Complete characterization of spin chains with two Ising symmetries

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Abstract – Spin chains with two Ising symmetries are the Jordan-Wigner duals of one-dimensional interacting fermions with particle-hole and time-reversal symmetry. From earlier works on Majorana chains, it is known that this class of models has 10 distinct topological phases. In this paper, we analyze the physical properties of the correspondent 10 phases of the spin model. In particular, thanks to a set of two non-commuting dualities, we determine the local and non-local order parameters of the phases. We find that 4 phases are topologically protected by the Ising symmetries, while the other 6 break at least one symmetry. Our study highlights the non-trivial relation between the topological classifications of interacting bosons and fermions.

Introduction. – Conventional phases of matter, like Ising ferromagnets, spontaneously break a symmetry and show a non-vanishing local order parameter. Phases that do not break any symmetry are called topological and include the quantum Hall phases [1–4], the topological insulators [5–7], and the Haldane phase [8–12]. In the last decade, these phases were classified and generalized within the framework of symmetry protected topological (SPT) phases [13–15].

In analogy to the classification of all crystal structures, the key question now is to associate specific materials or Hamiltonians to the different SPT phases, and to study their physical properties. Here we address this question for Ising spin chains.

The study of the topological phases of spin chains emanated from the detailed study of the Haldane phase. It was found that this phase can be protected by several symmetry groups \(G\): i) spin rotations, ii) time reversal, iii) lattice inversion, and iv) rotations of \(\pi\) around three orthogonal axes \((Z_2 \times Z_2)\) [19,20]. The cohomology group of these symmetries is \(Z_2\), indicating that these classes of models have only one topologically non-trivial phase, the Haldane phase [16,21,22].

A richer topological classification can be obtained by considering one-dimensional (1d) models with a symmetry group \(Z_2^T \times Z_2^T\), where \(Z_2^T\) is an anti-unitary \(Z_2\) symmetry. The correspondent cohomology group is \(Z_2 \times Z_2\), implying the existence of 4 topologically distinct phases that do not break any symmetry [14,18,23]. A simple realization of a \(Z_2^T \times Z_2^T\) symmetry is offered by spin chains with two Ising symmetries\(^1\)

\[
I_z: \sigma^z \rightarrow -\sigma^z \quad \text{and} \quad I_y: \sigma^y \rightarrow -\sigma^y.
\]

In addition to the 4 SPT phases, this model has 6 phases that break at least one symmetry, for a total number of 10 phases (see table 1). In this paper, we characterize the physical properties of these phases by providing exactly diagonalizable parent Hamiltonians, describing their local and non-local order parameters, and identifying their edge states.

\(^{1}\)The easiest way to see that Ising symmetries are anti-unitary is to note that they flip the sign of the canonical commutation relations of the spins. To better appreciate the distinction between the Ising symmetries of eq. (1) and the \(Z_2 \times Z_2\) symmetry associated with \(\pi\) rotations around the axes we observe that the term \(\sigma_i^z \sigma_{i+1}^z \sigma_{i+2}^z\) is symmetric under \(\pi\) rotations, but not under the Ising symmetries. And, vice versa, the term \(\sigma_i^z\) is symmetric only under the latter.
Table 1: The 10 phases of one-dimensional (1d) spin models with two Ising symmetries. The first 4 columns show the symmetries preserved by each group of phases and the resulting symmetry group $G$. The 5th column lists the 2-Borel-cohomology group $H^2_b[G, U(1)]$, whose number of elements is equal to the number of topologically distinct phases in 1d [14,17,18]. The 6th column lists the parent Hamiltonians, see eq. (2). The 8 phases listed in the first 3 rows ($m = -3, -1, 0$) can be mapped to the 8 phases of interacting fermions with $P$ and $T$ symmetries, see table 2. The last 2 phases correspond to fermionic phases that spontaneously break $T$.

