THE ESTIMATE OF THE DIFFERENCE OF INITIAL SUCCESSIVE COEFFICIENTS OF UNIVALENT FUNCTIONS

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Abstract. Let \( \mathcal{A} \) denote the family of all functions that are analytic in the unit disk \( \mathbb{D} := \{ z : |z| < 1 \} \) and satisfy \( f(0) = 0 = f'(0) - 1 \). Let \( S \) be the set of all functions \( f \in \mathcal{A} \) that are univalent in \( \mathbb{D} \). In this paper the sharp upper bounds of \( |a_3 - a_2| \) and \( |a_4 - a_3| \) for the functions \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) being in several subclasses of \( S \) are presented.

1. Introduction

Let \( \mathcal{A} \) denote the family of all functions that are analytic in the unit disk \( \mathbb{D} := \{ z : |z| < 1 \} \) and satisfy \( f(0) = 0 = f'(0) - 1 \). Let \( S \) be the set of all functions \( f \in \mathcal{A} \) that are univalent in \( \mathbb{D} \). Let \( S^* \) and \( K \) denote the subclasses of \( S \) consisting of starlike functions and convex functions, respectively. If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S \) then \( |a_n| \leq n \) and strict inequality holds for all \( n \) unless \( f \) is the Koebe function or one of its rotation. This is the famous conjecture of Bieberbach, first proposed by Bieberbach\[2\] in 1916 and finally proved by de Branges\[1\] in 1984. After Bieberbach conjecture was put forward, another coefficient problem which has attracted considerable attention is to estimate \( ||a_{n+1}| - |a_n|| \), the difference of the moduli of successive coefficients of a function \( f \in S \). Indeed, Hayman\[4\] proved \( ||a_{n+1}| - |a_n|| \leq A \) for \( f \in S \), where \( A \geq 1 \) is an absolute constant. Pommerenke\[17\] conjecture that \( ||a_{n+1}| - |a_n|| \leq 1 \) for \( f \in S^* \) which was proved by Leung\[6\]. Z. Ye also estimated the difference of the moduli of successive coefficients of certain univalent functions\[21, 22\].

In the present paper the upper bounds of \( |a_3 - a_2| \) and \( |a_4 - a_3| \) for \( f \) belonging to various subclasses of \( S \) are studied.

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2. Preliminaries

Let $\mathcal{P}$ denote the class of all functions $p(z)$ analytic and having positive real part on $\mathbb{D}$, with the form

$$P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$  

It is known that $|p_n| \leq 2$ for $p \in \mathcal{P}$ and $n = 1, 2, \cdots$ [2].

In the course of the subsequent discussion, we need to make use of the following lemmas.

**Lemma 1.** Let $-2 \leq p_1 \leq 2$ and $p_2, p_3 \in \mathbb{C}$. There exists a function $P \in \mathcal{P}$ with

$$P(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots$$  

if and only if

$$2p_2 = p_1^2 + x(4 - p_1^2)$$  

and

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - (4 - p_1^2)p_1 x^2 + 2(4 - p_1^2)(1 - |x|^2)y$$  

for some $x, y \in \mathbb{C}$ with $|x| \leq 1$ and $|y| \leq 1$.

Lemma 1 is due to Libera and Złotkiewicz[8], one can also find it in [7].

**Lemma 2.** For given real numbers $a, b, c$, let

$$Y(a, b, c) = \max_{z \in \mathbb{D}} \left( |a + bz + cz^2| + 1 - |z|^2 \right).$$  

If $a \geq 0$ and $c \geq 0$, then

$$Y(a, b, c) = \begin{cases} 
  a + |b| + c, & |b| \geq 2(1 - c) \\
  1 + a + \frac{b^2}{4(1 - c)}, & |b| < 2(1 - c)
\end{cases}$$

The maximum in the definition of $Y(a, b, c)$ is attained at $z = \pm 1$ in the first case according as $b = \pm |b|$.

Lemma 2 is due to R. Ohno and T. Sugawa[14], one can also find it in [7].

