Annihilation Rate of Heavy $0^{++}$ P-wave Quarkonium in Relativistic Salpeter Method

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Abstract

Two-photon and two-gluon annihilation rates of P-wave scalar charmonium and bottomonium up to third radial excited states are estimated in the relativistic Salpeter method. We solved the full Salpeter equation with a well defined relativistic wave function and calculated the transition amplitude using the Mandelstam formalism. Our model dependent estimates for the decay widths: $\Gamma(\chi_{c0} \to 2\gamma) = 3.78 \text{ keV}$, $\Gamma(\chi'_{c0} \to 2\gamma) = 3.51 \text{ keV}$, $\Gamma(\chi_{b0} \to 2\gamma) = 48.8 \text{ eV}$ and $\Gamma(\chi'_{b0} \to 2\gamma) = 50.3 \text{ eV}$. We also give estimates of total widths by the two-gluon decay rates: $\Gamma_{\text{tot}}(\chi_{c0}) = 10.3 \text{ MeV}$, $\Gamma_{\text{tot}}(\chi'_{c0}) = 9.61 \text{ MeV}$, $\Gamma_{\text{tot}}(\chi_{b0}) = 0.887 \text{ MeV}$ and $\Gamma_{\text{tot}}(\chi'_{b0}) = 0.914 \text{ MeV}$.

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I. INTRODUCTION

It is well known that two-photon or two-gluon annihilation rate of heavy quarkonium $c\bar{c}$ or $b\bar{b}$ is related to the wave function, so this process will be helpful to understand the formalism of inter-quark interactions, and can be a sensitive test of the potential model \[1\]. With a replacement of the photons by gluons, the final state becomes two gluon state, which will be helpful to give the information on total width of the corresponding quarkonium.

In previous letter \[2\], two-photon and two-gluon annihilation rates of $0^{-+}$ pseudoscalar $c\bar{c}$ and $b\bar{b}$ states are computed in the relativistic Salpeter method, good agreement of our predictions with other theoretical calculations and available experimental data is found. We also found the relativistic corrections are important and cannot be ignored. In this letter, we extend our previous analysis to include the P-wave $0^{++}$ scalar $c\bar{c}$ and $b\bar{b}$ states, present a relativistic calculation of these states decaying into two photons or two gluons.

For the theoretical estimates of the two-photon or two-gluon annihilation rate, we have various methods readily available in hand \[3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\]. First was the non-relativistic calculation, the corresponding decay width is related to the derivative of the non-relativistic P-wave function at the origin, this method will cause large uncertainty because in a full relativistic calculation the decay width is related to the full behavior of P-wave function which can be seen in this letter or in Ref. \[2\]. So the relativistic corrections is very important. In recent years, many authors try to focus on the relativistic corrections and there are already some versions of relativistic calculation, and they give improved results over the non-relativistic methods. In this letter, we give yet another relativistic calculation by the instantaneous Bethe-Salpeter method \[14\], which is a full relativistic method \[15\] with a well defined relativistic form of wave function.

In previous letter \[2\], we have pointed out that there are two sources of relativistic corrections; one is the correction in relativistic kinematics which appears in the decay amplitudes through a well defined form of relativistic wave function (related to the full behavior of wave function, not merely related to the derivative of the wave function at origin); the other relativistic correction comes via the relativistic inter-quark dynamics, which requires not only a well defined relativistic wave function but also a good relativistic formalism to describe the interactions among quarks. The Bethe-Salpeter equation and its instantaneous version, Salpeter equation, are well-known tools to describe relativistic bound states. In this letter we will solve the full Salpeter equation for $0^{++}$ state, and use the full Salpeter wave function to estimate the annihilation decay width of quarkonium.
The form of the wave function is also important in the calculation, since the corrections of the relativistic kinetics come mainly through it. We begin from the quantum field theory, analyze the parity and charge conjugation of bound state, and give a formula for the wave function that is in a relativistic form with definite parity and charge conjugation symmetry. Another important thing is how to use the relativistic wave function of bound state to obtain a relativistic transition amplitude, since a non-relativistic transition amplitude even with a relativistic wave function will lose the benefit of relativistic effects caused by a relativistic wave function. The Mandelstam formalism is well suited for the computation of relativistic transition amplitude, and we begin with this formulism to give a formula of the transition amplitude.

