LIEB–THIRRING TYPE ESTIMATES ON ISOLATED AND RESONANCE EIGENVALUES ON COMPLEX SUBPLANE

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Abstract

We consider non-self-adjoint Schrödinger operators $H = -\Delta + V$ acting in $L^2(\mathbb{R}^d)$, $d \geq 1$, with dilation analytic complex potentials $V$. We give a Lieb–Thirring type estimate on resonance eigenvalues of $H$ in the open complex sector. In order to obtain our desired estimate, we derive Lieb–Thirring type inequalities for isolated eigenvalues of $H$ which exist in the sector of the second or third quadrant of $\mathbb{C}$. The proofs are based on the method proposed in [17] (N. Someyama, "Number of Eigenvalues of Non-self-adjoint Schrödinger Operators with Dilation Analytic Complex Potentials," Reports on Mathematical Physics, Volume 83, Issue 2, pp.163-174 (2019).). So, the present paper could be considered a sequel to [17]. We also derive Lieb–Thirring type inequalities for isolated eigenvalues on each quadrant of $\mathbb{C}$ as their corollaries.

Keywords: non-self-adjoint Schrödinger operator, dilation analytic complex potential, Lieb–Thirring (type) inequality, complex isolated eigenvalue, resonance eigenvalue

1 Introduction

Let $d \geq 1$ be the dimension of Euclidean space. We consider the non-self-adjoint Schrödinger operator defined as the quasi-maximal accretive operator [7] acting in $L^2(\mathbb{R}^d)$:

$$H := H_0 + V, \quad H_0 := -\Delta$$

where the Laplacian $\Delta := \sum_{j=1}^{d} \partial^2/\partial x_j^2$ means the distributional derivative and $V$ is the dilation analytic complex potential (see Definition 1.1 for detailed definitions). We define the domain $D(H_0)$ of $H_0$ as the second-order Sobolev space $H^2(\mathbb{R}^d) := W^{2,2}(\mathbb{R}^d)$. The $L^2$-inner product and $L^2$-norm are defined by

$$(u,v) := \int_{\mathbb{R}^d} u(x) \overline{v(x)} \, dx, \quad \|u\|_{L^2(\mathbb{R}^d;\mathbb{C})} := (u,u)^{1/2}$$

respectively. Moreover, we consider the one-parameter unitary group $\{U(\theta) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d); \theta \in \mathbb{R}\}$ defined by

$$U(\theta)u(x) := e^{\theta/2}u(e^\theta x)$$

for $u \in L^2(\mathbb{R}^d)$. We put

$$H(\theta) := U(\theta)HU(\theta)^{-1} = e^{-2\theta}(H_0 + e^{2\theta}V_\theta) \quad (1.1)$$
$$V_\theta(x) := U(\theta)VU(\theta)^{-1} = V(e^\theta x) \quad (1.2)$$
and call $H(\theta)$ (resp. $V_{\theta}$) the *dilated Hamiltonian* (resp. *dilated potential*). We also call the transform by $U(\theta)$ such as \( (1.1) \) the *complex dilation*. We write
\[
\tilde{H}(\theta) := H_0 + e^{2\theta} V_{\theta}.
\] (1.3)

It is of course that $H(0) = \tilde{H}(0) = H$. Furthermore, we denote the real (resp. imaginary) part of $z \in \mathbb{C}$ by $\text{Re} \, z$ (resp. $\text{Im} \, z$).

**Definition 1.1** (\[17\]). $V$ is called the dilation analytic complex potential if it satisfies the followings: Let $d, \gamma \geq 1$.

i) The multiplication operator $V$ with the complex-valued measurable function $\mathbb{R}^d \ni x \mapsto V(x) \in \mathbb{C}$ obeying $V \in L^{\gamma+d/2}(\mathbb{R}^d, \mathbb{C})$.

ii) The operator $V$ is $H_0$-compact, that is, $\mathcal{D}(V) \supset \mathcal{D}(H_0) = H^2(\mathbb{R}^d)$ and $V(H_0 + 1)^{-1}$ is compact in $L^2(\mathbb{R}^d)$.

iii) The function $V_{\theta}$ with respect to $\theta \in \mathbb{R}$ has an analytic continuation into the complex strip
\[
\mathcal{S}_{\alpha} := \{ z \in \mathbb{C} : |\text{Im} \, z| < \alpha \}
\]
for some $\alpha > 0$ as an $L^{\gamma+d/2}(\mathbb{R}^d, \mathbb{C})$-valued function with respect to $x$.

iv) The function $V_{\theta}(H_0 + 1)^{-1}$ with respect to $\theta \in \mathbb{R}$ can be extended to $\mathcal{S}_{\alpha}$ as a $\mathcal{B}(L^2(\mathbb{R}^d))$-valued analytic function, where $\mathcal{B}(S)$ denotes the set of whole bounded operators on the set $S$.

We write the set of whole dilation analytic complex potentials by $\mathcal{D}(\mathcal{S}_{\alpha}; \mathbb{C})$ for convenience.

Since $U(\theta + \phi)$ and $U(\theta)$ are unitarily equivalent for any $\phi \in \mathbb{R}$, we can suppose that $\theta$ is the pure-imaginary number by setting $\phi = -\text{Re} \, \theta$. In other words, $H(\theta)$ does not depend on $\text{Re} \, \theta$ and $\sigma(H(\theta))$ is only dependent on $\text{Im} \, \theta$. $H(\theta)$ is a Kato’s type-(A) function (e.g. \[14\] [16]) which is operator-valued and analytic with respect to $\theta \in \mathcal{S}_{\alpha}$.

**Remark 1.1.** (1) The dilation analytic method originally introduced in \[2\] and it was defined for real potentials. We also call the dilation analytic method the *complex dilation method* or *complex scaling method*. This method and the now famous results derived by it were organized and customized in e.g. \[15\] [4]. Aguilar and Combes originally proposed dilation analytic potentials so as to give a sufficient condition for the absence of the singularly continuous spectrum of the Schrödinger operator (then, remark that the non-negative half line $[0, \infty)$ is the essential spectrum of it). More to say, the dilation analytic method is a natural factor that we consider and introduce complex potentials.

