Flavour physics and Lattice QCD: averages of lattice inputs for the Unitarity Triangle Analysis

V. Lubicz(1)(* ) and C. Tarantino(1)
(1) Dip. di Fisica, Università di Roma Tre, Via della Vasca Navale 84, I-00146 Rome, Italy.

Summary. — We review recent results of Lattice QCD calculations relevant for flavour physics. We discuss in particular the hadronic parameters entering the amplitudes of $K^0 - \bar{K}^0$, $D^0 - \bar{D}^0$ and $B^0 - \bar{B}^0$ mixing, the $B$- and $D$-meson decay constants and the form factors controlling $B$-meson semileptonic decays. On the basis of these lattice results, which are extensively collected in the paper, we also derive our averages of the relevant hadronic parameters.

PACS 12.38.Gc – Lattice QCD calculations.

The aim of this talk is to illustrate the status of the art of Lattice QCD calculations relevant for flavour physics and to provide averages of the hadronic quantities which are useful for phenomenological analysis of flavour physics, within or beyond the Standard Model. These averages will be used in particular by the UTfit collaboration in the unitarity triangle analysis (UTA) [1]-[4]. We believe, however, that the whole discussion given in the present contribution can be of more general interest. Even though the process of averaging lattice results unavoidably involves some degree of subjectivity, for all the quantities discussed in this work we also provide extensive collections of lattice results, as well as the references to the original papers. In addition, we try to enrich the discussion with some technical considerations about the accuracy of the various lattice calculations, which can be useful for the non-expert of the field in order to have a deeper view and to draw their conclusions.

The status of the art of Lattice QCD calculations is represented by unquenched simulations performed with either $N_f = 2$ or $N_f = 2 + 1$ dynamical quarks and values of the light (i.e. up/down) quark masses well below $m_s/2$, where $m_s$ is the physical strange quark mass. Both these features are particularly important to keep the lattice systematic uncertainties under control. On the one hand, it is well known that the quenched approximation introduces a systematic error which, besides depending on the specific quantity under investigation, is also difficult to evaluate (if not by performing the corresponding unquenched calculation). On the other hand, lattice simulations performed with sufficiently light values of the up and down quark masses ($m_{ud} < m_s/2$) allow to rely on the

(*) Speaker
predictions of chiral perturbation theory when performing the chiral extrapolations of the lattice results to the physical masses.

If the possibility of performing unquenched calculations at light quark masses represents a major advance of Lattice QCD in the last few years, it has also to be noted that these calculations have not always reached yet the same accuracy in controlling other sources of systematic effects as done in the more recent (and less expensive) quenched calculations. For instance, some of the unquenched results have not yet been extrapolated to the continuum limit, and non-perturbative renormalization techniques have not always been implemented. Moreover, while most of the quenched results have been derived by many lattice collaborations, often using different approaches, in several cases only few (if any) unquenched results are already available for a given quantity. For instance, the unquenched results for the kaon B-parameter have been only obtained so far at fixed (and rather large) values of the lattice spacing, whereas the matrix elements of the full basis of four-fermion operator relevant for \( K^0 - \bar{K}^0 \) mixing in generic New Physics models have been only obtained so far within the quenched approximation.

In what follows we illustrate in some detail the results for various lattice parameters and present our averages. We will not be able to discuss in the present contribution the whole set of hadronic parameters available from the lattice and which are relevant for flavour physics. In particular, we are not going to review the accurate lattice results for the quark masses, for the ratio of decay constants \( f_K/f_\pi \) and for the vector form factor controlling the semileptonic \( K_\ell 3 \) decays. For the latter two quantities, which allow the important determination of \( V_{us} \) (the Cabibbo angle), we refer to the review by A. Juttner at the Lattice’07 conference [5].

1. – The kaon B-parameter \( B_K \)

A collection of quenched and unquenched lattice results for \( B_K^{\text{MS}}(2 \text{ GeV}) \) [6]-[12] is shown in fig.1 as a function of the square of the lattice spacing \( a \). Among the quenched results we have selected all determinations which are \( O(a) \)-improved, i.e. that are affected by discretization errors of \( O(a^2) \) or smaller, and have been obtained at several values of the lattice spacing, thus allowing an extrapolation to the physical continuum limit. An additional feature of all results shown in fig.1 is that they have been obtained with lattice discretizations of the fermionic action such that the \( \Delta S = 2 \) four-fermion operator defining \( B_K \) is subject to multiplicative renormalization only. This allows to significantly increase the accuracy of the determination of the kaon B-parameter.

As illustrated in fig.1, the unquenched results for \( B_K \), represented by filled points in the plot, have been all obtained so far at a fixed value of the lattice spacing. Thus, at variance with the quenched determinations, the unquenched results cannot be extrapolated to the continuum limit, and the size of discretization errors affecting them cannot be easily estimated. On general grounds, we note that there is no reason to expect discretization errors in the unquenched case to have smaller size than those affecting the quenched results.

