CANONICAL WITT FORMAL SCHEME EXTENSIONS AND p-TORSION GROUPS

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Abstract. We study the n-th arithmetic jet space of the p-torsion subgroup attached to a smooth commutative formal group scheme. We show that the n-th jet space above fits in the middle of a canonical short exact sequence between a power of the formal scheme of Witt vectors of length n and the p-torsion subgroup we started with. This result generalizes a result of Buium on roots of unity.

1. Introduction

Buium in [8] introduced the theory of arithmetic jet spaces on (formal) abelian schemes over p-adic rings and showed that the jet spaces of an abelian scheme A are naturally affine fibrations over A. Since then the theory of arithmetic jet spaces has been developed in several articles such as [1], [9], [12], [6], [7], [16], and has found remarkable applications in diophantine geometry as in [8] and [13].

In this paper, we study the structure of the jet space functors associated to the p-torsion subgroup G[p∞] of a smooth commutative formal group scheme G over a fixed p-adic basis. Here we show that for any n, the n-th jet space J^n(G[p∞]) is canonically an extension of G[p∞] by a power of the unipotent formal group scheme \( \hat{W}_{n-1} \), where \( \hat{W}_{n-1} \) is \( \hat{A}^n \), the n-dimensional formal affine space endowed with the group scheme structure of the additive Witt vectors of length n. This generalizes results obtained by Buium in [11] for G the multiplicative group scheme.

Before stating our main result in detail, let us introduce some notation. Let K be a finite extension of \( \mathbb{Q}_p \) with ramification index e, uniformizer \( \pi \) and ring of integers \( \mathcal{O} \). We denote by k the residue field of \( \mathcal{O} \) and let q be its order. Then the identity map of \( \mathcal{O} \) is a lift of \( q \)-Frobenius. Fix a \( \pi \)-adically complete \( \pi \)-torsion free \( \mathcal{O} \)-algebra R with a lifting of Frobenius \( \phi \), i.e., an endomorphism of R such that \( \phi(r) - r^q \in \pi R \) for all \( r \in R \). As an example, consider the ring of restricted power series \( \mathcal{O}(x) \) with \( \phi \) the \( \mathcal{O} \)-algebra endomorphism given by \( \phi(x) = x^q \). Let \( W_n \) be the functor of ramified Witt vectors of length \( n+1 \) (following Borger’s convention, details in §2.2).

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Let \( \text{Nilp}_R \) be the category of \( R \)-algebras on which \( \pi \) (or equivalently, \( p \)) is nilpotent. Its opposite category is a site with respect to the Zariski topology. Any adic \( R \)-algebra \( A \) with ideal of definition \( I \) containing \( \pi \) gives rise to a sheaf (of sets) \( \text{Spf}(A) \) such that

\[
\text{Spf}(A)(B) = \lim_{\rightarrow n} \text{Hom}_R(A/I^n, B),
\]

for any \( B \) in \( \text{Nilp}_R \). By a formal scheme over \( R \) we mean a sheaf on \( \text{Nilp}_R^{\text{op}} \) admitting an open cover by open subfunctors of the type \( \text{Spf}(A) \) for \( A \) as above.

Given a sheaf \( X \) on \( \text{Nilp}_R^{\text{op}} \) we define its \( n \)-th \( \pi \)-jet by

\[
J^n X(C) = X(W_n(C))
\]

for any \( C \in \text{Nilp}_R \). If \( X \) is a formal scheme over \( R \) the same is \( J^n X \) and it holds

\[
\text{Hom}_R(\text{Spf}(C), J^n X) = \text{Hom}_R(\text{Spf}(W_n(C)), X)
\]

[8], [4], [7], [2].

For a smooth commutative formal group scheme \( G \) over \( R \) there is a short exact sequence of formal group schemes

\[
0 \to N^nG \to J^nG \to G \to 0,
\]

which we call the **canonical Witt formal scheme extension** of \( G \) because of Theorem 3.18 below. Let \( G[p^\nu] \) be the \( p^\nu \)-torsion formal subgroup scheme of \( G \) and let \( G[p^\infty] \) denote the sheaf on \( \text{Nilp}_R^{\text{op}} \) such that \( G[p^\infty](C) = \lim_{\rightarrow \nu} G[p^\nu](C) \), for any \( C \in \text{Nilp}_R \). For each \( \nu \) the closed immersions \( G[p^\nu] \to G[p^{\nu+1}] \) induce closed immersions of \( \pi \)-jets \( J^n(G[p^\nu]) \to J^n(G[p^{\nu+1}]) \) and

\[
J^n G[p^\infty] = \lim_{\nu_0} J^n G[p^\nu]
\]

as sheaves on \( \text{Nilp}_R^{\text{op}} \); see Lemma [52].

Consider the natural projection map \( u: J^n(G[p^\infty]) \to G[p^\infty] \) and let \( N^n(G[p^\infty]) \) denote the kernel of \( u \). Buium in [11] Corollary 1.2] shows that if \( e = 1, \ p > 2 \) and \( G \) is the formal multiplicative group scheme over \( R \), then the sheaf \( N^n(G[p^\infty]) \) is representable by a formal \( R \)-scheme and is isomorphic to \( \hat{\mathbb{A}}^n \), the \( n \)-dimensional affine space over \( \text{Spf}(R) \).

In this paper, we will enrich Buium’s result and extend it to any smooth commutative formal \( R \)-group scheme \( G \) of relative dimension \( d \geq 1 \). In fact, in Theorem 3.18 we show that if \( p \geq e + 2 \) the kernel \( N^n(G[p^\infty]) \) is isomorphic to \( (\widehat{\mathbb{W}}_{n-1})^d \) where \( \widehat{\mathbb{W}}_{n-1} \) is \( \hat{\mathbb{A}}^n \) endowed with the group structure of Witt vectors of length \( n \). Further, we deduce the following result (see Theorem 3.18).

**Main Theorem.** Assume \( p \geq e + 2 \). Given a smooth commutative formal group scheme \( G \) of relative dimension \( d \) over \( R \), for any positive integer \( n \) the natural morphism \( J^n G \to G \) gives an exact sequence

\[
0 \to (\widehat{\mathbb{W}}_{n-1})^d \to J^n G[p^\infty] \to G[p^\infty] \to 0
\]
of sheaves on $\text{Nilp}^{\text{op}}_R$.

We remark that by Lemma 4.2 it is $J^n(G[p^\infty]) = (J^nG)[p^\infty]$ as sheaves on $\text{Nilp}^{\text{op}}_R$; hence there is no possible ambiguity in the above statement.

1.1. Plan of the paper. In Section 2 we recall the definition and properties of $\pi$-jets in the setting of formal schemes, with particular attention to the adjunction between jet algebras and Witt vectors (2.14).

In Section 3 we focus on the notion of shifted Witt vectors $W^+_n$, introduced by [7], and show that $W^+_n$ induces an adjoint functor to $N^n$ (Theorem 3.8). This is an important result, analogous to the adjunction formula that involves $W_n$ and $J^n$ (1.2). Then we show that given a smooth formal $R$-group scheme $G$ of dimension $d$, we have

i) For all $n > 0$, $N^nG \simeq J^{n-1}(N^1G)$; see Theorem 3.15.

ii) Assume $p \geq e+2$. Then there is a natural isomorphism of formal group schemes $N^nG \simeq (\hat{W}_{n-1})^d$; see Theorem 3.18.

The proof of the first fact reduces to a local computation in coordinates which is detailed in the Appendix section. Another important ingredient is the notion of lateral Frobenius introduced in [7]. Both results i) and ii) are generalized to the case of $m$-shifted Witt vectors in [17] by the third author.

In Section 4 we apply the previous results to the study of the sheaves $J^n(G[p^\infty])$ and $N^n(G[p^\infty])$ and deduce a statement similar to Theorem 3.18 where $G$ is replaced by $G[p^\infty]$, see Theorem 4.8.

In this paper all rings are assumed to be commutative with unit and $\text{Alg}_R$ denotes the category of $R$-algebras, i.e., of ring homomorphisms $R \to B$.