(2) $V_{\theta}$ has an analytic extension from $\mathcal{S}_{\alpha}$ to the closure $\overline{\mathcal{S}_{\alpha}}$ of $\mathcal{S}_{\alpha}$ and $H(\theta)$ can be extended from $\mathbb{R}$ to $\overline{\mathcal{S}_{\alpha}}$ with respect to $\theta$ as a $\mathcal{B}(L^2(\mathbb{R}^d))$-valued analytic function, but we do not need such assumptions in the present paper.

### 1.1 Complex Lieb–Thirring Type Inequalities

Throughout the present paper, we write $\sigma(T)$, $\sigma_d(T)$, $\sigma_{\text{ess}}(T)$ for the spectrum, discrete spectrum, essential spectrum of the closed operator $T$ respectively. Also, ‘isolated eigenvalues’ are simply abbreviated as ‘eigenvalues’. The algebraic multiplicity $m_\lambda(H)$ of $\lambda \in \sigma_d(H)$ is defined by
\[
m_\lambda(H) := \sup_{N \in \mathbb{N}} \left( \dim \ker(H - \lambda)^N \right).
\]
In estimating the sum of power of eigenvalues hereafter, we count the number of eigenvalues according to their algebraic multiplicities whether potentials are real or complex.

If \( V \) decays at infinity, it is well known that \( \sigma_d(H) \subset (-\infty, 0) \). Then, the Lieb–Thirring inequality for such a real potential \( V \in L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{R}) \) is well known (e.g. [11, 13, 14]) as the estimate on negative eigenvalues:

\[
\sum_{\lambda \in \sigma_d(H) \subset (-\infty, 0)} |\lambda|^\gamma \leq L_{\gamma,d} \|V\|_{L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{R})}^{\gamma+d/2}, \quad V_\pm := \frac{|V| + V}{2} \tag{1.4}
\]

where the dimension \( d \) obeys that

\[
\begin{align*}
\gamma &\geq 1/2 \quad \text{if } d = 1, \\
\gamma &> 0 \quad \text{if } d = 2, \\
\gamma &\geq 0 \quad \text{if } d \geq 3.
\end{align*}
\tag{1.5}
\]

Then, \( L_{\gamma,d} \) is a constant depending on \( d, \gamma \) and it is important for the accuracy of the estimate (see e.g. [5, 9, 10, 12]). In particular, (1.4) is well known as Cwikel–Lieb–Rozenbljum inequalities (e.g. [15, 19]) which are estimates on the number of negative eigenvalues of \( H \) if \( d \geq 3 \). Related to this, Frank, Laptev, Lieb and Seiringer [6] gave some Lieb–Thirring type inequalities for isolated eigenvalues of Schrödinger operators with any complex potentials on partial complex planes. The following inequality (1.7) is particularly the most fundamental result for complex Lieb–Thirring inequalities.

**Theorem 1.1** ([6]). Let \( d, \gamma \geq 1 \). Suppose \( V \in L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C}) \). We denote

\[
\mathcal{C}_\pm(\kappa) := \{z \in \mathbb{C} : |\text{Im } z| < \pm \kappa \text{Re } z\},
\tag{1.6}
\]

where these sets represent two sets, one for the upper sign and the other for the lower sign. Then, for any \( \kappa > 0 \),

\[
\sum_{\lambda \in \sigma_d(H) \cap \mathcal{C}_+^{(\kappa)}} |\lambda|^\gamma \leq C_{\gamma,d} \left(1 + \frac{2}{\kappa}\right)^{\gamma+d/2} \|V\|_{L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C})}^{\gamma+d/2} \tag{1.7}
\]

and

\[
\sum_{\lambda \in \sigma_d(H) \cap \mathcal{C}_-^{(\kappa)}} |\lambda|^\gamma \leq (1 + \kappa) L_{\gamma,d} \|\text{Re } V\|_{L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C})}^{\gamma+d/2}. \tag{1.8}
\]

Here \( S^c \) is the complement set of the set \( S \),

\[
C_{\gamma,d} := 2^{1+\gamma/2+d/4} L_{\gamma,d}
\]

and \( L_{\gamma,d} \) the constant of real Lieb–Thirring inequalities (1.4).

We can obtain the usual Lieb–Thirring inequality (1.4) by letting \( \kappa \downarrow 0 \) in (1.8). In other words, (1.8) is an inequality which extends (1.4). On the other hand, the Lieb–Thirring inequality for the eigenvalues on the complex left-half plane immediately holds from (1.7) by letting \( \kappa \to \infty \).

**Corollary 1.1** ([6]). Suppose \( V \in L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C}) \). For \( d, \gamma \geq 1 \), one has

\[
\sum_{\lambda \in \sigma_d(H) \cap \{z \in \mathbb{C} : \text{Re } z \leq 0\}} |\lambda|^\gamma \leq C_{\gamma,d} \|V\|_{L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C})}^{\gamma+d/2} \tag{1.9}
\]
Remark 1.2. 1) It is unclear whether Theorem 1.1 and Corollary 1.1 hold under (1.5). Furthermore the possibility of the complex Lieb–Thirring estimate on all eigenvalues (in \( \mathbb{C} \setminus [0, \infty) \)) of \( H \) with any complex potential is also still open.

2) The proofs by [6] of Theorem 1.1 and Corollary 1.1 enable us to replace \( H = -\Delta + V \) by \( H(A) := (-i\nabla + A)^2 + V \) with any real vector potential \( A \) and complex potential \( V \). (So, we can read Theorem 1.2 and Theorem 2.1-2.2 described later as results for \( H(A) \).) In addition, their proofs also enable us to replace \(|V(x)|\) in (1.7) and (1.9) by \( (\text{Re} V(x) - |\text{Im} V(x)|) / \sqrt{2} \). See [6] for details.

[17] shows that, if \( V \) is a dilation analytic complex potential, we can obtain the Lieb–Thirring type inequality for all eigenvalues (in \( \mathbb{C} \setminus [0, \infty) \)) of \( H \) as follows. On and after, we write \( i := \sqrt{-1} \).

