On Integer Balancing of Digraphs

Mohamed-Ali Belabbas* and Xudong Chen†

Abstract

A weighted digraph is balanced if the sums of the weights of the incoming and of the outgoing edges are equal at each vertex. We show that if these sums are integers, then the edge weights can be integers as well.

1 Introduction

Let $G = (V, E)$ be a strongly connected digraph on $n$ vertices, with vertex set $V$ and edge set $E$. We use $v_iv_j$ to denote an edge from $v_i$ to $v_j$. The digraph $G$ can have self-arcs. For a vertex $v_i$, let $N^{-}(v_i) := \{ v_j \in V \mid v_iv_j \in E \}$ and $N^{+}(v_i) := \{ v_k \in V \mid v_kv_i \in E \}$ be the sets of out-neighbors and in-neighbors of $v_i$, respectively.

We let $\mathbb{R}^{+}$ (resp. $\mathbb{Z}^{+}$) be the set of nonnegative real numbers (resp. non-negative integers). We assign $w_{ij} \in \mathbb{R}^{+}$ to edges $v_iv_j$, for $v_iv_j \in E$, and denote by $w \in \mathbb{R}_{|E|}^{+}$ the collection of these $w_{ij}$. We call $(G, w)$ a weighted digraph.

Definition 1. The weighted digraph $(G, w)$ is said to be balanced if, for each vertex, the inflow equals to the outflow:

$$u_i := \sum_{v_j \in N^{-}(v_i)} w_{ij} = \sum_{v_k \in N^{+}(v_i)} w_{ki}, \quad \forall v_i \in V. \quad (1)$$

We call $u_i$ the weight of vertex $v_i$.

The vector $u := (u_1, \ldots, u_n) \in \mathbb{R}^{n}_{+}$ is said to be feasible if there exists $w \in \mathbb{R}_{|E|}^{+}$ such that (1) holds.

Balanced digraphs have a host of applications in engineering and applied sciences, including the study of flocking behaviors [5], sensor networks and distributed estimation [2]. While balancing over the real numbers is acceptable in some scenarios, others such as traffic management and fractional packing, require integer balancing [3, 6, 1, 4, 7].

If all the $w_{ij}$ are integers, then clearly every $u_i$ is an integer. The question we are interested in is: given a feasible integer-valued $u$, can we find an integer-valued $w$? We show that the answer is affirmative:

*Coordinated Science Laboratory, University of Illinois, Urbana-Champaign. Email: belabbas@illinois.edu
†Department of Electrical, Computer, and Energy Engineering, University of Colorado, Boulder. Email: xudong.chen@colorado.edu
Theorem 1. Let $G$ be a strongly connected digraph and $u \in \mathbb{Z}^n$ be any feasible vector. Then, there exist nonnegative integers $w_{ij}$ such that (1) holds.

The result also applies to the case of weakly connected digraph $G$, based upon the fact that $(G, w)$ is balanced if and only if every strongly connected component of $(G, w)$ is balanced [4].

We provide below a constructive proof of Theorem 1.

2 Algorithm, Propositions, and Proofs

To proceed, we associate to the digraph $G = (V, E)$ an undirected bipartite graph $B = (X, Y, F)$ on $2n$ vertices, where $X \sqcup Y$ is the vertex set and $F$ is the edge set. Each of the two sets $X$ and $Y$ comprises $n$ vertices. The edge set $F$ is defined as follows: there is an edge $(x_i, y_j)$ in $B$ if $v_i v_j$ is an edge of $G$. See Fig. 1 for an illustration.

Note that the directed edges in $G$ are in one-to-one correspondence with the undirected edges in $B$. Thus, we can assign the edge weights $w_{ij}$, for $v_i v_j \in E$, to the edges $(x_i, y_j)$ in $B$.

The balance relation (1), when applied to the bipartite representation of $G$, is now turned into

$$ u_i = \sum_{y_j \in N(x_i)} w_{ij} = \sum_{x_k \in N(y_i)} w_{ki}, \quad \forall i = 1, \ldots, n. \quad (2) $$

If the above relations hold for some nonnegative real numbers $u_i$, then $(B, w)$ is said to be balanced with vertex weights $u_i$ for both $x_i$ and $y_i$. The following result is an immediate consequence of the above construction of $(B, w)$:

Lemma 1. The digraph $(G, w)$ is balanced if and only if $(B, w)$ is balanced.

