ABSTRACT. The article proved the upper bound of leading coefficient of characteristic polynomial of graded ideal in a ring of generalized polynomials.

Examples of such rings are as well the rings of commutative polynomials (for which the classical Bézout theorem holds), as some rings of differential operators. For a system of generalized homogeneous equations in small codimensions we obtain exact polynomial in $d$ estimates. In the general case, the estimate is double exponential in $\tau$: $O(d^{2^{\tau-1}})$, where $d$ is a maximal degree of generators of a graded ideal, $\tau$ is it’s codimension.

For systems of linear differential equations bounds of the same asymptotics, but by other methods, were obtained by D. Grigoryev in [2].

Keywords: Differential algebra, ring of generalized polynomials, graded ideal, characteristic polynomial, typical dimension, Bézout theorem.

1. INTRODUCTION

In algebraic geometry and commutative algebra many studies are devoted to the Hilbert polynomial. Differential dimension polynomial was introduced by E. Kolchin [4] and has the same important role in differential algebra. Estimations of its coefficients are the classic unsolved problems of differential algebra.

In recent years interest in computer algebra has increased, and one of its directions is the study of Gröbner bases. For polynomial ideals, this notion was studied quite fully, in particular upper and lower degree bounds of polynomials in the Gröbner basis by degrees of generators of the ideal (see, for example, [1]) are found. It is interesting, that the complexity of Gröbner basis computation and of task of finding the leading coefficient of Hilbert polynomial has different asymptotic order.

For rings of differential operators over a field, great success in such studies were reached by D. Grigoryev and A. Chistov (see [2], [3]). Here the situation is another, than for polynomial ideals, and is known only upper double exponential bound (as for Gröbner basis orders, and for the leading coefficient of dimension polynomial).
To generalize these results, we will consider rings, introduced in [5], see definition 4.1.4. Gröbner basis technique works for ideals of such (in the general case, non-commutative) rings, and the concept of characteristic polynomial is defined.

Our bound (Theorem 3) in a situation when degree of characteristic polynomial 1 less than the maximum possible (we will use the term codimension 1) coincides with Kolchin's result (see [4], p. 199).

By analogy with this linear estimate, Kolchin believed that in other codimension \( \tau \) the bound of the leading coefficient also is a polynomial of degree \( O(\tau) \). In the general case, this has not yet been disproved.

In codimensions 2 for a differential dimensional polynomial the bound is proved (see 5.6.7, [5]), and it coincides with that obtained in this paper for homogeneous systems generalized polynomials.

Now in codimensions 3, 4, and 5 (see Theorem 4) exact upper bounds are obtained for the first time. Note (see the example 6) that in codimension 3 the bound is achieved.

In the case when the characteristic polynomial of a homogeneous ideal in the ring of generalized polynomials is a constant, we obtain the upper double exponential bound of it by the number of generalized unknowns. This result is similar to the result of D. Grigoriev.

2. Preliminary facts.

One can find basic concepts and facts in [4, 9, 5].

Denote the set of integers by \( \mathbb{Z} \), non-negative integers by \( \mathbb{N}_0 \) and binomial coefficients \( \binom{s(s-1)...(s-m+1)}{m!} \) by \( \binom{s}{m} \).

For vector \( e = (j_1, \ldots, j_m) \in \mathbb{N}_0^m \), the order of \( e \) is defined by \( \text{ord} e = \sum_{k=1}^{m} j_k \). Note that any numerical polynomial \( v(s) \) can be written as \( v(s) = \sum_{i=0}^{d} a_i (s+i) \), where \( a_i \in \mathbb{Z} \). We call numbers \( (a_d, \ldots, a_0) \) standard coefficients of polynomial \( v(s) \).

**Definition 1.** (see [5], definition 2.4.9 or [6]). Let \( \omega = \omega(s) \) be a univariate numerical polynomial in \( s \) and let \( d = \text{deg} \omega \). The sequence of minimizing coefficients \( b(\omega) \) is the vector \( b(\omega) = (b_d, \ldots, b_0) \in \mathbb{Z}^{d+1} \) defined by induction on \( d \) as follows. If \( d = 0 \) (i.e. \( \omega \) is a constant), then \( b(\omega) = (\omega) \). Let \( d > 0 \) and \( \omega(s) = \sum_{i=0}^{d} a_i (s+i) \). Let \( v(s) = \omega(s+a_d)-(s+1)^d+a_d) + (s+d+1)^d \). Since \( \text{deg} v < d \), one may suppose that the sequence of minimizing coefficients \( b(v) = (b_k, \ldots, b_0) \) \( (0 \leq k < d) \) of the polynomial \( v(s) \) has been defined. To define the same for \( \omega \) we set \( b(\omega) = (a_d, 0, \ldots, 0, b_k, \ldots, b_0) \in \mathbb{Z}^{d+1} \).

Now we define the Kolchin dimension polynomial of a subset \( E \subset \mathbb{N}_0^m \).

