CLOSED NODAL LINES AND INTERIOR HOT SPOTS OF THE SECOND EIGENFUNCTION OF THE LAPLACIAN ON SURFACES

PEDRO FREITAS

ABSTRACT. We build a one-parameter family of $S^1$–invariant metrics on the unit disc with fixed total area for which the second eigenvalue of the Laplace operator in the case of both Neumann and Dirichlet boundary conditions is simple and has an eigenfunction with a closed nodal line. In the case of Neumann boundary conditions, we also prove that this eigenfunction attains its maximum at an interior point, and thus provide a counterexample to the hot spots conjecture on a simply connected surface. This is a consequence of the stronger result that within this family of metrics any given (finite) number of $S^1$–invariant eigenvalues can be made to be arbitrarily small, while the non–invariant spectrum becomes arbitrarily large.

1. Introduction

A conjecture of J. Rauch from 1974 states that the eigenfunction corresponding to the second eigenvalue of the Laplace operator on a domain with Neumann boundary conditions attains its maximum and minimum on the boundary. As has been pointed out in [4], for instance, this is related to the location of the points of maxima of solutions of the heat equation (hot spots) and it is basically equivalent to saying that, for most initial conditions, these hot spots move towards the boundary as time goes to infinity.

Recently, Burdzy and Werner [5] gave a counterexample to this conjecture on a domain in $\mathbb{R}^2$ with two holes and posed the question of whether it would still be possible to find a counterexample on a doubly connected domain, or whether the conjecture would hold in that case – note that it is not known if the conjecture holds on simply connected domains.

On the positive side, Kawohl [13] has shown that for the case of domains of the form $D \times (a, b)$, where $D \subset \mathbb{R}^{n-1}$ has a $C^{0,1}$ boundary, the values attained by a second eigenfunction on the domain are less than or equal to the values it takes on the boundary. More recently, Bañuelos and Burdzy proved that the conjecture holds in the case of some special domains which include, among others, convex domains with a line of symmetry [4].

This problem is related (although not necessarily equivalent) to the nonexistence of closed nodal lines of the second eigenfunction, where the nodal set of an eigenfunction is defined to be the closure of the subset of the domain where the eigenfunction vanishes. With the exception of Courant’s nodal domain theorem which states that the nodal set of a $k^{th}$ eigenfunction divides the domain into at most $k$ subregions [6], very little is known about the general structure of such sets –

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Courant’s result implies that a second eigenfunction always divides the domain into exactly two subregions and a conjecture of Payne from 1967 for the case of Dirichlet boundary conditions states that any such eigenfunction cannot have a closed nodal line \[18\]. This was proved by Payne in 1973 under some symmetry and convexity assumptions on the domain \[19\]. Within the last ten years there have been several developments regarding the existence or not of closed nodal lines of the second eigenfunction for the Dirichlet Laplacian – see \[2, 8, 10, 11, 12, 14, 15, 16, 20\]. Although it has been shown that the conjecture holds for convex domains \[2, 16\], again it is not known whether or not there exist simply connected sets where the second eigenfunction will have a closed nodal line. So far the only known counterexample requires the boundary of the domain to have at least three components \[10\].

Note that, in the case of the Neumann problem, it is not too difficult to prove that on simply connected domains the second eigenfunction cannot have a closed nodal line \[3, 13, 18\].

The question of whether or not results similar to these hold in the case of manifolds with boundary has been raised by S.T. Yau – see, for instance, Problem 45 in the Chapter Open problems in differential geometry in \[21\]. Note that in this case it is not difficult to find a counterexample to the closed nodal line conjecture if one considers doubly connected surfaces. For this, it is sufficient to think of long cylinders, for which the nodal line is a circle located halfway between the top and the bottom. Incidentally, this also justifies the statement that the hot spots and the closed nodal line problems are not necessarily equivalent, as in this case the maximum and minimum are still attained at the boundary. As far as we know, the only results for the case of surfaces are given in the book by Bandle \[3\], where it is shown that in the Neumann case and under some conditions on the metric the second eigenvalue is not simple and the maximum and minimum are attained at the boundary. It is stated in that book that one cannot expect for this to be true in general, but no counterexample is given.

