Strong field limit of the Born-Infeld $p$-form electrodynamics

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Abstract

We study the strong field limit of $p$-form Born-Infeld theory. It turns out that this limiting theory is a unique theory displaying the full symmetry group of the underlying canonical structure. Moreover, being a nonlinear theory, it possesses an infinite hierarchy of conservation laws.

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1 Introduction

The duality between strong and weak coupling regimes of the underlying theory has played in recent years very prominent role (see e.g. review in [1]). In the present paper we study the weak-strong field limit correspondence for the $p$-form Born-Infeld theory. Recently, Born-Infeld nonlinear electrodynamics (BI) [2] has found a beautiful applications in string theory and $p$-brane physics [3]. The motivation to study the corresponding $p$-form version of the BI theory comes also from the string theory where one considers extended objects ($p$-branes) coupled to a $p$-form gauge potential [4].

In the weak field limit of $p$-form BI theory one obtains a linear $p$-form Maxwell theory. The corresponding strong field limit is not so well known. It was studied in [5] for $p = 1$ under the name Ultra Born-Infeld theory (UBI). In the present paper we find the corresponding $p$-form UBI and study its properties. It turns out that this theory being nonlinear possesses very instructive features: it is invariant under the full conformal group in $(2p+2)$-dimensional Minkowski space-time. Us usual [6] the parity of $p$ plays a crucial role. However, in both cases (i.e. for $p$ odd and even) the corresponding $p$-form UBI displays the full canonical symmetry group of the underlying canonical structure, i.e. $SO(2,1)$ and $SO(1,1) \times Z_2$ symmetry for odd and even $p$ respectively. Moreover, for odd $p$ it has an infinite hierarchy of conservation laws. Therefore, the strong field limit of BI theory is even more symmetric than the Maxwell theory which is also conformally invariant and being linear has an infinite hierarchy of constants of motion.
2 Born-Infeld $p$-form theory

2.1 A general theory

Consider a general nonlinear $p$-form electrodynamics defined in $D = 2p + 2$ dimensional Minkowski space-time $\mathcal{M}^{2p+2}$ with the signature of the metric tensor $(-, +, ..., +)$. The corresponding field tensor $F = dA$ ($A$ denotes a $p$-form gauge potential) gives rise to the following relativistic and gauge invariants:

\begin{align}
S_p &= -\frac{1}{2(p+1)!} F_{\mu_1...\mu_{p+1}} F^{\mu_1...\mu_{p+1}} , \\
P_p &= -\frac{1}{2(p+1)!} F_{\mu_1...\mu_{p+1}} \star F^{\mu_1...\mu_{p+1}} ,
\end{align}

where the Hodge star operation in $\mathcal{M}^{2p+2}$ is defined by:

\begin{equation}
\star F_{\mu_1...\mu_{p+1}} = \frac{1}{(p+1)!} \eta_{\mu_1...\mu_{p+1}}^{\nu_1...\nu_{p+1}} F_{\nu_1...\nu_{p+1}} ,
\end{equation}

and $\eta^{\mu_1\mu_2...\mu_{2p+2}}$ is the covariantly constant volume form in the Minkowski space-time. The Hodge star satisfies $\star^2 = (-1)^p$ which implies the crucial difference between $p$-form theories with different parities of $p$. Having a Lagrangian $L_p = L_p(S_p, P_p)$ one introduces a $G$-tensor

\begin{equation}
G^{\mu_1...\mu_{p+1}} = -(p+1)! \frac{\partial L_p}{\partial F_{\mu_1...\mu_{p+1}}} .
\end{equation}

Eq. (2.4) defines the constitutive relation for the underlying $p$-forms electrodynamics. In the Maxwell theory one has $G(F) = F$ but in the general case this relation may be highly nonlinear.

Now one may define the electric and magnetic intensities and inductions in the obvious way:

\begin{align}
E_I &= F_{I0} , \\
B_I &= \frac{1}{(p+1)!} \epsilon_{IJK} F^{JK} , \\
D_I &= G_{I0} , \\
H_I &= \frac{1}{(p+1)!} \epsilon_{IJK} G^{JK} ,
\end{align}

where we introduced a $p$-index $I = (i_1 i_2 ... i_p)$ with $i_k = 1, 2, ..., 2p + 1$. $\epsilon_{i_1...i_p j_1...j_k} = \epsilon_{IJK}$ denotes the Lévi-Civita tensor in $2p + 1$ dimensional Euclidean space, i.e. a space-like hyperplane $\Sigma$ in the Minkowski space-time. Note, that $\epsilon^{i_1...i_{2p+1}} := \eta^{0i_1...i_{2p+1}}$.

