ALGEBRAIC MONTGOMERY-YANG PROBLEM
AND CASCADE CONJECTURE

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Abstract. The conjecture called algebraic Montgomery-Yang problem is still open for rational $\mathbb{Q}$-homology projective planes with cyclic quotient singularities having ample canonical divisor. All known such surfaces have a special birational behavior called a cascade. In this note, we establish algebraic Montgomery-Yang problem assuming the cascade conjecture, which claims that every rational $\mathbb{Q}$-homology projective planes with quotient singularities having ample canonical divisor admits a cascade.

1. Introduction

Motivated by works of Seifert([S]), Montgomery and Yang([MY]) and Petri([P]), Fintushel and Stern formulated the following conjecture.

Conjecture 1.1. [FS87] (Montgomery-Yang problem) Every pseudo-free differentiable circle action on the 5-dimensional sphere has at most three non-free orbits.

One way to approach this conjecture, as was already considered in [MY], [FS85] and [FS87], is to consider the orbit space as a 4-dimensional orbifold. Considering the case when the orbit space has a structure of a complex projective surface, Kollár formulated the following version of the conjecture.

Conjecture 1.2. [K] (Algebraic Montgomery-Yang problem)
Let $S$ be a $\mathbb{Q}$-homology projective plane with quotient singularities. If the smooth locus of $S$ is simply-connected, then $S$ has at most three singular points.

See Notation 2.1 for the definition of a $\mathbb{Q}$-homology projective plane. The conjecture turned out to be true when $S$ has a non-cyclic singular point([HK2]), $S$ is not a rational surface([HK4]), and $-K_S$ is nef([HK5] and [HK4] Lemma 3.6 (4))). Thus, it remained open only for rational $\mathbb{Q}$-homology projective planes with at worst cyclic quotient singularities having ample canonical divisor.

Several efforts have been devoted to construct rational $\mathbb{Q}$-homology projective planes with ample canonical divisor ([KM], [K], [HK3], [AL1], [AL2]). In fact, all of them admit a special birational behavior, called a cascade.

Definition 1.3. Let $S$ be a $\mathbb{Q}$-homology projective plane with quotient singularities. We say that $S$ admits a cascade if there exists a diagram as follows:

Date: January 10, 2021.
2010 Mathematics Subject Classification. Primary 14J25; Secondary 14J17, 14J26.
Key words and phrases. $\mathbb{Q}$-homology projective plane, algebraic Montgomery-Yang problem, cascade, $\mathbb{P}^1$-fibration.
\[ S' = S'_t \xrightarrow{\phi_1} S'_{t-1} \xrightarrow{\phi_{t-1}} \ldots \xrightarrow{\phi_1} S'_0 \]

where for each \( k \)

1. \( \phi_k \) is a blow-down,
2. \( \pi_k \) is the minimal resolution,
3. \( S_k \) is a \( \mathbb{Q} \)-homology projective plane, and
4. \( S_0 \) is a log del Pezzo surface of Picard number one.

In this case, we also say that \( S \) admits a cascade to \( S_0 \).

Intuitively speaking, the idea behind this definition is that \( K_{S_t} \) becomes more negative for each step of cascade, i.e., as \( t \) decreases. Thus, the study on rational \( \mathbb{Q} \)-homology projective plane with ample canonical divisor can be reduced to that on log del Pezzo surfaces of Picard number one.

We would like to pose the following conjecture, which is supported by every known example ([KM], [HK3], [AL1], [AL2]).

**Conjecture 1.4.** Every rational \( \mathbb{Q} \)-homology projective plane with quotient singularities such that the canonical divisor is ample admits a cascade.

Conjecture 1.4 implies that the rational \( \mathbb{Q} \)-homology projective plane with quotient singularities having ample canonical divisor constructed in [AL1, Theorem 8.2] attains the minimal volume, as was expected.

In addition, the main result of the paper is the following.

**Theorem 1.5.** Conjecture 1.4 implies Conjecture 1.2.

In fact, we prove the following stronger statement. Note that if Conjecture 1.4 is true, then there exists a \((-1)\)-curve \( E \) with \( E.D \leq 2 \) where \( D \) is the reduced exceptional divisor of the minimal resolution of \( S \).

**Theorem 1.6.** Under the assumption in Conjecture 1.2, we further assume that the canonical divisor is ample and there exists a \((-1)\)-curve \( E \) with \( E.D \leq 2 \) where \( D \) is the reduced exceptional divisor of the minimal resolution of \( S \). Then, \( S \) has at most three singular points.

The proof consists of arguments used in [HK4] together with a method of using \( \mathbb{P}^1 \)-fibration structure.

We work over the field \( \mathbb{C} \) of complex numbers.

2. **Preliminaries**

2.1. **Notation.** We first fix some notations, following [HK4], that will be used in the remaining of the paper. A normal projective surface with quotient singularities is called a \( \mathbb{Q} \)-homology projective plane if it has the same Betti numbers as the complex projective plane. It is said to be rational if it is birationally equivalent to the projective plane \( \mathbb{P}^2 \). It is also said to be of log general type if its canonical divisor is ample. We always denote by \( S \) a rational \( \mathbb{Q} \)-homology projective plane of log general type with 4 cyclic quotient singularities \( p_1, \ldots, p_4 \), unless otherwise stated. Let \( f : S' \to S \) be a minimal resolution of \( S \). Denote by \( D_i \) the reduced
part of the \( f \)-exceptional divisor \( f^{-1}(p_i) \) of \( p_i \) and by \( l_i \) the number of irreducible components of \( D_i \). Let \( D = D_1 + \ldots + D_4 \) and \( L = l_1 + \ldots + l_4 \).