**Definition 2.** Regard the following partial order on \( \mathbb{N}_0^m \): the relation \( (i_1, \ldots, i_m) \leq (j_1, \ldots, j_m) \) is equivalent to \( i_k \leq j_k \) for all \( k = 1, \ldots, m \). We consider a function \( \omega | E (s) \), that in a point \( s \) equals \( \text{Card} V_E(s) \),
where \( V_E(s) \) is the set of points \( x \in \mathbb{N}_0^m \) such that \( \text{ord} x \leq s \) and for every \( e \in E \) the condition \( e \leq x \) isn’t true. Then (see for example, [4], p.115, or [5], theorem 5.4.1) function \( \omega_E(s) \) for all sufficiently large \( s \) is a numerical polynomial. We call this polynomial the Kolchin dimension polynomial of a subset \( E \).

Not every numerical polynomial is a Kolchin dimensional polynomial for some set \( E \). The connection of these concepts is established in the following theorem.

**Theorem 1.** (see [6] and [5], proposition 2.4.10). The sequence of minimizing coefficients of dimension polynomial Kolchin consists of only non-negative integers. The converse is also true: if the sequence of minimizing coefficients of some numerical polynomial consists of non-negative numbers, then it is the Kolchin dimension polynomial of some set \( E \). We denote the set of such polynomials by \( W \).

Note that the set \( W \) is closed with respect to addition, difference

\[
\Delta_1 \omega(s) = \omega(s) - \omega(s-1)
\]

and positive shift: \( \omega(s) = \omega(s+j), j \in \mathbb{N} \). (see [5], propositions 2.4.13 and 2.4.22).

Let \( X = \{x_1, \ldots, x_m\} \) be a finite system of elements. By \( T = T(X) \) we denote the free commutative semigroup with unity (written multiplicatively), generated by the elements of \( X \). Elements of \( T \) will be called monomials. Let \( \theta \in T \), \( \theta = x_1^{e_1} \cdots x_m^{e_m} \). By the order of \( \theta \) we shall call the sum \( e_1 + \cdots + e_m \) that will be denoted by \( \text{ord} \theta \). Suppose, that the set of monomials is linearly ordered and for any \( \theta \in T \) the following conditions hold:

\[
1 \leq \theta;
\]

and if \( \theta_1 < \theta_2 \), then

\[
\theta \theta_1 < \theta \theta_2.
\]

In this case we shall say, that a ranking is defined on the set of monomials \( T \).

Let \( F \) be a field, \( P \) the vector \( F \)-space with the basis \( T = T(X) \). We define on \( P \) the function “taking the leader” in the following way: any \( g \) in \( P \) may be represented as a sum \( g = \sum_{\theta \in T} a_\theta \theta \), where only a finite number of coefficients \( a_\theta \in F \) are distinct from zero (such representation is unique up to the order of the terms). Among all monomials, present in this expression with nonzero coefficients, we choose the maximal with respect to the order introduced on the set \( T \). This monomial will be called the leader of \( g \in P \) and will be denoted by \( u_g \).

**Definition 3.** Let some ranking on the set of monomials \( T = T(X) \) be given and let \( P \) be the vector \( F \)-space with the basis \( T \). Suppose that \( P \) is a \( F \)-algebra, and \( u_{AB} = u_A u_B \) for all \( A, B \in P \). Furthermore,
suppose that $1\theta_1 \cdot 1\theta_2 = 1\theta_1\theta_2 \in P$ for any $\theta_1, \theta_2 \in T$; in particular, the generators $x_1, \ldots, x_m$ pairwise commute. Such ring we shall call the ring of generalized polynomials in the indeterminates $X = \{x_1, \ldots, x_m\}$.

**Example 1.** The ring of commutative polynomials over a field. Consider an arbitrary ranking on the set $X = \{x_1, \ldots, x_m\}$. Let $P$ be the algebra $F[x_1, \ldots, x_m]$ of polynomials in the commutative indeterminates $x_1, \ldots, x_m$ over a field $F$. It is easy to see, that the condition $u_{AB} = u_A u_B$ holds for all $A, B \in P$ and therefore we may treat $F[x_1, \ldots, x_m]$ as a ring of generalized polynomials in the indeterminates $x_1, \ldots, x_m$.

**Definition 4.** An operator $\partial$ on a ring $\mathbb{K}$ is called a derivation operator (or differentiation) iff $\partial(a + b) = \partial(a) + \partial(b)$ and $\partial(ab) = a\partial(b) + \partial(a)b$ for all $a, b \in \mathbb{K}$.

A commutative ring $\mathbb{K}$ with a basic set of derivation operators on $\mathbb{K}$ is called a differential ring.

**Definition 5.** Let $\mathcal{F}$ be a differential field and let $\Delta = \{\partial_1, \ldots, \partial_m\}$ be a basic set of derivation operators on $\mathcal{F}$. The ring $D = \mathcal{F}[\partial_1, \ldots, \partial_m]$ of skew polynomials in indeterminates $\partial_1, \ldots, \partial_m$ with coefficients in $\mathcal{F}$ and the commutation rules $\partial_i \partial_j = \partial_j \partial_i$, $\partial_i a = a \partial_i + \partial_i(a)$ for all $a \in \mathcal{F}$, $\partial_i, \partial_j \in \Delta$ is called a (linear) differential (or $\Delta$-) operator ring on $\mathcal{F}$.

