A note on eigenvalue bounds for non-compact manifolds

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Abstract
In this article we prove upper bounds for the Laplace eigenvalues \( \lambda_k \) below the essential spectrum for strictly negatively curved Cartan–Hadamard manifolds. Our bound is given in terms of \( k^2 \) and specific geometric data of the manifold. This applies also to the particular case of non-compact manifolds whose sectional curvature tends to \(-\infty\), where no essential spectrum is present due to a theorem of Donnelly/Li. The result stands in clear contrast to Laplacians on graphs where such a bound fails to be true in general.

KEYWORDS
Cheeger inequality, eigenvalues, Laplacian, negative curvature, Riemannian manifold

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1 | INTRODUCTION

In 1979 Donnelly and Li \cite{DonnellyLi} proved a criterion for discrete spectrum of the Laplacian on Riemannian manifolds in terms of decreasing sectional curvature. This complemented a result by Weyl for Schrödinger operators with increasing potential.

In particular, let \( M \) be a complete Riemannian manifold and let \( \Delta \) be the Laplacian. We denote by \( K_r \) the supremum of the sectional curvatures at points outside of \( B_r(x_0) \), the ball of radius \( r \) about some arbitrary base point \( x_0 \), that is

\[
K_r := \sup \{ K(\sigma) \mid \sigma \subset T_pM \text{ two-dimensional subspace}, p \in M \setminus B_r(x_0) \}.
\] (1.1)

Then the theorem of Donnelly/Li reads as follows.

**Theorem 1.1** (Donnelly/Li). Let \( M \) be a complete simply connected negatively curved Riemannian manifold. If \( K_r \to -\infty \) as \( r \to \infty \), then \( \Delta \) has purely discrete spectrum.

In this note we give an upper bound on the eigenvalues \( \lambda_k \) (listed with increasing order and counting multiplicities) in terms of \( k^2 \) and specific geometric data of the manifold. While this bound is a classical result in the case of compact manifolds, it stands in clear contrast to the case of Laplacians on graphs. Indeed, for graphs any asymptotics of eigenvalues can occur, see e.g. \cite{2}.

Our result is based on so-called improved Cheeger inequalities which were introduced in the setting of finite graphs in \cite{7}. A dimension-free version of these improved Cheeger inequalities in the manifold setting was derived in \cite{8} to prove an eigenvalue ratio result for closed weighted manifolds of nonnegative Bakry–Émery curvature. In this article, we discuss an application in the case of negative curvature: we use an adaption of the improved Cheeger inequalities for general...
non-closed manifolds (Theorem 2.1 below) to derive the following result on eigenvalues below the essential spectrum for strictly negatively curved Cartan–Hadamard manifolds:

**Theorem 1.2.** Let $M$ be a complete simply connected Riemannian manifold with strictly negative curvature, that is $K_0 < 0$ (with $K_r$ defined in (1.1)). Then, we have for the $L^2$-eigenvalues $\lambda_0 < \lambda_1 \leq \ldots$ of the Laplacian below the essential spectrum

$$\lambda_k \leq \frac{128\mu^2}{|K_0|(\dim(M) - 1)^2}k^2, \quad k \geq 1,$$

where

$$\mu = \inf_{r,s>0, x \in M} \frac{\text{vol}(B_{r+s}(x) \setminus B_s(x))}{r^2 \text{vol}(B_s(x))}.$$

**Remark 1.3.** Using the result of Cheng [4], one can obtain a different upper bound as follows. For a ball $B_r(x) \subset M$ with lower Ricci curvature bound larger than $(n - 1)R$ with $R < 0$, and $n = \dim(M)$, Cheng obtains for the Dirichlet eigenvalues of this ball

$$\lambda_k(B_r(x)) \leq \frac{n^2}{4}|R| + \frac{(1 + \pi^2)(1 + 2^4n)}{r^2}k^2$$

for odd dimensions and an estimate with somewhat better constants for the even-dimensional case and all $k \geq 0$, see [4, Corollary 2.3] and [3, Theorem 7, Chapter III]. (Note that Cheng proves this result for closed manifolds but his arguments work also without modification in the case of the compact manifold $B_r(x)$ with Dirichlet boundary conditions. Note also that under the assumptions of Theorem 1.2, we have $R \leq K_0$.) By domain monotonicity, [3, Corollary 1, Chapter I], we have for all eigenvalues $\lambda_k(M)$ of $M$ below the essential spectrum

$$\lambda_k(M) \leq \lambda_k(B_r(x)).$$

This yields an upper estimate with different geometric constants.

