DUAL $F$-SIGNATURE OF SPECIAL COHEN-MACAULAY MODULES OVER CYCLIC QUOTIENT SURFACE SINGULARITIES

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ABSTRACT. The notion of $F$-signature was defined by C. Huneke and G. Leuschke and this numerical invariant characterizes some singularities. This notion is extended to finitely generated modules by A. Sannai and is called dual $F$-signature. In this paper, we determine the dual $F$-signature of a certain class of Cohen-Macaulay modules (so-called “special”) over cyclic quotient surface singularities. Also, we compare the dual $F$-signature of a special Cohen-Macaulay module with that of its Auslander-Reiten translation. This gives a new characterization of the Gorensteinness.

1. INTRODUCTION

Throughout this paper, we suppose that $k$ is an algebraically closed field of prime characteristic $p > 0$. Let $(R, m, k)$ be a Noetherian local ring with $\text{char} R = p > 0$. Since $\text{char} R = p > 0$, we can define the Frobenius map $F : R \to R (r \mapsto r^p)$. For $e \in \mathbb{N}$, we can also define the $e$-times iterated Frobenius map $F^e : R \to R (r \mapsto r^{p^e})$. For any $R$-module $M$, we denote the module $M$ with its $R$-module structure pulled back via the $e$-times iterated Frobenius map $F^e$ by $^eM$. Namely, $^eM$ is just $M$ as an abelian group, and its $R$-module structure is defined by $r \cdot m = F^e(r)m = r^{p^e}m$ for all $r \in R, m \in M$. We say $R$ is $F$-finite if $^1R$ is a finitely generated $R$-module.

In order to investigate the properties of $R$, C. Huneke and G. Leuschke introduced the notion of $F$-signature.

Definition 1.1 ([HL]). Let $(R, m, k)$ be a reduced $F$-finite local ring of prime characteristic $p > 0$. For each $e \in \mathbb{N}$, decompose $^eR$ as follows

$$^eR \cong R^{\oplus a_e} \oplus M_e,$$

where $M_e$ has no free direct summands. We call $a_e$ the $e$-th $F$-splitting number of $R$. Then, the $F$-signature of $R$ is

$$s(R) := \lim_{e \to \infty} \frac{a_e}{p^{ed}},$$

if it exists, where $d := \text{dim} R$.

Note that K. Tucker showed its existence in a general situation ([Tuc]). As Kunz’s theorem ([Kun]) shows, this invariant measures the deviation from regularity (see also Theorem 1.4 (1)).

For a finitely generated $R$-module, A. Sannai extended the notion of $F$-signature as follows.

Definition 1.2 ([San]). Let $(R, m, k)$ be a reduced $F$-finite local ring of prime characteristic $p > 0$. For a finitely generated $R$-module $M$ and $e \in \mathbb{N}$, we set

$$b_e(M) := \max\{n \mid \exists \phi : ^eM \to M^{\oplus n}\},$$

2010 Mathematics Subject Classification. Primary 13A35, 13A50; Secondary 13C14, 16G70.

Key words and phrases. $F$-signature, dual $F$-signature, generalized $F$-signature, finite $F$-representation type, Auslander-Reiten quiver, cyclic quotient surface singularities, special Cohen-Macaulay modules.
and call it the \( e \)-th \( F \)-surjective number of \( M \). Then we call the limit

\[
s(M) := \lim_{e \to \infty} \frac{b_e(M)}{p^{ed}}
\]

dual \( F \)-signature of \( M \) if it exists, where \( d := \dim R \).

Remark 1.3. The dual \( F \)-signature of \( R \) coincides with the \( F \)-signature of \( R \). Because the morphism \( e_R \to R \oplus b_e(R) \) is split. Therefore, we use the same notation unless it causes confusion.

By using these invariants, we can characterize some singularities.

**Theorem 1.4** ([HL], [Yao2], [AL], [San]). Let \((R,m,k)\) be a \( d \)-dimensional reduced \( F \)-finite Noetherian local ring with \( \text{char} R = p > 0 \). Then we obtain

1. \( R \) is regular if and only if \( s(R) = 1 \).
2. \( R \) is strongly \( F \)-regular if and only if \( s(R) > 0 \).

In addition, we suppose \( R \) is Cohen-Macaulay with a canonical module \( \omega_R \), then

3. \( R \) is \( F \)-rational if and only if \( s(\omega_R) > 0 \).
4. \( s(R) \leq s(\omega_R) \).
5. \( s(R) = s(\omega_R) \) if and only if \( R \) is Gorenstein.

As the above theorem shows, the value of \( s(R) \) and \( s(\omega_R) \) have some pieces of information about singularities. How about the value of dual \( F \)-signature for other \( R \)-modules? Since the value of dual \( F \)-signature is not known except the case of two-dimensional Veronese subrings [San] and we don’t know the method for determining it, this question is so difficult for now. Therefore, in this paper, we determine the dual \( F \)-signature for a certain class of Cohen-Macaulay (CM) modules (so-called special CM modules) over cyclic quotient surface singularities. As we will see later, special CM modules are compatible with the geometry.

The study of special CM modules was started by the work of J. Wunram [Wun1, Wun2] (the definition of special CM modules appears in Section 3). For a finite subgroup \( G \) of \( SL(2,k) \) such that the order of \( G \) is invertible in \( k \), the McKay correspondence is very famous, that is, there is a one-to-one correspondence between non-trivial irreducible representations of \( G \) and irreducible exceptional curves on the minimal resolution of quotient surface singularities. When we intend to generalize this correspondence to a finite subgroup \( G \) of \( GL(2,k) \), this correspondence is no longer true. In fact, there are more irreducible representations than exceptional curves. However, if we choose some irreducible representations which is called special, then we again obtain one-to-one correspondence between irreducible special representations of \( G \) and exceptional curves [Wun2]. And a maximal CM module associated with a special representation is called a special CM module. For more about the special McKay correspondence, see also [Ish], [Ito] and [Rie].

