Reentrant phase transition of Born-Infeld-AdS black holes

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We investigate thermodynamic phase structure and critical behaviour of Born-Infeld (BI) black holes in an anti-de Sitter (AdS) space, where the charge of the system can vary and the cosmological constant (pressure) is fixed. We find that the BI parameter crucially affects the temperature of the black hole when the horizon radius, \( r_+ \), is small. We observe that depending on the value of the nonlinear parameter, \( \beta \), BI-AdS black hole may be identified as RN black hole for \( Q \geq Q_m \), and Schwarzschild-like black hole for \( Q < Q_m \), where \( Q_m = 1/(8\pi\beta) \) is the marginal charge. We analytically calculate the critical point \((Q_c, T_c, r_{+c})\) by solving the cubic equation and study the critical behaviour of the system. We also explore the behavior of Gibbs free energy for BI-AdS black hole. We find out that the phase behaviour of BI-AdS black hole depends on the charge \( Q \). For \( Q > Q_c \), the Gibbs free energy is single valued and the system is locally stable \((C_Q > 0)\), while for \( Q < Q_c \), it becomes multivalued and \( C_Q < 0 \). In the range of \( Q_c < Q < Q_0 \), a first order phase transition occurs between small black hole (SBH) and large black hole (LBH). Interestingly enough, in the range of \( Q_t \leq Q \leq Q_z \), a reentrant phase transition occurs between intermediate (large) black hole, SBH and LBH in Schwarzschild-type region. This means that in addition to the first order phase transition which separates SBH and LBH, a finite jump in Gibbs free energy leads to a zeroth order phase transition between SBH and intermediate black hole (LBH) where initiates from \( Q = Q_z \) and terminates at \( Q = Q_t \).

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I. INTRODUCTION

Phase transition is a most remarkable property in black hole thermodynamics which can relate a gravitational system to an accessible experimental system. The pioneering work on black holes phase transition was done by Hawking and Page who proved that a first-order phase transition occurs between thermal radiation and large black hole in the Schwarzschild anti-de-Sitter (AdS) spacetime [1]. After that, Witten showed that this phase transition can be related to a confinement-deconfinement phase transition in the quark gluon plasma [2]. Later, it has been claimed that similar to the Van der Waals phase transition, a charged AdS black hole undergoes a first-order phase transition between large black hole (LBH) and small black hole (SBH) [3]. While similarity between liquid-gas Van der Waals phase transition and small-large black hole phase transition in the background of Reissner-Nordstrom-AdS (RN-AdS) spacetime was provided by extending phase space of black hole thermodynamics [4, 5]. In an extended phase space, the variable cosmological constant is treated as the pressure, while its conjugate quantity is the volume of the system. In this approach the mass of black hole is treated as the enthalpy [6]. Critical behaviour of black holes in an extended phase space have been explored in various setups [7–15]. Recently, this approach has been utilized to present the emergence of superfluid-like phase transition for a class of AdS hairy black hole in Lovelock gravity [16]. Very recently, we disclosed a novel phase transition in dilaton black holes, in an extended phase space where the charge of the black hole is regarded as a fixed quantity [17].

The reentrant phase transition (RPT) is composed of two (or more) phase transition, which occurs by a monotonic variation of any thermodynamic variable so that the initial and final phase of system are macroscopically the same. This has previously been observed in a nicotine/water mixture [18], granular superconductors, liquid crystals, binary gases, ferroelectrics and gels (see [19] and the references therein). Recently, RPT has been discovered in the four-dimensional BI-AdS black hole and higher-dimensional singly spinning Kerr-AdS [20, 21]. These black holes, in certain range of pressure, undergo a large-small-large black hole phase transition where the latter “large” state refers to the intermediate black hole (IBH). This RPT is accompanied by a finite jump in the Gibbs free energy which referred to a zeroth-order phase transition. The zeroth-order phase transition has been seen in superfluidity and superconductivity [22] and recently in the charged dilaton black hole [17]. The studies on RPT in an extended phase space for different black holes have been carried out in [23–29]. Microscopic origin of black hole reentrant phase transitions has been explored in [30]. It was argued that the type of interaction between small black holes near large/small transition line, differs for usual and reentrant phase transitions. Indeed, for usual case, the dominant interaction is repulsive whereas for reentrant case one may encounter with an attractive interaction [30]. It was also shown that in reentrant phase transition case, the small black holes behave like a Bosonic gas whereas in the usual phase transition.

