A CLASS OF MATRIX-VALUED POLYNOMIALS GENERALIZING JACOBI POLYNOMIALS

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Abstract. A hierarchy of matrix-valued polynomials which generalize the Jacobi polynomials is found. Defined by a Rodrigues formula, they are also products of a sequence of differential operators. Each class of polynomials is complete, satisfies a two-step recurrence relation, integral inter-relations, and quasi-orthogonality relations.

1. Motivation

The understanding of matrix-valued orthogonal polynomials has advanced greatly in recent years, and a wealth of references can be found in the recent work of Barry Simon [1].

The polynomials introduced in the present paper generalize the Jacobi polynomials. A different generalization, yielding an orthogonal class, is due to Grünbaum [2].

The polynomials defined here were not sought-for. Rather they appear naturally in the forefront of a different problem, the study of the possibility to linearize (by an analytic change of variables) a differential equation whose linear part has only regular singular points. In a neighborhood of one such point equations are (generically) linearizable [3]. But this is no longer the case if the domain studied contains two such singularities. For example:

\[
\frac{du}{dx} = Mu + f(x,u) \quad \text{with } M = \frac{1}{x-1} A + \frac{1}{x+1} B
\]

(\(f\) collects all the nonlinear terms in \(u\)) is not necessarily equivalent to its linear part

\[
\frac{dw}{dx} = Mw
\]

for \(x\) in a domain in the complex plane including both singular points \(\pm 1\).

It turns out, however that for any nonlinear term \(f(x,u)\) there exists a unique \(\phi(u)\) so that the equation with the ”corrected” nonlinear part \(f(x,u) - \phi(u)\) is linearizable [4].

Besides its clear intrinsic interest, the problem of detecting linearizable equations is important also since linearizability and integrability turn out to be intimately connected [5].
Looking for an analytic change of variables $y = w + h(x, w)$ that transforms (2) into (1) one is lead to the study of the homological equation which boils down to equations of the form
\[
\partial_x h + d_w h M w - M h = g(x, w)/(x^2 - 1)
\]
where now all functions can be assumed homogeneous polynomials in $w$, of degree, say, $n$.

The main question is: for which functions $g$ does equation (3) possess a solution $h$ which is analytic in a domain containing both singularities $x = \pm 1$?

In the scalar case, when $w = w \in \mathbb{C}$, $h = h$, $g = g \in \mathbb{C}$, and $A = a, B = b \in \mathbb{C}$, it turns out that: equation (3) has an analytic solution if and only if the first coefficient in the Jacobi series in $\{P_k((n-1)a-1,(n-1)b-1)\}$ of $g$ is zero. Moreover, Jacobi series expansions are canonical to approaching the problem, since for $g(x) = P_k^{((n-1)a-1,(n-1)b-1)}(x)$ the unique analytic solution of (3) is $h = -\frac{1}{2x}P_k^{((n-1)a,(n-1)b)}(x)$.

In the linearization problem mentioned the double sequence of Jacobi polynomials $P_k^{(na,nb)}$ for $k \geq 0, n \geq 2$ is interconnected. Numerical expansions in Jacobi series yield (what appear to be) rapidly convergent series.\footnote{However, the proof of convergence of the series obtained, which did not use the Jacobi structure, turned out to be extremely delicate: a steepest descent method was used and small denominators had to be dealt with [6].}

The natural question was whether the polynomial structure found in the one-dimensional case survives in more dimensions.

The answer turns out affirmative - see (28), (29). These multidimensional polynomials share many properties common to the usual classes of orthogonal polynomials: each class is complete, satisfies a two-step recurrence relation, quasi-orthogonality relations exist, and there are integral inter-relations. It is the author’s belief that more that what is presented here does hold.

2. Main Results

2.1. Notations. Consider a Fuchsian differential equation in $\mathbb{C}^d$ with three singularities in the extended complex domain $\mathbb{C}$:
\[
y' = My \quad \text{where} \quad M = \frac{1}{x - 1} A + \frac{1}{x + 1} B, \quad (x \in \mathbb{C}, \ y \in \mathbb{C}^d)
\]
where $A$ and $B$ are $d \times d$ matrices so that
(i) $A + B$ is diagonalizable, and
(ii) $A + B$ satisfies the following nonresonance condition: the eigenvalues $\lambda_1, \ldots, \lambda_d$ of the matrix $A + B$ satisfy:
\[
n \cdot \lambda - \lambda_j \notin \mathbb{Z}_+ \quad \text{for all} \quad n \in \mathbb{N}^d, \ j = 1, \ldots, d
\]
These assumptions are, of course, satisfied by generic matrices.

Denote for simplicity:
\[
M_1 = A + B, \quad M_2 = A - B, \quad Q = x^2 - 1
\]
so that $QM = xM_1 + M_2$.