3. Main Results

Let $\mathcal{G}$ denote the class of functions $f$ from $\mathcal{A}$ satisfying the conditions

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \frac{3}{2}, \quad z \in \mathbb{D},$$

It is known that $\mathcal{G} \subset S$ and $\frac{1}{2} f''(0) = |a_2| \leq \frac{1}{2}$ for $f = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{G}$ [19, 15, 10, 5]. Now, let

$$\mathcal{G}(p) = \{ f \in \mathcal{G}, f''(0) = p \},$$

where $p$ is a given number satisfying $-1 \leq p \leq 1$. 
Theorem 1. Let $0 \leq p \leq 1$ and let $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ be in the class $S(p)$. Then the next sharp inequalities hold:

$$|a_3 - a_2| \leq \frac{1}{6}(-p^2 + 3p + 1)$$  \hspace{1cm} (5)

$$|a_4 - a_3| \leq \frac{1}{24}(1 - p^2)(3p + 4)$$  \hspace{1cm} (6)

Proof. Since

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \frac{3}{2}, \quad z \in \mathbb{D},$$

it is follows that

$$\text{Re} \left( 1 - 2\frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in \mathbb{D}.$$

We can put

$$1 - 2\frac{zf''(z)}{f'(z)} = P(z),$$

where $P$ is given by (1) and satisfy $\text{Re}P(z) > 0, z \in \mathbb{D}$. From the last relation we have

$$f'(z) - 2zf''(z) = P(z)f'(z).$$  \hspace{1cm} (7)

By using the Taylor representations for the functions $f$ and $P$ and comparing the coefficients of $z^n$ ($n = 1, 2, 3$) in both sides of (7), we obtain

$$a_2 = -\frac{p_1}{4}, \quad a_3 = -\frac{1}{12}p_2 - \frac{1}{6}a_2p_1, \quad a_4 = -\frac{1}{24}p_3 - \frac{1}{8}a_3p_1 - \frac{1}{12}a_2p_2.$$  \hspace{1cm} (8)

Since, $2a_2 = f''(0) = p$, we have $p_1 = -4a_2 = -2p$ by (8). In view of these facts and Lemma 1, we have

$$p_2 = 2(p^2 + (1 - p^2)x),$$

$$p_3 = -2p^3 - 4(1 - p^2)px + 2(1 - p^2)px^2 + 2(1 - p^2)(1 - |x|^2)y,$$  \hspace{1cm} (9)

where $x, y \in \mathbb{C}$ with $|x| \leq 1$ and $|y| \leq 1$.

From the relations (8) and (9) and by some simple calculations, we have

$$|a_3 - a_2| = \left| -\frac{1}{6}(1 - p^2)x - \frac{1}{2}p \right|$$

$$\leq \frac{1}{6}(1 - p^2) + \frac{1}{2}p = \frac{1}{6}(-p^2 + 3p + 1),$$

where equality occurs if $x = 1$. Also, we have

$$|a_4 - a_3| = \left| -\frac{1}{12}(1 - p^2)(1 - |x|^2)y + \frac{1}{24}(1 - p^2)p + \frac{1}{6}(1 - p^2)x - \frac{1}{12}(1 - p^2)px^2 \right|$$

$$\leq \frac{1}{12}(1 - p^2) \left( 1 - |x|^2 + \left| -\frac{1}{2}(p + 4)x + px^2 \right| \right)$$

$$\leq \frac{1}{12}(1 - p^2)Y(a, b, c),$$
where $Y(a, b, c)$ is given in (4) and with

$$a = 0, \ b = -\frac{1}{2}(p + 4), \ c = p.$$  

Since $0 \leq p \leq 1$, we have that $|b| \geq 2(1 - c)$. Then by using Lemma 2 we get

$$Y(a, b, c) = \frac{3}{2}p + 2.$$  

Therefore

$$|a_4 - a_3| \leq \frac{1}{12}(1 - p^2)Y(a, b, c) = \frac{1}{24}(1 - p^2)(3p + 4)$$

The equality holds for $x = -1$.

If we denote by

$$\mathcal{G}^+ = \bigcup_{0 \leq p \leq 1} \mathcal{G}_p = \{ f : f \in \mathcal{G}, f''(0) \geq 0 \},$$

then by using (5) and (6) and a simple calculation, we easily get

$$\sup_{f \in \mathcal{G}^+} |a_3(f) - a_2(f)| = \frac{1}{2}$$

and

$$\sup_{f \in \mathcal{G}^+} |a_4(f) - a_3(f)| = \frac{260 + 43\sqrt{43}}{2916} = 0.1858...$$

where $a_n(f)$ ($n = 2, 3, 4$) are the Taylor coefficients of $f(z)$. □

As usual, let $\mathcal{U}$ denote the set of all $f \in \mathcal{A}$ satisfying the condition

$$\left| \left( \frac{z}{f'(z)} \right)^2 f'(z) - 1 \right| < 1$$

for $z \in \mathbb{D}$. It is known that $\mathcal{U} \subset S[16]$. In recent years, the properties of $\mathcal{U}$ were studied in detail[11, 12, 13, 3]. Let

$$\mathcal{U}_p = \{ f \in \mathcal{U}, f''(0) = p \},$$

where $p$ is a given number with $-4 \leq p \leq 4$ (Noticing that for $f \in \mathcal{U}$, we have $|\frac{1}{2}f''(0)| = |a_2(f)| \leq 2$).

