_1) \]

\[ = 1 - \mathbb{P}(\Sigma_{i=1}^{N} \xi_i (A^{(i)}_1 x - b^{(i)}_1) \geq b^{(0)}_1 - A^{(0)}_1 x) \] (6)

\[ \geq 1 - \frac{B_1 + a^2}{(b^{(0)}_1 - A^{(0)}_1 x + a)^2}. \] (7)
We can get (7) from (6) using the generalization of the Chebyshev inequality. Thus, the theorem is proved. □

Using the generalization of the Chebyshev inequality directly does not yield a suitable form of the constraint. However, its form is similar to a conic constraint, which we analyze below.

Suppose that there is a mapping \( f : \text{for any } a, \sigma, x \in \mathbb{R}, x + a \neq 0 \)
\[
f(a, x, \sigma) = \frac{\sigma^2 + a^2}{(x + a)^2}.
\]
Since the denominator is a quadratic component, we may assume \( x > 0 \). Thus, for fixed \( x \) and \( \sigma \), denote \( f_{x, \sigma}(a) := f(a, x, \sigma) \). After making use of the properties of this function, we have
\[
\mathbb{P}(X - \mathbb{E}X \geq x) \leq \inf_{a \in \mathbb{R}} f_{x, \sigma}(a) = f_{x, \sigma}(\frac{\sigma^2}{x}) = \frac{\sigma^2}{\sigma^2 + x^2}.
\] (8)

**Corollary 1.** Thus, when \( m = 1 \), using Theorem 2.1 and Theorem 2.2, (8) can be utilized directly. The chance constraint \( \mathbb{P}(A_{1 \times n}(\xi)x \leq b(\xi)) \geq 1 - \epsilon \) can be approximated as
\[
\frac{B_1}{(b_1^{(0)} - A_1^{(0)}x)^2 + B_1} \leq \epsilon.
\] (9)

As a result, after transformation, the current second-order cone constraint (SOCC) is an approximation of primal chance constraint (1).

**2.2. Method based on Petrov exponential inequalities.** Petrov Exponential Inequalities are introduced in this section in order to relax single chance constraint.

**Theorem 2.3.** (Petrov Exponential Inequality [16]) Let \( X_j \in \mathbb{R}, j = 1, 2, \ldots, n \) be random variables such that \( \mathbb{E}X_j = 0 \). And there exists a positive constants \( H \) such that
\[
| \mathbb{E}X_j^m | \leq \frac{m!}{2} \sigma_j^2 H^{m-2}, j = 1, \ldots, n,
\]
for any integer \( m \geq 2 \), where \( \mathbb{E}X_j^2 = \sigma_j^2 \). Let \( B = \sum_{j=1}^n \sigma_j^2 \), then there exists a positive constant \( T \) such that for \( 0 \leq x \leq B/H \),
\[
\mathbb{P}(S_n \geq x) \leq e^{-x^2/(4B)}, \mathbb{P}(S_n \leq -x) \leq e^{-x^2/(4B)}
\]
and for \( x \geq B/H \),
\[
\mathbb{P}(S_n \geq x) \leq e^{-Tx^2/2}, \mathbb{P}(S_n \leq -x) \leq e^{-Tx^2/2}.
\]

**Theorem 2.4.** For \( m=1 \), by the Petrov Exponential Inequality (Theorem 2.3), there exist positive constants \( H \) and \( T \) such that chance constraint \( \mathbb{P}(A_{1 \times n}(\xi)x \leq b(\xi)) \geq 1 - \epsilon \) can be transformed into
\[
\begin{cases}
4B_1 \ln 1/\epsilon \leq (b_1^{(0)} - \sum_{i=1}^n a_1^{(0)} x_i)^2, \text{ when } 0 \leq b_1^{(0)} - \sum_{i=1}^n a_1^{(0)} x_i \leq B_1/H \\
\frac{1}{2} T(b_1^{(0)} - \sum_{i=1}^n a_1^{(0)} x_i) \geq \ln 1/\epsilon, \text{ when } b_1^{(0)} - \sum_{i=1}^n a_1^{(0)} x_i \geq B_1/H,
\end{cases}
\]
where \( B_1 = \sum_{i=1}^N (\sum_{j=1}^n a_{ij}^{(i)} x_j - b_1^{(i)})^2 \sigma_i^2 \).

