On the compactness of a nonlinear operator related to stream function-vorticity formulation for the Navier-Stokes equations

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Received June 16, 2017, Accepted August 17, 2017

Abstract

A compactness proof of a nonlinear operator related to stream function-vorticity formulation for the Navier-Stokes equations is presented. The compactness of the operator provides important information for fixed-point formulations, especially for computer-assisted proofs based on Schauder’s fixed-point theorem. Our idea for the compactness proof comes from books by Girault & Raviart and Ladyzhenskaia, and our principle would be also applied to convex polygonal regions.

Keywords compactness, nonlinear operator, Navier-Stokes equation

Research Activity Group Quality of Computations

1. Introduction

For each integer m, let $H^m(\Omega)$ be the $L^2$-Sobolev space of order m on a two-dimensional rectangular domain $\Omega = (a, b) \times (c, d)$. We define

$$H^2_0(\Omega) := \left\{ u \in H^2(\Omega) \mid u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega \right\}$$  \hspace{1cm} (1)

equipped with the inner product $(u, w)_{H^2_0(\Omega)} := (\Delta u, \Delta w)_{L^2}$ and the norm $\|u\|_{H^2_0(\Omega)} := \|\Delta u\|_{L^2}$, where $(\cdot, \cdot)_{L^2}$ and $\|\cdot\|_{L^2}$ stand for the inner product and the norm on $L^2(\Omega)$, respectively. The notation $\partial/\partial n$ in (1) indicates the normal derivative. Note that since $\Omega$ is convex, it holds that

$$\|\Delta u\|_{L^2}^2 = \|u_{xx}\|_{L^2}^2 + 2\|u_{xy}\|_{L^2}^2 + \|u_{yy}\|_{L^2}^2$$  \hspace{1cm} (2)

by partial integration [1, in the proof of Theorem 4.3.1.4]. Let $H^{-2}(\Omega)$ be the dual space of $H^2_0(\Omega)$ with the duality pairing $(\cdot, \cdot)$ and the norm

$$\|u\|_{H^{-2}} := \sup_{\theta \in H^2_0(\Omega)} \frac{\langle u, \theta \rangle}{\|\theta\|_{H^2_0(\Omega)}^2}$$  \hspace{1cm} (3)

The aim of this paper is to show the compactness of a nonlinear operator on $H^2_0(\Omega)$ defined by

$$F(w) := \Delta^{-2} \circ J(w, \Delta w),$$  \hspace{1cm} (4)

where $J$ stands for the bilinear form defined by

$$J(u, v) = u_x v_y - u_y v_x,$$  \hspace{1cm} (5)

which satisfies $J(w, \Delta w) \in H^{-2}(\Omega)$ for $w \in H^2_0(\Omega)$, and $\Delta^{-2} : H^{-2}(\Omega) \to H^2_0(\Omega)$ is a mapping from $\phi \in H^{-2}(\Omega)$ to the solution $u \in H^2_0(\Omega)$ satisfying

$$\begin{cases}
\Delta^2 u = \phi & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$  \hspace{1cm} (6)

When $\Omega$ is a rectangle, it is known [2] that for each $\phi \in H^{-2}(\Omega)$ there exists a solution $u \in H^2_0(\Omega)$ of (6).

The operator $F$ is the essential mapping of the fixed-point formulation by using stream functions for the two-dimensional Navier-Stokes equation [3–6], and its compactness provides useful information for existence proofs of the fixed points by using Schauder’s fixed-point theorem, for example.

For $H^2_0(\Omega) := \{ w \in H^2(\Omega) \mid u = 0 \text{ on } \partial \Omega \}$ with the norm $\|\nabla u\|_{L^2}$ and the dual space $H^{-1}(\Omega)$ of $H^2_0(\Omega)$, it is well-known that the restricted mapping $\Delta^{-2}|_{H^{-1}}$ is an isomorphism from $H^2(\Omega) \cap H^2_0(\Omega)$ onto $H^{-1}(\Omega)$ if $\Omega$ is a convex plane polygon [1, Corollary 7.3.2.5]; therefore, for a continuous operator $q : H^2_0(\Omega) \to H^{-1}(\Omega)$, the compactness of the operator $\Delta^{-2} q$ on $H^2_0(\Omega)$ can be checked by using the compact embedding $H^3(\Omega) \hookrightarrow H^2(\Omega)$. However, because $u \cdot v$ for $u \in H^1(\Omega), v \in H^{-1}(\Omega)$ is not an element of $H^{-1}(\Omega)$ but just of $H^{-2}(\Omega)$ [1, Theorem 1.4.4.2], $J(w, \Delta w)$ in (4) does not belong to $H^{-1}(\Omega)$ for $w \in H^2_0(\Omega)$. Therefore, the compactness of $F$ is not trivial.

