ON THE COHOMOLOGY OF ACTIONS OF GROUPS
BY BERNOULLI SHIFTS

by

SORIN POPA* and ROMAN SASYK

ABSTRACT. We prove that if $G$ is a countable, discrete group having infinite, normal subgroups with the relative property (T), then the Bernoulli shift action of $G$ on

$\prod_{g \in G} (X_0, \mu_0)_g$, for $(X_0, \mu_0)$ an arbitrary probability space, has first cohomology group isomorphic to the character group of $G$.

1. Introduction

Let $G$ be a countable discrete group and $\sigma$ a measure-preserving, free, ergodic action of $G$ on a probability space $(X, \mu)$. $\sigma$ induces an action (also denoted by $\sigma$) of $G$ on the abelian von Neumann algebra $A = L^\infty(X, \mu)$ by $\sigma_g(f) := f \circ \sigma_{g^{-1}}$. A common example of such actions are the Bernoulli shifts $\sigma$, defined by taking an arbitrary probability space $(X_0, \mu_0)$, then defining $(X, \mu) = \prod_{g \in G} (X_0, \mu_0)_g$, where $(X_0, \mu_0)_g$ are identical copies of $(X_0, \mu_0)$, and then letting $\sigma_g$ act on $(X, \mu)$ by $\sigma_g((x_h)_h) = (x_{g^{-1} h})_h$.

A 1-cocycle for a free, ergodic measure-preserving, action $\sigma$ of $G$ on a probability space $(X, \mu)$ is a map $w : G \to U(A)$ satisfying the relations $w_{gh} = w_g \sigma_g(w_h), \forall g, h \in G_0$ and $w_e = 1$, where $U(A)$ is the group of \mathbb{T} valued functions in $A = L^\infty(X, \mu)$. For example, any character $\gamma$ of $G$ gives a 1-cocycle for $\sigma$ by $w_g = \gamma(g)1, g \in G$. A 1-cocycle $w$ is co-boundary if there exists $v \in U(A)$ such that $w_g = v \sigma_g(v^*), \forall g$. Denote by $Z^1(\sigma)$ the set of all 1-cocycles and by $B^1(\sigma)$ the set of co-boundaries. $Z^1(\sigma)$ is clearly a commutative group under multiplication, with $B^1(\sigma)$ a subgroup. The corresponding quotient group $H^1(\sigma) = Z^1(\sigma)/B^1(\sigma)$ is called the first cohomology group of $\sigma$, and is clearly a conjugacy invariant for $\sigma$.

In the early 80’s Klaus Schmidt proved that the group $G$ has the property (T) of Kazhdan ([K]) if and only if $H^1(\sigma)$ is countable for any free, ergodic, measure preserving action $\sigma$ of $G$ ([S2]). He also showed that $G$ is amenable iff the Bernoulli shift actions of $G$ are non-strongly ergodic, and iff all measure-preserving actions of the group $G$ are non-strongly ergodic. Related to these results, Connes and Weiss proved that $G$ has the property (T) iff all its ergodic, free measure-preserving actions are strongly ergodic.

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In this paper we obtain the first actual computations of cohomology groups $H^1(\sigma)$, in the case $\sigma$ is a Bernoulli shift action and the group $G$ is weakly rigid in the following sense: $G$ contains infinite, normal subgroups $H \subset G$ such that $(G, H)$ has the relative property (T) of Kazhdan-Margulis([M], [dHV]), i.e., any representation of $G$ that weakly contains the trivial representation of $G$ must contain the trivial representation of $H$. Note that any group $G$ of the form $G = H \times \Gamma$ with $H$ an infinite group with the property (T) of Kazhdan is weakly rigid.

**Theorem.** If $G$ is a countable, weakly rigid discrete group and $\sigma$ is a Bernoulli shift action of $G$ then $H^1(\sigma)$ is equal to the character group of $G$.

**Corollary.** If $\Gamma$ is an arbitrary countable discrete abelian group, $G = SL(n, \mathbb{Z}) \times \Gamma$, for some $n \geq 3$, and $\sigma$ is a Bernoulli shift action of $G$ then $H^1(\sigma) = \hat{\Gamma}$.

We mention that the similar result for (purely) non-commutative Bernoulli shifts was obtained in ([Po]). In fact, to prove the above Theorem we will follow the line of arguments in ([Po]), with the commutativity allowing many simplifications.

