A computational algorithm for constructing a two-dimensional heat wave generated by a non-stationary boundary condition

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Abstract. The paper discusses solutions of the nonlinear heat equation, which have the form of a heat wave propagating on a zero background with a finite velocity. Such solutions are not typical for parabolic equations, and their existence is associated with the degeneration of the problem at the wave (zero) front. We propose a numerical algorithm for constructing a two-dimensional heat wave, symmetrical with respect to the origin, with a non-zero boundary condition defined on the moving boundary. The main difficulty of the new task is that at each time point a heat wave front (a domain boundary) is unknown. The solution is carried out in two stages. At first, we change the roles of unknown function and radial polar coordinate. For a new unknown function at each time point, we obtain a boundary value problem for the Poisson equation in a known region. The step-by-step solving of this problem by the method of boundary elements at a given time interval allows us to determine the law of the zero front moving. At second, we approximate the found zero front by an analytical function and construct a generalized self-similar solution. The developed algorithm is implemented and tested on a task set.

1. Introduction
We consider a nonlinear second order parabolic equation, which describes the process of heat propagation in the case of a power dependence of the thermal conductivity coefficient on temperature [1]. It is also referred as the porous medium equation [2]

\[ T_t = \text{div} (K(T)\nabla T), \quad K(T) = \alpha T^\sigma. \] (1)

Here \( T \) is an unknown function (temperature), \( t \) is time, \( K(T) \) is the thermal conductivity coefficient, \( \text{div}, \nabla \) are the divergence and gradient operators on spatial coordinates \( \alpha, \sigma \) are positive constants.

In addition to modeling heat conductivity [1], Eq. (1) is used to describe the filtration processes of ideal polytropic gas in porous media [2], as well as the migration of biological populations [3].

A lot of publications are devoted to the study of mathematical models of continuum mechanics based on Eq. (1). Among them, we especially note those of Zeldovich [4], Barenblatt et al. [5], Samarsky et al. [1], Sidorov [6], Kudryashov [7], Antontsev and Shmarev [8]. The difference in
the approaches to the study is due to a wide range of considered boundary value problems, as well as a large range of possible research objectives.

Our previous studies were devoted to the boundary-value problems with a moving boundary, the solutions of which have the form of a heat wave propagating on a cold (zero) background with a finite velocity. The primary directions of research were to prove existence and uniqueness theorems with the construction of analytical solutions [9, 10], as well as to find numerical solutions for a finite time interval [11, 12, 13, 14].

Here we consider a mathematical model of a two-dimensional heat wave generated by a nonzero condition defined on a moving boundary. The case of circular symmetry is investigated. Equation (1) is reduced to the following equation in polar coordinates by standard substitution $u = T^\sigma, t = \alpha t$

$$u_t = uu_{\rho\rho} + \frac{u_{\rho}^2}{\sigma} + \frac{uu_{\rho}}{\rho}.$$  

(2)

We previously studied Eq. (2) under the following types of boundary conditions:

$$u|_{\rho=a(t)} = 0,$$  

(3)

$$u|_{\rho=R} = f(t),$$  

(4)

$$u|_{\rho=b(t)} = g(t).$$  

(5)

We proved the existence and uniqueness theorems for an analytical solution [10] and presented step-by-step algorithms for solving by the boundary element method (BEM) at a given time interval [12].

In this study, we deal with problem (2), (5). Equation (5) sets a nonzero boundary condition on the moving boundary $\rho = b(t)$. For an approximate solution of problem (2), (5) we propose to use generalized self-similar solutions, which are presented in Sections 2 and 3. Section 4 proposes a numerical algorithm for the approximate finding of the law of moving of the zero front $\rho = a(t)$ under the given condition (5). In Section 5, for a special approximated function $\rho = a(t)$, we construct a generalized self-similar solution, which is taken as an approximate solution of the problem (2), (5).

