**W-algebra \( W(2, 2) \) and the vertex operator algebra \( L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0) \)**

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Abstract

In this paper the \( W \)-algebra \( W(2, 2) \) and its representation theory are studied. It is proved that a simple vertex operator algebra generated by two weight 2 vectors is either a vertex operator algebra associated to a highest irreducible \( W(2, 2) \)-module or a tensor product of two irreducible Virasoro vertex operator algebras. Furthermore, any rational, \( C_2 \)-cofinite and simple vertex operator algebra whose weight 1 subspace is zero and weight 2 subspace is 2-dimensional, and with central charge \( c = 1 \) is isomorphic to \( L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0) \).

1 Introduction

Motivated partially by the problem of classification of rational vertex operator algebras with central charge \( c = 1 \) and by the Frenkel-Lepowsky-Meurman's uniqueness conjecture on the moonshine vertex operator algebra \( V^{\natural} \) [FLM], we give a characterization of the vertex operator algebra \( L(1/2, 0) \otimes L(1/2, 0) \) in terms of the central charge and the dimensions of weights 1 and 2 subspaces in this paper. Here the \( L(1/2, 0) \) is the vertex operator algebra associated to the irreducible highest weight module for the Virasoro algebra with central charge 1/2 which is the smallest central charge among the discrete unitary series for the Virasoro algebra.

The classification of \( c = 1 \) rational conformal field theories at character level has been achieved in the physics literature under the assumption that the sum of the square of the norm of the irreducible characters is a modular function over the full modular group [K]. But the classification of rational vertex operator algebras with \( c = 1 \) remains an open and hard problem. If a vertex operator algebra \( V = \sum_{n \geq 0} V_n \) with \( \dim V_0 = 1 \) is rational and \( C_2 \)-cofinite, then \( V_1 \) is a reductive Lie algebra and its rank is less than or equal to the effective central charge \( \tilde{c} \) [DM2]. Also, the vertex operator subalgebra generated by \( V_1 \) is a tensor product of vertex operator algebras associated to integrable highest weight modules for affine Kac-Moody algebras and lattice vertex operator algebra [DM3]. In the case that \( c = \tilde{c} = 1 \), we can classify the vertex operator algebras with \( \dim V_1 \neq 0 \). Since \( V_1 \) is a reductive Lie algebra whose rank is less than or equal to 1, we immediately see

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that $v_1$ is either 1-dimensional or 3-dimensional, as a result, $V$ is isomorphic to a vertex operator algebra associated to a rank 1 lattice. So one can assume that $V_1 = 0$. The vertex operator algebra $L(1/2, 0) \otimes L(1/2, 0)$ has this property. So characterization of $L(1/2, 0) \otimes L(1/2, 0)$ can be regarged as a part of program of classification of rational vertex operator algebras with $c = 1$.

The vertex operator algebra $L(1/2, 0) \otimes L(1/2, 0)$ plays an important role in the study of the moonshine vertex operator algebra $V^\natural$. The moonshine vertex operator algebra $V^\natural$ which is fundamental in shaping the field of vertex operator algebra was constructed as a bosonic orbifold theory based on the Leech lattice [FLM]. The discovery of existence of $L(1/2, 0)^\otimes 48$ inside the moonshine vertex operator algebra $V^\natural$ [DMZ] opens a different way to study $V^\natural$. This leads to the theory of code and framed vertex operator algebras [M2], [DGH]. This discovery is also essential in a proof that $V^\natural$ is holomorphic [D], a new construction of $V^\natural$ [M3], proofs of weak versions of the Frenkel-Lepowsky-Meurman’s uniqueness conjecture on $V^\natural$ [DGL], [LY] and a study of $V^\natural$ in terms of conformal nets [KL]. There is no doubt that a characterization of $L(1/2, 0) \otimes L(1/2, 0)$ will be very helpful in the study of structure of $V^\natural$ and the Frenkel-Lepowsky-Meurman’s uniqueness conjecture.

The $W(2, 2)$ and its highest weight modules enter the picture naturally during our discussion on $L(1/2, 0) \otimes L(1/2, 0)$. The $W$-algebra $W(2, 2)$ is an extension of the Virasoro algebra and also has a very good highest weight module theory (see Section 2). Its highest weight modules produce a new class of vertex operator algebras. Contrast to the Virasoro algebra case, this class of vertex operator algebras are always irrational. From this point of view, this class of vertex operator algebras are not interesting.

The $W(2, 2)$ and associated vertex operator algebras are also closely related to the classification of simple vertex operator algebra with 2 generators. It is well known that each homogeneous subspace $V_n$ of a vertex operator algebra $V = \sum_{n \in \mathbb{Z}} V_n$ is some kind of algebra under the product $u \cdot v = u_{n-1} v$ for $u, v \in V_n$ where $u_m$ is the component operator of $Y(u, z) = \sum_{m \in \mathbb{Z}} u_m z^{-m-1}$. If a vertex operator algebra $V = \sum_{n \geq 0} V_n$ with dim $V_0 = 1$ is rational and $C_2$-cofinite, then $V_1$ and the vertex operator subalgebra generated by $V_1$ are well understood [DM2]. So it is natural to turn our attention to $V_2$. This is still a very hard problem even with $V_1 = 0$. A simple vertex operator algebra $V$ satisfying $V_1 = 0$ is called the moonshine type. The $V_2$ in this case is a commutative nonassociative algebra. The simple vertex operator algebras of the moonshine type with dim $V_2 = 2$ and generated by $V_2$ are also classified in this paper. There are two families of such algebras. One of this family consists the tensor product of two vertex operator algebras associated to the irreducible highest weight modules for the Virasoro algebra and the other family consists the vertex operator algebras associated the highest weight modules for the the $W$-algebra $W(2, 2)$.

