Two-loop Back-reaction in 2D Dilaton Gravity

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ABSTRACT

We calculate the two-loop quantum corrections, including the back-reaction of the Hawking radiation, to the one-loop effective metric in a unitary gauge quantization of the CGHS model of 2d dilaton gravity. The corresponding evaporating black hole solutions are analysed, and consistent semi-classical geometries appear in the weak-coupling region of the spacetime when the width of the matter pulse is larger then the short-distance cutoff. A consistent semi-classical geometry also appears in the limit of a shock-wave matter. The Hawking radiation flux receives non-thermal corrections such that it vanishes for late times and the total radiated mass is finite. There are no static remnants for matter pulses of finite width, although a BPP type static remnant appears in the shock-wave limit. Semi-classical geometries without curvature singularities can be obtained as well. Our results indicate that higher-order loop corrections can remove the singularities encountered in the one-loop solutions.
1. Introduction

The work on two-dimensional (2d) dilaton gravity models has shown that they possess the features one is interested to understand in a realistic gravitational collapse (for a review see [1]). Especially interesting is the CGHS model [2], which is soluble classically and it is a renormalizable 2d field theory. This raises a hope that the quantum theory may be tractable, so that the properties of quantum black holes could be understood. As far as the quantum solvability of the CGHS model is concerned, one can find the physical Hilbert space of states, which is the matter Fock space [3, 4, 5, 6, 7, 8]. This exact result allows one to show that the quantum theory is unitary, because the dynamics is generated by a free-field matter Hamiltonian, which can be easily promoted into a Hermitian operator acting on the physical Hilbert space [1]. Furthermore, one can show that evaporating black hole geometries appear in the semi-classical limit of a unitary gauge quantization of the CGHS model [6, 8], which shows that at least in 2d evaporating black holes can exist in a unitary quantum theory.

Although the results of [6, 8] imply unitary evolution for the quantum CGHS black hole, one would like to see what is the end-state geometry (i.e. is it a remnant or the black hole completely evaporates with the information being returned through the Hawking radiation [1]). For this one needs the exact effective quantum metric, which can be obtained either from the exact effective action, or from the expectation value of the metric operator. So far only the one-loop perturbative approximations of the effective metric have been obtained [2, 3, 10, 5], which are valid in the weak-coupling region of the spacetime. Especially interesting is the BPP geometry [10], which describes an evaporating black hole which ends up as a remnant. This same geometry arises at the one-loop level of the operator formalism [8], which is consistent with the unitarity. One is then interested to see what is the effect of the higher loop corrections for the BPP solution, and therefore calculating the two-loop corrections is the simplest thing to do. In this paper we calculate the two loop corrections to the effective metric by using the operator formalism of [8]. The advantage of the operator formalism over the conventional effective action approach is that it gives the spacetime dependence of the effective metric automatically, while in the effective action approach one has to solve the effective equations of motion, which in the most cases cannot be done explicitly. Also at two loops there is a large number of diagrams one would have to evaluate in order to obtain the contribution to the effective action.

In section 2 we review the operator formalism of [8]. In section 3 we review the BPP solution in the context of the operator formalism, since it is a starting point for
our perturbative calculation. In section 4 we calculate the two-loop corrections. In section 5 we examine the two-loop semi-classical geometry. In section 6 we present our conclusions.

2. The operator formalism

We start from the classical CGHS action \[2\]

\[
S = \int_M d^2x \sqrt{-g} \left[ e^{-\phi} \left( R + (\nabla \phi)^2 + 4\lambda^2 \right) - \frac{1}{2} (\nabla f)^2 \right] , \tag{2.1}
\]

where \(\phi\) is a dilaton scalar field, \(f\) is a matter scalar field, \(g, R\) and \(\nabla\) are determinant, curvature scalar and covariant derivative respectively, associated with a metric \(g_{\mu\nu}\) on a 2d manifold \(M\). The topology of \(M\) is that of \(\mathbb{R} \times \mathbb{R}\). The equations of motion can be solved in the conformal gauge \(ds^2 = -e^\rho dx^+ dx^-\) as

\[
e^{-\rho} = e^{-\phi} = -\lambda^2 x^+ x^- - F_+ - F_- , \quad f = f_+(x^+) + f_-(x^-) , \tag{2.2}
\]

where

\[
F_\pm = a_\pm + b_\pm x^\pm + \int^{x^\pm} dy \int^y dz T_{\pm\pm}(z) , \tag{2.3}
\]

and \(T_{\pm\pm}\) is the matter energy-momentum tensor

\[
T_{\pm\pm} = \frac{1}{2} \partial_\pm f \partial_\pm f . \tag{2.4}
\]

The residual conformal invariance can be fixed by a gauge choice \(\rho = \phi\), and the independent integration constants are \(a_+ + a_-\) and \(b_\pm\). An equivalent form of the solution (2.2) which is suitable for our purposes, is given by

\[
F_\pm = \alpha_\pm + \beta_\pm x^\pm + \int^{x^\pm} dy (x^\pm - y) T_{\pm\pm}(y) , \tag{2.5}
\]

where

\[
a_\pm = \alpha_\pm + \int_{\Lambda^\pm} dy T_{\pm\pm}(y) , \quad b_\pm = \beta_\pm - \int_{\Lambda^\pm} dy T_{\pm\pm}(y) . \tag{2.6}
\]

It is clear from the solution (2.2) that the independent dynamical degree of freedom is a free scalar field \(f\). This conclusion also comes out from a reduced phase space analysis \([6]\). Consequently the quantum theory is that of a free mass-less scalar quantum field \(f\), propagating on a flat background \(ds_f^2 = -dx^+ dx^-\) with the dilaton and the conformal factor operators given by (2.2) \([3, 8]\). The matter energy-momentum tensor operator is defined as

\[
T_{\pm\pm} = \frac{1}{2} \partial_\pm f \partial_\pm f . \tag{2.7}
\]
The normal ordering in (2.7) is chosen to be with respect to creation and annihilation operators associated to coordinates \(x^\pm\) by

\[
f_\pm(x^\pm) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{\sqrt{2\omega_k}} \left[ a_{\pm k} e^{-ikx^\pm} + a_{\mp k}^\dagger e^{ikx^\pm} \right],
\]

(2.8)

where \(\omega_k = |k|\).

The physical Hilbert space of the model is just a Fock space \(\mathcal{F}(a_k)\) constructed from \(a_k^\dagger\) acting on the vacuum \(|0\rangle\). The model is unitary because the dynamics is generated by a free-field Hamiltonian

\[
H = \int_{-\infty}^{\infty} dk \omega_k a_k^\dagger a_k + E_0,
\]

(2.9)

which is a Hermitian operator acting on \(\mathcal{F}\), where \(E_0\) is the vacuum energy. Consequently the states at \(t = \frac{1}{2}(x^+ + x^-) = \text{const.}\) surfaces are related by a unitary transformation

\[
\Psi(t_2) = e^{-iH(t_2 - t_1)}\Psi(t_1).
\]

(2.10)

One can also define the Heisenberg picture

\[
\Psi_0 = e^{iHt}\Psi(t), \quad A(t) = e^{iHt}Ae^{-iHt},
\]

(2.11)

which relates the covariant quantization to the canonical quantization. For example,

\[
f(t, x) = e^{iHt}f(x)e^{-iHt} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\omega_k}} \left[ a_k e^{i(kx - \omega_k t)} + a_k^\dagger e^{-i(kx - \omega_k t)} \right],
\]

(2.12)

where \(x = \frac{1}{2}(x^+ - x^-)\). Similarly, the operator expressions (2.2) are the Heisenberg picture operators.

