ROOT SYSTEMS AND DIAGRAM CALCULUS.
III. SEMI-COXETER ORBITS OF LINKAGE DIAGRAMS AND THE CARTER
THEOREM

RAFAEL STEKOLSHCHIK

Abstract. A diagram obtained from the Carter diagram $\Gamma$ by adding one root together with
its bonds such that the resulting subset of roots is linearly independent is said to be the linkage
diagram. Given a linkage diagram, we associate the linkage labels vector, which is introduced
like the vector of Dynkin labels. Similarly to the dual Weyl group, we introduce the group $W_\Lambda^\vee$
associated with $\Gamma$, and we call it the dual partial Weyl group. The linkage labels vectors connected
under the action of $W_\Lambda^\vee$ constitute the linkage system $\mathcal{L}(\Gamma)$, which is similar to the weight system
arising in the representation theory of the semisimple Lie algebras. The Carter theorem states that
every element of a Weyl group $W$ is expressible as the product of two involutions. We give the
proof of this theorem based on the description of the linkage system $\mathcal{L}(\Gamma)$ and semi-Coxeter orbits
of linkage labels vectors for any Carter diagram $\Gamma$. The main idea of the proof is based on the fact
that, with a few exceptions, in each semi-Coxeter orbit there is a special linkage diagram – called
unicolored, for which the decomposition into the product of two involutions is trivial.

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References
Gelfand requested that I review the H. Weyl - Van der Waerden papers on semisimple Lie groups. I found them very difficult to read, and I tried to find my own ways. It came to my mind that there is a natural way to select a set of generators for a semisimple Lie algebra by using simple roots (i.e., roots which cannot be represented as a sum of two positive roots). Since the angle between any two simple roots can be equal only to $\pi/2, 2\pi/3, 3\pi/4, 5\pi/6$, a system of simple roots can be represented by a simple diagram. An article was submitted to Matematicheskii Sbornik in October 1944, [Dy46]. Only a few years later, when recent literature from the West reached Moscow, I discovered that similar diagrams have been used by Coxeter for describing crystallographic groups.

E. B. Dynkin, Foreword in “Selected papers of E. B. Dynkin with commentary ”, [Dy00] p. 2

1. Introduction

1.1. The Carter theorem. In the present paper we give the proof of the Carter theorem on the decomposition of every element of any Weyl group $W$ into the product of two involutions.

**Theorem 1.1** ([Ca72], Theorem C). The following equivalent statements hold for the Weyl group:

(i) Every element of a Weyl group $W$ is expressible as the product of two involutions.

(ii) Every element of $W$ is contained in some dihedral subgroup.

(iii) For each element $w \in W$ there is an involution $i \in W$ such that $iwi = w^{-1}$.

**Corollary 1.2.** Every element of $W$ is conjugate to its inverse.

In [Sp74], Springer gives a proof of the Carter theorem for all finite Coxeter groups including the non-crystallographic cases $I_n$ (dihedral group), $H_3$ and $H_4$. Springer deduced the proof from the classification of so-called regular elements in the Coxeter groups and by inspection from the known character tables of the irreducible Weyl groups. [Sp74] §8.6, §8.7.

The proof given by Carter in [Ca72] uses the calculation of all conjugacy classes in the Weyl group. Our proof uses the classification of linkage systems and semi-Coxeter orbits for every Carter diagrams. The definitions of linkage systems and semi-Coxeter orbits will be given below in Section 1.2. The linkage systems for Carter diagrams from $C_4 \sqcup DE_4$ are presented in [St10.II], where $C_4$ is the class of Carter diagrams, each of which contains 4-cycle $D_4(a_1)$ as a subdiagram, and $DE_4$ is the class of Carter diagrams, each of which contains $D_4$ as a subdiagram, see Section 1.2.4. In this paper, we give the complete description of semi-Coxeter orbits for any $\Gamma \in C_4 \sqcup DE_4$. The linkage systems and semi-Coxeter orbits for $A_l$ and $B_l$ will be presented in [St11].

1.2. The Carter diagrams and semi-Coxeter elements.

1.2.1. Solid and dotted edges. The Carter diagram (= admissible diagram) [Ca72] §4] is the diagram $\Gamma$ satisfying two conditions:

(a) The nodes of $\Gamma$ correspond to a set of linearly independent roots.

(b) Each subgraph of $\Gamma$ which is a cycle contains even number of vertices.

Let $w = w_1w_2$ be the decomposition of $w$ into the product of two involutions. By [Ca72, Lemma 5] each of $w_1$ and $w_2$ can be expressed as a product of reflections corresponding to mutually
orthogonal roots:

\[ w = w_1 w_2, \quad w_1 = s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_k}, \quad w_2 = s_{\beta_1} s_{\beta_2} \ldots s_{\beta_h}, \quad \text{where} \quad k + h = l_C(w). \]  

(1.1)

For details, see \cite{Ca72, §4}, \cite{St10.I, §1.1}. We denote by \( \alpha\text{-set} \) (resp. \( \beta\text{-set} \)) the subset of roots corresponding to \( w_1 \) (resp. \( w_2 \)):

\[ \alpha\text{-set} = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}, \quad \beta\text{-set} = \{\beta_1, \beta_2, \ldots, \beta_h\}. \]  

(1.2)

Any coordinate from \( \alpha\text{-set} \) (resp. \( \beta\text{-set} \)) of the linkage labels vector we call \( \alpha\text{-label} \) (resp. \( \beta\text{-label} \)). The decomposition (1.1) is said to be the \textit{bicolored decomposition}.

For the Dynkin diagrams, a number of bonds for non-orthogonal roots describes the angle between roots, and the ratio of lengths of two roots. For the Carter diagrams, we add designation distinguishing acute and obtuse angles between roots. Recall, that for the Dynkin diagrams, all angles between simple roots are obtuse and a special designation is not necessary. A \textit{solid edge} indicates an obtuse angle between roots exactly as for simple roots in the case of Dynkin diagrams. A \textit{dotted edge} indicates an acute angle between the roots considered, see \cite{St10.I}. For examples of diagrams with dotted and solid edges, see Table 2.2.

1.2.2. \textit{Semi-Coxeter elements}. A conjugacy class of \( W \) which can be described by a connected Carter diagram with number of nodes equal to the rank of \( W \) is called a \textit{semi-Coxeter} class, \cite{CE72} (or, a \textit{primitive} conjugacy class, \cite{KP85}). Let us fix some basis of roots corresponding to the given Carter diagram \( \Gamma \):

\[ \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_h\}, \]  

(1.3)

where \( \alpha_i, \beta_j \) are roots (not necessarily simple) corresponding to \( \Gamma \). The element

\[ c = w_\alpha w_\beta, \quad \text{where} \quad w_\alpha = \prod_{i=1}^k s_{\alpha_i}, \quad w_\beta = \prod_{j=1}^h s_{\beta_j}, \]  

(1.4)

given in the basis (1.3) we call the semi-Coxeter element. It is the representative of the semi-Coxeter class. The dual semi-Coxeter element

\[ c^* = {}^t w_\alpha {}^t w_\beta, \quad \text{where} \quad w_\alpha = \prod_{i=1}^k s_{\alpha_i}^*, \quad w_\beta = \prod_{j=1}^h s_{\beta_j}^*, \]  

(1.5)

is used for the proof of the Carter theorem. Semi-Coxeter elements for diagrams \( D_l, D_l(a_k), E_l, E_l(a_k) \), where \( l \leq 7 \), are presented in Tables \ref{tab:A3} \cite{A3, A5}.

Note that roots (1.3) are not necessarily simple. If all roots (1.3) are simple, the Carter diagram \( \Gamma \) is a Dynkin diagram and the semi-Coxeter element (1.4) coincides with the corresponding Coxeter element.

1.2.3. \textit{Linkages, linkage diagrams and linkage systems}. Let \( w = w_1 w_2 \) be the bicolored decomposition of some element \( w \in W \), where \( w_1, w_2 \) are two involutions, associated, respectively, with \( \alpha\text{-set} \{\alpha_1, \ldots, \alpha_k\} \) and \( \beta\text{-set} \{\beta_1, \ldots, \beta_h\} \) of roots from the root system \( \Phi \), see (1.1), (1.2), and \( \Gamma \) be the Carter diagram associated with this bicolored decomposition. We consider the \textit{extension} of the root basis \( \Pi_w \) by means of the root \( \gamma \in \Phi \), such that the set of roots

\[ \Pi_w(\gamma) = \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_h, \gamma\} \]  

(1.6)
is linearly independent. Let us multiply \( w \) on the right by the reflection \( s_\gamma \) corresponding to \( \gamma \) and consider the diagram \( \Gamma' = \Gamma \cup \gamma \) together with new edges. These edges are
\[
\begin{cases}
\text{solid, for } (\gamma, \tau) = -1, \\
\text{dotted, for } (\gamma, \tau) = 1,
\end{cases}
\]
where \( \tau \) one of elements \( \{1, 0\} \). The diagram \( \Gamma' \) is said to be the linkage diagram, and the root \( \gamma \) is said to be the linkage or the \( \gamma \)-linkage. Consider vectors \( \gamma^\vee \) belonging to the dual space \( L^\vee \) and defined by \( \{1, 0\} \). Vector \( \{1, 0\} \) is said to be linkage labels vector or, for brevity, linkage labels. There is, clearly, the one-to-one correspondence between linkage labels vectors \( \gamma^\vee \) (with labels \( \gamma_i^\vee \in \{0, -1, 1\} \)) and simply-laced linkage diagrams (i.e., such linkage diagrams that \( (\gamma, \tau) \in \{0, -1, 1\} \)).

Let \( L \) be the linear space spanned by the roots associated with \( \Gamma \), The linkage labels vector is the element of the dual linear space \( L^\vee \). We denote the linkage labels vector by \( \gamma^\vee \). A certain group \( W_L^\vee \) named the dual partial Weyl group acts in the dual space \( L^\vee \). This group acts on the linkage label vectors, i.e., on the set of linkage diagrams:
\[
(w\gamma)^\vee = w^*\gamma^\vee,
\]
where \( w^* \in W_L^\vee \), see Proposition 2.9 from [St10.II]. The set of linkage diagrams (=linkage labels) under action of \( W_L^\vee \) constitute the diagram called the linkage system similarly to the weight system in the theory of representations of semisimple Lie algebras, see [Si81] p. 30, [St10.II] p. 4.

**Remark 1.3.** By abuse of language, we sometimes say linkages instead of linkage diagrams. Similarly, remembering only the algebraic nature of the linkage diagram, we use the term linkage label vector or linkage labels.

1.2.4. **Classes of Carter diagrams.** We divide all Carter diagrams to the following classes:

- **Simply-laced Carter diagrams:**
  1. DE4, Dynkin diagrams containing \( D_4 \) as a subdiagram,
  2. C4, Carter diagrams containing 4-cycle \( D_4(a_1) \) as a subdiagram,
  3. A, Dynkin diagrams \( A_l \),

- **Multiply-laced Carter diagrams:**
  4. BC, Dynkin diagrams \( B_l, C_l \),
  5. FG, Dynkin diagrams \( F_4, G_2 \), and the 4-cycle with two double bonds \( F_4(a_1) \).

