On the uncertainty in the $0\nu\beta\beta$ decay nuclear matrix elements

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The nuclear matrix elements $M^{0\nu}$ of the neutrinoless double beta decay ($0\nu\beta\beta$) are evaluated for $^{76}\text{Ge}$, $^{100}\text{Mo}$, $^{130}\text{Te}$, and $^{136}\text{Xe}$ within the Renormalized Quasiparticle Random Phase Approximation (RQRPA) and the simple QRPA. Three sets of single particle level schemes are used, ranging in size from 9 to 23 orbits. When the strength of the particle-particle interaction is adjusted so that the $2\nu\beta\beta$ decay rate is correctly reproduced, the resulting $M^{0\nu}$ values become essentially independent on the size of the basis, and on the form of different realistic nucleon-nucleon potentials. Thus, one of the main reasons for variability of the calculated $M^{0\nu}$ within these methods is eliminated.

\begin{equation}
\langle m_\nu \rangle = \sum_i^N |U_{ei}|^2 e^{i\alpha_i} m_i , \text{ (all } m_i \geq 0) , \tag{2}
\end{equation}

where $\alpha_i$ are the unknown Majorana phases. The elements of the mixing matrix $|U_{ei}|^2$ and the mass-squared differences $\Delta m^2$ can be determined in the oscillation experiments. Using the present knowledge of these quantities, (see, e.g., [6]) and limits on $T_{1/2}$, one could decide whether the neutrino mass pattern is degenerate, or follows the inverse or normal hierarchies. If, on the other hand, the existence of the $0\nu\beta\beta$ decay is proven and the value of $T_{1/2}$ is found, a relatively narrow range of absolute neutrino mass scale can be determined, independently of the phases $\alpha_i$ in most situations [4,6]. However, such an important insight is possible only if the nuclear matrix elements are accurately known.

The nuclear matrix element $M^{0\nu}$ is defined as

\begin{equation}
M^{0\nu} = \langle |f| - M^{0\nu}_F g_A^2 + M^{0\nu}_{GT} + M^{0\nu}_T |i \rangle \tag{3}
\end{equation}

where $|i\rangle$, $\langle |f\rangle$ are the wave functions of the ground states of the initial (final) nuclei. The explicit forms of the operators $M^{0\nu}_F$, $M^{0\nu}_{GT}$ and $M^{0\nu}_T$ are given in Ref. [7]. In comparison with most of previous $0\nu\beta\beta$ decay studies [8–11] the higher order terms of the nucleon current are also included in the present calculation, resulting in a suppression of the nuclear matrix element by about 30% [7]. Note that in the numerical calculation of $M^{0\nu}$ here the closure approximation is avoided and the unquenched values $g_A = 1.25$, $g_V = 1.0$ are used.

Two basic methods are used in the evaluation of $M^{0\nu}$, the quasiparticle random phase approximation (QRPA) with its various modifications and the nuclear shell model (NSM). These two approaches represent in some sense opposite extremes.

In the QRPA one can include essentially unlimited set of single-particle states, but only a limited subset of configurations (iterations of the particle-hole, respectively two-quasiparticle configurations). In the context of
QRPA several issues that have been raised, and deserve a systematic study:

- For realistic interactions the QRPA solutions are near the point of the so-called collapse and thus its applicability is questionable. Numerous attempts have been made to extend the method’s range of validity by partially avoiding the violation of the Pauli principle. Here we consider the simplest of them, the renormalized QRPA (RQRPA) [12,8]. We compare the results of RQRPA with those of the standard QRPA [9–11].
- The choice of the size of single-particle (s.p.) space is to some extent arbitrary, often dictated by convenience. What effect this choice has on the $M^{0\nu}$ values is the main thrust of the present work.
- There are various forms of the nucleon-nucleon potential that lead to somewhat different forms of the resulting $G$-matrix. By comparing the results obtained with three such potentials (Bonn-CD [16], Argonne [17], and Nijmegen [18]) we show that the resulting $M^{0\nu}$ are essentially identical, and independent of the choice of the realistic nucleon-nucleon potential.

In contrast, in the NSM one chooses a limited set of single-particle states in the vicinity of the Fermi level, and includes all (or most) configurations of the valence nucleons on these orbits in the evaluation of $M^{0\nu}$. The main open question in this approach is to determine the effects of the neglected single-particle states further away from the Fermi level. As shown below, we have also performed QRPA and RQRPA calculations with the set of s.p. states used usually in NSM. It appears that these methods, at least with the nucleon-nucleon interaction we used, are not applicable for such small s.p. bases.

