On new parameters concerning a generalization of the parallelogram law in Banach spaces

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Abstract

We shall introduce a new geometric constant $L'_Y(\lambda, X)$ based on a generalization of the parallelogram law. We first investigate some basic properties of this new coefficient. Next, it is shown that, for a Banach space, $L'_Y(\lambda, X)$ becomes 1 for some $\lambda_0 \in (0, 1)$ if and only if the norm is induced by an inner product. Moreover, some relations between other well-known geometric constants are studied. Finally, a sufficient condition which implies normal structure is presented.

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1 Introduction and preliminaries

It is well known that geometric constant plays an important role in the geometric theory of Banach spaces. It is effective to study the geometric properties of Banach spaces by means of some geometric constants, which is a powerful tool to help us quantitatively study spaces, see [1, 2, 7, 8, 11, 19, 22]. In addition, there are many well-known constants can show that how far the normed space departs from being an inner product space. It’s worth noting, however, calculation of the constant for some concrete spaces can be complex and challenging. But is worth paying for because it lets us have a better understanding of the space being studied, several results can be consulted in [13, 18].

We now give some definitions related to geometric constants. Let $X$ be a real Banach space with $\dim X \geq 2$ and denote by $S_X$ and $B_X$ the unit sphere and the unit ball, respectively.

The first norm characterization of inner product spaces was given by Fréchet [12] in 1935. He proved that a normed space $(X, \| \cdot \|)$ is an inner product space if and only if

$$
\|x + y\|^2 + \|y + z\|^2 + \|x + z\|^2 = \|x + y + z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2
$$

for all $x, y, z \in X$.

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The von Neumann-Jordan constant $C_{NJ}(X)$ for a Banach space $X$, is defined by [6]

$$C_{NJ}(X) = \sup \left\{ \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X, (x, y) \neq (0, 0) \right\}.$$ 

Moreover, the various properties of the constant $C_{NJ}(X)$ are given in [17, 21]:

(i) $1 \leq C_{NJ}(X) \leq 2$.

(ii) $X$ is a Hilbert space iff $C_{NJ}(X) = 1$.

(iii) $X$ is uniformly non-square iff $C_{NJ}(X) < 2$.

Gao first introduced and studied the constant [15]

$$E(X) = \sup\{\|x + y\|^2 + \|x - y\|^2 : x, y \in S_X\}.$$ 

Recall that the Banach space $X$ is called uniformly non-square [16] if there exists a $\delta \in (0, 1)$ such that for any $x_1, x_2 \in S_X$ either $\frac{\|x_1 + x_2\|}{2} \leq 1 - \delta$ or $\frac{\|x_1 - x_2\|}{2} \leq 1 - \delta$.

Takahashi [23] and Alonso et al. [3] studied the modified von Neumann–Jordan constant $C'_{NJ}(X)$, as follows:

$$C'_{NJ}(X) := \sup\left\{ \frac{\|x + y\|^2 + \|x - y\|^2}{4} : x, y \in S_X \right\}. $$

It easy to see, that $X$ is uniformly non-square if and only if $C'_{NJ}(X) < 2$.

The Clarkson modulus of convexity of a Banach space $X$ is the function $\delta_X : [0, 2] \to [0, 1]$ defined by [5]:

$$\delta_X(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in S_X, \|x - y\| \geq \epsilon \right\}.$$ 

M. M. Day gave the following well-known characterization, referred to as “the rhombus law”.

Lemma 1.1. [9] Let $(X, \| \cdot \|)$ be a real normed linear space. Then $\| \cdot \|$ derives from an inner product if and only if

$$\|x + y\|^2 + \|x - y\|^2 \sim 4$$

for all $x, y \in S_X$, where $\sim$ stands either for $\leq$ or $\geq$.

Drawing on methods in the literature [9], and using the concept of "Loewner ellipse", M. M. Day proved the following more general result [10] on the basis of what has already been proved: for each pair $x, y$ of points of $X$ with $\|x\| = \|y\| = 1$ there exists a number $\lambda$, depending on $x$ and $y$, such that $0 < \lambda < 1$ and

$$\|\lambda x + (1 - \lambda)y\|^2 + \lambda(1 - \lambda)\|x - y\|^2 \sim 1,$$

where $\sim$ stands either for $\leq$ or $\geq$, then $(X, \| \cdot \|)$ is an inner product space. M. M. Day’s result is significant because instead of relying on the parallelogram law, he describes the inner product space in terms of parameters $\lambda$ combined with points on the unit sphere.

