Ring structure, uniform expressions and intersection homology

Jonathan Fine

11 May 1998

Abstract

Although intersection homology lacks a ring structure, certain expressions (called uniform) in the intersection homology of an irreducible projective variety $X$ always give the same value, when computed via the decomposition theorem on any resolution $X_r \rightarrow X$. This paper uses uniform (and non-uniform) expressions to define what is believed to be the usual intersection homology (and its local-global variant) of a convex polytope (or a projective toric variety). Such expressions are generated by the facets, and so may lead to necessary numerical conditions on the flag vector. Most of the concepts, however, apply to more general algebraic varieties, and perhaps some other situations also.

1 Introduction

This is the third of a series of papers devoted to intersection homology and the combinatorics of convex polytopes. The first [9] gave a topological definition of the new local-global intersection homology groups, and also gave the expected value of their Betti numbers, for projective toric varieties (or equivalently convex polytopes). The second [10] constructed for each such group a complex which, if exact at all but a single specified location, would produce a vector space whose dimension is the expected value of the Betti number. More briefly, the paper gave both the Betti numbers and the topological definition, while the second gave the linear algebra construction for the homology groups. As proofs of these statements have not been sought, they are at present only conjectures.

The paper in some sense gives the geometric meaning of local-global homology, at least in the context of convex polytopes. It takes the view that homology arises as the result of studying the calculation of something, and of the properties of such a calculation. Historically, homology arose out of the study of intersection numbers (or, if one prefers, the Schubert calculus), and was then placed by Poincaré into its now well-known topological form. This approach, however, does not tell us what is the something that homology is calculating. This paper presents homology as a consequence of the study of volume, and its dependence on the conditions defining the object being studied. Such an approach works well for convex polytopes but not, as noted in the final section, for more general varieties.

This paper, together with its two companions, give the leading definitions for the theory of local-global (intersection) homology, and also gives conjectures for many of the main results that will hold, when this theory is applied to convex polytopes. (The major lack is any understanding of how the various homology groups should be equipped with a ring-like structure, that reduces to the usual one when the object is nonsingular.) The proofs are expected to be recursive, and involve...
a deeper knowledge of the subject than we have at present. The main purpose has been to present what are believed to be the landmarks, and not to describe the paths between them, nor to explore the intervening territory. Throughout convex polytopes have been studied, because that are the least complicated objects to which such a theory will apply. By and large, this paper can be read independently of the other two.

Convex polytopes satisfy subtle combinatorial inequalities, whose proof is essentially geometric in nature. The discovery of such results has been a central motive for these papers. (Other objects, such as graphs and hypergraphs, will similarly satisfy subtle combinatorial inequalities, but except for a few words in the final section, this matter is not taken up again in this paper. In [8, 11, 12] the author introduces what he believes is the proper concept of a flag vector for use in this new context.)

A central result in the combinatorics and geometry of convex polytopes is the proof of the necessity (Stanley [18]) and sufficiency (Billera and Lee [5]) of McMullen’s conjectured conditions [15] on the face vector of simple polytopes. This paper studies general polytopes. When $\Delta$ is a simple convex polytope there is a homology ring $H_* \Delta = \bigoplus_{i=0}^n H_i \Delta$, whose properties imply McMullen’s conditions. The main goal of this paper is to define a similar object for general polytopes. This new object will be based on the theory of intersection homology. Although formulated for convex polytopes, the definitions also apply to any irreducible projective variety.

In some sense the central problem is this. When $\Delta$ is simple the homology object is a ring $H_* \Delta$, generated over $H_n \Delta \cong R$ by $H_{n-1} \Delta$, or in other words the facets of $\Delta$. Although intersection homology does not supply a ring structure, one wishes nonetheless to make a similar statement in the general case. The proposed solution to this problem is the concept of a uniform expression. We will explain this first in the context of algebraic varieties.

Suppose that $X$ is an irreducible projective variety of real dimension $m = 2n$. Suppose also that $i + j = m$. In that case (middle perversity) intersection homology provides a pair of homology groups, which we will denote by $H_i X$ and $H_j X$, together with the pairing

$$\langle \cdot, \cdot \rangle : H_i X \otimes H_j X \to H_0 X \cong R$$

that is, by Poincaré duality, nondegenerate. If $X$ were nonsingular, then for any $i$ and $j$ there would be a product map

$$\langle \cdot, \cdot \rangle : H_i X \otimes H_j X \to H_k X$$

that gives the homology ring structure. Here $(m - k) = (m - i) + (m - j)$.

Now suppose that $X_1 \to X$ is a resolution of the singularities of $X$. According to the decomposition theorem [2], there is now an inclusion $H_i X \hookrightarrow H_i X_1$ of the (intersection) homology groups of $X$ into the ordinary homology of $X_1$ (which is the same as its intersection homology, because $X_1$ is nonsingular). The decomposition theorem is a deep result, and the exact nature of the inclusion it provides is not well understood. Nonetheless, it exists, and we can use it. (In fact, we must assume that $X_1 \to X$ is a resolution of $X$ together with what is known as a relatively ample line bundle. But this detail does not affect the exposition.)

We can now use the decomposition theorem, together with Poincaré duality, to provide a ring-like structure on $H_* X$. First suppose $i + j + k = m$, and that $\psi$, $\eta$ and $\xi$ are homology classes on $X$ with dimensions $i$, $j$ and $k$ respectively. Now let

$$X_1 \sim \psi \sim \eta \sim \xi \in H_0 X_1 \cong H_0 X \cong R$$

denote the result of lifting the classes to $X_1$, and then computing their intersection number. The value will of course depend on the choice of $X_1 \to X$. 
Now think of $\psi$ and $\eta$ as fixed, and $\psi$ as variable. The above expression is a linear function of $\psi$ and so, by Poincaré duality, it can be represented by an intersection homology class in $H_{m-k}X$, which we can denote by $\psi \smile_1 \eta$. As noted, it will in general depend on the choice of $X_1 \to X$. It does not seem likely that such a product will be associative, although it may have some quasi-associative properties.

Now say that an expression $\zeta = \psi \sim \eta$ (or more generally $\zeta = \psi_1 \sim \ldots \sim \psi_s$) in the (intersection) homology classes of $X$ is uniform if the evaluation of

$$X_r \sim \zeta \sim \xi \in H_0X_r \cong H_0X$$

does not depend on the choice of a resolution $X_r \to X$, where $\xi$ is an arbitrary cycle in $H_\bullet X$, of complementary dimension. The same concept will also be applied to sums of products of cycles, provided each term in the expression has the same degree. Clearly, it follows from Poincaré duality that each uniform expression determines an (intersection) homology class, provided we assume as we do that $X$ has at least one resolution $X_r \to X$.

