Width of satellite knot and its companion

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Abstract

In the paper we prove the conjecture by Alexander Zupan that \( w(K) \geq n^2 w(J) \) where \( w \) denote the width and \( K \) and \( J \) are satellite knot and its companion with winding number \( n \). Also we proved that for satellite knot with braid pattern, the equality holds.

1 Introduction

Width is an important invariant of knots defined by Gabai [2], and is used in many works of knots, see [2,3] for example. The definition of width is similar to bridge number knots, and it is believed that behaviors of width and bridge number are similar under operations of knots, especially connected sum and the operation of satellite knots.

The behavior under connected sum had been made clear:

\[
w(k_1) + w(k_2) - 2 \geq w(k_1 \# k_2) \geq \max\{w(k_1), w(k_2)\}.
\]

The upper bound is obvious and the lower bound is proved Sharlemann and Schultens [6] (The tightness of such boundaries are showed in [1,4]). In [6] a connected graph is defined for 3-manifold with planar presentation. We would use this object to prove our main theorem (theorem 1.1) that stated below.

In this paper we check the behavior of width under satelliting. In [5], Schultens prove that \( b(K) \geq nb(J) \), where \( b \) is the bridge number and \( n \) the wrapping number. In analogous to her result, Zupan presents two conjectures in [7,8], stating the same as \( w(K) \geq n^2 w(J) \) where \( w \) is the width of knot and \( K \) is a satellite knot with companion \( J \) and \( n \) refers to respectively winding number and wrapping number. In this paper we prove one of his conjecture, that is:

**Theorem 1.1.** If \( K \) is a satellite knot with company \( J \) and the winding number is \( n \), then \( w(K) \geq n^2 w(J) \).
Especially for satellite knot with braid pattern, we have:

**Corollary 1.2.** Suppose $K$ is a satellite knot with company $J$ and the winding number is $n$. If $K$ has braid pattern, then $w(K) = n^2 w(J)$.

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2 Preliminaries

In this paper, we use $K$, $J$, $L$ to denote the isotopic class of a knot, and $k$, $j$, $l$ a certain embedding belonging to the knot type $K$, $J$, or $L$.

**Definition 2.1.** (Satellite knots) Let $\hat{k} \subset \hat{V}$ a knot in a standard solid torus. Let $f : \hat{V} \to S^3$ be a knotted embedding. Let $V = f(\hat{V})$ and $j$ be the core of $V$. Then we call $k = f(\hat{k})$ the satellite knot with pattern $\hat{k}$ and companion $j$.

**Definition 2.2.** (Winding number) Let $k, \hat{k}, \hat{V}$ be defined as above, let $D$ be a meridian disks of $\hat{V}$, then define the winding number $n$ of satellite knot $k$ to be

$$n = |lk(\partial D, \hat{k})|.$$  

Let $h : S^3 \to \mathbb{R}$ be a Morse function which has exactly two critical points, denoted by $+\infty$ and $-\infty$. This function will be fixed throughout the paper.

**Definition 2.3.** (Width) Let $h$ the fixed Morse function. Fix a knot class $K$, Use $\mathcal{K}$ to denote the set of all embedding $k \in K$ such that $h|_k$ is Morse and critical points of $h|_k$ are all in different levels. For each $k \in \mathcal{K}$, suppose all critical values of $h|_k$ are $c_1 < ... < c_m$. Choose regular values $c_1 < r_1 < c_2 < r_2 < ... < c_{m-1} < r_{m-1} < c_m$ of $h$, for each $i$, let $\omega_i = |k \cap h^{-1}(r_i)|$. Define

$$w(k) = \sum_{i=1}^{m-1} \omega_i$$

and

$$w(K) = \min_{k \in \mathcal{K}} w(k)$$

to be respectively the width of $k$ and $K$. If $k \in \mathcal{K}$ such that $w(k) = w(K)$, we say that $k$ is in thin position. In this paper we call $\omega_i$ the contribution to width by level $c_i$ (also by the unique critical point in this level).
From now on, let $K$ be a satellite knot with knotted companion $J$ and winding number $n$. Let $j$ be a particular embedding and $V$ the tubular neighbourhood of $j$ and $T$ the boundary of $V$, $k \subset V$. Assume that $k, T$ are all in "Morse position", and critical points of $k$ and $T$ are all in distinct levels. Also, we call a horizontal circle in $T$ a level circle. Each level circle bounds two disks on a level sphere, which we call level disks. The level circles are called essential if they are essential in $T$.