In addition, after making some assumptions about matrix \( A \) and vector \( b \), the constraints can be guaranteed to be solvable which we omit the detailed analysis.
2.3. Method based on Kolmogorov inequalities. In this subsection, we discuss random variables when their bounds are limited by the sum of variances of r.v.s.

**Lemma 2.5. (Kolmogorov Inequalities [16])** Suppose $EX_i = 0$, $\sigma_i^2 = EX_i^2 < \infty$, and $|X_i| \leq cs_n$ a.s., $i = 1, 2, \ldots, n$, where, $c > 0$ is a constant and $s^2_n = \sum_{i=1}^{n} \sigma_i^2$.

For $x > 0$, we have

$$P(S_n/s_n \geq x) \leq \begin{cases} \text{exp}\left\{-\frac{x^2}{2}\left(1 - \frac{s^2}{2n}\right)\right\}, & xc \leq 1, \\ \text{exp}\left\{-\frac{x^2}{4}\right\}, & xc \geq 1. \end{cases}$$

For $m=1$, the chance constraint is

$$P(\sum_{i=1}^{N}(A_i^{(i)}x - b_i^{(i)})\xi_i \geq b_1^{(0)} - A_1^{(0)}x) \leq \epsilon.$$ 

To use Kolmogorov inequalities (Lemma 2.5), we assume $|A_i^{(i)}x - b_i^{(i)}\xi_i| \leq c\sqrt{\Xi_1}$, $i = 1, 2, \ldots, N$. From the left-hand-side of the chance constraint above, we have

$$P(\sum_{i=1}^{N}(A_i^{(i)}x - b_i^{(i)})\xi_i/\sqrt{\Xi_1} \geq y) \leq \begin{cases} \text{exp}\left\{-\frac{y^2}{2}\left(1 - \frac{s^2}{2n}\right)\right\}, & yc \leq 1, \\ \text{exp}\left\{-\frac{y^2}{4}\right\}, & yc \geq 1, \end{cases}$$

where, $y = (b_1^{(0)} - A_1^{(0)}x)/\sqrt{\Xi_1}$.

When $yc < 1$, let the right part less than or equal to $\epsilon$. Then,

$$y^3 - \frac{2}{c}y^2 - \frac{4}{c}ln\epsilon \leq 0. \quad (10)$$

Let $y = z + \frac{2}{3c}, p = -\frac{4}{3c}, q = -\frac{4}{c}ln\epsilon - \frac{16}{27c},$

$$z^3 + pz + q = 0. \quad (11)$$

and inequality (11) requires the existence of the solution of a cubic equation (10), whose discriminant is

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3. \quad (12)$$

Suppose that equation has three roots $z_1, z_2, z_3$. Using properties of cubic equation and Cardano’s method, we provide the following assumptions.

**Assumption 1. (i)** When $\epsilon = e^{-\frac{27}{8c}}$ and the three roots are real with two of them equal, we assume that $z_3$ is the largest root. **(ii).** When $e^{-\frac{27}{8c}} < \epsilon < 1$ and three distinct root are real, we assume $z_1 < z_2 < z_3$. **(iii).** When $0 < \epsilon < e^{-\frac{27}{8c}}$ and there is only one real root, we assume the real root is $z_1$.

**Theorem 2.6.** Using the Kolmogorov inequalities, the single chance constraint $P(A_{1 \times n}(\xi)x \leq b(\xi)) \geq 1 - \epsilon$ can be approximated by the following form.

For $yc \leq 1$, if $\epsilon = e^{-\frac{27}{8c}}$:

$$z \leq z_3;$$

if $e^{-\frac{27}{8c}} < \epsilon < 1:$

$$z \leq z_1 \text{ or } z_2 \leq z \leq z_3;$$

if $0 < \epsilon < e^{-\frac{27}{8c}}$:

$$z \leq z_1;$$

and for $yc \geq 1$,

$$z \geq 4c \cdot ln(1/\epsilon).$$
Here, \( y = \frac{(b_i^{(0)} - A_1^{(0)} x)}{\sqrt{B_1}}, z = y - \frac{2}{3\epsilon} \).