For \( \Gamma \in C4 \cup DE4 \), the linkage systems are described in [St10.II]. In this case, for \( l \leq 7 \), the linkage systems \( \mathcal{L}(\Gamma) \) looks as follows: every linkage diagram containing at least one non-zero \( \alpha \)-label (see Section 1.2.1) belongs to a certain 8-cell "spindle-like" linkage subsystem called loctet (= linkage octet). The loctets are the main construction blocks in every linkage system. If all \( \alpha \)-labels (resp. \( \beta \)-labels) of \( \gamma^\vee \) are zeros, the linkage diagram \( \gamma^\vee \) is said to be \( \beta \)-unicolored (resp. \( \alpha \)-unicolored) linkage diagram. Every linkage system is the union of several loctets and several \( \beta \)-unicolored linkage diagrams, see [St10.II] §3. In the case, where \( l > 7 \), the linkage systems for two infinite series \( D_l \) and \( D_l(a_k) \) are described as follows: for \( D_l(a_k) \) the linkage system looks as wind rose of linkages, see [St10.II] Fig. B.46-B.47]; for \( D_l \) the linkage system looks as the Carter diagram \( D_l(a_k) \), see [St10.II] Fig. B.48.

**Remark 1.4** (multiply-laced Carter diagrams FG). For any \( \Gamma \in FG \), the linkage system is trivial. Really, for the case \( G_2 \), there are maximum two linearly independent roots. Thus, the linkage system \( \mathcal{L}(G_2) \) is trivial, see [St10.I] Rem. 2.2. Further, for the multiply-laced 4-cycle \( F_4(a_1) \), there is no additional fifth edge, otherwise such a diagram contains an extended Dynkin diagram as a subdiagram, see [St10.II] §A.3.2]. The simple extension of \( F_4 \) leads to the subdiagram, which
is one of extended Dynkin diagrams $\tilde{F}_{41}, \tilde{F}_{42}$, see [St10.I, Example A.3], or $\tilde{C}D_n, \tilde{D}D_n$ that can not be. Any triangle extending $F_4$ is also moved to one of cases $\tilde{F}_{41}, \tilde{F}_{42}$, see [St10.II, §4.3]. □

In [St11], we will construct remaining cases of linkage systems for two infinite series $A_l$ and $B_l$.

2. The proof of the Carter theorem

2.1. Linear independency and reduced decomposition.

2.1.1. Reduced decomposition and the Carter length $l_C(w)$. Each element $w \in W$ can be expressed in the form

$$w = s_{\tau_1}s_{\tau_2}\ldots s_{\tau_k}, \quad \tau_i \in \Phi,$$

where $\Phi$ is the root system associated with the Weyl group $W$; $s_{\tau_i}$ are reflections in $W$ corresponding to not necessarily simple roots $\tau_i \in \Phi$. We denote by $l_C(w)$ the smallest value $k$ in any expression like (2.1). The Carter length $l_C(w)$ is always less than the classical length $l(w)$. The decomposition (2.1) is called reduced if $l_C(s_{\tau_1}s_{\tau_2}\ldots s_{\tau_k}) = k$, i.e., the number of reflections in (2.1) can not be decreased.

Lemma 2.1. [Ca72, Lemma 3] Let $\gamma, \tau_1, \tau_2, \ldots, \tau_k \in \Phi$. Then $s_{\tau_1}s_{\tau_2}\ldots s_{\tau_k}$ is reduced if and only if $\gamma, \tau_1, \tau_2, \ldots, \tau_k$ are linearly independent.

2.1.2. The basic conjugacy relation.

Lemma 2.2 (on conjugacy). Let $\{\tau_1, \ldots, \tau_n\}$ be the subset of linearly independent roots (not necessarily simple), $\tau_i \in \Phi$, and let $w \in W$ be the element, which is decomposed into the product of reflections $\{s_{\tau_1}, \ldots, s_{\tau_n}\}$.

1) If $\gamma$ such a root that $\{\gamma, \tau_1, \ldots, \tau_n\}$ are linearly independent then $\{w\gamma, \tau_1, \ldots, \tau_n\}$ are also linearly independent.

2) The following conjugacy relation holds for any integer $k$:

$$s_\gamma w \simeq s_w^k s_\gamma w.$$

In particular, for the semi-Coxeter element $c$, we have

$$s_\gamma c \simeq s_c s_\gamma c.$$

Proof. 1) Since $s_{\tau_i}(\tau_j) \in \{\tau_i, \tau_j\}$, and $s_{\tau_i}(\gamma) \in \{\gamma, \tau_i\}$, we have

$$w(\tau_1, \ldots, \tau_n) \subseteq \{\tau_1, \ldots, \tau_n\},$$

$$w_\gamma \in \{\gamma, \tau_1, \ldots, \tau_n\},$$

for some rational factors $a_i$. If $\{w_\gamma, \tau_1, \ldots, \tau_n\}$ are linearly dependent, we have

$$w_\gamma = \sum_{i=1}^n b_i \tau_i$$

for some rational $b_i$, i.e., $\gamma + \sum_{i=1}^n a_i \tau_i = \sum_{i=1}^n b_i \tau_i$

that contradicts to the linear independency of $\{\gamma, \tau_1, \ldots, \tau_n\}$. 


2) Let
\[ w = \prod_{i=1}^{m} s_{\tau_i}, \]  
where not necessarily all \( \tau_i \) are different. For example, it can be that \( m > n \). We have
\[ s_{\gamma} w = s_{\gamma} s_{\tau_1} \ldots s_{\tau_m} = s_{\tau_1} s_{\gamma} (s_{\tau_2} s_{\gamma}) s_{\tau_3} \ldots s_{\tau_m} = s_{\tau_1} s_{\tau_2} s_{\tau_3} \ldots s_{\tau_m} = \ldots = w s_{w-1, \gamma} \simeq s_{w-1, w} \simeq \cdots \simeq w s_{w-k, \gamma}, \]  
for \( k > 0 \).

By mapping \( \gamma' = w^{-k} \gamma \) we obtain from (2.4) also that
\[ s_{\gamma'} w \simeq s_{w} (w^{-k}) \gamma' \]
for any integer \( k > 0 \) and every root \( \gamma' \). Thus, (2.2) is proven. □

Eq. (2.2) is the basic relation in our proof of the Carter theorem, see Section 2.2.1.

2.2. The proof of Theorem 1.1

2.2.1. The induction step. Let \( c \) be the semi-Coxeter element associated with the Carter diagram \( \Gamma \) such that \( c \) has bicolored decomposition given by (1.4). The proof of the theorem is carried out by induction on the Carter length \( l_C(c) \) of the decomposition (1.4), see Section 2.1.1. For details, see [Ca72, St10.I, p. 4]. Suppose, \( \gamma \) is the root such that roots \( \{ \gamma, \alpha_1, \ldots, \beta_h \} \) are linearly independent. According to Lemma 2.2, heading 1) \( \{ c^{n} \gamma, \alpha_1, \ldots, \beta_h \} \) are also linearly independent.

We will show that
\[ s_{\gamma} c \]  
is also associated with a certain Carter diagram. \hspace{1cm} (2.5)

Of course, it suffices to prove the property (2.5) for any conjugate of \( s_{\gamma} c \). The property (2.5) gives us the induction step. According to Lemma 2.2 it suffices to find such an integer \( n \) that any conjugate of \( s_{c^{n} \gamma} c \) has the bicolored decomposition.

2.2.2. Semi-Coxeter orbits. We have \( (c^{n} \gamma)^{\vee} = (c^{n})^{\gamma} \gamma^{\vee} = (c^{\gamma})^{n} \gamma^{\vee} \) for any \( n \), see [St10.II Proposition 2.9]. Let us consider the sequence of linkages
\[ (c^{n} \gamma)^{\vee} = (c^{\gamma})^{n} \gamma^{\vee}, \quad n = 0, \pm 1, \pm 2, \ldots \]  
It is clear that (2.6) is the finite periodic sequence, see Tables A.3-A.5. This sequence is said to be the semi-Coxeter orbit. Remember, that the linkage diagram \( \gamma^{\vee} \) is said to be the \( \alpha \)-unicolored (resp. \( \beta \)-unicolored) linkage diagram if all \( \beta \)-labels (resp. \( \alpha \)-labels), i.e., coordinates corresponding to all \( \beta_i \) (resp. \( \alpha_i \)) of \( \gamma^{\vee} \) are zeros, [St10.II, p. 5]. Suppose, for some integer \( m \), the element \( (c^{\gamma})^{m} \gamma^{\vee} \) in semi-Coxeter orbit is a certain unicolored linkage diagram. Let \( (c^{\gamma})^{m} \gamma^{\vee} \) be, for example, \( \alpha \)-unicolored. Then \( (c^{m} \gamma, \beta_i) = 0 \) for all \( \beta \)-labels. This means that \( s_{c^{m} \gamma} \) commute with all \( s_{\beta_i} \). By (2.2) and (1.4)
\[ s_{\gamma} c = s_{c^{m} \gamma} c = s_{c^{m} \gamma} \prod_{i=1}^{k} s_{\alpha_i} \prod_{j=1}^{h} s_{\beta_j} \simeq \prod_{i=1}^{k} s_{\alpha_i} (\prod_{j=1}^{h} s_{\beta_j}) s_{c^{m} \gamma}. \]

The latter product is the bicolored decomposition, since \( (\prod_{j=1}^{h} s_{\beta_j}) s_{c^{m} \gamma} \) is involution. Thus, it suffices to prove that any semi-Coxeter orbit contains an unicolored linkage diagram.
2.2.3. Unicolored linkage diagrams and exceptional orbits. However, there are semi-Coxeter orbits containing no unicolored linkage diagrams. We call these orbits *exceptional semi-Coxeter orbits*. The total quantity of orbits for Carter diagrams from $\Gamma \in \mathbb{C}4 \coprod \mathbb{D}E4$ (for $l \leq 7$) is 140, the number of exceptional orbits is 24, see Table 2.1. Instead of 24 orbits it suffices to consider only 10 exceptional orbits, namely (1a), (2a), (2c), (3a), (4a), (4b), (5a), (6a), (7a), (7b), see Table 2.2, they are checked case-by-case in Section 2.3.

| The Carter diagram | Number of orbits | Lengths of orbits $^1$ | Number of linkages |
|--------------------|-----------------|------------------------|-------------------|
| $D_4(a_1)$        | 6               | 6                      | 4                 |
| $D_4$             | 6               | 6                      | 4                 |
| $D_5(a_1)$        | 6               | 2                      | 6                 |
| $D_6$             | 6               | 2                      | 6                 |
| $E_6(a_1)$        | 6               | -                      | 9                 |
| $E_6(a_2)$        | 10              | 4                      | 8                 |
| $E_6$             | 6               | -                      | 4                 |
| $D_6(a_1)$        | 10              | 4                      | 9                 |
| $D_6(a_2)$        | 14              | 2                      | 12                |
| $D_7$             | 8               | -                      | 7                 |
| $E_7(a_1)$        | 4               | -                      | 4                 |
| $E_7(a_2)$        | 6               | 2                      | 4                 |
| $E_7(a_3)$        | 4               | -                      | 3                 |
| $E_7(a_4)$        | 10              | 6                      | 9                 |
| $E_7$             | 4               | -                      | 3                 |
| $D_7(a_1)$        | 10              | -                      | 6                 |
| $D_7(a_2)$        | 10              | -                      | 4                 |
| $D_l$             | 14              | 4                      | 10                |
| $D_l(a_k)$, $l > 7$ | 2               | -                      | $2 \times (k + 1)$ | $2 \times (l - 1)$ |
| $D_l$, $l > 7$    | 2               | -                      | $2 \times (l - 1)$ | $2l$ |

Table 2.1. Number and lengths of semi-Coxeter orbits

**Remark 2.3.** We observe that the number of unicolored linkage diagrams in every semi-Coxeter orbit is equal to 0 (exceptional orbit) or 2, see Tables B.6 - B.19 (where unicolored linkage diagrams are framed by a rectangle). Of course, this fact requires *a priori* reasoning.

