PROPER MERGINGS OF STARS AND CHAINS ARE COUNTED BY SUMS OF ANTIDIAGONALS IN CERTAIN CONVOLUTION ARRAYS – THE DETAILS –

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Abstract. A proper merging of two disjoint quasi-ordered sets \( P \) and \( Q \) is a quasi-order on the union of \( P \) and \( Q \) such that the restriction to \( P \) or \( Q \) yields the original quasi-order again and such that no elements of \( P \) and \( Q \) are identified. In this article, we determine the number of proper mergings in the case where \( P \) is a star (i.e. an antichain with a smallest element adjoined), and \( Q \) is a chain. We show that the lattice of proper mergings of an \( m \)-antichain and an \( n \)-chain, previously investigated by the author, is a quotient lattice of the lattice of proper mergings of an \( m \)-star and an \( n \)-chain, and we determine the number of proper mergings of an \( m \)-star and an \( n \)-chain by counting the number of congruence classes and by determining their cardinalities. Additionally, we compute the number of Galois connections between certain modified Boolean lattices and chains.

1. Introduction

Given two quasi-ordered sets \((P, \leq_P)\) and \((Q, \leq_Q)\), a merging of \( P \) and \( Q \) is a quasi-order \( \leq \) on the union of \( P \) and \( Q \) such that the restriction of \( \leq \) to \( P \) or \( Q \) yields \( \leq_P \) respectively \( \leq_Q \) again. In other words, a merging of \( P \) and \( Q \) is a quasi-order on the union of \( P \) and \( Q \), which does not change the quasi-orders on \( P \) and \( Q \).

In [3] a characterization of the set of mergings of two arbitrary quasi-ordered sets \( P \) and \( Q \) is given. In particular, it turns out that every merging \( \leq \) of \( P \) and \( Q \) can be uniquely described by two binary relations \( R \subseteq P \times Q \) and \( T \subseteq Q \times P \). The relation \( R \) can be interpreted as a description, which part of \( P \) is weakly below \( Q \), and analogously the relation \( T \) can be interpreted as a description, which part of \( Q \) is weakly below \( P \). It was shown in [3] that the set of mergings forms a distributive lattice in a natural way. If a merging satisfies \( R \cap T^{-1} = \emptyset \), and hence if no element of \( P \) is identified with an element of \( Q \), then it is called proper, and the set of proper mergings forms a distributive sublattice of the previous one.

In [5], the author gave formulas for the number of proper mergings of (i) an \( m \)-chain and an \( n \)-chain, (ii) an \( m \)-antichain and an \( n \)-antichain and (iii) an \( m \)-antichain and an \( n \)-chain, see [5, Theorem 1.1]. The present article can be seen as a subsequent work which was triggered by the following observation: if we denote the number of proper mergings of an \( m \)-star (i.e. an \( m \)-antichain with a minimal element adjoined) and an \( n \)-chain by \( F_{\text{sc}}(m, n) \), then the first few entries

\[
\begin{array}{c|c|c|c|c|c|c|c}
\cline{3-8}
m & n & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 1 & & & & & & \\
2 & 3 & 2 & 3 & 1 & & & \\
3 & 6 & 9 & 6 & 2 & 1 & & \\
4 & 10 & 16 & 16 & 8 & 1 & & \\
5 & 15 & 25 & 35 & 25 & 10 & & \\
6 & 21 & 36 & 54 & 54 & 28 & & \\
\end{array}
\]

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of $F_{sc}(2,n)$ (starting with $n=0$) are

$$1, 12, 68, 260, 777, 1960, 4368, \ldots ,$$

and the first few entries of $F_{sc}(3,n)$ (starting with $n=0$) are

$$1, 24, 236, 1400, 6009, 20608, 59952, \ldots .$$

Surprisingly, these sequences are $[6, A213547]$ and $[6, A213560]$, respectively, and they describe sums of antidiagonals in certain convolution arrays. Inspired by this connection, we are able to prove the following theorem.

**Theorem 1.1.** Let $\mathcal{E}^\bullet_{m,n}$ denote the set of proper mergings of an $m$-star and an $n$-chain. Then,

$$|\mathcal{E}^\bullet_{m,n}| = \sum_{k=1}^{n+1} k^m (n - k + 2)^{m+1}.$$ 

The proof of Theorem 1.1 is obtained in the following way: after recalling the necessary notations and definitions in Section 2, we observe in Section 3 that the lattice $(\mathcal{E}^\bullet_{m,n}, \preceq)$ contains a certain quotient lattice, namely the lattice $(\mathcal{A}^\bullet_{m,n}, \preceq)$ of proper mergings of an $m$-antichain and an $n$-chain. The cardinality of $\mathcal{A}^\bullet_{m,n}$ was determined by the author in [5]. Then, in Section 4, we determine the cardinalities of the congruence classes of the lattice congruence generating $(\mathcal{A}^\bullet_{m,n}, \preceq)$ as a quotient lattice of $(\mathcal{E}^\bullet_{m,n}, \preceq)$, using a decomposition of $\mathcal{A}^\bullet_{m,n}$ by means of the bijection with monotone $(n+1)$-colorings of the complete bipartite digraph $\vec{K}_{m,m}$ described in [5, Section 5].

Using a theorem from Formal Concept Analysis which relates Galois connections between lattices to binary relations between their formal contexts, we are able to determine the number of Galois connections between certain modified Boolean lattices and chains in Section 5. The mentioned modified Boolean lattices and chains arise in a natural way, when considering proper mergings of stars and chains, thus we have decided to include this result in the present article.

2. Preliminaries

In this section we recall the basic notations and definitions needed in this article. For a detailed introduction to Formal Concept Analysis, we refer to [4].

2.1. **Formal Concept Analysis.** The theory of Formal Concept Analysis (FCA) was introduced in the 1980s by Rudolf Wille, see [7], as an approach to restructure lattice theory. The initial goal was to interpret lattices as hierarchies of concepts and thus to give meaning to the lattice elements in a fixed context. Such a formal context is a triple $(G, M, I)$, where $G$ is a set of so-called objects, $M$ is a set of so-called attributes and $I \subseteq G \times M$ is a binary relation that describes whether an object has an attribute. Given a formal context $\mathcal{K} = (G, M, I)$, we define two derivation operators

$$(\cdot)^I : \wp(G) \rightarrow \wp(M), \quad A \mapsto A^I = \{ m \in M \mid g I m \text{ for all } g \in A \},$$

$$(\cdot)^I : \wp(M) \rightarrow \wp(G), \quad B \mapsto B^I = \{ g \in G \mid g I m \text{ for all } m \in B \},$$

where $\wp$ denotes the power set. The notation $g I m$ is to be understood as $(g, m) \in I$. Let now $A \subseteq G$, and $B \subseteq M$. For brevity, if $g \in G$, then we write simply $g^I$
Instead of \( \{g\}^I \), and analogously if \( m \in M \), then we write \( m^I \) instead of \( \{m\}^I \). The pair \( b = (A, B) \) is called formal concept of \( K \) if \( A^I = B \) and \( B^I = A \). In this case, we call \( A \) the extent and \( B \) the intent of \( b \). It can easily be seen that for every \( A \subseteq G \), and \( B \subseteq M \), the pairs \( (A^I, A^I) \) and respectively \( (B^I, B^I) \) are formal concepts. Conversely, every formal concept of \( K \) can be written in such a way. We denote the set of all formal concepts of \( K \) by \( B(K) \), and define a partial order on \( B(K) \) by

\[
(A_1, B_1) \leq (A_2, B_2) \quad \text{if and only if} \quad A_1 \subseteq A_2 \quad \text{or equivalently} \quad B_1 \supseteq B_2.
\]

Let \( \mathcal{B}(K) \) denote the poset \( (B(K), \leq) \). The basic theorem of FCA (see \([4, \text{Theorem 3}]\)) states that \( \mathcal{B}(K) \) is a complete lattice, the so-called concept lattice of \( K \). Moreover, every complete lattice is a concept lattice.

Usually, a formal context is represented by a cross-table, where the rows represent the objects and the columns represent the attributes. The cell in row \( g \) and column \( m \) contains a cross if and only if \( g \vdash m \). For every context \( K = (G, M, I) \), there are two maps

\[
\gamma : G \to \mathcal{B}(K), \quad g \mapsto (g^I, g^I), \quad \text{and} \quad \mu : M \to \mathcal{B}(K), \quad m \mapsto (m^I, m^I),
\]

mapping each object, respectively attribute, to its corresponding formal concept. It is common sense in FCA to label the Hasse diagram of \( \mathcal{B}(K) \) in the following way: the node representing a formal concept \( b \in \mathcal{B}(K) \) is labeled with the object \( g \) (or with the attribute \( m \)) if and only if \( b = \gamma g \) (or \( b = \mu m \)). Object labels are attached below the nodes in the Hasse diagram, and attribute labels above. In this presentation, the extent (intent) of a formal concept corresponds to the labels weakly below (weakly above) this formal concept in the Hasse diagram of \( \mathcal{B}(K) \).

See Figures 1 and 2 for small examples.

2.2. Bonds and Mergings. Let \( K_1 = (G_1, M_1, I_1) \), and \( K_2 = (G_2, M_2, I_2) \) be formal contexts. A binary relation \( R \subseteq G_1 \times M_2 \) is called bond from \( K_1 \) to \( K_2 \) if for every object \( g \in G_1 \), the row \( g^R \) is an intent of \( K_2 \) and for every \( m \in M_2 \), the column \( m^R \) is an extent of \( K_1 \).

Now let \( (P, \vdash_P) \) and \( (Q, \vdash_Q) \) be disjoint quasi-ordered sets. Let \( R \subseteq P \times Q \), and \( T \subseteq Q \times P \). Define a relation \( \vdash_{R,T} \) on \( P \cup Q \) as

\[
p \vdash_{R,T} q \quad \text{if and only if} \quad p \vdash_P q \quad \text{or} \quad p \vdash_Q q \quad \text{or} \quad p R q \quad \text{or} \quad p T q,
\]

for all \( p, q \in P \cup Q \). The pair \( (R, T) \) is called merging of \( P \) and \( Q \) if \( (P \cup Q, \vdash_{R,T}) \) is a quasi-ordered set. Moreover, a merging is called proper if \( R \cap T^{-1} = \emptyset \). Since for fixed quasi-ordered sets \( (P, \vdash_P) \) and \( (Q, \vdash_Q) \) the relation \( \vdash_{R,T} \) is uniquely determined by \( R \) and \( T \), we refer to \( \vdash_{R,T} \) as a (proper) merging of \( P \) and \( Q \) as well. Let \( \circ \) denote the relational product.