**Example 2.** The ring of differential operators over a field. Let $\mathcal{F}$ be a $\Delta$-field, and let an arbitrary ranking be fixed on the set $T = T(\Delta)$. Then the ring $D = \mathcal{F}[\partial_1, \ldots, \partial_m]$ of linear differential operators over $\mathcal{F}$ (see definition 5) is a ring of generalized polynomials in the indeterminates $\partial_1, \ldots, \partial_m$.

**Example 3.** The ring of differential operators over a ring of polynomials. Let $\mathcal{F}$ be a $\Delta = \{\partial_1, \ldots, \partial_m\}$-field and $R$ a ring of commutative polynomials in the indeterminates $y_1, \ldots, y_n$ over $\mathcal{F}$. We define the derivation operators $\mathcal{D}^\prime = \{\partial_1^\prime, \ldots, \partial_m^\prime\}$ on $R$ in the following way. Set $\partial_i^\prime(f) = 0$ for all $i = 1, \ldots, m$, $j = 1, \ldots, n$. For any $1 \leq i \leq m$ we fix a number $1 \leq j \leq n$ (for different $i$ the corresponding indices $j$ may coincide) and set $\partial_i^\prime(f) = \partial_i(f)y_j$ for all $j = 1, \ldots, n$ and $f \in \mathcal{F}$. Then the ring $D_R$ of linear $\mathcal{D}^\prime$-operators over $R$ is a ring of generalized polynomials in the indeterminates $X = \{\partial_1^\prime, \ldots, \partial_m^\prime, y_1, \ldots, y_n\}$. Indeed, if we consider the ranking such that $\partial_i^\prime > y_j$ for all $i = 1, \ldots, m$, $j = 1, \ldots, n$, then the condition $u_f u_g = u_{fg}$ is fulfilled.

Let $D$ be a ring of generalized polynomials in the indeterminates $X = \{x_1, \ldots, x_m\}$ over a field $\mathcal{F}$ and $F$ the free $D$-module with the basis $B = \{f_1, \ldots, f_n\}$. The $\mathcal{F}$-vector space $F$ has as a basis the direct (Cartesian) product $T \times B$ of the sets $T = T(X)$ and $B$. This product
we shall call the **set of terms** of the module \( F \),
\[
T_F = \{ x_1^{i_1} \cdots x_m^{i_m} f_j \mid (i_1, \ldots, i_m) \in \mathbb{N}_0^m, \ j = 1, \ldots, n \}.
\)
We cannot multiply terms, but we can define the product of a term by
a monomial satisfying the following conditions:
1. for any term \( u \leq v, \ u, v \in T_F \), and for any monomial \( \theta \in T \) true \( \theta u \leq \theta v \).

**Definition 6.** A ranking will be called **orderly** if the condition
\( \text{ord} \ \theta_1 < \text{ord} \ \theta_2 (\theta_1, \theta_2 \in T) \) implies \( \theta_1 f_i < \theta_2 f_j \) for all \( 1 \leq i, j \leq n \).

**Example 4.** Let a ranking on the set \( T \) of monomials be given. We
shall order the terms \( T_F \): \( \theta_1 f_i < \theta_2 f_j \) if either \( i < j \) or \( i = j \) and \( \theta_1 < \theta_2 \). Such ranking on \( T_F \) is not orderly.

**Example 5.** Let a ranking on the set \( T \) be following: \( \theta_1 < \theta_2 \iff \text{ord} \ \theta_1 < \text{ord} \ \theta_2 \) or \( \text{ord} \ \theta_1 = \text{ord} \ \theta_2 \) and \( \theta_1 < \theta_2 \) with respect to lexicographic order on monomials. Let \( t_1, t_2 \in T_F \). We
set \( t_1 = \theta_1 f_i < t_2 = \theta_2 f_j \) if and only if either \( \text{ord} \ \theta_1 < \text{ord} \ \theta_2 \), or \( \text{ord} \ \theta_1 = \text{ord} \ \theta_2 \) and \( i < j \), or \( \text{ord} \ \theta_1 = \text{ord} \ \theta_2 \), \( i = j \) and \( \theta_1 < \theta_2 \). This
ranking is orderly. We shall call it **standard**.

In the submodule of the free module \( F \) over the ring of generalized
polynomials \( D \) a Gröbner basis exists:

**Definition 7.** (see definition 4.1.25, [5]). Let \( D \) be a ring of generalized
polynomials in indeterminates \( X = \{ x_1, \ldots, x_m \} \), \( F \) a free \( D \)-module.
Suppose that \( M \subseteq F \) is a submodule of \( F \), \( G \subset M \) is a finite set and < is a ranking on the set of terms \( T_F \). The set \( G \) is called a **Gröbner basis** of \( M \), if there exists for any nonzero \( f \in M \) a representation:
\[
f = \sum_{i=1}^{r} c_i \theta_i g_i, \ 0 \neq c_i \in \mathcal{F}, \ \theta_i \in T(\mathcal{X}), \ g_i \in \mathcal{G}, \ \theta_i u_{g_i} > \theta_{i+1} u_{g_{i+1}},
\]
that, in particular, implies
\[
\mathcal{U}_f = \theta_i u_{g_i}.
\]

We shall now consider graded modules over the ring of generalized
polynomials. Firstly we consider in \( T = T(\mathcal{X}) \) the subset
\[
T_s = \left\{ x_1^{i_1} \cdots x_m^{i_m} \mid \sum_{k=1}^{m} i_k = s, \ (i_1, \ldots, i_m) \in \mathbb{N}_0^m \right\},
\]
s \( \in \mathbb{Z} \) and \( T_s \) for all \( s < 0 \).

