THE LAWSON-YAU FORMULA AND ITS GENERALIZATION

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Abstract. The Euler characteristic of Chow varieties of algebraic cycles of a given degree in complex projective spaces was computed by Blaine Lawson and Stephen Yau by using holomorphic symmetries of cycles spaces. In this paper we compute this in a direct and elementary way and generalize this formula to the $l$-adic Euler-Poincaré characteristic for Chow varieties over any algebraically closed field. Moreover, the Euler characteristic for Chow varieties with certain group action is calculated. In particular, we calculate the Euler characteristic of the space of right quaternionic cycles of a given dimension and degree in complex projective spaces.

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1. Introduction

Let $\mathbb{P}^n$ be the complex projective space of dimension $n$ and let $C_{p,d}(\mathbb{P}^n)$ be the space of effective algebraic $p$-cycles of degree $d$ on $\mathbb{P}^n$. A fact proved by Chow and Van der Waerden is that $C_{p,d}(\mathbb{P}^n)$ carries the structure of a closed complex algebraic variety $\mathbb{P}^n$. Hence it carries the structure of a compact Hausdorff space. Denote by $\chi(C_{p,d}(\mathbb{P}^n))$ the Euler Characteristic of $C_{p,d}(\mathbb{P}^n)$. This number was computed in terms of $p$, $d$, and $n$ by Blaine Lawson and Stephen Yau explicitly, i.e.,

**Theorem 1.1 (Lawson-Yau, [1]).**

\[ \chi(C_{p,d}(\mathbb{P}^n)) = \binom{v_{p,n}+d-1}{d}, \quad \text{where} \quad v_{p,n} = \binom{n+1}{p+1}. \]

This equation [1] is called the **Lawson-Yau formula**. The original method of calculation is an application of a fixed point formula for a compact complex analytic space with a weakly holomorphic $S^1$-action.

Equivalently, if we define

\[ Q_{p,n}(t) := \sum_{d=0}^{\infty} \chi(C_{p,d}(\mathbb{P}^n)) t^d, \]

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then the Lawson-Yau formula may be restated as

\[(2) \quad Q_{p,n}(t) = \left( \frac{1}{1-t} \right)^{\binom{n+1}{p+1}}, \quad \text{ where } \chi(C_{p,0}(\mathbb{P}^n)) := 1.\]

A direct and elementary proof of the Lawson-Yau formula is given in section 2 by using the technique “pulling of normal cone” established in the book of Fulton (3), which was used by Lawson in proving his Complex Suspension Theorem (8).

As an application of this elementary method, we obtain an l-adic version of the Lawson-Yau formula:

**Theorem 1.2.** Let \(C_{p,d}(\mathbb{P}^n)_K\) be the space of effective \(p\)-cycles of degree \(d\) in \(\mathbb{P}^n_K\). For all \(l\) prime to \(\text{char}(K)\), we have

\[(3) \quad \chi(C_{p,d}(\mathbb{P}^n)_K, l) = \binom{v_{p,n}+d-1}{d}, \quad \text{ where } v_{p,n} = \binom{n+1}{p+1},\]

where \(\chi(X_K, l)\) denotes the \(l\)-adic Euler-Poincaré Characteristic of an algebraic variety \(X_K\) over \(K\).

The detailed explanation of notations in Theorem 1.2 as well as its proof is given in section 3.

We apply our method to the space of algebraic cycles with certain finite group action \(G\) to obtain the Euler characteristic of the \(G\)-invariant Chow varieties (see Theorem 4.5).

As an application, we obtain the Euler characteristic of the space of right quaternionic cycles of a given dimension and degree in complex projective spaces (see Corollary 5.2).

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2. AN ELEMENTARY PROOF OF THE LAWSON-YAU FORMULA

First we list a result we will use in our calculation. See (4), page 95.

**Lemma 2.1.** Let \(X\) be a complex algebraic variety (not necessarily smooth, compact, or irreducible) and let \(Y \subset X\) be a closed algebraic set with complement \(U\). Then \(\chi(X) = \chi(U) + \chi(Y)\).

Now we give a brief review of Lawson’s construction of the suspension map and some of its properties. The reader is referred to (3) for details. Fix a hyperplane \(\mathbb{P}^n \subset \mathbb{P}^{n+1}\) and a point \(P = [0 : \cdots : 0 : 1] \in \mathbb{P}^{n+1} - \mathbb{P}^n\). Let \(V \subset \mathbb{P}^n\) be any closed algebraic subset. The algebraic suspension of \(V\) with vertex \(P\) (i.e., cone over \(P\)) is the set

\[\Sigma_P V := \bigcup \{l \mid l \text{ is a projective line through } P \text{ and intersects } V\}.\]

Extending by linearity algebraic suspension gives a continuous homomorphism

\[(4) \quad \Sigma_P : C_{p,d}(\mathbb{P}^n) \to C_{p+1,d}(\mathbb{P}^{n+1})\]

for any \(p \geq 0\).