Given a physical state \(\Psi_0\), one can associate an effective metric to \(\Psi(t) = e^{-iHt}\Psi_0\) as

\[
e^{\phi_{\text{eff}}(t, x)} = \langle \Psi_0 | e^{\phi(t, x)} | \Psi_0 \rangle,
\]

(2.13)

where \(e^\phi\) is the inverse operator of the Heisenberg operator (2.2). The geometry which is generated by \(e^{\phi_{\text{eff}}}\) via \(ds^2 = -e^{\phi_{\text{eff}}}dx^+dx^-\) makes sense only in the regions of \(M\) where the quantum fluctuations are small. This will happen if

\[
\sqrt{|\langle e^{2\phi} \rangle - \langle e^\phi \rangle^2|} << \langle e^\phi \rangle.
\]

(2.14)

The effective conformal factor \(e^{\phi_{\text{eff}}}\) can be calculated perturbatively by using a series expansion [6]

\[
(-\lambda^2 x^+ x^- F)^{-1} = e^{\phi_0}(1 - e^{\phi_0}\delta F)^{-1} = e^{\phi_0} \sum_{n=0}^{\infty} e^{n\phi_0} \delta F^n,
\]

(2.15)
where $F_0$ is a c-number function, $e^{-\phi_0} = -\lambda^2 x^+ x^- - F_0$ and $\delta F = F - F_0$. Then
\[
\left\langle (-\lambda^2 x^+ x^- - F)^{-1} \right\rangle = e^{\phi_0} \sum_{n=0}^{\infty} e^{n\phi_0} \delta F^n .
\] (2.16)

A convenient choice for $F_0$ is
\[
F_0 = \langle F_+ \rangle + \langle F_- \rangle ,
\] (2.17)
since then the lowest order metric is a one-loop semi-classical metric
\[
e^{-\phi_0} = -\lambda^2 x^+ x^- - \langle F_+ \rangle - \langle F_- \rangle .
\] (2.18)

$\Psi_0$ is chosen such that it is as close as possible to a classical matter distribution $f_0(x^+)$ describing a left-moving pulse of matter. The corresponding classical metric is described by
\[
e^{-\rho} = \frac{M(x^+)}{\lambda} = \lambda^2 x^+ \Delta(x^+) = \lambda^2 x^+ x^- .
\] (2.19)

where
\[
M(x^+) = \lambda \int_{-\infty}^{x^+} dy y T^0_{++}(y) , \quad \lambda^2 \Delta = \int_{-\infty}^{x^+} dy T^0_{++}(y)
\] (2.20)
and $T^0_{++} = \frac{1}{2} \partial_+ f_0 \partial_+ f_0$. The geometry is that of a black hole of the mass
\[
M = \lim_{x^+ \to +\infty} M(x^+) ,
\] (2.21)
and the horizon is at
\[
x^- = -\Delta = -\lim_{x^+ \to +\infty} \Delta(x^+) .
\] (2.22)

In the limit of a shock-wave matter distribution, for which
\[
T^0_{++} = a \delta(x^+ - x^+_0) ,
\] (2.23)
we have
\[
M(x^+) = \lambda a x^+_0 \theta(x^+ - x^+_0) , \quad \Delta = \frac{a}{\lambda^2} .
\] (2.24)

The asymptotically flat coordinates $(\eta^+, \eta^-)$ at the past null infinity are given by
\[
\lambda x^+ = e^{\lambda \eta^+} , \quad x^- = -\Delta e^{-\lambda \eta^-} ,
\] (2.25)
while the asymptotically flat coordinates $(\sigma^+, \sigma^-)$ at the future null infinity satisfy
\[
\lambda x^+ = e^{\lambda \sigma^+} , \quad \lambda(x^- + \Delta) = -e^{-\lambda \sigma^-} .
\] (2.26)

The corresponding Penrose diagram is given in Fig. 1.
Note that a change of coordinates $x^\pm \rightarrow \xi^\pm$ defines a new set of creation and annihilation operators through

$$f_\pm = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{\sqrt{2\omega_k}} \left[ b_\pm k e^{-ik\xi^\pm} + b_\pm^\dagger e^{ik\xi^\pm} \right]. \quad (2.27)$$

The old and the new creation and annihilation operators are related by a Bogoluibov transformation

$$a_k = S^{-1} b_k S = \int_{-\infty}^\infty dq (b_q \alpha_{qk} + b_q^* \beta_{qk}^*) \quad (2.28)$$

and the new vacuum is given by $|0_\xi\rangle = S |0_x\rangle$. This allows us to take for $\Psi_0$ a coherent state

$$\Psi_0 = e^A |0^+_\eta\rangle \otimes |0^-_\eta\rangle \quad (2.29)$$

where $|0_\eta\rangle = |0^+_\eta\rangle \otimes |0^-_\eta\rangle$ is the vacuum for the coordinates (2.25), while

$$A = \int_0^\infty dk \left[ f_0(k) a_{-k}^\dagger - f_0^*(k) a_{-k} \right] \quad (3.20)$$

where $f_0(k)$ are the Fourier modes of $f_0(x^+)$.  

### 3. One-loop metric

By using the operator formalism we can calculate perturbatively the effective metric (2.13) via the expansion (2.16). The lowest order semi-classical metric will be given by the expression (2.18) which requires a calculation of the expectation values of the $T_{\pm\pm}$ operators [8]. One can show that

$$\langle \Psi_0 | T_{++} | \Psi_0 \rangle = -\frac{\kappa}{4(x^+)^2} + \frac{1}{2} \left( \frac{\partial f_0}{\partial x^+} \right)^2, \quad \langle \Psi_0 | T_{--} | \Psi_0 \rangle = -\frac{\kappa}{4(x^-)^2}, \quad (3.1)$$

where $\kappa = \frac{1}{24\pi}$, so that

$$e^{-\rho_0} = e^{-\phi_0} = C + b_\pm x^\pm - \lambda^2 x^+ x^- - \frac{\kappa}{4} \log |\lambda^2 x^+ x^-| - \frac{1}{2} \int_{\Lambda^+} dy^+ (x^+ - y^+) \left( \frac{\partial f_0}{\partial y^+} \right)^2. \quad (3.2)$$

The expression (3.2) can also be obtained as a solution of the equations of motion of an effective one-loop action [10]

$$S_{eff} = S_0 - \frac{\kappa}{4} \int d^2 x \sqrt{-g} \Box^{-1} R - \kappa \int d^2 x \sqrt{-g} (R \phi - (\nabla \phi)^2) \quad (3.3)$$

where $S_0$ is the CGHS action (2.1).

Note that a natural choice for $\Lambda_\pm$ is $\Lambda_\pm = -\infty$. This choice makes the constant $C$ infinite and $b_\pm = 0$. We will ignore this infinity, since from the effective action point
of view $C$ is a constant of integration whose value is determined from a requirement of having a consistent semi-classical geometry. By choosing $C = \frac{1}{4} \kappa [\log(\kappa/4) - 1]$ one can obtain a consistent semi-classical geometry [10]. In the case of the shock-wave matter this geometry is well defined in the $x^+ > 0, x^- < 0$ quadrant. In the dilaton-vacuum sector ($x^+ < x^+_0$) the solution (3.2) becomes static

$$e^{-\rho_0} = e^{-\phi_0} = C - \lambda^2 x^+ x^- - \frac{\kappa}{4} \log(\lambda^2 x^+ x^-) \quad ,$$

(3.4)

and it is defined for $\sigma \geq \sigma_{cr}$, where $\sigma = \log(-\lambda^2 x^+ x^-)$ is the static coordinate. At $\sigma = \sigma_{cr}$ there is a singularity, and this line is interpreted as a boundary of a strong coupling region. This is a common feature of the semi-classical metrics [9], and a consistent geometry can be defined for $\sigma \geq \sigma_{cr}$ by imposing a reflecting boundary conditions at $\sigma = \sigma_{cr}$. However, in the operator approach we do not impose reflecting boundary conditions. A consistent geometry is defined only in the regions of the spacetime where the metric fluctuations are small, and these coincide with the weak-coupling region $e^{-\phi_0} > 0$ [3]. Note that a curvature singularity occurs at $\sigma = \sigma_{cr}$ for $C < \frac{1}{4} \kappa [\log(\kappa/4) - 1]$. Hence this naked singularity will not appear for $C \geq \frac{1}{4} \kappa [\log(\kappa/4) - 1]$.

