HOMFLY POLYNOMIALS FOR PERIODIC KNOTS VIA STATE MODEL

JOONOH KIM AND KYOUNG-TARK KIM

ABSTRACT. We give criteria for oriented links to be periodic of prime order using the quantum SL(N)-invariant. The criteria are based upon an observation on the linking number between a periodic knot and its axis of the rotation.

1. INTRODUCTION

A link \( L \) in \( S^3 \) is called \( p \)-periodic \((p \in \mathbb{Z}_{\geq 2})\) provided that there is a homeomorphism \( g : S^3 \rightarrow S^3 \) of order \( p \) whose fixed point set \( \gamma \) is \( S^1 \) on \( S^3 \) with \( \gamma \cap L = \emptyset \). (See [19, Section 1].)

The homfly polynomial \( P_L(a, z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}] \) of an oriented link \( L \) is known to be defined uniquely by the following recursive relation:

\[
\begin{align*}
(i) & \quad P_{T_1}(a, z) = 1; \\
(ii) & \quad a^{-1}P_{L_+}(a, z) - aP_{L_-}(a, z) = zP_{L_0}(a, z),
\end{align*}
\]

where \( T_1 \) is the trivial knot, and \( L_+, L_- \), and \( L_0 \) are obtained from \( L \) which are identical with \( L \) except one given crossing as depicted in Figure 1 (see [14, Theorem 8.2.6, p. 105]).

The Jones polynomial \( V_L(t) \) of an oriented link \( L \) can be obtained from \( P_L(a, z) \) by substituting \( a = t \) and \( z = \sqrt{t} - \frac{1}{\sqrt{t}} \) (see [14, Proposition 8.2.8, p. 106]).

The Kauffman polynomial \( F_L(a, z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}] \) of an oriented link \( L \) is defined by

\[ F_L(a, z) = a^{-w(D)}\Lambda_D(a, z), \]

where \( D \) is a diagram of \( L \), \( w(D) \) is the writhe of \( D \), and the polynomial \( \Lambda_D(a, z) \) is a regular isotopy invariant of an unoriented link obtained by:

\[
\begin{align*}
(i) & \quad \Lambda_\varnothing(a, z) = 1; \\
(ii) & \quad \Lambda_{\bigcirclearrowleft}(a, z) + \Lambda_{\bigcirclearrowright}(a, z) = z(\Lambda_{\bigcirclearrowleft}(a, z) + \Lambda_{\bigcirclearrowright}(a, z)); \\
(iii) & \quad \Lambda_{\bigcirclearrowleft}(a, z) = a\Lambda_{\bigcirclearrowright}(a, z); \\
(iv) & \quad \Lambda_{\bigcirclearrowleft}(a, z) = a^{-1}\Lambda_{\bigcirclearrowright}(a, z).
\end{align*}
\]

The following are several known results for periodic knots or links. They can be used as criteria for non-periodicity of links.

**Theorem 1.1** (Przytycki [22, Theorem 1.2, Theorem 1.4]). Let \( p \) be a prime number and \( L \) an oriented \( p \)-periodic link. Then

\[
\begin{align*}
(i) & \quad P_L(a, z) \equiv P_L(a^{-1}, z) \mod (p, z^p); \\
(ii) & \quad F_L(a, z) \equiv F_L(a^{-1}, z) \mod (p, z^p).
\end{align*}
\]
Theorem 1.2 (Traczyk [24], [14, Exercise 10.1.6, p. 125]). Let $L$ be an oriented $p$-periodic link. Then $V_L(t) \equiv V_L(t^{-1}) \mod (p, t^p - 1)$.

Theorem 1.3 (Murasugi [19, Section 1]). If $K$ is an oriented $p^r$-periodic knot ($p$ is a prime and $r \in \mathbb{N}$), then the Alexander polynomial $\Delta_K(t)$ of $K$ satisfies

$$\Delta_K(t) \equiv f(t)^{p^r} (1 + t + t^2 + \cdots + t^{\lambda - 1})^{p^r - 1} \mod p$$

for some polynomial $f(t)$ and $\lambda \in \mathbb{Z}_{>0}$ with $\gcd(\lambda, p) = 1$.

In the following theorem, Traczyk used the degree-zero term $P_0(a)$ in $z$ of $P_K(a, z) = \sum_{i \geq 0} P_{2i}(a) z^{2i}$ for periodic knots.

Theorem 1.4 (Traczyk [25, Theorem 1.1], [20, Theorem 1 (a)]). If $K$ is a $p$-periodic knot ($p$ an odd prime) and the linking number of the periodic version of $K$ with the axis of the rotation is equal to $\lambda$, then in the polynomial $P_0(a) = \sum_{i \in \mathbb{Z}} c_{2i} a^{2i}$ we have $c_{2i} \equiv c_{2i+2} \mod p$ for all $i \in \mathbb{Z}$, except possibly when $2i + 1 \equiv \pm \lambda \mod p$.

Remark 1.5. The paper [25] used a different skein relation in [16], namely,

$$lP_{L_+}(l, m) + l^{-1} P_{L_-}(l, m) + mP_{L_0}(l, m) = 0.$$ 

Theorem 1.4 is a modified version [20] in conformity with our definition of $P_L(a, z)$ in variables $a$ and $z$.

The quantum SL($N$)-invariant $\mathcal{R}_L^{(N)}(q)$ of an oriented link $L$ (see [8, Eq. (3), p. 5] and [12, Remark in p. 171]) is defined by

$$\mathcal{R}_L^{(N)}(q) := \frac{q^N - q^{-N}}{q - q^{-1}} P_L(q^{-N}, q - q^{-1}) \in \mathbb{Z}[q^{\pm 1}].$$

In this paper we give criteria for periodic links using the quantum SL($N$)-invariant $\mathcal{R}_L^{(N)}(q)$ of an oriented link $L$. 

