STRUCTURE OF THE NUTTALL PARTITION FOR SOME
CLASS OF FOUR-SHEETED RIEMANN SURFACES

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ABSTRACT. The structure of a Nuttall partition into sheets of some class of
four-sheeted Riemann surfaces is studied. The corresponding class of multival-
ued analytic functions is a special class of algebraic functions of fourth order
generated by the function inverse to the Zhukovskii function. We show that in
this class of four-sheeted Riemann surfaces, the boundary between the second
and third sheets of the Nuttall partition of the Riemann surface, is completely
characterized in terms of an extremal problem posed on the two-sheeted Rie-
mann surface of the function \( w \) defined by the equation \( w^2 = z^2 - 1 \). In
particular, we show that in this class of functions the boundary between the
second and third sheets does not intersect both the boundary between the first
and second sheets and the boundary between the third and fourth sheets.

Bibliography: [34] titles.

Keywords: multivalued analytic functions, Riemann surface, Nuttall parti-
tion, Hermite–Padé polynomials, Green function, extremal problem

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1. INTRODUCTION AND THE STATEMENT OF THE MAIN RESULTS

1.1. It is well known that a global partition of a Riemann surface of an algebraic
function into “sheets” plays a key role in the asymptotic theory of Hermite–Padé
polynomials; see, first of all [19], and also [25], [13], [14]. Recently the well-known
Nuttall’s conjecture (see [19, § 3]), which has been open since 1984, was proved
in [13, § 5, Lemma 5]. This conjecture claimed that, for the so-called Nuttall part-
tition (with respect to some highlighted point; see [19], [13], [32]) of a compact
\( m \)-sheeted Riemann surface into sheets, the complement of the “topmost” closed
\( m \)th sheet is always connected; that is, it is always a domain on this Riemann
surface. In [31], this fact was used as a basis for the new approach to the imple-
mentation of the well-known Weierstrass program (see [5] and the bibliography

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given there) of effective continuation of a power series\(^1\) of a given germ of a multivalued analytic function. The verified Nuttall conjecture naturally suggests the more detailed study of the topological properties of the Nuttall partition of a Riemann surface into sheets. The need for this investigation stems, in particular, from the fact that the further progress in the study of asymptotic properties of the generalized Hermite–Padé polynomials (which were introduced in [14], see also [31]) requires a more detailed knowledge of the topological structure of the Nuttall partition of a compact Riemann surface into sheets. However, many properties of a Nuttall partition, which seem quite natural at first sight, are still not rigorously justified in the general case. The study of such properties would allow one, in particular, to extend the Stahl theory of the convergence of Padé approximants, which holds, \textit{inter alia}, for infinitely-valued analytic functions) to more general rational approximants of functions with the same properties constructed on the basis of Hermite–Padé polynomials.\(^2\) So, in parallel with the study of the asymptotic properties of Hermite–Padé polynomials pertaining to algebraic functions and the corresponding compact Riemann surfaces (see [19], [13], [14]), it is natural (in analogy with the Stahl theory) to investigate the asymptotic properties of Hermite–Padé polynomials for multivalued functions with finite number of singular points on the Riemann sphere, but with the infinitely-sheeted Riemann surface. In this direction, only the first steps have been made; see [21], [17], [32].

It is known (see, in the first place, [26], and also [19] and [2]) that according to the Stahl theory,\(^3\) with each germ \(f_{z_0}\) (considered at a point \(z_0 \in \hat{\mathbb{C}}\)) of a multivalued analytic function \(f\) with a finite number of singular points on the Riemann sphere \(\hat{\mathbb{C}}\) one can uniquely associate a hyperelliptic Riemann surface, whose first sheet\(^4\) corresponds to the uniquely defined (from the given germ \(f_{z_0}\)) Stahl domain, which is the maximal domain of convergence\(^5\) of the diagonal Padé approximants.

In the new approach of [31] to the solution of the problem of effective analytic continuation of a given multivalued function, when the Hermite–Padé polynomials are considered in lieu of the Padé polynomials, one should deal, instead of a single (multivalued) function \(f\) (or, in a different terminology, a family of functions \([1, f]\)), with the pair of functions \(f, f^2\) (or, in other words, with the family of functions \([1, f, f^2]\)). According to [31] (see also [19] and [13]), this approach in principle allows one to double (in comparison with the Stahl domain) the domain in which the values of the function \(f\) can be efficiently (that is, in terms of rational functions) recovered from a given germ. The corresponding generalizations within the framework of this new approach are given in [31] also for the family

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\(^1\)In other words, here one speaks about the effective summation of a power series beyond its convergence disk. By effective summation/continuation of a power series one means its summation/extension via an (infinite) sequence of rational functions, in which each member of the sequence is constructed directly from a finite number of coefficients of a given series.

\(^2\)Here and in what follows, we shall speak about Hermite–Padé polynomials of type I.

\(^3\)By the Stahl theory one usually means the series of results obtained by Stahl in 1985–1986s on the convergence of Padé approximants to multivalued analytic functions; see [26].

\(^4\)We assume, unless otherwise stated, that all the sheets of a Riemann surface are open. Note that it is well known that, for a Nuttall partition of a Riemann surface into sheets, these sheets may fail to be connected.

\(^5\)The convergence of Padé approximants is usually understood in the sense of the convergence in logarithmic capacity on compact subsets of the Stahl domain.
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[1, f, f^2, f^3] (see also [30], [34]). In this case, the domain of effective recovery of an analytic function increases already by three times in comparison with the Stahl domain. Namely, the corresponding Nuttall domain lies on the Riemann surface, and hence in a certain sense this domain is a three-sheeted covering of the Stahl domain, which lies on the Riemann sphere. A much more general case of the family [1, f, ..., f^m] \( (m \in \mathbb{N} \text{ is arbitrary}, f \text{ is an algebraic function of order } (m + 1)) \) was considered by Komlov [14].

All that was said above, was discussed in [31] on an example of multivalued functions \( f \) from the class introduced earlier by the author of the present paper (see [29]) — this being the class of functions of the form

\[
f(z) := \left[ \left( \frac{1}{\varphi(z)} \right) \left( B - \frac{1}{\varphi(z)} \right) \right]^{-1/2}, \quad z \in D := \mathbb{C} \setminus E, \quad E := [-1, 1],
\]

where \( \varphi(z) := z + (z^2 - 1)^{1/2} \) is the function inverse to the Zhukovskii function, \( 1 < A < B < \infty \); here and what follows we choose the branch of the function \( (\cdot)^{1/2} \) such that \((z^2 - 1)^{1/2}/z \to 1 \) as \( z \to \infty \) (for details, see [29] and § 1.2 below).

However, for an arbitrary multivalued function (in particular, for an algebraic functions of order \( \geq 4 \)), the question of asymptotic properties of the Hermite–Padé polynomials for the family \([1, f, f^2] \) remains open in the general case. The corresponding results in this direction have been obtained to date only in very special cases; see, for example, [21], [15], [24]). One of the main reasons is as follows. In the Stahl theory, the domain of convergence of Padé approximants is completely characterized by the property that the boundary of this domain has the smallest capacity in the class of all the domains admissible for a given germ of a multivalued function. So, the maximal Stahl domain corresponds to the solution of a certain extremal problem (the problem of an admissible compact set of minimal capacity). For the family \([1, f, f^2] \) (or, in a different terminology, for the pair of functions \( f, f^2 \)), where the function \( f \) lies in the same class, which was considered by Stahl, all the attempts to pose the corresponding extremal problem in a sufficiently general case have proved futile.

We can mention several extreme problems associated with the study of the asymptotic behavior of Hermite–Padé polynomials, but which are suitable only for certain special cases and extending only slightly the Stahl theory to the setting of Hermite–Padé polynomials (see [1], [21], [4], [31], [32], [24]). Of course, the case with four functions \([1, f, f^2, f^3] \) is even more involved. In this setting, some or other characterization of the Nuttall partition of a Riemann surface into sheets in terms of an appropriate extremal problem (even in sufficiently simple cases when the structure of the partition of a Riemann surface into sheets is fairly simple and quite natural) would be of great value from the point of view of the further development of the Stahl theory and its extension to the Hermite–Padé polynomials setting.

