Study of a Class of Four Dimensional Nonsingular Cosmological Bounces

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Abstract

We study a novel class of nonsingular time-symmetric cosmological bounces. In this class of four dimensional models the bounce is induced by a perfect fluid with a negative energy density. Metric perturbations are solved in an analytic way all through the bounce. The conditions for generating a scale invariant spectrum of tensor and scalar metric perturbations are discussed.

1 Introduction

Bouncing universes become attractive and fashionable periodically. Bouncing models can solve the horizon problem with a long period of slow contraction \((a \sim (-t)^p\) with \(0 < p < 1\)) as opposed to inflationary models which undergo a period of quasi-exponential expansion. Models with a bounce play an important role in string cosmology (see [1] and references therein). It is therefore interesting to study how metric perturbations are generated in bouncing models paying particular attention to what happens close to the bounce, even in a simplified model.

In this paper we study a hydrodynamical toy model where the bounce is induced by a negative energy perfect fluid. Negative energy density fluids seem ideas quite wild - or desperate [2], but they have already been considered as toy models [3, 4]. Moreover, negative energy fluid have been recently connected to brane models [5], where the interaction between bulk and brane can induce a

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negative energy component in the brane through the projection of the 5d Weyl
tensor.

A two field description of a bounce is also very pedagogical for the treat-
ment of cosmological perturbations through a bounce. In such a bounce, both
the Hubble parameter $H$ and its time derivative $\dot{H}$ cross zero. Only the latter
phenomena occurs in an expanding universe. Indeed, the Hubble parameter
changes slope during the transition from an accelerated to a decelerated expan-
sion (from inflation to the standard big bang era, i.e. during reheating) and
viceversa (from the matter dominated era to the present accelerated stage).
The vanishing of $H$ is instead a peculiarity of a bounce. It turns out that the
treatment of cosmological perturbations during a bounce is much more compi-
cated than during a stage which only includes $\dot{H} = 0$.

The paper is organized as follows. In section 2 we introduce a novel class of
bouncing universe. To our knowledge this class generalizes the case of a radiation
dominated universe where the bounce is obtained with a massless scalar field
with negative energy density [2, 3, 4]. In section 3 we study both in analytic
and numerical ways the evolution of gravitational waves in such a background.
The equation can be solved analytically with the use of spheroidal functions, as
Sahni did for gravitational waves in a radiation-matter transition [6]. In section
4 we describe the problems in treating cosmological scalar perturbations during
a bounce. In section 5 we study the evolution of scalar fluctuations and in
section 6 we conclude.

2 Background

We consider a bouncing universe which satisfies the Einstein equation:

$$H^2 = \frac{1}{3M_{pl}^2} \left[ \frac{\rho_+}{a^m} - \frac{\rho_-}{a^n} \right], \quad (1)$$

where $8\pi G = M_{pl}^{-2}$, $m < n$ and $\rho_+, \rho_-$ are constants. From this last relation it
is obvious that $\rho_-$ is important only close to the bounce. The two fluids have
ratios between pressure and energy densities $w_+ = m/3 - 1$ and $w_- = n/3 - 1$,
respectively.

By assuming

$$n = 2(m - 1) \rightarrow w_- = 2w_+ + \frac{1}{3} \quad (2)$$

the scale factor $a$ in terms of the conformal time is:

$$a(\eta) = \epsilon \left( 1 + \frac{\eta^2}{\eta_0^2} \right)^{\alpha} \quad (3)$$

with

$$\epsilon = \left( \frac{\rho_-}{\rho_+} \right)^\frac{1}{n-m} \quad (4)$$
\[ \alpha = \frac{1}{1 + 3w_+} = \frac{1}{n - m} = \frac{1}{m - 2} \quad (5) \]

\[ \eta_0^2 = 12\alpha^2 M_{pl}^2 \frac{\rho_+}{\rho_+^2} = \frac{12}{(n - m)^2} \frac{M_{pl}^2}{\rho_+} \epsilon^{n - m} \quad (6) \]

This class of solutions covers many different cases of physical contraction, each of them bounced in an expansion phase by a negative energy density fluid with a different equation of state. The parameter \( \epsilon \) regulates the magnitude of the scale factor at the bounce and the time scale \( \eta_0 \) sets the duration of the bounce. Contraction driven by dust, radiation, free scalar field and ultra-stiff equation of state \((w_+ >> 1)\) are given by \( \alpha = 1, 1/2, 1/4 \) and \( \alpha \sim 0 \), respectively.