Now, let $u \in \mathbb{Z}^n$ be a feasible vector and $w \in \mathbb{R}^{|E|}$ be such that (2) is satisfied. In the sequel, we refer to elements of $\mathbb{R}^+ \setminus \mathbb{Z}^+$ as decimal numbers.

Every cycle in $B$ has an even number of edges, and the number is at least 4. A cycle in $B$ does not correspond to a (directed) cycle in $G$, as illustrated in Fig. 1. Instead, if $x_{\alpha_1} y_{\beta_1} \cdots x_{\alpha_p} y_{\beta_p} x_{\alpha_1}$ is a cycle in $B$, then each vertex $v_{\alpha_i}$
in $G$ has two outgoing edges $v_{\alpha_i}v_{\beta_i}$ and $v_{\alpha_i}v_{\beta_{i-1}}$ (with $\beta_0$ identified with $\beta_p$) while each vertex $v_{\beta_i}$ has two incoming edges $v_{\alpha_i}v_{\beta_i}$ and $v_{\alpha_{i+1}}v_{\beta_i}$ (with $\alpha_{p+1}$ identified with $\alpha_1$).

We next introduce the following definition:

**Definition 2.** An edge in $(B, w)$ is called decimal if its weight is a decimal number. A cycle $C$ in $(B, w)$ is completely decimal if all its edges are decimal.

Given a balanced $(B, w)$, we aim to obtain a set of integer edge weights $w^*_{ij} \in \mathbb{Z}_+$ that satisfy (1). We present below an algorithm that does so in a finite number of steps:

**Algorithm 1:**

1. If $(B, w)$ does not contain a completely decimal cycle, then the algorithm is terminated. Otherwise, select a completely decimal cycle in $B$:

   \[ C = x_{\alpha_1}y_{\beta_1} \cdots x_{\alpha_p}y_{\beta_p}x_{\alpha_1}. \]

2. For the selected cycle $C$, find an edge whose weight has the smallest decimal part. Without loss of generality, we assume that the edge is $x_{\alpha_1}y_{\beta_1}$ and the decimal part is $\epsilon := w_{\alpha_1\beta_1} - \lfloor w_{\alpha_1\beta_1} \rfloor$. Update the weights along the cycle as follows:

   \[ w_{\alpha_i\beta_i} \leftarrow w_{\alpha_i\beta_i} - \epsilon \]

   \[ w_{\beta_i\alpha_{i+1}} \leftarrow w_{\beta_i\alpha_{i+1}} + \epsilon \]

   for $1 \leq i \leq p$,

   \[ (3) \]

   where we identify $x_{\alpha_{p+1}}$ with $x_{\alpha_1}$. All the other edge weights remain unchanged.

Note that one can easily obtain a decimal cycle in $(B, w)$, if one exists. Let $x_i$ be a vertex incident to a decimal edge. Denote by $\lambda(x_i)$ the number of decimal edges incident to $x_i$. Since the vertex weight $u_i$ of $x_i$ is integer-valued, then clearly $\lambda(x_i) \geq 2$. Now fix a decimal edge $(x_i, y_j)$ in $(B, w)$. By the above arguments, $\lambda(y_j) \geq 2$. Thus, there exists another decimal edge incident to $y_j$, say $(y_j, x_k)$ and, similarly, $\lambda(x_k) \geq 2$. Iterating this procedure, we will return to some previously encountered vertex $x_\ell$, since $B$ is finite. By construction, the vertices obtained in the process yield a completely decimal cycle.

Theorem 1 is then a direct consequence of the following result:

**Theorem 2.** Let $(B, w)$ be a balanced bipartite graph with integer-valued vertex weights $u$ satisfying (2). Then, Algorithm 1 terminates in a finite number of steps and returns a nonnegative integer-valued solution $w^*$ to (2), with $u^* = u$.

We establish below Theorem 2 and start with the following proposition:

**Proposition 3.** Let $(B, w)$ be balanced with vertex weights $u$ and $C$ be a completely decimal cycle in $(B, w)$. Denote by $(B, w')$ the bipartite graph obtained after a one-step update on $w$ described by (3). Let $u'$ be the vertex weights associated with $(B, w')$. Then, $(B, w')$ is balanced with $u' = u$.
Proof. If a vertex $x_i$ does not belong to $C$, then none of the edges incident to it are updated. Hence, the summation $\sum_{y_j \in N(x_i)} w_{ij}$ is unchanged. The same argument applies to any vertex $y_j$ that does not belong to $C$.