Set

\[T_{p+1,d}(\mathbb{P}^{n+1}) := \left\{ c = \sum n_i V_i \in C_{p+1,d}(\mathbb{P}^{n+1}) \mid \dim(V_i \cap \mathbb{P}^n) = p, \forall i \right\}\]
Proposition 2.2. The subset \( T_{p+1,d}(\mathbb{P}^{n+1}) \subset T_{p+1,d}(\mathbb{P}^{n+1}) \) is Zariski open. Moreover, the image \( \Sigma_P : C_{p,d}(\mathbb{P}^n) \rightarrow C_{p+1,d}(\mathbb{P}^{n+1}) \) is included in \( T_{p+1,d}(\mathbb{P}^{n+1}) \) and \( \Sigma_P(C_{p,d}(\mathbb{P}^n)) \subset T_{p+1,d}(\mathbb{P}^{n+1}) \) is a strong deformation retract.

In particular, their Euler characteristics coincide, i.e., we have

\[
\chi(C_{p,d}(\mathbb{P}^n)) = \chi(T_{p+1,d}(\mathbb{P}^{n+1})).
\]

By Proposition 2.2 \( B_{p+1,d}(\mathbb{P}^{n+1}) \) is a closed subset of \( C_{p+1,d}(\mathbb{P}^{n+1}) \). From the definition,

\[
B_{p+1,d}(\mathbb{P}^{n+1}) = \{ c = \sum n_i V_i \in C_{p+1,d}(\mathbb{P}^{n+1}) | V_i \subset \mathbb{P}^n, \text{ for some } i \},
\]

i.e., there is at least one irreducible component lying in the fixed hyperplane \( \mathbb{P}^n \).

Lemma 2.3. \( B_{p+1,d}(\mathbb{P}^{n+1}) = \bigcup_{i=1}^d B_{p+1,d}(\mathbb{P}^{n+1})_i \), where \( \bigcup \) means disjoint union and

\[
B_{p+1,d}(\mathbb{P}^{n+1})_i = \left\{ c \in B_{p+1,d}(\mathbb{P}^{n+1}) \bigg| c = \sum n_k V_k + \sum m_j W_j, \right. \\
\left. V_k \subset \mathbb{P}^n, \forall k, \dim(W_j \cap \mathbb{P}^n) = p, \forall j, \right. \\
\left. \deg(\sum n_k V_k) = i, \right. \\
\left. \text{and } \deg(\sum m_j W_j) = d-i. \right\}
\]

For each \( i \), \( B_{p+1,d}(\mathbb{P}^{n+1})_i = C_{p+1,i}(\mathbb{P}^n) \times T_{p+1,d-i}(\mathbb{P}^{n+1}) \).

Proof. Clear from the definition of \( B_{p+1,d}(\mathbb{P}^{n+1})_i \). \( \square \)

From Lemma 2.3 we have

\[
\chi(B_{p+1,d}(\mathbb{P}^{n+1})_i) = \chi(C_{p+1,i}(\mathbb{P}^n)) \cdot \chi(T_{p+1,d-i}(\mathbb{P}^{n+1})).
\]

Hence we get

\[
\chi(B_{p+1,d}(\mathbb{P}^{n+1})) = \sum_{i=1}^d \chi(B_{p+1,d}(\mathbb{P}^{n+1})_i)
\]

(by inclusion-exclusion principle)

\[
= \sum_{i=1}^d \chi(C_{p+1,i}(\mathbb{P}^n)) \cdot \chi(T_{p+1,d-i}(\mathbb{P}^{n+1}))
\]

\[
= \sum_{i=1}^d \chi(C_{p+1,i}(\mathbb{P}^n)) \cdot \chi(C_{p,d-i}(\mathbb{P}^n)),
\]

(by equation 5)

Therefore we have the following result:

Proposition 2.4. For any integer \( p \geq 0 \) and \( d \geq 1 \), we have the following recursive formula

\[
\chi(C_{p+1,d}(\mathbb{P}^{n+1})) = \chi(C_{p,d}(\mathbb{P}^n)) + \sum_{i=1}^d \chi(C_{p+1,i}(\mathbb{P}^n)) \cdot \chi(C_{p,d-i}(\mathbb{P}^n)),
\]

where \( \chi(C_{q,0}(\mathbb{P}^N)) = 1 \) for integers \( N \geq q \geq 0 \). In particular, when \( d = 1 \), equation 6 is just the combinatorial identity \( \binom{n+q}{p+2} = \binom{n+1}{p+1} + \binom{n+1}{p+2} \).

To compute \( \chi(C_{p,d}(\mathbb{P}^n)) \), it is enough to identify the initial values.

Lemma 2.5. \( \chi(C_{0,d}(\mathbb{P}^n)) = \binom{n+d}{d} \).

