THE CP-MATRIX COMPLETION PROBLEM

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Abstract. A symmetric matrix $C$ is completely positive (CP) if there exists an entrywise nonnegative matrix $B$ such that $C = BB^T$. The CP-completion problem is to study whether we can assign values to the missing entries of a partial matrix (i.e., a matrix having unknown entries) such that the completed matrix is completely positive. We propose a semidefinite algorithm for solving general CP-completion problems, and study its properties. When all the diagonal entries are given, the algorithm can give a certificate if a partial matrix is not CP-completable, and it almost always gives a CP-completion if it is CP-completable. When diagonal entries are partially given, similar properties hold. Computational experiments are also presented to show how CP-completion problems can be solved.

1. Introduction

A matrix is partial if some of its entries are missing. The matrix completion problem is to study whether we can assign values to the missing entries of a partial matrix such that the completed matrix satisfies certain properties, e.g., it is positive semidefinite or an Euclidean distance matrix. This problem has wide applications, as shown in [3, 14, 28, 32, 36]. We refer to Laurent’s survey [31] and the references therein. Interesting applications include the Netflix problem [29, 37], global positioning [13], multi-task learning [1, 2, 4], etc. The motivation of this paper is to study whether or not a partial matrix can be completed to a matrix that is completely positive.

A real $n \times n$ symmetric matrix $C$ is completely positive (CP) if there exist nonnegative vectors $u_1, \cdots, u_m$ in $\mathbb{R}^n$ such that

$$C = u_1 u_1^T + \cdots + u_m u_m^T,$$

where $m$ is called the length of the decomposition (1.1). The smallest $m$ in the above is called the CP-rank of $C$. If $C$ is complete positive, we call (1.1) a CP-decomposition of $C$. Clearly, a matrix $C$ is completely positive if and only if $C = BB^T$ for an entrywise nonnegative matrix $B$. Hence, a CP-matrix is not only positive semidefinite but also nonnegative entrywise.

CP-matrices have wide applications in mixed binary quadratic programming [11], approximating stability numbers [16], max clique problems [43, 47], single quadratic constraint problems [44], standard quadratic optimization problems [7], and general quadratic programming [45]. Some NP-hard problems can be formulated as linear optimization problems over the cone of CP-matrices (cf. [22, 27, 34]). We refer to [12, 13, 29, 31, 32, 36] for more details.
to [5][6][8][10][12][17][19] for the work in this field. These important applications motivate us to study the so-called CP-completion problem. Let
\[ E = \{(i_k, j_k) \mid 1 \leq i_k \leq j_k \leq n, k = 1, \cdots, l\}. \]
be an index set of pairs. A partial symmetric matrix \( A \) is called an \( E \)-matrix if its entries \( A_{ij} \) are given for all \((i, j) \in E\), while \( A_{ij} \) are missing for \((i, j) \notin E\). The CP-completion problem is to study whether we can assign values to the missing entries of an \( E \)-matrix such that the completed matrix is completely positive. If such an assignment exists, we say that the \( E \)-matrix is CP-completable; otherwise, we say that it is not CP-completable (or non-CP-completable).

A symmetric matrix can be identified by a vector that consists of its upper triangular entries. Similarly, an \( E \)-matrix \( A \) can be identified as a vector \( a \in \mathbb{R}^E \), such that \( a_{ij} = A_{ij} \) for all \((i, j) \in E\). (The symbol \( \mathbb{R}^E \) stands for the space of all real vectors indexed by pairs in \( E \).) For a matrix \( F \), we denote by \( F|_E \) the vector in \( \mathbb{R}^E \) whose \((i, j)\)-entry is \( F_{ij} \), for all \((i, j) \in E\). Clearly, an \( E \)-matrix \( A \) is CP-completable if and only if there exists a CP-matrix \( C \) such that \( a = C|_E \).

For example, consider the \( E \)-matrix \( A \) given as (1.2):
\[
\begin{bmatrix}
2 & 3 & 0 & * \\
3 & 6 & 3 & 0 \\
* & 0 & 3 & 2 \\
\end{bmatrix},
\]
where * means that the entry there is missing, throughout the paper. The index set \( E \) is
\[ \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}, \]
and the identifying vector \( a \) of \( A \) is
\[ (2, 3, 0, 6, 0, 6, 3, 2). \]
If we assign the missing entry \( A_{14} \) a value, say, \( t \), the determinant of \( A \) is \(-27(t+1)\). So, \( A \) can not be positive semidefinite for any \( t > -1 \). This implies that the \( E \)-matrix \( A \) is not CP-completable.

For another example, consider the \( E \)-matrix \( A \) given as:
\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & * \\
\end{bmatrix}.
\]
The index set \( E \) is \{\((1, 1), (1, 2), (1, 3), (2, 2), (2, 3)\}\} and the identifying vector \( a \) of \( A \) is \((1, 1, 1, 1, 1)\). Since
\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T
\]
is completely positive, we know that this \( E \)-matrix is CP-completable.

Note that if a diagonal entry of a CP-matrix is zero, then all the entries in its row or column are zeros. Without loss of generality, we can assume that all the given entries of an \( E \)-matrix are nonnegative and all its given diagonal entries are strictly positive. Otherwise, it can be reduced to a smaller \( E' \)-matrix with some index set \( E' \).
An \(E\)-matrix \(A\) is called a partial CP-matrix if every principal submatrix of \(A\), whose entry indices are all from \(E\), is completely positive. When all the diagonal entries are given, the specification graph of an \(E\)-matrix is defined as the graph whose vertex set is \(\{1, 2, \ldots, n\}\) and whose edge set is \(\{(i, j) \in E : i \neq j\}\). A symmetric partial matrix is called a matrix realization of a graph \(G\) if its specification graph is \(G\). It is shown in [3,20] that a partial CP-matrix realization of a connected graph \(G\) is CP-completable if and only if \(G\) is a block-clique graph. Under some conditions, a partial CP-matrix, whose specification graph \(G\) contains cycles, is CP-completable if and only if all the blocks of \(G\) are cliques or cycles [21]. These results assume that all the diagonal entries are given and the specification graphs satisfy certain combinatorial properties.

