The B-Exponential Map: A Generalization of the Logistic Map, and Its Applications In Generating Pseudo-random Numbers

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July 13, 2006
Abstract

A 1-dimensional generalization of the well known Logistic Map is proposed. The proposed family of maps is referred to as the B-Exponential Map. The dynamics of this map are analyzed and found to have interesting properties. In particular, the B-Exponential Map exhibits robust chaos for all real values of the parameter $B \geq e^{-4}$.

We then propose a pseudo-random number generator based on the B-Exponential Map by chaotically hopping between different trajectories for different values of $B$. We call this BEACH (B-Exponential All-Chaotic Map Hopping) pseudo-random number generator. BEACH successfully passes stringent statistical randomness tests such as ENT, NIST and Diehard. An implementation of BEACH is also outlined.
1 Introduction

A function which has the same domain space and range space will be called a map. A chaotic map $x_{n+1} = F(x_n)$ is typically a non-linear discrete dynamical iteration equation, which exhibits some sort of chaotic behavior. Chaos is characterized by deterministic nonlinearity, non-periodicity, sensitive dependence on initial conditions, boundedness and topological transitivity \[1\]. One of the well known 1-dimensional iterative maps which exhibits chaotic properties is the Logistic Map \[1\]. The Logistic Family is defined by the iteration $x_{n+1} = ax_n(1 - x_n)$. The parameter $a$ controls the dynamics of this map. The map attains full chaos at $a = 4$ since it becomes surjective at this value. Henceforth, we shall refer to the map $x_{n+1} = 4x_n(1 - x_n)$ as the Logistic Map. The Logistic Map was first proposed as a demographical model and was later applied to study population dynamics of species considering the twin effects of reproduction and starvation. The parameter $a$ is a positive number and represents a combined rate for reproduction and starvation.

In this report, we propose a generalization of the Logistic Map. The generalized family of maps is referred to as the B-Exponential Map. Characteristics of the B-Exponential Map such as return maps, bifurcation diagram, Lyapunov exponents, Schwarzian derivatives are investigated. The investigations reveal interesting properties, in particular we show that the B-Exponential Map exhibits robust chaos for a large range of $B$.

One of the interesting features of Chaotic systems from an application point of view is that they statistically mimic random white noise while remaining deterministic. This property along with others (such as topological transtivity and Ergodicity, robust chaos) provide the necessary requirements for Chaotic systems to design strong pseudo-random number generators (PRNG). The B-Exponential Map, exhibits these properties (as we shall soon show) and hence can be utilized to build a PRNG. An implementation of the B-Exponential Map as a PRNG is outlined.

2 The B-Exponential Map

The B-Exponential Map $GL(B, x)$ is defined as follows.

$$GL(B, x) = \frac{B - xB^x - (1 - x)B^{1-x}}{B - \sqrt{B}} \quad 0 \leq x \leq 1 \text{ and } B \in \mathbb{R}^+.$$  \hspace{1cm} (1)

Here, $B$ is the adjustable parameter. Note that $x_{n+1} = GL(B, x_n)$ is the iteration function. A plot of $GL(B, x)$ is shown in Figures \[1\] and \[2\] for different values of $B$. $GL(B, x)$ is unimodal for $e^{-4} \leq B < \infty$.

Consider the interval $[0, 1]$. $GL(B, x)$ is a linear combination of $f(x) = xB^x$ and $f(1 - x) = (1 - x)B^{1-x}$ which are both single-hump maps for all $B$. The critical point of $f(x)$ is $\frac{1}{\ln(B)}$ and that of $f(1 - x)$ is $1 + \frac{1}{\ln(B)}$. If $B > 1$, these are negative and greater than one respectively. Hence, we know that $GL(B, x)$ is unimodal for $B > 1$. For $B < 1$, these two critical points lie within $[0, 1]$ on either sides of $x = 0.5$. Consider $0 < B < e^{-4}$. We show in Appendix A that in this case, we lose surjectivity, $x = 0.5$ is a local minimum and hence $GL(B, x)$ is no longer unimodal. For all $B \geq e^{-4}$, we know that $x = 0.5$ is a local maximum (at $B = e^{-4}$, it is a point of inflexion). We know that $GL(B, 0) = 0$ and $GL(B, 0.5) = 1$. Assume that there is a critical point in $[0, 0.5)$ and it is a local maximum.
In such a case, there has to be another critical point in $[0, 0.5]$. This would mean that $GL(B, x)$ will have at least 5 critical points in $[0, 1]$ owing to symmetry around $x = 0.5$. Hence, $GL(B, x)$ is unimodal for $B \geq e^{-4}$.

