High-dimensional free-space quantum key distribution using spin, azimuthal, and radial quantum numbers

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Abstract

The dimension of the state space for information encoding offered by the transverse structure of light is usually limited by the finite size of apertures. The widely used orbital angular momentum (OAM) modes in free-space communications cannot achieve the theoretical maximum transmission capacity unless the radial degree of freedom is multiplexed into the protocol. Here we experimentally demonstrate a free-space quantum key distribution (QKD) protocol that encodes information in all three transverse degrees of freedom: spin, azimuthal, and radial quantum numbers of photons. Our protocol presents an important step towards the goal of reaching the capacity limit of the finite-sized apertures in free-space QKD link. We realized 8-dimensional information encoding onto single photons, and extension to a higher dimension is straightforward. Our protocol relies on a scheme that can sort photons in two mutually unbiased bases, each involving three transverse degrees of freedom. In addition to boosting the transmission rate, the spatial mode sorting scheme developed for this QKD protocol may be useful to other applications that involve spatial modes of photons.
INTRODUCTION

Within the past few decades the orbital angular momentum (OAM) modes have received extensive attention \[1\text{--}6\] and are widely applied in various information technologies, including quantum teleportation \[7\], optical communications \[8\], and quantum key distribution \[9\]. Compared to the intrinsically bounded polarization state space, the OAM modes offer an infinite-dimensional Hilbert space for information encoding and therefore can be used to increase the transmission rate of a communication link. However, the finite size of apertures in a realistic system usually constrains the dimension of OAM state space that can be accessed. On the other hand, the OAM modes only account for azimuthal variations in the transverse plane, and it has been shown that these modes cannot reach the capacity limit of a communications link without including the radial degree of freedom \[10\]. It is thus highly desirable to multiplex the remaining radial degree of freedom to increase the transmission rate for both classical and quantum communications.

A complete and orthonormal basis that incorporates both radial and azimuthal variations can be constituted by Laguerre-Gaussian (LG) modes. The LG modes are characterized by a radial quantum number \(p\) and an OAM quantum number \(\ell\), and arbitrary paraxial field can be described by these modes \[11\]. Notably, because of small divergence angle and intrinsic rotational symmetry, LG modes have been shown to be good approximations to the eigen propagation modes of circular apertures \[12\]. Moreover, it is apparent that the transmission rate of a communications system can be further increased by using both azimuthal and radial degrees of freedom, and it has been shown that the radial index \(p\) can potentially mitigate the power loss when the receiver has a limited aperture size \[13\]. In addition, different from the vortex phase structure related to the OAM index, the radial index corresponds to a radial, amplitude-only distribution. While the phase structure can be strongly distorted by atmospheric turbulence \[14\], the intensity pattern of the transmitted beam can remain recognizable for a 1.6 km free-space link \[4\], and an intensity pattern recognition accuracy can be higher than 98\% for a 3 km link \[15\] and 80\% for a 143 km link \[16\], which suggests that the amplitude structure associated with a nonzero radial index may be helpful to turbulence mitigation \[17\]. However, while recent advances have shown how to efficiently measure \(p\) and \(\ell\) \[18\text{--}21\], the sorting in mutually unbiased basis of LG modes required by a QKD protocol has not previously been realized. Besides the usefulness in QKD, the capability of sorting in the mutually unbiased bases can be helpful to other quantum applications such as certifying high-dimensional entanglement \[22\]. In the following we present how to build a superposition
mode sorter and provide the experimental demonstration of a QKD protocol employing all three transverse degrees of freedom.

RESULTS

Construction of mutually unbiased bases

To develop a BB84 protocol one needs at least two sets of bases that are mutually unbiased: Every element in each basis is a uniform superposition of elements in another basis, which guarantees that measurement in the wrong basis reveals no information of the measured state. Here we first examine each transverse degree of freedom individually and then demonstrate how to construct two mutually unbiased bases for QKD. The first degree of freedom employed in QKD protocol is the polarization of photons [23], which provides a two-dimensional state space spanned by horizontal $|H\rangle$ and vertical $|V\rangle$ polarizations. While polarization is the most widely used degree of freedom in QKD, the azimuthal structures of light have recently seen increasing usage due to their direct relation with the OAM of photons [9]. Each photon can carry an OAM of $l\hbar$ [11] and the corresponding OAM state can be written as $|l\rangle_\ell$, where $l$ is an arbitrary integer and the subscript $\ell$ denotes an OAM state. In this work we restrict ourselves to an OAM sub-space that is spanned by $|\pm 2\rangle_\ell$. Finally we use the radial quantum number $p$ of LG functions as another independent resource for information encoding [24]. Again we restrict ourselves to a sub-space that is spanned by the two lowest radial indices $p = 0, 1$. The direct product of these three sub-spaces constitutes an eight-dimensional Hilbert space that is used as our first basis for QKD. The elements in this basis are $\{|H, 0, -2\rangle_\ell, |H, 0, 2\rangle_\ell, \cdots, |V, 1, 2\rangle_\ell\}$. We refer to this basis as the main basis. The above-specified choices of bases for azimuthal and radial degrees of freedom allows us to employ recent advances in sorting LG modes according to both OAM and radial indices [18–21].