The following proposition plays a major role in our article. For the reader’s convenience, we present the proof as follows:
Proposition 1.2. A normed space \((X, \| \cdot \|)\) is an inner product space if and only if for each \(x, y \in S_X\) there exists \(\lambda\) such that \(\lambda \in (0, 1)\) and

\[
\|\lambda x + (1 - \lambda) y\|^2 + \lambda (1 - \lambda) \|x - y\|^2 = 1.
\]

Proof. If \(X\) is an inner product space. Observe that for \(\lambda \in (0, 1)\) and \(x, y \in S_X\) we can obtain

\[
\|\lambda x + (1 - \lambda) y\|^2 + \lambda (1 - \lambda) \|x - y\|^2 \\
= \lambda^2 \|x\|^2 + 2\lambda (1 - \lambda) \langle x, y \rangle + (1 - \lambda)^2 \|y\|^2 \\
+ \lambda (1 - \lambda) (\|x\|^2 - 2\langle x, y \rangle + \|y\|^2) \\
= 1.
\]

For the second part of the proof, using the result in [10, p. 95, Theorem 1] that we can get the result.

Proposition 1.3. A normed space \((X, \| \cdot \|)\) is an inner product space if and only if

\[
\|\lambda x + (1 - \lambda) y\|^2 + \lambda (1 - \lambda) \|x - y\|^2 = 1
\]

for \(x, y \in S_X\) and all \(\lambda \in [0, 1]\).

Proof. The proof is a direct consequence of Proposition 1.2, so we omit the proof.

The paper is organized as follows: we introduced the constant \(L'_Y(\lambda, X)\) in the next section and its connection between Hilbert spaces is investigated. Furthermore, the relationship between the constant \(L'_Y(\lambda, X)\) and other well-known constants is emphasized in terms of nontrivial inequalities.

In the last section, we establish a new necessary condition for Banach spaces have normal structure in the form of \(L'_Y(\lambda, X)\).

2 The constant \(L'_Y(\lambda, X)\)

Motivated by the general characterization of inner product spaces by M. M. Day, we introduce a new geometric constant \(L'_Y(\lambda, X)\) in a Banach space \(X\).

From now on, we will consider only Banach spaces of dimension at least 2. Now we shall introduce the constant: for \(\lambda \in [0, 1]\)

\[
L'_Y(\lambda, X) = \sup \{ \|\lambda x + (1 - \lambda) y\|^2 + \lambda (1 - \lambda) \|x - y\|^2 : x, y \in S_X \}.
\]

Noted that

\[
L'_Y\left(\frac{1}{2}, X\right) = \frac{1}{4} E(X) = C'_{NJ}(X).
\]

Proposition 2.1. Suppose that \(X\) is a Banach space. Then

\[
1 \leq L'_Y(\lambda, X) \leq -4\lambda^2 + 4\lambda + 1.
\]
Proof. Let $y = x$, we have

$$\|\lambda x + (1 - \lambda)y\|^2 + \lambda(1 - \lambda)\|x - y\|^2 = 1.$$ 

The latter assertion can be derived from the following estimate

$$\|\lambda x + (1 - \lambda)y\|^2 + \lambda(1 - \lambda)\|x - y\|^2 \leq (\|\lambda x\| + \|\lambda y\|)^2 + 2\lambda(1 - \lambda)(\|x\|^2 + \|y\|^2) = -4\lambda^2 + 4\lambda + 1.$$ 

The next proposition will play an important role in the subsequent proof.

**Proposition 2.2.** Suppose that $X$ is a Banach space. Then

$$L_Y'(\lambda, X) = L_Y'(1 - \lambda, X).$$

**Proof.** By the definition of $L_Y'(\lambda, X)$, we see that

$$L_Y'(1 - \lambda, X) = \sup \{\| (1 - \lambda)x + \lambda y\|^2 + \lambda(1 - \lambda)\|x - y\|^2 : x, y \in S_X \},$$

as desired.

