Abstract

This paper introduces conditions on the symmetric and skew-symmetric parts of time-dependent bilinear forms that imply a parabolic Harnack inequality for appropriate weak solutions of the associated heat equation, under natural assumptions on the underlying space. In particular, these local weak solutions are locally bounded and Hölder continuous.

Keywords: Parabolic Harnack Inequalities, Dirichlet Forms, Heat Kernel.

Introduction

This paper is concerned with parabolic Harnack inequalities for weak solutions of the heat equation associated with non-symmetric, time-dependent, closed bilinear forms that are local, regular and generalize non-symmetric Dirichlet forms.

As observed by Moser ([22, 23, 24]), the parabolic Harnack inequality implies that weak solutions of the heat equation are locally bounded and Hölder continuous. Further, Nash ([25]) and later Aronson [11] developed heat kernel estimates and other results. See also [2, 27, 8, 32, 26].

For purely second order divergence form operators (with no time dependence) on complete Riemannian manifolds, Grigor’yan [11] and Saloff-Coste [29] observed that the parabolic Harnack inequality is equivalent to the volume doubling property and the Poincaré inequality. This characterization of the parabolic Harnack inequality has proved very useful in the development of analysis on rough spaces including spaces equipped with a sub-Riemannian structure ([14, 30]), Lipschitz manifolds, Alexandrov spaces ([15]), polytopal complexes and Gromov-Hausdorff limits of Riemannian manifolds ([37]).

Biroli and Mosco [3] and Sturm [36] extended these ideas in symmetric strongly local, regular Dirichlet spaces equipped with a non-degenerate intrinsic distance. The paper [36] proves a parabolic Harnack inequality for local weak solutions of the heat equation associated with symmetric, time-dependent,
strongly local Dirichlet forms that are all uniformly comparable to a fixed symmetric strongly local regular Dirichlet form that satisfies the doubling property and the Poincaré inequality and defines a metric that induces the original topology of the space.

The work [34] contains further developments towards the parabolic Harnack inequality for a class of (non-symmetric) Dirichlet forms. However, we have encountered difficulty in justifying applying [34] to general weak solutions whose local boundedness is not known, a priori.

The aim of the present work is to deduce the parabolic Harnack inequality from the volume doubling property and the Poincaré inequality in the context of non-symmetric forms. In doing so, we were in part motivated by applications to the study of the heat kernel with Dirichlet boundary condition and a boundary Harnack principle in inner uniform domains. See [17, 16].

Aronson and Serrin [2] developed the theory of parabolic Harnack inequalities for quasi-linear divergence form equations having the proper structure. This includes time-dependent linear equations in divergence form with uniformly elliptic second order term, first and zero order terms with bounded coefficients, that is,

$$\partial_t u(t, x) = \sum_{i,j} \partial_j (a_{i,j}(t, x) \partial_i u(t, x)) + \sum_i b_i(t, x) \partial_i u(t, x) + \sum_j \partial_j (d_j(t, x) u(t, x)) + c(t, x) u(t, x),$$

(0.1)

to be interpreted in the weak sense and where $a_{i,j}, b_i, d_j$ and $c$ are bounded measurable functions with, $\forall \xi \in \mathbb{R}^n, \sum_{i,j} a_{i,j}(t, x) \xi_i \xi_j \geq \epsilon |\xi|^2$, $\epsilon > 0$. If the lower order coefficients $b_i, d_j$ and $c$ all vanish, then the weak solutions satisfy a global scale invariant parabolic Harnack inequality even when $a_{i,j}$ is time-dependent and not necessarily symmetric. This provides a two-sided Aronson heat kernel estimate that is global in time and space. One goal of this paper is to obtain a similar result in the context of Dirichlet spaces. See Corollary 3.7.

On $\mathbb{R}^n$, the Moser iteration technique makes use of derivation-type properties (the linearity and chain rule of the differential operator $\partial_i$) as well as the Cauchy-Schwarz inequality. For symmetric strongly local regular Dirichlet forms, these properties are available due to the existence of an energy measure. In order to treat non-symmetric forms, we need to impose additional structural hypotheses.

We introduce hypotheses (Assumption 0) on the structure of the form that provide us with a decomposition

$$\mathcal{E}(f, g) = \mathcal{E}^*(f, g) + \mathcal{E}^{\text{sym}}(fg, 1) + \mathcal{L}(f, g) + \mathcal{R}(f, g)$$

of $\mathcal{E}$ into a symmetric strongly local part $\mathcal{E}^*$, a symmetric zero order part, a left-strongly local part $\mathcal{L}$ and a right-strongly local part $\mathcal{R}$. A similar - but different - decomposition was obtained in [13], where a (non-local) form is decomposed into a left-strongly local diffusion part, a jump part, and a killing part. Thus, in their decomposition, $\mathcal{R}$ is implicitly contained in $\mathcal{L}$ and in the killing part (which is
different from our symmetric zero order part). The purpose of making $\mathcal{R}$ explicit in our decomposition is that it allows us to use derivation-type properties of the different parts. For instance, $\mathcal{L}$ satisfies a chain rule in the first argument, while $\mathcal{R}(f,g) = -\mathcal{L}(g,f)$ satisfies a chain rule in the second argument.

A further hypothesis (Assumption 1 and 2) constitutes Cauchy-Schwarz-type inequalities for $\mathcal{L}$, $\mathcal{R}$ and the symmetric zero order part. Naturally, these are satisfied when the form corresponds to a second order differential operator as in [0.1].

We fix a symmetric strongly local Dirichlet space $(X, \mu, \mathcal{E}, D(\mathcal{E}))$ satisfying the volume doubling property and the Poincaré inequality. We assume that the Dirichlet form defines a metric $d$ that induces the original topology on $X$. We further assume that $(X, d)$ is a complete metric space. In this space, we consider equations that generalize (0.1). These equations are associated with time-dependent, possibly non-symmetric, local closed bilinear forms $\mathcal{E}_t$ with domain $D(\mathcal{E}_t) = D(\mathcal{E})$ for all $t$.

We impose the structural hypothesis described above on each of the forms $(\mathcal{E}_t, D(\mathcal{E}))$. This allows us to implement the Moser iteration technique in the context of time-dependent non-symmetric forms. In particular, in Section 2.2 we prove the a priori boundedness of local weak solutions and mean value estimates. Thus, we complete the reasoning in [34] (see Remark 1.18). In Section 2.3, we prove the parabolic Harnack inequality for non-symmetric forms, which is not covered in [34, 36].

The proof of the boundedness of local weak solutions utilizes an approximation of the local weak solution by bounded functions, following [2]. Our hypotheses on the forms $\mathcal{E}_t$ allow us to carry over the approximation argument to the context of non-symmetric forms. In this sense, our hypotheses are more restrictive than those in [34]. On the other hand, we do not require that our forms are non-symmetric Dirichlet forms, so in that sense our hypotheses are less restrictive. We carefully explain how these hypotheses allow us to follow [2] and treat local weak solutions without an a priori assumption on their local boundedness.

The results obtained in this paper involve two types of assumptions. The first type concerns the structure of the forms $\mathcal{E}_t$. They are introduced in Sections 1.2 and 1.4 as Assumptions 0, 1 and 2. The second type of assumptions concerns the underlying space, these are introduced in Section 2.1 as Assumptions 3 and 4.

The main results (Harnack inequality and Hölder continuity of weak solutions) are stated in Theorem 2.14, Corollary 2.15 and Corollary 2.17. Applications to heat kernels are described in Section 3.
Radon measure on $X$. On this space, we will consider bilinear forms that generalize (non-symmetric) local Dirichlet forms.

### 1.1 The model form

Throughout this paper, we fix a symmetric, strongly local, regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(X, \mu)$ with energy measure $\Gamma$. In particular,

$$\mathcal{E}(f, g) = \int d\Gamma(f, g), \quad \forall f, g \in D(\mathcal{E}).$$

Recall that the energy measure $\Gamma$ satisfies a chain rule: For any $v, u_1, u_2, \ldots, u_m \in D(\mathcal{E}) \cap L^\infty(X, \mu)$, and $\Phi \in C^1(\mathbb{R}^m)$ with $\Phi(0) = 0$, we have $\Phi(u) \in D(\mathcal{E}) \cap L^\infty(X, \mu)$ and

$$d\Gamma(\Phi(u), v) = \sum_{i=1}^m \Phi_{x_i}(\tilde{u}) d\Gamma(u_i, v), \quad (1.1)$$

where $\Phi_{x_i} := \partial\Phi/\partial x_i$ and $\tilde{u}$ is a quasi-continuous version of $u$, see [9, (3.2.27) and Theorem 3.2.2]. Note that in [9] the energy measure is denoted as $\frac{1}{2} \mu^c_{u,v}$. Further, $\Gamma$ satisfies a sort of Cauchy-Schwarz inequality (cf. [9, Lemma 5.6.1])

$$\left| \int fg \, d\Gamma(u, v) \right| \leq \left( \int f^2 d\Gamma(u, u) \right)^{\frac{1}{2}} \left( \int g^2 d\Gamma(v, v) \right)^{\frac{1}{2}}, \quad (1.2)$$

for any $f, g$ bounded Borel measurable and $u, v \in D(\mathcal{E})$, or for any $f, v \in D(\mathcal{E}) \cap L^\infty(X)$, $g, u \in D(\mathcal{E})$. By [9, Theorem 1.4.2], $D(\mathcal{E}) \cap L^\infty(X, \mu)$ is an algebra. Hence, inequality (1.2) together with a Leibniz rule ([9, Lemma 3.2.5]) implies that

$$\int d\Gamma(fg, fg) \leq 2 \int f^2 d\Gamma(g, g) + 2 \int g^2 d\Gamma(f, f), \quad f, g \in D(\mathcal{E}) \cap L^\infty(X). \quad (1.3)$$

Here, on the right hand side, quasi-continuous versions of $f$ and $g$ must be used. Because the domain of $\mathcal{E}$ plays a fundamental role, we set

$$\mathcal{F} := D(\mathcal{E}) \quad \text{and} \quad \|f\|_\mathcal{F} := \left( \int |f|^2 \, d\mu + \int d\Gamma(f, f) \right)^{\frac{1}{2}}.$$

In our context, the space $\mathcal{F}$ plays the role of the first Sobolev space. By definition, the (essential) support of $f \in L^2(X, d\mu)$ is the support of the measure $|f| d\mu$. For an open set $U \subset X$, we set

$$\mathcal{F}_c(U) := \{ f \in D(\mathcal{E}) : \text{The support of } f \text{ is compact in } U \},$$

$$\mathcal{F}_{loc}(U) := \{ f \in L^2_{loc}(U) : \forall \text{ compact } K \subset U, \exists \int K \in D(\mathcal{E}), f|_K = f|_K \text{ a.e.} \}.$$
Note that $\Gamma(f,g)$ can be defined for $f, g \in \mathcal{F}_{loc}(X)$ by virtue of [9] Corollary 3.2.1. For any $v, u_1, \ldots, u_m \in \mathcal{F}_{loc}(X) \cap L^\infty_c(X,\mu)$ and $\Phi \in C^1(\mathbb{R}^m)$, we have $\Phi(u) \in \mathcal{F}_{loc}(X) \cap L^\infty_c(X,\mu)$ and the chain rule (1.1) holds. For convenience, we set $\mathcal{F} := \mathcal{F}_c(X)$ and $\mathcal{F}_{loc} := \mathcal{F}_{loc}(X)$. We will use this notation throughout. One fundamental assumption for the results of this paper is that all other bilinear forms on $L^2(X,\mu)$ that we will consider will share with $\mathcal{E}$ the same domain $\mathcal{F}$.

### 1.2 Basic structural assumptions on forms

In this section, we introduce some basic notation and definitions regarding bilinear forms. Let $(\mathcal{E},D(\mathcal{E}))$ be a (possibly non-symmetric) bilinear form on $L^2(X,\mu)$. Let

$$\mathcal{E}_{sym}(f,g) = \frac{1}{2} (\mathcal{E}(f,g) + \mathcal{E}(g,f))$$

be the symmetric part of $\mathcal{E}$ and

$$\mathcal{E}_{skew}(f,g) = \frac{1}{2} (\mathcal{E}(f,g) - \mathcal{E}(g,f))$$

the skew-symmetric part.

Recall that $(\mathcal{E},D(\mathcal{E}))$ is local if $\mathcal{E}(f,g) = 0$ for any pair $f,g \in D(\mathcal{E})$ with compact disjoint supports. The form $(\mathcal{E},D(\mathcal{E}))$ is strongly local if $\mathcal{E}(f,g) = 0$ for any pair $f,g \in D(\mathcal{E})$ with compact supports with $f$ constant on a neighborhood of the support of $g$ or vice versa. We say that $1$ is locally in the domain of $\mathcal{E}$ if for any compact set $K \subset X$ there is a function $f_K \in D(\mathcal{E})$ with compact support and such that $f_K = 1$ in a neighborhood of $K$. If that is the case and $\mathcal{E}$ is local then $\mathcal{E}(u,1)$ and $\mathcal{E}(1,u)$ are well defined for any function $u \in D(\mathcal{E})$ with compact support. Indeed, assuming that the support of $u$ is $K$, set $\mathcal{E}(u,1) = \mathcal{E}(u,f_K)$ and note that the result is independent of the choice of the function $f_K \in D(\mathcal{E})$ which has compact support and equals 1 on a neighborhood of $K$.

**Example 1.1.** On $X = \mathbb{R}$, for any choice of $k_1, k_2 \in \mathbb{N}$, the form $(f,g) \mapsto \mathcal{E}(f,g) = \int f^{(k_1)}g^{(k_2)}dx$, where $f^{(k)}$ denotes the $k$-th derivative of $f$ and $f,g \in C^\infty(\mathbb{R})$, is local. It is strongly local if and only if $(k_1,k_2) \neq (0,0)$. However, if $|k_1 - k_2|$ is odd, the symmetric part of the form is degenerate.

**Definition 1.2.** Assume that $(\mathcal{E},D(\mathcal{E}))$ is local and that $1$ is locally in $D(\mathcal{E})$. Define the bilinear forms $\mathcal{L} = \mathcal{L}_{\mathcal{E}}$ and $\mathcal{R} = \mathcal{R}_{\mathcal{E}}$ by

$$\mathcal{L}(u,v) = \frac{1}{4} \left[ \mathcal{E}(uv,1) - \mathcal{E}(1,uv) + \mathcal{E}(u,v) - \mathcal{E}(v,u) \right],$$

$$\mathcal{R}(u,v) = \frac{1}{4} \left[ \mathcal{E}(1,uv) - \mathcal{E}(uv,1) + \mathcal{E}(u,v) - \mathcal{E}(v,u) \right] = -\mathcal{L}(v,u),$$

for any $u,v \in D(\mathcal{E})$ with $uv$ having compact support and $uv \in D(\mathcal{E})$.

**Remark 1.3.** (i) Without further assumption, it is not clear that there are many $u,v \in D(\mathcal{E})$ such that $uv \in D(\mathcal{E})$. We will use this definition only in cases where there are plenty of such $u,v$. 

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(ii) The locality of $\mathcal{E}$ implies that $\mathcal{L}$ is left-strongly local, i.e. $\mathcal{L}(u,v) = 0$ if $u,v$ have compact support and $u$ is constant on a neighborhood of the support of $v$. Moreover, for any $u,v \in D(\mathcal{E})$ with $uv$ of compact support and $uv \in D(\mathcal{E})$, 
$$\mathcal{E}^{skew}(u,v) = \mathcal{L}(u,v) + \mathcal{R}(u,v).$$

**Example 1.4.** In the case $(f,g) \mapsto \mathcal{E}(f,g) = \int (f^{(k_1)}g^{(k_2)} + fg)dx$, $f,g \in C_\infty^\infty(\mathbb{R})$, we have 
$$\mathcal{E}^{sym}(f,g) = \int \left(\frac{1}{2}(f^{(k_1)}g^{(k_2)} + f^{(k_2)}g^{(k_1)}) + fg\right)dx,$$
$$\mathcal{E}^{skew}(f,g) = \int \frac{1}{2}(f^{(k_1)}g^{(k_2)} - f^{(k_2)}g^{(k_1)})dx.$$

If $k_1 = 1, k_2 = 0$, then $\mathcal{L}(f,g) = \frac{1}{2} \int f'gdx$. If $k_1 = 2, k_2 = 0$, then $\mathcal{L}(f,g) = \frac{1}{2} \int (f''g + f'g')dx$. Anticipating on the definition of a “chain rule skew form” given below, note that the skew-symmetric part of $\mathcal{E}$ satisfies a chain rule in the case $k_1 = 1, k_2 = 0$ but not in the case $k_1 = 2, k_2 = 0$.

