Heat Kernel Estimate in a Conical Singular Space

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Abstract
Let \((X, g)\) be a product cone with the metric \(g = dr^2 + r^2 h\), where \(X = C(Y) = (0, \infty)_r \times Y\) and the cross section \(Y\) is a \((n-1)\)-dimensional closed Riemannian manifold \((Y, h)\). We study the upper boundedness of heat kernel associated with the operator 
\[ L^V = -\Delta_{g} + V_0 r^{-2}, \]
where \(-\Delta_{g}\) is the positive Friedrichs extension Laplacian on \(X\) and \(V = V_0(y) r^{-2}\) and \(V_0 \in C^\infty(Y)\) is a real function such that the operator \(-\Delta_{h} + V_0 + (n-2)^2/4\) is a strictly positive operator on \(L^2(Y)\). The new ingredient of the proof is the Hadamard parametrix and finite propagation speed of wave operator on \(Y\).

Keywords
Heat kernel · Metric cone · Hadamard parametrix · Schrödinger operator

Mathematics Subject Classification
42B37 · 35Q40

1 Introduction

Let \((Y, h)\) be a \((n-1)\)-dimensional closed Riemannian manifold, we consider the product cone \(X = C(Y) = (0, \infty)_r \times Y\) and the metric \(g = dr^2 + r^2 h\). The product cone is an incomplete manifold, however, one can complete it to \(C^*_c(X) = C(Y) \cup P\) where \(P\) is its cone tip, see Cheeger [6, 7]. Let \(-\Delta_{g}\) denote the positive Friedrichs’ self-adjoint extension of Laplace-Beltrami operator from the domain \(C^\infty_c(X)\) that consist of the compactly supported smooth functions on the interior of the metric cone. One can write
\[ -\Delta_{g} = -\partial^2_r - \frac{n-1}{r} \partial_r + \frac{-\Delta_{h}}{r^2}, \] (1.1)
where $-\Delta_h$ is the positive Laplacian on the closed Riemannian manifold $Y$, see [8, p. 302] and [21, Theorem 2.1]. The heat kernel associated with the operator $-\Delta_g$ has been investigated, we refer to Mooers [21] and Nagase [22] for asymptotic expansion, to Li [18] for upper boundedness and to Coulhon-Li [4] for lower boundedness.

In this paper, we consider the heat kernel associated with the Schrödinger operator

$$L_V = -\Delta_g + V, \quad V = V_0(y)r^{-2}, \quad (1.2)$$

where $V_0(y)$ is a smooth function on the section $Y$ such that the operator $-\Delta_h + V_0 + (n-2)^2/4$ is a strictly positive operator on $L^2(Y)$ space. The decay of the perturbation potential considered is scaling critical and is closely related to the angular momentum as $r \to \infty$, hence the Schrödinger operator $L_V$ has attracted interest from other topics. For examples, we refer to [3, 26] for the asymptotical behavior of the Schrödinger propagator, to [14, 17] for the Riesz transform, to [11, 29–31] for the Strichartz estimates and the restriction estimates. In the present paper, we focus on the upper boundedness of heat kernel. More precisely, we prove

**Theorem 1.1** Let $L_V$ be the operator on metric cone of dimension $n \geq 2$ given in $(1.2)$ and suppose \( \{\lambda_k, \varphi_k\}_{k=0}^\infty \) to be the eigenvalues and eigenfunctions of the operator $-\Delta_h + V_0(y) + (n-2)^2/4$, which satisfies

$$\begin{cases}
( -\Delta_h + V_0(y) + (n-2)^2/4 ) \varphi_k(y) = \lambda_k \varphi_k(y) \\
\int_Y |\varphi_k(y)|^2 dy = 1,
\end{cases} \quad (1.3)$$

and the eigenvalues $\{\lambda_k\}_{k=0}^\infty$ enumerated such that

$$0 < \lambda_0 \leq \lambda_1 \leq \cdots \quad (1.4)$$

repeating each eigenvalue as many times as its multiplicity. Then, for $t > 0$ and $(r, y), (s, y') \in X$, the heat kernel can be written as

$$e^{-tL_V}(r, y; s, y') = e^{-\frac{r^2 + s^2}{4t}}(rs)^{-\frac{n-2}{2}} \sum_{k \in \mathbb{N}} \varphi_k(y)\varphi_k(y') I_{\mu_k} \left( \frac{rs}{2t} \right), \quad (1.5)$$

where $\mu_k = \sqrt{\lambda_k}$ and $I_\mu$ is the modified Bessel function of the first kind of order $\mu$. Furthermore, there exist positive constants $c$ and $C$ such that

$$\left| e^{-tL_V}(r, y; s, y') \right| \leq C \left[ \min \left\{ 1, \left( \frac{rs}{2t} \right) \right\} \right]^{-\sigma} t^{-\frac{n}{2}} e^{-\frac{d^2((r, y), (s, y'))}{2t}}, \quad (1.6)$$

where $\sigma = \frac{n-2}{2} - \mu_0$. Here, $d((r, y), (s, y'))$ is the distance between two points $(r, y), (s, y') \in X$

$$d((r, y), (s, y')) = \begin{cases}
\sqrt{r^2 + s^2 - 2rs \cos(d_h(y, y'))}, & d_h(y, y') \leq \pi; \\
r + s, & d_h(y, y') \geq \pi,
\end{cases} \quad (1.7)$$
and \( d_h \) is the distance on the section \( Y \).

**Remark 1.1** From (1.6), the square root of the smallest eigenvalue \( \lambda_0 \) plays a non-trivial role. In particular, when \( Y = \mathbb{S}^{n-1} \) and \( V_0(y) = a \) with \( a \geq -(n - 2)^2/4 \), then

\[
\sigma = \frac{n - 2}{2} - \sqrt{\frac{(n - 2)^2}{4}} + a,
\]

which is positive when \(-(n - 2)^2/4 \leq a < 0\) while is nonpositive when \( a \geq 0 \). Hence if \(- (n - 2)^2/4 \leq a < 0\), then

\[
\left[ \min \left\{ 1, \left( \frac{r}{2t} \right) \right\} \right]^{-\sigma} \leq C \left[ \min \left\{ \frac{r}{\sqrt{t}}, 1 \right\} \right]^{-\sigma} \left[ \min \left\{ \frac{s}{\sqrt{t}}, 1 \right\} \right]^{-\sigma},
\]

together with (1.6) shows

\[
|e^{-t(-\Delta + a|z|^2)}(z, z')| \leq C \left[ \min \left\{ \frac{|z|}{\sqrt{t}}, 1 \right\} \right]^{-\sigma} \left[ \min \left\{ \frac{|z'|}{\sqrt{t}}, 1 \right\} \right]^{-\sigma} t^{-\frac{n}{2}} e^{-\frac{|z-z'|^2}{ct}},
\]

which consists with the results of Liskevich-Sobol [19] and Milman-Semenov [20].

**Remark 1.2** The proof is based on the Hadamard parametrix and finite propagation speed of wave operator on \( Y \), which is a bit different from [18] due to the perturbation of the potential \( V \). In [18], Li used a theorem of Grigor’yan [12, Theorem 1.1] which claims that if the heat kernel \( H(t, r, y; s, y') \) on Riemannian manifolds satisfies on-diagonal bounds

\[
H(t, r, y; r, y) \lesssim t^{-\frac{n}{2}}, \quad H(t, s, y'; s, y') \lesssim t^{-\frac{n}{2}},
\]

then we have

\[
H(t, r, y; s, y') \lesssim t^{-\frac{n}{2}} \exp \left( - \frac{d(r, y; s, y')^2}{ct} \right).
\]

However, we do not know whether the analogs hold or not for the Laplacian with the perturbation of Hardy type potential. Therefore, we have to use a different argument instead.

**Remark 1.3** As an application of the estimate (1.6) of the heat kernel, one can use the argument of [15] and the Hardy inequality in [3] to establish the Littlewood-Paley theory, Sobolev embedding associated with the operator \( \mathcal{L}_V \). For example, we can prove the Mikhlin Multipliers estimates: Suppose \( m : [0, \infty) \rightarrow \mathbb{C} \) satisfies

\[
|\partial^j m(\lambda)| \lesssim \lambda^{-j} \quad \text{for all} \quad 0 \leq j \leq 3 + 3[\frac{n}{4}]. \tag{1.8}
\]

Then \( m(\sqrt{\mathcal{L}_V}) \) is a bounded operator on \( L^p(X) \) provided that either

\[ \square \] Springer
• $\mu_0 > (n-2)/2$ and $1 < p < \infty$, or
• $0 < \mu_0 \leq \frac{n-2}{2}$ and $p_0 < p < p_0' := \frac{n}{\sigma}$.