**Definition 8.** Let \( D \) be a ring of generalized polynomials over a field
\( K \) in the indeterminates \( X = \{ x_1, \ldots, x_m \} \) We suppose the ranking of
\( T = T(\mathcal{X}) \) to be orderly, and
\[
D_s = \left\{ \sum_{\theta \in T_s} a_{\theta} \theta \mid a_{\theta} \in \mathcal{F} \text{ and almost all coefficients are equal to 0} \right\}
\]
The ring $D$ will be called **graded** if

$$D = \bigoplus_{s \in \mathbb{N}_0} D_s$$

and

$$D_s D_r \subseteq D_{s + r}$$

for all $s, r \in \mathbb{N}_0$.

The rings (examples 1, 4), with standard ranking (see example 5), are graded. The ring of differential operators over field $\mathcal{F}$ (example 6) isn’t graded, if there are non-constant elements in field $\mathcal{F}$.

**Definition 9.** Let $D$ be a graded ring of generalized polynomials over a field $\mathcal{F}$. A $D$-module $M$ will be called **graded**, if for any $s \in \mathbb{N}_0$ a $\mathcal{F}$-subspace $M_s$ of $M$ is defined such $M = \bigoplus_{s \in \mathbb{N}_0} M_s$ and

$$D_s M_r \subseteq M_{s + r}$$

for all $s, r \in \mathbb{N}_0$. The elements of $M_s$ will be called the homogeneous elements of degree $s$.

**Definition 10.** Let $D M$ be a finitely generated module over a ring of generalized polynomials and $M = \bigoplus_{s \in \mathbb{N}_0} M_s$ be a grading of $M$. The function $\phi^g_M$, whose value at any $s \in \mathbb{N}_0$ is equal to $\dim_\mathcal{F} M_s$ will be called the characteristic function of the graded module $M$.

**Theorem 2.** (see [5], theorem 4.3.20.) Let $D$ be a graded ring of generalized polynomials over a field in the indeterminates $X = \{x_1, \ldots, x_m\}$, $D M$ be a graded module and $\{m_1, \ldots, m_n\}$ be a finite set of its generators such that $m_i \in M_{(\alpha)}$. Then there exist sets $E_i \subset \mathbb{N}_0^m$ ($i = 1, \ldots, n$) such that for all large enough $s$ the characteristic function of $M$ is equal to

$$\phi^g_M(s) = \Delta_1 \sum_{i=1}^n \omega_{E_i}(s - \alpha_i),$$

(2)

where $\omega_{E_i}(s)$ is the Kolchin dimension polynomial of the matrix $E$ (see theorem 2, equation [7]).

As follows from the proof, the sets $E_i$ correspond to leaders of a homogeneous Gröbner basis of relations between generators (syzygy module). It is easy to see that for sufficiently large $s$ the function $\phi^g_M(s)$ is polynomial.

We denote it by $\omega_M(s)$ and call the characteristic polynomial graded finitely generated module $D M$. Its degree $d(M) = \deg(\omega_M)$ is called (generalized) type of module $M$, the difference $(m - 1 - d)$ - (generalized) codimension, and the standard leading (nonzero) coefficient $\tau_d(M)$ - (generalized) type dimension.

Graded modules over a ring of generalized polynomials have properties similar to properties differential modules: $d(M) < m$ and $a_{m-1}(\omega_M) = \text{rk}_D M$. 

Let $F$ be a free $D$-module with generators $f_1, \ldots, f_n$. Each element of $f \in F$ is represented as $f = \sum_{1 \leq j \leq n} \theta_j f_j$, where $\theta_j \in D$. Denote by $\text{ord}_f f = \text{ord} \theta_1$ and $\text{ord} f = \max_{1 \leq i \leq n}(\text{ord} \theta_i)$.

Consider the following grading on $F$: $F_s = \sum_{i=1}^n T_s f_i$. Let $H$ be the submodule of the module $F$ generated by elements of $\Sigma \subset H$, and $\text{ord}_f h \leq e_j$ for all $j = 1, \ldots, n$, $h \in \Sigma$. The induced grading arises on the module $H$: $H_s = H \cap F_s$. The factor module $F/H$ can also be regarded as graded: $(F/H)_s = (F_s/H_s)$, and $\omega_{F/H}(s) = \omega_F(s) - \omega_H(s) = \sum_{i=1}^n (s^m - 1) - \omega_M(s)$. Sometimes this polynomial called the characteristic polynomial of a system of generalized polynomial equations $\Sigma$ (or a system of $D$-equations) and denoted by $\omega_{\Sigma}$.

