1/4 is the new 1/2: Interaction-induced Quantum Anomalous and Spin Hall Mott Insulators

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(Dated: October 2022)

We introduce interactions into two general models for quantum spin Hall physics. Although the traditional picture is that such physics appears when the two lower spinful bands are occupied, that is, half-filling, we show that in the presence of strong interactions, the quarter-filled state instead exhibits the quantum spin Hall effect with spin Chern number $C_s = 1$. A topological Mott insulator is the underlying cause lying outside the standard $\mathbb{Z}_2$ topological classification. An intermediate interacting regime that exhibits a quantum anomalous Hall effect emerges when the lower band is ‘flat’ so that the interaction is large compared to the bandwidth but remains comparable to or smaller than the topological gap. This state transitions to the quantum spin Hall effect once the interactions are sufficiently large. We show that this intermediate regime is consistent with the simultaneous observation in transition metal dichalcogenide moiré materials of a quantum anomalous Hall phase at quarter filling and a quantum spin Hall effect at half-filling. The valley coherence seen experimentally can be captured in a bi-layer extension of our results.

INTRODUCTION

Although topological insulators[1–18] represent a new class of bulk insulating materials with gapless conducting edges, their physics is completely entailed by the band theory of non-interacting electrons. The new twist is that should two atoms reside in each unit cell, the standard insulating gap that obtains at half-filling, full lower band, does not tell the whole story when spin-orbit coupling[1–3] is present. As long as time-reversal invariance is maintained, two spinful counter-propagating edge modes exist and exhibit a quantized conductance proportional to $e^2/h$, thereby giving rise to a non-zero quantum spin-Hall (QSH) effect in two dimensions. Within the Kane-Mele (KM)[1, 2] and Bernevig-Hughes-Zhang (BHZ)[3] models, the QSH effect obtains only at half-filling. This physics is robust to perturbations that yield only a smooth deformation[18] of the Hamiltonian. Additionally, the quantum anomalous Hall (QAH) effect, that is, the existence of a quantized Hall conductance with zero net magnetic field, also requires half-filling of the Haldane model[19]. As the QAH effect breaks time-reversal symmetry while the QSH effect does not, it is difficult for them to be realized in the same material.

However, recently, both of these effects[20, 21] have been observed in the same material in direct contrast to predictions of the standard non-interacting models[1–3]. In the AB-moiré-stacked transition metal dichalcogenide (TMD) bi-layer MoTe$_2$/WSe$_2$[20, 21], the QSH insulator is observed at $\nu = 2$ with the QAH effect residing at $\nu = 1$. In terms of the 4-band KM/BHZ model, $\nu = 2$ and $\nu = 1$ correspond to half-filling and quarter-filling, respectively. Numerous theories[22–33] have been put forth in this context, and the most recent experiment[34] shows that both valleys contribute to the QAH effect and hence valley coherence rather than valley polarization is the operative mechanism. The striking deviation from the standard theory raises the question: can interactions drive either of these transitions away from half- to quarter-filling in the KM/BHZ model?

It is this question that we address here. We show quite generally that interactions shift the QSH effect to quarter filling with a decrease of the spin Chern number by a factor of two. Further, we find that in the flat-band limit with an intermediate interaction strength, an anomaly occurs at quarter-filling that is consistent with the QAH effect. Once the interaction exceeds the sum of the bandwidths and the topological gap, the QAH effect reverts to the standard QSH effect. Since this is a generic conclusion on the most general models proven to undergird the QSH effect, we analyze the experiments[21, 34] in this context. Our model yields a quarter-filled QAH effect which coexists with a QSH effect at half-filling as is seen experimentally. However, we note that an extension beyond the 4-band model is required to explain the valley coherence observed for the QAH state in AB stacked materials. The valley coherence can be captured from a straightforward bi-layer extension with 8 bands (two copies of KM).

At the outset, we illustrate why quarter-filling is special in the presence of interactions. With interactions, singly and doubly occupied states are no longer degenerate. As a result, the two spin-filtered states at each edge cannot both contribute to the conductance. Only the one that lies in the purely singly occupied band comes into play. This effectively changes the topologically relevant filling from 1/2 to 1/4. The gap at 1/4 filling will now be entirely due to the interactions, and the doubly occupied band is pushed up to high energy. Because only the singly occupied edge states are relevant, the edge current is reduced by a factor of two relative to that in the non-interacting system. It is this picture that we demonstrate here by two independent methods: an exactly solvable model and corroborating determinantal quantum Monte Carlo (DQMC) calculations with the Hubbard interaction. Both indicate that
the strongly correlated quarter-filled KM and BHZ models are QSH Mott insulators with a helical edge current half the non-interacting value. Although this model lacks any details of the moiré potential, we find great resonance with the experiments because when the lower band is flat, a QAH effect obtains and turns into the QSH effect once the interactions exceed a critical strength. This result is encouraging as it lends credence to the ultimate experimental applicability of this work.

Most studies on the KM-Hubbard[35–47] and the BHZ-Hubbard[48–53] models focused on the half-filled system and found a transition from a QSH insulator to a topologically trivial anti-ferromagnetic Mott insulator as the interaction strength \( U \) increases. If the non-interacting system is a topologically trivial band insulator to begin with, there could be a correlated QSH phase at intermediate \( U \)[48, 53] which enhances the effective spin-orbit coupling. This correlated QSH insulator can be characterized by the \( \mathbb{Z}_2 \) invariant, essentially the same as the non-interacting QSH insulator. Interactions also lie at the heart of fractional topological insulators[4, 11, 54–56] built from fractional Chern insulators[57–61] which resemble the fractional quantum Hall effect but with no net magnetic field. Such phases appear at a fractional filling in a flat-band \( \Delta \gg W \) (where \( \Delta \) is the non-interacting topological gap and \( W \) is the bandwidth) and requires nearest-neighbor interactions. A recent study on the strongly interacting spinful Haldane model[62] demonstrates that a Chern Mott insulator originates at quarter-filling with Chern number \( C = \pm 1 \). This physics arises as a general consequence of an interplay between Mottness and topology.

Motivated by Refs. [21, 34, 62], we explore the general phenomena that emerge from the interplay between Mottness and the QSH effect in the context of the KM and BHZ models. To demonstrate that the quarter-filled state is a topological Mott insulator with a strongly correlated QSH effect, we construct an analytically solvable Hamiltonian for a general interacting QSH system and numerically solve both the BHZ-Hubbard and KM-Hubbard Hamiltonians using DQMC and obtain essentially the same result for sufficiently large interactions. This QSH Mott insulator is beyond[63] the description of the \( \mathbb{Z}_2 \) invariance which necessarily yields \( C_s = 0 \) or 2.

