Teleportation with indistinguishable particles.*

Luca Marinatto†
Department of Theoretical Physics of the University of Trieste, and
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Italy,
and

Tullio Weber‡
Department of Theoretical Physics of the University of Trieste, and
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Italy.

Abstract

We analyze in a critical way the mathematical treatment of a quantum teleportation experiment performed with photon particles, showing that a symmetrization operation over both the polarization and spatial degrees of freedom of all the particles involved is necessary in order to reproduce correctly the observed experimental data.

Key words: Teleportation, Entanglement, Identical particles.

PACS: 0.3.65.Bz

1 Introduction.

One of the most characteristic features of Quantum Mechanics, the one that according to Schrödinger words “enforces its entire departure from classical line of thoughts” [1], is represented by the Entanglement. This property, also commonly known as Quantum Non-Separability, consists in the impossibility of attributing to the constituent subsystems of some composite quantum system well-defined state vectors. Thus, in this very frequent physical situation, which always arises as a consequence of an interaction between the subsystems, it is impossible to ascribe them objective properties (or elements of reality). Entangled states played an important role in the development of the Foundations of Quantum Mechanics since the very first times of its history: in fact the incompleteness argument of EPR is essentially based on the properties of an entangled couple of elementary particles. Moreover, Bell’s inequalities, proving the unavoidable non-local features of every conceivable theory which gives the same probabilistic predictions of the Orthodox Quantum Mechanics, exploit the correlations between the outcomes of measurement processes performed on entangled states. In the present days the interest of the scientific community towards the entanglement has considerably increased since it has been acknowledged

*Work supported in part by Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Italy
†e-mail: marinatto@ts.infn.it
‡e-mail: weber@ts.infn.it
as an essential resource in the development of the quantum theories of Information and Com-
putation. In fact, the possibility of performing successful teleportation procedures of arbitrary
and unknown quantum states [2], the possibility of transmitting two bits of classical information
by manipulating only one qubit [3], and the possibility of implementing quantum algorithms
able to solve in a more efficient way hard computational problems [4], are mainly due to the
non-local correlation properties displayed by the entangled states and constitute only a few of a
larger number of examples which undoubtly prove the usefulness of entanglement. In this paper
we focus our attention on the analysis of the mathematical description of the first experimental
realization of the quantum teleportation process [5, 6], in which an arbitrary polarization state
carried by a photon is transmitted and later reconstructed over a distant spatial location by
means of an entangled quantum channel, corresponding to two more other photon particles. In
order to reproduce correctly the observed experimental data, we will see that a symmetrization
procedure over both the polarization and spatial degrees of freedom of all the indistinguishable
particles involved is needed. The analysis of the experiment shows that the identity of the par-
ticles, and the consequent and compulsory introduction of the spatial part of the Hilbert space,
which is not present in the original work of Bennett et al. [2] since no explicit mention was made
about the nature of the particles, has not been exhaustively stressed in the usual discussions
about the subject.

2 Usual teleportation scheme.

The purpose of the teleportation process consists in making a perfect copy of an arbitrary and
even unknown quantum state at a distant location, in accordance with the special relativity
principles and the no-cloning theorem. This process is successfully achieved by means of the
peculiar features displayed by certain maximally entangled quantum states, which exhibit non-
local correlations between the outcome results of measurement operations. The experiment
described in [5] succeeded in realizing a teleportation scheme permitting the transmission of
an arbitrary polarization state of a photon between two distant spatial regions. It involves the
production of a couple of entangled photons by means of a parametric down conversion process,
a particular kind of Bell-measurement performed by a beamsplitter used as a Bell-state analyzer
and, finally, a classical communication between the two distant parties, which allows the receiver
to successfully reconstruct the initial quantum state.

Before analyzing this experiment, it is worthwhile reviewing briefly the main mathematical
ingredients of the original teleportation scheme devised by Bennett et al. [2]. First of all we
note that they make the tacit assumption that the quantum physical systems involved are
not identical, the global state vector representing the whole system being not symmetrized or
antisymmetrized.