As follows from the theorem 2 (put $a_i = 0$ $(i = 1, \ldots, n)$, $M = F/H$), characteristic polynomial of a system of generalized polynomial equations can be calculated as in the differential case: (Theorem 4.3.5 [5]):

$$\omega_{\Sigma}(s) = \sum_{j=1}^n \Delta_j \omega_{E_j}(s), \quad (3)$$

where $E_j \subset \mathbb{N}_0^n$.

We are interested in following

**Question 1.** How to estimate the typical dimension of $\Sigma$ in known orders $e_1, \ldots, e_n$?

Firstly this question was asked in differential algebra by J. Ritt for ordinary differential systems. Later E. Kolchin decided this problem in a codimension for nonlinear systems. His bound (see [4], p. 199) is as follows: typical differential dimension $a_{m-1}$ of the system $\Sigma$ does not exceed $e_1 + \cdots + e_n$.

In codimension 2, such a result is known (see 5.6.7, [5]):

Let $n = 1$, then $a_{m-2}(\omega_{\Sigma}) \leq e_1^2$.

Both of these results are also true for systems of homogeneous generalized polynomial equations.

3. **Basic results.**

So, for systems of generalized homogeneous polynomial equations in codimensions 1 and 2, the classical Bézout theorem holds. If the codimension is greater than 2, in the general case this is not true. Consider an example.

**Example 6.** Let $k \in \mathbb{N}$, $\mathcal{F} = \mathbb{C}(x_1, x_2, x_3)$, $n = 1$, $D = \mathcal{F}\{\partial_1, \partial_2, \partial_3, y_1\}$ —

ring of differential operators over ring of polynomials in one variable $y_1$, (see the example [4] over the field $\mathcal{F}$, $m = 4$), $\Sigma = \{\partial^k_1 f_1, (\partial^k_2 + x_1 \partial^k_3)f_1\}$. 

Let’s we have the orderly rank \(\partial_1 > \partial_2 > \partial_3 > y_1\). One can find a homogeneous Gröbner basis of the ideal \([\Sigma]\). It consists of elements 
\[ G = \{ \partial_1^k f_1, (\partial_2^k + x_1 \partial_2^k) f_1, \partial_1^{-1} \partial_2^k y_1 f_1, \partial_1^{k-2} \partial_3^k y_2 f_1, ..., \partial_1^{k-i} \partial_3^k y_1^i f_1, ..., \partial_3^k y_1^i f_1 \}. \]

From here according to the equation \([3]\), \(\omega_{[\Sigma]} = \Delta_1 \omega_E\), where
\[
E = \begin{pmatrix}
k & 0 & 0 & 0 \\
0 & k & 0 & 0 \\
k-1 & 0 & k & 1 \\
\ldots & \ldots & \ldots & \ldots \\
k-i & 0 & ik & i \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & k^2 & k
\end{pmatrix}.
\]

One of the main methods for calculating the dimension polynomial of a matrix is the using of the formula (see \([5]\), Theorem 2.2.10):
\[
\omega_E(s) = \omega_{E_{nc}} + \omega_H(s - \text{ord}(e)),
\]
where \(e \in \mathbb{N}_0^m\), \(H\) is the matrix obtained by subtracting the vector \(e\) from each row of \(E\) (negative numbers are replaced by zeros).

Apply the formula \([4]\) \(e = (0, k, 0, 0)\) \(k\) times, we obtain \(\omega_E = k \omega_{E_1}\), here
\[
E_1 = \begin{pmatrix}
k & 0 & 0 \\
k-1 & k & 1 \\
\ldots & \ldots & \ldots \\
k-i & ik & i \\
\ldots & \ldots & \ldots \\
0 & k^2 & k
\end{pmatrix}.
\]

By Theorem 2.2.17 (see \([5]\)) we have:
\(\Delta_1(\omega_{E_1}) = \omega_{E_2} + \omega_{E_3}\), where \(E_2, E_3\) are the matrices, obtained by deletion, respectively, second and third columns of the matrix \(E_1\). Applying corollary 2.3.21 (see \([5]\)), we get \(\omega_{E_2} = 1+2+\cdots+k = k(k+1)/2\) and \(\omega_{E_3} = k(1+2+\cdots+k) = k^2(k+1)/2\) whence \(\omega_E = k^2(k+1)/2\). If the classical Bézout theorem holds for the system \(\Sigma\), we would have to have \(\omega_{[\Sigma]} = k^2(k+1)/2 \leq k^3\) (the system has codimension 3, while it has 2 homogeneous generators), which is wrong.

**Theorem 3.** Let \(D\) be a graded ring of generalized polynomials over the field \(\mathcal{F}\) in indeterminates \(X = \{x_1, \ldots, x_m\}\), \(F = \bigoplus_{i=1}^{m} D\) - free graded \(D\)-module with generators \(f_1, \ldots, f_n\), \(\Sigma \subset F\) is a system of homogeneous \(D\)-equations. Let be \(\text{ord}_{f_i} h \leq e_j\) for any \(h \in \Sigma\).

