Fluctuation limits of the super-Brownian motion with a single point catalyst

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Abstract

We prove a fluctuating limit theorem of a sequence of super-Brownian motions over \( \mathbb{R} \) with a single point catalyst. The weak convergence of the processes on the space of Schwarz distributions is established. The limiting process is an Ornstein-Uhlenbeck type process solving a Langevin type equation driven by a one-dimensional Brownian motion.

AMS Subject Classifications (2000): Primary 60J80; Secondary 60F05

Key words and Phrases: super-Brownian motion, single point catalyst, fluctuation limit, Ornstein-Uhlenbeck type process

1 Introduction

In recent years, there has been growing interest in the study of branching systems in singular media. Although from the viewpoint of applications some of the models can be artificial, they give useful insight into the behavior of more realistic systems. The extremely simple case of the single non-random branching catalyst described by a Dirac function was introduced in Dawson and Fleischmann (1994). The model has been studied extensively since then; see, e.g., Dawson et al. (1995), Fleischmann and Le Gall (1995) and Fleischmann and Xiong (2006).

In the present paper, we study the fluctuation limits of the single point catalytic super-Brownian motion (SBM) with small branching. Our limit theorem shows that the asymptotic fluctuating behavior of the processes around the Lebesgue measure can be approximated by a Schwarz distribution-valued Ornstein-Uhlenbeck type process. We also show that the Ornstein-Uhlenbeck type process solves a Langevin type equation driven by a one-dimensional Brownian motion. The pathwise uniqueness for the stochastic equation

1 Supported by NSFC and CJSP grants.

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is established by an explicit construction of the solution. To prove the weak convergence of the fluctuating processes on the path space, we first give an extension of a tightness criterion in Ethier and Kurtz (1986). The results of this work extend those of Li (2009) and Li and Zhang (2006) on Dawson-Watanabe superprocesses with immigration; see also Bojdecki and Gorostiza (1986), Gorostiza and Li (1998) and Dawson et al. (1989) for some earlier results.

2 Single point catalytic SBM

Let $C(\mathbb{R})$ be the Banach space of bounded continuous functions on $\mathbb{R}$ endowed with supremum norm $\|\cdot\|$. Write $C_0(\mathbb{R})$ for the space of functions in $C(\mathbb{R})$ vanishing at infinity. Let $C^2(\mathbb{R})$ denote the space of smooth functions on $\mathbb{R}$ with continuous derivatives up to the second order belonging to $C(\mathbb{R})$. We fix a constant $p > 1$ and let $h_p(x) = (1 + x^2)^{-p/2}$ for $x \in \mathbb{R}$. Let $C_p(\mathbb{R})$ denote the set of continuous functions $f \in C_0(\mathbb{R})$ satisfying $|f| \leq \text{const} \cdot h_p$ and let $C_p(\mathbb{R})^+$ be the subset of its nonnegative elements. Let $M_p(\mathbb{R})$ be the space of $\sigma$-finite measures $\mu$ on $\mathbb{R}$ satisfying $\int_\mathbb{R} h_p d\mu < \infty$. Write $\langle \mu, f \rangle = \int_\mathbb{R} f d\mu$ for $\mu \in M_p(\mathbb{R})$ and $f \in C_p(\mathbb{R})$. The topology on $M_p(\mathbb{R})$ is defined by the convention:

$$\mu_n \to \mu \text{ if and only if } \langle \mu_n, f \rangle \to \langle \mu, f \rangle \text{ for all } f \in C_p(\mathbb{R}).$$

We denote the Lebesgue measure on $\mathbb{R}$ by $\lambda$, which clearly belongs to $M_p(\mathbb{R})$. Let $(P_t)_{t \geq 0}$ be the transition semigroup of the one-dimensional standard Brownian motion $\xi$ generated by $A := \Delta/2$ and let $\sigma > 0$ be a constant. Let $p(t, x, y) = p(t, y - x)$ denote the transition density of the Brownian motion. A time-homogeneous Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P_\mu)$ with state space $M_p(\mathbb{R})$ is called a **SBM with single point catalyst** at $c \in \mathbb{R}$ if it has transition semigroup $(Q_t)_{t \geq 0}$ given by

$$\int_{M_p(\mathbb{R})} e^{-\langle \nu, f \rangle} Q_t(\mu, d\nu) = \exp \{-\langle \mu, V_t f \rangle\}, \quad (2.1)$$

where $f \in C_p(\mathbb{R})^+$ and $v(t, x) := V_t f(x)$ is the unique positive solution of the integral evolution equation

$$v(t, x) = P_t f(x) - \frac{\sigma^2}{2} \int_0^t p(t - s, c - x)v(s, c)^2 ds, \quad t \geq 0, x \in \mathbb{R}. \quad (2.2)$$

The following theorems are generalizations of the results in Dawson and Fleischmann (1994). In particular, the existence of the single point catalytic SBM is a consequence of the first theorem.

**Theorem 2.1** The time-homogeneous Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P_\mu)$ determined by equation (2.2) via the Laplace transition functional (2.1) can be constructed on the space $C([0, \infty), M_p(\mathbb{R}))$ of continuous $M_p(\mathbb{R})$-valued trajectories satisfying $X_t(\{c\}) = 0$ for all $t > 0$. The following expectation and covariance formulas hold:

$$\mathbf{E}_\mu(X_t, f) = \langle \mu P_t, f \rangle, \quad (2.3)$$
\[
\text{Var}_F(X_t, f) = \sigma^2 \int_\mathbb{R} \mu(dx) \int_0^t p(t-s, c-x)P_s f(c)^2ds,
\] (2.4)

where \( \mu \in M_p(\mathbb{R}) \) and \( f \in C_p(\mathbb{R}) \).