![Skein triple](image1)

**Figure 1.** Skein triple

![Crossing signs](image2)

**Figure 2.** Crossing signs
\[ a \rightarrow b \xrightarrow{+1} b \Rightarrow w(v) = q^{-q^{-1}} \quad (1) \ a = c, \ b = d, \ a > b; \]

\[ a \rightarrow b \xrightarrow{+1} a \Rightarrow w(v) = q \quad (2) \ a = b = c = d; \]

\[ a \rightarrow b \xrightarrow{+1} a \Rightarrow w(v) = 1 \quad (3) \ a = d, \ b = c, \ a \neq b; \]

\[ a \rightarrow b \xrightarrow{-1} a \Rightarrow w(v) = q^{-1} - q \quad (4) \ a = c, \ b = d, \ a < b; \]

\[ a \rightarrow b \xrightarrow{-1} a \Rightarrow w(v) = q^{-1} \quad (5) \ a = b = c = d; \]

\[ a \rightarrow b \xrightarrow{-1} a \Rightarrow w(v) = 1 \quad (6) \ a = d, \ b = c, \ a \neq b. \]

Figure 3. Rules for splice and projection with vertex weights

2. Quantum SL(N)-invariant and its quiver state model

Throughout the paper, the letters \( L \) and \( D \) always stand for a link and its diagram, respectively.

We recall a Yang–Baxter state model of \( \mathcal{P}^{(N)}(q) \) for an oriented link \( L \). (See e.g. [12, Remark in p. 171] for detailed theory.) For \( N \in \mathbb{Z}_{\geq 2} \), we set

\[ \mathcal{N} := \{-N + 1, -N + 3, \ldots, N - 3, N - 1\}. \quad (|\mathcal{N}| = N) \]

Then we obtain the following description of \( \mathcal{P}^{(N)}(q) \):

\[ \mathcal{P}^{(N)}(q) = q^{-w(D); N} \langle D \rangle. \]

Here \( w(D) \) is the writhe of \( D \) (=the sum of crossing signs in \( D \) as shown in Figure 2) and \( \langle D \rangle \) is defined by

\[ \langle D \rangle := \sum_{\sigma} \langle D | \sigma \rangle q^{||\sigma||}, \quad (\sigma \text{ runs over all states on } D) \]

where a state \( \sigma \), the product \( \langle D | \sigma \rangle \) of vertex weights, and the state norm \( ||\sigma|| \) will be explained now.
Fix a diagram \( D \) of \( L \). We first see that \( D \) corresponds to a planar quiver (=directed multigraph) together with crossing data, say

\[
G(D) = (V(D), A(D), C(D)),
\]

where the vertex set \( V(D) \) is the set of crossings in \( D \), the arrow set \( A(D) \) is the set of oriented arcs in \( D \), and \( C(D) \) is the function \( V(D) \to \{ \pm 1 \} \) for crossing signs given as in Figure 2. Note that every vertex in \( V(D) \) has indegree 2 and outdegree 2. Now fix \( N \in \mathbb{Z}_{\geq 2} \). A state (or an \( N \)-state) on \( D \) is a pair \( (G(D), \eta) \) such that \( \eta : A(D) \to \mathcal{J}_N \) is an arrow labeling (or coloring) function satisfying the following local condition:

For each \( v \in V(D) \), its adjacent arrow-labels \( a, b, c, d \in \mathcal{I}_N \) fulfill one of the six conditions in Figure 3.

If \( \sigma = (G(D), \eta) \) is a state on \( D \), then we remove each vertex \( v \in V(D) \) by the rules in Figure 3 so as to obtain the vertex weight \( w(v) \) of \( v \). The quantity \( \langle D|\sigma \rangle \) is defined by

\[
\langle D|\sigma \rangle := \prod_{v \in V(D)} w(v) \in \mathbb{Z}[q^\pm 1].
\]

If every vertices are removed according to the rules in Figure 3, then we get a diagram of planar loops with some flat crossings (graphical crossings). It turns out (easy to verify) that the resulting diagram consists entirely of simple closed oriented loops, called component loops of \( \sigma \). The norm \( \|\sigma\| \) of \( \sigma \) is now defined by

\[
\|\sigma\| := \sum_\ell \text{label}(\ell) \cdot \text{rot}(\ell),
\]

where the summation is over all component loops \( \ell \) of \( \sigma \), \( \text{label}(\ell) \) is the assigned (coherent) arrow-label for \( \ell \), and \( \text{rot}(D) \) is defined by

\[
\text{rot}(\ell) := \begin{cases} +1 & \text{if the orientation of } \ell \text{ is counterclockwise;} \\ -1 & \text{if the orientation of } \ell \text{ is clockwise.} \end{cases}
\]

The definition of \( \text{rot}(\ell) \) makes sense because \( \ell \) is simply closed.

### 3. A CONGRUENCE OF \( \mathcal{P}_L^{(N)}(q) \) FOR PERIODIC LINKS

In what follows we occasionally use the next easy lemma.

**Lemma 3.1.** Let \( i, j \in \mathbb{Z} \). If \( i \equiv j \mod p \), then \( q^i \equiv q^j \mod (q^p - 1) \).

**Proof.** Assume that \( i > j \) with \( i = j + kp \) for some positive integer \( k \). Then

\[
q^i - q^j = ((q^p)^k - 1)q^j = (q^p - 1)(q^{p(k-1)} + \cdots + q^p + 1)q^j.
\]

The following is our main congruence of \( \mathcal{P}_L^{(N)}(q) \) for a periodic link.