It will be assumed, without loss of generality, that the matrix $M_1$ is diagonal.

Let $Y$ be a fundamental matrix of solutions of (4), therefore

$$Y' = MY \quad \text{and} \quad (Y^{-1})' = -Y^{-1}M$$

Denote by $P_n$ the space of vector-valued polynomials in $w = (w_1, \ldots, w_d)$, homogeneous, degree $n$:

$$P_n = \left\{ q : q(w) = \sum_{m \in \mathbb{N}^d, |m| = n} q_m w^m, \ q_m \in \mathbb{C}^d \right\}$$

where $|m| = m_1 + \ldots + m_d$, $w^m = w_1^{m_1} \ldots w_d^{m_d}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$.

2.2. The class of polynomials: definition. The polynomial in $x$ degree $k$, $P_k(x)$, is, for every $x$, the linear operator on $P_n$ defined by

$$P_k(x)q(w) = Y(x) \left. \frac{d^k}{dt^k} \right|_{t=x} \left[ Q(t)^k Y^{-1}(t) q \left( Y(t)Y^{-1}(x)w \right) \right]$$

Formula (7) can be compactly written as follows. Denoting by $W(x)$ the linear operator on $P_n$:

$$W(x)q(w) = Y(x)^{-1} q \left( Y(x)w \right)$$

(which gives $q$ in new, $x$-dependent coordinates in $\mathbb{C}^d$) the definition (7) is

$$P_k(x) = W(x)^{-1} \frac{d^k}{dx^k} \left[ Q(x)^k W(x) \right]$$

which is a Rodrigues formula.

The fact that $P_k(x)$ are polynomials in $x$, degree $k$, is shown in Proposition 1.

Denote by $D_1, D_2$ the following linear operators on $P_n$:

$$D_i q(w) = dq(w) M_i w - M_i q(w), \quad i = 1, 2$$

where $M_i$ are defined in (6).

**Proposition 1.** The operators $P_k(x)$ are polynomials in $x$ degree $k$.

Moreover, they can be written as the composition

$$P_k = A_1 A_2 \ldots A_k$$

where

$$A_j = x (2j + D_1) + D_2 + Q \partial_x$$

Note that the action of first operator $A_k$ when applied to $q(w)$ (which does not depend on $x$) equals $[x (2j + D_1) + D_2]q(w)$.
we obtain
\[
\frac{d}{dt} \left[ Q(t)^k Y(t)^{-1} q(Y(t)u) \right] = Q(t)^{k-1} Y(t)^{-1} (2kt + tD_1 + D_2) q(Y(t)u)
\]
\[
\equiv Q(t)^{k-1} Y(t)^{-1} r(t, Y(t)u) \quad \text{where } r(t, w) = A_k q(w)
\]
Then
\[
\frac{d^2}{dt^2} \left[ Q(t)^k Y(t)^{-1} q(Y(t)u) \right] = \frac{d}{dt} \left[ Q(t)^{k-1} Y(t)^{-1} r(t, Y(t)u) \right]
\]
\[
= Q(t)^{k-2} Y(t)^{-1} [2(k+1)t + tD_1 + D_2 + Q\partial_t] r(t, Y(t)u)
\]
\[
= Q(t)^{k-2} Y(t)^{-1} A_{k-1} r(t, Y(t)u) = Q(t)^{k-2} Y(t)^{-1} A_{k-1} A_k q(Y(t)u)
\]
The $k$ derivatives in (7) can be calculated in this way recursively, yielding (10) and completing the proof of Proposition 1.

**Remark 1.** The same definition (7) can be used to define $P_k(x)$ as linear operators on the space of $\mathbb{C}^d$-valued formal series in $w$, or on the space of convergent such series.

### 2.3. The polynomials (7) generalize the Jacobi polynomials

Consider the one-dimensional case: $d = 1$. Equation (4) and a fundamental solution are, in this scalar case,
\[
y' = \left( \frac{a}{x-1} + \frac{b}{x+1} \right) y \quad \text{and} \quad y(x) = (x-1)^a (x+1)^b
\]
The homogeneous polynomials degree $n$ are just multiples of $w^n$: $q(w) = c w^n$ and formula (7) gives
\[
P_k(x)q = \frac{1}{y(x)^{n-1}} \frac{d}{dx} \left[ Q(x)^k y(x)^{n-1} \right] q
\]
which is the Rodrigues formula for the Jacobi polynomial $P_k^{(n-1)a,(n-1)b}(x)$ (up to a multiplicative factor).