The paper is organized as follows. We define and study the $W$-algebra $W(2, 2)$ in Section 2. In particular we use the bilinear form on Verma modules $V(c, h_1, h_2)$ to determine the irreducible quotient modules $L(c, h_1, h_2)$ for $W(2, 2)$ for most $c$ and $h_i$. In Section 3 we classify the simple vertex operator algebras of the moonshine type generated by 2 weight 2 vectors. The section 4 is devoted to the characterization of rational vertex
operator algebra $L(1/2, 0) \otimes L(1/2, 0)$. The main idea is to use the modular invariance of the graded characters of the irreducible modules $[Z]$ to control the growth of the graded dimensions of the vertex operator algebra.

2 $W$-algebra $W(2, 2)$

The $W$-algebra $W(2, 2)$ considered in this paper is an infinite dimensional Lie algebra with generators $L_m, W_m, C$ for $m \in \mathbb{Z}$ and Lie bracket

$$ [L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0}C, $$

$$ [L_m, W_n] = (m - n)W_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0}C, $$

$$ [W_m, W_n] = 0 $$

for $m, n \in \mathbb{Z}$ where $C$ is a central element. In this section we study the highest weight modules for this algebra and the corresponding vertex operator algebras.

Let $c, h_1, h_2 \in \mathbb{C}$ and we denote by $V(c, h_1, h_2)$ the highest weight module for $W(2, 2)$ with central charge $c$ and highest weight $(h_1, h_2)$. Then $V(c, h_1, h_2) = U(W(2, 2))/I_{c,h_1,h_2}$ where $I_{c,h_1,h_2}$ is the left ideal of the universal enveloping algebra $U(W(2, 2))$ generated by $L_m, W_m, C - c, L_0 - h_1$ and $W_0 - h_2$ for positive $m$. The $V(c, h_1, h_2)$ can also be realized as induced module as in the case of Virasoro algebra. It is standard that $V(c, h_1, h_2)$ has a unique maximal submodule $J(c, h_1, h_2)$ so that $L(c, h_1, h_2) = V(c, h_1, h_2)/J(c, h_1, h_2)$ is an irreducible highest weight module. As in the case of Virasoro algebra, there is a unique invariant symmetric bilinear form $(,)$ on $V(c, h_1, h_2)$ such that

$$ (L_m u, v) = (u, L_m v), \quad (W_m u, v) = (u, W_m v), $$

$$(1, 1) = 1 $$

where $1 = 1 + I_{c,h_1,h_2}$. Moreover, the radical of this bilinear form is exactly the maximal submodule $J(c, h_1, h_2)$.

Let $X$ be a proper submodule of $V(c, h_1, h_2)$. Then $X$ is a submodule of $J(c, h_1, h_2)$ and the bilinear form $(,)$ on $V(c, h_1, h_2)$ induces a symmetric invariant bilinear from $(,)$ on the quotient module $V(c, h_1, h_2)/X$.

As in the classical case we need to answer the basic question: What is $J(c, h_1, h_2)$? We first consider the case $(c, h_1, h_2) = (c, 0, 0)$. Clearly, $L(0,0,0) = \mathbb{C}$. So we now assume that $c \neq 0$. Note that $U(W(2, 2))L_{-1}1 + U(W(2, 2))W_{-1}1$ is a proper submodule of $V(c, 0, 0)$.

Theorem 2.1 If $c \neq 0$ then $J(c, 0, 0) = U(W(2, 2))L_{-1}1 + U(W(2, 2))W_{-1}1$ and $L(c, 0, 0)$ has basis

$$ S = \{W_{-m_1} \cdots W_{-m_s}L_{-n_1} \cdots L_{-n_t} | m_1 \geq \cdots \geq m_s > 1, n_1 \geq \cdots \geq n_t > 1 \} $$

where $1$ is the canonical highest weight vector of $L(c, 0, 0)$.
Proof: Set $V(c,0,0) = V(c,0,0)/(U(W(2,2))L_{-1}1 + U(W(2,2))W_{-1}1$. Then $S$ forms a basis of $V(c,0,0)$ by PBW theorem. Let $S_n$ be the subset of $S$ consisting of weight $n$ (with respect to the operator $L_0$) vectors:

$$W_{-m_1} \cdots W_{-m_s}L_{-n_1} \cdots L_{-n_t}1$$

with $m_1 \geq \cdots \geq m_s > 1$, $n_1 \geq \cdots \geq n_t > 1$ and $\sum_i m_i + \sum_j n_j = n$. So it is enough to show that $S_n$ is linearly independent in $L(c,0,0)$. The idea is to prove that the determinant of $((u,v))_{u,v \in S_n}$ is nonzero.

To see how the determinant argument works we first consider two subsets $S_0,n$ and $S_{n,0}$ of $S_n$ where $S_0,n$ consists of vectors $L_{-n_1} \cdots L_{-n_t}1$ for $n_1 \geq \cdots \geq n_t > 1$ and $S_{n,0}$ consists of vectors $W_{-m_1} \cdots W_{-m_s}1$ for positive integers $m_1 \geq \cdots \geq m_s > 1$.

Let

$$P_n = \{(m_1, m_2, \ldots, m_s) | s \geq 1, m_1 \geq m_2 \geq \cdots \geq m_s > 1, \sum m_i = n\}$$

which is a set of partitions of $n$ without 1. We define a total order on $P_n$ so that if $(m_1, \ldots, m_s) > (p_1, \ldots, p_t)$ if and only if there exists $1 \leq k \leq s$ such that $m_i = p_i$ for $i < k$ and $m_k > p_k$. Note that each vector in $S_{0,n}$ or $S_{n,0}$ is associated to a partition in $P_n$. We can label the corresponding vectors in $A$ and $B$ by using this order. Let $p_n = |P_n|$ be the cardinality of $P_n$. Then we can write the vectors in $S_{0,n}$ by $\{u_1^{0,n}, \ldots, u_{p_n}^{0,n}\}$ so that if $i < j$ then the partition corresponding to $u_i^{0,n}$ is less than the partition corresponding to $u_j^{0,n}$. We denote the elements in $B$ by $\{v_1^{n,0}, \ldots, v_{p_n}^{n,0}\}$ with the same order.