Our idea for the compactness proof comes from Girault and Raviart [7, eq.(2.28); p.294] and Ladyzhenskaia [8, p.117], and our principle would be also applied to convex polygonal regions. We will discuss general domains in more detail in a separate article.
The next section is devoted to some lemmas about the bilinear form $J$, and the compactness of a nonlinear operator $F$ is shown in Section 3.

2. Some properties of $J$

In this section we collect a number of properties of $J$.

Lemma 1 For $u \in H^2_0(\Omega)$ and $v, w \in H^1(\Omega)$ it holds that

$$ (J(u, v), w)_{L^2} = (J(w, u), v)_{L^2} = -(J(u, w), v)_{L^2}. \quad (7) $$

Proof By using a partial integration, it holds that

$$ (J(u, v), w)_{L^2} = (u_x v_y - u_y v_x, w)_{L^2} $$

and, from (8),

$$ (J(u, v), w)_{L^2} = (u_x v_y - u_y v_x, w)_{L^2} $$

and partial integration leads to

$$ (\Delta v, u)_{L^2} = (\Delta v, \Delta u)_{L^2}. $$

Therefore, duality pairings: $\Delta^2 u, J(u, \Delta u) \in H^{-2}(\Omega)$ for $u \in H^2_0(\Omega)$ can be denoted by

$$ (\Delta^2 u, v) := (\Delta u, \Delta v)_{L^2}, \quad \forall v \in H^2_0(\Omega), \quad (9) $$

$$ (J(u, \Delta u), v) := (J(v, u), \Delta u)_{L^2}, \quad \forall v \in H^2_0(\Omega), \quad (10) $$

respectively. Concerning nonlinear term (10), the following lemma is essential for our main result.

Lemma 2 For $u, v \in H^2_0(\Omega)$, it is true that

$$ (J(v, u), \Delta u)_{L^2} = (J(u, v_x), u_x)_{L^2} + (J(v_y, u_y), u_y)_{L^2}. \quad (11) $$

Proof The idea of the proof comes from [7, eq. (2.28); p. 294]. Since

$$ (J(v, u), \Delta u)_{L^2} = (v_x u_y - v_y u_x, u_{xx} + u_{yy})_{L^2} $$

by using partial integration, the first term of the right-hand side in (12) is

$$ (v_x u_y, u_y)_{L^2} = -(v_{xy} u_y + v_y u_{yy}, u_y)_{L^2} $$

and the second term of the right-hand side in (12) is

$$ (v_y u_x, u_{xx})_{L^2} = (v_{xy} u_x + v_y u_{xx}, u_x)_{L^2} $$

then

$$ (v_y u_x, u_{xx})_{L^2} = -\frac{(v_{xy} u_x)_{L^2} + (v_y u_{xx})_{L^2}}{2} $$

These imply

$$ (J(v, u), \Delta u)_{L^2} = (v_x u_y, u_y)_{L^2} + (v_y u_x, u_{xx})_{L^2} $$

Therefore, applying partial integration again, we have

$$ (v_y u_x, u_{xx})_{L^2} = (v_{xy} u_x + v_y u_{xx}, u_x)_{L^2} $$

and

$$ (v_x u_y, u_y)_{L^2} = (v_{xy} u_y + v_y u_{yy}, u_y)_{L^2} $$

which is the desired result.

(QED)

Concerning (9), the following lemma indicates the boundedness of the operator $\Delta^{-2} : H^{-2}(\Omega) \to H^2_0(\Omega)$.