Theorem 1.2 ([17]). Suppose that \( V \in D(\mathcal{S}; \mathbb{C}) \), \( \alpha \in (\pi/4, \pi/2) \). For \( d, \gamma \geq 1 \), one has

\[
\sum_{\lambda \in \sigma_d(H)} |\lambda|^\gamma \leq C_{\gamma,d} \sum_{\pm} \|V_{\pm i\pi/4}\|_{L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C})}^{\gamma+d/2},
\]

More precisely, if we write \( \mathbb{C}_+ \) (resp. \( \mathbb{C}_- \)) for the upper-half (resp. lower-half) complex plane, we have

1) the estimate on eigenvalues on \( \mathbb{C}_+ \):

\[
\sum_{\lambda \in \sigma_d(H) \cap (\mathbb{C}_+ \cup (-\infty, 0))} |\lambda|^\gamma \leq C_{\gamma,d} \|V_{i\pi/4}\|_{L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C})}^{\gamma+d/2},
\]

2) the estimate on eigenvalues on \( \mathbb{C}_- \):

\[
\sum_{\lambda \in \sigma_d(H) \cap (\mathbb{C}_- \cup (-\infty, 0))} |\lambda|^\gamma \leq C_{\gamma,d} \|V_{-i\pi/4}\|_{L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C})}^{\gamma+d/2}.
\]

The above theorems and corollary indicate that improving \( L_{\gamma,d} \) is an important study to increase the accuracy of Lieb–Thirring estimates for complex potentials.

1.2 Lieb–Thirring Type Estimates on Resonance Eigenvalues

Definition 1.2 (e.g. [15]). For \( \theta \in \mathcal{S} \), elements of the complex subset

\[ \sigma_{\text{res}}(H|\theta) := \sigma_d(H(\theta)) \setminus \sigma_d(H) \]

are called resonance eigenvalues of \( H \) under complex dilation with \( \theta \in \mathcal{S} \).

Remark 1.3. Resonances of \( H \) are sometimes defined as isolated and non-real eigenvalues of \( H(\theta) \). We can find that definition in [16] for instance.

One of the reasons to study the eigenvalue estimates for complex potentials is to estimate the resonance eigenvalues or those number. For instance, the following resonance estimate is known, where \( \mathbb{R}_+ := (0, \infty) \).
Theorem 1.3. Let $V$ be a dilation analytic real potential and let $d, \gamma \geq 1$. Fix $\phi \in [0, \pi]$ and $\eta \in [0, \phi]$. Then, for any $\varepsilon \in (0, \phi - \eta]$, the resonances of $H$ in the complex sector $e^{i[-\eta,0]} \mathbb{R}_+$ satisfy that
\[
\sum_{\mu \in e^{i[-\eta,0]} \mathbb{R}_+} |\mu|^\gamma \leq C_{\gamma,d,\phi,\eta,\varepsilon} \left( \left| \text{Im}(e^{i(\eta+\varepsilon)}V_{i\phi/2}) \right|^\gamma + \frac{\gamma+d/2}{L_{\gamma,d}(\mathbb{R}^d; \mathbb{C})} \right) \tag{1.10}
\]
where
\[C_{\gamma,d,\phi,\eta,\varepsilon} := \frac{L_{\gamma,d}}{(\sin \varepsilon)^\gamma (\sin(\phi - \eta - \varepsilon))^d}.
\]
In the right-hand side of (1.10), it represents that $V_{\pm i\phi/2}(x) = V(e^{\pm i\phi/2}x)$ for $x \in \mathbb{R}^d$ respectively, just as when $V$ is complex.

2 Main Results and Proofs

We will prove Theorem 2.2 which is our main theorem in the same way as the proof of Theorem 1.2. For that reason, we recall some results in [17]. Hereafter, $\mathbb{C}^+$ (resp. $\mathbb{C}^-$) denotes the complex upper-half (resp. lower-half) plane.

Proposition 2.1 ([17]). Suppose $V \in D(\mathcal{S}_\alpha; \mathbb{C})$. Then,
\[
\sigma_{\text{ess}}(\tilde{H}(i\phi)) = \sigma_{\text{ess}}(H_0) = [0, \infty),
\]
\[
\sigma_{\text{ess}}(H(i\phi)) = e^{-2i\phi}[0, \infty)
\]
for any $i\phi \in \mathcal{S}_\alpha$.

Lemma 2.1 ([17]). Suppose that $V \in D(\mathcal{S}_\alpha; \mathbb{C})$, $0 < \alpha \leq \pi/2$. Then,
\[
\sigma_a(H) \cap \mathbb{C}^\pm = \sigma_a(H(i\phi)) \cap \mathbb{C}^\pm
\]
for any $i\phi \in \mathcal{S}_\alpha \cap \mathbb{C}^\pm$, where the two symbols $\pm$ correspond arbitrarily to each other.

We write $\{\lambda(i\phi)\}$ for the eigenvalues of $H(i\phi)$. It is shown (e.g. [7, 15, 17]) that each $\lambda(i\phi) \in \sigma_a(H(i\phi))$ is given by the branch of one or several analytic functions as Puiseux series. Then, they can be written as
\[
\tilde{\lambda}(i\phi) = e^{2i\phi}\lambda(i\phi) \tag{2.1}
\]
by virtue of (1.1) and (1.3), if $\{\tilde{\lambda}(i\phi)\}$ are eigenvalues of $\tilde{H}(i\phi)$.

Lemma 2.2 ([17]). Suppose that $V \in D(\mathcal{S}_\alpha; \mathbb{C})$, $\alpha > \pi/4$. Then,
\[
m_\lambda(H) = m_{\lambda(i\phi)}(H(i\phi)) = m_{\tilde{\lambda}(i\phi)}(\tilde{H}(i\phi)) \tag{2.2}
\]
for any $i\phi \in \mathcal{S}_\alpha \cap \mathbb{C}^\pm$.

Remark 2.1. (1) It is well known [7, 15] that Proposition 2.1 and Lemma 2.1 hold for real $V$. Moreover Lemma 2.2 is the same.