In this section, we explain some results concerning the B-Exponential Map.

![Figure 1](image1.png)

Figure 1: The B-exponential map for various values of $B$. The blue curves are are the maps for $B < 1$, and the red ones for $B > 1$, the black curve is the logistic map. All the curves are for $B \geq e^{-4}$.

![Figure 2](image2.png)

Figure 2: The B-exponential map for various values of positive $B < e^{-4}$.
The B-Exponential Map is a generalization of the Logistic Map because of the following property:

\[
\lim_{B \to 1} GL(B, x) = 4x(1 - x).
\]

This interesting property can be derived by applying L'Hospital's rule. The derivation is given in Appendix A.

The B-exponential map is concave for a wide range of \( B \). We state and prove the following properties.

**Theorem 1** The B-Exponential Map is topologically conjugate to the Logistic Map for \( e^{-4} \leq B < \infty \).

**Proof:** There is only one critical point for \( GL(B, x) \) at \( x = 0.5 \) for \( e^{-4} \leq B < \infty \). We can define a diffeomorphism of \( GL(B, x) \) to the line \( 2x \) (standard Tent-map) for \( 0 \leq x < 0.5 \) by associating the appropriate heights. Similarly, we can define another diffeomorphism to the line \( 2 - 2x \) for \( 0.5 \leq x \leq 1 \). This establishes that \( GL(B, x) \) is topologically conjugate to the standard Tent-map which we know to be topological conjugate to the Logistic Map. Thus, by transitive relation of topological conjugacy, we know that \( GL(B, x) \) is topologically conjugate to the Logistic Map \( (e^{-4} \leq B < \infty) \).

The Lyapunov exponents of the B-Exponential map appears to be constant for all \( B \geq e^{-4} \) and equal to ln2. The Lyapunov exponents are defined as follows [1]:

\[
\lambda(B) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} \left| \frac{d}{dx} GL(B, x) \right|_{x=x_t}.
\]

It is a difficult task to evaluate the above expression analytically. We resort to numerical estimation of the Lyapunov exponent. The plot of Lyapunov exponents of the B-Exponential Map is shown in Figure 7. As it can be seen, the Lyapunov exponent is ln2 for \( B \geq e^{-4} \). The number of iterations we used for computing the exponent is 10,000.

**Theorem 2** The B-Exponential Map is chaotic for all real \( B \geq e^{-4} \).

**Proof:** There is no universal definition of Chaos. We use the following conditions as necessary and sufficient for Chaos:

- Determinism: \( GL(B, x) \) is a deterministic map of \([0, 1] \to [0, 1]\) for \( B \geq e^{-4} \).
- Surjective and Boundedness: \( GL(B, x) \) is bounded for all \( 0 \leq x \leq 1 \). \( GL(B, x) \) is surjective on \([0, 1]\) for \( B \geq e^{-4} \) (see Appendix A).
- Sensitive dependence on initial conditions: \( GL(B, x) \) exhibits sensitive dependence on initial conditions (continuously). This is characterized by positive Lyapunov exponents. \( GL(B, x) \) seems to have a Lyapunov exponent of ln2=0.6931 (almost everywhere) for every \( B \geq e^{-4} \) (refer to Figure 7).
- Positive Topological Entropy: The symbolic dynamics of \( GL(B, x) \) is such that all possible transitions (0 to 0, 0 to 1, 1 to 0 and 1 to 1) are achieved. Here the Markov partitions are 0 (if \( 0 \leq x < 0.5 \)) and 1 (if \( 0.5 \leq x \leq 1 \)). Hence topological entropy is ln2 which is positive.
• Topological transitivity: Successive iterations of $GL(B, x)$ mixes the domain. For every pair of open sets $A, B \subseteq [0, 1]$, there is a $k > 0$ such that $T^k(A) \cap B \neq \emptyset$.

• Periodic points are dense in $[0, 1]$: This follows by the fact that $GL(B, x)$ is topological conjugate to the Logistic Map.

2.1 Robust Chaos

Robust Chaos is defined by the absence of periodic windows and coexisting attractors in some neighborhood of the parameter space $\mathbb{R}$. Barreto had conjectured that robust chaos may not be possible in smooth unimodal one-dimensional maps. This was shown to be false with counter-examples by Andrecut and Banerjee. Banerjee demonstrates the use of robust chaos in a practical example in electrical engineering. Andrecut provides a general procedure for generating robust chaos in smooth unimodal maps.