2. Preliminaries

Let $G$ be a discrete group and $\sigma$ a measure preserving action of $G$ on a standard probability measure space $(X, \mu)$. The action it implements on the abelian von Neumann algebra $A = L^\infty(X, \mu)$, still denoted by $\sigma$, preserves the integral and thus extends to an action (or unitary representation) $\sigma$ of $G$ on the Hilbert space $L^2(X, \mu)$. We denote by $U(A)$ the group of unitary elements in $A$. Besides the notion of 1-cocycles for $\sigma$ defined in the introduction we need the following:

**2.1. Definition.** A weak 1-cocycle for the action $\sigma$ is a function $w : G \to U(A)$ satisfying $w_{gh} = w_g \sigma_g(w_h) \mod \mathbb{C}, \forall g, h \in G$, and $w_e = 1$. A weak cocycle $w$ is a weak coboundary if there exist a unitary $u$ in $A$ such that $w_g = u \sigma_g(u^*) \mod \mathbb{C}, \forall g \in G$. Note that if $w$ is a weak 1-cocycle for $\sigma$ and $v \in U(A)$ then $w'_g = vw_g \sigma_g(v^*), g \in G$ is also a weak 1-cocycle for $\sigma$. Two weak 1-cocycles $w, w'$ for which there exists $v$ as above are called equivalent.

**2.2. Remarks.** 1°. Let $w$ be a weak 1-cocycle for $\sigma$ and denote by $\gamma(g, h) \in \mathbb{T}$ the scalar satisfying $w_{gh} = \gamma(g, h)w_g \sigma_g(w_h), \forall g, h$. Condition $w_e = 1$ then implies $\gamma(e, g) = \gamma(g, e) = 1, \forall g \in G$. Also, the associativity relation $w_g(w_hw_k) = (w_gw_h)w_k$ entails

$$\gamma(g, h)\gamma(gh, k) = \gamma(g, hk)\gamma(h, k), \forall g, h, k.$$  

A function $\gamma : G \times G \to \mathbb{T}$ that verifies the previous conditions is called a scalar valued (or $\mathbb{T}$-valued) 2-cocycle for the group $G$. Thus any weak 1-cocycle $w$ for the action $\sigma$ has associated a scalar 2-cocycle $\gamma = \gamma_w$. 
2°. If the weak 1-cocycle $w$ is a weak coboundary and $w_g = \lambda_g \nu \sigma_g(v^*)$ with $\lambda_g \in \mathbb{C}$ then $\lambda_{gh} = \gamma(g,h)\lambda_g\lambda_h$. In particular if $w$ is a genuine 1-cocycle then $\lambda$ follows a character of $G$.

2.3. Lemma. Let $w$ be a weak 1-cocycle for the action $\sigma$, with scalar 2-cocycle $\gamma$.

1°. For $g \in G$ and $\xi \in L^2(X,\mu)$ denote $\sigma_g^w(\xi) := w_g^* \sigma_g(\xi)$. Then $\sigma^w$ is a projective representation of $G$ on $L^2(X,\mu)$ with scalar 2-cocycle $\gamma$. 

2°. Let $H\bar{S}$ the space of Hilbert-Schmidt operators on $L^2(X,\mu)$ and for each $T$ in $H\bar{S}$ denote $\tilde{\sigma}^w_g(T) := \sigma^w_g T \sigma^w_g$. Then $\tilde{\sigma}^w$ is a unitary representation of $G$ on $H\bar{S}$.

3°. If we identify an element $T \in H\bar{S}$ with an element $a$ of $L^2(X,\mu) \otimes L^2(X,\mu) \simeq L^2(X \times X, \mu \times \mu)$ in the usual way, then $\tilde{\sigma}^w_g(T) = 1 \otimes w_g \cdot (\sigma_g \otimes \sigma_g)(a) \cdot w_g^* \otimes 1$.

Proof. 1°. We have:

$$\sigma_g^w \sigma_h^w(\xi) = \sigma_g^w(\sigma_h(\xi)) w_h^* = \sigma_g(\sigma_h(\xi)) \sigma_g^w(\xi) w_h^* = \sigma_{gh}(\xi) \gamma(g,h) w_{gh}^* = \gamma(g,h) \sigma_{gh}(\xi),$$

showing that $\sigma^w$ is a projective representation.