(2) As we can see from the proofs in [17], Lemma 2.1, 2.2 still hold even if ‘$\mathbb{C}^\pm$’ is replaced with ‘any subset of $\mathbb{C}^\pm$’ in each statement. However, in order to replace ‘$\mathbb{C}^\pm$’ by ‘any subset of’ the left-half complex plane’, we must keep in mind the range of $\alpha$ (see Theorem 2.1 and that proof for details).
2.1 New Complex Lieb–Thirring Type Estimates

We give an important theorem for our main theorem. The following result serves as a lemma to prove Theorem 2.2.

**Theorem 2.1.** Let $d, \gamma \geq 1$. Suppose that $V \in D(\mathcal{A}; \mathbb{C})$ with $\alpha \in (\pi/4, \pi/2)$. Then, one has, for any $\kappa > 0$,

$$
\sum_{\lambda \in \sigma_{d}(H) \cap \mathcal{U}_{\kappa}} |\lambda|^{\gamma} \leq (1 + \kappa) L_{\gamma,d} \left\| (\text{Re} \ V_{\pm i(\frac{\pi}{4} - \frac{1}{2} \text{Arctan} \kappa)})^{\gamma + d/2} \right\|_{L_{\gamma + d/2}(\mathbb{R}^{d}; \mathbb{C})} \tag{2.3}
$$

where these represent two inequalities, one for the upper sign and the other for the lower sign, and

$$
\mathcal{U}_{\kappa} := \{ z \in \mathbb{C} : \pi/2 < \arg z < \pi/2 + 2 \text{Arctan} \kappa \}, \quad \mathcal{U}_{-\kappa} := \{ z \in \mathbb{C} : -\pi/2 - 2 \text{Arctan} \kappa < \arg z < -\pi/2 \}.
$$

**Proof.** Fix $\kappa > 0$ arbitrarily. We prove only for eigenvalues in $\mathcal{U}_{\kappa}$. The same can be said for them in $\mathcal{U}_{-\kappa}$. We write $\lambda$ for an eigenvalue of $H$ and denote the complex left-half plane by $\mathbb{C}_{\leq}$. We can first show that $\lambda(i\phi) = \lambda$ for any $i\phi \in \mathcal{A} \cap \mathbb{C}_{\leq}$ as well as Lemma 2.1. We can next show, from (2.1), that $\lambda(i\phi) = e^{2\phi} \lambda$ for any $i\phi \in \mathcal{A} \cap \mathbb{C}_{\leq}$. We can also see (2.2) for any $i\phi \in \mathcal{A} \cap \mathbb{C}_{\leq}$ as well as Lemma 2.2. Thus, we should estimate $\{\tilde{\lambda}(i\phi)\}$ instead of $\{\lambda\}$, because of these facts and (2.1). Let us set $\phi = \frac{\pi}{4} - \frac{1}{2} \text{Arctan} \kappa$. It follows, from the above, that

$$
e^{i(2\phi)} (\sigma_{d}(H) \cap \mathcal{U}_{\kappa}) = \sigma_{d}(\tilde{H}(i\phi)) \cap \mathcal{U}_{-\kappa}.$$

by recalling (1.8) for $\mathcal{U}_{-\kappa}$. So, we have

$$
\sum_{\lambda \in \sigma_{d}(H) \cap \mathcal{U}_{\kappa}} |\lambda|^{\gamma} = \sum_{\lambda \in \sigma_{d}(H) \cap \mathcal{U}_{\kappa}} |e^{2i\phi} \lambda|^{\gamma} = \sum_{\tilde{\lambda}(i\phi) \in \sigma_{d}(H(i\phi)) \cap \mathcal{U}_{-\kappa}} |\tilde{\lambda}(i\phi)|^{\gamma} \leq (1 + \kappa) L_{\gamma,d} \left\| (\text{Re} \ V_{i\phi})^{\gamma + d/2} \right\|_{L_{\gamma + d/2}(\mathbb{R}^{d}; \mathbb{C})}.
$$

Hence, this completes the proof. \hfill \blacksquare

We write $\mathbb{C}_{II}$ (resp. $\mathbb{C}_{III}$) for the second (resp. third) quadrant of $\mathbb{C}$. Because of Theorem 2.1, we can easily know Lieb–Thirring type inequalities for eigenvalues on $\mathbb{C}_{II}$ or $\mathbb{C}_{III}$ as follows.

**Corollary 2.1.** Let $d, \gamma \geq 1$. Suppose that $V \in D(\mathcal{A}; \mathbb{C})$ with $\alpha \in (\pi/4, \pi/2)$. Then,

1) **Eigenvalue estimate on $\mathbb{C}_{II}$:**

$$
\sum_{\lambda \in \sigma_{d}(H) \cap \mathbb{C}_{II}} |\lambda|^{\gamma} \leq 2 L_{\gamma,d} \left\| (\text{Re} \ V_{i\phi/8})^{\gamma + d/2} \right\|_{L_{\gamma + d/2}(\mathbb{R}^{d}; \mathbb{C})},
$$

2) **Eigenvalue estimate on $\mathbb{C}_{III}$:**

$$
\sum_{\lambda \in \sigma_{d}(H) \cap \mathbb{C}_{III}} |\lambda|^{\gamma} \leq 2 L_{\gamma,d} \left\| (\text{Re} \ V_{-i\phi/8})^{\gamma + d/2} \right\|_{L_{\gamma + d/2}(\mathbb{R}^{d}; \mathbb{C})}.
$$

**Proof.** It is obvious from (2.3), since we have $\kappa = 1$ by setting $\text{Arctan} \kappa = \pi/4$. \hfill \blacksquare
2.2 Estimates on Complex Resonance Eigenvalues for Complex Potentials

We now would like to estimate the complex eigenvalues which appear newly by complex dilation. We focus on eigenvalues of $H(i\phi)$ appear in open complex sector $\{z \in \mathbb{C} : -2\phi < \arg z < 0\}$. For convenience, let us call them complex resonance eigenvalues of $H$ hereinafter. The following result is our main theorem. The idea of that proof is the way which can be called ‘double complex dilation’. We denote

$$V_{\theta_1, \ldots, \theta_n}(x) := [U(\theta_1) \cdot \ldots \cdot U(\theta_2)U(\theta_1)VU(\theta_2)^{-1}U(\theta_2)^{-1} \cdot \ldots \cdot U(\theta_n)^{-1}elu](x)$$

$$H(\theta_1, \ldots, \theta_n) := U(\theta_n) \cdot \ldots \cdot U(\theta_2)U(\theta_1)HU(\theta_2)^{-1}U(\theta_2)^{-1} \cdot \ldots \cdot U(\theta_n)^{-1}H$$

for any $n \in \mathbb{N}$. The same applies to $\tilde{H}(\theta_1, \ldots, \theta_n)$. 