In Sec. II, we give the transition amplitude in Mandelstam formulism and corresponding wave function with a well defined relativistic form. In Sec. III, the full Salpeter equation is solved, and the mass spectra and numerical value of wave function are obtained. Then the two-photon decay width and full width of heavy $0^{++}$ quarkonium are estimated. In Sec. III, short discussions and a summary are also given.

II. THEORETICAL DETAILS

According to the Mandelstam formalism, the relativistic transition amplitude of a quarkonium decaying into two photons (see figure 1) can be written as:

$$T_{2\gamma} = i\sqrt{3} (ie_q)^2 \int \frac{d^4q}{(2\pi)^4} \text{tr} \left\{ \chi(q) [\hat{p}_2 S(p_1 - k_1) \hat{p}_1 + \hat{p}_1 S(p_1 - k_2) \hat{p}_2] \right\},$$

where $k_1, k_2; \varepsilon_1, \varepsilon_2$ are the momenta and polarization vectors of photons; $e_q = \frac{2}{3}$ for charm quark and $e_q = \frac{1}{3}$ for bottom quark; $p_1$ and $p_2$ are the momentum of constitute quark and antiquark; $\chi(q)$ is the quarkonium Bethe-Salpeter wave function with the total momentum $P$ and relative momentum $q$, related by

$$p_1 = \alpha_1 P + q, \quad \alpha_1 \equiv \frac{m_1}{m_1 + m_2},$$

$$p_2 = \alpha_2 P - q, \quad \alpha_2 \equiv \frac{m_2}{m_1 + m_2},$$

where $m_1 = m_2$ is the constitute quark mass of $c$ or $b$.

Since $p_{10} + p_{20} = M$, the approximation $p_{10} = p_{20} = \frac{M}{2}$ is a good choice for the equal mass system \[9,17,18\]. Having this approximation, we can perform the integration over $q_0$
to reduce the expression, with the notation of Salpeter wave function $\Psi(q) = \int \frac{dq}{2\pi} \chi(q)$, to
\[
T_{2\gamma} = \sqrt{3} (ee_q)^2 \int \frac{d\vec{q}}{(2\pi)^3} \text{tr} \left\{ \Psi(\vec{q}) \left[ \frac{1}{p_1 - \vec{k}_1 - m_1} \vec{\gamma}_1 \frac{1}{p_1 - \vec{k}_2 - m_1} \vec{\gamma}_2 \right] \right\}. \tag{2}
\]
With this relativistic amplitude, the two photon decay width can be written as
\[
\Gamma_{2\gamma} = \frac{T_{2\gamma}^2}{16 \pi M}. \tag{3}
\]
The general form for the relativistic wave function of scalar state $J^{PC} = 0^{++}$ can be written as 8 terms constructed by momentum $P$, $q$ and Dirac matrix $\gamma$, because of the approximation of instantaneous, 4 terms with $P \cdot q$ become zero, with further constraint from Salpeter equation, the relativistic Salpeter wave function $\Psi(\vec{q})$ for $0^{++}$ state with a definite parity (+) and charge conjugation (+) can be written as:
\[
\Psi(\vec{q}) = \varphi_1(\vec{q}) \vec{g} + \varphi_2(\vec{q}) \gamma_0 \vec{g} - \frac{q^2}{m_1} \varphi_1(\vec{q}). \tag{4}
\]
The wave function $\varphi_1(\vec{q})$, $\varphi_2(\vec{q})$ and bound state mass $M$ can be obtained by solving the full Salpeter equation with the constituent quark mass as input, and they should satisfy the normalization condition:
\[
\int \frac{d\vec{q}}{(2\pi)^3} \frac{8 \omega_1 q^2}{m_1} \varphi_1(\vec{q}) \varphi_2(\vec{q}) = 2M, \tag{5}
\]
where $\omega_1 = \sqrt{m_1^2 + q^2}$.

The two gluon decay width of quarkonium can be easily obtained from the two photon decay width, with a simple replacement in the photon decay width formula
\[
e_q^4 \alpha^2 \longrightarrow \frac{2}{9} \alpha_s^2. \tag{6}
\]

III. NUMERICAL RESULTS AND DISCUSSIONS

We will not show the details of Solving the full Salpeter equation, only give the final results, interested readers can find the the detail technique in Ref. [19].