In fact, they consider an unknown two-level quantum state $|\psi\rangle_1 = \alpha|0\rangle_1 + \beta|1\rangle_1$ (where
$\{0, 1\}$ are a complete set of orthonormal vectors) coupled with a maximally entangled sin-
glet state of two other quantum systems constituting a so-called quantum channel, $|\omega\rangle_{23} =
1/\sqrt{2} (|01\rangle_{23} - |10\rangle_{23})$. By performing a Bell-state measurement on the initial state $|\psi\rangle_1 |\omega\rangle_{23}$
onto the four maximally entangled vectors describing the first and the second particle

\[
\begin{align*}
|\phi^\pm_{12}\rangle &= \frac{1}{\sqrt{2}} \left[ |00\rangle_{12} \pm |11\rangle_{12} \right], \\
|\psi^\pm_{12}\rangle &= \frac{1}{\sqrt{2}} \left[ |01\rangle_{12} \pm |10\rangle_{12} \right].
\end{align*}
\] (2.1)

one obtains four possible and equiprobable wave function reduction processes, whose resulting quantum states are:

\[
\begin{align*}
|\psi^+\rangle_1 |\omega\rangle_{23} &\rightarrow |\phi^+\rangle_{12} [-\beta|0\rangle_3 + \alpha|1\rangle_3], \\
&\rightarrow |\phi^-\rangle_{12} [+\beta|0\rangle_3 + \alpha|1\rangle_3], \\
&\rightarrow |\psi^+\rangle_{12} [-\alpha|0\rangle_3 + \beta|1\rangle_3], \\
&\rightarrow |\psi^-\rangle_{12} [-\alpha|0\rangle_3 - \beta|1\rangle_3].
\end{align*}
\] (2.2)

Knowing the Bell-state which has been obtained, one immediately sees that it is possible to get the initial quantum state $|\psi\rangle$, which had to be teleported, by performing suitable unitary manipulations on the third particle.

However, this mathematical treatment of the teleportation process ceases to be valid when considering quantum states which describe indistinguishable physical systems, this situation being the one described in [5]. In fact, in such a case the quantum mechanical formalism for identical particles forces one to consider state vectors with definite symmetry properties under arbitrary permutations of the three particles involved. Although this mathematical requirement appears as obvious, no care is given to the experimental impossibility of distinguishing between particle one, two and three. This indistinguishability descends from the fact that, both in the parametric down conversion process and in the Bell-state measurement within the analyzer, the wave functions of all the particles overlap each other. Moreover, as clearly stated in [3], the considered experimental set up is able to perform a Bell-measurement onto the only Bell state $|\psi^-\rangle$, which is discriminated from the other three by means of a position measurement within an interferometric apparatus. It is therefore necessary to introduce explicitly also the spatial part of the state vector describing the whole system of the three photon particles, in addition to its polarization part, and the symmetrization procedure must be extended to these degrees of freedom too.

It is our purpose to show that, if one follows the description of the teleportation experiment given in [5], it is not possible to perform a successful reconstruction of an unknown polarization state of a photon, this fact being due to the assumption of the distinguishability of one of the three particles involved and to the absence of the spatial part of the state vectors of all the particles, part which should be present from the very beginning of the mathematical description. In fact, the initial vector describing the three photons is once again the one chosen by Bennett et al., i.e. the one composed of the polarization state $|\psi\rangle_1$ of particle 1, which must be teleported, and of an entangled couple of photons described by the usual singlet state $|\omega\rangle_{23}$. It can be written in a more suitable way in terms of the vectors belonging to the Bell basis of equations (2.1) as follows:

\[
|\psi\rangle_1 |\omega\rangle_{23} = \frac{1}{2} \left[ |\phi^+\rangle_{12} [-\beta|0\rangle_3 + \alpha|1\rangle_3] + |\phi^-\rangle_{12} [-\alpha|0\rangle_3 + \beta|1\rangle_3] + |\psi^+\rangle_{12} [-\alpha|0\rangle_3 + \beta|1\rangle_3] - |\psi^-\rangle_{12} [-\alpha|0\rangle_3 + \beta|1\rangle_3] \right].
\] (2.3)

