A Numerical Method for Pricing Discrete Double Barrier Option by Lagrange Interpolation on Jacobi Node

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Abstract

In this paper, a rapid and high accurate numerical method for pricing discrete single and double barrier knock-out call options is presented. According to the well-known Black-Scholes framework, the price of option in each monitoring date could be calculate by computing a recursive integral formula upon the heat equation solution. We have approximated these recursive solutions with the aim of Lagrange interpolation on Jacobi polynomials node. After that, an operational matrix, that makes our computation significantly fast, has been driven. The most important feature of this method is that its CPU time dose not increase when the number of monitoring dates increases. The numerical results confirm the accuracy and efficiency of the presented numerical algorithm.

Keywords: Double and single barrier options, Black-Scholes model, Option pricing, Jacobi polynomials

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1. Introduction

Barrier options play a key role in financial markets where the most important problem is the so called option valuation problem, i.e. to compute a fair value for the option, i.e. the premium. The Nobel Prize-winning Black-Scholes option valuation theory motivates using classical numerical methods for partial differential equations (PDE’s) \cite{1}. In computational Finance numerous nonstandard numerical methods are proposed and successfully applied for pricing options \cite{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}. Numerical methods are often preferred to closed-form solutions as it they could me more easily extended or adapted to satisfy all the financial requirements of the option contracts and continuously changing conditions imposed by financial institutions and over-the-counter market for controlling trading of derivatives.

Kunitomo and Ikeda \cite{13} obtained general pricing formulas for European double barrier options with curved barriers but like for a variety of path-dependent options and corporate securities most formulas are obtained for restricted cases as continuous monitoring or single barrier \cite{5}. The discrete monitoring is essential as the trading year is considered to consist of 250 working days and a week of 5 days. Thus, taking for one year \( T = 1 \), the application of barriers occurs with a time increment of 0.004 daily and 0.02 weekly.

For discrete barrier options there are some analytical solutions. For example, Fusai reduces the problem of pricing one barrier option to a Wiener-Hopf integral equation \cite{3}. Several other different contracts with discrete time monitoring are characterized by updating the initial conditions, such as Parisian options and occupation time derivatives \cite{14}. We remark that although most real contracts specify fixed times for monitoring the asset, academic researchers have focused mainly on continuous time monitoring models as the analysis of fixed barriers could be treated mathematically using some techniques such as the reflection principle \cite{15}. For example, using the reflection principle in Brownian motions, Li expresses the solution in general as summation of an infinite number of normal distribution functions for standard double barrier options, and in many non-trivial cases the solution consists of finite terms \cite{16}.

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2. The Pricing Model

We assume that the stock price process $S_t$ follows the Geometric Brownian motion:

$$ \frac{dS_t}{S_t} = rdt + \sigma dB_t $$

where $S_0$, $r$ and $\sigma$ are initial stock price, risk-free rate and volatility respectively. We consider the problem of pricing knock-out discrete double barrier call option, i.e. a call option that becomes worthless if the stock price touches either lower or upper barrier at the predetermined monitoring dates:

$$ 0 = t_0 < t_1 < \cdots < t_M = T. $$
We assume that monitoring dates are equally spaced, i.e.; \( t_m = m \tau \) where \( \tau = \frac{T}{m} \). If the barriers are not touched in monitoring dates, the pay off at maturity time is \( \max(S_T - E, 0) \), where \( E \) is exercise price. The price of option is defined discounted expectation of pay off at the maturity time. Based on the Black-Scholes framework, the option price \( \mathcal{P}(S, t, m - 1) \) as a function of stock price at time \( t \in (t_{m-1}, t_m) \), satisfies in the following partial differential equations

\[
-\frac{\partial \mathcal{P}}{\partial t} + rS \frac{\partial \mathcal{P}}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \mathcal{P}}{\partial S^2} - r\mathcal{P} = 0, \tag{1}
\]

subject to the initial conditions:

\[
\mathcal{P}(S, t_0, 0) = (S - E) \mathbf{1}_{\{S \geq E \}}
\]

\[
\mathcal{P}(S, t_m, 0) = \mathcal{P}(S, t_m, m - 1) \mathbf{1}_{\{S \leq U \}}; \quad m = 1, 2, \ldots, M - 1,
\]

where \( \mathcal{P}(S, t_m, m - 1) := \lim_{t \to t_m} \mathcal{P}(S, t, m - 1) \).