$^1$Explanation to the column. For example, expression $6 \times 20 + 2 \times 4 + 10 + 4$ in the line $D_7(a_1)$ means that the total number of linkage diagrams in the linkage system for the Carter diagram $D_7(a_1)$ is divided into the sum of 6 orbits each of 20 elements, 2 orbits each of 4 elements, one orbit containing 10 elements and one orbit containing 4 elements.
2.2.4. **Semi-Coxeter orbits for infinite series $D_l(a_k)$ and $D_l$.** It is convenient to imagine a semi-Coxeter orbit for $D_l$ (resp. $D_l(a_k)$) as a cosine wave that runs in one direction and then returns in the opposite direction with a shift in the phase by half a period, see Fig. 2.1 and Fig. 2.2.

Diagram $D_l$. We have one long orbit – the red wave in the horizontal direction, and one 2-element orbit – the blue orbit in the vertical direction, see Fig. 2.1. There are two cases: $l = 2p + 2$ and $l = 2p + 1$. For $l = 2p + 2$, the linkage labels vector $\gamma_{\alpha_p}^+$ (resp. $\gamma_{\alpha_p}^-$) is the vector with the unit in the place $\alpha_p^+$ (resp. $\alpha_p^-$) and zeros in remaining places, see Fig. 2.1. They are two unicolored linkages for the long semi-Coxeter orbit (red wave). The blue orbit consists of following two unicolored linkages: $\gamma_{\alpha_2^+} = \{0, 1, -1, 0, \ldots, 0\}$, $\gamma_{\alpha_2^-} = -\gamma_{\alpha_2^+}$ with the only non-zero coordinates in coordinates $\alpha_2$ and $\alpha_3$. Notations of $\gamma_{\alpha_2^+}^+$, $\gamma_{\alpha_2^-}^+$ are retained as in [St10.II, Fig. B.48]. For $l = 2p + 1$, the linkage labels vector $\gamma_{\beta_p}^+$ (resp. $\gamma_{\beta_p}^-$) is the vector with the unit in the place $\beta_p^+$ (resp. $\beta_p^-$) and zeros in remaining places, see Fig. 2.1. The blue orbit is the same as in the case $l = 2p + 2$. Linkages $\gamma_{\alpha_p}^+$ and $\gamma_{\alpha_p}^-$ (see Remark 1.3) for $l = 2p + 2$, and linkages $\gamma_{\beta_p}^+$ and $\gamma_{\beta_p}^-$ for $l = 2p + 1$ are the same linkages as $\gamma_{\tau_{l-3}}^+$ and $\gamma_{\tau_{l-3}}^-$ in [St10.II, Fig. B.48].

**Figure 2.1.** Two semi-Coxeter orbits of $D_l$, one of length $2(l - 1)$ (= Coxeter number), one of length 2.
Diagram $D_l(a_k)$. Here, we have two long orbits: one red wave in the horizontal direction, and one blue wave in the vertical direction, see Fig. 2.2. The linkage labels vector $\gamma_{\tau_{k-1}^+}$ (resp. $\gamma_{\tau_{k-2}^-}$, $\gamma_{\tau_{l-k-2}^-}$) is the vector with the unit on the place $\tau_{k-1}^+$ (resp. $\tau_{k-2}^-$, $\tau_{l-k-2}^-$) and zeros on remaining places, see Fig. 2.2. As above, these vectors are unicolored, see [St10.II, Fig. B.46-B.47].

Two semi-Coxeter orbits of $D_l(a_k)$ are of lengths $2(k + 1)$ and $2(l - k - 1)$. For the left (resp. right) branch of $D_l(a_k)$, there are two options for endpoints: $\alpha_p$ or $\beta_p$ (resp. $\alpha_q$ or $\beta_q$). Thus, from the view of endpoints there are 4 options for the Carter diagram $D_l(a_k)$:

$$\{\alpha_p, \alpha_q\}, \quad \{\alpha_p, \beta_q\}, \quad \{\beta_p, \alpha_q\}, \quad \{\beta_p, \beta_q\}.$$ 

In Fig. 2.2 we depict only one from 4 options for $D_l(a_k)$ and its linkage system $L(D_l(a_k))$. 

---

**Figure 2.2.** Two semi-Coxeter orbits of $D_l$: one of length $2(k + 1)$, one of length $2(l - k - 1)$.
2.3. Exceptional semi-Coxeter orbits.

|   | The Carter diagram $\Gamma$ | Total orbits | Representatives of exceptional orbits in the linkage system $\mathcal{L}(\Gamma)$ |
|---|-----------------------------|--------------|--------------------------------------------------------------------------------|
| 1 | $D_4(a_1)$                  | 6            | (1a) $\{ -1, 0, 0, 1 \}$ (1b) $\{ 0, 1, 0, -1 \}$ |
| 2 | $D_6(a_1)$                  | 10           | (2a) $\{ 0, -1, 0, 0, -1, 0 \}$ (2b) $\{ 0, 0, -1, 0, 1, 0 \}$ (2c) $\{ 0, 0, 1, 0, 0, -1 \}$ (2d) $\{ 0, 1, 0, 0, 0, -1 \}$ |
| 3 | $D_6(a_2)$                  | 14           | (3a) $\{ 0, 0, -1, 0, 0, 1 \}$ (3b) $\{ 0, 0, 1, 0, -1, 0 \}$ (3c) $\{ 0, -1, 0, 0, 0, -1 \}$ (3d) $\{ 0, 1, 0, 0, -1, 0 \}$ |
| 4 | $E_6(a_2)$                  | 10           | (4a) $\{ -1, 0, 0, 0, 0, -1 \}$ (4b) $\{ 0, 0, -1, 0, 0, 0 \}$ |
| 5 | $E_7(a_2)$                  | 6            | (5a) $\{ -1, 0, 0, 0, 0, 1 \}$ |
| 6 | $E_7(a_4)$                  | 10           | (6a) $\{ -1, 0, 0, 0, 0, 1, 0 \}$ (6b) $\{ 0, 0, -1, 0, 0, -1, 0 \}$ (6c) $\{ 0, 0, 1, 0, -1, 0, 0 \}$ |
| 7 | $D_7$                       | 14           | (7a) $\{ 0, 0, -1, 0, 0, 1 \}$ (7b) $\{ -1, 0, 0, 0, 1, -1, 0 \}$ |

Table 2.2. Exceptional orbits
2.3.1. Diagram $D_4(a_1)$. Case (1a). For the Carter diagram $D_4(a_1)$, there are 2 exceptional semi-Coxeter orbits with representatives (1a) and (1b). These cases are similar, see Table 2.2. We consider only (1a).

Case (1a).

\[ w = s_{\alpha_1} s_{\alpha_2} s_{\beta_1} s_{\beta_2} s_{\gamma} = s_{\alpha_2} s_{\beta_2} s_{\beta_1} s_{\beta_2 + \alpha_1} s_{\gamma} \sim (s_{\beta_2} s_{\beta_1})(s_{\beta_1 + \beta_2 + \alpha_1} s_{\gamma}). \]

**Figure 2.3.** The linkage diagram (1a) for the Carter diagram $D_4(a_1)$

Thus, the linkage diagram (1a) from Table 2.2 is equivalent to the Carter diagram $D_5(a_1)$. This case was also considered in [St10.I, Lemma 1.8].

2.3.2. Diagram $D_6(a_1)$. Cases (2a) and (2c). For the diagram $D_6(a_1)$, there are 4 exceptional semi-Coxeter orbits (2a), (2b), (2c), (2d). Cases (2a) and (2b) are similar; cases (2c) and (2d) are also similar. We consider only (2a) and (2c).

Case (2a). Here, we have

\[ w = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\gamma} = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\beta_2} s_{\beta_3} s_{\gamma} \sim (s_{\alpha_1} s_{\alpha_2} s_{\alpha_3})(s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\gamma}). \]

**Figure 2.4.** The linkage diagram (2a) for the Carter diagram $D_6(a_1)$

Hence, the linkage diagram (2a) from Table 2.2 is equivalent to the Carter diagram $E_7(a_1)$.

Case (2c). This case is reduced to the exception case (4a) in the exceptional orbit for $E_6(a_2)$:

\[ w = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\gamma} = (s_{\alpha_1} s_{\alpha_2} s_{\alpha_3})(s_{\beta_1} s_{\beta_2} s_{\beta_3} + s_{\beta_1 + \gamma}) = (s_{\alpha_1} s_{\alpha_2} s_{\alpha_3}) s_{\beta_2 + \gamma} = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\beta_1} s_{\beta_2} s_{\beta_3} \sim (s_{\alpha_1} s_{\alpha_2} s_{\alpha_3})(s_{\beta_1} s_{\beta_2} s_{\beta_3}). \]

**Figure 2.5.** The linkage diagram (2c) for the Carter diagram $D_6(a_1)$
2.3.3. Diagram $D_6(a_2)$. Case (3a). For the Carter diagram $D_6(a_2)$, there are 4 exceptional semi-Coxeter orbits with representatives (3a), (3b), (3c), (3d). These cases are similar to each other, see Table 2.2. We consider only (3a).

Case (3a). Here, we have

$$w = s_{a_1} s_{a_2} s_{a_3} s_{a_4} s_{b_1} s_{b_2} s_{b_3} s_{b_4} = s_{a_1} s_{b_2} s_{a_3} s_{a_4} s_{b_1} s_{b_3} s_{b_4} = s_{b_2} s_{a_3} s_{a_4} (s_{a_1} s_{a_2} s_{a_3} s_{a_4}) s_{b_1} s_{b_3} = (s_{a_1} s_{a_2} s_{a_3}) s_{b_1} s_{b_3} (s_{b_2} s_{a_3} s_{a_4} + s_{b_2} s_{b_3} s_{b_4} + s_{b_2} s_{b_3} s_{b_4}).$$

Therefore, the linkage diagram (3a) from Table 2.2 is equivalent to the Carter diagram $E_7(a_2)$.

![Figure 2.6](image)

**FIGURE 2.6.** The linkage diagram (3a) for the Carter diagram $D_6(a_2)$

**Remark 2.4** (on s-permutation). Let $s_{a}$ and $s_{b}$ be two adjacent reflections in the decomposition of $w$:

$$w = \ldots s_{a} s_{b} \ldots.$$  

If decomposition (2.7) is written down in one of the equivalent forms

$$w = \ldots s_{a} s_{b} \ldots = \ldots s_{a} (s_{b} s_{a}) s_{a} \ldots,$$

$$w = \ldots s_{a} s_{b} \ldots = \ldots s_{b} s_{a} (s_{b} s_{a}) \ldots,$$

we say that elements $s_{a}$ and $s_{b}$ are s-permuted. The linkage diagram related to (2.7) is respectively changed. The corresponding transformation of the word $w$ and related linkage diagram we call the s-permutation. In [Sa1.1, §1.4.1], we considered s-permutation in the framework of equivalent transformations of connection diagrams.