The number of derivatives required in (1.8) is far from sharp, we do not purchase the
the sharpness here. More precisely, for the cases that $\mu_0 > (n-2)/2$, one has the
exactly Gaussian upper bounds of heat kernel, since the factor
$[\min\left\{1, (\frac{r}{\sigma})^2\right\}]^{-\sigma} \leq 1$
in (1.6). By using Alexopoulos [1, Theorem 6.1], we have $m(\sqrt{L_v})$ is a bounded
operator on $L^p(X)$ for $1 < p < \infty$. When $0 < \mu_0 \leq \frac{n-2}{2}$, one can closely follow
the argument of Killip, Miao, Visan, Zheng and the last author [15, Theorem 4.1] to
prove that $m(\sqrt{L_v})$ is $L^p(X)$ for $p_0 < p < p_0'$. In particular, the semigroup $e^{-tL_v}$
is a multiplier satisfying (1.8), hence it is bounded operator on $L^p(X)$ with above $p$.

2 Construction of the Heat Kernel

In this section, in spirit of functional calculus in Cheeger-Taylor [8, 9], we construct the
heat kernel associated with the operator $L_v$. Since the metric $g = dr^2 + r^2h(y, dy)$,
we write the operator $L_v$ in the coordinate $(r, y)$,

$$L_v = -\partial_r^2 - \frac{n-1}{r} \partial_r + \frac{1}{r^2} \left( -\Delta_h + V_0(y) \right), \quad (2.1)$$

where $\Delta_h$ is the Laplace-Beltrami operator on $(Y, h)$.

From classical spectral theory, the spectrum of $-\Delta_h + V_0(y) + (n-2)^2/4$ is formed
by a countable family of real eigenvalues $\{\lambda_k\}_{k=0}^\infty$ enumerated such that

$$0 < \lambda_0 \leq \lambda_1 \leq \cdots, \quad (2.2)$$

where we repeat each eigenvalue as many times as its multiplicity, and $\lim_{k\to\infty} \lambda_k = +\infty$. Let $\{\varphi_k(y)\}$ be the eigenfunctions of $-\tilde{\Delta}_h = -\Delta_h + V_0(y) + (n-2)^2/4$, that is

$$\begin{cases} (-\Delta_h + V_0(y) + (n-2)^2/4)\varphi_k(y) = \lambda_k \varphi_k(y), \\
\int_Y |\varphi_k(y)|^2 \, dy = 1. \end{cases} \quad (2.3)$$

Therefore, we obtain an orthogonal decomposition of the $L^2(Y)$ in a sense that

$$L^2(Y) = \bigoplus_{k \in \mathbb{N}} \mathcal{H}^k, \quad \mathbb{N} = \{0, 1, 2, \cdots\},$$

where

$$\mathcal{H}^k = \text{span}\{\varphi_k\}.$$
Define the orthogonal projection $\pi_k$ of $f$ onto $\mathcal{H}^k$ by
\begin{equation}
\pi_k f = \int_Y f(r, y') H_k(y, y') dh, \quad f \in L^2(X),
\end{equation}
where $dh$ is the measure on $Y$ under the metric $h$ and the kernel
\begin{equation}
H_k(y, y') = \varphi_k(y) \overline{\varphi_k(y')},
\end{equation}
then
\begin{equation}
f(r, y) = \sum_{k \in \mathbb{N}} \pi_k f = \sum_{k \in \mathbb{N}} a_k(r) \varphi_k(y), \quad a_k(r) = \int_Y f(r, y') \overline{\varphi_k(y')} dh.
\end{equation}

Let $\mu = \mu_k = \sqrt{\lambda_k}$, for $f \in L^2(X)$, as [2, Page 523], we define the Hankel transform of order $\mu$
\begin{equation}
(\mathcal{H}_\mu f)(\rho, y) = \int_0^\infty (\rho \rho)^{-\frac{n-2}{2}} J_\mu(\rho) f(r, y) r^{n-1} dr,
\end{equation}
where the Bessel function of order $\mu$ is given by
\begin{equation}
J_\mu(r) = \frac{(r/2)^\mu}{\Gamma(\mu + 1/2) \Gamma(1/2)} \int_{-1}^1 e^{isr} (1 - s^2)^{(2\mu-1)/2} ds, \quad \mu > -1/2, \ r > 0,
\end{equation}
which satisfies the following equation
\[ r^2 \frac{d^2}{dr^2} (J_\mu(r)) + r \frac{d}{dr} (J_\mu(r)) + (r^2 - \mu^2) J_\mu(r) = 0. \]

Following the [25, (8.45)], for well-behaved functions $F$ (say Borel measured functions), we have by the functional calculus
\begin{equation}
F(\mathcal{L}_V) f(r, y) = \int_0^\infty \int_0^{2\pi} K(r, y, s, y') f(s, y') s^{n-1} ds \ dh(y'),
\end{equation}
where the kernel
\[ K(r, y, s, y') = (rs)^{-\frac{n-2}{2}} \sum_{k \in \mathbb{N}} \varphi_k(y) \overline{\varphi_k(y')} K_{\mu_k}(r, s) \]
\[ = (rs)^{-\frac{n-2}{2}} \sum_{k \in \mathbb{N}} H_k(y, y') K_{\mu_k}(r, s), \]
and
\[
K_{\mu_k}(r, s) = \int_{0}^{\infty} F(\rho^2) J_{\mu_k}(r \rho) J_{\mu_k}(s \rho) \rho \, d\rho.
\]  
(2.10)

In particular, \( F(\rho^2) = e^{-t \rho^2} \), by using Weber’s second exponential integral [27, Section 13.31 (1)], we obtain
\[
K_{\mu_k}(r, s) = \int_{0}^{\infty} e^{-t \rho^2} J_{\mu_k}(r \rho) J_{\mu_k}(s \rho) \rho \, d\rho = (2t)^{-1} e^{-\frac{r^2 + s^2}{4t}} I_{\mu_k}\left(\frac{rs}{2t}\right), \quad t > 0,
\]  
(2.11)

where \( I_{\mu}(x) \) is the modified Bessel function of the first kind in series version
\[
I_{\mu}(x) = \sum_{j=0}^{\infty} \frac{1}{j! \Gamma(\mu + j + 1)} \left(\frac{x}{2}\right)^{\mu + 2j},
\]
or in the integral representation
\[
I_{\mu}(x) = \frac{1}{\sqrt{\pi} \Gamma(\mu + \frac{1}{2})} \left(\frac{x}{2}\right)^{\mu} \int_{-1}^{1} e^{-\tau^2} (1 - \tau^2)^{\mu - \frac{1}{2}} \, d\tau.
\]
Therefore we obtain (1.5)
\[
e^{-t \mathcal{L}V}(r, y; s, y') = (2t)^{-1} e^{-\frac{r^2 + s^2}{4t}} (rs)^{-\frac{n-2}{2}} \sum_{k \in \mathbb{N}} \varphi_k(y) \overline{\varphi_k(y')} I_{\mu_k}\left(\frac{rs}{2t}\right).
\]

3 The Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. To this end, by observing (1.7) and scaling \((r, s)\), it suffices to show
\[
\left| e^{-\frac{r^2 + s^2}{4t}} (rs)^{-\frac{n-2}{2}} \sum_{k \in \mathbb{N}} \varphi_k(y) \overline{\varphi_k(y')} I_{\mu_k}\left(\frac{rs}{2t}\right) \right|
\leq C \left[ \min \left\{ 1, \left(\frac{rs}{2}\right)^{\frac{2}{\sigma}} \right\} \right]^{\sigma} e^{-\frac{d^2((r, r), (s, s'))}{c}},
\]  
(3.1)

which is the consequence of the following lemma

Lemma 3.1 Let \( z = \frac{1}{2} rs, \sigma = \frac{n-2}{2} - \mu_0 \) and \( \delta = d_h(y, y') \), then there exist constants \( C \) and \( N \) only depending on \( n \) such that
• either for $0 < z \leq 1$,
\[
|z^{-\frac{n-2}{2}} \sum_{k \in \mathbb{N}} \phi_k(y) \overline{\phi_k(y')} I_{\mu k}(z)| \leq C z^{-\sigma} \times \begin{cases} e^{z \cos \delta}, & 0 \leq \delta \leq \pi, \\ 1 & \pi \leq \delta, \end{cases}
\] (3.2)

• or for $z \gtrsim 1$
\[
|z^{-\frac{n-2}{2}} \sum_{k \in \mathbb{N}} \phi_k(y) \overline{\phi_k(y')} I_{\mu k}(z)| \\
\leq C \times \begin{cases} e^{z \cos \delta} + z^N e^{z \cos (\frac{1}{2} \epsilon_0)}, & 0 \leq \delta \leq \frac{1}{2} \epsilon_0, \\ z^N e^{z \cos \delta}, & \frac{1}{2} \epsilon_0 \leq \delta \leq \pi, \\ e^\frac{z}{2} & \pi \leq \delta, \end{cases}
\] (3.3)

where $\epsilon_0$ is the injectivity radius of the manifold $Y$ with $0 < \epsilon_0 \leq \pi$.