**Theorem 2.2** There is a version of \( X \) such that there exists a jointly continuous random field \( x = \{x_t(z) : t > 0, z \neq c\} \) satisfying

\[ X_t(dz) = x_t(z)dz \quad \text{for all } t > 0, \ P_\mu\text{-a.s.} \]

The random field \( x \) has the Laplace transforms

\[ E_\mu \exp \left\{ -\sum_{i=1}^k x_t(z_i)\theta_i \right\} = \exp \left\{ -\langle \mu, u(t) \rangle \right\}, \quad t > 0, \theta_i \geq 0, z_i \neq c, 1 \leq i \leq k, \]

where \( u(t, x) \geq 0 \) solves

\[ u(t, x) = \sum_{i=1}^k \theta_i p(t, z_i - x) - \frac{\sigma^2}{2} \int_0^t p(t-s, c-x)u(s, c)^2ds, \quad t > 0. \]

Moreover, the function \( f \) in formulas (2.3) and (2.4) can be replaced by Dirac function \( \delta_z \) for any \( z \neq c \).

By the sample path continuity of the single point catalytic SBM, we may introduce the occupation time process \( Y = \{Y_t : t \geq 0\} \) related to \( X \), defined by

\[ \langle Y_t, f \rangle = \int_0^t \langle X_s, f \rangle ds, \quad f \in C_p(\mathbb{R})^+. \]

Of course, by the integration, \( Y \) is smoother than \( X \), and

\[ y_t(z) := \int_0^t x_s(z)ds, \quad t \geq 0, z \neq c, \quad (2.5) \]

yields a density field of \( Y \), which is \( P_\mu\text{-a.s.} \) jointly continuous on \( \mathbb{R}_+ \times \{z \neq c\} \). The next result shows that we have a everywhere jointly continuous occupation density.

**Theorem 2.3** There is a version of \( X \) such that the density field \( y \) of \( Y \) defined by (2.5) extends continuously to all of \( \mathbb{R}_+ \times \mathbb{R} \). Moreover,

\[ E_\mu \exp \left\{ -\sum_{i=1}^k y_t(z_i)\theta_i \right\} = \exp \left\{ -\langle \mu, u(t) \rangle \right\}, \quad t \geq 0, \theta_i \geq 0, z_i \in \mathbb{R}, 1 \leq i \leq k, \]

where \( u(t, x) \geq 0 \) solves

\[ u(t, x) = \sum_{i=1}^k \theta_i \int_0^t p(t-s, z_i - x) \, ds - \frac{\sigma^2}{2} \int_0^t p(t-s, c-x)u(s, c)^2ds, \quad t \geq 0. \]
Moreover, for \( s \leq t \) and \( z \in \mathbb{R} \) the following expectation and variance formulas hold:

\[
E_{\mu} y(t, z) = \int_{\mathbb{R}} \mu(dx) \int_0^t p(s, z - x)ds,
\]

\[
\text{Var}_{\mu} y(t, z) = \sigma^2 \int_{\mathbb{R}} \mu(dx) \int_0^t p(s, c - x) \left[ \int_s^t p(u - s, z - c)du \right]^2 ds.
\]

We call \( y(t, z) \) the occupation density of the single point catalytic SBM at \( z \in \mathbb{R} \) during the time period \([0, t] \). Set \( \mathcal{D}_p(A) = \{ f \in C_p(\mathbb{R}) \cap C^2(\mathbb{R}) : Af \in C_p(\mathbb{R}) \} \).

**Theorem 2.4** For all \( f \in \mathcal{D}_p(A) \),

\[
M_t(f) := \langle X_t, f \rangle - \langle X_0, f \rangle - \int_0^t \langle X_s, Af \rangle ds,
\]

\( t \geq 0, \)

is a continuous martingale with quadratic variation process

\[
\langle M(f) \rangle_t := \sigma^2 f^2(c)y_t(c),
\]

\( t \geq 0. \)

**Proof of Theorems 2.1.** We here give a simple construction of the catalytic SBM by summing up an infinite sequence of processes taking values of finite measures. This construction is also useful in deriving some properties of the catalytic SBM. For any \( \mu \in M_p(\mathbb{R}) \) we can find a sequence of finite measures \( \{ \mu_i \}_{i \geq 1} \) such that \( \mu = \sum_{i=1}^{\infty} \mu_i \).

For each \( i \geq 1 \) let \( X_i = \{ X_i(t) : t \geq 0 \} \) be a single point catalytic SBM with initial measure \( \mu_i \). We assume the sequence of processes \( X_i, i \geq 1 \) are defined on the same probability space and are independent. Then we can define a single point catalytic SBM \( X = \{ X(t) : t \geq 0 \} \) with initial measure \( \mu \) by

\[
X(t) = \sum_{i=1}^{\infty} X_i(t),
\]

\( t \geq 0. \) (2.8)

For \( n \geq k \geq 1 \) it is easy to see that

\[
X_{k,n}(t) = \sum_{i=k}^{n} X_i(t),
\]

\( t \geq 0 \)

is a continuous finite measure-valued catalytic SBM with initial state \( \mu_{k,n} := \sum_{i=k}^{n} \mu_i \). By applying Theorem 2.4 to the process \( X_{k,n}(t) \) we have

\[
E \left[ \sup_{0 \leq s \leq t} \langle X_{k,n}(s), h_p \rangle \right]^2 \leq 2\langle \mu_{k,n}, h_p \rangle^2 + 16\sigma^2 h^2_p(c) \left( \int_{\mathbb{R}} \mu_{k,n}(dx) \int_0^t p(s, c - x)ds \right)
\]

\[
+ 4t \int_0^t \left( \sigma^2 \int_{\mathbb{R}} \mu_{k,n}(dx) \int_0^s p(s - u, c - x)P_uAh_p(c)^2du + \langle \mu_{k,n}P, Ah_p \rangle^2 \right)ds.
\]

The right hand side tends to zero as \( k, n \to \infty \). This implies that \( \{ X(t) : t \geq 0 \} \) can be realized in \( C([0, \infty), M_p(\mathbb{R})) \). The moment formulas (2.3) and (2.4) follow from
Theorems 1.2.1 and 1.2.4 in Dawson and Fleischmann (1994) and the construction (2.5).

\[\square\]