**Proposition 2.3.** Let $X$ be a Banach space. Then $L_Y'(\lambda, X)$ is continuous on $[0, 1]$.

**Proof.** We distinguish two cases.

**Case 1.** $\forall \lambda \in [\frac{1}{2}, 1]$.

Given any $\epsilon > 0$, there is a $\delta_0 = \frac{\epsilon}{4} > 0$. When $0 < \delta < \delta_0$, we conclude that the following estimate

$$\| (\lambda + \delta)x + (1 - (\lambda + \delta))y\|^2 + (\lambda + \delta)(1 - (\lambda + \delta))\|x - y\|^2$$

$$- \|\lambda x + (1 - \lambda)y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$$

$$= \| (\lambda + \delta)x + (1 - (\lambda + \delta))y\|^2 - \|\lambda x + (1 - \lambda)y\|^2$$

$$+ \delta(1 - 2\lambda - \delta)\|x - y\|^2$$

$$\leq [\| (\lambda + \delta)x + (1 - (\lambda + \delta))y\| - \|\lambda x + (1 - \lambda)y\|]$$

$$[\| (\lambda + \delta)x + (1 - (\lambda + \delta))y\| + \|\lambda x + (1 - \lambda)y\|]$$

$$\leq \| (\lambda + \delta)x + (1 - (\lambda + \delta))y - (\lambda x + (1 - \lambda)y)\|$$

$$[ (\lambda + \delta)\|x\| + (1 - (\lambda + \delta))\|y\| + \lambda\|x\| + (1 - \lambda)\|y\|]$$

$$= 2\delta\|x - y\| \leq 4\delta < \epsilon.$$
Then we obtain
\[
\left| L'_Y(\lambda + \delta, X) - L'_Y(\lambda, X) \right| \\
= \sup \left\{ \|\lambda x + (1 - \lambda y)\|^2 + \lambda(1 - \lambda)\|x - y\|^2 : x, y \in S_X \right\} \\
- \sup \left\{ \|\lambda x + (1 - \lambda y)\|^2 + \lambda(1 - \lambda)\|x - y\|^2 : x, y \in S_X \right\}
\]
\[
\leq \sup \left\{ \|\lambda x + (1 - \lambda y)\|^2 + \lambda(1 - \lambda)\|x - y\|^2 : x, y \in S_X \right\} < \epsilon,
\]
which implies that
\[
\left| L'_Y(\lambda + \delta, X) - L'_Y(\lambda, X) \right| < \epsilon.
\]

Case 2. \( \forall \lambda \in [0, \frac{1}{2}] \).

Let \( \bar{\lambda} = 1 - \lambda \in (\frac{1}{2}, 1] \). It is clear that \( L'_Y(\bar{\lambda}, X) \) is continuous on \( (\frac{1}{2}, 1] \). This means that for every \( \epsilon > 0 \) there exists a \( \delta_0 > 0 \) such that
\[
\left| L'_Y(\bar{\lambda}_2, X) - L'_Y(\bar{\lambda}_1, X) \right| < \epsilon \text{ if } |\bar{\lambda}_2 - \bar{\lambda}_1| < \delta_0, \bar{\lambda}_1, \bar{\lambda}_2 \in (\frac{1}{2}, 1].
\]

Note also that
\[
L'_Y(\lambda, X) = L'_Y(1 - \lambda, X) = L'_Y(\bar{\lambda}, X),
\]
we get
\[
\left| L'_Y(\lambda_2, X) - L'_Y(\lambda_1, X) \right| = \left| L'_Y(\bar{\lambda}_2, X) - L'_Y(\bar{\lambda}_1, X) \right| < \epsilon.
\]
This completes the proof.

\[\text{Theorem 2.4. If } X \text{ is a Hilbert space, then } L'_Y(\lambda, X) = 1.\]

\[\textbf{Proof.} \text{ Assume } X \text{ is a Hilbert space, using Proposition 1.3 we have} \]
\[
\|\lambda x + (1 - \lambda y)\|^2 + \lambda(1 - \lambda)\|x - y\|^2 = 1,
\]
for \( x, y \in S_X \) and all \( \lambda \in [0, 1] \), as desired.