Observe that if $(m_1, \ldots, m_s) \in P_n$ and $m \geq m_1$ then

$$L_m W_{-m_1} \cdots W_{-m_s}1 = \frac{m^3 - m}{12} \frac{\partial}{\partial W_{-m}} W_{-m_1} \cdots W_{-m_s}1 .$$

This implies immediately that if $i > j$ $(u_i^t, v_j^n) = 0$. Clearly, $(u_i^t, v_j^n) \neq 0$. This shows that both $S_{0,n}$ and $S_{n,0}$ are linearly independent in $L(c,0,0)$.

Let

$$u = W_{-m_1} \cdots W_{-m_s}L_{-n_1} \cdots L_{-n_t}1, \quad v = W_{-p_1} \cdots W_{-p_a}L_{-q_1} \cdots L_{-q_b}1 \in S_n .$$

Clearly, $(u,v) = 0$ if $\sum m_i > \sum q_j$ or $\sum p_i > \sum n_j$. If $\sum m_i = \sum q_j$ and $\sum n_i = \sum p_j$ we have

$$(u,v) = (W_{-m_1} \cdots W_{-m_s}1, L_{-q_1} \cdots L_{-q_b}1),$$

$$(W_{-p_1} \cdots W_{-p_a}1, L_{-n_1} \cdots L_{-n_t}1) ,$$

which follows from the computation

$$(W_{-m_1} \cdots W_{-m_s}1, L_{-q_1} \cdots L_{-q_b}1)1 = L_{q_b} \cdots L_{q_1}W_{-m_1} \cdots W_{-m_s}1 ,$$

$$(W_{-m_1} \cdots W_{-m_s}1, L_{-n_1} \cdots L_{-n_t}1) .$$

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where we have used the fact that $W_m$ commute with each other.

We now fix nonnegative integers $d$ and $e$ such that $d + e = n$. Let $(m_1, ..., m_s) \in P_d$ and $(n_1, ..., n_t) \in P_e$. Assume that $u^d_i = L_{-m_1} \cdots L_{-m_s} 1$ and $u^e_j = L_{-n_1} \cdots L_{-n_t} 1$ (see the argument above for the definition of $u^d_i$). Set

$$u^{d,e}_{i,j} = W_{-m_1} \cdots W_{-m_s} L_{-n_1} \cdots L_{-n_t} 1$$

for $i = 1, ..., p_d$ and $j = 1, ..., p_e$ and let $S_{d,e}$ the set consisting of these vectors. Then $S_{n-1,1} = S_{1,n-1} = \emptyset$ and

$$S_n = \cup_{d \neq 1, n-1} S_{d,n-d}.$$

From the discussion above, for any $u \in S_{d,e}$ and $v \in S_{d_1,e_1}$ with $d, d_1 \neq 1$ we have $(u, v) = 0$ if $d > e_1$ or $d_1 > e$.

Again from the discussion on $S_{0,n}$ and $S_{n,0}$ we can relabel the vectors in $S_{d,n-d}$ in two different ways as $\{u^{d,n-d}_i | i = 1, ..., p_dp_e\}$ and $\{v^{d,n-d}_i | i = 1, ..., p_dp_e\}$ so that $(u^{d,n-d}_i, v^{n-d,d}_j) = 0$ if $i > j$ and $(u^{d,n-d}_i, v^{n-d,d}_i)$ is nonzero for all $i$.

Let

$$A = \begin{pmatrix}
A_{0,0} & A_{0,2} & A_{0,3} & \cdots & A_{0,n-3} & A_{0,n-2} & A_{0,n} \\
A_{2,0} & A_{2,2} & A_{2,3} & \cdots & A_{2,n-3} & A_{2,n-2} & A_{2,n} \\
A_{3,0} & A_{3,2} & A_{3,3} & \cdots & A_{3,n-3} & A_{3,n-2} & A_{3,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
A_{n-3,0} & A_{n-3,2} & A_{n-3,3} & \cdots & A_{n-3,n-3} & A_{n-3,n-2} & A_{n-3,n} \\
A_{n-2,0} & A_{n-2,2} & A_{n-2,3} & \cdots & A_{n-2,n-3} & A_{n-2,n-2} & A_{n-2,n} \\
A_{n,0} & A_{n,2} & A_{n,3} & \cdots & A_{n,n-3} & A_{n,n-2} & A_{n,n}
\end{pmatrix}$$

where $A_{d,e}$ is a submatrix defined by

$$A_{d,e} = \{(u^{d,n-d}_i, v^{n-d,d}_j)\}$$

with obvious ranges for $i, j$. Clearly, $A_{d,e} = 0$ if $d > e$. So $A$ is an upper triangular matrix with nonzero diagonals. Thus $A$ is nondegenerate. As a result $S_n$ is linearly independent. This finishes the proof. □

**Remark 2.2** Although $W(2, 2)$ is an extension of the Virasoro algebra, the representation theory for $W(2, 2)$ is different from that for the Virasoro algebra in a fundamental way. For $W(2, 2)$, the structure of $L(c, 0, 0)$ for $c \neq 0$ is uniform and simple. But for the Virasoro algebra, the situation is totally different. Let $L(c, h)$ be the irreducible highest weight module for the Virasoro algebra with central charge $c$ and highest weight $h$. In the case $c = 1 - (p - q)^2/pq$ where $p, q$ are two coprime positive integers $1 < p < q$, then $L(c, 0) = \tilde{V}(c, 0)$ where $\tilde{V}(c, 0) = V(c, 0)/U(Vir)L_{-1}v$ and $v$ is a nonzero highest weight vector of the Verma module $V(c, 0)$ (see [FF]). The structure of $L(c_{s,t}, 0)$ is much more complicated. On the other hand, from the point of view of vertex operator algebra, $L(c_{s,t}, 0)$ is a rational vertex operator algebra for all $c_{s,t}$ but $L(c, 0)$ is not if $c \neq c_{s,t}$ (see [FZ] and [W]).
Next we discuss the vertex operator algebras associated to the highest weight modules for $W(2, 2)$. Let $1$ be the canonical highest weight vector of $V(c, 0, 0)$. From the axiom of vertex operator algebra we must mod out the submodule generated by $L_{-1}1$. From the commutator relation between $L_m$ and $W_n$ we also know that $W_n$ cannot be the component operators of a vertex operator associated to a weight one vector. This forces $W_{-1}1 = 0$ if there is a vertex operator algebra structure. So by Theorem 2.1, $L(c, 0, 0)$ is the only quotient of $V(c, 0, 0)$ which may have a structure of vertex operator algebra.