**Remark 3.1** If the injectivity radius $\epsilon_0$ of the manifold $Y$ is larger than $\pi$, we do not need to introduce the cutoff function $\chi$ in the Step 1 below, and we can follow the same argument to obtain, for $z \gtrsim 1$
\[
|z^{-\frac{n-2}{2}} \sum_{k \in \mathbb{N}} \phi_k(y) \overline{\phi_k(y')} I_{\mu k}(z)| \leq C \times \begin{cases} e^{z \cos \delta}, & 0 \leq \delta \leq \pi, \\ e^{\frac{z}{2}} & \pi \leq \delta, \end{cases}
\] (3.4)

which is better than (3.3). Hence, we omit this easier case in the following argument.

We shall postpone the proof of Lemma 3.1 for a moment and first see how we can use it to prove (3.1).

**Proof of (3.1)** We divide into two cases $z \leq 1$ and $z \geq 1$ where $z = \frac{r s}{2}$. When $z \leq 1$, we recall the distance (1.7) and put (3.2) into the left hand side of (3.1) to obtain
\[
\left| e^{-\frac{r^2+s^2}{2} (\nabla_s)} \sum_{k \in \mathbb{N}} \phi_k(y) \overline{\phi_k(y')} I_{\mu k}(z) \right| \leq C \left( \frac{r s}{2} \right)^{-\sigma} e^{-\frac{\rho((r,s),(y,y'))}{c_k}}. \] (3.5)

Next, we consider the case that $z \geq 1$. In the subcase that $0 \leq \delta \leq \frac{1}{2} \epsilon_0$ of (3.3), we have
\[
\text{LHS of (3.1)} \lesssim e^{-\frac{r^2-2r \cos \delta + s^2}{4} + z^N e^{-\frac{r^2-2r \cos (\frac{1}{2} \epsilon_0)+s^2}{4}}}
\lesssim e^{-\frac{r^2-2r \cos \delta + s^2}{8}} (1 + z^N e^{-\frac{z}{4} (1-\cos (\frac{1}{2} \epsilon_0))}) \lesssim e^{-\frac{\rho((r,s),(y,y'))}{c_k}}.
\]

In the subcase that $\frac{1}{2} \epsilon_0 \leq \delta \leq \pi$ of (3.3), we have
\[
\text{LHS of (3.1)} \lesssim z^N e^{-\frac{r^2-2r \cos \delta + s^2}{4}} \lesssim z^N e^{-\frac{z}{4} (1-\cos \delta)} e^{-\frac{r^2-2r \cos \delta + s^2}{8}}.
\]
Since \( \frac{1}{2} \epsilon_0 \leq \delta \leq \pi \), one has \( 1 - \cos \delta \geq \epsilon^2/4 > 0 \). Hence, for \( z \geq 1 \), no matter how large \( N \) is, there exists a constant \( C \) independent of \( z \) such that

\[
\text{LHS of (3.1)} \leq C e^{-\frac{\epsilon^2 - 2r \cos \delta + \frac{r^2}{8}}{8}} \lesssim e^{-\frac{d^2((r, y), (s, y'))}{c}}.
\]

In the last subcase \( \delta \geq \pi \), we have

\[
\text{LHS of (3.1)} \leq C e^{-\frac{r^2 + \frac{r^2}{8}}{8}} e^{-\frac{r^2 - 2r \cos \delta + \frac{r^2}{8}}{8}} \leq C e^{-\frac{(r+y)^2}{16}}.
\]

Therefore we prove (3.1) once we could prove (3.2) and (3.3), which are our main tasks from now on. To this end, we first claim that

\[
|H_k(y, y')| \leq \|\varphi_k(y)\|_{L^\infty(Y)}^2 \leq C(1 + \lambda_k)^{\frac{n-2}{2}} \leq C(1 + k)^{\frac{n-2}{n-1}}, \quad (3.6)
\]

where we used the eigenfunction estimate (see [24, (3.2.5)-(3.2.6)]) and the Weyl's asymptotic formula (e.g. see [28])

\[
\lambda_k \sim (1 + k)^{\frac{1}{n-1}}, \quad k \geq 1, \implies \mu_k \sim (1 + k)^{\frac{1}{n-1}}. \quad (3.7)
\]

\( \square \)

### 4 The Proof of Lemma 3.1

In this section, we give the proof of Lemma 3.1, we shall prove (3.2) and (3.3) separately.

**The proof of (3.2)** For the case \( 0 \leq \delta \leq \pi \), we first notice that \( e^z \lesssim e^{z \cos \delta} \) if \( 0 \leq z \leq 1 \), then the modified Bessel function satisfies

\[
|I_\mu(z)| \leq \sqrt{\pi} e^{z} \left(\frac{z}{2}\right)^\mu \Gamma(\mu + \frac{1}{2}) \lesssim e^{z \cos \delta} \frac{z^{\mu_0}}{2^{\mu} \Gamma(\mu + \frac{1}{2})}, \quad \mu \geq \mu_0. \quad (4.1)
\]

Therefore, from (3.6), we have

\[
\text{LHS of (3.2)} \lesssim z^{-\frac{n-2}{2}} e^{z \cos \delta} \sum_{k \in \mathbb{N}} (1 + k)^{\frac{n-2}{n-1}} \frac{z^{\mu_0}}{2^{\mu_k} \Gamma(\mu_k + \frac{1}{2})}.
\]

Note that \( \mu_k \geq \mu_0 \) and \( \mu_k = \sqrt{\lambda_k} \sim (1 + k)^{\frac{1}{n-1}} \), then the summation converges. Hence we show

\[
\text{LHS of (3.2)} \lesssim e^{z \cos \delta} z^{\mu_0 - \frac{n-2}{2}}, \quad 0 \leq \delta \leq \pi, \quad (4.2)
\]
as desired. For the case that $\delta \geq \pi$, we replace (4.1) by

$$|I_\mu(z)| \leq \sqrt{\pi} e^{\frac{z}{2}} \frac{(\frac{z}{2})^{\mu_0}}{\Gamma(\mu + \frac{1}{2})} \lesssim \frac{z^{\mu_0}}{2^\mu \Gamma(\mu + \frac{1}{2})}, \quad \mu \geq \mu_0.$$ 

Therefore, as before the summation converges, we obtain

$$\text{LHS of (3.2)} \lesssim z^{\mu_0 - \frac{n-2}{2}},$$

which complete the proof of (3.2).

\[ \square \]

The proof of (3.3) For $z \gtrsim 1$, we need the integral representation (see [27], [22, p. 419] or [10, (10.32.4)]) of the modified Bessel function

$$I_\mu(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos(\tau)} \cos(\mu \tau) d\tau - \frac{\sin(\mu \pi)}{\pi} \int_0^\infty e^{-z \cosh(\tau)} e^{-\tau \mu} d\tau. \quad (4.3)$$

We divide into three steps to prove (3.3).

**Step 1:** we consider the case $0 \leq \delta \leq \frac{1}{2} \epsilon_0$ which is the most difficult case. We first introduce a function $\chi \in C_c^\infty([0, \pi])$ such that

$$\chi(\tau) = \begin{cases} 1, & \tau \in [0, \frac{1}{2} \epsilon_0], \\ 0, & \tau \in [\epsilon_0, \pi]. \end{cases} \quad (4.4)$$

**Lemma 4.1** For fixed $0 \leq \delta \leq \pi$ and $m \geq 0$, let $\chi$ be in (4.4), we have

$$\frac{1}{\pi} \int_0^\pi e^{z(\cos(\tau - \cos \delta))} \cos(\mu \tau) d\tau - \frac{\sin(\mu \pi)}{\pi} \int_0^\infty e^{-z(\cosh(\tau + \cos \delta))} e^{-\tau \mu} d\tau$$

$$= \frac{1}{\pi} \int_0^\pi e^{z(\cos(\tau - \cos \delta))} \chi(\tau) \cos(\mu \tau) d\tau$$

$$+ \frac{(-1)^m}{\pi} \int_0^\pi \left( \frac{\partial}{\partial \tau} \right)^m (e^{z(\cos(\tau - \cos \delta))} (1 - \chi(\tau))) \frac{\cos(\mu \tau)}{\mu \tau} d\tau$$

$$- \frac{\sin(\mu \pi)}{\pi} \int_0^\infty \left( \frac{\partial}{\partial \tau} \right)^m (e^{-z(\cosh(\tau + \cos \delta))}) \frac{e^{-\tau \mu}}{\mu \tau} d\tau. \quad (4.5)$$