As observed by Andrecut, robust chaos implies a kind of ergodicity or good mixing properties of the map. This makes it very beneficial for cryptographic purposes. The absence of windows would mean that the these maps can be used in hardware implementation as there would be no fragility of chaos with noise induced variation of the parameters. We shall demonstrate that the B-Exponential Map exhibits robust chaos and this property makes it highly beneficial for generating pseudo-random number generators.

**Theorem 3** The B-Exponential Map exhibits robust chaos $B \geq e^{-4}$.

**Proof:** We know that $GL(B \geq e^{-4}, x)$ has only one critical point at $x = 1/2$. We numerically find that the Schwarzian derivative which is given by

$$S(f)(x) = \frac{f''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2 \text{ whenever } f'(x) \neq 0.$$  \hspace{1cm} (4)

is negative (Figures 3 and 4). Since $GL(B, x)$ is smooth and unimodal in the range $[e^{-4}, \infty]$, we invoke the theorem in [11] to say that there can be at most one attracting periodic orbit with the critical point in its basin of attraction. Since $x = 0.5$ is the only critical point and it ends in the value 0 after two iterates, we end up on an unstable fixed point (one can verify that $|GL'(B, 0)| > 1$). Hence, we infer that $GL(B, x)$ does not have any attracting periodic orbits for $B$ in $[e^{-4}, \infty]$.

It is interesting to observe that one can derive a generalization of the standard Tent Map by noting that there is a conjugacy between the Tent Map and Logistic Map. The Generalized Tent Map is described in Appendix A. By means of topological conjugacy, one could potentially generate a number of maps, all of which exhibit robust chaos. These could also be used for designing PRNG, but we shall restrict our attention to the B-Exponential Map in this report.

The bifurcation diagram of the B-Exponential Map is shown in Figures 5 and 6. It is clear from the bifurcation diagram that the map is chaotic for a large range of $B$. This property is very interesting. This is a unique continuous map in that respect. There is full chaos, with surjective mapping, for an infinite range of $B$. A property such as this can be very useful in generating pseudo-random numbers. We explore this application in Section 3.

Another interesting property of the B-exponential Map tends to a constant function (with value 1) as $B$ tends to $\infty$, for all $x$. 


Figure 3: Schwarzian derivative of $GL(B, x)$ plotted for $B = 0$ to $10^6$.

Figure 4: Schwarzian derivative of $GL(B, x)$ for $B = 1000$. It can be observed that the value is always negative.
Figure 5: The bifurcation diagram of the B-exponential map for B ranging from near 0 to 10000. The transition to chaos at $e^{-4}$ is not visible because of the large range of B.

Figure 6: The bifurcation diagram of the B-exponential map for B ranging from close to 0 to 0.1. The breakdown of chaos is clearly visible at $e^{-4}$, i.e about 0.018316.
2.2 The Numerator Term

The numerator term of the B-exponential map, \( G(B, x) = B - xB^x - (1 - x)B^{1-x} \) has interesting properties. It becomes topologically conjugate to the logistic map at \( B = \phi^2 \) where \( \phi \) is the golden mean (\( \phi = \frac{1 + \sqrt{5}}{2} \)). Thus, it becomes fully chaotic at \( B = \phi^2 \). The bifurcation diagram of the numerator term is shown in Figure 8. Figure 9 shows the period doubling route to chaos of the numerator term.

Figure 7: A plot of the Lyapunov exponents of the B-exponential map against the parameter B.

Figure 8: Bifurcation diagram of the numerator term \( G(x) \). The end of chaos can be seen at \( B = \phi^2 \approx 2.6180 \).
Figure 9: A blown up version of the bifurcation diagram of the numerator term $G(x)$. Full-chaos is achieved at $B = \phi^2 \approx 2.6180$.

3 A Pseudo-random Number Generator based on the B-Exponential Map

A random number is a number chosen as if by chance from some specified distribution such that selection of a large set of these numbers reproduces the underlying distribution [14]. An ideal random number generator is a discrete memoryless information source that generates equiprobable symbols. In this report, the phrase random numbers refers to uniformly distributed random numbers. Pseudo-random number generators are algorithms implemented on digital systems that can generate sequences of numbers which are random-like in their statistical properties. Due to limitations in computation and precision, pseudo-random number sequences are necessarily periodic. Sequences generated by pseudo-random number generator algorithms are expected to have large periods and pass a number of statistical randomness tests.