In terms of $(E, B, D, H)$ the field equations $dF = 0$ and $d\star G = 0$ read:

\begin{align}
\partial_0 B^I &= (-1)^p \frac{1}{p!} \epsilon^{IkJ} \partial_k E_J , \\
\partial_0 D^I &= \frac{1}{p!} \epsilon^{IkJ} \partial_k H_J .
\end{align}

They are supplemented by the following constraints (Gauss laws):

\begin{equation}
\partial_i B^{i...} = \partial_i D^{i...} = 0 .
\end{equation}

The dynamical properties of the $p$-form electrodynamics is fully described by the energy-momentum tensor $T^\mu_\nu$:

\begin{equation}
T^\mu_\nu = \frac{1}{p!} F^{\mu I} G^\nu_I + g^\mu_\nu L_p ,
\end{equation}

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or in components:

\[
T^{00} = \frac{1}{p!} E \cdot D - L_p ,
\]

(2.13)

\[
T^{0k} = (-1)^{p+1} (E \times H)^k ,
\]

(2.14)

\[
T^{k0} = (-1)^{p+1} (D \times B)^k ,
\]

(2.15)

\[
T^{kl} = -\frac{1}{(p-1)!} \left( E^{k\ldots l}D^{\ldots i} + H^{k\ldots l}B^{\ldots i} \right) + \delta^{kl} \left( \frac{1}{p!} H \cdot B - L_p \right) ,
\]

(2.16)

where we introduced a convenient notation: \( E \cdot D := E^I D_I \). Moreover,

\[
(E \times H)^k = \frac{1}{(p!)^2} \epsilon^{kIJ} E_I H_J .
\]

(2.17)

Note, that

\[
(E \times H)^k = (-1)^p (H \times E)^k .
\]

Now, observe that dynamical equations (2.9)-(2.10) have already canonical form. The Hamiltonian \( \mathcal{H}_p = T^{00} \) and the corresponding Poisson bracket for the canonical variables \((D^I, B^J)\) reads:

\[
\{D^I(x), B^J(y)\}_p = \epsilon^{kIJ} \partial_k \delta^{(2p+1)}(x-y) ,
\]

(2.18)

(all other brackets vanish).

### 2.2 Canonical symmetries

#### 2.2.1 \( p \) odd

The \( p \)-form theory based on \( L_p = L_p(S_p, P_p) \) is obviously relativistically invariant. As is well known in the Hamiltonian framework this invariance is equivalent to the symmetry of the energy-momentum tensor. Let us introduce the following scalar quantities built out of the canonical variables \((D^I, B^J)\):

\[
\alpha = \frac{1}{2p!} (D \cdot D + B \cdot B) ,
\]

(2.19)

\[
\beta = \frac{1}{2p!} (D \cdot D - B \cdot B) ,
\]

(2.20)

\[
\gamma = \frac{1}{p!} D \cdot B .
\]

(2.21)

Now, the condition \( T^{0k} = T^{k0} \) which is equivalent to

\[
(E \times H)^k = (D \times B)^k ,
\]

(2.22)

implies the following equation for \( \mathcal{H}_p \):

\[
(\partial_\alpha \mathcal{H}_p)^2 - (\partial_\beta \mathcal{H}_p)^2 - (\partial_\gamma \mathcal{H}_p)^2 = 1 .
\]

(2.23)

Eq. (2.23) has a hyperbolic \( SO(2,1) \) symmetry. It turns out that the group \( SO(2,1) \) has a natural representation on the level of a canonical structure for a general \( p \)-form theory.
For \( p \) odd the canonical structure defined in (2.18) is invariant under:

1) duality \( SO(2) \) rotations:
\[
\begin{align*}
D^I & \rightarrow D^I \cos \varphi - B^I \sin \varphi , \\
B^I & \rightarrow D^I \sin \varphi + B^I \cos \varphi ,
\end{align*}
\]
(2.24)

2) hyperbolic \( SO(1,1) \) rotations:
\[
\begin{align*}
D^I & \rightarrow D^I \cosh \varphi + B^I \sinh \varphi , \\
B^I & \rightarrow D^I \sinh \varphi + B^I \cosh \varphi ,
\end{align*}
\]
(2.25)

