Abstract

We propose a model for the quark masses and mixings based on an $A_4$ family symmetry. Three scalar $SU(2)$ doublets form a triplet of $A_4$. The three left-handed-quark $SU(2)$ doublets are also united in a triplet of $A_4$. The right-handed quarks are singlets of $A_4$. The $A_4$-symmetric scalar potential leads to a vacuum in which two of the three scalar $SU(2)$ doublets have expectation values with equal moduli. Our model makes an excellent fit of the observed $|V_{ub}/V_{cb}|$. The symmetry $CP$ is respected in the charged gauge interactions of the quarks.

1 Introduction

In the standard $SU(2) \times U(1)$ gauge model for the electroweak interactions, the Yukawa couplings of the fermions to the unique scalar $SU(2)$ doublet of the model are completely arbitrary—as a matter of fact, those couplings make up almost all the free parameters of the Standard Model (SM). As a consequence of this fact, the SM leaves the quark masses and mixings unpredicted; although the model accounts for the existence of quark masses and mixings, their actual values remain arbitrary in the context of the SM.

Several more complex models have tried to overcome this shortcoming of the SM. However, most of those models are in reality Ansätze: instead of deriving the structure of the Yukawa couplings from some underlying symmetry of a self-consistent gauge theory, they simply assume the Yukawa couplings to have some aesthetically appealing pattern or texture. A model should instead rely on some flavour (family) symmetry.
If the flavour symmetry is Abelian, then all its irreducible representations (irreps) are one-dimensional and the symmetry can at most force some Yukawa couplings to vanish. A non-Abelian flavour symmetry can also force non-vanishing Yukawa couplings to be interrelated among themselves through definite Clebsch–Gordan factors. Since there are three families of quarks, a most desirable non-Abelian flavour symmetry ought to have three-dimensional irreps, in order to achieve full unification of the three generations and, hence, to achieve a minimal number of independent Yukawa couplings. The smallest discrete group with a three-dimensional irrep is $A_4$, the group of the even permutations of four objects. The group $A_4$ has 12 group elements, one triplet irrep $3$ and three inequivalent singlet irreps $1$, $1'$, and $1''$ (the $1$ is the trivial representation, the irreps $1'$ and $1''$ are complex-conjugate of each other). This group has, in the last few years, been used in many models for the lepton masses and mixings [3]. It has also been used in models for the quark sector [4], or for both quarks and leptons simultaneously [5].

In this paper we suggest a model for the Yukawa couplings of the quarks based on an $A_4$ family symmetry. Our model has three scalar gauge-$SU(2)$ doublets united in a $3$ of $A_4$. The left-handed-quark $SU(2)$ doublets are also united in a $3$ of $A_4$. In each electric-charge sector, the three right-handed quarks are in a $1 \oplus 1' \oplus 1''$ of $A_4$. Thus, our model achieves a high degree of simplicity and, even, uniqueness, because it treats the three families of quarks in the same way and it treats both electric-charge sectors in the same way. Furthermore, our model does not require $A_4$ to be broken anywhere in the Lagrangian, not even through soft terms—the breaking of $A_4$ is solely spontaneous. This, too, adds to the simplicity of the model.

Surprisingly, our model is able to predict the mixing parameter $|V_{ub}/V_{cb}|$ (where $V$ is the quark mixing, or CKM, matrix) fully right: it predicts $|V_{ub}/V_{cb}| \approx 0.088$, in agreement with the usual averages of the various phenomenological analyses. On the other hand, our model also leads to a null violation of the discrete symmetry $CP$ in the charged gauge interactions of the quarks; thus, the observed $CP$ violation, for instance in $K^0–\bar{K}^0$ mixing, or in $B^0_d$ decays, should in the context of our model be explained through scalar-mediated interactions, including flavour-changing neutral Yukawa interactions.

The plan of our paper is the following. In section 2 we derive the form of the quark Yukawa-coupling matrices. In section 3 we study the scalar potential and the ensuing vacuum. In section 4 we write down the quark mass matrices and demonstrate that in our model there is no $CP$ violation in the CKM matrix. In section 5 we explain the method that we used in the numerical analysis and give some fits and results. A short summary is provided in section 6.