2.3.4. Diagram $E_6(a_2)$. Cases (4a), (4b). These two cases are different.

Case (4a). First, reflections $s_{b_3}$ and $s_{a_4} s_{a_3}$ are s-permuted. The new connection between $b_2$ and $b_3 - a_1 + a_3$ appears, see Fig. 2.7(b). After that, $s_{b_2}$ and $s_{a_2} s_{a_3}$ are s-permuted. The new connection disappears, see Fig. 2.7(c).

$$w = s_{a_1} s_{a_2} s_{a_3} s_{b_1} s_{b_2} s_{b_3} s_{b_4} = s_{a_2} s_{b_3} - a_1 + a_3 s_{a_1} s_{a_3} s_{b_1} s_{b_2} s_{b_3} s_{b_4} = s_{b_3} s_{a_1} - a_3 s_{a_2} s_{a_3} s_{b_1} s_{b_2} s_{b_3} s_{b_4} s_{b_2},$$

since $s_{a_2}$ and $s_{b_2} - a_1 + a_3$ commute, see Fig. 2.7(b).

$$w = s_{b_3} s_{a_1} - a_1 + a_3 s_{a_2} s_{b_2} s_{b_3} s_{b_4} = s_{b_3} s_{a_1} - a_1 + a_3 s_{b_2} - a_2 + a_3 s_{a_2} s_{a_3} s_{b_1} s_{b_2} s_{b_3} s_{b_4} s_{b_2},$$

since $s_{a_1}$ and $s_{b_2} - a_2 + a_3$ commute, see Fig. 2.7(c). Further,

$$w = s_{b_3} - a_1 + a_3 s_{b_2} - a_2 + a_3 s_{a_2} s_{a_3} s_{b_1} s_{b_2} s_{b_3} s_{b_4} = s_{b_3} - a_1 + a_3 s_{b_2} - a_2 + a_3 s_{a_2} s_{a_3} s_{b_1} s_{b_2} s_{b_3} s_{b_4} s_{b_2},$$

see Fig. 2.7(d). Hence, the linkage diagram (4a) from Table 2.2 is equivalent to the Carter diagram $E_7(a_3)$.

**Case (4b).**

$$w = s_{a_1} s_{a_2} (s_{a_3} s_{b_1} s_{b_2} s_{b_3}) s_{b_4} = s_{a_1} s_{a_2} s_{b_1} s_{b_2} s_{b_3} s_{a_3} + s_{a_1} s_{a_2} s_{b_1} s_{b_2} s_{b_3} s_{a_4},$$

where

$$(a_3 + b_1 + b_2 + b_3, a_1) = (b_3, a_1) + (b_1, a_1) = 0,$$

$$(a_3 + b_1 + b_2 + b_3, a_2) = (b_2, a_2) + (b_1, a_2) = 0.$$
Thus, \( w \) is described by the diagram Fig. 2.8 (b). Further, 

\[
w = s_{\alpha_1} s_{\alpha_2} s_3 s_2 s_3 s_{\alpha_3} s_1 + \beta_1 + \beta_2 + \beta_3 s_\gamma = s_{\alpha_1} s_{\alpha_2} s_3 s_2 s_3 s_{\alpha_3} s_1 + (\beta_1 + \beta_2 + \beta_3) s_\gamma,
\]

see Fig. 2.8 (c). Hence, the linkage diagram (4b) from Table 2.2 is equivalent to the Carter diagram \( E_7 (a_3) \).
Further, reflections $\alpha_2$ and $s_{\beta_1}s_{\beta_2}$ are $s$-permuted. Then the new connection disappears:

$$w = s_{\alpha_1} + \beta_1 - \beta_3 + \gamma s_{\alpha_2} + \gamma s_{\alpha_3} + s_{\beta_1}s_{\beta_2}$$

where

$$(\alpha_1 + \beta_1 - \beta_3 + \gamma, \alpha_2 - \beta_2 + \beta_1) = (\alpha_2, \beta_1) + (\beta_1, \beta_1) + (\beta_1, \alpha_1) = -\frac{1}{2} + 1 - \frac{1}{2} = 0.$$

Further,

$$w = s_{\alpha_1} + \beta_1 - \beta_3 + \gamma s_{\alpha_2} + \gamma s_{\alpha_3} + s_{\beta_1}s_{\beta_2}s_\gamma = s_{\alpha_1} + \beta_1 - \beta_3 + \gamma s_{\alpha_2} + \gamma s_{\alpha_3} + s_{\beta_1}s_{\beta_2}s_\gamma \equiv s_{\alpha_1} + \beta_1 - \beta_3 + \gamma s_{\alpha_2} + \gamma s_{\alpha_3} \equiv s_{\alpha_1} + \beta_1 - \beta_3 + \gamma s_{\alpha_2} + \gamma s_{\alpha_3} \equiv$$

$$s_{\alpha_1} + \beta_1 - \beta_3 + \gamma s_{\alpha_2} + \gamma s_{\alpha_3},$$

see Fig. 2.10(d). In the last step, $s_{\beta_1}$ is replaced by $s_{-\beta_1}$, the corresponding diagram is depicted in Fig. 2.10(e). Thus, the linkage diagram (5a) from Table 2.2 is equivalent to the Carter diagram $E_8(a_5)$.

2.3.6. Diagram $E_7(a_4)$. Case (6a). For the Carter diagram $E_7(a_4)$, there are 6 exceptional semi-Coxeter orbits with representatives (6a), (6b), (6c), (6d), (6e), (6f), that are similar to each other, see Table 2.2. Let us consider (6a).

Case (6a). First, reflections $s_{\beta_3}$ and $s_{\alpha_1}s_{\alpha_3}$ are $s$-permuted. Two new connections $\{\beta_3 - \alpha_3 + \alpha_1, \beta_2\}$ and $\{\beta_3 - \alpha_3 + \alpha_1, \beta_1\}$ appear:

$$w = s_{\alpha_2}(s_{\alpha_1}s_{\alpha_3}s_{\beta_3})s_{\beta_1}s_{\beta_2}s_\gamma = s_{\alpha_2}s_{\beta_3} - \alpha_3 + \alpha_1 s_{\alpha_1}s_{\alpha_3} + s_{\beta_1}s_{\beta_2}s_\gamma,$$

since $\alpha_2$ and $\beta_3 - \alpha_3 + \alpha_1$ commute, see Fig. 2.10(b). After that, reflections $s_{\beta_3}$ and $s_{\alpha_2}s_{\alpha_3}$ are $s$-permuted. Then the connection $\{\beta_3 - \alpha_3 + \alpha_1, \beta_2\}$ disappears, and the connection $\{\beta_2 - \alpha_2 + \alpha_3, \beta_1\}$ appears:

$$w = s_{\beta_3} - \alpha_3 + \alpha_1 s_{\alpha_1}s_{\alpha_3}s_{\beta_2} - \alpha_2 + \alpha_3 s_{\alpha_2}s_{\alpha_2} - \alpha_3 s_{\beta_2}s_\gamma =$$

$$s_{\beta_3} - \alpha_3 + \alpha_1 s_{\alpha_1}s_{\alpha_3}s_{\beta_2} - \alpha_2 + \alpha_3 s_{\alpha_2}s_{\alpha_2} - \alpha_3 s_{\beta_2}s_\gamma,$$

since $s_{\alpha_1}$ and $s_{\beta_2} - \alpha_2 + \alpha_3$ commute, see Fig. 2.10(c). We have

$$(\beta_3 - \alpha_3 + \alpha_1, \beta_2 - \alpha_2 + \alpha_3) = (\beta_3, \alpha_3) - (\alpha_3, \alpha_3) - (\alpha_3, \beta_2) = \frac{1}{2} - 1 + \frac{1}{2} = 0.$$
Further, reflections $s_{\beta_3 - \alpha_3 + \alpha_1}s_{\beta_2 - \alpha_2 + \alpha_3}$ and $s_{\beta_4}$ are $s$-permuted:

$$w = s_{\beta_3 - \alpha_3 + \alpha_1}s_{\beta_2 - \alpha_2 + \alpha_3}s_{\alpha_3}s_{\alpha_2}s_{\alpha_4}s_{\beta_3}s_{\beta_2}s_{\beta_4}s_{\gamma} \cong$$

$$(s_{\beta_4}s_{\beta_3 - \alpha_3 + \alpha_1}s_{\beta_2 - \alpha_2 + \alpha_3})s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\gamma} = s_{\beta_3 - \alpha_3 + \alpha_1}s_{\beta_2 - \alpha_2 + \alpha_3}s_{\varphi}s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\beta_3}s_{\gamma},$$

where $\varphi = \beta_4 - (\beta_3 - \alpha_3 + \alpha_1) - (\beta_2 - \alpha_2 + \alpha_3) = \beta_4 - \beta_3 - \beta_2 - \alpha_1 + \alpha_2$.

Then

$$(\varphi, \alpha_1) = (\beta_4 - \beta_3 - \alpha_1, \alpha_1) = \frac{1}{2} + \frac{1}{2} - 1 = 0,$$

$$(\varphi, \alpha_2) = (\beta_4 - \beta_2 + \alpha_2, \alpha_2) = \frac{1}{2} - \frac{1}{2} + 1 = 0,$$

i.e., connections $\{\varphi, \alpha_1\}$ and $\{\varphi, \alpha_2\}$ disappear, see Fig. 2.10(d). Hence, the linkage diagram (6a) from Table 2.2 is equivalent to the Carter diagram $E_8(\alpha_7)$, see Fig. 2.10(c).

2.3.7. Diagram $D_7$. Cases (7a), (7b). These 2 cases are different.

Case (7a). We have

$$w = s_{\gamma}s_{\alpha_1}s_{\alpha_2}s_{s_{\alpha_3}}s_{\beta_1}s_{\beta_2}s_{\beta_3} = s_{\alpha_3}(s_{\gamma + \alpha_3}s_{\alpha_1}s_{\alpha_2}s_{\alpha_4}s_{\beta_1})s_{\beta_2}s_{\beta_3}.$$  

Let $s$-permute $s_{\gamma + \alpha_3}s_{\alpha_1}s_{\alpha_2}s_{s_{\alpha_3}}$ and $s_{\beta_1}$:

$$w = s_{\alpha_3}s_{\beta_1 + \alpha_1 + \alpha_2 + \gamma + \alpha_3}(s_{\gamma + \alpha_3}s_{\alpha_1}s_{\alpha_2}s_{s_{\alpha_3}})s_{\beta_2}s_{\beta_3}s_{\beta_4}s_{\gamma} \cong$$

$$(s_{\beta_2}s_{\beta_3}s_{\alpha_3}s_{\beta_1 + \alpha_1 + \alpha_2 + \gamma + \alpha_3})(s_{\gamma + \alpha_3}s_{\alpha_1}s_{\alpha_2}s_{s_{\alpha_3}}).$$

\[\text{Figure 2.11. The linkage diagram (7a) for the Carter diagram } D_7\]

Since

$$(\beta_1 + \alpha_1 + \alpha_2 + \gamma + \alpha_3, \alpha_3) = (\beta_1 + \gamma + \alpha_3, \alpha_3) = 1 - \frac{1}{2} - \frac{1}{2} = 0,$$

$$(\beta_1 + \alpha_1 + \alpha_2 + \gamma + \alpha_3, \beta_2) = (\gamma + \alpha_2, \beta_2) = \frac{1}{2} - \frac{1}{2} = 0,$$

we get the bicolored decomposition, see Fig. 2.11. Since $s_{\beta_1 + \alpha_1 + \alpha_2 + \gamma + \alpha_3} = s_{-(\beta_1 + \alpha_1 + \alpha_2 + \gamma + \alpha_3)}$ and $s_{\alpha_3} = s_{-\alpha_3}$ we can change $\beta_1 + \alpha_1 + \alpha_2 + \gamma + \alpha_3$ to the opposite vector $-(\beta_1 + \alpha_1 + \alpha_2 + \gamma + \alpha_3)$ and $\alpha_3$ to $-\alpha_3$. Thus, we obtain the last diagram in Fig. 2.11. The dotted edge $\{\gamma + \alpha_3, \beta_2\}$ (i.e., the property “be dotted edge”) can be moved to any edge of the square. Hence, we get the Carter diagram $E_8(\alpha_2)$.

