A VARIATIONAL APPROACH TO COMPLEX HESSIAN EQUATIONS IN $\mathbb{C}^n$

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Abstract. Let $\Omega$ be a $m$-hyperconvex domain of $\mathbb{C}^n$ and $\beta$ be the standard Kähler form in $\mathbb{C}^n$. We introduce finite energy classes of $m$-subharmonic functions of Cegrell type, $\mathcal{E}_p^m(\Omega)$, $p > 0$ and $\mathcal{F}_m(\Omega)$. Using a variational method we show that the degenerate complex Hessian equation $(dd^c \varphi)^m \wedge \beta^{n-m} = \mu$ has a unique solution in $\mathcal{E}_1^m(\Omega)$ if and only if every function in $\mathcal{E}_1^m(\Omega)$ is integrable with respect to $\mu$. If $\mu$ has finite total mass and does not charge $m$-polar sets, then the equation has a unique solution in $\mathcal{F}_m(\Omega)$.

1. Introduction

Let $\Omega$ be a bounded domain of $\mathbb{C}^n$ and $m$ be an integer such that $1 \leq m \leq n$. We consider complex $m$-Hessian equations of the form

$$(dd^c \varphi)^m \wedge \beta^{n-m} = \mu,$$

where $\beta := dd^c |z|^2$ is the standard Kähler form in $\mathbb{C}^n$ and $\mu$ is a positive Radon measure.

The border cases $m = 1$ and $m = n$ correspond to the Laplace equation which is a classical subject and the complex Monge-Ampère equation which was studied intensively in the recent years by many authors.

The complex $m$-Hessian equation was first studied by Li [21]. He used the well-known continuity method to solve the non-degenerate Dirichlet problem for equation (1.1) (where the data is smooth and we seek for smooth solutions) in strongly $m$-pseudoconvex domains. One of its degenerate counterparts was studied by Blocki [4]. More precisely, he solved the homogeneous equation with continuous boundary data and initiated a potential theory for this equation. Recently, Abdullaev and Sadullaev [30] also considered $m$-polar sets and $m$-capacity for $m$-subharmonic functions. When the right-hand side $\mu$ has density in $L^p(\Omega)$ ($p > n/m$) Dinew and Kołodziej proved in [9] that given a continuous boundary data, the Dirichlet problem of equation (1.1) has a unique continuous solution. The Hölder regularity of the solution has been recently studied by Nguyen Ngoc Cuong [27]. He also showed how to construct solutions from subsolutions [26]. A viscosity approach to this equation has been developed in [24] which generalize results in [34] and [11].

The real Hessian equation is a classical subject which was studied intensively in the recent years. The reader can find a survey for this in [33]. It was explained in [9] that real and complex Hessian equations are very different and direct adaptations of the real methods to the complex setting often fails.
The corresponding complex $m$-Hessian equation on compact Kähler manifolds has been studied by many authors. It has the following form

$$\left(\omega + dd^c \varphi\right)^m \wedge \omega^{n-m} = \mu,$$

where $(X, \omega)$ is a compact Kähler manifold of dimension $n$ and $1 \leq m \leq n$ and $\mu$ is a positive Radon measure.

When $\mu = f \omega^n$, $f > 0$ is a smooth function satisfying the compatibility condition $\int_X f \omega^n = \int_X \omega^n$, this is a generalization of the well-known Calabi-Yau equation [31]. In [19], Kokarev gave some conditions on the measure $\mu$ and on the holomorphic sectional curvature of the metric so that equation (1.2) has a $\omega$-plurisubharmonic solution. In general, if $\varphi$ solves equation (1.2) the form $\omega + dd^c \varphi$ is not positive. This lack of positivity prevents one from copying the proof of Yau’s Theorem without assuming a positivity condition on the holomorphic bisectional curvature. Hou, Ma, Wu [15], and Jbilou [17] independently proved that equation (1.2) has a smooth solution provided this positivity condition. Another effort from Hou, Ma and Wu [16] showed that one can obtain a $C^2$ estimate if a gradient estimate holds. As suggested by these authors, this estimate can be used in some blow-up analysis. This blow-up analysis reduces the problem of solving equation (1.2) to a Liouville-type theorem for $m$-subharmonic functions in $\mathbb{C}^n$ which was recently proved by Dinew and Kołodziej [10] and the solvability of equation (1.2) is thus confirmed on any compact Kähler manifold.

When $0 \leq f \in L^p(X, \omega^n)$ for some $p > n/m$, Dinew and Kołodziej recently proved that (1.2) admits a unique continuous weak solution. The result also holds when the right-hand side $f = f(x, \varphi)$ depends on $\varphi$ (see [23]).

To deal with more singular measures (measures of finite energy), the variational method developed in [5] is a powerful method. However, applying this method to the complex Hessian equation (1.2) need further studies on the local Dirichlet problem and on the regularizing process which are not yet available and seem to be very difficult.

As a matter of fact, it is interesting to first develop this approach for the complex Hessian equation in the flat case, i.e the case when the metric is $\beta$. This is the aim of this paper.

The paper is organized as follows. In section 2, we recall basic facts about $m$-subharmonic functions and the complex $m$-Hessian operators. At the end of section 2 we give a connection between the m-polarity and the Hausdorff measure of a set. Using this one can find examples of $m$-polar sets ($m < n$) which are not pluripolar. In section 3, we study finite energy classes of $m$-subharmonic functions inspired by [7, 8]. An $m$-subharmonic function $\varphi$ belongs to the class $E^1_m(\Omega)$ if the Hessian measure $H_m(\varphi) = (dd^c \varphi)^m \wedge \beta^{n-m}$ is well-defined and with respect to which $\varphi$ is integrable. The class $F^a_m(\Omega)$ consists of non-positive $m$-subharmonic functions whose Hessian measures are well-defined, of finite total mass and do not charge $m$-polar sets. In section 4, we develop a variational approach inspired by [5] (see also [2]) to solve equation (1.1) with a "finite energy" right-hand side.

The main results are the followings.

**Theorem 1.** Let $\mu$ be a positive Radon measure in $\Omega$, an $m$-hyperconvex domain. Then $E^1_m(\Omega) \subset L^1(\Omega, \mu)$ if and only if there exists a unique $\varphi \in E^1_m(\Omega)$ such that $(dd^c \varphi)^m \wedge \beta^{n-m} = \mu$. 
To prove this result we use a variational method introduced in [5]. Our result generalizes the result in [2]. Using this and following [8] we also get:

**Theorem 2.** Let \( \mu \) be a positive Radon measure in an \( m \)-hyperconvex domain \( \Omega \) such that \( \mu(\Omega) < +\infty \) and \( \mu \) does not charge \( m \)-polar sets. Then there exists a unique \( \varphi \in \mathcal{F}_m^+(\Omega) \) such that \((dd^c\varphi)^m \wedge \beta^{n-m} = \mu\).

2. Preliminaries

2.1. \textbf{m-subharmonic functions and the Hessian operator.} In the whole paper, \( \beta \) denotes the standard Kähler form in \( \mathbb{C}^n \). In this section we summarize basic facts about \textit{m}-subharmonic functions and the Hessian operator which will be used in the next sections. Most of these results can be found in [26, 27], [30] or can be proved similarly as in the case of plurisubharmonic functions (see for example [18], [20]).

**Definition 2.1.** Let \( \alpha \) be a real \((1, 1)\)-form in \( \Omega \), a domain of \( \mathbb{C}^n \). We say that \( \alpha \) is \textit{m}-positive in \( \Omega \) if the following inequalities hold

\[
\alpha^j \wedge \beta^{n-j} \geq 0, \quad \forall j = 1, \ldots, m.
\]

Let \( T \) be a current of bidegree \((n-k, n-k)\), with \( k \leq m \). Then \( T \) is called \textit{m}-positive if for all \textit{m}-positive \((1, 1)\)-forms \( \alpha_1, \ldots, \alpha_k \), we have

\[
\alpha_1 \wedge \ldots \wedge \alpha_k \wedge T \geq 0.
\]

**Definition 2.2.** A function \( u : \Omega \rightarrow \mathbb{R} \cup \{-\infty\} \) is called \textit{m}-subharmonic if it is subharmonic and the current \( dd^c u \) is \textit{m}-positive. The class of all \textit{m}-subharmonic functions in \( \Omega \) will be denoted by \( \mathcal{SH}_m(\Omega) \).

**Definition 2.3.** Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \). Then \( \Omega \) is called \textit{m}-hyperconvex if there exists a continuous \textit{m}-subharmonic function \( \varphi : \Omega \rightarrow \mathbb{R}^- \) such that \( \{ \varphi < c \} \in \Omega \), for every \( c < 0 \).

From now on, we always assume that \( \Omega \) is \textit{m}-hyperconvex.

We list in the following proposition some elementary facts on \textit{m}-subharmonicity.

**Proposition 2.4.** (i) If \( u \in C^2 \) smooth then \( u \) is \textit{m}-subharmonic if and only if the form \( dd^c u \) is \textit{m}-positive in \( \Omega \).

(ii) If \( u, v \in \mathcal{SH}_m(\Omega) \) then \( \lambda u + \mu v \in \mathcal{SH}_m(\Omega), \forall \lambda, \mu > 0 \).

(iii) If \( u \) is \textit{m}-subharmonic in \( \Omega \) then the standard regularization \( u \ast \chi_\epsilon \) are also \textit{m}-subharmonic in \( \Omega_\epsilon := \{ x \in \Omega / d(x, \partial \Omega) > \epsilon \} \).

(iv) If \( (u_i) \subset \mathcal{SH}_m(\Omega) \) is locally uniformly bounded from above then \( \sup u_i^* \in \mathcal{SH}_m(\Omega) \), where \( v^* \) is the upper semi continuous regularization of \( v \).

(v) \( \text{PSH}(\Omega) = \mathcal{SH}_n(\Omega) \subset \ldots \subset \mathcal{SH}_m(\Omega) \subset \ldots \subset \mathcal{SH}_1(\Omega) = \mathcal{SH}(\Omega) \).

(vi) Let \( \emptyset \neq U \subset \Omega \) be a proper open subset such that \( \partial U \cap \Omega \) is relatively compact in \( \Omega \). If \( u \in \mathcal{SH}_m(\Omega) \), \( v \in \mathcal{SH}_m(U) \) and \( \limsup_{x \to y} v(x) \leq u(y) \) for each \( y \in \partial U \cap \Omega \) then the function \( w \), defined by

\[
w(z) = \begin{cases} u(z) & \text{if } z \in \Omega \setminus U, \\ \max(u(z), v(z)) & \text{if } z \in U \end{cases}
\]

is \textit{m}-subharmonic in \( \Omega \).

For locally bounded \textit{m}-subharmonic functions \( u_1, \ldots, u_p \) (with \( p \leq m \)) we can inductively define a closed \textit{m}-positive current (following Bedford and Taylor [3]).
Lemma 2.5. Let $u_1, \ldots, u_k$ (with $k \leq m$) be locally bounded $m$-subharmonic functions in $\Omega$ and let $T$ be a closed $m$-positive current of bidegree $(n-p, n-p)$ (with $p \geq k$). Then we can define inductively a closed $m$-positive current
\[ dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_k \wedge T, \]
and the product is symmetric, i.e.
\[ dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_p \wedge T \leq dd^c u_{\sigma(1)} \wedge dd^c u_{\sigma(2)} \wedge \ldots \wedge dd^c u_{\sigma(k)} \wedge T, \]
for every permutation $\sigma : \{1, \ldots, k\} \rightarrow \{1, \ldots, k\}$.
In particular, the Hessian measure of $u \in SH_m(\Omega) \cap L^\infty_{locc}$ is defined to be
\[ H_m(u) = (dd^c u)^m \wedge \beta^{n-m}. \]

Proof. See [30]. \qed

Theorem 2.6. Let $(u_0^0), \ldots, (u_k^0)$ be decreasing sequences of $m$-subharmonic functions in $\Omega$ converging to $u_0, \ldots, u_k \in SH_m(\Omega) \cap L^\infty_{locc}(\Omega)$ respectively. Let $T$ be a closed $m$-positive current of bidegree $(n-p, n-p)$ (with $p \geq k$) on $\Omega$. Then
\[ u_0^0 dd^c u_1^0 \wedge \ldots \wedge dd^c u_k^0 \wedge T \rightarrow u_0 dd^c u_1 \wedge \ldots \wedge dd^c u_k \wedge T, \]
weakly in the sense of currents.

Proof. See [30]. \qed

One of the most important properties of $m$-subharmonic functions is the quasicontinuity. Every $m$-subharmonic function is continuous outside an arbitrarily small open subset. The $m$-Capacity is used to measure the smallness of these sets.

Definition 2.7. Let $E \subset \Omega$ be a Borel subset. The $m$-capacity of $E$ with respect to $\Omega$ is defined to be
\[ \text{Cap}_m(E, \Omega) := \sup \left\{ \int_E H_m(\varphi) \mid \varphi \in SH_m(\Omega), 0 \leq \varphi \leq 1 \right\}. \]
The $m$-Capacity shares the same elementary properties as the Capacity introduced by Bedford and Taylor.