A $W(2, 2)$-module $M$ is restricted if for any $w \in M$, $L_m w = W_m w = 0$ if $m$ is sufficiently large. Recall the weak module, admissible module and ordinary module from [DLM1].

**Theorem 2.3** Assume that $c \neq 0$. Then

1. There is a unique vertex operator algebra structure on $L(c, 0, 0)$ with vacuum $1$ and Virasoro element $\omega = L_{-2}1$. Moreover, $L(c, 0, 0)$ is generated by $\omega$ and $x = W_{-2}1$ such that $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ and $Y(x, z) = \sum_{n \in \mathbb{Z}} W_n z^{-n-2}$.

2. If $M$ is a restricted $W(2, 2)$-module with central charge $c$, then $M$ is a weak $L(c, 0, 0)$-module such that $Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ and $Y_M(x, z) = \sum_{n \in \mathbb{Z}} W_n z^{-n-2}$. In particular, any quotient module of $V(c, h_1, h_2)$ is an ordinary module for $L(c, 0, 0)$.

3. Any irreducible admissible $L(c, 0, 0)$-module is ordinary.

4. $\{L(c, h_1, h_2)|h_i \in \mathbb{C}\}$ gives a complete list of irreducible $L(c, 0, 0)$-modules up to isomorphism.

**Proof:** (1) and (2) are fairly standard following from the local system theory (see [L2], [LL]). (3) and (4) follow from that fact that any irreducible admissible module for $L(c, 0, 0)$ is an irreducible highest weight module for $W(2, 2)$. □

We now turn our attention to the Verma module $V(c, h_1, h_2)$ in general. As in general highest weight module theory, we want to know when $V(c, h_1, h_2) = L(c, h_1, h_2)$ is irreducible.

**Theorem 2.4** The Verma module $V(c, h_1, h_2)$ is irreducible if and only if $\frac{m^2 - 1}{12} c + 2 h_2 \neq 0$ for any nonzero integer $m$.

**Proof:** As in the proof of Theorem 2.1 we use the determinant of the invariant bilinear form to prove the result. Note that

$$V(c, h_1, h_2) = \oplus_{n \geq 0} V(c, h_1, h_2)_{h_1+n}$$

where $V(c, h_1, h_2)_n$ has a basis consisting of vectors

$$W_{-m_1} \cdots W_{-m_s} L_{-n_1} \cdots L_{-n_t} 1$$

where $m_1 \geq \cdots \geq m_s > 0$, $n_1 \geq \cdots \geq n_t > 0$, $\sum m_i + \sum n_j = n$. As in the proof of Theorem 2.1 we still denote this set by $S_n$. Define a matrix $A_n = ((u, v))_{u, v \in S_n}$. Then $V(c, h_1, h_2) = L(c, h_1, h_2)$ if and only if $\det A_n \neq 0$ for all $n > 0$. 

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Note that if \( m \geq m_1 \geq \cdots \geq m_s > 0 \)
\[
L_m W_{-m_1} \cdots W_{-m_s} = \left( \frac{m^3 - m}{12} c + 2m h_2 \right) \frac{\partial}{\partial W_m} W_{-m_1} \cdots W_{-m_s}. 
\]
Following the proof of Theorem 2.1 we see immediately that if \( \frac{m^2-1}{12} c + 2h_2 \neq 0 \) for all \( 0 \neq m \in \mathbb{Z} \) then \( \det A_n \neq 0 \) and if \( \frac{m^2-1}{12} c + 2h_2 = 0 \) for some \( 0 < m \), then \( \det A_m = 0 \). The proof is complete. \( \square \)

It is definitely interesting to determine the \( J(c, h_1, h_2) \) if \( \frac{m^2-1}{12} c + 2h_2 = 0 \) for some nonzero integer \( m \). But this will be a problem which has nothing to do with the characterization of \( L(1/2, 0) \otimes L(1/2, 0) \) in this paper. We will not go in this direction further.

### 3 Vertex operator algebras of the moonshine type

In this section we classify the simple vertex operator algebras \( V \) of the moonshine type such that \( V \) is generated by \( V_2 \) and \( V_2 \) is 2-dimensional.

Note that \( V_0 = \mathbb{C} \mathbf{1} \) is 1-dimensional for the moonshine type vertex operator algebra \( V \) and \( V_n = 0 \) if \( n < 0 \) \([\text{DGL}]\). Since \( V_1 = 0 \) and \( V_0 \) is 1-dimensional, there is a unique symmetric, nondegenerate invariant bilinear from \( (,)_0 \) on \( V \) such that \( (1, 1) = 1 \) (see \([\text{L1}]\)). Then for any \( u, v, w \in V \)
\[
(Y(u, z)v, w) = (v, Y(e^{L(1)z}(-z^{-2})^{L(0)}u, z^{-1})w)
\]
and
\[
(u, v)_1 = \text{Res}_{z^{-1}}(e^{L(1)z}(-z^{-2})^{L(0)}u, z^{-1})v.
\]
In particular, the restriction of the form to each homogeneous subspace \( V_n \) is nondegenerate and
\[
(u_{n+1}v, w) = (v, u_{-n+1}w)
\]
for all \( u, v \in V_2 \) and \( w \in V \).