Case (7b). Let $s$-permute $s_{\gamma}s_{\alpha_2}$ and $s_{\beta_1}$ as follows:
with semi-Coxeter orbits are depicted in Fig. B.13-B.19. Let \( L \) commute. Examples of semi-Coxeter orbits are presented in Appendix B, where the orbits are self-opposite orbit opposite to themselves, such an orbit is said to be the linkage system. We call such a semi-Coxeter orbit the 2.4. Finding semi-Coxeter orbits. 2.4.1. How to find semi-Coxeter orbits? We use two ways to find semi-Coxeter orbits. The first one is the matrix approach:

1. Calculation of powers of dual semi-Coxeter elements \( e^* \), see Tables A.3 - A.5.
2. Applying \( (e^*)^k \) to any unicolored linkage diagram \( \gamma \) until finding the period of \( e^* \) on this linkage.
3. Search any new linkage from the corresponding linkage system, preferably unicolored and back to step 2). The linkage systems for all Carter diagrams from DE4 and C4 are in [St10.III].

The second way is the diagram approach. We find all semi-Coxeter orbits as a closed cycles in the linkage diagram. We call such a semi-Coxeter orbit the \( \mathbf{e^*-cycle} \). The link connecting \( \gamma \) and \( e \gamma \) we call the \( \mathbf{e^*-transition} \). Every \( \mathbf{e^*-cycle} \) consists of \( \mathbf{e^*-transitions} \) \( \gamma \rightarrow c \gamma \). Each \( \mathbf{e^*-transition} \) consists of 2 passages, one after the other: \( \gamma \rightarrow t w_\beta \gamma \) and \( t w_\beta \gamma \rightarrow t w_\alpha w_\beta \gamma \), where

\[
\begin{align*}
\ell w_\alpha &= \prod_{i=1}^k s_{\alpha_i}^*, \\
\ell w_\beta &= \prod_{j=1}^h s_{\beta_j}^*.
\end{align*}
\]

Reflections \( s_{\beta_j}^* \) and \( s_{\alpha_i}^* \) act on the linkage diagrams in the linkage system \( \mathcal{L}(\Gamma) \). The order of actions of \( s_{\beta_j}^* \) within \( \ell w_\beta \) (resp. \( s_{\alpha_i}^* \) within \( \ell w_\beta \)) does not matter since all \( s_{\beta_j}^* \) (resp. \( s_{\alpha_i}^* \)) mutually commute. Examples of semi-Coxeter orbits are presented in Appendix B where the orbits are differed by colors or bold and dotted lines. Let \( \mathcal{L}(\Gamma) \) be the linkage system for the Carter diagram of \( \Gamma \). Note that for any linkage \( \gamma \) in \( \mathcal{L}(\Gamma) \), we have \( -\gamma \) in \( \mathcal{L}(\Gamma) \), since \( \mathcal{B}^L_\gamma(-\gamma) = \mathcal{B}^L_\gamma(\gamma) \) and \( \mathcal{B}^L_\gamma(\gamma) < 2 \iff \gamma \in \mathcal{L}(\Gamma), \)

see [St10.III] Theorem 2.14. Two orbits are said to be the \textit{opposite orbits} if for every linkage \( \gamma \) in one of the orbits there exists the linkage \( -\gamma \) in another one. There are some orbits which are opposite to themselves, such an orbit is said to be the \textit{self-opposite orbit}.

For Carter diagrams D4, D5(a1), D5(a1), D5, E6(a1), E6(a2), E6, the figures of linkage systems with semi-Coxeter orbits are depicted in Fig. B.13-B.19.
### APPENDIX A. The dual semi-Coxeter element for the Carter diagrams

| Diagram | Transpose semi-Coxeter Element $\mathbf{e}$ | Dual semi-Coxeter Element $\mathbf{e}^* = (\mathbf{e}^{-1})^T$ | Order of $\mathbf{e}^*$ |
|---------|---------------------------------------------|-------------------------------------------------|------------------|
| $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a_1 \downarrow f_1 \\
 a_2 \downarrow f_2 \\
 a_3 \downarrow f_3
\end{array}
\end{array}
\end{array}$ | $\begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & -1 & 1 \\
-1 & 1 & -1 & 0 \\
-1 & 1 & 0 & -1
\end{bmatrix}$ | $\begin{bmatrix}
-1 & 0 & -1 & -1 \\
0 & 1 & 1 & -1 \\
1 & -1 & 1 & 0 \\
1 & 1 & 0 & 1
\end{bmatrix}$ | 4 |
| $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a_1 \downarrow f_1 \\
 a_2 \downarrow f_2 \\
 a_3 \downarrow f_3
\end{array}
\end{array}
\end{array}$ | $\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & -1 \\
1 & 0 & 2 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 & 1 & 0 \\
0 & 1 & -1 & 1 & 1 & 0 \\
0 & 0 & -1 & 1 & 1 & 0
\end{bmatrix}$ | $\begin{bmatrix}
-1 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & -1 & 1 & 0 \\
0 & 0 & -1 & -1 & 1 & 0 \\
1 & 1 & 1 & 1 & 2 & 1 \\
0 & -1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1
\end{bmatrix}$ | 12 |
| $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a_1 \downarrow f_1 \\
 a_2 \downarrow f_2 \\
 a_3 \downarrow f_3
\end{array}
\end{array}
\end{array}$ | $\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & -1 \\
1 & 1 & 0 & 1 & 1 & -1 \\
0 & 0 & 2 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 & 1 & 0 \\
0 & 1 & -1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}$ | $\begin{bmatrix}
-1 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & -1 & 1 & 0 \\
0 & 0 & -1 & -1 & 1 & 0 \\
1 & 1 & 1 & 1 & 2 & 0 \\
0 & -1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1
\end{bmatrix}$ | 6 |
| $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a_1 \downarrow f_1 \\
 a_2 \downarrow f_2 \\
 a_3 \downarrow f_3
\end{array}
\end{array}
\end{array}$ | $\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 2 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 & 1 & 0 \\
0 & 1 & -1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}$ | $\begin{bmatrix}
-1 & 0 & 0 & 0 & -1 & -1 \\
0 & -1 & 0 & -1 & 1 & 0 \\
0 & 0 & -1 & -1 & 1 & 0 \\
1 & 1 & 1 & 2 & 0 & 1 \\
0 & -1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1
\end{bmatrix}$ | 6 |

Table A.3. The dual semi-Coxeter element $\mathbf{e}^*$ for $l < 7$
| The Carter diagram | The transpose semi-Coxeter element $\mathfrak{e}$ | The dual semi-Coxeter element $\mathfrak{e}^* = \mathfrak{e}^{-1}$ | Order of $\mathfrak{e}$ |
|--------------------|--------------------------------|-------------------------------------------------|-----------------|
| $\varphi_2, \varphi_3$ | $\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 2 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$ | 14 |
| $\xi_1, \xi_2$ | $\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 2 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$ | 12 |
| $\xi_1, \xi_2$ | $\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$ | 30 |
| $\xi_1, \xi_2$ | $\begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & 0 & -1 & -1 & 1 \\ -1 & 1 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}$ | 6 |
| $\varphi_2, \varphi_3$ | $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & 2 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$ | 20 |
| $\varphi_2, \varphi_3$ | $\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & 2 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$ | 24 |

Table A.4 (cont.) The dual semi-Coxeter element $\mathfrak{e}^*$ for $l = 7$
| The Carter diagram | The transpose semi-Coxeter element 'c | The dual semi-Coxeter element c* = 'c⁻¹ | Order of 'c |
|--------------------|-------------------------------------|----------------------------------------|-------------|
| ![Diagram](image) | ![Matrix](matrix1) | ![Matrix](matrix2) | 6 |
| ![Diagram](image) | ![Matrix](matrix3) | ![Matrix](matrix4) | 8 |
| ![Diagram](image) | ![Matrix](matrix5) | ![Matrix](matrix6) | 12 |
| ![Diagram](image) | ![Matrix](matrix7) | ![Matrix](matrix8) | 10 |
| ![Diagram](image) | ![Matrix](matrix9) | ![Matrix](matrix10) | 18 |
| ![Diagram](image) | ![Matrix](matrix11) | ![Matrix](matrix12) | 12 |

*Table A.5. (cont.) The dual semi-Coxeter element c*
APPENDIX B. Semi-Coxeter orbits

Recall, that orbits of dual semi-Coxeter element acting on the linkage diagrams are said to be semi-Coxeter orbits, see Section 2.2.2

B.1. Semi-Coxeter orbits for $D_l(a_i)$, $E_l(a_i)$, $D_l$, $E_l$, where $l < 7$.

| $\gamma^\vee$ | Orbit 1 (red, $L_{12}$) | Orbit 2 (green, $L_{13}$) | Orbit 3 (blue, $L_{23}$) |
|--------------|-------------------------|-------------------------|-------------------------|
| $\gamma^\vee$ | 0, 0, -1, 0             | 0, -1, 0, 0             | -1, 0, 0, 0             |
| $c^\ast \gamma^\vee$ | 0, 0, 1, 0             | 0, 1, 0, -1             | 1, 0, 0, -1             |
| $(c^\ast)^2 \gamma^\vee$ | 1, 1, 0, -1         | 1, 0, 0, -1             | 0, 1, 1, -1             |
| $(c^\ast)^3 \gamma^\vee$ | 0, 0, 1, 0             | 0, 1, 0, 0              | 1, 0, 0, 0              |
| $(c^\ast)^4 \gamma^\vee$ | 0, 0, -1, 1           | 0, -1, 0, 1             | -1, 0, 1, 0             |
| $(c^\ast)^5 \gamma^\vee$ | -1, -1, 0, 1       | -1, 0, -1, 1            | 0, -1, -1, 1            |

Table B.6. $D_4$, there exist 6 semi-Coxeter orbits. All orbits are self-opposite.

Figure B.13. $D_4$, 6 semi-Coxeter orbits, three of length 6, three of length 2.

\footnote{Here and below in all tables unicolored linkage labels vectors are framed by a rectangle.}
Table B.7. \( D_4(a_1) \), there exist 6 semi-Coxeter orbits. All orbits are self-opposite. Orbits 1 – 4 contain unicolored linkage diagrams.

Figure B.14. The linkage system of \( D_4(a_1) \), three components, 24 linkage diagrams, 3 components
Table B.8. \( D_5(a_1) = D_5(a_2) \), 6 semi-Coxeter orbits. All orbits contain unicolored linkage diagrams. Orbits 1, 2 have opposite orbits (in bottom component). Orbits 3, 4 are self-opposite.