The comparison between quenched and unquenched determinations of \( B_K \) obtained with the same lattice action and at similar values of the lattice spacing does not show any evidence of the quenching effect within the present uncertainties. Moreover, the comparison between \( N_f = 0 \) and \( N_f = 2 \) results obtained with the same lattice action but at different values of the lattice spacing, see for instance the case of the DBW2-DWF action (empty and filled circles in the plot), suggests that quenched and unquenched results could be affected by similar discretization errors. This observation has motivated
Fig. 1. – Lattice QCD results for $B_{K}^{\overline{MS}}(2 \text{ GeV})$ [6]-[12] plotted as a function of the lattice spacing square. All results selected in the plot are O(α)-improved and required only multiplicative renormalization of the relevant $ΔS = 2$ four-fermion operator. Filled points are unquenched results obtained with either $N_f = 2$ or $N_f = 2+1$ dynamical quarks. Empty points are quenched results. Our average is shown by the horizontal band.

the RBC collaboration to consider their $N_f = 0$ and $N_f = 2$ results together in order to perform a combined extrapolation to the continuum limit [7]. By taking into account the scale dependence suggested by the quenched results, we quote as final average for $B_K$ the value

$B_{K}^{\overline{MS}}(2 \text{ GeV}) = 0.55 ± 0.05$ ,

which corresponds to the renormalization group invariant parameter

$\hat{B}_K = 0.75 ± 0.07$ .

Clearly, a better estimate of discretization errors in the unquenched case will be obtained once the results of unquenched simulations performed at different values of the lattice spacing, which are currently in progress, will be available.

Our average in eq. (1) is shown by the horizontal band in fig.1. It can be compared with the average $B_{K}^{\overline{MS}}(2 \text{ GeV}) = 0.58 ± 0.03 ± 0.06$ quoted by Dawson at Lattice’05 [13], where the central value and the first error corresponded to the average of the quenched results in the continuum limit, while the second error was an estimate of the quenching effect. The recent unquenched results for $B_K$ allow to remove the latter uncertainty and slightly decrease the previous central value.

2. – The $B$-mesons decay constants $f_{B_s}$ and $f_B$

The unquenched lattice QCD results for the $B$-mesons decay constants $f_{B_s}$ and $f_B$ and for their ratio $f_{B_s}/f_B$ [14]-[20] are shown in fig.2. They have been obtained by treating the $b$ quark on the lattice with two different approaches, either FNAL [21] or NRQCD.
Fig. 2. – Lattice QCD results for the $B$-mesons decay constants $f_{B_s}$ (top left), $f_B$ (top right) and the ratio $f_{B_s}/f_B$ (bottom) obtained with $N_f = 2$ and $N_f = 2 + 1$ unquenched simulations [14]-[20]. A star in the legends labels preliminary results. Our averages are shown by the vertical bands.

Our average for the $B_s$ decay constant,

$\begin{equation}
    f_{B_s} = 245 \pm 25 \text{ MeV},
\end{equation}$

is based on the results shown in the upper plot of fig.2. The lattice determination of $f_B$ is more delicate, because its value is enhanced by chiral logs effects relevant at low quark masses. In order to properly account for these effects, simulations at light values of the quark mass (typically $m_{ud} < m_s/2$) are required. For this reason, we derive our average for $f_B$ and for the ratio $f_{B_s}/f_B$ by taking into account only the results HPQCD’05 [19] and FNAL/MILC’07 [20] (see fig.2), which use the MILC gauge field configurations generated at light quark masses as low as $m_s/8$. In this way we obtain

$\begin{equation}
    f_{B_s}/f_B = 1.21 \pm 0.04.
\end{equation}$

Finally, we derive directly from the previous averages of $f_{B_s}$ and $f_{B_s}/f_B$ the estimate of
the $B$-meson decay constant

$$f_B = 200 \pm 20 \text{ MeV}.$$ (5)

3. The $D$-mesons decay constants $f_{Ds}$ and $f_D$

The CKM matrix elements $V_{cs}$ and $V_{cd}$, which control the rate of leptonic $D_s^+ \to \ell^+ \nu_\ell$ and $D^+ \to \ell^+ \nu_\ell$ decays, are well constrained by the unitarity of the CKM matrix, which predicts $|V_{cs}| \simeq 1 - \lambda^2/2$ and $|V_{cd}| \simeq \lambda$ (up to $O(\lambda^4)$ corrections), where $\lambda$ is the Wolfenstein parameter. By using these constraints, the measurements of the leptonic decay rates then allow a determination of the $D$-mesons decay constants, $f_{Ds}$ and $f_D$.

The current experimental averages, as evaluated in ref. [22], are

$$f_{Ds}^{\text{EXP.}} = 273 \pm 10 \text{ MeV} , \quad f_D^{\text{EXP.}} = 205.8 \pm 8.9 \text{ MeV} , \quad (f_{Ds}/f_D)^{\text{EXP.}} = 1.33 \pm 0.07 .$$ (6)

The unquenched lattice QCD results for these constants [14, 16, 20, 23, 24] are shown in fig. 3, where they are also compared with the experimental averages given in eq. (6).

Among the various lattice determinations, which show overall a very good consistency among each other, it deserves to be noted the result of the HPQCD collaboration [24], which quotes $f_{Ds}$ and $f_D$ with the impressive accuracy of about 1% and 2% respectively. Remarkably, while the result for $f_D$ of ref. [24] is in good agreement with the experimental average, the prediction for $f_{Ds}$ differs from the experimental value by approximately 3 standard deviations. This discrepancy has been interpreted in [25] as an evidence for New Physics effects in leptonic $D_s$ decay.