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\[ ^6 \text{A corresponding general result would give a natural extension of Stahl’s theorem to the case of Hermite–Padé polynomials for the family } [1, f, f^2] \text{ (in place of the family } [1, f]). \]

\[ ^7 \text{Up to a set of zero capacity.} \]

\[ ^8 \text{We recall that a domain } G \ni z_0 \text{ on the Riemann sphere is called admissible for a given germ } f_{z_0} \text{ if this germ extends from the point } z_0 \text{ to this domain as a meromorphic (single-valued analytic) function.} \]

\[ ^9 \text{Of course, the case of a hyperelliptic function is excluded from consideration.} \]
This is precisely the purpose of the present paper — for a function of the form (2) to characterize the Nuttall partition of the corresponding four-sheeted Riemann surface in terms of the extremal problem posed on the two-sheeted Riemann surface of the function \( w^2 = z^2 - 1 \) (more precisely, here we speak about the characterization of the boundary between the second and third sheets of the Nuttall partition). Namely, in the present paper we consider some class of multivalued analytic functions which have only second-order branch points. Moreover, it is assumed that geometrically the branch points are symmetric with respect to the real line. In other words, all \( \alpha_j \in \{1/2, -1/2\} \) in representation (2). In the present paper, this class will be denoted by \( \mathcal{Z} \). In accordance with the general result of [13], under the Nuttall partition (with respect to the point at infinity \( z = \infty(1) \)) of the corresponding Riemann surface into sheets, the complement of the fourth (last, the “topmost”) sheet is always connected. Furthermore, from Komlov’s general results [14] it follows that using the Hermite–Padé polynomials (of type I and II, respectively, and the generalized ones; see [31] and [14]) the values of a function \( f \in \mathcal{Z} \) can be recovered on the first, second, and third sheets of the Riemann surface \( \mathcal{R}_4(f) \). In the present paper, under certain geometric constraints on the arrangement of branching points of a function \( f \) from the class \( \mathcal{Z} \), we give a characterization of the boundary between the second and third sheets of its Riemann surface \( \mathcal{R}_4(f) \) in terms of a certain extremal problem posed now not on the Riemann sphere \( \hat{\mathbb{C}} \), but on the two-sheeted Riemann surface \( \mathcal{R}_2(w) \) of the function \( w \), which is given by the equation \( w^2 = z^2 - 1 \). Note that this extremal problem calls for the characterization of an “admissible” compact set located on the Riemann surface \( \mathcal{R}_2(w) \) and having the property that the Green function of its complement (assuming that this complement is connected) maximizes the Robin constant at one of the two points at infinity located on the Riemann surface \( \mathcal{R}_2(w) \). Thus, this extremal problem is quite analogous to the classical problem of an admissible compact set of minimal capacity, which was posed (and solved) by Stahl on the Riemann sphere.

In the last §4, we compare the extremal Problem 1 considered here and the extremal problem of [30] on the existence of a three-sheeted Nuttall-partitioned Riemann surface associated with a given multivalued analytic function.

Thus, in the present paper, we further extend the new approach (which was proposed by the author in [30]) to the study of asymptotic properties of Hermite–Padé polynomials for multivalued functions. This approach is based on the extremal equilibrium problem posed not on the Riemann sphere, but instead on the Riemann surface \( \mathcal{R}_2(w) \) (further advances in this problem were made in [33], [12]).

To conclude our introduction, we comment on the condition to the effect that all the exponents in (2) are \( \pm 1/2 \). It is well known that the first results (obtained prior to the completion of the general Stahl theory) on the convergence of Padé approximants in the class of multivalued analytic functions were obtained by Nuttall for the class of hyperelliptic functions and their natural generalizations; that is, for functions having only second-order branch points; see [18], [19], and also [28], [2], [3].

It worth pointing out again that the problem of effective recovery of a function \( f \in \mathcal{Z} \) on three Nuttall sheets of its Riemann surface \( \mathcal{R}_4(f) \), which was considered in [31], is a very particular case of the general problem of the recovery of the values of an algebraic function \( f \) of order \( m + 1 \) on the first \( m \)-Nuttall sheets of
its Riemann surface $\mathcal{R}_{m+1}(f)$ (this problem was considered by Komlov in [14]). From the properties of the Nuttall partition of the Riemann surface $\mathcal{R}_{4}(f)$ into sheets, which we obtain below, it follows that the results of [14] on the convergence of rational approximants constructed on the basis of Hermite–Padé polynomials are also valid in the case considered here.

1.2. We will now give the necessary definitions and notation.

Let, as above, $\varphi(z) := z + (z^2 - 1)^{1/2}$ be the function inverse to the Zhukovskii function, which is meromorphic and single-valued in the domain $D := \hat{\mathbb{C}} \setminus E$, $E := [-1, 1]$, and which maps the domain $D$ onto the exterior $U := \{ \zeta \in \hat{\mathbb{C}} : |\zeta| > 1 \}$ of the unit disk $D := \{ \zeta \in \mathbb{C} : |\zeta| < 1 \}$. Here and in what follows, we choose the branch of the function $(\cdot)^{1/2}$ such that $(z^2 - 1)^{1/2} / z \to 1$ as $z \to \infty$.

Consider the class $\mathcal{P}$ of functions $f$ defined for $z \in D$ by the explicit representation

$$f(z) = \prod_{j=1}^{2p} \left( A_j - \frac{1}{\varphi(z)} \right)^{\alpha_j}, \quad z \in D,$$

(2)

where $p \in \mathbb{N}$ is a natural number, all the exponents $\alpha_j \in \{1/2, -1/2\}$, $\sum_{j=1}^{2p} \alpha_j = 0$ (mod $\mathbb{Z}$), all the quantities $A_j$ are pairwise different, $|A_j| > 1$, and the set $\{A_1, \ldots, A_{2p}\}$ is symmetric with respect to the real line. The function $f$ of the form (2) has the following properties:

1) to the function $f(z)$, $z \in D$, there corresponds the germ $f_{\infty}$ holomorphic at the point $z = \infty$;

2) $f$ is an algebraic function of fourth order;

3) the function $f(z)$, $z \in D$, extends analytically along any path lying on the Riemann sphere $\hat{\mathbb{C}}$ and not intersecting the set of points $\Sigma = \Sigma(f) = \{ \pm 1, a_j, j = 1, \ldots, 2p \}$, where $a_j = (A_j + 1/A_j)/2 \notin E$.

4) each point of the set $\Sigma$ is a second-order branch point of the function $f$.

By $\mathcal{R}_4(f)$ we denote the four-sheeted Riemann surface of the function $f \in \mathcal{P}$; the Riemann surface $\mathcal{R}_4(f)$ has genus $p - 1$.

Note that the class of functions of the form (2) is a subclass of the more general class studied in [31] and [29].

Consider the two-sheeted Riemann surface $\mathcal{R}_2(w)$ of an algebraic function $w$ defined by the equation $w^2 = z^2 - 1$. A point $z \in \mathcal{R}_2(w)$ on this Riemann surface is the pair $z = (z, w)$. Let $\pi_2$ be the corresponding canonical projection of $\mathcal{R}_2(w)$ onto the Riemann sphere $\hat{\mathbb{C}}$: $\pi_2: \mathcal{R}_2(w) \to \hat{\mathbb{C}}, \pi_2(z) = z$. We set $\Gamma := \{ z \in \mathcal{R}_2(w) : \pi_2(z) \in E \}$ (that is, $\Gamma = \pi_2^{-1}(E)$). Let $\infty^{(1)} \in \pi_2^{-1}(\infty)$ be a point of the set $\pi_2^{-1}(\infty)$ such that $w/z \to 1$ as $z \to \infty^{(1)}$; let $\infty^{(2)}$ be the second point of the set $\pi_2^{-1}(\infty)$ ($w/z \to -1$ as $z \to \infty^{(2)}$). Let $\mathcal{R}_2^{(1)}$ and $\mathcal{R}_2^{(2)}$ be, respectively, the first and second (open) sheets of the Riemann surface $\mathcal{R}_2(w)$, $z^{(1)} = (z, (z^2 - 1)^{1/2})$ and let $z^{(2)} = (z, -(z^2 - 1)^{1/2})$ be points lying, respectively, on the first and second sheets.

Let $\Phi(z) := z + w$ and $g(z, \infty^{(1)}, \infty^{(2)}) := \log |z + w| = \log |z \pm (z^2 - 1)^{1/2}| = \log |\Phi(z)|$, $z \in \mathcal{R}_2(w)$, be the bipolar Green function for the Riemann surface $\mathcal{R}_2(w)$ (see [6], [7]). Note that $g(z, \infty^{(1)}, \infty^{(2)}) \equiv 0$ for $z \in \Gamma$.

So, $g(z^{(1)}, \infty^{(1)}, \infty^{(2)}) = g_E(z, \infty)$ is the Green function for the domain $D = \hat{\mathbb{C}} \setminus E$ (note that $E$ is a compact Stahl set for $f \in \mathcal{P}$, $D$ is the corresponding Stahl domain). For $z = z^{(2)}$ we have $\log |z + w| = \log |z - (z^2 - 1)^{1/2}| = -\log |z+$
(z^2 - 1)^{1/2}$, and hence the function $\eta(z) := -g(z, \infty^{(1)}, \infty^{(2)})$ defines a Nuttall partition of the Riemann surface $\mathcal{R}_2(w)$ into sheets; see [19, § 3], [13]. Namely, $\eta(z^{(1)}) < \eta(z^{(2)})$ for $z \in D$.

Let $f_\infty$ be a germ of a function $f \in \mathcal{Z}$ defined in the domain $D$ under the above conditions. The germ $f_\infty$ is lifted to the point $z = \infty^{(1)}$ lying on the first sheet $\mathcal{R}_2^{(1)}$ of the Riemann surface $\mathcal{R}_2(w)$. The corresponding germ $f_\infty^{(1)}$ extends from the point $z = \infty^{(1)}$ to the entire first sheet of the Riemann surface $\mathcal{R}_2(w)$. From the first sheet to the second sheet this germ extends as a multivalued function. So, in order to define a single-valued meromorphic extension of $f_\infty^{(1)}$ on the second sheet, one needs to introduce an appropriate family of admissible compact sets ("cuts") $K$ lying on the second sheet, $K \subset \mathcal{R}_2(w)$. Namely, we consider compact sets such that:

1) the compact set $K = K^{(2)} \subset \mathcal{R}_2^{(2)}(w)$;
2) the compact set $K$ does not separate the Riemann surface $\mathcal{R}_2(w)$; that is, the complement of $K$ is connected, $\mathcal{R}_2(w) \setminus K = D(K)$ is a domain on $\mathcal{R}_2(w)$, and $\infty^{(1)} \in D(K)$;
3) the germ $f_\infty^{(1)} = f_\infty$ extends from the point $z = \infty^{(1)}$ in the domain $D(K)$ as a meromorphic (single-valued analytic) function $f \in \mathcal{M}(D(K))$.