Changing notation to emphasize the fact that we now make an important extra hypothesis, we consider a bilinear form $\mathcal{E}_*$ whose domain $D(\mathcal{E}_*)$ is equal to the domain $\mathcal{F}$ of our model form $(\mathcal{E}, \mathcal{F})$. Since the model form is a symmetric strongly local regular Dirichlet form, $\mathcal{F}$ has many good properties including the fact that $1 \in \mathcal{F}_ec$ and that $\mathcal{F}_e \cap L^\infty(X, \mu)$ is an algebra and is dense in the Hilbert space $(\mathcal{F}, \| \cdot \|_x)$. We will use freely the notation $\mathcal{E}^{sym}_*, \mathcal{E}^{skew}_*, \mathcal{L} = \mathcal{L}_{\mathcal{E}_*}$ and $\mathcal{R} = \mathcal{R}_{\mathcal{E}_*}$. Note that, by locality of $\mathcal{E}_*$, $\mathcal{L}(u,v)$ and $\mathcal{R}(u,v)$ are well-defined for any $u \in \mathcal{F}_loc \cap L^\infty(X, d\mu)$ and $v \in \mathcal{F}_e \cap L^\infty(X, d\mu)$.

**Definition 1.5.** Assuming $\mathcal{E}_*$ is local with $D(\mathcal{E}_*) = \mathcal{F}$, we say that $\mathcal{E}^{skew}_*$ is a chain rule skew form relative to $\mathcal{F}$ if the following two properties hold:

(i) For any $u,v,f \in \mathcal{F} \cap C_c(X)$, we have 
$$\mathcal{L}(uf,v) = \mathcal{L}(u,fv) + \mathcal{L}(f,uv).$$

(ii) Let $v, u_1, u_2, \ldots, u_m \in \mathcal{F} \cap C_c(X)$ and $u = (u_1, \ldots, u_m)$. For any $\Phi \in C^2(X)$, 
$$\mathcal{L}(\Phi(u), v) = \sum_{i=1}^m \mathcal{L}(u_i, \Phi_{x_i}(u)v).$$

**Remark 1.6.** A chain rule skew form can include a term associated with the second order part of the infinitesimal generator if that second order part is not symmetric. See Example 1.11 below.

**Remark 1.7.** When $\mathcal{E}_*$ is a non-symmetric local regular Dirichlet form, its skew-symmetric part is a chain rule skew form with respect to the domain of the form itself, see [13, Theorems 3.2 and 3.8]. Note that $\mathcal{L}(u, fv)$ is the same as $\frac{1}{2}\langle L(u, v), f \rangle$ in their notation.
Note that the following structural assumptions refer to the domain \( \mathcal{F} \) of the model form \( \mathcal{E} \).

**Assumption 0.** The form \( (\mathcal{E}_*, D(\mathcal{E}_*)) \) is local, its domain \( D(\mathcal{E}_*) \) is \( \mathcal{F} \) and:

(i) The form \( \mathcal{E}_* \) satisfies

\[
\forall f, g \in \mathcal{F}, \quad |\mathcal{E}_*(f, g)| \leq C_* \|f\|_\mathcal{F} \|g\|_\mathcal{F},
\]

and, for all \( f, g \in \mathcal{F} \) with \( fg \in \mathcal{F}_c \),

\[
|\mathcal{E}_*(fg, 1)| + |\mathcal{E}_*(1, fg)| \leq C_* \|f\|_\mathcal{F} \|g\|_\mathcal{F},
\]

for some constant \( C_* \in (0, \infty) \).

(ii) The symmetric bilinear form \( \mathcal{E}_*^{\text{sym}}(f, g) := \mathcal{E}_{\text{sym}}^*(f, g) - \mathcal{E}_{\text{sym}}^*(fg, 1) \), defined for \( f, g \in \mathcal{F} \) with \( fg \in \mathcal{F}_c \), extends to a regular strongly local symmetric Dirichlet form with domain \( \mathcal{F} \). Let \( \Gamma_* \) be the energy measure of \( \mathcal{E}_*^{\text{sym}} \).

(iii) The skew-symmetric part \( \mathcal{E}_*^{\text{skew}} \) is a chain rule skew form with respect to \( \mathcal{F} \).

**Remark 1.8.**

(i) Under Assumption 0(i), the form \( \mathcal{E}_* \) as well as each of the components \( \mathcal{E}_*^{\text{sym}}, \mathcal{E}_*^{\text{sym}} \) is continuous on \( \mathcal{F} \times \mathcal{F} \). Further \( (f, g) \mapsto \mathcal{E}_*(fg, 1) \) and \( (f, g) \mapsto \mathcal{E}_*^{\text{sym}}(fg, 1) \) extend continuously to \( \mathcal{F} \times \mathcal{F} \).

(ii) Under Assumption 0, the chain rule for \( \mathcal{L} \) can be deduced from the Leibniz rule, by the same method as in [13, Theorem 3.4].

(iii) Suppose Assumption 0 is satisfied. Then the Leibniz rule extends to functions \( u, v, f \in \mathcal{F} \) with \( uv, uf, vf \in \mathcal{F} \) and \( uvf \in \mathcal{F}_c \).

The chain rule for \( \mathcal{L} \) of Definition 1.5 extends to functions \( v, u_1, \ldots, u_m \in \mathcal{F} \cap L^\infty(X, d\mu) \) for any \( \Phi \in C^2(\mathbb{R}^m) \). The chain rule (1.1) for \( \Gamma \) extends to functions \( u_1, \ldots, u_m \in \mathcal{F} \) and \( v \in \mathcal{F} \cap L^\infty(X, d\mu) \) for any \( \Phi \in C^1(\mathbb{R}^m) \) with \( \Phi(0) = 0 \) and \( \Phi_i \) uniformly bounded on \( \mathbb{R}^m \) for all \( i \). See [9, (3.2.28)].

(iv) Note that, under Assumption 0, \( f \mapsto \|f\|_\mathcal{F} \) and \( f \mapsto (\mathcal{E}_*(f, f) + \int |f|^2 d\mu)^{1/2} \) are two equivalent norms on \( \mathcal{F} \).

(v) Assume further that

\[
|\mathcal{E}_*^{\text{sym}}(f^2, 1)| \leq C \|f\|_2 \|f\|_\mathcal{F},
\]

for any \( f \in \mathcal{F} \cap \mathcal{C}_* \), where \( \|f\|_2 = (\int |f|^2 d\mu)^{1/2} \). Then there exists \( \lambda \in \mathbb{R} \) such that \( \mathcal{E}_* + \lambda \langle \cdot, \cdot \rangle_\mu \) is a coercive closed form. In addition, this form is positivity preserving. See [20] and Proposition 1.9. In fact, the form \( (\mathcal{E}_*, \mathcal{F}) \) itself is closed and positivity preserving.
Assume that \((E_\ast, \mathcal{F})\) is a (non-symmetric) local regular Dirichlet form. In order for Assumption 0 to be satisfied, we need to assume that the strongly local part of \(E_\ast^{sym}\) is itself a regular Dirichlet form with domain \(\mathcal{F}\). This puts restrictions on its skew-symmetric part \(E_\ast^{skew}\) and on the zero order symmetric part \(E_\ast^{sym}(fg,1)\).

**Proposition 1.9** (Strong version of the locality properties). Under Assumption 0, the following holds:

(i) If \(u, v \in \mathcal{F}\) are such that \(uv = 0\) \(\mu\)-a.e. then \(E_\ast(u,v) = 0\).

(ii) If \(u, v \in \mathcal{F}\) are such that there exists \(c \in \mathbb{R}\) such that \((u - c)v = 0\) \(\mu\)-a.e. then \(E_\ast(u,v) = 0\) and \(L(u,v) = 0\).

**Proof.** See the proof of [5, Theorems 2.4.2 and 2.4.3]. The essential point here is that the common domain of these forms is \(\mathcal{F}\), the domain of a regular Dirichlet form, and the fact that these forms are continuous on \(\mathcal{F} \times \mathcal{F}\) thanks to Assumption 0(i).

**Example 1.10.** On \(X = \mathbb{R}\) equipped with Lebesgue measure \(dx\), let \(a, b, c\) be bounded measurable functions and consider

\[ E_\ast(f,g) = \int f'g'\,dx + \int a(f'g - g'f)\,dx + \int b(f'g + fg')\,dx + \int cfg\,dx \]

with domain the first Sobolev space \(\mathcal{F} = W^{1,2}(\mathbb{R})\). Then \(E_\ast\) obviously satisfies Assumption 0. Assume that the distributions \(a', b'\) are signed Radon measures (obviously, this is not always the case!). The form \(E_\ast\) is not a Dirichlet form in general. Indeed, for \(E_\ast\) to be a Dirichlet form it is necessary that

\[ c \geq b' - a' \quad \text{and} \quad c \geq b' + a'. \]

If \(\gamma\) is a non-negative Radon measure such that \(c + \gamma \geq b' - a' \quad \text{and} \quad c + \gamma \geq b' + a'\) then \((f,g) \mapsto E_\ast(f,g) + \gamma(fg)\) is a Dirichlet form on \(L^2(\mathbb{R}, dx)\) but its domain \(\mathcal{F} \cap L^2(\mathbb{R}, \gamma)\) will, in general, be smaller than the first Sobolev space \(\mathcal{F}\).

**Example 1.11.** On Euclidean space \(X = \mathbb{R}^n\), consider the form

\[ E_\ast(f,g) = \int \left( \sum_{i,j=1}^n a_{i,j} \partial_i f \partial_j g + \sum_{i=1}^n b_i \partial_i f g + \sum_{i=1}^n f d_i \partial_i g + c f g \right) \,dx, \]

with coefficients \(a = (a_{i,j})\), \(b = (b_i)\), \(d = (d_i)\), \(c\) satisfying

(i) \(\sum_{i,j=1}^n |a_{i,j} - a_{j,i}| \leq M\) for some \(M > 0\),

(ii) there are positive constants \(k_0, K_0\) such that \(k_0|\xi|^2 \leq \sum_{i,j} a_{i,j} \xi_i \xi_j \leq K_0|\xi|^2\) for all \(\xi \in \mathbb{R}^n\).
Set \( \tilde{a}_{i,j} := (a_{i,j} + a_{j,i})/2 \) and \( \bar{a}_{i,j} = (a_{i,j} - a_{j,i})/2 \). Then the symmetric part of \( E \) is

\[
E^\text{sym}_*(f,g) = \int \sum_{i,j=1}^n \tilde{a}_{i,j} \partial_i f \partial_j g \, dx + \int \sum_{i=1}^n \frac{b_i + d_i}{2} \partial_i f g \, dx
\]

\[
+ \int \sum_{i=1}^n f \frac{b_i + d_i}{2} \partial_i g \, dx + \int c f g \, dx,
\]

while the skew-symmetric part of \( E \) is

\[
E^\text{skew}_*(f,g) = \int \sum_{i,j=1}^n \bar{a}_{i,j} \partial_i f \partial_j g \, dx + \int \sum_{i=1}^n \frac{b_i - d_i}{2} \partial_i f g \, dx
\]

\[
+ \int \sum_{i=1}^n f \frac{b_i - d_i}{2} \partial_i g \, dx.
\]

The symmetric part can be written as \( E^\text{sym}_*(f,g) = E^*_s(f,g) + E^\text{sym}_*(fg,1) \), where \( E^*_s \) is the symmetric strongly local part

\[
E^*_s(f,g) = \int \sum_{i,j=1}^n \tilde{a}_{i,j} \partial_i f \partial_j g \, dx.
\]

The skew-symmetric part can be written as \( E^\text{skew}_*(f,g) = \mathcal{L}(f,g) + \mathcal{R}(f,g) \) with

\[
\mathcal{L}(f,g) = \int \sum_{i,j=1}^n \frac{\bar{a}_{i,j}}{2} \partial_i f \partial_j g \, d\mu + \int \sum_{i=1}^n \frac{b_i - d_i}{2} \partial_i f g \, d\mu.
\]

In the context of this paper, the coefficients \((a_{i,j}), (b_i), (d_i), c\) can be allowed to be functions of the time-space variable \((t,x)\) so that the form \( E^*_s \) above would also depend on \( t \). In this example, if all coefficients are bounded measurable, the underlying domain \( F \) is the first Sobolev space and Assumption 0 is satisfied (with respect to that space).

**Remark 1.12.** Condition (i) and (ii) in the above example is equivalent to

(ii') There are positive constants \( k_0, K_0 \) such that

\[
|\sum_{i,j} a_{i,j} \xi_i \xi_j| \leq K_0 |\xi||\zeta| \quad \text{for all } \xi, \zeta \in \mathbb{R}^n
\]

and

\[
k_0 |\xi|^2 \leq \sum_{i,j} a_{i,j} \xi_i \xi_j \quad \text{for all } \xi \in \mathbb{R}^n.
\]
1.3 Some algebraic computations

Let $\mathcal{E}$ be a form satisfying the structural hypotheses of Assumption 0. For a non-negative function $u \in \mathcal{F}_{\text{loc}}(X)$ and a positive integer $n$ let

$$u_n := u \wedge n.$$ 

We will apply the following lemmas only in the case when $d\Gamma_*(\psi, \psi) \leq c\,d\mu$ for some constant $c \in (0, \infty)$.

**Lemma 1.13.** Let $p \in \mathbb{R}$, $\psi \in \mathcal{F}_r(X) \cap L^\infty(X)$, and $0 \leq u \in \mathcal{F}_{\text{loc}}(X) \cap L^2_{\text{loc}}(X, d\Gamma_*(\psi, \psi))$. Assume one of the following hypotheses.

(i) $p \geq 2$,

(ii) $u$ is locally uniformly positive.

Then $uu_n^{p-2} \in \mathcal{F}_{\text{loc}}(X)$, $uu_n^{p-2} \in \mathcal{F}_r(X)$, and for any $k > 0$ it holds

$$(1 - p)\mathcal{E}(u, uu_n^{p-2}) \leq 4k \int u^2 uu_n^{p-2} d\Gamma_*(\psi, \psi)$$

$$+ \left( \frac{|1 - p|^2}{k} + (1 - p) \right) \int \psi^2 uu_n^{p-2} d\Gamma_*(u, u)$$

$$- ((1 - p)^2 + (1 - p)) \int \psi^2 uu_n^{p-2} d\Gamma_*(u_n, u_n).$$

**Remark 1.14.** The above lemma implies that also the functions $uu_n^{p-2}$, $uu_n^{p-1}$, $uu_n^{p/2}$ are in $\mathcal{F}_{\text{loc}}(X)$, and the functions $uu_n^{p-2}$, $uu_n^{p-1}$, $uu_n^{p/2}$ are in $\mathcal{F}_r(X)$.

**Proof.** Approximating $u$ by $u_m$ and using (1.1), (1.3) and the strong locality, we find that $uu_n^{p-2} \in \mathcal{F}_{\text{loc}}(X)$ and $uu_n^{p-2} \in \mathcal{F}_r(X)$.

