Polynomials can be used to represent real-world situations, and their roots have real-world meanings when they are real numbers. The fundamental theorem of algebra tells us that every nonconstant polynomial \( p \) with complex coefficients has a complex root. However, no analogous result holds for guaranteeing that a real root exists to \( p \) if we restrict the coefficients to be real. Let \( n \geq 1 \) and \( P_n \) be the vector space of all polynomials of degree \( n \) or less with real coefficients. In this article, we give explicit forms of polynomials in \( P_n \) such that all of their roots are real. Furthermore, we present explicit forms of linear transformations on \( P_n \) which preserve real roots of polynomials in a certain subset of \( P_n \).

The organizing of this article is as follows. In Section 2, we review some basic definitions and properties about \( q \)-factorial as well as polynomials and their applications. In Section 3, we present explicit forms of polynomials and show that they always have all real roots. In Section 4, we define a subset \( S \) of \( P_n \) and linear transformations on \( P_n \). Then, we prove that they preserve real roots of all polynomials in \( S \). In Section 5, we give a conclusion of our results.

2. Preliminary

In this section, we introduce \( q \)-factorial and recall basic theorems of polynomials and their applications.

2.1. \( q \)-Factorial

Definition 1. For nonnegative integers \( n \) and \( k \), the number of combinations of \( n \) objects, taken \( k \) at a time, is given by
\[
\frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2) \ldots (n-k+1)}{k!}.
\]  \hfill (1)

By convention, \(\binom{n}{0}\) is defined to be 1, and \(\binom{n}{k}\) is defined to be 0 for \(n < k\).

**Definition 2.** For a nonnegative integer \(n\), the \(q\)-analog of \(n\) is defined to be
\[
[q]^n = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}.
\]  \hfill (2)

By convention, \([0]^n = 0\) is defined to be 0. Note that \(\lim_{q \to 1} [n]^n = n\).

**Definition 3.** For a nonnegative integer \(n\), the \(q\)-factorial of \(n\) denoted by \([n]_q\) is defined to be
\[
[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q.
\]  \hfill (3)

By convention, \([0]_q! = 1\) is defined to be 1. Note that \(\lim_{q \to 1} [n]_q! = n!\), and
\[
[n]_q! = \frac{1 - q^n}{1 - q} \frac{1 - q^{n-1}}{1 - q} \cdots \frac{1 - q^2}{1 - q} \frac{1 - q}{1 - q} = (1 + q + q^2 + \cdots + q^{n-1}) \cdot (1 + q + q^2 + \cdots + q^{n-2}) \cdots (1 + q)(1).
\]  \hfill (4)

**Example 1.** Consider \(n = 4\); we have
\[
[4]_4! = (1 + 1 + 1^2 + 1^3)(1 + 1 + 1^2)(1 + 1) = 24 = 4!,
\]
\[
[4]_2! = (1 + 2 + 2^2 + 2^3)(1 + 2 + 2^2)(1 + 2) = 315.
\]  \hfill (5)

### 2.2. Polynomials

**Definition 4.** An element \(w\) in \(\mathbb{C}\) is said to be a root of a given polynomial \(p\) over \(\mathbb{C}\) if \(p(w) = 0\). The set of all roots of \(p\) is denoted by \(Z(p)\).

**Theorem 1** (the fundamental theorem of algebra; see [7]).

*Every nonconstant polynomial \(p\) with complex coefficients has a complex root.*

This is a remarkable statement; however, no analogous result holds for guaranteeing that a real root exists to \(p\) if we restrict the coefficients to be real numbers.

The following theorem which we use to prove throughout our main results provides a sufficient condition for the existence of all real roots.

**Theorem 2** (see [8]). Let \(p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0\) be a polynomial of degree \(n \geq 2\) with positive real coefficients. If
\[
a_i^2 - 4a_ia_{i+1} > 0 \quad \text{for all } i = 1, 2, 3, \ldots, n - 1,
\]
then \(Z(p) \subseteq \mathbb{R}\).

There are a variety of different applications of polynomials that we can look at. The following examples show real roots of polynomials apply to real-world situations.

