Twisted cohomology pairings of knots III; triple cup products

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Abstract

Given a representation of a link group, we introduce a trilinear form, as a topological invariant. We show that, if the link is either hyperbolic or a knot with malnormality, then the trilinear form equals the pairing of the (twisted) triple cup product and the fundamental relative 3-class. Further, we give some examples of the computation.

Keywords

Cup product, Bilinear form, knot, twisted Alexander polynomial, group homology, quandle

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1 Introduction

This paper examines topological invariants of trilinear forms, while the previous papers \([N2]\) in this series discussed of bilinear forms. In general, bilinear form arising from Poincaré duality is a powerful method, as in algebraic surgery theory and classification theorems of some manifolds. In contrast, there are not so many studies of trilinear forms. However, some 3-forms and trilinear cup products appear in 3-dimensional geometry together with topological information (see, e.g., \([CGO, M, L, S, Tur1]\)). For example, we mention interesting observations from the Chern-Simons invariant (or \(\phi^3\)-theory) of the form

\[
\frac{k}{2\pi} \int_M \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).
\]

We give the definition of trilinear pairing (see (1)) in a general situation where the coefficients are arbitrary. Let \( M \) be a compact 3-manifold with toroidal boundary with orientation
3-class \([Y, \partial Y] \in H_3(Y, \partial Y; \mathbb{Z}) \cong \mathbb{Z}\). Choose a group homomorphism \(\pi_1(Y) \to G\), a right \(G\)-module \(M\), and a \(G\)-invariant trilinear function \(\psi : M^3 \to A\) over a ring \(A\). Then, we can define the composite map

\[
H^1(Y, \partial Y; M)^{\otimes 3} \xrightarrow{\sim} H^3(Y, \partial Y; M^{\otimes 3}) \xrightarrow{\langle \cdot, [Y, \partial Y] \rangle} M^{\otimes 3} \xrightarrow{\psi} A.
\]

Here \(M\) is regarded as the local coefficient of \(Y\) via \(f\), and the first map \(\sim\) is the cup product, and the second (resp. third) is defined by the pairing with \([Y, \partial Y]\) (resp. \(\psi\)). In contrast to this definition, the 3-form \((1)\) is considered to be something uncomputable. Actually, it seems hard to concretely deal with the 3-class \([Y, \partial Y]\) and the cup products.

This paper addresses the link case where \(Y\) is the 3-manifold which is obtained from the 3-sphere by removing an open tubular neighborhood of a link \(L\), i.e., \(Y = S^3 \setminus \nu L\). In fact, if \(L\) is a hyperbolic link, we obtain a diagrammatic method of computing the trilinear pairings. To be precise, in Section 2.2 starting from a link diagram, we define invariants of trilinear forms, and show (Theorem 2.5) that the invariant is equal to \((1)\), if \(L\) is a hyperbolic link. In addition, we also show a similar theorem in the torus case (see Theorem 2.6). The point in the theorem is that, in the computation, we do not need describing \([Y, \partial Y]\) and cup products; thus, this computation is not so hard. In fact, we give some examples; see §4. In addition, as an application (Theorem 3.1), when \(Y\) is a 3-fold covering space of \(S^3\) branched along a hyperbolic link \(L\) and \(M\) is a trivial coefficient, we give a diagrammatic computation of the trilinear pairing \((1)\).

This paper is organized as follows. Section 2 formulates the trilinear forms in terms of the quandle cocycle invariants, and states the main theorems. Section 3 discusses a relation to 3-fold branched coverings. Section 4 describes some computations. Section 5 gives the proofs of the theorems.

**Notation.** Every link \(L\) is smoothly embedded in the 3-sphere \(S^3\) with orientation. We write \(E_L\) for the 3-manifold which is obtained from \(S^3\) by removing an open neighborhood of \(L\).

### 2 Results; diagrammatic formulations of the trilinear forms

Our purpose in this section is to give a link invariant of trilinear form (Theorem 2.3), and to state the main results in §2.2. For this purpose, §2.1 starts by reviewing colorings, and formulates some link-invariants of linear forms.

Thorough this section, we fix a group \(G\) and a right \(G\)-module \(M\) over a ring \(A\).

#### 2.1 Preliminary; the formulations of the first cohomology

We need some notation from [IIJO, N2] before proceeding. Denote \(M \times G\) by \(X\). Further, define a binary operation on \(X\) by

\[
\triangleleft : (M \times G) \times (M \times G) \longrightarrow M \times G,
(a, g, b, h) \longmapsto (a - b) \cdot h + b, \ h^{-1}gh,
\]

which was first introduced in [IIJO, Lemma 2.2], and satisfies “the quandle axiom”. Furthermore, we choose a link \(L \subset S^3\) with a group homomorphism \(f : \pi_1(S^3 \setminus L) \to G\).
Next, we review colorings. Choose an oriented diagram $D$ of $L$. Then, it follows from the Wirtinger presentation of $D$ that the homomorphism $f$ is regarded as a map $\{\text{arcs of } D\} \rightarrow G$. Furthermore, a map $C : \{\text{arcs of } D\} \rightarrow X$ is an $X$-coloring if it satisfies $C(\alpha_{\tau}) \triangleleft C(\beta_{\tau}) = C(\gamma_{\tau})$ at each crossings of $D$ illustrated as Figure 1. It is worth noticing that the set of all colorings is regarded as a subset of the direct product $X^\alpha_D$, where $\alpha_D$ is the number of arcs of $D$. Let $\text{Col}_X(D_f)$ denote the set of all $X$-colorings over $f$, that is,

$$\text{Col}_X(D_f) := \{ C \in (M \times G)^{\alpha_D} | C \text{ is an } X\text{-coloring, } p_G \circ C = f \},$$

(3)

where $p_G$ is the projection $X = M \times G \rightarrow G$. Then, we can easily verify from the linear operation (2) that $\text{Col}_X(D_f)$ is made into an abelian subgroup of $M^{\alpha(D)}$, and that the diagonal subset $M_{\text{diag}} \subset M^{\alpha_D}$ is a direct summand in $\text{Col}_X(D_f)$. Denoting another summand by $\text{Col}_{\text{red}}^X(D_f)$, we have a decomposition $\text{Col}_X(D_f) \cong \text{Col}_{\text{red}}^X(D_f) \oplus M_{\text{diag}}$.

