INVARIANT TENSORS AND THE CYCLIC SIEVING PHENOMENON

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Abstract. The problem of finding the orbit structure of the promotion map acting on standard tableaux with rectangular shape was solved in [Rho10] using Springer’s theory of regular elements and the Khazdan-Lusztig basis. This paper extends this result by replacing the vector representation of $SL(n)$ by any highest weight representation of a simple Lie algebra. The promotion map is defined using crystal graphs. The orbit structure is determined using Springer’s theory of regular elements and Lusztig’s theory of based modules.

1. Introduction

The cyclic sieving phenomenon was introduced in [RSW04]. There are two ingredients for this phenomenon, a finite set with an action of a cyclic group and a polynomial with non-negative integer coefficients.

Let $C$ be a finite cyclic group of order $n$ and let $C$ act on a finite set $X$. Let $P(q)$ be a polynomial in $q$ with non-negative integer coefficients such that $P(1) = |X|$.

Fix an embedding of $\omega : C \rightarrow \mathbb{C}^\times$. Then the following conditions on the triple $(X, P(q), C)$ are equivalent:

1. For every $c \in C$, $P(\omega(c)) = |\{ x \in X : c(x) = x \}|$
2. Define the coefficients $a_l$ by the expansion

$$P(q) = \sum_{l=0}^{n-1} a_l q^l \mod q^n - 1$$

Then $a_l$ is the number of $C$-orbits on $X$ such that the order of the stabiliser of an orbit representative divides $l$.

The triple $(X, P(q), C)$ is said to exhibit the cyclic sieving phenomenon if these conditions are satisfied.

The polynomial $P \mod q^n - 1$ encodes the orbit structure and so a polynomial $P$ always exists. However in the examples the polynomials $P$ that arise are usually of interest for other reasons, for example they may be generating functions of a statistic on $X$. The polynomials that arise are also usually given in terms of $q$-integers and Gauss binomial...
coefficients. Define

\[ [n] = \frac{q^n - 1}{q - 1} \quad [n!] = [n][n - 1] \ldots [2][1] \]

\[
\exp_q(z) = \sum_{n \geq 0} \frac{z^n}{[n]!}
\]

In fact since \( P(1) = |X| \) we can take \( P \) to be a \( q \)-analogue of \( |X| \).

The starting point for this work was the paper \[ \text{Rho10} \]. This paper applied the theory of the cyclic sieving phenomenon and representation theory to understand the orbit structure of the promotion map acting on rectangular standard tableaux. Promotion is a combinatorial map defined on the set of linear extensions of a finite poset. The promotion map was originally defined in \[ \text{Sch72} \] and is reviewed in \[ \text{Sta09} \]. The motivating example is standard tableaux of fixed shape which can be viewed as the set of linear extensions of a poset whose elements are the boxes of the shape. In general the orbit structure of the promotion map is mysterious.

For rectangular standard tableaux there is additional structure. Let \( V \) be the vector representation of \( \text{SL}(n) \). The space of invariant tensors in \( \otimes^r V \) is 0 unless \( r = nk \) for \( k > 0 \). In this case the space of invariant tensors is an irreducible representation of the symmetric group. The paper \[ \text{Rho10} \] used the Khazdan-Lusztig basis of this representation with Springer’s theory of regular elements to find a polynomial which satisfies the cyclic sieving phenomenon for promotion acting on standard tableaux of shape \( k^n \). This polynomial is the \( q \)-analogue of the hook length formula. This determines the orbit structure of promotion.

The next development was the paper \[ \text{PPR09} \] which gives a simpler proof of this result for \( n = 2 \) and \( n = 3 \). The innovation is to replace the Khazdan-Lusztig basis by a simpler basis. For \( n = 2 \) this basis is the basis constructed using the graphical calculus in \[ \text{FK97} \] (which coincides with the Khazdan-Lusztig basis) and for \( n = 3 \) this basis is the basis constructed using the graphical calculus in \[ \text{Kup96} \].

The aim of this paper is to extend this by replacing the vector representation of \( \text{SL}(n) \) by any highest weight representation \( V \) of a simple Lie algebra. Then for each \( r > 0 \) we consider the space of invariant tensors in \( \otimes^r V \). The fundamental operation on this vector space is the rotation map. This is constructed only using the structure of a spherical category on the category of representations. In graphical notation this is given in \[ \text{2} \]. This vector space also has an action of the symmetric group which is constructed using the structure of a symmetric monoidal category. In particular the long cycle generates an action of the cyclic group. The rotation map and the long cycle map are the same, up to a prescribed sign. If the sign is \(+1\) then they are equal. If the sign is \(-1\) we tensor by the sign representation. This defines an action of the symmetric group such that the long cycle acts by rotation.
Then we also define a combinatorial promotion map using crystal graphs. Let $X$ be the crystal graph. Then the analogue of the space of invariant tensors is the set of copies of the trivial crystal in $\otimes^r X$. Then promotion is an operator on this set and is the analogue of the rotation map. There is no analogue of the action of the symmetric group for this set. Then the problem is to understand the orbit structure of promotion.

This problem is solved by finding a polynomial which exhibits the cyclic sieving phenomenon. This polynomial is given by an application of Springer’s theory of regular elements. In order to apply this theory we need to know that there exists a basis of the space of invariant tensors which is preserved as a set by rotation. If a graphical calculus exists for the representation $V$ then we have a diagrammatic basis with this property. However a graphical calculus has only been developed in a small number of special cases. A general construction of a basis with this property is given by Lusztig’s theory of based modules which is developed in [Lus93, Part IV].

The polynomial given by Springer’s theory of regular elements is the fake degree polynomial. This is essentially the principal specialisation of the Frobenius character of the space of invariant tensors considered as a representation of the symmetric group. This reduces the problem of finding the polynomial to the problem of finding this Frobenius character. Although this is a natural problem I am not aware of any previous work on this problem. In this paper I discuss a small number of examples where these Frobenius characters are known.

The vector representation of $\text{SL}(n)$ has already been discussed. The first example we discuss are the representations of $\text{SL}(2)$. In this example we have a graphical calculus but no result on the Frobenius character (except for the two and three dimensional representations). The next examples are the vector representations of $\text{SO}(n)$ (and $\text{Sp}(n)$). In this example we have an explicit basis of the space of invariant tensors and a result on the Frobenius character of the invariant tensors. Then we discuss the adjoint representations of $\text{SL}(n)$ and $\text{GL}(n)$. For $\text{GL}(n)$ we again have an explicit basis of the space of invariant tensors and the generating function for the Frobenius characters. Finally we discuss the fundamental representation of $G_2$ and the spin representation of $\text{Spin}(7)$ (of type $B_3$). In these examples we have a graphical calculus but no result on the Frobenius characters. For $G_2$ the graphical calculus is given in [Kup96] and [Wes07]. For Spin(7) the graphical calculus is given in [Wes08].

2. Temperley-Lieb

The paradigm for this is the simplest representation. Namely the two-dimensional representation $V$ of $\text{SL}(2)$. This case is also covered in [PPR09].
The dimension of the space of invariant tensors in $\otimes^m V$ is 0 if $m$ is odd. The dimension of the space of invariant tensors in $\otimes^{2r} V$ is the Catalan number $C(r)$. Here we describe three sets for each $r$ such that each set has order $C(r)$. We also construct bijections between these. Any of these sets can then be taken as a basis of the space of invariant tensors.

**Definition 2.1.** A Temperley-Lieb diagram consists of $2r$ points lying on a horizontal line $L$ connected by $r$ noninteresting arcs lying in the half plane below $L$.

**Definition 2.2.** A Dyck path is a sequence $a_0 = 0, a_1, \ldots, a_{2r} = 0$ such that $|a_{i+1} - a_i| = 1$ for $0 \leq i \leq 2r - 1$ and $a_i \geq 0$ for $0 \leq i \leq 2r$.

**Definition 2.3.** A standard tableaux of shape $(r, r)$ is a partition of $\{1, \ldots, 2r\}$ into two subsets of size $r$ say $a_1 < \cdots < a_r$ and $b_1 < \cdots < b_r$ such that $a_i > b_i$ for $1 \leq i \leq r$.

Each of these three sets has an inclusion in the set of words in the two letters $U, D$. Given a Temperley-Lieb diagram the $i$-th letter of the word is defined by considering the arc with one endpoint at $i$. Let the other endpoint of this arc be $j$. Then the $i$-th letter is $U$ if $j > i$ and is $D$ if $j < i$. Given a Dyck path the $i$-th letter of the word is $U$ if $a_{i+1} - a_i = 1$ and is $D$ if $a_{i+1} - a_i = -1$. Given a standard tableaux of shape $(r, r)$ the $i$-th letter of the word is $U$ if $i \in \{a_1, \ldots, a_r\}$ and is $D$ if $i \in \{b_1, \ldots, b_r\}$.