The main purpose of this note is to present examples of simply–connected surfaces with boundary for which both the closed nodal line and the hot spots conjectures fail. In other words, for these surfaces the eigenfunctions associated with the second eigenvalue of the Laplace operator in both the Neumann and the Dirichlet case have closed nodal lines which do not touch the boundary. Furthermore, in the Neumann case, this eigenfunction can be chosen in such a way that it attains its maximum at an interior point. This shows that simple connectivity by itself is not a sufficient condition for the above conjectures to hold.

These counterexamples are corollaries to the stronger result which asserts the existence of a family of \(S^1\)-invariant metrics with constant area on the unit disc for which the first \(m\) eigenvalues are simple, for any given positive integer \(m\). For \(S^1\)-invariant metrics, the spectrum can be divided into an invariant part corresponding to the spectrum of the Laplacian acting on \(S^1\)-invariant functions, and a non–invariant part. The counterexamples given here are then made possible by the fact that we are able to choose a family of metrics for which the invariant and the non–invariant parts of the spectrum may be separated. More precisely, for any positive integer \(m\) we have that the first \(m\) invariant eigenvalues can be made to be arbitrarily small, while the first non–invariant eigenvalue becomes arbitrarily large.
2. Main result

The main result from which the counterexamples are then a straightforward consequence is the following

**Theorem 1.** There exists a one–parameter family $\mathcal{M}$ of smooth $S^1$–invariant metrics on the unit disc with positive curvature and total area $\pi$, such that given any positive integer $m$ and real number $\varepsilon$ there exists a subset $\mathcal{M}_\varepsilon$ of $\mathcal{M}$ with $\varepsilon$ on an open interval for which the first $m$ $S^1$–invariant eigenvalues of the Laplace operator both with Neumann and Dirichlet boundary conditions are smaller than $\varepsilon$. On the other hand, the non–invariant spectrum remains uniformly bounded away from zero in $\mathcal{M}$ and becomes arbitrarily large in $\mathcal{M}_\varepsilon$ as $\varepsilon$ goes to zero.

Since the Gaussian curvature is positive, these surfaces can actually be isometrically embedded in $\mathbb{R}^3$ [17]. As we shall see in Section 4, the curvature of such metrics can be made to be arbitrarily close to zero except at the centre of the disc.

From the proof of the theorem, it follows that the first $m$ invariant eigenvalues must be simple, and, in the case of the second Neumann eigenvalue, that the corresponding (invariant) eigenfunction is strictly monotone along radial lines and changes sign. We thus obtain the following

**Corollary 2.** Given any positive integer $m$ there exists a family $\mathcal{M}$ of $S^1$–invariant metrics on the unit disc with positive curvature and total area $\pi$, for which the first $m$ eigenvalues of the Laplace operator with both Neumann and Dirichlet boundary conditions are simple. In both cases, the eigenfunctions corresponding to the $j^{\text{th}}$ eigenvalue ($j = 2, \ldots, m$) are also invariant by $S^1$ and have $j - 1$ nodal lines which are closed disjoint circumferences dividing the disc into $j$ nodal domains. In the Neumann case the second eigenfunction may be chosen in such a way that its (strict) maximum is attained at the origin.

The estimates obtained in the proof allow us to write a more quantitative version of this result which we mention in Section 4.

Counterexamples to this type of conjectures have usually been obtained by exhibiting a domain for which the second eigenvalue can be proven to be simple [3, 10, 15]. If the domain has some symmetry, it will then be inherited by the eigenfunction and hence also by the nodal set. In order to prove the above results, we shall use a variation of this technique and consider one–parameter families of metrics on the disc for which we are able to prove that the $m^{\text{th}}$ invariant eigenvalue is smaller than the first non–invariant eigenvalue.