2. Arithmetic jets

2.1. Conventions. Let $R$ be the base ring fixed in the introduction. Given a formal scheme $X$ and a fixed point $x : \text{Spf}(R) \to X$, one can consider the fibre of $x$ under the natural map $J^nX \to X$, which is the closed formal subscheme $N^nX = (J^nX)_x = J^nX \times_X \text{Spf}(R)$. Note that if $G$ is a formal group scheme then $J^nG$ is naturally a formal group scheme too and we set $N^nG = (J^nG)_x = \ker(J^nG \to G)$ to be the fibre along the unit section $\varepsilon$.

If $X$ is a functor on $\text{Alg}_R$, we will usually denote by $\hat{X}$ the restriction of $X$ to $\text{Nilp}_R$. Let $R(x_1, \ldots, x_n)$ be the $\pi$-adic completion of the $R$-polynomial algebra in $n$ variables.

Let $\hat{G}_a := \text{Spf}(R(x))$ be the additive formal group scheme over $R$. Note that this formal group scheme should not be confused with the $(x)$-adic formal group $G^\text{for}_a = \text{Spf}(R[[x]])$, the formal completion of $G_a$ along the zero section.
If \( F \in R[[x,y]] \) is a commutative formal group law of dimension \( g \), let \( F\{n\} \) be the formal group law given by \( \pi^{-n}F(\pi^nx,\pi^ny) \), for any \( n \geq 1 \). Note that \( F\{n\} \) endows \( \text{Spf}(R(x)) \) with a structure of formal group scheme over \( R \).

If \( B \) is an \( R \)-algebra, \( \rho = \rho_B : R \to B \) always denote the corresponding ring homomorphism. If the context is clear, we will write \( r \) in place of \( \rho(r) \in B \).

2.2. Witt vectors over \( R \). In the following pages \( W_n \) denotes the functor of \( \pi \)-typical Witt vectors of length \( n+1 \) on \( R \)-algebras. Hence, for any \( R \)-algebra \( B \), the ring \( W_n(B) \) is always considered with its natural \( R \)-algebra structure, which depends on \( \phi \). We explain this briefly.

As functor on \( \mathcal{O} \)-algebras \( W_n \) coincides with the so-called functor of ramified Witt vectors of length \( n+1 \) (see [15], [3]). Let \( w : W_n(R) \to \prod_{i=0}^n R \) be the ghost map. Then for any Witt vector \( a = (a_0, \ldots, a_n) \), \( w(a) = (w_0(a), \ldots, w_n(a)) \) where \( w_i \) are the ghost polynomials

\[
w_i = x^i_0 + \pi x^{i-1}_1 + \cdots + \pi^ix_i.
\]

Since \( R \) has a lifting of Frobenius \( \phi \), by the universal property of Witt vectors there exists a ring homomorphism of \( \mathcal{O} \)-algebras \( \exp_\delta \) making the following diagram commute (see [2] (2.9))

\[
\begin{array}{ccc}
R & \xrightarrow{\exp_\delta} & W_n(R) \\
\downarrow^{(\phi^0,\phi,\ldots,\phi^n)} & & \downarrow^w \\
\prod_{i=0}^n R & & W_n(R)
\end{array}
\]

Let \( B \) be an \( R \)-algebra. Then \( W_n(B) \) is naturally endowed with the \( R \)-algebra structure

\[
R \xrightarrow{\exp_\delta} W_n(R) \xrightarrow{W_n(\rho_B)} W_n(B).
\]

In [7, §3.2] the authors give an equivalent construction of the functor \( W_n \). The ghost map \( w \) in (2.2) is \( \mathcal{O} \)-linear, but not \( R \)-linear in general, if the ring \( \prod_{i=0}^n R \) is endowed with the direct product \( R \)-module structure. It is then preferable to change the \( R \)-module structure on the product ring so that \( w \) becomes \( R \)-linear. Let \( \phi^n B \) denote the ring \( B \) with the \( R \)-algebra structure induced by \( \rho_B \circ \phi^n : R \to B \), and let

\[
\prod_\phi^n(B) := \prod_{i=0}^n (\phi^n B)
\]
be the direct product algebra. Its underlying ring is $\prod_i B$ and there is a commutative diagram of $R$-algebras

\[
\begin{array}{c}
R \xrightarrow{\varphi} W_n(R) \xrightarrow{W_n(\rho_B)} W_n(B) \\
\downarrow_{(\text{id}, \phi, \ldots, \phi^n)} \downarrow w & \downarrow w \\
\prod^n \phi(R) \xrightarrow{\prod^n(\rho_B)} \prod^n \phi(B)
\end{array}
\]

Then Frobenius and Verschiebung maps can be described in terms of ghost components as in the case of ramified Witt vectors, with caution when considering the $R$-algebra structure. As for example, the Frobenius ring homomorphism $F: W_n(B) \to W_{n-1}(B)$ described in terms of ghost components as the left shift is $\phi$-semilinear. We prefer then to write it as the homomorphism of $R$-algebras

\[
F: W_n(B) \to W_{n-1}(\phi B)
\]

corresponding to the homomorphism of $R$-algebras

\[
F_w: \prod^n \phi(B) \to \prod^{n-1} \phi(B), \quad (b_0, \ldots, b_n) \mapsto (b_1, \ldots, b_n).
\]

Similarly, the Verschiebung map $V: W_n(B) \to W_{n+1}(B)$ is described on ghost components as the right shift multiplied by $\pi$. Clearly it is $\mathcal{O}$-linear but not $R$-linear in general. We prefer then to write it as the homomorphism of $R$-modules

\[
V: W_n(\phi B) \to W_{n+1}(B)
\]

corresponding to the homomorphism of $R$-modules

\[
V_w: \prod^n \phi(\phi B) \to \prod^{n+1} \phi(B), \quad (b_0, \ldots, b_n) \mapsto (0, \pi b_0, \ldots, \pi b_n).
\]

Then $FV$ is multiplication by $\pi$ on $W_n(\phi B)$.

Since $\phi$ might not be invertible, one can not write $B$ in place of $\phi B$ in (2.5) and (2.7). However, since $\phi$ is the identity on $\mathcal{O}$, the $\mathcal{O}$-module structure on $B$ and $\phi B$ are the same.

**Remark 2.9.** If $B$ is a $\pi$-adic $R$-algebra (by this we mean $\pi$-adically complete and separated) then the same is $W_n(B)$ for any $n$. The proof works as in [19, Proposition 3].

### 2.3. Shifted Witt vectors

We recall the construction of 0-shifted Witt vectors as introduced in [7] and [17]. Here we simply refer to them as shifted Witt vectors. The general theory of $m$-shifted Witt vectors is developed in [17].

Let $B$ be an $R$-algebra and set-theoretically define

\[
W_n^+(B) := R \times_B W_n(B) \simeq R \times W_{n-1}(B).
\]
Also define the product ring
\[ \prod_{n}^+(B) := R \times \prod_{i=1}^{n} (\phi^i B) = R \times \prod_{i=1}^{n-1} (\phi^i B), \]
where \( \prod_{n}^+(B) \) was introduced in (2.3). Note that there is an isomorphism of \( R \)-algebras
\[ \prod_{n}^+ B := R \times \prod_{i=1}^{n-1} (\phi^i B) \cong R \times B \prod_{i=1}^{n} (\phi^i B), \]
mapping \((r, b_1, \ldots, b_n)\) to the element \((r, (r, b_1, \ldots, b_n))\).

Define the \textit{apriori} set-theoretic map \( w^+ : W_{n}^+(B) \rightarrow \prod_{n}^+(B) \) given by
\[ w_i = r^{q_i} + \pi b_1^{q_i-1} + \cdots + \pi b_i, \]
for all \( i = 0, \ldots, n \). Then note that \( W_{n}^+(B) \) naturally is endowed with the Witt ring structure of addition and multiplication making \( w^+ \) a ring homomorphism. Hence we have the following commutative diagram
\[
\begin{array}{ccc}
W_{n}^+(B) & \xrightarrow{w^+} & \prod_{n}^+(B) \\
\downarrow & & \downarrow \\
W_n(B) & \xrightarrow{w} & \prod_{n}^0(B)
\end{array}
\]
where \( \text{pr}_0 \) is the projection onto the 0-th component. The \( R \)-algebra \( W_{n}^+(B) \) was denoted by \( \tilde{W}_n(B) \) in [7, §4] and by \( W_{0n}(B) \) in [17].