A critical review of the lattice study of ref. [24] is not possible yet, since important details on the analysis are not given in the paper, being postponed to a longer publi-
Fig. 4. – Lattice QCD results for $B^0_{s d}(m_b)$ and $B^0_{d s}(m_b)$ (left) and for their ratio $B^0_{d s}/B^0_{s d}$ (right) [17], [28]-[32]. A star in the legend labels preliminary results. Our averages are shown by the vertical bands.

cation [24]. Of particular relevance, in this respect, are the details of the Bayesian fits implemented to perform the combined continuum and chiral extrapolation, as well as the knowledge of the precise functional form assumed for this extrapolation. The large number of free parameters introduced in the fit (specifically, 45 parameters with 28 data points) is the source of some concern, particularly in view of the remarkable accuracy claimed on the final results. We also mention that the use of the so called “fourth root trick” in dynamical simulations of staggered quarks, which the results of ref. [24] are based on, is controversial. This approach has been the subject of intensive theoretical and numerical investigation in the last few years. Despite the valuable progresses achieved in our understanding of the rooting procedure, the dispute on its validity has not been, and will likely never be, completely resolved (for extensive discussions, see e.g. the recent reviews [26, 27] at the Lattice conferences).

Waiting for a confirmation of the precise predictions of ref. [24] by other lattice calculations, in deriving averages for the $D$-mesons decay constants we choose to quote uncertainties which are larger than those given in ref. [24], and comparable in size to those presented by the other results. Thus, we quote as our final averages

$$f_{D_s} = 250 \pm 15 \text{ MeV} \quad , \quad f_D = 212 \pm 14 \text{ MeV} \quad , \quad f_{D_s}/f_D = 1.18 \pm 0.03 ,$$  

which are the values also shown by the vertical bands in fig.3.

4. – Matrix elements for $B^0_{d/s} - \bar{B}^0_{d/s}$ mixing

A collection of quenched and unquenched lattice results for the $B_d$ and $B_s$ bag parameters and for their ratio [17], [28]-[32] is shown in fig.4. We firstly observe that the dependence on the light quark mass, that should allow to distinguish between $B_d$ and $B_s$, is practically invisible. Moreover, the unquenched results, obtained with $N_f = 2$ and $N_f = 2 + 1$ dynamical quarks, tends to be slightly lower than the quenched determinations, though still well compatible within the errors. On the basis of the unquenched
results we quote the average

\[ B_{B_d}^{\text{MS}}(m_b) = B_{B_s}^{\text{MS}}(m_b) = 0.80 \pm 0.08 , \]

which is illustrated by the vertical band in fig.4 (left) and corresponds to the renormalization group invariant parameters

\[ \hat{B}_{B_d} = \hat{B}_{B_s} = 1.22 \pm 0.12 . \]

For the ratio $\hat{B}_{B_s}/\hat{B}_{B_d}$, where statistical and systematic uncertainties partially cancel (see fig.4 right), we quote

\[ \hat{B}_{B_s}/\hat{B}_{B_d} = 1.00 \pm 0.03 . \]

Combining the above results with the averages quoted for the $B$-mesons decay constants, eqs. (3)-(5), we then obtain

\[ f_{B_s} \sqrt{\hat{B}_{B_s}} = 270 \pm 30 \text{ MeV} , \quad f_{B_d} \sqrt{\hat{B}_{B_d}} = 225 \pm 25 \text{ MeV} , \]

\[ \xi = \frac{f_{B_s} \sqrt{\hat{B}_{B_s}}}{f_{B_d} \sqrt{\hat{B}_{B_d}}} = 1.21 \pm 0.04 . \]

It is important to compare the average of $f_{B_s} \sqrt{\hat{B}_{B_s}}$ of eq. (11), obtained from $f_{B_s}$ and $B_{B_s}$ separately, with the direct calculation of $f_{B_s} \sqrt{\hat{B}_{B_s}}$ from the whole $B_s$-mixing matrix element. For the latter, the unquenched results obtained in JLQCD’03 [17] and HPQCD’06 [31] are $f_{B_s} \sqrt{\hat{B}_{B_s}} = 245 \pm 10^{+19}_{-17}$ MeV and $f_{B_s} \sqrt{\hat{B}_{B_s}} = 281 \pm 21$ MeV respectively, well consistent with the average of eq. (11).

5. – $V_{cb}$ exclusive

The lattice results for the zero momentum transfer form factors of $B \to D^* l \nu$ and $B \to D l \nu$ semileptonic decays [33]-[38], denoted as $F(1)$ and $G(1)$ respectively, are shown in fig.5. In the infinite mass limit of both the charm and bottom quarks these form factors are normalized to unity, up to small radiative corrections. Fig.5 shows that the percent level accuracy required to evaluate the $1/m_Q$ corrections to the static limit has been reached by the lattice calculations. Notice that the form factors for both the $B \to D^* l \nu$ and $B \to D l \nu$ decays have been determined by the FNAL and ROMA-TOV collaborations using different techniques, and the agreement between the results is reassuring. From a phenomenological point of view, the semileptonic decay in the vector channel, $B \to D^* l \nu$, plays a privileged role, since the experimental accuracy in the determination of the decay rate is higher by almost a factor 10 than the one reached for $B \to D l \nu$ decays.