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case, they behave like a quantum anyon gas \cite{30}. It is worth noting that in all references mentioned above, the charge of black hole, \( Q \), is regarded as a fixed parameter and the cosmological constant (pressure) is treated as thermodynamic quantity which can vary.

In a quite different viewpoint, it was argued that one can think of variation of charge, \( Q \), of a black hole and keep the cosmological constant as a fixed parameter \cite{31}. According to this perspective, the critical behavior occurs in \( Q^2 - \Psi \) plane, where \( \Psi = 1/2r_+ \) is the conjugate of \( Q^2 \) \cite{31}. In this approach, the thermodynamic process is carried out in phase space where the black hole quantities are varied in a fixed AdS background geometry. Treating the charge of black hole as a natural external variable affects the thermodynamic behaviors of gravitational system e.g., it can lead to interesting critical phenomena \cite{31}. This alternative viewpoint for description of phase transition and critical behavior, relevant response function clearly signifies stable and unstable regime. In addition, it leads to small-large black hole phase transition for charged AdS black holes, and completes the analogy with van der Waals fluid system \cite{31}.

In this paper, following \cite{31}, we investigate thermodynamic phase structure and critical behaviour of BI-AdS black holes in a phase space, where the cosmological constant (pressure) is taken to be constant. It was argued that the \((3+1)\)-dimensional BI-AdS black hole implies a peculiar phase transition in an extended phase space such as a reentrant phase transition \cite{20}. As we shall see the BI parameter crucially affects the temperature of the black hole when the horizon radius is small. We shall analytically calculate the critical point by solving the cubic equation which is missed in the previous studies (see e.g., \cite{32}). Also, we observe the interesting reentrant phase transition that occurs for certain range of charge of black hole. Our work differs from previous studies on phase transition in BI-AdS black hole \cite{32-39}, in which they do not study the reentrant phase transition of BI-AdS black holes, while in the present work we focus on the reentrant phase transition of such system. The investigation on phase transition of BI-AdS black holes has also been performed at the fixed electric potential (the grand canonical ensemble) in Ref. \cite{40}.

The framework of our paper is as follows. In section II, we review the basic thermodynamic of BI-AdS black hole in four dimensional spacetime. In section III, we study the critical behavior of BI-AdS black hole in \( T - r_+ \) plane with fixed cosmological constant. In section IV, we investigate the Gibbs free energy in order to find the phase transition in the system. We summarize our results in section V.

\section{II. REVIEW ON BI-ADS BLACK HOLE}

The well known action of four-dimensional Einstein gravity in the presence of BI nonlinear electrodynamics is \cite{41}

\begin{equation}
S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - 2\Lambda + 4\beta^2 \left( 1 - \sqrt{1 + \frac{F_{\mu\nu}F^{\mu\nu}}{2\beta^2}} \right) \right],
\end{equation}

where \( R \) is the Ricci scalar curvature, \( \Lambda \) is the cosmological constant which relates to AdS radius as \( \Lambda = -3/L^2 \) and the constant \( \beta \) is the BI parameter with the dimension of mass that relates to the string tension as \( \beta = 1/(2\pi\alpha') \) \cite{42}. Here, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic tensor field where \( A_\mu \) is the vector potential. The static and spherically symmetric metric of spacetime is

\begin{equation}
ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2,
\end{equation}
in which, \( d\Omega \) is the metric of an unit 2-sphere with volume \( \omega = 4\pi \) and the metric function \( f(r) \) is given by \cite{39, 43, 44}

