REMARKS ON THE DYNAMICS OF THE HOROCYCLE FLOW
FOR HOMOGENEOUS FOLIATIONS BY HYPERBOLIC
SURFACES

FERNANDO ALCALDE CUESTA AND FRANÇOISE DAL’BO

Abstract. This article is a first step towards the understanding of the dynamics of the horocycle flow on foliated manifolds by hyperbolic surfaces. This is motivated by a question formulated by M. Martínez and A. Verjovsky on the minimality of this flow assuming that the natural affine foliation is minimal too. We have tried to offer a simple presentation, which allow us to update and shed light on the classical theorem proved by G. A. Hedlund in 1936 on the minimality of the horocycle flow on compact hyperbolic surfaces. Firstly, we extend this result to the product of $\text{PSL}(2, \mathbb{R})$ and a Lie group $G$, which places us within the homogeneous framework investigated by M. Ratner. Since our purpose is to deal with non-homogeneous situations, we do not use the Ratner’s famous Orbit-Closure Theorem, but we give an elementary proof. We show that this special situation arises for homogeneous Riemannian and Lie foliations, reintroducing the foliation point of view. Examples and counter-examples take an important place in our work, in particular, the very instructive case of the solvable manifold $T^3_A$. Our aim in writing this text is to offer to the reader an accessible introduction to a subject that was intensively studied in the algebraic setting, although there still are unsolved geometric problems.

1. Introduction and motivation

In this paper, we start by focusing our attention on the following subgroups

$$U = \{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \} \quad \text{and} \quad B = \{ \begin{pmatrix} \lambda & t \\ 0 & \lambda^{-1} \end{pmatrix} \mid t \in \mathbb{R}, \lambda \in \mathbb{R}_+^\ast \}$$

of the group $\text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})/\{\pm \text{Id}\}$. We also consider a connected Lie group $G$ and the natural right actions of $U$ and $B$ on the product $\text{PSL}(2, \mathbb{R}) \times G$ where every element of $\text{PSL}(2, \mathbb{R})$ acts trivially on the second factor $G$. We discuss the minimality of the right actions of $U$ and $B$ induced on the left quotient $X = \Gamma\backslash\text{PSL}(2, \mathbb{R}) \times G$ by a cocompact discrete subgroup of $\text{PSL}(2, \mathbb{R}) \times G$. Recall that an action is said to be minimal if all the orbits are dense.

In the case where $G$ is trivial, assuming $\Gamma$ is torsion-free, the quotient $X = \Gamma\backslash\text{PSL}(2, \mathbb{R})$ becomes the unitary tangent bundle $T^1S$ to the compact hyperbolic surface $S = \Gamma\backslash\mathbb{H}$ obtained from the Poincaré half-plane $\mathbb{H}$. In 1936, G. A. Hedlund [19] proved that the horocycle flow on $X$ is minimal (for an elementary proof, see [15]). In our context, this theorem can be reformulated as follows:

Hedlund’s Theorem. The right $U$-action on $X = \Gamma\backslash\text{PSL}(2, \mathbb{R})$ is minimal.

2010 Mathematics Subject Classification. 37D40, 37C85, 57R30.

1
On the contrary, if $X$ is not compact, M. Kulikov \cite{22} constructed an infinitely generated Fuchsian group without non-empty $U$-minimal sets. In the case of non-uniform lattices of $PSL(2, \mathbb{R})$, like the modular subgroup $PSL(2, \mathbb{Z})$, the $U$-orbits are dense or periodic. Actually, it is known from \cite{11} that the $U$-action on $X$ is minimal if and only if $X$ is compact.

When $G$ is not trivial, even assuming $X$ is compact, the $U$-action may be non-minimal. This is the case for example when $G = PSL(2, \mathbb{R})$ and $\Gamma$ is the product of two cocompact Fuchsian groups. However, in this setting, we prove the following criterium:

**Theorem 1.** Let $G$ be a connected Lie group and $\Gamma$ be a discrete cocompact subgroup of $PSL(2, \mathbb{R}) \times G$. Then the right $U$-action on $X = \Gamma \backslash PSL(2, \mathbb{R}) \times G$ is minimal if and only if the corresponding $PSL(2, \mathbb{R})$-action is minimal.

