The present paper, the first of a series, represents the first part of an investigation of abstract convolution equations. A preliminary communication [8] appeared already in the Soviet Doklady in 1974.

The aim of these investigations is to develop a functional-analytic theory of Hörmander's results on convolution equations. It is obvious that such a theory must contain two essential parts. The first task is to find a suitable abstract analogue of the notion of "singular support" of a distribution. This line of research started with the 1966 paper [5] and was pursued further in [11], [1] and [8], [8']. The second step consist in formulating criteria for

\[ F' = (\lim E_n)' \quad \text{or} \quad F = \lim F \cap E_n \]

where \( E_n \) is a sequence of Fréchet spaces and \( F \subseteq E = \lim E_n \). Results in this direction have been obtained in [9].

We shall use the following terminology and notation. An \( F_0 \) space will be a locally convex space the topology of which is given by a sequence of pseudonorms; it follows that a separated and complete \( F_0 \) space is a Fréchet space.

Given two topologies \( u_1 \) and \( u_2 \) on a set \( T \) we say that \( u_1 \) is coarser than \( u_2 \) or that \( u_2 \) is finer than \( u_1 \) if \( u_1 \subseteq u_2 \). In other words, a finer topology has more open sets and gives, accordingly, smaller closures. We shall denote by \( u_1 \lor u_2 \) the topology generated by the union \( u_1 \cup u_2 \), in other words, the coarsest topology which is finer than both \( u_1 \) and \( u_2 \).

\[(1.1) \quad \text{Lemma. Let } F \text{ be a linear space and } w_1 \text{ and } w_2 \text{ two convex topologies on } F. \text{ Let } u = w_1 \lor w_2. \text{ Then } (F, u)' = (F, w_1)' + (F, w_2)' .\]

Proof. The mapping \( x \mapsto [x, x] \) is an algebraically and topologically isomorphic injection of \( (F, w_1 \lor w_2) \) into \( (F, w_1) \oplus (F, w_2) \). Its adjoint mapping takes the pair \( [f_1, f_2] \in (F, w_1) \oplus (F, w_2) \) into its sum.

\[(1.2) \quad \text{Proposition. Let } (E_1, u_1), (E_2, u_2), (E_3, u_3) \text{ be three } F_0 \text{ spaces. Let }\]

\[
T: (E_1, u_1) \mapsto (E_3, u_3),
\]

\[
A: (E_1, u_1) \mapsto (E_2, u_2)
\]
be two continuous linear mappings. Let $U$ be a fixed closed absolutely convex neighborhood of zero in $(E_1, u_1)$. Denote by $u$ the topology on $E_1$ generated by the set $U$ and suppose that $(E_1, u)$ is a normed space.

Then the following conditions are equivalent:

1° $A'E_2' \subset T'E_3' + (E_1, u)'$.

2° $A$ is continuous from $(E_1, u \vee T^{-1}u_3)$ into $(E_2, u_2)$; in other words: if $x_n \to 0$ in $(E_1, u)$ and $Tx_n \to 0$ then $Ax_n \to 0$.

3° If $x_n$ is a sequence such that $x_n$ is Cauchy in $(E_1, u)$ and $Tx_n$ is Cauchy in $(E_3, u_3)$ then there exists a sequence $x'_n$ such that $x'_n \to x_n \to 0$ in $(E_1, u)$, $Tx'_n \to Tx_n \to 0$ in $(E_3, u_3)$ and $Ax'_n$ is Cauchy in $(E_2, u_2)$; furthermore, if $z_n \to 0$ in $(E_1, u)$, $Tz_n \to 0$ in $(E_3, u_3)$ and $Az_n$ is Cauchy in $(E_2, u_2)$ then $Az_n \to 0$ in $(E_2, u_2)$.