\[\text{Remark 2.5. For any Banach space, we have} \]
\[
L'_Y(0, X) = L'_Y(1, X) = 1
\]
and hence \( X \) is a Hilbert space cannot be derived from \( L'_Y(\lambda, X) = 1 \). But, we can get the following proposition if we exclude \( \lambda = 0, 1 \).

\[\text{Proposition 2.6. Let } X \text{ be a Banach space. Then the following conditions are equivalent:} \]
(i) \( L'_Y(\lambda, X) \) is a convex function.
(ii) \( L'_Y(\lambda, X) = 1 \) for all \( \lambda \in (0, 1) \).
(iii) \( L'_Y(\lambda, X) = 1 \) for some \( \lambda_0 \in (0, 1) \).
(iv) \( X \) is an inner product space.
Proof. (i)⇒(ii). From the fact \( L_Y'(\lambda, X) \) is convex function, we obtain
\[
1 \leq L_Y'(\lambda, X) \\
= L_Y'((1 - \lambda) \cdot 0 + \lambda \cdot 1, X) \\
\leq (1 - \lambda)L_Y'(0, X) + \lambda L_Y'(1, X) \\
= 1
\]
for all \( \lambda \in [0, 1] \), and hence \( L_Y'(\lambda, X) = 1 \) for all \( \lambda \in [0, 1] \).

(ii)⇒(iii). Obvious.

(iii)⇒(iv). Applying Proposition 1.2, as desired.

(iv)⇒(i). If \( X \) is an inner product space, using Theorem 2.4 we have \( L_Y'(\lambda, X) = 1 \) for all \( \lambda \in [0, 1] \).

This completes the proof. \( \blacksquare \)

Example 2.7. Consider \( X \) be \( \mathbb{R}^2 \) with the norm defined by
\[
\|(x_1, x_2)\| = (|x_1|^3 + |x_2|^3)^{\frac{1}{3}},
\]
then \( L_Y'(\lambda, X) \) be not a convex function. Indeed, take
\[
x_0 = (1, 0), \ y_0 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).
\]
It is clear that \( \|x_0\|_3 = \|y_0\|_3 = 1, \) and
\[
L_Y'(\frac{1}{2}, X) \geq \left\|\frac{x_0 + y_0}{2}\right\|^2 + \frac{1}{4}\|x_0 - y_0\|^2.
\]
By elementary calculus, we can easily get
\[
\left\|\frac{x_0 + y_0}{2}\right\|^2 + \frac{1}{4}\|x_0 - y_0\|^2 > 1
\]
and hence
\[
L_Y'(\frac{1}{2}, X) > 1 = \frac{1}{2} L_Y'(0, X) + \frac{1}{2} L_Y'(1, X).
\]

Theorem 2.8. Let \( X \) be a Banach space, \( \epsilon \in [0, 2] \). Then
\[
\frac{1}{4} \epsilon^2 + |2\lambda - 1|(1 - \delta_X(\epsilon))\epsilon + (1 - \delta_X(\epsilon))^2 \leq L_Y'(\lambda, X) \\
\leq (2\lambda + |1 - 2\lambda| - 2\lambda \delta_X(\epsilon))^2 + \lambda(1 - \lambda)\epsilon^2.
\]

Proof. Let \( \delta_X(\epsilon) = \alpha \). We can deduce that \( \exists \ x_n, y_n \in S_X, \|x_n - y_n\| = \epsilon, \) such that
\[
\lim_{n \to \infty} \|x_n + y_n\| = 2(1 - \alpha).
\]
Furthermore,
\[
\|\lambda x_n + (1 - \lambda)y_n\| = \left\|\frac{1}{2}(x_n + y_n) + (\lambda - \frac{1}{2})(x_n - y_n)\right\| \\
\geq \frac{1}{2}\|x_n + y_n\| - |2\lambda - 1|\epsilon.
\]
which implies that
\[ L'_Y(\lambda, X) \geq \frac{1}{4} \left\| x_n + y_n \right\| - |2\lambda - 1| \epsilon^2 + \lambda(1 - \lambda) \epsilon. \]

Let \( n \to \infty \), as desired.