**Example 2.4.** The Temperley-Lieb diagram and standard tableau associated to the word $UUUDDUDD$ are:

```
1 2 3 6
4 5 7 8
```

Let $T(r)$ be the set of Temperley-Lieb diagrams. These can be drawn in a disc instead of a half plane and this defines rotation of a diagram. This generates an action of the cyclic group of order $2r$. This raises the problem of finding a polynomial which exhibits the cyclic sieving phenomenon for this action. This problem is solved using additional structure on this set which arises from representation theory.

Let $N(r)$ be the vector space with basis $T(r)$. Then we define an action of the symmetric group $S(2r)$ on $N(r)$ using the relations

```
\[ \text{circle} = -2 \]
```

\[ \text{cross} = - \text{circle} - \text{circle} \]
The action of a permutation on a Temperley-Lieb diagram is given by putting the string diagram on the top edge of the triangle and then simplifying using these rules. This gives a linear combination of Temperley-Lieb diagrams.

Then this action has the property that rotation is given by the action of the long cycle in $S(2r)$. This representation of $S(2r)$ is irreducible and corresponds to the partition $2^r$ (which is conjugate to $[r, r]$). The fake degree polynomial is

$$P(r) = q^{r(r-1)} \frac{1}{[r+1]} \left[ \begin{array}{c} 2r \\ r \end{array} \right]$$

It then follows from Theorem 4.3 that $(T(r), P(r), C(r))$ exhibits the cyclic sieving phenomenon.

These fake degree polynomials and their reductions mod $q^{2r} - 1$ are:

| $r$ | $P(r)$ |
|-----|--------|
| 0   | 1      |
| 1   | $1 + q$ |
| 2   | $q^4 + q^2$ |
| 3   | $q^{12} + q^{10} + q^9 + q^8 + q^6$ |

For $r = 2$ this says that there is one orbit of order two. For $r = 3$ this says that there is one orbit of order two and one of order three.

There is also a combinatorial operation called promotion. This operates on standard tableaux. We will only consider rectangular shapes.

**Definition 2.5.** Given a standard tableau remove the 1 from the top left corner to leave an empty box. Then decrease the entry in every other box by 1. Now we slide the empty box from the top left corner to the bottom right corner. Look to the right of the empty box and below the empty box. If these are both boxes in the tableau then slide the smaller of the two entries into the empty box. If one of these is a box in the tableau then slide the entry in that box into the empty box. If neither of these boxes is in the tableau then the empty box has reached the bottom right corner. After $r$ slides the empty box will reach the bottom right corner. Then put $2r$ in the bottom right corner.

This operation can be understood in terms of words as follows. Given a word which corresponds to a Temperley-Lieb diagram we remove the first letter which must be a $U$ finally we change a $D$ to a $U$ and then we add $D$ to the end of the word. The rule that specifies which $D$ is changed to a $U$ is that it is the last $D$ such that the word before the $D$ corresponds to a Temperley-Lieb diagram.

This agrees with promotion. A standard tableau with an empty box can still be regarded as a word. Removing the first letter $U$ corresponds to removing the 1 from the top left corner and decreasing the entry in every other box by 1. Moving the empty box to the right does not change the word. Moving the empty box down from the top row to the
bottom row corresponds to changing a $D$ to a $U$. Putting $2r$ in the bottom right corner corresponds to adding $D$ to the end of the word.

Next we show that promotion corresponds to rotation. First we extend the graphical calculus. A triangular diagram is a triangle with $2r$ points on the top edge and points on the left and right edges. These points are connected in pairs by noncrossing arcs and we require that that every point on the left or right edge is connected to a point on the top edge. Then there is a bijection between triangular diagrams and words of length $2r$ in $U$ and $D$. This is a reformulation of the graphical calculus in [FK97].

Given a word in the two letters $U$ and $D$ we construct a triangular diagram.

The edges of the triangle will be referred to as the top edge the left edge and the right edge. The two letters $U$ and $D$ correspond to the two triangles

Given a word we start by putting the sequence of triangles in a horizontal line. Then we fill in the triangle using the following four diamonds.

This gives a Temperley-Lieb diagram precisely when there are no boundary points on the left or right edges of the triangle.

**Example 2.6.** Applying promotion to the word $UUDD$ gives the word $UDUD$. The corresponding Temperley-Lieb diagrams and standard tableaux are

The two intermediate words are $UDD$ and $UDU$. The corresponding Temperley-Lieb diagrams and standard tableaux are

3. Promotion

Promotion was originally defined in [Sch72]. This definition was then applied to rectangular standard tableaux. Here we take this example of
promotion and interpret it in terms of crystal graphs. This interpretation then extends to the crystal graph of any finite dimensional highest weight representation. Given a crystal graph, $X$, then promotion is a map of the copies of the trivial crystal graph in the tensor powers of $X$.

This then raises the problem of determining the order of this map and the orbit structure. The order for $X^r$ is $r$. The orbit structure is determined by finding a polynomial which satisfies the cyclic sieving phenomenon.

3.1. **Crystal graphs.** Here we give the definition of a crystal graph and the tensor product rule. These were introduced in [Kas90]. A good introduction to the theory of crystal graphs is [HK02].

**Definition 3.1.** A crystal graph is a set $B$, functions $H, D$ from $B$ to dominant weights and functions

$$e_i: \{A|H_A(i) > 0\} \rightarrow \{A|D_A(i) > 0\}$$
$$f_i: \{A|D_A(i) > 0\} \rightarrow \{A|H_A(i) > 0\}$$

These are required to satisfy a number of conditions.

- The maps $e_i$ and $f_i$ are inverse bijections.
- $H_{e_i A}(i) = H_A(i) - 1$ and $D_{e_i A}(i) = D_A(i) + 1$
- $D_{f_i A}(i) = D_A(i) - 1$ and $H_{f_i A}(i) = H_A(i) + 1$

Then we define the weight of $A$ to be $\omega(A) = H_A - D_A$.

It is usual to represent this as a labelled directed graph. The set $B$ is the set of vertices. There is an arrow from $a$ to $b$ labelled $i$ iff $e_i(a) = b$.

The tensor product rule for crystal graphs is

$$D_{A \otimes B}(i) = \begin{cases} D_B(i) & \text{if } H_B(i) \geq D_A(i) \\ D_B(i) + D_A(i) - H_B(i) & \text{if } D_A(i) \geq H_B(i) \end{cases}$$

$$H_{A \otimes B}(i) = \begin{cases} H_A(i) + H_B(i) - D_A(i) & \text{if } H_B(i) \geq D_A(i) \\ H_A(i) & \text{if } D_A(i) \geq H_B(i) \end{cases}$$

This is illustrated in the two triangles below.

Hence $\omega(A \otimes B) = \omega(A) + \omega(B)$. The maps $e_i$ and $f_i$ are defined by

$$e_i(A \otimes B) = \begin{cases} e_i A \otimes B & \text{if } D_A(i) \geq H_B(i) \\ A \otimes e_i B & \text{if } H_B(i) > D_A(i) \end{cases}$$

$$f_i(A \otimes B) = \begin{cases} f_i A \otimes B & \text{if } D_A(i) > H_B(i) \\ A \otimes f_i B & \text{if } H_B(i) \leq D_A(i) \end{cases}$$
An important property of this tensor product is that it is associative. The unit for the tensor product is the crystal graph with one element, \(A\), with \(H_A = 0\), \(D_A = 0\).

For \(\alpha\) and \(\beta\) integral dominant weights we define

\[\alpha B\beta = \{x \in B | H(x) = \alpha, D(x) = \beta\}\]

In this paper we are interested in the tensor powers of a given crystal. There are two ways to view this. Let \(A\) be the vertices of the crystal. Then the vertices of the \(r\)-th tensor are words of length \(r\) in \(A\). The usual way to regard a word is as a function \(u: \{1, 2, \ldots, r\} \to A\). The alternative is to take the sets \(\{f^{-1}(a) | a \in A\}\) as a partition of \(\{1, 2, \ldots, r\}\) into subsets indexed by \(A\). This way of representing a word labels each vertex of the crystal graph by a subset of \(\{1, 2, \ldots, r\}\). Then a raising or lowering operator, \(e_i\), can be seen as sliding one element of a subset along an edge of the crystal graph with label \(i\).

**Example 3.2.** Take the crystal graph with two vertices \(U\) and \(D\) connected by an arrow. Then the vertices of the \(r\)-th tensor power are words of length \(r\) which we identify with triangular diagrams with \(r\) points on the top edge. The lowering operator takes the last point on the left edge of the triangle (if there is one) and moves it to the right edge. Similarly the raising operator takes the last point on the right edge of the triangle (if there is one) and moves it to the left edge.

