Coherent States for
Discrete Spectrum Dynamics

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Abstract

Coherent states for general systems with discrete spectrum, such as the bound states of the hydrogen atom, are discussed. The states in question satisfy: (1) continuity of labeling, (2) resolution of unity, (3) temporal stability, and (4) an action identity. This set of reasonable physical requirements uniquely specify coherent states for the (bound state portion of the) hydrogen atom.

1 Introduction

Coherent states for the harmonic oscillator have long been known and their properties have frequently been taken as models for defining coherent states for other systems. Interest has long existed in coherent states for the hydrogen atom, at least for the bound state portion, and there have been a number of attempts in the past to define such states. We shall approach this problem by adopting four postulates. The first two are standard when dealing with coherent states, while the third and fourth are rather more physical in nature dealing with a specific Hamiltonian operator.

The first two postulates are given by:

(1) Continuity of labeling
(2) Resolution of unity

Indeed, these postulates are the minimal requirements generally accepted at present to characterize coherent states. The remaining two postulates refer to a specific Hamiltonian \( \mathcal{H} \geq 0 \), which, in units where \( \hbar = 1 \), we assume is non-degenerate and fulfills \( \mathcal{H} \ket{n} = E_n \ket{n} = \omega e_n \ket{n} \). For convenience we confine our attention to Hamiltonians with an infinite number of bound states, \( 0 \leq n < \infty \).
We set $\lim_{n \to \infty} E_n = E^*$; cases where $E^* < \infty$ and where $E^* = \infty$ are both of interest. Note that the $e_n$ are dimensionless numbers which satisfy the property that $0 = e_0 < e_1 < e_2 < \cdots$.

The appropriate coherent states are defined by the expression

$$|J, \gamma\rangle \equiv M(J)^{-1} \sum_{n=0}^{\infty} (J^{n/2} e^{-i e_n \gamma / \sqrt{\rho_n}} |n\rangle),$$

for suitable $J \geq 0$ and $-\infty < \gamma < \infty$. The coefficients $\rho_n$ are chosen as the moments of a distribution $\rho(J) \geq 0$ in the manner

$$\rho_n = \int_0^{J^*} J^n \rho(J) dJ, \quad \rho_0 = 1.$$

Here $M$ denotes a normalization constant chosen so that $\langle J, \gamma|J, \gamma\rangle = 1$, namely $M(J)^2 = \sum_{n=0}^{\infty} J^n / \rho_n$, and we denote by $J^*$ the radius of convergence for this series. Cases where $J^* < \infty$ and $J^* = \infty$ are both of interest.

For the third postulate we choose

(3) Temporal stability

which means that the time evolution of a coherent state remains a coherent state for all time. In particular, for the case at hand,

$$e^{-i H t} |J, \gamma\rangle = \sum_{n=0}^{\infty} (J^{n/2} e^{-i e_n \gamma / \sqrt{\rho_n}} e^{-i e_n \omega t} |n\rangle) = |J, \gamma + \omega t\rangle.$$

We observe that the expressions $J(t) = J$ and $\gamma(t) = \gamma + \omega t$ are reminiscent of the time dependence of action-angle variables in classical mechanics. When expressed in canonical action-angle variables, the classical action functional becomes

$$I = \int [J(t) \dot{\gamma}(t) - \omega J(t)] dt.$$

Recall [2, 4] that the quantum action functional restricted just to coherent states, namely

$$I = \int [i \langle J(t), \gamma(t) | (d/dt)|J(t), \gamma(t)\rangle - \langle J(t), \gamma(t) | \mathcal{H} |J(t), \gamma(t)\rangle] dt,$$

also expresses the classical action. Thus our fourth postulate is the

(4) Action identity

which ensures that $J$ and $\gamma$ are action-angle variables, and requires that

$$\langle J, \gamma | \mathcal{H} |J, \gamma\rangle = \omega J = \omega (\Sigma_n e_n J^n / \rho_n) / (\Sigma_m J^m / \rho_m).$$

To satisfy the action identity requires that

$$\rho_n = e_n e_{n-1} e_{n-2} \cdots e_1.$$
For the harmonic oscillator, \( e_n = n, \rho_n = n! \), \( E^* = J^* = \infty, \rho(J) = e^{-J} \), \( M(J)^2 = e^{J} \), and therefore
\[
|J, \gamma\rangle = e^{-J/2} \sum_{n=0}^{\infty} \left( \frac{J^{n/2} e^{-i n \gamma}}{\sqrt{n!}} \right) |n\rangle \equiv |z\rangle ,
\]
which is just a standard canonical coherent state with \( z \equiv J^{1/2} e^{-i \gamma} \). In this case it suffices to choose \(-\pi < \gamma \leq \pi\).

We conclude with a brief discussion of a one-dimensional hydrogen-atom analog with spectrum \( E_n = E_0 - \omega/(n+1)^2 \), i.e., \( e_n = 1 - 1/(n+1)^2 \). It follows in this case that
\[
|J, \gamma\rangle = M(J)^{-1} \sum_{n=0}^{\infty} \left( \frac{\sqrt{(2n+2)}/(n+2)} {J^{n/2} e^{-i \gamma (1-1/(n+1)^2)}} \right) |n\rangle .
\]
Here
\[
M(J)^2 = [J(1-J)]^{-1} + J^{-2} \ln(1-J) , \quad 0 \leq J < J^* = 1.
\]
For the system at hand these states satisfy the four cited postulates presented in this paper.

In addition, for this example, it is noteworthy that
\[
\langle J, \gamma | \mathcal{H}^2 | J, \gamma \rangle = \omega^2 J^2 + \omega^2 v(J) ,
\]
where \( 0 < v(J) < 6(1-J) \). Thus as \( J \to 1 \), \( v(J) \to 0 \), indicative of a very narrow distribution.

An earlier letter \[3\] discussed the (bound state portion of the) hydrogen-atom coherent states along the lines of this note without, however, introducing the action identity. (Some other analyses of hydrogen-atom coherent states are referenced in that letter as well).

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References

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