On preparation of the W-states from atomic ensembles

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A scheme, where three atomic ensembles can be prepared in the states of the W-class via Raman
type interaction of strong classical field and a projection measurement involved three single-photon
detectors and two beamsplitters, are considered. The obtained atomic entanglement consists of the
Dicke or W-states of each of the ensembles.

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I. INTRODUCTION

An atom of lambda-configuration seems to be attractive with respect to state preparation, because one of its
transitions allows us to spy upon the atom behavior, say, by detecting of the emitted photons. This is projection
measurement, that is a way to obtained a quantum system in some state, particulary entangled. In fact, the entanglement generated through the Bell-state measurement on two atomic samples has been demonstrated experimentally [1].

We focus on the states from the W-class, introduced in ref. 2, where some particular members and their generalizations belong to family of the symmetric Dicke states 3. They are interesting because of its robustness with respect to loss of a particle 4. In ref. 5 a set of projection measurements has been proposed to produce the W-state of atomic ensembles, that consist of lambda-atoms. The W-state, generated in this way, has been considered as an intermediate step, for example, in teleportation 7. Also during evolution there are some entangled states of light to be close to ZSA (Zero Sum Amplitude) states introduced in ref. 5.

In this paper we present a scheme for preparing the W-and symmetric Dicke states. It contrast to scheme discussed in ref. 5, it is more simple because of the introduced projection measurement, that doesn’t involve preparation of EPR pair as an intermediate step, for example. Indeed, the obtained W-entanglement of atomic ensembles is represented by the W-states of atoms in each ensemble and can be particulary suitable as the quantum channel for teleportation 8. Also during evolution there are some entangled states of light to be close to ZSA (Zero Sum Amplitude) states introduced in ref. 8.

Our paper is organized as follows. First, we introduce hamiltonian with local field operators to describe interaction of light with atomic ensembles spatially separated. Next, we consider a family of W-states obtained while evolution and projection measurement.

II. HAMILTONIAN

To describe interaction of light with atomic ensembles spatially separated, we consider a hamiltonian, in which local field operators are introduced. In the dipole approximation it has the form

\[ H = -\hbar^{-1}\hat{\vartheta}, \]

\[ \hat{\vartheta} = \sum_{m,l} \sqrt{\frac{\hbar \omega_m}{2\varepsilon_0 a^3}} A_{ml} \exp(-i\omega_m t + iml) d_l - h.c. \]  

(1)

In 11 local field operators are represented by packets of plain waves

\[ A_{ml} = \frac{1}{\sqrt{M}} \sum_{k \sim m} a_k \exp(-i(\omega_k - \omega_m)t + i(k - m)l), \]

(2)

which wave vectors \( k \sim m \) lie in a band \( m_s - \pi/a \leq k_s < m_s + \pi/a, \Delta k_s = 2\pi/L, s = x, y, z, \) where \( L^3 \) is a normalized volume, which modes are described by operators \( a_k, a_k^\dagger \) \( [a_k; a_k^\dagger] = \delta_{kk'} \). Packets are localized in a space sell \( a^3 = (L/M)^3 \), which position is given by vector \( l \). The local field operators obey commutation relations \( [A_{ml}, A_{ml'}^\dagger] = \delta_{mm'}\delta_{ll'} \) and describe creation and annihilation of photon in a point \( l \). In hamiltonian of interaction 9 atoms are presented by their dipole momentum operator \( d_l \), that is a total momentum of all atoms inside a sell. Assuming for simplicity, that there is only one atom in any of the \( M \) space cells, one finds, that \( d_l \) is the dipole momentum of single atom, located in \( l \) and summing over \( l \) is summing over atoms.

Consider three atomic ensembles \( A, B, C \) spatially separated and consisted of the \( \Lambda \)-atoms with a ground level \( 0 \) and exited levels \( 1 \) and \( 2 \) as it is shown in fig.1. Assuming a Raman type interaction of strong classical field with atomic transition \( 0 \rightarrow 2 \), that results in week waves scattered from \( 2 \rightarrow 1 \). We shall describe light scattered through the introduced local field operators. Then hamiltonian of interaction can be obtained from 10, where summation over \( l \), or atoms, is rewritten as a sum over ensembles and a sum over atoms inside of each ensemble. We shall neglect spatial behavior of all fields inside any ensemble, assuming, that light interacts with such family of atoms.