By denoting \( E^* = \ln \left( \frac{E}{S} \right) \), \( \mu = r - \frac{\sigma^2}{2} \), \( \theta = \ln \left( \frac{E^*}{S} \right) \) and \( \delta = \max \{ E^*, 0 \} \), we define \( g_m(z) \) as following recursive formula:

\[
g_1(z) = \int_0^0 k(z, \tau) g_0(\xi) d\xi \tag{2}
\]

\[
g_m(z) = \int_0^0 k(z, \tau) g_{m-1}(\xi) d\xi; \quad m = 2, 3, \ldots, M \tag{3}
\]

where

\[
g_0(z) = Le^{-\alpha\tau} \left( e^{\alpha} - e^{\delta} \right) \mathbf{1}_{\{0 \leq \xi \leq \delta \}},
\]

\[
k(z, \tau) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{z^2}{4t}}. \tag{5}
\]

It could be shown that the price of the knock-out discrete double barrier option can be obtain as follows (see [41]):

\[
\mathcal{P}(S_0, t_M, M - 1) = e^{\alpha \tau + \beta} g_M(z_0/\theta) \tag{6}
\]

where \( z_0 = \log \left( \frac{E}{S} \right) \).

3. Jacobi Polynomials

Let \( w^{(\alpha, \beta)}(x) = (1 - x)^\alpha (1 + x)^\beta \), \( \alpha, \beta > -1 \), and \( L_{\nu_{\alpha, \beta}}^2(-1, 1) \) be Hilbert space with the following inner product and norm:

\[
< f, g >_{\nu_{\alpha, \beta}} = \int_{-1}^1 f(x)g(x)w(x)dx, \tag{7}
\]

\[
\|f\|_{\nu_{\alpha, \beta}} = \sqrt{< f, f >_{\nu_{\alpha, \beta}}}. \tag{8}
\]

The Jacobi polynomials, \( J_i^{(\alpha, \beta)}(x) \) are orthogonal polynomials in \( L_{\nu_{\alpha, \beta}}^2(-1, 1) \), i.e;

\[
\int_{-1}^1 J_i^{(\alpha, \beta)}(x) J_j^{(\alpha, \beta)}(x)w(x)dx = \delta_{ij}, \tag{9}
\]

where \( \lambda_i = |J_i^{(\alpha, \beta)}|^2 \). These polynomials, that set an orthogonal basis in \( L_{\nu_{\alpha, \beta}}^2(-1, 1) \), satisfy in following three-term recurrence relation:

\[
J_0^{(\alpha, \beta)}(x) = 1, \quad J_1^{(\alpha, \beta)}(x) = \frac{1}{2} (\alpha + \beta + 2) x + \frac{1}{2} (\alpha - \beta) \tag{10}
\]

\[
J_{i+1}^{(\alpha, \beta)}(x) = (\alpha_i^{(\alpha, \beta)} x - b_i^{(\alpha, \beta)} J_i^{(\alpha, \beta)}(x)) - c_i^{(\alpha, \beta)} J_{i-1}^{(\alpha, \beta)}(x) \tag{11}
\]
where:

\[
\alpha_i^{(\alpha, \beta)} = \frac{(2i + \alpha + \beta + 1)(2i + \alpha + \beta + 2)}{2(i + 1)(n + \alpha + \beta + 1)}
\]

\[
b_i^{(\alpha, \beta)} = \frac{(\beta^2 - \alpha^2)(2n + \alpha + \beta + 1)}{2(i + 1)(n + \alpha + \beta + 1)(2n + \alpha + \beta)}
\]

\[
c_i^{(\alpha, \beta)} = \frac{(n + \alpha)(n + \beta)(2n + \alpha + \beta + 2)}{(i + 1)(n + \alpha + \beta + 1)(2n + \alpha + \beta)}
\]

4. Pricing by orthogonal Lagrange interpolation

In this section we consider \(\Pi_n\) as space of all polynomials with degree less or equal to \(n\), set points \(\{x_i^{\alpha, \beta}\}_{i=0}^n\) as roots of \((n + 1)\)-th Jacobi polynomial \(J_n^{(\alpha, \beta)}\) that are shifted to \([0, \theta]\) and \(I_n^{\alpha, \beta} : C[0, \theta] \rightarrow \Pi_n\) as orthogonal polynomial interpolation projection operator, that is defined as follows:

\[
I_n^{\alpha, \beta}(f) = \sum_{i=0}^{n} f(x_i^{\alpha, \beta})L_i(x)
\]

where \(L_i(x)\) is the \(i\)-th Lagrange polynomial basis function defined on \(\{x_i^{\alpha, \beta}\}_{i=0}^n\):

\[
L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j^{\alpha, \beta}}{x_i^{\alpha, \beta} - x_j^{\alpha, \beta}}
\]