Proof of Theorems 2.2, 2.3 and 2.4. The existence and the characterizations of the Laplace transforms of the density fields \( x = \{ x_t(z) : t > 0, z \neq c \} \) and \( y = \{ y_t(z) : t > 0, z \in \mathbb{R} \} \) follow from the construction (2.5) of the catalytic SBM. The moment formulas can be derived from the Laplace transforms. The continuity properties of the fields follow from the three lemmas given below. The martingale problem characterization also follows from the construction (2.5). We leave the details to the reader. \[\square\]

Now we give three lemmas which were already used in the above proof. These are modifications of Lemmas 2.6.2, 3.2.1 and 3.2.2 in Dawson and Fleischmann (1994). The proofs are similar to theirs and are omitted here. Fix \( f \in C_p(\mathbb{R}) \) and set

\[ u_\theta(t, x) := \theta P_t f(x) - v_\theta(t, x), \quad t \geq 0, x \in \mathbb{R}, \theta \geq 0, \]

where \( v_\theta(t, x) = V_t(\theta f)(x) \). We denote by \( u_\theta^{(k)} \) the \( k \)th derivative of \( u_\theta \) with respect to \( \theta \) taken at \( \theta = 0 \). For \( T \geq 0 \) and \( f \in C_p(\mathbb{R}) \) put

\[ \| Pf \|_T = \sup \{|P_r f(c)| : 0 \leq r \leq T}\].

**Lemma 2.5** For each \( k \geq 2 \) there is a constant \( c_k > 0 \) so that the power series \( \sum_{k \geq 2} c_k \theta^k \) has a positive radius of convergence and that

\[ |u_\theta^{(k)}(t, x)| \leq 2^{1-k} k! c_k \sigma^{2(k-1)} \| Pf \|_T^{k(k-1)/2} \exp \left\{ - \frac{|x|^2}{2t} \right\}, \]

where \( x \in \mathbb{R}, 0 \leq t \leq T \) and \( k \geq 2 \).

Set \( Z_t = X_t - X_0 P_t \). Then we have \( \mathbb{E}_\mu Z_t = 0 \).

**Lemma 2.6** For each \( k \geq 2 \) there exists a constant \( C_k \geq 0 \) such that

\[ |\mathbb{E}_\mu \langle Z_t, f \rangle^k| \leq C_k t^{k/4} \| Pf \|_T^k \sum_{i=1}^{k-1} \left\langle \mu, \exp \left\{ - \frac{|t|^2}{2t} \right\} \right\rangle^i, \]

where \( 0 \leq t \leq T, \mu \in M_p(\mathbb{R}) \) and \( f \in C_p(\mathbb{R}) \).

**Lemma 2.7** Let \( k \geq 1, T > 0 \) and \( \mu \in M_p(\mathbb{R}) \). Then there exists a constant \( C_k(T, \mu) \geq 0 \) such that

\[ \mathbb{E}_\mu \langle Z_{t+h} - Z_t, f \rangle^{2k} \leq C_k(T, \mu) \left( \| P_h f - f \|_T^{2k} + h^{k/2} \| Pf \|_T^k \right), \]

where \( 0 \leq t \leq t + h \leq T \) and \( f \in C_p(\mathbb{R}) \).
A tightness criterion based on the martingale problems were given in Ethier and Kurtz (1986, p.145). However, the martingales considered there have absolutely continuous increasing processes. In this section, we give a generalized version of the result. Although the proof is similar to that of Ethier and Kurtz (1986), we give it here for reader’s convenience.

Let \( E \) be a metric space and let \( \bar{C}(E) \) be the space of bounded and uniformly continuous functions on \( E \). (Since \( E \) is an arbitrary metric space, the bounded support property does not make any restriction.) For each index \( \alpha \), let \( X_\alpha \) be a process with sample paths in \( D([0, \infty), E) \) defined on a probability space \( (\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha) \) and adapted to a filtration \( \{\mathcal{F}_t\} \). Let \( A_\alpha \) be an increasing process which is adapted to the filtration \( \{\mathcal{F}_t\} \) and satisfies

\[
\lim_{\delta \to 0} \sup_{\alpha} E \left[ \sup_{0 \leq r \leq T} |A_\alpha(r + \delta) - A_\alpha(r)| \right] = 0. \tag{3.1}
\]

Let \( \mathcal{L}_\alpha \) denote the Banach space of real-valued \( \{\mathcal{F}_t\} \)-progressive processes with norm \( \|Y\| = \sup_{t \geq 0} E[|Y(t)|] < \infty \). Given \( T \geq 0 \) and \( h \in \mathcal{L}_\alpha \), define

\[
\|h_\alpha\|_{p,T} = \left[ \int_0^T |h_\alpha(t)|^p dA_\alpha(t) \right]^{1/p}
\]

for \( 0 < p < \infty \) and define \( \|h\|_{\infty,T} = \text{ess sup}_{0 \leq t \leq T} |h(t)| \). Let

\[
\mathcal{A}_\alpha = \left\{(Y, Z) \in \mathcal{L}_\alpha \times \mathcal{L}_\alpha : Y(t) - \int_0^t Z(s) dA_\alpha(s) \text{ is an } \{\mathcal{F}_t\}-\text{martingale}\right\}
\]

Let \( Q \) denote the set of rational numbers. Then we have

**Theorem 3.1** Suppose that \( C_\alpha \) is a subalgebra of \( \bar{C}(E) \). Let \( D \) be the collection of \( f \in \bar{C}(E) \) such that for every \( \varepsilon > 0 \) and \( T > 0 \) there exist \( (Y_\alpha, Z_\alpha) \in \mathcal{A}_\alpha \) with

\[
\sup_{\alpha} E \left[ \sup_{t \in [0,T] \cap Q} |Y_\alpha(t) - f(X_\alpha(t))| \right] < \varepsilon \tag{3.2}
\]

and

\[
\sup_{\alpha} E[\|Z_\alpha\|_{p,T}^p] < \infty \quad \text{for some } p \in (1, \infty]. \tag{3.3}
\]

If \( C_\alpha \) is contained in the closure of \( D \) (in the sup norm), then \( \{f \circ X_\alpha\} \) is tight in \( D([0, \infty), \mathbb{R}) \) for each \( f \in C_\alpha \).