Since \( w_+ > w_+ \) the bounce is unavoidable in Eq. (1) and it does not depend on initial conditions.

Important relations are:

\[ w_+ = \frac{1}{3\alpha} - \frac{1}{3}, \quad w_- = \frac{2}{3\alpha} - \frac{1}{3} \quad (7) \]

\[ \mathcal{H} = \frac{2\alpha \eta}{\eta^2 + \eta_0^2} = \frac{2\alpha x}{\eta_0(1 + x^2)} \quad (8) \]

\[ \mathcal{H}' = \frac{2\alpha}{\eta_0} \frac{\eta_0^2 - \eta^2}{(\eta^2 + \eta_0^2)^2} = 2\alpha \frac{1 - x^2}{\eta_0^2(1 + x^2)^2} \quad (9) \]

\[ \frac{a''}{a} = \mathcal{H}' + \mathcal{H}^2 = a^2 \frac{\mathcal{R}}{6} = \frac{2\alpha}{\eta_0^2(1 + x^2)^2} \left[ 1 + (2\alpha - 1)x^2 \right] \quad (10) \]

where \( x \equiv \eta/\eta_0 \). We observe that \( \rho_- \) cannot be simply related to the curvature of the spatial sections since \( m \geq 3 \) to avoid negative instability for fluid fluctuations \(^1\). However, the above solution can be extended to non flat spatial sections, as for the particular case \( \alpha = 1/2 \) \(^5\). The Hubble law in Eq. (1) in general is:

\[ H^2 = \frac{1}{3M_{pl}^2} \left[ \frac{\rho_+}{a^m} - \frac{\rho_+}{a^n} \right] - \frac{K}{a^2} \quad (11) \]

where \( K = 0, 1, -1 \) corresponds to flat, closed and open spatial sections. For \( K = 1 \) the solution for the scale factor is:

\[ a(\eta) = \left( \frac{\rho_+}{6M_{pl}^2} \right)^\alpha \left[ 1 - c_1 \cos \left( \frac{\eta}{\alpha} \right) \right]^\alpha \quad (12) \]

\(^1\)Scalar metric fluctuations have been studied during a bounce in \(^{[10]}\) for a universe filled by a massive scalar field with closed spatial sections.
Figure 1: The scale factor $a$ as function of $x$ for $\alpha = 1/4, 1/2, 3/4, 1$ (from bottom to up, $\epsilon = 1$).

Figure 2: The curvature term $a''/a$ as a function of $x$ for $\alpha = 1/2, 1$ and $\eta_0 = 1$. 
Figure 3: The curvature term $a''/a$ as a function of $x$ for $\alpha = 0.1, 0.2$ and $\eta_0 = 1$.

Figure 4: The curvature term $a''/a$ as a function of $\alpha$ and $x$. 

where \( c_1 = \sqrt{1 - 12M_{pl}^2 \rho_0 / \rho_+^2} \). For \( K = -1 \) the solution for the scale factor is:

\[
a(\eta) = \left( \frac{\rho_+}{6M_{pl}^2} \right)^{\alpha} \left[ c_2 \cosh \left( \frac{\eta}{\alpha} \right) - 1 \right]^\alpha
\]

(13)

where \( c_2 = \sqrt{1 + 12M_{pl}^2 \rho_0 / \rho_+^2} \). Also for the open and closed solutions \( \alpha \) is given by Eq. (5).

### 3 Gravitational Waves

As usual, the amplitude of gravitational waves \( h \) satisfies the following equation:

\[
h'' + 2H h' + k^2 h = 0.
\]

(14)

By neglecting \( k^2 \) the infinite wavelength solution is:

\[
h \sim A + B \int \frac{d\eta}{a^2} = A + \eta_0 B x F_1 \left( \frac{1}{2} ; 2 \alpha ; \frac{3}{2} ; -x^2 \right)
\]

(15)

In particular, for a dust dominated collapse (\( \alpha = 1 \)) we get

\[
h_{\text{mat}} \sim A_{\text{mat}} + B_{\text{mat}} \frac{\eta_0}{e^2} \left( \frac{x}{2(x^2 + 1)} + \frac{\arctan x}{2} \right),
\]

