Anonymization with Worst-Case Distribution-Based Background Knowledge

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Abstract

Background knowledge is an important factor in privacy preserving data publishing. Distribution-based background knowledge is one of the well studied background knowledge. However, to the best of our knowledge, there is no existing work considering the distribution-based background knowledge in the worst case scenario, by which we mean that the adversary has accurate knowledge about the distribution of sensitive values according to some tuple attributes. Considering this worst case scenario is essential because we cannot overlook any breaching possibility. In this paper, we propose an algorithm to anonymize dataset in order to protect individual privacy by considering this background knowledge. We prove that the anonymized datasets generated by our proposed algorithm protects individual privacy. Our empirical studies show that our method preserves high utility for the published data at the same time.

1 Introduction

Privacy preserving data publishing is an important topic in the literature of privacy for very pragmatic reasons. As an example, AOL did not take sufficient precaution and encountered some undesired consequences. A dataset about search logs was published in 2006. Later AOL realized that a single 62 year old woman living in Georgia can be re-identified from the search logs by some New York Times reporters. The search logs were withdrawn and two employees responsible for releasing the search logs were fired.

Example 1 (Data Publishing) Suppose a table T like Table 1 is to be anonymized for publication. Table T has two kinds of attributes, (1) the quasi-identifier (QI) attributes and (2) the sensitive attribute. (1) The QI attributes can be used as an identifier in the table. In our example, the QI attributes are Nationality and Zipcode. Attribute Name is just for discussion and is not used for publication. [17] points out that in a real dataset, about 87% of individuals can be uniquely identified by some QI attributes with a publicly available external table such as a voter registration list. An example of a voter registration list is shown in Table 2. The sensitive attribute contains some sensitive values. In our example, the sensitive attribute is “Disease” containing sensitive values such as Heart Disease and HIV. Assume that each tuple in the table is owned by an individual and each individual owns at most one tuple.

Our target is to anonymize T and publish the anonymized dataset T* like Table 3 to satisfy some privacy requirements. A typical anonymization is described as follows. T is horizontally partitioned into multiple tuple groups. Let P be a resulting group. We give a unique ID called GID to P and all tuples in P are said to have the same GID value. An anonymization defines a function β on each P to form an anonymized group (in short, A-group) such that the linkage between the QI attributes and the sensitive attribute in the A-group is broken. One way to break the linkage is bucketization, forming two tables, called the QI table (Table 3(a)) and the sensitive table (Table 3(b)): P is projected on all QI attributes and attribute GID to form the QI table, and on the sensitive attribute and attribute GID to form the sensitive table. Therefore, a table T is anonymized to a dataset T* if T* is formed by first partitioning T into a number of groups, then forming an A-group from each partition by β and finally inserting each A-group into T*.

¹There are many sources of such an external table. Most municipalities sell population registers that include the identifiers of individuals along with basic demographics; examples include local census data, voter lists, city directories, and information from motor vehicle agencies, tax assessors, and real estate agencies [15]. From [17], it is reported that a city’s voter list in two diskettes was purchased for twenty dollars, and was used to re-identify medical records.
Example 2 (Background Knowledge) Consider $L_1$ in Table 3. In $L_1$, Heart Disease and Flu are values of the sensitive attribute Disease. Since most individuals can be reidentified by the QI attributes with a publicly available external table such as voter registration list, if we are given the voter registration list as shown in Table 2 it is easy to figure out that the two tuples in $L_1$ correspond to Alex and Bob. From $L_1$, it seems that each of the two individuals, Alex and Bob, in this group has a 50% chance of linking to Heart Disease (Flu). The reason why the chance is interpreted as 50% is that the analysis is based on this group without any additional information.

Suppose we are given a probability distribution as shown in Table 3. The distribution of attribute set {"Nationality"} consists of the probabilities that a Japanese, an American or a French is linked to "Heart Disease" (and "Not Heart Disease"). For example, the probability that American is linked to Heart Disease is 0.1 and the probability that Japanese is linked to Heart Disease is 0.003. With this distribution, the adversary can say that Bob, being a Japanese, has less chance of having Heart Disease. S/he can deduce that Alex, being an American, has a higher chance of having Heart Disease. The intended 50% threshold is thus violated.

Hence background knowledge has important impact on privacy preserving data publishing. Recent works [9, 12, 14, 6, 18] start to focus on modeling background knowledge. Distribution-based background knowledge is one type of the well-known background knowledge which is used in the state-of-the-art privacy model, t-closeness. Distribution-based background knowledge [9, 12] is the information related to the distribution about sensitive information in data. There are at least two kinds of distribution-based background knowledge, namely dataset based distribution and QI based distribution. The dataset based distribution is the distribution of the values in the sensitive attribute according to the entire dataset [9]. The QI based distribution is the distribution of the values in the sensitive attribute restricted to individuals with the same values on some QI attributes [12].

Example 3 (Distribution-based background knowledge) Suppose that there are 100,000 individuals in the dataset $T$ and with 6,000 individuals linking to "Heart Disease". The probability that an individual $t$ in the dataset is linked to "Heart Disease" is 0.06. The dataset based distribution has been considered by [9].

In this paper we consider QI based distribution [12]. Some well-known examples of such knowledge are the facts that Japanese seldom suffer from Heart Disease [13] and male individual cannot be linked to ovarian cancer [10]. For example, the distribution of the sensitive attribute according to Japanese may be encoded as \{Japanese:"Heart Disease", 0.003, (Japanese:"Flu", 0.21), ...\} where (Japanese:$x$, $p$) denotes that the probability that a Japanese is linked to a value $x$ is $p$. 

| Name | Nationality | Zipcode | Disease |
|------|-------------|---------|---------|
| Alex | American    | 55501   | Heart Disease |
| Bob  | Japanese    | 55502   | Flu     |
| Japanese | Japanese | 55503   | Flu     |
| Japanese | Japanese | 55504   | Flu     |

| Name | Nationality | Zipcode | GID |
|------|-------------|---------|-----|
| Alex | American    | 55501   | L_1 |
| Bob  | Japanese    | 55502   | L_2 |
| Japanese | Japanese | 55503   | L_3 |
| Japanese | Japanese | 55504   | L_4 |

| Nationality | Zipcode | Disease |
|-------------|---------|---------|
| American    | 55501   | Heart Disease |
| Japanese    | 55502   | Flu     |
| Japanese    | 55503   | Flu     |
| Japanese    | 55504   | Flu     |

Table 1. An example

Table 2. Voter registration list

Table 3. A 2-diverse dataset anonymized from Table 1

For example, Table 1 is anonymized to Table 3 by bucketization. Such an anonymization is commonly adopted in the literature of data publishing [20, 21, 14, 18, 10].

