Schurifying quasi-hereditary algebras

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Abstract
We study new classes of quasi-hereditary and cellular algebras which generalize Turner’s double algebras. Turner’s algebras provide a local description of blocks of symmetric groups up to derived equivalence. Our general construction allows one to “schurify” any quasi-hereditary algebra $A$ to obtain a generalized Schur algebra $S_A(n,d)$ which we prove is again quasi-hereditary if $d \leq n$. We describe decomposition numbers of $S_A(n,d)$ in terms of those of $A$ and the classical Schur algebra $S(n,d)$. In fact, it is essential to study schurifications of superalgebras $A$, in which case the construction of the schurification involves a non-trivial full rank sub-lattice $T_A^\alpha(n,d) \subseteq S_A(n,d)$.

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1 | INTRODUCTION

The goal of this paper is to define new quasi-hereditary algebras, and hence new highest weight categories, from old. Starting with a quasi-hereditary algebra $A$, we obtain a generalized Schur algebra $S_A^A(n,d)$ which we prove is again quasi-hereditary if $d \leq n$. The procedure of passing from $A$ to $S_A^A(n,d)$ is sometimes referred to as “schurification” of $A$. We describe decomposition numbers of $S_A^A(n,d)$ in terms of those of $A$ and the classical Schur algebra $S(n,d)$. In fact, it is essential to study schurifications of superalgebras, in which case the construction involves a non-trivial choice of a sub-lattice $T_A^\alpha(n,d) \subseteq S_A(n,d)$. In the purely even case we have $T_A^\alpha(n,d) = S_A^A(n,d)$.

To describe our results more precisely, let $\k$ be a commutative domain of characteristic 0, $A$ be a $\k$-superalgebra, and $\alpha \subseteq A_0$ be a subalgebra. The associated generalized Schur algebra $T_A^\alpha(n,d)$
was defined in [13] as a certain full sublattice in the algebra of (super)invariants:

\[ T^A_a(n, d) \subseteq S^A(n, d) := (M_n(A) \otimes d) \otimes_d. \]

Extending scalars to a field \( \mathbb{k} \) of characteristic 0 produces the same algebras: \( T^A_a(n, d)_{\mathbb{k}} = S^A(n, d)_{\mathbb{k}} \). However, importantly, extending scalars to a field \( \mathbb{F} \) of positive characteristic will in general yield non-isomorphic algebras \( T^A_a(n, d)_{\mathbb{F}} \) and \( S^A(n, d)_{\mathbb{F}} \) of the same dimension. It turns out that in many situations it is the more subtly defined algebra \( T^A_a(n, d)_{\mathbb{F}} \) that plays an important role.

In [12], we defined the notion of a based quasi-hereditary algebra. If \( \mathbb{k} \) is a complete local Noetherian ring, then a \( \mathbb{k} \)-algebra is based quasi-hereditary if and only if it is split quasi-hereditary in the sense of Cline, Parshall, and Scott [2].

The first main result of this paper is that under some reasonable assumptions, the algebra \( T^A_a(n, d) \) is based quasi-hereditary if \( A \) is based quasi-hereditary and \( d \leq n \). This is proved by generalizing Green’s work in [8] for the classical Schur algebra \( S^k(n, d) \). Green constructs a basis of codeterminants, and then proves (in effect) that this gives \( S^k(n, d) \) the structure of a based quasi-hereditary algebra. Similarly, we define a set of generalized codeterminants and prove that these form a basis for \( T^A_a(n, d) \). This gives \( T^A_a(n, d) \) the structure of a based quasi-hereditary algebra.

Given a ring homomorphism \( \mathbb{k} \to \mathbb{F} \), where \( \mathbb{F} \) is a field of arbitrary characteristic, we define a quasi-hereditary \( \mathbb{F} \)-algebra by extending scalars:

\[ T^A_a(n, d)_{\mathbb{F}} := T^A_a(n, d) \otimes_{\mathbb{k}} \mathbb{F}. \]

Our second main result describes (under some constraints) the decomposition numbers of standard \( T^A_a(n, d)_{\mathbb{F}} \)-modules in terms of those of \( A_{\mathbb{F}} \), the classical Schur algebra \( S(n, d)_{\mathbb{F}} \), and Littlewood–Richardson coefficients.

Our third main result describes conditions on \( A \) under which \( T^A_a(n, d) \) is known to be indecomposable, allowing for a classification of the blocks of \( T^A_a(n, d) \) in most cases.

Our motivation comes from Turner’s double algebras [5, 19–21], which arise as schurifications \( T^Z_{\delta}(n, d) \) of zigzag superalgebras \( Z \). As conjectured in [21] and proved in [6], Turner’s algebras \( T^Z_{\delta}(n, d) \) can be considered as a “local” object replacing wreath products of Brauer tree algebras in the context of the Broué abelian defect group conjecture for blocks of symmetric groups with non-abelian defect groups. We expect that various versions of generalized Schur algebras will be appearing in local descriptions of blocks of group algebras and other algebras arising in classical representation theory.

As an application of the general techniques developed in this paper, we construct an explicit cellular basis of \( T^Z_{\delta}(n, d) \). To achieve this goal, we first construct quasi-hereditary algebras \( T^Z_{\delta}(n, d) \), where \( Z \) is a quasi-hereditary cover of \( \tilde{Z} \) known as extended zigzag superalgebra. Special cases of the main results of this paper describe the quasihereditary structure and decomposition numbers of \( T^Z_{\delta}(n, d) \). We then use an idempotent truncation technique to describe a cellular structure and the corresponding decomposition numbers of \( T^Z_{\delta}(n, d) \). We formulate an explicit conjecture for RoCK blocks of classical Schur algebras in terms of the generalized Schur algebras \( T^Z_{\delta}(n, d) \), see Conjecture 7.61.

We now describe the contents of the paper in more detail. In Section 2 we recall the definition and basic results on based quasi-hereditary superalgebras \( A \) with heredity data \( I, X, Y \). In Section 3 we set up the combinatorics of colored alphabets and describe the poset of \( I \)-colored multipartitions \( \Lambda^I_+(n, d) \). For any \( \lambda \in \Lambda^I_+(n, d) \), we define the sets \( \text{Std}^X(\lambda) \) and \( \text{Std}^Y(\lambda) \) of \( X \)-colored
and $Y$-colored standard tableaux. In Section 4 we recall the definition and basic results on the
generalized Schur algebra $T^A_d(n,d)$, and prove some multiplication lemmas.

In Section 5, we define generalized codeterminants. For every $\lambda \in \Lambda^I_d(n,d)$, $S \in \text{Std}^X(\lambda)$,
$T \in \text{Std}^Y(\lambda)$, we define the corresponding codeterminant $B^\lambda_{S,T}$ as a product $\lambda_S \lambda_T$ of certain
elements of $T^A_d(n,d)$. We show that the codeterminants form a basis for $T^A_d(n,d)$, using a generalization of Woodcock’s “straightening” argument in [22] to prove spanning, and the super RSK correspondence to show independence.

In Section 6, we prove the first main theorem of the paper, which appears as Theorem 6.6.

**Theorem 1.** Let $d \leq n$ and $A$ be a based quasi-hereditary $\mathbb{k}$-superalgebra with $a$-conforming
heredity data $I, X, Y$. Then $T^A_d(n,d)$ is a based quasi-hereditary $\mathbb{k}$-superalgebra with heredity data
$\Lambda^I_d(n,d), X, Y$.

We then go on to describe the standard modules $\Delta(\lambda)$ over $T^A_d(n,d)$, as well as idempotent
truncations and involutions of these algebras. In the final Section 7 we focus on decomposition
numbers of standard modules over $T^A_d(n,d)_{\mathbb{F}}$. We define a certain explicit set $\Lambda^D_+(n)$ of multipartitions depending on the decomposition matrix $D$ of $A_{\mathbb{F}}$, an explicit statistics $\text{deg}$ on $\Lambda^D_+(n)$, certain classical Littlewood–Richardson coefficients $c^{(i)}_{\lambda \nu}$ and $c^{(i)}_{\nu \lambda}$, and products $d_{\gamma,\mu}^{\text{cl}}$ of decomposition numbers for the classical Schur algebra over $\mathbb{F}$. Then our second main theorem, which appears as
Theorem 7.51, is as follows ($\mathcal{J}(A)$ stands for the Jacobson radical of $A$).

**Theorem 2.** Suppose that $(A, a)$ is a unital pair and assume that $A_{\mathcal{J}} \subset \mathcal{J}(A)$. Then for $\lambda, \mu \in \Lambda^I_d(n,d)$, the corresponding graded decomposition number is given by

$$d_{\lambda,\mu} = \sum_{\gamma \in \Lambda^I_d(n)} \sum_{\nu \in \Lambda^D_+(n)} d_{\gamma,\mu}^{\text{cl}} \text{deg}(\nu) \left( \prod_{i \in I} c^{(i)}_{\lambda \nu} c^{(i)}_{\nu \lambda} \right).$$

After specialization $A := Z$ and appropriate idempotent truncation, Theorem 2 yields the formula of Turner [21, Corollary 134], cf. [3, Theorem 6.2], [14, Corollary 10], [10, Theorem 4.1], see
Remark 7.60 for further comments. As an application of Theorem 2, we prove our third main
theorem, which appears as Theorem 7.54 in the body of the paper.

**Theorem 3.** Suppose that $(A, a)$ is a unital pair and $A_{\mathcal{J}} \subset \mathcal{J}(A)$. Moreover, suppose that $A$ is indecomposable, and $|I| > 1$. Then $T^A_d(n,d)$ is indecomposable.

This, coupled with the decomposition result described in Lemma 7.53, allows one to classify
the blocks of $T^A_d(n,d)$ in many cases. In fact, we prove in Theorem 7.54 a slightly stronger result,
giving indecomposability conditions for $T^A_d(n,d)$, where $(\bar{A}, \bar{a})$ is an idempotent truncation of
the unital pair $(A, a)$.

## 2 | QUASI-HEREDITARY ALGEBRAS

Throughout the paper $\mathbb{k}$ is a commutative domain of characteristic 0.
2.1 Based quasi-hereditary algebras

We begin by reviewing theory of quasi-hereditary algebras in the language of [12].

Let \( V = \bigoplus_{n \in \mathbb{Z}, \varepsilon \in \mathbb{Z}/2} V^n \) be a graded \( \mathbb{k} \)-supermodule. We set \( V^n := V^n_0 \oplus V^n_1 \) and \( V_\varepsilon := \bigoplus_{n \in \mathbb{Z}} V^n_\varepsilon \). An element \( v \in V \) is called homogeneous if \( v \in V^n_\varepsilon \) for some \( \varepsilon \) and \( n \). For \( \varepsilon \in \mathbb{Z}/2, n \in \mathbb{Z} \) and a set \( S \) of homogeneous elements of \( V \), we write

\[
S_\varepsilon := S \cap V_\varepsilon \quad \text{and} \quad S^n_\varepsilon := S \cap V^n_\varepsilon. \tag{2.1}
\]

Let

\[
R := \mathbb{Z}[q, q^{-1}][t]/(t^2 - 1), \tag{2.2}
\]

and denote the image of \( t \) in the quotient ring by \( \pi \), so that \( \pi^\varepsilon \) makes sense for \( \varepsilon \in \mathbb{Z}/2 \). For \( v \in V^n_\varepsilon \), we write

\[
\deg(v) := q^n \pi^\varepsilon. \tag{2.3}
\]

For \( v \in V_\varepsilon \), we also write \( \bar{v} := \varepsilon \). A map \( f : V \to W \) of graded \( \mathbb{k} \)-supermodules is called \textit{homogeneous} if \( f(V^n_\varepsilon) \subseteq W^n_\varepsilon \) for all \( m \) and \( \varepsilon \).

For a free \( \mathbb{k} \)-module \( W \) of finite rank \( d \), we write \( d = \dim W \). A graded \( \mathbb{k} \)-supermodule \( V \) is free of finite rank if each \( V^n_\varepsilon \) is free of finite rank and we have \( V^n = 0 \) for almost all \( n \). Let \( V \) be a free graded \( \mathbb{k} \)-supermodule of finite rank. A \textit{homogeneous basis} of \( V \) is a \( \mathbb{k} \)-basis all of whose elements are homogeneous. The \textit{graded dimension} of \( V \) is

\[
\dim_{q^\pi} V := \sum_{n \in \mathbb{Z}, \varepsilon \in \mathbb{Z}/2} (\dim V^n_\varepsilon)q^n \pi^\varepsilon \in R. \]

A (not necessarily unital) \( \mathbb{k} \)-algebra \( A \) is called a \textit{graded \( \mathbb{k} \)-superalgebra}, if \( A \) is a graded \( \mathbb{k} \)-supermodule and \( A^n A^m \subseteq A^{n+m} \) for all \( \varepsilon, \delta \) and \( n, m \). By a graded \( A \)-supermodule we understand an \( A \)-module \( V \) which is a graded \( \mathbb{k} \)-supermodule and \( A^n V^m \subseteq V^{n+m} \) for all \( \varepsilon, \delta \) and \( n, m \). We denote by \( A-\text{mod} \) the category of all finitely generated graded \( A \)-supermodules and homogeneous \( A \)-homomorphisms. All ideals, submodules, and so on, are assumed to be homogeneous. Given \( V \in A-\text{mod} \), \( n \in \mathbb{Z} \) and \( \varepsilon \in \mathbb{Z}/2 \), we denote by \( q^n \pi^\varepsilon V \) the graded \( A \)-supermodule which is the same as \( V \) as an \( A \)-module but with \( (q^n \pi^\varepsilon V)_m^\delta = V^{m-n}_{\delta+\varepsilon} \).

For a partially ordered set \( I \) and \( i \in I \), we let

\[
I^{>i} := \{ j \in I \mid j > i \} \quad \text{and} \quad I^{\geq i} := \{ j \in I \mid j \geq i \}. \tag{2.4}
\]

\textbf{Definition 2.5.} Let \( A \) be a graded \( \mathbb{k} \)-superalgebra. \textit{Heredity data} on \( A \) consist of a finite partially ordered set \( I \) and finite sets \( X = \bigsqcup_{i \in I} X(i) \) and \( Y = \bigsqcup_{i \in I} Y(i) \) of homogeneous elements of \( A \) with distinguished \textit{initial elements} \( e_i \in X(i) \cap Y(i) \) for each \( i \in I \). For \( i \in I \), we set

\[
A^{\geq i} := \text{span}(x y \mid j \in I^{\geq i}, x \in X(j), y \in Y(j)).
\]
We require that the following axioms hold:

(a) \( B := \{ xy \mid i \in I, x \in X(i), y \in Y(i) \} \) is a basis of \( A \) with cardinality \( \sum_{i \in I} |X(i)| \cdot |Y(i)| \);
(b) For all \( i \in I, x \in X(i), y \in Y(i) \) and \( a \in A \), we have
\[
ax \equiv \sum_{x' \in X(i)} l^x_{x'}(a)x' \pmod{A^{>i}} \quad \text{and} \quad ya \equiv \sum_{y' \in Y(i)} r^y_{y'}(a)y' \pmod{A^{>i}}
\]
for some \( l^x_{x'}(a), r^y_{y'}(a) \in \mathbb{k} \);
(c) For all \( i, j \in I \) and \( x \in X(i), y \in Y(i) \) we have
\[
xe_i = x, \quad e_i x = \delta_{x,e_i} x, \quad e_i y = y, \quad ye_i = \delta_{y,e_i} y
\]
\[
e_j x = x \text{ or } 0, \quad ye_j = y \text{ or } 0.
\]

If \( A \) is endowed with heredity data \( I, X, Y \), we call \( A \) based quasi-hereditary (with respect to the poset \( I \)), and refer to \( B \) as a heredity basis of \( A \).

Remark 2.6. The notion of a based quasi-hereditary algebra is closely related to that of a split quasi-hereditary algebra developed in \([2, 4, 18]\) for algebras over an arbitrary Noetherian commutative unital ring \( \mathbb{k} \). In fact, if \( \mathbb{k} \) is complete local Noetherian, which is sufficient for most applications, the two notions are equivalent. We refer the reader to \([12]\) for the proof of a slightly stronger statement.

We now record some basic results on a based quasi-hereditary algebra \( A \) with heredity data as in Definition 2.5. The proofs can be found in \([12]\). Denote
\[
B(i) := \{ xy \mid x \in X(i), y \in Y(i) \} \quad (i \in I).
\]

**Lemma 2.8** \([12, \text{Lemmas } 2.5, 2.6]\). If \( \Omega \) is an upper set in \( I \), then
\[
A(\Omega) := \text{span} \left( \bigcup_{i \in \Omega} B(i) \right)
\]
is an ideal in \( A \). Moreover, if \( \Theta \) is another upper set in \( I \), we have \( A(\Omega)A(\Theta) \subseteq A(\Omega \cap \Theta) \).

**Lemma 2.10** \([12, \text{Lemmas } 2.7, 2.8]\). Let \( i, j \in I \) and \( x \in X(i), y \in Y(i) \).

(i) \( e_i e_j = \delta_{i,j} e_i \)
(ii) If \( j \notin i \), then \( e_j x = ye_j = 0 \).
(iii) \( yx \equiv f_i(y, x)e_i \pmod{A^{>i}} \) for some \( f_i(y, x) \in \mathbb{k} \) with \( f_i(e_i, e_i) = 1 \) and \( f_i(y, x) = 0 \) unless \( \deg(x) \deg(y) = 1 \).

Fix \( i \in I \) and denote \( \bar{A} := A/A^{>i}, \bar{a} := a + A^{>i} \in \bar{A} \) for \( a \in A \). By inflation, \( \bar{A} \)-modules will be automatically considered as \( A \)-modules. The standard module \( \Delta(i) \) and the right standard module \( \Delta^{\text{op}}(i) \) are defined as \( \Delta(i) := \bar{A} \bar{e}_i \) and \( \Delta^{\text{op}}(i) := \bar{e}_i \bar{A} \). We have that \( \Delta(i) \) and \( \Delta^{\text{op}}(i) \) are free \( \mathbb{k} \)-modules with bases \( \{ v_x \mid x \in X(i) \} \) and \( \{ w_y \mid y \in Y(i) \} \), respectively, and the actions
\[
au_x = \sum_{x' \in X(i)} l^x_{x'}(a)v_{x'} \quad \text{and} \quad w_y a = \sum_{y' \in Y(i)} r^y_{y'}(a)w_{y'} \quad (a \in A).
\]
In particular, denoting \( v_i := v_{e_i} \), we have \( e_j v_i = v_{e_j} \) \( e_j \Delta(i) \neq 0 \) implies \( j \leq i \), and for all \( x \in X(i) \) we have \( x v_i = v_x, e_i v_x = \delta_{x,0} v_x \). We have a bilinear pairing \( (\cdot, \cdot)_i : \Delta(i) \times \Delta^\op(i) \to k \) satisfying \( (v_x, w_y)_i = f_{i}(y, x) \) with \( \text{rad} \Delta(i) \) being a submodule of \( \Delta(i) \).

Let \( k \) be a field. Then \( L(i) := \Delta(i)/\text{rad} \Delta(i) \) is an irreducible \( A \)-module and

\[
\{q^n \pi^\varepsilon L(i) \mid i \in I, n \in \mathbb{Z}, \varepsilon \in \mathbb{Z}/2\}
\]

is a complete set of non-isomorphic irreducible graded \( A \)-supermodules. Recalling the ring \( R \) from (2.2), the bigraded decomposition numbers

\[
[\Delta(i) : L(j)]_{q,\pi} = d_{i,j}(q, \pi) := \sum_{n \in \mathbb{Z}, \varepsilon \in \mathbb{Z}/2} d_{i,j}^{n,\varepsilon} q^n \pi^\varepsilon \in R \quad (i, j \in I),
\]

are determined from

\[
d_{i,j}^{n,\varepsilon} := [\Delta(i) : q^n \pi^\varepsilon L(j)] \quad (n \in \mathbb{Z}, \varepsilon \in \mathbb{Z}/2).
\]

Then \( d_{ii}(q, \pi) = 1 \), and \( d_{i,j}(q, \pi) \neq 0 \) implies \( j \leq i \).

### 2.2 Additional properties of quasi-hereditary algebras

Let \( A \) be a based quasi-hereditary \( k \)-superalgebra with heredity data \( I, X, Y \). We continue reviewing results from [12] that will be needed later.

A homogeneous anti-involution \( \tau \) on \( A \) is called standard if for all \( i \in I \) there is a bijection \( X(i) \xrightarrow{\sim} Y(i), x \mapsto y(x) \) such that \( y(e_i) = e_i \) and \( \tau(x) = y(x) \). For a standard anti-involution \( \tau \), we have \( \tau(xy(x')) = x'y(x) \) for all \( i \in I, x, x' \in X(i) \). If \( \tau \) is a standard anti-involution on \( A \), then \( B \) is a cellular basis of \( A \) with respect to \( \tau \).

If \( e \in A \) is a homogeneous idempotent, we consider the idempotent truncation \( \tilde{A} := e A e \), and denote \( \tilde{a} := e a e \in \tilde{A} \) for \( a \in A \). We say that \( e \) is adapted (with respect to \( I, X, Y \)) if for all \( i \in I \) there exist subsets \( \tilde{X}(i) \subseteq X(i) \) and \( \tilde{Y}(i) \subseteq Y(i) \) such that for all \( i \in I \) and \( x \in X(i), y \in Y(i) \) we have:

\[
ex = \begin{cases} x & \text{if } x \in \tilde{X}(i), \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad ye = \begin{cases} y & \text{if } y \in \tilde{Y}(i), \\ 0 & \text{otherwise}. \end{cases}
\]

Set \( \tilde{I} := \{ i \in I \mid \tilde{X}(i) \neq \emptyset \neq \tilde{Y}(i) \} \). We refer to \( \tilde{I}, \tilde{X}, \tilde{Y} \) as the \( e \)-truncation of \( I, X, Y \). We say that \( e \) is strongly adapted (with respect to \( I, X, Y \)) if it is adapted and \( e e_i = e_i e = e_i \) for all \( i \in \tilde{I} \).

**Lemma 2.13** [12, Lemma 4.4]. Let \( e \in A \) be an adapted idempotent.

(i) If \( \tau \) is a standard anti-involution of \( A \) such that \( \tau(e) = e \), then \( \{ xy \mid i \in \tilde{I}, x \in \tilde{X}(i), y \in \tilde{Y}(i) \} \) is a cellular basis of \( \tilde{A} \) with respect to the restriction \( \tau|_{\tilde{A}} \).

(ii) If \( e \) is strongly adapted, then \( \tilde{A} \) is based quasi-hereditary with heredity data \( \tilde{I}, \tilde{X} := \bigsqcup_{i \in \tilde{I}} \tilde{X}(i), \tilde{Y} := \bigsqcup_{i \in \tilde{I}} \tilde{Y}(i) \).
Lemma 2.14 [12, Lemma 4.7]. Let $\mathbb{k}$ be a field, and $e \in A$ be an adapted idempotent.

(i) $eL(i) = 0$ if and only if $e\Delta(i) \subseteq \text{rad} \Delta(i)$.

(ii) $eL(i) = 0$ if and only if $yex \in A^{>i}$ for all $x \in X(i)$ and $y \in Y(i)$.

(iii) $eL(i) = 0$ if and only if $yix \in A^{>i}$ for all $x \in X(i)$ and $y \in Y(i)$.

(iv) $eL(i) = 0$ for all $i \in I \setminus I$. In particular, there exists a subset $I' \subseteq I$ such that $\{eL(i) \mid i \in I'\}$ is a complete and irredundant set of irreducible $A$-modules up to isomorphism.

We now turn to more subtle additional properties of heredity data, which have to do with the super-structure. Symbols $X_0, Y_0$ are understood in the sense of (2.1).

Definition 2.15. Suppose that $\mathfrak{a} \subseteq A_0$ is a subalgebra. The heredity data $I, X, Y$ of $A$ are $\mathfrak{a}$-conforming if $I, X_0, Y_0$ is heredity data for $\mathfrak{a}$. If, in addition, $A$ is unital and $\mathfrak{a}$ is a unital subalgebra, that is, $1_\mathfrak{a} = 1_A$, we call $(A, \mathfrak{a})$ a unital pair.

If the heredity data $I, X, Y$ of $A$ are $\mathfrak{a}$-conforming, then $\mathfrak{a}$ is recovered as $\mathfrak{a} = \text{span}(xy \mid i \in I, x \in X(i)_0, y \in Y(i)_0)$, so sometimes we will just speak of conforming heredity data. Even though in some sense $\mathfrak{a}$ is redundant in the definition of conformity, it is often convenient to use it. For example, we will deal with generalized Schur algebras $T^A_{\mathfrak{a}}(n, d)$, which only depend on $A$ and $\mathfrak{a}$, but not on $I, X, Y$.

Lemma 2.16 [12, Corollary 2.24]. Suppose that $\mathbb{k}$ is a local ring with the maximal ideal $\mathfrak{m}$ and the quotient field $\mathbb{F} = \mathbb{k}/\mathfrak{m}$. Then:

(i) $A/\mathfrak{m}A \cong A \otimes_{\mathbb{k}} \mathbb{F}$ is based quasi-hereditary $\mathbb{F}$-superalgebra.

(ii) For each $i \in I$, denote the corresponding canonical irreducible $A/\mathfrak{m}A$-module by $L_{A/\mathfrak{m}A}(i)$ and denote by $L_A(i)$ the $A$-module obtained from $L_{A/\mathfrak{m}A}(i)$ by inflation. Then

$$\{q^n\pi\varepsilon L_A(i) \mid i \in I, n \in \mathbb{Z}, \varepsilon \in \mathbb{Z}/2\}$$

is a complete and irredundant set of irreducible graded $A$-supermodules up to a homogeneous isomorphism.

If $\mathbb{k}$ is a local ring, we call $A$ basic if the modules $L_{A/\mathfrak{m}A}(i)$ are 1-dimensional as $\mathbb{F}$-vector spaces, equivalently if the modules $L_A(i)$ are free of rank 1 as $\mathbb{k}$-modules.

If the heredity data $I, X, Y$ of $A$ are $\mathfrak{a}$-conforming, then by definition $\mathfrak{a}$ is also based quasi-hereditary and has its own standard $\mathfrak{a}$-modules $\Delta_\mathfrak{a}(i)$ and simple $\mathfrak{a}$-modules $L_\mathfrak{a}(i)$. Recall the definition of the ideals $A(\Omega)$ from (2.9). The following theorem is proved in [12, Theorem 4.13].

Theorem 2.17. Let $\mathbb{k}$ be local and $A$ be a based quasi-hereditary graded $\mathbb{k}$-superalgebra with $\mathfrak{a}$-conforming heredity data $I, X, Y$. Suppose that $(A, \mathfrak{a})$ is a unital pair. Then there exists a-conforming heredity data $I, X', Y'$ with the same ideals $A(\Omega)$ and $\mathfrak{a}(\Omega)$ and such that the new initial elements $\{e'_i \mid i \in I\}$ are primitive idempotents in a satisfying $e_i e'_i = e'_i e_i = e_i^2$ and $e'_i \equiv e_i \pmod{\mathfrak{a}^{>i}}$ for all $i \in I$. Moreover, setting $f := \sum_{i \in I} e'_i, \tilde{A} := fAf$ and $\tilde{\mathfrak{a}} := fA\mathfrak{a}f$, we have the following.

