Developments in the Khintchine-Meinardus probabilistic method for asymptotic enumeration

Boris L. Granovsky∗ and Dudley Stark†

Abstract

A theorem of Meinardus provides asymptotics of the number of weighted partitions under certain assumptions on associated ordinary and Dirichlet generating functions. The ordinary generating functions are closely related to Euler’s generating function \( \prod_{k=1}^{\infty} S(z^k) \) for partitions, where \( S(z) = (1 - z)^{-1} \). By applying a method due to Khintchine, we extend Meinardus’ theorem to find the asymptotics of the coefficients of generating functions of the form \( \prod_{k=1}^{\infty} S(a_k z^k)^{b_k} \) for sequences \( a_k, b_k \) and general \( S(z) \). We also reformulate the hypotheses of the theorem in terms of generating functions. This allows us to prove rigorously the asymptotics of Gentile statistics and to study the asymptotics of combinatorial objects with distinct components.

1 Introduction

Meinardus [10] proved a theorem about the asymptotics of weighted partitions with weights satisfying certain conditions. His result was extended

∗Department of Mathematics, Technion-Israel Institute of Technology, Haifa, 32000, Israel, e-mail:mar18aa@technunix.technion.ac.il
†School of Mathematical Sciences, Queen Mary, University of London, London E1 4NS, United Kingdom, e-mail:D.S.Stark@maths.qmul.ac.uk

2000 5Mathematics Subject Classification: 05A16, 60F99, 81T25
Keywords and phrases: Meinardus’ theorem, Asymptotic enumeration, Dirichlet generating functions, Models of ideal gas and of quantum field theory.
to the combinatorial objects called assemblies and selections in [4] and to
Dirichlet generating functions for weights, with multiple singularities in [6].
In this paper, we extend Meinardus’ theorem further to a general frame-
work, which encompasses a variety of models in physics and combinatorics,
including previous results.

Let \( f \) be a generating function of a nonnegative sequence \( \{c_n, n \geq 0, c_0 = 1\} \):

\[
f(z) = \sum_{n \geq 0} c_n z^n,
\]

with radius of convergence 1. As an example, consider the number of weighted
partitions \( c_n \) of size \( n \), determined by the generating function identity

\[
\sum_{n=0}^{\infty} c_n z^n = \prod_{k=1}^{\infty} \left(1 - z^k\right)^{-b_k}, \quad |z| < 1,
\]

for some sequence of real numbers \( b_k \geq 0, k \geq 1 \). When \( b_k = 1 \) for all \( k \geq 1 \),
then \( c_n \) is the number of integer partitions. Meinardus [10] proved a theorem
giving the asymptotics of \( c_n \) under certain assumptions on the sequence \( \{b_k\} \).

The generating function in (2) may be expressed as \( \prod_{k=1}^{\infty} (S(z^k))^{b_k} \), where
\( S(z) = (1 - z)^{-1} \). This observation allows the following generalization. Let
\( f \) in (1) be of the form:

\[
f(z) = \prod_{k=1}^{\infty} (S(a_k z^k))^{b_k},
\]

with given sequences \( 0 < a_k \leq 1, b_k \geq 0, k \geq 1, \) and a given function \( S(z) \).

This is a particular case of the class of general multiplicative models,
introduced and studied by Vershik ([15]). In the setting ([3]), in the case
of weighted partitions, a combinatorial meaning can be attributed to the
parameters \( a_k, b_k \). Namely, if \( b_k = 1 \), then \( a_k \) can be viewed as a properly
scaled number of colours for each component of size \( k \), such that given \( l \)
components of size \( k \), the total number of colourings is \( a_k^l \). On the other
hand, if \( a_k = 1 \), then given \( l \) components of size \( k \), the total number of
colourings equals the number of distributions of \( b_k \) indistinguishable balls
among \( l \) cells, so that in this model \( b_k \) has a meaning of a scaled number of
types prescribed to a component of size \( k \).

Yakubovich ([17]) derived the limit shapes for models (3) in the case \( a_k = 1, k \geq 1, \) under some analytical conditions on \( S \) and \( b_k \). Note that past
versions [4]–[6] of Meinardus’ theorem deal with the asymptotics of \( c_n, \ n \to \infty \), when \( a_k = 1, \ k \geq 1 \), for three cases of the function \( S \), corresponding to the three classic models of statistical mechanics, which are equivalent to the three aforementioned models in combinatorics. Our objective in this paper is to derive the asymptotics \( c_n, \ n \to \infty \), in the general framework [3].

The assumptions above [1] and [3] on the sequence \( c_n \) imply that \( S(0) = 1 \), that \( S(z) \) can be expanded in a power series with radius of convergence \( \geq 1 \) and non-negative coefficients \( d_j \):

\[
S(z) = \sum_{j=0}^{\infty} d_j z^j , \quad (4)
\]

with \( d_0 = 1 \), and that \( \log S(z) \) can be expanded as

\[
\log S(z) = \sum_{j=1}^{\infty} \xi_j z^j \quad (5)
\]

with radius of convergence 1. From [3] and [5] one can express the coefficients \( \Lambda_k \) of the power series for the function \( \log f(z) \), with radius of convergence 1:

\[
\log f(z) = \sum_{k=1}^{\infty} \Lambda_k z^k, \quad \Lambda_k = \sum_{j|k} b_j a_j^{k/j} \xi_{k/j} . \quad (6)
\]

We define the Dirichlet generating function for the sequence \( \Lambda_k \):

\[
D(s) = \sum_{k=1}^{\infty} \Lambda_k k^{-s} , \quad (7)
\]

which by virtue of [6] admits the following presentation

\[
D(s) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} b_k \xi_j a_j^k (jk)^{-s} , \quad (8)
\]

as long as \( \Re s \) is large enough so that the double Dirichlet series in [8] converges absolutely. If \( a_k = a, \ 0 < a \leq 1 \) for all \( k \geq 1 \), then \( D(s) \) can be factored as

\[
D(s) = D_b(s) D_{\xi,a}(s) , \quad (9)
\]
where

\[ D_{\xi,a}(s) = \sum_{j=1}^{\infty} a_j \xi_j j^{-s} \]

and

\[ D_{b}(s) = \sum_{k=1}^{\infty} b_k k^{-s}. \]  \( \tag{10} \)

The greater generality of (3) than in previous versions of Meinardus’ theorem will allow novel applications. The proof of Theorem 1 stated below, is a substantial modification of the method used in [4, 5, 6].

We suppose that \( \Lambda_k \) and \( D(s) \) satisfy conditions (I) – (III), which are modifications of the three original Meinardus’ conditions in [10].

**Condition (I).** The Dirichlet generating function \( D(s) \), \( s = \sigma + it \) is analytic in the half-plane \( \sigma > \rho_r > 0 \) and it has \( r \geq 1 \) simple poles at positions \( 0 < \rho_1 < \rho_2 < \ldots < \rho_r \), with positive residues \( A_1, A_2, \ldots, A_r \) respectively. It may also happen that \( D(s) \) has a simple pole at 0 with residue \( A_0 \). (If \( D(s) \) is analytic at 0, we take \( A_0 = 0 \)). Moreover, there is a constant \( 0 < C_0 \leq 1 \), such that the function \( D(s), s = \sigma + it \), has a meromorphic continuation to the half-plane

\[ \mathcal{H} = \{ s : \sigma \geq -C_0 \} \]

on which it is analytic except for the above \( r \) simple poles.

**Condition (II).** There is a constant \( C_1 > 0 \) such that

\[ D(s) = O \left( |t|^{C_1} \right), \quad t \to \infty \]

uniformly for \( s = \sigma + it \in \mathcal{H} \).

**Condition (III).**

The following property of the parameters \( a_k, b_k \) holds:

\[ b_k a_k^{l_0} \geq C_2 k^{\rho_r - 1}, \quad k \geq 1, \quad C_2 > 0, \]  \( \tag{11} \)

where

\[ l_0 := \min\{ j > 0 : d_j > 0 \}. \]  \( \tag{12} \)

Moreover, if \( l_0 > 1 \) then for \( \delta_n \) as defined below in [28], for some fixed \( \epsilon > 0 \) and for large enough \( n \),

\[ 2 \sum_{k=1}^{\infty} \Lambda_k e^{-k\delta_n} \sin^2(\pi k\alpha) \geq \left( 1 + \frac{\rho_r}{2} + \epsilon \right) |\log \delta_n|, \]  \( \tag{13} \)

\[ (2l_0)^{-1} \leq |\alpha| \leq 1/2, \quad l_0 > 1, \]
where \( \Lambda_k \) is as defined in (3).