Let \( D \) be a \( \mathbb{Q} \)-divisor whose support consists of a chain of rational curves with the dual graph of the form \( \circ^{n_1} \ldots \circ^{n_4} \) where each weight \( n_i \) denotes the self-intersection number of the corresponding irreducible component \( D_i \). Then, we introduce the following notation. (See [HK4, Section 2] for more extensive explanation about the notation.)

1. We denote by \( q_D \) the order of the local fundamental group of the singular point obtained by contracting the divisor \( D \).
2. We denote by \( u_{k,D} \) the order of the local fundamental group of the singular point obtained by contracting the first \( k - 1 \) irreducible components of \( D \).
3. We denote by \( v_{k,D} \) the order of the local fundamental group of the singular point obtained by contracting the last \( l - k \) irreducible components of \( D \).
4. For convenience, we also denote \( v_{1,D} \) by \( q_{1,D} \) and \( u_{l,D} \) by \( q_{l,D} \).
5. \( tr_D = -\sum_{i=1}^{l} D_i^2 \).

We will omit the subscript \( D \) in (5) whenever there is no confusion in the context.

2.2. Known results and easy consequences. We first summarize known results mainly from [HK4].

**Theorem 2.1.** [HK] Let \( S \) be a \( \mathbb{Q} \)-homology projective plane with quotient singularities. Then, \( S \) has at most 5 singular points and it has exactly 5 singular points if and only if the singularities of \( S \) are of type \( 3A_1 + 2A_3 \) and the minimal resolution of \( S \) is an Enriques surface. In particular, if \( S \) is rational, then \( S \) has at most 4 singular points.

**Corollary 2.2.** Assume the situation in Notation 2.1. If \( E \) is a \((-1)\)-curve on \( S' \), then \( E.D \geq 2 \). Moreover, if \( E.D = 2 \), then \( E \) intersects \( D \) at two different points.

**Proof.** Since \( S \) has Picard number one, \( E.D \geq 1 \). If \( E.D = 1 \), then by blowing up the intersection point of \( E \) and \( D \), we get a minimal resolution of another rational \( \mathbb{Q} \)-homology projective plane with 5 singular points, a contradiction by Theorem 2.1.

Consider the case \( E.D = 2 \). Assume that \( E \) intersects \( D \) at one point \( p \) with multiplicity 2. If \( p \) is an intersection of two different irreducible components of \( D \), by blowing up \( p \), we get a minimal resolution of another rational \( \mathbb{Q} \)-homology projective plane with 5 singular points, a contradiction by Theorem 2.1. Now, we may assume that \( E \) intersects an irreducible component \( D \) of \( D \) with multiplicity two. By blowing up the intersection point of \( E \) and \( D \), and then again by blowing up the intersection point of the proper transform of \( E \) and \( D \), we get a minimal resolution of another \( \mathbb{Q} \)-homology projective plane which has 6 quotient singularities, a contradiction by Theorem 2.1.

Now, we consider possible orders for the local fundamental groups.

**Lemma 2.3.** [HK4] Lemma 5.2 and Lemma 5.3] Let \( S \) be as in Notation 2.1. Assume that \( H_1(S^{sm}, \mathbb{Z}) \) is trivial. Then, the orders of the local fundamental groups of each singular point is either \((2, 3, 7, 19)\) or \((2, 3, 5, q)\) for some positive integer \( q \) with \( \gcd(q, 30) = 1 \). Moreover, the singularity type must be \( A_1 + A_2 + \frac{1}{7}(1, 1) + \frac{1}{19}(1, 9) \) in the first case and the order 3 singularity must be of type \( \frac{1}{3}(1, 1) \) in the latter case.
We have more precise information on the configuration of \((-1)\)-curves on \(S'\) in the first case.

**Lemma 2.4.** [HK4, Lemma 5.6] Let \(S\) be a rational \(\mathbb{Q}\)-homology projective plane of log general type with exactly four cyclic quotient singular points \(p_1, p_2, p_3, p_4\) of orders \((2, 3, 7, 19)\). Let \(D\) be the reduced \(f\)-exceptional divisor on \(S'\), and \(E\) be any \((-1)\)-curve on \(S'\). Then, \(E.D \geq 2\) and the equality holds if and only if \(E.f^{-1}(p_i) = 0\) for \(i = 1, 2, 3\), \(E.f^{-1}(p_4) = 2\) and \(E\) does not meet an end component of \(f^{-1}(p_4)\).

We also have more information in the latter case.

**Lemma 2.5.** [HK4, Lemma 5.4 and Lemma 5.5] Let \(S\) be a rational \(\mathbb{Q}\)-homology projective plane of log general type with exactly four cyclic quotient singular points \(p_1, p_2, p_3, p_4\) of orders \((2, 3, 5, q)\). Assume that the order 3 singularity is of type \(\frac{1}{3}(1, 1)\). Then, we have

1. \(L \geq 11\) and \(L = 11\) if and only if \(S\) has singularities of one of the following types:
   a. \(A_1 + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 1) + A_8\)
   b. \(A_1 + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2) + \frac{1}{19}(1, 17)\)
   c. \(A_1 + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2) + \frac{1}{13}(1, 13)\)
   d. \(A_1 + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2) + \frac{1}{13}(1, 19)\)
2. \(q \geq 20\) except the case \(A_1 + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 1) + A_8\).
3. Let \(D\) be the reduced \(f\)-exceptional divisor on \(S'\), and \(E\) be any \((-1)\)-curve on \(S'\). Then, \(E.D \geq 2\) and the equality holds if and only if \(E.f^{-1}(p_i) = 0\) for \(i = 1, 2, 3\) and \(E.f^{-1}(p_4) = 2\).