For a given germ $f_\infty$ of the function $f \in \mathcal{Z}$, the class of all compact sets $K \subset \mathcal{R}_2(w)$ satisfying the above conditions 1)–3) will be denoted by $\mathcal{R}(f_\infty)$. Compact sets from this class will be called admissible. It is clear that the family $\mathcal{R}(f_\infty)$ is nonempty.

It is also clear that the compact set $K \in \mathcal{R}(f_\infty)$ is a nonpolar set (for this concept on a Riemann surface, see [7]). So, the Green function $g_K(z, \infty^{(1)})$ for the domain $D(K)$ with singularity at the point $z = \infty^{(1)}$ is well defined; see [7], [8], [9]. At the point $z = \infty^{(1)}$ we introduce the local coordinate $\xi$. For an arbitrary compact set $K \in \mathcal{R}(f_\infty)$, we have (in terms of this coordinate)

$$g_K(z, \infty^{(1)}) = \log \frac{1}{|\xi|} + \gamma(K) + o(1), \quad \text{as} \quad z \to \infty^{(1)}. \quad (3)$$

The constant $\gamma(K)$ in (3), which depends on the local coordinate $\xi$, is called the Robin constant (with respect to this coordinate) at the point $z = \infty^{(1)}$.

In the class of compact sets $\mathcal{R}(f_\infty)$, we pose the following extremal problem.

**Problem 1.** Prove the existence and characterize an admissible compact set $F \in \mathcal{R}(f_\infty)$ such that

$$\gamma(F) = \sup_{K \in \mathcal{R}(f_\infty)} \gamma(K). \quad (4)$$

It is clear that even though the quantity $\gamma(F)$ depends on the local coordinate $\xi$, the extremal compact set (provided it exists) does not depend on the local coordinate.

The following result holds.

**Theorem 1.** There exists a unique compact set $F \in \mathcal{R}(f_\infty)$ satisfying condition (4), that is, $\gamma(F) = \max_{K \in \mathcal{R}(f_\infty)} \gamma(K)$. The compact set $F$ consists of a finite number of analytic arcs and is completely characterized by the following...
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\[ \frac{\partial g_F(z, \infty^{(1)})}{\partial n^+} = \frac{\partial g_F(z, \infty^{(1)})}{\partial n^-}, \quad z \in F^\circ, \]

where \( F^\circ \) is the union of all open arcs whose closures comprise the compact set \( F \), and \( \partial / \partial n^\pm \) are the normal derivatives at a point \( z \in F^\circ \) from the opposite sides of \( F^\circ \).

Note that the case \( F \ni \infty^{(2)} \) is not excluded from consideration.

In what follows, in the proof of Theorem 2 we shall assume that the following “general position” condition is satisfied: the compact set \( F \) consists precisely of \( p \) pairwise disjoint analytic arcs, that is, the compact set \( F \) contains no Chebotarev points. This assumption is quite analogous to that adopted in the first papers on asymptotic properties of Padé polynomials; see [18], [19], [28]. Note that this assumption is equivalent to saying that all zeros of the polynomial \( V_{m-1} \), which appear in a characterization of the Stahl compact set in terms of a variational method, are of even multiplicity; see (8).

Theorem 1 follows in part from the general results of the book [23, Ch. 8, § 4]. However in [23] the proof of a more general result is based on the Schiffer variational method. Nevertheless, below we shall give our independent proof of Theorem 1, because from this proof it will be possible to derive an additional information on the structure of the compact set \( F \) and its relation to some quadratic differential. Namely, it will be shown that the compact set \( F \) consists of the critical trajectories of this quadratic differential. This information will be required below in the proof of Theorem 2. Theorem 1 will be proved in § 2.

Note that Problem 1 is quite similar to the classical problem on an admissible compact set of minimal capacity from the Stahl theory of Padé approximation. In the classical case, for an extremal compact set \( S \) on the Riemann sphere, the quantity \( e^{-\gamma(S)} \) is the capacity of this compact set (with respect to the point at infinity \( z = \infty \)). It is well known (see [26], and also [2]) that the Stahl compact set \( S \) satisfies the characteristic relation

\[ \frac{g_S(z, \infty)}{\partial n^+} = \frac{g_S(z, \infty)}{\partial n^-}, \quad z \in \hat{S}^\circ, \]

which is quite similar to (5). Here, \( \hat{S}^\circ \) is the union of open arcs whose closures comprise the Stahl compact set \( S \) and \( g_S(z, \infty) \) is the Green function for the domain \( \hat{C} \setminus S \). We recall that the Stahl compact set \( S \) (an admissible compact set of minimal capacity) consists of the closures of critical trajectories of the quadratic differential. Namely, the following relations hold:

\[ S = \left\{ z \in \mathbb{C} : -\frac{V_{m-2}(z)}{B_m(z)} \, dz^2 > 0 \right\}, \]

where \( B_m(z) := \prod_{b \in \Sigma} (z - b) \), \( \Sigma = \sum \) is the set of singular points of a multivalued function \( f \in \mathcal{H}(\infty), \# \Sigma = m < \infty, \deg V_{m-2} = m - 2, V_{m-2}(z) = z^{m-2} + \cdots = \prod_{j=1}^{m-2} (z - v_j), v_j \) are the Chebotarev points of the compact set \( S \).

For the definition of the \( S \)-property, see [10], [11], [20], [22] and there references given there.
So, with a given germ $f_\infty$ of a multivalued analytic function $f$ with finite number of branching points on the Riemann sphere one associates in a unique way a (unique) two-sheeted hyperelliptic Riemann surface $R_2(f_\infty)$ defined by the equation $w^2 = V_{m-2}(z)/B_m(z)$. Since by the above the rational function $1/(V_{m-2}(z)/B_m(z))$ is uniquely defined from the germ $f_\infty$, the function $w$ is also uniquely defined from the original germ. This hyperelliptic Riemann surface is known as the (Stahl) associated surface with the germ $f_\infty$. With the exception of the case when the original germ $f_\infty$ is a germ of a hyperelliptic function, $f_\infty$ extends not on the entire associated Riemann surface, but also to the first sheet of this surface. For a further extension of this germ $f_\infty$ as a single-valued analytic function on the second sheet of this Riemann surface, we need to organize the corresponding cuts (in fact, the family of such cuts forms an admissible compact set, and the corresponding family of compact sets forms the family of admissible compact sets). Nevertheless, it turns out the so-called strong asymptotics of the Padé polynomials is characterized precisely in terms pertaining to this two-sheeted Riemann surface $R_2(f_\infty)$ (see [19], [28], [3]). So, our approach, in which properties of extremal compact sets pertaining to Hermite–Padé polynomials are studied with the help of the results obtained earlier in the Stahl theory and its further advances made by Stahl himself and other researchers (see [28], [30], [32], [12]), is also quite natural. The new approach proposed in [30] has proved instrumental in delivering, for the class of functions of the form (2), some new and previously available results related to the Hermite–Padé polynomials in terms of the scalar equilibrium problem (posed on a two-sheeted Riemann surface), rather than in terms of the generally accepted equilibrium problem (posed on the Riemann sphere). This scalar approach was further advanced in [29], [30], [32], [12] for a pair of functions forming a Nikishin system (cf. [21]). In particular, this also pertains to the pair of functions $f, f^2$, which, as was shown by the author of the present paper in [29], forms a Nikishin system (under a minimal extension of this classical concept). With this approach, the extremal problems on the corresponding two-sheeted Riemann surface were posed and solved (see also §4 below). This approach leads naturally to the (now three-sheeted) Riemann surface which is Nuttall-associated with the original germ $f_\infty$. Namely, this germ $f_\infty$, as lifted to the point $z = \infty^{(1)}$ as a germ $f_\infty^{(1)}$, extends as a (single-valued) meromorphic function from this point to the domain defined as the complement of the “topmost” third sheet. However, for further single-valued extension of the germ $f_\infty^{(1)}$ to the third sheet of this Riemann surface, corresponding cuts are required (cf. [21] and [25]).

In the present paper, we generally adhere to the scalar approach, which, however, is developed in a slightly different situation. Namely, we formulate and solve the extremal problem now for the family of four functions $[1, f, f^2, f^3]$, where

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11Note that the polynomials $V_{m-2}$ and $B_m$ may have common zeros. Hence $V_{m-2}/B_m = V_{m-2}/B_m^{*}$, where now the polynomials $V_{m-2}$ and $B_m^{*}$ are relatively prime polynomials. Moreover, the polynomial $V_{m-2}$ may have zeros of even multiplicity. As a result, the two-sheeted Riemann surface $R_2(f_\infty)$ is defined in fact by the quadratic equation $w^2 = \tilde{V}_{m-2}/B_m^{*}$, where the polynomial $\tilde{V}_{m-2}$ is obtained from the polynomial $V_{m-2}$ by removing the zeros of even multiplicity.

12The fact that this complement is a domain is a part of the well-known Nuttall conjecture [19] of 1984. This conjecture was proved in 2017 in [13].
Let \( \mathcal{R}_4(f) \) be the four-sheeted Riemann surface of a function \( f \in \mathcal{Z} \) and let \( \pi_4 : \mathcal{R}_4(f) \to \mathbb{C} \) be the corresponding canonical projection (see Fig. 1 in the case \( p = 1 \) and \( 1 < A < B \) in (2)).

