HILBERT SPACES OF ENTIRE FUNCTIONS AND TOEPLITZ QUANTIZATION OF EUCLIDEAN PLANES

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Abstract. The theory of Toeplitz quantization presented in our previous paper is extended and further developed to include diverse and interesting non-commutative realizations of the classical Euclidean plane. This is done using Hilbert spaces of entire functions, where polynomials in one complex variable form a dense subspace. The complex coordinate naturally acts as an unbounded multiplication operator generating, together with its adjoint, a highly non-commutative *-algebra of operators. The Toeplitz operators are then geometrically constructed as special elements from this algebra; they are associated to the symbols from another quadratic non-commutative algebra, which is interpretable as polynomials over a plane to be quantized. Such a conceptual framework promotes interesting non-trivial conditions on the initial scalar product. These are analyzed in detail. Various illustrative examples are computed.

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1. Introduction

This paper is a continuation of our previous work [10], where we have provided a detailed analysis of geometric, algebraic and analytic aspects for a coherent states quantization of the Manin $q$-plane via Toeplitz operators. A general method for such Toeplitz quantizations has been introduced and extensively studied by the second author in [19, 20].

Here we would like to further expand on these principal dimensions by exploring their deeper interrelations and establishing a link with the Hilbert spaces of entire functions [3, 6]. On one hand this allows the use of a rich array of techniques from complex analysis, and on the other it provides an elegant framework for defining quantum versions of classical fundamental geometrical objects—such as points, coordinates, observables and transformations.

Our context fits well into the principal quantum non-commutative geometry foundations as developed by Connes [8] and Woronowicz [22] with a variety of interesting non-commutative algebras, representable by operators that appear naturally in a Hilbert space. And moreover, our approach is particularly in resonance with the pioneering formulation of quantum geometry by Prugovečki [15], where the concept of point is quantized by morphing it into a special wave function. These ‘wave functions of points’ can then be interpreted as coherent states, and in our framework they are an integral part of the Hilbert space geometry, which directly emerges from the reproducing kernel.

The reproducing kernel is a primary and defining structural part of Hilbert spaces of holomorphic functions defined over an open domain $\Omega \subset \mathbb{C}$. This is a map $K: \Omega \times \Omega \to \mathbb{C}$, antiholomorphic in the first coordinate and holomorphic in the second and exhibiting an appropriate matrix complete positivity condition. Such a structure gives us the possibility of speaking of ‘quantized’ points, these being the vectors of the Hilbert space representing the points of $\Omega$. This association is obtained from the kernel map by $\Omega \ni w \mapsto \langle \bar{w}, \ast \rangle \in \mathcal{H}$. By taking scalar products these point wave functions evaluate in the points $w$ the functions of $\mathcal{H}$. And by normalizing them we obtain the corresponding coherent states. We shall here almost exclusively deal with entire functions, which correspond to taking the domain $\Omega = \mathbb{C}$.

Let us outline the contents of the paper. In the next section in a series of steps we establish our principal geometric and algebraic setup. We begin with a detailed analysis of scalar products in the space $\mathbb{C}[z]$ of complex polynomials in a complex variable $z$. Every such scalar product $\langle \rangle$ leads to a Hilbert space $\mathcal{H}$, namely the completion of $\mathbb{C}[z]$ with respect to $\langle \rangle$. So $\mathbb{C}[z]$ is dense in $\mathcal{H}$. We would like to capture in terms of the operators on $\mathcal{H}$ some basic idea of a plane, be it classical or quantum. This naturally leads to interpreting the complex coordinate $z$ as an appropriate multiplication operator $Z$ in $\mathcal{H}$, so that on polynomials $\phi$ it acts as $Z: \phi(z) \mapsto z\phi(z)$. We would also like to be able to consider the operator representation of the conjugate coordinate $\bar{z}$ as an adjoint operator $Z^\ast$ so that an interesting operator calculus involving $Z$ and $Z^\ast$ can be defined, reflecting the geometrical idea of an underlying plane-like space. An important technical problem here is to find the most effective context for defining $Z$ and $Z^\ast$, including their domains. From consideration of this problem a number of non-trivial conditions for the initial scalar product on $\mathbb{C}[z]$ naturally emerge. We shall call them Harmony Properties, and we shall number them from 0 to 3.
Harmony Zero treats the very basic structure at the level of polynomials, while Harmony One provides a particularly elegant and geometrically natural answer by postulating that all the elements of the completion $\mathcal{H}$ of $\mathbb{C}[z]$ are interpretable as entire functions. The definition of the operator $Z$ then can be extended from the polynomials to its natural domain, in the same way as the complex coordinate multiplication operator $Z: \psi(z) \mapsto z\psi(z)$ with dense domain $D(Z)$ consisting of all $\psi(z) \in \mathcal{H}$ for which $z\psi(z) \in \mathcal{H}$. We shall establish a variety of important properties for the operator $Z$ and its adjoint $Z^*$. In particular $Z$ is closed and the point wave functions associated to classical points $w \in \mathbb{C}$ are the unique (modulo non-zero scalar multiples) eigenvectors for $Z^*$ with the eigenvalues $\bar{w}$. This implies that the spectra of $Z$ and $Z^*$ coincide with the whole $\mathbb{C}$; so they are unbounded operators.

However, this harmony property still does not ensure the existence of an algebraically effective operational setting involving the operators $Z$ and $Z^*$ so that we can meaningfully compose them in arbitrary combinations.

Our next harmony property addresses this, by postulating the continuity of $Z^*$ in the normal topology of $\mathcal{H}$ (namely, the topology of uniform convergence on compact sets) and thereby putting $Z$ and $Z^*$ in a kind of balanced relationship. As we shall explain, this provides a common acting space $\mathcal{W}$ for them, consisting of special vectors emerging from the very geometry of $\mathcal{H}$. The point wave functions are always in $\mathcal{W}$, and for polynomials to be included, we shall need the topological half of Harmony Zero.

In such a framework the operators $Z$ and $Z^*$ generate a fundamental $*$-algebra $\mathcal{A}$, in terms of which the underlying quantum space is brought to life. We shall prove that $\mathcal{A}$ is quite non-commutative, as it is always with a trivial centralizer in a pretty large algebra $\mathcal{O}(\mathbb{C})$ of normally continuous linear transformations of entire functions. We shall also prove that, under certain general additional assumptions on interchageability of order of the product of $Z^*$ and $Z$, the underlying quantum space is always ‘pointless’. In other words there will be no characters on $\mathcal{A}$.

We shall then proceed with interpreting the constructed framework as a natural destination for a Toeplitz quantization of appropriate quadratic algebras $\mathbb{C}[z, \bar{z}, Q]$. These algebras are generated by coordinates $z$ and $\bar{z}$, and equipped with a quadratic flip-over type relation $Q$ between these coordinates. This relation can be trivial, in which case we are dealing with the classical complex plane $\mathbb{C}$, twisted commutative as in the case of the Heisenberg-Weyl algebra or the Manin $q$-plane, or a more elaborate quadratic relation telling us how to exchange $z$ and $\bar{z}$.

The main construction here is that of an extended space $\mathcal{R}$ equipped with a non-degenerate, but not necessarily positive definite form $\langle \cdot | \cdot \rangle$, such that the common action space $\mathcal{W}$ for the operators of $\mathcal{A}$ together with its scalar product $\langle \cdot | \cdot \rangle$ can be singled out via a symmetric idempotent $\Pi$ acting on $\mathcal{R}$, and such that the algebra $\mathbb{C}[z, \bar{z}, Q]$ acts symmetrically on $\mathcal{R}$ with $\mathcal{W}$ being the cyclic $Z$-invariant subspace. The Toeplitz operator with the symbol $f \in \mathbb{C}[z, \bar{z}, Q]$ is then defined as $\Pi f \Pi$ interpreted as an element of $\mathcal{A}$.

It is worth mentioning that the principal algebraic steps in a sense mirror those of the Stinespring construction for completely positive maps between C*-algebras. As we shall see in detail, however, we are in the context of unbounded operators and a key positivity condition is not always fulfilled. On the other hand when $\langle \cdot | \cdot \rangle$ is positive – and this will be our final harmony condition – we can close $\mathcal{R}$ in a
Hilbert space $\mathcal{F}$ and the relationship between $\mathbb{C}[z, \bar{z}, Q]$ and $\mathcal{A}$ established by the Toeplitz quantization turns out to have a particular richness.

We then proceed with some more detailed calculations of a number of concrete examples by analyzing the coherent states, their resolution of the identity and the Toeplitz quantization interpretation. We next analyze four principal types for the algebras $\mathbb{C}[z, \bar{z}, Q]$, and we also discuss their possible realizations in Hilbert spaces of analytic (but not necessarily entire) functions. As we shall see, the Heisenberg-Weyl algebra and its $q$-variations are the only types allowing a realization with entire functions. On the other hand the Manin $q$-plane is naturally realizable in a space of Laurent series, whose common domain of definition is $\mathbb{C} - \{0\}$. Some of these spaces have the unit disk or the half plane as their common domain of definition for the holomorphic functions, and these provide a natural setting for quantizing the classical hyperbolic plane [9]. In all these calculations basic identities involving hypergeometric series and $q$-special functions (such as the triple product identity, the $q$-binomial theorem, the second Euler identity, $q$-versions of exponential, logarithm and Gamma functions) naturally appear. The classical treaties [2] and [13] provide an excellent in-depth exposition of these identities.

In Section 3 we focus on some general algebraic aspects of Toeplitz quantization. We explain how the construction of the extended algebra and module with the projector and the quadratic form can be performed over a large class of algebras $\mathcal{C}$ equipped with a flip-over operator $\sigma: \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$, where $\mathcal{C}$ is the conjugate algebra of $\mathcal{C}$. This includes as a special case our main protagonist—the polynomials $\mathbb{C}[z]$—and also multi-dimensional versions with several complex variables.

Finally, in Section 4 some concluding remarks are made. The paper ends with two extensions exhibiting perhaps some interest on their own.

In the first appendix, we provide a simple geometrical interpretation for the positivity of the canonical quadratic form and of the Stieltjes moment condition. And in the second appendix some formulas and illustrative examples of Hilbert spaces of holomorphic functions are collected, including the classical structures such as Segal-Bargmann spaces and their $q$-variations, de Branges spaces including the Paley-Wiener space, and the Bergman spaces of square integrable holomorphic functions over bounded domains. For all these spaces a similar quantization scheme for their underlying classical domain can be established.

We use standard notation such as $\mathbb{N}$ for the set of non-negative integers and $\mathbb{C}$ for the complex plane.

2. Quantization Via Hilbert Spaces of Entire Functions

Polynomials Generating a Hilbert Space

The central object for our considerations will be the polynomial algebra $\mathbb{C}[z]$ in one complex variable $z$. It is a complex infinite dimensional vector space, in which the monomials $\{z^n | n \in \mathbb{N}\}$ form a natural linear basis. For every $n \in \mathbb{N}$ the polynomials of degree $\leq n$ form an $(n+1)$-dimensional subspace, denoted by $\mathbb{C}_n[z]$.

The polynomials $\mathbb{C}[z]$ are included as a subalgebra in the larger algebra of all entire functions, denoted by $\mathbb{H}(\mathbb{C})$. In the normal topology we have

\begin{equation}
\overline{\mathbb{C}[z]} = \mathbb{H}(\mathbb{C}),
\end{equation}

where $\overline{A}$ denotes the topological closure of $A$. 
Non-constant polynomials can also be understood as entire functions \( \phi: \mathbb{C} \to \mathbb{C} \) preserving the complex infinity point \( \infty \). Under composition polynomials form a non-commutative unital semi-group. An algebraic counterpart to this geometrical view is the interpretation of polynomials as unital algebra homomorphisms \( \phi: \mathbb{C}[z] \to \mathbb{C}[z] \). This linear action of \( \phi \) on \( \mathbb{C}[z] \), which is called substitution, is specified on basis elements by \( \mathbb{C}[z] \ni z^n \mapsto \phi(z)^n \in \mathbb{C}[z] \).

Of particular interest are those polynomials whose corresponding homomorphism is invertible, which we call the invertible polynomials. They are precisely the linear transformations \( z \mapsto az + b \) where \( a, b \in \mathbb{C} \) with \( a \neq 0 \). They provide all the orientation preserving symmetries of the Euclidean plane \( \mathbb{C} \). If \( a = 1 \) we have translations given by \( b \) (with no fixed point if \( b \neq 0 \)) and amplitwists around the unique fixed point \( b/(1-a) \) if \( a \neq 1 \) with rotation factor given by the phase of \( a \) and similarity factor given by \( |a| \). (We continue assuming \( a \neq 0 \).) It is well known that all the automorphisms of the algebra \( \mathbb{C}[z] \) are of this form, and this is the same as all the holomorphic automorphisms of the complex plane \( \mathbb{C} \).

There is a canonical anti-linear involutive algebra automorphism \( J \) of \( \mathbb{C}[z] \) defined by \( J(z) = \bar{z} \) and which just conjugates the coefficients of the polynomials.

The multiplication in \( \mathbb{C}[z] \) can be interpreted as the left regular representation, where \( \mathbb{C}[z] \) acts by linear operators in \( \mathbb{C}[z] \). The constant polynomial \( 1 \in \mathbb{C}[z] \) is both cyclic and separating for this representation. Viewed in terms of the left regular representation, \( \mathbb{C}[z] \) is a maximal commutative subalgebra of linear operators in \( \mathbb{C}[z] \): any linear operator \( l: \mathbb{C}[z] \to \mathbb{C}[z] \) commuting with left multiplication by \( z \) (and hence by induction and linearity with left multiplication by any polynomial) is left multiplication by \( l(1) \in \mathbb{C}[z] \).

Among other important operators acting in \( \mathbb{C}[z] \) is the complex differentiation \( \partial/\partial z \), and hence all differential operators whose coefficients are polynomial in \( z \). All these operators—multiplication, substitution, differentiation—are continuous (which will always mean in this context with respect to the normal topology on \( \mathbb{C}[z] \)) induced by its inclusion in \( H(\mathbb{C}) \), and they naturally extend to the whole space \( H(\mathbb{C}) \). We shall denote by \( \mathcal{O}(\mathbb{C}) \) the algebra of all normal continuous linear transformations of \( \mathbb{C}[z] \) to itself:

\[
\mathcal{O}(\mathbb{C}) := \{ \tau: \mathbb{C}[z] \to \mathbb{C}[z] \mid \tau \text{ is linear and normal continuous} \}.
\]

Every \( \tau \in \mathcal{O}(\mathbb{C}) \) is uniquely extendible to a normal continuous linear transformation \( \tau: H(\mathbb{C}) \to H(\mathbb{C}) \). We can say equivalently that \( \mathcal{O}(\mathbb{C}) \) consists of continuous linear transformations of \( H(\mathbb{C}) \), which preserve \( \mathbb{C}[z] \). So \( \mathcal{O}(\mathbb{C}) \) is a subalgebra of the algebra of all continuous linear transformations of \( H(\mathbb{C}) \), which will be denoted by \( \mathcal{O}(\mathbb{C}) \):

\[
\mathcal{O}(\mathbb{C}) := \{ \sigma: H(\mathbb{C}) \to H(\mathbb{C}) \mid \sigma \text{ is linear and normal continuous} \}.
\]

The algebra \( H(\mathbb{C}) \) acts on itself via its left regular representation in terms of which \( H(\mathbb{C}) \) is a maximal commutative subalgebra of \( \mathcal{O}(\mathbb{C}) \).

The dual space \( H(\mathbb{C})^\ast \) consists by definition of all normally continuous complex linear functionals \( f: H(\mathbb{C}) \to \mathbb{C} \). This is the same as saying normally continuous functionals on \( \mathbb{C}[z] \), since each of the latter uniquely extends by normal continuity to \( H(\mathbb{C}) \). The elements \( f \in H(\mathbb{C})^\ast \) are in one-to-one correspondence with the complex sequences \( f_n = f(z^n) \) satisfying the following geometric boundedness property: there exists a constant \( \Lambda > 1 \) such that for all \( n \in \mathbb{N} \) we have

\[
(2.2) \quad |f_n| \leq \Lambda^n.
\]
In addition to this very basic structure, we shall assume that $\mathbb{C}[z]$ is equipped with a positive definite scalar product $\langle | \rangle$, which is anti-linear in its first entry and linear in the second. It is completely specified by the infinite matrix
\begin{equation}
S_\infty = (s_{nm}) \quad \text{where} \quad s_{nm} = \langle z^n | z^m \rangle \quad \text{for } n, m \in \mathbb{N}.
\end{equation}
This matrix is required to be hermitian and moreover to be strictly positive in the sense that the partial matrices
\begin{equation}
S_n = \begin{pmatrix}
s_{00} & \cdots & s_{0n} \\
\vdots & \ddots & \vdots \\
s_{n0} & \cdots & s_{nn}
\end{pmatrix}
\end{equation}
determining the scalar product in the subspaces $\mathbb{C}_n[z]$ are strictly positive matrices for every $n \in \mathbb{N}$. For a hermitian matrix this will be the case if and only if all the numbers
\begin{equation}
d_n = \det(S_n)
\end{equation}
are strictly positive real numbers. In particular, $S_n$ is invertible and $S_n^{-1}$ is strictly positive, too.

**Remark 2.1.** All our constructions will be independent of rescalings of the scalar product. So we can always normalize $\langle | \rangle$ by fixing a value for $s_{00} > 0$, for example by postulating $s_{00} = 1$.

The following is a straightforward result.

**Proposition 2.1.** The involution $J$ is an isometry if and only if all the coefficients $s_{nm}$ are real. \hfill \Box

By using the Gram-Schmidt procedure to orthonormalize the naturally ordered sequence of monomials $z^n$, we obtain an orthonormal sequence of polynomials $\phi_n(z)$ such that the degree of $\phi_n(z)$ is $n$ with its highest order coefficient being positive. Also $\phi_{n+1}(z) \perp \mathbb{C}_n[z]$. The space $\mathbb{C}_n[z]$ is spanned by $\phi_0(z), \ldots, \phi_n(z)$, and
\begin{equation}
B := \left\{ \phi_n(z) \mid n \geq 0 \right\}
\end{equation}
is an orthonormal vector space basis in the whole space $\mathbb{C}[z]$.

Explicitly, these canonical orthonormal polynomials are given by
\begin{equation}
\phi_n(z) = \frac{1}{\sqrt{d_n d_{n-1}}} \begin{vmatrix}
s_{00} & \cdots & s_{0n} \\
\vdots & \ddots & \vdots \\
\cdot & \cdots & \cdot \\
s_{n-10} & \cdots & s_{n-1n}
\end{vmatrix}
\end{equation}
with the extra definition $d_{-1} := 1$.

**Remark 2.2.** When appropriate we shall also write simply $|n\rangle = \phi_n(z)$.

The following polynomial identity holds:
\begin{equation}
\sum_{k=0}^{n} \phi_n(w)\phi_n(z) = \sum_{i,j=0}^{n} z^i [S_n^{-1}]_{ij} \bar{w}^j.
\end{equation}

The correspondence between the scalar products $\langle | \rangle$ on $\mathbb{C}[z]$ and systems $\phi_n(z)$ is in fact one-to-one. If a system $\phi_n(z)$ of polynomials satisfying $\deg \phi_n(z) = n$ and
with positive highest-order coefficients is given, then there exists a unique scalar product $\langle \rangle$ on $\mathbb{C}[z]$ orthonormalizing the monomials $z^n$ into $\phi_n(z)$.

Let us check how the scalar product matrix $S_\infty$ transforms under holomorphic symmetries of $\mathbb{C}$. Again these are precisely the linear maps $z \mapsto az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$. We shall consider separately rotations, scalings and translations.

First, under the rotations $z \mapsto uz$ around 0 by a unitary complex number $u$ the coefficients $s_{nm}$ transform as
\begin{equation}
(2.9) \quad s_{nm} \leadsto u^{m-n}s_{nm}.
\end{equation}

In particular the scalar product will be invariant under all of these transformations if and only if $s_{nm} = 0$ for all $n \neq m$. Second, the scaling $z \mapsto rz$ with $r > 0$ gives
\begin{equation}
(2.10) \quad s_{nm} \leadsto s_{nm} r^{n+m}.
\end{equation}

Finally, the transformation rule for translations $z \mapsto z + b$ is
\begin{equation}
(2.11) \quad s_{nm} \leadsto \sum_{k=0}^{n} \sum_{l=0}^{m} \binom{n}{k} \binom{m}{l} \bar{b}^{n-k} b^{m-l} s_{kl}.
\end{equation}

The transformations (2.9), (2.10) and (2.11) are easy to verify. From this it is also easy to see that every transformation which is not a rotation around some point of $\mathbb{C}$ affects non-trivially the scalar product.