In order to prove the main theorem, we need two steps. First, we construct a connected graph and prove that there is a unique loop $l$ in the graph. Also $l$ can be decomposed under connected sum and $j$ is a summand. Second, we determine the relationship between $w(K)$ and $w(L)$.

### 3 Construct the connected graph

Now we construct a graph from $V$. Let $c_1 < c_2 < \ldots < c_m$ denote all the critical values of $h_{|T}$. Let each component of $V - h^{-1}(c_1) - h^{-1}(c_m)$ correspond to a vertex of the graph. Arcs are built in the most obvious ways. To be detailed, draw a directed edge between two vertices $v_1$ and $v_2$ if $v_1, v_2$ are separated by level $h^{-1}(c_i)$ in $V$, and the two component can be connected by a small monotone decreasing arc within a small product neighbourhood of $h^{-1}(c_i) \cap \text{int}(V)$. The direction of the edge is from the upper one to the lower one with respect to $h$. We denote the graph by $\Gamma(V)$. See figure 1 as the graph for a companion of trefoil.
Note that since we cut off critical levels, each component can be seen as a product of a planar surface $S$ with the open interval $(c_i, c_{i+1})$, where $c_i, c_{i+1}$ are the two critical values that bounds the component from below and above. We call each horizontal surface $S \times r$ a base surface of the vertex. In this paper we would use the word "vertex” to denote either a vertex in $\Gamma(V)$ or the component it corresponds to in $V$.

Remark 3.1. The connected graph was defined in Sharlemann and Schultens’s paper [6]. We use it here to see which critical points of $T$ are necessary while which are not. In [6], connected graph is defined for general 3-manifold, while in this paper we fixed our attention on the solid torus.

To realize $\Gamma(V)$ in the interior of $V$. For each vertex, it corresponds to a product $S \times (c_i, c_{i+1})$, choose a point in $S \times \{c_i + c_{i+1}/2\}$ and then connect this point with other chosen points that are in adjacent vertices by monotone increasing or decreasing arcs. Thus we get a 1-complex, see figure 2 as an example. We can construct the 1-complex in the interior of $V$.

From now on, when we use a loop to denote both a simple loop in the graph (regardless of the direction of edge) and a simple closed curve in the 1-complex.

Now consider about the direction of the edge in $\Gamma(V)$. Given a simple loop $l$ (if exists) in $\Gamma(V)$, if a vertex has the start points of both its two edges (that are in the loop), we call it a maximal vertex according to $l$. If a vertex has two end points, we call it a minimal vertex. If it has one end point and one start point, we call it vertical vertex. We use critical vertex to denote both minimal and maximal vertices. Corresponding to each critical vertex, there is a critical level around which the two directed edge are built. we call the height of this critical level the height of the critical vertex. For example, in figure 1 there are 4 critical vertices,
We call the base surface of a vertical vertex a vertical base surface. Also the vertices adjacent to a critical vertex are all vertical.

Now we show that $\Gamma$ has and only has one loop so the definition above make sense and they will be defined for $\Gamma(V)$, not just for a loop. First, we need some prepares.

**Definition 3.2 (Preferred level surface).** Let $S$ be a base surface. If $\partial S$ has and only has 1 essential boundary (in $T$), then we call $S$ a preferred level surface or simply P-surface. See figure 3 as an example, or seen in the whole solid torus, it may looks like the left picture in figure 4.

**Remark 3.3.** P-disks have very good properties. If we start from the innermost inessential circle and attach small disks that are on $T = \partial V$ along each inessential boundary of a P-surface $S$ one by one. Then push the attached disks slightly into the interior of $V$. So we get a properly embedded meridian disk $\hat{S}$ of $V$. By construction, $S \cap k = \hat{S} \cap k$. So in many cases we can consider P-surfaces just as meridian disks. See figure 4 as an example.

**Lemma 3.4.** If $T$ has an essential level circle $s$, so $s$ bounds two closed level disks $D_1$ and $D_2$, then there are disjoint P-surfaces $S_1$ and $S_2$ so that $S_1 \subset D_1$ and $S_2 \subset D_2$.

