Pullback Attractors of Non-autonomous Stochastic Degenerate Parabolic Equations on Unbounded Domains

Andrew Krause and Bixiang Wang
Department of Mathematics, New Mexico Institute of Mining and Technology
Socorro, NM 87801, USA
Email: akrause@nmt.edu, bwang@nmt.edu

Abstract

This paper is concerned with pullback attractors of the stochastic $p$-Laplace equation defined on the entire space $\mathbb{R}^n$. We first establish the asymptotic compactness of the equation in $L^2(\mathbb{R}^n)$ and then prove the existence and uniqueness of non-autonomous random attractors. This attractor is pathwise periodic if the non-autonomous deterministic forcing is time periodic. The difficulty of non-compactness of Sobolev embeddings on $\mathbb{R}^n$ is overcome by the uniform smallness of solutions outside a bounded domain.

Key words. Pullback attractor; random attractor; periodic attractor; $p$-Laplace equation.

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1 Introduction

In this paper, we study random attractors of the non-autonomous stochastic $p$-Laplace equation defined on $\mathbb{R}^n$. Suppose $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ is a metric dynamical system where $(\Omega, \mathcal{F}, P)$ is a probability space and $\{\theta_t\}_{t \in \mathbb{R}}$ is a measure-preserving transformation group on $\Omega$. Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, consider the stochastic equation defined for $x \in \mathbb{R}^n$ and $t > \tau$,

$$\frac{\partial u}{\partial t} + \lambda u - \text{div}(|\nabla u|^{p-2}\nabla u) = f(t, x, u) + g(t, x) + \alpha \eta(\theta_t \omega) u + \varepsilon h(x) \frac{dW}{dt},$$  

where $p \geq 2$, $\alpha > 0$, $\lambda > 0$, $\varepsilon > 0$, $f$ is a time dependent nonlinearity, $g$ and $h$ are given functions, $\eta$ is a random variable and $W$ is a Wiener process on $(\Omega, \mathcal{F}, P)$. The $p$-Laplace equation has been used to model a variety of physical phenomena. For instance, in fluid dynamics, the motion of non-Newtonian fluid is governed by the $p$-Laplace equation with $p \neq 2$. This equation is also used to study flow in porous media and nonlinear elasticity (see [28] for more details).
In this paper, we will investigate the asymptotic behavior of solutions of the \( p \)-Laplace equation (1.1) driven by deterministic as well as stochastic forcing. If \( f \) and \( g \) do not depend on time, then we call (1.1) an autonomous stochastic equation. In the autonomous case, the existence of random attractors of (1.1) has been established recently in [22, 23, 24] by variational methods under the condition that the growth rate of the nonlinearity \( f \) is not bigger than \( p \). This result has been extended in [37] to the case where \( f \) is non-autonomous and has a polynomial growth of any order.

Note that in all papers mentioned above, the \( p \)-Laplace equation is defined in a bounded domain where compactness of Sobolev embeddings is available. As far as we are aware, there is no existence result on random attractors for the stochastic \( p \)-Laplace equation defined on unbounded domains. The goal of this paper is to overcome the non-compactness of Sobolev embeddings on \( \mathbb{R}^n \) and prove the existence and uniqueness of random attractors for (1.1) in \( L^2(\mathbb{R}^n) \). More precisely, we will show by a cut-off technique that the tails of solutions of (1.1) are uniformly small outside a bounded domain for large times. We then use this fact and the compactness of solutions in bounded domains to establish the asymptotic compactness of solutions in \( L^2(\mathbb{R}^n) \). By the asymptotic compactness and absorbing sets of the equation, we can obtain the existence and uniqueness of random attractors. This random attractor is pathwise periodic if \( f(t,x,u) \) and \( g(t,x) \) are periodic in \( t \).

It is worth mentioning that the definition of random attractor for autonomous stochastic systems was initially introduced in [16, 18, 30]. Since then, such attractors for autonomous stochastic PDEs have been studied in [6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 30, 31] in bounded domains, and in [4, 33, 34] in unbounded domains. For non-autonomous stochastic PDEs, the reader is referred to [1, 5, 12, 17, 23, 24, 27, 35, 36, 37] for existence of random attractors.

The rest of this article consists of three sections. In the next section, we prove the well-posedness of (1.1) in \( L^2(\mathbb{R}^n) \) under certain conditions. Section 3 is devoted to uniform estimates of solutions for large times which include the estimates on the tails of solutions outside a bounded domain. In the last section, we derive the existence and uniqueness of random attractors for the non-autonomous stochastic equation (1.1).

The following notation will be used throughout this paper: \( \| \cdot \| \) for the norm of \( L^2(\mathbb{R}^n) \) and \((\cdot,\cdot)\) for its inner product. The norm of \( L^p(\mathbb{R}^n) \) is usually written as \( \| \cdot \|_p \) and the norm of a Banach space \( X \) is written as \( \| \cdot \|_X \). The symbol \( c \) or \( c_i \) \( (i = 1, 2, \ldots) \) is used for a general positive number which may change from line to line.
Finally, we recall the following inequality which will be used frequently in the sequel:

\[ \|u\|_p^p \le \frac{q-p}{q-2} \|u\|^2 + \frac{p-2}{q-2} \|u\|_q^q, \]

(1.2)

where \(2 < p < q\) and \(u \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\).

\section{Cocycles Associated with Degenerate Equations}

In this section, we first establish the well-posedness of equation (1.1) in \(L^2(\mathbb{R}^n)\), and then define a continuous cocycle for the stochastic equation. This step is necessary for us to investigate the asymptotic behavior of solutions.

Let \((\Omega, \mathcal{F}, P)\) be the standard probability space where \(\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}\), \(\mathcal{F}\) is the Borel \(\sigma\)-algebra induced by the compact-open topology of \(\Omega\) and \(P\) is the Wiener measure on \((\Omega, \mathcal{F})\). Denote by \(\{\theta_t\}_{t \in \mathbb{R}}\) the family of shift operators given by

\[ \theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t) \quad \text{for all } \omega \in \Omega \text{ and } t \in \mathbb{R}. \]

From [2] we know that \((\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})\) is a metric dynamical system. Given \(\tau \in \mathbb{R}\) and \(\omega \in \Omega\), consider the following stochastic equation defined for \(x \in \mathbb{R}^n\) and \(t > \tau\),

\[ \frac{\partial u}{\partial t} + \lambda u - \text{div}(|\nabla u|^{p-2}\nabla u) = f(t, x, u) + g(t, x) + \alpha\eta(\theta_t\omega)u + \varepsilon h(x) \frac{dW}{dt} \]

(2.1)

with initial condition

\[ u(\tau, x) = u_\tau(x), \quad x \in \mathbb{R}^n, \]

(2.2)

where \(p \ge 2\), \(\alpha > 0\), \(\lambda > 0\), \(\varepsilon > 0\), \(g \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^n))\), \(h \in H^2(\mathbb{R}^n)\), \(\eta\) is an integrable tempered random variable and \(W\) is a two-sided real-valued Wiener process on \((\Omega, \mathcal{F}, P)\). We assume the nonlinearity \(f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}\) is continuous and satisfies, for all \(t, s \in \mathbb{R}\) and \(x \in \mathbb{R}^n\),

\[ f(t, x, s) \le -\gamma|s|^q + \psi_1(t, x), \]

(2.3)

\[ |f(t, x, s)| \le \psi_2(t, x)|s|^{q-1} + \psi_3(t, x), \]

(2.4)

\[ \frac{\partial f}{\partial s}(t, x, s) \le \psi_4(t, x), \]

(2.5)

where \(\gamma > 0\) and \(q \ge p\) are constants, \(\psi_1 \in L^1_{\text{loc}}(\mathbb{R}, L^1(\mathbb{R}^n))\), \(\psi_2 \in L^q_{\text{loc}}(\mathbb{R}, L^q(\mathbb{R}^n))\), \(\psi_3 \in L^q_{\text{loc}}(\mathbb{R}, L^q(\mathbb{R}^n))\). From now on, we always assume \(h \in H^2(\mathbb{R}^n) \cap W^{2,q}(\mathbb{R}^n)\) and use \(p_1\) and \(q_1\) to denote the conjugate exponents of \(p\) and \(q\), respectively. Since \(h \in H^2(\mathbb{R}^n) \cap W^{2,q}(\mathbb{R}^n)\) and \(q \ge p\), by [12] we find \(h \in W^{2,p}(\mathbb{R}^n)\).
To define a random dynamical system for (2.1), we need to transfer the stochastic equation to a pathwise deterministic system. As usual, let \( z \) be the random variable given by:

\[
z(\omega) = -\lambda \int_{-\infty}^{0} e^{\lambda \tau} \omega(\tau) d\tau, \quad \omega \in \Omega.
\]

It follows from [2] that there exists a \( \theta_t \)-invariant set \( \tilde{\Omega} \) of full measure such that \( z(\theta_t \omega) \) is continuous in \( t \) and \( \lim_{t \to \pm \infty} \frac{|z(\theta_t \omega)|}{|t|} = 0 \) for all \( \omega \in \tilde{\Omega} \). We also assume \( \eta(\theta_t \omega) \) is pathwise continuous for each fixed \( \omega \in \tilde{\Omega} \). For convenience, we will denote \( \tilde{\Omega} \) by \( \Omega \) in the sequel. Let \( u(t, \tau, \omega, u_\tau) \) be a solution of problem (2.1)-(2.2) with initial condition \( u_\tau \) at initial time \( \tau \), and define

\[
v(t, \tau, \omega, v_\tau) = u(t, \tau, \omega, u_\tau) - \varepsilon h(x) z(\theta_t \omega) \quad \text{with} \quad v_\tau = u_\tau - \varepsilon h z(\theta_\tau \omega).
\]

By (2.1) and (2.6), after simple calculations, we get

\[
\frac{\partial v}{\partial t} - \text{div} \left( |\nabla (v + \varepsilon h(x) z(\theta_t \omega))|^{p-2} \nabla (v + \varepsilon h(x) z(\theta_t \omega)) \right) + \lambda v
\]

\[
= f(t, x, v + \varepsilon h(x) z(\theta_t \omega)) + g(t, x) + \alpha \eta(\theta_\omega) v + \alpha \varepsilon \eta(\theta_t \omega) z(\theta_t \omega) h,
\]

with initial condition

\[
v(\tau, x) = v_\tau(x), \quad x \in \mathbb{R}^n.
\]

In what follows, we first prove the well-posedness of problem (2)-(8) in \( L^2(\mathbb{R}^n) \), and then define a cocycle for (2.1)-(2.2).