In order to show (1.5), we first consider the case that $u$ is bounded. Write

$$(1 - p)\mathcal{E}(u, uu_n^{p-2}) = 2(1 - p) \int \psi uu_n^{p-2} d\Gamma_*(u, \psi)$$

$$+ (1 - p) \int \psi^2 d\Gamma_*(u, uu_n^{p-2}).$$

The first integral on the right hand side can be estimated using Cauchy-Schwarz inequality. Due to the Leibniz and chain rule and the strong locality, we have

$$(1 - p) \int \psi^2 d\Gamma_*(u, uu_n^{p-2})$$

$$= (1 - p) \int \psi^2 d\Gamma_*(u, (u - u_n)n^{p-2}) + (1 - p) \int \psi^2 d\Gamma_*(u_n, uu_n^{p-1})$$

$$= (1 - p) \int n^{p-2} \psi^2 d\Gamma_*(u, u - u_n) - (1 - p)^2 \int uu_n^{p-2} \psi^2 d\Gamma_*(u_n, u_n)$$

$$= (1 - p) \int uu_n^{p-2} \psi^2 d\Gamma_*(u, u) - ((1 - p)^2 + 1 - p) \int uu_n^{p-2} \psi^2 d\Gamma_*(u_n, u_n).$$

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Hence, 

\[ R \]

Then \[ L \] rule for \[ W \] bounded.

\[ E \]

\[ u \]

\[ X, d \]

\[ \Gamma \]

\[ R \]

\[ L \]

\[ X \]

\[ d \]

\[ \Gamma \]

\[ X \]

\[ d \]

\[ \Gamma \]

\[ X \]

\[ d \]

Lemma 1.15. Let \( p \in \mathbb{R}, \psi \in \mathcal{F}_s(X) \cap L^\infty(X) \) and \( 0 \leq u \in \mathcal{F}_{loc}(X) \cap L^2_{\text{loc}}(X, d\Gamma_s(\psi, \psi)) \). Assume either of the following hypotheses:

(i) \( p \geq 2 \),

(ii) \( p \neq 0 \) and \( u \) is locally uniformly positive.

Then

\[ E_s^{\text{shw}} (u, uu_n^{-2} \psi^2) = E_s^{\text{shw}} (uu_n^{-2}, uu_n^{-2} \psi^2) + \frac{2 - p}{p} E_s^{\text{shw}} (u_n^{p/2}, u_n^{p/2} \psi^2) \]

\[ + \frac{2 - p}{p} E_s^{\text{shw}} (u_n^p, 1). \]

Proof. Approximating \( u \) by \( u_m \) if necessary, it suffices to prove the assertion for bounded \( u \). We have

\[ E_s^{\text{shw}} (u, uu_n^{-2} \psi^2) = \mathcal{L}(u, uu_n^{-2} \psi^2) + \mathcal{R}(u, uu_n^{-2} \psi^2) \]

\[ = \mathcal{L}(u, uu_n^{-2} \psi^2) - \mathcal{L} (uu_n^{-2}, uu_n^{-2} \psi^2) + \mathcal{R} (u^2 uu_n^{-2}, uu_n^{-2} \psi^2). \]

We consider the three terms on the right hand side separately. By the Leibniz rule for \( \mathcal{R} \) (in the second argument) and the fact that \( \mathcal{R}(f, g) + \mathcal{L}(g, f) = E_s^{\text{shw}} (f, g) \) by definition, we have

\[ \mathcal{R}(u^2 uu_n^{-2}, uu_n^{-2} \psi^2) = \mathcal{R}(u^2 uu_n^{-2}, uu_n^{-2} \psi^2) + \mathcal{R}(uu_n^{-2}, uu_n^{-2} \psi^2) + \mathcal{L}(uu_n^{-2}, uu_n^{-2} \psi^2) \]

Due to the strong left-locality of \( \mathcal{L} \) and the fact that \( u = u_n + (u - u_n) \), we have

\[ \mathcal{L}(u, uu_n^{-2} \psi^2) = \mathcal{L}(u_n, uu_n^{-2} \psi^2) + \mathcal{L}(u - u_n, uu_n^{-2} \psi^2) \]

\[ = \mathcal{L}(u_n, uu_n^{-1} \psi^2) + \mathcal{L}(u - u_n, uu_n^{-2} \psi^2). \]

Observe that, by the locality and bilinearity of \( \mathcal{L} \),

\[ \mathcal{L}((u - u_n) uu_n^{-2}, uu_n^{-2} \psi^2) = \mathcal{L}(u - u_n, uu_n^{-2} uu_n^{-2} \psi^2) \]

\[ = \mathcal{L}(u - u_n, uu_n^{-2} \psi^2). \]

Hence,

\[ -\mathcal{L}(uu_n^{-2}, uu_n^{-2} \psi^2) = -\mathcal{L}(uu_n^{-1}, uu_n^{-2} \psi^2) - \mathcal{L}((u - u_n) uu_n^{-2}, uu_n^{-2} \psi^2) \]

\[ = -\mathcal{L}(uu_n^{-1}, uu_n^{-2} \psi^2) - \mathcal{L}(uu_n^{-2}, uu_n^{-2} \psi^2) \]

\[ = -\mathcal{L}(uu_n^{-2}, uu_n^{-2} \psi^2) + \mathcal{L}(u_n, uu_n^{-1} \psi^2) - \mathcal{L}(u - u_n, uu_n^{-2} \psi^2) \]

\[ = \mathcal{L}(u_n, uu_n^{-1} \psi^2) - \mathcal{L}(u - u_n, uu_n^{-2} \psi^2). \]
\[-(p - 1)\mathcal{L}(u_n, uu_n^{p - 2}\psi^2) - \mathcal{L}(u - u_n, uu_n^{p - 2}\psi^2), \tag{1.7}\]

where, in the last equality, we applied the chain rule for \(\mathcal{L}\) with \(\Phi(x) = x^p\) for \(x \geq 0\) and \(\Phi(x) = 0\) for \(x < 0\) (for \(p \neq 2\) and \(\Phi(x) = x^2\) for \(p = 2\)). Combining (1.6) and (1.7), and applying the Leibniz rule and the chain rule, we obtain

\[
\mathcal{L}(u, uu_n^{p - 2}\psi^2) - \mathcal{L}(uu_n^{p - 2}, u\psi^2) = (2 - p)\mathcal{L}(u_n, uu_n^{p - 2}\psi^2) = \frac{2(2 - p)}{p} \mathcal{L}(uu_n^{p/2}, uu_n^{p/2}\psi^2) = \frac{2 - p}{p} \mathcal{E}^{\text{skew}}(u_n\psi^2, 1) + \frac{2 - p}{p} \mathcal{E}^{\text{skew}}(uu_n^{p/2}, uu_n^{p/2}\psi^2).
\]

\[\blacksquare\]

**Remark 1.16.** For \(p < 2\), we can not apply the chain rule as in the proofs of Lemma 1.13 and Lemma 1.15, and \(u_n^{p/2}\) may not be in \(\mathcal{F}_{\text{loc}}\), unless \(u\) is locally uniformly positive.

### 1.4 Assumptions on the forms

In this section, we consider families of time-dependent forms each of which is of the type introduced in Assumption 0.

For every \(t \in \mathbb{R}\), let \((\mathcal{E}_t, \mathcal{F})\) be a (possibly non-symmetric) local bilinear form. Throughout, we assume further that for every \(f, g \in \mathcal{F}\) the map \(t \mapsto \mathcal{E}_t(f, g)\) is measurable and that, for each \(t\), \(\mathcal{E}_t\) satisfies the structural hypotheses introduced in Assumption 0. In particular, for every \(t \in \mathbb{R}\),

\[\mathcal{E}_t(f, g) = \mathcal{E}_t^{\text{sym}}(f, g) - \mathcal{E}_t^{\text{sym}}(fg, 1),\]

\(f, g \in \mathcal{F},\ fg \in \mathcal{F}_e\), extends to \(\mathcal{F} \times \mathcal{F}\) as a symmetric regular strongly local Dirichlet form with domain \(\mathcal{F}\) and energy measure \(\Gamma_t\).

**Assumption 1.**

(i) There is a constant \(C_1 \in [1, \infty)\) so that for all \(t \in \mathbb{R}\) and all \(f, g \in \mathcal{F} \cap \mathcal{C}_c(X)\),

\[C_1^{-1} \int f^2 d\Gamma(g, g) \leq \int f^2 d\Gamma_t(g, g) \leq C_1 \int f^2 d\Gamma_t(g, g),\]

where \(\Gamma_t\) is the energy measure of \(\mathcal{E}_t\).

(ii) There are constants \(C_2, C_3 \in [0, \infty)\) so that for all \(t \in \mathbb{R}\) and all \(f \in \mathcal{F} \cap \mathcal{C}_c(X)\),

\[|\mathcal{E}_t^{\text{sym}}(f^2, 1)| \leq 2 \left(\int f^2 d\mu\right)^{\frac{1}{2}} \left(C_2 \int d\Gamma(f, f) + C_3 \int f^2 d\mu\right)^{\frac{1}{2}}.\]
(iii) There are constants $C_4, C_5 \in [0, \infty)$ such that for all $t \in \mathbb{R}$ and all $f, g \in \mathcal{F} \cap C_c(X)$,

$$|\mathcal{E}^{skew}_t(f, fg^2)| \leq 2 \left( \int f^2 d\Gamma(g, g) \right)^{\frac{1}{2}} \left( C_4 \int g^2 d\Gamma(f, f) + C_5 \int f^2 g^2 d\mu \right)^{\frac{1}{2}}.$$

Assumption 2. There are constants $C_6, C_7 \in [0, \infty)$ such that for all $t \in \mathbb{R}$,

$$|\mathcal{E}^{skew}_t(f, f^{-1}g^2)| \leq 2 \left( \int d\Gamma(g, g) \right)^{\frac{1}{2}} \left( C_6 \int g^2 d\Gamma(\log f, \log f) \right)^{\frac{1}{2}} + 2 \left( \int d\Gamma(g, g) + \int g^2 d\Gamma(\log f, \log f) \right)^{\frac{1}{2}} \left( C_7 \int g^2 d\mu \right)^{\frac{1}{2}},$$

for all $g \in \mathcal{F} \cap L^\infty(X)$ and all $0 \leq f \in \mathcal{F}_{loc}$ with $f + f^{-1} \in L^\infty_{loc}(X)$.

Remark 1.17. (i) Assumption 1(i) holds if and only if for all $t \in \mathbb{R}$ and all $f \in \mathcal{F} \cap C_c(X)$,

$$C_1^{-1} \mathcal{E}(f, f) \leq \mathcal{E}^{*}_t(f, f) \leq C_1 \mathcal{E}(f, f).$$

See, e.g., [21].

(ii) When we apply Assumption 1 and 2 in computations, we will often make use of the elementary inequality $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$ for $a, b > 0$.

(iii) A consequence of Assumption 1 is that

$$\mathcal{E}^{*}_t(f, f) = \mathcal{E}^{skew}_t(f, f) + \alpha \int f^2 d\mu,$$

where $\alpha = (C_1 C_2 + 2 C_4^{1/2})$, for all $t \in \mathbb{R}$ and all $f \in \mathcal{F} \cap C_c(X)$. Hence, the form $\mathcal{E}(f, g) + \alpha \int f g d\mu$ is a coercive closed form with domain $\mathcal{F}$. Further, this form (hence also $\mathcal{E}_t$) preserves positivity (i.e., the associated semigroup preserves positivity). See [20].

(iv) The forms $\mathcal{E}_t$ satisfy the above assumptions if and only if the adjoint forms $\mathcal{E}^{*}_t(f, g) := \mathcal{E}_t(g, f)$ satisfy them.

(v) If Assumption 1(iii) is satisfied with $C_4 = 0$, then Assumption 2 is satisfied with $C_6 = 0$. To see this, apply Assumption 1(iv) to $\mathcal{E}^{skew}_t(f, f^{-1}g^2) = \mathcal{E}^{skew}_t(f, f^{-1}f^2 g^2)$.

(vi) Assumptions 1 and 2 are satisfied by the classical forms on Euclidean space associated with Example 1.11 when all coefficients are bounded. The constants $C_4, C_6$ can be taken to be equal to 0 only if $(a_{i,j})$ is symmetric, and $C_2, C_5, C_7$ can be taken to be equal to 0 only if $b_i = d_i = 0$ for all $i$ (i.e., if there is no drift term).
Remark 1.18. Let \( \psi \in \mathcal{F}_c(X) \cap L^\infty(X) \), and \( 0 \leq u \in \mathcal{F}_{\text{loc}}(X) \) with \( u + u^{-1} \in L^\infty_{\text{loc}}(X) \). Let \( 0 \neq p \in \mathbb{R} \). Then it follows from Assumption \( \textbf{H} \) and Lemma 1.13 and Lemma 1.15 that there is a constant \( k \geq 1 \) such that for all \( t \in \mathbb{R} \),

\[
\frac{1 - p}{2} \mathcal{E}_t(u, u^{p-1} \psi^2) \leq \frac{1}{k} \int u^pd\Gamma(\psi, \psi) - \frac{1}{k} \frac{(1 - p)^2}{p^2} \int \psi^2d\Gamma(u^{p/2}, u^{p/2}) \\
+ k(p^2 + 1) \int u^p \psi^2d\mu.
\]

This is the strong uniform parabolicity condition (SUP) in \( \textbf{[34]} \) except for the zero order term \( (p^2 + 1) \int u^p \psi^2d\mu \). Note that a first order term in the non-symmetric form \( \mathcal{E}_t \) creates the factor \( (p^2 + 1) \), rather than the factor \( \pm(p - 1) \) stated in \( \textbf{[34]} \).

Since we do not know a priori that local weak solutions are locally bounded (we prove this in Corollary \( \textbf{2.3} \)), the condition (SUP) appears to be not sufficient to deduce mean value estimates, contrary to the statements in \( \textbf{[34]} \). Furthermore, the approximation argument given in \( \textbf{[34]} \) does not seem to work: \( \mathcal{E}(u, u^{p-1} \psi^2) \) should be approximated by \( \mathcal{E}(u, uu^{p-2} \psi^2) \) and not by \( \mathcal{E}(u, u^{p-1} \psi^2) \). The correct approximation was already applied in \( \textbf{[2]} \).

In addition, condition (SUP) as stated in \( \textbf{[34]} \) makes no sense in the case \( p = 0 \). Assumption \( \textbf{2} \) is the correct requirement to treat the case \( p = 0 \), which is essential to deduce the parabolic Harnack inequality from the mean value estimates (cf. Section 2.3).

Definition 1.19. Suppose Assumption \( \textbf{H} \) is satisfied.

\[ D(L_t) = \{ f \in \mathcal{F} : g \mapsto \mathcal{E}_t(f, g) \text{ is continuous w.r.t. } ||\cdot||_{L^2(X)} \text{ on } \mathcal{F}_c(X) \}. \]

For \( f \in D(L_t) \), let \( L_tf \) be the unique element in \( L^2(X) \) such that

\[ -\int L_tf \psi d\mu = \mathcal{E}_t(f, \psi) \quad \text{for all } g \in \mathcal{F}_c(X). \]

Then we say that \((L_t, D(L_t))\) is the infinitesimal generator of \((\mathcal{E}_t, \mathcal{F})\) on \( X \).

See, e.g., \( \textbf{[6], Section IV.2, [19]} \). Note that, by Remark 1.17(iii), for each fixed \( t \), the semigroup generated by \( L_t \) is positivity preserving.

In Section 1.6 and Section 2.3 we will need the following Lemma (as well as simple variants that are omitted). Let \( \varepsilon > 0 \) and set \( u_\varepsilon := u + \varepsilon \).