(16)

and for a radiation dominated collapse (\( \alpha = 1/2 \))

\[
h_{\text{rad}} \sim A_{\text{rad}} + B_{\text{rad}} \frac{\eta_0}{e^2} \arctan x,
\]

(17)

Instead, for a contraction driven by a free scalar field, we get:

\[
h_{\phi} \sim A_{\phi} + B_{\phi} \frac{\eta_0}{e^2} \arcsinh x.
\]

(18)

The presence of \( e^2 \) at the denominator indicates that the amplitude of gravitational waves blows up in the limit of singular bounces, i.e. \( \epsilon \to 0 \). This fact is not surprising and is related to the normalization condition between the two independent solutions for the modes \( W(h, h^*) = i/a^2 \). Fluctuations become singular in a singularity.

It is interesting to note that the B mode (odd with respect to \( x \to -x \)) tends to a constant far from the bounce, only for \( \alpha \geq 1/2 \). For \( \alpha < 1/2 \) the amplitude of gravitational waves grows outside the Hubble radius, even in the expanding phase. This result is not completely surprising, since for slow expansion (\( a(\eta) \propto (\eta)^p \) with \( p < 1/2 \)), free scalar fields (and henceforth gravitational waves) are a superposition of a constant and growing mode (but
Figure 5: Behaviour of a long-wavelength gravitational wave across the bounce. The plots are for $\alpha = .1, .25, .5, 1$ (from bottom to top).

proportional to $k\eta$). Here one has this effects for a different range of parameters, probably because of the bounce.

These results indicate that it is potentially important to know the $k$ dependence of $B$, in order to predict the spectrum of gravitational waves far from the bounce. The contribution of the $B$ mode to $h$ right at the bounce is zero in this infinite wavelength approximation.

With the evolution of the scale factor in Eq. (3), the equation for $h$ can be solved analytically. The trick is to put the above equation in the form of the differential equation of spheroidal wave function \cite{7}. It is useful to introduce then:

$$x = \eta/\eta_0 \quad \tilde{h} = h(1 + x^2)^{\alpha - 1/2}. \quad (19)$$

After a little algebra one gets:

$$\frac{d^2 \tilde{h}}{dx^2} + \frac{2x}{1 + x^2} \frac{d\tilde{h}}{dx} + \left[ k^2 \eta_0^2 - \frac{2\alpha(2\alpha - 1)}{1 + x^2} + \frac{(2\alpha - 1)^2}{(1 + x^2)^2} \right] \tilde{h} = 0. \quad (20)$$

We note that the above equation for $k\eta_0 = 0$ reduces to the Legendre differential equation \cite{7, 8}. The solution to Eq. (20) is given by the radial oblate spheroidal function \cite{8}. The above equation can be obtained by the transformation $z \rightarrow \pm ix$ in the equation in prolate coordinates \cite{7, 8, 6}:

$$(1 - z^2) \frac{d^2 \tilde{h}}{dz^2} - 2z \frac{d\tilde{h}}{dz} + \left[ \gamma^2 (1 - z^2) + \nu(\nu + 1) - \frac{\mu^2}{(1 - z^2)} \right] \tilde{h} = 0, \quad (21)$$

where $\mu, \nu, \theta$ are numerical coefficients. In many books only the case with integer $\mu, \nu$ is treated \cite{7, 8}, however $\mu, \nu, \gamma$ can be even complex numbers \cite{9}.

The solution for $h$ is:

$$h = c \sqrt{\eta_0 \gamma} (1 + x^2)^{1/2 - \alpha} S_{2\alpha - 1}^{2\alpha - 1}(3)(x, \gamma). \quad (22)$$
where (the following series is convergent only when $|z| > 1$ [9], to have convergence at all times one needs a different expression):

\[
S^{\mu (3)}_{\nu}(x, \gamma) = \left( \frac{\pi}{2\gamma x} \right)^{1/2} \left( \frac{x^2}{x^2 + 1} \right)^{\mu/2} \sum_{r=0}^{\infty} \frac{a^\mu_{\nu, r}(\gamma)}{S^\nu(\gamma)} H^{(1)}_{\nu + 1/2 + 2r}(\gamma z) \tag{23}
\]

\[
\gamma = -ik\eta_0, \quad z = -ix, \quad \gamma z = -k\eta \tag{24}
\]

\[
\mu = \nu = 2\alpha - 1 \tag{25}
\]