4° If $x_n$ is a sequence such that $x_n$ is Cauchy in $(E_1, u)$ and $Tx_n$ is Cauchy in $(E_3, u_3)$ then there exists a sequence $x'_n$ such that $x'_n \to x_n \to 0$ in $(E_1, u)$ and $Ax'_n$ is Cauchy in $(E_2, u_2)$; at the same time, if $z_n \to 0$ in $(E_1, u)$, $Tz_n \to 0$ in $(E_3, u_3)$ and $Az_n$ is Cauchy in $(E_2, u_2)$ then $Az_n \to 0$ in $(E_2, u_2)$.

Proof. According to lemma (1,1) we have

$$(E_1, u \vee T^{-1}u_3)' = (E_1, u)' + T'E_3'.$$

Condition 1° may thus be restated as follows: the mapping $A$ is continuous in the weak topologies corresponding to $u \vee T^{-1}u_3$ and $u_2$. All spaces in question being $F_0$ spaces weak and strong continuity coincides. This establishes the equivalence of 1° and 2°.

For the rest of the proof, it will be convenient to introduce some notation. Let $T_0$ and $A_0$ be the mapping from $(E_1, u)$ respectively into $(E_3, u_3)$ and $(E_2, u_2)$ which coincide with $T$ and $A$ as mappings of linear spaces, hence $T = T_0v$ and $A = A_0v$ where $v$ is the injection of $(E_1, u_1)$ into $(E_1, u)$.

Denote by $G(T_0)$ and $G(A_0)$ their graphs in $(E_1, u) \times (E_3, u_3)$ and $(E_1, u) \times (E_2, u_2)$. Denote by $A^\Box$ the mapping of $G(T_0)$ into $G(A_0)$ defined as follows

$$A^\Box[x, Tx] = [x, Ax].$$

We set

$$T^\sim = TP_1 \mid G(T_0) = P_2T^\Box, \quad A^\sim = AP_1 \mid G(T_0) = P_2A^\Box.$$ 

The implications $2° \to 3° \to 4°$ are immediate. Suppose now that condition 4° is satisfied.

Consider the set $M \subset (E, u)^\wedge \times E_3^\wedge \times E_2^\wedge$ defined as follows: The triple $[e_1, e_3, e_2]$ belongs to $M$ if and only if $[e_1, e_3] \in G(T_0)^\sim$ and at the same time $[e_1, e_2] \in G(A_0)^\sim$. Here the closures are taken in the completions of the spaces in question. It follows from the definition of the set $M$ that it is closed in $(E_1, u)^\wedge \times E_3^\wedge \times E_2^\wedge$.
implies $e_2 = 0$. The set $M$ is, therefore, the graph of a mapping from $G(T_0)^-$ into $E_2^\dagger$. Hence the mapping $A^\dagger$ is closable. Let us show that the domain of $M$ is the whole of $G(T_0)^-$. Indeed, let $[e_1, e_3] \in G(T_0)^-$. It follows that there exists a sequence $x_n \in E_1$ such that $x_n \to e_1$ in $(E_1, u)^\dagger$ and $Tx_n \to e_3 \in (E_3, u_3)^\dagger$. According to 4° there exists a sequence $x_n' \in E$ such that $x_n' - x_n \to 0$ in $(E_1, u)$ and $Ax_n'$ is a Cauchy sequence in $E_2$. It follows that $x_n' \to e_1$ in $(E_1, u)$ and $Ax_n' \to e_2$ for a suitable $e_2 \in (E_2, u_2)^\dagger$ so that $[e_1, e_2] \in G(A_0)^-$; hence $[e_1, e_3, e_2] \in M$. To sum up; the closure of $A^\dagger$ is again a mapping and is defined on the whole of $G(T_0)^-$. It follows from the closed graph theorem that $A^\dagger$ is continuous so that $A^\dagger = P_2 A^\dagger$ is continuous as well. We complete the proof by proving the implication 4° $\to$ 1°.