For later use, we give a formula for computing \(K_S^2\).

**Lemma 2.6.** [HK4, Section 3] Let \(S\) be as in Notation 2.1. Then, we have

\[
K_S^2 = 9 - L + \sum_{p \in \text{Sing}(S)} \left( t_r - 2t_p - 2 + \frac{q_1 + q_{l,p} + 2}{q_p} \right).
\]

**Proof.** Every surface quotient singular point is log terminal. So, up to numerical equivalence, we can write

\[
K_{S'} \equiv_{\text{num}} f^*K_S - \sum_{p \in \text{Sing}(S)} D_p
\]

where \(D_p = \sum (a_jA_j)\) is an effective \(\mathbb{Q}\)-divisor with \(0 \leq a_j < 1\) supported on \(f^{-1}(p) = \cup A_j\) for each singular point \(p\). Then,

\[
K_{S'}^2 = K_S^2 + \sum_p D_pK_{S'} - \sum_p D_p^2.
\]

Since \(K_{S'}^2 = 12 - (3 + L) = 9 - L\) by Noether’s formula and for each \(p\)

\[
D_p^2 = 2t_p - t_r + 2 - \frac{q_1 + q_{l,p} + 2}{q_p}
\]

by [HK4, Lemma 3.1 (3)], the result follows. \(\square\)

**Corollary 2.7.** Let \(S\) be a rational \(\mathbb{Q}\)-homology projective plane of log general type with exactly four cyclic quotient singular points \(p_1, p_2, p_3, p_4\) of orders \((2, 3, 5, q)\). Assume that \(p_2\) is of type \(\frac{1}{3}(1, 1)\). Then, we have
Corollary 2.9. Let $\text{Following Notation 2.1}$, we have 
Proof. 

Theorem 2.8. Let $\text{Theorem 2.8.}$ 

Corollary 2.11. Let $\text{Corollary 2.11.}$ \[ \square \] 

In the above, the subscript $D_4$ is omitted for $tr$, $l$, $q_1$, $q_i$, and $q$. 

Recall that the orbifold Euler number is defined as follows: 
\[ e_{\text{orb}}(S) = e_{\text{top}}(S) - \sum_{p \in \text{Sing}(S)} \left(1 - \frac{1}{|G_p|}\right) \] 
where $|G_p|$ denotes the order of the local fundamental group of $p$. Then, we have 

**Theorem 2.8.** \[ \text{Mc} \] Let $S$ be a normal projective surface with quotient singularities. If $K_S$ is nef, then 
\[ K_S^2 \leq 3e_{\text{orb}}(S). \] 

**Corollary 2.9.** Let $S$ be as in Lemma \[ \text{Lemma 2.7} \] Then, \[ 0 < K_S^2 \leq \frac{3}{q} + \frac{1}{10}. \] In particular, \[ 0 < K_S^2 \leq \frac{1}{2}. \] 

**Proof.** Since $K_S$ is ample, $K_S^2 > 0$. Now the result follows from Lemma \[ \text{Lemma 2.6} \] (2) and Lemma \[ \text{Lemma 2.7} \] (3). \[ \square \] 

As an application, we can prove a nonexistence of a $\mathbb{Q}$-homology projective plane with special singular points for later use. 

**Corollary 2.10.** There exists no rational $\mathbb{Q}$-homology projective plane of log general type with four cyclic quotient singular points of type $A_1 + \frac{1}{3}(1,1) + \frac{1}{2}(1,1) + \frac{1}{2l+1}(1,1)$ for any $l \geq 1$. 

**Proof.** By Lemma \[ \text{Lemma 2.5} \] (1), $L \geq 11$, so $l \geq 8$. Since the singularity $\frac{1}{2l+1}(1,1)$ has the resolution graph of the form $\begin{array}{cccc} \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \end{array} \cdots$, by Lemma \[ \text{Lemma 2.6} \] $K_S^2 < 0$, a contradiction to Lemma \[ \text{Lemma 2.9} \] \[ \square \] 

**Corollary 2.11.** Let $S$ be a rational $\mathbb{Q}$-homology projective plane with four cyclic quotient singular points of type $A_1 + \frac{1}{3}(1,1) + A_4 + [2, a, b, 2, c, 2]$ where the last one is the singularity type for the singular point whose dual graph is of the form $\begin{array}{cccc} \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \end{array} \cdots$ where $a, b$ and $c$ are integers greater than 1. Assume that $a + b + c = 10$ or $a + b + c = 11$. Then, $S$ is not of log general type. 

