LOCAL AND GLOBAL STRONG SOLUTION TO THE STOCHASTIC 3-D INCOMPRESSIBLE ANISOTROPIC NAVIER-STOKES EQUATIONS

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ABSTRACT. Considering the stochastic 3-D incompressible anisotropic Navier-Stokes equations, we prove the local existence of strong solution in $H^2(T^3)$. Moreover, we express the probabilistic estimate of the random time interval for the existence of a local solution in terms of expected values of the initial data and the random noise, and establish the global existence of strong solution in probability if the initial data and the random noise are sufficiently small.

1. Introduction. Let us first recall the 3-D incompressible anisotropic Navier-Stokes equations:

$$
\begin{cases}
\partial_t u + u \cdot \nabla u - \nu_h \Delta_h u - \nu_3 \partial_3^2 u + \nabla P = 0, \\
\nabla \cdot u = 0, \\
u|_{t=0} = u_0,
\end{cases}
$$

(ANS)

where $u(t, x)$ and $P(t, x)$ denote the fluid velocity and the pressure, respectively, the viscosity coefficients $\nu_h$ and $\nu_3$ are two constants satisfying $\nu_h > 0$, $\nu_3 \geq 0$, $x = (x_h, x_3) \in T^2 \times T$ (or $\mathbb{R}^2 \times \mathbb{R}$) and $\Delta_h = \partial_1^2 + \partial_2^2$. Systems of this type appear in geophysical fluids (see for instance [10]). In fact, in order to model turbulent diffusion, physicists often consider a diffusion term of the form $-\nu_h \Delta_h - \nu_3 \partial_3^2$, where $\nu_h$ and $\nu_3$ are empirical constants, and $\nu_3$ is much smaller than $\nu_h$ in most applications. For a more complete discussion, we refer to the book of J. Pedlosky [35], Chapter 4. In the particular case of the so-called Ekman boundary layers for rotating fluid (see for instance [13, 14, 21]), $\nu_3 = \epsilon \nu_h$ and $\epsilon$ is a very small parameter.

The mathematical theory of the system (ANS) is first studied by J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier in [9] and D. Iftimie in [23], they proved...
that such a system is locally wellposed for the initial data in the anisotropic Sobolev space
\[ H^{0, \frac{1}{2} + \varepsilon} = \left\{ u \in L^2(\mathbb{R}^3); \| u \|^2_{H^{0, \frac{1}{2} + \varepsilon}} = \int_{\mathbb{R}^3} |\xi_3|^{1+2\varepsilon} |\hat{u}(\xi_3)|^2 d\xi < \infty \right\}, \]
for some \( \varepsilon > 0 \), and is global wellposed if the initial data are small enough in the sense that
\[ \| u_0 \|_{L^2} \| u_0 \|_{H^{0, \frac{1}{2} + \varepsilon}} \leq c\nu_h \]
for some sufficiently small constant \( c \). After that, M. Paicu [32], J. -Y. Chemin and P. Zhang [11], T. Zhang and D. Y. Fang [38] obtained the similar results in the scaling invariant spaces \( B^{0, \frac{1}{2}}, B^{-\frac{1}{2}} \) and \( B_p^{-1+\frac{1}{2}, \frac{1}{2}} \), respectively. M. Paicu obtained the global wellposedness of the periodic anisotropic Navier-Stokes equations in [33]. T. Zhang [37], M. Paicu and P. Zhang [34] proved the global wellposedness provide only the initial horizontal data are sufficiently small in \( B^{0, \frac{1}{2}} \) and \( B^{-\frac{1}{2}} \), respectively.

In the meantime, the stochastic 3-D incompressible anisotropic Navier-Stokes equations are gaining more and more interest in fluid mechanical research. On the one hand, the stochastic model has the ability to model small perturbations (numerical, empirical, and physical uncertainties) or the thermodynamic fluctuations present in fluid flows. On the other hand, it is an appropriate way to describe turbulence in the statistical sense (see [18] Chapter 4 and references cited therein).

The system of stochastic 3-D incompressible anisotropic Navier-Stokes equations reads
\[
\begin{align*}
\begin{cases}
    du + (u \cdot \nabla u - \nu_h \Delta_h u - \nu_3 \partial_3^2 u + \nabla P)dt = fdW, \\
    \nabla \cdot u = 0, \\
    u|_{t=0} = u_0,
\end{cases}
\end{align*}
\]
where \( W \) is a cylindrical Wiener progress, and \( fdW \) can be written formally in the expansion \( \sum_{k \geq 1} f_k dW_k \), where \( W_k \) are a collection of 1-D independent Brownian motions.

When \( \nu_h = \nu_3 = 0 \), the system (1.1) turns into a stochastic 3-D incompressible Euler system. J. U. Kim [26] and R. Mikulevicius [31] considered the local solutions with additive noise in a weighted Hölder space and \( H^{5/2 + \varepsilon}(\mathbb{R}^3) \), respectively. N. Glatt-Holtz and V. Vicol established the local existence of the pathwise solution in a bounded domain and showed the regularizing effect of the linear multiplicative noise in [19].

When \( \nu_h = \nu_3 = \nu > 0 \), such a system is a stochastic 3-D incompressible Navier-Stokes system, whose first result can be traced back to the pioneering work of A. Bensoussan and R. Temam [2] in 1973, after that, there have been a lot of studies on the existence and uniqueness of the solution for the stochastic Navier-Stokes system, refer to [4, 12, 28] and the references therein. M. Capinski and S. Peszat [7] obtained the local existence and uniqueness of the strong solution in the three-dimensional bounded domain with the sufficiently regular initial data. N. Glatt-Holz and M. Ziane [20] obtained the local existence and uniqueness of the strong solution for the stochastic Navier-Stokes equations in two-dimensional or three-dimensional bounded domains forced by a multiplicative noise when the initial data are in \( H^1 \), and they proved the global existence in the two-dimensional case. J. U. Kim [27] obtained the local and global existence of the strong solution for the three-dimensional stochastic Navier-Stokes equations when the initial data are in \( H^{\alpha + 1/2}(\mathbb{R}^3), 0 < \alpha < \frac{1}{2} \). Recently, we obtain a similar result in the space
$\dot{B}^{d/p-1}_{p,r}(\mathbb{R}^d)$ $(d \geq 2)$ in [15]. On the other hand, many authors considered the martingale solutions to the stochastic Navier-Stokes equations, see e.g. [6, 8, 16, 17, 29, 30].

For the anisotropic case, only H. Bessaih and A. Mille [3] considered the stochastic modified 3-D anisotropic Navier-stokes equations, which add a Brinkman-Forchheimer type term $a|u|^{2a}u$ $(a > 0, \alpha > 1)$. They proved the existence and uniqueness of global weak solutions in $\mathbb{R}^3$ and gave a large deviations principle.

In this paper, we shall show that the Cauchy problem (1.1) admits a local strong solution in $H^2(\mathbb{T}^3)$ under some conditions. The required regularity $s = 2$ is smaller than $s > 5/2$ in J. U. Kim [26] due to the existence of $\Delta_h u$. Furthermore, if the initial data and the external random force are sufficiently small, we establish the existence of global strong solutions in probability.

The manuscript is organized as follows. In Section 2, we introduce some mathematics backgrounds. Section 3 contains the precise definition of strong solution to (1.1), along with the statement of our main result. In Section 4, we construct the strong solution. We prove Theorem 3.2 in Section 5 and prove a useful lemma in Appendix.

2. Preliminaries. We start by introducing the notation and some basic facts used in the paper.

2.1. Analytic framework. We denote the usual Sobolev space and the anisotropic Sobolev space as follows (see [1]): given any nonnegative numbers $s$ and $s'$,

$$H^s(\mathbb{T}^3) := \{ f \in L^2(\mathbb{T}^3) \mid \sum_{k \in \mathbb{Z}^3} (1 + |k|^2)^s \hat{f}(k)^2 < \infty \},$$

$$H^{s,s'}(\mathbb{T}^3) := \{ f \in L^2(\mathbb{T}^3) \mid \sum_{k \in \mathbb{Z}^3} (1 + |k_h|^2)^s (1 + |k_3|^2)^{s'} \hat{f}(k)^2 < \infty \},$$

where $k = (k_h, k_3)$, and $\hat{f} (\cdot)$ is the Fourier transform of $f$. We denote by $\langle \cdot, \cdot \rangle_s$ (resp., $\langle \cdot, \cdot \rangle_{s,s'}$) and $\| \cdot \|_{H^s}$ (resp., $\| \cdot \|_{H^{s,s'}}$) the inner product and the norm of $H^s(\mathbb{T}^3)$ (resp., $H^{s,s'}(\mathbb{T}^3)$) respectively. We define, for $s \geq 0$,

$$H^s := \{ f \in H^s(\mathbb{T}^3) \mid \nabla \cdot f = 0 \},$$

which is a closed subspace of $H^s(\mathbb{T}^3)^3$. The symbol $\Pi$ stands for the projection $H^s(\mathbb{T}^3)^3 \to H^s$,

which can be explicitly expressed by means of the Fourier transform. For each $f = (f^1, f^2, f^3) \in H^s(\mathbb{T}^3)^3$,

$$(\Pi f)^j = \hat{f}^j - \frac{\xi_j}{|\xi|^2} \sum_{k=1}^3 \xi_k \hat{f}^k, \quad j = 1, 2, 3.$$

When $s = 0$, we also denote $L^2$ by $H^0$. Throughout the paper, we write $\mathbb{T}^3 = \mathbb{T}_h^3 \times \mathbb{T}_v$. The components of the three-dimensional vector field $f$ are denoted $(f^h, f^3)$, and it is understood that $\nabla f^h \triangleq (\partial_1, \partial_2)$ and $\text{div} f = \partial_1 f^1 + \partial_2 f^2$. Finally, the notation $X_h$ (resp., $X_v$) means that $X_h$ is a function space over $\mathbb{T}_h^2$ (resp., $\mathbb{T}_v$).

A function space over $\mathbb{T}^3$ is simply denoted by $X$. For instance, $L^p \triangleq L^p(\mathbb{T}^3)$, $L^p_h \triangleq L^p(\mathbb{T}_h^2)$, and $L^p_v \triangleq L^p(\mathbb{T}_v)$.