The previous paper [N2] gave a topological meaning of the coloring sets as follows:

**Theorem 2.1 ([N2]).** Let $E_L$ be a link complement in $S^3$ as in §1. Regard the $G$-module $M$ as a local system of $E_L$ via $f : \pi_1(E_L) \rightarrow G$. Then, there are isomorphisms

$$\text{Col}_X(D_f) \cong H^1(E_L; \partial E_L; M) \oplus M, \quad \text{Col}_{\text{red}}^X(D_f) \cong H^1(E_L, \partial E_L; M).$$

(4)

Furthermore, let us review shadow colorings [CKS, IJO]. A shadow coloring is a pair of a coloring $C$ over $f$ and a map $\lambda$ from the complementary regions of $D$ to $M$, satisfying the condition depicted in the right side of Figure 1 for every arcs. Let $\text{SCol}_X(D_f)$ denote the set of shadow colorings of $D$ such that the unbounded exterior region is assigned by $0 \in M$. Notice that, by the coloring rules, assignments of the other regions are uniquely determined from the unbounded region, and admit, therefore, a shadow coloring; we thus obtain a bijection

$$\text{Col}_X(D_f) \cong \text{SCol}_X(D_f).$$

(5)

\[ \begin{array}{ccc}
\alpha_{\tau} & \beta_{\tau} & \gamma_{\tau} \\
& C(\alpha_{\tau}) \triangleleft C(\beta_{\tau}) = C(\gamma_{\tau}) & \\
& \delta & \lambda(R') = (\lambda(R) - b) \cdot h + b. \\
\end{array} \]

Figure 1: The coloring conditions at each crossing $\tau$ and around each arcs.

### 2.2 Invariants of trilinear forms

In addition, we will explain Definition 2.2 below, and show Theorem 2.3.

For this, we need two things: first, we take three $G$-modules $M_1$, $M_2$, $M_3$ and the associated $X_i = M_i \times G$. Let $A$ be an abelian group. On the other hand, we prepare a trilinear map $\psi : M_1 \times M_2 \times M_3 \rightarrow A$ over $\mathbb{Z}$ satisfying the $G$-invariance, that is,

$$\psi(a_1 \cdot g, a_2 \cdot g, a_3 \cdot g) = \psi(a_1, a_2, a_3),$$

(6)

holds for any $a_i \in M_i$ and $g \in G$. 

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Next, let us consider the map \( X_1 \times X_2 \times X_3 \to A \) by the formula
\[
((b_1, g_1), (b_2, g_2), (b_3, g_3)) \mapsto \psi((b_1 - b_2) \cdot (1 - g_2), b_2 - b_3, b_3 - b_3 \cdot g_3^{-1}),
\]
for \( a_i \in M_i \) and \( g_1, g_2, g_3 \in G \). This map was first defined in [4] Corollary 4.6. Furthermore, given three shadow colorings \( S_i \in SCol_{X_i}(D_f) \) with \( i \leq 3 \) and each crossing \( \tau \) of \( D \), we can find assignments as illustrated in Figure 2. Inspired by the formula (7), we define a weight of \( \tau \) to be
\[
W_{\psi, \tau}(S_1, S_2, S_3) := \psi((a_1 - b_1)(1 - g^{\epsilon_\tau}), b_2 - c_2, c_3 - c_3 \cdot h^{-1}) \in A,
\]
where \( \epsilon_\tau \in \{ \pm 1 \} \) is the sign of \( \tau \).

![Figure 2: Colors around a crossing with respect to three shadow colorings.](image)

**Definition 2.2.** Given a \( G \)-invariant trilinear map \( \psi : M_1 \times M_2 \times M_3 \to A \), we define a trilinear map
\[
\mathcal{T}_\psi : \prod_{i=1}^{3} SCol_{X_i}(D_f) \to A; \quad (S_1, S_2, S_3) \mapsto \sum_\tau W_{\psi, \tau}(S_1, S_2, S_3),
\]
where \( \tau \) runs over all the crossings of \( D \).

The point is that, given a diagram \( D \), we can diagrammatically deal with the trilinear \( \mathcal{T}_\psi \) by definitions; see [4.4.3] for examples.

Next, we now show the invariance of \( \mathcal{T}_\psi \) up to trilinear equivalence:

**Theorem 2.3.** Let two diagrams \( D \) and \( D' \) differ by a Reidemeister move. There is a canonical isomorphism \( B_i : SCol_{X_i}(D_f) \cong SCol_{X_i}(D'_f) \), for which the equality \( \mathcal{T}_\psi = \mathcal{T}'_\psi \circ (B_1 \otimes B_2 \otimes B_3) \) holds as a map.

In particular, the equivalence class of the trilinear map \( \mathcal{T}_\psi \) depends on only the homomorphism \( f : \pi_1(S^3 \setminus L) \to G \) and the input data \( (M_1, M_2, M_3, \psi) \).