Proposition 2.8. i) $\text{Cap}_m(E_1, \Omega) \leq \text{Cap}_m(E_2, \Omega)$ if $E_1 \subset E_2$,
ii) $\text{Cap}_m(E, \Omega) = \lim_{j \to \infty} \text{Cap}_m(E_j, \Omega)$ if $E_j \uparrow E$,
iii) $\text{Cap}_m(E, \Omega) \leq \sum \text{Cap}_m(E_j, \Omega)$ for $E = \cup E_j$.

The following results can be proved by repeating the arguments in [20].

Theorem 2.9. Every $m$-subharmonic function $u$ defined in $\Omega$ is quasi-continuous. This means that for any positive number $\epsilon$ one can find an open set $U \subset \Omega$ with $\text{Cap}_m(U, \Omega) < \epsilon$ and such that $u$ restricted to $\Omega \setminus U$ is continuous.

Theorem 2.10. Let $\{u_k^j\}_{j=1}^\infty$ be a locally uniformly bounded sequence of $m$-subharmonic functions in $\Omega$ for $k = 1, 2, \ldots, N \leq m$ and let $u_k^j \uparrow u_k \in SH_m(\Omega) \cap L^\infty_{locc}$ almost everywhere as $j \to \infty$ for $k = 1, 2, \ldots, N$. Then
\[ dd^c u_1^j \wedge \ldots \wedge dd^c u_N^j \wedge \beta^{n-m} \rightarrow dd^c u_1 \wedge \ldots \wedge dd^c u_N \wedge \beta^{n-m}. \]
Theorem 2.11 (Integration by parts). Let \( u, v \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega) \) such that 
\[
\lim_{z \to \partial \Omega} u(z) = \lim_{z \to \partial \Omega} v(z) = 0. \]
Then 
\[
\int_{\Omega} udd^c v \wedge T = \int_{\Omega} vdd^c u \wedge T,
\]
where \( T = dd^c \varphi_1 \wedge \ldots \wedge dd^c \varphi_{m-1} \wedge \beta^{n-m} \) with \( \varphi_1, \ldots, \varphi_{m-1} \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega) \). The equality is understood in the sense that if one of the two terms is finite then so is the other, and they are equal.

Theorem 2.12 (Maximum principle). If \( u, v \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega) \) then 
\[
\mathbb{P}_{(u>v)} H_m(\max(u, v)) = \mathbb{P}_{(u>v)} H_m(u).
\]

Theorem 2.13 (Comparison principle). Let \( u, v \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega) \) such that 
\[
\liminf_{z \to \partial \Omega} (u(z) - v(z)) \geq 0.
\]
Then 
\[
\int_{\{u<v\}} (dd^c v)^m \wedge \beta^{n-m} \leq \int_{\{u<v\}} (dd^c u)^m \wedge \beta^{n-m}.
\]

Definition 2.14. For a subset \( E \) of a domain \( \Omega \subset \mathbb{C}^n \) we define the relative \( m \)-extremal function of \( E \) by 
\[
u_{m,E,\Omega} := \sup \{ u \in SH_m(\Omega) / u < 0, \text{ and } u \leq -1 \text{ on } E \}.
\]

It is easy to see that \( \nu_{m,E,\Omega} \) is \( m \)-subharmonic in \( \Omega \).

Proposition 2.15. i) If \( E_1 \subset E_2 \) then \( u_{E_2} \leq u_{E_1} \).
ii) If \( E \subset \Omega_1 \subset \Omega_2 \) then \( u_{m,E,\Omega_2} \leq u_{m,E,\Omega_1} \).
iii) If \( K_j \downarrow K \), with \( K_j \) compact in \( \Omega \) then \( (\lim u_{m,K_j})^* = u_{m,K,\Omega}^* \).

As in [18] we have the following formula for the \( m \)-extremal functions of concentric balls.

Lemma 2.16. Let \( 0 < r < R \) and set \( a = \frac{n}{m} > 1 \). The \( m \)-extremal function of \( B(r) \) with respect to \( B(R) \) is given by 
\[
u_{m,B(r),B(R)} = \max \left( \frac{R^{2-2a} - \|z\|^{2-2a}}{r^{2-2a} - R^{2-2a}}, -1 \right).
\]

Proposition 2.17. If \( E \subset \Omega \) then one has \( \lim_{z \to w} u_{m,E,\Omega}(z) = 0 \) for any \( w \in \partial \Omega \).

Proposition 2.18. Let \( K \subset \Omega \) be a compact subset which is the union of closed balls, then \( u_K^* = u_K \) is continuous. In particular, if \( K \subset \Omega \) is an arbitrary compact set and \( \varepsilon < \text{dist}(K, \partial \Omega) \), then \( u_{K_\varepsilon} \) is continuous, where \( K_\varepsilon = \{ z \in \Omega / \text{dist}(z, K) \leq \varepsilon \} \).

Sketch of proof: From Lemma 2.16 we know that the \( m \)-extremal function of balls is continuous. Let \( K \) be a compact set which is a union of closed balls, \( K = \cup B_j \) and let \( u \) be its \( m \)-extremal function. Since \( u \leq u_{m,B_j,\Omega} \), it is easy to see that \( u^* \equiv -1 \) on \( K \). The same arguments as in [18, Proposition 4.5.3] show that \( u \) is continuous in \( \Omega \).

Definition 2.19. The outer \( m \)-capacity of a Borel set \( E \subset \Omega \) is defined by 
\[
\text{Cap}_m^*(E,G) := \inf \{ \text{Cap}_m(G, \Omega) / E \subset G, G \text{ is an open subset of } \Omega \}.
\]
Theorem 2.20. If $E \Subset \Omega$ is a Borel subset then
\[
\text{Cap}^*_m(E, \Omega) = \int_{\Omega} H_m(u^*_{m,E,\Omega}), \quad \text{and } \text{Cap}^*_m(E, \Omega) = \text{Cap}_m(E, \Omega) \text{ if } E \text{ is compact.}
\]

We compute the $m$-Capacity of the concentric balls.

Example 2.21. For every $0 < r < R$ we have
\[
\text{Cap}_m(B(r), B(R)) = \frac{2^n(n-m)}{m. n!(r^2 - 2a - R^2 - 2a)^m}.
\]

Definition 2.22. Let $\Omega$ be an open set in $\mathbb{C}^n$, and let $\mathcal{U} \subset S\text{H}_m(\Omega)$ be a family of functions which is locally bounded from above. Define
\[
u(z) = \sup\{v(z) / v \in \mathcal{U}\}.
\]
Sets of the form $\mathcal{N} = \{z \in \Omega / \nu(z) < u^*(z)\}$ and all their subsets are called $m$-negligible.

Definition 2.23. A set $E \subset \mathbb{C}^n$ is called $m$-polar if $E \subset \{v = -\infty\}$ for some $v \in S\text{H}_m(\mathbb{C}^n)$ and $v \neq -\infty$.

Theorem 2.24. Let $E \Subset \Omega$. Then $E$ is $m$-negligible $\Leftrightarrow$ $E$ is $m$-polar $\Leftrightarrow$ $\text{Cap}^*_m(E, \Omega) = 0$.

2.2. $m$-polarity of sets with small Hausdorff measure. In this section, following [22] we give a sufficient condition for a set being $m$-polar using Hausdorff measure. We then give examples of $m$-polar sets ($m < n$) which are not pluripolar.

Definition 2.25. A function $h : [0, 1) \to \mathbb{R}^+$ is called measuring function if it is increasing and $\lim_{r \to 0} h(r) = h(0) = 0$.

For a measuring function $h$, the $h$-Hausdorff measure of $E \subset \mathbb{C}^n$ is defined by (see [22], [25], [32])
\[
\Lambda_h(E) := \lim_{\delta \to 0} \left(\inf \sum_{j} h(r_j)\right),
\]
where the infimum is taken over all coverings of $E$ by balls $B_j$ of radii $r_j \leq \delta$.

Theorem 2.26. Let $H(r) = r^{2n-2m}$ ($n > m \geq 1$). Then every subset $E \subset \mathbb{C}^n$ satisfying $\Lambda_H(E) < +\infty$ is $m$-polar.

Proof. We can assume that $E \Subset B = B(0, R)$. Suppose that $E$ is not $m$-polar. Then $\text{Cap}^*_m, B(E) > 0$ and $u := u^*_{m,E,B} \neq 0$. Set $E_1 := \{z \in E / u(z) = -1\}$. Then by Theorem 2.24, $E \setminus E_1$ is $m$-polar and hence
\[
\int_{E \setminus E_1} H_m(u) = 0.
\]
Therefore, as $H_m(u)$ is a regular Borel measure and $E_1$ is a Borel set, we can find a compact set $K \subset E_1$ such that $\int_{K} H_m(u) > 0$.

We claim that for every bounded open set $\Omega \supset K$, $\text{Cap}_m(K, \Omega) \leq C.\Lambda_H(K)$, where $C > 0$ is a constant independent of $\Omega$. Indeed, let $\delta := \text{dist}(K, \Omega)$ and fix $\epsilon \in (0, 1)$ such that $\epsilon < \delta/4$. We cover $K$ by open balls $B(z_j, r_j)$ such that $r_j < \epsilon$. We may assume that $B(z_j, r_j) \subset \Omega$. From (2.21) and after simple computations we get
\[
\text{Cap}_m(B(z_j, r_j), B(z_j, \delta/2)) \leq C_j^{2n-2m},
\]
where $C > 0$ is a constant depending only on $n, m$. Using this and the monotonicity and subadditivity of the outer $m$-capacity we get

$$\text{Cap}_m^*(K, \Omega) \leq \sum_j \text{Cap}_m^*(B(z_j, r_j), \Omega) \leq \sum_j \text{Cap}_m^*(B(z_j, r_j), B(z_j, \delta/2)) \leq C \sum_j H(r_j).$$

Now, the claim follows by taking the infimum over all such coverings and letting $\epsilon \to 0$.

For each $\delta > 0$ set $\Omega_\delta := \{ z \in B / \text{dist}(z, K) < \delta \}$. Since $u$ is continuous at every point on $K$ and $u(x) = -1$ for every $x \in K$, we get

$$c(\delta) := \text{osc}_{\Omega_\delta}(u) \to 0 \quad \text{as} \quad \delta \to 0.$$

For any $z \in \Omega_\delta$ we have $0 \leq \frac{u(z)+1}{c(\delta)} \leq 1$. Thus,

$$\text{Cap}_m^*(K, \Omega_\delta) \geq \text{Cap}_m(K, \Omega_\delta) \geq \int_K H_m \left( \frac{u+1}{\epsilon (\delta)} \right) = \frac{1}{\epsilon (\delta)^m} \int_K H_m(u).$$

We then get $\text{Cap}_m^*(K, \Omega_\delta) \to +\infty$ as $\delta \to 0$. The Claim yields $\Lambda_H(E) = +\infty$. □

**Example 2.27.** Assume that $1 \leq m < n$ and let $E$ be the Cantor set constructed in $[22]$ (see also $[1], [6], [25]$) and $E^n = E \times E \times \ldots \times E$ ($n$ times). Since $\lim_{r \to 0} H(r)/[\log(1/r)]^{-n} = 0$, we can choose the sequence $(\ell_j)$ defining $E$ such that $E^n$ is not pluripolar but $\Lambda_H(E^n) = 0$. This implies that $E^n$ is $m$-polar in view of Theorem 2.26.

### 3. Finite energy classes

In this section we study finite energy classes of $m$-subharmonic functions in $m$-hyperconvex domains. They are generalizations of Cegrell’s classes $[7, 8]$ for plurisubharmonic functions.

**3.1. Definitions and properties.** In pluripotential theory one of the most important steps is to regularize singular plurisubharmonic functions. It can be easily done locally by convolution with a smooth kernel. The following theorem explains how to do it globally in a $m$-hyperconvex domain. Let $\mathcal{S}H_m^-(\Omega)$ denote the class of non-positive functions in $\mathcal{S}H_m(\Omega)$.

**Theorem 3.1.** For each $\varphi \in \mathcal{S}H_m^-(\Omega)$ there exists a sequence $(\varphi_j)$ of $m$-sh functions verifying the following conditions:

(i) $\varphi_j$ is continuous on $\Omega$ and $\varphi_j \equiv 0$ on $\partial \Omega$;

(ii) each $H_m(\varphi_j)$ has compact support;

(iii) $\varphi_j \downarrow \varphi$ on $\Omega$.

**Proof.** If $B$ is a closed ball in $\Omega$ then by Proposition 2.18, the $m$-extremal function $u_{m,B,\Omega}$ is continuous on $\Omega$ and $\text{supp}(H_m(u)) \subseteq \Omega$. We can follow the lines in $[8$, Theorem 2.1]. □

**Definition 3.2.** We let $\mathcal{E}^0_m(\Omega)$ denote the class of bounded functions in $\mathcal{S}H_m^-(\Omega)$ such that $\lim_{z \to \partial \Omega} \varphi(z) = 0$ and $\int_{\Omega} H_m(\varphi) < +\infty$.

For each $p > 0$, $\mathcal{E}^p_m(\Omega)$ denote the class of functions $\varphi \in \mathcal{S}H_m(\Omega)$ such that there exists a decreasing sequence $(\varphi_j) \subseteq \mathcal{E}^0_m(\Omega)$ satisfying

(i) $\lim_j \varphi_j = \varphi$, in $\Omega$ and
Combining the above two inequalities we obtain the result.

If we require moreover that \( \sup_j \int_{\Omega} H_m(\varphi_j) < +\infty \) then, by definition, \( \varphi \) belongs to \( F_m^p(\Omega) \).

