Quantum Correlations, Local Interactions and Error Correction

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(February 21, 2007)

We consider the effects of local interactions upon quantum mechanically entangled systems. In particular we demonstrate that non-local correlations cannot increase through local operations on any of the subsystems, but that through the use of quantum error correction methods, correlations can be maintained. We provide two mathematical proofs that local general measurements cannot increase correlations, and also derive general conditions for quantum error correcting codes. Using these we show that local quantum error correction can preserve nonlocal features of entangled quantum systems. We also demonstrate these results by use of specific examples employing correlated optical cavities interacting locally with resonant atoms. By way of counter example, we also describe a mechanism by which correlations can be increased, which demonstrates the need for non-local interactions.
I. INTRODUCTION

In the last year or so, much of the interest in quantum information theory has been directed towards two related subjects: firstly in analysing so called purification procedures \cite{1, 2} and secondly in exploring the idea of quantum error correction \cite{3, 9}, as well as examining the connections between the two \cite{10}. Purification procedures are based on Gisin’s original proposal \cite{1} to use ‘local filters’ to increase correlations between two entangled quantum subsystems. Following this a number of other schemes have been designed for the purpose of local purification \cite{3}. All of these have one idea in common: they all rely on some form of classical communication on which subsequent post-selection is based. This means that if we start with an ensemble of \(N\) pairs of particles in a mixed state, the final pure state will invariably have fewer particles. This will be seen as a consequence of the fact that local operations (i.e. generalised filters) cannot increase correlations. However, although the increase in correlations cannot be achieved, an error correction procedure can always be applied locally, which will maintain the entanglement.

We introduce the necessary information–theoretic background in section 2. In section 3 we present a simple model of atoms interacting ‘locally’ with two entangled cavities and give a number of feedback schemes by which the correlations might possibly be increased, without using any classical communication and post-selection. We show that each of these
schemes fails, and we link this to the impossibility of superluminal propagation of any signal. At the end of this section we briefly show how non–local interactions can easily be used to increase correlations. Section 4 presents two rigorous proofs of the impossibility of increasing correlations locally. In section 5 we derive general conditions for error correcting codes, which are then used in section 6 to show that local error correction, in fact, preserves correlations and entanglement. Using these considerations we then present a simple example of how to encode two cavities against a single amplitude error on either cavity using four atoms.

II. THEORETICAL BACKGROUND

In this section we introduce the information–theoretical background necessary to understand the results in this paper. We summarise the basic definitions and mathematical framework relevant to the problem, and define the key concepts and quantities which are used to characterise entanglement between systems (for a more elaborate discussion of quantum information theory see references [11,12]).

A. Information Measures

In this subsection we introduce various classical information measures [13]. Quantum analogues are then defined in the following subsection. Fundamental to our understanding of correlations is the measure of uncertainty in a given probability distribution.
Definition 1. The uncertainty in a collection of possible states with corresponding probability distribution \( p_i \) is given by an entropy:

\[
H(p) := -\sum_i p_i \ln p_i
\]  

(1)
called Shannon's entropy. We note that there is no Boltzman constant term in this expression, as there is for the physical entropy, since \( k_B \) is by convention set to unity.

We need a means of comparing two different probability distributions, and for this reason we introduce the notion of relative entropy.

Definition 2. Suppose that we have two probability distributions, \( p_i \) and \( q_i \). The Shannon relative entropy between these two distributions is defined as

\[
D(p \| q) := -\sum_i p_i \ln \frac{p_i}{q_i}.
\]  

(2)

This function is a good measure of the ‘distance’ between \( p_i \) and \( q_i \), even though, strictly speaking, it is not a mathematical distance since \( D(p \| q) \neq D(q \| p) \). Its information-theoretic significance becomes apparent through the notion of mutual information.

Definition 3. The Shannon mutual information between two random variables \( A \) and \( B \), having a joint probability distribution \( p_{i,\alpha} \), and marginal probability distributions \( p_i \) and \( p_{\alpha} \) is defined as

\[
I_S(A : B) := D(p_{i,\alpha} \| p_i p_{\alpha}) = H(p_i) + H(p_{\alpha}) - H(p_{i,\alpha})
\]  

(3)
where the Latin indices refer to $A$ and Greek indices to $B$.

The Shannon mutual information, as its name indicates, measures the quantity of information conveyed about the random variable $A$ ($B$) through measurements of the random variable $B$ ($A$). Written in the above form, the Shannon mutual information represents the ‘distance’ between the joint distribution and the product of the marginals; loosely speaking it determines how far the joint state is away from the product state, and is hence suitable as a measure of the degree of correlations between the two random variables. We now show how the above measure can be used to determine correlations between two ‘entangled’ quantum systems.

**B. Quantum Correlations and Entanglement**

The general state of a quantum system is described by its density matrix $\hat{\rho}$. If $\hat{A}$ is an operator pertaining to the system described by $\hat{\rho}$, then by the spectral decomposition theorem $\hat{A} = \sum_i a_i \hat{P}_i$, where $\hat{P}_i$ is the projection onto the state with the eigenvalue $a_i$. The probability of obtaining the eigenvalue $a_j$ is given by $p_j = \text{Tr}(\hat{P}_j \hat{\rho}) = \text{Tr}(\hat{\rho} \hat{P}_j)$. The uncertainty in a given observable can now be expressed through the Shannon entropy. However, to determine the uncertainty in the state as a whole we use the ‘von Neumann’ entropy.
Definition 4. The von Neumann entropy of a quantum system described by a density matrix \( \hat{\rho} \) is defined as

\[
S(\hat{\rho}) := -Tr(\hat{\rho} \ln \hat{\rho})
\]

(4)

The Shannon entropy is equal to the von Neumann entropy only when it describes the uncertainties in the values of a particular set of observables, called Schmidt observables \[ \text{(This is the set of observables that possesses the same spectrum as the density matrix describing the state).} \]

Definition 5. The two quantum systems \( A \) and \( B \) are said to be entangled if their joint state cannot be expressed as a convex sum of the direct products of the individual states; otherwise they are disentangled (a convex sum of direct products is a sum of the form \( \sum_i p_i \hat{\rho}_A^i \otimes \hat{\rho}_B^i \), where indices \( A \) and \( B \) refer to the first and the second subsystem respectively, and \( \sum_j p_j = 1 \)).

The prime example of the entangled state is the EPR–type state between the two 2–state systems, \( A \) and \( B \):

\[
|\psi_{AB}\rangle = \alpha |0\rangle_A |1\rangle_B + \beta |1\rangle_A |0\rangle_B .
\]

(5)

which obviously cannot be expressed as a direct product of the individual states, unless either \( \alpha \) or \( \beta \) equals zero.
To quantify the degree of correlations between the two quantum systems, we introduce the \textit{von Neumann mutual information} via the notion of the reduced density matrix. If the joint state of the two quantum systems is $\hat{\rho}_{AB}$, then the reduced density matrices of the subsystems $A$ and $B$ are given by

$$\hat{\rho}_A := \text{Tr}_B \hat{\rho}_{AB} \quad \hat{\rho}_B := \text{Tr}_A \hat{\rho}_{AB}$$ (6)

In analogy with the Shannon relative entropy between two probability distributions, we define the so called von Neumann relative entropy, as a measure of ‘distance’ between two density matrices.