**Proof.** We first focus on Reidemeister move of type III; see Figure 3. Then, considering the correspondence in Figure 3 with \( x_i, y_i, z_i \in X_i \), we have the bijection \( B_i \). Moreover, we suppose that the left region is colored by \( r_i \in M_i \). Thus, it is enough to show the desired equality. For this, take \( a_i, b_i, c_i \in M_i \) and \( g, h, k \in G \) such that \( x_i = (a_i, g), y_i = (b_i, h), z_i = (c_i, k) \in X_i \). Then, the sum from the left side is, by definition and examining the figure, computed as
\[
\psi((r_1 - a_1)(1 - g), a_2 - c_2, c_3(1 - k^{-1})) + \psi((r_1 g - a_1 g + a_1 - b_1)(1 - h), b_2 - c_2, c_3(1 - k^{-1}))
\]
\[
+ \psi((r_1 - a_1) k(1 - k^{-1} g k), (a_2 - b_2) k, (b_3 k - c_3 k + c_3)(1 - k^{-1} h^{-1} k)).
\]
On the other hand, the sum from the right side is formulated as
\[
\psi((r_1 - a_1)(1 - g), a_2 - b_2, b_3(1 - h^{-1})) + \psi((r_1 - b_1)(1 - h), b_2 - c_2, c_3(1 - k^{-1}))
\]
Then, an elementary calculation from (6) can show the two sums are equal. However, since the calculation is a little tedious, we omit the detail.

Finally, the required equality concerning Reidemeister moves of type I immediately follows from \(\psi(0, y, z) = 0\), and the invariance of type II is clear by a similar discussion.

\[
\begin{align*}
\mathcal{T}_1 & \mathcal{T}_2 \\
\mathcal{T}_3 & \mathcal{T}_4
\end{align*}
\]

Figure 3: The 1:1-correspondence associated with a Reidemeister move of type III.

Remark 2.4. In this way, the construction for trilinear forms is applicable to not only tame links in \(S^3\), but also handlebody-knots \(\mathcal{H}_g\) in \(S^3\). In fact, as a similar discussion to [IIJO], we can easily check that the trilinear form is invariant with respect to the diagrammatic moves of handlebody-knots; see [IIJO] Figures 1 and 2) for the moves.

2.3 Topological meaning of the trilinear forms

As mentioned in the introduction, we will show (Theorems 2.5 and 2.6) that the trilinear forms of some links are equal to the trilinear pairings (The proofs of the theorems appear in §5).

Theorem 2.5. Let \(M_1, M_2, M_3\) be \(G\)-modules as in Definition 2.2. Furthermore, choose a fundamental class \([E_L, \partial E_L]\) in \(H_3(E_L, \partial E_L; \mathbb{Z}) \cong \mathbb{Z}\).

We assume that \(L\) is either a hyperbolic link or a prime knot which is neither a cable knot nor a torus knot. Then, via the identification (4), the trilinear form \(\mathcal{T}_\psi\) is equal to the following composite map:

\[
\bigotimes_{i: 1 \leq i \leq 3} H^1(E_L, \partial E_L; M_i) \xymatrix{\ar[r]^-{\sim} &} H^3(E_L, \partial E_L; M_1 \otimes M_2 \otimes M_3) \xymatrix{\ar[r]^-{\psi\bullet \cdot [E_L, \partial E_L]} &} A. \tag{8}
\]

In addition, we mention the torus knot, although we need a condition. More precisely,

Theorem 2.6. Let \(M_1, M_2, M_3, \psi, \) and \([E_L, \partial E_L]\) be as above.

Assume that \(L\) is the \((m, n)\)-torus knot. Then, the trilinear form \(\mathcal{T}_\psi\) is equal to the composite \(\bigotimes\) modulo the integer \(nm \in \mathbb{Z}\).

As a concluding remark, while the triple cup product of a link often is considered to be speculative and uncomputable, it become computable from only a link diagram without describing \([E_L, \partial E_L]\) and any triangulation in \(S^3 \setminus L\).
Remark 2.7. Finally, we compare the trilinear forms in Definition 2.2 with the existing results on “the quandle cocycle invariants”, in detail. Briefly speaking, the link invariant in [CKS] is constructed from a quandle $X$ and a map $\Phi : X^3 \to A$ which satisfy “the quandle cocycle condition”, and is defined to be a certain map $\mathcal{J}_\Phi : \text{SCol}_X(D) \to A$. Then, we note that our trilinear form is a trilinearization of the quandle cocycle invariants with respect to quandles of the form $X = M \times G$. To be precise, if $M = M_1 = M_2 = M_3$, we can see that the associated invariant $\mathcal{J}_\Phi : \text{SCol}_X(D) \to A$ is equal to the composite $\mathcal{T}_\psi \circ (\triangle \times \text{id}) \circ \triangle$ by definitions. In conclusion, the theorems also suggest topological meanings of the quandle cocycle invariants with $X = M \times G$.

3 Relation to 3-fold branched coverings

Although we considered relative cohomology, for $n \in \mathbb{Z}_{\geq 0}$ and a closed 3-manifold $N$, let us consider the triple cup product

$$H^1(N; \mathbb{Z}/n\mathbb{Z})^\otimes 3 \xrightarrow{\sim} H^3(N; \mathbb{Z}/n\mathbb{Z}) \xrightarrow{(\ast, [N, \partial N])} \mathbb{Z}/n,$$

where the coefficient module $\mathbb{Z}/n$ is trivial. Although there are studies of this map (see, e.g., [CGO], [Tur]), there are few examples of the computation. As an application of the theorems above, this section gives a recover of the triple cup products of $N$, when $N$ is a 3-fold cyclic covering of $S^3$ branched over a link.

To state Theorem 3.1 we need some terminology. Let $G$ be $\mathbb{Z}/3 = \langle t | t^3 = 1 \rangle$. Consider the epimorphism $f : \pi_1(S^3 \setminus L) \to G$ which sends every meridian to $t$, and the associated 3-fold cyclic branched covering $\widetilde{C}_L \to S^3$.