In this paper, we study general CP-completion problems in a unified framework. If an \(E\)-matrix is not CP-completable, how can we get a certificate for this? If it is CP-completable, how can we get a CP-completion and a CP-decomposition for the completed matrix? To the best knowledge of the authors, there exists few work on solving general CP-completion problems.

The paper is organized as follows. In Section 2, we give a semidefinite algorithm for solving general CP-completion problems, after an introduction of truncated moment problems. Its basic properties are also studied. In Section 3, we study properties of CP-completion problems when some diagonal entries are missing. Computational results are given in Section 4. Finally, we conclude the paper in Section 5 with some applications and discussions about future work.

2. A semidefinite algorithm for CP-completion

Recently, Nie [38] proposed a semidefinite algorithm for solving \(A\)-truncated \(K\)-moment problems (\(A\)-TKMPs), which are generalizations of classical truncated \(K\)-moment problems (cf. [24]). In this section, we show how to formulate CP-completion problems in the framework of \(A\)-TKMP, and then propose a semidefinite algorithm to solve them.

2.1. CP-completion as \(E\)-TKMP. First, we characterize when an \(E\)-matrix is CP-completable. Let

\[
\Delta = \{x \in \mathbb{R}^n : x_1 + \cdots + x_n - 1 = 0, x_1 \geq 0, \ldots, x_n \geq 0\}
\]

be the standard simplex in \(\mathbb{R}^n\). For convenience, denote the polynomials:

\[
h(x) := x_1 + \cdots + x_n - 1, \quad g_1(x) := x_1, \ldots, g_n(x) := x_n.
\]

Let \(A\) be an \(E\)-matrix with the identifying vector \(a \in \mathbb{R}^E\). We have seen that \(A\) is CP-completable if and only if \(a = C|_E\) for some CP-matrix \(C\). Note that every nonnegative vector is a multiple of a \(v\) in the simplex \(\Delta\). So, in view of (1.1), \(A\) is CP-completable if and only if there exist vectors \(v_1, \ldots, v_m \in \Delta\) and \(\rho_1, \ldots, \rho_m > 0\) such that

\[
C = \rho_1 v_1 v_1^T + \cdots + \rho_m v_m v_m^T \quad \text{and} \quad a = C|_E.
\]

Let \(N\) be the set of nonnegative integers. For \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\), denote \(|\alpha| := \alpha_1 + \cdots + \alpha_n\). Let \(\mathbb{N}_0^n := \{\alpha \in \mathbb{N}^n : |\alpha| \leq n\}\). Denote

\[
\mathcal{E} := \{\alpha \in \mathbb{N}_0^n : |\alpha| \leq n\},
\]

where \(e_i\) is the \(i\)-th unit vector. For instance, when \(n = 3\) and \(E = \{(1, 2), (2, 2), (2, 3)\}\), then \(\mathcal{E} =\{(1, 1, 0), (0, 2, 0), (0, 1, 1)\}\). The degree \(\deg(\mathcal{E}) := \max\{|\alpha| : \alpha \in \mathcal{E}\}\) is
two for all \( E \). Since there is a one-to-one correspondence between \( E \) and \( E \), we can also index the identifying vectors \( a \in \mathbb{R}^E \) of \( E \)-matrices as

\[
a = (a_\alpha)_{\alpha \in \mathcal{E}} \in \mathbb{R}^E, \quad a_{\alpha} = a_{ij} \quad \text{if} \quad \alpha = e_i + e_j.
\]

(\( \mathbb{R}^E \) denotes the space of real vectors indexed by elements in \( \mathcal{E} \).) We call such \( a \) an \( E \)-truncated moment sequence. A measure \( \mu \) for \( a \) is called \( \Delta \) for \( a \). A measure is called \( \Delta \)-representing for \( a \). A measure is called finitely atomic if its support consists of at most \( m \) distinct points. We refer to \( \Delta \) for representing measures of truncated moment sequences.

Hence, by \((2.2)\), an \( E \)-matrix \( A \), with the identifying vector \( a \in \mathbb{R}^E \), is CP-completable if and only if \( a \) admits an \( m \)-atomic \( \Delta \)-measure, i.e.,

\[
a = \rho_1[v_1]_E + \cdots + \rho_m[v_m]_E,
\]

with each \( v_i \in \Delta \) and \( \rho_i > 0 \). In the above, we denote

\[
[v]_E := (v^\alpha)_{\alpha \in \mathcal{E}}.
\]

In other words, CP-completion problems are equivalent to \( \mathcal{E} \)-T\( \Delta \)MPs with \( \mathcal{E} \) and \( \Delta \) given by \((2.3)\) and \((2.4)\) respectively.

2.2. A semidefinite algorithm. We present a semidefinite algorithm for solving CP-completion problems, by formulating them as \( \mathcal{E} \)-T\( \Delta \)MPs. To describe it, we need to introduce localizing matrices. Denote

\[
\mathbb{R}[x]_E := \text{span}\{x^\alpha : \alpha \in \mathcal{E}\}.
\]

We say that \( \mathbb{R}[x]_E \) is \( \Delta \)-full if there exists a polynomial \( p \in \mathbb{R}[x]_E \) such that \( p|_\Delta > 0 \) (cf. \((12)\)). Let \( \mathbb{R}[x]_d := \text{span}\{x^\alpha : \alpha \in \mathbb{N}^n_0\} \). An \( \mathcal{E} \)-tms \( y \in \mathbb{R}^E \) defines an \( \mathcal{E} \)-Riesz function \( \mathcal{L}_y \) acting on \( \mathbb{R}[x]_E \) as

\[
\mathcal{L}_y(\sum_{\alpha \in \mathcal{E}} p_\alpha x^\alpha) := \sum_{\alpha \in \mathcal{E}} p_\alpha y_\alpha.
\]