**Remark 2.3.** If the scalar product is invariant under the rotations (2.9) ($\Leftrightarrow$ the matrix $S_\infty$ is diagonal), we simplify the notation by defining a sequence
\begin{equation}
(2.12) \quad s_n := s_{nn} \quad \text{for } n \in \mathbb{N}.
\end{equation}

The canonical orthonormal polynomials are then simply
\begin{equation}
(2.13) \quad \phi_n(z) = z^n \sqrt{s_n} \quad \text{for } n \in \mathbb{N}.
\end{equation}

In our previous paper [10], we only dealt with such scalar products.

Let $\mathcal{H}$ be the Hilbert space obtained by completing the incomplete pre-Hilbert space $\mathbb{C}[z]$ relative to $\langle \rangle$. Then the subspace $\mathbb{C}[z]$ is dense in $\mathcal{H}$, and the set $B$ in (2.6) is an orthonormal basis for $\mathcal{H}$. Of course, $\mathcal{H}$ is not unique, but it is unique up to a unique isometric isomorphism which is the identity on $\mathbb{C}[z]$. One possible explicit construction of $\mathcal{H}$ is as the set of all formal infinite series $\sum_{n \in \mathbb{N}} a_n \phi_n(z)$ with coefficients $a_n \in \mathbb{C}$ satisfying $\sum_{n \in \mathbb{N}} |a_n|^2 < \infty$. In this setting the dense subspace $\mathbb{C}[z]$ is identified as the set of all formal sums for which only finitely many coefficients are non-zero. This clearly shows that $\mathbb{C}[z]$ is not equal to $\mathcal{H}$. Other realizations of $\mathcal{H}$ will be considered later.

There exist four important, mutually subtly related, conditions for the scalar product $\langle \rangle$ and consequently for the resulting Hilbert space $\mathcal{H}$. These conditions put this simple framework into special harmony with operator theory and complex analysis, thereby establishing an elegant context for constructing quantum models of the complex Euclidean plane $\mathbb{C}$. We shall call them Harmony Properties and assign them numbers 0, 1, 2 and 3.

Harmony Zero is a dual thing, namely a symbiosis of both an algebraic and a topological condition. It provides the simplest, most straightforward *-algebraic structure emerging from the space of polynomials and their scalar product $\langle \rangle$, and
at the same time it maintains an elementary topological resonance with complex analysis.

But first we shall say that an operator \( S : \mathbb{C}[z] \rightarrow \mathbb{C}[z] \) is formally adjointable with respect to \( \langle | \rangle \) if there exists (a necessarily unique) operator \( T : \mathbb{C}[z] \rightarrow \mathbb{C}[z] \) satisfying \( \langle \phi | S \psi \rangle = \langle T \phi | \psi \rangle \) for all \( \phi, \psi \in \mathbb{C}[z] \), in which case we say that \( T \) is the formal adjoint of \( S \). Also, for \( \phi, \psi \in \mathbb{C}[z] \) we recall the Dirac notation \( \langle \phi | \psi \rangle \), a linear functional on \( \mathbb{C}[z] \), which is continuous in the \( \langle | \rangle \) (or norm) topology on \( \mathbb{C}[z] \).

**Definition 2.4.** We shall say that **Harmony Zero** (or simply H0) holds if:

- The multiplication operator by \( z \) is formally adjointable in \( \mathbb{C}[z] \) with respect to the inner product \( \langle | \rangle \).
- For every \( \phi \in \mathbb{C}[z] \) the linear functional \( \langle \phi | \rangle \) is continuous in the normal topology of \( \mathbb{C}[z] \).

We shall refer to the first condition as Algebraic H0 and to the second condition as Topological H0. The reader is advised to carefully note the distinction between the norm topology and the normal topology. In the context of Algebraic H0 we let \( z \) denote the operator of multiplication \( z : \mathbb{C}[z] \rightarrow \mathbb{C}[z] \) and denote its formal adjoint by \( z^* : \mathbb{C}[z] \rightarrow \mathbb{C}[z] \). The operator \( z^* \) should not be confused with the conjugate complex variable \( \bar{z} \).

Here are some useful reformulations of these properties. As for Analytical H0 let us consider the infinite matrix \( z_{nm} = \langle n | z | m \rangle := \langle \phi_n(z) | z \phi_m(z) \rangle \). (More Dirac notation.) This represents the multiplication operator by \( z \) in the basis \( B \). By the construction of \( B \), we have that if \( n - m \geq 2 \), then \( z_{nm} = 0 \). So each column of this matrix has only finitely many non-zero entries. Moreover, \( z_{nm} > 0 \) if \( n - m = 1 \). The formal adjointability of \( z \) is equivalent to the statement that every row of this matrix has only finitely many non-zero entries. If so, then the formal adjoint operator \( z^* : \mathbb{C}[z] \rightarrow \mathbb{C}[z] \) to the multiplication operator by \( z \) in \( \mathbb{C}[z] \) is represented in \( B \) by the corresponding adjoint matrix.

As for Topological H0 let us observe first that the normal continuity of all the dual vectors \( \langle \phi | \rangle \) is equivalent to the continuity of only all the basis vectors, that is of \( \langle n | = \langle \phi_n(z) | \rangle \) for all \( n \in \mathbb{N} \). In accordance with (2.2) this property can be expressed as saying that for every \( m \in \mathbb{N} \) there exists a real number \( \Lambda_m \) such that we have the system of inequalities

\[
|s_{mn}| \leq (\Lambda_m)^n \quad \text{for all } n \in \mathbb{N}.
\]

In particular this is the case if the rows (\( \leftrightarrow \) the columns) of \( S_\infty \) possess only finitely many non-zero entries.

**Remark 2.5.** The two components of H0, the algebraic and the topological, do not entangle strongly within this basic polynomial context. Their mutual correlations are manifested through the higher harmony levels.

**Remark 2.6.** Harmony Zero always holds for diagonal scalar products. Indeed, in this case a quick calculation shows that

\[
z | n \rangle = \left( \frac{s_{n+1}}{s_n} \right)^{1/2} | n + 1 \rangle \quad \forall n \geq 0.
\]
This is a kind of creation operator with the only non-zero matrix entries being those immediately below the main diagonal. It is formally adjointable in $\mathbb{C}[z]$ and its formal adjoint, the annihilation operator, is given by

\begin{equation}
(z^*)^n |n\rangle = \left(\frac{s_n}{s_{n-1}}\right)^{1/2} |n-1\rangle \quad \forall n \geq 1,
\end{equation}

together with $(z^*)^0 |0\rangle = 0$. As for the topological part of $H_0$ that follows from the above observation on non-zero entries of $S_\infty$ and comparing with (2.14). Or it follows explicitly from the formula

\begin{equation}
\langle n|φ(z)⟩ = \frac{s_1^{1/2}}{n!} \frac{∂^n}{∂z^n} φ(z) \bigg|_{z=0},
\end{equation}

which tells us that all the $\langle n|$ are indeed continuous in the normal topology of $\mathbb{C}[z]$.

The algebraic part of $H_0$ ensures the existence of a basic $*$-algebra $A$.

**Definition 2.7.** $A$ is defined to be the algebra of linear operators acting on $\mathbb{C}[z]$ generated by the operators $z$ and $z^*$. Specifically, the elements of $A$ are finite linear combinations of monomials in $z$ and $z^*$, which are by definition all of the finite products whose factors are either $z$ or $z^*$ in all possible orders.

Clearly, $A$ is a $*$-algebra, with its $*$-structure being the formal adjoint operation.

The existence of the formal adjoint implies that every operator in $A$, whose domain $\mathbb{C}[z]$ is dense in $\mathcal{H}$, is closable in $\mathcal{H}$. (See Theorem VIII.1 in [14].) Moreover,

\begin{equation}
a ⊂ \overline{a} \quad \text{and} \quad \overline{a^*} ⊃ a^* \quad \text{for every} \quad a ∈ A.
\end{equation}

In the second formula the $*$ in $\overline{a^*}$ means the Hilbert space adjoint operator in $\mathcal{H}$ of the densely defined operator $\overline{a}$. Also, note that the closure $\overline{a}$ is always strictly greater than $a$.

Depending on $⟨|⟩$ the $*$-algebra $A$ can be very simple or arbitrarily complicated. The simplest situation is when $z = z^*$, in which case the operator of multiplication by $z$ with dense domain $\mathbb{C}[z]$ is symmetric as a densely defined operator acting in $\mathcal{H}$. This defines the classical context of orthogonal polynomials over $\mathbb{R}$. In this case the scalar product can always be represented in the form

\begin{equation}
⟨\varphi, \psi⟩ = \int_\mathbb{R} \overline{φ(t)} \psi(t) \, dμ(t) \quad \text{for all} \quad \varphi, \psi ∈ \mathbb{C}[z]
\end{equation}

with respect to a finite measure $μ$ on $\mathbb{R}$. Indeed, if $z$ is symmetric, then all the numbers $s_{nm} = ⟨z^n z^m⟩ = ⟨0|z^{n+m}|0⟩$ are real. This implies that $J$ is isometric, and in particular it extends to an anti-linear isometry of the whole $\mathcal{H}$. So we have an anti-linear isometry $J$ preserving $z$. This symmetry situation implies that $z$ is extendible to a self-adjoint operator in $\mathcal{H}$, as $J$ exchanges $\ker(\lambda - z^*)$ and $\ker(\lambda - z^*)$ for every non-real $\lambda ∈ \mathbb{C}$, and so both defect indices of $z$ are the same. Let us now take any self-adjoint extension $\tilde{z}$ of $z$ and consider its spectral measure $E$. This is a projector-valued measure, that is its values $E(\Lambda)$ on Borel subsets $\Lambda$ of $\mathbb{R}$ are orthogonal projectors in $\mathcal{H}$ so that

$$\tilde{z} = \int_\mathbb{R} t \, dE(t).$$

The desired standard measure $μ$ reproducing the scalar product $⟨|⟩$ is defined by

$$μ(\Lambda) = ⟨0|E(\Lambda)|0⟩.$$
for Borel subsets \( \Lambda \) of \( \mathbb{R} \).

**Remark 2.8.** Here the topological part of H0 is equivalent to the compactness of the support of the measure \( \mu \).

**Proposition 2.2.** This classical context of orthogonal polynomials is equivalent, modulo linear transformations of the complex variable \( z \), to the commutativity of the *-algebra \( \mathcal{A} \).

**Proof.** If \( z^* \) commutes with \( z \), then it belongs to \( \mathbb{C}[z] \) in the sense that \( z^* = \phi(z) \), where \( \phi \) is a polynomial of degree at least 1. Here \( z \) is viewed in the same way as \( z^* \), namely as an operator in \( \mathbb{C}[z] \). But then

\[
z = z^{**} = \phi(z)^* = (J\phi)(z^*) = (J\phi)[\phi(z)].
\]

So the composition of \( J\phi \) and \( \phi \) is the identity map on \( \mathbb{C} \). This is possible only if

\[
(2.20) \quad \phi(z) = az + b \quad \text{for} \ a, b \in \mathbb{C} \quad \text{satisfying} \quad |a| = 1, \ ab + b = 0.
\]

Let \( \sqrt{a} \) denote one fixed value for the square root of \( a \). Using the conditions on \( a, b \) in (2.20) and \( z^* = \phi(z) = az + b \), one then readily calculates that

\[
\left( \sqrt{a}z + \frac{1}{2} \frac{b}{\sqrt{a}} \right)^* = \sqrt{a}z + \frac{1}{2} \frac{b}{\sqrt{a}}.
\]

Therefore the substitution \( z \mapsto \sqrt{a}z + b/(2\sqrt{a}) \) does the trick. \( \square \)

We can extend (2.19) in a sense to our non-commutative context by introducing a canonical ‘integration functional’ on \( \mathcal{A} \) defined by

\[
\int F := \langle 0|F|0 \rangle = \langle \phi_0|F\phi_0 \rangle = \langle 0|F \rangle \quad \forall F \in \mathcal{A},
\]

where \( \phi_0 = |0\rangle = 1 \) is the first element in the basis (2.6). In particular we can write

\[
(2.22) \quad \langle \rho|\phi \rangle = \int \rho^*|\phi \quad \text{for} \ \rho, \phi \in \mathbb{C}[z],
\]

where on the right side the polynomials \( \rho, \phi \) are interpreted as elements of \( \mathcal{A} \).

With the *-algebra \( \mathcal{A} \) we have the very basic algebraic structure for crafting the idea of a quantized Euclidean plane. We would like to interpret the operators \( z \) and \( z^* \) as the quantized complex coordinate and its conjugate. But the structure still lacks certain geometrical and analytical contents, which are fundamental for a truly quantum interpretation. For instance, we would expect both \( z \) and \( z^* \) to be unbounded operators with their spectra being equal to the whole plane \( \mathbb{C} \). And we also want to be able to construct coherent states associated to every complex number. And this is not always the case within the framework of Harmony Zero.

Our next harmony property provides these additional geometrical and analytical dimensions by linking the theory to Hilbert spaces of entire functions.

**Definition 2.9.** We shall say that Harmony One (or simply H1) holds (for \( \mathcal{H} \) or for \( \langle \rangle \) on \( \mathbb{C}[z] \)) if \( \mathcal{H} \) can be realized as a Hilbert space of entire functions on \( \mathbb{C} \).

**Remark 2.10.** Properties H0 and H1 are logically independent. However, in diverse interesting examples they will happily work together.
Harmony One is a symbiosis of two more elementary conditions, which are related to continuity and injectivity. The algebra of entire functions $H(\mathbb{C})$ possesses its natural normal topology of uniform convergence on compact sets. All polynomials are entire functions. Firstly, we require that the inclusion map $\mathbb{C}[z] \to H(\mathbb{C})$ be continuous, where $\mathbb{C}[z]$ is considered equipped with the $\langle \cdot \rangle$-induced topology. If so, the inclusion extends to a continuous map $\mathcal{H} \to H(\mathbb{C})$. Secondly, we require that this extended map be injective. In such a way all the elements of $\mathcal{H}$ are viewable as entire functions.

If, on the other hand, we only have the continuity but not injectivity, the situation allows an elegant geometrical ‘renormalization’. In this case there is a non-trivial Hilbert space completion of $\mathbb{C}^*$ satisfying Harmony One, and (orthogonally project $\mathbb{C}$ as entire functions.

**Proposition 2.3.** There is a canonical one-to-one correspondence between the scalar products $\langle \cdot \rangle$ on $\mathbb{C}[z]$ for which the above continuity property holds and triplets $(\mathcal{K}, \langle \cdot \rangle^\sim, F)$, where $\mathcal{K}$ is a separable Hilbert space, $\langle \cdot \rangle^\sim$ a scalar product on $\mathbb{C}[z]$ satisfying Harmony One, and $F : \mathbb{C}[z] \to \mathcal{K}$ a completely discontinuous linear map in the sense that $D(F^*) = \{0\}$. The space $\mathcal{H}$ is realized as $\mathcal{H}^\sim \oplus \mathcal{K}$, where $\mathcal{H}^\sim$ is the Hilbert space completion of $\mathbb{C}[z]$ relative to $\langle \cdot \rangle^\sim$. In terms of this identification, the $\langle \cdot \rangle^\sim$-isometric inclusion of $\mathbb{C}[z]$ into $\mathcal{H}$ is $p(z) \mapsto p(z) \oplus F[p(z)]$.  

**Remark 2.11.** The map $F$ has the wildest possible behavior for a linear map: its composition with any non-zero continuous functional on $\mathcal{K}$ is discontinuous. This wildness ensures that the graph of $F$, which realizes the embedding of $\mathbb{C}[z]$ into $\mathcal{H}$, is everywhere dense in $\mathcal{H}$.

To see why this is so, note that $\Gamma(F^*) = [U(\Gamma(F))]^\perp$, where $\Gamma(F)$ is the graph of $F$ in the inner product space $\mathbb{C}[z] \oplus \mathcal{K}$ and $U : \mathbb{C}[z] \oplus \mathcal{K} \to \mathcal{K} \oplus \mathbb{C}[z]$ is the inner product preserving map $(f,g) \mapsto (-g,f)$. Similarly, $\Gamma(F^*)$ is the graph of $F^*$. Therefore $\Gamma(F)$ dense implies that $\Gamma(F^*) = 0$ and thus $D(F^*) = \{0\}$. But $D(F^*) = \{ \kappa \in \mathcal{K} \mid \phi \mapsto \langle \kappa | F \phi \rangle \}$ is a continuous map $\mathbb{C}[z] \to \mathcal{K}$. Moreover, $F$ composed with a continuous functional on the Hilbert space $\mathcal{K}$ has exactly the form $\phi \mapsto \langle \kappa | F \phi \rangle$ for some $\kappa \in \mathcal{K}$. So $D(F^*) = \{0\}$ implies that this composition is continuous only for the zero functional.

Interestingly, such an extreme discontinuity naturally emerges in the study of an important continuity property. If this continuity is granted, then the injectivity property holds for $\langle \cdot \rangle$ if and only if $\mathcal{K} = \{0\}$ which is itself equivalent to $\langle \cdot \rangle = \langle \cdot \rangle^\sim$.

**Remark 2.12.** On the other hand under some appropriate additional symmetry conditions on the scalar product, the injectivity of the extended map $\mathcal{H} \to H(\mathbb{C})$ automatically holds. As we shall see, this includes our primary scenario when the monomials $z^n$ are mutually orthogonal.

If Harmony One holds, then for all $z \in \mathbb{C}$ the linear functional $l_z : \mathcal{H} \to \mathbb{C}$ defined for all $f \in \mathcal{H}$ by $l_z(f) := f(z)$ (called evaluation at $z$) is continuous with respect
to the norm topology of $\mathcal{H}$. To see that this is so, note that uniform convergence on compact sets implies convergence on the compact, singleton set $\{z\}$, that is, point-wise convergence. And that tells us that $l_z : \mathcal{H}(\mathbb{C}) \to \mathbb{C}$, defined by the same formula as above, is continuous in the normal topology. But the inclusion map $\mathcal{H} \hookrightarrow \mathcal{H}(\mathbb{C})$ is continuous by Harmony One. So the composition of these two maps is continuous, as claimed. This in turn implies that $\mathcal{H}$ is a reproducing kernel Hilbert space. So in accordance with the general theory of Hilbert spaces of entire functions (for example, see [6]) there exists a unique reproducing kernel $K : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ of $\mathcal{H}$ given by the series

$$K(\bar{w}, z) = \sum_{n=0}^{\infty} \phi_n(w) \phi_n(z),$$

which converges absolutely and uniformly on compact sets of $\mathbb{C} \times \mathbb{C}$. The following are the characteristic properties of $K$:

- For every $z \in \mathbb{C}$ and $f \in \mathcal{H}$ we have the Reproducing Property:
  $$f(z) = \int_{\mathbb{C}} K(\bar{w}, z) f(w) dw$$

- For each $w \in \mathbb{C}$ the function $\mathbb{C} \ni z \mapsto K(\bar{w}, z)$ is an element in $\mathcal{H}$.

It follows in this case from (2.8) that the sequence of inverse matrices $S_n^{-1}$, viewed naturally in $M_\infty(\mathbb{C})$, converges entry-by-entry as $n \to \infty$ to an infinite hermitian matrix $S_\infty^{-1}$ so that

$$K(\bar{w}, z) = \sum_{i,j=0}^{\infty} z^i[S_\infty^{-1}]_{ij} \bar{w}^j,$$

a series which is normally convergent on $\mathbb{C} \times \mathbb{C}$.

**Remark 2.13.** An additional explanation for this important inversion formula is, perhaps, in order here. We know that the partial sums, the left hand side of (2.8), converge normally on $\mathbb{C} \times \mathbb{C}$ to the reproducing kernel $K(\bar{w}, z)$. On the other hand, the reproducing kernel is expandable into a double power series in $\bar{w}$ and $z$, normally convergent on the whole $\mathbb{C} \times \mathbb{C}$. In particular, the coefficients of the partial sums, the right-hand side of (2.8), converge as $n \to \infty$ to the coefficients of the expansion of the reproducing kernel, and (2.24) indeed holds.