**Theorem 2.2.** Let $d, \gamma \geq 1$. Suppose that $V \in \mathcal{D}(\mathcal{A}; \mathbb{C})$ with $\alpha \in (\pi/4, \pi/2)$. Then, complex resonance eigenvalues of $H$ are estimated as

$$\sum_{\mu \in \sigma_{\text{res}}(H(i\phi)) \setminus [0, \infty)} |\mu|^\gamma \leq (1 + \tan \phi) L_{\gamma,d} \left\| \left( \Re V_{i(\frac{3}{2} \phi - \frac{\pi}{4})} \right) \right\|_{L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C})}$$

for any $i\phi \in \mathcal{A}$.

**Proof.** This proof is similar to the proofs of Theorem 2.1 and Theorem 2.2. The key to proof is to apply Theorem 2.1 as $\kappa = \tan \phi$. Then, Lemma 2.2 and Theorem 2.2 (2.5) and (2.1) imply that

$$e^{-i \pi/2} \sigma_{\text{res}}(H(i\phi)) = e^{-i \pi/2} \left[ \sigma_d(H(i\phi)) \cap \{z \in \mathbb{C} : -2\phi < \arg z < 0\} \right]$$

$$= \sigma_d(H(i\phi, -i\pi/4)) \cap \mathcal{U}_{-\tan \phi}$$

$$= \sigma_d(\tilde{H}(i(\phi - \frac{\pi}{4}))) \cap \mathcal{U}_{-\tan \phi}.$$ 

Hence, it follows that

$$\sum_{\mu \in \sigma_{\text{res}}(H(i\phi))} |\mu|^\gamma \leq \sum_{\tilde{\mu}(i(\phi - \frac{\pi}{4})) \in \sigma_d(H(i(\phi - \frac{\pi}{4}))) \cap \mathcal{U}_{-\tan \phi}} |\tilde{\mu}(i(\phi - \frac{\pi}{4}))|^\gamma$$

$$\leq (1 + \tan \phi) L_{\gamma,d} \left\| \left( \Re V_{i(\phi - \frac{\pi}{4})} \right) \right\|_{L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C})}$$

by applying (2.3) for eigenvalues in $\mathcal{U}_{-\alpha}$. 

**Remark 2.2.** We write $\sigma_p(T)$ for the point spectrum of the closed operator $T$. If $V$ is a dilation analytic real potential, the spectral decomposition theorem implies that

$$\sigma_p(H) \cap (0, \infty) = \sigma_d(H(i\phi)) \cap (0, \infty)$$

(2.5)
for \( \phi \in (0, \min\{\alpha, \pi/2\}) \) (e.g. [15]). In this sense, embedded eigenvalues (in the essential or absolutely continuous spectrum \([0, \infty)\) of \(H\) are invariant under complex dilation. (In case of dilation analytic complex potentials, we cannot however use the spectral decomposition theorem and we have no idea if the same is true.) Thus, all eigenvalues which appear newly by complex dilation belong to \( \{z \in \mathbb{C} : -2\phi < \arg z < 0\} \) if embedded eigenvalues of \(H\) exist.

We derived Corollary 2.1 by complex dilation, but we can produce the following results by double complex dilation and Corollary 2.1. Here, \( \mathbb{C}_1 \) (resp. \( \mathbb{C}_{IV} \)) denotes the first (resp. fourth) quadrant of \( \mathbb{C} \).

**Proposition 2.2.** Let \( d, \gamma \geq 1 \). Suppose that \( V \in D(\mathcal{S}_\alpha; \mathbb{C}) \) with \( \alpha \in (\pi/4, \pi/2) \). Then,

1) **Eigenvalue estimate on \( \mathbb{C}_1 \):**

\[
\sum_{\lambda \in \sigma_d(H) \cap \mathbb{C}_1} |\lambda|^\gamma \leq 2L_{\gamma,d} \left\| (\text{Re} V_{3\pi i/8}) - \|^{\gamma+d/2}_{L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C})} \right. ,
\]

2) **Eigenvalue estimate on \( \mathbb{C}_{IV} \):**

\[
\sum_{\lambda \in \sigma_d(H) \cap \mathbb{C}_{IV}} |\lambda|^\gamma \leq 2L_{\gamma,d} \left\| (\text{Re} V_{-3\pi i/8}) - \|^{\gamma+d/2}_{L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C})} \right. .
\]

**Proof.** We should apply Corollary 2.1 to \( \tilde{\lambda}(i\pi/4) = i\lambda \in \mathbb{C}_H \) if \( \lambda \in \mathbb{C}_1 \). In fact, we have 1) as follows:

\[
\sum_{\lambda \in \sigma_d(H) \cap \mathbb{C}_1} |\lambda|^\gamma \leq 2L_{\gamma,d} \left\| (\text{Re} V_{i\pi/8, i\pi/4}) - \|^{\gamma+d/2}_{L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C})} \right. ,
\]

\[
= 2L_{\gamma,d} \left\| (\text{Re} V_{3\pi i/8}) - \|^{\gamma+d/2}_{L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C})} \right. .
\]

2) can be shown in the same way. \( \square \)

We write \( \mathbb{C}_\triangleright \) for the right-half complex plane. Proposition 2.2 immediately derives the following estimate.

**Corollary 2.2.** Let \( d, \gamma \geq 1 \). Suppose that \( V \in D(\mathcal{S}_\alpha; \mathbb{C}) \) with \( \alpha \in (\pi/4, \pi/2) \). Then, the eigenvalues of \( H \) on \( \mathbb{C}_\triangleright \setminus [0, \infty) \) are estimated as follows:

\[
\sum_{\lambda \in \sigma_d(H) \cap \mathbb{C}_\triangleright} |\lambda|^\gamma \leq 2L_{\gamma,d} \left\| (\text{Re} V_{\pm3\pi i/8}) - \|^{\gamma+d/2}_{L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C})} \right. .
\]

Thus, we can obtain an estimate on all eigenvalues different form Theorem 1.2 as follows.