### 3.2. Promotion

Let \(C\) be an irreducible Cartan matrix of finite type and let \(\omega\) be a dominant integral weight. Then associated to \(\omega\) is a crystal graph \(X(\omega)\). This can be constructed from a crystal basis of the representation \(V(\omega)\) but can also be constructed directly, for example, by using the Littelmann path operators defined in [Lit95].

Let \(X = X(\omega)\) be the crystal graph of \(V(\omega)\). This graph is connected. Let \(x^{hi}\) be the unique element in \(\omega X_\omega\) and let \(x^{lo}\) be the unique element in \(0 X_\omega\). These correspond to the highest weight vector and the lowest weight vector in \(V(\omega)\).

Consider a word \(w\) in \(0 X_\omega^r\). Then the first letter is \(x^{lo}\) and the last letter is \(x^{hi}\). Then omitting the first letter gives a map \(0 X^{r-1}_0 \to \omega X^{r-1}_0\) and omitting the last letter gives a map \(0 X^{r-1}_\omega \to 0 X^{r-1}_\omega\). These maps are both bijections. The inverse maps are given by \(w \mapsto x^{hi} w\) and \(w \mapsto wx^{lo}\), respectively.

There is also a bijection \(\omega X^{r-1}_0 \to 0 X^{r-1}_\omega\). Let \(w \in \omega X^{r-1}_0\). Then the connected component of \(X^{r-1}\) which contains \(w\) is isomorphic to \(X\) where \(w\) is identified with \(x^{hi}\). Then \(w\) is mapped to the unique element of \(0 X^{r-1}_\omega\) in this connected component. This is the element identified with \(x^{lo}\).

**Definition 3.3.** The promotion map \(P\) is the composite of the bijections

\[P: 0 X_0^r \to \omega X_0^{r-1} \to 0 X^{r-1}_\omega \to 0 X_0^r\]
Example 3.4. Promotion can be seen to be rotation, see [PPR09, Theorem 1.4]. Given a Temperley-Lieb diagram the construction of promotion is to drop the first letter which must be \( U \). This gives a triangular diagram with one point on the left edge. The lowering operator moves this point to the right edge. Then we add a \( D \) to the end of the word and this gives a Temperley-Lieb diagram.

This also corresponds to promotion defined using sliding on standard tableaux.

3.3. Jeu-de-taquin. In this section we take \( V \) to be the vector representation of \( \mathfrak{sl}(n) \). We show that the rotation, \( P \), defined in Definition 3.3 gives the promotion.

For the crystal graph we take the set \( X = \{1, 2, \ldots, n\} \) and the operators \( e_i \) are given by \( e_i(i + 1) = i \) for \( 1 \leq i \leq n - 1 \) and \( e_i(j) \) is not defined if \( j \neq i + 1 \). The vertices of \( X^r \) are words of length \( r \) in the elements of \( X \). For \( i \leq j \leq n - 1 \) and \( w \in X^r \), \( e_i(w) \) is obtained from \( w \) by changing one letter \( i + 1 \) in \( w \) to \( i \). The rule which specifies which letter to change is given in [Lot02, §5.5]. The rule is to take the word in these two letters ignoring the other letters in the word. Then apply the rule from section 2.

There are several ways of defining standard tableaux given in [Sta99, 7.10.3]. One of these are lattice permutations which are also known as Yamanouchi words or ballot sequences. Given a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) this is the set \( \mathfrak{o}X^r_{\lambda'} \) where \( \lambda' \) is the integral dominant weight \((\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n)\). In particular \( \mathfrak{o}X^r_0 \) is identified with standard tableaux of shape \( kn \) for \( r = kn \).

The sliding move is defined for tableaux by Definition 2.5. In this paper we only consider rectangular tableaux. The moves which move a number horizontally do not change the word. When a number is moved from row \( i + 1 \) to row \( i \) this letter is changed from \( i + 1 \) to \( i \). This corresponds to the operator \( e_i \) on the word; to see this, note that \( e_i \) is defined using the letters \( i \) and \( i + 1 \) (ignoring the others), that sliding only depends on rows \( i \) and \( i + 1 \) and that these agree when there are only two letters or rows.

This shows that the sequence of tableaux or words created by sliding is the crystal graph isomorphic to \( X \) and so the promotion map is the rotation map defined in Definition 3.3.

Here is an example. We start with the tableau, lattice word and flow diagram

\[
\begin{array}{cccc}
1 & 2 & 3 \\
4 & 5 & 7 \\
6 & 8 & 9 \\
\end{array}
\]

111223233
After promotion we have the tableau, lattice word and flow diagram

\[
\begin{array}{cccc}
1 & 2 & 6 \\
3 & 4 & 8 \\
5 & 7 & 9
\end{array}
\quad 112231323
\]

The three intermediate tableaux, words of length eight and flow diagrams are

\[
\begin{array}{ccc}
\begin{array}{ccc}
1 & 5 \\
2 & 3 & 7 \\
4 & 6 & 8
\end{array} & \begin{array}{ccc}
1 & 3 & 5 \\
2 & 1 & 7 \\
4 & 6 & 8
\end{array} & \begin{array}{ccc}
1 & 3 & 5 \\
2 & 6 & 7 \\
4 & 8
\end{array}
\end{array}
\quad 11223233 & 11223133 & 112231323
\]

Here is a second example. We start with the tableau, lattice word and flow diagram

\[
\begin{array}{cccc}
1 & 4 & 6 \\
2 & 5 & 7 \\
3 & 9 & 11 \\
8 & 10 & 12
\end{array}
\quad 123121243434
\]

After promotion we have the tableau, lattice word and flow diagram

\[
\begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 & 6 \\
7 & 8 & 10 \\
9 & 11 & 12
\end{array}
\quad 121212334344
\]

The four intermediate words of length eleven are

\[
\begin{array}{cccc}
2 & 3 & 1 & 2 \\
1 & 3 & 1 & 2 \\
1 & 2 & 1 & 2 \\
1 & 2 & 1 & 2
\end{array}
\quad \begin{array}{cccc}
2 & 4 & 3 & 4 \\
1 & 3 & 4 & 3 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4
\end{array}
\]

The four intermediate tableaux are

\[
\begin{array}{cccc}
\begin{array}{ccc}
2 & 4 \\
1 & 3 & 5 \\
6 & 7 & 9 \\
8 & 10 & 11
\end{array} & \begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 \\
6 & 7 \\
8 & 10 & 11
\end{array} & \begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & 9 \\
6 & 7 \\
8 & 10 & 11
\end{array} & \begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & 9 \\
6 & 7 & 11 \\
8 & 10
\end{array}
\end{array}
\]

3.4. **Exceptional.** In this section we show how sliding and promotion work for the seven dimensional representation of the exceptional simple Lie algebra $G_2$. The weight vectors for the seven dimensional
representation is

\[
\begin{array}{c}
\text{representation is} \\
\end{array}
\]

The crystal graph is a chain and so we let \( X = \{1, 2, \ldots, 7\} \). The crystal graph is then given by

\[
\begin{array}{c}
\text{The crystal graph is a chain and so we let } X = \{1, 2, \ldots, 7\}. \text{ The crystal graph is then given by} \\
\end{array}
\]

The intention is that these diagrams should be superimposed. In particular this graph is a chain so in the examples below we list the vertices and omit the arrows.

As a first example we take the word 147 whose planar trivalent graph has a single trivalent vertex and so is invariant under rotation. We remove the initial 1 to get 47. Then we apply the lowering operators to get the following sequence of words

\[
\begin{array}{c}
47 \rightarrow 37 \rightarrow 27 \rightarrow 26 \rightarrow 16 \rightarrow 15 \rightarrow 14 \\
\end{array}
\]

Then we add the 7 at the end to get 147. In terms of sliding the word 147 is represented by

\[
\begin{array}{c}
\text{Then we add the 7 at the end to get 147. In terms of sliding the word 147 is represented by} \\
\end{array}
\]
Then the sequence of words is represented by the sequence

As a second example we start with the word 123747. We remove the initial 1 to get 23747. Then we apply the lowering operators to get the following vertices of a crystal graph

Then we add the 7 at the end to get 126267. This gives 123747 $\mapsto$ 126267.

As a second example we start with the word 126267. We remove the initial 1 to get 26267. Then we apply the lowering operators to get the following vertices of a crystal graph
Then we add the 7 at the end to get 141567. This gives $126267 \mapsto 141567$.

4. Theory

In this section we give an account of the results that solve the problem of finding a polynomial so that promotion with this polynomial exhibits the cyclic sieving phenomenon.

4.1. Fake degree polynomials. Let $W$ be a finite complex reflection group. Let $V$ be the reflection representation of $W$ and denote the ring of polynomial valued functions on $V$ by $\mathbb{C}[V]$. Let $\mathbb{C}[V]^W_+$ be the subring invariant under the action of $W$. Consider the quotient $\mathbb{C}[V]/\mathbb{C}[V]^W_+$ as a graded $W$-module. Then as a $W$-module this is isomorphic to the regular representation of $\mathbb{C}W$.