Lemma 1.20. Suppose Assumption \( \textbf{H} \) is satisfied. Let \( p \in \mathbb{R} \). Let \( \psi \in \mathcal{F}_c(X) \cap L^\infty(X) \) and \( 0 \leq u \in \mathcal{F}_{\text{loc}}(X) \). Suppose \( u \) is locally bounded. Then for any \( t \in \mathbb{R} \), \( k \geq 1 \),

\[
|\mathcal{E}_t(\varepsilon, u_\varepsilon^{p-1} \psi^2)| \leq \frac{4}{k} \int u_\varepsilon^p d\Gamma(\psi, \psi) + \frac{(p - 1)^2}{k} \int \psi^{2p-2}d\Gamma(u_\varepsilon, u_\varepsilon) \\
+ 2(C_2 + C_3^2 + C_5)k \int u_\varepsilon^p \psi^2d\mu.
\]
Proof. If $p = 1$ the assertion trivial, so let us consider the case $p \neq 1$. Observe that
\[ |E_{t}(\varepsilon, u_{\varepsilon}^{-1}\psi^{2})| = \varepsilon|E_{t}^{\text{skew}}(1, u_{\varepsilon}^{-1}\psi^{2}) + E_{t}^{\text{sym}}(1, u_{\varepsilon}^{-1}\psi^{2})|. \]
We apply Assumption 1(iii) with $f = 1$ and $g = u_{\varepsilon}^{-1}\psi$.
\[
\varepsilon|E_{t}^{\text{skew}}(1, u_{\varepsilon}^{-1}\psi^{2})| \leq 2 \left( \int \varepsilon d\Gamma(u_{\varepsilon}^{-1}\psi, u_{\varepsilon}^{-1}\psi) \right)^{\frac{1}{2}} \left( C_{5} \int \varepsilon u_{\varepsilon}^{-1}\psi^{2} \, d\mu \right)^{\frac{1}{2}}
\]
\[
\leq \frac{2}{k} \int \varepsilon u_{\varepsilon}^{-1}\psi \, d\Gamma(\psi, \psi) + \frac{2}{k} \int \varepsilon \psi^{2} \, d\Gamma(u_{\varepsilon}^{-1}, u_{\varepsilon}^{-1})
\]
\[
+ kC_{5} \int \varepsilon u_{\varepsilon}^{-1}\psi^{2} \, d\mu
\]
\[
\leq \frac{2}{k} \int u_{\varepsilon}^{p-1}\psi \, d\Gamma(\psi, \psi) + \frac{2}{k} \frac{(p-1)^{2}}{4} \int \psi^{2} u_{\varepsilon}^{-2} \, d\Gamma(u_{\varepsilon}, u_{\varepsilon})
\]
\[
+ kC_{5} \int u_{\varepsilon}^{p}\psi^{2} \, d\mu,
\]
for any $k > 0$. Here we used (1.3), the chain rule for $\Gamma$ and the fact that $\varepsilon \leq u_{\varepsilon}$.

Next, we apply Assumption 1(ii) with $f = u_{\varepsilon}^{\frac{p-1}{2}}\psi$.
\[
\varepsilon|E_{t}^{\text{sym}}(1, u_{\varepsilon}^{-1}\psi^{2})|
\]
\[
\leq 2 \left( \int \varepsilon u_{\varepsilon}^{p-1}\psi^{2} \, d\mu \right)^{\frac{1}{2}} \left( C_{2} \int \varepsilon d\Gamma(u_{\varepsilon}^{-1}\psi, u_{\varepsilon}^{-1}\psi) + C_{3} \int \varepsilon u_{\varepsilon}^{-1}\psi^{2} \, d\mu \right)^{\frac{1}{2}}.
\]
Now the right hand side can be estimated using (1.3), the chain rule for $\Gamma$ and the fact that $\varepsilon \leq u_{\varepsilon}$. \qed

1.5 Local weak solutions

For a time interval $I$ and a separable Hilbert space $H$, let $L^{2}(I \rightarrow H)$ be the Hilbert space of those functions $v : I \rightarrow H$ such that
\[
\|v\|_{L^{2}(I \rightarrow H)} = \left( \int_{I} \|v(t)\|_{H}^{2} \, dt \right)^{1/2} < \infty.
\]

We say that a function $v \in L^{2}(I \rightarrow H)$ has a distributional time-derivative that can be represented by a function in $L^{2}(I \rightarrow H)$, if there exists $v' \in L^{2}(I \rightarrow H)$ such that for all smooth compactly supported functions $\phi : I \rightarrow H$ we have
\[
\int_{H} \left( \frac{\partial}{\partial t} \phi(t), v(t) \right) \, dt = - \int_{H} (v'(t), \phi(t)) \, dt.
\]
Let $W^{1}(I \rightarrow H) \subset L^{2}(I \rightarrow H)$ be the Hilbert space of those functions $v : I \rightarrow H$ in $L^{2}(I \rightarrow H)$ whose distributional time derivative $v'$ can be represented by functions in $L^{2}(I \rightarrow H)$, equipped with the norm
\[
\|v\|_{W^{1}(I \rightarrow H)} = \left( \int_{I} \|v(t)\|_{H}^{2} + \|v'(t)\|_{H}^{2} \, dt \right)^{1/2} < \infty.
\]

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Identifying $L^2(X, \mu)$ with its dual space and using the dense embeddings $\mathcal{F} \subset L^2(X, \mu) \subset \mathcal{F}'$, we set

$$\mathcal{F}(I \times X) = L^2(I \to \mathcal{F}) \cap W^1(I \to \mathcal{F}'),$$

where $\mathcal{F}'$ is the dual space of $\mathcal{F}$. We note that it is well known (and easy to see since $L^2(X, d\mu)$ is separable) that $L^2(I \to L^2(X, d\mu))$ can be identified with $L^2(I \times X, dt \times d\mu)$. Indeed, continuous functions with compact support in $I \times X$ are dense in both spaces and the two norms coincide on these functions.

We recall the following fact from [28, Lemma 10.4],

$$L^2(I \to \mathcal{F}) \cap W^1(I \to \mathcal{F}') \subset C(\tilde{I} \to L^2(X, \mu)).$$

Therefore, a function $u \in \mathcal{F}(I \times X)$ can be considered as a continuous path $t \mapsto u(t, \cdot)$ in $L^2(X, \mu)$.

Let $U \subset X$ be open. Let

$$\mathcal{F}_{loc}(I \times U)$$

be the set of all functions $u : I \times U \to \mathbb{R}$ such that for any open interval $J$ that is relatively compact in $I$, and any open subset $A$ relatively compact in $U$, there exists a function $u^\sharp \in \mathcal{F}(I \times X)$ such that $u^\sharp = u$ a.e. in $J \times A$.

Let

$$\mathcal{F}_c(I \times X) = \{ u \in \mathcal{F}(I \times X) : \text{the support of } u \text{ is compact in } I \times U \},$$

where the support of $u$ is taken as an element of $L^2(I \times X, dt \times d\mu)$.

**Definition 1.21.** Let $I$ be an open interval and $U$ open. Set $Q = I \times U$. A function $u : Q \to \mathbb{R}$ is a local weak solution of the heat equation $\frac{\partial}{\partial t} u = L_t u$ in $Q$, if

(i) $u \in \mathcal{F}_{loc}(Q)$,

(ii) For any open interval $J$ relatively compact in $I$,

$$\forall \phi \in \mathcal{F}_c(Q), \int_J \langle \frac{\partial}{\partial t} u, \phi \rangle_{\mathcal{F}', \mathcal{F}} \, dt + \int_J E_t(u(t, \cdot), \phi(t, \cdot)) \, dt = 0. \quad (1.8)$$

**Remark 1.22.** We will abuse notation in writing $\int \frac{\partial}{\partial t} u \phi \, d\mu$ for the pairing $\langle \frac{\partial}{\partial t} u, \phi \rangle_{\mathcal{F}', \mathcal{F}}$.

**Definition 1.23.** Let $I$ be an open interval and $U \subset X$ open. Set $Q = I \times U$. A function $u : Q \to \mathbb{R}$ is a local weak subsolution of $\frac{\partial}{\partial t} u = L_t u$ in $Q$, if

(i) $u \in \mathcal{F}_{loc}(Q)$,

(ii) For any open interval $J$ relatively compact in $I$,

$$\forall \phi \in \mathcal{F}_c(Q), \phi \geq 0, \int_J \langle \frac{\partial}{\partial t} u, \phi \rangle_{\mathcal{F}', \mathcal{F}} \, dt + \int_J E_t(u(t, \cdot), \phi(t, \cdot)) \, dt \leq 0. \quad (1.9)$$
We also write $\frac{\partial}{\partial t} u \leq L_t u$ weakly in $Q$ to indicate that a function $u$ is a local weak subsolution in $Q$.

A function $u$ is called a local weak supersolution if $-u$ is a local weak subsolution.

Let $L^2_{\text{loc}}(I \to \mathcal{F}; U)$ be the space of all functions $u : I \to \mathcal{F}$ such that for any open interval $J$ relatively compact in $I$, and any open subset $A$ relatively compact in $U$, there exists a function $u^\Delta \in L^2(I \to \mathcal{F})$ such that $u^\Delta = u$ a.e. in $J \times A$.

**Lemma 1.24.** Suppose that for any compact set $K \subset U$ there exists a cut-off function $\psi \in \mathcal{F}$ such that $0 \leq \psi \leq 1$, $\psi = 1$ on $K$, $\psi$ has compact support in $U$, and $\psi f \in \mathcal{F}$ for all $f \in \mathcal{F}$. A function $u : I \to \mathcal{F}$ is a local weak solution of $\frac{\partial}{\partial t} u = L_t u$ on $Q = I \times U$ if and only if

(i) $u \in L^2_{\text{loc}}(I \to \mathcal{F}; U)$,

(ii) $-\int_I \left\langle \frac{\partial}{\partial t} \phi, u \right\rangle_{\mathcal{F}', \mathcal{F}} dt + \int_I E_t(u(t, \cdot), \phi(t, \cdot)) dt = 0$,

for all $\phi \in \mathcal{F}_c(Q)$.

**Proof.** See [7, Lemma 5.1].

**Remark 1.25.** Assumptions A1-A2 of Section 2.1 below imply existence of cut-off functions as needed in Lemma 1.24.

### 1.6 Estimates for subsolutions and supersolutions

Let $d$ be a metric on $X$ inducing the original topology, and assume that open balls $B(x, r) = \{y \in X : d(x, y) < r\}$ are relatively compact.

Fix parameters $\tau > \tau' > 0$. Let $B = B(x, r) \subset X$, $s \in \mathbb{R}$. For $\sigma \in (0, 1)$, set

\[
\begin{align*}
\sigma B &= B(x, \sigma r), \\
I &= (s - \tau r^2, s + \tau r^2), \\
I^-_\sigma &= (s - \sigma \tau r^2, s + \sigma \tau r^2), \\
I^+_\sigma &= (s - \tau' r^2, s + \sigma \tau r^2), \\
Q &= Q(\tau, x, s, r) = I \times B(x, r), \\
Q^-_\sigma &= I^-_\sigma \times \sigma B, \\
Q^+_\sigma &= I^+_\sigma \times \sigma B.
\end{align*}
\]

The parameter $\tau'$ is introduced to make sure that functions, which are locally $L^2$-integrable over $I$, can be integrated over $I^-_\sigma$ or $I^+_\sigma$.

Let $0 < \sigma' < \sigma < 1$ and $\omega = \sigma - \sigma'$. Let $\psi \in \mathcal{F}_c(B)$ be such that $0 \leq \psi \leq 1$, $\text{supp}(\psi) \subset \sigma B$, $\psi = 1$ in $\sigma' B$, and $d\Gamma(\psi, \psi) \leq c d\mu$ for some $c \in (0, \infty)$. Let
\( \chi \) be a smooth function of the time variable \( t \) such that \( 0 \leq \chi \leq 1, \chi = 0 \) in \((-\infty, s - \sigma r^2)\), \( \chi = 1 \) in \((s - \sigma r^2, \infty)\) and \( 0 \leq \chi' \leq 2/(\omega r^2) \). Let \( d\mu = d\mu \times dt \). Recall that for \( u \in F_{\text{loc}}(X) \) we set \( u_n = u \wedge n \).

**Lemma 1.26.** Suppose Assumption 1 is satisfied. Let \( p \geq 2 \). Let \( u \in F_{\text{loc}}(Q) \) be a non-negative subsolution of the heat equation for \( L_t \) in \( Q \), that is, \( \frac{\partial}{\partial t} u \leq L_t u \) weakly in \( Q \). Suppose \( \int_{Q_t^*} u^p d\mu dt < \infty \). Then there are \( a_1 = a_1(C_1) \in (0, 1) \), and \( A_1, A_2 \in [0, \infty) \) depending on \( C_1 - C_5 \) such that

\[
\sup_{t \in I_{\sigma}^*} \int u^p \psi^2 d\mu + a_1 \int_{Q_t^*} \psi^2 d\Gamma(u^{p/2}, u^{p/2}) dt \\
\leq A_1(p^2 + 1) \int_{Q_t^*} u^p d\Gamma(\psi, \psi) dt \\
+ \left( A_2(C_2 + C_3^{1/2} + C_5) + \frac{2}{\omega r^2} \right) (p^2 + 1) \int_{Q_t^*} u^p \psi^2 d\mu dt.
\]

**Proof.** We follow [2]. Let \( u_n := u \wedge n \).

\[
\mathcal{H}(u) = \begin{cases} 
\frac{1}{p} u^2 u_n^{-2}, & \text{if } u \leq n \\
\frac{1}{2} u^2 u_n^{-2} + \frac{4}{p} \left( \frac{1}{p} - \frac{1}{2} \right), & \text{if } u > n.
\end{cases}
\]

Since \( \frac{\partial}{\partial t} u \leq L_t u \) weakly in \( Q \), we have for any \( t_0 \in I_{\sigma}^* \) and \( J = (s - \frac{1 + \sigma^2}{\omega} r^2, t_0) \),

\[
\int_J \int \frac{\partial}{\partial t}(\mathcal{H}(u)\psi^2) d\mu dt \\
= \int_J \int \frac{\partial u}{\partial t} u u_n^{-2} \psi^2 d\mu dt + \int_J \int \chi' \mathcal{H}(u)\psi^2 d\mu dt \\
\leq - \int_J \mathcal{E}_t(u, u u_n^{-2}) \psi^2 dt + \int_J \int \chi' \mathcal{H}(u)\psi^2 d\mu dt \\
\leq - \int_J \mathcal{E}_t(u, u u_n^{-2}) \psi^2 dt - \int_J \mathcal{E}_t^{\text{sym}}(u, u u_n^{-2} \psi^2) dt \\
- \int_J \mathcal{E}_t^{\text{sym}}(u^2 u_n^{-2} \psi^2, 1) dt + \int_J \int \frac{1}{\omega r^2} u^2 u_n^{-2} \psi^2 d\mu dt.
\]

By Lemma 1.13 and Assumption 1(i), we have for any \( k_1 > p - 1 \),

\[
- \mathcal{E}_t(u, u u_n^{-2} \psi^2) \\
\leq \frac{4k_1}{p - 1} \int u^2 u_n^{-2} \chi d\Gamma_t(\psi, \psi) - \left( 1 - \frac{p - 1}{k_1} \right) \int u_n^{-2} \chi \psi^2 d\Gamma_t(u, u) \\
- (p - 2) \int u_n^{-2} \chi \psi^2 d\Gamma_t(u_n, u_n) \\
\leq \frac{4k_1 C_1}{p - 1} \int u^2 u_n^{-2} \chi d\Gamma_t(\psi, \psi) - \frac{1}{k_1} \left( 1 - \frac{p - 1}{k_1} \right) \int u_n^{-2} \chi \psi^2 d\Gamma_t(u, u) \\
- (p - 2) C_1^{-1} \int u_n^{-2} \chi \psi^2 d\Gamma_t(u_n, u_n).
\]
By strong locality, the chain rule for $\Gamma$, \[1.3\], and because $p^2/4 \geq 1$, we have
\[
\int \psi^2 \chi d\Gamma(u_n^p u^{\frac{p-2}{2}}, u_n^p u^{\frac{p-2}{2}}) = \int u_n^{p-2} \psi^2 \chi d\Gamma(u-u_n, u-u_n) + \int \psi^2 \chi d\Gamma(u_n^p, u_n^p) \\
= 3 \int u_n^{p-2} \psi^2 \chi d\Gamma(u, u) + 2 \int u_n^{p-2} \psi^2 \chi d\Gamma(u_n, u_n) \\
+ \frac{p^2}{4} \int u_n^{p-2} \psi^2 \chi d\Gamma(u_n, u_n) \\
\leq 3 \int u_n^{p-2} \psi^2 \chi d\Gamma(u, u) + \frac{3p^2}{4} \int u_n^{p-2} \psi^2 \chi d\Gamma(u_n, u_n). \\
\]

(1.13)