**Theorem 2.** Let $0 \leq p \leq 4$ and let $f(z) = z + a_2z^2 + a_3z^3 + ...$ be in the class $\mathcal{U}_p$. Then we have the following sharp inequalities:

$$|a_3 - a_2| \leq \begin{cases} 1 + \frac{p}{4}(2 - p), & 0 \leq p \leq 2 \\ 1 + \frac{p}{4}(p - 2), & 2 \leq p \leq 4. \end{cases} \quad (10)$$

$$|a_4 - a_3| \leq \begin{cases} \frac{1}{4}(-p^3 + 6p^2 - 8p + 8), & 0 \leq p \leq 2 \\ \frac{1}{8}(p^3 - 2p^2 + 8p - 8), & 2 \leq p \leq 4. \end{cases} \quad (11)$$
Proof. If \( f \in \mathcal{U} \), then
\[
\left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < 1, \, |z| < 1.
\]

It is equivalent to
\[
\text{Re} \left( 2 \left( \frac{f(z)}{z} \right)^2 \frac{1}{f'(z)} - 1 \right) > 0, \quad z \in \mathbb{D}.
\]

So, we can put
\[
2 \left( \frac{f(z)}{z} \right)^2 \frac{1}{f'(z)} - 1 = P(z),
\]
where \( P \) is given by (1) and satisfy \( \text{Re} P(z) > 0, \, z \in \mathbb{D} \). From the last relation we have
\[
2 \left( \frac{f(z)}{z} \right)^2 - f'(z) = P(z)f'(z). \tag{12}
\]

By using the relation (12) and the Taylor expansions of functions \( f \) and \( P \), we obtain
\[
p_1 = 0, \quad a_3 = a_2^2 - \frac{1}{2} p_2, \quad a_4 = -\frac{1}{4} p_3 - \frac{1}{2} a_2 p_2 + a_2 a_3. \tag{13}
\]

Since \( 2a_2 = p \), we have \( a_2 = \frac{p}{2} \). Also, since \( p_1 = 0 \), it follows from Lemma 1 that
\[
p_2 = 2x, \quad p_3 = 2(1 - |x|^2)y \tag{14}
\]
for some \( x, y \in \mathbb{C} \) with \( |x| \leq 1 \) and \( |y| \leq 1 \).

By using the all previous facts, we obtain that
\[
a_3 = \frac{1}{4} p^2 - x, \quad a_4 = \frac{1}{8} p^3 - px - \frac{1}{2} (1 - |x|^2)y.
\]

Now, we have
\[
|a_3 - a_2| = \left| -x + \frac{1}{4} p^2 - \frac{p}{2} \right| \leq 1 + \frac{p}{4} |p - 2|,
\]
or equivalently,
\[
|a_3 - a_2| \leq \begin{cases} 
1 + \frac{p}{4} (2 - p), & 0 \leq p \leq 2 \\
1 + \frac{p}{4} (p - 2), & 2 \leq p \leq 4.
\end{cases}
\]

Also we have
\[
|a_4 - a_3| = \left| \frac{1}{8} p^3 - px - \frac{1}{2} (1 - |x|^2)y - \frac{1}{4} p^2 + x \right|
\]
\[
\leq \frac{1}{2} \left( 1 - |x|^2 + \left| \frac{1}{4} p^2 (p - 2) + 2(1 - p)x \right| \right)
\]
\[
\leq \frac{1}{2} Y(a, b, c),
\]