The normalization factor is fixed by requiring that for $x \to \infty$ (and then $|k\eta| \to \infty$):

\[
h \sim e^{-ik\eta} \frac{a\sqrt{2k}}{a}. \tag{26}
\]

\[
\lim_{x \to \infty} S^{\mu (3)}_{\nu}(x, \gamma) = \left( \frac{\pi}{2\gamma x} \right)^{1/2} H^{(1)}_{\nu + 1/2}(\gamma z) \tag{27}
\]

In order to understand the behaviour for large wavelengths we follow Sahni [6] and we take the limit for $\gamma \to 0$ [7]:

\[
\lim_{\gamma \to 0} S^{\mu (3)}_{\nu}(x, \gamma) = \gamma^{\nu/2} \left( \#_1 P^\mu_{\nu}(x) + \#_2 Q^\mu_{\nu}(x) \right) + \frac{\#_3}{\gamma^{\nu}+1} Q^\mu_{\nu}(x) \tag{28}
\]

where $P^\mu_{\nu}(x)$, $Q^\mu_{\nu}(x)$ are respectively the associate Legendre functions of the first and second kind [7] and $\#_1$, $\#_2$, $\#_3$ are numerical coefficients. Gravitational waves have finite amplitude right at the bounce since $P^\mu_{\nu}(0)$, $Q^\mu_{\nu}(0)$ are finite.

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**Figure 6:** Behaviour of the phase of a long-wavelength gravitational wave across the bounce for $\alpha = 1$. 

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\[\text{This limit identifies the infrared modes at the bounce, i.e. } k^2 << a''/a \text{ at } \eta \sim 0.\]
Figure 7: A snapshot of the gravitational wave amplitude as a function of $\log_{10}k\eta_0$ at $x = 30$ for $\alpha = 1/2$. The other detail of the simulation are $x_i = 10^5$, $\epsilon = 10^3$ and a initial vacuum spectrum for gravitational waves. The linear fit is $-0.8 - 0.5(\log_{10}k\eta_0 + 3.4)$.

(see Eqs. 8.6.1 and 8.6.2 in [8]). This leads to a spectrum for gravitational waves in the infrared:

$$\lim_{\theta \rightarrow 0} |h|^2 \sim \frac{1}{k^{2\alpha+1}}.$$  

The above result is in formal agreement with Sahni [6].

We note that the phases of gravitational waves are not constant while outside the Hubble radius, as shown in Fig. (6). Phases are locked to a constant value when the wavelength of a mode exceeds the Hubble radius during inflation: this phenomenon has been dubbed "decoherence without decoherence" [11]. Models based on a contraction which aim to produce a scale invariant spectrum seem to not have this peculiarity.

The final spectrum of gravitational waves far from the bounce is scale invariant for a dust contraction ($\alpha = 1$), and it is a vacuum spectrum for a radiation contraction ($\alpha = 1/2$), as we can see from Figs. (7,8).

4 Equations for Scalar Perturbations

We now review the equations of motion of scalar perturbations in the longitudinal gauge [12] with two hydrodynamical fluids. The energy and momentum constraints are respectively:

$$-\Delta \Phi + 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 (\delta\rho_+ - \delta\rho_-)$$

$$\Phi' + \mathcal{H}\Phi = 4\pi G a^2 \frac{k}{k} [\rho_+(1 + w_+ v_+) - \rho_-(1 + w_- v_-)$$

The spatial part of the Einstein equations lead to:

$$\Phi'' + 3\mathcal{H}\Phi' + [2\mathcal{H}' + \mathcal{H}^2] \Phi = 4\pi G a^2 (\delta p_+ - \delta p_-)$$
Figure 8: A snapshot of the gravitational wave amplitude as a function of $\log_{10} k_{\eta 0}$ at $x = 30$ for $\alpha = 1$. The other details the same of the previous figure. The linear fit is $2.78 - 1.5(\log_{10} k_{\eta 0} + 3.4)$.