For each $w \in \mathbb{C}$ its point wave function is defined for all $z \in \mathbb{C}$ by

$$w(z) := K(\bar{w}, z) \quad \text{or more simply by} \quad [w] := K(\bar{w}, \cdot).$$

We have that $[w] = w \in \mathcal{H}$ by the second characteristic property of $K$. (That both $w \in \mathbb{C}$ and $w \in \mathcal{H}$ is an abuse of notation which will be be resolved by context.) Also by the reproducing property of $K$ we obtain

$$\langle [w]|\psi \rangle = \langle w(z)|\psi(z) \rangle = \psi(w)$$

for every $\psi = \psi(z) \in \mathcal{H}$ and $w \in \mathbb{C}$. In particular, for the point wave functions associated to $v, w \in \mathbb{C}$ we obtain

$$\langle [v]|[w] \rangle = \langle v(z)|w(z) \rangle = K(\bar{w}, v) \quad \text{and} \quad \| [w] \|^2 = \| w(z) \|^2 = K(\bar{w}, w).$$

We claim that a point wave function $[w]$ can not be identically equal to zero. In fact, by (2.25) the set of dormant points defined as $\{w \in \mathbb{C} \mid [w] = 0\}$ is equal to

$$\{w \in \mathbb{C} \mid \psi(w) = 0 \quad \forall \psi \in \mathcal{H}\}.$$
But $\mathcal{H}$ contains all the elements in $\mathbb{C}[z]$, that is, all polynomials. And there is no complex number that is the common zero of all polynomials. So, in this setting there are no dormant points and consequently $[w] \neq 0$ for all $w \in \mathbb{C}$. However, in other more general reproducing kernel Hilbert spaces there are dormant points.

The following is a standard result, which goes much beyond $[w] \neq 0$.

**Lemma 2.1.** The set of vectors $\{[w] \mid w \in \mathbb{C}\}$ in $\mathcal{H}$ is linearly independent.

**Proof.** A set of vectors is linearly independent if and only if all of its finite subsets are linearly independent. So let $w_1, \ldots, w_n$ be $n$ distinct points in $\mathbb{C}$. We have to show that the vectors $[w_1], \ldots, [w_n]$ are linearly independent. So suppose that

$$
\sum_{j=1}^{n} \lambda_j [w_j] = 0
$$

for some $\lambda_j \in \mathbb{C}$. We have to prove that $\lambda_j = 0$ for all $j$. Now for all $\psi \in \mathcal{H}$ we have

$$
0 = \left( \sum_{j=1}^{n} \lambda_j [w_j] \right) |\psi\rangle = \sum_{j=1}^{n} \lambda_j \langle [w_j] |\psi\rangle = \sum_{j=1}^{n} \lambda_j \psi(w_j).
$$

Since the points $w_1, \ldots, w_n$ are distinct, for each $1 \leq i \leq n$ there exists a polynomial $p_i(z) \in \mathbb{C}[z] \subset \mathcal{H}$ such that $p_i(w_j) = \delta_{ij}$, the Kronecker delta, for each $1 \leq j \leq n$. (These are the basis Lagrange polynomials for the points $w_1, \ldots, w_n$.) Taking $\psi = p_i$ in (2.27) shows that $\lambda_i = 0$ for every $1 \leq i \leq n$. □

Let us consider the linear span (finite linear combinations) of all the point wave functions:

$$
\mathcal{L} := \left\{ \sum_{w \in \mathbb{C}}^* c_w w(z) \right\}.
$$

This is a fundamental object directly emerging from the reproducing kernel. From (2.25) we conclude that there is no non-zero vector in $\mathcal{H}$ that is orthogonal to all of the point wave functions. Consequently, the vector subspace $\mathcal{L}$ is dense in $\mathcal{H}$ in the norm topology and hence also dense in $\mathcal{H}$ in the normal topology. Moreover, because of (2.26) both the scalar product and the norm restricted to $\mathcal{L}$ are completely determined by the reproducing kernel.

The reproducing kernel formula is a key to expressing Harmony One in terms of the canonical orthonormal basis.

**Proposition 2.4.** Harmony One holds if and only if the series

$$
\sum_{n=0}^{\infty} |\phi_n(z)|^2
$$

is normally convergent on $\mathbb{C}$, and in addition the orthonormal polynomials $\phi_n(z)$ are $\infty$-linearly independent in the sense that all infinite linear combinations

$$
\sum_{n=0}^{\infty} c_n \phi_n(z) \quad \text{with} \quad \sum_{n=0}^{\infty} |c_n|^2 < +\infty,
$$

understood as entire functions, uniquely determine their coefficients $c_n$.

**Proof.** The $\Rightarrow$ part is clear as in this case (2.29) normally converges to $K(\bar{z}, z)$, and the functions (2.30) are faithful representations of the vectors of $\mathcal{H}$.

The $\Leftarrow$ part is a little subtler. If (2.29) is normally convergent on $\mathbb{C}$, then we can define the function $K(\bar{w}, z)$ by the series (2.23) for which it is easy to see that
it converges normally on \( \mathbb{C} \times \mathbb{C} \) in this case. Moreover, this function satisfies by construction the primary matrix positivity condition (as discussed in Appendix B).

So it is the reproducing kernel function for a Hilbert space \( \mathcal{H} \) of entire functions. Let \( \mathcal{L} \) be its dense linear subspace of finite linear combinations of point wave functions. The formula
\[
\mathcal{L} \ni w(z) \mapsto \sum_{n=0}^{\infty} \phi_n(w) \phi_n \in \mathcal{H}
\]
for all \( w \in \mathbb{C} \) defines an isometric embedding of \( \mathcal{L} \) into \( \mathcal{H} \leftrightarrow \ell^2(\mathbb{N}) \). This isometric embedding extends to an isometric embedding of \( \mathcal{H} \) into \( \mathcal{H} \). Using this embedding, let us calculate the orthocomplement of \( \mathcal{H} \) in \( \mathcal{H} \), namely
\[
\mathcal{H}^\perp = \mathcal{L}^\perp = \bigcap_{w \in \mathbb{C}} [w(z)]^\perp.
\]
Taking into account that
\[
\langle w(z) | \sum_{n=0}^{\infty} c_n \phi_n \rangle = \sum_{n=0}^{\infty} c_n \phi_n(w)
\]
we see that this orthocomplement consists precisely of infinite square summable decompositions of 0 into \( \infty \)-linear combinations of the functions \( \phi_n(z) \) as in (2.30). The orthocomplement will be trivial (equivalently \( \mathcal{H} = \mathcal{H}^\perp \)) if and only if (2.30) faithfully represents the vectors of \( \mathcal{H} \leftrightarrow \ell^2(\mathbb{N}) \).

Remark 2.14. In fact the normal convergence of the series (2.29) is equivalent to the continuity part of Harmony One, while the faithfulness of (2.30) is equivalent to the injectivity part.

The point wave functions give us corresponding coherent states, since they can be normalized. By using some more Dirac notation, there is a natural embedding
\[
\mathbb{C} \ni w \mapsto |w(z)\rangle \langle w(z)| = \frac{[|w\rangle \langle w|]}{||w||^2} \in \text{CP}(\mathcal{H})
\]
into the complex projective space CP(\( \mathcal{H} \)), which is naturally identified as the space of pure states of \( \mathcal{H} \). In the context given by \( \mathcal{H} \) the coherent states \([|w\rangle \langle w|]/||w||^2\) can be interpreted as the quantum counterparts of classical points in the plane \( \mathbb{C} \). In the literature the corresponding unit vectors \([w]/||w|| \in \mathcal{H} \) are also called coherent states.

**Proposition 2.5.** The induced metric on \( \mathbb{C} \) from CP(\( \mathcal{H} \)) via the above map is given by
\[
ds^2 = dw d\bar{w} \left\{ \frac{\partial^2}{\partial \bar{w} \partial w} \log K(w, w) \right\}.
\]

**Proof.** The complex projective space, viewed as the set of rank 1 projectors as above, is a subset of the space of Hilbert-Schmidt operators acting in \( \mathcal{H} \), which itself is a Hilbert space relative to the scalar product defined by
\[
\langle X|Y \rangle := \frac{1}{2} \text{Tr}(X^*Y).
\]
By definition, the metric on $\mathbb{C}P(\mathcal{H})$ is the corresponding induced metric. And in the coordinates $w, \bar{w}$ we obtain
\[
\frac{1}{2} \text{Tr} \left\{ \left( \frac{|w + dw\rangle \langle w + dw|}{\langle w + dw| w + dw\rangle} - \frac{|w\rangle \langle w|}{\langle w| w\rangle} \right)^2 \right\} \sim \int \frac{d\bar{w} dw}{K(w, w)^2}
\]
where we have disregarded the higher order terms in $dw$ and $d\bar{w}$. The formula (2.32) follows elementarily.

**Remark 2.15.** The above formula is valid for all reproducing kernel Hilbert spaces of analytic functions over arbitrary domains $\Omega$ in $\mathbb{C}$ as long as the point vectors $w(z)$ are all non-zero. In other words, no point $w \in \Omega$ is a common zero for all the elements of $\mathcal{H}$. This is a wise thing to assume always, since in the contrary case we can simply remove the discrete set of dormant points from $\Omega$. However, as we already showed, there are no dormant points in the setting of this paper.

**Remark 2.16.** The factor $1/2$ in the definition (2.33) is justified by the simple resulting expression for the induced metric in $\mathbb{C}$. Another justification is that in this case for unit vectors $\varphi, \psi$ the square of the distance between the one-dimensional projectors $|\psi\rangle \langle \psi|$ and $|\varphi\rangle \langle \varphi|$ has a direct physical meaning: the value is $1 - |\langle \varphi| \psi \rangle|^2$, which is the quantum mechanical probability of not transitioning from one of the states $\varphi, \psi$ to the other. Intuitively, this probability of non-transition (or more accurately, its square root) is measuring how far apart the states determined by $\varphi, \psi$ are from each other.

**Proposition 2.6.** Assume that Harmony One holds. Then every transformation $\tau \in \text{O}(\mathbb{C})$ naturally induces a closed linear operator, denoted by $\hat{\tau}$, acting in $\mathcal{H}$ on the not necessarily dense domain defined by
\[
\text{D}(\hat{\tau}) := \left\{ \psi(z) \in \mathcal{H} \mid \tau[\psi(z)] \in \mathcal{H} \right\}
\]
on which $\hat{\tau}$ acts by $\hat{\tau}: \psi(z) \mapsto \tau[\psi(z)]$.

**Remark 2.17.** The operator $\hat{\tau}$ is the maximal restriction of $\tau$ within $\mathcal{H}$. Its domain can be trivial, consisting of $\{0\}$ only. The most interesting situations occur when its domain is dense in $\mathcal{H}$ so that the adjoint operator $\hat{\tau}^*$ is defined in $\mathcal{H}$. Then, because of the density of $\mathcal{H}$ in $\text{H}(\mathbb{C})$ in the normal topology, $\tau$ is completely fixed by $\hat{\tau}$. This includes the maximal compatibility $\tau(\mathcal{H}) \subseteq \mathcal{H}$, in other words $\text{D}(\hat{\tau}) = \mathcal{H}$, which by the closed graph theorem, implies the boundedness of $\hat{\tau}$.

**Proof.** The closeness follows from the continuity of $\tau$ in the normal topology. The graph $G(\tau)$ of $\tau$ is closed in $\text{H}(\mathbb{C}) \oplus \text{H}(\mathbb{C})$. The natural inclusion map $\mathcal{H} \oplus \mathcal{H} \mapsto \text{H}(\mathbb{C}) \oplus \text{H}(\mathbb{C})$ is continuous. So the inverse image of $G(\tau)$, which is precisely the graph of $\hat{\tau}$, is closed in $\mathcal{H} \oplus \mathcal{H}$.

In examples many important operators $\tau$ will come from $\text{o}(\mathbb{C})$. In particular, we see that such operators include all the multiplication operators by polynomials in $z$ (i.e., by the elements in $\mathbb{C}[z]$, which includes multiplication by $z$), the derivative
operator $\partial/\partial z$ and all linear differential operators with polynomial coefficients in $z$ and also the substitution operators by polynomials $\phi(z)$.

In what follows we use the notation of the previous proposition to define the symbol $Z := \tilde{\partial}$, that is, the closed operator of multiplication by the coordinate $z$ acting in a domain of $\mathcal{H}$ defined by

$$D(Z) := \left\{ \psi(z) \in \mathcal{H} \mid z\psi(z) \in \mathcal{H} \right\}. \tag{2.35}$$

So $Z$ acts by $Z: \psi(z) \mapsto z\psi(z)$. The operator $Z$ is intrinsically related to the point wave functions $w(z)$ as we shall now explain.

Let us observe for all $w \in \mathbb{C}$ we have that

$$w(z)^{\perp} = \left\{ \psi(z) \in \mathcal{H} \mid \psi(w) = 0 \right\} = (z-w)\mathbb{C}[z], \tag{2.36}$$

where the superscript line on the right denotes the closure in the norm topology of $\mathcal{H}$. Indeed, the first equality follows from the property $\langle \psi \mid w \rangle = w\langle \psi \mid [w] \rangle$ of the point wave functions, and the second equality means that the functions from $\mathcal{H}$ vanishing at $w$ can always be arbitrarily well approximated in $\mathcal{H}$ by polynomials with the same property.

**Proposition 2.7.** For every $w \in \mathbb{C}$ its associated point function $[w] = w(z)$ belongs to the domain of the adjoint operator $Z^\ast$. In particular, $Z^\ast$ is densely defined. Moreover, $Z^\ast[w] = \bar{w}[w]$ and

$$\ker(\bar{w} - Z^\ast) = \mathbb{C}[w]. \tag{2.37}$$

The operators $Z$ and $Z^\ast$ are unbounded. Every complex number $w$ is an eigenvalue of the annihilation operator $Z^\ast$ of multiplicity one and with associated eigenvector being the normalized point wave function (equivalently, coherent state) $[\bar{w}] / \| [\bar{w}] \|$. \[\]

**Proof.** Take $w \in \mathbb{C}$. From the principal kernel equation (2.25) applied twice we find

$$\langle [w] | Z \psi \rangle = w\psi(w) = w\langle [w] | \psi \rangle = \langle \bar{w} [w] | \psi \rangle$$

for every $\psi \in D(Z)$ and every $w \in \mathbb{C}$. Therefore $[w]$ belongs to the domain of $Z^\ast$ and $Z^\ast[w] = \bar{w}[w]$. In particular $D(Z^\ast) \supseteq \mathcal{L}$ (cp. (2.28)), and so $Z^\ast$ is densely defined in $\mathcal{H}$. Since $[w] \neq 0$, it follows that $\bar{w}$ is an eigenvalue of $Z^\ast$ for every $w \in \mathbb{C}$. So the spectrum of $Z^\ast$ is $\mathbb{C}$. So $Z^\ast$ is an unbounded operator, which in turn implies that $Z$ is also an unbounded operator. In particular $D(Z) \neq \mathcal{H}$ and $D(Z^\ast) \neq \mathcal{H}$.

Having shown that every point in $\mathbb{C}$ is an eigenvalue of $Z^\ast$, we now identify its multiplicity. Using a standard identity for the adjoint operator we have

$$\ker(\bar{w} - Z^\ast)^{\perp} = \text{im}(w - Z) = [w]^{\perp},$$

where the second identity follows from (2.36). We conclude that $\ker(\bar{w} - Z^\ast) = \mathbb{C}[w]$, which shows (2.37). Now $\mathbb{C}[w]$ is a one dimensional subspace because $[w] \neq 0$. And this shows that each eigenvalue of $Z^\ast$ has multiplicity one. \[

**Remark 2.18.** There is a certain complementarity between the two basic spaces $\mathbb{C}[z]$ and $\mathcal{L}$: The space $\mathbb{C}[z]$ is $Z$-invariant and the space $\mathcal{L}$ is $Z^\ast$-invariant. In a variety of interesting examples we encounter one of the following two special configurations:

$$\mathbb{C}[z] \cap \mathcal{L} = \mathbb{C} \tag{2.38}$$

$$\mathbb{C}[z] \subseteq \mathcal{L} \tag{2.39}.$$
In general, the finite dimensional spaces $C_n[z]$ can only contain a finite number of the linearly independent point wave functions $w(z)$. (Cp. Lemma 2.1.) Hence, at most countably many of them will be in $C[z]$.

Since $Z^*$ is densely defined and $Z$ is closed, $Z^{**} = Z = Z$. So $Z$ and $Z^*$ are adjoints of each other.

**Proposition 2.8.** The subspace $L$ of $H$ is a core for the operator $Z^*$. In other words $Z^*$ is the closure of its restriction $Z^*|L: L \rightarrow L$, that is $Z^*|L = Z^*$.

**Proof.** Let us compute the orthocomplement of the graph $G(Z^*|L)$ in the graph $G(Z^*)$. It consists of the pairs $(\psi, Z^*\psi)$ for some $\psi \in D(Z^*)$ which are orthogonal to all the pairs $([w], [w])$ where $w \in C$. In other words $\langle [w]|\psi \rangle + \langle w|[Z^*\psi] \rangle = 0$. But this means $\psi(w) + w[Z^*\psi](w) = 0$. We now claim under this hypothesis on $\psi$ that $Z^*\psi \in D(Z)$. This is because $C \ni w \mapsto wZ^*\psi(w) = -\psi(w)$ is the function $-\psi \in H$. It then follows from the definition of $Z$ that $(Z[Z^*\psi](w) = wZ^*\psi(w)$ holds for all $w \in C$ for this particular vector $\psi$.

Next, this says that $\psi(w) + (Z[Z^*\psi])(w) = 0$ for all $w \in C$, and therefore $\psi$ satisfies $\psi + Z^*\psi = 0$. By functional analysis $1 + ZZ^*$ is invertible with its inverse defined on the whole $H$ and bounded. It follows that $\psi = 0$, and so the orthocomplement is the zero subspace. Thus $G(Z^*|L)$ is dense in $G(Z^*)$. \[\square\]

**Remark 2.19.** Another way to see this is by considering the adjoint of $Z^*|L$. For $\psi$ in the domain of this adjoint, that is $\psi \in D((Z^*|L)^*)$, we have that the function
\[C \ni w \mapsto w\psi(w) = \langle Z^*[w]|\psi \rangle = \langle [w]|(Z^*)^*\psi \rangle = \langle (Z|L)^*\psi \rangle(w)\]
is in $H$, which shows us that in fact $\psi \in D(Z)$ and $Z \subset (Z^*|L)^*$. And therefore $Z^* \subset (Z^*|L)^* = Z|L$. The opposite inclusion is trivial, and so $Z^* = Z|L$. Interestingly, a similar statement does not in general hold for the operator $Z$. In general $Z$ will be strictly greater than the closure of its restriction on $C[z]$.

The above proposition extends to all the powers of $Z^*|L$, as
\[(Z^*|L)^{n*} = Z_n,\]
where $Z_n: \psi(z) \mapsto z^n\psi(z)$ is the operator associated to the multiplication by $z^n$. In general, this operator will be a non-trivial extension of $Z^n$ although in many important examples $Z_n = Z^n$. We refer to [5] for an elegant and self-contained geometrically oriented exposition of the theory of unbounded operators in a Hilbert space.

As we have already mentioned, the polynomials $\phi \in C[z]$ can be viewed as unital homomorphisms of $C[z]$ into itself given for $p \in C[z]$ by the ‘substitution map’ $p \mapsto p \circ \phi$, where $\circ$ denotes composition of functions. Since every such a map is continuous in the normal topology, by Proposition 2.6 we can associate to it a closed operator $\hat{\phi}$, which is obviously densely defined, by $\psi \mapsto \psi \circ \phi$ for all $\psi \in H$ satisfying $\psi \circ \phi \in H$.