## 2 The Yukawa couplings

The gauge symmetry of the model is $SU(2) \times U(1)$. There are three scalar $SU(2)$ doublets $\phi_j$ ($j = 1, 2, 3$) with hypercharge $1/2$. They form a triplet $3$ of the flavour symmetry $A_4$. There are three left-handed-quark $SU(2)$ doublets $Q_{Lj}$ with hypercharge $1/6$. They are

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1 The converse of this statement also holds: any pattern of vanishing Yukawa couplings may be enforced by an Abelian flavour symmetry with an adequate spectrum of scalars [1].

2 A useful list of all the discrete non-Abelian groups with 31 or less group elements is provided in [2].
united in another 3 of $A_4$. There are three right-handed-quark $SU(2)$ singlets $n_{Rj}$ with hypercharge $-1/3$ and three right-handed-quark $SU(2)$ singlets $p_{Rj}$ with hypercharge $2/3$. The $n_{R1}$ and $p_{R1}$ are $A_4$-invariant (they are 1’s of $A_4$), the $n_{R2}$ and $p_{R2}$ are 1’s of $A_4$, and the $n_{R3}$ and $p_{R3}$ are 1’’s of $A_4$. This means that there exist two non-commuting transformations $T_1$ and $T_2$,

\[
T_1 : \begin{cases} 
\phi_1 \to \phi_2 \to \phi_3 \to \phi_1, \\
Q_{L1} \to Q_{L2} \to Q_{L3} \to Q_{L1}, \\
n_{R2} \to \omega n_{R2}, n_{R3} \to \omega^2 n_{R3}, \\
p_{R2} \to \omega p_{R2}, p_{R3} \to \omega^2 p_{R3}, 
\end{cases} \tag{1}
\]

\[
T_2 : \begin{cases} 
\phi_2 \to -\phi_2, \phi_3 \to -\phi_3, \\
Q_{L2} \to -Q_{L2}, Q_{L3} \to -Q_{L3}, 
\end{cases} \tag{2}
\]

under which the Lagrangian is invariant. In equation (1), $\omega = \exp(2i\pi/3) = \sqrt[3]{-1 + i\sqrt{3}}/2$.

The 3 is a real representation of $A_4$. Indeed, the scalar $SU(2)$ doublets with hypercharge $-1/2$

\[
\tilde{\phi}_j \equiv i\tau_2 \phi_j^* 
\]

transform under $T_1$ and $T_2$ in exactly the same way as the $\phi_j$, as is obvious from equations (1) and (2).

Given this spectrum of fields and their transformation laws under both the gauge symmetry $SU(2) \times U(1)$ and the flavour symmetry $A_4$, the quark Yukawa Lagrangian is

\[
\mathcal{L}_{Yukawa} = -y_1 \left( \overline{Q_{L1}} \phi_1 + \overline{Q_{L2}} \phi_2 + \overline{Q_{L3}} \phi_3 \right) n_{R1} \\
- y_2 \left( \overline{Q_{L1}} \phi_1 + \omega \overline{Q_{L2}} \phi_2 + \omega^2 \overline{Q_{L3}} \phi_3 \right) n_{R2} \\
- y_3 \left( \overline{Q_{L1}} \phi_1 + \omega^2 \overline{Q_{L2}} \phi_2 + \omega \overline{Q_{L3}} \phi_3 \right) n_{R3} \\
- y_4 \left( \overline{Q_{L1}} \tilde{\phi}_1 + \overline{Q_{L2}} \tilde{\phi}_2 + \overline{Q_{L3}} \tilde{\phi}_3 \right) p_{R1} \\
- y_5 \left( \overline{Q_{L1}} \tilde{\phi}_1 + \omega \overline{Q_{L2}} \tilde{\phi}_2 + \omega^2 \overline{Q_{L3}} \tilde{\phi}_3 \right) p_{R2} \\
- y_6 \left( \overline{Q_{L1}} \tilde{\phi}_1 + \omega^2 \overline{Q_{L2}} \tilde{\phi}_2 + \omega \overline{Q_{L3}} \tilde{\phi}_3 \right) p_{R3} + \text{H.c.,} \tag{4}
\]

the six Yukawa couplings $y_{1-6}$ being in general complex.