**Definition 3.3.** We let \( F_m(\Omega) \) denote the class of functions \( u \in SH_m^-(\Omega) \) such that there exists a sequence \( (u_j) \subset E_m^0(\Omega) \) decreasing to \( u \) in \( \Omega \) and

\[
\sup_j \int_{\Omega} H_m(u_j) < +\infty.
\]

**Definition 3.4.** We define the \( p \)-energy \( (p > 0) \) of \( \varphi \in E_m^0(\Omega) \) by

\[
E_p(\varphi) := \int_{\Omega} (-\varphi)^p H_m(\varphi).
\]

If \( p = 1 \) we drop the index and denote by \( E(\varphi) = E_1(\varphi) \).

We generalize Hölder inequality in the following lemma. When \( m = n \) it is a result of Persson [28]. Our proof uses the same idea.

**Lemma 3.5.** Let \( u, v_1, \ldots, v_m \in E_m^0(\Omega) \) and \( p \geq 1 \). We have

\[
\int_{\Omega} (-u)^p dd^c v_1 \wedge \ldots \wedge dd^c v_m \wedge \beta^{n-m} \leq D_p(E_p(u))^{-\frac{1}{m-p}} E_p(v_1)^{-\frac{1}{m-\alpha(p,m)}} \ldots E_p(v_m)^{-\frac{1}{m-\alpha(p,m)}},
\]

where \( D_1 = 1 \) and for each \( p > 1 \), \( D_p := p^\alpha(p,m)/(p-1) \), where

\[
\alpha(p,m) = (p+2)\left(\frac{p+1}{p}\right)^{m-2} - p - 1.
\]

**Proof.** Let

\[
F(u, v_1, \ldots, v_m) = \int_{\Omega} (-u)^p dd^c v_1 \wedge \ldots \wedge dd^c v_m \wedge \beta^{n-m}, \quad u, v_1, \ldots, v_m \in E_m^0(\Omega).
\]

Thanks to [28, Theorem 4.1] it suffices to prove that

\[
(3.1) \quad F(u, v, v_1, \ldots, v_{m-1}) \leq a(p) F(u, u, v_1, \ldots, v_{m-1})^{-\frac{1}{m-p}} F(v, v, v_1, \ldots, v_{m-1})^{-\frac{1}{m-p}},
\]

where \( a(p) = 1 \) if \( p = 1 \) and \( a(p) = p^{\frac{1}{m-p}} \) if \( p > 1 \). Set \( T = dd^c v_1 \wedge \ldots \wedge dd^c v_{m-1} \wedge \beta^{n-m} \). When \( p = 1 \), (3.1) becomes

\[
\int_{\Omega} (-u)^p dd^c v \wedge T \leq \left( \int_{\Omega} (-u)^p dd^c u \wedge T \right)^{\frac{1}{2}} \left( \int_{\Omega} (-v)^p dd^c v \wedge T \right)^{\frac{1}{2}},
\]

which is the Cauchy-Schwarz inequality. In the case \( p > 1 \), integrating by parts we get

\[
\int_{\Omega} (-u)^p dd^c v \wedge T \leq p \int_{\Omega} (-u)^{p-1} (-v) dd^c u \wedge T.
\]

By using Hölder inequality we obtain

\[
\int_{\Omega} (-u)^p dd^c v \wedge T \leq p \left( \int_{\Omega} (-u)^p dd^c u \wedge T \right)^{\frac{1}{p}} \left( \int_{\Omega} (-v)^p dd^c v \wedge T \right)^{\frac{1}{p}}.
\]

Now, interchanging \( u \) and \( v \) we get

\[
\int_{\Omega} (-v)^p dd^c u \wedge T \leq p \left( \int_{\Omega} (-u)^p dd^c v \wedge T \right)^{\frac{1}{p}} \left( \int_{\Omega} (-v)^p dd^c v \wedge T \right)^{\frac{1}{p}}.
\]

Combining the above two inequalities we obtain the result. \( \square \)
Thanks to Lemma 3.5 we can bound \( \int_{\Omega}(u_0)^p(dd^c u_1 \wedge ... \wedge dd^c u_m \wedge \beta^{n-m}) \) by 
\( E_p(u_j), \ j = 0,...,m \) if \( p \geq 1 \). To get similar estimates when \( p \in (0,1) \) we can follow the lines in [14]:

**Lemma 3.6.** Let \( u, v \in E^0_m(\Omega) \) and \( 0 < p < 1 \). If \( T \) is a closed \( m \)-positive current of type \( T = dd^c u_1 \wedge ... \wedge dd^c u_m \wedge \beta^{n-m} \), where \( u_j \in SH_m(\Omega) \cap L^\infty_{loc} \), then
\[
\int_{\Omega}(-u)^p(dd^c v)^k \wedge T \leq 2 \int_{\Omega}(-u)^p(dd^c u)^k \wedge T + 2 \int_{\Omega}(-v)^p(dd^c v)^k \wedge T.
\]

**Proof.** The same as in the proof of [14, Proposition 2.5].

**Proposition 3.7.** Let \( 0 < p < 1 \). There exists \( C_p > 0 \) such that
\[
0 \leq \int_{\Omega}(-\varphi_0)^p dd^c \varphi_1 \wedge ... \wedge dd^c \varphi_m \wedge \beta^{n-m} \leq C_p \max_{0 \leq j \leq m} E_p(\varphi_j),
\]
for all \( 0 \geq \varphi_0, \ldots, \varphi_m \in E^0_m(\Omega) \).

**Proof.** See [14, Proposition 2.10]

From Lemma 3.5 and Proposition 3.7 we easily get the following result.

**Corollary 3.8.** Let \( (u_j) \) be a sequence in \( E^0_m(\Omega) \) and \( p > 0 \). Assume also that \( \sup_j E_p(u_j) < +\infty \). Then
\[
u = \sum_{j=1}^{\infty} 2^{-j} u_j \text{ belongs to } E^p_m(\Omega).
\]

From the above facts, we can prove the convexity the classes \( E^p_m(\Omega), E_m(\Omega) \) by the same way as in [7, 8].

**Theorem 3.9.** By \( E \) we denote one of the classes \( E^0_m(\Omega), F_m(\Omega), E^p_m(\Omega), F^p_m(\Omega) \), \( p > 0 \). They are convex and moreover, if \( v \in E \), \( u \in SH_m(\Omega) \), \( U \geq u \), then \( u \in E \).

3.2. **Definition of the complex Hessian operator and basic properties.** In this section we prove that the complex Hessian operator \( H_m \) is well-defined for functions in \( F_m(\Omega) \) and in \( E^p_m(\Omega) \), \( p > 0 \). We follow the arguments in [8].

As in [8], continuous functions in \( E^\infty_m(\Omega) \) can be considered as test functions.

**Lemma 3.10.** \( C^\infty(\Omega) \subset E^0_m(\Omega) \cap C(\Omega) \) \( - E^0_m(\Omega) \cap C(\Omega) \).

**Theorem 3.11.** Let \( u^p \in F_m(\Omega), p = 1,\ldots, m \) and \( (g^p_j)_{j} \subset E^0_m(\Omega) \) such that \( g^p_j \downarrow u^p \), \( \forall p \). Then the sequence of measures
\[
dd^c g^1_j \wedge dd^c g^2_j \wedge ... \wedge dd^c g^m_j \wedge \beta^{n-m}
\]
converges weakly to a positive Radon measure which does not depend on the choice of the sequences \((g^p_j)\). Then we define \( \dd^c u^1 \wedge ... \wedge dd^c u^m \wedge \beta^{n-m} \) to be this weak limit.

**Proof.** See [8, Theorem 4.2].

It is convenient to use the notation \( H_m(u_1, ..., u_m) := \dd^c u^1 \wedge ... \wedge dd^c u^m \wedge \beta^{n-m} \).

**Definition 3.12.** A function \( u \) belongs to the class \( F^u_m(\Omega) \) if \( u \in F_m(\Omega) \) and \( H_m(u) \) vanishes on \( m \)-polar sets.
Corollary 3.13. Let $u_1, \ldots, u_m \in \mathcal{F}_m(\Omega)$ and $u_1^j, \ldots, u_m^j$ be sequences of functions in $\mathcal{E}_m^0(\Omega) \cap C(\Omega)$ decreasing to $u_1, \ldots, u_m$ respectively such that
\[ \sup_{j,p} \int_{\Omega} H_m(u_p^j) < +\infty. \]
Then for each $\varphi \in \mathcal{E}_m^0(\Omega) \cap C(\Omega)$ we have
\[ \lim_{j \to +\infty} \int_{\Omega} \varphi dd^c u_1^j \wedge \ldots \wedge dd^c u_m^j \wedge \beta^{n-m} = \int_{\Omega} \varphi dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge \beta^{n-m}. \]

Proof. It is clear that
\[ (3.2) \quad \sup_{j} \int_{\Omega} dd^c u_1^j \wedge \ldots \wedge dd^c u_m^j \wedge \beta^{n-m} < +\infty. \]
Fix $\epsilon > 0$ small enough and consider $\varphi_{\epsilon} = \max(\varphi, -\epsilon)$. The function $\varphi - \varphi_{\epsilon}$ is continuous and compactly supported in $\Omega$. It follows from Theorem 3.11 that
\[ \lim_{j \to +\infty} \int_{\Omega} (\varphi - \varphi_{\epsilon}) dd^c u_1^j \wedge \ldots \wedge dd^c u_m^j \wedge \beta^{n-m} = \int_{\Omega} (\varphi - \varphi_{\epsilon}) dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge \beta^{n-m}. \]
Observe also that $|\varphi_{\epsilon}| \leq \epsilon$. By using (3.2), we get the result. \hfill \Box

Corollary 3.14. Assume that $(u_j) \subset \mathcal{E}_m^0(\Omega)$ decreases to $u$ such that
\[ \sup_{j} \int_{\Omega} H_m(u_j) < +\infty \]
Then for every $h \in \mathcal{E}_m^0(\Omega)$ we have the weak convergence
\[ hH_m(u_j) \rightharpoonup hH_m(u). \]

Proof. For every test function $\chi$ the function $h \chi$ is upper semicontinuous. Thus,
\[ \lim_{j \to +\infty} \int_{\Omega} (-h)\chi H_m(u_j) \geq \int_{\Omega} (-h)\chi H_m(u). \]
Let $\Theta$ be any cluster point of this the sequence $(-h)H_m(u_j)$. From the above inequality we infer that $\Theta \geq (-h)H_m(u)$. Moreover, it follows from Corollary 3.13 that the sequence $\int_{\Omega} (-h)H_m(u_j)$ increases to $\int_{\Omega} (-h)H_m(u)$. This implies that the total mass of $\Theta$ is less than or equal to the total mass of $(-h)H_m(u)$ and hence these measures are equal. \hfill \Box

Theorem 3.15. Let $u_1, \ldots, u_m \in \mathcal{F}_m^p(\Omega)$, $p > 0$ and $(g_1^j), (g_2^j), \ldots, (g_m^j) \subset \mathcal{E}_m^0(\Omega)$ be such that $g_k^j \downarrow u_k, \forall k = 1, \ldots, m$ and
\[ \sup_{j,k} E_p(g_k^j) < +\infty. \]
Then the sequence of measures $dd^c g_1^j \wedge dd^c g_2^j \wedge \ldots \wedge dd^c g_m^j \wedge \beta^{n-m}$ converges weakly to a positive Radon measure which does not depend on the choice of the sequences $(g_k^j)$. We then define $dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge \beta^{n-m}$ to be this weak limit.

Proof. Since the problem is local we can assume that $u_1, \ldots, u_m \in \mathcal{F}_m^p(\Omega)$. Indeed, let $K$ be a compact subset of $\Omega$. For each $j \in \mathbb{N}, k = 1, \ldots, m$ consider
\[ h_k^j := \sup \{ u \in \mathcal{S} \mathcal{H}_m(\Omega) / u \leq g_k^j \text{ on } K \}. \]
Then by using a standard balayage argument we see that $H_m(h_k^j)$ is supported in $K$. It follows that $h_k^j$ decreases to $v_k \in \mathcal{F}_m^p(\Omega)$. Observe also that $v_k = u_k$ on $K$. 

...
Now, fix $h \in E^0_m(\Omega)$. Then
\[ \int_\Omega hdd^c g_1^2 \wedge dd^c g_2^2 \wedge ... \wedge dd^c g_m^2 \wedge \beta^{n-m} \]
is decreasing to a finite number. Thus the limit $\lim_j \int_\Omega hdd^c g_1^2 \wedge dd^c g_2^2 \wedge ... \wedge dd^c g_m^2 \wedge \beta^{n-m}$ exists for every $h \in E^0_m(\Omega)$. In view of Lemma 3.10, this implies the weak convergence of the sequence
\[ dd^c g_1^2 \wedge dd^c g_2^2 \wedge ... \wedge dd^c g_m^2 \wedge \beta^{n-m}. \]
To prove the last statement it suffices to follow the lines in [8, Theorem 4.2]. \hfill \Box

### 3.3. Integration by parts and the comparison principle

In this section we prove that integration by parts is valid in the classes $E^p_m(\Omega)$, $p > 0$ and $F_m(\Omega)$. Following a classical argument of Cegrell in [8] we prove that the comparison principle holds in $E^p_m(\Omega)$ for $0 < p \leq 1$. To prove the comparison principle in $E^p_m(\Omega)$, $p > 1$ and in $F_m(\Omega)$ we need strong convergence results which will be established in the next section.