By Lemma \[1.15\] Assumption 1(iii), \[1.3\] and \[1.3\], we have for any $k_2, k_3, k_4 > 0$,
\[
- \mathcal{E}_t^{\text{show}}(u, uu_n^{p-2} \chi^2) \\
= - \mathcal{E}_t^{\text{show}}(u_n^{p-2}, uu_n^{p-2} \chi^2) - \frac{2-p}{p} \mathcal{E}_t^{\text{show}}(u_n^{p/2}, uu_n^{p/2}) \\
- \frac{2-p}{p} \mathcal{E}_t^{\text{show}}(u_n^p \chi^2, 1) \\
\leq k_2 \int u_n^{p-2} \chi d\Gamma(u, u) + \left( k_3 \left( \frac{2-p}{p^2} + \frac{2}{k_4} \right) \right) \int u_n^p \chi d\Gamma(u, u) \\
+ \frac{C_4}{k_2} \int \psi^2 \chi d\Gamma(u_n^{p-2}, uu_n^{p-2} \chi^2) + \left( \frac{C_4}{k_3} + \frac{2}{k_4} \right) \int \psi^2 \chi d\Gamma(u_n^{p/2}, uu_n^{p/2}) \\
+ \frac{C_5}{k_2} \int u_n^{p-2} \psi^2 \chi d\mu + \left( \frac{C_5}{k_3} + \frac{C_5 k_4 (2-p)^2}{p^2} \right) \int u_n^p \psi^2 \chi d\mu \\
\leq k_2 \int u_n^{p-2} \chi d\Gamma(u, u) + \left( k_3 \left( \frac{2-p}{p^2} + \frac{2}{k_4} \right) \right) \int u_n^p \chi d\Gamma(u, u) \\
+ \frac{3C_4}{k_2} \int u_n^{p-2} \psi^2 \chi d\Gamma(u, u) + \frac{p^2}{4} \left( \frac{3C_4}{k_2} + \frac{C_4}{k_3} + \frac{2}{k_4} \right) \int u_n^{p-2} \psi^2 \chi d\Gamma(u_n, u_n) \\
+ \frac{C_5}{k_2} \int u_n^{p-2} \psi^2 \chi d\mu + \left( \frac{C_5}{k_3} + \frac{C_5 k_4 (2-p)^2}{p^2} \right) \int u_n^p \psi^2 \chi d\mu.
\]

By Assumption 1(ii), \[1.3\] and \[1.13\], we have for any $k_5 > 0$,
\[
- \mathcal{E}_t^{\text{sym}}(u_n^{p-2} \chi^2, 1) \leq \frac{2}{k_5} \int u_n^{p-2} \chi d\Gamma(u, u) + \frac{2}{k_5} \int \psi^2 \chi d\Gamma(u_n^{p-2}, uu_n^{p-2} \chi^2) \\
+ \left( C_2 k_5 + 2C_1^{1/2} \right) \int u_n^{p-2} \psi^2 \chi d\mu \\
= \frac{2}{k_5} \int u_n^{p-2} \chi d\Gamma(u, u) + \frac{6}{k_5} \int u_n^{p-2} \psi^2 \chi d\Gamma(u_n, u_n) \\
+ \frac{3p^2}{2k_5} \int u_n^{p-2} \psi^2 \chi d\Gamma(u_n, u_n).
\]
Then there are \( \eta \) and \( \eta \) finishes the proof.

**Proof.** We follow [2]. Let \( \eta : \eta : (1 + \eta, 2] \) for some small \( \eta > 0 \). Let \( u \in \mathcal{F}_{\infty}(Q) \) be a non-negative subsolution of the heat equation for \( L_t \) in \( Q \), that is, \( \frac{\partial}{\partial t} u \leq L_t u \) weakly in \( Q \). Suppose that \( u \) is locally bounded. Then there are \( a_1 = a_1(C_1) \in (0, 1) \), and \( A_1, A_2 \in [0, \infty) \) depending on \( C_1 \) - \( C_5 \) and \( \eta \) such that

\[
\begin{align*}
\sup_{t \in I_{t_0}} \int H(u) \psi^2 d\mu + (p - 2) & \left( \frac{3C_4}{k_3} + \frac{2}{k_4} + \frac{6}{k_5} \right) \int I_{t_0} \int \chi \psi^2 u_{t_0}^{p - 2} d\Gamma(u_{t_0}, u_{t_0}) dt \\
+ \left( \frac{1}{C_1} - \frac{p - 1}{C_1k_1} - \frac{3C_4}{k_2} - \frac{6}{k_5} \right) \int I_{t_0} \int u_{t_0}^{p - 2} \chi^2 d\Gamma(u, u) dt
\end{align*}
\]

Hence, inequality (1.12) gives

\[
\begin{align*}
\sup_{t \in I_{t_0}} \int H(u) \psi^2 d\mu & + \left( \frac{p - 2}{C_1} - \frac{p^2}{4} - \frac{3C_4}{k_3} + \frac{2}{k_4} + \frac{6}{k_5} \right) \int I_{t_0} \int \chi \psi^2 u_{t_0}^{p - 2} d\Gamma(u_{t_0}, u_{t_0}) dt \\
+ \left( \frac{1}{C_1} - \frac{p - 1}{C_1k_1} - \frac{3C_4}{k_2} - \frac{6}{k_5} \right) \int I_{t_0} \int u_{t_0}^{p - 2} \chi^2 d\Gamma(u, u) dt
\end{align*}
\]

Appropriate choices of \( k_1, k_2, k_3, k_4, k_5 \) allow us to let \( n \) tend to infinity. This finishes the proof. \( \square \)

**Lemma 1.27.** Suppose Assumption [1] is satisfied. Let \( p \in (1 + \eta, 2] \) for some small \( \eta > 0 \). Let \( u \in \mathcal{F}_{\infty}(Q) \) be a non-negative subsolution of the heat equation for \( L_t \) in \( Q \), that is, \( \frac{\partial}{\partial t} u \leq L_t u \) weakly in \( Q \). Suppose that \( u \) is locally bounded. Then there are \( a_1 = a_1(C_1) \in (0, 1) \), and \( A_1, A_2 \in [0, \infty) \) depending on \( C_1 \) - \( C_5 \) and \( \eta \) such that

\[
\begin{align*}
\sup_{t \in I_{t_0}} \int u^p \psi^2 d\mu + a_1 \int I_{t_0} \int \psi^2 d\Gamma(u^{p/2}, u^{p/2}) dt \\
\leq A_1(p^2 + 1) \int Q_{t_0} u^p d\Gamma(\psi, \psi) dt \\
+ \left( A_2(C_2 + C_3^{1/2} + C_5) + \frac{2}{\omega \tau^{1/2}} \right) (p^2 + 1) \int Q_{t_0} u^p \psi^2 d\mu dt.
\end{align*}
\]

**Proof.** We follow [2]. Let \( u_\varepsilon := u + \varepsilon \) for \( \varepsilon > 0 \). Since \( \frac{\partial}{\partial t} u \leq L_t u \) weakly in \( Q \), we have for any \( t_0 \in I_{t_0} \), \( J = (s - \frac{1 + \varepsilon}{20}, t_0) \),

\[
\begin{align*}
\frac{1}{p} \int J \int \frac{\partial}{\partial t_r}(u_{t_0}^{\varepsilon} \psi^2) d\mu dt & = \int J \int \frac{\partial}{\partial t_r} u_{t_0}^{\varepsilon - 1} \psi^2 d\mu dt + \int J \int \frac{\chi}{p} u_{t_0}^{\varepsilon - 1} \psi^2 d\mu dt
\end{align*}
\]

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\begin{align*}
&\leq -\int J E_t(u, u_{\varepsilon}^{p-1} \chi \psi^2) dt + \int J \int \frac{\chi'}{p} u_{\varepsilon}^{p} \psi^2 d\mu \ dt \\
&\leq -\int J E_t(u_{\varepsilon}, u_{\varepsilon}^{p-1} \chi \psi^2) dt - \int J E_{t-\varepsilon}^{\text{sym}}(u_{\varepsilon}^{\psi^2}, 1) dt \\
&\quad - \int J E_{t-\varepsilon}^{\text{skew}}(u_{\varepsilon}, u_{\varepsilon}^{p-1} \chi \psi^2) dt + \int J \int \frac{\chi'}{p} u_{\varepsilon}^{p} \psi^2 d\mu \ dt \\
&\quad + \int J E_t(\varepsilon, u_{\varepsilon}^{p-1} \chi \psi^2) dt. \quad (1.15)
\end{align*}

By Lemma 1.13 and Assumption 1(i), we have for any $k_1 > 0$,
\begin{align*}
- E_t(u_{\varepsilon}, u_{\varepsilon}^{p-1} \chi \psi^2)
&\leq \frac{4k_1}{p-1} \int u_{\varepsilon}^{p} \chi d\Gamma_t(\psi, \psi) - (p-1) \left( 1 - \frac{1}{k_1} \right) \int u_{\varepsilon}^{p-2} \chi \psi^2 d\Gamma_t(u_{\varepsilon}, u_{\varepsilon}) \\
&\leq \frac{4k_1C_1}{p-1} \int u_{\varepsilon}^{p} \chi d\Gamma(\psi, \psi) - (p-1) C_1^{-1} \left( 1 - \frac{1}{k_1} \right) \int u_{\varepsilon}^{p-2} \chi^2 d\Gamma(u_{\varepsilon}, u_{\varepsilon}).
\end{align*}

By Assumption 1(ii) and (1.3), we have for any $k_2 > 0$,
\begin{align*}
|E_{t-\varepsilon}^{\text{sym}}(u_{\varepsilon}^{\chi \psi^2}, 1)| &\leq \frac{2}{k_2} \int u_{\varepsilon}^{p} \chi d\Gamma(\psi, \psi) + \frac{p^2}{2k_2} \int \psi^2 u_{\varepsilon}^{p-2} \chi d\Gamma(u_{\varepsilon}, u_{\varepsilon}) \\
&\quad + (C_2 k_2 + 2C_3^{1/2}) \int u_{\varepsilon}^{p} \chi \psi^2 d\mu.
\end{align*}

By Lemma 1.15, Assumption 1(iii), and (1.3), we have for any $k_3, k_4 > 0$,
\begin{align*}
- E_{t-\varepsilon}^{\text{skew}}(u_{\varepsilon}, u_{\varepsilon}^{p-1} \chi \psi^2)
&= \frac{2}{p} E_{t-\varepsilon}^{\text{skew}}(u_{\varepsilon}^{p/2}, u_{\varepsilon}^{p/2} \chi \psi^2) - \frac{2p}{p} E_{t-\varepsilon}^{\text{skew}}(u_{\varepsilon}^{p \chi \psi^2}, 1) \\
&\leq \left( k_3^2 \frac{4}{p^2} + \frac{2}{k_4} \right) \int u_{\varepsilon}^{p} \chi d\Gamma(\psi, \psi) + \left( \frac{C_4}{k_3} + \frac{2}{k_4} \right) \int \psi^2 \chi d\Gamma(u_{\varepsilon}^{p/2}, u_{\varepsilon}^{p/2}) \\
&\quad + \left( C_5 + C_6 (2-p)^2 \right) \frac{k_4}{p^2} \int u_{\varepsilon}^{p} \psi^2 \chi d\mu.
\end{align*}

By Lemma 1.20, we have for any $k_5 \geq 1$,
\begin{align*}
|E_t(\varepsilon, u_{\varepsilon}^{p-1} \chi \psi^2)| &\leq \frac{4}{k_5} \int u_{\varepsilon}^{p} \chi d\Gamma(\psi, \psi) + \frac{(p-1)^2}{k_5} \int \psi^2 u_{\varepsilon}^{p-2} \chi d\Gamma(u_{\varepsilon}, u_{\varepsilon}) \\
&\quad + 2(C_2 + C_3^{1/2} + C_5) k_5 \int u_{\varepsilon}^{p} \psi^2 \chi d\mu.
\end{align*}

Hence, inequality (1.15) gives
\[ \sup_{t \in I_\varepsilon} \int \frac{1}{p} u_{\varepsilon}^{p} \psi^2 d\mu. \]
Proof. Let \( u_\varepsilon := u + \varepsilon \) for \( \varepsilon > 0 \). Before we consider the cases \( p < 0 \) and \( p \in (0, 1 - \eta) \) separately, we first show some estimates that hold for any \( p \).

Applying Assumption 1(iii) and [1.3], we obtain that for any \( k_1, k_2, k_3 > 0 \),
\[
2|E_{\varepsilon}^\text{skew}(u_\varepsilon^{p/2}, u_\varepsilon^{p/2}\chi\psi^2)| + |2 - p||E_{\varepsilon}^\text{skew}(u_\varepsilon^{p^2}, 1)| + |p|E_{\varepsilon}^\text{skew}(\varepsilon, u_\varepsilon^{p-1}\chi\psi^2)| \\
\leq \left( \frac{4k_1}{k_2} + \frac{2}{k_3} \right) \int u_\varepsilon^2 \chi d\Gamma(\psi, \psi) \\
+ \left( \frac{C_4}{k_1} + \frac{2}{k_2} + \frac{p - 1)^2}{p^2} \right) \int \psi^2 \chi d\Gamma(u_\varepsilon^{p/2}, u_\varepsilon^{p/2})
\]

(1.18)
+ C_5 \left( \frac{1}{k_1} + |2 - p|^2 k_2 + |p|^2 k_3 \right) \int u_\varepsilon^p \chi^2 d\mu.

By Assumption 1(ii) and (1.3), we have for any $k_4 > 0$,
\[ |p| |\mathcal{E}_t^{sym}(u_\varepsilon^{p-1}\chi^{p^2}, 1)| = |p| \left| \mathcal{E}_t^{sym}(u_\varepsilon^{p-1}\chi^{p^2}, 1) - \varepsilon \mathcal{E}_t^{sym}(u_\varepsilon^{p-1}\chi^{p^2}, 1) \right| \]
\[ \leq \frac{4}{k_4} \int u_\varepsilon^p \chi d\Gamma(\psi, \psi) \]
\[ + \frac{2}{k_4} \left( 1 + \frac{(p - 1)^2}{p^2} \right) \int \psi^2 \chi d\Gamma(u_\varepsilon^{p/2}, u_\varepsilon^{p/2}) \]
\[ + 2(C_2 k_4 p^2 + 2C_3^{1/2}|p|) \int u_\varepsilon^p \chi^2 d\mu. \]

By Lemma 1.13 and by Assumption 1(i), we have for any $k_5 \geq 1$,
\[ |p| \mathcal{E}_t(u_\varepsilon, u_\varepsilon^{p-1}\chi^{p^2}) \leq 4k_5 C_1 \frac{|p|}{|p - 1|} \int u_\varepsilon^p \chi d\Gamma(\psi, \psi) \]
\[ - \left( 1 - \frac{1}{k_5} \right) \frac{|p - 1|}{|p|} \frac{4}{C_1} \int \psi^2 \chi d\Gamma(u_\varepsilon^{p/2}, u_\varepsilon^{p/2}). \]

Now we consider the case $p < 0$. Since $-\frac{\partial}{\partial t} u \leq -Lu$ and by Lemma 1.15 and Assumption 1 we have for any $t_0 \in I_\sigma$, and $J = (s - \frac{1 + \sigma \tau r^2}{2}, t_0)$,
\[ \int_J \left( \frac{\partial}{\partial t} \right) \int u_\varepsilon^p \chi \psi^2 d\mu \ dt \]
\[ = p \int_J \int \frac{\partial u}{\partial t} u_\varepsilon^{p-1} \chi \psi^2 d\mu dt + \int \chi' u_\varepsilon^p \psi^2 d\mu dt \]
\[ \leq p \int_J \mathcal{E}_t(u, u_\varepsilon^{p-1}\chi^{p^2}) dt + \int_J \int \chi' u_\varepsilon^p \psi^2 d\mu dt \]
\[ \leq |p| \int_J \mathcal{E}_t(u_\varepsilon, u_\varepsilon^{p-1}\chi^{p^2}) dt + \int_J \int \chi' u_\varepsilon^p \psi^2 d\mu dt \]
\[ + 2 \int_J \mathcal{E}_t^{s1}(u_\varepsilon^{p/2}, u_\varepsilon^{p/2}\chi^{p^2}) dt + |2 - p| \int_J \mathcal{E}_t^{s1}(u_\varepsilon^{p\chi^{p^2}}, 1) dt \]
\[ + |p| \int_J \mathcal{E}_t^{s1}(\varepsilon, u_\varepsilon^{p-1}\chi^{p^2}) dt + |p| \int_J \mathcal{E}_t^{s1}(u_\varepsilon^{p-1}\chi^{p^2}, 1) dt. \]

The proof of (1.19) can now be finished similarly to the proof of Lemma 1.26 and Lemma 1.27 by using the estimates (1.18), (1.19), and (1.21).