**Proposition 2.9.** Suppose that $\phi \in C[z]$. Then the domain of the adjoint operator $\hat{\phi}^*$ contains the space $L$. We have for every $w \in C$ that
\[(2.40) \quad \hat{\phi}^*([w]) = [\phi(w)].\]

In particular, the subspace $L$ is $\hat{\phi}^*$-invariant.
Thus applying the kernel equation (2.25) we find for every $\psi \in \mathcal{D}(\hat{\phi})$ that
\[
|w|\langle\hat{\phi}\psi\rangle = |w|\langle\psi(\phi)\rangle = \psi(\phi(w)) = |\langle\phi(w)\rangle\rangle = |\langle\phi\rangle\rangle|\psi|.
\]
Thus $|w| \in \mathcal{D}(\hat{\phi}^*)$ and (2.40) holds. \hfill $\square$

**Remark 2.20.** The above argument is extendible without essential change from polynomials to entire functions, understood as unital homomorphisms of $\mathcal{H}(\mathbb{C})$ into itself, continuous in the normal topology. The only subtlety is that we have to explicitly postulate the density of $\mathcal{D}(\hat{\phi})$ in $\mathcal{H}$.

The point wave functions are special case of an important class of vectors in $\mathcal{H}$, which correspond to the linear functionals on $\mathcal{H}$ continuous in the normal topology.

**Definition 2.21.** A vector $\xi \in \mathcal{H}$ is called normal if its dual vector, the linear functional $\langle\xi|: \psi \mapsto \langle\xi|\psi\rangle$, is continuous in the normal topology of $\mathcal{H}$.

We shall write $\mathcal{W}$ for the set of all normal vectors of $\mathcal{H}$. Clearly, $\mathcal{W}$ is a linear subspace of $\mathcal{H}$ and $\mathcal{L} \subseteq \mathcal{W}$. So $\mathcal{W}$ is dense in $\mathcal{H}$.

**Remark 2.22.** In the framework of Harmony One, the topological part of Harmony Zero can be rephrased as the inclusion $\mathbb{C}[z] \subseteq \mathcal{W}$, that is, the polynomials are normal vectors.

**Lemma 2.2.** If polynomials are normal, then the infinite matrix $S_{\infty}$ is invertible and
\[
S_{\infty}^{-1} = S_{\infty}^{-1}.
\]
In particular, the extended scalar product $\langle|\rangle: \mathbb{C}[z] \times \mathcal{H}(\mathbb{C}) \rightarrow \mathbb{C}$ is non-degenerate.

**Proof.** From the expansion (2.24) we find for all $k \in \mathbb{N}$ and $w \in \mathbb{C}$ that
\[
\langle z^k | w(z) \rangle = \langle z^k | \sum_{i,j \geq 0} z^i [S_{\infty}]_{ij} \bar{w}^j \rangle = \sum_{i,j \geq 0} [S_{\infty}]_{ki} [S_{\infty}^{-1}]_{ij} \bar{w}^j = \bar{w}^k,
\]
where the second equality follows from the hypothesis that polynomials are normal. And thus $S_{\infty}S_{\infty} = 1_{\infty}$. Because of the hermicity of $S_{\infty}$ and $S_{\infty}^{-1}$ we also have $S_{\infty}S_{\infty} = 1_{\infty}$. \hfill $\square$

**Proposition 2.10.** Let a transformation $\tau \in \mathcal{O}(\mathbb{C})$ be such that $\mathcal{D}(\hat{\tau}) = \mathcal{H}$, where $\hat{\tau}$ is defined in Proposition 2.4. Then $\mathcal{W} \subseteq \mathcal{D}(\hat{\tau}^*)$ and, in particular, $\hat{\tau}^*$ is densely defined. Moreover, $\hat{\tau}^*(\mathcal{W}) \subseteq \mathcal{W}$.

**Proof.** For every $\xi \in \mathcal{W}$ the map $\mathcal{D}(\hat{\tau}) \ni \psi \mapsto \langle\xi|\hat{\tau}\psi\rangle \in \mathbb{C}$, being the composition of the inclusion $\mathcal{D}(\hat{\tau})$ into $\mathcal{H}(\mathbb{C})$, the normally continuous tranformation $\tau$ and normally continuous scalar product with $\xi$, is normally continuous and in particular $\langle|\rangle$-continuous. Therefore such vectors $\xi$ are in the domain of the adjoint operator for $\hat{\tau}$. The same initial map now viewed as $\mathcal{D}(\hat{\tau}) \ni \psi \mapsto \langle\hat{\tau}^*\xi|\psi\rangle \in \mathbb{C}$ for being normally continuous, and since $\mathcal{D}(\hat{\tau})$ is dense in $\mathcal{H}$, extends by continuity to a normally continuous functional on the whole $\mathcal{H}$, given by the same formula. It follows that $\hat{\tau}^*\xi \in \mathcal{W}$. \hfill $\square$

In particular we see that $\mathcal{W} \subseteq \mathcal{D}(Z^*)$. So if Topological H0 holds, then $\mathbb{C}[z]$ is also in the domain of $Z^*$. And if in addition Algebraic H0 holds (so that we have full H0), then $Z^* \supset z^*$, where $z$ denotes the operator of multiplication by $z$ acting on $\mathbb{C}[z]$.
Proposition 2.11. If a transformation \( \tau \in \mathcal{O}(\mathbb{C}) \) satisfies \( D(\hat{\tau}) = \mathcal{H} \) (as in the previous proposition) and \( \hat{\tau}^* \) is also continuous in the normal topology of \( \mathcal{H} \), then \( \mathcal{W} \subseteq D(\hat{\tau}) \) and \( \hat{\tau}(\mathcal{W}) \subseteq \mathcal{W} \).

Proof. Since \( \hat{\tau}^* \) is normally continuous, it extends by continuity to a transformation \( \varrho \in \mathcal{O}(\mathbb{C}) \). Clearly \( \hat{\varrho} \supseteq \hat{\tau}^* \) and so \( \hat{\tau} = \hat{\varrho}^{**} \supseteq \hat{\varrho}^* \). In particular, \( D(\hat{\varrho}^*) \subseteq D(\hat{\varrho}) \). And by the previous proposition \( \mathcal{W} \subseteq D(\hat{\varrho}) \). Combining these two inclusions gives \( \mathcal{W} \subseteq D(\hat{\tau}) \). Since \( \mathcal{W} \) is \( \hat{\varrho}^* \)-invariant (again by the previous proposition), \( \mathcal{W} \) is also invariant for its extensions such as \( \hat{\tau} \). □

A Minimal Effective Operational Setting

Our next harmony condition quite naturally emerges when looking for the most effective setting for constructions involving the fundamental unbounded operators \( Z \) and \( Z^* \).

Definition 2.23. We shall say that Harmony Two (or simply H2) holds if the operator \( Z^* \) is continuous in the normal topology of \( \mathcal{H} \).

Before going further, some important observations are due. It is clear that H2 requires H1 as it builds on top of it. However, H2 is logically independent of H0.

The operator \( Z \) comes from a map in \( \mathcal{O}(\mathbb{C}) \), and hence it is in \( \mathcal{O}(\mathbb{C}) \). However, in general, the adjoint operator \( Z^* \) will not be continuous in the normal topology. Therefore, there is an inherent contextual asymmetry between the two operators \( Z \) and \( Z^* \). This symmetry is imposed by this new harmony property. In a sense, Harmony Two is a ‘grown up’ version of Algebraic H0, which provides the simplest common setting for a strictly polynomial version of symmetry between the operators \( Z \) and \( Z^* \).

For \( Z^* \) to be normally continuous it is sufficient for the restriction \( Z^*|\mathcal{L} \) to be normally continuous, since \( \mathcal{L} \) is dense in the normal topology in \( \mathcal{H} \) and so also in \( D(Z^*) \). If \( Z^* \) is normally continuous, then it uniquely extends by continuity to \( \mathcal{H}(\mathbb{C}) \), and so it is interpretable as a transformation of \( \mathcal{O}(\mathbb{C}) \). We shall use the inverted symbol \( z \) to denote this transformation \( z : \mathcal{H}(\mathbb{C}) \rightarrow \mathcal{H}(\mathbb{C}) \).

Therefore, if H2 holds, there is a naturally emerging subalgebra \( \mathcal{A} \) of \( \mathcal{O}(\mathbb{C}) \) defined as the set of all (in general, non-commutating) polynomials in \( z \) and \( \bar{z} \).

Proposition 2.12. In the framework of Harmony Two the space \( \mathcal{W} \) is both \( Z \) and \( Z^* \)-invariant. Hence, it is invariant for all the operators in \( \mathcal{A} \). The algebra \( \mathcal{A} \) is faithfully realized in \( \mathcal{W} \) and is equipped with a natural \( * \)-structure induced by the formal adjoint operation on \( \mathcal{W} \).

Proof. A direct application of Proposition 2.11 implies that \( \mathcal{W} \) is contained in the domains of \( Z \) and of \( Z^* \), and it is a common invariant subspace of these operators. When restricted to \( \mathcal{W} \) the mutual adjointness of \( Z \) and \( Z^* \) becomes the formal adjointness of their restrictions. The entire algebra \( \mathcal{A} \) is representable by restrictions in \( \mathcal{W} \). Since \( \mathcal{W} \) is norm dense in \( \mathcal{H} \) it is also normally dense in \( \mathcal{H}(\mathbb{C}) \), which means that the representation of \( \mathcal{A} \) in \( \mathcal{W} \) is faithful. □

Remark 2.24. The same argument extends to all operators \( \tau \in \mathcal{O}(\mathbb{C}) \) having densely defined \( \hat{\tau} \) whose adjoint \( \hat{\tau}^* \) is normally continuous.
The algebra $\mathcal{A}$ is highly non-commutative. One manifestation of this would be the absence of characters (⇒ one-dimensional *-representations) of $\mathcal{A}$. Quantum mechanically this means that $\mathcal{A}$, viewed as an algebra of observables, does not admit dispersion-free states. And geometrically, the interpretation is that the ‘underlying’ quantum Euclidean plane, whose ‘functions’ are the elements of $\mathcal{A}$, possesses no classical points.

Any finite linear combination of the monomials $z^k\bar{z}^l$ is called a Wick polynomial. The word ‘polynomial’ is to be understood as a polynomial expression of an element in $\mathcal{A}$ and not as an element in an abstract polynomial algebra. As we shall see two such distinct expressions can be equal.

**Proposition 2.13.** In the framework of Harmony Two, if $z$ and $z$ satisfy that $zz$ is expressible as a Wick polynomial, then the algebra $\mathcal{A}$ has no characters.

**Proof.** Let us assume to the contrary, that $\kappa: \mathcal{A} \to \mathbb{C}$ is a character and define $w := \kappa(z) \in \mathbb{C}$. (Recall that here $z \in \mathcal{A}$ is the operator of multiplication by $z$ acting on $\mathcal{H}(\mathbb{C})$.) Because $\kappa$ is a *-representation, we have $\kappa(z^k\bar{z}^l) = w^k\bar{w}^l$. Since property H2 holds, by Proposition 2.12 the point wave function $|w\rangle$ belongs to the domain of all the polynomials in $Z$ and $Z^\ast$. Moreover, for all $k, l \in \mathbb{N}$ we have

$$\langle |w\rangle |Z^kZ^l|w\rangle \rangle = \langle Z^k[w]|Z^l[w]\rangle = w^k\bar{w}^l||w||^2 = \kappa(z^k\bar{z}^l)||w||^2.$$

Write $zz = p(z, z)$, where $p$ is a Wick polynomial. By using the previous equation and linearity to get the second equality below, we then obtain

$$\langle |w\rangle |Z^*Z|w\rangle \rangle = \langle |w\rangle |p(Z, Z^*|w\rangle \rangle = \kappa(p(z, z))||w||^2 = \kappa(zz)||w||^2 = |w|^2||w||^2.$$

The last equality holds because $\kappa$ is a *-representation. Note that $|w\rangle$ is in the domain of the polynomial $Z - w$. And hence we see that

$$||z - w|w||^2 = \langle (Z - w)|w\rangle|(Z - w)|w\rangle \rangle = \langle |w\rangle |Z^*Z|w\rangle \rangle - \langle Z|w\rangle |w\rangle \rangle \rangle - \langle |w\rangle |Z|w\rangle \rangle \rangle + \langle |w\rangle |w\rangle \rangle \rangle \rangle \rangle = |w|^2||w||^2 - |w|^2||w||^2 - |w|^2||w||^2 + |w|^2||w||^2 = 0.$$

where in the two middle terms on the second line we used $Z^*|w\rangle = \bar{w}|w\rangle$. In other words $(z - w)|w\rangle = 0$ in $\mathcal{H}$, which contradicts the fact that the holomorphic function $|w\rangle$ is non-zero. $\square$

**Remark 2.25.** The ‘flipping’ condition given by a Wuck polynomial for the product $zz$ is essential. The phenomenon is illustrated, in the last part of this section by an example where $\mathcal{A}$ is an extension of the Manin $q$-plane by the infinite matrix algebra $M_{\infty}(\mathbb{C})$.

**Definition 2.26.** We shall say that Harmony Three (or simply H3) holds if both Topological H0 and H2 hold.

Harmony Three holds if and only if $1 \in \mathcal{W}$, in the framework of Harmony Two.

**Proposition 2.14.** In the framework of Harmony Three, the only eigenvectors of $z$ in $\mathcal{H}(\mathbb{C})$ are those of $Z^*$ in $\mathcal{H}$ or, in other words, the point wave functions $w(z)$ modulo scalar multiples.
Proof. Let us assume that \( \Psi(z) \in H(\mathbb{C}) \) satisfies \( z\Psi(z) = \bar{w}\Psi(z) \) for some \( w \in \mathbb{C} \). This means that \( \langle (z - w)\phi(z) | \Psi(z) \rangle = 0 \) for every \( \phi(z) \in \mathbb{C}[z] \). Here we have extended the scalar product by normal continuity to \( \mathbb{C}[z] \times H(\mathbb{C}) \). The extended \( |\rangle \) is non-degenerate. This leaves only one degree of freedom for \( \Psi(z) \). \( \square \)

**Proposition 2.15.** In the framework of Harmony Three, the commutant of \( \mathcal{A} \) in \( \Omega(\mathbb{C}) \) is trivial. In particular, the center of the algebra \( \mathcal{A} \) is trivial: \( Z(\mathcal{A}) = \mathbb{C} \).

**Proof.** If something from \( \Omega(\mathbb{C}) \) commutes with \( z \) then it must be of a multiplicative form \( \psi(z) \mapsto \Theta(z)\psi(z) \), where \( \Theta(z) \in H(\mathbb{C}) \). If \( \Theta(z) \) in addition commutes with \( z \) then \( z[\Theta(z)w(z)] = \bar{w}\Theta(z)w(z) \). Proposition 2.14 implies that \( \Theta(z)w(z) \) must be proportional to \( w(z) \) which is possible only if \( \Theta(z) \) is a constant function. \( \square \)

**Projection Construction**

The following list recapitulates the symbols used for the diverse operators related intrinsically to the complex coordinate \( z \).

\[
\begin{align*}
z & \quad \text{The multiplication operator } \psi(z) \mapsto z\psi(z) \text{ acting in } \mathbb{C}[z], \mathcal{W} \text{ or } H(\mathbb{C}). \\
\mathbb{C} \times \mathbb{Z} & \quad \text{Induced } z\text{-multiplication in } H \text{ with } D(Z) = \{ \psi(z) \in H \mid z\psi(z) \in H \}. \\
\mathbb{Z}^* & \quad \text{The adjoint operator for the operator } Z \text{ in } H. \\
z^* & \quad \text{The continuous extension of } z^* \text{ to the whole } H(\mathbb{C}). \\
\overline{z} & \quad \text{The formal adjoint of the multiplication operator } z \text{ in } \mathbb{C}[z] \text{ and } \mathcal{W}. \\
\end{align*}
\]

These symbols are all related to \( H \) and \( \mathcal{A} \). Following [10], we shall now interpret the algebra \( \mathcal{A} \) as an algebra of Toeplitz operators for the Toeplitz quantization of another, a priori unrelated, algebra, which will be assumed to be a quadratic polynomial *-algebra generated by \( z \) and \( \bar{z} \). So \( z \) is a true multi-personality creature here!

The basic example is the standard commutative *-algebra \( \mathbb{C}[z, \bar{z}] \) of polynomials in \( z \) and \( \bar{z} \). Its only defining relation is simply \( z\bar{z} = \bar{z}z \), and its *-structure is the anti-linear map which exchanges \( z \) and \( \bar{z} \). The product between \( \mathbb{C}[\bar{z}] \) and \( \mathbb{C}[z] \) induces a natural decomposition

\[
(2.42) \quad \mathbb{C}[z, \bar{z}] \leftrightarrow \mathbb{C}[\bar{z}] \otimes \mathbb{C}[z] \quad \bar{z}^nz^m \leftrightarrow \bar{z}^n \otimes z^m.
\]

A variety of interesting non-commutative variations of this emerge, if we fix a hermitian \( 2 \times 2 \) matrix

\[
Q = \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix}
\]

so that \( q_{10} = q_{01} \) and \( q_{00}, q_{11} \in \mathbb{R} \), and define \( \mathbb{C}[z, \bar{z}, Q] \) to be the *-algebra generated by \( z \) and \( \bar{z} \) together with the following relation:

\[
(2.43) \quad z\bar{z} = \bar{z}z + (1 \bar{z}) \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix} \begin{pmatrix} 1 \\ z \end{pmatrix}
\]

where also \( q_{11} \neq -1 \). The *-structure, as in the commutative case, is anti-linear and exchanges \( z \) and \( \bar{z} \). We can think of \( Q \) as being a quadratic form on \( \mathbb{C}^2 \). We recover the commutative case if \( Q = 0 \).

These non-commutative polynomials keep valid the basic decomposition (2.42) or, in other words for every \( Q \) we have an isomorphism of vector spaces, but not of algebras

\[
(2.44) \quad \mathbb{C}[z, \bar{z}, Q] \leftrightarrow \mathbb{C}[\bar{z}] \otimes \mathbb{C}[z] \quad \bar{z}^nz^m \leftrightarrow \bar{z}^n \otimes z^m
\]
by the composition-diamond lemma. (See [18].) In terms of this identification \(1 \otimes 1\) is the unit element of the algebra, and \(\mathbb{C}[z]\) and \(\mathbb{C}[\bar{z}]\) are mutually conjugate unital subalgebras of \(\mathbb{C}[z, \bar{z}, Q]\). The basis elements \(\bar{z}^n z^m\) of \(\mathbb{C}[z, \bar{z}, Q]\) are called anti-Wick monomials. Consequently, every element in \(\mathbb{C}[z, \bar{z}, Q]\) has a unique expression as a anti-Wick polynomial, that is as a finite linear combination of the anti-Wick monomials.

**Remark 2.27.** The coordinates \(\bar{z}\) and \(z\) appear symmetrically at this point of the theory, since the condition \(q_{11} \neq -1\) ensures the presence of non-zero coefficients of both \(\bar{z} z\) and \(\bar{z} z\) in the defining relation (2.43). Then the composition-diamond lemma implies that there is also an ‘opposite’ vector space isomorphism, but not an algebra isomorphism, \(\mathbb{C}[z, \bar{z}, Q] \leftrightarrow \mathbb{C}[z] \otimes \mathbb{C}[\bar{z}]\) given by \(z^n \bar{z}^m \leftrightarrow z^n \otimes \bar{z}^m\). The basis elements \(z^n \bar{z}^m\) of \(\mathbb{C}[z, \bar{z}, Q]\) are called Wick monomials. Symmetrically, every element in \(\mathbb{C}[z, \bar{z}, Q]\) is uniquely expressible as a Wick polynomial.

It follows that the product of \(\mathbb{C}[z, \bar{z}, Q]\) is encoded in the ‘flipping’ rules for monomials \(z^n\) and \(\bar{z}^m\) which can be written as finite sums

\[
(2.45) \quad z^n \bar{z}^m = \sum_{k,l \geq 0} n \hat{\otimes}_l^k \bar{z}^l z^k
\]

for an array with 4 indices \(n \hat{\otimes}_l^k\) of complex numbers which can only have non-zero values for \(0 \leq k \leq n\) and \(0 \leq l \leq m\), since the relation (2.43) has no terms of order greater than 2 in \(z\) and \(\bar{z}\). Polynomial expressions whose terms are all monomials of the form \(\bar{z}^l z^k\) as in (2.45) are said to be reduced in abstract ring theory and in anti-Wick order in quantum theory.