A fundamental (standard) ingredient in the proof is the reduction of the original two–dimensional problem to a sequence of one–dimensional problems, which we achieve by means of standard polar coordinates. Since we are interested in estimates for higher eigenvalues, we then perform a change of variables in order to avoid function weights in the orthogonality conditions which appear in the corresponding variational formulations – see Section 3. At this point we should remark that it is also possible to proceed by means of a different technique using symplectic coordinates. An example of this can be found in [1], where the behaviour of the invariant spectrum for $S^1$–invariant metrics on $S^2$ was studied. We shall briefly indicate in Section 3 how these two different coordinate systems are related in this case.

Regarding the choice of metrics, we point out that with the proper parametrization in isothermic coordinates (see Section 3), the eigenvalue problem on the
surface becomes equivalent to that on a flat disc of inhomogeneous density, and so
our results also apply in that case. On the other hand, this suggests that the we
build the family $\mathcal{M}$ by making the density much higher close to the centre than
near the boundary, so that there is a strong resistance to the movement of the hot
spots towards the outside regions of the disc. We remark that in the opposite case
where the (radially symmetric) density increases as we move away from the centre,
it is known that the hot spots conjecture holds [3].

3. Abstract surfaces

In this section we collect the main facts about abstract surfaces that will be
needed in the paper. We shall follow the exposition in the book by Bande [3] very
closely, and begin by introducing the concept of an abstract surface $S$. We then
derive the expressions that will be used in the sequel, which include the Laplace–
Beltrami operator in $S$ given in conformal coordinates and the variational formu-
lation for the problem. We shall also indicate the expression for the Gaussian
curvature of $S$.

Let $D$ be a domain in the $(x,y)$–parameter plane, and $d\sigma^2$ the Riemannian
metric in $D$ defined by the quadratic form

$$d\sigma^2 = E(x,y)dx^2 + 2F(x,y)dxdy + G(x,y)dy^2.$$

**Definition 3.1.** A domain $D \subset \mathbb{R}^2$ with the Riemannian metric $d\sigma$ is called an
abstract surface and will be denoted by $S = (D, d\sigma)$. $S$ is said to be in its isothermic
(or conformal) representation if $d\sigma^2 = p(x,y)ds^2$ ($E = G = p$ and $F = 0$), where
$ds$ denotes the linear element of the Euclidean plane.

The eigenvalue problems that we are interested in are thus

$$\Delta_S u + \gamma u = 0 \quad \text{in } D,$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ or } u = 0 \quad \text{on } \partial D,$$

where $\Delta_S$ denotes the Laplace–Beltrami operator on $S$ and $\nu$ the conormal deriv-
ative. We shall denote the eigenvalues of the Neumann and Dirichlet problems by
$\mu_j$ and $\lambda_j$, $j = 1, \ldots$, respectively, and always assume that they are written in
increasing order.

The Laplace–Beltrami operator is now given by

$$\Delta_S = \frac{1}{W} \left[ \frac{\partial}{\partial x} \left( \frac{G \frac{\partial}{\partial x} - F \frac{\partial}{\partial y}}{W} \right) + \frac{\partial}{\partial y} \left( \frac{E \frac{\partial}{\partial y} - F \frac{\partial}{\partial x}}{W} \right) \right],$$

where $W = \sqrt{EG - F^2}$. In the case where $S$ is in its isothermic representation,
this expression simplifies to

$$\Delta_S = \frac{1}{p(x,y)} \Delta,$$
where $\Delta$ now denotes the usual Laplacian operator in $\mathbb{R}^2$. This means that the eigenvalue problem \eqref{3.1} on $S$ becomes

$$\frac{1}{p(x, y)} \Delta u + \gamma u = 0 \quad \text{in } D,$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ or } u = 0 \quad \text{on } \partial D,$$

As mentioned in the Introduction, this is equivalent to the problem of an inhomogeneous membrane whose density is given by the function $p$.