In order to perform the Bell-measurement and complete in such a way the teleportation procedure, we must project this state onto one of the four Bell-states. This task is accomplished
by means of the use of a beamsplitter which is able to discriminate the only state $|\psi^-\rangle$ from
the remaining three, and this happens with probability $1/4$. This Bell-state analyzer acts on
the spatial degrees of freedom of the particles 1 and 2 impinging on it, leaving unchanged their
polarization state, and producing the following effect: if the particles are described by a
antisymmetric spatial state vector, they emerge separately in two different outputs beyond the
beamsplitter, while, if the spatial part is symmetric, they emerge both in the same output.

It is only at this point of the treatment that the authors of [5] consider that the two photons
1 and 2 impinging onto the beamsplitter are identical particles and, consequently, that the Bell
vectors must display an overall symmetry property on the polarization and the spatial degrees
of freedom:

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle)[|A\rangle|B\rangle + |B\rangle|A\rangle],$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle)[|A\rangle|B\rangle - |B\rangle|A\rangle],$$

$$|\phi^+_S\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)[|A\rangle|B\rangle + |B\rangle|A\rangle],$$

$$|\phi^-_S\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle - |1\rangle|1\rangle)[|A\rangle|B\rangle + |B\rangle|A\rangle],$$

(2.4)

where we have indicated with $|A\rangle$ and $|B\rangle$ two non-overlapping (i.e. orthogonal) spatial regions.
A closer inspection of these states reveals that a joint detection of particles 1 and 2 by two
detectors placed in two different outputs beyond the beamsplitter, identify uniquely, according
to what we have said before, the second of the above states, being the only one described by an
antisymmetric spatial part. The authors therefore conclude that only in this case it is possible to
affirm that the teleportation procedure has been successfully achieved, and that the polarization
part of photon 3, after the measurement, is described by the state $\alpha|0\rangle + \beta|1\rangle$.

We claim that this procedure is clearly not correct, since photon 2 is considered distinguish-
able from photon 3, even if they have been produced in a parametric down converter which
undoubtedly has mixed their spatial wave functions.

Let us try to reformulate the whole analysis by considering particles 2 and 3 indistinguishable
and by adding the spatial degrees of freedom for all the particles. We then consider the following
normalized state vector $|\psi\rangle_1$, describing a photon located in the bounded spatial region $|A\rangle$,
whose polarization state we want to teleport to another distant spatial region $|C\rangle$:

$$|\psi\rangle_1 = [\alpha|0\rangle_1 + \beta|1\rangle_1]|A\rangle_1 ,$$

(2.5)

and the quantum channel $|\phi\rangle$ consisting again of an entangled couple of indistinguishable pho-
tons, whose describing vector however displays explicitly the overall symmetry property both
on the polarization and on the spatial degrees of freedom:

$$|\phi\rangle_{23} = \frac{1}{2}(|0\rangle_2|1\rangle_3 - |1\rangle_2|0\rangle_3)[|B\rangle_2|C\rangle_3 - |C\rangle_2|B\rangle_3] .$$

(2.6)

The polarization part of the state $|\phi\rangle_{23}$ is the usual singlet state, while the spatial part describes
two particles having reached two distant and non-overlapping regions $|B\rangle$ and $|C\rangle$. This state is
clearly totally entangled, since we cannot attribute any objective property to the single photons:
we can only say that the probability of finding a particle (without being able to say which one of the two) with a particular polarization (|0⟩ or |1⟩) in a precise region of space (|B⟩ or |C⟩) is always 1/2.