\textit{Definition 6.} Given two density matrices $\hat{\rho}_A$ and $\hat{\rho}_B$ the \textit{von Neumann relative entropy} is defined as:

$$D(\hat{\rho}_A \| \hat{\rho}_B) := -\text{Tr} \hat{\rho}_A \ln \frac{\hat{\rho}_A}{\hat{\rho}_B}$$ (7)

(where $\ln \frac{\hat{\rho}_A}{\hat{\rho}_B} := \ln \hat{\rho}_A - \ln \hat{\rho}_B$).

The degree of correlation between the two quantum subsystems is given by the von Neumann mutual information, defined by analogy with the Shannon mutual information via the concept of relative entropy.

\textit{Definition 7.} \textit{The von Neumann mutual information} between the two subsystems $\hat{\rho}_A$ and $\hat{\rho}_B$ of the joint state $\hat{\rho}_{AB}$ is defined as

$$I_N(\hat{\rho}_A : \hat{\rho}_B ; \hat{\rho}_{AB}) := D(\hat{\rho}_{AB} \| \hat{\rho}_A \otimes \hat{\rho}_B)$$ (8)
\[ = S(\hat{\rho}_A) + S(\hat{\rho}_B) - S(\hat{\rho}_{AB}) \]  \hspace{1cm} (9)

From this we can see that the state in eq. (5) is maximally correlated when \(|\alpha|^2 = |\beta|^2 = \frac{1}{2}\), whereas the correlations are minimal for either \(\alpha = 0\) or \(\beta = 0\), i.e. when the state is disentangled.

In this paper we mainly focus on two systems in a joint pure state \([14]\) in which case the entropy of the overall state, \(S(\hat{\rho}_{AB})\), is zero, and the reduced entropies are equal \([15]\). In this case, there are no classical uncertainties, and then the degree of correlation is purely quantum mechanical. This is then also called the degree of entanglement. However, for mixed states it is at present not possible to separate entirely quantum from classical correlations and a good measure of entanglement does not exist (although steps towards resolution of this problem are being taken, e.g. \([16]\)), which is the reason why we use the von Neumann mutual information throughout.

C. Entropic Properties

In this subsection we present without proofs several properties of entropy which will be used in the later sections \([17]\). These are:

1. *additivity:* \[ S(\hat{\rho}_A \otimes \hat{\rho}_B) = S(\hat{\rho}_A) + S(\hat{\rho}_B); \]  \hspace{1cm} (10)

2. *concavity:* \[ S \left( \sum_i \lambda_i \hat{\rho}_i \right) \geq \sum_i \lambda_i S(\hat{\rho}_i); \]  \hspace{1cm} (11)
3. **strong subadditivity**: \[ S(\hat{\rho}_{ABC}) + S(\hat{\rho}_{B}) \leq S(\hat{\rho}_{AB}) + S(\hat{\rho}_{BC}). \] (12)

(where \(\hat{\rho}_B = \text{Tr}_{AC} \hat{\rho}_{ABC}\) and similarly for the others).

It is also worth mentioning that the consequence of the strong subadditivity is the so called weak subadditivity described by the Araki–Lieb inequality [15]: \(S(\hat{\rho}_{AB}) \leq S(\hat{\rho}_{A}) + S(\hat{\rho}_{B})\). This asserts that there is less uncertainty in the joint state of any two subsystems than if the two subsystems are considered separately. We now turn to describing two equivalent ways of *complete measurement*.

**D. Complete Measurement**

In this subsection we present two different ways to describe the dynamical evolution of a quantum system. First we can look at the joint unitary evolution of the system, \(S\), and its environment, \(E\). The environment can be a similar quantum system to the one we observe, or much larger: we leave this choice completely open in order to be as general as possible. Let the joint ‘\(S+E\)’ state initially be disentangled, \(|\psi\rangle_S|\psi\rangle_E\), after which we apply a unitary evolution \(\hat{U}_{SE}\) on ‘\(S+E\)’ resulting in the state

\[ \hat{U}_{SE}|\psi\rangle_S|\psi\rangle_E \] (13)

Since we are interested in the system’s evolution only, to obtain its final state, \(\hat{\rho}_S\), we have to trace over the environment, i.e.

\[ \hat{\rho}_S = \text{Tr}_E(\hat{U}_{SE}|\psi\rangle_S\langle\psi|_S \otimes |\psi\rangle_E\langle\psi|_E \hat{U}_{SE}^\dagger) \] (14)
Another way to obtain the same result is to exclude the environment from the picture completely by defining operators of the ‘complete measurement’ \[18\]

\[
\sum_i \hat{A}^i \hat{A}^i = \hat{I}
\]

which act on the system alone, and therefore to be equivalent to the above system’s evolution must satisfy

\[
\sum_i \hat{A}^i \ket{\psi}_S \bra{\psi}_S \hat{A}^i = \hat{\rho}_S.
\]

Let us now derive the necessary form of \(\hat{A}\)’s using eq. (13). Let an orthonormal basis of \(E\) be \(\{\ket{\phi}_E\}\). Then,

\[
\hat{A}^i = \bra{\phi}^i_E \hat{U}_{SE} \ket{\psi}_E
\]

It can easily be checked that the above \(\{\hat{A}^j\}\)’s satisfy the completeness relations in eq. (13). Since the choice of basis for \(E\) is not unique, then neither is the choice of complete measurement operators. In fact, there is an infinite number of possibilities for the operators \(\{\hat{A}^j\}\). Note that the dimension of the complete measurement, \(\hat{A}\), is in general different to the dimension of the observed system, and in fact equal to the dimension of \(E\).

III. ATOM–CAVITY MODELS

We present here a simple model which aims to increase the quantum correlations between two entangled subsystems. The model we present employs a technique of perform-
ing ‘local’ complete measurements. By this, we mean that when the two quantum systems are entangled we perform complete measurements on either subsystem separately, while not interacting directly with the other subsystem. We may regard this result to be counter-intuitive — it does not seem at first sight possible that purely local operations could increase the non-local quantum features. There have been many schemes devised whereby correlations can be increased by local measurements on an ensemble of systems combined with classical communication, followed by a procedure of post-selection. Indeed, the model presented here can also be adapted readily to represent such a scheme. However, we verify that by local measurement alone, and without post-selection based upon classical communication, the correlations do not increase. In the next section we present two proofs that this is, in fact, a general result.

The models used to demonstrate this are of the ‘cavity QED’ type, and are both easy to understand physically and simple to analyse analytically. A good outline of cavity QED is given in [19]. We consider two optical cavities, the field states of which are entangled number states (for simplicity)

\[ |\Psi\rangle_{AB} = \alpha |n\rangle_A |m\rangle_B + \beta |n'\rangle_A |m'\rangle_B, \]  

where the subscripts ‘A’ and ‘B’ refer to the two cavities, and without loss of generality, we assume that |\alpha| > |\beta|. This is a pure state but is not maximally entangled. The aim is to produce the state:
\[ |\Phi\rangle_{AB} = \frac{1}{\sqrt{2}} (|n\rangle_A|m\rangle_B + |n'\rangle_A|m'\rangle_B), \]  
(19)
i.e. we have made \( \alpha = \beta = \frac{1}{\sqrt{2}} \), which is maximally entangled.