| $I_x$ | $I_y$ | $I_xI_y$ | Symmetry group $G$ | Cohomology group $[14]$ | Parent Hamilton |
|-------|-------|----------|-------------------|------------------------|-----------------|
| yes   | yes   | yes      | $Z_2^2 \times Z_2^2$ | $Z_2 \times Z_2$       | $m = 0, \pm 2, 4$ |
| yes   | no    | no       | $Z_2^2$           | $Z_2$                  | $m = 1, -3$     |
| no    | yes   | no       | $Z_2^2$           | $Z_2$                  | $m = -1, 3$     |
| no    | no    | yes      | $Z_2$            | $Z_1$                  | nematic         |
| no    | no    | no       | $Z_1$            | $Z_1$                  | canted          |

From non-interacting fermions to spin chains. –

Our strategy is based on the Jordan-Wigner transformation of known topological phases of 1d fermions. This transformation was first introduced to solve the quantum Ising model, by mapping it to a 1d system of non-interacting fermions [24,25]. At a critical value of the Ising coupling, the model undergoes a quantum phase transition, where the ground-state degeneracy changes from 1 (paramagnet) to 2 (ferromagnet). In the fermionic language, this is due to the appearance of zero-energy Majorana edge modes [26,27]. This fermionic system is currently referred to as a Kitaev chain and has become very popular due to its implications for topological quantum computing (see ref. [28] for an introduction).

Further topological phases of 1d fermions can be obtained by stacking in parallel several Kitaev chains [29,30]. Reordering the site indexes, this is equivalent to a single chain with longer-range couplings (see fig. 1):

$$H_{0}^{(m)} = \sum_{i} (\psi_i^\dagger \psi_{i+m} + \psi_{i+m}^\dagger \psi_i) + \text{sign}(m)(\psi_i^\dagger \psi_{i+m} + \psi_{i+m}^\dagger \psi_i).$$

(2)

Here the topological index $m \in \mathbb{Z}$ determines the number and flavor of the Majorana modes at each edge\(^2\). Due to the presence of zero-energy Majorana modes, each eigenstate of eq. (2) is $2^{|m|}$-fold degenerate\(^3\).

The models (2) have both time-reversal ($T^2 = +1$) and particle-hole ($P^2 = +1$) symmetries. Thus, according to

\(^2\)Majorana fermions appear in two flavors, often denoted by $a$ and $b$: one is the time-reversed conjugate of the other. Note that negative $m$'s correspond to Kitaev chains with inverted site indexes ($i \rightarrow -i$).

\(^3\)The phase $m$ has $2|m|$ Majorana modes ($m$ per side), which can be combined in a set of $|m|$ complex fermions. Note that in the non-interacting case, each eigenstate has the same degeneracy.

Fig. 1: Real-space representation of topological phases of non-interacting fermions in 1d, eq. (2): the trivial phase ($m = 0$) is site factorizable. The phases with $m = \pm 1$ are Kitaev chains with one Majorana mode at each edge. The phases with $|m| > 1$ can be constructed by stacking in parallel several Kitaev chains. For the phase $m = 2$, two equivalent graphical representations are given.

The classification of topological phases of non-interacting fermions (topological insulators and superconductors), they belong to the class BDI [5–7,31,32]: In 1d, this symmetry group gives rise to $\mathbb{Z}$ distinct topological phases, one for each possible value of $m$.

We now use eq. (2) to construct parent Hamiltonians for the topological phases of spin chains. By applying the Jordan-Wigner transformation, one obtains the integrable Hamiltonians:

$$H_0^{(m)} = -\sum_{i} \left\{ \sigma_i^x \left( \prod_{j=1}^{i-1-m} \sigma_j^z \right) \sigma_i^{z+m}, \quad m > 0, \right\}$$

$$\left\{ \sigma_i^y \left( \prod_{j=1}^{i-1-m} \sigma_j^z \right) \sigma_i^{y+m}, \quad m < 0, \right\}$$

where $\sigma_i^x, \sigma_i^y, \sigma_i^z$ are Pauli matrices. These Hamiltonians were first introduced by Suzuki [33] and are known as generalized XY models, or generalized cluster Ising models [30,34–42]. The $m = 2$ model, $H(2)^{(2)} = -\sum_i \sigma_i^x \sigma_i^y \sigma_i^{z+2}$, is of particular interest for quantum computing: Its ground state, known as the one-dimensional cluster state [43], offers the possibility to realize universal one-qubit gates via measurement based quantum computations [44–46]. The Hamiltonians of the models with $-4 < m \leq 4$ are explicitly shown in the 2nd column of table 2. Importantly, the $T$ and $P$ symmetries of the fermionic Hamiltonians imply the two Ising symmetries of eq. (1) for the spin models.