\begin{equation}
f(r) = 1 + \frac{r^2}{L^2} - \frac{m}{r} + \frac{2\beta^2 r^2}{3} \left( 1 - \sqrt{1 + \frac{16\pi^2 Q^2}{\beta^2 r^4}} \right) + \frac{64\pi^2 Q^2}{3r^2} 2F_1 \left[ \frac{1}{4}, \frac{5}{4}; \frac{1}{4}; -\frac{16\pi^2 Q^2}{\beta^2 r^4} \right],
\end{equation}

where \( 2F_1(a, b, c, z) \) is the hypergeometric function, and \( m \) is an integration constant which is related to the mass of the black hole via \( M = m/8\pi \), and \( Q \) is the electric charge of the black hole per unit volume \( \omega \). The non-zero component of the gauge potential can be written as

\begin{equation}
A_t(r) = -\frac{4\pi Q}{r} 2F_1 \left[ \frac{1}{4}, \frac{5}{4}; \frac{1}{4}; -\frac{16\pi^2 Q^2}{\beta^2 r^4} \right].
\end{equation}

In the limit \( \beta \to \infty \), the metric function (3) and the gauge potential (4) reduce to ‘Reissner-Nordstrom’ (RN)-AdS black holes \cite{5, 31}. The Hawking temperature of BI-AdS black hole is given by

\begin{equation}
T = \frac{1}{4\pi r_+} + \frac{3r_+}{4\pi L^2} + \frac{\beta^2 r_+^2}{2\pi} \left( 1 - \sqrt{1 + \frac{16\pi^2 Q^2}{\beta^2 r_+^4}} \right),
\end{equation}

where for large \( \beta \), it can be expanded as

\begin{equation}
T = \frac{1}{4\pi r_+} \left( 1 + \frac{3r_+^2}{L^2} - \frac{16\pi^2 Q^2}{r_+^4} \right) + \frac{16\pi^4 Q^4}{r_+^4} + \mathcal{O} \left( \frac{1}{\beta^4} \right),
\end{equation}

where, \( r_+ \) is the event horizon which is defined by the largest positive real root of \( f(r_+) = 0 \). Note that the first term in the right hand side of the above expression is the RN-AdS black hole temperature and the next term is the leading order BI correction to the temperature. The electric potential at infinity with respect to the event horizon is given by

\begin{equation}
U = -\frac{4\pi Q}{r_+} 2F_1 \left[ \frac{1}{4}, \frac{5}{4}; \frac{1}{4}; -\frac{16\pi^2 Q^2}{\beta^2 r_+^4} \right].
\end{equation}
The other thermodynamic quantities associated with the Born-Infeld black hole are \[20\]
\[
S = \frac{r^3}{4}, \quad P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi L^2}, \quad V = \frac{r^3}{3},
\]
\[
\mathcal{B} = \frac{\beta r^3}{6\pi} \left( 1 - \sqrt{1 + \frac{16\pi^2 Q^2}{\beta^2 r^4}} + \frac{4\pi Q^2}{3\beta r^2} \right) \times_2 F_1 \left[ \frac{1}{4}, \frac{5}{4} ; \frac{1}{4} ; -\frac{16\pi^2 Q^2}{\beta^2 r^4} \right],
\]
where \(S\), \(P\) and \(V\) are entropy, pressure and volume, respectively. Also, \(\mathcal{B}\) is a quantity conjugate to \(\beta\) that is interpreted as the BI vacuum polarization \[20\]. It is easy to show that the thermodynamic quantities of BI black hole satisfy the first law of thermodynamic and Smarr formula, respectively, \[20\]
\[
dM = T dS + U dQ + V dP + \mathcal{B} d\beta,
\]
\[
M = 2TS + UQ - 2VP - \mathcal{B} \beta.
\]
It is notable that mass \((M)\), charge \((Q)\), entropy \((S)\) and volume \((V)\) are written per unit volume \(\omega\). In the next section, we study the critical behavior of BI-AdS black hole while the cosmological constant is fixed and the electric charge of black hole can vary.