**Theorem 3.2.** Let \( p \) be a prime. Suppose that \( L = \bigcup_{j=1}^m L_j \) is a \( p \)-periodic oriented link of \( m \) components with diagram \( D = \bigcup_{j=1}^m D_j \) and the axis \( \gamma \) of
the rotation. Then we have
\[ |\mathcal{P}_L^{(N)}(q)| = \sum_{\phi \in \mathcal{J}_N^m} \prod_{j=1}^m q^{\lambda_j} \cdot \phi(j) \mod (p, q^p - 1), \]
where \( \mathcal{J}_N^m \) is the set of all functions from \( \{1, \ldots, m\} \) into \( \mathcal{J}_N \), and \( \lambda_j \) is the linking number between \( L_j \) and \( \gamma \).

A criterion for non-periodicity is an immediate corollary:

**Corollary 3.3.** Suppose \( p \) is a prime and \( L = \bigcup_{j=1}^m L_j \) is an oriented link. If
\[ |\mathcal{P}_L^{(N)}(q)| \neq \sum_{\phi \in \mathcal{J}_N^m} \prod_{j=1}^m q^{\psi(j)} \cdot \phi(j) \mod (p, q^p - 1) \]
for any function \( \psi : \{1, \ldots, m\} \to \{0, \ldots, p-1\} \), then \( L \) is not \( p \)-periodic.

**Proof of Theorem 3.2.** We denote by \( \zeta \) the rotation around \( \gamma \) through the angle \( 2\pi/p \), and assume without loss of generality that \( D \) and \( G(D) \) are \( p \)-periodic configurations, i.e., symmetric under \( \zeta \).

Let \( \sigma \) be an \( N \)-state on \( D \). Then either \( \zeta(\sigma) \neq \sigma \) or \( \zeta(\sigma) = \sigma \). If \( \sigma \) is not symmetric under \( \zeta \) (i.e., \( \zeta(\sigma) \neq \sigma \)), then the orbit of \( \sigma \) under the action of \( \zeta \) consists of \( p \) distinct but congruent states \( \sigma, \zeta(\sigma), \ldots, \zeta^{p-1}(\sigma) \). Since all these congruent states contribute the same value toward \( \langle D \rangle \), we have \( \sum_{i=0}^{p-1} \langle D | \zeta^i(\sigma) \rangle q^{||\sigma||} \equiv 0 \mod p \). If \( \sigma \) is symmetric under \( \zeta \) (i.e., \( \zeta(\sigma) = \sigma \)) and \( \sigma \) has a vertex of weight \( \pm(q - q^{-1}) \), then \( \langle D | \sigma \rangle \) has a factor \( \pm(q - q^{-1})^p \) whence zero modulo \( (p, q^p - 1) \). On the other hand, if \( \sigma \) is symmetric under \( \zeta \) and \( \sigma \) has no vertex of weight \( \pm(q - q^{-1}) \) (i.e., each vertex of \( \sigma \) has weight \( q^{\pm 1} \) or \( 1 \)), then \( \langle D | \sigma \rangle = q^{pk} \) for some \( k \in \mathbb{Z} \), and hence \( \langle D | \sigma \rangle \equiv 1 \mod (q^p - 1) \).

When \( \sigma \) has no vertex of weight \( \pm(q - q^{-1}) \), we say that \( \sigma \) is “proper”. From the previous argument we see that
\[ \langle D \rangle = \sum_{\sigma : \text{all}} \langle D | \sigma \rangle q^{||\sigma||} \equiv \sum_{\sigma : \text{proper \& \ symmetric}} q^{||\sigma||} \mod (p, q^p - 1). \]

Next we calculate the norm of a proper symmetric state. Let \( C_\sigma \) be the set of component loops of a state \( \sigma \). We consider the subset \( C_\sigma' := \{ \ell \in C_\sigma : \text{there is } \gamma \text{ inside } \ell \} \). If \( \sigma \) is symmetric under \( \zeta \), then
\[ ||\sigma|| = \sum_{\ell \in C_\sigma} \text{label}(\ell) \cdot \text{rot}(\ell) \equiv \sum_{\ell \in C_\sigma'} \text{label}(\ell) \cdot \text{rot}(\ell) \mod p. \]

Here we note from Lemma 3.1 that the quantity \( q^{||\sigma||} \) is determined up to modulo \( (q^p - 1) \) whenever the exponent \( ||\sigma|| \) is determined up to modulo \( p \).

For the calculation of \( ||\sigma|| \) we claim the following lemma.

**Lemma 3.4.** If a vertex of a proper state is a self-crossing (i.e., it is not a crossing between two distinct link components), then it cannot have vertex weight 1.
Figure 4. Typical crossings in a proper state \((a \neq b)\)

**Proof of Lemma 3.4.** The proof is done by contradiction: Assume that a proper state has the crossing of the third (or fourth) diagram in Figure 4. If we chase \(D\) (not \(G(D)\)) starting from the crossing along the upper right arc (labeled \(a\)), then we arrive at the crossing again along the lower right arc (labeled \(b\)) by the assumption. However it is impossible because chasing across the crossings of weights \(q^\pm 1\) or 1 do not alter an arrow-label. (Consider all crossing types of \(D\) depicted in Figure 4.) □

**Corollary 3.5.** There is a one-to-one correspondence \(\sigma \mapsto \varphi_\sigma\) between the set of all proper states on \(D = \bigcup_{j=1}^m D_j\) and the set \(\mathcal{I}_N\).