2. Exact solutions

Exact solutions of nonlinear equations of mathematical physics are of interest both for the development of the theory of differential equations and for verifying numerical methods.

To construct exact solutions of Eq. (2) with a heat wave type, we use the Clarkson-Kruskal Direct Method [15]. According to this method, the solution is sought as

$$u = \lambda(t)w(z), \quad z = a_0(t) + \frac{\rho}{a_1(t)}.$$  

(6)

After substituting of Eq. (6) in Eq. (2) and reduction of similar terms, we obtain the following equation:

$$ww'' + \frac{1}{\sigma}(w')^2 + \left[\frac{w}{z-a_0} + \frac{a_1(a_1'z + a_0'a_1 - a_1'a_0)}{\lambda(t)}\right]w' - \frac{a_1^2(t)\lambda'(t)}{\lambda^2(t)}w = 0.$$  

(7)

It is easy to see that the necessary condition for separation of variables $t$ and $z$ and Eq. (7) becomes an ordinary differential equation (ODE) for the function $a_0 = \text{const}$. Such solutions are sometimes called generalized self-similar solutions [15].

**Proposition 1.** Equation (2) has solutions of the form (6) in the following cases:

1. $a_0 = \text{const}, a_1(t) = C_1e^{C_2t}$. 


2. $a_0 = \text{const}$, $a_1(t) = C_3(t + C_4)^\lambda$.
Moreover, $\lambda(t) = a_1a_1'$, and $w(z)$ satisfy the following ODE:

$$ww'' + \frac{1}{\sigma}(w')^2 + \left(\frac{w}{z - a_0} + z - a_0\right)w' - \eta w = 0,$$

(8)

where $\eta = 2$ for the first case and $\eta = (2\omega - 1)/\omega$ for the second one, $C_i = \text{const}$, $i = 1, \ldots, 4$.

The proof of the proposition is carried out by integrating a system of ordinary differential equations. It is obtained if $a_0 = \text{const}$ (see above), and the factors in front of $w$ and $w'$ is equated to constants in the left-hand side of Eq. (7). The correctness of the proposition can also be verified by direct substitution in Eq. (7) of the corresponding expressions for $\lambda$, $a_0$, $a_1$.

In order for the obtained generalized self-similar solutions to have the type of heat wave, it is necessary to supplement Eq. (8) with Cauchy conditions. Let $a(t) = -a_1(t)$. Then the condition given on the heat wave front (3) takes the form $w(a_0 - 1) = 0$. We require that the heat flow at the heat wave front be non-zero. Let us put $z = a_0 - 1$, $w = 0$ in Eq. (8), then $w'|_{z=a_0-1} = -\sigma$.

Thus, we have for Eq. (8) the following Cauchy conditions:

$$w(a_0 - 1) = 0, \quad w'(a_0 - 1) = -\sigma,$$

(9)

From the above reasoning, the following statement holds:

**Proposition 2.** Problem (2), (3) has generalized self-similar solutions (6), where $a_0$, $a_1$, $\lambda$ are found from the condition of Proposition 1, and $w$ satisfies Cauchy problem (8), (9).

Note that for problem (8), (9) classical theorems of existence and uniqueness of the solution are not satisfied, since for $z = a_0 - 1$ the factor in front of the highest derivative vanishes. Nevertheless, the existence and uniqueness of a solution in the class of analytical functions follows from the theorem previously proved by the authors [10]. Find a solution to problem (8), (9) in the form of an analytical formula is possible only in some cases. In the general case, a continuous solution can be obtained by iteration using the boundary element method. An extensive computational experiment shows a fairly high accuracy of such solutions. The resulting generalized self-similar solutions to problem (2), (3) having form (6) are not exact in the full sense; however, they can certainly be used as a reference for testing numerical algorithms.