**Definition 2.1.** Given \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( v_\tau \in L^2(\mathbb{R}^n) \), let \( v(\cdot, \tau, \omega, v_\tau) : [\tau, \infty) \to L^2(\mathbb{R}^n) \) be a continuous function with \( v \in L^p_{\text{loc}}([\tau, \infty), W^1_p(\mathbb{R}^n)) \cap L^q_{\text{loc}}([\tau, \infty), L^q(\mathbb{R}^n)) \) and \( \frac{\partial v}{\partial t} \in L^p_{\text{loc}}([\tau, \infty), (W^1_p)^*) + L^q_{\text{loc}}([\tau, \infty), L^q(\mathbb{R}^n)) + L^q_{\text{loc}}([\tau, \infty), L^q(\mathbb{R}^n)) \). We say \( v \) is a solution of (2)-(8) if \( v(\tau, \tau, \omega, v_\tau) = v_\tau \) and for every \( \xi \in W^{1, p}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \),

\[
\frac{d}{dt}(v, \xi) + \int_{\mathbb{R}^n} |\nabla (v + \varepsilon h z(\theta_t \omega))|^{p-2} \nabla (v + \varepsilon h z(\theta_t \omega)) \cdot \nabla \xi dx + (\lambda - \alpha \eta(\theta_t \omega)) (v, \xi)
\]

\[
= \int_{\mathbb{R}^n} f(t, x, v + \varepsilon h z(\theta_t \omega)) \xi dx + (g(t, \cdot), \xi) + \alpha \varepsilon \eta(\theta_t \omega) z(\theta_t \omega) (h, \xi)
\]

in the sense of distribution on \( [\tau, \infty) \).

Next, we prove the existence and uniqueness of solutions of (2)-(8) in \( L^2(\mathbb{R}^n) \). To this end, we set \( \mathcal{O}_k = \{ x \in \mathbb{R}^n : |x| < k \} \) for each \( k \in \mathbb{N} \) and consider the following equation defined in \( \mathcal{O}_k \):

\[
\frac{\partial v_k}{\partial t} - \text{div} \left( |\nabla (v_k + \varepsilon h(x) z(\theta_t \omega))|^{p-2} \nabla (v_k + \varepsilon h(x) z(\theta_t \omega)) \right) + \lambda v_k
\]
\[ = f(t, x, v_k + \varepsilon h(x)z(\theta t\omega)) + g(t, x) + \alpha \eta(\theta t\omega)v_k + \alpha \varepsilon \eta(\theta t\omega)z(\theta t\omega)h, \quad (2.9) \]

with boundary condition

\[ v_k(t, x) = 0 \quad \text{for all} \ t > \tau \text{ and} \ |x| = k \quad (2.10) \]

and initial condition

\[ v(\tau, x) = v_\tau(x) \quad \text{for all} \ x \in \mathcal{O}_k. \quad (2.11) \]

By the arguments in [37], one can show that if (2.3)-(2.5) are fulfilled, then for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), system (2) has a unique solution \( v_k(\cdot, \tau, \omega, v_\tau) \) in the sense of Definition 2.1 with \( \mathbb{R}^n \) replaced by \( \mathcal{O}_k \). Moreover, \( v_k(t, \tau, \omega, v_\tau) \) is \( (\mathcal{F}, \mathcal{B}(L^2(\mathcal{O}_k))) \)-measurable with respect to \( \omega \in \Omega \). We now investigate the limiting behavior of \( v_k \) as \( k \to \infty \). For convenience, we write \( V_k = W^{1,p}(\mathcal{O}_k) \) and \( V = W^{1,p}(\mathbb{R}^n) \). Let \( A : V_k \to V_k^* \) be the operator given by

\[ (A(v_1), v_2)(V_k^*, V_k) = \int_{\mathcal{O}_k} |\nabla v_1|^{p-2}\nabla v_1 \cdot \nabla v_2 \, dx, \quad \text{for all} \ v_1, v_2 \in V_k, \quad (2.12) \]

where \((\cdot, \cdot)(V_k^*, V_k)\) is the duality pairing of \( V_k^* \) and \( V_k \). Note that \( A \) is a monotone operator as in [32] and \( A : V \to V^* \) is also well defined by replacing \( \mathcal{O}_k \) by \( \mathbb{R}^n \) in (2.12). The following uniform estimates on \( v_k \) are useful.

**Lemma 2.2.** Suppose (2.3)-(2.5) hold. Then for every \( T > 0, \tau \in \mathbb{R}, \omega \in \Omega \) and \( v_\tau \in L^2(\mathbb{R}^n) \), the solution \( v_k(t, \tau, \omega, v_\tau) \) of system (2) has the properties:

\[ \{v_k\}_{k=1}^\infty \text{ is bounded in } L^\infty(\tau, \tau + T; L^2(\mathcal{O}_k)) \bigcap L^q(\tau, \tau + T; L^p(\mathcal{O}_k)) \bigcap L^p(\tau, \tau + T; V_k), \]

\[ \{A(v_k + \varepsilon h z(\theta t\omega))\}_{k=1}^\infty \text{ is bounded in } L^{p_1}(\tau, \tau + T; V_k^*) \text{ with } \frac{1}{p_1} + \frac{1}{p} = 1, \]

\[ \{f(t, x, v_k + \varepsilon h z(\theta t\omega))\}_{k=1}^\infty \text{ is bounded in } L^{q_1}(\tau, \tau + T; L^{q_1}(\mathcal{O}_k)), \quad \frac{1}{q_1} + \frac{1}{q} = 1, \]

and

\[ \left\{ \frac{dv_k}{dt} \right\} \text{ is bounded in } L^{p_1}(\tau, \tau + T; V_k^*) + L^2(\tau, \tau + T; L^2(\mathcal{O}_k)) + L^{q_1}(\tau, \tau + T; L^{q_1}(\mathcal{O}_k)). \]

**Proof.** By (2) we get

\[ \frac{1}{2} \frac{d}{dt} ||v_k||^2 + \int_{\mathcal{O}_k} |\nabla (v_k + \varepsilon h z(\theta t\omega))|^{p-2}\nabla (v_k + \varepsilon h z(\theta t\omega)) \cdot \nabla v_k \, dx + \lambda ||v_k||^2 \]

\[ = \int_{\mathcal{O}_k} f(t, x, v_k + \varepsilon h z(\theta t\omega))v_k \, dx + (g(t), v_k) + \alpha \eta(\theta t\omega)||v_k||^2 + \alpha \varepsilon \eta(\theta t\omega)z(\theta t\omega)(h, v_k). \quad (2.13) \]
For the second term on the left-hand side of (2), by Young’s inequality we obtain
\[
\int_{\mathcal{O}_k} |\nabla (v_k + \varepsilon \text{h}z(\theta t \omega))|^{p-2} \nabla (v_k + \varepsilon \text{h}z(\theta t \omega)) \cdot \nabla v_k \, dx
\]
\[
= \int_{\mathcal{O}_k} |\nabla (v_k + \varepsilon \text{h}z(\theta t \omega))|^p \, dx - \int_{\mathcal{O}_k} |\nabla (v_k + \varepsilon \text{h}z(\theta t \omega))|^{p-2} \nabla (v_k + \varepsilon \text{h}z(\theta t \omega)) \cdot \nabla \varepsilon \text{h}z(\theta t \omega) \, dx
\]
\[
\geq \frac{1}{2} \int_{\mathcal{O}_k} |\nabla (v_k + \varepsilon \text{h}z(\theta t \omega))|^p \, dx - c_1 |\varepsilon \text{h}z(\theta t \omega)|^p \|\nabla \varepsilon h\|^p_p. \tag{2.14}
\]

For the first term on the right-hand side of (2), by (2.3) and (2.4) we get
\[
\int_{\mathcal{O}_k} f(t, x, v_k + \varepsilon \text{h}z(\theta t \omega)) \, v_k \, dx
\]
\[
= \int_{\mathcal{O}_k} f(t, x, v_k + \varepsilon \text{h}z(\theta t \omega))(v_k + \varepsilon \text{h}z(\theta t \omega)) \, dx - \varepsilon(\theta t \omega) \int_{\mathcal{O}_k} f(t, x, v_k + \varepsilon \text{h}z(\theta t \omega)) h(x) \, dx
\]
\[
\leq -\gamma \int_{\mathcal{O}_k} |v_k + \varepsilon \text{h}z(\theta t \omega)|^q \, dx + \int_{\mathcal{O}_k} \psi_1(t, x) \, dx
\]
\[
+ \int_{\mathcal{O}_k} \psi_2(t, x) |v_k + \varepsilon \text{h}z(\theta t \omega)|^q |\varepsilon \text{h}z(\theta t \omega)| \, dx + \int_{\mathcal{O}_k} \psi_3(t, x) |\varepsilon \text{h}z(\theta t \omega)| \, dx
\]
\[
\leq -\frac{\gamma}{2} \|v_k + \varepsilon \text{h}z(\theta t \omega)\|^q_q + \|\psi_1(t)\|_1 + \|\psi_2(t)\|_{q_1} + c_2 \int_{\mathcal{O}_k} |\varepsilon \text{h}z(\theta t \omega)|^q \, dx. \tag{2.15}
\]

By Young’s inequality we obtain
\[
\int_{\mathcal{O}_k} g(t, x) v_k \, dx + \alpha \varepsilon \eta(\theta t \omega) z(\theta t \omega) \int_{\mathcal{O}_k} h(x) v_k \, dx
\]
\[
\leq \frac{4}{\lambda} |\alpha \varepsilon \eta(\theta t \omega) z(\theta t \omega)|^2 \|h\|^2 + \frac{4}{\lambda} \|g(t)\|^2 + \frac{\lambda}{8} \|v_k\|^2. \tag{2.16}
\]

It follows from (2.2) that
\[
\frac{d}{dt} \|v_k\|^2 + \frac{7}{4} \lambda \|v_k\|^2 + \int_{\mathcal{O}_k} |\nabla (v_k + \varepsilon \text{h}z(\theta t \omega))|^p \, dx + \gamma \int_{\mathcal{O}_k} |v_k + \varepsilon \text{h}z(\theta t \omega)|^q \, dx
\]
\[
\leq 2\alpha \eta(\theta t \omega) \|v_k\|^2 + c_3 (|\varepsilon z(\theta t \omega)|^p + |\varepsilon z(\theta t \omega)|^q + |\alpha \varepsilon \eta(\theta t \omega) z(\theta t \omega)|^2)
\]
\[
+ c_4 (\|g(t)\|^2 + \|\psi_1(t)\|_1 + \|\psi_3(t)\|_{q_1}). \tag{2.17}
\]