2.4. Method based on Hoeffding inequality. In this subsection, we assume that there are known upper and lower bounds of r.v.s. In addition, information about the variances of random variables is not required, unlike the methods above.

**Lemma 2.7.** (Hoeffding Inequality [16]) Suppose that there exists an \( a_i \leq b_i \) for all \( i = 1, 2, \ldots, n, a_i \leq \xi_i \leq b_i, i = 1, 2, \ldots, n \). Then for any \( x > 0 \)

\[
\mathbb{P}(\eta - \mu \geq x) \leq \exp\{-2n^2x^2/\sum_{i=1}^{n}(b_i - a_i)^2\}. \tag{13}
\]

Here, \( \eta = \frac{1}{n}\sum_{i=1}^{n} \eta_i, \mu = \mathbb{E}\eta_i, \mu = \mathbb{E}\eta_i \).

Suppose that \( \xi_i \) in the primal chance constraint (1) satisfies \( \mathbb{E}\xi_i = 0, \xi_i \in [a_i, b_i], i = 1, 2, \ldots, N \) and variances of r.v. are not required. Using Hoeffding inequality (Lemma 2.7), we find that

\[
\mathbb{P}(A(\xi) \times x \geq b(\xi)) = \mathbb{P}((A_1^{(0)} + \sum_{i=1}^{N} \xi_i A_1^{(i)}) x \geq b_1^{(0)} + \sum_{i=1}^{N} \xi_i b_1^{(i)})
\]

\[
= \mathbb{P}(\sum_{i=1}^{N} (A_1^{(i)} x - b_1^{(i)}) \xi_i \geq b_1^{(0)} - A_1^{(0)} x)
\]

\[
\leq \exp\left\{-\frac{2n^2(b_1^{(0)} - A_1^{(0)} x)^2}{\sum_{i=1}^{N}(b_i - a_i)^2(A_1^{(i)} x - b_1^{(i)})^2}\right\}.
\]

Letting the right-hand-side of the inequality above less than or equal to \( \epsilon \), we arrive at the following theorem.

**Theorem 2.8.** When \( m = 1 \) and \( \mathbb{E}\xi_i = 0, \xi_i \in [a_i, b_i], i = 1, \ldots, N \), the single chance-constraint \( \mathbb{P}(A(\xi) \times x \leq b(\xi)) \geq 1 - \epsilon \) can be approximated by

\[
2n^2(b_1^{(0)} - A_1^{(0)} x)^2 \geq \ln(1/\epsilon) \sum_{i=1}^{N}(b_i - a_i)^2(A_1^{(i)} x - b_1^{(i)})^2.
\]

3. Joint chance-constrained linear programming. We have analyzed single chance constraint containing linear inequality with perturbation (\( m=1 \)). In this section, we consider chance constraints with \( m > 1 \), which are called joint chance constraints. Probability inequalities above cannot be used directly. Thus, we first discuss how to reduce the number of inequalities contained in the joint chance constraints.

**Lemma 3.1.** [16] Suppose \( K_1, K_2, \ldots, K_{n+1} \) are \( n+1 \) events, then

\[
\mathbb{P}\left(\bigcup_{i=1}^{n} K_i\right) \leq \Sigma_{i=1}^{n} P(K_i) - \Sigma_{i=2}^{n} P(K_1 K_i), \tag{14}
\]

\[
\mathbb{P}\left(\bigcup_{i=1}^{n+1} K_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{n} K_i\right) + \mathbb{P}(K_{n+1}) - \mathbb{P}\left(\bigcup_{i=1}^{n} K_i K_{n+1}\right). \tag{15}
\]

**Corollary 2.** By using Lemma 2.5, we have

\[
\mathbb{P}\left(\bigcup_{i=1}^{n} K_i\right) \leq \Sigma_{i=1}^{n} P(K_i) - \frac{1}{n} \Sigma_{i \neq j} P(K_i K_j), \tag{16}
\]
\[ P^\left( \bigcup_{i=1}^{n+1} K_i \right) = \frac{1}{n+1} \left( \sum_{j=1}^{n+1} P^\left( \bigcup_{i \neq j} K_i \right) + \sum_{j=1}^{n+1} P(K_j) - \sum_{j=1}^{n+1} P^\left( \bigcup_{i \neq j} K_j K_i \right) \right). \]  

(17)