Chaotic dynamical systems exhibit unpredictability, ergodicity and mixing properties. This suggests that chaotic maps can be used in generating random numbers. The relationship between chaos and cryptography have been discussed by Kocarev [7]. Various one-dimensional chaotic maps have been proposed for generating random numbers, e.g: PL1D [9], LOGMAP [8], etc. In their study of the Logistic Map as a random number generator, Pathak and Rao [8] propose the logistic map as a pseudo-random number generator which has a period of about $10^8$ when implemented in double precision. This period is quite small when compared to many other ‘good’ random number generators in the literature. They conjecture that such a period is due to the fact that the value of $a$ in $y = ax(1 - x)$ becomes slightly less than 4 (which corresponds to full chaos), because of which the map goes into periods. The fact that full chaos exists only for a small set of parameters is a major hindrance in using chaotic maps as pseudo-random number generators. The limi-
tions of computation and precision cause the parameters to deviate from full chaos values and this result in periodicity. Another major disadvantage of using chaotic maps directly is that the successive points are strongly correlated. This shows up in the 2-dimensional phase space which will be the plot of the mapping function.

To overcome some of the problems mentioned above, one of the strategies might be to take iterates from different maps and use them as a sequence of random numbers. As shown in Theorem 3, the B-exponential Map shows full chaos for all values of $B$ greater than $e^{-4}$. This provides us with a theoretically infinite number of maps to choose from. We hop from map to map, picking iterates on each hop to generate random numbers. Figure 10 shows the 10th return maps for a large number of B’s $> e^{-4}$. It can be seen that the maps almost fill the space even with just ten iterates. This gives evidence that an algorithm based on map hopping on $B$ has the potential to yield a uniformly distributed set of numbers between 0 and 1.

![Figure 10: The tenth return map for various values of $B \geq e^{-4}$.](image)

### 3.1 BEACH

We propose a PRNG based on B-Exponential Map with the name BEACH (B-Exponential All-Chaotic map-Hopping). As the name suggests, the pseudo-random number generator is based on the principle of hopping from map to map to extract numbers for the generator. Such a scheme has been studied by Rowlands [12] and Zhang [13]. Their methods were limited by the choice of maps and the kind of hopping mechanism. MMOHOCC of Zhang [13] uses a finite number of arbitrarily predefined chaotic maps. They use pre-defined hopping patterns to extract points from the trajectories. We propose a different hopping mechanism, one that is deterministically chaotic. We also have the advantage of choosing from
a very large number of fully developed chaotic maps. Zhang’s MMOHOCC also has the problem of not having robust chaos for any of their maps. The maps they use (Chebyshev and Logistic) are not full-chaos for all values of the parameter. They use the fully chaotic value of $a = 4$ for the Logistic Map. However, such a method would have the draw-back of not being fully chaotic when implemented in hardware. It is impossible to maintain a constant value for a parameter exactly in hardware owing to noise. Another problem they have to worry about is the presence of periodic orbits for some values of the parameters even in the chaotic regime (this is indicated by the windows in the bifurcation diagrams). Our method eliminates all these problems.

We showed in Theorem 4 that B-exponential Map exhibits robust chaos for all values of $B \geq e^{-4}$. In other words, for no value of $B \geq e^{-4}$ does the B-exponential map settle into periodic orbits (except for a measure zero set of initial conditions). The BEACH uses this property to parametrically generate many maps exhibiting full chaos. In the following section, we discuss the BEACH algorithm.

### 3.2 The Algorithm

The seed to the BEACH pseudo-random number generating algorithm is any number between 0 and 1. This number forms the initial value of the iteration. Let $x_0$ denote the seed. We assume that the seed itself is generated using a random procedure like the movement of the mouse, the speed of typing on the keyboard or some physical characteristic (like heat dissipation) in the hardware. In BEACH, each random number is picked from a particular map. The maps are generated parametrically using a sequence of $B$’s, $\{B_1, B_2, ..., B_M\}$ where $M$ is the number of maps we wish to use for hopping. In our implementation, we pick one iterate from each of the maps. Thus $M$ becomes equal to the length of the pseudo-random number sequence we intend to generate (however the $B$s are not necessarily distinct though there are a potentially infinite number of them). Let $x_n$ denote the $n^{th}$ iterate of the map corresponding to $B = B_m$. Since BEACH takes one value from each iterate, the sequence of random numbers will be $\{x_1, x_2, x_3, ..., x_m\}$. 