This array must obey a number of consistency conditions in order to represent a unital associative *-algebra structure on \(\mathbb{C}[z, \bar{z}, Q]\). At first, by conjugating the above formula we arrive at

\[
(2.46) \quad n \hat{\otimes}_k^l = m \hat{\otimes}_{nk}^{ml}.
\]

Then, the requirement for the existence of a unit translates into

\[
(2.47) \quad k \hat{\otimes}_l^0 = \delta_{nk} \delta_{ml} \quad 0 \hat{\otimes}_k^m = \delta_{nk} \delta_{ml}.
\]

More generally, \(z^n\) and \(\bar{z}^m\) commute if and only if

\[
(2.48) \quad k \hat{\otimes}_l^n = \delta_{nk} \delta_{ml}
\]

for all \(k, l \geq 0\). Finally, the following charming and easy to remember convolution identities hold, where \(\times\) denotes the product of complex numbers:

\[
(2.49) \quad u^+ v^+ \hat{\otimes}_k^l = \sum_{a + b = k} u^a \hat{\otimes}_s^a \times b \hat{\otimes}_s^m \quad u^+ v^+ \hat{\otimes}_k^l = \sum_{a + b = l} n \hat{\otimes}_a^n \times b \hat{\otimes}_b^v.
\]

They reflect the fact that both \(\mathbb{C}[z]\) and \(\mathbb{C}[\bar{z}]\) are unital subalgebras of \(\mathbb{C}[z, \bar{z}]\). In particular, by inductively applying these identities, we see that the array \(n \hat{\otimes}_k^l\) is completely determined by \(Q\) because

\[
(2.50) \quad 1 \hat{\otimes}_0^0 = q_{00} \quad 1 \hat{\otimes}_1^0 = q_{01} \quad 1 \hat{\otimes}_0^1 = q_{10} \quad 1 \hat{\otimes}_1^1 = 1 + q_{11}
\]

which is the same as the basic generating relation (2.43).

In order to clarify different roles of the same space, we shall write \(Q\) for \(\mathbb{C}[z, \bar{z}, Q]\) as a vector space on which the same algebra acts via its left regular representation. In a similar manner, we shall write \(P\) for \(\mathbb{C}[z]\) understood as the corresponding
representation space. Clearly $\mathcal{P}$ is a subalgebra of $\mathcal{Q}$. Also, every element in $\mathcal{Q}$ can be written uniquely as a finite sum of polynomials homogeneous in $\bar{z}$, namely as $\sum_n a_n z^n \varphi_n$ with $a_n \in \mathbb{C}$ and $\varphi_n \in \mathcal{P}$. Singling out the subalgebra $\mathcal{P} = \mathbb{C}[z]$ for special consideration, instead of $\mathbb{C}[\bar{z}]$ which we could have done, breaks the previously mentioned symmetry between $z$ and $\bar{z}$.

We shall assume here that H3 holds for $\mathcal{H}$.

**Proposition 2.16.** There exists a unique quadratic form $\langle | \rangle$ on $\mathcal{Q}$ which extends the scalar product on $\mathcal{P}$ and such that the left regular representation of $\mathbb{C}[z, \bar{z}, \mathcal{Q}]$ on $\mathcal{Q}$ is symmetric with respect to that form. It is determined by

$$\langle z^n \varphi \bar{z}^m \psi \rangle = \sum_{k,l \geq 0} \hat{n} \hat{m} (Z^l \varphi | Z^k \psi)$$

for $n, m \in \mathbb{N}$ and $\varphi, \psi \in \mathcal{P}$.

**Proof.** Let us suppose that $\langle | \rangle$ exists. Then

$$\langle z^n \varphi \bar{z}^m \psi \rangle = \langle \varphi | z^n \bar{z}^m \psi \rangle = \sum_{k,l \geq 0} \hat{n} \hat{m} \langle \varphi | z^l \bar{z}^k \psi \rangle = \sum_{k,l \geq 0} \hat{n} \hat{m} \langle Z^l \varphi | Z^k \psi \rangle$$

so indeed (2.51) holds and hence $\langle | \rangle$ must be unique. To prove its existence, we define it on $\bar{z}$-homogenous polynomials by (2.51), extending it to $\mathcal{Q} \times \mathcal{Q}$ by using bi-additivity. From (2.46) it follows that $\langle | \rangle$ extends $\langle | \rangle$. It is also hermitian symmetric because of (2.46)

$$\langle z^n \varphi \bar{z}^m \psi \rangle^* = \sum_{k,l \geq 0} \hat{n} \hat{m} \langle Z^k \psi | Z^l \varphi \rangle = \sum_{k,l \geq 0} \hat{m} \hat{n} \langle Z^l \varphi | Z^k \psi \rangle = \langle z^n \varphi \bar{z}^m \psi \rangle.$$

So we have a quadratic form on $\mathcal{Q}$. We next have to prove that $z$ and $\bar{z}$ are mutually formally adjoint with respect to $\langle | \rangle$. Again, we verify this on polynomials homogeneous in $\bar{z}$ by using (2.45), (2.49) and (2.51):

$$\langle z^n \varphi \bar{z}^m \psi \rangle = \sum_{k,l \geq 0} \hat{n} \hat{m}^{a+b} \langle Z^l \varphi | Z^k \psi \rangle = \sum_{k,s,a,b \geq 0} \{ \hat{n}^{a+b} \times \hat{m} \} \langle Z^{a+b} \varphi | Z^k \psi \rangle = \sum_{k,s,a,b \geq 0} \{ \hat{s} \hat{a} \hat{n} \hat{b} \} \langle Z^{a+b} \varphi | Z^k \psi \rangle = \sum_{a,s \geq 0} \hat{a} \hat{s} \langle z^s \bar{z}^a \varphi | \bar{z}^m \psi \rangle = \langle z^n \varphi | \bar{z}^m \psi \rangle.$$

So $z$ and $\bar{z}$ are formally adjoint relative to $\langle | \rangle$. Since they generate $\mathbb{C}[z, \bar{z}, \mathcal{Q}]$ this property extends to the whole polynomial algebra. In other words we get that

$$\langle F \varphi | \psi \rangle = \langle \varphi | \overline{F} \psi \rangle$$

for every $F \in \mathbb{C}[z, \bar{z}, \mathcal{Q}]$ and $\varphi, \psi \in \mathcal{Q}$.

**Remark 2.28.** The form $\langle | \rangle$ will not be positive in general. The positivity problem for the form, in the context of the commutative algebra $\mathbb{C}[z, \bar{z}]$, relates to the moment problem — the existence of a finite measure $\mu$ on $\mathbb{C}$ reproducing the scalar product via the standard $L^2$ inner product

$$\langle f | g \rangle = \int_C \bar{f} g \, d\mu(z, \bar{z}).$$

It is easy to see that if a measure $\mu$ on $\mathbb{C}$ reproduces the scalar product $\langle | \rangle$ on $\mathcal{P}$, then it automatically reproduces the extended form $\langle | \rangle$ on $\mathbb{C}[z, \bar{z}]$, and in particular it will be positive (possibly not strictly positive). On the other hand there exist interesting examples where $\langle | \rangle$ is strictly positive, even though there will be no
underlying measure. As we shall see, such exotic situations do not occur if the scalar product \( \langle \rangle \) on \( P \) is diagonal.

In resonance with this, we can introduce an integration functional for the algebra \( \mathbb{C}[z, \bar{z}, Q] \). We shall use the same generic integration symbol as for \( A \), and similarly to (2.21) define

\[
\int F = \langle 0|F\rangle. \ \forall F \in \mathbb{C}[z, \bar{z}, Q].
\]

(Recall the Dirac notation is \( |0\rangle = \phi_0(z) = 1 \in \mathbb{C}[z, \bar{z}, Q] \). This is a hermitian functional evaluating to \( s_{nm} = \langle z^n|z^m \rangle \) on the monomials \( \bar{z}^n z^m \). Moreover,

\[
\langle \rho|\phi\rangle = \int \rho \phi
\]

for every \( \rho, \phi \in \mathbb{C}[z, \bar{z}, Q] \).

**Proposition 2.17.** The following conditions are equivalent:

— The space \( P \) is orthocomplementable in \( Q \) or in other words there exists a subspace \( P^\perp \) such that

\[
Q = P \oplus P^\perp.
\]

— There exists a linear map \( \Pi: Q \to Q \) satisfying

\[
\Pi^2 = \Pi \ \ \text{im}(\Pi) = P \ \ \langle \Pi \psi|\varphi\rangle = \langle \psi|\Pi \varphi\rangle
\]

for every \( \varphi, \psi \in Q \).

— Algebraic H0 holds. In other words \( P \) is \( Z^* \)-invariant and \( Z^*P = z^* \) the formal adjoint of \( z \) in \( P \).

If any of these (and hence all of these) hold, then \( \ker(\Pi) = P^\perp \) and in particular \( \Pi \) is the projection associated to the orthogonal decomposition of \( Q \). Moreover, there is a commutative diagram

\[
\begin{array}{ccc}
Q & \xrightarrow{\bar{z}} & Q \\
\uparrow & & \downarrow \Pi \\
P & \xrightarrow{z^*} & P,
\end{array}
\]

where the left vertical arrow is the inclusion map and the top horizontal arrow is multiplication by \( \bar{z} \).

**Proof.** If (2.56) holds, then \( \Pi \) is defined to be the associated projection on \( P \) and (2.57) is immediate. On the other hand if (2.57) holds, then the kernel of \( \Pi \) is an orthocomplement of \( P \) and so (2.56) holds.

If \( \Pi \) exists as in (2.57), we define \( z^* \) by (2.58). Then for \( \varphi, \psi \in P \) we have

\[
\langle \varphi|z^*\psi\rangle = \langle \varphi|\Pi\bar{z}\psi\rangle = \langle \Pi\varphi|\bar{z}\psi\rangle = \langle \varphi|\bar{z}\psi\rangle = \langle z\varphi|\psi\rangle = \langle z\varphi|\psi\rangle.
\]

So the operator \( z^* \) is indeed the formal adjoint of \( z \) in \( P \) and so Algebraic H0 holds.

Finally, assume Algebraic H0. So the operator \( z^* \) exists (i.e., \( Z^* \) preserves \( P \)). Then we can define \( \Pi \) on \( \bar{z} \)-homogeneous polynomials by

\[
\Pi(z^n\psi) = z^n\psi \in P
\]
for $\psi \in \mathcal{P}$ and $n \in \mathbb{N}$, and extend it to $\mathcal{Q}$ by additivity. By construction $\Pi$ is an idempotent projecting on $\mathcal{P}$. It is also symmetric. Indeed if $\varphi$ is also from $\mathcal{P}$, then
\[
\langle \Pi(\bar{z}^n \varphi) | \bar{z}^m \psi \rangle = \langle \bar{z}^m z^n \varphi | \psi \rangle = \langle \bar{z}^m \varphi | z^m \psi \rangle =
\]
and by bi-additivity the identity extends to the whole $\mathcal{Q}$. Therefore all the identities in (2.57) have been proved.

Here is an explicit way to calculate the projection map $\Pi$ in terms of the canonical orthonormal polynomials $\phi_n(z) \in \mathbb{C}[z] \subset \mathbb{C}[z, \bar{z}, \mathcal{Q}]$.

**Proposition 2.18.** Assume that $\Pi$ exists as in Proposition 2.17. We then have for every $F \in \mathbb{C}[z, \bar{z}, \mathcal{Q}]$ that
\[
(2.60) \quad \Pi(F) = \sum_{k=0}^{\infty} \langle \phi_k(z) | F\varphi_k(z) \rangle,
\]
There are only finitely many non-zero terms in this infinite sum.

**Proof.** For $F \in \mathcal{P} = \mathbb{C}[z]$ equation (2.60) reduces to saying that the polynomials $\phi_n(z)$ are an orthonormal basis of the vector space $\mathbb{C}[z]$, in which case only finitely many terms are non-zero. For $F \in \mathcal{P}^\perp = \ker(\Pi)$ the left side of (2.60) is zero and every term on the right side is also zero since $\phi_n(z) \in \mathcal{P}$. Then (2.60) follows for every $F \in \mathcal{P} \oplus \mathcal{P}^\perp = \mathbb{C}[z, \bar{z}, \mathcal{Q}]$ by additivity.

**Remark 2.29.** In Dirac notation (which technically does not apply, since we are not in a Hilbert space setting) we can write (2.60) as
\[
(2.61) \quad \Pi = \sum_{k=0}^{\infty} |k\rangle \langle k|. \tag{2.61}
\]
The quadratic form $\langle | \rangle$ will fail in general to be non-degenerate. In other words, we might encounter a non-trivial null-space $\mathcal{N} = \mathcal{Q}^\perp$. This next result is immediate.

**Proposition 2.19.** The null-space $\mathcal{N}$ is an invariant subspace for the left regular representation of $\mathbb{C}[z, \bar{z}, \mathcal{Q}]$ in $\mathcal{Q}$. Moreover $\Pi$, if it exists, maps $\mathcal{N}$ into $\{0\}$. Thus, the whole module structure, quadratic form and $\Pi$ naturally project down to the factor space $\mathcal{R} = \mathcal{Q}/\mathcal{N}$, preserving all the basic formulas. The projected $\langle | \rangle$ is non-degenerate. We have $\mathcal{P} \cap \mathcal{N} = \{0\}$, and so $\mathcal{P}$ is naturally a subspace of $\mathcal{R}$.

**Definition 2.30.** For any $f \in \mathbb{C}[z, \bar{z}, \mathcal{Q}]$ we define the *Toeplitz operator with symbol* $f$, denoted as $T(f) = T_f : \mathcal{P} \to \mathcal{P}$, by $T_f \phi := \Pi(f \phi)$ for all $\phi \in \mathcal{P}$. Notice that the product $f \phi$ of the two elements $f, \phi \in \mathbb{C}[z, \bar{z}, \mathcal{Q}]$ is again an element in the algebra $\mathbb{C}[z, \bar{z}, \mathcal{Q}]$. Then the projection $\Pi$ maps this product to an element of $\mathcal{P}$. In this way we obtain a linear map $T : \mathbb{C}[z, \bar{z}, \mathcal{Q}] \to \mathcal{A}$, the *-algebra of operators generated by $z$ and $z^*$. We say that $T$ is the *Toeplitz quantization*.

Even though $T$ is a linear map from one algebra to another, it is not expected nor desired to be multiplicative, that is, a map of algebras. We will come back to this point. However, $T$ does preserve the identity element, namely $T_1$ is the identity operator of $\mathcal{P}$.

As a direct consequence of the above definition of $T$, we find that
\[
(2.62) \quad T(z^n \bar{z}^m) = z^n \bar{z}^m.
\]
So the recipe to calculate the operators $T_f$ is quite simple and can be used as an alternative definition of $T$. Just replace $z$ and $\bar{z}$ by their counterparts $z$ and $z^*$ in the polynomial expression for $f$, assuming that $f$ is written in anti-Wick form, that is, all $\bar{z}$’s are moved to the left of $z$’s. This is the reason for saying that Toeplitz quantization is an anti-Wick quantization. Equation (2.62) works out so nicely in part because the definition of $z^*$ in diagram (2.58) now reads as $T(\bar{z}) = z^*$. On the other hand the identity $T(z) = z$ follows immediately from the fact that $\mathcal{P} = \mathbb{C}[z]$ is $z$-invariant. Equation (2.62) also shows that all the operators $T_f$ are indeed in $\mathcal{A}$.

**Remark 2.31.** In general, $T$ will not be surjective. Its image is the $\mathbb{C}[\bar{z}]$-$\mathbb{C}[z]$ bimodule in $\mathcal{A}$ generated by $1 \in \mathcal{A}$. So $\mathcal{A}$ is always generated by the image of $T$. In this context, we shall refer to $\mathcal{A}$ as the algebra of Toeplitz operators.

If we trivially extend the operators in $\mathcal{P}$ to $\mathbb{R}$ by requiring that they vanish on $\mathcal{P}^\perp$, then we can write

$$T_f = \Pi f \Pi$$

where on the right side $f$ is interpreted as the left regular representation operator. The expression on the right side of (2.63) is known as the compression by the projection $\Pi$ of the operator defined by $f$.

We see that $T$ intertwines the $*$-structures on $\mathbb{C}[z, \bar{z}, Q]$ and $\mathcal{A}$. It also connects the integration functionals on both algebras.

**Proposition 2.20.** We have

$$\int T = \int.$$

**Proof.** It is sufficient to check the identity on the monomials $\bar{z}^n z^m$. And then both the left and the right hand side evaluate to $\langle z^n | z^m \rangle = s_{nm}$. □

**Lemma 2.3.** For every $\rho, \phi \in \mathcal{P}$ and $n \in \mathbb{N}$ we have

$$\langle \bar{z}^n \rho (\bar{z} - z^*) \phi \rangle = \langle \rho | T(z^n \bar{z}) \phi \rangle - \langle \rho | z^n z^* \phi \rangle.$$

In particular, the map $T$ is multiplicative only in the trivial scenario $\mathbb{R} = \mathcal{P}$.

**Proof.** We compute

$$\langle \bar{z}^n \rho (\bar{z} - z^*) \phi \rangle = \langle \bar{z}^n \rho \bar{z} \phi \rangle - \langle \bar{z}^n \rho z^* \phi \rangle = \langle \rho | z^n \bar{z} \phi \rangle - \langle \rho | z^n z^* \phi \rangle = \langle \rho | T(z^n \bar{z}) \phi \rangle - \langle \rho | z^n z^* \phi \rangle.$$

We see that $T(z^n \bar{z}) = z^n z^*$ if and only if $\bar{z}$ acts as $z^*$ on $\mathcal{P}$. Since $\mathcal{P}$ is cyclic for $\bar{z}$ in $\mathbb{R}$ this is only possible when the two spaces coincide. □

**Remark 2.32.** Therefore, the complement $\mathcal{P}^\perp$ of $\mathcal{P}$ in $\mathbb{R}$ can be viewed as a subtle measure of difference between $\mathbb{C}[z, \bar{z}, Q]$ and $\mathcal{A}$. It is also worth observing that $(\bar{z} - z^*) \phi \in \mathcal{P}^\perp$ always.
The Fifth Element

To sum things up so far, our construction produces a canonical \( \mathbb{C}[z, \bar{z}, Q] \) representation space \( \mathcal{R} \) extending \( \mathcal{P} \) and equipped with a non-degenerate hermitian form \( \langle \cdot | \cdot \rangle \). If Algebraic H0 holds, there is a symmetric idempotent \( \Pi \) projecting \( \mathcal{R} \) onto \( \mathcal{P} \) and realizing the operators \( z \) and \( \bar{z} \) of \( \mathbb{C}[z, \bar{z}, Q] \) by \( \Pi \). This is our main algebro-geometric setting for the Toeplitz quantization.

The whole construction can be performed without essential changes with \( \mathcal{W} \) instead of \( \mathcal{P} \). The advantage is that \( \mathcal{W} \) is always invariant under both \( z \) and \( \bar{z} \), as far as we stay within Harmony Two. In this case the space \( \mathcal{Q} \) is redefined as

\[
(2.66) \quad \mathcal{Q} = \mathbb{C}[z, \bar{z}, Q] \otimes_{\mathbb{C}[z]} \mathcal{W} \leftrightarrow \mathbb{C}[\bar{z}] \otimes \mathcal{W}.
\]

This brings us naturally to our final harmony property. It is about a mutual relationship between the two principal algebras \( \mathbb{C}[z, \bar{z}, Q] \) and \( \mathcal{A} \), related via the Toeplitz quantization.

Definition 2.33. We shall say that Harmony Four (or simply H4) holds if the form \( \langle \cdot | \cdot \rangle \) is positive.

In the framework of H4 the space \( \mathcal{R} \) is a pre-Hilbert space and can be completed into a Hilbert space \( \mathcal{J} \). Moreover, \( \mathcal{H} \) can be viewed as a subspace of \( \mathcal{J} \). The projection \( \Pi \) extends to an orthogonal projection \( \Pi : \mathcal{J} \rightarrow \mathcal{H} \), where we use the same notation.

Remark 2.34. This is all much in the spirit of the Stinespring construction [21] for completely positive maps between C*-algebras.