Proof. Because of (14) in Lemma 3.1 and the symmetry of \( K_i \),

\[ P\left( \bigcup_{i=1}^{n} K_i \right) \leq \sum_{i=1}^{n} P(K_i) - \sum_{i \neq j} P(K_j K_i), \text{ for } j = 1, 2, \ldots, n. \]

Thus,

\[ \sum_{j=1}^{n} P\left( \bigcup_{i=1}^{n} K_i \right) \leq \sum_{j=1}^{n} \sum_{i=1}^{n} P(K_i) - \sum_{i \neq j} \sum_{j=1}^{n} P(K_j K_i). \]

Dividing both sides of this inequality by \( n \) yields (16).

We can prove (17) similarly.

3.1. Chebyshev inequality. Suppose that \( D_i \) are independent quantities with \( \bigcup_{i=1}^{j-1} D_i \).

\[ P\left( \bigcup_{i=1}^{j} D_i \right) \leq P\left( \bigcup_{i=1}^{j-1} D_i \right) + P(D_j) - P(D_j \bigcup_{i=1}^{j-1} D_i) \]

\[ = P\left( \bigcup_{i=1}^{j-1} D_i \right) + P(D_j) - P(D_j \bigcup_{i=1}^{j-1} D_i) \]

\[ \leq \epsilon_{j-1} + \frac{B_j + a^2}{(b_j^{(0)} - A_j^{(0)}) x + a)^2} - \epsilon_{j-1} \frac{B_j + a^2}{(b_j^{(0)} - A_j^{(0)}) x + a)^2}. \]

(19)

Let the right-hand-side of (19) is less than or equal to \( \epsilon_j \).

Theorem 3.2. The joint chance-constrained linear inequalities

\[ P\left( \bigcap_{i=1}^{m} D_i \right) \geq 1 - \epsilon \]

can be approximated by \( m \) solvable constraints, which are, for \( j = 1, 2, \ldots, m \),

\[ \frac{B_j + a^2}{(b_j^{(0)} - A_j^{(0)}) x + a)^2} \leq \frac{\epsilon_j - \epsilon_{j-1}}{1 - \epsilon_{j-1}}. \]
Review that the primal chance constraint is $P(A_{m \times n}(\xi)x \leq b(\xi)) \geq 1 - \epsilon$. After transforming it, the following constraints provide approximation to the primal chance constraint. For $j = 1, 2, \ldots, n$,

$$\frac{B_j}{(b_j^{(0)} - A_j^{(0)})x^2 + B_j} \leq \frac{\epsilon_j - \epsilon_{j-1}}{1 - \epsilon_{j-1}}$$

### 3.2. Petrov exponential inequalities.

Suppose that $D_j^c$ are independent quantities with $\bigcup_{i=1}^{j-1} D_i^c$. According to Petrov Exponential Inequalities in Theorem 2.3 and Lemma 3.1, for $j = 1, 2, \ldots, m$,

$$P\left(\bigcup_{i=1}^{j} D_i^c\right) \leq P\left(\bigcup_{i=1}^{j-1} D_i^c\right) + P(D_j^c) - P(D_j^c\bigcup_{i=1}^{j-1} D_i^c)$$

$$\leq \epsilon_{j-1} + e^{-(b_j^{(0)} - A_j^{(0)})x^2/(4B_j)} - \epsilon_{j-1}e^{-(b_j^{(0)} - A_j^{(0)})x^2/(4B_j)}, \quad (22)$$

for $0 \leq b_j^{(0)} - A_j^{(0)}x \leq B_j/H$ and

$$P\left(\bigcup_{i=1}^{j} D_i^c\right) \leq P\left(\bigcup_{i=1}^{j-1} D_i^c\right) + P(D_j^c) - P(D_j^c\bigcup_{i=1}^{j-1} D_i^c)$$

$$\leq \epsilon_{j-1} + e^{-(b_j^{(0)} - A_j^{(0)})x^2/2} - \epsilon_{j-1}e^{-(b_j^{(0)} - A_j^{(0)})x^2/2}, \quad (23)$$

for $b_j^{(0)} - A_j^{(0)}x \geq B_j/H$. Setting the right-hand-sides of (22) and (23) to be less than or equal to $\epsilon$, then for any $j = 1, 2, \ldots, m$,

$$\left\{ \begin{array}{l}
-(b_j^{(0)} - A_j^{(0)})x^2/(4B_j) \leq \ln \frac{\epsilon_j - \epsilon_{j-1}}{1 - \epsilon_{j-1}}, \quad \text{when } 0 \leq b_j^{(0)} - A_j^{(0)}x \leq B_j/H