Two-level atoms are sent, one at a time, through cavity A and interact with that individual cavity field, via the Jaynes-Cummings Hamiltonian \([20]\), for a pre-determined time period. After each atom passes through the cavity, a measurement is made which projects the atomic state into either the ground state or the excited state. Due to the entanglement developed between the atom and the field in cavity A during the interaction, this measurement also collapses the joint cavity A – cavity B field state into a different superposition, one with either the same number of photons in cavity A, or with one extra photon respectively. By successively sending atoms through the cavity for interaction periods determined from the state of the previously measured atom, a feedback mechanism can be set up whereby one might expect to optimise the probability of achieving the state defined in eq.(19). Similar schemes have been used on single cavities for quantum state-engineering \([21]\).

We also consider extensions to this procedure. Firstly, we mention procedures for interacting locally with both cavities, the qualitative results of which are the same. And secondly, we give two examples of non-local interactions, which give quite different results to the above local procedures.
A. Cavity Models With Local Feedback

The first model involves sending atoms through cavity \(A\) only, a schematic of which is given in Fig.1; we assume the initial joint cavity field state is given by (18). The first atom is in the excited state, and so the initial atom-field state is

\[
(\alpha |n\rangle_A |m\rangle_B + \beta |n'\rangle_A |m'\rangle_B) \otimes |e\rangle_A. \tag{20}
\]

After interaction for a time \(t_1\), determined from the atomic time of flight, the joint atom-field state becomes

\[
(\alpha a_n(t_1)|n\rangle_A |m\rangle_B + \beta a_{n'}(t_1)|n'\rangle_A |m'\rangle_B) \otimes |e\rangle_A \\
+ (\alpha b_n(t_1)|n+1\rangle_A |m\rangle_B + \beta b_{n'}(t_1)|n'+1\rangle_A |m'\rangle_B) \otimes |g\rangle_A \tag{21}
\]

where the coefficients are given by

\[
a_n(t_1) = \cos \left( \frac{R nt_1}{2} \right), \quad a_{n'}(t_1) = \cos \left( \frac{R_{n'}t_1}{2} \right), \quad b_n(t_1) = -i \sin \left( \frac{R nt_1}{2} \right), \quad b_{n'}(t_1) = -i \sin \left( \frac{R_{n'}t_1}{2} \right),
\]

and

\[
R_i = 2g\sqrt{i+1} = R_o\sqrt{i+1}.
\]

We now arrange that the velocity of the atom, and hence the interaction time with the field, is such that

\[
\alpha a_n(t_1) = \beta a_{n'}(t_1), \tag{22}
\]

in which case the joint atom-field state becomes
\[
\alpha a_n(t_1) (|n\rangle_A|m\rangle_B + |n\rangle_A|m\rangle_B) \otimes |e\rangle_A \\
+ (\alpha b_n(t_1)|n+1\rangle_A|m\rangle_B + \beta b_n(t_1)|n'\rangle_A|m'\rangle_B) \otimes |g\rangle_A
\]

From this we see that if we measure the atom in the excited state, the resulting cavity field state is maximally entangled. The probability of measuring the excited atomic state is

\[
P_1(e) = 2|\alpha a_n(t_1)|^2.
\]

If we were to prepare a whole ensemble of cavities in precisely the same initial state (20), then after measurement on all of the ensemble members, we would have prepared approximately \((100 \times P_1(e))\)% of the cavities in the maximally entangled state (19). We can discard all the cavities for which we measured the atom in the ground state, and we will have a whole sub-ensemble of cavities for which the entanglement has increased. This is the post-selection procedure mentioned earlier, and always requires that measurements on the whole ensemble be ‘thrown away’ in order to increase the entanglement of a sub-ensemble.

What we wish to do here is to increase the entanglement on an individual pair of entangled cavities. Instead of performing one measurement on an ensemble of cavities, we keep performing a number of measurements on this single pair until we achieve our aim. When the atom is measured in the excited state, we are there. If the outcome of the
atomic state measurement was $|g\rangle_A$, the final cavity field state would be the corresponding field state in eq. (23), which is still entangled, but not maximally so. We can now use this field state as a new initial entangled cavity field. In this way, we would hope that it is just a matter of sending through ‘enough’ atoms until the desired state is reached.

Since the field state corresponding to a ground state measurement involves the $(n+1)$ Fock state, sending through another excited atom allows the possibility of generating an $(n+2)$ Fock state, which takes us further away from the initial state (20). We thus send through a ground-state atom, which can remove the extra photon.

Using, therefore, this as the starting field-state, we define the ‘new’ $\alpha$ and $\beta$ as

$$
\alpha' = \frac{\alpha b_n(t_1)}{\sqrt{(\alpha b_n(t_1))^2 + (\beta b_{n'}(t_1))^2}}, \quad \beta' = \frac{\beta b_{n'}(t_1)}{\sqrt{(\alpha b_n(t_1))^2 + (\beta b_{n'}(t_1))^2}}.
$$

and the joint atom-field state after sending through a ground state atom for time $t_2$, such that $b_n(t_2)\alpha' = b_{n'}(t_2)\beta'$, becomes

$$
(a_n(t_2)\alpha'|n + 1\rangle_A|m\rangle_B + a_{n'}(t_2)\beta'|n' + 1\rangle_A|m'\rangle_B) \otimes |g\rangle_A
$$

$$
+ b_n(t_2)\alpha' (|n\rangle_A|m\rangle_B + |n'\rangle_A|m'\rangle_B) \otimes |e\rangle_A
$$

As before, if the atom is measured in the excited state, then the cavities are left in the maximally entangled field state, once normalised, as desired. The probability for this measurement is

$$
P_2(e) = 2|b_n(t_2)\alpha'|^2.
$$
It is worth noting at this point that the state of the field, after measuring a ground state atom, is in itself less entangled than the initial state \[18\]. This is a direct consequence of the concave property of entropy when applied to either reduced density matrix. Namely, the fact that in one case, when registering an excited atom, the field becomes more entangled than previously (i.e. the entropy of either reduced system is greater after the interaction), implies that the entanglement of the other field state, when we register a ground atom, is ‘smaller’ than previously (i.e. the entropy is smaller than before the interaction). This can be quantified as follows. Let the reduced field state after the interaction be

\[
\hat{\rho}'_A = p\hat{\rho}'_{A1} + (1 - p)\hat{\rho}'_{A2}
\]  

(28)

where \(\hat{\rho}'_A\) is the reduced density matrix for cavity A formed from eq.(23), and \(\hat{\rho}'_{A1}, \hat{\rho}'_{A2}\) are the parts of \(\hat{\rho}'_A\) corresponding to the measurement of an excited or ground state atom respectively. Now using the concave property eq.(11) we see that

\[
S(\hat{\rho}_A) = S(\hat{\rho}'_A) \geq pS(\hat{\rho}'_{A1}) + (1 - p)S(\hat{\rho}'_{A2})
\]  

(29)

where the first equality follows from the fact that the reduced density matrix does not change during this interaction, which can readily be derived for this example, and is shown generally in the next section. It follows that

\[
S(\hat{\rho}_A) \geq p(S(\hat{\rho}_A) + \Delta) + (1 - p)S(\hat{\rho}'_{A2})
\]  

(30)
where $\Delta$ is the amount by which the entropy (and hence entanglement) of the reduced subsystem is constructed to increase upon measurement of $|e\rangle$, by arranging atomic interaction times. So,

$$S(\hat{\rho}_A) - S(\hat{\rho}'_{A2}) \geq \frac{p\Delta}{1 - p} > 0 \quad (31)$$

Hence, it is immediately seen that

$$S(\hat{\rho}_A) > S(\hat{\rho}'_{A2}) \quad (32)$$

and the result (in this case) is proven.