**Proposition 2.1** ([3, Proposition 2]). Let \( (P, \vdash_P) \) and \( (Q, \vdash_Q) \) be disjoint quasi-ordered sets, and let \( R \subseteq P \times Q \), and \( T \subseteq Q \times P \). The pair \( (R, T) \) is a merging of \( P \) and \( Q \) if and only if all of the following properties are satisfied:

1. \( R \) is a bond from \( (P, P, \not\vdash_P) \) to \( (Q, Q, \not\vdash_Q) \),
2. \( T \) is a bond from \( (Q, Q, \not\vdash_Q) \) to \( (P, P, \not\vdash_P) \),
is a poset again if and only if \( \preceq P \) and \( \preceq Q \) are both antisymmetric and \( R \cap T^{-1} = \emptyset \).

Moreover, the relation \( \preceq_{R,T} \) as defined in (2) is antisymmetric if and only if \( \preceq P \) and \( \preceq Q \) are both antisymmetric and \( R \cap T^{-1} = \emptyset \).

In the case that \( P \) and \( Q \) are posets, this proposition implies that \( (P \cup Q, \preceq_{R,T}) \) is a poset again if and only if \( (R, T) \) is a proper merging of \( P \) and \( Q \). Denote the set of mergings of \( P \) and \( Q \) by \( \mathfrak{M}_{P,Q} \), and define a partial order on \( \mathfrak{M}_{P,Q} \) by

\[
(R_1, T_1) \preceq (R_2, T_2) \quad \text{if and only if} \quad R_1 \subseteq R_2 \text{ and } T_1 \supseteq T_2.
\]

It was shown in [3, Theorem 1] that \( (\mathfrak{M}_{P,Q}, \preceq) \) is a lattice, where \((\emptyset, Q \times P)\) is the unique minimal element, and \((P \times Q, \emptyset)\) the unique maximal element. Moreover, it follows from [3, Theorem 2] that \( (\mathfrak{M}_{P,Q}, \preceq) \) is distributive. Let \( \mathfrak{M}^\bullet_{P,Q} \subseteq \mathfrak{M}_{P,Q} \) denote the set of all proper mergings of \( P \) and \( Q \). It was also shown in [3] that \( (\mathfrak{M}^\bullet_{P,Q}, \preceq) \) is a distributive sublattice of \( (\mathfrak{M}_{P,Q}, \preceq) \).

2.3. \textit{m-Stars.} Let \( A = \{a_1, a_2, \ldots, a_m\} \) be a set. An \textit{m-antichain} is a poset \( a = (A, \preceq_a) \), satisfying \( a_i \preceq_a a_j \) if and only if \( i = j \) for all \( i, j \in \{1, 2, \ldots, m\} \). Consider the set \( S = A \cup \{s_0\} \), and define a partial order \( \preceq_S \) on \( S \) as follows: \( s \preceq_S s' \) if and only if either \( s = s' \) or \( s = s_0 \) for all \( s, s' \in S \). The poset \( s = (S, \preceq_S) \) is called an \textit{m-star}. (That is, an \textit{m-star} is an \textit{m-antichain} with a smallest element adjoined. See Figure 1 for an example.) We are interested in the formal concepts of the contradiordinal scale of an \textit{m-star}, namely the formal concepts of the formal context \((S, S, \not\simeq_S)\). It is clear that \((\emptyset, S)\) is a formal concept of \((S, S, \not\simeq_S)\), and we notice further that, for every \( B \subseteq S \setminus \{s_0\} \) (considered as an object set), we have \( B \not\simeq_S = S \setminus (B \cup \{s_0\}) \). Since the object \( s_0 \) satisfies \( s_0 \not\simeq_S = S \setminus \{s_0\} \), we conclude further that \( B \not\simeq_S = B \cup \{s_0\} \). Thus, \((S, S, \not\simeq_S)\) has precisely \( 2^m + 1 \) formal concepts, namely

\[
(\emptyset, S) \quad \text{and} \quad (B \cup \{s_0\}, S \setminus (B \cup \{s_0\})) \quad \text{for} \quad B \subseteq S \setminus \{s_0\}.
\]

2.4. \textit{n-Chains.} Let \( C = \{c_1, c_2, \ldots, c_n\} \) be a set. An \textit{n-chain} is a poset \( c = (C, \preceq_c) \) satisfying \( c_i \preceq_c c_j \) if and only if \( i \leq j \) for all \( i, j \in \{1, 2, \ldots, n\} \). (See Figure 2 for an example.) Clearly, the corresponding contradiordinal scale \((C, C, \not\simeq_c)\) has precisely \( n + 1 \) formal concepts, namely

\[
(\{c_1, c_2, \ldots, c_{i-1}\}, \{c_i, c_{i+1}, \ldots, c_n\}) \quad \text{for} \quad i \in \{1, 2, \ldots, n + 1\}.
\]
Figure 2. A 4-chain, its incidence table and the corresponding contraordinal scale.

\[ \begin{array}{cccccc}
  c_1 & c_2 & c_3 & c_4 \\
  \times & \times & \times & \times \\
  \times & \times & \times & \times \\
  \times & \times & \times & \times \\
  \times & \times & \times & \times \\
\end{array} \]

Figure 3. The first four rows and six columns of the convolution array of \( u_2 \) and \( v_2 \).

\[
\begin{array}{cccccc}
  i = 1 & i = 2 & i = 3 & i = 4 & i = 5 & i = 6 \\
  j = 1 & 1 & 8 & 34 & 104 & 259 & 560 \\
  j = 2 & 4 & 25 & 88 & 234 & 524 & 1043 \\
  j = 3 & 9 & 52 & 170 & 424 & 899 & 1708 \\
  j = 4 & 16 & 89 & 280 & 674 & 1384 & 2555 \\
\end{array}
\]

In the case \( i = n + 1 \), the set \( \{c_1, c_{i+1}, \ldots, c_n\} \) is to be interpreted as the empty set and in the case \( i = 1 \), the set \( \{c_1, c_2, \ldots, c_{i-1}\} \) is to be interpreted as the empty set.) See for instance [5, Section 3.1] for a more detailed explanation.

2.5. **Convolutions.** Let \( u = (u_1, u_2, \ldots, u_k) \) and \( v = (v_1, v_2, \ldots, v_k) \) be two vectors of length \( k \). The convolution of \( u \) and \( v \) is defined as

\[
U \ast V = \sum_{i=1}^{k} u_i \cdot v_{k-i+1}.
\]

In this article, we are interested in the convolutions of two very special vectors, given by functions \( u_m(h) = h^m \) and \( v_m(i, h) = (i - 1 + h)^m \). Define the convolution array of \( u_m \) and \( v_m \) as the rectangular array whose entries \( a_{i,j} \) are defined as

\[
a_{i,j} = \left( u_m(1), u_m(2), \ldots, u_m(j) \right) \ast \left( v_m(i, 1), v_m(i, 2), \ldots, v_m(i, j) \right)
\]

\[
= \sum_{k=1}^{i} u_m(k) \cdot v_m(i, j-k+1)
\]

\[
= \sum_{k=1}^{i} (k(i + j - k))^m
\]

See Figure 3 for an illustration. In the cases \( m = 2 \) and \( m = 3 \) we recover [6, A213505] and [6, A213558] respectively. However, we are not interested in the whole convolution array, but in the sums of the antidiagonals. Define

\[
C(m, n) = \sum_{l=1}^{n} a_{l,n-l+1}
\]

\[
= \sum_{l=1}^{n} \sum_{k=1}^{n-l+1} (k(n - k + 1))^m
\]
\[= \sum_{k=1}^{n} k^m (n - k + 1)^{m+1}\]
to be the sum of the \(n\)-th antidiagonal of the convolution array of \(u_m\) and \(v_m\). The first few entries of the sequence \(C(2, n)\) (starting with \(n = 0\)) are

\[0, 1, 12, 68, 260, 777, 1960, 4368, \ldots,\]

see [6, A213547], and the first few entries of the sequence \(C(3, n)\) (starting with \(n = 0\)) are

\[0, 1, 24, 236, 1400, 59952, \ldots,\]

see [6, A213560]. In view of (4), proving Theorem 1.1 is equivalent to showing that

\[(5) \quad |\mathcal{E}_{m,n}^\bullet| = C(m, n + 1).
\]

3. Embedding \(\mathcal{E}_{m,n}^\bullet\) into \(\mathcal{E}_{m,n}\)

In order to prove Theorem 1.1, we make use of the following observation. Let \(\mathcal{E}_{m,n}^\bullet\) denote the set of proper mergings of an \(m\)-antichain and an \(n\)-chain.

**Proposition 3.1.** The lattice \((\mathcal{E}_{m,n}^\bullet, \preceq)\) is a quotient lattice of \((\mathcal{E}_{m,n}, \preceq)\).

Let \(a = (A, =_a), s = (S, \leq_s), \) and \(e = (C, \leq_e)\) be an \(m\)-antichain, an \(m\)-star and an \(n\)-chain, respectively, as defined in Sections 2.3 and 2.4. If we consider the restriction \((S \setminus \{s_0\}, \leq_s)\) we implicitly understand the partial order \(\leq_s\) to be restricted to the ground set \(A = S \setminus \{s_0\}\). Hence, we identify the posets \((S \setminus \{s_0\}, \leq_s)\) and \((A, =_a)\). If we write \(S = \{s_0, s_1, \ldots, s_m\}\), then we identify \(s_i = a_i\) for \(i \in \{1, 2, \ldots, m\}\).

Now let \((R, T) \in \mathcal{E}_{m,n}^\bullet\) and consider the restrictions \(R = R \cap (A \times C), \) and \(T = T \cap (C \times A)\). Further, if \((R, T) \in \mathcal{E}_{m,n}^\bullet\), then define a pair of relations \((R_0, T_0)\) with \(R_0 \subseteq S \times C\) and \(T_0 \subseteq C \times S\) in the following way:

\[
\begin{align*}
s &\ R_0 \ c_j &\text{if and only if} &\begin{cases} 
s = s_0 &\text{and there exists some } i \in \{1, 2, \ldots, m\} \\
s = a_i &\text{for some } i \in \{1, 2, \ldots, m\} \text{ and } c_j \ T_0 \ a_i.\end{cases}
\end{align*}
\]

We notice that \(T\) and \(T_0\) coincide as sets, but they differ as cross-tables, since \(T_0\) has an additional (but empty) column. \(R_0\) can be viewed as a copy of the cross-table of \(R\), where the union of the rows of \(R\) is added again as first row. Now let us define two maps

\[(6) \quad \eta : \mathcal{E}_{m,n}^\bullet \to \mathcal{E}_{m,n}^\bullet, \quad (R, T) \mapsto (\overline{R}, \overline{T}), \quad \text{and} \]

\[(7) \quad \xi : \mathcal{E}_{m,n}^\bullet \to \mathcal{E}_{m,n}^\bullet, \quad (R, T) \mapsto (R_0, T_0).\]

See Figure 4 for an illustration. We have to show that \(\eta \) and \(\xi\) are well-defined.

