An Explicit Second Order Scheme for Decoupled Anticipated Forward Backward Stochastic Differential Equations

Yabing Sun¹ and Weidong Zhao²,*

¹College of Science, National University of Defense Technology, Changsha, Hunan 410073, China.
²School of Mathematics & Finance Institute, Shandong University, Jinan, Shandong 250100, China.

Received 27 November 2019; Accepted (in revised version) 20 February 2020.

Abstract. The Feynman-Kac formula and the Lagrange interpolation method are used in the construction of an explicit second order scheme for decoupled anticipated forward backward stochastic differential equations. The stability of the scheme is rigorously proved and error estimates are established. The scheme has the second order accuracy when weak order 2.0 Taylor scheme is employed to solve stochastic differential equations. Numerical tests confirm the theoretical findings.

AMS subject classifications: 60H35, 65C20, 60H10
Key words: Anticipated forward backward stochastic differential equations, explicit scheme, error estimate, second order convergence.

1. Introduction

Let \((Ω, ℱ, ℋ, P)\) be a complete filtered probability space with the filtration \(ℱ = \{ℱ_t\}_{t≥0}\) of the \(m\)-dimensional Brownian motion \(W = (W_t)_{t≥0}\). We consider the decoupled anticipated forward backward stochastic differential equation (AFBSDE) on \((Ω, ℱ, ℋ, P)\)

\[
\begin{align*}
    dX_s &= b(s, X_s)ds + σ(s, X_s)dW_s, & s \in [0, T + K], \\
    dY_s &= f(s, X_s, Y_s, Z_s, X_{s+γ(s)}, Y_{s+δ(s)}, Z_{s+ζ(s)})ds - Z_s dW_s, & s \in [0, T), \\
    Y_t &= Q_t, & t \in [T, T + K], \\
    Z_t &= P_t, & t \in [T, T + K],
\end{align*}
\]

(1.1)

where the function

\[
f : [0, T] \times ℝ^d \times ℝ^p × ℝ^{p×m} × L^2(ℱ; ℝ^d) × L^2(ℱ; ℝ^p) × L^2(ℱ; ℝ^{p×m}) \rightarrow L^2(ℱ; ℝ^p)
\]

*Corresponding author. Email addresses: sunybly@163.com (Y. Sun), wdzhao@sdu.edu.cn (W. Zhao).
Explicit Second-Order Scheme for AFBSDEs

with $r, r', r'' \in [s, T + K]$ is referred to as the generator of the anticipated backward stochastic differential equation (ABSDE), and $Q_t, P_t \in \mathcal{F}_t$ are terminal conditions such that

$$
\mathbb{E} \left[ \int_T^{T+K} |Q_t|^2 + |P_t|^2 dt \right] < +\infty,
$$

$b(t, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma(t, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ the drift and diffusion coefficients, $\eta(\cdot), \delta(\cdot)$ and $\zeta(\cdot)$ continuous $\mathbb{R}^+$ valued functions defined on $[0, T]$ and satisfying the conditions:

(i) There is a constant $K \geq 0$ such that for all $s \in [0, T]$ the inequalities

$$
s + \eta(s) \leq T + K, \quad s + \delta(s) \leq T + K, \quad s + \zeta(s) \leq T + K
$$

hold.

(ii) There is a constant $L_0 \geq 0$ such that for all $t \in [0, T]$ and for any nonnegative integrable function $g$ the inequality

$$
\int_t^T g(s + h(s)) ds \leq L_0 \int_t^{T+K} g(s) ds,
$$

where

$$
h(s) = \eta(s), \quad h(s) = \delta(s) \quad \text{or} \quad h(s) = \zeta(s),
$$

holds.

A triplet $(X_t, Y_t, Z_t)$ is called an $L^2$-adapted solution of the Eqs. (1.1) if it is $\mathcal{F}_t$-adapted, square integrable and satisfies (1.1). It is worth noting that the generator of ABSDEs depends on the conditional expectation of the future path of the solution.

An ABSDE was first introduced as a duality equation for stochastic differential delay equations (SDDEs) by Peng and Yang in [18], where the existence and uniqueness of the solutions of ABSDE are established and a stochastic control problem is solved. In the last decade, AFBSDEs attracted substantial interest because of wide applications in stochastic optimal control problems with delay [4, 10, 14, 15], stochastic differential games with delay [17, 35] and mean-field problems [5, 8]. However, exact solutions of AFBSDEs are rarely available. Therefore, it is important to develop numerical methods for AFBSDEs. We note that there is a vast literature on numerical methods for forward backward stochastic differential equations (FBSDEs) — cf. [2, 3, 7, 16, 21, 24–30, 32–34] and references therein. On the other hand, numerical methods for AFBSDEs are not well developed. In this work, we focus on a numerical method for decoupled AFBSDEs with generators containing a conditional expectation. An explicit numerical scheme for solving the decoupled mean-field FBSDEs with generators containing an expectation is studied in [22]. Here we propose an explicit second order scheme for decoupled AFBSDEs, rigorously prove the stability of the method, and obtain theoretical error estimates. Note that the scheme has the second order convergence rate if the weak order-two Taylor scheme is used for solving forward SDEs. We also carry out a number of numerical tests to verify theoretical findings. The tests show that our numerical scheme is stable, effective, and can achieve the second-order accuracy.
Remark 1.1. We start with an example of ABSDE arising in stochastic optimal control. Let \( \delta > 0 \) be a constant. Consider a stochastic control problem where the laws of a controlled process belong to a family of equivalent measures whose densities satisfy the SDDEs

\[
dX^a_t = \left( a(s, \alpha_s)X^a_s + b(s - \delta, \alpha_{s-\delta})X^a_{s-\delta} \right) ds + X^a_s \sigma(s, \alpha_s) dW_t, \quad s \in [t, T + \delta],
\]
\[
X^a_t = 0, \quad s \in [t - \delta, t),
\]
\[
X^a_T = 1,
\]

where \( \{\alpha_s, s \in [-\delta, T + \delta]\} \) is an adapted feasible continuous control taking values in a compact subset \( U \) of \( \mathbb{R}^k \). Let \( l \) be the running cost and \( Q(s), s \in [T, T + \delta] \) the terminal condition. The problem is to maximise the objective functional

\[
J[\alpha] = E \left[ X^a_T Q(T) + \int_T^{T+\delta} X^a_s Q(s) b(s - \delta, \alpha_{s-\delta}) ds + \int_0^T X^a_t l(s, \alpha_s) ds \right]
\]

over all feasible control processes \( \alpha \). If \( a, b, \sigma, l \) and \( Q \) satisfy certain regularity conditions and \( (Y^a, Z^a) \) is the solution of the ABSDE

\[
-dY^a_t = f^a(t, Y^a_t, Z^a_t, Y^a_{t+\delta}) dt - Z^a_t dW_t, \quad t \in [0, T],
\]
\[
Y^a_t = Q(t), \quad t \in [T, T + \delta]
\]

with

\[
f^a(t, y, z, \eta) = a(t, \alpha_t) y + \sigma(t, \alpha_t) z + b(t, \alpha_t) E[\eta|\mathcal{F}_t] + l(t, \alpha_t)
\]

for \( \eta \in L^2(\mathcal{F}_T, \mathbb{R}), r \in [t, T + \delta] \), then we have \( J[\alpha] = Y^a_0 \). For more details, the readers can consult [18].

If in the Eqs. (1.1) we have \( \zeta(t) = 0 \) for \( t \in [0, T] \), then it was already shown that the terminal condition of \( Z_t \) for \( t \in [T, T + K] \) is not needed. In this case, the Eqs. (1.1) have the form

\[
dX_s = b(s, X_s) ds + \sigma(s, X_s) dW_s, \quad s \in [0, T + K],
\]
\[
dY_s = f(s, X_s, Y_s, Z_s, X_{s+\eta(s)}, Y_{s+\delta(s)}) ds - Z_s dW_s, \quad s \in [0, T),
\]
\[
Y_t = Q_t, \quad t \in [T, T + K],
\]

where

\[
f(s, x, y, z, \eta, \xi): [0, T] \times \mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R}^{r \times m} \times L^2(\mathcal{F}_T; \mathbb{R}^p) \times L^2(\mathcal{F}_T; \mathbb{R}^p) \to L^2(\mathcal{F}_T; \mathbb{R}^p)
\]

with \( r, r' \in [s, T + K] \). Here \( L^2(\mathcal{F}_T; \mathbb{R}^d) \) denotes the set of \( \mathcal{F}_T \)-measurable \( \mathbb{R}^d \)-valued square integrable random variables.

2. Preliminaries

In this section we introduce the Feynman-Kac formula and discrete backward Gronwall inequalities.
2.1. Feynman-Kac formula

In order to find the numerical solution of the AFBSDE (1.1), we write it in the form

\[ \begin{align*}
  dX_s &= b(s,X_s)ds + \sigma(s,X_s)dW_s, & s \in [0, T + K], \\
  dY_s &= \mathbb{E} \left[ f \left( s, X_s, Y_s, Z_s, X_{s+\eta(s)}, Y_{s+\delta(s)}, Z_{s+\zeta(s)} \right) \right] ds - Z_s dW_s, & s \in [0, T), \\
  Y_t &= \varphi(t, X_t), & t \in [T, T + K], \\
  Z_t &= (\varphi' \sigma)(t, X_t), & t \in [T, T + K],
\end{align*} \tag{2.1} \]

where

\[ f(s, x, y, z, x_1, y_1, z_1) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R}^{p \times m} \times \mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R}^{p \times m} \to \mathbb{R}^p, \]

\[ \varphi(t, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^p \text{ is a two deterministic function and} \]

\[ \varphi'_x(t, x) = \begin{pmatrix}
  \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \cdots & \frac{\partial \varphi_1}{\partial x_d} \\
  \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_2} & \cdots & \frac{\partial \varphi_2}{\partial x_d} \\
  \vdots & \vdots & \ddots & \vdots \\
  \frac{\partial \varphi_p}{\partial x_1} & \frac{\partial \varphi_p}{\partial x_2} & \cdots & \frac{\partial \varphi_p}{\partial x_d}
\end{pmatrix}. \]

We assume that the function \( f \) satisfies the following conditions:

(H1) \( \) There exists a constant \( L > 0 \), such that for all \( s \in [0, T] \), \( x, \bar{x}, x_1, \bar{x}_1 \in \mathbb{R}^d \), \( y, y_1, \bar{y}, \bar{y}_1 \in \mathbb{R}^p \) and \( z, z_1, \bar{z}, \bar{z}_1 \in \mathbb{R}^{p \times m} \),

\[ \begin{align*}
  |f(s, x, y, z, x_1, y_1, z_1) - f(s, \bar{x}, \bar{y}, \bar{z}, \bar{x}_1, \bar{y}_1, \bar{z}_1)| &
  \leq L (|x - \bar{x}| + |y - \bar{y}| + |z - \bar{z}| + |x_1 - \bar{x}_1| + |y_1 - \bar{y}_1| + |z_1 - \bar{z}_1|).
\end{align*} \]

(H2) \( \) For any fixed \( x, x_1 \in \mathbb{R}^d \), \( y, y_1 \in \mathbb{R}^p \), \( z, z_1 \in \mathbb{R}^{p \times m} \), the function \( f(s, x, y, z, x_1, y_1, z_1) \) is continuous.

We recall that under the assumptions (H1),(H2), the AFBSDE (2.1) admits a unique adapted solution \( (X_t, Y_t, Z_t) \), cf. [5, 8, 18].

Let \( C^{l,k} \) be the set of functions \( \phi(t, x, y) \) with uniformly bounded partial derivatives \( \partial^{l_1}_t \phi \) and \( \partial^{k_1}_x \partial^{k_2}_y \phi \) for \( l_1 \leq l \) and \( k_1 + k_2 \leq k \). Supposing that the AFBSDE (2.1) has a unique adapted solution, we have the following lemma.