In what follows the $0\nu\beta\beta$ decay nuclear matrix elements $M^{0\nu}$ for $^{76}$Ge, $^{100}$Mo, $^{130}$Te, and $^{136}$Xe are evaluated. These nuclei are most often considered as candidate sources for the next generation of the experimental search for $0\nu\beta\beta$ decay. For each of them three choices of the s.p. basis are considered. The smallest one has 9 levels (oscillator shells $N=3,4$) for $^{76}$Ge, and 13 levels (oscillator shells $N=3,4$ plus the $f + h$ orbits from $N = 5$) for $^{100}$Mo, $^{130}$Te, and $^{136}$Xe. For the intermediate size s.p. base the $N = 2$ shell in $^{76}$Ge and $^{100}$Mo are added, and for $^{130}$Te and $^{136}$Xe also the $p$ orbits from $N = 5$. Finally, the largest s.p. space [8] contains 21 levels for $^{76}$Ge and $^{100}$Mo (all states from shells $N = 1 − 5$), and 23 levels for $^{130}$Te and $^{136}$Xe ($N = 1 − 5$ and $i$ orbits from $N = 6$). Thus the smallest set corresponds to $1\hbar\omega$ particle-hole excitations, and the largest to about $4\hbar\omega$ excitations. The s.p. energies have been calculated with the Coulomb corrected Woods-Saxon potential.

It is well known that the residual interaction is an effective interaction that depends on the size of the single-particle basis. Hence, when the basis is changed, the interaction should be modified as well. Here we propose a rather simple way to accomplish the needed renormalization.

There are three important ingredients in QRPA and RQRPA. First, the pairing interaction has to be included by solving the corresponding gap equations. Within the BCS method the strength of the pairing interaction depends on the size of the s.p. basis. As usual, we multiply the pairing part of the interaction by a factor $g_{pair}$ whose magnitude is adjusted, for both protons and neutrons, such that the pairing gap is correctly reproduced, separately for the initial and final nuclei.

Second, QRPA equations of motion contain a block corresponding to the particle-hole interaction, renormalized by an overall strength parameter $g_{ph}$. That parameter is typically adjusted by requiring that the energy of some chosen collective state, often the giant GT resonance, is correctly reproduced. We find that the calculated energy of the giant GT state is almost independent of the size of the s.p. basis and is well reproduced with $g_{ph} \approx 1$. Accordingly, we use $g_{ph} = 1$ throughout, without adjustment.

Finally, QRPA equations of motion contain a block corresponding to the particle-particle interaction, renormalized by an overall strength parameter $g_{pp}$. (The importance of the particle-particle interaction for the $\beta\beta$ decay was recognized first in Ref. [19], and for the $\beta\beta$ decay in [9,9].) It is well known that the decay rate for both modes of $\beta\beta$ decay depends sensitively on the value of $g_{pp}$. In the following we use this property to find the value of $g_{pp}$ for each of the possible s.p. bases. The value of the parameter $g_{pp}$ is fixed in each case so that the known half-life of the $2\nu\beta\beta$ decay is correctly reproduced. The $2\nu\beta\beta$ half-lives and average matrix elements collected in Table 1 of Ref. [4] are used, where the original references to the corresponding experiments can be found. A similar compilation of the $2\nu\beta\beta$ data can be found in Ref. [20]. The resulting adjusted values of $g_{pp}$ are shown in Table I for both the RQRPA and QRPA methods. One can see that as the basis increases, the effective $g_{pp}$ decreases $^2$, as expected.

The adjustment of $g_{pp}$ is a crucial point of the present work. Several studies of the sensitivity of the nuclear matrix elements $M^{0\nu}$ to various modifications of the QRPA method as well as to the number of s.p. states and values

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1Brueckner reaction matrix elements with these nucleon-nucleon potentials were calculated and used in Refs. [13–15].

2For $^{136}$Xe we use the lower limit on the $2\nu$ half-life for the adjustment of $g_{pp}$; hence further adjustment might be needed when the $2\nu$ half-life becomes known.
of $g_{pp}$ were made in the recent past [8,21,22]. Typically, these studies concluded that the values of $M^{0\nu}$ vary substantially depending on all of these things. Here we come to a different conclusion: by requiring that the known $2\nu\beta\beta$ decay half-life is correctly reproduced, and adjusting the parameter $g_{pp}$ accordingly, we remove much of the sensitivity on the number of single-particle states, on the $NN$ potential employed, and even on whether RQRPA or just simple QRPA methods are used.

Clearly, the chosen procedure of finding the effective interaction is rather crude. Ideally, properly evaluated effective Hamiltonian, as well as the corresponding effective $\beta\beta$ operator should be used in each case. However, as shown below, the chosen procedure appears to be sufficient to stabilize the values of the nuclear matrix elements $M^{0\nu}$. Note, that we did not use the quenched value of the axial current coupling constant $g_A$, as is often done in studies of ordinary $\beta$ decay. We are convinced that with the quenched $g_A$ the basic conclusion of our work will be similar, even though the values of $g_{pp}$ in Table I would be of course different.