We conclude this section with some bounds on the nonlinear terms, which will be used throughout the rest of the work.
Lemma 2.1. There exist a constant $C$ such that

1. for $u \in H^2_0$, $\nabla_h u \in H^2$

\[
\langle u \cdot \nabla u, u \rangle_2 \leq C(\|u\|_{H^2} \|\nabla_h u\|_{H^{0.1}} + \|u\|_{H^2} \|\nabla_h u\|_{H^{0.1}}), \quad (2.1)
\]

2. for $u \in H^1_0$, $\nabla_h u \in H^1$

\[
\|\langle \partial_3 (u \cdot \nabla u), \partial_3 u \rangle_0\| \leq C\|\partial_3 u\|_{L^2} \|\nabla_h u\|_{L^2} \|\nabla_h u\|_{L^\infty L^2}. \quad (2.2)
\]

Proof. Similar to the proof in Theorem 6.2 and Proposition 6.3 in [1] where Bahouri etc. consider the case of $H^{0.1}(\mathbb{R}^3)$, we can prove this lemma. For the convenience of readers, we give some details in Appendix.

\[\square\]

2.2. Stochastic framework. The driving process $W$ is a cylindrical Wiener processes defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with a complete, right continuous filtration, and taking values in a separable Hilbert space $\mathcal{H}$. More specifically, $W$ is given by a formal expansion

\[
W(t) = \sum_{k \geq 1} e_k W_k(t).
\]

Here $\{W_k\}_{k \geq 1}$ is a family of mutually independent real-valued Brownian motions with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and $\{e_k\}_{k \geq 1}$ is an orthonormal basis of $\mathcal{H}$.

Consider now another separable Hilbert space $X$. We denote the collection of Hilbert-Schmidt operators, the set of all bounded operators $f$ from $\mathcal{H}$ to $X$ such that $\|f\|_{L^2(\mathcal{H}, X)} := \sum_k |\langle f e_k, x \rangle_X|^2 < \infty$, by $L^2(\mathcal{H}, X)$. Given an $X$ valued predictable process $f \in L^2(\Omega; L^2_{loc}([0, \infty), L^2(\mathcal{H}, X)))$ and taking $f_k = f e_k$, one may define the (Itô) stochastic integral

\[
M_t := \int_0^t f(t) dW = \sum_k \int_0^t f_k(t) dW_k \quad (2.3)
\]

as an element in $\mathcal{M}_X$, that is, the space of all $X$ valued square integrable martingales.

The process $\{M_t\}_{t \geq 0}$ has many desirable properties. Most notably for the analysis here, the Burkholder-Davis-Gundy inequality holds, which in the present context takes the form

\[
\mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_0^t f(t) dW(t)^r \right| \right) \leq C \mathbb{E} \left( \int_0^T \|f(t)\|_{L^2(\mathcal{H}, X)}^2 dt \right)^{r/2}, \quad (2.4)
\]

valid for any $r \geq 1$, and where $C$ is an absolute constant depending only on $r$. For an extended treatment, we refer the reader to [12] and [24].

3. Main result. Let us first state the definition of local strong solution.

Definition 3.1. Fix a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{W})$ and $u_0$ an $H^2_\sigma$-valued $\mathcal{F}_0$ measurable random variable. Suppose that $f \in L^2(\Omega; L^2_{loc}([0, \infty), L^2(\mathcal{H}, H^2)))$ is progressively measurable, $\nu_h > 0$, $\nu_3 \geq 0$.

(i) A pair $(u, \tau)$ is called a local strong solution of (1.1) if the following conditions are satisfied:

1. $u$ is right continuous progressively measurable process adapt to $\{\mathcal{F}_t\}_{t \geq 0}$, and $u \in L^2(\Omega, L^\infty_{loc}([0, \infty) : H^2))$, $\nabla_h u \in L^2(\Omega; L^2_{loc}([0, \infty); H^2))$. 


2. \( \tau(\omega) \) is a stopping time with respect to \( \mathcal{F}_t \) such that
\[
\tau(\omega) = \lim_{N \to \infty} \tau_N(\omega)
\]
for almost all \( \omega \),
where we define
\[
\tau_N(\omega) = \begin{cases}
\inf \{0 \leq t < \infty : \|u(t, \omega)\|_{L^{H,1}}^2 + \int_0^t \|\nabla u(s, \omega)\|_{L^{H,c,1}}^2 \, ds \geq N \}, \\
\infty, & \text{if the above set } \{ \ldots \} \text{ is empty};
\end{cases}
\]
(3.1)
3. \( u(t, x) \in C([0, \tau(\omega)); H^2_0) \) for almost all \( \omega \in \Omega \), and the following holds \( \mathbb{P} \)-a.s.,
\[
\langle u(t \wedge \tau_N), \phi \rangle_0 - \langle u_0, \phi \rangle_0 = \int_0^{t \wedge \tau_N} \langle 
abla \Delta u + \nu_3 \partial_3^2 u - u \cdot \nabla u, \phi \rangle_0 \, ds
+ \left( \int_0^{t \wedge \tau_N} f(s) \, d\mathbb{W}(s), \phi \right)_0
\]
for all \( 0 \leq t < \infty \) and all \( \phi \in L^2_{\sigma} \).

(ii) We say the local strong solutions are unique if, given any pair \( (u^{(1)}, \tau^{(1)}) \), \( (u^{(2)}, \tau^{(2)}) \) of local strong solutions,
\[
\mathbb{P} \left( \{ u^{(1)}(0) = u^{(2)}(0) \} \right) = 0; \forall t \in [0, \tau^{(1)} \wedge \tau^{(2)}) \right) = 1.
\]

Our main result is stated as follows.

**Theorem 3.2.** Fix a stochastic basis \( (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{W}) \) and \( u_0 \) an \( H^2 \)-valued \( \mathcal{F}_0 \) measurable random variable. Suppose that \( f \in L^2(\Omega; L^2_{\text{loc}}([0, \infty); L^2(\mathbb{U}, H^2))) \) is progressively measurable, \( \nu_3 > 0, \nu_3 \geq 0 \). Then, there is a unique local strong solution of (1.1), and for all \( 0 < \delta < 1 \),
\[
\mathbb{P}(\{\tau \geq \delta\}) \geq 1 - C\delta \left( \mathbb{E} \left( \|u_0\|_{L^2}^2 \right) + \mathbb{E} \left( \int_0^\delta \|f(s)\|_{L^2(\mathbb{U}, H^2)}^2 \, ds \right) \right),
\]
(3.4)
where \( C \) denotes a positive constant independent of \( u \) and \( \delta \). In addition, if \( f \in L^2(\Omega; L^2([0, \infty); L^2(\mathbb{U}, H^{0,1})) \), there exists a constant \( C^* \) independent of \( u_0 \) and \( f \) such that
\[
\mathbb{P}(\{\tau = \infty\}) \geq 1 - C^*(\alpha^{\frac{1-2\zeta}{2}} + \beta^{\frac{1-2\zeta}{2}} + \alpha),
\]
(3.5)
where
\[
\alpha = \mathbb{E} \left( \|u_0\|_{L^2}^2 \right) + \mathbb{E} \left( \int_0^\infty \|f(s)\|_{L^2(\mathbb{U}, L^2)}^2 \, ds \right),
\]
\[
\beta = \mathbb{E} \left( \|\partial_3 u_0\|_{L^2}^2 \right) + \mathbb{E} \left( \int_0^\infty \|\partial_3 f(s)\|_{L^2(\mathbb{U}, L^2)}^2 \, ds \right),
\]
and \( \zeta > 0 \) is a sufficiently small constant.

**Remark 1.** For any \( \varepsilon > 0 \), if
\[
\alpha < \min \left\{ \frac{\varepsilon}{2C^*}, \left( \frac{\varepsilon}{2C^*} \right)^{\frac{2}{2\zeta}} \right\},
\]
then we have
\[
\mathbb{P}(\{\tau = \infty\}) \geq 1 - \varepsilon.
\]
Hence, in order to obtain the global existence in probability of the strong solution, we only need the \( L^2 \) norm of the initial data and external force are sufficiently small. We extend the result in [9] to the stochastic case.
Remark 2. Because of the existence of the stopping time $\tau_N$, we should consider the solution in $H^{a,b}(a > 0, b > \frac{1}{2})$ to ensure that $\tau_N(u^n) \to \tau_N(u)$. In the other hand, we need control $\|\nabla u\|_{L^\infty}$ in the proof of $u \in C([0, \tau); H^{a,b})(a > 0, b > \frac{1}{2})$ (see Lemmas 4.33 and 4.34). Therefore, it is a good choice to consider the solution in $H^{0,\frac{1}{2}+}$, possibly this regularity could be achieved by a density-stability argument, see [19] Section 7.

In the deterministic case, Chemin etc. [9] consider the solution in the space $H^{0,\frac{1}{2}+}$, which use the fact $u \in \tilde{L}^\infty([0, T]; H^{0,\frac{1}{2}+})$ and

$$2^{ks}(\Delta^k u \cdot \nabla u)|\Delta^k u|_{L^2} \leq \|u\|_{H^{\frac{1}{2}+}}^2 \|\nabla u\|_{H^0,s}$$

for $s \geq \frac{1}{2}$. There are some technical difficulties to extend the result in [9] to stochastic case. We will consider this problem in our further work.

Remark 3. If the noise is multiplicative, i.e. $f = f(u)$, we could have a similar result under appropriate assumptions of $f$. The proof is more complicate: we first establish the existence of martingale solutions by the stochastic compactness method, then obtain the existence of the strong solution by the uniqueness of martingale solutions and Yamada-Watanabe theorem, see [19] Section 6 for example.

We shall prove Theorem 3.2 in several steps, our procedure to construct a solution is classical, see [26]. Without loss of generality, we take $\nu_h = 1, \nu_3 = 0$ in the rest of the paper.

Step 1. Introduce a cut-off function to control the nonlinear convection term in (1.1). We consider the Galerkin approximation system (4.5), and obtain a global smooth solution for almost all sample point $\omega$.