From Theorem 3.11 and Corollary 3.13 we prove the integration by parts formula for functions in $F_m(\Omega)$.

**Theorem 3.16.** Integration by parts is allowed in $F_m(\Omega)$, more precisely
\[ \int_\Omega u dd^c v \wedge T = \int_\Omega v dd^c u \wedge T, \]
where $u, v, \varphi_1, ..., \varphi_{m-1} \in F_m(\Omega)$ and $T = dd^c \varphi_1 \wedge ... \wedge dd^c \varphi_{m-1} \wedge \beta^{n-m}$ and the equality means that if one of the two terms are finite then they are equal.

**Proof.** Let $u_j, v_j, \varphi^j_1, ..., \varphi^j_{m-1}$ be sequences in $E^0_m(\Omega) \cap C(\Omega)$ decreasing to $u, v, \varphi_1, ..., \varphi_{m-1}$ respectively such that their total mass are uniformly bounded:
\[ \sup_j \int_\Omega dd^c v_j \wedge T_j < +\infty, \quad \sup_j \int_\Omega dd^c u_j \wedge T_j < +\infty, \]
where $T_j = dd^c \varphi^j_1 \wedge ... \wedge dd^c \varphi^j_{m-1} \wedge \beta^{n-m}$. Theorem 3.11 gives us that $dd^c u_j \wedge T_j \rightharpoonup dd^c u \wedge T$. For each fixed $k \in \mathbb{N}$ and any $j > k$ we have
\[ \int_\Omega v_k dd^c u_k \wedge T_k \geq \int_\Omega v_k dd^c u_j \wedge T_j \geq \int_\Omega v_j dd^c u_j \wedge T_j. \]
We then infer that the sequence of real numbers $\int_\Omega v_j dd^c u_j \wedge T$ decreases to some $a \in \mathbb{R} \cup \{-\infty\}$. By letting $j \to +\infty$ and using Corollary 3.13 we get
\[ \int_\Omega v_k dd^c u \wedge T \geq a, \]
from which we obtain $\int_\Omega v dd^c u \wedge T \geq a$. For each fixed $k$ we also have
\[ \int_\Omega v dd^c u \wedge T \leq \int_\Omega v_k dd^c u \wedge T = \lim_{j \to +\infty} \int_\Omega v_k dd^c u_j \wedge T_j \leq \int_\Omega v_k dd^c u_k \wedge T_k. \]
This implies that $\int_\Omega v dd^c u \wedge T = a$, from which the result follows. \hfill \Box
Let \( u \in \mathcal{E}_p^m(\Omega), p > 0 \). It is clear form the definition that the \( H_m(u) \) does not charge \( m \)-polar sets. One expects that \( H_m(u) \) is dominated by the Capacity. The following results tell us more about that.

**Lemma 3.17.** Let \( U \) be an open subset of \( \Omega \) and \( \varphi \in \mathcal{E}_m^0(\Omega), p \geq 1 \). Then

\[
\int_U H_m(\varphi) \leq \text{Cap}_m(U) \frac{p}{p+m} \text{Cap}_p(\varphi) \frac{1}{p+m}.
\]

**Proof.** We can suppose that \( U \) is relatively compact in \( \Omega \). Denote by \( u = u_{m,U,\Omega} \) the \( m \)-extremal function of \( U \) in \( \Omega \). Then \( u \in \mathcal{E}_m^0(\Omega) \) and \( u = -1 \) in \( U \). From Lemma 3.5 we have

\[
\int_U H_m(\varphi) \leq \int_\Omega (-u)^p H_m(\varphi) \leq \text{Cap}_p(u) \frac{p}{p+m} \text{Cap}_p(\varphi) \frac{1}{p+m}
\]

\[
\leq \left( \int_\Omega H_m(u) \right) \frac{p}{p+m} \text{Cap}_p(\varphi) \frac{1}{p+m} = \text{Cap}_m(U) \frac{p}{p+m} \text{Cap}_p(\varphi) \frac{1}{p+m}.
\]

\( \square \)

**Lemma 3.18.** Let \( U \subset \Omega \) be an open subset and \( \varphi \in \mathcal{E}_m^0(\Omega), 0 < p \leq 1 \). Then for each \( \epsilon > 0 \) small enough we have

\[
\int_U H_m(\varphi) \leq 2(\text{Cap}_m(U))^{1-m\epsilon} + 2\text{Cap}_m(U)^{p\epsilon} \text{Cap}_p(\varphi).
\]

**Proof.** Without loss of generality we can assume that \( U \subseteq \Omega \). Let \( u \) be the \( m \)-extremal function of \( U \) with respect to \( \Omega \). Put \( a = \text{Cap}_m(U) = \int_\Omega H_m(u) \). If \( a = 0 \), we are done. Thus, we can assume that \( a > 0 \). By applying Lemma 3.6 we obtain

\[
\int_U H_m(\varphi) \leq a^{p\epsilon} \int_\Omega (-u/a)^p H_m(\varphi) \leq 2a^{p\epsilon} \text{Cap}_p(u/a^\epsilon) + 2a^{p\epsilon} \text{Cap}_p(\varphi) \leq 2a^{1-m\epsilon} + 2a^{p\epsilon} \text{Cap}_p(\varphi).
\]

\( \square \)

The following result is the so-called **maximum principle**.

**Theorem 3.19.** Let \( u_1, ..., u_m \in \mathcal{E}_m^p(\Omega), p > 0 \) and \( v \in \mathcal{S}H_m(\Omega) \). Then

\[
\mathbb{I}_AH_m(u_1, ..., u_m) = \mathbb{I}_A H_m(\max(u_1, v), ..., \max(u_m, v)),
\]

where \( A = \cap_{j=1}^m \{ u_j > v \} \) and \( H_m(u_1, ..., u_m) = dd^c u_1 \wedge ... \wedge dd^c u_m \wedge \beta^{n-m} \).

**Proof.** Let \( (u_1^j, ..., u_m^j) \) be sequences in \( \mathcal{E}_m^p(\Omega) \) decreasing to \( u_1, ..., u_m \) respectively as in the definition of \( \mathcal{E}_m^p(\Omega) \). We can assume that they are continuous in \( \Omega \). Set \( v_k^j := \max(u_k^j, v), k = 1, ..., m \). Then since the set \( A_j := \cap_{k=1}^m \{ u_k^j > v \} \) is open, we get

\[
\mathbb{I}_A H_m(u_1^j, ..., u_m^j) = \mathbb{I}_A H_m(v_1^j, ..., v_m^j).
\]

Set \( w^j := \min(u_1^j, ..., u_m^j) \) and \( u := \min(u_1, ..., u_m) \). Consider \( \psi_j := \max(w^j - v, 0) \). Then \( \psi_j \downarrow \psi := \max(u - v, 0) \), all of them are quasi-continuous.

Fix \( \delta > 0 \) and set \( g_j := \frac{\psi_j}{\psi_j + \delta}, g = \frac{\psi}{\psi + \delta} \). By multiplying (3.3) with \( g_j \) we obtain

\[
g_j H_m(u_1^j, ..., u_m^j) = g_j H_m(v_1^j, ..., v_m^j).
\]
Now, let \( \chi \in C^\infty_0(\Omega) \) be a test function and fix \( \epsilon > 0 \). By Theorem 2.9, there exists an open subset \( U \subset \Omega \) such that \( \text{Cap}_m(U) < \epsilon \), and there exist \( \varphi_j, \varphi \) continuous functions in \( \Omega \) which coincide with \( \psi_j, \psi \) respectively on \( K := \Omega \setminus U \). The monotone convergence \( \psi_j \downarrow \psi \) implies that \( \varphi_j \) converges uniformly to \( \varphi \) on \( K \cap \text{Supp} \chi \), which in turn implies the uniform convergence of \( h_j = \frac{\varphi_j}{\varphi_j + \delta} \) on \( K \cap \text{Supp} \chi \) to \( h = \frac{\varphi}{\varphi + \delta} \).

In the next arguments, we let \( C \) denote a positive constant which does not depend on \( j, \epsilon \). Since \( g_j, h_j \) are uniformly bounded, Lemma 3.17 and Lemma 3.18 give us

\[
\int_{\Omega} | \chi g_j H_m(u_1^j, \ldots, u_m^j) - \int_{\Omega} \chi h_j H_m(u_1^j, \ldots, u_m^j) | \leq C \int_{U} H_m(u_1^j, \ldots, u_m^j) \leq C \epsilon^q,
\]

where \( q \) is some positive constant. The last inequality follows since \( H_m(u_1, \ldots, u_m) \leq H_m(u_1 + \ldots + u_m) \) and since \( E_m^\infty(\Omega) \) is convex. We also obtain

\[
\int_{\Omega} | \chi g H_m(u_1, \ldots, u_m) - \int_{\Omega} \chi h H_m(u_1, \ldots, u_m) | \leq C \int_{U} H_m(u_1 + \ldots + u_m) \leq C \epsilon^q.
\]

Moreover, since \( h \) is continuous on \( \Omega \) and \( H_m(u_1^j, \ldots, u_m^j) \to H_m(u_1, \ldots, u_m) \), we get

\[
\lim_{j \to +\infty} \int_{\Omega} \chi h(H_m(u_1^j, \ldots, u_m^j) - H_m(u_1, \ldots, u_m)) = 0.
\]

Hence, we obtain

\[
\limsup_{j \to +\infty} \left| \int_{\Omega} \chi h_j H_m(u_1^j, \ldots, u_m^j) - \int_{\Omega} \chi h H_m(u_1, \ldots, u_m) \right| \leq \limsup_{j \to +\infty} \int_{\Omega} \chi | h_j - h | H_m(u_1^j, \ldots, u_m^j).
\]

Since \( h_j \) converges uniformly to \( h \) on \( K \cap \supp \chi \), we have

\[
\int_{\Omega} \chi | h_j - h | H_m(u_1^j, \ldots, u_m^j) = \int_{K} \chi | h_j - h | H_m(u_1^j, \ldots, u_m^j)
+ \int_{K} \chi | h_j - h | H_m(u_1^j, \ldots, u_m^j)
\leq C \int_{U} H_m(u_1^j, \ldots, u_m^j)
+ \| h_j - h \|_{L^\infty(K \cap \supp \chi)} \int_{\Omega} \chi H_m(u_1^j, \ldots, u_m^j).
\]

From the two inequalities above we get

\[
\limsup_{j \to +\infty} \left| \int_{\Omega} \chi g_j H_m(u_1^j, \ldots, u_m^j) - \int_{\Omega} \chi g H_m(u_1, \ldots, u_m) \right| \leq C \epsilon^q,
\]

where \( q \) is some positive constant. From (3.5), (3.6) and (3.7), we see that

\[
\limsup_{j} \left| \int_{\Omega} \chi g_j H_m(u_1^j, \ldots, u_m^j) - \int_{\Omega} \chi g H_m(u_1, \ldots, u_m) \right| \leq C \epsilon^q.
\]

We then see that \( g_j H_m(u_1^j, \ldots, u_m^j) \to g H_m(u_1, \ldots, u_m) \). In the same way, we get

\[
g_j H_m(v_1^j, \ldots, v_m^j) \to g H_m(v_1, \ldots, v_m),
\]

and hence \( g H_m(u_1, \ldots, u_m) = g H_m(v_1, \ldots, v_m) \). The result follows by letting \( \delta \) go to zero. \( \square \)
We now return to the integration by parts formula in the class $\mathcal{E}_m^p(\Omega)$, $p > 0$. We first need the following convergence result.

**Corollary 3.20.** Let $u_1, \ldots, u_m \in \mathcal{E}_m^p(\Omega)$, $p > 0$ and $u_1', \ldots, u_m'$ be sequences of functions in $\mathcal{E}_m^0(\Omega)$ decreasing to $u_1, \ldots, u_m$ respectively such that

$$\sup_{j,k} \int_{\Omega} (-u'_k)^p H_m(u'_k) < +\infty.$$ 

Then for each $\varphi \in \mathcal{E}_m^0(\Omega)$ we have

$$\lim_{j \to +\infty} \int_{\Omega} \varphi \, dd^c u'_1 \wedge \ldots \wedge dd^c u'_m \wedge \beta^{n-m} = \int_{\Omega} \varphi \, dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge \beta^{n-m}.$$ 

**Proof.** Without loss of generality we can assume that the right hand side is finite. Since if it is $-\infty$ then the equality is obvious. We can also assume that $-1 \leq \varphi \leq 0$. We will use a truncation argument. For each $k, j \in \mathbb{N}$ set $u_{l,k} := \max(u_{l,k}^0, k\psi)$, and $u_{l,k} := \max(u_l, k\psi)$, $l = 1, \ldots, m$. Here we set $\psi := -(-\varphi)^q$, where $q = \min(1, 1/p)$.

We claim that, for any $k$,

$$\lim_{j \to +\infty} \int_{\Omega} (-\varphi) H_m(u_{l,k}^0, \ldots, u_{m,k}^0) = \int_{\Omega} (-\varphi) H_m(u_1, \ldots, u_m).$$

Indeed, the inequality "$\geq"$ follows from the fact that the sequence of Hessian measures converges and $-\varphi$ is lower semi continuous. Moreover, it follows from Theorem 2.11 that we can integrate by parts in the right hand side, which implies the inequality "$\leq"$. Thus, the Claim is proved.