![Diagram](image1)

Figure B.15. The linkage system of \( D_5(a_1) \), 42 linkage diagrams, 3 components
Table B.9. $D_5$, there exist 6 semi-Coxeter orbits. All orbits contain unicolored linkage diagrams. Orbits 1, 2 have opposite orbits (in 2nd $E$-type component). Orbits 3, 4 are self-opposite.

| $\gamma$ | Orbit 1 (red, $E$-type) | Orbit 2 (blue, $E$-type) | Orbit 3 (red, $D$-type) | Orbit 4 (blue, $D$-type) |
|---|---|---|---|---|
| $\gamma^V$ | $0, 0, -1, 0, 0$ | $0, 0, 1, 0, -1$ | $0, 0, 0, 0, -1$ | $1, 0, -1, 0, 0$ |
| $(c^*)\gamma^V$ | $0, 0, 1, -1, 0$ | $0, 1, -1, 0, 0$ | $0, 1, 0, -1, 0$ | $-1, 0, 1, 0, 0$ |
| $(c^*)^2\gamma^V$ | $1, 1, 0, -1, 1$ | $0, -1, 1, 0, 1$ | $1, 0, 0, -1, 0$ | $0, 1, 0, 1, 0$ |
| $(c^*)^3\gamma^V$ | $0, 1, 1, -1, 0$ | $0, 0, -1, 1, -1$ | $0, 1, 0, 0, -1$ | $-1, 0, -1, 0, 0$ |
| $(c^*)^4\gamma^V$ | $1, 0, 0, 0, 0$ | $-1, 0, 0, 0, 1$ | $0, 0, 0, 0, 1$ | $0, -1, 0, 1, 0$ |
| $(c^*)^5\gamma^V$ | $0, -1, 1, 1, 0$ | $-1, 1, 0, 0, 1$ | $-1, 0, 0, 0, 1$ | $0, -1, 0, 0, 1$ |
| $(c^*)^6\gamma^V$ | $1, -1, 0, 1, 0$ | $1, 0, 0, -1, 1$ | $0, -1, 0, 0, 0$ | $0, 0, 0, 0, -1$ |

Figure B.16. The linkage system of $D_5$, 42 linkage diagrams, 3 components
Table B.10. $E_6(a_1)$, there exist 6 semi-Coxeter orbits, each of length 9. Every orbit contains the $\beta$-unicolored linkage diagram $\gamma^\vee$. Orbits 1, 2, 3 have opposite orbits (starting from $-\gamma^\vee$)

| $a_1$ | Orbit 1 (red) | Orbit 2 (green) | Orbit 2 (blue) |
|-------|---------------|-----------------|----------------|
| $\gamma^\vee$ | $0, 0, 0, 0, -1$ | $0, 0, 0, 1, 0$ | $0, 0, 0, -1, 1$ |
| $c^*\gamma^\vee$ | $0, 0, 1, -1, -1, 0$ | $0, 1, -1, 0, 1$ | $0, -1, 0, 1, -1$ |
| $(c^*)^2\gamma^\vee$ | $1, 0, 1, -1, 0, -1$ | $0, 0, -1, 1, 0, 0$ | $-1, 0, 1, 0, 1$ |
| $(c^*)^3\gamma^\vee$ | $0, 1, 1, -1, 0, 0$ | $-1, -1, 0, 1, -1, 0$ | $1, 0, -1, 1, 0$ |
| $(c^*)^4\gamma^\vee$ | $1, 0, 0, 0, 0, 0$ | $0, 1, 0, 0, 0, 0$ | $-1, 1, 0, 0, 0$ |
| $(c^*)^5\gamma^\vee$ | $-1, 0, 1, 0, 0, 0$ | $0, 1, 0, -1, -1, 0$ | $1, -1, 0, -1, 0$ |
| $(c^*)^6\gamma^\vee$ | $0, -1, 1, 1, 0, 1$ | $1, 1, 0, -1, 0, 0$ | $-1, 0, 1, 0, -1$ |
| $(c^*)^7\gamma^\vee$ | $-1, 0, -1, 1, 1, 0$ | $0, 1, 0, -1, -1, 0$ | $1, 0, -1, 0, 1$ |
| $(c^*)^8\gamma^\vee$ | $0, 0, -1, 0, 0, 1$ | $0, -1, 1, 0, -1, 0$ | $0, 1, 0, 0, 1, -1$ |

Figure B.17. The linkage system of $E_6(a_1)$, two components, 54 linkage diagrams, 6 loctets
### Table B.11. \( E_6(a_2) \), there exist 10 semi-Coxeter orbits. Orbits 1 – 5 have opposite orbits. Only orbits 1, 2, 5 contain unicolored linkage diagrams.

| Orbit 1 (blue) | Orbit 2 (green) |
|---------------|-----------------|
| \( \gamma^V \) | \( -1, 0, 0, 0, -1 \) | \( 1, 0, 0, 0, 0 \) |
| \( c^* \gamma^V \) | \( 0, 1, 0, -1, 1, 0 \) | \( 0, -1, 1, 0, 0, -1 \) |
| \( (c^*)^2 \gamma^V \) | \( 1, 1, 0, -1, 0, 1 \) | \( -1, 0, 1, 0, 0, -1 \) |
| \( (c^*)^3 \gamma^V \) | \( 1, 0, 0, 0, 0 \) | \( 0, 0, 0, 0, 0, 1 \) |
| \( (c^*)^4 \gamma^V \) | \( -1, 0, 1, 0, -1 \) | \( 1, 0, -1, 0, 1 \) |
| \( (c^*)^5 \gamma^V \) | \( -1, -1, 0, 1, 0 \) | \( 0, 1, -1, 1, 0 \) |

| Orbit 3 (red) | Orbit 4 (brown) | Orbit 5 (turquoise) |
|---------------|-----------------|---------------------|
| \( \gamma^V \) | \( -1, 0, 0, 0, -1 \) | \( 0, 0, 0, 1, -1 \) |
| \( c^* \gamma^V \) | \( 0, 1, -1, 1, 0 \) | \( -1, 1, 0, 0, 0 \) |
| \( (c^*)^2 \gamma^V \) | \( 1, 0, 1, 0, 0 \) | \( 1, -1, 0, 0, -1 \) |
| \( (c^*)^3 \gamma^V \) | \( 0, 0, 1, 0, 0 \) | \( -1, 0, 1, 0, 0 \) |
| \( (c^*)^4 \gamma^V \) | \( 0, -1, -1, 1, 0 \) | \( -1, 0, -1, 1, 1 \) |

**Figure B.18.** The linkage system of \( E_6(a_2) \), two components, 54 linkage diagrams, 6 loctets.
Table B.12. $E_6$, there exist 6 semi-Coxeter orbits: four of length 12, and two of length 3. Orbits 1, 2, 3 have opposite orbits lying in the second component.

| Orbit 1 (red) | Orbit 2 (blue) | Orbit 3 (green) |
|---------------|---------------|-----------------|
| $\gamma^\wedge$ | $0, 0, 0, 0, 0, -1$ | $0, -1, 0, 1, 0$ | $1, -1, 0, 0, 0$ |
| $e^*\gamma^\wedge$ | $0, 1, 0, -1, 0, 0$ | $-1, 0, 1, 0, 0$ | $-1, 1, 0, 1, -1$ |
| $(e^*)^2\gamma^\wedge$ | $1, 0, -1, -1, 0$ | $1, 0, -1, 0, -1$ | $0, 0, 0, 0, -1$ |
| $(e^*)^3\gamma^\wedge$ | $1, 1, 0, -1, 0, -1$ | $0, 1, -1, 1, 0$ | $0, 1, 0, 0, -1$ |
| $(e^*)^4\gamma^\wedge$ | $0, 1, 0, -1, 0, 0$ | $0, 1, 0, -1, 0, 1$ | $0, 0, 1, 0, 0$ |
| $(e^*)^5\gamma^\wedge$ | $1, 0, 0, 0, 0$ | $1, 0, 0, -1, 0, 1$ | $0, 0, 1, 0, 0$ |
| $(e^*)^6\gamma^\wedge$ | $0, 0, 0, 1, 0$ | $0, 0, 1, 0, 0$ | $0, 0, 1, 0, 0$ |
| $(e^*)^7\gamma^\wedge$ | $-1, 0, 1, 0, 0$ | $0, 1, -1, 0, 0, 0$ | $0, 1, -1, 0, 0, 0$ |
| $(e^*)^8\gamma^\wedge$ | $0, -1, -1, 0, 1$ | $0, -1, 1, 0, 1$ | $0, -1, 0, 0, -1$ |
| $(e^*)^9\gamma^\wedge$ | $-1, -1, 0, 1, 0$ | $0, -1, 1, 0, 1$ | $0, -1, 0, 0, -1$ |
| $(e^*)^{10}\gamma^\wedge$ | $-1, -1, 0, 1, 0$ | $-1, 0, 0, 0, 1, 1$ | $-1, 0, 0, 0, 1, 1$ |
| $(e^*)^{11}\gamma^\wedge$ | $0, -1, 0, 0, 1$ | $0, -1, 0, 1, -1, 0$ | $0, -1, 0, 1, -1, 0$ |

Figure B.19. The linkage system of $E_6$, two components, 54 linkage diagrams, 6 loctets.
Here is the table as plain text:

| $\gamma^\vee$ | 0, 0, 0, 0, 0, 0 | 0, 0, 0, 0, 0, 0 | 0, -1, 0, 0, 0, 0 | -1, 1, 0, 0, 0, 0 |
|---------------|------------------|------------------|------------------|------------------|
| $e^*\gamma^\vee$ | 1, 0, 0, -1, 0, 0 | 0, 0, 0, 0, 0, 0 | 0, 0, 0, 0, 0, 0 | 0, 0, 0, 0, 0, 0 |
| $(e^*)^2\gamma^\vee$ | 0, 1, -1, 0, 0, 0 | 1, 0, 0, 0, 0, 0 | 1, 0, 0, 0, 0, 0 | 0, 1, 0, 0, 0, 0 |
| $(e^*)^3\gamma^\vee$ | 1, 0, 0, 0, -1, 0 | 1, 0, 0, 0, 0, 0 | 0, -1, 0, 0, 0, 0 | 0, 0, 0, 0, 0, 0 |
| $(e^*)^4\gamma^\vee$ | 0, 1, 0, 0, 0, 1 | 0, 1, 0, 0, 0, 0 | -1, 1, 0, 0, 0, 0 | 1, 0, 0, 0, 0, 0 |
| $(e^*)^5\gamma^\vee$ | 1, 0, 0, 1, 0, 0 | 0, -1, 0, 1, 0, 0 | 0, 0, 0, 1, 0, 0 | 0, 1, 0, 1, 0, 0 |
| $(e^*)^6\gamma^\vee$ | 0, 1, -1, 1, 0, 0 | -1, 0, -1, 1, 0, 0 | -1, 0, 1, 1, 0, 0 | 0, 1, 0, 0, 1, 0 |
| $(e^*)^7\gamma^\vee$ | -1, 0, 1, 1, 1, 1 | 0, -1, 0, 1, 1, 0 | 0, -1, 0, 1, 1, 0 | 0, 1, 0, 1, 1, 1 |