The equality is a special case of MacDonald formula \( \langle 12 \rangle \).
Proof of Lemma 2.5. Now we give an independent proof for MacDonald formula in this special case. We can write
\[ C_{0,d}(\mathbb{P}^{n+1}) = C_{0,d}(\mathbb{C}^{n+1}) \prod B_{0,d}(\mathbb{P}^{n+1}), \]
where \( C_{0,d}(\mathbb{C}^{n+1}) \subset C_{0,d}(\mathbb{P}^{n+1}) \) contains effective 0-cycles \( c \) of degree \( d \) such that no points in \( c \) lying in the fixed hyperplane \( \mathbb{P}^{n} \) and \( B_{0,d}(\mathbb{P}^{n+1}) \) is the complement of \( C_{0,d}(\mathbb{C}^{n+1}) \) in \( C_{0,d}(\mathbb{P}^{n+1}) \). It is easy to see that \( C_{0,d}(\mathbb{C}^{n+1}) \) is contractible. We can write \( B_{0,d}(\mathbb{P}^{n+1}) = \prod_{i=1}^{d} B_{0,d}(\mathbb{P}^{n+1})_{i} \), as in Lemma 2.3 where \( B_{0,d}(\mathbb{P}^{n+1})_{i} \) contains 0-cycles \( c \) of degree \( d \) on \( \mathbb{P}^{n+1} \) in which there are exact \( i \) points (count multiplicities) lying in \( \mathbb{P}^{n} \), hence \( B_{0,d}(\mathbb{P}^{n+1})_{i} = C_{0,i}(\mathbb{P}^{n}) \times C_{0,d-i}(\mathbb{C}^{n+1}) \). In particular, \( \chi(B_{0,d}(\mathbb{P}^{n+1})) = \sum_{i=1}^{d} \chi(C_{0,i}(\mathbb{P}^{n})) \).
Therefore, we have
\[ \chi(C_{0,d}(\mathbb{P}^{n+1})) = 1 + \sum_{i=1}^{d} \chi(C_{0,i}(\mathbb{P}^{n})). \]
The first formula in the lemma follows from this by induction. \( \square \)

Proof of Theorem 1.2. Equation (4) together with Lemma 2.5 is equivalent to the following recursive functional equation with initial values
\[ Q_{p+1,n+1}(t) = Q_{p+1,n}(t) \cdot Q_{p,n}(t), \]
\[ Q_{0,m}(t) = (\frac{1}{1-t})^{m+1}. \]
From this, we get the equation (2) by induction on \( n \) and hence the Lawson-Yau formula (1). \( \square \)

Example 2.6. For divisors of degree \( d \) in \( \mathbb{P}^{n} \), we have the formula \( \chi(C_{p,d}(\mathbb{P}^{p+1})) = \binom{p+d+1}{d} \).

From equation (4), we have
\[ Q_{p,p+1}(t) = Q_{p,p}(t) \cdot Q_{p-1,p}(t) = \frac{1}{1-t} \cdot Q_{p-1,p}(t) \]
since \( C_{p,d}(\mathbb{P}^{p}) \) contains exactly one degree \( d \) cycle and so \( \chi(C_{p,d}(\mathbb{P}^{p})) = 1 \), i.e., \( Q_{p,p}(t) = \frac{1}{1-t} \). By the fact that \( Q_{0,1}(t) = (\frac{1}{1-t})^{2} \) and induction on \( p \), we get \( Q_{p,p+1}(t) = (\frac{1}{1-t})^{p+2} \). Hence \( \chi(C_{p,d}(\mathbb{P}^{p+1})) = \binom{p+d+1}{d} \).
Alternatively, this formula follows directly from the fact that \( C_{p,d}(\mathbb{P}^{p+1}) \) is the moduli space of hypersurfaces of degree \( d \) in \( \mathbb{P}^{p+1} \) and hence it is a complex projective space of dimension \( \binom{p+d+1}{d} - 1 \). To see this, we choose a basis for the monomials of degree \( d \) in \( p+2 \) variables and then associate a point in this projective space to the hypersurface whose defining equation is given by the coordinates of that point (cf. [2]).

3. The \( l \)-adic Euler-Poincaré Characteristic for Chow varieties

In the section, the Lawson-Yau formula is generalized to an algebraically closed field \( K \) with arbitrary characteristic \( \text{char}(K) \geq 0 \). Let \( l \) be a positive integer prime to \( \text{char}(K) \).

For a variety \( X \) over \( K \), let \( H^i(X, \mathbb{Z}_l) \) be the \( l \)-adic cohomology group of \( X \). Set \( H^i(X, \mathbb{Q}_l) := H^i(X, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \). Denote by \( \beta^i(X, l) := \dim_{\mathbb{Q}_l} H^i(X, \mathbb{Q}_l) \) the \( i \)-th \( l \)-adic Betti number of \( X \). The \( l \)-adic Euler Characteristic is defined by \( \chi(X, l) := \)
\[ \sum (-1)^i \beta_i(X, l) \]. Similarly, let \( H^*_i(X, \mathbb{Z}_l) \) be the \( l \)-adic cohomology group of \( X \) with compact support. Set \( \beta_i(X, l) := \dim_{\mathbb{Z}_l} H^*_i(X, \mathbb{Q}_l) \) the \( i \)-th \( l \)-adic Betti number of \( X \) with compact support and \( \chi_c(X, l) := \sum (-1)^i \beta_i(X, l) \) the \( l \)-adic Euler-Poincaré Characteristic with compact support. Note that \( \chi_c(X, l) \) is independent of the choice of \( l \) prime to \( \text{char}(K) \) (See, e.g., [8] or [5]).

By using the method in the last section we will deduce the \( l \)-adic version of Lawson-Yau formula (see Theorem 1.2).

From the proof of equation (2) above, we need similar results for Lemma 2.1-2.5, Proposition 2.2, and the homotopy invariance of \( l \)-adic Euler-Poincaré Characteristics.