(i) $f$ is strongly adapted with respect to $(I, X', Y')$, so that $\tilde{A}$ is based quasi-hereditary with heredity data $(I, X', Y')$.

(ii) $(I, X', Y')$ is $\tilde{\mathfrak{a}}$-conforming;

(iii) $\tilde{\mathfrak{a}}$ is basic and if $A_F \subseteq J(A)$ then $\tilde{A}$ is a basic as well;
(iv) the functors

\[ F_A : A\text{-mod} \to \tilde{A}\text{-mod}, \ V \mapsto fV \quad \text{and} \quad F_a : a\text{-mod} \to \tilde{a}\text{-mod}, \ V \mapsto fV \]

are equivalences of categories, such that

\[ F_A(L_A(i)) \cong L_{\tilde{A}}(i), \quad F_A(\Delta_A(i)) \cong \Delta_{\tilde{A}}(i), \quad F_a(L_a(i)) \cong L_{\tilde{a}}(i), \quad F_a(\Delta_a(i)) \cong \Delta_{\tilde{a}}(i). \]

3 | COMBINATORICS

We fix \( n \in \mathbb{Z}_{>0}, d \in \mathbb{Z}_{\geq 0}, \) and a based quasi-hereditary graded \( k \)-superalgebra \( A \) with \( a \)-conforming heredity data \( I, X, Y \) and the corresponding heredity basis \( B = \bigsqcup_{i \in I} B(i). \) Let

\[ B_a := \{xy \mid i \in I, \ x \in X(i)_0, \ y \in Y(i)_0\} \quad \text{and} \quad B_i := \{xy \mid i \in I, \ x \in X(i)_1, \ y \in Y(i)_1\}, \]

so that

\[ B = B_a \sqcup B_i \sqcup B_1. \quad (3.1) \]

We now review the theory developed in [13] following [5].

Without loss of generality, we assume that

\[ I = \{0, 1, \ldots, \ell\} \]

with the total order

\[ 0 < 1 < \ldots < \ell \quad (3.2) \]

refining the fixed partial order on \( I. \)

3.1 | Compositions and partitions

We set \( N := \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}, \quad N^I := I \times \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \) and refer to the elements of \( N \) and \( N^I \) as nodes. Define a partial order \( \leq \) on \( N \) and \( N^I \) as follows: \( (r, s) \leq (r', s') \) if and only if \( r \leq r' \) and \( s \leq s' \), and

\[ (i, r, s) \leq (i', r', s') \] if and only if \( i = i', r \leq r' \) and \( s \leq s'. \quad (3.3) \]

We denote by \( \Lambda(n) \) (respectively, \( \Lambda(I) \)) the set of all compositions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) (respectively, \( (\lambda_0, \lambda_1, \ldots, \lambda_\ell) \)) with non-negative integer parts. For such a composition \( \lambda \), we denote by \( |\lambda| \) the sum of its parts, and set

\[ \Lambda(n, d) := \{\lambda \in \Lambda(n) \mid |\lambda| = d\}, \quad \Lambda(I, d) := \{\lambda \in \Lambda(I) \mid |\lambda| = d\}. \]
We denote by $0 \in \Lambda(n,0)$ the composition with all zero parts. The Young diagram of $\lambda \in \Lambda(n,d)$ is

$$[\lambda] := \{(r,s) \in \mathbb{N} \mid s \leq \lambda_r\}.$$ 

We define $\Lambda^I(n,d)$ to be the set of tuples $\lambda = (\lambda^{(0)},\lambda^{(1)},\ldots,\lambda^{(\ell)})$ of compositions $\lambda^{(i)} \in \Lambda(n)$ such that $|\lambda| := \sum_{i \in I} |\lambda^{(i)}| = d$. For $\lambda \in \Lambda^I(n,d)$, we set

$$||\lambda|| := (|\lambda^{(0)}|,\ldots,|\lambda^{(\ell)}|) \in \Lambda(I,d).$$ 

Let $\lambda = (\lambda^{(0)},\ldots,\lambda^{(\ell)}) \in \Lambda^I(n,d)$. We denote by

$$[\lambda] = [\lambda^{(0)}] \sqcup \cdots \sqcup [\lambda^{(\ell)}] \subset \mathbb{N}^I$$

the Young diagram of $\lambda$, where

$$[\lambda^{(i)}] = \{(i,r,s) \in \mathbb{N}^I \mid s \leq \lambda^{(i)}_r\} \quad (i \in I).$$ 

Let $\preceq$ be the usual *dominance partial order* on $\Lambda(n,d)$, that is,

$$\lambda \preceq \mu \quad \text{if and only if} \quad \sum_{r=1}^{s} \lambda_r \leq \sum_{r=1}^{s} \mu_r \quad \text{for all} \quad s = 1,\ldots,n.$$ 

We denote by $\preceq_I$ the following partial order on $\Lambda(I,d)$:

$$\lambda \preceq_I \mu \quad \text{if and only if} \quad \sum_{j \geq i} \lambda_j \leq \sum_{j \geq i} \mu_j \quad \text{for all} \quad i \in I.$$ (3.4)

We denote by $\preceq$ the partial order on $\Lambda^I(n,d)$ defined as follows:

$$\lambda \preceq \mu \quad \text{if and only if} \quad \text{either} \quad ||\lambda|| <_I ||\mu|| \quad \text{or} \quad ||\lambda|| = ||\mu|| \quad \text{and} \quad \lambda^{(i)} \preceq \mu^{(i)} \quad \text{for all} \quad i \in I.$$ (3.5)

We label the nodes of $[\lambda]$ with numbers $1,\ldots,d$ going from left to right along the rows, starting with the first row of $[\lambda^{(0)}]$, then going along the second row of $[\lambda^{(0)}]$, and so on until the $n$th row of $[\lambda^{(0)}]$, then along the first row of $[\lambda^{(1)}]$, the second row of $[\lambda^{(1)}]$, and so on. For $1 \leq k \leq d$, we denote the $k$th node of $\lambda$ by $N_k(\lambda)$.

Let $i \in I$ and $d_i := |\lambda^{(i)}|$. We can also label the nodes of $[\lambda^{(i)}]$ with numbers $1,\ldots,d_i$ going from left to right along the rows, starting with the first row of $[\lambda^{(i)}]$, then going along the second row of $[\lambda^{(i)}]$, and so on. For $1 \leq k \leq d_i$, we denote the $k$th node of $\lambda^{(i)}$ by $N_k(\lambda^{(i)})$. Note that $N_k(\lambda^{(i)}) = N_{k+d_0+\ldots+d_{i-1}}(\lambda)$ for $1 \leq k \leq d_i$.

The *row stabilizer* of $\lambda^{(i)}$ is the subgroup $\mathfrak{S}_{\lambda^{(i)}} \leq \mathfrak{S}_{d_i}$, consisting of all $\sigma \in \mathfrak{S}_{d_i}$ such that for all $k = 1,\ldots,d_i$ we have that $N_k(\lambda^{(i)}), N_{\sigma k}(\lambda^{(i)})$ are in the same row of $[\lambda^{(i)}]$. The *row stabilizer* of $\lambda$ is the subgroup $\mathfrak{S}_\lambda \leq \mathfrak{S}_d$, consisting of all $\sigma \in \mathfrak{S}_d$ such that for all $k = 1,\ldots,d$ we have that $N_k(\lambda), N_{\sigma k}(\lambda)$ are in the same row of some component $[\lambda^{(i(k))}]$. We have $\mathfrak{S}_\lambda \cong \mathfrak{S}_{\lambda^{(0)}} \times \cdots \times \mathfrak{S}_{\lambda^{(\ell)}}$. 
If \( \lambda \in \Lambda(n) \) and \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \), we say that \( \lambda \) is a partition and write \( \lambda \in \Lambda_+(n) \). If \( \lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(i)}) \in \Lambda^i(n, d) \) is such that each \( \lambda^{(i)} \) is a partition, we say that \( \lambda \) is a multiset and write \( \lambda \in \Lambda^i_+(n, d) \). If \( \mu \in \Lambda(n) \), there is a unique partition \( \mu_+ \in \Lambda_+(n, d) \) obtained from \( \mu \) by permuting its parts.

### 3.2 Words and triples

Let \( d \in \mathbb{Z}_{\geq 0} \). For a set \( S \), the elements of \( S^d \) are referred to as words (of length \( d \)). The words are usually written as

\[
s_1 s_2 \cdots s_d \in S^d.
\]

For \( s \in S^d \) and \( t \in S^c \) we denote by \( st \in S^{d+c} \) the concatenation of \( s \) and \( t \). For \( s \in S \), we denote \( s^d := s \cdots s \in \mathbb{Z}^d \).

The symmetric group \( \mathfrak{S}_d \) acts on the right on \( S^d \) by place permutations:

\[
(s_1 \cdots s_d) \sigma = s_{\sigma 1} \cdots s_{\sigma d}.
\]

For \( s, t \in S^d \), we write \( s \sim t \) if \( s \) and \( t \) are in the same \( \mathfrak{S}_d \)-orbit.

More generally, if \( S_1, \ldots, S_k \) are sets, then \( \mathfrak{S}_d \) acts on \( S_1^d \times \cdots \times S_k^d \) diagonally:

\[
(s^1, \ldots, s^k) \sigma = (s^1 \sigma, \ldots, s^k \sigma).
\]

We write

\[
(s^1, \ldots, s^k) \sim (t^1, \ldots, t^k)
\]

if \( (s^1, \ldots, s^k) \) and \( (t^1, \ldots, t^k) \) are in the same \( \mathfrak{S}_d \)-orbit. If \( S \subseteq S_1^d \times \cdots \times S_k^d \) is a \( \mathfrak{S}_d \)-invariant subset, we denote by \( S/\mathfrak{S}_d \) a complete set of the \( \mathfrak{S}_d \)-orbit representatives in \( S \) and we identify \( S/\mathfrak{S}_d \) with the set of all \( \mathfrak{S}_d \)-orbits on \( S \).

Let \( P = P_0 \sqcup P_1 \) be a set of non-zero homogeneous elements of the algebra \( A \), and \( \text{Tri}^P(n, d) \) be the set of all triples

\[
(\mathbf{p}, \mathbf{r}, \mathbf{s}) = (p_1 \cdots p_d, r_1 \cdots r_d, s_1 \cdots s_d) \in P^d \times [1, n]^d \times [1, n]^d
\]

such that for any \( 1 \leq k \neq l \leq d \) we have \( (p_k, r_k, s_k) = (p_l, r_l, s_l) \) only if \( p_k \in P_0 \). The diagonal \( \mathfrak{S}_d \)-action on \( P^d \times [1, n]^d \times [1, n]^d \) preserves \( \text{Tri}^P(n, d) \) so that we can choose the corresponding set \( \text{Tri}^P(n, d)/\mathfrak{S}_d \) of \( \mathfrak{S}_d \)-orbit representatives and identify it with the set of all \( \mathfrak{S}_d \)-orbits on \( \text{Tri}^P(n, d) \) as in the previous paragraph.

We fix a total order “\(<\)” on \( P \times [1, n] \times [1, n] \). Then we also have a total order on \( \text{Tri}^P(n, d) \) defined as follows: \( (\mathbf{p}, \mathbf{r}, \mathbf{s}) < (\mathbf{p}', \mathbf{r}', \mathbf{s}') \) if and only if there exists \( l \in [1, d] \) such that \( (p_k, r_k, s_k) = (p'_k, r'_k, s'_k) \) for all \( k < l \) and \( (p_l, r_l, s_l) < (p'_l, r'_l, s'_l) \). Denote

\[
\text{Tri}^P_0(n, d) = \{(\mathbf{p}, \mathbf{r}, \mathbf{s}) \in \text{Tri}^P(n, d) \mid (\mathbf{p}, \mathbf{r}, \mathbf{s}) \leq (\mathbf{p}, \mathbf{r}, \mathbf{s}) \sigma \text{ for all } \sigma \in \mathfrak{S}_d\}.
\]
For \((p, r, s) \in \text{Tri}^B(n,d), p' \in P^d\) and \(\sigma \in \mathfrak{S}_d\), we define
\[
\langle p, r, s \rangle := \# \{(k, l) \in [1, d]^2 | k < l, p_k, p_l \in P_1, (p_k, r_k, s_k) > (p_l, r_l, s_l)\},
\]
\[
\langle p, p' \rangle := \# \{(k, l) \in [1, d]^2 | k > l, p_k, p'_l \in P_1\},
\]
\[
\langle \sigma; p \rangle := \# \{(k, l) \in [1, d]^2 | k < l, \sigma^{-1}k > \sigma^{-1}l, p_k, p_l \in P_1\}.
\]

Let \((b, r, s) \in \text{Tri}^B(n,d)\). For \(b \in B\) and \(r, s \in [1, n]\), we denote
\[
[b, r, s]_b^{\, r,s} := \# \{k \in [1, d] | (b_k, r_k, s_k) = (b, r, s)\}, \tag{3.7}
\]
\[
[b, r, s]^! := \prod_{b \in B, r, s \in [1, n]} [b, r, s]_b^{\, r,s}!, \quad [b, r, s] ! := \prod_{b \in B, r, s \in [1, n]} [b, r, s]_b^{\, r,s}! \tag{3.8}
\]
\[
[b, r, s]_a^{\, r,s} := \prod_{b \in B_a, r, s \in [1, n]} [b, r, s]_b^{\, r,s}!, \quad [b, r, s]_c^{\, r,s} := \prod_{b \in B_c, r, s \in [1, n]} [b, r, s]_b^{\, r,s}! \tag{3.9}
\]

### 3.3 Colored Letters and Tableaux

For \(r, s \in \mathbb{Z}\) we denote \([r, s] := \{ t \in \mathbb{Z} | r \leq t \leq s\}\). We introduce colored alphabets
\[
\mathcal{A}_X := [1, n] \times X \quad \text{and} \quad \mathcal{A}_{X(i)} := [1, n] \times X(i),
\]
so that \(\mathcal{A}_X = \bigsqcup_{i \in I} \mathcal{A}_{X(i)}\). The colored alphabets \(\mathcal{A}_Y\) and \(\mathcal{A}_{Y(i)}\) are defined similarly. We think of elements of \(\mathcal{A}_X\) as \(X\)-colored letters, and often write \(l^x\) instead of \((l, x) \in \mathcal{A}_X\). If \(L = l^x \in \mathcal{A}_X\), we denote
\[
\text{let}(L) := l \quad \text{and} \quad \text{col}(L) := x.
\]

For all \(i \in I\), we fix arbitrary total orders “\(<\)” on the sets \(\mathcal{A}_{X(i)}\) which satisfy \(r^x < s^x\) if \(r < s\) (in the standard order on \([1, n]\)). Similarly we fix total orders on the sets \(\mathcal{A}_{Y(i)}\) with \(r^y < s^y\) if \(r < s\).

All definitions of this subsection which involve \(X\) have obvious analogs for \(Y\). Let \(\lambda = (\lambda^{(i)}_0, \ldots, \lambda^{(i)}_d) \in \Lambda^I(n,d)\). Fix \(i \in I\) and let \(d_i := |\lambda^{(i)}_0|\).

An \(X(i)\)-colored \(\lambda^{(i)}\)-tableau is a function \(T : [\lambda^{(i)}] \to \mathcal{A}_{X(i)}\) such that the following condition holds:

- if \(M \neq N\) are nodes in the same row of \([\lambda^{(i)}]\), then \(T(M) = T(N)\) implies \(\text{col}(T(M)) \in X(i)_0\).

We denote the set of all \(X(i)\)-colored \(\lambda^{(i)}\)-tableaux by \(\text{Tab}^{X(i)}(\lambda^{(i)})\).

Recall the partial order \((3.3)\) on the nodes of \(\lambda^{(i)}\) and a fixed total order on \(\mathcal{A}_{X(i)}\). Let \(T \in \text{Tab}^{X(i)}(\lambda^{(i)})\). Then \(T\) is called row standard if the following condition holds.

- If \(M < N\) are nodes in the same row of \([\lambda^{(i)}]\), then \(T(M) \leq T(N)\).

On the other hand, \(T\) is called column standard if the following condition holds.

- If \(M < N\) are nodes in the same column of \([\lambda^{(i)}]\), then \(T(M) \leq T(N)\) and the equality is allowed only if \(\text{col}(T(M)) \in X(i)_1\).
Finally, \( T \) is called \textit{standard} if it is both row and column standard. Denote
\[
\text{Rst}^{X(i)}(\lambda(i)) := \{ T \in \text{Tab}^{X(i)}(\lambda(i)) \mid T \text{ is row standard} \},
\]
\[
\text{Cst}^{X(i)}(\lambda(i)) := \{ T \in \text{Tab}^{X(i)}(\lambda(i)) \mid T \text{ is column standard} \},
\]
\[
\text{Std}^{X(i)}(\lambda(i)) := \{ T \in \text{Tab}^{X(i)}(\lambda(i)) \mid T \text{ is standard} \}.
\]
Recalling the idempotents \( e_i \in X(i) \cap Y(i) \), the \textit{initial} \( \lambda(i) \)-tableau \( T^{\lambda(i)} \) is
\[
T^{\lambda(i)} : [\lambda(i)] \rightarrow \mathcal{A}_X, \ (i, r, s) \mapsto r^i.
\]
Note that \( T^{\lambda(i)} \) is in both \( \text{Std}^{X(i)}(\lambda(i)) \) and \( \text{Std}^{Y(i)}(\lambda(i)) \).
Tableaux \( S, T \in \text{Tab}^{X(i)}(\lambda(i)) \) are called \textit{row equivalent} if there exists \( \sigma \in \mathfrak{S}_{\lambda(i)} \) such that for all \( k = 1, \ldots, d_i \), we have \( S_k = T_{\sigma(k)} \). The following is clear:

\textbf{Lemma 3.10.} For every \( T \in \text{Tab}^{X(i)}(\lambda(i)) \), there exists a unique \( S \in \text{Rst}^{X(i)}(\lambda(i)) \) which is row equivalent to \( T \).

For a function \( T : [\lambda] \rightarrow \mathcal{A}_X \) and \( i \in I \), we set \( T^{(i)} := T |_{[\lambda^{(i)}]} \) to be the restriction of \( T \) to \( [\lambda^{(i)}] \subseteq [\lambda] \). We write \( T = (T^{(0)}, \ldots, T^{(I)}) \), keeping in mind that the restrictions \( T^{(i)} \) determine \( T \) uniquely. An \( X \)-\textit{colored} \( \lambda \)-tableau is a function \( T : [\lambda] \rightarrow \mathcal{A}_X \) such that the restrictions \( T^{(i)} \) are \( X(i) \)-colored \( \lambda^{(i)} \)-tableaux for all \( i \in I \). We denote the set of all \( X \)-colored \( \lambda \)-tableaux by \( \text{Tab}^X(\lambda) \).

Let \( T \in \text{Tab}^X(\lambda) \). Then \( T \) is called \textit{row standard} (respectively, \textit{column standard}, \textit{standard}) if so are all the \( T^{(i)} \) for \( i = 0, \ldots, I \). We use the notation \( \text{Rst}^X(\lambda), \text{Cst}^X(\lambda) \) and \( \text{Std}^X(\lambda) \) to denote the sets of all row standard, column standard, and standard \( X \)-colored \( \lambda \)-tableaux, respectively. For example, we have the \textit{initial} \( \lambda \)-tableau \( T^{\lambda} = (T^{\lambda(0)}, \ldots, T^{\lambda(I)}) \in \text{Std}^X(\lambda) \cap \text{Std}^Y(\lambda) \). We denote
\[
T_k := T(N_k(\lambda)) \in \mathcal{A}_X \quad (1 \leq k \leq d), \tag{3.14}
\]
\[
L^T := T_1 \cdots T_d \in \mathcal{A}^d_X, \tag{3.15}
\]
\[
L^{\lambda} := L^{T^{\lambda}} = L^{\lambda(0)} \cdots L^{\lambda(I)}. \tag{3.16}
\]
Tableaux \( S, T \in \text{Tab}^X(\lambda) \) are called \textit{row equivalent} if there exists \( \sigma \in \mathfrak{S}_\lambda \) such that for all \( k = 1, \ldots, d \), we have \( S_k = T_{\sigma(k)} \). The following is clear:

\textbf{Lemma 3.17.} For every \( T \in \text{Tab}^X(\lambda) \), there exists a unique \( S \in \text{Rst}^X(\lambda) \) which is row equivalent to \( T \).
The notions introduced in this section generalize the classical notion of a standard tableau which we now recall. Given \( \lambda \in \Lambda_+(n, d) \), a classical \( \lambda \)-tableau is a function \( T : [\lambda] \to [1, n] \). A classical \( \lambda \)-tableau \( T \) is called standard if whenever \( M < N \) are nodes in the same row of \( [\lambda] \), then \( T(M) \leq T(N) \), and whenever \( M < N \) are nodes in the same column of \( [\lambda] \), then \( T(M) < T(N) \).

Recall the notation introduced in (2.7). Let

\[
\text{Std}_2(I, n, d) := \{ (\lambda, S, T) \mid \lambda \in \Lambda_+^I(n, d), S \in \text{Std}^X(\lambda), T \in \text{Std}^Y(\lambda) \}.
\]

**Lemma 3.18.** There is a bijection between the sets \( \text{Tri}^B(n, d)/\mathbb{S}_d \) and \( \text{Std}_2(I, n, d) \).

**Proof.** We first prove a one-color version of the claim. Fix \( i \in I \), and define

\[
\Lambda_+^i(n, d) := \{ \lambda \in \Lambda_+^I(n, d) \mid \lambda^{(j)} = \delta_{i,j}\lambda^{(i)} \text{ for all } j \in I \}.
\]

In other words, \( \Lambda_+^i(n, d) \) is the subset of multipartitions concentrated in the \( i \)th component. The colored alphabets \( \mathcal{A}_X(i) \) and \( \mathcal{A}_Y(i) \), with orders chosen in §3.3, are alphabets in the terminology of [15]. The set of signed two-row arrays on \( \mathcal{A}_X(i) \) and \( \mathcal{A}_Y(i) \) described in [15, Definition 4.1] can be seen to be in bijection with \( \text{Tri}^B(i)(n, d)/\mathbb{S}_d \), via the assignment

\[
\begin{bmatrix}
r_1^{x_1} & r_2^{x_2} & \cdots & r_d^{x_d} \\
s_1^{y_1} & s_2^{y_2} & \cdots & s_d^{y_d}
\end{bmatrix} \mapsto ((x_1y_1) \cdots (x_dy_d), r_1 \cdots r_d, s_1 \cdots s_d)\mathbb{S}_d.
\]

It is proved in [15, Theorem 4.2] (translating the main result of [1] from the context of the fourfold algebra) that the set of signed two-row arrays on \( \mathcal{A}_X(i) \) and \( \mathcal{A}_Y(i) \) is in bijection with the set (in the language of [15]) of pairs of same-shape super semistandard Young tableaux on \( \mathcal{A}_X(i) \) and \( \mathcal{A}_Y(i) \). In view of [15, Definition 2.2] and §3.3, this latter set is in bijection with

\[
\text{Std}_2(i, n, d) := \{ (\lambda, S, T) \mid \lambda \in \Lambda_+^i(n, d), S \in \text{Std}^X(i)(\lambda^{(i)}), T \in \text{Std}^Y(i)(\lambda^{(i)}) \}.
\]

Thus, for all \( i \in I \) and \( d \in \mathbb{Z}_{\geq 0} \), there is a bijection between \( \text{Tri}^B(i)(n, d)/\mathbb{S}_d \) and \( \text{Std}_2(i, n, d) \).

Now note that \( \text{Tri}^B(n, d)/\mathbb{S}_d \) is in bijection with the set

\[
\bigcup_{d_0, \ldots, d_r \in \mathbb{Z}_{\geq 0}} \left( \prod_{i \in I} \text{Tri}^B(i)(n, d_i)/\mathbb{S}_{d_i} \right),
\]

and restriction of tableaux gives a bijection between \( \text{Std}_2(I, n, d) \) and

\[
\bigcup_{d_0, \ldots, d_r \in \mathbb{Z}_{\geq 0}} \left( \prod_{i \in I} \text{Std}_2(i, n, d_i) \right),
\]

so the bijection in the general case follows from the one-color case. \( \square \)
4 | GENERALIZED SCHUR ALGEBRAS

We continue to work with a fixed $d \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{>0}$, and based quasi-hereditary graded $k$-superalgebra $A$ with $a$-conforming heredity data $I, X, Y$ and the corresponding heredity basis

\[ B = B_a \sqcup B_c \sqcup B_1 = \bigsqcup_{i \in I} B(i), \]

as in (3.1) and (2.7). Define the structure constants $\kappa_{a,c}^b$ of $A$ from

\[ ac = \sum_{b \in B} \kappa_{a,c}^b b \quad (a, c \in A). \quad (4.1) \]

More generally, for $b = b_1 \cdots b_d \in B^d$ and $a = a_1 \cdots a_d$, $c = c_1 \cdots c_d \in A^d$, we define

\[ \kappa_{a,c}^b := \kappa_{a_1,c_1}^{b_1} \cdots \kappa_{a_d,c_d}^{b_d}. \quad (4.2) \]

Throughout the section, we also use the following notation.

- $H$ denotes the set of all non-zero homogeneous elements of $A$.
- For $0 \leq c \leq d$ we denote by $(d-c,c) \not\in \mathbb{S}_d$ the set of the shortest coset representatives for $(\mathbb{S}_{d-c} \times \mathbb{S}_c) \setminus \mathbb{S}_d$. More generally for a composition $\delta = (d_1, \ldots, d_q)$, we have the standard parabolic subgroup

\[ \mathbb{S}_\delta := \mathbb{S}_{d_1} \times \cdots \times \mathbb{S}_{d_q} \leq \mathbb{S}_d \]

and the set $\delta \not\in \mathbb{S}_d$ of the shortest coset representatives for $\mathbb{S}_\delta \setminus \mathbb{S}_d$.

4.1 | The algebras $S^A(n, d)$ and $T^A_a(n, d)$

The matrix algebra $M_n(A)$ is naturally a superalgebra. For $r, s \in [1, n]$ and $a \in A$, we denote

\[ \xi_{r,s}^a := a E_{r,s} \in M_n(A). \]

There is a right action of $\mathbb{S}_d$ on $M_n(A)^{\otimes d}$ by (super)algebra automorphisms, such that for all $a_1, \ldots, a_d \in H$, $r_1, s_1, \ldots, r_d, s_d \in [1, n]$ and $\sigma \in \mathbb{S}_d$, we have

\[ (\xi_{r_1,s_1}^{a_1} \otimes \cdots \otimes \xi_{r_d,s_d}^{a_d})^\sigma = (-1)^{\langle a, r, s \rangle} \xi_{r_1,s_1}^{a_1} \otimes \cdots \otimes \xi_{r_d,s_d}^{a_d}. \]

The algebra $S^A(n, d)$ is defined as the algebra of invariants

\[ S^A(n, d) := (M_n(A)^{\otimes d})^{\mathbb{S}_d}. \]

For $(a, r, s) \in \text{Tri}_H(n, d)$, we define elements

\[ \xi_{r,s}^a := \sum_{(c,t,u)-(a,r,s)} (-1)^{\langle a,r,s \rangle + \langle c,t,u \rangle} \xi_{t_1,u_1}^{c_1} \otimes \cdots \otimes \xi_{t_d,u_d}^{c_d} \in S^A(n, d). \quad (4.3) \]

We will occasionally also use the alternative notation (in settings where we wish to avoid nested subscripts):

\[ \xi(a, r, s) := \xi_{r,s}^a. \quad (4.4) \]

By [13, Lemma 3.3], \{ $\xi_{r,s}^b$ \ | \ $(b, r, s) \in \text{Tri}^B(n, d)/\mathbb{S}_d$ \} is a basis of $S^A(n, d)$. 