**Proof.** Since \( D \) is actually closed, we may assume that \( C_\alpha \subset D \). Let \( \varepsilon > 0 \) and \( T > 0 \). For \( f \in C_\alpha \) we have \( (Y_\alpha, Z_\alpha) \in \mathcal{A}_\alpha \) such that (3.2) and (3.3) hold. Since \( f^2 \in C_\alpha \), there are \( (Y'_\alpha, Z'_\alpha) \in \mathcal{A}_\alpha \) such that

\[
\sup_{\alpha} E \left[ \sup_{t \in [0,T+1] \cap Q} |Y'_\alpha(t) - f(X_\alpha(t))^2| \right] < \varepsilon
\]
and
\[ \sup_{\alpha} \mathbb{E}[\|Z'_\alpha\|_{p',T}^p] < \infty \quad \text{for some} \quad p' \in (1, \infty). \]

Let \( 0 < \delta < 1 \). For each \( t \in [0, T] \cap Q \) and \( u \in [0, \delta] \cap Q \) we have

\[
\mathbb{E} \left[ (f(X_\alpha(t + u)) - f(X_\alpha(t)))^2 \bigg| \mathcal{F}^\alpha_t \right] \\
= \mathbb{E} \left[ (f(X_\alpha(t + u))^2 - f(X_\alpha(t))^2 \bigg| \mathcal{F}^\alpha_t \right] \\
- 2f(X_\alpha(t)) \mathbb{E} \left[ (f(X_\alpha(t + u)) - f(X_\alpha(t))) \bigg| \mathcal{F}^\alpha_t \right] \\
\leq 2\mathbb{E} \left[ \sup_{t \in [0,T+1] \cap Q} \left| f(X_\alpha(t))^2 - Y'_\alpha(t) \right| \bigg| \mathcal{F}^\alpha_t \right] \\
+ 4\|f\| \mathbb{E} \left[ \sup_{t \in [0,T+1] \cap Q} \left| f(X_\alpha(t)) - Y_\alpha(t) \right| \bigg| \mathcal{F}^\alpha_t \right] \\
+ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \int_t^{t+\delta} |Z'_\alpha(s)|dA_\alpha(s) \bigg| \mathcal{F}^\alpha_t \right] \\
+ 2\|f\| \mathbb{E} \left[ \sup_{0 \leq t \leq T} \int_t^{t+\delta} |Z_\alpha(s)|dA_\alpha(s) \bigg| \mathcal{F}^\alpha_t \right].
\]

It follows that

\[
\mathbb{E} \left[ (f(X_\alpha(t + u)) - f(X_\alpha(t)))^2 \bigg| \mathcal{F}^\alpha_t \right] \leq \mathbb{E} \left[ \gamma_\alpha(\delta) \bigg| \mathcal{F}^\alpha_t \right]. \tag{3.4}
\]

where

\[
\gamma_\alpha(\delta) = 2\sup_{t \in [0,T+1] \cap Q} \left| f(X_\alpha(t))^2 - Y'_\alpha(t) \right| + 4\|f\| \sup_{t \in [0,T+1] \cap Q} \left| f(X_\alpha(t)) - Y_\alpha(t) \right| \\
+ \sup_{0 \leq t \leq T} \int_t^{t+\delta} |Z'_\alpha(s)|dA_\alpha(s) + 2\|f\| \sup_{0 \leq t \leq T} \int_t^{t+\delta} |Z_\alpha(s)|dA_\alpha(s). \tag{3.5}
\]

Note that the inequality (3.4) actually holds for all \( 0 \leq t \leq T \) and \( 0 \leq u \leq \delta \) by the right continuity of \( X_\alpha \). Let \( 1/p + 1/q = 1 \) and \( 1/p' + 1/q' = 1 \). By (3.5) and Hölder’s inequality we have

\[
\sup_{\alpha} \mathbb{E}[\gamma_\alpha(\delta)] \leq 2(1 + 2\|f\|\varepsilon + B(\delta, T)^{1/2} \sup_{\alpha} \mathbb{E}^{T,T+1}_{\alpha}[\|Z'_\alpha\|_{p',T+1}^{p'}] \\
+ 2\|f\|B(\delta, T)^{1/2} \sup_{\alpha} \mathbb{E}^{T,T+1}_{\alpha}[\|Z_\alpha\|_{p,T+1}^{p}],
\]

where

\[
B(\delta, T) = \sup_{\alpha} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| A_\alpha(t + \delta) - A_\alpha(t) \right| \right].
\]

Then we may select \( \varepsilon \) depending on \( \delta \) in such a way that

\[
\lim_{\delta \to 0} \sup_{\alpha} \mathbb{E}[\gamma_\alpha(\delta)] = 0.
\]

Therefore, \( \{f \circ X_\alpha\} \) is tight in \( D([0, \infty), \mathbb{R}) \) by Theorem 8.6 in Ethier and Kurtz (1986, pp.137-138). \( \Box \)
4 A fluctuation limit theorem

For each integer \( k \geq 1 \), let \( \{ X_k(t) : t \geq 0 \} \) be the single point catalytic super-Brownian motion characterized by (2.1) and (2.2) with \( \sigma^2/2 \) replaced by \( \sigma^2/2k^2 \). Let \( \{ x_i^k(z) : t > 0, z \neq c \} \) and \( \{ y_i^k(z) : t > 0, z \in \mathbb{R} \} \) be the corresponding density and occupation density fields. For simplicity, we assume \( X_k(0) = \lambda \), so

\[
E(X_k(t), f) = \langle \lambda P_t, f \rangle = \langle \lambda, f \rangle, \quad t \geq 0, f \in C_p(\mathbb{R}).
\]

We define a centered signed-measure-valued Markov process \( \{ Z_k(t) : t \geq 0 \} \) by

\[
Z_k(t) := k(X_k(t) - \lambda), \quad t \geq 0.
\]

Then \( E(Z_k(t), f) = 0 \) for each \( f \in C_p(\mathbb{R}) \).

