Operator product expansion and non-perturbative renormalization

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It has been recently proposed to use the operator product expansion to evaluate the expectation values of renormalized operators without the need of a direct computation of the relevant renormalization constants. We test the viability of this idea in the two-dimensional non-linear \(\sigma\)-model discussing the non-perturbative renormalization of the energy-momentum tensor.

The Operator Product Expansion (OPE) consists in rewriting, for \(y \to x\),
\[
A(x)B(y) \to \sum_C W^{AB}_C(x-y)C(y) .
\]

(1)

Here \(A\), \(B\), and \(C\) are operators; \(W^{AB}_C(x-y)\) are \(c\)-number functions (Wilson coefficients), singular when \(|x-y| \to 0\), that can be computed, for example, in perturbation theory. Dimensional analysis tells us that the leading contribution for \(y \to x\) is due to the operators \(C\)'s in Eq. (1) that have the lowest dimension. Notice that the OPE is an operator relation and therefore, for any matrix element \(\langle \phi | A(x)B(y) | \phi \rangle\), the coefficients \(W^{AB}_C(x-y)\) are independent of the states \(|\psi\rangle\) and \(|\phi\rangle\). This also means that if one wants to use the OPE, one should go through all the subtleties in the definition of the operators in quantum field theory (regularization, renormalization, \ldots).

The authors of reference \(\ref{1}\) have proposed to consider, for lattice applications, the OPE in the particular case in which \(A\) and \(B\) are the components of a conserved current \(J_\mu\). This case is particularly simple because the components of \(J_\mu\) do not need to be renormalized. This means that
\[
[J_\mu(x)J_\nu(0)]_{\text{MS}}(\mu) = [J_\mu(x)J_\nu(0)]_{\text{L}}(1/a) ,
\]

(2)

where the currents are expressed in terms of the bare fields and the coupling constant on the lattice is related to the coupling constant \(g\) in the continuum at the scale \(\mu\) by a renormalization factor
\[
g_L(1/a) = Z^\mu_g (g(\mu), \mu a) g(\mu) .
\]

(3)

Therefore, the lattice computation of \(\langle \phi | J_\mu(x)J_\nu(0) | \phi \rangle\) provides the same expectation value as in the continuum, say in the MS-scheme, at the scale \(\mu\). One can then use Eq. (1) to estimate the matrix elements of the operators \(C\)'s that dominate the r.h.s. of (1) (in the continuum at the scale \(\mu\), bypassing all the troubles of lattice perturbation theory (large deviations from 1 of the \(Z\)'s, large corrections, improvements, \ldots)). The recipe is the following. One evaluates on the lattice the matrix elements \(\langle \psi | J_\mu(x)J_\nu(0) | \phi \rangle\) in the range \(1 < |x| < \xi\) (where \(\xi\) is as usual the correlation length) and then uses the OPE in the continuum (at the scale \(\mu\) to fit the (now) numbers \(\langle \psi | C(0) | \phi \rangle\). One needs the Wilson coefficients \(W^{(\mu)}_C(x)\). The idea is to use their expression computed in perturbation theory in the continuum.

Assuming \(C(0)\) to be multiplicatively renormalized, one can then use
\[
\langle \psi | C(0) | \phi \rangle = Z_C \langle g_L(1/a), \mu a \rangle \langle \psi | C'_{\text{L}}(0) | \phi \rangle .
\]

(4)

In this way, by measuring \(\langle \psi | C'_{\text{L}}(0) | \phi \rangle\) on the lattice, we obtain a non-perturbative determination of \(Z_C\).

The discussion in \(\ref{1}\) ends with a question: is it feasible? That is, does it exist, in practice, a numerically accessible window for \(|x|\) where, for a truncation of the sum over \(C\), the OPE applies accurately enough to extract the expectation values of the \(C\)'s?

To perform a check we need a case in which we already know the matrix elements \(\langle \psi | C(0) | \phi \rangle\), so
that we can verify the correctness of our results.

Our example, given the experience that we have accumulated along the past years, will refer to the $O(N)$ $\sigma$-model in two dimensions with lattice action

$$S = \frac{1}{2gL} \sum_{x, \mu} (\partial_\mu \sigma)^2$$

(5)

with $\sigma \in S^{N-1}$ and $(\partial_\mu f)_x = f_{x+\mu} - f_x$. The fields are related to their continuum version by

$$\sigma_x = \sqrt{Z^L} \sigma(x).$$

(6)

All the renormalization factors $Z_i^L$ have an expansion of the form

$$Z_i^L(g, \mu, a) = \sum_{l=0}^{\infty} g^l \sum_{n=0}^l c^{(l)}_n \ln^n \mu a.$$  

(7)

For example $Z_i^g$ and $Z_i^L$ are known up to four loops [3].