The two duality transformations. – To characterize the phases of the cluster Ising models, it is useful to introduce two duality transformations [34,38]. These transformations reflect the symmetry of the model: they map a magnetic field in the $x$-direction to Ising couplings in the $z$- and $y$-direction, respectively. Having introduced the string operator

$$S_i^z = \prod_{j \leq i} \sigma_j^z,$$

we can formally define the duality transformations

$$D_z: \sigma_i^x \rightarrow \sigma_i^z \sigma_i^{z+1}, \quad \sigma_i^y \rightarrow S_i^z;$$

(5)
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Fig. 2: Configuration-space representation of topological phases of spin-(1/2) chains, eq. (3). The phases are connected by the duality transformations $D_z$ and $D_y$, eqs. (7) and (8). The odd-$m$ phases (green) are characterized by a local order parameter, and the even-$m$ phases (light blue) by a non-local order parameter — see table 2 for more details. The 8 phases shown in this graph cannot be adiabatically connected even in the presence of integrability-breaking terms.

\[
D_y: \sigma_i^y \rightarrow \sigma_i^y \sigma_{i+1}^y, \quad \sigma_i^y \rightarrow S_i^z. \tag{6}
\]

By definition, $D_z$ and $D_y$ map $H^{(0)}$, respectively, to $H^{(1)}$ and $H^{(-1)}$. What happens if we now apply $D_z$ to $H^{(1)}$ and $H^{(-1)}$? By using the identity $\sigma_i^y = i\sigma_i^z\sigma_i^x$, we obtain that $D_z$ transforms $\sigma_i^y \sigma_i^y \rightarrow -\sigma_i^z \sigma_{i+1}^z$, or equivalently $H^{(1)} \rightarrow -H^{(2)}$, and vice versa. By extending this argument to finite $m \neq 0$, we find the relations summarized in fig. 2:

\[
D_z: H_0^{(2m)} \rightarrow -H_0^{(1-2m)}, \tag{7}
\]
\[
D_y: H_0^{(-2m)} \rightarrow -H_0^{(-1+2m)}. \tag{8}
\]

This is the first main result of this paper: by a subsequent alternation of $D_z$ and $D_y$ we can generate all the $\mathbb{Z}$ phases of the non-interacting model.

We now use the duality transformations to derive the order parameters of the different phases. Our starting point is the ferromagnetic model $H^{(1)}$, whose order parameter is $O^{(1)} \equiv \sigma^z$. By applying subsequently the duality transformations (5) and (6), we obtain the order parameters listed in the 3rd column of table 2. This list allows us to detect which symmetries are spontaneously broken in each phase (4th column). In general, we find that two phases break the same symmetry if and only if their $m$ indices differ by a multiple of 4. In particular, the phases $m = 1$ and $m = -3$ are ferromagnets in the $z$-direction: both their order parameters break the symmetry $I_z: \sigma^z \rightarrow -\sigma^z$.

The phases with even $m$'s have non-local (string) order parameter. String orders were extensively discussed in the context of topological phases of spins [21,47–51], bosons [19], and fermions [52]. In particular, it was shown that string orders exist only in systems with unitary symmetries [50]. Indeed, our model is symmetric under rotations of $\pi$ around the $x$-axis (which is equivalent to the subsequent application of the two Ising symmetries (1)).

