ALGEBRAIC MODELS OF CHANGE OF GROUPS IN RATIONAL STABLE EQUIVARIANT HOMOTOPY THEORY

J.P.C. GREENLEES

Abstract. Shipley and the author have given an algebraic model for free rational $G$-spectra for a compact Lie group $G$. In the present note we describe, at the level of homotopy categories, the algebraic models for induction, restriction and coinduction relating free rational $G$-spectra and free rational $H$-spectra for a subgroup $H$ of $G$.

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1. Introduction

1.A. Change of groups for spectra. We are concerned with change of groups functors, so we suppose that $i : H \rightarrow G$ is the inclusion of a subgroup in a compact Lie group $G$. The restriction functor $i^* : G\text{-spectra} \rightarrow H\text{-spectra}$ has a left adjoint $i_*$ and a right adjoint $i_!$ defined on $H\text{-spectra} Y$ by $i_*(Y) = G_+ \wedge_H Y$ (induction) and $i_!(Y) = F_H(G_+, Y)$ (coinduction). Altogether we have an adjoint triple

$$
\begin{array}{c}
G\text{-spectra} \\
\xrightarrow{i_*} \\
\xleftarrow{i_!}
\end{array}
\rightarrow H\text{-spectra} ,
$$

and this passes to homotopy categories.

There are a number of cases where categories of rational $G$-spectra have been shown to be equivalent to algebraic categories, and one may then ask for the algebraic counterparts of the change of groups functors. The present paper considers the case of free spectra, where the algebraic models are at their simplest. This gives considerable insight in the general case, since the nature of the models in the more general case is that they are built from one contributions at each closed subgroup, each of which has the character of the free model (see [14, 9] for examples of this).

1.B. The model for free spectra. The homotopy category of free $G$-spectra is the full subcategory of spectra $X$ for which the map $EG_+ \wedge X \rightarrow X$ is an equivalence. The above adjoint triple passes to free spectra and gives the adjoint triple that we study here.
For any compact Lie group $G$, there is an algebraic model of free rational $G$-spectra $[10][11]$. If $G$ is connected this is the category of differential graded (DG) torsion modules over the polynomial ring $H^*(BG)$ (rational coefficients understood throughout). More generally if $G$ has identity component $G_e$ and component group $\overline{G} = G/G_e \cong \pi_0(G)$ we note that $\overline{G}$ acts via conjugation on $H^*(BG_e)$ and we may form the twisted group ring $H^*(BG_e)[\overline{G}]$. A torsion $H^*(BG_e)[\overline{G}]$-module is one which is torsion as a module over $H^*(BG_e)$ (i.e., its localization at any non-maximal prime ideal is trivial). There is a Quillen equivalence $[11]$ free-$G$-spectra/Q $\simeq$ DG-torsion-$H^*(BG_e)[\overline{G}]$-modules; in fact, a different Quillen equivalence was given in $[10]$ in the connected case, and the author expects this too to extend to the general case.

1.C. Contribution. The purpose of the present note is to identify the algebraic models of the change of groups functors, at least at the level of the homotopy category. We expect that a more precise analysis will show that (with a suitable choice of models) the change of groups Quillen adjunctions in topology correspond (under suitably chosen Quillen equivalences with the algebraic models) to the Quillen adjunctions we identify in algebra.

In Theorem 5.2 we will identify the algebraic model of the above adjoint triple, but there are two significant lessons along the way. Firstly, the right algebraic models are less easy to guess than the author expected, even at the crudest level. Secondly, we show that the algebraic models depend on which of the two Quillen equivalences (i.e., from $[10]$ or from $[11]$) are used.

The point of writing this elementary note separately is that the proofs are simple formal manipulations whilst the conclusions significantly illuminate the wider landscape. Of course one expects this to be helpful in lifting these results to the model categorical level, but the usefulness goes beyond that. For example, it was helpful in guiding the search for certain functors in $[14]$ and in identifying the algebraic model for toral spectra $[9]$. The author found it illuminated phenomena from $[2][3][4]$ and that it was a useful guide in dealing with models of categories of modules over profinite groups. Finally, it explained a delicate point in comparing the results of $[10]$ and $[11]$.