where
\[
Y(a, b, c) = \frac{1}{2} \left( 1 - |x|^2 + \left| \frac{1}{4} p^2 (p - 2) + 2(1 - p)x \right| \right).
\]
where $Y(a,b,c)$ is given in (4). Since
\[
\left| \frac{1}{4} p^2(p - 2) + 2(1 - p)x \right| = \left| \frac{1}{4} p^2(2 - p) + 2(p - 1)x \right|,
\]
we can put $a = \frac{1}{4} p^2(2 - p)$, $b = 2(p - 1)$, $c = 0$ in case $0 \leq p \leq 2$ and $a = \frac{1}{4} p^2(p - 2)$, $b = 2(1 - p)$, $c = 0$ in case $2 \leq p \leq 4$. We have that $|b| \leq 2(1 - c)$ in the first case and $|b| \geq 2(1 - c)$ in the second case. By Lemma 2 we have
\[
Y(a,b,c) = \begin{cases} 
1 + \frac{1}{4} p^2(2 - p) + \frac{4(p - 1)^2}{4}, & 0 \leq p \leq 2 \\
\frac{1}{4} p^2(p - 2) + 2(p - 1), & 2 \leq p \leq 4
\end{cases}
\]
and therefore
\[
|a_4 - a_3| \leq \begin{cases} 
\frac{1}{4} (-p^3 + 6p^2 - 8p + 8), & 0 \leq p \leq 2 \\
\frac{1}{8} (p^3 - 2p^2 + 8p - 8), & 2 \leq p \leq 4.
\end{cases}
\]

Now, let
\[
U^+ = \bigcup_{0 \leq p \leq 4} U_p = \{ f : f \in U, f''(0) \geq 0 \}.
\]
Then, in view of (10) and (11), we easily get
\[
\sup_{f \in U^+} |a_3(f) - a_2(f)| = 3
\]
and
\[
\sup_{f \in U^+} |a_4(f) - a_3(f)| = 7.
\]

□

For a long time, the research on Bazilevic functions has attracted the attention of many scholars[20, 9, 23]. R. Singh[20] considered a subclass $B_1(\alpha)$ of Bazilevic functions. $f \in B_1(\alpha)$ if $f \in A$ and
\[
\Re \left\{ \left( \frac{f(z)}{z} \right)^{\alpha-1} f'(z) \right\} > 0, z \in \mathbb{D}, \alpha \geq 0.
\]

It is well-known that $B_1(\alpha)(\alpha \geq 0)$ is the subclass of $S$. For $\alpha = 1$ we have the class $R$ defined by the condition
\[
\Re \{ f'(z) \} > 0, z \in \mathbb{D}.
\]
Further, let denote by $B(2)$ and $B(3)$ the classes given from $B_1(\alpha)$ for $\alpha = 2$ and $\alpha = 3$, i.e. the classes of $A$ satisfying the next conditions
\[
\Re \left\{ \frac{f(z)f'(z)}{z} \right\} > 0, z \in \mathbb{D}
\]
and
\[
\Re \left\{ \left( \frac{f(z)}{z} \right)^2 f'(z) \right\} > 0, z \in \mathbb{D}.
\]
respectively. Also, let
\[
\mathcal{R}_p = \{ f \in \mathcal{B}, f''(0) = p \},
\]
\[
\mathcal{B}_p^{(2)} = \{ f \in \mathcal{B}^2, f''(0) = p \},
\]
\[
\mathcal{B}_p^{(3)} = \{ f \in \mathcal{B}^3, f''(0) = p \}.
\]

**Theorem 3.** Let \(0 \leq p \leq 2\) and let \(f(z) = z + a_2 z^2 + a_3 z^3 + \ldots\) be in the class \(\mathcal{R}_p\). Then we have the next sharp inequalities:

\[
|a_3 - a_2| \leq \frac{1}{6}(4 + 3p - 2p^2),
\]

(15)

\[
|a_4 - a_3| \leq \begin{cases} 
\frac{1}{18}(13 - 4p), & 0 \leq p \leq \frac{5}{3} \\
\frac{1}{12}(-3p^3 + 4p^2 + 9p - 8), & \frac{5}{3} \leq p \leq 2.
\end{cases}
\]

(16)

**Proof.** Since \(f \in \mathcal{R}_p\), we can put
\[
f'(z) = P(z),
\]

where \(P\) is given by (1) with \(\text{Re}P(z) > 0, z \in \mathbb{D}\). By using the Taylor representations for the functions \(f\) and \(P\) and comparing the coefficients of \(z^n (n = 1, 2, 3)\) in both sides of (17), we obtain
\[
a_2 = \frac{1}{2}p_1, \quad a_3 = \frac{1}{3}p_2, \quad a_4 = \frac{1}{4}p_3.
\]

(18)

Since \(2a_2 = f''(0) = p\), it follows from (18) that \(p_1 = 2a_2 = p\) and \(|p| \leq 2\). By using Lemma 1, we have
\[
p_2 = \frac{1}{4}[p^2 + (4 - p^2)x],
\]

\[
p_3 = \frac{1}{4}[p^3 + 2(4 - p^2)px - (4 - p^2)px^2 + 2(4 - p^2)(1 - |x|^2)y]
\]

(19)

for some \(x, y \in \mathbb{C}\) with \(|x| \leq 1\) and \(|y| \leq 1\).