In order to state our main result, we need some more notations, which were also used in \([6]\). Define the finite set

\[
\tilde{\Upsilon}_r = \left\{ \sum_{k=0}^{r-1} \tilde{d}_k (\rho_r - \rho_k) : \tilde{d}_k \in \mathbb{Z}_+, \sum_{k=0}^{r-1} \tilde{d}_k \geq 2 \right\} \cap (0, \rho_r + 1],
\]

where we have set \( \rho_0 = 0 \) and let \( \mathbb{Z}_+ \) denote the set of nonnegative integers. Let \( 0 < \alpha_1 < \alpha_2 < \ldots < \alpha_{|\tilde{\Upsilon}_r|} \leq \rho_r + 1 \) be all ordered numbers forming the set \( \tilde{\Upsilon}_r \). Clearly, \( \alpha_1 = 2(\rho_r - \rho_{r-1}) \), if the set \( \tilde{\Upsilon}_r \) is not empty. We also define the finite set

\[
\Upsilon_r = \tilde{\Upsilon}_r \cup \{ \rho_r - \rho_k : k = 0, 1, \ldots, r - 1 \},
\]

observing that some of the differences \( \rho_r - \rho_k, k = 0, \ldots, r - 1 \) may fall into the set \( \tilde{\Upsilon}_r \). We let \( 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_{|\Upsilon_r|} \) be all ordered numbers forming the set \( \Upsilon_r \).

**Theorem 1** Suppose conditions (I) – (III) are satisfied. Suppose that \( c_n \) has ordinary generating function of the form (3), where \( 0 < a_k \leq 1 \) and \( b_k \geq 0, k \geq 1 \), that (11) is satisfied for a constant \( C_2 > 0 \), and that

\[
\frac{d^2}{d\delta^2} \log S(e^{-\delta}) > 0, \quad \delta > 0.
\]

We then have, as \( n \to \infty \),

\[
c_n \sim Hn^{-\frac{2\rho_r - 2A_0}{2(\rho_r + 1)}} \exp \left( \sum_{l=0}^{r} P_l n^{\rho_l/(\rho_r + 1)} + \sum_{l=0}^{r} \hat{h}_l \sum_{s: \lambda_s \leq \rho_l} K_{s,l} n^{\rho_l - \lambda_s/(\rho_r + 1)} \right),
\]

where \( H, P_l, \hat{h}_l \) and \( K_{s,l} \) are constants. In particular, if \( r = 1 \), then \( K_{s,l} = 0 \) for all \( s \) and \( l \),

\[
P_1 = \left( 1 + \frac{1}{\rho_1} \right) (A_1 \Gamma(\rho_1 + 1))^{1/(\rho_1 + 1)}
\]

and

\[
H = e^{\Theta - \gamma A_0} \left( 2\pi(1 + \rho_1) \right)^{-1/2} (A_1 \Gamma(\rho_1 + 1))^{1-2A_0/(2(\rho_1 + 1))},
\]

where

\[
\Theta := \lim_{s \to 0} (D(s) - A_0 s^{-1})
\]

and \( \gamma \) is Euler’s constant.
Theorem 1 generalizes the results in [4, 6] and implies the results therein, including expansive weighted partitions, for which \( S(x) = (1 - x)^{-1}, \ a_k = 1, k \geq 1 \) and \( b_k = k^{r-1}, k \geq 1 \) for some \( r > 0 \).

**Example** This example shows that (11) is not implied by the other hypotheses of Theorem 1. Let \( a_k = 1 \) for all \( k \), let \( b_k = k^{\rho_1 - 1} \) where \( \rho_1 > 0 \), and let \( \xi_k = k^{\rho_2 - 1} \), where \( 0 < \rho_1 < \rho_2 \). Then, \( D_{\xi,1}(s) = \zeta(s+1-\rho_2), D_b(s) = \zeta(s+1-\rho_1) \), where \( \zeta \) is the Riemann zeta function, and \( D(s) = D_b(s)D_{\xi,1}(s) \) has poles at \( \rho_1 \) and \( \rho_2 \). Moreover, \( S(z) = \exp \left( \sum_{k=1}^{\infty} k^{\rho_1-1} z^k \right) \) has radius of convergence 1 and it is easy to check that (15) is satisfied.

Theorem 1 is proven in Section 2. In the remaining two sections, we focus on two novel applications implied by Theorem 1. In Sections 3 and 4 we apply our results to the asymptotic enumeration of Gentile statistics and expansive selections with \( a_k = k^{-q} \). The latter generalizes previous results for polynomials over a finite field.

## 2 Proof of Theorem 1

As in [4-6], the proof of Theorem 1 is based on the Khintchine type representation (9)

\[
c_n = e^{n\delta} f_n(e^{-\delta}) \mathbb{P}(U_n = n), \ n \geq 1,
\]

where \( \delta > 0 \) is a free parameter,

\[
f_n = \prod_{k=1}^{n} S(a_k z^k)^{b_k}
\]

is the truncation of (3), and the \( U_n, n \geq 1 \) are integer-valued random variables with characteristic functions defined by

\[
\phi_n(\alpha) = \mathbb{E} \left( e^{2\pi i \alpha U_n} \right) = \prod_{k=1}^{n} \left( \frac{S \left( a_k e^{2\pi i k\alpha - k\delta} \right)}{S(a_k e^{-k\delta})} \right)^{b_k}, \ n \geq 1, \ \alpha \in \mathbb{R}.
\]

Khintchine established (18) for the three basic models of statistical mechanics. For general multiplicative measures (18) was stated in equation (4) of [4].

The first step in the proof is to find the asymptotics of \( F(\delta) := \log f(e^{-\delta}) \), as \( \delta \to 0 \).
Lemma 1

(i) As $\delta \to 0^+$,

$$\mathcal{F}(\delta) = \exp\left(\sum_{l=0}^{r} h_l \delta^{-\rho_l} - A_0 \log \delta + M(\delta; C_0)\right), \quad (21)$$

where $\rho_0 = 0$,

$$h_l = A_l \Gamma(\rho_l), \quad l = 1, \ldots, r,$$

$$h_0 = \Theta - \gamma A_0,$$

$$M(\delta; C_0) = \frac{1}{2\pi i} \int_{-C_0-i\infty}^{-C_0+i\infty} \delta^{-s} \Gamma(s) D(s) ds = O(\delta^{C_0}), \quad \delta \to 0,$$

where $\Theta$ is as in (17).

(ii) The asymptotic expressions for the derivatives

$$\left(\log \mathcal{F}(\delta)\right)^{(k)}$$

are given by the formal differentiation of the logarithm of (21), with

$$(M(\delta; C_0))^{(k)} = O(\delta^{C_0-k}), \quad k = 1, 2, 3, \quad \delta \to 0.$$

Proof

We use the fact that $e^{-u}$, $u > 0$, is the Mellin transform of the Gamma function:

$$e^{-u} = \frac{1}{2\pi i} \int_{v-i\infty}^{v+i\infty} u^{-s} \Gamma(s) ds, \quad u > 0, \quad \Re(s) = v > 0. \quad (22)$$

Applying (22) with $v = \rho_r + \epsilon$, $\epsilon > 0$ we have

$$\log \mathcal{F}(\delta) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} b_k \xi_j a_k^j \delta^{-\epsilon} \Gamma(s)$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} b_k \xi_j a_k^j \delta^{-\epsilon} \Gamma(s)$$

$$= \frac{1}{2\pi i} \int_{\epsilon + \rho_r - i\infty}^{\epsilon + \rho_r + i\infty} \delta^{-s} \Gamma(s) D(s) ds,$$

$$\quad (23)$$
where we have used (6) and (7) at (23). Next, we apply the residue theorem for the integral (23), in the complex domain $-C_0 \leq \Re(s) \leq \rho_r + \epsilon$, with $0 \leq C_0 < 1, \epsilon > 0$. By virtue of condition (I), the integrand in (23) has $r$ simple poles at $\rho_l > 0$, $l = 1, \ldots, r$. The corresponding residues at $s = \rho_l$ are equal to: $A_0 \delta - \rho_l \Gamma(\rho_l)$, $l = 1, \ldots, r$.