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as a whole. Then Hamiltonian takes the form

\[ H = -i\hbar^{-1} \sum_{x=A,B,C} \partial_x, \]

\[ \partial_x = \Omega(S_{20}(x) \exp(-i(\omega - \omega_{20})t + ikl_x) - h.c.) \]

\[ + \sum_m \epsilon_m A_{mx} \exp(-i(\omega_m - \omega_{2m})t + iml_x) S_{21}(x) - h.c.), \]

where \( \Omega \) is normalized amplitude of the classical field of frequency \( \omega \) and wave vector \( k \), \( \epsilon_m = d\sqrt{\hbar\omega_m/2\epsilon_0a^3} \), \( d \) is a dipole momentum, \( S_{pq}(x) = \sum_{a \in x} |p\rangle_a \langle q| \), \( p, q = 0, 1, 2 \) is atomic operator of the ensemble \( x \), which position is \( 1_x \). In (3), local operators \( A_{mx}, A_{mx}^\dagger \) describe creation and annihilation of photon from the ensemble \( x \) or in point \( l_x \), where \( m \) is the photon wave vector. The obtained Hamiltonian allows to consider evolution of atoms and light scattered.

For simplicity we will take into account only single mode toward resonance scattering. It results in one-dimension problem, where \( \omega \approx \omega_{20}, \omega_m \approx \omega_{21} \) and \( k \approx m \). In this approximation Hamiltonian (3) reads

\[ \partial_x = \Omega(S_{20}(x) \exp(i kl_x) - h.c.) \]

\[ + \epsilon(A_x \exp(i kl_x) S_{21}(x) - h.c.). \]

Let the initial total state be a vacuum in a sense that all atoms are in its ground levels and all photons are in vacuum state

\[ \Psi_0 = |000\rangle_{ABC} \otimes |0\rangle_f, \]

where \( |000\rangle_{ABC} = |0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C, |0\rangle_x = |0\rangle \otimes \cdots \otimes |0\rangle = |0\rangle^\otimes N_x, N_x \) is total number of atoms of an ensemble \( x = A, B, C, |0\rangle_f \) is the field vacuum.

**III. W-STATES**

Collection of the light Fock states \( \{|n_1,n_2,n_3\rangle\} \) is a complete set and suitable to describe a measurement including three single-photon detectors. For considering evolution of wave function due from Hamiltonian (4), we will focus on three terms of the expansion of wave function due from Hamiltonian (4), we will focus on three single-photon detectors. For considering evolution of atoms and light scattered.

Using the theory of perturbation over interaction in the first non-vanishing order coefficients in \( \Psi_{(1)} \) take the form

\[ \Psi_{(1)} = \sum_{x,y=A,B,C} |1_x, 1_y\rangle \Psi(1_x, 1_y). \]

and

\[ \Psi(1_x) = \langle 1_x | S | \Psi_0 \rangle \]

\[ = -(1/2)(\hbar t^2)^2 \Omega e \langle 1_x | A_x^2 S_{10}(x) | \Psi_0 \rangle. \]

Also we find

\[ \Psi(1_x, 1_y) = \langle 1_x, 1_y | S | \Psi_0 \rangle \]

\[ = (3/4)(\hbar t^2)^2 \langle 1_x | A_x^2 S_{10}(x) | 1_x \rangle \Psi(1_x, 1_y) \]

and

\[ \Psi(2_x) = \langle 2_x | S | \Psi_0 \rangle \]

\[ = (3/4)(\hbar t^2)^2 \langle 2_x | A_x^2 S_{10}(x) | 2_x \rangle \Psi(2_x) \]