Step 2. Fix $T > 0$, and obtain energy estimates of approximate solutions on the time interval $[0, T]$. By means of these estimates which depend only on the original regularity of the initial data and random noise before approximation, we can construct strong solutions of the modified equation (4.3). We introduce a stopping time whose ultimate goal is to remove the cut-off function later on.

Step 3. Pass $T \to 0$, and then, pass $N \to \infty$ where $N$ is a parameter in the cut-off function such that $N = \infty$ makes the cut-off function an identity map. The limit function will be a desired solution.

Step 4. Prove (3.4) and (3.5) by stopping time, energy estimates and Chebyshev’s inequality.

4. Construction of the strong solution. In the section, we will construct the strong solution for fixed $N \geq 1$ and $T > 0$. The main result of this section reads as follows.

Theorem 4.1. Fix $N \geq 1$ and $T > 0$, there is a stopping time $\hat{\tau}_N$ and a function $u$, which is unique on $[0, \hat{\tau}_N]$, satisfying the following properties:

(i) $\hat{\tau}_N = \left\{ \begin{array}{ll}
\inf \{0 \leq t \leq T \mid \|u(t)\|_{H^{0,1}}^2 + \int_0^t \|\nabla u(s)\|_{H^{0,1}}^2 ds \geq N \}, \\
T, \text{ if the set } \{ \ldots \} \text{ is empty;}
\end{array} \right.$

(ii) $u(t \wedge \hat{\tau}_N) \in C([0, T]; H^2)$ a.s., it is $H^2$-valued progressively measurable, and it satisfies $u(\cdot \wedge \hat{\tau}_N) \in L^2(\Omega; L^\infty(0, T; H^2))$, $\nabla u(\cdot \wedge \hat{\tau}_N) \in L^2(\Omega; L^2(0, T; H^2))$;
(iii) It holds $\mathbb{P}$-a.s.

$$(u(t \wedge \hat{T}_N), \phi)(0) - (u_0, \phi)(0) = \int_0^{t \wedge \hat{T}_N} \langle \Delta_h u(s) - u(s) \cdot \nabla u(s), \phi \rangle_0 ds + \left\langle \int_0^{t \wedge \hat{T}_N} f(s) dW(s), \phi \right\rangle_0,$$

for all $0 \leq t < \infty$ and all $\phi \in L^2_\sigma$.

Now, we begin to prove Theorem 4.1. In Section 4.1, we construct the global smooth solution of the Galerkin approximation (4.5). In Section 4.2, we obtain some uniform estimates of the approximate solution, and construct the strong solutions of smooth solution of the Galerkin approximation (4.5). In Section 4.3, we construct the strong solutions of (4.5) by the compactness argument in Section 4.3. In Section 4.4, we define the suitable stopping time $\hat{T}_N$ to remove the cut-off function and show the uniqueness of $u$ on $[0, \hat{T}_N]$. At last, we show $u \in C([0, \hat{T}_N]; H^2)$ by considering $\|u(t \wedge \hat{T}_N)\|_{H^2}$.

4.1. The Galerkin approximation. We introduce a suitable cut-off operator applied to $u$ and consider the following modified system:

$$\begin{cases}
  du + \varphi_{N,u} \Pi(u \cdot \nabla) u dt - \Delta_h u dt = \Pi f dW, \\
  u(0) = u_0,
\end{cases}$$

where $\varphi_{N,u}(t) = \varphi_N \left( \|u(t)\|_{H^{0,1}}^2 + \int_0^t \| \nabla_h u(s) \|_{H^{0,1}}^2 ds \right)$, and

$$\varphi_N(y) = \begin{cases} 1, & \forall y \leq N, \\
 1 + N - y, & \text{for } N \leq y \leq N + 1, \\
 0, & \forall y \geq N + 1, \end{cases}$$

for each integer $N \geq 1$. Considering the orthonormal basis $\{\psi_m\}_{m=1}^\infty$ of the space $L^2_\sigma$ formed by trigonometric functions and set

$$X_n = \text{span} \{\psi_1, \ldots, \psi_n\}$$

with the associated projection $P_n : L^2 \rightarrow X_n$.

Fix $N \geq 1$, $n \geq 1$, we consider the following Galerkin approximation scheme for (4.3):

$$\begin{cases}
  du^n + \varphi_{N,u^n} P_n \Pi(u^n \cdot \nabla) u^n dt - \Delta_h u^n dt = P_n \Pi f dW, \\
  u^n(0) = P_n u_0.
\end{cases}$$

For each $T > 0$, there exists a unique solution of (4.5) in $L^2(\Omega; C([0, T]; X_n))$ as a consequence of the equivalence of norms on $X_n$, the Banach fixed point argument, and the map $\omega \mapsto u^n$ from $\Omega$ into $C([0, T]; X_n)$ is $\mathcal{F}_T$ measurable. See, for example, [5] Section 3.1 and [26] Proposition 2.4 for further details.

4.2. The uniform estimates. We apply the Itô’s formula in $H^2$ to (4.5), and using that $P_n$ is self-adjoint on $X_n$, we obtain

$$d\|u^n\|_{H^2}^2 + 2\|\nabla_h u^n\|_{H^2}^2 dt = -2 \varphi_{N,u^n}(u^n \cdot \nabla u^n, u^n)_{L^2} dt + \|P_n \Pi f\|_{L^2([0, T]; H^2)}^2 dt + 2(f, u^n)_{L^2} dW.$$  

By (2.1) and Young’s inequality, we have

$$\left| \int_0^t \varphi_{N,u^n}(u^n \cdot \nabla u^n, u^n)_{L^2} ds \right| \leq \frac{1}{2} \int_0^t \|\nabla_h u^n(s)\|_{H^2}^2 ds + C \int_0^t \|u^n\|_{H^2}^2 R^n(s) ds,$$
where \( R^n(s) = \varphi_{N,u^n}(\|u^n\|_{H^{0.1}}^2 + \|\nabla_h u^n\|_{H^{0.1}}^2 + \|\nabla_h u^n\|_{H^{0.1}}^2) \), and satisfies
\[
\int_0^T R^n(s) ds \leq C(N + 1)(T + 1) \tag{4.8}
\]
owing to the definition of \( \varphi_{N,u^n} \). Denote
\[
Y^n(t) = \sup_{0 \leq s \leq t} \|u^n\|_{H^2}^2 + \int_0^t \|\nabla_h u^n(s)\|_{H^2}^2 ds,
\]
and introduce a stopping time \( \sigma_{n,K} \) for \( n, K = 1, 2, \ldots \) by
\[
\sigma_{n,K} = \begin{cases} \inf \{0 \leq t < T : Y^n(t) \geq K\}, \\ T, \text{ if the set } \{\ldots\} \text{ is empty.} \end{cases}
\]
Then for any \( 0 \leq t \leq T \), we have
\[
Y^n(t \wedge \sigma_{n,K}) \leq 2\|u^n_0\|_{H^2}^2 + C \int_0^t Y^n(s \wedge \sigma_{n,K}) R^n(s) ds
\]
\[
+ \sup_{0 \leq t \leq T \wedge \sigma_{n,K}} 2 \left| \int_0^t (f(s), u^n(s))_2 dW(s) \right|
\]
\[
+ \int_0^{T \wedge \sigma_{n,K}} \|f(s)\|_{L^2(u,H^2)}^2 ds. \tag{4.9}
\]
By the Grönwall inequality, in view of (4.8), it holds that \( \mathbb{P} \)-a.s.
\[
Y^n(T \wedge \sigma_{n,K}) \leq C_{N,T} \|u_0\|_{H^2}^2 + C_{N,T} \sup_{0 \leq t \leq T \wedge \sigma_{n,K}} \left| \int_0^t (f(s), u^n(s))_2 dW(s) \right|
\]
\[
+ C_{N,T} \int_0^{T \wedge \sigma_{n,K}} \|f(s)\|_{L^2(u,H^2)}^2 ds, \tag{4.10}
\]
for some constants \( C_{N,T} \) independent of \( n \) and \( K \). Furthermore, by the Burkholder-Davis-Gundy inequality (2.4), we obtain
\[
C_{N,T} \mathbb{E} \sup_{0 \leq t \leq T \wedge \sigma_{n,K}} \left| \int_0^t (f(s), u^n(s))_2 dW(s) \right|
\]
\[
\leq C_{N,T} \mathbb{E} \left( \int_0^{T \wedge \sigma_{n,K}} \|u^n(s)\|_{H^2}^2 \|f(s)\|_{L^2(u,H^2)}^2 ds \right)^{1/2}
\]
\[
\leq \frac{1}{2} \mathbb{E} \left( \sup_{0 \leq t \leq T \wedge \sigma_{n,K}} \|u^n(t)\|_{H^2}^2 \right) + C_{N,T} \mathbb{E} \left( \int_0^{T \wedge \sigma_{n,K}} \|f(s)\|_{L^2(u,H^2)}^2 ds \right). \tag{4.11}
\]
Then taking the mathematical expectation in (4.10), (4.11) implies
\[
\mathbb{E}(Y^n(T \wedge \sigma_{n,K})) \leq C_{N,T} \left( \mathbb{E}(\|u_0\|_{H^2}^2) + \mathbb{E} \left( \int_0^{T \wedge \sigma_{n,K}} \|f(s)\|_{L^2(u,H^2)}^2 ds \right) \right). \tag{4.12}
\]
By passing \( K \to \infty \), we obtain
\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} \|u^n\|_{H^2}^2 + \int_0^T \|\nabla_h u^n(s)\|_{H^2}^2 ds \right) \leq C_{N,T}. \tag{4.13}
\]
4.3. The existence of the solutions for the modified system. We consider the set
\[ \Omega_1 = \bigcup_{L=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} (\|u^n\|_{C([0,T];H^2)} + \|\nabla_h u^n(\cdot)\|_{L^2([0,T];H^2)} \leq L). \]
By (4.13), we see that \( \mathbb{P}(\Omega_1) = 1 \). Setting
\[ M(t) = \int_0^t \Pi f(s) dW, \quad M^n(t) = \int_0^t \Pi P_n f(s) dW, \]
and \( \Omega_2 \) be the set of all \( \omega \) for which \( n \to \infty \),
\[ M^n \to M \text{ in } C([0,T];H^2_\sigma), \]
\[ u^n(0) \to u_0 \text{ in } H^2_\sigma, \]
and
\[ (4.5) \text{ holds in the sense of } \mathcal{D}'([0,T] \times T^3), \text{ for all } n \geq 1. \]
We define
\[ \Omega^* = \Omega_1 \cap \Omega_2. \]
Then, \( \mathbb{P}(\Omega^*) = 1 \). Fix any \( \omega^* \in \Omega^* \), there exist some \( L = L(\omega^*) \), and a subsequence denoted by \( \{u^{n_j}(\omega^*)\} \) such that
\[ \|u^{n_j}\|_{C([0,T];H^2)} + \|\nabla_h u^{n_j}\|_{L^2([0,T];H^2)} \leq L, \text{ for all } n_j. \]
(4.14)
Since it holds that
\[ \frac{\partial}{\partial t} (u^{n_j} - M^{n_j}) = \Delta_h u^{n_j} - \varphi_{N,u^n} P_n \Pi (u^{n_j} \cdot \nabla) u^{n_j} \]
in the sense of \( \mathcal{D}'((0,T) \times T^3) \), then
\[ \left\| \frac{\partial}{\partial t} (u^{n_j} - M^{n_j}) \right\|_{L^2([0,T];L^2)} \leq C_T (\|u^{n_j}\|_{C([0,T];H^2)}^2 + 1), \]
(4.15)
where \( C_T \) is a positive constant independent of \( u^{n_j} \) and \( \omega^* \). By virtue of (4.14)-(4.15), we can further extract a subsequence still denoted by \( \{u^{n_j}\} \) such that for a function \( u \),
\[ u^{n_j} \to u \text{ weak star in } L^\infty(0,T;H^2_\sigma), \]
(4.16)
and
\[ u^{n_j} \to u \text{ strongly in } C([0,T];H^2_\sigma), \]
(4.17)
\[ u^{n_j} \cdot \nabla u^{n_j} \to u \cdot \nabla u \text{ strongly in } C([0,T];H^{-1}), \]
(4.18)
because of [36] Corollary 8 and the embedding \( H^2(T^3) \hookrightarrow H^1(T^3) \) is compact.
Moreover, according to Fatou’s Lemma, we know \( \nabla_h u \) and \( \nabla_h u^{n_j} \) are bounded in \( L^2([0,T];H^2) \), combining with (4.17), we obtain
\[ \nabla_h u^{n_j} \to \nabla_h u \text{ strongly in } L^2([0,T];H^1). \]
(4.19)
Thus, we see that
\[ \varphi_{N,u^{n_j}} \to \varphi_{N,u} \text{ strongly in } C([0,T]). \]
This, together with (4.17) and (4.18), yields that
\[ du + \varphi_{N,u} \Pi (u \cdot \nabla u) - \Delta_h u = \Pi f dW, \]
(4.20)
holds in the sense of \( \mathcal{D}'((0,T) \times T^3) \), at \( \omega^* \), and we also have
\[ u \in L^\infty([0,T];H^2_\sigma) \cap C([0,T];H^1_\sigma), \quad \nabla_h u \in L^2([0,T];H^2). \]
(4.21)
4.4. The uniqueness and the stopping time. Let $u_1$ and $u_2$ be two functions which satisfy (4.20), (4.21) and $u(0) = u_2(0) = u_0$, for the same fixed $\omega^* \in \Omega^*$. Define