**Proof of Lemma 4.1** is a variant of [22, (5.30)]. We prove this lemma by using induction argument and the argument of [22]. We first verify $m = 1$. By integration by parts, we have

$$\frac{1}{\pi} \int_0^\pi e^{z(\cos(\tau - \cos \delta))} \cos(\mu \tau) d\tau - \frac{\sin(\mu \pi)}{\pi} \int_0^\infty e^{-z(\cosh(\tau + \cos \delta))} e^{-\tau \mu} d\tau$$

$$= \frac{1}{\pi} \int_0^\pi e^{z(\cos(\tau - \cos \delta))} \chi(\tau) \cos(\mu \tau) d\tau + \frac{1}{\pi} (e^{z(\cos(\tau - \cos \delta))} (1 - \chi(\tau))) \frac{\sin(\mu \tau)}{\mu} \bigg|_{\tau = 0}$$

$$+ \frac{(-1)^1}{\pi} \int_0^\pi \left( \frac{\partial}{\partial \tau} \right) (e^{z(\cos(\tau - \cos \delta))} (1 - \chi(\tau))) \frac{\sin(\mu \tau)}{\mu} d\tau.$$
\[
+ \frac{\sin(\mu \pi)}{\pi} \left( e^{-z(cosh \tau + cos \delta)} \right) \frac{e^{-\tau \mu}}{\mu} \bigg|_{\tau=0}^{\infty} \\
- \frac{\sin(\mu \pi)}{\pi} \int_0^\infty \left( \frac{\partial}{\partial \tau} \right) \left( e^{-z(cosh \tau + cos \delta)} \right) \frac{e^{-\tau \mu}}{\mu} d\tau.
\]

Note the boundary term

\[
\frac{1}{\pi} \left( e^{z(cos \tau - cos \delta)} (1 - \chi(\tau)) \right) \sin(\mu \tau) \bigg|_{\tau=0}^{\tau=\pi} \\
+ \frac{\sin(\mu \pi)}{\pi} \left( e^{-z(cosh \tau + cos \delta)} \right) \frac{e^{-\tau \mu}}{\mu} \bigg|_{\tau=0}^{\infty} \\
= \frac{1}{\pi} \left( e^{z(cos \tau - cos \delta)} \right) \sin(\mu \tau) \bigg|_{\tau=0}^{\tau=\pi} \\
- \frac{\sin(\mu \pi)}{\pi} \left( e^{-z(cosh \tau + cos \delta)} \right) \frac{e^{-\tau \mu}}{\mu} \bigg|_{\tau=\tau}^{\tau=0} = 0.
\]

By integration by parts again, we have

\[
\frac{1}{\pi} \int_0^\pi e^{z(cos \tau - cos \delta)} \cos(\mu \tau) d\tau - \frac{\sin(\mu \pi)}{\pi} \int_0^\infty e^{-z(cosh \tau + cos \delta)} e^{-\tau \mu} d\tau \\
= \frac{1}{\pi} \int_0^\pi e^{z(cos \tau - cos \delta)} \chi(\tau) \cos(\mu \tau) d\tau \\
+ \frac{1}{\pi} \left( \frac{\partial}{\partial \tau} \right) \left( e^{z(cos \tau - cos \delta)} \right) \left( 1 - \chi(\tau) \right) \frac{\cos(\mu \tau)}{\mu^2} \bigg|_{\tau=0}^{\tau=\pi} \\
+ \frac{(-1)}{\pi} \int_0^\pi \left( \frac{\partial}{\partial \tau} \right)^2 \left( e^{z(cos \tau - cos \delta)} \right) \left( 1 - \chi(\tau) \right) \frac{\cos(\mu \tau)}{\mu^2} d\tau \\
+ \frac{\sin(\mu \pi)}{\pi} \left( \frac{\partial}{\partial \tau} \right) \left( e^{-z(cosh \tau + cos \delta)} \right) \frac{e^{-\tau \mu}}{\mu^2} \bigg|_{\tau=0}^{\tau=\infty} \\
- \frac{\sin(\mu \pi)}{\pi} \int_0^\infty \left( \frac{\partial}{\partial \tau} \right) \left( e^{-z(cosh \tau + cos \delta)} \right) \frac{e^{-\tau \mu}}{\mu^2} d\tau.
\]

Again we observe that the boundary term

\[
\frac{1}{\pi} \left( \frac{\partial}{\partial \tau} \right) \left( e^{z(cos \tau - cos \delta)} \right) \left( 1 - \chi(\tau) \right) \frac{\cos(\mu \tau)}{\mu^2} \bigg|_{\tau=0}^{\tau=\pi} \\
+ \frac{\sin(\mu \pi)}{\pi} \left( \frac{\partial}{\partial \tau} \right) \left( e^{-z(cosh \tau + cos \delta)} \right) \frac{e^{-\tau \mu}}{\mu^2} \bigg|_{\tau=0}^{\infty} \\
= \frac{1}{\pi} \left( \frac{\partial}{\partial \tau} \right) \left( e^{z(cos \tau - cos \delta)} \right) \left( 1 - \chi(\tau) \right) \frac{\cos(\mu \tau)}{\mu^2} \bigg|_{\tau=\pi}^{\tau=\pi} \\
- \frac{\sin(\mu \pi)}{\pi} \left( \frac{\partial}{\partial \tau} \right) \left( e^{-z(cosh \tau + cos \delta)} \right) \frac{e^{-\tau \mu}}{\mu^2} \bigg|_{\tau=0}^{\tau=0}
\]
vanishes due to the fact $\sin \pi = \sinh 0 = 0$. Therefore, we have proved (4.5) with $m = 1$. Now, we assume (4.5) holds for $m = k$, that is,

$$\frac{1}{\pi} \int_0^\pi e^{z(\cos \tau - \cos \delta)} \cos(\mu \tau) d\tau - \frac{\sin(\mu \pi)}{\pi} \int_0^\infty e^{-z(\cosh \tau + \cos \delta)} e^{-\tau \mu} d\tau$$

$$= \frac{1}{\pi} \int_0^\pi e^{z(\cos \tau - \cos \delta)} \chi(\tau) \cos(\mu \tau) d\tau$$

$$- \frac{(-1)^k}{\pi} \int_0^\pi \left( 1 - \chi(\tau) \right) \frac{\partial}{\partial \tau} \left( e^{z(\cos \tau - \cos \delta)} \right) \frac{\cos(\mu \tau)}{\mu^{2k+1}} d\tau$$

we aim to prove (4.5) when $m = k + 1$. To this end, it suffices to check the boundary terms vanish. Indeed,

$$\left. \frac{(-1)^k}{\pi} \left( 1 - \chi(\tau) \right) \frac{\cos(\mu \tau)}{\mu^{2k+1}} \right|_{\tau = 0} = \left. \frac{\sin(\mu \pi)}{\pi} \frac{\partial}{\partial \tau} \left( e^{z(\cos \tau - \cos \delta)} \right) \frac{e^{-\tau \mu}}{\mu^{2k+1}} \right|_{\tau = 0}$$

$$= \left. \frac{(-1)^k}{\pi} \left( 1 - \chi(\tau) \right) \frac{\cos(\mu \tau)}{\mu^{2k+1}} \right|_{\tau = \pi}$$

and

$$\left. \frac{(-1)^{k+1}}{\pi} \left( 1 - \chi(\tau) \right) \frac{\cos(\mu \tau)}{\mu^{2k+2}} \right|_{\tau = 0} = \left. \frac{\sin(\mu \pi)}{\pi} \frac{\partial}{\partial \tau} \left( e^{z(\cos \tau - \cos \delta)} \right) \frac{e^{-\tau \mu}}{\mu^{2k+2}} \right|_{\tau = 0}$$

$$= \left. \frac{(-1)^{k+1}}{\pi} \left( 1 - \chi(\tau) \right) \frac{\cos(\mu \tau)}{\mu^{2k+2}} \right|_{\tau = \pi}$$

where we use the facts derived from [22, Pag. 420]

$$\left. (-1)^{k+1} \frac{\partial}{\partial \tau} \left( e^{z(\cos \delta - \cos \tau)} \left( 1 - \chi(\tau) \right) \right) \right|_{\tau = \pi} = \left. \frac{\partial}{\partial \tau} \left( e^{z(\cos \delta + \cos \tau)} \right) \right|_{\tau = 0},$$

and

$$\left. (-1)^{k+1} \left( 1 - \chi(\tau) \right) \frac{e^{z(\cos \delta - \cos \tau)} \left( 1 - \chi(\tau) \right)}{\mu^{2k+2}} \right|_{\tau = \pi}.$$
Let $P = \sqrt{-\Delta_h + V_0(y) + \frac{(n-2)^2}{4}}$, then the left hand side of (3.3) can be regarded as the operator

$$\frac{\partial}{\partial \tau} z^2 \int_0^\pi \left( e^{z (\cos \tau - \cos \delta)} \chi(\tau) \cos(\tau P) \right) d\tau$$

The $\cos(\tau P) f$ in the first term is the unique solutions of wave equation

$$\left\{ \begin{array}{l}
(\partial_t^2 + (-\Delta_h + V_0(y) + \frac{(n-2)^2}{4}) u = 0, \\
u|_{t=0} = f, \quad \partial_t u|_{t=0} = 0.
\end{array} \right. \quad (4.7)$$

By the finite speed of propagation [5, Theorem 3.3], $\cos(\tau P)(y, y')$ vanishes if $\tau < d_h(y, y')$ where $d_h$ denotes the distance in $Y$, then (4.6) equals

$$e^{z \cos \delta} z^2 \int_0^\pi \left( e^{z (\cos \tau - \cos \delta)} \chi(\tau) \cos(\tau P) \right) d\tau$$

Then, the first case of (3.3) is a consequence of the following lemma.