For the form factor $F(1)$, an unquenched determination with $N_f = 2 + 1$ dynamical quarks has been recently obtained by the FNAL group, and it is in good agreement with
the quenched estimates by the FNAL and ROMA-TOV collaborations (see fig.5 right). Thus, we quote the unquenched result as the best estimate of the form factor, namely

\[ F(1) = 0.924 \pm 0.022 \]

In the case of \( B \to D^* l\nu \) decays, the unquenched result for \( G(1) \) by the FNAL collaboration has been only presented at the Lattice 2004 conference [38] and it has not been submitted for publication. Therefore, in this case, we prefer to quote as our final estimate the value

\[ G(1) = 1.060 \pm 0.035 , \]

obtained by combining the unquenched and the two quenched determinations by the FNAL and ROMA-TOV collaborations, also shown in fig.5 (left).

In order to extract \( V_{cb} \), we use the HFAG experimental averages \( |V_{cb}| \cdot F(1) = (36.18 \pm 0.55) \cdot 10^{-3} \) and \( |V_{cb}| \cdot G(1) = (42.3 \pm 4.5) \cdot 10^{-3} \) [39], thus obtaining

\[ |V_{cb}| (\text{excl.}) = (39.2 \pm 1.1) \cdot 10^{-3} , \quad \text{from } B \to D^* l\nu \]
\[ |V_{cb}| (\text{excl.}) = (39.9 \pm 4.4) \cdot 10^{-3} , \quad \text{from } B \to D l\nu . \]

Clearly, the determination of \( V_{cb} \) is dominated by the result obtained from \( B \to D^* l\nu \) decays, which we thus take as the best estimate for the CKM matrix element.

6. – \( V_{ub} \) exclusive

Lattice QCD results for the form factor controlling the exclusive semileptonic \( B \to \pi l\nu \) decay are usually considered to evaluate the integrated decay rate in the large momentum transfer region, which is the one directly accessible to lattice calculations. In fig.6 we present a compilation of both quenched and unquenched lattice results [38], [40]-[44] for the quantity \( FF(q^2 > 16 \text{ GeV}^2) \equiv \Gamma(q^2 > 16 \text{ GeV}^2)/|V_{ub}|^2 \). They have been obtained using different approaches to treat the \( b \) quark (relativistic QCD, NRQCD and FNAL).
A very good agreement among all the results is observed, and the effect of the quenching error is not visible so far within the present uncertainties.

By averaging the lattice results shown in fig. 6 we obtain

\begin{equation}
FF(q^2 > 16 \text{ GeV}^2) = 2.04 \pm 0.40 \text{ ps}^{-1},
\end{equation}

which can be combined with the experimental branching fraction \(BF(q^2 > 16 \text{ GeV}^2) = (0.38 \pm 0.03 \pm 0.03) \cdot 10^{-4}\) quoted for \(B^0\) by HFAG \([39]\) and with the lifetime \(\tau_{B^0} = 1.530 \pm 0.009 \text{ ps}\) to obtain

\begin{equation}
|V_{ub}| \text{ (excl.)}_{LQCD} = (35.4 \pm 4.0) \cdot 10^{-4}.
\end{equation}

For these decays also the information coming from QCD sum rules on the small momentum transfer region can be considered. Ref. \([45]\) provides \(FF(q^2 < 16 \text{ GeV}^2) = 5.44 \pm 1.43 \text{ ps}^{-1}\) which, combined with the HFAG average, \(BF(q^2 < 16 \text{ GeV}^2) = (0.95 \pm 0.05 \pm 0.05) \cdot 10^{-4}\) and the \(B^0\) lifetime quoted above, gives \(|V_{ub}| \text{ (excl.)}_{QCD-SR} = (34.7 \pm 4.8) \cdot 10^{-4}\). The more recent and updated analysis of ref. \([46]\) quotes

\begin{equation}
|V_{ub}| \text{ (excl.)}_{QCD-SR} = (35 \pm 4 \pm 2 \pm 1) \cdot 10^{-4},
\end{equation}

in good agreement with the previous QCD sum rules result and the lattice determination. We thus quote as our final average

\begin{equation}
|V_{ub}| \text{ (excl.)} = (35.0 \pm 4.0) \cdot 10^{-4}.
\end{equation}

7. – Matrix elements for \(K^0 - \bar{K}^0\) mixing: full basis of four-fermion operators

In models of physics beyond the Standard Model, the effective Hamiltonian which describes the \(K^0 - \bar{K}^0\) mixing amplitude involves in general the complete basis of \(\Delta S = 2\)
four-fermion operators. A common choice for this basis is constituted by the operators

\begin{align}
O_1 &= \bar{s}^\alpha \gamma_\mu (1 - \gamma_5) d^\alpha \bar{s}^\beta \gamma_\mu (1 - \gamma_5) d^\beta , \\
O_2 &= \bar{s}^\alpha (1 - \gamma_5) d^\alpha \bar{s}^\beta (1 - \gamma_5) d^\beta , \\
O_3 &= \bar{s}^\alpha (1 - \gamma_5) d^\beta \bar{s}^\beta (1 - \gamma_5) d^\alpha , \\
O_4 &= \bar{s}^\alpha (1 - \gamma_5) d^\beta \bar{s}^\beta (1 + \gamma_5) d^\beta , \\
O_5 &= \bar{s}^\alpha (1 - \gamma_5) d^\beta \bar{s}^\beta (1 + \gamma_5) d^\alpha ,
\end{align}

(19)

where \( \alpha \) and \( \beta \) are colour indices, together with the operators \( \tilde{O}_{1,2,3} \) obtained from \( O_{1,2,3} \) with the exchange \( \gamma_5 \to -\gamma_5 \). In chirally invariant renormalization schemes, the operators \( \tilde{O}_i \) have the same matrix elements of the \( O_i \) and, for this reason, they will not be further discussed in what follows.