**Remark 2.** In the one-dimensional case the operators $D_{1,2}$ of (9) are multiplication by $(n-1)(a+b)$, and $(n-1)(a-b)$ respectively, and formula (10) gives the following representation for Jacobi polynomials (up to a numerical factor):
\[
P_k^{(\alpha,\beta)}(x) = A_1 A_2 \ldots A_k 1
\]
where $A_j = (2j + \alpha + \beta)x + (\alpha - \beta) + Q\partial_x$

Formula (12) provides the following elegant way to deduce the second order operator for which Jacobi polynomials are eigenvalues: noting that
\[
\partial_x A_j = (\alpha + \beta + 2j) + A_{j+1} \partial_x
\]
we obtain
\[
A_1 \partial_x P_k^{(\alpha,\beta)} = A_1 \partial_x A_1 A_2 \ldots A_k = (\alpha + \beta + 2) P_k^{(\alpha,\beta)} + A_1 A_2 \partial_x A_3 \ldots A_k
\]
\[
= \ldots = [(\alpha + \beta + 2) + \ldots + (\alpha + \beta + 2k)] P_k^{(\alpha,\beta)} = k(\alpha + \beta + k + 1) P_k^{(\alpha,\beta)}
\]
2.4. The multidimensional commutative case. If the matrices $A$ and $B$ are simultaneously diagonalizable:

$$A = \text{diag } [a_i]_{i=1,...,d} \quad \text{and} \quad B = \text{diag } [b_i]_{i=1,...,d}$$

then the matrix $Y$ is

$$Y = \text{diag } \left[ (x - 1)^{a_i}(x + 1)^{b_i} \right]_{i=1,...,d}$$

and the action of the operators (7) on the canonical basis of $P_n$: $b_{m,j} = w^m e_j$, $j = 1, \ldots, d$, $|m| = n$ is

$$P_k(x) b_{m,j} = \frac{1}{W_{m,j}(x)} \frac{d^k}{dx^k} \left[ Q(x)^k W_{m,j}(x) \right] b_{m,j}$$

where

$$W_{m,j}(x) = (x - 1)^{m \cdot a_j - a_j}(x + 1)^{m \cdot b_j - b_j}$$

therefore $P_k(x)$ are diagonal operators, with entries Jacobi polynomials degree $k$.

This is a family of matrix-valued orthogonal polynomials with respect to the inner product

$$< P_l, P_k > := \int_{-1}^{1} \text{Tr} \left\{ P_l(x) W(x) P_k(x) \right\} dx$$

$$= \sum_{|m|=n,j=1,...,d} \int_{-1}^{1} P_l^{(m \cdot a_j, m \cdot b_j)}(m \cdot a_j) W_{m,j} P_k^{(m \cdot a_j, m \cdot b_j)}(m \cdot b_j) dx$$

in the cases when the integrals exist.

2.5. The non-commutative linear case. For $n = 1$, the homogeneous polynomials degree one are linear transformations: $q(w) = qw$ where $q$ is a matrix. Formula (15) is, in this case,

$$P_k(x) q = Y(x) \frac{d^k}{dx^k} \left[ Q(x)^k Y^{-1}(x) q Y(x) \right] Y^{-1}(x)$$

which implies

$$\text{Tr} \left\{ P_k(x) q \right\} = P_k^{(0,0)}(x) \text{Tr} q$$

where $P_k^{(0,0)}(x) = P_k(x)$ are the Legendre polynomials.

2.6. Properties of the generalized polynomials (7).

2.6.1. Rodrigues formula. The definition (8) of $P_k(x)$ is a Rodrigues formula.

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3 Appropriate conditions relating $a, b, n, d$ will ensure existence of the integrals.
2.6.2. A two-step recurrence relation.

**Proposition 2.** The operators $P_k(x)$ defined by (7) satisfy the following two-step recurrence relation

\begin{equation}
 x P_k(x) = P_{k+1}(x) \alpha_k + P_k(x) \beta_k + P_{k-1}(x) \gamma_k
\end{equation}

where $\alpha_k, \beta_k, \gamma_k$ are linear operators on $P_n$, and in particular

\begin{equation}
 \alpha_k = (D_1 + 2k + 1)^{-1}(D_1 + 2k + 2)^{-1}(D_1 + k + 1)
\end{equation}

**Proof of Proposition 2.**

Note the identities

\begin{equation}
 A_j(x, r) = x A_j(r) + Q r
\end{equation}

and

\begin{equation}
 Q A_j(r) = A_{j-1}(Q r)
\end{equation}

for any $r = r(x, \cdot) \in P_n$

which by iteration give

$$A_1 A_2 \cdots A_k(x, q) = A_1 A_2 \cdots A_{k-1}(x A_k q) + A_1 A_2 \cdots A_{k-1}(Q q)$$