Theorem 3.1. Let $M_1$, $M_2$, and $M_3$ be $\mathbb{Z}[t^{\pm 1}]/(n, t^2 + t + 1)$. Let $p : \mathbb{Z}[t^{\pm 1}]/(n, t^2 + t + 1) \to \mathbb{Z}/n$ be the map which sends $a + tb$ to $a$. Set up the map $\psi_0 : M^3 \to \mathbb{Z}/n$ which takes $(x, y, z)$ to $xyz$. As in Theorem 2.3, assume that $L$ is either a hyperbolic link or a prime knot which is neither a cable knot nor a torus knot.

Then, there is an isomorphism $\text{Col}^\text{red}_{X_i}(D_f) \cong H^1(\widetilde{C}_L; \mathbb{Z}/n\mathbb{Z})$ such that the trilinear map $\mathcal{T}_{\psi_0}$ is equivalent to (9) with $N = \widetilde{C}_L$.

Proof of Theorem 3.1 We first show the isomorphism $\text{Col}^\text{red}_{X_i}(D_f) \cong H^1(\widetilde{C}_L; \mathbb{Z}/n\mathbb{Z})$. Let $R$ be the ring $\mathbb{Z}[t]/(n, t^2 + t + 1)$. By Theorem 2.1, we have $\text{Col}^\text{red}_{X_i}(D_f) \cong H^1(E_L; \partial E_L; M)$. Notice that $H^1(\partial E_L; M)$ is annihilated by $1 - t$. Since $1 - t$ and $1 + t + t^2$ are coprime, we have

$$H^1(E_L; \partial E_L; M) \cong H^1(E_L; M) \cong \text{Hom}_{R\text{-mod}}(H_1(E_L, M), R).$$

(10)

Let $\widetilde{E}_L \to E_L = S^3 \setminus L$ be the 3-fold covering. Then, by Shapiro’s Lemma (see, e.g., [Bro]), the canonical inclusion $i : \mathbb{Z}/n \to R$ yields the isomorphisms:

$$H^*(\widetilde{E}_L; \mathbb{Z}/n) \cong H^*(E_L; \mathbb{Z}[t]/(n, t^3 - 1)) \cong H^*(E_L; R) \oplus H^*(E_L; \mathbb{Z}[t]/(n, t - 1)).$$

(11)

Here, the second isomorphism is obtained from the ring isomorphism $\mathbb{Z}[t]/(n, t^3 - 1) \cong R \oplus \mathbb{Z}[t]/(n, t - 1)$. Let $i : \widetilde{E}_L \hookrightarrow C_L$ be the inclusion. According to [Kaw, Theorem 5.5.1], the homology $H_1(C_L; \mathbb{Z})$ is annihilated by $1 + t + t^2$, and the induced map $i_* : H_1(\widetilde{E}_L; \mathbb{Z}) \to H_1(C_L; \mathbb{Z})$
is a splitting surjection. Thus, dually, the induced map $i^* : H^1(\tilde{C}_L; \mathbb{Z}/n) \to H^1(\tilde{E}_L; \mathbb{Z}/n)$ is injective and the image is isomorphic to $H^1(E_L; R)$. In summary, we obtained the desired isomorphism.

We will complete the proof. By (11), we have a splitting injection $\mathcal{S} : H^*(\tilde{E}_L, \partial \tilde{E}_L; R) \to H^*(E_L, \partial E_L; \mathbb{Z}/n)$. Take the canonical maps $j : (\tilde{E}_L, \partial \tilde{E}_L) \to (\tilde{C}_L, \tilde{C}_L \setminus \tilde{E}_L)$ and $k : (C_L, \emptyset) \to (\tilde{C}_L, \tilde{C}_L \setminus \tilde{E}_L)$. Then, we have the commutative diagrams on the cup products:

$\begin{align*}
H^1(E_L, \partial E_L; M)^{\otimes 3} &\xrightarrow{\psi_0 \circ \sigma} H^3(E_L, \partial E_L; M) \xrightarrow{(\cdot, [E_L, \partial E_L])} R \\
H^1(\tilde{E}_L, \partial \tilde{E}_L; \mathbb{Z}/n)^{\otimes 3} &\xrightarrow{\sim} H^3(\tilde{E}_L, \partial \tilde{E}_L; \mathbb{Z}/n) \xrightarrow{\sim} \mathbb{Z}/n \\
H^1(\tilde{C}_L, \tilde{C}_L \setminus \tilde{E}_L; \mathbb{Z}/n)^{\otimes 3} &\xrightarrow{\sim} H^3(\tilde{C}_L, \tilde{C}_L \setminus \tilde{E}_L; \mathbb{Z}/n) \xrightarrow{(\cdot, [\tilde{C}_L, \tilde{C}_L \setminus \tilde{E}_L])} \mathbb{Z}/n \\
H^1(\tilde{C}_L; \mathbb{Z}/n)^{\otimes 3} &\xrightarrow{\sim} H^3(\tilde{C}_L; \mathbb{Z}/n) \xrightarrow{(\cdot, [\tilde{C}_L])} \mathbb{Z}/n.
\end{align*}$

Here, the vertical maps $j^*$ are isomorphisms by the excision axiom. Moreover, by the discussion in the above paragraph, the composite $k^* \circ (j^*)^{-1} \circ \mathcal{S}$ is an isomorphism from $H^1(E_L, \partial E_L; M)$. Hence, since $p \circ i : \mathbb{Z}/n \to \mathbb{Z}/n$ is an isomorphism, the following two composites are equivalent:

$p \circ \psi_0 \circ (\cdot, [E_L, \partial E_L]) \circ \sim, \quad (\cdot, [\tilde{C}_L]) \circ \sim.$

By Theorem 2.5, the left hand side is equal to the trilinear map $\mathcal{T}_{p \circ \psi_0}$. Hence, $\mathcal{T}_{p \circ \psi_0}$ is equivalent to (9) with $N = \tilde{C}_L$ as desired.