Since the lower horizontal arrows in (2.11) are homomorphisms of \( R \)-algebras, the same are the upper horizontal arrows. Hence the left hand square in (2.11) is a diagram of \( R \)-algebras and, up to the above identifications, it can be illustrated as
\[
\begin{array}{ccc}
R \times W_{n-1}(\phi B) & \xrightarrow{w^+} & R \times \prod_{n-1}^0(\phi B) \\
\downarrow & & \downarrow \\
W_n(B) & \xrightarrow{w} & \prod_{n}^0(B)
\end{array}
\]
where we have written \( r \) in place of \( \rho(r) \) in \( B \) and \( w_i \) are the ghost polynomials in (2.1).

2.4. Prolongation sequences. For any formal schemes \( Y \) and \( Z \) over \( \text{Spf}(R) \) we say that \((u, \delta) : Z \rightarrow Y \) is a prolongation if \( u : Z \rightarrow Y \) is a morphism of formal schemes over \( \text{Spf}(R) \) and \( \delta : \mathcal{O}_Y \rightarrow u_* \mathcal{O}_Z \) is a \( \pi \)-derivation on the sheaves (cf. Appendix A). Then a sequence of formal schemes \( T^* = \{ T^n \}_{n=0}^{\infty} \) is a \textit{prolongation sequence} if for each \( n, (u_n, \delta_n) : T^{n+1} \rightarrow T^n \) is a prolongation of formal group schemes over \( \text{Spf}(R) \) satisfying
\( u_{n-1} \circ \delta_n = \delta_{n-1} \circ u_n \) and making the following diagram commute

\[
\begin{array}{ccc}
R & \xrightarrow{u_n} & O_Z \\
\downarrow^{\delta} & & \downarrow^{\delta} \\
R & \xrightarrow{\delta} & O_Y
\end{array}
\]

A morphism of prolongation sequences \( T^* \rightarrow P^* \) is a system of morphisms of formal schemes \( f_n: T^n \rightarrow P^n \) that satisfy the expected commutations: \( f_n \circ u_n = u_n \circ f_{n+1} \) and \( f_n \circ \delta_n = \delta_n \circ f_{n+1} \). For each \( n \), let \( S^n = \text{Spf}(R) \). Then the fixed \( \pi \)-derivation \( \delta \) on \( R \) makes \( S^* \) into a prolongation sequence. Let \( \mathcal{C}_{S^*} \) denote the category of prolongation sequences defined over \( S^* \).

### 2.5. Jet spaces

Given a formal scheme \( X \) over \( S^0 = \text{Spf}(R) \), Buium constructs the canonical prolongation sequence \( J^*X = \{ J^nX \}_{n=0}^{\infty} \) where \( J^0X = X \) and by [9, Proposition 1.1], \( J^*X \) satisfies the following universal property: for any \( T^* \) in \( \mathcal{C}_{S^*} \) it is

\[(2.13) \quad \text{Hom}_{S^0}(T^0, X) = \text{Hom}_{S^*}(T^*, J^*X).\]

Moreover, by [5] and [2] we have the following functorial description:

\[(2.14) \quad J^nX(C) = X(W_n(C))\]

for any \( C \in \text{Nilp}_R \). In particular in the affine case with \( X = \text{Spf}(A) \) and \( J^nX = \text{Spf}(J_nA) \), we have a natural adjunction

\[(2.15) \quad \Theta: \text{Hom}_R(J_nA, C) \xrightarrow{\sim} \text{Hom}_R(A, W_n(C))\]

such that \( u_0 \circ \Theta(g) = g \circ \iota \) with \( \iota : A \rightarrow J_nA \) the natural morphism.

Here we make the above adjunction explicit when \( X = \text{Spf}(R(x)) \) is the formal affine line over \( \text{Spf}(R) \). Let \( A = R(x) \). Then \( J^nX = \text{Spf}(J_nA) \) and

\[(2.15) \quad J_nA = R(x, x', \ldots, x^{(n)}) = R(p_0, \ldots, p_n)\]

where \( x, x', \ldots, x^{(n)} \) are the Buium-Joyal coordinates and \( p_0, \ldots, p_n \) are the Witt coordinates and they satisfy \( p_0 = x, p_1 = x' \) while the general relation between the above two coordinate systems can be found in [2] Proposition 2.10.

If \( g \in \text{Hom}_R(J_nA, C) \), then \( \Theta(g) \in \text{Hom}_R(A, W_n(C)) \) is determined by

\[(2.15) \quad \Theta(g)(x) = (g(p_0), \ldots, g(p_n)).\]

Note that when \( G = \hat{G}_a \) we have the following isomorphism of formal group schemes

\[(2.16) \quad J^n\hat{G}_a = \hat{W}_n.\]

where \( \hat{W}_n \) is \( \hat{A}^{n+1} \) endowed with the additive group structure of Witt vectors of length \( n + 1 \).
3. The Kernel as a \( \pi \)-jet space

For any sheaf of groups \( G \) on \( \text{Nilp}_R\) one defines

\[
N^n G := \ker(J^n G \xrightarrow{\cdot u} G)
\]

where \( u \) is the natural morphism. Scope of this section is to take a closer look at the kernel \( N^n G \) in the case \( G \) is representable by a smooth formal scheme. Since the kernel is the fibre at the unit section, we will first consider more general fibres.

Let \( X \) be a smooth formal scheme over \( R \) with a marked point \( a \) and let \( u: U \to \text{Spf}(\mathbb{A}) = \text{Spf}(R[\![x]\!] ) \) be an \( \acute{e} \)tale chart around \( a \), where \( x \) denotes here a finite family of indeterminates. Hence \( U \) is an open affine formal subscheme of \( X \), \( u \) is \( \acute{e} \)tale, \( a \) factors through \( u \) and \( u \circ a \) is the zero section \( 0 \) of the affine space \( \mathbb{A} \). By \([10, \text{Proposition 3.13} & \text{Corollary 3.16}] \) (see also \([2, \text{Proposition 3.12}] \)), we have \( J^n U \cong J^n \mathbb{A} \times \mathbb{A} U \) for all \( n \) and hence

\[
(3.1) \quad N^n X = J^n X \times_{X,a} \text{Spf}(R) = J^n U \times_{U,a} \text{Spf}(R) = J^n \mathbb{A} \times J^n \mathbb{A} \;
\]

in particular, \( N^n X \) is formal affine and isomorphic to \( N^n U \). Up to shrinking \( U \), we may assume \( U = \text{Spf}(A) \) with \( A \) a \( \pi \)-adically complete separated \( R \)-algebra. Then the point \( a: \text{Spf}(R) \to U \) induces an \( R \)-algebra morphism \( \varepsilon: A \to R \). Let \( J^n U = \text{Spf}(J_n A) \). It is immediate to check that

\[
(3.2) \quad N^n U = \text{Spf}(J_n A \otimes_{A,\varepsilon} R).
\]

3.1. Adjunction. Let \( \text{Alg}_R^+ \) denote the category of augmented (commutative) \( R \)-algebras. Its objects are commutative \( R \)-algebras \( A \) together with an augmentation \( \varepsilon \), i.e. an \( R \)-algebra morphism \( \varepsilon: A \to R \); morphisms in \( \text{Alg}_R^+ \) are morphisms of \( R \)-algebras \( h: A_1 \to A_2 \) respecting augmentations, i.e. \( \varepsilon_2 \circ h = \varepsilon_1 \). For any \( (A, \varepsilon) \) in \( \text{Alg}_R^+ \) we define the \( R \)-algebra

\[
(3.3) \quad N_n A := J_n A \otimes_{A,\varepsilon} R.
\]

Note that shifted Witt vectors yield objects in \( \text{Alg}_R^+ \). Indeed, let \( B \) be an \( R \)-algebra and let \( w_0^+: W_n^+(B) \to R \) denote the projection onto the first component, i.e., the composition of the upper horizontal arrows in \((2.11)\). Then \( W_n^+(B) \) together with \( w_0^+ \) is an augmented \( R \)-algebra. Hence the above construction defines a functor

\[
W_n^+: \text{Alg}_R \to \text{Alg}_R^+ : B \mapsto (W_n^+(B), w_0^+),
\]

on the category of \( R \)-algebras.