**Corollary 2.3** (cf. [17]). Let \( d, \gamma \geq 1 \). Suppose that \( V \in D(\mathcal{S}_\alpha; \mathbb{C}) \) with \( \alpha \in (\pi/4, \pi/2) \). Then, one has

\[
\sum_{\lambda \in \sigma_d(H)} |\lambda|^\gamma \leq C_{\gamma,d} \left\| V \|^{\gamma+d/2}_{L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C})} + 2L_{\gamma,d} \sum_{\pm} \left\| (\text{Re} V_{\pm3\pi i/8}) - \|^{\gamma+d/2}_{L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C})} \right. . \tag{2.6}
\]

**Proof.** The desired estimate follows by combining (1.9) and (2.6). \( \square \)
2.3 On Number of Isolated and Complex Resonance Eigenvalues

It is important to count and estimate the number of eigenvalues of differential operators in mathematical physics. The Schrödinger operators are no exception. Since

$$\sum |\lambda|^\gamma \geq (\text{the lowest absolute eigenvalue})^\gamma \cdot (\text{the number of eigenvalues}),$$

the number of eigenvalues can be immediately estimated as Lieb–Thirring type. However, we need the (sufficient) conditions for the lowest absolute eigenvalue not to accumulate at 0 (thus they are positive) for that purpose. For instance, X. P. Wang [18] derived the following condition:

**Proposition 2.3.** Let $d, \gamma \geq 1$. We denote the number of complex resonance eigenvalues under complex dilation with $\theta$ (resp. isolated eigenvalues in $\Omega$) of $H$ by $N_d(H; \Omega)$ (resp. $N_d(H; \Omega)$). Suppose that the potential $V$ satisfies the followings:

a) $V \in D(J_\alpha; \mathbb{C})$ with $\alpha \in (\pi/4, \pi/2)$.

b) Wang’s condition; the function $\mathbb{R}^d \ni x \mapsto V(x) \in \mathbb{C}$ obeys that $\text{Im} V(x) < 0$ for any $x \in \mathbb{R}^d$ and that

$$|\text{Re} V(x)|, |\text{Im} V(x)| \lesssim (x)^{-\rho}$$

outside the sufficiently large sphere in $\mathbb{R}^d$, where $\langle x \rangle := (1 + |x|^2)^{1/2}$ and $\rho > 1$.

Then, we have

1) the number of isolated eigenvalues in the sectors $\mathcal{U}_{\pm \alpha}$:

$$N_d(H; \mathcal{U}_{\pm \alpha}) \leq (1 + \tan \phi) \tilde{L}_{\gamma, d} \left\| (\text{Re} V_{\frac{\pi}{2} + \phi - \frac{\pi}{2}}) - \right\|_{L^{\gamma + d/2}(\mathbb{R}^d; \mathbb{C})}^{\gamma + d/2},$$

2) the number of isolated eigenvalues in $\mathcal{C}_1$:

$$N_d(H; \mathcal{C}_1) \leq 2 \tilde{L}_{\gamma, d} \left\| (\text{Re} V_{3\pi i/8}) - \right\|_{L^{\gamma + d/2}(\mathbb{R}^d; \mathbb{C})}^{\gamma + d/2},$$

3) the number of isolated eigenvalues in $\mathcal{C}_II$:

$$N_d(H; \mathcal{C}_II) \leq 2 \tilde{L}_{\gamma, d} \left\| (\text{Re} V_{i\pi/8}) - \right\|_{L^{\gamma + d/2}(\mathbb{R}^d; \mathbb{C})}^{\gamma + d/2},$$

4) the number of isolated eigenvalues in $\mathcal{C}_{III}$:

$$N_d(H; \mathcal{C}_{III}) \leq 2 \tilde{L}_{\gamma, d} \left\| (\text{Re} V_{-i\pi/8}) - \right\|_{L^{\gamma + d/2}(\mathbb{R}^d; \mathbb{C})}^{\gamma + d/2},$$

5) the number of isolated eigenvalues in $\mathcal{C}_{IV}$:

$$N_d(H; \mathcal{C}_{IV}) \leq 2 \tilde{L}_{\gamma, d} \left\| (\text{Re} V_{-3\pi i/8}) - \right\|_{L^{\gamma + d/2}(\mathbb{R}^d; \mathbb{C})}^{\gamma + d/2},$$

6) (cf. [17]) the number of all isolated eigenvalues:

$$N_d(H; \mathcal{C}) \leq \tilde{C}_{\gamma, d} \left\| V \right\|_{L^{\gamma + d/2}(\mathbb{R}^d; \mathbb{C})}^{\gamma + d/2} + 2 \tilde{L}_{\gamma, d} \sum_{\pm} \left\| (\text{Re} V_{\pm 3\pi i/8}) - \right\|_{L^{\gamma + d/2}(\mathbb{R}^d; \mathbb{C})}^{\gamma + d/2},$$
7) the number of complex resonance eigenvalues:

\[ N_{cr}(H|\phi) \leq (1 + \tan \phi)\tilde{L}_{\gamma,d} \left\| \left( \mathrm{Re} V_i\left( \frac{3}{2} - \frac{\pi}{2} \right) \right) \right\|_{L^\gamma+d/2(\mathbb{R}^d;\mathbb{C})} \]

for any \( \phi \in S_\alpha \). Here

\[ \tilde{L}_{\gamma,d} := \inf_{\lambda \in \sigma_d(H)} |\lambda|\gamma, \quad \tilde{C}_{\gamma,d} := \inf_{\lambda \in \sigma_d(H)} |\lambda|\gamma, \quad \hat{L}_{\gamma,d} := \inf_{\lambda \in \sigma_{\text{res}}(H|\phi) \setminus [0,\infty)} |\mu|\gamma. \]

\section{Appendix}

We are interested in how the complex dilation affects the accuracy of eigenvalue estimates. In this appendix, we investigate the \( L^p \)-norms of dilated potentials via examples.