# 2 | PROOF OF OUR MAIN RESULT

We introduce the following notation. For a Riemannian manifold $M$ let $\text{vol}$ be its volume measure and $d$ the Riemannian distance. For a Borel set $A \subseteq M$ the boundary measure $\text{vol}^+(A)$ is defined as

$$\text{vol}^+(A) = \liminf_{r \to 0} \frac{\text{vol}(O_r(A)) - \text{vol}(A)}{r},$$

where $O_r(A) = \{ x \in M \mid d(x, a) \leq r \text{ for some } a \in A \}$. If $A$ has positive volume and finite boundary measure, we let

$$\phi(A) = \frac{\text{vol}^+(A)}{\text{vol}(A)}$$

and $\phi(A) = \infty$ otherwise. The Cheeger constant of a non-compact Riemannian manifold $M$ is defined as (see [3, p. 95])

$$h = h(M) = \inf_{A \subseteq M} \phi(A).$$

We deduce our main result, Theorem 1.2 above, from the following result for general manifolds which was shown in the setting of closed manifolds, [8, Theorem 1.6]. The basic idea of the proof is an extension of the methods of [7, Lemma 4, Proposition 2] developed for finite graphs to prove the so-called improved Cheeger inequalities.

**Theorem 2.1.** Let $M$ be a complete Riemannian manifold. Then, we have for the $L^2$-eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots$ of the Laplacian below the essential spectrum

$$h^2\lambda_k \leq 128k^2\lambda_0^2, \quad k \geq 1.$$
The proof of this theorem is based on an estimate which was proven for compact manifolds in [8, Theorem 3.1]. Although the proof carries over we recall the proof here for the convenience of the reader. To this end let \( f \geq 0 \) be a function on \( M \) that is supported on a set of positive measure and define

\[
\phi(f) = \inf_{t \geq 0} \phi(M_f(t)),
\]

where \( M_f(t) = \{ x \in M \mid f(x) > t \} \) is the level set of \( f \) for \( t \in \mathbb{R} \). Furthermore, we denote the \( L^p \) norm by \( \| \cdot \|_p \) for \( p \in [1, \infty] \). The following proposition is the essential ingredient in the proof of Theorem 2.1.

**Proposition 2.2** (Non-compact version of Theorem 3.1 [8]). Let \( M \) be a complete Riemannian manifold with \( L^2 \)-eigenvalues \( \lambda_0 \leq \lambda_1 \leq \ldots \) of the Laplacian below the essential spectrum and let \( f \geq 0 \) be a bounded Lipschitz function in \( L^2(M) \). Then,

\[
\phi(f) \leq 8\sqrt{2} \frac{k}{\sqrt{\lambda_k}} \frac{\|\nabla f\|_2^2}{\|f\|_2^2}, \quad k \geq 1.
\]

**Proof.** Here we sketch the core arguments of the proof. For more details we refer the reader to [8]. We assume \( |\nabla f| \in L^2(M) \) since otherwise the asserted inequality is trivial.

For a finite set \( \vartheta \subset \mathbb{R} \), let \( \psi_\vartheta : \mathbb{R} \to \mathbb{R} \) be defined by

\[
\psi_\vartheta(s) = \arg\min_{t \in \vartheta} |s - t|,
\]

\( \eta_\vartheta : \mathbb{R} \to \mathbb{R} \) be defined by

\[
\eta_\vartheta(s) = |s - \psi_\vartheta(s)|,
\]

and \( \eta_\vartheta,f : M \to [0, \infty) \)

\[
\eta_\vartheta,f = \eta_\vartheta \circ f = |f - \psi_\vartheta \circ f|
\]

be the difference of \( f \) and its approximation \( \psi_\vartheta \circ f \). Note that we have \( 0 \leq \eta_\vartheta,f \leq f \).