We now prove a key result: \( N_n \) and \( W_n^+ \) is a pair of adjoint functors.

**Theorem 3.4.** For any augmented \( R \)-algebra \((A, \varepsilon)\) and any \( R \)-algebra \( B \) there is a natural bijection

\[
(3.5) \quad \Theta^+: \text{Hom}_R(N_n A, B) \xrightarrow{\sim} \text{Hom}_{\text{Alg}_R^+}(A, W_n^+(B)).
\]
Proof. Let \( \iota : A \to J_n A \) denote the natural morphism. Then
\[
\text{Hom}_R(N_n A, B) = \{ g \in \text{Hom}_R(J_n A, B) \mid g \circ \iota = \rho_B \circ \varepsilon \}
\]
\[
= \{ f \in \text{Hom}_R(A, W_n(B)) \mid w_0 \circ f = \rho_B \circ \varepsilon \}
\]
\[
= \{ f^+ \in \text{Hom}_R(A, W_n^+(B)) \mid w_0^+ \circ f^+ = \varepsilon \}
\]
where the first equality follows by (3.3), the second by (2.14) taking \( f = \Theta(g) \), the third by definition of \( W_n^+(B) \) in (2.10).
\[\square\]

By Remark 2.9 an analogous adjunction holds when working with the category of augmented formal \( R \)-algebras \( \text{fAlg}^+_{R} \). We make this fact explicit.

Example 3.6. Let \( A = R(x) \) and \( \varepsilon(x) = 0 \). By (2.15)
\[
N_n A = J_n A \hat{\otimes}_{A, \varepsilon} R = R(x', \ldots, x^{(n)}) = R(p_1^+, \ldots, p_n^+),
\]
where \( p_i^+ = R[x', x, \ldots, x^{(n)}] \) denotes the polynomial \( p_i \in R[x, x', \ldots, x^{(n)}] \) evaluated at \( x = 0 \). Then the formal counterpart of (3.3) works as follow: given \( g \in \text{Hom}_R(N_n A, B) \), then \( \Theta^+(g) \) maps \( x \) to \( (0, g(p_1^+), \ldots, g(p_n^+)) \).

The higher dimensional case is analogous. Let \( A = R(\mathbf{x}) \) with \( \mathbf{x} \) a collection of \( r \) indeterminates \( \{x_1, \ldots, x_r\} \) and let \( \varepsilon \) be the zero section. Then \( J_n A \simeq R(p_0^+, p_1, \ldots, p_n) \) where \( p_i \) denotes a collection of polynomials \( \{p_{i,1}, \ldots, p_{i,r}\} \) and \( p_{i,j} \in R[x_j, x_j', \ldots, x_j^{(n_j)}] \) plays the role of \( p_i \) in (2.15). Then
\[
N_n A \simeq R(p_0^+, \ldots, p_n) \hat{\otimes}_{R(\mathbf{x}), \varepsilon} R \simeq R(p_1^+, \ldots, p_n^+),
\]
where \( p_i^+ \) denotes the collection of polynomials \( \{p_{i,1}^+, \ldots, p_{i,r}^+\} \) with \( p_{i,j}^+ \) obtained by evaluating \( p_{i,j} \) at \( x_j = 0 \). Finally for a homomorphism \( g : N_n A \to B \), \( \Theta^+(g) \) maps \( x \) to \( (0, g(p_1^+), \ldots, g(p_n^+)) \).

We can now describe the functor \( N^n X \) on \( R \)-algebras as done in (1.2) for \( J^n X \).

Theorem 3.8. Let \( X \) be a smooth formal scheme over \( R \) with a marked point \( x \) and let \( B \) be in \( \text{Nilp}_R \). Then
\[
N^n X(B) = \text{Hom}_{R\text{-pt}}(\text{Spf}(W_n^+(B)), X),
\]
where on the right we are considering morphisms of \( R \)-pointed formal schemes.

Proof. The result is clearly true if \( X \) is affine by Theorem 3.4. For the general case, assume first that \( X \) is an \( R \)-scheme and consider the following diagram

\[
\begin{array}{ccc}
J^n X & \xrightarrow{u} & \text{Spec}(W_n(B)) \\
\downarrow^g & & \downarrow^{w_0} \text{Spec}(B) \xrightarrow{\iota} \text{Spec}(W_n(B)) \\
& & \downarrow^f \text{Spec}(R) \xrightarrow{w_0^+} \text{Spec}(W_n^+(B)) \\
& & \downarrow^x \text{Spec}(W_n^+(B)) \xrightarrow{f^+} X
\end{array}
\]
where \( w_0 \) is induced by the projection on the first component on algebras, \( \rho \) is the structure morphism and \( u \) is the natural map. Note that \( \text{Spec}(W_n^+(B)) \) is the push-out of \( w, \rho \) in the category of all schemes [15, 07RS]. Then \( N^n X(B) = X(W_n^+(B)) \). Indeed,

\[
N^n X(B) = \{ g \in \text{Hom}_R(\text{Spec}(B), J^n X) \mid u \circ g = x \circ \rho \} \\
= \{ f \in \text{Hom}_R(\text{Spec}(W_n(B)), X) \mid f \circ w_0 = x \circ \rho \} \\
= \{ f^+ \in \text{Hom}_R(\text{Spec}(W_n^+(B)), X) \mid f^+ \circ w_0^+ = x \} \\
= \text{Hom}_{R-p}(\text{Spec}(W_n^+(B)), X).
\]

If \( X \) is a formal scheme, then the above holds for all schemes \( X \times \text{Spec} R/(\pi^n) \) and thus one concludes. \( \square \)

3.2. A special case. Let \( G \) be a formal group scheme over \( R \) and denote by \( G^{\text{for}} \) the formal completion of \( G \) along the unit section. Let \( \mathcal{F} \in R[x_1, \ldots, x_r, y_1, \ldots, y_r] \) be the formal group law on \( G^{\text{for}} \), \( \mathcal{F}^\phi \) the one obtained by acting on the coefficients of \( \mathcal{F} \) by \( \phi \) and \( \mathcal{F}^\phi\{1\} \) := \( \pi^{-1}\mathcal{F}^\phi(\pi x, \pi y) \), where \( \pi x := (\pi x_1, \ldots, \pi x_r) \). By [5, Lemma 2.2] it is \( N^1G \cong \mathcal{F}^\phi\{1\} \) as formal \( R \)-schemes. We give below a direct computation of this fact.

**Lemma 3.9.** Let the notation be as above. Then the formal group law on the formal completion of \( N^1G \) at the origin is isomorphic to \( \mathcal{F}^\phi\{1\} \).

**Proof.** As seen in Remark 3.6, we may write \( G^{\text{for}} = \text{Spec}(R[x]) \) and \( N^1G = \text{Spec}(R(p_1^+)) \) = \( \text{Spec}(R(x^\prime)) \). Let \( \delta: R[R, x, y] \to R[R, x, y, x^\prime, y^\prime] \) be the \( \pi \)-derivation compatible with that of \( R \) and such that \( \delta(x) = x^\prime, \delta(y) = y^\prime \). If \( \mathcal{F}(x, y) \) is the formal group law of \( G \), the formal group law of \( N^1G \) is \( \delta(\mathcal{F}(x, y)) \) evaluated at \( x = 0, y = 0 \). Write \( \mathcal{F}(x, y) = \sum a_{\alpha, \beta} x^\alpha y^\beta \) with \( \alpha, \beta \) varying in \( \mathbb{N}^r - \{0\} \), \( x^\alpha := x_1^{\alpha_1}x_2^{\alpha_2} \ldots x_r^{\alpha_r} \), and the coefficients of the monomials of degree 1 equal to 1. By induction, applying the usual rules of \( \pi \)-derivations, one checks that

\[
\delta(\mathcal{F}(x, y))|_{x=0, y=0} = \sum_{\alpha, \beta} \delta(a_{\alpha, \beta} x^\alpha y^\beta)|_{x=0, y=0} = \\
\sum_{\alpha, \beta} \pi^{-1}\phi(a_{\alpha, \beta})(\pi x^\alpha)(\pi y^\beta) = \pi^{-1}\mathcal{F}^\phi(\pi x, \pi y) = \mathcal{F}^\phi\{1\}.
\]

\( \square \)