### EXACTLY SOLVABLE MODEL FOR INTERACTING QUANTUM SPIN HALL INSULATORS

The usual way to introduce a QSH insulator[3, 64] is by considering the Hamiltonian,

\[
H_{\text{QSH}} = \sum_{k,\sigma} \Phi^\dagger(k) \begin{pmatrix}
\hbar^\text{QAH}(k) & 0 \\
0 & \hbar^\text{QAH}(-k)
\end{pmatrix} \Phi(k).
\]

Here \( \Phi^\dagger = \{c_{1x,\sigma}, c_{1y,\sigma}, c_{2x,\sigma}, c_{2y,\sigma}\} \) is a two-component spinor, where \( c_{1/2} \) stands for different orbitals or sub-lattices, respectively. Eq. (1) means that the spin-up and -down electrons are described by a quantum anomalous Hall (QAH) Hamiltonian \( \hbar^\text{QAH}(k) = \hbar_0(k) \tau^\sigma \) (\( \tau^\sigma \) is the Pauli matrix for orbital/sublattice space) and its TR conjugate counterpart \( \hbar^\text{QAH}_\ast(-k) \) with opposite chirality. As a result, the system is TR invariant and the half-filled case can be topologically trivial and non-trivial insulator, categorized as a \( \mathbb{Z}_2 \) invariant. In general, this Hamiltonian can be diagonalized into

\[
H_{\text{QSH}} = \sum_{k,\sigma} \left[ (\varepsilon_{+,k,\sigma} - \mu)n_{+,k,\sigma} + (\varepsilon_{-,k,\sigma} - \mu)n_{-,k,\sigma} \right],
\]

where \( \mu \) is the chemical potential and

\[
\varepsilon_{\pm,k,\sigma} = \hbar_0(k) \pm \sqrt{\hbar^2_{\pm,\sigma}(k) + \hbar^2_{\pm,\sigma}(k) + \hbar^2_{\pm,\sigma}(k)}
\]

represents the upper (+) and lower (−) bands for each spin. For concreteness, we consider the BHZ model[3]:

\[
\hbar_0(k) = 0, \quad \hbar_{x,\sigma}(k) = \sigma t \sin(k_x), \quad \hbar_{y,\sigma}(k) = t \sin(k_y), \quad \hbar_{z,\sigma}(k) = M + t \cos(k_x) + t \cos(k_y),
\]

where \( t = 1 \) sets the energy scale. The spin-up and -down electrons have the same dispersion but different wave functions with opposite chirality. For \( 2 > M > 0 \) (or \( -2 < M < 0 \)), the half-filled system is a QSH insulator[3] with Chern number \( C_0 = C_{+0} + C_{-0} = 0 \) and spin Chern number \( C_{s0} = C_{+0} - C_{-0} = 2 \) (or \( -2 \)) related to the spin Hall conductance[3, 10], where the subscript 0 means no interaction. \( |M| > 2 \) corresponds to a topologically trivial band insulator with Chern numbers \( C_{s0} = C_{0} = 0 \).

To flesh out the physics outlined in the introduction regarding the quarter-filled state, we now introduce the Hatsugai-Kohmoto (HK) momentum-space interaction[65–67] into the Hamiltonian,

\[
H = \sum_{k,\sigma} \left[ (\varepsilon_{+,k,\sigma} - \mu)n_{+,k,\sigma} + (\varepsilon_{-,k,\sigma} - \mu)n_{-,k,\sigma} \right]
\]

\[
+ U \sum_k \left( n_{+,k,\uparrow} n_{+,k,\downarrow} + n_{-,k,\downarrow} n_{-,k,\uparrow} \right).
\]

This interaction introduces Mottness by tethering double occupancy to \( k \)-space rather than the usual real space as in the well known Hubbard model. As we will show, both yield the same physics for strong interactions. The reason for this consilience[67] as that both models break the underlying \( \mathbb{Z}_2 \) (distinct from the classification scheme for topological insulators) symmetry of the non-interacting Fermi surface[68]. As the interaction commutes with the kinetic term, the original non-interacting wave function is untouched and momentum \( k \) remains a good quantum number. Therefore, the interacting Green function can be written down analytically[62, 66] as

\[
G_{\pm,k,\sigma}(i\omega_n) = \frac{1 - (n_{\pm,k,\sigma})}{\omega + \mu - (\varepsilon_{\pm,k,\sigma} + U)}
\]

The Green function immediately reveals the effect of the correlations. The non-interacting lower and upper bands which
were degenerate for spin-up and -down electrons split into singly and doubly occupied sub-bands as a result of Mottness. In the following, we use the abbreviation LSB and LDB for lower singly and doubly occupied sub-bands respectively, and likewise USB and UDB for the upper bands. The energy of the LSB and USB remains at the non-interacting value, while the LDB and UDB move up by a value equal to $U$. For a large enough $U$, the quarter-filled system emerges as an insulator with the LSB filled up. To rigorously calculate the corresponding spin Chern number, we cannot use the simple formula $C_s = C_{\uparrow} - C_{\downarrow}$ because the interaction mixes the spin channels, leading to a huge degeneracy ($2^{N_s}$) in the ground state ($N_s$ is the number of unit cells). Instead, we can calculate the average $C_s$ for all microscopic states,

$$C_s = \frac{1}{2N_s} \sum_\Omega \left( C_\uparrow(\Omega) - C_\downarrow(\Omega) \right) = \bar{C}_\uparrow - \bar{C}_\downarrow,$$

(7)

Here $\bar{C}_\uparrow(\Omega) = (1/2^{N_s}) \sum_{\Omega'} C_{\uparrow}(\Omega')$. Since the HK interaction does not blend the non-interacting wave-functions, we can use the non-interacting result ($C_{\uparrow0}$, $C_{\downarrow0}$ and $C_{s0}$) without having to calculate the Berry curvature explicitly. We start from the microscopic states with $N_\uparrow = 1$. The summation over their spin-up contribution picks out only those momentum states with up spin: $\sum_{\Omega (N_\uparrow = 1)} C_\uparrow(\Omega) = C_{\uparrow0}$. Similarly, $\sum_{\Omega (N_\uparrow = 2)} C_\uparrow(\Omega) = \left( N_c - 1 \right) C_{\uparrow0} = \left( N_c - 1 \right) C_{\uparrow0}$. Then in general, $\sum_{\Omega (N_\uparrow = n)} C_\uparrow(\Omega) = \left( N_c - 1 \right)^n C_{\uparrow0}$ and hence,

$$C_\uparrow = \frac{1}{2N_c} \sum_{N_\uparrow = 1}^{N_c} \sum_{\Omega (N_\uparrow)} C_\uparrow(\Omega) = \frac{\sum_{N_\uparrow = 1}^{N_c} (N_c - 1)^{N_\uparrow - 1}}{2N_c} C_{\uparrow0} = \frac{C_{\uparrow0}}{2}. $$

Likewise $\bar{C}_\downarrow = C_{\downarrow0}/2$ so that $C_s = C_{s0}/2$. This can be understood from the perspective that each momentum state is singly occupied by either spin-up or -down electron, or equivalently, half spin-up and half spin-down electrons on average. Similarly, the LDB has the same $C_s$, while the USB and UDB have the opposite $C_s$. In short, the quarter-filled system is a QSH Mott insulator with a spin Chern number $C_s = C_{s0}/2$ should the interaction exceed the bandwidth. Hence, our work provides a definitive platform for realizing odd spin Chern numbers which are explicitly excluded in non-interacting spinful systems[63].