The order parameters $O^{(m)}$ can be used to rewrite the generalized cluster models (3) in an appealing way:

\[
H_0^{(m)} = -\sum_i O_i^{(m)} O_{i+1}^{(m)}. \tag{9}
\]

Because the order parameters $O^{(m)}$ at different sites commute, $[O_i^{(m)}, O_j^{(m)}] = 0$, these operators form a complete set of integrals of motion, with eigenvalues $\pm 1$ (see footnote 5). Note that, in a system of size $L$, the Hamiltonian (9) has $L - |m|$ terms. This leads to a ground-state degeneracy of $2^{|m|}$, in agreement with the fermionic analysis. This degeneracy has two distinct origins: symmetry-breaking and edge states. For example, the ground state of $H^{(m=1)}$ is doubly degenerate due to the spontaneous breaking of the $I_z$ symmetry. The ground state of $H^{(m=-3)}$ has additionally two edge modes, $\sigma_i^y$ and $\sigma_{i+1}^y$, which commute with all $O_i^{(m)}$ and lead to a total degeneracy of $2^3 = 8$ (see footnote 6). As expected, these edge states are invariant under the unbroken $I_y$ symmetry.

The duality transformations $D_z$ and $D_y$ are not only convenient theoretical tools to describe the SPT phases, but also have practical implications in the context of quantum computers. These many-body quantum systems are not characterized by an Hamiltonian, whose ground state can be reached by cooling to low temperatures. Instead, a quantum computer is initially prepared in a simple product state, usually $|\psi\rangle = \prod_i |0\rangle$, on which one-qubit and two-qubit unitary operations can be applied. Reference [53] demonstrated experimentally that using Hadamard (H) and controlled-Z (CZ) gates, one can prepare the ground state of the cluster Ising model, $m = 2$. Their algorithm involves two steps: in the first step one applies H gates to each qubit independently and obtains the ground state of the $m = 0$ Hamiltonian. Next, one applies CZ gates on all neighboring spins to obtain the ground state of the $m = 2$ cluster Ising model. This second step is equivalent to the transformation $D_z D_y$, which indeed transforms the Hamiltonian $H_0^{(0)}$ to $H_0^{(2)}$ [54].

Effects of integrability-breaking terms. — The generalized cluster models of eq. (2) have an infinite number of local order parameters $(O^{(m)}$ with odd $m$). However, when we consider the general class of models with two Ising symmetries, the number of order parameters must be reduced to two, one for each symmetry. To understand how this happens, we consider the effects of generic perturbations that preserve the two Ising symmetries (1). Under the Jordan-Wigner transformation, these terms are mapped to local interactions among the fermions: For example, $\sigma_i^z \sigma_{i+1}^z$ is mapped to the quartic term $\psi_i \psi_i \psi_{i+1} \psi_{i+1}$. In general, these fermionic terms preserve both $T$ (they have a real coefficient) and $P$ (they include an even number of fermionic operators). This type of interactions was

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Footnote:

4This can be verified either by a direct evaluation, or by noting that the order parameters at different sites are generated by the application of series of duality transformations to $\sigma^z$ operators belonging to different sites.

5Their eigenvalues can be determined by noting that $(O_i^{(m)})^1 = O_i^{(m)}$ and $(O_i^{(m)})^2 = 1$.

6Note that the physics of edge modes is not captured by the duality transformations (7) and (8), which only apply to infinite systems with no boundaries.
Table 2: Physical properties of 8 phases of spin chains with two Ising symmetries. In the 2nd and 3rd columns, each operator acts on a neighboring site: for example $H_0^{(m)} = - \sum_i \sigma^x_i \sigma^z_{i+1} \sigma^x_{i+2} \sigma^z_{i+3} \sigma^y_{i+4}$ and $O^{(3)} = \sigma^y_i \sigma^z_{i+1} \sigma^y_{i+2} \sigma^z_{i+3} \sigma^x_{i+4}$. Here $\lambda$ is defined in eq. (4). The 4th column indicates which symmetry is broken by the order parameter: for example the $m = 3$ phase breaks the $I_y$: $\sigma^y \rightarrow -\sigma^y$ symmetry. The last two columns refer to non-integrable models within the same phase: the topologically non-trivial phases are characterized by one zero-energy mode per edge.