Thus, it is enough to prove that

$$\left| \int_{\Omega} (-\varphi) H_m(u_{l,k}^0, \ldots, u_{m,k}^0) - \int_{\Omega} (-\varphi) H_m(u_1, \ldots, u_m) \right| \leq \epsilon(k),$$

where $0 < \epsilon(k) \to 0$ as $k \to +\infty$ and (of course) $\epsilon(k)$ does not depend on $j$. If we can prove it then the same estimate holds for the limit functions $u_1, \ldots, u_m$ and we are done.

In the following arguments we use $C_1, C_2, \ldots$ to denote positive constants that do not depend on $j, k$. By Theorem 3.19 and since $\varphi$ is bounded it suffices to estimate

$$\int_{\{u_l' \leq k\psi\}} (-\varphi) H_m(u_{l,k}^0, \ldots, u_{m,k}^0), \quad l = 1, \ldots, m.$$ 

But we can bound this term by using Lemma 3.5 (for $p \geq 1$), Proposition 3.7 (for $0 < p < 1$) and the fact that the $p$-energy of these functions are uniformly bounded:

$$\int_{\{u_l' \leq k\psi\}} (-\varphi) H_m(u_{l,k}^0, \ldots, u_{m,k}^0) \leq \frac{1}{k^p} \int_{\Omega} (-u_{l,k}^0)^p H_m(u_{l,k}^0, \ldots, u_{m,k}^0) \leq \frac{C_1}{k^p}.$$ 

In the last step the constant $C_1$ depends on the $p$-energy of $u_{l,k}^0, l = 1, \ldots, m$. Note also that $u_{l,k}^0 \geq u_l^0$. Now, for functions $u, v \in \mathcal{E}_m(\Omega)$ such that $u \leq v$ we always have $E_p(v) \leq C_2 E_p(u)$ (where $C_2$ does not depend on $u, v$). To see this we can use integration by parts (if $p < 1$) or use Lemma 3.5. This explains why the constant $C_1$ in the above estimate does not depend on $k$. Thus, the proof is complete.  \[\square\]
**Theorem 3.21.** Integration by parts is allowed in $\mathcal{E}_m^p(\Omega), p > 0$. More precisely, assume that $u, v \in \mathcal{E}_m^p(\Omega)$ and $T$ is a closed $m$-positive current of type $T = \text{dd}^c \varphi_1 \wedge ... \text{dd}^c \varphi_{m-1} \wedge \beta^{n-m}$, where $\varphi_j \in \mathcal{E}_m^p(\Omega), \forall j$. Then

$$\int_\Omega u \text{dd}^c v \wedge T = \int_\Omega v \text{dd}^c u \wedge T,$$

where the equality means that if one of the two terms is finite then so is the other and they are equal.

**Proof.** Thanks to Corollary 3.20 the same arguments as in the proof of Theorem 3.16 can be used here. \(\square\)

**Theorem 3.22.** Let $u, v \in \mathcal{E}_m^p(\Omega), p > 0$ (or $\mathcal{F}_m(\Omega)$) such that $u \leq v$ on $\Omega$. Then

$$\int_\Omega H_m(u) \geq \int_\Omega H_m(v).$$

**Proof.** Let $(u_j), (v_j)$ be two sequences in $\mathcal{E}_m^0(\Omega)$ decreasing to $u, v$ as in the definition of $\mathcal{E}_m^p(\Omega)$. Fix $h \in \mathcal{E}_m^0(\Omega) \cap C(\Omega)$. We can suppose that $u_j \leq v_j, \forall j$. Integrating by parts we get

$$\int_\Omega (-h)H_m(v_j) \leq \int_\Omega (-h)H_m(u_j).$$

Corollary 3.13 and Corollary 3.20 then yield

$$\lim_{j \to \infty} \int_\Omega (-h)H_m(v_j) = \int_\Omega (-h)H_m(v), \quad \text{and} \quad \lim_{j \to \infty} \int_\Omega (-h)H_m(u_j) = \int_\Omega (-h)H_m(u).$$

Combining them we obtain

$$\int_\Omega (-h)H_m(v_j) \leq \int_\Omega (-h)H_m(u).$$

The result follows by letting $h$ decrease to $-1$. \(\square\)

**Theorem 3.23.** If $u \in \mathcal{E}_m^p(\Omega), p > 0$, then $E_p(u) := \int_\Omega (-u)^p H_m(u) < +\infty$. If $(u_0^1), ... (u_0^n)$ are sequences in $\mathcal{E}_m^0(\Omega)$ decreasing to $u_0, ..., u_m \in \mathcal{E}_m^p(\Omega)$ respectively then

$$\int_\Omega (-u_0^j) \text{dd}^c u_0^1 \wedge ... \text{dd}^c u_0^j \wedge \beta^{n-m} \geq \int_\Omega (-u_0) \text{dd}^c u_1 \wedge ... \text{dd}^c u_m \wedge \beta^{n-m}.$$

**Proof.** Let $(u_j)$ be a sequence in $\mathcal{E}_m^0(\Omega)$ decreasing to $u$ and having uniformly bounded $p$-energy. Then

$$\int_\Omega (-u)H_m(u) \leq \liminf_{j \to +\infty} \int_\Omega (-u_j)H_m(u_j) < +\infty.$$

We now prove the second statement. We can assume that the sequences have uniformly bounded $p$-energy. It follows from Theorem 3.15 that

$$T_j := \text{dd}^c u_0^1 \wedge ... \text{dd}^c u_0^j \wedge \beta^{n-m} \rightarrow T := \text{dd}^c u_1 \wedge ... \text{dd}^c u_m \wedge \beta^{n-m}.$$

Furthermore since $(-u_0^j) \uparrow (-u_0)$ and since all of them are lower semicontinuous, we have

$$\liminf_{j} \int_\Omega (-u_0^j)T_j \geq \int_\Omega (-u_0^j)T.$$
Thus, it suffices to prove that
\[
\int_{\Omega} (-u_0)T_j \leq \int_{\Omega} (-u_0)T, \forall j.
\]
But it can be easily seen by integrating by parts thanks to Theorem 3.21. The proof is thus complete. \(\square\)

**Theorem 3.24.** If \(0 < p \leq 1\) and \(u, v \in \mathcal{E}^p_m(\Omega)\) then
\[
\int_{\{u>v\}} H_m(u) \leq \int_{\{u>v\}} H_m(v).
\]

**Proof.** Fix \(h \in \mathcal{E}^0_m(\Omega) \cap C(\Omega)\). The measure \(H_m(v)\) does not charge \(m\)-polar sets. We can easily show that for almost every \(r\),
\[
\int_{\{v=ru\}} (-h)H_m(v) = 0.
\]
This allows us to restrict ourself to the case \(\int_{\{u=v\}} (-h)H_m(v) = 0\). From Theorem 3.19, we get
\[
\mathbb{I}_{\{u>v\}} H_m(u) = \mathbb{I}_{\{u>v\}} H_m(\max(u, v)), \text{ and } \mathbb{I}_{\{u<v\}} H_m(v) = \mathbb{I}_{\{u<v\}} H_m(\max(u, v)).
\]
Furthermore, as in the proof of Theorem 3.22, we can prove that
\[
\int_{\Omega} (-h)H_m(\max(u, v)) \leq \int_{\Omega} (-h)H_m(u).
\]
From this we get
\[
\int_{\{u>v\}} (-h)H_m(u) = \int_{\{u>v\}} (-h)H_m(\max(u, v)) \leq \int_{\Omega} (-h)H_m(\max(u, v)) + \int_{\{u<v\}} hH_m(\max(u, v)) \leq \int_{\Omega} (-h)H_m(v) + \int_{\{u<v\}} hH_m(v) = \int_{\{u>v\}} (-h)H_m(v).
\]
The above arguments hold since all terms are finite. This is no longer true if \(p > 1\). Now, letting \(h \downarrow -1\) we obtain the result. \(\square\)

**Remark 3.25.** We proved in the above arguments that
\[
\int_{\{u>v\}} (-h)H_m(v) \leq \int_{\{u>v\}} (-h)H_m(u)
\]
if \(u, v \in \mathcal{E}^p_m(\Omega), 0 < p \leq 1\) and \(h \in \mathcal{E}^0_m(\Omega) \cap C(\Omega)\). Thanks to the regularization theorem (Theorem 3.1) it also holds for every \(h \in \mathcal{S}H^-_m(\Omega)\).

**Theorem 3.26.** Let \(u, v \in \mathcal{E}^p_m(\Omega), 0 < p \leq 1\), such that \(H_m(u) \geq H_m(v)\). Then \(u \leq v\) in \(\Omega\).

**Proof.** See [7, Theorem 4.5]. \(\square\)
4. The variational approach

In this section we use a variational method to solve the equation $H_m(u) = \mu$, where $\mu$ is a positive Radon measure. We characterize the range of $H_m(u)$ when $u$ runs in $E^m_\infty(\Omega)$.

Our results are direct generalizations of the classical case of plurisubharmonic functions (see [2], [7, 8]). The variational approach for the complex Monge-Ampère equation was first introduced in [5].

4.1. The energy functional. We recall some useful results obtained from previous sections. For $\varphi \in E^1_m(\Omega)$, we define its energy by $E(\varphi) := \int_\Omega (-\varphi) H_m(\varphi)$.

- If $0 \geq u_j \downarrow u$ and $u \in E^1_m(\Omega)$, then by Theorem 3.23, we have $E(u_j) \uparrow E(u)$.
- If $u, v \in E^1_m(\Omega)$ and $u \leq v$ then $E(u) \geq E(v)$.

**Lemma 4.1.**
(i) If $(u_j) \subset E^1_m(\Omega)$ then $(\sup_j u_j)^* \in E^1_m(\Omega)$.

(ii) If $(u_j) \in E^1_m(\Omega)$ such that $\sup_j E(u_j) < +\infty$ and $u_j \downarrow u$, then $u \in E^1_m(\Omega)$.

(iii) For each $C > 0$, $E^1_{m,C} := \{ u \in E^1_m(\Omega) : E(u) \leq C \}$ is convex and compact in $\mathcal{SH}_m(\Omega)$.

**Proof.**
(i) Let $(\varphi_j)$ be a sequence of continuous functions in $E^0_m(\Omega)$ decreasing to $\varphi := (\sup_j u_j)^*$. Since $u_j \leq \varphi_j$, we have $\sup_j E(\varphi_j) < +\infty$, which implies that $\varphi \in E^0_m(\Omega)$.

(ii) Let $(\varphi_j)$ be a sequence in $E^0_m(\Omega) \cap C(\Omega)$ decreasing to $u$. Set $\psi_j := \max(u_j, \varphi_j)$. Then $\psi_j \in E^1_m(\Omega)$, $\forall j$ and $E(\psi_j) \leq E(u_j)$. Thus, $u \in E^1_m(\Omega)$.

(iii) Let $(u_j)$ be a sequence in $E^1_{m,C}$. Since $\sup_j E(u_j) < +\infty$, $(u_j)$ can not go uniformly to $-\infty$ in $\Omega$. Thus, there exists a subsequence (still denoted by $(u_j)$) converging to $u \in \mathcal{SH}_m(\Omega)$ in $L^1_{\text{loc}}(\Omega)$. Set

$$\varphi_j := (\sup_{k \geq j} u_k)^* \in E^1_m(\Omega), \forall j.$$  

Then $\varphi_j \downarrow u$ and $\sup_j E(\varphi_j) \leq C$. In view of (ii), we have $u \in E^1_m(\Omega)$, and since $(-\varphi_j) \uparrow (-u)$, all of them being lower semicontinuous we get, for each fixed $k \in \mathbb{N}$,

$$C \geq \liminf_{j \to \infty} \int_\Omega (-\varphi_j) H_m(\varphi_j) \geq \liminf_{j \to \infty} \int_\Omega (-\varphi_k) H_m(\varphi_j) \geq \int_\Omega (-\varphi_k) H_m(u).$$

By monotone convergence Theorem we see that $E_1(u) \leq C$. This means $u \in E^1_{m,C}$. □

**Lemma 4.2.** Let $\mu$ be a positive Radon measure in $\Omega$ such that $\mu(\Omega) < +\infty$ and $\mu$ does not charge m-polar sets. Let $(u_j)$ be a sequence in $\mathcal{SH}^m_\infty(\Omega)$ which converges in $L^1_{\text{loc}}$ to $u \in \mathcal{SH}_m(\Omega)$. If $\sup_j \int_\Omega (-u_j)^2 d\mu < +\infty$ then $\int_\Omega u_j d\mu \to \int_\Omega u d\mu$.