Note that the estimates (1.18), (1.19), (1.21) hold in fact for any non-negative function $\chi$ of the time-variable. For the case $p \in (0, 1 - \eta)$, let $\chi$ be such that $0 \leq \chi \leq 1$, $\chi = 0$ in $(s + \sigma \tau r^2, \infty)$, $\chi = 1$ in $(-\infty, s + \sigma \tau r^2)$, and $0 \geq \chi' \geq -2/((\omega \tau r^2)$.

Since $-\frac{\partial}{\partial t} u \leq -Lu$ and by Lemma 1.15 and Assumption 1 we have for any $t_0 \in I_\sigma$, and $J = (t_0, s + \tau r^2)$,
\[ - \int_J \left( \frac{\partial}{\partial t} \right) \int u_\varepsilon^p \chi \psi^2 d\mu \ dt \]
\[-p \int_J \int \frac{\partial \mu}{\partial t} u_{\varepsilon}^{p-1} \chi \psi^2 d\mu dt + \int \chi' u_{\varepsilon} \psi^2 d\mu dt \leq p \int_J \mathcal{E}_t(u, u_{\varepsilon}^{p-1} \chi \psi^2) dt - \int \chi' u_{\varepsilon} \psi^2 d\mu dt \leq p \int_J \mathcal{E}_t(u, u_{\varepsilon}^{p-1} \chi \psi^2) dt - \int \chi' u_{\varepsilon} \psi^2 d\mu dt + 2 \int |\mathcal{E}_t^{\text{skew}}(u_{\varepsilon}^{p/2}, u_{\varepsilon}^{p/2} \chi \psi^2)| dt + |2 - p| \int_J |\mathcal{E}_t^{\text{skew}}(u_{\varepsilon}^{p} \chi \psi^2, 1)| dt + p \int_J |\mathcal{E}_t^{\text{sym}}(u_{\varepsilon}^{p-1} \chi \psi^2, 1)| dt.\]

The proof of (1.17) can now be finished similarly to the proof of Lemma 1.20 and Lemma 1.27 by using the estimates (1.18), (1.19), and (1.21).

2 Parabolic Harnack inequality

2.1 Assumptions on the underlying space

The intrinsic distance \(d := d_{\mathcal{E}}\) induced by \((\mathcal{E}, \mathcal{F})\) is defined as

\[d_{\mathcal{E}}(x, y) := \sup \{ f(x) - f(y) : f \in \mathcal{F}_{\text{loc}}(X) \cap C(X), d\gamma(f, f) \leq d\mu \},\]

for all \(x, y \in X\), where \(C(X)\) is the space of continuous functions on \(X\). Consider the following properties of the intrinsic distance that may or may not be satisfied. They are discussed in [35, 33].

The intrinsic distance \(d\) is finite everywhere and defines

the original topology of \(X\). \hspace{1cm} (A1)

\((X, d)\) is a complete metric space. \hspace{1cm} (A2)

Note that if (A1) holds true, then (A2) is by [35] Theorem 2] equivalent to

\[\forall x \in X, r > 0, \text{ the open ball } B(x, r) \text{ is relatively compact in } (X, d).\] \hspace{1cm} (A2')

Moreover, (A1)-(A2) imply that \((X, d)\) is a geodesic space, i.e. any two points in \(X\) can be connected by a minimal geodesic in \(X\). See [35] Theorem 1]. If (A1) and (A2) hold true, then by [33] Proposition 1],

\[d_{\mathcal{E}}(x, y) = \sup \{ f(x) - f(y) : f \in \mathcal{F} \cap C_{\text{loc}}(X), d\gamma(f, f) \leq d\mu \}, \quad x, y \in X.\]

It is sometimes sufficient to consider property (A2) only on an open, connected subset \(Y\) of \(X\), that is,

\[\text{For any ball } B(x, 2r) \subset Y, B(x, r) \text{ is relatively compact}. \hspace{1cm} (A2-Y)\]

Note that an open set \(Y\) such that \(\overline{Y}\) is complete in \((X, d)\) automatically satisfies (A2-Y), see, e.g., [36] Lemma 1.1(i)].
Definition 2.1. \((\mathcal{E}, \mathcal{F})\) satisfies the volume doubling property on \(Y\), if there exists a constant \(D_Y \in (0, \infty)\) such that for every ball \(B(x, 2r) \subset Y\),
\[ V(x, 2r) \leq D_Y V(x, r), \tag{VD} \]
where \(V(x, r) = \mu(B(x, r))\) denotes the volume of \(B(x, r)\).

Definition 2.2. \((\mathcal{E}, \mathcal{F})\) satisfies the Poincaré inequality on \(Y\) if there exists a constant \(P_Y \in (0, \infty)\) such that for any ball \(B(x, 2r) \subset Y\),
\[ \forall f \in \mathcal{F}, \int_{B(x, r)} |f - f_B|^2 d\mu \leq P_Y r^2 \int_{B(x, 2r)} d\Gamma(f, f), \tag{PI} \]
where \(f_B = \frac{1}{V(x, r)} \int_{B(x, r)} f d\mu\) is the mean of \(f\) over \(B(x, r)\).

Definition 2.3. \((\mathcal{E}, \mathcal{F})\) satisfies the localized Sobolev inequality on \(Y\) if there exist constants \(\nu > 2\) and \(S_Y > 0\) such that for any ball \(B = B(x, r) \subset Y\), we have
\[ \left( \int_B |\nabla_{x, r} f|^2 d\mu \right)^{\frac{\nu - 2}{\nu}} \leq S_Y \frac{r^2}{V(x, r)^{2/\nu}} \left( \int_B d\Gamma(f, f) + r^{-2} \int_B f^2 d\mu \right), \tag{2.1} \]
for all \(f \in \mathcal{F}(B)\).

Theorem 2.4. Let \((X, \mu, \mathcal{E}, \mathcal{F})\) be as above and \(Y \subset X\). If (A1), (A2-Y), volume doubling and Poincaré inequality hold on \(Y\), then the localized Sobolev inequality holds on \(Y\).

Proof. See [36, Theorem 2.6]. \(\square\)

In what follows we will consider the following assumptions where \(Y\) is a fixed open subset of \(X\).

Assumption 3. The form \((\mathcal{E}, \mathcal{F})\) satisfies (A1), (A2-Y), and the Sobolev inequality on \(Y\).

Assumption 4. The form \((\mathcal{E}, \mathcal{F})\) satisfies (A1), (A2-Y), volume doubling on \(Y\) and the Poincaré inequality on \(Y\).

Remark 2.5. Assumption 3 implies the volume doubling property on \(Y\). See, e.g., [31, Theorem 5.2.1].

2.2 Mean value estimates

We follow [2] and [31]. In this section we suppose that Assumption 4 holds true and that Assumption 3 is satisfied on the open set \(Y\). We use the notation of Section 1.6. Let \(A_1\) and \(A_2\) be large enough so that the estimates of Section 1.6 hold with these constants. Let \(A'_2 = A_2(C_2 + C_3^{1/2} + C_5)\). Fix a ball \(B = B(x, r) \subset Y\). Let \(\psi\) be the cut-off function
\[ \psi(y) = \max \{0, (\sigma r - d_E(x, y))/\omega r\} \wedge 1. \tag{2.2} \]
Then $0 \leq \psi \leq 1$, supp($\psi$) $\subset \sigma B$, $\psi = 1$ in $\sigma' B$, and by [33, Section 4.2] $\psi \in \mathcal{F}_c(B)$ and 
\[ d\Gamma(\psi, \psi) \leq (\omega r)^{-2}d\mu. \]

**Theorem 2.6.** Suppose Assumptions 1 and 3 are satisfied. Let $p > 1 + \eta$ for some small $\eta > 0$. Fix a ball $B = B(x, r)$ with $B(x, 2r) \subset Y$ and $\tau > 0$. Then there exists a constant $A = A(\tau, \nu, \eta, C_1 - C_3)$ such that, for any real $s$, any $0 < \delta < \delta' \leq 1$, and any non-negative function $u \in \mathcal{F}_{loc}(Q)$ with $\frac{\partial u}{\partial t} \leq L_i u$ weakly in $Q = Q(\tau, x, s, r)$, we have
\[ \sup_{Q_x} \{u^p\} \leq \frac{A S_Y^{\nu/2}[(A_1 + A_2 \tau r^2)(\nu^2 + 1)]^{1+\nu/2}}{(\delta' - \delta)^{2+\nu/2} Y(x, r)} \int_{Q_x} u^p d\bar{\mu}, \quad (2.3) \]

where $\nu, S_Y$ are the constants of the localized Sobolev inequality [2,4].

**Proof.** For simplicity, we assume that $\tau = \delta' = 1$. First, consider the case $p \geq 2$. Let $E(B) = S_Y V(x, r)^{-2/\nu}$ be the Sobolev constant for the ball $B$ given by [2,4], and set $\beta = \nu/\nu - 2$. By the Hölder inequality, we have for any $v \in \mathcal{F}_c(B)$,
\[ \int_B v^{2(1+2/\nu)} d\mu \leq \left( \int_B v^{2\beta} d\mu \right)^{1/\beta} \left( \int_B v^2 d\mu \right)^{2/\nu}. \]

So, [2,4] gives
\[ \int_B v^{2(1+2/\nu)} d\mu \leq E(B) \left( \int_B d\Gamma(v, v) + r^{-2} \int_B v^2 d\mu \right) \left( \int_B v^2 d\mu \right)^{2/\nu}. \quad (2.4) \]

Let $w = \psi u$. Since $u$ is a local weak subsolution on $Q$, $u \in L^2_{loc}(I \to \mathcal{F}; B)$. Hence, $w \in L^2_{loc}(I \to \mathcal{F}_c(B); B)$. In particular, for almost every $t \in I$, $v = w(t, \cdot)$ is in $\mathcal{F}_c(B)$ and satisfies [2,4]. Integrating over $I_{\nu}$, and applying Hölder inequality, we get
\[ \int_{I_{\nu}} \int_B w^{2\theta} d\mu dt \leq E(B) \left( \int_{I_{\nu}} \int_B d\Gamma(w, w) dt + r^{-2} \int_{I_{\nu}} \int_B w^2 d\mu dt \right) \sup_{t \in I_{\nu}} \left( \int_B w^2 d\mu \right)^{2/\nu}, \quad (2.5) \]

where $\theta = 1 + 2/\nu$. Note that the right hand side is finite by Lemma 1.26 (applied with $p = 2$). Thus, $w^\theta \in L^2(I_{\nu} \to L^2(B))$. Applying Lemma 1.26 with $p = 2\theta$, we obtain that $w^\theta \in L^2(I_{\nu} \to \mathcal{F}_c(B))$ for some $0 < \theta'' < \nu'$. In particular, $u^\theta \in L^2(I_{\nu} \to \mathcal{F}(\sigma'' B))$. Similarly (using a cut-off function that takes the value 1 on $\sigma B$), one can show that $w^\theta \in L^2(I_{\nu} \to \mathcal{F}_c(B))$. Hence, $w = \psi u^\theta$ is in $L^2(I_{\nu} \to \mathcal{F}_c(B))$ and satisfies [2,5].
By an inductive argument we obtain that, for any \( q \geq 1 \), \( \psi u^{pq/2} \) is in 
\( L^2(I^{-}\sigma \rightarrow \mathcal{F}c(B)) \) and satisfies \( \psi u^{pq/2} \) is in 
\( L^2(I^{-}\sigma \rightarrow \mathcal{F}c(B)) \) and satisfies (2.5). Recall that 
\( d\Gamma(\psi, \psi) \leq (\omega r)^{-2} d\mu \). Thus,
\[
\int \int_{Q^{-}\sigma} u^{pqd} d\mu = \int \int_{Q^{-}\sigma} (\psi u^{pq/2})^{2d} d\mu
\]
\[
\leq E(\sigma B) \left( \int_{I^{-}\sigma} \int_{\sigma B} d\Gamma(\psi u^{pq/2}, \psi u^{pq/2}) dt + r^{-2} \int_{I^{-}\sigma} \int_{B} \psi^{2} u^{pq} d\mu \right)
\]
\[
\left( \sup_{t \in I^{-}\sigma} \int_{J_{\sigma B}} \psi^{2} u^{pq} d\mu \right)^{\nu/2}
\]
\[
\leq E(\sigma B) \left( Cq^{2} (\omega r)^{-2} [(A_{1} + A_{2} r^{2})(p^{2} + 1)] \int \int_{Q^{-}\sigma} u^{pq} d\mu \right)^{\theta},
\] (2.6)

for some constant \( C > 0 \). Observe the different roles of \( p \) (which is fixed) and \( q \) (which will be absorbed in the constant \( A^{i+1} \) below).

Set \( \omega_{i} = (1 - \delta)2^{-i} \) so that \( \sum_{i=1}^{\infty} \omega_{i} = 1 - \delta \). Set also \( \sigma_{0} = 1 \), \( \sigma_{i+1} = \sigma_{i} - \omega_{i} = 1 - \sum_{j=1}^{i} \omega_{j} \).
Applying (2.6) with \( q = q_{i} = \theta^{i} \), \( \sigma = \sigma_{i} \), \( \sigma' = \sigma_{i+1} \), we obtain
\[
\int \int_{Q^{-}\sigma_{i+1}} u^{pq^{i+1}} d\mu
\]
\[
\leq E(B) \left( A^{i+1} [(1 - \delta)r]^{-2} [(A_{1} + A_{2} r^{2})(p^{2} + 1)] \int \int_{Q^{-}\sigma_{i}} u^{pq} d\mu \right)^{\theta},
\]
where the constant \( A \) depends on \( \theta \). Hence,
\[
\left( \int \int_{Q^{-}\sigma_{i+1}} u^{pq^{i+1}} d\mu \right)^{\theta^{-1-i}}
\]
\[
\leq E(B) \sum_{j=1}^{\infty} \sum_{i=1}^{j} \sum_{(j+1)\theta^{-j}} \left( [(1 - \delta)r]^{-2} [(A_{1} + A_{2} r^{2})(p^{2} + 1)] \right)^{\theta^{-j}} \int_{Q} u^{p} d\mu,
\]
where all the summations are taken from 0 to \( i \). Letting \( i \) tend to infinity, we obtain
\[
\sup_{Q^{-}\sigma} \{ u^{p} \} \leq E(B)^{v/2} \left( A[(1 - \delta)r]^{-2} [(A_{1} + A_{2} r^{2})(p^{2} + 1)]^{1+v/2} \right) \| u \|_{p,Q}^{p}.
\]

As \( E(B) = SYV(x, r)^{-2i/v^{-2}} \), this yields (2.3).

At this stage of the proof, Corollary 2.8 already follows. Thus, in the case 
\( 1 + \eta < p < 2 \) the assertion can be proved similarly, by using Lemma [27] and 
Corollary 2.8.

**Remark 2.7.** For \( 0 < p < 1 + \eta \), an estimate similar to (2.3) for subsolutions 
holds true with constants depending on \( p \). See [31, Theorem 2.2.3, Theorem 
5.2.9].
Corollary 2.8. Suppose Assumptions 1 and 3 are satisfied. Any non-negative local weak subsolution \( u \) of \( \frac{\partial}{\partial t} u = L_t u \) on \( Y \) is locally bounded. In particular, any local weak solution of \( u \) of \( \frac{\partial}{\partial t} u = L_t u \) on \( Y \) is locally bounded.