1.D. A mismatch. A moment’s thought shows that it is not straightforward to write down the models. Suppose for the present that $G$ and $H$ are connected and write

\[ \theta = i^* : H^*(BG) \rightarrow H^*(BH) \]

for the induced map in cohomology. The ring homomorphism $\theta$ gives a restriction functor $\theta^* : H^*(BH)$-modules $\rightarrow$ $H^*(BG)$-modules with left adjoint $\theta_*$ and right adjoint $\theta_!$ given on $H^*(BG)$-modules $M$ by $\theta_*(M) = H^*(BH) \otimes_{H^*(BG)} M$ (extension of scalars) and $\theta_!(M) = \text{Hom}_{H^*(BG)}(H^*(BH), M)$ (coextension of scalars), altogether giving

\[ \begin{array}{ccc}
H^*(BH)\text{-modules} & \xrightarrow{\theta_*} & H^*(BG)\text{-modules} \\
\theta_! & & \theta^* 
\end{array} \]

and these functors pass to torsion modules and again give an adjoint triple.

We note that in topology two of these functors go from $H$-spectra to $G$-spectra, whereas in algebra two functors go from $H^*(BG)$-modules to $H^*(BH)$-modules. It is clear these two triples cannot correspond to each other, and that we must find other functors in algebra to
model the functors in topology. Our technique in the various cases is to choose the most accessible of the three functors \( i_\ast, i^\ast \) and \( i_! \) and find its algebraic model. The models of the other two follow by taking adjoints.

1.E. **Conventions.** This paper is a contribution to rational equivariant stable homotopy theory, so we are concerned with algebraic models of categories of rational equivariant spectra. We work throughout at the level of homotopy categories and our conclusions are independent of models provided well-behaved change of groups functors exist. In particular our results apply to orthogonal spectra [16] and to the spectra of [15]. Given a choice of model for \( G \)-spectra, there are numerous models for free \( G \)-spectra, and we choose to localize with respect to non-equivariant homotopy. Similarly, models for rational spectra are obtained by localizing with respect to rational homotopy.

Since the algebraic model is only available rationally, our notation will leave rationalization implicit: all algebras are assumed to be \( \mathbb{Q} \)-algebras and all cohomology has coefficients in \( \mathbb{Q} \).

2. **The case of finite groups**

We suppose first that \( G \) and \( H \) are finite. There are two obvious equivalences

\[
\text{free-}G\text{-spectra} \simeq \text{DG-mod-}\mathbb{Q}G,
\]
given by hom or tensor Morita equivalences. We are using the fact that we have an equivalence \( \mathbb{Q}G \simeq F(G_+, G_+)^G \) identifying the endomorphism ring of the generator \( G_+ \) with an Eilenberg-MacLane spectrum. Now we may use the hom Morita equivalence and let a \( G \)-spectrum \( X \) be mapped to the right \( \mathbb{Q}G \)-module \( E_G X := F(G_+, X)^G \), or we may use the tensor Morita equivalence in which it is mapped to the left \( \mathbb{Q}G \)-module \( G_+ \wedge_G X \), which is made into a right \( \mathbb{Q}G \)-module in the usual way. These are naturally equivalent.

Now the inclusion \( i : H \longrightarrow G \) induces a map \( \overline{i} : \mathbb{Q}H \longrightarrow \mathbb{Q}G \) of group rings, and hence restriction \( \overline{i}^\ast : \text{mod-}\mathbb{Q}G \longrightarrow \text{mod-}\mathbb{Q}H \) with left adjoint \( \overline{i}_! \), induction and right adjoint coinduction as usual. We observe

\[
E_{Hi^\ast}(X) = F(H_+, i^\ast X)^H = \overline{i}^\ast F(G_+, X)^G = \overline{i}_! E_G X.
\]

We conclude that in this case, the spectrum level restrictions \( i_\ast \vdash i^\ast \vdash i_! \) correspond to the functors \( \overline{i}_\ast \vdash \overline{i}^\ast \vdash \overline{i}_! \) just as expected.