To implement secure QKD we need an additional capability of measuring photons in at least one conjugate basis. The conjugate basis is mutually unbiased with respect to the main basis, and can be constructed through a direct product of respective complementary bases of polarization, azimuthal, and radial sub-spaces [25]. The complementary basis of polarization sub-space can be built as

$$|D\rangle = \frac{|H\rangle + |V\rangle}{\sqrt{2}}, \quad |A\rangle = \frac{|H\rangle - |V\rangle}{\sqrt{2}},$$

(1)

where $|D\rangle$ and $|A\rangle$ denote the diagonal and anti-diagonal polarizations respectively. We use a
FIG. 1. Spatial representations of elements in the main and conjugate bases. The normalized intensity distribution of each spatial mode is presented. The upper right image is the corresponding experimental (Exp.) record of the simulated (Sim.) image for comparison. Each mode can have the two different polarizations as shown at the upper left corner. Scale bar, 1 mm.

similar choice for OAM states and define

\[ |\ell_D\rangle = \frac{|-2\rangle_\ell + |2\rangle_\ell}{\sqrt{2}}, \quad |\ell_A\rangle = \frac{|-2\rangle_\ell - |2\rangle_\ell}{\sqrt{2}}, \]  

(2)

The complementary basis for radial modes is taken to be

\[ |p_L\rangle = \frac{|0\rangle_p + i |1\rangle_p}{\sqrt{2}}, \quad |p_R\rangle = \frac{|0\rangle_p - i |1\rangle_p}{\sqrt{2}}, \]  

(3)

where the subscript \( L \) and \( R \) follow the notation of left- and right-handed circular polarization. We choose this definition because such states are easier to generate experimentally, but we stress that this will not make any fundamental change to the QKD protocol. With all these definitions, the elements in the complementary basis, referred to as conjugate basis, are \{\( |D, p_L, \ell_D\rangle, |D, p_L, \ell_A\rangle, \ldots, |A, p_R, \ell_A\rangle \}\}. Spatial intensity distributions of these modes are given in Fig. 1 (see Supplementary Section I for more details).
FIG. 2. *Schematic of the radial superposition mode sorter.* (A), conceptual schematic of a $d$-dimensional radial superposition mode sorter. (B), two-dimensional ($d = 2$) realization to sort $|p_L\rangle$ and $|p_R\rangle$. The unitary transformation $\hat{U}_{1 \rightarrow 0}$ is realized by SLMs. The modes $|p_L\rangle$ and $|p_R\rangle$ are directed to different output ports as indicated at the last beamsplitter.

**Construction of a superposition mode sorter**

Having established the two mutually unbiased bases, we need to perform coherent detection in each of these bases. For the main basis, devising a coherent detection strategy is straightforward. A sequence of a polarizing beamsplitter (PBS), a radial mode sorter [18], and an OAM sorter [19] can losslessly project input photons onto elements in the main basis. However, each state in the conjugate basis is a superposition of different LG modes, and no effective sorter for these states has been reported to our knowledge. To address this problem, we develop a generic, scalable scheme for sorting such superposition states and experimentally realize it as a part of our QKD protocol. For simplicity we only focus on the radial degree of freedom thereafter, although the same scheme can be directly applied to other degrees of freedom.

A conceptual schematic for a radial superposition mode sorter is shown in Fig. 2. For a $d$-dimensional Hilbert space spanned by radial modes $|m\rangle_p$, where $m \in \{0, 1, \ldots, d - 1\}$, the
FIG. 3. **Schematic of the QKD protocol involving three degrees of freedom.** (A), common-path radial mode sorter. Different colors represent different polarizations as the inset indicates. The mode $|1_p, 2\ell\rangle$ is shown as an example, and its polarization is turned from 45 degrees to 135 degrees. (B), radial mode converter transforming $|1_p, 2\ell\rangle$ mode to $|0_p, 2\ell\rangle$ mode. Detailed phase profiles on SLMs are given in the supplement. (C), experimental setup for 8-D QKD involving polarization, azimuthal quantum number, and radial quantum number of photons. Relay lenses to keep the spatial modes from diffraction are omitted for simplicity. See Methods for more details.