Since the mapping

$$A^\dagger = P_2 A^\dagger : [x, Tx] \mapsto Ax$$

is continuous from $G(T_0)$ into $(E_2, u_2)$, it follows that, for each $e_2' \in (E_2, u_2)'$ the function

$$[x, Tx] \mapsto \langle Ax, e_2' \rangle$$

is continuous on $G(T_0)$. Hence there exist two functionals $e_1' \in (E_1, u)'$ and $e_3' \in (E_3, u_3)'$ such that

$$\langle Ax, e_2' \rangle = \langle x, e_1' \rangle + \langle Tx, e_3' \rangle = \langle x, e' + T'e_3' \rangle$$

whence

$$A' e_2' = e' + T' e_3' \in (E_1, u)' + T'E_3'.$$

This proves 1°.

Conditions 3° and 4° may be restated in the form of statements about domains of definition of certain mappings. We shall use the following notation. If $G$ is the graph of a mapping from $F_1$ into $F_2$ we shall denote by $D(G^-)$ the projection on $F_1^\dagger$ of the closure $G^-$ in $F_1^\dagger \times F_2^\dagger$. The set $D(G^-)$ will be called the domain of definition of $G^-$; of course, in the general case, $G^-$ need not be the graph of a mapping from $F_1^\dagger$ into $F_2^\dagger$.

First of all, let us notice that the second part of conditions 3° and 4° asserts that the mapping $A^\dagger$ is closable. Using this fact, condition 3° assumes the following form

5° The mapping $A$ is closable and $G(T_0)^- \subset D(G(A^\dagger)^-)$. 

Since clearly $D(G(T_-)^-) = G(T_0)^-$, we have the following equivalent form of 3°

6° The mapping $A^\dagger$ is closable and $D(G(T_-)^-) \subset D(G(A^\dagger)^-)$. 

Let us turn to condition 4°. Its second part may be interpreted as the closability of $A^\dagger$. In view of this condition 4° may be restated in each of the two following equivalent forms.

7° The mapping $A^\dagger$ is closable and $G(T_0)^- \subset D(G(A^\dagger)^-)$. 

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8° The mapping $A^\square$ is closable and
\[ D(G(T_0)^-) \subseteq D(G(A_0)^-) . \]

In the sequel we shall often identify $G(T_0)$ with the space $(E_1, \bar{u})$ where $\bar{u} = u \vee T^{-1}u_3$. Accordingly, $T^-$ and $A^-$ will be taken as the mappings $T$ and $A$ considered as mappings of $(E_1, \bar{u})$ into $(E_3, u_3)$ and $(E_2, u_2)$ respectively.

\begin{enumerate}
\item[(1,3)] \textbf{Proposition.} The following conditions are equivalent:
\item[1°] If $x_n \in U$ and $Tx_n \to 0$ then $Ax_n$ tends to zero in the weak topology of $E_2$.
\item[2°] For every $\varepsilon > 0$, the set $A'E_2'$ is contained in $T'E_3' + \varepsilon U^0$.
\item[3°] The mapping $A^-\varepsilon$ is continuous and
\[ \Ker T^- \subseteq \Ker A^- \varepsilon . \]
\item[4°] The mapping $A^-\varepsilon$ is continuous and $\Ker T^- \subseteq \Ker (T_0 \oplus A_0)^\varepsilon$.
\item[5°] The mapping $A^-\varepsilon$ is continuous and if $\xi \in (E, u)^\varepsilon$ annihilates $(E, u)' \cap T'E_3'$ then $\xi$ annihilates $(E, u)' \cap (T'E_3' + A'E_2')$.
\item[6°] The mapping $A^-\varepsilon$ is continuous and the subspace $(E, u)' \cap (T'E_3' + A_2'E_2')$ is contained in the closure of $(E, u)' \cap T'E_3$ in the strong topology of the space $(E, u)'$.
\item[7°] The weak topology on $E_1$ generated by $A'E_2'$ is coarser than that generated by $T'E_3$ when restricted to $U$; in other words
\[ \sigma(E_1, A'E_2') \cap U = \sigma(E_1, T'E_3') . \]
\item[8°] The weak topology on $E_1$ generated by $A'E_2'$ is coarser than the topology $T^{-1}u_3$ when restricted to $U$; in other words if $W$ is an arbitrary neighbourhood of zero in the topology $\sigma(E_1, A'E_2')$ then there exists a neighbourhood of zero $U_3$ in $(E_3, u_3)$ such that
\[ U \cap T^{-1}U_3 \subseteq W . \]
\end{enumerate}