**Lemma 3.2.** If \((R, T) \in \mathcal{E}_{m,n}^\bullet\), then \((\overline{R}, \overline{T}) \in \mathcal{E}_{m,n}^\bullet\).
Proof. Write \( A = S \setminus \{s_0\} \), and let \((R, T) \in \mathcal{G}_{m,n}^*\). We need to show that \((\overline{R}, \overline{T})\) satisfies the conditions from Proposition 2.1. First of all, we want to show that \(\overline{R}\) is a bond from \((A, A, \neq_a)\) to \((C, C, \leq_c)\), and we know that \(R \subseteq S \times C\) is a bond from \((S, S, \not\leq_s)\) to \((C, C, \leq_c)\). By construction, \(\overline{R} \subseteq A \times C\), and we have \(a_i^R = a_i^\overline{R}\) for \(i \in \{1, 2, \ldots, m\}\), thus every row of \(\overline{R}\) is an intent of \((C, C, \leq_c)\). Now let \(c \in C\). By definition, we know that \(c^R\) is an extent of \((S, S, \not\leq_s)\). It follows from the reasoning in Section 2.3 that either \(c^R = \emptyset\) or \(c^R = B \cup \{s_0\}\) for some \(B \subseteq A\). Hence, \(c^\overline{R} = \emptyset\) or \(c^\overline{R} = B\) for some \(B \subseteq A\). Since \((A, \neq_a)\) is an antichain, the contraordinal scale \((A, A, \neq_a)\) is known to be isomorphic to the formal context of the Boolean lattice with \(2^m\) elements, and \(c^\overline{R}\) is thus an extent of this context. The fact that \(\overline{T}\) is a bond from \((C, C, \leq_c)\) to \((A, A, \neq_a)\) follows analogously.

It is easy to see that \((\overline{R} \circ \overline{T}) \subseteq (R \circ T)\) and \((\overline{T} \circ \overline{R}) \subseteq (T \circ R)\), proving the remaining two conditions. \(\square\)

Lemma 3.3. If \((R, T) \in \mathcal{G}_{m,n}^*\), then \((R \circ T, T \circ R) \in \mathcal{G}_{m,n}^*\).
Proof. Let $S = A \cup \{s_0\}$, where $A = \{a_1, a_2, \ldots, a_m\}$ is the ground set of the antichain $a = (A, \neq)$. For every $i \in \{1, 2, \ldots, m\}$, we have $a_i^{R_0} = a_i^R$. Since $R$ is a bond from $(A, A, \neq)$ to $(C, C, \not\geq)$, we find that $a_i^1 R c_j$ implies $a_i^1 R c_k$ for all $k \geq j$. Hence, $s_0^{R_0} = a_i^R$ for some $a_i \in A$, and thus every row of $R_0$ is an intent of $(C, C, \not\geq)$. If $c \in C$, then by construction $c^{R_0} = \emptyset$ or $c^{R_0} = c^R \cup \{s_0\}$, and thus every column of $R_0$ is an extent of $(S, S, \not\leq)$. For every $i \in \{1, 2, \ldots, m\}$, we have $a_i^{T_0} = a_i^T$ and $s_0^{T_0} = \emptyset$. Hence, every column of $T_0$ is an extent of $(C, C, \not\geq)$. Moreover, for $c \in C$, we have $c^{T_0} = c^T$, and thus every row of $T_0$ is an intent of $(S, S, \not\leq)$.

Consider the relational product $R_0 \circ T_0$, and let $(s, s') \in R_0 \circ T_0$. By definition, there exists some $c \in C$ with $s R_0 c$ and $c T_0 s'$. By construction, $s' \neq s_0$, and for every pair $(s, s') \in R_0 \circ T_0$ with $s \neq s_0$, we have $(s, s') \in R \circ T$, and thus $s = s'$, since $R \circ T$ is contained in $=a$. This is, however, a contradiction to $R \cap T^{-1} = \emptyset$. Thus, $R_0 \circ T_0$ can only contain pairs of the form $(s_0, s')$. These pairs satisfy $s_0 \leq s s'$ by definition of the order relation $\leq_S$, and we conclude that $R_0 \circ T_0$ is contained in $\leq$. Now let $(c, c') \in T_0 \circ R_0$, and let $s \in S$ with $c T_0 s$ and $s R_0 c'$. By construction, $T_0$ does not contain a pair of the form $(c, s_0)$, and if $s \neq s_0$, then $c \leq s c'$ since $T \circ R$ is contained in $\leq$, which completes the proof. \hfill \Box

Let us collect some properties of $\eta$ and $\xi$.

**Lemma 3.4.** The map $\eta$ is surjective, and the map $\xi$ is injective.

Proof. Let $(R, T) \in \mathfrak{A} M_{m,n}^*$, and let $(R_0, T_0) = \xi(R, T)$. By construction, $R_0$ arises from $R$ by adding elements of the form $(s_0, \cdot)$, and $T_0 = T$. Consider $(R_0, T_0) = \eta(R_0, T_0)$. By construction, $R_0$ contains all elements in $R_0$, except those of the form $(s_0, \cdot)$, and analogously for $T_0$. Thus, $(R_0, T_0) = (R, T)$, and we conclude that $\eta \circ \xi = \text{Id}_{\mathfrak{A} M_{m,n}^*}$.

Suppose there exists $(R, T) \in \mathfrak{A} M_{m,n}^*$ with $(R, T) \notin \text{Im}(\eta)$. By definition, we have $\xi(R, T) \in \mathfrak{C} M_{m,n}$, and thus $\eta(\xi(R, T)) \in \mathfrak{A} M_{m,n}$. We have shown in the previous paragraph that $\eta(\xi(R, T)) = (R, T)$, which contradicts $(R, T) \notin \text{Im}(\eta)$. Thus, $\eta$ is surjective.

Now let $(R_1, T_1), (R_2, T_2) \in \mathfrak{A} M_{m,n}$ with $\xi(R_1, T_1) = \xi(R_2, T_2)$. Since $\eta$ is a map, this implies that $\eta(\xi(R_1, T_1)) = \eta(\xi(R_2, T_2))$, and we obtain with the reasoning in the first paragraph that $(R_1, T_1) = (R_2, T_2)$. Thus, $\xi$ is injective. \hfill \Box

**Proposition 3.5.** The maps $\eta$ and $\xi$ defined in (6) and (7) are order-preserving lattice-homomorphisms.

Proof. Let us start with $\eta$, and let $(R_1, T_1), (R_2, T_2) \in \mathfrak{C} M_{m,n}$ be two proper mergings of an $m$-star and an $n$-chain, satisfying $(R_1, T_1) \preceq (R_2, T_2)$. This means by definition of $\preceq$, see (3), that $R_1 \subseteq R_2$ and $T_1 \supseteq T_2$. By definition of $\eta$, we have $R_i = R_i \setminus \{s_0^R\}$ and $T_i = T_i \setminus \{s_0^T\}$ for $i \in \{1, 2\}$. Thus, it follows immediately that $(R_i, T_i) \preceq (R_2, T_2)$.

For showing that $\eta$ is a lattice-homomorphism, we need to show that it is compatible with the lattice operations. This means, we need to show that for every $(R_1, T_1), (R_2, T_2) \in \mathfrak{C} M_{m,n}$, we have

$$\eta((R_1, T_1) \lor (R_2, T_2)) = \eta((R_1, T_1)) \lor \eta((R_2, T_2))$$

and
\[ \eta((R_1, T_1) \land (R_2, T_2)) = \eta((R_1, T_1)) \land \eta((R_2, T_2)). \]

It was shown in [3, Theorem 1] that
\[ (R_1, T_1) \lor (R_2, T_2) = (R_1 \cup R_2, T_1 \cap T_2), \quad \text{and} \]
\[ (R_1, T_1) \land (R_2, T_2) = (R_1 \cap R_2, T_1 \cup T_2). \]

Thus, we have to show that
\[ (R_1 \cup R_2, T_1 \cap T_2) = (R_1 \cup R_2, T_1 \cap T_2), \quad \text{and} \]
\[ (R_1 \cap R_2, T_1 \cup T_2) = (R_1 \cap R_2, T_1 \cup T_2). \]

Since \( \overline{\cdot} \) is a restriction operator, these equalities are trivially satisfied.

Let now \((R_1, T_1), (R_2, T_2) \in \mathcal{C}_{m,n}^*\) be two proper mergings of an \(m\)-antichain and an \(n\)-chain, satisfying \((R_1, T_1) \preceq (R_2, T_2)\). By construction, \((T_i)_o = T_i\) (considered as sets) for \(i \in \{1, 2\}\). Moreover, for \(i \in \{1, 2\}\), the set \((R_i)_o\) is obtained from \(R_i\) by adding pairs \((s_o, c)\) for all \(c \in C\) satisfying \(a_j \leq R_i, c_k \leq s_o\) for some \(a_j \in A\).

If \(R_1 \subseteq R_2\), then it is clear that \((R_2)_o\) has at least as many additional relations as \((R_1)_o\), hence implying \((R_1)_o \subseteq (R_2)_o\). This proves \(((R_1)_o, (T_1)_o) \preceq ((R_2)_o, (T_2)_o), \]

which implies that \(\xi\) is order-preserving.

With the reasoning from above, showing that \(\xi\) is a lattice-homomorphism reduces to showing that for every \((R_1, T_1), (R_2, T_2) \in \mathcal{C}_{m,n}^*\), we have
\[ ((R_1 \cup R_2)_o, (T_1 \cap T_2)_o) = ((R_1)_o \cup (R_2)_o, (T_1)_o \cap (T_2)_o), \quad \text{and} \]
\[ ((R_1 \cap R_2)_o, (T_1 \cup T_2)_o) = ((R_1)_o \cap (R_2)_o, (T_1)_o \cup (T_2)_o). \]

Since by construction \((T_i)_o = T_i\) for \(i \in \{1, 2\}\), we can restrict our attention to the relations \(R_1\) and \(R_2\), and it is sufficient to focus on the behavior of \(s_0^R\) and \(s_0^R\), since the other rows remain unchanged. Clearly, \((s_0, c) \in (R_1 \cup R_2)_o\) is equivalent to the existence of some \(a \in A\) with \(a R_1 c\) or \(a R_2 c\), which means that \((s_0, c) \in (R_1)_o \cup (R_2)_o\). Similarly, \((s_0, c) \in (R_1 \cap R_2)_o\) is equivalent to the existence of some \(a \in A\) with \(a R_1 c\) and \(a R_2 c\), which means that \((s_0, c) \in (R_1)_o \cap (R_2)_o\), and we are done. \(\square\)

**Proof of Proposition 3.1.** Lemma 3.4 and Proposition 3.5 imply that \(\eta\) is a surjective lattice homomorphism from \((\mathcal{C}_{m,n}^*, \preceq)\) to \((\mathcal{C}_{m,n}^*, \preceq)\). Then, the Homomorphism Theorem for lattices, see for instance [1, Theorem 6.9], implies the result. \(\square\)

A consequence of Proposition 3.1 is that for \((R, T) \in \mathcal{C}_{m,n}^*\), the fiber \(\eta^{-1}(R, T)\) is an interval in \((\mathcal{C}_{m,n}^*, \preceq)\), and all the fibers of \(\eta\) are disjoint. We will use this property for the enumeration of the proper mergings of an \(m\)-star and an \(n\)-chain in the next section. Figure 5 shows the lattice of proper mergings of a 3-star and a 1-chain, and the shaded edges indicate how the lattice of proper mergings of a 3-antichain and a 1-chain arises as a quotient lattice.