A small amount of simple algebra applied to eq.(22) shows that whatever the initial values of $\alpha$ and $\beta$, the ratio

$$\frac{\min(\alpha, \beta)}{\max(\alpha, \beta)} \quad (33)$$

always decreases unless $n = n'$ i.e. the cavities are not entangled in the first place (a ratio equal to unity implies maximal entanglement). We thus have that $|\alpha'| > |\alpha|$ and $|\beta'| < |\beta|$. It is readily seen from this, and the fact that $|a_i(t)| < 1$ and $|b_i(t)| < 1$, that

$$P_2(e)_{\text{max}} = 2|\beta'|^2 < P_1(e)_{\text{max}} = 2|\beta|^2 \quad (34)$$

Thus, there are two effects each time an atom is sent through the cavity — the first is that the probability of detecting an atom in the excited state, and hence collapsing the field state to the maximally entangled form, on average decreases with each atom
that goes through; and the second is that the field-state if the atom is measured in the ground state becomes successively more disentangled. The effect is to make it successively more likely that the field will become completely disentangled, rather than completely entangled, which was the original aim. This can be seen mathematically by adding up the probabilities of detecting an atom in the excited state after sending through exactly \( N \) atoms. If the probability of detection in state \( |e\rangle \) after the \( i \)-th atom is \( a_i \), and the corresponding probability for \( |g\rangle \) is \( b_i \), then the probability of detection in \( |e\rangle \) after \( N \)-atoms is

\[
a_0 + b_0a_1 + b_0b_1a_2 + b_0b_1b_2a_3 + b_0b_1b_2b_3a_4 + \ldots + b_0\ldots b_{N-1}a_N
\]

\[
= (1 - b_0) + b_0(1 - b_1) + b_0b_1(1 - b_2) + b_0b_1b_2(1 - b_3) + \ldots + b_0\ldots b_{N-1}(1 - b_N)
\]

\[
= 1 - \prod_{i=0}^{N} b_i
\]

The above product term is always less than unity since each and every \( b_i \) is individually less than unity, and similarly is always positive since all \( b_i \) are individually positive, so the probability of detection of \( |e\rangle \) after \( N \)-atoms is less than unity. In the limit of \( N \to \infty \), it can be verified by a computer program that the above product always tends to the value of \( 2|\beta|^2 \). This result has the following consequence. In the limit \( N \to \infty \) we either register a maximally entangled state or a completely disentangled state. However we could arrange the atom cavity interaction time to be such that this happens when the first atom goes through the cavity. In this case it can be easily shown that the probability for the maximally entangled state to be registered (i.e. measuring the excited atomic
state) is exactly $2|\beta|^2$. Thus, no matter how many atoms we send through the cavity (one or infinitely many), the highest probability of reaching the maximally entangled state is *always* less than unity. We thus see that this scheme cannot increase correlations between two entangled systems.

We note also that we do not have to aim to achieve maximum entanglement for the particular initial state given by eq. (20). We could continue to send, for example, excited atoms and simply hope to achieve increased entanglement for *any* state. However, the same arguments given above also show that we cannot increase the entanglement of *both* field states corresponding to the two atomic measurement outcomes, as eq. (32) shows.

We should note that if it was possible to increase entanglement by the above local scheme, we would have a means of superluminal communication. Namely, the sender of the message could change the entanglement by operating locally on his cavity which could then be detected on the other end by the receiver in possession of the other cavity. The communication would then proceed as follows: two participants would initially share a number of not maximally entangled cavities. Then, if the sender does nothing on one of his cavities, this could represent ‘logical zero’, whereas if the sender maximally entangled the cavities this would represent ‘logical one’. After sharing the entangled sets of cavities, the two participants could travel spatially as far away from each other as desired. In this way, they would be able to communicate, through the above binary code, at a speed effectively instantaneously (only the time to actually prepare the binary states, and to
measure them at the other end). Therefore, we see that the impossibility of locally increasing the correlations is closely related to Einstein’s principle of causality. This is a curious consequence of quantum mechanics, the postulates of which contain no reference to special relativity. Indeed, this could be turned upside down, and viewed as one reason why the above (or any similar) scheme would not work.

We thus find that the above scheme cannot increase correlations by local actions on one cavity alone. We might expect to compensate for this by sending independent atoms through both cavities, and arranging a feedback mechanism based upon classically communicating the knowledge of each state to the other side. In this way, we approach more closely the scheme of classical communication with post selection [1], but hope to replace the post-selection procedure with that of sending through multiple atoms until we achieve success. We would also expect to avoid superluminal communications since the method inherently involves classical communication between the two observers. The analysis for this problem is very similar to that given above for the one-atom model, except that there is much more freedom to choose which state to measure and how to optimise it. Following through a similar reasoning as in the single atom model, it is readily deduced that there is no way in this scheme to increase correlations. There are numerous variations on this above scheme: maximising the probability of detection in $|e\rangle_A|e\rangle_B$, minimising the rate of change of $\alpha$ and $\beta$, and so on, but the basic fact that the probability is never identically unity for any number of atoms remains the same.
B. Increasing Entanglement Non-Locally

We now present two simple examples showing how a nonlocal operation can increase and, in fact, create correlations and entanglement. The procedures described here can be used to prepare initially entangled states.

1. Method 1

Suppose that the two cavities, A and B, start disentangled in the state:

$$|\phi_{cav}\rangle_{AB} = \frac{1}{\sqrt{2}}|0\rangle_A|0\rangle_B .$$  \hfill (36)

Let us send an entangled atomic pair through the cavities, each atom going through one cavity only, with the initial atomic state:

$$|\phi_{atom}\rangle_{AB} = \frac{1}{\sqrt{2}}(|e\rangle_A|g\rangle_B + |g\rangle_A|e\rangle_B) .$$  \hfill (37)

After the interaction for the same time $t$ the joint state will be:

$$|\psi_{joint}\rangle_{AB} = -b^2_0(t)|g\rangle_A|g\rangle_B \left\{ \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B) \right\}$$

$$+ a^2_0(t) \left\{ \frac{1}{\sqrt{2}}(|e\rangle_A|g\rangle_B + |g\rangle_A|e\rangle_B) \right\} |0\rangle_A|0\rangle_B .$$ \hfill (38)

Therefore, by simply setting $b_0(t) = 0$ we end up with certainty in the maximally entangled field state. Hence nonlocal interactions can, as expected, increase and create correlations and entanglement. The difference between this scheme and the previous two
is that entanglement is being transferred to the cavities, from the atoms. This allows the cavity entanglement to ‘increase’, but at the expense of the entanglement of the atoms.