The concept of uniform expression can now be used to make precise the statement that the homology object $H_\bullet \Delta$ of a general convex polytope $\Delta$ is generated, as in the simple case, by $H_{n-1}\Delta$. In fact, because the inclusion provided by the decomposition theorem is in general still rather obscure, we will reverse the process, and define the (intersection) homology groups $H_i\Delta$ of a general $\Delta$ to be certain uniform expressions, modulo the relations induced by evaluation (or Poincaré duality). This will be done in the next section.

Expressions that are not uniform can also be useful. They are capable of recording information about the singularities of $X$. (Uniform expressions are by definition incapable of doing this.) Consider for example the expression

$$(X_1 - X_2) \sim \psi \sim \eta \sim \xi \in H_0X,$$

where $X_1 \to X$ and $X_2 \to X$ are two distinct resolutions of $X$. If $\psi \sim \eta$ is not uniform then this expression can be non-zero. Its value will be concentrated, so to speak, along the region of $X$ at which the two resolutions differ. In §3 a scheme will be presented that records such information in what are believed to be variant forms of the local-global intersection homology groups that the author has introduced elsewhere [9, 10].

2 Volume and homology

There is a connection between the homology and the volume of a simple convex polytope, that deserves to be more widely known. It leads to a presentation of the homology theory, equivalent to the usual one [3], which is close in spirit to the concept of uniform expressions.

Throughout $\Delta$ will be an $n$-dimensional convex polytope in an $n$-dimensional affine space, described by an irredundant system

$$\alpha_i v \geq 0 \quad i = 1, \ldots, f_{n-1}$$

of affine linear inequalities, one for each facet $\delta_i$ of $\Delta$. This writes $\Delta$ as an intersection of half-spaces. We now subject the half-spaces to a small displacement, and consider how the volume changes. In other words, let $\Delta(\epsilon)$ be the convex polytope

$$\alpha_i v \geq \epsilon_i \quad i = 1, \ldots, f_{n-1}$$

where the $\epsilon_i$ are close to zero.
This process of $\epsilon$-variation will in general change the combinatorial structure of $\Delta$. The cone (or pyramid) on a square is an example of this. It will be studied further, in the course of this paper. If the $\epsilon$-variation process does not change the combinatorics (for $\epsilon$ close to zero of course) then $\Delta$ will be said to be *simple*. This happens exactly when at each vertex there are exactly $n$ facets (or equivalently $n$ edges), which is the usual definition of simple.

Now suppose that $\Delta$ is a simple polytope. From this it follows without any real difficulty that the volume $\text{vol}(\epsilon)$ of $\Delta(\epsilon)$ is a polynomial of degree $n$ in the linear quantity $\epsilon$, at least for $\epsilon$ small. (To prove this, decompose $\Delta(\epsilon)$ into pyramids, one for each facet. The height of each such pyramid will be linear in $\epsilon$, while by induction the area of the base will be a polynomial of degree $n - 1$ in $\epsilon$. For general polytopes the volume $\text{vol}(\Delta)$ will be given by one of a number of polynomials, one for each simple structure reached via small $\epsilon$-variation.)

The polynomial $\text{vol}(\Delta)$ is perhaps complicated, but it is far from arbitrary. It has a great deal of structure. For example, to know the top degree term $\text{vol}_\Delta$ is equivalent to knowing the homology ring $H_\bullet \Delta$, a fact that we will explain shortly. The lower degree terms give information about the Chern classes of $\Delta$ (or of the projective toric variety $P_\Delta$ associated to $\Delta$). This we do not need, and so will not discuss further.

The top degree term $\text{vol}_\Delta$ can be written as a symmetric multilinear form, also to be denoted by $\text{vol}_\Delta$, in $n$ variables $\alpha_1, \ldots, \alpha_n$. Each $\alpha_i$ is a (small) $\epsilon$-variation $\Delta$. The simplest case is where each $\alpha_i$ is a displacement of a single facet of $\Delta$. By linearity, these determine the general case, which is where the $\alpha_i$ are formal sums of displacements of individual facets. In any case, let $T^1 \Delta$ denote the linear span of all such $\alpha_i$. It has dimension $f_{n-1}$, the number of facets, and represents *thickenings* or $\epsilon$-displacements of the facets.

Now let $T^i \Delta$ denote the $i$-fold tensor product of $T^1 \Delta$ with itself. As already noted, $\text{vol}_\Delta$ can be thought of as a symmetric linear function on $T^n \Delta$. Now let $i + j = n$ and consider the map

$$\text{vol}_\Delta : T^i \Delta \otimes T^j \Delta \to \mathbb{R}$$

induced by $T^i \Delta \otimes T^j \Delta \cong T^n \Delta$. This is a pairing of vector spaces. Now form the null spaces

$$N^i \Delta = \{ \psi \in T^i \mid \text{vol}_\Delta(\psi, \eta) = 0 \text{ for all } \eta \in T^j \}$$

and $N^j \Delta$ similarly, and then form the quotient spaces $H^i = T^i / N^i$, $H^j = T^j / N^j$. The pairing now descends to give a nondegenerate (or perfect) pairing

$$H^i \Delta \otimes H^j \Delta \to \mathbb{R}$$

between a pair of vector spaces.

This definition of the (co)homology spaces $H^i \Delta$ is equivalent to the usual one. (To simplify matters, we will identify the cohomology group $H^i \Delta$ with the homology group $H_j \Delta$, where $i + j = n$. Thus, cohomology is homology, but indexed by codimension as a superscript, rather than dimension as a subscript.) The key facts are this. First, the (co)homology is generated by the facets. This means that $T^i \Delta$ will generate $H^i \Delta$. Second, Poincaré duality holds. This means that passing from (1) to (2) will produce (co)homology in the usual sense. Third, the volume of $\Delta$ and the volume of the toric variety $P_\Delta$ agree, provided $P_\Delta$ exists. More exactly, $\Delta$ determines on $P_\Delta$ a ‘hyperplane class’ or Kähler form $\omega$ such that

$$\text{vol}_{P_\Delta} = \int_{P_\Delta} \omega^n$$

is proportional to $\text{vol}_\Delta$. This is true for all $\Delta$, and so respects $\epsilon$-variation. Agreement here is enough to force agreement everywhere.
One can look at this in another way. Suppose $H^1 \Delta$ is known, and that it is generated by $H^1 \Delta$. The induced map
\[ H^1 \Delta \otimes \ldots \otimes H^1 \Delta \rightarrow H^n \Delta \cong \mathbb{R} \] (3)
on the $n$-fold tensor product is more-or-less equivalent to vol$_\Delta$ on $T^n \Delta$. (In fact, $H^1 \Delta$ is $T^1 \Delta$ modulo certain relations.) Because of Poincaré duality (and generation by $H^1 \Delta$), using $i + j = n$ to break this up into a pairing will produce $H^i$ and $H^j$. The induced map is however equivalent to knowing a homogeneous polynomial on $H^1 \Delta$ (or $T^1 \Delta$). In this way, the $\epsilon$-variation volume polynomial and the homology theory determine each other.