As we stated before, an algebraic version of Proposition 2.2 over any algebraically closed field \( K \) was proved by Friedlander (cf. [2], Prop.3.2). Lemma 2.3 is a purely algebraic result and hence it holds for any algebraically closed field \( K \). An algebraic version of the first statement in Lemma 2.5 follows from a corresponding result of (5).

An algebraic version of the second statement in Lemma 2.6 holds over any algebraically closed field. Therefore, the key part to prove Theorem 1.2 is the following algebraic version of Lemma 2.1 and the homotopy invariance of \( l \)-adic cohomology.

**Lemma 3.1.** Let \( X \) be an algebraic variety (not necessarily smooth, compact, or irreducible) over an algebraically closed field \( K \) and let \( Y \subset X \) be a closed algebraic set with complement \( U \). Then \( \chi(X, l) = \chi(U, l) + \chi(Y, l) \) for any positive integer \( l \) prime to \( \text{char}(K) \).

**Proof.** This lemma follows from the long localization exact sequence for \( l \)-adic cohomology and the following result proved by Laumon (independently by Gabber). \( \square \)

**Proposition 3.2 (5).** For any algebraic variety \( X \) over an algebraically closed field \( K \) and integer \( l \) prime to \( \text{char}(K) \), we have \( \chi(X, l) = \chi_c(X, l) \).

To prove Theorem 1.2 we also need the following definition (cf. [2], page 61).

**Definition 1.** A proper morphism \( g : X' \to X \) of locally noetherian schemes is said to be a **bicontinuous algebraic morphism** if it is a set theoretic bijection and if for every \( x \in X \) the associated map of residue fields \( K(x) \to K(g^{-1}(x)) \) is purely inseparable. A **continuous algebraic map** \( f : X \to Y \) is a pair \( (g : X' \to X, f' : X \to Y) \) in which \( g \) is a bicontinuous algebraic morphism (and \( f \) is a morphism).

**Lemma 3.3.** Let \( F : X \times \mathbb{A}^1_K \to Y \) be a continuous algebraic map such that \( F(-, 0) = f(-) \) and \( F(-, 1) = g(-) \). Then the pullback \( f^* : H^i(Y, \mathbb{Z}_l) \to H^i(X, \mathbb{Z}_l) \) is equal to \( g^* : H^i(Y, \mathbb{Z}_l) \to H^i(X, \mathbb{Z}_l) \).

**Proof.** It is essentially proved in [2]. Prop. 2.1. The map \( F : X \times \mathbb{A}^1_K \to Y \) induces a map in cohomology \( F^* : H^i(Y, \mathbb{Z}_l) \to H^i(X \times \mathbb{A}^1_K, \mathbb{Z}_l) \). By the usual homotopy invariance of étale cohomology, one has the homotopy invariance of \( l \)-adic cohomology, i.e., \( H^i(X \times \mathbb{A}^1_K, \mathbb{Z}_l) \cong H^i(X, \mathbb{Z}_l) \). Therefore, we get the equality of \( f \) and \( g \) by the restriction of \( F \) to \( X \times 0 \) and \( X \times 1 \). \( \square \)

**Corollary 3.4.** Let \( i : Y \subset X \) be an algebraically closed subset. Let \( F : X \times \mathbb{A}^1_K \to X \) be a continuous algebraic map such that \( F(-, 0) = id_X \), \( F(x, 1) \circ i = id_Y \) and \( F(y, t) = y \) for \( y \in Y \). Then \( i^* : H^i(X, \mathbb{Z}_l) \to H^i(Y, \mathbb{Z}_l) \) is an isomorphism.
The detailed computation is given below.

Proof of Theorem 1.2. Set \( \tilde{Q}_{p,n}(t) := \sum_{d=0}^{\infty} \chi(C_{p,d}(\mathbb{P}^n))K^d \),

\[
T_{p+1,d}(\mathbb{P}^{n+1})_K := \left\{ c = \sum n_i V_i \in C_{p+1,d}(\mathbb{P}^{n+1})_K \mid \dim(V_i \cap \mathbb{P}^n_K) = p, \forall i \right\}
\]

and

\[
B_{p+1,d}(\mathbb{P}^{n+1})_K := \left\{ c = \sum n_i V_i \in C_{p+1,d}(\mathbb{P}^{n+1})_K \mid V_i \subset \mathbb{P}^n_K, \text{for some } i \right\}
\]

By Corollary 3.3 and the algebraic version of Proposition 2.2 we have

\begin{equation}
\chi(C_{p,d}(\mathbb{P}^n)_K) = \chi(T_{p+1,d}(\mathbb{P}^{n+1})_K).
\end{equation}

From the Künneth formula for l-adic cohomology (cf. [13]), the algebraic version of Lemma 2.3 and Equation (8), we get

\begin{equation}
\chi(C_{p+1,d}(\mathbb{P}^{n+1})_K) = \chi(C_{p,d}(\mathbb{P}^n)_K) + \sum_{i=1}^{d} \chi(C_{p+1,i}(\mathbb{P}^n)_K) \cdot \chi(C_{p,d-i}(\mathbb{P}^n)_K),
\end{equation}

for integers \( p \geq 0 \) and \( d \geq 1 \).