Usually Eq. (30) times the total sound speed $c_s^2$:

$$c_s^2 = \frac{\dot{\rho}}{\rho} = \frac{\dot{p} - \dot{\rho}}{\rho + \dot{\rho}}$$  \hspace{1cm} (33)

is subtracted by Eq. (32) to obtain:

$$\Phi'' + 3(1 + c_s^2)H\Phi' + \left[c_s^2 k^2 + 2H' + H^2 + 3c_s^2 H^2\right] \Phi = 4\pi G a^2 \delta S,$$  \hspace{1cm} (34)

where $\delta S$ are the total non adiabatic pressure perturbations:

$$\delta S = \sum_i \delta p_i - c_s^2 \sum_i \delta \rho_i.$$  \hspace{1cm} (35)

In problems with a bounce the sound speed $c_s^2$ as defined in Eq. (33) becomes singular 3. This is connected with the violation of the null energy condition (henceforth NEC) $\rho + p \geq 0$, which occurs not right at the bounce ($\eta = 0$), but when

$$\dot{H} = -\frac{1}{2M_{\text{pl}}^2} (\rho + p) = -\frac{1}{2M_{\text{pl}}^2} \sum_i (1 + w_i) \rho_i,$$  \hspace{1cm} (36)

vanishes, i.e. when $\eta = \eta_0/2$ in this class of models.

This problem has been already encountered in the study of cosmological perturbations during reheating [14]. In a scalar field driven cosmology $\dot{H}$ vanishes when the field bounces, i.e. $\dot{\phi} = 0$, otherwise it remains always negative. The equation for $\Phi$ has singularities [14]. In [14, 15] the problem was approached 3Recently a paper which studies numerically the spectrum of $\Phi$ fluctuations during a fictitious bounce driven by a single perfect fluid appeared [13]. We think that the toy model used in [13] is inconsistent. Infact the equation of motion of $\Phi$ used in [13] contains a regular and ad hoc quantity which substitues the physical $c_s^2$, which becomes singular during the bounce.
by using the Mukhanov variable $Q(= \delta \dot{\phi} + \dot{\phi} \Phi / H)$ whose evolution is regular during the oscillation of a scalar field, despite the NEC violation. The regularity of the equations which involve the $Q$s persist also in the multifield case [15].

When a bounce is present the equations governing the evolutions of the $Q$s become singular in $H = 0$ (see [15] for scalar fields). This would be true also for perfect fluids. In fact, for a system of perfect fluids characterized by $w_F$ and $c_{F}^2$, the Mukhanov variables $Q_F$s would satisfy:

$$\ddot{Q}_F + 3H\dot{Q}_F + \left[ c_{F}^2 \frac{k^2}{a^2} + \frac{3}{2} H(1 - w_F) + \frac{9}{4} H^2(1 - w_F^2) \right] Q_F = \frac{1}{H} \times (Q_s, \dot{Q}_s \text{ of other components}).$$

We note that in presence of just one component, only a dust contraction can generate a scale invariant spectrum, as for the scalar field case [16].

We therefore turn to different variables in the next section.

5 Evolution of Scalar Perturbations

In the appendix of [4] Peter and Pinto-Neto suggest a possible way to deal with metric perturbations during a bounce. They suggest to split the Newtonian potential $\Phi$ in two components

$$\Phi = \Phi_+ + \Phi_-$$

where each one satisfies:

$$\Phi_i'' + 3(1 + w_i)\mathcal{H}\Phi_i' + \left[ w_i k^2 + 2\mathcal{H} + \mathcal{H}^2 + 3w_i\mathcal{H}^2 \right] \Phi_i = 0$$

with $i = +, -$. This splitting is valid under the assumption that non adiabatic pressure perturbations in the two components are absent:

$$\delta p_i = c_i^2 \delta \rho_i \quad \text{with} \quad c_i^2 = w_i$$

It is important to note that the above equation(s) are regular and uncoupled.