| $m$ | $H_0^{(m)}$ : parent Hamiltonian | $O^{(m)}$ : order parameter | Broken symmetry | GS degeneracy | Num. edge modes |
|-----|---------------------------------|-----------------------------|----------------|---------------|---------------|
| 4   | $\sigma^x \sigma^y \sigma^z \sigma^x \sigma^z$ | $S^x \sigma^y \sigma^x \sigma^z$ | $\emptyset$ | 4             | 1             |
| 3   | $\sigma^x \sigma^y \sigma^z \sigma^x \sigma^z$ | $\sigma^y \sigma^x \sigma^z$ | $I_y$ | 8             | 1             |
| 2   | $\sigma^x \sigma^y \sigma^z$ | $S^y \sigma^x \sigma^z$ | $\emptyset$ | 4             | 1             |
| 1   | $\sigma^x \sigma^z$ | $\sigma^z$ | $I_z$ | 2             | 0             |
| 0   | $\sigma^x$ | $S^x$ | $\emptyset$ | 1             | 0             |
| $-1$ | $\sigma^y \sigma^x$ | $\sigma^y$ | $I_y$ | 2             | 0             |
| $-2$ | $\sigma^y \sigma^x \sigma^y$ | $S^x \sigma^x \sigma^y$ | $\emptyset$ | 4             | 1             |
| $-3$ | $\sigma^y \sigma^x \sigma^y \sigma^y$ | $\sigma^y \sigma^x \sigma^y$ | $I_z$ | 8             | 1             |

shown to adiabatically connect two phases if (and only if) their topological indexes differ by an integer multiple of 8. Thus, the number of topologically distinct phases is reduced to 8 [29,55]. In the spin language, this leads to the 8 distinct phases listed in table 2, and labeled by $m = -3,...,4$. The number and the symmetry of these phases coincide with the expectations based on the classification presented in table 1.

Interactions affect the order parameters as well and, in general, couple all order parameters with the same symmetry. To exemplify this effect, let us consider the parent Hamiltonian $H^{(1)}$ whose ground state is a perfect ferromagnet in the $z$-direction $|\psi_{GS}\rangle = \prod_i |\sigma^z_i = +1\rangle$. According to the first-order perturbation theory, the effect of the interaction term $V_2 = \sigma^x_i \sigma^z_{i+2}$ is proportional to $(O^{(-3)}_i) \approx \langle \psi_{GS}|\sigma^y_i \sigma^x_{i+1} \sigma^y_{i+2} V_2 |\psi_{GS}\rangle + \text{c.c.} \neq 0$. This calculation shows that interaction can lead to a finite order parameter $O^{(-3)}$ in the phase $m = 1$.

To study the fate of the order parameter $O^{(1)} = \sigma^z$ in the phase $m = -3$, we consider a model that interpolates between the parent Hamiltonians $H^{(1)}_0$ and $H^{(-3)}_0$:

$$H(\lambda, V) = \sum_i (1 - \lambda) \sigma^x_i \sigma^z_{i+1} + \lambda \sigma^y_i \sigma^x_{i+1} \sigma^z_{i+2} \sigma^y_{i+3} + V \left( \sigma^x_i \sigma^y_{i+1} \sigma^z_{i+2} \sigma^y_{i+3} + \sigma^y_i \sigma^z_{i+1} \sigma^y_{i+2} \sigma^z_{i+3} \right).$$

(10)

Here $V$ is a term that breaks the integrability of the model, but preserves the two Ising symmetries. Figure 3 shows the numerically calculated excitation gap and the order parameter $O^{(1)}$ as a function of $\lambda$ (see footnote 7).

The key observation is that although the excitation gap (necessarily) closes when going from one phase to the other, the order parameter does not vanish at the transition. This finding shows that the phases $m = 1$ and $m = -3$ are locally indistinguishable, and their distinction is of topological nature. Thanks to the duality transformations, we can extend this argument to any two phases with $m$ indexes that differ by a multiple of 4. We deduce that as expected, two phases share the same local order parameters, if and only if they break the same symmetry.