**Proof.** Since $\int_\Omega u_j d\mu$ is bounded it suffices to prove that every cluster point is $\int_\Omega u d\mu$. Without loss of generality we can assume that $\int_\Omega u_j d\mu$ converges. Since the sequence $u_j$ is bounded in $L^2(\mu)$, one can apply Banach-Saks theorem to extract a subsequence (still denoted by $u_j$) such that

$$\varphi_N := \frac{1}{N} \sum_{j=1}^N u_j.$$
converges in $L^2(\mu)$ and $\mu$-almost everywhere to $\varphi$. Observe also that $\varphi_N \to u$ in $L^1_{loc}$. For each $j \in \mathbb{N}$ set
\[ \psi_j := (\sup_{k \geq j} \varphi_k)^*. \]
Then $\psi_j \downarrow u$ in $\Omega$. But $\mu$ does not charge the $m$-polar set $\{(\sup_{k \geq j} \varphi_k)^* > \sup_{k \geq j} \varphi_k\}$. We thus get $\psi_j = \sup_{k \geq j} \varphi_k$ $\mu$-almost everywhere. Therefore, $\psi_j$ converges to $\varphi$ $\mu$-almost everywhere hence $u = \varphi$ $\mu$-almost everywhere. This yields
\[ \lim_j \int_{\Omega} u_j d\mu = \lim_j \int_{\Omega} \varphi_j d\mu = \int_{\Omega} u d\mu. \]

**Lemma 4.3.** The functional $E : E_m^1(\Omega) \to \mathbb{R}$ is lower semicontinuous.

*Proof.* Suppose that $u, u_j \in E_m^1(\Omega)$ and $u_j$ converges to $u$ in $L^1_{loc}(\Omega)$. We are to prove that $\liminf_j E(u_j) \geq E(u)$. For each $j \in \mathbb{N}$, the function
\[ \varphi_j := (\sup_{k \geq j} u_k)^* \]
belongs to $E_m^1(\Omega)$ and $\varphi_j \downarrow u$. Hence $E(\varphi_j) \uparrow E(u)$. We also have $E(u_j) \geq E(\varphi_j)$ from which the result follows. \hfill \Box

**Definition 4.4.** Let $\mu$ be a positive Radon measure in $\Omega$. The functional $F_\mu : E_m^1(\Omega) \to \mathbb{R}$ is defined by
\[ F_\mu(u) = \frac{1}{m+1} E(u) + L_\mu(u), \]
where $L_\mu(u) = \int_{\Omega} u d\mu$. We say that $F_\mu$ is proper (with respect to $E$) if $F_\mu \to +\infty$ whenever $E \to +\infty$.

**Definition 4.5.** For each $p > 0$, let $\mathcal{M}_p$ denote the set of all positive Radon measures $\mu$ in $\Omega$ such that $E_m^p(\Omega) \subset L^p(\mu)$. 

**Proposition 4.6.** Let $\mu$ be a positive Radon measure on $\Omega$ and $p > 0$. Then $\mu \in \mathcal{M}_p$ if and only if there exists a positive constant $C = C(p) > 0$ such that
\[ \int_{\Omega} (-u)^p d\mu \leq C E_p(u)^{\frac{p}{m+p}}, \forall u \in E_m^p(\Omega). \]

*Proof.* The "if" statement is evident. To prove the "only if", suppose by contradiction that $\mu \in \mathcal{M}_p$ and there exists a sequence $(u_j) \subset E_m^p(\Omega)$ such that
\[ \int_{\Omega} (-u_j)^p d\mu \geq 4^{jp} E_p(u_j)^{\frac{p}{m+p}}. \]
For simplicity we can assume that $E_p(u_j) = 1$, $\forall j$. By Corollary 3.8, $v = \sum_{j=1}^{\infty} 2^{-j} v_j$ belongs to $E_m^p(\Omega)$. But
\[ \int_{\Omega} (-v)^p d\mu \geq \int_{\Omega} (-2^{-j} v_j)^p d\mu \geq 2^{jp} \to +\infty, \]
which contradicts $E_m^p(\Omega) \subset L^p(\mu)$. \hfill \Box
Remark 4.7. If \( u, v \in E^p_m(\Omega) \) and \( (u_j), (v_j) \subset E^p_m(\Omega) \) decrease to \( u, v \) respectively, then by Lemma 3.5 and Proposition 3.7 we have
\[
\int_{\Omega} (-u)^p H_m(v) \leq \liminf_j \int_{\Omega} (-u_j)^p H_m(v_j) < +\infty.
\]
Thus, \( E^p_m(\Omega) \subset L^p(H_m(v)) \) and by Proposition 4.6 there exists \( C_v > 0 \) such that
\[
\int_{\Omega} (-u)^p H_m(v) \leq C_v E_p(u)^{\frac{1}{m}} \mu, \quad \forall u \in E^p_m(\Omega).
\]
It is not clear how to obtain this inequality directly by using Hölder inequality.

Lemma 4.8. If \( u, v \in E^1_m(\Omega) \) then \( E(u + v)^{\frac{1}{m+1}} \leq E(u)^{\frac{1}{m+1}} + E(v)^{\frac{1}{m+1}} \). Moreover, if \( \mu \in M_1 \) then \( \mathcal{F}_\mu \) is convex and proper.

Proof. It follows from Lemma 3.5 that
\[
E(u + v) \leq E(u)^{\frac{1}{m+1}} + E(v)^{\frac{1}{m+1}} \mu + E(u + v)^{\frac{1}{m+1}} \mu,
\]
which implies that \( E^{\frac{1}{m+1}} \) is convex since it is homogeneous of degree 1. So, \( E \) is also convex. If \( \mu \) belongs to \( M_1 \), there exists \( A > 0 \) such that
\[
\|u\|_{L^{\frac{1}{\mu}}(\mu)} \leq A E(u)^{\frac{1}{m}} , \quad \text{for every } u \in E^1_m(\Omega).
\]
We thus obtain
\[
\mathcal{F}_\mu(u_j) = \frac{1}{m+1} E(u_j) - \|u_j\|_{L^{\frac{1}{\mu}}(\mu)} \geq \frac{1}{m+1} E(u_j) - A E(u_j)^{\frac{1}{m}} \to \infty.
\]

Let \( u : \Omega \to \mathbb{R} \cup \{-\infty\} \) be an upper semicontinuous function. Suppose that there exists \( w \in E^1_m(\Omega) \) such that \( w \leq u \). We define the projection of \( u \) on \( E^1_m(\Omega) \) by
\[
P(u) := \sup\{v \in E^1_m(\Omega) / v \leq u\}.
\]
Using a standard balayage argument, we obtain the following result.

Lemma 4.9. Let \( u : \Omega \to \mathbb{R} \) be a continuous function. Suppose that there exists \( w \in E^1_m(\Omega) \) such that \( w \leq u \). Then
\[
\int_{\{P(u) < u\}} H_m(P(u)) = 0.
\]

Proof. Without loss of generality we can assume that \( w \) is bounded. From Choquet’s lemma, there exists an increasing sequence \( (u_j) \subset E^1_m(\Omega) \cap L^\infty(\Omega) \) such that
\[
(\lim_j u_j)^* = P(u).
\]
Let \( x_0 \in \{P(u) < u\} \). Since \( u \) is continuous, there exists \( \epsilon > 0, r > 0 \) such that
\[
P(u)(x) < u(x_0) - \epsilon < u(x), \quad \forall x \in B = B(x_0, r).
\]
For each fixed \( j \), by approximating \( u_j|_{\partial B} \) from above by a sequence of continuous functions on \( \partial B \) and by using [9, Theorem 2.10], we can find a function \( \varphi_j \in SH_m(B) \) such that \( \varphi_j = u_j \) on \( \partial B \) and \( H_m(\varphi_j) = 0 \) in \( B \). The comparison principle gives us that \( \varphi_j \geq u_j \) in \( B \). The function \( \psi_j \), defined by \( \psi_j = \varphi_j \) in \( B \) and \( \psi_j = u_j \) in \( \Omega \setminus B \), belongs to \( E^1_m(\Omega) \cap L^\infty(\Omega) \). For each \( x \in \partial B \) we have \( \varphi_j(x) = u_j(x) \leq
Let $M$ be a positive constant which depends only on $m$, and let $u, v \in E^1_m(\Omega)$ be a sequence decreasing to $u$, then for each $t < 0$, we have
\[
\lim_{t \to 0} \int_{\Omega} h_t(\partial^c u)^k \wedge (\partial^c P(u + tv))^{m-k} \wedge \beta^{n-m} = 0
\]
In particular,
\[
\lim_{t \to 0} \int_{\Omega} \frac{P(u + tv) - u}{t} \partial^c u \wedge (\partial^c P(u + tv))^{m-k} \wedge \beta^{n-m} = \int_{\Omega} v H_m(u).
\]

Proof. An easy computation shows that $h_t$ is decreasing in $t$ and $0 \leq h_t \leq -v$. For each fixed $s > 0$ we have
\[
\lim_{t \to 0} \int_{\Omega} h_t(\partial^c u)^k \wedge (\partial^c P(u + tv))^{m-k} \wedge \beta^{n-m}
\]
\[
\leq \lim_{t \to 0} \int_{\Omega} h_s(\partial^c u)^k \wedge (\partial^c P(u + tv))^{m-k} \wedge \beta^{n-m}
\]
\[
= \int_{\Omega} h_s(\partial^c u)^m \wedge \beta^{n-m} \leq \int_{\{P(u + sv) - sv < u\}} (-v)(\partial^c u)^m \wedge \beta^{n-m}.
\]
Let $u_k \in E^0_m(\Omega) \cap C(\Omega)$ be a sequence decreasing to $u$ such that
\[
\int_{\{P(u + sv) - sv < u\}} (-v)(\partial^c u)^m \wedge \beta^{n-m} \leq 2 \int_{\{P(u_k + sv) - sv < u_k\}} (-v)(\partial^c u)^m \wedge \beta^{n-m}.
\]
Taking into account Remark 3.25 and Lemma 4.9 we can conclude that
\[
\int_{\{P(u_k + sv) - sv < u\}} (-v)(\partial^c u)^m \wedge \beta^{n-m}
\]
\[
\leq \int_{\{P(u_k + sv) - sv < u_k\}} (-v)(\partial^c (P(u_k + sv) - sv))^{m} \wedge \beta^{n-m}
\]
\[
\leq -sM \to 0, \quad \text{as} \ s \to 0.
\]
Here, $M$ is a positive constant which depends only on $m$, $||v||$, and $\int_{\Omega} v H_m(u + v)$. Equality (4.2) follows from equality (4.1). The proof is thus complete. □

Lemma 4.10. Let $u, v \in E^1_m(\Omega)$ and suppose that $v$ is continuous. For each $t < 0$, we define
\[
h_t = \frac{P(u + tv) - tv - u}{t}.
\]
Then for each $0 \leq k \leq m$,
\[
\lim_{t \to 0} \int_{\Omega} h_t(\partial^c u)^k \wedge (\partial^c P(u + tv))^{m-k} \wedge \beta^{n-m} = 0.
\]

Proof. Let $u, v \in E^1_m(\Omega)$ and suppose that $v$ is continuous, then for each $t < 0$, we have
\[
h_t = \frac{P(u + tv) - tv - u}{t}.
\]
It follows from Theorem 2.10 that $H_m(\psi_j) \to H_m(P(u))$. Therefore,
\[
H_m(P(u))(B) \leq \liminf_{j \to +\infty} H_m(\psi_j)(B) = 0,
\]
from which the result follows. □
Proof. If $t > 0$, $P(u + tv) = u + tv$. It is easy to see that

$$\frac{d}{dt}igg|_{t=0^+} E(P(u + tv)) = \int_\Omega (-v)H_m(u).$$

To compute the left-derivative observe that

$$\frac{1}{t} \left( \int_\Omega (-P(u + tv))(dd^c P(u + tv))^m \wedge \beta^{n-m} - \int_\Omega (-u)(dd^c u)^m \wedge \beta^{n-m} \right)$$

$$= \sum_{k=0}^m \int_\Omega \frac{u - P(u + tv)}{t}(dd^c u)^k \wedge (dd^c P(u + tv))^{m-k} \wedge \beta^{n-m}.$$ 

It suffices to apply Lemma 4.10. \qed

4.2. Resolution. In this section we use the variational formula established above to solve the equation $H_m(u) = \mu$ in finite energy classes of Cegrell type, where $\mu$ is a positive Radon measure. Our main results represented in the introduction follow from these theorems. The following lemma is important for the sequel.

Lemma 4.12. Let $\mu$ be a positive Radon measure such that $F_\mu$ is proper and lower semicontinuous on $E^1_m(\Omega)$. Then there exists $\varphi \in E^1_m(\Omega)$ such that

$$F_\mu(\varphi) = \inf_{\psi \in E^1_m(\Omega)} F_\mu(\psi).$$

Proof. Let $(\varphi_j) \subset E^1_m(\Omega)$ be such that

$$\lim_j F_\mu(\varphi_j) = \inf_{\psi \in E^1_m(\Omega)} F_\mu(\psi) \leq 0.$$

From the properness of the functional $F_\mu$, we obtain $\sup_j E(\varphi_j) < +\infty$. It follows that the sequence $(\varphi_j)$ forms a compact subset of $E^1_m(\Omega)$. Hence there exists a subsequence (still denoted by $(\varphi_j)$) such that $\varphi_j$ converges to $\varphi$ in $L^1_{\text{loc}}(\Omega)$. Since $F_\mu$ is lower semicontinuous we have

$$\liminf_{j \to \infty} F_\mu(\varphi_j) \geq F_\mu(\varphi).$$

We then deduce that $\varphi$ is a minimum point of $F_\mu$ on $E^1_m(\Omega)$. \qed

We now prove a Dirichlet principle.

Theorem 4.13. Let $\varphi \in E^1_m(\Omega)$ and $\mu \in M_1$. Then $H_m(\varphi) = \mu \iff F_\mu(\varphi) = \inf_{\psi \in E^1_m(\Omega)} F_\mu(\psi)$.