-\frac{1}{2}(b_j^{(0)} - A_j^{(0)})x \leq \ln \frac{\epsilon_j - \epsilon_{j-1}}{1 - \epsilon_{j-1}}, \quad \text{when } b_j^{(0)} - A_j^{(0)}x \geq B_j/H.
\end{array} \right. \quad (24)$$

### 3.3. Hoeffding inequality.

According to *Hoeffding* inequality in Lemma 2.7, we have

$$P(D_j^c) \leq \exp\left\{\frac{-2n^2(b_j^{(0)} - A_j^{(0)})x^2}{N \sum_{i=1}^{j-1}(b_i - a_i)^2(A_j^{(i)}x - b_j^{(i)})^2}\right\}.$$

Suppose that $D_j^c$ are independent quantities with $\bigcup_{i=1}^{j-1} D_i^c$. According to Lemma 3.1, for any $j = 1, 2, \ldots, m$, similarly we have

$$P\left(\bigcup_{i=1}^{j} D_i\right) \leq \epsilon_{j-1} + (1 - \epsilon_{j-1})\exp\left\{\frac{-2n^2(b_j^{(0)} - A_j^{(0)})x^2}{N \sum_{i=1}^{j-1}(b_i - a_i)^2(A_j^{(i)}x - b_j^{(i)})^2}\right\}. \quad (24)$$

Setting the right-hand-sides of (24) to be less than or equal to $\epsilon_j$, then for any $j = 1, 2, \ldots, m$,

$$\exp\left\{\frac{-2n^2(b_j^{(0)} - A_j^{(0)})x^2}{N \sum_{i=1}^{j-1}(b_i - a_i)^2(A_j^{(i)}x - b_j^{(i)})^2}\right\} \leq \frac{\epsilon_j - \epsilon_{j-1}}{1 - \epsilon_{j-1}}, \quad (25)$$
which means
\[ 2n^2(b_j^{(0)} - A_j^{(0)}x)^2 \geq \ln\left(\frac{1 - \epsilon_{j-1}}{\epsilon_j - \epsilon_{j-1}}\right) \sum_{i=1}^N (b_i - a_i)^2 (A_i^{(0)}x - b_i^{(0)})^2. \]

4. Further explanation of the application of probability inequalities. Above we provided methods and analysis for handling chance constraints with probability inequalities. When \( m = 1 \), we can use the probability inequalities to transform the single chance constraint with restrictions on the expectations and variances of the r.v.s. For the more general cases with \( m > 1 \), we cannot use these inequalities directly. Therefore, in this section we provide explanations on how to transform joint chance-constrained linear inequalities and analyze parameters which will appear after the transformation in this section.

4.1. A assignment method for joint chance-constrained linear inequalities. Suppose that there exists a decreasing sequence \( \{\epsilon_m\} \) satisfying (18), \( \epsilon = \epsilon_m \geq \epsilon_{m-1} \geq \ldots \geq \epsilon_1 > \epsilon_0 = 0 \). For \( \{\epsilon_m\} \), let
\[ \epsilon_j = \frac{j}{m} \epsilon, \quad j = 0, 1, 2, \ldots, m. \]
The following chance-constrained programming problem will be considered:
\[ \begin{align*} 
\min & \quad f(x) \\
\text{s.t.} & \quad \mathbb{P}(A_{m \times n}(\xi)x \leq b(\xi)) \geq 1 - \epsilon, \\
\end{align*} \]
where \( f(x) : \mathbb{R}^n \rightarrow \mathbb{R} \) is a continuously differentiable convex function.