**Proof.** Following Notation \[ \text{Notation 2.1} \] we have $q_1 = 8abc - 8ab - 4bc - 8a + 6a + 4b + 2c - 1$, 
$q_2 = 8abc - 4ab - 8bc - 8ca + 2a + 4b + 6c - 1$ and $q = 16abc - 16ab - 16bc - 16a - 12a + 16b + 12c - 8$. Thus, we see that 
\[ \frac{q_1 + q_2 + 2}{q} - 1 = \frac{4(b - 1)(a + c - 2)}{q}. \] 
If $a + b + c = 11$, then by Lemma \[ \text{Lemma 2.6} \] Corollary \[ \text{Corollary 2.7} \] and Corollary \[ \text{Corollary 2.9} \] we have 
\[ \frac{8}{q} + \frac{1}{3} \leq K_S^2 = \frac{4(b - 1)(a + c - 2)}{q} + \frac{1}{3} \leq \frac{3}{q} + \frac{3}{20}. \]
which is a contradiction. If \( a + b + c = 10 \), then by \text{Lemma 2.6} and \text{Corollary 2.7} we have
\[
K_S^2 = \frac{4(b-1)(a+c-2)}{q} - \frac{2}{3}.
\]
In this case, one can compute that \( K_S^2 < 0 \), which is a contradiction. \( \square \)

2.3. A curve-detecting formula. We present a useful formula for detecting the existence of a \((-1)\)-curve.

\textbf{Proposition 2.12. \textit{[HK3] Proposition 4.2}} Let \( C \) be a smooth irreducible curve on \( S' \). Then, there exists a rational number \( m_E \) satisfying both of the following.

1. \( C.K_{S'} = m_C K_S^2 - \sum_{p} \left( \frac{v_{j,p} + u_{j,p}}{q_p} \right) A_{j,p} \).
2. If, for each \( p \in \text{Sing}(S) \), \( C \) has a non-zero intersection number with at most 2 components of \( f^{-1}(p) \), i.e., \( A_{j,p} = 0 \) for \( j \neq s_p, t_p \) for some \( s_p \) and \( t_p \) with \( 1 \leq s_p < t_p \leq \ell_p \) where the rational curves \( A_{1,p}, A_{2,p}, \ldots, A_{l_p} \) form the tree of exceptional divisor of \( f^{-1}(p) \), then
\[
C^2 = m_C K_S^2 - \sum_p \left( \frac{v_{s,p} u_{s,p}}{q_p} (CA_{s,p})^2 + \frac{v_{t,p} u_{t,p}}{q_p} (CA_{t,p})^2 + \frac{2v_{t,p} u_{s,p}}{q_p} (CA_{s,p})(CA_{t,p}) \right).
\]

\textbf{Remark 2.13.} We omit the subscript \( C \) of \( m_E \) if there is no confusion in the context.

One can determine the positivity of \( K_S \) simply by looking at the sign of \( m \) for an irreducible curve \( C \) on \( S' \).

\textbf{Lemma 2.14. \textit{[HK3] Lemma 2.6}}

1. \( K_S \) is ample if\( m_C > 0 \) for all irreducible curves \( C \) not contracted by \( f \) if\( m_C > 0 \) for an irreducible curve \( C \) not contracted by \( f \).
2. \( K_S \) is numerically trivial if\( m_C = 0 \) for all irreducible curves \( C \) not contracted by \( f \) if\( m_C = 0 \) for an irreducible curve \( C \) not contracted by \( f \).
3. \( -K_S \) is ample if\( m_C < 0 \) for all irreducible curves \( C \) not contracted by \( f \) if\( m_C < 0 \) for an irreducible curve \( C \) not contracted by \( f \).

We present some applications of the formula that will be used later.

\textbf{Lemma 2.15.} Let \( S \) be a \( \mathbb{Q} \)-homology projective plane with four singular points of type \( A_1 + A_2 + \frac{1}{2}(1,1) + \frac{1}{3}(1,9) \). Let \( D \) be the reduced \( f \)-exceptional divisor on \( S' \), and \( E \) be any \((-1)\)-curve on \( S' \). Assume that \( K_S \) is ample. Then, \( E.D \geq 3 \).

\textbf{Proof.} By \text{Lemma 2.4} we may assume that \( E.D = E.D_4 = 2 \). Since \( K_S \) is ample, by \text{Lemma 2.14} \( m > 0 \). By \text{Proposition 2.12} we have
\[
1 < \sum (1 - \frac{v_j + u_j}{q_j}).
\]
Since \( E.D_4 = 2 \), this is impossible by the computation in Table I. \( \square \)

\textbf{Lemma 2.16.} Let \( S \) be a \( \mathbb{Q} \)-homology projective plane with cyclic quotient singularities, and \( S' \) be its minimal resolution. Assume that there exists a \((-1)\)-curve \( E \) on \( S' \) such that the dual graph of \( D + E \) forms a cycle where \( D \) is the \( f \)-exceptional divisor of a singular point of \( S \). Then, \( K_S \) is not ample.
Proof. By Proposition 2.12, we have
\[ m_E K^2 = 1 - \frac{q_1 + q_2 + 2}{q} \quad \text{and} \quad 0 \leq m_E^2 K_S^2 = \frac{q_1 + q_2 + 2}{q} - 1. \]

Now Lemma 2.14 implies that \( K_S \) is not ample. \( \Box \)

**Lemma 2.17.** Let \( S \) be a \( \mathbb{Q} \)-homology projective plane with exactly four cyclic quotient singular points \( p_1, p_2, p_3, p_4 \) of orders \( (2, 3, 5, q) \) with \( q \geq 2 \).

1. Assume that \( K_S \) is ample. If \( p_4 \) is a rational double point, then the singularities of \( S \) are of type \( A_1 + \frac{1}{5}(1, 1) + \frac{1}{5}(1, 1) + A_8 \) and \( E.D \geq 3 \) for any \(-1\)-curve \( E \).