There are many privacy models in the literature such as k-anonymity [17], l-diversity [13], t-closeness [9], (k, e)-anonymity [21], Injector [10] and m-confidentiality [18]. For illustration, let us consider a simplified setting of the l-diversity model [13] as a privacy requirement for published data $T^*$. An A-group is said to be l-diversify or satisfy l-diversity if in the A-group the number of occurrences of any sensitive value is at most $1/l$ of the group size. A table satisfies l-diversity (or it is l-diversify) if all A-groups in it are l-diversify. Suppose that Table 1 is anonymized to Table 3. Consider the A-group with GID equal to $L_1$ which corresponds to the first two tuples in QI table (Table 3(a)) and the first two tuples in sensitive table (Table 3(b)). In the following, we simply refer to the A-group with GID equal to $L_1$ by $L_1$. Since $L_1$ contains two tuples, the group size of $L_1$ is equal to 2. Since the number of occurrences of any sensitive value (i.e., 1) is at most $1/2$ of the group size, $L_1$ satisfies 2-diversity. Similarly, $L_2$ and $L_3$ satisfy 2-diversity. Thus, Table 3 satisfies 2-diversity. The intention of 2-diversity is that each individual cannot be linked to a disease with a probability of more than 0.5 without any additional information.

However, this table does not protect individual privacy sufficiently if we consider background knowledge.

In Table 3, consider the A-group with GID equal to $L_1$. From $L_1$, it seems that each of the two individuals, Alex and Bob, in this group has a 50% chance of linking to Heart Disease (Flu). The reason why the chance is interpreted as 50% is that the analysis is based on this group without any additional information.
If the QI based background knowledge is accurate, we say that we have the worse case scenario. Considering the worst-case scenario is essential in data publishing [14, 6, 18] because it gives the maximal protection [11]. To the best of our knowledge, there is no existing work considering the worst-case QI based distribution.

There is only one work [12] closely related to ours. However, [12] considers the QI based distribution background knowledge with uncertainty. Specifically, in [12], the uncertainty of the background knowledge is denoted by an input parameter $B$. Conceptually, if $B$ is equal to 0, then the adversary has the clearest understanding about background knowledge which corresponds to the worst-case background knowledge. However, if $B$ is set to 0 in the model proposed by [12], then the background knowledge is undefined. Also [12] adopts a brute force approach in the anonymization by checking the breaching probability of anonymized groups. There are two disadvantages on this approach. The first problem is that the breaching probability is hard to compute and therefore approximation is needed in their method, which sacrifices the correctness. The second problem is that the breaching probability is not monotone in that an A-group that violates privacy may be split into two groups that preserve privacy. Therefore, even though Mondrian [8] is adopted as their algorithm, it does not guarantee an optimal solution in spite of the effort in exhaustive search in each iteration in the top-down processing. Our solution will overcome both of these problems.

Building on previous works, we propose a new method to handle the worse case background knowledge. The essence of our method is the following. We observe that privacy is breached whenever an individual in an A-group has a much higher chance of linking to a sensitive value compared with another individual in the A-group according to the QI based distribution. Based on this observation, we propose a solution which generates a dataset such that all individuals in each A-group have “similar” chances of linking to any sensitive value in the group, according to the distribution. For example if we form a group with an American and a Canadian, linking to heart disease and flu, and suppose the probabilities of Americans and Canadians being linked to heart disease and to flu are similar. Since they have “similar” chances, it is not possible for the adversary to pinpoint any linkage of an individual to a sensitive value with a higher chance. At the same time, our methods can maintain high utility for the published table.

Our contributions can be summarized as follows. Firstly, to the best of our knowledge, we are the first to handle the worst-case QI based distribution. Secondly, we derive an interesting and useful theoretical property and based on this property, we propose an algorithm which generates a dataset protecting individual privacy in the presence of the worst-case QI based distribution. Finally, we have conducted experiments which shows that our proposed algorithm is efficient and incurs low information loss.

## 2 Problem Definition

Let $T$ be a table. We assume that one of the attributes is a sensitive attribute $X$ where some values of this attribute should not be linkable to any individual. These values are called sensitive values. The value of the sensitive attribute of a tuple $t$ is denoted by $t.X$. A quasi-identifier (QI) is a set of attributes of $T$, $A_1, A_2, ..., A_q$, that may serve as identifiers for some individuals. Each tuple in the table $T$ is related to one individual and no two tuples are related to the same individual. With publicly available voter registration lists (like Table 2), the QI values can often be used to identify a unique individual [17] [18].

There are two common approaches for anonymization, which generates $T^*$ from $T$. One is generalization by generalizing all QI values in each A-group to the same value. The other is bucketization, which we have illustrated in the previous section. For the ease of illustration, we focus on bucketization. The discussion for generalization is similar. With anonymization, there is a mapping which maps each tuple in $T$ to an A-group in $T^*$. For example, the first tuple $t_1$ in Table 1 is mapped to A-group $L_1$.

The aim of privacy preserving data publishing is to deter any attack from the adversary on linking an individual to a certain sensitive value. Specifically, the data publisher would try to limit the probability of such a linkage that can be established.

In the literature [20] [18] [10] [9], it is assumed that the knowledge of an adversary includes (1) the published dataset $T^*$, (2) a publicly available external table $T^e$ such as a voter registration list that maps QIs to individuals [17] [13] and (3) some background knowledge. We also follow these assumptions in our analysis. We focus on the QI based distribution as background knowledge.

The QI based distribution for the attribute set {“Nationality”} is described in Table 4. Each probability in the table is called a global probability. The sample space for each such discrete probability distribution consists of the possible assignments of the sensitive values such as $x$ to an individual with the particular nationality. For nationality $s$, the sample space is denoted by $Ω_s$.

Each possible value in attribute “Nationality” in our example is called a signature. There are three possible

| p(t) | Heart Disease | Not Heart Disease |
|------|---------------|-------------------|
| American | 0.11 | 0.89 |
| Japanese | 0.003 | 0.997 |
| French | 0.05 | 0.95 |

Table 4. A QI based distribution “Nationality” for our motivating example
signatures in our example: “Japanese”, “American” and “French”. In general, there can be other attribute sets, such as \{“Nationality”, “Zipcode”\}, with their corresponding QI-based distributions. We define the signature and the QI-based distribution for a particular attribute set \( A \) as follows.