**Lemma 4.2** For $0 \leq \delta \leq \frac{1}{2} \epsilon_0$ and $z \geq 1$, there exists a constant $N$ such that

$$\left| \int_0^\pi \left( e^{z (\cos \tau - \cos \delta)} \chi(\tau) \cos(\tau P)(y, y') \right) d\tau \right| \lesssim 1, \quad (4.9)$$

and

$$\left| \int_0^\pi \left( e^{z (\cos \tau - \cos \delta)} \chi(\tau) \cos(\tau P)(y, y') \right) d\tau \right| \lesssim z^N e^{z (\cos(\frac{1}{2} \epsilon_0) - \cos \delta)}. \quad (4.10)$$
Proof of Lemma 4.2 We first prove (4.10). Notice that $z \geq 1$, by the chain rule, we obtain

$$\left| \left( \frac{\partial}{\partial \tau} \right)^{2m} \left( e^{-z (\cos \delta - \cos \tau)} (1 - \chi (\tau)) \right) \right| \leq C e^{-z (\cos \delta - \cos \tau)} z^{2m}, \quad (4.11)$$

and

$$\left| \left( \frac{\partial}{\partial \tau} \right)^{2m} \left( e^{-z (\cos \delta + \cosh \tau)} \right) \right| \leq C e^{-z (\cos \delta + 1/2 \cosh \tau)} z^{2m}. \quad (4.12)$$

Therefore, by (3.6) and the support of $1 - \chi$, we show that the LHS of (4.10) is bounded by

$$z^{2m - \frac{n-2}{2}} \sum_{k \in \mathbb{N}} (1 + k) \frac{n-2}{n-1} \mu_k^{-2m} \left( \int_0^\pi e^{-z (\cos \delta - \cos \tau)} d\tau + e^{-z \cos \delta} \int_0^\infty e^{-\frac{1}{2} \cosh \tau} d\tau \right)$$

$$\lesssim z^{2m - \frac{n-2}{2}} e^{z (\cos \frac{1}{2} \epsilon_0) - \cos \delta} \sum_{k \in \mathbb{N}} (1 + k) \frac{n-2}{n-1} \mu_k^{-2m}.$$ 

Noting $\mu_k = \sqrt{\lambda_k} \sim (1 + k)^{1/(n-1)}$ again, we choose $m$ large enough to ensure that

$$\sum_{k \in \mathbb{N}} (1 + k) \frac{n-2}{n-1} \mu_k^{-2m} \lesssim 1.$$ 

Therefore, we obtain (4.10) by choosing $N = 2n - 3$.

We next prove (4.9). Due to the compact support of $\chi$, for $|\tau| \leq \epsilon_0$, it suffices to prove

$$\left| z^{-\frac{n-2}{2}} \int_\delta^{\epsilon_0} e^{z (\cos \tau - \cos \delta)} \chi (\tau) \cos (\tau P)(y, y') d\tau \right| \lesssim 1. \quad (4.13)$$

To prove this, we shall use the Hadamard parametrix for $\partial^2_t - \Delta h + V_0(y) + \frac{(n-2)^2}{4}$. First let’s briefly recall that the Hadamard parametrix for $\partial^2_t - \Delta h$ (see e.g., [24, Theorem 3.1.5]) says that if $\tau$ is smaller than the injectivity radius of $Y$,

$$\left( \cos \tau \sqrt{-\Delta_h} \right)(y, y') = \sum_{\nu=0}^N w_\nu(y, y') W_\nu(\tau, y, y') + R_N(\tau, y, y') \quad (4.14)$$

where $w_\nu \in C^\infty (Y \times Y)$,

$$W_0(\tau, y, y') = (2\pi)^{-(n-1)} \int_{\mathbb{R}^{n-1}} e^{i d_h(y, y') \xi_1} \cos |\xi| d\xi, \quad (4.15)$$
while for \( \nu = 1, 2, \ldots \), \( W_\nu(\tau, y, y') \) is a finite linear combination of Fourier integrals of the form

\[
\int_{\mathbb{R}^{n-1}} e^{i \theta h(y, y') \xi_1} e^{\pm i \tau |\xi|} \alpha_\nu(|\xi|) \, d\xi, \quad \text{with} \quad \alpha_\nu(\tau) = 0,
\]

for \( \tau \leq 1 \) and \( \partial^j_\tau \alpha_\nu(\tau) \lesssim \tau^{-\nu-j} \),

(4.16)

and, if \( N_0 \) is given, then if \( N \) is large enough,

\[
|\partial^j_\tau R_N(\tau, y, y')| \leq C, \quad 0 \leq j \leq N_0,
\]

(4.17)

for a fixed constant \( C \). Furthermore, the leading coefficient \( w_0(y, y') \) reflects the geometry of \((Y, h)\). Specifically, if we are in geodesic normal coordinates about \( y \)

\[
w_0(0, y') = (\det h_{ij}(y'))^{-1/4}.
\]

The other coefficients in (4.14) are also well behaved, if \( N_0 \) is fixed

\[
|\partial^\beta_\gamma w_\nu(y, y')| \leq C, \quad |\beta|, \nu \leq N_0,
\]

(4.18)

for some uniform constant \( C \) (depending on \( \tilde{g} \) and \( N_0 \)). Actually, if we are in geodesic normal coordinates about \( y \), where \( y \) becomes the origin, then \( w_\nu(0, y') \) are defined recursively by the following transport equations (see [24, Chapter 2.4]),

\[
\rho w_0 = 2\langle x, \nabla_x w_0 \rangle, \quad w_0(0) = 1
\]

as well as \( w_\nu(y'), \nu = 1, 2, 3.. \) so that

\[
2\nu w_\nu - \rho w_\nu + 2\langle x, \nabla_x w_\nu \rangle - 2\Delta_{\tilde{g}} w_{\nu-1} = 0,
\]

where \( \rho \in C^\infty(Y) \) is a fixed function that depend on the metric of \( Y \).

Similarly, in the construction of Hadamard parametrix for \( \partial_t^2 - \Delta_h + V_0(y) + \frac{(n-2)^2}{4} \), where \( V_0 \in C^\infty(Y) \). If \( P = \sqrt{-\Delta_h + V_0(y) + \frac{(n-2)^2}{4}} \), we still have

\[
(\cos \tau P)(y, y') = \sum_{\nu=0}^N \tilde{w}_\nu(y, y') W_\nu(\tau, y, y') + \tilde{R}_N(\tau, y, y'),
\]

(4.19)

where \( W_\nu \) is defined as in (4.15)-(4.16), while \( \tilde{w}_\nu(0, y') \) satisfy the modified transport equations

\[
\rho \tilde{w}_0 = 2\langle x, \nabla_x \tilde{w}_0 \rangle, \quad \tilde{w}_0(0) = 1,
\]

which is same as before, as well as \( \tilde{w}_\nu(x), \nu = 1, 2, 3.. \) so that

\[
2\nu \tilde{w}_\nu - \rho \tilde{w}_\nu + 2\langle x, \nabla_x \tilde{w}_\nu \rangle - 2\Delta_{\tilde{g}} \tilde{w}_{\nu-1} + 2(V_0 + \frac{(n-2)^2}{4}) \cdot \tilde{w}_{\nu-1} = 0.
\]
In the case $V_0 \in C^\infty(Y)$, by arguing as in [24, Chapter 2.4], one can see that, if we are in geodesic normal coordinates about $y$

$$\tilde{w}_0(0, y') = w_0(0, y') = (\det h_{ij}(y'))^{-1/4}.$$ 

And if $N_0$ is fixed, the other coefficients satisfy

$$|\partial^\beta_{y, y'} \tilde{w}_v(y, y')| \leq C_{V_0}, \ |\beta|, v \leq N_0,$$ (4.20)

for some uniform constant $C_{V_0}$ (depending on $\tilde{g}$, $N_0$ and $V_0$). And similarly if $N_0$ is given, then if $N$ is large enough,

$$|\partial^j \tilde{R}_N(\tau, y, y')| \leq C_{V_0}, \ 0 \leq j \leq N_0,$$ (4.21)

for a fixed constant $C_{V_0}$. We also refer the reader to Hörmander [13, §17.4] for the parametrix of a general second order differential operator with low order perturbations.