Let operator \(\mathcal{K} : L^2([0, \theta]) \rightarrow L^2([0, \theta])\) is defined as follows:

\[
\mathcal{K}(g)(z) := \int_{0}^{\theta} \kappa(z - \xi, \tau)g(\xi)d\xi.
\]

where \(\kappa\) is defined in (5). According to the definition of operator \(\mathcal{K}\), equations (2) and (3) can be rewritten as below:

\[
g_1 = \mathcal{K}g_0
\]

\[
g_m = \mathcal{K}g_{m-1} \quad m = 2, 3, ..., M
\]

We denote

\[
\mathcal{g}_{1,n} = I_n^{\alpha, \beta}\mathcal{K}(g_0)
\]

\[
\mathcal{g}_{m,n} = I_n^{\alpha, \beta}\mathcal{K}(\mathcal{g}_{m-1}) = \left(I_n^{\alpha, \beta}\mathcal{K}\right)^{(m)}(g_0) , \quad m \geq 2.
\]

where \(I_n^{\alpha, \beta}\mathcal{K}\) is as follows:

\[
(I_n^{\alpha, \beta}\mathcal{K})(g) = I_n^{\alpha, \beta}\left(\mathcal{K}(g)\right).
\]

Since, \(\mathcal{g}_{m,n} \in \Pi_n\) for \(m \geq 1\), we can write

\[
\mathcal{g}_{m,n} = \sum_{i=0}^{n} a_{mi}L_i(z) = \Phi'_n(x)G_m,
\]

where \(G_m = [a_{m0}, a_{m1}, \ldots, a_{mn}]\) and \(\Phi_n = [L_m, L_m, \ldots, L_m]'\). From equation (21) we obtain

\[
\mathcal{g}_{m,n} = (I_n^{\alpha, \beta}\mathcal{K})^{(m-1)}(\mathcal{g}_{1,n}).
\]
Since $\Pi_n$ is a finite dimensional linear space, thus the linear operator $L_{n}^{a\beta}K \text{ on } \Pi_n$ could be considered as a $n \times n$ matrix $K$. Consequently equation (22) can be written as following matrix operator form

$$
\tilde{g}_{mn} = \Phi_n^{a} K^{m-1} G_1.
$$

For evaluation of the option price by (23), it is enough to calculate the matrix operator $K$ and the vector $G_1$. It is easy to check (see [41]) that:

$$
G_1 = [a_{11}, a_{12}, \cdots, a_{1n}]',
$$

$$
K = (k_{ij})_{n \times n}
$$

where

$$
a_{1i} = \int_0^\theta \kappa(x_i^{a\beta} - \xi, \tau)g_0(\xi)d\xi, \ 0 \leq i \leq n.
$$

$$
k_{ij} = \int_0^\theta \kappa(x_i^{a\beta} - \xi, \tau)L_{j-1}(\xi)d\xi.
$$

Therefore, the price of the knock-out discrete double barrier option can be estimated as follows:

$$
\mathcal{P}(S_0, t_M, M - 1) \simeq e^{\gamma \Delta \tau} \tilde{g}_{M,n}(z_0)
$$

where $z_0 = \log\left(\frac{K}{M}\right)$ and $\tilde{g}_{M,n}$ from (23). The matrix form of relation (23) implies that the computational time of presented algorithm be nearly fixed when monitoring dates increase. Actually, the complexity of our algorithm is $O(n^2)$ that dose not depend on number of monitoring dates.

5. Numerical Result

In the current section, the presented method in previous section for pricing knock-out call discrete double barrier option is compared with some other methods. The numerical results are obtained from the relation (24) with $n$ basis functions. The Source code has been written in MATLAB 2015 on a 3.2 GHz Intel Core i5 PC with 8 GB RAM.