To visualize how this phase emerges, we plot the band structure in Fig. 1 for varying $U$. With $M = 1$, the bandwidth for the lower and upper BHZ bands is $W_{\downarrow} = 2$ and $\Delta = 2$ is the topological gap. The non-interacting lower band has $C_{s0} = 2$, while the upper band carries the opposite spin Chern number. We separate the non-interacting lower and upper bands into LSB (red-unmeshed), LDB (red-meshed), USB (green-unmeshed) and UDB (green-meshed). As derived above, the red and green sub-bands have the spin Chern number $C_s = 1$ and $-1$ respectively. Turning on the interaction causes the doubly occupied sub-bands to increase in energy while the singly occupied sub-bands remain unchanged. For small interactions $W_- > U > 0$ (Fig. 1a), the band structure only slightly departs from the non-interacting case. As $U$ increases to $W_- + W_+ + \Delta \geq U > W_-$ (Fig. 1b), the same-color sub-bands fully separate, leading to a gap opening at quarter-filling. Then both the 1/4- and 3/4-filled systems become a QSH Mott insulator with a spin Chern number $C_s = 1$, while the 1/2-filled case becomes a conductor. Upon further increasing $U$ to $U > W_- + W_+ + \Delta$ (Fig. 1c), the 1/2-filled state becomes a topologically trivial Mott insulator. All the while, the QSH Mott insulator at 1/4- and 3/4-fillings persists with a gap equal to $\Delta$. For a different $M$, the intermediate panel b may change, while panel c is always valid for a large enough $U$. This indicates that generally in the presence of strong interactions, the system becomes a QSH Mott insulator at 1/4- and 3/4-filling with spin Chern number $C_s = C_{s0}/2$ and a topologically trivial Mott insulator at 1/2-filling.

We distinguish the physics of the quarter-filled QSH Mott insulator from the non-interacting half-filled QSH insulator through the standard schematic picture[11, 18] shown in Fig. 2. In the non-interacting case, the QSH effect appears when the chemical potential crosses the Kramers edge pair an odd number of times (Fig. 2a)[16]. In the QSH Mott insulator, the odd number of crossing still holds, but the Kramers pairs are singly occupied at the same momentum, carrying only half the weight of their non-interacting counterpart, represented as the dashed line in Fig. 2b. The non-interacting QSH insulator have decoupled helical edge currents (Fig. 2c)[11]. However, the spin-momentum-locking edge currents for QSH Mott insulator are highly degenerate due to the singly occupied condition for each momentum, depicted symbolically by the blue and red dashed lines on opposite edges along the same direction connected by a black dashed curve in Fig. 2d.

To summarize, we have been able to characterize the transition between a metal and a QSH Mott insulator at quarter-filling based on an analytically solvable Hamiltonian Eq. (5). This QSH Mott insulator has a spin Chern number $C_s = \pm 1$ that is beyond the description based on the standard $Z_2$ topological invariant. We will now extend this argument to the
FIG. 2: The comparison between non-interacting QSH insulator (a,c) and QSH Mott insulator (b,d) in the electronic dispersion between two boundary Kramers degenerate points $\Gamma_a$ and $\Gamma_b$, and edge current, respectively. For the non-interacting QSH insulator, the solid lines represent the states with degeneracy between singly and doubly occupied states\[11, 18\]. For the QSH Mott insulator, the dashed lines represent only singly occupied states. The correlation between spin-up and -down electrons are shown through the black dashed curves.

Hubbard model and show that the same conclusion arises. Consequently, the 1/4-filled QSH Mott insulator emerges as a general feature of strongly correlated topological matter.

**HUBBARD INTERACTION**

Now it is natural to ask whether the essence and universality of the physics from Hamiltonian Eq. (5) remains if we replace the HK-type interaction with the more well known Hubbard interaction. In the presence an external magnetic field, The Hamiltonian for the BHZ-Hofstadter-Hubbard (BHZ-HH) model is

$$H = t \sum_{i,\sigma} [\exp(i\phi_{i,i+\hat{x}})c_{i,\sigma}^\dagger c_{i+\hat{x},\sigma} + \exp(i\phi_{i+\hat{y}i})c_{i,\sigma}^\dagger c_{i+\hat{y},\sigma} + \text{h.c.}] - \mu \sum_{i,\sigma} n_{i,\sigma} (9)$$

where $t = 1$ (energy scale) and $\tau_{\alpha}$ is a Pauli matrix in the orbital basis. We again set $M = 1$. The phase factor $\exp(i\phi_{i,j})$ which arises from the standard Peierls substitution contains the effect of the external magnetic field, which is introduced to measure the magnetic response of the incompressible states at high temperature. Here $\phi_{i,j} = \frac{2\pi}{\Phi_0} \int_{\gamma_i} A \cdot dl$, where $\phi_0 = e/h$ is the magnetic flux quantum, the vector potential $A = (x\hat{y} - y\hat{x})B/2$ (symmetric gauge), and the integration is along a straight-line path. It is the spin Chern number that describes a QSH insulator. To measure this quantity, we use a spin-dependent (SD) magnetic field inspired by cold-atom experiments\[69, 70\], namely $\phi_{i,j} \to \sigma \phi_{i,j}$. The corresponding Hamiltonian is referred as BHZ-HH-SD model.

We use the DQMC method\[71–73\] to simulate these models on an $N_{\text{site}} = 6 \times 6$ cluster (two orbitals per site) with modified periodic boundary conditions\[74\]. A single-valued wave function requires the flux quantization condition $\Phi/\Phi_0 = n_f/N_c$ (with $n_f$ an integer). We focus on the inverse temperature $\beta = 3/4$ restricted by the sign problem (see supplement).