The scalar doublets

\[
\phi_j = \left( \begin{array}{c} \phi_j^+ \\ \phi_j^0 \\
\end{array} \right), \quad \tilde{\phi}_j = \left( \begin{array}{c} \phi_j^{0*} \\ -\phi_j^- \\
\end{array} \right) \tag{5}
\]

are assumed to have vacuum expectation values (VEVs)

\[
\langle 0 | \phi_j^0 | 0 \rangle = v_1 e^{-i\alpha/2}, \quad \langle 0 | \phi_2^0 | 0 \rangle = v_2 e^{i\beta/2}, \quad \langle 0 | \phi_3^0 | 0 \rangle = v_3, \tag{6}
\]

where $v_{1,2,3}$ are, without loss of generality, real and non-negative. Since $\overline{Q_{Lj}} = (p_{Lj}, \ n_{Lj})$, the quark mass matrices, defined through

\[
\mathcal{L}_{\text{mass}} = - \left( \begin{array}{ccc} n_{L1} & n_{L2} & n_{L3} \\
\end{array} \right) M_n \left( \begin{array}{ccc} n_{R1} & n_{R2} & n_{R3} \\
\end{array} \right) - \left( \begin{array}{ccc} p_{L1} & p_{L2} & p_{L3} \\
\end{array} \right) M_p \left( \begin{array}{ccc} p_{R1} & p_{R2} & p_{R3} \\
\end{array} \right) + \text{H.c.,} \tag{7}
\]

3
are

\[
M_n = D \begin{pmatrix} y_1 v_1 & y_2 v_1 & y_3 v_1 \\ y_1 v_2 & \omega y_2 v_2 & \omega^2 y_3 v_2 \\ y_1 v_3 & \omega^2 y_2 v_3 & \omega y_3 v_3 \end{pmatrix}, \quad (8)
\]

\[
M_p = D^* \begin{pmatrix} y_4 v_1 & y_5 v_1 & y_6 v_1 \\ y_4 v_2 & \omega y_5 v_2 & \omega^2 y_6 v_2 \\ y_4 v_3 & \omega^2 y_5 v_3 & \omega y_6 v_3 \end{pmatrix}, \quad (9)
\]

where

\[
D \equiv \text{diag} \left( e^{-i\alpha/2}, e^{i\beta/2}, 1 \right). \quad (10)
\]

Let the unitary matrices \( U_{L,R}^{n,p} \) satisfy

\[
U_L^n \begin{pmatrix} y_1 v_1 & y_2 v_1 & y_3 v_1 \\ y_1 v_2 & \omega y_2 v_2 & \omega^2 y_3 v_2 \\ y_1 v_3 & \omega^2 y_2 v_3 & \omega y_3 v_3 \end{pmatrix} U_R^n = \text{diag} \left( m_d, m_s, m_b \right), \quad (11)
\]

\[
U_L^p \begin{pmatrix} y_4 v_1 & y_5 v_1 & y_6 v_1 \\ y_4 v_2 & \omega y_5 v_2 & \omega^2 y_6 v_2 \\ y_4 v_3 & \omega^2 y_5 v_3 & \omega y_6 v_3 \end{pmatrix} U_R^p = \text{diag} \left( m_u, m_c, m_t \right). \quad (12)
\]

Then, the quark mixing (CKM) matrix is

\[
V = U_L^p D^2 U_L^n. \quad (13)
\]

One may absorb the phases of \( y_{1,2,3} \) in the overall phases of the three rows of \( U_L^n \), and similarly absorb the phases of \( y_{4,5,6} \) in the matrix \( U_R^p \). Those six phases are therefore unphysical. Thus, this model for the quark masses and mixings has ten parameters: the two phases \( \alpha \) and \( \beta \) in the diagonal matrix \( D^2 \), and the eight real quantities

\[
|y_1| v_3, \quad \frac{y_2}{y_1}, \quad \frac{y_3}{y_1}, \quad |y_4| v_3, \quad \frac{y_5}{y_4}, \quad \frac{y_6}{y_4}, \quad |v_2| v_1, \quad \frac{v_3}{v_3}. \quad (14)
\]

As we shall see in the next section, the \( A_4 \)-symmetric scalar potential is so constrained that these ten parameters reduce to only eight.