**Definition 4.1.** Let $\lambda$ be a simple $W$-module. The fake degree polynomial of $\lambda$ is $f_q(\lambda) = \sum_{i \geq 0} a_i q^i$ where $a_i$ is the multiplicity of $\lambda$ in degree $i$ in $\mathbb{C}[V]/\mathbb{C}[V]^W_+$.

Then $f_q(\lambda) \in \mathbb{Z}[q]$ and $f_1(\lambda) = \dim(\lambda)$.

This can be extended to any $W$-module by linearity. Let $\mu$ be a $W$-module and let $m_\lambda$ be the multiplicity of $\lambda$ in $\mu$. Then define $f_q(\mu)$ by $f_q(\mu) = \sum_{\lambda} m_\lambda f_q(\lambda)$.

**Definition 4.2.** An element $w \in W$ is called regular if there exists an eigenvector $v$ for $w$ in the reflection representation of $W$ such that $v$ does not lie on any of the reflecting hyperplanes for the reflections in $W$. 
The main examples of regular elements are the Coxeter elements in a Weyl group.

**Theorem 4.3.** Let $W$ be a finite complex reflection group and let $w \in W$ be a regular element. Let $\mu$ be a $W$-module together with a basis $X$ such that $w(X) = X$. Let $C$ be the cyclic subgroup of $W$ generated by $w$ and put $P(q) = f_q(\mu)$. Then the triple $(X, P(q), C)$ exhibits the cyclic sieving phenomenon.

**Proof.** Springer showed in [Spr74, Proposition 4.5] that if $w \in W$ is a regular element and $v \in V$ is an associated eigenvector with $w.v = \omega v$ then $\chi^\lambda(w) = f_\omega(\lambda)$ for $\lambda$ irreducible. This gives the theorem for $\lambda$ an irreducible representation. The theorem then follows for all representations by complete reducibility. □

As an application of this take $W$ to be a symmetric group considered as a finite Weyl group. Take $w \in W$ to be the long cycle. This is a Coxeter element and so is regular.

The irreducible representations are indexed by partitions and the fake degree polynomial is given by a $q$-analogue of the Frame-Thrall-Robinson hook length formula.

For a partition $\lambda$, define $b(\lambda)$ by $b(\lambda) = \sum_{i=1}^{k} (i-1)\lambda_i$. The fake degree polynomial $f_q(\lambda) \in \mathbb{Z}[q]$ is given by

$$f_q(\lambda) = q^{b(\lambda)} \frac{[n]!}{\prod_{(i,j) \in D(\lambda)}[h(i,j)]}$$

where $D(\lambda) = \{(i,j) | 1 \leq i, 1 \leq j \leq \lambda_i \}$ and $h(i,j)$ is the hook length of $(i,j) \in D(\lambda)$.

Then it follows from the definition that for all $r \geq 1$,

$$[r]! = \sum_{|\lambda|=r} \dim(\lambda)f_q(\lambda)$$

An alternative definition, given in [Sta99, 7.19.11 Proposition] is

$$f_q(\lambda) = \sum_T q^{\text{maj}(T)}$$

where the sum is over standard tableaux of shape $\lambda$ and $\text{maj}(T)$ is the major index of $T$.

Before we give an application of Theorem 4.3 we make some remarks. The first is that if a symmetric group acts on a set then we can take the set itself as the basis of the permutation representation. Then clearly the conditions of Theorem 4.3 are satisfied. In this case the fake degree polynomial is given by a specialisation of the cycle index series. This is explained in [RSW04, §6].
4.2. Invariant tensors. Let $V$ be a finite dimensional representation of a group, Lie group or Lie algebra. Then for $r \geq 1$, $\otimes^r V$ is also a representation. In the group case the action is the diagonal action. For a Lie algebra $\mathfrak{g}$ the tensor product of representations $\rho_1$ and $\rho_2$ is the representation $\rho$ given by 

$$\rho(g) = \rho_1(g) \otimes 1 + 1 \otimes \rho_2(g)$$

for all $g \in \mathfrak{g}$.

The symmetric group $S(r)$ also acts on $\otimes^r V$. The usual action is to define the action of the permutation $\sigma$ by

$$\sigma(v_1 \otimes \cdots \otimes v_r) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}$$

An alternative is to tensor with the sign representation so the action of the permutation $\sigma$ is given by

$$\sigma(v_1 \otimes \cdots \otimes v_r) = (-1)^{\ell(\sigma)}v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}$$

where $\ell(\sigma)$ is the length of $\sigma$. Both actions commute with the action of the group or Lie algebra.

For any representation $V$ we have the space of invariants in $V$. This is $\text{Hom}(\mathbb{C}, V)$ where $\mathbb{C}$ is the trivial representation. This is the maximal submodule with trivial action. We also have the space of co-invariants. This is the maximal quotient module with trivial action. Let $A$ be the group algebra or universal enveloping algebra. Then there is a homomorphism $A \to \mathbb{C}$ whose kernel is the augmentation ideal, $A_+$. The space of co-invariants is the quotient $V \to V/A_+$. There is a canonical map from the space of invariants to the space of coinvariants given by the composition

$$\text{Hom}(\mathbb{C}, V) \to V \to V/V A_+$$

and if $V$ is completely reducible then this is an isomorphism. The space of co-invariant tensors for $V$ is denoted by $N(V)$.

Then, for $r \geq 1$, the action of the symmetric group $S(r)$ on $\otimes^r V$ induces an action of $S(r)$ on the subspace of invariant tensors and on the quotient space of co-invariant tensors. These actions are determined by the properties that the inclusion map $\text{Hom}(\mathbb{C}, \otimes^r V) \to \otimes^r V$ and the projection map $\otimes^r V \to N(\otimes^r V)$ are homomorphisms of representations of $S(r)$.

Let $C$ be a Cartan matrix of finite type. Then associated to $C$ is a semi-simple Lie algebra, $\mathfrak{g}(C)$, and the universal enveloping algebra, $U(\mathfrak{g}(C))$. The category of finite dimensional representations a semi-simple abelian category. The simple objects are indexed by the integral dominant weights and we denote the simple object associated to $\lambda$ by $V(\lambda)$. Also associated to $C$ is $U_q(\mathfrak{g}(C))$, the Drinfeld-Jimbo quantised enveloping algebra.

Here we will apply the theory of based modules. This theory is developed in [Lus93, Part IV] and a summary is given in [Kan98]. This
theory is elementary in the sense that the methods are elementary and in particular do not use any algebraic geometry. The only difficulty is the result that every $V(\lambda)$ has a basis which gives a based module. Lusztig proves this by showing that the canonical basis has this property and he constructs the canonical basis using perverse sheaves. An alternative, elementary, construction is given by following [Jan96].

Let $P$ be the weight lattice and $Q \subset P$ the root lattice and let $P^+$ be the dominant weights and $Q^+$ the positive linear combinations of the simple roots. Let $\theta \in Q$ be the highest root (that is, the highest weight of the adjoint representation). Then each coset of $Q \subset P$ has a unique representative, $\lambda$, such that $\lambda \in P^+$, $\lambda \neq 0$ and $\theta - \lambda \in Q^+$. These weights are also characterised by the property that for each such $\lambda$, the weights of $V(\lambda)$ are the orbit of $\lambda$ under the action of the Weyl group (each with multiplicity 1) and the zero weight. For each such $\lambda$, $V(\lambda)$ can be constructed directly as a based module. Then $V(\lambda)$ can be constructed as a based module for any $\lambda \in P^+$ using the tensor product of based modules and the properties of based modules.

The theory of based modules is used to prove the following:

**Theorem 4.4.** Let $C$ be a Cartan matrix of finite type and let $V(\lambda)$ be a highest weight representation. For $r \geq 1$, the space $(\otimes^{2p,\lambda}\text{sign}) \otimes N(\otimes^r V)$ has a basis which is fixed by the long cycle in $S(r)$.

**Proof.** Based modules are defined in [Lus93, 27.1.2]. The first result is that, for any $\lambda \in P^+$, by taking the canonical basis, $V(\lambda)$ becomes a based module. This is stated in [Lus93, 27.1.4].

The main construction for based modules is the tensor product construction. This is defined and shown to be associative in [Lus93, 27.3]. In particular, for all $\lambda \in P^+$ and all $r > 0$, $\otimes^r V(\lambda)$ has a basis which makes it a based module.

Let $(M, B)$ be any based module. Then $M$ has a decomposition into isotypic components $M = \oplus_{\lambda} M[\lambda]$. Corresponding to this is a partition $B = \sqcup_{\lambda} B[\lambda]$ which is defined in [Lus93, 27.2.1]. Let $\pi: M \to N(M)$ be the projection onto the space of coinvariants. Then [Lus93, 27.2.5, 27.2.6] shows that for $b \in B$, $\pi(b) = 0$ iff $b \notin B[0]$ and $\{\pi(b) : b \in B[0]\}$ is a basis of $N(M)$. In particular, for all $\lambda \in P^+$ and all $r > 0$, we have a basis of $N(\otimes^r V(\lambda))$ which we denote by $B$.