Note that the phase function in (4.15) and (4.16) are essentially the same, so we can rewrite the first term on the right side of (4.14) as the following

$$\cos(\tau P)(y, y') = K_N(\tau, y, y') + R_N(\tau; y, y'), \ \tau < \epsilon_0,$$ (4.22)

where $R_N(\tau, y, y')$ satisfies (4.21) for large enough $N$, and

$$K_N(\tau, y, y') = (2\pi)^{n-1} \int_{\mathbb{R}^{n-1}} e^{i dh_{\xi}(y, y') 1^{\xi} - \xi} a(\tau, y, y'; |\xi|) \cos(\tau |\xi|) d\xi,$$ (4.23)

where $1 = (1, 0, \ldots, 0)$ and $a \in S^0$ zero order symbol satisfies

$$|\partial^\alpha_{\tau, y, y'} \partial^k \rho a(\tau, y, y'; \rho)| \leq C_{\alpha, k, V_0} (1 + \rho)^{-k}.$$ (4.24)

Now we shall return to the proof of (4.13), first, if we choose $N$ large enough such that

$$|R_N(\tau, y, y)| \leq 1, \ \tau \leq \epsilon_0,$$

then it is easy to see

$$|z^{-\frac{n-2}{2}} \int_{-\delta}^{\pi} e^{\zeta(\cos \tau - \cos \delta)} \chi(\tau) R_N(\tau; y, y') d\tau| \leq C z^{-\frac{n-2}{2}} \lesssim 1.$$ (4.25)

Therefore it suffices to prove

$$|z^{-\frac{n-2}{2}} \int_{-\delta}^{\pi} e^{\zeta(\cos \tau - \cos \delta)} \chi(\tau) \int_{\mathbb{R}^{n-1}} e^{i dh_{\xi}(y, y') 1^{\xi} - \xi} a(\tau, y, y'; |\xi|) \cos(\tau |\xi|) d\xi d\tau| \leq C.$$ (4.26)
From \[23, \text{Theorem 1.2.1}\], we also note that

\[
\int_{\mathbb{R}^{n-2}} e^{i \mathcal{R}(y, y') \rho} 1^\omega \, d\omega \, d\tau = \sum_{\pm} a_\pm(\rho \mathcal{R}(y, y')) e^{\pm i \rho \mathcal{R}(y, y')}, \tag{4.27}
\]

where

\[
|\partial^k_r a_\pm(r)| \leq C_k (1 + r)^{-\frac{n-2}{2}} - k, \quad k \geq 0. \tag{4.28}
\]

Then we are reduced to estimate the integral

\[
\int_\pi^{-n-2} e^{z(\cos \tau - \cos \delta)} \chi(\tau) \int_{\mathbb{R}^{n-1}} e^{i \mathcal{R}(y, y') 1^\xi} a(\tau, y, y'; |\xi|) \cos(\tau |\xi|) d\xi d\tau
\]

\[
= \int_\pi^{-n-2} e^{z(\cos \tau - \cos \delta)} \chi(\tau) \int_0^{\infty} a_\pm(\rho \mathcal{R}(y, y')) e^{\pm i \rho \mathcal{R}(y, y')} a(\tau, y, y'; \rho) \cos(\tau \rho) \rho^{n-2} d\rho d\tau.
\]

To prove (4.13) when we replace \(\cos(\tau P)\) by \(K_N\), we shall divide the discussion into two cases.

**Case 1.** \(z \leq 2 \delta^{-2} = 2d_h^{-2}(y, y').\) In this case, we do not need to make use of the finite propagation speed property, by the above argument using Hadamard parametrix, our goal is to show

\[
\int_0^{\pi} e^{z(\cos \tau - \cos \delta)} \chi(\tau) \int_0^{\infty} a_\pm(\rho \mathcal{R}(y, y')) e^{\pm i \rho \mathcal{R}(y, y')} a(\tau, y, y'; \rho) \cos(\tau \rho) \rho^{n-2} d\rho d\tau \lesssim 1.
\]

Since the above integral is even in \(\tau\), we are further reduced to showing that

\[
\int_\mathbb{R} e^{z(\cos \tau - \cos \delta)} \chi(\tau) \int_0^{\infty} b(\rho \mathcal{R}(y, y')) a(\tau, y, y'; \rho) \cos(\tau \rho) \rho^{n-2} d\rho d\tau \lesssim 1, \tag{4.29}
\]

where we set \(b(\rho \mathcal{R}(y, y')) = a_\pm(\rho \mathcal{R}(y, y')) e^{\pm i \rho \mathcal{R}(y, y')}\).

Let us fix a Littlewood-Paley bump function \(\beta \in C_0^\infty((1/2, 2))\) satisfying

\[
\sum_{\ell=-\infty}^{\infty} \beta(2^{-\ell} s) = 1, \quad s > 0,
\]

and we set

\[
\beta_0(s) = \sum_{\ell \leq 0} \beta(2^{-\ell} |s|) \in C_0^\infty((-2, 2)).
\]
Subcase 1.1 the sub-case that $|\tau| \leq 4z^{-1/2}$. In this case, we want to show that
\[
\int_{\mathbb{R}} e^{z(\cos \tau - \cos \delta)} \chi(\tau) \left( \beta_0(z^{1/2} \tau) + \beta(2^{-1/2} z^{1/2} \tau) \right) \\
\times \int_{0}^{\infty} b(\rho d_h(y, y') a(\tau, y, y'; \rho) \cos(\tau \rho) \rho^{n-2} d \rho d \tau \lesssim 1. \quad (4.30)
\]

If we also have $\rho \leq 4z^{1/2}$, then we do not do any integration by parts, note that by using the condition $z \leq 2 \delta^{-2}$, the power on the exponential $z(\cos \tau - \cos \delta) \lesssim 1$ as long as $|\tau| \leq 4z^{-1/2}$. Thus the integral in (4.30) is always bounded by
\[
z^{-\frac{n-2}{2} z^{-1/2} z^{n-1} \lesssim 1.}
\]

If on the other hand, we have $\rho \geq 4z^{1/2}$, we do integration by parts in $d \tau$, then each time we gain a factor of $\rho^{-1}$ from the function $\cos(\tau \rho)$, and we at most lose a factor of $z \sin \tau$ or $z^{1/2}$, which is always less than $z^{1/2}$ up to a constant, so after integration by parts $N$ times for $N \geq n$, the integral in (4.30) is bounded by
\[
z^{-\frac{n-2}{2} z^{-1/2} z^{n-1} \lesssim 1}
\]

Subcase 1.2 $\tau \approx 2^j z^{-1/2}$, $j \geq 2$ and $2^j \lesssim \epsilon_0 z^{1/2}$. In this case, we want to show that
\[
\int_{-\infty}^{+\infty} e^{z(\cos \tau - \cos \delta)} \chi(\tau) \beta(z^{1/2} 2^{-j} |\tau|) \\
\times \int_{0}^{\infty} b(\rho d_h(y, y') a(\tau, y, y'; \rho) \cos(\tau \rho) \rho^{n-2} d \rho d \tau \lesssim e^{-j}. \quad (4.31)
\]

which would give us desired bounds after summing over $j$. Since $\tau \approx 2^j z^{-1/2}$ and $z \leq 2d_h^{-2}(y, y')$ imply that $|\tau| \geq 2d_h(y, y')$, it is straightforward to check that $\cos \tau - \cos \delta \approx -\tau^2$ in this case, which implies
\[
e^{z(\cos \tau - \cos \delta) \lesssim e^{-2j}}.
\]

Now we can repeat the previous argument, if in this case we have $\rho \leq 2^{-j} z^{1/2}$, then we do not do any integration by parts, the integral in (4.31) is always bounded by
\[
z^{-\frac{n-2}{2} z^{-1/2} z^{n-1} \lesssim 1}
\]

If on the other hand, we have $\rho \geq 2^{-j} z^{1/2}$, we do integration by parts in $d \tau$, then each time we gain a factor of $\rho^{-1}$ from the function $\cos(\tau \rho)$, and we at most lose a
factor of $z \sin \tau \lesssim 2^{j} z^{1/2}$, so after integration by parts $N$ times for $N \geq n$, the integral in (4.31) is bounded by

$$e^{-2^{j} z^{-n/2} z^{-1/2} 2^{j} z^{N/2} 2^{N} \int_{2^{-j} z^{1/2}}^{\infty} \rho^{n-2^{-N}} d \rho} \lesssim e^{-j}.$$  