Lemma 3 It holds that

$$ \|\Delta^{-2} \psi\|_{H^2_0} \leq \|\psi\|_{H^{-2}}, \quad \forall \psi \in H^{-2}(\Omega). \quad (14) $$

Proof The case for $\psi = 0$ is trivial. For each $0 \neq \psi \in H^{-2}(\Omega)$ from (9) and (3) we have

$$ \|\Delta^{-2} \psi\|_{H^2_0}^2 = (\Delta \Delta^{-2} \psi, \Delta \Delta^{-2} \psi)_{L^2} $$
\[
\psi, \Delta^{-2} \psi \quad \text{and} \quad \psi, \Delta^{-2} \psi/\|\Delta^{-2} \psi\|_{H^0} \times \|\Delta^{-2} \psi\|_{H^0},
\]
\[
\leq \|\psi\|_{H^{-1}} \|\Delta^{-2} \psi\|_{H^0},
\]
as desired. (QED)

Now for an \( L^4 \)-Lebesgue space endowed with the norm
\[
\|u\|_{L^4} := \left( \int_{\Omega} |u(x)|^4 \, dx \right)^{1/4},
\]
it is well-known that the embedding \( H^0_0(\Omega) \hookrightarrow L^4(\Omega) \) holds and there exists an embedding constant \( c_4 > 0 \) such that
\[
\|\psi\|_{L^4} \leq c_4 \|\nabla \psi\|_{L^2}, \quad \forall \psi \in H^1_0(\Omega).
\]

For example, if \( \Omega = (0,1)^2 \), \( c_4 \) can be taken as \( 1/\pi \) [9]. We define
\[
\|\nabla u\|_{L^4} := \sqrt{\|u_x\|_{L^4}^2 + \|u_y\|_{L^4}^2}.
\]
Note that the definition of the norm \( \|\nabla u\|_{L^4} \) by (17) is different from the usual \( L^4 \)-vector norm. Using \( c_4 > 0 \) satisfying (16) we obtain the following upper bound of norm estimation.

**Lemma 4** It holds that
\[
\|\nabla u\|_{L^4} \leq c_4 \|u\|_{H^0}, \quad \forall u \in H^0_0(\Omega).
\]

**Proof** For each \( u \in H^0_0(\Omega) \), since \( \Omega \) is a rectangle, \( u_x \) and \( u_y \) belong to \( H^0_0(\Omega) \); then from (16) and (2), we obtain
\[
\|\nabla u\|_{L^4}^2 = \|u_x\|_{L^4}^2 + \|u_y\|_{L^4}^2 \leq c_4^2 \|u\|_{H^0}^2
\]
\[
= c_4^2 \|u\|_{H^0}^2.
\]
(QED)

The final lemma in this section bounds differences in the nonlinear function \( J(u, \Delta u) \). Although we use only (20) in the main theorem, the estimation (19) would be useful for detailed norm estimations for the Navier-Stokes equations in future work.

**Lemma 5** For \( u, v, w, x, y \in H^0_0(\Omega) \), it holds that
\[
(J(v, u), \Delta u)_{L^2} - (J(v, w), \Delta w)_{L^2}
\]
\[
\leq \|v\|_{H^0} \|\Delta u - \Delta w\|_{L^1} \times \sqrt{\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla (u + w)\|_{L^2}^2}
\]
\[
\leq 2c_4 \|v\|_{H^0} \|\nabla (u - w)\|_{L^2} \|\nabla (u + w)\|_{L^2}.
\]