3. Approximate constructing of an invariant heat wave

The construction of a generalized self-similar solution (6) is possible only for certain types of functions defining the law of motion of the zero front. However, let us consider the possibility of constructing such a solution in the case when the function $\rho = a(t)$ is close to a power or exponential law corresponding to the conditions of Proposition 1. Suppose that the function $\rho = a(t)$ is determined experimentally at some points in time, and the experimental results are approximated with sufficient accuracy by a function that satisfies the conditions of Proposition 1. It can be assumed that solution (6) to Cauchy problem (8), (9) will be an approximate solution to problem (2), (3).

As a typical example, we consider problem (2), (3) when the zero-front moving is given by a segment of the Taylor series of the function $\rho = e^{at}$:

$$\rho = a_N(t) = \sum_{n=0}^{N} \frac{t^n}{n!},$$

(10)

For the values of this function on the segment $t \in [0, 1]$, at $t_k = kh$, $k = 0, 1, \ldots, K$, $h = 1/K$, the least squares method gives us an approximation function $\rho = C_1e^{C_2t}$. Using this approximation as the law of motion of the zero front, we construct a generalized self-similar solution (6), which will be an approximate solution of problem (2), (3), (10). The accuracy of the solution obtained
is evaluated by comparing it with a step-by-step solution based on the BEM. Figure 1 shows the solutions for $N = 3$, $K = 10$, 20. Similar results for $N = 4$ are shown in Figure 2. In all calculations, hereinafter, for simplicity, it is assumed that $a_0 = 1$. From the figures it can be seen that with an increase in the degree of a segment of a series (10), the time interval increases, where the generalized self-similar solution is close to the BEM one.

Computational results indicate the stability of generalized self-similar solutions of the form (6) with respect to small deviations of the law of motion of the zero front. Consequently, a generalized self-similar approach can be applied to approximate solving boundary value problems for Eq. (2).

**Figure 1.** The comparison of numerical solutions for $N = 3$.

**Figure 2.** The comparison of numerical solutions for $N = 4$.

4. Approximate constructing of the zero front
We now turn to problem (2), (5). To find the unknown law of moving of the zero front $\rho = a(t)$, we change the role of the unknown function $u$ and spatial variable $\rho$. Such a replacement was previously used in the papers of Sidorov [6], as well as in the works of the authors. This change of variables is correct in the case of a monotonic function $f(t)$, then at the current time $t = t_k$ the function $u(t_k, \rho)$ is reversible. For the inverse function $\rho = \rho(t_k, u)$ Eq. (2) and the boundary condition (5) take the form:

$$\rho u^2 = u \frac{\rho_u}{\sigma} - \frac{u \rho_u^2}{\rho},$$

$$\rho|_{u=g(t)} = b(t).$$

According to the approach proposed in [11, 12, 14], we solve problem (11), (12) in time steps. At the time moment $t_k = kh$, where $h$ is the step size, we consider the boundary value problem in the domain $u \in [L_0, L_k]$, $L_0 = g(0)$, $L_k = g(t_k)$, for the Poisson equation

$$\rho uu = \frac{1}{u} \left( \rho_t \rho_u^2 + \frac{\rho_u}{\sigma} \right) + \frac{\rho_u^2}{\rho},$$

$$\rho|_{u=L_k} = g(t_k).$$

The unknown zero front for the original problem corresponds to the values of the unknown function $\rho(t_k, u)$ with $u = L_0$:

$$\rho|_{u=L_0} = a(t_k).$$

Condition (15) arrive us at

$$\rho(t)|_{u=0} = \frac{\partial \rho}{\partial n}|_{u=0} = \frac{1}{\sigma a'(t_k)}.$$
where \( q(\rho) \) is the flow for the function \( \rho(t, u) \).

An iterative algorithm for solving the problem (13), (14), (16) by the boundary element method allows us to find at each step a continuous function \( \rho = \rho(t_k, u) \), which is inverse to the solution of the original problem. At the same time, we found the zero front \( \rho = a(t_k) \) at each moment \( t = t_k \) on a given time interval. For further research, the table function obtained in this way can be approximated by a continuous function \( \rho = a(t) \).