Then the following statements are true:
- if the codimension of the system is 0, then typical differential dimension does not exceed \(n\);
- if the codimension of the system \(\Sigma\) is 1, then \(\tau_d(\Sigma) \leq e_1 + \text{ dots } + e_n\);
- if the codimension of \(d(\Sigma)\) is 2, then \(\tau_d(\Sigma) \leq (e_1 + \cdots + e_n) \max_{i=1}^{n} e_i + \prod_{i<j} e_i e_j \leq (e_1 + \ldots e_n)^2\).
This bound is being achieved, see an example from [7].

First we prove the lemma.

**Lemma 1.** Suppose that under the conditions of the theorem 3 the generalized type of the system \( \Sigma \) is greater than 1. Then

\[
\omega_{\Sigma}(s) = \sum_{i=1}^{n} \left( \binom{s + m - 1}{m - 1} - \binom{s + m - 1 - \tilde{e}_i}{m - 1} \right) - w(s),
\]

(5)

where \( w(s + e) \in W, e = \max_{i=1}^{n}(e_i), \tilde{e}_i \leq e_i. \)

**Proof.** We denote by \( H \) the submodule of the \( D \)-module \( F \) generated by the system \( \Sigma \). \( D \) is an Ore ring and, since the codimension of \( \Sigma \) is greater than 1, \( \text{rk}_D F/H = 0 \), whence \( \text{rk}_D H = n \). We choose \( n \) \( D \)-independent equations from \( \Sigma \), and let \( D \)-factor module \( F \) by these equations (we denote them by \( \Sigma_1 \)). We have the exact sequence of graded \( D \)-modules:

\[
0 \to N \to M \to F/H \to 0.
\]

(6)

We can assume that \( N \) is a graded submodule of the module \( M \), generated by the equations \( \Sigma \setminus \Sigma_1 \), and let \( \alpha^i \) - the degrees of these generators. Gradings, associated with the choice of homogeneous generators \( (N_s = D_{s-\alpha^i}g_i) \) and grading, induced by \( M \) \( (N_s = M_s \cap N) \), coincide.

Let \( e = \max(e_1, \ldots, e_n) \). From the theorem 2 it follows that \( \omega_N(s + e) \in W \). Indeed, the generators \( g_i \) of the module \( N \) have degrees \( \alpha^i \) not greater than \( e_i \) and from the formula (2) we get that \( \omega_N(s + e) = \sum_{i=1}^{k} \omega_i(s + e - \alpha^i) \), where \( \omega_i \in W \) (here \( k = \text{Card}(\Sigma) - n \)). Because \( \alpha^i \leq e_i \), keeping in mind closed \( W \) relatively positive shift and summation we get that \( \omega_N(s + e) \in W \).

To calculate \( \omega_M = \omega_{\Sigma_1} \), we use \( D \)-independence of the equations \( \Sigma_1 \). Proof of Lemma 5.8.2 ([5]) can also be done for a system of generalized polynomial equations, therefore we have: \( J(\Sigma_1) \neq \infty \), where \( J \) is the Jacobi number of matrix \( (\text{ord}_f h_{ij})_{i,j=1}^{n} \), \( h_i \in \Sigma_1 \). Choosing the final diagonal sum of the matrix and renumbering the equations \( \Sigma_1 \), since \( \text{ord}_f h_{ij} \neq \infty \), we get

\[
\omega_{\Sigma_1}(s) = \omega_F(s) - \sum_{i=1}^{n} \left( s + m - 1 - \text{ord}_i h_i \right) \left( \binom{s + m - 1}{m - 1} - \binom{s + m - 1 - \tilde{e}_i}{m - 1} \right),
\]

where \( \tilde{e}_i = \text{ord}_f h_i \leq e_i \).

To prove the lemma it remains to use the equality \( \omega_M = \omega_N + \omega_{F/H} \), obtained from the sequence (6).

□
We return to the proof of the theorem.

Proof. The case \(d(\Sigma) = m - 1\) follows from Theorems 2

Let \(d(\Sigma) = m - 2\). It follows from the lemma that \(\sum_{i=1}^{n} ((s+1)-(s+1-\hat{e}_i)) - \Delta_{1}^{n-2}\omega_\Sigma(s) \in W'\) (since \(W\) is closed relative to the operation \(\Delta_1\), see the formula (11)). From here \(\sum_{i=1}^{n} \hat{e}_i - \tau_d(\Sigma) \in W\). Because minimizing coefficients of a polynomial from \(W\) are non-negative, we immediately get that \(\tau_d(\Sigma) \leq \sum_{i=1}^{n} \hat{e}_i \leq \sum_{i=1}^{n} e_i\).