For finite groups, we note that this applies more generally; the group homomorphism \( i : H \longrightarrow G \) need not be a monomorphism. We may pull back a free \( G \)-spectrum to obtain an \( H \)-spectrum. If \( i \) is not a monomorphism the resulting spectrum need not be free, but we can make it so, and the functor \( i^\ast : \text{free-}G\text{-spectra} \longrightarrow \text{free-}H\text{-spectra} \) is induced by restriction along the homomorphism \( \overline{i} : \mathbb{Q}H \longrightarrow \mathbb{Q}G \). The left and right adjoint of \( \overline{i}^\ast \) are extension and coextension of scalars, and by Maschke’s Theorem these are automatically homotopy invariant.

3. **The case of connected groups**

Now suppose that \( G \) and \( H \) are connected. It turns out that the equivalence from [10] and [11] give different algebraic models for change of groups functors. We will describe three equivalences between topology and algebra, the first two give the same models for change of groups, and the third (namely that of [10]) gives a different model.
3.A. **Eilenberg-Moore correspondents.** This subsection discusses the equivalence of [11]. In this case, the equivalence

\[
\text{free-}G\text{-spectra} \simeq \text{DG-torsion-}H^*(BG)\text{-modules}
\]

is a composite of two equivalences. The first is a change of rings along \(S \to DEG_+ =: C^*(EG)\), where \(S\) is the sphere spectrum. Since \(G\)-spectra are simply \(S\)-modules, and since the change of rings map is a non-equivariant equivalence, cellularization with respect to \(G_+\) gives a Quillen equivalence free-\(G\)-spectra \(\simeq \text{cell-}C^*(EG)\)-mod-\(G\)-spectra (see [12, Section 4] for full details). In the second we pass to fixed points to give cell-\(C^*(EG)\)-mod-\(G\)-spectra \(\simeq \text{cell-}C^*(BG)\)-mod-spectra, since \(C^*(EG)^G \simeq C^*(BG)\) [13, Section 8]. Since we are working rationally, \(C^*(BG)\) is formal and we will work with \(H^*(BG)\) rather than \(C^*(BG)\)-throughout. In the first we have only changed the ring we are working over, so this plays no direct role in the change of groups. In the second equivalence we need to make a short manipulation.

**Lemma 3.1.** \(i^!\) corresponds to \(\theta^*\).

**Proof:** We calculate

\[
(i^!(DEH_+ \wedge X))^G = F_H(G_+, DEH_+ \wedge Y)^G = i^*(DEH_+ \wedge Y)^H.
\]

We will infer the correspondents of \(i^*\) and \(i_*\) in Subsection 3.E below.

3.B. **Koszul correspondents I.** This case is closely related to that of [10] described in Subsection 3.C, but significantly different.

In this case the equivalence

\[
\text{free-}G\text{-spectra} \simeq \text{DG-torsion-}H^*(BG)\text{-modules}
\]

is again a composite of two equivalences. The first is the Morita equivalence free-\(G\)-spectra \(\simeq \text{mod-}C_*(G)\) [5, 17], where \(C_*(G)\) is now the group ring spectrum of \(G\) over the rational sphere spectrum (literally \(Q \wedge G_+\), so that we could have continued with the notation \(QG\) from the case of finite groups); the equivalence is given again by taking a \(G\)-spectrum \(X\) to the right \(C_*(G)\)-module \(E_G(X) = F(G_+, X)^G\). The second equivalence is between modules over \(C_*(G)\) and modules over \(C^*(BG) = F(BG_+, Q)\); this is another Morita equivalence, where a \(C_*(G)\)-module \(P\) is taken to \(E'_G(P) = \text{Hom}_{C_*(G)}(Q, P)\), where we note that this is a module over \(\text{Hom}_{C_*(G)}(Q, Q) = \text{Hom}_{C_*(G)}(C_*(EG), C_*(EG)) \simeq C^*(BG)\).

