Local Properties of Self-Dual Harmonic 2-forms on a 4-Manifold

Ko Honda

May 27, 1997

Abstract

We will prove a Moser-type theorem for self-dual harmonic 2-forms on closed 4-manifolds, and use it to classify local forms on neighborhoods of singular circles on which the 2-form vanishes. Removing neighborhoods of the circles, we obtain a symplectic manifold with contact boundary - we show that the contact form on each $S^1 \times S^2$, after a slight modification, must be one of two possibilities.

1 Introduction

This paper is a study of generic self-dual (SD) harmonic 2-forms $\omega$ near its zero set. Let $M^4$ be a closed, oriented 4-manifold with $b^+_2 > 0$. Then, for a pair $(\omega, g)$ consisting of a generic metric $g$ and a self-dual harmonic 2-form $\omega$ with respect to $g$, $(\omega, g)$ represents a section of $\Lambda^+_g \to M$, which is transverse to the zero section. Here $\Lambda^+_g$ is the subbundle of $\Lambda^2 T M \to M$ whose fiber over a point $p \in M$ is $\Lambda^+_g(p) = \{ \omega \mid {}^g_\ast \omega = \omega \}$. In particular, the zeros of $\omega$ are disjoint embedded circles. Since $\omega \wedge \omega = \omega \wedge {}^\ast \omega$, $\omega$ is nondegenerate at $p$ if and only if $\omega(p) \neq 0$. That is, $\omega$ is closed and symplectic away from the union of circles $C$, and is identically 0 on $C$.

We also have a relative version of the previous discussion, which is the following theorem (cf. [1]):

**Theorem 1** Let $(\omega_0, g_0)$ and $(\omega_1, g_1)$ be harmonic forms transverse to $\Lambda^+_{g_0}$ and $\Lambda^+_{g_1}$, respectively. If there exists a path $(\omega_t, g_t)$ of harmonic forms $\omega_t$ with respect to $g_t$ such that $\omega_t \neq 0$ for all $t \in [0, 1]$, then there exists a $G_\delta$-set of perturbations $\{(\tilde{\omega}_t, \tilde{g}_t)\}$ of this path, fixing endpoints, such that $\{(\tilde{\omega}_t, \tilde{g}_t)\}$ has regular zeros in $M \times [0, 1]$.

**Note:** The conditions for the theorem are minimal. The space $\{(\omega, g) \mid g \in \text{Met}^k(M), {}^*\omega = \omega, \Delta_g \omega = 0\}$ is diffeomorphic to $\mathbb{R}^{b^+_2} \times \text{Met}^k(M)$, where $\text{Met}^k(M)$ is the space of $C^k$-metrics on...
M, and, as long as $b^+_2 > 1$, we can always connect $(\omega_0, g_0)$ to $(\omega_1, g_1)$ via a cobordism such that $\omega_t \neq 0$ for all $t \in [0, 1]$. In the case $b^+_2 = 1$, as long as $(\omega_0, g_0)$ and $(\omega_1, g_1)$ lie on the same side of the real line, there exists a cobordism.

We briefly outline the contents of the paper. In Section 3, we will discuss a version of Moser’s theorem (Theorem 2) which applies to our singular symplectic forms. In Section 4, we classify local normal forms of the singular symplectic forms near an $S^1$, with an eye towards global results, and in the last section we discuss the induced contact structures on the boundaries of $N(S^1)$. These remarks lay the groundwork for the Floer homology of singular symplectic forms, which we hope to return to in a subsequent paper.

2 Almost complex structures

Observe first that we can define an almost complex structure $J$ on $M - C$.

Proposition 1 If $\omega$ is a self-dual harmonic 2-form which is nondegenerate on a connected set $M - C$, then there exists a unique almost complex structure $J$ compatible with $\omega$ and $g$ on $M - C$, where $g$ is conformally equivalent to $g$.