For $x^+ > x^+_0$ one obtains an evaporating black hole solution

$$e^{-\rho_0} = e^{-\phi_0} = C + \frac{M}{\lambda} - \lambda^2 (x^- + \Delta) - \frac{\kappa}{4} \log(-\lambda^2 x^+ x^-) \quad .$$

(3.5)

The corresponding Hawking radiation flux at the future null-infinity is determined in the operator formalism by evaluating

$$\langle \Psi_0 | T_{-\sigma}(\xi^-) | \Psi_0 \rangle \quad ,$$

(3.6)

where $T_{-\sigma}(\xi^-)$ is normal ordered with respect to the asymptotically flat coordinates $\xi^\pm$ of the metric (3.5) at the future null-infinity [3, 4]. The $\xi^\pm$ coordinates turn out to be the same as the “out” coordinates (2.26) of the classical black hole solution, so that

$$2\pi \langle T_{-\sigma}(\xi^-) \rangle = \frac{\lambda^2}{48} \left[ 1 - (1 + \Delta e^{\lambda \sigma^-})^{-2} \right] \quad .$$

(3.7)

The expression (3.7) corresponds to a thermal Hawking radiation, with $T_H = \frac{\lambda}{2\pi}$ [4, 11]. The Hawking radiation shrinks the apparent horizon of the solution (3.5), so that the apparent horizon line meets the curvature singularity in a finite proper time, at

$$x^+_i = \frac{1}{\lambda^2 \Delta} \left( -\kappa/4 + e^{1+\frac{4}{\kappa}(C+M/\lambda)} \right) \quad , \quad x^-_i = \frac{-\Delta}{1 - \frac{\kappa}{4} e^{-1-\frac{4}{\kappa}(C+M/\lambda)}} \quad .$$

(3.8)
The curvature singularity then becomes naked for \( x^+ > x_i^+ \). However, a static solution (3.4) of the form

\[
e^{-\rho_0} = e^{-\phi_0} = \hat{C} - \lambda^2 x^+ (x^- + \Delta) - \frac{\kappa}{4} \log(-\lambda^2 x^+ (x^- + \Delta))
\]

(3.9)
can be continuously matched to (3.5) along \( x^- = x_i^- \) if \( \hat{C} = \frac{1}{4}\kappa[\log(\kappa/4) - 1] \). A small negative energy shock-wave emanates from that point, and for \( x^- > x_i^- \) the Hawking radiation stops, while the static geometry (3.9) has a null ADM mass. There is again a critical line \( \tilde{\sigma} = \tilde{\sigma}_{cr} \), corresponding to a singularity of the geometry (3.9). Note that the scalar curvature of (3.9) is bounded at \( x^- = x_i^- \), and the singularity comes from the pathological behavior of \( e^{-\phi_0} \), which becomes ill-defined for \( x^- > x_i^- \). This singularity can be interpreted as the boundary of the region where higher order corrections become important. The spatial geometry of the remnant (3.9) is that of a semi-infinite throat, extending to the strong coupling region. The Penrose diagram of the one-loop geometry is given in Fig. 2.

Note that in the operator approach the spacetime is always \( \mathbb{R}^2 \), with the geometry of Fig. 2 defined in the region where the metric fluctuations and higher order corrections are small. The one-loop approximation (2.18) is valid in the regions where

\[
e^{2\phi_0} \left| \langle \delta F^2 \rangle \right| << 1
\]

(3.10)
The condition (3.10) will certainly break down for \( e^{-\phi_0} = 0 \), which also determines the position of the curvature singularities of the one-loop metric. Therefore one can see directly in the operator approach why the one-loop singularities represent a border of the strong coupling region. Also note that (3.10) is perturbatively equivalent to the condition for small quantum fluctuations (2.14), so that a well-defined geometry appears in the region \( e^{-\phi_0} > 0 \), which coincides with the weak-coupling region. This situation can be best described by the Kruskal diagram of Fig. 3. The region to the left of the curve ABCD in Fig. 3 corresponds to the strong-coupling region.

The operator formalism can also resolve problems connected with a shift in the classical horizon due to quantum corrections. From (3.8) it appears as if the classical horizon \( x^- = -\Delta \) has been shifted to a new position \( x^- = x_i^- < -\Delta \). One can calculate the corresponding Bogoliubov coefficients, and due to this shift, they will be described by incomplete Gamma functions in the late time approximation \([12]\), in contrast to the ordinary Gamma functions when there is no shift \([11]\). As a result, the Bogoliubov coefficients will have a different asymptotic behavior for large frequencies from the standard case, and as a consequence the Hawking flux will diverge \([12]\). However, at the one-loop order \( \langle T \rangle \) is given by (3.7), which is finite. The way out of
this paradox is provided by the fact that we are working in a quantum gravity theory, and therefore the space-time geometry fluctuates. Hence the position of the horizon fluctuates, and consistency requires that the effective horizon must be at $x^- \geq -\Delta$.

4. Two-loop corrections

In order to calculate the two-loop corrections it will be useful to redefine $F$ as 

$$e^{-\phi} = \alpha + \beta_x x^\pm - \lambda^2 x^+ x^- - \int_{\Lambda^\pm}^{x^\pm} dy(x-y) \langle 0_\eta | T_{ \pm \pm } (y) | 0_\eta \rangle$$

$$- \frac{1}{2} \int_{\Lambda^\pm}^{x^\pm} dy^+(x^+ - y^+) (T_{ \pm \pm } (y) - \langle 0_\eta | T_{ \pm \pm } (y) | 0_\eta \rangle)$$

$$= C - \lambda^2 x^+ x^- - \frac{\kappa}{4} \log |\lambda^2 x^+ x^-| - F_+ - F_- \ , \quad (4.1)$$

so that the new $F_{ \pm }$ are given as

$$F_{ \pm } = \int_{\Lambda^\pm}^{x^\pm} dy(x-y)T_{ \pm \pm } (y) \ , \quad (4.2)$$

where $\tilde{T} = T - \langle 0_\eta | T | 0_\eta \rangle$ and $\langle 0_\eta | \tilde{T} | 0_\eta \rangle = 0$. Another convenient redefinition is to rescale $f(x)$ to $\sqrt{2\pi f(x)}$.

The left-moving sector gives corrections due to the matter quantum fluctuations $\Psi$. We introduce the following operator ordering

$$: T_{ ++ } (x_1) T_{ ++ } (x_2) := T_{ ++ } (x_1) T_{ ++ } (x_2) - \langle 0_\eta | T_{ ++ } (x_1) T_{ ++ } (x_2) | 0_\eta \rangle \ , \quad (4.3)$$

so that

$$\langle \Psi_0 | : T_{ ++ } (x_1) T_{ ++ } (x_2) : | \Psi_0 \rangle = - \frac{1}{8 \pi^2} \frac{\partial f_0}{\partial x_1} \frac{\partial f_0}{\partial x_2} \partial_{x_1} \partial_{x_2} \log |\eta(x_1) - \eta(x_2)| \ , \quad (4.4)$$

where

$$\langle 0_\eta | f(x_1) f(x_2) | 0_\eta \rangle = - \frac{1}{2} \log |\eta^+(x_1) - \eta^+(x_2)| - \frac{1}{2} \log |\eta^-(x_1) - \eta^-(x_2)| \quad (4.5)$$

has been used $\S$.