For the first step, it follows as in the finite case that \(i^*\) corresponds to \(\overline{\tau}^*\). The more interesting step is the second.

**Lemma 3.2.** \(i^!\) corresponds to \(\theta^*\).

**Proof:** Since \(i^!\) corresponds to \(\overline{\tau}^!\), it suffices to deal with modules over the chains on groups. For a \(C_*(H)\)-module \(Q\), we calculate

\[
E'_G(\overline{\tau}!Q) = \text{Hom}_{C_*(G)}(Q, \overline{\tau}!Q) = \theta^* \text{Hom}_{C_*(H)}(Q, Q).
\]

We will infer the correspondents of \(i^*\) and \(i_*\) in Subsection 3.E below.
3.C. **Koszul correspondents II.** This case was treated in [10], but we recap here to show the difference from the case of Subsection 3.B.

In this case the equivalence

free-$G$-spectra $\simeq$ DG-torsion-$H^*(BG)$-modules

is again a composite of two equivalences. The first is the Morita equivalence free-$G$-spectra $\simeq$ mod-$C_*(G)$, precisely as in Subsection 3.B and it follows as before that $i^*$ corresponds to $i^!$.

However for the second equivalence we use the reverse Morita equivalence [5, 17]. This time, a right $C_*(G)$-module $P$ is taken to $T_G(P) = P \otimes C_*(G)$, where we note that this is a left module over $\text{Hom}_{C_*(G)}(Q, Q) = \text{Hom}_{C_*(G)}(C_*(EG), C_*(EG)) \simeq C^*(BG)$.

**Lemma 3.3.** $i_*$ corresponds to $\theta^*$.

**Proof:** For a $C_*(H)$-module $Q$, we calculate

$T_G(i_*Q) = (Q \otimes_{C_*(H)} C_*(G)) \otimes_{C_*(G)} C_*(H) Q = \theta^*(Q \otimes C_*(H) Q)$.

\[\square\]

We will infer the correspondents of $i^*$ and $i_1$ in Subsection 3.E below.

3.D. **Iterated adjoints of restriction of scalars.** We consider a map $\theta : R \to S$ of commutative rings. The example we have in mind is $R = H^*(BG), S = H^*(BH)$ and $\theta = i^*$.

The ring homomorphism induces a restriction of scalars $\theta^*$, which always has a left adjoint $\theta_*$ defined by $\theta_*(M) = S \otimes_R M$ and a right adjoint $\theta^!$ defined by $\theta^!(M) = \text{Hom}_R(S, M)$. If $S$ is small as an $R$-module there are further adjoints. To describe these, we first define the relative dualizing complex

$D(R|S) = \text{Hom}_R(S, R)$,

which we note is an $(R, S)$-bimodule.

**Lemma 3.4.** If $S$ is small as an $R$ module we have a string of adjunctions

$\theta^! \dashv \theta_* \dashv \theta^* \dashv \theta^!$

defined on $R$-modules $M$ and $S$-modules $N$ by

- $\theta^!(N) = D(R|S) \otimes_S N$
- $\theta_* (M) = S \otimes_R M$
- $\theta^*(N)$ is $N$ with restricted action
- $\theta^!(M) = \text{Hom}_R(S, M)$
- $\theta^!(N) = \text{Hom}_S(D(R|S), N)$

**Remark 3.5.** The 2014 PhD thesis of M. Abbasirad [1] investigates the model categorical underpinnings of this.

**Proof:** We write $DM = \text{Hom}_R(M, R)$ for the Spanier-Whitehead duality functor, and note that when $M'$ is small the natural maps $M' \to DDM'$ and $DM' \otimes_R M \to \text{Hom}_R(M', M)$ are equivalences. If $S$ is small over $R$ then the case $M' = S$ gives

$\theta_*(M) = S \otimes_R M \simeq \text{Hom}_R(DS, R) \otimes_R M \simeq \text{Hom}_R(DS, M)$

and

$\theta^!(M) = \text{Hom}_R(S, M) \simeq DS \otimes_R M.$
Since $\mathbb{D}(R|S) = DS$, the results follow from the usual Hom-tensor adjunction. □

The relevant special case is as follows.