Given a QI attribute set \( A \) with \( q \) attributes \( A_1, ..., A_q \). A signature \( s \) of \( A \) is a set of attribute-value pairs \((A_1, v_1), ..., (A_q, v_q)\) which appear in the published dataset \( T^* \), where \( A_i \) is a QI attribute and \( v_i \) is a value. A tuple \( t \) in \( T^* \) is said to match \( s \) if \( t.A_i = v_i \) for all \( i = 1, 2, ..., q \). For example, a signature \( s \) can be \{\{“Nationality”, “American”\}, \{“Zipcode”, “55501”\}\} if the attribute set \( A \) is \{\{“Nationality”, “Zipcode”\}\}. For convenience, we often drop the attribute names, and thus we have \{\{“American”, “55501”\}\} for the above signature. The first tuple in Table 3a matches \{“American”\} but the second does not.

Given an attribute set \( A \), the QI-based distribution \( G \) of \( A \) contains a set of entries \((s : x, p)\) for each possible signature \( s \) of \( A \), where \( p \) is equal to \( p(s : x) \) which denotes the probability that a tuple matching signature \( s \) is linked to \( x \). For example, \( G \) may contain \{\{“Japanese”: “Heart Disease”, 0.003\} and \{“American”: “Heart Disease”, 0.1\\}. This involves two sample spaces \( \Omega_{\text{Japanese}} \) and \( \Omega_{\text{American}} \).

**Definition 1 (r-robustness)** Given the QI-based distribution, a dataset \( T^* \) is said to satisfy r-robustness (or \( T^* \) is r-robust) if, for any individual \( t \) and any sensitive value \( x \), the probability that \( t \) is linked to \( x \), \( p(t : x) \), does not exceed \( 1/r \).

We will discuss about the sample space for \( p(t : x) \) and derive a formula for \( p(t : x) \) in Section 3. In this paper, we are studying the following problem: given a dataset \( T \), generate an anonymized dataset \( T^* \) from \( T \) which satisfies \( r \)-robustness and at the same time minimizing the information loss. There have been different definitions for information loss in the literature. In our experiments, we shall adopt the measurement of accuracy in query results from \( T^* \) versus that from \( T \).

# 3 Probability Formulation

For the sake of illustration, in this section, we consider a certain attribute set \( A \) and a sensitive value \( x \). We will consider any attribute set and any sensitive value in Section 4.

Suppose there are \( m \) possible signatures for attribute set \( A \), namely \( s_1, s_2, ..., s_m \). Let \( G \) be the background knowledge consisting of the set of all QI based distributions. In \( G \), the probability that \( s_i \) is linked to a sensitive value \( x \) is given by \( p(s_i : x) \).

Given \( G \), the formula for \( p(t : x) \), the probability that a tuple \( t \) is linked to sensitive value \( x \), is derived below.

In the following, we consider the anonymized dataset \( T^* \). Suppose \( t \) belongs to A-group \( L_k \) in \( T^* \). For the ease of reference, let us summarize the notations that we use in Table 6. We shall need the following definitions.

**Definition 2 (Possible World)** Consider an A-group \( L_k \) with \( N \) tuples, namely \( t_1, t_2, ..., t_N \), with corresponding values in sensitive attribute \( X \) of \( \gamma_1, \gamma_2, ..., \gamma_N \). A possible world \( w \) for \( L_k \) is a possible assignment mapping the tuples in set \( \{t_1, t_2, ..., t_N\} \) to values in multi-set \( \{\gamma_1, \gamma_2, ..., \gamma_N\} \) in \( L_k \).

Given an A-group \( L_k \) with a set of tuples and a multi-set of the values in \( X \). Considering all possible worlds, we form a sample space. More precisely, the sample space \( \Omega_{w|L_k} \) consists of all the possible assignments of the sensitive values in \( L_k \) to the \( N \) tuples in \( L_k \). For each such possible world \( w \), according to the QI based distribution \( G \) based on attribute set \( A \), we can determine the probability \( p(w|L_k) \) that \( w \) occurs given \( L_k \).

**Definition 3 (Primitive Events, Projected Events)** A mapping \( t : x \) from an individual or tuple \( t \) to a value \( x \) in the set of sensitive attributes is called a primitive event. Suppose \( t \) matches signature \( s \). Let us call an event for the corresponding signature, \( s : x \), a projected event for \( t \). Note that this projected event belongs to sample space \( \Omega_s \).

A primitive event is an event in the sample space \( \Omega_{w|L_k} \). The probability of such an event, \( p(t : x) \), is the probability of interest for the adversary. The probability of the projected event, \( p(s : x) \), is in the QI based distribution \( G \).

Similar to [13][20][18], we assume that the linkage of a value in \( X \) to an individual is independent of the linkage of a value in \( X \) to another individual. For example, whether an American suffers from Heart Disease is independent of whether a Japanese suffers from Heart Disease. Thus, for a
possible world \(w\) for \(L_k\), the probability that \(w\) occurs given \(L_k\) is proportional to the product of the probabilities of the corresponding projected events for the tuples \(t_1, \ldots, t_N\) in \(L_k\), we shall denote this product as \(p(w)\):

\[
p(w) = p_1, w \times p_2, w \times \ldots \times p_N, w
\]

where \(p_{j,w}\) is the probability that \(t_j\) is linked to a value in the sensitive attribute specified in \(w\). Suppose \(t_j\) matches signature \(s_t\). If \(t_j\) is linked to \(x\) in \(w\), then \(p_{j,w} = p(s_t : x)\).

Let the set of all the possible worlds for \(L_k\) be \(\mathcal{W}_k\). The sum of probabilities of all the possible worlds given \(L_k\) must be 1, since they form the sample space \(\Omega_{w|L_k}\). Hence, the probability of \(w\) given \(L_k\) is given by:

For \(w \in \mathcal{W}_k\), we have

\[
p(w|L_k) \propto \frac{p(w)}{\sum_{w' \in \mathcal{W}_k} p(w')}
\]

Our objective is to find the probability that an individual \(t_j\) in \(L_k\) is linked to a sensitive value \(x\). This is given by the sum of the probabilities \(p(w|L_k)\) of all the possible worlds \(w\) where \(t_j\) is linked to \(x\).

\[
p(t_j : x) = \sum_{w \in \mathcal{W}_k^{t_j : x}} p(w|L_k)
\]

where \(\mathcal{W}_k^{t_j : x}\) is a set of all possible worlds \(w\) in \(\mathcal{W}_k\) in which \(t_j\) is assigned value \(x\).