### 4. Enumerating Proper Mergings of Stars and Chains

In order to enumerate the proper mergings of an \(m\)-star and an \(n\)-chain, we investigate a decomposition of the set of proper mergings of an \(m\)-antichain and an \(n\)-chain, and determine for every \((R, T) \in \mathcal{C}_{m,n}^*\) the number of elements in the fiber \(\eta^{-1}(R, T)\).
Figure 5. The lattice of proper mergings of a 3-star and a 1-chain, where the nodes are labeled with the corresponding proper merging. The 1-chain is represented by the black node, and the 3-star by the (labeled) white nodes. The highlighted edges and vertices indicate the congruence classes with respect to the lattice homomorphism $\eta$ defined in (6).
4.1. Decomposing the Set \( \mathfrak{M}^*_{m,n} \). Denote by \( \mathfrak{M}^*_{m,n}(k_1,k_2) \) the set of proper mergings \( (R,T) \in \mathfrak{M}^*_{m,n} \) satisfying the following condition: \( k_1 \) is the minimal index such that there exists some \( j_1 \in \{1,2,\ldots,m\} \) with \( a_{j_1} R c_{k_1} \), and \( k_2 \) is the maximal index such that there exists some \( j_2 \in \{1,2,\ldots,m\} \) with \( c_{k_2} T a_{j_2} \). By convention, if \( R = \emptyset \), then we set \( k_1 := n + 1 \), and if \( T = \emptyset \), then we set \( k_2 := 0 \). Let \( \{j \} \) denote the disjoint set union.

**Lemma 4.1.** If \( (R,T) \in \mathfrak{M}^*_{m,n}(k_1,k_2) \) is a proper merging of \( a \) and \( c \), then \( k_1 > k_2 \). Moreover we have

\[
\mathfrak{M}^*_{m,n} = \bigcup_{k_1=1}^{n+1} \bigcup_{k_2=0}^{k_1-1} \mathfrak{M}^*_{m,n}(k_1,k_2).
\]

**Proof.** Let \( (R,T) \in \mathfrak{M}^*_{m,n} \). Denote by \( \preceq_{R,T} \) the order relation induced by the proper merging \( (R,T) \) on the set \( A \cup C \). Assume that \( k_1 \leq k_2 \). This means that there exist elements \( a_{j_1}, a_{j_2} \in A \) with \( a_{j_1} \preceq_{R,T} c_{k_1} \) and \( a_{j_2} \preceq_{R,T} c_{k_2} \). If \( k_1 = k_2 \), then \( c_{k_1} = c_{k_2} \), and this implies that \( a_{j_1} \preceq_{R,T} a_{j_2} \) (since \( a \) is an antichain) which is a contradiction to \( (R,T) \) being a proper merging. If \( k_1 < k_2 \), we have \( c_{k_1} < c_{k_2} \), and thus \( a_{j_1} \preceq_{R,T} c_{k_1} \). This is a contradiction to \( R \circ T \) being contained in \( =_a \).

It is clear that the values \( k_1 \) and \( k_2 \) are uniquely determined, and thus the result follows. \( \square \)

For later use, we will decompose \( \mathfrak{M}^*_{m,n}(k_1,k_2) \) even further. Let \( (R,T) \in \mathfrak{M}^*_{m,n}(k_1,k_2) \). It is clear that there exists a maximal index \( l \in \{0,1,\ldots,k_2\} \) such that \( c_l \triangleright R a \) for all \( a \in A \). (The case \( l = 0 \) is to be interpreted as the case where there exists no \( c_l \) with the desired property.) Denote by \( \mathfrak{M}^*_{m,n}(k_1,k_2,l) \) the set of proper mergings \( (R,T) \in \mathfrak{M}^*_{m,n}(k_1,k_2) \) with \( l \) being the maximal index such that \( c_l \triangleright R a_j \) for all \( j \in \{1,2,\ldots,m\} \). Similarly to Lemma 4.1, we can show that

\[
\mathfrak{M}^*_{m,n}(k_1,k_2) = \bigcup_{l=0}^{k_2} \mathfrak{M}^*_{m,n}(k_1,k_2,l),
\]

and we obtain

\[
|\mathfrak{M}^*_{m,n}| = \sum_{k_1=1}^{n+1} \sum_{k_2=0}^{k_1-1} \sum_{l=0}^{k_2} |\mathfrak{M}^*_{m,n}(k_1,k_2,l)|.
\]

4.2. Determining the Cardinality of \( \mathfrak{M}^*_{m,n}(k_1,k_2,l) \). In [5, Section 5], the author has investigated the number of proper mergings of an \( m \)-antichain and an \( n \)-chain, and has constructed a bijection from these proper mergings to monotone \( (n+1) \)-colorings of the complete bipartite graph \( \tilde{K}_{m,m} \). This bijection is essential for determining the cardinality of \( \mathfrak{M}^*_{m,n}(k_1,k_2,l) \). Let us recall this construction briefly. Let \( V \) be the vertex set of a complete bipartite digraph \( \tilde{K}_{m,m} \) partitioned into sets \( V_1 \) and \( V_2 \) such that \( |V_1| = |V_2| = m \), and such that the set \( \tilde{E} \) of edges of \( \tilde{K}_{m,m} \) satisfies \( \tilde{E} = V_1 \times V_2 \). A monotone \( n \)-coloring of \( \tilde{K}_{m,m} \) is now a map \( \gamma : V \rightarrow \{1,2,\ldots,n\} \) satisfying \( \gamma(v) \leq \gamma(v') \) for all \( (v,v') \in \tilde{E} \).

Let \( a \) and \( c \) denote an \( m \)-antichain and an \( n \)-chain, respectively, as defined in Sections 2.3 and 2.4. For \( (R,T) \in \mathfrak{M}^*_{m,n} \), we construct a coloring \( \gamma_{(R,T)} \) of \( \tilde{K}_{m,m} \) as
follows

(9) \( \gamma_{(R,T)}(v_i) = n + 1 - k \) if and only if \[
\begin{aligned}
v_i & \in V_1 \quad \text{and } a_i \ R \ c_j \ \text{for all } j \in \{k+1,k+2,\ldots,n\}, \\
v_i & \in V_2 \quad \text{and } c_j \ T \ a_i \ \text{for all } j \in \{1,2,\ldots,k\},
\end{aligned}
\]
where \( i \in \{1,2,\ldots,m\} \). It is the statement of [5, Theorem 5.6] that this defines a bijection between \( \mathfrak{A}_{m,n}^* \) and the set of monotone \((n+1)\)-colorings of \( \tilde{K}_{m,m} \). The next lemma describes how the monotone \((n+1)\)-coloring of \( \tilde{K}_{m,m} \) induced by \((R,T)\) is influenced by the parameters \( k_1, k_2 \) and \( l \).

**Lemma 4.2.** Let \((R,T) \in \mathfrak{A}_{m,n}^*(k_1,k_2,l)\). The monotone \((n+1)\)-coloring \( \gamma_{(R,T)} \) of \( \tilde{K}_{m,m} \) as defined in (9) satisfies

\[
1 \leq \gamma_{(R,T)}(v) \leq n + 2 - k_1 \quad \text{if } v \in V_1, \quad \text{and} \\
n + 1 - l \geq \gamma_{(R,T)}(v) \geq n + 1 - k_2 \quad \text{if } v \in V_2,
\]
and there is at least one vertex \( v^{(1)} \in V_1 \) with \( \gamma_{(R,T)}(v^{(1)}) = n + 2 - k_2 \), and there is at least one vertex \( v^{(2)} \in V_2 \) with \( \gamma_{(R,T)}(v^{(2)}) = n + 1 - k_2 \), and at least one vertex \( v^{(2)} \in V_2 \) with \( \gamma_{(R,T)}(v^{(2)}) = n + 1 - l \).

**Proof.** Assume that there exists some \( t \in \{1,2,\ldots,m\} \) such that the vertex \( v_t \in V_1 \) satisfies \( \gamma_{(R,T)}(v_t) = k > n + 2 - k_1 \). In view of (9), this means that \( a_i \ R \ c_j \) for all \( j \in \{n + 2 - k,n + 3 - k,\ldots,n\} \), in particular \( a_i \ R \ c_{n+2-k} \). We have \( n + 2 - k < n + 2 - (n + 2 - k_1) = k_1 \), and thus \( c_{n+2-k} \leq c_{k_1} \) which contradicts the minimality of \( k_1 \). If all \( v \in V_1 \) have \( \gamma_{(R,T)}(v) \leq n + 2 - k_1 \), then we obtain a contradiction to the minimality of \( k_1 \) in an analogous way. The argument for the vertices in \( V_2 \) works similar. Note that we have to consider both bounds \( k_2 \) and \( l \). \( \square \)

The next two lemmas determine the cardinality of \( \mathfrak{A}_{m,n}^*(k_1,k_2,l) \) for every valid triple \((k_1,k_2,l)\) by enumerating the corresponding monotone colorings of \( \tilde{K}_{m,m} \). Note that the number of possible ways to color the vertex set \( V_1 \) depends on the parameters \( m,n \) and \( k_1 \), while the number of possible ways to color the vertex set \( V_2 \) depend on the parameters \( m,k_2 \) and \( l \). For a fixed choice of indices \( k_1, k_2 \) and \( l \), denote by \( F_{V_1}(m,n,k_1) \) the number of possible colorings of \( V_1 \), and denote by \( F_{V_2}(m,k_2,l) \) the number of possible colorings of \( V_2 \).

**Lemma 4.3.** For \( k_1 \in \{1,2,\ldots,n+1\} \), we have

\[
F_{V_1}(m,n,k_1) = (n + 2 - k_1)^m - (n + 1 - k_1)^m.
\]

**Proof.** Let \( V = V_1 \cup V_2 \) be the vertex set of \( \tilde{K}_{m,m} \) where \( V_1, V_2 \) are maximal disjoint independent sets of \( \tilde{K}_{m,m} \). Recall that we want to count the possible colorings of \( \tilde{K}_{m,m} \) such that the vertices in \( V_1 \) have color at most \( n + 2 - k_1 \) and there is at least one vertex in \( V_1 \) having color exactly \( n + 2 - k_1 \).