**Corollary 3.6.** With $\theta = i^*: H^*(BG) \to H^*(BH)$, the relative dualizing module is given by

$$\mathbb{D}(G|H) = \text{Hom}_{H^*(BG)}(H^*(BH), H^*(BG)) \simeq \Sigma^{L(G/H)}H^*(BH).$$

Accordingly, the adjoint functors

$$\theta! \dashv \theta_* \dashv \theta^* \dashv \theta!$$

are defined on $H^*(BG)$-modules $M$ and $H^*(BH)$-modules $N$ by

- $\theta!(N) = \Sigma^{L(G/H)}N$
- $\theta_*(M) = H^*(BH) \otimes_{H^*(BG)} M$
- $\theta^*(N)$ is $N$ with restricted action
- $\theta_!(M) = \text{Hom}_{H^*(BG)}(H^*(BH), M)$
- $\theta^!(N) = \Sigma^{-L(G/H)}N$.

**Remark 3.7.** If we only take into account the module structures over $H^*(BG)$ and $H^*(BH)$, then the suspension by $L(G/H)$ is simply the integer suspension by dim$(G/H)$ because $G$ and $H$ are connected. We have written it using tangent spaces to record the functoriality in $G$ and $H$. This will be important when we treat the case of disconnected groups.

**Proof:** Note that $H^*(BG)$ is a polynomial ring and by Venkov's theorem $H^*(BH)$ is a finitely generated $H^*(BG)$-module, so the condition of Lemma 3.4 is satisfied.

The identification of the dualizing module is given in [14, Theorem 6.8] using stable equivariant homotopy. □

3.E. **The Eilenberg-Moore correspondents of [11]**. We showed in Lemma 3.1 above that $i_!$ corresponds to $\theta^*$. It follows that the left adjoints of $i_!$ and $\theta^*$ correspond, so that $i^*$ corresponds to $\theta_*$. Taking left adjoints again, it follows that $i_*$ corresponds to the left adjoint of $\theta_*$.

Altogether we find

$$\text{free-}G\text{-spectra} \underbrace{i_*}_i \rightarrow \text{free-}H\text{-spectra},$$

which corresponds to

$$\text{torsion-}H^*(BG)\text{-modules} \underbrace{\theta^!}_\theta \rightarrow \text{torsion-}H^*(BH)\text{-modules}.$$

Of course this applies equally to the Koszul I equivalence described in Subsection 3.B using Lemma 3.2.
3.F. The Koszul II correspondents of [10]. We showed in Lemma 3.3 that $i_*$ corresponds to $\theta^*$. It follows that the left and right adjoints of $i_*$ correspond to the left and right adjoints of $\theta^*$.

Altogether we find

$$
\text{free-}G\text{-spectra} \overline{\longrightarrow} \text{free-}H\text{-spectra} \overline{\longrightarrow}
$$
corresponds to

$$
\text{DG-torsion-}H^*(BG)\text{-modules} \overline{\longrightarrow} \text{DG-torsion-}H^*(BH)\text{-modules} \overline{\longrightarrow}
$$

4. Twisted group rings

Before turning to the case of arbitrary compact Lie groups, it is worth recording some general facts about the algebra of twisted group rings.