Here is a result that will be used to help do an example in the next section. Clearly, an $\epsilon$-variation that is in fact a rigid bodily translation of $\Delta$ will not change its volume. Thus, vol$_\Delta(\alpha_1, \ldots, \alpha_n)$ will be zero if any one of the $\alpha_i$ is of this form. (Each $\alpha_i$ is a formal sum of facet displacements.) Next, just as
\[ 2\langle x|y \rangle = \|x + y\|^2 - \|x\|^2 - \|y\|^2 \] (4)
turns a quadratic form into a bilinear form, so a similar process will produce the multilinear form vol$_\Delta$ from the polynomial form. It now follows that if each $\alpha_i$ is a displacement of a single facet $\delta_i$, and these facets $\delta_i$ have empty common intersection, then vol($\alpha_1, \ldots, \alpha_n$) is zero. For $n = 2$, this is because the $\delta_1$ and $\delta_2$ displacements do not, so to speak, interfere with each other, and so (3) will be zero. A similar argument holds in general.

In the usual approach Poincaré duality asserts that a pairing between already defined homology spaces is nondegenerate. In the present approach, duality is part of the definition. Poincaré duality then becomes a characterization of the nullspaces $N^i$ and $N^j$. It asserts that they are generated by the rigid displacement and empty intersection results stated in the previous paragraph.

Finally, in the present approach, the ring structure is automatic. Clearly there is a map $T^i \otimes T^j \rightarrow T^{i+j}$, and it is obvious that $N^i \otimes T^j$ and $T^j \otimes N^j$ both lie in $N^{i+j}$. Hence the tensor product descends to give a multiplication $H^i \otimes H^j \rightarrow H^{i+j}$ on (co)homology. Because all this is based on the well-defined concept of volume, there is no need to move cycles into general position and so forth.

3 Uniform expressions

When $\Delta$ is not simple, the polytopes produced by $\epsilon$-variation will in general have different combinatorial structures. The cone (or pyramid) on a square is the simplest example. Suppose that $\Delta$ is this polytope. Let the points of the compass $N$, $S$, $E$ and $W$ denote the triangular facets, and let $B$ denote the square base. Use the same symbols to denote thickenings of these facets, so chosen that $N - S$, $E - W$ and $N + E - B$ each represent a thickening due to a rigid displacement. (Such displacements do not change the volume.) Normalise so that $N \sim E \sim B$ has volume one. Now let $\Delta_{NS}$ denote the simple polytope that results from ‘squeezing in’ the $E$ and $W$ faces (or ‘letting out’ the $N$ and $S$ faces). Thus, $\Delta_{NS}$ is a ‘ridge roof’ polytope, with the ridge running $NS$. Similarly, let $\Delta_{EW}$ be the other ridge roof polytope obtained from $\Delta$ by the $\epsilon$-variation process.

As before, let $N$, $S$, $E$, $W$ and $B$ denote the previous thickened facets on $\Delta_{NS}$ and $\Delta_{EW}$, as well as on $\Delta$. That $N - S$, $E - W$ and $N + E - B$ represent trivial thickenings is still true. In the language of algebraic varieties, $\Delta_{NS}$ and $\Delta_{EW}$ are two different resolutions of $\Delta$. The $H_2$ groups of the three polytopes are naturally isomorphic, and this isomorphism is the inclusion given by the decomposition theorem. Analogous results will hold for all convex polytopes. That $\epsilon$-variation does not change ‘the geometry in codimension one’ allows us to use facets as a starting point for the use of the decomposition theorem.
On \( \Delta \), there are five generators for \( H_2 \Delta \) (the facets) and three relations \((N - S = E - W = N + E - B = 0)\). This leaves a two dimensional space, with generators \( N \) and \( E \) say. We will now look for uniform expressions. In other words, adapting the notation of the previous section, we seek \( \alpha \) in \( T^2 \Delta \) such that

\[
\Delta_{NS} \smallfrown N \smallfrown \alpha = \Delta_{EW} \smallfrown N \smallfrown \alpha
\]

and so on for the other facets. More exactly we seek an \( \alpha \) and a \( \beta \), both with this uniform property, that provide a basis for \( H_1 \) that is dual to the \{ \( N, E \) \} basis for \( H_2 \).

Here is an easy solution. Take \( \alpha = W \smallfrown B \) and \( \beta = S \smallfrown B \). In every case the intersection or homology calculation can be performed entirely on the base \( B \) of \( \Delta \), and so is insensitive to the choice of \( \Delta_{NS} \) or \( \Delta_{EW} \). Thus, \( \alpha \) and \( \beta \) are uniform expressions, that provide the sought for dual basis.

This solution is perhaps too easy. Duality is in essence a local matter. About the apex of \( \Delta \) both \( N \) and \( E \) represent non-trivial local cycles. However, because \( N + E - B \) is trivial, and because \( B \) is not local to the apex, the local cycles due to \( N \) and \( E \) are equal but opposite. We therefore would like a uniform expression that is both local to the apex and dual to \( N \).

The quantity \( \eta = E \smallfrown W - N \smallfrown S \) is a solution to this harder local problem. The verification proceeds as follows. First

\[
\Delta_{NS} \smallfrown N \smallfrown (E \smallfrown W - N \smallfrown S) = \Delta_{NS} \smallfrown N \smallfrown E \smallfrown W - 0 = 1
\]

because on \( \Delta_{NS} \) the facets \( N \) and \( S \) do not meet. Next

\[
\Delta_{EW} \smallfrown N \smallfrown (E \smallfrown W - N \smallfrown S) = 0 - \Delta_{EW} \smallfrown N \smallfrown N \smallfrown S
\]

because on \( \Delta_{EW} \) the facets \( E \) and \( W \) do not meet. To compute \( N \smallfrown N \smallfrown S \) on \( \Delta_{EW} \) use \( N = B - E \) to obtain

\[
\Delta_{EW} \smallfrown B \smallfrown N \smallfrown S - \Delta_{EW} \smallfrown E \smallfrown N \smallfrown S = 0 - 1 = -1
\]

(because \( B, N \) and \( S \) have empty intersection on \( \Delta_{EW} \)) and thus the equation

\[
\Delta_{NS} \smallfrown N \smallfrown \eta = \Delta_{EW} \smallfrown N \smallfrown \eta
\]

holds. The verification that

\[
\Delta_{NS} \smallfrown E \smallfrown \eta = \Delta_{EW} \smallfrown E \smallfrown \eta
\]

is left to the reader. (It also follows because \( N + E = B \), and any evaluation involving \( B \) is automatically uniform.)