Our proof of Theorem \ref{thm:main} does not use the Ratner’s famous Orbit-Closure Theorem \cite{27}, see also \cite{15} and \cite{21} for an overview. In fact, some ideas will be applied in a non-homogeneous context.

In the second part of this paper, we adopt a foliation point of view, which is natural in the previous context. For any connected Lie group $G$, the horizontal foliation of $PSL(2, \mathbb{R}) \times G$ by the fibres of the projection on the second factor $G$ is invariant by the action of $\Gamma$ and so induces a foliation on $X = \Gamma \backslash PSL(2, \mathbb{R}) \times G$ whose leaves are the orbits of the right $PSL(2, \mathbb{R})$-action. This action gives rise to a $G$-Lie foliation as defined in \cite{17} and \cite{25}. Before recalling this definition, let us explain how to construct some examples. Let $H$ be a connected Lie group with a surjective morphism $\rho : H \to G$ and let $\Gamma$ be a cocompact discrete subgroup of $H$. As before, the foliation by the fibres of $\rho$ induces a foliation of $X = \Gamma \backslash H$ whose leaves are the orbits of the right action of the kernel $K$ of $\rho$. They are diffeomorphic to $K \cap \Gamma \backslash K$. According to \cite{14}, any $G$-Lie foliation constructed by this method is called homogeneous.

In general, as stated in a theorem by E. Fédida \cite{12}, $G$-Lie foliations are characterised as follows. Given a discrete group $\Gamma$ acting freely and properly discontinuously on a smooth manifold $\tilde{M}$, a group homomorphism $h : \Gamma \to G$ and a surjective smooth submersion $\rho : \tilde{M} \to G$ with connected fibres that is $\Gamma$-equivariant (i.e. $\rho(\gamma x) = h(\gamma)f(x)$ for all $\gamma \in \Gamma$ and for all $x \in \tilde{M}$), the foliation $\tilde{F}$ by the fibres of $\rho$ induces a foliation $F$ of $M = \Gamma \backslash \tilde{M}$, called $G$-Lie foliation, whose leaves are quotients of the fibres of $\rho$ by the kernel of $h$. We say $F$ is a $G$-Lie foliation by hyperbolic surfaces if $M$ is endowed with a complete Riemannian metric whose restriction to each leaf has hyperbolic conformal type. Actually, according to \cite{8} and \cite{30}, we can assume that each restriction has constant negative curvature, namely each leaf is a hyperbolic surface. We define the unitary tangent bundle $X = T^1_+F$ as the vector bundle whose fibre $T^1_xF$ at $x \in M$ is the unitary tangent space $T^1_xL$ to the leaf $L_x$ passing through $x$. Notice that $X$ admits a natural right $PSL(2, \mathbb{R})$-action, which is only continuous in general.

For instance, we can consider the homogeneous $PSL(2, \mathbb{R})$-Lie foliation $F$ whose unitary tangent bundle $X = T^1F$ is the quotient of $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ by a cocompact discrete subgroup $\Gamma$. If $\Gamma$ is irreducible, the projections of $\Gamma$ on each factor are dense and hence the natural right $PSL(2, \mathbb{R})$-action on $X$ is minimal. By applying Theorem \ref{thm:main} we deduce that the horocycle flow on $X = T^1F$ is also minimal.
More generally, we can derive from this theorem the following generalisation of Hedlund’s Theorem in the spirit of the work of M. Martínez and A. Verjovsky \[23\] on which we comment below:

**Theorem 2.** Let \(X = T^1F\) be the unitary tangent bundle of a homogeneous \(G\)-Lie foliation \(F\) by hyperbolic surfaces of a compact manifold. If \(F\) is minimal, then the right \(U\)-action on \(X\) is minimal.