The \( V_2 \) is a commutative and associative algebra with the product \( ab = a_1 b \) for \( a, b \in V_2 \) and identity \( \frac{1}{2} \) (cf. \([\text{FLM}]\)) The \( V_2 \) is called Griess algebra of \( V \). Note that for \( a, b \in V_2 \) we have \( (a, b) = a_2 b \). Moreover, the form on \( V_2 \) is associative. That is, \( (ab, c) = (a, bc) \) for \( a, b, c \in V_2 \).

**Theorem 3.1** Let \( V \) be a simple vertex operator algebra of the moonshine type with central charge \( c \neq 0 \) such that \( V \) is generated by \( V_2 \) and \( V_2 \) is 2-dimensional. Then \( V \) is isomorphic to \( L(c_1, 0) \otimes L(c_2, 0) \) for some nonzero complex number \( c_1, c_2 \) such that \( c_1 + c_2 = c \) if \( V_2 \) is semisimple, and isomorphic to \( L(c, 0, 0) \) if \( V_2 \) is not semisimple.

**Proof:** If \( V_2 \) is a 2-dimensional semisimple commutative associative algebra with the identity \( \omega/2 \). Then \( \omega = \omega^1 + \omega^2 \) so that \( \omega^1/2 \) and \( \omega^2/2 \) are the primitive idempotents. It follows from \([\text{M1}]\) that \( \omega^1 \) and \( \omega^2 \) are Virasoro vectors. Let
\[
Y(\omega^j, z) = \sum_{n \in \mathbb{Z}} L^j(n) z^{-n-2}
\]
for $i = 1, 2$. Then

$$[L^i(m), L^i(n)] = (m - n)L^i(m + n) + \frac{m^3 - m}{12}\delta_{m+n,0}c_i$$

for all $m, n \in \mathbb{Z}$ where $c_i \in \mathbb{C}$ is the central charge of $\omega^i$. Since $\frac{\omega^i}{2}, \frac{\omega^j}{2} = \delta_{i, j}$ we see that $(\omega^i)_3\omega^2 = (\omega^i, \omega^2) = 0$ by using the invariant property of the bilinear form. This implies that

$$[L^1(m), L^2(n)] = 0$$

for all $m, n \in \mathbb{Z}$ and $c_1 + c_2 = c$. Then $V = \langle \omega^1 \rangle \otimes \langle \omega^2 \rangle$ where $\langle \omega^i \rangle$ is the vertex operator subalgebra of $V$ generated by $\omega^i$ (with a different Virasoro vector). Note that $\langle \omega^i \rangle$ is a quotient of $\tilde{V}(c_i, 0)$. Since $V$ is simple we immediately have that $\langle \omega^i \rangle$ is isomorphic to $L(c_i, 0)$. As a result, $V$ is isomorphic to $L(c_1, 0) \otimes L(c_2, 0)$ in this case.

It remains to deal with the case that $V_2$ is not semisimple. Then the Jacobson radical $J$ of $V_2$ is 1-dimensional. Assume that $J = \mathbb{C}x$. Then $x^2 = 0$ and $(x, x) = (\omega/2, x^2) = 0$. Using the skew symmetry $Y(x, z)x = e^{L(-1)x}Y(x, -z)x$ we see that

$$x_0x = -x_0x + L(-1)x_1x = -x_0x + L(-1)x^2 = -x_0x.$$

This implies $x_0x = 0$. As a consequence, we see the component operators $x_n$ of $Y(x, z)$ commute with each other. That is, $[x_m, x_n] = 0$ for all $m, n \in \mathbb{Z}$.

Note that $(\omega, \omega) = L(2)\omega = c$. Since the form $(\cdot, \cdot)$ on $V_2$ is nondegenerate, we may choose $x$ so that $(\omega, x) = c$. Set $W(m) = x_{m+1}$ for $m \in \mathbb{Z}$. Then we have the following commutator formula

$$[L(m), W(n)] = (m - n)W(m + n) + \frac{m^3 - m}{12}\delta_{m+n,0}c.$$ 

This exactly says that the operators $L(m), W(m), c$ generate a copy of $W(2, 2)$ and $V$ is an irreducible highest weight module for $W(2, 2)$. So in this case, $V$ is isomorphic to $L(c, 0, 0)$, as desired. □

**Remark 3.2** Theorem 3.1 is the main reason we introduce and study the Lie algebra $W(2, 2)$ and its highest weight modules. The vertex operator algebra $L(c, 0, 0)$ will be used in the next section when we characterize the rational vertex operator algebra $L(1/2, 0) \otimes L(1/2, 0)$.

## 4 Characterization of $L(1/2, 0) \otimes L(1/2, 0)$

In this section we give a characterization for the vertex operator algebra $L(1/2, 0) \otimes L(1/2, 0)$.

We first recall some basic facts about a rational vertex operator algebra following from [DLM1]. A vertex operator algebra $V$ is called rational if any admissible module is completely reducible. It is proved in [DLM1] (also see [Z]) that if $V$ is rational then there
are only finitely many irreducible admissible modules $M^1, ..., M^k$ up to isomorphism such that

$$M^i = \oplus_{n \geq 0} M^i_{\lambda_i + n}$$

where $\lambda_i \in \mathbb{Q}$, $M^i_{\lambda_i} \neq 0$ and each $M^i_{\lambda_i + n}$ is finite dimensional (see [AM] and [DLM2]). Let $\lambda_{\text{min}}$ be the minimum of $\lambda_i$’s. The effective central charge $\tilde{c}$ is defined as $c - 24\lambda_{\text{min}}$.

A vertex operator algebra is called $C_2$-cofinite if $C_2(V)$ has finite codimension where

$$C_2(V) = \langle u - 2v | u, v \in V \rangle.$$

For each $M^i$ we define the $q$-character of $M^i$ by

$$\text{ch}_q M^i = q^{-c/24} \sum_{n \geq 0} (\dim M^i_{\lambda_i + n}) q^{n + \lambda_i}.$$

Then $\text{ch}_q M^i$ converges to a holomorphic function on the upper half plane if $V$ is $C_2$-cofinite [Z]. Using the modular invariance result from [Z] and results on vector valued modular forms from [KM] we have (see [DM2])

**Lemma 4.1** Let $V$ be rational and $C_2$-cofinite. For each $i$, the coefficients of $\eta(q)^{\tilde{c}} \text{ch}_q M^i$ satisfy the polynomial growth condition where

$$\eta(q) = q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

We also need some basic facts about the highest weight modules for the Virasoro algebra (see [FF], [FQS], [GKO], [FZ], [W]).