Proof: Any 2-form $\omega$ can be written, with respect to the metric $g$, as

$$\omega = \lambda_1 e_1 e_2 + \lambda_2 e_3 e_4,$$

with $e_1, ..., e_4$ orthonormal and positively oriented at a point $p \in M - C$.

For $\omega$ to be self-dual, $\lambda_1 = \lambda_2$. Hence,

$$\omega = \lambda (e_1 e_2 + e_3 e_4).$$

This $\lambda$ is well-defined up to sign: Simply consider $\frac{1}{2} \omega \wedge \omega = \lambda^2 e_1 ... e_4 = \lambda^2 dv_g$, with $dv_g$ the volume form for $g$. Since $\lambda^2$ is only dependent on $\omega$ and $g$, we can determine $\lambda$ up to sign. However, taking advantage of $M - C$ being connected, we may fix $\lambda$ on all of $M - S$ so that $\lambda > 0$.

We then set $J : e_1 \mapsto e_2, e_2 \mapsto -e_1, e_3 \mapsto e_4, e_4 \mapsto -e_3$. This definition is equivalent to the following: Let $\tilde{g} = \lambda g$, and define $J$ such that $\tilde{g}(x, y) = \omega(Jx, y)$. Hence we see that if there is a $J$ compatible with $\omega$ and $\tilde{g}$, it must be unique. Thus $J$ is compatible with $\omega$ and $\tilde{g} = \lambda g$ on $M - C$. \qed

Observe that $\omega$ is defined on all of $M$ and is zero on $C$, $\tilde{g}$ can be defined on all of $M$ and is zero on $C$, but is not smooth on $C$, while $J$ is defined only on $M - C$.

Let $\{(\omega_t, g_t)\}$ be a regular cobordism. As in the previous proposition, we can define $\lambda_t$ uniformly over $\bigcup_{t \in [0, 1]} (M \times \{t\} - C_t)$ and get a family $\{ (\omega_t, \tilde{g}_t, J_t) \}$, which is compatible where defined.
3 Moser argument for self-dual harmonic 2-forms

Consider $M^4$ as above. Let $\{\omega_t\}$ be a generic family of self-dual harmonic 2-forms such that

(i) $[\omega_t] \in H^2(M; \mathbb{R})$ is constant.

(ii) The sets $C_t = \{x \in M | \omega_t(x) = 0\}$ are all $S^1$'s; hence via a diffeomorphism, we may assume that $C = C_t$ is a fixed $S^1$.

(iii) $[\omega_t] \in H^2(M, C; \mathbb{R})$ does not vary with $t$.

If we assume that $C$ is contractible, then we are asking for the following:

(iii') Let $\Omega$ be an oriented surface with $\partial \Omega = C$. Then $\int_{\Omega} \omega_t$ does not vary with $t$.

Then we have the following:

Theorem 2 There exists a 1-parameter family of $C^1$-diffeomorphisms of $M$, which is smooth away from $C$, and takes $(M - C, \omega_0) \sim (M - C, \omega_1)$ symplectically.

This generalizes the classical

Theorem 3 (Moser) Let $\{\omega_t\}$ be a family of symplectic forms on a closed manifold $M$. Provided $[\omega_t] \in H^2(M; \mathbb{R})$ is fixed, there is a 1-parameter family of diffeomorphisms $\phi_t$ such that $\phi_t^* \omega_t = \omega_0$.

Proof: (Moser) Let $\eta_t$ be a 1-parameter family of 1-forms such that $\frac{d\omega_t}{dt} = d\eta_t$. Thus, if we define $X_t$ such that $i_X \omega_t = \eta_t$, then $L_{X_t} \omega_t = (i_{X_t} \circ d + d \circ i_{X_t}) \omega = d\eta_t$, which, integrated, gives a 1-parameter family $\phi_t$ such that $\phi_t^* \omega_t = \omega_0$. \hfill $\Box$

Proof: (Theorem 2) The point here is to find a suitable $\eta_t$ such that $\frac{d\omega_t}{dt} = d\eta_t$ and $\eta_t|_C = 0$. Fix some $\tilde{\eta}_t$ such that $\frac{d\omega_t}{dt} = d\tilde{\eta}_t$. We shall find a function $f_t$ on $M$ such that $\tilde{\eta}_t = df_t$ "up to first order" near $C$.