**Lemma 2.1.** Let function \( f(s, x, y, z, x_1, y_1, z_1) \) be continuous with respect to \( s \) and uniformly Lipschitz continuous with respect to \( (x, y, z, x_1, y_1, z_1) \). Assume \( \eta, \delta \) and \( \zeta \) are continuous with respect to \( s \), and \( \varphi \in C^{1,2+d}_{b} \) for some \( \alpha \in (0, 1) \). Then the solution \( (Y_t, Z_t) \) of the ABSDE in (2.1) can be represented as

\[ \begin{align*}
  Y_t &= u(t, X_t), \\
  Z_t &= (u'_x \sigma)(t, X_t),
\end{align*} \tag{2.2} \]
where \(u(t, x)\) is the smooth solution of the following nonlocal PDE:

\[
\begin{align*}
    u_t(t, x) + (u'_x b)(t, x) + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k=1}^{m} (u'_{x_i} \sigma_{ik} \sigma_{jk})(t, x) \\
    + \mathbb{E} \left[ f \left( t, x, u(t, x), (u'_x \sigma)(t, x), X_{t+\eta(t)}^{t, x}, u(t+\delta(t), X_{t+\delta(t)}^{t, x}) \right) \\
    \right] = 0
\end{align*}
\]

(2.3)

with the terminal condition \(u(s, x) = \varphi(s, x)\) for \(s \in [T, T+K]\), where

\[
    u_t(t, x) = \left( \frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}, \ldots, \frac{\partial u_d}{\partial t} \right)^T, \\
    u''_{x_i, x_j}(t, x) = \left( \frac{\partial^2 u_1}{\partial x_i \partial x_j}, \frac{\partial^2 u_2}{\partial x_i \partial x_j}, \ldots, \frac{\partial^2 u_d}{\partial x_i \partial x_j} \right)^T.
\]

Furthermore, if \(\delta, \zeta \in C^{1+k,2k}_b, f \in C^{1+k,2k+2k}_b, \) and \(\varphi \in C^{1+k,3+2k}_b\) for an \(\alpha \in (0,1)\), then we have

\[
    u \in C^{1+k,3+2k}_b, \quad k = 1, 2, \ldots
\]

The expression in (2.2) is the celebrated Feynman-Kac formula.

2.2. Discrete backward Gronwall inequalities

Let us recall two backward discrete Gronwall inequalities, which are needed in the stability analysis.

**Lemma 2.2.** Let \(N\) and \(M\) be nonnegative integers and \(\Delta t\) any positive number. Assume that \(\{\eta_n\}, n = N-1, N-2, \ldots, 0\) satisfies the inequality

\[
    |\eta_n| \leq \beta + a\Delta t \sum_{j=n+1}^{N+M} |\eta_j| \tag{2.4}
\]

with positive constants \(\alpha\) and \(\beta\) and set \(M_\eta := \max_{N \leq j \leq N+M} |\eta_j|, T^+ := (N + M)\Delta t\). Then for \(n = N-1, N-2, \ldots, 0\),

\[
    |\eta_n| \leq e^{\alpha T^+} \left( \beta + a(M + 1)\Delta t M_\eta \right).
\]

**Lemma 2.3.** Let \(N, M\) and \(N_0\) be positive integers, \(\Delta t\) a positive number, \(T^+ := (N + M)\Delta t\) and \(\{a_n\}, \{b_n\}, \{R^n\}, n = 0, 1, \ldots, N + M\) be nonnegative sequences such that

\[
    a_n + C_1 \Delta t b_n \leq (1 + C_2 \Delta t)a_{n+1} + C'_2 \Delta t b_{n+1} + C_3 \Delta t \sum_{i \in \mathcal{A}_n} (a_{n+i} + b_{n+i}) + R^n
\]

for \(0 \leq n \leq N-1\), where \(\mathcal{A}_n = \{i_1^n, \ldots, i_n^n\}\) is an index set with \(i_j^n \in \{1, 2, \ldots, M + N - n\}\) for \(1 \leq j \leq l_n\) and satisfies

\[
    n(\mathcal{A}_n) \leq N_0 < \infty
\]
Explicit Second-Order Scheme for AFBSDEs

with \( n(\mathcal{A}_n) = l_n \) being the number of the elements in \( \mathcal{A}_n \), and \( C_1, C_2, C'_2 \) and \( C_3 \) are positive constants satisfying \( C_1 - C'_2 - C_3 N_0 \geq C_4 \) for a constant \( C_4 > 0 \). Then for \( n = N - 1, \ldots, 1, 0 \) the following inequalities hold:

\[
a_n + \Delta t \sum_{i=n}^{N-1} b_i \leq C \left( a_N + \max_{N \leq i \leq N+M} \{a_i\} + \max_{N \leq i \leq N+M} \{b_i\} + \sum_{i=0}^{N-1} R_i \right),
\]

where \( C \) is a positive constant depending on \( T^*, N_0, C_1, C'_2, C_2 \) and \( C_3 \).

**Proof.** Taking into account Lemma 2.2 and using the inequality \( M \Delta t \leq T^* \), we can follow the proof of [34, Lemma 4.2].

\[\square\]

\section{3. Numerical Scheme}

Given positive integers \( N \) and \( M \), we consider the partition

\[0 = t_0 < t_1 < \cdots < t_N = T < t_{N+1} < \cdots < t_{N+M} = T + K\]

of the time interval \([0, T + K]\), and for each \( n = 0, 1, \ldots, M + N - 1 \) we set

\[
\Delta t_n := t_{n+1} - t_n, \quad \Delta t := \max_{0 \leq n \leq N+M-1} \Delta t_n, \\
\Delta W_{t_n,s} := W_s - W_{t_n}, \quad \Delta W_t := W_{t_{n+1}} - W_{t_n},
\]

where \( t_n \leq s \leq t_{n+1} \). Suppose the time partition satisfies the inequality

\[
\frac{\max_{0 \leq n \leq N+M-1} \Delta t_n}{\min_{0 \leq n \leq N+M-1} \Delta t_n} \leq c_0
\]

with a constant \( c_0 \geq 1 \).

\subsection{3.1. Numerical schemes for solving SDEs}

Here we introduce Itô-Taylor type schemes for solving SDEs [12], considering the case \( d = m = 1 \) for simplicity.

Let \( X^n \) be an approximation of the solution \( X_t \) of SDE in (2.1) at time \( t = t_n \). We set

\[
b^n = b(t_n, X^n), \quad \sigma^n = \sigma(t_n, X^n), \\
b'_{n} = b'_n(t_n, X^n), \quad \sigma'_n = \sigma'_n(t_n, X^n), \\
b''_{n} = b''_n(t_n, X^n), \quad \sigma''_n = \sigma''_n(t_n, X^n),
\]

and introduce three Itô-Taylor schemes for one-dimensional SDEs — viz.
1. The Euler scheme:

\[ X_{n+1}^{t,x} = X_n + b^n \Delta t_n + \sigma^n \Delta W_n. \]  

(3.1)

2. The Milstein scheme:

\[ X_{n+1}^{t,x} = X_n + b^n \Delta t_n + \sigma^n \Delta W_n + \frac{1}{2} \sigma^n \sigma_x^n ((\Delta W_n)^2 - \Delta t_n). \]  

(3.2)

3. The weak order 2.0 Itô-Taylor scheme:

\[
\begin{align*}
X_{n+1}^{t,x} &= X_n^{t,x} + b^n \Delta t_n + \sigma^n \Delta W_n + \frac{1}{2} \sigma^n \sigma_x^n ((\Delta W_n)^2 - \Delta t_n) \\
&
\quad + \frac{1}{2} \left( b^n_t + b^n_x \sigma^n + \frac{1}{2} b^n_{xx}(\sigma^n)^2 \right) (\Delta t_n)^2 + b^n_x \sigma^n \Delta Z_n \\
&
\quad + \frac{1}{2} \left( \sigma^n_t + \sigma^n_x b^n + \frac{1}{2} \sigma^n_{xx}(\sigma^n)^2 \right) (\Delta W_n \Delta t_n - \Delta Z_n),
\end{align*}
\]

(3.3)

where

\[ \Delta Z_n = \Delta W_n \Delta t_n - \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} d r d W_s. \]

### 3.2. Reference equations for ABSDEs

Let \((X^{t,x}_s, Y^{t,x}_s, Z^{t,x}_s)\) be an adapted solution of the AFBSDE (2.1) with the forward SDE starting from the time-space point \((t, x)\), i.e. \((X^{t,x}_s, Y^{t,x}_s, Z^{t,x}_s)\) satisfies the AFBSDE

\[
\begin{align*}
X^{t,x}_s &= x + \int_t^s b(s, X^{t,x}_r) d r + \int_t^s \sigma(s, X^{t,x}_r) d W_r, \\
Y^{t,x}_s &= \varphi(T, X^{t,x}_T) + \int_s^T \mathbb{E} \left[ f \left( r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r, X^{t,x}_{r+\delta(r)}, Y^{t,x}_{r+\delta(r)}, Z^{t,x}_{r+\delta(r)} \right) \bigg| \mathcal{F}_r \right] d r \\
&
\quad - \int_s^T Z^{t,x}_r d W_r.
\end{align*}
\]

Using the notation

\[ f^{t,x}_s = f \left( s, X^{t,x}_s, Y^{t,x}_s, Z^{t,x}_s, X^{t,x}_{s+\delta(s)}, Y^{t,x}_{s+\delta(s)}, Z^{t,x}_{s+\delta(s)} \right), \]

we write

\[
\begin{align*}
X^{t,x}_{t_n^{x}} &= x + \int_{t_n}^{t_{n+1}} b(s, X^{t,x}_s) d s + \int_{t_n}^{t_{n+1}} \sigma(s, X^{t,x}_s) d W_s, \\
Y^{t,x}_{t_n^{x}} &= Y^{t,x}_{t_n^{x}} + \int_{t_n}^{t_{n+1}} \mathbb{E} \left[ f^{t,x}_s \big| \mathcal{F}_s \right] d s - \int_{t_n}^{t_{n+1}} Z^{t,x}_s d W_s,
\end{align*}
\]

(3.4)  

(3.5)
for $0 \leq n \leq N - 1$. Noting that $\mathcal{F}_t \subseteq \mathcal{F}_s$ for $t \leq s \leq T$, we take conditional mathematical expectation $\mathbb{E}_t^X[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t, X_t = x]$ of the Eq. (3.5), thus obtaining

$$
Y_{t_n} = \mathbb{E}_t^X \left[ f_{t_{n+1}} \right] + \int_{t_n}^{t_{n+1}} \mathbb{E}_t^X \left[ f_s \right] ds,
$$

where $R_{n+1}^n = \mathbb{E}_t^X \left[ f_{t_{n+1}} \right] + \frac{1}{2} \Delta t \mathbb{E}_t^X \left[ f_{t_{n+1}} \right] + \frac{1}{2} \Delta t \mathbb{E}_t^X \left[ f_{t_{n+1}} \right] + R_{n+1}^n,$ (3.6)

where $R_{n+1}^n$ is the approximation error

$$
R_{n+1}^n = \int_{t_n}^{t_{n+1}} \left( \mathbb{E}_t^X \left[ f_s \right] - \frac{1}{2} \mathbb{E}_t^X \left[ f_{t_{n+1}} \right] - \frac{1}{2} \mathbb{E}_t^X \left[ f_{t_{n+1}} \right] \right) ds.
$$

Since the points $t_n + \delta(t_n)$, $t_n + \zeta(t_n)$, $t_{n+1} + \delta(t_{n+1})$ and $t_{n+1} + \zeta(t_{n+1})$, in general, do not belong to the time grid, numerical methods are needed to approximate the values of $(Y_t, Z_t)$ at these points. In this work, the Lagrange interpolation polynomials of order one are used. Assume that there are nonnegative integers $n_\delta$ and $n_\zeta$ such that

$$
t_n + \delta(t_n) \in [t_{n+n_\delta}, t_{n+n_\delta+1}], \quad t_n + \zeta(t_n) \in [t_{n+n_\zeta}, t_{n+n_\zeta+1}]
$$

for $n = 0, 1, \ldots, N$. Now we define

$$
Y_{t_n}', Y_{t_n}', Z_{t_n}', Z_{t_n}', R_{n+1}^n = (1 - \delta_n)Y_{t_n}', \delta_n Y_{t_n}', (1 - \zeta_n)Z_{t_n}', \zeta_n Z_{t_n}',
$$

where

$$
\delta_n = \frac{t_n + \delta(t_n) - t_{n+n_\delta}}{\Delta t_n}, \quad \zeta_n = \frac{t_n + \zeta(t_n) - t_{n+n_\zeta}}{\Delta t_n}.
$$

For $k = n$ or $n + 1$, we let

$$
f_{t_k} = f \left( t_k, X_{t_k}, Y_{t_k}', Z_{t_k}', X_{t_k} + \mathbb{E}_{t_k} \left[ f_{t_{n+1}} \right], Y_{t_k}', Z_{t_k}', \right).
$$

It follows from (3.6) that

$$
Y_{t_n} = \mathbb{E}_t^X \left[ f_{t_{n+1}} \right] + \frac{1}{2} \Delta t \mathbb{E}_t^X \left[ f_{t_{n+1}} \right] + \frac{1}{2} \Delta t \mathbb{E}_t^X \left[ f_{t_{n+1}} \right] + R_{n+1}^n + R_{n+1}^n,
$$

where

$$
R_{n+1}^n = \frac{1}{2} \Delta t \mathbb{E}_t^X \left[ f_{t_{n+1}} \right] - f_{t_{n+1}}, \quad \frac{1}{2} \Delta t \mathbb{E}_t^X \left[ f_{t_{n+1}} \right] - f_{t_{n+1}}
$$

To determine $Y_{t_n}'$, we can approximate $Y_{t_n}'$ in $f_{t_{n+1}}$ by the right rectangle formula. This yields

$$
Y_{t_n}' = \mathbb{E}_t^X \left[ f_{t_{n+1}} \right] + \Delta t \mathbb{E}_t^X \left[ f_{t_{n+1}} \right] + R_{n+1}^n,
$$