Let \( f_{\infty} \in \mathcal{H}(\infty) \) be the above germ of a function \( f \in \mathcal{Z} \) (that is, \( f_{\infty} \) extends holomorphically from the point \( z = \infty \) to the Stahl domain \( D = \hat{\mathbb{C}} \setminus E \)). We shall assume that the first (open) sheet \( \mathcal{R}_4^{(1)} \) of the Riemann surface \( \mathcal{R}_4(f) \) is chosen so that \( \mathcal{R}_4^{(1)} \simeq D \) and the mapping \( \pi_4 : \mathcal{R}_4^{(1)} \to D \) is biholomorphic. Let \( \infty^{(1)} := \pi_4^{-1}(\infty) \cap \mathcal{R}_4^{(1)} \). Then the germ \( f_{\infty} \) is lifted to the point \( \infty^{(1)} \in \mathcal{R}_4^{(1)} \) and extends everywhere to the Riemann surface \( \mathcal{R}_4 \) as a single-valued meromorphic function. In what follows, we will identify the germs \( f_{\infty} \) and \( f_{\infty^{(1)}} \), retaining the above notation \( f_{\infty} \).

Let us now define the global partition of the Riemann surface \( \mathcal{R}_4(f) \) into sheets as follows. The Green function \( g_\mathcal{R}(z, \infty^{(1)}) \) for the domain \( \mathcal{D}_1 := \mathcal{R}_2(w) \setminus \mathcal{F} \) (the complement of the extremal compact set \( \mathcal{F} = \mathcal{F}^{(2)} \subset \mathcal{R}_2(w) \)) has symmetric boundary (5), and moreover, by the above assumption, the boundary consists of a finite number of disjoint analytic arcs. Besides, by the definition of the Green function, we have \( g_\mathcal{R}(z, \infty^{(1)}) = 0 \) for \( z \in \mathcal{F} \). By arranging cuts on \( \mathcal{R}_2(w) \) we may consider these arcs as two-sided arcs (with the exception of the endpoints \( \mathcal{F} \setminus \mathcal{F}^0 \)). From the surface \( \mathcal{P}_1 \simeq \mathcal{R}_2(w) \setminus \mathcal{F} \), we construct a four-sheeted Riemann surface \( \mathcal{R}_4 = \mathcal{R}_4(f_{\infty}) \) as follows. Consider the second copy \( \mathcal{P}_2 \) of this surface (\( \mathcal{P}_2 \simeq \mathcal{R}_2(w) \setminus \mathcal{F} \)) and “glue” it together with the first copy by identifying the opposite sides ("edges") of the new cuts on these two copies of the surface \( \mathcal{R}_2(w) \setminus \mathcal{F} \). We have \( \mathcal{P}_1 \simeq \mathcal{P}_2 \), and hence the points \( z_1 \in \mathcal{P}_1 \) and \( z_2 \in \mathcal{P}_2 \) are in a one-to-one correspondence, which we denote by the sign "\( \simeq \)"; \( z_1 \simeq z_2 \). Note that \( \pi_2(z_1) = \pi_2(z_2) \).

In accordance with the above “general position” assumption to the effect that the arcs comprising the compact set \( \mathcal{F} \) are disjoint, it is clear that the four-sheeted Riemann surface \( \mathcal{R}_4(f_{\infty}) \) thus obtained coincides with the Riemann surface \( \mathcal{R}_4(f) \). Thus, the original germ \( f_{\infty} \) extends to the entire surface \( \mathcal{R}_4 \) as a single-valued meromorphic function. We also note that the fact the four-sheeted Riemann surface \( \mathcal{R}_4 \) thus constructed coincides with the Riemann surface \( \mathcal{R}_4(f) \) of the function \( f \in \mathcal{Z} \) can be also derived directly from the proof of Theorem 1.

Let us now define the global partition of the Riemann surface \( \mathcal{R}_4(f) \) into sheets as follows. In accordance with the definition of the Green function, we have \( g_\mathcal{R}(z, \infty^{(1)}) = 0 \) for \( z \in \mathcal{F} \). From this equality and the symmetry condition (5) of the \( S \)-compact set \( \mathcal{F} \) it follows that the Green function \( g_\mathcal{R}(z, \infty^{(1)}) \) extends to a harmonic function on the second part of the Riemann surface \( \mathcal{R}_4(f) \) with the same values but with different sign ("minus" instead of "plus"). Namely,

\[
g_\mathcal{F}(z_2, \infty^{(1)}) = -g_\mathcal{F}(z_1, \infty^{(1)}) \quad \text{for} \quad z_1 \in \mathcal{P}_1, \quad z_2 \in \mathcal{P}_2.
\]
Thus, the compact set $F$, provided that it is composed of precisely $p$ disjoint analytic arcs and these arcs are considered as two-sided cuts (with two "edges") on the Riemann surface $\mathcal{R}_2(w)$, is the boundary between the second and third sheets of the Riemann surface $\mathcal{R}_1(f)$. The boundary between the first and second sheets goes along the curve $\Gamma^{(1,2)}$ such that $\pi_4(\Gamma^{(1,2)}) = E$. The boundary between the third and fourth sheets goes along the curve $\Gamma^{(3,4)}$ such that $\pi_4(\Gamma^{(3,4)}) = E$.

Let us summarize. The boundary between the second and third sheets goes along the family of $p$ closed disjoint curves $\Gamma^{(2,3)}$ such that $\pi_4(\Gamma^{(2,3)}) = \pi_2(F) = F$, $E \cap F = \emptyset$. Note that with this definition it is quite possible that both points $\infty^{(2)}$, $\infty^{(3)}$ lie in $\Gamma^{(2,3)}$. Note also that the boundary of the compact set $F$ has no common points with the compact set $\Gamma = \partial \mathcal{R}_2^{(1)}(w) = \partial \mathcal{R}_2^{(2)}(w)$, and moreover, $\Gamma^{(2,3)}$ is disjoint both from $\Gamma^{(1,2)}$ and from $\Gamma^{(3,4)}$ (cf. the assumption in [14]).

It is clear that by performing the above procedure of continuation of the Green function $g_E(z, \infty^{(1)})$ from the surface $\mathcal{R}_2 \simeq \mathcal{R}_2(w) \setminus F$ to the entire Riemann surface $\mathcal{R}_4(f)$ we get the bipolar Green function $g(z, \infty^{(1)}, \infty^{(4)})$ defined on the Riemann surface $\mathcal{R}_4(f)$. This function has logarithmic singularities at the points $z = \infty^{(1)}$ and $z = \infty^{(4)}$ and is normalized by the condition $g(z, \infty^{(1)}, \infty^{(4)}) \equiv 0$ for $z \in \Gamma^{(2,3)}$ (see [6], [7]). The fact that the Nuttall partition of a Riemann surface into sheets can be related to the (zero, with appropriate normalization) level line of the bipolar Green function for the Riemann surface was known earlier only for the case of Padé polynomials and the two-sheeted Riemann surface associated with the original germ. In this classical case, the bipolar Green function splits the Riemann surface into two sheets.

It is easily checked that the Green function $g(z) := g_E(z, \infty)$, $z = z^{(4)} \in \mathcal{R}_2^{(4)}(w)$, also extends to the four-sheeted Riemann surface $\mathcal{R}_4(f)$. Indeed, $g(z) = g(z, \infty^{(1)}, \infty^{(2)})$ is the bipolar Green function for the Riemann surface $\mathcal{R}_2(w)$. Since the points of the compact set $F$ are not singular for this function, one can define, for a give partition of $\mathcal{R}_4(f)$ into sheets,

$$
g(z^{(3)}) : = g(z^{(2)}), \quad z \in \hat{\mathbb{C}},
$$

$$
g(z^{(4)}); = g(z^{(1)}), \quad z \in \hat{\mathbb{C}} \setminus E \quad (10)
$$

with the given partition of the Riemann surface. The function $g(z)$ thus obtained has logarithmic singularities at all four\footnote{If $\infty \in F$, then $\infty^{(2)} = \infty^{(3)}$ and in this case, it is necessary, at this point at infinity lying on the Riemann surface $\mathcal{R}_4(f)$, to introduce the corresponding local coordinate and then argue as in the paper [13].} points of the set $\pi_4^{-1}(\infty)$: $\infty^{(1)}$, $\infty^{(2)}$, $\infty^{(3)}$, $\infty^{(4)}$.

Now the following definition is correct. For $z \in \mathcal{R}_4(f)$, we put

$$
u(z) := -2g_E(z, \infty^{(1)}) - g(z), \quad z \notin \pi_4^{-1}(\infty). \quad (11)
$$

The following result holds.

**Theorem 2.** Let $u(z)$, $z \in \mathcal{R}_4(f)$, be the function defined by (11).

1) The following asymptotic formulas hold:

$$
u(z) = -3 \log |z| + O(1), \quad z \to \infty^{(1)},
$$

$$
u(z) = \log |z| + O(1), \quad (12)$$
as \( z \) tends to any of the points of the set \( \pi^{-1}_4(\infty) \setminus \{1, 2, 3, 4\} \).

2) For the above partition of the Riemann surface \( \mathcal{R}_4(f) \) into sheets,
\[
u(z^{(1)}) < \nu(z^{(2)}) < \nu(z^{(3)}) < \nu(z^{(4)}) \quad \text{for} \quad z = \pi_4(z^{(j)}) \notin E \cup F, \quad j = 1, 2, 3, 4.
\]

Together relations (12) and (13) mean that the function \( \nu(z) \), as defined by (11) and which is harmonic on \( \mathcal{R}_4(f) \setminus \pi^{-1}_4(\infty) \), coincides up to a constant to the real part of the Abelian integral used in the definition of the Nuttall partition of the Riemann surface into sheets (see [19], [13]).