According to the Petrov Exponential Inequalities and results in section 3, we can transform the chance-constrained programming problem \((P_0)\) to the following form:
\[ \begin{align*} 
\min & \quad f(x) \\
\text{s.t.} & \quad g_j(x) \geq 0, \quad j = 1, 2, \ldots, m \\
\end{align*} \]
where
\[ g_j(x) = \begin{cases} 
(b_j^{(0)} - A_j^{(0)}x)^2 \geq 4B_j \ln \frac{m - (j-1)\epsilon}{\epsilon}, & \text{when } 0 \leq b_j^{(0)} - A_j^{(0)}x \leq B_j/H; \\
\frac{T}{\epsilon} (b_j^{(0)} - A_j^{(0)}x) \geq \ln \frac{m - (j-1)\epsilon}{\epsilon}, & \text{when } b_j^{(0)} - A_j^{(0)}x \geq B_j/H.
\end{cases} \]

Similarly, according to Theorem 3.2 and the Chebyshev inequality, we can transform the chance-constrained programming problem \((P_0)\) to the following form:
\[ \begin{align*} 
\min & \quad f(x) \\
\text{s.t.} & \quad (\frac{m}{\epsilon} - j)B_j \leq (b_j^{(0)} - A_j^{(0)}x)^2, \quad j = 1, 2, \ldots, m.
\end{align*} \]

Similarly, by using the Hoeffding inequality, we can transform the chance-constrained programming problem \((P_0)\) to the following form:
\[ \begin{align*} 
\min & \quad f(x) \\
\text{s.t.} & \quad 2n^2(b_j^{(0)} - A_j^{(0)}x)^2 \geq \ln\left(\frac{m}{\epsilon} - (j - 1)\right) \sum_{i=1}^N (b_i - a_i)^2 (A_i^{(0)}x - b_i^{(0)})^2, \quad j = 1, 2, \ldots, m.
\end{align*} \]

Remarks. The optimal value got by our methods is an upper bound of the optimal value of primal chance constrained problem \((P_0)\) no matter under what kinds of r.v. moment assumptions.
4.2. Parameter analysis of Petrov exponential inequalities. The chance constraint can be transformed into familiar quadratic or second-order cone constraints by utilizing probability inequalities. However, if the transformation is based on Petrov Exponential Inequalities, parameters $T$ and $H$ will appear in the constraints.

From the proof of the Petrov Exponential Inequalities ([16], P69, 7.2b Proof), we know

$$T \leq \frac{1}{2H}.$$  

Let $T = 1/(2H)$ ensure that there exists a solution to the optimization problem. Parameter $T$ is a function of $H$. Thus, we only need to analyze parameter $H$.

From Theorem 2.3, parameter $H$ is a constant that satisfies the moment inequalities of the r.v.s. The scale of $H$ depends on the properties of the r.v.s.

To use the Petrov Exponential Inequalities, assume that $P(\|A_i x - b_i\| \leq 1)$ is sufficiently small, and for all $i = 1, 2, \ldots, n$, there exists an $H_i$ such that

$$|E\xi_i^m| \leq \frac{m!}{2} \sigma_i^2 H_i^{m-2}, \forall m \geq 2.$$  

The r.v.s that satisfy the inequalities above must be subgaussian variables.

i. Random variables $\xi_i$ are bounded. If r.v.s are bounded, we have

$$H \geq \max_i \{C_i\} = \max_i \{\inf_j \{C_{ij} \in \mathbb{R} \mid C_{ij} \geq |\xi_j|\}\}.$$  

Thus, if the r.v.s have bounds, the possible values of the parameter $H$ are known.

ii. Random variables $\xi_i$ are normal. Here, $\xi_i \sim N(0, \sigma_i^2)$, but we do not know if they are independent. However, for all $m \geq 3$,

$$E\xi_i^m = \begin{cases} 0, & \text{if } m \text{ is even}, \\ (m-1)!! \sigma_i^2, & \text{if } m \text{ is odd}. \end{cases}$$  

Thus, we know

$$H_i \geq \sup \left\{ \left( \frac{2^{n/2} - n/2}{n/2!} \right)^{1/(n-2)} \right\} \frac{1}{2}, \quad n \geq 4, n \in \mathbb{Z}.$$  

Thus, even if the $\xi_i$ are not independent, we can use Petrov Exponential Inequalities to solve the problem.

5. Numerical experiments. Several methods for chance constraint optimization problems have been presented. In the section, we provide a simple example to examine the performance of these methods. Since optimization problems about chance-constrained linear inequalities are complicated, we start to solve a problem when $m = 1$ and then give further numerical experiment based on the same problem when $m > 1$.