The simulation results for the BHZ-HH-SD and BHZ-HH models are presented in Fig. 3 with $U = 0$ and $U = 12$. It is instructive to first identify the signature of a QSH insulator in the presence of an external magnetic field without the interaction (first row). We first calculate the compressibility,

$$\chi = \beta \chi_c = \frac{\beta}{N} \sum_{i,j} [(n_i n_j) - \langle n_i \rangle \langle n_j \rangle],$$

where the orbital and spin summations are implied in $n_i$. For any density, the inverse slope of the leading straight-line incompressible valley that extends to the zero-field limit\[62\] provides the Chern number. In our choice of parameters ($M = 1$), the non-interacting 1/2-filled system is a QSH insulator with $C_s = 2$. In the BHZ-HH-SD model, the minus sign in the magnetic field changes the chirality of the spin-down edge currents, namely $C_s^\text{SD} = -C_s$. Thus, the “SD” Chern number measured in the BHZ-HH-SD model $C_s^\text{SD} = C_s^\text{SD} + C_{\perp}^\text{SD} = C_\perp - C_\parallel$ is equal to the spin Chern number in the corresponding BHZ-HH model. In Fig. 3a, the “SD” Chern number $C_s^\text{SD} = 2$ obtains as the inverse slope of the leading green straight line indicating the spin Chern number $C_s = 2$ in the corresponding BHZ-HH model. This method overcomes the drawback of the simple additivity formula for $C_s$ when the spin channels are mixed by the interaction and works well even when $k$ is no longer a good quantum number in an interacting system. With a strong electron-electron interaction ($U = 12$ in Fig. 3e), two (red) straight lines appear from the zero-field 1/4— and 3/4—filled BHZ-HH-SD system, whose inverse slope gives an “SD” Chern number $C_s^\text{SD} = 1$. This indicates that the corresponding zero-field 1/4— and 3/4—filled BHZ-HH systems emerge as a QSH insulator with $C_s = 1$ while the 1/2—filled system becomes a topologically trivial Mott insulator with $C_s = 0$.

Next we focus on the BHZ-HH model which is experimentally more accessible. Again, we first compute the compressibility. In the non-interacting case (Fig. 3b), there is a short middle vertical straight line at low fields which indicates a Chern number $C = 0$. This state bifurcates into two lines or equivalently two Landau levels (LLs) at higher magnetic flux. This pair of zero-mode LLs is a reliable fingerprint for a QSH insulator observed in experiments\[12\]. In the presence of strong correlation (Fig. 3f), similar pairs of zero-mode Landau levels with sharper slope appear at 1/4— and 3/4—filling.
FIG. 3: DQMC results for the BHZ-HH-SD and BHZ-HH models at $U = 0$ (first row) and $U/t = 12$ (second row). The first column shows the compressibility $\chi$ as a function of magnetic flux and electron density for the BHZ-HH-SD model. The second to fourth columns (in the first two rows) show $\chi$, the spin susceptibility $\chi_s$ and the magnetization $\langle m_z \rangle$ respectively, for the BHZ-HH model. The temperature is $\beta = 3/t$.

Consequently, we find a QSH Mott insulator at quarter-filling with $C_s = C_{so}/2$ under strong Hubbard interaction, in agreement with the conclusion from Eq. (5) with HK-type interactions. We also show that the assignment of the Chern number for the Hubbard model is independent of system size (see supplement). This is a non-trivial result as the interactions in the Hubbard model mix momenta whereas they do not in the HK model. The fact that a ground state property (namely the Chern number) agrees across both platforms tells us that the HK and Hubbard models are in complete agreement as to the ground state of the system. This can be understood by the fact that the HK and Hubbard interaction are governed by the same fixed point[67]. The interaction-induced 1/4-filled QSH state is also present in the KM-Hofstadter-Hubbard (KM-HH) model on a honeycomb lattice. Our DQMC study on the KM-HH model (see supplement) produces essentially the same results as Fig. 3 the BHZ-HH model despite slight difference in $m_z$ (see supplement) due to different lattice geometry. Therefore, we expect these behaviors to be universal for a general QSH system under strong Hubbard interaction.

signaling the QSH effect. The vertical line extends over the whole range of magnetic flux at $\langle n \rangle = 2$ corresponds to a topologically trivial Mott insulator. We also calculate the spin susceptibility defined as

$$\chi_s = \sum_r S(r) - N m_z^2 = \frac{1}{N} \sum_{i,r} [(S_i^z S_{i+r}^z) - \langle S_i^z \rangle \langle S_{i+r}^z \rangle]$$

where $m_z = \sum \langle S_i^z \rangle / N$ is the magnetization per spin. The non-interacting spin susceptibility is related to the compressibility by $\chi_s = \chi/(4\beta)$ as shown in Fig. 3(b,c). For $U = 12$ in Fig. 3(f,g), nevertheless, the peaks in the spin susceptibility correspond to the dips in the compressibility that arise from zero-mode LLs at quarter filling. It is also interesting to compute the magnetization. A maximized moment at $\langle n \rangle = 2$, indicated by the dark feature in panel Fig. 3 (d), is the key feature of the QSH effect for the non-interacting system. In the interacting system, the moment is now maximized (dark feature with different colors in Fig. 3 (h)) at the LLs (Fig. 3 (f)) emanating from $\langle n \rangle = 1$ and $\langle n \rangle = 3$ corresponding to 1/4- and 3/4-filling, respectively. This is consistent with the assignment of the QSH effect from the compressibility and spin susceptibility. At zero field, although the magnetic moment vanishes, the spin correlation (Fig. 3(g)) peaks at $\langle n \rangle = 1, 2, 3$ at $U = 12t$. From the real-space spin pattern (see supplement), we find ferromagnetic correlations for $\langle n \rangle = 1, 3$ and antiferromagnetic correlation for $\langle n \rangle = 2$ at a lower temperature. To corroborate our findings, we conducted a finite-size analysis (see supplement) and confirm that the same QSH Mott insulator survives in system sizes as large as $N_{site} = 12 \times 12$ with little finite-size effect and hence our results are valid in the thermodynamic limit.
EXPERIMENTAL REALIZATION