Table B.13. $D_6(a_1)$, there exist 10 semi-Coxeter orbits: nine of length 8, and one of length 4. Pairs of orbits $\{1, 2\}$, $\{3, 4\}$, $\{6, 7\}$, $\{8, 9\}$ are pairs of opposite orbits. Orbits 5 and 10 are self-opposite.
Table B.14. $D_6(a_2)$, there exist 14 semi-Coxeter orbits: orbits 1 – 12 are of length 6, and orbits 13, 14 are of length 2. All orbits are self-opposite.
| $v_s$ | Orbit 1 | Orbit 2 | Orbit 3 | Orbit 4 |
|-------|---------|---------|---------|---------|
| $v_s$ | 0, 0, -1, 0, 0, 0 | -1, 0, 0, 1, 0, 0 | 1, -1, 0, 0, 0, 0 | -1, 0, 0, 0, 0, 0 |
| $\gamma^V$ | 0, 0, -1, 0, -1, 0 | 1, 0, 0, -1, -1, 1 | -1, 1, 0, 0, 0, -1 | 1, 0, 0, -1, 0, 0 |
| $(e^*)^2 \gamma^V$ | 1, 1, 0, 0, -1, -1 | 0, 0, 1, 0, 0, -1 | 1, 0, 0, 1, -1, 0 | 0, 1, 0, 0, -1, -1 |
| $(e^*)^3 \gamma^V$ | 0, 1, 1, -1, -1 | 0, 1, -1, 0, -1 | 0, 1, 1, -1, -1, 0 | 1, 1, 0, 1, -1, 1 |
| $(e^*)^4 \gamma^V$ | 1, 1, 0, 0, -1, 0 | 0, 0, 1, 0, -1, 1 | 1, 0, 0, 1, 0, -1 | 0, 1, 1, 0, -1, 0 |
| $(e^*)^5 \gamma^V$ | 0, 0, 1, 0, 0, 0 | 1, 0, 0, -1, 0, 0 | -1, 1, 0, 0, 0, 0 | 1, 0, 1, 0, 0, 0 |
| $(e^*)^6 \gamma^V$ | 0, 0, -1, 0, 1, 0 | -1, 0, 1, 1, -1 | 1, -1, 0, 0, 0, 1 | -1, 0, 0, 1, 0, 0 |
| $(e^*)^7 \gamma^V$ | -1, -1, 0, 0, 1, 1 | 0, 0, -1, 0, 1, 1 | -1, 0, 0, -1, 1, 0 | 0, -1, -1, 0, 1, 1 |
| $(e^*)^8 \gamma^V$ | 0, -1, -1, -1, 1 | 0, -1, 1, -1, 0 | 0, -1, -1, 1, -1 | -1, -1, 0, -1, 1, 1 |
| $(e^*)^9 \gamma^V$ | -1, -1, 0, 0, 1, 0 | 0, -1, 1, 1, -1 | -1, 0, 0, -1, 0, 1 | 0, -1, -1, 0, 1, 0 |

| Orbit 5 | Orbit 6 | Orbit 7 | Orbit 8 |
|---------|---------|---------|---------|
| $\gamma^V$ | 0, 0, -1, 1, 0, 0 | 0, -1, 1, 0, 0, 0 | 0, 0, 0, 0, 0, 0 | 1, 0, -1, 0, 0, 0 |
| $(e^*)^2 \gamma^V$ | 0, 0, 1, -1, 1, 0 | 0, 1, 1, 1, 1, 0 | 0, 0, 1, 0, -1, 0 | 0, 1, 0, 1, 0, 0 |
| $(e^*)^3 \gamma^V$ | 1, 0, 0, 0, -1, 1 | 0, 0, 1, 1, -1, 0 | 0, 1, 0, 0, -1, 0 | 1, 0, 0, -1, 1, 0 |
| $(e^*)^4 \gamma^V$ | -1, 1, 0, 1, 0, 1 | 1, 1, 1, 0, -1, 1 | 0, 1, 1, -1, 1, 0 | 0, -1, -1, 0, 1, 1 |
| $(e^*)^5 \gamma^V$ | 0, 0, 1, 0, -1, 1 | 0, 0, 1, 1, -1, 0 | 1, 0, 0, 0, -1, 1 | 0, 1, 0, 1, 0, 0 |
| $(e^*)^6 \gamma^V$ | 0, 0, -1, 0, 0, 1 | 0, -1, 1, 0, -1, 1 | 0, 0, 0, 1, -1, 0 | 0, 1, 1, 0, 0, 1 |
| $(e^*)^7 \gamma^V$ | -1, -1, 0, 0, 0, 1 | -1, 0, 1, 1, 0, 0 | 0, -1, 1, 0, 1, 0 |

| Orbit 9 | Orbit 10 |
|---------|---------|
| $\gamma^V$ | 0, 1, -1, 1, 0, 0 | 1, -1, 0, 1, 0, 0 |
| $(e^*)^2 \gamma^V$ | 0, 1, -1, 1, 0, 0 | 1, -1, 0, 1, 0, 0 |

Table B.15. $D_8$: there exist 10 semi-Coxeter orbits: 7 of length 10, and three orbits of length 2, all orbits contain unicorolored linkage diagrams. All orbits are self-opposite.
B.2. Semi-Coxeter orbits for $D_7(a_1)$, $E_7(a_1)$, $D_7$, $E_7$.

| $\tilde{s}_1 \tilde{s}_2 \tilde{s}_3 \tilde{s}_4 \tilde{s}_5 \tilde{s}_6 \tilde{s}_7$ | Orbit 1 | Orbit 2 | Orbit 3 | Orbit 4 |
|---|---|---|---|---|
| $E_7(a_1)$ | $\gamma^v$ | $0, 0, 0, 0, 1, -1, 0$ | $0, 0, 0, 0, 0, -1, 0$ | $0, 0, 0, 0, -1, 0, 0$ | $0, 0, 0, 0, 0, 1, 0$ |
| $e^* \gamma^v$ | $0, 1, 0, -1, 1, 0$ | $0, 1, 1, -1, 0, -1$ | $0, 0, 1, -1, 0, 0$ | $-1, 1, 0, 0, 0, -1$ |
| $(e^*)^2 \gamma^v$ | $1, 0, 0, 0, -1, 1$ | $1, 0, 0, 0, 0, -1$ | $0, 1, -1, 0, 0, 0$ | $1, 0, -1, 1, 0, 1$ |
| $(e^*)^3 \gamma^v$ | $0, 0, 1, -1, 0, 0$ | $1, 0, 1, -1, 0, -1$ | $1, 0, 0, 0, 0, -1$ | $0, 1, 0, 0, 0, -1$ |
| $(e^*)^4 \gamma^v$ | $0, 0, 0, 0, 0, -1$ | $1, 0, 0, 0, 0, -1$ | $0, 0, 1, 0, 0, -1$ | $1, 0, 0, -1, 0, 0$ |
| $(e^*)^5 \gamma^v$ | $0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 1$ | $0, 0, 0, 0, 0, 1, 0$ |
| $(e^*)^6 \gamma^v$ | $0, 0, 0, 0, 0, -1$ | $0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 0, 0$ |
| $(e^*)^7 \gamma^v$ | $0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, -1$ | $0, 0, 0, 0, 0, -1$ | $0, 0, 0, 0, 0, -1, 0$ |

**Table B.16.** $E_7(a_1)$, there exist 4 semi-Coxeter orbits, each of which of length 14, all orbits are self-opposite

| $\tilde{s}_1 \tilde{s}_2 \tilde{s}_3 \tilde{s}_4 \tilde{s}_5 \tilde{s}_6 \tilde{s}_7$ | Orbit 1 | Orbit 2 | Orbit 3 (orbit 5 is opposite) | Orbit 4 (orbit 6 is opposite) (no unicolored) |
|---|---|---|---|---|
| $E_7(a_2)$ | $\gamma^v$ | $0, -1, 1, 0, -1, 0, 0$ | $0, 0, 0, 0, 1, -1, 0$ | $0, 0, 0, 0, -1, 0, 0$ |
| $e^* \gamma^v$ | $0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 0$ | $0, 0, 1, -1, 0, 0, 0$ | $1, 0, -1, 0, 0, -1$ |
| $(e^*)^2 \gamma^v$ | $0, 0, 0, 0, 0, 0$ | $0, 0, 1, 0, 0, 0, 0$ | $0, 0, 0, 1, 0, 0, 0$ | $0, 0, 1, 0, 0, 0, 0$ |
| $(e^*)^3 \gamma^v$ | $0, 0, 0, 0, 0, 0, 0$ | $0, 0, 1, 0, 0, 0, 0$ | $0, 0, 0, 1, 0, 0, 0$ | $0, 0, 1, 0, 0, 0, 0$ |
| $(e^*)^4 \gamma^v$ | $0, 0, 0, 0, 0, 0, 0$ | $0, 0, 1, 0, 0, 0, 0$ | $0, 0, 0, 1, 0, 0, 0$ | $0, 0, 1, 0, 0, 0, 0$ |
| $(e^*)^5 \gamma^v$ | $0, -1, 0, 0, -1, 0, 0$ | $0, 0, 0, 0, 0, 0, 0$ | $0, 0, 0, -1, 0, 0, 0$ | $1, 0, 0, 0, 0, 0, -1$ |
| $(e^*)^6 \gamma^v$ | $0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 1$ | $1, 0, 0, 0, 0, 0, 0$ |
| $(e^*)^7 \gamma^v$ | $0, 0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 0, 1$ | $1, 0, 0, 0, 0, 0, 0$ |
| $(e^*)^8 \gamma^v$ | $0, 0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 0, 0$ |
| $(e^*)^9 \gamma^v$ | $0, 0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 0, 0$ |
| $(e^*)^{10} \gamma^v$ | $0, 0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 0, 0$ | $0, 0, 0, 0, 0, 0, 0$ |

**Table B.17.** $E_7(a_2)$, 6 semi-Coxeter orbits. Orbits 1 and 2 are self-opposite. Orbits 3 and 5 (resp. 4 and 6) are opposite
Table B.18. $E_7(a_3)$, there exist 4 semi-Coxeter orbits, one of length 30, two of length 10 and one of length 6. All orbits are self-opposite.
| Orbit 1 | Orbit 2 | Orbit 3 | Orbit 4 |
|--------|--------|--------|--------|
| (no unicolored) | (no unicolored) | (no unicolored) | (no unicolored) |

| $\gamma^\vee$ | 0, 0, 0, 0, 0, 1 | 0, 0, 0, 0, 0, -1 | 0, 0, 0, 0, -1, 0 | -1, 0, 0, 0, 1, 0 |
| $c^\ast \gamma^\vee$ | 1, -1, 0, 0, -1, -1 | 1, 0, -1, 0, -1, 0 | 0, -1, 1, 0, -1, 1 | 0, 1, -1, -1, 0 |
| $(c^\ast)^2 \gamma^\vee$ | 1, -1, 0, 0, 0, 1 | 1, 0, -1, 0, -1, 0 | 0, -1, 1, 0, -1, 0 | 1, 0, 1, -1, 0, 0 |
| $(c^\ast)^3 \gamma^\vee$ | 0, 0, 0, 0, 0, -1 | 0, 0, 0, 0, 1, 0 | 0, 0, 0, 1, 0, 0 | 1, 0, 0, 0, -1, 0 |
| $(c^\ast)^4 \gamma^\vee$ | -1, 1, 0, 1, -1, -1 | -1, 0, 1, 0, -1, 0 | 0, 1, -1, 0, 1, -1 | 0, -1, 1, 1, 0, 0 |
| $(c^\ast)^5 \gamma^\vee$ | -1, 1, 0, 0, 0, -1 | -1, 0, 1, 0, 1, 0 | 0, 1, -1, 0, 1, 0 | -1, 0, -1, 1, 0, -1 |

| Orbit 5 | Orbit 6 | Orbit 7 | Orbit 8 |
|--------|--------|--------|--------|
| (no unicolored) | (no unicolored) | (no unicolored) | (no unicolored) |