Suppose that there is a convex quadratic programming problem with linear constraints as follows:

$$\begin{align*}
\min & \quad x^T Q x + p^T x + C \\
\text{s.t.} & \quad Ax \leq b,
\end{align*}$$  \hspace{1cm} (25)$$  

where $x \in \mathbb{R}^n$ is decision vector, $p \in \mathbb{R}^n$ is a constant vector, $C \in \mathbb{R}$ is a constant and $Q \in \mathbb{R}^{n \times n}$ is a positive semi-definite matrix.
Then, let $n = 2$, 
\[ Q = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}, \quad p = \begin{pmatrix} -1 \\ 4/3 \end{pmatrix}, \quad C = 19/12, \]
and 
\[ A = \begin{pmatrix} 2 & -2 \\ 0.8 & 1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 0.5 \\ 0.3 \end{pmatrix}. \]

It is easily to know the optimization value of this programming problem is 0.55 with optimal decision vector $x^* = (0.2, -0.8)^T$.

Since now we are considering $m = 1$, we perturb one of the linear inequality constraints. Suppose that $\xi_1, \xi_2 \in \mathbb{R}$ are r.v.s. The primal quadratic programming (25) becomes a program containing Single Chance – Constrained Linear Inequalities as follows
\[
\begin{align*}
\min & \quad x^T Q x + p^T x + C \\
\text{s.t.} & \quad (0.8 1)x \leq 0.5, \\
& \quad (1 0)x \leq 0.3, \\
& \quad \mathbb{P}((2 - 2) + \xi_1(1 0) + \xi_2(0 1)x \leq 2) \geq 1 - \epsilon.
\end{align*}
\]

We use the Gurobi solver to solve this programming problem after we applied the methods mentioned earlier. The results are summarized as follows:

| Inequality Type | $\mathbb{E}(\xi_1, \xi_2)$ | $\text{Var}(\xi_1, \xi_2)$ | $\text{bound}(\xi_1, \xi_2)$ | $x^*$ | optimal value |
|-----------------|-----------------|-----------------|-----------------|------|--------------|
| Chebyshev       | (0,0)           | $\mathbb{O}(0.1^2, 0.2^2)$ | $\mathbb{O}(0.2627, -0.0504)^T$ | (0.2627, -0.5084)$^T$ | 0.7634       |
| Petrov          | (0,0)           | $\mathbb{O}(0.1^2, 0.2^2)$ | $\mathbb{O}(\pm 0.3, \pm 0.3)$ | (0.2510, -0.5528)$^T$ | 0.7287       |
| Hoeffding       | (0,0)           | -               | $\mathbb{O}(\pm 0.3, \pm 0.3)$ | (0.2040, -0.6678)$^T$ | 0.6584       |

Experiment above provided methods’ effect when $m = 1$. Then, on the basis of the same programming problem, we add perturbation to the whole linear inequalities in order to change them to a chance constraint containing several inequalities. Let $\xi_1, \xi_2 \in \mathbb{R}$ be random variables such that quadratic programming problem (25) becomes the program containing Joint Chance – Constrained Linear Inequalities as following.
\[
\begin{align*}
\min & \quad x^T Q x + p^T x + C \\
\text{s.t.} & \quad \mathbb{P}(A_0 + \xi_1 A_1 + \xi_2 A_2 \leq b) \geq 1 - \epsilon, \\
& \quad A_0 = \begin{pmatrix} 2 & -2 \\ 0.8 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 0.5 \\ 0.3 \end{pmatrix}.
\end{align*}
\]

Next, we transform the chance constraint into a solvable constraint based on methods aiming to deal with joint chance-constrained linear inequalities. After transformation, the results are as follows with the help of solver GUROBI in CVX.