**III. CRITICAL BEHAVIOR OF BI-ADS BLACK HOLE**

In this section, we investigate the critical behavior of BI-AdS black hole in which the charge of black hole can vary whereas the cosmological constant is a fixed parameter. In order to do this, we can study the behavior of the specific heat at constant charge
\[
C_Q = T \left( \frac{dS}{dT} \right)_Q,
\]
where, the positive (negative) sign of this quantity indicates the local thermodynamic stability (instability). Note that Eq. (12) is also calculated at fixed \(L\) (i.e. cosmological constant) and \(\beta\). To see the behavior of \(C_Q\), we plot the curves of the temperature versus the event horizon for different values of the charge in Fig. 1. As one can see, for small \(r_+\), the behavior of the temperature strongly depends on the charge of the black holes. Hence, we expand the Hawking temperature for small \(r_+\) as
\[
T = \frac{2\beta}{r_+} (Q_m - Q) + \frac{r_+ (3 + 2L^2 \beta^2)}{4\pi L^2} - \frac{\beta^3 r_+^3}{16\pi^2 Q} + \mathcal{O} \left( r_+^4 \right),
\]
where \(Q_m = 1/(8\pi\beta)\) is the ‘marginal charge’. Also, the large \(r_+\) limit of the temperature is \(3r_+/(4\pi L^2)\) which is independent of the charge. Hence for large \(r_+\), the temperature increases linearly with increasing \(r_+\). Depending on the value of \(\beta\), BI-AdS black hole may be identified as follows: For \(Q \geq Q_m\), black hole is RN type. In this case, we have an extremal black hole because temperature crosses over from zero with decreasing \(r_+\). For \(Q < Q_m\), black hole is ‘Schwarzschild-like’ \(S\) type. Similar to Schwarzschild solution, black hole does not exist in the region of low temperature. According to Fig. 1, the rightmost (leftmost) branch of isocharge is related to large black hole which is locally stable (unstable), i.e. positive (negative) sign of specific heat at constant charge.

Now, we turn to obtain the critical point (where continuous phase transition occurs) with respect to the above mentioned for BI-AdS black hole. In Fig. 1, isocharge diagrams show that, for constant \(L\) and \(Q = Q_c\), the critical point is an inflection point which can be characterized by
\[
\frac{dT}{dr_+} \bigg|_{Q_c} = 0, \quad \frac{\partial^2 T}{\partial r_+^2} \bigg|_{Q_c} = 0.
\]
Note that for \(S\)-type black hole, the critical point must be occurred in the right branch of \(Q_c\) which is locally stable. Using the temperature formula in Eq. (5), the expressions of Eq. (14) are, respectively, written as
\[
\frac{1}{x} - 2x + \left( 1 + \frac{3}{2\beta^2 L^2} - \frac{1}{2\beta^2 r_{+c}^4} \right) = 0,
\]
\[
x^3 - x^2 - \frac{1}{2x} + \frac{1}{4\beta^2 r_{+c}^4} = 0,
\]
where
\[
x = \left( 1 + \frac{16\pi^2 Q^2}{\beta^2 r_{+c}^4} \right)^{-1}.
\]
Since the critical quantities are real positive values, we should have the following constraint on $x$,

$$0 \leq x \leq 1.$$  \hspace{1cm} (18)

Combining Eqs. (15) and (16) gives the cubic equation

$$x^3 + px + q = 0,$$  \hspace{1cm} (19)

where

$$p = -\frac{3}{2}, \quad q = \frac{1}{2} \left(1 + \frac{3}{2\beta^2 L^2}\right).$$  \hspace{1cm} (20)

Due to fact that $q$ is real and $p < 0$, the cubic equation has three or one real roots. Three roots exist if $4p^3 + 27q^2 \leq 0$, i.e.

$$\beta \geq \beta_0 = \sqrt{\frac{3}{2} \left(1 + \sqrt{2}\right)} / L \approx 1.9098 / L.$$  \hspace{1cm} (21)

In this case, the roots are written as

$$x_k = \sqrt{2} \cos \left(\frac{1}{3} \arccos \left[ -\frac{\sqrt{2}}{2} \left(1 + \frac{3}{2\beta^2 L^2}\right) \right] - \frac{2\pi k}{3}\right),$$  \hspace{1cm} (22)

$k = 0, 1, 2$.