For \( z \in \mathbb{R}^{N_{2k}} \) and \( q \in \mathbb{R}[x]_{2k} \), the \( k \)-th localizing matrix of \( q \) generated by \( z \) is the symmetric matrix \( L_q^{(k)}(z) \) satisfying

\[
\mathcal{L}_z(qp^2) = \text{vec}(p)^T(L_q^{(k)}(z))\text{vec}(p) \quad \forall p \in \mathbb{R}[x]_{[\deg(q)/2]}.
\]

In the above, \( \text{vec}(p) \) denotes the coefficient vector of \( p \) in the graded lexicographical ordering, and \([t]\) denotes the smallest integer that is not smaller than \( t \). In particular, when \( q = 1 \), \( L_q^{(k)}(z) \) is called a \( k \)-th order moment matrix and denoted as \( M_k(z) \). We refer to \( \Delta \) for more details about localizing and moment matrices.

Let \( g_0(x) := 1 \) and \( g_{n+1}(x) := 1 - \|x\|_2^2 \). Since \( \Delta \subseteq B(0,1) := \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\} \), we can also describe \( \Delta \) equivalently as

\[
\Delta = \{x \in \mathbb{R}^n : h(x) = 0, g(x) \geq 0\}.
\]
Consider the linear optimization problem is equivalent to investigating whether a \((\text{cf.}[38])\). By (2.5), determining whether an 

\[
\begin{aligned}
\mathbf{L}^{(2)}_{x_1^4 + x_2^1}(z) &= \begin{bmatrix}
z(1,0) + z(2,0) - z(0,0) & z(2,0) + z(1,1) - z(0,1) & z(1,1) + z(0,2) - z(0,0) \\
z(2,0) + z(1,1) - z(1,0) & z(3,0) + z(2,1) - z(2,0) & z(2,1) + z(1,2) - z(1,1) \\
z(1,1) + z(0,2) - z(0,1) & z(2,1) + z(1,2) - z(1,1) & z(1,2) + z(0,3) - z(0,2)
\end{bmatrix}, \\
M_2(z) &= \mathbf{L}^{(2)}_{x_1^1}(z) = \begin{bmatrix}
z(0,0) & z(0,1) & z(1,0) & z(0,2) \\
z(0,1) & z(1,1) & z(2,0) & z(1,2) \\
z(0,2) & z(1,2) & z(2,1) & z(1,3) \\
z(0,3) & z(1,3) & z(2,2) & z(1,4)
\end{bmatrix}, \\
L^{(2)}_{x_1^4 + x_2^1}(z) &= \begin{bmatrix}
z(1,0) & z(2,0) & z(0,0) & z(1,1) \\
z(2,0) & z(3,0) & z(2,1) & z(2,1) \\
z(1,1) & z(2,1) & z(0,3) & z(1,2) \\
z(1,0) & z(2,0) & z(1,2) & z(0,4)
\end{bmatrix}
\end{aligned}
\]

As shown in [38], a necessary condition for \(z \in \mathbb{R}^{n_d^e}\) to admit a \(\Delta\)-measure is

\[
L_h^{(k)}(z) = 0, \quad \text{and} \quad L_{g_j}^{(k)}(z) \geq 0, \quad j = 0, 1, \ldots, n + 1.
\]

(In the above, \(X \succeq 0\) means that \(X\) is positive semidefinite.) If, in addition to

\[
\text{(2.8)} \quad \text{rank } M_{k-1}(z) = \text{rank } M_k(z),
\]

then \(z\) admits a unique \(\Delta\)-measure, which is \(\text{rank } M_k(z)\)-atomic (cf. Curto and Fialkow [15]). We say that \(z\) is flat if \((2.8)\) and \((2.9)\) are both satisfied.

Given two tms' \(y \in \mathbb{R}^{n_d^e}\) and \(z \in \mathbb{R}^{n_d^e}\), we say \(z\) is an extension of \(y\), if \(d \leq e\) and \(y_\alpha = z_\alpha\) for all \(\alpha \in \mathbb{N}_d^e\). We denote by \(z|_E\) the subvector of \(z\), whose entries are indexed by \(\alpha \in E\). For convenience, we denote by \(z|_d\) the subvector \(z|_{n_d^E}\). If \(z\) is flat and extends \(y\), we say \(z\) is a flat extension of \(y\). Note that an \(\mathcal{E}\)-tms \(a \in \mathbb{E}\) admits a \(\Delta\)-measure if and only if it is extendable to a flat tms \(z \in \mathbb{R}^{n_d^e}\) for some \(k\) (cf. [38]). By (2.9), determining whether an \(\mathcal{E}\)-tms \(A\) is \(\text{CP-completable}\) or not is equivalent to investigating whether \(a\) has a flat extension or not.

Let \(d > 2\) be an even integer. Choose a polynomial \(R \in \mathbb{R}[x]_d\) and write it as

\[
R(x) = \sum_{\alpha \in \mathbb{N}_d^e} R_\alpha x^\alpha.
\]