On the other hand, we can deduce that the following estimate
\[ \|\lambda x + (1 - \lambda)y\| \leq \lambda\|x + y\| + |1 - 2\lambda|\|y\| \leq 2\lambda + |1 - 2\lambda| - 2\lambda \delta(\epsilon). \]

Then we obtain
\[ \|\lambda x + (1 - \lambda)y\| + (1 - \lambda)\|x - y\| \leq (2\lambda + |1 - 2\lambda| - 2\lambda \delta(\epsilon))^2 + \lambda(1 - \lambda) \epsilon^2. \]

This completes the proof. \( \Box \)

**Example 2.9.** Consider \( X \) be \( c_0 = \{ (x_i) : \lim_{i \to \infty} |x_i| = 0 \} \) be equipped with the norm defined by
\[ \|x\| = \sup_{1 \leq i < \infty} |x_i| + \left( \sum_{i=1}^{\infty} \frac{|x_i|^2}{4^i} \right)^{1/2}. \]

It is known that \( \delta_X(2) = 1 \). Then we have
\[ L'_Y(\lambda, X) \leq (2\lambda + |1 - 2\lambda| - 2\lambda \delta_X(\epsilon))^2 + \lambda(1 - \lambda) \epsilon^2 = 1. \]

Thus \( L'_Y(\lambda, X) = 1 \) directly from the estimate of \( L'_Y(\lambda, X) \).

**Theorem 2.10.** Let \( X \) be a Banach space, \( \lambda \in [0, 1] \). Then
\[ L'_Y(\lambda, X) \geq 2 \min\{\lambda, 1 - \lambda\} C'_{NJ}(X), \]
and
\[ C'_{NJ}(X) \geq \left( 1 - \frac{|1 - 2\lambda|}{2\sqrt{\lambda(1 - \lambda)}} \right) L'_Y(\lambda, X). \]

**Proof.** Using the equality fact that
\[ \lambda x + (1 - \lambda)y = \frac{1}{2}(x + y) + \frac{2\lambda - 1}{2}(x - y), \]
we obtain
\[ L'_Y(\lambda, X) \geq \|\lambda x + (1 - \lambda)y\|^2 + \lambda(1 - \lambda)\|x - y\|^2 \]
\[ \geq \left( \frac{1}{2}\|x + y\| - \frac{1 - 2\lambda}{2} \|x - y\| \right)^2 + \lambda(1 - \lambda)\|x - y\|^2 \]
\[ = \frac{1}{4}\|x + y\|^2 + \frac{(1 - 2\lambda)^2}{4}\|x - y\|^2 - \frac{1 - 2\lambda}{2} \|x + y\| \|x - y\| \]
\[ + \lambda(1 - \lambda)\|x - y\|^2 \]
\[ \geq \frac{1}{4}\left(\|x + y\|^2 + \|x - y\|^2\right) - \frac{1 - 2\lambda}{2} \left( \|x + y\|^2 + \|x - y\|^2 \right). \]
This shows that

\[ L'_Y(\lambda, X) \geq 2 \min\{\lambda, 1 - \lambda\} C'_{NJ}(X). \]

For the second part of the proof, let \( L'_Y(\lambda, X) = \omega \). We can deduce that \( \exists x_n, y_n \in S_X \), such that

\[ \lim_{n \to \infty} (\|\lambda x_n + (1 - \lambda)y_n\|^2 + \lambda(1 - \lambda)\|x_n - y_n\|^2) = \omega. \]

For the convenience of presentation, we set

\[ \alpha_n = \lambda x_n + (1 - \lambda)y_n, \quad \beta_n = \sqrt{\lambda(1 - \lambda)}(x_n - y_n), \]

it is clear that

\[ \lim_{n \to \infty} (\|\alpha_n\|^2 + \|\beta_n\|^2) = \omega. \]

Furthermore, we can deduce that

\[ x_n = \alpha_n + \sqrt{\frac{1 - \lambda}{\lambda}} \beta_n, \quad y_n = \alpha_n - \sqrt{\frac{\lambda}{1 - \lambda}} \beta_n. \]