The initial state vector of the three particles is therefore the following one:

\[ |\Omega\rangle_{123} = |\psi\rangle_1 |\phi\rangle_{23} = \frac{1}{2} \left[ \alpha |0A\rangle_1 |0B\rangle_2 |1C\rangle_3 - \alpha |0A\rangle_1 |0C\rangle_2 |1B\rangle_3 \\
+ \beta |1A\rangle_1 |0B\rangle_2 |1C\rangle_3 - \beta |1A\rangle_1 |0C\rangle_2 |1B\rangle_3 \\
- \alpha |0A\rangle_1 |1B\rangle_2 |0C\rangle_3 + \alpha |0A\rangle_1 |1C\rangle_2 |0B\rangle_3 \\
- \beta |1A\rangle_1 |1B\rangle_2 |0C\rangle_3 + \beta |1A\rangle_1 |1C\rangle_2 |0B\rangle_3 \right]. \tag{2.7} \]

Incidentally we note that in the state (2.7) the photon labelled with the index 1 is kept distinguishable from the remaining two until it impinges upon the beamsplitter.

In order to complete the teleportation procedure by a measurement operation, it is again necessary to express the states of the photons located in regions A and B in terms of the Bell-states (2.4). Yet, these states are no longer sufficient to express completely the state (2.7), since it has no symmetry properties with respect to the exchange of particles 1 and 2. In order to overcome this difficulty, one has to resort to other four states:

\[ |\psi_+\rangle_A = \frac{1}{2} \left[ |0\rangle|1\rangle + |1\rangle|0\rangle \right] \left[ |A\rangle|B\rangle - |B\rangle|A\rangle \right], \]
\[ |\psi_-\rangle_A = \frac{1}{2} \left[ |0\rangle|1\rangle - |1\rangle|0\rangle \right] \left[ |A\rangle|B\rangle + |B\rangle|A\rangle \right], \]
\[ |\phi_+\rangle_A = \frac{1}{2} \left[ |0\rangle|0\rangle + |1\rangle|1\rangle \right] \left[ |A\rangle|B\rangle - |B\rangle|A\rangle \right], \]
\[ |\phi_-\rangle_A = \frac{1}{2} \left[ |0\rangle|0\rangle - |1\rangle|1\rangle \right] \left[ |A\rangle|B\rangle - |B\rangle|A\rangle \right]. \tag{2.8} \]

However, these new states are totally antisymmetric and therefore cannot correspond to the physical states of the photons which enter the interferometric apparatus of the Bell-state analyzer.

In any case, the two sets of vectors (2.4) and (2.8) taken together are necessary and sufficient in order to express the state of the particles located in regions |A⟩ and |B⟩ of equation (2.7) in terms of vectors with a spatial part with definite symmetry. In fact we have:

\[ |0A⟩|0B⟩ = \frac{1}{2} \left[ |\phi_+⟩_S + |\phi_-⟩_S + |\phi_+⟩_A + |\phi_-⟩_A \right], \]
\[ |0A⟩|1B⟩ = \frac{1}{2} \left[ |\psi_+⟩_S + |\psi_-⟩_S + |\psi_+⟩_A + |\psi_-⟩_A \right], \]
\[ |1A⟩|0B⟩ = \frac{1}{2} \left[ |\psi_+⟩_S - |\psi_-⟩_S + |\psi_+⟩_A - |\psi_-⟩_A \right], \]
\[ |1A⟩|1B⟩ = \frac{1}{2} \left[ |\phi_+⟩_S - |\phi_-⟩_S + |\phi_+⟩_A - |\phi_-⟩_A \right]. \tag{2.9} \]

If we insert these vectors into equation (2.7), a straightforward rearrangement gives:

\[ |\Omega⟩_{123} = \frac{1}{4} \left[ (|φ_+⟩_{12} + |φ_−⟩_{12}) \left( \alpha |1C⟩_3 - \beta |0C⟩_3 \right) + (|φ_−⟩_{12} + |φ_+⟩_{12}) \left( \alpha |1C⟩_3 + \beta |0C⟩_3 \right) \\
+ (|φ_+⟩_{13} + |φ_−⟩_{13}) \left( \alpha |1C⟩_2 - \beta |0C⟩_2 \right) + (|φ_−⟩_{13} + |φ_+⟩_{13}) \left( \alpha |1C⟩_2 + \beta |0C⟩_2 \right) \\
\right]. \]
\[
-|\psi_S^+\rangle_{12} + |\psi_A^+\rangle_{12} [\alpha|0C\rangle_3 - \beta|1C\rangle_3] - [|\psi_S^-\rangle_{12} + |\psi_A^-\rangle_{12} [\alpha|0C\rangle_3 + \beta|1C\rangle_3] \\
-|\psi_S^+\rangle_{13} + |\psi_A^+\rangle_{13} [\alpha|0C\rangle_2 - \beta|1C\rangle_2] - [|\psi_S^-\rangle_{13} + |\psi_A^-\rangle_{13} [\alpha|0C\rangle_2 + \beta|1C\rangle_2] .
\]