Omitting terms which are of higher order in chiral perturbation theory, the \( B \)-parameters are introduced using the expressions

\begin{align}
\langle \bar{K}^0 | O_1 (\mu) | K^0 \rangle &= \frac{8}{3} M_K^2 \tilde{f}_K B_1^{sd} (\mu) , \\
\langle \bar{K}^0 | O_2 (\mu) | K^0 \rangle &= -\frac{5}{3} \left( \frac{M_K}{m_s (\mu) + m_d (\mu)} \right)^2 M_K^2 \tilde{f}_K B_2^{sd} (\mu) , \\
\langle \bar{K}^0 | O_3 (\mu) | K^0 \rangle &= \frac{1}{3} \left( \frac{M_K}{m_s (\mu) + m_d (\mu)} \right)^2 M_K^2 \tilde{f}_K B_3^{sd} (\mu) , \\
\langle \bar{K}^0 | O_4 (\mu) | K^0 \rangle &= 2 \left( \frac{M_K}{m_s (\mu) + m_d (\mu)} \right)^2 M_K^2 \tilde{f}_K B_4^{sd} (\mu) , \\
\langle \bar{K}^0 | O_5 (\mu) | K^0 \rangle &= \frac{2}{3} \left( \frac{M_K}{m_s (\mu) + m_d (\mu)} \right)^2 M_K^2 \tilde{f}_K B_5^{sd} (\mu) ,
\end{align}

(20)

where \( B_1^{sd} (\mu) = B_K (\mu) \).

The matrix elements of the full operator basis for \( K^0 - \bar{K}^0 \) mixing have been computed, at present, only in three lattice studies [47, 48, 49], all performed within the quenched approximation. Refs. [47] and [48] use tree-level improved Clover and overlap fermions respectively, and both implement non-perturbative renormalization with the RI-MOM method. The results of ref. [49] are obtained by using domain wall fermions and renormalizing the operators with one-loop perturbation theory. They have been presented at the Lattice 2006 conference but a final analysis has not been published. The values of the \( B \)-parameters from refs. [47]-[49] are collected in table I (top) \(^{(1)}\). In table I (bottom) we collect, instead, the corresponding values of the ratios of the non Standard Model to the Standard Model matrix elements,

\begin{equation}
R_i^{sd} (\mu) = \frac{\langle \bar{K}^0 | O_i (\mu) | K^0 \rangle}{\langle K^0 | O_i (\mu) | K^0 \rangle}, \quad i = 2, \ldots 5 .
\end{equation}

\(^{(1)}\) We have selected from ref. [47] the results obtained at the largest values of the lattice coupling, and from ref. [49] those obtained with the Iwasaki gauge action on the larger volume. In the latter case, we add to the statistical error the uncertainty coming from the difference between the results obtained with the Iwasaki and the Wilson plaquette gauge actions, which provides an estimate of \( \mathcal{O}(a^2) \) discretization effects.
From the results shown in table I we observe that the differences among the three determinations of refs. [47]-[49] are typically larger than the quoted uncertainties. Clearly, new lattice studies of these matrix elements would be necessary in order to clarify the situation. The observation that the discrepancies are much more pronounced in the case of the ratios $R_{sd}$ might suggest that some of the systematic errors cancel, at least in part, in the determination of the $B$-parameters. For this reason, we average the results for the $B$-parameters and quote:

\begin{align}
B_{sd}^1 &= 0.7(2) , \quad B_{sd}^2 = 1.0(4) , \quad B_{sd}^3 = 0.9(2) , \quad B_{sd}^4 = 0.6(1) , \\
B_{sd}^1 &= 1.9(6) , \quad B_{sd}^2 = 12(3) , \quad B_{sd}^3 = 2.6(9) , \\
R_{sd}^2 &= -7(2) , \quad R_{sd}^3 = 1.9(6) , \quad R_{sd}^4 = 12(3) , \quad R_{sd}^5 = 2.6(9) , \\
R_{sd}^2 &= 1.9(6) , \quad R_{sd}^3 = 12(3) , \quad R_{sd}^4 = 2.6(9) , \quad R_{sd}^5 = 2.6(9) .
\end{align}

in the RI-MOM scheme at the scale $\mu = 2$ GeV. For the parameter $B_{sd}^1 = B_K$ we refer instead to the average given in eq. (1) which, translated into the RI-MOM scheme at $\mu = 2$ GeV, corresponds to $B_{sd}^1 = B_K = 0.54(5)$.