**Case 2.** $z \geq 2\delta^{-2} = 2d_{h}^{-2}(y, y')$. In this case, we need to make use of the finite propagation speed property, in order to avoid the blow up of the function $e^{z(\cos \tau - \cos \delta)}$ when $\tau$ is close to 0. We choose a smooth cut off function $\eta(\tau) = \beta_{0}(z\delta|\tau - \delta|) + \sum_{\ell \geq 1} \beta(z\delta 2^{-\ell}(\tau - \delta))$. It is easy to see that $\text{supp } \eta \subset (\delta - 2(z\delta)^{-1}, +\infty)$ which is a subset of $(0, \infty)$ since $(z\delta)^{-1} < \delta/2$ in our case. Also note that $\eta(\tau) \equiv 1$, $\forall \tau \geq \delta$, by using the finite propagation speed property of $\cos(\tau P)$ and Hadamard parametrix as above, we are reduced to show that

$$z^{-n/2} \int_{0}^{\pi} e^{z(\cos \tau - \cos \delta)} \chi(\tau) \eta(\tau) \int_{0}^{\infty} b(\rho d_{h}(y, y')) a(\tau, y, y'; \rho) \cos(\tau \rho) \rho^{n-2} d \rho d \tau \lesssim 1.$$  

**Subcase 2.1** $|\tau - \delta| \leq (z\delta)^{-1} < \frac{\delta}{2}$. In this case, we want to show that

$$z^{-n/2} \int_{0}^{\pi} e^{z(\cos \tau - \cos \delta)} \chi(\tau) \beta_{0}(z\delta|\tau - \delta|) \times \int_{0}^{\infty} b(\rho d_{h}(y, y')) a(\tau, y, y'; \rho) \cos(\tau \rho) \rho^{n-2} d \rho d \tau \lesssim 1. \quad (4.32)$$  

If we also have $\rho \leq z\delta$, then we do not do any integration by parts, note that since in this case $\tau \approx \delta$, we have $\cos \tau - \cos \delta = 2 \sin(\frac{\delta + \tau}{2}) \sin(\frac{\delta - \tau}{2}) \approx \delta(\delta - \tau)$, the power on the exponential $z(\cos \tau - \cos \delta) \lesssim 1$ as long as $|\tau - \delta| \leq (z\delta)^{-1}$. Thus the integral in (4.32) is always bounded by

$$z^{-n/2} (z\delta)^{-1} (z\delta)^{n/2 + 1} \delta^{-n/2} \lesssim 1,$$

where we used the fact that $b(\rho d_{h}(y, y')) \leq C(\rho \delta)^{-n/2}$.  

If on the other hand, we have $\rho \geq z\delta$, we do integration by parts in $d\tau$, then each time we gain a factor of $\rho^{-1}$ from the function $\cos(\tau \rho)$, and we at most lose a factor of $z \sin \tau \lesssim z\delta$, so after integration by parts $N$ times for $N \geq n$, the integral in (4.32) is bounded by

$$z^{-n/2} (z\delta)^{-1} (z\delta)^{n/2} \int_{z\delta}^{\infty} \rho^{-n-2^{-N}} d \rho \lesssim 1.$$
Subcase 2.2 $\tau - \delta \approx 2^j (z\delta)^{-1}$, $j \geq 1$ and $2^j \lesssim \varepsilon_0 z\delta$. In this case, we want to show that

$$z^{-\frac{n-2}{2}} \int_{-\infty}^{+\infty} e^{z(\cos \tau - \cos \delta)} \chi(\tau) \beta(z\delta 2^{-j} (\tau - \delta))$$

$$\times \int_0^{+\infty} b(\rho d\eta(y, y')) a(\tau, y, y'; \rho) \cos(\tau \rho) \rho^{n-2} d\rho d\tau \lesssim e^{-j}, \quad (4.33)$$

which would give us desired bounds after summing over $j$. In this case, it is straightforward to check that $z(\cos \tau - \cos \delta) = 2z \sin(\frac{\delta + \tau}{2}) \sin(\frac{\delta - \tau}{2}) \lesssim -2j$ in this case, which implies

$$e^{z(\cos \tau - \cos \delta)} \lesssim e^{-2j}.$$

Now we can repeat the previous argument. To begin with, we shall further assume that

(i) $\tau \leq 2\delta$, in other words $2^j (z\delta)^{-1} \lesssim \delta$.

In this case, if we have $\rho \leq 2^{-j} z\delta$, then we do not do any integration by parts, the integral in (4.33) is always bounded by

$$z^{-\frac{n-2}{2}} 2^j (z\delta)^{-1} (2^{-j} z\delta)^{\frac{n-2}{2} + 1} \delta^{-\frac{n-2}{2}} e^{-2j} \lesssim e^{-j}.$$

On the other hand, if we have $\rho \geq 2^{-j} z\delta$, we do integration by parts in $d\tau$, then each time we gain a factor of $\rho^{-1}$ from the function $\cos(\tau \rho)$, and we at most lose a factor of $z \sin \tau \lesssim z\delta$, so after integration by parts $N$ times for $N \geq n$, the integral in (4.33) is bounded by

$$e^{-2j} z^{-\frac{n-2}{2}} 2^j (z\delta)^{-1} (2^{-j} z\delta)^{\frac{n-2}{2} + 1} \delta^{-\frac{n-2}{2}} e^{-2j} \lesssim e^{-j}.$$

(ii) $\tau \geq 2\delta$, in other words $2^j (z\delta)^{-1} \gtrsim \delta$.

In this case, if we have $\rho \leq \delta^{-1} 2^j \approx z\tau$, then we do not do any integration by parts, the integral in (4.33) is always bounded by

$$z^{-\frac{n-2}{2}} 2^j (z\delta)^{-1} (\delta^{-1} 2^j)^{\frac{n-2}{2} + 1} \delta^{-\frac{n-2}{2}} e^{-2j} \lesssim e^{-j},$$

where we used the fact that $z^{-n/2} \delta^{-n} \lesssim 1$.

On the other hand, if we have $\rho \geq \delta^{-1} 2^j$, we do integration by parts in $d\tau$, then each time we gain a factor of $\rho^{-1}$ from the function $\cos(\tau \rho)$, and we at most lose a factor of $z \sin \tau \lesssim \delta^{-1} 2^j$ or $z\delta 2^{-j} \lesssim \delta^{-1}$, so after integration by parts $N$ times for $N \geq n$, the integral in (4.33) is bounded by

$$e^{-2j} z^{-\frac{n-2}{2}} 2^j (z\delta)^{-1} (\delta^{-1} 2^j)^{\frac{n-2}{2} + 1} \delta^{-\frac{n-2}{2}} e^{-2j} \lesssim e^{-j}.$$

This finishes the proof of (4.13) and thus complete the proof of Lemma 4.2. \hfill \square
Step 2: we consider the case $\frac{1}{2}\epsilon_0 \leq \delta \leq \pi$. In this case, it is much easier than the above step. Indeed, we do not need the cut function $\chi$ and we modify the above argument. By integration by parts and arguing as Lemma 4.1, for fixed $0 \leq \delta \leq \pi$ and $m \geq 0$, we compute that

\[
\begin{align*}
e^{-z \cos \delta} I_\mu(z) &= \frac{1}{\pi} \int_0^\pi e^{z (\cos \tau - \cos \delta)} \cos(\mu \tau) d\tau \\
&= \frac{\sin(\mu \pi)}{\pi} \int_0^\infty e^{-z (\cos \tau + \cos \delta)} e^{-\tau \mu} d\tau \\
&= \frac{(-1)^m}{\pi} \int_0^\pi \left( \frac{\partial}{\partial \tau} \right)^{2m-1} \left( e^{z (\cos \tau - \cos \delta)} \right) \frac{\sin(\mu \pi)}{\mu^{2m-1}} d\tau \\
&= \frac{\sin(\mu \pi)}{\pi} \int_0^\infty \left( \frac{\partial}{\partial \tau} \right)^{2m-1} \left( e^{-z (\cosh \tau + \cos \delta)} \right) \frac{e^{-\tau \mu}}{\mu^{2m}} d\tau \\
&= \frac{\sin(\mu \pi)}{\pi} \int_0^\infty \left( \frac{\partial}{\partial \tau} \right)^{2m} \left( e^{-z (\cos \tau + \cos \delta)} \right) \frac{e^{-\tau \mu}}{\mu^{2m}} d\tau. \quad (4.34)
\end{align*}
\]