2°. If $T \in H\bar{S}$ then

$$\tilde{\sigma}_g^w \tilde{\sigma}_h^w(T) = \sigma_g^w T \sigma_h^w = \gamma(g,h) \sigma_{gh}^w(\xi) \gamma(\xi) \sigma_{gh}^w = \tilde{\sigma}_{gh}^w(T).$$

3°. Since the space $H\bar{S}$ of Hilbert-Schmidt operators on $L^2(X,\mu)$ is isomorphic to $L^2(X,\mu) \otimes L^2(X,\mu)$ via the identification $x \otimes y^*(\xi) = (\xi|x) y$, we have:

$$\tilde{\sigma}_g^w(x \otimes y^*)(\xi) = (\sigma_g^w(\xi)|x) \sigma_g^w(y) = (\xi|\sigma_g^w(x)) \sigma_g^w(y) w_g^* = (\sigma_g^w(x) w_g^*) \otimes (\sigma_g^w(y))^*(\xi) = (1 \otimes w_g)(\sigma_g \otimes \sigma_g)(y^*)(w_g^* \otimes 1)(\xi).$$

Then extend by linearity. \qed

2.4. Lemma. With the notations of Lemma 2.3, the following are equivalent:

1. $\tilde{\sigma}^w$ contains a copy of the trivial representation.

2. $\sigma^w$ has a non-trivial, invariant finite dimensional subspace $\mathcal{H}_0 \subset L^2(X,\mu)$.

3. There exist $a \neq 0$ in $L^2(X) \otimes L^2(X)$ such that $(1 \otimes w_g)(\sigma_g \otimes \sigma_g)(a) \cdot (w_g^* \otimes 1) = a$.

Proof. (1) $\Rightarrow$ (2). First note that the action $\tilde{\sigma}^w$ can be extended to all $\mathcal{B}(L^2(X,\mu))$. Also note that if $T \in H\bar{S}$ is fixed by $\tilde{\sigma}^w$, so is $T^*$. Thus the trace class operator $TT^* \in \mathcal{B}(L^2(X,\mu))$ is fixed by $\tilde{\sigma}^w$. By the Borel functional calculus, all the spectral projections of $TT^*$ are fixed by $\tilde{\sigma}^w$. As they have finite trace, it follows that they are projections on finite dimensional subspaces. Choose $\mathcal{H}_0$ a non-trivial finite dimensional subspace of $L^2(X,\mu)$ corresponding to some spectral projection $P$ of $TT^*$, and note that $\tilde{\sigma}_g^w(P)$ is the projection of $L^2(X,\mu)$ onto $\sigma_g^w(\mathcal{H}_0)$. Then as $\tilde{\sigma}_g^w(P) = P$, it follows that $\sigma_g^w(\mathcal{H}_0) = \mathcal{H}_0$. Thus $\mathcal{H}_0$ is an invariant subspace for $\sigma^w$.
(2) ⇒ (1). If \( H_0 \) is an invariant subspace for \( \sigma^w \), take \( P \) the finite rank projection of \( L^2(X, \mu) \) onto \( H_0 \). Then \( P \) is invariant for the action \( \tilde{\sigma}^w \).

(2) ⇔ (3). Trivial by previous lemma.

Recall that the measure preserving action \( \sigma \) of \( G \) on the probability space \((X, \mu)\) is weakly mixing if and only if the only finite dimensional subspace of \( L^2(X, \mu) \) invariant to \( \sigma \) is \( \mathbb{C} \) (see e.g. [BMe]).

### 2.5. Lemma

Assume \( \sigma \) is weakly mixing and let \( w \) be a weak 1-cocycle for \( \sigma \). Then \( \tilde{\sigma}^w \) contains a copy of the trivial representation if and only if \( w \) is a weak coboundary. Moreover, if this is the case, then the unitary element \( u \in L^\infty(X, \mu) \) with \( w_g = u \sigma_g(u^*) \) mod \( \mathbb{C} \), \( \forall g \in G \), is unique up to a scalar multiple.

**Proof.** If \( w_g = \lambda_g u^* \sigma_g(u), \forall g \in G \), then \( \sigma_g^w(\mathbb{C}u) = \mathbb{C} \sigma_g(u) w_g^* = \mathbb{C}u \), thus \( H_0 = \mathbb{C}u \) is \( \sigma^w \)-invariant. Thus, by Lemma 2.4, the orthogonal projection onto \( H_0 \) is a fixed point for \( \tilde{\sigma}^w \).

Conversely, let \( H_0 \subset L^2(X, \mu) \) be a \( \sigma^w \)-invariant finite dimensional subspace. Choose an orthonormal basis \( \{ \xi_1, \ldots, \xi_n \} \) of \( H_0 \) and note that \( \{ \eta_i \} = \{ \sigma_g(\xi_i) w_g^* \} \) is also an orthonormal basis of \( H_0 \). But an easy computation shows that \( \Sigma_i \xi_i \xi_i^* = \Sigma_i \eta_i \eta_i^* \in L^1(X, \mu) \) for any two orthonormal basis of \( H_0 \). Thus, since

\[
\sigma_g(\Sigma_i \xi_i \xi_i^*) = \Sigma_i (\sigma_g(\xi_i) w_g^*) (w_g \sigma_g(\xi_i^*)) = \Sigma_i \eta_i \eta_i^* = \Sigma_i \xi_i \xi_i^*
\]

and since \( \sigma \) is ergodic on \((X, \mu)\), it follows that \( \Sigma_i \xi_i \xi_i^* \in \mathbb{C} \). In particular, all \( \xi_i \) are bounded elements, \( \xi_i \in L^\infty(X, \mu) \).