\textbf{Proof.} Suppose that condition 1° is satisfied and that a positive number $\varepsilon$ is given. Let us prove that $A'(E_2, u_2)' < T'(E_3, u_3)' + \varepsilon U^0$. If not, then there exists a $g_0' \in \varepsilon U^0 + T'W_n^0$ such that, for each $n$, the point $A'g_0'$ lies outside the set $\varepsilon U^0 + T'W_n^0$ where $W_n$ runs over a fundamental system of neighbourhoods of zero in $(E_3, u_3)$. The sets $\varepsilon U^0 + T'W_n^0$ being $\sigma((E_1, u_1)', E_1)$ compact, there exists, for each natural number $n$, an element $x_n \in E_1$ such that $\langle x_n, \varepsilon U^0 + T'W_n^0 \rangle \leq \varepsilon$ and $\langle x_n, A'g_0' \rangle > \varepsilon$.

In particular, $\langle x_n, U^0 \rangle \leq 1$ whence $x_n \in U^{00} = U$ and $\langle T x_n, W_n^0 \rangle \leq 1$ so that $T x_n \in W_n$. It follows from condition 1° that $Ax_n$ tends to zero weakly in $(E_2, u_2)$; however, $\langle Ax_n, g_0' \rangle = \langle x_n, A'g_0' \rangle > \varepsilon$ which is a contradiction. This proves condition 2°.
Now assume condition 2°. It follows that $A'E'\subset T'E'+(E_1, u)' = (E_1, \tilde{u})'$ so that $A$ is continuous as a mapping of $(E_1, \tilde{u})$ into $(E_2, u_2)$. Suppose now that $\xi \in (E_1, \tilde{u})'' = ((E_1, \tilde{u})', (E_1, u_1), (E_1, \tilde{u}))'$ is given and that $T''\xi = 0$. It follows that $\langle \xi, T'(E_3, u_3) \rangle = 0$. Now let us denote by $P\xi$ the restriction of $\xi$ to $(E_1, u)$. Since $\xi$ is bounded on the polar $B^0$ of some set $B$ bounded in $(E_1, \tilde{u})$, $\xi$ is bounded on $U^0$ since $B \subset \lambda U$ for some $\lambda$. It follows that $P\xi$ may be considered as an element of the second dual of the normed space $(E_1, u)$. Let $\beta$ be a number greater than $|P\xi|$, the norm of $P\xi$ in $(E_1, u)'$. 

Now let $g' \in (E_2, u_2)'$ and a positive $\varepsilon$ be given. According to our assumption, there exists an $f' \in (E_3, u_3)'$ and an $x' \in (E_1, u)'$ such that $A'g' = T'f' + x'$ and $|x'| < \varepsilon\beta^{-1}$. It follows that $\langle \xi, A'g' \rangle = \langle \xi, T'f' \rangle + \langle \xi, x' \rangle = \langle P\xi, x' \rangle$ whence $|\langle \xi, A'g' \rangle| \leq \varepsilon$. Since $\varepsilon$ was an arbitrary positive number, we have proved that $\langle \xi, A'g' \rangle = 0$ for every $g' \in (E_2, u_2)'$ or, in other words that $A'\xi = 0$.