$$
\hat{\tau}_{N,i} = \begin{cases} 
\inf \left\{ 0 \leq t \leq T \mid \|u_i(t)\|_{H^\alpha_0}^2 + \int_0^t \|\nabla_h u_i(s)\|_{H^\beta_0}^2 ds \geq N \right\}, & i = 1, 2, \\
T, & \text{if the set } \{ \ldots \} \text{ is empty},
\end{cases}
$$

(4.22)

If $\hat{\tau}_{N,1} \wedge \hat{\tau}_{N,2} = 0$, it is obvious that $\hat{\tau}_{N,1} = \hat{\tau}_{N,2}$. Now, we assume $\hat{\tau}_{N,1} \wedge \hat{\tau}_{N,2} > 0$. Since

$$u_i \in C((0, T]); H^\alpha_0), \quad \nabla_h u_i \in L^2([0, T]; H^1), \quad i = 1, 2,$$

and

$$\frac{\partial}{\partial t} (u_1 - u_2) + \Pi(u_1 \cdot \nabla)u_1 - \Pi(u_2 \cdot \nabla)u_2 + \Delta_h(u_1 - u_2) = 0$$

in the sense of $D'(0, \hat{\tau}_{N,1} \wedge \hat{\tau}_{N,2} \times T^3)$, we get

$$\frac{1}{2} \frac{d}{dt} \|\tilde{u}(t)\|_{L^2}^2 + \|\nabla_h \tilde{u}(t)\|_{L^2}^2 = \langle u_2 \cdot \nabla \tilde{u}, \tilde{u} \rangle_0 + \langle \tilde{u} \cdot \nabla u_1, \tilde{u} \rangle_0,$$

(4.23)

where $\tilde{u} = u_1 - u_2$. We know $\langle u_2 \cdot \nabla \tilde{u}, \tilde{u} \rangle_0 = 0$ according to the divergence-free condition of $u_2$. Now we begin to estimate $\langle \tilde{u} \cdot \nabla u_1, \tilde{u} \rangle_0$.

$$\langle \tilde{u} \cdot \nabla u_1, \tilde{u} \rangle_0 = \langle \tilde{u} \cdot \nabla u_1, \tilde{u} \rangle_0 + \langle \tilde{u} \cdot \partial_h u_1, \tilde{u} \rangle_0.$$

(4.24)

For $\langle \tilde{u} \cdot \nabla u_1, \tilde{u} \rangle_0$, by the Sobolev embedding inequality, Hölder’s inequality and Young’s inequality, we have

$$|\langle \tilde{u} \cdot \nabla u_1, \tilde{u} \rangle_0| \leq C\|\tilde{u}\|_{L^2}^2 \|\nabla_h u_1\|_{L^2} \leq C\|\tilde{u}\|_{L^2} \|\tilde{u}\|_{H^{1.0}} \|\nabla_h u_1\|_{H^{0.1}} \leq C\|\tilde{u}\|_{L^2} \|\tilde{u}\|_{L^2} \|\nabla_h \tilde{u}\|_{L^2} \|\nabla_h u_1\|_{H^{0.1}} \leq \frac{1}{4} \|\nabla_h \tilde{u}\|_{L^2}^2 + C\|\tilde{u}\|_{L^2}^2 (\|\nabla_h u_1\|_{H^{0.1}} + \|\nabla_h u_1\|_{H^{0.1}}).$$

(4.25)

Similarly, for $\langle \tilde{u} \cdot \partial_h u_1, \tilde{u} \rangle_0$, using $\partial_h \tilde{u}^3 = -\text{div}_h \tilde{u}^3$, we have

$$|\langle \tilde{u} \cdot \partial_h u_1, \tilde{u} \rangle_0| \leq C\|\tilde{u}^3\|_{L^2} \|\nabla_h u_1\|_{L^2} \leq C\|\tilde{u}^3\|_{H^{0.1}} \|\nabla_h u_1\|_{H^{0.1}} \leq C\|\tilde{u}^3\|_{L^2} \|\tilde{u}\|_{L^2} \|\partial_h u_1\|_{L^2} \leq C\|\tilde{u}^3\|_{L^2} \|\tilde{u}\|_{L^2} \|\partial_h u_1\|_{L^2} \|\partial_h u_1\|_{L^2} \|\partial_h u_1\|_{L^2} \leq C\|\nabla_h \tilde{u}\|_{L^2} \|\tilde{u}\|_{L^2} \|\tilde{u}\|_{L^2} \|\nabla_h \tilde{u}\|_{L^2} A,$$

(4.26)

where $A = \|u_1\|_{H^{0.1}} \|\nabla_h u_1\|_{H^{0.1}}$. By the Young’s inequality, we obtain

$$|\langle \tilde{u} \cdot \partial_h u_1, \tilde{u} \rangle_0| \leq \frac{1}{4} \|\nabla_h \tilde{u}\|_{L^2}^2 + \frac{1}{4} \|\tilde{u}\|_{L^2}^2 (1 + A^2).$$

(4.27)

Combining (4.23)-(4.27), we obtain

$$\|\tilde{u}(t)\|_{L^2}^2 + \int_0^t \|\nabla_h \tilde{u}(s)\|_{L^2}^2 ds \leq C \int_0^t \|\tilde{u}(s)\|_{L^2}^2 (\|\nabla_h u_1\|_{H^{0.1}} + \|\nabla_h u_1\|_{H^{0.1}}^2 + 1 + A^4).$$

(4.28)

By the definition of $\hat{\tau}_{N,i}$ ($i = 1, 2$), for any $t \in \hat{\tau}_{N,1} \wedge \hat{\tau}_{N,2}$, we have

$$\int_0^t (\|\nabla_h u_1\|_{H^{0.1}} + \|\nabla_h u_1\|_{H^{0.1}}^2 + 1 + A^4) ds \leq C_{N,T}.$$
Therefore, invoking the Grönwall’s inequality, we obtain 
\[ u_1(t) = u_2(t), \text{ for all } t \in [0, \hat{\tau}_{N,1} \wedge \hat{\tau}_{N,2}], \]
Thus, \( \hat{\tau}_{N,1} = \hat{\tau}_{N,2} \).

Hence, for each \( \omega \in \Omega^* \), \( \hat{\tau}_N \) associated with a limit function \( u \) (through the above procedure) of a certain subsequence \( \{u^n\} \) is determined uniquely, and also the limit function \( u \) itself is unique on the interval \([0, \hat{\tau}_N] \). By (4.20), It holds that
\[
\langle u(t \wedge \hat{\tau}_N), \phi \rangle_0 - \langle u_0, \phi \rangle_0 = \int_0^{t \wedge \hat{\tau}_N} \left( \Delta_h u(s) - u(s) \cdot \nabla u(s), \phi \right)_0 ds
+ \left( \int_0^{t \wedge \hat{\tau}_N} f(s) dW(s), \phi \right)_0 ,
\]
for all \( 0 \leq t < \infty \) and all \( \phi \in L^2_\omega \). Moreover, we have that \( \hat{\tau}_N \) is a stopping time, and \( u(t \wedge \hat{\tau}_N) \) is measurable, and
\[
E \left( \|u(\cdot \wedge \hat{\tau}_N)\|_{L^\infty(0,T;H^2)} + \|\nabla_h u(\cdot \wedge \hat{\tau}_N)\|_{L^2(0,T;H^2)}^2 \right) \leq C_{N,T},
\]
where \( C_{N,T} \) is the same constant as in (4.13). The details of proofs could be found in [26] Lemma 3.2 and Lemma 3.3, we do not repeat them here.