**Example 2.** Gravity is roughly constant on the earth’s surface with an acceleration of \(g = 32\text{ ft/sec}^2\), and the height of a rigid object in free fall at time \(t\) is modeled by Newton’s equation of motion:
\[
p(t) = \frac{1}{2}gt^2 + v_0t + p_0,
\]  \hfill (7)

where \(t\) is the elapsed time in seconds, \(v_0\) is the initial velocity in feet/second, and \(p_0\) is the initial height of the object above the ground level in feet. When \(v_0\) and \(p_0\) are known, we have \(p(t_0) = 0\) if \(t_0\) is a root of \(p(t)\). This means that the object will take \(t_0\) seconds to hit the ground. Note that since \(t\) represents time, only nonnegative real roots apply.

In [9], the special case \(v_0 = 0\) is used to translate height measurements of the moving object in the image to metric units in 3D world coordinates and derive relations for the case of rigid objects and then for articulated motion to estimate a person’s height from the video.

**Example 3.** Chemical equilibrium is a state in which the rate of the forward reaction equals the rate of the backward reaction. Consider chemical equilibrium in the gaseous system \(\text{Cl}_2 + 2\text{NO} \rightleftharpoons 2\text{NOCl}\) described by
\[
A + 2B = 2C.
\]  \hfill (8)

In [10], if \(P = 1\) bar and the equilibrium constant \(K_p = 70.23\) bar at 500 K, the expression for \(K_p\) can be written in the equation
\[
f(x) = x^3 - 3x^2 + 3.173x - 0.9422 = 0,
\]  \hfill (9)

where \(x\) is the number of moles of \(A\) reacted at equilibrium from an initial state consisting of 1 mol of \(A\), 2 mol of \(B\), and 2 mol of \(C\). Since \(f(x)\) has two complex roots and one real root \(x_0 = 0.479764\), only the real root is a chemical root of \(f(x)\).

### 3. Real Root Polynomials

Fisk [11] observed that if the coefficients of a polynomial are decreasing sufficiently rapidly, then all of the roots of the polynomial are real numbers. Motivated by this observation, we construct polynomials in the following forms and use Theorem 2 to show that they have all real roots.

**Proposition 1.** Let \(p\) be a polynomial in one of the following forms. Then, \(Z(p) \subseteq \mathbb{R}\).

1. \(p(x) = \sum_{i=0}^{n} \alpha^{-i}x^i, \alpha > 4\)
2. \(p(x) = \sum_{i=0}^{n} \alpha^{-i}x^i, \alpha > 2\)
3. \(p(x) = \sum_{i=0}^{n} x^i/[i]_q!, q \geq 4\)

**Proof.** First, note that if \(\alpha > 4\), then for \(1 \leq i \leq n - 1\),
\[
(\alpha - \gamma) \cdot (\alpha - \gamma') = \alpha^2 - 4\alpha \cdot \gamma + \gamma'^2 > 0.
\]

Similarly, if \(a > 2\), then for \(1 \leq i \leq n - 1\),
\[
(\alpha - \gamma) \cdot (\alpha - (-i)^{1/2}) = \alpha^{2i} - 4\alpha \cdot \gamma + \gamma'^2 > 0.
\]

\[
(11)
\]

Also, if \(q \geq 4\), then for \(1 \leq i \leq n - 1\), \([i+1]_q > q[i]_q\) and hence,
\[
\left(\frac{1}{[i]_q}\right)^2 - \frac{4}{[i-1]_q} \cdot \left(\frac{1}{[i+1]_q}\right)^2 > 0.
\]

\[
(12)
\]

Therefore, by Theorem 2, all roots of \(p\) in the above forms are real. \(\square\)

**Example 4.** By Proposition 1,
\[
\sum_{i=0}^{4} 5^{-i/2} x^i = 1 + x + \frac{1}{5} x^2 + \frac{1}{125} x^3 + \frac{1}{15625} x^4,
\]
\[
\sum_{i=0}^{3} 3^{-i/2} x^i = 1 + \frac{1}{3} x + \frac{1}{81} x^2 + \frac{1}{19683} x^3,
\]
\[
\sum_{i=0}^{5} [i]_3^{-1/2} x^i = 1 + x + \frac{x^2}{2} + \frac{x^3}{105} + \frac{x^4}{8925} + \frac{x^5}{3043425}
\]

have all real roots.