**Proof of Corollary 3.5.** By the lemma if \(\sigma\) is a proper state, all arcs of \(D_j\) have the same arrow-label. We define this label as \(\varphi_\sigma(j)\). Conversely, from any \(\varphi \in \mathcal{I}_N\), we get a unique arrow-labeling function by refining \(\varphi\). It is naturally consistent with the conditions (2), (3), (5), (6) in Figure 3. □

If \(\sigma\) is a proper symmetric state, then \(C_\sigma^\mathcal{Y} = \bigcup_{I \in \mathcal{I}_N} E_I\) where

\[
E_I := \{ \ell \in C_\sigma^\mathcal{Y} : \ell \text{ has label } I \} = \left\{ \ell \in C_\sigma^\mathcal{Y} : \ell \text{ lies in } \bigcup_{k \in \varphi_\sigma^{-1}(I)} D_k \right\}.
\]

From the previous result \(\|\sigma\| \equiv \sum_{\ell \in C_\sigma^\mathcal{Y}} \text{label}(\ell) \cdot \text{rot}(\ell) \mod p\), we get

\[
\|\sigma\| = \sum_{I \in \mathcal{I}_N} I \cdot \sum_{\ell \in E_I} \text{rot}(\ell) = \sum_{I \in \mathcal{I}_N} I \cdot \sum_{k \in \varphi_\sigma^{-1}(I)} \lambda_k = \sum_{j=1}^m \varphi_\sigma(j) \lambda_j \mod p.
\]

If a proper state \(\sigma\) is not symmetric, then the quantities \(\sum_{j=1}^m \varphi_\xi^{-1}(\sigma)(j) \lambda_j\) are the same for all \(k \in \{0, \ldots, p-1\}\) thanks to the symmetry of \(D\) under \(\xi\). Therefore \(\sum_{k=0}^{p-1} q^{\sum_{j=1}^m \varphi_\xi^{-1}(\sigma)(j) \lambda_j} \equiv 0 \mod p\) and so we obtain from Eq. (†)

\[
\langle D \rangle \equiv \sum_{\sigma : \text{proper}} q^{\sum_{j=1}^m \varphi_\sigma(j) \lambda_j} = \sum_{\varphi \in \mathcal{I}_N} q^{\sum_{j=1}^m \varphi(j) \lambda_j} \mod (p, q^p - 1).
\]

Since \(L\) is \(p\)-periodic, \(p\) divides \(w(D)\) whence

\[
\mathcal{P}_L^{(N)}(q) = q^{-w(D)N} \langle D \rangle \equiv \langle D \rangle \mod (q^p - 1).
\]

as desired. The proof of Theorem 3.2 is now complete. □
| $N$ | $r$  | $s$  | $\mathcal{P}_L^{(N)}(q)$ | $(Q_1, Q_2, Q_3)$ | Note  |
|-----|------|------|--------------------------|--------------------|-------|
| even| even | even | odd                      | odd + even + even  | knots |
| odd | even | even | even                    | even + even + even | knots |
| even| odd  | odd  | even                    | odd + even + odd   |       |
| odd | odd  | odd  | even                    | even + odd + odd   |       |

**Figure 5.** Table for the parities of exponents in $\mathcal{P}_L^{(N)}(q)$

## 4. More Congruences for Periodic Links

In this section we change the ideal $(p, q^p - 1)$ in Theorem 3.2. We first establish the following two (easy) observations; we have included proofs for the sake of completeness.

**Lemma 4.1.** Let $c > 0$ be an odd integer and $A$ a commutative ring. Put $R := A[q^{\pm 1}]$ and $R = R_0 \oplus R_1$ where $R_0 := \bigoplus Aq^{2k}$ and $R_1 := \bigoplus Aq^{2k+1}$. If $f \in R_0$ or $f \in R_1$, then $f \in (q^c + 1)$ and $f \in (q^{2c} - 1)$ whenever $f \in (q^c - 1)$.

**Proof.** Suppose that $f \in R_0$ and $0 \neq f \in (q^c - 1)$ with $\deg f = d$. Then there is $g = \sum_{j=-n}^{d-c} a_jq^j \in R$ such that $f = g(q^c - 1)$. Since $-a_{d-c}q^{d-c} \in R_1$ appears in the product of $g$ and $q^c - 1$ it should be canceled by $a_{d-2c}q^{d-2c}$ in $a_{d-2c}q^{d-2c}(q^c - 1)$. Thus $d - 2c \geq -n$ and $a_{d-c} = a_{d-2c}$ so that

$h := a_{d-c}q^{d-c}(q^c - 1) + a_{d-2c}q^{d-2c}(q^c - 1) = a_{d-c}q^{d-2c}(q^c - 1)(q^c + 1)$

is in $R_0$. If $f - h = 0$, then the proof is done. If $f - h \neq 0$, then the reverse induction on $\deg g$ finishes the proof: The quotient $g' = f - h = g'(q^c - 1)$ is $g' = g - (a_{d-c}q^{d-c} + a_{d-2c}q^{d-2c})$. The case $f \in R_1$ is similar. □

**Proposition 4.2.** Let $p$ be an odd prime. Suppose $R = Z[q^{\pm 1}]$ in Lemma 4.1.

If $f \in R_0$ or $f \in R_1$, then we have $f \in (p, q^p + 1)$ and $f \in (p, q^{2p} - 1)$ whenever $f \in (p, q^p - 1)$.