3 The scalar potential

The most general renormalizable scalar potential invariant under the symmetry \( A_4 \) is

\[
V = \mu \left( \phi_1^4 \phi_1 + \phi_2^4 \phi_2 + \phi_3^4 \phi_3 \right) + \lambda_1 \left( \phi_1^4 \phi_1 + \phi_2^4 \phi_2 + \phi_3^4 \phi_3 \right)^2
\]

\[
+ \lambda_2 \left[ \left( \phi_1^4 \phi_1 \right) \left( \phi_2^4 \phi_2 \right) \left( \phi_3^4 \phi_3 \right) + \left( \phi_1^4 \phi_2 \right) \left( \phi_2^4 \phi_3 \right) + \left( \phi_1^4 \phi_3 \right) \left( \phi_2^4 \phi_1 \right) \left( \phi_3^4 \phi_1 \right) \right]
\]

\[
+ (\lambda_3 - \lambda_2) \left[ \left( \phi_1^4 \phi_2 \right) \left( \phi_3^4 \phi_1 \right) + \left( \phi_1^4 \phi_3 \right) \left( \phi_2^4 \phi_2 \right) + \left( \phi_2^4 \phi_1 \right) \left( \phi_3^4 \phi_1 \right) \right]
\]

\[
+ \frac{\lambda_4}{2} \left\{ e^{i\epsilon} \left[ \left( \phi_1^4 \phi_2 \right)^2 + \left( \phi_2^4 \phi_3 \right)^2 + \left( \phi_3^4 \phi_1 \right)^2 \right] + \text{H.c.} \right\}, \quad (15)
\]

4
where $\mu$ and $\lambda_{1-4}$ are real. The phase $\epsilon$ is arbitrary.

We define $v \equiv \sqrt{v_1^2 + v_2^2 + v_3^2}$ and

$$\begin{align*}
\theta_1 & \equiv \epsilon - \beta, \\
\theta_2 & \equiv \epsilon - \alpha, \\
\theta_3 & \equiv \epsilon + \alpha + \beta.
\end{align*}$$

Then,

$$V_0 \equiv \langle 0 | V | 0 \rangle = \mu v^2 + \lambda_1 v^4 + \lambda_3 \left( v_1^2 v_2^2 + v_2^2 v_3^2 + v_3^2 v_1^2 \right) + \lambda_4 \left( v_1^2 v_2^2 \cos \theta_3 + v_2^2 v_3^2 \cos \theta_1 + v_3^2 v_1^2 \cos \theta_2 \right).$$

The equations for vacuum stability are

$$\begin{align*}
\frac{\partial V_0}{\partial v_1} &= 0 = \mu + 2 \lambda_1 v^2 + \lambda_3 \left( v_2^2 + v_3^2 \right) + \lambda_4 \left( v_2^2 \cos \theta_3 + v_3^2 \cos \theta_2 \right), \\
\frac{\partial V_0}{\partial v_2} &= 0 = \mu + 2 \lambda_1 v^2 + \lambda_3 \left( v_1^2 + v_3^2 \right) + \lambda_4 \left( v_1^2 \cos \theta_3 + v_3^2 \cos \theta_1 \right), \\
\frac{\partial V_0}{\partial v_3} &= 0 = \mu + 2 \lambda_1 v^2 + \lambda_3 \left( v_1^2 + v_2^2 \right) + \lambda_4 \left( v_1^2 \cos \theta_2 + v_2^2 \cos \theta_1 \right), \\
\frac{\partial V_0}{\partial \alpha} &= 0 = \lambda_4 v_1^2 \left( -v_2^2 \sin \theta_3 + v_3^2 \sin \theta_2 \right), \\
\frac{\partial V_0}{\partial \beta} &= 0 = \lambda_4 v_2^2 \left( -v_1^2 \sin \theta_3 + v_3^2 \sin \theta_1 \right).
\end{align*}$$