3) \( R^* \)-scaling
\[
\begin{align*}
D^I & \rightarrow e^\lambda D^I , \\
B^I & \rightarrow e^{-\lambda} B^I .
\end{align*}
\]
(2.26)

The easiest way to find the corresponding generators for (2.24)-(2.26) is to use a two potential formulation [6], [7]. Let us introduce a \( p \)-form potential \( \Phi^I \) for \( D^I \):
\[
D^I = \epsilon^{IJK} \partial_k \Phi^J ,
\]
(2.27)

in analogy to
\[
B^I = \epsilon^{IJK} \partial_k \Phi^J ,
\]
(2.28)

where both \( A^I \) and \( \Phi^I \) are in the transverse gauge, i.e. \( \partial_i A^{i\cdots} = \partial_i \Phi^{i\cdots} = 0 \). Defining \( A^I_{(a)} = (A^I, \Phi^I) \) and \( B^I_{(a)} = (B^I, D^I) \) the corresponding generators may be written as follows:
\[
\begin{align*}
G_1 & = \frac{1}{2p!} \int d^{2p+1} x \left( A_{(1)} \cdot B_{(1)} + A_{(2)} \cdot B_{(2)} \right) , \\
G_2 & = \frac{1}{2p!} \int d^{2p+1} x \left( A_{(1)} \cdot B_{(1)} - A_{(2)} \cdot B_{(2)} \right) , \\
G_3 & = \frac{1}{2p!} \int d^{2p+1} x \left( A_{(1)} \cdot B_{(2)} + A_{(2)} \cdot B_{(1)} \right) .
\end{align*}
\]
(2.29-2.31)

For the Maxwell \( p \)-form theory one has \( H_{p,\text{Maxwell}}^\alpha = \alpha \) and (2.23) is trivially satisfied. As is well known [8] Maxwell theory is invariant under (2.24), i.e. \( \{ H_{p,\text{Maxwell}}, G_1 \} = 0 \) and, therefore, \( G_1 \) defines a constant of motion. Its physical meaning will be clarified in subsection 2.3.

### 2.2.2 \( p \) even

For even \( p \) the situation is very different. The invariant \( P_p \) defined in (2.2) vanishes and therefore the general Hamiltonian \( \mathcal{H}_p \) does not depend upon \( \gamma \) defined in (2.21). Eq. (2.23) reduces now to
\[
(\partial_\alpha \mathcal{H}_p)^2 - (\partial_\beta \mathcal{H}_p)^2 = 1 .
\]
(2.32)

Eq. (2.33) displays the \( SO(1,1) \) symmetry which is now realized by (2.26). Neither (2.24) nor (2.25) are implementable as canonical transformations [8] and (2.26) is generated by:
\[
G_4 = \frac{1}{2p!} \int d^{2p+1} x \left( A_{(1)} \cdot B_{(2)} - A_{(2)} \cdot B_{(1)} \right) .
\]
(2.33)

However, in this case we have two additional discrete \( Z_2 \) symmetries:
\[
D^I \rightarrow B^I , \quad B^I \rightarrow D^I ,
\]
(2.34)

and
\[
D^I \rightarrow -B^I , \quad B^I \rightarrow -D^I .
\]
(2.35)

Now, Maxwell theory is only \( Z_2 \times Z_2 \)-invariant, i.e. w.r.t. (2.34) and (2.35).
2.3 $p$-photons

For odd $p$ the canonical generator $G_1$ defined in (2.29) is constant in time (in the Maxwell theory). To find the physical interpretation of this quantity let us introduce a complex notation:

$$F^I = D^I + iB^I,$$
$$G^I = E^I + iH^I.$$  

The dynamical equations (2.9)-(2.10) rewritten in terms of $(F^I, G^J)$ have the following form:

$$i\partial_0 F^I = \frac{1}{p!} \epsilon^{IkJ} \partial_k G^J.$$  

Moreover, let

$$V^I = Z^I + iA^I,$$  

i.e. $F^I = \epsilon^{IkJ} \partial_k V^J$. Now, introduce a Fourier representation for $V(x)$:

$$\tilde{V}_I(k) = \int d^{2p+1}x e^{-ikx} V_I(x).$$

The transverse gauge $\partial_i V^{i\cdots}(x) = 0$ implies $k_l \tilde{V}^{i\cdots}(k) = 0$. Let $\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(N_p)}$ form an orthonormal basis of $p$-forms on 2$p$-dimensional plane perpendicular to $k$, i.e. $\omega^{(i)} \cdot \omega^{(j)} = \delta^{ij}$. The number $N_p$ reads

$$N_p = \left( \frac{2p}{p} \right),$$

and it is equal to the number of degrees of freedom for a $p$-form theory [8]. Now, for any $\alpha \in \{1, 2, ..., N_p\}$ there exists exactly one index $\alpha'$ such that

$$(\omega^{(\alpha)} \times \omega^{(\alpha')})^I = \frac{k^I}{k}.$$  

Therefore, let us define a complex basis:

$$e^{(\alpha)} = \frac{1}{\sqrt{2}} (\omega^{(\alpha)} + i\omega^{(\alpha')}),$$  

where $\alpha$ runs from 1 to $N_p/2$. Note, that the complex basis $e^{(\alpha)}$ satisfies:

$$\epsilon^{IJ} k_l e^{(\alpha)}_I e^{(\beta)}_J = -ik e^{(\alpha)l}$$

and obviously: $\overline{e^{(\alpha)} \cdot e^{(\beta)}} = \delta^{\alpha\beta}$ and $e^{(\alpha)} \cdot e^{(\beta)} = 0$, where $\overline{a}$ denotes a complex conjugation of $a$.

Now, decomposing the Fourier transform of $V_I(x)$:

$$\tilde{V}_I(k) = \sum_{\alpha=1}^{N_p/2} \left( e_+^{(\alpha)} f_+^{(\alpha)}(k) + e_-^{(\alpha)} f_-^{(\alpha)}(k) \right),$$

and inserting into (2.29) one obtains:

$$G_1 = \frac{1}{2p!} \int d^{2p+1}x V_I(x) e^{ikJ} \partial_k \tilde{V}_J(x)$$

$$= \frac{1}{2p!(2\pi)^{2p+1}} \int d^{2p+1}k \sum_{\alpha=1}^{N_p/2} \left( |f_+^{(\alpha)}(k)|^2 - |f_-^{(\alpha)}(k)|^2 \right).$$


Note, that the energy of the Maxwell field reads:

\[
\mathcal{H}_p = \frac{1}{2p!} \int d^{2p+1}x \left( D \cdot D + B : B \right) = \frac{1}{2p!} \int d^{2p+1}x F(x) \cdot \overline{F(x)} \\
= \frac{1}{2p!} \int d^{2p+1}x \epsilon^{IJ} \partial_k V_J(x) \epsilon_{IJL} \partial^L V^L(x) \\
= \frac{1}{2p!(2\pi)^{2(2p+1)}} \int d^{2p+1}k k^2 \sum_{\alpha=1}^{N_p/2} \left( |f_+^{(\alpha)}(k)|^2 + |f_-^{(\alpha)}(k)|^2 \right). \tag{2.44}
\]

Now, \( \mathcal{H}_p \) measures the sum of intensities of all possible polarizations. On the other hand \( G_1 \) measures the intensity of right-handed polarizations minus the intensity of left-handed polarizations.

In the quantum theory of a \( p \)-form Maxwell field they correspond to different polarization states of “\( p \)-photons” \( (N_p/2 \) left-handed defined by \( e^{(\alpha)} \) and \( N_p/2 \) right-handed defined by \( e^{(\alpha')} \)).

Remarkably, (2.44) is valid in the linear (Maxwell) case only but (2.43) holds for any \( p \)-form theory and defines a constant of motion for any duality invariant theory. Neither \( G_2 \) nor \( G_3 \) have any clear physical interpretation.

2.4 Born-Infeld model

The Born-Infeld \( p \)-form theory is defined by the following Lagrangian:

\[
L^{(BI)}_p = \frac{b^2}{p!} \left( 1 - \sqrt{1 - 2b^{-2}p!S_p - (b^{-2}p!P_p)^2} \right), \tag{2.45}
\]

where \( b \) denotes a generalized fundamental parameter of Born and Infeld [2]. In terms of \( E^I \) and \( B^I \) our two basic invariants read:

\[
S_p = \frac{1}{2p!} (E \cdot E - B \cdot B), \\
P_p = \begin{cases} \frac{1}{p} E \cdot B, & \text{for odd } p, \\ 0, & \text{for even } p. \end{cases}
\]