Condition (iii) implies that there exists an $f_t$ on $C$ such that $i^* \tilde{\eta}_t = df_t$, where $i : C \to M$ is the inclusion, i.e. $i^* \tilde{\eta}_t$ is exact. In particular, assuming (iii') we have

\[ \int_C i^* \tilde{\eta}_t = \int_{\partial \Omega} i^* \tilde{\eta}_t = \int_{\Omega} d\tilde{\eta}_t = \int_{\Omega} \frac{d\omega_t}{dt} = 0. \]

In order to extend $f_t$ to a neighborhood $N(C)$ of $C$, first observe that there is only one orientable rank 3 bundle over $S^1$ ($\pi_1(BSO(3)) = 0$ implies $S^1 \to BSO(3)$ is homotopically trivial) and hence $N(C) \simeq C \times D^3$. Choose coordinates $(\theta, x_1, x_2, x_3)$ such that $d\theta, dx_1, dx_2, dx_3$ at $(\theta, 0)$ are orthonormal.
Setting
\[ f_t(\theta, x_1, x_2, x_3) = f_t(\theta, 0) + \sum_i \tilde{\eta}_i(\theta, 0)x_i + \frac{1}{2} \sum_{i,j} \frac{\partial \tilde{\eta}_i}{\partial x_j}(\theta, 0)x_ix_j \]
on \(N(C)\), where \(\tilde{\eta}_t = \tilde{\eta}_\theta \, d\theta + \sum_i \tilde{\eta}_i \, dx_i\), we have
\[ df_t(\theta, x_1, x_2, x_3) = \frac{\partial f_t}{\partial \theta}(\theta, 0) \, d\theta + \sum_i \frac{\partial \tilde{\eta}_i}{\partial \theta}(\theta, 0)x_i \, d\theta \]
\[ + \ \sum_i \tilde{\eta}_i(\theta, 0)dx_i + \frac{1}{2} \sum_{i,j} \frac{\partial \tilde{\eta}_i}{\partial x_j}(\theta, 0)(x_i dx_j + x_j dx_i) \]
up to first order in the \(x_i\)’s. Now observing that
\[ \frac{\partial f}{\partial \theta}(\theta, 0) = \tilde{\eta}_\theta(\theta, 0), \] (1)
\[ d\tilde{\eta}_t(\theta, 0) = 0, \] (2)
and that Equation 3 gives
\[ \frac{\partial \tilde{\eta}_\theta}{\partial x_i}(\theta, 0) = \frac{\partial \tilde{\eta}_i}{\partial \theta}(\theta, 0), \]
\[ \frac{\partial \tilde{\eta}_i}{\partial x_j}(\theta, 0) = \frac{\partial \tilde{\eta}_j}{\partial x_i}(\theta, 0), \]
we obtain
\[ df_t(\theta, x) = \left( \tilde{\eta}_\theta(\theta, 0) + \sum_i \frac{\partial \tilde{\eta}_i}{\partial x_i}(\theta, 0)x_i \right) d\theta \]
\[ + \sum_i \left( \tilde{\eta}_i(\theta, 0) + \sum_j \frac{\partial \tilde{\eta}_j}{\partial x_j}(\theta, 0)x_j \right) dx_i \]
\[ = \tilde{\eta}_\theta(\theta, x) d\theta + \sum_i \tilde{\eta}_i(\theta, x) dx_i \]
up to first order in \(x\).

Damping \(f_t\) out to 0 outside \(N(C)\), we arrive at \(\eta_t = \tilde{\eta}_t - df_t\). Finally, we obtain the vector field \(X_t\) such that \(i_{X_t} \omega_t = \eta_t\). \(X_t\) will then give rise to a 1-parameter family of symplectomorphisms, away from \(C\), once we establish that \(X_t \to 0\) rapidly enough as \(p \to C\) \((p \in M)\).