Multiplying (2) by \(e^{\frac{\varepsilon}{2} \lambda(s-t)} f_{\theta} \eta(\theta t \omega) \, dr\), and then integrating from \(\tau\) to \(t\), we get
\[
\|v_k(t, \tau, \omega, v_{\tau})\|^2 + \int_{\tau}^{t} e^{\frac{\varepsilon}{2} \lambda(s-t)} f_{\theta} \eta(\theta t \omega) \, dr \int_{\mathcal{O}_k} |\nabla (v_k(s, \tau, \omega, v_{\tau}) + \varepsilon \text{h}z(\theta s \omega))|^p_p \, dx \, ds
\]
\[
+ \gamma \int_{\tau}^{t} e^{\frac{\varepsilon}{2} \lambda(s-t)} f_{\theta} \eta(\theta t \omega) \, dr \int_{\mathcal{O}_k} |v_k(s, \tau, \omega, v_{\tau}) + \varepsilon \text{h}z(\theta s \omega)|^q_q \, dx \, ds
\]
\begin{equation}
\leq c_3 \int_{\tau}^{t} e^{\frac{1}{2}\lambda (s-t)-2a f_t^s \eta (\theta t \omega) ds} (|\varepsilon z(\theta t \omega)|^p + |\varepsilon z(\theta t \omega)|^q + |\alpha \varepsilon \eta (\theta t \omega) z(\theta t \omega)|^2) ds
\end{equation}

\begin{equation}
+c_4 \int_{\tau}^{t} e^{\frac{1}{2}\lambda (s-t)-2a f_t^s \eta (\theta t \omega) ds} (\|g(s)\|^2 + \|\psi_1(s)\|_1 + \|\psi_3(s)\|_q^q) ds
\end{equation}

By (2) we get

\{v_k\} is bounded in \(L^\infty (\tau, \tau + T; L^2(\Omega_k)) \cap L^q(\tau, \tau + T; L^q(\Omega_k)) \cap L^p(\tau, \tau + T; V_k).\) (2.19)

By (2.4) and (2.19) we obtain

\{f(t, x, v_k + \varepsilon h z(\theta t \omega))\}_{k=1}^{\infty} is bounded in \(L^{q_1}(\tau, \tau + T; L^{q_1}(\Omega_k)).\) (2.20)

By (2.12) and (2.19) we get

\{A(v_k + \varepsilon h z(\theta t \omega))\}_{k=1}^{\infty} is bounded in \(L^{p_1}(\tau, \tau + T; V^*_k).\) (2.21)

By (2.19)-(2.21) it follows from (2) that

\(\frac{dv_k}{dt}\) is bounded in \(L^{p_1}(\tau, \tau + T; V^*_k) + L^2(\tau, \tau + T; L^2(\Omega_k)) + L^{q_1}(\tau, \tau + T; L^{q_1}(\Omega_k))\),

which completes the proof.

The next lemma is concerned with the well-posedness of (2)-(2.8) in \(L^2(\mathbb{R}^n)\).

**Lemma 2.3.** Suppose (2.3)-(2.5) hold. Then for every \(\tau \in \mathbb{R}, \omega \in \Omega\) and \(v_\tau \in L^2(\mathbb{R}^n)\), problem (2)-(2.8) has a unique solution \(v(t, \tau, \omega, v_\tau)\) in the sense of Definition 2.1. In addition, \(v(t, \tau, \omega, v_\tau)\) is \((F, B(L^2(\mathbb{R}^n)))\)-measurable in \(\omega\) and continuous in \(v_\tau\) in \(L^2(\mathbb{R}^n)\) and satisfies

\begin{equation}
\frac{d}{dt}\|v(t, \tau, \omega, v_\tau)\|^2 + 2(\lambda - \alpha \eta (\theta t \omega)) \|v\|^2 + 2\|\nabla(v + \varepsilon h z(\theta t \omega))\|^p
\end{equation}

\begin{equation}
= 2\varepsilon z(\theta t \omega) \int_{\mathbb{R}^n} |\nabla(v + \varepsilon h z(\theta t \omega))|^{p-2} \nabla(v + \varepsilon h z(\theta t \omega)) \cdot \nabla h dx
\end{equation}

\begin{equation}
+ 2 \int_{\mathbb{R}^n} f(t, x, v + \varepsilon h z(\theta t \omega)) \omega dx + 2(g(t), v) + 2\alpha \varepsilon \eta (\theta t \omega) z(\theta t \omega)(h, v)
\end{equation}

for almost all \(t \geq \tau\).
Proof. Let $T > 0$, $t_0 \in [\tau, \tau + T]$ and $v_k(t, \tau, \omega, v_\tau)$ be the solution of system (2)-(2.11) defined in $O_k$. Extend $v_k$ to the entire space $\mathbb{R}^n$ by setting $v_k = 0$ on $\mathbb{R}^n \setminus O_k$ and denote this extension still by $v_k$. By Lemma 2.2 we find that there exist $\bar{v} \in L^2(\mathbb{R}^n)$, $v \in L^\infty(\tau, \tau + T; L^2(\mathbb{R}^n)) \cap L^p(\tau, \tau + T; V) \cap L^q(\tau, \tau + T; L^q(\mathbb{R}^n))$, $\chi_1 \in L^q(\tau, \tau + T; L^q(\mathbb{R}^n))$, $\chi_2 \in L^q(\tau, \tau + T; V^*)$ such that, up to a subsequence,

\begin{align}
v_k &\to v \text{ weak-star in } L^\infty(\tau, \tau + T; L^2(\mathbb{R}^n)), \\
v_k &\to v \text{ weakly in } L^p(\tau, \tau + T; V) \text{ and } L^q(\tau, \tau + T; L^q(\mathbb{R}^n)), \\
A(v_k + \varepsilon h z(\theta_\tau \omega)) &\to \chi_2 \text{ weakly in } L^{p_1}(\tau, \tau + T; V^*), \\
f(t, x, v_k + \varepsilon h z(\theta_\tau \omega)) &\to \chi_1 \text{ weakly in } L^{q_1}(\tau, \tau + T; L^{q_1}(\mathbb{R}^n)),
\end{align}

and

\begin{equation}
v_k(t_0, \tau, \omega, v_\tau) \to \bar{v} \text{ weakly in } L^2(\mathbb{R}^n). \tag{2.27}\end{equation}

On the other hand, by the compactness of embedding $W^{1,p}(O_k) \hookrightarrow L^2(O_k)$ and Lemma 2.2 we can choose a further subsequence (not relabeled) by a diagonal process such that for each $k_0 \in \mathbb{N}$,

\begin{equation}
v_k \to v \text{ strongly in } L^2(\tau, \tau + T; L^2(O_{k_0})). \tag{2.28}\end{equation}

By (2) and (2.23)-(2.26) one can show that for every $\xi \in V \cap L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$,

\begin{align}
\frac{d}{dt}(v, \xi) &+ (\chi_2, \xi)(V^*, V) + (\lambda - \alpha \eta(\theta_\tau \omega))(v, \xi) \\
&= (\chi_1, \xi)L^1(\tau, \beta_1) + (g(t), \xi) + \alpha \varepsilon \eta(\theta_\tau \omega) z(\theta_\tau \omega)(h, \xi) \tag{2.29}
\end{align}

in the sense of distribution. By (2.29) we find

\begin{equation}
\frac{dv}{dt} = -\chi_2 + \chi_1 - (\lambda - \alpha \eta(\theta_\tau \omega))v + g + \alpha \varepsilon \eta(\theta_\tau \omega) z(\theta_\tau \omega)h \tag{2.30}\end{equation}

in $L^{p_1}(\tau, \tau + T; V^*) + L^{q_1}(\tau, \tau + T; L^{q_1}(\mathbb{R}^n)) + L^2(\tau, \tau + T; L^2(\mathbb{R}^n))$, which along with the fact $v \in L^\infty(\tau, \tau + T; L^2(\mathbb{R}^n)) \cap L^p(\tau, \tau + T; V) \cap L^q(\tau, \tau + T; L^q(\mathbb{R}^n))$ implies (see, e.g., [28]) that $v \in C([\tau, \tau + T], L^2(\mathbb{R}^n))$ and

\begin{equation}
\frac{1}{2} \frac{d}{dt} \|v\|^2 = \langle \frac{dv}{dt}, v \rangle_{(V^* + L^1 + L^2, V \cap L^q \cap L^2)} \text{ for almost all } t \in (\tau, \tau + T). \tag{2.31}\end{equation}

By (2.23)-(2.26), we can argue as in [37] to show that

\begin{align}
\chi_2 &= A(v + \varepsilon h z(\theta_\tau \omega)), \\
\chi_1 &= f(t, x, v + \varepsilon h z(\theta_\tau \omega)), \\
v(\tau) &= v_\tau \text{ and } v(t_0) = \bar{v}. \tag{2.32}\end{align}
By (2.29) and (2.32) we find that \( v \) is a solution of problem (2)-(2.8) in the sense of Definition 2.1. On the other hand, by (2.30) and (2.32) we see that \( v \) satisfies energy equation (2.22).

We next prove the uniqueness of solutions. Let \( v_1 \) and \( v_2 \) be the solutions of (2) and \( \tilde{v} = v_1 - v_2 \). Then we have

\[
\frac{d\tilde{v}}{dt} + A(v_1 + \varepsilon h(z(\theta t))) - A(v_2 + \varepsilon h(z(\theta t))) + \lambda \tilde{v} = \alpha \eta(\theta t) \tilde{v} + f(t, x, v_1 + \varepsilon h(z(\theta t))) - f(t, x, v_2 + \varepsilon h(z(\theta t))),
\]

which along with (2.5) and the monotonicity of \( A \) yields, for all \( t \in [\tau, \tau + T] \),

\[
\frac{d}{dt} \| \tilde{v} \|^2 \leq 2\alpha \eta(\theta t) \| \tilde{v} \|^2 + 2 \int_{\mathbb{R}^n} \psi_4(t, x) |\tilde{v}|^2 dx \leq c \| \tilde{v} \|^2
\]

for some positive constant \( c \) depending on \( \tau, T \) and \( \omega \). By Gronwall’s lemma we get, for all \( t \in [\tau, \tau + T] \),

\[
\| v_1(t, \tau, \omega, v_{1,\tau}) - v_2(t, \tau, \omega, v_{2,\tau}) \|^2 \leq e^{c(t-\tau)} \| v_{1,\tau} - v_{2,\tau} \|^2.
\]

(2.33)

So the uniqueness and continuity of solutions in initial data follow immediately.

Note that (2.27), (2.32) and the uniqueness of solutions imply that the entire sequence \( v_k(t_0, \tau, \omega, v_\tau) \to v(t_0, \tau, \omega, v_\tau) \) weakly in \( L^2(\mathbb{R}^n) \) for every fixed \( t_0 \in [\tau, \tau + T] \) and \( \omega \in \Omega \). By the measurability of \( v_k(t, \tau, \omega, v_\tau) \) in \( \omega \), we obtain the measurability of \( v(t, \tau, \omega, v_\tau) \) directly.

The following result is useful when proving the asymptotic compactness of solutions.

**Lemma 2.4.** Let (2.3) - (2.5) hold and \( \{v_n\}_{n=1}^{\infty} \) be a bounded sequence in \( L^2(\mathbb{R}^n) \). Then for every \( \tau \in \mathbb{R}, \ t > \tau \) and \( \omega \in \Omega \), there exist \( v_0 \in L^2(\tau, t; L^2(\mathbb{R}^n)) \) and a subsequence \( \{v(\cdot, \tau, \omega, v_{n_m})\}_{m=1}^{\infty} \) of \( \{v(\cdot, \tau, \omega, v_n)\}_{n=1}^{\infty} \) such that \( v(s, \tau, \omega, v_{n_m}) \to v_0(s) \) in \( L^2(\mathcal{O}_k) \) as \( m \to \infty \) for every fixed \( k \in \mathbb{N} \) and for almost all \( s \in (\tau, t) \).