For our purposes we made use of the non-perturbative determination of $Z_i^g$ given in [3] for the case $N = 3$.

The Noether currents related to the $O(N)$-invariance, which is preserved also on the lattice, are

$$j^{a,b}_\mu(x) = \frac{1}{g} (\sigma^a_2 \partial_\mu \sigma^b_2 - \sigma^b_2 \partial_\mu \sigma^a_2);$$

(8)

and their singlet product is

$$j_\mu(x) \cdot j_\nu(0) = \sum_{a,b} j^{a,b}_\mu(x) j^{a,b}_\nu(0).$$

(9)

In the continuum, using the OPE and averaging over a circle, we obtain

$$\frac{g^2}{2} j_1(x) \cdot j_0(0) =$$

$$= g^2 \int \frac{d\theta}{4\pi} j_1(r \cos \theta, r \sin \theta) \cdot j_0(0,0)$$

$$= \frac{1}{g} [\partial_1 \sigma \cdot \partial_0 \sigma]_{MS} (0) W_{C}^{10}(r) + O(r^2)$$

$$= g T_{10}^{MS} (0) W_{C}^{10}(r) + O(r^2)$$

(10)

where $T_{10}^{MS}$ is the continuum energy-momentum tensor (EMT), which is exactly conserved, a property that is not shared by its lattice counterpart. It follows that

$$\int \frac{g^2}{2} T_{10}^{MS} (x, t) dx = p,$$

(11)

where $p$ is the momentum, so that, in a strip of size $L$, the expectation value between states of momentum $p$ is

$$\frac{\langle p | T_{10}^{MS} (0) | p \rangle}{\langle p | p \rangle} = \frac{p}{L}.$$  

(12)

The Wilson coefficient is given by

$$W_{C}^{10}(r) = 1 + \frac{5(N - 2)}{8\pi} g$$

$$\frac{N - 2}{4\pi} g (\gamma + \ln \pi^2 \mu^2 r^2) + O(g^2).$$

We have performed a Monte Carlo simulation on a lattice of size $L \times T = 128 \times 256$ with periodic boundary conditions at $g_{10}^{-1} = 1.54$, which corresponds to $\xi = 13.632(6)$. We measure:

a) the correlation function in the one-particle sector with momentum $p$

$$C(p, 2t) = \frac{1}{L} \sum_{x,y} \langle \sigma_{-t,x} \cdot \sigma_{t,y} \rangle e^{ip(x-y)}$$

$$\approx \frac{Z(p)}{2\omega(p)} e^{-2\omega(p)};$$

(14)

b) the correlation function of the out-of-diagonal entry of the lattice EMT with two one-particle operators

$$T_{10}(p, 2t) =$$

$$= \frac{1}{gL} \sum_{x,y} \langle (\partial_1 \sigma_{0,0} \cdot \partial_0 \sigma_{0,0}) (\sigma_{-t,x} \cdot \sigma_{t,y}) \rangle$$

$$\times e^{ip(x-y)}$$

$$\approx \frac{Z(p)}{2\omega(p)} \frac{\langle p | T_{10}(0) | p \rangle}{\langle p | p \rangle} e^{-2\omega(p)};$$

(15)

c) the correlation between the products of two currents in the singlet sector and two one-particle operators

$$\mathcal{I}_{12}(z; p, 2t) =$$

$$= \frac{1}{L} \sum_{x,y} \langle (\partial_{12} \sigma_{0,0} \cdot \partial_{20} \sigma_{0,0}) (\sigma_{-t,x} \cdot \sigma_{t,y}) \rangle e^{ip(x-y)}$$

$$\approx \frac{Z(p)}{2\omega(p)} \frac{\langle p | \partial_{12} \sigma_{0,0} \cdot \partial_{20} \sigma_{0,0} \cdot \sigma_{-t,x} \cdot \sigma_{t,y} \rangle}{\langle p | p \rangle} e^{-2\omega(p)}.$$  

(17)
Moreover we need a lattice version of the angular average

\[ \overline{f}(r) = \frac{1}{N(r)} \sum_z f(z) \Xi(z) \]

\[ N(r) = \sum_z \Xi(z) \]

\[ \Xi(z) = \theta(|z| - r + \frac{1}{2}) \theta(r + \frac{1}{2} - |z|) \] (18)

Let us now discuss the Monte Carlo data. In Fig. 1 we plot the angular average \( g^2 L I_{01}(z; p, 20)/2 \) for various distances \( z \) and various momenta \( p = n \pi/L \) for \( n = 1, 2, 3 \).

We also looked at the OPE directly on the lattice, i.e. using the Wilson coefficient computed in lattice perturbation theory resumming the first two leading logs, the expectation value \( T_{10}(p, 2t) \), and the non-perturbative renormalization constant of the lattice EMT. In this case we do not observe a reasonable window where the OPE is verified.

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