A standard counting argument shows that there are precisely \((n + 2 - k_1)^m \) ways to color the \( m \) vertices of \( V_1 \) with colors in \( \{1,2,\ldots,n+2-k_1\} \). Since we require that at least one vertex has color \( n + 2 - k_1 \), we have to exclude the cases where every vertex is colored \( \leq n + 1 - k_1 \). The same counting argument shows
that there are \((n + 1 - k_1)^m\)-many such colorings. Hence the number of ways to
color the vertices of \(V_1\) with the given restrictions is precisely \((n + 2 - k_1)^m - (n + 1 - k_1)^m\) as desired.

\[\ □\]

**Lemma 4.4.** Let \(k_1 \in \{1, 2, \ldots, n + 1\}\). For \(k_2 \in \{0, 1, \ldots, k_1 - 1\}\) and \(l \in \{0, 1, \ldots, k_2\}\), we have

\[
F_{V_2}(m, k_2, l) = \begin{cases} 
1, & k_2 = l, \text{ or } \\
(k_2 - l + 1)^m - 2(k_2 - l)^m + (k_2 - l - 1)^m, & \text{otherwise.}
\end{cases}
\]

**Proof.** Let \(V = V_1 \cup V_2\) be the vertex set of \(\mathbf{K}_{m,m}\) where \(V_1, V_2\) are maximal disjoint
independent sets of \(\mathbf{K}_{m,m}\). Recall that we want to count the possible colorings of
\(\mathbf{K}_{m,m}\) such that the vertices in \(V_2\) have colors in \{n + 1 - k_2, n + 2 - k_2, \ldots, n + 1 - l\} with at least one vertex having color exactly \(n + 1 - k_2\), and at least one vertex
having color exactly \(n + 1 - l\).

If \(k_2 = l\), it follows from Lemma 4.2 that every vertex in \(V_2\) has color \(n + 1 - k_2 = n + 1 - l\). There is obviously only one possibility.

So let \(l < k_2\). With the same standard counting argument as in the proof
of the previous lemma, we notice that there are precisely \((k_2 - l + 1)^m\) ways to
color the \(m\) vertices of \(V_2\) with colors in \{n + 1 - k_2, n + 2 - k_2, \ldots, n + 1 - l\}.
Since we require to color at least one vertex with color \(n + 1 - k_2\) and at least
one vertex with color \(n + 1 - l\), we have to subtract the cases where all vertices
have color \(\geq n + 2 - k_2\) and the cases where all vertices have color \(\leq n - l\).
However, we subtract the cases where all vertices have a color in \{n + 2 - k_2, n + 3 - k_2, \ldots, n - l\} twice, so we have to add these again. Thus, with an analogous
counting argument as before, we obtain

\[
F_{V_2}(m, k_2, l) = (k_2 - l + 1)^m - 2(k_2 - l)^m + (k_2 - l - 1)^m,
\]
as desired. \[\ □\]

Every proper merging in \(\mathbf{K}_{m,n}(k_1, k_2, l)\) corresponds to a monotone coloring
of \(\mathbf{K}_{m,m}\) where the colors respect the restrictions described in Lemma 4.2. Since
\(k_1 > k_2\) (see Lemma 4.1) we notice that the largest possible color for \(V_1\) is strictly
smaller than the smallest possible color for \(V_2\), and we obtain

\[
\big| \mathbf{K}_{m,n}(k_1, k_2, l) \big| = F_{V_1}(m, n, k_1) \cdot F_{V_2}(m, k_2, l).
\]

(10)

4.3. **Determining the Cardinality of the Fibers.** We have seen in Section 3 that
\((\mathbf{K}_{m,n}, \leq)\) is a quotient lattice of \((\mathbf{S}_{m,n}, \leq)\). Thus, every proper merging of an
\(m\)-antichain and an \(n\)-chain corresponds to a set of proper mergings of an \(m\)-star
and an \(n\)-chain (namely the corresponding fiber under the lattice homomorphism \(\eta\), and these sets are pairwise disjoint. Thus, if we can determine the number of
elements in each fiber, then we can determine the number of all proper mergings
of an \(m\)-star and an \(n\)-chain.

Let \((R, T) \in \mathbf{S}_{m,n}\) be a proper merging of an \(m\)-star and an \(n\)-chain. In the
following, we write for some \(j \in \{1, 2, \ldots, n\}\) simply “\(s_0 \leq_{R,T} c_j\)” to mean that
we create a pair of relations \((R', T)\) from \((R, T)\) by setting

\[
R' = R \cup \{ (s_0, c_j), (s_0, c_{j+1}), \ldots, (s_0, c_n) \}.
\]
Similarly, we write \( c_j \leq_{R,T} s_0 \) for some \( j \in \{1, 2, \ldots, n\} \) to mean that we create a new pair of relations \((R, T')\) from \((R, T)\) by setting
\[
T' = T \cup \{(c_1, s_i), (c_2, s_i), \ldots, (c_j, s_i)\}, \text{ for all } i \in \{0, 1, \ldots, m\}.
\]

For \( c \in C \), the operations \( s_0 \leq_{R,T} c \) respectively \( c \leq_{R,T} s_0 \) can be understood as adding a covering relation \((s_0, c)\) respectively \((c, s_0)\) to the proper merging \((R, T)\) and applying transitive closure. Thus, it is not immediately clear that these operations yield a merging of an \( m\)-star and an \( n\)-chain at all. The next lemma determines the number of proper mergings we can generate from the image under the map \( \xi \) of a proper merging of an \( m\)-antichain and an \( n\)-chain.

**Lemma 4.5.** Let \((R, T) \in \mathcal{MC}_{m,n}(k_1, k_2, l)\). Then \(|\eta^{-1}(R, T)| = k_1(l + 1) - (l + 1)^2\).

**Proof.** By construction, we have \( \xi(R, T) = (R_0, T_0) \in \mathcal{MC}(R, T) \), and \( s_0 \leq_{R_0, T_0} c_k \) for all \( k \in \{k_1, k_1 + 1, \ldots, n\} \). Thus, performing \( c_j \leq_{R_0, T_0} s_0 \) for some \( j \geq k_1 \) would simply do nothing. Performing \( c_j \leq_{R_0, T_0} s_0 \) for some \( j \geq k_1 \) adds in particular the relation \((c_j, a_k)\) to \( T_0 \) for all \( k \in \{1, 2, \ldots, m\} \). Since \((R, T) \in \mathcal{MC}_{m,n}(k_1, k_2, l)\), we can assume that there exists some \( i \in \{1, 2, \ldots, m\} \) such that \( a_i \in R_0 c_k\), and thus in particular \( a_i \in R_0 c_j \). Thus we have \( c_j T_0 a_i \) and \( a_i R c_j\), which is a contradiction to \((R, T')\) being a proper merging. Hence, we can only create new proper mergings from \((R_0, T_0)\) by applying the operations \( s_0 \leq_{R_0, T_0} c_j \) or \( c_j \leq_{R_0, T_0} s_0 \) for some \( j \in \{1, 2, \ldots, k_1 - 1\} \).

If we perform \( c_j \leq_{R_0, T_0} s_0 \) for some \( j \in \{k_2 + 1, k_2 + 2, \ldots, k_1 - 1\} \), then we obtain a proper merging \((R_0, T'_0)\) which contains the relations \((c_j, a_i)\) for all \( i \in \{1, 2, \ldots, m\} \). Hence, \( \eta(R_0, T'_0) \neq (R, T) \), and thus \((R_0, T'_0) \notin \eta^{-1}(R, T)\). However, we can perform \( s_0 \leq_{R_0, T_0} c_j \) for every \( j \in \{k_2 + 1, k_2 + 2, \ldots, k_1 - 1\} \) without problems. This gives us \((k_1 - k_2 - 1)\)-many new proper mergings in \( \eta^{-1}(R, T)\).

With the same reasoning as before, we see that performing \( c_j \leq_{R_0, T_0} s_0 \) for some \( j \in \{1 + 1, 1 + 2, \ldots, k_2\} \) yields a proper merging \((R_0, T'_0) \notin \eta^{-1}(R, T)\), but we can apply \( s_0 \leq_{R_0, T_0} c_j \) for every such \( j \), giving us \((k_2 - 1)\)-many new proper mergings in \( \eta^{-1}(R, T)\).

Now let \( j \in \{1, 2, \ldots, l\} \). Performing \( c_j \leq_{R_0, T_0} s_0 \) works fine in this case, and we obtain a proper merging \((R_0, T'_0)\). Additionally, we can now perform \( s_0 \leq_{R_0, T'_0} c_i \) for every \( i \in \{j + 1, j + 2, \ldots, k_1 - 1\} \) to obtain a new proper merging from \((R_0, T'_0)\). Note the new subscript \( R_0, T'_0 \) in the operator! (Suppose that we perform \( s_0 \leq_{R_0, T'_0} c_i \) for some \( i \in \{1, 2, \ldots, j\} \). Then we had \( s_0 R'_0 c_i \in T'_0 \) which is a contradiction to \((R_0', T_0')\) being a proper merging. Performing \( s_0 \leq_{R_0, T'_0} c_i \) for some \( i \in \{k_1, k_1 + 1, \ldots, l\} \) would yield \((R_0', T_0') = (R_0, T_0)\).) Thus, for every \( j \in \{1, 2, \ldots, l\} \) we obtain \((k_1 - j)\)-many new proper mergings in \( \eta^{-1}(R, T)\). Finally, we can also perform \( s_0 \leq_{R_0, T_0} c_{j} \) to obtain a new proper merging \((R_0', T_0) \in \eta^{-1}(R, T)\). However, we cannot perform \( c_i \leq_{R_0, T_0} s_0 \) for any \( i \in \{1, 2, \ldots, n\} \), because we would either obtain a contradiction or a proper merging we have already counted. Hence, this case gives us \( l \) new proper mergings in \( \eta^{-1}(R, T)\).
Now we just have to add all the possibilities and obtain
\[ |\eta^{-1}(R, T)| = 1 + (k_1 - k_2 - 1) + (k_2 - l) + \left( \sum_{j=1}^{l} k_1 - j \right) + l \]
\[ = k_1(l + 1) - l(l + 1) \right) \]
\[ = k_1(l + 1) - \left( \frac{l + 1}{2} \right), \]
as desired. \( \square \)

Now we are set to enumerate the proper mergings of an \( m \)-star and an \( n \)-chain.