Thus, the partition of the Riemann surface \( \mathcal{R}_4(f) \) into sheets \( \mathcal{R}_4(1)(f), \mathcal{R}_4(2)(f), \mathcal{R}_4(3)(f), \mathcal{R}_4(4)(f) \), which was constructed with the use of the extremal problem 1, is a Nuttall partition.

2. Proof of Theorem 1

So, given an \( f \in \mathcal{Z} \) defined by (2), suppose that the above assumptions on the geometric arrangement of branching points and the corresponding exponents are satisfied.

We consider the two-sheeted Riemann surface \( \mathcal{R}_2(w) \) of the function \( w \) defined by \( w^2 = z^2 - 1 \). We shall assume that the Riemann surface \( \mathcal{R}_2(w) \) is realized as a two-sheeted covering of the Riemann surface \( \mathcal{C} \) using the explicitly given uniformization:
\[
z = \frac{1}{2} \left( \zeta + \frac{1}{\zeta} \right), \quad w = \frac{1}{2} \left( \zeta - \frac{1}{\zeta} \right), \quad \zeta \in \hat{\mathbb{C}}.
\]

Accordingly, the first sheet \( \mathcal{R}_2(1)(w) \), on which \( w = (z^2 - 1)^{1/2}/z \rightarrow 1 \) as \( z \rightarrow \infty \), corresponds to the exterior \( \mathcal{U}_\zeta := \{ \zeta : |\zeta| > 1 \} \) of the unit disk \( \mathbb{D}_\zeta := \{ \zeta : |\zeta| < 1 \} \) in the \( \zeta \)-plane, and the second sheet \( \mathcal{R}_2(2)(w) \), on which \( w = -(z^2 - 1)^{1/2}/z \rightarrow -1 \) as \( z \rightarrow \infty \), corresponds to the unit disk \( \mathbb{D}_\zeta \) itself. By a point \( z \) on the Riemann surface \( \mathcal{R}_2(w) \), \( z \in \mathcal{R}_2(w) \), we shall mean the pair \( z := (z, w) = (z, \pm(z^2 - 1)^{1/2}) \).

The canonical projection \( \pi_2 : \mathcal{R}_2(w) \rightarrow \hat{\mathbb{C}} \) is defined by \( \pi_2(z) := z \). By the point \( z = \infty(1) \in \mathcal{R}_2(1)(w) \) we mean the point on the Riemann surface \( \mathcal{R}_2(w) \) such that \( \pi_2(\infty(1)) = \infty \) and \( w/z \rightarrow 1 \) as \( z \rightarrow \infty(1) \). Similarly, for \( z = \infty(2) \) we have \( \pi(\infty(2)) = \infty \) and \( w/z \rightarrow -1 \) as \( z \rightarrow \infty(2) \). A passage from the first sheet \( \mathcal{R}_2(1)(w) \) to the second sheet \( \mathcal{R}_2(2)(w) \) proceeds along the cut closed interval \( E \). Here we assume as usual that the interval has to edges (the upper and the lower ones) and that the sheets are “glued” by identifying crosswisely the edges of the cuts; that is, by identifying the upper edge of one cut with the lower edge of the other cut, and vice versa. It can be easily shown that the above partition into sheets is a Nuttall partition. Indeed, let
\[
G(z) := \int_{-1}^{z} \frac{dt}{\sqrt{t^2 - 1}} = \log(z + w) = \log(z \pm (z^2 - 1)^{1/2})
\]
be an Abelian integral of the third kind with purely imaginary periods and logarithmic singularities only at the points \( z = \infty(1) \) and \( z = \infty(2) \). Hence \( \eta_2(z) := -\text{Re} G(z) \) is a harmonic function on the Riemann surface \( \mathcal{R}_2(w) \setminus \{\infty(1), \infty(2)\} \),
\[
\eta_2(z) = \mp \log |z| + O(1), \quad z \rightarrow \infty(1) \quad \text{or} \quad z \rightarrow \infty(2),
\]
\[ \eta_2(z^{(1)}) < \eta_2(z^{(2)}), \quad z \in D. \]  

Moreover, \( \eta_2(z) = 0 \) for \( z \in \Gamma, \Gamma := \pi_2^{-1}(E) \). So, \(-\eta_2(z^{(1)}) = g_E(z, \infty)\) is the Green function of the domain \( D \). The given germ \( f_\infty \in \mathcal{H}(\infty) \) of a function \( f \in \mathcal{L}, f \in \mathcal{H}(D) \), is lifted to the point \( z = \infty^{(1)} \in \mathcal{R}_2(w) \) and extends to the entire first sheet \( \mathcal{R}_2^{(1)}(w) \) as a single-valued holomorphic function (recall that \( \pi_2(\mathcal{R}_2^{(1)}(w)) = D = \mathbb{C} \setminus E \)). A further single-valued extension of this function to the entire second sheet of the Riemann surface \( \mathcal{R}_2^{(2)}(w) \) is hindered by the branch points \( a_j^{(2)} \in \mathcal{R}_2^{(2)}(w) \) such that \( \pi_2(a_j^{(2)}) = a_j, j = 1, \ldots, 2p, a_j = (A_j + 1/A_j)/2 \).

In order that such a single-valued analytic (meromorphic) extension of the germ \( f_\infty \) be possible one should make appropriate cuts on the second sheet \( \mathcal{R}_2^{(2)}(w) \). Above in §1 we introduced the corresponding class of admissible compact sets \( \mathcal{R}(f_\infty) \).

Since the Riemann surface \( \mathcal{R}_2(w) \) is of zero genus, Theorem 1 can be reduced to the planar case and to the corresponding compact set of minimal capacity on the plane. Indeed, the uniformization of \( \mathcal{R}_2(w) \) is defined using the Zhukovskii function (14). Here, to the first sheet of \( \mathcal{R}_2^{(1)}(w) \) there corresponds the exterior \( U := \mathbb{C} \setminus \mathbb{D} \) of the unit disk \( \mathbb{D} = \{ \zeta : |\zeta| < 1 \} \), and to the second sheet, the unit disk \( \mathbb{D} \). So, in this case, the quantity \( 1/\zeta \) is the local coordinate \( \xi \) alluded to above. With this uniformization, to each compact set \( K \in \mathcal{R}(f_\infty) \) admissible for the germ \( f_\infty \) there corresponds an admissible compact set \( \tilde{K} \in \mathcal{R}(\tilde{f}_\infty), \tilde{K} \subset \mathbb{D}_\xi \), for the germ \( \tilde{f}_\infty \in \mathcal{H}(\infty) \) corresponding to \( f_\infty \). The Green function is invariant with respect to conformal mappings of the domain. The Robin constant changes accordingly. Hence Problem 1 on the maximum of the Robin constant \( \gamma(K) \) on the class of compact sets \( K \in \mathcal{R}(f_\infty) \) is equivalent to the maximization problem of the Robin constant \( \gamma(\tilde{K}) \) on the class of admissible compact sets \( \tilde{K} \subset \mathbb{D}_\xi, \tilde{K} \in \mathcal{R}(\tilde{f}_\infty) \); that is, it is equivalent to the Stahl problem on compact set of minimal capacity.

Under transformation (14), a function \( f(z) \in \mathcal{L} \) of the form (2) is transformed to the function

\[ \tilde{f}(\zeta) = \prod_{j=1}^{2p} \left( A_j - \frac{1}{z + w} \right)^{\alpha_j} = \prod_{j=1}^{2p} \left( A_j - \frac{1}{\zeta} \right)^{\alpha_j}, \]  

where all \( \alpha_j = \pm 1/2 \). Since \( A_j \in \mathbb{U} \) for all \( j \), all singular points of the function \( \tilde{f} \) have the form \( \zeta = A_j = 1/A_j \in \mathbb{D} \). Thus, to the extremal compact set \( F \) for the function \( f \) on the Riemann surface \( \mathcal{R}_2(w) \) there corresponds an admissible compact set of minimal capacity \( \hat{F} \) for the function \( \tilde{f} \), which is given by \( \hat{f}_\infty \in \mathcal{H}(\hat{\mathbb{C}} \setminus \hat{F}) \) and at the point \( \zeta = 0 \) the function \( f \) has a pole). By the well-known properties of a compact set of minimal capacity, we have \( \hat{F} \subset \mathbb{D} \) (more precisely, the compact set \( \hat{F} \) lies in the convex hull of the set \( \{ \hat{A}_j = 1/A_j, j = 1, \ldots, 2p \} \) and \( \hat{F} \) consists of a finite number of analytic arcs (which are trajectories of the quadratic differential), does not split the complex
plane, and has the $S$-property (6). Moreover (see (7)),
\[
\tilde{F} = \left\{ \zeta \in \mathbb{C} : \text{Re} \int_{A_1}^{\zeta} \sqrt{\frac{V_{2p-2}(t)}{B_{2p}(t)}} \, dt = 0 \right\},
\]
where $B_{2p}(t) := \prod_{j=1}^{2p} (t - \tilde{A}_j)$, $V_{2p-2}(t) := (t - v_1) \ldots (t - v_{2p-2})$ is the corresponding Chebotarev polynomial, $v_j$, $j = 1, \ldots, 2p - 2$, are the Chebotarev points of the compact set $\tilde{F}$. All these properties of a compact set of minimal capacity are well known in the general case and have been obtained already by Stahl in 1985 (see [26], [27], and also [2]).