Consider the linear optimization problem

\[
\min_{z} \sum_{\alpha \in \mathbb{N}_d^e} R_\alpha z_\alpha 
\text{ s.t. } z|_E = a, z \in \Upsilon_d(\Delta),
\]

where \(\Upsilon_d(\Delta)\) is the set of all tms' \(z \in \mathbb{R}^{n_d^e}\) admitting \(\Delta\)-measures. Note that \(\Delta\) is a compact set. It is shown in [38] that if \(\mathbb{R}[x]_E\) is \(\Delta\)-full, then the feasible set of (2.10) is compact convex and (2.10) has a minimizer for all \(R\). If \(\mathbb{R}[x]_E\) is not \(\Delta\)-full, we need to choose \(R\) which is positive definite on \(\Delta\), to guarantee that (2.10) has a minimizer. Therefore, to get a minimizer of (2.10), it is enough to solve (2.10) for a generic positive definite \(R\), no matter whether \(\mathbb{R}[x]_E\) is \(\Delta\)-full or not. For this reason, we choose \(R \in \Sigma_{n,d}\), where \(\Sigma_{n,d}\) is the set of all sum of squares polynomials.
in $n$ variables with degree $d$. Since $\Upsilon_d(\Delta)$ is typically quite difficult to describe, we relax it by the cone

$$\Gamma_k(h, g) := \left\{ z \in \mathbb{R}^{N_2} \mid L_h^{(k)}(z) = 0, L_g^{(k)}(z) \succeq 0, j = 0, 1, \ldots, n + 1 \right\},$$

with $k \geq d/2$ an integer. The $k$-th order semidefinite relaxation of $(2.10)$ is

$$\text{(SDR)}_k : \left\{ \min_z \sum_{\alpha \in \mathbb{N}^d} R_\alpha z_\alpha \right\} \text{s.t. } \begin{array}{l} z \vert_E = a, z \in \Gamma_k(h, g). \end{array}$$

Based on solving the hierarchy of (2.12), our semidefinite algorithm for solving CP-completion problems is as follows.

**Algorithm 2.1.** A semidefinite algorithm for solving CP-completion problems.

**Step 0:** Choose a generic $R \in \Sigma_{n,d}$, and let $k := d/2$.

**Step 1:** Solve (2.12). If (2.12) is infeasible, then $a$ doesn’t admit a $\Delta$-measure, i.e., $A$ is not CP-completable, and stop. Otherwise, compute a minimizer $z^{*,k}$. Let $t := 1$.

**Step 2:** Let $w := z^{*,k}|_{2t}$. If the rank condition (2.9) is not satisfied, go to Step 4.

**Step 3:** Compute the finitely atomic measure $\mu$ admitted by $w$:

$$\mu = \lambda_1 \delta(u_1) + \cdots + \lambda_m \delta(u_m),$$

where $m = \text{rank} M_t(w)$, $u_i \in \Delta$, $\lambda_i > 0$, and $\delta(u_i)$ is the Dirac measure supported on the point $u_i$ ($i = 1, \ldots, m$). Stop.

**Step 4:** If $t < k$, set $t := t + 1$ and go to Step 2; otherwise, set $k := k + 1$ and go to Step 1.

**Remark 2.2.** Algorithm 2.1 is a specialization of Algorithm 4.2 in [38] to CP-completion. Denote $[x]_d := (x^\alpha)_{\alpha \in \mathbb{N}^d}$. We choose $R = [x]_d^{T/2} J^{T/2} [x]_d^{T/2}$ in (2.12), where $J$ is a random square matrix obeying Gaussian distribution. We check the rank condition (2.9) numerically with the help of singular value decompositions [23]. The rank of a matrix is evaluated as the number of its singular values that are greater than or equal to $10^{-6}$. We use the method in [25] to get a $m$-atomic $\Delta$-measure for $w$.

2.3. Some properties of Algorithm 2.1. We first show some basic properties of Algorithm 2.1 which are from [38 Section 5].

**Theorem 2.3.** Algorithm 2.1 has the following properties:

1) If (2.12) is infeasible for some $k$, then $a$ admits no $\Delta$-measures and the corresponding $E$-matrix $A$ is not CP-completable.

2) If the $E$-matrix $A$ is not CP-completable and $\mathbb{R}[x]_E$ is $\Delta$-full, then (2.12) is infeasible for all $k$ big enough.

3) If the $E$-matrix $A$ is CP-completable, then for almost all generated $R$, we can asymptotically get a flat extension of $a$ by solving the hierarchy of (2.12). This gives a CP-completion of $A$.

**Remark 2.4.** Under some general conditions, which is almost sufficient and necessary, we can get a flat extension of $a$ by solving the hierarchy of (2.12), within finitely many step (cf. [38]). This always happens in our numerical experiments. So, if an $E$-matrix $A$ with the identifying vector $a \in \mathbb{R}^2$ is CP-completable, then we can asymptotically get a flat extension of $a$ for almost all $R \in \Sigma_{n,d}$ by Algorithm...
Moreover, it can often be obtained within finitely many steps. After getting a flat extension of $a$, we can get a $m$-atomic $\Delta$-measure for $a$, which then produces a CP-completion of $A$, as well as a CP-decomposition.

When $\mathbb{R}[x]_E$ is $\Delta$-full, Algorithm 2.1 can give a certificate for the non-CP-completability. However, if it is not $\Delta$-full and $A$ is not CP-completable, it is not clear whether there exists a $k$ such that (2.12) is infeasible. This is an open question, to the best knowledge of the authors. We now characterize when $\mathbb{R}[x]_E$ is $\Delta$-full.

**Proposition 2.5.** Suppose $E = \{(i_k,j_k) | 1 \leq i_k \leq j_k \leq n, k = 1, \ldots, l\}$. Then, $\mathbb{R}[x]_E$ is $\Delta$-full if and only if $\{(i,i) : 1 \leq i \leq n\} \subseteq E$.

**Proof.** We first prove the sufficient condition. If $\{(i,i), 1 \leq i \leq n\} \subseteq E$, then $\{(2,0,\ldots,0),(0,2,\ldots,0),\ldots,(0,0,\ldots,2)\} \subseteq E$, so we have $x_i^2 \in \mathbb{R}[x]_E$ for all $1 \leq i \leq n$. Hence for any $x \in \Delta$, there exists a polynomial $p(x) = \sum_{i=1}^n x_i^2 \in \mathbb{R}[x]_E$ such that $p(x) > 0$. Thus $\mathbb{R}[x]_E$ is $\Delta$-full.