Let \(d(\Sigma) = m - 3\). As above, we use the operator \(\Delta_{1}^{m-3}\) to expression (5). We get:

\[
\Delta_{1}^{m-3} \left( \sum_{i=1}^{n} \left( \frac{s + m - 1}{m - 1} \right) - \left( \frac{s + m - 1 - \hat{e}_i}{m - 1} \right) \right) - \tau_d(\Sigma) = w'(s),
\]

whence

\[
\left( \sum_{i=1}^{n} \left( \frac{s + 2}{2} \right) - \left( \frac{s + 2 - \hat{e}_i}{2} \right) \right) - \tau_d(\Sigma) = w'(s), \ w'(s + e) \in W
\]

and \(\sum_{i=1}^{n} (\hat{e}_i(s + 1) - \left( \frac{\hat{e}_i}{2} \right)) - \tau_d(\Sigma) = w'(s)\). Let the minimizing coefficients of the polynomial \(w'(s + e)\) be equal to \((b_1, b_0)\). Then \(w'(s + e) = b_1(s + 1) - \left( \frac{b_1}{2} \right) + b_2\), and \(b_1 \geq 0, b_2 \geq 0\). We have:

\[
\tau_d(\Sigma) = (\sum_{i=1}^{n} \hat{e}_i - b_1(s + 1) - \sum_{i=1}^{n} (\hat{e}_i - \left( \frac{\hat{e}_i}{2} \right) )) + eb_1 + \left( \frac{b_1}{2} \right) - b_0.
\]

Equating the coefficient in \(s\) to the right side of the equality to zero, we obtain \(b_1 = \sum_{i=1}^{n} \hat{e}_i\), whence \(\tau_d(\Sigma) \leq (\sum_{i=1}^{n} \hat{e}_i) e - \sum_{i=1}^{n} (\hat{e}_i/2) + (\sum_{i=1}^{n} \hat{e}_i) = \prod_{i<j} \hat{e}_i \hat{e}_j + (\sum_{i=1}^{n} \hat{e}_i) e \leq (\sum_{i=1}^{n} e_i) \max_{i=1}^{n} e_i + \prod_{i,j=1,ij}^{n} e_i e_j\)

Further we consider homogeneous ideals in the ring of generalized polynomials, i.e. case \(n = 1\). Let an ideal be generated by elements of order no higher than \(e\). If the generalized type of an ideal is 2, then from the theorem follows that its typical dimension does not exceed \(e^2\), i.e. the classical Bézout theorem holds.

Theorem 4. Let \(D\) be a graded ring of generalized polynomials over the field \(F\) in determinates \(X = \{x_1, \ldots, x_m\}\), \(\Sigma \subset D\) is a system of homogeneous \(D\)-equations. Let be \(\text{ord} \ h \leq e\) for any \(h \in \Sigma\).

Then the following bounds are true:

if the codimension of \(\Sigma\) is 3, then \(\tau_d(\Sigma) \leq e^2(e + 1)^2/2\) (according to the example this estimate is achieved);

if the codimension of \(\Sigma\) is 4, then \(\tau_d(\Sigma) \leq e^2(e + 1)^2(3e^4 + 6e^3 + 11e^2 + 8e + 8)/24\);

if the codimension of \(\Sigma\) is 5, then \(\tau_d(\Sigma) \leq e^2(e + 1)^2(288 + 480e + 952e^2 + 1264e^3 + 1592e^4 + 1648e^5 + 1529e^6 + 1174e^7 + 775e^8 + 420e^9 + 183e^{10} + 54e^1 + 9e^2)(e + 1)^2/1152\)
in any codimension $\tau > 0$ the generalized typical dimension $\tau_d(\Sigma)$ does not exceed $O(e^{2^{\tau-1}})$.

Proof. is based on the lemma and the fact that minimizing coefficients are multivalued from the set $W$ are non-negative. For $\Sigma'$, we choose the element $\Sigma$ in maximal order, let it be $e$. Then, in the equation (5) $n = 1, \tilde{e}_1 = e$.

Consider the case of codimension 3. We apply the operator $\Delta^m_{\tau}$ to both sides of the equality (5).

$$\tau_d(\Sigma) = \left( \left( \frac{s+3}{3} \right) - \left( \frac{s+3-e}{3} \right) \right) - w'(s), \ w'(s+e) \in W$$

Let the sequence of minimizing coefficients of the polynomial $w'$ is $(b_2, b_1, b_0)$. According to the definition we can explicitly express the standard coefficients $w'$ through the numbers $b_2, b_1, b_0$ and find the coefficients of 'shifted' the polynomial $w'(s+e)$. Equating the coefficients at $s^2$, $s$ on the right side of the equation to 0, we get: $b_2 = e, b_1 = e^2$ and

$$\tau_d(\Sigma) = \left( \left( \frac{s+3}{3} \right) - \left( \frac{s+3-e}{3} \right) \right) - \left( \left( \frac{s+3-e}{3} \right) - \left( \frac{s+3-2e}{3} \right) \right) - \left( \left( \frac{s+2-2e}{2} \right) - \left( \frac{s+2-2e-b_1}{2} \right) \right) - b_0.$$  

Substituting $s = -1$, we get $\tau_d(\Sigma) \leq e^2(e+1)^2/2$.

Bounds in any codimension are calculated in the same way. Each time we will receive a polynomial in $e$.