Example 1. In the first example, the pricing of knock-out call discrete double barrier option is considered with the following parameters: $r = 0.05$, $\sigma = 0.25$, $T = 0.5$, $S_0 = 100$, $E = 100$, $U = 120$ and $L = 80, 90, 95, 99, 99.5$. In table 1, numerical results of presented method with Milev numerical algorithm [3], Crank-Nicholson [22], trinomial, adaptive mesh model (AMM) and quadrature method QUAD-K200 as benchmark [23] are compared for various number of monitoring dates. In addition, it can be seen that CPU time of presented method is fixed against increases of monitoring dates.

| $\alpha$ | $\beta$ | $-0.8$ | $-0.5$ | $0$ | $0.5$ | $0.8$ |
|----------|---------|--------|--------|-----|------|------|
| $-0.8$   | 8.6074e-06 | 8.1718e-06 | 2.2103e-05 | 1.8770e-05 | 9.5120e-06 |
| $-0.5$   | 8.7606e-06 | 7.8929e-06 | 1.2852e-05 | 3.8600e-05 | 4.6071e-05 |
| 0        | 2.7040e-05 | 2.5788e-05 | 2.2103e-05 | 5.3726e-05 | 9.3608e-05 |
| 0.5      | 9.1438e-05 | 9.5461e-05 | 9.2619e-05 | 8.2564e-05 | 1.0667e-04 |
| 0.8      | 1.4675e-04 | 1.6600e-04 | 1.7498e-04 | 1.6536e-04 | 1.5456e-04 |

Table 1: The maximum norm error for $n = 25$ of example (1) with $L = 95$ and $M = 125$. 

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Example 2. In this example, the parameters of knock-out call discrete double barrier option is considered as $r = 0.05$, $\sigma = 0.25$, $T = 0.5$, $E = 100$, $U = 110$ and $L = 95$. In table (3) the option price for different spot prices are evaluated and compared with Milev numerical algorithm [5], Crank-Nicholson [42] and the Monte Carlo (MC) method with $10^7$ paths [44].
Example 3. \textit{Due to the fact that the probability of crossing upper barrier during option’s life when } \( \text{U} \geq 2E \text{ is too small, the price of discrete single down-and-out call option can be estimated by double ones by setting upper barrier greater than } 2E \text{ (for more details see[5]). Now, we consider a discrete single down-and-out call option with the following parameters: } r = 0.1, \sigma = 0.2, T = 0.5, S_0 = 100, E = 100 \text{ and } L = 95 \text{, } 99.5 \text{, } 99.9 \text{. The price is estimated by double ones with } U = 2.5E. The numerical results are shown in table 4 and compared with Fusai’s analytical formula[3], the Markov chain method (MCh)[2] and the Monte Carlo method (MC) with } 10^8 \text{ paths[28] that shows the validity of presented method in this case.}
Example 4. In this example we estimate the price of continuous monitoring call barrier down and out option, \( P_c \), with discrete ones, \( P_{d_m} \), using the following formula [18]:

\[
P_c(L) = P_{d_m}(L e^{\delta \sqrt{\Delta t}}),
\]

where \( \beta = \frac{\zeta(1/2)}{\sqrt{2\pi}} \approx 0.5826 \) with \( \zeta \) the Riemann zeta function. The parameters of this problem is considered as \( r = 0.1, \sigma = 0.3, T = 0.2, E = 100, S = 100 \). In table (5) the option price for different lower barriers are evaluated and compared with continuous monitoring price that is obtained in [18]. As we can see, this estimations is accurate except when the barrier is close to the spot price.

| L | Countinous Barrier n=25 | n=50 | (IR17) | MCH | MC (st.error) |
|---|--------------------------|------|--------|-----|---------------|
| 95 | 25 | 6.308 | 6.307 | 6.308 | 6.306 | 6.308 |
| 99.5 | 25 | 6.185 | 6.185 | 6.185 | 6.182 | 6.185 |
| 99.9 | 25 | 5.808 | 5.808 | 5.808 | 5.809 | 5.808 |
| 95 | 125 | 5.277 | 5.277 | 5.277 | 5.277 | 5.277 |
| 99.5 | 125 | 4.398 | 4.396 | 4.397 | 4.398 | 4.397 |
| 99.9 | 125 | 3.060 | 3.067 | 3.067 | 3.059 | 3.059 |
| CPU | 0.038 s | 0.051 s | 0.038 s | 0.051 s |

Table 5: Single barrier option pricing with continuous monitoring of Example (4). \( T = 0.2, r = 0.1, \sigma = 0.3, S_0 = 100, E = 100, U = 250 \).

6. Conclusion and remarks

In this article, we used the Lagrange interpolation on Jacobi polynomial nodes for pricing discrete single and double barrier options. In section 4 we obtained a matrix relation (23) for solving this problem. Numerical results verify that computational time is fixed when the number of monitoring dates increase.

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