**Example 4** Consider an A-group \(L_k\) in a published table \(T^*\). Suppose there are four tuples, \(t_1, t_2, t_3\) and \(t_4\), with the \(X\) values of \(x, x, y, y\) in \(L_k\). Suppose the published table \(T^*\) satisfies 2-diversity.

Consider the QI based distribution \(G\) based on a certain QI attribute set \(A\) which contains two possible signatures \(s_1\) and \(s_2\). Table 5(a) shows the four global probabilities, namely \(p(s_1 : x) = 0.5, p(s_1 : y) = 0.5, p(s_2 : x) = 0.2, p(s_2 : y) = 0.8\).

Suppose \(t_1, t_2, t_3\) and \(t_4\) match signatures \(s_1, s_3, s_2\) and \(s_2\), respectively. There are six possible worlds \(w\) as shown in Table 5(b). For example, the first row is the possible world \(w_1\) with mapping \(\{t_1 : x, t_2 : x, t_3 : y, t_4 : y\}\). The table also shows the values \(p(w)\) of the possible worlds.

Take the first possible world \(w_1\) for illustration. From the QI based distribution in Table 5(a), \(p(s_1 : x) = 0.5\) and \(p(s_2 : y) = 0.8\). Hence, \(p(w_1) = 0.5 \times 0.5 \times 0.8 \times 0.8 = 0.16\). The sum of \(p(w)\) of all possible worlds from Table 5(b) is equal to 0.16 + 0.04 + 0.04 + 0.04 + 0.04 + 0.01 = 0.33. Consider \(w_1\) again. Since \(p(w_1) = 0.16, p(w_1|L_k) = 0.16/0.33 = 0.48\).

Suppose the adversary is interested in the probability that \(t_1\) is linked to \(x\). We obtain \(p(t_1 : x)\) as follows. \(w_1, w_2\) and \(w_3\), as shown in Table 5(b), contain “\(t_1 : x\)”. Thus, \(p(t_1 : x)\) is equal to the sum of the probabilities \(p(w_1|L_k), p(w_2|L_k)\) and \(p(w_3|L_k)\). \(p(t_1 : x) = 0.48 + 0.12 + 0.12 = 0.72\). Note that this is greater than 0.5, the intended upper bound for 2-diversity that an individual is linked to a sensitive value.

### 4 Algorithm for Data Publishing

Given the formulation of \(p(t : x)\), a naive approach for \(r\)-robustness is to adopt some known anonymization algorithm \(A\) and replace the probability measure in \(A\) by \(p(t : x)\). However, the complexity of computing \(p(t : x)\) is very high given the exponential number of possible worlds. Moreover, \(r\)-robustness is not monotone in the sense that an \(A\)-group that violates \(r\)-robustness may be split into small groups that are \(r\)-robust, while known top-down algorithms are based on monotone privacy conditions.

This section presents an algorithm for generating an \(r\)-robust table that overcome the above problems. Section 4.1 first presents an important theoretical property for this problem. Section 4.2 then describes our proposed algorithm, ART.

**4.1 Theoretical Property**

In Section 1 we observe that privacy is breached easily whenever an individual in an \(A\)-group has a much higher chance of linking to a sensitive value compared with another individual in the \(A\)-group. For example, consider the \(A\)-group \(L_1\) in Table 3. From the QI-based distribution (Table 4), it is more likely that American is linked to Heart Disease compared with Japanese, we can deduce that Alex, an American, has Heart Disease with higher probability. Note that the global probability of American linking to Heart Disease, denoted by \(f_1\), is 0.1 and the global
probability of Japanese linking to Heart Disease, denoted by \( f_2 \), is 0.003. The difference in the global probabilities is \( 0.1 - 0.003 = 0.097 \). Since the A-group size is small, the difference gives some information to aid privacy breach. The difference in the global probabilities and the A-group size are the properties of the A-group.

In the following, we have a theorem on the relationship between the privacy guarantee and the properties of an A-group \( L \). Consider a tuple \( t_v \) in an A-group \( L \). We want to show that, if the properties of \( L \) satisfy some conditions, the privacy of \( t_v \) can be guaranteed (i.e., \( p(t_v : x) \leq 1/r \)). The conditions essentially limits the deviations in the global probabilities in terms of the group size.

In the following we consider the QI based distribution \( G \) on a certain attribute set \( A \). The algorithm to be described later will consider multiple attribute sets.

**Definition 4 (Greatest Probability Deviation \( \triangle \))** Let \( L \) be an A-group in \( T^* \) with tuples \( t_1, t_2, \ldots, t_N \) where \( N \) is the group size and \( N \geq r \). Let \( x \) be a sensitive value that appears once in \( L \). Without loss of generality, suppose tuple \( t_v \) matches signature \( s_v, v \in [1, N] \). Thus, tuple \( t_v \) has the QI based probability (or global probability) linking to \( x \) in \( L \) equal to \( p(s_v : x) = f_v \).

Let \( f_{\text{max}} \) be the greatest global probabilities in \( L \) (i.e., \( f_{\text{max}} = \max_{v \in [1, N]} f_v \)). The probability deviation of \( t_v \) given \( f_{\text{max}} \) is \( \triangle_v = f_{\text{max}} - f_v, \ v \in [1, N] \).

Let us give some examples to illustrate the above notations. In our running example of \( L_1 \), the group size \( N \) is equal to 2. In \( L_1 \), the first tuple (Alex) is \( t_1 \) and the second tuple (Bob) is \( t_2 \). Let \( s_1 = \{ \text{"American"} \} \) and \( s_2 = \{ \text{"Japanese"} \} \). Thus, \( f_1 = 0.1 \) and \( f_2 = 0.003 \). We know that \( t_1 \) matches \( s_1 \) and \( t_2 \) matches \( s_2 \). Since \( f_{\text{max}} \) is the greatest global probabilities in \( L \), \( f_{\text{max}} \) is equal to 0.1 (because \( f_1 = 0.1 \) and \( f_2 = 0.003 \)). Thus, \( \triangle_1 = f_{\text{max}} - f_1 = 0.1 - 0.1 = 0 \) and \( \triangle_2 = f_{\text{max}} - f_2 = 0.1 - 0.003 = 0.097 \).