We prove the necessary condition by contradiction. Suppose there exists some $i_0 \in \{1, \ldots, n\}$ such that $(i_0,i_0) \notin E$. For any polynomial $p(x) \in \mathbb{R}[x]_E$, $p(x)$ is a linear combination of all the monomials of degree 2 except $x_{i_0}^2$. Let $c = (0, \ldots, 0, 1_{i_0}, 0, \ldots, 0)^T \in \Delta$ be a constant vector, then $p(c) = 0$ holds for all $p(x) \in \mathbb{R}[x]_E$. Hence, there does not exist any polynomial $p(x) \in \mathbb{R}[x]_E$ such that $p(x) \Delta > 0$. This contradicts the fact that $\mathbb{R}[x]_E$ is $\Delta$-full. The proof is completed. \(\square\)

**Remark 2.6.** Proposition 2.5 shows that $\mathbb{R}[x]_E$ is $\Delta$-full if and only if all the diagonal entries are given. In such case, Algorithm 2.1 can determine whether an $E$-matrix $A$ can be completed to a CP-matrix or not. If $A$ is CP-completable, Algorithm 2.1 can give a CP-completion with a CP-decomposition. If $A$ is not CP-completable, then it can give a certificate, i.e., (2.12) is infeasible for some $k$.

### 3. CP-completion with missing diagonal entries

When all the diagonal entries are given, which is equivalent to that $\mathbb{R}[x]_E$ is $\Delta$-full, the properties of Algorithm 2.1 are summarized in Theorem 2.3 and Remark 2.6. In this section, we study properties of CP-completion problems when some diagonal entries are missing.

#### 3.1. All diagonal entries are missing

Consider $E$-matrices with all the diagonal entries missing, i.e., $(i,i) \notin E$ for all $i$. In such case, CP-completion is relatively simple, as shown below.

**Proposition 3.1.** Let $A$ be an $E$-matrix whose entries are all nonnegative. If all the diagonal entries of $A$ are missing, then $A$ has a CP-completion.

**Proof.** The matrix

$$C = \sum_{(i,j) \in E} A_{ij}(e_i + e_j)(e_i + e_j)^T$$

is clearly completely positive, because each $A_{ij} \geq 0$. It is easy to check that $C$ is a CP-completion of $A$. \(\square\)

**Remark 3.2.** By Proposition 3.1 every nonnegative $E$-matrix is CP completable, when all diagonal entries are missing. In such case, Algorithm 2.1 typically gives a CP-decomposition whose length is much smaller than the length in the proof of...
Proposition 3.1, which is the cardinality of $E$. This is an advantage of Algorithm 2.1.

3.2. Diagonal entries are partially missing. We consider $E$-matrices whose diagonal entries are not all missing, i.e., the set $E$ contains at least one but not all of $(1, 1), \ldots, (n, n)$. Suppose the diagonal entry indices in $E$ are $(i_1, i_1), \ldots, (i_r, i_r)$. Let $\hat{E} = \{(i, j) \in E : i, j \in \{i_1, \ldots, i_r\}\}$.

Let $A$ be an $E$-matrix. An $\hat{E}$-matrix $P$ is called the maximum principal submatrix of $A$ if $P_{ij} = A_{ij}$ for all $(i, j) \in \hat{E}$. If $P$ is CP-completable, we say that $A$ is partially CP-completable. Clearly, if $A$ is CP-completable, then $P$ is also CP-completable. This immediately leads to the following proposition.

**Proposition 3.3.** If the maximum principal submatrix of an $E$-matrix $A$ is not CP-completable, then $A$ is not CP-completable.

**Remark 3.4.** The converse of Proposition 3.3 is not necessarily true. For example, consider the $E$-matrix $A$ given as

\[
\begin{pmatrix}
1 & 1 & 2 \\
1 & 1 & 3 \\
2 & 3 & *
\end{pmatrix}
\]

The determinant of $A$ is always $-1$, no matter what the $(3, 3)$-entry is. So, it cannot be positive semidefinite, and hence $A$ is not CP-completable. However, its maximum principal submatrix $P$ is completely positive, so $A$ is partially CP-completable.

Though the $E$-matrix $A$ in (3.2) is not CP-completable, we can show that there exists a sequence $\{A_k\}$ of CP-completable $E$-matrices such that their identifying vectors converge to the one of $A$.

**Theorem 3.5.** Suppose the maximum principal submatrix of an $E$-matrix $A$, with the identifying vector $a \in \mathbb{R}^E$, is CP-completable. Then there exists a sequence of CP-completable $E$-matrices $\{A_k\}$, whose identifying vectors converge to $a$.

**Proof.** If all the diagonal entries are given, the theorem is clearly true.

First, we assume exactly one diagonal entry is missing. Without loss of generality, we assume $A$ is given in the following form

\[
\begin{pmatrix}
A' & \cdots & A_{n-1,n} \\
\vdots & \ddots & \vdots \\
A_{n,1} & \cdots & A_{n,n-1} \\
\end{pmatrix}
\]

where $A'$ is the maximum principal submatrix of $A$. If some of the entries $A_{i,n}, A_{n,i}$ ($i = 1, \ldots, n-1$) are missing, we assign the constant value 1 to them. The matrix $A'$ is CP-completable, by assumption, and all its diagonal entries are given. Consider the following sequence of $E$-matrices:

\[
\begin{pmatrix}
A' + \varepsilon_k I_{n-1} & \cdots & A_{1,n} \\
\vdots & \ddots & \vdots \\
A_{n,1} & \cdots & A_{n,n-1} \\
\end{pmatrix}, \quad k = 1, 2, \ldots,
\]
where \( I_{n-1} \) is the identity matrix of order \( n-1 \) and \( 0 < \varepsilon_k \to 0 \) as \( k \to \infty \). Let \( \overline{A'} \) be a CP-completion of \( A' \), and

\[
A_k = \begin{bmatrix} A' & 0 \\ 0^T & 1 \end{bmatrix} + \sum_{1 \leq i \leq n-1} (\sqrt{\varepsilon_k} e_i + \sqrt{\varepsilon_k}^{-1} A_{i,n} e_n)(\sqrt{\varepsilon_k} e_i + \sqrt{\varepsilon_k}^{-1} A_{i,n} e_n)^T,
\]

where \( 0 \) is the zero vector. Clearly, \( A_k \) is a CP-completion of the matrix in (3.4), and the sequence \( \{A_k\}_E \) converges to \( a \) as \( k \to +\infty \).