(3.13)
where

\[ R_{yr}^n = \int_{t_n}^{t_{n+1}} \left( E_{t_n}^x \left[ f_{t_n}^{x, y} \right] - E_{t_n}^x \left[ f_{t_{n+1}}^{x, y} \right] \right) ds + \Delta t_n E_{t_n}^x \left[ f_{t_{n+1}}^{x, y} - f_{t_{n+1}}^{x, y, L} \right]. \]  \hspace{1cm} (3.14)

Let

\[ \tilde{Y}_{t_n}^{y, x} = E_{t_n}^x \left[ Y_{t_n+1}^{y, x} \right] + \Delta t_n E_{t_n}^x \left[ f_{t_{n+1}}^{y, x, L} \right], \]  \hspace{1cm} (3.15)

and define

\[ \tilde{Y}_{t_n+\delta(t_n)}^{y, x, L} = (1 - \delta_n) \tilde{Y}_{t_n}^{y, x} + \delta_n Y_{t_{n+1}}^{y, x}, \]  \hspace{1cm} (3.16)

when \( t_n + \delta(t_n) \in [t_n, t_{n+1}) \). Using the notations

\[ \tilde{f}_{t_n, \delta_0}^{x, y, L} = f \left( t_n, X_{t_n}^{x, y}, \tilde{Y}_{t_n}^{x, y}, Z_{t_n}^{x, y} \right), \]  \hspace{1cm} (3.17)

along with the Eqs. (3.11) and (3.17), we arrive at the following reference equation for finding \( Y_{t_n}; \)

\[ Y_{t_n}^{y, x} = E_{t_n}^x \left[ Y_{t_{n+1}}^{y, x} \right] + \frac{1}{2} \Delta t_n E_{t_n}^x \left[ f_{t_{n+1}}^{y, x, L} I_{\{n=0\}} + f_{t_{n+1}}^{y, x, L} I_{\{n\geq 1\}} \right] \]
\[ + \frac{1}{2} \Delta t_n E_{t_n}^x \left[ f_{t_{n+1}}^{y, x, L} \right] + R_{y}^n, \]  \hspace{1cm} (3.18)

where \( I \) is the indicator function, \( R_{y}^n = R_{y1}^n + R_{y2}^n + R_{y3}^n \) and

\[ R_{y3}^n = \frac{1}{2} \Delta t_n E_{t_n}^x \left[ f_{t_n}^{y, x, L} I_{\{n=0\}} - \tilde{f}_{t_n, \delta_0}^{y, x, L} I_{\{n\geq 1\}} \right]. \]  \hspace{1cm} (3.19)

Considering the term \( \Delta \tilde{W}_{t_n, \tilde{\eta}} \) defined by

\[ \Delta \tilde{W}_{t_n, \tilde{\eta}} = 2 \Delta W_{t_n, \tilde{\eta}} - \frac{3}{\Delta t_n} \int_{t_n}^{t} (r - t) dW_r, \]

we note that \( \Delta \tilde{W}_{t_n, \tilde{\eta}} \) is a new Brownian motion such that

\[ E_{t_n}^x \left[ \Delta \tilde{W}_{t_n, t_{n+1}} \right] = 0, \quad E_{t_n}^x \left[ (\Delta \tilde{W}_{t_n, t_{n+1}})^2 \right] = \Delta t_n. \]

Multiplying both sides of (3.5) by \( \Delta \tilde{W}_n = \Delta \tilde{W}_{t_n, t_{n+1}} \) and taking the conditional expectation \( E_{t_n}^x \left[ \cdot \right] \) of the resulting equation gives

\[ 0 = E_{t_n}^x \left[ Y_{t_{n+1}}^{y, x, \Delta \tilde{W}_n} \right] + \int_{t_n}^{t_{n+1}} E_{t_n}^x \left[ f_{s}^{y, x, \Delta \tilde{W}_n} \right] ds - E_{t_n}^x \left[ \int_{t_n}^{t_{n+1}} Z_{s}^{y, x, \Delta \tilde{W}_n} ds \cdot \tilde{W}_n \right]. \]

This equation leads to a reference equation for finding \( Z_{t_n, \tilde{\eta}} \), viz.

\[ \frac{1}{2} \Delta t_n Z_{t_n}^{y, x} = E_{t_n}^x \left[ Y_{t_{n+1}}^{y, x, \Delta \tilde{W}_n} \right] + \Delta t_n E_{t_n}^x \left[ f_{t_{n+1}}^{y, x, L} \Delta \tilde{W}_n \right] + R_{z}^n. \]  \hspace{1cm} (3.20)
where \( R^n_z = R^n_{z1} + R^n_{z2} + R^n_{z3} \) and
\[
R^n_{z1} = \int_{t_n}^{t_{n+1}} \left( \mathbb{E}^{X^n}_{t_n} \left[ f^n_{t_{n+1}} \Delta \tilde{W}^n_{t_{n+1}} \right] - \mathbb{E}^{X^n}_{t_n} \left[ f^n_{t_{n+1}} \Delta \tilde{W}^n_n \right] \right) ds,
\]
\[
R^n_{z2} = \frac{1}{2} \Delta t_n Z^n_{t_n} - \mathbb{E}^{X^n}_{t_n} \left[ \int_{t_n}^{t_{n+1}} Z^n_{s} f^n_s dW_s \cdot \tilde{W}^n_n \right],
\]
\[
R^n_{z3} = \Delta t_n \mathbb{E}^{X^n}_{t_n} \left[ (f^n_{t_{n+1}} - f^n_{t_{n+1}}) \Delta \tilde{W}^n_n \right].
\]

### 3.3. Explicit scheme for ABSDEs

Let \((X^n, Y^n, Z^n)\) be a numerical approximation of solution \((X_t, Y_t, Z_t)\) of the AFBSDE (2.1) at the time level \( t = t_n, n = N + M, \ldots, 1, 0 \). We also set
\[
f^n := f \left( t_n, X^n, Y^n, Z^n, X^n_{\eta}, Y^n_{\eta}, Z^n_{\zeta} \right), \quad n = N, \ldots, 1,
\]
where \( X^n_{\eta} \) is the approximation of \( X_t \) at the time \( t = t_n + \eta(t_n) \), and let
\[
Y^n_\delta := (1 - \delta_n) Y^{n+\delta} + \delta_n Y^{n+\delta+1},
\]
\[
Z^n_\zeta := (1 - \zeta_n) Z^{n+\zeta} + \zeta_n Z^{n+\zeta+1}
\]
with \( \delta_n \) and \( \zeta_n \) defined by (3.9). Taking into account the Eqs. (3.15), (3.18), (3.20), we introduce the following explicit scheme for the AFBSDE (2.1).

**Scheme 3.1.** Given an initial value \( X_0, \) terminal conditions \( Y^{N+i} \) and \( Z^{N+i}, \) \( i = 0, 1, \ldots, M, \) we determine random variables \( Y^n = Y^n(X^n), Z^n = Z^n(X^n), n = N - 1, \ldots, 0 \) from the equations
\[
\frac{1}{2} \Delta t_n Z^n = \mathbb{E}^{X^n}_{t_n} \left[ Y^{n+1} \Delta \tilde{W}^n_n \right] + \Delta t_n \mathbb{E}^{X^n}_{t_n} \left[ f^{n+1} \Delta \tilde{W}^n_n \right],
\]
\[
\tilde{Y}^n = \mathbb{E}^{X^n}_{t_n} \left[ Y^{n+1} \right] + \Delta t_n \mathbb{E}^{X^n}_{t_n} \left[ f^{n+1} \right],
\]
\[
Y^n = \mathbb{E}^{X^n}_{t_n} \left[ Y^{n+1} \right] + \frac{1}{2} \Delta t_n \mathbb{E}^{X^n}_{t_n} \left[ \tilde{f}^{n}_{\delta_0} \mathbbm{1}_{\{n=0\}} + \tilde{f}^{n}_{\delta_1} \mathbbm{1}_{\{n=1\}} \right] + \frac{1}{2} \Delta t_n \mathbb{E}^{X^n}_{t_n} \left[ f^{n+1} \right],
\]
where \( X^n \) and \( X^n_{\eta} \) are solved by the Itô-Taylor schemes,
\[
\tilde{f}^{n}_{\delta_0} = f \left( t_n, X^n, \tilde{Y}^n, Z^n, X^n_{\eta}, \tilde{Y}^n_{\delta}, Z^n_{\zeta} \right), \quad \tilde{f}^{n}_{\delta_1} = f \left( t_n, X^n, \tilde{Y}^n, Z^n, X^n_{\eta}, Y^n_{\eta}, Z^n_{\zeta} \right)
\]
and \( \tilde{Y}^n_\delta = (1 - \delta_n) \tilde{Y}^n + \delta_n Y^{n+1} \).

**Remark 3.1.** \( X^n_{\eta} \) is the approximation value of \( X_t \) at time \( t = t_n + \eta(t_n) \). However, in general, \( t_n + \eta(t_n) \) does not belong to the time grid \( \{ t_k, k = 0, 1, \ldots, N + M \} \). In this case, when \( t_n + \eta(t_n) \in [t_{n+n_0}, t_{n+n_1}] \), we first solve \( \{X^{n+i}, i = 1, \ldots, n_0\} \) using standard schemes for SDEs and then give \( X^n_{\eta} \) by the Euler scheme
\[
X^n_{\eta} = X^{n+n_0} + b(t_{n+n_0}, X^{n+n_0})(t_n + \eta(t_n) - t_{n+n_0}) + \sigma(t_{n+n_0}, X^{n+n_0})(W_{t_n + \eta(t_n)} - W_{t_{n+n_0}}).
\]
Since the local error of the Euler scheme is \((\Delta t_n)^2\), it does not affect the accuracy of the schemes for SDEs presented in Subsection 3.1.
Moreover, we consider the terms

\[ Y^{N+i}_\varepsilon = Y^{N+i} + \varepsilon^{N+i}_y, \]
\[ Z^{N+i}_\varepsilon = Z^{N+i} + \varepsilon^{N+i}_z, \]
\[ f_\varepsilon(x, y, z, x_1, y_1, z_1) = f(x, y, z, x_1, y_1, z_1) + \varepsilon_f. \]

Moreover, we consider the terms

\[ f^{n,\varepsilon}(t_n, X^n_\varepsilon, Y^n_\varepsilon, Z^n_\varepsilon, X^n_{\varepsilon,\delta}, Z^n_{\varepsilon,\delta}), \]
\[ \tilde{f}^{n,\varepsilon}_{50}(t_n, X^n_\varepsilon, \tilde{Y}^n_\varepsilon, Z^n_\varepsilon, X^n_{\varepsilon,\delta}, Z^n_{\varepsilon,\delta}), \]
\[ \tilde{f}^{n,\varepsilon}_{51}(t_n, X^n_\varepsilon, \tilde{Y}^n_\varepsilon, Z^n_\varepsilon, X^n_{\varepsilon,\delta}, Z^n_{\varepsilon,\delta}), \]

where \( \tilde{Y}^n_\varepsilon, Y^n_\varepsilon \) and \( Z^n_\varepsilon \) are the solutions of the perturbed Scheme 4.1 below,

\[ Y^n_{\varepsilon,\delta} = (1 - \delta_n) Y^{n+n_\delta} + \delta_n Y^{n+n_\delta+1}, \quad n_\delta \geq 1, \]
\[ \tilde{Y}^n_{\varepsilon,\delta} = (1 - \delta_n) \tilde{Y}^n_\varepsilon + \delta_n \tilde{Y}^n_{\varepsilon,\delta}, \quad n_\delta = 0, \]

and \( Z^n_{\varepsilon,\zeta} = (1 - \zeta_n) Z^{n+n_\zeta} + \zeta_n Z^{n+n_\zeta+1}, \quad n_\zeta \geq 0. \)

**Scheme 4.1.** Given random variables \( X_0 \) and

\[ (Y^{N+i}_\varepsilon, Z^{N+i}_\varepsilon) = (Y^{N+i} + \varepsilon^{N+i}_y, Z^{N+i} + \varepsilon^{N+i}_z), \quad i = 0, 1, \ldots, M, \]

determine \( (Y^n_\varepsilon, Z^n_\varepsilon), n = N - 1, \ldots, 0 \) from the equations

\[ \frac{1}{2}\Delta t_n Z^n_\varepsilon = \mathbb{E}^x_{t_n} \left[ Y^{n+1}_\varepsilon \Delta W_n \right] + \Delta t_n \mathbb{E}^x_{t_n} \left[ (f^{n+1,\varepsilon} + \varepsilon_f^{n+1}) \Delta W_n \right], \]
\[ \tilde{Y}^n_\varepsilon = \mathbb{E}^x_{t_n} \left[ Y^{n+1}_\varepsilon \right] + \Delta t_n \mathbb{E}^x_{t_n} \left[ f^{n+1,\varepsilon} + \varepsilon_f^{n+1} \right], \]
\[ Y^n_\varepsilon = \mathbb{E}^x_{t_n} \left[ Y^{n+1}_\varepsilon \right] + \frac{1}{2}\Delta t_n \mathbb{E}^x_{t_n} \left[ (\tilde{f}^{n,\varepsilon}_{50} + \varepsilon^{n}_f) I_{n_3 = 0} + (f^{n,\varepsilon}_{51} + \varepsilon^{n}_f) I_{n_3 \geq 1} \right] \]
\[ + \frac{1}{2}\Delta t_n \mathbb{E}^x_{t_n} \left[ f^{n+1,\varepsilon} + \varepsilon^{n+1}_f \right]. \]

Let \( \varepsilon^n_y, \varepsilon^n_z \) and \( \varepsilon^n_\gamma \) refer to the perturbation errors – i.e.

\[ \varepsilon^n_y = Y^n_\varepsilon - Y^n, \quad \varepsilon^n_z = Z^n_\varepsilon - Z^n, \quad \varepsilon^n_\gamma = \tilde{Y}^n_\varepsilon - \tilde{Y}^n. \]
Definition 4.1. We call Scheme 3.1 stable if for any \( \epsilon_0 > 0 \) and \( n = N - 1, \ldots, 1, 0 \), there is an \( r_0 > 0 \) such that
\[
\mathbb{E} \left[ |e^n_y|^2 + \Delta t \sum_{i=n}^{N-1} |e^i_z|^2 \right] < \epsilon_0
\]
provided that
\[
\mathbb{E} \left[ |e^{N+i}_y|^2 + |e^{N+i}_z|^2 \right] < r_0, \quad |\epsilon_f| < r_0
\]
for all \( i = 0, 1, \ldots, M \).