**Proof.** Suppose $T$ an essential circle $s_0$ on level $h^{-1}(r)$, one of the two closed level circles it bounds is $D'_0$. We may first assume that one component of $D'_0 \cap V$ is a planar surface containing $s_0$ (else one component of $D'_0 \cap V$ is just the circle $s_0$ itself), and denote this planar surface by $S_0$. Then if other boundary component of $S_0$ are all inessential circles, we finish the proof. If one of the other components, say $s_1$ is essential, let $D'_1$ be the level disk bounded by $s_1$ within the disk $D'_0$, then there must occur another essential level circle in the interior of $D'_1$, which we denote by $s_2$. If $s_2$ do not exist, we would construct a compressible disk in the exterior of $V$, which contradicts the fact that $V$ is knotted. Now repeat the same
argument with $s_2$, we would get that the ”innermost level circle” must bounds a P-surface. The other side of $s_0$ is just the same. □

**Corollary 3.5.** Every essential level circle is a meridian in $T$.

**Proof.** Since there is a P-surface that do not intersect the level circle according to lemma 3.4, the circle is isotopic to the essential boundary of a P-surface (which is meridian by remark 3.3) in $T$. □

**Lemma 3.6.** Let $V$ be a solid torus, $l$ be a loop (if exists) in $\Gamma(V)$, then every vertical base surface with respect to $l$ has at least one essential boundary.

**Proof.** If not, we can construct a 2-sphere form the vertical base surface in the same way as constructing a meridian disk from P-disk in remark 3.3. Then $l$ intersects this 2-sphere transversely at 1 time, Which contradicts to the fact that this 2-sphere is separating. □

**Corollary 3.7.** $\Gamma(V)$ is not a tree and there do exist P-surface.

**Proof.** The first part follows proposition 2.3 in [6], since every essential level circle is meridian in $V$ and thus cannot bound meridian disks in the exterior of $V$. The second part is a direct result from first part, lemma 3.4 and lemma 3.6. □

**Lemma 3.8.** $\Gamma(V)$ has an unique loop.

**Proof.** Existence of a loop is in Corollary 2.7. Now to prove the uniqueness. Suppose there are two different loops, then all possibilities of part of 1-complex is pictured in figure 5.

We claim that for each loop $l$ in $\Gamma(V)$ (for example, in case 1, there are three loop $a_1 \cup a_2, a_2 \cup a_3, a_3 \cup a_1$), there is a P-surface intersecting $l$ transversely at 1 point.

To prove this claim, if it is not the case, there are two possibilities: one is that $l$ does not intersects any P-surface; the other is that there is a P-surface intersect $l$
transversely at 2 points (by construction, more point are not possible). In second situation the P-surface is a base surface of a critical vertex, so we move up or down to get a parallel P-surface that do not intersect $l$. So in both case we can find a P-surface disjoint from $l$, say $S_0$. Now we start from an arbitrary vertical vertex $v$ with base surface $S$, by lemma 3.6, $S$ has at least 1 essential boundary and by construction it intersects $l$ transversely at 1 point. We will construct a 2-sphere from $S$. First, we construct a meridian disk $\hat{S}_0$ from $S_0$, then we cancel all the inessential boundaries of $S$ as the same in remark 3.3 (and avoid intersecting $\hat{S}_0$).

Second, label all essential boundaries with $s_1, s_2, ..., s_m$. We would cancel them one by one by attaching disks alone them. We can rearrange the index $1, 2, ..., m$ so that $s_{i+1}$ is "farther away" from $\partial \hat{S}_0$ than $s_i$ on $T$. To describe definitely, by corollary 3.5 they are all meridians so there is a suitable order, suppose it is just $s_1, ..., s_m$, so that there exists a collection of annulus $A_1, ..., A_m$ such that $\partial A_i = \partial \hat{S}_0 \cup s_i$ and $A_i \subset A_{i+1} \subset T$ for each $i = 1, ..., m - 1$.

Now let

$$S_1 = S \cup A_1 \cup \hat{S}_0.$$  

Push the part of $S_1$ in $T$ slightly into the interior of $V$ resulting in $S'_1$. $S'_1$ is a properly embedded planar surface that has one less essential boundary than $S$ but the intersection with $l$ do not change. Then repeat this process using $A_2, ..., A_m$ and $\hat{S}_0$ to cancel all the essential boundaries one by one. See figure 6 as an example. At last we would get a 2-sphere.

Since the attached disks are all disjoint from $l$, the 2-sphere we have already constructed must intersect $l$ transversely 1 time, and this is a contradiction. Thus $l$ would intersect a p-surface transversely at 1 point and so for every P-surface.

However, in case 1, we can show the contradiction since the claim above indicates each loop is a generator of $H_1(V) = \mathbb{Z}$. In case 2, there is a P-surface

Figure 5: The 1-complex
Figure 6: Cancel essential boundaries one by one
disjoint from the vertex $v$ due to lemma 3.4. So this P-surface cannot intersect
both loop, which is a contradiction to the claim above. In case 3, any P-surface
can lead to the same contradiction. □

Since there is a unique loop, we use $l$ to denote it and its isotopic class is $L$.