**Remark 1.5.** All irreducible representations of a finite subgroup of \( SL(2,k) \) are special, thus we can recover the McKay correspondence in the original sense from the special one.

For a cyclic quotient singularity, a special CM module takes a simple form as follows. (For more details on terminologies, see Section 2 and 3)

Suppose \( R \) is the invariant subring of \( S = k[[x,y]] \) under the action of a cyclic group \( \frac{1}{n}(1,a) \). In this situation, a non-free indecomposable special CM \( R \)-module is described as \( M_i = R^{ii} + Ry^i \) (i.e., it is minimally 2-generated). Then we have the value of dual \( F \)-signature as follows.
Theorem 1.6 (= 3.8). For any non-free indecomposable special CM R-module $M_i$, we have

$$s(M_i) = \begin{cases} 
\frac{\min(i_t, j_t) + 1}{n} & \text{(if } i_t \neq j_t) \\
\frac{2i_t + 1}{2n} & \text{(if } i_t = j_t) 
\end{cases}$$

Moreover, by paying attention to special CM modules and its Auslander-Reiten translation, we characterize the Gorensteiness.

Proposition 1.7 (= 4.2). Let $R$ be a quotient surface singularity. (Note that we don’t restrict to a cyclic case.) Suppose $M$ is an indecomposable special CM $R$-module. Then we have

$$s(M) \leq s(\tau(M)).$$

Moreover, $R$ is Gorenstein if and only if $s(M) = s(\tau(M))$. Here, $\tau(M)$ stands for the Auslander-Reiten translation of $M$.

Remark 1.8. Since $\tau(R) \cong \omega_R$ in our situation, this proposition is an analogue of Theorem 1.4 (4), (5). But it says that this characterization is obtained by not only the comparison between $R$ and $\omega_R$ but also the comparison between a special CM module and its AR translation.

The structure of this paper is as follows. In order to determine the dual $F$-signature, we need the notion of generalized $F$-signature and Auslander-Reiten quiver. So we prepare them in Section 2. In Section 3 we determine the dual $F$-signature of special CM modules over cyclic quotient surface singularities and give several examples. In Section 4, we compare special CM modules with its Auslander-Reiten translation by using the dual $F$-signature and characterize the Gorensteiness. Note that the statements appearing in Section 4 hold not only for cyclic quotient surface singularities but also for any quotient surface singularities.

2. Preliminary

2.1. Generalized $F$-signature of invariant subrings. Let $G$ be a finite subgroup of $\text{GL}(d, k)$ which contains no pseudo-reflections and $S := k[[x_1, \ldots, x_d]]$ be a power series ring. We assume that the order of $G$ is coprime to $p = \text{char } k$. We denote the invariant subring of $S$ under the action of $G$ by $R := S^G$. In order to determine the dual $F$-signature of a finitely generated $R$-module $M$, we have to know about the structure of $eM$ (for instance, the direct sum decomposition of $eM$, the asymptotic behavior of the multiplicities of direct summands, etc). To achieve this, we use the results of generalized $F$-signature of the invariant subrings [HN].

For a positive characteristic Noetherian ring, K. Smith and M. van den Bergh introduced the notion of finite $F$-representation type [SVdB]. This notion is a characteristic $p$ analogue of the notion of finite representation type. The definition of finite $F$-representation type is the following.

Definition 2.1 ([SVdB]). We say that $R$ has finite $F$-representation type (or FFRT for short) by $\mathcal{N}$ if there exists a finite set $\mathcal{N}$ of isomorphism classes of indecomposable finitely generated $R$-modules, such that for every $e \in \mathbb{N}$, the $R$-module $eR$ is isomorphic to a finite direct sum of elements of $\mathcal{N}$.

For example, a power series ring $S$ has FFRT by $\{S\}$ (cf. Kunz’s theorem [Kun]) and FFRT is inherited by a direct summand [SVdB]. Thus, the invariant subring $R$ also has FFRT. More explicitly, we have the next proposition.
Proposition 2.2 ([SVdB]). Let $V_0 = k, V_1, \ldots, V_{n-1}$ be the complete set of irreducible representations of $G$ and we set $M_t := (S \otimes_k V_t)^G$ ($t = 0, 1, \ldots, n-1$). Then $R$ has finite $F$-representation type by the finite set $\{M_0 \cong R, M_1, \ldots, M_{n-1}\}$.

Thus we can write $eR$ as follows.

$$eR \cong R^\oplus c_{0,e} \oplus M_1^\oplus c_{1,e} \oplus \cdots \oplus M_{n-1}^\oplus c_{n-1,e}.$$ 

Remark 2.3. We can see that each $M_t$ is an indecomposable maximal Cohen-Macaulay (= MCM) $R$-module and $M_t \neq M_s$ ($s \neq t$) under the assumption $G$ contains no pseudo-reflections. And the multiplicities $c_{i,e}$ are determined uniquely in that case. For more details refer to [HN Section 2].

Moreover, since the invariant subring $R$ has FFRT, the limit $\lim_{e \to \infty} \frac{c_{t,e}}{pde}$ ($t = 0, 1, \ldots, n-1$) exists [SVdB, Yao1]. So we can define the limit $s(R, M_t) := \lim_{e \to \infty} \frac{c_{t,e}}{pde}$ and call it the (generalized) $F$-signature of $M_t$. The value of $s(R, M_t)$ is determined by M. Hashimoto and the author as follows.

Theorem 2.4 ([HN]). For $t = 0, 1, \ldots, n-1$, one has

$$s(R, M_t) = \frac{\text{rank}_R M_t}{|G|}.$$  

Remark 2.5. In the case of $t = 0$ is also due to [WY]. And a similar result holds for a finite subgroup scheme of $\text{SL}_2$ [HS].

We also obtain the next statement as a corollary.