Note from the definition, it is as follows:

**Lemma 4.5** [13, Lemma 3.4]. If \((a', r', s') \sim (a, r, s)\) are elements of \(\text{Tri}^H(n, d)\), then

\[
\xi_{a',r',s'} = (-1)^{\langle a, r, s \rangle + \langle a', r', s' \rangle} \xi_{a,r,s}.
\]

For \((a, p, q), (c, u, v) \in \text{Tri}^H(n, d)\) and \((b, r, s) \in \text{Tri}^B(n, d)\), the structure constants \(f_{a,p,q,c,u,v}^{b,r,s}\) are defined from

\[
\xi_{a,p,q} \xi_{c,u,v} = \sum_{(b,r,s) \in \text{Tri}^B(n,d)/\mathcal{S}_d} f_{a,p,q,c,u,v}^{b,r,s} \xi_{b,r,s}.
\]  \(\text{(4.6)}\)

**Proposition 4.7** [5, (6.14)]. Let \((a, p, q), (c, u, v) \in \text{Tri}^H(n, d)\) and \((b, r, s) \in \text{Tri}^B(n, d)\). Then

\[
f_{a,p,q,c,u,v}^{b,r,s} = \sum_{a',c',t} (-1)^{\langle a, p, q \rangle + \langle c, u, v \rangle + \langle a', r, t \rangle + \langle c', t, s \rangle + \langle b, r, s \rangle} \kappa_{b,a',c',c'},
\]

where the sum is over all \(a', c' \in H^d\) and \(t \in [1, n]\) such that \((a', r, t) \sim (a, p, q)\) and \((c', t, s) \sim (c, u, v)\).

We set

\[
\eta_{b,r,s}^b := [b, r, s]^t \xi_{r,s},
\]  \(\text{(4.8)}\)

and

\[
T^A_a(n,d) := \text{span} \left\{ \eta_{r,s}^b \mid (b, r, s) \in \text{Tri}^B(n, d) \right\}.
\]

This is a basis of \(T^A_a(n,d)\). It is proved in [13, Proposition 3.12] that \(T^A_a(n,d) \subseteq S^A(n,d)\) is a \(k\)-subalgebra. Moreover, it is a unital algebra if \((A, a)\) is a unital pair. Sometimes we call the algebra \(T^A_a(n,d)\) a generalized Schur (super)algebra.

**Proposition 4.9** [13, Proposition 4.11]. The algebra \(T^A_a(n,d)\) depends only on the subalgebra \(a\), and not on the choice of the basis \(B\).

**Lemma 4.10** [13, Lemma 3.10]. Let \(a_1, \ldots, a_d \in a \cup \hat{A}_1\) and \(r, s \in [1, n]^d\). Then \(\xi_{r,s}^a \in T^A_a(n,d)\).

**4.2 | Coproduct**

If \(T = (b, r, s) \in \text{Tri}^B(n, d)\), we write

\[
\xi_T := \xi_{b,r,s}, \quad \eta_T := \eta_{b,r,s}, \quad [T]^t_c := [b, r, s]^t_c, \quad T \sigma := (b, r, s)\sigma,
\]

and so on.

If \(d = d_1 + d_2\), \(T^1 = (b^1, r^1, s^1) \in \text{Tri}^B(n, d_1)\) and \(T^2 = (b^2, r^2, s^2) \in \text{Tri}^B(n, d_2)\), we denote

\[
T^1 T^2 := (b^1 b^2, r^1 r^2, s^1 s^2) \in B^d \times [1, n]^d \times [1, n]^d.
\]
Recall the notation (3.6), and let \( \mathcal{T} \in \text{Tri}_0^B(n, d) \). For \( 0 \leq l \leq d \) define
\[
\text{Spl}_l(\mathcal{T}) := \{ (\mathcal{T}^1, \mathcal{T}^2) \in \text{Tri}_0^B(n, l) \times \text{Tri}_0^B(n, d - l) \mid \mathcal{T}^1 \mathcal{T}^2 \sim \mathcal{T} \},
\]
and set \( \text{Spl}(\mathcal{T}) := \bigsqcup_{0 \leq l \leq d} \text{Spl}_l(\mathcal{T}) \). For \( (\mathcal{T}^1, \mathcal{T}^2) \in \text{Spl}(\mathcal{T}) \), let \( \sigma^\mathcal{T}_{\mathcal{T}^1, \mathcal{T}^2} \) be the unique element of \( \mathcal{T}^{(l,d-l)} \) such that \( \mathcal{T} \sigma^\mathcal{T}_{\mathcal{T}^1, \mathcal{T}^2} = \mathcal{T}^1 \mathcal{T}^2 \).

It is well known that \( \bigoplus_{d \geq 0} M_n(A)^{\otimes d} \) is a supercoalgebra with the coproduct
\[
\nabla : M_n(A)^{\otimes d} \to \bigoplus_{l=0}^{d} M_n(A)^{\otimes l} \otimes M_n(A)^{\otimes (d-l)}
\]
\[
\xi_1 \otimes \cdots \otimes \xi_d \mapsto \sum_{l=0}^{d} (\xi_1 \otimes \cdots \otimes \xi_l) \otimes (\xi_{l+1} \otimes \cdots \otimes \xi_d),
\]
see, for example, [5, §3.3]. Let
\[
S^A(n) := \bigoplus_{d \geq 0} S^A(n, d) \quad \text{and} \quad T_a^A(n) := \bigoplus_{d \geq 0} T_a^A(n, d).
\] (4.11)

**Lemma 4.12** [5, (6.12)] [13, Corollary 3.24]. If \( \mathcal{T} = (b, r, s) \in \text{Tri}_0^B(n, d) \) then
\[
\nabla(\xi^\mathcal{T}) = \sum_{(\mathcal{T}^1, \mathcal{T}^2) \in \text{Spl}(\mathcal{T})} (-1)^{\langle \sigma^\mathcal{T}_{\mathcal{T}^1, \mathcal{T}^2};b \rangle} \xi_{\mathcal{T}^1} \otimes \xi_{\mathcal{T}^2},
\]
\[
\nabla(\eta^\mathcal{T}) = \sum_{(\mathcal{T}^1, \mathcal{T}^2) \in \text{Spl}(\mathcal{T})} (-1)^{\langle \sigma^\mathcal{T}_{\mathcal{T}^1, \mathcal{T}^2};b \rangle} \frac{[\mathcal{T}^1]}{[\mathcal{T}^1][\mathcal{T}^2]} \eta_{\mathcal{T}^1} \otimes \eta_{\mathcal{T}^2},
\]
with \( \frac{[\mathcal{T}]}{[\mathcal{T}^1][\mathcal{T}^2]} \in \mathbb{Z} \). In particular, we have that \( S^A(n) \) and \( T_a^A(n) \) are sub-supercoalgebras of \( \bigoplus_{d \geq 0} M_n(A)^{\otimes d} \).

**4.3 Star-product**

Given \( \xi_1 \in M_n(A)^{\otimes d} \) and \( \xi_2 \in M_n(A)^{\otimes e} \), we define
\[
\xi_1 \ast \xi_2 := \sum_{\sigma \in \{(d,e)\}} (\xi_1 \otimes \xi_2)^\sigma.
\] (4.13)

This \( \ast \)-product makes \( \bigoplus_{d \geq 0} M_n(A)^{\otimes d} \) into an associative supercommutative superalgebra. Moreover,

**Lemma 4.14** [13, Corollary 4.4]. We have that \( S^A(n) \) and \( T_a^A(n) \) are subsuperalgebras of \( \bigoplus_{d \geq 0} M_n(A)^{\otimes d} \) with respect to the \( \ast \)-product. Moreover, with respect to the coproduct \( \nabla \) and the product \( \ast \), \( S^A(n) \) and \( T_a^A(n) \) are superbialgebras.

**Lemma 4.15** [13, Lemma 4.2]. For \( (b, r, s) \in \text{Tri}^B(n, d) \) and \( (c, t, u) \in \text{Tri}^B(n, e) \), we have
\[(i) \quad \xi_{b,r,s} \ast \xi_{c,t,u} = \frac{[bc,rt,su]}{[b,r,s][c,t,u]} \xi_{bc},
(ii) \quad \eta_{b,r,s} \ast \eta_{c,t,u} = \frac{[bc,rt,su]}{[b,r,s][c,t,u]} \eta_{bc},
\]

where \([bc,rt,su]\) and \([b,r,s][c,t,u]\) are integers, and the right-hand sides of (i) and (ii) are taken to be zero when \((bc,rt,su) \notin \text{Tri}^B(n, d + e)\).

There is a special case where we can guarantee that the coefficients in the right-hand sides of the expressions from Lemma 4.15 are equal to 1. To describe it, let \(q \in \mathbb{Z}_{>0}\) and \(\delta = (d_1, \ldots, d_q)\) be a composition of \(d\). Suppose that for each \(m = 1, \ldots, q\), we are given

\[(a^{(m)}, r^{(m)}, s^{(m)}), (c^{(m)}, t^{(m)}, u^{(m)}) \in \text{Tri}_H(n, d_m).\]

We write \(a^{(m)} = a_1^{(m)} \cdots a_{d_m}^{(m)}\), \(r^{(m)} = r_1^{(m)} \cdots r_{d_m}^{(m)}\), and so on. Let \(a = a^{(1)} \cdots a^{(q)}\), \(r = r^{(1)} \cdots r^{(q)}\), and so on. We write \(a = a_1 \cdots a_d\), \(r = r_1 \cdots r_d\), and so on. The triple \((a, r, s)\) is called \(\delta\)-separated if \(1 \leq m \neq l \leq q\) implies \((a_t^{(m)}, r_t^{(m)}, s_t^{(m)}) \neq (a_u^{(l)}, r_u^{(l)}, s_u^{(l)})\) for all \(1 \leq t \leq d_m\) and \(1 \leq u \leq d_l\). Note that we then automatically have \((a, r, s) \in \text{Tri}_H(n, d)\).

**Lemma 4.16** [13, Lemma 4.6]. If \((a, r, s)\) is \(\delta\)-separated, then

\[\xi_{a,r,s} = \xi_{a^{(1)},r^{(1)},s^{(1)}} \ast \cdots \ast \xi_{a^{(q)},r^{(q)},s^{(q)}} \quad \text{and} \quad \eta_{a,r,s} = \eta_{a^{(1)},r^{(1)},s^{(1)}} \ast \cdots \ast \eta_{a^{(q)},r^{(q)},s^{(q)}}.\]

**Lemma 4.17** [13, Lemma 4.7]. Let \((a, r, s)\) and \((c, t, u)\) be \(\delta\)-separated and suppose that

\[\left(\xi_{a^{(1)},r^{(1)},s^{(1)}} \otimes \cdots \otimes \xi_{a^{(q)},r^{(q)},s^{(q)}}\right) \left(\xi_{c^{(1)},t^{(1)},u^{(1)}} \otimes \cdots \otimes \xi_{c^{(q)},t^{(q)},u^{(q)}}\right) = 0\]

whenever \(\sigma\) and \(\sigma'\) are distinct elements of \(\delta \mathcal{D}\). Then

\[\xi_{a,r,s} \xi_{c,t,u} = \pm \left(\xi_{a^{(1)},r^{(1)},s^{(1)}} \xi_{c^{(1)},t^{(1)},u^{(1)}} \ast \cdots \ast \xi_{a^{(q)},r^{(q)},s^{(q)}} \xi_{c^{(q)},t^{(q)},u^{(q)}}\right).\]

Moreover, if \(a_1, \ldots, a_d\) or \(c_1, \ldots, c_d\) are all even, then the sign in the right-hand side is +.

### 4.4 Idempotents

Let \(\lambda \in \Lambda(n,d)\). Set

\[\mathbf{l}^\lambda := 1^d_1 \cdots n^d_n \in [1,n]^d.\]

If \(e \in A\) is an idempotent, define

\[\xi(\lambda,e) := \xi_{\mathbf{l}^\lambda,e}^d = \xi_{1^d_1,1^d_1}^{e_1} \ast \cdots \ast \xi_{n^d_n,n^d_n}^{e_n} \in S^A(n,d),\]

where the last equality follows from Lemma 4.16. The element \(\xi(\lambda,e)\) was denoted by \(\xi_{\lambda}^e\) in [13, §5A], where we have noted that it is an idempotent. Note using Lemma 4.10 that \(\xi(\lambda,e) \in T_\lambda^A(n,d)\) if \(e \in a\). If \(A\) is unital, we denote as \(\xi(\lambda) := \xi(\lambda,1_A)\). Then \(1_{S^A(n,d)} = \sum_{\lambda \in \Lambda(n,d)} \xi(\lambda)\) is an orthogonal idempotent decomposition. If the pair \((A, a)\) is unital, then \(\xi(\lambda) \in T_\lambda^A(n,d)\) for all \(\lambda \in \Lambda(n,d)\).
Let $\lambda = (\lambda^{(0)}, \ldots, \lambda^{(\ell)}) \in \Lambda^I(n, d)$ with $\|\lambda\| = (d_0, \ldots, d_\ell)$. We define
\begin{align*}
I^{(i)} := & I^{(0)} \cdots I^{(\ell)} \in [1, n]^d, \\
e_{\lambda} := & \xi_{\lambda^{(0)}, \lambda^{(0)}} \cdots \xi_{\lambda^{(\ell)}, \lambda^{(\ell)}} = \xi(\lambda^{(0)}, e_0) \ast \cdots \ast \xi(\lambda^{(\ell)}, e_\ell),
\end{align*}
where the last equality follows from Lemma 4.16. We have noted in [13, §5A] that $e_{\lambda}$ is an idempotent.

For $b \in B^d$ and $r \in [1, n]^d$, we define multicompositions $\alpha(b, r) = (\alpha^{(0)}, \ldots, \alpha^{(\ell)})$ and $\beta(b, r) = (\beta^{(0)}, \ldots, \beta^{(\ell)})$ in $\Lambda^I(n)$ via
\begin{align*}
\alpha^{(i)} (s) := & \#\{k \in [1, d] \mid r_k = s \text{ and } e_i b_k = b_k\}, \\
\beta^{(i)} (s) := & \#\{k \in [1, d] \mid r_k = s \text{ and } b_k e_i = b_k\}.
\end{align*}
(4.22)

Note that $\alpha(b, r) \in \Lambda^I(n, f)$ and $\beta(b, r) \in \Lambda^I(n, f')$ for some $0 \leq f, f' \leq d$. The following result follows easily from Lemma 4.17:

**Lemma 4.23.** Let $\lambda \in \Lambda^I(n, d)$ and $(b, r, s) \in \text{Tri}^B(n, d)$. Then
\begin{align*}
e_{\lambda} e_{b, r, s} := & \delta_{\lambda, \alpha(b, r)} e_{b, r, s}, \\
\epsilon_{r, s} e_{\lambda} := & \delta_{\lambda, \beta(b, r)} \epsilon_{r, s}.
\end{align*}
(4.23)

**4.5 | Multiplication lemma**

Throughout the subsection we fix $i \in I$. If $u = (u_1, \ldots, u_d) \in [1, n]^d$ and $x = x_1 \cdots x_d \in X(i)^d$, we denote
\begin{align*}
u^x := & u_1^{x_1} \cdots u_d^{x_d} \in \mathcal{A}_d X(i).
\end{align*}
Recall the total order “$<$” on $\mathcal{A}_d X(i)$ from §3.3. This total order induces the lexicographical order “$<$” on $\mathcal{A}_d X(i)$.

**Lemma 4.24.** Let $g, f \in \mathbb{Z}_{\geq 0}$ with $d = g + f$, and $r, s, t \in [1, n]$ with $s \neq t$. Set
\begin{align*}
y := & e_1^d, \quad r := r^d, \quad s := s^g, \quad t := t^f.
\end{align*}
Let
\begin{align*}
p := & p_1 \cdots p_g \in [1, n]^g, \quad q = q_1 \cdots q_f \in [1, n]^f, \\
x_1^1 = & x_1^1 \cdots x_g^1 \in X(i)^g, \quad x_1^2 = x_1^2 \cdots x_f^2 \in X(i)^f
\end{align*}
be such that $(x_1^1 x_1^2, pq, r) \in \text{Tri}^{X(i)}(n, d)$ and
\begin{align*}
q_1^1 \leq & \cdots \leq q_f^1 \leq p_1^1 \leq \cdots \leq p_g^1.
\end{align*}
Then \((x^1x^2, pq, st) \in \text{Tri}^B(i)(n, d)\) and
\[
\xi_{pq,r}^x y_{r,st} = \xi_{pq,st}^x + (*),
\]
where (*) is a linear combination of \(\xi_{u,st}^x\) with \(u^x \sim p^x q^x^2\) and \(u^x < p^x q^x^2\).

**Proof.** The property \((x^1x^2, pq, st) \in \text{Tri}^B(i)(n, d)\) easily follows from the assumption \((x^1x^2, pq, r) \in \text{Tri}^X(i)(n, d)\). By Proposition 4.7, we have
\[
\xi_{pq,r}^x y_{r,st} = \sum_{(b,u,v) \in \text{Tri}^B(n,d)/\mathfrak{S}_d} f_{b,u,v} \xi_{u,v}^b,
\]
for
\[
f_{b,u,v} = \sum_{a',c',t'} (-1)^{(x^1x^2,pq,r) + (yr,st) + (a',u,t') + (c',t',v) + (a',c')} \kappa_{a',c'},
\]
where the last sum is over all triples \((a', c', t') \in H^d \times H^d \times [1,n]^d\) such that \((a', u, t') \sim (x^1x^2, pq, r)\) and \((c', t', v) \sim (y, r, st)\). It follows that \(t' = r, c' = y\) and \(a' \in X(i)^d\). Observing that \(\kappa_{a',y} = \delta_{b,a'}\), we now deduce that
\[
f_{b,u,v} = \begin{cases} 0 & \text{if } v \not\sim st, \\ \sum_{a'} (-1)^{(x^1x^2,pq,r) + (a',u,r) \delta_{b,a'}} & \text{if } v \sim st, \end{cases}
\]
where the sum is over all \(a' \in X(i)^d\) such that \((a', u) \sim (x^1x^2, pq)\).

Since the sum in (4.25) is over orbit representatives, by the previous paragraph we may assume that \(v = st\) and \((b, u) \sim (x^1x^2, pq)\). Moreover, acting if necessary with the stabilizer \(\mathfrak{S}_g \times \mathfrak{S}_f\) of \(st\), we may assume that \(b = x_1 \cdots x_d\) for some permutation \((x_1, \ldots, x_d)\) of \((x_1^1, x_1^2, x_2^1, \ldots, x_2^d)\) such that \(u_{x_1}^1 \leq \cdots \leq u_{x_g}^g\) and \(u_{x_{g+1}}^{x_g+1} \leq \cdots \leq u_{x_d}^d\). It follows that \(u^x \sim p^x q^x^2\), \(u^x < p^x q^x^2\) and the equality \(u^x = p^x q^x^2\) is only possible if \((b, u, v) = (x^1x^2, pq, st)\). In the latter case, the formula in the previous paragraph yields \(f_{x^1x^2,pq,st}^x = 1\), completing the proof. \(\square\)

## 5 | CODETERMINANTS

We continue to work with a fixed \(d \in \mathbb{Z}_{\geq 0}\), \(n \in \mathbb{Z}_{>0}\), and based quasi-hereditary graded \(k\)-superalgebra \(A\) with a-conforming heredity data \(I,X,Y\) and the corresponding heredity basis \(B\). Fix \(\lambda = (\lambda^{(0)}, \ldots, \lambda^{(\ell)}) \in \Lambda^I(n, d)\).

### 5.1 | Single-colored codeterminants

Throughout the subsection we fix \(i \in I\) and set
\[
\mu := \lambda^{(i)}, \quad c := |\mu|.
\]
Recall the notation (2.7) and let
\[ T_a^A(n, c)_i := \text{span}(\eta^b_{p, q} \ | \ (b, p, q) \in \text{Tri}^B(i)(n, c)) \subseteq T_a^A(n, c). \]

Recall the combinatorial notions introduced in §3.3. Let \( S \in \text{Tab}^X(i)(\mu) \) and \( T \in \text{Tab}^Y(i)(\mu) \) with
\[ L^S = r_1^x \cdots r_c^x \in \mathcal{A}_X(i) \quad \text{and} \quad L^T = s_1^y \cdots s_c^y \in \mathcal{A}_Y(i), \]
see (3.12). We define
\[ x_S := x_1 \cdots x_c \in X(i)^c, \quad l^S := r_1 \cdots r_c \in [1, n]^c, \]
\[ y_T := y_1 \cdots y_c \in Y(i)^c, \quad l^T := s_1 \cdots s_c \in [1, n]^c. \]

For the initial \( \mu \)-tableau \( T^\mu \), set \( l^\mu := l^{\mu y} = 1^{\mu_1} \cdots n^{\mu_n} \). This agrees with (4.18). We now define
\[ \mathcal{X}_S := \mathcal{X}^{x_S}_{x_S^\mu}, \quad \mathcal{Y}_T := \mathcal{Y}^{y_T}_{y_T^\mu}, \quad B^\mu_{S, T} := \mathcal{X}_S \mathcal{Y}_T. \]

We refer to the elements \( B^\mu_{S, T} \) as codeterminants of color \( i \), cf. [8]. Note that \( X(i), Y(i) \subseteq B_a \cup B_1 \) so \( \mathcal{X}_S = \eta^x_{x, x} \in T_a^A(n, c) \) and \( \mathcal{Y}_T = \eta^y_{y, y} \in T_a^A(n, c) \). Therefore \( B^\mu_{S, T} \in T_a^A(n, c) \). Since \( xy \in B(i) \) whenever \( x \in X(i) \) and \( y \in Y(i) \), it now follows that \( B^\mu_{S, T} \in T_a^A(n, c) \).

We refer to \( \mu \) as the shape of the codeterminant \( B^\mu_{S, T} \). A codeterminant \( B^\mu_{S, T} \) is called dominant if \( \mu \in \Lambda_+\times(n, c) \), that is, if its shape is a partition. A codeterminant \( B^\mu_{S, T} \) is called standard if it is dominant and \( S \in \text{Std}^X(i)(\mu) \) and \( T \in \text{Std}^Y(i)(\mu) \).

**Lemma 5.1.** If \( S, S' \in \text{Tab}^X(i)(\mu) \) are row equivalent, then \( \mathcal{X}_S = \pm \mathcal{X}_{S'} \). If \( T, T' \in \text{Tab}^Y(i)(\mu) \) are row equivalent, then \( \mathcal{Y}_T = \pm \mathcal{Y}_{T'} \).

**Proof.** This follows from Lemma 4.5. \( \square \)

For \( w \in \mathfrak{S}_n \) and \( \nu = (\nu_1, \ldots, \nu_n) \in \Lambda(n) \), we define
\[ w\nu := (\nu_{w^{-1}1}, \ldots, \nu_{w^{-1}n}) \in \Lambda(n). \] (5.2)

Note that the rows of the Young diagram \([\nu]\) are obtained by a permutation of the rows of the Young diagram \([w\nu]\), and this permutation of rows defines a bijection \( \varphi_w : [w\nu] \to [\nu] \). If \( S : [\nu] \to \mathcal{A} \) is a function from the Young diagram of \( \nu \) to some set \( \mathcal{A} \), we denote
\[ wS := S \circ \varphi_w. \] (5.3)

With this notation we have

**Lemma 5.4.** If \( w \in \mathfrak{S}_n \) then \( B^\mu_{S, T} = \pm B^\mu_{wS, wT} \).

**Proof.** We can write
\[ x^S = x_1^S \cdots x_n^S, \quad l^S = l_1^S \cdots l_n^S, \quad y^T = y_1^T \cdots y_n^T, \quad l^T = l_1^T \cdots l_n^T, \]
where the words $x^S_k, l^S_k, y^T_k, l^T_k$ have length $\mu_k$ for $k = 1, \ldots, n$. For $m = 1, \ldots, n$, denote $k_m := w^{-1} m$. Note that $w \mu = (\mu_{k_1}, \ldots, \mu_{k_n})$ and

$$x^{wS} = x^S_{k_1} \cdots x^S_{k_n}, \quad l^{wS} = l^S_{k_1} \cdots l^S_{k_n}, \quad y^{wT} = y^T_{k_1} \cdots y^T_{k_n}, \quad l^T = l^T_{k_1} \cdots l^T_{k_n}.$$  

We now get, using the alternative notation (4.4),

$$B^\mu_{S,T} = \frac{\xi(x^S, l^S, y^T, l^T)}{\xi(l^S, l^T)} = \xi(x^S, l^S, 1^{\mu_1}, y^T, 1^{\mu_1}, l^T) \cdots \xi(x^S, l^S, n^{\mu_n}, y^T, n^{\mu_n}, l^T) = \pm \xi(x^{wS}, l^{wS}, y^{wT}, l^{wT}) = \pm B^{w\mu}_{wS, wT},$$

where the first and the last equations are by definition, the second and the penultimate equations come from Lemma 4.17, the third equation is by the supercommutativity of the $*$-product, and the fourth equality holds by Proposition 4.7. \hfill \Box

Note that we can always pick $w$ in the previous lemma so that $w \mu \in \Lambda_+(n, c)$. So, when working with codeterminants $B^\mu_{S,T}$, we can usually assume that $\mu \in \Lambda_+(n, c)$. In addition, in view of Lemmas 5.1 and 3.10, we can usually assume that $S$ and $T$ are row standard. For example, the following result shows that for $n \geq c, T_A(\mathfrak{a})_i$ is spanned by dominant codeterminants of color $i$ corresponding to row standard tableaux.

**Proposition 5.5.** Let $n \geq c$, and $(b, p, q) \in \mathrm{Tri}_B(i)(n, c)$. Then $\eta^{b}_{p,q} = \pm B^\mu_{S,T}$ for some $\mu \in \Lambda_+(n, c)$, $S \in \mathrm{Rst}^X(i)(\mu)$ and $T \in \mathrm{Rst}^Y(i)(\mu)$.

**Proof.** In view of Lemmas 5.1 and 5.4, it suffices just to prove that $\eta^{b}_{p,q} = \pm B^\mu_{S,T}$ for some $\mu \in \Lambda(n, c), S \in \mathrm{Tab}^X(i)(\mu)$ and $T \in \mathrm{Tab}^Y(i)(\mu)$.