**Lemma 4.6.** For \( m, n \in \mathbb{N} \), we have \( F_{\star}(m, n) = C(m, n + 1) \), where \( C \) is defined in (4).

**Proof.** Putting (8), (10) and Lemmas 4.3–4.5 together, we obtain
\[ (11) \quad F_{\star}(m, n) = \sum_{(R,T) \in \mathcal{M}_{n,m}} |\eta^{-1}(R, T)| \]
\[ = \sum_{k_1=1}^{n+1} \sum_{k_2=0}^{k_1-1} \sum_{l=0}^{k_2} |\eta^{-1}(R, T)| \]
\[ = \sum_{k_1=1}^{n+1} \sum_{k_2=0}^{k_1-1} \sum_{l=0}^{k_2} F_{V_1}(m,n,k_1) \cdot F_{V_2}(m,k_2, l) \cdot \left( k_1(l + 1) - \left( \frac{l + 1}{2} \right) \right) \]
\[ = \sum_{k_1=1}^{n+1} F_{V_1}(m,n,k_1) \sum_{k_2=0}^{k_1-1} \sum_{l=0}^{k_2} F_{V_2}(m,k_2, l) \cdot \left( k_1(l + 1) - \left( \frac{l + 1}{2} \right) \right). \]
The proof that this last sum equals \( C(m, n + 1) \) is not very difficult, but rather technical and longish. Thus we have decided to provide this proof in every detail in Appendix A. \( \square \)

**Proof of Theorem 1.1.** This follows from Lemma 4.6. \( \square \)

**Remark 4.7.** The presented proof of Theorem 1.1 is obtained by counting the proper mergings of an \( m \)-star and an \( n \)-chain in a rather na"{i}ve way, and the conversion of the na"{i}ve counting formula into the desired formula is rather longish. Christian Krattenthaler proposed a family of objects that are also counted by \( C(m, n + 1) \): let \( V_1, V_2, \) and \( V_3 \) be disjoint sets with cardinalities \( |V_1| = k_1, |V_2| = k_2, \) and \( |V_3| = k_3, \) and denote by \( \vec{K}_{k_1,k_2,k_3} \) the directed graph \( (V, E) \) whose vertex set is \( V = V_1 \cup V_2 \cup V_3, \) and whose set of edges is \( E = (V_1 \times V_2) \cup (V_2 \times V_3). \) A monotone \((n + 1)\)-coloring of a directed graph is an assignment of at most \( n + 1 \) different numbers to the vertices of the graph such that the numbers weakly increase along directed edges. A standard counting argument shows that the number of monotone \((n + 1)\)-colorings of \( \vec{K}_{m+1,1,m} \) is precisely \( C(m,n+1). \) A much more elegant, and perhaps much simpler proof of Theorem 1.1 could thus be obtained by solving the following problem.
Problem 4.8 (Solved by Jonathan Farley, May 2013). Construct a bijection between the set $\mathcal{C}_{m,n}$ of proper mergings of an $m$-star and an $n$-chain, and the set $\Gamma_{n+1}(\tilde{K}_{m+1,1,m})$ of monotone $(n+1)$-colorings of $\tilde{K}_{m+1,1,m}$.

4.4. Update: Jonathan Farley’s Solution of Problem 4.8. Recently, Jonathan Farley [2] has solved Problem 4.8. We will briefly explain his bijection in this section. Let $\mathcal{P} = (P, \leq_P)$ and $\mathcal{Q} = (Q, \leq_Q)$ be two posets. We say that $\mathcal{P}$ is bounded if it has a unique minimal element, denoted by $0_P$, and a unique maximal element, denoted by $1_P$, and likewise for $\mathcal{Q}$. The ordinal sum of $\mathcal{P}$ and $\mathcal{Q}$ is the poset $\mathcal{P} \oplus \mathcal{Q} = (P \cup Q, \leq)$, with $p \leq q$ if and only if either (i) $p, q \in P$ and $p \leq_P q$, (ii) $p, q \in Q$ and $p \leq_Q q$, or (iii) $p \in P$ and $q \in Q$. If $\mathcal{P}$ has a unique maximal element $1_P$, and if $\mathcal{Q}$ has a unique minimal element $0_Q$, then the coalesced ordinal sum of $\mathcal{P}$ and $\mathcal{Q}$ is the ordinal sum of $\mathcal{P}$ and $\mathcal{Q}$ with $1_P$ and $0_Q$ identified, and will be denoted by $\mathcal{P} \oplus_\mathcal{C} \mathcal{Q}$. Now let $\zeta : \mathcal{P} \rightarrow \mathcal{Q}$ be a map from $\mathcal{P}$ to $\mathcal{Q}$. We say that $\zeta$ is order-preserving if $p \leq_P p'$ implies $\zeta(p) \leq_Q \zeta(p')$. Further, if $\mathcal{P}$ and $\mathcal{Q}$ are bounded, we say that $\zeta$ is bound-preserving if $\zeta(0_P) = 0_Q$ and $\zeta(1_P) = 1_Q$.

Let $a_k$ denote an antichain with $k$ elements, let $c_i$ denote a chain with $k$ elements, and let $B_k$ denote the Boolean lattice with $2^k$ elements. For two posets $\mathcal{P}$ and $\mathcal{Q}$, let $OP(\mathcal{P}, \mathcal{Q})$ denote the set of order-preserving maps from $\mathcal{P}$ to $\mathcal{Q}$, and if $\mathcal{P}$ and $\mathcal{Q}$ are lattices, let $BP(\mathcal{P}, \mathcal{Q})$ denote the set of bound-preserving lattice homomorphisms from $\mathcal{P}$ to $\mathcal{Q}$.

It is easy to see that

$$\Gamma_{n+1}(\tilde{K}_{m+1,1,m}) \cong OP(a_{m+1} \oplus a_1 \oplus a_m, c_{n+1}).$$

Using Priestley’s Representation Theorem For Distributive Lattices, see for instance [1, Theorem 11.23], we conclude that

$$|OP(a_{m+1} \oplus a_1 \oplus a_m, c_{n+1})| = |BP(c_{n+2}, B_m \oplus \mathcal{C} B_1 \oplus \mathcal{C} B_{m+1})|.$$

Since $c_{n+2}$ is a chain, we find

$$|BP(c_{n+2}, B_m \oplus \mathcal{C} B_1 \oplus \mathcal{C} B_{m+1})| = |OP(c_{n+2}, B_m \oplus \mathcal{C} B_1 \oplus \mathcal{C} B_{m+1})|,$$

and if we forget about the bounds, we obtain

$$|OP(c_{n+2}, B_m \oplus \mathcal{C} B_1 \oplus \mathcal{C} B_{m+1})| = |OP(c_n, (B_m \setminus \{0_{B_m}\}) \oplus B_0 \oplus B_{m+1})|.$$

We notice that order-preserving maps from a chain to a poset $\mathcal{P}$ are in bijection with multichains of $\mathcal{P}$. Clearly, to every multichain in $\mathcal{P} \oplus \mathcal{Q}$, we can associate a unique multichain in $\mathcal{Q} \oplus \mathcal{P}$, by exchanging the corresponding components. Hence, the order of the summands does not really play a role, and we obtain

$$|OP(c_n, (B_m \setminus \{0_{B_m}\}) \oplus B_0 \oplus B_{m+1})| = |OP(c_n, B_0 \oplus B_{m+1} \oplus \mathcal{C} B_m)|.$$

The next step is to construct a bijection from $OP(c_n, B_0 \oplus B_{m+1} \oplus \mathcal{C} B_m)$ to $\mathcal{C}_{m,n}$. For that, let $X = \{x\}$, let $Y$ be a poset which is order-isomorphic to the Boolean lattice whose elements are subsets of $\{0, 1, 2, \ldots, m\}$ (via the map $\varphi_Y$), and let $Z$ be a poset which is order-isomorphic to the Boolean lattice whose elements are subsets of $\{1, 2, \ldots, m\}$ (via the map $\varphi_Z$), and let $\mathcal{P}_{1,m+1,m} = X \oplus Y \oplus \mathcal{C} Z$. Now let $\zeta \in OP(c_n, \mathcal{P}_{1,m+1,m})$, and suppose that $\zeta(c_i) = d_i$ for all $i \in \{1, 2, \ldots, n\}$. Define a proper merging $(R, T) \zeta \in \mathcal{C}_{m,n}$ as follows:

1. if $d_i = x$, then $(c_i, s) \in T$ for all $s \in S$,
(2a) if \(d_i \in Y \setminus Z\), then \((s_0, c_i) \in R\) if and only if \(0 \in \varphi_Y(d_i)\),
(2b) if \(d_i \in Y \setminus Z\), then \((c_j, s_j) \in T\) for all \(j \in \{1, 2, \ldots, k\}\) if and only if \(j \notin \varphi_T(d_i)\), and
(3) if \(d_i \in Z\), then \((s_0, c_i) \in R\) and \((s_j, c_i) \in R\) for all \(j \in \{1, 2, \ldots, k\}\) if and only if \(j \in \varphi_Z(d_i)\).

It was shown by Farley [2] that this construction is indeed a bijection. See Appendix C for an illustration.

5. Counting Galois Connections between Chains and Modified Boolean Lattices

In the spirit of [5, Sections 3.4 and 5.2], we can use the enumeration formula for the proper mergings of an \(m\)-star and an \(n\)-chain to determine the number of Galois connections between \(\mathcal{P}(C, C, \geq_c)\) and \(\mathcal{P}(S, S, \geq_s)\). In particular, we prove the following proposition within this section.

**Proposition 5.1.** Let \(s = (S, \leq_s)\) be an \(m\)-star and let \(c = (C, \leq_c)\) be an \(n\)-chain. The number of Galois connections between \(\mathcal{P}(C, C, \geq_c)\) and \(\mathcal{P}(S, S, \geq_s)\) is \(\sum_{k=1}^{n+1} k^m\).

We have seen in Section 2.4 that \(\mathcal{P}(C, C, \geq_c)\) is isomorphic to an \((n + 1)\)-chain, and the reasoning in Section 2.3 implies that \(\mathcal{P}(S, S, \geq_s)\) can be constructed as follows: let \(B_m\) denote the Boolean lattice with \(2^m\) elements. Replacing the bottom element of \(B_m\) by a 2-chain yields a lattice which we call \(m\)-balloon, and we denote it by \(B_m^{(1)}\). Figure 6 shows the Hasse diagram of \(B_4^{(1)}\). The labels attached to some of the nodes indicate how \(B_4^{(1)}\) arises as the concept lattice of the contraordinal scale of the 4-star shown in Figure 1.

**Remark 5.2.** The construction of \(B_m^{(1)}\) can be generalized easily, by replacing the bottom element of \(B_m\) by an \((l + 1)\)-chain for some \(l > 1\). We call the corresponding lattice an \((m, l)\)-balloon, and denote it by \(B_m^{(l)}\). However, the case \(l > 1\) is not considered further in this article, even though it can be considered as the concept lattice of the contraordinal scale of the poset that arises from an \(m\)-star by replacing the unique bottom element by an \(l\)-chain.