4 Examples as diagrammatic computations

4.1 General situation for the trefoil knot and the figure eight knot

![Image of the trefoil knot and the figure eight knot](image)

Figure 4: The trefoil knot and the figure eight knot

We will compute the trilinear forms $\mathcal{T}_\psi$ associated with some homomorphisms $f : \pi_L \to G$, where $L$ is either the trefoil knot or the figure eight knot.

As a simple example, we will focus on the trefoil knot $3_1$. Let $D$ be the diagram of $K$ as illustrated in Figure 3. Note the Wirtinger presentation $\pi_L \cong \langle \alpha, \beta \mid \alpha \beta \alpha = \beta \alpha \beta \rangle$. Then, we can easily see that a correspondence $\mathcal{C} : \{\alpha, \beta, \gamma\} \to X$ with

$$\mathcal{C}(\alpha) = (a_i, g), \quad \mathcal{C}(\beta) = (b_i, g), \quad \mathcal{C}(\gamma) = (c_i, g) \in M_i \times G$$
is an $X$-coloring $C$ over $f : \pi_L \to G$, if and only if it satisfies the four equations

$$g = f(\alpha), \ h = f(\beta), \ ghg = hgh,$$

(12)

$$c_i = a_i \cdot h + b_i \cdot (1 - h),$$

(13)

$$(a_i - b_i) \cdot (1 - g + hg) = (a_i - b_i) \cdot (1 - h + gh) = 0.$$  

(14)

Furthermore, given a $G$-invariant linear form $\psi$, the sum $\mathcal{T}_\psi$ is equal to

$$\psi(-a_1 \cdot (1 - g), \ a_2 - b_2, \ a_3 \cdot (1 - h^{-1})) + \psi(-b_1 \cdot (1 - h), \ b_2 - c_2, \ c_3 \cdot (1 - h^{-1}g^{-1}h))$$

$$+ \psi(-c_1 \cdot (1 - h^{-1}gh), \ c_2 - a_2, \ a_3 \cdot (1 - g^{-1})),$$

by definition. Then, by canceling out $c_i$ by using (13) and (12), we can easily obtain the resulting computation: for $((a_i, g), (b_i, h)) \in \text{SCol}_X(D_f) \subset M_i^2,$

$$\mathcal{T}_\psi((a_1, b_1), (a_2, b_2), (a_3, b_3)) = \psi((a_1 - b_1)g^{-1}, \ (a_2 - b_2) \cdot h, \ a_3 - b_3) \in A.$$  

(15)

Next, we will compute $\mathcal{T}_\psi$ of the figure eight knot. However, the computation can be done in a similar way to the trefoil case. So we only describe the outline.

Let $D$ be the diagram with arcs as illustrated in Figure 1. Similarly, we can see that a correspondence $C : \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \to X$ with $C(\alpha_i) = (x_i, z_i) \in M_i \times G$ is an $X$-coloring $C$ over $f : \pi_L \to G$, if and only if it satisfies the following equations:

$$z_i = f(\alpha_i), \ z_2^{-1}z_1z_2 = z_1^{-1}z_2^{-1}z_1z_2z_1^{-1}z_2z_1 \in G,$$

(16)

$$x_3 = (x_1 - x_2) \cdot z_2 + x_2, \quad x_4 = (x_2 - x_1) \cdot z_1 + x_1,$$

(17)

$$(x_1 - x_2) \cdot (z_1 + z_2 - 1) = (x_1 - x_2) \cdot (1 - z_2^{-1})z_1z_2 = (x_1 - x_2) \cdot (1 - z_1^{-1})z_2z_1 \in M.$$  

(18)

Accordingly, it follows from (17) that the set Col$_X(D_f)$ is generated by $x_1, x_2$.

Given a $G$-invariant trilinear form $\psi$, it can be seen that the trilinear form $\mathcal{T}_\psi((x_1, x_2), (x_1', x_2'), (x_1'', x_2''))$ is expressed as

$$\psi((x_1 - x_2)z_1z_2^{-1}, \ x_2' - x_1', \ (x_1'' - x_2'')(1 - z_2^{-1})) + \psi((x_1 - x_2)z_2^{-1}z_1, \ (x_1' - x_2')(1 - z_1), \ (x_1'' - x_2'')(1 - z_2^{-1})z_1).$$

**Remark 4.1.** Here, we should give some examples from concrete $M$ and $\psi$. In particular, the author attempted to get non-trivial trilinear form $\mathcal{T}_\psi$ when $G$ is a Lie group and $M$ is a representation of $G$. However, even if $G = SL_2(\mathbb{R})$ or $G = SL_2(\mathbb{C})$ and $M = \mathbb{C}^2$ or $\mathbb{C}^3$, the author computed the resulting $\mathcal{T}_\psi$ equal to zero. Indeed, the author could not find non-trivial examples of $\mathcal{T}_\psi$ except those in (13).

Thus, it is a problem to find non-trivial examples of $\mathcal{T}_\psi$ from representations with respect to Lie groups.
4.2 The \((m, m)\)-torus link \(T_{m,m}\)

We also calculate the trilinear form \(\mathcal{T}_\psi\) concerning the \((m, m)\)-torus link, following from Definition 2.2. These calculations will be useful in the paper \([N4]\), which suggests invariants of “Hurewitz equivalence classes”.