Now let \( \Delta \) be an arbitrary convex polytope, as usual of dimension \( n \). In the simple case there was a single (multilinear) volume form \( \text{vol}_\Delta \), whose evaluation on \( T^i \otimes T^j \) (with \( i + j = n \)) induced the homology groups. In the general case we will define \( \text{vol}_\Delta \) to be the set \{ \( \text{vol}_r \) \} of multilinear forms, due to all the simple polytopes \( \Delta_r \) that arise from \( \Delta \) as a result of \( \epsilon \)-variation. We now seek subspaces \( U^i \) and \( U^j \) of \( T^i \) and \( T^j \), such that \( \text{vol}_\Delta \) is on \( U^i \otimes U^j \) a single valued function.

The task now is to define \( U^i \) and \( U^j \). Of course, once say \( U^i \) is known, one can define \( U^j \) to be all \( \xi \) in \( T^j \) such that \( \text{vol}_\Delta(\eta, \xi) \) is single valued, for any \( \eta \) in \( U^i \). One can then as in the simple case quotient by the null spaces \( N^i \) and \( N^j \) of \( U^i \otimes U^j \to \mathbb{R} \) to obtain a perfect pairing on \( H^i \) and \( H^j \).

The correct choice of the \( U^i \) and \( U^j \) is somewhat delicate. One would like in some sense to choose them so that \( H^i \) and \( H^j \) are as large as possible. In this way, they will contain as much information as they are able to. Enlarging \( U^i \), if it does not change \( U^j \), might enlarge \( H^i \). But
it might reduce $U^j$, and thus enlarge $N^i \subseteq U^i$. As noted, once say $U^i$ is known, the rest of the construction follows in a mechanical manner. Thus, the definition of $U^i$ is the central question of this section.

For $i = 1$ the correct value of $U^1$ is already known. It is the whole of $T^1$. Thus, $U^{n-1}$ consists of all $\xi$ in $T^{n-1}$ such that $\text{vol}_\Delta(\eta, \xi)$ is single valued, for any $\eta$ in $T^1$. For this construction, and indeed the whole of the uniform expression programme, to be correct, the resulting homology groups should have the desired dimension, which in this case is $f_{n-1} - n$. In other words, there should be enough uniform $\xi$.

In the present case of $U^{n-1}$ we can use the strong Lefschetz theorem to find such $\xi$. Choice of a point $p$ interior to $\Delta$ will determine a thickening $\omega_p \in T^1 \Delta$, namely the displacement that would take each of the facets from $p$ to where they actually are. Changing $p$ will add a rigid (and so trivial) displacement to $\omega_p$. We will think of $\omega_p$ as the unique hyperplane class or Lefschetz element $\omega$ in $T^1 \Delta$, although it is only when in $H^1 \Delta$ that it becomes unique.

The Lefschetz element $\omega$ has an important local property, and an important global property. The local property is that it is, in the language of algebraic geometry, a slice or a section through a variety. As a result, it is locally trivial, in the sense that at each vertex there is a (unique) trivial displacement that agrees with $\omega$ on the facets through that vertex. We will let $S^1$ denote all such locally trivial systems of thickened facets. Similarly, $S^i$ will be the $i$-fold tensor product of $T^1$. If $\Delta$ is simple, then $S^1 = T^1$, and vice versa. Even though in general two intersection homology classes cannot be multiplied together, they always can be, provided one of them is locally trivial.

The global property is the strong Lefschetz theorem. This tells us that for $i + j = n$ and $i \geq j$ the map

$$\omega^{j-i} : H^j \Delta \to H^i \Delta$$

is an isomorphism. This is at present known only for simple polytopes, general polytopes with rational vertices, and some other special cases. But it is reasonable to assume that it is true in general.

As noted, $U^1$ is $T^1$. Now let $U^{n-1}$ be the expressions (in $T^{n-1}$) that are uniform when paired against $U^1$. The first property of $\omega$ allows us to conclude that $\omega^{n-2} \xi$ is a uniform expression, for any $\xi \in U^1 = T^1$. (This follows because $\Delta$ is simple along its faces of dimension $n - 2$.) The second property allows us to conclude that there are enough such elements of $U^{n-1}$, to obtain dual spaces $H^1$ and $H^{n-1}$ of dimension $f_{n-1} - n$. (This requires either Poincaré duality for intersection homology, or better some further facts about the strong Lefschetz theorem. The required facts are, for $U^1$ and $U^{n-1}$, a consequence of Minkowski’s results [17] on facet area and ‘outward normal vectors’.) Thus, it is already known that at least in this case the uniform expression approach gives the correct answer. There is unlikely to be an easy proof of this result.

We now turn to $U^2$ and $U^{n-2}$, or more generally $U^i$ and $U^j$, for $i + j = n$ and $i \leq j$. We can suppose that $U^{i-1}$ is already known. If $\xi$ is to lie in $U^i$, the product

$$\xi \sim \omega^k \sim \eta \in T^n$$

must also be uniform, for any $\eta$ in $U^{i-1}$. Here, $k$ is $n - 2i + 1$. We will assume that this necessary condition is also sufficient, or in other words that it produces the correct definition of $U^i$.

The space $U^j$ of complementary dimension can be defined just as before, to be the expressions that are uniform when paired with expressions in $U^i$. This completes the definition except for the case $i = j$ (and $n$ is even). In this case $U^i$ and $U^j$ has better be equal. But this question arises already, with the other $U^j$. If $U^j$ has been correctly defined then expressions such as (5) must again be uniform, where now $\xi$ and $\eta$ are in $U^i$, and $k$ is $n - 2i$. This then is a wished for property of $U^j$. If it fails to hold, then the uniform expression programme also fails. The equality of $U^i$ and
$U^j$ for $i = j$ is a special case of this. Finally, the definitions of $U^0$ and $U^n$, and of $U^1$ and $U^{n-1}$, are special cases of this general scheme.

## 4 Local-global intersection homology

By design, the uniform expressions of the previous section are insensitive to the choice of a resolution (simple $\epsilon$-variation) of the object being studied. The definition of this section will record information about how the value of a (non-uniform) expression depends on the resolution chosen.