**4. Real Root Preserving Transformations**

Let \(n \geq 3\) and \(S\) be the set of polynomials satisfying conditions in Theorem 2, i.e., \(S = \{ p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \mid a_i \geq 0 \text{ for } 0 \leq i \leq n \text{ and } a_i^2 - 4a_{i-1}a_{i+1} > 0 \text{ for } 1 \leq i \leq n - 1 \}.\) Clearly, \(S\) is a subset of \(P_n\), and all of the roots of polynomials in \(S\) are real numbers.

**Proposition 2.** The following linear transformations on \(P_n\) preserve real roots of polynomials in \(S\).

\((1)\) \(x^i \mapsto \beta x^i\) for \(\beta > 0\)

\((2)\) \(x^i \mapsto \binom{n}{i} x^i\)

\((3)\) \(x^i \mapsto \beta x^i\)

\((4)\) \(x^i \mapsto x^i!\)

\((5)\) \(x^i \mapsto x^i/(n-i)!\)

\((6)\) \(x^i \mapsto x^i/(i)!\)

\((7)\) \(x^i \mapsto x^i/[i]_q!\) for \(q > 0\)

**Proof.** Suppose \(p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0\) is in \(S\). Then, for \(1 \leq i \leq n - 1\), we have \(a_i^2 - 4a_{i-1}a_{i+1} > 0\) for each linear transformation, we show that it preserves real roots of \(p\) using Theorem 2.

\((1)\) Let \(\beta > 0\). Define \(T_1(r(x)) = \beta r(x)\) for \(r(x) \in P_n\). It is obvious that \(T_1\) preserves real roots of all polynomials in \(P_n\).

\((2)\) Define
\[
T_2(r(x)) = \binom{n}{i} x^i \mapsto \beta \binom{n}{i} x^i
\]
\[
(14)
\]

\[
\text{for } r(x) = k_n x^n + k_{n-1} x^{n-1} + \cdots + k_2 x^2 + k_1 x + k_0 \in P_n. \text{ Then, for } 1 \leq i \leq n - 1,
\]
\[
\binom{n}{i} a_i - 4 \binom{n}{i-1} a_{i-1} \binom{n}{i+1} a_{i+1} = \frac{n!^2}{(n-i)! (i-1)! (n-i-1)! (i+1)!}
\]
\[
\left(\frac{(n-i+1)(i+1)}{(n-i)!}\right) a_i - 4a_{i-1}a_{i+1}
\]
\[
(15)
\]

Since \((n-i+1)(i+1) > (n-i)i\), we have \(((n-i+1)(i+1)/(n-i)) a_i^2 - 4a_{i-1}a_{i+1} > 0\), and hence, \(Z(T_2 p) \subseteq \mathbb{R}\).

\((3)\) Define
\[
T_3(r(x)) = n k_n x^{n-1} + (n-1) k_{n-1} x^{n-2} + \cdots + 2k_2 x + k_1,
\]
\[
(16)
\]}
for \( r(x) = k_n x^n + k_{n-1} x^{n-1} + \cdots + k_2 x^2 + k_1 x + k_0 \) 
\( \in P_n \). Then, for \( 2 \leq i \leq n - 2 \),
\[
(ia_i)^2 - 4(i - 1)a_{i-1}(i + 1)a_{i+1} = (i^2 - 1) \left[ \frac{i^2}{i^2 - 1} - 4a_{i-1}a_{i+1} \right] > 0,
\]
and hence, \( Z(T_3)p \subseteq \mathbb{R} \).

(4) Define
\[
T_4(r(x)) = \frac{k_n x^n}{n!} + \frac{k_{n-1} x^{n-1}}{(n-1)!} + \cdots + \frac{k_2 x^2}{2!} + \frac{k_1 x}{1!} + \frac{k_0}{0!},
\]
(18)

and hence, \( Z(T_4)p \subseteq \mathbb{R} \).