While the interactions in ultracold atoms in optical traps\cite{75} can be adjusted to mimic the physics here, the most obvious synergy is with the moiré TMD experiments\cite{20, 21, 34} discussed previously. Since the experiments are in the flat-band limit, we artificially set the bandwidth of the lower band to nearly zero and repeated our DQMC simulation with Hubbard interaction. The results are striking. In Fig. 4(a) ($U = 2t \approx \Delta \gg W_-$), we show that $\langle n \rangle > 1.2$ is essentially non-interacting and the standard Landau fan for the QSH effect (with $C_s = 2$ as shown in Fig. S6(e) in supplement) remains from $\langle n \rangle > 2$ at zero field. What is new here at $\langle n \rangle \lesssim 1$ is that non-trivial interacting physics obtains (see Fig. 4(a)), in particular a QAH state with unit Chern number whose sign is determined by that of the magnetic field. Panel Fig. 4(c) for the compressibility illustrates that when $U = 12t$, which exceeds $W_- + W_+ + \Delta$, the non-interacting QSH Landau fan vanishes for $1 < \langle n \rangle < 3$ and most strikingly, a new Landau level emerges corresponding to the mirror image of the QAH state that terminates at $\langle n \rangle = 1$. The presence of both Landau components completes the QSH Mott state at quarter filling. The magnetization (Fig. 4(d)) shows a more dramatic change than does the compressibility, namely it vanishes at $\langle n \rangle = 2$ as a result of the topological trivial Mott insulator. Further, the magnetization splits into peaks on either side of $\langle n \rangle = 1$. This physics in Fig. 4(b) is only present in the flat-band limit when $U$ is much larger than the bandwidth but comparable to the topological gap. Consequently, our theoretical work here is consistent with the sudden onset of the QAH state.

However, we cannot make direct contact with the observation of valley coherence\cite{34} within a single-layer KM model in which spin-valley locking obtains. Note relaxing the spin-valley locking constraint of the KM model by reversing the spins in one of the bands relative to the other, as indicated in the experiment\cite{34} (see supplement), would lead to a contradiction with a non-zero Chern number per spin in the band insulator limit. That is, the moiré band structure of AB stacked MoTe$_2$/WSe$_2$ bilayer can not be captured by a strict four-band model such as the KM model. The remedy is to construct an eight-band model (details in supplement) consisting of two copies of the KM model, one for each layer with an effective voltage difference between the layers. For completeness, we recomputed the compressibility for the bilayer flat-band KM-HH model. Clearly shown in Fig. 4(e) is the QAH at $\langle n \rangle = 1$, the QSH at $\langle n \rangle = 2$ and $\langle n \rangle = 4$. The emergent possibly QAH state at $\langle n \rangle = 3$ deserves further study (discussion in supplement). The accompanying magnetization in Fig. 4(f) is also consistent with these assignments. Within the eight-band model, spin polarization requires layer coherence because the interaction does not commute with the interlayer hopping and since the same spin is assigned to different valleys in each layer, layer coherence necessarily entails valley coherence. Hence, a simple two-layer extension of our results is sufficient to account for the QAH effect in TMD moiré systems. This reasoning motivates first principles calculations to determine how the 8-band model should be tailored to apply to specific moiré materials.

FIG. 4: DQMC results for the flat-band generalized KM-HH model at $U = 2t, \beta = 8/t$ (first row) and $U = 12t, \beta = 3/t$ (second row), and bilayer KM-HH model at $U = 1.5t, \beta = 12/t$ with an interlayer hopping $t_\perp = 0.3t$ and voltage difference between the two layers of $V = 0.4t$. The first and second columns show the compressibility and magnetization, respectively, as a function of magnetic flux and electron density.

DISCUSSION

Interactions play a non-trivial role in topology in two distinct ways. First, they lead to QSH Mott insulator with a spin Chern number of $C_s = 1$ at quarter filling in the BHZ and KM models. The generality of this behavior is supported by an exact solution of a general interacting QSH Hamiltonian with HK interactions and DQMC simulations of two QSH prototypical models with Hubbard interactions on square and honeycomb lattices. This agreement speaks to Mott physics being described by a k-space quartic fixed point exemplified by the HK model as pointed out earlier\cite{67}. We point out
that while the 1/4-filled state is a topological Mott insulator, it falls outside[63] the $\mathbb{Z}_2$ topological insulator classification. In the $\mathbb{Z}_2$ classification, the $\mathbb{Z}_2$ topological invariant $\nu_{\mathbb{Z}_2}$ is related to whether or not an odd or even number $N_K$ of Kramers pairs of edge modes intersects the Fermi energy; that is, $N_K = \nu_{\mathbb{Z}_2} \mod 2$. In a strongly correlated system, however, each of the QSH lower and upper bands separate into singly and doubly occupied sub-bands. When the chemical potential falls between the singly and doubly occupied sub-bands, the Kramers pair of edge modes are also singly occupied and highly degenerate with 50% spin-up and 50% spin-down electrons on average. Therefore, each pair carries only half the weight of its non-interacting counterpart. This gives rise to a generalized $\mathbb{Z}_2$ invariant being $\nu_{\mathbb{Z}_2} = 1/2$. This result is also supported by the relation $\nu_{\mathbb{Z}_2} = C_s/2$ when the system conserves perpendicular spin $S_z$. The QSH Mott insulator is also qualitatively distinct from the fractional topological insulator[4, 11, 54–56] driven by at least nearest-neighbor interactions. The fractional topological insulator usually consists of two decoupled fractional Chern insulators with opposite spins. However, in QSH Mott insulator, spin-up and spin-down electrons are correlated to form the inseparable singly occupied states. Second, we showed that in the flat-band limit, a QAH state exists also at 1/4-filling. In the double-layer extension of this model, the QAH state exhibits valley coherence as is seen experimentally in moiré TMD materials[34].

Acknowledgements We thank Taylor L. Hughes, Edwin W. Huang, Kin Fai Mak and Kam Tuen Law for useful discussions. We also thank P. Armitage for help with the pithy title. This work was supported by the Center for Quantum Sensing and Quantum Materials, a DOE Energy Frontier Research Center, grant DE-SC0021238 (P. M. B. E. F., and P.W.P.). PWP also acknowledges NSF DMR-2111379 for partial funding of the HK work which led to these results. The DQMC calculation of this work used the Extreme Science and Engineering Discovery Environment (XSEDE) expanse supercomputer through the research allocation TG-PHY220042, which is supported by National Science Foundation grant number ACI-1548562[76].