In order to derive from the average values of the $B$-parameters the ratios $R_{sd}^i$, a determination of the strange and down quark masses is required. We use, at $\mu = 2$ GeV, $(m_s^{MS} + m_d^{MS}) = 135 \pm 18$ MeV, corresponding to $(m_s^{MS} + m_d^{MS}) = 110 \pm 15$ MeV. In this way we find

\begin{align}
R_{sd}^2 &= -11 \pm 4 , \quad R_{sd}^3 = 3.1 \pm 1.5 , \quad R_{sd}^4 = 17 \pm 6 , \quad R_{sd}^5 = 3.8 \pm 1.3 .
\end{align}

8. – Matrix elements for $B_{d/s}^0 - \bar{B}_{d/s}^0$ mixing: full basis of four-fermion operators

In the case of $B$ mesons, besides the matrix element of the operator $O_1$ which enters the mixing amplitude, also the operator $O_2$ (sometimes denoted as $O_8$) is of interest in the Standard Model, since it contributes to the theoretical prediction of the lifetime difference $\Delta \Gamma$ of the neutral $B$ mesons. As in the case of $K^0 - \bar{K}^0$ mixing, in generic New Physics models the effective Hamiltonian describing $\Delta B = 2$ transitions receives contribution from the full basis of four fermions operators. Here we define the operators
Table II. - $B$-parameters for the full basis of four-fermions operators in $B_0^s - \bar{B}_0^s$ mixing. The results are given at the scale $\mu = m_b$ in the $\overline{\text{MS}}$ scheme of ref. [50] for $B_1$-$B_3$ and of ref. [51] for $B_4$-$B_5$.

|       | $B_1^{bs}$ | $B_2^{bs}$ | $B_3^{bs}$ | $B_4^{bs}$ | $B_5^{bs}$ |
|-------|------------|------------|------------|------------|------------|
| Ref. [29] | 0.88(5)    | 0.84(4)    | 0.91(9)    | 1.15(6)    | 1.74(7)    |
| Ref. [30] | 0.86(5)    | 0.86(5)    | —          | —          | —          |
| Ref. [31] | 0.76(11)   | 0.84(13)   | 0.90(14)   | —          | —          |

Note that the larger errors quoted by ref. [31] on the $B$-parameters, with respect to refs. [29, 30], are due to the fact that the $B$-parameters are extracted in [31] from the values of the corresponding matrix elements, rather than directly calculated from the more precise ratios of 2- and 3-point correlation functions. On the other hand, in order to evaluate from the $B$-parameters the full matrix elements of four-fermion operators, which are eventually the quantities of physical interest, the results of refs. [29, 30] for the $B$-parameters have to be combined with the value of the decay constant $f_{B_s}$, which thus increases the final uncertainty.

The $SU(3)$ breaking effects for the full basis of $B$-parameters have been computed in ref. [29], that quotes

\begin{equation}
(B_i^{bs} / B_i^{bd})|_{i=1,...,5} = \{0.99(2), 1.01(2), 1.01(3), 1.01(2), 1.01(3)\} .
\end{equation}

These ratios are well compatible with unity within the errors. Thus, we choose to quote common averages for $B_d$ and $B_s$ mixing.

The results collected in table II show a very good agreement among the central values of the three (two) determinations of $B_2$ ($B_3$). On the other hand, the comparison of several lattice determinations for the $B$-parameter $B_1 = B_{B_d}$ has led us to quote in eq. (8) a final uncertainty on this quantity of about 10%. Therefore, we choose to add the same 10% of systematic uncertainty also to the final estimates of $B_2$ and $B_3$, as well as to those of $B_4$ and $B_5$ for which only a single (and rather old) lattice calculation exists.

\(^{(2)}\) Ref. [30] quotes final results for the $B_2$ parameter only, although both the matrix elements of the operators $O_2$ and $O_3$ have been evaluated, since they mix under renormalization.
In this way, we quote

\[(25)\quad B_b^{q} = 0.85(10) \quad , \quad B_s^{q} = 0.90(13) \quad , \quad B_s^{4q} = 1.15(13) \quad , \quad B_s^{5q} = 1.74(19) ,\]

for both \( q = d, s \). The results in eq. (25) are given at the scale \( \mu = m_b \) in the \( \overline{\text{MS}} \) scheme of ref. [50] for \( B_2 - B_3 \) and of ref. [51] for \( B_4 - B_5 \). For the parameter \( B_{bs}^{4s} = B_{bs} \) we refer instead to the average given in eq. (8) (in the same \( \overline{\text{MS}} \) scheme of ref. [50]).

The values of the \( B \)-parameters in eq. (25), combined with the average of the \( B_s \)-meson decay constant of eq. (3), allow to evaluate the quantities

\[(26) \quad R_i(m_b) = \left( \frac{M_{B_i}}{m_b(m_b) + m_s(m_b)} \right) f_{B_i} \sqrt{B_{bs}^i(m_b)} , \quad i = 2, \ldots , 5 .\]

The squares \( R^2_i \) are proportional to the corresponding operator matrix elements. Using \( m_b^{\overline{\text{MS}}}(m_b) = 4.21(8) \text{ GeV} \) and \( m_s^{\overline{\text{MS}}}(m_b) = 87(12) \text{ MeV} \) (which corresponds to \( m_s^{\overline{\text{MS}}}(2 \text{ GeV}) = 105(15) \text{ MeV} \)), we obtain

\[ R_2 = 282 \pm 34 \text{ MeV} , \quad R_3 = 290 \pm 37 \text{ MeV} , \quad R_4 = 328 \pm 39 \text{ MeV} , \quad R_5 = 404 \pm 47 \text{ MeV} . \]

For \( R_2 \) and \( R_3 \) these results are well consistent with those quoted by the unquenched calculation of ref. [31], namely \( R_2 = 295 \pm 22 \text{ MeV} \) and \( R_3 = 305 \pm 23 \text{ MeV} \).