**Corollary 3.9.** $w(L) \geq w(J)$

*Proof.* First by the claim in lemma 3.8, $L$ can be seen as a connected sum of $J$ and
another knot. Then the inequality is a direct result from corollary 6.5 in [6]. □

## 4 Calculating width

**Lemma 4.1.** Let $n$ be the winding number of satellite knot $k$. Then each vertical
vertex’s base surface intersects $k$ at least $n$ times.

*Proof.* First choose an arbitrary vertical base surface $S$ in a vertex $V$. Choose a
P-surface $S_0$ disjoint from $v$ by lemma 3.4. Then cut $V$ along $S_0$ we get a 3-ball,
whose boundary is the union of part of $T$ and two meridian disk of $V$, say $D_+, D_-$. Since winding number of $k$ is $n$, there must be $n$ arcs so that each arc is part of $k$
and one end point of it is in $D_+$ and the other in $D_-$. Now since the uniqueness
of loop in $\Gamma(V)$, $S$ would separate $D_+$ and $D_-$. So it must intersect $k$ at least $n$
times. □

Now we should modify $l$ into a suitable position so that it is much easier to
find the relationship between $w(k)$ and $w(L)$.
In each critical vertex $v$ of $\Gamma(V)$, we move up or down the part of $l$ that is in $v$. If $v$ is a critical vertex of height $r$, when constructing $l$, we chose a point $P$ in the middle horizontal level of $v$ and connects it with two points $Q_1, Q_2$ in adjacent vertices by two monotone decreasing arc. Now we choose the saddle point $P'$ of $T$ in the level $h^{-1}(r)$ instead of $P$ and connect $P'$ with $Q_1, Q_2$ by two monotone decreasing arc. This process can be realized as an isotopy of $l$. So the resulting knot $l'$ is also in the knot type $L$. Do the same thing for all critical vertex of $\Gamma(V)$ (Note that critical vertices are defined according to the unique loop $l$) It is not difficult to make $l'$ to be in the Morse position while not change the height of its critical points. See figure 7 as an example.

Now we can bounds $w(K)$ from below using $w(l')$:

**Lemma 4.2.** Let $k$ be a satellite knot with companion $j$ and winding number $n$. $j$ has a regular neighbourhood $V$ that cotains $k$. Then construct $\Gamma(V)$ and the modified loop $l'$, we have:

$$w(k) \geq n^2 w(l').$$

*Proof.* Let $k \subset V$ be a satellite knot with winding number $n$. Construct $\Gamma(V)$ and $l'$ as described above.

Now suppose all the critical values of $l'$ are $c_1 < \ldots < c_m$, and $c_i$ corresponds to a critical vertex $v_i$. Suppose the level $c_i$ would contribute $\omega_i$ to the width of $l'$. That is

$$w(l') = \sum_{i=1}^{m} \omega_i.$$  

We use the tree related to a vertex $v$ to denote the subgraph of $\Gamma(V)$ whose vertices are $v$ and vertices that are not in $l$ but are connected to $v$. Since $l$ is unique, this is really a tree and any two tree do not have common vertices. Let the tree related to $v_i$ donated by $T_i$. Any arc that is part of $k$ and in $T_i$ must have both of its end points in the two vertical base surfaces adjacent to $v_i$. See figure 8 as an example.
By definition of $\omega_i$, for small enough $\epsilon_i$, the level $h^{-1}(c_i + \epsilon_i)$ will intersect $l'$ at $\omega_i$ points. Each point is in a vertical vertex and any two such vertex are different vertex (since a vertical base surface only intersects $l'$ at 1 points). So there are exactly $\omega_i$ vertical base surfaces in the level $h^{-1}(c_i + \epsilon_i)$ (but we do not know how many critical base surface). By lemma 4.1, this level would contribute at least $n\omega_i$ to the width of $k$.

Our idea is to modify $k$ in $m$ similar steps. Let the result knot be $k'$. Then we would calculate the difference between $w(k)$ and $w(k')$ and at last evaluate $w(k')$. The key observation is that for each arc in a tree $T$, only one critical point is necessary (as the end points of arc are in the same level), more critical points, though cannot be canceled by isotopy, could only increase width. Choose a small enough $\epsilon$ so that for each $i$, $h^{-1}(c_i - \epsilon, c_i + \epsilon)$ contains no other critical points of $T$ and $k$.