By the Laurent expansions at $s = 0$ of the Gamma function $\Gamma(s) = \frac{1}{s} - \gamma + \ldots$, and the function $D(s) = \frac{A_0}{s} + \Theta + \ldots$, the integrand $\delta^{-s}D(s)\Gamma(s)$ may also have a pole at $s = 0$, which is a simple one with residue $\Theta$, if $A_0 = 0, \Theta \neq 0$, and is of a second order with residue $\Theta - \gamma A_0 - A_0 \log \delta$, if $A_0 \neq 0, \Theta \neq 0$. In the case $A_0 = \Theta = 0$, the integrand $\delta^{-s}D(s)\Gamma(s)$ is analytic at $s = 0$.

Applying condition (II) shows that the integral of the integrand $\delta^{-s}\Gamma(s)D(s)$, over the horizontal contour $-C_0 \leq \Re(s) \leq \epsilon + \rho_r, \Im(s) = t$, tends to zero, as $t \to \infty$, for any fixed $\delta > 0$. This gives the claimed formulae (21), where the remainder term $M(\delta; C_0)$ is the integral taken over the vertical contour $-C_0 + it, -\infty < t < \infty$. This proves (i).

In order to prove (ii), one differentiates the logarithm of (21) with respect to $\delta$ and estimates the remaining integral in the same way as above. \[ \blacksquare \]

We will need the following bound on $b_k$.

**Proposition 1** Let the double series $D(s)$ defined by (8) converge absolutely in the half-plane $\Re(s) > \rho$, for some $\rho > \rho_r$. Then the following bound holds

$$ b_k a_k^{j_0} = o(k^\rho), \quad k \to \infty, $$

(24)

where $j_0 = \min\{j \geq 1 : \xi_j \neq 0\}$.

**Proof** The assumed absolute convergence of the double series in (8) implies the absolute convergence of the iterated series

$$ \sum_{j=1}^{\infty} \frac{\xi_j}{j^\rho} \sum_{k=1}^{\infty} \frac{b_k a_k^j}{k^\rho}. $$

Consequently,

$$ \sum_{k=1}^{\infty} \frac{b_k a_k^j}{k^\rho} < \infty, \quad \text{for all} \quad j \geq 1 : \xi_j \neq 0. $$

Hence,

$$ \frac{b_k a_k^j}{k^\rho} \to 0, \quad k \to \infty, \quad \text{for all} \quad j \geq 1 : \xi_j \neq 0. $$
The latter implies (24)

In the probabilistic approach initiated by Khintchine, the free parameter \( \delta = \delta_n \) is chosen to be the solution of the equation

\[
\mathbb{E} U_n = n, \ n \geq 1.
\] (25)

The equation for \( \delta_n \) can be written as

\[
\left( -\log (f_n(e^{-\delta})) \right)'_{\delta = \delta_n} = n, \ n \geq 1.
\] (26)

For each \( n \geq 1 \), the function \( \left( -\log (f_n(e^{-\delta})) \right)'_{\delta} \) is decreasing for all \( \delta > 0 \) because of (15). Moreover, setting \( \delta = C n^{\frac{1}{\rho+1}} \), \( C > 0 \) we have

\[
\left( -\log (f_n(e^{-\delta})) \right)'_{\delta} = \left( -\log F(\delta) \right)'_{\delta} - \sum_{k=n+1}^{\infty} \left( -b_k \log S(a_k e^{-k\delta}) \right)'_{\delta}
\]

\[
= \left( -\log F(\delta) \right)'_{\delta} - O \left( \sum_{k=n+1}^{\infty} b_k \xi_{j_0} a_k^{j_0} e^{-\delta k j_0} k j_0 \right)
\]

\[
\sim C n^{\frac{1}{\rho+1}} \rho_r h_r n,
\] (27)

where the step before the last is because for the chosen \( \delta \) we have \( n \delta = C n^{\frac{1}{\rho+1}} \to \infty, n \to \infty \), because of Lemma (ii) and because of the fact that for \( k \geq n + 1 \),

\[
\left( -\log S(a_k e^{-k\delta}) \right)'_{\delta} \sim \xi_{j_0} a_k^{j_0} e^{-\delta k j_0} k j_0, \ n \to \infty,
\]

where \( j_0 \) as in (24), while the last step follows from Lemma (ii) and (24). The right hand side of (27) is \( > n \), if \( C > (\rho_r h_r)^{-(\rho_r+1)} \) and \( \leq n \) otherwise. This and (15) say that for a sufficiently large \( n \), (26) has a unique solution \( \delta_n \), which satisfies

\[
\delta_n \sim (\rho_r h_r)^{-(\rho_r+1)} n^{\frac{1}{\rho+1}}, \ n \to \infty.
\] (28)

We proceed to find an asymptotic expansion for \( \delta_n \) by using a refinement of the scheme of Proposition 1 of [6]. We call any \( \tilde{\delta}_n \), such that

\[
\left( -\log f_n(e^{-\delta}) \right)'_{\delta = \tilde{\delta}_n} - n \to 0, \ n \to \infty
\] (29)
an asymptotic solution of (26). We will show that it is sufficient for (29) that \( \tilde{\delta}_n \) obeys the condition

\[
( - \log F(\delta) )'_{\delta = \tilde{\delta}_n} - n \to 0, \quad n \to \infty. \tag{30}
\]

By Lemma 1 we have

\[
( - \log F(\delta) )'_{\delta = \delta_n} \sim h_r \rho_r \delta^{-\rho_r - 1}, \quad \delta \to 0,
\]

so that (30) implies

\[
\tilde{\delta}_n \sim (h_r \rho_r)^{\frac{1}{\rho_r + 1}} n^{-\frac{1}{\rho_r + 1}}, \quad n \to \infty. \tag{31}
\]

Next we have for all \( n \geq 1 \)

\[
\log f_n(e^{-\tilde{\delta}_n}) = \log F(\tilde{\delta}_n) - \sum_{k=n+1}^{\infty} b_k \log S(a_k e^{-k \delta_n}). \tag{32}
\]

Applying the same argument as in (27), we derive the bound

\[
\sum_{k=n+1}^{\infty} \left( - b_k \log S(a_k e^{-k \delta}) \right)'_{\delta = \tilde{\delta}_n} = o(1), \quad n \to \infty. \tag{33}
\]

Now, (32) and (33) show that (30) implies (29). We will now demonstrate that the error of approximating the exact solution \( \delta_n \) by the asymptotic solution \( \tilde{\delta}_n \) is of order \( o(n^{-1}) \). By the definitions of \( \delta_n, \tilde{\delta}_n \) we have

\[
\left( - \log f_n(e^{-\delta}) \right)'_{\delta = \delta_n} - \left( - \log f_n(e^{-\delta}) \right)'_{\delta = \tilde{\delta}_n} = \epsilon_n, \quad \epsilon_n \to 0, \quad n \to \infty. \tag{34}
\]

Next, applying the Mean Value Theorem, we obtain

\[
\left| \left( - \log f_n(e^{-\delta_n}) \right)' - \left( - \log f_n(e^{-\tilde{\delta}_n}) \right)' \right| = \left| (\delta_n - \tilde{\delta}_n) \left( \log f_n(e^{-u_n}) \right)' \right|,
\]

where

\[
u_n \in [\min(\delta_n, \tilde{\delta}_n), \max(\delta_n, \tilde{\delta}_n)].
\]

By (34), the left hand side of (35) tends to 0, as \( n \to \infty \), while, by virtue of (28), (31),

\[
\left( \log f_n(e^{-u_n}) \right)' \sim \rho_r (\rho_r + 1) h_r (\delta_n)^{-\rho_r - 2} = O(n^{\frac{\rho_r + 2}{\rho_r + 1}}), \tag{36}
\]

10
Combining (35) with (36), gives the desired estimate
\[ \left| \delta_n - \tilde{\delta}_n \right| = o(n^{-1}). \]  

(37)

An obvious modification of the argument in (27) allows also to conclude that
\[ \sum_{k=n+1}^{\infty} b_k \log S(a_k e^{-k\delta_n}) \to 0, \quad n \to \infty. \]

As a result,
\[ f_n(e^{-\delta_n}) \sim F(\delta_n), \quad n \to \infty. \]

The latter relation will be used for derivation of the asymptotics of the second factor in (18).