By Corollary 3.3, we have \( \chi(C_{0,d}(A_K^n)) = 1 \), where \( A_K^n \) is the \( n \)-dimensional affine space over \( K \). Hence the computation in the proof of Lemma 2.3 works when \( \mathbb{C} \) is replaced by any algebraically closed field \( K \) and we get the formula for \( \chi(C_{0,d}(\mathbb{P}^n)_K) \):

\begin{equation}
\chi(C_{0,d}(\mathbb{P}^n)_K) = \binom{n+1}{d+1}.
\end{equation}

From the fact that \( C_{p,d}(\mathbb{P}^{p+1})_K \) is the moduli space of hypersurfaces of degree \( d \) in \( \mathbb{P}^{p+1}_K \) and hence it is a projective space over \( K \) of dimension \( (p+d+1) - 1 \),

\begin{equation}
\chi(C_{p,d}(\mathbb{P}^{p+1})_K) = \binom{p+d+1}{d}.
\end{equation}

From the definition of \( \tilde{Q}_{p,n}(t) \) and Equation (10)-(11), we get

\[ \tilde{Q}_{p+1,n+1}(t) = \tilde{Q}_{p+1,n}(t) \cdot \tilde{Q}_{p,n}(t), \]

\[ \tilde{Q}_{0,m}(t) = \left( \frac{1}{t} \right)^{m+1}, \]

\[ \tilde{Q}_{q+1}(t) = \left( \frac{1}{t} \right)^{q+2}. \]

From Equation (12), we complete the proof of Theorem 1.2 by induction on \( n \). \( \square \)

4. ALGEBRAIC CYCLES WITH GROUP ACTION

In this section, we apply our method to the space of algebraic cycles with certain finite group action \( G \) to obtain the Euler characteristic of the \( G \)-invariant Chow varieties. Let \( G \) be a finite group of the automorphism of \( \mathbb{P}^n \). By passing to an appropriate group extension we can always assume that \( \rho : G \to U_{n+1} \) and that its action on \( \mathbb{P}^n \) comes from the linear action of \( U_{n+1} \) on homogeneous coordinates. The action of \( G \) on \( \mathbb{P}^n \) induces actions on the Chow varieties \( C_{p,d}(\mathbb{P}^n) \).

Denote by

\[ C_{p,d}(\mathbb{P}^n)_G := \{ c \in C_{p,d}(\mathbb{P}^n) : g^* c = c, \forall g \in G \} \]
the $G$-invariant subset of $C_{p,d}(\mathbb{P}^n)$. Since $G$ is a finite group of the automorphism of $\mathbb{P}^n$, it induces an automorphism of $C_{p,d}(\mathbb{P}^n)$. Hence $C_{p,d}(\mathbb{P}^n)^G$ is a closed complex subvariety of $C_{p,d}(\mathbb{P}^n)$.

Choose homogeneous coordinates $\mathbb{C}^{n+2} = \mathbb{C}^{n+1} \oplus \mathbb{C}$ for $\mathbb{P}^{n+1} = \Sigma \mathbb{P}^n$ and extend the fixed linear representation $\rho : G \to U_{n+1}$ to a representation $\tilde{\rho} : G \to U_{n+1}$ by setting $\tilde{\rho} = \rho \oplus \lambda \cdot \text{id}_\mathbb{C}$, where $\lambda \in \mathbb{C}^*$ is a fixed complex number. The construction was given in [9], where $\lambda$ is chosen to be 1.

Set $T_{p+1,d}(\mathbb{P}^{n+1})^G := \{ c = \sum n_i V_i \in C_{p+1,d}(\mathbb{P}^{n+1})^G | \dim(V_i \cap \mathbb{P}^n) = p, \forall i \}$ and $B_{p+1,d}(\mathbb{P}^{n+1})^G = C_{p+1,d}(\mathbb{P}^{n+1})^G - T_{p+1,d}(\mathbb{P}^{n+1})^G$.

The following proposition proved by Lawson and Michelsohn in [9] will be used in our calculation.

**Proposition 4.1 ([9])**. For each $p \geq 0$, $T_{p+1,d}(\mathbb{P}^{n+1})^G \subset C_{p+1,d}(\mathbb{P}^{n+1})^G$ is Zariski open. Moreover, the image

$$\Sigma : C_{p,d}(\mathbb{P}^n)^G \to C_{p+1,d}(\mathbb{P}^{n+1})^G$$

is included in $T_{p+1,d}(\mathbb{P}^{n+1})^G$ and $\Sigma(C_{p,d}(\mathbb{P}^n)^G) \subset T_{p+1,d}(\mathbb{P}^{n+1})^G$ is a strong deformation retract.

In particular, their Euler characteristics coincide, i.e., we have

$$\chi(C_{p,d}(\mathbb{P}^n)^G) = \chi(T_{p+1,d}(\mathbb{P}^{n+1})^G).$$

**Remark 4.2.** In [9], Lawson and Michelsohn consider the extension $\tilde{\rho} = \rho \oplus \lambda \cdot \text{id}_\mathbb{C}$ only for the case $\lambda = 1$. However, their proof works for all $\lambda \in \mathbb{C}^*$ without changing anything except that 1 is replaced by $\lambda$ in $\tilde{\rho}$.