Let,

$$f(B_m, x_n) = \frac{B_m - x_n B_m^{x_n} - (1 - x_n)B_m^{1-x_n}}{B_m - \sqrt{B_m}}$$  \hspace{1cm} (5)

Where $B_m$ is the $m^{th}$ value of the sequence $\{B_1, B_2, ..., B_M\}$. This sequence of $B$’s can be generated in many ways. We only need to ensure that successive $B$’s are not sufficiently close with a high frequency so that any two consecutive maps differ considerably. One way of varying $B$ is by using the Logistic Map. Alternatively, $B$ can also be varied using a simple, weak PRNG like Linear Congruential Generator. Such a scheme ensures that successive $B$’s are not close to each other on the real line for most of the times. We could also vary $B$ using the orbit of BEACH itself. We use the Logistic Map for generating $B$’s, with slight modifications for effective implementation. We explain the modifications in the subsequent sections. Although varying $B$ according to the logistic map does not give a random sequence of $B$’s, it is sufficient for the purpose of hopping maps. The logistic map is periodic because of limitations in precision. But this does not bring about periodicity in the pseudo-random number sequence because for each of the $B$’s, the previous iterate is different.
For a particular $B_m$, $f(B_m, x_{m-1})$ is iterated on $x_{m-1}$, keeping $B_m$ constant. $R$ such iterations make sure that $x_m$ is considerably different from $x_{m-1}$, especially for $x_m$’s close to the fixed points of the $B_m$ map. Now, the $R^{th}$ iterate is extracted as the new random number $x_m$. The same is repeated for a new, updated value of $B$, $B_{m+1}$, i.e $f(B_{m+1}, x_m)$ is iterated on $x_m$, $R$ times, with $B_{m+1}$ remaining constant, and the $R^{th}$ iterate will be $x_{m+1}$ and so on. $R$ is chosen such that there is a good trade-off between time of computation and departure from previous iterate. A value of $R = 20$ should be sufficient to ensure a good departure from $x_{m-1}$ even when $x_{m-1}$ is close to the fixed point of $B_m$. The pseudo-code of BEACH is given below. We limit the value of $B$ because for large values of $B$, as seen from Figure 1, the maps become flat in shape. This may result in periodicity or fixed points owing to limitations of precision on a computer (values very near to 1 may be rounded off to 0 in the next iterate).

\[
\begin{align*}
x_{old} &= \text{SEED}; \\
y_{old} &= \text{SEED}; \\
&\text{for } \text{1(i=1;} i < \text{LEN/32;} i++) \\
&\quad y_{new} = 4y_{old}(1 - y_{old}); \\
&\quad y_{old} = y_{new}; \\
&\quad \text{for } \text{2(j=1;} j < R; j++) \\
&\quad\quad x_{new} = \frac{B - x_{old}B^{x_{old} - (1 - x_{old})B^{1 - x_{old}}}}{B - \sqrt{B}}; \\
&\quad\quad x_{old} = x_{new}; \\
&\quad\quad x_{newint} = x_{new} \times 2^{52}; \\
&\quad \text{end for } 2 \\
&\quad \text{if } 1(y_{new} \leq \frac{1}{\text{BLIMIT}}) \\
&\quad \quad \text{if } 2(x_{new} \geq \frac{1}{\text{BLIMIT}}) \\
&\quad\quad\quad y_{new} = x_{new}; \\
&\quad\quad\quad \text{else} \\
&\quad\quad\quad y_{new} = \frac{1}{\text{BLIMIT}}; \\
&\quad\quad \text{end if } 2 \\
&\quad \text{end if } 1 \\
&\text{end for } 1
\end{align*}
\]

The SEED is the initial random seed input to the program. In this particular implementation, SEED cannot be initialized to 0.75 since this is a fixed point of the Logistic Map. The other disallowed seeds are 0 and 1 for obvious reasons. LEN is the number of bits which we wish to generate using the PRNG. BLIMIT refers to the maximum value of $B$ allowed.

### 3.3 Period of BEACH

It is very hard to analytically determine the period of BEACH. Theoretically, robust chaos implies that there are no stable periodic orbits and we also know that the measure of periodic orbits is zero (in full chaos). However, when implemented on a computer, all orbits are periodic owing to limited precision. Since we have implemented BEACH in double precision arithmetic, the number of chaotic maps available for hopping is around $10^{300}$. As we are hopping in a chaotic fashion, consecutive maps from which random numbers are
extracted will be considerably different. Given this, we believe that the period of BEACH will be at least \(10^{300}\).

4 Implementation

The implementation of the algorithm was written in ANSI C in double precision. Theoretically, a chaotic iteration may visit any point in the interval \((0, 1)\). This poses some problems in practical implementations. If an iterate goes very close to zero, less than the precision, it will be truncated to zero. We use a zero-trap to prevent such a thing from happening. The zero-trap traps values which are very close to zero and replaces them with arbitrary iterates of the Logistic Map. These are the same iterates which are used to generate \(B\).