Proof. Assume first that $H_m(\varphi) = \mu$. Let $\psi \in E^1_m(\Omega)$. By Lemma 3.5 and Hölder inequality we get

$$\int_\Omega (-\psi)H_m(\varphi) \leq E(\psi)^{1/(1+m)}E(\varphi)^{m/(1+m)} \leq \frac{1}{m+1}E(\psi) + \frac{m}{m+1}E(\varphi).$$

We then easily obtain $F_\mu(\psi) \geq F_\mu(\varphi)$.

Now, assume that $\varphi$ minimizes $F_\mu$ on $E^1_m(\Omega)$. Let $\psi$ be a continuous function in $E^1_m(\Omega)$ and consider the function $g(t) = E(P(\varphi + t\psi)) + \mathcal{L}_\mu(\varphi + t\psi)$, $t \in \mathbb{R}$. Since $P(\varphi + t\psi) \leq \varphi + tv$, we have that

$$g(t) \geq F_\mu(P(\varphi + tv)) \geq F_\mu(\varphi) = g(0), \forall t.$$
It follows that $g$ attains its minimum at $t = 0$, hence $g'(0) = 0$. Since $\mu \in \mathcal{M}_1$, $\mathcal{L}_\mu$ is finite on $\mathcal{E}^1_m(\Omega)$ which implies that

$$\frac{d}{dt}\mathcal{L}_\mu(\varphi + t\psi) = \mathcal{L}_\mu(\varphi).$$

This coupled with Lemma 4.11 yields

$$\int_\Omega \psi H_m(\varphi) = \int_\Omega \psi d\mu.$$  

The test function $\psi$ is taken arbitrarily, so it follows that $\mu = H_m(\varphi)$. $\Box$

**Theorem 4.14.** Let $\mu$ be a positive Radon measure such that $\mathcal{E}^1_m(\Omega) \subset L^1(\Omega, \mu)$. Then there exists a unique $u \in \mathcal{E}^1_m(\Omega)$ such that $H_m(u) = \mu$.

**Proof.** The uniqueness follows from the comparison principle.

We prove the existence. Suppose first that $\mu$ has compact support $K \Subset \Omega$, and let $h_K := h^*_{m,K,\Omega}$ denote the $m$-extremal function of $K$ with respect to $\Omega$. Set

$$\mathcal{M} = \left\{ \nu \geq 0 \mid \text{supp}(\nu) \subset K, \int_\Omega (-\varphi)^2 d\nu \leq C.E(\varphi)^{\frac{2}{m+1}} \text{ for every } \varphi \in \mathcal{E}^1_m(\Omega) \right\},$$

where $C$ is a fixed constant such that $C > 2E(h_K)^{\frac{2}{m+1}}$. For each compact $L \subset K$, we have $h_K \leq h_L$. We deduce that $E(h_L) \leq E(h_K)$. Therefore, for every $\varphi \in \mathcal{E}^1_m(\Omega)$, we have

$$\int_\Omega (-\varphi)^2 H_m(h_L) \leq 2\|h_L\| \int_\Omega (-\varphi)(dd^c \varphi) \wedge (dd^c h_L)^{m-1} \wedge \omega^{n-m}$$

$$\leq 2 \left( \int_\Omega (-\varphi) H_m(\varphi) \right)^{\frac{2}{m+1}} \left( \int_\Omega (-h_L) H_m(h_L) \right)^{\frac{m+1}{m+1}}$$

$$\leq C.E(\varphi)^{\frac{2}{m+1}}.$$  

This implies that $H_m(h_L) \in \mathcal{M}$ for every compact $L \subset K$.

Put $T = \sup\{\nu(\Omega) / \nu \in \mathcal{M}\}$. We claim that $T < +\infty$. In fact, since $\Omega$ is $m$-hyperconvex, there exists $h \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}(\Omega)$ such that $K \Subset \{h < -1\} \Subset \Omega$. For each $\nu \in \mathcal{M}$, we have

$$\nu(K) \leq \int_K (-h) d\nu \leq C.E(h)^{\frac{2}{m+1}},$$

from which the claim follows.

Fix $\nu_0 \in \mathcal{M}$ such that $\nu_0(\Omega) > 0$. Let $\mathcal{M}'$ denote the set of all probability measures $\nu$ in $\Omega$ supported in $K$ such that

$$\int_\Omega (-\varphi)^2 d\nu \leq \left( \frac{C}{T} + \frac{C}{\nu_0(\Omega)} \right) E(\varphi)^{\frac{2}{m+1}}, \quad \forall \varphi \in \mathcal{E}^1_m(\Omega).$$

Then, for each $\nu \in \mathcal{M}$ and $\varphi \in \mathcal{E}^1_m(\Omega),$

$$\int_\Omega (-\varphi)^2 \frac{(T - \nu(\Omega))d\nu_0 + \nu(\Omega)d\nu}{Tv_0(\Omega)} \leq \frac{T - \nu(\Omega)}{Tv_0(\Omega)} \int_\Omega (-\varphi)^2 d\nu_0 + \frac{1}{T} \int_\Omega (-\varphi)^2 d\nu$$

$$\leq \left( \frac{C}{Tv_0(\Omega)} + \frac{C}{T} \right) E(\varphi)^{\frac{2}{m+1}}$$

$$\leq \left( \frac{C}{\nu_0(\Omega)} + \frac{C}{T} \right) E(\varphi)^{\frac{2}{m+1}}.$$
From this we infer that
\[
\frac{(T - \nu(\Omega))\nu_0 + \nu(\Omega)\nu}{T \nu(\Omega)} \in \mathcal{M}', \text{ for every } \nu \in \mathcal{M}.
\]

We conclude that \( \mathcal{M}' \) is (non empty) convex and weakly compact in the space of probability measures. It follows from a generalized Radon-Nykodim Theorem [29] that there exists a positive measure \( \nu \in \mathcal{M}' \) and a positive function \( f \in L^1(\nu) \) such that \( \mu = f \, d\nu + \nu_s \), where \( \nu_s \) is orthogonal to \( \mathcal{M}' \). Observe also that every measures orthogonal to \( \mathcal{M}' \) is supported in some \( m \)-polar set since \( H_m(h_L) \in \mathcal{M} \) for each \( L \subseteq K \). We then deduce that \( \nu_s \equiv 0 \) since \( \mu \) does not charge \( m \)-polar sets.

From Lemma 4.2, Lemma 4.8 we see that for each \( \lambda \in \mathcal{M}' \), the functional \( \mathcal{F}_\lambda \) is proper and lower semicontinuous. For each \( j \in \mathbb{N} \) set \( \mu_j = \min(f, j)\nu \). Then \( \mathcal{L}_{\mu_j} \) is also continuous on \( \mathcal{E}'_1(\Omega) \) and \( \mathcal{F}_{\mu_j} \) is proper since \( \mu_j \leq j\nu \). Therefore, by Lemma 4.12 and Theorem 4.13, there exists \( u_j \in \mathcal{E}'_1(\Omega) \) such that \( H_m(u_j) = \mu_j \). It is clear from the comparison principle that \( \{u_j\} \) decreases to a function \( u \in \mathcal{E}'_m(\Omega) \) which solves \( H_m(u) = \mu \).

It remains to treat the case when \( \mu \) does not have compact support. Let \( \{K_j\} \) be an exhaustive sequence of compact subsets of \( \Omega \) and consider \( \mu_j = \chi_{K_j} \, d\mu \). Let \( u_j \in \mathcal{E}'_1(\Omega) \) solve \( H_m(u_j) = \mu_j \). Observe also that \( \{u_j\} \) decreases to \( u \in \mathcal{SH}_m(\Omega) \). It suffices to prove that \( \sup_j \mathcal{E}(u_j) < +\infty \). Since \( \mu \in \mathcal{M}_1 \), we have
\[
\mathcal{E}(u_j) = \int_{\Omega} (-u_j)H_m(u_j) = \int_{K_j} (-u_j)\, d\mu \leq \int_{\Omega} (-u_j)\, d\mu \leq A\mathcal{E}(u_j)^{1/3}.
\]
This implies that \( \mathcal{E}(u_j) \) is uniformly bounded, hence \( u \) belongs to \( \mathcal{E}'_1(\Omega) \) and the result follows.

\[\Box\]

5. Some applications

Lemma 5.1. Let \( \mu \) be a positive Radon measure having finite mass \( \mu(\Omega) < +\infty \). Assume that \( \mu \leq H_m(\psi) \), where \( \psi \) is a bounded \( m \)-sh function in \( \Omega \). Then there exists a unique function \( \varphi \in \mathcal{E}'_m(\Omega) \) such that \( \mu = H_m(\varphi) \).

Proof. Without loss of generality, we can assume that \(-1 \leq \psi \leq 0 \). Consider \( h_j = \max(\psi, jh) \), where \( h \in \mathcal{E}'_m(\Omega) \) is an exhaustion function of \( \Omega \). Let \( A_j := \{z \in \Omega \mid jh < -1\} \). From Theorem 4.14, there exists \( (\varphi_j)_j \subset \mathcal{E}_m(\Omega) \) such that \( H_m(\varphi_j) = \mathbb{1}_{A_j} \, \mu \), \( \forall j \). Thus,
\[
0 \geq \varphi_j \geq h_j \geq \psi, \text{ and } \varphi_j \downarrow \varphi \in \mathcal{E}'_m(\Omega).
\]
\[\Box\]

Using this lemma we can prove the comparison principle for the classes \( \mathcal{E}'_m(\Omega) \) with \( p > 1 \).

Theorem 5.2. If \( p > 1 \) and \( u, v \in \mathcal{E}_m^p(\Omega) \) then
\[
\int_{\{u > v\}} H_m(u) \leq \int_{\{u > v\}} H_m(v).
\]

Proof. Fix \( 0 \neq h \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega) \). Assume first that \( v \) is bounded and vanishes on the boundary. Let \( K_j \) be an exhaustion sequence of compact subsets of \( \Omega \). Use
Lemma 5.1 to solve \( H_m(v_j) = \mathbb{I}_K H_m(v) \). Then \( v_j \downarrow v \). Now, the arguments of the proof of Theorem 5.2 can be applied to yield
\[
\int_{\{ u > v_j \}} (-h)H_m(u) \leq \int_{\{ u > v_j \}} (-h)H_m(v_j) = \int_{\{ u > v_j \} \cap K_j} (-h)H_m(v).
\]
Letting \( j \to +\infty \) we get
\[
\int_{\{ u > v \}} (-h)H_m(u) \leq \int_{\{ u > v \}} (-h)H_m(v).
\]
It remains to remove the assumption on \( v \). For each \( k \in \mathbb{N} \) set
\[
\varphi_k := \max \left( v, -k(-h)^{1/p} \right).
\]
Since \( \varphi_k \) is bounded and vanishes on \( \partial \Omega \), by the above arguments we get
\[
\int_{\{ u > \varphi_k \}} (-h)H_m(u) \leq \int_{\{ u > \varphi_k \}} (-h)H_m(\varphi_k).
\]
Set \( A_k := \{ v > -k(-h)^{1/p} \} \) and \( B_k := \{ v \leq -k(-h)^{1/p} \} \). On \( B_k \) we have \(-h \leq \frac{(-\varphi_k)^p}{k^p} \). It then follows that
\[
\int_{B_k} (-h)H_m(\varphi_k) \leq \frac{1}{k^p} E_p(\varphi_k) \leq \frac{C}{k^p} E_p(v),
\]
where \( C > 0 \) does not depend on \( k \).
It follows from Theorem 3.19 that \( H_m(\varphi_k) = H_m(v) \) on \( A_k \). We thus get
\[
\int_{\{ u > \varphi_k \}} (-h)H_m(u) \leq \int_{\{ u > \varphi_k \} \cap A_k} (-h)H_m(v) + \frac{C}{k^p} E_p(v).
\]
It suffices now to let \( k \to +\infty \).

Now we prove a decomposition theorem of Cegrell type.

**Theorem 5.3.** Let \( \mu \) be a positive measure in \( \Omega \) which does not charge \( m \)-polar sets. Then there exists \( \varphi \in \mathcal{E}_m^0(\Omega) \) and \( 0 \leq f \in L^1(\mathcal{H}_m(\varphi)) \) such that \( \mu = f \cdot H_m(\varphi) \).

**Proof.** We first assume that \( \mu \) has compact support. By applying Theorem 4.14 we can find \( u \in \mathcal{E}_m^1(\Omega) \) and \( 0 \leq f \in L^1(\mathcal{H}_m(u)) \) such that \( \mu = f \cdot H_m(u) \), and \( \text{supp}(H_m(u)) \in \Omega \). Consider
\[
\psi = (-u)^{-1} \in S \mathcal{H}_m(\Omega) \cap L^\infty_{\text{loc}}(\Omega).
\]
Then \( (-u)^{-2m} H_m(u) \leq H_m(\psi) \). Since \( H_m(u) \) has compact support in \( \Omega \), we can modify \( \psi \) in a neighborhood of \( \partial \Omega \) such that \( \psi \in \mathcal{E}_m^0(\Omega) \). It follows from Lemma 5.1 that
\[
(-u)^{-2m} H_m(u) = H_m(\varphi), \ \varphi \in \mathcal{E}_m^0(\Omega).
\]
This gives us \( \mu = f(-u)^{-2m} H_m(\varphi) \).