Let $b = b_1 \cdots b_c$, $p = p_1 \cdots p_c$, $q = q_1 \cdots q_c$. For $x \in X(i), y \in Y(i)$ and $r, s \in [1, n]$, recalling (3.7), denote

$$m^{x,y}_{r,s} := [b, p, q]^{xy}_{r,s}.$$  

Note that $m^{x,y}_{r,s} \leq 1$ if $x \neq y$. Put

$$Q := \{(x, y, r, s) \mid x \in X(i), \quad y \in Y(i), \quad r, s \in [1, n], \quad m^{x,y}_{r,s} \neq 0\}.$$  

Pick a total order on $Q$ and write

$$Q = \{(x_1, y_1, r_1, s_1) < \cdots < (x_t, y_t, r_t, s_t)\}.$$
To every \((x, y, r, s) \in Q\), we associate a composition \(v_{r,s}^{x,y}\) of \(m_{r,s}^{x,y}\) as follows:

\[
v_{r,s}^{x,y} = \begin{cases} (m_{r,s}^{x,y}) & \text{if } \bar{x} = \bar{y} = \bar{0}, \\ (1m_{r,s}^{x,y}) & \text{otherwise.} \end{cases}
\]

Now we define \(\mu \in \Lambda(n, c)\) as the concatenation

\[
\mu := v_{r_1,s_1}^{x_1,y_1} \ldots v_{r_t,s_t}^{x_t,y_t} u \in \Lambda(n, c).
\]

where \(u \in \mathbb{Z}_{\geq 0}\) is chosen so that \(\mu\) has \(n\) parts. Let \(S\) be the \(\mu\)-tableau which associates to the nodes of each \(v_{r,s}^{x,y}\) the value \(r^x \in \mathcal{X}(i)\) and let \(T\) be the \(\mu\)-tableau which associates to the nodes of each \(v_{r,s}^{x,y}\) the value \(s^y \in \mathcal{Y}(i)\).

Let \(\delta = (m_{r_1,s_1}^{x_1,y_1}, \ldots, m_{r_t,s_t}^{x_t,y_t}) \in \Lambda(t, c)\). We can decompose any word \(l\) of length \(c\) as the concatenation \(l = l_1 \ldots l_t\), where, for \(1 \leq u \leq t\), the length of the word \(l_u\) is \(m_{r_u,s_u}^{x_u,y_u}\). We will apply this to the words \(b, p, q, x_S, l_S, y_T, l_T, l_\mu\). Note that the triples \((x_S, l_S, l_\mu)\) and \((y_T, l_\mu, l_T)\) satisfy the assumptions of Lemma 4.17, so

\[
B_{S,T}^{\mu} = \mathcal{E}_x^{x_S} \mathcal{E}_y^{y_T} = \pm (\xi_{r_1,s_1}^{x_1,y_1} \mathcal{E}_u^{x_u,y_u} \mathcal{E}_v^{y_u,y_v} \mathcal{E}_w^{y_w,y_w}) * \ldots * (\xi_{r_t,s_t}^{x_t,y_t} \mathcal{E}_u^{x_u,y_u} \mathcal{E}_v^{y_u,y_v} \mathcal{E}_w^{y_w,y_w}). \tag{5.6}
\]

Let \(1 \leq u \leq t\). Denote \(m := m_{r_u,s_u}^{x_u,y_u}\) Then

\[
x_u^S = x_u^m, \quad t_u^S = r_u^m, \quad y_u^T = y_u^m, \quad l_u^T = s_u^m.
\]

If \(\bar{x}_u = \bar{y}_u = \bar{0}\), then \(l_u^\mu = v^m\) for some \(1 \leq v \leq n\), and in this case, using (4.3), we get

\[
\xi_{r_u,s_u}^{x_u,y_u} = \mathcal{E}_u^{x_u,y_u} = \mathcal{E}_u^{x_u,y_u} = \eta_{p_u,q_u}^b.
\]

If \(\bar{x}_u\) and \(\bar{y}_u\) are not both \(\bar{0}\), then \(l_u^\mu = (v, v + 1, \ldots, v + m - 1)\) for some \(v\), and in this case, using (4.3) and (4.8), we get

\[
\xi_{r_u,s_u}^{x_u,y_u} = \mathcal{E}_u^{x_u,y_u} = \mathcal{E}_u^{x_u,y_u} = \mathcal{E}_u^{x_u,y_u} = \mathcal{E}_u^{x_u,y_u} = \mathcal{E}_u^{x_u,y_u} = \eta_{p_u,q_u}^b.
\]

So, using (5.6), in all cases we have

\[
B_{S,T}^{\mu} = \pm \eta_{p_1,q_1}^b * \ldots * \eta_{p_t,q_t}^b = \pm \eta_{p,q}^b
\]

where we have applied Lemma 4.16 for the last equality, using the fact that \((b, p, q)\) is \(\delta\)-separated. \qed
5.2 | Straightening

We continue with the set-up of the previous subsection. In particular, \( i \in I \) is fixed, \( \mu := \lambda^{(i)} \) and \( c := |\mu| \). Recall the lexicographical order “<” on \( \mathcal{A}^{c}_{X(i)} \) from §4.5. We have a similarly defined lexicographical order on \( \mathcal{A}^{c}_{Y(i)} \).

The following is an analog of the main lemma in [22]. In the lemma we denote by < the lexicographic order on partitions.

**Theorem 5.7.** Suppose that \( n \geq c \) and \( \mu \in \Lambda_+(n, c) \). Let \( S \in \text{Rst}^{X(i)}(\mu) \setminus \text{Std}^{X(i)}(\mu) \), \( T \in \text{Rst}^{Y(i)}(\mu) \setminus \text{Std}^{Y(i)}(\mu) \). Then:

(i) there exists \( \lambda \in \Lambda(n, c) \) with \( \lambda_+ > \mu \) such that

\[
\pm \xi^S_{\lambda, \mu} = \pm \xi^S_{\hat{S}, \lambda} + \sum_{S' \in \text{Rst}^{X(i)}(\mu), L < L'} c_{S'} \xi^{S'}_{\hat{S}, \lambda}
\]

for some \( c_{S'} \in \mathbb{k} \);

(ii) there exists \( \nu \in \Lambda(n, c) \) with \( \nu_+ > \mu \) such that

\[
\pm \xi^T_{\hat{\nu}, \mu} = \pm \xi^{T}_{\nu, \mu} + \sum_{T' \in \text{Rst}^{Y(i)}(\mu), L' < L} c_{T'} \xi^{T'}_{\nu, \mu}
\]

for some \( c_{T'} \in \mathbb{k} \).

**Proof.** By left–right symmetry it suffices to prove (i). In this proof, for \( L, L' \in \mathcal{A}^{c}_{X(i)} \), we write \( L \rightarrow L' \) if \( L > L' \) or \( L = L' = r^x \) with \( x \in X(i) \). Since \( S \) is not column standard, there exists some \( (i, a, b) \in [\mu] \) such that \( (i, a + 1, b) \in [\mu] \) and \( S(i, a, b) \rightarrow S(i, a + 1, b) \). Let \((a, b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \) be the lexicographically smallest pair with such property.

For \( 1 \leq t \leq n \), we consider the \( t \)th row of the Young diagram \([\mu] \):

\[
[\mu]_t := \{(i, t, s) \mid s \leq \mu_t \}.
\]

For \( t = 1, \ldots, n \), we define set partitions \([\mu]_t = E_t \cup F_t \) as follows. For \( t < a \), we set

\[
E_t := [\mu]_t, \quad F_t := \emptyset.
\]

For \( t = a \), we set

\[
E_a := \{(i, a, s) \in [\mu]_a \mid s < b - 1 \}, \quad F_a := [\mu]_a \setminus E_a.
\]

For \( t > a \), we set

\[
E_t := \{M \in [\mu]_t \mid S(N) \rightarrow S(M) \text{ for all } N \in F_{t-1} \}, \quad F_t := [\mu]_t \setminus E_t.
\]

For all \( 1 \leq t \leq n \), we let \( e_t := |E_t| \) and \( f_t := |F_t| \).

To define \( \lambda \in \Lambda(n, c) \), we consider two cases.
Case 1: \( b = 1 \). In this case, we have \( E_a = \emptyset \), and we set

\[
\lambda_t = \begin{cases} 
\mu_t & \text{if } t < a, \\
\mu_a + e_{a+1} & \text{if } t = a, \\
e_{t+1} + f_t & \text{if } a < t \leq n,
\end{cases}
\]

where \( e_{n+1} \) is interpreted as 0.

Case 2: \( b > 1 \). In this case we set

\[
\lambda_t = \begin{cases} 
\mu_t & \text{if } t < a, \\
 e_t + f_{t-1} & \text{if } a \leq t \leq n,
\end{cases}
\]

where \( f_0 \) is interpreted as 0.

Note that \( |\lambda| = |\mu| = c \). This is clear for the case \( b = 1 \). If \( b > 1 \), then the assumptions \( c \leq n \) and \( \mu \in \Lambda_+(n, c) \) imply \( \mu_n = 0 \), so \( f_n = 0 \), giving the claim.

We next claim that \( \lambda_+ > \mu \). Indeed, we have \( \lambda_t = \mu_t \) for \( t < a \). If \( b = 1 \) then \( \lambda_a > \mu_a \). If \( b > 1 \), then \( \lambda_{a+1} > \mu_a \). So the claim follows in all cases.

Given any word \( r \) of length \( c \), we can write it as concatenation \( r = r_1 \ldots r_n \) such that the length of the word \( r_k \) is \( \lambda_k \) for all \( k = 1, \ldots, n \). For \( 1 \leq k \leq n \), we have \( l_k = k^{\lambda_k} \) and

\[
l_k^\mu = \begin{cases} 
k^{\mu_k} & \text{if } k < a, \\
(k - 1)^{f_{k-1}}k^{\mu_k} & \text{if } k \geq a \text{ and } b > 1, \\
k^{f_k}(k + 1)^{e_{k+1}} & \text{if } k \geq a \text{ and } b = 1.
\end{cases}
\]

So, by Lemma 4.17, we have, using the alternative notation (4.4),

\[
\xi x^S \xi e^S \xi f^S \mathrm{Fr}(t, e, f) = \pm \left( \xi(x^S_1, l^S_1, 1^{\lambda_1})\xi(e^\lambda_1, 1^{\lambda_1}, l^\mu_1) \right) \ast \cdots \ast \left( \xi(x^S_n, l^S_n, n^{\lambda_n})\xi(e^\lambda_n, n^{\lambda_n}, l^\mu_n) \right).
\]

Let \( 1 \leq k \leq n \). By Lemma 4.24,

\[
\xi(x^S_k, l^S_k, k^{\lambda_k})\xi(e^\lambda_k, k^{\lambda_k}, l^\mu_k) = \xi x^S_{k, \mu_k} + X_k,
\]

where \( X_k \) is a linear combination of some \( \xi x^S_{u_1 \ldots u_n} \) with \( u^x \sim (l^S_k)^x_{k} \) and \( u^x < (l^S_k)^x_{k} \).

By the definition of \( \lambda \), the triple \( (x^S, l^S, l^\mu) \in \text{Tri}^{(i)}(n, c) \) is \( \lambda \)-separated. So by Lemma 4.16, we have

\[
\xi x^S_{l^S_1 \ldots l^S_n} \ast \cdots \ast \xi x^S_{l^S_n l^S_1} = \pm \xi x^S_{l^S_1 \ldots l^S_n},
\]

which gives us the required leading term. On the other hand, if we have \( u^x_k \leq (l^S_k)^x_{k} \) for all \( k = 1, \ldots, n \), with at least one inequality being strict, then by Lemma 4.15,

\[
\xi x^S_{u_1 l^S_1} \ast \cdots \ast \xi x^S_{u_n l^S_n} = \pm c \xi x^S_{u_1 \ldots u_n l^S}.
\]
for some $c \in k$. Note that $u_1^{x_1} \cdots u_1^{x_1} < (l^S)^{x^S}$. Moreover, there is a tableau $S' \in \text{Rst}^{X(i)}(\mu)$ such that $(x_{S'}^{x'_1} \cdots x_{S'}^{x'_n}, u_1^{x_1} \cdots u_n^{x_n}, l^\mu) \sim (x_1^{x_1} \cdots x_n^{x_n}, u_1^{x_1} \cdots u_n^{x_n}, l^\mu)$. Then $\xi_{u_1^{x_1} \cdots u_n^{x_n}, l^\mu} = \pm c \xi_{l^S, l^\mu}$ and

$$L_{S'}^{x_{S'}^{x'_1} \cdots x_{S'}^{x'_n}} \leq u_1^{x_1} \cdots u_1^{x_1} < (l^S)^{x^S} = L^S,$$

as required.

**Lemma 5.8.** For $\lambda \in \Lambda(n, c)$ and $T \in \text{Rst}^{Y(i)}(\mu)$, we have

$$\xi_{l^\mu, l^T}^{e_i} \xi_{l^\mu, l^T}^{y} = \sum_{T' \in \text{Rst}^{Y(i)}(\lambda)} c_{T'} \xi_{l^\mu, l^T'}^{y},$$

for some $c_{T'} \in k$.

**Proof.** By Proposition 4.7, we have that $\xi_{l^\mu, l^T}^{e_i} \xi_{l^\mu, l^T}^{y}$ is a linear combination of terms of the form $\xi_{l^\mu, l^T}^{y}$ for some $l \in [1, n]^c$ and $y \in Y(i)^c$ with $(y, l', l) \in \text{Tri}^{B(i)}(n, c)$. Each of these terms equals $\pm \xi_{l^\mu, l^T'}^{y}$ for some $T' \in \text{Rst}^{Y(i)}(\lambda)$.

Let $\lambda, \lambda' \in \Lambda(n, c)$ and $S \in \text{Rst}^{X(i)}(\lambda), S' \in \text{Rst}^{X(i)}(\lambda'), T \in \text{Rst}^{Y(i)}(\mu)$, and $T' \in \text{Rst}^{Y(i)}(\lambda')$. We write

$$(\lambda, S, T) \geq (\lambda', S', T')$$

(5.9)

if $\lambda \triangleright \lambda'$, or $\lambda = \lambda'$, $L^S \leq L^{S'}$, $L^T \leq L^{T'}$.

**Theorem 5.10.** Let $n \geq c$, $S \in \text{Rst}^{X(i)}(\mu)$ and $T \in \text{Rst}^{Y(i)}(\mu)$. Then $B_{S, T}^{x_i}$ is a linear combination of standard codeterminants $B_{S', T'}^{y}$ such that $(\lambda, S', T') \geq (\mu, S, T)$.

**Proof.** By Lemmas 5.4, 5.1, and 3.10, we may assume that $\mu \in \Lambda_+(n, c)$. If $S$ and $T$ are standard, we are done. Otherwise we may assume by symmetry that $S$ is not standard. Let $U$ be the set of all triples $(\lambda, S', T')$ such that $\lambda \in \Lambda(n, c)$, $S' \in \text{Rst}^{X(i)}(\lambda)$, $T' \in \text{Rst}^{Y(i)}(\lambda)$, and either $\lambda \triangleright \mu$ or $\lambda = \mu$, $T' = T$, $L^{S'} < L^S$. Using induction on the partial order (5.9) and Lemma 5.4, we see that it suffices to prove

$$B_{S, T}^{x_i} = \sum_{(\lambda, S', T') \in U} c_{\lambda, S', T'} B_{S', T'}^{y}$$

(5.11)

for some $c_{\lambda, S', T'} \in k$.

By Theorem 5.7, there exists $\lambda \in \Lambda_+(n, c)$ with $\lambda_+ \triangleright \mu$ such that

$$\xi_{l^\mu, l^T}^{x_i} \xi_{l^\mu, l^T}^{e_i} + \sum_{S' \in \text{Rst}^{X(i)}(\mu), L^{S'} < L^S} c_{S'} \xi_{l^{S'}, l^T}^{x_i} \xi_{l^{S'}, l^T}^{e_i}.$$

Multiplying on the right with $\xi_{l^{S'}, l^T}^{y}$ yields

$$B_{S, T}^{x_i} = \pm \xi_{l^\mu, l^T}^{x_i} \xi_{l^\mu, l^T}^{e_i} \xi_{l^{S'}, l^T}^{y} + \sum_{S' \in \text{Rst}^{X(i)}(\mu), L^{S'} < L^S} c_{S'} B_{S', T'}^{y}.$$


It remains to note, using Lemma 5.8, that we can write \( \xi^x_S \xi^\epsilon_{\mu, \lambda} \xi^y_T \) as a linear combination of codeterminants of shape \( \lambda \).

\[ \square \]

### 5.3 Multicolored codeterminants

Recall that in the beginning of the section, we have fixed \( \lambda \in \Lambda^I(n, d) \). Let

\[ \| \lambda \| = (d_0, \ldots, d_r). \]

Recalling the notation of §5.1, for \( S = (S(0), \ldots, S(\ell)) \in \text{Tab}^X(\lambda) \) and \( T = (T(0), \ldots, T(\ell)) \in \text{Tab}^Y(\lambda) \), we define

\[
\begin{align*}
x^S &= x^{S(0)} \cdots x^{S(\ell)} \in X^d, \\
y^T &= y^{T(0)} \cdots y^{T(\ell)} \in Y^d, \\
l^S &= l^{S(0)} \cdots l^{S(\ell)} \in [1, n]^d, \\
l^T &= l^{T(0)} \cdots l^{T(\ell)} \in [1, n]^d,
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{X}_S &= \xi^x_S \xi^\epsilon_{\mu, \lambda}, \\
\mathcal{Y}_T &= \xi^y_T \xi^\epsilon_{\mu, \lambda}, \\
B^{\lambda}_{S,T} &= \mathcal{X}_S \mathcal{Y}_T,
\end{align*}
\]

where, in agreement with (4.20), we set

\[
\begin{align*}
\ell^\lambda &= l^{T(\ell)} \cdots l^{T(0)},
\end{align*}
\]

We refer to the elements \( B^{\lambda}_{S,T} \) as **codeterminants**. As for singled colored codeterminants, it is easy to see that \( \mathcal{X}_S, \mathcal{Y}_T, B^{\lambda}_{S,T} \in T^A_{a}(n, d) \).

We refer to \( \lambda \) as the **shape** of the codeterminant \( B^{\lambda}_{S,T} \). A codeterminant \( B^{\lambda}_{S,T} \) is called **dominant** if \( \lambda \in \Lambda^I(n, d) \), that is, if its shape is a multipartition. A codeterminant \( B^{\lambda}_{S,T} \) is called **standard** if it is dominant and \( S \in \text{Std}^X(\lambda) \) and \( T \in \text{Std}^Y(\lambda) \).

The following lemma will allow us to use the theory of single-colored codeterminants developed in the previous subsections.

**Lemma 5.12.** We have

\[
B^{\lambda}_{S,T} = \pm B^{(0)}_{S(0),T(0)} \ast \cdots \ast B^{(\ell)}_{S(\ell),T(\ell)},
\]

and the sign is + if all \( x^S_k \) or all \( y^T_k \) are even.

**Proof.** By Lemmas 4.16 and 4.17, we have

\[
B^{\lambda}_{S,T} = \mathcal{X}_S \mathcal{Y}_T = (\mathcal{X}_S^{(0)} \ast \cdots \ast \mathcal{X}_S^{(\ell)}) (\mathcal{Y}_T^{(0)} \ast \cdots \ast \mathcal{Y}_T^{(\ell)})
\]

\[
= \pm (\mathcal{X}_S^{(0)} \mathcal{Y}_T^{(0)}) \ast \cdots \ast (\mathcal{X}_S^{(\ell)} \mathcal{Y}_T^{(\ell)}) = \pm B^{(0)}_{S(0),T(0)} \ast \cdots \ast B^{(\ell)}_{S(\ell),T(\ell)},
\]

and the sign claim also follows from Lemma 4.17. \( \square \)
For \( \mathbf{b} = b_1 \ldots b_d \in B^d \), we write

\[
\| \mathbf{b} \| = \mu
\]

(5.13)

if \( \mu_i = \# \{ k \in [1, d] \mid b_k \in B(i) \} \) for all \( i \in I \).

**Proposition 5.14.** Let \( n \geq d \), and \((\mathbf{b}, \mathbf{p}, \mathbf{q}) \in \text{Tri}^B(n, d) \). Then \( \eta^b_{pq} = \pm B^\mu_{ST} \) for some \( \mu \in \Lambda^I_+(n, d) \) with \( \| \mu \| = \| \mathbf{b} \| \), \( \mathbf{S} \in \text{Rst}^X(\mu) \) and \( \mathbf{T} \in \text{Rst}^Y(\mu) \).

**Proof.** This follows from Proposition 5.5 and Lemmas 5.12 and 4.16.

Let \( \lambda, \mu \in \Lambda(n, d), \mathbf{S} \in \text{Rst}^X(\lambda), \mathbf{T} \in \text{Rst}^Y(\lambda) \) and \( \mathbf{T'} \in \text{Rst}^Y(\mu) \). Recalling (5.9), we write \((\lambda, \mathbf{S}, \mathbf{T}) \geq (\mu, \mathbf{S'}, \mathbf{T'})\) if \((\lambda(i), S(i), T(i)) \geq (\mu(i), S'(i), T'(i))\) for all \( i \in I \).

**Theorem 5.15.** Let \( n \geq d \), \( \lambda \in \Lambda^I(n, d), \mathbf{S} \in \text{Rst}^X(\lambda) \) and \( \mathbf{T} \in \text{Rst}^Y(\lambda) \). Then \( B^\mu_{ST} \) is a linear combination of \( B^\mu_{S'T'} \), such that \( \mu \in \Lambda^I_+(n, d), \mathbf{S'} \in \text{Std}^X(\mu), \mathbf{T'} \in \text{Std}^Y(\mu) \) and \((\mu, \mathbf{S'}, \mathbf{T'}) \geq (\lambda, \mathbf{S}, \mathbf{T})\).

**Proof.** This follows from Lemma 5.12 and Theorem 5.10.

**Corollary 5.16.** Let \( n \geq d \). Then the standard codeterminants

\[
\{ B^\lambda_{ST} \mid \lambda \in \Lambda^I_+(n, d), \mathbf{S} \in \text{Std}^X(\lambda), \mathbf{T} \in \text{Std}^Y(\lambda) \}
\]

span \( T_a^A(n, d) \).

**Proof.** The result follows from Proposition 5.14 and Theorem 5.15.

**Theorem 5.17.** Let \( n \geq d \). Then the standard codeterminants

\[
\{ B^\lambda_{ST} \mid \lambda \in \Lambda^I_+(n, d), \mathbf{S} \in \text{Std}^X(\lambda), \mathbf{T} \in \text{Std}^Y(\lambda) \}
\]

form a \( k \)-basis of \( T_a^A(n, d) \).

**Proof.** By Lemma 3.18, there exists a bijection between the indexing set for the standard codeterminants and the indexing set for the standard basis of \( T_a^A(n, d) \). Since the standard codeterminants span \( T_a^A(n, d) \) by Corollary 5.16, the result follows since \( k \) is a domain.

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### 6 | QUASI-HEREDITARY STRUCTURE ON \( T_a^A(n, d) \)

We continue working with a fixed \( d \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}_{>0} \), and based quasi-hereditary graded \( k \)-superalgebra \( A \) with \( a \)-conforming heredity data \( I, X, Y \). Recall the order \( \leq \) on \( \Lambda^I(n, d) \) defined in (3.5).
6.1 Heredity basis

Throughout the subsection we fix $\lambda \in \Lambda^{l}(n,d)$ with $\|\lambda\| = (d_0, \ldots, d_r)$. Recall the idempotent $e_{\lambda}$ defined in (4.21). It is easy to see that

$$e_{\lambda} = X_{\tau^{\lambda}} = Y_{\tau^{\lambda}} = B_{T_{\lambda}, T_{\lambda}}^{\lambda}.$$  (6.1)

Let $S \in \text{Tab}^{X}(\lambda)$ and $T \in \text{Tab}^{Y}(\lambda)$. Recalling (4.22), define

$$\alpha^{S} := \alpha(x^{S}, t^{S}), \quad \beta^{T} := \beta(y^{T}, t^{T}).$$  (6.2)

The following results should be compared to Definition 2.5(c).

Lemma 6.3. Let $\lambda \in \Lambda^{l}_{+}(n,d)$, $\mu \in \Lambda^{l}(n,d)$, $S \in \text{Std}^{X}(\lambda)$ and $T \in \text{Std}^{Y}(\lambda)$. Then:

(i) $X_{S} e_{\mu} = \delta_{\lambda, \mu} X_{S}$ and $e_{\mu} Y_{T} = \delta_{\lambda, \mu} Y_{T};$

(ii) $e_{\lambda} X_{S} = \delta_{S, T_{\lambda}} X_{S}$ and $Y_{T} e_{\lambda} = \delta_{T_{\lambda}, T_{\lambda}} Y_{T}.$

(iii) $e_{\mu} X_{S} = \delta_{\mu, \alpha^{S}} X_{S}$ and $Y_{T} e_{\mu} = \delta_{\mu, \beta^{T}} Y_{T}.$

Proof. (i) and (iii) follow easily from Lemma 4.23.

(ii) We prove the first equality, the second one being similar. By Lemma 4.23, we have

$$e_{\lambda} X_{S} = \delta_{\lambda, \alpha^{S}} X_{S}.$$  

So it suffices to prove that $\lambda = \alpha^{S}$ if and only if $S = T_{\lambda}$. If $S = T_{\lambda}$, then $\alpha^{S} = \alpha_{T_{\lambda}}^{\lambda} = \lambda$.

Conversely, if $\lambda = \alpha^{S}$, then $\|\lambda\| = \|\alpha^{S}\|$ and it follows using Definition 2.5(c) that for all $i \in I$ every entry of $S^{(i)}$ is of the form $r_{\lambda}^{i}$ for some $r \in [1, d]$. Fix $i \in I$. For $r = 1, \ldots, n$, let $v_{r} := \# \{ a \in [1, d] \mid \text{let}(S^{(i)}_{a}) = r \}$. Let $S$ be the $\lambda^{(i)}$-tableau obtained from $S^{(i)}$ by replacing each entry $r^{(i)}_{\lambda}$ with $r$. Then $S$ is a classical standard $\lambda^{(i)}$-tableau of weight $v$. So $v \leq \lambda^{(i)}$, and $v = \lambda^{(i)}$ if and only if $S^{(i)} = T_{\lambda}^{(i)}$.

Since $i$ is arbitrary, we have proved that $S = T_{\lambda}^{\lambda}$. \[\square\]

If $\lambda \in \Lambda^{l}_{+}(n,d)$, we denote

$$T_{A}^{\lambda}(n,d)_{+} := \text{span}\{B_{S, T}^{\mu} \mid \mu \in \Lambda^{l}_{+}(n,d), \mu \geq \lambda, S \in \text{Std}^{X}(\mu), T \in \text{Std}^{Y}(\mu)\},$$

$$T_{A}^{\lambda}(n,d)_{+} := \text{span}\{B_{S, T}^{\mu} \mid \mu \in \Lambda^{l}_{+}(n,d), \mu > \lambda, S \in \text{Std}^{X}(\mu), T \in \text{Std}^{Y}(\mu)\}.$$

Proposition 6.4. Let $n \geq d$ and $\lambda \in \Lambda^{l}_{+}(n,d)$. Then $T_{A}^{\lambda}(n,d)_{+}$ is the two-sided ideal of $T_{A}^{\lambda}(n,d)$ generated by $\{e_{\mu} \mid \mu \in \Lambda^{l}_{+}(n,d), \mu \geq \lambda\}.$

Proof. Let $J$ be the two-sided ideal of $T_{A}^{\lambda}(n,d)$ generated by $\{e_{\mu} \mid \mu \in \Lambda^{l}_{+}(n,d), \mu \geq \lambda\}$. If $\mu \in \Lambda^{l}_{+}(n,d), \mu \geq \lambda$, $S \in \text{Std}^{X}(\mu)$ and $T \in \text{Std}^{Y}(\mu)$, then by Lemma 6.3(i), we have $B_{S, T}^{\mu} = X_{S} Y_{T} = \delta_{\lambda, \alpha^{S}} X_{S} Y_{T} \in J$, so $T_{A}^{\lambda}(n,d)_{+} \subseteq J$.