As stated earlier, \(B\) is generated using the Logistic Map recursion. The initial value for the recursion on \(B\) is set equal to the seed of the generator. The B-Exponential Map is quite flat for large values of \(B\), as shown in Figure 1. It is desirable to limit the value of \(B\) for generating pseudo-random numbers. We choose the upper limit of \(B\) as 10,000. Hence, \(B\) is extracted 10,000 times from the iterate of the Logistic Map with the seed as the initial value. If an iterate is lesser than \(10^{-4}\), it is replaced by an iterate of the B-Exponential Map. If the iterate of B-Exponential Map is less than \(10^{-4}\), the Logistic Map iterate is set to \(10^{-4}\). Thus, we ensure that \(B\) does not exceed 10,000. The B-exponential map iteration gives a value between 0 and 1, in double precision in our implementation. To convert this to integers, we multiply the iterate by \(2^{52}\) (similar to Zhang’s method).

The 2-dimensional phase space of 3,000 random numbers generated using BEACH is shown in Figure 14 (i.e. \(x_n\) vs. \(x_{n+1}\)). The 3-dimensional phase space (i.e. \(x_n\) vs. \(x_{n+1}\) vs. \(x_{n+2}\)) is shown in Figure 15. It can be seen from the 2-dimensional phase space that the numbers seem to randomly fill the entire area between \(0\) and \(2^{32}\) and the \(2^{32}\) by \(2^{32}\) by \(2^{32}\) cube in the 3-dimensional phase space. This seems to provide a strong evidence that the pseudo-random numbers are not correlated in either 2 or 3-dimensions. Weak generators like the linear congruential generator, distinctly settle in patterns in the 3-dimensional phase space.

5 Randomness Evaluation of BEACH pseudo-random number generator

The BEACH pseudo-random number generator was tested using 3 standard tests- The National Institute of Standards in Technology’s Statistical Test Suite (NIST) [15, 16], George Marsaglia’s Diehard Battery of tests [18], and the ENT test [17]. The BEACH pseudo-random number generator successfully passed all the 15 tests of NIST and all the 18 tests of Diehard Battery.

5.1 Entropy, Chi-square and Mean

Entropy (Shannon entropy) is defined as \(H(X) = - \sum_x P(x) \log_2 P(x)\), where \(P(x)\) is the probability that the random variable \(X\) is in the state \(x\). Thus, entropy is a measure of the information density of the data. We created a binary file with the random numbers (taken as 32 bit integers). Thus, the entropy of such a file would be defined based on the states 0
and 1. An optimal compression of the file using the ENT Pseudo-random Number Sequence Test Program (by John Walker) [17], resulted in an entropy of 1 per symbol. Thus, the program was unable to compress the file. This gives strong evidence that BEACH is a good pseudo-random number generator. This is supported by the fact that the file also passed the Lempel-Ziv Compression test which is a part of the NIST Statistical Testing Suite.

The chi-square test is a very basic test of randomness. Knuth [6] gives a detailed treatment of the chi-square test. The chi-square distribution is for a sequence file and expressed as an absolute number and a percentage which indicates how frequently a truly random sequence would exceed the value calculated. This percentage is a measure of the randomness. If the percentage is less than 1% or greater than 99%, then the sequence is not random. Percentages between 90% and 95% and 5% and 10% indicate the sequence is “almost suspect” [6]. Sequences generated by BEACH were within 25% to 75% consistently.

The mean of 1 billion bit sequences was consistently at 0.5 for 1 bit word length and 127.5 for 8 bit word length. The serial correlation was also very low, of the order of $10^{-5}$ for a billion bit sequence. In addition to this, ENT program carried out Monte Carlo Value of Pi test. Each successive sequence of 24 bits are used as X and Y co-ordinates within a square. If the distance of the randomly-generated point is less than the radius of the circle inscribed within the square, the 24-bit sequence is considered a hit. The percentage of hits is used to calculate the value of $\pi$. For very large streams (this approximation converges very slowly), the value will approach the correct value of $\pi$ if the sequence is close to random. For BEACH, the error percentage was almost 0.00%, consistently. For

Figure 11: The 2-dimensional phase space of a sequence of 3000 integers generated using BEACH.
Figure 12: The 3-dimensional phase space of a sequence of 3000 integers generated using BEACH.
Table 1: Results of ENT on 3 bitstreams.

| Length | Entropy (per bit) | Chi-square distribution(%) | Arithmetic mean | Monte Carlo value of $\pi$ (error %) | Serial correlation coeff |
|--------|-------------------|----------------------------|-----------------|-------------------------------------|------------------------|
| 100 Mb | 1.000000          | 50.00                      | 0.5000          | 0.01                                | 0.000151               |
| 500 Mb | 1.000000          | 50.00                      | 0.5000          | 0.00                                | 0.000024               |
| 1 Gb  | 1.000000          | 75.00                      | 0.5000          | 0.01                                | 0.000035               |

the complete ENT test results, visit [http://mahesh.shastry.googlepages.com/beach](http://mahesh.shastry.googlepages.com/beach). The following table gives the value of these parameters for different lengths of random bits.