Now, fix \( k \in \mathbb{N} \) for the rest of the proof and let \( t_0 = 0 \). Assume \( t_0 < t_1 < \ldots < t_{j-1} \) are given. If there is \( t \geq t_{j-1} \) such that

\[
\|\eta_{|t_{j-1},t),f}1_{f^{-1}(t_{j-1},t]}\|_2^2 = \frac{1}{k \lambda_k} \|\nabla f\|_2^2 = C_0,
\]

then let \( t_j \) be the smallest such \( t \geq t_{j-1} \). Otherwise, let \( t_j = \|f\|_\infty \). Observe that

\[
f_j = \eta_{|t_{j-1},t_j),f}1_{f^{-1}(t_{j-1},t_j)}, \quad j \geq 1,
\]

are positive disjointly supported Lipschitz functions which are trivial whenever \( t_j = \|f\|_\infty \). Moreover, \( f_j \in L^2 \) since \( 0 \leq f_j \leq f \) and \( f \in L^2(M) \). Furthermore, by the reverse triangle inequality we have \( |f_j(x) - f_j(y)| \leq |f(x) - f(y)|, \) \( x, y \in M \). Therefore, as the supports of the \( f_j \) are disjoint, we obtain

\[
\sum_{j=1}^\infty |\nabla f_j|^2 \leq |\nabla f|^2
\]

and therefore, \( |\nabla f_j| \in L^2(M) \) whenever \( |\nabla f| \in L^2(M) \). By completeness of the Riemannian manifold, the Laplacian is essentially selfadjoint. Thus, the \( f_j \)'s are included in the form domain of the Laplacian since \( f_j, |\nabla f_j| \in L^2(M) \). We show the following claim.
Claim: $t_{2k} = \| f \|_{\infty}$.

In the case $t_{2k} < \| f \|_{\infty}$, we infer by the arguments above and by the fact that in this case $\| f_j \|_2^2 = C_0$ for all $j = 1, \ldots, 2k$

$$\sum_{j=1}^{2k} \left( \frac{\| \nabla f_j \|_2^2}{\| f_j \|_2^2} \right) \leq \frac{1}{C_0} \| \nabla f \|_2^2 = k\lambda_k.$$ 

By the assumption $t_{2k} < \| f \|_{\infty}$, the functions $f_j$ are non-zero and therefore non-constant. Thus, there exist at least $k + 1$ of the $f_j$'s such that

$$\frac{\| \nabla f_j \|_2^2}{\| f_j \|_2^2} \leq \lambda_k.$$ 

Hence, the inequality above for $k + 1$ orthogonal functions stands in contradiction the Min-Max-Principle and the claim is proven.

So let $\theta = \{ 0 = t_0 < t_1 \leq \cdots \leq t_{2k} = \| f \|_{\infty} \}$. By (2.1) and what we have shown above, we obtain

$$\| f - \psi_\theta \circ f \|_2^2 = \sum_{j=1}^{2k} \| \eta_{\{t_j, t_{j+1}\}} f \|_2^2 \leq \frac{2}{\lambda_k} \| \nabla f \|_2^2.$$ 

(2.2)

In order to estimate the $L^2$ norm of $f - \psi_\theta \circ f$ from below, we observe that the function $h : M \to \mathbb{R}$

$$h(x) = \int_0^{f(x)} \eta_\theta(t) \, dt$$

has the same level sets as $f$ and therefore,

$$\phi(f) = \phi(h) \leq \frac{\| \nabla h \|_1}{\| h \|_1},$$

where the last inequality follows from the area formula and the co-area inequality [1, Lemma 3.2] (for more details see [8, Lemma 2.4]). Firstly, we find by the fundamental theorem of calculus and the chain rule and secondly by the Cauchy–Schwarz inequality and $\eta_{\theta, f} = f - \psi_\theta \circ f$ that

$$\| \nabla h \|_1 = \| \nabla f (\eta_{\theta} \circ f) \|_1 \leq \| \nabla f \|_2 \| f - \psi_\theta \circ f \|_2.$$ 

Thirdly, is it elementary to estimate

$$h \geq \frac{1}{8k} f^2$$

by choosing $t_j \leq f(x) \leq t_{j+1}$ for $x \in M$ and estimating

$$h(x) \geq \frac{1}{4} \left( \sum_{i=0}^{j-1} (t_{i+1} - t_i)^2 + (f(x) - t_j)^2 \right) \geq \frac{1}{8k} \left( \sum_{i=0}^{j-1} (t_{i+1} - t_i) + (f(x) - t_j) \right)^2 = \frac{1}{8k} f(x)^2.$$

These considerations together with (2.2) yield

$$\phi(f) \leq \frac{\| \nabla h \|_1}{\| h \|_1} \leq \frac{8k \| \nabla f \|_2 \| f - \psi_\theta \circ f \|_2}{\| f \|_2^2} \leq \frac{8 \sqrt{2} \sqrt{k}}{\sqrt{\lambda_k}} \frac{\| \nabla f \|_2^2}{\| f \|_2^2},$$

which finishes the proof.