Corollary 2.6 ([HN]). Suppose an MCM $R$-module $M_t$ decomposes as

$$eR \cong M_t^\oplus d_{t,e} \oplus M_1^\oplus d_{1,e} \oplus \cdots \oplus M_{n-1}^\oplus d_{n-1,e}.$$ 

Then, for all $s, t = 0, \ldots, n-1$, we obtain

$$s(M_s, M_t) = \lim_{e \to \infty} \frac{d_{t,e}}{pde} = \left(\frac{\text{rank}_R M_t}{\text{rank}_R M_s}\right) \cdot s(R, M_s) = \frac{(\text{rank}_R M_t) \cdot (\text{rank}_R M_s)}{|G|}.$$ 

Remark 2.7. In dimension two, it is known that an invariant subring $R$ is of finite representation type, that is, it has only finitely many non-isomorphic indecomposable MCM $R$-modules $\{R, M_1, \ldots, M_{n-1}\}$. From Corollary 2.6, every indecomposable MCM $R$-modules appear in $eR$ as a direct summand for sufficiently large $e$. Thus, the additive closure $\text{add}_R(eM_t)$ coincides with the category of MCM $R$-modules $\text{CM}(R)$. So we use several results so-called Auslander-Reiten theory to $\text{add}_R(eM_t)$ (see the next subsection).

One of the aim of this paper is to determine the dual $F$-signature of special CM modules over cyclic quotient surface singularities. Thus, in the rest of Section 2 and Section 3, we suppose that $G$ is a cyclic group as follows.

$$G := \langle \sigma = \left(\begin{array}{cc} \zeta_n & 0 \\ 0 & \zeta_a^n \end{array}\right) \rangle,$$

where $\zeta_n$ is a primitive $n$-th root of unity, $1 \leq a \leq n-1$, and $\gcd(a, n) = 1$. We denote the cyclic group $G$ as above by $\frac{1}{n}(1, a)$. Let $S := k[[x, y]]$ be a power series ring and we assume that $n$ is coprime to $p = \text{char } k$. We denote the invariant subring of $S$ under the action of $G$ by $R := S^G$.

Since $G$ is an abelian group, any irreducible representations of $G$ are described by

$$V_t : \sigma \mapsto \zeta_n^{-t} \quad (t = 0, 1, \ldots, n-1).$$
In the case of

\[ \text{consists of morphisms } \Hom_{R} \text{ the Auslander-Reiten } \phi \]

\[ \text{In addition, we define the submodule } \text{singularity, then there exists the AR sequence ending in } \]

\[ \text{And we set, } \]

\[ \text{This exact sequence is called fundamental sequence of } \]

\[ \text{the AR sequence is unique up to isomorphism, if it exists. And if } A \text{ is an isolated singularity, then there exists the AR sequence ending in } M, \text{ where } M \text{ is a non-free indecomposable MCM } A \text{-module } [\text{Aus2}]. \]

Therefore, in our case (namely } R \text{ is the invariant subring of } S \text{ under the action of } G = \frac{1}{n}(1, a)) \text{, there exists the AR sequence ending in } M_{t} \text{ for any indecomposable MCM } R \text{-modules } M_{t} = (S \otimes_{k} V_{t})^{G} (t \neq 0). \]

In our situation, the AR sequence ending in } M_{t} (t \neq 0) \text{ is}

\[ 0 \to N \xrightarrow{f} L \xrightarrow{\psi} M \to 0 \]

\[ \text{In the case of } t = 0, \text{ there is the exact sequence } \]

\[ 0 \to \omega_{R} \to M_{t} \oplus M_{t-a} \to R \to k \to 0. \]

\[ \text{This exact sequence is called fundamental sequence of } R. \]

We call the left term of these sequences the Auslander-Reiten (=AR) translation and denote by } \tau(M_{t}) \text{ and it is known that } \tau(M_{t}) \equiv (M_{t} \otimes_{R} \omega_{R})^{**}, \text{ where } (-)^{*} = \Hom_{R}(\_, R) \text{ is the } R \text{-dual functor } [\text{Aus1}]. \text{ Sometimes we denote the next term of } \tau(M_{t}) \text{ by } E_{M_{t}} \text{. Namely, } \tau(M_{t}) = M_{t-a} \text{ and } E_{M_{t}} = M_{t-1} \oplus M_{t-a} \text{ for } t = 0, 1, \cdots, n - 1 \text{ in our situation.} \]

Next, we prepare some notions to define the Auslander-Reiten quiver.

\[ \text{Definition 2.10 (Irreducible morphism). Let } M \text{ and } N \text{ be MCM } R \text{-modules. We decompose } M \text{ and } N \text{ into indecomposable modules as } M = \bigoplus \bigoplus_{i} M_{i}, N = \bigoplus \bigoplus_{j} N_{j} \text{ and decompose } \psi \in \Hom_{R}(M, N) \text{ along this decomposition as } \psi = (\psi_{ij} : M_{i} \to N_{j}|_{ij}) \text{. Then we define the submodule } \text{rad}^{R}_{R}(M, N) \subset \Hom_{R}(M, N) \text{ as follows.} \]

\[ \psi \in \text{rad}^{R}_{R}(M, N) \text{ def } \text{no } \psi_{ij} \text{ is an isomorphism} \]

\[ \text{In addition, we define the submodule } \text{rad}_{R}^{2}(M, N) \subset \Hom_{R}(M, N). \text{ The submodule } \text{rad}_{R}^{2}(M, N) \text{ consists of morphisms } \psi : M \to N \text{ such that } \psi \text{ decomposes as } \psi = g \circ f, \]

\[ M \xrightarrow{\psi} N \]

\[ f \xrightarrow{\psi} g \]

\[ Z \xrightarrow{\psi} Z \]
where $Z$ is an MCM $R$-module, $f \in \text{rad}_R(M, Z)$, $g \in \text{rad}_R(Z, N)$. We call a morphism $\psi : M \to N$ irreducible if $\psi \in \text{rad}_R(M, N) \setminus \text{rad}_R^2(M, N)$. Set
\[
\text{Irr}_R(M, N) := \text{rad}_R(M, N) / \text{rad}_R^2(M, N),
\]
then $\text{Irr}_R(M, N)$ is a vector space over $k$.

By using these notions, we define the Auslander-Reiten quiver.