\subsection{3.1 \( \| V \|_{L^\gamma+d/2(\mathbb{R}^d;\mathbb{C})} \) v.s. \( \| V \|_{L^\gamma+d/2(\mathbb{R}^d;\mathbb{C})} \)}

We first argue the comparison of values of \( \| V \|_{L^2(\mathbb{R}^d;\mathbb{C})} \) and \( \| V_i\phi \|_{L^2(\mathbb{R}^d;\mathbb{C})} \).

The following example gives us invariance for \( L^p \)-norms under complex dilation, but it is so trivial. Recall that the real potential \( V \) which belongs to \( L^p(\mathbb{R}^d) \) for \( p > \max\{2-\varepsilon, d/2\} \) with any \( \varepsilon > 0 \) is \( H_0 \)-compact. That is, we only need to show that \( V \in L^p(\mathbb{R}^d) \) if \( p \geq 2 \) and \( p > d/2 \) in order to verify that \( V \) is \( H_0 \)-compact, as is well known.

\begin{proposition}
Let \( d \geq 1 \) and \( \gamma \geq \max\{2-d/2,1\} \). Suppose that \( V \) is the finite complex well on \( \mathbb{R}^d \):

\[ V(x) = \begin{cases} 0, & |x| \leq a \\ ih, & |x| > a \end{cases} \]

where \( a, h > 0 \). Then, \( V \in D(S_\alpha;\mathbb{C}) \) for any \( \phi \in S_\alpha \) and one has

\[ \| V \|_{L^\gamma+d/2(\mathbb{R}^d;\mathbb{C})} = \| V_i\phi \|_{L^\gamma+d/2(\mathbb{R}^d;\mathbb{C})}. \]

We feel that the complex dilation may increase the norm of the potential in general. (One of such examples can be actually seen in \cite{17}. See also Proposition 3.3.) However, the following example gives us that our feeling is not always true.

\begin{proposition}
Let \( d \geq 1 \) and \( \gamma \geq \max\{2-d/2,1\} \). Suppose that the potential \( V \) is defined as a multiplication operator with a Gauss-type function \( V(x) = e^{-cx^2}, \quad c \in \{ z \in \mathbb{C} : \mathrm{Re} z > 0 \} \)

on \( \mathbb{R}^d \). Then, \( V \in D(S_\alpha;\mathbb{C}) \) for any \( \phi \in S_\alpha \) obeying

\[ (\mathrm{Re} c) \cos 2\phi > (\mathrm{Im} c) \sin 2\phi, \]

and the followings hold:

1) If \( \mathrm{Re} c \geq (\mathrm{Re} c) \cos 2\phi - (\mathrm{Im} c) \sin 2\phi \), then one has

\[ \| V \|_{L^{\gamma+1/2}(\mathbb{R}^d;\mathbb{C})} \leq \| V_i\phi \|_{L^{\gamma+1/2}(\mathbb{R}^d;\mathbb{C})}. \]

\end{proposition}
2) If \( c \leq (\text{Re } c) \cos 2\phi - (\text{Im } c) \sin 2\phi \), then one has
\[
\|V_{i\phi}\|_{L^{q+1/2}(\mathbb{R}; \mathbb{C})} \leq \|V\|_{L^{q+1/2}(\mathbb{R}; \mathbb{C})}.
\]

**Proof.** It is not difficult to see that \( V \) is dilation analytic on \( \mathcal{S}_\alpha \) for all \( \gamma \geq \max\{2-d/2, 1\} \) with any \( d \geq 1 \). It is however sufficient to prove this proposition for \( d = 1 \) by virtue of the exponential law. We assume (3.3). Then, \( V_{i\phi} \in L^{q+1/2}(\mathbb{R}; \mathbb{C}) \) and we have
\[
\|V\|_{L^{q+1/2}(\mathbb{R}; \mathbb{C})}^{q+1/2} = \int_{-\infty}^{\infty} |e^{-c|x|^2}|^{q+1/2} dx = \left( \frac{\pi}{(\text{Re } c)(\gamma + 1/2)} \right)^{1/2},
\]
\[
\|V_{i\phi}\|_{L^{q+1/2}(\mathbb{R}; \mathbb{C})}^{q+1/2} = \int_{-\infty}^{\infty} |e^{-c(e^{i\phi}x)^2}|^{q+1/2} dx = \left( \frac{\pi}{\{(\text{Re } c)\cos 2\phi - (\text{Im } c)\sin 2\phi\}(\gamma + 1/2)} \right)^{1/2}.
\]

Hence, the proof of this theorem completes. \( \square \)

### 3.2 On Monotonicity of \( \|V_{i\phi}\|_{L^{q+1/2}(\mathbb{R}; \mathbb{C})} \)

We finally investigate whether \( \|V_{i\phi}\|_{L^{q+1/2}(\mathbb{R}; \mathbb{C})} \) is monotonic with respect to the dilation angle \( \phi \). We feel that the more complex dilation we give, the bigger the values of norms of dilation analytic potentials may be. In fact, we can give many examples that affirm our feeling as follows.

**Proposition 3.3** (cf. [17]). Let \( d = 1 \) and \( \gamma \geq 3/2 \). We define the potential \( V \) as a multiplication operator by
\[
V(x) = \frac{c}{(1 + x^2)^s}, \quad s > \frac{1}{2\gamma + 1}, \quad c \in \mathbb{C}.
\]

Then, \( V \in D(\mathcal{S}_\alpha; \mathbb{C}) \) and \( \{\|V_{i\phi}\|_{L^{q+1/2}(\mathbb{R}; \mathbb{C})}\}_{\phi \in [0, \pi/2]} \) is always monotone increasing.