Then we have

\[
C'_{NJ}(X) \geq \frac{\|x_n + y_n\|^2 + \|x_n - y_n\|^2}{4} \\
= \frac{\left\|2\alpha_n + \frac{1 - 2\lambda}{\sqrt{\lambda(1 - \lambda)}} \beta_n\right\|^2 + \frac{1}{\lambda(1 - \lambda)} \|\beta_n\|^2}{4} \\
\geq \frac{2\|\alpha_n\|^2 - \frac{|1 - 2\lambda|}{\sqrt{\lambda(1 - \lambda)}} \|\beta_n\|^2 + \frac{1}{\lambda(1 - \lambda)} \|\beta_n\|^2}{4} \\
= \frac{4(\|\alpha_n\|^2 + \|\beta_n\|^2) + \frac{2(2\lambda - 1)^2}{\lambda(1 - \lambda)} \|\beta_n\|^2 - \frac{4|1 - 2\lambda|}{\sqrt{\lambda(1 - \lambda)}} ||\alpha_n\||\|\beta_n\||}{4} \\
\geq \left(1 - \frac{|1 - 2\lambda|}{2\sqrt{\lambda(1 - \lambda)}}\right)(\|\alpha_n\|^2 + \|\beta_n\|^2)
\]

and let \( n \to \infty \), as desired.

**Theorem 2.11.** For a Banach space \( X \) the following assertions are equivalent:

(i) \( C'_{NJ}(X) = 2 \).

(ii) \( L'_Y(\lambda, X) = -4\lambda^2 + 4\lambda + 1 \) for all \( \lambda \in [0, 1] \).

(iii) \( L'_Y(\lambda, X) = -4\lambda^2 + 4\lambda_0 + 1 \) for some \( \lambda_0 \in (0, 1) \).

**Proof.** (i)\(\Rightarrow\) (ii). Since \( C'_{NJ}(X) = 2 \), we deduce that there exists \( x_n, y_n \in S_X \) such that

\[ \|x_n + y_n\| \to 2, \quad \|x_n - y_n\| \to 2 \ (n \to \infty). \]

This means that there exists \( x_n, y_n \in S_X \) such that

\[ \|\lambda x_n + (1 - \lambda)y_n\| = \|((x_n + y_n) + (\lambda - 1)x_n - \lambda y_n\| \\
\geq 2 - (1 - \lambda) - \lambda = 1. \]
So we can deduce that there exists \( x_n, y_n \in S_X \) such that
\[
\|\lambda x_n + (1 - \lambda)y_n\| \to 1, \quad \|x_n - y_n\| \to 2 \quad (n \to \infty)
\]
which implies that \( L'_Y(\lambda, X) = -4\lambda^2 + 4\lambda + 1 \).

(ii) \( \Rightarrow \) (iii). Obvious.

(iii) \( \Rightarrow \) (i). If \( C_{NJ}(X) < 2 \), then there exists \( \delta > 0 \), such that for any \( x, y \in S_X \) either
\[
\|\frac{x + y}{2}\| \leq 1 - \delta
\]
or\[
\|\frac{x - y}{2}\| \leq 1 - \delta.
\]

We first consider the case \( \|\frac{x + y}{2}\| \leq 1 - \delta \). Applying the convexity of the unit sphere, we can obtain that \( \|\lambda x + (1 - \lambda)y\| < 1 \) for all \( \lambda \in (0, 1) \), and hence
\[
\|\lambda x + (1 - \lambda)y\|^2 + \lambda(1 - \lambda)\|x - y\|^2 < -4\lambda^2 + 4\lambda + 1
\]
for all \( \lambda \in (0, 1) \).

In the case \( \|\frac{x - y}{2}\| \leq 1 - \delta \), we have
\[
\|\lambda x + (1 - \lambda)y\|^2 + \lambda(1 - \lambda)\|x - y\|^2 \leq 1 + \lambda(1 - \lambda)[2(1 - \delta)]^2
\]
for all \( \lambda \in (0, 1) \).