The corresponding \( D^I \) and \( H^I \) fields read:

\[
D^I = \frac{1}{l_p} \left( E^I + b^{-2}P_p B^I \right), \\
H^I = \frac{1}{l_p} \left( B^I - b^{-2}P_p E^I \right),
\]

with \( l_p = \sqrt{1 - 2b^{-2}p!S_p - (b^{-2}p!P_p)^2} \). From (2.13) one easily gets the corresponding Hamiltonian:

\[
\mathcal{H}^{(BI)}_p = \frac{b^2}{p!} \left( \sqrt{1 + b^{-2}(D \cdot D + B \cdot B) + b^{-4} [(D \cdot D)(B \cdot B) - \epsilon_p (D \cdot B)^2]} - 1 \right), \tag{2.46}
\]

where

\[
\epsilon_p = \begin{cases} 1, & \text{odd } p, \\ 0, & \text{even } p. \end{cases}
\]

Note, that for any \( p \)-forms \( D \) and \( B \)

\[
(p!)^2 |D \times B|^2 = (D \cdot D)(B \cdot B) + (-1)^p (D \cdot B)^2.
\]
Therefore, for odd $p$

$$H_p^{(BI)} = b^2 \left( \frac{1}{p!} \left( 1 + b^{-2}(D \cdot D + B \cdot B) + b^{-4}(p!)^2|D \times B|^2 - 1 \right) \right) . \tag{2.47}$$

In terms of $(\alpha, \beta, \gamma)$ the BI Hamiltonian (2.46) reads:

$$H_p^{(BI)} = b^2 \left( \sqrt{1 + b^{-2}\alpha + b^{-4}(\alpha^2 - \beta^2 - \epsilon_p \gamma^2)} - 1 \right) , \tag{2.48}$$

and satisfies (2.23) or (2.32) for odd or even $p$ respectively.

It is easy to see that $p$-form BI is invariant under:

- dual $SO(2)$ rotations (2.24) for odd $p$,
- $Z_2 \times Z_2$ transformations (2.34) and (2.33) for even $p$,

exactly as Maxwell theory.

### 3 Strong field limit

The crucial property of the Born-Infeld model is that the magnitude of $E^I$ and $B^I$ fields is bounded by the value of the critical parameter $b$, i.e. $|F^{p_1...p_{p+1}}| < b$. Therefore, we are not able to study the strong field limit of this model in the Lagrangian framework. Actually, performing the $b \to 0$ limit in the Born-Infeld Lagrangian (2.45) one obtains $|P_p|$ which defines a trivial theory. However, in the Hamiltonian framework the $D^I$ field is not bounded and the question about the strong field limit is well posed. The same property displays the relativistic particle’s dynamics: the particle’s velocity is always bounded $|\mathbf{v}| < c$ contrary to its momentum $\mathbf{p}$ (or energy $E$) and the ultrarelativistic limit is defined by $|\mathbf{p}| \gg mc$ (or $E \gg mc^2$). The particle’s Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = c\sqrt{\mathbf{p}^2 + (mc)^2 + V(\mathbf{q})} , \tag{3.1}$$

tends in the ultrarelativistic limit to

$$H^U(\mathbf{p}) = c|\mathbf{p}| \tag{3.2}$$

which implies

$$\dot{\mathbf{p}} = 0 , \quad \mathbf{v} = c\mathbf{n} , \tag{3.3}$$

with $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$, i.e. one has a free evolution of photons. By analogy we call the strong field limit of BI theory the Ultra Born-Infeld theory (UBI) [3].

#### 3.1 $p$ odd

Performing $b \to 0$ limit in (2.46) one gets

$$H_p^{(UBI)} = |D \times B| . \tag{3.4}$$

Therefore, the constitutive relations are given by:

$$E_I = p! \frac{\delta H_p^{(UBI)}}{\delta D^I} = \epsilon_{kIJ}n^kB^J , \tag{3.5}$$

$$H_I = p! \frac{\delta H_p^{(UBI)}}{\delta B^I} = -\epsilon_{kIJ}n^kB^J , \tag{3.6}$$
where \( n^k \) stands for the unit \((2p+1)\)-vector in the direction of the generalized Poynting vector:

\[
n^k = \frac{(D \times B)^k}{|D \times B|}.
\]  

(3.7)

The dynamical equations (2.9)-(2.10) have the following form:

\[
\partial_0 B^I = - \delta_{[i,j]}^k \partial_k (n^j B^I),
\]