**Proposition 4.2** Let $c$ be a complex number.

(i). $\bar{V}(c,0)$ is a vertex operator algebra and $L(c,0)$ is a simple vertex operator algebra.

(ii) The following are equivalent: (a) $\bar{V}(c,0) = L(c,0)$, (b) $c \neq c_{s,t} = 1 - 6(s - t)^2/st$ for all coprime positive integers $s, t$ with $1 < s < t$, (c) $L(c,0)$ is not rational. In this case, the $q$-graded character of $L(c,0)$ is equal to $\prod q^{-c/24} n + \lambda_i$ and the coefficients grow faster than any polynomials.

(iii) The following are equivalent: (a) $\bar{V}(c,0) \neq L(c,0)$, (b) $c = c_{s,t}$ for some $s, t$, (c) $L(c,0)$ is rational.

From now on we assume that $V$ is a rational and $C_2$-cofinite vertex operator algebra of the moonshine type such that $c = \tilde{c} = 1$ and $\dim V_2 = 2$. We have already mentioned in Section 3 that $V_2$ is a commutative algebra with identity $\omega$. The assumption that $\dim V_2 = 2$ makes $V_2$ an associative algebra.

**Lemma 4.3** The $V_2$ is a semisimple associative algebra. That is, $V_2$ is a direct sum of two ideals isomorphic to $\mathbb{C}$.

**Proof:** Suppose that $V_2$ is not semisimple. Recall from the proof of Theorem 3.1 that the Jacobson radical $J = \mathbb{C}x$ is one-dimensional. We assume that $(\omega, x) = 1$. Then the
Lemma 4.4
If \( c \neq 0 \), then the coefficients of \( \text{ch}_q L(c, 0) \) do not satisfy the polynomial growth condition.

**Proof:** If \( c \neq c_{s, t} \) for any coprime integers \( 1 < s < t \) then \( \text{ch}_q L(c, 0) = \frac{q^{-c/24}}{\prod_{n>1}(1-q^n)} \) by Proposition 4.2 and the result is clear. We now assume that \( c = c_{s, t} \) for some \( s, t \). Suppose that the coefficients of

\[
\text{ch}_q L(c_{s, t}, 0) = q^{-c/24} \sum_{n \geq 0} a_n q^n
\]
satisfy the polynomial growth condition. Then there exists a positive integer \( A \) and \( \alpha \) such that \( a_n \leq An^\alpha \) for all \( n \geq 0 \).

Let \( m \) be a positive integer such that \( m \geq \alpha \). Then

\[
\frac{1}{(1-q)^{m+1}} = \sum_{n \geq 0} \binom{-m-1}{n} (-1)^n q^n
\]
where

\[
\binom{-m-1}{n} = \frac{(-m-1)(-m-2) \cdots (-m-n)}{n!} = \binom{m+n}{m} (-1)^n.
\]
Thus
\[ \frac{1}{(1 - q)^{m+1}} = \sum_{n \geq 0} \binom{m + n}{m} q^n. \]
Since \( \binom{m + n}{m} \) is greater than \( \frac{m^n}{m!} \) we see that
\[ q^{c/24} \text{ch}_q L(c, 0) \leq m! A \frac{1}{(1 - q)^{m+1}} \]
as formal power series.

We next prove that there exists a positive integer \( k \) such that \( kc_{s,t} \neq c_{s_1,t_1} \) for any coprime integers \( 1 < s_1 < t_1 \). To see this we need to examine the equation
\[ 1 - \frac{6(s_1 - t_1)^2}{st} = k(1 - \frac{6(s - t)^2}{st}) \]
which is equivalent to
\[ st(13s_1t_1 - 6s_1^2 - 6t_1^2) = s_1t_1k(13st - 6s^2 - 6t^2). \]
Then both \( s_1 \) and \( t_1 \) are factors of \( 6st \). So there are only finitely many \( s_1, t_1 \) satisfy this equation. This implies that such \( k \) exists.

Consider vertex operator algebra \( L(c, 0)^{\otimes k} \) which contains a vertex operator subalgebra
\[ \tilde{V}(ck, 0) = L(ck, 0) \text{ as } ck \neq c_{s_1,t_1} \text{ for any } s_1, t_1. \]
So
\[ q^{ck/24} \text{ch}_q L(ck, 0) \leq q^{ck/24} (\text{ch}_q L(c, 0)^{\otimes k})^k \leq (m! A)^k \frac{1}{(1 - q)^{mk}} \]
and the coefficients of \( q^{ck/24} \text{ch}_q L(ck, 0) \) satisfy the polynomial growth condition.

On the other hand we know from Proposition 4.2 that
\[ q^{ck/24} \text{ch}_q L(ck, 0) = \frac{1}{\prod_{n>1}(1 - q^n)} \]
whose coefficients satisfy the exponential growth condition. This is a contradiction. The proof is complete. \( \square \)

**Lemma 4.5** Let \( \omega^i \) and \( c_i \) be as before. Then \( c_i = c_{s_i,t_i} \) for some coprime integers \( 1 < s_i < t_i \) and \( \langle \omega^i \rangle \) is isomorphic to \( L(c_{s_i,t_i}, 0) \) for \( i = 1, 2 \).