The corresponding complementary basis can be defined as

$$|\tilde{n}\rangle_p = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \exp\left(-i\frac{2\pi m \tilde{n}}{d}\right) |m\rangle_p,$$

where $\tilde{n} \in \{0, 1, \cdots, d-1\}$ is the superposition mode index, and a bar above the number indicates that this is an element in the complementary basis. We first apply a radial mode sorter to direct different radial mode components towards distinct, non-overlapping ports [18]. Then a mode converter is used to transform individual radial modes to the same state $|0\rangle_p$, which enables effective
interference between these otherwise orthogonal modes. The relative phase carried by each mode as in Eq. (4) can be viewed as a discrete, tilted wavefront, and subsequently a discrete Fourier transform performed by a quantum $F$-gate can direct photons to different output port indexed by the superposition mode index $\tilde{n}$ (see Supplementary Section II for derivation). A two-dimensional realization of this scheme to sort $|p_L\rangle$ and $|p_R\rangle$ is shown in Fig. 2b as an intuitive example. A binary radial mode sorter is by the scheme in ref. [18]. The mode converter can be realized by spatial light modulator (SLM) to transform the component $|1\rangle_p$ to $|0\rangle_p$ [26, 27], and the phase imparted to the SLMs can be determined by a nonlinear fitting algorithm (Supplementary Section IV). Afterwards a beamsplitter recombines the modes and acts as a binary quantum $F$-gate [28]. It can be easily seen that $|p_L\rangle$ and $|p_R\rangle$ are directed to different output ports due to interference at the last beamsplitter. In our experiment, we use polarization-sensitive SLMs to develop a stabilized, common-path radial mode sorter and converter as shown in Fig. 3a and b, respectively. More details about these common-path realizations are presented in Supplementary Section III and IV. We note that this scheme should also be applicable to other spatial modes such as Hermite-Gaussian modes given the existence of a corresponding mode sorter [29] and converter [26], which can be useful to realize super-resolution imaging [30].

**Implementation of QKD**

Having constructed the sorters for both mutually unbiased bases, we next demonstrate the implementation of an eight-dimensional QKD protocol involving all three degrees of freedom. A schematic diagram of the setup is presented in Fig. 3c. A He-Ne laser is modulated by an acousto-optic modulator (AOM) and an attenuator to generate optical pulses in a weak, coherent state. Computer-generated holograms imparted on SLM are used to modulate the spatial distribution of photons [31]. A combination of PBS and half-wave plate (HWP) is used to measure the polarization states in both mutually unbiased bases. Then an OAM mode sorter consisting of an un-wrapper and a phase corrector is used to sort the two OAM eigenmodes $|−2\rangle_\ell$ and $|2\rangle_\ell$ [19]. An additional beamsplitter (BS) is needed to recombine the separated modes so as to sort OAM superposition states $|\ell_D\rangle$ and $|\ell_A\rangle$. The sorting mechanism for the radial degree of freedom follows the scheme presented in Fig. 2. After the polarization state is determined, the common-path radial mode sorter can be used to map different radial modes to different polarizations, therefore one can use a PBS to detect the radial quantum number. A radial mode converter needs to be
FIG. 4. **Experimental results of 8-D QKD protocol.** (A), conditional probability matrix measured with $\mu = 0.1$ pulses. Each element in this matrix represents the probability that Bob detects the corresponding symbol conditioned on the symbol sent by Alice. (B), the original, encrypted, and decrypted image transmitted between Alice and Bob.

Inserted to sort the radial superposition modes $|p_L\rangle$ and $|p_R\rangle$. All sorted photons are collected by multi-mode fibers (MMFs) and detected by single-photon avalanche photodiodes (APDs). We note that since only four APDs are available at the time of performing experiment, at Bob’s side the data is collected for elements in each basis separately and combined later. More details about the experimental setup can be found in Methods and Supplementary Section V.

Based on this setup, Alice generates a sequence of random bits and individually sends them to Bob in randomly chosen bases. Due to the nonzero cross-talk Alice and Bob need to implement error correction [32] and privacy amplification [9] to correct the errors in the shared key while maintaining the security. Then Alice transmits the message encrypted by the key via a classical
channel. Due to the speed limitation of our setup, Alice and Bob share a short key and repeat it to match the length of message. In our experiment the message to be sent is the shield logo of the University of Rochester. Bob then decrypted the image using the shared random key and the reconstructed image is presented in Fig. 4.

**DISCUSSION**

To evaluate the performance of our QKD link, we measure the conditional probability of detection for each mode detected at Bob’s side as a function of the mode sent by Alice, and the result is presented in Fig. 4. The cross-talk ranges from 6.0% to 16.7%, with an average of 11.7% well below the 8-D QKD error threshold 24.7% [33]. Here the cross-talk is the probability that the photon triggers the wrong APDs when Bob is detecting in the correct basis. The mutual information can be calculated to be $I_{AB} = 2.15$ bits per sifted photon and the secure key rate of the proposed protocol is $R_{net} = 0.39$ bits/s when 16 APDs are working together (see Supplementary Section V). We note that the speed of our implementation is mainly limited by 60 Hz frame rate of our SLM. Alternative rapid generation mechanisms such as digital micromirror devices (DMD) can be used to generate spatial modes at the speed of 4 kHz [9, 31]. This enhancement can lead to an immediate improvement of two orders of magnitude in the secure key rate without any modification to the rest of the system. Also we use a coherent state of low average photon number $\mu = 0.1$ to provide security against a photon number splitting attack (see Supplementary Section VI). However, one can use decoy state [34] or high-brightness single-photon source to further enhance the communication rate. In addition, because we display a static phase on the SLM, the SLM used in our experiment can be readily replaced by passive, polarization-dependent liquid crystal retarder to realize a scalable, low-cost sorter.