**Definition 2.11** (Auslander-Reiten quiver). The Auslander-Reiten (=AR) quiver of $R$ is an oriented graph whose vertices are indecomposable MCM $R$-modules $R, M_1, \cdots, M_{n-1}$ with $\dim_k \text{Irr}_R(M_s, M_t)$ arrows from $M_s$ to $M_t$ ($s, t = 0, 1, \cdots, n - 1$).

It is known that $\dim_k \text{Irr}_R(M_s, M_t)$ is equal to the multiplicity of $M_s$ in the decomposition of $E_{M_t}$. From (2.2) and (2.3), for $t = 0, 1, \cdots, n - 1$ there is an arrow from $M_{t-1}$ to $M_t$ and from $M_{t-a}$ to $M_t$, that is, $\dim_k \text{Irr}_R(M_{t-1}, M_t) = 1$ and $\dim_k \text{Irr}_R(M_{t-a}, M_t) = 1$.

**Remark 2.12.** Since $S \cong R \oplus M_1 \oplus \cdots \oplus M_{n-1}$, each MCM $R$-modules $M_t$ is an $R$-submodule of $S$. And we can take a morphism $\cdot x$ (resp. $\cdot y$) as a basis of 1-dimensional vector space $\text{Irr}_R(M_{t-1}, M_t)$ (resp. $\text{Irr}_R(M_{t-a}, M_t)$).

\[
\begin{align*}
M_{t-1} &= \{ f \in S \mid \sigma \cdot f = \zeta_n^{t-1} f \} \\ M_t &= \{ f \in S \mid \sigma \cdot f = \zeta_n^t f \} \\
M_{t-a} &= \{ f \in S \mid \sigma \cdot f = \zeta_n^{t-a} f \} \\ M_t &= \{ f \in S \mid \sigma \cdot f = \zeta_n^t f \}
\end{align*}
\]

**Example 2.13.** Let $G = \frac{1}{3}(1, 3)$ be a cyclic group of order 7. The irreducible representations of $G$ are
\[
V_t : \sigma \mapsto \zeta_7^{-t} \quad (t = 0, \cdots, 6),
\]
where $\zeta_7$ is a primitive 7-th root of unity. Then the AR quiver of $R$ is described as follows. For simplicity, we only describe subscripts as vertices.

\[
\begin{array}{cccccccc}
0 & \longrightarrow & 3 & \longrightarrow & 6 & \longrightarrow & \cdots & \longrightarrow & 1 & \longrightarrow & 4 & \longrightarrow & 0 \\
| & | & | & | & | & & | & | & | & | & | \\
6 & \longrightarrow & 2 & \longrightarrow & 5 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 3 & \longrightarrow & 6 \\
| & | & | & | & | & & | & | & | & | & | \\
5 & \longrightarrow & 4 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 2 & \longrightarrow & 5 \\
| & | & | & | & | & & | & | & | & | & | \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
| & | & | & | & | & & | & | & | & | & | \\
2 & \longrightarrow & 5 & \longrightarrow & 1 & \longrightarrow & \cdots & \longrightarrow & 3 & \longrightarrow & 6 & \longrightarrow & 2 \\
| & | & | & | & | & & | & | & | & | & | \\
1 & \longrightarrow & 4 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 2 & \longrightarrow & 5 & \longrightarrow & 1 \\
| & | & | & | & | & & | & | & | & | & | \\
0 & \longrightarrow & 3 & \longrightarrow & 6 & \longrightarrow & \cdots & \longrightarrow & 1 & \longrightarrow & 4 & \longrightarrow & 0 \\
\end{array}
\]

where the left and right hand sides are identified, moreover the top and bottom row are identified.

**Remark 2.14.** For each diagram $\begin{array}{ccc}
a & \longrightarrow & b \\
\nw & \downarrow & \wedge \\
\v & \longrightarrow & d
\end{array}$, if $b \neq 0$ then $0 \to M_c \to M_a \oplus M_d \to M_b \to 0$ is the AR sequence ending in $M_b$. And any diagram is commute $\begin{array}{ccc}
a & \longrightarrow & b \\
\nw & \downarrow & \wedge \\
\v & \longrightarrow & d
\end{array}$ from Remark 2.12.
3. Dual $F$-Signature of Special CM Modules

In this section, we introduce the notion of special CM modules and determine the dual $F$-signature of them. Firstly, we recall the definition of special CM modules and the properties of them.

**Definition 3.1** ([Wun2]). For an MCM $R$-module $M$, we call $M$ special if $(M \otimes_R \omega_R)/\text{tor}$ is also an MCM $R$-module.

In other words, let $\varphi$ be the natural morphism $M \otimes_R \omega_R \to (M \otimes_R \omega_R)^{**}$, then $M \otimes_R \omega_R / \text{Ker} \varphi$ is also an MCM $R$-module if and only if $M$ is a special CM $R$-module. Also, it is known that $M \otimes_R \omega_R / \text{Ker} \varphi \cong \tau(M) \cong (M \otimes_R \omega_R)^{**}$.

Therefore, $M$ is a special CM $R$-module if and only if $\varphi$ is a surjection.

For a cyclic group $G = \frac{1}{n}(1, a)$, we can describe special CM-modules as follows. Firstly, we consider the Hirzebruch-Jung continued fraction expansion of $n/a$,

$$\frac{n}{a} = \alpha_1 \frac{1}{\alpha_2 \frac{1}{\alpha_3 \cdots}} := [\alpha_1, \alpha_2, \ldots, \alpha_r],$$

and then we introduce the notion of $i$-series and $j$-series (cf. [Wem], [Wun1]).