It is very interesting to see that the contribution of $\Phi_-$ to $\Phi$ decays in the expanding phase. The reason for this behaviour is that $\Phi_-$ does not have a constant mode in the long wavelength limit far from the bounce, as instead $\Phi_+$ does. In order to see this, it is sufficient to evaluate

$$2\mathcal{H}' + \mathcal{H}^2 + 3w_i\mathcal{H}^2 = -3(w - w_i)\mathcal{H}^2 .$$

For $i = -$ we get:

$$a^2 \frac{\rho_{+}}{3M_{pl}^2}(n - m) ,$$
Figure 9: Behaviour of a long-wavelength mode of $\Phi_-$ across the bounce. After a transient decay around the bounce, $\Phi_-$ does not stabilize at a constant value, as also seen analytically. The plots are for $\alpha = 0.1, 0.25, 0.5, 1$ (from bottom to top after the bounce).

which is a strictly positive quantity at all times. For $i = +$ we get a vanishing quantity at leading order far from the bounce:

$$w_+ \left( 3H^2 - a^2 \frac{\rho_+}{M_{pl}^2} \right) + w_- a^2 \frac{\rho_-}{M_{pl}^2}.$$

The above result implies that the growing mode of $\Phi_-$ during the contracting phase does not match to the growing mode of $\Phi$ in the expanding phase. This result is confirmed by numerical analysis, as shown in Fig. (9).

Therefore, the $-\$ component is important only close to the bounce both at the background level and at the linear level.

We now turn to the analysis of $\Phi_+$ and we repeat the previous treatment for long wavelengths. For $i = +$ we get a vanishing frequency at leading order far from the bounce:

$$w_+ \left( 3H^2 - a^2 \frac{\rho_+}{M_{pl}^2} \right) + w_- a^2 \frac{\rho_-}{M_{pl}^2}.$$

Now we would like to recast Eq. (39) in a spheroidal equation form. By introducing:

$$u_i = (1 + x^2)^{-\delta} \Phi_i,$$

where $\delta = 1/2 - 3\alpha(1 + w_i)/2$. The equation for $u_i$ is:

$$u_i'' + \frac{2x}{1 + x^2} u_i' + \left[ w_i k^2 \eta_0^2 + \frac{b_i}{1 + x^2} + \frac{c_i}{(1 + x^2)^2} \right] u_i = 0. \quad (46)$$

where

$$b_i = 4\alpha(\alpha(1 + 3w_i) - 1) + 6\alpha\delta(1 + w_i). \quad (47)$$
Figure 10: Behaviour of a long-wavelength mode of $\Phi_+$ across the bounce. After a transient decay around the bounce, $\Phi_+$ stabilizes at a constant value, as also seen analytically. The plots are for $\alpha = 0.1, 0.25, 0.5, 1$ (from bottom to top after the bounce).

\begin{equation}
c_i = 4\alpha(1 + 2\delta) + 4\delta^2
\end{equation}

When $i = +$, we get $b_+ = -2\alpha(2\alpha + 1)$, $c_+ = 4\alpha(1 + \alpha)$, leading to:

\begin{equation}
\nu_+ = 2\alpha \quad \mu_+ = \sqrt{4\alpha(1 + \alpha)}.
\end{equation}

The solution is therefore

\begin{equation}
\Phi_+ = \frac{N}{(1 + x^2)^{\alpha}} S_{2\alpha}^{(1)}(x, \gamma)
\end{equation}

where $N \sim \mathcal{O}(\gamma^{-1/2})$ is fixed by imposing:

\begin{equation}
\Phi \sim \frac{1}{k^{3/2}}
\end{equation}

on short wavelengths far from the bounce. Indeed,

\begin{align*}
\lim_{-k\eta \to \infty} \lim_{x \to \infty} \Phi_+ &= \lim_{-k\eta \to \infty} \lim_{x \to \infty} \frac{N}{(1 + x^2)^{\alpha}} \left( \frac{\pi}{2\gamma x} \right)^{1/2} H_{2\alpha + 1/2}^{(1)}(\gamma x) \\
&= \frac{N}{(1 + x^2)^{\alpha}} \left( \frac{1}{\gamma^2 x^2} \right)^{1/2} e^{-ik\eta}
\end{align*}

For large wavelengths, by using Eq. (29) one has:

\begin{equation}
\lim_{\gamma \to 0} |\Phi_+|^2 \sim \frac{1}{k^{3+4\alpha}}.
\end{equation}

A nearly scale invariant spectrum for $\Phi_+$ emerges in the expanding phase for $\alpha \sim 0$. For all the other relevant values of $\alpha$ the spectrum for $\Phi_+$ in the expanding phase seems too red to agree with observations (for $\alpha = 1/2$ $\Phi_+ \sim k^{-5/2}$ in agreement with [4]).
6 Discussions and Conclusions

We have studied cosmological perturbations through a non-singular bounce in a simple class of with two perfect fluid toy models. A normal fluid drives the contraction/expansion, while a negative energy density fluid models the bounce. The equations of state of the two fluids are linked in a particular way described by Eq. (2).