Compared to a QKD protocol with OAM encoding only [9], our protocol employs the slowly divergent LG modes and are thus practical for free-space links with finite-sized apertures [12]. In a realistic free-space channel, atmospheric turbulence can lead to modal cross-talk and reduce the transmission rate. A recent experiment [4] has suggested the potential of LG modes in a free-space channel in an urban environment. While employing a large number of OAM modes for a 1.6 km link would require complex adaptive optics in more than one plane to mitigate turbulence [4], the recognizability of the intensity pattern at the receiver aperture suggests the potential of multiplexing the radial index, which can be subject to future study. On the other hand, the large
communication bandwidth offered by LG modes [35] can be obtained with less difficulty for a smaller distance and thus provides benefits to short-range optical interconnects [36].

In conclusion, we provide an experimental demonstration of a QKD protocol which encodes information using all possible transverse degrees of freedom, i.e. polarization, radial, and OAM modes, with a resulting transmission of 2.15 bits per sifted photon. A sorting scheme for superposition spatial modes is implemented to enable this 8-D protocol and can find direct application in other fields such as super-resolution imaging and high-dimensional entanglement certification. We believe our demonstration opens up a way to fully exhaust the information resources of finite-sized apertures and therefore reach the capacity limit of a communication channel. The slowly divergent LG beams also make this protocol promising for a free-space communication network.

MATERIALS AND METHODS

Experimental setup

A 5 mW 633 nm He-Ne laser (Melles Griot) is used as our light source and modulated by an AOM (1205C-2, Isomet) to generate pulses of duration 200 ns. The average photon number in each pulse is attenuated by neutral density filters and crossed polarizers to $\mu = 0.1$. The SLM used by Alice is a Cambridge Correlator SDE1024 phase-only SLM with a resolution of 1024×768 pixels and a pixel size of 9 $\mu$m to generate LG modes of beam waist radius $w_0 = 462.3$ $\mu$m at the first diffraction order [31]. The SLMs at Bob’s side are Hamamatsu X10468 phase-only SLMs with a resolution of 792×600 pixels and a pixel size of 20 $\mu$m. The un-wrapper and phase corrector of the OAM sorter is made of poly methyl methacrylate (PMMA). The Hamamatsu SLM at Bob’s side is used twice by reflecting the modulated beam back to a different area on the same SLM, so the radial mode sorter and radial mode converter are in fact realized by a single SLM, respectively. The single-photon APDs are Perkin Elmer SPCMAQRH-14-FC modules. The typical dark count rate of an APD is 50 counts/s and the quantum efficiency is $\eta = 0.65$. The output electric pulses from APDs are transmitted to a field programmable gate array (FPGA, Altera DE2) to perform photon counting. The multi-mode fiber (M31L02, Thorlabs) has a 62.5/125 $\mu$m core/cladding diameter with 0.275 NA. Since the output of the OAM mode sorter has a small size, we use a 10X beam expander (GBE10-A, Thorlabs) to expand the beam. Due to the misalignment of the OAM sorter, the sorted mode has slight side lobes which can lead to cross-talk. In the experiment we use
an iris to block the side lobes. This blockage leads to a slight loss of 8%. The transmission of the whole link is measured to be 15%, where the loss mainly comes from SLMs, lenses, and mirrors. In our experiment we use Winnow error reconciliation protocol to perform error correction and Hash function for privacy amplification. The image sent by Alice is of 80×80 pixels.

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SUPPLEMENTARY MATERIALS

I. DEFINITION OF MUTUALLY UNBIASED BASES

In our experiment three degrees of freedom are used to encode information on single photons. Three transverse properties, i.e. polarization, radial index \( p \), and OAM index \( \ell \), form a 8-D Hilbert space and two mutually unbiased bases (MUBs) based on these degrees of freedom are defined in Table I. The spatial expressions of states in two MUBs are explicitly shown in terms of LG modes with equal weight in another basis.