**Proof.** Let \( T \) be a sufficiently large number such that \( t \in (\tau, \tau + T) \). Following the proof of (2.28), we can show that there exists \( \tilde{v} \in L^2(\tau, \tau + T; L^2(\mathbb{R}^n)) \) such that, up to a subsequence,

\[
v(\cdot, \tau, \omega, v_n) \to \tilde{v} \text{ strongly in } L^2(\tau, \tau + T; L^2(\mathcal{O}_k)) \text{ for every } k \in \mathbb{N}.
\]

Thus, for \( k = 1 \), there exist a set \( I_1 \subseteq [\tau, \tau + T] \) of measure zero and a subsequence \( v(\cdot, \tau, \omega, v_{n_1}) \) such that

\[
v(s, \tau, \omega, v_{n_1}) \to \tilde{v}(s) \text{ in } L^2(\mathcal{O}_1) \text{ for all } s \in [\tau, \tau + T] \setminus I_1.
\]
Similarly, for $k = 2$, there exist a set $I_2 \subseteq [\tau, \tau + T]$ of measure zero and a subsequence $v(\cdot, \tau, \omega, v_{n_2})$ of $v(\cdot, \tau, \omega, v_{n_1})$ such that

$$v(s, \tau, \omega, v_{n_2}) \to \bar{v}(s) \in L^2(O_2) \quad \text{for all} \quad s \in [\tau, \tau + T] \setminus I_2.$$ 

Repeating this process we find that for each $k \in \mathbb{N}$, there exist a set $I_k \subseteq [\tau, \tau + T]$ of measure zero and a subsequence $v(\cdot, \tau, \omega, v_{n_k})$ of $v(\cdot, \tau, \omega, v_{n_{k-1}})$ such that

$$v(s, \tau, \omega, v_{n_k}) \to \bar{v}(s) \in L^2(O_k) \quad \text{for all} \quad s \in [\tau, \tau + T] \setminus I_k.$$ 

Let $I = \bigcup_{k=1}^{\infty} I_k$. Then by a diagonal process, we infer that there exists a subsequence (which is still denoted by $v(\cdot, \tau, \omega, v_{n_k})$) such that

$$v(s, \tau, \omega, v_{n_k}) \to \bar{v}(s) \in L^2(O_k) \quad \text{for all} \quad s \in [\tau, \tau + T] \setminus I \quad \text{and} \quad k \in \mathbb{N}. \quad (2.34)$$ 

Note that $I$ has measure zero and $t \in (\tau, \tau + T]$, which along with $(2.34)$ completes the proof. \qed

Based on Lemma 2.3, we can define a continuous cocycle for problem (2.1)-(2.2) in $L^2(\mathbb{R}^n)$. Let $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be a mapping given by, for every $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$ and $u_\tau \in L^2(\mathbb{R}^n)$,

$$\Phi(t, \tau, \omega, u_\tau) = v(t + \tau, \tau, \theta_{-\tau}\omega, v_\tau) + \varepsilon h(x)z(\theta_t\omega), \quad (2.35)$$

where $v$ is the solution of system (2)–(2.8) with initial condition $v_\tau = u_\tau - \varepsilon h(x)z(\omega)$ at initial time $\tau$. Note that (2.6) and (2.35) imply

$$\Phi(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau), \quad (2.36)$$

where $u$ is a solution of (2.1)-(2.2) in some sense. Since the solution $v$ of (2)–(2.8) is $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^n)))$-measurable in $\omega$ and continuous in initial data in $L^2(\mathbb{R}^n)$, we find that $\Phi(t, \tau, \omega, u_\tau)$ given by (2.35) is also $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^n)))$-measurable in $\omega$ and continuous in $u_\tau$ in $L^2(\mathbb{R}^n)$. In fact, one can verify that $\Phi$ is a continuous cocycle on $L^2(\mathbb{R}^n)$ over $(\Omega, \mathcal{F}, P, \{\theta\}_{t \in \mathbb{R}})$ in the sense that for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $t, s \in \mathbb{R}^+$,

(i) $\Phi(\cdot, \tau, \cdot) : \mathbb{R}^+ \times \Omega \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(L^2(\mathbb{R}^n)), \mathcal{B}(L^2(\mathbb{R}^n)))$-measurable.

(ii) $\Phi(0, \tau, \omega, \cdot)$ is the identity map on $L^2(\mathbb{R}^n)$.

(iii) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s\omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$. 

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(iv) $\Phi(t, \tau, \omega, \cdot) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is continuous.

Note that the cocycle property (iii) of $\Phi$ can be easily proved by (2.35) and the properties of the solution $v$ of the pathwise deterministic equation (2.7)-(2.8). Our goal is to establish the existence of random attractors of $\Phi$ with an appropriate attraction domain. To specify such an attraction domain, we consider a family $D = \{D(\tau, \omega) \subseteq L^2(\mathbb{R}^n) : \tau \in \mathbb{R}, \omega \in \Omega\}$ of bounded nonempty sets such that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{s \to -\infty} e^{\lambda s + 2\alpha \int_0^s \eta(\theta_r \omega) dr} \|D(\tau + s, \theta_s \omega)\|^2 = 0,$$

where $\|S\| = \sup_{u \in S} \|u\|_{L^2(\mathbb{R}^n)}$ for a nonempty bounded subset $S$ of $L^2(\mathbb{R}^n)$. In the sequel, we will use $D$ to denote the collection of all families with property (2.37):

$$D = \left\{ D = \left\{ D(\tau, \omega) \subseteq L^2(\mathbb{R}^n) : \tau \in \mathbb{R}, \omega \in \Omega \right\} : D \text{ satisfies } (2.37) \right\}.$$

(2.38)

We will construct a $D$-pullback attractor $A = \{A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$ for $\Phi$ in $L^2(\mathbb{R}^n)$ in the following sense:

(i) $A$ is measurable and $A(\tau, \omega)$ is compact for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

(ii) $A$ is invariant, that is, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Phi(t, \tau, \omega, A(\tau, \omega)) = A(\tau + t, \theta_t \omega), \quad \forall t \geq 0.$$

(iii) For every $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$ and for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t \to \infty} \text{dist}_{L^2(\mathbb{R}^n)}(\Phi(t, \tau - t, \theta_{-t} \omega, B(\tau - t, \theta_{-t} \omega)), A(\tau, \omega)) = 0,$$

where $\text{dist}_{L^2(\mathbb{R}^n)}$ is the Hausdorff semi-distance between two sets in $L^2(\mathbb{R}^n)$.

We will apply the following result from [35] to show the existence of $D$-pullback attractors for $\Phi$. Similar results on existence of random attractors can be found in [3, 12, 16, 18, 23, 30].

**Proposition 2.5.** Let $D$ be the collection given by (2.38). If $\Phi$ is $D$-pullback asymptotically compact in $L^2(\mathbb{R}^n)$ and $\Phi$ has a closed measurable $D$-pullback absorbing set $K$ in $D$, then $\Phi$ has a unique $D$-pullback attractor $A$ in $L^2(\mathbb{R}^n)$ which is given by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$A(\tau, \omega) = \Omega(K, \tau, \omega) = \bigcup_{B \in D} \Omega(B, \tau, \omega),$$

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where $$\Omega(K) = \{ \Omega(K, \tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \}$$ is the $$\omega$$-limit set of $$K$$.

If, in addition, there is $$T > 0$$ such that $$\Phi(t, \tau + T, \omega, \cdot) = \Phi(t, \tau, \omega, \cdot)$$ and $$k(\tau + T, \omega) = k(\tau, \omega)$$ for all $$t \in \mathbb{R}^+, \tau \in \mathbb{R}$$ and $$\omega \in \Omega$$, then the attractor $$A$$ is pathwise $$T$$-periodic, i.e., $$A(\tau + T, \omega) = A(\tau, \omega)$$ for all $$\tau \in \mathbb{R}$$ and $$\omega \in \Omega$$.

We remark that the $$\mathcal{F}$$-measurability of the attractor $$A$$ was given in [36] and the measurability of $$A$$ with respect to the $$P$$-completion of $$\mathcal{F}$$ was given in [35]. For our purpose, we further assume the following condition on $$g$$, $$\psi_1$$ and $$\psi_3$$: for every $$\tau \in \mathbb{R},$$

$$\int_{-\infty}^{\tau} e^{\lambda s} \left( \|g(s, \cdot)\|^2 + \|\psi_1(s, \cdot)\|_{L^1(\mathbb{R}^n)} + \|\psi_3(s, \cdot)\|_{q_1}^{q_1(1)} \right) ds < \infty. \quad (3.1)$$

## 3 Uniform Estimates of Solutions

This section is devoted to uniform estimates of solutions of (2.1) and (2) which are needed for proving the existence of random attractors for $$\Phi$$. When deriving uniform estimates, the following positive number $$\alpha_0$$ is useful:

$$\alpha_0 = \frac{1}{8(1 + |E(\eta)|)} \lambda. \quad (3.1)$$

**Lemma 3.1.** Let $$\alpha_0$$ be the positive number given by (3.1). Suppose (2.3)-(2.5) and (2.39) hold. Then for every $$\alpha \leq \alpha_0$$, $$\sigma \in \mathbb{R}$$, $$\tau \in \mathbb{R}$$, $$\omega \in \Omega$$ and $$D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}$$, there exists $$T = T(\tau, \omega, D, \sigma, \alpha) > 0$$ such that for all $$t \geq T$$, the solution $$v$$ of problem (2)-(2.5) satisfies

$$\|v(\sigma, \tau - t, \theta_{-\tau} \omega, v_{\tau-t})\|^2 + \int_{\tau-t}^{\sigma} e^{\frac{\lambda}{2} (s-\sigma) - \alpha_0} f_{s-\tau}^{s} \eta(\theta_{s-\tau} \omega) ds \|v(s, \tau - t, \theta_{-\tau} \omega, v_{\tau-t})\|^2 ds \leq M,$$

where $$v_{\tau-t} \in D(\tau - t, \theta_{-\tau} \omega)$$ and $$M = M(\tau, \omega, \sigma, \alpha, \varepsilon)$$ is given by

$$M = c \int_{-\infty}^{\sigma-\tau} e^{\frac{\lambda}{2} (s-\sigma-\tau) - \alpha_0} f_{s-\tau}^{s} \eta(\theta_{s-\tau} \omega) ds \left( |\varepsilon z(\theta_{s-\tau} \omega)|^p + |\varepsilon z(\theta_{s-\tau} \omega)|^q + |\alpha \varepsilon \eta(\theta_{s-\tau} \omega) z(\theta_{s-\tau} \omega)|^2 \right) ds \leq M,$$

with $$c$$ being a positive constant independent of $$\tau$$, $$\omega$$, $$D$$, $$\alpha$$ and $$\varepsilon$$. 