The expression (4.4) is still divergent when $x_1 \to x_2$ since

$$\partial_{x_1} \partial_{x_2} \log |\eta^+(x_1) - \eta^+(x_2)| = \frac{1}{(x_1-x_2)^2} + \frac{1}{6} D_x (\eta) + O(x_1-x_2) \ , \quad (4.6)$$

where $D_x (\eta) = \frac{\partial^3 \eta}{\partial^3 \eta^2} - \frac{2}{3} \left( \frac{\partial^2 \eta}{\partial^2 \eta^2} \right)^2$ is a Schwarzian derivative. Expression (4.6) can be regularized by subtracting $(x_1 - x_2)^{-2}$ from $\partial_1 \partial_2 \log (\eta_1 - \eta_2)$, which corresponds to using a new ordering $\S$

$$: T_{ ++ } (x_1) T_{ ++ } (x_2) := T_{ ++ } (x_1) T_{ ++ } (x_2) - \langle 0_\eta | T_{ ++ } (x_1) T_{ ++ } (x_2) | 0_\eta \rangle$$

$$- \langle 0_x | [A, T_{ ++ } (x_1)] [A, T_{ ++ } (x_2)] | 0_x \rangle \ . \quad (4.7)$$
This ordering gives for \( \langle \delta F_+^2 \rangle \)
\[
- \frac{1}{8\pi^2} \prod_{i=1}^{2} \int_{x_0^+}^{x_{0}^+ + \epsilon} dx_i (x^+ - x_i) \left( (x_1 x_2)^{-1} (\log(x_1/x_2))^2 - \frac{1}{(x_1 - x_2)^2} \right) \frac{\partial f_0}{\partial x_1} \frac{\partial f_0}{\partial x_2}. \tag{4.8}
\]

Now we will calculate (4.8) for the shock-wave matter distribution (2.23). In order to make the calculation simple and well defined, we take a matter pulse
\[
\partial_+ f_0(x^+) = \sqrt{\frac{2\alpha}{\epsilon}} \left[ \theta(x^+ - x_0^+) - \theta(x^+ - x_0^+ - \epsilon) \right] \tag{4.9}
\]
where \( \epsilon \) is the width of the pulse. When \( \epsilon \to 0 \) (4.9) gives \( T_{++} \) for the shock-wave. Let \( x^+ > x_0^+ + \epsilon \), then \( \langle \delta F_+^2 \rangle \) is given by
\[
- \frac{a}{4\epsilon \pi^2} \int_{x_0^+}^{x_0^+ + \epsilon} \int_{x_0^+}^{x_0^+ + \epsilon} dx_1 dx_2 (x^+ - x_1)(x^+ - x_2) \left( \frac{1}{x_1 x_2 (\log(x_1/x_2))^2} - \frac{1}{(x_1 - x_2)^2} \right). \tag{4.10}
\]
By using the expansion (4.6), we obtain
\[
\langle \delta F_+^2 \rangle = - \frac{a}{4\epsilon \pi^2} \int_{x_0^+}^{x_0^+ + \epsilon} \int_{x_0^+}^{x_0^+ + \epsilon} dx_1 dx_2 (x^+ - x_1)(x^+ - x_2) \left( \frac{1}{12 x_1 x_2} + O(x_1 - x_2) \right) \]
\[
= - \frac{a}{4\epsilon \pi^2} (x^+ - x_0^+)^2 \left( \frac{\log(1 + \epsilon/x_0^+)^2}{12} + O(\epsilon^4) \right). \tag{4.11}
\]
Since \( \epsilon \) is small, we have
\[
\langle \delta F_+^2 \rangle \approx - \frac{a\epsilon}{48\pi^2} (x^+ - x_0^+)^2 \theta(x^+ - x_0^+) . \tag{4.12}
\]
Note that in the limit \( \epsilon \to 0 \) (4.12) will vanish, so that there is no two-loop correction in the right-moving sector for the shock-wave. In the case of arbitrary pulses
\[
\langle \delta F_+^2 \rangle = C_+(x^+)^2 + C'_+ x^+ + C''_+ \tag{4.13}
\]
for \( x^+ > x_0^+ + \epsilon \). Note that
\[
C_+ = \int_{x_0^+}^{x_0^+ + \epsilon} \int_{x_0^+}^{x_0^+ + \epsilon} dx_1 dx_2 \langle : T_{++}(x_1) T_{++}(x_2) : \rangle, \tag{4.14}
\]
in accordance with (4.12).

One can get a similar result by using a \( \zeta \)-function regularization. One starts from the formula
\[
\int_0^\infty dk \, k e^{ik(\eta_1 - \eta_2)} = - \frac{1}{(\eta_1 - \eta_2)^2} , \quad \eta_1 - \eta_2 \neq 0 , \tag{4.15}
\]
which can be used to rewrite (4.4) as
\[
\frac{1}{8\pi^2} \int_0^\infty dk \, k e^{ik(\eta_1 - \eta_2)} \partial_{\eta_1} \partial_{\eta_2} \partial_{\eta_1} f_0 \partial_{\eta_2} f_0 , \tag{4.16}
\]
so that
\[
\langle \delta F_+^2 \rangle = \frac{1}{8\pi^2} \prod_{i=1}^{2} \int_{-\infty}^{\eta^+} d\eta_i (e^{\lambda (\eta^+ - \eta_i)} - 1) \int_{0}^{\infty} dk \ k e^{\delta k (\eta^+ - \eta_2)} \frac{\partial f_0}{\partial \eta_1} \frac{\partial f_0}{\partial \eta_2}
\]
\[
= \frac{1}{8\pi^2} \int_{0}^{\infty} dk \ k |\mathcal{F}(k, \eta^+)|^2 \ , \quad (4.17)
\]
where
\[
\mathcal{F}(k, \eta^+) = \int_{-\infty}^{\eta^+} d\eta e^{ik\eta} (e^{\lambda (\eta^+ - \eta)} - 1) \frac{\partial f_0}{\partial \eta} \ . \quad (4.18)
\]
By comparing (4.13) to (4.17), one obtains for the pulse (4.9)
\[
C_+ = \frac{a}{4\pi^2 \epsilon} \int_{0}^{\infty} \frac{dk}{k} \left[ \omega^{-2} + \nu^{-2} - \frac{2}{\omega \nu} \cos k(\omega - \nu) \right] \quad (4.19)
\]
where \(\omega = \log \lambda (x_0^+ + \epsilon)\) and \(\nu = \log \lambda x_0^+\). When \(\epsilon \to 0\), we get
\[
C_+ = \frac{a}{4\pi^2} \int_{0}^{\infty} \frac{dk}{k} \frac{\epsilon}{\omega^2 (x_0^+)^2} + O(\epsilon^2) \ . \quad (4.20)
\]
The infinite integral \(\int_{0}^{\infty} kdk\) can be replaced by a sum \(\sum_{n=1}^{\infty} n\), which can be regularized by using the \(\zeta\) function
\[
\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \ , \quad \text{Re} \ s > 1 \ , \quad (4.21)
\]
in the following way
\[
\int_{0}^{\infty} kdk = \lim_{N \to \infty} \int_{0}^{N} kdk = \lim_{N \to \infty} \frac{N^2}{2}
\]
\[
= \lim_{N \to \infty} \left( \frac{1}{2} N(N + 1) - \frac{1}{2} N \right) = \lim_{N \to \infty} \left( \sum_{n=1}^{N} n - \frac{1}{2} \sum_{n=1}^{N} 1 \right)
\]
\[
= \sum_{n=1}^{\infty} n - \frac{1}{2} \sum_{n=1}^{\infty} 1 = \zeta(-1) - \frac{1}{2} \zeta(0) = -\frac{1}{12} - \frac{1}{4} \ . \quad (4.22)
\]
Therefore
\[
C_+ = -\frac{ae}{12\pi^2 x_0^2} \log^{-2}(\lambda x_0^+) \ , \quad (4.23)
\]
again a small negative constant, and \(\lim_{\epsilon \to 0} C_+ = 0\). Note that a finite result in (4.23) has been obtained by analytical continuation of \(\zeta(s)\) to \(s = -1\) and \(s = 0\), which is the same as if an infinite number has been subtracted from (4.20). But this is what has been explicitly done in the regularization (4.8), so that it is no surprise that a similar result was obtained.