Having fixed the parameters of the effective Hamiltonian we can proceed and evaluate the $0\nu\beta\beta$ nuclear matrix elements $M^{0\nu}$ and then the corresponding half-life (we list the phase-space factors in Table II in units of $10^{-25}$ years for $\langle m_{\nu} \rangle$ in eV.) In evaluating $M^{0\nu}$ we take the short-range correlations into account in the standard way, i.e. by multiplying the operators with the square of the correlation function [23]

$$f(r) = 1 - e^{-ar^2} (1 - br^2); \quad a = 1.1 \, \text{fm}^{-2}, \quad b = 0.68 \, \text{fm}^{-2}.$$  

(4)

Note that the effect of short-range correlations reduces the matrix elements $M^{0\nu}$ by a factor of about two, in agreement with other evaluations.

### TABLE I. Values of the effective particle-particle strength parameter $g_{pp}$ for the different nuclei and different sizes of the single-particle space. The values in col. 3 correspond to the minimal s.p. space, in column 4 to the intermediate one, and in col. 5 to the largest s.p. space. In each of these there are three entries corresponding to the G-matrix based on the Bonn, Argonne, and Nijmegen potentials (in that order). For every considered nucleus the upper line was obtained in RQRPA and the lower one in the simple QRPA. In column 2 we give the corresponding half-life of the $2\nu\beta\beta$ decay used in the adjustment (see text, for $^{136}\text{Xe}$ the lower limit is used).

| Nucleus | $T_{1/2}^{2\nu}$ (in $10^{-20}$ y) | min. s.p. space | interm. s.p. space | largest s.p. space |
|---------|----------------------------------|-----------------|-------------------|-------------------|
| $^{100}\text{Ge}$ | 13.0 | 0.99 1.12 1.07 | 0.88 1.00 0.95 | 0.79 0.88 0.84 |
| $^{100}\text{Mo}$ | 0.08 | 1.21 1.35 1.30 | 1.09 1.21 1.17 | 1.00 1.10 1.07 |
| $^{136}\text{Te}$ | 27.0 | 0.97 1.10 1.05 | 0.90 1.01 0.97 | 0.84 0.94 0.90 |
| $^{136}\text{Xe}$ | 8.1 | 0.82 0.93 0.89 | 0.77 0.87 0.83 | 0.72 0.82 0.78 |

As pointed out earlier, in the nuclear shell model an even smaller set of single-particle states is used corresponding to $0\hbar\omega$. This choice reflects the practical computational limitations in handling the extremely large number of possible configurations, while it seems to be sufficient to describe the spectroscopy of low-lying nuclear states. In the NSM evaluation of the $\beta\beta$ decay rates [24] four s.p. orbits ($f_5/2, p_3/2, p_1/2, g_9/2$) were used.

### TABLE II. Values of the nuclear matrix element $|M^{0\nu}|$ for the different nuclei and different sizes of the single-particle space. In column 2 is corresponding phase space factor $G^{0\nu}$. For the explanation of the notation in columns 3-5 see the caption of Table I.

| Nucleus | $G^{0\nu}$ (in $10^{-25}$ y eV$^{-2}$) | min. s.p. space | interm. s.p. space | largest s.p. space |
|---------|--------------------------------------|-----------------|-------------------|-------------------|
| $^{76}\text{Ge}$ | 0.30 | 2.41 2.35 2.35 | 2.52 2.44 2.47 | 2.32 2.34 2.35 |
| $^{100}\text{Mo}$ | 2.19 | 1.08 1.08 1.05 | 1.12 1.14 1.08 | 1.28 1.34 1.27 |
| $^{136}\text{Te}$ | 2.12 | 1.42 1.32 1.34 | 1.40 1.33 1.32 | 1.17 1.13 1.14 |
| $^{136}\text{Xe}$ | 2.26 | 1.05 0.99 0.99 | 1.09 1.03 1.04 | 0.92 0.83 0.86 |

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for $^{76}$Ge and a five orbits ($d_{5/2}, d_{3/2}, s_{1/2}, g_{7/2}, h_{11/2}$) for
$^{130}$Te and $^{136}$Xe. These s.p. sets are free of the spurious
center-of-mass states, but obviously miss a large part of
the GT strength as well as of the strength corresponding
to the higher multipoles. In order to describe GT transi-
tions between low-lying states in the NSM, it is necessary
to quench the corresponding strength. This is most con-
veniently formally achieved by using $g_A = 1.0$ instead of
the free nucleon value of $g_A = 1.25$. We follow this pre-
scription in our attempt to use this smallest s.p. space,
and only there.