Let \( C^\infty(\mathbb{R}) \) be the set of bounded infinitely differentiable functions on \( \mathbb{R} \) with bounded derivatives. Let \( \mathcal{S}(\mathbb{R}) \subset C^\infty(\mathbb{R}) \) denote the Schwartz space of rapidly decreasing functions. That is, a function \( f \in \mathcal{S}(\mathbb{R}) \) is infinitely differentiable and for every \( k \geq 0 \) and every \( n \geq 0 \) we have

\[
\lim_{|x| \to \infty} |x|^n \left| \frac{d^k}{dx^k} f(x) \right| = 0.
\]

The topology of \( \mathcal{S}(\mathbb{R}) \) is defined by the increasing sequence of norms \( \{ p_0, p_1, p_2, \cdots \} \) given by

\[
p_n(f) = \sum_{0 \leq k \leq n} \sup_{x \in \mathbb{R}} (1 + |x|^n/2) \left| \frac{d^k}{dx^k} f(x) \right|.
\]

Let \( \mathcal{S}'(\mathbb{R}) \) be the dual space of \( \mathcal{S}(\mathbb{R}) \) endowed with the strong topology. Then both \( \mathcal{S}(\mathbb{R}) \) and \( \mathcal{S}'(\mathbb{R}) \) are nuclear spaces; see, e.g., Treves (1967, p.514 and p.530). Thus we can regard \( \{ Z_k(t) : t \geq 0 \} \) as a process taking values from \( \mathcal{S}'(\mathbb{R}) \).

**Lemma 4.1** For any \( t \geq 0 \) and \( f \in C_p(\mathbb{R}) \) we have

\[
E[\langle Z_k(t), f \rangle^2] = \sigma^2 \int_\mathbb{R} \lambda(dx) \int_0^t p(t-s, c-x) P_s f(c)^2 ds.
\]

**Proof.** By Theorem 2.1 we have

\[
E[\langle Z_k(t), f \rangle^2] = \text{Var}(X_k(t), kf) = \sigma^2 \int_\mathbb{R} \lambda(dx) \int_0^t p(t-s, c-x) P_s f(c)^2 ds
\]

Thus (4.1) follows. \( \square \)

**Lemma 4.2** For any \( t \geq 0 \) and \( f \in \mathcal{S}(\mathbb{R}) \) we have

\[
\sup_{k \geq 1} E \left[ \sup_{0 \leq s \leq t} \langle Z_k(s), f \rangle^2 \right] \leq 8\sigma^2 f(c)^2 t + 2t\sigma^2 \int_0^t ds \int_\mathbb{R} \lambda(dx) \int_0^s p(s-u, c-x) P_u Af(c)^2 du.
\]
Proof. By Theorem 2.4 we have

$$\langle X_k(t), f \rangle = \langle \lambda, f \rangle + M_k(t, f) + \int_0^t \langle X_k(s), Af \rangle ds,$$

where \{M_k(t, f) : t \geq 0\} is a continuous martingale with increasing process

$$\langle M_k(f) \rangle_t = \frac{\sigma^2}{k^2} f(c)^2 y^k_t(c). \quad (4.2)$$

It is easy to show that for any \( f \in \mathcal{S}(\mathbb{R}) \), \( Af \in \mathcal{S}(\mathbb{R}) \) and \( \langle \lambda, Af \rangle = 0 \). Then we get

$$\langle Z_k(t), f \rangle = k M_k(t, f) + \int_0^t \langle Z_k(s), Af \rangle ds. \quad (4.3)$$

By Doob’s inequality,

$$\begin{align*}
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \langle Z_k(s), f \rangle^2 \right] &\leq 2k^2 \mathbb{E} \left[ \sup_{0 \leq s \leq t} |M_k(s, f)|^2 \right] + 2 \mathbb{E} \left[ \int_0^t |\langle Z_k(s), Af \rangle| ds \right]^2 \\
&\leq 8\sigma^2 \mathbb{E} \left[ \int_0^t f(c)^2 dy^k_s(c) \right] + 2t \int_0^t \mathbb{E} \left[ (\langle Z_k(s), Af \rangle)^2 \right] ds \\
&\leq 8\sigma^2 f(c)^2 \mathbb{E} [y^k_t(c)] + 2t\sigma^2 \int_0^t ds \int_0^s \lambda(dx) \int_0^x p(s-u, c-x) P_u Af(c)^2 du \\
&= 8\sigma^2 f(c)^2 t + 2t\sigma^2 \int_0^t ds \int_0^s \lambda(dx) \int_0^x p(s-u, c-x) P_u Af(c)^2 du.
\end{align*}$$

That gives the desired estimate. \(\square\)

**Lemma 4.3** For any \( G \in C^2(\mathbb{R}) \) and \( f \in D_p(A) \) we have

$$G(\langle Z_k(t), f \rangle) = \int_0^t G'(\langle Z_k(s), f \rangle) \langle Z_k(s), Af \rangle ds + \frac{\sigma^2}{2} \int_0^t G''(\langle Z_k(s), f \rangle) f(c)^2 dy^k_s(c) + \text{mart.}$$

Proof. By (4.2), (4.3) and Itô’s formula, it is easy to see that

$$G(\langle Z_k(t), f \rangle) = \int_0^t G'(\langle Z_k(s), f \rangle) \langle Z_k(s), Af \rangle ds + \frac{\sigma^2}{2} \int_0^t G''(\langle Z_k(s), f \rangle) f(c)^2 dy^k_s(c) + \text{local mart.}$$

Since the local martingale in the above equality is actually a square-integrable martingale, we obtain the desired equality. \(\square\)
Lemma 4.4 As \( k \to \infty \), \( \{y^k_t(c) : t \geq 0\} \) converges in distribution on \( C([0,\infty),\mathbb{R}_+) \) to \( \{t : t \geq 0\} \).

Proof. From the moment formula (2.6) we have

\[
E[y^k_t(c)] = \int \lambda(dx) \int_0^t p(s, z-x)ds = t, \quad t \geq 0.
\]  

(4.4)

Using (2.7) it is not hard to show that

\[
\text{Var}[y^k_t(z)] = \sigma^2 \int \lambda(dx) \int_0^t p(s, c-x) \left[ \int_s^t p(u, z-c)du \right]^2 ds.
\]  

(4.5)

Then \( y^k_t(c) \) converges in probability to \( t \) for each fixed \( t > 0 \). Consequently, \( \{y^k_t(c) : t \geq 0\} \) converges in finite dimensional distributions to deterministic process \( \{t : t \geq 0\} \) as \( k \to \infty \).