To analyse the stability of Scheme 3.1, we write the Eqs. (4.1) as
\[
\begin{align*}
\frac{1}{2} \Delta t_n Z^n_x & = \mathbb{E}^X_{t_n} \left[ Y^{n+1}_\xi \Delta \tilde{W}_n \right] + \Delta t_n \mathbb{E}^X_{t_n} \left[ f^{n+1, \epsilon} \Delta \tilde{W}_n \right] + R^n_{x, n}, \\
Y^n_x & = \mathbb{E}^X_{t_n} \left[ Y^{n+1}_\eta \right] + \Delta t_n \mathbb{E}^X_{t_n} \left[ f^{n+1, \epsilon} \right] + \tilde{R}^n_{x, y}, \\
Y^n_z & = \mathbb{E}^X_{t_n} \left[ Y^{n+1}_q \right] + \frac{1}{2} \Delta t_n \mathbb{E}^X_{t_n} \left[ f^{n, \epsilon}_{\delta_0} I_{(n_0 = 0)} + f^{n, \epsilon}_{\delta_1} I_{(n_0 \geq 1)} \right] \\
& + \frac{1}{2} \Delta t_n \mathbb{E}^X_{t_n} \left[ f^{n+1, \epsilon} \right] + R^n_{x, y},
\end{align*}
\]
where
\[
\begin{align*}
\tilde{R}^n_{x, y} & = \Delta t_n \mathbb{E}^X_{t_n} \left[ e^{n+1}_f \right], \\
R^n_x & = \Delta t_n \mathbb{E}^X_{t_n} \left[ e^{n+1, \epsilon} \Delta \tilde{W}_n \right], \\
R^n_y & = \frac{1}{2} \Delta t_n \mathbb{E}^X_{t_n} \left[ e^{n, \epsilon}_{\delta_0} I_{(n_0 = 0)} + e^{n, \epsilon}_{\delta_1} I_{(n_0 \geq 1)} \right] + \frac{1}{2} \Delta t_n \mathbb{E}^X_{t_n} \left[ e^{n+1}_f \right].
\end{align*}
\]
Subtracting (3.21)-(3.23) from (4.3)-(4.5) yields
\[
\begin{align*}
\frac{1}{2} \Delta t_n e^n_x & = \mathbb{E}^X_{t_n} \left[ e^{n+1}_y \Delta \tilde{W}_n \right] + \Delta t_n \mathbb{E}^X_{t_n} \left[ (f^{n+1, \epsilon} - f^{n+1}) \Delta \tilde{W}_n \right] + R^n_x, \\
e^n_y & = \mathbb{E}^X_{t_n} \left[ e^{n+1}_y \right] + \Delta t_n \mathbb{E}^X_{t_n} \left[ f^{n+1, \epsilon} - f^{n+1} \right] + \tilde{R}^n_y, \\
e^n_y & = \mathbb{E}^X_{t_n} \left[ e^{n+1}_y \right] + \frac{1}{2} \Delta t_n \mathbb{E}^X_{t_n} \left[ (f^{n, \epsilon}_{\delta_0} - f^{n}_{\delta_0}) I_{(n_0 = 0)} + (f^{n, \epsilon}_{\delta_1} - f^{n}_{\delta_1}) I_{(n_0 \geq 1)} \right] \\
& + \frac{1}{2} \Delta t_n \mathbb{E}^X_{t_n} \left[ f^{n+1, \epsilon} - f^{n+1} \right] + R^n_{x, y}.
\end{align*}
\]
The representations (4.6)-(4.8) are called the perturbation error equations of Scheme 3.1. They are used to study the stability of Scheme 3.1.

Theorem 4.1. Assume that \( f(t, x, y, z, x_1, y_1, z_1) \) is uniformly Lipschitz continuous with respect to \( (x, y, z, x_1, y_1, z_1) \) and let \( L \) be a Lipschitz constant. If
\[
M_{x, y} = \max_{N \leq i \leq N+M} \mathbb{E}[|e^n_y|^2], \quad M_{x, z} = \max_{N \leq i \leq N+M} \mathbb{E}[|e^n_z|^2],
\]
then for sufficiently small time step $\Delta t$, the inequalities

\[
\mathbb{E}[|\varepsilon^n_{y}|^2] + \Delta t \sum_{i=n}^{N-1} \mathbb{E}[|\varepsilon^n_{i}|^2] \\
\leq C \left( M_{x,y} + M_{z} + \sum_{i=0}^{N-1} \frac{\mathbb{E}[(R^i_{x,y})^2 + (\Delta t)^2|R^i_{x,y}|^2 + |R^i_{x,z}|^2]}{\Delta t} \right), \quad n = 0, 1, \ldots, N - 1
\]

hold with a constant $C$ that depends on $c_0, L, T$ and $K$.

Proof: The Lipschitz-condition implies

\[
|f^n - f^n| \leq L (|\varepsilon^n_{x}| + |\varepsilon^n_{y}| + (1 - \delta_n)|\varepsilon^n_{y}| + \delta_n|\varepsilon^n_{y}|) + (1 - \xi_n)|\varepsilon^n_{z}| + \xi_n|\varepsilon^n_{z}|, \quad n = 0, 1, \ldots, N - 1
\]

(4.10)

\[
|\bar{f}_0 - \bar{f}_0| \leq L (|\varepsilon^n_{y}| + |\varepsilon^n_{y}| + (1 - \delta_n)|\varepsilon^n_{y}| + \delta_n|\varepsilon^n_{y}|) + (1 - \xi_n)|\varepsilon^n_{z}| + \xi_n|\varepsilon^n_{z}|, \quad n = 0, 1, \ldots, N - 1
\]

(4.11)

\[
|\bar{f}_1 - \bar{f}_1| \leq L (|\varepsilon^n_{y}| + |\varepsilon^n_{y}| + (1 - \delta_n)|\varepsilon^n_{y}| + \delta_n|\varepsilon^n_{y}|) + (1 - \xi_n)|\varepsilon^n_{z}| + \xi_n|\varepsilon^n_{z}|, \quad n = 0, 1, \ldots, N - 1
\]

(4.12)

Since $I_{(n=0)} + I_{(n=1)} = 1$, the Eq. (4.8) and inequalities (4.10)-(4.12) give

\[
|\varepsilon^n_{y}| \leq \mathbb{E}_{t_n}^{X_n}[\varepsilon^{n+1}_{y}] + \frac{1}{2} \Delta t L \mathbb{E}_{t_n}^{X_n}\left(|\varepsilon^n_{y}| + |\varepsilon^n_{z}| \right) + \mathbb{E}_{t_n}^{X_n}\left[(1 - \delta_n)\left(|\varepsilon^{n+1}_{y}| + |\varepsilon^n_{y}|I_{(n=0)} + |\varepsilon^n_{y}|I_{(n=1)} + \delta_n|\varepsilon^{n+1}_{y}| \right) + \mathbb{E}_{t_n}^{X_n}\left[(1 - \xi_n)|\varepsilon^{n+1}_{z}| + \xi_n|\varepsilon^{n+1}_{z}| \right)
\]

It follows that

\[
|\varepsilon^n_{y}| \leq \mathbb{E}_{t_n}^{X_n}[\varepsilon^{n+1}_{y}] + \frac{1}{2} \Delta t L \mathbb{E}_{t_n}^{X_n}\left(|\varepsilon^n_{y}| + |\varepsilon^n_{z}| \right) + \mathbb{E}_{t_n}^{X_n}\left[(1 - \delta_n)\left(|\varepsilon^{n+1}_{y}| + |\varepsilon^n_{y}|I_{(n=0)} + |\varepsilon^n_{y}|I_{(n=1)} + \delta_n|\varepsilon^{n+1}_{y}| \right) + \mathbb{E}_{t_n}^{X_n}\left[(1 - \xi_n)|\varepsilon^{n+1}_{z}| + \xi_n|\varepsilon^{n+1}_{z}| \right)
\]

(4.13)
Analogously, using (4.7), we obtain

\[ |e^n_j| \leq \mathbb{E}^{X^n} \left[ |e^{n+1}_y| \right] + \Delta t_n L \left( \mathbb{E}^{X^n} \left[ |e^{n+1}_y| \right] + \mathbb{E}^{X^n} \left[ \left| e^{n+1+(n+1)c} \right| \right] \right) + \left( e^{n+1+(n+1)c} \right) + |\tilde{R}^n_{ey}|. \]  

(4.14)

Assuming that \( \Delta t_n L \leq 1 \) and inserting (4.14) into (4.13) yields

\[ |e^n| \leq |\mathbb{E}^{X^n} [e^{n+1}]| + \Delta t_n L \left( \mathbb{E}^{X^n} \left[ \left| e^{n+1} \right| \right] + \mathbb{E}^{X^n} \left[ \left| e^{n+1} \right| \right] \mathbb{I}_{\left\{ n \geq 1 \right\}} + |e^{n+1}| + |e^{n+1}_{n+1}| \right) \]

\[ + 4 \mathbb{E}^{X^n} \left[ \left| e^{n+1} \right| + \left| e^{n+1} \right| \right] + 2 \mathbb{E}^{X^n} \left[ \left| e^{n+1} \right| + \left| e^{n+1} \right| \right] \]

\[ + 2 \mathbb{E}^{X^n} \left[ \left| e^{n+1} \right| + \left| e^{n+1} \right| \right] + 2 |\tilde{R}^{n}_{ey}| + |R^{n}_{ey}|. \]

Without loss of generality, we can suppose that \( n_\delta \geq 1 \) and apply the inequalities

\[ (a + b)^2 \leq a^2 + b^2 + \gamma \Delta t a \leq \frac{1}{\gamma} b^2, \quad \left( \sum_{i=1}^{n} a_i \right)^2 \leq n \sum_{i=1}^{n} a_i^2, \]

so that

\[ |e^n|^2 \leq (1 + \gamma \Delta t) |\mathbb{E}^{X^n} [e^{n+1}]|^2 + 7 \left( 1 + \frac{1}{\gamma \Delta t} \right) \]

\[ \times \left( \Delta t_n L \left( 4 \mathbb{E}^{X^n} \left[ \left| e^{n+1} \right| \right] + \mathbb{E}^{X^n} \left[ \left| e^{n+1} \right| \right] \mathbb{I}_{\left\{ n \geq 1 \right\}} + |e^{n+1}| + |e^{n+1}_{n+1}| \right) \right) \]

\[ + 32 \mathbb{E}^{X^n} \left[ \left| e^{n+1} \right| + \left| e^{n+1} \right| \right] + 8 \mathbb{E}^{X^n} \left[ \left| e^{n+1} \right| + \left| e^{n+1} \right| \right] \]

\[ + 4 |e^{n+1}| + 8 |e^{n+1} + (n+1)c| + 4 |\tilde{R}^{n}_{ey}| + |R^{n}_{ey}|^2. \]  

(4.15)