First to describe the modifications. Begin from $c_1$, do a modification and result in $k_1$ and then do the similar process for $c_2, c_3, \ldots, c_n$ and get in sequence $k_2, \ldots, k_m = k'$. For each $i$, after the modification for $c_{i-1}$, if $v_i$ is a minimal vertex, then we focus on the 3-ball which is bounded by $h^{-1}(c_i)$ and contains $-\infty$. First remove all the parts of $k$ that are in this ball but not in the tree $T$. Then move up the remaining part into the region $h^{-1}(c_i - \epsilon, c_i)$ by an isotopy and also preserve the relative height. Then attach back what had been removed in their previous manner, see figure 8 as an example (The level is actually a 2-sphere but in the figure we just use horizontal straight line for convenience). After small perturbs we can avoid possible self intersections created by move up and attach back while not changing height. For maximal vertex, we should use the ball containing $+\infty$ and move down.
We do not care whether \( k_i \) and \( k_{i-1} \) are in the same isotopic class. So we need not bother to check whether this modification is a knot isotopy and just picture a linked part of \( k \) in figure 8. However, we prefer two properties of the result knot \( k' \). Firstly, for each \( i \), and small enough \( \varepsilon_i > 0 \), let \( S_i \) denote the disjoint union of all vertical base surface on \( h^{-1}(c_i + \varepsilon_i) \) (be careful, only vertical base surface, not including critical base surface), since we do not change \( k \) in any vertical vertex, \( k \cap S_i = k' \cap S_i \). Secondly, before the modification, there are no critical points of \( k \) in the region \( h^{-1}(c_i - \varepsilon, c_i + \varepsilon) \), and after that, the region contains only those critical points that are in the tree \( T_i \) and have been moved up or down.

To calculate the difference between \( w(k) \) and \( w(k') \). Suppose \( v_i \) is a minimal critical vertex. Suppose there are \( n \geq n_i \) arcs below level \( h^{-1}(c_i) \) and are in the tree (at least \( 2n \) points in two vertical base surface and each arc has its end points on level \( h^{-1}(c_i) \)) and suppose there are \( m_i \) maximal points below level \( h^{-1}(c_i) \), then there are \( m_i + n_i \) minimal points below level \( h^{-1}(c_i) \). Without mention, we would only discuss critical points below level \( h^{-1}(c_i) \). Pick \( m_i \) pairs \( (P_{i,j}, P'_{i,j}), j = 1, 2, \ldots, m_i \), where \( P_{i,j} \) is a maximal point in tree \( T_i \) and \( P'_{i,j} \) lies below \( P_{i,j} \); \( P'_{i,j} \) is the first minimal point that below \( P_{i,j} \) and differ from \( P'_{i,j-1} \), \( P'_{i,j-1} \ldots, P'_{i,1} \). Denote all the other minimal points that are not in the pair by \( Q_{i,1}, \ldots, Q_{i,n_i} \). Suppose during the modification, \( P_{i,j} \) pass through \( x_{i,j} \) maximal points and \( y_{i,j} \) minimal points and \( P'_{i,j} \) pass through \( (x_{i,j} + u_{i,j}) \) maximal points and \( (y_{i,j} + v_{i,j}) \) minimal points, and \( Q_{i,j} \) pass through \( h_{i,j} \) maximal points. For maximal vertical, just exchange all min and max.

Then the relation between \( w(k_{i-1}) \) \((k_0 = k)\) and \( w(k_i) \) is:

\[
w(k_{i-1}) = w(k_i) - \sum_{j=1}^{m_i} (4u_{i,j} + 4x_{i,j} - 4y_{i,j}) + \sum_{j=1}^{n_i} 4h_{i,j}, \tag{1}
\]

So the relations between \( w(k) \) and \( w(k') \) is:

\[
w(k) = w(k') - \sum_{i=1}^{m} \left( \sum_{j=1}^{m_i} (4u_{i,j} + 4x_{i,j} - 4y_{i,j}) + \sum_{j=1}^{n_i} 4h_{i,j} \right) \tag{2}
\]