One can easily verify that the constraint in Eq.(18) is violated by $x_2$. Also, by calculating the critical quantities from $x_1$, we find that $r+/c$ occurs in the unstable branch of critical isocharge for S-type black hole. For $\beta < \beta_0$ case, one root is given by

$$x_3 = -\sqrt{2} \cosh \left(\frac{1}{3} \arccosh \left[ -\frac{\sqrt{2}}{2} \left(1 + \frac{3}{2\beta^2 L^2}\right) \right] \right),$$  \hspace{1cm} (23)

which violates the condition in Eq. (18). Consequently, the critical behavior of BI-AdS black hole can be observed only for $\beta \geq \beta_0$. With $x_0$ at hand, the critical quantities are obtained as

$$r_-^2 = 2\beta^2 \left(\frac{1}{x_0} + x_0 - 2x_0^2\right),$$  \hspace{1cm} (24)

$$Q_c^{-2} = 64\pi^2 \beta^2 \left(1 - x_0^2\right) \left(1 + 2x_0^2\right)^2,$$  \hspace{1cm} (25)

$$T_c = \frac{\sqrt{x_0} \left[3 + 2L^2 \beta^2 \left(1 - x_0\right) \left(1 + 2x_0 + 2x_0^2\right)\right]}{4\pi L^2 \sqrt{2} \left(1 - x_0^2\right) \left(1 + 2x_0^2\right)}.$$  \hspace{1cm} (26)

The critical charge is larger than $Q_m$ when

$$\beta > \beta_1 = \sqrt{\frac{3}{2(\sqrt{6\sqrt{3} - 9} - 1)}} / L \approx 2.8871 / L.$$  \hspace{1cm} (27)

Thus for $\beta \geq \beta_1$, the critical behavior takes place in the RN-type of black hole. On the other hand, for $\beta_0 \leq \beta < \beta_1$, the critical point occurs for S-type black hole at the right branch of $Q = Q_c$.

Expanding the critical quantities of BI-AdS black hole for large $\beta$ (RN-type), lead to

$$r_+^c = \frac{L}{\sqrt{6}} - \frac{7}{24\sqrt{6}\beta^2} + O\left(\frac{1}{\beta^4}\right),$$

$$Q_c = \frac{L}{24\pi} + \frac{7}{576 \pi L \beta^2} + O\left(\frac{1}{\beta^4}\right),$$

$$T_c = \frac{1}{L\pi} \sqrt{\frac{2}{3} - \frac{1}{12\sqrt{6\pi L^3 \beta^2}}} + O\left(\frac{1}{\beta^4}\right).$$  \hspace{1cm} (28)

As expected, the first terms on the right-hand side of Eqs.(28) reproduces the critical values of RN-AdS black hole [31]. The leading BI correction to critical point is identified by the last terms in above equations. To determine the possible phase transition in the system, we study the Gibbs free energy in the next section.

**IV. GIBBS FREE ENERGY**

To find out phase transition and classify its type, we need to explore the behavior of thermodynamic potential (partition function) of BI-AdS black hole. This is due to the fact that the thermodynamic potential determines the globally stable state at equilibrium process. For fixed temperature $T$, pressure $P$ ($\Lambda$) and charge $Q$, the Gibbs free energy is thermodynamic potential which is computed from the Euclidean action with appropriate boundary term in the canonical ensemble [5, 20]. In this case, such a global stable state is corresponding to the lowest Gibbs free energy. Since the mass of black hole is identified as the enthalpy Eq. (10), one obtains the Gibbs free energy per unit volume $\omega$ by Legendre transformation [20]

$$G(T, Q, P) = M - TS = \frac{r_+}{16\pi} - \frac{r_+^3}{16\pi L^2} - \frac{r_+^3 \beta^2}{24\pi} \times \left(1 - \sqrt{1 + \frac{16\pi^2 Q^2}{\beta^2 r_+^4}}\right) + \frac{8\pi Q^2}{3r_+^4} F \left[\left[\begin{array}{c} 1, 1, 5, \frac{16\pi^2 Q^2}{16\pi^2 Q^2}\end{array}\right] \right],$$  \hspace{1cm} (29)