Moreover, employing the Holder’s inequality and the inequality

\[ (a + b)^2 \leq (1 + \beta) a^2 + \left( 1 + \frac{1}{\beta} \right) b^2, \quad \beta > 0 \]

in (4.6) gives

\[ \frac{1}{4} \left( \Delta t_n \right)^2 |e^n|^2 \leq (1 + \beta) |\mathbb{E}^{X^n} [e^{n+1} \Delta W_n]|^2 + 2 \left( 1 + \frac{1}{\beta} \right) \]

\[ \times \left( \Delta t_n \right)^2 |\mathbb{E}^{X^n} [\left| e^{n+1} - f^{n+1} \right|]\mathbb{E}^{X^n} [\left| \Delta W_n \right|^2] + |\tilde{R}^{n}_{ey}|^2. \]  

(4.16)

It follows from (4.16) and the estimate

\[ \left| \mathbb{E}^{X^n} [e^{n+1} \Delta W_n] \right|^2 = \left| \mathbb{E}^{X^n} [e^{n+1} - \mathbb{E}^{X^n} [e^{n+1}]] \Delta W_n] \right|^2 \]

\[ \leq \Delta t_n \left( \mathbb{E}^{X^n} \left| e^{n+1} \right|^2 - \mathbb{E}^{X^n} \left| e^{n+1} \right|^2 \right), \]
Taking the expectations of (4.15) and (4.18), we obtain
\[
E_{t_n}^{x_n} \left[ |f^{n+1} - f^n|^2 \right] \leq 6L^2 \left( E_{t_n}^{x_n} \left[ |e_n^{n+1}|^2 + |e_z^{n+1}|^2 \right] + E_{t_n}^{x_n} \left[ |e_z^{n+1} + (n+1)\delta_1|^2 \right] + E_{t_n}^{x_n} \left[ |e_z^{n+1} + (n+1)\delta_1|^2 + |e_z^{n+1} + (n+1)\delta_1|^2 \right] \right),
\]
that
\[
\frac{1}{4}(\Delta t_n)^2 |e_z^n|^2 \leq (1 + \beta)\Delta t_n \left( E_{t_n}^{x_n} \left[ |e_y^{n+1}|^2 \right] - |E_{t_n}^{x_n} \left[ e_y^{n+1} \right] |^2 \right) + 12 \left( 1 + \frac{1}{\beta} \right) \left( L^2(\Delta t_n)^3 \left( E_{t_n}^{x_n} \left[ |e_y^{n+1}| + |e_z^{n+1}| \right] + E_{t_n}^{x_n} \left[ |e_y^{n+1} + (n+1)\delta_1|^2 \right] + E_{t_n}^{x_n} \left[ |e_z^{n+1} + (n+1)\delta_1|^2 + |e_z^{n+1} + (n+1)\delta_1|^2 \right] \right) \right) + \frac{12(c_0)^2}{\beta(\Delta t)} |R_n^{t_n}|^2 \right). \quad (4.17)
\]
Dividing (4.17) by \((1 + \beta)(\Delta t_n)^2/\Delta t\) yields
\[
\frac{\Delta t}{4(1 + \beta)} |e_z^n|^2 \leq c_0 \left( E_{t_n}^{x_n} \left[ |e_y^{n+1}|^2 \right] - |E_{t_n}^{x_n} \left[ e_y^{n+1} \right] |^2 \right) + \frac{12L^2(\Delta t)^2}{\beta} \left( E_{t_n}^{x_n} \left[ |e_y^{n+1}| + |e_z^{n+1}| \right] + E_{t_n}^{x_n} \left[ |e_y^{n+1} + (n+1)\delta_1|^2 \right] + E_{t_n}^{x_n} \left[ |e_z^{n+1} + (n+1)\delta_1|^2 + |e_z^{n+1} + (n+1)\delta_1|^2 \right] \right) + \frac{12(c_0)^2}{\beta(\Delta t)} |R_n^{t_n}|^2. \quad (4.18)
\]
Taking the expectations of (4.15) and (4.18), we obtain
\[
E \left[ |e_y^{n+1}|^2 \right] \leq (1 + \gamma \Delta t)E \left[ E_{t_n}^{x_n} \left[ |e_y^{n+1}|^2 \right] \right] + 7 \left( 1 + \frac{1}{\gamma \Delta t} \right) \left( (\Delta t L)^2 \left( 4E \left[ |e_y^{n+n_1}|^2 + |e_y^{n+n_1}|^2 \right] + 4E \left[ |e_y^{n+n_1}|^2 + |e_z^{n+n_1}|^2 \right] + 8E \left[ |e_y^{n+1+n_1}\delta_1|^2 + |e_y^{n+1+n_1}\delta_1|^2 \right] + 4E \left[ |e_z^{n+1}|^2 \right] + 8E \left[ |e_z^{n+1+n_1}\delta_1|^2 + |e_z^{n+1+n_1}\delta_1|^2 \right] \right) \right) + \frac{12L^2(\Delta t)^2}{\beta} \left( E \left[ |e_y^{n+1}| + |e_z^{n+1}| \right] + E \left[ |e_y^{n+1+n_1}\delta_1|^2 + |e_y^{n+1+n_1}\delta_1|^2 \right] + \frac{12(c_0)^2}{\beta(\Delta t)} E \left[ |R_n^{t_n}|^2 \right] \right), \quad (4.19)
\]
\[
\frac{\Delta t}{4(1 + \beta)} E \left[ |e_z^n|^2 \right] \leq c_0 \left( E \left[ |e_y^{n+1}|^2 \right] - E \left[ E_{t_n}^{x_n} \left[ |e_y^{n+1}|^2 \right] \right] \right) \right) \right) + \frac{12L^2(\Delta t)^2}{\beta} \left( E \left[ |e_y^{n+1}| + |e_z^{n+1}| \right] + E \left[ |e_y^{n+1+n_1}\delta_1|^2 + |e_y^{n+1+n_1}\delta_1|^2 \right] + E \left[ |e_z^{n+1+n_1}\delta_1|^2 + |e_z^{n+1+n_1}\delta_1|^2 \right] \right) \right) + \frac{12(c_0)^2}{\beta(\Delta t)} E \left[ |R_n^{t_n}|^2 \right]. \quad (4.20)
\]
Multiplying (4.19) by \(c_0\) and adding the resulting inequality to (4.20) gives

\[
c_0 \mathbb{E} \left[ |e_y^n|^2 \right] + \frac{\Delta t}{4(1 + \beta)} \mathbb{E} \left[ |e_z^n|^2 \right]
\leq c_0 \left( 1 + \left( \gamma + 224L^2 \Delta t + \frac{224L^2}{\gamma} + \frac{12L^2 \Delta t}{c_0 \beta} \right) \Delta t \right) \mathbb{E} \left[ |e_y^{n+1}|^2 \right]
+ \left( 224c_0 L^2 \Delta t + \frac{224c_0 L^2}{\gamma} + \frac{12L^2 \Delta t}{\beta} \right) \Delta t \mathbb{E} \left[ |e_z^{n+1}|^2 \right]
+ \left( 28c_0 L^2 \Delta t + \frac{28c_0 L^2}{\gamma} \right) \Delta t \mathbb{E} \left[ |e_y^{n+1}|^2 + |e_z^{n+1}|^2 \right]
+ \left( 28c_0 L^2 \Delta t + \frac{28c_0 L^2}{\gamma} \right) \Delta t \mathbb{E} \left[ |e_y^{n+1}|^2 + |e_z^{n+1}|^2 \right]
+ \left( 56c_0 L^2 \Delta t + \frac{56c_0 L^2}{\gamma} + \frac{12L^2 \Delta t}{\beta} \right) \Delta t \mathbb{E} \left[ |e_y^{n+1}|^2 + |e_z^{n+1}|^2 \right]
+ \left( 56c_0 L^2 \Delta t + \frac{56c_0 L^2}{\gamma} + \frac{12L^2 \Delta t}{\beta} \right) \Delta t \mathbb{E} \left[ |e_y^{n+1}|^2 + |e_z^{n+1}|^2 \right]
+ \left( 56c_0 L^2 \Delta t + \frac{56c_0 L^2}{\gamma} + \frac{12L^2 \Delta t}{\beta} \right) \Delta t \mathbb{E} \left[ |e_y^{n+1}|^2 + |e_z^{n+1}|^2 \right]
+ \left( 28c_0 L^2 \Delta t + \frac{28c_0 L^2}{\gamma} \right) \Delta t \mathbb{E} \left[ |e_y^{n+1}|^2 \right]
+ \left( 28c_0 L^2 \Delta t + \frac{28c_0 L^2}{\gamma} \right) \Delta t \mathbb{E} \left[ |e_y^{n+1}|^2 \right]
+ \frac{12(c_0)^2 \mathbb{E} \left[ |R_{xy}^n|^2 \right]}{\Delta t}
\]

This can be written as

\[
c_0 \mathbb{E} \left[ |e_y^n|^2 \right] + C_1 \Delta t \mathbb{E} \left[ |e_z^n|^2 \right] \leq c_0 (1 + C_2 \Delta t) \mathbb{E} \left[ |e_y^{n+1}|^2 \right] + C_2 \Delta t \mathbb{E} \left[ |e_z^{n+1}|^2 \right]
+ C_3 \Delta t \sum_{i \in A_n} \left( \mathbb{E} \left[ |e_y^{n+i}|^2 \right] + \mathbb{E} \left[ |e_z^{n+i}|^2 \right] \right) + R^n \tag{4.21}
\]

with the index set \(A_n = \{n_\delta, n_\delta + 1, (n + 1)_\delta + 1, (n + 1)_\delta + 2, (n + 1)_\zeta + 1, (n + 1)_\zeta + 2\}\), the constants

\[
C_1 = \frac{1}{4(1 + \beta)} - 28c_0 L^2 \left( \Delta t + \frac{1}{\gamma} \right),
C_2 = \gamma + 224L^2 \Delta t + \frac{224L^2}{\gamma} + \frac{12L^2 \Delta t}{c_0 \beta}.
\]
\[ C_2' = 224c_0 L^2 \Delta t + \frac{224c_0 L^2}{\gamma} + \frac{12L^2 \Delta t}{\beta}, \]
\[ C_3 = 56c_0 L^2 \Delta t + \frac{56c_0 L^2}{\gamma} + \frac{12L^2 \Delta t}{\beta}, \]

and the remainder
\[ R^n = \frac{12(c_0)^2}{\beta} \frac{E[|R^n_{xy}|^2]}{\Delta t} + 28c_0 L^2 \left( \Delta t + 1 \right) \frac{E[|R^n_{xy}|^2 + (\Delta t)^2 |\tilde{R}^n_{xy}|^2]}{\Delta t}. \]

We now choose \( \beta = 1 \), a sufficiently large \( \gamma \) and a sufficiently small \( \Delta t_0 \), and let \( \gamma \geq \gamma_0, \quad 0 < \Delta t \leq \Delta t_0 \) such that
\[ C_1 - C_2' - 6C_3 > C^* > 0. \]

Now it follows from (4.21) and Lemma 2.3 with \( N_0 = 6 \) that for any \( n = 0, 1, \ldots, N - 1 \) we have
\[ E[|\epsilon^n_{xy}|^2] + \Delta t \sum_{i=n}^{N-1} E[|\epsilon^n_{xy}|^2] \leq C \left( E[|\epsilon^n_{xy}|^2] + M_{xy} + M_{xz} + \sum_{i=0}^{N-1} R^i \right), \]
which leads to (4.9).