2. There exists no \(-1\)-curve \( E \) with \( E.D = E.D_1 = 2 \) where \( D_1 \) is an end component of \( D_4 \).

**Proof.** (1) If \( p_3 \) is of type \( \frac{1}{5}(1, 1) \), then \( tr_{D_4} \geq 3l_{D_4} - 8 \) by Lemma 2.14. But, since \( p_4 \) is a rational double point, \( tr_{D_4} = 2l_{D_4} \). Thus, we have \( l \leq 8 \), hence \( L \leq 11 \). Then, by Lemma 2.5 (1), \( p_4 \) is of type \( A_1 + \frac{1}{5}(1, 1) + \frac{1}{5}(1, 1) + A_8 \). If \( E.D \leq 2 \), then by Lemma 2.5 \( E.D_4 = 2 \). Since \( D_4 \) is an exceptional divisor of a rational double point, \( m_E K_S^2 = -1 \) by Proposition 2.12. By Lemma 2.14, \( -K_S \) is ample, a contradiction.

If \( p_3 \) is not of type \( \frac{1}{5}(1, 1) \), then by a similar argument as before using Lemma 2.7, we have \( L = 8 \) or \( 9 \), a contradiction.

(2) Assume that such a curve \( E \) exists. Let \( n_1 := -D_1^2 \). By Proposition 2.12
\[ 0 < m_E K_S^2 = \frac{4q_1}{q} - 1 = \frac{4q_1 - q}{q} = \frac{(4 - n_1)q_1 + q_1}{q}, \]
hence \( n_1 \leq 4 \). By contracting \( E \), we get a minimal resolution \( \tilde{S}' \) of another rational \( \mathbb{Q} \)-homology projective plane \( \tilde{S} \). We use the same notation for the new surfaces and the curves lying on it with the bar above the letter. We do not put the bar for \( q_1 \), \( q_2 \) and \( q \) for readability since it does not cause any confusion. Let \( \tilde{C} := \tilde{D}_1 \).

Consider the case \( n_1 = 4 \). Then, \( \tilde{C}^2 = 0 \). By Proposition 2.12, we have
\[ m_{\tilde{C}} K_{\tilde{S}}^2 = 1 - \frac{q_1 + 1}{q} = \frac{q - q_1 - 1}{q} \quad \text{and} \quad m_{\tilde{C}}^2 K_{\tilde{S}}^2 = \frac{q_1}{q}. \]

We claim that \( q - q_1 - 1 > 0 \). If not, then \( q = q_1 + 1 \), so \( p_4 \) is a rational double point. By Lemma 2.7, \( K_{\tilde{S}}^2 \neq 0 \), so \( K_S \) is not numerically trivial. But since \( m_{\tilde{C}} K_{\tilde{S}}^2 = 0 \), Lemma 2.14 gives a contradiction.

Now, we see that
\[ m_{\tilde{C}} = \frac{q_1}{q - q_1 - 1} > 0, \]
thus \( K_{\tilde{S}} \) is ample by Lemma 2.14. Note that since \( q = 4q_1 - q_{1,2} > 3q_1 \), so \( q \geq 3q_1 + 1 \), hence \( q - q_1 - 1 \geq 2q_1, \) Moreover, since \( 4q_1 > q \),
\[ K_{\tilde{S}}^2 = \frac{(q - q_1 - 1)^2}{q q_1} \geq \frac{4q_1^2}{qq_1} > 1, \]

**Table 1.**

| \( j \) | \( 3, 2, 2, 2, 2, 2, 2, 2 \) |
|---|---|
| \( 1 - \frac{v_j + u_j}{q} \) | \( 9/19 \) | \( 8/19 \) | \( 7/19 \) | \( 6/19 \) | \( 5/19 \) | \( 4/19 \) | \( 3/19 \) | \( 2/19 \) | \( 1/19 \) |
a contradiction by Lemma 2.7.

Consider the case $n_1 = 3$. Then, $C^2 = 1$. By Proposition 2.12 we have

$$m_C K_S^2 = \left(1 - \frac{q_1 + 1}{q}\right) - 1 = -\frac{q_1 + 1}{q} \quad \text{and} \quad m_C^2 K_S^2 = 1 + \frac{q_1}{q} = \frac{q + q_1}{q}.$$ 

So, $m = -\frac{q + q_1}{q_1 + 1}$, hence, by Lemma 2.14 $S$ is a log del Pezzo surface of Picard number one. Moreover,

$$K_S^2 = \frac{(q_1 + 1)^2}{q(q + q_1)}.$$ 

By [HK2] Lemma 3, the integer $30qK_S^2$ is a square number, thus so is $30(q_1 + q)$. Then, $gcd(q, 30) \neq 1$. Indeed, if otherwise, such a surface does not exist by [HK5], as a solution to the algebraic Montgomery-Yang problem for log del Pezzo surfaces of Picard number one. Thus, $q$ is divisible by 2, 3, or 5. It is easy to see that none of them is possible since $gcd(q, q_1) = 1$. For instance, if $q$ is even, so is $q_1$ since $30(q_1 + q)$ is a square. But this is a contradiction since $gcd(q, q_1) = 2 \neq 1$.

Consider the case $n_1 = 2$. By Corollary 2.7, it is not hard to see that

$$q_1 + q_2 + 2 \equiv \bar{q}_1 + \bar{q}_2 + 2 \quad \text{mod} \quad \frac{q}{q}.$$ 

We claim that this is equivalent for $p_4$ to be a rational double point. Then, since $L \geq 11$ by Lemma 2.5, this leads to a contradiction by Corollary 2.7. For example, if $p_3$ is of type $\frac{1}{3}(1, 1)$, then since $tr = 3l - 7$, we have $l = 7$, hence $L \leq 10$, a contradiction by Corollary 2.5.