**Proof.** Consider the canonical homomorphism

$$R \rightarrow \overline{R} := Z_p[q^{\pm 1}]; \quad g = \sum a_iq^i \mapsto \overline{g} = \sum \overline{a_i}q^i,$$

where $Z_p = \{0, \ldots, p-1\}$ is the ring of integers modulo $p$. Suppose $f \in (p, q^p - 1) = (p) + (q^p - 1)$. Then $\overline{f} \in (\overline{1}q^p - \overline{1}) \subseteq \overline{R}$. Since $\overline{f} \in \overline{R}_0$ or $\overline{f} \in \overline{R}_1$ we have $\overline{f} \in (\overline{1}q^p + \overline{1}) \subseteq \overline{R}$ by Lemma 4.1 for $A = Z_p$. This means (from $\overline{R} = R/(p)$) that there exist $f_1 \in (p) \subseteq R$ and $f_2 \in (q^p + 1) \subseteq R$ such that $f = f_1 + f_2 \in (p, q^p + 1)$. The case $f \in (p, q^{2p} - 1)$ is similar. □

An algebraic result induced from Proposition 4.2 and Theorem 3.2 is:

**Corollary 4.3.** Let $p$ be an odd prime. Suppose that both the LHS and the RHS of the congruence in Theorem 3.2 lie in either $R_0$ or $R_1$ where $R = Z[q^{\pm 1}]$. Then we can replace the ideal $(p, q^p - 1)$ by $(p, q^p + 1)$ or $(p, q^{2p} - 1)$.
It is known that \( r+s \) is always even for each monomial part \( ca^r z^s \) (\( c, r, s, \in \mathbb{Z} \)) in a homfly polynomial \( P_L(a, z) \). By (1.1) the term \( ca^r z^s \) changes into \( cQ_1 Q_2 Q_3 \) in \( \mathcal{P}_L(N)(q) \), where \( Q_1 = (q^{N-1} + q^{N-3} + \cdots + q^{-N+1}) \), \( Q_2 = q^{-N} \), and \( Q_3 = (q^{-q^{-1}})^s \). Thus the parity of exponents in \( \mathcal{P}_L(N)(q) \) is described in the table of Figure 5. On the other hand, the RHS of the congruence in Theorem 3.2 depends on \( N \) and \( \lambda_j \)'s. Note that, whenever \( N \) is odd (i.e., all \( I \in \mathcal{I}_N \) are even), the parities for the LHS and the RHS coincide automatically. In this case we can change the ideal by Corollary 4.3.

The preceding argument gives:

**Corollary 4.4.** Let \( p \) be an odd prime and we keep the same notation as in Theorem 3.2. If \( N \) is odd, then we have up to either modulo \((p,q^p+1)\) or modulo \((p,q^{2p}+1)\)

\[
\mathcal{P}_L(N)(q) \equiv \sum_{\varphi \in \mathcal{I}_N^m} \prod_{j=1}^m q^{\lambda_j, \varphi(j)}. 
\]  

The proof of the next theorem (which is, by Corollary 4.4, meaningful only when \( N \) is even) is based on a different approach.

**Theorem 4.5.** Let \( p \) be an odd prime and we keep the same notation as in Theorem 3.2. Then we have

\[
\mathcal{P}_L(N)(q) \equiv \pm \sum_{\varphi \in \mathcal{I}_N^m} \prod_{j=1}^m q^{\lambda_j, \varphi(j)} \mod (p,q^p+1). 
\]  

**Proof.** The proof is a modified version of that of Theorem 3.2. As in the previous proof, we use the same notation, terminology, and assumption (that \( D \) and \( G(D) \) are symmetric under \( \zeta \)).

Let \( \sigma \) be an \( N \)-state on \( D \). If \( \sigma \) is not symmetric under \( \zeta \), then the congruent states \( \sigma, \zeta(\sigma), \ldots, \zeta^{p-1}(\sigma) \) give zero contribution modulo \( p \) toward the state model. If \( \sigma \) is symmetric but not proper, then \( \langle D|\sigma \rangle \) has a factor \( \pm (q^{-q^{-1}})^p \) whence zero modulo \((p,q^p+1)\).

We now assume that \( \sigma \) is proper. We see from Lemma 3.4 that each “self-crossing” of a link component is not a vertex of weight 1 but a vertex of weight \( q^{\pm 1} \) depending only on its crossing sign. Since the number of “link-crossings” between two distinct components are even, the parity (modulo 2) of an integer \( k \) for which \( \langle D|\sigma \rangle = q^k \) is determined completely by the diagram \( D \) (i.e., the crossing signs of the self-crossings in \( D \)), and does not depend on having a specific choice of a proper state \( \sigma \) on \( D \). We have now

\[
\langle D \rangle = \sum_{\sigma: \text{all}} \langle D|\sigma \rangle q^{||\sigma||} \equiv \pm \sum_{\sigma: \text{proper \& symmetric}} q^{||\sigma||} \mod (p,q^p+1).
\]

In order to compute \( q^{||\sigma||} \), we first decompose \( C_\sigma = C_\sigma^Y \cup B_\sigma^+ \cup B_\sigma^- \) for any (not necessarily, proper symmetric) state \( \sigma \), where

\[
B_\sigma^\pm := \{ \ell \in C_\sigma \setminus C_\sigma^Y : \text{rot}(\ell) = \pm 1 \}. 
\]
Proof. Let \( \sigma \) be an odd, and \( I \) be even) because all elements \( \sigma \) is independent on \( \sigma \). Similarly, since \( \sum_{\ell \in C_\sigma} \) is the sum \( \sum_{j=1}^{m} \lambda_j \) of linking numbers, it is also independent on \( \sigma \). Thus, \( |B_\sigma^+| - |B_\sigma^-| \) is independent on \( \sigma \). If \( \sigma \) is symmetric, then both \( |B_\sigma^+| \) and \( |B_\sigma^-| \) become multiple of \( p \) and \( t_\sigma := (|B_\sigma^+| + |B_\sigma^-|)/p \) has the same parity with \( (|B_\sigma^+| - |B_\sigma^-|)/p \). Thus, whenever \( \sigma \) is symmetric, we have

\[
\sum_{\ell \in B_\sigma^+ \cup B_\sigma^-} \text{label}(\ell) \cdot \text{rot}(\ell) = p(\pm I_1 \pm I_2 \pm \cdots \pm I_t) \quad \text{for some } I_j \in \mathcal{I}_N.
\]