**Remark 4.1.** Theorem 4.1 means that Scheme 3.1 is stable.

### 5. Error Estimates

Let us respectively write \( \bar{Y}_t, \tilde{t}_n \) and \( \tilde{Y}_t, \bar{t}_n \) for \( Y_{t_n}, X^\xi \), \( Z_{t_n}, X^\xi \) and \( Y_{\bar{t}_n}, X^\xi \), i.e.
\[ \bar{Y}_t_n = Y_{t_n}(X^n), \quad \tilde{t}_n = Z_{t_n}(X^n), \quad \bar{t}_n = \tilde{Y}_{t_n}(X^n). \]

Analogously, if \( k_1 = n_\delta \) or \( n_\delta + 1 \) and \( k_2 = n_\zeta \) or \( n_\zeta + 1 \), we let
\[ \tilde{Y}_{t_n+k_1} = Y_{t_n+k_1}(X^n_\delta), \quad \tilde{Z}_{t_n+k_2} = Z_{t_n+k_2}(X^n_\zeta), \]
and \( \bar{Y}_{t_n} = \bar{Y}_{t_n}(X^n_\delta) \) if \( n_\delta = 0 \). Now we can consider the terms
\[ \bar{f}_{t_n} = f(t_n, X^n, \bar{Y}_{t_n}, \bar{Z}_{t_n}, X^n, \bar{Y}_{t_n}^L + \delta(t_n), \tilde{Z}_{t_n} + \zeta(t_n)), \]
\[ \bar{f}_{t_n,\bar{t}} = f(t_n, X^n, \bar{Y}_{t_n}, \bar{Z}_{t_n}, X^n, \bar{Y}_{t_n}^L + \delta(t_n), \tilde{Z}_{t_n} + \zeta(t_n)), \]
\[ \bar{f}_{t_n,\tilde{t}} = f(t_n, X^n, \bar{Y}_{t_n}, \bar{Z}_{t_n}, X^n, \bar{Y}_{t_n}^L + \delta(t_n), \tilde{Z}_{t_n} + \zeta(t_n)), \]

where
\[ \bar{Y}_{t_n} + \delta(t_n) = (1 - \delta_n) \bar{Y}_{t_n} + \delta_n \bar{Y}_{t_n+1}, \]
\[ \bar{Y}_{t_n} + \delta(t_n) = (1 - \delta_n) \bar{Y}_{t_n+1} + \delta_n \bar{Y}_{t_n+2}, \]
\[ \tilde{Z}_{t_n} + \zeta(t_n) = (1 - \zeta_n) \tilde{Z}_{t_n+2} + \zeta_n \tilde{Z}_{t_n+3}. \]
In order to proceed, we write the reference equations (3.15), (3.18) and (3.20) in the form

$$\frac{1}{2} \Delta t_n \tilde{Z}_t = \mathbb{E}_{t_n}^{X^n} \left[ \tilde{Y}_{t_{n+1}} \Delta \tilde{W}_n \right] + \Delta t_n \mathbb{E}_{t_n}^{X^n} \left[ \tilde{f}_{t_{n+1}} L \Delta \tilde{W}_n \right] + R_z^n + \tilde{R}_z^n,$$

$$\tilde{Y}_{t_n} = \mathbb{E}_{t_n}^{X^n} \left[ \tilde{Y}_{t_{n+1}} \right] + \Delta t_n \mathbb{E}_{t_n}^{X^n} \left[ \tilde{f}_{t_{n+1}} L \right] + \tilde{R}_{y r}^n,$$

$$\tilde{Y}_{t_n} = \mathbb{E}_{t_n}^{X^n} \left[ \tilde{Y}_{t_{n+1}} \right] + \frac{1}{2} \Delta t_n \mathbb{E}_{t_n}^{X^n} \left[ \tilde{f}_{t_{n+1}} L \tilde{Y}_{t_{n+1}} \right] + \tilde{R}_{y r}^n,$$

where $\tilde{R}_z^n, \tilde{R}_{y r}^n$ and $\tilde{R}_y^n$ are weak errors of the Itô-Taylor schemes used to solve SDEs — i.e.

$$\tilde{R}_z^n = \mathbb{E}_{t_n}^{X^n} \left[ (Y_{t_{n+1}} - \tilde{Y}_{t_{n+1}}) \Delta \tilde{W}_n \right] + \Delta t_n \mathbb{E}_{t_n}^{X^n} \left[ (f_{t_{n+1}}^{X^n} L - \tilde{f}_{t_{n+1}} L) \Delta \tilde{W}_n \right],$$

$$\tilde{R}_{y r}^n = \mathbb{E}_{t_n}^{X^n} \left[ Y_{t_{n+1}} - \tilde{Y}_{t_{n+1}} \right] + \frac{1}{2} \Delta t_n \mathbb{E}_{t_n}^{X^n} \left[ f_{t_{n+1}}^{X^n} L - \tilde{f}_{t_{n+1}} L \right],$$

$$\tilde{R}_y^n = \mathbb{E}_{t_n}^{X^n} \left[ Y_{t_{n+1}} - \tilde{Y}_{t_{n+1}} \right] + \frac{1}{2} \Delta t_n \mathbb{E}_{t_n}^{X^n} \left[ f_{t_{n+1}}^{X^n} L - \tilde{f}_{t_{n+1}} L \right] + \frac{1}{2} \Delta t_n \mathbb{E}_{t_n}^{X^n} \left[ (f_{t_{n+1}}^{X^n} L - \tilde{f}_{t_{n+1}} L) \sum_{n=0}^{\infty} \tilde{Y}_{t_{n+1}} \right] + \left( f_{t_{n+1}}^{X^n} L - \tilde{f}_{t_{n+1}} L \right) \sum_{n=0}^{\infty} \tilde{Y}_{t_{n+1}}.$$

Considering the solution $(Y_t, Z_t)$ of AFBSDE (2.1) at the point $(t_n, X^n)$ as the solution of Scheme 4.1, we define the numerical errors by

$$e_y^n = \tilde{Y}_{t_n} - Y^n, \quad e_z^n = \tilde{Z}_{t_n} - Z^n,$$

and replacing $(R_z^n, R_{y r}^n, R_y^n)$ in Theorem 4.1 by $(\tilde{R}_z^n + R_z^n, \tilde{R}_{y r}^n + R_{y r}^n, \tilde{R}_y^n + R_y^n)$ leads to the following result.

**Theorem 5.1.** Let $M_{y r} = \max_{N \leq i \leq N+M} \mathbb{E}[|e_y^n|^2]$ and $M_{y z} = \max_{N \leq i \leq N+M} \mathbb{E}[|e_z^n|^2]$. Then under the assumptions of Theorem 4.1, the inequalities

$$\mathbb{E} \left[ |e_y^n|^2 \right] + \Delta t \sum_{i=n}^{N-1} \mathbb{E} \left[ |e_z|^2 \right] \leq C \left( M_{y r} + M_{y z} + \frac{1}{\Delta t} \sum_{i=n}^{N-1} \mathbb{E} \left[ |R_y^i|^2 + |\tilde{R}_y^i|^2 + (\Delta t)^2 |\tilde{R}_{y r}^i|^2 + |R_z^i|^2 + |\tilde{R}_z^i|^2 \right] \right),$$

$$0 \leq n \leq N - 1$$

hold with a constant $C$ that depends on $c_0, L, T$ and $K$.

According to Theorem 5.1, we are left with estimating the truncation errors $R_z^n, R_{y r}^n$ and the errors $\tilde{R}_z^n, \tilde{R}_{y r}^n, \tilde{R}_y^n$ of the Itô-Taylor schemes for SDEs. To proceed, let us introduce additional assumptions — cf. [12]. Let $c_2^{2\beta+2}$ be set of all $2\beta + 2$ continuous differentiable functions with at most polynomial growth.
Assumption 5.1. The functions $b$ and $\sigma$ are jointly $L^2$-measurable in $(t, x) \in [0, T+K] \times \mathbb{R}^d$, uniformly Lipschitz continuous and linear growth bounded — i.e. there are constants $L_1 > 0, L_2 > 0$ such that
\[
|b(t, x)| + |\sigma(t, x)| \leq L_1(1 + |x|), \\
|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq L_2|x - y|,
\]
for all $t, s \in [0, T + K]$ and $x, y \in \mathbb{R}^d$.

Assumption 5.2. If $X^{n+1}$ is the approximation solution determined by Itô-Taylor schemes, then there are numbers $r_1, r_2, p, \gamma > 0$ such that for any $g \in C^2_p$ and $n = 0, 1, \ldots, N + M$ the following inequalities hold:
\[
\begin{align*}
|E[g(X_{t_n}) - g(X^n)]| &\leq C_g(\Delta t)^\beta, \\
|E^{X^n}_{t_n}[g(X^{t_{n+1}}_{t_n}) - g(X^{n+1})]| &\leq C_g(1 + |X^n|^{2r_1})(\Delta t)^{\beta + 1}, \\
|E^{X^n}_{t_n}[(g(X^{t_{n+1}}_{t_n}) - g(X^{n+1})) \Delta \hat{W}_n]| &\leq C_g(1 + |X^n|^{2r_2})(\Delta t)^{\gamma + 1}
\end{align*}
\]
with a constant $C_g > 0$ independent of $\Delta t$.

We note that number $\beta$ above is called the global weak convergence order of the scheme for SDEs.

Now we are ready to estimate the error terms of Theorem 5.1.

Lemma 5.1. If $b, \sigma \in C_b^{2,5}$, $\eta, \delta, \zeta \in C_b^2$, $f \in C_b^{2,5,5,5,5,5}$, $\varphi \in C_b^{2,5+\alpha}$ for an $\alpha \in (0, 1)$ and Assumption 5.1 holds, then for sufficiently small time step $\Delta t$ we have
\[
|R^n| \leq C(1 + |X^n|^4)(\Delta t)^3, \quad n = 0, 1, \ldots, N - 1,
\]
where constant $C$ depends on $L_1, L_2, T, K$ and on the upper bounds of the derivatives of $b$, $\sigma$, $\eta$, $\delta$, $\zeta$, $f$ and $\varphi$.

Proof. Without loss of generality, we assume that $\eta(t) = \delta(t) = \zeta(t)$ for $0 \leq t \leq T$. By Lemma 2.1, we have
\[
Y_t = u(t, X_t), \quad Z_t = (u'_x \sigma)(t, X_t),
\]
where $u$ is the smooth solution of the nonlocal PDE (2.3). Since $\mathcal{F}_{t_n} \subseteq \mathcal{F}_t$ for $t_n \leq s$, the Markov property gives
\[
E^{X_{t_n}}_{t_n}[f_{s, t_n, X}] = E^{X_{t_n}}_{t_n}\left[E\left[f(s, X_{s}^s, u(s, X_{s}^s), (u'_x \sigma)(s, X_{s}^s), X_{s}^s, X_{s+\delta(s)}^s) | \mathcal{F}_s\right]\right]
\]
\[
= E^{X_{t_n}}_{t_n}\left[E\left[f(s, p, u(s, p), (u'_x \sigma)(s, p), p + X_{s+\delta(s)}^s, p + X_{s+\delta(s)}^s) | \mathcal{F}_{s}\right]\right].
\]
Define the function $F(s, p) : [0, T] \times \mathbb{R} \to \mathbb{R}$ by

$$F(s, p) := \mathbb{E} \left[ f \left( s, p, u(s, p), (u'_x \sigma)(s, p), p + X^s p_{s+\delta(s)} \right) \right]$$

It follows from the conditions of Lemma 5.1 that

$$u \left( s + \delta(s), p + X^s p_{s+\delta(s)} \right), (u'_x \sigma)(s + \delta(s), p + X^s p_{s+\delta(s)}) \right)$$

where $\rho^p_{\delta}(\cdot)$ is the probability density function of the random variable $X^s p_{s+\delta(s)}$. Thus

$$\mathbb{E}^x_n \left[ f^s_{s+\delta(s)} \right] = \mathbb{E}^x \left[ F \left( s, X^s_{s+\delta(s)} \right) \right].$$

It follows from the conditions of Lemma 5.1 that $F(t, x) \in C^{2,4}_b$. Now the Eqs. (3.7), (5.6) and the Itô-Taylor expansion yield (5.5), cf. [23,29,33].

**Lemma 5.2.** If $b, \sigma \in C^{1,3}_b$, $\eta, \delta, \zeta \in C^{1,3}_b$, $f \in C^{1,3,3,3,3,3,3}_b$ and $\varphi \in C^{1,3+\alpha}_b$ for an $\alpha \in (0, 1)$, then for sufficiently small time step $\Delta t$ we have

$$|R^n_{y,2}| \leq C(\Delta t)^3, \quad n = 0, 1, \ldots, N - 1,$$

where constant $C$ depends on the upper bounds of the derivatives of $b, \sigma, \eta, \delta, \zeta, f$ and $\varphi$.

**Proof:** The inequalities (5.7) follow from (3.8),(3.10),(3.12) and the accuracy of the Lagrange interpolation method.

**Lemma 5.3.** If $b, \sigma \in C^{1,3}_b$, $\eta, \delta, \zeta \in C^{1,3}_b$, $f \in C^{1,3,3,3,3,3,3}_b$ and $\varphi \in C^{1,3+\alpha}_b$ for an $\alpha \in (0, 1)$, then for sufficiently small time step $\Delta t$, we have

$$|R^n_{y,3}| \leq C(\Delta t)^3, \quad n = 0, 1, \ldots, N - 1,$$

where constant $C$ depends on the upper bounds of the derivatives of $b, \sigma, \eta, \delta, \zeta, f$ and $\varphi$.

**Proof:** Using the Eqs. (3.13),(3.15) and (3.17), we get

$$\left| f^{t_{n+1}, L}_{t_n} - f^{t_{n+1}, L}_{t_n, \delta_0} 1_{\{\delta_0 = 0\}} - f^{t_{n+1}, L}_{t_n, \delta_1} 1_{\{\delta_0 \geq 1\}} \right| \leq 2C|\mathbb{R}^n_{y,3}|.$$
The cases for $\tilde{R}_{y_{t-1}}^n$ and using (5.3), we obtain

$$|R_{y_{t-1}}^n| \leq C(\Delta t)^2,$$  \hspace{1cm} (5.10)

and combining (3.19), (5.9) and (5.10) gives (5.8).

The terms $R_{z_{t-1}}^n, R_{z_{t-2}}^n$ and $R_{z_{t-3}}^n$ arising in Lemmas 5.4 and 5.5 can be estimated analogously — cf. [22, 33].