But since

\[
\sigma_g(\xi_i \xi_j^*) = \sigma_g(\xi_i) w_g^* w_g \sigma_g(\xi_j^*) = \sigma_g^w(\xi_i) (\sigma_g^w(\xi_j))^*,
\]

the finite dimensional subspace \( H_0 \cdot H_0^* \) of \( L^\infty(X, \mu) \) spanned by \( \{ \xi_i \xi_j^* \}_{i,j=1}^n \) is \( \sigma \)-invariant. Since \( \sigma \) is weakly mixing, it follows that \( H_0^\sigma H_0^* = \mathbb{C} \). Thus \( \xi_i \xi_j^* \in \mathbb{C}1, \forall i, j \), which implies that \( n = 1 \) and \( \xi_1 \) is a scalar multiple of a unitary element.

Finally, if \( w_g = u^* \sigma_g(u) \mod \mathbb{C} \) and \( w_g = u^* \sigma_g(u^*) \mod \mathbb{C} \), then \( u^* u^* = \sigma_g(u^* u^*) \mod \mathbb{C} \), i.e. the subspace \( \mathbb{C} u^* u^* \) is invariant to \( \sigma \), implying that \( u^* = u \mod \mathbb{T} \). □

### 3. The main result

Let \((X_0, \mu_0)\) be a nontrivial probability space and \( G \) an infinite discrete group. Denote \((X, \mu) := \prod_{g \in G}(X_0, \mu_0)\) and let \( \sigma \) be the action of \( G \) on \((X, \mu)\) by \( G \)-Bernoulli shifts, i.e., \( \sigma_g((x_h)_h) = (x_{g^{-1}h})_h \). This action is well known to be mixing. Note that if \( H \subset G \) is a subgroup of \( G \) then \( \sigma_H \) is a \( H \)-Bernoulli shift. Also, note that the (diagonal) product of two \( G \)-Bernoulli shifts is a \( G \)-Bernoulli shift.
Recall from ([dHV]) that an inclusion of discrete groups $H \subset G$ has the relative property $(T)$ if the following condition holds true:

3.0. There exist a finite set of elements $g_1, g_2, \ldots, g_n$ in $G$ and $\epsilon > 0$ such that for any unitary representation $\pi$ of $G$ on a Hilbert space $\mathcal{H}$ which has a unit vector $\xi$ with $\|\pi(g_i)\xi - \xi\|_\mathcal{H} < \epsilon$ for all $1 \leq i \leq n$, there exists a unit vector fixed by $\pi|_\mathcal{H}$.

By a result of Jolissaint ([Jo]), the above condition is equivalent to the following:

3.0'. Given any $\epsilon > 0$ there exist a finite set of elements $g_1, g_2, \ldots, g_n$ in $G$ and $\delta > 0$ such that for any unitary representation $\pi$ of $G$ on a Hilbert space $\mathcal{H}$ that has a unit vector $\xi$ such that $\|\pi(g_i)\xi - \xi\|_\mathcal{H} < \delta$ for all $1 \leq i \leq n$ then $\|\pi(g)\xi - \xi\|_\mathcal{H} < \epsilon$ for all $g \in H$.

3.1. Theorem. Let $G$ be a countable discrete group and $H \subset G$ a subgroup with the relative property $(T)$. Given any weak 1-cocycle $w$ for a $G$-Bernoulli shift $\sigma$, $w|_H$ is a weak coboundary.

Proof. We first prove the case when $(X_0, \mu_0)$ is non atomic, thus isomorphic to $(\mathbb{T}, \lambda)$, the torus with its Haar measure.

Denote by $A$ the abelian von Neumann Algebra $L^\infty(X, \mu)$. By Lemma 2.5, it is sufficient to prove that there exists $u \in \mathcal{U}(A \otimes A)$ such that $\tilde{\sigma}_h^w(u) = u, \forall h \in H$. We’ll prove this in the Lemmas 3.2-3.5 below.