From the definition,

$$B_{p+1,d}(\mathbb{P}^{n+1})^G = \{ c = \sum n_i V_i \in C_{p+1,d}(\mathbb{P}^{n+1})^G | V_i \subset \mathbb{P}^n, \text{for some } i \},$$

i.e., there is at least one irreducible component lying in the fixed $G$-invariant hyperplane $\mathbb{P}^n$.

**Lemma 4.3.** $B_{p+1,d}(\mathbb{P}^{n+1})^G = \Pi_{i=1}^d B_{p+1,d}(\mathbb{P}^{n+1})^G_i$, where $\Pi$ means disjoint union and

$$B_{p+1,d}(\mathbb{P}^{n+1})^G_i = \begin{cases} 
\{ c \in B_{p+1,d}(\mathbb{P}^{n+1})^G | c = \sum n_k V_k + \sum m_j W_j, \\
V_k \subset \mathbb{P}^n, \forall k, \\
\dim(W_j \cap \mathbb{P}^n) = p, \forall j, \\
\deg(\sum n_k V_k) = i, \\
\text{and } \deg(\sum m_j W_j) = d - i. \} 
\end{cases}$$

For each $i$, $B_{p+1,d}(\mathbb{P}^{n+1})^G_i = C_{p+1,i}(\mathbb{P}^n)^G \times T_{p+1,d-i}(\mathbb{P}^{n+1})^G$.

**Proof.** An algebraic cycle $c \in B_{p+1,d}(\mathbb{P}^{n+1})^G_i$ may be written as $c = \sum n_k V_k + \sum m_j W_j$ as the formal sum of irreducible varieties, where $V_k \subset \mathbb{P}^n$ and $\dim(W_j \cap \mathbb{P}^n) = p + 1$. Since $c$ is $G$-invariant, we have $\sum n_k V_k$ and $\sum m_j W_j$ are $G$-invariant. To see this, recall that $G$ is identified with the subgroup of the unitary group $U_{n+1}$. Suppose $g^*(V_i) = W_j$ for some $g \in G$ and $i,j$. Since $\mathbb{P}^n$ is $G$-invariant, we have $\mathbb{P}^n = g^*(\mathbb{P}^n) \subset g^*(V_i) = W_j$. This contradicts to the assumption that $\dim(W_j \cap \mathbb{P}^n) = p + 1$. Similar for $\sum m_j W_j$.

Now the lemma follows from the definition of $B_{p+1,d}(\mathbb{P}^{n+1})^G_i$. \qed
From Lemma 4.3, we have
\[ \chi(B_{p+1,d}(\mathbb{P}^{n+1})) = \chi(C_{p+1,d}(\mathbb{P}^n)^G) \cdot \chi(T_{p+1,d-i}(\mathbb{P}^{n+1})^G). \]
Hence we get
\[ \chi(B_{p+1,d}(\mathbb{P}^{n+1})) = \sum_{i=1}^{d} \chi(B_{p+1,d}(\mathbb{P}^{n+1})) \]
(by inclusion-exclusion principle)
\[ = \sum_{i=1}^{d} \chi(C_{p+1,i}(\mathbb{P}^n)^G) \cdot \chi(T_{p+1,d-i}(\mathbb{P}^{n+1})^G) \]
\[ = \sum_{i=1}^{d} \chi(C_{p+1,i}(\mathbb{P}^n)^G) \cdot \chi(C_{p,d-i}(\mathbb{P}^n)^G). \]

Therefore we have the following result:

**Proposition 4.4.** For any integer \( p \geq 0 \) and \( d \geq 1 \), we have the following formula
\[
\chi(C_{p+1,d}(\mathbb{P}^{n+1})) = \chi(C_{p,d}(\mathbb{P}^n)^G) + \sum_{i=1}^{d} \chi(C_{p+1,i}(\mathbb{P}^n)^G) \cdot \chi(C_{p,d-i}(\mathbb{P}^n)^G),
\]
where \( \chi(C_{q,0}(\mathbb{P}^N)^G) = 1 \) for integers \( N \geq q \geq 0 \).

From our construction, we know that if the representation \( \rho : G \subset U_{n+1} \) is diagonalizable, i.e., up to a linear transformation, \( \rho = \oplus_{i=1}^{n+1} \chi_i \cdot \text{id}_{\mathbb{C}} \), then equation (13) gives us a recursive formula. In these cases, the Euler characteristic is calculated explicitly as follows.

**Theorem 4.5.** Let \( \rho : G \subset U_{n+1} \) be a diagonalizable representation. The Euler characteristic of Chow variety of \( G \)-invariant cycles \( \chi(C_{p,d}(\mathbb{P}^n)^G) \) is given by the formula
\[
\chi(C_{p,d}(\mathbb{P}^n)^G) = \binom{v_{p,n} + d - 1}{d}, \quad \text{where} \quad v_{p,n} = \binom{n+1}{p+1}.
\]