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Proof. Using energy equation (2.22) and following the proof of (2), we obtain
\[
\frac{d}{dt} \|v\|^2 + \frac{7}{4} \lambda \|v\|^2 + \int_{\mathbb{R}^n} |\nabla (v + \epsilon h z(\theta_1 \omega))|^p dx + \gamma \int_{\mathbb{R}^n} |v + \epsilon h z(\theta_1 \omega))|^q dx
\leq 2\alpha \eta(\theta_1 \omega) \|v\|^2 + c_3 \left( |\varepsilon(z(\theta_1 \omega)|^p + |\varepsilon(z(\theta_1 \omega)|^q + |\alpha \varepsilon \eta(\theta_1 \omega) z(\theta_1 \omega)|^2 \right)
\]
\[
+ c_4 \left( \|g(t)\|^2 + \|\psi_1(t)\|_1 + \|\psi_3(t)\|_{q_1}^q \right).
\] (3.2)

Multiplying (3) by \(e^{\frac{3}{4} \lambda (t-s)} \int_{s}^{t} \eta(\theta_s \omega) ds\), and then integrating from \(t - \tau\) to \(\sigma\) with \(\sigma \geq t - \tau\), we get,
\[
\|v(\sigma, t - \tau, \omega, v_{t-\tau})\|^2 + \frac{\lambda}{2} \int_{t-\tau}^{\sigma} e^{\frac{3}{4} \lambda (s-\tau)} \int_{s}^{t} \eta(\theta_s \omega) ds \|\nabla (v(s, t - \tau, \omega, v_{t-\tau}))\|^p ds
\]
\[
+ \int_{t-\tau}^{\sigma} e^{\frac{3}{4} \lambda (s-\tau)} \int_{s}^{t} \eta(\theta_s \omega) ds \|\nabla (v(s, t - \tau, \omega, v_{t-\tau}) + \epsilon h z(\theta_s \omega))\|^q ds
\]
\[
+ \gamma \int_{t-\tau}^{\sigma} e^{\frac{3}{4} \lambda (s-\tau)} \int_{s}^{t} \eta(\theta_s \omega) ds \|v(s, t - \tau, \omega, v_{t-\tau}) + \epsilon h z(\theta_s \omega))\|^q ds
\]
\[
\leq e^{\frac{3}{4} \lambda (t-\tau-\sigma)} \int_{t-\tau}^{\sigma} \eta(\theta_s \omega) ds \|v_{t-\tau}\|^2
\]
\[
+ c_3 \int_{t-\tau}^{\sigma} e^{\frac{3}{4} \lambda (s-\tau-\sigma)} \int_{s}^{t} \eta(\theta_s \omega) ds \left( |\varepsilon_z(\theta_s \omega)|^p + |\varepsilon z(\theta_s \omega)|^q + |\alpha \varepsilon \eta(\theta_s \omega) z(\theta_s \omega)|^2 \right) ds
\]
\[
+ c_4 \int_{t-\tau}^{\sigma} e^{\frac{3}{4} \lambda (s-\tau-\sigma)} \int_{s}^{t} \eta(\theta_s \omega) ds \left( \|g(s)\|^2 + \|\psi_1(s)\|_1 + \|\psi_3(s)\|_{q_1}^q \right) ds.
\] (3.3)

Replacing \(\omega\) with \(\theta_{-\tau} \omega\) in (3), we get
\[
\|v(\sigma, t - \tau, \theta_{-\tau} \omega, v_{t-\tau})\|^2 + \frac{\lambda}{2} \int_{t-\tau}^{\sigma} e^{\frac{3}{4} \lambda (s-\tau-\sigma)} \int_{s}^{t} \eta(\theta_s \omega) ds \|v(s, t - \tau, \theta_{-\tau} \omega, v_{t-\tau})\|^2 ds
\]
\[
+ \int_{t-\tau}^{\sigma} e^{\frac{3}{4} \lambda (s-\tau-\sigma)} \int_{s}^{t} \eta(\theta_s \omega) ds \|\nabla (v(s, t - \tau, \theta_{-\tau} \omega, v_{t-\tau}) + \epsilon h z(\theta_s \omega))\|^q ds
\]
\[
+ \gamma \int_{t-\tau}^{\sigma} e^{\frac{3}{4} \lambda (s-\tau-\sigma)} \int_{s}^{t} \eta(\theta_s \omega) ds \|v(s, t - \tau, \theta_{-\tau} \omega, v_{t-\tau}) + \epsilon h z(\theta_s \omega))\|^q ds
\]
\[
\leq e^{\frac{3}{4} \lambda (t-\tau-\sigma)+2\alpha f_{t-\tau}^{s-\tau} \eta(\theta_s \omega) ds} \|v_{t-\tau}\|^2
\]
\[
+ c_3 \int_{t-\tau}^{\sigma} e^{\frac{3}{4} \lambda (s+\tau-\sigma)} \int_{s}^{t} \eta(\theta_s \omega) ds \left( |\varepsilon_z(\theta_s \omega)|^p + |\varepsilon z(\theta_s \omega)|^q + |\alpha \varepsilon \eta(\theta_s \omega) z(\theta_s \omega)|^2 \right) ds
\]
\[
+ c_4 \int_{t-\tau}^{\sigma} e^{\frac{3}{4} \lambda (s+\tau-\sigma)} \int_{s}^{t} \eta(\theta_s \omega) ds \left( \|g(s+\tau)\|^2 + \|\psi_1(s+\tau)\|_1 + \|\psi_3(s+\tau)\|_{q_1}^q \right) ds.
\] (3.4)

By the ergodicity of \(\eta\), (3.1) and (2.39) one can verify that for all \(\alpha \leq \alpha_0\),
\[
\int_{-\infty}^{\sigma} e^{\frac{3}{4} \lambda (s+\tau-\sigma)} \int_{s}^{t} \eta(\theta_s \omega) ds \left( \|g(s+\tau)\|^2 + \|\psi_1(s+\tau)\|_1 + \|\psi_3(s+\tau)\|_{q_1}^q \right) ds < \infty.
\] (3.5)
Similarly, by the temperedness of $\eta$ and $z$, we can prove that for all $\alpha \leq \alpha_0$,
\[
\int_{-\infty}^{\sigma-\tau} e^{\frac{1}{2} \lambda(s+\tau-\sigma)-2\alpha} f_{s-r} \eta(\theta, \omega) dr (|\varepsilon z(\theta, \omega)|^p + |\varepsilon z(\theta, \omega)|^q + |\alpha \varepsilon \eta(\theta, \omega) z(\theta, \omega)|^2) ds < \infty. \tag{3.6}
\]
Since $v_{\tau-t} \in D(\tau - t, \theta_{-t} \omega)$ and $D \in \mathcal{D}$, by (2.44)-(2.48) we obtain
\[
e^{\frac{1}{2} \lambda(\tau-t-\sigma)+2\alpha} f_{\tau-t} \eta(\theta, \omega) dr \|v_{\tau-t}\|^2 \\
\leq e^{\frac{1}{2} \lambda(\tau-t-\sigma)+2\alpha} f_{\tau-t} \eta(\theta, \omega) dr \|D(\tau - t, \theta_{-t} \omega)\|^2 \to 0,
\]
as $t \to \infty$. Therefore, there exists $T = T(\tau, \omega, D, \sigma, \alpha) > 0$ such that for all $t \geq T$,
\[
\int_{\tau-t}^{\sigma-\tau} e^{\frac{1}{2} \lambda(s+\tau-\sigma)-2\alpha} f_{s-r} \eta(\theta, \omega) dr \left(\|g(s + \tau)\|^2 + \|\psi_1(s + \tau)\|_1 + \|\psi_3(s + \tau)\|_{q_1}^q \right) ds,
\]
which along with (3.6) concludes the proof.

By Lemma 3.1 we obtain the following estimates.

**Lemma 3.2.** Suppose (2.36)-(2.5) and (2.39) hold. Then for every $\alpha \leq \alpha_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \alpha) > 0$ such that for all $t \geq T$ and ,

the solution $v$ of problem (2.4)-(2.8) satisfies

\[
\|v(\tau, \tau - t, \theta_{-t} \omega, v_{\tau-t})\|^2 + \int_{\tau-t}^{\tau} e^{\frac{1}{2} \lambda(s+\tau-\sigma)-2\alpha} f_{s-r} \eta(\theta, \omega) dr \|v(s, \tau - t, \theta_{-t} \omega, v_{\tau-t})\|^2 ds \\
+ \int_{\tau-t}^{\tau} e^{\frac{1}{2} \lambda(s-\tau)-2\alpha} f_{s-r} \eta(\theta, \omega) dr \|\nabla(v(s, \tau - t, \theta_{-t} \omega, v_{\tau-t}) + \varepsilon h(x) z(\theta_{s-\tau} \omega))\|^p ds \\
+ \int_{\tau-t}^{\tau} e^{\frac{1}{2} \lambda(s-\tau)-2\alpha} f_{s-r} \eta(\theta, \omega) dr \|v(s, \tau - t, \theta_{-t} \omega, v_{\tau-t}) + \varepsilon h(x) z(\theta_{s-\tau} \omega)\|_{q_1}^q ds \leq R(\tau, \omega, \alpha, \varepsilon), \tag{3.7}
\]

where $v_{\tau-t} \in D(\tau - t, \theta_{-t} \omega)$ and $R(\tau, \omega, \alpha, \varepsilon)$ is given by

\[
R(\tau, \omega, \alpha, \varepsilon) = c \int_{-\infty}^{0} e^{\frac{1}{2} \lambda s - 2\alpha} f_{c} \eta(\theta, \omega) dr (|\varepsilon z(\theta, \omega)|^p + |\varepsilon z(\theta, \omega)|^q + |\alpha \varepsilon \eta(\theta, \omega) z(\theta, \omega)|^2) ds
\]
\[
+ c \int_{-\infty}^{0} e^{\frac{1}{2} \lambda s - 2\alpha} f_{c} \eta(\theta, \omega) dr (\|g(s + \tau)\|^2 + \|\psi_1(s + \tau)\|_1 + \|\psi_3(s + \tau)\|_{q_1}^q) ds, \tag{3.8}
\]
with $c$ being a positive constant independent of $\tau$, $\omega$, $D$, $\alpha$ and $\varepsilon$. In addition, we have

\[
\lim_{t \to \infty} e^{\frac{1}{2} \lambda + 2\alpha} f_{\tau-t} \eta(\theta, \omega) dr R(\tau - t, \theta_{-t} \omega, \alpha, \varepsilon) = 0. \tag{3.9}
\]
Proof. (3.7) and (3.8) are special cases of Lemma 3.1 for \( \sigma = \tau \). We now prove (3.9). By (3.8) we have

\[ R(\tau - t, \theta_{-t \omega}; \alpha, \varepsilon) \]

\[ = c \int_{-\infty}^{-t} e^{\frac{2}{3} \lambda s - 2 \alpha} \int_{0}^{r_0} \eta(\theta_{s - t \omega}) dr \left( |\varepsilon z(\theta_{s - t \omega})|^{\nu_1} + |\varepsilon z(\theta_{s - t \omega})|^{\nu} + |\alpha \varepsilon \eta(\theta_{s - t \omega}) z(\theta_{s - t \omega})|^{\nu_1} \right) ds \]

\[ + c \int_{-\infty}^{-t} e^{\frac{2}{3} \lambda s - 2 \alpha} \int_{0}^{r_0} \eta(\theta_{s - t \omega}) dr \left( |\varepsilon z(\theta_{s - t \omega})|^{\nu_1} + |\varepsilon z(\theta_{s - t \omega})|^{\nu} + |\alpha \varepsilon \eta(\theta_{s - t \omega}) z(\theta_{s - t \omega})|^{\nu_1} \right) ds \]

\[ = c \int_{-\infty}^{-t} e^{\frac{2}{3} \lambda (t + s) - 2 \alpha} \int_{s-t}^{t} \eta(\theta_{s - t \omega}) dr \left( |\varepsilon z(\theta_{s - t \omega})|^{\nu_1} + |\varepsilon z(\theta_{s - t \omega})|^{\nu} + |\alpha \varepsilon \eta(\theta_{s - t \omega}) z(\theta_{s - t \omega})|^{\nu_1} \right) ds \]