By the finite speed of propagation and similar argument as above, it suffices to prove

Lemma 4.3 For $\frac{1}{2}\epsilon_0 \leq \delta \leq \pi$ and $z \geq 1$, there exists a constant $N$ such that

\[
\begin{align*}
&\left| z^{-\frac{n-2}{2}} \sum_{k \in \mathbb{N}} \varphi_k(y) \overline{\varphi_k(y')} \mu_k^{-2m} \left( \int_0^\pi \left( \frac{\partial}{\partial \tau} \right)^{2m} \left( e^{z (\cos \tau - \cos \delta)} \right) \cos(\mu \mu_k) d\tau \\
&- \sin(\pi \mu_k) \int_0^\infty \left( \frac{\partial}{\partial \tau} \right)^{2m} \left( e^{-z (\cosh \tau + \cos \delta)} \right) e^{-\tau \mu_k} d\tau \right) \right| \lesssim z^N. \quad (4.35)
\end{align*}
\]

Proof of Lemma 4.3 Notice that $z \geq 1$, we further obtain

\[
\left| \left( \frac{\partial}{\partial \tau} \right)^{2m} \left( e^{-z (\cos \delta - \cos \tau)} \right) \right| \leq C e^{-z (\cos \delta - \cos \tau)} z^{2m}, \quad (4.36)
\]

and

\[
\left| \left( \frac{\partial}{\partial \tau} \right)^{2m} \left( e^{-z (\cos \delta + \cosh \tau)} \right) \right| \leq C e^{-z (\cos \delta + \frac{1}{2} \cosh \tau)} z^{2m}. \quad (4.37)
\]

Therefore, by (3.6), we show that the LHS of (4.35) is bounded by

\[
\begin{align*}
z^{2m-\frac{n-2}{2}} \sum_{k \in \mathbb{N}} (1 + k)^{\frac{n-2}{2m}} \mu_k^{-2m} \left( \int_0^\infty e^{-z (\cos \delta - \cos \tau)} d\tau + e^{-z \cos \delta} \int_0^\infty e^{-z \frac{1}{2} \cosh \tau} d\tau \right). \quad (4.38)
\end{align*}
\]
Note $\mu_k = \sqrt{\lambda_k} \sim (1 + k)^{\frac{1}{n-1}}$ again, we choose $m$ large enough to ensure that
\[
\sum_{k \in \mathbb{N}} (1 + k)^{\frac{n-2}{n-1}} \mu_k^{-2m} \lesssim 1.
\]

Therefore, we obtain (4.35) when $\frac{1}{2} \epsilon_0 \leq \delta \leq \pi$ by choosing $N = 2n - 3$. $\square$

Step 3: We consider the last case that $\delta > \pi$. Arguing as above, to prove (3.1), it suffices to prove

**Lemma 4.4** *For $\pi \leq \delta$ and $z \geq 1$, we have the estimate*

\[
\left| z^{-\frac{n-2}{2}} \sum_{k \in \mathbb{N}} \varphi_k(y) \varphi_k(y') \mu_k^{-2m} \left( \int_0^\pi \left( \frac{\partial}{\partial \tau} \right)^{2m} \left( e^{-(\cos \tau - 1)} \right) \cos(\tau \mu_k) d\tau \right) \right| \lesssim e^{-z}. \quad (4.39)
\]

**Proof** By the finite speed of propagation, the first term of (4.39) vanishes due to $\delta > \pi$. Arguing as before, it is easy to see

\[
\text{LHS of (4.39)} \lesssim z^{-\frac{n-2}{2}} z^{-2m} e^{-\frac{5}{2}z} \sum_{k \in \mathbb{N}} (1 + k)^{\frac{n-2}{n-1}} \mu_k^{-2m} \int_0^\infty e^{-\frac{z}{2} \cosh \tau} d\tau \lesssim C_2 e^{-z}.
\]

We choose $m$ large enough to ensure the summation converges, and the factor $z^{-2m}$ can be absorbed by $e^{-\frac{5}{2}z}$.

$\square$

This finishes the proof of all three cases in (3.3).

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**References**

1. Alexopoulos, G.: Spectral multipliers for Markov chains. J. Math. Soc. Japan 56, 833–852 (2004)
2. Burq, N., Planchon, F., Stalker, J., Tahvildar-Zadeh, A.S.: Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential. J. Funct. Anal. 203, 519–549 (2003)
3. Carron, G.: Le saut en zéro de la fonction de décalage spectral. J. Funct. Anal. 212, 222–260 (2004)
4. Couhon, T., Li, H.: Estimations inférieures du noyau de la chaleur sur les variétés coniques et transformée de Riesz. Arch. Math. 83, 229–242 (2004)
5. Coulhon, T., Sikora, T.: Gaussian heat kernel upper bounds via the Phragmén-Lindelöf theorem. Proc. Lond. Math. Soc. (3) 96, 507–544 (2008)
6. Cheeger, J.: On the spectral geometry of spaces with cone-like singularities. Proc. Natl. Acad. Sci. U.S.A. 76, 2103–2106 (1979)
7. Cheeger, J.: Spectral geometry of singular Riemannian spaces. J. Differ. Geom. 18, 575–657 (1983)
8. Cheeger, J., Taylor, M.: On the diffraction of waves by conical singularities, I. Commun. Pure Appl. Math. 35, 275–331 (1982)
9. Cheeger, J., Taylor, M.: On the diffraction of waves by conical singularities, II. Commun. Pure Appl. Math. 35, 487–529 (1982)
10. https://dlmf.nist.gov/10.32
11. Gao, X., Zhang, J., Zheng, J.: Restriction estimates in a conical singular space: wave equation. J. Fourier Anal Appl. 28, 44 (2022). https://doi.org/10.1007/s00041-022-09941-7
12. Grigor’yan, A.: Gaussian upper bounds for the heat kernel on arbitrary manifolds. J. Differ. Geom. 45, 33–52 (1997)
13. Hörmander, L.: The Analysis of Linear Partial Differential Operators III: Pseudo-differential Operators. Springer, Berlin (1985)
14. Hassell, A., Lin, P.: The Riesz transform for homogeneous Schrödinger operators on metric cones. Rev. Mat. Iberoamericana 30, 477–522 (2014)
15. Killip, R., Miao, C., Visan, M., Zhang, J., Zheng, J.: Sobolev spaces adapted to the Schrödinger operator with inverse-square potential. Math. Z. 288, 1273–1298 (2018)
16. Li, H.: $L^p$-estimates for the wave equation on manifolds with conical singularities. Math. Z. 272, 551–575 (2012)
17. Li, H.: La transformation de Riesz sur les variétés coniques. J. Funct. Anal. 168, 145–238 (1999)
18. Li, H.: Estimations du noyau de la chaleur sur les variétés coniques et ses applications. Bull. Sci. Math. 124, 365–384 (2000)
19. Liskevich, V., Sobol, Z.: Estimates of integral kernels for semigroups associated with second order elliptic operators with singular coefficients. Potential Anal. 18, 359–390 (2003)
20. Milman, P.D., Semenov, Yu.A.: Global heat kernel bounds via desingularizing weights. J. Funct. Anal. 212, 373–398 (2004)
21. Mooers, E.: Heat kernel asymptotics on manifolds with conic singularities. J. Anal. Math. 78, 1–36 (1999)
22. Nagase, M.: The fundamental solutions of the heat equations on Riemannian spaces with cone-like singular points. Kodai Math. J. 7, 382–455 (1984)
23. Sogge, C.D.: Fourier Integrals in Classical Analysis, Cambridge Tracts in Mathematics, vol. 105. Cambridge University Press, Cambridge (1993)
24. Sogge, C.D.: Hangzhou Lectures on Eigenfunctions of the Laplacian. Princeton University Press, Princeton (2014)
25. Taylor, M.: Partial Differential Equations, vol. II. Springer, Berlin (1996)
26. Wang, X.: Asymptotic expansion in time of the Schrödinger group on conical manifolds. Ann. Inst. Fourier 56, 1903–1945 (2006)
27. Watson, G.N.: A Treatise on the Theory of Bessel Functions, 2nd edn. Cambridge University Press, Cambridge (1944)
28. Yau, S.T.: Nonlinear Analysis in Geometry, Monographie No. 33 de l’Enseignement Mathématique, Série des Conférences de l’Union Mathématique Internationale, No. 8
29. Zhang, J.: Linear restriction estimates for Schrödinger equation on metric cones. Commun. PDE 40, 995–1028 (2015)
30. Zhang, J., Zheng, J.: Global-in-time Strichartz estimates and cubic Schrödinger equation in a conical singular space. arXiv:1702.05813
31. Zhang, J., Zheng, J.: Strichartz estimates and wave equation in a conic singular space. Math. Ann. 376, 525–581 (2020)

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