**Proof.** It is easy to see that \( V \in L^{q+1/2}(\mathbb{R}; \mathbb{C}) \) and \( V \) is \( H_0 \)-compact, if \( s > 1/(2\gamma + 1) \) and \( \gamma \geq 3/2 \). Since
\[
|V_{i\phi}(x)|^{q+1/2} = \frac{|c|^{q+1/2}}{(x^4 + 2(\cos 2\phi)x^2 + 1)^{s(2\gamma + 1)}} \leq \frac{C_\gamma}{x^{4s(2\gamma + 1)}}
\]
for a suitable constant \( C_\gamma > 0 \) depending on \( \gamma \), we also have \( V_{i\phi} \in L^{q+1/2}(\mathbb{R}; \mathbb{C}) \) because of \( s > 1/(2\gamma + 1) \). It is not difficult to see that \( V \) is dilation analytic from the above. We now consider the function \( F(\phi) := 2x^2 \cos 2\phi + (x^4 + 1) \) with respect to \( \phi \) by fixing \( x \in \mathbb{R} \). Since \( F \) is monotone decreasing on \( [0, \pi/2] \), the proof of this proposition completes from (3.6). \( \square \)

**Proposition 3.4.** Let \( d = 1 \) and \( \gamma \geq 3/2 \). We consider a complex potential \( V \) defined as a multiplication operator by
\[
V(x) = \begin{cases} 
e^{-cx} & \text{if } x \geq 0; \\ e^{cx} & \text{if } x < 0, \end{cases}
\]
where \( c \in \mathbb{C} \). Then, \( V \in D(\mathcal{S}_\alpha; \mathbb{C}) \) and \( \{\|V_{i\phi}\|_{L^{q+1/2}(\mathbb{R}; \mathbb{C})}\}_{\phi \in [0, \pi/2]} \) is monotone increasing on \( [0, \pi/2] \).
Proof. It is not difficult to see that $V$ is dilation analytic as before. We first consider the case of $x \geq 0$. If we denote $a := \text{Re } c > 0$ and $b := \text{Im } c > 0$, we obtain

$$|V_{i\phi}(x)|^{\gamma + 1/2} = \exp \left( - (a \cos \phi - b \sin \phi)(\gamma + 1/2)x \right)$$

$$= \exp \left( - \sqrt{a^2 + b^2} \cos(\phi + A)(\gamma + 1/2)x \right)$$  \hspace{1cm} (3.6)$$

where $\cos A := \frac{a}{\sqrt{a^2 + b^2}}$ and $\sin A := \frac{b}{\sqrt{a^2 + b^2}}$. Since $0 \leq \phi < \pi/2$, $0 < A < \pi/2$ and $x \geq 0$, the cosine function $F(\phi) = \cos(\phi + A)$ is monotone decreasing. Since $0, \cos(\phi + A)$ is monotone decreasing, $\{\|V_{i\phi}\|_{L^{\gamma + 1/2}(0,\infty; \mathbb{C})}\}_\phi$ is monotone increasing. We next consider the case of $x < 0$. We obtain

$$|V_{i\phi}(x)|^{\gamma + 1/2} = \exp \left( (a \cos \phi - b \sin \phi)(\gamma + 1/2)x \right)$$

$$= \exp \left( \sqrt{a^2 + b^2} \cos(\phi + A)(\gamma + 1/2)x \right)$$

as well as (3.6). Since $0 \leq \phi < \pi/2$, $0 < A < \pi/2$ and $x < 0$, the cosine function $F(\phi) = \cos(\phi + A)$ is monotone decreasing, $\{\|V_{i\phi}\|_{L^{\gamma + 1/2}((-\infty,0; \mathbb{C})}\}_\phi$ is monotone increasing. Hence, we have obtained the proof.

As the above, we feel that the norms of dilated potentials may have monotonically increasing properties. We can however see that our feeling is not always true as follows.

**Proposition 3.5.** Let $d = 1$ and $\gamma \geq 3/2$. For $V \in D(\mathcal{S}_\alpha; \mathbb{C})$ defined as a multiplication operator with a Gauss-type function (3.2) for any $i\phi \in \mathcal{S}_\alpha$ obeying (3.3), the following hold:

1) If $\text{Im } c > 0$, then $\{\|V_{i\phi}\|_{L^{\gamma + 1/2}(\mathbb{R}; \mathbb{C})}\}_\phi$ is monotone increasing.

2) If $\text{Im } c < 0$ and $\phi \in [0,p)$ (resp. $[p,\pi/2)$), then $\{\|V_{i\phi}\|_{L^{\gamma + 1/2}(\mathbb{R}; \mathbb{C})}\}_\phi$ is monotone decreasing (resp. monotone increasing). Here

$$p := \frac{1}{2} \arctan \left( \frac{-\text{Im } c}{\text{Re } c} \right).$$  \hspace{1cm} (3.7)$$

Proof. We consider the function $F(\phi) := (\text{Re } c) \cos 2\phi - (\text{Im } c) \sin 2\phi$ with respect to $\phi \in [0,\pi/2)$. Remark $\text{Re } c > 0$. Since we have $F'(\phi) = -2(\text{Re } c) \sin 2\phi + (\text{Im } c) \cos 2\phi$, we obtain the critical point $p$ defined by (3.7) by solving $F'(\phi) = 0$.

1) We assume $\text{Im } c > 0$. Then, $p < 0$,

$$F(0) = \text{Re } c > 0 \quad \text{and} \quad \lim_{\phi \uparrow \pi/2} F(\phi) = -a < 0.$$  \hspace{1cm} (3.8), (3.9)$$

Thus, $F$ is monotone decreasing on $[0, \pi/2)$. Hence, (3.4) implies that $\{\|V_{i\phi}\|_{L^{\gamma + 1/2}(\mathbb{R}; \mathbb{C})}\}_\phi$ is monotone increasing on $[0, \pi/2)$.

2) We assume $\text{Im } c < 0$. Then, $p > 0$, (3.8), (3.9) and

$$F(p) = (\text{Re } c) \cos \left( \arctan \left( \frac{-\text{Im } c}{\text{Re } c} \right) \right) - (\text{Im } c) \sin \left( \arctan \left( \frac{-\text{Im } c}{\text{Re } c} \right) \right) > 0$$

because of $\text{Re } c > 0$. Thus, $F$ is monotone decreasing on $[0,p)$ and is monotone increasing on $[p,\pi/2)$. Hence, (3.4) implies that $\{\|V_{i\phi}\|_{L^{\gamma + 1/2}(\mathbb{R}; \mathbb{C})}\}_\phi$ is monotone increasing on $[0,p)$ and is monotone decreasing on $[p,\pi/2)$. 

\hfill \Box
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