Using (10) and (21) relation (17) follows if we have

$$A_k x - kQ = A_k A_{k+1} \alpha_k + A_k \beta_k + \gamma_k$$

which expanded yields an identity of quadratic polynomials in $x$, and by identifying the coefficients we obtain the following equations for $\alpha_k, \beta_k, \gamma_k$:

\begin{equation}
 k + 1 + D_1 = (2k + 1 + D_1) (2k + 2 + D_1) \alpha_k
\end{equation}

\begin{equation}
 D_2 = [(4k + 2) D_2 + D_1 D_2 + D_2 D_1] \alpha_k + (2k + D_1) \beta_k
\end{equation}

\begin{equation}
 k - 1 = [D_2^2 - (2k + 2 + D_1)] \alpha_k + D_2 \beta_k + \gamma_k
\end{equation}

The system (22)-(23) has a unique solution $\alpha_k, \beta_k, \gamma_k$ due to the nonresonance condition (5). \hfill \Box

2.6.3. The dominant coefficient.

**Proposition 3.** The coefficient of $x^k$ in $P_k(x)$ is

\begin{equation}
 (D_1 + k + 1) (D_1 + k + 2) \cdots (D_1 + 2k)
\end{equation}

which is invertible under the assumptions of [2.7].
Proof. Retaining only the dominant coefficients in the representation (10), (11) we have:
\[ P_k(x) = [x(2 + D_1) + x^2 \partial_x] \ldots [x(2k - 2 + D_1) + x^2 \partial_x] \ [x(2k + D_1)] + O(x^{k-1}) \]
which yields
\[ P_k(x) = (D_1 + k + 1)(D_1 + k + 2) \ldots (D_1 + 2k) x^k + O(x^{k-1}) \]
giving (25).

Under the assumptions and notations of (i), (ii) of §2.1, we can assume \( M_1 \) diagonal, and the operator \( D_1 \) has the eigenvector/values:
\[ D_1 b_{m,j} = (m \cdot \lambda - \lambda_j) b_{m,j} \]
Then \( D_1 + j \) is invertible for all \( j \in \mathbb{Z}_+ \) due to the assumption (ii), and therefore (25) is invertible.

2.6.4. Completeness. The set \( P_k(x), k = 0, 1, 2, \ldots \) is complete in \( \mathcal{P}_n[x] \) in the following sense:

**Proposition 4.** For any \( f = f(x, w) \in \mathcal{P}_n[x] \) polynomial, homogeneous degree \( n \) in \( w \) and degree \( k \) in \( x \) there exist \( q_0, \ldots, q_k \in \mathcal{P}_n \) so that
\[ f(x, w) = \sum_{j=0}^{k} P_j(x) q_j(w) \]
and the representation (26) is unique.

**Proof.** The decomposition (26) follows easily by induction on \( k \), relying on the fact that the dominant coefficients (25) are invertible.

2.6.5. Integral inter-relation. Consider the following variation of the Fuchsian equation (1):
\[ \tilde{M} = M - \frac{2x}{(n - 1)Q} I \]
and let \( \tilde{Y} \) be a corresponding fundamental matrix: \( \tilde{Y}' = \tilde{M} \tilde{Y} \). Let \( \tilde{P}_k \) denote the polynomials associated by (7).

We have the following analogue of an integral relation for Jacobi polynomials (see [7] §22.13.1)
\[ P_k(x) q(w) = Q(x)^{-1} Y(x) \int_{-1}^{x} Y(t)^{-1} \tilde{P}_{k+1}(t) q(Y(t)Y(x)^{-1}w) \ dt \]
or, in differential form,
\[ \partial_x P_k q + d_w(P_k q) M w - M P_k q = Q^{-1} \tilde{P}_{k+1} q \]
or, in operator notation,
\[ (xD_1 + D_2 + Q \partial_x) P_k = \tilde{P}_{k+1} \]
This can be easily seen due to the fact that ˜\(P_k\) also satisfy
\[
\dot{P}_{k+1}(x)q(w) = Q(Y(x)) \frac{d^{k+1}}{dt^{k+1}} \left( Q(t)^k Y^{-1}(t) q(Y(t)Y^{-1}(x)w) \right)
\]
(which follows by a short calculation using the fact that \(q(w)\) is a homogeneous polynomial
in \(w\), degree \(n\).)

2.6.6. Orthogonality. The following quasi-orthogonality relations hold:

\[
\int_{-1}^{1} P_j(x) W(x) P_k(x) \, dx = 0 \quad \text{for} \quad j < k
\]

and

\[
\int_{-1}^{1} W(x) P_j(x) P_k(x) \, dx = 0 \quad \text{for} \quad j > k
\]

provided that the integrals exist.

\textit{Proof.} For any index \(k\) we have
\[
[W(x) P_k(x)]q(w) = Y(x)^{-1} P_k(x)q(Y(x)w) = \frac{d^k}{dx^k} \left[ Q(x)^k Y^{-1}(x) q(Y(x)w) \right]
\]
and \((31),(32)\) follow using integration by parts (which holds for matrix multiplication). \(\Box\)

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