2. Method 2

This method involves only one atom, first interacting with one cavity and then with the other. This type of “entanglement generation” has been analysed in a number of other places [22]. Let the initial state of ‘atom+fields’ be:

\[ |e⟩|0⟩_A|0⟩_B \]  

(39)

After interaction between the atom and the cavity A for time \( t_1 \) the state is

\[ (a_o(t_1)|e⟩|0⟩_A + b_o(t_1)|g⟩|1⟩)|0⟩_B . \]  

(40)

The atom now interacts with the cavity B for time \( t_2 \) after which the final state is

\[ a_o(t_1)a_o(t_2)|e⟩|0⟩_A|0⟩_B + a_o(t_1)b_o(t_2)|g⟩|0⟩_A|1⟩_B + b_o(t_1)|g⟩|1⟩_A|0⟩_B . \]  

(41)

Choosing \( a_o(t_2) = 0 \) and \( a_o(t_1) = b_o(t_2) = \frac{1}{\sqrt{2}} \) the above reduces to:

\[ |g⟩\frac{1}{\sqrt{2}}(|0⟩_A|1⟩_B + |1⟩_A|0⟩_B) , \]  

(42)

which is the desired, maximally entangled state of the field. Thus, the method achieves an entangled cavity state by creating an entangled atom-cavity state, and transferring this to the two cavities alone.
IV. LOCAL INTERACTIONS CANNOT INCREASE CORRELATIONS

The central problem addressed in this paper, and described in the specific examples in the previous section, is summarised in the following theorem:

**Theorem.** Correlations do not increase during local complete measurements carried on two entangled quantum systems.

We present here two quite separate, but mathematically rigorous proofs of this theorem, the first using the notion of entropy, the second using the ideas of complete measurements and conditional entropy as a measure of relative information.

First, we show that, as mentioned in subsection 3.1, no local complete measurement on subsystem $A$ can change the reduced density matrix of $B$, and vice versa. Let us perform a complete measurement on $A$, defined by $\sum_i \hat{A}_i^\dagger \hat{A}_i \otimes I_B = \hat{I}$ where the identity in the direct product signifies that the other subsystem does not undergo any interaction. Let the overall state of $'A+B'$ be described by $\hat{\rho}$. Then after $A$ has undergone a complete measurement, $B$’s reduced density matrix is given by:

$$
\hat{\rho}_B' = tr_A \left\{ \sum_i \hat{A}_i \otimes I_B \hat{\rho} \hat{A}_i^\dagger \otimes I_B \right\} \\
= \sum_i tr_A\{\hat{\rho}\hat{A}_i^\dagger \hat{A}_i \otimes I_B\} = tr_A\{\hat{\rho}\sum_i \hat{A}_i^\dagger \hat{A}_i \otimes I_B\} = tr_A\{\hat{\rho}\} = \hat{\rho}_B .
$$

(43)

Therefore the equality in eq. (29) is now justified. We note also that the $\hat{A}$’s in the above equation can be unitary operators, since $\hat{U}^\dagger \hat{U} = \hat{I}$. We use this result in Proof.1 below.
**A. Proof 1:**

This proof is due to Partovi [23], who proved it as a general result, rather than applied it to increasing correlations by local operations. Consider three quantum systems $A$, $B$, $C$, initially in the state described by a density matrix of the form: $\hat{\rho}_{ABC}(0) = \hat{\rho}_{AB}(0)\hat{\rho}_C(0)$, i.e. A and B are initially correlated and both completely independent from C. We are now going to let $B$ and $C$ interact and evolve unitarily for time $t$, resulting in the state $\hat{\rho}_{ABC}(t)$. The partial trace is defined in the usual fashion, e.g. $\hat{\rho}_{AB}(t) = \text{tr}_C\hat{\rho}_{ABC}(t)$, and similarly for all the other subsystems. Now we use the strong subadditivity [23] applied to $A + B + C$ at time $t$ to obtain

$$S_{ABC}(t) + S_B(t) \leq S_{AB}(t) + S_{BC}(t).$$

(44)

But $S_{ABC}(t) = S_{ABC}(0)$, as the whole system evolves unitarily. Also, $S_{ABC}(0) = S_{AB}(0) + S_C(0)$, since at the beginning $C$ is independent from $A$, $B$. $A$ is only a spectator in the evolution of $B$ and $C$, so that, as shown above, $S_A(t) = S_A(0)$, $S_{BC}(t) = S_{BC}(0)$. Finally, there are no correlations between $B$ and $C$ at the beginning, implying: $S_{BC}(0) = S_B(0) + S_C(0)$. Invoking the definition in eq.(3) for quantum correlations, and using the above properties and strong subadditivity in eq.(44), we arrive at the following

$$I(\hat{\rho}_A : \hat{\rho}_B; \hat{\rho}_{AB})(t) \leq I(\hat{\rho}_A : \hat{\rho}_B; \hat{\rho}_{AB})(0)$$

(45)

Adding another system $D$ to interact with $A$ locally would lead to the same conclusion, hence completing the proof.
B. Proof 2:

This proof is a quantum analogue of the well known classical result that can loosely be stated as ‘Markovian processes cannot increase correlations’ \cite{12,13}. We will now describe the interactions of $A + B$ with $C$ and $D$ in terms of complete measurements performed on $A + B$. Let the state of $A + B$ be initially described by the density operator $\hat{\rho}$, whose diagonal elements, $\rho_{ii}$, give the probabilities of being in various states, depending on the basis of the density matrix. Let this state undergo a complete measurement, described by operators $\hat{A}^j$, such that

$$\sum \hat{A}^\dagger j \hat{A}^j = \hat{I}.$$ \hspace{1cm} (46)

The new diagonal elements are then:

$$\rho'_{ii} = \sum_{nlm} A^a_{il} \rho_{lm} A^{\dagger n}_{mi}.$$ \hspace{1cm} (47)

Let us introduce a relative information measure (defined in sec.2) to $\rho_{ii}$: to each value of $\rho_{ii}$ we assign a nonnegative number $a_{ii}$. We now wish to compare the distance \cite{12} between $\hat{\rho}$ and $\hat{a}$ before and after ($\hat{\rho}'$ and $\hat{a}'$) the complete measurement, $\hat{A}$. The distance after the measurement is:

$$\sum_i \frac{\rho'_i}{a_i} \log \frac{\rho'_i}{a_i} = \sum_i \left( \sum_{nlm} A^a_{il} \rho_{lm} A^{\dagger n}_{mi} \right) \log \frac{\sum_{nlm} A^a_{il} \rho_{lm} A^{\dagger n}_{mi}}{\sum_{nlm} A^a_{il} a_{lm} A^{\dagger n}_{mi}} \leq \sum_i \left( \sum_{nlm} A^a_{il} \rho_{lm} A^{\dagger n}_{mi} \right) \log \frac{A^a_{il} \rho_{lm} A^{\dagger n}_{mi}}{A^a_{il} a_{lm} A^{\dagger n}_{mi}}.$$
\[
\sum_i \left( \sum_{nlm} A^n_{il} \rho_{lm} A^{*n}_{mi} \right) \log \frac{\rho_{lm}}{a_{lm}} = \sum_{lm} \rho_{lm} \left( \sum_{in} A^n_{il} A^{*n}_{mi} \right) \log \frac{\rho_{lm}}{a_{lm}} = \sum_{lm} \rho_{lm} \delta_{lm} \log \frac{\rho_{lm}}{a_{lm}} = \sum_i \rho_{il} \log \frac{\rho_{il}}{a_{il}},
\]

(48)

where for the inequality in the second line we have used one of the consequences of the concave property of the logarithmic function [12,17], and in the fifth line we used the completeness relation given in eq. (46). This implies that the distance between the density matrix distribution and the relative information measure decreases by making a complete measurement. If we now consider the particular case where \(\hat{a}\) is taken to be a distribution generated by the direct product of the reduced density matrices (i.e. if we assume no correlations), then the result above implies that the full density matrix becomes ‘more like’ the uncorrelated density matrix. From this, the theorem immediately follows.