On \(N(C)\),
\[ \omega_t = L_1(\theta, x)(d\theta dx_1 + dx_2 dx_3) \]
\[ + L_2(\theta, x)(d\theta dx_2 + dx_3 dx_1) \]
\[ + L_3(\theta, x)(d\theta dx_3 + dx_1 dx_2) \]
\[ + Q, \] (3)
where $L_i(\theta, x) = \sum_j L_{ij}(\theta) x_j$ and $Q$ consists of forms in $d\theta$ and $dx_i$, whose coefficients are quadratic or higher in the $x_i$. In terms of matrices, $\omega_t$ corresponds to

$$A = \begin{pmatrix} 0 & L_1 & L_2 & L_3 \\ -L_1 & 0 & L_3 & -L_2 \\ -L_2 & -L_3 & 0 & L_1 \\ -L_3 & L_2 & -L_1 & 0 \end{pmatrix} + \tilde{Q},$$

where $\tilde{Q}$ has quadratic or higher terms in the $x_i$ and the matrix is with respect to basis $\{d\theta, dx_1, dx_2, dx_3\}$. $i_\omega \omega_t = \eta_t$ then becomes

$$(a_\theta \ a_1 \ a_2 \ a_3) A = (\eta_\theta \ \eta_1 \ \eta_2 \ \eta_3)$$

with $X_t = a_\theta d\theta + \sum_i a_i dx_i$. Thus,

$$(a_\theta \ a_1 \ a_2 \ a_3) = (\eta_\theta \ \eta_1 \ \eta_2 \ \eta_3) A^{-1}$$

$$= \frac{(\eta_\theta \ \eta_1 \ \eta_2 \ \eta_3)}{L_1^2 + L_2^2 + L_3^2} \begin{pmatrix} 0 & -L_1 & -L_2 & -L_3 \\ L_1 & 0 & L_3 & L_2 \\ L_2 & L_3 & 0 & -L_1 \\ L_3 & -L_2 & L_1 & 0 \end{pmatrix}$$

up to first order in $x$. This means that $|X_t| < k|x|$ near $C$; hence, as $x \to 0$, $|\phi_1(\theta, x) - \phi_0(\theta, x)| \to 0$, where $\phi_t$ is the flow such that $\frac{d\phi}{dt} = X_t$. This concludes our proof. \qed

4 Local normal forms

On a neighborhood $N(C) = C \times D^3$ of $C$, $\omega$ can be written as in Equation 3. If $\omega$ is generic, then it is transverse to the zero section of $\Lambda^+$, and $(L_{ij}(\theta))$ is nondegenerate for all $\theta$.

Lemma 1 $(L_{ij}(\theta))$ is symmetric and traceless.

Proof: By comparing 0th order terms in the $x_i$, $d\omega = 0$ implies

$$\frac{\partial L_1}{\partial x_1} + \frac{\partial L_2}{\partial x_2} + \frac{\partial L_3}{\partial x_3} = 0,$$

$$\frac{\partial L_2}{\partial x_3} - \frac{\partial L_3}{\partial x_2} = 0, \quad \frac{\partial L_3}{\partial x_1} - \frac{\partial L_1}{\partial x_3} = 0, \quad \frac{\partial L_1}{\partial x_2} - \frac{\partial L_2}{\partial x_1} = 0.$$

$(L_{ij}(\theta))$ thus has a basis $\{v_1(\theta), v_2(\theta), v_3(\theta)\}$ of eigenvectors for each $\theta$ (though the $v_i$ are not necessarily continuous in $\theta$). Since $(L_{ij}(\theta))$ is traceless, either two of the eigenvalues are
positive and the remaining is negative for all $\theta$, or vice versa. Hence, $(L_{ij}(\theta))$ gives rise to a splitting of $\mathbb{R}^3 \times S^1 \rightarrow S^1$ into a real line bundle over $S^1$ and a rank 2 vector bundle over $S^1$. Such splittings are classified by homotopy classes of $S^1$ into $\mathbb{RP}^2$, and $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2$. Hence,

**Proposition 2** There exist two splittings of $\mathbb{R}^3 \times S^1 \rightarrow S^1$, the oriented one and the unoriented one.