Define the notations
\[ \hat{h}_l = \rho_l h_l, \quad l = 1, \ldots, r, \]
and
\[ \hat{h}_0 = -A_0. \]

By (ii) of Lemma 1 we have
\[ \left( -\log F(\delta) \right)' = \sum_{l=0}^r \hat{h}_l \delta^{-\rho_l-1} + \left( M(\delta; C_0) \right)'. \]

(38)

This is exactly the starting point of the analysis of \( \tilde{\delta}_n \) in Proposition 1 of [6]. We may therefore apply Proposition 1 of [6] and (37) and conclude that
\[ \delta_n = \left( \hat{h}_r \right)^{1/r} n^{-1/r} + \sum_{s=1}^{\lceil \gamma_r \rceil} \tilde{K}_s n^{-1/r} + o(n^{-1}), \]

(39)

where \( \tilde{K}_s \) do not depend on \( n \), and the powers \( \lambda_s \) are as defined in (14). We now analyze the three factors in the representation (18) when \( \delta = \delta_n \).

(i) It follows from (39) that the first factor of (18) equals
\[ e^{n \delta_n} = \exp \left\{ \left( \hat{h}_r \right)^{1/r} n^{-1/r} + \sum_{s: \lambda_s \leq \rho} \tilde{K}_s n^{\rho - \lambda_s} + \epsilon_n \right\}, \]

(40)
where $\lambda_s \in \Upsilon_r$ and $\epsilon_n \to 0$.

(ii) By an argument similar to the one for the proof of (33) we conclude that

$$
(\log f_n(e^{-\delta}))^{(k)}_{\delta = \delta_n} = (\log F(\delta))^{(k)}_{\delta = \delta_n} + \epsilon_k(n),
$$

for $k = 1, 2, 3$, where $\epsilon_k(n) = o(1)$. For $l = 0, 1, \ldots, r$

$$(\delta_n)^{-\rho_l} = (\hat{h}_r)^{-\rho_l + 1}_n + \sum_{s: \lambda_s \leq \rho_l} K_{s,l} \frac{\rho_l - \lambda_s}{\rho_r + 1} + \epsilon_n(l),$$

where $\epsilon_n(l) = o(1), l = 1, 2, \ldots r$, and where the coefficients $K_{s,l}$ are obtained from the binomial expansion for $(\delta_n)^{-\rho_l}$, based on (39) and the definition (14) of the set $\Upsilon_r$. Consequently, substituting $\delta = \delta_n$ into (21) gives

$$
\log f_n(e^{-\delta_n}) = \sum_{l=0}^r \hat{h}_l (\hat{h}_r)^{-\rho_l + 1}_n + \sum_{l=0}^r \hat{h}_l \sum_{s: \lambda_s \leq \rho_l} K_{s,l} \frac{\rho_l - \lambda_s}{\rho_r + 1} + \left(\frac{A_0}{\rho_r + 1} \log n - \frac{A_0}{\rho_r + 1} \log \hat{h}_r\right) + \epsilon_n.
$$

(iii) The following estimate is central to our arguments.

Proposition 2 Recall that $\phi_n(\alpha)$ is defined by (20) and that $l_0, j_0$ are defined by (12), (24) respectively. Then we have for all $\alpha \in \mathcal{R}$,

$$
\log |\phi_n(\alpha)| = \log |\phi_n(\alpha; \delta_n)| = -2 \sum_{k=1}^{\infty} \Lambda_k e^{-k\delta_0} \sin^2(\pi k \alpha) + \epsilon_n
$$

$$
\leq - \frac{2d_{l_0}}{S^2(e^{-1/8\delta_0})} \sum_{k=(8\delta_0)^{-1}}^n b_k a_{l_0}^k e^{-\delta_0 l_0 k} \sin^2(\pi \alpha l_0 k),
$$

where $\epsilon_n \to 0$ as $n \to \infty$. 

12
Proof We write \( \log \left| \phi_n(\alpha) \right|, \alpha \in \mathbb{R} \), as

\[
\log \left| \phi_n(\alpha) \right| = \frac{1}{2} \sum_{k=1}^{n} b_k \left\{ \log S(a_k e^{2\pi i k \alpha - k \delta_n}) + \log S(a_k e^{-2\pi i k \alpha - k \delta_n}) - 2 \log S(a_k e^{-k \delta_n}) \right\}
\]

\[
= \frac{1}{2} \sum_{k=1}^{\infty} b_k \left\{ \log S(a_k e^{2\pi i k \alpha - k \delta_n}) + \log S(a_k e^{-2\pi i k \alpha - k \delta_n}) - 2 \log S(a_k e^{-k \delta_n}) \right\}
\]

\[
+ O \left( \sum_{k=n+1}^{\infty} b_k \xi_{j_0} a_k e^{-k \delta_n} \eta_0 \right) \quad (45)
\]

\[
= \frac{1}{2} \sum_{k=1}^{\infty} b_k \left\{ \log S(a_k e^{2\pi i k \alpha - k \delta_n}) + \log S(a_k e^{-2\pi i k \alpha - k \delta_n}) - 2 \log S(a_k e^{-k \delta_n}) \right\}
\]

\[
+ O \left( \sum_{k=n+1}^{\infty} k^j e^{-k \delta_n j_0} \right) \quad (46)
\]

\[
= \frac{1}{2} \sum_{k=1}^{\infty} b_k \sum_{j=1}^{\infty} \xi_j a_k^j e^{-jk \delta_n} \left( e^{2\pi i j k \alpha} + e^{-2\pi i j k \alpha} - 2 \right) + \epsilon_n \quad (47)
\]

\[
= -2 \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} b_k \xi_j a_k^j e^{-jk \delta_n} \sin^2(\pi j k \alpha) + \epsilon_n
\]

\[
= -2 \sum_{k=1}^{\infty} \Lambda_k e^{-k \delta_n} \sin^2(\pi k \alpha) + \epsilon_n, \quad (48)
\]

where \( \epsilon_n \to 0, n \to \infty \), (45) and (46) use (24), (47) uses (28) and (48) follows from (6).

As for the inequality (44), defining \( \tau = \delta_n - 2\pi i \alpha, \alpha \in \mathbb{R} \), we have

\[
\log \left| \phi_n(\alpha) \right| = \Re \left( \log f_n(e^{-\tau}) - \log f_n(e^{-\delta_n}) \right)
\]

\[
= \frac{1}{2} \sum_{k=1}^{n} b_k \log \frac{|S(a_k e^{-k \tau})|^2}{S^2(a_k e^{-k \delta_n})}
\]

\[
= \frac{1}{2} \sum_{k=1}^{n} b_k \log \left( 1 - \frac{S^2(a_k e^{-k \delta_n}) - |S(a_k e^{-k \tau})|^2}{S^2(a_k e^{-k \delta_n})} \right)
\]

\[
\leq -\frac{1}{2} \sum_{k=1}^{n} b_k \frac{S^2(a_k e^{-k \delta_n}) - |S(a_k e^{-k \tau})|^2}{S^2(a_k e^{-k \delta_n})}, \quad (49)
\]
where the last inequality is because \( S^2(a_k e^{-k\delta_n}) - |S(a_k e^{-k\tau})|^2 \geq 0 \), for all \( \alpha \in \mathcal{R} \) and because \( \log(1 - x) \geq -x, \ 0 < x < 1 \). Recalling (4) and (12), we obtain for all \( \alpha \in \mathcal{R} \),

\[
S^2(a_k e^{-k\delta_n}) - |S(a_k e^{-k\tau})|^2 = 4 \sum_{0 \leq l, m < \infty} d_l d_m a_k^{l+m} e^{-(l+m)k\delta_n} \sin^2((l - m)\pi \alpha_k)
\]

\[
\geq 4d_{l_0}a_{k_0}^l e^{-\delta_{n,l_0}k} \sin^2(\pi \alpha_{l_0}k), \ k = 1, 2, \ldots,
\]

which allows to continue (49), arriving at the desired bound:

\[
\log |\phi_n(\alpha)| \leq -2d_{l_0} \sum_{k=1}^n b_k a_{k_0}^l e^{-\delta_{n,l_0}k} \sin^2(\pi \alpha_{l_0}k)
\]

\[
\leq -2d_{l_0} \sum_{k=(8l_0\delta_n)^{-1}}^n b_k a_{k_0}^l e^{-\delta_{n,l_0}k} \sin^2(\pi \alpha_{l_0}k)
\]

\[
\leq - \frac{2d_{l_0}}{S^2(e^{-1/8l_0})} \sum_{k=(8l_0\delta_n)^{-1}}^n b_k a_{k_0}^l e^{-\delta_{n,l_0}k} \sin^2(\pi \alpha_{l_0}k),
\]

where the last step is because \( d_l \geq 0, \ l = 1, 2, \ldots \), because \( 0 < a_k \leq 1 \) and because \( 1 \leq S(z) < \infty \) is monotonically increasing in \( 0 \leq z < 1 \).