3.2. Lemma. There exists a continuous action $\alpha$ of $\mathbb{R}$ on $A \otimes A \simeq L^\infty(X \times X, \mu \times \mu)$, by automorphisms preserving the integral over $\nu \times \nu$, such that:

(3.2.1). $\alpha$ commutes with the Bernoulli shift $\tilde{\sigma} = \sigma \otimes \sigma$.

(3.2.1). $\alpha_1(A \otimes \mathbb{C}) = \mathbb{C} \otimes A$.

Proof. Denote $A_0 = L^\infty(\mathbb{T}, \lambda)$, $\tilde{A}_0 = A_0 \otimes A_0$ and $\tau_0$ the functional on $\tilde{A}_0$ given by the integral over $\lambda \times \lambda$. We first construct a continuous action $\beta : \mathbb{R} \rightarrow \text{Aut}(\tilde{A}_0, \tau_0)$ such that $\beta_1(A_0 \otimes \mathbb{C}) = \mathbb{C} \otimes A_0$.

Let $u$ (resp. $v$) be a Haar unitary generating $A_0 \otimes \mathbb{C} \simeq L^\infty(\mathbb{T}, \lambda)$ (resp. $\mathbb{C} \otimes A_0$). Thus, $u, v$ is a pair of generating Haar unitaries for $A_0$, i.e., $\{u^n v^m\}_{n, m \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\tilde{A}_0, \tau_0) \simeq L^2(\mathbb{T}, \lambda) \otimes L^2(\mathbb{T}, \lambda)$. We need to construct the action $\beta$ so that $\beta_1(u) = v$.

Note that given any other pair of generating Haar unitaries $u', v'$ for $\tilde{A}_0$, the map $u \mapsto u, v \mapsto v'$ extends to a $\tau_0$-preserving automorphism of $\tilde{A}_0$. Also, note that $v, uv$ is a pair of generating Haar unitaries for $A_0$. Thus, in order to get $\beta$, it is sufficient to find a continuous action $\beta' : \mathbb{R} \rightarrow \text{Aut}(\tilde{A}_0, \tau_0)$ such that $\beta_1'(v) = uv$. 

Let $h \in \hat{A}_0$ be a self-adjoint element such that $\exp(2\pi i h) = u$. It is easy to see that for each $t$, $u$ and $\exp(2\pi i t h)v$ is a pair of Haar unitaries. Denote by $\beta'_t$ the automorphism $u \mapsto u, v \mapsto \exp(2\pi i t h)v$. We then clearly have $\beta'_t \beta'_s = \beta'_{t+s}$, $\forall t, s \in \mathbb{R}$ and $\beta'_1(v) = uv$.

Finally, we take $\alpha$ to be the product action $\alpha_t = \bigotimes_{g \in G}(\beta'_t)_g, t \in \mathbb{R}$. Since $\alpha$ acts identically on the components of the product of the $G$-shifts, it commutes with $\sigma$. Also, $\alpha_1$ flips $A \otimes \mathbb{C}$ onto $\mathbb{C} \otimes A$ because each $(\beta'_1)_g$ takes $(A_0)_g \otimes \mathbb{C}$ onto $\mathbb{C} \otimes (A_0)_g$. □

For the next lemma, note that if $K$ is a convex subset of the von Neumann algebra $A\overline{\otimes}A = L^\infty(X \times X, \mu \times \mu)$ which is bounded in the norm $\| \cdot \| = \| \cdot \|_\infty$, then its closure $K$ in the $w$-operator topology on $A\overline{\otimes}A$ coincides with its closure in the norm $\| \cdot \|_2$ on $L^2(X \times X, \mu \times \mu)$ (with $A\overline{\otimes}A \supset K$ regarded as a subset of this Hilbert space).

3.3. Lemma. For each $t \in \mathbb{R}$ let $x_t$ be the (unique) element of minimal norm-2 in $K_t := \overline{\sigma(t)\|w_h \otimes 1\|_2\{ (w_h \otimes 1)\alpha_t(w_h^* \otimes 1) \}}_{h \in H}$. Then $x_t \in A\overline{\otimes}A$ and it satisfies the following conditions:

1°. $(w_h \otimes 1)\tilde{\sigma}(h)x_t = x_t\alpha_t(w_h \otimes 1), \forall h \in H$.

2°. $x_t x_t^* \in \mathbb{C} \otimes \mathbb{C}$.