9. – Matrix elements for \( D^0 - \bar{D}^0 \) mixing

The matrix elements of the full basis of four-fermion operators for \( D^0 - \bar{D}^0 \) mixing can be obtained as a byproduct from ref. [29]. This work provides quenched results for the \( B \)-parameters corresponding to heavy-light meson masses equal to \( M_D = 1.75(9) \text{ GeV} \) and \( M_P = 2.02(10) \text{ GeV} \) respectively. By interpolating between the two sets of results, one obtains for the physical \( D \)-mesons the values \( B_{cu}^{i} |_{i=1, \ldots, 5} = \{ 0.85(2), 0.82(3), 1.07(5), 1.10(2), 1.37(3) \} \), in the RI-MOM scheme at the scale \( \mu = 2.8 \text{ GeV} \). The \( B \)-parameter \( B_D = B_1^{cu} \) has been also computed in ref. [52], always within the quenched approximation and using non-perturbative RI-MOM renormalization, with domain wall fermions. The result is affected by a large uncertainty: they obtain \( B_1^{cu} = 0.85(2)(11) \) in the \( \overline{\text{MS}} \) scheme at \( \mu = 2.0 \text{ GeV} \). Translated into the RI-MOM scheme at the scale \( \mu = 2.8 \text{ GeV} \), this value corresponds to \( B_1^{cu} = 0.81(2)(10) \), well consistent with the result of ref. [29].

By considering that the results of ref. [29] are obtained from a single quenched lattice calculation, we add in the final averages a systematic uncertainty of 10%, as we have done in the case of \( B^0 - \bar{B}^0 \) mixing. In this way we obtain

\[(28) \quad B_1^{cu} = 0.85(9) \quad , \quad B_2^{cu} = 0.82(9) \quad , \quad B_3^{cu} = 1.07(12) \quad , \quad B_4^{cu} = 1.10(11) \quad , \quad B_5^{cu} = 1.37(14) ,\]

in the RI-MOM scheme at the scale \( \mu = 2.8 \text{ GeV} \).