Let \(L\) be the \((m, m)\)-torus link \(T_{m,m}\) with \(m \geq 2\), and let \(\alpha_1, \ldots, \alpha_m\) be the arcs depicted in Figure 4. Furthermore, let us identity \(a_{i+m}\) with \(a_i\) of period \(m\). By Wirtinger presentation, we have a presentation of \(\pi_L\) as

\[
\langle a_1, \ldots, a_m | a_1 \cdots a_m = a_m a_1 a_2 \cdots a_{m-1} = a_{m-1} a_m a_1 \cdots a_{m-2} = \cdots = a_2 \cdots a_m a_1 \rangle.
\]

Given a homomorphism \(f : \pi_L \to G\) with \(f(\alpha_i) \in G\), let us discuss \(X\)-colorings \(C\) over \(f\). Then, concerning the coloring condition on the \(\ell\)-th link component, it satisfies the equation

\[
\big( \cdots (C(\alpha_{\ell}) \triangleleft C(\alpha_{\ell+1}) \triangleleft \cdots) \big) \triangleleft C(\alpha_{\ell+m-1}) = C(\alpha_\ell), \quad \text{for any } 1 \leq \ell \leq m. \tag{19}
\]

With notation \(C(\alpha_i) : = (x_i, z_i) \in X\), this equation (19) reduces to a system of linear equations

\[
(x_{\ell-1} - x_\ell) + \sum_{\ell \leq j \leq \ell + m - 2} (x_j - x_{j+1}) \cdot z_{j+1} z_{j+2} \cdots z_{m+\ell} = 0 \in M, \quad \text{for any } 1 \leq \ell \leq m. \tag{20}
\]

Conversely, we can easily verify that, if a map \(C : \{\text{arcs of } D\} \to X\) satisfies the equation (20), then \(C\) is an \(X\)-coloring. Denoting the left side in (20) by \(\Gamma_{f,k}(\vec{x})\), consider a homomorphism

\[
\Gamma_f : M^m \to M^m; \quad (x_1, \ldots, x_m) \mapsto (\Gamma_{f,1}(\vec{x}), \ldots, \Gamma_{f,m}(\vec{x})�).
\]

To conclude, the set \(\text{Col}_X(D_f)\) coincides with the kernel of \(\Gamma_f\).

Next, we precisely formulate the resulting trilinear form.

**Proposition 4.2.** Let \(f : \pi_1(S^3 \setminus T_{m,m}) \to G\) be as above. Let \(\psi : M^3 \to A\) be a \(G\)-invariant linear functions. Then, the trilinear form \(\mathcal{T}_\psi : \text{Ker}(\Gamma_f)^{\otimes 3} \to A\) sends \((w_1, \ldots, w_m) \otimes (x_1, \ldots, x_m) \otimes (y_1, \ldots, y_m)\) to

\[
\sum_{\ell=1}^{m-1} \sum_{k=1}^{m-1} \psi(w_\ell \cdot (1 - z_\ell)) \cdot \hat{z}_{\ell+1} z_{\ell+1} \cdots \sum_{j=1}^{k} (x_{j+\ell} - x_{j+\ell}) \cdot \hat{z}_{j+\ell} \hat{z}_{j+\ell} z_{j+\ell+1} \cdot y_{k+\ell} \cdot (1 - z_{k+\ell}^{-1}). \tag{21}
\]

Here, for \(s \leq t\), we use notation \(\hat{z}_{s:t} : = z_s z_{s+1} \cdots z_t\) and \(\hat{z}_{s+1:s} : = 1 \in G\).

The formula is obtained by direct calculation and definitions.

4.3 Examples of Theorem 3.1

We will give some examples in Theorem 3.1. Thus, we should suppose the situation of Theorem 3.1 as follows. Let \(G = \mathbb{Z}/3 = \langle t | t^3 = 1 \rangle\), and \(f : \pi_1(S^3 \setminus L) \to \mathbb{Z}/3\) be the map which sends every meridian to \(t\). Furthermore, take \(M_i = A = \mathbb{Z}[t]/(n, t^2 + t + 1)\) for some \(n \in \mathbb{Z}_{\geq 0}\), let \(\psi_0 : M_1 \times M_2 \times M_3 \to A\) send \((x, y, z)\) to \(xyz\).

In this paragraph, we focus on only knots, \(K\), such that \(H^1(E_K, \partial E_K; A) \cong H^1(\tilde{B}_K; \mathbb{Z}/n)\) is isomorphic to either \(A\) or 0. We will write the trilinear map \(\mathcal{T}_{\psi_0}\) as a cubic polynomial with respect to \(a, b, c \in (H^1(E_K, \partial E_K; \mathbb{Z}/n))^3\). Then, we give the resulting computation of \(\mathcal{T}_{\psi_0}\), when \(K\) is a prime knot with crossing number < 7. The list of the computation is as follows:
5 Proofs of the theorems

We will complete the proofs of Theorems 2.5–2.6 in §5.3. While the statements were described in terms of ordinary cohomology, the proof will be done via the group cohomology. For this purpose, in §5.1, we review the relative group homology.

5.1 Preliminary; Review of relative group cohomology

We will spell out the relative group (co)homology in the non-homogeneous terms. Throughout this subsection, we fix a group Γ and a homomorphism \( f : Γ \to G \). Then, we have the action of \( G \) on the right \( G \)-module \( M \) via \( f \).

Let \( C^n_\text{gr}(Γ; M) \) be \( \text{Map}(Γ^n, M) \). For \( φ ∈ C^n_\text{gr}(Γ; M) \), define the coboundary \( ∂^n(φ) ∈ C^{n+1}_\text{gr}(Γ; M) \) by the formula

\[
∂^n(φ)(g_1, \ldots, g_{n+1}) = φ(g_2, \ldots, g_{n+1}) + \sum_{1 ≤ i ≤ n}(-1)^i φ(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) + (-1)^n φ(g_1, \ldots, g_n)g_{n+1}.
\]

Furthermore, we set subgroups \( K_j \) and the inclusions \( \iota_j : K_j \hookrightarrow Γ \), where the index \( j \) runs over \( 1 ≤ j ≤ m \). Then, we can define the mapping cone of \( \iota_j \): More precisely,

\[
C^n(Γ, K_\mathcal{J}; M) := \text{Map}(Γ^n, M) ⊕ (⊕_j \text{Map}((K_j)^{n-1}, M)).
\]

For \( (h, k_1, \ldots, k_m) ∈ C^n(Γ, K_\mathcal{J}; M) \), let us define \( ∂^n(h, k_1, \ldots, k_m) \) in \( C^{n+1}(Γ, K_\mathcal{J}; M) \) by

\[
∂^n(h, k_1, \ldots, k_m)(a, b_1, \ldots, b_m) = (∂^n h(a), h(b_1) - ∂^{n-1} k_1(b_1), \ldots, h(b_m) - ∂^{n-1} k_m(b_m)),
\]

where \( (a, b_1, \ldots, b_m) ∈ Γ^{n+1} × K_1^n × \cdots × K_m^n \). Then we have a complex \( (C^*(Γ, K_\mathcal{J}; M), ∂^*) \), and can define the cohomology.