Here is an example. Let $\Delta$ be the cone on a square, and let $\Delta_{\text{NS}}, \Delta_{\text{EW}}, N, S, E, W$ and $B$ be as before. Now consider the expression $N \cong S \cong E$. This expression is not uniform. On $\Delta_{\text{NS}}$ it evaluates to zero, because the $N$ and $S$ facets do not meet. On $\Delta_{\text{EW}}$ it is 1. (Replace $N$ by the equivalent cycle $B - W$. On $\Delta_{\text{EW}}$ the facets $W$ and $E$ do not meet. Moreover, $B \cong S \cong E$ is equal to 1.)

The task is to organise such information in a sensible way. The author has already defined a theory of local-global intersection homology \cite{9,10} which will, usually behind the scenes, guide the definitions that follow. This theory provides, for each $n$-polytope, a system of $F_{n+2}$ ‘Betti numbers’, of which $F_{n+1}$ are linearly independent. These ‘Betti numbers’ are organised into $F_n$ sequences or strings. The Lefschetz operator $\omega$ goes from the homology group at one location in a string to the previous one (or next, depending on the indexing scheme). Here, $F_i$ is the $i$-th Fibonacci number. Bayer and Billera \cite{1} showed that the flag vectors of $n$-polytopes span a space whose dimension is $F_{n+1}$. The ‘Betti numbers’ are a re-encoding of information in the flag vector.

Now let $\Delta$ be any 3-polytope, and let $\Delta_1$ be any resolution (by which we mean of course, a simple $\epsilon$-variation). Now think of the expression $\Delta_1 \cong \alpha \cong \beta \cong \gamma$, $\alpha, \beta, \gamma \in U^1 = T^1$ as a linear function of $\gamma$. It will of course vanish for $\gamma \in N^1$, and so by Poincaré duality determines a class in $H_1\Delta = H^2\Delta$. (In the first section, this class was denoted by $\alpha \sim_1 \beta$.)

If $\Delta_1$ were replaced by another resolution $\Delta_2$, a different class might result. Now consider the expression

$$(\Delta_1 - \Delta_2) \cong \alpha \cong \beta \cong \gamma, \quad \alpha, \beta, \gamma \in U^1 = T^1$$

as a linear function of $\gamma$. Again it represents a class in $H_1\Delta$, but in contrast to the previous case it has an important geometric property. It is concentrated about, or local to, the locus in $\Delta$ above which the two resolutions $\Delta_1$ and $\Delta_2$ differ. In the present case (3-polytopes) this locus will be a finite set of points. Thus, global cycles that are constructed in this way satisfy certain locality properties. This is why they are called local-global cycles. They are neither purely local or purely global, but partake in the properties of both.

Now consider the octahedron. Such a local-global cycle can be found at each of its six vertices. The octahedron has $(1, 5, 5, 1)$ as its Betti numbers, and so there must be, when considered as global cycles, a relation between these six local-global cycles. (In fact there are two such relations.) These relations do not however respect the local aspect of the given local-global cycles. It is important, particularly in higher dimensions, to have a satisfactory definition of the equivalence of local-global cycles.

The solution to this problem is to introduce a dual theory. In other words, a bilinear pairing will be produced. As in the previous section the quotients by the null-spaces will be called the local-global homology groups. In contrast to the previous case, where $H^1$ and $H^2$ were paired with each other, here the pairing will be between compact and open local-global cycles. They are markedly different objects. The cycles just recently introduced are compact.
We continue to study the case $n = 3$. As already noted, any global cycle (thickened facet) $\gamma \in U^1 = T^1$ can be paired with a compact local-global cycle such as

$$(\Delta_1 - \Delta_2) \sim \alpha \sim \beta$$

(6)

but that there will not in general be enough such cycles. We can obtain more by relaxing the properties that $\gamma$ satisfies. This involves an understanding of how intersection numbers (products of homology cycles) can be calculated.

Suppose $\Delta$ is simple. Previously the intersection number of $n$ thickened facets was defined using the already well-defined polynomial $\text{vol} \Delta(\varepsilon)$. There is another method, which has been used in the examples. It is to use equivalence of cycles to move them into 'general position', and to then compute the well-defined volume of their common intersection (as thickened facets). This is the traditional method. It does not rely on the volume polynomial. That it produces a well-defined product is a consequence of two facts. The first is that cycles can always be moved into general position. This ensures that the product can be calculated in every case. The second fact is that however the calculation is performed, the same answer results.

The dual theory of open local-global cycles will be developed using this 'general position' approach to the pairing. When a product of $n$ thickened facets is in general position, the intersection takes place at a vertex. Suppose now that instead of having a global thickened facet $\gamma$, one has at each vertex $v$ a thickened facet $\gamma_v$. This is a more general concept, for $\gamma$ at $v$ need not be the same thickened facet as $\gamma$ at $w$. Such will be called a (vertex centered) system of open local cycles, as will be formal sums of such. Each (formal sum of) global thickened facets will determine such a system, although in general the converse is plainly false.

Recall that an intersection pairing will exist as a consequence of the two properties of computability and consistency. The first is essentially a local matter. To move a class from a vertex, add to it a suitable cycle that is equivalent to zero. This process can be used to produce a perhaps ill-defined pairing between compact cycles on $\Delta$ and vertex-centered open local cycles of complementary dimension, as defined in the previous paragraph. The cycles equivalent to zero are as in the usual theory, but understood as open local cycles.

In general, such a product will not be consistent. However, we wish to apply it not to all compact cycles, but only to those that arise in a particular way, namely as compact local-global cycles. Suppose such a set of cycles is given. For an open local cycle to give a consistent answer when evaluated against such a family of compact cycles is a global property of open local cycles. Such a cycle will be called, as one would expect, an open local-global cycle. The global cycles (of appropriate dimension) have this property. Because the open cycles are being paired with a restricted set of compact cycles, there will in general be many open cycles that do not arise from a global cycle. In fact, for $n = 3$ the consistency condition turns out to be vacuous. (In higher dimensions it is more subtle.)

There are some points that should be clarified. First, when computing the compact-open local-global pairing, any moving that is to be applied on the compact side should respect the local-global origin of the cycles. In other words, a cycle such as $[\delta]$ should be moved only by moving $\alpha$ and $\beta$, and not by the arbitrary moving of elements in $H_1\Delta_1$ and $H_1\Delta_2$. Second, on the open side the local model should be a uniform expression, and not an arbitrary product of thickened facets. (This seems to be demanded on aesthetic and logical grounds. This author does not know if this will affect the final outcome.) The third and final point is that the relation between compact and open is similar to that between $U^i$ and $U^j$, for $i + j = n$ and $i \geq j$. As noted, enlarging $U^i$ might reduce $U^j$ and hence reduce $H^i$. On the other hand, it might not change $U^j$, and can thus enlarge $H^i$. This phenomenon helps explain some of the subtle differences between various similar-looking local-global groups.
We will now assume that whenever a system of compact local-global cycles is defined, it will be paired with the space of open local-global cycles that it determines (as with $U^3$ and $U^5$), to produce both the compact and the open local-global groups. (Recall that this is done by factoring out the nullspaces.)