**Proof:** First we note that as formal power series, \( \text{ch}_q U \leq \text{ch}_q V \). Let \( U^i = \langle \omega^i \rangle \). Then \( U = U^1 \otimes U^2 \) and \( \text{ch}_q U^1 \text{ch}_q U^2 \leq \text{ch}_q V \). Since \( \text{ch}_q U^i \geq \text{ch}_q L(c_i, 0) \) for \( i = 1, 2 \) we have \( \text{ch}_q L(c_1, 0) \text{ch}_q (c_2, 0) \leq \text{ch}_q V \). So
\[ \eta(q) \text{ch}_q U^1 \text{ch}_q U^2 \leq \eta(q) \text{ch}_q V, \quad \eta(q) \text{ch}_q L(c_1, 0) \text{ch}_q L(c_2, 0) \leq \eta(q) \text{ch}_q V \]
as functions for \( q \in (0, 1) \).
Assume that \( \text{ch}_q U^1 = \prod_{n>1} q^{-c_1/24} (1-q^n) \). Then
\[
\eta(q) \text{ch}_q U \geq \eta(q) \prod_{n>1} q^{-c_1/24} \text{ch}_q L(c_2, 0)
\]
as functions for \( q \in (0, 1) \). That is,
\[
\eta(q) \text{ch}_q U \geq q^{c_2/24} \text{ch}_q L(c_2, 0).
\]
From the proof of Lemma 4.4 we see that if the coefficients of \( (1-q) \text{ch}_q L(c_2, 0) \) satisfy the polynomial growth condition, so does the coefficients of \( \text{ch}_q L(c_2, 0) \). But this is impossible by Lemma 4.4. But \( q^{c_2/24} (1-q) \text{ch}_q L(c_2, 0) \leq \eta(q) \text{ch}_q V \) as functions for \( q \in (0, 1) \) and the coefficients of \( \eta(q) \text{ch}_q V \) satisfy the polynomial growth condition. This is a contradiction.

By Proposition 4.2 we see immediately that \( c_i = c_{s_i, t_i} \) for some \( s_i, t_i \) and \( \langle \omega_i \rangle \) is isomorphic to \( L(c_{s_i, t_i}) \) for \( i = 1, 2 \).

Lemma 4.6 Let \( c_i = c_{s_i, t_i} \) as in Lemma 4.5. Then both \( c_1 \) and \( c_2 \) are \( \frac{1}{2} \).

Proof: We need to solve the equation
\[
1 - \frac{6(s_1 - t_1)}{s_1 t_1} + 1 - \frac{6(s_2 - t_2)}{s_2 t_2} = 1
\]
for two pairs of coprime integers \( 1 < s_i < t_i \). That is,
\[
\frac{s_1}{t_1} + \frac{t_1}{s_1} + \frac{s_2}{t_2} + \frac{t_2}{s_2} = \frac{25}{6}.
\]
Let \( x = \frac{s_1}{t_1} \) and \( y = \frac{s_2}{t_2} \). Then we the equation becomes
\[
x + \frac{1}{x} + y + \frac{1}{y} = \frac{25}{6}.
\]

The following argument using the elliptic curve is due to N. Elkies and we thank him and A. Ryba for communicating the solution to us. The equation \( x + \frac{1}{x} + y + \frac{1}{y} = \frac{25}{6} \) gives an elliptic curve. Multiply the equation by \( 6xy \) to get
\[
E : 6xy^2 + 6x^2 y + 6x + 6y = 25.
\]
Putting one of the Weierstrass points at infinity yields the curve
\[
y^2 + XY = X^3 - 1070X + 7812
\]
which has rank 0 over \( \mathbb{Q} \). So every rational points in \( E \) is a torsion point. So \( E/\mathbb{Q} \) has at most 16 torsion points. Note that the curve has 8 obvious symmetries, generated by the involutions taking \((x, y)\) to \((1/x, y), (x, 1/y), \) and \((y, x)\). Here are the rational points in \( E \): four from \( (\frac{3}{7}, \frac{3}{7}) \), four from \((1, \frac{3}{7})\), four from \((-1, 6)\) and 4 from infinity.

Since we assume that \( 1 < s_i < t_i \) and \( s_i, t_i \) are coprime, we immediately see that the only solution interesting to us is \( (\frac{3}{7}, \frac{3}{7}) \). This is, \( c_i = \frac{1}{2} \) for \( i = 1, 2 \).

Here is a characterization of \( L(1/2, 0) \otimes L(1/2, 0) \).
Theorem 4.7 If $V$ is a rational and $C_2$-cofinite vertex operator algebra of the moonshine type such that $c = \tilde{c} = 1$ and $\dim V_2 = 2$, then $V$ is isomorphic to $L(1/2, 0) \otimes L(1/2, 0)$.

Proof: By Lemmas 4.5 and 4.6, the vertex operator subalgebra $U$ generated by $V_2$ of $V$ is isomorphic to $L(1/2, 0) \otimes L(1/2, 0)$ which is rational has 9 inequivalent irreducible modules $L(1/2, h_1) \otimes L(1/2, h_2)$ for $h_i \in \{0, 1/2, 1, 1/4, 1/16\}$ (see [DMZ], [W]). Thus $V$ is a direct sum of irreducible $L(1/2, 0) \otimes L(1/2, 0)$-modules. Note that $h_1 + h_2 \in \mathbb{Z}$ if and only if $h_1 = h_2 = 0$ or $h_1 = h_2 = 1/2$. So only $L(1/2, 0) \otimes L(1/2, 0)$ and $L(1/2, 1/2) \otimes L(1/2, 1/2)$ can possibly occur in $V$ as $L(1/2, 0) \otimes L(1/2, 0)$-modules. Using the assumption that $\dim V_0 = 1$ and $V_1 = 0$ gives the result that $V$ is isomorphic to $L(1/2, 0) \otimes L(1/2, 0)$. $\square$

We end this paper with the following conjecture which strengthens Theorem 4.7.

Conjecture 4.8 If $V$ is simple, rational and $C_2$ cofinite vertex operator algebra of the moonshine type with $c = \tilde{c} = 1$ and $\dim V_2 > 1$, then $V$ is isomorphic to $L(1/2, 0) \otimes L(1/2, 0)$.