REFERENCES

[1] M. Bona et al. [UTfit Collaboration], these proceedings.
[2] M. Bona et al. [UTfit Collaboration], JHEP 0610 (2006) 081 [arXiv:hep-ph/0606167].
[3] M. Bona et al. [UTfit Collaboration], Phys. Rev. Lett. 97 (2006) 151803 [arXiv:hep-ph/0605213].
[4] The UTfit Collaboration, http://www.utfit.org
[5] A. Juttner, PoS LATTICE2007 (2007) 014 [arXiv:0711.1239 [hep-lat]].
[6] S. Aoki et al. [JLQCD Collaboration], Phys. Rev. Lett. 80 (1998) 5271 [arXiv:hep-lat/9710073].
[7] Y. Aoki et al., Phys. Rev. D 73 (2006) 094507 [arXiv:hep-lat/0508011].
[8] Y. Nakamura, S. Aoki, Y. Taniguchi and T. Yoshie [CP-PACS Collaboration], arXiv:0803.2569 [hep-lat].
[9] Y. Aoki et al., Phys. Rev. D 72 (2005) 094507 [arXiv:hep-lat/0411006].
[10] S. Aoki et al. [JLQCD Collaboration], arXiv:0801.4186 [hep-lat].
[11] E. Gamiz, S. Collins, C. T. H. Davies, G. P. Lepage, J. Shigemitsu and M. Wingate [HPQCD Collaboration], Phys. Rev. D 73 (2006) 114502 [arXiv:hep-lat/0603023].
[12] D. J. Antonio et al. [RBC Collaboration], Phys. Rev. Lett. 100 (2008) 032001 [arXiv:hep-ph/0702042].
[13] C. Dawson, PoS LAT2005 (2006) 007.
[14] A. Ali Khan et al. [CP-PACS Collaboration], Phys. Rev. D 64 (2001) 034505 [arXiv:hep-lat/0010009].
[15] A. Ali Khan et al. [CP-PACS Collaboration], Phys. Rev. D 64 (2001) 054504 [arXiv:hep-lat/0103020].
[16] C. Bernard et al. [MILC Collaboration], Phys. Rev. D 66 (2002) 094501 [arXiv:hep-ph/0206016].
[17] S. Aoki et al. [JLQCD Collaboration], Phys. Rev. Lett. 91 (2003) 212001 [arXiv:hep-ph/0307039].
[18] M. Wingate, C. T. H. Davies, A. Gray, G. P. Lepage and J. Shigemitsu, Phys. Rev. Lett. 92 (2004) 162001 [arXiv:hep-ph/0311130].
[19] A. Gray et al. [HPQCD Collaboration], Phys. Rev. Lett. 95 (2005) 212001 [arXiv:hep-lat/0507015].
[20] C. Bernard et al. [Fermilab Lattice and MILC Collaborations], PoS LATTICE2007 (2007) 370
[21] A. X. El-Khadra, A. S. Kronfeld and P. B. Mackenzie, Phys. Rev. D 55 (1997) 3933 [arXiv:hep-lat/9604004].
[22] J. L. Rosner and S. Stone, arXiv:0802.1043 [hep-ex].
[23] C. Tarantino [ETM Collaboration], talk at Lattice'08, http://conferences.jlab.org/lattice2008
[24] E. Follana, C. T. H. Davies, G. P. Lepage and J. Shigemitsu [HPQCD Collaboration and UKQCD Collaboration], Phys. Rev. Lett. 100 (2008) 062002 [arXiv:0706.1726 [hep-lat]].
[25] B. A. Dobrescu and A. S. Kronfeld, Phys. Rev. Lett. 100 (2008) 241802 [arXiv:0803.0512 [hep-ph]].
[26] S. R. Sharpe, PoS LAT2006 (2006) 022 [arXiv:hep-lat/0610094].
[27] A. S. Kronfeld, PoS LATTICE2007 (2007) 016 [arXiv:0711.0699 [hep-lat]].
[28] L. Lellouch and C. J. D. Lin [UKQCD Collaboration], Phys. Rev. D 64 (2001) 094501 [arXiv:hep-ph/0110126].
[29] D. Becirevic, V. Gimenez, G. Martinelli, M. Papinutto and J. Reyes, JHEP 0204 (2002) 025 [arXiv:hep-lat/0110091].
[30] S. Aoki et al. [JLQCD Collaboration], Phys. Rev. D 67 (2003) 014506 [arXiv:hep-lat/0208038].
[31] E. Dalgic et al., Phys. Rev. D 76 (2007) 011501 [arXiv:hep-lat/0610104].
[32] C. Albertus et al. [RBC and UKQCD Collaborations], PoS LATTICE2007 (2007) 376
[33] S. Hashimoto, A. S. Kronfeld, P. B. Mackenzie, S. M. Ryan and J. N. Simone, Phys. Rev. D 66 (2002) 014503 [arXiv:hep-ph/0110253].
[34] G. M. de Divitiis, R. Petronzio and N. Tantalo, arXiv:0807.2944 [hep-lat].
[35] J. Laiho [Fermilab Lattice and MILC Collaborations], PoS LATTICE2007 (2006) 358 [arXiv:0710.1111 [hep-lat]].
[36] S. Hashimoto, A. X. El-Khadra, A. S. Kronfeld, P. B. Mackenzie, S. M. Ryan and J. N. Simone, Phys. Rev. D 61 (2000) 014502 [arXiv:hep-ph/9906376].
[37] G. M. de Divitiis, E. Molinaro, R. Petronzio and N. Tantalo, Phys. Lett. B 655 (2007) 45 [arXiv:0707.0582 [hep-lat]].
[38] M. Okamoto et al., Nucl. Phys. Proc. Suppl. 140 (2005) 461 [arXiv:hep-lat/0409116].
[39] The Heavy Flavor Averaging Group (HFAG), http://www.slac.stanford.edu/xorg/hfag.
[40] K. C. Bowler et al. [UKQCD Collaboration], Phys. Lett. B 486 (2000) 111 [arXiv:hep-lat/9911011].
[41] A. Abada, D. Becirevic, P. Boucaud, J. P. Leroy, V. Lubicz and F. Mescia, Nucl. Phys. B 619 (2001) 565 [arXiv:hep-lat/0011065].
[42] A. X. El-Khadra, A. S. Kronfeld, P. B. Mackenzie, S. M. Ryan and J. N. Simone, Phys. Rev. D 64 (2001) 014502 [arXiv:hep-ph/0101023].
[43] S. Aoki et al. [JLQCD Collaboration], Phys. Rev. D 64 (2001) 114505 [arXiv:hep-lat/0106024].
[44] E. Dalgic, A. Gray, M. Wingate, C. T. H. Davies, G. P. Lepage and J. Shigemitsu, Phys. Rev. D 73 (2006) 074502 [Erratum-ibid. D 75 (2007) 119906] [arXiv:hep-lat/0601021].
[45] P. Ball and R. Zwicky, Phys. Rev. D 71 (2005) 014015 [arXiv:hep-ph/0406232].
[46] G. Duplancic, A. Khodjamirian, T. Mannel, B. Melic and N. Offen, JHEP 0804 (2008) 014 [arXiv:0801.1796 [hep-ph]].
[47] A. Donini, V. Gimenez, L. Giusti and G. Martinelli, Phys. Lett. B 470 (1999) 233 [arXiv:hep-lat/9910017].
[48] R. Babich, N. Garron, C. Hoelbling, J. Howard, L. Lellouch and C. Rebbi, Phys. Rev. D 74 (2006) 073009 [arXiv:hep-lat/0605016].
[49] Y. Nakamura et al. [CP-PACS Collaboration], PoS LAT2006 (2006) 089 [arXiv:hep-lat/0610075].
[50] M. Beneke, G. Buchalla, C. Greub, A. Lenz and U. Nierste, Phys. Lett. B 459 (1999) 631 [arXiv:hep-ph/9808385].
[51] A. J. Buras, M. Misiak and J. Urban, Nucl. Phys. B 586 (2000) 397 [arXiv:hep-ph/0005183].
[52] H. W. Lin, S. Ohta, A. Soni and N. Yamada, Phys. Rev. D 74 (2006) 114506 [arXiv:hep-lat/0607035].