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The Quantum Spin Hall Mott Insulator at Quarter-Filling

In this section, we take a close look at the zero-mode Landau levels as a signature of the quantum spin Hall (QSH) effect in the Bernevig-Hughes-Zhang-Hofstadter-Hubbard (BHZ-HH) model. In Fig. S1, we replot Fig. 3(b) and (f) from the main text for the compressibility as a function of chemical potential instead of density for $U = 0$ around half-filling (Fig. S1(b)) and $U = 12$ around quarter-filling (Fig. S1(d)). In Fig. S1(a) and (c), the black lines show the zero-mode Landau levels at high field, while the red dashed lines show their extrapolation to zero field. In the non-interacting case (Fig. S1(a)), the QSH effect with spin Chern number $C_s = 2$ at zero field and half-filling preserves time-reversal symmetry, hence giving rise to a short vertical dip. Time-reversal symmetry is broken upon turning on an external magnetic field. At a critical magnetic flux, the vertical dip bifurcates into a V-shaped region corresponding to the zero-mode Landau levels, with inverse slope $= \pm 2$. The V-shape with opposite Chern numbers arises from particle-hole symmetry. As the magnetic field decreases, the left and right Landau-level components do not cross at high field representing gaps at distinct ranges of chemical potential shown in Fig. S1(b). As we reduce the magnetic field, they tend to merge at a finite small field and approach the non-interacting zero-field limit. A similar situation applies to quarter-filling at $U = 12$ shown in Fig. S1(d). As long as both left and right Landau components are present, their finite gaps merge at finite critical magnetic flux $\phi_c$ and the merged gap stays at quarter-filling as magnetic flux further decreases to zero. This confirms the vertical dips in Fig. S1(c) at quarter-filling and $|\phi| \leq \phi_c$, and hence the QSH effect under strong correlation.

**FIG. S1:** Panels (a) and (c) are the copy of Fig.3(b) and (f) in the main text respectively. Panel (b) shows the non-interacting compressibility around half-filling as a function of magnetic flux and chemical potential. Panel (d) shows the compressibility at $U = 12$ around quarter-filling as a function of magnetic flux and chemical potential.
FINITE-SIZE ANALYSIS ON THE BHZ-HH MODEL

To corroborate our finding of a QSH Mott insulator at quarter-filling, we conducted a finite-size analysis on the compressibility of the BHZ-HH-SD (Fig. S2(a)) and the BHZ-HH models (Fig. S2(b)) at low hole and electron density respectively enough to cover the feature from 3/4- and 1/4-fillings with $U = 12$ as allowed by the sign problem. In Fig. S2(c), we present the BHZ-HH-SD compressibility at different magnetic fluxes $\phi/\phi_0 = 4/36, 8/36, 16/36$ and system sizes $N_{\text{site}} = L^2$ with $L = 6, 9, 12$. The collapse of the curves regardless of system size (panels (a) and (b)) suggests that the dip feature at 1/4- and 3/4-filling is fundamental and survives the thermodynamic limit. As a consequence, it makes sense to extract a spin Chern number. We find that for all the system sizes studied, $C_s = 1$ as depicted in Fig. S2(c).

![Finite-size analysis (U = 12)](image)

FIG. S2: Panel (a) and (b) shows the compressibility at low hole and electron density for BHZ-HH-SD and BHZ-HH models respectively, at varying cluster sizes $N_{\text{site}} = L \times L$ under different magnetic fluxes as labeled. They share the same legend. Panel (k) presents the spin Chern number extracted for different cluster sizes. The temperature is $\beta = 3/t$. 
LOWER TEMPERATURE RESULTS FOR BHZ-HH-SD AND BHZ-HH MODEL

To explore lower-temperature physics, we are restricted to low density and a weaker interaction $U = 8$. In Fig. S3, we show the compressibility at $\phi/\phi_0 = 16/36$ for BHZ-HH-SD and BHZ-HH models, though at low hole and electron density respectively, enough to capture the interested feature from quarter-fillings. In either case, the dips (representing the $C_{SD} = 1$ valley for BHZ-HH-SD model in Fig. S3(a) and the zero-mode Landau levels for BHZ-HH model in panel Fig. S3(b)) become more evident as the temperature decreases, leading to an incompressible state at lower temperature.

![Graphs showing compressibility vs hole density for different inverse temperatures.](image)

**FIG. S3:** The compressibility at low hole and electron density for BHZ-HH-SD and BHZ-HH models respectively, under a magnetic flux $\phi/\phi_0 = 16/36$ with different inverse temperatures $\beta = 3, 4, 5$. The interaction strength is $U = 8$. 
DETERMINANTAL QUANTUM MONTE CARLO STUDY ON THE KANE-MELE MODEL WITH HUBBARD INTERACTION

We use determinantal quantum Monte Carlo (DQMC) method KM-Hofstadter-Hubbard (KM-HH) model on a honeycomb lattice. The underlying Hamiltonian here is

\[
H = -\sum_{i,j} t_{ij} \exp(i\phi_{ij}) c^\dagger_{i\sigma} c_{j\sigma} - \mu \sum_{i,\sigma} n_{i\sigma} + U \sum_{i} (n_{i\uparrow} - \frac{1}{2})(n_{i\downarrow} - \frac{1}{2}),
\]

(S1)

where \( t_{ij} \) contains the nearest neighbor hopping \( t = 1 \) as the energy scale) and next nearest neighbor hopping \( \sigma t' e^{\pm i\psi} \) as the spin-orbit coupling with \( \pm i\psi \) following the convention in the Haldane model[1]. If we set \( \psi = \pi/2 \), the hopping term reduces to the original KM model[2, 3]. Without loss of generality, we set \( t'/t = 0.3 \). Then the non-interacting half-filled system is a QSH insulator with \( C_{\text{SO}} = 2 \). With a spin-dependent (SD) magnetic field \( (\phi_{ij} \rightarrow \sigma \phi_{ij}) \), the corresponding Hamiltonian is referred as the KM-HH-SD model. We carry out DQMC simulations for both KM-HH-SD and KM-HH models in a \( N_{\text{site}} = 6 \times 6 \times 2 \) cluster. The factor of 2 is for two sub-lattices. The periodic boundary conditions are then tailored for the honeycomb lattice[4].

![FIG. S4: DQMC results for the KM-HH-SD and KM-HH models at \( U/t = 0 \) (first row) and \( U/t = 12 \) (second row). The first column shows the compressibility \( \chi \) as a function of magnetic flux and electron density for the KM-HH-SD model. The second to fourth columns show \( \chi \), spin susceptibility \( \chi_s \) and magnetization \( \langle m_z \rangle \) respectively, for the KM-HH model. The temperature is \( \beta = 3/t \).