First suppose $R$ is a commutative $\mathbb{Q}$-algebra with an action of a finite group $A$, so that we may form the twisted group ring $R[A]$. If $R$ is a $k$-algebra, and $M_1, M_2$ are $R[A]$-modules

$$
\text{Hom}_{R[A]}(M_1, M_2) = \text{Hom}_R(M_1, M_2)^A
$$

Now suppose that $\overline{\iota} : B \rightarrow A$ is a homomorphism of finite groups. We want to observe that extension and coextension of scalars along $\overline{\iota} : \mathbb{Q}B \rightarrow \mathbb{Q}A$ (which are group theoretic induction and coinduction when $\overline{\iota}$ is a monomorphism) are compatible with the action on $R$. This is straightforward once we observe we have a map $\overline{\iota} : R[A] \rightarrow R[B]$.

Lemma 4.1. Suppose $N$ is an $R[B]$-module.

- The natural map is an isomorphism $\mathbb{Q}A \otimes_{\mathbb{Q}B} N \cong RA \otimes_{RB} N$, so that the two possible meanings of $\overline{\iota}_* N$ are compatible.
- The natural map is an isomorphism $\text{Hom}_{\mathbb{Q}B}(\mathbb{Q}A, N) \cong \text{Hom}_{RB}(RA, N)$, so that the two possible meanings of $\overline{\iota}_! N$ are compatible.
- We have an adjoint triple

$$
R[A]\text{-modules} \overline{\longrightarrow} \overline{\longrightarrow} S[B]\text{-modules} \overline{\longrightarrow} \overline{\longrightarrow}
$$

Now suppose given a map $\theta_e : R \rightarrow S$ of commutative rings. We suppose given an action of a finite group $A$ on $R$ and a finite group $B$ on $S$, and that these are compatible via a homomorphism $\overline{\iota} : B \rightarrow A$ of finite groups in the sense that $\theta_e(\overline{\iota}(b)r) = b\theta_e(r)$. We may then write $\theta = (\theta_e, \overline{\iota})$ for the pair of structure maps.

Lemma 4.2. We have an adjoint triple

$$
R[A]\text{-modules} \overline{\longrightarrow} \overline{\longrightarrow} S[B]\text{-modules} \overline{\longrightarrow} \overline{\longrightarrow}
$$

The functors are defined as follows.
• The functor $\theta_* = \theta^e_* : R[A]\text{-modules} \rightarrow S[B]\text{-modules}$ is defined by the formula $\theta_*(M) = S \otimes_R M$, where the $B$-action is diagonal, using $i$ on the second factor (so we might write $\theta_* = i^*\theta^e_*$).

• The left adjoint of $\theta_*$ is the functor $\theta^!_*$ defined by $\theta^!(N) = \overline{i}_*\theta^!_e N$.

• The right adjoint of $\theta_*$ is the functor $\theta^*_*$ defined by $\theta^*(N) = \overline{i}_\theta^*e_* N$.

**Proof:** In view of the fact that the space of $R[A]$-maps is just the $A$-equivariant $R$-maps, and similarly for $S[B]$, this is just a case of checking that the adjunctions defined above are equivariant. □

5. The general case

For a general compact Lie group $G$ and subgroup $H$, we have a diagram

\[ \begin{array}{ccccccccc}
1 & \rightarrow & G^e & \rightarrow & G & \rightarrow & \overline{G} & \rightarrow & 1 \\
\uparrow & & \downarrow i_e & & \downarrow i & & \downarrow \overline{i} & & \\
1 & \rightarrow & H^e & \rightarrow & H & \rightarrow & \overline{H} & \rightarrow & 1
\end{array} \]

We note that $\overline{i}$ need not be a monomorphism.

We observe that with $R = H^*(BG^e)$, $A = \overline{G}$, $S = H^*(BH^e)$, $B = \overline{H}$ and $\theta_e = i^* : H^*(BG^e) \rightarrow H^*(BH^e)$ and $\theta = (\theta_e, \overline{i})$, we obtain an instance of the twisted group ring context described in the Section 4. The compatibility of $\theta_e$ and $\overline{i}$ arises since the action of $B = \overline{H}$ on $H^e$ by conjugation is compatible with the action of $A = \overline{G}$ by conjugation on $G^e$, and the action of $G^e$ on $H^*(BG^e)$ is trivial.