(3.8)

\[
\partial_0 D^I = - \delta_{[i,j]}^k \partial_k (n^j D^I),
\]

(3.9)

with \( \delta_{[i,j]}^{kl} = \delta_{[i}^k \delta_{j]}^l \). The remarkable property of \( p \)-form UBI is a structure of the energy-momentum tensor. One easily finds:

\[
T^0k = T^{k0} = \mathcal{H}^{(UBI)}_{p} n^k,
\]

(3.10)

\[
T^{kl} = \mathcal{H}^{(UBI)}_{p} n^k n^l.
\]

(3.11)

These relations were already derived by Bialynicki-Birula [5] for \( p = 1 \). But they hold for any odd \( p \). In terms of \( (\alpha, \beta, \gamma) \) the UBI Hamiltonian (3.4) reads:

\[
\mathcal{H}^{(UBI)}_{p} = \sqrt{\alpha^2 - \beta^2 - \gamma^2},
\]

(3.12)

and displays the full \( SO(2,1) \) symmetry of (2.23), i.e. \( p \)-form UBI is invariant under: (2.24), (2.25) and (2.26).

Moreover, for any \( p \) the trace of \( T^{\mu\nu} \) for UBI vanishes and, therefore, UBI is invariant under the conformal group in \( \mathcal{M}^{2p+2} \). This last property follows from the fact that UBI does not contain a dimensional parameter.

Note, that after performing the Legendre transformation the UBI Lagrangian vanishes: \( L^{(UBI)}_{p} = D \cdot E - \mathcal{H}^{(UBI)}_{p} \equiv 0 \). It does not mean, however, that the theory is trivial (we already know that it is not). Vanishing of \( L^{(UBI)}_{p} \) denotes the presence of Lagrangian constraints. It is easy to see that the constitutive relations imply \( S_p = 0 \) and \( P_p = 0 \). Therefore, the UBI action has the following form:

\[
W^{(UBI)}_{p} = \int d^{2p+2} x (\Lambda_1 S_p + \Lambda_2 P_p),
\]

(3.13)

where \( \Lambda_1 \) and \( \Lambda_2 \) are Lagrange multipliers. Now, (3.13) implies

\[
D^I = \Lambda_1 E^I + \Lambda_2 B^I,
\]

and hence \( |D \times B| = \Lambda_1 B \cdot B \) (since \( E \cdot E = B \cdot B \)), which gives \( \Lambda_1 = |D \times B|/B \cdot B \). Moreover, \( D \cdot B = \Lambda_2 B \cdot B \) and hence \( \Lambda_2 = D \cdot B/B \cdot B \). Inserting

\[
E^I = \frac{1}{\Lambda_1} D^I - \frac{\Lambda_2}{\Lambda_1} B^I,
\]

\[
H^I = \frac{\Lambda_1^2 + \Lambda_2^2}{\Lambda_1^2} B^I - \frac{\Lambda_2}{\Lambda_1} D^I,
\]

into field eqs. (2.9)–(2.9) one gets (3.8)–(3.9).
3.2 \( p \) even

Now, due to \((2.46)\) the UBI Hamiltonian simplifies to

\[
\mathcal{H}_{p}^{(UBI)} = \frac{1}{p!} |D||B| ,
\]

(3.14)
giving rise to the following constitutive relations:

\[
E^I = \frac{|B|}{|D|} D^I ,
\]

(3.15)
\[
H^I = \frac{|D|}{|B|} B^I .
\]

(3.16)
The corresponding stress tensor reads:

\[
T^{kl} = \mathcal{H}_{p}^{(UBI)} \left( \delta^{kl} - p \left( |D|^{-2} D^k D^l + |B|^{-2} B^k B^l \right) \right) .
\]

(3.17)

Now, dynamical equations \((2.9)-(2.10)\) read:

\[
\partial_0 B^I = \frac{1}{p!} \epsilon^{IKJ} \partial_k \left( \frac{|B|}{|D|} D_J \right) ,
\]

(3.18)
\[
\partial_0 D^I = \frac{1}{p!} \epsilon^{IKJ} \partial_k \left( \frac{|D|}{|B|} B_J \right) .
\]

(3.19)

Note, that \((3.14)\) rewritten in terms of \((\alpha, \beta)\) has \(SO(1,1)\)–invariant form:

\[
\mathcal{H}_{p}^{(UBI)} = \sqrt{\alpha^2 - \beta^2} .
\]

(3.20)
The theory displays the full symmetry of the canonical structure, i.e. it is invariant under \((2.26)\) generated by \(G_4\) and under \(Z_2 \times Z_2\) defined in \((2.34)-(2.35)\). Obviously, \(T^{\mu}_{\mu} = 0\) and the theory is conformally invariant. The Lagrangian structure may be derived from the following action

\[
W_{p}^{(UBI)} = \int d^{2p+2}x \Lambda S_p ,
\]

(3.21)

\((\Lambda - \text{Lagrange multiplier})\) in analogy to \((3.13)\).