Therefore we get

\[ e^{\frac{2}{3} \lambda t + 2 \alpha} \int_{-t}^{0} \eta(\theta_{s - t \omega}) dr R(\tau - t, \theta_{-t \omega}; \alpha, \varepsilon) \]

\[ = c \int_{-\infty}^{-t} e^{\frac{2}{3} \lambda s - 2 \alpha} \int_{0}^{r_0} \eta(\theta_{s - t \omega}) dr \left( |\varepsilon z(\theta_{s - t \omega})|^{\nu_1} + |\varepsilon z(\theta_{s - t \omega})|^{\nu} + |\alpha \varepsilon \eta(\theta_{s - t \omega}) z(\theta_{s - t \omega})|^{\nu_1} \right) ds \]

\[ + c \int_{-\infty}^{-t} e^{\frac{2}{3} \lambda s - 2 \alpha} \int_{0}^{r_0} \eta(\theta_{s - t \omega}) dr \left( |\varepsilon z(\theta_{s - t \omega})|^{\nu_1} + |\varepsilon z(\theta_{s - t \omega})|^{\nu} + |\alpha \varepsilon \eta(\theta_{s - t \omega}) z(\theta_{s - t \omega})|^{\nu_1} \right) ds. \] \hfill (3.10)

Since the integrals in (3.8) are convergent, by (3.10) we obtain \( e^{\frac{2}{3} \lambda t + 2 \alpha} \int_{-t}^{0} \eta(\theta_{s - t \omega}) dr R(\tau - t, \theta_{-t \omega}; \alpha, \varepsilon) \to 0 \) as \( t \to \infty \). This completes the proof.

Next, we derive uniform estimates on the tails of solutions of (2.3)–(2.8) outside a bounded domain. These estimates are crucial for proving the asymptotic compactness of solutions on unbounded domains.

**Lemma 3.3.** Suppose (2.3)–(2.5) and (2.39) hold. Then for every \( \nu > 0 \), \( \alpha \leq \alpha_0 \), \( \varepsilon > 0 \), \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( D \in \mathcal{D} \), there exists \( T = T(\tau, \omega, D, \alpha, \varepsilon, \nu) > 0 \) and \( K = K(\tau, \omega, D, \alpha, \varepsilon, \nu) \geq 1 \) such that for all \( t \geq T \) and \( \sigma \in [\tau - 1, \tau] \), the solution \( v \) of (2.3)–(2.8) satisfies

\[ \int_{|x| \geq K} |v(\sigma, t, \theta_{-t \omega}, v_{\tau - t})|^2 dx \leq \nu, \]

where \( v_{\tau - t} \in D(\tau - t, \theta_{-t \omega}) \). In addition, \( T(\tau, \omega, D, \alpha, \varepsilon, \nu) \) and \( K(\tau, \omega, D, \alpha, \varepsilon, \nu) \) are uniform with respect to \( \varepsilon \in (0, 1] \).
Proof. Let $\rho$ be a smooth function defined on $\mathbb{R}^+$ such that $0 \leq \rho(s) \leq 1$ for all $s \in \mathbb{R}^+$, and

$$\rho(s) = \begin{cases} 0 & \text{for } 0 \leq s \leq 1; \\ 1 & \text{for } s \geq 2. \end{cases}$$

Multiplying (2) by $\rho(|x|^2/k^2)v$ and then integrating over $\mathbb{R}^n$ we get

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^2 dx - \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \text{div}\left(|\nabla (v + \varepsilon \eta t \omega)|^{p-2} \nabla (v + \varepsilon \eta t \omega)\right) v dx$$

$$= (\alpha \varepsilon \eta t \omega - \lambda) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^2 dx + \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) f(t, x, v + \varepsilon \eta t \omega) v dx$$

$$+ \alpha \varepsilon \eta t \omega \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) h dx + \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) g(t, x) v dx. \quad (3.11)$$

For the term involving the divergence we have

$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \text{div}\left(|\nabla (v + \varepsilon \eta t \omega)|^{p-2} \nabla (v + \varepsilon \eta t \omega)\right) v dx$$

$$= - \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |\nabla (v + \varepsilon \eta t \omega)|^p dx$$

$$+ \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \text{div} (v + \varepsilon \eta t \omega) |\nabla (v + \varepsilon \eta t \omega)| \cdot \nabla (v + \varepsilon \eta t \omega) dx$$

$$- \int_{\mathbb{R}^n} \rho'\left(\frac{|x|^2}{k^2}\right) \frac{2x}{k^2} \cdot \nabla (v + \varepsilon \eta t \omega) |\nabla (v + \varepsilon \eta t \omega)|^{p-2} v dx$$

$$\leq - \frac{1}{2} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |\nabla (v + \varepsilon \eta t \omega)|^p dx + c_1 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |\nabla (v + \varepsilon \eta t \omega)|^p dx$$

$$- \frac{1}{2} \int_{k \leq |x| \leq 2k} \rho'\left(\frac{|x|^2}{k^2}\right) \frac{2x}{k^2} \cdot \nabla (v + \varepsilon \eta t \omega) |\nabla (v + \varepsilon \eta t \omega)|^{p-2} v dx$$

$$\leq c_1 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |\nabla (v + \varepsilon \eta t \omega)|^p dx + \frac{c_2}{k} \|v\|^p + \|\nabla (v + \varepsilon \eta t \omega)\|^p. \quad (3.12)$$

As in (2), for the nonlinearity $f$ we have

$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) f(t, x, v + \varepsilon \eta t \omega) v dx$$

$$\leq \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) f(t, x, v + \varepsilon \eta t \omega) (v + \varepsilon \eta t \omega) dx + \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |f(t, x, v + \varepsilon \eta t \omega)| \varepsilon \eta t \omega dx$$

$$\leq - \gamma \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v + \varepsilon \eta t \omega| q dx + \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \psi_1(t, x) dx$$

$$+ \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \psi_2(t, x) |v + \varepsilon \eta t \omega|^{q-1} \varepsilon \eta t \omega dx + \frac{c_3}{k} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \psi_3(t, x) \varepsilon \eta t \omega dx.$$
\[ \leq -\frac{\gamma}{2} \int_\mathbb{R}^n \rho(\frac{|x|^2}{k^2})|v + \varepsilon h z(\theta_1 \omega)|q \, dx + \int_\mathbb{R}^n \rho(\frac{|x|^2}{k^2})(|\psi_1(t, x)| + |\psi_3(t, x)|^q) \, dx \\
+ c_3 \int_\mathbb{R}^n \rho(\frac{|x|^2}{k^2})\varepsilon h z(\theta_1 \omega)|q \, dx. \quad (3.13) \]

Note that
\[
\alpha \varepsilon \eta(\theta_1 \omega)z(\theta_1 \omega) \int_\mathbb{R}^n \rho(\frac{|x|^2}{k^2})h v \, dx + \int_\mathbb{R}^n \rho(\frac{|x|^2}{k^2})g(t, x) v \, dx \\
\leq c_4 \int_\mathbb{R}^n \rho(\frac{|x|^2}{k^2})|\alpha \varepsilon \eta(\theta_1 \omega)z(\theta_1 \omega)|^2 \, dx \\
+ c_5 \int_\mathbb{R}^n \rho(\frac{|x|^2}{k^2})|g(t, x)|^2 \, dx + \frac{3}{8} \int_\mathbb{R}^n \rho(\frac{|x|^2}{k^2})|v|^2 \, dx. \quad (3.14) \]

It follows from (3.13) that
\[
\frac{d}{dt} \int_\mathbb{R}^n \rho(\frac{|x|^2}{k^2})|v|^2 \, dx + (\frac{5}{4} \lambda - 2\alpha \eta(\theta_1 \omega)) \int_\mathbb{R}^n \rho(\frac{|x|^2}{k^2})|v|^2 \, dx \\
\leq \frac{c_7}{k} (\|v\|^p_p + \|\nabla(v + \varepsilon h z(\theta_1 \omega))\|^p_p) \\
+ c_7 \int_\mathbb{R}^n \rho(\frac{|x|^2}{k^2})(|g(t, x)|^2 + |\psi_1(t, x)| + |\psi_3(t, x)|^q) \, dx \\
+ c_7 \int_\mathbb{R}^n \rho(\frac{|x|^2}{k^2})(|\nabla \varepsilon z(\theta_1 \omega)|^p + |\varepsilon h z(\theta_1 \omega)|^q + |\alpha \varepsilon \eta(\theta_1 \omega)z(\theta_1 \omega)|^2) \, dx. \quad (3.15) \]

Since \( h \in H^1(\mathbb{R}^n) \cap W^{1, q}(\mathbb{R}^n) \) with \( 2 \leq p \leq q \), we find that for every \( \nu > 0 \), there exists a \( K_1 = K_1(\nu) \geq 1 \) such that for all \( k \geq K_1 \),
\[
c_7 \int_\mathbb{R}^n \rho(\frac{|x|^2}{k^2})(|\nabla \varepsilon z(\theta_1 \omega)|^p + |\varepsilon h z(\theta_1 \omega)|^q + |\alpha \varepsilon \eta(\theta_1 \omega)z(\theta_1 \omega)|^2) \, dx \\
= c_7 \int_{|x| \geq k} \rho(\frac{|x|^2}{k^2})(|\nabla \varepsilon z(\theta_1 \omega)|^p + |\varepsilon h z(\theta_1 \omega)|^q + |\alpha \varepsilon \eta(\theta_1 \omega)z(\theta_1 \omega)|^2) \, dx \\
\leq \nu (|\varepsilon z(\theta_1 \omega)|^p + |\varepsilon z(\theta_1 \omega)|^q + |\alpha \varepsilon \eta(\theta_1 \omega)z(\theta_1 \omega)|^2). \quad (3.16) \]