**Theorem 1** Let \( r \) be the privacy parameter in \( r \)-robustness where \( r > 1 \). Following the symbols in Definition 4 if for all \( v \in [1, N] \),

\[
\triangle_v \leq \frac{(N - r)f_{\text{max}}}{f_{\text{max}}(r - 1)/(1 - f_{\text{max}}) + (N - 1)} \tag{4}
\]

then for all \( v \in [1, N] \), \( p(t_v : x) \leq 1/r \)

**Proof:** The proof is given in the appendix.

**Definition 5 (\( \triangle, \triangle_{\text{max}} \))** \( \triangle_{\text{max}} \) is defined to be the R.H.S. of Inequality (4). That is,

\[
\triangle_{\text{max}} = \frac{(N - r)f_{\text{max}}}{f_{\text{max}}(r - 1)/(1 - f_{\text{max}}) + (N - 1)}
\]

Define \( \triangle = \max_{v \in [1, N]} \{ \triangle_v \} \)

| \( N \) | \( r \) | \( f_{\text{max}} \) | \( \triangle_{\text{max}} \) |
|----|----|----|----|
| 3  | 2  | 0.1 | 0.097 |
| 3  | 2  | 0.3 | 0.1235 |
| 3  | 2  | 0.5 | 0.1667 |
| 3  | 2  | 0.9 | 0.0818 |
| 4  | 2  | 0.3 | 0.1759 |
| 6  | 2  | 0.3 | 0.2211 |
| 6  | 4  | 0.3 | 0.1537 |
| 6  | 4  | 0.3 | 0.0955 |

**Table 7. Values of \( \triangle_{\text{max}} \) with some chosen values of \( N, r \) and \( f_{\text{max}} \)**

Hence, \( \triangle \) is the greatest difference in the global probabilities linking to \( x \) in an A-group. Note that \( \triangle \geq 0 \). In our running example, since \( \triangle_1 = 0 \) and \( \triangle_2 = 0.097 \), we have \( \triangle = \max\{0, 0.097\} = 0.097 \).

Consider another example. If an A-group \( L \) contains three tuples matching \( s_1, s_2 \) and \( s_3 \) with the global probabilities \( f_1 = 0.1, f_2 = 0.08 \) and \( f_3 = 0.09 \). Then, \( N = 3 \) and \( f_{\text{max}} = 0.1 \). \( \triangle = 0.1 - 0.08 = 0.02 \). Suppose \( r = 2 \). The R.H.S. of (4) is \( \triangle_{\text{max}} = (3 - 2)/0.1 \times (2 - 1)/(1 - 0.1) = 0.0474 \). Since \( \triangle < 0.0474 \), from Theorem 1 for all tuples \( t_v \) in \( L \), \( p(t_v : x) \leq 1/r \) where \( r = 2 \).

Let us consider the effects of the values of \( f_{\text{max}} \) and \( N \) to understand the physical meaning of Theorem 1. If \( f_{\text{max}} = 1 \) or \( f_{\text{max}} = 0 \), then \( \triangle \leq 0 \). Hence, the QI based distributions of all tuples in \( L \) should be the same to guarantee privacy.

Table 7 shows the values of \( \triangle_{\text{max}} \) with some chosen values of \( N, r \) and \( f_{\text{max}} \). It can be seen that \( \triangle_{\text{max}} \) is small when \( f \) is near the extreme values of 0 or 1, since the global probability of a tuple is more pronounced.

Consider Inequality (4). If \( N \rightarrow \infty \), then \( \triangle \leq f_{\text{max}} \). Since \( f_{\text{max}} \) is the greatest possible global probability in \( L \), it means that \( \triangle \) can be any feasible value (i.e., \( 0 \leq \triangle \leq f_{\text{max}} \)). Therefore, when the A-group is extremely large, under Theorem 1 there will be no privacy breach. When \( N = r, \triangle \leq 0 \). That is, the global probabilities of all tuples in \( L \) should be equal. Otherwise, there may be a privacy breach. Furthermore, \( N \) has the following relation with \( \triangle_{\text{max}} \).

**Lemma 1** \( \triangle_{\text{max}} \) is a monotonic increasing function on \( N \).

**Proof:** Let \( f = f_{\text{max}}, \frac{d\triangle_{\text{max}}}{dN} = \frac{(r - 1)\times x(r - 1) + (r - 1)\times f}{(r - 1)\times x(r - 1) + (N - 1)^2} \geq 0 \)

From the above, in order to guarantee \( p(t_v : x) \leq 1/r \), we can increase the size \( N \) of the A-group \( L \). With a greater value of \( N \), the upper bound \( \triangle_{\text{max}} \) increases, and the constraint as dictated by Inequality (4) is relaxed, making it easier to reach the guarantee.
4.2 Algorithm ART

Based on Theorem 1, we propose an Algorithm generating \( r \)-Robust Table called ART. If an A-group \( L \) satisfies the inequality in Theorem 1 with respect to attribute set \( A \) and, in \( L \), each sensitive value occurs at most once, we say that \( L \) satisfies the QI based distribution bound condition with respect to \( A \). Otherwise, \( L \) violates the QI based distribution bound condition.

In the algorithm, initially, each individual forms an independent A-group. The algorithm repeatedly looks for any A-group such that there exists an attribute set \( A \) where it violates the QI based distribution bound condition with respect to \( A \). Such a group is merged with other existing groups so that the resulting group satisfies the condition. After merging, the number of tuples in \( L \), \( N \), is increased. Then, by Lemma 1, \( \triangle_{\text{max}} \) is also increased. The constraint by Inequality 1 is relaxed and it is more likely to satisfy the QI based distribution bound condition. When a final solution is reached, each individual is linked to any sensitive value with probability at most \( 1/r \).

Specifically, algorithm ART involves two major steps.

- **Step 1** (Individual A-group Formation): For each tuple \( t \) in the table \( T \), we form an A-group \( L \) containing \( t \) only.

- **Step 2** (Merging): For each sensitive value \( x \), while there exists an A-group \( L \) and an attribute set \( A \) such that \( L \) violates the QI based distribution bound condition with respect to \( A \), we find a set \( L \) of A-groups such that, after merging all A-groups in \( L \) with \( L \), the merged A-group satisfies the QI based distribution bound condition with respect to any attribute set \( A \).