4.5. The continuity of \( u(t \wedge \hat{\tau}_N) \) in \( H^2 \). Let \( \rho_\varepsilon = \rho_\varepsilon(x), \varepsilon > 0 \), be the Friedich modifier, then it holds 
\[
u(t \wedge \hat{\tau}_N) \ast \rho_\varepsilon - u_0 \ast \rho_\varepsilon = \int_0^{t \wedge \hat{\tau}_N} \left( \Delta_h u - \Pi(u(s) \cdot \nabla u(s)) \right) \ast \rho_\varepsilon ds
+ \int_0^{t \wedge \hat{\tau}_N} \Pi f(s) \ast \rho_\varepsilon dW(s)
\]
for all \( 0 \leq t < \infty \) and all \( \varepsilon > 0 \), for each \( \omega \in \Omega^* \) (the convolution must be understood on the whole \( \mathbb{R}^2 \), where each term in above equality is extended by periodicity). Hence, by Ito’s formula,
\[
\|u(t_2 \wedge \hat{\tau}_N) \ast \rho_\varepsilon\|^2_{H^2} - \|u(t_1 \wedge \hat{\tau}_N) \ast \rho_\varepsilon\|^2_{H^2}
= -\int_{t_1 \wedge \hat{\tau}_N}^{t_2 \wedge \hat{\tau}_N} 2\|\nabla_h u(s) \ast \rho_\varepsilon\|_{H^1} ds
- \int_{t_1 \wedge \hat{\tau}_N}^{t_2 \wedge \hat{\tau}_N} 2\|u(s) \cdot \nabla u(s) \ast \rho_\varepsilon, u(s) \ast \rho_\varepsilon\|_{H^1} ds
+ \int_{t_1 \wedge \hat{\tau}_N}^{t_2 \wedge \hat{\tau}_N} 2\langle f(s) \ast \rho_\varepsilon \rangle \ast \rho_\varepsilon dW(s)
+ \int_{t_1 \wedge \hat{\tau}_N}^{t_2 \wedge \hat{\tau}_N} \left\| \Pi f(s) \ast \rho_\varepsilon \right\|^2_{L^2(U,H^2)} ds
\]
for all \( 0 \leq t_1 < t_2 < \infty \), and \( \varepsilon = \varepsilon_n = \frac{1}{n}, n = 1, 2, \ldots \), for each \( \omega \in \Omega^* \), where \( \Omega \subset \Omega^* \) with \( P(\Omega \setminus \bar{\Omega}) = 0 \). We need the following version of the Friedlich Lemma and the commutator estimate, which proofs could be found in [22] and [25], respectively.

Lemma 4.2. Let \( u \in \dot{W}^{1,\infty} \) and \( v \in L^2 \). It holds that
\[
\| (u \partial_j v) \ast \rho_\varepsilon - u (\partial_j v \ast \rho_\varepsilon) \|_{L^2} \leq C \| \nabla u \|_{L^\infty} \| v \|_{L^2}
\]
for some constant \( C > 0 \) independent of \( \varepsilon > 0 \), \( u \) and \( v \). For each fixed \( u \) and \( v \), the left-hand side tends to zero as \( \varepsilon \to 0 \).
Lemma 4.3. Let $u \in H^s \cap \dot{W}^{1,\infty}$ and $v \in H^{s-1} \cap L^{\infty}$. It holds that
\[
\sum_{|\alpha| \leq s} \|\partial^\alpha (u - v)\|_{L^2} \leq C \left( \|\nabla u\|_{L^\infty} \|v\|_{H^{s-1}} + \|u\|_{H^s} \|v\|_{L^\infty} \right) \tag{4.34}
\]
for some constant $C > 0$ independent of $u$ and $v$.

Then, by (4.33), (4.34) and (2.1), we have
\[
\left| \left( u \cdot \nabla u \right) \ast \rho_\varepsilon, u \ast \rho_\varepsilon \right|_2 \leq C \|u\|_{H^2} \left( \sum_{|\alpha| \leq 2} \left( \|\partial^\alpha (u \cdot \nabla u) \|_{L^2} + \|\partial^\alpha \nabla u \ast \rho_\varepsilon, \partial^\alpha u \ast \rho_\varepsilon \|_{L^2} \right) \right.
\]
\[
\left. + \left( u \cdot \nabla (\partial^\alpha u \ast \rho_\varepsilon), \partial^\alpha u \ast \rho_\varepsilon \right)_2 \right) \leq C \|u\|_{H^2}^2 (\|u\|_{H^2} + \|\nabla u\|_{H^2}), \tag{4.35}
\]
where the following inequality is used,
\[
\|\nabla u\|_{L^\infty} \leq C \|\nabla u\|_{H^{\frac{3}{2}}} + \|\nabla \nabla u\|_{H^{\frac{3}{2}}} \leq C (\|u\|_{H^2} + \|\nabla u\|_{H^2}). \tag{4.36}
\]
Therefore, by Young’s inequality, we see
\[
\lim_{\varepsilon \to 0} \left| \int_{t_1 \wedge \tau_N}^{t_2 \wedge \tau_N} \left( u(s) \cdot \nabla u(s) \right) \ast \rho_\varepsilon, u(s) \ast \rho_\varepsilon \, ds \right| \leq C \int_{t_1 \wedge \tau_N}^{t_2 \wedge \tau_N} \left( \|u(s)\|_{H^2}^2 + \|u(s)\|_{H^2}^3 + \|\nabla u(s)\|_{H^2}^2 \right) ds \tag{4.37}
\]
for all $0 \leq t_1 < t_2 < \infty$, and all $\omega \in \hat{\Omega}$, for some constants $C > 0$. By the Burkholder-Davis-Gundy inequality (2.4), (4.30), $f \in L^2(\Omega; L^2_{\text{loc}}([0, \infty); L^2(\Omega; H^2)))$ and the identity
\[
\langle f \ast \rho_\varepsilon, u \ast \rho_\varepsilon \rangle_2 = \langle f \ast \rho_\varepsilon \ast \rho_\varepsilon, u \rangle_2,
\]
we have
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \left| \int_0^{t \wedge \tau_N} 2\langle f(s) \ast \rho_\varepsilon, u(s) \ast \rho_\varepsilon \rangle_{2} dW(s) - \int_0^{t \wedge \tau_N} 2\langle f(s), u(s) \rangle_{2} dW(s) \right| \right) \leq C \mathbb{E} \left( \sup_{0 \leq t \leq T} \|u(t \wedge \tau_N)\|_{H^2} \left( \int_0^T \|f(s) \ast \rho_\varepsilon \ast \rho_\varepsilon - f(s) \|_{L^2(\Omega; H^2)}^2 ds \right)^{1/2} \right) \to 0, \text{ as } \varepsilon \to 0.
\]
Therefore, there is a subsequence of $\{\varepsilon_n\}$ still denoted by $\{\varepsilon_n\}$ and a subset $\hat{\Omega} \subset \Omega$ such that $\mathbb{P}(\Omega \setminus \hat{\Omega}) = 0$, and
\[
\int_0^{t \wedge \tau_N} 2\langle f(s) \ast \rho_\varepsilon, u(s) \ast \rho_\varepsilon \rangle_{2} dW(s) \to \int_0^{t \wedge \tau_N} 2\langle f(s), u(s) \rangle_{2} dW(s) \tag{4.38}
\]
in $C([0, T])$, as $\varepsilon_n \to 0$, for each $\omega \in \hat{\Omega}$. By virtue of (4.37), (4.38) and the fact that $u(t \wedge \tau_N) \in L^\infty_{\text{loc}}(0, \infty; H^2)$, $\nabla u(\cdot \wedge \tau_N) \in L^2_{\text{loc}}(0, \infty; H^2)$ for each $\omega \in \Omega^*$, we can
pass \( \varepsilon = \varepsilon_n \to 0 \) in (4.32) to arrive at

\[
||u(t_2 \wedge \hat{\tau}_N)||_{H^2}^2 - ||u(t_1 \wedge \hat{\tau}_N)||_{H^2}^2 \leq C \int_{t_1 \wedge \hat{\tau}_N}^{t_2 \wedge \hat{\tau}_N} (||u(s)||_{H^2}^2 + ||u(s)||_{H^2}^4 + \|\nabla_h u(s)\|_{H^2}^2) \, ds \\
+ \int_{t_1 \wedge \hat{\tau}_N}^{t_2 \wedge \hat{\tau}_N} 2\langle f(s), u(s) \rangle_2 \, dW(s) + \int_{t_1 \wedge \hat{\tau}_N}^{t_2 \wedge \hat{\tau}_N} \|f(s)\|^2_{L^2(\Omega)} \, ds \tag{4.39}
\]

for all \( 0 \leq t_1 < t_2 < \infty \), and all \( \omega \in \hat{\Omega} \). Since for each \( \omega \in \Omega^* \), \( u(\cdot \wedge \hat{\tau}_N) \in C([0, T]; H^2_\sigma) \), \( u(\cdot \wedge \hat{\tau}_N) \in L^\infty(0, T; H^2_\sigma) \) and \( \nabla_h u \in L^2(0, T; H^2) \), (4.39) implies that

\[ u(\cdot \wedge \hat{\tau}_N) \in C([0, T]; H^2_\sigma) \]

for almost all \( \omega \). The proof of Theorem 4.1 is therefore completed. \( \square \)