Note, that for Cauchy data satisfying

\[
D \cdot B = 0 .
\]

(3.22)
one has

\[
|D \times B| = \frac{1}{p!} |D||B| ,
\]

(3.23)
and the Hamiltonian \((3.14)\) has the same form as in \((3.4)\). The constitutive relations \((3.15)-(3.16)\)
are equivalent to:

\[
E_I = \epsilon_{kl} n^k B^J ,
\]

(3.24)
\[
H_I = \epsilon_{kl} n^k D^J ,
\]

(3.25)
with \(n_k\) defined in \((3.7)\) and the stress tensor \((3.17)\) may be rewritten as in \((3.11)\).
4 Fluid dynamics and new constants of motion

In this section we generalize the observation made in [5] for any odd $p$. Observe that due to (3.10)-(3.11), $T^\mu\nu$ may be written in the following form:

$$T^\mu\nu = \mathcal{H}_p^{(UBI)} U^\mu U^\nu ,$$

(4.1)

where the $(2p+2)$-velocity $U^\mu = (1, n^k)$ satisfies $U^\mu U_\mu = 0$ (for even $p$ it holds for the Cauchy data satisfying (3.22)). Such a theory describes a dust of particles moving with the speed of light in $\mathcal{M}^{2p+2}$ – “$p$-photons”. It is easy to show that both the continuity equation

$$\partial_\mu (\mathcal{H}_p^{(UBI)} U^\mu) = 0 ,$$

(4.2)

and the Euler equation

$$U^\nu \partial_\nu U^\mu = 0$$

(4.3)

are satisfied. Moreover, one easily proves that due to (4.2) and (4.3) the following infinite set of continuity equations hold:

$$\partial_\mu \left( \mathcal{H}_p^{(UBI)} U^\mu U^{i_1} U^{i_2} ... U^{i_k} \right) = 0 .$$

(4.4)

They give rise to the following hierarchy of conserved quantities:

$$K^{i_1...i_k} = \int d^{2p+2}x \mathcal{H}_p^{(UBI)} U^{i_1} U^{i_2} ... U^{i_k} .$$

(4.5)

All these quantities are in involution, i.e.

$$\{K^{i_1...i_k}, K^{j_1...j_l}\}_p = 0 .$$

(4.6)

The only exception is $p = 0$. In this case $|U^1| = 1$ and all $K^{1...1}$ are equal (up to a sign). But now the conformal group is infinite dimensional and one has still an infinite number of constants of motion.

For other relation between Born-Infeld theory and fluid dynamics see e.g. [9].

5 Self-dual field

Note, that the Maxwell eqs. for even $p$ have the following form:

$$\partial_0 B^I = \frac{1}{p!} \epsilon^{IkJ} \partial_k D_J ,$$

(5.1)

$$\partial_0 D^I = \frac{1}{p!} \epsilon^{IkJ} \partial_k B_J ,$$

(5.2)

and differ from the field eqs. of UBI (3.18)-(3.19) by the presence of the scaling parameter $|D|/|B|$. Now, let us introduce chiral and anti-chiral combinations:

$$V^I_\pm = \frac{1}{\sqrt{2}} (D^I \pm B^I) .$$

(5.3)