Second, when two or more diagonal entries are missing, the proof is same as in the above. We omit it here for cleanness. \( \square \)

**Remark 3.6.** Theorem 3.5 implies that the set of all CP-completable \( E \)-matrices is not closed, if some diagonal entries are missing.

When an \( E \)-matrix has only one given diagonal entry, there is a nice property of CP-completion.

**Proposition 3.7.** Let \( A \) be a nonnegative \( E \)-matrix. If only one diagonal entry of \( A \) is given and it is positive, then \( A \) is CP-completable.

**Proof.** Without loss of generality, we assume the first diagonal is given and positive. Let \( \tilde{n} \) be the number of the given entries in the first row. So, \( 1 \leq \tilde{n} \leq n \). If \( \tilde{n} = 1 \), let

\[
C = A_{11} e_1 e_1^T + \sum_{2 \leq i < j \leq n, (i,j) \in E} A_{ij} (e_i + e_j)(e_i + e_j)^T.
\]

Then \( C \) is a CP-completion of \( A \). Otherwise, if \( \tilde{n} > 1 \), let

\[
C = \sum_{1 \leq i < j \leq n, (i,j) \in E} \left( \sqrt{\frac{A_{11}}{\tilde{n} - 1}} e_1 + \sqrt{\frac{\tilde{n} - 1}{A_{11}}} A_{1j} e_j \right) \left( \sqrt{\frac{A_{11}}{\tilde{n} - 1}} e_1 + \sqrt{\frac{\tilde{n} - 1}{A_{11}}} A_{1j} e_j \right)^T + \sum_{2 \leq i < j \leq n, (i,j) \in E} A_{ij} (e_i + e_j)(e_i + e_j)^T.
\]

It can be easily checked that \( C \) is a CP-completion of \( A \). \( \square \)

**Remark 3.8.** 1) If an \( E \)-matrix is not partially CP-completable, a certificate (i.e., the relaxation (2.12) is infeasible) can be obtained by applying Algorithm 2.1 to its maximum principle submatrix. 2) If an \( E \)-matrix is partially CP-completable, then Algorithm 2.1 can give a CP-completion, up to an arbitrarily tiny positive perturbation (applied to given diagonal entries).

4. Numerical experiments

In this section, we present numerical experiments for solving CP-completion problems by using Algorithm 2.1. We use software GloptiPoly 3 [26] and SeDuMi [46] to solve semidefinite relaxations in (2.12). We choose \( d = 4 \) and \( k = 2 \) in Step 0 of Algorithm 2.1.

**Example 4.1.** Consider the \( E \)-matrix \( A \) given as (cf. [3] Exercise 2.57):

\[
A = \begin{bmatrix}
b & 3 & 0 & * \\
3 & 6 & 3 & 0 \\
0 & 3 & 6 & 3 \\
* & 0 & 3 & b
\end{bmatrix},
\]

(4.1)
where \( b \geq 0 \) is a parameter. For a symmetric nonnegative matrix of order \( n \leq 4 \), it is completely positive if and only if it is positive semidefinite (cf. [33]), i.e., all its principal minors are nonnegative. Let \( c = A_{14} \), the missing value. Then \( A \) is completely positive if and only if

\[
b \geq 0, c \geq 0, 2b - 3 \geq 0, b - 2 \geq 0, 2b^2 - 3b - 2c^2 \geq 0, (b - 2)^2 - (c + 1)^2 \geq 0.
\]

The above is satisfiable if and only if \( b \geq 3 \), i.e., \( A \) is CP-completable if and only if \( b \geq 3 \). When \( b = 3 \), \( A \) is CP-completable only for \( c = 0 \).

We choose \( b = 3 \) and apply Algorithm 2.1. It terminates at Step 3 with \( k = 3 \), and gives the CP-completion

\[
A = \begin{bmatrix}
3 & 3 & 0 & 0.0000 \\
3 & 6 & 3 & 0 \\
0 & 3 & 6 & 3 \\
0.0000 & 0 & 3 & 3
\end{bmatrix}
= \sum_{i=1}^{3} \rho_i u_i^T u_i T,
\]

where \( u_i \) and \( \rho_i \) are given in Table 1.

**Example 4.2.** Consider the \( E \)-matrix \( A \) given as:

\[
\begin{bmatrix}
* & 4 & 1 & 2 & 2 \\
4 & * & 0 & 1 & 3 \\
1 & 0 & * & 1 & 2 \\
2 & 1 & 1 & * & 1 \\
2 & 3 & 2 & 1 & *
\end{bmatrix}
\]

(4.2)

All its diagonal entries are missing. By Proposition 3.1, we know \( A \) is CP-completable. We apply Algorithm 2.1. It terminates at Step 3 with \( k = 3 \), and gives the CP-completion:

\[
\begin{bmatrix}
5.8127 & 4 & 1 & 2 & 2 \\
4 & 4.6697 & 0 & 1 & 3 \\
1 & 0 & 2.2682 & 1 & 2 \\
2 & 1 & 1 & 0.9087 & 1 \\
2 & 3 & 2 & 1 & 4.7740
\end{bmatrix}
= \sum_{i=1}^{3} \rho_i u_i^T u_i T,
\]

where \( u_i \) and \( \rho_i \) are shown in Table 2. The length of the CP-decomposition is 3.
which is much shorter than 9 given by (3.1). This shows an advantage of Algorithm 2.1.