It is essentially proved in [K] that if $V$ is a rational vertex operator algebra such that $\sum_i |\chi_i(q)|^2$ is modular invariant where $\chi_i(q)$ are the $q$-character of the irreducible $V$-module, then the $q$-character of $V$ is equal to the character of one of the following vertex operator algebras $V_L$, $V_L^+$ and $V_{Z\alpha}^G$ where $L$ is any positive definite even lattice of rank 1, $V_L^+$ is the fixed points of the automorphism of $V$ lifted from the $-1$ isometry of $L$, and $Z\alpha$ is the root lattice of type $A_1$ such that $(\alpha, \alpha) = 2$ and $G$ is a finite subgroup of $SO(3)$ isomorphic to $A_4$, $S_4$ or $A_5$. It is widely believed that $V_L$, $V_L^+$ and $V_{Z\alpha}^G$ should give a complete list of simple and rational vertex operator algebras with $c = \tilde{c} = 1$. One can easily find counter example if $c \neq \tilde{c}$. It is clearly from the construction that if $V$ is one of these vertex operator algebras of the moonshine type then $\dim V_2 = 2$. This should be a very strong evidence for the conjecture 4.8. We remark that the assumption that $\sum_i |\chi_i(q)|^2$ is modular invariant in [K] is still an open problem in mathematics.

References

[AM] G. Anderson and G. Moore, Rationality in conformal field theory, Comm. Math. Phys. 117 (1988), 441-450.

[D] C. Dong, Representations of the moonshine module vertex operator algebra, Contemp. Math. 175 (1994), 27-36.

[DGH] C. Dong, R. Griess Jr. and G. Hoehn, Framed vertex operator algebras, codes and the moonshine module, Comm. Math. Phys. 193 (1998), 407-448.

[DGL] C. Dong, R. Griess Jr. and C. Lam, Uniqueness results of the moonshine vertex operator algebra, American Journal of Math. 129 (2007), 583-609.

[DLM1] C. Dong, H. Li and G. Mason, Twisted representations of vertex operator algebras, Math. Ann. 310 (1998), 571-600.
C. Dong, H. Li and G. Mason, Modular invariance of trace functions in orbifold theory and generalized moonshine, *Comm. Math. Phys.* **214** (2000), 1-56.

C. Dong, H. Li, G. Mason and P. Montague, The radical of a vertex operator algebra, in: *Proc. of the Conference on the Monster and Lie algebras at The Ohio State University, May 1996*, ed. by J. Ferrar and K. Harada, Walter de Gruyter, Berlin- New York, 1998, 17-25. York, 1998.

C. Dong and G. Mason, Holomorphic vertex operator algebras of small central charges, *Pacific J. Math.* **213** (2004), 253-266.

C. Dong and G. Mason, Rational vertex operator algebras and the effective central charge, *International Math. Research Notices* **56** (2004), 2989-3008.

C. Dong and G. Mason, Integrability of $C_2$-cofinite vertex operator algebras, *International Math. Research Notices* **2006** (2006), Article ID 80468, 15 pages.

C. Dong, G. Mason and Y. Zhu, Discrete series of the Virasoro algebra and the moonshine module, *Proc. Symp. Pure. Math., American Math. Soc.* **56** II (1994), 295-316.

B. Feigin and D. Fuchs, Verma Modules over the Virasoro Algebra, *Lect. Notes in Math.* **1060**, Springer, (1984), 230-245.

I. B. Frenkel, Y. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, *Memoirs American Math. Soc.** 104*, 1993.

I. B. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, *Pure and Applied Math.*, Vol. **134**, Academic Press, 1988.

I. B. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, *Duke Math. J.* **66** (1992), 123-168.

D. Friedan, Z. Qiu and S. Shenker, Details of the non-unitarity proof for highest weight representations of Virasoro Algebra, *Comm. Math. Phys.* **107** (1986), 535-542.

M. Gaberdiel and A. Neitzke, Rationality, quasirationality and finite W-algebras, *Comm. Math. Phys.** 238* (2003), 305-331.

P. Goddard, A. Kent and D. Olive, Unitary representations of the Virasoro Algebra and super-Virasoro algebras *Comm. Math. Phys.** 103* (1986), 105-119.

Y. Kawahigashi and R. Longo, Local conformal nets arising from framed vertex operator algebras, *Adv. Math.** 206* (2006), 729-751.

E. Kiritsis, Proof of the completeness of the classification of rational conformal field theories with $c = 1$, *Phys. Lett.** B217* (1989), 427-430.
[KM] M. Knopp and G. Mason, On vector-valued modular forms and their Fourier coefficients, *Acta Arith.* 110 (2003), 117-124.

[LY] C. Lam and H. Yamauchi, A characterization of the moonshine vertex operator algebra by means of Virasoro frame, arXiv:math/0609718

[L1] H. Li, Symmetric invariant bilinear forms on vertex operator algebras, *J. Pure and Appl. Math.* 96 (1994), 279-297.

[L2] H. Li, Local system of vertex operators, vertex superalgebras and modules, *Pure and Appl. Math.* 109 (1996), 143-195

[LL] J. Lepowsky and H. Li, Introduction to Vertex Operator Algebras, and Their Representations, *Progress in Math.* Vol. 227, Birkhäuser, Boston 2004.

[M1] M. Miyamoto, Griess algebras and conformal vectors in vertex operator algebras, *J. Algebra* 179 (1996), 523-548.

[M2] M. Miyamoto, Binary codes and vertex operator (super)algebras, *J. Algebra* 181 (1996), 207-222.

[M3] M. Miyamoto, A new construction of the Moonshine vertex operator algebra over the real number field, *Ann. of Math.* 159 (2004), 535–596.

[W] W. Wang, Rationality of Virasoro vertex operator algebras, *International Math. Research Notices*, 71 (1993), 197-211.

[X] F. Xu, New braided endmorphisms from conformal inclusions, *Comm. Math. Phys.* 192 (1998), 347-403.

[Z] Y. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* 9 (1996), 237-302.