For our purposes it will be instructive to regularize (4.19) when \(\epsilon\) is small but finite by a momentum cutoff. In that case
\[
C_+ = \frac{a}{\pi^2 \epsilon \nu^2} \int_{0}^{\Lambda} \frac{dk}{k} \sin^2 \frac{ek}{2\lambda x_0^+} + O(\epsilon) \ . \quad (4.24)
\]
When $\Lambda$ is becoming large, we have
\[
C_+ = \frac{a}{2\pi^2 \epsilon \nu^2} \log \frac{\epsilon \gamma \Lambda}{\lambda x_0^+} + O(\Lambda^{-1}) ,
\]
(4.25)
where $\gamma$ is the Euler-Mascheroni constant. Therefore $C_+$ is negative if the width of the pulse is less than the short distance cutoff $l_c = \lambda x_0^+ / \gamma \Lambda$ and otherwise $C_+$ is positive.

The right-moving sector determines the back-reaction of the Hawking radiation [6, 8]. In this case we use a point-splitting method for regularizing the operator products. It amounts to calculating an appropriate limes ($x_2^+ \rightarrow x_1^-$) of the expression [8]
\[
\langle 0_\eta | \prod_{i=1}^{2n} \partial_{x_i^-} \delta f(\eta(x_i)) | 0_\eta \rangle .
\]
(4.26)
The expression (4.26) can be calculated by using Wick’s theorem and by using the expression for a two-point function (4.5). A normal ordering can be defined by an appropriate subtraction of the terms $(x_i^+ - x_j^-)^{-2}$ and $\partial f_k \partial f_l$ from the expression (4.26) before taking the limes, such that one obtains a regular expression after taking the limes. A useful formula for doing this is (4.6).

In the $n = 2$ case, one can define
\[
4\pi^2 : T_{--}(x_1)T_{--}(x_2) := \lim_{x_3 \rightarrow x_1} \lim_{x_4 \rightarrow x_2} \frac{1}{4} \left( \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_3} \frac{\partial f}{\partial x_2} \frac{\partial f}{\partial x_4} + \frac{1}{2} \frac{\partial f}{\partial x_2} \frac{\partial f}{\partial x_4} + \frac{1}{4} \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_3} + \frac{1}{4} \frac{\partial f}{\partial x_2} \frac{\partial f}{\partial x_4} \right) \]
(4.27)
where $x_i$ denotes $x_i^-$. This ordering gives
\[
\langle 0_\eta | T_{--}(x_1)T_{--}(x_2) : | 0_\eta \rangle = \frac{1}{32\pi^2} \left( \partial_{x_1} \partial_{x_2} \log |\eta^-(x_1) - \eta^-(x_2)| \right)^2
+ \langle T_{--}(x_1) \rangle \langle T_{--}(x_2) \rangle .
\]
(4.28)
The expression (4.28) is still divergent when $x_1 \rightarrow x_2$, and it will be the source of divergence in
\[
\langle \delta F^2 \rangle = \frac{1}{32\pi^2} \int_{\Lambda^-}^{x_1^-} \int_{\Lambda^-}^{x_2^-} dx_1 dx_2 (x_1^- - x_1)(x_2^- - x_2) \left( \partial_{x_1} \partial_{x_2} \log |\eta^-(x_1) - \eta^-(x_2)| \right)^2 .
\]
(4.29)
One way to regularize (4.28) is by changing the definition (4.27) [6, 8]. This amounts to
\[
\langle 0_\eta | T_{--}(x_1)T_{--}(x_2) : | 0_\eta \rangle = \frac{1}{32\pi^2} \left( \partial_{x_1} \partial_{x_2} \log |\eta^-(x_1) - \eta^-(x_2)| - \frac{1}{(x_1^- - x_2^-)^2} \right)^2
+ \langle T_{--}(x_1) \rangle \langle T_{--}(x_2) \rangle ,
\]
(4.30)
which is finite for \( x_1 \to x_2 \) due to (4.6). Consequently \( \langle \delta F^2 \rangle \) is given by

\[
\frac{1}{32\pi^2} \prod_{i=1}^{n} \int_{\Lambda_-}^{x_i} \ dx_i (x_i - x_i) \left( \partial_{x_1} \partial_{x_2} \log |\eta^-(x_1) - \eta^-(x_2)| - \frac{1}{(x_1 - x_2)^2} \right)^2 ,
\]

(4.31)

which is finite for \( \Lambda_- \) finite. However, it is natural to take \( \Lambda_- = -\infty \). In that case the integral (4.31) is independent of \( x^- \), which can be seen by making a substitution

\[
x_i = x^- e^{t_i}
\]

(4.32)

so that

\[
\langle \delta F^2 \rangle = \frac{1}{32\pi^2} \int_0^\infty \int_0^\infty \ dt_1 \ dt_2 e^{t_1 + t_2} (1 - e^{t_1})(1 - e^{t_2}) \left( e^{-\frac{(t_1 + t_2)}{t_1 - t_2}} - (e^{t_1} - e^{t_2})^{-2} \right)^2 .
\]

(4.33)

The integral (4.33) is a divergent constant, and the way it arises is similar to the one-loop constant \( C \). Hence we will consider (4.33) as a finite constant, whose value is determined from some self-consistency requirement, and therefore

\[
\langle \delta F^2 \rangle = C_.
\]

(4.34)

Note that the result (4.34) can be also obtained with the \( \zeta \)-function regularization. Namely, by making the substitution (4.32) in the integral (4.29) we obtain

\[
\langle \delta F^2 \rangle = \frac{1}{32\pi^2} \int_0^\infty \int_0^\infty \ dt_1 \ dt_2 (1 - e^{-t_1})(1 - e^{-t_2})(t_1 - t_2)^{-4} .
\]

(4.35)

By repeated differentiation of (4.14) with respect to \( \eta \) we obtain

\[
\frac{1}{(t_1 - t_2)^4} = \frac{1}{6} \int_0^\infty \ dk \ k^3 \ e^{ik(t_1 - t_2)} ,
\]

(4.36)

so that

\[
C_\cdot = \lim_{l \to \infty} \frac{1}{6 \cdot 32\pi^2} \int_0^\infty \ dk k^3 |J_l(k)|^2
\]

(4.37)

where

\[
J_l(k) = \int_0^l \ dt (1 - e^{-t}) e^{ikt} .
\]

(4.38)

For large \( l \) we have

\[
J_l(k) = \frac{e^{ikl} - 1}{ik} + \frac{1}{ik - 1} + O(e^{-l})
\]

(4.39)

so that

\[
C_\cdot = \lim_{l \to \infty} \frac{1}{6 \cdot 32\pi^2} \int_0^\infty \ dk k^3 \left( \frac{k^2 + 2}{k^2(1 + k^2)} - \frac{2 \sin kl}{k(1 + k^2)} - \frac{2 \cos kl}{k^2(1 + k^2)} \right)
\]

\[
= \lim_{l \to \infty} \frac{1}{6 \cdot 32\pi^2} \left( \int_0^\infty \ dk \ k^2 + 2 - 2 \int_0^\infty \ dx \frac{x \sin x}{l^2 + x^2} - \int_0^\infty \ dx \frac{2x \cos x}{l^2 + x^2} \right)
\]

\[
= \frac{1}{6 \cdot 32\pi^2} \int_0^\infty \ dk \ (1 + \frac{1}{k^2 + 1}) .
\]