Next we compare the rotation map and the action of the long cycle in the representation $(\otimes^{2p,\lambda}\text{sign}) \otimes N(\otimes^r V)$. In diagram notation these are defined by

$$
(2) \quad \begin{array}{c}
\begin{array}{c}
\includegraphics[scale=0.5]{rotation_map}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\includegraphics[scale=0.5]{action_of_long_cycle}
\end{array}
\end{array}
$$
These can be seen to be equal from the isotopy

\[ \begin{array}{c}
\begin{array}{c}
\text{U}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{U}
\end{array}
\end{array} = (-1)^{(2\rho, \lambda)} \begin{array}{c}
\begin{array}{c}
\text{U}
\end{array}
\end{array} \]

Let \( P \) be the endomorphism of \( N(\otimes^r V(\lambda)) \) defined in [Lus93, 28.2.2]. Then \( P \) preserves the basis \( B \) by [Lus93, 28.2.4] (which is proved in [Lus93, 28.2.8]. This map \( P \) is identified with the action of the long cycle in [Lus93, 28.2.9].

The conclusion of this discussion is the main theorem:

**Theorem 4.5.** Let \( \lambda \) be any dominant integral weight for a Cartan matrix of finite type and let \( r > 0 \). Then using crystal graphs we have a finite set \( X \) and the promotion map \( X \to X \). Take \( P \) to be the fake degree polynomial of the representation \( (\otimes(2\rho, \lambda) \text{sign}) \otimes N(\otimes^r V) \). Then promotion generates an action of the cyclic group, \( C \), of order \( r \) and the triple \( (X, P, C) \) exhibits the cyclic sieving phenomenon.

### 4.3. Plethysms.

For Theorem 4.5 to be useful we need to be able to calculate the fake degree polynomial. Here we explain how the Frobenius character of the representation \( N(\otimes^r V) \) can be computed in simple examples.

The polynomial \( P(r) \) is determined by the character of the representation \( N(\otimes^r M) \). This can be calculated using plethysms. For a partition \( \lambda \) let \( U(\lambda) \) be the associated irreducible representation of \( S(|\lambda|) \) and let \( S^\lambda \) be the associated Schur functor. Then the decomposition of \( \otimes^r M \) as a representation of \( S^r \times \text{GL}(M) \) is

\[
\otimes^r M \cong \bigoplus_{\lambda : |\lambda| = r} U(\lambda) \otimes S^\lambda(M)
\]

Now regard \( S^\lambda(M) \) as a \( g \)-module and take the decomposition into highest weight representations

\[
S^\lambda(M) \cong \bigoplus_{\omega} A(\lambda, \omega) \otimes V(\omega)
\]

where the sum is over the dominant integral weights. Here \( A(\lambda, \omega) \) is a vector space and we put \( a(\lambda, \omega) = \dim A(\lambda, \omega) \).

Then the decomposition of \( \otimes^r M \) as a representation of \( C S(r) \otimes U(g(C)) \) is

\[
\otimes^r M \cong \bigoplus_{\lambda, \omega} A(\lambda, \omega) \otimes U(\lambda) \otimes V(\omega)
\]

Taking \( \omega \) to be the zero weight, this shows that the decomposition of \( N(\otimes^r M) \) is

\[
\bigoplus_\lambda A(\lambda, 0) \otimes U(\lambda)
\]
Define $P(r)$ by

$$P(r) = \begin{cases} \sum_{\lambda: |\lambda|=r} a(\lambda, 0)f_q(\lambda) & \text{if } \langle 2\rho, \lambda \rangle \text{ is even} \\ \sum_{\lambda: |\lambda|=r} a(\lambda, 0)f_q(\overline{\lambda}) & \text{if } \langle 2\rho, \lambda \rangle \text{ is odd} \end{cases}$$

This is the fake degree polynomial of $(\otimes^2(2\rho,\lambda) \text{ sign}) \otimes N(\otimes^r V)$ as a representation of $S(r)$.

An algorithm for computing the numbers $a(\lambda, \omega)$ is implemented in the LiE package, [vLCL92]. This package is no longer supported but it has been incorporated into Magma, [BCP97] and Sage, [S+10]. The calculations in this paper used these two packages.

5. SL(2)

This section gives the results for the representations of SL(2). Let $[k]$ be the irreducible representation of weight $k$ and dimension $k + 1$. The dimension of the space of invariant tensors in $\otimes^n [k]$ is the coefficient of $t^n x^{kn}$ in

$$\frac{(1 - x^2)^2}{(1 - x^2) - t(1 - x^{2k+2})}$$

It then follows from [Sta99] 6.3.3 Theorem] that for fixed $k$ the generating function for these coefficients is algebraic. For $k = 1$ we have the Catalan numbers and this case was discussed in Section 2. For $k = 2$ we have the Riordan numbers and these are discussed below in Section 5.1. These seem to be the only two cases that have been studied in detail. Here we discuss promotion using the graphical calculus in [FK97].

A basis for the invariant tensors in $\otimes^n [k]$ is given by taking a subset of the set of Temperley-Lieb diagrams with $kn$ boundary points. The $kn$ boundary points are grouped in $n$ blocks of $k$ points. For $0 \leq i \leq n-1$ the points in block $i$ are the $k$ points $\{ki + j | 1 \leq j \leq k\}$. Then we require that all arcs have endpoints in distinct blocks. This set has an action of the cyclic group of order $n$. Let $P$ be the rotation of Temperley-Lieb diagrams. This does not preserve this subset. However $P^k$ does and this generates an action of a cyclic group of order $n$.

Next we look at this from the point of view of crystal graphs. The vertices of the crystal graph are $\{x_i | 0 \leq i \leq k \}$. The basis of the invariant tensors is the set of words such that if we substitute $D^i U^{k-i}$ for $x_i$ then we get a Dyck path.

The promotion operator is constructed by dropping the first letter which must be $x_0$. This means dropping the $U^k$ at the beginning of the Dyck path. Then we apply the lowering operator $k$ times. Then we add $D^k$ to the end of the Dyck path. This gives a Dyck path which corresponds to a basis element.

The graphical calculus makes it clear that these two maps, the rotation and the promotion are the same. Here we describe this graphical
calculus. This is equivalent to the graphical calculus in [FK97]. The letter $x_i$ is represented by the diagram

This is the same diagram that represents $D^i U^{k-i}$. Then we fill in the diamonds. The promotion operator is constructed by dropping the $x_0$ at the beginning of the word. This gives a triangle with $k$ points on the left boundary edge. The lowering operator moves the last point on the left boundary edge to the right boundary edge. Doing this $k$ times gives a diagram with no points on the left boundary edge and $k$ points on the right boundary edge. Then we add the triangle $x_k$ to the end. This gives the diagram of an invariant tensor.

5.1. Three dimensional. The three dimensional representation is the first example of two other cases that we will return to. This is the adjoint representation of $SL(2)$ and is also the vector representation of $SO(3)$.

The dimension of the spaces of invariant tensors are:

| $r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|---|---|---|---|---|---|---|---|---|---|
| 0   | 1 | 0 | 1 | 1 | 3 | 6 | 15 | 36 | 91 | 232 |

These are known as Riordan numbers and are sequence A005043 in [Slo08].

**Conjecture 5.1.** The Frobenius character of $N(\otimes^r V)$ is $\sum \lambda s_\lambda$ where the sum is over partitions of $r$ into three parts such that either all parts are even or all parts are odd.

The following table shows the fake degrees of these symmetric functions and gives the reduction mod $q^n - 1$.

| $r$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|---|---|
| 0   | 1 | 1 |
| 1   | 0 | 0 |
| 2   | 1 | 1 |
| 3   | $q^3$ | 1 |
| 4   | $q^4 + q^2 + 1$ | $q^2 + 2$ |
| 5   | $q^7 + q^6 + 2q^5 + q^4 + q^3$ | $q^4 + q^2 + q + 2$ |

These fake degrees are $q$-analogues of the Riordan numbers.

6. Classical groups

For $n \geq 3$, take $g = so(n)$ and $V$ to be the vector representation.

For $\mu = (\mu_1, \ldots, \mu_n)$ a partition and $p \geq 0$ let $\lambda(\mu; p)$ be the partition $\lambda = (2\mu_1 + p, \ldots, 2\mu_n + p)$. 
Conjecture 6.1. For $r \geq 1$, the Frobenius character of $N(\otimes^{2r} V)$ as a representation of $S(2r)$ is
\[
\sum_{(\mu, p)} s_{\lambda(\mu; p)}
\]
where the sum is over $p \geq 0$ and partitions $\mu$ such that $\mu$ has at most $n$ parts and $2|\mu| + np = r$.