\[
\text{LG}_p^\ell(r, \phi) = \sqrt{\frac{2p!}{\pi (p + |\ell|)!}} \frac{1}{w_0} \left( \frac{\sqrt{2}r}{w_0} \right)^{|\ell|} \times \exp\left( -\frac{2r^2}{w_0^2} \right) L_{\ell}^{|\ell|} \left( \frac{2r^2}{w_0^2} \right) e^{i\ell \phi},
\]

where \( p \) and \( \ell \) are the radial and azimuthal index and \( L_{\ell}^{|\ell|} \) is the generalized Laguerre polynomial. The polarization is represented by unit vector \( \hat{x} \) (horizontal polarization \( |H\rangle \)) and \( \hat{y} \) (vertical polarization \( |V\rangle \)). We note that some normalization factors are ignored for simplicity.

| Main basis | Spatial representation | Conjugate basis | Spatial representation |
|------------|-----------------------|----------------|-----------------------|
| \(|H, 0_p, -2_\ell\)| \( \text{LG}_0^{-2} \hat{x} \) | \(|D, p_L, \ell_D\rangle \) | \( \text{LG}_0^{-2} + \text{LG}_0^{+2} + i\text{LG}_1^{-2} + i\text{LG}_1^{+2} \) (\( \hat{x} + \hat{y} \)) |
| \(|H, 0_p, +2_\ell\)| \( \text{LG}_0^{+2} \hat{x} \) | \(|D, p_L, \ell_A\rangle \) | \( \text{LG}_0^{-2} - \text{LG}_0^{+2} + i\text{LG}_1^{-2} - i\text{LG}_1^{+2} \) (\( \hat{x} + \hat{y} \)) |
| \(|H, 1_p, -2_\ell\)| \( \text{LG}_1^{-2} \hat{x} \) | \(|D, p_R, \ell_D\rangle \) | \( \text{LG}_0^{-2} + \text{LG}_0^{+2} - i\text{LG}_1^{-2} - i\text{LG}_1^{+2} \) (\( \hat{x} + \hat{y} \)) |
| \(|H, 1_p, +2_\ell\)| \( \text{LG}_1^{+2} \hat{x} \) | \(|D, p_R, \ell_A\rangle \) | \( \text{LG}_0^{-2} - \text{LG}_0^{+2} - i\text{LG}_1^{-2} + i\text{LG}_1^{+2} \) (\( \hat{x} + \hat{y} \)) |
| \(|V, 0_p, -2_\ell\)| \( \text{LG}_0^{-2} \hat{y} \) | \(|A, p_L, \ell_D\rangle \) | \( \text{LG}_0^{-2} + \text{LG}_0^{+2} + i\text{LG}_1^{-2} + i\text{LG}_1^{+2} \) (\( \hat{x} - \hat{y} \)) |
| \(|V, 0_p, +2_\ell\)| \( \text{LG}_0^{+2} \hat{y} \) | \(|A, p_L, \ell_A\rangle \) | \( \text{LG}_0^{-2} - \text{LG}_0^{+2} + i\text{LG}_1^{-2} - i\text{LG}_1^{+2} \) (\( \hat{x} - \hat{y} \)) |
| \(|V, 1_p, -2_\ell\)| \( \text{LG}_1^{-2} \hat{y} \) | \(|A, p_R, \ell_D\rangle \) | \( \text{LG}_0^{-2} + \text{LG}_0^{+2} - i\text{LG}_1^{-2} - i\text{LG}_1^{+2} \) (\( \hat{x} - \hat{y} \)) |
| \(|V, 1_p, +2_\ell\)| \( \text{LG}_1^{+2} \hat{y} \) | \(|A, p_R, \ell_A\rangle \) | \( \text{LG}_0^{-2} - \text{LG}_0^{+2} - i\text{LG}_1^{-2} + i\text{LG}_1^{+2} \) (\( \hat{x} - \hat{y} \)) |

TABLE I. Definition of two MUBs used in the experiment. Each mode in one basis is a superposition of all modes with equal weight in another basis.
II. MECHANISM OF SUPERPOSITION MODE SORTER

Consider a $d$-dimensional Hilbert space spanned by radial modes $|m\rangle_p$ with $m \in \{0, 1, \ldots, d-1\}$ as described in the manuscript. A complementary basis of this Hilbert space can be constructed as

$$|\bar{n}\rangle_p = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \exp \left( -\frac{i2\pi m\bar{n}}{d} \right) |m\rangle_p,$$

where $\bar{n} \in \{0, 1, \ldots, d-1\}$ is the superposition mode index, and a bar above the number indicates that this is an element in complementary basis. The port of each state is labelled by another distinct ket $|k\rangle$, where $k \in \{0, 1, \ldots, d-1\}$. Initially, all superposition modes are located at the same port, which can be expressed as $|\bar{n}\rangle_p |0\rangle$. Then a radial mode sorter is operated to direct different radial mode components towards distinct, non-overlapping ports and the following transform can be achieved

$$|\bar{n}\rangle_p |0\rangle = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \exp \left( -\frac{i2\pi m\bar{n}}{d} \right) |m\rangle_p |0\rangle \rightarrow \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \exp \left( -\frac{i2\pi m\bar{n}}{d} \right) |m\rangle_p |m\rangle.$$  