5. An invariant heat wave construction in the case of a nonzero boundary regime defined on a moving manifold

Based on the results of the computational experiment described in Section 3, we now construct an approximate solution of problem (2), (5), as a generalized self-similar solution (6). Here we use the approximation of the law of zero front moving \( \rho = a(t) \), built in steps in time in Section 4.

Let us illustrate the proposed approach with examples. As a first example, consider problem (2), (5) for

\[
\begin{align*}
    b(t) &= 0.5(1 + e^t), \\
    g(t) &= U(t, \rho)|_{\rho=0.5(1+e^t)},
\end{align*}
\]

where \( U(t, \rho) \) is generalized self-similar solution (6) for \( a(t) = e^t \), found as a solution of Cauchy problem (8), (9) by BEM. Stepwise solving of problem (13), (14), (16), (17) on the interval \( t \in [0, 1] \) allows us to determine the position of the zero front at time points \( t_k = kh \), \( k = 0, 1, \ldots, K \), \( h \) is time step, \( h = 1/K \). The obtained data are approximated by the function \( \rho = C_1e^{C_2t} \) using the least squares method. As an approximate solution of problem (2), (5), (17) we use a solution of the form (6) for \( a(t) = C_1e^{C_2t} \) for the found values of the parameters of the approximating function. Table 1 shows the values of the parameters \( C_1, C_2 \) and standard deviation \( \epsilon \) for different numbers of steps at the first stage of the solving. Figure 3 illustrates comparison of approximate solutions with the reference solution \( U(t, \rho) \). With an increase in the number of steps, the approximate solution tends to the reference one.

| \( K \) | \( C_1 \) | \( C_2 \) | \( \epsilon \) | \( C_3 \) | \( C_4 \) | \( \omega \) | \( \epsilon \) |
|---|---|---|---|---|---|---|---|
| 10 | 1.0004227 | 1.005316 | 0.003468 | 1.019098 | 0.9796 | 0.9871 | 0.002984 |
| 20 | 1.0003852 | 1.002483 | 0.001267 | 1.006263 | 0.9934 | 0.9958 | 0.001501 |

As a second example, we consider problem (2), (5) for

\[
\begin{align*}
    b(t) &= 1 + 0.5t, \\
    g(t) &= V(t, \rho)|_{\rho=1+0.5t},
\end{align*}
\]

where \( V(t, \rho) \) is generalized self-similar solution (6) for the power function \( a(t) = 1 + t \). The values of \( \rho = a(t_k) \) obtained by the step-by-step solution are approximated by the power function \( \rho = C_3(t + C_4)\omega \). As an approximate solution of problem (2), (5), (18) we use a solution of the form (6) for the found parameters of the approximating function.

Table 1 shows the values of the parameters \( C_3, C_4, \omega \) and standard deviation \( \epsilon \) for different numbers of steps. Figure 4 illustrates comparison of approximate solutions with the reference solution \( V(t, \rho) \).

The calculations show that the generalized self-similar approach is applicable for constructing a heat wave generated by a non-zero boundary regime defined on a moving manifold. The proposed algorithm can be used if the zero-front motion found by BEM is approximated with sufficient accuracy by a power or exponential function.
6. Conclusion

The study shows that the generalized self-similar solutions of the considered equation, having the type of a heat wave, are stable with respect to small perturbations of the law of a heat wave moving. In the case of an arbitrary non-zero boundary condition defined on a moving manifold, this property allows one to construct approximations of heat wave fronts based on generalized self-similar solutions. The performed computational experiment shows a fairly high accuracy of such solutions.

A thought-provoking direction for further research is to obtain qualitative and quantitative estimates of the stability domains of generalized self-similar solutions. These assessments allow us to determine the scope of applicability of the proposed approach.

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