We reject possible solutions to these equations in which one of the VEVs vanishes, and also solutions in which $v_1 = v_2 = v_3$. Then, equations (23) and (24) yield

$$\sin \theta_j = kv_j^2,$$

where $k$ is a real constant with dimension $M^{-2}$. Subtracting equations (21) and (22) from equation (20), one obtains

$$\begin{align*}
(v_2^2 - v_1^2) \lambda_3 + \left[ \left( v_2^2 - v_1^2 \right) \cos \theta_3 + v_3^2 \left( \cos \theta_2 - \cos \theta_1 \right) \right] \lambda_4 &= 0, \\
(v_3^2 - v_1^2) \lambda_3 + \left[ \left( v_3^2 - v_1^2 \right) \cos \theta_2 + v_2^2 \left( \cos \theta_3 - \cos \theta_1 \right) \right] \lambda_4 &= 0.
\end{align*}$$

Equations (26) constitute a Cramer system for $\lambda_3$ and $\lambda_4$. The Cramer determinant must vanish and one hence obtains

$$\sum_{j=1}^{3} a_j \cos \theta_j = 0,$$

where

$$\begin{align*}
a_1 & \equiv v_3^4 - v_1^4 + v_1^2 v_2^2 - v_1^2 v_3^2, \\
a_2 & \equiv v_1^4 - v_3^4 + v_2^2 v_3^2 - v_1^2 v_2^2, \\
a_3 & \equiv v_2^4 - v_1^4 + v_1^2 v_3^2 - v_2^2 v_3^2.
\end{align*}$$
satisfy

\[ \sum_{j=1}^{3} a_j = 0. \]  (31)

Equation (27) together with equation (25) imply

\[ 0 = -\lambda \left( a_1 \sqrt{1 - k^2 v_1^4}, \ a_2 \sqrt{1 - k^2 v_2^4}, \ a_3 \sqrt{1 - k^2 v_3^4} \right) \]

\[ = 4k^2 \left[ 4k^2 v_1^4 v_2^4 v_3^4 - \lambda \left( v_1^2, v_2^2, v_3^2 \right) \right] \left( v_1^2 - v_2^2 \right)^2 \left( v_1^2 - v_3^2 \right)^2 \left( v_2^2 - v_3^2 \right)^2, \]  (32)

where

\[ \lambda (a, b, c) \equiv -a^4 - b^4 - c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2. \]  (33)

Equation (32) has five solutions:

1. \( k = 0, \)
2. \( k^2 = \lambda \left( v_1^2, v_2^2, v_3^2 \right) / (4v_1^4 v_2^4 v_3^4), \)
3. \( v_1 = v_2, \)
4. \( v_1 = v_3, \)
5. \( v_2 = v_3. \)

It is easy to explore in detail solutions 1 and 2 and to show that they require \( \lambda_3 = \pm \lambda_4 \) (solution 1 furthermore needs \( \epsilon = 0 \)). Thus, those solutions require a non-trivial constraint on the parameters of the potential, and that constraint is in general unstable under renormalization. Those solutions should therefore be discarded. The remaining solutions 3–5 are equivalent, since the three scalar doublets \( \phi_j \) form a triplet of \( A_4 \). We shall use for definiteness solution 3: \( v_1 = v_2 \) and \( \theta_1 = \theta_2, \) i.e. \( \alpha = \beta. \) Equations (20)–(24) then reduce to only three equations,

\[ 0 = \mu + 2\lambda_1 \left( 2v_1^2 + v_3^2 \right) + \lambda_3 \left( v_1^2 + v_3^2 \right) + \lambda_4 \left[ v_1^2 \cos (\epsilon + 2\alpha) + v_3^2 \cos (\epsilon - \alpha) \right], \]  (34)

\[ 0 = \mu + 2\lambda_1 \left( 2v_1^2 + v_3^2 \right) + 2\lambda_3 v_1^2 + 2\lambda_4 v_3^2 \cos (\epsilon - \alpha), \]  (35)

\[ 0 = v_1^2 \sin (\epsilon + 2\alpha) - v_3^2 \sin (\epsilon - \alpha), \]  (36)

which determine the three quantities \( v_1, v_3, \) and \( \alpha. \)