\hspace{1cm} (2.10)

Omitting for a moment the fact that part of the Bell-states are non physical, it is instructive to see why it is not possible to perform a successful teleportation in region C of the polarization state \(\alpha|0\rangle + \beta|1\rangle\). A joint detection of particle 1 and of one of the remaining particles (2 or 3, being they indistinguishable) in the two different outputs located beyond the beamsplitter, causes the collapse onto the state:

\[
|\Omega\rangle_{123} \rightarrow \frac{1}{2\sqrt{2}} \left[ - |\psi_S^-\rangle_{12} [\alpha|0C\rangle_3 + \beta|1C\rangle_3] - |\psi_S^-\rangle_{13} [\alpha|0C\rangle_2 + \beta|1C\rangle_2] \\
+ |\phi_A^+\rangle_{12} [\alpha|1C\rangle_3 - \beta|0C\rangle_3] + |\phi_A^+\rangle_{13} [\alpha|1C\rangle_2 - \beta|0C\rangle_2] \\
+ |\phi_A^-\rangle_{12} [\alpha|1C\rangle_3 + \beta|0C\rangle_3] + |\phi_A^-\rangle_{13} [\alpha|1C\rangle_2 + \beta|0C\rangle_2] \\
- |\psi_A^-\rangle_{12} [\alpha|0C\rangle_3 - \beta|1C\rangle_3] - |\psi_A^-\rangle_{13} [\alpha|0C\rangle_2 - \beta|1C\rangle_2] \right],
\]

where all the kets describing particles in A and B have an antisymmetric spatial part. It is therefore apparent that, also in this non physical situation, we cannot conclude anything about the polarization part of the photon located in region \(|C\rangle\), and that the teleportation process cannot be achieved.

We conclude this section by stressing that it is not correct to forget the indistinguishability of particles 2 and 3. However, even if we take into account that they are identical quantum systems, and we add the spatial degrees of freedom from the beginning, it is not possible to express the global state only in terms of the symmetric Bell-states \([2.4]\).

### 3 Teleportation with indistinguishable particles.

We show now that a symmetric polarization procedure performed over both the spatial and the polarization degrees of freedom of all the particles involved, allows one to achieve a successful teleportation process, at least in 1/4 of all the cases, in complete agreement with the tested experimental data. Therefore, we assume as initial state vector of the teleportation scheme the state

\[
|\tilde{\Omega}\rangle_{123} = S \left[ |\psi\rangle_1|\phi\rangle_{23} \right],
\]

where the operator \(S\) performs a complete symmetrization over all the particles and makes the state totally symmetric. In such a way the symmetric vectors \([2.4]\) become again sufficient for describing the states of the particles located in regions \(|A\rangle\) and \(|B\rangle\). Inserting the set of vectors \([2.4]\) into expression (3.1), the state vector of the system exhibits the following mathematical form:

\[
|\tilde{\Omega}\rangle_{123} = \frac{1}{\sqrt{12}} \left[ |\phi_S^+\rangle_{12} [\alpha|1C\rangle_3 - \beta|0C\rangle_3] + |\phi_A^+\rangle_{13} [\alpha|1C\rangle_2 - \beta|0C\rangle_2] + |\phi_S^\pm\rangle_{23} [\alpha|1C\rangle_1 - \beta|0C\rangle_1] \\
+ |\phi_S^+\rangle_{12} [\alpha|1C\rangle_3 + \beta|0C\rangle_3] + |\phi_A^+\rangle_{13} [\alpha|1C\rangle_2 + \beta|0C\rangle_2] + |\phi_S^\pm\rangle_{23} [\alpha|1C\rangle_1 + \beta|0C\rangle_1] \\
+ |\psi_S^+\rangle_{12} [\beta|1C\rangle_3 - \alpha|0C\rangle_3] + |\psi_A^+\rangle_{13} [\beta|1C\rangle_2 - \alpha|0C\rangle_2] + |\psi_S^\pm\rangle_{23} [\beta|1C\rangle_1 - \alpha|0C\rangle_1] \right]
\]
\[ -|\psi^-\rangle_{12}[\alpha|0C\rangle_3 + \beta|1C\rangle_3] - |\psi^-\rangle_{13}[\alpha|0C\rangle_2 + \beta|1C\rangle_2] - |\psi^-\rangle_{23}[\alpha|0C\rangle_1 + \beta|1C\rangle_1] \].