Let us prove that condition 3° implies 1°. Let $x_n \in U$ and suppose that $Tx_n \to 0$. Denote by $M$ the set of all elements of the sequence $x_n$. Since $M$ is bounded in $(E_1, \tilde{u})$ and $A$ is continuous, the set $AM$ is bounded in $(E_2, u_2)$. Let $g' \in (E_2, u_2)'$ be given and suppose that $\langle Ax_n, g' \rangle$ does not tend to zero. The sequence $\langle Ax_n, g' \rangle$ being bounded, there exists a subsequence $y_n$ of the sequence $x_n$ such that $\langle Ay_n, g' \rangle$ converges to a limit different from zero. Since $y_n \in M$ there exists a cluster point $\eta$ of the sequence $y_n$ in the topology $\sigma((E_1, \tilde{u})', (E_1, \tilde{u}))$. Let us prove that $T''\eta = 0$. Indeed, if $f' \in (E_3, u_3)'$ is given, the product $\langle \eta, T'f' \rangle$ is cluster point of the sequence $\langle y_n, T'f' \rangle = \langle Ty_n, f' \rangle \to 0$. It follows that $\langle \eta, T'f' \rangle = 0$. Since $f'$ was arbitrary we have $T''\eta = 0$. It follows from our assumption that $A'\eta = 0$ so that, in particular, $\langle \eta, A'g' \rangle = 0$.

Now $\langle \eta, A'g' \rangle$ is a cluster point of the sequence $\langle y_n, A'g' \rangle$ because $A$ is continuous. This sequence, however, tends to a limit different from zero, a contradiction. This proves condition 1° hence the equivalence of the first three conditions.

Conditions 5° and 6° are equivalent by the Hahn-Banach theorem. Let us prove the implications 2° $\to$ 5° $\to$ 1°.

Suppose 2° satisfied. It follows from Proposition (1,2) that $A$ is continuous. Consider a $\xi \in (E_1, u)^*$ which annihilates $T'E_3 \cap (E_1, u)'$. Suppose that $e' = T'e_3' + A'e_2' \in (E_1, u)'$ and let $\varepsilon > 0$ be given. According to 2°, we have a decomposition

$$A'e_2' = T'f_3' + g$$

where $g \in (E_1, u)'$ and $|g| < \varepsilon|\xi|^{-1}$ if $\xi \neq 0$. Hence

$$e' = T'e_3' + T'f_3' + g.$$ 

Since $e', g \in (E_1, u)'$, we have $T'(e_3' + f_3') \in (E_1, u)'$ so that, by our assumption, $\langle \xi, T'(e_3' + f_3') \rangle = 0$. Hence $|\langle \xi, e' \rangle| = |\langle \xi, g \rangle| \leq |\xi||g| < \varepsilon$. Since $\varepsilon$ was an arbitrary positive number, $\langle \xi, e' \rangle = 0$ and 5° is established.
Now assume 5° satisfied and let \( x_n \in U, \quad Tx_n \to 0 \). Let \( e_2' \in (E_2, u_2)' \) be given. Since \( \tilde{A} \) is continuous, there exists, by Proposition (1,2), a decomposition

\[
A'e_2' = T'e_3' + f
\]

with \( f \in (E_1, u)' \). It follows that \( f \in (A'E_2' + T'E_3') \cap (E_1, u)' \). Suppose that \( \langle Ax_n, e_2' \rangle \) does not tend to zero. Then \( \langle x_n, f \rangle \) does not tend to zero. Otherwise we would have \( \langle Ax_n, e_2' \rangle = \langle x_n, A'e_2' \rangle = \langle x_n, T'e_3' \rangle + \langle x_n, f \rangle \to 0 \) which is a contradiction. Therefore there exists a cluster point \( \xi \in (E_1, u)' \) such that \( \langle \xi, f \rangle \neq 0 \). If \( h \in T'E_3' \cap (E_1, u)' \) then \( h = T'e_3' \) for a suitable \( e_3' \in E_3' \).