To complete this section we will exhibit, for $n \leq 6$, the appropriate spaces of compact local-global cycles. They all have the general form $(\Delta_1 - \Delta_2)\eta$, where $\eta$ is a non-uniform expression. As already noted, for $n = 3$ we take $\eta = \alpha \sim \beta \in T^2$. This gives, as promised $4 + 1 = 5 = F_5$ homology groups. (The 4 comes from $H_0, \ldots, H_4$.) For $n = 4$ we use $T^2, T^2 \otimes S^1$ and $T^3$ to obtain $5 + 3 = 8$ homology groups.

Now for $n = 5$. We use $T^2, T^2 \otimes S^1$ and $T^2 \otimes S^2$. This corresponds to a ‘string’ of Betti numbers. Because the Lefschetz element $\omega$ is an element of $S^1$, it acts on this string of compact local-global groups. There will also be an adjoint map on the open groups. We also have $T^3$ and $T^2 \otimes S^1$ as another string. (This group will be reconsidered shortly.) And there is also $T^4$. This gives 6 local-global groups organised into 3 strings. There are also $H_0, \ldots, H_5$. As the general theory, as already noted, gives 13 organised into 5 strings, we have missed one of the local-global groups.

A certain amount of thought shows that the missing group consists of compact local 2-cycles. Taking $\eta \in T^3$ is the way to produce such a cycle, but this must be distinguished from the previous use of $T^3$. There, the $T^3$ represents a ‘one-dimensional family of local 1-cycles’, and such will usually have a non-empty intersection against the generic element of $S^1$, whereas such will always miss a local 2-cycle. Thus, the ‘missing’ group will be produced by $T^3 \cap (S^1)^\perp$. (By this is meant formal sums $\eta$ of $\alpha_1 \sim \alpha_2 \sim \alpha_3$ such that $\eta \sim \xi$ is zero for any open $\xi$ that is locally of the form $S^1 \otimes T^1$. At this point it make no sense to impose global conditions of $\xi$.)

The first use of $T^3$ is not, according to the local-global theory, quite as it should be. The cone on a simple 4-polytope will in general have non-trivial such $T^3$ items, but will not of course have any ‘one-dimensional family of local 1-cycles’. The solution is to impose an additional condition, besides consistency, on the open cycles that are used. The condition is that if $\Delta_1$ and $\Delta_2$ differ only over isolated points of $\Delta$, then the open local cycle $\xi$ should vanish against $(\Delta_1 - \Delta_2) \sim \eta$, for any $\eta$ in $T^3$. A similar condition should be imposed for $T^2$ when $n = 4$, but in that case it is vacuous.

For $n = 6$ we will have all the $n = 5$ items, both as themselves, and multiplied by $S^1$. In other words, there is $T^2, \ldots, T^2 \otimes S^3$ and $T^3, T^3 \otimes S^1, T^3 \otimes S^2$ and $T^4, T^4 \otimes S^1$. The missing groups from $n = 5$ produces

$$T^3 \cap (S^2)^\perp, (T^3 \otimes S^1) \cap (S^2)^\perp$$

for $n = 6$. (This last requires some thought.) There will also be $T^5$ and $T^4 \cap (S^1)^\perp$. Altogether this is 13 local-global groups organised into 6 strings. Conditions similar to those formulated in the previous paragraph should again be imposed on open cycles. Add to this the $H_0, \ldots, H_5$ string to obtain 20 groups in 7 strings. This is short, by a string containing a single group. This ‘missing’ item will be defined in the next section.

### 5 Second order local-global homology

In the previous section it was seen how suitable expressions of the form $(\Delta_1 - \Delta_2) \sim \eta$ can be made to represent compact local-global homology classes. These, together with the uniform expressions of §3, were enough in dimension at most 5 to produce the $F_{n+2}$ homology groups that are demanded by the general combinatorial structure of the local-global theory. We also saw that for $n = 6$, the
count was one short. This section will define the missing group. (The previous sections dealt with order zero and order one respectively.)

First let $\Delta$ be the cone on a square. This polytope has dimension three, and in some sense is the first non-simple polytope. On it there is a non-trivial compact local-global cycle. Now consider the product $\Delta \times \Delta$. This has dimension six. On it one can form the product $\eta$ of the local-global cycles on its factors. Once this has been suitably understood, it will provide an example of a second-order local-global cycle.

It is natural to stratify $\Delta$ into the apex (the only non-simple point) and the rest. In the same way, $\Delta \times \Delta$ can be stratified into $\{0\} \times \{0\}$, $\{0\} \times \Delta$, $\Delta \times \{0\}$ and the rest. Here ‘0’ denotes the apex of $\Delta$. The cycle $\eta$ will in some sense be local to the ‘apex’ $\{0\} \times \{0\}$ of $\Delta \times \Delta$. It represents a local-global cycle that is local to $\{0\} \times \Delta$ (or to $\Delta \times \{0\}$) that can further be made local to $\{0\} \times \{0\}$. In fact, there will be two such cycles, one for each factor in $\Delta \times \Delta$. Although equivalent when regarded as first-order local-global cycles, they will be inequivalent when regarded as second-order such.

For simplicity, let $\Delta_a$ and $\Delta_b$ denote the two simple polytopes formed from $\Delta$ by the process of $\epsilon$-variation. Now let $\Delta_{aa}$, ..., $\Delta_{bb}$ be the corresponding resolutions of $\Delta \times \Delta$. The alternating sum

$$\Delta_\ld = \Delta_{aa} - \Delta_{ab} - \Delta_{ba} + \Delta_{bb}$$

has the interesting property that above the whole of $\Delta \times \Delta$, except for the apex $\{0\} \times \{0\}$, it is so to speak zero. Consider, for example, the cycle $\Delta_\ld \leadsto \eta$, for any suitable (non-uniform) $\eta$. This will be a compact cycle that is, in some sense, concentrated at the apex $\{0\} \times \{0\}$. (Throughout this section we will take $\eta$ to be as follows. Each thickened facet $\alpha$ on $\Delta$ determines thickened facets $\alpha \times \Delta$ and $\Delta \times \alpha$ on $\Delta \times \Delta$. We will have $\eta$ be $(\psi \times \Delta) \leadsto (\Delta \times \psi)$ where $\psi$ is say $N \leadsto E$ on $\Delta$, or any other non-uniform element of $T^2 \Delta$.)