What is rather remarkable is that we have the following:

**Theorem 4** There exist SD harmonic 2-forms for the product metric on $S^1 \times D^3$ for both types of splittings:

(A) $\omega_A = x_1(d\theta dx_1 + dx_2 dx_3) + x_2(d\theta dx_2 + dx_3 dx_1) - 2x_3(d\theta dx_3 + dx_1 dx_2) = *_3 \mu + d\theta \wedge \mu,$

where $\mu = d(\frac{1}{2}(x_1^2 + x_2^2) - x_3^2)$, and $*_3$ is the $*$-operator for the flat metric on $D^3$. Here, $(L_{ij}(\theta)) = \text{diag}(1, 1, -2)$, with fixed positive and negative eigenspaces. Note that $\omega_A$ is $S^1$-invariant.

(B) $\omega_B = (x_1 \cos \theta + x_2 \sin \theta)e^{x_3}(d\theta dx_1 + dx_2 dx_3) + (x_1 \sin \theta - x_2 \cos \theta)e^{x_3}(d\theta dx_2 + dx_3 dx_1) + R(-x_1(d\theta dx_1 + dx_2 dx_3) + x_3(d\theta dx_3 + dx_1 dx_2)),$

with $0 < R < 1$. Here,

$$(L_{ij}(\theta)) = \begin{pmatrix}
\cos \theta - R & \sin \theta & 0 \\
\sin \theta & -\cos \theta & 0 \\
0 & 0 & R
\end{pmatrix}.$$ 

Observe that there are two positive eigenvectors, one of which is $(0, 0, 1)$. It is a direct computation to show that $d\omega_B = 0$ and that the splitting of $\mathbb{R}^3$ given by $(L_{ij}(\theta))$ is the unoriented one.

We can alternatively construct an $\omega_B$, which is not the same as the one above, but arises more naturally. Starting with

$$\omega = x_1(d\theta dx_1 + dx_2 dx_3) + x_2(d\theta dx_2 + dx_3 dx_1) - 2x_3(d\theta dx_3 + dx_1 dx_2)$$

6
on $[0, 2\pi] \times D^3$, we glue $\phi : \{2\pi\} \times D^3 \to \{0\} \times D^3$ via

$$
\begin{align*}
\theta & \mapsto \theta - 2\pi \\
x_1 & \mapsto x_1 \\
x_2 & \mapsto -x_2 \\
x_3 & \mapsto -x_3.
\end{align*}
$$

$\phi^* \omega = \omega$, and obtain an $\omega_B$ on $S^1 \times S^2$ corresponding to the unoriented splitting.

**Theorem 5** Given an SD harmonic 2-form $\omega$, there exists a 1-parameter family of perturbations $\{\omega_t\}$, local near $C$, such that $\omega_0 = \omega$, $\omega_1|_{N(C)}$ is one of the two local forms as in Theorem 4 (up to $\pm \omega$), and $[\omega_t] \in H^2(M; \mathbb{R})$ is independent of $t$.

**Proof:** Given $\omega$, consider the neighborhood $N(C) = C \times D^3$ of one of the circles. Assume we are in case (A). Case (B) is identical. After an orthonormal change of frame, we may write

$$
\omega = (L_{11}(\theta)x_1 + L_{12}(\theta)x_2)(d\theta dx_1 + dx_2 dx_3) + (L_{21}(\theta)x_1 + L_{22}(\theta)x_2)(d\theta dx_2 + dx_3 dx_1) + \lambda_3(\theta)x_3(d\theta dx_3 + dx_1 dx_2) + Q,
$$

with, say, $(L_{ij}(\theta))_{1 \leq i,j \leq 2}$ positive definite and $\lambda_3(\theta) < 0$. Here, the $L_{ij}(\theta)$ and $\lambda_3(\theta)$ are differentiable in $\theta$.