Theorem 2.9. Suppose Assumptions 1 and 3 are satisfied. Let \( p \in (-\infty, 0) \). Fix a ball \( B = B(x, r) \) with \( B(x, 2r) \subset Y \) and let \( \tau > 0 \). Then there exists a constant \( A = A(\tau, \nu, C_1 - C_5) \) such that, for any real \( s \), any \( 0 < \delta < \delta' \leq 1 \), and any non-negative, locally bounded function \( u \in \mathcal{F}_{\infty}(Q) \) with \( \frac{\partial}{\partial t} u \geq L_t u \) weakly in \( Q = \mathcal{Q}(\tau, x, s, r) \), we have

\[
\sup_{Q^+_{\tau}} u^p \leq \frac{A S^{1/2}_Y \left( (A_1 + A_2 \tau^2) (p^2 + 1) \right)^{1+\nu/2}}{(\delta - \delta')^{2+\nu^2} V(x, r)} \int_{Q^+_{\tau}} u^p \, d\bar{\mu}.
\]

The above theorem can be proved analogously to the proof of Theorem 2.6, by applying Lemma 1.28 instead of Lemma 1.26.

Theorem 2.10. Suppose Assumptions 1 and 3 are satisfied. Fix a ball \( B = B(x, r) \) with \( B(x, 2r) \subset Y \) and let \( \tau > 0 \). Then there exists a constant \( A = A(p_0, \tau, \nu, C_1 - C_5) \) such that, for any real \( s \), \( 0 < \delta < \delta' \leq 1 \), \( 0 < p < p_0/\theta \), and any non-negative, locally bounded function \( u \in \mathcal{F}_{\infty}(Q) \) with \( \frac{\partial}{\partial t} u \geq L_t u \) weakly in \( Q = \mathcal{Q}(\tau, x, s, r) \), we have

\[
\left( \int_{Q^+_{\tau}} u^{p_0} \, d\bar{\mu} \right)^{\frac{1}{p_0}} \leq \left[ \frac{A S^{1/2}_Y (A_1 + A_2 \tau^2) (p^2 + 1)^{1+\nu/2}}{(\delta - \delta')^{2+\nu^2} \mu(B)} \right]^{\frac{1}{p}} \left( \int_{Q^+_{\tau}} u^p \, d\bar{\mu} \right)^{\frac{1}{p}}.
\]

Proof. We follow [31, Theorem 5.2.17]. For simplicity, we assume for the proof that \( \tau = \delta' = 1 \). Let \( 0 < p < p_0/\theta \). By Lemma 1.28 we have

\[
\sup_{L^+_{\tau}} \left\{ \int_B \psi^2 \, d\mu + a_1 \int_{Q^+_{\tau}} \psi^2 \, dT(u^p, u^p/2) \right\} \leq A_1 (p^2 + 1) \int_{Q^+_{\tau}} \psi^2 \, dT(\psi, \psi) dt + A_2 (p^2 + 1) \int_{Q^+_{\tau}} u^p \, d\bar{\mu}.
\]

Similar to the proof of Theorem 2.6 but with the cylinders \( Q^-_{\tau} \) replaced by \( Q^+_{\tau} \), we find that for any \( 0 < \beta < p_0/\theta \),

\[
\int_{Q^+_{\tau}} u^\beta \, d\bar{\mu} \leq E(B) \left[ \frac{A}{(r\omega)^2} (A_1 + A_2 \tau^2) (\beta^2 + 1) \int_{Q^+_{\tau}} u^p \, d\bar{\mu} \right]^\theta,
\]

where \( E(B) = S_Y r^2 V(x, r)^{-2/\nu} \).

Define \( p_i = p_0 \theta^{-i} \). We first prove the claim for these values of \( p \), using the same iteration as in the proof of [31, Theorem 2.2.5]. Set \( \sigma_0 = 1 \) and \( \sigma_{i-1} - \sigma_i = 2^{-i} (1 - \delta) \). Fix \( i \geq 1 \), and apply (2.7) with \( \beta = p_i \theta^{i-1} \), \( j = 1, 2, \ldots, i \), \( \sigma = \sigma_{i-1}, \sigma' = \sigma_i \). This yields for all \( j = 1, \ldots, i \) (note that \( A \) may change from line to line),

\[
\int_{Q^+_{\tau}} u^{p_i} \, d\bar{\mu} \leq E(B) \left[ A_j^j [(1 - \delta)r]^{-2} [(A_1 + A_2 \tau^2) (p_i^2 + 1)] \int_{Q^+_{\tau}} u^{p_i \theta^j} \, d\bar{\mu} \right]^\theta.
\]
Hence,
\[
\int_{Q^+} w_p^\rho \, d\mu \leq C \left( \int_{Q} w_p^\rho \, d\mu \right)^{\rho^*},
\]
where
\[
C = E(B)^{r^{i+1}} \sum_{j=0}^{i+1} \frac{\theta^j}{\theta - 1} \left( \frac{1}{\theta} + \frac{1}{\rho^*} \right),
\]
Observe that
\[
\sum_{j=0}^{i} \theta^j = \frac{\theta^i - 1}{\theta - 1} = (\nu/2)(p_0/\rho_i - 1),
\]
and
\[
\sigma_i = 1 - \left( \sum_{j=1}^{i} 2^{-j} \right) (1 - \delta) > \delta.
\]
Thus,
\[
\left( \int_{Q^+} w_p^\rho \, d\mu \right)^{\frac{1}{\rho^*}} \leq C\left( \int_{Q} w_p^\rho \, d\mu \right)^{\frac{1}{\rho^*}},
\]
where
\[
C' = \left( E(B) A \left( [(1 - \delta) r]^{-i+1} (A_1 + A_2 r^2) (p_0^2 + 1) \right) \right)^{\frac{1}{\rho^*}}.
\]
To obtain the inequality for any \( p \in (0, \rho_0/\theta) \), see [31, Theorem 2.2.5].

2.3 Parabolic Harnack inequality

**Theorem 2.11.** Suppose that Assumptions 1, 2 and 4 are satisfied. Then \((E,F)\) satisfies the weighted Poincaré inequality on \( Y \). That is, there exists a constant \( C \in (0, \infty) \) such that for any ball \( B(x, r) \subset Y \),
\[
\forall f \in F, \quad \int_{B(x, r)} |f - f_B|^2 \psi^2 \, d\mu \leq C \omega_1 r^2 \int_{B(x, r)} \psi^2 \, d\Gamma(f, f), \tag{2.8}
\]
where \( f_B = \int_{B(x, r)} f \psi^2 \, d\mu / \int_{B(x, r)} \psi^2 \, d\mu \) is the mean of \( f \) over \( B(x, r) \), and \( \psi(z) = \max\{0, (1 - d(x, z))/r\} \in F(B) \).

**Proof.** See [31, Corollary 2.5].
Let

\[ C_8 := C_2 + C_3^{1/2} + C_5 + C_7. \]

**Lemma 2.12.** Suppose that Assumptions 1, 2, and 4 are satisfied. Let \( \tau > 0 \) and \( \delta, \eta \in (0, 1) \). For any real \( s \), any \( B = B(x, r) \) with \( B(x, 2r) \subset Y \), and any non-negative, locally bounded function \( u \in \mathcal{F}_{w}(Q) \) with \( \frac{\partial}{\partial r} u \geq L_1 u \) weakly in \( Q \), there is a constant \( c \) depending on \( u, \tau, \delta, \eta \) and on an upper bound on \( r \), and constant \( C \in (0, \infty) \) depending on \( C_1 - C_7 \) such that for all \( \lambda > 0 \),

\[
\bar{\mu}(\{(t, z) \in K_+ : \log u(z) < -\lambda - c\}) \leq C(1 + C_8 r^2)\mu(B)\lambda^{-1}
\]

and

\[
\bar{\mu}(\{(t, z) \in K_- : \log u(z) > -\lambda - c\}) \leq (1 + C_8 r^2)\mu(B)\lambda^{-1},
\]

where \( K_+ = (s, s + \eta \tau^2) \times \delta B \) and \( K_- = (s - (1 - \eta)\tau^2, s) \times \delta B \).

**Proof.** For simplicity, assume that \( \tau = 1 \). Let \( \psi(z) = \max\{0, (1 - d(x, z))/r'\} \in \mathcal{F}_{c}(B) \), where \( r' > 0 \) is slightly smaller than \( r \). Note that \( d\Gamma(\psi, \psi) \leq cr^{-2}d\mu \).

Because of the Cauchy-Schwarz inequality (1.2) and Assumptions 1, 2, and Lemma 1.20, we get

\[
E_t(\log(u), \log(u)) = \int 2\psi d\Gamma_t(\log(u), \log(u)) - \int \psi^2 d\Gamma_t(\log(u), \log(u))
\]

\[
\leq k \int d\Gamma(\psi, \psi) - \frac{1}{k} \int \psi^2 d\Gamma(\log(u), \log(u)),
\]

for some constant \( k > 1 \). Hence, applying the chain rule for \( \Gamma \), the Cauchy-Schwarz inequality (1.2), Assumptions 1, 2, and Lemma 1.20 we get

\[
-\frac{\partial}{\partial t} \int \log u \psi^2 d\mu = -\int \frac{\partial u}{\partial t} u^{-1} \psi^2 d\mu
\]

\[
\leq E_t(u, u^{-1} \psi^2)
\]

\[
\leq E_t(u, u^{-1} \psi^2) + E_{t, \text{sym}}(u, u^{-1} \psi^2) + E_{t, \text{sym}}(\psi^2, 1)
\]

\[
- E_t(\psi, u^{-1} \psi^2)
\]

\[
\leq k' \int d\Gamma(\psi, \psi) - \frac{1}{k'} \int \psi^2 d\Gamma(u, \log u)
\]

\[
+ k'C_8 \int \psi^2 d\mu,
\]

for some \( k' > 1 \). For the remaining part of the proof, we follow the line of arguments in [31] Proof of Lemma 5.4.1]. Let

\[
W = -\frac{\int \log u \psi^2 d\mu}{\int \psi^2 d\mu}.
\]

By the weighted Poincaré inequality of Theorem 2.14 it holds

\[
\int | - \log u - W |^2 \psi^2 d\mu \leq C_{\text{wPI}} r^2 \int \psi^2 d\Gamma(u, \log u).
\]
This and (2.9) give
\[
\frac{\partial}{\partial t} W + \frac{1}{Cr^2 \mu(B)} \int_{\delta B} | - \log u - W |^2 \psi^2 d\mu \leq C' r^{-2} + k'C_8, 
\]
for some constants \( C, C' > 0 \). Writing
\[
\bar{w}(t, z) = - \log u - (C + k'C_8 r^2) r^{-2}(t - s),
\]
\[
\bar{W}(t) = W(t) - (C' + k'C_8 r^2) r^{-2}(t - s),
\]
we obtain
\[
\frac{\partial}{\partial t} \bar{W} + \frac{1}{Cr^2 \mu(B)} \int_{\delta B} | \bar{w} - \bar{W} |^2 \psi^2 d\mu \leq 0. \tag{2.10}
\]
Set
\[
\Omega^-_t(\lambda) = \{z \in \delta B : \bar{w}(t, z) < -\lambda + \bar{W}(s)\},
\]
\[
\Omega^+_t(\lambda) = \{z \in \delta B : \bar{w}(t, z) > \lambda + \bar{W}(s)\}.
\]
Then, if \( t > s \),
\[
\bar{w}(t, z) - \bar{W}(t) \geq \lambda + \bar{W}(s) - \bar{W}(t) > \lambda
\]
in \( \Omega^+_t(\lambda) \), because \( \frac{\partial}{\partial t} \bar{W} \leq 0 \). Using this in (2.10) we obtain
\[
\frac{\partial}{\partial t} \bar{W} + \frac{1}{Cr^2 \mu(B)} | \lambda + \bar{W}(s) - \bar{W}(t)|^2 \mu(\Omega^+_t(\lambda)) \leq 0,
\]
or, equivalently,
\[
\mu(\Omega^+_t(\lambda)) \leq -Cr^2 \mu(B) \frac{\partial}{\partial t} (| \lambda + \bar{W}(s) - \bar{W}(t)|^{-1}).
\]
Integrating from \( s \) to \( s + \eta r^2 \), we obtain
\[
\bar{\mu}(\{(t, z) \in K_+ : \bar{w}(t, z) > \lambda + \bar{W}(s)\}) \leq Cr^2 \mu(B) \lambda^{-1},
\]
and hence
\[
\bar{\mu}(\{(t, z) \in K_+ : \log u_c(t, z) + (C' r^{-2} + k'C_8)(t - s) < -\lambda - \bar{W}(s)\}) \leq Cr^2 \mu(B) \lambda^{-1}.
\]
Finally,
\[
\bar{\mu}(\{(t, z) \in K_+ : \log u_c(t, z) < -\lambda - \bar{W}(s)\}) \leq \bar{\mu}(\{(t, z) \in K_+ : \log u_c(t, z) + (C + k'C_8 r^2) r^{-2}(t - s) < -\lambda/2 - \bar{W}(s)\})
\]
\[
+ \bar{\mu}(\{(t, z) \in K_+ : (C' + k'C_8 r^2) r^{-2}(t - s) > \lambda/2\}) \leq C''(1 + C_8 r^2) r^2 \mu(B) \lambda^{-1}.
\]
This proves the first inequality in Lemma 2.12. By a similar argument, using \( \Omega^-_t \) instead of \( \Omega^+_t \), we obtain the second inequality. \( \square \)
Fix a parameter $\tau > 0$. Let $B = B(x, r) \subset X$, $s \in \mathbb{R}$. Fix $\delta \in (0, 1)$, set

$$\delta B = B(x, \delta r),$$

$$Q = (s - \tau r^2, s + \tau r^2) \times B,$$

$$Q_- = (s - \delta \tau r^2, s - \frac{\delta}{2} \tau r^2) \times \delta B,$$

$$Q'_- = (s - \tau r^2, s - \frac{\delta}{2} \tau r^2) \times \delta B,$$

$$Q'_+ = (s + \frac{\delta}{2} \tau r^2, s + \delta \tau r^2) \times \delta B.$$

**Theorem 2.13.** Suppose Assumptions 1, 2 and 4 are satisfied. Let $\nu > 2$ be as in (2.1) and $p_0 \in (0, 1 + 2/\nu)$. Then there exists a constant $A$ such that, for any real $s$, any $B = B(x, r)$ with $B(x, 2r) \subset Y$, and any locally bounded function $u \in F_{loc}(Q)$ with $\frac{\partial}{\partial t} u \geq L_t u$ weakly in $Q$, we have

$$\left(\frac{1}{\mu(Q_-)} \int_{Q_-} u^{p_0} \, d\mu\right)^{1/p_0} \leq A \inf_{Q'_+} u.$$

The constant $A$ depends only on $\tau$, $\delta$, $D_Y$, $P_Y$, $p_0$, $C_1 - C_7$, and an upper bound on $(C_2 + C_3^{1/2} + C_5 + C_7)r^2$.

**Proof.** This follows from Theorem 2.10, Lemma 2.12 and an abstract Lemma similar to [31, Lemma 2.2.6] by the line of reasoning presented in [31, Proof of Theorem 5.4.2].

**Theorem 2.14.** Suppose Assumptions 1, 2 and 4 are satisfied. Then the family $(E_t, F)$ satisfies the parabolic Harnack inequality on $Y$. That is, there is a constant $H_Y = H_Y(\tau, \delta, D_Y, P_Y, C_1 - C_7, (C_2 + C_3^{1/2} + C_5 + C_7)r^2)$ such that for any $s \in \mathbb{R}$, $B(x, r)$ with $B(x, 2r) \subset Y$, and any non-negative local weak solution $u \in F_{loc}(Q)$ of $\frac{\partial}{\partial t} u = L_t u$ in $Q$, we have

$$\sup_{Q_-} u \leq H_Y \inf_{Q'_+} u.$$

**Proof.** Follows from Corollary 2.8, Theorem 2.6 and Theorem 2.13.

**Corollary 2.15.** Suppose Assumptions 1, 2 and 4 are satisfied globally on $Y = X$ with $C_2 = C_3 = C_5 = C_7 = 0$. Then the family $(E_t, F)$ satisfies a scale-invariant parabolic Harnack inequality on $X$. That is, there is a constant $H$ such that for any $s \in \mathbb{R}$, $B(x, r) \subset X$, and any non-negative local weak solution $u \in F_{loc}(Q)$ of $\frac{\partial}{\partial t} u = L_t u$ in $Q$, we have

$$\sup_{Q_-} u \leq H \inf_{Q'_+} u.$$

The constant $H$ depends only on $\tau$, $\delta$, $D_X$, $P_X$, $C_1$, $C_4$, $C_6$. 32
Remark 2.16. The hypothesis $C_2 = C_3 = C_5 = C_7 = 0$ is satisfied for second-order differential operators with no lower order terms. The second order term may be non-symmetric.