Now we prove the claim. It is easy to see that $q_1 = \bar{q}_1$, $q_1 = 2\bar{q}_1 - \bar{q}_{1, i}$ and $q = 2\bar{q} - \bar{q}_1$. Thus,

$$\frac{q_1 + q_2 + 2}{q} \equiv \frac{\bar{q}_1 + \bar{q}_2 + 2}{\bar{q}} = \frac{\bar{q} + 2\bar{q}_1 - \bar{q}_{1, i} + 2}{2\bar{q} - \bar{q}_1} - \frac{\bar{q}_1 + \bar{q}_2 + 2}{\bar{q}}.$$ 

Multiplying $\bar{q}(2\bar{q} - \bar{q}_1)$, by using the well-known identity $\bar{q}_1\bar{q}_1 = \bar{q}\bar{q}_{1, i} + 1$, we can easily see that the above expression becomes

$$(\bar{q} - \bar{q}_1) - 1)^2$$

which equals to 0 if and only if $p_4$ is a rational double point. $\square$

3. SINGULAR FIBERS OF A $\mathbb{P}^1$-FIBRATION

We study singular fibers of a $\mathbb{P}^1$-fibration on the minimal resolution of some rational $\mathbb{Q}$-homology projective planes. See [M] or [GMM] for basics about $\mathbb{P}^1$-fibations on rational surfaces.

In this section, we always denote by $S'$ the minimal resolution of a rational $\mathbb{Q}$-homology projective plane of log general type with 4 cyclic singularities of orders $(2, 3, 5, q)$ where $q \geq 20$ and the order 3 singularity is of type $\frac{1}{3}(1, 1)$. Assume that $S'$ admits a $\mathbb{P}^1$-fibration $\Phi : S' \to \mathbb{P}^1$ that has at most four horizontal components; at most one of them being a 2-section and the rests being ordinary sections. Assume further that $D_1, D_2$ and $D_3$ belong to fiber components of $\Phi$.

**Lemma 3.1.** Let $F$ be a singular fiber of $\Phi$ containing a connected component of $\mathcal{D}$, say $\mathcal{D}_k$, where $k \neq 4$. Let $E$ be a $(-1)$-curve in $F$ intersecting $\mathcal{D}_k$. Then, we have the following.

1. $E.(F - E) \leq 2$.
2. $E$ intersects a horizontal component of $\Phi$. 

Proof. (1) If \( E, (F - E) \geq 3 \), then \( F^2 > 0 \), a contradiction.

(2) By Lemma 2.3, \( E.D \geq 3 \). By (1), there exists a component of \( D \) that is a horizontal component of \( \Phi \) intersecting \( E \).

\[ \square \]

Lemma 3.2. Let \( F \) be the singular fiber of \( \Phi \) containing the \((-3)\)-curve \( D_2 \). Then, \( \Phi \) has a 2-section and we have the following two cases for the configuration of \( F \).

1. \( F = E_1 + A + 2E_2 + B \) whose dual graph is the form \(-1\) \(-3\) \(-1\) \(-2\)
\[ E_1 \ A \ E_2 \ B \]
where \( B \) is a component of \( D_1 \) or \( D_4 \).

2. \( F = E_1 + A + 2E_2 + B \) whose dual graph is the form \(-1\) \(-3\) \(-1\) \(-2\) \(-4\) \(-1\) \(-2\)
\[ C_1 \ E_1 \ E_2 \ C_2 \]
where \( C_1 \) intersects \( E_2 \), or \( B \) is obtained by a finite sequence of blowups at the intersection point of two irreducible curves in the previous chain of \( B \) one of them being a \((-1)\)-curve.

In both cases, we have \( s_1, E_1 = s_2, E_1 = 1 \) and \( s_2, E_2 = 1 \) where \( s_1 \) and \( s_2 \) are sections of \( \Phi \); and \( s \) is a 2-section of \( \Phi \).

Proof. Let \( t \) be the number of \((-1)\)-curves in \( F \) intersecting \( D_2 \). Note that \( 1 \leq t \leq 3 \) since otherwise we have \( F^2 \neq 0 \). If \( t = 1 \), then \( E \) has multiplicity 3, thus it should intersect a horizontal component of \( \Phi \) with multiplicity at least 3, a contradiction.

If \( t = 3 \), then \( F \) is of the form \( D_2 + E_1 + E_2 + E_3 \) where \( E_1, E_2 \) and \( E_3 \) are disjoint \((-1)\)-curves intersecting the \((-3)\)-curve \( D_2 \). By Lemma 2.3 (3), each of \( E_1, E_2 \) and \( E_3 \) should intersect at least two more components of \( D \) that are horizontal components of \( \phi \). This is a contradiction since \( \Phi \) has at most 4 horizontal components and at most one 2-section.

Thus, \( t = 2 \). Now, the configuration of \( F \) is of the form (2) in the statement. Note that \( E_1 \) has multiplicity one and \( E_2 \) has multiplicity two. Moreover, \( B^2 = -2 \).

If \( B \) consists of one irreducible component, we arrive at (1). In this case, \( B \) is not a component of \( D_3 \) since every irreducible component of \( D_3 \) belongs to one fiber. \( \square \)

From now on, assume that \( S' \) admits a \( \mathbb{P}^1 \)-fibration \( \Phi : S' \to \mathbb{P}^1 \) that has three horizontal components consisting of two sections \( s_1 \) and \( s_2 \); and a 2-section \( s \).