**Proof.** The theorem follows from Proposition 4.3 and the following initial values identities:
\[
\chi(C_{0,d}(\mathbb{P}^n)^G) = \binom{n+d}{d}
\]
As before, we can write
\[
C_{0,d}(\mathbb{P}^{n+1})^G = C_{0,d}(\mathbb{C}^{n+1})^G \prod B_{0,d}(\mathbb{P}^{n+1})^G,
\]
where \( C_{0,d}(\mathbb{C}^{n+1})^G \subset C_{0,d}(\mathbb{P}^{n+1})^G \) contains effective \( G \)-invariant 0-cycles \( c \) of degree \( d \) such that no points in \( c \) lying in the fixed hyperplane \( \mathbb{P}^n \) and \( B_{0,d}(\mathbb{P}^{n+1})^G \) is the complement of \( C_{0,d}(\mathbb{C}^{n+1})^G \) in \( C_{0,d}(\mathbb{P}^{n+1})^G \). We claim that \( C_{0,d}(\mathbb{C}^{n+1})^G \) is contractible. To see this, note that \( \mathbb{P}^{n+1} - \mathbb{P}^n = \mathbb{C}^{n+1} \) and Let \( \phi_t : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} \) denote scalar multiplication by \( t \in \mathbb{C} \). The family of maps \( \phi_t \) induces a family of maps \( \phi_{t*} : C_{0,d}(\mathbb{C}^{n+1}) \to C_{0,d}(\mathbb{C}^{n+1}) \) since the multiplication by \( t \in \mathbb{C} \) is \( G \)-invariant. From the definition, the map \( \phi_{t*} = \text{id} \) and \( \phi_{0*} \) is a constant map.

We can write
\[
B_{0,d}(\mathbb{P}^{n+1})^G = \prod_{i=1}^{d} B_{0,d}(\mathbb{P}^{n+1})^G_i
\]
as in Lemma 4.3 where \( B_{0,d}(\mathbb{P}^{n+1})^G_i \) contains \( G \)-invariant 0-cycles \( c \) of degree \( d \) on \( \mathbb{P}^{n+1} \) in which there are \( \text{exact} \ i \) points (count multiplicities) lying in \( \mathbb{P}^n \), hence \( B_{0,d}(\mathbb{P}^{n+1})^G_i = C_{0,i}(\mathbb{P}^n)^G \times C_{0,d-i}(\mathbb{C}^{n+1})^G \). In particular, \( \chi(B_{0,d}(\mathbb{P}^{n+1})^G) = \sum_{i=1}^{d} \chi(C_{0,i}(\mathbb{P}^n)^G). \)
The Lawson-Yau formula and its generalization

Therefore, we have

$$\chi(C_{0,d}(\mathbb{P}^{n+1})^G) = 1 + \sum_{i=1}^{d} \chi(C_{0,i}(\mathbb{P}^{n})^G).$$

Now the formula in the lemma follows from this by induction.

\[\square\]

**Remark 4.6.** By a carefully checking the proof of Theorem 4.6 in [9] and the proof of Theorem 4.5 above, we observe that if the linear representation $\rho : G \to U_{n+1}$ is diagonalizable, then the conclusion in Theorem 4.5 holds even if there is no assumption of finiteness of $G$. More precisely, let $T_{n+1} \subset U_{n+1}$ be the maximal torus and let $G$ be any subgroup of $T_{n+1}$. Then The Euler characteristic of Chow variety of $G$-invariant cycles $\chi(C_{p,d}(\mathbb{P}^{n})^G)$ is given by the formula

$$\chi(C_{p,d}(\mathbb{P}^{n})^G) = \binom{v_{p,n}+d-1}{d}, \quad \text{where } v_{p,n} = \binom{n+1}{p+1}.$$ 

This explains Theorem 4.1, one of the main results in [11], on the invariance of Euler characteristic of a compact complex analytic space and that of the fixed-point set under a holomorphic $S^1$-action in the important case for Chow varieties over $\mathbb{C}$.

5. **The Euler Characteristic for the space of right-quaternionic cycles**

Let $\mathbb{H}$ denote the quaternions with standard basis $1, i, j, k$, and let $\mathbb{C}^2 \cong \mathbb{H}$ be the canonical isomorphism given by $(u, v) \mapsto u + v$. This gives us a canonical complex isomorphism $\mathbb{C}^{2n} \cong \mathbb{H}^n$.

Under this identification *right* scalar multiplication by $j$ in $\mathbb{H}^n$ becomes the complex linear map

$$J : \mathbb{C}^{2n} \to \mathbb{C}^{2n}, \quad J(u_1, ..., u_n, v_1, ..., v_n) = (-v_1, ..., -v_n, u_1, ..., u_n).$$

This induces a holomorphic map $\tilde{J} : \mathbb{P}^{2n-1} \to \mathbb{P}^{2n-1}$ with $\tilde{J} = Id$. Note that the fixed point set of $\tilde{J}$ is a pair of disjoint $\mathbb{P}^{n-1}$. The involution $\tilde{J}$ carries algebraic subvarieties of $\mathbb{P}^{2n-1}$ to themselves and induces a holomorphic involution

$$\tilde{J}^* : C_{p,d}(\mathbb{P}^{2n-1}) \to C_{p,d}(\mathbb{P}^{2n-1})$$

for all $p$ and $d$.

**Remark 5.1.** The construction is an analog to the one given in [10], where a *left* scalar multiplication by $j$ on $\mathbb{H}^n$ was checked in detail. We consider the right multiplication here since the induced map $\tilde{J}$ is a holomorphic.