**Example 4.3.** Consider the $E$-matrix $A$ given as

$$
\begin{bmatrix}
6.1232 & 4.1232 & 1.1233 & 2.1231 & 2.3321 \\
4.1232 & * & 0 & 1.0987 & 3.2873 \\
1.1233 & 0 & 3.2318 & 1.2332 & 2.1232 \\
2.1231 & 1.0987 & 1.2332 & * & 1.1232 \\
2.3321 & 3.2873 & 2.1232 & 1.1232 & *
\end{bmatrix}
$$

(4.3)

By Proposition 2.5, the space $\mathbb{R}[x]E$ is not $\Delta$-full. We apply Algorithm 2.1. It terminates at Step 3 with $k = 4$, and gives the CP-completion:

$$
\begin{bmatrix}
6.1232 & 4.1232 & 1.1233 & 2.1231 & 2.3321 \\
4.1232 & 5.5494 & 0 & 1.0987 & 3.2873 \\
1.1233 & 0 & 3.2318 & 1.2332 & 2.1232 \\
2.1231 & 1.0987 & 1.2332 & 1.0430 & 1.1232 \\
2.3321 & 3.2873 & 2.1232 & 1.1232 & 3.6641 \\
\end{bmatrix} = \sum_{i=1}^{4} \rho_i u_i^T,
$$

where $u_i$ and $\rho_i$ are shown in Table 3. This also shows a nice property of Algorithm 2.1: if it exists, a CP-completion can be found, even if $\mathbb{R}[x]E$ is not $\Delta$-full.

**Example 4.4.** Consider the $E$-matrix $A$ given as:

$$
\begin{bmatrix}
* & 7 & 1 & 3 & 9 & * \\
* & 5 & 8 & 5 & 3 \\
5 & * & 2 & 2 & * \\
3 & 8 & 3 & 1 & 4 \\
9 & 5 & 2 & 1 & * \\
* & 3 & 4 & 1 & *
\end{bmatrix}
$$

(4.4)

Only one diagonal entry is given. By Proposition 3.7, (4.4) is CP-completable. We apply Algorithm 2.1. It terminates at Step 3 with $k = 5$, and gives the CP-completion

$$
\begin{bmatrix}
11.3758 & 7 & 1 & 3 & 9 & 6.1225 \\
7 & 32.5203 & 5 & 8 & 5 & 3 \\
1 & 5 & 4.2013 & 2 & 2 & 2.5314 \\
3 & 8 & 2 & 3 & 1 & 4 \\
9 & 5 & 2 & 1 & 11.1114 & 1 \\
6.1225 & 3 & 2.5314 & 4 & 1 & 10.8581 \\
\end{bmatrix} = \sum_{i=1}^{9} \rho_i u_i^T,
$$

where $u_i$ and $\rho_i$ are listed in Table 4.
Example 4.5. Consider the $E$-matrix $A$ given as (cf. [5, Example 1.35]):

\[
\begin{bmatrix}
1 & 1 & * & * & 0 \\
1 & 1 & 1 & * & * \\
* & 1 & 1 & 1 & * \\
* & * & 1 & 1 & 1 \\
0 & * & * & 1 & 1
\end{bmatrix}.
\]

(4.5)

It is shown in [5] that (4.5) is not CP-completable. We apply Algorithm 2.1 to verify this fact. It terminates at Step 1 with $k = 3$ as (2.12) is infeasible, which confirms that (4.5) is not CP-completable.

Example 4.6. Consider the $E$-matrix $A$ given as:

\[
\begin{bmatrix}
1 & 1 & 2 & * & 4 \\
1 & 1 & 3 & * & 3 \\
2 & 3 & 3 & 3 & * \\
* & * & 3 & 2 & * \\
4 & 3 & * & * & *
\end{bmatrix}.
\]

(4.6)

By Proposition 2.5 the set $\mathbb{R}[x]_E$ is not $\Delta$-full. We apply Algorithm 2.1 to solve this CP-completion problem. It terminates at Step 1 with $k = 1$, because (2.12) is infeasible. This shows that the $E$-matrix $A$ is not CP-completable. By this example, we can see that Algorithm 2.1 might get a certificate for non-CP-completable, even if $\mathbb{R}[x]_E$ is not $\Delta$-full.

Example 4.7. Consider the $E$-matrix $A$ given as in Remark 3.4:

\[
\begin{bmatrix}
1 & 1 & 2 \\
1 & 1 & 3 \\
2 & 3 & * 
\end{bmatrix}.
\]

(4.7)

We have already seen that $A$ is not CP-completable. We apply positive perturbations to $A$ as follows:

\[
\begin{bmatrix}
1 + 10^{-l} & 1 \frac{2}{10} \\
1 \frac{2}{10} & 3 + 10^{-l} \\
2 & 3 \\
\end{bmatrix}, \quad l = 1, 2, \ldots
\]

(4.8)

\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
$i$ & $u_i$ & $\rho_i$ \\
\hline
1 & (0.0000, 0.0458, 0.5132, 0.0224, 0.4187, 0.0000)$^T$ & 1.9950 \\
2 & (0.0000, 0.4881, 0.2702, 0.1191, 0.1225, 0.0000)$^T$ & 5.6983 \\
3 & (0.0434, 0.5268, 0.1914, 0.1203, 0.1183, 0.0000)$^T$ & 41.6816 \\
4 & (0.0000, 0.0000, 0.3857, 0.1581, 0.0000, 0.4562)$^T$ & 11.2576 \\
5 & (0.1508, 0.5697, 0.0000, 0.1599, 0.0000, 0.1197)$^T$ & 44.0288 \\
6 & (0.1929, 0.5399, 0.0000, 0.1050, 0.1622, 0.0000)$^T$ & 17.9222 \\
7 & (0.2977, 0.0000, 0.0324, 0.1558, 0.0000, 0.5141)$^T$ & 29.2892 \\
8 & (0.4287, 0.0842, 0.0000, 0.0000, 0.4871, 0.0000)$^T$ & 11.0745 \\
9 & (0.4121, 0.0000, 0.0306, 0.0000, 0.4875, 0.0697)$^T$ & 29.4268 \\
\hline
\end{tabular}
\caption{The points $u_i$ and weights $\rho_i$ in Example 4.4.}
\end{table}
By the proof of Theorem 3.5, we know that (4.8) is CP-completable for all $l$. For $l = 1, 2, 3$, Algorithm 2.1 produces the following CP-completions:

$$A_1 = \begin{bmatrix} 1.1 & 1 & 2 \\ 1 & 1.1 & 3 \\ 2 & 3 & 10.9524 \end{bmatrix} = 2 \sum_{i=1}^{2} \lambda_i u_i u_i^T,$$

$$A_2 = \begin{bmatrix} 1.01 & 1 \\ 1 & 1.01 & 3 \\ 2 & 3 & 56.2189 \end{bmatrix} = 2 \sum_{i=1}^{2} \rho_i v_i v_i^T,$$

and

$$A_3 = \begin{bmatrix} 1.001 & 1 \\ 1 & 1.001 & 3 \\ 2 & 3 & 487.2967 \end{bmatrix} = 4 \sum_{i=1}^{4} \sigma_i \omega_i \omega_i^T.$$

The points and their weights are shown in Tables 5 and 6.

| $i$ | $u_i$ | $\lambda_i$ | $v_i$ | $\rho_i$ |
|-----|-------|-------------|-------|---------|
| 1   | (0.1254, 0.1881, 0.6866)$^T$ | 23.2350 | (0.0327, 0.0490, 0.9183)$^T$ | 66.6636 |
| 2   | (0.6190, 0.3810, 0.0000)$^T$ | 1.9174 | (0.5124, 0.4876, 0.0000)$^T$ | 3.5753 |

Table 5. The points and weights for $A_1$ and $A_2$ in Example 4.7.

| $i$ | $\omega_i$ | $\sigma_i$ |
|-----|-------------|------------|
| 1   | (0.5012, 0.4987, 0.0001)$^T$ | 3.7527 |
| 2   | (0.0682, 0.0696, 0.8623)$^T$ | 11.6502 |
| 3   | (0.0027, 0.0048, 0.9925)$^T$ | 485.9000 |
| 4   | (0.4545, 0.3039, 0.2487)$^T$ | 0.0023 |

Table 6. The points and weights for $A_3$ in Example 4.7.

When $l \geq 4$, the resulting semidefinite relaxations (2.12) are ill-conditioned, and the semidefinite programming solver SeDuMi has trouble to solve them accurately. This is because this $E$-matrix is not CP-completable, but it has an arbitrarily tiny perturbation that is CP-completable.

5. Conclusions and Discussions

This paper proposes a semidefinite algorithm (i.e., Algorithm 2.1) for solving general CP-completion problems. When all the diagonal entries are given, Algorithm 2.1 can give a certificate for non-CP-completable. If a partial matrix is CP-completable, Algorithm 2.1 almost always gets a CP-completion, as well as a CP-decomposition. When some diagonal entries are missing, if the maximum principal submatrix is not CP-completable, then a certificate for non-CP-completable can be obtained; if it is CP-completable, then Algorithm 2.1 also almost always gives a CP-completion.

CP-completion has wide applications (cf. [5]). Here we show one in probability theory. Let $x$ be a random vector in $\mathbb{R}^n$. Suppose its expectation $\mathbb{E}x = b$ and partial entries of its covariance matrix are known, say, for an index set $E$ the entries

$$X_{ij} = \mathbb{E}[(x_i - b_i)(x_j - b_j)]$$
with \((i,j) \in E\) are known. We want to investigate for what values of \(X_{ij}\) \(((i,j) \in E)\) the density function of \(x\) is supported in the nonnegative orthant \(\mathbb{R}^n_+\). This question is basic and natural, because many statistical quantities are positive in the world. Interestingly, this question can be formulated as a CP-completion problem. From the expression of \(X_{ij}\), we can see that

\[
\mathbb{E}(x_i x_j) = X_{ij} + b_i b_j.
\]

Let \(A\) be the \(E\)-matrix such that \(A_{ij} = X_{ij} + b_i b_j\) for all \((i,j) \in E\). Whether the random vector \(x\) has a density function supported in \(\mathbb{R}^n_+\) or not is basically equivalent to whether the following partial matrix

\[
\begin{bmatrix}
* & b^T \\
 b & A
\end{bmatrix}
\]

is CP-completable or not. In the above, \(A\) is also a partial matrix; only the entries \(A_{ij}\) with \((i,j) \in E\) are known.

CP-completion also has applications in nonconvex quadratic optimization. Let \(E\) be an index set. A typical quadratic optimization problem is

\[
\begin{align*}
\min & \sum_{(i,j) \in E} a_{ij} x_i x_j \\
\text{s.t.} & p(x) = 1, \ x \geq 0,
\end{align*}
\]

where \(p(x) = \sum_{(i,j) \in E} p_{ij} x_i x_j\) is a given polynomial. To solve this nonconvex optimization problem globally, we need to characterize the cone

\[
\mathcal{C}_E = \{ y \in \mathbb{R}^E : y = C|_E \text{ for some CP-matrix } C \in \mathbb{R}^{n \times n} \}.
\]

It can be shown that (5.1) is equivalent to the linear convex problem

\[
\begin{align*}
\min & \sum_{(i,j) \in E} a_{ij} y_{ij} \\
\text{s.t.} & \sum_{(i,j) \in E} p_{ij} y_{ij} = 1, \ y \in \mathcal{C}_E.
\end{align*}
\]

To design efficient numerical methods for solving (5.2), we need to check the memberships in \(\mathcal{C}_E\). This is equivalent to solving CP-completion problems.

In this paper, we mainly focus on determining whether a partial matrix is CP-completable or not. However, we did not discuss the question of how to get a CP-completion whose CP-rank is minimum. This question is hard, and there exists few work about it, to the best knowledge of the authors. This is an interesting future work.

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