3.3. Lateral Frobenius. Let \( X \) be a formal \( R \)-scheme. As \( n \) varies, the \( \pi \)-jet spaces \( J^nX \) form an inverse system of formal schemes and, more precisely, a prolongation sequence, whence a lifting of Frobenius \( \phi_J \) exists on the limit. Clearly, the transition maps \( u = u_n^{n+1}: J^{n+1}X \to J^nX \) induce homomorphisms \( N^{n+1}X \to N^nX \), but the image of \( \phi \) restricted to \( N^{n+1}X \) is not necessarily contained in \( N^nX \) and hence \( \phi \) does not induce a lifting of Frobenius on the sequence of the kernels. For this reason, the notion of lateral Frobenius was introduced and studied in [5, 17].
On shifted Witt vectors, the lateral Frobenius $F^+$ is defined as the homomorphism of $R$-algebras making the following diagram commute, where vertical identifications are meant as sets, and $F$ is the Frobenius on Witt vectors recalled in (2.5). The homomorphism $F^+$ then corresponds to the homomorphism of $R$-algebras via ghost map, i.e., it makes the following diagram commute. Then, the homomorphism $F^+$ induces via (2.14) a natural morphism called again lateral Frobenius. It is showed in [7, Theorem 4.3] that $f$ is a lift of Frobenius and satisfies

$$
\phi \circ \phi \circ u = \phi \circ u \circ f,
$$

where $u$ denotes the immersion $N^m X \to J^m X$ and $\phi$ denotes the Frobenius morphism $J^m X \to J^{m-1} X$ for any $m$.

**Remark 3.13.** For later use, note that the element $(0, b) = (0, b_1, \ldots, b_n) \in W_n^+(B)$ traces in (3.12) the following images

$$
(0, b_1, \ldots, b_n) \xrightarrow{\pi} (0, \pi b_1 + \pi^2 b_2, \ldots, \pi b_{n-1}(b)) \xrightarrow{\pi} (0, \pi b_1, \pi b_2, \ldots, \pi b_{n-1}(b)).
$$

Hence $\pi c_1 = \pi b_1 + \pi^2 b_2$ implies $c_1 = b_1 + \pi b_2$ (for $B$ without $\pi$-torsion and hence for any $R$-algebra $B$ by standard arguments). By recursion one sees

$$(F^+)^i(0, b_1, \ldots, b_n) = (0, b_1^i + \pi b_2^i - 1 + \cdots + \pi b_1, \ldots) \in W_{n-i}^+(B)$$

for any $i < n$.

If $G$ is smooth over $R$, the same are $J^n G$ and $N^n G$ for all $n$. As seen in the previous section, $N^n G = \text{Spf}(N_n A)$ is an affine space over $R$. In particular the $R$-algebras $N_n A$ are $\pi$-torsion free and therefore the lateral Frobenius homomorphisms


\[ \varphi^*: N_nA \to N_{n+1}A \] induce a unique \( \pi \)-derivation \( \Delta \) on the prolongation sequence \( N_\ast A := \{ N_nA \}_{n=1}^\infty \).

In order to describe \( N^n \) as a jet functor we need a preparation lemma.

**Lemma 3.14.** Let \( \mathbb{A} = \text{Spf}(R(\mathbf{x})) \) with \( \mathbf{x} \) a collection of \( r \) indeterminates, and choose the origin as marked point. Let \( \varphi^*: \mathbb{A} \to N^{n-1}\mathbb{A} \) denote the \( i \)-th fold composition of lateral Frobenius for any \( i \leq n \). Then \( \varphi^* \) induces an homomorphism of \( \pi \)-adic \( R \)-algebras

\[
(\varphi^*)^*: R(\{ p_1^+, \ldots, p_{n-1}^+ \}) \to R(\{ p_1^+, \ldots, p_n^+ \})
\]
such that

\[
(\varphi^*)^*(p_i^+) = (p_i^+)^{q^{i-1}} + \pi(p_i^+)^{q^{i-2}} + \cdots + \pi^{i-1}p_i^+.
\]

**Proof.** Recall from Example 3.6 that \( N^n\mathbb{A} = \text{Spf}(N_nR(\mathbf{x})) \cong \text{Spf}(R(\{ p_1^+, \ldots, p_n^+ \})) \). Then by definition of \( \varphi \) and (3.7) with \( B = N_nR(\mathbf{x}) \) we have a commutative diagram of rings

\[
\begin{array}{ccc}
\text{Hom}_R(R(\{ p_1^+, \ldots, p_n^+ \}), R(\{ p_1^+, \ldots, p_{n-1}^+ \})) & \xrightarrow{\Theta^+} & \text{Hom}_{R\text{Alg}}(R(\mathbf{x}), W_n^+(R(\{ p_1^+, \ldots, p_n^+ \}))) \\
\downarrow & & \downarrow \\
\text{Hom}_R(R(\{ p_1^+, \ldots, p_{n-1}^+ \}), R(\{ p_1^+, \ldots, p_n^+ \})) & \xrightarrow{\Theta^+} & \text{Hom}_{R\text{Alg}}(R(\mathbf{x}), W_n^+(R(\{ p_1^+, \ldots, p_n^+ \})))
\end{array}
\]

By Remark 3.13 the identity map on \( R(\{ p_1^+, \ldots, p_n^+ \}) \) traces the following images

\[
\begin{array}{ccc}
\text{id} & \xrightarrow{\varphi^*} & \mathbf{x} \\
\downarrow & & \downarrow \\
\varphi^* & \xrightarrow{\varphi} & (0, p_1^+, \ldots, p_n^+)
\end{array}
\]

The conclusion follows by the explicit description of the map \( \Theta^+ \) as in the lines below (3.7). \( \square \)

We can now prove the main result of this section. This is a particular case of [17, Theorem 1.3] for which we give a shorter proof.

**Theorem 3.15.** Let \( G \) be a smooth formal group scheme over \( R \). Then for all \( n \) we have

\[ N^nG \cong J^{n-1}(N^1G). \]

**Proof.** Let \( \mathbb{A} = \text{Spf}(R(\mathbf{x})) \) be an \( \acute{e} \text{tale} \) coordinate system around the identity section of \( G \). We have seen in (3.1) and Example 3.10 that \( N^n\mathbb{A} = \text{Spec} N_nA \) with \( N_nA \cong R(\{ p_1^+, \ldots, p_n^+ \}) \). Now by Lemma 3.14 the lateral Frobenius satisfies

\[
(\varphi^*)^i(p_i^+) = (p_i^+)^{q^{i-1}} + \pi(p_i^+)^{q^{i-2}} + \cdots + \pi^{i-1}p_i^+.
\]

for all \( i = 0, \ldots, n \). Hence by Theorem A.8 we have \( N_nA \cong J_{n-1}(N_1A) \) and we are done. \( \square \)
Examples 3.16. (1) Assume $G = \hat{\mathcal{G}}_a = \text{Spf}(R(x))$ with the comultiplication mapping $x$ to $x \otimes 1 + 1 \otimes x$. Then $J^1 \hat{\mathcal{G}}_a = \text{Spf}(R(x, x'))$ where the group law is described by

$$x \mapsto x \otimes 1 + 1 \otimes x,$$

$$x' \mapsto x' \otimes 1 + 1 \otimes x' + C_\pi(x \otimes 1, 1 \otimes x)$$

with $C_\pi(X, Y) = \frac{X^s + Y^s - (X + Y)^s}{s} \in \mathcal{O}[X, Y]$. Hence $N^1 \hat{\mathcal{G}}_a = \text{Spf}((x')) = \hat{\mathcal{G}}_a$.

By Theorem 3.15 and equation (2.16) one concludes $N^n \hat{\mathcal{G}}_a = J^{n-1}(\hat{\mathcal{G}}_a) = \hat{W}_{n-1}$.