5. The proof of Theorem 3.2.

5.1. The existence of local strong solutions. The procedure is classical, we refer the reader to [26] Section 3.2-Section 3.3 for more details. Let \( u_{N,k} \) be the solution obtained in Theorem 4.1 with \( T = k, k = 1, 2, \ldots \), and \( \hat{\tau}_{N,k} \) be a stopping time defined by

\[
\hat{\tau}_{N,k} = \begin{cases} 
\inf \{0 \leq t \leq k \|u_{N,k}(t)\|_{H^2}^2 + \int_0^t \|\nabla_h u_{N,k}(s)\|_{H^2}^2 \, ds \geq N\}, \\
k, \text{ if the set } \{\ldots\} \text{ is empty.}
\end{cases} 
\tag{5.1}
\]

There is some subset \( \Omega_{N,k} \) such that \( \mathbb{P}(\Omega \setminus \Omega_{N,k}) = 0 \), and \( u_{N,k} \) and \( \hat{\tau}_{N,k} \) satisfy the properties (i)-(iii) in Theorem 4.1 for all \( k = 1, 2, \ldots \), for each \( \omega \in \Omega_{N,k} \). Let \( \Omega_0 = \bigcap_{N,k=1}^\infty \Omega_{N,k}, \) then \( \mathbb{P}(\Omega_0) = 1 \). We define, for each \( \omega \in \Omega_0 \),

\[
\tau_{N} = \lim_{k \to \infty} \hat{\tau}_{N,k}, \\
\tau = \lim_{N \to \infty} \tau_{N},
\]

and

\[ u(t) = \lim_{N \to \infty} \lim_{k \to \infty} u_{N,k}(t \wedge \hat{\tau}_{N,k}), \text{ for each } 0 \leq t < \tau. \]

By the uniqueness of \( u_{N,k} \) on \([0, \tau_{N,k}]\), \( u, \tau_{N} \) and \( \tau \) is well defined. It is easy to see that

\[ u \in C([0, \tau]; H^2_\sigma), \quad \nabla_h u \in L^2_{\text{loc}}([0, \tau]; H^2), \text{ for each } \omega \in \Omega_0. \]

We set

\[ u(t) = 0, \text{ if } t \geq \tau. \]

Then, \( u(\cdot) \) is right continuous on \([0, \infty)\), and \( u(t) \) is \( \mathcal{F}_t \)-measurable for each \( 0 \leq t < \infty \). Consequently, it is \( H^2_\sigma \)-valued progressively measurable, and it satisfies

\[ u \in L^2(\Omega; L^\infty_{\text{loc}}(0, \infty; H^2_\sigma)), \quad \nabla_h u \in L^2(\Omega; L^2_{\text{loc}}(0, \infty; H^2)). \]

We also have

\[
\tau_{N} = \begin{cases} 
\inf \{0 \leq t < \infty \|u(t)\|^2_{H^0,1} + \int_0^t \|\nabla_h u(s)\|^2_{H^0,1} \, ds \geq N\}, \\
\infty, \text{ if the set } \{\ldots\} \text{ is empty},
\end{cases}
\]

it follows from (4.2) that

\[
\langle u(t \wedge \tau_{N}), \phi \rangle_0 - \langle u_0, \phi \rangle_0 = \int_0^{t \wedge \tau_{N}} \langle \Delta_h u(s) - u(s) \cdot \nabla u(s), \phi \rangle_0 \, ds
\]
Therefore, combining (5.4)-(5.6), we get
\[ \sum_{0 \leq t < \infty} \phi \in L^2, \text{and all } N \geq 1, \text{ for each } \omega \in \Omega_0. \text{ This } (u, \tau) \text{ is a local strong solution of } (1.1) \text{ according to Definition 3.1.} \]

5.2. The estimate of \( \mathbb{P}\{ \tau > \delta \} \). Let \((u, \tau)\) be the solution obtained in Section 5.1. Since \( u(\cdot \wedge \tau) \in C([0, \infty); H^2) \) and \( \nabla_h u(\cdot \wedge \tau) \in L^2_{loc}([0, \infty); H^2) \) a.s., we can define
\[
T_N = \inf \{ 0 \leq t < \infty \mid \| u(t \wedge \tau) \|^2_{H^2} + \int_0^{t \wedge \tau} \| \nabla_h u(s) \|^2_{H^2} ds \geq N \},
\]
then, \( T_N \) is a stopping time and \( T_N \leq \tau_N \) a.s.. By Itô’s formula in \( H^2 \) for (1.1), we derive that
\[
\begin{align*}
\| u(t \wedge T_N) \|^2_{H^2} + 2 \int_0^{t \wedge T_N} \| \nabla_h u(s) \|^2_{H^2} ds &= \| u_0 \|^2_{H^2} + 2 \int_0^{t \wedge T_N} \langle u(s) \cdot \nabla u(s), u(s) \rangle_{H^2} ds + 2 \int_0^{t \wedge T_N} \langle u(s), f(s) \rangle_{H^2} ds + \int_0^{t \wedge T_N} \| \Pi f(s) \|^2_{L_2(\Omega; H^2)} ds \\
&\leq C \int_0^{t \wedge T_N} \| u(s) \|^2_{H^2} + C \int_0^{t \wedge T_N} \| \nabla_h u(s) \|^2_{H^2} ds + C N \int_0^t \| u(s \wedge T_N) \|^2_{H^2} ds.
\end{align*}
\]
for all \( 0 \leq t < \infty \), for almost all \( \omega \). By means of (2.1) and Young’s inequality, we have
\[
\begin{align*}
2 \int_0^{t \wedge T_N} \langle u(s) \cdot \nabla u(s), u(s) \rangle_{H^2} ds &\leq C \int_0^{t \wedge T_N} \| u_0 \|^2_{H^2} + C \int_0^{t \wedge T_N} \| \nabla_h u(s) \|^2_{H^2} ds + C N \int_0^t \| u(s \wedge T_N) \|^2_{H^2} ds.
\end{align*}
\]
By the Burkholder-Davis-Gundy inequality (2.4), we obtain
\[
\begin{align*}
\mathbb{E} \left( \sup_{0 \leq \eta \leq t} \int_0^{t \wedge \eta} \langle u(s), f(s) \rangle_{H^2} ds \right) &\leq C \mathbb{E} \left( \int_0^{t \wedge T_N} \| u(\eta \wedge T_N) \|^2_{H^2} + \| f(s) \|^2_{L_2(\Omega; H^2)} ds \right)^{1/2} \\
&\leq \frac{1}{2} \mathbb{E} \left( \int_0^t \| u(s \wedge T_N) \|^2_{H^2} ds + \mathbb{E} \left( \int_0^t \| f(s) \|^2_{L_2(\Omega; H^2)} ds \right) \right).
\end{align*}
\]
Therefore, combining (5.4)-(5.6), we get
\[
\begin{align*}
\mathbb{E} \left( \sup_{0 \leq \eta \leq t} \| u(\eta \wedge T_N) \|^2_{H^2} + \int_0^{t \wedge T_N} \| \nabla_h u(s) \|^2_{H^2} ds \right) &\leq C \mathbb{E} \left( \| u_0 \|^2_{H^2} \right) + C \mathbb{E} \left( \int_0^t \| u(s \wedge T_N) \|^2_{H^2} ds \right) + C \mathbb{E} \left( \int_0^t \| f(s) \|^2_{L_2(\Omega; H^2)} ds \right).
\end{align*}
\]
(5.7)
By the Grönwall’s inequality, it from (5.7) that, for any $0 < \delta < 1$,
\[
\mathbb{E}\left( \sup_{0 \leq t \leq \delta} \| u(t \wedge T_N) \|_{L^2}^2 + \int_0^{\delta \wedge T_N} \| \nabla_h u(s) \|_{H^2}^2 ds \right) 
\leq C \left( \mathbb{E} \left( \| u_0 \|_{H^2}^2 \right) + \mathbb{E} \left( \int_0^\delta \| f(s) \|_{L^2(\Omega;H^2)}^2 ds \right) \right) \exp(CN\delta),
\]
(5.8)
where $C$ denotes positive constant independent of $u$, $N$ and $\delta$. By the definition of $T_N$, one has
\[
\{ \omega | T_N(\omega) \leq \delta \} \subset \left\{ \omega | \sup_{0 \leq t \leq \delta} \| u(t \wedge T_N) \|_{H^2}^2 + \int_0^{\delta \wedge T_N} \| \nabla_h u(s) \|_{H^2}^2 ds \geq N \right\}.
\]
By Chebyshev’s inequality, it follows from (5.8) that
\[
P(\{ T_N \leq \delta \}) \leq \frac{C \exp(CN\delta)}{N} \left( \mathbb{E} \left( \| u_0 \|_{H^2}^2 \right) + \mathbb{E} \left( \int_0^\delta \| f(s) \|_{L^2(\Omega;H^2)}^2 ds \right) \right). \quad (5.9)
\]
Choose an integer $N > 0$ such that
\[
\frac{1}{N + 1} \leq \delta < \frac{1}{N}.
\]
Thus, we have
\[
P(\tau > \delta) \geq P(\{ T_N > \delta \} \geq 1 - C\delta \left( \mathbb{E} \left( \| u_0 \|_{H^2}^2 \right) + \mathbb{E} \left( \int_0^\delta \| f(s) \|_{L^2(\Omega;H^2)}^2 ds \right) \right),
\]
where $C$ denotes positive constant independent of $u$ and $\delta$.

5.3. The estimate of $P(\{ \tau = \infty \})$. Let $(u, \tau)$ be the solution obtained in Section 5.1. We introduce another stopping time as the following:
\[
\sigma_\xi = \left\{ \begin{array}{ll}
\inf \{ 0 \leq t < \infty | \| \partial_\gamma u(t \wedge \tau) \|_{L^2} + \int_0^{t \wedge \tau} \| \partial_\gamma \nabla_h u(s) \|_{L^2}^2 ds \geq \xi \} & \text{if the set } \{ \ldots \} \text{ is empty}, \\
\tau, & \text{otherwise}.
\end{array} \right.
\]
(5.10)
First, we have the following lemma.