We prove the reverse inclusion by downward induction on \(\leq\). Suppose that the result has been proved for all $\nu > \lambda$, and let $\eta \in J$. By the inductive assumption, we may assume that $\eta = \eta_{1} e_{\lambda} \eta_{2}$ for some $\eta_{1}, \eta_{2} \in T_{A}^{\lambda}(n,d)$. By Lemma 4.23, we may assume that $\eta_{1}$ is of the form $\eta_{T_{s}}^{b}$ for $(b, r, s) \in \text{Tn}^{B}(n,d)$ with $\beta(b, s) = \lambda$, and $\eta_{2}$ is of the form $\eta_{t_{u}}^{c}$ for $(c, t, u) \in \text{Tn}^{B}(n,d)$ with $\alpha(c, t) = \lambda.$
Recalling the notation (5.13), Definition 2.5, and Proposition 4.7, we now deduce that either \( \eta \in T_a^A(n, d)^{>\|\lambda\|} \) or \( \|b\| = \|\lambda\| = \|c\| \). In the latter case, using Definition 2.5(c) and Proposition 4.7, we see that \( \eta_1 \epsilon_\lambda \eta_2 \neq 0 \) only if \( \eta_1 \) is of the form \( X_S \) for some \( S \in \text{Tab}^X(\lambda) \) and \( \eta_2 \) is of the form \( Y_T \) for some \( T \in \text{Tab}^Y(\lambda) \), that is, we may assume that \( \eta = B^A_{S, T} \). By Theorem 5.15 and Lemma 5.12, \( B^A_{S, T} \) is a linear combination of standard codeterminants \( B^A_{S', T'} \) with \( \mu \geq \lambda \). Thus \( \eta \in T_a^A(n, d)^{\geq \lambda} \).

\[ \eta X_S \equiv \sum_{S' \in \text{Std}^X(\lambda)} l^S_{S'}(\eta) X_{S'} \pmod{T_a^A(n, d)^{>\lambda}}, \]
\[ Y_T \eta \equiv \sum_{T' \in \text{Std}^Y(\lambda)} r^T_{T'}(\eta) Y_{T'} \pmod{T_a^A(n, d)^{>\lambda}} \]

for some \( l^S_{S'}(\eta), r^T_{T'}(\eta) \in k \).

**Proposition 6.5.** Let \( n \geq d, \lambda \in \Lambda^I_+(n, d), S \in \text{Std}^X(\lambda), T \in \text{Std}^Y(\lambda), \) and \( \eta \in T_a^A(n, d) \). Then

\[ \eta X_S \equiv \sum_{S' \in \text{Std}^X(\lambda)} l^S_{S'}(\eta) X_{S'} \pmod{T_a^A(n, d)^{>\lambda}}, \]
\[ Y_T \eta \equiv \sum_{T' \in \text{Std}^Y(\lambda)} r^T_{T'}(\eta) Y_{T'} \pmod{T_a^A(n, d)^{>\lambda}} \]

for some \( l^S_{S'}(\eta), r^T_{T'}(\eta) \in k \).

**Proof.** We prove the first equality, the second one being similar. By Proposition 6.4, \( X_S = X_S e_\lambda \) belongs to the ideal \( T_a^A(n, d)^{>\lambda} \). So we can write

\[ \eta X_S \equiv \sum_{S' \in \text{Std}^X(\lambda), T \in \text{Std}^Y(\lambda)} l^{S, T}_{S', T}(\eta) B^A_{S', T} \pmod{T_a^A(n, d)^{>\lambda}} \]

for some \( l^{S, T}_{S', T}(\eta) \in k \). Multiplying on the right by \( e_\lambda \) and using Lemma 6.3(ii), we see that \( l^{S, T}_{S', T}(\eta) = 0 \) unless \( T = T^\lambda \), in which case \( B^A_{S', T} = X_{S'} \). \( \square \)

The partial order “\( \leq \)” on \( \Lambda^I(n, d) \) restricts to a partial order on the subset \( \Lambda^I_+(n, d) \subseteq \Lambda^I(n, d) \). For each \( \lambda \in \Lambda^I_+(n, d) \), set

\[ \mathcal{X}(\lambda) = \{ X_S \mid S \in \text{Std}^X(\lambda) \}, \quad \mathcal{Y}(\lambda) = \{ Y_T \mid T \in \text{Std}^Y(\lambda) \} \]

Define

\[ \mathcal{X} := \bigsqcup_{\lambda \in \Lambda^I_+(n, d)} \mathcal{X}(\lambda), \quad \mathcal{Y} := \bigsqcup_{\lambda \in \Lambda^I_+(n, d)} \mathcal{Y}(\lambda). \]

**Theorem 6.6.** Let \( n \geq d \) and \( A \) be a based quasi-hereditary \( k \)-superalgebra with \( a \)-conforming heredity data \( I, X, Y \). Then \( T_a^A(n, d) \) is a based quasi-hereditary \( k \)-superalgebra with heredity data \( \Lambda^I_+(n, d), \mathcal{X}, \mathcal{Y} \) and initial elements \( e_\lambda = X_{T_2} = Y_{T_2} \) for all \( \lambda \in \Lambda^I_+(n, d) \).

**Proof.** The property (a) of Definition 2.5 follows from Theorem 5.17, the property (b) of Definition 2.5 follows from Proposition 6.5, and the property (c) of Definition 2.5 follows from Lemma 6.3. \( \square \)

**Remark 6.7.** The assumption \( n \geq d \) in Theorem 6.6 is necessary. In its absence, we can only sometimes guarantee cellularity, see Lemma 6.25. See Remark 7.57 for a counterexample to quasi-heredity when \( n < d \).

**Remark 6.8.** In view of [13, Remark 5.17], Theorem 6.6 should be compared to the main result of [7], which claims that the wreath product algebra \( A \hat{\otimes} \mathcal{S}_d \) is cellular if \( A \) is cyclic cellular.
Remark 6.9. While Theorem 6.6 claims that $T^A_{a}(n, d)$ is a based quasi-hereditary superalgebra with heredity data $\Lambda^I_+(n, d), \mathcal{X}, \mathcal{Y}$, it does not claim that in general the heredity data are conforming. However, $\Lambda^I_+(n, d), \mathcal{X}, \mathcal{Y}$ would be conforming under some natural additional assumptions on the heredity data $I, X, Y$ of $A$ which hold in most interesting examples, see [12, §4.4]. We consider such a situation in the following lemma.

Lemma 6.10. Suppose that $A$ possesses a $(\mathbb{Z}/2 \times \mathbb{Z}/2)$-grading $A = \bigoplus_{\varepsilon, \delta \in \mathbb{Z}/2} A_{\varepsilon, \delta}$ and heredity data $I, X, Y$ such that the following conditions hold:

1. $A_{\varepsilon, \delta} A_{\varepsilon', \delta'} \subseteq A_{\varepsilon + \varepsilon', \delta + \delta'}$ for all $\varepsilon, \delta, \varepsilon', \delta' \in \mathbb{Z}/2$;
2. For all $\varepsilon \in \mathbb{Z}/2$, we have $A_{\varepsilon} = \bigoplus_{\varepsilon'' = \varepsilon} A_{\varepsilon', \varepsilon''}$;
3. $X_{\varepsilon} \subseteq A_{\varepsilon, \bar{0}}$ and $Y_{\varepsilon} \subseteq A_{\bar{0}, \varepsilon}$ for all $\varepsilon \in \mathbb{Z}/2$.

Then we have the following.

(i) The heredity data $I, X, Y$ is $a$-conforming for $a = A_{\bar{0}, \bar{0}}$.
(ii) The $(\mathbb{Z}/2 \times \mathbb{Z}/2)$-grading on $A$ induces a $(\mathbb{Z}/2 \times \mathbb{Z}/2)$-grading on $T^A_{a}(n, d)$ which, with the heredity data $\Lambda^I_+(n, d), \mathcal{X}, \mathcal{Y}$, satisfies axioms (1)–(3).

(iii) The heredity data $\Lambda^I_+(n, d), \mathcal{X}, \mathcal{Y}$ are conforming.

Proof. Claim (i) is easy to see, and (iii) will likewise follow from (ii). For the proof of (ii), write the $\mathbb{Z}/2 \times \mathbb{Z}/2$-degree of a homogeneous element $a \in A$ as $(a^{(1)}, a^{(2)}) \in \mathbb{Z}/2 \times \mathbb{Z}/2$. The $(\mathbb{Z}/2 \times \mathbb{Z}/2)$-grading on $A$ induces such a grading on $T^A_{a}(n, d)$, where

$$((\xi^{b}_{r,s})^{(1)}, (\xi^{b}_{r,s})^{(2)}) = (b^{(1)}_{1} + \cdots + b^{(1)}_{d}, b^{(2)}_{1} + \cdots + b^{(2)}_{d}) \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

(6.11)

for all $(b, r, s) \in \text{Tri}^b(n, d)$. We have by condition (2) on $A$ that

$$\tilde{\xi}^{b}_{r,s} = b_{1} + \cdots + b_{d} = (b^{(1)}_{1} + b^{(2)}_{2}) + \cdots + (b^{(1)}_{d} + b^{(2)}_{d}) = (\xi^{b}_{r,s})^{(1)} + (\xi^{b}_{r,s})^{(2)},$$

so the $\mathbb{Z}/2 \times \mathbb{Z}/2$-grading on $T^A_{a}(n, d)$ satisfies condition (2) as well.

Elements of $\mathcal{X}$ are of the form $\xi^{x}_{r,s}$ for some $(x, r, s) \in \text{Tri}^x(n, d)$, so by (6.11) and condition (3) on $A$, we have that $(\lambda^{(1)}, \lambda^{(2)}) = ((\lambda^{(1)}_{x}), \bar{0}) = (\mathcal{X}_{\varepsilon}, \bar{0})$. Thus $\mathcal{X}_{\varepsilon} \subseteq T^A_{a}(n, d)_{\varepsilon, \bar{0}}$. We similarly have $\mathcal{Y}_{\varepsilon} \subseteq T^A_{a}(n, d)_{\bar{0}, \varepsilon}$, so the $\mathbb{Z}/2 \times \mathbb{Z}/2$-grading on $T^A_{a}(n, d)$ satisfies condition (3) as well, which completes the proof of (ii).

□

Remark 6.12. We sometimes refer to the process of passing from $A$ to $T^A_{a}(n, d)$ as schurifying $A$. If there is no problem with conformity, as discussed in Remark 6.9, one can schurify iteratively. For example, starting with $k$ this produces interesting quasi-hereditary algebras which could be considered as Schur algebra analogues of iterated wreath products of symmetric groups.

We complete this subsection with a technical result needed for future reference. Given $\mu = (\mu^{(0)}, \ldots, \mu^{(c)}) \in \Lambda^I_+(n, d)$ and recalling the notation $\mu_+$ from §3.1, let $\lambda_+ \in \Lambda^I_+(n, d)$ be defined from $\lambda^{(i)}_+ := \mu^{(i)}_+$ for all $i \in I$. Recall the notation (4.22).

Lemma 6.13. Let $n \geq d$ and $(x, r, s) \in \text{Tri}^x(n, d)$. Then $\beta(x, s) \in \Lambda^I_+(n, d)$, and $\xi^{x}_{r,s} \in T^A_{a}(n, d)_{\beta(x,s)+}$.
Proof. We denote \( \mu := \beta(x, s) \) and \( \lambda := \beta(x, s)_+ \). We can write \( \xi^x_{r,s} = \pm \lambda_S = \pm B^\mu_{S,T\mu} \) for some \( S \in \text{Tab}^x(\mu) \). By Lemmas 5.12 and 5.4, we now deduce that \( \xi^x_{r,s} = \pm B^\lambda_{S',T'} \) for some \( S', T' \). The result now follows from Theorem 5.15.

6.2 Standard modules over generalized Schur algebras

Recall the coproduct \( \nabla : T^A_a(n) \to T^A_a(n) \otimes T^A_a(n) \) defined in §4.2. In view of coassociativity of \( \nabla \), we also have a well-defined homomorphism \( \nabla^m : T^A_a(n) \to T^A_a(n) \otimes^m \) for any \( m \geq 2 \), with \( \nabla^2 = \nabla \). Restricting \( \nabla^{\ell+1} \) from \( T^A_a(n) \) to \( T^A_a(n, d) \subset T^A_a(n) \) gives a map:

\[
\nabla^{\ell+1} : T^A_a(n, d) \to \bigoplus_{(d_0, \ldots, d_{\ell}) \in \Lambda(I, d)} T^A_a(n, d_0) \otimes \cdots \otimes T^A_a(n, d_{\ell}).
\]

Let \( \delta := (d_0, \ldots, d_{\ell}) \in \Lambda(I, d) \). The natural projections \( T^A_a(n) \to T^A_a(n, d_i) \) for all \( i \in I \) induce the natural projection

\[
\pi_\delta : T^A_a(n) \otimes^{\ell+1} \to T^A_a(n, d_0) \otimes \cdots \otimes T^A_a(n, d_{\ell}).
\]

Then we have an algebra homomorphism

\[
\nabla_\delta := \pi_\delta \circ \nabla^{\ell+1} : T^A_a(n, d) \to T^A_a(n, d_0) \otimes \cdots \otimes T^A_a(n, d_{\ell}).
\]

If \( V_i \in T^A_a(n, d_i) \)-mod for all \( i \in I \), we use \( \nabla_\delta \) to define a structure of \( T^A_a(n, d) \)-modules on

\[
\bigotimes_{i \in I} V_i = V_0 \otimes \cdots \otimes V_{\ell}.
\]

Let now \( \lambda = (\lambda^{(0)}, \ldots, \lambda^{(\ell)}) \in \Lambda^I_+(n, d) \) with

\[
\delta := \|\lambda\| = (d_0, \ldots, d_{\ell}) \in \Lambda(I, d).
\]

For \( i \in I \), we let

\[
\lambda^{(i)} := (0, \ldots, 0, \lambda^{(i)}, 0, \ldots, 0) \in \Lambda^I(n, d_i),
\]

with \( \lambda^{(i)} \) in the \( i \)th position. Recalling (4.19), we have \( e_{\lambda^{(0)}} = \xi(\lambda^{(i)}, e_i) \in T^A_a(n, d_i) \). Denote

\[
T^A_a(n, d_i) := T^A_a(n, d_i) / T^A_a(n, d_i)^{>\lambda^{(0)}},
\]

\[
T^A_a(n, \delta) := T^A_a(n, d_0) \otimes \cdots \otimes T^A_a(n, d_{\ell}),
\]

\[
T^A_a(n, \delta)^{>\lambda} := \sum_{i=0}^\ell T^A_a(n, d_i) \otimes \cdots \otimes T^A_a(n, d_i)^{>\lambda^{(0)}} \otimes \cdots \otimes T^A_a(n, d_{\ell}),
\]

so that \( T^A_a(n, \delta) / T^A_a(n, \delta)^{>\lambda} \cong T^A_a(n, \delta) \). With this notation we have the following.
Lemma 6.15. If $S = (S^{(0)}, \ldots, S^{(c)}) \in \text{Std}^X(\lambda)$ and $T = (T^{(0)}, \ldots, T^{(c)}) \in \text{Std}^Y(\lambda)$, then

\[
\nabla \delta(A_S) \equiv A_{S^{(0)}} \otimes \cdots \otimes A_{S^{(c)}} \pmod{T^A(n, \delta)^{>\lambda}},
\]

\[
\nabla \delta(A_T) \equiv A_{T^{(0)}} \otimes \cdots \otimes A_{T^{(c)}} \pmod{T^A(n, \delta)^{>\lambda}}.
\]

In particular,

\[
\nabla \delta(e_\lambda) \equiv e_{\lambda^{(0)}} \otimes \cdots \otimes e_{\lambda^{(c)}} \pmod{T^A(n, \delta)^{>\lambda}}.
\]

Proof. We prove the result for $A$, the proof for $A$ being similar. Recall the set $\text{Tri}_0^X(n, d)$ from 3.6, which depends on the choice of a total order on $X \times [1, n] \times [1, n]$ with $x \in X(i), x' \in X(i')$. Letting $(x, r, s), (x', r', s') \in X \times [1, n] \times [1, n]$ such that $(x, r, s) < (x', r', s')$ if and only if one of the following holds: (a) $i > i'$; (b) $i = i'$ and $s < s'$; (c) $i = i'$, $s = s'$ and $r_x < (r_x')$ in our fixed total order on $A_X(i)$, cf. § 3.3.

Since $S \in \text{Std}^X(\lambda)$, we have that $(x_S, l_S, l_\lambda) \in \text{Tri}_0^X(n, d)$. Applying Lemma 4.12 to $A_S = \xi x_S l_S, l_\lambda$, we get $\nabla \delta(A_S) = \xi x_S^{(0)} l_S^{(0)}, l_\lambda^{(0)} \otimes \cdots \otimes \xi x_S^{(c)} l_S^{(c)}$, where (*) is a linear combination of terms of the form $\xi x_0^{(i)}, r_0, s_0 \otimes \cdots \otimes \xi x_0^{(c)}, r_c, s_c$ such that $(x_0^{(i)}, r_0, s_0) \in T^X(n, d_i)$ for all $i \in I$, and for at least one $i \in I$ we have that not all entries of $x_0^{(i)}$ belong to $X(i)$. By choosing the smallest such $i$ (with respect to $<$) for each $\xi x_0^{(i)}, r_0, s_0 \otimes \cdots \otimes \xi x_0^{(c)}, r_c, s_c$ and using Lemma 6.13, we deduce that $\xi x_0^{(i)}, r_0, s_0 \otimes \cdots \otimes \xi x_0^{(c)}, r_c, s_c \in T^A(n, d_i) \otimes \cdots \otimes T^A(n, d_c) \pmod{T^A(n, \delta)^{>\lambda}}$. It remains to note that $\xi x_0^{(i)}, r_0, s_0 \otimes \cdots \otimes \xi x_0^{(c)}, r_c, s_c = A_{S^{(i)}}$ for all $i \in I$. □

Lemma 6.16. We have $\nabla \delta(T^A(n, d)^{>\lambda}) \subseteq T^A(n, \delta)^{>\lambda}$.

Proof. By Proposition 6.4, $T^A(n, d)^{>\lambda}$ is the two-sided ideal of $T^A(n, d)$ generated by all $e_\mu$ with $\mu > \lambda$. So it suffices to prove that for all $\nabla \delta(e_\mu) \in T^A(n, \delta)^{>\lambda}$ for all $\mu > \lambda$. By the second statement of Lemma 6.15, we have $\nabla \delta(e_\mu) \equiv e_{\mu^{(0)}} \otimes \cdots \otimes e_{\mu^{(c)}} \pmod{T^A(n, \delta)^{>\lambda}}$ and hence modulo $T^A(n, \delta)^{>\lambda}$. It remains to observe that $e_{\mu^{(0)}} \otimes \cdots \otimes e_{\mu^{(c)}} \in T^A(n, \delta)^{>\lambda}$. □

Let $n \geq d$. In Theorem 6.6, we have established that $T^A(n, d)$ is a based quasi-hereditary $k$-superalgebra. By the general theory of §2.1, we have standard modules $\{\Delta(\lambda) \mid \lambda \in \Lambda^+_k(n, d)\}$. Each standard module $\Delta(\lambda)$ has basis $\{v_S \mid S \in \text{Std}^X(\lambda)\}$ such that, denoting $v_\lambda := v_{T^n \lambda}$, we have $A_S v_\lambda = v_S$. In particular $e_\lambda v_\lambda = v_\lambda$. We also have a bilinear pairing

\[
(\cdot, \cdot)_\lambda : \Delta(\lambda) \times \Delta^{op}(\lambda) \rightarrow k.
\]

If $k$ is a field, the quotient $L(\lambda)$ of $\Delta(\lambda)$ by the radical of $(\cdot, \cdot)_\lambda$ is an irreducible $T^A(n, d)$-module, and $\Lambda^+_k(n, d)$ is a complete and irredundant set of irreducible $T^A(n, d)$-modules.

Theorem 6.17. Let $\lambda = (\lambda^{(0)}, \ldots, \lambda^{(c)}) \in \Lambda^+_k(n, d)$. Then:
(i) \( \Delta(\lambda) \cong \bigotimes_{i \in I} \Delta(\lambda^{(i)}) \) and \( \Delta^{op}(\lambda) \cong \bigotimes_{i \in I} \Delta^{op}(\lambda^{(i)}) \).

(ii) \textit{Under the isomorphisms of (i), the pairing} \((\cdot, \cdot)_\lambda\) \textit{corresponds to the tensor product of the pairings} \((\cdot, \cdot)_{\lambda^{(i)}}\) \textit{over} \(i \in I\).

\textbf{Proof.} (i) We prove (i) for \( \Delta(\lambda) \), the proof for \( \Delta^{op}(\lambda) \) being similar. Denote \( \bar{\mathcal{T}}A (n, \delta) := \mathcal{T}A (n, \delta) / \mathcal{T}A (n, \delta) > \lambda \) and write \( \bar{\eta} := \eta + \mathcal{T}A (n, \delta) > \lambda \) for \( \eta \in \mathcal{T}A (n, \delta) \). Moreover, for all \( i \in I \), denote \( \bar{T}_a (n, \delta_i) := T_a (n, \delta_i) / T_a (n, \delta_i) > \lambda(i) \) and \( \bar{\eta} := \eta + T_a (n, \delta_i) > \lambda(i) \) for \( \eta \in T_a (n, \delta_i) \). Then \( \bar{\varsigma}_{S(i)} = \bar{\varsigma}_{S(i)} \).

Recall from §2.1 that \( \Delta(\lambda) = \bar{\mathcal{T}}A (n, \delta) \bar{e}_\lambda \) and \( \Delta(\lambda^{(i)}) = \bar{T}_a (n, \delta_i) \bar{e}_{\lambda^{(i)}} \) for all \( i \in I \). By the second statement of Lemma 6.15, we now have that

\[ e_{\lambda} (\bar{v}_{\lambda^{(0)}} \otimes \cdots \otimes \bar{v}_{\lambda^{(r)}}) = (e_{\lambda^{(0)}} v_{\lambda^{(0)}}) \otimes \cdots \otimes (e_{\lambda^{(r)}} v_{\lambda^{(r)}}) = \bar{v}_{\lambda^{(0)}} \otimes \cdots \otimes \bar{v}_{\lambda^{(r)}}. \]

So it follows from Lemma 6.16 that there is a \( T_a (n, \delta) \)-module homomorphism \( \varphi : \Delta(\lambda) \to \Delta(\lambda^{(0)}) \otimes \cdots \otimes \Delta(\lambda^{(r)}) \) which maps \( \bar{v}_{\lambda} \) onto \( \bar{v}_{\lambda^{(0)}} \otimes \cdots \otimes \bar{v}_{\lambda^{(r)}} \). Moreover, by the first statement of Lemma 6.15, for \( S = (S^{(0)}, \ldots, S^{(r)}) \in \text{Std}^Y (\lambda) \), we have

\[ \varphi (\lambda_S (\bar{v}_{\lambda^{(0)}} \otimes \cdots \otimes \bar{v}_{\lambda^{(r)}})) = (\lambda_S^{(0)}) \bar{v}_{\lambda^{(0)}} \otimes \cdots \otimes (\lambda_S^{(r)}) \bar{v}_{\lambda^{(r)}} = \bar{v}_{S^{(0)}} \otimes \cdots \otimes \bar{v}_{S^{(r)}}, \]

so \( \varphi \) is surjective. Now, \( \varphi \) is injective by dimensions.

(ii) It is sufficient to prove that for any \( S = (S^{(0)}, \ldots, S^{(r)}) \in \text{Std}^X (\lambda) \) and \( T = (T^{(0)}, \ldots, T^{(r)}) \in \text{Std}^Y (\lambda) \), we have

\[ (v_S, v_T)_\lambda = \prod_{i \in I} (v_{S(i)}^T, v_{T(i)})_{\lambda^{(i)}}. \]

By definition,

\[ \mathcal{Y}_T \lambda_S \equiv (v_S, v_T)_\lambda e_{\lambda} \mod T_a (n, \delta)^{> \lambda}. \]

Applying the isomorphism \( \varphi \) and using Lemma 6.15, we get

\[ \mathcal{Y}_{T^{(0)}} \lambda_{S^{(0)}} \otimes \cdots \otimes \mathcal{Y}_{T^{(r)}} \lambda_{S^{(r)}} \equiv (v_S, v_T)(e_{\lambda^{(0)}} \otimes \cdots \otimes e_{\lambda^{(r)}}) \mod T_a (n, \delta)^{> \lambda}. \]

But the left-hand side is congruent to

\[ \prod_{i \in I} (v_{S(i)}, v_{T(i)})_{\lambda^{(i)}} (e_{\lambda^{(0)}} \otimes \cdots \otimes e_{\lambda^{(r)}}) \mod T_a (n, \delta)^{> \lambda}, \]

so indeed \( (v_S, v_T)_\lambda = \prod_{i \in I} (v_{S(i)}, v_{T(i)})_{\lambda^{(i)}} \). \qed

\section*{6.3 \textbf{Anti-involution}}

Let \( \tau \) be a homogeneous anti-involution on \( A \). Then \( \tau \) induces a homogeneous anti-involution

\[ \tau_n : M_n (A) \to M_n (A), \quad \xi r, s \mapsto \xi r, s \tau, \]

which in turn induces an anti-involution

\[ \tau_{n,d} : S^A (n, d) \to S^A (n, d), \quad \xi r, s \mapsto \xi r, s \tau, \quad (6.18) \]
where for a tuple $a = a_1 \cdots a_d$ of homogeneous elements, we have denoted

$$a^\tau := \tau(a_1) \cdots \tau(a_d).$$

If $\tau(a) = a$, then $\tau_{n,d}$ restricts to the involution of $T^A_a(n,d)$. Moreover, if $\tau(B_a) = B_a$, $\tau(B_c) = B_c$ and $\tau(B_1) = B_1$, then we have

$$\tau_{n,d} : T^A_a(n,d) \to T^A_a(n,d), \; \eta^b_{r,s} \mapsto \eta^{b^\tau}_{s,r}.$$  \tag{6.19}

Now, suppose in addition that $\tau$ is a standard anti-involution on $A$ with $y(x) = \tau(x)$, see §2.2. In §3.3, to define standard tableaux we have fixed arbitrary total orders on all $\mathcal{X}(i)$ and $\mathcal{Y}(i)$. Note that the map $r^x \mapsto r^{y(x)}$ induces a bijection between $\mathcal{X}(i)$ and $\mathcal{Y}(i)$. Let us choose the total order on $\mathcal{X}(i)$ and $\mathcal{Y}(i)$ so that this bijection is an isomorphism of totally ordered sets.

Let $\lambda \in \Lambda^i(n,d)$. Given a tableau $S \in \text{Tab}^X(\lambda)$ we define a tableau $S^\tau \subset \text{Tab}^Y(\lambda)$ via $S^\tau_k = r^{y(x)}$ if $S_k = r^x$ for all $k = 1, \ldots, d$. Due to the choice of total orders made in the previous paragraph, we have that $S \mapsto S^\tau$ is a bijection between $\text{Tab}^X(\lambda)$ and $\text{Tab}^Y(\lambda)$, which restricts to a bijection between $\text{Std}^X(\lambda)$ and $\text{Std}^Y(\lambda)$.