(2) Assume $G = \hat{\mathcal{G}}_m = \text{Spf}(R(x, y)/(xy - 1))$ with the comultiplication mapping $x$ to $x \otimes 1 + 1 \otimes 1$. Then $J^1 \hat{\mathcal{G}}_m = \text{Spf}(R(x, y, x', y')/(xy - 1, \delta(xy))) = \text{Spf}(R(x, x^{-1}, x'))$ with the group law described by

$$x \mapsto x \otimes 1 + 1 \otimes x,$$

$$x' \mapsto x' \otimes x^t + x^t \otimes x' + \pi x' \otimes x'. $$

Hence $N^1 \hat{\mathcal{G}}_m = \text{Spf}(R(x', y')/(x' + y' + \pi x' y')) = \text{Spf}(R(x'))$ and the group law on the latter maps $x'$ to $x' \otimes 1 + 1 \otimes x' + \pi x' \otimes x'$, i.e., $N^1 \hat{\mathcal{G}}_m = \mathcal{G}_m^{\text{for}}\{1\}$ as formal group schemes. Now, $\mathcal{G}_m^{\text{for}}\{1\}$, as formal group law, has invariant differential $(1 + \pi T)^{-1}dT$ and the corresponding logarithm is

$$\pi^{-1} \log(1 + \pi T) = \sum_{j \geq 1} \frac{(-\pi)^{j-1}}{j} T^j = T + \sum_{j \geq 2} a_j T^j \in K[T].$$

Assume $p \geq e + 1$. We prove that $\pi^{-1} \log(1 + \pi T) \in \mathcal{O}[T]$ and indeed in $\mathcal{O}(T)$. It suffices to check that $v_{\pi}(a_j) \geq 0$ tends to infinity as $j$ tends to infinity. Let $r > 0$ and note that $v_{\pi}(a_{pr}) = p^r - 1 - er \geq 0$ since

$$p^r - 1 = (p-1)(p^{r-1} + \ldots + 1) \geq er.$$

Further $v_{\pi}(a_{pr}) < v_{\pi}(a_{pr+1})$ and for $p^r \leq j < p^{r+1}$ we have

$$v_{\pi}(a_{pr}) = p^r - 1 - er \leq j - 1 - v_{\pi}(j) = v_{\pi}(a_j).$$

Hence $\pi^{-1} \log(1 + \pi T) \in \mathcal{O}(T)$ and it defines a morphism of formal group schemes $\mathcal{G}_m^{\text{for}}\{1\} \rightarrow \hat{\mathcal{G}}_a$. It is an isomorphism under the stronger hypothesis that $p \geq e + 2$. Indeed the inverse of $\pi^{-1} \log(1 + \pi T)$ is

$$\pi^{-1}(\exp(\pi T) - 1) = \sum_{j \geq 1} \frac{(\pi T)^j}{j!} \in K[[T]],$$

and the $\pi$-adic valuation of the $j$-th coefficient is

$$v_{\pi}(j!/j!) = j - v_{\pi}(j!) = j - e \cdot \frac{j - s_p(j)}{p-1} = \frac{j(p-1-e) + es_p(j)}{p-1},$$

where $s_p(j)$ denotes the sum of the digits in the base-$p$ expansion of $j$. Clearly if $p \geq e + 2$ this valuation tends to infinity as $j$ tends to infinity and hence
\[ \pi^{-1}(\exp(\pi T) - 1) \in \mathcal{O}(T). \] Then, if \( p \geq e + 2 \), one concludes that \( N^1\hat{G}_m \simeq \hat{G}_a \) and, with arguments as in [11], that \( N^n\hat{G}_m \simeq \hat{W}_{n-1} \).

The next result is an extension of [8, Lemma 2.3].

**Lemma 3.17.** Let \( \mathcal{F} \) be a commutative formal group law over \( R \) of dimension \( d \). If \( n(p - 1) \geq e + 1 \) then \( \mathcal{F}\{n\} \simeq (\hat{G}_a)^d \) as formal group schemes over \( R \).

**Proof.** In [8, Lemma 2.3] \( R \) is a complete discrete valuation ring with algebraically closed residue field. The proof in our hypothesis works the same. Indeed Buium applies results in [15] that are valid for any \( \mathbb{Z}_p \)-algebra and the key-point is showing that the coefficients of the logarithm and exponential series of \( \mathcal{F}\{n\} \) over \( R[1/p] \) are indeed in \( \pi R \). This is done by explicit estimates for the \( \pi \)-valuation of those coefficients. \( \square \)

**Theorem 3.18.** Let \( G \) be a smooth commutative formal group scheme of relative dimension \( d \) over \( \text{Spf}(R) \). Assume \( p \geq e + 2 \). Then there is a natural isomorphism of formal group schemes

\[ N^nG \simeq (\hat{W}_{n-1})^d. \]

**Proof.** Let \( G^{\text{for}} \) be the formal completion of \( G \) along the unit section \( \text{Spf}(R) \to G \). Let \( \mathcal{F} \in R[[x_1, \ldots, x_d, y_1, \ldots, y_d]] \) be the formal group law on \( G^{\text{for}} \), \( \mathcal{F}^\phi \) the one obtained by acting the coefficients of \( \mathcal{F} \) by \( \phi \) and \( \mathcal{F}^\phi \{1\} : = \pi^{-1}\mathcal{F}^\phi(\pi x, \pi y) \). By Lemma 3.9 (see also [8, Lemma 2.2]) it is \( N^1G \simeq \mathcal{F}^\phi \{1\} \) as formal group schemes. Note that since \( \phi(\pi) = \pi \) it is \( \mathcal{F}^\phi \{1\} = (\mathcal{F}\{1\})^\phi \). Now by hypothesis and Lemma 3.17 we have \( \mathcal{F}\{1\} \simeq (\hat{G}_a)^d \). Hence \( \mathcal{F}^\phi \{1\} \simeq ((\hat{G}_a)^d)^\phi = (\hat{G}_a)^d \) and hence \( N^1G \simeq (\hat{G}_a)^d \). By Theorem 3.15 and definition of \( J^{n-1} \) it is

\[ N^nG \simeq J^{n-1}(\hat{G}_a)^d \simeq (J^{n-1}\hat{G}_a)^d \simeq (\hat{W}_{n-1})^d. \] \( \square \)

**Remark 3.19.** Assume \( R = \mathcal{O} \), \( p > 2 \) and let \( G \) be as in the previous theorem. Then passing to limit on \( n \) we have a short exact sequence

\[ 0 \to N^\infty G(k) \to J^\infty G(k) = G(\mathcal{O}) \to G(k) \to 0 \]

where \( G(\mathcal{O}) \to G(k) \) is the reduction map, and we recover the fact that the kernel of the reduction map is isomorphic to \( \mathcal{O}^d = W(k)^d \) as groups.

### 4. \( p \)-POWER TORSION

For any formal commutative \( R \)-group scheme \( G \) let \( G[p^\nu] \) denote the kernel of the multiplication by \( p^\nu \) on \( G \) and let \( G[p^\infty] \) be the sheaf on \( \text{Nilp}_R^{\text{op}} \) such that

\[ G[p^\infty](C) = \lim_{\nu} G[p^\nu](C) \]
for any $C$ in $\text{Nilp}_R$. Note that $G[p^\infty]$ is a sheaf since the above colimit commutes with equalizers $[15]$ 04AX. If $G$ is a formal torus or a formal abelian scheme $G[p^\infty]$ is $p$-divisible, but not in general.

For each $\nu > 0$ the closed immersions $G[p^\nu] \hookrightarrow G[p^{\nu+1}]$ induce closed immersions of $\pi$-jets $J^n(G[p^\nu]) \hookrightarrow J^n(G[p^{\nu+1}])$ $[9$ Proposition 1.7], $[2$ Lemma 3.8, Theorem 3.9]. We can naturally pass to limit on $\nu$.

Lemma 4.2. Let the notation be as above. Then

$$J^n(G[p^\infty]) = \lim_{\nu} J^n(G[p^\nu]) = J^n(G)[p^\infty]$$

as sheaves on $\text{Nilp}_R^{op}$.

Proof. Recall (1.1) and (4.1). For any $C$ in $\text{Nilp}_R$ it is

$$J^n(G[p^\infty])(C) = G[p^\infty](W_n(C)) = \lim_{\nu} G[p^\nu](W_n(C)) = \lim_{\nu} J^n(G[p^\nu])(C).$$

Whence the first isomorphism. The second one follows by the fact that $J^n$ is left exact (being a right adjoint) and hence

$$\lim_{\nu} J^n(G[p^\nu])(C) = \lim_{\nu} ((J^n G)[p^\nu])(C) = (J^n(G)[p^\infty])(C).$$

\[\square\]

Recall that $N^n F$ is defined as $\ker(J^n F \to F)$ for any sheaf of groups on $\text{Nilp}_R^{op}$. We then deduce from Lemma 4.2 the following result.