It appears that it is impossible to describe the $2\nu\beta\beta$
decay in such s.p. space using QRPA or RQRPA, and the
nucleon-nucleon potentials employed in this work. One
would have to re-normalize the particle-particle block too
much, with $g_{pp} \sim 2.0$, unlike the rather modest renor-
malization shown in Table I. With such large value of
$g_{pp}$, the interaction is too far removed from the $G$-matrix
used in the rest of this work. Therefore, one cannot expect
to obtain sensible $0\nu$ matrix elements. In fact, we
obtained very small matrix elements in this case for $^{130}$Te
and $^{136}$Xe, while, perhaps accidentally, for $^{76}$Ge they are
in a crude agreement with the NSM result [24].

We list the results with the three larger single-particle
bases in Table II which represents the most significant
part of the present work. As one can see by inspecting
the entries, one can draw two important conclusions:

- The resulting $M^{0\nu}$ do not depend noticeably on the
  form of the nucleon-nucleon potential used. That is
  not an unexpected result.

- Even more importantly, with our choice of $g_{pp}$ the
  results are also essentially independent on the size of
  the s.p. basis. This is a much less obvious and
  rather pleasing conclusion. It can be contrasted
  with the result one would get for a constant $g_{pp}$
  independent on the size of the s.p. basis. The
  values of $M^{0\nu}$ differ then between the small and
  large bases by a factor of two or more.

The effect of the $g_{pp}$ adjustment is illustrated in Fig.
1, showing that our procedure leads to almost constant
$M^{0\nu}$ matrix elements. On the other hand, by choosing a
fixed value of $g_{pp}$ the resulting $M^{0\nu}$ matrix elements
for 9 and 21 s.p. levels would differ substantially.

The entries in Table II are relatively close to each
other. To emphasize this feature, each calculated value is
averaged over the three potentials and the three choices
of the s.p. space is evaluated, as well as its variance $\sigma$

$$
\sigma^2 = \frac{1}{N-1} \sum_{i=1}^{N} (M_{i}^{0\nu} - \langle M^{0\nu} \rangle)^2, \quad (N = 9).
$$

These quantities (with the value of $\sigma$ in parentheses)
are shown in Table III. Not only is the variance subst-
tially less than the average value, but the results of QRPA, al-
beit slightly larger, are quite close to the RQRPA values.
The averaged nuclear matrix elements for both methods
and their variance are shown in Fig. 2.

Combining the average $\langle M^{0\nu} \rangle$ with the phase-space
factors listed in Table II the expected half-lives (for
RQRPA and $\langle m_\nu \rangle = 50$ meV, the scale of neutrino masses
suggested by oscillation experiments) are also shown in
Table III. These predicted half-lives are a bit longer (par-
ticularly for the last three nuclei on our list) then vari-
ous QRPA calculations usually predict. They are faster,
however, then the shell model results of Ref. [24].

**TABLE III.** Averaged $0\nu\beta\beta$ nuclear matrix elements
$\langle M^{0\nu} \rangle$ and their variance $\sigma$ (in parentheses) evaluated in the
RQRPA and QRPA. In column 4 the $0\nu\beta\beta$ half-lives evaluated
with the RQRPA average nuclear matrix element and for
the $\langle m_\nu \rangle = 50$ meV are shown.

| Nucleus | RQRPA | QRPA | $T_{1/2}$ ($10^{27} y$ for $\langle m_\nu \rangle = 50$ meV) |
|---------|-------|------|-----------------|
| $^{76}$Ge | 2.40(0.07) | 2.68(0.06) | 2.3 |
| $^{130}$Te | 1.16(0.11) | 1.28(0.09) | 1.4 |
| $^{136}$Xe | 1.29(0.11) | 1.35(0.13) | 1.1 |
| $^{136}$Xe | 0.98(0.09) | 1.03(0.08) | 1.9 |
FIG. 2. Average nuclear matrix elements $\langle M^{0\nu} \rangle$ and their variance for both methods and for the four considered nuclei.

Given the average nuclear matrix elements in Table III and the phase space factors in Table II one can find a limit (or actual value) of the effective neutrino mass $\langle m_\nu \rangle$ from any limit (or value) of $T_{1/2}$ from

$$\langle m_\nu \rangle \ (\text{eV}) = \left( T_{1/2} \times G^{0\nu} \right)^{-1/2} \times \frac{1}{\langle M^{0\nu} \rangle}. \quad (6)$$

In conclusion, we have developed a “practical” way of stabilizing the values of the $0\nu\beta\beta$ nuclear matrix elements against their variation caused by the modification of the nucleon-nucleon interaction potential and the chosen size of the single-particle space. We have also shown that the procedure yields very similar matrix elements for the QRPA and RQRPA variants of the basic method. Even though we cannot guarantee that this basic method is trustworthy, we have eliminated, or at least greatly reduced, the arbitrariness commonly present in the published calculations.

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