**Proposition 6.20.** Let $n \geq d$ and $A$ be a based quasi-hereditary $\kappa$-superalgebra with standard anti-involution $\tau$. Then, considering $T^A_a(n,d)$ as a based quasi-hereditary algebra as in Theorem 6.6, the involution $\tau_{n,d}$ of $T^A_a(n,d)$ is standard, with $\tau(A_X) = Y_S^r$ for all admissible $S$.

**Proof.** The first statement follows from the second one, which in turn follows using (6.19). \hfill $\Box$

### 6.4 Idempotent truncation

Let $e \in a$ be an idempotent. Set $\bar{A} := eAe$ and $\bar{a} := eae$. Recalling (4.19), we can associate to $e$ the idempotent

$$\xi^e := \sum_{\lambda \in \Lambda^i(n,d)} \xi(\lambda,e) \in T^A_a(n,d).$$

**Lemma 6.21** [13, Lemma 5.12]. We have:

(i) $S^\xi(n,d) = \xi^e S^A(n,d) \xi^e$.

(ii) $T^\xi_a(n,d) = \xi^e T^A_a(n,d) \xi^e$.

Suppose now that $e$ is adapted with respect to $I, X, Y$, with the corresponding $e$-truncation $I, X, Y$, see §2.2. For a subset $J \subset I$, we consider $\Lambda^I_+(n,d)$ as the subset of $\Lambda^i_+(n,d)$ as follows:

$$\Lambda^I_+(n,d) = \{ \lambda \in \Lambda^i_+(n,d) \mid \lambda^{(i)} = \emptyset \text{ if } i \not\in J \}.$$  

In particular, we have $\Lambda^I_+(n,d) \subset \Lambda^i_+(n,d)$.

**Lemma 6.22.** Let $n \geq d$ and $e \in a$ be an idempotent.

(i) If $e$ is adapted with respect to the heredity data $I, X, Y$ of $A$ with $e$-truncation $\bar{I}, \bar{X}, \bar{Y}$, then $\xi^e$ is adapted with respect to the heredity data $\Lambda^I_+(n,d), \bar{X}, \bar{Y}$ of $T^A_a(n,d)$, with $\xi^e$-truncation...
\[\Lambda^I_+(n,d), \tilde{\mathcal{X}}, \tilde{\mathcal{Y}}, \text{ where} \]

\[\tilde{\mathcal{X}} = \bigsqcup_{\lambda \in \Lambda^I_+(n,d)} \{\lambda^*_S \mid S \in \text{Std}^Y(\lambda)\},\]

\[\tilde{\mathcal{Y}} = \bigsqcup_{\lambda \in \Lambda^I_+(n,d)} \{\mathcal{Y}_T \mid T \in \text{Std}^Y(\lambda)\}.\]

(ii) If \(e\) is strongly adapted, then so is \(\xi^e\).

Proof.

(i) Let \(x = x_1 \cdots x_d \in X^d\). For \(k \in [1,d]\), we have \(ex_k = \delta_{\{x_k \in X\}} x_k\), whence \(\xi^e r_{r,s} = \xi^e \delta_{\{x_{1,\cdots,d}\}} r_{r,s}\).

In particular, for \(S \in \text{Std}^X(\lambda)\), we have \(\xi^e \lambda^*_S = \delta_{\{S \in \text{Std}^X(\lambda)\}} \lambda^*_S\). Similarly for \(T \in \text{Std}^Y(\lambda)\), we have \(\mathcal{Y}_T \xi^e = \delta_{\{T \in \text{Std}^Y(\lambda)\}} \mathcal{Y}_T\). Thus \(\xi^e\) is adapted with \(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}\) as in the statement of the lemma.

It remains to show that \(\Lambda^I_+(n,d) = \Lambda^I_+(n,d)\). Note that \(\Lambda^I_+(n,d) \subseteq \Lambda^I_+(n,d)\) since if \(\lambda \in \Lambda^I_+(n,d) \setminus \Lambda^I_+(n,d)\) and \(S \in \text{Std}^X(\lambda)\), then \(x^S \notin X^d\). To prove the converse inclusion we just need to observe, using the fact that \(n \geq d\), that for every \(\lambda \in \Lambda^I_+(n,d)\) there exist \(S \in \text{Std}^X(\lambda)\) and \(T \in \text{Std}^Y(\lambda)\).

(ii) If \(e\) is strongly adapted, then for any \(\lambda \in \Lambda^I_+(n,d)\), we have \(ee_{\lambda} = e_{\lambda} = e_{\lambda}e\), which implies the claim. \(\square\)

In the following result we consider \(T^A_{\tilde{\mathcal{A}}}(n,d)\) as the subalgebra of \(\xi^e T^A_{\tilde{\mathcal{A}}}(n,d)\xi^e \subseteq T^A_{\tilde{\mathcal{A}}}(n,d)\) using Lemma 6.21. Recall the anti-involution \(\tau_{n,d}\) from Proposition 6.20.

Proposition 6.23. Let \(\tau\) be a standard anti-involution on \(A\) and \(e \in \mathfrak{a}\) be a adapted \(\tau\)-invariant idempotent. Then \(T^A_{\tilde{\mathcal{A}}}(n,d)\) is a cellular algebra with cellular basis

\[\{C^A_{S,T} \mid \lambda \in \Lambda^I_+(n,d), S, T \in \text{Std}^X(\lambda)\},\]

where we have set \(C^A_{S,T} := B^A_{S,T} r_{r,s}\).

Proof. We use Lemma 6.22(i), which shows that \(\xi^e\) is an adapted idempotent with \(\xi^e\)-truncation \(\Lambda^I_+(n,d), \tilde{\mathcal{X}}, \tilde{\mathcal{Y}}\). By Proposition 6.20, \(\tau_{n,d}\) is a standard anti-involution on \(T^A_{\tilde{\mathcal{A}}}(n,d)\) with \(\tau(B^A_{S,T}) = B^A_{T,S}\). Since \(\xi^e\) is obviously \(\tau_{n,d}\)-invariant, \([12, \text{Lemma 4.4(ii)}]\) implies the result. \(\square\)

Remark 6.24. Let \(\tilde{\mathcal{A}}\) be a cellular algebra with cellular basis \(\tilde{B}\) and a subalgebra \(\tilde{a} \subseteq \tilde{\mathcal{A}}\). The following question was raised in \([12, \text{Remark 4.6}]:\) is there a based quasi-hereditary algebra \(A\) with heredity basis \(B\), a standard anti-involution \(\tau\), and \(\tau\)-invariant adapted idempotent \(e\) such that \(\tilde{A} = e Ae, \tilde{a} = eae,\) and \(\tilde{B}\) is the \(e\)-truncation of \(\tilde{B}\)? If such \(A\) exists, which at least happens in many examples, it follows from the previous proposition that \(T^A_{\tilde{\mathcal{A}}}(n,d)\) is cellular.

Note that in the following lemma we do not require that \(n \geq d\). While it is not in general true that \(T^A_{\tilde{\mathcal{A}}}(n,d)\) is quasi-hereditary, the lemma shows that at least it is cellular under some natural assumptions.
Lemma 6.25. Suppose that \((A,\alpha)\) is a unital pair. If \(A\) possesses a standard anti-involution, then the algebra \(T^A_{\alpha}(n, d)\) is cellular.

Proof. We may assume that \(n < d\), in which case, by \([13, \text{Lemma 5.15(ii)}]\), we can realize \(T^A_{\alpha}(n, d)\) as the idempotent truncation \(\xi^d_n T^A_{\alpha}(d, d) \xi^d_n\), where \(\xi^d_n\) is a \(\tau_{n,d}\)-invariant idempotent. Now we apply \([12, \text{Lemma 4.4(ii)}]\).

If \(e \in A\) is an adapted idempotent, then by Lemma 2.14, there is a subset \(I' \subseteq I\) with \(eL(i) \neq 0\) if and only if \(i \in I'\).

Lemma 6.26. Let \(n \geq d\). Suppose that \(\kappa\) is a field, \(A_{1} \subseteq J(A)\), and \(e \in \alpha\) is an idempotent adapted with respect to the heredity data \(I, X, Y\) of \(A\). Then \(\xi e L(\lambda) \neq 0\) if and only if \(\lambda \in \Lambda'^{I'}(n, d)\). In particular, \(\{\xi e L(\lambda) \mid \lambda \in \Lambda'^{I'}(n, d)\}\) is a complete and irredundant set of irreducible \(\xi e T^A_{\alpha}(n, d) \xi e\)-modules up to isomorphism.

Proof. By Lemma 6.22, \(\xi e\) is an adapted idempotent, which will be used repeatedly without further reference.

If \(\lambda \in \Lambda'^{I'}(n, d)\), then there exists \(i \in I \setminus I'\) such that \(\lambda^{(i)} \neq \emptyset\). By Lemma 2.14, the condition \(i \notin I'\) implies that \(yex \in A_{>i}\) for all \(x \in X(i)\) and \(y \in Y(i)\). So we have \(V_{T} \xi e X_{S} \in T^A_{\alpha}(n, d) \lambda_{T}^{=\lambda}\) for all \(S \in \text{Std}^{\lambda}(\lambda)\) and \(T \in \text{Std}^{\lambda}(\lambda)\). Hence \(\xi e L(\lambda) = 0\) by Lemma 2.14 again.

In the other direction, assume that \(\lambda \in \Lambda'^{I'}(n, d)\) with \(d_{i} := |\lambda^{(i)}|\). By Lemmas 2.14 and 2.10(iii), for every \(i \in I'\) there exists \(x_{i} \in X(i)\) and \(y_{i} \in Y(i)\) such that \(y_{i} x_{i} \equiv \kappa \epsilon_{i} \text{ (mod } A^{>i})\), for some \(\kappa \neq 0\). Then \(y_{i} L(i) \neq 0\), and since \(A_{1} \subseteq J(A)\), we have that \(x_{i}, y_{i} \in A_{0}\). Then there exists \(S \in \text{Std}^{\lambda}(\lambda)\), \(T \in \text{Std}^{\lambda}(\lambda)\) such that \(S_{k}^{(i)} = T_{k}^{(i)} = \text{let}(T_{k}^{(i)})\), \(\text{col}(S_{k}^{(i)}) = x_{i}\), and \(\text{col}(T_{k}^{(i)}) = y_{i}\), for all \(i \in I'\) and \(k \in [1, d_{i}]\). Applying Theorem 6.17(ii), we have that \(V_{T} X_{S} \equiv \kappa_{d_{0}} \cdots \kappa_{d_{e}} \epsilon_{\lambda} \neq 0\) (mod \(T^A_{\alpha}(n, d) \lambda_{T}^{=\lambda}\)), so \(\xi e L(\lambda) \neq 0\) by Lemma 2.14.

### 7 | Decomposition Numbers

Let again \(\kappa\) be a commutative integral domain of characteristic zero, and suppose that we are given a ring homomorphism \(\kappa \to F\), where \(F\) is a field of characteristic \(p > 0\). An important example is when \((\mathbb{L}, \kappa, \mathbb{K})\) is a modular system and \(F = \mathbb{K}\) or \(\mathbb{L}\), that is, \(\kappa\) is a complete discrete valuation ring of characteristic 0, and either \(F\) is the field of fractions of \(\kappa\) or \(F\) is the residue class field of \(\kappa\).

Throughout the section we assume that \(d \leq n\). Recall from Theorem 6.6 that, starting with a based quasi-hereditary \(\kappa\)-superalgebra \(A\) with \(\alpha\)-conforming heredity data, we have constructed a based quasi-hereditary \(\kappa\)-superalgebra \(T^A_{\alpha}(n, d)\).

#### 7.1 | Set-up

Define

\[
A_{F} := A \otimes_{\kappa} F \quad \text{and} \quad T^A_{\alpha}(n, d)_{F} := T^A_{\alpha}(n, d) \otimes_{\kappa} F.
\]

As \(T^A_{\alpha}(n, d)_{F}\) is a based quasi-hereditary \(\kappa\)-superalgebra with heredity data \(\Lambda^I_{\alpha}(n, d), X, Y\), it is immediate that \(T^A_{\alpha}(n, d)_{F}\) is a based quasi-hereditary \(F\)-superalgebra with heredity data...
\[ \Lambda_+(n, d), X_\varphi, Y_\varphi, \] where \( X_\varphi = \{ x \otimes 1 \mid x \in X \}, Y_\varphi = \{ y \otimes 1 \mid x \in X \} \subseteq T^A(n, d) \). Normally we will drop indices and for example write \( X \) for \( X_\varphi \), and so on.

**Remark 7.1.** If \( F \) has characteristic \( p = 0 \), then \( T^A(n, d) \cong T^A(n, d) \). However, if \( p > 0 \), then in general \( T^A(n, d) \) is a “wrong object”; in particular, it does not have to be quasi-hereditary, and we might have \( \dim T^A(n, d) < \dim T^A(n, d) \) due to the presence of factorials in (4.8).

Let \( \lambda \in \Lambda_+(n, d) \). The standard objects \( \Delta^F(\lambda) \) and \( \Delta^{op}(\lambda) \) over \( T^A(n, d) \) are obtained by extending scalars from \( k \) to \( F \) from the standard objects \( \Delta(\lambda) \) and \( \Delta^{op}(\lambda) \) over \( T^A(n, d) \), and the pairing \( (\cdot, \cdot)_\lambda : \Delta(\lambda) \times \Delta^{op}(\lambda) \to F \) is obtained from the pairing \( (\cdot, \cdot)_\lambda \) also by extending scalars. So for the irreducible \( T^A(n, d) \)-modules \( L(\lambda) := \Delta(\lambda) / \text{rad}(\cdot, \cdot)_\lambda \) we have from Theorem 6.17:

**Lemma 7.2.** If \( \lambda = (\lambda^{(0)}, \ldots, \lambda^{(r)}) \in \Lambda_+(n, d) \), then \( L(\lambda) \cong \otimes_{i \in I} L(\lambda^{(i)}) \).

We would like to describe the decomposition numbers

\[ d^F_{\lambda, \mu} := [\Delta(\lambda) : L(\mu)] \]

of the quasi-hereditary algebra \( T^A(n, d) \) in terms of the decomposition numbers \( d^F_{i,j} \) of \( A^F \) and the decomposition numbers \( d_{\lambda, \mu}^{cl,F} \) of the classical Schur algebra \( S(n, d) \).

From now on, until the end of the section, since \( F \) is fixed, we drop “\( F \)” from all the indices and write \( T^A(n, d) = T^A(n, d) \), \( \Delta(\lambda) = \Delta(\lambda) \), \( d_{\lambda, \mu}^{cl} = d_{\lambda, \mu}^{cl,F} \), and so on.

### 7.2 Classical characters

Let us write the operation \( (\mu, \nu) \mapsto \mu + \nu \) on the monoid \( \Lambda(n) \) multiplicatively, and let \( \mathbb{Z}\Lambda(n) \) be the corresponding monoid algebra with coefficients in \( \mathbb{Z} \). So the product \( st \) is defined for any \( s, t \in \mathbb{Z}\Lambda(n) \). Moreover, if \( s \in \mathbb{Z}\Lambda(n, d) \) and \( t \in \mathbb{Z}\Lambda(n, e) \), then \( st \in \mathbb{Z}\Lambda(n, d + e) \). We identify

\[ \mathbb{Z}(\Lambda(n)^m) = (\mathbb{Z}\Lambda(n))^\otimes m \quad (m \in \mathbb{Z}_{>0}). \quad (7.3) \]

with \( (\mu^1, \ldots, \mu^m) \) on the left corresponding to \( \mu^1 \otimes \cdots \otimes \mu^m \) on the right.

Let \( \lambda \in \Lambda_+(n, d) \). For \( \mu \in \Lambda(n, d) \), the *Kostka number* \( k_{\lambda, \mu} \) is the number of the classical standard \( \lambda \)-tableaux of weight \( \mu \). We denote by \( \Delta^{cl}(\lambda) \) and \( L^{cl}(\lambda) \) the standard and the irreducible module with high weight \( \lambda \) over the classical Schur algebra \( S(n, d) \), see [9]. We have the classical decomposition numbers

\[ d_{\lambda, \mu}^{cl} := [\Delta^{cl}(\lambda) : L^{cl}(\mu)] \quad (\lambda, \mu \in \Lambda_+(n)), \quad (7.4) \]

which is interpreted as zero if \( |\lambda| \neq |\mu| \). For \( \lambda, \mu \in \Lambda_+(n) \), we will also need the products of the classical decomposition numbers:

\[ d_{\lambda, \mu}^{cl} := \prod_{i \in I} d_{\lambda^{(i)}, \mu^{(i)}}^{cl}. \quad (7.5) \]
For the classical formal characters we have

\[
\mathbf{s}_\lambda := \text{ch} \Delta^d(\lambda) = \sum_{\mu \in \Lambda(n, d)} k_{\lambda, \mu} \cdot \mu \in \mathbb{Z}(n, d), \quad (7.6)
\]

\[
\bar{s}_\lambda := \text{ch} L^\circ(\lambda) = \sum_{\mu \in \Lambda(n, d)} \bar{k}_{\lambda, \mu} \cdot \mu \in \mathbb{Z}(n, d), \quad (7.7)
\]

where \( k_{\lambda, \mu} \) are the Kostka numbers and the weight multiplicities \( \bar{k}_{\lambda, \mu} := \dim L(\lambda) \mu \) are not known in general if \( p > 0 \). Note that

\[
\mathbf{s}_\lambda = \sum_{\mu \in \Lambda_+(n)} d^\circ_{\lambda, \mu} \bar{s}_\mu. \quad (7.8)
\]

For \( \mu^1 \in \Lambda_+(n, d_1), \ldots, \mu^m \in \Lambda_+(n, d_m) \), we can write

\[
\mathbf{s}_{\mu^1} \cdots \mathbf{s}_{\mu^m} = \sum_{\lambda \in \Lambda_+(n)} c^\lambda_{\mu^1, \ldots, \mu^m} S_{\lambda}, \quad (7.9)
\]

where \( c^\lambda_{\mu^1, \ldots, \mu^m} \) are the classical Littlewood–Richardson coefficients. We have \( c^\lambda_{\mu^1, \ldots, \mu^m} = 0 \) unless \( d = d_1 + \cdots + d_m \). Denoting by \( \underline{\mu} = (\mu^k)_{k \in [1, m]} \) the unordered tuple of the partitions \( \mu^1, \ldots, \mu^m \), we define

\[
c^\lambda_{\underline{\mu}} := c^\lambda_{\mu^1, \ldots, \mu^m}, \quad (7.10)
\]

which makes sense by the commutativity of tensor product. The following follows easily from the definitions.

**Lemma 7.11.** Let \( \lambda \in \Lambda_+(n) \), and \( u_1, \ldots, u_m \in \mathbb{Z}_{\geq 0} \) for some \( m \geq 1 \). Suppose that we are given a tuple \( \underline{\mu} = (\mu^r_s)_{1 \leq r \leq m, 1 \leq s \leq u_r} \) of partitions in \( \Lambda_+(n) \). Then

\[
c^\lambda_{\underline{\mu}} = \sum_{\chi = (\chi^r_s)_{1 \leq r \leq m}} c^\lambda_{\chi^1, \ldots, \chi^m} \prod_{r=1}^m c^{\chi^r_{u_r}}_{\mu^r_1, \ldots, \mu^r_{u_r}}. \]

Let \( \mu \in \Lambda_+(n, e) \) and \( \lambda \in \Lambda_+(n, e + d) \) for some \( d, e \in \mathbb{Z}_{\geq 0} \) be such that \([\mu] \subseteq [\lambda] \). Denote

\[
[\lambda / \mu] := [\lambda] \setminus [\mu].
\]

An even (respectively, odd) standard \( \lambda / \mu \)-tableau is a function \( T: [\lambda / \mu] \to [1, n] \) such that whenever \( M < N \) are nodes in the same row of \([\lambda / \mu] \), then \( T(M) \leq T(N) \) (respectively, \( T(M) < T(N) \)), and whenever \( M < N \) are nodes in the same column of \([\lambda / \mu] \), then \( T(M) < T(N) \) (respectively, \( T(M) \leq T(N) \)). Let \( St^0(\lambda / \mu) \) (respectively, \( St^1(\lambda / \mu) \)) denote the set of all even (respectively, odd) standard \( \lambda / \mu \)-tableaux.

Let \( \varepsilon \in \mathbb{Z}/2 \) and \( T \in St^\varepsilon(\lambda / \mu) \). The weight of \( T \) is the composition

\[
\omega^T = (\omega^T_1, \ldots, \omega^T_n) \in \Lambda(n, d)
\]

with

\[
\omega^T_r := \# \{ N \in [\lambda / \mu] \mid T(N) = r \} \quad (r = 1, \ldots, n).
\]
We note that, if \( \lambda \in \Lambda_+(n,d) \), \( \mu = \emptyset \), and \( T \) is a classical (even) standard \( \lambda \)-tableau as defined in §3.3, then \( \omega^T \) corresponds with the classical notion of the weight of \( T \).

Define

\[
\mathbf{s}^\varepsilon_{\lambda / \mu} := \sum_{T \in \text{St}_t(\lambda / \mu)} \omega^T \in \mathbb{Z} \Lambda(n, d).
\]

For \( \nu \in \Lambda_+(n, d) \) we denote

\[
\nu^\varepsilon := \begin{cases} 
\nu & \text{if } \varepsilon = \overline{0}, \\
\nu^\text{transpose} & \text{if } \varepsilon = \overline{1}.
\end{cases}
\]

As \( d \leq n \), we can interpret \( \nu^\varepsilon \) as an element of \( \Lambda_+(n, d) \) again.

**Lemma 7.12.** Let \( \mu \in \Lambda_+(n, e) \) and \( \lambda \in \Lambda_+(n, e + d) \) for some \( d, e \in \mathbb{Z}_{\geq 0} \) be such that \([\mu] \subseteq [\lambda]\). Then

\[
\mathbf{s}^\varepsilon_{\lambda / \mu} = \sum_{\nu \in \Lambda_+(n, d)} \mathbf{c}^\lambda_{\mu, \nu \varepsilon} \mathbf{s}_\nu.
\]

**Proof.** Follows from \([16, (3.8),(5.2),(5.3),(5.6),(5.13)]\). \(\square\)

We need the following generalization of the above. Let \( \lambda \in \Lambda_+(n, d) \) and

\[
d = (d_1, \ldots, d_m) \in \Lambda(m, d)
\]

for some \( m \geq 1 \). We denote by \( \Lambda^\lambda_{+,d} \) the set of all tuples \( \bar{\mu} = (\mu^1, \ldots, \mu^m) \) such that

\[
\mu^t \in \Lambda_+(n, d_1 + \cdots + d_t) \text{ for } t = 1, \ldots, m, \quad \text{and} \quad [\mu^1] \subseteq \cdots \subseteq [\mu^m] = [\lambda].
\]

We will usually write \( \bar{\mu} = \mu^m / \cdots / \mu^1 \) instead of \( \bar{\mu} = (\mu^1, \ldots, \mu^m) \), and interpret \([\mu^0]\) as \( \emptyset \). Denote

\[
\Lambda^\lambda_{+,m} := \bigcup_{d \in \Lambda(m, d)} \Lambda^\lambda_{+,d}.
\]

(7.13)

Let \( \bar{\mu} = \mu^m / \cdots / \mu^1 \in \Lambda^\lambda_{+,m} \) and \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \in (\mathbb{Z} / 2)^m \). A standard \( \bar{\mu} \)-tableau of parity \( \varepsilon \) is a function \( \bar{T} : [\lambda] \to [1, n] \) such that \( \bar{T}_t := \bar{T}_{[\mu^t / \mu^{t-1}]} \in \text{St}_{\varepsilon_t}(\mu^t / \mu^{t-1}) \) for all \( t = 1, \ldots, m \). Let \( \text{St}_{\varepsilon}(\bar{\mu}) \) denote the set of all standard \( \bar{\mu} \)-tableaux of parity \( \varepsilon \). The weight of a tableau \( \bar{T} \in \text{St}_{\varepsilon}(\bar{\mu}) \) is

\[
\omega^{\bar{\mu}, \bar{T}} := (\omega^{t_1}, \ldots, \omega^{t_m}) \in \Lambda(n)^{\times m}.
\]

(7.14)

Now, define

\[
\mathbf{s}^\varepsilon_{\lambda,m} := \sum_{\bar{\mu} \in \Lambda^\lambda_{+,m}, \bar{T} \in \text{St}_{\varepsilon}(\bar{\mu})} \omega^{\bar{\mu}, \bar{T}} \in \mathbb{Z}(\Lambda(n)^{\times m}).
\]

(7.15)
If \( \nu = (\nu^1, ..., \nu^m) \in \Lambda(n)^m \) with \( |\nu^k| < n \) for all \( k = 1, ..., m \), and \( \varepsilon = (\varepsilon_1, ..., \varepsilon_m) \in (\mathbb{Z}/2)^m \), we denote
\[

\nu^\varepsilon := ((\nu^1)^{\varepsilon_1}, ..., (\nu^m)^{\varepsilon_m}) \in \Lambda(n)^m.
\]

Recall (7.3). Using Lemma 7.12, we have the following.

**Corollary 7.16.** Let \( \lambda \in \Lambda_+(n, d) \) and \( \varepsilon = (\varepsilon_1, ..., \varepsilon_m) \in (\mathbb{Z}/2)^m \). Then
\[

s_{\lambda, m}^\varepsilon = \sum_{\nu = (\nu^1, ..., \nu^m) \in \Lambda_+(n)^m} c_{\nu}^\lambda s_{\nu^1} \otimes \cdots \otimes s_{\nu^m}.
\]

### 7.3 Characters

Let \( V \in A - \text{mod} \) and \( W \in T^A(n, d) - \text{mod} \). If \( e_i V \) is free of finite rank as a graded \( k \)-supermodule for all \( i \in I \), we say that \( V \) has free weight spaces. Similarly, if \( e_\lambda W \) is free of finite rank as a graded \( k \)-supermodule for all \( \lambda \in \Lambda^I(n, d) \), we say that \( W \) has free weight spaces. Suppose that \( V \) and \( W \) have free weight spaces. Although in general \( \bigoplus_{i \in I} e_i V \subsetneq V \) and \( \bigoplus_{\mu \in \Lambda^I(n, d)} e_\mu W \subsetneq W \), we define the (bigraded) characters:
\[

\text{ch}^q_{\pi} V := \sum_{i \in I} (\dim_q e_i V) i \in RI,
\]
\[

\text{ch}^q_{\pi} W := \sum_{\mu \in \Lambda^I(n, d)} (\dim_q e_\mu W) \mu \in RA^I(n, d).
\]

**Lemma 7.17** [13, Lemma 5.10]. Suppose that \( W_1 \in T^A(n, d_1) - \text{mod} \) and \( W_2 \in T^A(n, d_2) - \text{mod} \) have free weight spaces. We consider \( W_1 \otimes W_2 \) as a \( T^A(n, d_1 + d_2) \)-supermodule via the coproduct \( \Delta \). Then \( W_1 \otimes W_2 \) has free weight spaces, and
\[

\text{ch}_{\pi}(W_1 \otimes W_2) = \text{ch}_{\pi}(W_1) \text{ ch}_{\pi}(W_2).
\]

As in (7.3), we always identify
\[

RA^I(n) = (RA(n))^\otimes I,
\]
with \( \mu = (\mu^{(0)}, ..., \mu^{(I)}) \) on the left corresponding to \( \mu^{(0)} \otimes \cdots \otimes \mu^{(I)} \) on the right.