4 The mass matrices and CP conservation

We saw in the previous section that, just as we had advertised at the end of section 2, consideration of the most general \( A_4 \)-invariant scalar potential actually reduces the ten
parameters of our model to only eight, since \( v_2 = v_1 \) and \( \beta = \alpha \). The quark mass matrices of our model are therefore

\[
M_n = \text{diag} \left( e^{-i\alpha/2}, e^{i\alpha/2}, 1 \right) \begin{pmatrix} a & b & c \\ a & \omega b & \omega^2 c \\ r a & \omega^2 r b & \omega r c \end{pmatrix}
\]

and

\[
M_p = \text{diag} \left( e^{i\alpha/2}, e^{-i\alpha/2}, 1 \right) \begin{pmatrix} f & g & h \\ f & \omega g & \omega^2 h \\ r f & \omega^2 r g & \omega r h \end{pmatrix}
\]

where \( r \equiv v_3/v_1 \) and \( a, b, c, f, g, \) and \( h \) are real and positive. Then,

\[
H_n \equiv M_n M_n^\dagger = \begin{pmatrix} x & ye^{-i\alpha} & ry e^{-i\alpha/2} \\ ye^{i\alpha} & x & ry^{*} e^{i\alpha/2} \\ ry^{*} e^{-i\alpha/2} & ry e^{i\alpha/2} & r^2 x \end{pmatrix}
\]

and

\[
H_p \equiv M_p M_p^\dagger = \begin{pmatrix} z & w e^{-i\alpha} & rw e^{i\alpha/2} \\ we^{i\alpha} & z & rw^{*} e^{-i\alpha/2} \\ rw^{*} e^{-i\alpha/2} & rw e^{i\alpha/2} & r^2 z \end{pmatrix}
\]

where \( x \equiv a^2 + b^2 + c^2 \) and \( z \equiv f^2 + g^2 + h^2 \) are real, while \( y \equiv a^2 + \omega b^2 + \omega^2 c^2 \) and \( w \equiv f^2 + \omega g^2 + \omega^2 h^2 \) are complex. Now, computing the commutator of \( H_p \) and \( H_n \) one finds that it is of the form

\[
[H_p, H_n] = \begin{pmatrix} -n & 0 & -m \\ 0 & n & -m^* \\ m^* & m & 0 \end{pmatrix}
\]

hence \( \det [H_p, H_n] = 0 \). Therefore, in this model there is no \( CP \) violation in the quark mixing matrix, i.e. the Jarlskog observable \( J \) vanishes.

### 5 Numerical procedure and results

We have performed a global \( \chi^2 \) analysis of the quark mass matrices given in the previous section—equations (37) and (38)—by employing the downhill simplex method [8]. Table 1 specifies in its first two columns the observable quantities \( O_i \) in the form

\[
O_i = \bar{O}_i \pm \sigma_i
\]

where \( \bar{O}_i \) is the experimental mean value of \( O_i \) and \( \sigma_i \) is the square root of its variance. The index \( i = 1, \ldots, 9 \) labels the nine observables given in Table 1. Writing \( \mathbf{x} \) for the set of the eight parameters of our model \( (a, b, c, f, g, h, r, \) and \( \alpha) \), and \( P_i(\mathbf{x}) \) for the resulting predictions for each of the observables, one constructs the \( \chi^2 \) function

\[
\chi^2(\mathbf{x}) = \sum_{i=1}^{9} \left[ \frac{P_i(\mathbf{x}) - \bar{O}_i}{\sigma_i} \right]^2
\]

The global minimum of \( \chi^2 \) represents the best possible fit of the model predictions to the experimental data.
| Observable | Experimental value | Model prediction | Pull |
|------------|-------------------|-----------------|------|
| $m_d$ [MeV] | 5 ± 2             | 4.977           | $-1.2 \times 10^{-2}$ |
| $m_s$ [MeV] | 95 ± 25           | 90.545          | $-1.8 \times 10^{-1}$ |
| $m_b$ [MeV] | 4200 ± 70         | 4200.79         | $+1.1 \times 10^{-2}$ |
| $m_u$ [MeV] | 2.25 ± 0.75       | 2.250           | $+1.9 \times 10^{-5}$ |
| $m_c$ [MeV] | 1250 ± 90         | 1250.498        | $+5.5 \times 10^{-3}$ |
| $m_t$ [GeV] | 172.5 ± 2.7       | 172.497         | $-1.2 \times 10^{-3}$ |
| $\sin \theta_{12}$ | 0.2243 ± 0.0016 | 0.22431         | $+8.1 \times 10^{-3}$ |
| $\sin \theta_{23}$ | 0.0413 ± 0.0015 | 0.04139         | $+6.0 \times 10^{-2}$ |
| $\sin \theta_{13}$ | 0.0037 ± 0.0005 | 0.003627        | $-1.5 \times 10^{-1}$ |