| $\gamma^\vee$ | 0, 0, -1, 0, 0, -1 | 0, 1, 1, -1, 0, 0 | 0, -1, 0, 0, -1, 0 | -1, -1, 0, 1, -1, 0 |
| $c^\ast \gamma^\vee$ | 1, 0, 0, -1, 0, 1 | 0, 0, 1, 0, 1, 0 | 0, 0, 1, 0, -1, 0 | 0, 1, 0, 0, 0, 0 |
| $(c^\ast)^2 \gamma^\vee$ | 1, 0, 1, -1, 0, 0 | 0, -1, 0, 1, 0, 1 | 0, 1, 1, -1, 0, 0 | 1, 0, -1, 0, -1, 0 |
| $(c^\ast)^3 \gamma^\vee$ | 0, 0, 1, 0, 0, 1 | 0, -1, -1, 1, 0, 0 | 0, 1, 0, 1, 0, 0 | 1, 1, 0, -1, 0, 0 |
| $(c^\ast)^4 \gamma^\vee$ | -1, 0, 0, 1, 0, 0, -1 | 0, 0, -1, 0, 1, 0 | 0, 0, -1, 1, 0, -1 | 0, 1, 0, 0, 0, -1 |
| $(c^\ast)^5 \gamma^\vee$ | -1, 0, -1, 1, 1, 0, 0 | 0, 1, 0, -1, 0, 0 | 0, -1, -1, 1, 0, 0 | -1, 0, 1, 0, 1, 0 |

| Orbit 9 | Orbit 10 |
|--------|--------|
| (no unicolored) | (no unicolored) |

| $\gamma^\vee$ | 0, 1, 0, -1, 1, 0 | 0, 0, 0, 0, 0, 1 |
| $c^\ast \gamma^\vee$ | 1, 0, 0, -1, 0, 0 | 0, 0, 0, -1, 0, -1 |
| $(c^\ast)^2 \gamma^\vee$ | 1, 0, 0, 0, 0, 1 |
| $(c^\ast)^3 \gamma^\vee$ | 0, -1, 0, 1, -1, 0 |
| $(c^\ast)^4 \gamma^\vee$ | -1, -1, 1, 0, 0, 0 |
| $(c^\ast)^5 \gamma^\vee$ | -1, 0, 0, 0, 0, -1 |

Table B.19. $\mathbb{E}_7(a_4)$, 10 semi-Coxeter orbits, nine of length 6, one of length 2
| Orbit 1 | Orbit 2 | Orbit 3 | Orbit 4 |
|--------|--------|--------|--------|
| $\gamma^\nu$ | 0, 0, 0, -1, 0, 0, 0 | -1, 0, 1, 0, 0, 0, 0 | 1, -1, 0, 0, 0, 0, 0 | 0, 1, -1, 0, 0, 0, 0 |
| $e^*\gamma^\nu$ | 0, 0, 1, 0, 0, -1 | 1, 0, -1, 0, 0, -1 | 0, 1, 0, 0, 0, 1, -1 | 0, -1, 1, 1, 0, 0, 0 |
| $(e^*)^2\gamma^\nu$ | 0, 1, 0, 0, -1, 0, 0 | 0, 0, 1, 0, -1, 1, 0 | 0, 0, 0, 1, 0, -1, 0 |
| $(e^*)^3\gamma^\nu$ | 1, 0, 1, 0, -1, -1, 0 | 0, 1, 0, 0, -1, -1 | 1, 0, 0, -1, -1, 0, 1 |
| $(e^*)^4\gamma^\nu$ | 1, 1, 0, 0, -1, 0, -1 | 1, 0, 0, 1, -1, 1, 0 | 0, 1, 0, 0, 0, -1 |
| $(e^*)^5\gamma^\nu$ | 0, 1, 1, 1, -1, 0, -1 | 0, 1, 1, -1, -1, 0, 0 | 0, 1, -1, 1, 0, 0, -1 |
| $(e^*)^6\gamma^\nu$ | 1, 1, 0, 0, -1, 0, 0, -1, 0, 1, 0, -1, 0 |
| $(e^*)^7\gamma^\nu$ | 1, 0, 1, 0, -1, 0, 0 | 0, 1, 0, 0, -1, 1, 0 | 1, 0, 0, -1, 0, -1, 0 |
| $(e^*)^8\gamma^\nu$ | 0, 1, 0, 0, 0, 0, -1 | 0, 0, 1, 0, 0, -1, 0 | 0, 0, 1, 0, 1, -1 |
| $(e^*)^9\gamma^\nu$ | 0, 0, 0, 1, 0, 0, 0, -1, 0, 1, 0, 0, 0, 0, 0 |
| $(e^*)^{10}\gamma^\nu$ | 0, 0, -1, 0, 0, 1 | -1, 0, 1, 0, 0, 1, 0 | 1, -1, 0, 0, 0, -1, 1 |
| $(e^*)^{11}\gamma^\nu$ | 0, -1, 0, 0, 0, 1, 0, 0 | 0, -1, 0, 1, -1, 1, 0 | 0, 0, 0, -1, 0, 1, 0 |
| $(e^*)^{12}\gamma^\nu$ | -1, 0, -1, 0, 1, 1, 0 | 0, -1, 0, 0, 0, -1, 1 | 0, 0, 0, 1, 1, 0, -1 |
| $(e^*)^{13}\gamma^\nu$ | -1, -1, 0, 0, 1, 0, 1 | -1, 0, 0, -1, 0, 1, 0 | 0, -1, 0, 0, 0, 1 |
| $(e^*)^{14}\gamma^\nu$ | -1, 0, -1, -1, 1, 1, 0 | 0, -1, -1, 1, -1, 0, 0 | 0, -1, 1, -1, 0, 1 |
| $(e^*)^{15}\gamma^\nu$ | -1, -1, 0, 0, 1, 1, 0 | -1, 0, 0, -1, 0, 1, 1 | 0, -1, 0, 1, -1, 0 |
| $(e^*)^{16}\gamma^\nu$ | -1, 0, -1, 0, 1, 0, 0 | -1, 0, 0, 0, 1, -1, 0 | 1, 0, 0, 1, 0, 1, 0 |
| $(e^*)^{17}\gamma^\nu$ | 0, -1, 0, 0, 0, 1 | 0, 0, -1, 0, 0, 1, 0 | 0, 0, 0, -1, 0, -1, 1 |

Table B.20. $E_7$, 4 semi-Coxeter orbits, three of length 18 and one of length 2. All orbits are self-opposite. All orbits contain $\beta$-unicolored linkage diagrams.
| Orbit 1 | Orbit 2 | Orbit 3 | Orbit 4 |
|--------|--------|--------|--------|
| c γ^ν | 0, 0, 1, 0, 0, 0 | 1, -1, 0, 0, 0, 0 | 0, -1, 0, 1, 0, 0 | -1, 0, 1, 0, 0, 0 |
| γ^ν | 0, 0, -1, 0, 1, 0 | -1, 1, 0, 0, 0, 1 | 0, 1, 0, -1, 1, 1 | 1, 0, -1, 1, 0, 1 |
| (c^*)^2 γ^ν | -1, 0, -1, 0, 1, 0 | 0, 0, -1, -1, 1, 0 | 0, 1, 0, 0, 0, -1 | -1, 1, 0, 0, 0, 0 |
| (c^*)^4 γ^ν | -1, -1, 0, -1, 1, -1 | -1, -1, 1, 0, 1, -1 | 1, -1, 0, 1, 0, -1 | -1, -1, 0, 1, 0, -1 |
| (c^*)^6 γ^ν | -1, -1, 0, 0, 1, 0 | 0, -1, 0, -1, 0, 1 | 0, 1, 0, -1, 0, 1 | -1, -1, 0, 1, 0, 1 |
| (c^*)^8 γ^ν | 0, 1, 0, 0, -1, 0 | 1, 0, -1, 0, 0, -1 | 0, 0, -1, 1, 1, 0 | 0, 0, -1, 1, 1, 0 |
| (c^*)^10 γ^ν | 1, 0, 1, 0, -1, 1 | 0, 1, 1, -1, 1, 0 | 0, -1, 0, 0, 0, -1 | 0, -1, 0, 0, 0, 0 |
| (c^*)^11 γ^ν | 0, -1, 0, 1, -1, 0 | -1, -1, 0, 0, -1, 1 | 0, 1, -1, 1, -1, 1 | 0, 1, -1, 1, -1, 1 |
| (c^*)^12 γ^ν | -1, -1, 0, 0, 1, 0 | 0, -1, -1, 1, 0, 0 | 0, -1, 1, 0, 0, 0 | 0, -1, 1, 0, 0, 0 |
| (c^*)^13 γ^ν | -1, 0, -1, 1, 1, 1 | -1, -1, 1, 1, 1 | 1, -1, 1, 0, 1, -1 | 0, -1, 0, 0, 0, 1 |
| (c^*)^14 γ^ν | -1, 0, -1, 0, 1, 0 | 0, -1, -1, 0, 0, 1 | 0, 1, 0, 0, 0, -1 | 0, 1, 0, -1, 0, 1 |
| (c^*)^15 γ^ν | -1, 0, 1, 0, -1, 0 | -1, 0, 1, 0, 0, -1 | 0, 0, -1, 1, -1, 0 | 0, 1, 0, -1, 0, -1 |
| (c^*)^16 γ^ν | 0, 0, 1, 0, -1, 0 | 1, -1, 0, 0, 0, -1 | 0, -1, 0, 1, 0, -1 | 0, -1, 0, 1, 0, -1 |
| (c^*)^17 γ^ν | 1, 1, 0, 0, -1, 1 | 0, 1, 1, -1, 1, 0 | 0, 0, -1, 0, 0, 1 | 0, 0, -1, 0, 1, 1 |
| (c^*)^18 γ^ν | 1, 1, 0, -1, -1, 0 | 1, 1, -1, -1, 0, 0 | -1, 1, 0, -1, 0, 1 | -1, 1, 0, -1, 0, 1 |
| (c^*)^19 γ^ν | 1, 0, 1, 0, -1, 1 | 0, 1, 1, 1, 0, -1 | 0, -1, 0, 1, 0, -1 | 0, -1, 0, 1, 0, -1 |

Table B.21. \( D_7(a_1) \), there exist 10 semi-Coxeter orbits. Semi-Coxeter orbits 1-4 (three of length 20 and one of length 4) belong to the first \( E \)-type component. Every orbit out of 1-4 has the opposite one (7-10) lying in the second \( E \)-type component. Orbits 5 and 6 are self-opposite. All orbits contain \( \alpha \)-unicolored or \( \beta \)-unicolored linkage diagrams.
Table B.22. \( \text{D}_7(a_2) \), there exist 10 semi-Coxeter orbits. Semi-Coxeter orbits 1-4 (two of length 24 and two of length 8) belong to the first \( E \)-type component. Every orbit out of 1-4 has the opposite one (7-10) lying in the second \( E \)-type component. Orbits 5 and 6 are self-opposite. All orbits contain \( \alpha \)-unicolored or \( \beta \)-unicolored linkage diagrams.
**Table B.23.** $D_7$, there exist 14 semi-Coxeter orbits. Semi-Coxeter orbits 1-6 (5 of length 12 and one of length 4) belong to the first $E$-type component. Every orbit out of 1-6 has the opposite one (9-14) lying in the second $E$-type component. Orbits 7 and 8 are self-opposite. Orbits 1, 3 (and opposite to these orbits, i.e., 9, 11) do not contain unicolored linkage diagrams.
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