By (3.13) we find that there exists \( K_2 = K_2(\nu) \geq K_1 \) such that for all \( k \geq K_2 \),
\[
\frac{d}{dt} \int_\mathbb{R}^n \rho(\frac{|x|^2}{k^2})|v|^2 \, dx + (\frac{5}{4} \lambda - 2\alpha \eta(\theta_1 \omega)) \int_\mathbb{R}^n \rho(\frac{|x|^2}{k^2})|v|^2 \, dx \\
\leq \nu (\|v\|^p_p + \|\nabla(v + \varepsilon h z(\theta_1 \omega))\|^p_p) \\
+ c_7 \int_{|x| \geq k} (|g(t, x)|^2 + |\psi_1(t, x)| + |\psi_3(t, x)|^q) \, dx \\
+ \nu (|\varepsilon z(\theta_1 \omega)|^p + |\varepsilon z(\theta_1 \omega)|^q + |\alpha \varepsilon \eta(\theta_1 \omega)z(\theta_1 \omega)|^2). \quad (3.17) \]
Multiplying (3) by $e^{\frac{5}{2} \lambda t - 2 \sigma \int_0^t \eta(\theta, \omega) d\tau}$, and integrating from $t - \tau$ to $\tau$ with $\sigma \geq \tau - t$, we get
\[
\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v(\sigma, \tau - t, \omega, v_{\tau-t})|^2 \, dx \leq e^{5 \lambda (\tau - t - \sigma) - 2 \alpha \int_{\tau-t}^\tau \eta(\theta, \omega) \, d\tau} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v_{\tau-t}|^2 \, dx \\
+ \nu \int_{\tau-t}^\sigma e^{5 \lambda (\tau - t - \sigma) - 2 \alpha \int_{\tau-t}^\tau \eta(\theta, \omega) \, d\tau} \left( \|v(\sigma, \tau - t, \omega, v_{\tau-t})\|_p^p + \|\nabla (v + \varepsilon h z(\theta, \omega))\|_p^p \right) \, d\sigma \\
+ \nu \int_{\tau-t}^{\sigma - \tau} e^{5 \lambda (s + \tau - \sigma) - 2 \alpha \int_{s}^{\tau - \tau} \eta(\theta, \omega) \, d\tau} \left( \|\varepsilon z(\theta, \omega)\|_p^p + \|\varepsilon z(\theta, \omega)\|_q^q + |\alpha \varepsilon \eta(\theta, \omega) z(\theta, \omega)|^2 \right) \, ds \\
+ c_7 \int_{\tau-t}^{\sigma - \tau} \int \left. \frac{5}{2} \lambda (s - \sigma) - 2 \alpha \int_{s}^{\tau - \tau} \eta(\theta, \omega) \, d\tau \right| \left( |g(s + \tau, x)|^2 + |\psi_1(s + \tau, x)| + |\psi_3(s + \tau, x)|^q \right) \, dx \, ds. \tag{3.18}
\]
Replacing $\omega$ with $\theta_t \omega$ in (4), after simple calculations, we get for all $k \geq K_2$ and $\sigma \in [\tau - 1, \tau]$,
\[
\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v(\sigma, \tau - t, \theta_t \omega, v_{\tau-t})|^2 \, dx \leq e^{5 \lambda (\tau - t - \sigma) + 2 \alpha \int_{\tau-t}^\tau \eta(\theta, \omega) \, d\tau} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v_{\tau-t}|^2 \, dx \\
+ \nu \int_{\tau-t}^\sigma e^{5 \lambda (s + \tau - \sigma) - 2 \alpha \int_{\tau-t}^{s + \tau} \eta(\theta, \omega) \, d\tau} \left( \|v(s + \tau, \theta_t \omega, v_{\tau-t})\|_p^p + \|\nabla (v + \varepsilon h z(\theta, \omega))\|_p^p \right) \, d\sigma \\
+ \nu \int_{\tau-t}^{\sigma - \tau} e^{5 \lambda (s - \tau - \sigma) - 2 \alpha \int_{s}^{\tau - \tau} \eta(\theta, \omega) \, d\tau} \left( \|\varepsilon z(\theta, \omega)\|_p^p + \|\varepsilon z(\theta, \omega)\|_q^q + |\alpha \varepsilon \eta(\theta, \omega) z(\theta, \omega)|^2 \right) \, ds \\
+ c_7 \int_{\tau-t}^{\sigma - \tau} \int \left. \frac{5}{2} \lambda (s - \tau - \sigma) - 2 \alpha \int_{s}^{\tau - \tau} \eta(\theta, \omega) \, d\tau \right| \left( |g(s + \tau, x)|^2 + |\psi_1(s + \tau, x)| + |\psi_3(s + \tau, x)|^q \right) \, dx \, ds. \tag{3.19}
\]
Since $v_{\tau-t} \in D(\tau - t, \theta_t \omega)$, we see that for every $\nu > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\alpha > 0$, there exists a $T_1(\tau, \omega, D, \alpha, \nu) > 0$ such that for every $t \geq T_1$ and $\sigma \in [\tau - 1, \tau]$,
\[
e^{\frac{5}{2} \lambda + 2 \alpha \int_{-1}^0 \eta(\theta, \omega) \, d\tau} e^{\frac{5}{2} \lambda t - 2 \alpha \int_{-1}^\tau \eta(\theta, \omega) \, d\tau} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v_{\tau-t}|^2 \, dx \\
+ \nu e^{\frac{5}{2} \lambda + 2 \alpha \int_{-1}^0 \eta(\theta, \omega) \, d\tau} e^{\frac{5}{2} \lambda t - 2 \alpha \int_{-1}^\tau \eta(\theta, \omega) \, d\tau} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v_{\tau-t}|^2 \, dx \\
+ \nu e^{\frac{5}{2} \lambda + 2 \alpha \int_{-1}^0 \eta(\theta, \omega) \, d\tau} \int_{-\infty}^0 e^{\frac{5}{2} \lambda s - 2 \alpha \int_{s}^{\tau - \tau} \eta(\theta, \omega) \, d\tau} \left( \|\varepsilon z(\theta, \omega)\|_p^p + \|\varepsilon z(\theta, \omega)\|_q^q + |\alpha \varepsilon \eta(\theta, \omega) z(\theta, \omega)|^2 \right) \, ds \\
+ c_8 \int_{-\infty}^0 \int \left. e^{\frac{5}{2} \lambda s - 2 \alpha \int_{s}^{\tau - \tau} \eta(\theta, \omega) \, d\tau} |g(s + \tau, x)|^2 + |\psi_1(s + \tau, x)| + |\psi_3(s + \tau, x)|^q \right) \, dx \, ds. \tag{3.20}
\]
as \( k \to \infty \). Therefore, there exists \( K = K_3(\tau, \omega, \alpha, \nu) \geq K_2 \) such that for all \( k \geq K_3 \),

\[
c_8 \int_{-\infty}^{0} \int_{|x| \geq k} e^{\frac{\lambda s - 2 \alpha}{K_0}} \eta(\theta, \omega) dx ds \leq \nu. \tag{3.21}
\]

Note that

\[
\|v(s, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|_p^p \leq 2^p \left( \|v(s, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|_p^p + \|\nabla z(\theta_{t+\tau}\omega)\|_p^p \right),
\]

which along with (1.2) and Lemma 3.2 shows that there exists \( T_2 = T_2(\tau, \omega, D, \alpha, \nu) \geq T_1 \) such that for all \( t \geq T_2 \),

\[
\int_{\tau - t}^{\tau} e^{\frac{\lambda (s-\tau) - 2 \alpha}{K_0}} \eta(\theta, \omega) ds \left( \|v(s, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|_p^p + \|\nabla (v + \varepsilon z)(\theta_{t+\tau}\omega)\|_p^p \right) ds \leq c_9 R(\tau, \omega, \alpha, \varepsilon) + c_{10} \int_{-\infty}^{0} e^{\frac{\lambda s - 2 \alpha}{K_0}} \eta(\theta, \omega) ds |\varepsilon z(\theta_{\tau}\omega)|^p ds, \tag{3.22}
\]

where \( R(\tau, \omega, \alpha, \varepsilon) \) is the number given by (3.8). It follows from (3.21)-(3.22) that for all \( k \geq K_3 \), \( t \geq T_2 \) and \( \sigma \in [\tau - 1, \tau] \),

\[
\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v(\sigma, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})|^2 dx \leq 2\nu + \nu c_{11} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v(\sigma, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})|^2 dx + \nu c_{12} \int_{-\infty}^{0} e^{\frac{\lambda s - 2 \alpha}{K_0}} \eta(\theta, \omega) ds \left( |\varepsilon z(\theta_{\tau}\omega)|^p + |\varepsilon z(\theta_{\tau}\omega)|^q + |\alpha \varepsilon \eta(\theta_{\tau}\omega) z(\theta_{\tau}\omega)|^2 \right) ds. \tag{3.23}
\]

Note that \( \rho\left(\frac{|x|^2}{k^2}\right) = 1 \) when \( |x|^2 \geq 2k^2 \). This along with (3.9) concludes the proof. \( \Box \)

The asymptotic compactness of solutions of equation (2) is given below.

**Lemma 3.4.** Suppose (2.3), (2.5) and (2.39) hold. Then for every \( \alpha \leq \alpha_0, \varepsilon > 0, \tau \in \mathbb{R}, \omega \in \Omega \) and \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D \), the sequence \( v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}) \) has a convergent subsequence in \( L^2(\mathbb{R}^n) \) provided \( t_n \to \infty \) and \( v_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega) \).