Proof. 1°. Since $w_h \sigma(h) w_k = w_{hk}$, mod $\mathbb{C}$, and the actions $\tilde{\sigma}, \alpha$ commute, it follows that for all $h, k \in G$ we have

$$(w_k \otimes 1)\tilde{\sigma}(h)(w_h \otimes 1)\alpha_t(w_h^* \otimes 1)\alpha_t(w_k^* \otimes 1) = (w_{kh} \otimes 1)\alpha_t(w_{kh}^* \otimes 1)$$

showing that for each fixed $k \in H$ the unitary operator on $L^2(X \times X, \mu \times \mu) = L^2(A\overline{\otimes}A)$ given by $x \mapsto (w_k \otimes 1)\tilde{\sigma}(k)x\alpha_t(w_k^* \otimes 1)$ takes $K_t$ into itself. Thus, by the uniqueness of the element of minimal norm $\| \cdot \|_2$ in $K_t$, it follows that $x_t = (w_k \otimes 1)\tilde{\sigma}(k)(x_t\alpha_t(w_k^* \otimes 1), \forall k \in H$.

2°. From the proof of 1° and the commutativity of $A\overline{\otimes}A$ it follows that for $k \in H$ we have

$$\tilde{\sigma}(k)(x_t x_t^*) = (w_k \otimes 1)\tilde{\sigma}(k)(x_t x_t^*)(w_k^* \otimes 1) = x_t x_t^*.$$ 

But since $\sigma|_H$ is weakly mixing, $\tilde{\sigma}|_H$ is ergodic and thus $x_t x_t^*$ follows a scalar. □

3.4. Lemma. Assume $(G, H)$ has the relative property $(T)$. If $x_t$ are defined as in Lemma 3.3, then there exists $t_0 > 0$ such that $x_t \neq 0$ and $u_t = x_t/\|x_t\|$ is a unitary element in $A\overline{\otimes}A$ for all $t \in [0, t_0]$.

Proof. Let $\epsilon > 0$. Let $g_1, \ldots, g_n \in G$ and $\delta > 0$ be given by condition (3.0'). By the continuity of the action $\alpha_t$, there exists $t_0 > 0$ such that if $0 < t \leq t_0$ then

$$\| (w_{g_t} \otimes 1)\alpha_t(w_{g_t}^* \otimes 1) - 1 \|_2 < \delta, \forall i.$$ 

Fix $t \in (0, t_0]$. Since the action $\tilde{\sigma}$ commutes with the automorphism $\alpha_{t}$, it follows that $\tilde{\sigma}_g \times \alpha^n_t$ implements an action of $G \times \mathbb{Z}$ on $A\overline{\otimes}A$ which preserves the functional $\tau$ given by the integral over $\mu \times \mu$. 
3.5. Lemma. There exists $u \in \mathcal{U}(\mathcal{A} \rtimes \mathcal{G})$ such that

$$\tilde{\sigma}^w_h(u) = u, \forall h \in H.$$

Proof. Choose $n \in \mathbb{N}$ such that $1/n < t_0$, where $t_0$ is by 3.4. With $u_t$ defined as in Lemma 3.4, we let $u = u_{1/n}a_{1/n}(u_{1/n}) \cdots a_{1/n}^{n-1}(u_{1/n})$. By 3.3.1° we have $(w_h \otimes 1)\tilde{\sigma}_h(u_{1/n}) = u_{1/n}a_{1/n}(w_h \otimes 1)$, which by applying on both sides $(a_{1/n})^k = a_{k/n}$, $k = 1, 2, ..., n - 1$, gives

$$a_{k/n}(w_h \otimes 1)\tilde{\sigma}_h(a_{k/n}(u_{1/n})) = a_{k/n}(u_{1/n})a_{(k+1)/n}(w_h \otimes 1).$$

By applying this repeatedly to $u$, we get

$$(w_h \otimes 1)\tilde{\sigma}_h(u) = u\alpha_1(w_h \otimes 1) = u(1 \otimes w_h), \forall h \in H,$$

or equivalently $\tilde{\sigma}^w_h(u) = u, \forall h \in H.$

This ends the proof of the nonatomic case. For the atomic case we need the following:

Lemma 3.6. Suppose $(X_0, \mu_0)$ is an atomic probability space. There exists an embedding of $L^\infty(X_0, \mu_0)$ into $L^\infty(\mathbb{T}, \lambda)$ with a sequence of diffuse von Neumann subalgebras $(B_n)_{n \in \mathbb{N}}$ of $L^\infty(\mathbb{T}, \lambda)$ such that $B_{n+1} \subseteq B_n$ and $L^\infty(X_0, \mu_0) = \bigcap_{n \in \mathbb{N}} B_n$. 
Proof. Identify $L^\infty(\mathbb{T}, \lambda)$ with $\bigotimes_{n \geq 0} L^\infty(X_n, \mu_n)$, where $(X_n, \mu_n) = (X_0, \mu_0), \forall n \geq 0$. Also, identify the initial algebra $L^\infty(X_0, \mu_0)$ with $L^\infty(X_0, \mu_0) \bigotimes_1^\infty 1 \subset L^\infty(\mathbb{T}, \lambda)$ and put

$$B_n = L^\infty(X_0, \mu_0)(\mathbb{C} \bigotimes_1^n) \bigotimes_{j=n+1}^\infty L^\infty(X_j, \mu_j).$$

Then $B_n$ are clearly diffuse and $\cap_n B_n = L^\infty(X_0, \mu_0)$.