in which $r_+ = r_+(T, Q, L)$ and $L = L(P)$. In the following, we shall assume the charge of BI-AdS black hole can vary (parametric changes in the Gibbs free energy in Eq.(29)), while the pressure do not take different values. It is worth mentioning that since the pressure and $\beta$ are fixed, one may consider the above equation as the Helmholtz free energy. However, the results of this section are not changed.

The behavior of Gibbs free energy for $\beta \approx 3.5 \in [\beta_1, \infty)$ and $\beta \approx 2.84 \in [\beta_0, \beta_1)$ are illustrated in Figs. 2 and 4, respectively. As one can see from Figs. 2 and 4, BI-AdS black hole has a different and complex behavior for various charge $Q$. In Fig. 2, there is a critical point at $Q = Q_c$ in the region of RN-type. For $Q > Q_c$, the Gibbs free energy is single valued and is locally stable ($C_Q > 0$) every where, which is shown by the solid blue
and (0.05 0.30 0.60 0.20 0.70 0.45 0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.0 0.005 0.010 0.015 0.020 0.025 0.030)

FIG. 2: Gibbs free energy as a function of temperature for different values of charge and $\beta \geq \beta_1$. For $Q \geq Q_m$, the phase transition behavior is reminiscent of RN black hole. Reentrant phase transition (LBH/SBH/LBH) takes place for $Q_t \leq Q \leq Q_z$. For $Q < Q_t$, the lower (upper) branch is globally stable (unstable). The positive (negative) sign of $C_Q$ is denoted by the blue solid (dash red) line. Note that curves are shifted for clarity. We fix $L = 1$ and $\beta = 3.5$.

SBH and LBH, a finite jump in Gibbs free energy leads to a zeroth order phase transition between SBH and intermediate black hole (LBH) where initiates from $Q = Q_z$ and terminates at $Q = Q_t$. The zeroth order phase transition earlier seen in superfluidity and superconductivity in [22] and recently has been found in charged dilaton black hole [17]. Finally, for $Q < Q_t$, only LBH exists that is stable.

Especially, let us consider the case of $Q \approx 0.0101 \in [Q_t, Q_z]$ in Fig. 3. Decreasing the radius of the event horizon, black hole follows the lower solid blue curve until it joints to the upper solid blue curve in $G-T$ plane (see the inset in Fig. 3). In this position, black hole enters to the left solid blue curve with a first order LBH/SBH phase transition which is identified by gold line in $T-r_+$ plane. If we continue decreasing $r_+$, a zeroth order phase transition occurs between SBH and intermediate black hole (LBH) which is accompanied by a finite jump in $G$ at the end of solid blue curve. The zeroth order phase transition is displayed by purple line in Fig. 3. Finally, with decreasing $r_+$, black hole follows the solid blue line until the end. In case of $\beta \approx 2.84 \in [\beta_0, \beta_1]$, Fig. 4, the behavior of Gibbs free energy is similar to the mentioned above while all the phase transition takes place in region $Q < Q_m$, i.e., S-type black hole. The corresponding phase diagram of BI-AdS black hole for $\beta \approx 3.5 \in [\beta_1, \infty)$ and $\beta \approx 2.84 \in [\beta_0, \beta_1)$ are depicted in the $Q-T$ diagram in Figs. 5 and (6), respectively. As is seen from Fig. 5, the critical point is highlighted by a black spot and the first order phase transition separation small and large black hole in the region of $Q_t < Q < Q_c$ which
is marked by the gold curve. For $Q_t < Q < Q_2$, along with the fist order SBH/LBH, a zeroth order phase transition separates SBH/LBH (SBH/LBH) which is identified by purple curve. Hence, we have LBH/SBH/LBH phase transition in the region of $Q_t < Q < Q_2$, which is the reentrant phase transition. There is no phase transition in the charge range $Q > Q_2$ as well as $Q < Q_1$. Also, no BH region corresponds to no black hole solution for region of parameter space.

is marked by large BH region for $5 < Q < 4$, and a second-order (black solid circle). The critical point happens in the RN-type black hole. The inset demonstrates the presence of a large/small/large (intermediate) black hole reentrant phase transition for a small range of charge. The marginal charge is shown by black horizontal line. At charges below the marginal charge, no BH region corresponds to no black hole solution for region of parameter space.