(S3)

After that, a mode converter is applied sequentially and can transform $|m\rangle_p$ to $|0\rangle_p$, which can be expressed as

$$\frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \exp \left( -\frac{i2\pi m\bar{n}}{d} \right) |m\rangle_p |m\rangle \rightarrow \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \exp \left( -\frac{i2\pi m\bar{n}}{d} \right) |0\rangle_p |m\rangle.$$  

(S4)

The following quantum $F$-gate performs a discrete Fourier transform according to the port index of each state, which can be expressed as

$$\hat{F} [ |m\rangle_p |k\rangle ] = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \exp \left( \frac{i2\pi jk}{d} \right) |m\rangle_p |j\rangle.$$  

(S5)

Then we apply the quantum $F$-gate to the converted mode and we can have
\[ \hat{F} \left[ \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \exp \left( \frac{-i2\pi m \bar{n}}{d} \right) |0\rangle_p |m\rangle \right] \]

\[ = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \exp \left( \frac{-i2\pi m \bar{n}}{d} \right) \hat{F} \left[ |0\rangle_p |m\rangle \right] \]

\[ = \frac{1}{d} \sum_{j=0}^{d-1} \sum_{m=0}^{d-1} \exp \left( \frac{i2\pi m(j - \bar{n})}{d} \right) |0\rangle_p |j\rangle \]

\[ = \sum_{j=0}^{d-1} \delta(j - \bar{n}) |0\rangle_p |j\rangle = |0\rangle_p |\bar{n}\rangle . \]

(S6)

In summary, the superposition mode \( |\bar{n}\rangle_p \) will finally be directed to \( \bar{n} \)-th output port, which can be expressed as \( |\bar{n}\rangle_p |0\rangle \rightarrow |0\rangle_p |\bar{n}\rangle \).

### III. ANALYSIS AND RESULTS OF MODE SORTER

The experimental setup to characterize the mode sorter for main basis and conjugate basis is presented in Fig. [S1]. We emphasize that to detect states in main basis the radial mode converter in dashed box is not necessary and should be removed. The common-path radial mode sorter is composed of two polarization-sensitive spatial light modulators (SLMs), each with a spherical lenses attached. The spherical lens is imaged onto the SLM by a 4-\( f \) system in the experiment. A quadratic phase pattern which is equivalent to a lens of focal length 0.62 m is imprinted on both SLMs. Both spherical lenses have a focal length of \( f_1 = 1.5 \) m. The distance between the injected mode beam waist and the first SLM is \( z = 0.44 \) m, and the separation between two SLMs is \( 2z \). The output plane of the radial mode sorter is of distance \( z \) after the last SLM, and the first SLM in the mode converter should be placed at this output plane. The Fourier transform lens in radial mode converter is 1 m. The phase patterns on SLMs in the converter are presented in the next section.

In this setup two polarizations are employed as two arms of a Mach-Zenhder interferometer, and the injected mode is 45 degree polarized. This is permissible because we have sorted the polarization of photons before these mode sorters, so the injected photons have a deterministic polarization and one can use a half wave-plate (HWP) to rotate polarization to desired angle. For the radial mode sorter, since vertically polarized light is not affected by SLMs, one can check that two spherical lenses perform a Fourier transform to it [S1]. Horizontal polarization, however, is modulated by both SLMs. Each SLM with the attached lens becomes a lens of focal length 0.44 m,
exactly equal to the propagation distance $z$. Hence, horizontally polarized light experiences two consecutive Fourier transforms. If we treat vertically polarized beam as the reference arm, then horizontally polarized beam will gain a mode-dependent Gouy phase of $\exp[-i(2p + |\ell|)\pi/2]$ due to its extra Fourier transform. One can check that this phase is $-\pi$ for $|0_p, \pm 2\ell\rangle$ and 0 for $|1_p, \pm 2\ell\rangle$ [S1]. Since the OAM index we use in our protocol is either -2 or 2, so the absolute value $|\ell|$ is the same for all modes and thus does not have any influence on the sorting of radial index. In the following we just omit the OAM index for simplicity. In the experiment we add a constant

![Diagram of radial mode sorter results](image)