As presented in the previous paragraph, the expression $\Delta_\ld \leadsto \eta$ represents a first-order local-global cycle. Let us now present it as a second order such. To do this we introduce

$$\Delta_a = \Delta_{aa} - \Delta_{ba}, \quad \Delta_b = \Delta_{ab} - \Delta_{bb}$$

and then consider both $\Delta_a \leadsto \eta$ and $\Delta_b \leadsto \eta$. Each of these expressions represents a first-order local-global cycle on $\Delta \times \Delta$. The difference

$$(\Delta_a - \Delta_b) \leadsto \eta$$

is again a local-global cycle, but now it is local to the apex $\{0\} \times \{0\}$.

The differences between the various local-global cycles that can be constructed from $\Delta_\ld$ and $\eta$ are subtle, and somewhat as in the previous section they only become apparent once the dual open theory has been defined. For order zero ‘open’ cycles, the basic model is a (thickened) facet lying on $\Delta$. For the first order theory, the basic model is a facet passing through a vertex. For the second order theory, the basic model will be a facet that contains a flag. Here, a flag is for example a vertex lying on a 3-face. The ‘Betti numbers’ are in some clever way counting how many flags there are of each type.

Now once again consider the difference $\Delta_a - \Delta_b$. We do this not as the formal sum $\Delta_\ld$, but as an expression in its own right. In other words, it can more exactly be thought of as an ordered pair $(\Delta_a, \Delta_b)$, that is to be treated in a particular way. The quantity $\Delta_a$ has a face of $\Delta \times \Delta$ naturally associated to it, namely the face $\{0\} \times \Delta$ along which the two components $\Delta_{aa}$ and $\Delta_{ba}$ differ. The same applies to $\Delta_b$. Now note that $\Delta_a$ and $\Delta_b$ are so to speak concentrated over or around the face $\{0\} \times \Delta$, and so it makes sense that they should be paired with products of thickened facets that pass through this face.
As already noted, this pairing will vanish over all of $\Delta \times \Delta$, except for the apex $\{0\} \times \{0\}$. One way to record this fact is to observe that $\Delta, \sim \eta \sim \alpha \sim \beta$ will always be zero, if it is known that $\alpha$ lies in $S^1$. This is true both as a local and a global statement.

For the open cycles, there is a detail missing. Although the face $\{0\} \times \Delta$ has entered the discussion, the apex $\{0\} \times \{0\}$ has not. It does so in the following way. Let $\alpha$ and $\beta$ be the thickened facets $\Delta \times N$ and $N \times \Delta$ on $\Delta \times \Delta$. (Any of the four triangular facets of $\Delta$ could just as well have been chosen.) Now consider the object that is the flag $(\{0\} \times \{0\} \subset \{0\}) \times \Delta$, together with the (open) expression $\alpha \sim \beta$. Note that expression has certain vanishing or incidence properties with respect to the flag part of the object. Both $\alpha$ and $\beta$ pass through $\{0\} \times \{0\}$, while $\beta$ also contains $\{0\} \times \Delta$. As in [10], these properties can be presented in an abstract form.

This study of $\Delta \times \Delta$ has led to a compact cycle $(\Delta_{a} - \Delta_{b}) \sim \eta$, and an open cycle (described in the previous paragraph), that have a non-zero pairing with each other. By writing down the abstract properties that these cycles satisfy, a definition of the second-order local-global cycles for $n = 6$ will be obtained.

At this point, it is well to stop. The reader who is already familiar with [9] and [10], especially the former, will appreciate that there are subtleties in the dimension of flags to be used, that have yet to manifest themselves. For the other readers, there is no easy way to explain further, other than to appeal to the principles already announced in [9].

It has not been the goal of this section, to produce a complete and rigorous definition of the higher-order local-global homology. Rather, it has been to develop the concepts in a fairly natural manner, up to the point where all the major features have been exposed. The goal has been more to show the existence of a such an approach, than to exhibit it in a formal and rigorous manner.

### 6 Summary and conclusions

This section discusses the following. First, application to the combinatorics of general convex polytopes. Second, application to more general algebraic varieties. Third, application of the volume polynomial approach to other situations, such as the Voronoi polytope of a positive definite quadratic form. Finally, some remarks are made on how the conjectures implicit in this paper might be proved.

If $\Delta$ is a simple polytope, the known facts regarding $H_{\bullet} \Delta$ imply numerical conditions on the face vector that are, in addition to being necessary, are also sufficient for the existence of a simple polytope with given flag vector. These facts are generation by the (thickened) facets, the ring structure, the strong Lefschetz theorem, and a formula for the Betti numbers in terms of the face vector (and vice versa). The numerical conditions are implicit in the properties of $H_{\bullet} \Delta$. The proof of necessity requires both the strong Lefschetz theorem and some result in monomial rings and the like. The proof of sufficiency is a matter of finding a suitable ingenious construction, and showing that it gives a polytope with the required face vector.

For general polytopes, results about the homology object $H_{\bullet} \Delta$ will again produce necessary conditions on this time the flag vector of $\Delta$. It is still true that $H_{\bullet} \Delta$ is generated, in some sense, by the possibly thickened facets of $\Delta$. This is of course true for the expressions $\eta, \psi$ and so forth, whether uniform or not. It also seems to be true for the simple $\epsilon$-variations $\Delta_{\epsilon}$. Each ordering of the facets of $\Delta$ will determine the combinatorial type of such a $\Delta_{\epsilon}$. Simply move the first facet outward until it is in general position, then the second, then the third, and so on. Even if not all $\Delta_{\epsilon}$ arise in this way, perhaps a ‘spanning set’ will so arise. The nature of any ring-like structure that might exist (in the simple case this gives the ‘pseudo-power inequalities’) is not so obvious. Proof of the strong Lefschetz theorem and formulae for the Betti numbers are likely, in general,
to be deep results. (It follows from Bayer’s example \cite[5?]{Bayer} that such is unlikely to hold for all
the local-global homology groups. There seems to be a strong analogy or connection between the
$\epsilon$-variation process and the construction of the ‘secondary polytope’ \cite{Bayer, Bier}.)

Nonetheless, it is true that results of this nature regarding $H_{\ast}\Delta$ will imply, as in the simple
case, necessary numerical conditions on the $f$-vector of $\Delta$. Even if such results are only conjectural,
the result will be numerical conjectures on $f\Delta$. This would be an advance, for we are at present
without even any plausible conjectures for the conditions on $f\Delta$ in dimension greater than 3.