Combining (18) with (19), we obtain
\[
|a_3 - a_2| = \left| \frac{1}{6}(4 - p^2)x - \frac{p}{6}(3 - p) \right|
\]
\[
\leq \frac{1}{6}(4 - p^2) + \frac{p}{6}(3 - p)
\]
\[
= \frac{1}{6}(4 + 3p - 2p^2)
\]

where equality occurs if \(x = -1\). Similarly, we have
\[
|a_4 - a_3|
\]
\[
= \frac{1}{16}[p^2 + 2(4 - p^2)px - (4 - p^2)px^2 + 2(4 - p^2)(1 - |x|^2)y] - \frac{1}{6}[p^2 + (4 - p^2)x]
\]
\[
\leq \frac{1}{8}(4 - p^2) \left[ 1 - |x|^2 + \left| \frac{p^2(8/3 - p)}{2(4 - p^2)} + (4/3 - p)x + \frac{p}{2}x^2 \right| \right]
\]
\[
\leq \frac{1}{8}(4 - p^2)Y(a,b,c),
\]
where \( Y(a,b,c) \) is given in (4) with
\[
a = \frac{p^2(8/3 - p)}{2(4 - p^2)}, \quad b = 4/3 - p, \quad c = \frac{1}{2}p, \tag{4}
\]
(for \( p = 2 \), we have directly that \( |a_4 - a_3| = \frac{1}{6} \)).

Noticing that for \( p \in [0,2] \), \( |b| \leq 2(1 - c) \) is equivalent \( 0 \leq p \leq \frac{5}{3} \), by Lemma 2 we have
\[
Y(a,b,c) = \begin{cases}
1 + \frac{p^2(8/3 - p)}{2(4 - p^2)} + \frac{(4/3 - p)^2}{4(1-p/2)}, & 0 \leq p \leq \frac{5}{3} \\
\frac{p^2(8/3 - p)}{2(4 - p^2)} + p - \frac{4}{3} + \frac{1}{4}p, & \frac{5}{3} \leq p < 2.
\end{cases}
\]

Hence
\[
|a_4 - a_3| \leq \begin{cases}
\frac{1}{18}(13 - 4p), & 0 \leq p \leq \frac{5}{3} \\
\frac{1}{12}(-3p^3 + 4p^2 + 9p - 8), & \frac{5}{3} \leq p \leq 2.
\end{cases}
\]

If we denote by
\[
\mathbb{R}^+ = \bigcup_{0 \leq p \leq 2} \mathbb{R}_p = \{ f : f \in \mathbb{R}, f''(0) \geq 0 \},
\]
then in view of (15) and (16), we easily get
\[
\sup_{f \in \mathbb{R}^+} |a_3(f) - a_2(f)| = \frac{41}{48}
\]
and
\[
\sup_{f \in \mathbb{R}^+} |a_4(f) - a_3(f)| = \frac{13}{18}.
\]

**Theorem 4.** Let \( 0 \leq p \leq \frac{4}{3} \) and let \( f(z) = z + a_2z^2 + a_3z^3 + \ldots \) be in the class \( B^2_p \). Then we have the next sharp inequalities:
\[
|a_3 - a_2| \leq \frac{1}{16}(-7p^2 + 8p + 8), \quad 0 \leq p \leq \frac{4}{3}. \tag{20}
\]
\[
|a_4 - a_3| \leq \begin{cases}
\frac{1}{2560}(-85p^3 - 400p^2 - 260p + 1424), & 0 \leq p \leq \frac{6}{5} \\
\frac{1}{80}(-5p^3 - 10p^2 - 4p + 40), & \frac{6}{5} \leq p \leq \frac{4}{3}.
\end{cases} \tag{21}
\]