The study of scalar metric fluctuations during a non-singular bounce is rather tricky, but it can be approached, even analytically, as this example shows. Following [4], one could find variables whose evolution equation is regular through the bounce. We do not think that the approach proposed in [4] is unique in order to evolve cosmological scalar perturbations through a bounce. One could define a fictitious total sound speed \( \tilde{c}_s^2 \) which is regular through the bounce and asymptotically behaves like the sound speed of the dominant component. There is a possibility that in this way also the equation of motion for isocurvature perturbations is regularized. It remains to study the coupling of \( \Phi \) to isocurvature perturbations case by case. We observe that this trick of a fictitious total sound speed has been used in [13] in order to evolve \( \Phi \) and ignoring isocurvature perturbations. This of course is equivalent to study only half of the system and neglect the possible interaction between isocurvature and adiabatic mode.

The metric fluctuations induced by the fluid which drives the contraction \( \Phi_+ \) go through the bounce and seed the constant (growing) mode in the expansion. The metric fluctuations induced by the exotic fluid \( \Phi_- \) are important only close to the bounce and decay with time in the expanding phase. However, we note that the latter have a spectrum which is steeper in the infrared than the constant mode. This means that the largest piece of \( \Phi \) in the contracting phase matches to a decaying mode in the expanding phase [21]. The picture which emerges seems the following: in presence of a second entity which is responsible for the bounce, both the growing and decaying mode of metric fluctuations seeded by the dominant fluid which drives the contraction match to the growing mode of \( \Phi \) in the expanding phase.

In the expanding phase, after the bounce, gravitational waves have a scale invariant spectrum for a dust contraction as originally discovered by Starobinsky [17]. This is also what we find here (see also [16, 18]). In this two field model, after the bounce, scalar fluctuations with a (nearly, slightly red-tilted) scale invariant spectrum are generated by a very slow contraction, corresponding to a contracting fluid with an ultra-stiff equation of state (\( w_+ \gg 1 \)). In this picture, the scale invariant spectrum for sub-Hubble metric fluctuations \( \Phi \) is adiabatically transferred to super-Hubble wavelengths by the slow contraction, but remains imprinted in the subsequent expanding phase because of the bounce induced by a second entity. The amplitude of \( \Phi_+ \) in agreement with observations could be obtained by tuning the parameters of the model. In the same way \( \Phi_- \), hopefully, should be kept in the linear regime during the bounce. This possibility could be of interest for the Ekpyrotic scenario [19], without the need
of a singularity. The model obtained would be similar to the idea of hybrid inflation in the context of inflationary models. As in hybrid inflation [20] one field drives inflation and the other terminates the accelerating stage, in this case one component would drive the contraction and the other would be responsible for the bounce.

It is possible that the above conclusions are due to the particular relation among the equations of state of the two fluids or to the time symmetry of the model, but this seems unlikely. The absence of intrinsic non adiabatic pressure perturbations (i.e. the assumption of two perfect fluids) may be a more important issue. The danger could be a coupling (it may be just of gravitational origin) of $\Phi_-$ to $\Phi_+$. In such a case a scale invariance of metric fluctuations in the expanding phase may be very difficult to obtain. In any case this toy model shows how the physics at the bounce can alter the predictions about cosmological perturbations also in the PBB scenario [1].

It remains to connect this study with a prescription for matching perturbation through a non-singular bounce in a multifield case.

**Note Added:** While this paper was almost completed, two papers which studied metric perturbations during a bounce appeared [22, 23]. Gasperini, Giovannini and Veneziano [22] studied a single field non singular bounce induced by a non local potential and found that the growing mode of $\Phi$ in the contracting phase (not supported by a growing mode of $\zeta$) does not match to the growing of $\Phi$ in the expanding phase [24, 21]. Previous studies with a bounce induced by loop corrections (with or without a local potential) [25, 26] agree with [22]. This result could be related to the single field assumption. Tolley, Turok and Steinhardt [23] studied cosmological perturbations through a singular bounce.

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