The idea of Step 2 is to keep the \( \triangle \) value in \( L \) with respect to \( A \) unchanged or only slightly increased after merging. At the same time, we also make sure that each merged A-group contains at most one \( x \) for any sensitive value \( x \). Before going into the details of Step 2, we need to define a new term. Given an A-group \( L \), another A-group \( L' \) is called a closest A-group with respect to \( L \) if, after merging \( L' \) and \( L \), the increase in the value of \( \triangle \) with respect to any attribute set \( A \) is the smallest among all possible A-groups.

**Definition 6 (Closest A-group)** Suppose \( \triangle_{\text{before},A} \) represents \( \triangle \) with respect to an attribute set \( A \) in \( L \) and \( \triangle_{\text{after},A}(L, L') \) represents \( \triangle \) with respect to an attribute set \( A \) in the A-group obtained by merging \( L \) and \( L' \).

Let \( D_A(L, L') = \triangle_{\text{after},A}(L, L') - \triangle_{\text{before},A} \).

Let \( D(L, L') = \sum_A D_A(L, L') \).

\( L' \) is a closest A-group with respect to \( L \) if \( D(L, L') = \min_{L''} \{ D(L, L'') \} \).

We are ready to describe Step 2 in details. Let \( Y(L) \) be the set of sensitive values which appear in an A-group \( L \). Given an A-group, it is easy to derive \( \triangle \) and \( f_{\text{max}} \). Note that \( r \) is a user parameter. After we know \( \triangle, f_{\text{max}} \) and \( r \), we can derive the expected minimum size of \( L \) based on the QI based distribution bound condition with respect to \( A \), denoted by \( N_o \). By replacing \( N \) with \( N_o \) and changing the subject of Inequality 1 in the QI based distribution condition to \( N_o \), we have

\[
N_o \geq \frac{(f_{\text{max}}(r-1)\triangle)/(1-f_{\text{max}}) - \triangle + rf_{\text{max}}}{(f_{\text{max}} - \Delta)}
\]

Let us choose a smallest integer \( N' \) such that the above inequality holds. We calculate \( N' \) for every attribute set \( A \) and choose the greatest values of \( N' \) as our final \( N' \). If the total number of tuples in \( L, N \), is smaller than \( N' \), then we have to choose additional \( N' - N \) tuples to be merged with \( L \). We choose a closest A-group \( L' \) with respect to \( L \) where \( L' \) does not contain any sensitive value in \( Y(L) \). \( L' \) is merged with \( L \), and \( \triangle, f \) and \( N' \) are updated accordingly. If the updated \( N \) value is still smaller than \( N' \), then we repeatedly continue the above process.

**Theorem 2** Any table \( T^* \) generated by Algorithm ART is \( r \)-robust.

5 Empirical Study

A Pentium IV 2.2GHz PC with 1GB RAM was used to conduct our experiment. The algorithm was implemented in C/C++. We adopted the publicly available dataset, Adult Database, from the UCIrvine Machine Learning Repository [14]. This dataset (5.5MB) was also adopted by [13, 18]. We used a configuration similar to [13] [18]. The records with unknown values were first eliminated resulting in a dataset with 45,222 tuples (5.4MB). Nine attributes were chosen as the quasi-identifier and the sensitive attribute, respectively. Similar to [18], in attribute “Education”, all values representing the education levels before “secondary” (or “9th-10th”) such as “1st-4th”, “5th-6th” and “7th-8th” are regarded as a sensitive value set where an adversary checks whether each individual is linked to this set more than \( 1/r \), where \( r \) is a parameter.

There are 3.46% tuples with education levels before “secondary”. Since there is a set \( G \) of multiple QI based distributions \( G \), we can calculate \( p(t : x) \) for different \( G \)'s and different \( x \)'s. We take the greatest such value to report as the probability that individual \( t \) is linked to some sensitive value since this corresponds to the worst case privacy breach.
We compared our proposed algorithm ART with four algorithms, Anatomy [20], MASK [18], Injector [10] and t-closeness [9]. They are selected because they consider l-diversity or similar privacy requirements, so we need only set l = r. We are interested to know the overhead required in our approach in order to achieve r-robustness. When we compared ART with Anatomy, we set l = r. When we compared it with MASK, the parameters k and m used in MASK are set to r. For Injector, the parameters minConf, minExp and l are set to 1, 0.9 and r, respectively, which are the default settings in [10]. For t-closeness, similar to [9], we set t = 0.2. We evaluate the algorithms in terms of four measurements: (1) execution time, (2) relative error ratio, (3) the proportion of problematic tuples among all sensitive tuples and (4) the average value of ∆.

(1) Execution time: We measured the execution time of algorithms. (2) Relative error ratio: As in [20, 18, 10], we measure the error by the relative error ratio in answering an aggregate query. We adopt both the form of the aggregate query and the parameters of the query dimensionality qd and the expected query selectivity s from [20, 18, 10]. For each evaluation in the case of two anonymized tables, we performed 10,000 queries and then reported the average relative error ratio. By default, we set s = 0.05 and qd to be the QI size. (3) Proportion of problematic tuples among all sensitive tuples: According to the probability formulation in Section 3, according to the anonymized table generated by all algorithms, we can calculate the probability that a tuple is linked to a sensitive value set. If the tuple has the probability > 1/r, it is said to be a problematic tuple. The tuples linking to sensitive values in the original table are called sensitive tuples. In our experiments, we measure the proportion of problematic tuples among all sensitive tuples. (4) Average value of ∆: More formally, the average value of ∆ is evaluated with respect to every attribute set A containing large samples. Consider a sensitive value x. With respect to a certain attribute set A, the average value of ∆ denoted by \( H_A \) is equal to \( \frac{1}{u} \sum_{L \in T^*} \Delta_L \), where u is the total number of A-groups in \( T^* \) and \( \Delta_L \) is the greatest difference in the global probability linking to a sensitive value x with respect to A in an A-group L. Let B be the set of all attribute sets A containing large samples. With respect to every attribute set in B, the average value of ∆ is equal to \( \frac{1}{u'} \sum_{A \in B} H_A \). We perform the same steps for every sensitive value x and take the average as the reporting average value of ∆. For each measurement, we conducted the experiments 100 times and took the average.