Two points maybe made about it. First, any atomic states \( S_{10}^m(x)|0\rangle_x, e = 1, 2, m = 1, \ldots, N_x \) are known as Dicke states, described \( m \) excited atoms from \( N_x \). They read

\[ S_{10}^m(x)|0\rangle_x = \sqrt{C_{N,m}} |m, N\rangle, \]

where we omit the ensemble index \( x \), and the normalized ket is introduce

\[ |m, N\rangle = 1/\sqrt{C_{N,m}} \sum_z P_z |1\rangle^\otimes m \otimes |0\rangle^\otimes (N-m), \]

where \( P_z \) is set of \( C_{N,m} = N!/m!(N - m)! \) distinguished permutations of particles. For particular case \( m = 1, 2 \) one finds the states of the W-class

\[ W_1 = |1, N\rangle \]

\[ = 1/\sqrt{N}(|110\ldots0\rangle + |011\ldots0\rangle + \cdots + |00\ldots1\rangle), \]

\[ W_2 = |2, N\rangle \]

\[ = \sqrt{2/N(N-1)}(|110\ldots0\rangle + |011\ldots0\rangle + \cdots + |00\ldots11\rangle). \]

Second, in Eq. (9), (10) and (11) the Fock states of light \( A_x^0 |0\rangle_f \), \( A_x^2 |0\rangle_f \) and \( A_x^2 |0\rangle_f \) are localized states field, that describe one photon, emitted from atomic ensemble \( x \), two photons each of which from \( x \) and \( y \), and a pair of quanta from \( x \). Indeed, they look as W-states. The reason is that in accordance with (2) the local operators \( A_x \) is a superposition of \( M \) operators \( a_k \)

\[ A_x^0 |0\rangle_f = \exp(i m l_x) |\bar{W}\rangle, \]

(16)
where
\[
|\tilde{W}\rangle = \frac{1}{\sqrt{M}} \sum_{k=-m}^{M} \exp(-i k \ell_x) a_k^\dagger |0\rangle
\]
\[
= e^{i\zeta} \sum_{q=0}^{M-1} C_q |0\rangle \otimes |1\rangle \otimes (M-q-1). \tag{17}
\]
In Eq (17) the phase factor and coefficients read ζ = (π/2 - m)ℓ_x, and C_q = exp(-i 2π q k_x / L) / \sqrt{M}. Does the state |\tilde{W}\rangle belong to the W-class? Note, if M ≫ 1, then in (17) sum of coefficients C_q is
\[
\sum_q C_q \propto \exp(i y) \sin(y) / y, \quad y = π L / a.
\]
This sum is equal to zero, if y ≠ 0, then maybe it is a reason, that the considered state is close to the ZSA-states, introduced in ref. [8].

As result the Ψ(11) takes the form
\[
\Psi_{(11)} = -(t^2 \hbar^{-4} \Omega k/2)(\sqrt{N_A} |100\rangle_f \otimes |W_100\rangle + \sqrt{N_B} |010\rangle_f \otimes |W_010\rangle + \sqrt{N_C} |001\rangle_f \otimes |00W_1\rangle), \tag{18}
\]
where |W_1\rangle is given by (14). This is a superposition state that is obtained after a photon is emitted from one of the three ensembles. Equation (13) tells, if an ensemble emits one photon, then their atoms are prepared in the W-state. Also, when two photons arise from two ensembles, one concludes, that both ensembles are in the W-state. It is described by the wave function Ψ_{(1,1)}, that has the form
\[
\Psi_{(1,1)} = (t^2 \hbar^{-4} \Omega k)^{2/3} 4! (\sqrt{N_A N_B} |110\rangle_f \otimes |W_1 W_0\rangle + \sqrt{N_A N_C} |101\rangle_f \otimes |W_0 W_1\rangle + \sqrt{N_B N_C} |011\rangle_f \otimes |0W_1 W_1\rangle). \tag{19}
\]
Atomic states of the W_2 type are presented in Ψ_{(2)}, that reads
\[
\Psi_{(2)} = (t^2 \hbar^{-4} \Omega k)^{2/3} 4! (\sqrt{N_A(N_A - 1)} |200\rangle_f \otimes |W_200\rangle + \sqrt{N_B(N_B - 1)} |020\rangle_f \otimes |0W_20\rangle + \sqrt{N_C(N_C - 1)} |002\rangle_f \otimes |00W_2\rangle), \tag{20}
\]
where |W_2\rangle is given by (15).