In the case considered here all the branch points of the function $\tilde{f}$ are of second order. In this setting, the existence and description of a compact set of minimal capacity was given already by Nuttall [18], [19]. In particular, in the case of general position, all the zeros of the polynomial $V_{2p-2}$ are of even multiplicity, all $v_j \neq \tilde{A}_k$ for $j = 1, \ldots, 2p - 2$, $k = 1, \ldots, 2p$, and the compact set $\tilde{F}$ consists of $p$ disjoint analytic arcs that pairwisely connect points of the set $\{ \tilde{A}_j, j = 1, \ldots, 2p \}$. Moreover, the compact set $\tilde{F}$ has the classical $S$-property; that is, the corresponding Green function satisfies the relation of the form (6). By the invariance of the Green function with respect to a conformal mapping it follows that the Green function corresponding to the compact set $\tilde{F}$ satisfies (5). Hence the compact set $\tilde{F}$ has the required $S$-property.

So, on the second sheet of the Riemann surface $\mathcal{R}_2(w)$ we have obtained a system of analytic arcs comprising the compact set $\tilde{F}$, not splitting the Riemann surface $\mathcal{R}_2(w)$, having the $S$-property (5), and such that $f_\infty \in \mathcal{M}(\mathcal{R}_2(w) \setminus \tilde{F})$. The $S$-property (5) is a direct consequence of the $S$-property from (6), because the Green function is invariant with respect to a conformal mapping.

3. PROOF OF THEOREM 2

The first part of Theorem 2 is a direct consequence of Definition (11) of the function $u(z)$ and properties of the functions $g(z)$ and $g_F(z, \infty^{(1)})$.

Let us now prove the second part of Theorem 2.

We proceed as follows. Consider the function
\[
u(z) := -2g_F(z, \infty^{(1)}) - g(z), \quad z \in \mathcal{R}_2(w) \setminus \tilde{F},
\]
where $g(z) = \log |z + w| = \log |\zeta|$. The function $u(z)$ is harmonic in the domain $\mathcal{R}_2(w) \setminus \tilde{F}$, with the exception of the points at infinity $z = \infty^{(1)}$ and $z = \infty^{(2)}$, where it behaves as follows:
\[
u(z) = \begin{cases} -3 \log |z| + O(1), & z \to \infty^{(1)}, \\ \log |z| + O(1), & z \to \infty^{(2)} \end{cases}
\]
(19)

(here and in what follows we assume for simplicity that $\infty^{(2)} \notin \tilde{F}$; otherwise a more careful consideration is required quite similar to that conducted in [13]).

We set, as before
\[
\Gamma = \{ z \in \mathcal{R}_2(w) : u(z^{(1)}) = u(z^{(2)}) \};
\]
(20)

here when writing $z^{(1)} \in \mathcal{R}_2^{(1)}(w)$ and $z^{(2)} \in \mathcal{R}_2^{(2)}(w)$ we mean, as in the above, a Nuttall partition of the Riemann surface $\mathcal{R}_2(w)$ into sheets, $\pi_2(\Gamma) = E$. The compact set $\Gamma$ is a closed arc on the Riemann surface $\mathcal{R}_2(w)$ passing through the
points $z = \pm 1$, not intersecting the compact set $F$, and splitting $R_2(w)$ into two domains, of which one contains $F$, and the other one, the point $z = \infty^{(1)}$.

For $z \notin E \cup F$, we set
\[
v_1(z) := u(z^{(2)}) - u(z^{(1)}).
\]
By (11), using properties of the functions $g(z)$ and $g_{\hat{F}}(z, \infty^{(1)})$ and taking into account the symmetry of the compact set $F$ with respect to the real line, we have
\[
v_1(z) = -2g_{\hat{F}}(z^{(2)}, \infty^{(1)}) - g(z^{(2)}) + 2g_{\hat{F}}(z^{(1)}, \infty^{(1)}) + g(z^{(1)})
= 2g_{\hat{F}}(z^{(1)}, \infty^{(1)}) - 2g_{\hat{F}}(z^{(2)}, \infty^{(1)}) + 2g(z^{(1)}).
\]
This implies the following properties of the function $v_1$:

1) $v_1(z)$ is a harmonic function in the domain $\mathbb{C} \setminus (E \cup F)$ and is continuous in $\hat{\mathbb{C}}$;
2) $v_1(z) \equiv 0$ for $z \in E$;
3) $v_1(z) = 4 \log |z| + O(1)$ as $z \to \infty$, $z \notin F$;
4) $v_1(z) > 0$ for $z \in F \setminus \infty$.

From properties 1)–4) of the function $v_1$ we get the inequality $v_1(z) > 0$ for $z \in \mathbb{C} \setminus E$, and therefore, the inequality
\[
u(z^{(1)}) < u(z^{(2)}) \text{ for } z \in \hat{\mathbb{C}} \setminus E.
\]
We set $v_2(z) := u(z^{(3)}) - u(z^{(2)})$, $z \notin E \cup F$. By definition (11) and using the properties of the function $g(z)$ we have $v_2(z) = -2g_{\hat{F}}(z^{(3)}, \infty^{(1)}) + 2g_{\hat{F}}(z^{(2)}, \infty^{(1)}) = 4g_{\hat{F}}(z^{(2)}, \infty^{(1)}) > 0$. Therefore,
\[
u(z^{(2)}) < u(z^{(3)}) \text{ for } z \in \hat{\mathbb{C}} \setminus (E \cup F).
\]

Finally, let us prove that
\[
u(z^{(3)}) < u(z^{(4)}) \text{ for } z \in \hat{\mathbb{C}} \setminus (E \cup F).
\]  
(21)
Indeed, we set
\[
v_3(z) := u(z^{(4)}) - u(z^{(3)})
= 2g_{\hat{F}}(z^{(3)}, \infty^{(1)}) + g(z^{(3)}) - 2g_{\hat{F}}(z^{(4)}, \infty^{(1)}) - g(z^{(4)}).
\]
By definition we have
\[
g_{\hat{F}}(z^{(3)}, \infty^{(1)}) = -g_{\hat{F}}(z^{(2)}, \infty^{(1)}) \text{ for } z \notin F,
g_{\hat{F}}(z^{(4)}, \infty^{(1)}) = -g_{\hat{F}}(z^{(1)}, \infty^{(1)}) \text{ for } z \notin F
\]
\[
g(z^{(3)}) = g(z^{(2)}), \quad g(z^{(4)}) = g(z^{(1)}).
\]
Hence
\[
v_3(z) = u(z^{(4)}) - u(z^{(3)})
= -2g_{\hat{F}}(z^{(2)}, \infty^{(1)}) + 2g_{\hat{F}}(z^{(1)}, \infty^{(1)}) + g(z^{(2)}) - g(z^{(1)})
= 2g_{\hat{F}}(z^{(1)}, \infty^{(1)}) - 2g_{\hat{F}}(z^{(2)}, \infty^{(1)}) - 2g(z^{(1)}).
\]
Hence the function $v_3(z), z \in \hat{\mathbb{C}} \setminus (E \cup F)$, has the following properties:

1) the function $v_3$ is harmonic in the domain $\mathbb{C} \setminus (E \cup F)$ and is continuous on the plane $\hat{\mathbb{C}}$;
2) $v_3(z) \equiv 0$ for $z \in E$;
3) $v_3(z) = 2g_{\hat{F}}(z^{(1)}, \infty^{(1)}) - 2g(z^{(1)}) > 0$ for $z \in F$;
4) for $\infty \notin F$, the function $v_3$ is harmonic near the point $z = \infty$; if $\infty \in F$, then $v_3$ is a continuous function at the point $\infty \in F$ and is harmonic in $U \setminus F$, where $U$ is some neighborhood of the point $z = \infty$.

Indeed, properties 1), 2) and 4) of the function $v_3$ are clear. Let us verify property 3). Consider the function

$$v_4(z) := 2g_F(z^{(1)}, \infty^{(1)}) - 2g(z^{(1)}), \quad z \in \mathbb{C} \setminus F.$$ 

We have $v_4(z) = v_3(z)$ for $z \in F$. The function $v_4(z)$ is continuous on $\mathbb{C}$ and by properties of the functions $g_F(z, \infty^{(1)})$ and $g(z)$ it extends as a harmonic to the neighborhood of the point $z = \infty$. The compact set $\Gamma$ is an admissible set for Problem 1. Since the compact set $F$ lies on the second sheet of the Riemann surface $\mathcal{R}_2(w)$, $F \subset \mathcal{R}_2^{(2)}(w)$, we have $g_F(z, \infty^{(1)}) > 0$ for $z \in E$ and $g(z) \equiv 0$ for $z \in \Gamma$. It is clear that for the function $v_4(z)$ we have $v_4(z) > 0$ for $z \in E$ and $v_4(z)$ is a harmonic function in $\mathbb{C} \setminus E$. Therefore, $v_4(z) > 0$ everywhere in $\mathbb{C}$, and hence, also on the compact set $F$. So, for $z \in F$, we have $v_3(z) = v_4(z) > 0$. This proves property 3) of the function $v_3$. Hence $v_3(z) > 0$ for $z \notin E \cup F$. Now the required inequality $u(z^{(3)}) < u(z^{(4)})$ for $z \notin E \cap F$ follows from the definition of the function $v_3(z)$.