Note, however, that there are \(\text{PSL}(2,\mathbb{R})\)-Lie foliations which are not homogeneous \[18\]. A natural question arises if we replace \(G\) with the quotient \(G/G_0\) by a closed Lie subgroup \(G_0\): does Theorem 2 remains valid for these more general foliations? They are transversely homogeneous foliations \[7\] whose structure can be described in a similar way to that of the \(G\)-Lie foliations. If \(G_0\) is compact, we can construct by averaging a left-invariant Riemannian metric on \(G\) that is also invariant by the right action of \(G_0\). Then the distance between two \(\text{PSL}(2,\mathbb{R})\)-orbits in \(X = \Gamma\backslash \text{PSL}(2,\mathbb{R}) \times G/G_0\) remains locally constant and therefore the right \(\text{PSL}(2,\mathbb{R})\)-action on \(X\) defines a Riemannian foliation according to \[24\] and \[25\]. Using Molino’s theory \[24\], we extend Theorem 2 to this context in Corollary 3.6.

In the third part of this paper, we show that, on the contrary, Theorem 2 does not hold for general transversely homogeneous foliations where \(G_0\) is a non-compact closed Lie subgroup of \(G\):

**Theorem 3.** There are minimal transversely homogeneous foliations of compact manifolds \(X = \Gamma\backslash \text{PSL}(2,\mathbb{R}) \times G/G_0\) such that neither the \(U\)-action nor the \(B\)-action are minimal. Moreover, there is such an example admitting an unique \(B\)-minimal set which is not \(U\)-minimal.

As mentioned before, the problem of generalising Hedlund’s Theorem for compact foliated manifolds by hyperbolic surfaces has been discussed by M. Martínez and A. Verjovsky in several versions of their article \[23\]. Theorems 1 and 2 give an affirmative answer to the initial conjecture for homogeneous Lie foliations, also valid for homogeneous Riemannian foliations, while Theorem 3 gives a negative answer in the transversely homogeneous setting. Nevertheless, according to the second version of \[23\], the question can be reformulated in the following way:

**Martínez-Verjovsky’s Question.** Let \(X = T^1F\) be the unitary tangent bundle of a compact foliated manifold whose leaves are hyperbolic surfaces. Is it true that the right \(U\)-action on \(X\) is minimal if and only if the right \(B\)-action is minimal?

We complete the paper with some comments on this question.

**Acknowledgements.** We thank Bertrand Deroin and Gaël Meigniez for their accurate comments. This work has been partially supported by the Ministry of Science and Innovation - Government of Spain (Grant MTM2010-15471) and IEMath Network CN 2012/077.

2. Proof of Theorem 1

Let \(G\) be a connected Lie group. Let \(\Gamma\) be any discrete subgroup of \(\text{PSL}(2,\mathbb{R}) \times G\) acting on this Lie group by left translation. We denote by \(p_1\) and \(p_2\) the first and second projection of \(\text{PSL}(2,\mathbb{R}) \times G\) onto \(\text{PSL}(2,\mathbb{R})\) and \(G\) respectively. Any
The proof of Proposition 2.1 uses hyperbolic geometry. Let $H$ be diffeomorphic to the linear space $E^d$ of $P\text{SL}_R$. By duality, the right $F$-action on $X$ is minimal if and only if the action of $\Gamma$ on the quotient $P\text{SL}_R/F \times G$ by left translation is minimal. In particular, the right $P\text{SL}_R$-action on $X$ is minimal if and only if $p_2(\Gamma) = G$. If the right $F$-action is minimal, then $p_1(\Gamma)$ acts minimally on $P\text{SL}_R/F$ and $p_2(\Gamma) = G$. For $F = B$, we prove:

**Proposition 2.1.** Let $G$ be a connected Lie group and let $\Gamma$ be a discrete subgroup of $P\text{SL}_R \times G$. Then the right $B$-action on $X$ is minimal if and only if the following two properties hold:

(i) $p_1(\Gamma)$ acts minimally on $P\text{SL}_R/B$,
(ii) $p_2(\Gamma) = G$, or equivalently the right $P\text{SL}_R$-action on $X$ is minimal.