**Definition 3.2.** For $n/a = [\alpha_1, \alpha_2, \ldots, \alpha_r]$, we define the $i$-series and the $j$-series as follows.

$$i_0 = n, \quad i_1 = a, \quad i_t = \alpha_{t-1}i_{t-1} - i_{t-2} \quad (t = 2, \ldots, r + 1),$$

$$j_0 = 0, \quad j_1 = 1, \quad j_t = \alpha_{t-1}j_{t-1} - j_{t-2} \quad (t = 2, \ldots, r + 1).$$

**Remark 3.3.** By the construction method of the $i$-series and the $j$-series, it is easy to see

- $i_t \equiv j_ia \pmod{n}$,
- $i_0 = n > i_1 = a > i_2 > \cdots > i_r = 1 > i_{r+1} = 0$,
- $j_0 = 0 < j_1 = 1 < j_2 = \alpha_1 < \cdots < j_r < j_{r+1} = n$,
- $i_t/i_{t+1} = \alpha_{t+1} - \frac{1}{\alpha_{t+2} - \frac{1}{\alpha_{t+3} - \cdots}} = [\alpha_{t+1}, \alpha_{t+2}, \ldots, \alpha_r]$,
- $j_{t+1}/j_t = \alpha_t - \frac{1}{\alpha_{t-1} - \frac{1}{\alpha_{t-2} - \cdots}} = [\alpha_t, \alpha_{t-1}, \ldots, \alpha_1]$.

By using the $i$-series and the $j$-series, we can characterize special CM $R$-modules.

**Theorem 3.4** ([Wun1]). For a cyclic group $G = \frac{1}{n}(1, a)$ with $n/a = [\alpha_1, \alpha_2, \ldots, \alpha_r]$, special CM $R$-modules are $M_i$ $(t = 0, 1, \ldots, r)$. Moreover, the minimal generators of $M_i$ are $x^i$ and $y^i$ for $i = 1, \ldots, r$.

**Example 3.5.** Let $G = \frac{1}{7}(1, 3)$ be a cyclic group of order 7. The Hirzebruch-Jung continued fraction expansion of $7/3$ is

$$\frac{7}{3} = 3 - \frac{1}{2 - \frac{1}{2}} = [3, 2, 2],$$
and the $i$-series and the $j$-series are described as follows.

$$i_0 = 7, \quad i_1 = 3, \quad i_2 = 2, \quad i_3 = 1, \quad i_4 = 0,$$

$$j_0 = 0, \quad j_1 = 1, \quad j_2 = 3, \quad j_3 = 5, \quad j_4 = 7.$$

Thus, the special CM modules are $R, M_1, M_2, M_3$ and these are described explicitly

$$R = k[[x^7, x^4y, xy^2, y^7]]$$

$$M_1 = Rx + Ry^5$$

$$M_2 = Rx^2 + Ry^3$$

$$M_3 = Rx^3 + Ry.$$

By using the AR quiver, we can reinterpret these results as follows.

By the above arguments, in order to construct the surjection $G \to H$, we may ignore them. Also, there are morphisms from vertices which described by double arrows in Figure 2. Thus, in order to investigate the surjection $\bullet \to \odot$, we can only discuss horizontal direction arrows from $\bullet$ to $\odot$ and the vertical direction from $\bullet$ to $\odot$ (Figure 3).

Minimal generators of $M_\star$ are generated by the morphisms from $0$ (which located in dotted vertices of Figure 2) to $\star$. Considering the composition of such a morphism and $\star \to \odot$

$$R \to M_\star \to M_2 (1 \to \delta \to x^{m_1}y^{m_2} \delta),$$

where $\delta$ is a minimal generator of $M_\star$ and $m_1 \geq 1, m_2 \geq 1$. Then it is easy to see that the image of the morphism $\star \to \odot$ is in $mM_2$. Thus we may ignore them.

Thus, in order to investigate the surjection $\epsilon M_2 \to M_2^{\mu h}$, we may only discuss the MCM $R$-modules located in the horizontal direction from $M_2$ to $R$ and the vertical direction from $M_2$ to $R$ (Figure 4).

In general, the number of minimal generators of special CM $R$-module $M_i$ is two and minimal generators take a form like $x^a, y^b$ (cf. Theorem 3.4). Thus, it is equivalent to there is no “0” in dotted vertices area of Figure 3. By the above arguments, in order to construct the surjection $cM_i \to M_i^{\mu h}$, we may only discuss horizontal direction arrows from $R$ to $M_i$ and vertical direction arrows from $R$ to $M_i$. And we consider sets of subscripts of vertices $\mathcal{F}_i = \{0, 1, \cdots, i_t - 1\}$ and $\mathcal{G}_i = \{i_t - a, \cdots, i_t - j_1a \equiv 0\}$. It is easy to see that $|\mathcal{F}_i| = i_t, |\mathcal{G}_i| = j_1$. 
Lemma 3.6 ([Wun1]). Let $\beta$ be same as above. Then $\tilde{\beta}$ is described as

$$\tilde{\beta} = d_1 j_1 + d_2 j_2 + \cdots + d_r j_r,$$

where $(j_1, \cdots, j_r)$ is the $j$-series associated with $\frac{1}{n}(1, a)$.

Lemma 3.7. Let the notation be same as above, then $\mathcal{F}_t \cap \mathcal{G}_i = \{0\}$ as a set of subscripts of vertices.
Proof. It is trivial that $0 \in \mathcal{F}_i \cap \mathcal{G}_i$ by the definition of $\mathcal{F}_i$ and $\mathcal{G}_i$. Thus, it suffices to show there is no pair $(m_1, m_2) \in \mathbb{Z}_{>0}^2$ such that $m_1 \equiv m_2 d \pmod{n}$, where $1 \leq m_1 \leq i - 1$ and $1 \leq m_2 \leq j_i - 1$. Assume that there exists such a pair $(m_1, m_2)$. Then we replace the set $\mathcal{F}_i$ by $\mathcal{F}_i \setminus \{0\}$ (resp. $\mathcal{G}_i \setminus \{0\}$). Since $1 \leq m_1 \leq i - 1$ and $i_r > i_{r+1}$ (cf. Remark 3.3), $d_1 = \cdots = d_l = 0$ and there exists $\lambda$ such that $r + 1 \leq \lambda \leq r$ and $d_\lambda \neq 0$. From Lemma 3.6 we obtain $m_2 = d_1 j_1 + d_2 j_2 + \cdots + d_r j_r$. Thus,

$$m_2 = d_{t+1} j_{t+1} + \cdots + d_r j_r \geq j_\lambda > j_t.$$  

This contradicts $m_2 \leq j_t - 1$.  