In particular, for $n > r$, this is independent of $n$.

Proposition 6.2. For $n > r \geq 1$, the Frobenius character of $N(\otimes^{2r} V)$ as a representation of $S(2r)$ is given by
\[
\sum_{\mu: |\mu|=r} s_{2\mu}
\]
where $2\mu = \lambda(\mu; 0)$.

Proof. Let $\{2r\}$ be a set with $2r$ elements. Let $X(r)$ be the set of partitions of $\{2r\}$ into $r$ disjoint subsets each with two elements. These are known as perfect matchings. Then $|X(r)| = 1 \ldots (2r - 1)$. Let $S(2r)$ be the group of permutations of $\{2r\}$. Then $S(2r)$ acts on $X(r)$.

For $n > r$ there is a basis of $N(\otimes^{2r} V)$ indexed by $X(r)$ such that the inclusion $X(r) \to N(\otimes^{2r} V)$ is $S(2r)$-equivariant. This result can be found in [Bra37] and [Wey39]. Each perfect matching determines a homomorphism of $\mathfrak{so}(n)$-modules $\otimes^{2r} V \to \mathbb{C}$ and so an element of $N(\otimes^{2r} V)$. The map determined by the matching $(i_1, j_1) \ldots (i_r, j_r)$ is
\[
v_1 \otimes \cdots \otimes v_{2r} \mapsto \langle v_{i_1}, v_{j_1} \rangle \ldots \langle v_{i_r}, v_{j_r} \rangle
\]

The Frobenius character of this representation is the plethysm $h_r[h_2]$. One way to see this is given in [Sta99] A.2.9 Example]. The action of $S(2r)$ on $X(r)$ is transitive and the stabiliser is the wreath product $S(r) \rtimes C_2^r$. Hence this representation is obtained by inducing the trivial representation of $S(r) \rtimes C_2^r$. This can also be understood using the theory of combinatorial species. A comprehensive account of this theory is given in [BLL98]. From this point of view perfect matchings can be seen as a species. This species is the composite of two functors. The first functor defines a unique structure on the set with two elements and no other structure. The cycle index series of this species is $h_2$. The other functor has cycle index series $h_r$.

The plethysm $h_r[h_2]$ was evaluated in [Lit06] and is also discussed in [Sta99] Exercise 7.28]. The result is
\[
\sum_{r \geq 0} h_r[h_2] = \prod_{i \leq j} (1 - x_i x_j)^{-1} = \sum_{\lambda} s_{2\lambda}
\]
**Proposition 6.3.** For $r \geq 1$, $(X(r), P(r), C)$ exhibits the cyclic sieving phenomenon where $C$ is the action of the cyclic group of order $2r$ acting on $X(r)$ by rotation and $P(r)$ is given by

$$P(r) = \sum_{\mu : |\mu| = r} f_q(2\mu)$$

**Example 6.4.** For $r = 1$, $X(r)$ has 1 element and $P(1) = 1$. The Brauer diagram of the only perfect matching on two points is

Example 6.5. For $r = 2$, $X(r)$ has 3 elements. The two partitions of 2 are $(2)$ and $(1, 1)$. Then we have

$$f_q(4) = 1 \quad f_q(2, 2) = q^2\frac{[4]}{2} = 1 + q^2 + q^4 \equiv 2 + q^2 \mod q^4 - 1$$

This corresponds to one orbit of size 2 and one orbit of size one. The Brauer diagrams and the perfect matchings are

Example 6.6. For $r = 3$, $X(3)$ has 15 elements. There are three partitions of 3. The polynomials $f_q(2\lambda)$ are given in the following table

|   | (3)          | (2, 1)       | (1, 1, 1) |
|---|--------------|--------------|-----------|
| 1 | $q^{[3][3]}$ | $q^{[6][6]}$ | $q^{[10][12]}$ |

The polynomial $P(3)$ is

$$1 + q^2 + q^3 + 2q^4 + 3q^5 + 3q^6 + q^7 + 2q^8 + q^9 + q^{10} + q^{12}$$

Then reducing modulo $q^6 - 1$ gives

$$5 + q + 3q^2 + 2q^3 + 3q^4 + q^5$$

There is one orbit of size one. This consists of the perfect matching $(1, 4)(2, 5)(3, 6)$. There is one orbit of size 2. This is

$$(1, 2)(3, 4)(5, 6), (1, 6)(2, 3)(4, 5)$$

. There are two orbits of size 3. These are

$$(1, 6)(2, 5)(3, 4), (1, 2)(3, 6)(4, 5), (1, 4)(2, 3)(5, 6)$$

$$(1, 3)(2, 5)(4, 6), (1, 5)(2, 4)(3, 6), (1, 4)(2, 6)(3, 5)$$

There is one orbit of size 6. This is

$$(1, 3)(2, 4)(5, 6), (1, 6)(2, 4)(3, 5), (1, 2)(3, 5)(4, 6)$$

$$(1, 5)(2, 3)(4, 6), (1, 5)(2, 6)(3, 4), (1, 3)(2, 6)(4, 5)$$

**Example 6.7.** For $n = 4$, $X(n)$ has 105 elements. There are five partitions of 4. The polynomials $f_q(2\lambda)$ are given in the following table
The polynomial $P(4)$ is

$$1 + q^2 + q^3 + 3q^4 + 2q^5 + 5q^6 + 4q^7 + 8q^8 + 6q^9 + 9q^{10} + 7q^{11} + 11q^{12} + 7q^{13} + 9q^{14} + 6q^{15} + 8q^{16} + 4q^{17} + 5q^{18} + 2q^{19} + 3q^{20} + q^{21} + q^{22} + q^{24}$$

Then reducing modulo $q^8 - 1$ gives

$$18 + 10q + 15q^2 + 10q^3 + 17q^4 + 10q^5 + 15q^6 + 10q^7$$

This corresponds to 10,5,2,1 orbits of size 8,4,2,1 respectively.

For $n \geq 1$, take $g = \mathfrak{sp}(2n)$ and $V$ to be the vector representation.

**Conjecture 6.8.** For $r \geq 1$, the character of the space of invariant tensors of $\otimes^{2r} V$ as a representation of $S(2r)$ is

$$\sum_{\mu} s_{\mu}$$

where the sum is over partitions $\mu$ such that $\mu$ has at most $2n$ parts and $|\mu| = r$.

In particular, for $n > r$, this is independent of $n$ and is given by

$$\sum_{\mu : |\mu| = r} s_{\mu}$$

Note that in (4) it makes sense to take the sum over partitions $\mu$ such that $\mu$ has at most $2n + 1$ parts and $|\mu| = r$. It seems reasonable to speculate that this can be interpreted using the character theory of the odd symplectic groups given in [Pro88].

### 7. Adjoint

Take $g = \mathfrak{sl}(n)$ and $V$ to be the adjoint representation, so $V$ is the vector space of $n \times n$ matrices with trace zero. Here we record the decomposition of the invariant tensors in $\otimes^r V$ as a representation of the symmetric group $S(r)$.

Then for $r = 2, 3, 4$ we have

| 2 | 3 | 1^3 | 4 | 2,2 | 2,1^2 |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 1 | 0 |
| 1 | 1 | 1 | 2 | 1 |
| 2 | 2 | 1 |

For $r = 2$ this records the fact that the adjoint representation has an invariant bilinear form. This is the Killing form. For $r = 3$ this records the fact that there is an invariant anti-symmetric trilinear form and an invariant symmetric trilinear form. The invariant anti-symmetric trilinear form is given by

$$u \otimes v \otimes w \mapsto \langle [u, v], w \rangle = \langle u, [v, w] \rangle$$
The invariant symmetric trilinear form is given by

\[(6) \quad u \otimes v \otimes w \mapsto \text{Tr}(uvw) + \text{Tr}(vuw)\]

The polynomials for \(n \geq r\) and their reductions modulo \(q^r - 1\) are:

\[
\begin{array}{c|c}
1 & 1 \\
1 + q^3 & 2 \\
2 + 2q^2 + q^3 + 3q^4 + q^5 & 5 + q + 2q^2 + q^3 \\
\end{array}
\]

For \(r = 5\) we have

\[
\begin{array}{cccccccc}
[5] & [4, 1] & [3, 2] & [3, 1^2] & [2^2, 1] & [2, 1^3] & [1^5] \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 2 & 1 & 1 & 1 \\
1 & 1 & 2 & 3 & 1 & 1 & 1 \\
2 & 1 & 2 & 3 & 1 & 1 & 1 \\
\end{array}
\]

The polynomial for \(n \geq 5\) computed from this is

\[2 + q + 3q^2 + 6q^3 + 7q^4 + 9q^5 + 7q^6 + 5q^7 + 2q^8 + q^9 + q^{10}\]