From the table, we conclude that these methods, which are based on different probability inequalities, give adequate solutions to the stochastic programming problem with chance-constrained linear inequalities.
Table 2. Comparison of different probability inequalities under 0.05 confidence level

| Inequality Type | $E(\xi_1, \xi_2)$ | Var($\xi_1, \xi_2$) | bound($\xi_1, \xi_2$) | $x^*$ | optimization value |
|-----------------|--------------------|---------------------|----------------------|-------|-------------------|
| Chebyshev (0,0) | (0.1^2, 0.2^2)    | 0.1                 | (0.1584, -0.1573)^T  | 1.3251|
| Petrov (0,0)    | (0.1^2, 0.2^2)    | ±0.3               | [0.1472, -0.5511]^T  | 1.1078|
| Hoeffding (0,0) | (0.1^2, 0.2^2)    | ±0.3               | [0.1472, -0.5511]^T  | 1.1078|

In addition, after transformation, the form of the constraints is quadratic or conic constraint. And by observing, when $\xi_i$’s coefficient matrices and vectors

\[
\| A_i \|_2 \to 0, \| b_i \|_2 \to 0, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m,
\]

the constraints would degenerate into the primal constraint $Ax \leq b$. The following two graphs show the changes of optimal value by different methods solving problem (1) and (2), respectively.

Figure 1. Changes of the optimal value in single chance constrained program

In Figure 1 and Figure 2, ‘Robust linear’ denotes the method from [4] and $\theta \geq 0$ represents the perturbation of the constraints while using different methods as follows:

- Chebyshev inequality: Let $\theta$ be such that $\sqrt{Var(\xi_i)} = \sigma_i = 0.3\theta$, for $i = 1, 2$.
- Petrov inequalities: Let $\theta$ be such that $\sqrt{Var(\xi_i)} = \sigma_i = 0.3\theta$, for $i = 1, 2$.
- Hoeffding inequalities: Let $\theta$ be such that $\xi_i \in [-0.3\theta, 0.3\theta]$, for $i = 1, 2$.
- Robust linear: Let $\theta$ be such that $\xi_i \in [-0.3\theta, 0.3\theta]$, for $i = 1, 2$.

Obviously, the differences between primal optimal value and optimal value after transformation would increase when the perturbation of the constraints increases.

The advantages of our methods contain saving time and making the programs solvable all the time compared to the SAA methods since it need to transform the primal program to a large-scale programs. Besides, our methods spend two orders of magnitude less time than SAA method when the number of required samples is 100 in the example.
6. **Conclusion.** To summarize, different probability inequalities should be applied to deal with chance-constrained linear inequalities depending on the known information about random variables, such as their expectation, variance and bounds.

- **Chebyshev inequality:** Chance constraint with known expectation and variance of random variables \( \inf_{\xi \sim (0, \Sigma)} \mathbb{P}(A(\xi)x \leq b(\xi)) \geq 1 - \epsilon \). The problem can be transformed into a second order conic programming (SOCP) problem.

- **Petrov exponential inequalities:** Chance constraint with known expectation, variance and bounds of random variables \( \inf_{\xi \sim (0, \Sigma), |\xi| \leq c} \mathbb{P}(A(\xi)x \leq b(\xi)) \geq 1 - \epsilon \). The problem can be transformed into a 0-1 mixed programming problem. After distinguishing indicative function, it is linear programming (LP) problem or SOCP problem.

- **Kolmogorov inequality:** Chance constraint with known expectation, variance and bounds of random variables \( \inf_{\xi \sim (0, \Sigma), |\xi| \leq c} \mathbb{P}(A(\xi)x \leq b(\xi)) \geq 1 - \epsilon \). The problem can be transformed into a 0-1 quadratic programming problem.

- **Hoeffding inequality:** Chance constraint with known expectation and bounds of random variables \( \inf_{\xi \sim (0, \Sigma), |\xi| \leq c} \mathbb{P}(A(\xi)x \leq b(\xi)) \geq 1 - \epsilon \). The problem can be transformed into a SOCP problem.

On the other hand, we also provide a thinking regarding how to handle joint chance-constrained linear inequalities (chance constraint containing several inequalities).

\[
ITEM_j \leq \frac{\epsilon_j - \epsilon_{j-1}}{1 - \epsilon_{j-1}},
\]

where \( ITEM_j \) is an item without uncertainty, which we could obtain it from \( \mathbb{P}(D_j^c) \) after inequality scaling.

The paper aims to give alternate methods to transform the chance constrained programs into solvable programs.
Acknowledgments. The authors would like to thank the editorial board and anonymous referees for the valuable constructive comments and suggestions on an earlier version of this paper.

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Received November 2020; revised April 2021; early access August 2021.

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