(4.40)
The last integral in (4.40) can be rewritten as

\[
I = \lim_{N \to \infty} \left( \int_0^N kdk + \frac{1}{2} \int_1^\infty \frac{dx}{x} \exp(-e^{-N}x) \right)
\]

\[
= \lim_{N \to \infty} \left[ \frac{1}{2} N(N+1) - \frac{1}{2} \gamma \right] = \sum_{n=1}^\infty n - \frac{1}{2} \gamma = \zeta(-1) - \frac{1}{2} \gamma ,
\]

(4.41)

where \( \gamma \) is the Euler-Mascheroni constant and we used

\[
\int_1^\infty e^{-\epsilon x} \frac{dx}{x} = -\text{Ei}(-\epsilon) = -\gamma - \log \epsilon + O(\epsilon) .
\]

(4.42)

Therefore we obtain

\[
C_- = -\frac{1}{12 \cdot 32\pi^2} \left( \frac{1}{6} + \gamma \right) ,
\]

(4.43)

a negative constant.

5. Two-loop semi-classical geometry

The analysis in the previous section implies that

\[
\left\langle \delta F^2 \right\rangle = C_- + C_+(x^+ - x_0^+)^2 \theta(x^+ - x_0^+) ,
\]

(5.1)

for narrow matter pulses. We have also shown that a regularization exists such that \( C_- \) is a negative constant, while \( C_+ \) depends on the shape of the matter pulse. Clearly the constants \( C_\pm \) are regularization dependent. This regularization dependence is not a problem, since from the effective action approach point of view \( C_\pm \) can be considered as integrals of motion (the two-loop equations of motion will be of higher order in spacetime derivatives, and therefore new integration constants will appear). Therefore the range of \( C_\pm \) could be determined from some appropriate consistency condition. Also, in the following analysis we will show that the corresponding geometries are essentially determined by the sign of \( C_+ \). Note that in the case of arbitrary pulses of compact support \( \left\langle \delta F^2 \right\rangle \) is quadratic in \( x^+ \) for \( x^+ \) outside of the support interval (see (4.13)), so that (5.1) is a good effective approximation in the general case. However, \( C_+ \) can be then of both signs, positive or negative, depending on the matter pulse profile, which can be seen from (4.14) and (4.25).

The two-loop metric can be written as

\[
ds^2 = -e^{\phi_2} dx^+ dx^- ;
\]

\[
e^{\phi_2} = e^{\phi_0} \left[ 1 + e^{2\phi_0} (C_- + C_+(x^+ - x_0^+)^2 \theta(x^+ - x_0^+)) \right] ,
\]

(5.2)
where $e^{\phi_0}$ is the one-loop dilaton solution

\[ e^{-\phi_0} = C - a(x^+ - x_0^+)\theta(x^+ - x_0^+) - \lambda^2 x^+ x^+ - \frac{\kappa}{4} \log|\lambda^2 x^+ x^-| \ . \quad (5.3) \]

We introduce new constants $\alpha$ and $\beta$ such that $C_- = -\alpha^2$ and $C_+ = \pm \beta^2$. As in the one-loop case, the relevant quadrant is $x^+ \geq 0, x^- \leq 0$. For $x^+ < x_0^+$ the solution (5.2) is static, and it can be written as

\[ e^{\phi_2} = e^{\phi_0} \left[ 1 - e^{2\phi_0} \alpha^2 \right] , \quad e^{-\phi_0} = C + e^\sigma - \frac{k}{4}\sigma \ , \quad (5.4) \]

where $\sigma = \log(-\lambda^2 x^+ x^-)$ is the static coordinate. The solution (5.4) describes a two-loop corrected dilaton vacuum. The corresponding scalar curvature will diverge on the curve

\[ e^{-2\phi_0} - \alpha^2 = 0 \ . \quad (5.5) \]

The equation (5.5) has two solutions $e^{-\phi_0} = \pm \alpha$, which correspond to the one-loop singularity lines where $C$ is replaced by $C \mp \alpha$. Since the shock-wave intersects first the $C - \alpha$ line ($\alpha > 0$), this will be the critical line. The semi-classical geometry will be defined for $\sigma \geq \sigma_c$, where $C + e^{\sigma_c} - \frac{k}{4}\sigma_c = \alpha$. The curvature singularity will be absent for $C \geq \alpha - \frac{k}{4}(1 - \log \frac{k}{4})$. Therefore if we want to avoid a naked singularity at two loops, the BPP choice of the one-loop $C$ has to be increased to

\[ C = \alpha + \frac{k}{4}(\log \frac{k}{4} - 1) \ . \quad (5.6) \]

Note that in the quadrants $x^+ x^- \geq 0$ naked singularities are present for any value of $C$. However, for the observer located at the right spatial infinity, these are located in the strong coupling region $e^{-\phi_0} \leq 0$, where the two-loop approximation breaks down, and therefore can be ignored. Also note that for $C_- = +\alpha^2$ no curvature singularities appear, and the only singularity comes from $\phi$ becoming complex for $e^{-\phi_0} < 0$.

For $x^+ > x_0^+$ the solution (5.2) becomes

\[ e^{\phi_2} = e^{\phi_0} \left[ 1 + e^{2\phi_0}(-\alpha^2 + C_+(x^+ - x_0^+)^2) \right] , \quad (5.7) \]

where

\[ e^{-\phi_0} = C + \frac{M}{\lambda} - \lambda^2 x^+(x^- + \Delta) - \frac{k}{4} \log(-\lambda^2 x^+ x^-) \ . \quad (5.8) \]

It describes a two-loop corrected evaporating black hole geometry. The curvature singularity is given by the curve

\[ e^{-2\phi_0} - \alpha^2 + C_+(x^+ - x_0^+)^2 = 0 \ . \quad (5.9) \]
The curve (5.9) can be parameterized as
\[
x^+x^- = -e^{\sigma} \quad \text{with } \frac{\Delta C_\sigma + \sqrt{\alpha^2(\Delta^2 + C_+)} - C_+ C_\sigma^2}{\Delta^2 + C_+} = 0.
\]
where \( C_\sigma = C + e^{\sigma} - \frac{k}{4}\sigma \), and we have set \( \lambda = 1 \). When \( C_+ = -\beta^2 \), the relevant branch of (5.10) is a curve which starts from \((x_0^+, -k/4x_0^+)\) and it goes to \( x^+ = \infty \) with an asymptote \( x^- = -\Delta - \beta \). When \( C_+ = \beta^2 \), (5.9) is a closed curve, which starts from \((x_0^+, -k/4x_0^+), \) extends to \( x^+ = x_0^+ + (\alpha \Delta / \beta)(\Delta^2 + \beta^2)^{-\frac{1}{2}} \), where it turns back and ends up at \( x^+ = x_0^+ \) line.