Note that \( (\pm I_1 \pm I_2 \pm \cdots \pm I_t) \) has the constant parity for all symmetric \( \sigma \). If \( N \) is odd, i.e., \( I_j \)’s are even, then this fact is trivial. However, if \( N \) is even, i.e., \( I_j \)’s are odd, then we may use the fact that the parity of \( t_\sigma \) is independent on a choice of \( \sigma \). Since

\[
||\sigma|| = \sum_{\ell \in C_\sigma} \text{label}(\ell) \cdot \text{rot}(\ell) + \sum_{\ell \in B_\sigma^+ \cup B_\sigma^-} \text{label}(\ell) \cdot \text{rot}(\ell),
\]

we obtain

\[
\sum_{\sigma: \text{proper } & \text{symmetric}} q^{|\sigma|} = \pm \sum_{\sigma: \text{proper } & \text{symmetric}} q^{\sum_{\ell \in C_\sigma} \text{label}(\ell) \cdot \text{rot}(\ell)} \mod (p, q^p + 1).
\]

As in the proof of Theorem 3.2 we have

\[
\sum_{\ell \in C_\sigma} \text{label}(\ell) \cdot \text{rot}(\ell) = \sum_{I \in \mathcal{I}_N} I \cdot \sum_{\ell \in E_I} \text{rot}(\ell) = \sum_{I \in \mathcal{I}_N} I \cdot \sum_{k \in \varnothing}\sum_{\gamma \in \varnothing} \lambda_k = \sum_{j=1}^{m} \varnothing(\gamma) \lambda_j.
\]

Since \( \sum_{k=0}^{p-1} \sum_{\ell \in C_\sigma} \text{label}(\ell) \cdot \text{rot}(\ell) \equiv 0 \mod p \) for a proper and asymmetric \( \sigma \),

\[
\sum_{\sigma: \text{proper } & \text{symmetric}} q^{\sum_{\ell \in C_\sigma} \text{label}(\ell) \cdot \text{rot}(\ell)} = \pm \sum_{\sigma: \text{proper}} q^{\sum_{j=1}^{m} \varnothing(\gamma) \lambda_j} = \pm \sum_{\varnothing \in \mathcal{I}_N^m} q^{\sum_{j=1}^{m} \varnothing(\gamma) \lambda_j} \mod (p, q^p + 1).
\]

Finally, since \( w(D) \equiv 0 \mod p \), we have

\[\mathcal{P}_L^{(N)}(q) = q^{-w(D)} \cdot \langle D \rangle \equiv \langle D \rangle \mod (q^p + 1).\]

Gathering the pieces of the above congruences, the proof is done. \( \square \)

**Proposition 4.6.** Suppose \( N \) is even. If all \( \lambda_j \)’s are simultaneously odd or even, then the \( \pm \)-sign of the congruence in Theorem 4.5 is determined by the parity of exponents in \( \mathcal{P}_L^{(N)}(q) \). If \( L \) is a knot, then the \( \pm \)-sign is determined by the parity of \( \lambda = \lambda_1 \). Moreover, if the sign is “+”, then the ideal \( (p, q^p + 1) \) in Theorem 4.5 can be replaced by \( (p, q^2p - 1) \).

**Proof.** Let \( f, g \in \mathbb{Z}[q^{\pm 1}] \) be \( f := \mathcal{P}_L^{(N)}(q) \) and \( g := \sum_{\varnothing \in \mathcal{I}_N^m} \prod_{j=1}^{m} q^{\lambda_j} \varnothing(j) \). If all \( \lambda_j \)’s are odd (resp. even), then the parity of exponents in \( g \) is odd (resp. even) because all \( j \in \mathcal{I}_N \) are odd. Consider \( \bar{f}, \bar{g} \in \mathbb{Z}_p[q^{\pm 1}] \) as in the proof of Proposition 4.2. Then \( \bar{f} \equiv \pm \bar{g} \mod (\bar{q}^p + 1) \) by Theorem 4.5 Since \( p \) is odd, the \( \pm \)-sign is therefore determined by the parity of the exponent in \( \bar{f} \). (See Figure 5) Note that if \( L \) is a knot, then the exponents in \( \bar{f} \) are always
odd so that odd \( \lambda \) (resp. even \( \lambda \)) yields the plus sign (resp. minus sign). Finally, the next proposition completes the proof.

**Proposition 4.7.** Suppose \( f \in \mathbb{Z}[q^{\pm 1}] \). If \( f \in (p, q^p - 1) \cap (p, q^p + 1) \), then \( f \in (p, q^{2p} - 1) \).

**Proof.** Again consider \( \tilde{f} \in \mathbb{Z}_p[q^{\pm 1}] \). Then \( \tilde{f} \in (\tilde{1}q^p - \tilde{1}) \cap (\tilde{1}q^p + \tilde{1}) \). Since the resultant \( \text{Res}(\tilde{1}q^p - \tilde{1}, \tilde{1}q^p + \tilde{1}) = \tilde{2} \) is not zero, there is no common root of \( \tilde{1}q^p - \tilde{1} \) and \( \tilde{1}q^p + \tilde{1} \) in an algebraic closure of \( \mathbb{Z}_p \). Thus \( \tilde{f} \in (\tilde{1}q^{2p} - \tilde{1}) \) which implies that \( f \in (p, q^{2p} - 1) \), as desired. □

**Remark 4.8.** We do not know whether Theorem 4.5 is better than Theorem 3.2. So far, we could not find an example which turns out to be a non-\( p \)-periodic link using Theorem 4.5 but not possibly by Theorem 3.2.