Recall that \( e_\lambda \Delta(\lambda) = Fv_\lambda \) and \( e_\mu \Delta(\lambda) \neq 0 \) implies \( \mu \leq \lambda \). Similar properties hold for \( L(\lambda) \).

Therefore:

**Lemma 7.19.** The subsets \( \{ \text{ch}^q_{\pi} \Delta(\lambda) \mid \lambda \in \Lambda^I_+(n, d) \} \) and \( \{ \text{ch}^q_{\pi} L(\lambda) \mid \lambda \in \Lambda^I_+(n, d) \} \) are \( R \)-linearly independent in \( RA^I(n, d) \).

For \( i, j \in I \), we set
\[

jX(i) := \{ x \in X(i) \mid e_j x = x \} \quad \text{and} \quad jX := \bigsqcup_{i \in I} jX(i).
\]
We have \( jX(i) = \{e_i\} \) and \( jX(i) = \emptyset \) unless \( j \leq i \). Each \( jX(i) \) splits as unions
\[
jX(i) = jX(i)_0 \cup jX(i)_1 = \bigcup_{\varepsilon \in \mathbb{Z}/2, n \in \mathbb{Z}} jX(i)^n_{\varepsilon}, \tag{7.21}
\]
see (2.1). We set
\[
X'(i) := \bigsqcup_{j \in I} jX(i) = jX(i) \cup jX(i-1) \cup \cdots \cup jX(i) \subseteq X(i),
\]
\[
X' := \bigsqcup_{i \in I} X'(i) = \bigsqcup_{j \in I} jX(i).
\tag{7.22}
\]
Recall the basis \( \{v_S \mid S \in \text{Std}^X(\lambda)\} \) of \( \Delta(\lambda) \) from §6.2. Since \( e_i x = 0 \) for all \( x \in X \setminus X' \) and all \( i \), we have the following.

**Lemma 7.23.** We have \( e_{\mu} v_S = 0 \) for all \( \mu \in \Lambda^L(n, d) \), unless \( S \in \text{Std}^X'(\lambda) \).

Let \( A \) be basic, that is, all irreducible \( A \)-modules \( L(i) \) are 1-dimensional. In this case we have \( 1_A = \sum_{i \in I} e_i \), \( X' = X \), and \( 1_{T^A(n,d)} = \sum_{\mu \in \Lambda^L(n,d)} e_\mu \). Moreover, for all \( i \in I \), we have
\[
\chi^2_n L(i) = i \quad \text{and} \quad \chi^2_n \Delta(i) = \sum_{j \leq i} d_{i,j}(q, \pi) \cdot j. \tag{7.24}
\]
In view of (7.24), we have
\[
|jX(i)^n_{\varepsilon}| = d_{i,j}^{n,\varepsilon} := \sum_{n \in \mathbb{Z}} d_{i,j}^{n,\varepsilon} \quad \text{and} \quad |jX(i)^n_{\varepsilon}| = d_{i,j}^{n,\varepsilon}.\]

**Lemma 7.25.** Let \( A \) be basic and \( i \in I \). Then \( \text{rad} \Delta(i) = \text{span}\{v_x \mid x \in X(i) \setminus \{e_i\}\} \).

*Proof.* As the heredity data are basic, it follows that the codimension of \( \text{rad} \Delta(i) \) in \( \Delta(i) \) is 1, which implies the lemma since \( v_i \not\in \text{rad} \Delta(i) \). \( \square \)

### 7.4 Characters of standard and irreducible modules

We first study the characters of standard \( T^A(n,d) \)-modules. Let \( \lambda \in \Lambda^L(n,d) \). For \( S \in \text{Std}^X(\lambda) \), recall \( \alpha^S \in \Lambda^L(n,d) \) from (6.2) so that \( \alpha^S = \langle \alpha^{(0)}, \ldots, \alpha^{(r)} \rangle \) with
\[
\alpha^{(i)} := \#\{k \in [1,d] \mid \text{let}(S_k) = r \ \text{and} \ \text{col}(S_k) \in jX\}. \tag{7.26}
\]
Recalling (2.3), the *bi-degree* of \( S \) is defined to be
\[
\text{deg}(S) := \text{deg}(x_1) \cdots \text{deg}(x_d) \in R. \tag{7.27}
\]
Using Lemmas 6.3(iii) and 7.23, we deduce that for any \( \mu \in \Lambda^L(n,d) \) we have
\[
\dim e_\mu \Delta(\lambda) = \sum_{S \in \text{Std}^X(\lambda), \alpha^S = \mu} \text{deg}(S) = \sum_{S \in \text{Std}^X(\lambda), \alpha^S = \mu} \text{deg}(S).\]
Therefore:

**Lemma 7.28.** We have

\[ \text{ch}_q^\pi \Delta(\lambda) = \sum_{S \in \text{Std}^X(\lambda)} \text{deg}(S) \cdot \alpha^S. \]

Now we turn to characters of irreducible \( T_A^{\lambda}(n, d) \)-modules. Let \( \text{Std}^X_0(\lambda) \subseteq \text{Std}^X(\lambda) \) be the set of all standard \( \lambda \)-tableaux \( S = (S^{(0)}, \ldots, S^{(e)}) \) such that for all \( i \in I \), the entries of \( S^{(i)} \) are of the form \( r_{e_i} \). Since the elements \( e_i \) are even, replacing every entry \( r_{e_i} \) with \( r \) yields a bijection between \( \text{Std}^X_0(\lambda) \) and \( \text{St}_{\lambda(0)} \times \cdots \times \text{St}_{\lambda(e)} \). Recalling (7.18) and (7.7), we define

\[ s_\lambda := s_{\lambda(0)} \otimes \cdots \otimes s_{\lambda(e)} \in RA^I(n), \quad (7.29) \]

\[ \bar{s}_\lambda := \bar{s}_{\lambda(0)} \otimes \cdots \otimes \bar{s}_{\lambda(e)} \in RA^I(n). \quad (7.30) \]

**Lemma 7.31.** Let \( A \) be basic. Then \( \text{ch}_q^\pi L(\lambda) = \bar{s}_\lambda \).

*Proof.* In view of Lemma 7.2, we may assume that \( \lambda = (0, \ldots, 0, \lambda^{(i)}, 0, \ldots, 0) \) with \( \lambda^{(i)} \in \Lambda(n, d) \) in the \( i \)th component for some \( i \in I \). Recall the elements \( \lambda_S = \xi_{\lambda_S}^x \) from §5.3. By definition, the module \( \Delta(\lambda) \) is an \( F \)-subspace of \( T_A^{\lambda}(n, d) / T_A^{\lambda}(n, d)^{A>\lambda} \) with basis \( \{ \xi_{\lambda_S}^x \mid S \in \text{Std}^X(\lambda) \} \), where we write \( \xi := \xi + T_A^{\lambda}(n, d)^{A>\lambda} \) for \( \xi \in T_A^{\lambda}(n, d) \).

If \( x^S = x_1 \cdots x_d \) and \( x_k \neq e_i \) for some \( k \in [1, d] \), then \( x_k \in \text{rad} \Delta(i) \) by Lemma 7.25, and it follows from the definition of the pairing \((\cdot, \cdot)_\lambda \) that \( \xi_{\lambda_S}^x \in \text{rad} \Delta(\lambda) \). Let now \( x^S = e_i^d \), that is, \( S \in \text{Std}^X_0(\lambda) \). Since \( (v_S, w_T) \) is the coefficient of \( e_2^d \xi_{\lambda_S}^x \) in \( \bar{Y}_T \bar{R}_S = \xi_{\lambda_S}^{x^T} \xi_{\lambda_S}^{x^S} \), we have that \( (v_S, w_T) = 0 \) unless \( y^T = e_i \). But then by Proposition 4.7, using \( e_i^2 = e_i \), we obtain

\[ \xi_{\lambda_S}^{x^d} = \sum_{r,s} c_{r,s} \xi_{\lambda_S}^{x^d}, \]

where the coefficients \( c_{r,s} \) are determined from

\[ \xi_{\lambda_S}^{x^d} = \sum_{r,s} c_{r,s} \xi_{\lambda_S}^{x^d}, \]

in the classical Schur algebra \( S(n, d) \). The result follows.

**Corollary 7.32.** Let \( A \) be basic. Then \( s_\lambda = \sum_{\mu \in \Lambda^I(n)} q^{\mu} \text{ch}_n^\lambda L(\mu) \).

*Proof.* This follows immediately from Lemma 7.31 and (7.8).

### 7.5 One-colored standard characters

We continue with the set-up of §7.4. Throughout the subsection, we fix \( i \in I \) and \( \lambda \in \Lambda^I(n, d) \) of the form \( \lambda = \lambda^{(i)} \), see (6.14). We identify the Young diagram of \( \lambda^{(i)} \) and the Young diagram of \( \lambda^{(i)} \)
via the map $(r, s) \mapsto (i, r, s)$ on nodes. In the same way we also identify the sets $\text{Std}^{X'(i)}(\lambda(i))$ and $\text{Std}^{X'(i)}(\lambda(i))$ so that we have a function $\text{deg} : \text{Std}^{X'(i)}(\lambda(i)) \to \mathbb{R}$ as defined in (7.27).

Recalling (7.22), we list the elements of $X'(i)$ as

$$X'(i) = \{x_{i,1} = e_i, x_{i,2}, \ldots, x_{i,t_i}\}, \quad (7.33)$$

so that elements of $iX(i)_0$ precede elements of $iX(i)_1$ precede elements of $i^{-1}X(i)_0$, ..., precede elements of $0X(i)_1$. Define the segments

$$j\Omega(i) := \{u \in [1, t_i] \mid x_{i,u} \in jX(i)\},$$

so that $[1, t_i] = i\Omega(i) \cup i^{-1}\Omega(i) \cup \cdots \cup 0\Omega(i)$. We order $X'(i)$ so that $x_{i,1} < x_{i,2} < \cdots < x_{i,t_i}$. Now pick an order on $\mathcal{S}^{X(i)}$ with $r^x < s^{x'}$ if and only if $x < x'$ or $x = x'$ and $r < s$.

Let $S \in \text{Std}^{X'(i)}(\lambda(i))$. For $t \in [1, t_i]$, let $\lambda_{S,\leq t} \in \Lambda^+(n)$ be the partition such that

$$[\lambda_{S,\leq t}] = \{N \in [\lambda(i)] \mid \text{col}(S(N)) = x_{i,u} \text{ for some } u \leq t\}.$$

Recalling (7.13),

$$\tilde{\lambda}_S := \left(\lambda_{S,\leq t_i} / \lambda_{S,\leq t_i-1} / \cdots / \lambda_{S,\leq 1}\right) \in \Lambda^{\lambda(i),t_i}_+.$$

Denote

$$\xi_t := (\bar{x}_{i,1}, \bar{x}_{i,2}, \ldots, \bar{x}_{i,t_i}) \in (\mathbb{Z}/2)^t. \quad (7.34)$$

Then the map

$$\text{let} \circ S : [\lambda(i)] \to [1, n], \ N \mapsto \text{let}(S(N))$$

is a standard $\tilde{\lambda}_S$-tableau of parity $\xi_t$, see § 7.2. The map

$$f : S \mapsto (\tilde{\lambda}_S, \text{let} \circ S)$$

is easily seen to be a bijection between $\text{Std}^{X'(i)}(\lambda(i))$ and the set

$$P := \{(\tilde{\lambda}, \tilde{T}) \mid \tilde{\lambda} \in \Lambda^{\lambda(i),t_i}_+, \tilde{T} \in \text{St}_2(\tilde{\lambda})\}. \quad (7.35)$$

For every $(\tilde{\lambda} = \lambda^{t_i} / \cdots / \lambda^1, \tilde{T}) \in P$, we define

$$\text{deg}_i(\tilde{\lambda}) := \prod_{u=1}^{t_i} \text{deg}(x_{i,u})^{\lambda_u - |\lambda_u - 1|} \quad (7.36)$$

$$\alpha^{(\tilde{\lambda}, \tilde{T})} := (\alpha^{(0)}, \ldots, \alpha^{(r)}) \in \Lambda^I(n, d), \quad (7.37)$$

where

$$\alpha_r^{(j)} := \#\{N \in [\lambda(i)] \mid \tilde{T}(N) = r \text{ and } N \in [\lambda^u] \setminus [\lambda^{u-1}] \text{ for } u \in j\Omega(i)\}. \quad (7.38)$$

Note that $\alpha^{(0)} = \cdots = \alpha^{(i-1)} = 0$. It follows from the definitions that $\text{deg}(S) = \text{deg}(\tilde{\lambda}_S)$ and $\alpha^S = \alpha^{(f(S))}$. Thus we have the following.
Lemma 7.38. The map

\[ f : \text{Std}^X(i)(\lambda(i)) \to P, \quad S \mapsto (\tilde{\lambda}_S, \text{let}_S) \]

is a bijection such that \( \deg(S) = \deg_S(\tilde{\lambda}_S) \) and \( \alpha^S = \alpha^f(S) \).

For \( \nu = (\nu^1, ..., \nu^t_i) \in \Lambda(n)_i \), we define

\[
\deg^{(i)}(\nu) := \prod_{u=1}^t \deg(x_{i,u})^{|\nu^u|} \in R, \\
(j\nu) := (\nu^j)_{i \in j\Omega(i)} \in \Lambda(n)_{i\Omega(i)}^{j} \\
\chi(\nu) := (\chi^{(0)}, ..., \chi^{(t_i)}) \in \Lambda^I(n),
\]

where we have set \( \chi^{(j)} := \sum_{i \in j\Omega(i)} \nu^j \) for all \( j \in I \). We extend \( \chi \) by linearity to a function

\[
\chi : \mathbb{Z}(\Lambda(n)_i) = (\mathbb{Z}\Lambda(n))^{\otimes t_i} \to \mathbb{Z}\Lambda^I(n).
\]

From the Littlewood–Richardson rule (7.9), we get the following lemma.

Lemma 7.42. Let \( \nu = (\nu^1, ..., \nu^t_i) \in \Lambda_+(n)_i \). Then

\[
\chi(s_{\nu_1} \otimes ... \otimes s_{\nu_t}) = \sum_{\gamma = (\gamma^{(0)}, ..., \gamma^{(t_i)}) \in \Lambda_+^I(n)} \left( \prod_{j \in I} c_{j\nu}^{\gamma(j)} \right) s_{\gamma}.
\]

Let \( (\bar{\lambda} = \lambda^1 / ... / \lambda^t, T) \in P \). If \( \omega^{\bar{\lambda}, T} = (\omega^1, ..., \omega^t) \), then for all \( u \in [1, t_i] \) we have \( |\lambda^u| - |\lambda^{u-1}| = |\omega^u| \). So, comparing with (7.36), we deduce

\[
\deg_{i}(\bar{\lambda}) = \deg^{(i)}(\omega^{\bar{\lambda}, T}).
\]

Taking into account (7.37), we also have the following.

Lemma 7.44. For \( (\bar{\lambda}, T) \in P \), we have \( \alpha^{(\bar{\lambda}, T)} = \chi(\omega^{\bar{\lambda}, T}) \).

Proposition 7.45. We have

\[
\text{ch}_\pi \Delta(\lambda(i)) = \sum_{\gamma \in \Lambda_+^I(n)} \sum_{\nu \in \Lambda_+(n)_i} c_{\gamma\nu}^{\lambda(i)} \deg^{(i)}(\nu) \left( \prod_{j \in I} c_{j\nu}^{\gamma(j)} \right) s_\gamma.
\]

Proof. We have

\[
\text{ch}_\pi \Delta(\lambda(i)) = \sum_{S \in \text{Std}^X(i)(\lambda(i))} \deg(S) \alpha^S = \sum_{(\bar{\lambda}, T) \in P} \deg(\bar{\lambda}) \alpha^{(\bar{\lambda}, T)}
\]
\[
\begin{align*}
&= \sum_{\omega \in \Lambda(n)^i} \left( \sum_{(\lambda, T) \in P, \omega^L T = \omega} \operatorname{deg}^{(i)}(\omega) \right) \chi(\omega) \\
&= \sum_{\gamma = (\gamma^1, \ldots, \gamma^i) \in \Lambda_+ (n)^i} c_{\gamma^i}^{(i)} \operatorname{deg}^{(i)}(\gamma) \chi(s_{\gamma^1} \otimes \cdots \otimes s_{\gamma^i}) \\
&= \sum_{\gamma \in \Lambda_+ (n)^i} c_{\gamma}^{(i)} \operatorname{deg}^{(i)}(\gamma) \sum_{\gamma \in \Lambda_+(n)} \left( \prod_{j \in I} c_{j^i}^{(j)} \right) s_{\gamma},
\end{align*}
\]

where we have used Lemma 7.28 for the first equality, Lemma 7.38 for the second equality, Lemma 7.44 and (7.43) for the third equality, (7.35), (7.15) and Corollary 7.16 for the fourth equality, and Lemma 7.42 for the fifth equality. □

## 7.6 Standard characters and decomposition numbers

We continue with the notation of §7.5. In addition, we denote

\[ \Lambda^i_{+} (n) := \{ \gamma = (\gamma_0, \ldots, \gamma_\ell) \mid \gamma_0, \ldots, \gamma_\ell \in \Lambda^i_+ (n) \}. \]

Thus for \( \gamma = (\gamma_0, \ldots, \gamma_\ell) \in \Lambda^i_{+} (n) \), each \( \gamma_i \) looks like \( \gamma_i = (\gamma_i^{(0)}, \ldots, \gamma_i^{(\ell)}) \) with \( \gamma_i^{(j)} \in \Lambda_+ (n) \).

Given a multipartition \( \nu = (\nu_x)_{x \in X} \in \Lambda^X_+ (n) \) and recalling (7.33), we associate to \( \nu \) the tuple

\[ \nu = (\nu_{x_0}, \ldots, \nu_{x_\ell}) \in \Lambda_+ (n)^{t_0} \times \Lambda_+ (n)^{t_1} \times \cdots \times \Lambda_+ (n)^{t_\ell} \]

of multipartitions \( \nu_i := (\nu_{x_{i,1}}, \ldots, \nu_{x_{i,t_i}}) \in \Lambda_+ (n)^{t_i} \). We denote

\[ \deg(\nu) := \prod_{x \in X'} \deg(x)^{\nu_x} = \prod_{i \in I} \deg^{(i)}(\nu_i). \quad (7.46) \]

For \( i, j \in I \), \( \lambda = (\lambda^{(0)}, \ldots, \lambda^{(\ell)}) \in \Lambda^i_+ (n) \) and \( \gamma = (\gamma_0, \ldots, \gamma_\ell) \in \Lambda^i_{+} (n) \) we observe:

\[ c_{\lambda}^{(i)} = c_{\lambda^{(i)}}^{(i)} \quad \text{and} \quad c_{\gamma}^{(j)} = c_{\gamma^{(j)}}^{(j)}. \quad (7.47) \]

**Lemma 7.48.** For \( \nu \in \Lambda^X_+ (n) \), we have

\[ \sum_{\nu = (\nu_0, \ldots, \nu_\ell) \in \Lambda^X_+ (n)} \prod_{i \in I} \left( \prod_{j \in I} c_{j^i}^{(j)} \right) s_{\nu_i} = \sum_{\mu \in \Lambda_{+}^i (n)} \left( \prod_{j \in I} c_{\mu^{(j)}}^{(j)} \right) s_{\mu}. \]

**Proof.** In view of (7.47) and (7.9), the left-hand side equals

\[ \sum_{\nu \in \Lambda^X_+ (n)} \left( \prod_{i, j \in I} c_{j^i}^{(j)} \right) \left( \prod_{i \in I} s_{\nu_i} \right) \]
\[
\sum_{\gamma \in \Lambda_+^{\ast}(n)} \left( \prod_{i,j \in I} c_{\gamma_{x_i}x_{jx_i}}^{(j)} \right) \sum_{\mu \in \Lambda_+^+(n)} \left[ \prod_{i \in I} c_{\gamma_i}^{(i)} \gamma_{i}^{(i)} \ldots \gamma_{i}^{(i)} \right] s_{\mu} \]

\[
= \sum_{\mu \in \Lambda_+^+(n)} \sum_{\gamma \in \Lambda_+^{\ast}(n)} \left( \prod_{i,j \in I} c_{\gamma_{x_i}x_{jx_i}}^{(j)} \right) \left( \prod_{j \in I} c_{\gamma_j^{(j)}}^{(j)} \right) s_{\mu},
\]

which equals the right-hand side thanks to Lemma 7.11.

\[\square\]

**Proposition 7.49.** For \(\lambda \in \Lambda_+^I(n, d)\), we have

\[
\text{ch}_q \Delta(\lambda) = \sum_{\mu \in \Lambda_+^+(n)} \sum_{\varpi \in \Lambda X'} \deg(\varpi) \left( \prod_{i \in I} c_{\lambda_{(i)}}^{(i)} c_{\varpi_{(i)}}^{(i)} \right) s_{\mu}.
\]

**Proof.** The result follows from the computation

\[
\text{ch}_q \Delta(\lambda) = \text{ch}_q \left( \bigotimes_{i \in I} \Delta(\lambda_{(i)}) \right)
\]

\[
= \prod_{i \in I} \text{ch}_q \Delta(\lambda_{(i)})
\]

\[
= \sum_{\gamma \in \Lambda_+^{\ast}(n)} \sum_{\mu \in \Lambda_+^+(n)} \left( \prod_{i \in I} c_{\gamma_{x_i}x_{jx_i}}^{(j)} \right) \left( \prod_{j \in I} c_{\gamma_j^{(j)}}^{(j)} \right) s_{\gamma_1}.
\]

Now we arrive at a formula for decomposition numbers when \(A\) is basic.

**Corollary 7.50.** Let \(A\) be basic and \(\lambda, \mu \in \Lambda_+^I(n, d)\). Then

\[
d_{\lambda, \mu} = \sum_{\gamma \in \Lambda_+^I(n)} \sum_{\nu \in \Lambda_+^I(n)} d_{\gamma, \nu}^I \deg(\varpi) \left( \prod_{i \in I} c_{\lambda_{(i)}}^{(i)} c_{\varpi_{(i)}}^{(i)} \right).
\]
Proof. This follows from Proposition 7.49, Corollary 7.32, and Lemma 7.19. □

7.7 Decomposition numbers for non-basic algebras

Recall from (2.12) the decomposition numbers \( d_{i,j}^{n,e}(n) \) for the algebra \( A \). Define

\[
\Lambda^D_+(n) = \prod_{i,j \in I, m \in \mathbb{Z}, \varepsilon \in \mathbb{Z}/2} \Lambda_+(n)^{d_{i,j}^{m,e}}.
\]

We consider elements \( \nu \in \Lambda^D_+(n) \) as multipartitions \( (\nu(i,j,m,\varepsilon,t)) \) with components \( \nu(i,j,m,\varepsilon,t) \in \Lambda_+(n) \) indexed by \( i, j \in I, m \in \mathbb{Z}, \varepsilon \in \mathbb{Z}/2, t \in [1, d_{i,j}^{m,e}] \).

Let \( \nu \in \Lambda^D_+(n) \). We define \( \overline{\nu} = (\overline{\nu}(i,j,m,\varepsilon,t)) \) via \( \overline{\nu}(i,j,m,\varepsilon,t) := (\nu(i,j,m,\varepsilon,t))^{\varepsilon} \); that is, the conjugate partition if \( \varepsilon = 1 \), or the unchanged partition if \( \varepsilon = 0 \). For \( i \in I \), we define the multipartitions

\[
\nu_i := (\nu(j,i,m,\varepsilon,t))_{j \in I, m \in \mathbb{Z}, \varepsilon \in \mathbb{Z}/2, t \in [1, d_{j,i}^{m,e}]},
\]

\[
i\overline{\nu} := (\overline{\nu}(i,j,m,\varepsilon,t))_{j \in I, m \in \mathbb{Z}, \varepsilon \in \mathbb{Z}/2, t \in [1, d_{i,j}^{m,e}]},
\]

Finally, define

\[
\deg(\nu) = \prod_{i,j \in I, m \in \mathbb{Z}, \varepsilon \in \mathbb{Z}/2, t \in [1, d_{i,j}^{m,e}]} (q^m \pi^{-\varepsilon})^{\nu(i,j,m,\varepsilon,t)}|_{\nu(i,j,m,\varepsilon,t)}.
\]

Then we have the following.

Theorem 7.51. Suppose that \((A, a)\) is a unital pair and assume that \( A_1 \subset J(A) \). Then for \( \lambda, \mu \in \Lambda^D_+(n, d) \), the corresponding decomposition number is given by

\[
d_{\lambda,\mu} = \sum_{\gamma \in \Lambda^D_+(n)} \sum_{\nu \in \Lambda_+(n)} d_{\gamma,\mu}^{cl}(\nu) \deg(\nu) \left( \prod_{i \in I} c_i^{\gamma(i)} \xi_i^{\nu(i)} \right).
\]

Proof. By Theorem 2.17, there exists \( a\)-conforming heredity data \( I, X', Y' \) with the same ideals \( A(\Omega) \) and such that the new initial elements \( \{e'_i \mid i \in I\} \) are primitive idempotents in \( a \). Note that \( T_{A_1}^A(n, d) \) depends only on \( a \), see Proposition 4.9. Moreover, it is easy to see that the standard modules defined using the new heredity data are the same as the ones defined using the original heredity data since the ideals \( A(\Omega) \) have not changed. We may and will assume that the idempotents \( \{e_i \mid i \in I\} \) are primitive in \( a \).

We now set \( f := \sum_{i \in I} e_i, \tilde{A} := fAf \) and \( \tilde{a} := fAf \). Then by Theorem 2.17(i), \( f \) is strongly adapted so that \( \tilde{A} \) is based quasi-hereditary with \( \tilde{a}\)-conforming heredity data \( I, \tilde{X}, \tilde{Y} \) which is the \( f \)-truncation of \( I, X, Y \). Moreover, by Theorem 2.17(iii), \( \tilde{A} := fAf \) and \( \tilde{a} := fAf \) are basic. By Theorem 2.17(iv), \( F : A-\text{mod} \rightarrow \tilde{A}-\text{mod}, \ V \mapsto fV \) is an equivalence of categories with \( F(L_{A(i)}) \cong L_{\tilde{A}(i)} \) and \( F(\Delta_{A}(i)) \cong \Delta_{\tilde{A}}(i) \). In particular we have the equality of decomposition numbers \( [\Delta_{A}(i) : q^m \pi^{-\varepsilon} L_{A}(j)] = [\Delta_{\tilde{A}}(i) : q^m \pi^{-\varepsilon} L_{\tilde{A}}(j)] \) for all \( i, j, m, \varepsilon \).