Finally, we note that in this case, the area of $S$ and the Gaussian curvature are given by

$$A_p(S) = \int_D p(x, y) \, dx \, dy \quad \text{and} \quad K = -\frac{1}{2p} \Delta [\log(p)],$$

respectively. The expression for the total curvature is then

$$\omega(S) = -\frac{1}{2} \int_D \Delta [\log(p(x, y))] \, dx \, dy.$$

From this point on we shall concentrate on the Dirichlet problem, and mention the necessary changes for the Neumann case in Section 4.

We consider the eigenvalue problem \eqref{3.2} in the case where the function $p$ is radially symmetric, that is, $p = p(r)$. In polar coordinates and after separation of variables, the eigenvalue problem \eqref{3.2} then reduces to the sequence of one-dimensional eigenvalue problems

$$\left(r \varphi' + \frac{k^2}{r} \varphi \right)' - \lambda r p(r) \varphi = 0, \quad r \in (0, 1),$$

where $k^2$, $k = 0, 1, \ldots$, are the eigenvalues of the Laplacian on the circle. For $k = 0$ we have Neumann boundary conditions at 0 and Dirichlet at 1, that is $\varphi'(0) = \varphi(1) = 0$. This corresponds to the invariant spectrum of the original problem, and the associated eigenfunctions are the radially symmetric functions $\varphi$.

Note that the $i$th eigenfunction of the one-dimensional problem has $i - 1$ zeros on $(0, 1)$, and thus the corresponding eigenfunction of the original problem divides the disc into $i$ nodal domains.

For positive values of $k$ we have Dirichlet boundary conditions at both ends of the interval, and this now gives the non-invariant part of the spectrum which is made up of eigenvalues with multiplicity two. In this case, two linearly independent eigenfunctions are given by $\varphi(r) \cos(k\theta)$ and $\varphi(r) \sin(k\theta)$.

To obtain the variational formulation corresponding to the eigenvalue problems above we consider the space $C^1(0, 1)$ with the inner product defined by

$$\langle \varphi, \psi \rangle = \int_0^1 rp(r) \varphi(r) \psi(r) + r \varphi'(r) \psi'(r) \, dr.$$ 

Let now $H^1_0$ be the Sobolev space which is obtained as the closure of $C_0^\infty(0, 1)$ with respect to the norm induced by the above inner product. Then the eigenvalues of the spectral problem \eqref{3.3} when $k = 0$ are given by

$$\lambda^0_j = \lambda^0_j(p) = \inf_{\varphi \in H^1_0} \frac{\int_0^1 r |\varphi'(r)|^2 \, dr}{\int_0^1 rp(r) \varphi^2(r) \, dr}, \quad j = 1, \ldots.$$
where $H^1_{0,1}$ is the closure with respect to the above norm of the space of $C^\infty$ functions on $[0,1)$ with compact support on this interval,

$$H^1_{0,j} = H^1_{0,j-1} \cap \left\{ \varphi : \int_0^1 rp(r)\varphi_{j-1}(r)\varphi(r)dr = 0 \right\}, \quad j = 2, \ldots,$$

and $\varphi_{j}$ is the eigenfunction corresponding to $\lambda^0_j$.

In the case of the eigenvalues for each of the remaining problems we have

$$\lambda^k_j = \lambda^k_j(p) = \inf_{\varphi \in H^1_{1,j}} \frac{\int_0^1 r[\varphi'(r)]^2dr + k^2 \int_0^1 \frac{\varphi^2(r)}{r}dr}{\int_0^1 rp(r)\varphi^2(r)dr}, \quad j = 1, \ldots, k = 1, \ldots,$$

and where now the spaces $H^1_{1,j}$ are defined in a similar way as above but starting with $H^1_{0,1}$.