To evaluate \( w(k') \), we divide it into two parts: the contribution by \( Q \) points and the contribution by pairs there may be more points so we could only find lower bound for it. For \( Q \) points, first suppose \( Q_{i,1}, \ldots, Q_{i,n_i} \) are in tree related to a minimal vertex. Note we have choose pairs so that the minimal point is the first minimal point below the maximal point, so above the highest \( Q \) points, or between two adjacent \( Q \) points, there are at most a few (or just nothing) pairs. This implies the highest \( n \) \( Q \) points would contribute to width by

\[
\sum_{j=1}^{n} (2n\omega_i - 2(j - 1)). \tag{3}
\]
For maximal vertex it is very similar but we would from the maximal level to go up (2 more base surface, corresponding to \(2n\) in (4)) to calculate width and we say contribution use a level just above the critical point (corresponding to -2 in (4)) so the first \(n\) Q points contribute to width by

\[
\sum_{j=1}^{n} (2n\omega_i + 2n - 2(2(j - 1)))
\]  

(4)

Note that there are exactly same number of minimal and maximal vertices, then all Q points contribute to width no less than

\[
\sum_{i=1}^{m} 2n^2\omega_i
\]  

(5)

At last we show evaluate the contribution by pairs. Suppose before modification for \(v_i\), \(P_{i,j}\) contributes \(f_{i,j}\) to width, and \(P'_{i,j}\) contributes \(g_{i,j}\) to width. Then after \(i\)-th modification, the pair contribute to width by

\[
f_{i,j} + g_{i,j} + 4y_{i,j} + 2v_{i,j} - 4x_{i,j} - 2u_{i,j}
\]  

(6)

Yet this contribution may decrease after preceding modifications. For example, after the modification for \(v_i\), suppose below a pair there are \(a_i\) minimal points and \(b_i\) maximal points, then this contributes to width by \(2a_i - 2b_i + 2\). Suppose the \((i + 1)\)-th modification move up \(d_i\) minimal points points so that they are no longer below the pair. So after \((i + 1)\)-th modification, the pair contributes to width by \(4a_i - 4b_i + 2 - 4d_i\), which is smaller. Since during the modification for \(v_i\), one critical points must pass both points of a pair in trees having less index, One can prove that the decrease in contribution by pairs caused by the move for \(v_i\) is no more than

\[
\sum_{j=1}^{m_i} (2u_{i,j} + 2v_{i,j}) + \sum_{j=1}^{n_i} 4h_{i,j}.
\]  

(7)

See figure 9 for detail.

Combining (6), (7), the contribution by pairs is no less than:

\[
\sum_{i=1}^{m} \sum_{j} (f_{i,j} + g_{i,j} + 4y_{i,j} + 2v_{i,j} - 4x_{i,j} - 2u_{i,j}) - \sum_{i=1}^{m} \sum_{j=1}^{m_i} (2u_{i,j} + 2v_{i,j}) + \sum_{j=1}^{n_i} 4h_{i,j}
\]  

(8)

By (2), (5), (8), we finally get:

\[
w(K) = w(k) \geq \sum_{i=1}^{m} 2n^2\omega_i = n^2w(l).
\]  

(9)
Finally it comes to our main result:

**Theorem.** (1.1) Let \( K \) be a satellite knot with companion \( J \) with winding number \( n \), Then :

\[
w(K) \geq n^2 w(J).
\]

**Proof.** Choose the particular embedding \( k,j \) so that \( k \subset V \) is in thin position. Construct \( \Gamma(V) \) and the modified \( l' \). Then by corollary 3.9 and lemma 4.1, we have:

\[
w(K) = w(k) \geq n^2 w(l') \geq n^2 w(L) \geq n^2 w(J).
\]

**Corollary.** (1.2) Let \( K \) be a satellite knot with companion \( J \) and winding number \( n \), if \( K \) has braid pattern, then :

\[
w(K) = n^2 w(J).
\]

**Proof.** By theorem 1.1, we only need to show that \( w(k) \leq n^2 w(J) \). First let \( j \) be an embedding in thin position. Then choose regular neighbourhood \( V \) of \( j \) so that \( V \) can be decomposed into three basic parts as pictured in figure 10. The existence of such neighbourhood is obvious, or at least one could refer to Schultens’s paper [5] to find a proof.

Let \( c_1 < c_2 < \ldots < c_m \) be all the saddle critical values of \( T = \partial V \). Then we can embed \( k \) so that for each minimal saddle value \( c_i \) there are exactly \( n \) minimal point of \( k \) that below \( c_i \) but higher than \( c_1, \ldots, c_{i-1} \), and similar for maximal saddle
value. Since $K$ has braid pattern, we can also make $k$ have no other critical points. Then calculate directly the width of $k$, we have:

$$w(K) \leq w(k) = n^2w(j) = n^2w(J).$$

\[ \square \]

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