The asymptotics of the third factor of (18) are given by a local limit theorem, using condition (III).

**Theorem 2 (Local Limit Theorem).**

Let the random variable \( U_n \) be defined as in (19), (20). Then

\[
P(U_n = n) \sim \frac{1}{\sqrt{2\pi \text{Var}(U_n)}} \sim \frac{1}{\sqrt{2\pi K_2}} (\delta_n)^{1+\rho_r/2}
\]

\[
\sim \frac{1}{\sqrt{2\pi K_2}} \left( \hat{h}_{r,\rho_r/2} \right)^{2+\rho_r} n^{-\frac{2+\rho_r}{2+\rho_r}}, \ n \to \infty,
\]

for a constant \( K_2 = h_r \rho_r (\rho_r + 1) \).

**Proof** We take \( \delta = \delta_n \) in (20) and define

\[
\alpha_0 = (\delta_n)^{\rho_r/2} \log n. \quad (50)
\]
We write

\[ \mathbb{P}(U_n = n) = \int_{-1/2}^{1/2} \phi_n(\alpha)e^{-2\pi i n\alpha} d\alpha = I_1 + I_2, \]

where

\[ I_1 = \int_{-\alpha_0}^{\alpha_0} \phi_n(\alpha)e^{-2\pi i n\alpha} d\alpha \]

and

\[ I_2 = \int_{-\alpha_0}^{-1/2} \phi_n(\alpha)e^{-2\pi i n\alpha} d\alpha + \int_{1/2}^{\alpha_0} \phi_n(\alpha)e^{-2\pi i n\alpha} d\alpha. \]

The proof has two parts corresponding to evaluation of the integrals \( I_1 \) and \( I_2 \), as \( n \to \infty \).

**Part 1: Integral \( I_1 \).** Defining \( B_n \) and \( T_n \) by

\[ B_n^2 = \left( \log f_n(e^{-\delta}) \right)^{''\prime} \delta = \delta_n \quad \text{and} \quad T_n = -\left( \log f_n(e^{-\delta}) \right)^{''\prime} \delta = \delta_n \]

for \( n \) fixed we have the expansion

\[ \phi_n(\alpha)e^{-2\pi i n\alpha} = \exp \left( 2\pi i \alpha (\mathbb{E} U_n - n) - 2\pi^2 \alpha^2 B_n^2 + O(\alpha^3) T_n \right) \]

\[ = \exp \left( -2\pi^2 \alpha^2 B_n^2 + O(\alpha^3) T_n \right), \quad \alpha \to 0, \]

where the second equation is due to (25). By virtue of (21) and (41) we derive from (51) that the main terms in the asymptotics for \( B_n^2 \) and \( T_n \) depend on the rightmost pole \( \rho_r \) only:

\[ B_n^2 \sim K_2(\delta_n)^{-\rho_r-2}, \]

\[ T_n \sim K_3(\delta_n)^{-\rho_r-3}, \quad n \to \infty \]

where \( K_2 = h_r \rho_r(\rho_r + 1) \) and \( K_3 = h_r \rho_r(\rho_r + 1)(\rho_r + 2) \) are obtained from (51) and Lemma 1. Therefore,

\[ B_n^2 \alpha_0^2 \to \infty, \quad T_n \alpha_0^3 \to 0, \quad n \to \infty. \]

Consequently, in the same way as in the proof of local theorem in [4],

\[ I_1 \sim \frac{1}{\sqrt{2\pi B_n^2}}, \quad n \to \infty, \]

and it is left to show that

\[ I_2 = o(I_1), \quad n \to \infty. \]
Part 2: Integral $I_2$. We rewrite the upper bound in (44) in Proposition 2 as

$$\log |\phi_n(\alpha)| \leq -CV_n(\alpha), \quad \alpha \in \mathbb{R},$$

where $C > 0$ is a constant and

$$V_n(\alpha) := \sum_{k=(8l_0\delta_n)^{-1}}^{n} b_k a_k^{l_0} e^{-\delta_n l_0 k} \sin^2(\pi \alpha l_0 k).$$

We split the interval of integration $[\alpha_0, 1/2]$ into subintervals:

$$[\alpha_0, (2\pi)^{-1}\delta_n] \cup [(2\pi)^{-1}\delta_n, 1/2] \quad \text{if } l_0 = 1$$

and

$$[\alpha_0, (2\pi l_0)^{-1}\delta_n] \cup [(2\pi l_0)^{-1}\delta_n, (2l_0)^{-1}] \cup [(2l_0)^{-1}, 1/2] \quad \text{if } l_0 > 1.$$

Our goal is to bound, as $n \to \infty$, the function $V_n(\alpha)$ from below in each of the subintervals. Firstly, we show that on the first two subintervals for $l_0 \geq 1$, the desired bound is implied by the assumption (11) in condition (III).

In the first subinterval $[\alpha_0, (2l_0)^{-1}\delta_n], \ l_0 \geq 1$ we will use the inequality

$$\sin^2(\pi x) \geq 4 \|x\|^2, \ x \in \mathbb{R},$$

where $\|x\|$ denotes the distance from $x$ to the nearest integer, i.e.

$$\|x\| = \begin{cases} \{x\} & \text{if } \{x\} \leq 1/2; \\ 1 - \{x\} & \text{if } \{x\} > 1/2. \end{cases}$$

(see [2] for the proof of (55)). By (11) and (55), we then have

$$V_n(\alpha) \geq 4e^{-1/2} \sum_{k=(4l_0\delta_n)^{-1}}^{(2l_0\delta_n)^{-1}} C_2 k^{\rho r-1} \|\alpha l_0 k\|^2, \ \alpha \in \mathbb{R}, \ l_0 \geq 1. \quad (56)$$

In the first subinterval,

$$\|\alpha l_0 k\| = \alpha l_0 k, \ 1 \leq k \leq (2l_0\delta_n)^{-1}, \ l_0 \geq 1,$$
so that (56) produces
\[
V_n(\alpha) \geq 4C_2e^{-1/2l_0^2\alpha_0^2} \sum_{k=(4l_0\delta_n)^{-1}} (2l_0\delta_n)^{-1} k^{\rho_+ + 1} \\
\sim 4C_2e^{-1/2}(\rho_+ + 2)^{-1}l_0^2\alpha_0^2((2l_0\delta_n)^{-\rho_+} - (4l_0\delta_n)^{-\rho_+ - 2}), \quad n \to \infty.
\]
By (50), (44) this gives the desired bound in the first subinterval:
\[
\log |\phi_n(\alpha)| \leq -C \log^2 n, \quad C > 0, \quad n \to \infty.
\]
(57)
For the second subinterval we will apply the argument in the proof of Lemma 1 in [4]. Given \(\alpha \in \mathbb{R}\), define \(P\) by
\[
P = P(\alpha, \delta_n) = \left[\frac{1 + |\alpha|\delta_n^{-1}}{2|\alpha|}\right] \geq 1,
\]
where \([x]\) denotes the integer part of \(x\) and the inequality holds for \(n\) large enough. This supplies the bound
\[
\sum_{k=(8\delta_n)^{-1}}^{P} \sin^2(\pi k\alpha) \geq \frac{\delta_n^{-1}}{8},
\]
provided
\[
\frac{\delta_n}{2\pi} \leq \alpha \leq 1/2.
\]
(60)
Observing, that by definition (58), \(n > P \geq (2\delta_n)^{-1} > (8\delta_n)^{-1}\) for \(n\) large enough we rewrite (59) as
\[
\sum_{k=(8\delta_n)^{-1}}^{P} \sin^2(\pi kl_0\alpha) \geq \frac{\delta_n^{-1}}{8}, \quad l_0 \geq 1
\]
(61)
for
\[
(2\pi l_0)^{-1} \delta_n \leq \alpha \leq (2l_0)^{-1}, \quad l_0 \geq 1.
\]
(62)
Then we have for \(\alpha \in \mathbb{R}\) and \(l_0 \geq 1,
\[
\sum_{k=(8l_0\delta_n)^{-1}}^{n} b_ka_k^{l_0} e^{-\delta_n l_0k} \sin^2(\pi \alpha l_0k) \geq \sum_{k=(8l_0\delta_n)^{-1}}^{P} C_2k^{\rho_+-1} e^{-\delta_n l_0k} \sin^2(\pi \alpha l_0k) \\
\geq C_2e^{-P\delta_n} \sum_{k=(8\delta_n)^{-1}}^{P} k^{\rho_+-1} \sin^2(\pi \alpha l_0k) \\
:= Q(\alpha).
\]
In order to get the needed lower bound on $Q(\alpha)$, we take into account that for all $\alpha$ obeying (62), $\frac{1}{2} < P\delta_0 < \frac{1}{2}(1 + 2\pi l_0) := d$. Applying (61), we distinguish between the following two cases: 

(i) $0 < \rho_r < 1$ and 

(ii) $\rho_r \geq 1$.