FIG. 5: Phase diagram corresponding to Fig.(2). Three different transition are indicated: zeroth-order (purple curve), first-order (gold curve) and a second-order (black solid circle). The critical point happens in the RN-type black hole. The inset demonstrates the presence of a large/small/large (intermediate) black hole reentrant phase transition for a small range of charge. The marginal charge is shown by black horizontal line. At charges below the marginal charge, no BH region corresponds to no black hole solution for region of parameter space.

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V. SUMMARY

To summarize, we have investigated charged black holes in the background of AdS spacetime and in the presence of BI nonlinear electrodynamics. We have proposed a new viewpoint towards phase space of black holes by assuming that the cosmological constant, which is regard as the pressure of the system, is a fixed quantity, while in contrast the charge of the black hole can vary. Recently, it was argued that such an alternative view towards phase space of charged AdS black holes can successfully lead to a natural response function which measures how the size of a black hole ($\Psi \sim r_+^{-1}$) changes with its charge. It was shown that this alternative view more importantly leads to a natural correspondence of the charged AdS black hole with the Van der Waals fluid and the associated small-large black hole phase transition without a need for extended phase space [31].

First, we explored critical behavior of BI-AdS black holes by studying the behavior of the specific heat at constant charge, $C_Q$, which its sign indicates the local thermodynamic stability/instability of the system. We find that for small horizon radius $r_+$, the behavior of the temperature strongly depends on the charge of the black holes. We observed that depending on the value of the nonlinear parameter, $\beta$, BI-AdS black hole may be identified as RN black hole for $Q \geq Q_m$, and Schwarzschild-like black hole for $Q < Q_m$, where $Q_m = 1/(8\pi\beta)$ is the marginal charge. Besides, similar to Schwarzschild solution, black hole does not exist in the region of low temperature. We analytically calculated the critical point $(Q_c, T_c, r_+)$ by solving the cubic equation and studied the critical behavior of the system. We have also explored the behavior of Gibbs free energy for BI-AdS black hole. We find out that BI-AdS black hole has a different and complex phase behavior depending on the value of the charge $Q$ of the system. Indeed, there is a critical point at $Q = Q_c$ in the region of RN-type black holes. For $Q > Q_c$, the Gibbs free energy is single valued and the system is locally stable ($C_Q > 0$). However, for $Q < Q_c$, we have $C_Q < 0$ and the system experiences thermally unstable phase. We observed that for $Q_2 < Q < Q_c$, a first order phase transition occurs between SBH and LBH which is accompanied by a discontinuity in the slope of Gibbs free energy at transition point. An interesting phenomenon emerges for $Q$ in the range of $Q_t < Q < Q_2$, where a reentrant phase transition can be occurred between intermediate (large) BH, SBH and LBH in Schwarzschild-type region. This implies
that in addition to the first order phase transition which separates SBH and LBH, a finite jump in Gibbs free energy leads to a zeroth order phase transition between SBH and intermediate black hole (LBH) where initiates from \( Q = Q_s \) and terminates at \( Q = Q_t \).

Finally, it is important to mention that in the present work we only studied the phase behavior of BI-AdS black holes. This investigation can also be extended to other BI-like nonlinear electrodynamics such as Exponential and Logarithmic nonlinear electrodynamics. Besides, it is worth disclosing the effects of power parameter of the power-Maxwell electrodynamics on the reentrant phase transition of black holes in the background of AdS spaces with variable charge. These issues are now under investigation and the results will be appeared in the near future.

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