V. CONDITIONS FOR QUANTUM ERROR–CORRECTING CODES

We now describe an alternative way of manipulating quantum states which can be described using the language of quantum computation [24]. Quantum computation involves unitary operations and measurement on ‘quantum bits’, or qubits. A classical bit represents one of two distinguishable states, and is denoted by a ‘0’ or a ‘1’. On the other
hand, a qubit is in general a superposition of the two states, $\alpha |0\rangle + \beta |1\rangle$, and its evolution is governed by the laws of quantum mechanics, for which a closed system evolves unitarily. This is reflected in the nature of the elementary gates, which must be reversible, i.e. a knowledge of the output allows inference of the input. The practical realisation of a qubit can be constructed from any two-state quantum system e.g. a two-level atom, where the unitary transformations are implemented through interaction with a laser. An advantage of quantum computation lies in the fact that the input can be in a coherent superposition of qubit states, which are then simultaneously processed. The computation is completed by making a measurement on the output. However, a major problem is that the coherent superpositions must be maintained throughout the computation. In reality, the main source of coherence loss is due to dissipative coupling to an environment with a large number of degrees of freedom, which must be traced out of the problem. This loss is often manifested as some form of spontaneous decay, whereby quanta are randomly lost from the system. Each interaction with, and hence dissipation to, the environment can be viewed in information theoretic terms as introducing an error in the measurement of the output state. There are, however, techniques for ‘correcting’ errors in quantum states \[3 \text{–} 5\]. The basic idea of error-correction is to introduce an excess of information, which can then be used to recover the original state after an error. These quantum error correction procedures are in themselves quantum computations, and as such also susceptible to the same errors. This imposes limits on the nature of the ‘cor-
rection codes’, which are explored in this section. In the context of the present paper, we will use error-correction as a way of maintaining coherence between entangled cavities, described in the next section.

First we derive general conditions which a quantum error correction code has to satisfy and are, in particular, less restricting than those previously derived in \[6\]. Our derivation is an alternative to that in \[8\], which also arrives at the same conditions. We use the notation of Ekert and Macchiavello \[8\]. Assume that \(q\) qubits are encoded in terms of \(n\) qubits to protect against a certain number of errors, \(d\). We construct \(2^q\) code-words, each being a superposition of states having \(n\) qubits. These code-words must satisfy certain conditions, which are derived in this section. There are three basic errors \[6\] (i.e. all other errors can be written as a combination of those): amplitude, \(\hat{A}\), which acts as a NOT gate; phase, \(\hat{P}\), which introduces a minus sign to the upper state; and their combination, \(\hat{A}\hat{P}\). A subscript shall designate the position of the error, so that \(\hat{P}_{1001}\) means that the first and the fourth qubit undergo a phase error.

We consider an error to arise due to the interaction of the system with a ‘reservoir’ (\textit{any} other quantum system), which then become entangled. This procedure is the most general way of representing errors, which are not restricted to discontinuous ‘jump’ processes, but encompass the most general type of interaction. Error correction is thus seen as a process of disentangling the system from its environment back to its original state. The operators \(\hat{A}\) and \(\hat{P}\) are constructed to operate only on the system, and are defined in
the same way as the operators for a complete measurement described in subsection 2.4, eq. (17). In reality, each qubit would couple independently to its own environment, so the error on a given state could be written as a direct product of the errors on the individual qubits. A convenient error basis for a single error on a single qubit is \( \{ \hat{I}, \hat{\sigma}_i \} \), where the \( \hat{\sigma}_i \)'s are the Pauli matrices. In this case, the error operators are Hermitian, and square to the identity operator, and we assume this property for convenience throughout the following analysis.

In general the initial state can be expressed as

\[
|\psi_i\rangle = \sum_{k=1}^{2^q} c_k |C^k\rangle |R\rangle
\]

(49)

where the \( |C^k\rangle \) are the code–words for the states \( |k\rangle \) and \( |R\rangle \) is the initial state of the environment. The state after a general error is then a superposition of all possible errors acting on the above initial state

\[
|\psi_f\rangle = \sum_{\alpha\beta} \hat{A}_\alpha \hat{P}_\beta \sum_k c_k |C^k\rangle |R_{\alpha,\beta}\rangle.
\]

(50)

where \( |R_{\alpha,\beta}\rangle \) is the state of the environment. Note that \( |R_{\alpha,\beta}\rangle \) depends only on the nature of the errors, and is independent of the code–words \( \{ |C^k\rangle \} \). The above is, in general, not in the Schmidt form, i.e. the code–word states after the error are not necessarily orthogonal (to be shown) and neither are the states of the environment. Now, since we have no information about the environment, we must trace it out using an orthogonal basis for the environment \( \{ |R_n\rangle, n = 1, d \} \). The resulting state is a mixture of the form
\[ \hat{\eta} = \sum_n |\psi_n \rangle \langle \psi_n|, \] where

\[ |\psi_n \rangle = \sum_{\alpha \beta} x_n^{\alpha \beta} \hat{A}_\alpha \hat{P}_\beta \sum_k c_k |C^k \rangle, \]

and \[ x_n^{\alpha \beta} = \langle R_n |R_{\alpha \beta} \rangle. \] To detect an error, one then performs a measurement on the state \[ \hat{\eta} \] to determine whether it has an overlap with one of the following subspaces

\[ \mathcal{H}_{\alpha \beta} = \{ \hat{A}_\alpha \hat{P}_\beta |C^k \rangle, k = 1, \ldots, 2^q \}. \]

The initial space after the error is given by the direct sum of all the above subspaces,
\[ \mathcal{H} = \sum_{\alpha \beta} \oplus \mathcal{H}_{\alpha \beta}. \] Each time we perform an overlap and obtain a zero result, the state space \[ \mathcal{H} \] reduces in dimension, eliminating that subspace as containing the state after the error. Eventually, one of these overlap measurements will give a positive result which is mathematically equivalent to projecting onto the corresponding subspace. The state after this projection is then given by the mixture

\[ |\psi_{\text{proj}_{\alpha \beta}} \rangle = \sum_{k l} \sum_{\gamma \delta} x_n^{\gamma \delta} \hat{A}_\alpha \hat{P}_\beta |C^k \rangle \langle C^k| \hat{P}_\beta \hat{A}_\alpha \hat{A}_\gamma \hat{P}_\delta |C^l \rangle c_l. \]

The successful projection will effectively take us to the state generated by a superposition of certain types of error. One might expect that to distinguish between various errors the different subspaces \[ \mathcal{H}_{\alpha \beta} \] would have to be orthogonal. However, we will show that this is not, in fact, necessary.