### 5.2 NIST Statistical Test Suite

BEACH random numbers also passed the NIST Statistical Test Suite [15]. The input was given as an ASCII file consisting of 1’s and 0’s. The pseudo-random number sequence passed all the 15 tests of the NIST Suite. NIST checked the uniformity of p-values of 1000 streams of 1 million bits each and returns a p-value of the p-values. The p-values of all the 15 tests for each of the 1000 streams were greater than 0.01 which is the limit for passing a test. It was found that the p-values of p-values were all higher than 0.001. This meant that the p-values are all uniform. Passing of a test in NIST Suite implies a confidence level of 99%. In other words, when the p-value is more than the passing level, the test is considered passed with a confidence level of 99%. The details of all the 15 the tests and their interpretation is given in [16]. The histogram of p-values for the Block Frequency Test are given in Figure 13. A plot of the proportion of bit-streams passing the NIST tests is given in Figure 14 with the 2 thick lines indicating the passing values.

![Figure 13: The histogram of p-values of the block-frequency test of NIST Suite. It indicates that the p-values are uniform.](image-url)
5.3 Diehard Battery of Tests

The Diehard Battery of Tests of George Marsaglia are collectively considered to be one of the most stringent tests for randomness. Ten streams of 1 billion bits each were generated using ten different random seeds. Each of the seed was chosen randomly from 10 equally spaced intervals of (0,1). The criteria for passing a Diehard test is that the p-value should not be 0 or 1 up to 6 decimal places. BEACH passed all the 18 tests recommended in the Diehard Battery. The test results are tabulated in the table below. The full results, for all the seeds, of the Diehard Tests are available at [http://mahesh.shastry.googlepages.com/beach](http://mahesh.shastry.googlepages.com/beach).

A very few of the results were close to 0.9999. For those seeds, longer sequences were tested. The results of the Diehard Tests for longer sequences showed that such p-values close to 1.00000 were one-off occurrences. We have found that BEACH passes all the tests of Diehard and NIST for extremely large sequences (up to 10 Gb).

6 Summary and Future Research Directions

We have for the first time explored the utility of robust chaos for the generation of pseudo-random numbers. By chaotically hopping across maps which are fully chaotic, BEACH appears to be a strong candidate for pseudo-random number generators for cryptographic purposes with very long periods. The fact that BEACH successfully passes stringent statistical tests even for extremely long sequences is a testimony to this statement. The existence of one of the following, a secure pseudo-random number generator, a secure block encryption algorithm, and a secure one-way function, implies that the other two also exist [7].

![Proportion of templates passing the test](image.png)

Figure 14: The proportion of templates passing the test. The black lines indicate the threshold for the passing value. The threshold for passing is given approximately by \( \alpha \pm 3\sqrt{\alpha(1-\alpha)/M} \) here, \( \alpha = 0.01 \) is the passing value. \( M \) is the number of bitstreams.
B-Exponential Map

Table 2: Results of Diehard battery of tests on 10 bitstreams of 1 Gb each. The names of the tests are given in Appendix B.

This makes a strong case for the search of strong pseudo-random number generators and we believe that robust chaos will have an important role to play. Our future research would focus on developing a strong stream cipher based on BEACH. To this end, we wish to perform security analysis on BEACH.

Acknowledgements

We would like to thank Sutirth Dey of the Jawaharlal Nehru Centre for Advanced Scientific Research for stimulating discussions on the B-Exponential Map. Nithin Nagaraj would like to express his sincere gratitude to the Department of Science and Technology Ph.D. fellowship program and to Timothy Poston, National Institute of Advanced Studies for discussions on topological conjugacy.

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Appendix

A Derivation of certain results

A.1 Derivation of generalization of the Logistic Map

To prove that

\[ \lim_{B \to 1} GL(B, x) = 4x(1 - x). \]  

i.e.;

\[ \lim_{B \to 1} \frac{B - xB^x - (1 - x)B^{1-x}}{B - \sqrt{B}} = 4x(1 - x). \]  

Consider the LHS; by applying L'Hospital’s rule;

\[ \lim_{B \to 1} \frac{B - xB^x - (1 - x)B^{1-x}}{B - \sqrt{B}} = \lim_{B \to 1} \frac{1 - x^2B^{x-1} - (1 - x)^2B^{-x}}{1 - 1/2\sqrt{B}} = 4x(1 - x). \]  