**Proof.** From the definition of the class \( B^2_p \), we can put
\[
\frac{f(z)f'(z)}{z} = P(z), \tag{22}
\]
where $\text{Re}P(z) > 0, z \in \mathbb{D}$, and $P$ is given by (1). By using the Taylor representations for the functions $f$ and $P$ and comparing the coefficients of $z^n (n = 1, 2, 3)$ in both sides of (22), we obtain

$$a_2 = \frac{1}{3} p_1, a_3 = \frac{1}{4} p_2 - \frac{1}{2} a_2^2, a_4 = \frac{1}{5} p_3 - a_2 a_3. \tag{23}$$

Since $2a_2 = f''(0) = p$ and $|p| \leq 2$, it follows from (23) that $p_1 = 3a_2 = \frac{3}{2} p$ and $|p| \leq \frac{4}{3}$. In view of these facts and Lemma 1, we have

$$p_2 = \frac{9}{8} (p^2 + \left(\frac{16}{9} - p^2\right) x),$$

$$p_3 = \frac{3}{32} \left(3p^3 + 6\left(\frac{16}{9} - p^2\right) px - 3\left(\frac{16}{9} - p^2\right) px^2 + 4\left(\frac{16}{9} - p^2\right)(1 - |x|^2)y\right) \tag{24}$$

for some $x, y \in \mathbb{C}$ with $|x| \leq 1$ and $|y| \leq 1$. Combining (23) with (24), we obtain

$$|a_3 - a_2| = \frac{9}{32} (\frac{16}{9} - p^2) x + \frac{5}{32} p^2 - \frac{p}{2}$$

$$\leq \frac{9}{32} (\frac{16}{9} - p^2) + \frac{5}{32} p \left|p - \frac{16}{5}\right|$$

$$= \frac{1}{16} (-7p^2 + 8p + 8),$$

where equality occurs if $x = -1$. Similarly, we have

$$|a_4 - a_3| = \left|\frac{29}{320} p^3 - \frac{5}{32} p^2 + \left(\frac{16}{9} - p^2\right) \left(\frac{63}{320} px - \frac{27}{160} px^2 + \frac{9}{40} (1 - |x|^2) y - \frac{9}{32} x\right)\right|$$

$$\leq \frac{16 - 9p^2}{40} \left[1 - x^2 + \left|\frac{p^2 (50 - 29p)}{8(16 - 9p^2)} + \frac{1}{8} (10 - 7p)x + \frac{3}{4} px^2\right|\right]$$

$$\leq \frac{16 - 9p^2}{40} Y(a, b, c),$$

where $Y(a, b, c)$ is given in (4) with

$$a = \frac{p^2 (50 - 29p)}{8(16 - 9p^2)}, b = \frac{1}{8} (10 - 7p), c = \frac{3}{4} p.$$

(for $p = \frac{4}{3}$, we have directly that $|a_4 - a_3| = \frac{17}{270}$).

Since $p \in [0, 4/3]$, $|b| \leq 2(1 - c)$ is equivalent to $0 \leq p \leq \frac{6}{5}$, by Lemma 2, we have

$$Y(a, b, c) = \begin{cases} 1 + \frac{p^2 (50 - 29p)}{8(16 - 9p^2)} + \frac{(10 - 7p)^2}{64(4 - 3p)}, & 0 \leq p \leq \frac{6}{5} \\ \frac{p^2 (50 - 29p)}{8(16 - 9p^2)} + \frac{1}{8} (10 - 7p) + \frac{3}{4} p, & \frac{6}{5} \leq p < \frac{4}{3}. \end{cases}$$

Therefore

$$|a_4 - a_3| \leq \begin{cases} \frac{250}{2500} (-85p^3 - 400p^2 - 260p + 1424), & 0 \leq p \leq \frac{6}{5} \\ \frac{1}{80} (-5p^3 - 10p^2 - 4p + 40), & \frac{6}{5} \leq p < \frac{4}{3}. \end{cases}$$
Let

$$\mathcal{B}^{(2)+} = \bigcup_{0 \leq p \leq \frac{1}{4}} \mathcal{B}_{p}^{(2)} = \{ f : f \in \mathcal{B}^{(2)}, f''(0) \geq 0 \}.$$  

Then by using (20) and (21) we easily get

$$\sup_{f \in \mathcal{B}^{(2)+}} |a_3(f) - a_2(f)| = \frac{9}{14} = 0.64...$$

and

$$\sup_{f \in \mathcal{B}^{(2)+}} |a_4(f) - a_3(f)| = \frac{1424}{2560} = 0.556...$$

□

**THEOREM 5.** Let $0 \leq p \leq 1$ and let $f(z) = z + a_2z^2 + a_3z^3 + ...$ be in the class $\mathcal{B}_p^3$. Then we have the next sharp inequalities:

1. $|a_3 - a_2| \leq \frac{1}{20}(-11p^2 + 10p + 8)$. \hspace{1cm} (25)
2. $|a_4 - a_3| \leq \begin{cases} \frac{1}{600}(-53p^3 - 174p^2 - 24p + 272), & 0 \leq p \leq \frac{2}{3} \\ \frac{1}{120}(-25p^3 - 30p^2 + 8p + 48), & \frac{2}{3} \leq p \leq 1. \end{cases}$ \hspace{1cm} (26)

**Proof.** The hypothesis $f \in \mathcal{B}_p^3$ implies that there exists a function $P$, defined by (1) and satisfying $\text{Re}P(z) > 0, z \in \mathbb{D}$, such that

$$\left(\frac{f(z)}{z}\right)^2 f'(z) = P(z).$$ \hspace{1cm} (27)

By using the Taylor representations for the functions $f$ and $P$ and comparing the coefficients of $z^n (n = 1, 2, 3)$ in both sides of (27), we obtain

$$a_2 = \frac{1}{4}p_1, \quad a_3 = \frac{1}{5}p_2 - a_2^2, \quad a_4 = \frac{1}{6}p_3 - 2a_2a_3 - \frac{1}{3}a_3^2.$$ \hspace{1cm} (28)

Since $2a_2 = f''(0) = p$ and $|p_1| \leq 2$, by (28) we have $p_1 = 4a_2 = 2p$ and $|p| \leq 1$. By using these facts and Lemma 1, we get

$$\begin{align*}
p_2 &= 2[p^2 + (1 - p^2)x], \\
p_3 &= 2p^3 + 4(1 - p^2}px - 2(1 - p^2)px^2 + 2(1 - p^2)(1 - |x|^2)y
\end{align*}$$ \hspace{1cm} (29)

for some $x, y \in \mathbb{C}$ with $|x| \leq 1$ and $|y| \leq 1$.

Combining (28) with (29), we obtain

$$|a_3 - a_2| = \left| \frac{2}{5}(1 - p^2)x - \frac{3}{20}p(10/3 - p) \right| \leq \frac{1}{20}(-11p^2 + 10p + 8),$$
where equality occurs if $x = -1$. Similarly, we also have

$$|a_4 - a_3| = \left| \frac{1}{6} p_3 - 2a_2 a_3 - \frac{1}{3} a_2^3 - a_3 \right|$$

$$\leq \frac{1}{3} (1 - p^2) \left[ 1 - |x|^2 + \left| \frac{18p^2 - 17p^3}{40(1 - p^2)} \right| + \frac{2}{5} (3 - 2p)x + px^2 \right]$$

$$\leq \frac{1}{3} (1 - p^2) Y(a, b, c),$$

where $Y(a, b, c)$ is given in (4) with

$$a = \frac{18p^2 - 17p^3}{40(1 - p^2)}, \quad b = \frac{2}{5} (3 - 2p), \quad c = p.$$

(for $p = 1$ we have directly that $|a_4 - a_3| = \frac{1}{120}$).

Since for $p \in [0, 1]$, $|b| \leq 2(1 - c)$ is equivalent to $0 \leq p \leq \frac{2}{3}$, by using Lemma 2 we have

$$Y(a, b, c) = \begin{cases} 1 + \frac{p^2(18 - 17p)}{40(1 - p^2)} + \frac{(3 - 2p)^2}{25(1 - p)}, & 0 \leq p \leq \frac{2}{3} \\ \frac{p^2(18 - 17p)}{40(1 - p^2)} + \frac{6}{5} p + \frac{2}{3}, & \frac{2}{3} \leq p < 1. \end{cases}$$

And therefore

$$|a_4 - a_3| \leq \begin{cases} \frac{1}{600} (-53p^3 - 174p^2 - 24p + 272), & 0 \leq p \leq \frac{2}{3} \\ \frac{1}{120} (-25p^3 - 30p^2 + 8p + 48), & \frac{2}{3} \leq p \leq 1. \end{cases}$$

Let

$$\mathcal{B}^{(3)} = \bigcup_{0 \leq p \leq 1} \mathcal{B}^{(3)}_p = \{ f : f \in \mathcal{B}^{(3)}, f''(0) \geq 0 \}.$$