For $\alpha$ in (62), we have in case (i),

$$Q(\alpha) \geq C_2 e^{-P\delta_0} P\rho_r \delta_n \geq \frac{C_2}{8} e^{-d\delta_0} P\delta_n \rho_r \delta_n \geq \frac{C_2}{8} d\delta_0 \delta_n \rho_r := C_3 \delta_n \rho_r,$$

and in case (ii),

$$Q(\alpha) \geq C_2 e^{-P\delta_0} (\delta_n - \rho_r + 1) \delta_n \geq C_2 e^{-d\delta_0} \delta_n \delta_n \rho_r := C_4 \delta_n \rho_r.$$

Finally, combining this with (63) gives the desired upper bound on $\log |\phi_n(\alpha)|$ for all $\alpha$ in (62) and $n$ sufficiently large:

$$\log |\phi_n(\alpha)| \leq -C\delta_n^{-\rho_r}, \; C > 0. \quad (63)$$

Remark: If $l_0 > 1$, then $\sin^2(\pi\alpha l_0 k) = 0, \; k \geq 1$ when $\alpha = l_0^{-1} \leq 1/2$, so that in the third subinterval $[(2l_0)^{-1}, 1/2]$ the above bounds are not applicable.

In the third subinterval $[(2l_0)^{-1}, 1/2], \; l_0 > 1$ we apply (13) in condition (III). By (43) and (13) we have for $n$ large enough,

$$|\phi_n(\alpha)| \leq \delta_n^{-(1+\varepsilon)/\rho_r}, \; \varepsilon > 0. \quad (64)$$

Comparing the bounds (57), (63), (64) with the asymptotics (53), (52) proves (64).

Finally, to completely account for the influence of all $r+1$ poles $\rho_0, \rho_1, \ldots, \rho_r$, we present the sum of the expressions (40), (42) obtained in (i),(ii) for the first two factors in the representation (18) in the following form:

$$n\delta_n + \log f_n(e^{-\delta_n}) = \sum_{l=0}^{r} P_l n^{\frac{\rho_l}{\rho_r+1}} + \sum_{l=0}^{r} h_l \sum_{s, \lambda_s \leq \rho_l} K_s, l n^{\frac{\rho_l - \lambda_s}{\rho_r+1}}$$

$$+ \left( \frac{A_0}{\rho_r + 1} \log n - \frac{A_0}{\rho_r + 1} \log \hat{h}_r \right) + \varepsilon_n,$$

where $P_l$ denotes the resulting coefficient of $n^{\frac{\rho_l}{\rho_r+1}}$.

If $r = 1$, then (26), (38) produce

$$n = \hat{h}_1 \delta_n^{-\rho_r - 1} + \hat{h}_0 \delta_n^{-1} + O(\delta_n^{c_0-1}) + \varepsilon(n),$$

18
with $\varepsilon(n) \to 0$, $n \to \infty$, which is analogous to equation (54) of [4]. The previous equation can be inverted as in [4], giving

$$
\delta_n = \hat{h}_{1 \rho_1 + 1}^{-1} n^{-\frac{1}{\rho_1 + 1}} + \frac{\hat{h}_0}{\rho_1 + 1} n^{-1} + O(n^{-1 - \beta}),
$$

(65)

where

$$
\beta = \begin{cases}
\frac{C_0}{\rho_1 + 1}, & \text{if } \rho_1 \geq C_0 \\
\frac{\rho_1}{\rho_1 + 1}, & \text{otherwise}
\end{cases}
$$

Substituting (65) into the previous asymptotic estimates of the three factors in (18), obtained in (i)-(iii), results in the values $P_1$ and $H$ as stated in theorem.

### 3 Gentile statistics

Gentile statistics are a model arising in physics [3, 12, 14], which counts partitions of an integer $n$ with no part occurring more than $\eta - 1$ times, where $\eta \geq 2$ is a parameter. When $\eta = 2$, Fermi-Dirac statistics are obtained and when $\eta = \infty$, Bose-Einstein statistics, with uniform weights $b_k = 1$, $k \geq 1$ result. As far as we know, no rigorous derivation of the asymptotics of Gentile statistics has previously been given, although Theorem 3 below was anticipated in approximation (23) of [12]. In this work we derive the aforementioned theorem as a special case of our Theorem 1.

The Gentile statistics are the Taylor coefficients of the generating function

$$
f(z) = \prod_{k=1}^{\infty} \frac{1 - z^{\eta k}}{1 - z^k}, \quad |z| < 1, \quad \eta \geq 2 \text{ is an integer.}
$$

We remark that there is another natural interpretation of the Gentile statistics, which is the number of integer partitions with no part size divisible by $\eta$, but where part sizes can now appear an unlimited number of times. Gentile statistics fit into the framework (3) of Theorem 1 with

$$
S(z) = \frac{1 - z^\eta}{1 - z}, \quad |z| < 1, \quad \eta \geq 2 \text{ is an integer}
$$

and $a_k = b_k = 1$, $k \geq 1$. 
Theorem 3  Gentile statistics have asymptotics

$$c_n \sim \sqrt{\frac{\kappa}{4\pi\eta}} n^{-3/4} e^{2\kappa \sqrt{n}},$$

where

$$\kappa = \sqrt{\zeta(2)(1 - \eta^{-1})}, \quad \eta \geq 2 \text{ is an integer.}$$

Proof  We will show that all the conditions of Theorem 1 are satisfied for Gentile statistics. In order to show that (15) holds for $\eta > 1$, we calculate

$$\frac{d^2}{d\delta^2} \log S(e^{-\delta}) = e^\delta \left( \frac{e^\delta}{(e^\delta - 1)^2} - \frac{\eta^2 e^{\eta\delta}}{(e^{\eta\delta} - 1)^2} \right).$$

We have

$$\frac{d}{d\eta} \frac{\eta^2 e^{\eta\delta}}{(e^{\eta\delta} - 1)^2} = \frac{\eta e^{\eta\delta} g(\eta\delta)}{(e^{\eta\delta} - 1)^3},$$

where $g(x) = e^x(2 - x) - (2 + x)$. Taking the derivative of $g$ produces

$$g'(x) = e^x(1 - x) - 1 < 0 \text{ for } x > 0,$$

which, together with $g(0) = 0$, implies that $g(x) < 0$ for $x > 0$. Combining this with the fact that $\frac{d^2}{d\delta^2} \log S(e^{-\delta}) = 0$, if $\eta = 1$, we conclude that (15) holds, for all $\eta > 1$.

It remains to be shown that conditions (I) – (III) are satisfied for the model considered. We have

$$\log f(z) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{z^{jk}}{j} - \frac{z^{jk\eta}}{j} \right), \quad |z| < 1,$$

and so, by (9), (10) and (8),

$$D(s) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{(jk)^{-s}}{j} - \frac{(jk\eta)^{-s}}{j} \right)$$

$$= \zeta(s) \zeta(s + 1)(1 - \eta^{-s}).$$

Conditions (I) and (II) are satisfied because of the analytic continuation of the Riemann zeta function and the well known bound

$$\zeta(x + iy) = O(|y|^C), \quad y \to \infty,$$  \hspace{1cm} (66)
for a constant $C > 0$, uniformly in $x$. It is easy to check that $l_0 = 1$ and $b_k a_k = 1 = k^{\rho_1 - 1}$, where $\rho_1 = 1$, and so (III) is satisfied. Hence condition (III) is satisfied. Moreover,

$$r = 1, \quad \rho_0 = 0, \quad \rho_1 = 1, \quad A_0 = \lim_{s \to 0} sD(s) = 0, \quad A_1 = \zeta(2)(1 - \eta^{-1}),$$

$$\Theta = \lim_{s \to 0} D(s) = \zeta(0) \log \eta.$$  

By the argument preceding Proposition 1 this says that the integrand $\delta_n^s \Gamma(s) D(s)$ has a simple pole at $s = 0$ with residue $\Theta = \zeta(0) \log \eta$ and a simple pole at $s = 1$ with residue $\zeta(2)(1 - \eta^{-1})\delta_n^{-1}$. As a result, in the case considered $\delta_n = \hat{h}_1^{1/2} n^{-1/2} - 2^{-1}\hat{h}_0 n^{-1} + O(n^{-\frac{C_0}{2}})$ and we arrive at the claimed asymptotic formula for $c_n$. 