**Proof** By using (11) we have
\[
(J(v, u), \Delta u)_{L^2} - (J(v, w), \Delta w)_{L^2}
\]
\[
= (J(v, u_x), u_x)_{L^2} + (J(v, u_y), u_y)_{L^2}
\]
\[
- (J(w, v_x), w_x)_{L^2} - (J(w, v_y), w_y)_{L^2}
\]
\[
= (J(u - w, v_x), u_x)_{L^2} + (J(w, v_x), u_x)_{L^2}
\]
\[
+ (J(u - w, v_y), u_y)_{L^2} + (J(w, v_y), u_y)_{L^2}
\]
\[
- (J(w, v_x), w_x)_{L^2} - (J(w, v_y), w_y)_{L^2}
\]
\[
= (J(u - w, v_x), u_x)_{L^2} + (J(w, v_x), (u - w)_x)_{L^2}
\]
\[
+ (J(u - w, v_y), u_y)_{L^2} + (J(w, v_y), (u - w)_y)_{L^2}
\]
\[
- (w_x, v_x)_{L^2} - (w_y, v_y)_{L^2} - ((u - w)_x v_y, u_y)_{L^2}
\]
\[
+ ((u - w)_x v_y, u_y)_{L^2} + ((u - w)_y v_x, u_x)_{L^2}
\]
\[
+ ((u - w)_x v_y, (u - w)_y)_{L^2} - ((u - w)_x v_y, (u - w)_y)_{L^2}
\]
\[
= -v_{xx}(u - w)_x + w_x(u - w)_x + v_y(u - w)_y + w_y(u - w)_y
\]
\[
+ v_{xy}(u - w)_x (u + w)_x + (u - w)_y (u + w)_y
\]
\[
+ (u - w)_x (u + w)_x - (u - w)_y (u + w)_y,
\]
\[
\quad \text{the Cauchy-Schwarz and the Hölder inequalities}
\]
\[
\text{imply}
\]
\[
(J(v, u), \Delta u)_{L^2} - (J(v, w), \Delta w)_{L^2}
\]
\[
\leq \|v_{xx}\|_{L^2} \|w_x(u - w)_x\|_{L^2}
\]
\[
+ \|v_{yy}\|_{L^2} \|w_y(u - w)_y\|_{L^2}
\]
\[
+ \|v_{xy}\|_{L^2} \|w_x w_y(u - w)_x(u + w)_y\|_{L^2}
\]
\[
\leq \left( \|v_{xx}\|_{L^2}^2 + \|v_{yy}\|_{L^2}^2 + \|v_{xy}\|_{L^2}^2 \right)^{1/2}
\]
\[
\times \left( \|w_x(u - w)_x\|_{L^2}^2 + \|w_y(u - w)_y\|_{L^2}^2 + \|w_x w_y(u - w)_x(u + w)_y\|_{L^2}^2 \right)^{1/2}
\]
\[
\quad \text{It can also be shown that}
\]
\[
\|w_x(u - w)_x + w_y(u - w)_y\|_{L^2}
\]
\[
\leq \left( \|w_x\|_{L^2} \|w\|_{L^4} \|u - w\|_{L^2} \|u + w\|_{L^2} \right)^{1/2}
\]
\[
\leq \left( \|w_x\|_{L^2}^2 + \|w_y\|_{L^2}^2 \right)^{1/2} \|w\|_{L^4} \|u - w\|_{L^2} \|w\|_{L^4} \|u + w\|_{L^2} \|w\|_{L^4}
\]
\[
= \|\nabla (u - w)\|_{L^2} \|\nabla (u + w)\|_{L^2},
\]
\[
|w_x(u - w)_x + w_y(u - w)_y|_{L^2}
\]
\[
\leq \left( \|w_x\|_{L^2} \|w\|_{L^4} \|u - w\|_{L^2} \|u + w\|_{L^2} \right)^{1/2}
\]
\[
\leq \left( \|w_x\|_{L^2}^2 + \|w_y\|_{L^2}^2 \right)^{1/2} \|w\|_{L^4} \|u - w\|_{L^2} \|w\|_{L^4} \|u + w\|_{L^2} \|w\|_{L^4}
\]
\[
= \|\nabla (u - w)\|_{L^2} \|\nabla (u + w)\|_{L^2},
\]
and
\[
\|w_x(u - w)_x + w_y(u - w)_y\|_{L^2}
\]
\[
\leq \left( \|w_x\|_{L^2} \|w\|_{L^4} \|u - w\|_{L^2} \|u + w\|_{L^2} \right)^{1/2}
\]
\[
\leq \left( \|w_x\|_{L^2}^2 + \|w_y\|_{L^2}^2 \right)^{1/2} \|w\|_{L^4} \|u - w\|_{L^2} \|w\|_{L^4} \|u + w\|_{L^2} \|w\|_{L^4}
\]
\[
= \|\nabla (u - w)\|_{L^2} \|\nabla (u + w)\|_{L^2}.
\]
Therefore it is true that
\[
(J(v, u), \Delta u)_{L^2} - (J(v, w), \Delta w)_{L^2}
\]
converges because

\[ H \]

\[ f \]

\[ (\|u_x\|_{L^2}^2 + \|w_y\|_{L^2}^2) \]

\[ \Rightarrow \]

\[ = \|v\|_{H^2}^2 \]

\[ \|u\|_{H^2}^2 \]

\[ \times (\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla (u + w)\|_{L^2}^2)^{1/2}, \]

which is (19). Moreover, from (18) we obtain

\[ \|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla (u + w)\|_{L^2}^2 \leq c_2^2 \left( \|u\|_{H^2}^2 + \|w\|_{H^2}^2 + \|u + w\|_{H^2}^2 \right) \]

\[ \leq 2c_2^2 \left( \|u\|_{H^2}^2 + \|w\|_{H^2}^2 \right), \]

and hence (20) holds.