Now, take a 1-parameter family $\omega_t = (1 - t)\omega + t\omega_A$ on $N(C)$. After shrinking $N(C)$ if necessary, $\omega_t$ is symplectic on $N(C)$ away from $C$. Using a local version of Moser’s theorem (see the proof of Theorem 2), we see that there exists a $C^1$-diffeomorphism

$$
\phi : (N_0(C), \omega) \simto (N_1(C), \omega_A),
$$

where $N_0(C)$, $N_1(C)$ are small neighborhoods of $C$, $\phi = id$ on $C$, and $\phi$ is a smooth map away from $C$. Hence $\phi$ allows us to remove $(N_0(C), \omega)$ and graft on $(N_1(C), \omega_A)$. We can perform this operation through a 1-parameter family $\omega_t$, and hence there exists a global family $\omega_t$ on $M$ with $\omega_0 = \omega$ and $\omega_1|_{N(C)} = \omega_A$. Moreover, the perturbation can be performed in an arbitrarily small neighborhood of $C$ and without altering the cohomology class. \(\square\)

In essence, Theorem 5 tells us that, in studying the singular circles of $\omega$, we may assume that the zeros are either (A) or (B).
5 Contact structures on the boundaries

In this section we investigate the boundary properties of \( \omega_A \) and \( \omega_B \). More precisely, we have

**Theorem 6** There exist contact forms \( \lambda_A \) and \( \lambda_B \) on \( \partial N(C) = S^1 \times S^2 \) such that \( d\lambda_A = \omega_A \) and \( d\lambda_B = \omega_B \) on \( S^1 \times S^2 \).

**Proof:** (A) For example, consider the following \( S^1 \)-invariant contact 1-form

\[
\lambda = -\frac{1}{2}(x_1^2 + x_2^2 - 2x_3^2)d\theta + x_2x_3dx_1 - x_1x_3dx_2
\]
on \( N(C) \). We then compute that \( d\lambda = \omega \) on \( N(C) \) and

\[
\sum_i x_i dx_i \wedge \lambda \wedge d\lambda = \left( \frac{1}{2}(x_1^2 + x_2^2)(x_1^2 + x_2^2 + 2x_3^2) + 2x_3^3 \right) d\theta dx_1 dx_2 dx_3.
\]

Since \( S^1 \times S^2 = \{ \sum_i x_i^2 = 1 \} \) is a leaf of \( \sum_i x_i dx_i \), \( i^*_{S^1 \times S^2}(\lambda \wedge d\lambda) \neq 0 \) if and only if \( \lambda \wedge d\lambda \wedge \sum_i x_i dx_i \neq 0 \) on \( S^1 \times S^2 \). Here \( i^*_{S^1 \times S^2} \) is the inclusion \( S^1 \times S^2 \to S^1 \times D^3 \). Noting that \( \lambda \wedge d\lambda \wedge \sum_i x_i dx_i = 0 \) if and only if \( x_1 = x_2 = x_3 = 0 \), we have that \( i^*_{S^1 \times S^2}(\lambda) \) is a contact 1-form on \( S^1 \times S^2 \) with \( di^*_{S^1 \times S^2}(\lambda) = i^*_{S^1 \times S^2} \). Thus, \( (M - N(S^1), \omega) \) is a symplectic manifold with contact boundary.

(B) Consider the 1-form

\[
\lambda = -\frac{1}{2}(x_1^2 + x_2^2 - 2x_3^2)d\theta + x_2x_3dx_1 - x_1x_3dx_2
\]
on \([0, 2\pi] \times D^3\). \( d\lambda = \omega \), and \( \phi^*\lambda = \lambda \), where \( \phi \) was the glueing map of Theorem 4, so we glue together a contact 1-form \( \lambda_B \) such that \( d\lambda_B = \omega_B \). The rest is the same as (A). \( \square \)

Let us now describe the orbits of the Reeb vector fields.