5. Criteria for Non-periodicity of Knots

Throughout this section \( K \) denotes a knot. The following result is a case \( m = 1 \) for knots in Theorem 3.2 and Theorem 4.5.

**Theorem 5.1.** Suppose that \( K \) is \( p \)-periodic with diagram \( D \) and the axis \( \gamma \), and that \( \lambda \) is the linking number between \( K \) and \( \gamma \). Then we have

\[
\mathcal{P}_K^{(N)}(q) \equiv \sum_{I \in \mathcal{I}_N} q^{\lambda I} \mod (p, q^p - 1)
\]

and

\[
\mathcal{P}_K^{(N)}(q) \equiv \pm \sum_{I \in \mathcal{I}_N} q^{\lambda I} \mod (p, q^p + 1).
\]

**Corollary 5.2.** If, for any integer \( k \in \{0, \ldots, p-1\} \),

\[
\mathcal{P}_K^{(N)}(q) \neq \sum_{I \in \mathcal{I}_N} q^{kI} \mod (p, q^p - 1),
\]

then \( K \) is not \( p \)-periodic. If, for any integer \( k \in \{0, \ldots, 2p-1\} \),

\[
\mathcal{P}_K^{(N)}(q) \neq \pm \sum_{I \in \mathcal{I}_N} q^{kI} \mod (p, q^p + 1),
\]

then \( K \) is not \( p \)-periodic.

It is sometimes useful to combine our criteria with existing results like the next corollary suggests.

**Corollary 5.3.** Suppose that we already have a set \( \mathcal{C} \) of possible candidate for linking number: If

\[
\mathcal{P}_K^{(N)}(q) \neq \sum_{I \in \mathcal{I}_N} q^{kI} \mod (p, q^p - 1) \quad \text{for all } k \in \mathcal{C}
\]

or

\[
\mathcal{P}_K^{(N)}(q) \neq \pm \sum_{I \in \mathcal{I}_N} q^{kI} \mod (p, q^p + 1) \quad \text{for all } k \in \mathcal{C},
\]

then \( K \) is not \( p \)-periodic.
Fix an odd prime \( p \). Then, by reducing modulo \((p, q^p - 1)\), any Laurent polynomial \( f = \sum_i a_i q^i \in \mathbb{Z}[q^{\pm 1}] \) gives rise to a unique representative which we refer to as the “normal form” of \( f \) with respect to \((p, q^p - 1)\):

\[
f \equiv \sum_{-p/2 < j < p/2} b_j q^j \pmod{(p, q^p - 1)},
\]

where \( b_j \in \{0, 1, \ldots, p-1\} \) is congruent to \( \sum_{i=j+p} a_i \) modulo \( p \) for each \( j \) with \(-p/2 < j < p/2\). When we say two Laurent polynomials in \( \mathbb{Z}[q^{\pm 1}] \) are congruent modulo \((p, q^p - 1)\), we mean both the normal forms coincide.

We have the following observation on lower bounds of \( p \) for \( p \)-periodicity.

**Corollary 5.4.** Suppose that \( \mathcal{P}_k^{(N)}(q) \) is not equal to \( \sum_{I \in \mathcal{I}_N} q^{k I} \) for all \( k \in \mathbb{N} \). Then there is \( n \in \mathbb{N} \) such that \( K \) is not \( p \)-periodic for all odd prime \( p \geq n \).

**Proof.** Let \( \mathcal{P} \) be the set of all prime divisors of coefficients of \( \mathcal{P}_k^{(N)}(q) \). Set \( n := 1 + \max(\mathcal{P} \cup \{2 \deg \mathcal{P}_k^{(N)}(q)\}) \). If \( p \geq n \), then \( \mathcal{P}_k^{(N)}(q) \) itself is its normal form. Thus, \( \mathcal{P}_k^{(N)}(q) \) cannot be congruent to \( \sum_{I \in \mathcal{I}_N} q^{k I} \) for some \( k \) after modulo \((p, q^p - 1)\) by the assumption. \( \square \)

We next talk about how Theorem 5.1 itself provides the possible values of linking number. Suppose \( k \) is an integer such that \( |\{k \cdot I \mod p : I \in \mathcal{I}_N\}| = |\mathcal{I}_N| = \mathbb{N} \). For such \( k \), the number of terms in the normal form of \( \sum_{I \in \mathcal{I}_N} q^{k I} \) is \( N \), and we address the question which integer \( k' \) with \( 1 \leq k' \leq p-1 \) satisfies the following identity of sets:

\[
\{k' \cdot I \mod p : I \in \mathcal{I}_N\} = \{|k \cdot I \mod p : I \in \mathcal{I}_N\}.
\]

Important cases where we can easily determine such \( k' \) are the following.

**Proposition 5.5.** Suppose \( N \in \{2, 3\} \) and \( k \in \mathbb{Z} \) such that the number of terms in the normal form of \( \sum_{I \in \mathcal{I}_N} q^{k I} \) is \( N \). Then, there are exactly two choices for \( k' \in \mathbb{Z} \) with \( 1 \leq k' \leq p-1 \) satisfying Eq. (5.1). In particular, if \( k' \) and \( k'' \) are such two choices, then \( \{\pm k' \mod p\} = \{\pm k'' \mod p\} \).