(QED)

3. Main theorem and corollary

This section is devoted to our main result and its extension.

Theorem 6 The operator \( F \) defined by (4) is compact on \( H_0^2(\Omega) \).

Proof In order to prove the compactness of \( F \), we show that, for every bounded sequence \( \{w_n\} \subset H_0^2(\Omega) \), the sequence \( \{F(w_n)\} \subset H_0^2(\Omega) \) has a convergent subsequence. We use a generic symbol \( C > 0 \) which stands for boundedness independent of \( \{w_n\} \).

For each bounded sequence \( \{w_n\} \subset H_0^2(\Omega) \subset H^2(\Omega) \), because the embedding

\[ H^2(\Omega) \hookrightarrow W^{1,4}(\Omega) := \left\{ u \in L^4(\Omega) \mid \nabla u \in (L^1(\Omega))^2 \right\} \]

is compact [10, Theorem 6.3, Part I], there exists a subsequence \( \{w_{n_i}\} \) of \( \{w_n\} \) which converges in \( W^{1,4}(\Omega) \).

Since a convergent sequence is a Cauchy sequence, we can say

\[ \|\nabla (w_{m_i} - w_n)\|_{L^2} \to 0 \quad (m_i, n_i \to \infty), \quad \text{(21)} \]

Using (21), if we confirm that \( \{F(w_{m_i})\} \) is a Cauchy sequence in \( H_0^2(\Omega) \), then we can conclude that \( \{F(w_n)\} \) converges because \( H_0^2(\Omega) \) is a Banach space. From (4), (14), (3), (10), (20) of Lemma 5, the boundedness of \( \{w_{m_i}\} \) and (21), we obtain

\[ \|F(w_{m_i}) - F(w_n)\|_{H_0^2} \]

\[ = \|J(w_{m_i}, \Delta w_{m_i}) - J(w_n, \Delta w_n)\|_{H_0^2} \]

\[ \leq \|J(w_{m_i}, \Delta w_{m_i}) - J(w_n, \Delta w_n)\|_{H^{-2}} \]

\[ = \sup_{\theta \in H_0^2(\Omega)} \left( \langle J(\theta, w_{m_i}), \Delta w_{m_i} \rangle_{L^2} - \langle J(\theta, w_n), \Delta w_n \rangle_{L^2} \right) \]

\[ \leq \sup_{\|\theta\|_{H_0^2}^2 = 1} \left( \sqrt{2c_4} \|\theta\|_{H_0^2} \|\nabla (w_{m_i} - w_n)\|_{L^4} \right) \]

\[ \leq \|\nabla (w_{m_i} - w_n)\|_{L^2} \to 0 \quad (m_i, n_i \to \infty), \]

which shows that \( \{F(w_{m_i})\} \) is a Cauchy sequence.

(QED)

Finally, we show a corollary of the main theorem.

Corollary 7 Let \( A : H^{-2}(\Omega) \to H_0^2(\Omega) \) be a bounded operator such that

\[ \|Au\|_{H_0^2} \leq M \|u\|_{H^{-2}}, \quad \forall u \in H^{-2}(\Omega) \quad \text{(22)} \]

for a bound \( M > 0 \); then the operator defined by

\[ T(w) := A \circ J(w, \Delta w) : H_0^2(\Omega) \to H_0^2(\Omega) \quad \text{(23)} \]

is compact.

Proof The proof is the same as the proof of main theorem after replacing

\[ \|T(w_{m_i}) - T(w_n)\|_{H_0^2} \leq M \|J(w_{m_i}, \Delta w_{m_i}) - J(w_n, \Delta w_n)\|_{H^{-2}} \]

by (22).

(QED)

In computer-aided proofs with the Newton-type fixed-point formulation, \( A \) is taken as the inverse of a linearized operator of a given nonlinear equation [6]. This corollary assures the compactness of the nonlinear operator involved in such a formulation.

Acknowledgments

We would like to thank the referee for many helpful insights and comments. This work was supported by Grants-in-Aid from the Ministry of Education, Culture, Sports, Science and Technology of Japan (Nos. JP15K05012, JP15H03637) and CREST, JST.

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