With the help of this proposition we are now in the position to prove Theorem 2.1.
Proof of Theorem 2.1. We observe that for any $n$ we have
\[ \phi(f) \leq \phi(f \wedge n), \]
where $f \wedge n = \min\{f, n\}$. Moreover, by the proposition above we have
\[ \phi(f \wedge n) \leq 8 \sqrt{\frac{2}{k}} \frac{\int_M |\nabla f \wedge n|^2 \, d\text{vol}}{\int_M |f \wedge n|^2 \, d\text{vol}}. \]
Since $\phi(f) \leq \phi(f \wedge n)$, $|\nabla(f \wedge n)| \leq |\nabla f|$ and $\int_M |f \wedge n|^2 \, d\text{vol} \to \int_M |f|^2 \, d\text{vol} = 1$, $n \to \infty$, we conclude
\[ \phi(f) \leq 8 \sqrt{\frac{2}{k}} \sqrt{\frac{\int_M |\nabla f|^2 \, d\text{vol}}{\int_M |f|^2 \, d\text{vol}}}. \]
We choose $f$ to be an eigenfunction to $\lambda_0$. Then, $f$ is a Lipschitz function in $L^2(M)$ with a definite sign which can be chosen to be positive. Then, by the definition of the Cheeger constant and the proposition above, we have
\[ h \sqrt{\lambda_k} \leq 8 \sqrt{2k \lambda_0}, \]
which finishes the proof. \[\square\]

Finally, we present the proof of our main result, Theorem 1.2 in the Introduction.

Proof of Theorem 1.2. Let us first derive $h^2 \geq (\dim(M) - 1)^2 |K_0|$: In the definition of the Cheeger constant, we can restrict ourselves to sets $A$ with smooth boundary. Let $A \subset M$ be such a set, let $\hat{x} \in M$ be a point with positive distance to $A$, and let $d_{\hat{x}} : M \to [0, \infty)$ be the distance function to $\hat{x}$. Then $d_{\hat{x}}$ is a smooth function on $A$ (since the exponential map $exp_{\hat{x}} : T_{\hat{x}}M \to M$ is a diffeomorphism). By the Laplacian Comparison Theorem (see, e.g., [6, (3)]), we have
\[ \Delta_M d_{\hat{x}}(x) \geq (\dim(M) - 1) \sqrt{-K_0} \coth(\sqrt{-K_0} d_{\hat{x}}(x)). \]
This implies that $\Delta_M d_{\hat{x}}(x) \geq (\dim(M) - 1) \sqrt{|K_0|}$ for all $x \in A$ and, therefore, on the one hand,
\[ \int_A \Delta_M d_{\hat{x}} \, d\text{vol} \geq (\dim(M) - 1) \sqrt{|K_0|} \text{vol}(A), \]
and, on the other hand, using the Gauss Divergence Theorem,
\[ \int_A \Delta_M d_{\hat{x}} \, d\text{vol} = \int_{\partial A} \langle \text{grad} d_{\hat{x}}, \nu \rangle \, d\text{vol}_{\partial A} \leq \text{vol}^+(A), \]
where $\nu$ is the outward unit normal vector of $\partial A$. Combining both inequalities leads to the proof of the above estimate of the Cheeger constant $h$.

Furthermore, let $\eta = (1 - d(\cdot, B_s(x))/r)$. Then,
\[ \lambda_0 \leq \frac{\int_M |\nabla\eta|^2 \, d\text{vol}}{\int_M |\eta|^2 \, d\text{vol}} = \frac{\text{vol}(B_{r+s}(x) \setminus B_s(x))}{r^2(\text{vol}(B_s(x)) + \int_{B_{r+s}(x)\setminus B_s(x)}(r - d(y, B_s(x))^2 \, d\text{vol}(y))} \]
\[ \leq \frac{\text{vol}(B_{r+s}(x) \setminus B_s(x))}{r^2\text{vol}(B_s(x))}. \]
Hence, combining this with Proposition 2.1 we conclude the statement of the theorem. \[\square\]
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