(5) Define
\[
T_5(r(x)) = \frac{k_n x^n}{0!} + \frac{k_{n-1} x^{n-1}}{1!} + \cdots + \frac{k_2 x^2}{(n-2)!} + \frac{k_1 x}{(n-1)!} + \frac{k_0}{n!}
\]
(20)

for \( r(x) = k_n x^n + k_{n-1} x^{n-1} + \cdots + k_2 x^2 + k_1 x + k_0 \) 
\( \in P_n \). Then, for \( 1 \leq i \leq n - 1 \), we have
\[
\left( \frac{a_i}{(n - i)!} \right)^2 - \frac{4a_{i-1}}{(n - i - 1)!} \frac{a_{i+1}}{(n - i + 1)!} = \frac{1}{(n - i - 1)!^2 (n - i + 1)(n - i)} \cdot \left[ \frac{n - i + 1}{n - i} a_i^2 - 4a_{i-1}a_{i+1} \right] > 0.
\]
(21)

Thus, \( Z(T_5)p \subseteq \mathbb{R} \).

(6) Define
\[
T_6(r(x)) = \frac{k_n x^n}{n!(n-n)!} + \frac{k_{n-1} x^{n-1}}{(n-1)!(n-n)!} + \cdots + \frac{k_2 x^2}{2!(n-2)!} + \frac{k_1 x}{1!(n-1)!} + \frac{k_0}{0!(n-0)!}
\]
(22)

for \( r(x) = k_n x^n + k_{n-1} x^{n-1} + \cdots + k_2 x^2 + k_1 x + k_0 \) 
\( \in P_n \). Then, for \( 1 \leq i \leq n - 1 \),
\[
\left( \frac{a_i}{i!(n - i)!} \right)^2 - \frac{4a_{i-1}}{(i - 1)!(n - i - 1)!} \frac{a_{i+1}}{(i + 1)!(n - i + 1)!} = \frac{1}{(i + 1)(i - 1)!^2 (n - i + 1)(n - i)} \cdot \left[ \frac{(i + 1)(n - i + 1)}{i(n - i)} a_i^2 - 4a_{i-1}a_{i+1} \right].
\]
(23)

Therefore, \( Z(T_6)p \subseteq \mathbb{R} \).

(7) Let \( q > 0 \). Define
\[
T_7(r(x)) = \frac{k_n x^n}{[n]_q} + \frac{k_{n-1} x^{n-1}}{[n-1]_q} + \cdots + \frac{k_2 x^2}{[2]_q} + \frac{k_1 x}{[1]_q} + \frac{k_0}{[0]_q},
\]
(24)

for \( r(x) = k_n x^n + k_{n-1} x^{n-1} + \cdots + k_2 x^2 + k_1 x + k_0 \) 
\( \in P_n \). Then, for \( 1 \leq i \leq n - 1 \),
\[
\left( \frac{a_i}{[i]_q} \right)^2 - 4a_{i-1} \frac{a_{i+1}}{[i-1]_q} \frac{a_{i+1}}{[i+1]_q} = \frac{1}{[i-1]_q [i]_q [i+1]_q} \] 

\[
\cdot \left[ [i+1]_q [a_{i+1}^2 - 4a_{i-1} a_{i+1}] \right].
\]

(25)

Since \( [i+1]_q / [i]_q = (1 + q + q^2 + \ldots + q^{i-1} + q^i) / (1 + q + q^2 + \ldots + q^i) > 1 \), we have \( [i+1]_q / [i]_q a_i^2 - 4a_{i-1} a_{i+1} > 0 \). Hence, \( Z(T_7 p) \subseteq \mathbb{R} \).

5. Conclusion

Real-world situations that cannot be modeled using a linear function can be approximated using polynomials, and this article gives three explicit forms of real root polynomials and seven explicit forms of real root preserving linear transformations of polynomials that guarantee real-world interpretations and understanding.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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