The first stage of the equivalence

\[ \text{free-$G$-spectra} \simeq \text{DG-torsion-$H^*(BG^e)[\overline{G}]$-modules} \]

is again extension of scalars along $S \rightarrow DEG^+ =: C^*(EG)$. However the second stage is now to take $G^e$ fixed points (rather than $G$-fixed points) and to consider the value in free $\overline{G}$-spectra.

**Lemma 5.1.** $i_!$ corresponds to $\theta^*_*$. **Proof:** The invariant of an $H$-spectrum $Y$ is the $\overline{H}$-spectrum $(Y \wedge DEH_+)^{H_e}$ (or rather its homotopy groups as an $\overline{H}$-module. The invariant of the $G$-spectrum $i_! Y$ is the $\overline{G}$-spectrum

\[(F_H(G^+, Y) \wedge DEG_+)^{G^e} = F_H(G^+, Y \wedge DEH_+)^{G^e}\]

(or rather its homotopy groups as a $\overline{G}$-module). Now if $T$ is inflated from $\overline{G}$ we have

\[ [T, F_H(G^+, Y^t)^{G^e}]^G = [T, F_H(G^+, Y^t)]^G = [T, Y^t]^H = [T, (Y^t)^{H_e}]^{\overline{H}} \]

as required. □

This allows us to give the general case.
Theorem 5.2. Under the equivalence of [11], the adjoint triple

\[
\text{free-}G\text{-spectra} \xrightarrow{i_*} \text{free-}H\text{-spectra},
\]

corresponds to the adjoint triple

\[
\text{DG-torsion-}H^*(BG_e)[G]\text{-modules} \xrightarrow{\theta^*} \text{DG-torsion-}H^*(BH_e)[H]\text{-modules}.
\]

These algebraic models of the change of groups morphisms are given explicitly by

- \(\theta^!(N) = Q[G] \otimes Q[H] \Sigma^{L/G/H} N\),
- \(\theta_* (M) = H^*(BH_e) \otimes_{H^*(BG_e)} M\),
- \(\theta^* (N) = \text{Hom}_{Q[H]}(Q[G], N)\),

where \(\mathcal{H}\) acts on \(L(G/H)\) by differentiating the conjugation action. □

Example 5.3. We take \(G = SO(3), H = O(2)\). Now \(H^*(BG) = H^*(BSO(3)) = Q[d]\), \(H^*(BH_e) = H^*(BSO(2)) = Q[c]\) with \(\mathcal{H} = W\) of order 2, acting on \(Q[c]\) by \(c \mapsto -c\). The map \(\tilde{\eta} : QW \to Q\) is the augmentation. The inclusion of identity components induces \(\theta_e : Q[d] \to Q[c]\) which is given by mapping \(d\) to \(c^2\). Remembering that cohomological degrees are negative, the relative dualizing module is given by

\[
\mathbb{D}(G_e|H_e) = \text{Hom}_{Q[d]}(Q[c], Q[d]) \cong c^{-1} \cdot Q[c] \cong \Sigma^2 \what{Q} \otimes Q[c].
\]

Free \(SO(3)\)-spectra are modelled by torsion \(Q[d]\)-modules, and free \(O(2)\)-spectra are modelled by torsion \(Q[c][W]\)-modules. The forgetful functor \(i^*\) is modelled by

\[
\theta^* (N) = \Sigma^2 (\what{Q} \otimes N)_W
\]

the induction functor is modelled by

\[
\theta^! (N) = Q[c] \otimes Q[d] M,
\]

and the coinduction functor is modelled by

\[
\theta^* (N) = N^W.\quad \Box
\]

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School of Mathematics and Statistics, Hicks Building, Sheffield S3 7RH. UK.
E-mail address: j.greenlees@sheffield.ac.uk