Classification

The algebra \( \mathbb{C}[z, \bar{z}, Q] \) falls into one of four distinguished classes, corresponding to four canonical forms of \( Q \), obtained after making a linear substitution \( z \mapsto az + b \) where \( a, b \in \mathbb{C} \) and \( a \neq 0 \). Under such a substitution the matrix \( Q \) transforms as

\[
(2.67) \quad \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix} \mapsto \begin{pmatrix} q(\bar{b}, b)/|a|^2 & (q_{01} + q_{11}\bar{b})/a \\ (q_{10} + q_{11}b)/a & q_{11} \end{pmatrix}
\]

where \( q(\bar{b}, b) = q_{00} + q_{01}b + q_{10}\bar{b} + q_{11}|b|^2 \) as follows from the defining relation (2.43).

Let us briefly describe these classes.

Principal Flipping Type. This is defined by \( q = 1 + q_{11} \neq 0, 1 \). After the appropriate linear substitution, we arrive at

\[
(2.68) \quad z\bar{z} = q\bar{z}z + h \quad h \in \{-|q|, 0, 1\} \quad Q = \begin{pmatrix} h & 0 \\ 0 & q_{11} \end{pmatrix}.
\]

There are 6 important special subcases. The first one is given by \( h = 0 \). This is the Manin \( q \)-plane. A detailed analysis of this example and the general theory of its Toeplitz quantization can be found in [10].

The classical part of the corresponding quantum space (given by the characters of \( \mathbb{C}[z, \bar{z}, Q] \)) is just one point—the character evaluating to 0 on \( z \) and \( \bar{z} \). An important part of the analysis is the question of realizability of the algebra by operators in a Hilbert space, the case \( q > 0 \) is realizable, and we can always assume \( 0 < q < 1 \) since if \( q > 1 \) then by interchanging \( z \) and \( \bar{z} \) we have \( q \mapsto 1/q \). As we shall explain
below, there is a natural realization of this algebra in a Hilbert space of holomorphic functions in $\mathbb{C} - \{0\}$. The case $q < 0$ is, clearly, not realizable by operators in a Hilbert space.

Next, there are two essentially different cases with $q > 0$ and $h = -q, 1$. If $0 < q < 1$ and $h = -q$ or $q > 1$ and $h = 1$ then we obtain equivalent realizations of a $q$-variant of the standard quantum plane. If, on the other hand, $0 < q < 1$ and $h = 1$ or equivalently $q > 1$ and $h = -q$, then the algebra will represent a Poincaré model for a quantum hyperbolic plane [9]. The horizon of infinity is traced by a classical circle centered at $0$ of radius $\sqrt{h/1 - q}$. These can be morphed into

$$\bar{z}z = qz\bar{z} + 1 - q$$

with $0 < q < 1$, to fit the unitary disk $\mathbb{D}$.

If $q$ is negative, there are also two essentially different situations. The first one is given by $h = q$ and corresponds to a character-free $*$-algebra non-realizable in a Hilbert space. If $h = 1$ then we are again within the quantum hyperbolic planes, the structure can be morphed into the same generating expression as above, but now with $-1 < q < 0$.

**Parabolic Type.** This corresponds to $q_{11} = 0$ with $q_{01} = \frac{q_{00}}{q_{10}} \neq 0$, which can be transformed into

$$(2.69) \quad \bar{z}z = z\bar{z} + z + \bar{z} \quad Q = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

The classical points are naturally labeled by purely imaginary numbers. The generators $z$ and $\bar{z}$ linearly span a non-commutative 2-dimensional Lie algebra. Abstractly, there is only one such a structure and one concrete realization is the (complexified) Lie algebra of the group $t \mapsto at + b$ where $a, b \in \mathbb{R}$ with $a > 0$. These are orientation preserving affine transformations of $\mathbb{R}$. The algebra $\mathbb{C}[z, \bar{z}, Q]$ is then viewable as the universal envelope of this Lie algebra.

**Orthodox Quantum Plane.** This corresponds to $q_{11} = q_{01} = q_{10} = 0$ and $q_{00} \neq 0$. In this case $q_{00}$ can be scaled to $-1$ or $1$ and we obtain the Heisenberg-Weyl algebra

$$(2.70) \quad z\bar{z} - \bar{z}z = \pm 1 \quad Q = \pm \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$  

**Classical Euclidean Plane.** The classical commutative polynomial algebra $\mathbb{C}[z, \bar{z}]$ is obtained when $q_{11} = q_{01} = q_{10} = q_{00} = 0$.

**Diagonal Scalar Product**

In this subsection we will assume that the initial scalar product is diagonal, in which case all calculations significantly simplify. As we have explained already, property $H0$ always holds in this context. Also, recall the definition (2.12) of the sequence $s_n := s_{nn} = \langle z^n | z^n \rangle > 0$, which characterizes $\langle | \rangle$ in this case.

**Lemma 2.4.** In terms of the sequence $s_n$, the quadratic form $\langle | \rangle$ is determined by

$$(2.71) \quad \langle \bar{z}^i z^j | \bar{z}^n z^m \rangle = \sum_{k \geq 0} \frac{i^k}{\sqrt{n+1}} \frac{n!}{k!} \frac{n^k}{k!} s_{k+m}.$$
Proof. This is straightforward from the definitions of \( s_n \) and the 4-array \( \hat{\Diamond} \):

\[
\langle z^i \bar{z}^j | z^n \bar{z}^m \rangle = \sum_{k,l \geq 0} i^{k+l} s_{j+l} \delta_{j+l,k+m} \sum_{k \geq 0} i^{k} n^{k+m} s_k \delta_{k,m-j}.
\]

We have also used the fact that the \(^*\)-operation exchanges \( z \) and \( \bar{z} \).

\( \square \)

Remark 2.35. In particular the quadratic form for the commutative polynomial algebra \( \mathbb{C}[z, \bar{z}] \) is determined by

\[
\langle z^i \bar{z}^j | z^n \bar{z}^m \rangle = s_i + m \delta_{i,j} - m, \tag{2.72}
\]

where we used (2.48). This also follows easily by direct verification without appealing to the above Lemma. Notice even in this highly simplified case that the vector space basis \( \bar{z}^n z^m \) of \( \mathbb{C}[z, \bar{z}] \) is not orthogonal. Formula (2.72) with \( s_n = n! \) is readily available in the setting of Bargmann’s seminal paper [3], even though it is not explicitly given there.

Let us now discuss Harmony One and Two. As for the interpretability of all the elements of \( \mathcal{H} \) as entire functions, this is not always possible. It requires a special asymptotic behavior of the \( (\langle \rangle) \) defining sequence \( s_n \).

Proposition 2.21. A necessary and sufficient condition for Harmony One is that

\[
\lim_{n \to \infty} s_n^{1/n} = +\infty. \tag{2.73}
\]

In this case the reproducing kernel for \( \mathcal{H} \) is given by

\[
K(\bar{w}, z) = \sum_{n \geq 0} \frac{\bar{w}^n z^n s_n}{s_n}. \tag{2.74}
\]

Proof. The formula (2.74) for the reproducing kernel is a special case of (2.23) with \( \phi_n(z) \) given by (2.13). Let us now apply Proposition 2.4. The series (2.29) becomes

\[
\sum_{n=0}^{\infty} |\phi_n(z)|^2 = \sum_{n=0}^{\infty} \frac{|z|^2 s_n}{s_n}.
\]

For this to be normally convergent on \( \mathbb{C} \) a necessary and sufficient condition is indeed (2.73), as the Cauchy-Hadamard formula reveals. Next, the \( \infty \)-linear independence always holds here, because all power series encode the values of their coefficients.

\( \square \)

Proposition 2.22. A necessary and sufficient condition for Harmony Two, in the framework of Harmony One, is that the sequence

\[
t_n = \left( \frac{s_{n+1}}{s_n} \right)^{1/n} \tag{2.75}
\]

be bounded.

Proof. If \( Z^* \) is normally continuous by H2, then by using (2.13) and (2.16) we have

\[
\sum_{n \geq 0} c_n z^n \longrightarrow \sum_{n \geq 0} (c_{n+1} s_{n+1}/s_n) z^n.
\]
consistently defines the extension of $Z^*$ on the whole $\mathbb{H}(\mathbb{C})$. This means that the radius of convergence for the resulting power series on the right must always be infinite, that is, the same as the radius of convergence of the initial power series on the left. This property can be rephrased in terms of the sequences as follows. For any sequence of complex numbers $r_n$ converging to 0, the sequence $r_n t_n$ must also converge to zero. This means that the sequence $t_n$ belongs to the multiplier algebra of the algebra $C_0(\mathbb{N})$ of all sequences having limit zero. This multiplier algebra is precisely all bounded sequences $B(\mathbb{N})$. So $t_n$ must be bounded.

Conversely, it is a matter of a direct verification that the boundedness of $t_n$ implies that the above formula defines a (necessarily unique) continuous extension of $Z^*$ on the whole $\mathbb{H}(\mathbb{C})$. □

**Proposition 2.23.** A particularly effective scenario for the applicability of the above criterion occurs when

\[
\lim s_{n+1}/s_n = +\infty \quad \lim s_n s_{n+2}/s_{n+1}^2 < +\infty.
\]

**Proof.** A direct application of the classical convergence criterion. If $s_{n+1}/s_n$ is convergent then $s_n^{1/n}$ is convergent with the same limit. Similarly if the second sequence is convergent then $t_n$ will be convergent with the same limit. □

**The Two Frameworks Intersection**

Let us now elaborate more on the special situation, interesting in its own light, where the quadratic algebra $\mathbb{C}[z, \bar{z}, Q]$ is itself viewable as the Toeplitz operator algebra $A$ for some Hilbert space of entire functions $\mathcal{H}$ satisfying Harmony Two.

The Heisenberg-Weyl algebra can be viewed in this way, via the Segal-Bargmann space, where $s_n = n!$ and $K(\bar{w}, z) = \exp(\bar{w}z)$. (See [3].) Another class of examples is given by the $q$-variation

\[
z\bar{z} = q\bar{z}z - q \quad 0 < q < 1
\]

of the Heisenberg-Weyl algebra. Indeed, if we define the scalar product by the sequence

\[
s_n = q^k \left(1 + 1/q + \cdots + 1/q^{k-1}\right)
\]

then

\[
s_{n+1}/s_n = 1 + 1/q + \cdots + 1/q^n \sim \infty \quad \frac{s_n s_{n+2}}{s_{n+1}^2} = \frac{1 - q^{n+2}}{q 1 - q^{n+1}} \sim \frac{1}{q}
\]

and thus $\mathcal{H}$ holds. A direct calculation then reveals that $z \mapsto Z$ and $\bar{z} \mapsto Z^*$ extends to an isomorphism between $\mathbb{C}[z, \bar{z}, Q]$ and $A$.

Furthermore, it turns out that the corresponding reproducing kernel is given by

\[
K(\bar{w}, z) = E_q(\bar{w}z)
\]

where $E_q$ is the $q$-exponential function. Let us recall ([3]–Appendix II) that the $q$-exponential function is defined as

\[
E_q(z) = (qz - z|q)_\infty = \sum_{n=0}^{\infty} (1 - q)^n (q^n)_n q^{[2]} z^n = \sum_{n=0}^{\infty} \frac{z^n}{q(n)}
\]
where
\[(a|q)_n = \prod_{k=0}^{n-1} (1 - q^k a) \quad (a|q)_\infty = \prod_{k=0}^{\infty} (1 - q^k a). \tag{2.81}\]

It is also worth recalling ([2]—Chapter 10, Section 2) the second Euler $q$-identity
\[(a|q)_n = (a|q)_\infty = \prod_{k=0}^{\infty} (1 - q^k a) \tag{2.82}\]
valid for $|q| < 1$ and all $z \in \mathbb{C}$.

Let us now calculate the induced metric on $\mathbb{C}$. Applying the formula (2.32) we obtain
\[ds^2 = \sum_{n=0}^{\infty} \frac{(1 - q)n(z|z)n}{(1 + (1 - q)n|z|^2)^2}. \tag{2.83}\]

With the help of a $q$-logarithm function
\[\log_q(1 + z) = (1 - q) \sum_{n \geq 0} \frac{z^q^n}{1 + z^q^n} \tag{2.84}\]
where the first formula is valid for arbitrary $z \in \mathbb{C}$ and the second in the unit disk $|z| < 1$, the metric can also be expressed as
\[ds^2 = \log_q'(1 + (1 - q)|w|^2) \, dw \, d\bar{w}. \tag{2.85}\]

We see that in the limit $q \to 1^-$ this reproduces the classical Euclidean metric, and the first quantum correction looks elliptic
\[ds^2 \approx (1 - \frac{1}{1 + q}|w|^2) \, dw \, d\bar{w} \quad |w|^2 \ll \frac{1}{1 - q}. \tag{2.86}\]

We present more formulae related to these spaces in Appendix B.

**Proposition 2.24.** Modulo equivalence transformations of the quadratic form $Q$, the only algebras $\mathbb{C}[z, \bar{z}, Q]$ realizable as a Toeplitz operators algebra $A$ are precisely the Heisenberg-Weyl algebra and its above described $q$-variants.

**Proof.** As we have just explained, the Heisenberg-Weyl algebra and $q$-variations are realizable as some $A$. To complete the proof, let us observe that the *-algebra $A$ can not possess characters, in accordance with Proposition 2.13. This effectively excludes all other algebras $\mathbb{C}[z, \bar{z}, Q]$. \qed

Let us now consider the Manin $q$-plane
\[z \bar{z} = q \bar{z} z \quad 0 < q < 1. \tag{2.87}\]

The formula (2.71) for the quadratic form \[\langle \rangle\] simplifies into
\[\langle z^i \bar{z}^j | z^n \bar{z}^m \rangle = q^{in} s_{i+m} \delta_{i-j, n-m} \tag{2.88}\]
a $q$-deformed version of (2.72).

There is only one classical point here. It is the unique character evaluating to 0 on $z$ and $\bar{z}$, which refers to the center 0 of the classical plane. The algebra therefore can not be realized within our principal framework. But if we remove 0
from consideration and allow Laurent series with singularity in 0 as constituents of
our Hilbert space \( H \), then a faithful representation is possible.

Let us consider the space \( \mathcal{R} \) of generalized polynomials with all integer powers of
\( z \), and define the scalar product by requiring mutual orthogonality of the monomials
\( z^n \) and also
\[
\langle z^n | z^n \rangle = q^{-\binom{n}{2}}
\]
for all \( n \in \mathbb{Z} \). If we interpret \( z \) as the multiplication operator by the coordinate
\( z \) and \( \overline{z} \) as its formal adjoint, then the above non-commutation relation for the Manin
\( q \)-plane is fulfilled, and we have a faithful representation.

This space \( \mathcal{R} \) closes into a Hilbert space \( \mathcal{J} \) of holomorphic functions over \( \mathbb{C} - \{0\} \).
Its reproducing kernel is given by
\[
K(\overline{w}, z) = \sum_{n \in \mathbb{Z}} q^{\binom{n}{2}} (\overline{w}z)^n.
\]

Let us recall ([2]–Chapter 10, Section 4) the classical triple product identity
\[
\sum_{n \in \mathbb{Z}} (-1)^n q^{\binom{n}{2}} z^n = (z|q)_\infty (q/z|q)_\infty (q|q)_\infty
\]
which holds for \( z \neq 0 \) and \( |q| < 1 \). Applying this to our reproducing kernel, we
obtain
\[
K(\overline{w}, z) = (-\overline{w}z|q)_\infty (-q/\overline{w}z|q)_\infty (q|q)_\infty.
\]

The induced metric on \( \mathbb{C} - \{0\} \) is calculated by applying (2.32) to this infinite
product. The first two symbols on the right side of (2.92) are transformed by the
logarithm into infinite sums over \( \mathbb{N} \), which after applying the derivation \( \partial^2/\partial w \partial \overline{w} \)
merge into a single sum over \( \mathbb{Z} \). The third symbol \( (q|q)_\infty \) does not contribute to
the metric, as it is a multiplicative constant transformed by \( \log \) into an additive
constant. Explicitly, we get
\[
ds^2 = \sum_{n \in \mathbb{Z}} q^n \frac{dw d\overline{w}}{(1 + q^n \overline{w}w)^2}.
\]

We see that the unique classical point represented by 0 acts as a true geometrical
singularity, as the metric diverges when \( |w| \) tends to 0. On the other hand, far away
from 0 the metric becomes asymptotically Euclidean.

Let \( \Pi : \mathcal{J} \rightarrow \mathcal{J} \) be the orthogonal projector on the positive part, the Hilbert
space \( \mathcal{H} \) generated by the standard polynomials \( \mathbb{C}[z] \). Clearly \( \Pi(\mathcal{R}) = \mathcal{P} \) and in
such a way, for the sequence \( s_n \) given by the positive part of (2.89) we obtain a
non-commutative system satisfying Harmony Three. We can see this explicitly from
\[
\frac{s_{n+1}}{s_n} = q^n \quad \frac{s_{n+1}^2}{s_n s_{n+2}} = q.
\]

This construction, which went ‘backwards’ relative to our main considerations,
ensures that Harmony Four holds, too. The space \( \mathcal{H} \) consists of all entire functions
of \( \mathcal{J} \) with the reproducing kernel given by
\[
K(\overline{w}, z) = \sum_{n \geq 0} q^{\binom{n}{2}} (\overline{w}z)^n.
\]
The induced metric is hence
\[(2.95)\]
\[ds^2 = \sum_{n \geq 0} \frac{q^n \, dw \, d\bar{w}}{(1 + q^n \bar{w}w)^2} = \log_q'(1 + |w|^2) \frac{dw \, d\bar{w}}{1 - q}.\]

The operators $Z$ and $Z^*$ satisfy the following non-commutation relation
\[(2.96)\]
\[ZZ^* = qZ^*Z - q|0\rangle\langle 0|.\]

From this it is easy to see that the whole matrix algebra $M_\infty(\mathbb{C})$ is included in $\mathcal{A}$, and that the above relation is in fact a generating relation for $\mathcal{A}$. The following short exact sequence holds:
\[(2.97)\]
\[0 \to M_\infty(\mathbb{C}) \to \mathcal{A} \to \mathbb{C}[z, \bar{z}, Q] \to 0.\]

So, basically, this Toeplitz quantization smooths out the singularity at $0$. However, the change appears ‘mild’: The classical point at $0$ remains as the unique character of $\mathcal{A}$. This is a counterexample for the possible removal of the generators condition in Proposition 2.13.

It is also interesting to observe that the constructed non-commutative system can be viewed as the Toeplitz quantization of the classical Euclidean plane in the sense that harmony property $H4$ holds in this context, too. This follows from the positivity analysis for the Stieltjes moment condition presented in Appendix A together with this explicit calculation of the determinant generated by the full defining sequence:
\[(2.98)\]
\[\det \begin{vmatrix} q^{-\frac{(i+j)}{2}} \end{vmatrix} = q^{-\frac{\binom{n}{2}}{2}} \frac{4n^5}{3} \prod_{k=1}^{n-1} (1 - q^k)^{n-k}.\]

Here the integer indexes $i, j$ run from $0$ to $n - 1$.

A different Toeplitz quantization of the Manin $q$-plane is obtained if we choose the sequence (2.78). Then, as explained in Appendix B property $H4$ holds, too. It is interesting to observe that one and the same non-commutative system is viewed as a quantization of the classical plane as well as of the Manin $q$-plane.