So, for the function $u(z) = -2g_F(z, \infty^{(1)}) - g(z)$ we see that $u(z)$ is a harmonic function in the domain $\mathcal{R}_4(f) \setminus \{\infty^{(1)}, \infty^{(2)}, \infty^{(3)}, \infty^{(4)}\}$. Moreover, by the above we have

$$u(z) = \begin{cases} 
-3 \log |z| + O(1), & z \to \infty^{(1)}, \\
\log |z| + O(1), & z \to \infty^{(2)}, \\
\log |z| + O(1), & z \to \infty^{(3)}, \\
\log |z| + O(1), & z \to \infty^{(4)}, 
\end{cases}$$

and further, under the above partition of the Riemann surface $\mathcal{R}_4(f)$ into sheets, the following inequalities hold:

$$u(z^{(1)}) < u(z^{(2)}) < u(z^{(3)}) < u(z^{(4)}). \quad (22)$$

So, the partition of the Riemann surface $\mathcal{R}_4(f)$ into sheets, which we have introduced with the help of the Green function $g_F(z, \infty^{(1)})$ corresponding to the extremal compact set of minimal capacity $F \subset \mathcal{R}_2(w)$, is shown to be a Nuttall partition.

Theorem 2 is proved.

4. Some concluding remarks

4.1. It is natural, while remaining in the same class of multivalued functions $\mathcal{F}$, to compare the solution of Problem 1, as obtained in terms of the Nuttall partition of the four-sheeted Riemann surface $\mathcal{R}_4(f)$ of a function $f \in \mathcal{F}$, and the existence of a three-sheeted Riemann surface with a Nuttall partition $\mathcal{R}_3(f_{\infty})$ associated with a given germ $f_{\infty} \in \mathcal{H}(\infty)$; see [19], [21], [32]. The latter problem will be referred to as Problem 2; the solutions of these two problems will be denoted by $\Gamma_4^{(2,3)}$ (above in the present paper it was denoted by $\mathcal{F}$) and $\Gamma_3^{(2,3)}$, respectively. The natural question is whether these compact sets can be equal. More precisely, the question of course should be put like this: may the canonical projections

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14More precisely, this property is satisfied for the compact set $\overline{\mathcal{R}}_2^{(2)}(w) \supset \Gamma$, which is the closure of the second sheet of the Riemann surface $\mathcal{R}_2(w)$ of the function $w^2 = z^2 - 1$. 
Γ_4^{(2,3)} := π_4(Γ_4^{(2,3)}) and Γ_3^{(2,3)} := π_3(Γ_3^{(2,3)}) coincide with each other? It is easily seen that this is indeed so in the real-case situation, that is, when in (2) all (pairwise different) quantities A_j lie in \( \mathbb{R} \). In the case when in \( p = 1 \) and \( 1 < A_1 < A_2 \) in (2), the pair of functions \( f, f^2 \) forms a Nikishin system (see [29]).

The four-sheeted Riemann surface corresponding to this function \( f \) is depicted in Fig. 1. In this case, \( π_4(Γ_4^{(2,3)}) = π_3(Γ_3^{(2,3)}) = [a_1, a_2] \), where \( a_j = (A_j + 1/A_j)/2 \). The question of whether \( π_4(Γ_4^{(2,3)}) \) and \( π_3(Γ_3^{(2,3)}) \) may coincide for a function \( f \in \mathcal{Z} \) in some other (nonreal) case remains open from the theoretical point of view. Numerical experiments surely show that such a coincidence should not be expected if the reality condition is violated (see Fig. 2).

We recall (see [32], [12]) that the compact set \( Γ_3^{(2,3)} \) is the solution of the following potential theory “max-min”-problem for the nonstandard potential with harmonic external field.

**Problem 2.** Let \( \mathcal{R}(f_∞) \) be the above family of admissible compact sets \( K = K(2) \subset \mathcal{R}_3^{(2)} \) for the germ \( f_∞ \), that is,

1) the set \( Ω(K) := \mathcal{R}_3(w) \setminus K \ni ∞^{(1)} \) is a domain on the Riemann surface \( \mathcal{R}_2(w) \) of the function \( w^2 = z^2 - 1 \);

2) \( f \in \mathcal{M}(Ω(K)) \).

Let \( μ \in M_1(K) \) be a unit Borel measure supported in the compact set \( K \). We set \( Φ(z) := z + w \). Let

\[
P^μ(z) := \int_K \log \frac{|1 - 1/(Φ(z)Φ(t))|}{|z - t|^2} \, dμ(t),
\]

be the potential of the measure \( μ \), and let

\[
J_V(μ) := \int_K P^μ(z) \, dμ(z) + 2 \int_K V(z) \, dμ(z)
\]

be the energy of the measure \( μ \) corresponding to this potential and the external field \( V(z) := −\log |Φ(z)| \).

Let a measure \( λ_K \in M_1(K) \) be the solution of the extremal problem

\[
J_V(λ_K) = \min_{μ \in M_1(K)} J_V(μ).
\]

Then (see [32]) the “max-min-problem”

\[
J_V(λ_\widehat{F}) = \max_{K \in \mathcal{K}(f_∞)} J_V(λ_K) = \max_{K \in \mathcal{R}(f_∞)} \min_{μ \in M_1(K)} J_V(μ)
\]

has a unique (in the class \( \mathcal{R}(f_∞) \)) solution \( \widehat{F} \in \mathcal{R}(f) \) in the class of compact sets \( K \in \mathcal{R}(f_∞) \). The extremal compact set \( \widehat{F} \) has the following \( S \)-property:

\[
\frac{∂(P^λ_\widehat{F}(z) + V(z))}{∂n^+} = \frac{∂(P^λ_\widehat{F}(z) + V(z))}{∂n^-}, \quad z \in \widehat{F}^o;
\]

here \( \widehat{F}^o \) is the family of open arcs whose closures constitute \( \widehat{F} \).

Moreover (see [29]),

\[
\frac{1}{n^*} \chi_{Q_{n,*}} \rightarrow π_2(λ_\widehat{F}), \quad n \rightarrow ∞.
\]

where the measure \( π_2(λ_\widehat{F}) \) is defined as \( π_2(λ_\widehat{F})(e) = λ_\widehat{F}(e) \) for any \( e \subset \mathcal{R}_2^{(2)}(w) \), \( e = π_2(e) \).
Note that the extremal measure $\lambda_K$ is a (unique) equilibrium measure; that is,
\begin{equation}
P^\lambda_K(z) + V(z) \equiv w_K = \text{const}, \quad z \in K.
\end{equation}
The facts that in (26) the identity holds on the entire compact set $K$ and $\supp \lambda_K = K$ were proved in [33].

The extremal problems 1 and (24) are different. It is natural to assume that the corresponding extreme compact sets are also different. More precisely, one may assume that in general $\pi_3(\Gamma_3^{(2,3)}) \neq \pi_4(\Gamma_4^{(2,3)})$, where $\Gamma_3^{(2,3)} = F$, $\Gamma_4^{(2,3)} = F$. The fact that in some cases these two sets coincide follows from the example presented in Fig. 1. This case corresponds to the choice of the parameters $p = 1$ and $1 < A_1 < A_2$ in (2). With this choice, $\pi_3(\Gamma_3^{(2,3)}) = \pi_4(\Gamma_4^{(2,3)}) = [a_1, a_2]$. At present, in the general case we can only resort to numerical experiments.

For example, in Fig. 2 we show the zeros (dark blue, red and black points) of three Hermite–Padé polynomials of type I $Q_{300,j}$, $j = 0, 1, 2$ (for the family $[1, f, f^2]$, see (29)). From the calculated eight Hermite–Padé polynomials of type I $q_{299,j}$, $j = 0, 1, 2, 3$ (for the family $[1, f, f^2, f^3]$, see (30)), we evaluated new (nonstandard) Hermite–Padé polynomials, which were introduced in [31] (see also [14]). The zeros of these new polynomials localize the projection $\pi_4(\Gamma_4^{(2,3)})$ on the Riemann sphere $\hat{\mathbb{C}}$ of the compact set $\Gamma_4^{(2,3)}$, which is the boundary between the second and third Nuttall sheets of the four-sheeted Riemann surface $\mathcal{R}_4(f)$ of a function $f \in \mathcal{E}$. The compact set $\Gamma_4^{(2,3)} = F$ is the solution of problem 1. In Fig. 2, these zeros of the new (nonstandard) Hermite–Padé polynomials are shown by pale blue points. It is clear that the compact sets $\pi_3(\Gamma_3^{(2,3)})$ and $\pi_4(\Gamma_4^{(2,3)})$ differ from each other. The red points located on the real line are the zeros of the Padé polynomials of order 100. They correspond to the closed interval $[-1, 1] =: \Delta$. Correspondingly, this is the projection $\pi_2(\Gamma_2^{(1,2)})$ of the boundary $\Gamma_2^{(1,2)}$ between the first and second sheets of the Riemann surface $\mathcal{R}_2(w)$ of the function $w^2 = z^2 - 1$.