Lemma 3.3. Let \( F \) be a singular fiber of \( \Phi \) containing a connected component \( D_k \) of \( D \) where \( k \neq 4 \). Let \( E \) be a \((-1)\)-curve in \( F \) intersecting \( D_k \), and \( t \) be the number of all such \((-1)\)-curves. Then, we have the following.

1. We have \( 1 \leq t \leq 4 \), and \( t = 4 \) if and only if every \((-1)\)-curve in \( F \) has multiplicity one.

2. The multiplicity of \( E \) in \( F \) is at most 2.

3. There exists at most one \((-1)\)-curve of multiplicity 2 in \( F \).

Proof. (1) Since \( F \) is a singular fiber of \( \Phi \), \( t \geq 1 \). Since \( \Phi \) has three horizontal components with one 2-section, we have the result by Lemma 3.1 (2).

(2) It follows by Lemma 3.1 (2) since any horizontal component of \( \Phi \) has multiplicity at most two.

(3) It follows by Lemma 3.1 (2) since \( \Phi \) has exactly one 2-section. \( \square \)

Now we describe the configuration of possible singular fibers of \( \Phi \) together with the horizontal components.
Lemma 3.4. Let $F$ be a singular fiber of $\Phi$ containing $D_3$. Then, we have the following two cases.

1. $p_3$ is of type $A_4$ and $F = E_1 + A_1 + A_2 + A_3 + A_4 + E_2$ whose dual graph is of the form $\frac{1}{-1} - \frac{-2}{2} - \frac{-2}{2} - \frac{-2}{2} - \frac{-1}{1}$. Here, $s_1.E_1 = s_2.E_1 = 1$ and $s.E_2 = 2$.

2. $p_3$ is of type $\frac{1}{-1}(1, 2)$ and $F = E_1 + A_1 + A_2 + 2E_2 + B$ whose dual graph is of the form $\frac{-1}{E_1} - \frac{-2}{A_1} - \frac{-3}{A_2} - \frac{-1}{A_3} - \frac{-2}{A_4}$. Here, $s_1.E_1 = s_2.E_1 = 1$ and $s.E_2 = 1$.

Proof. Let $t$ be the number of $(-1)$-curves in $F$ intersecting $D_3$.

Assume that $p_3$ is of type $\frac{1}{-1}(1, 1)$. By Lemma 3.3 (1), $t \leq 4$. If $t = 1$, then the unique $(-1)$-curve intersecting $D_3$ has multiplicity 5, a contradiction to Lemma 3.3 (2). Similar argument shows that $t \neq 2$. If $t = 3$, then the three $(-1)$-curves intersecting $D_3$ have multiplicity 1, 2 and 2, respectively. This gives a contradiction by Lemma 3.3 (3). If $t = 4$, then similar argument leads to a contradiction by Lemma 3.3 (1).

Assume that $p_3$ is of type $A_4$. Let $E$ be a $(-1)$-curve intersecting

$$D_3 = \frac{-2}{A_1} - \frac{-2}{A_2} - \frac{-2}{A_3} - \frac{-2}{A_4}.$$

Then, $E.A_2 = E.A_3 = 0$ since otherwise we have $F^2 > 0$. Similarly, we have $t \leq 2$. If $t = 1$, then $E$ has multiplicity 5, a contradiction to Lemma 3.3 (2). If $t = 2$, then we arrive at the case (1) in the statement.

Assume that $p_3$ is of type $\frac{1}{-1}(1, 2)$. Then, $D_3 = \frac{-2}{A_1} - \frac{-3}{A_2}. By Lemma 3.3 (1), t \leq 4$.

Furthermore, $t \neq 4$ since $F^2 = 0$. If $t = 1$, then the unique $(-1)$-curve in $F$ has multiplicity 5, a contradiction to Lemma 3.3 (2).

Assume that $t = 3$. Let $E_1$, $E_2$ and $E_3$ be the $(-1)$-curves in $F$. Then, the support of $F$ consists of $E_1$, $E_2$, $E_3$ and $D_3$. Moreover, $E_i$ has multiplicity one for each $i = 1, 2, 3$. This is a contradiction by Lemma 3.1 since $\Phi$ has 3 horizontal components with only one 2-section.

Assume that $t = 2$. Then, $F$ is of the form $\frac{-1}{E_1} - \frac{-2}{A_1} - \frac{3}{A_2} - \frac{-1}{A_3} - \frac{-2}{A_4}$ where $B$ is a tree of rational curves with $B^2 = -2$. By Lemma 2.6 (3), we see that $E_1$ intersects two sections $s_1$ and $s_2$, and $E_2$ intersects the 2-section $s$. Thus, $B$ does not contain a component of $D_4$. Hence, $B = D_1$.

4. Proof of Theorem 1.6

Under the situation in Notation 2.1, we further assume that $S$ is of log general type and $H_1(S^{sm}, \mathbb{Z}) = 0$ where $S^{sm}$ denotes the smooth locus of $S$. Then, we may assume that the orders of the local fundamental groups of the singular points of $S$ is $(2, 3, 5, q)$ for some integer $q$ with $gcd(q, 30) = 1$ and the order 3 singular point is of type $\frac{1}{-1}(1, 1)$ by Lemma 2.8 and Lemma 2.15. By Lemma 2.8 and Lemma 2.17 we may assume that $L \geq 12$ and $q \geq 20$ except for the cases (b), (c) and (d) in Lemma 2.8. Let $D_1, \ldots, D_l$ be irreducible components of $D_4$ that form a chain of rational curves in this order.