**FIG. S1.** The radial mode sorter results. **a.** The experimental measurement of conditional probability for radial modes $|0\rangle_p$ and $|1\rangle_p$. **b.** The experimental measurement of conditional probability for complementary basis $|p_L\rangle$ and $|p_R\rangle$. **c.** Experimental intensity records of the two output ports of the radial mode sorter when the incident field is a superposition of $|0_p, -2\ell\rangle$ and $|1_p, -2\ell\rangle$. **d.** Experimental intensity records of the two output ports of the radial mode sorter when the incident field is a superposition of HG$_{00}$ and HG$_{20}$. Scale bar, 1 mm.
phase on SLMs and use a HWP so that \(|0\rangle_p \) is vertically polarized while \(|1\rangle_p \) becomes horizontally polarized. We first remove the converter and use a polarizing beamsplitter (PBS) to separate two radial modes. We characterize the performance of this radial mode sorter by measuring output power of two output ports of the PBS and the result is presented in Fig. S1. The cross-talk of the radial mode sorter is around 2.6\%, which is defined as the power in the wrong port divided by the total output power when a radial mode is injected. We inject a coherent superposition of \(|0_p, -2\ell\rangle \) and \(|1_p, -2\ell\rangle \) and the intensity distributions at two output ports of the PBS are presented in Fig. S1d. We note that this sorter can also be applied to Hermite-Gaussian (HG) modes as shown in Fig. S1e because both LG modes and HG modes are the eigenmodes of the fractional Fourier transform, and the Gouy phase for a HG mode is \(\exp[-i(m+n)\pi/2]\). As an illustration, the intensity records at the two output ports are given in Fig. S1d when the input field is a coherent superposition of HG\(_{00}\) and HG\(_{20}\) modes.

To sort the superposition of radial modes \(|p_L\rangle \) and \(|p_R\rangle \). Then we add a radial mode converter as in Fig. S1a. For \(|p_L\rangle \), its component \(|0\rangle_p \) becomes vertically polarized and \(|1\rangle_p \) becomes horizontally polarized due to the radial mode sorter, and then SLMs in the converter transform the horizontally polarized \(|1\rangle_p \) to \(|0\rangle_p \). In other words, \(|p_L\rangle \) goes through the following transformation

\[
|p_L\rangle |D\rangle = \frac{|0\rangle_p |D\rangle + i |1\rangle_p |D\rangle}{\sqrt{2}} \rightarrow \frac{|0\rangle_p |V\rangle + i |1\rangle_p |H\rangle}{\sqrt{2}} \rightarrow \frac{|0\rangle_p |V\rangle + i |0\rangle_p |H\rangle}{\sqrt{2}} = |0\rangle_p |R\rangle ,
\]

where \(|H\rangle , |V\rangle , |D\rangle , |R\rangle \) denote horizontal, vertical, diagonal, and right-handed circular polarization. It can be seen that \(|p_L\rangle \) becomes right-handed circularly polarized, and one can check that \(|p_R\rangle \) becomes left-handed circularly polarized after the same transformation. In the experiment we add a constant phase on SLM to turn both modes to be linearly polarized, and then use a HWP and a PBS to separate them. We measure the output power of two output ports of the PBS and the result is presented in Fig. S1. The cross-talk of superposition radial mode sorter is around 7.4\%, lower than the 2-D QKD error threshold 11.0\%. 

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IV. RADIAL MODE CONVERTER

In this section we discuss how to use two SLMs connected by a Fourier transform lens to realize mode conversion from $|1_p, \pm 2\ell\rangle$ to $|0_p, \pm 2\ell\rangle$ and the schematic is shown in Fig. S1a. Taking into account the fact that the converter only reshapes the radial profile of the mode, the phase written on SLMs should be a function of radius $r$ only. So the phase on first SLM can be decomposed to the polynomials of radius and we use nonlinear fitting algorithm to improve the conversion efficiency by adjusting coefficients of these polynomials. The second SLM is to cancel the residue phase of the converted mode because $|0_p, \pm 2\ell\rangle$ possesses a flat radial phase structure. We choose three polynomials $\{r^2, r^3, r^4\}$ in the algorithm. We tested that more polynomials (up to $r^7$) can provide negligible conversion efficiency improvement while costing much more time to run the code.

The phase distributions on two SLMs are shown in Fig. S2a and b. Before the fitting process, we erase the radial phase variation of $|1_p, \pm 2\ell\rangle$ induced by Laguerre polynomials and add this phase to the final result later. Also we ignore the azimuthal phase caused by the non-zero OAM.

FIG. S2. The radial mode converter. a. The radial phase profile on the first SLM. b. The radial phase profile on the second SLM. c. The normalized intensity of the incident field on the first SLM. d. The normalized intensity of the output field on the second SLM. The corresponding experimental results are shown on the upper right corner for a comparison. Scale bar, 1 mm.
index and omit the OAM index thereafter. The intensity distributions of the input and output fields are shown in Fig. [S2]: and d. It can be seen that the converted mode in Fig. [S2] is similar but not identical to $|0\rangle_p$ due to the non-unity conversion efficiency. The theoretical conversion efficiency in our simulation is $|p\langle 0 \rvert_{p'}|^2 = 82.7\%$ where $|0\rangle_p$ denotes the converted mode by SLMs. The corresponding cross-talk induced by this non-unit efficiency can be readily calculated to be 4.5%. In general, a mode conversion requires multi-plane iterations [S2], and therefore our implementation to transform $|1\rangle_p$ to $|0\rangle_p$ is not perfect. However, we note that this does not impose a fundamental problem on QKD protocol, as the experimental cross-talk is measured to be 7.4%, lower than the 2-D QKD error threshold 11.0%. Moreover, it is straightforward to cascade more SLMs to reduce the cross-talk.