Then reducing modulo \(q^5 - 1\) gives

\[12 + 8q + 8q^2 + 8q^3 + 8q^4 + 8q^5\]

For \(r = 6\) we have

\[
\begin{array}{cccccccc}
[6] & [5, 1] & [4, 2] & [3^2] & [4, 1^2] & [3, 2, 1] & [2^3] & [3, 1^3] & [2^2, 1^2] & [2, 1^4] \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
2 & 0 & 3 & 1 & 2 & 2 & 3 & 3 & 1 & 1 \\
3 & 1 & 5 & 1 & 3 & 4 & 5 & 4 & 2 & 2 \\
3 & 1 & 6 & 1 & 4 & 4 & 5 & 4 & 2 & 2 \\
4 & 1 & 6 & 1 & 4 & 4 & 5 & 4 & 2 & 2 \\
\end{array}
\]

The polynomial for \(n \geq 6\) computed from this is

\[4 + q + 7q^2 + 12q^3 + 21q^4 + 24q^5 + 38q^6 + 33q^7 + 37q^8 + 30q^9 + 25q^{10} + 14q^{11} + 13q^{12} + 4q^{13} + 2q^{14}\]

Then reducing modulo \(q^6 - 1\) gives

\[51 + 35q + 43q^2 + 39q^3 + 43q^4 + 35q^5\]

Next we give an alternative method for determining the character of the representation of the symmetric group on the space of invariant tensors. This method applies for \(n \geq r\).

First we consider the adjoint representation of \(\mathfrak{gl}(n)\) Let \(W\) be the vector representation, \(W^*\) the dual representation and \(V\) the adjoint representation. Then we identify \(V\) with \(W \otimes W^*\) and \(\otimes^r V\) with \(W \otimes W^* \otimes W \otimes \cdots \otimes W^*\). Define a directed matching to be a partition of the set \(\{1, 2, \ldots, 2r\}\) into \(r\) ordered pairs such that for each ordered pair \((i, j)\), \(i\) is odd and \(j\) is even. There is a bijection between perfect matchings and \(S(r)\). This maps the permutation \(\sigma\) to the directed matching \((1, 2\sigma(1)), \ldots, (2r - 1, 2\sigma(r))\).
For example, for $r = 1$ and $r = 2$, the Brauer diagrams and the permutations are

\begin{align*}
(1) & \quad (1, 2) & \quad (2, 1) \\
\end{align*}

Each permutation determines a homomorphism of $\mathfrak{gl}(n)$-modules $\otimes^r V \to \mathbb{C}$ and so an element of $N(\otimes^r V)$. The map determined by the permutation $\sigma$ is

$$v_1 \otimes \phi_1 \otimes v_2 \otimes \cdots \otimes \phi_r \mapsto \langle v_{i_1}, \phi_{\sigma(1)} \rangle \cdots \langle v_{i_r}, \phi_{\sigma(r)} \rangle$$

For $n \geq r$ this is a basis for the space of coinvariants. There is a canonical isomorphism $\otimes^r V \cong \text{End}(\otimes^r W)$. This induces an isomorphism $N(\otimes^r V) \cong \text{End}_{\mathfrak{gl}(n)}(\otimes^r W)$. This isomorphism induces a bijection between directed matchings and the image of $S(r)$ in $\text{End}_{\mathfrak{gl}(n)}(\otimes^r W)$. This shows that for $n \geq r$, we have a basis of $N(\otimes^r V)$ indexed by the set $S(r)$ such that the inclusion map $S(r) \to N(\otimes^r V)$ is $S(r)$-equivariant for the conjugation action of the group $S(r)$ on the set $S(r)$.

This shows that for $n \geq r$ there is a $C$-equivariant bijection between the Lusztig basis $\mathcal{B}(r)$ and the set $S(r)$ with $C$-action given by conjugation by the long cycle.

Now let $V$ be the adjoint representation of $\mathfrak{sl}(n)$. Then we have $W \otimes W^* \cong V \oplus \mathbb{C}$. Let $X(r) \subset S(r)$ be the set of permutations with no fixed point. These are known as derangements. Then, for $n \geq r$, we have a basis of $N(\otimes^r V)$ indexed by the set $X(r)$ such that the inclusion map $X(r) \to N(\otimes^r V)$ is $S(r)$-equivariant for the conjugation action of $S(r)$ on $X(r)$. In terms of the directed matchings, a directed matching is a derangement if it does not contain any ordered pair $(2i - 1, 2i)$ for $i \geq 1$. For example, in (7) only the third diagram corresponding to the permutation $(2, 1)$ is a derangement. For $r = 3$ there are two derangements; the Brauer diagrams and the permutations are

\begin{align*}
(2, 3, 1) & \quad (3, 1, 2) \\
\end{align*}

These correspond to the trilinear forms in (5) and (6).

This shows that for $n \geq r$ there is a $C$-equivariant bijection between the Lusztig basis $\mathcal{B}(r)$ and the set of derangements of $r$ letters with $C$-action given by conjugation by the long cycle.

### 7.1. Rencontré numbers

For $n \geq 0$ let $p_n(t)$ be the generating function for permutations with statistic the number of fixed points:

$$p_n(t) = \sum_{w \in S(n)} t^{\text{Fix}(w)}$$


Alternatively let $D_{n,k}$ be the number of permutations in $S(n)$ with $k$ fixed points. Then

$$p_n(t) = \sum_{k=0}^{n} D_{n,k} t^k$$

The numbers $D_n = D_{n,0}$ are known as derangement numbers and the numbers $D_{n,k}$ are known as rencontre numbers. The derangement numbers determine the rencontre numbers by the relation

$$D_{n,k} = \binom{n}{k} D_{n-k,0} \quad (9)$$

The derangement numbers are sequence [A008290] in [Slo08].

Then we have the following expression for the exponential generating function

$$\sum_{n \geq 0} p_n(t) \frac{z^n}{n!} = \frac{\exp(zt - z)}{1 - z}$$

$$= \frac{\exp(zt)}{1 - \sum_{n \geq 2} (n - 1) \frac{z^n}{n!}}$$

In particular putting $t = 1$ gives the exponential generating function for the number of permutations and putting $t = 0$ gives the exponential generating function for the number of derangements:

$$\sum_{n \geq 0} D_n \frac{z^n}{n!} = \frac{\exp(-z)}{1 - z}$$

The $q$-analogues of rencontre numbers are given in [Wac89] and [Lot02, Problems 11.6.2]. These are defined by

$$D_{n,k}(q) = \sum_{w} q^{\text{maj}(w)}$$

where the sum is over permutations in $S(n)$ with $k$ fixed points. It is shown that the $q$-derangement numbers are given by

$$D_{n,0}(q) = [n]! \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{q^k}{[k]!}$$

These determine the $q$-rencontre numbers by the relation

$$D_{n,k}(q) = \left[ \frac{n}{k} \right] D_{n-k,0} \quad (10)$$

The $q$-analogue of the exponential generating function is

$$\sum_{n \geq 0} \sum_{k=0}^{n} D_{n,k}(q) \frac{z^n}{n!} = \frac{\exp_q(zt)}{1 - \sum_{n \geq 2} (n - 1) \frac{z^n}{[n]!}}$$
This was extended in [SSW09]. Define the power series $H(z)$ by
\[ H(z) = \sum_{n \geq 0} h_n z^n \]
where $h_n$ is the complete homogeneous symmetric function of degree $n$. There they give the generating function for a sequence of symmetric functions
\[ \frac{(1 - qt)H(rz)}{H(qt) - qtH(z)} \]
Here we put $t = q^{-1}$ and $r = t$ in this generating function. This gives
\[ H(tz) \]
This is expanded as a power series in $z$ and then the coefficients of the powers of $z$ are polynomial in $t$ with coefficients symmetric functions.

This expansion defines the symmetric functions $F_{n,k}$ by
\[ \sum_{n \geq 0} \sum_{k=0}^n F_{n,k} t^k z^n = \frac{H(tz)}{1 - \sum_{n \geq 2} (n - 1) h_n z^n} \]
The symmetric functions $F_{n,0}$ determine the symmetric functions $F_{n,k}$ by the relation
\[ F_{n,k} = h_k F_{n-k,0} \]
The generating function for the $F_n = F_{n,0}$ is given by taking $t = 0$ this gives
\[ \sum_{n \geq 0} F_n z^n = \frac{1}{1 - \sum_{n \geq 2} (n - 1) h_n z^n} \]

**Proposition 7.1.** Let $X_{n,k}$ be the set of permutations in $S(n)$ with $k$ fixed points with the conjugation action of $S(n)$. Then $F_{n,k}$ is the Frobenius character or cycle index series of $X_{n,k}$.

This implies that $(X_{n,k}, D_{n,k}(q), C_{n,k})$ exhibits the cyclic sieving phenomenon where $C_{n,k}$ is the action of the cyclic group generated by a regular element (for example, the long cycle).