To determine the number of edge states in the SPT phases, we need to compare the degeneracy of the spin chains with open and closed boundary conditions. In the former case, the spin chains can be mapped to one-dimensional fermionic models. In these models, interactions are known to lift some of the Majorana degeneracies, and in particular, change the ground-state degeneracy of the fermionic model with 4 Majorana edge modes ($m = 4$) from $2^4 = 16$ to 4 [55,56]. The degeneracy of the phases with $|m| \leq 3$ are left unchanged, as summarized in table 2. In the case of periodic boundary conditions, no edge states are present and the ground-state degeneracy of the spin model is reduced to 1, for phases with no broken symmetry, or 2, if one Ising symmetry is broken. We conclude that in the topologically trivial phases ($m = 0, \pm 1$) the ground-state degeneracy of systems with open boundary conditions is equal to the case of periodic boundary condition. In contrast, in the topologically non-trivial phases ($|m| > 1$), the former is 4 times larger than the latter. This observation is consistent with the presence of spin-(1/2) edge modes in all three topologically non-trivial phases, $m = \pm 2, 4$.

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7The excitation gaps were computed by addressing the lowest states and selecting the energy of the first non-degenerate excited state. Note that at the transition, the ground-state degeneracy changes from 2 to 8. The order parameter is defined as the square root of the ground-state correlation functions for distant spins. Figure 3 shows the result for a chain of size $L = 128$. We verified that no significant finite-size effect is observed, with the exception of the critical point $\lambda = 0.5$. The calculations were performed using the ITensor C++ Library version 1.2, http://itensor.org.
Fig. 3: (a) Energy gap and (b) expectation value of the order parameter \(O^{(1)} = \sigma^x\) across the transition \(m = 1 \rightarrow m = -3\), for the integrable model \((V = 0)\), and in the presence of an integrability breaking term \((V = 0.3)\), see eq. (10). The phases \(m = 1\) and \(m = -3\) break the symmetry \(\sigma^x \rightarrow -\sigma^x\), but are topologically distinct thanks to the Ising symmetry \(I_y: \sigma^y \rightarrow -\sigma^y\).

**General classification.** We now connect the previous findings with the general classification of SPT phases of interacting bosons \([14,17]\). As mentioned in the introduction, the Ising symmetries of eq. (1) are anti-unitary, and can be written as the product of a unitary transformation (a \(\pi\) rotation around the \(z\)- or \(y\)-axis, respectively) and the spin time-reversal symmetry \((\vec{\sigma} \rightarrow -\vec{\sigma})\).

Indeed, under the Ising symmetry, the commutation relation of the spins operators changes sign and becomes \([\sigma^x, \sigma^y] = -2i\sigma^z\). Thus, the class of model under present consideration has two \(Z_2^T\) symmetries, or equivalently a \(Z_2^T \times Z_2^T\) symmetry.

This observation shows how to extend our analysis to generic models of interacting bosons, given in terms of bosonic creation and annihilation operators, \(b\) and \(b^\dagger\). To express the two Ising symmetries in the bosonic language, we consider the approximate mapping \(\sigma^x = 1 - 2b^\dagger b\), \(\sigma^y \approx i(b - b^\dagger)\), and \(\sigma^z \approx b + b^\dagger\). This mapping becomes exact in the hard-core limit, where \(b^\dagger b = 0.1\) and \(\sigma^z = \pm 1\). The hard-core constraint is equivalent to the addition of a term \(U(b^\dagger b - 1/2)^2\), with \(U \rightarrow \infty\), and thus, can be reached without affecting the symmetries of the model. Using this mapping we obtain that the two Ising symmetries map to the \(P\) and \(T\) bosonic symmetries, where \(T\) is the time-reversal symmetry (i.e., the requirement that the Hamiltonian can be written using creation and annihilation operators with real prefactors only) and \(P\) is the parity (i.e., the conservation of the parity of the total number of bosons). As shown in table 1, one-dimensional models with two Ising symmetries include 10 distinct phases (see table 1): When both symmetries are preserved, one obtains 4 topological phases \([14,18]\). We identify these phases as the \(m = 0, \pm 2\), and 4 phases of table 2. An interesting question involves the identification of the Haldane phase of spin-1 chains, whose Heisenberg point belongs to our class of models. As shown by ref. \([29]\), this model is topologically equivalent to 4 parallel Kitaev chains\(^8\), i.e., to the parent Hamiltonian \(H_0^{(4)}\). This result can be rationalized as follows: the Heisenberg model is symmetric under any rotation, and in particular under the rotation of \(\pi/2\) around the \(x\)-axis: This symmetry interchanges \(z\) with \(y\) and corresponds to \(m \rightarrow -m\). Recalling that the phases \(m\) and \(m = 8\) are topologically equivalent, we arrive to the conclusion that the Heisenberg model must belong to the \(m = 4\) phases, where \(m = 8 = -4\).