With $L^\infty(X_0, \mu_0) \subset B_n \subset L^\infty(\mathbb{T}, \lambda)$ as in Lemma 3.6, denote $A = \bigotimes_{g \in G} L^\infty(\mathbb{T}, \lambda)_g$ with its subalgebras $A_0 = \bigotimes_{g \in G} L^\infty(X_0, \mu_0)_g$ and $A_n = \bigotimes_{g \in G}(B_n)_g, n \geq 1$.

The $G$-Bernoulli shift $\sigma$ on $A_0$ extends to $G$-Bernoulli shifts on $A$ and $A_n$, $n \geq 1$, still denoted $\sigma$. If $w : G \to \mathcal{U}(A_0)$ is a weak 1-cocycle for $\sigma$ as a $G$-Bernoulli shift action on $A_0$, then $w$ can also be regarded as a weak 1-cocycle for the $G$-Bernoulli shift action on $A_n, n \geq 1$. The non atomic case of Theorem 3.1 implies that $w|_H$ is a weak coboundary for $\sigma|_H$ as an action on $A_n$. Thus, for each $n \geq 1$ there exists a unitary element $u_n \in A_n$ such that $w_h = u_n \sigma_h(u_n^*) \mod \mathbb{C}$. By Lemma 2.5, $u_n$ is unique up to a scalar multiple. Since $A_{n+1} \subset A_n$ and $\bigcap_{n \in \mathbb{N}} A_n = A_0$, it follows that $\mathbb{C}u_n = \mathbb{C}u_{n+1}$ and finally $\mathbb{C}u_n \in A_0$ for all $n \geq 1$. Thus $w|_H$ is a weak 1-cocycle for the action $\sigma|_H$ on $A_0$. □

4. Applications

As in the introduction, a group $G$ is called weakly rigid if it contains infinite, normal subgroups $H \subset G$ such that the pair $(G, H)$ has the relative property $(T)$.

4.1. Theorem. If $G$ is a weakly rigid group then any weak 1-cocycle for a $G$-Bernoulli shift is a weak coboundary.

Proof. By hypothesis, there exists an infinite normal subgroup $H \subset G$ such that $(G, H)$ has the relative property $(T)$. If $w$ is a weak 1-cocycle for the $G$-Bernoulli shift $\sigma$, then by Theorem 3.1 there exists $v \in \mathcal{U}(A)$ such that $w_h = v \sigma_h(v^*), \mod \mathbb{C}, \forall h \in H$.

Let $w'_h = v^* w_g \sigma_g(v)$. Then $w'$ is a weak 1-cocycle for $\sigma$ and it satisfies $w_h \in T1, \forall h \in H$.

For $a \in A$, denote by $L_a \in \mathcal{B}(L^2(X, \mu))$ the (left) multiplication operator given by $L_a(\xi) = a \xi, \forall \xi \in L^2(X, \mu)$. Then we have

$$L_{w'_g} \sigma_g L_{w'_h} \sigma_h(\xi) = w'_g \sigma_g(w'_h) \sigma_g \sigma_h(\xi) = w'_g \sigma_g \sigma_h(\xi) \mod \mathbb{T}.$$ 

Thus

$$(L_{w'_g} \sigma_g)(L_{w'_h} \sigma_h) = L_{w'_g} \sigma_h \mod \mathbb{T}.$$ 

Similarly

$$(L_{w'_g} \sigma_g)^* = L_{w'_{g^{-1}}} \sigma_{g^{-1}} \mod \mathbb{T}.$$
This implies
\[(Lw'_g\sigma_g)(Lw'_h\sigma_h)(Lw'_g\sigma_g)^* = w'_{ghg^{-1}}\sigma_{ghg^{-1}} \pmod{T},\]
for all \(g, h \in G\). Since \(w'_h\) are scalars for \(h \in H\) and \(ghg^{-1} \in H\), \(\forall g\), this further implies
\[Lw'_g\sigma_{ghg^{-1}}Lw'_g = (Lw'_g\sigma_g)\sigma_h(Lw'_g\sigma_g)^* = \sigma_{ghg^{-1}} \pmod{T}.