A.2 B-exponential map loses surjectivity at 1/e^4

For surjectivity, the local maximum of the map in the interval \( x = (0, 1) \) should be equal to 1. Using this condition, we show that the map loses surjectivity for \( B < e^{-4} \). The condition for extremum of \( GL(x) \) with respect to \( x \) is:

\[ \frac{\partial GL(B, x)}{\partial x} = 0. \]  

Now, we know that the B-exponential map is symmetric and concave. Hence, \( x = 0.5 \) should be a maximum or a minimum point. Thus, we get:

\[ \frac{\partial GL(B, x)}{\partial x} \bigg|_{x=0.5} = 0 \quad \forall \ B \in \mathbb{R}^+. \]  

i.e:

\[ \frac{\partial}{\partial x} \left( \frac{B - xB^x - (1 - x)B^{1-x}}{B - \sqrt{B}} \right) \bigg|_{x=0.5} = 0 \quad \forall \ B \in \mathbb{R}^+. \]  

But, when 0.5 is a local minimum in any interval, then \( GL(B, 0.5) = 1 \) will be a local minimum and the map will no longer be surjective. We use this condition to find \( B \) for which the map loses surjectivity.

\[ \frac{\partial^2 GL(B, x)}{\partial x^2} \bigg|_{x=0.5} > 0. \]  

Applying this, we get for \( B < e^{-4} \), B-exponential map is not surjective.

This also shows that the second derivative of the function with respect \( x \), is negative for all values of \( x \) only when \( 0 < B < e^{-4} \).
A.3 The B-exponential Map becomes constant at 1 as B tends to $\infty$

To find:

$$\lim_{B \to \infty} \frac{B - xB^x - (1 - x)B^{1-x}}{B - \sqrt{B}}.$$ \hspace{1cm} (13)

rearranging the terms, we get:

$$\lim_{B \to \infty} \frac{1 - xB^{(x-1)} - (1 - x)B^{-x}}{1 - 1/\sqrt{B}} = 1.$$ \hspace{1cm} (14)

Hence proved.

A.4 Generalized Tent Map

We know that the Logistic Map is conjugate to the Tent Map [1]. We can use the explicit conjugacy map on $GL(B, x)$ to obtain a generalization of the Tent Map. We know that the conjugate map $C(x)$ is given by [1]:

$$C(x) = \left(1 - \cos \pi x\right) = \sin^2\left(\frac{\pi x}{2}\right).$$ \hspace{1cm} (15)

The Generalized Tent Map is given by:

$$GT(B, x) = \frac{2}{\pi} \sin^{-1}\left(\frac{B - \sin^2\left(\frac{\pi x}{2}\right)B^{\sin^2\left(\frac{\pi x}{2}\right)} - (1 - \sin^2\left(\frac{\pi x}{2}\right))B^{(1-\sin^2\left(\frac{\pi x}{2}\right))}}{B - \sqrt{B}}\right)^{\frac{1}{2}}.$$ \hspace{1cm} (16)

with the following property:

$$\lim_{B \to 1} GT(B, x) = 1 - 2\left|x - \frac{1}{2}\right|.$$ \hspace{1cm} (17)

The Generalized Tent Map is plotted below for a few values of B.

![Figure 15: The Generalized Tent Map for three values of B. Black: $B = 2$, Red: $B = 200$, Green: $B = 2 \times 10^{10}$.](image)

The Generalized Tent Map also exhibits robust chaos property (since it is topologically conjugate to the B-Exponential Map) and could also be used for generating PRNG.
List of tests included in NIST and Diehard suites

B.1 NIST tests:
1. Frequency
2. Block-Frequency
3. Cumulative-Sums
4. Runs
5. Longest-Run
6. Rank
7. FFT
8. Nonperiodic-Templates
9. Overlapping-Templates
10. Universal
11. Approximate Entropy
12. Random-Excursions
13. Random-Excursions-Variant
14. Serial
15. Linear-Complexity

B.2 Diehard tests:
1. Birthday Spacings Test
2. Overlapping 5-Permutation Test
3. Binary Rank Test For $31 \times 31$ Matrices and $32 \times 32$ Matrices
4. Binary Rank Test For $6 \times 8$ Matrices
5. Bitstream Test
6. Tests OPSO, OQSO And DNA
7. Count-The-1’S Test On A Stream Of Bytes
8. Count-The-1’S Test For Specific Bytes
9. Parking Lot Test
10. Minimum Distance Test
11. 3Dspheres Test
12. Squeeze Test
13. Overlapping Sums Test
14. Runs Test
15. Craps Test