We now observe the submodule consisting of 1-cocycles \( Z^1(Γ, K_\mathcal{J}; M) \). Let us define the semi-direct product \( M × G \) by

\[
(a, g) * (a', g') := (a ∗ g' + a', gg'), \quad \text{for } a, a' ∈ M, \quad g, g' ∈ G.
\]

Let \( \text{Hom}_f(Γ, M × G) \) be the set of group homomorphisms \( Γ \to M × G \) over the homomorphism \( f \). Consider a map

\[
Z^1(Γ, K_\mathcal{J}; M) \to \text{Hom}_f(Γ, M × G) ⊕ M^m; \quad (h, y_1, \ldots, y_m) \mapsto (γ \mapsto (h(γ), f(γ)), y_1, \ldots, y_m).
\]
Lemma 5.1 ([N2, Lemma 5.2]). This map gives an isomorphism between $Z^1(\Gamma, K_\partial; M)$ and the following set:

$$\{ (f, y_1, \ldots, y_m) \in \text{Hom}_G(\Gamma, M \rtimes G) \oplus M^m \mid \tilde{f}(h_j) = (y_j - y_j \cdot h_j, f_j(h_j)), \text{ for any } h_j \in K_j \}. $$

Moreover, the image of $\partial^1$, i.e., $B^1(\Gamma, K_\partial; M)$, is equal to the subset $\{ (\tilde{f}_a, a, \ldots, a) \}_{a \in M}$. Here, for $a \in M$, the map $\tilde{f}_a : \Gamma \to M \rtimes G$ is defined as a homomorphism which sends $\gamma$ to $(a - a \cdot \gamma, f(\gamma))$. In particular, if $K_\partial$ is non-empty, $B^1(\Gamma, K_\partial; M)$ is a direct summand of $Z^1(\Gamma, K_\partial; M)$.

Furthermore, we review the cup product. When $K_\partial$ is the empty set, the product of $u \in C^p(\Gamma; M)$ and $v \in C^q(\Gamma; M')$ is defined to be $u \smile v \in C^{p+q}(\Gamma; M \otimes M')$ given by

$$(u \smile v)(g_1, \ldots, g_{p+q}) := (-1)^{pq}(u(g_1, \ldots, g_p)g_{p+1} \cdots g_{p+q}) \otimes v(g_{p+1}, \ldots, g_{p+q}). \quad (22)$$

Furthermore, if $K_\partial$ is not empty, for two elements $(f, k_1, \ldots, k_m) \in C^p(\Gamma, K_\partial; M)$ and $(f', k'_1, \ldots, k'_m) \in C^q(\Gamma, K_\partial; M')$, let us define the cup product to be the formula

$$(f \smile f', k_1 \smile k'_1, \ldots, k_m \smile k'_m) \in C^{p+q}(\Gamma, K_\partial; M \otimes M').$$

This formula yields a bilinear map, by passage to cohomology.

Finally, we observe another complex. Consider the module of the form

$$C^m_{\text{red}}(\Gamma) := \{ (c_1, \ldots, c_m) \in \text{Map}(\mathbb{Z}[\Gamma^n], M^m) \mid c_1 + c_2 + \cdots + c_m = 0 \in \text{Map}(\mathbb{Z}[\Gamma^n], M) \}.$$ 

Then, this complex canonically has an inclusion into the direct sum of $C^n(\Gamma, K_j)$:

$$P_n : C^m_{\text{red}}(\Gamma) \longrightarrow \bigoplus_{j: 1 \leq j \leq m} C^n(\Gamma, K_j).$$

Then, we define a quotient complex, $D^n(\Gamma, K_\partial; M)$, to be the cokernel of $P_n$. Then, $C^n(\Gamma, K_\partial; M)$ is isomorphic to $D^n(\Gamma, K_\partial; M)$, since the kernel of the inclusions $\oplus_{j=1}^m C^n(\Gamma, K_j) \to C^n(\Gamma, K)$ is the image of $P_n$.

**Remark 5.2.** We give a natural relation to usual cohomology. Take the Eilenberg-MacLane spaces of $\Gamma$ and of $K_j$, and consider the map $(t_j)_* : K(K_j, 1) \to K(\Gamma, 1)$ induced by the inclusions. Then the relative homology $H_n(\Gamma, K_\partial; M)$ is isomorphic to the homology of the mapping cone of $\cup_j K(K_j, 1) \to K(\Gamma, 1)$ with local coefficients. Further, the cup product $\smile$ above coincides with that on the singular cohomology groups.

We mention the case where $L$ is either a knot or a hyperbolic link. Then, the complement $S^3 \setminus L$ is known to be an Eilenberg-MacLane space. Since we only use $\Gamma$ as $\pi_1(S^3 \setminus L)$ in this paper, we may discuss only the relative group cohomology.

5.2 Review; results of the previous papers [N2] and [N5].

Throughout this section, we denote $\pi_1(S^3 \setminus L)$ by $\pi_L$, and the union of the fundamental groups of the boundaries of $S^3 \setminus L$ by $\partial \pi_L$, for brevity. Let $m = \#L$, and choose a diagram $D$. 