The apparent horizon curve is given by the equation \( \partial_+ \phi_2 = 0 \), which can be rewritten as
\[
x^- + \Delta + \frac{k}{4\lambda^2 x^+} = -\frac{2C_+(x^+ - x_0^+)e^{-\phi_0}/\lambda^2}{e^{-2\phi_0} - 3\alpha^2 + 3C_+(x^+ - x_0^+)^2}.
\]
When \( C_+ < 0 \), this curve starts from \((x_0^+, -\Delta - k/4x_0^+)\), intersects \( x^- = -\Delta - \beta \), and then asymptotically approaches this line as \( x^+ \to \infty \). This behavior implies that the apparent horizon must intersect the curvature singularity at some \((x_i^+, x_i^-)\), so that for \( x^+ > x_i^+ \) there is a naked singularity. If \( C_+ > 0 \), then (5.11) is a closed curve, which starts from \((x_0^+, -\Delta - k/4x_0^+)\), extends to \( x^+ \approx x_0^+ + 5/4\beta \), and it comes back to \( x^+ = x_0^+ \) line. The naked singularity will appear in this case, but it will be located in the strong coupling region \( e^{-\phi_0} \leq 0 \), and therefore it can be ignored. Note that this does not happen in the \( C_+ < 0 \) case, where the naked singularity lies outside the strong coupling region (see Figs 4 and 5). As we are going to see later, this behavior is correlated with the behavior of the Hawking flux.

When \( \beta = 0 \), the equation (5.9) becomes a shock wave singularity equation (5.5), and the relevant root of that equation is \( e^{-\phi_0} - \alpha = 0 \). This solution describes a singularity line of the one-loop metric with a smaller ADM mass \( C + \frac{M}{\lambda} - \alpha \). For a minimal allowed value of \( C \) (5.6), this becomes the one-loop metric with \( C = \frac{k}{4}(\log \frac{k}{4} - 1) \). For a very small or vanishing \( \beta \), the \( \beta \) terms can be neglected in (5.11) and one obtains the one-loop apparent horizon line equation. Therefore the intersection point of the apparent horizon line and the curvature singularity line for the shock-wave at two-loops is given by the one-loop expressions (3.8). The change in the intersection point (3.8) when \( \beta \neq 0 \) can be evaluated perturbatively, and it is given by
\[
\frac{\delta x_i^+}{a} = -\frac{C_+ (x_i^+ - x_0^+)^2}{2\alpha ax_i^+} x_i^- + O(C_+^2)
\]
and
\[
\frac{\delta x_i^-}{a} = -\frac{2\lambda^2 C_+ (x_i^+ - x_0^+)^2}{\kappa \alpha} x_i^- x_i^+ + O(C_+^2).
\]

\( \frac{\partial}{\partial x_i} \)
Therefore when $C_+ < 0$ a naked singularity will appear in the region $x^- > x_i^-$, unless we impose an appropriate boundary condition. In the shockwave case ($C_+ = 0$), one can impose a static solution (5.4) for $x^- > x_i^-

\begin{align}
\phi^2 &= \phi_0 \left[1 - e^{2\phi_0 \alpha^2}\right], \\
\phi^2 &= \hat{C} - \lambda^2 x^+(x^- + \Delta) - \frac{\alpha}{4} \log(-\lambda^2 x^+(x^- + \Delta)),
\end{align}

which can be continuously matched to (5.7) at $x^- = x_i^-$ if

$$\hat{C} = \alpha - \frac{\kappa}{4}(1 - \log \frac{\kappa}{4}).$$

The metric (5.13) with the value of $\hat{C}$ given by (5.14) does not have a curvature singularity at $\sigma = \frac{1}{\chi} \log(-\lambda^2 x^+(x^- + \Delta)) = \sigma_{cr}$, and the naked singularity is removed. In this case $\sigma = \sigma_{cr}$ corresponds to a curve where $e^{-\phi_0}$ is ill-defined (branch-point singularity), in a complete analogy with the one-loop case. However, when $C_+ = -\beta^2 \neq 0$, there is no static two-loop dilaton vacuum solution which can be continuously matched to (5.7). The best one can do is to take

$$\phi^2 = \phi_0 \left[1 - e^{2\phi_0 \alpha^2(\beta^2 + (\alpha \Delta - \beta x^- - \beta^2)^{-\frac{1}{2}})}\right],$$

for $x^- \geq x_i^-$ and $x^+ \geq x_i^+$, where $\phi_0$ is given by (5.13) and (5.14). Still, the naked singularity remains. Hence the two-loop corrections make the one-loop remnant unstable. Note that when $C_+ = \beta^2$, there is no curvature singularity for $x^+ > x_0^+ + (\alpha \Delta - \beta x^- - \beta^2)^{-\frac{1}{2}}$, and therefore in this case there is no need for a sewing procedure. By examining the Hawking flux, we will see that this solution has a well-defined flux of emitted particles at future null infinity.

In order to calculate the Hawking flux we need the asymptotically flat coordinates $(\sigma^+, \sigma^-)$ at $I^\pm_R$. These are given by

$$\lambda x^+ = e^{\lambda \sigma^+}, \quad \lambda(x^- + \Delta) = -e^{-\lambda \tilde{\sigma}},$$

where

$$\tilde{\sigma} = \sigma - \frac{C_+}{2\lambda^3} e^{2\lambda \tilde{\sigma}}.$$

This can be written as

$$\sigma^- = -\frac{1}{\lambda} \log[-\lambda(\Delta + x^-)] + \frac{C_+}{2\lambda^3} (\Delta + x^-)^{-2},$$

so that a non-zero correction appears at two loops. When $C_+ < 0$, the coordinate change (5.16) is well-defined for $x^- < -\Delta - \frac{\beta}{\lambda^2}$, otherwise it is two-valued. When $C_+ > 0$, then this coordinate singularity is absent. A direct consequence of (5.18) is
that the Hawking flux at $I^+$ will not have the thermal form (3.7), which can be seen by evaluating (3.6) for $\xi^- = \sigma^-$. This gives

$$2\pi \langle T_- \rangle |_{I^+} = -\frac{1}{24} \left( \frac{\eta'''}{\eta'} - \frac{3}{2} \left( \frac{\eta''}{\eta'} \right)^2 \right),$$

where the primes stand for derivatives with respect to $\sigma^-$ and $x^- = -\lambda \Delta e^{-\lambda \eta}$. One then obtains

$$2\pi \langle T_- \rangle |_{I^+} = T(y) = \frac{1}{24} \frac{y^4[y^4(-y\Delta + \frac{1}{2}\Delta^2) - C_+ P_4(y) + C_+^2 P_2(y)]}{(y - \Delta)^2(C_+ + y^2)^4},$$

where $y = x^- + \Delta$,

$$P_2(y) = -2y^2 + 3\Delta y - \frac{3}{2}\Delta^2, \quad P_4(y) = y^2(-4y^2 + 10\Delta y - 5\Delta^2),$$

and we have set $\lambda = 1$. In Fig. 6 we give plots of $T(y)$ versus $y$ for $C_+ < 0$ and $C_+ > 0$, respectively. The dotted line represents the classical (zero loops) Hawking flux

$$T_0(y) = \frac{1}{24} \frac{\frac{1}{2}\Delta^2 - y\Delta}{(y - \Delta)^2}. $$

When $C_+ < 0$, $T(y)$ is close to $T_0(y)$ for early times, but as $y \to 0$, i.e. for late times, $T(y)$ goes to zero and then it diverges to $-\infty$ at $y = -\beta$. Note that in this case there is a naked singularity in the weak coupling region. It has been also found at the one-loop order that naked singularities are accompanied by pathological Hawking fluxes [14], and it has been argued there that such a catastrophic divergence of the flux is a generic feature whenever naked singularities occur. On the other hand, when $C_+ > 0$, the naked singularity is absent, and $T(y)$ is finite and continuous in the semi-classical region. Moreover, it smoothly goes to zero for late times, so that the total radiated mass is finite