In view of (25) and (26), we easily get

$$\sup_{f \in \mathcal{B}^{(3)}+} |a_3(f) - a_2(f)| = \frac{113}{220} = 0.5136...$$

and

$$\sup_{f \in \mathcal{B}^{(3)}+} |a_4(f) - a_3(f)| = \frac{34}{75} = 0.4533.$$
Theorem 6. Let $0 \leq p \leq 1$ and let $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ be in the class $\Omega_p$. Then we have the following sharp inequalities:

$$|a_3 - a_2| \leq \frac{1}{4} + \frac{1}{2}p - \frac{1}{4}p^2, \quad 0 \leq p \leq 1. \quad (30)$$

$$|a_4 - a_3| \leq \left\{ \begin{array}{ll}
\frac{1}{12}(-16p^2 + 9p + 25), & 0 \leq p \leq \frac{1}{4} \\
\frac{1}{12}(-2p^3 + 3p^2 + 2p + 3), & \frac{1}{4} \leq p \leq 1.
\end{array} \right. \quad (31)$$

Proof. By the definition of $\Omega$, $f \in \Omega$ if and only if there exists a function $P(z)$ defined by (1) with $\text{Re}P(z) > 0, z \in \mathbb{D}$, such that

$$2[P(z) + 1][zf'(z) - f(z)] = z[P(z) - 1] \quad (32)$$

By using the Taylor representations for the functions $f$ and $P$ and comparing the coefficients of $z^n (n = 2, 3, 4)$ in both sides of (32), we obtain

$$a_2 = \frac{1}{4}p_1, \quad a_3 = \frac{1}{8}p_2 - \frac{1}{4}a_2p_1, \quad a_4 = \frac{1}{12}p_3 - \frac{1}{3}a_3p_1 - \frac{1}{6}a_2p_2. \quad (33)$$

Since $2a_2 = f''(0) = p$ and $|p_1| \leq 2$, by (33) we have $p_1 = 4a_2 = 2p$ and $|p| \leq 1$. In view of these facts and Lemma 1, we get

$$p_2 = 2[p^2 + (1 - p^2)x],$$

$$p_3 = 2p^3 + 4(1 - p^2)px - 2(1 - p^2)p^2x^2 + 2(1 - p^2)(1 - |x|^2)y \quad (34)$$

for some $x, y \in \mathbb{C}$ with $|x| \leq 1$ and $|y| \leq 1$.

Combining (33) with (34), we obtain

$$|a_3 - a_2| = \left| \frac{1}{4}(1 - p^2)x - \frac{1}{2}p \right|$$

$$\leq \frac{1}{4} + \frac{1}{2}p - \frac{1}{4}p^2,$$

where equality occurs if $x = -1$. Similarly, we also have

$$|a_4 - a_3| = \left| \frac{1}{6}(1 - p^2)(1 - |x|^2)y - \frac{1}{6}(1 - p^2)p^2x - \frac{1}{4}(1 - p^2)x \right|$$

$$\leq \frac{1}{6}(1 - p^2) \left[ 1 - |x|^2 + \frac{3}{2}x + px^2 \right]$$

$$\leq \frac{1}{6}(1 - p^2)Y(a, b, c),$$

where $Y(a, b, c)$ is given in (4) with

$$a = 0, \quad b = \frac{3}{2}, \quad c = p.$$
Since for $p \in [0, 1]$, $|b| \leq 2(1 - c)$ is equivalent to $0 \leq p \leq \frac{1}{4}$, by using Lemma 2 we have

$$Y(a, b, c) = \begin{cases} 1 + \frac{9}{16(1 - p)}, & 0 \leq p \leq \frac{1}{4} \\ \frac{3}{2} + p, & \frac{1}{4} \leq p \leq 1 \end{cases}$$

And therefore

$$|a_4 - a_3| \leq \begin{cases} \frac{1}{96}(-16p^2 + 9p + 25), & 0 \leq p \leq \frac{1}{4} \\ \frac{1}{12}(-2p^3 - 3p^2 + 2p + 3), & \frac{1}{4} \leq p \leq 1 \end{cases}$$

Let

$$\Omega^+ = \bigcup_{0 \leq p \leq 1} \Omega_p = \{f : f \in \Omega, f''(0) \geq 0\}.$$

In view of (30) and (31), we easily get

$$\sup_{f \in \Omega^+} |a_3(f) - a_2(f)| = \frac{1}{2}$$

and

$$\sup_{f \in \Omega^+} |a_4(f) - a_3(f)| = \frac{27 + 7\sqrt{21}}{216} = 0.2735\ldots$$

□

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