We conducted the experiments by varying four factors: (1) the QI size, (2) r, (3) query dimensionality qd and (4) selectivity s.

Figure 1 shows the results when r is set to 10. Figure 1(a) shows that the execution time increases with the QI size because the algorithms have to process more QI attributes. ART performs slower compared with Anatomy, MASK and t-closeness. Since ART requires to compute the QI based distribution with respect to every attribute set, when the QI size increases, the increase in the execution time of ART is larger.

Figure 1(b) shows that there is an increase in average relative error when the QI size increases because it is more difficult to form A-groups where the difference in QI based distributions among all tuples in an A-group is small when the QI size is larger. Since t-closeness is a global recoding and causes a lot of unnecessary generalizations, the average relative error is the largest. Since Injector tries to exclude some sensitive values in an A-group, its relative error is also small.

Figure 1(c) shows that the proportion of problematic tuples among sensitive tuples increases with QI size. With a larger QI size, there is a higher chance that individual privacy breaches due to more attributes which can be used to construct the QI based distributions. MASK has fewer privacy breaches compared with t-closeness, Anatomy and Injector because the side-effect of the minimization of QI values in each A-group adopted in MASK makes the difference in the QI based distribution among all tuples in each A-group smaller. Thus, the number of individual with privacy breaches is smaller. It is noted that there is no violation in ART.

In Figure 1(d), we include the theoretical bound of \( \Delta_{max} \) from Theorem 1 for comparison. We use the bound of ART as this theoretical bound because, compared with Anatomy and Injector, the size of A-groups formed in ART is largest (which yields the largest bound). Since the average value of \( \Delta \) of Anatomy and Injector are greater than this bound, they may have privacy breaches as shown in Figure 1(c). When the QI size increases, the average value of \( \Delta \) with
respect to every attribute set increases, as shown in Figure 1(d). With a larger QI size, during forming an A-group, we have to consider $\triangle$ with respect to more attribute sets. Thus, it is more likely that an A-group has a larger average value of $\triangle$ with respect to every attribute set. The average value of $\triangle$ is the largest in Anatomy and Injector, and the next two largest in MASK and $t$-closeness. This is because Anatomy and Injector does not take our QI based distribution directly into the consideration for merging but MASK and $t$-closeness do indirectly during the minimization of QI values. In Figure 1(d), although the average value of $\triangle$ of MASK is smaller than the theoretical bound of $\triangle$, it is possible to breach privacy as shown in Figure 1(c) because this evaluation only shows the average value and the actual $\triangle$ in some A-groups is larger than this bound.

We also conducted experiments when $r = 2$. For the sake of space, we did not show the figures. The results are also similar. But, the execution time and the average relative error are smaller. Since $r$ is smaller and thus $1/r$ is larger, the average value of $\triangle$ is larger when $r = 2$.

6 Related Work

With respect to attribute types considered for data anonymization, there are two branches of studying. The first branch is anonymization according to the QI attributes. A typical model is $k$-anonymity [2]. The other branch is the consideration of both QI attributes and sensitive attributes. Some examples are [13], [19], [9], [10] and [5]. In this paper, we focus on this branch. We want to check whether the probability that each individual is linked to any sensitive value is at most a given threshold.

$l$-diversity [13] proposes a model where $l$ is a positive integer and each A-group contains $l$ “well-represented” values in the sensitive attribute. For $t$-closeness [9], the distribution in each A-group in $T^*$ with respect to the sensitive attribute is roughly equal to the distribution of the entire table $T^*$.

In the literature, different kinds of background knowledge are considered [13], [14], [18], [12], [7], [10] and [1]. [14] considers another background knowledge in form of implications. [18] discovers that the minimality principle of the anonymization algorithm can also be used as background knowledge. [12] proposes to use the kernel estimation method to mine the background knowledge from the original table.

[10] finds that association rules can be mined from the original table and thus can be used for privacy protection during anonymization. In [1], the problem of privacy attack by adversarial association rule mining is investigated. However, as pointed out in [15], association rules used in [10] and [1] can contradict the true statistical properties. Also the solution in [1] is to invalidate the rules, but this will violate the data mining objectives of data publication.

7 Conclusion

In this paper, we consider the worst-case QI based distribution for privacy-preserving data publishing. Then, we derive a theoretical property and propose an algorithm which generates a dataset protecting individual privacy in the presence of the worst-case QI based distribution. Finally, we conducted experiments to show that our proposed algorithm is efficient and incurs low information loss. For future work, we plan to investigate how to anonymize the dataset with other kinds of background knowledge that may be possessed by the adversary.

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8 Appendix

Here we prove our main theorem. Let us recap a few notations.

| \( p(s_i : x) \) | probability that signature \( s_i \) is linked to \( x \) 
| \( f_i \) | a simplified notation for \( p(s_i : x) \) 
| \( f_{\text{max}} \) | maximum \( f_i \) value among all \( i \)'s 

Proof of Theorem 1: Let \( t_u \) be a tuple in \( L \) with the greatest global probability linking to \( x \) in \( L \) (i.e., for all tuples \( t_v \) in \( L \), \( f_v \geq f_u \)). Besides, \( f = f_u \).

Consider the set \( W_a \) of possible worlds where \( "t_u : x" \) occurs. Let \( t_v \) be a tuple such that \( \Delta_v = \max_{a \in \{1, N\}} \{\Delta_a\} \). Consider the set of possible worlds \( W_a \) where \( "t_u : x" \) occurs.

Consider also the set of possible worlds \( W_a \) where \( "t_u : x" \) occurs for an arbitrary \( t_w \) where \( t_w \neq t_v \). We first want to show that \( p(W_a) \geq p(W_v) \), where \( p(W_a) \) is the probability that any world in \( W_v \) occurs.

**Lemma 2** For \( a \in \{1, N\} \), \( p(W_a) \geq p(W_v) \).

Proof of Lemma 2: Since \( \Delta_v = \max_{a \in \{1, N\}} \{\Delta_a\} \), \( f_v \leq f_a \) and \( (1 - f_v) \geq (1 - f_a) \). Hence,

\[
f_a(1 - f_v) \geq f_v(1 - f_a)
\]

(6)

For a world \( w_v \in W_v \), \( p(w_v) = p_1(w_v) \times \ldots \times p_N(w_v) \).

For a world \( w_a \in W_a \), \( p(w_a) = p_1(w_a) \times \ldots \times p_N(w_a) \).