The Maxwell Hamiltonian rewritten in terms of $V_\pm$ has the following form

$$\mathcal{H}_p^{Maxwell} = \frac{1}{2p!} (V_+ \cdot V_+ + V_- \cdot V_-) ,$$

(5.4)
and the Maxwell eqs. read:
\[ V_I^\pm = \pm \frac{1}{p!} \epsilon^{lkj} \partial_k V_{J^\pm} , \] (5.5)
i.e. both components decouple and evolve independently. This property holds for the Maxwell theory only. However, if the initial condition is such that \( V_- = 0 \), i.e. \( D = B \) or \( V_+ = 0 \), i.e. \( D = -B \), then for any \( t \), \( V_-(t) = 0 \) or \( V_+(t) = 0 \) also for the UBI theory defined by (3.18)-(3.19).
This property holds for any \( p \)-form theory defined by a Hamiltonian satisfying (2.32) and invariant under \( Z_2 \times Z_2 \) given by (2.34)-(2.35). Therefore, for chiral (anti-chiral) data the dynamics always reduces to
\[ B_I^\pm = \pm \frac{1}{p!} \epsilon^{lkj} \partial_k B_{J^\pm} , \] (5.6)
i.e. it corresponds to (anti)self-dual field \([10]\). For (anti)self-dual data both Maxwell and UBI Hamiltonians read:
\[ H_{self}^p = \frac{1}{p!} B \cdot B , \] (5.7)
and (5.6) defines the Hamiltonian system with respect to the following canonical structures:
\[ \{ B_I(x), B_J(y) \} = \pm \epsilon^{lkj} \partial_k \delta^{(2p+1)}(x - y) . \] (5.8)

6 Conclusions

Note, that \( p \)-form UBI is uniquely defined by the symmetry group. Namely, the UBI Hamiltonian is a maximally symmetric solution of (2.23) or (2.32) for odd and even \( p \) respectively. Consider e.g. (2.23). Its \( SO(2,1) \)-symmetric solution is a function of one variable \( f = f(\tau) \) with
\[ \tau = \alpha^2 - \beta^2 - \gamma^2 . \]
One immediately shows that \( f = \sqrt{\tau} \) in agreement with (3.12).

For odd \( p \) the \( p \)-form theory is duality invariant (i.e. invariant under \( SO(2) \) rotations (2.24)) iff \([11]\)
\[ (\partial_\eta L_p)^2 - (\partial_\xi L_p)^2 = -1 , \] (6.1)
where
\[ \eta = \sqrt{S^2_p + P^2_p} , \]
\[ \xi = S_p . \]
Eq. (6.1) has the same form as (2.32). Now, the maximally symmetric solution to (6.1) reads \( L_p = |P_p| \), i.e. the corresponding theory is trivial. Note, that performing \( b \to 0 \) limit in the BI Lagrangian (2.43) we have also obtained \(|P_p|\).

The presence of an infinite hierarchy of constants of motion often implies complete integrability of the theory. This question for \( p = 1 \) UBI was posed in [5]. The answer is not known. It would be also interesting to find the corresponding quantum version of this theory.

Note added: While this paper was being completed I received unpublished notes [12], in which similar results were obtained. Moreover, in [12] the particles fluid of section 4 was generalized to \( p \)-brane fluids.

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References

[1] D. Olive, Nucl. Phys. (Proc. Suppl.) B 58 (1997) 43.
[2] M. Born and L. Infeld, Proc. Roy. Soc. A 144 (1934) 425.
[3] C.G. Callan and J.M. Maldacena, Nucl. Phys. B 513 (1998) 198; G.W. Gibbons, Nucl. Phys. B 514 (1998) 603.
[4] C. Teitelboim, Phys. Lett. B 167 (1986) 63, 69; M. Henneaux and C. Teitelboim, Found. Phys. 16 (1986) 593.
[5] I. Białyńnicki-Birula, *Nonlinear Electrodynamics: Variations on a Theme of Born and Infeld*, in *Quantum Theory of Fields and Particles*, Eds. B. Jancewicz and J. Lukierski, World Scientific, Singapore, 1983; Acta Phys. Pol. B 23 (1992) 553.
[6] S. Deser, A. Gomberoff, M. Henneaux and C. Teitelboim, Phys. Lett. B 400 (1997) 80.
[7] J.H. Schwarz and A. Sen, Nucl. Phys. B 411 (1994) 35.
[8] D. Chruściński, Rep. Math. Phys. 45 (2000) 121.
[9] R. Jackiw and A.P. Polychronakos, Comm. Math. Phys. 207 (1999) 107.
[10] M. Henneaux and C. Teitelboim, Phys. Lett. B 206 (1988) 650; I. Bengtsson and A. Kleppe, Int. J. Mod. Phys. A 12 (1997) 3397.
[11] G.W. Gibbons and D.A. Rasheed, Nucl. Phys. B 454 (1995) 18.
[12] G.W. Gibbons, (unpublished).