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Theorem 5.3 ([N2] Theorem 2.2]). Let $X$ be $M \times G$, as mentioned in [2]. Let $\kappa : X \to M \times G$ be a map which sends $(m, g)$ to $(m - mg, g)$. Given an $X$-coloring $C$ over $f$, consider a map \{arcs of $D$\} $\to M \times G$ which assigns $\alpha$ to $\kappa(C(\alpha))$. This assignment yields isomorphisms

$$\text{Col}_X(D_f) \cong Z^1(\pi_L, \partial \pi_L; M), \quad \text{Col}_X^\text{red}(D_f) \cong H^1(\pi_L, \partial \pi_L; M).$$

Next, we explain Theorem 5.4. Choose a relative 1-cocycle $\tilde{\theta} : \pi_L \to M \times \pi_L$ with $y_1, \ldots, y_m$. We define the subgroup $K_\ell$ to be

$$\{(y_\ell - y_\ell m_\ell b, m_\ell b) \in M \times \pi_L \mid a, b \in \mathbb{Z}^2 \}.$$ 

Furthermore, given a $G$-invariant trilinear map $\psi : M^3 \to A$, consider the map

$$\theta_\ell : (M \times \pi_1(S^3 \setminus L))^3 \to A;$$

$$((a, g), (b, h), (c, k)) \mapsto \psi((a + y_\ell - y_\ell g) \cdot hk, (b + y_\ell - y_\ell h) \cdot k, c + y_\ell - y_\ell k). \quad (23)$$

Then, we can easily check that each $\theta_\ell$ is a 3-cocycle in $C^3(M \times \pi_L; A)$. Then, the collection $\Psi := (\theta_1, \ldots, \theta_{#L})$ represents a relative 3-cocycle in $D^3(M \times \pi_L, \mathcal{K}; A)$.

Proposition 5.4 ([N3] Proposition 6.7]). Under the notation above, fix a shadow coloring $S_f$ corresponding the relative 1-cocycle $(\tilde{f}, y_1, \ldots, y_{#L})$.

If $L$ is either a hyperbolic link or a prime knot which is neither a cable knot nor a torus knot, as in Theorem 2.5, then the diagonal restriction of $\mathcal{T}_\psi$ is equal to the pairing of the 3-class $[E_L, \partial E_L]$ and the above 3-cocycle $\Psi$. To be precise,

$$\mathcal{T}_\psi(S_f, S_f, S_f) = \psi(\Psi, \tilde{f}, [E_L, \partial E_L]). \quad (24)$$

Furthermore, if $L$ is the $(m, n)$-torus knot, the same equality (24) holds modulo $mn$.

5.3 Proof of Theorem 2.5 trilinear pairing

Proof of Theorem 2.5. First, we observe (25) below. Consider a 0-cochain $\tilde{g} := (y_1, \ldots, y_{#L}) \in D^0(M \times \pi_L, M)$. Then, $\tilde{f} - \partial^0 \tilde{g}$ is represented by another 1-cocycle

$$C' := ((\tilde{f} - \tilde{g}), \ldots, (\tilde{f} - \tilde{g}_{#L})), (0, \ldots, 0)) \in D^1(M \times \pi_L, M),$$

where $\tilde{g}_\ell$ means a map $\pi_L \to M$ which takes $g$ to $y_1 - y_1 g$. Then, the 3-cocycle $\Psi$ explained in (23) is equal to the cup product $C' \cup C' \cup C'$, by definition. Hence, Proposition 5.4 implies

$$\mathcal{T}_\psi(S_f, S_f, S_f) = \psi(C' \cup C' \cup C', [E_L, \partial E_L]) = \psi(C \cup C \cup C, [E_L, \partial E_L]). \quad (25)$$

Finally, we will deal with non-diagonal parts, and complete the proof. Here, we define $M$ to be the direct product $M_1 \times M_2 \times M_3$, and consider the $j$-th inclusion

$$\iota_j : M_j \to M = M_1 \times M_2 \times M_3; \quad x \mapsto (\delta_{1j} x, \delta_{2j} x, \delta_{3j} x).$$

Thus, we can decompose $S_f$ as $(S_1, S_2, S_3) \in \text{Col}_{X_1}(D_f) \times \text{Col}_{X_2}(D_f) \times \text{Col}_{X_3}(D_f)$ componentwise. In addition, we define a $G$-invariant trilinear form

$$\overline{\psi} : M \times M \times M \to A; \quad ((a, b, c), (d, e, f), (g, h, i)) \mapsto \psi(a, e, f).$$
Then, the transformation of the coefficients \( \iota_1 \times \iota_2 \times \iota_3 \) yields a diagram

\[
\begin{array}{ccc}
\prod_{i=1}^{3} H^1(E, \partial E; M_i) & \xrightarrow{\sim} & H^3(E, \partial E; M_1 \times M_2 \times M_3) \\
\downarrow \iota_1 \times \iota_2 \times \iota_3, & & \downarrow \iota_1 \times \iota_2 \times \iota_3, \\
H^1(E, \partial E; M) & \xrightarrow{\Delta} & H^3(E, \partial E; M \times M \times M) \\
\end{array}
\]

\[ \psi \circ \langle \cdot, [E, \partial E] \rangle \to A \]

Here, the left bottom \( \sim \) is defined by \( a \mapsto a \sim a \sim a \). Then, we can verify the commutativity by definitions. By Proposition 5.4, the bottom arrow is equal to the left hand side in (24). Hence, the pullback to \( \prod_{i=1}^{3} H^1(E, \partial E; M_i) \) is equal to the trilinear \( T_\psi \) as desired. 

Proof of Theorem 2.6 Let \( L \) be the \((m, n)\)-torus knot. According to the latter part in Theorem 5.4 we need discussions modulo \( mn \). However, the proof runs well in the same manner.

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