Table 1: Experimental data and result of our best fit. The experimental data (average values and error bars) used in our numerical analysis are given in the second column. The data on the quark masses have been taken from [9]. The data on the quark mixing angles have been taken from [10]. The third column displays the values $P_i$ predicted by our model when the values of its parameters are those in equations (44). The fourth column shows the number of standard deviations from the mean values, $(P_i - \bar{O}_i) / \sigma_i$, computed using the data from the second column. The value $\chi^2 = 0.057$ is the sum of the squares of the numbers in the fourth column and is dominated by the pulls of $m_s$ and $\sin \theta_{13}$.

We found an excellent fit, with $\chi^2 = 0.057$, of our model to the nine input data specified in table 1. The input parameters of the fit are

$$
\begin{align*}
  a &= 40.75189, \\
  b &= 87.78761, \\
  c &= 2.347665, \\
  f &= 3941.127, \\
  g &= 515.0460, \\
  h &= 1.060808, \\
  r &= 43.37746, \\
  \alpha &= 0.2251660.
\end{align*}
$$

Other details of the fit are given in the third and fourth columns of table 1.

In order to test the variation of $\chi^2$ as a function of the value $\bar{O}_i$ of an observable quantity $O_i$, we substitute in the expression for $\chi^2(x)$ the term $\frac{(P_i(x) - \bar{O}_i)^2}{(\sigma_i)^2}$ by a term $\frac{(P_i(x) - \bar{O}_i)^2}{(0.01 \bar{O}_i)^2}$. The small error assigned to $\bar{O}_i$ in the denominator of this term guarantees that $O_i$ gets pinned down to the value $\bar{O}_i$.

Figure 1 depicts $\chi^2$ as a function of $r$, i.e. of the ratio of VEVs $v_3/v_1$. We read off from that figure that only for $40 \lesssim r \lesssim 52$ can good fits be obtained; thus, the range of the ratio of VEVs is severely constrained.

In figure 2 (left panel), the change of $\chi^2$ under variations of the quark-mixing observable $|V_{ub}/V_{cb}|$ is shown. There is a pronounced minimum of $\chi^2$ for $0.08 \lesssim |V_{ub}/V_{cb}| \lesssim 0.09$; this is in excellent agreement with the value obtained for that observable by the phe-
nomenological analyses. This remarkable result of our model provides a clear-cut prediction for $|V_{ub}/V_{cb}|$. Figure 2 (right panel) gives $\chi^2$ as a function of $|V_{td}/V_{ts}|$. We find in this case excellent fits whenever $0.14 \lesssim |V_{td}/V_{ts}| \lesssim 0.15$. Clearly, this result is correlated to our model’s prediction for $|V_{ub}/V_{cb}|$, since in our model $CP$ is conserved in quark mixing and therefore the CKM matrix is determined by only three parameters.

6 Conclusions

In this paper we have proposed a self-consistent model for the quark masses and mixings based on a family symmetry $A_4$. The Yukawa-coupling matrices of our model contain, at face value, ten parameters, but, when one considers the $A_4$-symmetric scalar potential in detail, one sees that two of those parameters actually disappear. In our model the family symmetry $A_4$ is not broken anywhere in the Lagrangian—its breaking is fully spontaneous. The model gives a perfect fit of the observed quark masses and mixing parameters, except for the fact that there is no $CP$ violation at all in the CKM matrix. The observed $CP$ violation should result in our model from scalar-mediated interactions, in particular flavour-changing neutral Yukawa interactions at tree level and also charged-scalar-mediated box diagrams at the one-loop level; a detailed study of those interactions should be the subject of a separate publication.

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Figure 1: $\chi^2$ as a function of the ratio of VEVs $r \equiv v_3/v_1$.

Figure 2: $\chi^2$ as a function of the CKM-matrix parameters $|V_{ub}/V_{cb}|$ and $|V_{td}/V_{ts}|$. 