Let $C_{p,d}(n) \subset C_{p,d}(\mathbb{P}^{2n-1})$ denote the $\tilde{J}$-fixed point set, i.e., the set of $\tilde{J}$-invariant algebraic $p$-cycles. An element $c \in C_{p,d}(n)$ is called a **right quaternionic cycle**. Since $\tilde{J}$ is a holomorphic involution, $C_{p,d}(n)$ is a closed complex algebraic set.

Since $J$ is diagonalizable, in fact,

$$J \sim \text{diag} \left\{ \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \ldots, \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \right\},$$

As an application of Theorem 4.5 we have the following result.
Corollary 5.2. For any $p \geq 0$, we have
\[
\chi(C_{p,d}(n)) = \chi(C_{p,d}(\mathbb{P}^{2n-1})) = \binom{v_{p,2n-1} + d - 1}{d}, \quad \text{where } v_{p,2n-1} = \binom{2n}{p+1}.
\]

Example 5.3. For $p = 0$, we have $\chi(C_{p,d}(n)) = \binom{2n+d-1}{d}$.

Alternatively, this can be seen in the following way. The set $C_{0,d}(n)$ can be decomposed into the disjoint union of quasi-projective algebraic varieties according to the number of fixed points of $J$ lying in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$, i.e., $C_{0,d}(n) = \bigcup_{i=0}^{d} \text{SP}^i(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}) \times \text{SP}^d(\mathbb{P}^{n-1})$, where $G(n) = \mathbb{P}^{2n-1} - (\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$ is a bundle over $\mathbb{P}^{n-1}$ with fibers $\mathbb{C}^n - \{0\}$. Hence we have
\[
\chi(C_{0,d}(n)) = \sum_{i=0}^{d} \chi(\text{SP}^i(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})) \cdot \chi(\text{SP}^d(\mathbb{P}^{n-1})).
\]

Since the Euler characteristic of $G(n)$ is zero, we get $\chi(\text{SP}^m(G(n))) = 0$ for all $m > 0$ by Macdonald formula (cf. [12]). Therefore, $\chi(C_{0,d}(n)) = \chi(\text{SP}^d(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}))$.

Note that
\[
\text{SP}^d(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}) = \bigcup_{i=0}^{d} \text{SP}^i(\mathbb{P}^{n-1}) \times \text{SP}^d(\mathbb{P}^{n-1}) - \bigcup_{i=0}^{d} \text{SP}^0(\mathbb{P}^{n-1}) \times \text{SP}^d(\mathbb{P}^{n-1}),
\]
where a 0-cycle $c = \sum n_i P_i + \sum m_j P_j^i \in \text{SP}^d(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$ of degree $d$ is written as the sum of two 0-cycles such that $P_i$ is in the first copy of $\mathbb{P}^{n-1}$ but $P_j^i$ is in the second copy of $\mathbb{P}^{n-1}$. By equation (16) and the fact that $\chi(C_{0,i}(\mathbb{P}^{n-1})) = \binom{n+i-1}{i}$, we get
\[
\chi(\text{SP}^d(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})) = \sum_{i=0}^{d} \chi(\text{SP}^i(\mathbb{P}^{n-1})) \cdot \chi(\text{SP}^{d-i}(\mathbb{P}^{n-1}))
\]
\[
= \sum_{i=0}^{d} \binom{n+i-1}{i} \cdot \binom{n+d-i-1}{d-i}
\]
\[
= \binom{2n+d-1}{d-1},
\]
where the last equality is obtained by comparing the coefficients of $t^d$ in the Taylor series of $\frac{1}{(1-t)^n} = \frac{1}{(1-t)^n} \cdot \frac{1}{(1-t)^{d-1}}$.

Example 5.4. For $d = 1$, we have $\chi(C_{p,1}(n)) = \binom{2n}{p+1}$.

Alternatively, this can be seen in the following way. The eigenvalues of $J$ are $\pm \sqrt{-1}$, each of them is of multiplicity $n$. Let $\{e_i\}_{1 \leq i \leq n}$ be the eigenvectors of the eigenvalue $\sqrt{-1}$ and $\{f_i\}_{1 \leq i \leq n}$ be the eigenvectors of the eigenvalue $-\sqrt{-1}$. The $J$-invariant $(p+1)$-complex vector space is spanned by $i$ eigenvectors from $\{e_i\}_{1 \leq i \leq n}$ and $p + 1 - i$ eigenvectors from $\{f_i\}_{1 \leq i \leq n}$. Therefore,
\[
C_{p,1}(n) = C_{p,1}(\mathbb{P}^{2n-1})^J = \prod_{1 \leq i \leq p+1} G(i, n) \times G(p+1-i, n),
\]
where $G(i, n) := G(i, \mathbb{C}^n)$ is the Grassmannian of $i$-dimensional complex linear subspaces in $\mathbb{C}^n$. Therefore,
\[
\chi(C_{p,1}(n)) = \chi(C_{p,1}(\mathbb{P}^{2n-1})^J) = \sum_{i=1}^{p+1} \chi(G(i, n)) \cdot \chi(G(p+1-i, n)),
\]
\[
= \sum_{i=1}^{p+1} \binom{n}{p+1-i} \cdot \binom{n}{p+1-i}
\]
\[
= \binom{2n}{p+1},
\]
where the last equality is obtained by comparing the coefficients of $t^{p+1}$ in the binomial expansion of $(1+t)^{2n} = (1+t)^n \cdot (1+t)^n$. 

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