**IV. PROJECTION MEASUREMENT**

Assuming a measurement performing on light scattered by atoms, where three detectors D_x, x = A, B, C collect photons, that come from ensembles x, so that detector D_A can’t see any photons from B and C etc. It follows from [13], that for example, the state |W_100\rangle can be prepared, when outcome of the measurement is |100\rangle, or there is a click of the detector D_A. This state, achieved with probability \(\text{Prob}(100) = (t^2 \hbar^{-4} \Omega k/2)^2 N_A\), is separable with respect to atomic ensembles. However a three-ensemble entanglement can be also prepared.

Consider a detection scheme, shown in fig.1. It includes two beamsplitters BS_1 and BS_2 with transmittances and reflectances c_k, s_k, c_k^2 + s_k^2 = 1, k = 1, 2. Let photons coming from the A, B and C ensembles are mixed by these beamsplitters before measuring, when photon from A and B are inputs of the first beamsplitter and the photon from C is mixed with one of the outputs of the first beamsplitter. Assuming, that beamsplitter can be described by Hamiltonian \(H = i \hbar k (a^\dagger b - ab^\dagger)\), where a and b are photonic operators, k is a coupling constant, we find, that BS_1 and BS_2 transform the single-photon Fock-state as
\[
a |100\rangle + b |010\rangle + c |001\rangle \rightarrow \phi = a |c_1 |100\rangle - s_1 (c_2 |010\rangle + s_2 |001\rangle) + b |s_1 |100\rangle + c_1 (c_2 |010\rangle - s_2 |001\rangle) + c |s_2 |010\rangle + c_2 |001\rangle). \tag{21}
\]
Indeed, if b = c = 0 outgoing is a state of the W-class. When \(c_1 = -s_1 = 1/\sqrt{2}\) and \(c_2 = \sqrt{2/3}\) output state of light to be measured by detectors takes the form
\[
\phi = 1/\sqrt{2}(a - b) |100\rangle + 1/\sqrt{3}(a + b + c) |010\rangle + (-a/\sqrt{6} - b/\sqrt{6} + c\sqrt{2/3}) |001\rangle. \tag{22}
\]
Using this transformation, one finds

\[ \Psi_{\{1\}} \rightarrow \Psi'_{\{1\}} = -(t^2 h^{-4} \Omega / 2) \]

\[ \langle 100 \rangle_f \otimes (\sqrt{N_A} |W_{100} \rangle - \sqrt{N_B} |0W_{1} \rangle) / \sqrt{2} \]

\[ + |010 \rangle_f \otimes (\sqrt{N_A} |W_{100} \rangle + \sqrt{N_B} |0W_{1} \rangle \]

\[ + \sqrt{N_C} |00W_{1} \rangle) / \sqrt{3} \]

\[ |001 \rangle_f \otimes (\sqrt{N_A} |W_{100} \rangle + \sqrt{N_B} |0W_{1} \rangle \]

\[ + 2 \sqrt{N_C} |00W_{1} \rangle) / \sqrt{6} \]. (23)

It follows from (23), that the states of EPR and W-class can be prepared. Suppose, any of ensembles has the same number of atoms \( N \). When outcome \( |100 \rangle \) or \( |010 \rangle \) is obtained, atomic ensembles are prepared in the states \( (|W_{100} \rangle - |0W_{1} \rangle) / \sqrt{2} \) and \( (|W_{100} \rangle + |0W_{1} \rangle + |00W_{1} \rangle) / \sqrt{3} \). It can be done with probability \( \text{Prob}(100) = \text{Prob}(010) = (t^2 h^{-4} \Omega / 2)^2 N \).

The similar way results in the W-state of the form \( (|W_{100} \rangle + |0W_{1} \rangle) / \sqrt{3} \). It needs \( c_1 = s_1 = c_2 = s_2 \) and outcome to be \( |101 \rangle \).

Manipulating number of atoms or another parameter a state of the form \( 1 / \sqrt{2} (|W_{100} \rangle + \sqrt{2} |0 \rangle \Psi^+) \), where \( \Psi^+ = (|W_{1} \rangle + |0W_{1} \rangle) / \sqrt{2} \), can be achieved. This three-partite entanglement of the W-class is unitary equivalent to the state of the GHZ-class and hence it is sufficient as a quantum channel for perfect teleportation [7].

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