The DQMC results are shown in Fig. S4. Comparing panels a and e, we find that the strong correlation induces an insulator at zero-field for \( 1/4 - \) and \( 3/4 - \) filling with a "SD" Chern number \( C_{\text{SD}} = 1 \), which is akin to that obtained for the spinful Haldane-Hofstadter-Hubbard model[5]. This means the zero-field \( 1/4 - \) and \( 3/4 - \) filled KM-HH systems emerge as a QSH Mott insulator with a spin Chern number \( C_s = 1 \). The compressibility and spin susceptibility for the KM-HH model in Fig. S4(b,c,f,g) are essentially the same as those for the BHZ-HH model in Fig. 3(b,c,f,g) of the main text. In the presence of strong electron-electron interaction, the compressibility displays a pair of zero-mode LLs at quarter-filling where the spin susceptibility has ridges. The relatively noticeable differences between the KM-HH and BHZ-HH models lie in the magnetization at the band edge as is evident from panels Fig. S4 (d) and (h) relative to Fig. 3(d) and (h) in the main text. Note the sign of the magnetization is not particularly important as a discrepancy already arises with the non-interacting case and hence likely arises from the difference in the lattices. In short, we observe the same QSH Mott insulator in the quarter-filled KM-HH model with large enough Hubbard interaction and adds to the ubiquity of this correlation-driven effect.
FIG. S5: DQMC results for the BHZ-HH (left column) and KM-HH (right column) models at zero magnetic field. Panels (a,b) and (c,d) show the charge and spin correlation respectively at varying interaction strengths $U = 2, 4, 8, 12$ (in the model-specific energy scale). They share the same legends.

FIG. S6: Band structure for flat band limit ($t' = 0.3, \psi = 2.54$) of the generalized KM model.

Although we have used a probe magnetic field to compute the Chern number in the QSH states, we close with a direct comparison of the zero-field limit of both the BHZ-HH and KM-HH models. Their charge correlation $\chi_c$ and spin correlation $\chi_s$ are presented in Fig. S5(a-d) for different $U$. For both models, as the interaction increases, the charge correlation develops a dip at $1/4$- and $3/4$-filling, representing a precursor of the low-temperature incompressible states, where the spin correlation acquires a peak.

Recently, experiments on AB-stacked transition metal dichalcogenide bilayers[6] observed a quantum anomalous Hall (QAH) effect when filling up the first moiré band corresponding to quarter-filling in the generalized KM-Hubbard model. Since the experimental system consists of flat bands, we take a parameter set ($t' = 0.3, \psi = 2.54$) giving rise to a flat lower band in the generalized KM model. The corresponding non-interacting band structure is shown in Fig. S6. With a flat lower band, we can access the strongly correlated physics with a small $U = 2$ ($\gg W - \approx 0.2$) for $\langle n \rangle \leq 1$, leaving the $\langle n \rangle > 1$ essentially non-interacting since $U < W$. For $\langle n \rangle > 1$, the non-interacting and interacting results for the corresponding KM-HH-SD and KM-HH models are shown in Fig. S7 at $\beta = 8/t$. Indeed comparing $U = 0$ (first row) and $U = 2$ (second row) in Fig. S7, there is no qualitative difference for $\langle n \rangle > 1$. But for $\langle n \rangle \leq 1$, the interacting case (Fig. S7(e-h)) shows a quantum anomalous Hall (QAH) effect with the Chern number changing sign with the magnetic flux, in complete agreement with experiment[6].
Compared to Fig. S4(f), only the left side of the zero-mode Landau levels show up in Fig. S7(f) as the QAH effect at this intermediate strength of $U$. The right component of the Landau level in Fig. S4(f-h) is missing because the physics for $\langle n \rangle > 1$ is essentially non-interacting. Recall the Landau feature for $\langle n \rangle < 1$ is correlation driven. The compressibility in the KM-HH-SD model Fig. S7(e) shows that the line indicating a unit spin Chern number stops at $\langle n \rangle$ slightly larger than 1, signaling the boundary between strongly and weakly correlated region. In this sense, the insulating state for $U = 2$ at 1/4-filling and zero field is a precursor of the QSH Mott insulator for large enough $U$ ($U \gg W_{+}(-)$ and $U \gg \Delta$).

Further, we compare the DQMC results for $U = 2$ and $U = 12$ in Fig. S8 at a higher temperature $\beta = 3/t$. Indeed, as $U$ becomes much larger than the bandwidth, we observe that emergent Landau levels appear on the right side of quarter-filling similar to Fig. S4 (f-h). Combined with the left component of the Landau levels, namely the QAH effect observed in experiment, they make the two zero-mode Landau levels as a full signature of the QSH insulator. Likewise in the magnetization (Fig. S8(c,f)), with a Mott gap opening at half-filling, the magnetic region above (and below) the non-interacting QSH insulator splits into two pieces. The left piece evolves into a ridge corresponding to the emergent right component of the Landau levels shown in the compressibility.

FIG. S7: DQMC results for the KM-HH-SD and KM-HH models at $U = 0$ (first row) and $U/t = 2$ (second row). The first column shows the compressibility $\chi$ as a function of magnetic flux and electron density for the KM-HH-SD model. The second to fourth columns show $\chi$, the spin susceptibility $\chi_s$ and the magnetization $m_z$ respectively, for the KM-HH model. The temperature is $\beta = 8/t$. 
FIG. S8: DQMC results for the flat-band KM-HH models at $U/t = 2$ (first row) and $U/t = 12$ (second row). The first to third columns show the compressibility $\chi$, the spin susceptibility $\chi_s$ and the magnetization $m_z$ respectively. The temperature is $\beta = 3/t$. 
The determinantal quantum Monte-Carlo (DQMC) simulation for the Bernevig-Hughes-Zhang-Hofstadter-Hubbard (BHZ-HH) and KM-HH model suffers from severe sign problems in certain doping densities. For this reason, our computation is limited to $\beta = 3$. The average sign for both models as a function of density at the zero field is shown in Fig. S9. The sign is more severe in the KM-HH model setup. To deal with that, we increased the number of runs for our simulations on the KM-HH model.

![Graph showing the average sign for BHZ-HH and KM-HH models with zero external magnetic flux at $U = 12$ and $\beta = 3$ in their individual energy scale. The other parameters are $M = 1$ for the BHZ-HH model and $t' = 0.3$ for the KM-HH model.](image.png)

**FIG. S9:** The average sign for the BHZ-HH and KM-HH model with zero external magnetic flux at $U = 12$ and $\beta = 3$ in their individual energy scale. The other parameters are $M = 1$ for the BHZ-HH model and $t' = 0.3$ for the KM-HH model.
In Fig. 3(g) of the main text, the quantum spin Hall (QSH) Mott insulator at quarter-filling features a peak in the spin correlation, while the non-interacting QSH insulator has a dip in the same quantity. To analyze the spin structure of the peak, we plot the zero-frequency real-space spin correlation $S(r, \omega = 0) = \frac{1}{N} \sum_i \int_0^\beta \langle S_i^z(r(\tau)) S_i^z(0) \rangle d\tau$ for both BHZ-HH and KM-HH models at quarter-filling $\langle n \rangle = 1$ and $U = 12$ in Fig. S10(a) and (b). They all exhibit a ferromagnetic correlation pattern independent of lattice geometry. Although there is no net magnetic moment as required by TR symmetry, a strong spin correlation exists at $Q = 0$, thereby making the system highly susceptible to magnetization when a B-field is turned on.