When solving the full Salpeter equation, we choose a phenomenological Cornell potential, there are some parameters in this potential including the constitute quark mass and one loop running coupling constant. Since we lack data of $0^{++}$ $c\bar{c}$ states to determine these parameters, we choose the same values as in Ref. [19] which were obtained by fitting the mass spectra for $0^-$ states, but only vary parameter $V_0$ to fit the mass of $\chi_{c0}$, so the following
parameters are adopted: \( a = e = 2.7183, \alpha = 0.06 \text{ GeV}, V_0 = -0.566 \text{ GeV}, \lambda = 0.2 \text{ GeV}^2, \Lambda_{QCD} = 0.26 \text{ GeV} \) and \( m_c = 1.7553 \text{ GeV} \). With this parameter set, we solve the full Salpeter equation and obtain the mass spectra shown in Table I. With the obtained wave function and Eq. (3), we calculate the two-photon decay width of \( c\bar{c} \ 0^{++} \) states, the results are shown in Table I. To give the numerical analysis of two gluons decay, we need to fix the value of the renormalization scale \( \mu \) in \( \alpha_s(\mu) \). In the case of charmonium, we choose the charm quark mass \( m_c \) as the energy scale and obtain the coupling constant \( \alpha_s(m_c) = 0.36 \) \[^{[19]}\]. The corresponding total decay width are also listed in Table I.

For the case of \( b\bar{b} \) system, to determine the parameters, we fit the available masses \( M_{\chi^{0}} = 9.8599 \text{ GeV}, M_{\chi'^{0}} = 10.2321 \text{ GeV} \), and we also fit the mass of \( \eta_b \) at 9.364 GeV \[^{[2]}\]. So the parameters are \( V_0 = -0.553 \text{ GeV}, \Lambda_{QCD} = 0.20 \) and \( m_b = 5.13 \text{ GeV} \), other parameters are same as in the case of \( c\bar{c} \). With this set of parameters, the coupling constant at scale of bottom quark mass is \( \alpha_s(m_b) = 0.232 \). The corresponding mass spectra, two-photon and total decay widths are shown in Table II.

| \( \chi_{c0}(1P) \) | \( 3.4159 \) | \( 3.78 \) | \( 10.3 \) |
| \( \chi_{c0}'(2P) \) | \( 3.8311 \) | \( 3.51 \) | \( 9.61 \) |
| \( \chi_{c0}''(3P) \) | \( 4.1324 \) | \( 2.86 \) | \( 7.83 \) |
| \( \chi_{c0}'''(4P) \) | \( 4.3694 \) | \( 2.42 \) | \( 6.62 \) |

We compare our predictions with recent other theoretical relativistic calculations and experimental results in Table III. Our results of \( \Gamma_{2\gamma}^{\chi_{c0}} \) agree with the estimates ofRefs.
TABLE II: Two-photon decay width and total width of P-wave $0^{++}$ bottomonium states, where the total width is estimated by the two-gluon decay width $\Gamma_{tot} \simeq \Gamma_{2\gamma}$.

| State      | Mass GeV | $\Gamma_{2\gamma}$ eV | $\Gamma_{tot}$ MeV |
|------------|----------|------------------------|---------------------|
| $\chi_{b0}(1P)$ | 9.8601   | 48.8                   | 0.887               |
| $\chi_{b0}^\prime(2P)$ | 10.2239  | 50.3                   | 0.914               |
| $\chi_{b0}^\prime\prime(3P)$ | 10.4970  | 44.7                   | 0.813               |
| $\chi_{b0}^\prime\prime\prime(4P)$ | 10.7192  | 41.9                   | 0.761               |

TABLE III: Recent theoretical and experimental results of two-photon decay width and total width.