Perhaps the best way to explore this problem is to focus on $n = 4$. This problem is hard enough
to be instructive, and easy enough to be accessible. Provided the conjectural structure of $H_{\ast}\Delta$
can be well understood in this case, it will be possible to extend the existing construction in the
simple case \cite{Bayer}, so that it deals with this new situation. Such is probably an appropriate starting
point for the study of the subtleties and complexities of local-global intersection homology in higher
dimensions. The complications should precisely satisfy the requirements of the proof process.

Now let $X$ be an irreducible projective algebraic variety. First, let $X$ be a Schubert variety, or
something similar. Such varieties have been extensively studied. The uniform expression approach
taken to the intersection homology of $P_{\Delta}$ should apply with significant change to this new situation.
In particular, the $\epsilon$-variation and study of volume method should still be valid. It may be that the
concepts introduced in this paper are related to results and methods already known and used in
this more complicated context.

Now let $X$ be any irreducible projective algebraic variety. The homology will no longer be
generated by the ‘facets’, and so the volume approach will no longer give as much information as
homology does. An elliptic curve is the simplest example of this. The volume approach is however
very attractive, and it would be nice if for such a general $X$ there were an $\epsilon$-variation (of something)
that would together with the volume analogue record all the information that homology does. This
can be thought of as a problem for nonsingular varieties only. One would also wish for this theory
to provide a ‘lifting’ for each resolution $X_i \to X$, similar to the decomposition theorem lifting that
exists for $\Delta_i$ and $\Delta$. Even without this, the decomposition theorem can be used to define the
concept of a uniform expression, and so in the same way produce local-global homology groups.

The general method of $\epsilon$-variation and volume can be applied in other situations, although with
what success is not yet clear. Here is an example. Let $Q$ be a positive definite quadratic form on
say $\mathbb{R}^n$. Inside $\mathbb{R}^n$ is the integer lattice $\mathbb{Z}^n$. The Voronoi polytope $\Delta_Q$ of $Q$ consists of all points of
$\mathbb{R}^n$ that are at least as close to the origin as they are to any other lattice point, where the distance
is measured using $Q$. Now vary $Q$, say by adding a quadratic form $\epsilon$ that is close to zero. This will
vary $\Delta_Q$ to $\Delta_Q(\epsilon)$. In this situation it is proper to study not only the volume of $\Delta_Q(\epsilon)$, but also the
area (or length or whatever) of its faces of the various dimensions. Provided the same programme
can be carried out, as has been done for simple polytopes, subtle combinatorial inequalities on $\Delta_Q$,
and thus $Q$, are likely to result.

One might also wish to apply similar methods to, say, centrally symmetric polytopes, or to
arrangements of hyperplanes in an affine space, or even to graphs and hypergraphs. The beauty
of the $\epsilon$-variation and volume approach, when or if it works, is that it produces the definition of
homology out of straightforward geometric concepts, and the requirements of consistency. It is as
if it provides a nucleus or seed, out of which an ingenious and appropriate homology theory might
emerge. But for such to be useful, the Betti numbers should be linear functions of the ‘flag vector’.

We close this paper by saying a few words about proofs. The local-global theory is of combin-
atorial interest only when it produces homology groups whose dimension is a linear function of
the flag vector. The prototype for the method of proof is in some sense McMullen’s elementary
and largely geometric proof \cite{McMullen} of strong Lefschetz for simple convex polytopes. What happens
here is that there is a package of properties, that is preserved as both the geometric realization
of a combinatorial type (continuous change) and the combinatorial type itself (discrete change) is varied, as a result of passing from one simple convex polytope to another. This package includes a strengthened version of the strong Lefschetz theorem, namely the ‘Riemann-Hodge-Minkowski’ inequalities. There is an induction built into the process.

Now consider the problem of proving that the order zero (or the usual middle perversity intersection homology) part of the theory has the predicted Betti numbers. This is true for polytopes with rational vertices, as a consequence of Deligne’s proof of the Weil conjectures. To prove such a result for general polytopes, a package and induction similar to that used by McMullen seems to be required. The author believes that the local-global intersection homology groups will provide the at present unknown part of this package. Again, this is a problem that can usefully be investigated in the case $n = 4$.

References

[1] M.M. Bayer and L.J. Billera, Generalized Dehn-Sommerville equations for polytopes, spheres, and Eulerian ordered sets, *Invent. Math.* 79 (1985), 143–157

[2] A. Belinson, J. Berstein and P. Deligne, *Faisceaux pervers*, Asterisque 100, Société mathématique de France, (1982)

[3] L.J. Billera, I. M. Gelfand and B. Sturmfels, Duality and minors of secondary polyhedra, *J. Combinatorial Theory (B)* 57 (1993), 258–268

[4] L.J. Billera and B. Sturmfels, Fiber polytopes, *Annals of Mathematics* 135 (1992), 527–549

[5] L.J. Billera and C. Lee, A proof of the sufficiency of McMullen’s conditions for $f$-vectors of simplicial polytopes, *J. Combinatorial Theory (A)* 31 (1981), 237–255

[6] V.I. Danilov, The geometry of toric varieties, *Russian Math. Surveys* 33 (1978), 97–154

[7] J. Fine, The Mayer-Vietoris and $IC$ Equations for Convex Polytopes, *Discrete Comput. Geom.* 13 (1995), 177–188

[8] J.Fine, On quantum topology, hypergraphs and flag vectors, preprint q-alg/9708001 (August 1997)

[9] _____ Local-global intersection homology, preprint alg-geom/9709011 (September 1997)

[10] _____ Convex polytopes and linear algebra, preprint alg-geom/9710001 (October 1997)

[11] _____ On shelling and flag vectors, preprint q-alg/9710002 (October 1997)

[12] _____ Graphs, flags and partitions, (in preparation)

[13] M. Goresky and R. MacPherson, Intersection homology theory, *Topology* 19 (1980), 135–162

[14] _____ Intersection homology II, *Invent. Math.* 72 (1983), 77–129

[15] P. McMullen, The number of faces of simplicial polytopes, *Israel J. Math.* 9 (1971), 559–570

[16] _____ On simple polytopes, *Invent. Math.* 113 (1993), 419–444
[17] H. Minkowski, *Allgemeine Lehreätze über die konvexe Polyeder*, Nachr. Ges. Wiss. Göttingen, (1897), 198–219

[18] R.P. Stanley, The number of faces of a simplicial convex polytope, *Adv. in Math.* 35 (1980), 236–238