Corollary 2.17. Suppose Assumptions 1, 2 and 3 are satisfied. Fix $\tau > 0$ and $\delta \in (0, 1)$. Then there exist $\beta \in (0, 1)$ and $H \in (0, \infty)$ such that for any $B(x, 2r) \subset Y$, any real $s$, any local weak solution of $\frac{\partial}{\partial t}u = L_t u$ in $Q = (s - \tau r^2, s + \tau r^2) \times B(x, r)$ has a continuous representative and satisfies

$$\sup_{(t,y),(t',y') \in Q'} \left\{ \frac{|u(t,y) - u(t',y')|}{(|t - t'|^{1/2} + d_B(y,y')^{1/2})} \right\} \leq \frac{H r^\beta}{r^\delta} \sup_Q |u|$$

where $Q' = (s - \delta \tau r^2, s + \delta \tau r^2) \times B(x, \delta r)$. The constant $H$ depends only on $\tau$, $\delta$, $D_Y$, $P_Y$, $C_1 - C_7$ and an upper bound on $(C_2 + C_3^{1/2} + C_5 + C_7)r^2$.

Proof. See, e.g., [31].

3 Applications

In this section, we continue to work under Assumptions 1 and 2 concerning the family of bilinear forms $\mathcal{E}_t$. We also assume that the sector condition holds uniformly in $t$ in the form

$$\forall f,g \in \mathcal{F}, \ |\mathcal{E}_t(f,g)| \leq C_* \|f\|_\mathcal{F} \|g\|_\mathcal{F}.$$ 

In addition, we assume that Assumption 3 is satisfied locally on $X$, that is, every point $x \in X$ has a neighborhood $Y_x = B(x, r_x)$ where Assumption 3 is satisfied with $Y = Y_x$. Recall from Remark 1.17(iii) that for all $t \in \mathbb{R}$ and all $f \in \mathcal{F}$, we have

$$\mathcal{E}_t(f, f) = \mathcal{E}_1^t(f, f) + \mathcal{E}_1^m(f^2, 1) \geq -\alpha \int f^2 d\mu,$$

where $\alpha = (C_1 C_2 + 2C_3^{1/2})$.

Proposition 3.1. For every $f \in L^2(X, \mu)$ there exists a unique weak solution $u$ of the initial value problem

$$\begin{align*}
\frac{\partial}{\partial t}u &= L_t u \quad \text{on } (s, \infty) \times X, \\
u(s, \cdot) &= f \quad \text{on } X.
\end{align*}$$

Proof. See [38] Chap. 3, Theorem 4.1 and Remark 4.3].

For any $s \leq t$ there exists a unique transition operator

$$T^s_t : L^2(X, \mu) \rightarrow L^2(X, \mu).$$
associated with \( L_t - \frac{\partial}{\partial t} \) such that for every \( f \in L^2(X, \mu) \) the unique solution \( u \) of (3.1)-(3.2) is given by \( u : t \mapsto T_t^f \). See, e.g., [34] Section 1.3 and 2.4] and [1] [26]. The map \( t \mapsto T_t^f \) is strongly continuous on \([s, \infty)\). Furthermore, \( ||T_t^s||_{2 \rightarrow 2} \leq e^{\alpha(t-s)} \) for some \( \alpha \geq 0 \) depending on \( C_1 - C_3 \), and \( T_t^s = T_t^s \circ T_s^r \) for any \( r \leq s \leq t \). Similarly, there exists a transition operator \((S^*)_t^s\) associated with \( L_t^s + \frac{\partial}{\partial t} \). These transition operators preserve positivity.

**Proposition 3.2.** There exists a measurable positive function \( p : \mathbb{R} \times X \times \mathbb{R} \times X \rightarrow [0, \infty) \) with the following properties:

(i) For every \( t > s \) \( \mu \)-a.e. \( x, y \in X \) and every \( f \in L^1(X, \mu) + \mathcal{L}^\infty(X, \mu) \),

\[
T_t^s f(y) = \int_X p(t,y,s,z) f(z) \mu(dz)
\]

and

\[
(S^*)_t^s f(x) = \int_X p(t,z,s,x) f(z) \mu(dz).
\]

(ii) For every \( s < \sigma < \tau \) and \( \mu \)-a.e. \( x \in X \) the function

\[
u : (t,y) \mapsto p(t,y,s,x)
\]

is a global solution of the equation \( L_t u = \frac{\partial}{\partial t} u \) on \((\sigma, \tau) \times X \) and for every \( t > \tau > s \) and \( \mu \)-a.e. \( y \in X \), the function

\[
u : (s,x) \mapsto p(t,y,s,x)
\]

is a global solution of the equation \( L_t^s u = -\frac{\partial}{\partial x} u \) on \((\sigma, \tau) \times X \).

(iii) For every \( s < r < t \) \( \mu \)-a.e. \( x, y \in X \),

\[
p(t,y,s,x) = \int_X p(t,y,r,z)p(r,z,s,x) \mu(dz).
\]

(iv) For every \( s < t \),

\[
\int_X \int_X p(t,y,s,x)^2 \mu(dx) \mu(dy) \leq e^{(-C_1^{-1}r + \alpha)(t-s)}
\]

where \( \lambda \) is the largest real such that \( E(f,f) \geq \lambda \|f\|_2^2 \) for all \( f \in \mathcal{F} \).

(v) \( p \) is locally bounded on the set \( \{ (t,y,s,x) : s < t \} \).

**Proof.** The proposition can be proved along the lines of [10] Lemma 3.7] and [34] Proposition 2.3].

**Theorem 3.3.** For all \( t > s \) and \( x, y \in X \),

\[
p(t,y,s,x) \leq C \exp \left( -\frac{(d(x,y)^2)}{C(t-s)} + \beta(t-s) \right) \frac{1}{V(x,\tau_x)^{\frac{1}{2}} V(y,\tau_y)^{\frac{1}{2}}},
\]

where \( \tau_x = \sqrt{t-s} \wedge r_x \), \( \tau_y = \sqrt{t-s} \wedge r_y \), and \( \beta = C_2 + C_3^{1/2} + C_5 + C_7 + \alpha \). The constants \( C, C' > 0 \) depend on \( C_1 - C_3 \), \( D_Y \), \( P_Y \) for \( Y = Y_x \) and \( Y = Y_y \).
Definition 3.4. For a ball $B = B(a, r) \subset X$, the Dirichlet-type forms on $B$ are defined as
\[
\mathcal{E}^D_B(f,g) = \mathcal{E}(f,g), \quad f, g \in D(\mathcal{E}^D_B),
\]
where the domain $D(\mathcal{E}^D_B) = \mathcal{F}^0(B)$ is defined as the closure of $\mathcal{F} \cap C(B)$ in $\mathcal{F}$ for the norm $\| \cdot \|_x$. Let $T^D_B(t,s)$ be the associated transition operator and $p^D_B(t,y,s,x)$ the Dirichlet propagator.

Theorem 3.5. Let $a \in X$ and $B = B(a, r_a/2)$.

(i) For any fixed $\epsilon \in (0, 1)$ there are constants $c, C \in (0, \infty)$ such that for any $x, y \in B(a, (1 - \epsilon) r_a/2)$ and $0 < \epsilon(t-s) \leq (r_a/2)^2$, the Dirichlet propagator $p^D_B$ is bounded below by
\[
p^D_B(t, y, s, x) \geq \frac{c}{V(x, \sqrt{t-s} \land R_a)} \exp \left( -C \frac{d(x,y)^2}{t-s} \right),
\]
where $R_a = d(x, \partial B)/2$.

(ii) For any fixed $\epsilon \in (0, 1)$ there are constants $c, C', C'' \in (0, \infty)$ such that for any $x, y \in B$, $t-s \geq (\epsilon r_a/2)^2$, the Dirichlet propagator $p^D_B$ is bounded above by
\[
p^D_B(t, y, s, x) \leq \frac{C}{V(a, r_a/2)} \exp \left( -\frac{c(t-s)}{r_a^2} + \alpha(t-s) \right)
\]
where $\alpha = C_2 + C_3^{1/2} + C_5 + C_7 + \alpha$.

(iii) There exist constants $c, C \in (0, \infty)$ such that for any $x, y \in B$, $t > s$, the Dirichlet propagator $p^D_B$ is bounded above by
\[
p^D_B(t, y, s, x) \leq C \frac{\exp \left( -\frac{c d(x,y)^2}{t-s} + \beta(t-s) \right)}{V(x, \sqrt{t-s} \land (r_a/2))^{1/2} V(y, \sqrt{t-s} \land (r_a/2))^{1/2}},
\]
where $\beta = C_2 + C_3^{1/2} + C_5 + C_7 + \alpha$.

All the constants $c, C$ above depend only on $C_1$-$C_3$, and on $D_Y$, $p_Y$, $C_4$-$C_7$ and an upper bound on $(C_2 + C_3^{1/2} + C_5 + C_7)r_a^2$ for $Y = B(a, r_a)$.

Proof. Statement (iii) follows from Theorem 3.3 and the set monotonicity of the kernel. To show the on-diagonal estimate in (i) we follow the proof of [31 Theorem 5.4.10]. Let $0 < \epsilon(t-s) \leq (r_a/2)^2$ and $x \in B(a, (1 - \epsilon) r_a/2)$. Let $r = \sqrt{t-s} \land R_a$. Let $\psi$ be a smooth function such that $0 \leq \psi \leq 1$, $\psi = 1$ on $B(x, r)$ and $\psi = 0$ on $X \setminus B(x, 2r)$. Define
\[
u(t,y) = \begin{cases} T^D_B(t,s)\psi(y) & \text{if } t > s, \\ \psi(y) & \text{if } t \leq s. \end{cases}
\]
One can show that \( u \) is a local weak solution of

\[
\hat{L}_t u = \frac{\partial}{\partial t} u \quad \text{on } Q' = (-\infty, +\infty) \times B(x,r),
\]

where

\[
\hat{L}_t = \begin{cases} 
L_t & \text{if } t > s, \\
L & \text{if } t \leq s.
\end{cases}
\]

Applying the parabolic Harnack inequality of Theorem 2.14 to \( u \) and then to \( p_B^D(\cdot, \cdot, s, z) \), we get

\[
1 = u(s, x) \leq C u(s + (t - s)/2, x)
\]

\[
= C \int p_B^D(s + (t - s)/2, x, s, z) \psi(z) \mu(dz)
\]

\[
\leq C \int_{B(x,2r)} p_B^D(s + (t - s)/2, x, s, z) \mu(dz)
\]

\[
\leq C^2 V(x,2r) p_B^D(s + (t - s)/2, x, s - (t - s)/2, x).
\]

Using volume doubling, we get

\[
p_B^D(t, x, s, x) \geq \frac{C'}{V(x,r)}.
\]

For the off-diagonal estimate, see the proof of [36, Theorem 4.8], and apply the parabolic Harnack inequality of Theorem 2.14.

To show (ii) we follow [12, Lemma 3.9 part 3]. Let \( r = (e r_a/4)^2 \). Then

\[
\sup_{y,z \in B} p_B^D(t, y, s, z) = \sup_{y,z \in B} \int p_B^D(t, y, (t + s)/2, x)p_B^D((t + s)/2, x, s, z) \mu(dx)
\]

\[
= \sup_{y,z \in B} \left( \int p_B^D((t + s)/2, x, s, z)^2 \mu(dx) \right)^{1/2}
\]

\[
= \left( \int p_B^D((t + s)/2, x, s, z)^2 \mu(dx) \right)^{1/2}
\]

\[
\leq \left\| T_B^D \left( t, \frac{t + s}{2} + r \right) \right\|_{2 \to \infty} \left\| (S_B^D)^* \left( (t + s)/2, s \right) \right\|_{2 \to \infty}
\]

\[
\leq \left\| T_B^D \left( \frac{t + s}{2} + r, \frac{t + s}{2} \right) \right\|_{2 \to \infty} \left\| (S_B^D)^* \left( \frac{t + s}{2}, s + r \right) \right\|_{2 \to \infty}
\]

\[
\exp \left( \frac{1}{2} \left( - (C_1)^{-1} \lambda_B + \alpha \right) (t - s)/4 \right)
\]

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\[ \leq \frac{C}{V(a, r_a/2)} \exp \left(-c \frac{t - s}{r_a^2} + \beta(t - s) \right) \]

for some constants \(c, C > 0\). Here \(\lambda_B\) is the lowest Dirichlet eigenvalue for the form \(\mathcal{E}\) on \(B\). By [12, Theorem 2.5], it holds \(\lambda_B \geq \text{const} \cdot r_a^{-2}\).

**Theorem 3.6.** Suppose Assumptions \([1, 2, 4]\) are satisfied. Then there are constants \(c, C, C' > 0\) such that for any point \(a \in X\), all \(x, y \in B(a, r_a/2)\) and \(t > s\), we have

\[ p(t, y, s, x) \geq c \frac{\exp \left(-C' \frac{d(x, y)^2}{t-s} - \frac{c'}{r_a^2} (t-s) \right)}{V(x, \sqrt{t-s} \land (r_a/2))} \]

The constants \(c, C, C'\) depend only on \(C_1 - C_3, D_Y, P_Y, C_4 - C_7\) and an upper bound on \((C_2 + C_3^{1/2} + C_5 + C_7) r_a^2\) for \(Y = B(a, r_a)\).

**Proof.** From Theorem 3.5(i) we obtain an on-diagonal bound for \(t - s < r_a^2\). The off-diagonal estimate (for any \(t > s\)) follows from the parabolic Harnack inequality.

The following corollary provides a global two-sided heat kernel bound for situations that generalize the model case of the equation

\[ \partial_t u = \sum_{i,j} \partial_j (a_{i,j}(t,x) \partial_i u) \]

on \(\mathbb{R}^n\) with bounded measurable uniformly elliptic but not necessarily symmetric \((a_{i,j})\).

**Corollary 3.7.** Suppose Assumptions \([1, 2, 4]\) are satisfied globally on \(Y = X\) with \(C_2 = C_3 = C_5 = C_7 = 0\). Then there are constants \(c, C, C', C'' > 0\) such that for any \(x, y \in X\) and \(t > s\), we have

\[ \frac{\exp \left(-C' \frac{d(x, y)^2}{t-s} \right)}{V(x, \sqrt{t-s})} \leq p(t, y, s, x) \leq \frac{C}{V(x, \sqrt{t-s})} \frac{\exp \left(-c \frac{d(x, y)^2}{t-s} \right)}{V(x, \sqrt{t-s})} \]

The constants \(c, C, C', C'' > 0\) depend only on \(C_1, C_4, C_6, D_X, P_X\).

Note that, under the assumption of \([3, 7]\) Corollary \([3, 10]\) provides assorted global time-space Hölder continuity estimates for the heat kernel.

**Remark 3.8.** For the sake of simplicity, in the results described above, we have not tried to capture the sharpest possible Gaussian upper bound as far as the constant in front of \(\frac{d(x, y)^2}{(t-s)}\) in the exponential Gaussian factor is concerned. The reason is that this question is rather unnatural and somewhat irrelevant in the present context of time-dependent forms. We note that, with the parabolic Harnack inequality of Theorem \([2, 14]\) established, it is possible to obtain more detailed Gaussian upper bounds in spirit of \([34]\) and \([31, Section 5.2.3]\) by following the line of reasoning used in these references.

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