**Lemma 5.1.** For any $\xi > 0$, $\sigma_\xi < \tau$ almost surely on the set $\{ \tau < \infty \}$.

**Proof.** Define $A_k^N = \{ \tau_N \leq \sigma_k \} \cap \{ \tau_N \leq k \}$ for $k \geq 1$ and $N \geq 1$. By the definition of $\tau_N$ and $\sigma_\xi$, we have
\[
A_k^N \subset \left\{ \| u(k \wedge \tau_N) \|_{L^2}^2 + \int_0^{k \wedge \tau_N} \| \nabla_h u(s) \|_{L^2}^2 ds \geq N - \xi \right\}.
\]
(5.11)
Applying Itô’s formula in $L^2$ for (1.1) and recall the divergence-free condition of $u$, we have
\[
\begin{aligned}
\| u(k \wedge \tau_N) \|_{L^2}^2 + 2 \int_0^{k \wedge \tau_N} \| \nabla_h u(s) \|_{L^2}^2 ds &= \| u_0 \|_{L^2}^2 + 2 \int_0^{k \wedge \tau_N} \langle u(s), f(s) \rangle_0 dW(s) \\
& \quad + \int_0^{k \wedge \tau_N} \| f(s) \|_{L^2(\Omega;L^2)}^2 ds.
\end{aligned}
\]
(5.12)
for almost all $\omega$. Therefore,
\[
E \left( \|u(k \wedge \tau_N)\|_{L^2}^2 + \int_0^{k \wedge \tau_N} \|\nabla_h u(s)\|_{L^2}^2 ds \right) 
\leq E \left( \|u_0\|_{L^2}^2 \right) + E \left( \int_0^\infty \|f(s)\|_{L^2(\Omega)}^2 ds \right).
\]
(5.13)

By Chebyshev’s inequality, when $N > \xi$, it from (5.11) that
\[
\mathbb{P}(A_k^N) \leq \frac{1}{N - \xi} \left( E \left( \|u_0\|_{L^2}^2 \right) + E \left( \int_0^\infty \|f(s)\|_{L^2(\Omega)}^2 ds \right) \right).
\]
(5.14)

Define $A_k = \{ \tau \leq \sigma_\xi \} \cap \{ \tau \leq k \}$. Then $A_k \subset A_k^N$ for all $N$ and then $\mathbb{P}(A_k) = 0$. Since $\{ \tau \leq \sigma_\xi \} \cap \{ \tau < \infty \} = \bigcup_{k=1}^\infty A_k$, then $\mathbb{P}(\{ \tau \leq \sigma_\xi \} \cap \{ \tau < \infty \}) = 0$. \hfill \Box

Now, we begin to prove (3.5). Applying $\partial_3$ and then Itô’s formula in $L^2$ to (1.1), we can derive that
\[
\|\partial_3 u(t \wedge \tau_N \wedge \sigma_\xi)\|_{L^2}^2 + 2 \int_0^{t \wedge \tau_N \wedge \sigma_\xi} \langle \partial_3 u \cdot \nabla u, \partial_3 u \rangle ds 
\leq \|\partial_3 u_0\|_{L^2}^2 + 2 \int_0^{t \wedge \tau_N \wedge \sigma_\xi} \langle \partial_3 u, \partial_3 f \rangle ds + \int_0^{t \wedge \tau_N} \|\partial_3 f(s)\|_{L^2(\Omega)}^2 ds
\]
for all $0 \leq t < \infty$, for almost all $\omega$. By (2.2), the Sobolev embedding inequality, the interpolation inequality and Young’s inequality, we have
\[
\left| 2 \int_0^{t \wedge \tau_N \wedge \sigma_\xi} \langle \partial_3 (u \cdot \nabla u), \partial_3 u \rangle ds \right| 
\leq C \int_0^{t \wedge \tau_N \wedge \sigma_\xi} \|\partial_3 u\|_{L^2} \|\partial_3 \nabla u\|_{L^2} \|\nabla_h u\|_{L^\infty L^2} ds
\leq C \int_0^{t \wedge \tau_N \wedge \sigma_\xi} \|\partial_3 u\|_{L^2} \|\partial_3 \nabla u\|_{L^2} \|\nabla_h u\|_{H^0 \frac{1}{2}} ds
\leq C \int_0^{t \wedge \tau_N \wedge \sigma_\xi} \|\partial_3 u\|_{L^2} \|\partial_3 \nabla u\|_{L^2} \|\nabla_h u\|_{L^2} \frac{2}{7 - \zeta} \|\nabla_h u\|_{L^2} \frac{1}{7 + \zeta} ds
\leq C \int_0^{t \wedge \tau_N \wedge \sigma_\xi} \|\partial_3 \nabla u\|_{L^2} ds + C \int_0^{t \wedge \tau_N \wedge \sigma_\xi} \left( \|\partial_3 u\|_{L^2}^2 + \|\partial_3 u\|_{L^2}^{7 - \zeta} \right) \|\nabla_h u(s)\|_{L^2} ds
\leq \int_0^{t \wedge \tau_N \wedge \sigma_\xi} \|\partial_3 \nabla u\|_{L^2}^2 ds + C(\xi + \xi^{\frac{2}{7}}) \int_0^{t \wedge \tau_N \wedge \sigma_\xi} \|\nabla_h u(s)\|_{L^2}^2 ds,
\]
(5.16)

where $\zeta$ is a sufficient small positive constant. We define
\[
\alpha = E \left( \|u_0\|_{L^2}^2 \right) + E \left( \int_0^\infty \|f(s)\|_{L^2(\Omega)}^2 ds \right),
\]
\[
\beta = E \left( \|\partial_3 u_0\|_{L^2}^2 \right) + E \left( \int_0^\infty \|\partial_3 f(s)\|_{L^2(\Omega)}^2 ds \right),
\]
Combining (5.13), (5.15) and (5.16), we have
\[ E \left( \| \partial_3 u(t \land \tau_N \land \sigma_\xi) \|_{L^2}^2 + \int_0^{t \land \tau_N \land \sigma_\xi} \| \partial_3 \nabla_h u(s) \|_{L^2}^2 ds \right) \leq C^* \left( \beta + (\xi + \xi \frac{2m}{2m-1}) E \int_0^{t \land \tau_N \land \sigma_\xi} \| \nabla_h u(s) \|_{L^2}^2 ds \right) \]
\[ \leq C^* \left( \beta + (\xi + \xi \frac{2m}{2m-1}) \alpha \right). \quad (5.17) \]

By Lemma 5.1, there exists some \( \hat{\Omega} \subset \Omega \) such that \( \mathbb{P}(\Omega \setminus \hat{\Omega}) = 0 \), and for each \( \omega \in \hat{\Omega} \), either \( \sigma_\xi(\omega) < \tau(\omega) < \infty \) or \( \tau_N(\omega) \uparrow \infty \). Therefore, for each \( 0 \leq t < \infty \) and \( \omega \in \hat{\Omega} \), we deduce that
\[ \lim_{N \to \infty} t \land \tau_N \land \sigma_\xi = t \land \sigma_\xi. \quad (5.18) \]

By Fatou’s Lemma, (5.17) and (5.18) imply
\[ E \left( \| \partial_3 u(t \land \sigma_\xi) \|_{L^2}^2 + \int_0^{t \land \sigma_\xi} \| \partial_3 \nabla_h u(s) \|_{L^2}^2 ds \right) \leq \beta + (\xi + \xi \frac{2m}{2m-1}) \alpha, \quad 0 \leq t < \infty. \]

(5.19)

Let us denote
\[ \mathcal{G} = \{ \sigma_\xi < \infty \}, \]
and \( \chi_\mathcal{G} \) is the characteristic function of the set \( \mathcal{G} \). By Fatou’s Lemma, for \( 0 \leq t < \infty \), we have
\[ \xi \mathbb{P}(\mathcal{G}) = E \left( \chi_\mathcal{G} \lim_{t \to \infty} \left( \| \partial_3 u(t \land \sigma_\xi) \|_{L^2}^2 + \int_0^{t \land \sigma_\xi} \| \partial_3 \nabla_h u(s) \|_{L^2}^2 ds \right) \right) \]
\[ \leq \inf_{t \to \infty} E \left( \chi_\mathcal{G} \left( \| \partial_3 u(t \land \sigma_\xi) \|_{L^2}^2 + \int_0^{t \land \sigma_\xi} \| \partial_3 \nabla_h u(s) \|_{L^2}^2 ds \right) \right) \]
\[ \leq C^* \left( \beta + (\xi + \xi \frac{2m}{2m-1}) \alpha \right). \]

Therefore
\[ \mathbb{P}(\mathcal{G}) \leq C^* \left( \xi^{-1} \beta + (1 + \xi \frac{2m}{2m-1}) \alpha \right). \]

By Lemma 5.1, we have
\[ \mathbb{P}(\{ \tau < \infty \}) = \mathbb{P}(\{ \tau < \infty \} \cap \mathcal{G}) \leq \mathbb{P}(\mathcal{G}). \]

Hence
\[ \mathbb{P}(\{ \tau = \infty \}) > 1 - C^* \left( \xi^{-1} \beta + (1 + \xi \frac{2m}{2m-1}) \alpha \right). \]

We choose \( \xi = \alpha \frac{1-2\xi}{2} \beta^{1-2\xi} \), then
\[ \mathbb{P}(\{ \tau = \infty \}) \geq 1 - C^* \left( \alpha \frac{1-2\xi}{2} \beta^{1+2\xi} + \alpha \right). \]

The proof of Theorem 3.2 is therefore completed. \( \Box \)

6. Appendix.

The proof of Lemma 2.1. We first recall the anisotropic Sobolev embedding inequality which would be used below. If \( 2 \leq p, q \leq \infty \) and \( 1 \leq m, n \leq \infty \) satisfy
\[ \begin{cases} \frac{1}{q} \geq \frac{1}{2} - \frac{m}{2}, & \text{if } m < 1, \\ q \in [2, \infty), & \text{if } m = 1, \quad \text{and} \quad \begin{cases} \frac{1}{p} \geq \frac{1}{2} - n, & \text{if } n < \frac{1}{2}, \\ p \in [2, \infty), & \text{if } n = \frac{1}{2}, \end{cases} \\ q = \infty, & \text{if } m > 1, \quad \begin{cases} \frac{1}{p} = \frac{1}{p} = \frac{1}{2}, \quad \text{if } n = \frac{1}{2}, \\ p = \infty, & \text{if } n > \frac{1}{2}, \end{cases} \end{cases} \]