Substituting \(h\) for \(ghg^{-1}\) and applying the first and last term of these equalities to the element \(\xi = w'_g \in L^2(X, \mu)\), it follows that \(\sigma_h(w'_g) \in \mathbb{C}w'_g, \forall h \in H, g \in G\). Since the action \(\sigma|_H\) is weakly mixing, it follows that \(Cw'_g = \mathbb{C}1\) for all \(g \in G\). Thus \(w_g = v\sigma_g(v^*)\), \(\mod{T}, \forall g \in G\), i.e., \(w\) is a weak coboundary. \(\square\)

4.2. Corollary. Under the same assumptions as in Theorem 4.1, if \(w\) is a genuine 1-cocycle then \(w\) is equivalent to a character of \(G\) and different characters give non equivalent 1-cocycles. In other words, \(H^1(\sigma) = \text{Char}(G)\).

Proof. Theorem 4.1 shows that there exist \(u \in \mathcal{H}\) such that \(w_g = \lambda_g u\sigma_g(u^*)\). On the other hand, by Remark 2.1, \(\lambda_g\) is a character of \(G\).

Moreover, if two characters \(\lambda_g, \lambda'_g\) are equivalent then there exists a unitary element \(u \in A\) such that \(\lambda_g 1 = \lambda'_g u\sigma_g(u^*), \forall g \in G\). Thus, \(\sigma_g(u) \in C\mathbb{C}, \forall g \in G\). But since \(\sigma\) is weakly mixing, the only finite dimensional \(\sigma\)-invariant subspace of \(A\) is \(C\mathbb{C}\), implying that \(u \in C\mathbb{C}\) and \(\lambda_g = \lambda'_g\). \(\square\)

4.3. Corollary. The first cohomology group \(H^1(\sigma)\) of a Bernoulli shift action \(\sigma\) of \(SL(n, \mathbb{Z})\), \(n \geq 3\) is trivial. More generally, if \(\Gamma\) is any abelian group, \(G = SL(n, \mathbb{Z}) \times \Gamma\) and \(\sigma\) is a \(G\)-Bernoulli shift, then \(H^1(\sigma) = \hat{\Gamma}\).

Proof. Indeed for \(n \geq 3\), \(SL(n, \mathbb{Z})\) has the property \(T\) of Kazhdan ([K]), and by the Nielsen Magnum theorem (see for instance [St]), for \(n \geq 3\) the commutator subgroup of \(G = SL(n, \mathbb{Z}) \times \Gamma\) is equal to \(SL(n, \mathbb{Z})\). Thus the group of characters of \(G\) is equal to \(\hat{\Gamma}\). \(\square\)

References

[BMe] B. Bekka, M. Meyer: "Ergodic Theory and Topological dynamics of group actions on Homogeneous Spaces", London Math Soc Lect. Notes 269, Cambridge University Press, 2000.

[CW] A. Connes, B. Weiss: Property (T) and asymptotically invariant sequences, Israel. J. Math. 37 (1980), 209-210.

[dHV] P. de la Harpe, A. Valette: "La propriété T de Kazhdan pour les groupes localement compacts", Astérisque 175, Soc. Math. de France (1989).

[Jo] P. Jolissaint: On the relative property T, preprint 2001.

[K] D. Kazhdan: Connection of the dual space of a group with the structure of its closed subgroups, Funct. Anal. and its Appl. 1 (1967), 63-65.
[M] G. Margulis: *Finitely-additive invariant measures on Euclidian spaces*, Ergodic. Th. and Dynam. Sys. 2 (1982), 383-396.

[MvN] F. Murray, J. von Neumann: *Rings of operators IV*, Ann. Math. 44 (1943), 716-808.

[Po] S. Popa: *Some rigidity results for non-commutative Bernoulli shifts*, MSRI preprint, 2001-005.

[S1] K. Schmidt: *Asymptotically invariant sequences and an action of SL(2, Z) on the 2-sphere*, Israel. J. Math. 37 (1980), 193-208.

[S2] K. Schmidt: *Amenability, Kazhdan’s property T, strong ergodicity and invariant means for ergodic group-actions*, Ergod. Th. & Dynam. Sys. 1 (1981), 223-236.

[St] R. Steinberg: *Some consequences of elementary relations of SL(n)*, Contemporary Math., 45 (1985), 335-350.

Math Dept UCLA, Los Angeles, CA 90095-155505

E-mail address: popa@math.ucla.edu, rsasyk@math.ucla.edu