4.2. Let us now consider the case of two intervals. Namely, let $\Delta_1 = [e_1, e_2]$ and $\Delta_2 = [e_3, e_4]$, where $e_1 < e_2 < e_3 < e_4$, and let $\varphi_{\Delta_j}$, $j = 1, 2$, be the functions inverse to the Zhukovskii function corresponding to these intervals, $|\varphi_{\Delta_j}(z)| > 1$ for $z \notin \Delta_j$. We set
\begin{equation}
f(z) = \prod_{j=1}^{2p} \left( \frac{A_j - 1}{\varphi_{\Delta_1}(z)} \right)^{\alpha_j} \prod_{k=1}^{2q} \left( B_k - \frac{1}{\varphi_{\Delta_2}(z)} \right)^{\beta_k}
\end{equation}
where $|A_j| > 1$, $|B_k| > 1$, $A_j = \overline{A}_s$ for all $j$ and for some $s \in \{1, \ldots, 2p\}$, $B_k = \overline{B}_s$ for some $\ell \in \{1, \ldots, 2q\}$ and all $A_k, B_j \notin \mathbb{R}$, $\alpha_j, \beta_k \in \{1/2, -1/2\}$, $\sum_{j=1}^{2p} \alpha_j + \sum_{k=1}^{2q} \beta_k = 0$ (mod $\mathbb{Z}$). We denote by $\mathcal{E}(\Delta_1, \Delta_2)$ the class of functions of the form (27) with the above conditions on $A_k$ and $B_j$. By the assumption, $|\varphi_{\Delta_j}(z)| > 1$ for $z \notin \Delta_j$. Hence $f \in \mathcal{H}(D)$, where $D := \hat{\mathbb{C}} \setminus (\Delta_1 \cup \Delta_2)$, and in particular, $f \in \mathcal{H}(\infty)$. The set of branching points of the multivalued function $f$ consists of the points $\pm 1$ and the points $a_j = \varphi_{\Delta_1}^{-1}(A_j) \notin \mathbb{R}$, $j = 1, \ldots, 2p$, $b_k = \varphi_{\Delta_2}^{-1}(B_k) \notin \mathbb{R}$, $k = 1, \ldots, 2q$. It is clear that, for $f \in \mathcal{E}(\Delta_1, \Delta_2)$, the Stahl compact set consists of two closed intervals, $S = \Delta_1 \cup \Delta_2$, and $D$ is the corresponding Stahl domain.
A natural question arises: in what terms should the limit distribution of the zeros of the Hermite–Padé polynomials of type I $Q_{n,j}$ (see (29)) for the family $[1, f, f^2]$ of functions $f$ from the class $\mathscr{P}(\Delta_1, \Delta_2)$ be characterized?

The conjecture is that instead of the Riemann surface of the function $w^2 = z^2 - 1$ we should now consider the Riemann surface of the function $w^2 = (z - e_1)(z - e_2)(z - e_3)(z - e_4)$. In this case, the general form of the potential (23), the external field, and the energy will be preserved. The $S$-compact set $\mathcal{F}$, which corresponds to the problem of the limit distribution of the zeros of the Hermite–Padé polynomials for the function $f \in \mathscr{P}(\Delta_1, \Delta_2)$, is as before characterized as the solution of an extremal problem of the form (24), while the $S$-property itself has the form (25).

It is worth pointing out that in (23)–(24) we speak about the $S$-compact set $\mathcal{F}^{(2,3)}$, whose existence is related to the existence of a three-sheeted Nuttall-partitioned Riemann surface $\mathcal{R}_3(f_\infty)$ associated with a given germ $f_\infty \in \mathcal{H}(\infty)$, $f \in \mathscr{P}(\Delta_1, \Delta_2)$. It is an open question whether there exists a four-sheeted Nuttall-partitioned Riemann surface $\mathcal{R}_4(f_\infty)$ associated with a germ $f_\infty \in \mathcal{H}(\infty)$. In the case $f \in \mathscr{P}(\Delta_1, \Delta_2)$ the Riemann surface of the function $f$ is 8-sheeted and the existing solution of Problem 1 apparently has nothing to do with the compact set $\Gamma_4^{(2,3)}$, which is the boundary between the second and third sheets of the Riemann surface $\mathcal{R}_4(f_\infty)$. Note that since the parameters $A_j$ and $B_k$ are real symmetric, the projection of the boundary $\Gamma_3^{(1,2)}$ between the first and second sheets of the Riemann surface $\mathcal{R}_4(f_\infty)$ always coincides with the union of the closed intervals $\Delta_1$ and $\Delta_2$. The same result also holds for the projection of the boundary $\Gamma_3^{(1,2)}$ between the first and second sheets of the Riemann surface $\mathcal{R}_4(f)$. Now let $p = q = 1$, $\alpha_1 = \alpha 2 = 1/2$, $\beta_1 = \beta_2 = -1/2$ in (27), that is,

$$f(z) = \left(\prod_{j=1}^{2} \left(A_j - \frac{1}{\varphi_{\Delta_1}(z)}\right) / \prod_{k=1}^{2} \left(B_k - \frac{1}{\varphi_{\Delta_2}(z)}\right)\right)^{1/2}.$$  (28)

For a function $f$ of the form (28), under certain values of $A_j$ and $B_k$ satisfying the above conditions, we have numerically found the zeros of the Hermite–Padé polynomials of type I $Q_{n,0}, Q_{n,1}, Q_{n,2}$, $\deg Q_{n,j} = n$, $n = 300$, for the family $[1, f, f^2]$ satisfying the relation

$$\left(Q_{n,0} + Q_{n,1}f + Q_{n,2}f^2\right)(z) = O\left(z^{-2n-2}\right), \quad z \to \infty,$$  (29)

and identified the zeros of the Hermite–Padé polynomials of type I $q_{n,0}, q_{n,1}, q_{n,2}, q_{n,3}$, $\deg q_{n,j} = n$, $n = 300$, for the family $[1, f, f^2, f^3]$ satisfying the relation

$$\left(q_{n,0} + q_{n,1}f + q_{n,2}f^2 + q_{n,3}f^3\right)(z) = O\left(z^{-3n-3}\right), \quad z \to \infty.$$  (30)

According to the available theoretical results and conjectures (see [19], [21], [32]), the zeros of the polynomials $q_{300,j}$ localize the compact set $\pi_3(\Gamma_3^{(2,3)})$, which is the projection of the compact set $\Gamma_3^{(2,3)}$ lying on the Riemann surface $\mathcal{R}_3(f_\infty)$ onto the Riemann sphere $\hat{\mathbb{C}}$. Here, $\mathcal{R}_3(f_\infty)$ is a three-sheeted Riemann surface with Nuttall partition into sheets associated with the germ $f_\infty$, $\Gamma_3^{(2,3)}$ is the boundary between its second and third sheets. In Fig. 3, these zeros of the polynomials $Q_{300,j}$, $j = 0, 1, 2$, are shown by dark blue, red, and black points. It is clearly seen that the compact set $\pi_3(\Gamma_3^{(2,3)})$ has four Chebotarev points of zero density.
From the calculated polynomials of type I \( q_{300,j} \) and \( q_{299,j} \), \( j = 0, 1, 2, 3 \), we calculate new (nonstandard) Hermite–Padé polynomials, which were introduced in [31] (see also [14]). The zeros of these polynomials localize the projection \( \pi_4(\Gamma_{4}^{(2,3)}) \) on the Riemann sphere \( \hat{\mathbb{C}} \) of the compact set \( \Gamma_{4}^{(2,3)} \), which is the boundary between the second and third Nuttall sheets of the four-sheeted Riemann surface \( \mathcal{R}_4(f_{\infty}) \) associated with the germ \( f_{\infty} \in \mathcal{H}(\infty) \) of a function \( f \in \mathcal{Z}(\Delta_1, \Delta_2) \). In Fig. 3, the zeros of the new (nonstandard) Hermite–Padé polynomials are shown by pale blue points. It is clearly seen that, first, the compact set \( \pi_4(\Gamma_{4}^{(2,3)}) \) contains four Chebotarev points, but they all have positive density. Second, the compact sets \( \pi_3(\Gamma_{3}^{(2,3)}) \) and \( \pi_4(\Gamma_{4}^{(2,3)}) \) differ from each other.

Next, the red points on the real line are the zeros of the Padé polynomials of order 100. They correspond to two closed intervals \( \Delta_1 \) and \( \Delta_2 \). The set \( \Delta_1 \cup \Delta_2 \) - is the projection \( \pi_2(\Gamma_{2}^{(1,2)}) \) of the boundary between the first and second sheets of the Riemann surface \( \mathcal{R}_2(w) \) of the function \( w^2 = (z - e_1)(z - e_2)(z - e_3)(z - e_4) \).

In Fig. 4 we show the same three sets, as in Fig. 3, but on a smaller scale. Each of the sets \( \pi_3(\Gamma_{3}^{(2,3)}) \) and \( \pi_4(\Gamma_{4}^{(2,3)}) \) splits the Riemann sphere \( \hat{\mathbb{C}} \) into two domains so that each of these domains contains precisely one closed interval \( \Delta_1 \) or \( \Delta_2 \). At the same time, if one considers the sets \( \Gamma_{3}^{(2,3)} \) and \( \Gamma_{4}^{(2,3)} \) on the Riemann surface \( \mathcal{R}_2(w) \) of the function \( w^2 = (z - e_1)(z - e_2)(z - e_3)(z - e_4) \), then it appears that the complement of each of such sets is not a domain on this Riemann surface.

This empirical fact should prove to be of utmost importance in generalizing the results obtained in [16] for a (real) Nikishin system (that is, for the case when the compact set \( F \) is the union of real closed intervals) to the more general complex case, in which the compact set \( F \) is not known \textit{a priori}; cf. [21], where the case of a complex Nikishin system is considered, but in the case when the complement of \( \Gamma_{3}^{(2,3)} = \pi_3(\Gamma_{3}^{(2,3)}) \) is connected.
Figure 1. Four-sheeted Riemann surface $\mathcal{R}_4(f)$ of the function $f$ of the form (1).
Figure 2
Figure 3
Figure 4
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