After having projected onto the subspace \[ \mathcal{H}_{\alpha \beta} \] we now have to correct the corresponding error by applying the operator \[ \hat{P}_\beta \hat{A}_\alpha \] onto \[ |\psi_{\text{proj}_{\alpha \beta}} \rangle \], since \[ \hat{P}_\beta \hat{A}_\alpha \hat{A}_\alpha \hat{P}_\beta = \hat{I}. \] In order to
correct for the error, the resulting state has to be proportional to the initial state of code–words in $|\psi_i\rangle$. This leads to the condition

$$\sum_{kl} \sum_{\gamma\delta} x_{\gamma\delta}^k |C^k\rangle \langle C^l| \hat{P}_\beta \hat{A}_\alpha \hat{A}_\gamma \hat{P}_\delta |C^l\rangle c_l = z^{\alpha\beta n} \sum_m c_m |C^m\rangle .$$

(54)

where $z^{\alpha\beta n}$ is an arbitrary complex number. Now we use the fact that all code words are mutually orthogonal, i.e. $\langle C^k|C^l\rangle = \delta_{kl}$, to obtain that

$$\sum_l \sum_{\gamma\delta} c_l x_{\gamma\delta}^k \langle C^k| \hat{P}_\beta \hat{A}_\alpha \hat{A}_\gamma \hat{P}_\delta |C^l\rangle = z^{\alpha\beta n} c_k$$

(55)

for all $k$ and arbitrary $c_k$. This can be written in matrix form as

$$\mathbf{F}^{\alpha\beta n} \mathbf{c} = z^{\alpha\beta n} \mathbf{c} ,$$

(56)

where the elements of the matrix $\mathbf{F}$ are given by

$$F_{kl}^{\alpha\beta n} := \sum_{\gamma\delta} x_{\gamma\delta}^k \langle C^k| \hat{P}_\beta \hat{A}_\alpha \hat{A}_\gamma \hat{P}_\delta |C^l\rangle .$$

(57)

As eq. (56) is valid for all $\mathbf{c}$ it follows that

$$\forall \ k, l, \quad F_{kl}^{\alpha\beta n} = z^{\alpha\beta n} \delta_{kl} .$$

(58)

However, we do not know the form of $x_{\gamma\delta}^k$’s as we have no information about the state of the environment. Therefore, for the above to be satisfied for any form of $x$’s we need each individual term in eq. (57) to satisfy

$$\langle C^k| \hat{P}_\beta \hat{A}_\alpha \hat{A}_\gamma \hat{P}_\delta |C^l\rangle = y^{\alpha\beta\gamma\delta} \delta_{kl}$$

(59)
where $y^{\alpha\beta\gamma\delta}$ is any complex number. From eqs. \[57,58,59\] we see that the numbers $x$, $y$, and $z$ are related through

$$\sum_{\gamma\delta} x_n^{\gamma\delta} y^{\alpha\beta\gamma\delta} = z^{\alpha\beta n}. \quad (60)$$

Eq. \[59\] is the main result in this section, and gives a general, and in fact the only, constraint on the construction of code-words, which may then be used for encoding purposes. If we wish to correct for up to $d$ errors, we have to impose a further constraint on the subscripts $\alpha$, $\beta$, $\gamma$, and $\delta$; namely, $\text{wt}(\text{supp}(\alpha) \cup \text{supp}(\beta))$, $\text{wt}(\text{supp}(\gamma) \cup \text{supp}(\delta)) \leq d$, where supp$(x)$ denotes the set of locations where the $n$-tuple $x$ is different from zero and $\text{wt}(x)$ is the Hamming weight \[25\], i.e. the number of digits in $x$ different from zero. This constraint on the indices of errors simply ensures that they do not contain more than $d$ logical ‘1’s altogether, which is, in fact, equivalent to no more than $d$ errors occurring during the process.

We emphasise that these conditions are the most general possible, and they in particular generalise the conditions in \[3\]. By substituting $z^{\alpha\beta\gamma\delta} = \delta_{\alpha\beta\gamma\delta}$ in eq.\[59\], we obtain the conditions

$$\langle C^k | \hat{P}_x \hat{A}_\alpha \hat{A}_\gamma \hat{P}_\delta | C^l \rangle = \delta_{\beta\delta} \delta_{\alpha\gamma} \delta_{kl} \quad (61)$$

given in \[3\]. These are therefore seen only as a special case of the general result in eq. \[58\].

Knill and Laflamme, who arrive at the same conditions as in eq. \[59\] \[8\], give no
example of a code that violates the conditions in eq. (54) but satisfies those of eq. (59). Such a code is given by Plenio et al [9], which by violating the conditions given in eq. (61), explicitly shows that they are not necessary, but merely sufficient.

VI. LOCAL ERROR CORRECTION PRESERVES CORRELATIONS

A. Theoretical Considerations

Imagine two initially entangled quantum systems $A$ and $B$ distributed between two spatially separated parties. Let, for the sake of simplicity, both $A$ and $B$ be two spin-$\frac{1}{2}$ particles in the initial EPR–like state

$$|\psi_{A+B}\rangle = \alpha|0\rangle_A|1\rangle_B + \beta|1\rangle_A|0\rangle_B$$  \hspace{1cm} (62)

where the first ket describes the system $A$ and the second the system $B$. Let both Alice particles be encoded locally (i.e. adding locally a certain number of auxiliary qubits and performing local unitary transformations to encode) in order to protect their own qubit against a desired number of errors. We suppose that they both use the same coding, with the code–words denoted by $|C^0\rangle$ and $|C^1\rangle$. After the encoding, the state is therefore

$$|\Psi_{A+B}\rangle = \alpha|C^0\rangle_A|C^1\rangle_B + \beta|C^1\rangle_A|C^0\rangle_B.$$  \hspace{1cm} (63)

Notice that the entanglement between the systems $A$ and $B$ is not changed by the encoding procedure, since local unitary operations do not change the spectrum of the reduced density matrices.
Let this state now be corrupted by errors, \( \hat{E} \), which are local in nature, after which we perform the above described projections in eq. (52) to obtain

\[
|\Psi_{A+B}'\rangle = \alpha \hat{E}_i |C^0\rangle_A \hat{E}_j |C^1\rangle_B + \beta \hat{E}_i |C^1\rangle_A \hat{E}_j |C^0\rangle_B .
\]  

(64)

We wish to show that the error does not change the value of the entanglement. For this we compute \( A \)'s reduced density matrix:

\[
\hat{\rho}_A = \text{tr}_B (|\Psi_{A+B}'\rangle\langle \Psi_{A+B}'|) = \langle C^0 |_B (|\Psi_{A+B}'\rangle\langle \Psi_{A+B}'|) |C^0\rangle_B + \langle C^1 |_B (|\Psi_{A+B}'\rangle\langle \Psi_{A+B}'|) |C^1\rangle_B
\]

\[
= \langle C^0 |_B \hat{E}_j |C^0\rangle_B \{ |\alpha|^2 \hat{E}_i |C^0\rangle_A \hat{E}_i + |\beta|^2 \hat{E}_i |C^1\rangle_A \hat{E}_i \} .
\]

(65)

which obviously has the same entropy as the original state in eqs. (61, 62) and eq. (63).