To complete our analysis of different types of quadratic algebras $\mathbb{C}[z, \bar{z}, Q]$, let us focus on the parabolic type given by (2.69). Although the framework of entire functions does not work here, the algebra admits an interesting realization within a Hilbert space of holomorphic functions in the positive half-plane $\Re(z) > 0$. Let us fix a positive number $h$. There exists a unique scalar product on the polynomial algebra $\mathbb{C}[z]$ such that the constant function $1 = |0\rangle$ has unit norm and
\[(2.99)\]
\[z^*|0\rangle = \frac{h}{2}|0\rangle.\]

If we define (using the Euler gamma function $\Gamma$) the polynomials as the shifted factorials
\[(2.100)\]
\[\xi_{\ell}(z) = (z + \frac{h}{2})_{\ell} \quad (z)^n := \prod_{j=0}^{n-1} (z + j) = \frac{\Gamma(z + n)}{\Gamma(z)},\]

then it is easy to see that
\[(2.101)\]
\[z^*\xi_{\ell}(z) = (\frac{h}{2} + n)\xi_{\ell}(z).\]
for every \( n \in \mathbb{N} \). Clearly \(|0\rangle = \xi_0(z)\) and these vectors form a basis of \( \mathbb{C}[z] \) with \( \mathbb{C}[z] = \text{span}\{\xi_0(z), \xi_1(z), \ldots, \xi_n(z)\} \). Furthermore, a direct calculation reveals
\[
\langle \xi_n(z) | \xi_m(z) \rangle = (-1)^n \frac{h^n}{(h/2 - z)^n}.
\]

Orthonormalization of these polynomials by the Gram-Schmidt procedure gives for the canonical orthonormal basis
\[
| n \rangle = \phi_n(z) = \prod_{j=0}^{n-1} \frac{z - j - h/2}{\sqrt{(j + 1)(h + j)}} = \frac{(-1)^n}{n!} \frac{h^n}{(h/2 - z)^n}.
\]

By using the Weierstrass product formula and Stirling asymptotic formula
\[
\frac{1}{\Gamma(z)} = z^{\gamma z} \prod_{n \geq 0} \left\{ (1 + \frac{z}{n}) e^{-z/n} \right\} \quad \Gamma(x) \sim \sqrt{2\pi} x^{x-1/2} e^{-x} \quad x \to +\infty
\]
it is easy to see that the kernel diagonal series is normally convergent on the half-plane \( \Re(z) > 0 \). The polynomials \( \phi_n(z) \) are also \( \infty \)-independent, so the conditions for Harmony One are here—although in this quite different context of the positive half-plane. Actually, the kernel can be explicitly summed to
\[
K(\bar{w}, z) = \frac{\Gamma(h)\Gamma(\bar{w} + z)}{\Gamma(\bar{w} + h/2)\Gamma(z + h/2)}.
\]

This follows directly from the classical Gauss summation formula
\[
\sum_{n \geq 0} \frac{(a)_n(b)_n}{n! (c)_n} = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}
\]
with \( a = h/2 - \bar{w} \), \( b = h/2 - z \) and \( c = h \). The induced metric \ref{eq:metric} on the positive half-plane is especially simple, namely
\[
ds^2 = dw d\bar{w} \times F'(\bar{w} + w)
\]
where \( F(z) = \Gamma'(z)/\Gamma(z) = [\log \Gamma(z)]' \) is the digamma function. Then taking into account the expansion around zero
\[
F(z) = -\frac{1}{z} - \gamma - \sum_{n \geq 1} (-1)^n \zeta(n + 1) z^n
\]
as well as the classical positive half-plane hyperbolic metric \( ds^2 = dz d\bar{z} / \Re(z)^2 \), we conclude that the quantum metric asymptotically behaves like the classical hyperbolic metric at the limit of the ‘infinity horizon’ of space represented by imaginary numbers. Not surprisingly, as a matter of fact, this ‘geometrical heaven’ is also interpretable as the classical part of this quantum space as has been observed in our initial purely algebraic analysis.

3. SOME ALGEBRAIC ASPECTS OF TOEPLITZ QUANTIZATION

In this section we shall describe an abstract algebraic underlying structure for the Toeplitz quantization. In particular, this will provide the foundations to fine-tune our principal construction of the extended space \( \mathcal{R} \) equipped with the quadratic form \( \langle \cdot | \cdot \rangle \) and the projection \( \Pi \), to Hilbert spaces of entire functions and harmony conditions, where it is natural to use the more elaborate module \( \mathcal{W} \), instead of polynomials.

Let us assume that \( \mathcal{P} \) is an everywhere dense linear subspace of a Hilbert space \( \mathcal{H} \). Let \( \mathcal{C} \) be a unital subalgebra of the *-algebra \( M(\mathcal{P}) \) of formally adjointable
linear operators in $\mathcal{P}$. Then the conjugate algebra $\bar{\mathcal{C}}$ is also of the same category. Let $\mathcal{A} \subseteq M(\mathcal{P})$ be the $^*$-subalgebra generated by $\mathcal{C}$ and $\bar{\mathcal{C}}$.

The space $\mathcal{P}$ can be viewed as a left $\mathcal{A}$-module, and in particular as both left $\mathcal{C}$-module as well as $\bar{\mathcal{C}}$-module. The algebra $\mathcal{A}$ can be purely algebraically described as the opposite algebra of $\mathcal{C}$ equipped with the conjugate vector space structure. In this interpretation, the $^*$-operation between $\mathcal{C}$ and $\bar{\mathcal{C}}$ is just the identity map. It naturally extends to $^*$-involutions on the vector spaces $\mathcal{C} \otimes \bar{\mathcal{C}}$ and $\bar{\mathcal{C}} \otimes \mathcal{C}$ by

\[(\alpha \otimes \bar{\beta})^* = \beta \otimes \bar{\alpha} \quad (\bar{\alpha} \otimes \beta)^* = \bar{\beta} \otimes \alpha.\]

In a similar way, the $^*$-operation extends to arbitrary tensor products of $\mathcal{C}$ and $\bar{\mathcal{C}}$.

Let $\mathcal{A} \subseteq \mathcal{M}(\mathcal{P})$ be the $^*$-subalgebra generated by $\mathcal{C}$ and $\bar{\mathcal{C}}$. The space $\mathcal{P}$ can be viewed as a left $\mathcal{A}$-module, and in particular as both left $\mathcal{C}$-module as well as $\bar{\mathcal{C}}$-module. The algebra $\bar{\mathcal{C}}$ can be purely algebraically described as the opposite algebra of $\mathcal{C}$ equipped with the conjugate vector space structure.

In this interpretation, the $^*$-operation between $\mathcal{C}$ and $\bar{\mathcal{C}}$ is just the identity map. It naturally extends to $^*$-involutions on the vector spaces $\mathcal{C} \otimes \bar{\mathcal{C}}$ and $\bar{\mathcal{C}} \otimes \mathcal{C}$ by

\[(\alpha \otimes \bar{\beta})^* = \beta \otimes \bar{\alpha} \quad (\bar{\alpha} \otimes \beta)^* = \bar{\beta} \otimes \alpha.\]

In a similar way, the $^*$-operation extends to arbitrary tensor products of $\mathcal{C}$ and $\bar{\mathcal{C}}$.

In particular, it mixes $\mathcal{C} \otimes \mathcal{C}$ and $\bar{\mathcal{C}} \otimes \bar{\mathcal{C}}$. Let $m : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ and $\bar{m} : \bar{\mathcal{C}} \otimes \bar{\mathcal{C}} \to \bar{\mathcal{C}}$ be the corresponding product maps. Clearly, we have this commutative diagram:

\[
\begin{array}{ccc}
\mathcal{C} \otimes \mathcal{C} & \xrightarrow{m} & \mathcal{C} \\
* & \downarrow & * \\
\bar{\mathcal{C}} \otimes \bar{\mathcal{C}} & \xrightarrow{\bar{m}} & \bar{\mathcal{C}}
\end{array}
\]

Let us assume that a linear map $\sigma : \mathcal{C} \otimes \bar{\mathcal{C}} \to \bar{\mathcal{C}} \otimes \mathcal{C}$ is given such that the diagram

\[
\begin{array}{ccc}
\mathcal{C} \otimes \bar{\mathcal{C}} & \xrightarrow{\sigma} & \bar{\mathcal{C}} \otimes \mathcal{C} \\
* & \downarrow & * \\
\mathcal{C} \otimes \bar{\mathcal{C}} & \xrightarrow{\sigma} & \bar{\mathcal{C}} \otimes \mathcal{C}
\end{array}
\]

is commutative. Furthermore, let us assume that the following pentagonal diagrams

\[
\begin{array}{ccc}
\mathcal{C} \otimes \mathcal{C} \otimes \bar{\mathcal{C}} & \xrightarrow{m \otimes \text{id}} & \mathcal{C} \otimes \bar{\mathcal{C}} \\
\downarrow \sigma & & \downarrow \sigma \\
\bar{\mathcal{C}} \otimes \mathcal{C} \otimes \bar{\mathcal{C}} & \xleftarrow{\text{id} \otimes \bar{m}} & \bar{\mathcal{C}} \otimes \bar{\mathcal{C}} \otimes \mathcal{C}
\end{array}
\]

are commutative too.

**Remark 3.1.** It is easy to see that the commutativity of each of these pentagonal diagrams implies that of the other, if we assume the conjugational symmetry (3.3).

The pentagonal diagrams as such, ensure that the space $\bar{\mathcal{C}} \otimes \mathcal{C}$ can be equipped with a natural associative product, such that

\[(\gamma \otimes \alpha)(\bar{\beta} \otimes \delta) = \gamma \sigma(\alpha \otimes \bar{\beta})\delta\]

and $1 \otimes 1$ is the unit element of this algebra. This implies that always $\sigma(1 \otimes 1) = 1 \otimes \alpha$ and $\sigma(1 \otimes \beta) = \beta \otimes 1$. In particular $\sigma(1 \otimes 1) = 1 \otimes 1$. The conjugational symmetry then is equivalent to the statement that the introduced $^*$ is the $^*$-structure on this algebra. Indeed, if $\sigma(\alpha \otimes \beta) = \sum_k \bar{\alpha}_k \otimes \beta_k$ then (3.3) is equivalent to

\[
\sigma(\beta \otimes \bar{\alpha}) = \sum_k \bar{\beta}_k \otimes \alpha_k \quad \leftrightarrow \quad [(1 \otimes \alpha)(\bar{\beta} \otimes 1)]^* = (1 \otimes \beta)(\bar{\alpha} \otimes 1),
\]

which in turn ensures that the $^*$-operation on $\bar{\mathcal{C}} \otimes \mathcal{C}$ is anti-multiplicative. Let us denote by $\mathcal{B}$ the resulting $^*$-algebra. By construction both $\mathcal{C}$ and $\bar{\mathcal{C}}$ are subalgebras of $\mathcal{B}$, and they generate $\mathcal{B}$ with the commutation rule given by $\sigma$. 

\[
\begin{array}{ccc}
\mathcal{C} \otimes \mathcal{C} & \xrightarrow{m} & \mathcal{C} \\
* & \downarrow & * \\
\bar{\mathcal{C}} \otimes \bar{\mathcal{C}} & \xrightarrow{\bar{m}} & \bar{\mathcal{C}}
\end{array}
\]
Let us now define an extended representation space
\[(3.6) \quad Q = B \otimes_{c} P \leftrightarrow \bar{C} \otimes P.\]
Because of the above identification, the space \(Q\) is a left \(B\)-module, in a natural way: the module structure is induced simply by left multiplication. By construction, the space \(P\) is naturally viewable as a \(C\)-submodule of \(Q\). The inclusion map is given by \(P \ni \psi \mapsto 1 \otimes \psi \in Q\). So \(P\) is cyclic for the \(B\)-module \(Q\).

The left \(\bar{C}\)-module structure on \(P\) is naturally expressed via the projection map \(\Pi: Q \rightarrow Q\) given by
\[(3.7) \quad \Pi(\bar{\alpha} \otimes \psi) = 1 \otimes (\bar{\alpha} \psi).\]
Clearly, the image of the idempotent \(\Pi\) is \(P\) and \(\Pi\) is \(\bar{C}\)-linear.

Lemma 3.1. The formula
\[(3.8) \quad \langle \bar{\alpha} \otimes \varphi | \bar{\beta} \otimes \psi \rangle = \sum_{k} \langle \beta_{k} \varphi, \alpha_{k} \psi \rangle\]
where \(\sigma(\alpha \otimes \bar{\beta}) = \sum_{k} \bar{\beta}_{k} \otimes \alpha_{k}\), defines a sesqui-linear form on \(Q\) which restricted on \(P\) reproduces the original scalar product. With respect of this form, the left \(B\)-module structure is symmetric.

Proof. Remember that \(\sigma(1 \otimes 1) = 1 \otimes 1\) so \(\langle 1 \otimes \varphi | 1 \otimes \psi \rangle = \langle \varphi, \psi \rangle\). Furthermore since \(\sigma(\beta \otimes \bar{\alpha}) = \sum_{k} \bar{\alpha}_{k} \otimes \beta_{k}\) so
\[\langle \bar{\beta} \otimes \psi | \bar{\alpha} \otimes \varphi \rangle = \sum_{k} \langle \alpha_{k} \psi, \beta_{k} \varphi \rangle = \left[\sum_{k} \langle \beta_{k} \varphi, \alpha_{k} \psi \rangle\right]^{*} = \langle \bar{\alpha} \otimes \varphi | \bar{\beta} \otimes \psi \rangle^{*}\]
and hence the hermitian symmetry of \(\langle | \rangle\). In order to prove that the action of \(B\) is symmetric, it is sufficient to check it for \(C\), or equivalently \(\bar{C}\). If \(c \in C\) then
\[\langle c \bar{\alpha} \otimes \varphi | \bar{\beta} \otimes \psi \rangle = \sum_{k} \langle u_{k} \varphi, v_{k} \psi \rangle = \langle \bar{\alpha} \otimes \varphi | c(\bar{\beta} \otimes \psi) \rangle^{*} \sum_{k} u_{k} \otimes v_{k} = \sigma(\alpha c \otimes \bar{\beta})\]
and we have applied the pentagonal symmetry for \(\sigma\) in twisting the product \(\alpha c\) with \(\bar{\beta}\).

Lemma 3.2. The projection \(\Pi\) is symmetric relative to \(\langle | \rangle\).

Proof. Remembering that \(\sigma\) classically flips \(1\) with everything, we obtain
\[\langle \Pi(\bar{\alpha} \otimes \varphi) | \bar{\beta} \otimes \psi \rangle = \langle 1 \otimes \bar{\alpha} \varphi | \bar{\beta} \otimes \psi \rangle = \langle \beta \bar{\alpha} \varphi | \psi \rangle\]
\[= \langle \bar{\alpha} \varphi | \bar{\beta} \psi \rangle = \langle \bar{\alpha} \otimes \varphi | \Pi(\bar{\beta} \otimes \psi) \rangle^{*}.\]
So \(\Pi\) is indeed symmetric. □

Let us now consider the null space of \(\langle | \rangle\). In general, it will be a non-trivial subspace of \(Q\). It is defined by
\[\mathcal{N} = \left\{ \eta \in Q \mid \langle \eta | Q \rangle = \{0\} \right\}.\]

Lemma 3.3. The space \(\mathcal{N}\) is \(B\)-invariant and moreover
\[\Pi(\mathcal{N}) = \{0\} = P \cap \mathcal{N}.\]
Proof. The invariance property is a direct consequence of the symmetry of the action of $\mathcal{B}$ with respect to $\langle|\rangle$. Since $\Pi$ is symmetric, it also follows that $\mathcal{N}$ is $\Pi$-invariant. On the other hand, $\Pi$ projects onto $\mathcal{P}$, and there $\langle|\rangle$ reduces to the initial strictly positive scalar product. □

Consequently, the whole structure naturally projects on the factor module

$$\mathcal{R} = \mathcal{Q}/\mathcal{N}$$

over the algebra $\mathcal{B}$. We shall use the same symbols $\langle|\rangle : \mathcal{R} \times \mathcal{R} \to \mathbb{C}$ and $\Pi : \mathcal{R} \to \mathcal{R}$ for the projected objects. Clearly, the projected $\langle|\rangle$ is non-degenerate and $\mathcal{P}$ is a $\mathcal{C}$-submodule of $\mathcal{R}$. Because of the symmetry of $\Pi$ we can write

$$\mathcal{R} = \mathcal{P} \oplus \mathcal{P}^\perp,$$

and $\mathcal{P}^\perp$ is also a $\mathcal{C}$-submodule of $\mathcal{R}$. To every element of $\mathcal{B}$ we can associate a linear operator from $\mathcal{A}$ via the symbol construction:

$$\mathcal{B} \ni b \mapsto \Pi b \Pi : \mathcal{P} \to \mathcal{P}.$$

Explicitly, if $b = \sum_k \bar{\alpha}_k \otimes \beta_k$, then simply

$$\Pi b \Pi \leftrightarrow \sum_k \bar{\alpha}_k \beta_k,$$

and in particular this is an element of $\mathcal{A}$.

Remark 3.2. These elements in (3.12) span the $\mathcal{C}$-$\mathcal{C}$ bimodule in $\mathcal{A}$ generated by $1 \in \mathcal{A}$. In general, this will be strictly smaller than $\mathcal{A}$.

The construction described here can be reversed, thereby giving its own abstract characterization, which is very much in resonance with the Stinespring construction (see [21]), which itself generalizes the GNS construction to the level of appropriate completely positive maps. We can start from the twisted product algebra $\mathcal{B} \leftrightarrow \mathcal{C} \otimes \mathcal{C}$ as above, represented by formally adjointable operators in a linear space $\mathcal{R}$ equipped with a not necessarily positive regular scalar product $\langle|\rangle$.

Let $\mathcal{P}$ be a linear subspace of $\mathcal{R}$ satisfying the following four structural properties:

- It is positive, in the sense that $\langle|\rangle$ reduces to a strictly positive scalar product on $\mathcal{P}$;
- It is orthocomplementable in $\mathcal{R}$ in the sense that
  $$\mathcal{R} = \mathcal{P} \oplus \mathcal{P}^\perp$$
  with respect to $\langle|\rangle$;
- It is $\mathcal{C}$-invariant;
- It is cyclic, in the sense that
  $$\mathcal{C}\mathcal{P} = \left\{ \sum c \psi \mid c \in \mathcal{C}, \psi \in \mathcal{P} \right\} = \mathcal{R}.$$

Proposition 3.1. Under the above assumptions, the entire system is naturally isomorphic to the one constructed above. □
4. Concluding Thoughts

Our Toeplitz quantization construction provides a kind of resonant bridge going between two, in principle very different, algebras. On one side we have a given polynomial quadratic *-algebra \( \mathbb{C}[z, \bar{z}, Q] \) generated by abstract symbols \( z \) and \( \bar{z} \). On the other there is a concrete non-commutative *-algebra of operators \( \mathcal{A} \), acting within a Hilbert space \( \mathcal{H} \) of entire functions, possessing a dense common invariant subspace \( \mathcal{W} \), and also viewable as transformations of the whole \( \mathcal{H}(\mathbb{C}) \), generated by the multiplication operator \( z \) and its companion \( \bar{z} \), interpretable as the adjoint of \( z \) in terms of \( \mathcal{H} \).

There is an extended space \( \mathcal{R} \), which is equipped with a non-necessarily positive non-degenerate scalar product \( \langle | \rangle \), on which the *-algebra \( \mathbb{C}[z, \bar{z}, Q] \) symmetrically acts. The common domain \( \mathcal{W} \) for the operators of \( \mathcal{A} \) is isometrically realized as an orthocomplementable subspace of \( \mathcal{R} \). Then \( \mathbb{C}[z, \bar{z}, Q] \) is morphed into \( \mathcal{A} \) with the help of the orthogonal projection \( \Pi: \mathcal{R} \rightarrow \mathcal{W} \). In this sense \( \mathcal{A} \) is interpreted as a Toeplitz quantization of \( \mathbb{C}[z, \bar{z}, Q] \).

At a first sight, it might come as a surprise that we can perform the construction on two a priori unrelated algebras such as \( \mathbb{C}[z, \bar{z}, Q] \) and \( \mathcal{A} \)—coming from two apparently quite different worlds.

On one hand, we believe that this reflects the flexibility and versatility of the construction, which as we have seen admits an elegant purely algebraic foundation. And after all, behind both algebras lies an intuitive idea of a quantum object resembling the classical Euclidean plane. On the other hand, it is natural to look for some additional geometrical context, in which the relationship between \( \mathbb{C}[z, \bar{z}, Q] \) and \( \mathcal{A} \) looks especially harmonic.

One such a context is given by our last harmony condition, which requires that \( \langle | \rangle \) be positive. As we have already mentioned, in the case of the standard commutative product in \( \mathbb{C}[z, \bar{z}] \) the positivity condition leads to its moment problem of finding a finite measure on \( \mathbb{C} \) reproducing the scalar product. Such a measure can always be taken to be radial for diagonal scalar products.