## 4 Asymptotic enumeration for distinct part sizes

Weighted partitions fit our framework with $S(z) = (1 - z)^{-1}, a_k = 1, k \geq 1$ and weights $b_k$. When $b_k = 1$, $k \geq 1$, Theorem gives the asymptotics of the number of partitions of $n$ obtained by Hardy and Ramanujan. If $S(z) = 1 + z, \quad a_k = 1, \quad b_k = k^{r-1}, \quad r > 0, \quad k \geq 1$, then $c_n$ enumerates weighted partitions having no repeated parts, called expansive selections. The asymptotics of expansive selections were also studied in [5].

In this section, we find the asymptotics of $c_n$ induced by the generating function

$$f(z) = \sum_{n=0}^{\infty} c_n z^n = \prod_{k=1}^{\infty} (1 + k^{-q} z^k), \quad |z| < 1, \quad q > 0.$$  

The model fits the setting with $S(z) = 1 + z, \quad b_k = 1, \quad a_k = k^{-q}, \quad k \geq 1$ and it can be considered as a colored selection with parameter $k^{-q}$ proportional to the number $m_k$ of colors of a component of size $k$, e.g. $m_k = y^k k^{-q}$, for some $y > 1$. A particular case of the model, when $q = 1$ was studied in Section 4.1.6 of [7] where it was proven, with the help of a Tauberian theorem, that in this case

$$\lim_{n \to \infty} c_n = e^{-\gamma} \quad (67)$$

and it was established the rate of convergence of $c_n, \quad n \to \infty$. Also, in [7] it was shown that $c_n$ is equal to the probability that a random polynomial
of order $n$ is a product of irreducible factors of different degrees. In [11], Section 11, it was demonstrated that $c_n$ can be treated as the probability that a random permutation on $n$ has distinct cycle lengths, and another proof of (67) was suggested.

Finally, note that in [11], (11.35), it is was shown that for $q = 2$, the generating function $f(z)$ can not be analytically continued beyond the unit circle.

**Theorem 4** Let

$$
\sum_{n=0}^{\infty} c_n z^n = \prod_{k=1}^{\infty} (1 + k^{-q} z^k), \quad |z| < 1.
$$

If $0 < q < 1$, then $c_n$ has asymptotics given by (16) with $r = \max\{j \geq 1 : 1 - qj > 0\}$ and $\rho_l = 1 - ql$, $l = 1, \ldots, r$.

If $q > 1$, then, for a constant $W(q) > 0$ depending only on $q$,

$$
c_n \sim W(q)n^{-q}, \quad n \to \infty.
$$

**Proof**

**The case** $0 < q < 1$.

We will apply Theorem 1. Assumption (15) is easy to verify. We have

$$
\log f(z) = \sum_{k \geq 1} \log \left(1 + \frac{z^k}{k^q}\right) = \sum_{k \geq 1} \sum_{j \geq 1} (-1)^{j-1} \frac{z^{kj}}{j^{kq}}, \quad |z| < 1
$$

and so, by (6) and (8),

$$
D(s) = D(s; q) = \sum_{k \geq 1} \sum_{j \geq 1} (-1)^{j-1} \frac{(kj)^{-s}}{jk^{qj}} = \sum_{k \geq 1} \sum_{j \geq 1} \frac{(-1)^{j-1}}{j^{s+1}k^{s+qj}}.
$$

We claim that the function $D(s; q)$ allows analytic continuation to the set $\mathbb{C}$ excepting for poles in $H_q := \{s = 1 - qj, \; j = 1, 2, \ldots, \; q < 1\}$. Changing the order of summation, we write

$$
D(s; q) = \sum_{j \geq 1} \sum_{k \geq 1} \frac{(-1)^{j-1}}{j^{s+1}k^{s+qj}} = \sum_{j \geq 1} \frac{(-1)^{j-1}}{j^{s+1}} \zeta(s + qj), \quad \Re(s) > 0, \; s \notin H_q.
$$

(68)
Note that
\[
\zeta(s + qj) = 1 + \sum_{n=2}^{\infty} \frac{1}{n^{s+qj}} := 1 + \Phi(s; q),
\]
where the function \( \Phi(s; q) \) is analytic for \( s \in \mathbb{C} \setminus H_q \), and moreover
\[
\Phi(s; q) = O(2^{-qj}) \quad j \to \infty, \quad q > 0,
\]
uniformly in \( s \) from any compact subset of \( \mathbb{C} \setminus H_q \). This implies that the series
\[
\sum_{j \geq 1} \frac{(-1)^{j-1}}{j^{s+1}} \Phi(s; q)
\]
converges absolutely and uniformly on any compact subset of \( \mathbb{C} \setminus H_q \). By the Weirstrass convergence theorem, this implies that the series above is analytic in the above indicated domain. Since the function
\[
\sum_{j \geq 1} \frac{(-1)^{j-1}}{j^{s+1}} = -(2^{-s} - 1)\zeta(s + 1)
\]
is analytic in \( \mathbb{C} \), our claim is proven. This allows to conclude that condition \( I \) of Theorem 1 holds with \( r = \max\{j \geq 1 : 1 - qj > 0\} \) simple poles \( \rho_l = 1 - ql, \ l = 1, \ldots, r \) and with \( 0 < C_0 < 1 \) defined by
\[
C_0 = \begin{cases} 
(r + 1)q - 1 - \epsilon, & 0 < \epsilon < (r + 1)q - 1, \text{ if } (r + 1)q \leq 2 \\
\text{any number in } (0, 1), & \text{if } (r + 1)q > 2.
\end{cases}
\]
Condition \( (II) \) follows from \((66)\) and \((68)\). Finally, \( l_0 = 1 \) in the case considered because \( S(z) = 1 + z \) and \( b_k a_k = k^{-q} = k^{-\rho_k} \) and so \( (III) \) is satisfied. Hence condition \( (III) \) is satisfied, by Lemma 1 in \( \text{[4]} \).

**The case \( q > 1 \).**

Theorem 1 is not applicable in this case, because all poles \( 1 - qj, j \geq 1, \quad q > 1 \) of the function \( D(s; q) \) in \((68)\), are negative. From
\[
f(z) = \prod_{k=1}^{\infty} (1 + z^kq^{-q}) = \sum_{n=1}^{\infty} c_n z^n, \quad |z| \leq 1, \quad q > 1
\]
we have
\[
f(1) = \prod_{k=1}^{\infty} (1 + k^{-q}) := W(q) < \infty, \quad q > 1, \quad (69)
\]
since the convergence of the infinite product in (69) is equivalent to the convergence of the series

\[ \sum_{k=1}^{\infty} k^{-q} < \infty, \quad q > 1. \]

By (69),

\[ \sum_{n=1}^{\infty} c_n := W(q). \]

Denoting

\[ W_n(q) = \prod_{k=1}^{n} (1 + k^{-q}) = \sum_{k=1}^{n} c_n, \quad q > 1, \]

implies the desired asymptotics for \( c_n \):

\[ c_n = W_n(q) - W_{n-1}(q) = W_{n-1}(q)n^{-q} \sim W(q)n^{-q}, \quad q > 1, \quad n \to \infty. \]

**Remark:** Comparing the asymptotics of \( c_n \) in the cases \( 0 < q < 1 \), \( q = 1 \) and \( q > 1 \) it is clearly seen that \( q = 1 \) is a point of phase transition.

In the remainder of this section we derive representations of the function \( W(q) \) in the case of rational \( q > 1 \). The infinite product

\[ F(z) := \prod_{k=1}^{\infty} \left(1 + \frac{z}{k^q}\right), \quad z \in \mathbb{C}, \quad q > 1, \quad (70) \]

is a Weierstrass representation of an entire function \( F \) with zeroes at \( \{ -k^q, \quad k = 1, 2, \ldots \} \). This follows from Theorem 5.12 in [1], since \( \sum_{k=1}^{\infty} k^{-q} < \infty, \quad q > 1 \).