As we are interested in higher eigenvalues, we have to take into account the orthogonality conditions which appear in the definition of the spaces $H^1_{1,j}$. Since these conditions become much simpler to handle if the inner product considered has a constant weight function instead of $rp(r,\delta)$, in order to show that $\lambda^m_p(p)$ is arbitrarily small within a family of metrics it is convenient to make a change of variables in the variational formulation of the problem. Thus, we want to find a function $r = r(z)$ and a constant $c$ such that

$$r(z)p(r(z))r'(z) = c$$

and the interval $(0,1)$ is mapped onto $(0,1)$. This will be the case if we take, for instance,

$$z(r) = \frac{1}{c} \int_0^r sp(s)ds \quad \text{and} \quad c = \int_0^1 sp(s)ds = \frac{A_p(S)}{2\pi}.$$

The Raleigh quotients in (3.4) and (3.5) then become

$$\frac{4\pi^2}{A^2_p(S)} \int_0^1 r^2(z)p(r(z))[\psi'(z)]^2dz + k^2 \int_0^1 \frac{[r^2(z)p(r(z))]^{-1}[\psi(z)]^2dz}{\int_0^1 \psi^2(z)dz},$$

for $k = 0, \ldots$. This transformation fixes both endpoints of the interval, and the inner product is changed accordingly to

$$\langle \phi, \psi \rangle = A_p(S) \int_0^1 \phi\psi dz + \frac{4\pi^2}{A^2_p(S)} \int_0^1 r^2(z)p(r(z))\phi'\psi' dz.$$
and isothermic coordinates for this particular case. To do this, we need the relation between \( r \) and \( z \) to be such that

\[
(3.8) \quad r^2 p(r) = \frac{1}{g(z)} \quad \text{and} \quad r'(z) = rg(z).
\]

Here \( g : (-1, 1) \to (0, +\infty) \) is a function of the form

\[
g(z) = \frac{1}{1 - z^2} + h(z),
\]

where \( h \) is a \( C^\infty[-1, 1] \) function such that \( g > 0 \), and it has been assumed that the area of the full surface is \( 4\pi \). We then obtain that the change of variables defined by

\[
r(z) = \sqrt{\frac{\varepsilon}{2 - \varepsilon}} \cdot \int_{\varepsilon}^{1} h(t) dt \sqrt{\frac{1 + z}{1 - z}} \quad \varepsilon > 0,
\]

transforms (3.4) into

\[
\int_{-1}^{1-\varepsilon} \frac{1}{g(z)} [\psi'(z)]^2 dz
\]

\[
\int_{-1}^{1-\varepsilon} \psi^2(z) dz,
\]

The function \( p \) is then given from (3.8) by

\[
p(r) = \frac{1}{r^2 g(z(r))},
\]

although it should be pointed out that in general it will not be possible to obtain an explicit expression for it in terms of \( r \). Note that because polar coordinates have a singularity at the origin, while symplectic coordinates have two singularities at both the North and South poles, we have to restrict ourselves to the interval \((-1, 1 - \varepsilon)\). In this case, the change of variables given by (3.6) has the advantage that it keeps the interval fixed.

As an example, consider the case of the standard sphere for which we have that \( h \) is the zero function and then

\[
p(r) = \frac{4\varepsilon(2 - \varepsilon)}{[\varepsilon + (2 - \varepsilon)r^2]^2},
\]
as expected.

4. Proof of Theorem 1

The first \( m \) eigenvalues of the original problem will be simple if and only if we have that \( \lambda_m^0 \) is smaller than \( \lambda_1^k \) for all positive \( k \). This follows from the fact that the invariant spectrum is the spectrum of a second order ordinary differential operator of the type in (3.3) with \( k = 0 \) and thus all its eigenvalues are simple. Clearly it is enough to ensure the above condition for \( k = 1 \), since \( \lambda_1^k < \lambda_1^{k+1} \) for all \( k \). To do this, we consider the one-parameter family of smooth metrics corresponding to

\[
p(r, \delta) = \frac{\alpha}{r^2 + \delta}, \quad \alpha = \frac{1}{\log \left( \frac{1 + \delta}{\delta} \right)},
\]
for positive $\delta$. The parameter $\alpha$ is included as a normalizing factor so that all elements in this family of metrics have fixed area equal to $\pi$.