**Lemma 5.4.** Under the conditions of Lemma 5.1 and sufficiently small time step $\Delta t$, we have

$$|R_{z_{t1}}^n| \leq C(\Delta t)^3, \quad |R_{z_{t2}}^n| \leq C(\Delta t)^3, \quad n = 0, 1, \ldots, N - 1,$$

where constant $C$ depends on $L_1, L_2$ and on the upper bounds of the derivatives of $b, \sigma, \eta, \delta, \zeta, f$ and $\varphi$.

**Lemma 5.5.** Under the conditions of Lemma 5.2 and sufficiently small time step $\Delta t$, we have

$$|R_{z_{t3}}^n| \leq C(\Delta t)^{7/2}, \quad n = 0, 1, \ldots, N - 1,$$

where constant $C$ depends on $b, \sigma, \eta, \delta, \zeta, f$ and $\varphi$.

Now we can estimate the error terms arising in the Itô-Taylor schemes for SDEs.

**Lemma 5.6.** Under the conditions of Lemma 5.1, Assumption 5.2 and sufficiently small time step $\Delta t$, for all $n = 0, 1, \ldots, N - 1$ we have

$$\begin{align*}
\mathbb{E}
\left[
|\tilde{R}_{y_{t}}^n|^2
\right] &\leq C \left(1 + \mathbb{E}[|X^n|^{4\beta+1}]\right)(\Delta t)^{2\beta+2}, \\
\mathbb{E}
\left[
|\tilde{R}_{y_{t-1}}^n|^2
\right] &\leq C \left(1 + \mathbb{E}[|X^n|^{4\beta+1}]\right)(\Delta t)^{2\beta+2}, \\
\mathbb{E}
\left[
|\tilde{R}_{y_{t-2}}^n|^2
\right] &\leq C \left(1 + \mathbb{E}[|X^n|^{4\beta+1}]\right)(\Delta t)^{2\beta+2},
\end{align*}$$

where constant $C > 0$ does not depend on $\Delta t$.

**Proof.** Consider for instance the term $\tilde{R}_{y_{t}}^n$. By Assumption 5.2, the scheme for SDEs has the global convergence order $\beta$, so that

$$\max_{0 \leq k \leq N+M-n} \mathbb{E}^{X_n}_{t_k} \left| g(Y^n_{t_{n+k}} - g(X^n_{t_{n+k}}) \right| \leq C(\Delta t)^\beta$$

for any $g \in C_p^{2\beta + 2}$, cf. [12]. Since $\lim_{\Delta t \to 0} n_\beta = +\infty$, it follows from (3.10), (5.1) that

$$\begin{align*}
\mathbb{E}^{X_n}_{t_k} \left| Y^n_{t_{n+1}} - Y^n_{t_{n+1}} \right| &\leq C(\Delta t)^{\beta+1}, \\
\mathbb{E}^{X_n}_{t_k} \left| T^n_{t_{n+1}} - T^n_{t_{n+1}} \right| &\leq C(\Delta t)^{\beta},
\end{align*}$$

and using (5.3), we obtain

$$\mathbb{E}
\left[
|\tilde{R}_{y_{t}}^n|^2
\right] \leq C(\Delta t)^{2\beta+2}.$$

The cases for $\tilde{R}_{y_{t-1}}^n$ and $\tilde{R}_{y_{t-2}}^n$ can be examined analogously.

Theorem 5.1 and Lemmas 5.1 - 5.6 allow to establish error estimates of Scheme 3.1.
Theorem 5.2. If \( b, \sigma \in C^{2,5}_b \), \( \eta, \delta, \zeta \in C^2_b \), \( f \in C^{2,5,5,5,5,5,5}_b \), \( \varphi \in C^{2,5+}_b \) for an \( \alpha \in (0, 1) \), \( M_{ey} + M_{ez} \leq C(\Delta t)^4 \), and Assumptions 5.1 and 5.2 hold, then for sufficiently small time step \( \Delta t \), we have

\[
\mathbb{E}
\left[

|Y_{tn} - Y^n|^2
\right]
+ \Delta t \sum_{i=n}^{N-1} \mathbb{E}
\left[

|Z_{ti} - Z^i|^2
\right]
\leq C \left((\Delta t)^4 + (\Delta t)^{2\beta} + (\Delta t)^{2\gamma}\right),
\]

where constant \( C \) depends on \( c_0, L_1, L_2, L, T, K \) and on the upper bounds of the derivatives of \( b, \sigma, \eta, \delta, \zeta, f \) and \( \varphi \).

Remark 5.1. In Euler and Milstein schemes for SDEs, the parameters \( \beta \) and \( \gamma \) in Assumption 5.2 are equal to 1. By Theorem 5.2, yields

\[
\mathbb{E}
\left[

|e^n_y|^2
\right]
+ \Delta t \sum_{i=n}^{N-1} \mathbb{E}
\left[

|e^i|^2
\right]
\leq C(\Delta t)^2,
\]

hence Scheme 3.1 has the first order of accuracy. On the other hand, for weak order 2.0 Taylor scheme, we have \( \beta = \gamma = 2 \), which yields

\[
\mathbb{E}
\left[

|e^n_y|^2
\right]
+ \Delta t \sum_{i=n}^{N-1} \mathbb{E}
\left[

|e^i|^2
\right]
\leq C(\Delta t)^4,
\]

showing the second order of accuracy of Scheme 3.1.

6. Numerical Experiments

In this section, we carry out numerical experiments aimed to verify the theoretical conclusions. For simplicity, we use uniform time partition with the partition numbers \( N = T/\Delta t \) and \( M = K/\Delta t \). Fixing an \( h > 0 \), we consider the space partition

\[ S_h = S_{1,h} \times S_{2,h} \times \cdots \times S_{d,h}, \]

where \( S_{j,h} \) is the partition of the real axis \( \mathbb{R} \), i.e.

\[ S_{j,h} := \left\{ x_i^j : x_i^j = ih, i = 0, \pm 1, \ldots, \pm \infty \right\} \]

for \( j = 1, 2, \ldots, d \). Although \( S_h \) involves infinite number of grid points, we are usually interested in the values of \( (Y_0, Z_0) \) in a finite interval. Therefore, we can only consider a finite partition where \( |i| \leq P_0 \) for a positive integer \( P_0 \), which can be very large and problem dependent.
6.1. The approximation of conditional expectation

Using the Markov property, Monte Carlo method and Gauss-Hermite quadrature rule, we show how to approximate the conditional expectation $E_{t_n}^X[f^{n+1}]$. For example, let us consider $E_{t_n}^X[f^{n+1}]$, assuming $d = m = 1$ for simplicity. Without loss of generality, we set $\eta(t) = \delta(t) = \zeta(t)$ and $\delta_{n+1} = 0$. Let $m = (n + 1)\delta$, then $X^{n+1}_0 = X^{n+1}_0$. We also denote by $X^{t_{n+1},X}_{n+1}$ the approximation of the solution $X^{t_{n+1},X}_{n+1}$ of SDEs starting from $(t_{n+1}, x)$. By the conditional expectation properties, we get

\[
E_{t_n}^X[f^{n+1}] = E_{t_n}^X[E[f^{n+1} \mid \mathcal{F}_{t_{n+1}}]]
\]

\[
= E_{t_n}^X[E(f(t_{n+1}, X^{n+1}, Y^{n+1}, X^{n+1+1+m}, Y^{n+1+1+m}, Z^{n+1+m}) \mid \mathcal{F}_{t_{n+1}})]
\]

\[
= E_{t_n}^X[E(f(t_{n+1}, X^{n+1}, Y^{n+1}(X^{n+1}), Z^{n+1}(X^{n+1}), X^{t_{n+1},X^{n+1}}_{n+1+m},
Y^{n+1+1+m}(X^{t_{n+1},X^{n+1}}_{n+1+m}), Z^{n+1+m}(X^{t_{n+1},X^{n+1}}_{n+1+m}), X^{t_{n+1},X^{n+1}}_{n+1+m}),
X^{t_{n+1},X^{n+1}}_{n+1+m})]
\]

\[
\approx \frac{1}{M} \sum_{k=1}^{M} E_{t_n}^X[f(t_{n+1}, X^{n+1}, Y^{n+1}(X^{n+1}), Z^{n+1}(X^{n+1}), X^{t_{n+1},X^{n+1}}_{n+1+m},
Y^{n+1+1+m}(X^{t_{n+1},X^{n+1}}_{n+1+m}), Z^{n+1+m}(X^{t_{n+1},X^{n+1}}_{n+1+m}), X^{t_{n+1},X^{n+1}}_{n+1+m})]
\]

(6.1)

where $M$ is the sample time and $X^{t_{n+1},X^{n+1}}_{n+1+m}$ the numerical approximation of $X^{t_{n+1},X^{n+1}}_{n+1+m}$ at the $k$-th sampling. Let $g_k(t, x) : [0, T] \times \mathbb{R} \to \mathbb{R}$ and

\[
g_k(t_{n+1}, X^{n+1}) := f(t_{n+1}, X^{n+1}, Y^{n+1}(X^{n+1}), Z^{n+1}(X^{n+1}), X^{t_{n+1},X^{n+1}}_{n+1+m},
Y^{n+1+1+m}(X^{t_{n+1},X^{n+1}}_{n+1+m}), Z^{n+1+m}(X^{t_{n+1},X^{n+1}}_{n+1+m}), X^{t_{n+1},X^{n+1}}_{n+1+m})
\]

(6.2)

Applying the Euler scheme to SDEs, we write

\[X^{n+1} = X^n + b_n \Delta t_n + \sigma_n \Delta W_n,\]

and the Gauss-Hermite quadrature rule gives

\[
E_{t_n}^X[\int g_k(t_{n+1}, X^{n+1}, b_n \Delta t_n + \sigma_n \sqrt{\Delta t_n} s) e^{-s^2/2} ds]
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_R g_k(t_{n+1}, X^{n+1}, b_n \Delta t_n + \sigma_n \sqrt{2\Delta t_n} p) e^{-p^2} dp
\]

\[
\approx \sum_{j=1}^{M_f} g_k(t_{n+1}, X^{n+1}, b_n \Delta t_n + \sigma_n \sqrt{2\Delta t_n} a_j) w_j,
\]

(6.3)
where \( \{a_j\}_{j=1}^{M_g} \) are the roots of the Hermite polynomial of degree \( M_g \) and \( \{w_j\}_{j=1}^{M_g} \) the corresponding weights, — cf. [1]. The Eqs. (6.1)-(6.3) yield

\[
\mathbb{E}_{t_n}^X [f^{n+1}] \approx \frac{1}{M_c} \sum_{k=1}^{M_c} \sum_{j=1}^{M_g} g_k(t_{n+1}, X^n_j) w_j,
\]

where \( X^n_j = X^n + b_n \Delta t_n + \sigma_n \sqrt{2\Delta t_n} a_j \). In general, the points \( X^n_j \) and \( X_{n+1+m,k}^{t_n+1} \) are not located on the grid, so that interpolation methods have to be used. Let \( I_h \) be the local interpolation operator such that \( I_h g(x) = g(x) \) for \( x \in S_h \). Defining

\[
I_h g_k(t_{n+1}, X^n_j) := f(t_{n+1}, X^n_j, I_h Y^{n+1}(X^n_j), I_h Z^{n+1}(X^n_j), X_{n+1+m,k}^{t_n+1})
\]

and taking into account (6.4) and (6.5), we obtain

\[
\mathbb{E}_{t_n}^X [f^{n+1}] \approx \frac{1}{M_c} \sum_{k=1}^{M_c} \sum_{j=1}^{M_g} I_h g_k(t_{n+1}, X^n_j) w_j.
\]

**Remark 6.1.** The local error terms for Scheme 3.1 contain the time discrete error \((\Delta t)^3\), interpolation error \(h^{r+1}\), Monte Carlo approximation error \(1/\sqrt{M_c}\), and the Gauss-Hermite quadrature approximation error \((\Delta t)^{M_g}\), where \( r \) refers to the degree of the interpolation polynomials used. In order to adjust the errors from time and space discretisations in this scheme, the time step \( \Delta t \), space step \( h \) and the numbers \( M_c \) and \( M_g \) should be chosen such that all the terms \( h^{r+1}, 1/\sqrt{M_c}, (\Delta t)^{M_g} \) are balanced and dominated by \((\Delta t)^3\). Here we set \( r = 3, h = (\Delta t)^{3/4} \) and choose \( M_c \) and \( M_g \) such that the errors of spatial approximation can be neglected. There are three computational complexity terms for the approximation of conditional expectations \( \mathbb{E}_t^X[-] \), viz. the complexity of Monte Carlo method \( N M_c N^{3/4} \), the complexity of Gauss-Hermite quadrature \( (M_g N^{3/4})^{d} \) and the complexity of cubic spline interpolation \( (N^{3/4})^d \). It follows that the computational complexity of Scheme 3.1 is

\[
\Theta(M_c(M_g)^d N^{2+3/d(1+2d)}).
\]

For more details, the readers are referred to [16, 28].