So we are now ready to state the main theorem.

**Theorem 3.8.** Let the notation be the same as above, then for any non-free special CM $R$-module $M_i$, one has

$$s(M_i) = \begin{cases} \frac{\min(i_t, j_t) + 1}{n} & (\text{if } i_t \neq j_t) \\ \frac{2i_t + 1}{2n} & (\text{if } i_t = j_t) \end{cases}.$$  

Proof. From (2.1), we may consider as

$$M_i \cong (R \oplus M_1 \oplus \cdots \oplus M_{n-1}) \oplus \mathbb{Z}^2.$$  

Firstly, we shall show in the case of $i_t > j_t$. If $\mathcal{G}_i \setminus \{0\} \neq \emptyset$, then we choose an element from $\mathcal{F}_i \setminus \{0\}$ (named it $f$) and also choose an element from $\mathcal{G}_i \setminus \{0\}$ (named it $g$). Note that $f \neq g$, from Lemma 3.7. By using the corresponding indecomposable MCM $R$-modules $M_f$ and $M_g$, we construct a surjection $M_f \oplus M_g \twoheadrightarrow M_i$.

Then we replace the set $\mathcal{F}_i \setminus \{0,f\}$ (resp. $\mathcal{G}_i \setminus \{0,g\}$) by the set $\mathcal{F}_i \setminus \{0\}$ (resp. $\mathcal{G}_i \setminus \{0\}$).

If $\mathcal{G}_i \setminus \{0\} \neq \emptyset$, then we repeat a similar process for the sets $\mathcal{F}_i \setminus \{0\}$ and $\mathcal{G}_i \setminus \{0\}$.

If $\mathcal{G}_i \setminus \{0\} = \emptyset$, then we construct a surjection by combining $0 \in \mathcal{G}_i$ and an element of $\mathcal{F}_i \setminus \{0\}$. Thus, we can obtain the total of $|\mathcal{G}_i| = j_t$ surjections through these processes. And there is the trivial surjection $M_0 \twoheadrightarrow M_i$. So the dual $F$-signature of $M_i$ is $s(M_i) = \frac{j_t}{n} + \frac{1}{n}$.

Similarily, we obtain $s(M_i) = \frac{i_t}{n} + \frac{1}{n}$ in the case of $i_t < j_t$.

In the case of $i_t = j_t$, we can obtain the total of $i_t - 1$ surjections by using a similar process as above. And we also obtain $\mathcal{F}_i \setminus \{0\} = \emptyset$ and $\mathcal{G}_i \setminus \{0\} = \emptyset$ at the same time. In addition to these surjections, we construct

$$M_i \twoheadrightarrow M_i \text{ and } R^{1/2} \oplus R^{1/2} \twoheadrightarrow M_i^{1/2}.$$
Thus, the dual $F$-signature of $M_i$ is
\[
s(M_i) = \frac{i_t - 1}{n} + \frac{1}{2n} = \frac{2i_t + 1}{2n}.
\]

**Example 3.9.** Let the notation be as in Example 3.5. Then, the dual $F$-signature of special CM modules are
\[
s(M_1) = \frac{2}{7}, \quad s(M_2) = \frac{3}{7}, \quad s(M_3) = \frac{2}{7}.
\]

Next, we give an example in the case $i_t = j_t$.

**Example 3.10.** Let $G = \frac{1}{8}(1,5)$ be a cyclic group of order 8. The Hirzebruch-Jung continued fraction expansion of $8/5$ is
\[
\frac{8}{5} = 2 - \frac{1}{3-1/2} = [2, 3, 2],
\]
and the $i$-series and the $j$-series are described as follows.
\[
i_0 = 8, \quad i_1 = 5, \quad i_2 = 2, \quad i_3 = 1, \quad i_4 = 0, \\
j_0 = 0, \quad j_1 = 1, \quad j_2 = 2, \quad j_3 = 5, \quad j_4 = 8.
\]
Thus, special CM modules are $R, M_1, M_2, M_5$. In this case, we have $i_2 = j_2$ and there exists the surjection as follows.

\[
0 \xrightarrow{\gamma} 5 \xrightarrow{\gamma} 2 \\
\uparrow x \quad \uparrow x \\
1 \\
\downarrow x \quad \downarrow x \\
0 \quad M_2 \\
M_1 \oplus M_5 \quad \rightarrow \quad M_2
\]
\[
\begin{array}{c}
R^{\oplus 1/2} \oplus R^{\oplus 1/2} \\
\rightarrow \\
M_2^{\oplus 1/2}
\end{array}
\]

Thus, the dual $F$-signature of $M_2$ is
\[
s(M_2) = \frac{1}{8} + \frac{1}{8} + \frac{1}{16} = \frac{5}{16}.
\]

**Example 3.11.** Let $G = \frac{1}{n}(1, n - 1) \subset \text{SL}(2, k)$ be a cyclic group of order $n$, that is, Dynkin type $A_{n-1}$. The Hirzebruch-Jung continued fraction expansion of $n/(n - 1)$ is
\[
\frac{n}{n-1} = 2 - \frac{1}{2 - \frac{1}{2 - \cdots}} = [2, 2, \cdots, 2],
\]
and the $i$-series and the $j$-series are described as follows.
\[
i_0 = n, \quad i_1 = n - 1, \quad i_2 = n - 2, \quad \cdots, \quad i_{n-1} = 1, \quad i_n = 0, \\
j_0 = 0, \quad j_1 = 1, \quad j_2 = 2, \quad \cdots, \quad j_{n-1} = n - 1, \quad j_n = n.
\]
Namely, $i_t = n - t, \quad j_t = t (t = 1, 2, \cdots, n - 1)$. As we mentioned in Remark 1.5, all irreducible representations of $G = \frac{1}{n}(1, n - 1) \subset \text{SL}(2, k)$ are special. Thus, any $M_t$ is a special CM module.
and the dual $F$-signature of $M_t$ is obtained by Theorem $3.8$

$$s(M_t) = \begin{cases} 
\frac{1}{n} + \frac{j_t}{n} = \frac{t+1}{n} & (\text{if } t < \frac{p}{2}) \\
\frac{1}{n} + \frac{t-1}{n} + \frac{1}{2n} = \frac{2t+1}{2n} & (\text{if } t = \frac{p}{2}) \\
\frac{1}{n} + \frac{i_t}{n} = \frac{n-t+1}{n} & (\text{if } t > \frac{p}{2}).
\end{cases}$$

About other Dynkin types (i.e. $D_n, E_6, E_7, E_8$), see [Nak].