Since \( h \in (E, u)' \), the number \( \langle \xi, h \rangle \) is a cluster point of the sequence \( \langle x_n, T'e_3' \rangle \). We have, however, \( \langle x_n, T'e_3' \rangle = \langle Tx_n, e_3' \rangle \to 0 \). Since \( h \) was an arbitrary element of the intersection \( T'E_3' \cap (E_1, u)' \), we see that \( \xi \) annihilates \( T'E_3' \cap (E_1, u)' \). It follows from our assumption that \( \xi \) annihilates \( (T'E_3' + A'E_2') \cap (E_1, u)' \), in particular, \( \xi \) annihilates \( f \). This is a contradiction.

Clearly the conditions 5° and 6° are equivalent by the Hahn-Banach theorem.

Let us prove now the equivalence of 4° and 5°. If \( S \) is linear mapping from a locally convex space \( P \) into another locally convex space \( Q \) and if \( \xi \in P' \) we write \( \xi \in \text{Ker} \; S' \) if and only if \( \xi \) annihilates the range of \( S' \). The range of \( S' \) is the set of all \( x' \in P' \) such that

\[
\langle Sx, y' \rangle = \langle x, x' \rangle
\]

for a suitable \( y' \in Q' \) and all \( x \in D(S) \). The equivalence of 4° and 5° will therefore be established if we show that

\[
R(T_0') = (E_1, u)' \cap T'E_3', \quad R((T_0 \oplus A_0)') = (E_1, u)' \cap (T'E_3' + A'E_2').
\]

First of all, \( x' \in R(T_0') \) if and only if there exists an \( e_3' \) such that

\[
\langle T_0x, e_3' \rangle = \langle x, x' \rangle
\]

for all \( x \in E_1 \); in other words if and only if \( x' = T'e_3' \) or \( x' \in (E_1, u)' \cap T'E_3' \). Similarly, \( x' \in R((T_0 \oplus A_0)') \) if and only if there exist \( e_3' \) and \( e_2' \) such that

\[
\langle T_0x, e_3' \rangle + \langle A_0x, e_2' \rangle = \langle x, x' \rangle
\]

for all \( x \in E_1 \); in other words if and only if \( x' = T'e_3' + A'e_2' \) or \( x' \in (E_1, u)' \cap \cap (T'E_3' + A'E_2') \).

This completes the proof of the equivalence of 4° and 5°.

To complete the proof, we intend to prove the implications 2° \( \to 7° \to 8° \to 1° \). First of all, the inclusion

\[
\sigma(E_1, T'E_3') \subset T^{-1}u_3
\]
is obvious. Hence $7° \rightarrow 8°$. Also, the implication $8° \rightarrow 1°$ is immediate. It remains to prove the implication $2° \rightarrow 7°$.

Suppose that $2°$ is satisfied and let us prove the following fact. If $x_0 \in U$ and if $V$ is a $\sigma(E, A'E_2')$ neighbourhood of $x_0$ then there exists a $\sigma(E, T'E_3')$ neighbourhood $W$ of $x_0$ such that $W \cap U \subset V$. First of all, there exist $f_1, \ldots, f_n \in E_2'$ such that $|\langle x - x_0, A'f_j \rangle| < 1$ for $j = 1, 2, \ldots, n$ implies $x \in V$. According to $2°$, each $A'f_j$ has a decomposition of the form

$$A'f_j = T'g_j + h_j$$

where $g_j \in E_3'$ and $h_j \in \frac{1}{2}U_0$. Denote by $W$ the set

$$W = \{x; |\langle x - x_0, T'g_j \rangle| < \frac{1}{2}\}.$$

If $x \in W \cap U$, we have, for each $j$

$$|\langle x - x_0, A'f_j \rangle| \leq |\langle x - x_0, T'g_j \rangle| + |\langle x - x_0, h_j \rangle| < 1$$

so that $x \in V$. The proof is complete.

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