(A) \( \omega_A \) is compatible with a metric \( \tilde{g} = \lambda g \), where \( g \) is the standard product metric on \( S^1 \times D^3 \). We can then write the compatible \( J \) satisfying \( \tilde{g}(x, y) = \omega(Jx, y) \) as \( J = \frac{1}{X}A \), where

\[
A = \begin{pmatrix}
0 & x_1 & x_2 & -2x_3 \\
-x_1 & 0 & -2x_3 & -x_2 \\
-x_2 & 2x_3 & 0 & x_1 \\
2x_3 & x_2 & -x_1 & 0
\end{pmatrix}
\]

represents \( \omega \) with respect to \( \{ \theta, x_1, x_2, x_3 \} \). Now, the Reeb vector field \( X \) for \( i^*_{S^1 \times S^2} \omega \) is given, up to multiple, by

\[
J(\sum_i x_i \frac{\partial}{\partial x_i}) = \frac{1}{\sqrt{x_1^2 + x_2^2 + 4x_3^2}}(x_1^2 + x_2^2 - 2x_3^2, -3x_2x_3, 3x_1x_3, 0)^T.
\]
Finally, $\lambda_A(X) = 1$ implies that

$$X = \frac{1}{f} \left[ \left( x_1^2 + x_2^2 - 2x_3^2 \right) \frac{\partial}{\partial \theta} - 3x_2x_3 \frac{\partial}{\partial x_1} + 3x_1x_3 \frac{\partial}{\partial x_2} \right],$$

with

$$f = -\frac{1}{2} \left[ (x_1^2 + x_2^2)(x_1^2 + x_2^2 + 2x_3^2) + 2x_3^4 \right].$$

Solving for the orbits, $x_1^2 + x_2^2$ and $x_3^2$ are fixed for each orbit, and hence,

$$x_1 = \sqrt{1 - r^2} \cos R_1(r)t,$$
$$x_2 = \sqrt{1 - r^2} \sin R_1(r)t,$$
$$x_3 = r,$$
$$\theta = R_2(r)t + c,$$

where $r$ is a constant, and $R_1$ and $R_2$ are functions of $r$.

In particular, the noteworthy closed orbits are $\{(0,0,1)\} \times S^1$, $\{(0,0,-1)\} \times S^1$, and $\{(x_1,x_2,0)\} \times S^1$, with $x_1^2 + x_2^2 = 1$ and $x_1$, $x_2$ fixed. These correspond to the stable and unstable gradient directions in the Morse theory of $\frac{1}{2}(x_1^2 + x_2^2 - 2x_3^2)$ near $(0,0,0)$. Moreover, the orbit $\{(0,0,1)\} \times S^1$ is nondegenerate, and so is the family $\{(x_1,x_2,0)\} \times S^1$. There are other closed orbits, but these do not seem to have any Morse-theoretic significance.

(B) We apply the previous considerations and work on $[0, 2\pi] \times S^1 / \sim$. There is one orbit $\{(0,0,\pm 1)\} \times S^1$, which is a double of the orbits for (A). Since $\phi$ identifies $(2\pi, (x_1,x_2,0)) \sim (0, (x_1,-x_2,0))$, we also have the doubled closed orbits $\{(x_1,\pm x_2,0)\} \times S^1$, with $x_2 \neq 0$, and the single closed orbits $\{(1,0,0)\} \times S^1$, $\{(-1,0,0)\} \times S^1$.

**Remark:** There is an example of a singularity of type (B) bounding a disk, which can be made to vanish.

**References**

[1] K. Honda, *Harmonic forms for generic metrics*, preprint.

[2] C. H. Taubes, *The Seiberg-Witten and Gromov invariants*, Math. Res. Letters 2 (1995), pp. 221-238.

[3] A. Weinstein, *Lectures on symplectic manifolds*, CBMS Regional Conference Series 29, 1977.

Mathematics Department, Princeton University, Princeton, NJ 08544

honda@math.princeton.edu