Note that \( p_{t_u,w_v} = f_u \) and \( p_{t_u,w_a} = f_a \).

Since there is only one \( x \) occurrence in \( L \), \( t_u \) is not assigned with \( x \) in any \( w_v \in W_v \). Let \( W'_a \) be a maximal subset of \( W_a \) where \( t_u \)'s are assigned to distinct \( X \) values. Obviously \( \sum_{w_a \in W'_a} p_{v,w_a} = 1 - f_a \).

\[
\sum_{w_a \in W'_a} (p_{a,u} \times p_{v,w_a}) = f_u(1 - f_v)
\]

(7)

Similarly, since \( t_u \) is not assigned with \( x \) in any \( w_v \in W_v \), we can find a maximal subset \( W'_v \) in \( W_v \) where \( t_v \)'s are assigned to distinct \( X \) values. We have \( \sum_{w_v \in W'_v} p_{v,w_v} = 1 - f_v \).

\[
\sum_{w_v \in W'_v} p_{v,w_a} = f_v(1 - f_a)
\]

(8)

From (6), (7), and (8).

\[
\sum_{w_a \in W'_a} p_{a,w_a} \times p_{v,w_a} \geq \sum_{w_v \in W'_v} p_{v,w_a} \times p_{a,w_v}
\]

(9)

For each \( w_a \in W'_a \) we can find a unique \( w_v \in W'_v \) so that \( f_v \) in \( w_a \) and \( f_a \) in \( W_v \) are assigned the same sensitive value. We say that \( w_a \) and \( w_v \) are matching. Let us further restrict \( W'_a \) based on \( W'_v \) in such a way that the matching world \( w_v \in W'_v \) for \( w_a \in W'_a \) has the same sensitive value assignments for the remaining tuples. It is obvious that we can always form such an \( W'_v \) from and any \( W_v \). For matching \( w_a \) and \( w_v \),

\[
\prod_{(i, e) \in E(a, e)} p_{i,w_a} = \prod_{(i, e) \in E(a, v)} p_{i,w_v}
\]

(10)

Furthermore, \( W_a \) can be partitioned into \( W'_a \)'s. and the union of the corresponding \( W'_a \) is equal to \( W_v \).

From (9) and (10), we conclude that

\[
\sum_{w_a \in W_a} p_{i,w_a} \times \ldots \times p_{N,w_a} \geq \sum_{w_v \in W_v} p_{i,w_v} \times \ldots \times p_{N,w_v}
\]

That is, \( \sum_{w_a \in W_a} p(w_a) \geq \sum_{w_v \in W_v} p(w_v) \).

Therefore, for \( a \in \{1, N\} \),

\[
p(W_a) \geq p(W_v)
\]

(11)

This completes the proof of Lemma 2.

**Lemma 3** If \( p(t_u : x) \leq 1/r \), then \( p(t_a : x) \leq 1/r \) for all \( a \in \{1, N\} \).

Proof of Lemma 3: By similar techniques used in the proof of Lemma 2 since \( f_a \geq f_a \) for all \( a \in \{1, N\} \), we derive that \( p(W_a) \geq p(W_v) \). Let \( K = \sum_{w \in W} p(w') \) where \( W \) is a set of all possible worlds. Since \( p(t_u : x) = p(W_a[L] = p(W_a)/K \)

and \( p(t_a : x) = p(W_v[L] = p(W_v)/K \), we have \( p(t_u : x) \geq p(t_a : x) \).

Thus, if \( p(t_u : x) \leq 1/r \), then, for all \( a \in \{1, N\} \), \( p(t_a : x) \leq 1/r \).

This completes the proof of Lemma 3.

**Lemma 3** suggest that, once \( p(t_u : x) \) is bounded \( 1/r \), all other probabilities \( p(t_a : x) \) in the \( A \)-group are also bounded. In the following, we focus on analyzing \( p(t_a : x) \) only (instead of all probabilities \( p(t_a : x) \)).

Consider \( p(t_u : x) \), which is equal to \( p(w_u|L) \). Let \( W \) be a set of all possible worlds. Let \( W^{(t_u=x)} \) be the set of all possible worlds with \( "t_u : x" \). By definition \( W^{(t_u=x)} = W_v \) and there are \( N \) such sets of worlds in \( W \). Also,

\[
p(W_u[L] = \frac{\sum_{w \in W^{(t_u=x)}} p(w)}{\sum_{w \in W} p(w)}
\]

\[
= \frac{\sum_{w \in W^{(t_u=x)}} p(w) + \sum_{w \in W^{(t_u=x)}} p(w')}{p(W_v) + \sum_{a \neq u} p(W_a)}
\]

By Lemma 2

\[
\sum_{a \neq u} p(W_a) \geq (N - 1)p(W_v)
\]

Hence,

\[
p(W_u[L] \leq p(W_v) + (N - 1)p(W_v)
\]

(12)

From the proof of Lemma 2 \( W_a \) and \( W_v \) can be partitioned into matching pairs of \( W'_a \) and \( W'_v \) where \( \sum_{w_a \in W'_a} p(w_a) = f_a(1 - f_a)C \) for some \( C \) and \( \sum_{w_v \in W'_v} p(w_v) = f_v(1 - f_v)C \).

Therefore, we can simplify Inequality (12) as follows.

\[
p(w_u|L) \leq \frac{f_a(1 - f_a)}{f_a(1 - f_a) + (N - 1) \times f_v(1 - f_v)}
\]

(13)
Consider the term \((N - 1) \times f_v(1 - f_u)\) in Inequality (13)

\[
(N - 1)(1 - f_u)f_v \\
= (N - 1)(1 - f)(f - \triangle_v) \\
= (r - 1)f(1 - f + \triangle_v) \times \frac{(N - 1)(1 - f)(f - \triangle_v)}{(r - 1)f(1 - f + \triangle_v)} \\
= (r - 1)f_u(1 - f_u) \times \frac{(N - 1)(1 - f)}{(r - 1)f(1 - f - \triangle_v - 1)}
\]

After substituting \(\triangle_v \leq (N - r)f/[1 - f] + (N - 1)\) into the above equation, with simple derivations, we obtain

\((N - 1) \times f_v(1 - f_u) \geq (r - 1) \times f_u(1 - f_v)\)

With the above inequality, Inequality (13) becomes

\[p(W_u|L) \leq 1/r\] (14)

This completes the proof of Theorem 1.