Before we enumerate the Galois connections between \(m\)-balloons and \((n + 1)\)-chains, we recall the definitions. A Galois connection between two posets \(\langle P, \leq_P \rangle\) and \(\langle Q, \leq_Q \rangle\) is a pair \((\varphi, \psi)\) of maps

\[
\varphi : P \to Q \quad \text{and} \quad \psi : Q \to P,
\]

satisfying

\[
\begin{align*}
p_1 \leq_P p_2 & \quad \implies \quad \varphi p_1 \geq_Q \varphi p_2, \\
q_1 \leq_Q q_2 & \quad \implies \quad \psi q_1 \geq_P \psi q_2, \\
p \leq_P \psi \varphi p & , \quad \text{and} \quad \varphi p \leq_Q \varphi \psi q,
\end{align*}
\]

for all \(p, p_1, p_2 \in P\) and \(q, q_1, q_2 \in Q\). Recall that, given formal contexts \(\mathbf{K}_1 = (G, M, I)\) and \(\mathbf{K}_2 = (H, N, J)\), a relation \(R \subseteq G \times H\) is called dual bond from \(\mathbf{K}_1\) to \(\mathbf{K}_2\) if for every \(g \in G\), the set \(g R\) is an extent of \(\mathbf{K}_2\) and for every \(h \in H\), the set \(h R\) is an extent of \(\mathbf{K}_1\). In other words, \(R\) is a dual bond from \(\mathbf{K}_1\) to \(\mathbf{K}_2\) if and
only if \( R \) is a bond from \( \mathcal{K}_1 \) to the dual\(^1\) context \( \mathcal{K}_2^d \). In the case, where the posets \( (P, \leq_p) \cong \mathfrak{P}(\mathcal{K}_1) \) and \( (Q, \leq_Q) \cong \mathfrak{P}(\mathcal{K}_2) \) are concept lattices, we can interpret the Galois connections between \( (P, \leq_p) \) and \( (Q, \leq_Q) \) as dual bonds from \( \mathcal{K}_1 \) to \( \mathcal{K}_2 \) as described in the following theorem.

**Theorem 5.3** ([4, Theorem 53]). Let \( (G, M, I) \) and \( (H, N, J) \) be formal contexts. For every dual bond \( R \subseteq G \times H \), the maps

\[
\varphi_R(X, X^I) = (X^R, X^R), \quad \text{and} \quad \psi_R(Y, Y^I) = (Y^R, Y^R),
\]

where \( X \) and \( Y \) are extents of \( (G, M, I) \) respectively \( (H, N, J) \), form a Galois connection between \( \mathfrak{P}(G, M, I) \) and \( \mathfrak{P}(H, N, J) \). Moreover, every Galois connection \( (\varphi, \psi) \) induces a dual bond from \( (G, M, I) \) to \( (H, N, J) \) by

\[
R_{(\varphi, \psi)} = \{(g, h) \mid \gamma g \leq \psi \gamma h\} = \{(g, h) \mid \gamma h \leq \varphi \gamma g\},
\]

where \( \gamma \) is the map defined in (1). In particular, we have

\[
\varphi_{R(\varphi, \psi)} = \varphi, \quad \psi_{R(\varphi, \psi)} = \psi, \quad \text{and} \quad R_{(\varphi, \psi, \psi)} = R.
\]

Since chains are self-dual, the previous theorem implies that every Galois connection between an \((n+1)\)-chain and an \(m\)-balloon corresponds to a bond from \((C, C, \not\leq_c)\) to \((S, S, \not\leq_s)\). In view of Proposition 2.1 this means that every Galois connection between an \((n+1)\)-chain and an \(m\)-balloon corresponds to a proper merging of \(s\) and \(c\) which is of the form \((\emptyset, T)\). These are relatively easy to enumerate as our next proposition shows.

**Proposition 5.4.** Let \( s \) be an \(m\)-star and let \( c \) be an \(n\)-chain. The number of proper mergings of \(s\) and \(c\) which are of the form \((\emptyset, T)\) is \(\sum_{k=1}^{n+1} k^m\).

**Proof.** Let \((\emptyset, T)\) be a proper merging of \(s\) and \(c\). Thus, \(T \subseteq C \times S\) is a bond from \((C, C, \not\leq_c)\) to \((S, S, \not\leq_s)\). This means, for every \(e \in C\), the row \(c^T\) is an intent of \((S, S, \not\leq_s)\), and thus must be either the set \(S\) or a set of the form \(S \setminus (B \cup \{s_0\})\) for some \(B \subseteq S \setminus \{s_0\}\). Moreover, for every \(s \in S\), the column \(s^T\) is

---

\(^1\)Let \( \mathcal{K} = (G, M, I) \) be a formal context. The dual context \( \mathcal{K}^d \) of \( \mathcal{K} \) is given by \((M, G, I^{-1})\) and satisfies \( \mathfrak{P}(\mathcal{K}^d) \cong \mathfrak{P}(\mathcal{K})^d \), where \( \mathfrak{P}(\mathcal{K})^d \) is the (order-theoretic) dual of the lattice \( \mathfrak{P}(\mathcal{K}) \).
an extent of \((C, C, \preceq_C)\), and thus must be of the form \(\{c_1, c_2, \ldots, c_{i-1}\}\) for some \(i \in \{1, 2, \ldots, n+1\}\). (The case \(i = 1\) is to be interpreted as the empty set.)

Since \(T\) is a bond from \((C, C, \preceq_C)\) to \((S, S, \preceq_S)\), we notice that if \(c_i T s_j\), then \(c_k T s_j\) for every \(k \in \{1, 2, \ldots, i\}\). In particular, if the \(i\)-th row of \(T\) is a full row, then every row above the \(i\)-th row is also a full row. Furthermore, if \(c_i T s_0\), then \(c_j T s_k\) for every \(k \in \{0, 1, \ldots, m\}\), since the only intent of \((S, S, \preceq_S)\) that contains \(\{s_0\}\) is itself.

Now let \(k \in \{1, 2, \ldots, n\}\) be the maximal index such that \(c_k^T = S\), and write \(C_{n-k} = \{c_{k+1}, c_{k+2}, \ldots, c_n\}\). We have just seen that this implies that \(c_i^T = S\) for \(j \leq k\), and \((c_j, s_0) \notin T\) for \(j > k\). Hence, \(T\) is a bond from \((C, C, \preceq_C)\) to \((S, S, \preceq_S)\) if and only if the restriction of \(T\) to \(C_{n-k} \times (S \setminus \{s_0\})\) is a bond from \((C_{n-k}, C_{n-k}, \preceq_C)\) to \((S \setminus \{s_0\}, S \setminus \{s_0\}, \preceq_S)\). See Figure 7 for an illustration.

![Figure 7. Illustration of the situation with \(k\) full rows in \(T\).](image)

The number \(g(m, n)\) of proper mergings of \(s\) and \(c\) which are of the form \((\emptyset, T)\) is now the sum over all proper mergings of \(s\) and \(c\) which are of the form \((\emptyset, T)\), and where the first \(k\) rows of \(T\) are full rows. We obtain

\[
g(m, n) = \sum_{k=0}^{n} (n-k+1)^m = \sum_{k=1}^{n+1} k^m,
\]

as desired.

\[\square\]

**Proof of Proposition 5.1.** This follows immediately from Proposition 5.4. \[\square\]

Appendix B lists the proper mergings of an 3-star and a 1-chain that are of the form \((\emptyset, T)\), and the corresponding Galois connections between an 3-balloon and a 2-chain.

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Appendix A. Proof of Lemma 4.6

Recall from (11) that putting (8), (10) and Lemmas 4.3–4.5 together, yields

\[
F_{ae}(m, n) = \sum_{k_1=1}^{n+1} F_{V_1}(m, n, k_1) \sum_{k_2=0}^{k_1-1} \sum_{l=0}^{k_2} F_{V_2}(m, k_2, l) \cdot \left( k_1(l+1) - \binom{l+1}{2} \right),
\]

where

\[
F_{V_1}(m, n, k_1) = (n+2-k_1)^m - (n+1-k_1)^m, \quad \text{and}
\]

\[
F_{V_2}(m, k_2, l) = \begin{cases} (k_2-l+1)^m - 2(k_2-l)^m + (k_2-l-1)^m, & \text{if } l < k_2 \\ 1, & \text{if } l = k_2. \end{cases}
\]

Recall further that

\[
C(m, n) = \sum_{k=1}^{n} k^m (n-k+2)^{m+1},
\]

and we want to show that \( F_{ae}(m, n) = C(m, n+1) \). Let us first focus on the term

\[
A(m, k_1, k_2) = \sum_{l=0}^{k_2-1} F_{V_2}(m, k_2, l) \cdot \left( k_1(l+1) - \binom{l+1}{2} \right).
\]

We can convince ourselves quickly that the following identities are true:

\[
k_1(l+1) - \binom{l+1}{2} = k_1(l+2) - \binom{l+2}{2} + l + 1 - k_1, \quad \text{and}
\]

\[
k_1(l+1) - \binom{l+1}{2} = k_1(l+3) - \binom{l+3}{2} + 2l + 3 - 2k_1.
\]