Next, denote the cycle by $C = x_{\alpha_1}y_{\beta_1} \cdots x_{\alpha_p}y_{\beta_p}x_{\alpha_1}$. Every vertex in $C$ is incident to exactly two consecutive edges in $C$. For any vertex $x_{\alpha_i}$ in the cycle, we have that

$$\sum_{\gamma \in N(x_{\alpha_i})} (w'_{\alpha_i\gamma} - w_{\alpha_i\gamma}) = (w'_{\alpha_i\beta_{i-1}} + w'_{\alpha_i\beta_i}) - (w_{\alpha_i\beta_{i-1}} + w_{\alpha_i\beta_i}).$$

We identify $\beta_0$ with $\beta_p$ for the case $i = 1$. By (3), the two expressions in parentheses on the right hand side of (4) are equal, so the difference is 0. The same arguments can be applied to vertices $y_{\beta_i}$.

Finally, because $\epsilon$ is the smallest decimal part of the weights on the edges along the cycle $C$, the updated edge weights are nonnegative. \[\square\]

We next have the following proposition:

**Proposition 4.** Let $(B, w)$ be a balanced bipartite graph, with integer-valued vertex weights. Then, the following statements are equivalent:

1. There is no decimal edge;
2. There is no completely decimal cycle;
3. The vector $w$ is integer-valued.

Proof. From the discussion after Algorithm 1, we see that 2 implies 1. Furthermore, it is clear that 1 implies 2 and that 2 implies 3. We show below that 2 implies 3.

Assuming 2 holds, let $F' \subseteq F$ be the collection of decimal edges. We show that $F'$ is an empty set. Suppose, to the contrary, that $F'$ is nonempty; then, we let $X' \subseteq X$ and $Y' \subseteq Y$ be the collections of vertices incident to the edges in $F'$. Consider the subgraph $B' = (X', Y', F')$ induced by $X' \cup Y'$. Because $(B, w)$ does not have a completely decimal cycle, $B'$ is acyclic. Denote by $B'_1, \ldots, B'_m$ the connected components of $B'$, each of which is a tree. Pick an arbitrary tree $B'_k$ and a leaf of $B'_k$, say $x_i$. On the one hand, there exists one and only one edge $(x_i, y_j)$ in $B'_k$ such that the weight $w_{ij}$ is decimal. By construction, this edge is also the only decimal edge in $B$ incident to $x_i$. On the other hand, since $(B, w)$ is balanced, we have that

$$w_{ij} = u_i - \sum_{y_{j'} \in N(x_i) \setminus \{y_j\}} w_{ij'}.$$ 

The right hand side of the above expression is integer-valued, which is a contradiction. \[\square\]

In fact, more can be said about decimal edges and completely decimal cycles:
Corollary 5. Let \((B, w)\) be a balanced bipartite graph, with integer-valued vertex weights. Then, every decimal edge belongs to a completely decimal cycle.

Proof. Suppose that \((x_i, y_j)\) is a decimal edge that is not contained in any decimal cycle; then, the weight \(w_{ij}\) will not be affected by executing Algorithm 1. On the other hand, when Algorithm 1 is terminated, there is no completely decimal cycle. By Prop. 4, there is no decimal edge, which is a contradiction. 

Finally, note that every one-step operation of Algorithm 1 reduces the number of completely decimal cycles by at least one. Thus, Theorem 2 follows as an immediate consequence of Propositions 3 and 4.

References

[1] Dimitri P Bertsekas. *Network optimization: continuous and discrete models*. Athena Scientific, 1998.

[2] Ruggero Carli, Alessandro Chiuso, Luca Schenato, and Sandro Zampieri. Distributed Kalman filtering based on consensus strategies. *IEEE Journal on Selected Areas in Communications*, 26(4):622–633, 2008.

[3] Naveen Garg and Jochen Koenemann. Faster and simpler algorithms for multicommodity flow and other fractional packing problems. *SIAM Journal on Computing*, 37(2):630–652, 2007.

[4] Loh Hooi-Tong. On a class of directed graphs—with an application to traffic-flow problems. *Operations Research*, 18(1):87–94, 1970.

[5] Ali Jadbabaie, Jie Lin, and A Stephen Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.

[6] Serge A Plotkin, David B Shmoys, and Éva Tardos. Fast approximation algorithms for fractional packing and covering problems. *Mathematics of Operations Research*, 20(2):257–301, 1995.

[7] Apostolos Rikos and Christoforos Hadjicostis. Distributed integer weight balancing in the presence of time delays in directed graphs. *IEEE Transactions on Control of Network Systems*, 5(3):1300–1309, 2017.