In the presence of strong correlation, at half-filling the non-interacting QSH order is destroyed and turns into a topologically trivial Mott insulator. We expect the Mott insulator to show anti-ferromagnetism (AF) robust to an external magnetic field in a bipartite lattice. However, Fig. 3(g) in the main text shows a peak in the spin correlation ($Q = 0$) at half-filling, which is inconsistent with this expectation. To further explore this issue, we look into its zero-frequency spin correlation in real and momentum space at two temperatures $\beta = 3$ and $\beta = 5$, shown in Fig. S11. Even at the relatively higher temperature of $\beta = 3$, the system already shows an AF pattern in the central region. As the temperature decreases, the AF region enlarges and further dominates the cluster. Thus, eventually at low enough temperature, AF order prevails in the Mott insulator, as expected. Note that here the smallest magnetic flux $\phi/\phi_0 = 1/36$ is used to reduce the finite-size effect at low temperature. This does not affect our conclusion because the influence from the magnetic field is negligible compared to the AF order.

![Figure S10](image_url)

**FIG. S10:** The static spin correlation at 1/4-filling in real spaces respectively for BHZ-HH (a) and KM-HH (b) models with $U = 12$ (in the model-specific energy scale) and zero magnetic field.
FIG. S11: The static spin correlation in real space and momentum space at half-filling for the BHZ-HH and KM-HH model at inverse temperature $\beta = 3$ (a-d) and $\beta = 5$ (e-h). The magnetic flux here is $\phi/\phi_0 = 1/36$ to reduce the finite-size effect.
BILAYER KM MODEL

In Fig. S12, comparing the schematic band structure of the KM model with that in the AB-stacked MoTe$_2$/WSe$_2$ heterobilayer[7] (blue and red colors are for Chen number $C = 1$ and $-1$ respectively), we find that the only difference is that they have opposite spin for the top Chern bands. This contrast leads to qualitatively different physics. Filling all four KM bands in Fig. S12(a) gives rise to a trivial band insulator, while filling all four moiré bands in Fig. S12(b) results in a double quantum spin Hall insulator with a spin Chern number $C_s = 4$ since $C_1 = 2$ and $C'_1 = -2$. The latter is inherently contradictory as non-trivial topology cannot result in the band insulator limit. Consequently, it is not possible to construct a four-band tight-binding model to describe this physics in AB-stacked heterobilayer. The principle for constructing a tight-binding model is that the system becomes a trivial band insulator if all bands are filled. Therefore, to describe the physics in the AB-stacked MoTe$_2$/WSe$_2$ heterobilayer, we use a bilayer KM model (with eight bands) whose Hamiltonian is

$$H = H_{KM_1} + H_{KM_2} + t_\perp \sum_{i,\sigma} (c_{1i\sigma}^\dagger c_{2i\sigma} + c_{2i\sigma}^\dagger c_{1i\sigma}) + V \sum_{i,\sigma} (c_{1i\sigma}^\dagger c_{1i\sigma} - c_{2i\sigma}^\dagger c_{2i\sigma}),$$  \hspace{1cm} (S2)

where $i$ is the site label for each layer, $\sigma$ represents the spin and the numbers 1 and 2 are layer indices, $V$ is the voltage difference between layers. $H_{KM_1}$ is the same as $H_{KM_2}$ with only different layer indices. In the flat-band limit, the inter-layer hopping $t_\perp$ lifts the degeneracy of the bottom (or top) orbitals between two KM layers while the Chern number for each band with a specific spin is unchanged, as shown in Fig. S13(a). Then the lower four bands are equivalent to the four moiré bands in the AB-stacked MoTe$_2$/WSe$_2$ heterobilayer (Fig. S12) for which the valley labels can be assigned. Without interaction, this model shows a QSH effect at quarter-filling ($\langle n \rangle = 2$) and a double QSH effect at half-filling ($\langle n \rangle = 4$) in Fig. S14(a-c) with an example flat-band parameter set $t'/t = 0.3$, $\psi = 2.54$, $t_\perp/t = 0.3$, $V/t = 0.4$ and $\beta = 12/t$.

We conduct the DQMC simulation for the corresponding bilayer flat-band KM-HH model at an intermediate Hubbard interaction $U = 1.5t$, $\beta = 12/t$ and present the compressibility, spin susceptibility and magnetization in Fig. S14(d-f). The system exhibits a QAH effect with spin polarization at 1/8-filling ($\langle n \rangle = 1$) and a QSH effect at quarter-filling ($\langle n \rangle = 1$), as expected given the similarity between single-layer and bilayer KM-HH models. Since the Hubbard interaction mixes the non-interacting bands and the resulting QAH state is spin polarized, this necessarily yields valley-coherence given the valley assignment in Fig. S13(a).

The presence of an emergent topologically non-trivial state at 3/8-filling ($\langle n \rangle = 3$) is special to the bilayer model. It has the feature of a QAH effect, namely the single Landau level in the compressibility (Fig. S14(d)) accompanied by the peak in the spin susceptibility (Fig. S14(e)). It is also likely to have the feature of a QSH effect (helical edge current). This obtains because the system can not be fully polarized at $\langle n \rangle = 3$. Some band must be doubly occupied, thereby explaining the white region in the magnetization in (Fig. S14(f)). Also, the helical currents from the $\langle n \rangle = 2$ QSH state probably play a role in the $\langle n \rangle = 3$ edge state because their edge dispersion extends to the upper bands, which would not be filled until $\langle n \rangle > 4$.

(a) KM model
(b) AB stacked MoTe$_2$/WSe$_2$ heterobilayer

![Diagram](image_url)

FIG. S12: The schematic band structures of KM model and AB-stacked MoTe$_2$/WSe$_2$ heterobilayer. Blue and red colors represent Chen number $C = 1$ and $-1$ respectively. $K$ and $K'$ are valley degrees of freedom.

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FIG. S13: The schematic band structure of the bilayer KM model. Blue and red colors represent Chen number $C = 1$ and $-1$ respectively.

![Bilayer KM model: band structure](image)

FIG. S14: DQMC results for the flat-band bilayer KM-HH models at $U = 0$ (first row) and $U/t = 1.5$ (second row). Each row shows the compressibility, spin susceptibility and magnetization, all as a function of density and magnetic flux. The parameters are $t = 1$, $t'/t = 0.3$, $\psi = 2.54$, $t_\perp/t = 0.3$, $V/t = 0.4$ and $\beta = 12/t$.

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