By Lemma 6.22(ii), we have a strongly adapted idempotent \( \xi_{f}^T \in T_{a}^A(n, d) \), and taking into account Lemma 6.21(ii), we have that \( \xi_{f}^T T_{a}^A(n, d) \xi_{f}^T = T_{a}^A(n, d) \) is a quasi-hereditary algebra with...
heredity data $\Lambda^+_t(n, d)$, $\mathcal{R}, \mathcal{P}$ for

$$
\mathcal{R} = \bigsqcup_{\lambda \in \Lambda^+_t(n, d)} \{ \chi_S \mid S \in \text{Std}^\nu(\lambda) \} \quad \text{and} \quad \mathcal{P} = \bigsqcup_{\lambda \in \Lambda^+_t(n, d)} \{ \chi_T \mid T \in \text{Std}^\nu(\lambda) \}.
$$

It follows that the functor $G : T^A_{\bar{a}}(n, d)_{-\text{mod}} \rightarrow T^A_{\bar{a}}(n, d)_{-\text{mod}}$, $W \mapsto \xi^f W$ is an equivalence of categories. It is clear that

$$
G(\Delta T^A_{\bar{a}}(n, d)(\lambda)) = \Delta T^A_{\bar{a}}(n, d)(\lambda) \quad \text{and} \quad G(L T^A_{\bar{a}}(n, d)(\lambda)) = L T^A_{\bar{a}}(n, d)(\lambda).
$$

for all $\lambda \in \Lambda^+_t(n, d)$, and so we have the equality of decomposition numbers

$$
[\Delta T^A_{\bar{a}}(n, d)(\lambda) : q^m \pi^z L T^A_{\bar{a}}(n, d)(\mu)] = [\Delta T^A_{\bar{a}}(n, d)(\lambda) : q^m \pi^z L T^A_{\bar{a}}(n, d)(\mu)]
$$

(7.52)

for all $\lambda, \mu, m, \varepsilon$.

From Corollary 7.50, we can compute the decomposition numbers in the right hand of (7.52) since $\bar{A}$ is basic. Moreover, it is easy to see that for basic algebras, the formula claimed in the theorem is equivalent to the formula of Corollary 7.50. Now the theorem follows from (7.52). □

### 7.8 Blocks of Schurifications

The following result, proved in [13, Lemma 4.8], shows that any algebra decomposition of $A$ yields an associated decomposition of $T^A_{\bar{a}}(n, d)$.

**Lemma 7.53.** Let $m \in \mathbb{Z}_{>0}$. For $t \in [1, m]$ assume that $(A_t, a_t)$ is a good pair. Write $A := \bigoplus_{t=1}^m A_t$ and $a := \bigoplus_{t=1}^m a_t$. Then we have

$$
S^A(n, d) \cong \bigoplus_{\nu \in \Lambda(m, d)} \bigotimes_{t=1}^m S^{A_t}(n, \nu_t) \quad \text{and} \quad T^A_{\bar{a}}(n, d) \cong \bigoplus_{\nu \in \Lambda(m, d)} \bigotimes_{t=1}^m T^A_{a_t}(n, \nu_t)
$$

as $\mathbb{k}$-superalgebras.

We now examine conditions under which $T^A_{\bar{a}}(n, d)$ is known to be indecomposable, thus showing that Lemma 7.53 describes a block decomposition of $T^A_{\bar{a}}(n, d)$ in terms of the blocks of $A$ in many important cases.

Throughout this subsection, let $e$ be an adapted idempotent for $A$ ($e = 1$ is allowed). Recall that $\{\bar{L}(i) := eL(i) \mid i \in \bar{I}\}$ is a complete set of simple modules for $\bar{A} := eAe$. We also have that $\bar{P}(i) := eP(i)$ is the projective cover of $\bar{L}(i)$ for all $i \in \bar{I}$.

For $i, j \in \bar{I}$, we will write $i \sim j$ if there exists a sequence $i = i_0, i_1, \ldots, i_m = j \in \bar{I}$ such that $\bar{P}(i_{t-1})$ and $\bar{P}(i_t)$ share a common composition factor, for every $t \in [1, m]$. Then $i \sim j$ if and only if $\bar{L}(i)$ and $\bar{L}(j)$ belong to the same block of $\bar{A}$, and thus, $\bar{A}$ is indecomposable if and only if $i \sim j$ for all $i, j \in \bar{I}$.

For all $i, j \in \bar{I}$, we have, using BGG reciprocity:

$$
[P(i) : L(j)] \neq 0 \iff [P(i) : L(j)] \neq 0 \iff (P(i) : \Delta(k)) \cdot [\Delta(k) : L(j)] \neq 0 \text{ for some } k \in I
$$
\[ \iff \forall (k) : L(i) \cdot [\Delta(k) : L(j)] \neq 0 \text{ for some } k \in I \]
\[ \iff [\Delta^\text{op}(k) : L^\text{op}(i)] \cdot [\Delta(k) : L(j)] \neq 0 \text{ for some } k \in I \]
\[ \iff d_{k,i}^\text{op}d_{k,j} \neq 0 \text{ for some } k \in I. \]

Then, for \(i, j \in \tilde{I}\), we have that \(\tilde{P}(i)\) and \(\tilde{P}(j)\) share a common composition factor if and only if there exist \(k, k' \in I, l \in \tilde{I}\) such that \(d_{k,i}^\text{op}d_{k,j} \neq 0\).

Then \(i \sim j\) if and only if there exist sequences \(i = i_0, \ldots, i_m = j, l_1, \ldots, l_m \in \tilde{I}, k_1, \ldots, k_m \in I, k'_1, \ldots, k'_m \in I\) such that \(d_{k_i, l_{i-1}}^\text{op}d_{k_i, l_i}d_{k'_i, l_{i-1}}^\text{op}d_{k'_i, l_i} \neq 0\) for all \(i \in [1, m]\).

**Theorem 7.54.** Suppose that \((A, \alpha)\) is a unital pair and \(A_\tau \subset J(A)\). Moreover, suppose that \(e\) is an adapted idempotent such that \(\overline{A} = eAe\) is indecomposable, and \(|\tilde{I}| > 1\). Then \(T^\lambda_{\overline{A}}(n, d)\) is indecomposable.

**Proof.** By the indecomposability assumption we have \(i \sim j\) for all \(i, j \in \tilde{I}\). We aim to show that \(\lambda \sim \mu\) for all \(\lambda, \mu \in \Lambda^\tilde{I}_+(n, d)\). We first prove a claim.

**Claim 1.** For \(\lambda, \alpha \in \Lambda^\tilde{I}_+(n)\), write \(\alpha \subseteq \lambda\) to indicate that \(\alpha^{(r)} \leq \lambda^{(r)}\) for all \(i \in I, r \in \mathbb{Z}_{>0}\). Let \(\alpha \in \Lambda^\tilde{I}_+(n, d - 1), \lambda, \mu \in \Lambda^\tilde{I}_+(n, d)\), with \(\alpha \subseteq \lambda, \mu\). Moreover, assume that \(|\lambda^{(i)}| = |\alpha^{(i)}| + 1\) and \(|\mu^{(j)}| = |\alpha^{(j)}| + 1\), for some \(i \neq j \in \tilde{I}\) such that \(d_{ij} \neq 0\) (respectively, \(d_{ij}^\text{op}\)). Then \(d_{\lambda, \mu} \neq 0\) (respectively, \(d_{\lambda, \mu}^\text{op} \neq 0\)).

**Proof of Claim 1.** Assume that \(d_{ij} \neq 0\); the proof that \(d_{ij}^\text{op} \neq 0\) implies \(d_{\lambda, \mu}^\text{op} \neq 0\) is similar. We have that \(d_{ij}^\text{op} \neq 0\) for some \(n \in \mathbb{Z}, \varepsilon \in \mathbb{Z}/2\). Let \(\nu\) be the element of \(\Lambda^\text{cl}_+(n, d)\) such that \(\nu^{(k, k, 0, 0, 1)} = \alpha^{(k)}\) for all \(k \in I\), \(\nu^{(i, j, n, \varepsilon, 1)} = (1)\), and all other components of \(\nu\) are empty. Then, noting that \(d_{\lambda, \mu}^\text{cl} = 1\) and \(\prod_{k \in I} c_{\lambda}^{(k)} c_{\mu}^{(k)} = 1\), Theorem 7.51 gives us that \(d_{\lambda, \mu} \neq 0\), as desired.

Now, for \(\lambda, \mu \in \Lambda^\tilde{I}_+(n, d)\), define

\[ \text{diff}(\lambda, \mu) := \sum_{k \in I} \sum_{r > 0} |\lambda^{(r)}_k - \mu^{(r)}_k|. \]

We will prove that \(\lambda \sim \mu\) by induction on \(\text{diff}(\lambda, \mu)\).

Assume \(\text{diff}(\lambda, \mu) = 2\) (the smallest non-trivial case). Then for some \(\alpha \in \Lambda^\tilde{I}_+(n, d - 1)\) we have \(\alpha \subseteq \lambda, \mu\), with \(|\lambda^{(i)}| = |\alpha^{(i)}| + 1\) and \(|\mu^{(j)}| = |\alpha^{(j)}| + 1\), for some \(i, j \in \tilde{I}\). Since \(i \sim j\) by assumption, there exist sequences

\[ i = i_0, \ldots, i_m = j, l_1, \ldots, l_m \in \tilde{I}, k_1, \ldots, k_m \in I, k'_1, \ldots, k'_m \in I \]

such that \(d_{k_i, l_{i-1}}^\text{op}d_{k_i, l_i}d_{k'_i, l_{i-1}}^\text{op}d_{k'_i, l_i} \neq 0\) for all \(i \in [1, m]\). Since \(|\tilde{I}| > 1\), we may moreover assume that we have chosen sequences such that \(i_{t-1} \neq i_t\) for all \(t \in [1, m]\).
For \( h \in I \), define \( h_\beta \in \Lambda^I_+(n,d) \) via

\[
h_\beta^{(h')} := \begin{cases} 
\alpha^{(h)}_r + 1 & \text{if } h' = h, r = 1 \\
\alpha^{(h')}_r & \text{otherwise.}
\end{cases}
\]

Define \( 0_\tau, \ldots, m_\tau \in \Lambda^I_+(n,d) \) by \( 0_\tau := \lambda, m_\tau := \mu \), and \( t_\tau := i_\beta \) for \( t \in [1, m-1] \). For \( t \in [1, m] \), define \( t_\eta \in \Lambda^I_+(n,d) \) via

\[
t_\eta := \begin{cases} 
\tau^{t-1} & \text{if } l_t = i_{t-1} \\
\tau^t & \text{if } l_t = i_t \\
i_t \beta & \text{otherwise.}
\end{cases}
\]

Furthermore, define \( t_\kappa, t_\kappa' \in \Lambda^I_+(n,d) \) via

\[
t_\kappa := \begin{cases} 
\tau^{t-1} & \text{if } k_t = i_{t-1} \\
t_\eta & \text{if } k_t = i_t \\
k_t \beta & \text{otherwise.}
\end{cases}
\]

\[
t_\kappa' := \begin{cases} 
\tau^t & \text{if } k'_t = i_t \\
t_\eta & \text{if } k'_t = l_t \\
k'_t \beta & \text{otherwise.}
\end{cases}
\]

Let \( t \in [1, m] \). By construction, we have that either \( t_\kappa = t^{-1}_\kappa \), or else \( t_\kappa, t^{-1}_\kappa \) satisfy the assumptions of Claim 1. Then, in either case we have \( d^{op}_{t_\kappa, t^{-1}_\kappa} \neq 0 \). Similarly, we have that the pairs \( (t_\kappa, t_\eta), (t_\kappa', t_\tau), (t'_\kappa, t_\eta) \) are either equal or satisfy the assumptions of Claim 1, so \( d_{t_\kappa, t_\eta}, d^{op}_{t_\kappa', t_\tau}, d_{t'_\kappa, t_\eta} \neq 0 \) as well. Therefore we have sequences

\[
\lambda = 0_\tau, \ldots, m_\tau = \mu \in \Lambda^I_+(n,d), \quad 0_\eta, \ldots, m_\eta \in \Lambda^I_+(n,d),
\]

\[
0_\kappa, \ldots, m_\kappa \in \Lambda^I_+(n,d), \quad 0_\kappa', \ldots, m_\kappa' \in \Lambda^I_+(n,d),
\]

such that \( d^{op}_{0_\kappa, t_{t^{-1}_\kappa}} d^{op}_{t_\kappa', t_\tau} d_{t'_\kappa, t_\eta} \neq 0 \) for all \( t \in [1, m] \). This proves that \( \lambda \sim \mu \).

Now for the induction step, assume \( \text{diff}(\lambda, \mu) = D > 2 \). There exists some \( i, j \in \hat{I}, r, s \in \mathbb{Z}_{>0} \) such that \( \lambda^{(i)}_r > \mu^{(i)}_r \) and \( \lambda^{(j)}_s < \mu^{(j)}_s \). Assume that \( r \) is maximal, and \( s \) is minimal such that these inequalities hold. Then define \( \rho \in \Lambda^I_+(n,d) \) via

\[
\rho^{(k)}_t := \begin{cases} 
\lambda^{(i)}_r - 1 & \text{if } t = r, k = i \\
\lambda^{(j)}_s + 1 & \text{if } t = s, k = j \\
\lambda^{(k)}_t & \text{otherwise.}
\end{cases}
\]

The maximality/minimality assumptions guarantee that \( \rho \) is in fact a multipartition. We have then that \( \text{diff}(\lambda, \rho) = 2 \) and \( \text{diff}(\rho, \mu) = D - 2 \). So by induction we have \( \lambda \sim \rho \sim \mu \), as desired. \( \square \)

Remark 7.55. The condition \( |\hat{I}| > 1 \) implies that \( \hat{A} \) is non-simple. If \( \hat{A} \) is simple, then \( T^A_\delta(n,d) \) is isomorphic to a classical Schur algebra, which is decomposable in general.
7.9 Decomposition numbers for the zigzag algebra

Fix $\ell' \geq 1$ and set

$$I := \{0, 1, \ldots, \ell\}, \quad J := \{0, \ldots, \ell' - 1\}.$$ 

Let $\Gamma$ be the quiver with vertex set $I$ and arrows $\{a_{j, j+1}, a_{j+1, j} \mid j \in J\}$ as in the picture:

The extended zigzag algebra $Z$ is the path algebra $\mathbb{k}\Gamma$ modulo in the following relations.

(i) All paths of length three or greater are zero.
(ii) All paths of length two that are not cycles are zero.
(iii) All length-two cycles based at the same vertex are equivalent.
(iv) $a_{\ell', \ell-1}a_{\ell-1, \ell'} = 0$.

Length zero paths yield the standard idempotents $\{e_i \mid i \in I\}$ with $e_ia_{i,j}e_j = a_{i,j}$ for all admissible $i$, $j$. The algebra $Z$ is graded by the path length: $Z = Z^0 \oplus Z^1 \oplus Z^2$. We also consider $Z$ as a superalgebra with $Z_0 = Z^0 \oplus Z^2$ and $Z_1 = Z^1$.

Define

$$c_j := a_{j, j+1}a_{j+1, j} \quad (j \in J).$$

The algebra $Z$ has an anti-involution $\tau$ with

$$\tau(e_i) = e_i, \quad \tau(a_{ij}) = a_{ji}, \quad \tau(c_j) = c_j.$$

We consider the total order on $I$ given by $0 < 1 < \cdots < \ell'$. For $i \in I$, we set

$$X(i) := \begin{cases} \{e_i, a_{i-1, i}\} & \text{if } i \neq 0, \\ \{e_0\} & \text{if } i = 0 \end{cases}, \quad Y(i) := \begin{cases} \{e_i, a_{i, i-1}\} & \text{if } i \neq 0, \\ \{e_0\} & \text{if } i = 0. \end{cases}$$

Finally, define $\mathfrak{z} := \text{span}(e_i \mid i \in I)$.

**Lemma 7.56.** The graded superalgebra $Z$ is a based quasi-hereditary algebra with $\mathfrak{z}$-conforming heredity data $I, X, Y$ and standard anti-involution $\tau$. If $\mathbb{k}$ is local, $Z$ is basic.

**Proof.** Follows immediately from definitions. \hfill \Box

Let $e := e_0 + \cdots + e_{\ell'-1} \in Z$. Note that $e$ is an adapted idempotent, and $\tau(e) = e$, so the zigzag algebra $\overline{Z} := eZe \subset Z$ is a cellular algebra with involution $\tau|_{\overline{Z}}$, and cellular basis

$$\overline{B} = \{xy \mid i \in I, x \in \overline{X}(i), y \in \overline{Y}(i)\},$$

where $\overline{X}(0) = \{a_{\ell'-1, \ell'}\}$, $\overline{Y}(0) = \{a_{\ell', \ell-1}\}$, and $\overline{X}(i) = X(i)$, $\overline{Y}(i) = Y(i)$ for all $i \in J$. The cell modules are $\{\overline{\Delta}(i) = e\Delta(i) \mid i \in I\}$. Note that $\mathfrak{z} = e\mathfrak{z} = \text{span}(e_i \mid i \in J)$. 


From now on let $d \leq n$. By Theorem 6.6, we have a based quasi-hereditary $\mathbb{k}$-superalgebra $T^\mathbb{Z}_\delta(n, d)$ with heredity data $\Lambda^I_+(n, d), \mathcal{X}, \mathcal{Y}$ and standard modules $\{\Delta(\lambda) \mid \lambda \in \Lambda^I_+(n, d)\}$.

**Remark 7.57.** As noted in Remark 6.7, the assumption $n \geq d$ in Theorem 6.6 is necessary, as it is easy to check explicitly that $T^\mathbb{Z}_\delta(1, 2)$ is not quasi-hereditary when $\ell = 1$.

By Lemmas 6.21(ii) and 6.23, $T^\mathbb{Z}_\delta(n, d) = \xi e T^\mathbb{Z}_\delta(n, d) \xi e$ is a cellular algebra with involution induced by $\tau$ and cell modules $\{\tilde{\Delta}(\lambda) = \xi e \Delta(\lambda) \mid \lambda \in \Lambda^I_+(n, d)\}$.

Recall that we have fixed a field $\mathbb{F}$ and a homomorphism $\mathbb{k} \to \mathbb{F}$. In particular, we have the algebras $Z_\mathbb{F}, Z^\mathbb{Z}_\mathbb{F}, T^\mathbb{Z}_\delta(n, d)_{\mathbb{F}}, T^\mathbb{Y}_\delta(n, d)_{\mathbb{F}}, \Delta(i)_{\mathbb{F}}, L(i)_{\mathbb{F}}, \Delta(\lambda)_{\mathbb{F}}, \tilde{\Delta}(\lambda)_{\mathbb{F}},$ and so on. Since $eL(0)_{\mathbb{F}} = 0$ and $eL(j)_{\mathbb{F}} = \tilde{L}(j)_{\mathbb{F}}$ for all $j \in J$, the simple $Z_\mathbb{F}$-modules are $\{\tilde{L}(j)_{\mathbb{F}} = eL(j)_{\mathbb{F}} \mid j \in J\}$. The following lemma is easily checked.

**Lemma 7.58.** Let $i, j \in I$. Then the graded decomposition numbers for $Z_\mathbb{F}$ are given by $d^i_{i,j}(q, \pi) = \delta_{i,j} + \delta_{i-1,j} q \pi$.

In view of Lemma 6.26, the irreducible $T^\mathbb{Z}_\delta(n, d)_{\mathbb{F}}$-modules $\{L(\lambda)_{\mathbb{F}} \mid \lambda \in \Lambda^I_+(n, d)\}$ give rise to the irreducible $T^\mathbb{Y}_\delta(n, d)_{\mathbb{F}}$-modules

$$\{\tilde{L}(\lambda)_{\mathbb{F}} = \xi e L(\lambda)_{\mathbb{F}} \mid \lambda \in \Lambda^I_+(n, d)\}.$$

Recalling the notation from §7.7 and Lemma 7.58, for $\nu \in \Delta^D_+(n, d)$ we may write

$$\nu = (\beta^{(0)}, \ldots, \beta^{(\ell)}, \alpha^{(0)}, \ldots, \alpha^{(\ell-1)}),$$

where $\beta^{(i)} = \nu(i, i, 0, \ddots, 1)$ and $\alpha^{(j)} = \nu(j+1, j, 1, \ddots, 1)$, for all $i \in I$ and $j \in J$. Setting $\beta := (\beta^{(i)})_{i \in I} \in \Lambda^I_+(n)$ and $\alpha := (\alpha^{(j)})_{j \in J} \in \Lambda^J_+(n)$, we identify $\Lambda^D_+(n)$ with $\Lambda^I_+(n) \times \Lambda^J_+(n)$ via $\nu \mapsto (\beta, \alpha)$. For all $\lambda, \mu \in \Lambda^I_+(n, d)$, define

$$\delta(\lambda, \mu) := \sum_{j \in J} j(|\alpha^{(j)}| - |\beta^{(j)}|).$$

**Lemma 7.59.** Let $n \geq d$. Then, for $\lambda, \mu \in \Lambda^I_+(n, d)$, the graded decomposition numbers for $T^\mathbb{Y}_\delta(n, d)_{\mathbb{F}}$ are given by the formula

$$d^\mathbb{Y}_{\lambda, \mu}(q, \pi) = \delta(\lambda, \mu) \sum_{\gamma, \beta \in \Lambda^I_+(n)} \sum_{\alpha \in \Lambda^I_+(n)} d^\mathbb{Y}_{\gamma, \mu}(\prod_{i \in I} c_{\beta^{(i)}, (\alpha(i-1))}^{\gamma^{(i)}, \alpha(i)}(q, \pi)), $n \geq d$.

where we formally impose that $\alpha^{(\ell-1)} = \alpha^{(\ell)} = \emptyset$ for $\alpha = (\alpha^{(0)}, \ldots, \alpha^{(\ell-1)}) \in \Lambda^I_+(n)$. Moreover, if $\mu \in \Lambda^I_+(n, d)$, then for the algebra $T^\mathbb{Z}_\delta(n, d)_{\mathbb{F}}$ we have

$$[\tilde{\Delta}(\lambda)_{\mathbb{F}} : L(\mu)_{\mathbb{F}}]_{q, \pi} = d^\mathbb{Y}_{\lambda, \mu}(q, \pi).$$
Proof. The second statement follows from the first since $T_{\lambda}^Z(n, d) = \xi c T_{\lambda}^Z(n, d) \xi^e$. For the first statement, by Theorem 7.51 we have

$$d_{\lambda, \mu} = \sum_{\gamma, \beta \in \Lambda_+^I(n)} \sum_{\alpha \in \Lambda_+^J(n)} d_{\gamma, \mu}^{\text{cl}} (q \pi)^{\alpha_0} \left( \prod_{i \in I} c_{\beta(i), (\alpha(i-1))_0}^{\alpha(i)} c_{\beta(i), (\alpha(i))}^{\mu(i)} \right),$$

For $\gamma, \beta \in \Lambda_+^I(n), \alpha \in \Lambda_+^J(n)$, and $i \in I$, we have $c_{\beta(i), (\alpha(i-1))_0}^{\alpha(i)} c_{\beta(i), (\alpha(i))}^{\mu(i)} = 0$ unless $|\beta(i)| + |\alpha(i-1)| = |\gamma(i)|$ and $|\beta(i)| + |\alpha(i)| = |\lambda(i)|$. But then this implies that

$$|\alpha| = \sum_{j \in J} j(|\lambda(j)| - |\gamma(j)|).$$

for all $\alpha \in \Lambda_+^J(n)$ which contribute to the sum. Now, noting that $d_{\gamma, \mu}^{\text{cl}} = 0$ unless $|\gamma(i)| = |\mu(i)|$ for all $i \in I$ gives the result. \hfill \Box

Remark 7.60. The generalized Schur algebra $T_{\lambda}^Z(n, d)$ is Morita equivalent to weight $d$ RoCK blocks of symmetric groups and the corresponding Hecke algebras, as conjectured by Turner [21] and proved in [6]. We conjecture that the Morita equivalence constructed in [6] sends cell modules to cell modules and behaves well on combinatorial labels. The evidence for the conjecture comes from the fact that the formula for $d_{\lambda, \mu}$ in Lemma 7.59 is equivalent to the formula obtained by Turner [21, Corollary 134] for decomposition numbers of Specht modules in RoCK blocks.

Note that when $d < \text{char } \mathbb{F}$ or $\text{char } \mathbb{F} = 0$, we have $d_{\lambda, \mu}^{\text{cl}} = \delta_{\lambda, \mu}$, so the formula in Lemma 7.59 may be simplified to

$$d_{\lambda, \mu} = (q \pi)^{\delta(\lambda, \mu)} \sum_{\alpha(-1), \ldots, \alpha(\ell), \beta(0), \ldots, \beta(\ell)} \left( \prod_{0 \leq i \leq \ell} c_{\beta(i), (\alpha(i+1))_0}^{\alpha(i)} c_{\beta(i), (\alpha(i))}^{\mu(i)} \right),$$

where the sum is over partitions $\alpha(-1), \ldots, \alpha(\ell), \beta(0), \ldots, \beta(\ell)$, with

$$|\alpha(i)| = \sum_{j=i+1}^\ell |\lambda(j)| - |\mu(j)| \quad \text{and} \quad |\beta(i)| = |\mu(i)| + \sum_{j=i+1}^\ell |\mu(j)| - |\lambda(j)|.$$

On the other hand, we have the formula obtained by Chuang-Tan [3, Theorem 6.2] and Leclerc-Miyachi [14, Corollary 10] (see also [10, Theorem 4.1]), for RoCK blocks of weight $d < \text{char } \mathbb{F}:

$$d_{\lambda, \mu}^{\text{RoCK}} = (q \pi)^{\delta(\lambda, \mu)} \sum_{\alpha(0), \ldots, \alpha(\ell+1), \beta(0), \ldots, \beta(\ell)} \left( \prod_{0 \leq i \leq \ell} c_{\beta(i), (\alpha(i+1))_0}^{\alpha(i)} c_{\beta(i), (\alpha(i))}^{\mu(i)} \right),$$

where the sum is over partitions $\alpha(0), \ldots, \alpha(\ell+1), \beta(0), \ldots, \beta(\ell)$, with

$$|\alpha(i)| = \sum_{j=0}^{i-1} |\lambda(j)| - |\mu(j)| \quad \text{and} \quad |\beta(i)| = |\mu(i)| + \sum_{j=0}^{i-1} |\mu(j)| - |\lambda(j)|.$$
and

\[ \delta^{\text{RoCK}}(\lambda, \mu) := \sum_{j=1}^{\ell} (\ell - j + 1)(|\lambda^{(j-1)}| - |\mu^{(j-1)}|). \]

After some manipulation and reindexing, we get

\[ d_{\lambda, \mu}^F = d_{\lambda', \mu'}^F, \]

where \( \lambda' := ((\lambda^{(\ell)})', \ldots, (\lambda^{(0)})') \) for all \( \lambda \in \Lambda_+^d(n) \). This notation coincides with the fact that if \( \lambda \) is a partition in a RoCK block with \( (\ell + 1) \)-quotient \( \lambda \), then \( \lambda' \) has \( (\ell + 1) \)-quotient \( \lambda' \), see [17, Lemma 1.1(2)].

For the following conjecture, we now consider the usual \( q \)-Schur algebra \( S_q(N, f) \). Let \( e \) be the corresponding quantum characteristic, see, for example, [11, §2.1]. Note that in the case \( q = 1 \), we have \( e = \text{char } \mathbb{F} \). To avoid trivial cases we assume that \( e > 0 \).

**Conjecture 7.61.** Let \( B \) be a weight \( d \) RoCK block of \( S_q(N, f) \). Let \( \ell' = e - 1 \) and \( Z \) be the corresponding extended zigzag algebra. For \( n \geq d \), the generalized Schur algebra \( T^Z_{\delta}(n, d) \) is Morita equivalent to \( B \).

This conjecture is in spirit of [21, Conjecture 178], although it is not clear to us whether Turner’s algebra \( Q_{\rho}(n, d) \) appearing in loc. cit. is isomorphic or even Morita equivalent to our algebra \( T^Z_{\delta}(n, d) \) in this case.

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