Our principal series of examples, as in [10], also comes from a diagonal scalar product on \( \mathbb{C}[z] \), defined by a positive sequence \( s_n = s_{nn} \) of squares of norms of mutually orthogonal \( z^n \). There is a natural generalization, which includes as a special case the parabolic quadratic algebra considered in Section 2, and which is also invariant under Euclidean transformations of \( \mathbb{C} \). If we require that the polynomials of the canonical orthonormal basis satisfy

\[
\phi_n(z) | \phi_{n+1}(z)
\]

for every \( n \in \mathbb{N} \), that is to say, that zeroes of \( \phi_n(z) \) are included (multiplicities counted) in zeroes of \( \phi_{n+1}(z) \), the whole scalar product is encoded in two sequences. These are a sequence of positive numbers \( s_n \) and a sequence of complex numbers \( \lambda_n \) so that

\[
\phi_n(z) = \frac{1}{\sqrt{s_n}}(z - \lambda_1) \cdots (z - \lambda_n)
\]

or equivalently

\[
z^n = \sum_{k=0}^{n} \sqrt{s_k} u_{n-k}(\lambda_1, \cdots, \lambda_{k+1}) \phi_k(z)
\]
sums its arguments and of degree \( u \)

The parabolic quadratic algebra example corresponds to \( \lambda_n = n-1+h/2 \). The algebraic part of Harmony Zero will be always satisfied, as \( Z^* \mathbb{C}_n[z] \subseteq \mathbb{C}_n[z] \) for every \( n \in \mathbb{N} \).

We have analyzed in this paper the ‘one-particle’ scenario, where there is only one complex variable \( z \) to quantize. In a very similar way, everything is extendible to several complex variables. The methods should also be extendible to classical subdomains in \( \mathbb{C}^n \). Here we would expect a natural emergence of non-Euclidean quantum geometries in resonance with the results of \cite{9}, where quantum hyperbolic planes are studied, and the interpretation of Poincaré models linked with the Hilbert spaces of analytic functions on the open unitary disk \( \mathbb{D} \).

Another very interesting and promising topic for future explorations is to consider multi-dimensional quantum Euclidean spaces defined via the described quantization, which preserve some important symmetries and develop differential geometry on such spaces. It is tempting to proceed in the spirit of \cite{11, 12}, where it was revealed that classical structures with discrete symmetries exhibit a ‘hidden’ quantum personality, which is describable in the elegant geometrical language of quantum principal bundles.

**Appendix A. Positivity and Stieltjes Moment Condition**

In this Appendix we shall discuss in more detail the positivity property for the quadratic form \( \langle | \rangle \), induced on the classical polynomial \(*\)-algebra \( \mathbb{C}[z, \bar{z}] \) by a diagonal scalar product \( \langle | \rangle \) on \( \mathbb{C}[z] \) with the defining sequence \( s_n \) given in (2.12).

The geometry of \( \mathbb{C}[z, \bar{z}] \) with such a \( \langle | \rangle \) is reflected in the accompanying figure. We start with an integer lattice. The white nodes correspond to the monomials \( z^k \bar{z}^l \) with \( k, l \geq 0 \). These monomials are a basis in the commutative \(*\)-algebra \( \mathbb{C}[z, \bar{z}] \). By allowing negative powers of \( z \) and \( \bar{z} \) such that \( k+l \geq 0 \), this algebra is included in an extended commutative \(*\)-algebra \( \mathbb{E}[z, \bar{z}] \). Such monomials are represented by grey nodes. To each node \( a = (k, l) \) we define its value to be the number \( |a| := k+l \).

In terms of this picture the scalar product \( \langle | \rangle \) is computed using (2.72) according to the following rule: if the nodes labeled by \( a \) and \( b \) share one of the parallel diagonal lines, then the scalar product is \( s_{(|a|+|b|)/2} \). Otherwise the monomials are orthogonal. This immediately reveals that \( \langle | \rangle \) will be positive if and only if its restriction to all of the diagonal subspaces is positive.

But the diagonal subspaces are of two canonical types: those consisting of nodes with even value and those consisting of nodes with odd value. So the positivity condition reduces to the positivity of the corresponding Gram matrices,

\[
\begin{pmatrix}
0 & s_1 & \cdots & s_n \\
 s_1 & s_2 & \cdots & s_{n+1} \\
 \vdots & \vdots & \ddots & \vdots \\
 s_n & s_{n+1} & \cdots & s_{2n}
\end{pmatrix}
\]

of even and odd types respectively. But this is exactly the Stieltjes moment condition for the existence of a measure \( \mu \) on \([0, \infty)\) satisfying

\[
s_n = \int_0^\infty t^n \, d\mu(t) \quad \forall n \in \mathbb{N}.
\]
And this is equivalent to the existence of a rotationally invariant measure $d\mu(z, \bar{z})$ on $\mathbb{C}$ reproducing the scalar product on $\mathbb{C}[z, \bar{z}]$ in the standard way:

$$\langle f | g \rangle = \int_{\mathbb{C}} f(z)g(z) d\mu(z, \bar{z}) = 0 = \langle f | \bar{f} \rangle.$$ 

Interestingly, we can look at all of this another way around and use the construction to prove the Stieltjes moment condition. Indeed, as is easily seen from the figure, the positivity of $\langle f | g \rangle$ on $\mathbb{C}[z, \bar{z}]$ is equivalent to the positivity of $\langle f | g \rangle$ on $E[z, \bar{z}]$. And such an extended positivity is equivalent, according to the theorem of Stochel–Szafraniec [17], to the existence of an underlying measure for the quadratic form. This measure, due to the rotational symmetry of the system, can be always taken to be purely radial.

**Appendix B. On Hilbert Spaces of Holomorphic Functions**

**Reproducing Kernels as Point Functionals**

We shall here review some general properties of Hilbert spaces of holomorphic functions, which are particularly relevant for our main considerations on Toeplitz quantization.

We refer to [1] for a lovely self-contained exposition of the theory of Hilbert function spaces, including those consisting of holomorphic functions. In dealing with these structures, it is optimal to assume that everything occurs in a mathematical universe in which linear functionals defined over Hilbert spaces are always continuous. This leads to interesting foundational issues in Functional Analysis, involving the axiom of choice, automatic continuity and constructibility of objects.
We refer to [16], especially Chapter 6 of the book, for an in-depth discussion of these conceptual roots.

Let \( \mathcal{H} \) be a Hilbert space consisting of holomorphic functions over a domain \( \Omega \). So \( \mathcal{H} \) is a complex vector subspace of \( \text{Hol}(\Omega) \). The structure of such a space is completely determined by its reproducing kernel function \( K : \Omega \times \Omega \rightarrow \mathbb{C} \), which in the main text we have met in the context \( \Omega = \mathbb{C} \) of entire functions. The primary properties we mentioned in that special context are valid for a general domains. In particular, the kernel is holomorphic in its second argument and anti-holomorphic in its first argument. And for every \( w \in \Omega \) we can define \( [w] = w(z) = K(\bar{w}, z) \) to be the corresponding ‘point wave function’. These functions belong to \( \mathcal{H} \) and reproduce the values of the functions from \( \mathcal{H} \) in the points \( w \) via the scalar product \( \langle w(z), \psi(z) \rangle = \psi(w) \). So the scalar product between two point wave functions \([v], [w]\) is the reproducing kernel of the points \( v, w \in \Omega \) taken in the opposite order \( K(\bar{w}, v) = \langle v(z)|w(z) \rangle = [w](v) = \overline{K(\bar{v}, w)} \).

If \( \Lambda \) is any subset of \( \Omega \) possessing an accumulation point in \( \Omega \), then the point wave functions \( \lambda(z) \) with \( \lambda \in \Lambda \) generate the whole Hilbert space \( \mathcal{H} \) or, in other words, the orthocomplement to all of them is \( \{0\} \).

And if \( \Lambda \) is a finite non-empty subset of \( \Omega \) we can construct a quadratic matrix \( K[\Lambda] \) whose entry \( (v, w) \) is \( K(\bar{w}, v) \), where now \( w, v \in \Lambda \). Such a matrix is always positive. It is the Gram matrix for the system of vectors \( \lambda(z) \) with \( \lambda \in \Lambda \). It turns out that conversely, any bi-anti-holomorphic function \( K \) on \( \Omega \times \Omega \) satisfying the positivity condition for its all matrices \( K[\Lambda] \) generates a Hilbert space of holomorphic functions.

We can naturally extend the notion of wave point function to include derivative operators. For every \( w \in \Omega \) and \( n \in \mathbb{N} \) let us define a function \( [w^n] = [w^n](z) \in \mathcal{H} \) by

\[
\langle [w^n]\psi \rangle = \frac{\partial^n \psi}{\partial z^n}(w) \quad \forall \psi \in \mathcal{H}.
\]

This makes sense since the right side is a norm continuous functional of \( \psi \in \mathcal{H} \). So we have included the original point wave functions as \( [w^0] = [w] \). In terms of the reproducing kernel we have

\[
[w^n](z) = \frac{\partial^n K}{\partial \bar{w}^n}(\bar{w}, z)
\]

and the classical Cauchy formula can be expressed as

\[
[w^n] = \frac{n!}{2\pi i} \oint \frac{dc}{(c-w)^{n+1}},
\]

where the integral is over a positively oriented simple curve in \( \Omega \) around \( w \). From the definition of these higher order point wave functions it follows that

\[
\langle [c^n][w^n] \rangle = \frac{\partial^{n+m}}{\partial \bar{w}^n \partial c^m} K(\bar{w}, c).
\]

**Lemma B.1.** For a given \( w \in \Omega \) the functions \( \{[w^n] : n \in \mathbb{N}\} \) span a dense linear subspace in \( \mathcal{H} \).

**Proof.** For a function from \( \mathcal{H} \), being orthogonal to all \( [w^n] \) with fixed \( w \) means vanishing at \( w \), together with all its derivatives. Because of the connectedness of \( \Omega \), such a holomorphic function must be zero identically. \( \square \)
Lemma B.2. For a given $w \in \Omega$ the functions $\{[w n] : n \in \mathbb{N}\}$ admit a biorthogonal system if and only if $\mathbb{C}[z] \subset \mathcal{H}$. In this case the corresponding biorthogonal system is given by $\{z - w)^n/n! : n \in \mathbb{N}\}$.

Proof. If $b_m(z)$ are functions biorthogonal to $[w n]$ then
\[
\langle [w n] | b_m \rangle = \delta_{nm} = \left. \frac{\partial^n}{\partial z^n} b_m(z) \right|_{z=w}
\]
and by expanding in Taylor series around $w$ we obtain $b_m(z) = (z - w)^m/m!$. In particular $\mathbb{C}[z] \subset \mathcal{H}$. Conversely, if polynomials are included in $\mathcal{H}$ then $b_m(z)$ defined by the same formula are clearly the biorthogonal system for $[w n](z)$. \[\square\]

We see that the inclusion of the polynomials $\mathbb{C}[z]$ in $\mathcal{H}$ is a natural structural property of $\mathcal{H}$. However, the space $\mathbb{C}[z]$ is not necessarily dense in $\mathcal{H}$. We can proceed in the same spirit here, as in the context of entire functions, and consider the array of harmony properties, related to the operator form of the complex coordinate $z$, and its adjoint operator, introduce the common invariant subspace $W$ consisting of normal vectors, and obtain a non-commutative $*$-algebra $\mathcal{A}$ which itself can be viewed as a form of quantization of the domain $\Omega$, and also as the algebra of the Toeplitz operators for the Toeplitz quantization of appropriate quadratic algebras. Here it is natural to assume the maximality of $\mathcal{H}$ relative to $\mathcal{H}$ in the sense that there is no common analytic extension beyond $\Omega$, valid for all the elements of $\mathcal{H}$. Of course, when $\Omega = \mathbb{C}$ this condition is trivial.

Segal-Bargmann Space and $q$-Deformations

Let us first consider the Hilbert space $L^2(\mathbb{C}, \rho)$ where $\rho(z) = \exp(-|z|^2)/\pi$. The Segal-Bargmann space then consists of all entire functions belonging to the above Hilbert space. The polynomials in $z$ are everywhere dense, and the monomials in $z$ are mutually orthogonal. The reproducing kernel is given by
\[
K(\bar{w}, z) = \exp(\bar{w}z).
\]

The defining sequence of weights (see (2.12)) is
\[
s_n = s_{nn} = n!
\]
as is also immediately visible from the expansion of the exponent in the reproducing kernel formula. Modulo normalizations this scalar product is the unique scalar product on $\mathcal{P} = \mathbb{C}[z]$ with respect to which $Z^* = \partial/\partial z$.

Let us now consider a $q$-deformation of the Segal-Bargmann space with the defining sequence given by the $q$-factorials \((2.78)\). As we have seen, the reproducing kernel is given by the $q$-exponential function $K(\bar{w}, z) = E_q(\bar{w}z)$. There are infinitely many generating measures on $\mathbb{C}$ for this reproducing kernel, which in particular means that $\langle \cdot \rangle$ is strictly positive on $\mathbb{C}[z, \bar{z}]$.

Here are two distinguished measures, both rotationally symmetric. The first one is discrete on the radial coordinate in $\mathbb{C}$ and based on the discrete $q$-integral
\[
\int_0^\infty f(t) \, d_qt = \sum_{n \in \mathbb{Z}} q^n f(q^n).
\]
A straightforward application of the $q$-integral formula
\[
\int_0^\infty \frac{t^n \, d_qt}{E_q(t/q)} = \frac{(q|q)_\alpha}{(q - 1|q)_\alpha(q/(q - 1)|q)_{-\alpha}} = \gamma_q(\alpha)
\]
together with the observation that \( l_q(\alpha) = s_\alpha \) for \( \alpha \in \mathbb{N} \), leads us to this radially

discrete density function

\[
\rho(z, \bar{z}) = \frac{1}{\pi \sqrt{\alpha}} \frac{1}{E_q(|z|^2/q)} \sum_{n \in \mathbb{Z}} q^n \delta(|z|^2 - q^n).
\]

We have used an extended definition for the \( q \)-symbols

\[
(z|q)_\alpha = \frac{(z|q)_\infty}{(q^n|q)_\infty}
\]

valid for arbitrary complex numbers \( \alpha \in \mathbb{C} \).

On the other hand, we can use the continuous variation

\[
\int_0^\infty \frac{t^{\alpha-1} dt}{(t|q)_\infty} = (q^{1-\alpha}|q)_\alpha \frac{\pi}{\sin(\pi \alpha)}
\]

of the integral \( (B.9) \). This for \( \alpha \in \mathbb{N} \) morphs into

\[
\int_0^\infty \frac{t^\alpha dt}{E_q(t/q)} = \log(1/q) \frac{1}{1/q - 1} s_\alpha
\]

and we naturally arrive to another density function, scaled by a positive factor and

degree of discrete \( \delta \)-terms

\[
\rho(z, \bar{z}) = \frac{1/q - 1}{\pi \log(1/q) E_q(|z|^2/q)}.
\]

In the limit \( q \rightarrow 1^- \) both measures become the standard Segal-Bargmann measure.

It is interesting to observe that from the sequence of weights we can see directly

that the form \( \langle | \rangle \) is strictly positive. Indeed, consider the following \( n \times n \) matrix determinant, with entries indexed by \( i, j = 0, \ldots, n-1 \) and composed of \( q \)-factorials:

\[
l_q(i + j + d) = q^{-\frac{n}{2}(d+\frac{2n-1}{3})} \prod_{k=0}^{d-1} \prod_{k=1}^{n-1} l_q(n+k) \left\{ \prod_{k=1}^{n-1} l_q(k) \right\}^2
\]

where \( d \in \mathbb{N} \). For a positive \( q \) these are all positive numbers (and quickly growing

very large if \( q \leq 1 \)). Here is the sequence of positive numbers whose partial products

generate the above determinants

\[
\lambda_n = q^{-n^2} l_q(n)^2 \quad \lambda_n = q^{-n^2-n d} l_q(n+d) l_q(n)^2
\]

corresponding to \( d = 0 \) and \( d \geq 1 \) respectively. From the analysis of the previous

Appendix it follows that the form \( \langle | \rangle \) must be strictly positive on \( \mathbb{C}[z, \bar{z}] \).

And here is a variation on the above sequences

\[
\lambda_n = q^{n^2} l_q(n)^2 \quad \lambda_n = q^{n^2} l_q(n+d) l_q(n)^2
\]

generating the determinants

\[
l_{\alpha}^{\prime}(i + j + d) = q^{\binom{n}{2} \left( d+n-1 \right)} \prod_{k=0}^{d-1} \prod_{k=1}^{n-1} l_q(n+k) \left\{ \prod_{k=1}^{n-1} l_q(k) \right\}^2
\]

via their partial products. The positivity of these numbers can be used to derive the

positivity of \( \langle | \rangle \) on the space \( \mathbb{C}[z, \bar{z}, Q] \) for the Manin \( q \)-plane, so in this scenario the

property H4 holds, too. Note that the parameter \( q \) is the same for both algebras.
Bergman Spaces

These spaces provide one of the earliest frameworks for quantizing Euclidean domains, via complex functions theory. They were introduced by Stefan Bergman [4] during the 20s of the 20th century.

Let us consider a bounded domain $\Omega$ equipped with its standard Euclidean measure. Let $\mathcal{H}$ be the space of all square integrable holomorphic functions in $\Omega$. It is easy to see that $\mathcal{H}$ is a closed subspace of $L^2(\Omega)$ and thus, in the induced scalar product, it becomes a Hilbert space of holomorphic functions.

The scalar product is given by

$$\langle f, g \rangle = \int_{\Omega} \overline{f(w)}g(w).$$

Because of the boundedness of $\Omega$, the multiplication operator by $z$ is bounded and defined on all $\mathcal{H}$. Together with its adjoint, it generates a non-commutative $C^*$-algebra representing a quantized domain $\Omega$.

We then have

$$\int_{\Omega} |w(z)|\langle w(z)| = 1: \mathcal{H} \rightarrow \mathcal{H}$$

where the integral is in the weak operator topology. This is a direct consequence of the definition of the space $\mathcal{H}$. It is worth noticing that in the special case of $\Omega = \mathbb{D}$ the unit disk, the reproducing kernel is given by

$$K(\bar{w}, z) = \frac{1}{\pi(1 - \bar{w}z)^2}$$

and the whole system is interpretable as a quantization of the Poincaré model of the hyperbolic plane.

The Toeplitz Extension

Another quantum model of the hyperbolic plane is obtained by considering the square summable power series [7]. The unit disc $\mathbb{D}$ as the common domain of holomorphicity, and the reproducing kernel is simply

$$K(\bar{w}, z) = \frac{1}{1 - \bar{w}z}$$

so that the monomials $z^n$ form an orthonormal basis in $\mathcal{H}$. The coordinate $z$ acts as the unilateral shift operator in $\mathcal{H}$.

Paley-Wiener & Euler Spaces

The Paley-Wiener space of index $a > 0$ is generated by the reproducing kernel

$$K(\bar{w}, z) = \frac{\sin[a(\bar{w} - z)]}{\pi(\bar{w} - z)}.$$ 

The elements of the space are the entire functions square-integrable over the reals, and the scalar product is given by

$$\langle f(z), g(z) \rangle = \int_{-\infty}^{\infty} \overline{f(t)}g(t) \, dt.$$
The space can be viewed as the image of $L^2[-a,a]$ via the complexified Fourier transform. More generally, if we consider an entire function $E(z)$ satisfying

$$|E(x - iy)| < |E(x + iy)|$$

where $x \in \mathbb{R}$ and $y > 0$, then the Hilbert space $\mathcal{H}(E)$ is given by the reproducing kernel

$$K(\bar{w}, z) = \frac{E^*(\bar{w})E(z) - E(\bar{w})E^*(z)}{2\pi i(\bar{w} - z)}.$$

The space $\mathcal{H}(E)$ consists of entire functions. They are all square integrable over the reals with the weight function $1/|E(t)|^2$ so that the scalar product is given by

$$\langle f(z), g(z) \rangle = \int_{-\infty}^{\infty} \frac{f(t)g(t)}{|E(t)|^2} \, dt.$$

The Paley-Wiener spaces are a special case when $E(z) = \exp(-iaz)$. We refer to [6] for a detailed study of these ‘Euler spaces’. They are a rich collection of examples of the Hilbert spaces of entire functions going beyond our harmony properties. Nevertheless, they allow similar non-commutative constructions and provide, with the geometrical interpretations appropriately refined, an interesting complementary framework for quantization of the classical Euclidean plane.
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