V. INFORMATION CAPACITY AND SECURE KEY RATE

In this section we calculate the information and secure key rate mainly following the procedure in [S3]. For our protocol, the two bases are mutually unbiased to each other, so Alice would send each symbols with equal probability as the BB84 protocol. Here we assume a uniform error rate for detecting each mode, so the mutual information can be expressed as

$$I_{AB} = \log_2 d + F \log_2 F + (1 - F) \log_2 \left( \frac{1 - F}{d - 1} \right),$$  

(S8)

where $d = 8$, the average error rate $\delta = 11.7\%$, and $F$ is the probability of correct measurement $F = 1 - \delta = 88.3\%$. With these numbers we can immediately get $I_{AB} = 2.15$ bits per sifted photon.

We assume the eavesdropper (Eve) performs the cloning-machine-based individual attack. The mutual information between Alice and Eve is

$$I_{AE} = \log_2 d + F_E \log_2 F_E + (1 - F_E) \log_2 \left( \frac{1 - F_E}{d - 1} \right),$$  

(S9)

where $F_E$ is the fidelity of Eve’s cloning machine and this quantity is

$$F_E = \frac{F}{d} + \frac{(d - 1)(1 - F)}{d} + \frac{2}{d} \sqrt{(d - 1)F(1 - F)}.$$  

(S10)

It can be readily calculated that $F_E = 0.43$ and therefore $I_{AE} = 0.40$ bits. For sufficient symbols the secure key $R$ can be expressed as

$$R_{net} = R_{sift} \cdot \left[ I_{AB} - \max (I_{AE}, I_{BE}) \right],$$  

(S11)
where \( R_{\text{sift}} \) is the sifted key rate and we assume \( \max(I_{\text{AE}}, I_{\text{BE}}) = I_{\text{AE}} \) for individual attack. The sifted key rate can be calculated as

\[
R_{\text{sift}} = \frac{1}{2} f_{\text{rep}} T_{\text{link}} \eta \mu,
\]

(S12)

here \( f_{\text{rep}} \) is the pulse repetition rate, \( T_{\text{link}} \) is the transmission of the link, and \( \eta \) is the quantum efficiency of the detector. In our experiment, \( T_{\text{link}} = (1 - 85\%) \cdot (1 - 8\%) \cdot 80\% = 0.11 \). Here 85\% is the transmission loss caused by SLMs, lenses, and mirrors, 8\% is the loss caused by the iris to mitigate the misalignment of the OAM sorter, and 80\% is the coupling efficiency of a multi-mode fiber. Due to the limited refreshing rate of the SLM, we have \( f_{\text{rep}} = 60 \) Hz. The quantum efficiency is \( \eta = 0.65 \) and the average photon number per pulse is \( \mu = 0.1 \), so we have \( R_{\text{sift}} = 0.22 \) photon/s and \( R_{\text{net}} = 0.39 \) bits/s. Here the calculation assumes 16 APDs are available and working simultaneously.

VI. PHOTON NUMBER SPLITTING (PNS) ATTACK

In our experiment, we attenuate a coherent laser to generate weak pulses with average photon number \( \mu = 0.1 \). However, it is likely to have more than one photon in a pulse. Under this circumstance, the Eve can split the extra photon and detect it when the basis is broadcast, which is the PNS attack. An inequality for security against such attack has been proposed and can be expressed as [S4]

\[
p_{\text{det}} > p_{\text{multi}},
\]

(S13)

where \( p_{\text{det}} \) is the probability that Bob will detect a photon, while \( p_{\text{multi}} \) is the probability of the pulse containing more than one photon. The photon number statistics of a coherent laser satisfies the Poisson distribution, which means that the probability of having \( n \) photon is \( P(n, \mu) = \mu^n e^{-\mu}/n! \).

So we can have

\[
p_{\text{multi}} = \sum_{n=2}^{\infty} P(n, \mu) \approx \frac{\mu^2}{2}.
\]

(S14)

In our implementation we have \( \mu = 0.1 \) and \( p_{\text{multi}} = 5 \times 10^{-3} \). The photon detection probability is \( p_{\text{det}} \approx \mu \eta T_{\text{link}} + p_{\text{dark}} \), Here \( p_{\text{dark}} \) is the probability of dark count. Then we can have \( p_{\text{det}} > \mu \eta T_{\text{link}} = 7.2 \times 10^{-3} > p_{\text{multi}} \). So our protocol is secure against the PNS attack.

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