4. Comparing with Auslander-Reiten translation

In this section, we compare the dual $F$-signature of special CM modules with its AR translation. It will give us a characterization of Gorensteiness (see Proposition $4.2$). As we mentioned in Section $1$, it is an analogue of Theorem $1.4$ (4), (5).

The statements appear in this section are valid for any quotient surface singularities. Therefore, we suppose that $G$ is a finite subgroup of $GL(2, k)$ which contains no pseudo-reflections and $S := k[[x, y]]$ be a power series ring. We assume that the order of $G$ is coprime to $p = \text{char } k$.

We denote the invariant subring of $S$ under the action of $G$ by $R := S^G$. Let $V_0 = v_1, \ldots, v_n$ be the complete set of irreducible representations of $G$ and set the indecomposable MCM $R$-modules $M_t := (S \otimes_k V_t)^G$ ($t = 0, 1, \cdots, n$).

**Lemma 4.1.** Let $M_t$ be an MCM $R$-module as above. Then we have

$$eM_t \cong (R^{\otimes d_{0,t}} \oplus M_1^{\otimes d_{1,t}} \oplus \cdots \oplus M_n^{\otimes d_{n,t}}) \oplus \mathbb{Z}^2 \cong e \tau(M_t)$$

(4.1)
on the order of $p^{2e}$, where $d_{i,t} = (\text{rank}_R M_t) \cdot (\text{rank}_R M_i)$ and $\tau$ stands for the AR translation.

Furthermore, we have

$$R^{\otimes d_{0,t}} \oplus M_1^{\otimes d_{1,t}} \oplus \cdots \oplus M_n^{\otimes d_{n,t}} \cong \tau(R)^{\otimes d_{0,t}} \oplus \tau(M_1)^{\otimes d_{1,t}} \oplus \cdots \oplus \tau(M_n)^{\otimes d_{n,t}}.$$

**Proof.** From Corollary $2.6$ we may consider as

$$eM_t \cong (R^{\otimes d_{0,t}} \oplus M_1^{\otimes d_{1,t}} \oplus \cdots \oplus M_n^{\otimes d_{n,t}}) \oplus \mathbb{Z}^2,$$

$$e \tau(M_t) \cong (R^{\otimes d_{0,t}} \oplus M_1^{\otimes d_{1,t}} \oplus \cdots \oplus M_n^{\otimes d_{n,t}}) \oplus \mathbb{Z}^2,$$

where $d_{i,t}' = (\text{rank}_R \tau(M_i)) \cdot (\text{rank}_R M_i)$. Since $\text{rank}_R M_t = \text{rank}_R \tau(M_t)$, it follows that $d_{i,t} = d_{i,t}'$ ($i = 0, 1, \cdots, n$). This implies (4.1).

Since the AR translation $\tau$ gives a bijection from the set of finitely many indecomposable MCM $R$-modules to itself, we set $\tau(M_i) = M_{\sigma(i)}$ ($i = 0, 1, \cdots, n$), where $\sigma$ is an element of symmetric group $S_{n+1}$. Then we have

$$R^{\otimes d_{0,t}} \oplus M_1^{\otimes d_{1,t}} \oplus \cdots \oplus M_n^{\otimes d_{n,t}} \cong M_{\sigma(0)}^{\otimes d_{\sigma(0),t}} \oplus M_{\sigma(1)}^{\otimes d_{\sigma(1),t}} \oplus \cdots \oplus M_{\sigma(n)}^{\otimes d_{\sigma(n),t}},$$

and

$$d_{\sigma(i),t} = (\text{rank}_R M_i) \cdot (\text{rank}_R M_{\sigma(i)}) = (\text{rank}_R M_t) \cdot (\text{rank}_R \tau(M_i))$$

$$= (\text{rank}_R M_t) \cdot (\text{rank}_R M_i) = d_{i,t}.$$ 

Thus,

$$M_{\sigma(0)}^{\otimes d_{\sigma(0),t}} \oplus M_{\sigma(1)}^{\otimes d_{\sigma(1),t}} \oplus \cdots \oplus M_{\sigma(n)}^{\otimes d_{\sigma(n),t}} = \tau(R)^{\otimes d_{0,t}} \oplus \tau(M_1)^{\otimes d_{1,t}} \oplus \cdots \oplus \tau(M_n)^{\otimes d_{n,t}}.$$
Proposition 4.2. Suppose $M_t$ is a special CM $R$-module. Then we have
\[ s(M_t) \leq s(\tau(M_t)). \]
Moreover, $R$ is Gorenstein if and only if $s(M_t) = s(\tau(M_t))$.

Proof. From Lemma 4.1 we may consider as
\[ \epsilon M_t \approx \epsilon \tau(M_t) \approx (R^{\oplus d_0} \oplus M_1^{\oplus d_1} \oplus \cdots \oplus M_n^{\oplus d_n}) \oplus 2e \]
when we discuss the asymptotic behavior on the order of $p^{2e}$, where $d_i = (\text{rank}_RM_i) \cdot (\text{rank}_RM_t)$.
In the rest of this proof, we discuss on this setting and for simplicity we identify $\epsilon M_t \approx \epsilon \tau(M_t)$ with $R^{\oplus d_0} \oplus M_1^{\oplus d_1} \oplus \cdots \oplus M_n^{\oplus d_n}$.