(6.1)
by the classical Sobolev embedding inequality, for \( u \in H^{m,n} \), we have
\[
\|u\|_{L^p L^q_k} \leq C\|u\|_{H^{m,n}},
\]
where \( C \) is a constant independent of \( u \). By the divergence-free condition of \( u \), we have
\[
\langle u \cdot \nabla u, u \rangle_2 = \sum_{1 \leq i,j,k \leq 3} \langle \partial_k u^i \partial_l u^j, \partial_k u^j \rangle_0 + \sum_{1 \leq i,j,k,l \leq 3} \langle \partial_k u^i \partial_l u^j, \partial_k u^j \rangle_0 + \langle \partial_k u^i \partial_l u^j, \partial_k u^j \rangle_0
\]
\[
\sum_{1 \leq i,j,k,l \leq 3} H_{i,j,k} + \sum_{1 \leq i,j,k,l \leq 3} I_{i,j,k,l}^1 + 2I_{i,j,k,l}^2.
\]
For estimating \( H_{i,j,k} \), we consider the following two cases:

1. \( i \neq 3 \) or \( k \neq 3 \)

\[
|H_{i,j,k}| \leq C\|\nabla u\|_{L^2} \|\nabla u\|^2_{L^2}
\]
\[
\leq C\|\nabla u\|_{H^{0,1}}^2 \|u\|^2_H.
\]

2. \( i = k = 3 \)

\[
|H_{3,j,3}| = |(\partial_3 u^i \partial_3 u^j, \partial_3 u^j_0)|
\]
\[
= |(\text{div} u^i \partial_3 u^j, \partial_3 u^j_0)|
\]
\[
\leq C\|\nabla u\|_{L^2} \|\nabla u\|^2_{L^2}
\]
\[
\leq C\|\nabla u\|_{H^{0,1}}^2 \|u\|^2_H.
\]

Now, we begin to estimate \( I_{i,j,k,l}^1 \):

1. \( i \neq 3 \)

\[
|I_{i,j,k,l}^1| \leq C\|\nabla^2 u\|^2_{L^2 L^2_k} \|\nabla u\|_{L^\infty L^2_k}
\]
\[
\leq C\|\nabla^2 u\|^2_{H^{0,\frac{1}{2}}} \|\nabla u\|_{H^{0,\frac{1}{2}}}
\]
\[
\leq C\|u\|_{H^2} \|\nabla^2 u\|_{H^{0,1}} \|\nabla u\|_{H^{0,1}}
\]
\[
\leq C\|u\|_{H^2} (\|u\|_{H^2} + \|\nabla u\|_{H^2}) \|\nabla u\|_{H^{0,1}}.
\]

2. \( i = 3 \), and \( k \neq 3 \) or \( l \neq 3 \)

\[
|I_{3,j,k,l}^1| \leq C\|\nabla \nabla u\|_{L^\infty L^2_k} \|\partial_3 u\|_{L^2 L^2_k} \|\nabla^2 u\|_{L^2}
\]
\[
\leq C\|\nabla \nabla u\|_{H^{0,\frac{1}{2}}} \|\partial_3 u\|_{H^{0,1}} \|u\|_{H^2}
\]
\[
\leq C\|\nabla u\|_{H^2} \|u\|_{H^0,1} + \|\nabla u\|_{H^{0,1}}.
\]

3. \( i = k = l = 3 \)

\[
|I_{3,j,3,3}^1| = |(\partial_3 u^i \partial_3 u^j, \partial_3 u^j_0)|
\]
\[
= |(\partial_3 \text{div} u^i \partial_3 u^j, \partial_3 u^j_0)|
\]
\[
\leq C\|\nabla \nabla u\|_{L^\infty L^2_k} \|\partial_3 u\|_{L^2 L^2_k} \|\nabla^2 u\|_{L^2}
\]
\[
\leq C\|\nabla \nabla u\|_{H^{0,\frac{1}{2}}} \|\partial_3 u\|_{H^{0,1}} \|u\|_{H^2}
\]
\[
\leq C\|\nabla u\|_{H^2} \|u\|_{H^0,1} + \|\nabla u\|_{H^{0,1}}.
\]

Then, we consider \( I_{i,j,k,l}^2 \).
Combining (6.3)-(6.11), we obtain
\[ |I_{i,j,k,l}^2| \leq C_1 \| \nabla u \|_{L^\infty L^2} \| \nabla^2 u \|_{L^2 L^2}^2 \leq C_2 \| \nabla u \|_{H^2} (\| u \|_{H^2} + \| \nabla u \|_{H^2}) \| \nabla^2 u \|_{L^2} \].

We have
\[ |u \cdot \nabla u, \partial^3 u \rangle \leq C (\| u \|_{H^2}^2 \| \nabla u \|_{H^3} + \| u \|_{H^2} \| \nabla u \|_{H^2} (\| u \|_{H^3} + \| \nabla u \|_{H^3})). \]

For estimating \( \langle u \cdot \nabla u, \partial^3 u \rangle \), we employ the method of [33]. Let \( u(x) = \int_{x_0}^{x} u(\cdot, x_3) \, dx_3 \), \( \tilde{u} = u - u \). By Hölder’s inequality and Poincaré’s inequality, there exists a constant \( C \), such that
\[ \| \partial_3 u \|_{L^2} \leq C \| \partial_3 u \|_{L^2}, \quad \| \partial_3 \tilde{u} \|_{L^2} \leq C \| \partial_3 u \|_{L^2}, \quad \| \partial_3 \tilde{u} \|_{H^{n,0}} \leq C \| \partial_3 \nabla u \|_{L^2}. \]

Now, we begin to estimate \( \langle \partial^3 (u \cdot \nabla u), \partial^3 u \rangle \). We obtain
\[ \langle \partial_3 (u \cdot \nabla u), \partial^3 u \rangle = \langle \partial_3 u^h \cdot \nabla u, \partial_3 u \rangle + \langle \partial_3 u^h \cdot \nabla u, \partial_3 u \rangle \]
\[ = \langle \partial_3 u^h \cdot \nabla \tilde{u}, \partial_3 u \rangle + \langle \partial_3 u^h \cdot \nabla \tilde{u}, \partial_3 u \rangle + \langle \partial_3 u^h \cdot \nabla \tilde{u}, \partial_3 u \rangle + \langle \partial_3 u^h \cdot \nabla \tilde{u}, \partial_3 u \rangle + \langle \partial_3 u^h \cdot \nabla \tilde{u}, \partial_3 u \rangle + \langle \partial_3 u^h \cdot \nabla \tilde{u}, \partial_3 u \rangle \]
\[ \quad + \langle \partial_3 u^h \cdot \nabla \tilde{u}, \partial_3 u \rangle + \langle \partial_3 u^h \cdot \nabla \tilde{u}, \partial_3 u \rangle + \langle \partial_3 u^h \cdot \nabla \tilde{u}, \partial_3 u \rangle \]
\[ =: \sum_{i=1}^{7} J_i, \]
and \( J_1 = J_2 = J_6 = 0 \) by using the divergence-free condition of \( u \). We have
\[ |J_3| \leq \| \partial_3 u \|_{L^2} \| \nabla \tilde{u} \|_{L^2 L^2} \| \partial_3 \tilde{u} \|_{L^2} \]
\[ \leq C \| \partial_3 u \|_{L^2} \| \nabla \tilde{u} \|_{L^2 L^2} \| \partial_3 \tilde{u} \|_{L^2} \]
\[ \leq C \| \partial_3 u \|_{L^2} \| \partial_3 \nabla u \|_{L^2} \| \nabla \tilde{u} \|_{L^2 L^2} \]
we obtain
\begin{align}
|J_5| & \leq \|\partial_3 \hat{u}\|_{L^2 H^1_{\gamma \lambda}} \|\nabla_h \hat{u}\|_{L^\infty L^2_{\gamma \lambda}} \\
& \leq C \|\partial_3 \hat{u}\|_{H^1_{\gamma \lambda}} \|\nabla_h \hat{u}\|_{L^\infty L^2_{\gamma \lambda}} \\
& \leq C \|\partial_3 \hat{u}\|_{L^2} \|\partial_3 \nabla_h u\|_{L^2} \|\nabla_h u\|_{L^\infty L^2_{\gamma \lambda}}.
\end{align}
(6.16)

For $J_7$, because the divergence-free condition of $u$, we have
\begin{align}
-J_7 &= \langle \text{div}_h \hat{u}, \partial_3 \hat{u}, \partial_3 \hat{u}\rangle_0 + 2 \langle \text{div}_h \hat{u}, \partial_3 \hat{u}, \partial_3 \hat{u}\rangle_0 + \langle \text{div}_h \hat{u}, \partial_3 \hat{u}, \partial_3 \hat{u}\rangle_0 \\
& := \sum_{i=1}^3 J_{7i}. \quad (6.17)
\end{align}

We have $J_7^3 = 0$, and as similar as $J_5$ and $J_4$,
\begin{align}
|J_{7i}| & \leq C \|\nabla_h \hat{u}\|_{L^\infty L^2_{\gamma \lambda}} \|\partial_3 \hat{u}\|_{L^2 H^1_{\gamma \lambda}} \\
& \leq C \|\partial_3 u\|_{L^2} \|\partial_3 \nabla_h u\|_{L^2} \|\nabla_h u\|_{L^\infty L^2_{\gamma \lambda}}, \quad (6.18)
\end{align}
\begin{align}
|J_{7j}| & \leq C \|\nabla_h \hat{u}\|_{L^\infty L^2_{\gamma \lambda}} \|\partial_3 \hat{u}\|_{L^2} \|\partial_3 u\|_{L^2} \\
& \leq C \|\partial_3 u\|_{L^2} \|\partial_3 \nabla_h u\|_{L^2} \|\nabla_h u\|_{L^\infty L^2_{\gamma \lambda}}. \quad (6.19)
\end{align}

Combining (6.13)-(6.19), we obtain
\begin{align}
|\langle \partial_3 (u \cdot \nabla u), \partial_3 u\rangle_0| & \leq C \|\partial_3 u\|_{L^2} \|\partial_3 \nabla_h u\|_{L^2} \|\nabla_h u\|_{L^\infty L^2_{\gamma \lambda}}.
\end{align}
(6.20)

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