If one of the two Ising symmetries is broken, the problem is equivalent to the class of models with a single \(Z_2^T\) symmetry, which has 2 distinct topological phases \([16,17,20-22,51]\). This observation explains why our model has two distinct phases with ferromagnetic order in the \(z\)-direction, \(m = 1\) and \(m = -3\) (2nd row of table 1). Because these two phases are topologically protected by an anti-unitary symmetry, they cannot be distinguished by both local and string order parameters \([20,50]\). A similar argument holds for the two ferromagnets in the \(y\)-direction (3rd row of table 1). These findings imply that the 8 phases listed in table 2 (corresponding to the 8 phases of interacting fermions with unbroken \(P\) and \(T\) symmetries) faithfully represent the first 3 rows of table 1 (phases of spin chains with at least one unbroken Ising symmetry).

The remaining two phases of table 1 can be constructed by considering fermionic models that spontaneously break \(T\). For example, this occurs if the operator \(O_T = i(\psi_1^\dagger + \psi_1)(\psi_{i+1}^\dagger + \psi_{i+1})\) acquires a finite expectation value. The presence of the \(P\) symmetry places these models in the topological class \(D\), which has 2 distinct topological phases. In the spin language, these two phases possess a finite expectation value of \(O_T = \sigma^x\sigma^y\). This gives rise to two possible phases: a nematic phase with one order parameter \(\langle \sigma^x\sigma^y_{i+1} \rangle \neq 0\), and a canted ferromagnet with two order parameters, \(\langle \sigma^x_i \rangle \neq 0\) and \(\langle \sigma^y_i \rangle \neq 0\). Note that the former phase has a residual \(Z_2\) symmetry (rotations of \(\pi\) around the \(x\)-axis), which is however insufficient to protect further topological phases.

**Summary and conclusions.** In summary, we discussed the general phase diagram of spin chains with two Ising symmetries. This class of models is equivalent, under the Jordan-Wigner transformation, to 1d fermions with \(P\) and \(T\) symmetries. Accordingly, we find that this class of models has 10 distinct phases. The physical properties of the 10 phases of the fermions and of the spins are, however, very different. In particular, in the fermionic language 2 phases spontaneously break \(T\), and the other 8

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\(^8\)Note that this result is in contradiction with an explicit statement of ref. \([57]\), who identified the Haldane phase with the cluster Ising model \(H_0^{(2)}\). This point requires further investigation.
are of topological nature and do not break any symmetry. In contrast, in the spin language only 4 phases preserve both Ising symmetries, 4 phases break one of them, and 2 phases break both (see table 1).

Although we have not explicitly used this property, the parent Hamiltonians (3) are translationally invariant. As shown in refs. [14,17,56], this symmetry can double the number of distinct phases. In the case of an Ising coupling $\sigma_i^z \sigma_{i+1}^z$, for example, translationally invariant models can be classified as ferromagnets (if the prefactor is negative) or anti-ferromagnets (if the prefactor is positive). Similarly, the number of parent Hamiltonians (3) can be doubled by considering positive prefactors. Possibly, this scenario can give rise to further SPT phases. Other interesting routes that are open by our work include the effects of periodic drives [42,58], disorder [59,60], or both [61,62], as well as the extension of our study to 2 dimensions, where the classification of SPT phases predicts several yet-to-be-explored topologically non-trivial phases.

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Additional remark: During the preparation of this manuscript we became aware of a related study now published in ref. [57].

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