(3.2)

Owing to the same considerations we have done in dealing with the previous teleportation scheme, we can state that a joint detection of two particles in the two different outputs beyond the beamsplitter (this case occurring exactly in 1/4 of the cases) forces the state to collapse onto the following manifold:

\[ |\tilde{\Omega}\rangle_{123} \rightarrow \frac{1}{\sqrt{3}} \left[ |\psi^-\rangle_{12}(\alpha|0C\rangle_3 + \beta|1C\rangle_3) + |\psi^-\rangle_{13}(\alpha|0C\rangle_2 + \beta|1C\rangle_2) + |\psi^-\rangle_{23}(\alpha|0C\rangle_1 + \beta|1C\rangle_1) \right]. \]

(3.3)

The polarization state describing the particle located in region \(|C\rangle\) (without knowing which of the three it really is, due to their indistinguishability) has exactly the same form as the one we want to teleport. We can therefore state that in region \(C\) there is with certainty a photon whose polarization state is \(\alpha|0\rangle + \beta|1\rangle\). On the contrary of what has happened in the previous section the teleportation scheme has been successfully realized thanks to the correct symmetrized form of the state describing the three particles, which cannot by no means be considered distinguishable.

4 Conclusions.

With the aim of shedding light on the mathematical treatment of the teleportation experiment involving identical particles, like the one described in [3], we have shown that a correct formulation of the subject, which turns out to be in perfect agreement with the observed experimental data, must include a symmetrization procedure over all the degrees of freedom of the particles involved. In fact, in the case of a teleportation procedure of an unknown polarization state of a photon, both the parametric down conversion process and the subsequent Bell-state analysis within an interferometer, forces the three particles to become definitely indistinguishable, due to the unavoidable overlapping of their spatial wavefunctions during the interaction. Moreover, in the described scheme the Bell-measurement is performed through a joint position detection of two particles beyond a beamsplitter. These two considerations necessarily imply that the spatial part of the wavefunction of the particles must be included explicitly in the initial state, and that the symmetrization procedure must be performed over both the polarization and the spatial degrees of freedom. Only in this way the theoretical treatment of the teleportation process is able to predict correctly the observed experimental behaviour.

References

[1] E.Schrödinger, Naturwissenschaften, 23, 807 (1935); English translation in Proc.Am.Philos.Soc., 124, 323 (1980);
[2] C.H.Bennett, G.Brassard, C.Crepeau, R.Josza, A.Peres and W.K.Wootters, Phys. Rev. Lett. 70, 1895 (1993);
[3] C.H.Bennett and S.J.Wiesner, Phys.Rev.Lett. 69, 2881 (1992);
[4] P.Shor, *Proc. of 35th Annual Symposium on the Foundations of Computer Science, IEEE Computer Society, Los Alamitos*, 124 (1994) and *S.I.A.M.Journal of Computing* **26**, 1484 (1997); L.Grover, *Proc. 28 annual ACM Symposium on the Theory of Computing, ACM Press New York*, 212 (1996);

[5] D.Bouwmeester, J-W.Pan, K.Mattle, M.Eible, H.Weinfurter and A.Zeilinger; *Nature* **390**, 575-579 (1997);

[6] D.Bouwmeester, H.Weinfurter and A.Zeilinger; *The Physics of Quantum Information*, Springer, 60-62 (2000).