Lemma 4.3. Let notation be as above. Then

$$N^n(G[p^\infty]) = \lim_{\nu} N^n(G[p^\nu]) = (N^n G)[p^\infty]$$

as sheaves on $\text{Nilp}_R^{op}$.

Proof. By Theorem 3.5 the functor $N^n$ is left exact; hence $(N^n G)[p^\nu] = N^n(G[p^\nu])$ and

$$(N^n G)[p^\infty] := \lim_{\nu} (N^n G)[p^\nu] = \lim_{\nu} N^n(G[p^\nu]).$$

Further, $(N^n G)[p^\infty] = \ker((J^n G)[p^\infty] \to G[p^\infty]) = \ker(J^n(G[p^\infty]) \to G[p^\infty]) = N^n(G[p^\infty]).$

\[\square\]

We say that a commutative formal $R$-group scheme is $\text{triangular}$ if it admits a finite filtration by formal subgroup schemes whose successive quotients are isomorphic to $\hat{G}_a$. It is called $\text{triangular of level}$ 0 in $[8$ p. 322]. The formal $R$-group scheme $\hat{W}_n$ is triangular.

Lemma 4.4. If $H$ is a triangular formal $R$-group scheme then $H[p^\infty] = H$ as sheaves on $\text{Nilp}_R^{op}$. 

Proof. We proceed by induction on the length $m$ of the filtration. The result is clearly true for $H = \hat{G}_a$ since $p$ is nilpotent in any object $C$ of $\text{Nilp}_R$. Assume $m > 1$. Then $H$ is extension of $\hat{G}_a$ by $G$ where $G$ is a triangular formal group scheme with a filtration of length $m - 1$. One concludes by induction hypothesis. \hfill $\square$

Moreover, we have the following stronger result.

**Theorem 4.5.** Assume $G$ is a smooth commutative formal $R$-group scheme and $p \geq e + 2$. Then $N^n(G[p^\infty]) = N^nG$ as sheaves on $\text{Nilp}^{op}_R$.

*Proof.* By Theorem 3.18, the formal $R$-group scheme $N^nG$ is triangular. Hence the result follows by Lemmas 4.3 and 4.4. \hfill $\square$

**Corollary 4.6.** Assume $p \geq e + 2$. The exact sequence (4.3) induces an exact sequence of formal groups

$$0 \to N^nG[p^\infty] \to J^nG[p^\infty] \to G[p^\infty] \to 0,$$

for any $n \geq 0$.

*Proof.* Only the right exactness needs to be proved. Let $s$ be a section of $G$ such that $p^m s = 0$. It lifts to a section $s'$ of $J^nG$. Now $p^n s'$ comes from a section of $N^nG$ and thus is $p$-power torsion by the previous theorem. Hence $s'$ is $p$-power torsion. \hfill $\square$

We are now ready to proof the Main Theorem stated in the introduction.

**Theorem 4.8.** Assume $p \geq e + 2$. Given a smooth commutative formal group scheme $G$ of relative dimension $d$ over $R$, for any positive integers $n$ the natural morphism $J^nG \to G$ gives an exact sequence

$$0 \to (\hat{W}_{n-1})^d \to J^nG[p^\infty] \to G[p^\infty] \to 0$$

as sheaves on $\text{Nilp}^{op}_R$.

*Proof.* This follows directly by applying Theorems 3.18 and Corollary 4.4. \hfill $\square$

We can now conclude the study of Examples 3.16.

**Examples 4.9.** (1) Assume $G = \hat{G}_a$. We have seen in Example 3.16 that $N^1\hat{G}_a = \text{Spf}(R(x'))$ is isomorphic to $\hat{G}_a = \text{Spf}(R(x))$ as formal group scheme mapping $x$ to $x'$ on algebras. In particular, for any $\nu \geq 1$, it is $N^1\hat{G}_a[p^\nu] \cong \hat{G}_a[p^\nu]$.

This result can be checked directly. Indeed $\hat{G}_a[p^\nu] = \text{Spf}(R(x)/(p^\nu x))$ and hence

$$J^1\hat{G}_a[p^\nu] = \text{Spf} \left( \frac{R(x,x')}{(p^\nu x, \delta(p^\nu x))} \right), \quad N^1\hat{G}_a[p^\nu] = \text{Spf} \left( \frac{R(x')}{(p^\nu x')} \right),$$
since \((p^n x)^q + \pi \delta(p^n x) = \phi(p^n x) = p^n \phi(x) = p^n (x^q + \pi x^q)\). Passing to limit on \(\nu\) we get an isomorphism \(N^1 \widehat{G}_m[p^\infty] \cong \widehat{G}_a[p^\infty]\).

(2) Assume \(G = \widehat{G}_m\). We have seen in Example 3.16(2) that if \(p \geq e + 2\) it is \(N^1 \widehat{G}_m \simeq \widehat{G}_a\). Hence \(N^1 \widehat{G}_m[p^\infty] \simeq \widehat{G}_a[p^\infty]\).

**Appendix A. Prolongation sequences of algebras**

Let for this section \(R\) be a flat \(O\)-algebra with a fixed \(\pi\)-derivation \(\delta\). In particular, it has a lift of Frobenius. Let \(u: B \to C\) be a morphism of \(R\)-algebras. Recall that if \(u: B \to C\) is a morphism of \(O\)-algebras, a \(\pi\)-derivation relative to \(u\) is a map of sets \(\partial: B \to C\) such that \(\partial(0) = 0 = \partial(1)\) and for any \(x, y \in B\)

\[
\begin{align*}
\partial(x + y) &= \partial(x) + \partial(y) + \frac{u(x)^q + u(y)^q - u(x + y)^q}{\pi} \\
\partial(xy) &= u(x)^q \partial(y) + u(y)^q \partial(x) + \pi \partial(x) \partial(y) .
\end{align*}
\]

(A.1) \hspace{1cm} (A.2)

Associated to \(\partial\) there is a lift of Frobenius (relative to \(u\)) \(\Psi: B \to C\) given by \(\Psi(x) = u(x)^q + \pi \partial(x)\) (see \([22, \S 3.1], [22, \S 1]\)).

Fix a positive integer \(r\). Let \(B_n = R[x_0, \ldots, x_n]\), where for each \(i \geq 0\), \(x_i\) denotes the \(r\)-tuple of variables \(x_{i,1}, \ldots, x_{i,r}\) and denote by \(u: B_n \to B_{n+1}\) the natural inclusions. Fix a prolongation sequence

\[
B = B_0 \overset{u_0}{\longrightarrow} B_1 \overset{u_1}{\longrightarrow} B_2 \to \ldots
\]

(A.3) i.e., for any \(n\) we fix a \(\pi\)-derivation (relative to \(u\)) \(\partial: B_n \to B_{n+1}\) such that

- \(\partial\) is compatible with the \(\pi\)-derivation of \(R\);
- \(\partial \circ u = u \circ \partial\) (and thus we avoided using subscripts).

The lift of Frobenius is then the homomorphism

\(\Psi: B_n \to B_{n+1}, \quad x_i \mapsto x_i^q + \pi \partial x_i\),

where we used a compact notation instead of writing \(x_{i,j} \mapsto x_{i,j}^q + \pi \partial x_{i,j}\) for all \(i \leq n\) and \(1 \leq j \leq r\). Note that we can handle all \(B_n\) together by introducing \(B_\infty = R[x_0, x_1, \ldots] = \bigcup_n B_n\) and again denoting by \(\partial\) the induced \(\pi\)-derivation associated with the identity on \(B_\infty\) and by \(\Psi\) the associated lift of Frobenius.