Further, by similar calculations as in Dawson and Fleischmann (1994, p.33), we can show that, for any \( T > 0 \)

\[
E[|y^k_{t+h}(c) - y^k_t(c)|^{2n}] \leq C_n(T, \lambda) h^{n/2}, \quad 0 < t \leq t + h \leq T,
\]

where \( C_n(T, \lambda) > 0 \) is a constant depending only on \( T, n \) and \( \lambda \). Therefore, \( \{y^k_t(c) : t \geq 0\}_{k \geq 1} \) is tight and \( \{y^k_t(c) : t \geq 0\} \) converges weakly to \( \{t : t \geq 0\} \) as \( k \to \infty \). \(\square\)

Lemma 4.5 For any \( T > 0 \),

\[
\lim_{\delta \to 0} \sup_{k \geq 1} E \left[ \sup_{0 \leq r \leq T} (y^k_{r+\delta}(c) - y^k_r(c)) \right] = 0.
\]

Proof. For each \( k \geq 1 \), since \( y^k_t(c) \) is continuous in \( t \) and \( \sup_{0 \leq r \leq T} (y^k_{r+\delta}(c) - y^k_r(c)) \) is decreasing in \( \delta \), we have

\[
\lim_{\delta \to 0} \sup_{0 \leq r \leq T} (y^k_{r+\delta}(c) - y^k_r(c)) = 0, \quad P_\lambda\text{-a.s.}
\]

We may assume \( 0 < \delta < 1 \), then

\[
\sup_{0 \leq r \leq T} (y^k_{r+\delta}(c) - y^k_r(c)) \leq y^k_{T+1}(c), \quad P_\lambda\text{-a.s.}
\]

In view of (4.4) we can use Fatou’s lemma to obtain

\[
\lim_{\delta \to 0} E \left[ \sup_{0 \leq r \leq T} (y^k_{r+\delta}(c) - y^k_r(c)) \right] = 0. \quad (4.6)
\]

Observe that for each fixed \( T > 0 \),

\[
\sup_{0 \leq r \leq T} |y^k_r(c) - r| \leq y^k_T(c) + T
\]
and the family \( \{y^k_t(c)\}_{k \geq 1} \) is uniformly integrable by (4.4) and (4.5). Then \( \{\sup_{0 \leq r \leq T} |y^k_r(c) - r|\}_{k \geq 1} \) is uniformly integrable. On the other hand, by Lemma 4.4 we have
\[
\sup_{0 \leq r \leq T} |y^k_r(c) - r| \to 0 \text{ in probability as } k \to \infty.
\]
It follows that
\[
\lim_{k \to \infty} E \left[ \sup_{0 \leq r \leq T} |y^k_r(c) - r| \right] = 0.
\]
Then for any given \( \varepsilon > 0 \), there exists \( K = K(\varepsilon) \geq 1 \) so that when \( k \geq K \),
\[
E \left[ \sup_{0 \leq r \leq T+1} |y^k_r(c) - r| \right] < \frac{\varepsilon}{4}.
\]
Thus for \( k \geq K \) and \( 0 < \delta < \varepsilon/2 \) we have
\[
E \left[ \sup_{0 \leq r \leq T} \left( y^k_{r+\delta}(c) - y^k_r(c) \right) \right] \\
\leq E \left[ \sup_{0 \leq r \leq T} |y^k_{r+\delta}(c) - (r + \delta)| \right] + E \left[ \sup_{0 \leq r \leq T} |y^k_r(c) - r| \right] + \delta \\
\leq 2E \left[ \sup_{0 \leq r \leq T+1} |y^k_r(c) - r| \right] + \frac{\varepsilon}{2} \leq \varepsilon.
\]
By (4.6) we can choose \( 0 < \delta_0 = \delta_0(K) < \varepsilon/2 \) so that when \( 0 < \delta < \delta_0 \),
\[
\sup_{1 \leq k \leq K} \mathbb{E} \left[ \sup_{0 \leq r \leq T} \left( y^k_{r+\delta}(c) - y^k_r(c) \right) \right] < \varepsilon.
\]
A combination of this and (4.7) completes the proof. \( \square \)

**Lemma 4.6** The sequence \( \{Z_k(t) : t \geq 0\}_{k \geq 1} \) is tight in \( C([0, \infty), \mathcal{S}'(\mathbb{R})) \).

**Proof.** We shall prove that for every \( f \in \mathcal{S}(\mathbb{R}) \) the sequence \( \{\langle Z_k(t), f \rangle : t \geq 0\} \) is tight in \( D([0, \infty), \mathbb{R}) \), so the result follows by a theorem of Mitoma (1983); see also Kallianpur and Xiong (1995, p.82). Let \( A_k(t) = t + \frac{\sigma^2}{2} y^k_t(c) \). By Lemma 4.3 for any \( G \in C^\infty(\mathbb{R}) \) we have
\[
G(\langle Z_k(t), f \rangle) = \int_0^t G'(\langle Z_k(s), f \rangle) \langle Z_k(s), Af \rangle ds \\
+ \frac{\sigma^2}{2} \int_0^t G''(\langle Z_k(s), f \rangle) f(c)^2 dy^k_t(c) + \text{mart.}
\]
\[
= \int_0^t \left[ G'(\langle Z_k(s), f \rangle) \langle Z_k(s), Af \rangle b_k(s) \\
+ G''(\langle Z_k(s), f \rangle) f(c)^2 h_k(s) \right] dA_k(s) + \text{mart.}
\]

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where \( b_k(s) \) and \( h_k(s) \) denote the densities of \( ds \) and \( \frac{\sigma^2}{2}dy^k_s(c) \) with respect to \( dA_k(s) \), respectively. From (4.4) it is elementary to see that

\[
\sup_{k \geq 1} \mathbb{E} \left[ \int_0^t \left| G'((Z_k(s), f)) (Z_k(s), Af) b_k(s) \right| \, ds + G''((Z_k(s), f)) f(c)^2 h_k(s) \right] < \infty.
\]