Thus, we can write

\[
A(m, k_1, k_2) = \sum_{l=0}^{k_2-1} (k_2-l+1)^m \cdot \left( k_1(l+1) - \binom{l+1}{2} \right) - 2 \sum_{l=0}^{k_2-1} (k_2-l)^m \cdot \left( k_1(l+1) - \binom{l+1}{2} \right) + \sum_{l=0}^{k_2-1} (k_2-l-1)^m \cdot \left( k_1(l+1) - \binom{l+1}{2} \right)
\]

\[
= \sum_{l=0}^{k_2-1} (k_2-l+1)^m \cdot \left( k_1(l+1) - \binom{l+1}{2} \right) - 2 \sum_{l=0}^{k_2-1} (k_2-l+1)^m \cdot \left( k_1(l+2) - \binom{l+2}{2} + l + 1 - k_1 \right) + \sum_{l=0}^{k_2-1} (k_2-l+2)^m \cdot \left( k_1(l+3) - \binom{l+3}{2} + 2l + 3 - 2k_1 \right).
\]
If we define \( \varphi(m,k_1,k_2,l) = (k_2 - l + 1)^m \cdot (k_1(l + 1) - \binom{l+1}{2} \cdot (l + 1 - k_1) \), then we obtain
\[
A(m,k_1,k_2) = \sum_{l=0}^{k_2-1} \varphi(m,k_1,k_2,l) - 2 \sum_{l=0}^{k_2-1} \varphi(m,k_1,k_2,l + 1) - 2 \sum_{l=0}^{k_2-1} (k_2 - l)^m \cdot (l + 1 - k_1) \\
+ \sum_{l=0}^{k_2-1} \varphi(m,k_1,k_2,l + 2) + \sum_{l=0}^{k_2-1} (k_2 - l - 1)^m \cdot (2(l + 1 - k_1) + 1) \\
= \sum_{l=0}^{k_2-1} \varphi(m,k_1,k_2,l) - 2 \sum_{l=0}^{k_2-1} \varphi(m,k_1,k_2,l + 1) + \sum_{l=0}^{k_2-1} \varphi(m,k_1,k_2,l + 2) \\
- 2 \sum_{l=0}^{k_2-1} (k_2 - l)^m \cdot (l + 1 - k_1) + \sum_{l=0}^{k_2-1} (k_2 - l - 1)^m \cdot (2(l + 1 - k_1) + 1) \\
= \sum_{l=0}^{k_2-1} \varphi(m,k_1,k_2,l) - 2 \sum_{l=0}^{k_2-1} \varphi(m,k_1,k_2,l + 1) + \sum_{l=0}^{k_2-1} \varphi(m,k_1,k_2,l + 2) \\
+ \sum_{l=0}^{k_2-1} (k_2 - l)^m \cdot (2k_1 - 2l - 2) + \sum_{l=0}^{k_2-1} (k_2 - l - 1)^m \cdot (2l + 3 - 2k_1).
\]

Let us now simplify the terms not involving \( \varphi \).
\[
\psi(m,k_1,k_2) = \sum_{l=0}^{k_2-1} (k_2 - l)^m \cdot (2k_1 - 2l - 2) + \sum_{l=0}^{k_2-1} (k_2 - l - 1)^m \cdot (2l + 3 - 2k_1) \\
= \left( k_2^m (2k_1 - 2) + (k_2 - 1)^m (2k_1 - 4) + \cdots + 1^m (2k_1 - 2k_2) \right) \\
+ \left( (k_2 - 1)^m (3 - 2k_1) + (k_2 - 2)^m (5 - 2k_1) + \cdots + 1^m (2k_2 - 1 - 2k_1) \right) \\
= k_2^m (2k_1 - 2) - (k_2 - 1)^m (k_2 - 2)^m - \cdots - 1^m \\
= k_2^m (2k_1 - 2) - \sum_{l=1}^{k_2-1} l^m.
\]

Applying this identity and shifting indices yields
\[
A(m,k_1,k_2) = \sum_{l=0}^{k_2-1} \varphi(m,k_1,k_2,l) - 2 \sum_{l=0}^{k_2-1} \varphi(m,k_1,k_2,l + 1) + \sum_{l=2}^{k_2+1} \varphi(m,k_1,k_2,l) \\
+ \psi(m,k_1,k_2) \\
= \varphi(m,k_1,k_2,0) - \varphi(m,k_1,k_2,1) - \varphi(m,k_1,k_2,k_2) + \varphi(m,k_1,k_2,k_2 + 1) \\
+ \psi(m,k_1,k_2) \\
= (k_2 + 1)^m k_1 - k_2^m (2k_1 - 1) - k_1 (k_2 + 1) + \binom{k_2 + 1}{2} \\
+ k_2^m (2k_1 - 2) - \sum_{l=1}^{k_2-1} l^m.
\]
= k_1 (k_2 + 1)^m - k_1 (k_2 + 1) + \binom{k_2 + 1}{2} - \sum_{l=1}^{k_2} l^m.

So far, we have shown that

\begin{equation}
F_{\text{sc}}(m, n) = \sum_{k_1=1}^{n+1} \left( (n + 2 - k_1)^m - (n + 1 - k_1)^m \right) \\
\quad \cdot \sum_{k_2=0}^{k_1-1} \left( k_1 (k_2 + 1)^m - k_1 (k_2 + 1) + \binom{k_2 + 1}{2} - \sum_{l=1}^{k_2} l^m \right) \\
\quad + k_1 (k_2 + 1) - \binom{k_2 + 1}{2}
\end{equation}

\begin{align*}
&= \sum_{k_1=1}^{n+1} \left( (n + 2 - k_1)^m - (n + 1 - k_1)^m \right) \\
&\quad \cdot \sum_{k_2=0}^{k_1-1} \left( k_1 (k_2 + 1)^m - \sum_{l=1}^{k_2} l^m + \right) \\
&= \sum_{k_1=1}^{n+1} \left( (n + 2 - k_1)^m - (n + 1 - k_1)^m \right) \\
&\quad \cdot \left( \sum_{k_3=0}^{k_1-1} k_1 (k_2 + 1)^m - \sum_{k_2=0}^{k_1-1} \sum_{l=1}^{k_2} l^m \right).
\end{align*}

We may now simplify the inner double sum:

\begin{equation}
\sum_{k_2=0}^{k_1-1} \sum_{l=1}^{k_2} l^m = 0 + \sum_{l=1}^{k_2} l^m + \sum_{l=1}^{k_2} \sum_{l=1}^{k_2} l^m + \ldots + \sum_{l=1}^{k_2} \sum_{l=1}^{k_2} \sum_{l=1}^{k_2} l^m \\
= k_1 0^m + (k_1 - 1) 1^m + (k_1 - 2) 2^m + \ldots + (k_1 - 1)^m \\
= \sum_{k_2=0}^{k_1-1} (k_1 - k_2) k_2^m.
\end{equation}

If this is substituted in (12), we obtain

\begin{align*}
&F_{\text{sc}}(m, n) = \sum_{k_1=1}^{n+1} \left( (n + 2 - k_1)^m - (n + 1 - k_1)^m \right) \cdot \sum_{k_2=0}^{k_1-1} k_1 (k_2 + 1)^m \\
&\quad - \sum_{k_1=1}^{n+1} \left( (n + 2 - k_1)^m - (n + 1 - k_1)^m \right) \cdot \sum_{k_2=0}^{k_1-1} (k_1 - k_2) k_2^m \\
&= \sum_{k_1=1}^{n+1} \left( (n + 2 - k_1)^m - (n + 1 - k_1)^m \right) \cdot \sum_{k_2=0}^{k_1-1} k_1 (k_2 + 1)^m \\
&\quad - \sum_{k_1=1}^{n+1} \left( (n + 2 - k_1)^m - (n + 1 - k_1)^m \right) \cdot \sum_{k_2=0}^{k_1-1} k_1 k_2^m
\end{align*}
Thus, substituting these in (13), we obtain

\[
F_{\infty}(m, n) = \sum_{k_1=1}^{n+1} \left( (n + 2 - k_1)^m - (n + 1 - k_1)^m \right) \cdot k_1^{m+1}
\]

\[
+ \sum_{k_1=1}^{n+1} \left( (n + 2 - k_1)^m - (n + 1 - k_1)^m \right) \cdot \sum_{k_2=0}^{k_1-1} k_2^{m+1}
\]

\[
= \sum_{k_1=1}^{n+1} \left( (n + 2 - k_1)^m - (n + 1 - k_1)^m \right) \cdot \sum_{k_2=0}^{k_1-1} k_2^{m+1}
\]

\[
= \sum_{k_1=1}^{n+1} \left( (n + 2 - k_1)^{m+1} - (n + 1 - k_1)^{m+1} \right)
\]

It is easy to check the identities

\[
\sum_{k_1=1}^{n+1} \left( (n + 2 - k_1)^m - (n + 1 - k_1)^m \right) \cdot k_1^{m+1} = \sum_{k_1=1}^{n+1} k_1^{m+1} \left( (n + 2 - k_1)^{m+1} - (n + 1 - k_1)^{m+1} \right),
\]

and

\[
\sum_{k_1=1}^{n+1} \left( (n + 2 - k_1)^m - (n + 1 - k_1)^m \right) \cdot \sum_{k_2=0}^{k_1-1} k_2^{m+1} = \sum_{k_1=1}^{n+1} k_1^{m+1} (n + 1 - k_1)^{m+1}.
\]

Thus, substituting these in (13), we obtain

\[
F_{\infty}(m, n) = \sum_{k_1=1}^{n+1} \left( (n + 2 - k_1)^m - (n + 1 - k_1)^m \right) \cdot k_1^{m+1}
\]

\[
+ \sum_{k_1=1}^{n+1} \left( (n + 2 - k_1)^m - (n + 1 - k_1)^m \right) \cdot \sum_{k_2=0}^{k_1-1} k_2^{m+1}
\]

\[
= \sum_{k_1=1}^{n+1} k_1^{m+1} \left( (n + 2 - k_1)^{m+1} - (n + 1 - k_1)^{m+1} \right) + \sum_{k_1=1}^{n+1} k_1^{m+1} (n + 1 - k_1)^{m+1}
\]

\[
= \sum_{k_1=1}^{n+1} k_1^{m+1} \left( (n + 2 - k_1)^{m+1} - (n + 1 - k_1)^{m+1} + (n + 1 - k_1)^{m+1} \right)
\]

\[
= \sum_{k_1=1}^{n+1} k_1^{m+1} (n + 2 - k_1)^{m+1}
\]

\[
= C(m, n + 1),
\]

as desired. \qed
Remark B.1. Let \((\emptyset, T)\) be a proper merging of an \(m\)-star \((S, \leq_S)\) and an \(n\)-chain \((C, \leq_C)\). In order to produce the corresponding Galois connection, we define a dual bond \(\hat{T}\) between \((S, S, \not\leq_S)\) and \((C, C, \not\leq_C)\) as follows: for every \(i \in \{1, 2, \ldots, n\}\), we define
\[
c_i^\dagger = \begin{cases} S \setminus \{c_{n+1-i}^T\} & \text{if } c_{n+1-i}^T \neq S, \\ \emptyset & \text{otherwise.} \end{cases}
\]
### Appendix C. Illustration of Farley’s Bijection

| $\xi \in \Omega P(c_n, P_{1,3,2})$ | $R_\xi$ | $T_\xi$ | $(\{s_0, s_1, s_2, c_1\}, \leq R_\xi, T_\xi)$ |
|---------------------------------|--------|--------|---------------------------------|
| ![Diagram 1](image1)           | ![Table 1](table1) | ![Diagram 2](image2) | ![Table 2](table2) |
| ![Diagram 3](image3)           | ![Table 3](table3) | ![Diagram 4](image4) | ![Table 4](table4) |
| ![Diagram 5](image5)           | ![Table 5](table5) | ![Diagram 6](image6) | ![Table 6](table6) |
| ![Diagram 7](image7)           | ![Table 7](table7) | ![Diagram 8](image8) | ![Table 8](table8) |
| ![Diagram 9](image9)           | ![Table 9](table9) | ![Diagram 10](image10) | ![Table 10](table10) |
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