It remains to consider the case \( \mu \) does not have compact support. Let \( \{ K_j \} \) be an exhaustive sequence of compact subsets of \( \Omega \). From previous arguments there exists \( u_j \in \mathcal{E}_m^0(\Omega) \) and \( f_j \in L^1(\mathcal{H}_m(u_j)) \) such that \( \mathbb{I}_{K_j} \mu = f_j H_m(u_j) \). Take a sequence of positive numbers \( \{ a_j \} \) satisfying \( \varphi := \sum_{j=1}^\infty a_j u_j \in \mathcal{E}_m^0(\Omega) \). The measure \( \mu \) is absolutely continuous with respect to \( H_m(\varphi) \). Thus,
\[
\mu = g H_m(\varphi) \text{ and } g \in L^1_{\text{loc}}(\mathcal{H}_m(\varphi)).
\]

□
Theorem 5.4. Let $\mu$ be a positive Radon measure on $\Omega$ such that $E^p_m(\Omega) \subset L^p(\mu), p > 0$. Then there exists a unique $\varphi \in E^p_m(\Omega)$ such that $H_m(\varphi) = \mu$.

Proof. The uniqueness follows from the comparison principle. Let us prove the existence result. Since $\mu$ does not charge $m$-polar sets, applying the decomposition theorem (Theorem 5.3) we get
\[ \mu = f H_m(u), \quad u \in E^0_m(\Omega), \quad 0 \leq f \in L^1_{\text{loc}}(H_m(u)). \]
For each $j$, use Lemma 5.1 to find $\varphi_j \in E^0_m(\Omega)$ such that
\[ H_m(\varphi_j) = \min(f, j) H_m(u). \]
By Proposition 4.6, $\sup_j E_p(\varphi_j) < +\infty$. Thus, the comparison principle gives us that $\varphi_j \downarrow \varphi \in E^0_m(\Omega)$ which solves $H_m(\varphi) = \mu$. \qed

6. PROOF OF THE MAIN RESULTS

6.1. Proof of Theorem 1.

Proposition 6.1. Let $u, v \in E^p_m(\Omega), p > 0$. There exist two sequences $(u_j), (v_j) \subset E^0_m(\Omega)$ decreasing to $u, v$ respectively such that
\[ \lim_{j \to +\infty} \int\Omega(-u_j)^p H_m(v_j) = \int\Omega(-u)^p H_m(v). \]
In particular, if $\varphi \in E^0_m(\Omega)$ then there exists $(\varphi_j) \subset E^0_m(\Omega)$ decreasing to $\varphi$ such that
\[ E_p(\varphi_j) \to E_p(\varphi). \]
Proof. Let $(u_j)$ be a sequence in $E^0_m(\Omega)$ decreasing to $u$ such that $\sup_j \int\Omega(-u_j)^p H_m(u_j) < +\infty$. Since $H_m(v)$ vanishes on $m$-polar sets Theorem 5.3 gives
\[ H_m(v) = f H_m(\psi), \quad \psi \in E^0_m(\Omega), \quad 0 \leq f \in L^1_{\text{loc}}(H_m(\psi)). \]
For each $j$, use Lemma 5.1 to find $v_j \in E^0_m(\Omega)$ such that
\[ H_m(v_j) = \min(f, j) H_m(\psi). \]
By the comparison principle $v_j \downarrow \varphi \in E^p_m(\Omega)$ which solves $H_m(\varphi) = H_m(v)$. It implies that $\varphi \equiv v$. We then have
\[ \int\Omega(-u)^p H_m(v) = \lim_j \int\Omega(-u_j)^p \min(f, j) H_m(\psi) = \lim_j \int\Omega(-u_j)^p H_m(v_j). \]
\qed

Theorem 1 is a consequence of the following result.

Theorem 6.2. Let $\mu$ be a positive Radon measure in $\Omega$ and $p > 0$. Then we have
\[ \mu = H_m(\varphi) \quad \text{with} \quad \varphi \in E^p_m(\Omega) \iff E^p_m(\Omega) \subset L^p(\Omega, \mu). \]
Proof. The implication $\Leftarrow$ has been proved in Theorem 4.14 and Theorem 5.4. Now, assume that $\mu = H_m(\varphi)$ with $\varphi \in E^p_m(\Omega)$. Let $\psi$ be another function in $E^p_m(\Omega)$. By Proposition 6.1 there exist sequences $(\varphi_j), (\psi_j)$ in $E^0_m(\Omega)$ having uniformly bounded energy such that
\[ \int\Omega(-\psi)^p H_m(\varphi) = \lim_j \int\Omega(-\psi_j)^p H_m(\varphi_j), \quad \lim_j E_p(\varphi_j) = E_p(\varphi), \quad \lim_j E_p(\psi_j) = E_p(\psi). \]
It suffices now to apply Lemma 3.5. \qed
6.2. Proof of Theorem 2.

**Proof.** We first prove the existence result. Since $\mu$ does not charge $m$-polar sets the decomposition theorem yields

$$\mu = fH_m(u), \ u \in C^0_m(\Omega), \ 0 \leq f \in L^1_{\text{loc}}(H_m(u)).$$

For each $j$ use Lemma 5.1 to find $\varphi_j \in C^0_m(\Omega)$ such that

$$H_m(\varphi_j) = \min(f, j)H_m(u).$$

Besides, $\sup_j \int_\Omega H_m(\varphi_j) \leq \mu(\Omega) < +\infty$. Thus, $\varphi_j \downarrow \varphi \in F_m(\Omega)$ in view of the comparison principle. The limit function $\varphi$ solves $H_m(\varphi) = \mu$ as required.

The uniqueness can be proved by the same ways as in [8, Lemma 5.14]. Assume that $\psi \in F_m(\Omega)$ solves $H_m(\psi) = \mu$. We are to prove that $\varphi = \psi$. Let $(K_j)$ be an exhaustive sequence of compact subsets of $\Omega$ such that $h_j = h_{m,K_j,\Omega}$ is continuous. For each $j$, the function $\psi_j := \max(\psi, jh_j)$ belongs to $C^0_m(\Omega)$, and $\psi_j \downarrow \psi$. Set $d_j := \psi_j - h_j = \max(\frac{\psi_j}{h_j} - h_j, 0)$. Then $d_j \leq H_{\{\psi > j, h_j\}}$ and $1 - d_j \downarrow 0$. For $s > j$, by the comparison principle we get

$$0 \leq d_j H_m(\max(\psi, s,h_j)) \leq H_{\{\psi > j, h_j\}} H_m(\max(\psi, s,h_j)) = H_m(\max(\psi, s,h_j)).$$

Letting $s$ tend to $+\infty$ and using Corollary 3.14 we get

$$d_j H_m(\psi) \leq H_{\{\psi > j, h_j\}} H_m(\max(\psi, j,h_j)) \leq H_m(\psi).$$

Recall that from the first part we have

$$\mu = fH_m(u), \ u \in C^0_m(\Omega), \ 0 \leq f \in L^1_{\text{loc}}(H_m(u)).$$

and $H_m(\varphi_p) = \min(f, p)H_m(u)$ for each $p \in \mathbb{N}$. For each $p, j$ we can find $\psi_j \in C^0_m(\Omega)$ such that

$$H_m(\psi_j) = (1 - d_j)H_m(\varphi_p).$$

Using (6.1) we get

$$H_m(\varphi_p) = d_j H_m(\varphi_p) + (1 - d_j)H_m(\varphi_p) \leq d_j H_m(\psi) + (1 - d_j)H_m(\varphi_p) \leq H_m(\psi_j) + H_m(\psi_j) \leq H_m(\psi_j) + H_m(\psi_j).$$

This coupled with the comparison principle yield $\varphi_p \geq \psi_j + \psi_j$. Letting $p \to +\infty$ we obtain $\varphi \geq v_j + \psi_j$, where $v_j \in F_m(\Omega)$ solves $H_m(v_j) = (1 - d_j)H_m(\varphi)$. Since $H_m(\varphi)$ does not charge $m$-polar sets, by monotone convergence theorem the total mass of $H_m(v_j)$ goes to 0 as $j \to +\infty$. This implies that $v_j$ increases to 0 and hence $\varphi \geq \psi$.

Now, we prove that $\varphi \leq \psi$. Let $\psi_j, t_j \in C^0_m(\Omega)$ such that $H_m(w_j) = d_j H_m(\psi_j)$ and $H_m(t_j) = (1 - d_j)H_m(\psi_j)$. Since $H_m(\varphi_p)$ increases to $H_m(\varphi)$, the comparison principle can be applied for $\varphi$ and $w_j$ which implies that $w_j \geq \psi$. But, applying
again the comparison principle for $t_j + w_j$ and $\psi_j$ we get $t_j + w_j \leq \psi_j$. Furthermore, the total mass of $H_m(t_j)$ can be estimated as follows

$$
\int_\Omega H_m(t_j) = \int_\Omega H_m(\psi_j) - \int_\Omega d_j H_m(\psi_j) \\
\leq \int_\Omega H_m(\psi) - \int_\Omega d_j^2 H_m(\psi) \\
\leq 2 \int_\Omega (1 - d_j) H_m(\psi) \to 0.
$$

This implies that $t_j$ converges in $m$-capacity to 0. Indeed, for every $\epsilon > 0$ and $m$-subharmonic function $-1 \leq \theta \leq 0$, by the comparison principle we have

$$
\epsilon^n \int_{\{t_j < -\epsilon\}} H_m(\theta) \leq \int_{\{t_j < -\epsilon\}} H_m(\epsilon \theta) \\
\leq \int_{\{t_j < -\epsilon\}} H_m(t_j) \leq \int_\Omega H_m(t_j) \to 0.
$$

Thus, we can deduce that $\varphi \leq \psi$ which implies the equality. \qed

We prove in the above uniqueness theorem that every $\psi \in F_m^a(\Omega)$ can be approximated from above by a sequence $\psi_j \in E_m^0(\Omega)$ such that $H_m(\psi_j)$ increases to $H_m(\psi)$. This type of convergence is strong enough to prove the comparison principle for the class $F_m^a(\Omega)$. We thus get

**Theorem 6.3.** The comparison principle is valid for functions in $F_m^a(\Omega)$.

7. Examples

**Lemma 7.1.** If $\varphi \in E_m^p(\Omega)$, $p > 0$ then $\operatorname{Cap}_m(\varphi < -t) \leq C_E p(\varphi, \frac{1}{m+p})$, where $C > 0$ is a constant depending only on $m$.

**Proof.** Without loss of generality we can assume that $\varphi \in E_m^0(\Omega)$. Fix $u \in \mathcal{S}H_m^-(\Omega)$ such that $-1 \leq u \leq 0$. Observe that, for any $t > 0$, $(\varphi < -2t) \subset (\varphi < tu - t) \subset (\varphi < -t)$.

Thus, by the comparison principle (Theorem 2.13) we have

$$
\int_{\{\varphi < -2t\}} H_m(u) \leq \frac{1}{t^m} \int_{\{\varphi < tu - t\}} H_m(tu - t) \leq \frac{1}{t^m} \int_{\{\varphi < tu - t\}} H_m(\varphi) \\
\leq \frac{1}{t^m} \int_{\{\varphi < -t\}} H_m(\varphi) \leq \frac{1}{t^m} \int_{\Omega} (-\varphi)^p H_m(\varphi).
$$

\qed

**Proposition 7.2.** Let $\mu = f dV$, where $0 \leq f \in L^p(\Omega, dv)$, $\frac{n}{m} > p > 1$. Then

$$
\mu = H_m(\varphi), \quad \varphi \in F_m^q(\Omega), \quad \forall q < \frac{nm(p-1)}{n - mp}.
$$

**Proof.** Fix $0 < r < n/(n-m)$. By Hölder’s inequality and [9, Proposition 2.1], there exists $C > 0$ depending only on $p, r, ||f||_p$ such that

$$
\mu(K) \leq C \operatorname{Vol}(K)^{\frac{1}{r}} \leq C \operatorname{Cap}_m(K)^{\frac{1}{m-1}}.
$$
Take $0 < q < \frac{nm(p-1)}{n-mp}$ and $u \in \mathcal{E}_m^q(\Omega)$. By Theorem 5.4 it suffices to show that $u \in L^q(\mu)$ which is, in turn, equivalent to showing that
\[
\int_1^{+\infty} \mu(u < -t^{1/q})dt < +\infty.
\]
The latter follows easily from (7.1) and Lemma 7.1, which completes the proof. □

The exponent $q(p) = \frac{nm(p-1)}{n-mp}$ is sharp in view of the following example.

**Example 7.3.** Consider $\varphi_\alpha = 1 - \|z\|^{-2\alpha}$, where $\alpha$ is a constant in $(0, \frac{n-m}{m})$. An easy computation shows that $\varphi_\alpha \in \mathcal{F}_m(\Omega)$ and
\[
H_m(\varphi_\alpha) = C.\|z\|^{-2m(\alpha+1)}dV = f_\alpha dV.
\]
Then
\[
\varphi_\alpha \in \mathcal{F}_m^q(\Omega) \iff q < \frac{n-m}{\alpha} - m,
\]
while
\[
f_\alpha \in L^p(\Omega, dV) \iff p < \frac{n}{m(\alpha + 1)}.
\]

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