By combining the above two cases, we can conclude that \( L'_Y(\lambda, X) < -4\lambda^2 + 4\lambda + 1 \). This is a contradiction and thus we complete the proof.

\[\blacksquare\]

**Corollary 2.12.** \( X \) is uniformly non-square if and only if \( L'_Y(\lambda, X) < -4\lambda_0^2 + 4\lambda_0 + 1 \) for some \( \lambda_0 \in (0, 1) \).

**Proof.** It can be directly concluded from Theorem 2.11 and the fact \( X \) is uniformly non-square if and only if \( C_{NJ}(X) < 2 \).

\[\blacksquare\]

### 3 \( L'_Y(\lambda, X) \) and normal structure

Next, we will see that the normal structure and the constant \( L'_Y(\lambda, X) \) have a close relationship.

**Definition 3.1.** [4] Let \( X \) be a real Banach space. We say that \( X \) has normal structure if for every closed bounded convex subset \( C \) in \( X \) that contains more than one point, there exists a point \( x_0 \in C \) such that
\[
\sup \{\|x_0 - y\| : y \in C\} < \sup \{\|x - y\| : x, y \in C\}.
\]

Normal structure is an important concept in fixed point theory [20]. A Banach space \( X \) is said to have weak normal structure if each weakly compact convex set \( K \) of \( X \) that contains more than one point has normal structure. Even more to the point, for Banach space \( X \) which is reflexive, the weak normal structure and normal structure coincide. Furthermore, every reflexive Banach space with normal structure has the fixed point property.

We begin by starting a lemma which will be our main tool.
Lemma 3.2. [14]. Let $X$ be a Banach space without weak normal structure, then for any $0 < \delta < 1$, there exist $x_1, x_2, x_3$ in $S_X$ satisfying

(i) $x_2 - x_3 = ax_1$ with $|a - 1| < \delta$;
(ii) $||x_1 - x_2|| - 1, ||x_3 - (-x_1)|| - 1 < \delta$; and
(iii) $\|\frac{x_1 + x_2}{2}\|, \|\frac{x_1 + (-x_1)}{2}\| > 1 - \delta$.

It is possible to understand the geometric sense of this lemma as follows: if $X$ does not have weak normal structure, then an inscribed hexagon exists in $S_X$ with an arbitrarily closed length of each side to 1, and at least four sides with an arbitrarily small distance to $S_X$.

Theorem 3.3. A Banach space $X$ with $L'_Y(\lambda, X) < -\lambda^2 + \lambda + 1$ for some $\lambda \in \left[\frac{1}{2}, 1\right]$ has normal structure.

Proof. Notice that, by Corollary 2.12, $L'_Y(\lambda, X) < -\lambda^2 + \lambda + 1$ implies that $X$ is uniformly non-square, and hence reflexive [16]. This means that normal structure and weak normal structure coincide.

Suppose $X$ does not have weak normal structure. For each $\delta > 0$, let $x_1, x_2$ and $x_3$ in $S_X$ satisfying the conditions in Lemma 3.2.

Then
\[
\|x_1 + \frac{1 - \lambda}{\lambda}x_2\| = \|(x_1 + x_2) - (1 - \frac{1 - \lambda}{\lambda})x_2\|
\geq \|x_1 + x_2\| - \|(1 - \frac{1 - \lambda}{\lambda})x_2\|
\geq 2 - 2\delta - (1 - \frac{1 - \lambda}{\lambda})
\]

and
\[
\lambda(1 - \lambda)\|x_1 - x_2\|^2 \geq \lambda(1 - \lambda)(1 - \delta)^2.
\]

Note that $\delta$ can be arbitrarily small, so we obtain
\[
L'_Y(\lambda, X) \geq \|\lambda x + (1 - \lambda)y\|^2 + \lambda(1 - \lambda)\|x - y\|^2
= \lambda^2\|x + \frac{1 - \lambda}{\lambda}y\|^2 + \lambda(1 - \lambda)\|x - y\|^2
\geq -\lambda^2 + \lambda + 1,
\]

which is a contradiction. This completes the proof.

Data Availability Statement

All type of data used for supporting the conclusions of this article is included in the article and also is cited at relevant places within the text as references.

Conflict of interest

The authors declare that they have no conflict of interest.
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