|           | $\Gamma_{X_c}^{\chi_c}$ keV | $\Gamma_{tot}^{\chi_c}$ MeV | $\Gamma_{X_b}^{\chi_b}$ keV | $\Gamma_{2\gamma}^{\chi_b}$ eV | $\Gamma_{tot}^{\chi_b}$ MeV | $\Gamma_{2\gamma}^{\chi_b}$ eV |
|-----------|-----------------------------|-----------------------------|-----------------------------|-------------------------------|-----------------------------|-------------------------------|
| Ours      | 3.78                        | 10.3                        | 3.51                        | 48.8                          | 0.887                       | 50.3                          |
| Crater[3] | 3.96, 3.34                  |                             |                             |                               |                             |                               |
| Gupta[8]  | 6.38                        | 13.44                       |                             | 80                            | 2.15                        |                               |
| Huang[9]  | 3.72±1.11                   | 12.5±3.2                    |                             |                               |                             |                               |
| Ebert[10] | 2.9                         | 1.9                         | 38                          | 29                            |                             |                               |
| Münz[11]  | 1.39±0.16                   | 1.11±0.13                   | 24±3                        | 26±2                          |                             |                               |
| ES35[20]  |                             | 9.8±1.0±0.1                 |                             |                               |                             |                               |
| CLEO[21]  | 3.76±0.65±0.41±1.69         |                             |                             |                               |                             |                               |
| PDG[22]   | 2.87±0.54                   | 10.4±0.7                    |                             |                               |                             |                               |
| BES[23]   |                             | 12.6±1.5+0.9−0.9±1.1        |                             |                               |                             |                               |

[3, 9, 21]: All the values of $\Gamma_{tot}^{X_c}$ listed in the table consist with each other. But there are large discrepancy in the results of $\chi_{c0}^\prime$, $\chi_{b0}$ and $\chi_{b0}^\prime$ from different methods, more theoretical calculations and experimental measurements are required for these channels.

To show the importance of the relativistic corrections, we give a estimate of non-relativistic results using the same method and with same parameters. The non-relativistic wave function can be written as $\varphi_1(q)(1+\gamma_0)\tilde{q}$, with this wave function we solve the positive part of Salpeter equation, and delete the contribution of negative part. The non-relativistic normalization condition is $\int d\tilde{q}8\varphi_1(\tilde{q})^2 = 2M (2\pi)^3$. When calculate the transition amplitude, we only keep the lowest order contribution and delete all the higher order contributions by counting the exponential number of $|\tilde{q}|$, the corresponding results are show in Table IV. One can see that the relativistic corrections are important especially for the $c\bar{c}$ system.
TABLE IV: Mass spectra and Two-photon decay width with and without relativistic corrections, where the "non-rel" means the non-relativistic results. The decay width is in unit KeV for $c\bar{c}$ system and eV for $b\bar{b}$ system.

| Mass | GeV  | non-rel | $\Gamma_{2\gamma}$ | non-rel |
|------|------|---------|---------------------|---------|
| $\chi_{c0}(1P)$ | 3.4159 | 3.3931 | 3.78 | 5.85 |
| $\chi'_{c0}(2P)$ | 3.8311 | 3.8267 | 3.51 | 5.47 |
| $\chi''_{c0}(3P)$ | 4.1324 | 4.1543 | 2.86 | 4.61 |
| $\chi''_{c0}(4P)$ | 4.3694 | 4.4205 | 2.42 | 3.94 |
| $\chi_{b0}(1P)$ | 9.8601 | 9.8471 | 48.8 | 58.3 |
| $\chi'_{b0}(2P)$ | 10.2239 | 10.2129 | 50.3 | 59.9 |
| $\chi''_{b0}(3P)$ | 10.4970 | 10.4906 | 44.7 | 54.0 |
| $\chi''_{b0}(4P)$ | 10.7192 | 10.7196 | 41.9 | 48.3 |

We comment that in this work we did not include the QCD radiative correction because we focus mainly on the relativistic corrections, though it is no doubt that the QCD correction is very important and interesting topic.

In summary, by solving the relativistic full Salpeter equation with a well defined form of wave function, we estimate two-photon decay rates: $\Gamma(\chi_{c0} \rightarrow 2\gamma) = 3.78$ keV, $\Gamma(\chi'_{c0} \rightarrow 2\gamma) = 3.51$ keV, $\Gamma(\chi_{b0} \rightarrow 2\gamma) = 48.8$ eV and $\Gamma(\chi'_{b0} \rightarrow 2\gamma) = 50.3$ eV, and the total decay widths: $\Gamma_{tot}(\chi_{c0}) = 10.3$ MeV, $\Gamma_{tot}(\chi'_{c0}) = 9.61$ MeV, $\Gamma_{tot}(\chi_{b0}) = 0.887$ MeV and $\Gamma_{tot}(\chi'_{b0}) = 0.914$ MeV.

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