**Proof.** If \( N = 2 \) (resp. \( N = 3 \)), then the set \( \{k \cdot I : I \in \mathcal{I}_N\} \) is equal to \( \{\pm k\} \) (resp. \( \{0, \pm 2k\} \)). Since \( \gcd(k, p) = 1 \) by the assumption, we can uniquely determine \( k' \in \mathbb{Z} \) with \( 1 \leq k' \leq p-1 \) such that \( k' \equiv k \mod p \). Clearly, another choice must be \( p - k' \). \( \square \)

Let \( K \) be a knot which we suspect being \( p \)-periodic. Suppose we already know \( \mathcal{P}_k^{(N)}(q) \) for every \( N \in \mathbb{N} \). Theoretically, for each \( N \in \mathbb{N} \), we can judge whether or not the congruence \( \mathcal{P}_k^{(N)}(q) \equiv \sum_{I \in \mathcal{I}_N} q^{k N I} \mod (p, q^p - 1) \) holds for some integer \( k_N \) depending on \( N \) with \( 0 \leq k_N \leq p - 1 \). (Here, the choice for \( k_N \) may not be unique.) Whenever this congruence holds for all elements of a fixed subset \( S \subseteq \mathbb{N} \), we can check further if there is an integer \( k \) such that \( \{\pm k \mod p\} = \{\pm k_N \mod p\} \) for all \( N \in S \) with some
choice of $k_N$, in which case we call $\pm k \mod p$ the “possible linking number of $K$ with respect to $S$”. (Note that the plus-minus sign for $\pm k \mod p$ corresponds to an orientation of $K$.) On the other hand, if there exists no such $k_N$ for some $N \in \mathbb{N}_{\geq 2}$, or there is a subset $S \subseteq \mathbb{N}_{\geq 2}$ such that the sets $\{ \pm k_N \mod p \}$ are not the same for all $N \in S$ with some $k_N$, then we can conclude that $K$ is not periodic. By Proposition 5.5, we often do this procedure with $S = \{2, 3\}$. Of course, the larger the size of $S$ becomes, the more reliable possible linking number we get.

Remark 5.6. There may be many ways to obtain a set $C$ of possible linking numbers (in order to apply Corollary 5.3). We can find $C$ from the previous argument, or from other theorems such as Theorem 1.4.

Suppose that one method gives $C$ and the other method gives $C'$ with $C \cap C' = \emptyset$. In this case we can say that $K$ is not $p$-periodic.

REFERENCES

[1] G. Burude, H. Zieschang, Knots, Walter de Gruyter & Co., Berlin, Second edition (2004).
[2] Q. Chen and T. Lê, Quantum Invariants of Periodic Links and Periodic 3-Manifold, arXiv:math/040835v2.
[3] D. Cooper, M. Culler, H. Gillet, D. D. Long, and P. B. Shalen, Plane curves associated to character varieties of 3-manifolds, J. Algebra. 118 (1994), 47-84.
[4] M. Culler and P. B. Shalen, Varieties of group representations and splittings of 3-manifolds, Ann. of Math. (2) 117 (1) (1983) 109-146.
[5] T. Dimofte and S. Gukov, Quantum Field Theory and the Volume Conjecture, Contemp. Math. 541, (2011), 41-67.
[6] S. Garoufalidis, On the characteristis and deformation varieties of a knot, Geometry and Topology Monographs, 7 (2004), 291-304.
[7] P. Gilmer, J. Kania-Bartoszynska and J. Przytycki, 3-Manifold Invariants and Periodicity of Homology Spheres, Algebr. Geom. Topol. 2 (2002), 825-842.
[8] S. Gukov, I. Sabery, Lectures on knot homology and quantum curves, arXiv:1211.6075v1.
[9] S. Gukov, P. Sulkowski, A-polynomial, B-model, and Quantization, JHEP 1202 (2012), 070.
[10] S. Gukov, Three-dimensional quantum gravity, Chern-Simons theory, and the A-polynomial, Commun. Math. Phys. 255 (2005) 577-627.
[11] K. Hikami, Difference equation of the colored Jones polynomial for torus knots, Internet J. Math. 15 (9) (2004).
[12] L. Kauffman, Knots and Physics, World Scientific (1991).
[13] L. Kauffman, State Models and the Jones Polynomial, Topology 26, (1987), 395-407.
[14] A. Kawauchi, A Survey of Knot Theory, Birkhäuser (1990).
[15] Knotinfo, http://www.indiana.edu/~knotinfo/
[16] W. B. R. Lickorish, K. C. Millett, A polynomial invariant of oriented links, Topology 26, no. 1 (1987), 107–141.
[17] T. Lê, The colored Jones polynomial and the A-polynomial of Knots, Advanced in Math. 207 (2006) 782-804.
[18] A. Lubotzky and A. Magid, Varieties of representations of finitely generated group, Mem. Amer. Math. Soc. 336 (1985), 1-117.
[19] K. Murasugi, On Periodic Knots, Comment. Math. Helv. 46 (1971), 162-174.
[20] J. Przytycki, *An elementary proof of the Traczyk-Yokota criteria for periodic knots*, Proc. Amer. Math. Soc. 123, no. 5 (1995), 1607–1611.

[21] J. Przytycki, *Fundamentals of Kauffman bracket skein modules*, Kobe J. Math 16 (1999), 45-66.

[22] J. Przytycki, *On Murasugi’s and Traczyk’s Criteria for Periodic Links*, Math. Ann. 283 (1989), 465-478.

[23] T. Takata, *The colored Jones polynomial and the A-polynomial for twist knows*, preprint, math.GT/0401068.

[24] P. Traczyk, *Knot has no period 7: A Criterion for Periodic Links*, Proc. Amer. Math. Soc. 108 (1990), 845-846.

[25] P. Traczyk, *Periodic Knots and the Skein Polynomial*, Invent. Math. 106 (1991), 73–84.

DEPARTMENT OF MATHEMATICS, PUSAN NATIONAL UNIVERSITY, BUSAN, KOREA

SOGANG RESEARCH TEAM FOR DISCRETE AND GEOMETRIC STRUCTURES, SOGANG UNIVERSITY, MAPO-GU, SEOUL 04107, KOREA