If the precise coefficients of this polynomial are not important, but only its degree in $e$, it is claimed to be $2^{\tau-1}$. Indeed, let $d$ be the generalized dimension of the system $\Sigma$, i.e. codimension $\tau$ of the polynomial $\omega_\Sigma$ is equal to $\tau = m - 1 - d$. Apply to (5) operator $\Delta^d_{\tau}$ (while $\Delta^d_{\tau} \omega_\Sigma$ is a polynomial of degree zero, i.e. constant $= \tau_d(\Sigma)$). Comparing the degrees, we get that the degree $w' = \Delta^d_{\tau} w$ is less than $\tau$. Let the minimizing coefficients of the polynomial $w'$ equal to $(0, \ldots, 0, b_{\tau-1}, \ldots, b_0)$. Replace in the resulting equation $s$ variable on $e$ and we have the following relation:

$$\tau_d(\Sigma) = \left( \frac{s+\tau+e}{\tau} \right) - \left( \frac{s+\tau}{\tau} \right) - w'(s).$$

We use the definition and get

$$\tau_d(\Sigma) = \left( \frac{s+\tau+e}{\tau} \right) - \left( \frac{s+\tau}{\tau} \right) - \sum_{k=\tau}^{\tau+1} \left( \frac{s+k-\sum_{j=1}^k b_j}{k} \right) - \left( \frac{s+k-\sum_{j=\tau}^{k-1} b_j}{k} \right).$$

(7)
Denote by \( c_i = \sum_{j=1}^{\tau-1} b_j \) and rewrite equation (7) in this form:

\[
\tau_d(\Sigma) = \left( s + \tau + \frac{e}{\tau} \right) - \left( s + \frac{\tau}{\tau} \right) - \sum_{k=\tau}^{1} \left( \frac{s + k - c_k}{k} - \frac{s + k - c_{k-1}}{k} \right),
\]

Using identity

\[
\left( s + k - 1 - a \right) = \left( s + k - a \right) - \left( s + k - 1 - a \right),
\]

transform the equation (8) to the form:

\[
\tau_d(\Sigma) = \left( s + \tau + \frac{e}{\tau} \right) - 2\left( s + \frac{\tau}{\tau} \right) + \sum_{k=\tau}^{2} \left( s + k - 1 - c_{k-1} \right) + (s+1-c_0).
\]

Take \( \Delta^{-1}_i \) from both sides of the equality (9). Will have: \( 0 = (s+1+e) - 2(s+1) + (s-c_{\tau-1}) + 1, \) \( P \times C'P \times C'P \times P \) \( c_{\tau-1} = e. \) By induction on \( i, \) we prove that for \( 1 \leq i < \tau - 1 \) it holds:

\[
c_i = O(e^{2^{(\tau-1)-i}}).
\]

Let \( c_j = O(e^{2^{(\tau-1)-j}}) \) for all \( j : i \leq j < \tau - 1. \)

Take \( \Delta^{-1}_i \) from both sides of the equality (9) and get:

\[
0 = \left( s + \tau - i + 1 + \frac{e}{\tau} \right) - 2\left( s + \frac{\tau - i + 1}{\tau - i + 1} \right) + \sum_{k=\tau}^{i-1} \left( s + k - i - c_{k-1} \right)
\]

Substituting \(-1\) instead of \( s, \) we get

\[
0 = \left( \frac{\tau - i + e}{\tau - i + 1} \right) - 2\left( \frac{\tau - i}{\tau - i + 1} \right) + \sum_{k=\tau}^{i+1} \left( \frac{k - i - 1 - c_{k-1}}{k - i + 1} \right) - c_{i-1}
\]

In the last sum we make the change \( j = k - i + 1 \) and get

\[
0 = \left( \frac{\tau - i + e}{\tau - i + 1} \right) + \sum_{j=2}^{i+1} \left( \frac{j - c_{i+j-2}}{j} \right) - c_{i-1}.
\]

Now we have a formula expressing \( c_{i-1} \) through \( c_i, ..., c_{\tau-1}: \)

\[
c_{i-1} = \sum_{j=2}^{\tau-i+1} O \left( \left( \frac{c_{i+j-2}}{j} \right) \right) = \sum_{j=2}^{\tau-i+1} O(c_{i+j-2}) = \sum_{j=2}^{\tau-i+1} O(e^{2^{(\tau-i)-j+1}}) =
\]

\[
O(e^{2^{\tau-1-i}}) + \sum_{j=3}^{\tau-i+1} O(e^{2^{\tau-i-j+1}}) = O(e^{2^{\tau-i}}) + \sum_{j=3}^{\tau-i+1} O(e^{2^{\tau-1-2^{\tau-i-j+1}}}) =
\]

\[
O(e^{2^{\tau-i}}) + \sum_{j=3}^{\tau-i} O(e^{2^{\tau-i}}) = O(e^{2^{\tau-i}})
\]
(we used the inductive assumption and the fact that \( j \leq 2^{j-1} \) for all \( j \geq 2 \)).

Substituting \( s = -1 \) into the equation (9), we obtain \( \tau_d(\Sigma) = O(c_1^2) - c_0 \leq O(2^{r-1}) \).

\( \square \)

It is not known whether the resulting double exponential typical dimension bound of graded ideal in a ring of generalized polynomials is being achieved. Note that it was proved in (8) that for degrees of elements in the Gröbner basis of the polynomial ideal double exponential bound is achieved from below.

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Moscow State University, Department of Mechanics and Mathematics, Leninskiy Gory, Moscow, Russia, 119991.

E-mail address: kondratieva@sumail.ru