$$M_{rad} = \int_{-\infty}^{0} dy d\sigma^- T(y) < \infty,$$

which are features not present at zero and one loop order. Actually, (5.23) is finite for the BPP solution, but this is done somewhat artificially, by taking (5.22) to be zero for $y > x^- + \Delta$. As a result of this discontinuous change in the flux, a shock-wave of negative energy emanates from $(x^+_i, x^-_i)$ (a thunder-pop), so that the total energy is conserved. The two-loop solution with $C_+ > 0$ has a desired property that the Hawking radiation turns off itself for late times, but then becomes negative, and it goes to zero at $y = 0$ (see Fig. 6). The appearance of this negative energy flux is a characteristic of a situation where all of the infalling matter gets out, since then
from the energy conservation it follows that \( M_{\text{rad}} = 0 \), which can be only achieved if \( T(y) \) becomes negative for late times. This behavior has been observed in the case of the one-loop BPP solution with reflecting boundary conditions, when \( M < M_{\text{cr}} \) so that a black hole is not formed, and hence all of the infalling matter gets out (it appears at \( x^+ = +\infty \), the right future null-infinity) [13]. In our case, the black hole forms and there are no reflecting boundary conditions, so that the infalling matter will reach the left future null-infinity \( (x^- = +\infty) \). This can be interpreted as matter getting out, provided there are no singularities at \( x^- = +\infty \). However, the line \( x^- = +\infty \) lays in the strong-coupling region, and without the knowledge of the full non-perturbative solution we can not say what exactly happens there. Still, all this indicates that the Hawking flux will deviate significantly from thermality for very late times. Also the behavior of \( T(y) \) is consistent with the creation of particle-antiparticle pairs, where particles reaching infinity give rise to positive \( T(y) \), while antiparticles carrying negative energy give rise to negative \( T(y) \) [13].

Note that (5.23) is only a function of \( \Delta \), \( C_+ = \beta^2 \) and \( \lambda \), and it does not depend on the incoming mass \( M \). Also \( M_{\text{rad}}(\beta, \Delta) \) diverges for small \( \beta \)'s, so that for a sufficiently small \( \beta \) one will have \( M_{\text{rad}} > M \), and these solutions will violate the energy conservation. This can only mean that the higher order corrections have not been taken into account. On the other hand, if one wants to have two-loop solutions which are consistent with the energy conservation, \( \beta \) should be larger (which means wider matter pulses, see (4.25)) so that \( M_{\text{rad}}(\beta, \Delta) < M \). Solutions with \( M_{\text{rad}} > 0 \) are consistent with the appearance of a remnant, while the \( M_{\text{rad}} = 0 \) solution is consistent with the complete evaporation of the black hole. When \( \beta = 0 \), then one has a BPP type solution, with a massless remnant. In that case

\[
M_{\text{rad}} = M - k \frac{\lambda \Delta}{4x^-} > M \quad , \quad (5.24)
\]

and the energy conservation is insured by emission of a thunder-pop, of negative energy \( k \frac{\lambda \Delta}{4x^-} \), which appears because the remnant metric and the black hole metric are not sewn up smoothly.

6. Conclusions

The two-loop corrections to the effective metric take a simple form (5.1), which is valid for arbitrary matter pulses when \( x^+ \) is outside of the pulse. The constants \( C_\pm \) are regularization dependent, and one can consider them as integrals of motion. The corresponding semi-classical geometries are essentially determined by the sign of \( C_+ \). When \( C_+ \geq 0 \), consistent semi-classical geometries appear, where consistency means
absence of naked singularities in the regions of the spacetime where the semi-classical approximation is valid and the total radiated mass is less than the infalling mass. The solutions with $C_+ = 0$ (shock-wave geometry) are special, in the sense that their geometry is essentially the same as the one-loop geometry, with the two-loop corrected static remnant appearing as the end-state. The Hawking flux is exactly the same as the one-loop flux, and a thunder-pop appears. However, when $C_+ > 0$, i.e., when the pulse has a reasonable width, the Hawking flux receives two-loop non-thermal corrections, and it becomes a continuous function of time. There is no remnant in the weak-coupling region, and the Hawking flux becomes negative for very late times (approaching zero) indicating particle-antiparticle pair creation. For sufficiently wide pulses one can have $M_{\text{rad}} < M$, so that the total energy is conserved, in agreement with the unitarity. Solutions with $M_{\text{rad}} > 0$ are consistent with a remnant appearing in the strong-coupling region. However, in order to check this, one would need the full nonperturbative solution, especially because $M_{\text{rad}}$ could then vanish, implying that all infalling matter gets out and the black hole completely evaporates. Note that appearance of remnants is somewhat unnatural in models where infalling matter does not couple to the gravitational field, and in order to insure the energy conservation one has to prevent the infalling matter to reach the future infinity. Apart from simply terminating the evolution by hand after some time $x_0^-$ (as is done in the BPP case), a more natural solution would be that a singularity in geometry appears at $x^- = +\infty$. Also note that when $C_+ > 0$ and $C_- > 0$ no curvature singularities appear at all, although singularities in $\phi$ remain ($\phi$ becomes complex in the strong-coupling region). Whether this can be interpreted in favor of the no-remnant scenario it is difficult to say, since at this moment we do not know what happens in the strong coupling region $e^{-\phi_0} \leq 0$. Still, it is very indicative that the two-loop corrections can cure the problems encountered in the lower loop approximations. This gives us a hope that the full non-perturbative solution will be well defined in the semi-classical regions of the spacetime (i.e. regions where the metric fluctuations are small), so that one can obtain a definite answer about the fate of the 2d black hole.

Note that the back-reaction effect on the Hawking radiation flux is such that the Hawking radiation must deviate significantly from thermality in the late stages of evaporation. This behavior is expected from general arguments, and it is very encouraging that it is recovered in a concrete model. An analysis of the Bogolibov coefficients confirms this [12], and it will be interesting to better understand the effect of the horizon fluctuations, which is described by the operator (2.22).

When $C_- < 0$, a naked singularity can appear in the dilaton vacuum sector if the one-loop constant $C < \alpha + \frac{5}{4} (\log \frac{k}{\Lambda} + 1)$. However, this singularity does not affect
the Hawking flux since it is independent of \( C_- \) and \( C \). Also the constant \( C \) does not depend on the incoming matter state, and hence it can be chosen freely, such that the naked singularity is absent. However, \( C_+ \) depends on the matter state (see (4.15)), and for narrow matter pulses \( C_+ \) is negative, resulting in the appearance of the naked singularity and the pathological behavior of the Hawking flux. The authors of [14] have conjectured that the back-reaction prevents naked singularities to form. The \( C_+ < 0 \) solution seem to be a counterexample to this conjecture. However, one has neglected the higher order corrections, so it is still possible that in the full non-perturbative solution the naked singularities do not appear in the semi-classical regions. Also it is very indicative from (4.25) that \( C_+ \) is positive for reasonable matter pulses, i.e. pulses which are wider than the short-distance cutoff \( l_c \), which can play the role of a 2d Planck length.

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Figure 1: Penrose diagram of the classical CGHS collapse geometry.
Figure 2: Penrose diagram of the one-loop semi-classical BPP geometry.
Figure 3: Kruskal diagram of the one-loop semi-classical BPP geometry.
Figure 4: Kruskal diagram of the two-loop semi-classical geometry when $C_+ < 0$. The straight dashed lines passing through D and G are $x^- = -\Delta$ and $x^- = -\Delta - \beta$ lines, respectively. The curve BD is the strong-coupling border $e^{-\phi_0} = 0$, and the apparent horizon curve FCG intersects the curvature singularity curve BCG in the weak-coupling semi-classical region $e^{-\phi_0} > 0$. 
Figure 5: Kruskal diagram of the two-loop semi-classical geometry when $C_+ > 0$. The apparent horizon curve FC intersects the curvature singularity curve BC inside the strong-coupling region, bordered by the curve BD.
Figure 6: Plots of the two-loop Hawking flux for $C_+ < 0$, $C_+ = 0$ and $C_+ > 0$. The vertical dashed line is $y = -\beta$ line.