On the other hand, starting with \(B = R[x_0]\) we have a prolongation sequence

\[
B = B_0 \overset{u_0}{\longrightarrow} J_1 B_0 \overset{u_1}{\longrightarrow} J_2 B_0 \to \ldots
\]

(A.4) with \(J_n B = R[x_0, x_0', \ldots, x_0^{(n)}]\) and \(x_0^{(i+1)} = \delta x_0^{(i)}\). Let

\(\Phi: J_n B \to J_{n+1} B, \quad x_0 \mapsto x_0^q + \pi x_0'\)

be the corresponding lift of Frobenius and define \(J_\infty B = \bigcup_n J_n B\).
By \[2\] (2.9) the restriction on the first component \(W(J_{\infty}B) \to J_{\infty}B\) admits a homomorphic section \(\exp\) such that the following diagram

\[
\begin{array}{ccc}
W(J_{\infty}B) & \overset{w}{\longrightarrow} & \prod_{i \in \mathbb{N}} B_{\infty} \\
\exp \uparrow & & \uparrow (id, \Phi, \Phi^2, \ldots) \\
J_{\infty}B & \overset{\mathrm{w}}{\longrightarrow} & \prod_{i \in \mathbb{N}} B_{\infty} \\
\end{array}
\]

commutes, with \(w\) the ghost map of ramified Witt vectors. Let \(\exp(x_0) = (z_0, z_1, z_2, \ldots)\) so that \(z_0 = x_0, z_1 = x'_0\) and for \(n > 1\)

\[
\Phi^n(x_0) = \Phi^n(z_0) = z_0^{q^n} + \pi z_1^{q^{n-1}} + \cdots + \pi^n z_n.
\]

We will show in Theorem \([A.8]\) that if the indeterminates \(x_0, x_1, \ldots\) satisfy the analogous property for \(\Psi\), i.e.,

\[
\Psi^n(x_0) = x_0^{q^n} + \pi x_1^{q^{n-1}} + \cdots + \pi^n x_n,
\]

then there is a unique isomorphism between the prolongation sequences \([A.3]\) and \([A.4]\). We start with a technical result. For brevity, let us define for any \(n > 1\) the following polynomial in \(2n - 2\) indeterminates

\[
H_n(x_0, \ldots, x_{n-2}; y_0, \ldots, y_{n-2}) := \sum_{i=0}^{n-2} \left[ \sum_{j=1}^{q^{n-1}-1} \pi^{i+j} \binom{q^{n-1}-1}{j} x_0^{q^{n-1}-1-j} y_i \right]
\]

**Lemma A.7.** Let \(B_*\) be the prolongation sequence in \([A.3]\) and assume that it satisfies \([A.6]\) for any \(n\). Then

\[
x_n = \partial x_{n-1} + H_n(x_0, \ldots, x_{n-2}; \partial x_0, \ldots, \partial x_{n-2})
\]

and hence \(x_n - \partial x_{n-1} \in \mathcal{O}[x_0, \ldots, x_{n-2}, \partial x_0, \ldots, \partial x_{n-2}]\) with trivial constant term. Similarly, in \(J_{\infty}B\) it is

\[
z_n = \delta z_{n-1} + H_n(z_0, \ldots, z_{n-2}; \delta z_0, \ldots, \delta z_{n-2})
\]
Proof. The second assertion is [2, Proposition 2.10]. The proof of the first one is similar and we write below the main steps:

\[
\sum_{i=0}^{n} \pi^i x_i^{q^{n-i}} = \Psi^n(x_0)
\]

\[
= \Psi(\Psi^{n-1}(x_0))
\]

\[
= \Psi \left( \sum_{i=0}^{n-1} \pi^i x_i^{q^{n-1-i}} \right)
\]

\[
= \sum_{i=0}^{n-1} \pi^i \Psi(x_i)^{q^n}
\]

\[
= \sum_{i=0}^{n-1} \pi^i (x_i^q + \pi \partial x_i)^{q^n}
\]

\[
= \sum_{i=0}^{n-1} \pi^i \left[ x_i^{q^{n-i}} + \sum_{j=1}^{n-i} \left( \frac{q^n}{j} \right) x_i^{q^{n-i-j}} \pi^j (\partial x_i)^j \right]
\]

\[
= \sum_{i=0}^{n-1} \pi^i x_i^{q^{n-i}} + \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} \pi^{i+j} \left( \frac{q^n}{j} \right) x_i^{q^{n-i-j}} (\partial x_i)^j
\]

with \(n_i = n - 1 - i\). Hence cancelling the common terms on both sides of the above equality we get

\[
\pi^n x_n = \pi^n \partial x_{n-1} + \sum_{i=0}^{n-2} \left[ q^{n-1-i} \sum_{j=1}^{n-i} \pi^{i+j} \left( \frac{q^n}{j} \right) x_i^{q^{n-i-j}} (\partial x_i)^j \right]
\]

and one can divide by \(\pi^n\).

We now prove that prolongation sequences \(B_n\) as above satisfying condition (A.6) for all \(n\) are unique up to unique isomorphism.

**Theorem A.8.** Let \(B_n\) be the prolongation sequence in (A.3). Assume that the indeterminates \(x_i\) satisfy (A.6) for all \(n\) and let \(z_i \in J_\infty R[x_0]\) be the elements defined just below (A.5). Then we have

(i) The inclusion \(R[z_0, \ldots, z_n] \to J_n R[x_0]\) is an isomorphism for any \(n\).

(ii) For any \(n \geq 0\) the \(R\)-algebra homomorphism \(h_n: J_n R[x_0] \to B_n, x_i^{(n)} \mapsto \partial^i x_0\) is an isomorphism and the following square

\[
\begin{array}{ccc}
R \left[ x_0, \ldots, x_n \right] & \xrightarrow{(u, \delta)} & R \left[ x_0, \ldots, x_n^{(n+1)} \right] \\
\downarrow h_n & & \downarrow h_{n+1} \\
R \left[ x_0, \ldots, x_n \right] & \xrightarrow{(u, \partial)} & R \left[ x_0, \ldots, x_{n+1} \right]
\end{array}
\]
Proof. The first assertion was proved in [2] Lemma 2.20 with \( z_i = P_i(x) \). Commutativity of the squares is immediate by definition of \( h_n \). We are then left to prove that \( h_n \) is an isomorphism. Note that \( J_nR[z_0, \ldots, z_n] = R[z_0, \ldots, z_n] \) by point [1] and \( B_n = R[x_0, \ldots, x_n] \).

If we prove that \( h_n(z_i) = x_i \), for all \( 0 \leq i \leq n \), the result is clear.

We proceed by strong induction on the subset \( \{(i, n), 0 \leq i \leq n\} \subset \mathbb{N}^2 \) totally ordered as follows:

\[(i_1, n_1) < (i_2, n_2) \text{ if } n_1 + i_1 < n_2 + i_2 \text{ or } n_1 + i_1 = n_2 + i_2 \text{ and } i_1 < i_2.\]

The picture below illustrates the order.

\[
\begin{array}{c}
(0, 4) \\
(0, 3) \\
(0, 2) \\
(0, 1) \\
(0, 0)
\end{array}
\]

\[
\begin{array}{c}
(1, 3) \\
(1, 2) \\
(1, 1)
\end{array}
\]

It follows immediately by definition of \( h_n \) that \( h_n(z_0) = h_n(x_0) = x_0 \) and \( h_n(z_1) = h_n(x_0) = \partial x_0 = x_1 \) for all \( n \geq 0 \). Hence the assertion \( h_n(z_i) = x_i \) is clear for \( i \leq 1 \) and any \( n \), in particular for the base step \( (0, 0) \). Assume then \( i > 1 \) and that \( h_s(z_j) = x_j \) for any \( (0, 0) \leq (j, s) < (i, n) \).

By Lemma A.7, the commutativity of the above square and the induction step, we have

\[
h_n(z_i) = h_n(\partial z_{i-1} + H_i(z_0, \ldots, z_{i-2}, \delta z_0, \ldots, \delta z_{i-2}))
\]

\[
= h_n(\partial z_{i-1}) + H_i(h_n(z_0), \ldots, h_n(z_{i-2}), h_n(\delta z_0), \ldots, h_n(\delta z_{i-2}))
\]

\[
= \partial(h_n-1(z_{i-1})) + H_i(x_0, \ldots, x_{i-2}, \partial h_n-1(z_0), \ldots, \partial h_n-1(z_{i-2}))
\]

\[
= \partial x_{i-1} + H_i(x_0, \ldots, x_{i-2}, \partial x_0, \ldots, \partial x_{i-2})
\]

\[
= x_i.
\]

\[
\square
\]

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