Since $M_t$ is special, the morphism $\varphi : M_t \otimes_R \omega_R \to (M_t \otimes_R \omega_R)^{**}$ is surjective. Let $b_e := b_e(M_t)$ be the $e$-th $F$-surjective number of $M_t$. Then there exists the surjection $\epsilon M_t \to M_t^{\oplus b_e}$. Applying the functor $(- \otimes_R \omega_R)$ and combining with $\varphi$, we obtain the surjection
\[ \epsilon M_t \otimes_R \omega_R \twoheadrightarrow (M_t \otimes_R \omega_R)^{\oplus b_e} \xrightarrow{\varphi^{\oplus b_e}} ((M_t \otimes_R \omega_R)^{**})^{\oplus b_e} \cong \tau(M_t)^{\oplus b_e}. \tag{4.2} \]
Since we consider as $\epsilon M_t \equiv R^{\oplus d_0} \oplus M_1^{\oplus d_1} \oplus \cdots \oplus M_n^{\oplus d_n}$, it follows that $\epsilon M_t \otimes_R \omega_R \cong \bigoplus_{i=0}^n (M_i \otimes_R \omega_R)^{\oplus d_i}$, and the surjection (4.2) induces the following commutative diagram.

\[ \begin{array}{ccc}
\bigoplus_{i=0}^n (M_i \otimes_R \omega_R)^{\oplus d_i} & \twoheadrightarrow & \tau(M_t)^{\oplus b_e} \\
\downarrow & & \downarrow \cong \\
\left( \bigoplus_{i=0}^n (M_i \otimes_R \omega_R)^{\oplus d_i} \right)^{**} & \twoheadrightarrow & (\tau(M_t)^{\oplus b_e})^{**}
\end{array} \]

Thus, the morphism
\[ \left( \bigoplus_{i=0}^n (M_i \otimes_R \omega_R)^{\oplus d_i} \right)^{**} \cong \bigoplus_{i=0}^n \tau(M_i)^{\oplus d_i} \twoheadrightarrow \tau(M_t)^{\oplus b_e} \]
is also surjective. From Lemma 4.1 we obtain $\epsilon \tau(M_t) \approx \bigoplus_{i=0}^n \tau(M_i)^{\oplus d_i}$. Thus, there exists the surjection $\epsilon \tau(M_t) \to \tau(M_t)^{\oplus b_e}$. This implies $s(M_t) \leq s(\tau(M_t))$.

If $R$ is Gorenstein, then $M_t \cong \tau(M_t)$. Thus $s(M_t) = s(\tau(M_t))$ holds. So we shall show the opposite direction. Assume that $R$ is not Gorenstein. Since $M_t$ is special, the number of minimal generators of $M_t$ is equal to $u := 2\text{rank}_RM_t$ [Wun2]. Thus, there exists the surjection $R^{\oplus b_e} \to M_t^{\oplus b_e}$ and induces the following commutative diagram.

\[ \epsilon M_t \twoheadrightarrow M_t^{\oplus b_e} \]

Applying the functor $(- \otimes_R \omega_R)^{**}$ to this commutative diagram, then we obtain the commutative diagram.

\[ \epsilon \tau(M_t) \approx (\epsilon M_t \otimes_R \omega_R)^{**} \twoheadrightarrow \tau(M_t)^{\oplus b_e} \]

\[ \begin{array}{ccc}
\epsilon \tau(M_t) & \twoheadrightarrow & \tau(M_t)^{\oplus b_e} \\
\downarrow & & \downarrow \psi_1 \\
\epsilon M_t \otimes_R \omega_R & \twoheadrightarrow & \tau(M_t)^{\oplus b_e}
\end{array} \]
Note that the morphism $\psi_1$ is surjective because the surjection $R \oplus d_0, e \twoheadrightarrow M_t \oplus b_e$ induces

$$\begin{align*}
\omega_R \oplus d_0, u & \twoheadrightarrow (M_t \otimes_R \omega_R) \oplus b_e \\
\cong & \quad (M_t \otimes_R \omega_R)^{**} \oplus b_e \\
\psi_1 & \twoheadrightarrow (M_t \otimes_R \omega_R)^{**} \oplus b_e
\end{align*}$$

and $\phi : M_t \otimes_R \omega_R \twoheadrightarrow (M_t \otimes_R \omega_R)^{**}$ is surjective. And this implies $\psi_2$ is also surjective. On the surjection

$$\omega_R \oplus d_0, u \twoheadrightarrow e \tau(M_t) \cong \bigoplus_{i=0}^n \tau(M_t) \oplus d_{i,}\psi_2 \twoheadrightarrow \tau(M_t) \oplus b_e,$$

the morphisms which go through $R$ don’t contribute to construct a surjection by Nakayama’s lemma. Thus,

$$\bigoplus_{i=0}^n \tau(M_t) \oplus d_{i,} / R \oplus d_{0,} e \twoheadrightarrow \tau(M_t) \oplus b_e,$$

is also surjective. In addition to this surjection, we can construct the surjection

$$R \oplus d_{0,} e \twoheadrightarrow \tau(M_t) \oplus d_{0,} e / v,$$

where $v$ is the number of minimal generators of $\tau(M_t)$. This implies

$$b_e(\tau(M_t)) \geq b_e + \frac{d_{i,}}{v} > b_e,$$

where $b_e(\tau(M_t))$ is the $e$-th $F$-surjective number of $\tau(M_t)$. Thus, $s(\tau(M_t)) > s(M_t)$.

Acknowledgements. The author is deeply grateful to Professor Mitsuyasu Hashimoto for giving him valuable advice and encouragements. He also thanks Akiyoshi Sannai for giving some comments about the dual $F$-signature and thanks Professor Ken-ichi Yoshida for suggesting the comparison between special CM modules and other modules (Section 4 is based on his suggestion).

The author is supported by Grant-in-Aid for JSPS Fellows (No. 26-422).

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