Note that \( W(q) = f(1) = F(1), \quad q > 1 \). We now show that in the case when \( q > 1 \) is a rational number, a modification of the argument in [16], p.238 allows us to decompose the value \( F(1) \) in (70) into a finite product of values of a canonic entire function of finite rank. (For the definition of a rank of entire function see Chapter XI in [1].) Let \( q = \frac{m_1}{m_2} \), where \( m_1 > m_2 \geq 1 \) are co-prime integers. We write

\[ 1 + k^{-\frac{m_1}{m_2}} = \prod_{l=1}^{m_1} k^{-\frac{1}{m_2}} - \alpha_l(m_1) = \prod_{l=1}^{m_1} \left(1 - \frac{\alpha_l(m_1)}{k^{\frac{1}{m_2}}}ight), \]
where
\[ \alpha_l(m_1) = \exp \left( \frac{\pi(2l-1)}{m_1} \right), \quad l = 1, \ldots, m_1 \]
are all \( m_1 \)-th roots of \(-1\), such that \( 0 < \arg(\alpha_l(m_1)) < 2\pi, \ l = 1, \ldots, m_1 \).

Consequently,
\[ W(q) = \prod_{k=1}^{\infty} \prod_{l=1}^{m_1} \left( 1 - \alpha_l(m_1) \right), \quad q = \frac{m_1}{m_2}, \quad (71) \]

Next, introduce the function
\[ \tilde{f}(z) := \prod_{k=1}^{\infty} \left( 1 + \frac{z}{k^{m_2}} \right) \exp \left( \sum_{p=1}^{m_2} \frac{(-z)^p}{k^{m_2} p} \right), \quad z \in \mathbb{C}, \quad (72) \]

which is a canonical form of an entire function of finite rank \( m_2 \) with zeroes \( \{-k^{\frac{1}{m_2}}, \ k = 1, 2, \ldots\}. \) Observing that \( \sum_{l=1}^{m_1} (\alpha_l(m_1))^p = 0, \ p = 1, \ldots, m_2, \) by the definition of \( \alpha_l(m_1), \ l = 1, \ldots, m_1, \) we derive from (71):
\[ W(q) = \prod_{l=1}^{m_1} \tilde{f}(\alpha_l(m_1)), \quad (73) \]

for rational \( q > 1. \)

For \( m_2 > 1, \) we will consider now the function
\[ \tilde{\Gamma}(z) := e^{Q(z)} \frac{1}{zf(z)}, \quad (74) \]

where \( Q(z) \) is a polynomial in \( z \) that will be defined below. The preceding discussion yields that \( \tilde{\Gamma} \) is a meromorphic function in \( \mathbb{C} \) with simple poles at \( (-k^{\frac{1}{m_2}}), \ k = 0, 1, \ldots, \) \( \). Now our purpose will be to obtain for the function \( \tilde{\Gamma} \) an analog of Gauss formula for gamma function. We recall the definition of generalized Euler constants:
\[
\gamma_\alpha = \lim_{n \to \infty} \left( \sum_{k=1}^{n} k^{-\alpha} - \int_{1}^{n} x^{-\alpha} \, dx \right) = \begin{cases} \sum_{k=1}^{n} \frac{1}{k} - \log n, & \text{if } \alpha = 1; \\ \sum_{k=1}^{n} \frac{1}{k^\alpha} - \frac{n^{1-\alpha} - 1}{1-\alpha}, & \text{if } 0 < \alpha < 1. \end{cases}
\]
(Note that $\gamma_1 = \gamma$ is the standard Euler constant). This allows to write the function $\frac{1}{zf(z)}$ in the following form:

$$
\frac{1}{zf(z)} = \lim_{n \to \infty} \frac{1}{z} \prod_{k=1}^{n} \left( \frac{k^{\frac{1}{m_2}}}{z + k^{\frac{1}{m_2}}} \right) \exp \left( - \sum_{p=1}^{m_2} \frac{(-z)^p}{k^{\frac{1}{m_2} p}} \right)
$$

$$
= \exp \left( - \sum_{p=1}^{m_2} \frac{(-1)^p z^p}{p} \gamma(p/m_2) \right)
$$

$$
\times \lim_{n \to \infty} \frac{(n!)^{\frac{1}{m_2}}}{\prod_{k=0}^{n} \left( z + k^{\frac{1}{m_2}} \right)} n^{-\frac{(z)^{m_2}}{m_2}} \exp \left( -m_2 \sum_{p=1}^{m_2-1} \frac{(-1)^p z^p}{p(m_2 - p)} \right).
$$

Setting now in (74) $Q(z) = \sum_{p=1}^{m_2} (-1)^p \frac{z^p}{p} \gamma(p/m_2)$, we arrive at the desired representation of the function

$$
\tilde{f}(z) = \lim_{n \to \infty} \frac{(n!)^{\frac{1}{m_2}}}{\prod_{k=0}^{n} \left( z + k^{\frac{1}{m_2}} \right)} n^{-\frac{(z)^{m_2}}{m_2}} \exp \left( -m_2 \sum_{p=1}^{m_2-1} \frac{(-1)^p z^p}{p(m_2 - p)} \right).
$$

Under $m_2 = 1$, (75) becomes the Gauss formula for the Gamma function.

In the case $q > 1$ is an integer, (73) conforms to the explicit expression for $W(q)$ in (16), p.238-239. In fact, after substituting in (72) $m_2 = 1$ and $p = 1$ we have

$$
\tilde{f}(z) = \prod_{k=1}^{\infty} \left( 1 + \frac{z}{k} \right) e^{-\frac{z}{k}}, \quad z \in \mathbb{C}
$$

and by the Weierstrass factorization theorem for the Gamma function,

$$
\tilde{f}(z) = \frac{e^{-\gamma z}}{\Gamma(1 + z)}, \quad z \in \mathbb{C}\\{-1, -2, \ldots\}.
$$

where $\gamma$ is Euler’s constant. Thus, when $q > 1$ is an integer,

$$
W(q) = \prod_{l=1}^{q} \left( \Gamma(1 - \alpha_l(q)) \right)^{-1}.
$$

Taking into account that the numbers $\alpha_l(q), \ l = 1, \ldots, q$ are pairwise conjugate and that $\Gamma(\bar{z}) = \Gamma(z), \ z \in \mathbb{C}$, the last expression can be written as
follows:

\[ W(q) = \prod_{l=1}^{[q/2]} \left( |\Gamma(1 - \alpha_l(q))|^2 \right)^{-1}, \quad q > 1. \]

References

[1] Conway, J.B. (1978). Functions of one complex variable, Springer-Verlag.

[2] Freiman, G., Granovsky B., (2002). Asymptotic formula for a partition function of reversible coagulation-fragmentation processes, Isr. J. Math., 130, 259-279.

[3] Gentile, G. (1941). Osservazione sopra le statistiche intermedie. Nuovo Cimento 17, 493.

[4] Granovsky, B., Stark, D. and Erlihson, M. (2008). Meinardus’ theorem on weighted partitions: Extensions and a probabilistic proof. Adv. Appl. Math. 41 307-328.

[5] Granovsky, B. and Stark, D. (2006), Asymptotic enumeration and logical limit laws for expansive multisets. J. Lon. Math. Soc. (2), 73 252-272.

[6] Granovsky, B. and Stark, D. (2012), A Meinardus theorem with multiple singularities. Comm. Math. Phys. 314 329-350.

[7] Greene, D. H. and Knuth, D. E. (1990). Mathematics for the Analysis of Algorithms. Third Edition. Birkhauser.

[8] Korevaar, J. (2004). Tauberian Theory. A century of development. Springer.

[9] Khinchin, A. I., (1960). Mathematical foundations of quantum statistics, Graylock Press, Albany, N.Y..

[10] Meinardus, G. (1954). Asymptotische Aussagen über Partitionen, Math. Z. 59 388-398.

[11] Odlyzko, A. M. (1995) Asymptotic enumeration methods. In Handbook of Combinatorics, R. Graham, M. Grötschel, and L. Lovász, Eds., vol. II, Elsevier, 1063-1299.
[12] Srivatsan, C. S., Murthy, M. V. N and Bhaduri, R. K. (2006), Gentile statistics and restricted partitions, Pramana, 66 485–494.

[13] Titchmarsh, E.C., (1939). The theory of functions. Oxford University Press.

[14] Tran, M. N., Murthy M. V. N. and Bhaduri, R. K., (2004) On the quantum density of states and partitioning an integer, Ann. Physics 311 204-219

[15] Vershik, A. (1996). Statistical mechanics of combinatorial partitions and their limit configurations. Funct. Anal. Appl. 30, 90-105.

[16] Whittaker, E.T. and Watson, G.N.(1940) A course of modern analysis, Cambridge.

[17] Yakubovich, Y. (2012). Ergodicity of multiplicative statistics. J. Comb. Theory, A 119, 1250-1279.