By Theorem 3.1 and Lemma 4.5, we infer that \( \{G((Z_k(t), f)) : t > 0\} \) is tight first in \( D([0, \infty), \mathbb{R}) \) and then in the closed subspace \( C([0, \infty), \mathbb{R}) \). By Lemma 4.2 and Chebyshev’s inequality we have

\[
\sup_{k \geq 1} \mathbb{P} \left[ \sup_{0 \leq s \leq t} |\langle Z_k(s), f \rangle| \geq \alpha \right] \to 0
\]
as \( \alpha \to \infty \). Then \( \{Z_k(t), f) : t > 0\} \) satisfies the compact containment condition and hence it is tight by Theorem 9.1 in Ethier and Kurtz (1986, p.142). □

**Lemma 4.7** Let \( \{Z_0(t) : t \geq 0\} \) be any limit point of \( \{Z_k(t) : t \geq 0\} \) in the sense of distributions on \( C([0, \infty), \mathcal{S}'(\mathbb{R})) \). Then for \( G \in C^\infty(\mathbb{R}) \) and \( f \in \mathcal{S}(\mathbb{R}) \) we have

\[
G(\langle Z_0(t), f \rangle) = \int_0^t G'(\langle Z_0(s), f \rangle) (Z_0(s), Af) \, ds + \frac{\sigma^2}{2} \int_0^t G''(\langle Z_0(s), f \rangle) f(c)^2 \, ds + \text{mart.}
\]

**Proof.** By passing to a subsequence and using the Skorokhod representation, we may assume \( \{Z_k(t) : t \geq 0\} \) and \( \{Z_0(t) : t \geq 0\} \) are defined on the same probability space and \( \{Z_k(t) : t \geq 0\} \) converges a.s. to \( \{Z_0(t) : t \geq 0\} \) in the topology of \( C([0, \infty), \mathcal{S}'(\mathbb{R})) \). From Lemma 4.3 we have

\[
G(\langle Z_k(t), f \rangle) = \int_0^t G'(\langle Z_k(s), f \rangle) (Z_k(s), Af) \, ds + \frac{\sigma^2}{2} \int_0^t G''(\langle Z_k(s), f \rangle) f(c)^2 \, ds
\]

\[
+ \frac{\sigma^2}{2} f(c)^2 \left( \int_0^t G''(\langle Z_k(s), f \rangle) \, ds \right)
\]

\[
+ \text{mart.}
\]

(4.8)

Let \( 0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = t \) be a partition of \( [0, t] \) so that \( \max_{1 \leq i \leq n} (s_i - s_{i-1}) \to 0 \) as \( n \to \infty \). Then we have

\[
\mathbb{E} \left[ \int_0^t G''(\langle Z_k(s), f \rangle) dy^k_s(c) \, ds \right]
\]

\[
= \mathbb{E} \left[ \lim_{n \to \infty} \sum_{i=1}^n G''(\langle Z_k(s_i), f \rangle) \left( (y^k_{s_i}(c) - y^k_{s_{i-1}}(c)) - (s_i - s_{i-1}) \right) \right]
\]

\[
\leq \|G''\| \mathbb{E} \left[ \liminf_{n \to \infty} \sum_{i=1}^n \left| (y^k_{s_i}(c) - y^k_{s_{i-1}}(c)) - (s_i - s_{i-1}) \right| \right]
\]

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Proposition 4.8
For every result. □

\[ C \]

Proof. By applying (4.10) to suitable truncations of the function \( \{ y_{s_i}^k(c) \} \) we have

\[ \lim_{k \rightarrow \infty} \sum_{i=1}^{n} E \left[ \left( y_{s_i}^k(c) - y_{s_{i-1}}^k(c) \right)^2 \right] \leq \| G'' \| \lim_{n \rightarrow \infty} \sum_{i=1}^{n} \left( \text{Var}(y_{s_i}^k(c) - y_{s_{i-1}}^k(c)) \right)^{1/2}. \] (4.9)

Using (2.7) it is not hard to show that

\[ \text{Var}(y_{s_i}^k(c) - y_{s_{i-1}}^k(c)) = \frac{\sigma^2}{k^2} \int_{\mathbb{R}} \lambda(dx) \int_{s_{i-1}}^{s_i} p(s, c - x) \left\{ \int_{s_{i-1}}^{s_i} p(u, c - x) \, du \right\} \, ds \]

\[ \leq \frac{2\sigma^2}{k^2\pi} (s_i - s_{i-1})^2. \]

Then the right hand side of (4.9) tends to zero as \( k \rightarrow \infty \). From (4.11) it is easy to show that for any \( f \in \mathcal{S}(\mathbb{R}) \), the sequence \( \{ (Z_k(s), f) \}_{k \geq 1} \) is uniformly integrable on \( \Omega \times [0, t] \) relative to the product measure \( P(d\omega)ds \). Letting \( k \rightarrow \infty \) in (4.8) we obtain the desired result. □

Proposition 4.8 For every \( \mu \in \mathcal{S}'(\mathbb{R}) \) there is a process \( \{ Z(t) : t \geq 0 \} \) with sample paths in \( C([0, \infty), \mathcal{S}'(\mathbb{R})) \) so that for \( G \in C^\infty(\mathbb{R}) \) and \( f \in \mathcal{S}(\mathbb{R}) \) we have

\[ G(\langle Z(t), f \rangle) = G(\langle \mu, f \rangle) + \int_0^t G'(\langle Z(s), f \rangle)\langle Z(s), Af \rangle ds \]

\[ + \frac{\sigma^2}{2} \int_0^t G''(\langle Z(s), f \rangle) f(c)^2 ds + \text{mart.} \] (4.10)

Proof. Let \( \{ Z_0(t) : t \geq 0 \} \) be the process mentioned in Lemma 4.7 and let \( Z(t) = P_t \mu + Z_0(t) \). Then (4.10) clearly holds. □

Proposition 4.9 Let \( \{ Z(t) : t \geq 0 \} \) be a solution to the martingale problem (4.10) with sample paths in \( C([0, \infty), \mathcal{S}'(\mathbb{R})) \). Then we have the Langevin type stochastic equation

\[ \langle Z(t), f \rangle = \langle \mu, f \rangle + \sigma B(t) f(c) + \int_0^t \langle Z(s), Af \rangle ds, \quad t \geq 0, f \in \mathcal{S}(\mathbb{R}), \] (4.11)

where \( \{ B_t : t \geq 0 \} \) is a standard one-dimensional Brownian motion.