The following table shows the first terms in this series of symmetric functions. The first row expresses these symmetric functions in terms of the complete homogeneous functions and the second row in terms of the Schur functions.

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | $h[2]$ | $2h[3]$ | $3h[4] + h[2,2]$ | $4h[5] + 4h[3,2]$ |
| 1 | 0 | $s[2]$ | $2s[3]$ | $4s[4] + s[3,1] + s[2,2]$ | $8s[5] + 4s[4,1] + 4s[3,2]$ |

Comparing this with the previous calculations we find we have different symmetric functions for $3 \leq n \leq 5$. These symmetric functions are given below for comparison:

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0 | $s[2]$ | $s[3] + s[1,1,1]$ | $2s[4] + 2s[2,2] + s[2,1,1]$ |
The conclusion is that these representations of $S(n)$ are different even though both representations have bases fixed by the action of the group and these sets are isomorphic as sets with an action of the cyclic group. Moreover the two fake degree polynomials are different even though they are equivalent modulo $q^n - 1$.

For $GL$ instead of $SL$ the bijection between the bases is an isomorphism of sets with an action of $S(n)$. The generating function for the symmetric functions is given by putting $t = 1$. This gives

$$H(z) = \frac{1}{1 - \sum_{n\geq 2}(n - 1)h_nz^n}$$

The $q$-analogue of the exponential generating function is

$$\sum_{n\geq 0}\sum_{k=0}^n D_{n,k}(q)\frac{z^n}{[n]!} = \frac{\exp_q(z)}{1 - \sum_{n\geq 2}(n - 1)\frac{z^n}{[n]!}}$$

8. Exceptional groups

For $G_2$ there two fundamental representations. One is the adjoint representation of dimension 14 and the other has dimension 7, which we refer to as the vector representation. We record the multiplicity of the trivial representation in $S^\lambda(V)$ for $|\lambda| = 2, 3$ for these two representations in the following table

|      | $[2]$ | $[1, 1]$ | $[3]$ | $[2, 1]$ | $[1, 1, 1]$ |
|------|-------|---------|------|---------|-----------|
| vector | 1     | 0       | 0    | 0       | 1         |
| adjoint | 1     | 0       | 0    | 0       | 1         |

This table just records the information that each representation has an invariant symmetric bilinear form and an invariant anti-symmetric trilinear form and there are no other invariant tensors of degree at most 3. For $r = 2$ this gives the polynomial 1. For $r = 3$ this gives the polynomial $q^3$. Then reducing modulo $q^3 - 1$ gives 1.

For $r = 4$ we have

|      | $[4]$ | $[3, 1]$ | $[2, 2]$ | $[2, 1, 1]$ | $[1, 1, 1, 1]$ |
|------|-------|---------|---------|------------|---------------|
| vector | 1     | 0       | 1       | 0          | 1             |
| adjoint | 1     | 0       | 2       | 0          | 0             |

This gives the polynomials

- vector: $1 + q^2 + q^4 + q^6$ vs $2 + 2q^2$
- adjoint: $1 + 2q^2 + 2q^4$ vs $3 + 2q^2$

This means that for the vector representation there are two orbits of size 2 and for the adjoint representation there are two orbits of size 2 and one orbit of size 1.

For $r = 5$ we have
This gives the polynomials

\[
\begin{align*}
\text{vector} & : q^3 + q^4 + 2q^5 + 2q^6 + 2q^7 + q^8 + q^9 \\
\text{adjoint} & : 2q^3 + 2q^4 + 4q^5 + 3q^6 + 3q^7 + q^8 + q^9
\end{align*}
\]

and the reductions modulo \(q^5 - 1\) are

\[
\begin{align*}
\text{vector} & : 2 + 2q + 2q^2 + 2q^3 + 2q^4 \\
\text{adjoint} & : 4 + 3q + 3q^2 + 3q^3 + 3q^4
\end{align*}
\]

This means that for the vector representation there are two free orbits and for the adjoint representation there are three free orbits and fixed point.

For \(r = 6\) we have

\[
\begin{array}{ccccccc}
\text{vector} & [6] & [4, 2] & [3, 2, 1] & [3, 1^2] & [2^3] & [2, 1^3] \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
\text{adjoint} & 0 & 0 & 0 & 2 & 0 & 1 & 0
\end{array}
\]

For the vector representation this gives the polynomial

\[
1 + q^2 + q^3 + 2q^4 + q^5 + 5q^6 + 2q^7 + 5q^8 + 4q^9 + 5q^{10} + 2q^{11} + 4q^{12} + q^{13} + q^{14}
\]

For the adjoint representation this gives the polynomial

\[
2 + 3q^2 + 3q^3 + 7q^4 + 5q^5 + 13q^6 + 7q^7 + 12q^8 + 8q^9 + 9q^{10} + 3q^{11} + 6q^{12} + q^{13} + q^{14}
\]

and the reductions modulo \(q^6 - 1\) are

\[
\begin{align*}
\text{vector} & : 10 + 3q + 7q^2 + 5q^3 + 7q^4 + 3q^5 \\
\text{adjoint} & : 21 + 8q + 16q^2 + 11q^3 + 16q^4 + 8q^5
\end{align*}
\]

For the vector representation this corresponds to 3,4,2,1 orbits of sizes 6,3,2,1 respectively. For the adjoint representation this corresponds to 8,8,3,2 orbits of sizes 6,3,2,1 respectively.

For the seven dimensional representation these agree with [Kup96]. The set \(X(r)\) is the set of non-positive trivalent planar graphs with \(r\) boundary points. A trivalent planar graph means a subset \(\Gamma\) of the lower half plane \(H\) such that every point of \(\Gamma\) has an open neighbourhood in \(H\) which is homeomorphic to one of the following three cases:

The centre point in the second case is called a trivalent vertex. In the third case the straight boundary is in the boundary of \(H\). Each of these points is a boundary point. Note that there is no restriction on the number of trivalent vertices of the graph or on the number of connected components. A trivalent planar graph is non-positive if it does not contain any of the following forbidden configurations:
The diagrams in this section and in section 3.3 are drawn with two types of edge; namely a single edge and a double edge. The conversion to non-positive trivalent planar graphs is given by making the following substitution for each double edge:

A curvature argument using the isoperimetric inequality for surfaces of non-positive curvature shows that for all \( r \geq 0 \), the set \( X(r) \) is finite. The numbers \(|X(r)|\) for \( 0 \leq r \leq 9 \) are given in the following table

| \( r \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|------|---|---|---|---|---|---|---|---|---|
| \(|X(r)|\) | 1 | 0 | 1 | 4 | 10 | 35 | 120 | 455 | 1792 |

This is sequence A059710 in [Slo08].

The cyclic group of order \( r \) acts on \( X(r) \) by rotation. Next we verify the previous calculations. For \( r = 0 \) the only diagram is the empty diagram. There is no diagram for \( r = 1 \). For \( r = 2, 3 \) there is one diagram. These are

**Example 8.1.** For \( r = 4 \) we have four diagrams. The two orbits of order two are:

**Example 8.2.** For \( r = 5 \) we have ten diagrams. There are two orbits of order five. One is the orbit
8.1. **Spin.** In this section we take $V$ to be the spin representation of $\mathfrak{so}(7)$. This has dimension eight. The diagrams for this example are given in [Wes08].

For $r = 2, 3$ the multiplicities of the trivial representation in $S^r(V)$ are:

| $r$ | Multiplicities |
|-----|----------------|
| 2   | [2], [1, 1]    |
| 3   | [3], [2, 1], [1, 1, 1] |

This records the information that $V$ has a symmetric invariant bilinear form and that there are no other invariant tensors in $\otimes^r V$ for $r = 2, 3$. In general, there are no non-zero invariant tensors in $\otimes^r V$ for $r$ odd.

For $r = 4$ we have

| $r$ | Multiplicities |
|-----|----------------|
| 4   | [4], [3, 1], [2, 2], [2, 1, 1], [1, 1, 1, 1] |

The fake degree polynomial and its reduction modulo $q^4 - 1$ are

$$1 + q^2 + q^4 + q^6 + 2 + 2q^2$$

This corresponds to two orbits of size 2.

For $r = 6$ we have

| $r$ | Multiplicities |
|-----|----------------|
| 6   | [6], [4, 2], [3, 1, 1], [2, 1, 1], [2, 1, 1] |

The fake degree polynomial is

$$1 + q^2 + q^3 + 2q^4 + q^5 + 4q^6 + 2q^7 + 4q^8 + 3q^9 + 4q^{10} + 2q^{11} + 3q^{12} + q^{13} + q^{14}$$

The reduction modulo $q^6 - 1$ is

$$8 + 3q + 6q^2 + 4q^3 + 6q^4 + 3q^5$$

This corresponds to 3, 3, 1, 1 orbits of sizes 6, 3, 2, 1 respectively.

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