### 6.2. Numerical tests

In our tests, we set the terminal time \( T = 1.0 \) and \( |Y_0 - Y^0| \) and \( |Z_0 - Z^0| \) denote the difference between the exact solution \((Y_t, Z_t)\) of the Eq. (2.1) at \( t = 0 \) and the numerical solution \((Y^n, Z^n)\) obtained by Scheme 3.1 at \( n = 0 \). The convergence rate (CR) with respect to the time step \( \Delta t \) is obtained by using the linear least square fitting of numerical errors.
Example 6.1. We consider the following anticipated FBSDE

\[ dX_t = bd_t + \sigma dW_t, \]

\[ -dY_t = \mathbb{E}\left[ \frac{1}{2} Y_t \sigma^2 - Z_t \frac{1 + b}{\sigma} - e^t \sin(t + X_t) \tan((1 + b) \delta(t)) \right] - \frac{e^{(\sigma^2/2)\delta(t)}Z_{t+\delta(t)} + e^{\delta(t)}\sin(t + \delta(t)) + X_t + b\delta(t))}{\sigma e^{-t} \cos((1 + b)\delta(t))} \mathbb{F}_t \] \[ dt - Z_t dW_t, \]

\[ Y_s = \sin(s + X_t) + e^t \cos(s + X_t), \quad s \in [T, T + K], \]

\[ Z_s = \sigma(\cos(s + X_t) - e^t \sin(t + X_t)), \quad s \in [T, T + K]. \]

It can be verified that the analytic solution yields second-order rate of convergence, consistent with the theoretical analysis.

Example 6.2. We now consider the anticipated FBSDE

\[ dX_t = \sin(t + X_t)dt + \cos(t + X_t)dW_t, \]

\[ -dY_t = \mathbb{E}\left[ \frac{1}{2} Y_t Z_t - \cos(t + X_t)(1 + \sin(t + X_t)) \right] + \frac{1}{20} \left( Y_{t+\delta(t)} + Z_{t+\delta(t)} - 1 \right)^3 \mathbb{F}_t \] \[ dt - Z_t dW_t, \]

\[ Y_s = \sin(s + X_t), \quad s \in [T, T + K], \]

\[ Z_s = \cos^2(s + X_t), \quad s \in [T, T + K]. \]

We test Scheme 3.1 for different functions of the above SDE is $X_t = X_0 + bt + \sigma W_t$. Thus, the errors caused by the Taylor schemes for solving SDEs vanish.

Table 1 shows numerical errors $|Y_0 - Y^0|$ and $|Z_0 - Z^0|$ and the convergence rates of Scheme 3.1 with the initial value $X_0 = 0.5$ and the coefficients $b = \sigma = 1$. We point out that Scheme 3.1 is stable and accurate for different functions $\delta(t)$. Moreover, it has the second-order rate of convergence, consistent with the theoretical analysis.

| $\delta(t) = 1/4$ | $\delta(t) = (1 + t^2)/8$ |
|------------------|------------------|
| $Na$ | $|Y_0 - Y^0|$ | $|Z_0 - Z^0|$ | $|Y_0 - Y^0|$ | $|Z_0 - Z^0|$ |
| 32 | 1.303E-03 | 3.944E-03 | 1.570E-03 | 3.165E-03 |
| 64 | 3.478E-04 | 9.964E-04 | 4.116E-04 | 7.931E-04 |
| 128 | 8.969E-05 | 2.506E-04 | 1.059E-04 | 1.989E-04 |
| 256 | 2.278E-05 | 6.283E-05 | 2.679E-05 | 4.974E-05 |
| 512 | 5.750E-06 | 1.573E-05 | 6.742E-06 | 1.243E-05 |
| CR | 1.958 | 1.993 | 1.967 | 1.998 |
Table 2: Example 6.2. Errors and convergence rates.

| N  | Euler | Milstein | Weak-2.0 |
|----|-------|----------|----------|
|    | $|Y_0 - Y^0|$ | $|Z_0 - Z^0|$ | $|Y_0 - Y^0|$ | $|Z_0 - Z^0|$ | $|Y_0 - Y^0|$ | $|Z_0 - Z^0|$ |
| 16 | 1.603E-02 | 9.473E-02 | 9.551E-03 | 7.698E-02 | 3.808E-04 | 1.265E-03 |
| 32 | 8.796E-03 | 5.030E-02 | 5.168E-03 | 4.029E-02 | 1.408E-04 | 3.601E-04 |
| 64 | 4.601E-03 | 2.594E-02 | 2.687E-03 | 2.062E-02 | 4.079E-05 | 9.418E-05 |
| 128| 2.353E-03 | 1.318E-02 | 1.370E-03 | 1.044E-02 | 1.092E-05 | 2.425E-05 |
| 256| 1.190E-03 | 6.645E-03 | 6.921E-04 | 5.251E-03 | 2.866E-06 | 6.614E-06 |
| CR | 0.941  | 0.960    | 0.949    | 0.970    | 1.780    | 1.905     |

It can be verified that the analytic solution yields

$$Y_t = \sin(t + X_t), \quad Z_t = \cos^2(t + X_t).$$

We now take $\delta(t) = \sqrt{2 - t^2}/16$ and $K = 1/16$. The Euler scheme (3.1), the Milstein scheme (3.2) and the weak order-2.0 Taylor scheme (3.3) are employed to solve SDEs. Table 2 shows the numerical errors $|Y_0 - Y^0|$ and $|Z_0 - Z^0|$ and the convergence rates of Scheme 3.1 with the initial value $X_0 = 0.0$. Thus for decoupled anticipated FBSDEs, Scheme 3.1 is accurate, stable and its accuracy depends on the Taylor schemes used to solve SDEs. In particular, Scheme 3.1 is convergent with order two when the weak order 2.0 Taylor scheme is used. This again confirms the theoretical analysis.

7. Conclusions

We propose an explicit scheme for solving decoupled anticipated forward backward stochastic differential equations. Rigorous stability analysis allows us to establish error estimates. They show that the method has the first order accuracy if the Euler or Milstein schemes are used and the second order accuracy if the weak order 2.0 Taylor scheme is used for solving SDEs. Numerical examples support the theoretical findings.

Acknowledgments

The authors would like to thank the referees for their valuable comments and suggestions which helped to improve this paper.

This research is partially supported by the Science Challenge Project (No. TZ2018001), by the National Key Research and Development Project (No. 2018YFA0703903), by the NSF of China (Nos. 11831010, 11871068), and by the China Postdoctoral Science Foundation (No. 2019TQ0073).

References

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover (1972).
[2] B. Bouchard, D. Possamai, X. Tan and C. Zhou, A unified approach to a priori estimates for supersolutions of BSDEs in general filtrations, Ann. Inst. Henri Poincare Probab. Stat. 54, 154–172 (2018).
[3] B. Bouchard, X. Tan, X. Warin and Y. Zou, Numerical approximation of BSDEs using local polynomial drivers and branching processes, Monte Carlo Methods Appl. 23, 241–263 (2017).
[4] L. Chen and Z. Wu, A type of generalized forward-backward stochastic differential equations and applications, Chin. Ann. Math. Ser. B 32, 279–292 (2011).
[5] S. Douissi, J. Wen and Y. Shi, Mean-field anticipated BSDEs driven by fractional Brownian motion and related stochastic control problem, Arxiv:1804.10482 (2019).
[6] Y. Fu, J. Yang and W. Zhao, Prediction-correction scheme for decoupled forward backward stochastic differential equations with jumps, East Asian J. Appl. Math. 6, 253–277 (2016).
[7] Y. Fu, W. Zhao and T. Zhou, Efficient spectral sparse grid approximations for solving multidimensional forward backward SDEs, Discrete Contin. Dyn. Syst. Ser. B 22, 3439–3458 (2017).
[8] T. Hao, Anticipated mean-field backward stochastic differential equations, Arxiv:1811.04359v1 (2018).
[9] P. Henry, N. Oudjane, X. Tan, N. Touzi and X. Warin, Branching diffusion representation of semilinear PDEs and Monte Carlo approximation, Ann. Inst. Henri Poincare Probab. Stat. 55, 184–210 (2019).
[10] J. Huang and J. Shi, Maximum principle for optimal control of fully coupled forward-backward stochastic differential delayed equations, ESAIM: Control, Optimisation and Calculus of Variations 18, 1073–1096 (2012).
[11] I. Kharroubi, N. Langrené and H. Pham, Discrete time approximation of fully nonlinear HJB equations via BSDEs with nonpositive jumps, Ann. Appl. Probab. 25, 2301–2338 (2015).
[12] P. E. Kloeden and E. Platen, Numerical Solution of Stochastic Differential Equations, Springer-Verlag (1992).
[13] B. Øksendal, Stochastic Differential Equations, Springer-Verlag (2003).
[14] B. Øksendal, A. Sulem and T. Zhang, Optimal control of stochastic delay equations and time-advanced backward stochastic differential equations, Adv. Appl. Prob. 43, 572–596 (2011).
[15] B. Øksendal, A. Sulem and T. Zhang, Optimal partial information control of SPDEs with delay and time-advanced backward SPDEs, Interdisciplinary Mathematical Sciences, 355–383 (2012).
[16] C. Pak, M. Kim and C. Rim, An efficient third-order scheme for BSDEs based on nonequidistant difference scheme, Numer. Algorithms (To appear).
[17] O. M. Pamen, Optimal control for stochastic delay systems under model uncertainty: a stochastic differential game approach, J. Optim. Theory Appl. 167, 998–1031 (2015).
[18] S. Peng and Z. Yang, Anticipated backward stochastic differential equations, Ann. Probab. 39, 877–902 (2009).
[19] D. Possamai, X. Tan and C. Zhou, Stochastic control for a class of nonlinear kernels and applications, Ann. Probab. 46, 551–603 (2018).
[20] Y. Sun, J. Yang and W. Zhao, Itô-Taylor schemes for solving mean-field stochastic differential equations, Numer. Math. Theor. Meth. Appl. 10, 798–828 (2017).
[21] Y. Sun and W. Zhao, New second-order schemes for forward backward stochastic differential equations, East Asian J. Appl. Math. 8, 399–421 (2018).
[22] Y. Sun and W. Zhao, An explicit second-order numerical scheme for mean-field forward backward stochastic differential equations, Numer. Algorithms 84, 253–283 (2020).
[23] Y. Sun, W. Zhao and T. Zhou, Explicit θ-scheme for solving mean-field forward backward stochastic differential equations, SIAM J. Numer. Anal. 56, 2672–2697 (2018).
[24] T. Tang, W. Zhao and T. Zhou, Deferred correction methods for forward backward stochastic
differential equations, Numer. Math. Theor. Meth. Appl. 10, 222–242 (2017).
[25] J. Yang, G. Zhang and W. Zhao, A first-order numerical scheme for forward-backward stochastic differential equations in bounded domains, J. Comput. Math. 36, 237–258 (2018).
[26] J. Yang, W. Zhao and T. Zhou, Explicit deferred correction methods for second-order forward backward stochastic differential equations, J. Sci. Comput. 79, 1409–1432 (2019).
[27] W. Zhao, L. Chen and S. Peng, A new kind of accurate numerical method for backward stochastic differential equations, SIAM J. Sci. Comput. 28, 1563–1581 (2006).
[28] W. Zhao, Y. Fu and T. Zhou, New kinds of high-order multistep schemes for coupled forward backward stochastic differential equations, SIAM J. Sci. Comput. 36, A1731–A1751 (2014).
[29] W. Zhao, Y. Li and L. Ju, Error estimates of the Crank-Nicolson scheme for solving backward stochastic differential equations, Int. J. Numer. Anal. Model 10, 876–898 (2013).
[30] W. Zhao, Y. Li and G. Zhang, A generalized θ-scheme for solving backward stochastic differential equations, Discrete Contin. Dyn. Syst. Ser. B 17, 1585–1603 (2012).
[31] W. Zhao, J. Wang and S. Peng, Error estimates of the θ-scheme for backward stochastic differential equations, Discrete Contin. Dyn. Syst. Ser. B 12, 905–924 (2009).
[32] W. Zhao, G. Zhang and L. Ju, A stable multistep scheme for solving backward stochastic differential equations, SIAM J. Numer. Anal. 48, 1369–1394 (2010).
[33] W. Zhao, W. Zhang and L. Ju, A numerical method and its error estimates for the decoupled forward-backward stochastic differential equations, Commun. Comput. Phys. 15, 618–646 (2014).
[34] W. Zhao, W. Zhang and L. Ju, A multistep scheme for decoupled forward-backward stochastic differential equations, Numer. Math. Theor. Meth. Appl. 9, 262–288 (2016).
[35] Y. Zhuang, Non-zero sum differential games of anticipated forward-backward stochastic differential delayed equations under partial information and application, Advances in Difference Equations 1, (2017).