Using the expression given in Section 3 we easily obtain

$$K(r, \delta) = \frac{2\delta}{r^2 + \delta} \log \left( \frac{1 + \delta}{\delta} \right),$$

and thus the curvature is positive and converges to 0 on $(0, 1]$ as $\delta$ goes to 0. On the other hand, the total curvature is given by

$$\omega(\delta) = \frac{2\pi}{1 + \delta}.$$

Using the change of variables given by (3.6) we obtain

$$r = \sqrt{\delta (e^{z/\alpha} - 1)},$$

and the variational problem (3.7) becomes

$$(4.1) \quad \lambda_j^k = \lambda_j^k(p) = \inf_{\psi \in H_{0,j}} \left( 4\alpha \int_0^1 \left( 1 - e^{-z/\alpha} \right) [\psi'(z)]^2 dz \right) +$$

$$+ k^2 \alpha \int_0^1 \frac{1}{\left( 1 - e^{-z/\alpha} \right)} \psi^2(z) dz \int_0^1 \psi^2(z) dz,$$

$$j = 1, \ldots, k = 0, \ldots.$$

We begin by noting that since $\alpha$ is positive the second term is always greater than or equal to $k^2/\alpha$ so that we have

$$\lambda_1^1(p) \geq \frac{1}{\alpha},$$

which becomes unbounded as $\delta$ goes to zero.

We shall now show that any finite number of invariant eigenvalues can be made to be arbitrarily small. We have that

$$\int_0^1 \frac{1 - e^{-z/\alpha}}{\psi^2(z) dz} \leq \int_0^1 [\psi'(z)]^2 dz \int_0^1 \psi^2(z) dz,$$

and so by the monotonicity principle derived from Poicaré’s principle it follows that

$$\lambda_j^0(p) \leq \alpha (2j - 1)^2 \pi^2, \quad j = 1, \ldots,$$

and thus, given a fixed eigenvalue $\lambda_j^0(p)$,

$$\lim_{\delta \to 0^+} \lambda_j^0(p) = 0.$$

The estimates obtained above allow us to conclude that if

$$\delta < \frac{1}{e^{(2j-1)\pi} - 1}, \quad j = 1, \ldots,$$

then the $j^{th}$ invariant eigenvalue is smaller than the first non–invariant eigenvalue. This is a very rough estimate, since the logarithmic term is taking into account
the asymptotic behaviour of the eigenvalues, while the crossing between the two eigenvalues as the parameter $\delta$ decreases must occur at a point larger than 1. Similar results can be obtained for higher eigenvalues.

Due to the fact that the Dirichlet and Neumann eigenvalues satisfy $\mu_j \leq \lambda_j$ for all integer $j$, the proof for the Neumann problem follows from the Dirichlet case. It is also possible to obtain some independent estimates by following essentially along the same lines as in the Dirichlet case, with only some minor changes. These have to do with the spaces considered, which should now be

$$H^1_j = H^1 \quad \text{and} \quad H^1_j = H^1_{j-1} \cap \left\{ \varphi : \int_0^1 \varphi_{j-1}(z) \varphi(z) \, dz = 0 \right\}$$

in the case of invariant eigenvalues, and $H^1_0$ in (3.5) is now the closure of the space of $C^\infty$ functions with compact support on $[0,1]$, with respect to the same norm as before. We may then proceed in a similar fashion to obtain the following estimate for the $j$th invariant Neumann eigenvalue:

$$\mu^0_j(p) \leq 4\alpha(j - 1)^2\pi^2.$$ 

From (3.3) with $k = 0$ we see that any solution $\varphi$ of that equation with precisely one zero on the interval $(0,1)$ and with $\varphi(0)$ positive will be strictly decreasing on this interval. Hence its maximum is attained at the origin.

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