Proof. By applying (4.10) to suitable truncations of the function \( G(z) = z \) we get

\[ \langle Z(t), f \rangle = \langle Z(0), f \rangle + M_t(f) + \int_0^t \langle Z(s), Af \rangle ds, \] (4.12)

where \( \{ M_t(f) \} \) is a local martingale. By Itô’s formula,

\[ \langle Z(t), f \rangle^2 = \langle Z(0), f \rangle^2 + 2 \int_0^t \langle Z(s), f \rangle \langle Z(s), Af \rangle ds + \langle M(f) \rangle_t + \text{local mart.} \]
On the other hand, if we apply (4.10) directly to suitable truncations of the function 
\[ G(z) = z^2, \]
then
\[ \langle Z(t), f \rangle^2 = \langle Z(0), f \rangle^2 + 2 \int_0^t \langle Z(s), f \rangle \langle Z(s), Af \rangle ds + \sigma^2 f(c)^2 t + \text{local mart.} \]

Comparing the above two equations we have
\[ \langle M(f) \rangle_t = \sigma^2 f(c)^2 t, \quad t \geq 0, f \in \mathcal{S}(\mathbb{R}). \]  
(4.13)

Clearly, (4.12) and (4.13) determine a continuous orthogonal martingale measure on 
\([0, \infty) \times \mathbb{R}\) with intensity \(\sigma^2 \delta_c(x) ds dx\). By El Karoui and Méléard (1990, Proposition II-1) we have
\[ M_t(f) = \sigma B_t f(c), \quad t \geq 0, f \in \mathcal{S}(\mathbb{R}). \]

for a standard one-dimensional Brownian motion \(\{B(t) : t \geq 0\}\). \(\square\)

**Proposition 4.10** Let \(\{Z(t) : t \geq 0\}\) be a solution to the stochastic equation (4.11) with sample paths in \(C([0, \infty), \mathcal{S}'(\mathbb{R}))\). Then we have a.s.
\[ \langle Z(t), f \rangle = \langle \mu, P_t f \rangle + \sigma \int_0^t P_{t-s} f(c) dB(s), \quad t \geq 0, f \in \mathcal{S}(\mathbb{R}). \]  
(4.14)

**Proof.** If \(\{Z(t) : t \geq 0\}\) is a solution of (4.11) with sample paths in \(C([0, \infty), \mathcal{S}'(\mathbb{R}))\), we have
\[
\int_0^t \langle Z(s), P_{t-s} f \rangle ds = \int_0^t \langle \mu, P_{t-s} f \rangle ds + \sigma \int_0^t P_{t-s} f(c) B(s) ds \\
+ \sigma \int_0^t ds \int_0^s \langle Z(u), P_{t-s} Af \rangle du \\
= \int_0^t \langle \mu, P_{t-s} f \rangle ds + \sigma \int_0^t P_{t-s} f(c) B(s) ds \\
+ \sigma \int_0^t du \int_u^t \langle Z(u), P_{t-s} Af \rangle ds \\
= \int_0^t \langle \mu, P_{t-s} f \rangle ds + \sigma \int_0^t P_{t-s} f(c) B(s) ds \\
- \int_0^t \langle Z(u), f \rangle du + \int_0^t \langle Z(u), P_{t-u} f \rangle du.
\]

It follows that
\[ \int_0^t \langle Z(s), f \rangle ds = \int_0^t \langle \mu, P_{t-s} f \rangle ds + \sigma \int_0^t P_{t-s} f(c) B(s) ds. \]

Consequently, we have
\[ \int_0^t \langle Z(s), Af \rangle ds = \int_0^t \langle \mu, P_{t-s} Af \rangle ds + \sigma \int_0^t P_{t-s} Af(c) B(s) ds. \]
where in the last equality we have used the formula of integration by parts. Then we use (4.11) again to see (4.14) holds. □

A combination of the above propositions shows that the Langevin type equation (4.11) has a pathwise unique solution and the martingale problem (4.10) is well-posed. Moreover, by (4.14) it is easy to show that \( \{Z(t) : t \geq 0\} \) is a Markov process with transition semigroup \((Q_t^c)_{t \geq 0}\) defined by

\[
\int_{\mathcal{S}'} e^{i\langle \nu, f \rangle} Q_t^c(\mu, d\nu) = \exp \left\{ i\langle \mu, P_t f \rangle - \frac{\sigma^2}{2} \int_0^t P_s f(c)^2 ds \right\}.
\]

(4.15)

A distribution-valued Markov process with transition semigroup in this form is usually called an Ornstein-Uhlenbeck type process. The process \( \{Z(t) : t \geq 0\} \) describes the asymptotic fluctuations of the single point catalytic SBM as the branching mechanisms are small. More precisely, we have the following theorem.

**Theorem 4.11** As \( k \to \infty \), the process \( \{Z_k(t) : t \geq 0\} \) converges weakly in \( C([0, \infty), \mathcal{S}'(\mathbb{R})) \) to the Ornstein-Uhlenbeck process \( \{Z(t) : t \geq 0\} \) with transition semigroup \((Q_t^c)_{t \geq 0}\) and \( Z(0) = 0 \).

*Proof.* By Lemma 4.6 the family \( \{Z_k(t) : t \geq 0\}_{k \geq 1} \) is tight in the space \( C([0, \infty), \mathcal{S}'(\mathbb{R})) \). Then the result follows from Lemma 4.7 and the well-posedness of the martingale problem. □

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