**Proof.** By Lemma 3.1 we find that there exists \( N_1 = N_1(\tau, \omega, D, \alpha) \geq 0 \) such that for all \( n \geq N_1 \),

\[
\|v(\tau - 1, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})\| \leq c_1. \tag{3.24}
\]

Applying Lemma 2.4 to the sequence \( v(\tau, \tau - 1, \theta_{-\tau}\omega, v(\tau - 1, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})) \), we find that there exist \( s_0 \in (\tau - 1, \tau) \), \( v_0 \in L^2(\mathbb{R}^n) \) and a subsequence (not relabeled) such that as \( n \to \infty \),

\[
v(s_0, \tau - 1, \theta_{-\tau}\omega, v(\tau - 1, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})) \to v_0 \quad \text{in } L^2(\mathbb{O}_k) \text{ for every } k \in \mathbb{N},
\]
that is, as $n \to \infty$,
\[ v(s_0, \tau - t_n, \theta_{-\tau} \omega, v_{0,n}) \to v_0 \quad \text{in } L^2(O_k) \text{ for every } k \in \mathbb{N}. \quad (3.25) \]

By (2.33) we get
\[ \|v(\tau, s_0, \theta_{-\tau} \omega, v(s_0, \tau - t_n, \theta_{-\tau} \omega, v_{0,n})) - v(\tau, s_0, \theta_{-\tau} \omega, v_0)\| \leq e^{c_1(\tau - s_0)} \|v(s_0, \tau - t_n, \theta_{-\tau} \omega, v_{0,n}) - v_0\|. \]

Since $s_0 \in (\tau - 1, \tau)$, we obtain
\[
\begin{align*}
\|v(\tau, s_0, \theta_{-\tau} \omega, v(s_0, \tau - t_n, \theta_{-\tau} \omega, v_{0,n})) - v(\tau, s_0, \theta_{-\tau} \omega, v_0)\|^2 &\leq e^{2c_1} \int_{|x| < k} |v(s_0, \tau - t_n, \theta_{-\tau} \omega, v_{0,n}) - v_0|^2 dx \\
&\quad + e^{2c_1} \int_{|x| \geq k} |v(s_0, \tau - t_n, \theta_{-\tau} \omega, v_{0,n}) - v_0|^2 dx \\
&\leq e^{2c_1} \int_{|x| < k} |v(s_0, \tau - t_n, \theta_{-\tau} \omega, v_{0,n}) - v_0|^2 dx \\
&\quad + 2e^{2c_1} \int_{|x| \geq k} (|v(s_0, \tau - t_n, \theta_{-\tau} \omega, v_{0,n})|^2 + |v_0|^2) dx. \quad (3.26)
\end{align*}
\]

Since $v_0 \in L^2(\mathbb{R}^n)$, given $\nu > 0$, there exists $K_1 = K_1(\nu) \geq 1$ such that for all $k \geq K_1$,
\[ 2e^{2c_1} \int_{|x| \geq k} |v_0|^2 ds \leq \nu. \quad (3.27) \]

On the other hand, by Lemma [5.3] there exist $N_2 = N_2(\tau, \omega, D, \alpha, \varepsilon, \nu) > 1$ and $K_2 = K_2(\tau, \omega, \alpha, \varepsilon, \nu) \geq K_1$ such that for all $n \geq N_2$ and $k \geq K_2$,
\[ 2e^{2c_1} \int_{|x| \geq k} |v(s_0, \tau - t_n, \theta_{-\tau} \omega, v_{0,n})|^2 dx \leq \nu. \quad (3.28) \]

By (3.25) we find that there exists $N_3 = N_3(\tau, \omega, D, \alpha, \varepsilon, \nu) \geq N_2$ such that for all $n \geq N_3$,
\[ e^{2c_1} \int_{|x| < K_2} |v(s_0, \tau - t_n, \theta_{-\tau} \omega, v_{0,n}) - v_0|^2 dx \leq \nu. \quad (3.29) \]

It follows from (3.25)-(3.29) that for all $n \geq N_3$,
\[ \|v(\tau, s_0, \theta_{-\tau} \omega, v(s_0, \tau - t_n, \theta_{-\tau} \omega, v_{0,n})) - v(\tau, s_0, \theta_{-\tau} \omega, v_0)\|^2 \leq 3\nu, \]

that is, for all $n \geq N_3$,
\[ \|v(\tau - t_n, \theta_{-\tau} \omega, v_{0,n}) - v(\tau, s_0, \theta_{-\tau} \omega, v_0)\|^2 \leq 3\nu. \]

Therefore, $v(\tau - t_n, \theta_{-\tau} \omega, v_{0,n})$ converges to $v(\tau, s_0, \theta_{-\tau} \omega, v_0)$ in $L^2(\mathbb{R}^n)$. This completes the proof. \[ \Box \]
4 Random Attractors

In this section, we prove the existence of $D$-pullback attractor for (2.1)-(2.2) in $L^2(\mathbb{R}^n)$ by Proposition 2.5. To this end, we need to establish the existence of $D$-pullback absorbing sets and the $D$-pullback asymptotic compactness of $\Phi$ in $L^2(\mathbb{R}^n)$. The existence of absorbing sets of $\Phi$ is given below.

**Lemma 4.1.** Suppose (2.3)-(2.5) and (2.39) hold. Then for every $\alpha \leq \alpha_0$ and $\varepsilon > 0$, the stochastic equation (2.1) with (2.2) has a closed measurable $D$-pullback absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$ which is given by

$$K(\tau, \omega) = \{u \in L^2(\mathbb{R}^n) : \|u\|^2 \leq 2\|\varepsilon h_z(\omega)\|^2 + 2R(\tau, \omega, \alpha, \varepsilon)\}, \quad (4.1)$$

where $R(\tau, \omega, \alpha, \varepsilon)$ is the number given by (3.8).

**Proof.** Let $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$. For every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, denote by

$$\widetilde{D}(\tau, \omega) = \{v \in L^2(\mathbb{R}^n) : v = u - \varepsilon h_z(\omega) \text{ for some } u \in D(\tau, \omega)\}. \quad (4.2)$$

Since $z$ is tempered, we find that the family $\widetilde{D} = \{\widetilde{D}(\tau, \omega), \tau \in \mathbb{R}, \omega \in \Omega\}$ belongs to $D$ provided $D \in D$. By (2.6) we have

$$u(\tau, \tau - t, \theta_{-t} \omega, u_{\tau-t}) = v(\tau, \tau - t, \theta_{-t} \omega, v_{\tau-t}) + \varepsilon h_z(\omega) \quad \text{with} \quad v_{\tau-t} = u_{\tau-t} - \varepsilon h_z(\theta_{-t} \omega). \quad (4.3)$$

Thus, if $u_{\tau-t} \in D(\tau - t, \theta_{-t} \omega) \in D$, then $v_{\tau-t} \in \widetilde{D}(\tau - t, \theta_{-t} \omega) \in D$. By Lemma 3.2 we find that there exists $T = T(\tau, \omega, D, \alpha, \varepsilon) > 0$ such that for all $t \geq T$,

$$\|v(\tau, \tau - t, \theta_{-t} \omega, v_{\tau-t})\|^2 \leq R(\tau, \omega, \alpha, \varepsilon),$$

where $R(\tau, \omega, \alpha, \varepsilon)$ is as in (3.8). By (4.3) we get for all $t \geq T$,

$$\|u(\tau, \tau - t, \theta_{-t} \omega, u_{\tau-t})\|^2 \leq 2\|\varepsilon h_z(\omega)\|^2 + 2R(\tau, \omega, \alpha, \varepsilon).$$

This along with (2.36) and (4.1) shows that for all $t \geq T$,

$$\Phi(t, \tau - t, \theta_{-t} \omega, D(\tau - t, \theta_{-t} \omega)) \subseteq K(\tau, \omega). \quad (4.4)$$

On the other hand, by (3.9) and the temperedness of $z$ we obtain

$$\lim_{t \to \infty} e^{-\frac{\alpha}{4}t + 2\alpha \int_{0}^{t} \eta(\theta_{r} \omega) dr} \|K(\tau - t, \theta_{-t} \omega)\| = 0. \quad (4.5)$$
By (4.4)-(4.5) we find that \( K \) given by (4.1) is a closed \( \mathcal{D} \)-pullback absorbing set of \( \Phi \) in \( \mathcal{D} \). Note that the measurability of \( K(\tau, \omega) \) in \( \omega \in \Omega \) follows from that of \( z(\omega) \) and \( R(\tau, \omega, \alpha, \varepsilon) \) immediately. This completes the proof. \( \square \)

The following is our main result regarding the existence of \( \mathcal{D} \)-pullback attractors of \( \Phi \).

**Theorem 4.2.** Suppose (2.3)-(2.5) and (2.50) hold. Then for every \( \alpha \leq \alpha_0 \) and \( \varepsilon > 0 \), the stochastic equation (2.1) with (2.2) has a unique \( \mathcal{D} \)-pullback attractor \( A = \{ A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \) in \( L^2(\mathbb{R}^n) \). In addition, if there is \( T > 0 \) such that \( f(t, x, s), g(t, x), \psi_1(t, x) \) and \( \psi_3(t, x) \) are all \( T \)-periodic in \( t \) for fixed \( x \in \mathbb{R}^n \) and \( s \in \mathbb{R} \), then the attractor \( A \) is also \( T \)-periodic.

**Proof.** We first prove that \( \Phi \) is \( \mathcal{D} \)-pullback asymptotically compact in \( L^2(\mathbb{R}^n) \); that is, for every \( \tau \in \mathbb{R}, \omega \in \Omega, \mathcal{D} \in \mathcal{D}, t_n \to \infty \) and \( u_{0,n} \in D(\tau - t_n, \theta - t_n \omega) \), we want to show that the sequence \( \Phi(t_n, \tau - t_n, \theta - t_n \omega, u_{0,n}) \) has a convergent subsequence in \( L^2(\mathbb{R}^n) \). Let \( v_{0,n} = u_{0,n} - \varepsilon h z(\theta - t_n \omega) \) and \( \overline{D} \) be the family given by (4.2). Since \( u_{0,n} \in D(\tau - t_n, \theta - t_n \omega) \), we find that \( v_{0,n} \in \overline{D}(\tau - t_n, \theta - t_n \omega) \in \mathcal{D} \). Therefore, by (4.3) and Lemma 3.4 we find that \( u(\tau, \tau - t_n, \theta - t_n \omega, u_{0,n}) \) has a convergent subsequence in \( L^2(\mathbb{R}^n) \). This together with (2.36) indicates that \( \Phi(t_n, \tau - t_n, \theta - t_n \omega, u_{0,n}) \) has a convergent subsequence, and thus it is \( \mathcal{D} \)-pullback asymptotically compact in \( L^2(\mathbb{R}^n) \). Since \( \Phi \) also has a closed measurable \( \mathcal{D} \)-pullback absorbing set \( K \) given by (4.1), by Proposition 2.5 we get the existence and uniqueness of \( \mathcal{D} \)-pullback attractor \( A \in \mathcal{D} \) of \( \Phi \) immediately.

Next, we discuss \( T \)-periodicity of \( A \). Note that if \( f \) and \( g \) are \( T \)-periodic in their first arguments, then the cocycle \( \Phi \) is also \( T \)-periodic. Indeed, in this case, for every \( t \in \mathbb{R}^+, \tau \in \mathbb{R} \) and \( \omega \in \Omega \), by (2.36) we have

\[
\Phi(t, \tau + T, \omega, \cdot) = u(t + \tau + T, \tau + T, \theta - \tau - T \omega, \cdot) = u(t + \tau, \tau, \theta - \tau \omega, \cdot) = \Phi(t, \tau, \omega, \cdot). \tag{4.6}
\]

In addition, if \( g(t, x), \psi_1(t, x) \) and \( \psi_3(t, x) \) are all \( T \)-periodic in \( t \), then by (3.8) and (4.1) we get \( K(\tau + T, \omega) = K(\tau, \omega) \) for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \). This along with (4.6) and Proposition 2.5 yields the \( T \)-periodicity of \( A \). \( \square \)

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