Let $E$ be a $(-1)$-curve on $S'$ with $E.D \leq 2$. By Corollary 2.2 we may assume that $E.D = 2$. By Lemma 2.6 $E$ intersects two components $D_i$ and $D_k$ of $D_4$ with $i \leq k$. 


4.1. The case $i = k$. The case $i = k = 1$ is considered in Lemma 2.17 (2) in a slightly more general context. By symmetry, we may assume that $1 < i = k < l$. Then, by contracting $E$ on $S'$, we get a minimal resolution of a rational $\mathbb{Q}$-homology projective plane with 5 quotient singularities, a contradiction to Theorem 2.1.

4.2. The case $i < k$. By [Z] Lemma 4.1, which can also be proven using Proposition 2.12 as mentioned in [HK5] Lemma 3.2, we see that $D_i^2 = -2$ or $D_k^2 = -2$.

4.2.1. The case $1 = i < k \leq l$. By Lemma 2.16 we may assume that $k < l$. If $D_i^2 \leq -3$ and $D_k^2 = -2$, then by contracting $E$, we get a minimal resolution of a rational $\mathbb{Q}$-homology projective plane with 5 quotient singularities, a contradiction to Theorem 2.1. This shows that $D_i^2 = -2$.

Assume that $D_k^2 < -2$. Contracting $E$ and then the new $(-1)$-curves, if they exist, which are images of the components of $D_4$, we arrive at the following three cases: (1) $D_1^2 = D_2^2 = -2$, (2) $i = k = 1$, or (3) $D_i^2 \leq -3$ and $D_k^2 = -2$. Here, we need some care since the new $\mathbb{Q}$-homology projective plane might not be of log general type. Case (3) is impossible by the same argument as in the beginning of 4.2.1, and Case (2) was treated in Lemma 2.17 (2).

Thus, it remains to consider the case $D_1^2 = D_2^2 = -2$. If $k = 2$, then by contracting $E$ and the image of $D_1$, we get a minimal resolution of another rational $\mathbb{Q}$-homology projective plane $T$. Since $D_1^2 = D_2^2 = -2$, we see that $K_2^2 < 0$ by Lemma 2.6 and Corollary 2.7 a contradiction. If $k > 2$, then $D_1 + 2E + D_k$ induces a $\mathbb{P}^1$-fibration $\Phi : S' \to \mathbb{P}^1$ which has only ordinary sections. Consider the fiber $F$ containing the $(-2)$-curve $D_2$. By Lemma 3.2, $\Phi$ has a 2-section, a contradiction.

4.2.2. The case $1 < i < k < l$. It is easy to see that $D_i^2 = D_k^2 = -2$ as in the beginning of 4.2.1. We claim that if $k = i + 1$, then $p_4$ is a rational double point, i.e., it is of type $A_1$. This leads to a contradiction by Lemma 2.17. To prove the claim, we first want to show that $D_{i-1}^2 = -2$. If otherwise, contracting $E$ and then the image of $D_i$, we get a minimal resolution of a rational $\mathbb{Q}$-homology projective plane with 5 quotient singularities, a contradiction to Theorem 2.1. By a similar argument, we see that $D_i^2 = D_2^2 = \ldots = D_{i-2}^2 = -2$, by symmetry, we also have $D_{k+1}^2 = D_{k+2}^2 = \ldots = D_l^2 = -2$.

Thus, we have $k \geq i + 2$. Then, $D_i + 2E + D_k$ induces a $\mathbb{P}^1$-fibration $\Phi : S' \to \mathbb{P}^1$. Since $D_i + 2E + D_k$ forms one singular fiber of $\Phi$, we see that there are at most 4 horizontal components, each of them is a section or possibly at most one double section of the $\mathbb{P}^1$-fibration $\Phi$. Note that the other three connected components $D_1 + D_2 + D_3$ form part of the fiber components. Take a fiber $F$ containing $D_2$. By Lemma 3.2, $\Phi$ has a 2-section. Then, the 2-section is $D_{i+1}$ and $k = i + 2$. Thus, $\Phi$ has exactly three horizontal components, two of them being ordinary sections and the remaining one being a 2-section.

Since $p_3$ is not of type $A(1, 1)$ by Lemma 3.4, $F$ has a component of $D_1$ or $D_4$ by Lemma 3.4. Since the $(-2)$-curve in $F$ does not intersect a horizontal component, $F$ does not contain a component of $D_4$, hence $F$ is of the form in Lemma 3.2 (1) with $B = D_1$. Let $G$ be the singular fiber of $\Phi$ containing $D_1$. Then, since $D_1$ and $D_2$ belong to the same singular fiber, we see that $F$ is of the form in Lemma 3.3 (1) by Lemma 2.16. In particular, $p_3$ is of type $A_4$. By Lemma 2.5 (1), $l \geq 6$. 
It remains to analyze the singular fibers with components from $D_4$. Since there is only one multiple section that is a 2-section, $D_4$ is of the form

$$-2 - a - b - c - 2 - 2 - a - b - c - 2 - 2$$

for some integers $a, b, c \geq 2$ with $a + b + c = 10$ or 11. Note that $D_1 + 2E + D_7$ forms the remaining singular fiber of $\Phi$ for some $(-1)$-curve $E$, and we may regard $s_1 = D_2$, $s_2 = D_6$ and $s = D_4$. This cannot happen by Corollary 2.11. This completes the proof.

Acknowledgements. This research was supported by the Samsung Science and Technology Foundation under Project SSTF-BA1602-03.

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