In the above derivation we used the relations in eq. (59) such that

\[
\langle C^0 |_B \hat{E}_i \hat{E}_j |C^0\rangle_B = \langle C^1 |_B \hat{E}_i \hat{E}_j |C^1\rangle_B
\]

(66)

\[
\langle C^0 |_B \hat{E}_i \hat{E}_j |C^1\rangle_B = 0 .
\]

(67)

Thus the entropy of the reduced density matrices of the initial pair of encoded systems, and of the systems after undergoing errors are both the same, indicating that the correlations and thus the entanglement do not change during the above described process.

By a process of introducing more local degrees of freedom into the problem, we are able to maintain non-local quantum correlations. So, in fact, this process does also involve discarding information, but is different to the post selection previously described. This
is so because all the error correcting particles are introduced locally, and do not form a part of the original ensemble.

**B. Example With Cavities**

We now present a simple example of how to locally preserve entanglement between two cavities in the state

\[ \alpha |0\rangle_A |1\rangle_B + \beta |1\rangle_A |0\rangle_B \]  

(68)

against a single amplitude error (action of \( \hat{\sigma}_x \) Pauli operator) on either cavity. For this purpose we *locally* introduce a pair of atoms to each cavity, all of which are in the ground state. These atoms interact identically with their respective cavities. We also allow errors to happen to the atoms, as long as there is no more than one error on either side, \( A \) or \( B \). We would like to implement the following interaction in order to encode the state against an amplitude error [4] (four additional atoms for each cavity are needed to correct against a general type of single error [5])

\[
|0\rangle_1 |g\rangle_1 |g\rangle_2 \longrightarrow |0\rangle_1 |g\rangle_1 |g\rangle_2
\]  

(69)

\[
|1\rangle_1 |g\rangle_1 |g\rangle_2 \longrightarrow |1\rangle_1 |e\rangle_1 |e\rangle_2
\]  

(70)

This is, in fact, an action of two ‘Control–Nots’ [26], with the control bit being the state of the cavity and the target bits being the atoms 1 and 2. We therefore perform identical
interactions on both cavities and their atoms. This is shown schematically in Fig.2. The state of the whole system (‘2 cavities + 4 atoms’) will be after the encoding procedure,
\[
\alpha |0\rangle_A |g, g\rangle_A |1\rangle_B |e, e\rangle_B + \beta |1\rangle_A |e, e\rangle_A |0\rangle_B |g, g\rangle_B
\] (71)
So all we need to know is how to implement a Control–Not operation between the cavity and one atom. This is done in the following way [27]. Let the atom be sent through the cavity, which in our case contains either one or no photons, interacting resonantly with the field. Let us in addition have a ‘classical’ light source (a laser) resonant with the dressed atom-field transition $|1\rangle |g\rangle \rightarrow |1\rangle |e\rangle$. Due to the vacuum Rabi splitting this will not be resonant with $|0\rangle |g\rangle \rightarrow |0\rangle |e\rangle$ which is precisely what we need. In this way the initial ‘cavity+atom’ state undergoes evolution of the form
\[
(\alpha |0\rangle + \beta |1\rangle) |g\rangle \rightarrow \alpha |0\rangle |g\rangle + \beta |1\rangle |e\rangle
\] (72)
which is a Control–Not gate. By repeated action of this gate we can create the state in eq. (71). Then if a single amplitude error occurs on either side (e.g. a spontaneous decay of the field) we can correct it by applying a unitary operation to the cavities to restore the original state, depending on the state of the four atoms [4].

Let us give a simple example of how this would work. Suppose that only the cavity $A$, after encoding, undergoes an amplitude error resulting in, after a small rearrangement, the joint ‘cavities + atoms+environment’ state of the form (eq. (50))
\[
(\alpha |0\rangle_A |1\rangle_B |g, g\rangle_A |e, e\rangle_B + \beta |1\rangle_A |0\rangle_B |e, e\rangle_A |g, g\rangle_B) |R_0\rangle
\]
\[
+ (\alpha|1\rangle_A|1\rangle_B|g, g\rangle_A|e, e\rangle_B + \beta|0\rangle_A|0\rangle_B|e, e\rangle_A|g, g\rangle_B)|R_1\rangle. \tag{73}
\]

To recover the original state we first have to decode the above state. This is just the inverse of encoding, i.e. we apply two Control-Nots as described above, resulting in the state

\[
(\alpha|0\rangle_A|1\rangle_B + \beta|1\rangle_A|0\rangle_B)|g, g\rangle_A|g, g\rangle_B)|R_0\rangle
+ (\alpha|1\rangle_A|1\rangle_B + \beta|0\rangle_A|0\rangle_B)|e, e\rangle_A|g, g\rangle_B)|R_1\rangle. \tag{74}
\]

In the second step we can make a measurement on the atoms and depending on the outcome apply an appropriate unitary transformation to the cavities. In this case we only have to consider cavity \( A \): if both of the atoms are in the ground state then we do nothing because the joint-cavity state remains unchanged, whereas if both of the atoms are excited we apply a NOT operation to cavity \( A \). This we do in a fashion similar to performing Control–Not. We could, for example, send an excited atom through the cavity and tune the external laser to the dressed transition \(|0\rangle|e\rangle \leftrightarrow |1\rangle|e\rangle\). In this way we recover the state in eq. (68). We emphasise that the form in eq. (74) is incomplete since the terms arising from all the other amplitude errors are missing (corresponding to the cavity \( B \) and the atoms); however, it can easily be checked that the above scheme would also accommodate for this.
VII. CONCLUSION

In this paper we presented simple models to demonstrate that correlations cannot be increased by any form of local complete measurement. The consequence of this is that any purification procedure has to represent a post-selection of the original ensemble to be purified. Classical communication is an essential precursor to the post-selection procedure — we cannot post-select without classical communication, but the post-selection procedure is necessary to prepare the maximally entangled subset. We then presented two general proofs of this fact. Additionally, we derived general conditions which have to be satisfied by quantum error correction codes, which can be used to protect a state against an arbitrary number of errors. We then showed that we can locally ‘protect’ the entanglement by standard quantum error correction schemes, such that the correlations (and therefore the entanglement) are preserved under any type of complete measurement, which can be viewed as an error in this context. We gave a simple example of how to encode two cavities against a single amplitude error. Thus, local error correction can protect nonlocal features of entangled quantum systems, which otherwise cannot be increased by any type of local actions which exclude classical communication and post-selection.
Acknowledgements

The authors thank P.L. Knight, and V.V. and M.B.P thank A. Barenco, A. Ekert and C. Macchiavello, for useful discussions on the subject of this paper. This work was supported by the European Community, the UK Engineering and Physical Sciences Research Council and by a Feodor-Lynen grant of the Alexander von Humboldt foundation.

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FIGURE CAPTIONS

Figure 1: The experimental setup for local interactions: two cavities are initially entangled in a state of the form $|\Psi\rangle_{AB} = \alpha|n\rangle_A|m\rangle_B + \beta|n'\rangle_A|m'\rangle_B$: and atoms are sent through cavity $A$ only.

Figure 2: The encoding network for protecting against amplitude errors is shown in the upper diagram: the encircled cross denotes a NOT operation while a dot denotes a control bit, together making a Control–Not operation. The atoms are initially in their ground states, and the order in which the gates are executed is irrelevant. The lower diagram gives a truth table for the Control–Not operation; here, ‘C’ and ‘T’ represent control and target bits respectively.