The proof of Proposition 2.1 uses hyperbolic geometry. Let $\mathbb{H} = \{z \in \mathbb{C}/\text{Im} \ z > 0\}$ be the Poincaré half-plane equipped with the hyperbolic distance $d$. The action of $P\text{SL}_R$ on $\mathbb{H}$ by isometries extends to a $P\text{SL}_R$-action on its boundary $\partial \mathbb{H} = \mathbb{R} \cup \{\infty\}$. This action is conjugated to the right $P\text{SL}_R$-action on $P\text{SL}_R/B$. Since $SL_2(\mathbb{R})$ acts transitively on $\mathbb{R}^2 - \{0\}$ and $U$ is the stabiliser of the vector $e_1 = (1, 0)$, the homogeneous manifolds $P\text{SL}_R/U$ and $P\text{SL}_R/B$ are diffeomorphic to the linear space $E = \mathbb{R}^2/\{\pm \text{Id}\}$ and the projective line $\mathbb{R}P^1$ respectively. Before we prove Proposition 2.1 we state the following key lemma:

**Lemma 2.2.** Let $\{f_n\}_{n \geq 0}$ be a sequence of elements of $P\text{SL}_R$. If for some $z \in \mathbb{H}$, there are points $\xi^+$ and $\xi^-$ in $\partial \mathbb{H}$ such that

$$\lim_{n \to +\infty} f_n(z) = \xi^+ \quad \text{and} \quad \lim_{n \to +\infty} f_n^{-1}(z) = \xi^-,$$

then for every point $\xi \neq \xi^-$ in $\mathbb{H} \cup \partial \mathbb{H}$, we have:

$$\lim_{n \to +\infty} f_n(\xi) = \xi^+.$$  

**Proof.** For each point $\xi \in \mathbb{H}$, we have $\lim_{n \to +\infty} f_n(\xi) = \xi^+$ since $d(f_n(\xi), f_n(z)) = d(\xi, z)$. For $\xi \neq \xi^-$ in $\partial \mathbb{H}$, we choose $\xi' \in \partial \mathbb{H}$ different from $\xi$ and $\xi^-$ and a geodesic $\alpha : \mathbb{R} \to \mathbb{H}$ joining $\xi$ to $\xi'$, that is, $\xi = \lim_{t \to -\infty} \alpha(t)$ and $\xi' = \lim_{t \to +\infty} \alpha(t)$. If we denote by $\alpha_n = f_n \alpha$ the geodesic joining $f_n(\xi)$ to $f_n(\xi')$, then $d(f_n^{-1}(z), \alpha(t)) = d(\xi, \alpha(t))$ for all $t \in \mathbb{R}$. Since $\lim_{n \to +\infty} f_n^{-1}(z) = \xi^-$ and $\xi^-$ is different from $\xi$ and $\xi'$, we have $\lim_{n \to +\infty} d(z, \alpha_n(t)) = +\infty$ for all $t \in \mathbb{R}$. It follows that the sequence of geodesic $\alpha_n$ converges to a point $\xi \in \partial \mathbb{H}$. This implies that $\lim_{n \to +\infty} f_n(\alpha(t)) = \xi$ for all $t \in \mathbb{R}$. Now, since $\alpha(t)$ belongs to $\mathbb{H}$, we have $\lim_{n \to +\infty} f_n(\alpha(t)) = \xi^+$ and hence $\lim_{n \to +\infty} f_n(\xi) = \xi^+$. $\Box$

**Proof of Proposition 2.1** By duality, it is enough to prove the action of $\Gamma$ on $\partial \mathbb{H} \times G$ is minimal when $p_1(\Gamma)$ acts minimally on $\partial \mathbb{H}$ and $p_2(\Gamma) = G$. This second condition allows us to choose a non stationary sequence $\{g_n\}_{n \geq 0}$ in $p_2(\Gamma)$ that converges to the identity element 1 of $G$. Then there is a sequence $\{f_n\}_{n \geq 0}$ in $P\text{SL}_R$ such that $\gamma_n = (f_n, g_n) \in \Gamma$ for all $n \geq 0$. Since $\Gamma$ is discrete, this sequence $\{f_n\}_{n \geq 0}$ is not bounded. Thus, without loss of generality, we can assume that the sequences
Thus, if we prove that \( x^D \) with \( z \) step, \( \Gamma(\xi, g) \) for all \( g \in G \). More generally, assuming that \( \xi \neq f(\xi^-) \) for some \( f \in p_1(\Gamma) \) and replacing \( \gamma_n \) with \( \gamma' \gamma_n(\gamma')^{-1} \) where \( \gamma' = (f, g') \in \Gamma \), we have:

\[
(f(\xi^+), g) = \lim_{n\to+\infty} (f_{n}(\xi), g_n g) = \lim_{n\to+\infty} \gamma_n(\xi, g) \in \overline{\Gamma(\xi, g)}
\]

for all \( g \in G \). Using the minimality of the action of \( p_1(\Gamma) \) on \( \partial \mathbb{H} \), we get \( \partial \mathbb{H} \times \{g\} \subseteq \overline{\Gamma(\xi, g)} \) for all \( g \in G \). Now, since \( p_2(\Gamma) = G \), it follows that \( \overline{\Gamma(\xi, g)} = \partial \mathbb{H} \times G \). Thus, assume that \( \xi = f(\xi^-) \) for some \( f \in p_1(\Gamma) \). Since \( p_1(\Gamma) \) acts minimally on \( \partial \mathbb{H} \) and contains unbounded sequences like \( \{f_n\}_{n \geq 0} \), either \( p_1(\Gamma) \) is dense in \( PSL(2, \mathbb{R}) \) or \( p_1(\Gamma) \) is a Fuchsian group of first kind (i.e. having \( \partial \mathbb{H} \) as limit set). This implies that there exists \( \gamma' = (f', g') \in \Gamma \) such that the sequence \( (f')^k(\xi^+) \) converges to a point \( \xi' \notin p_1(\Gamma)\xi^- \) when \( k \) goes to \( +\infty \) and \( (f')^k(\xi^-) \neq \xi \) for all \( k \geq 0 \). So the sequence \( (\gamma')^k = ((f')^k, (g')^k) \in \Gamma \) verifies:

\[
\lim_{k \to +\infty} (\gamma')^k \gamma_n(\gamma')^{-k} (\xi, g) = ((f')^k(\xi^+), g)
\]

and therefore \( (\xi', g) \) belong to \( \overline{\Gamma(\xi, g)} \). Since \( \xi' \notin p_1(\Gamma)\xi^- \), according to the previous step, \( \Gamma(\xi', g) \) is dense in \( \partial \mathbb{H} \times G \) and hence \( \Gamma(\xi, g) \) is also dense. \( \square \)

Theorem 1 really concerns cocompact discrete subgroups. Before we deal with this case, let us introduce the notion of semi-parabolic element of the Lie group \( PSL(2, \mathbb{R}) \times G \). Thus, we say that \( (f, g) \in PSL(2, \mathbb{R}) \times G \) is semi-parabolic if \( f \) is conjugated in \( PSL(2, \mathbb{R}) \) to an element \( u \neq Id \in U \). The existence of semi-parabolic elements in \( \Gamma \) is related to the behaviour of the right \( D \)-action on \( X \) where

\[
D = \{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \} / \lambda > 0 \} \quad \text{and} \quad D^+ = \{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \} / \lambda > 1 \}
\]

are the diagonal group and its strictly positive cone.

**Lemma 2.3.** If \( \Gamma \) contains a semi-parabolic element, then there are divergent positive semi-orbits with respect the right \( D^+ \)-action on \( X \).

**Proof.** Assume that \( \Gamma \) contains a semi-parabolic element \( \gamma = (fu^{-1}, g) \) where \( u \in U - \{Id\} \), \( f \in PSL(2, \mathbb{R}) \) and \( g \in G \). Given \( g' \in G \), we set \( x = \Gamma(f, g') \in X \) and we prove that \( xD^+ \) diverges. Suppose on the contrary that the sequence \( \{xd_n\}_{n \geq 0} \) converges for some non-bounded sequence \( \{d_n\}_{n \geq 0} \) in \( D^+ \). Put

\[
d_n = \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_n^{-1} \end{pmatrix}
\]

such that \( \lambda_n \to +\infty \). Also write

\[
u = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}
\]

with \( t \neq 0 \). By hypothesis, there exists a sequence \( \{\gamma_n\}_{n \geq 0} \) in \( \Gamma \) such that \( \gamma_n(f, g')d_n \) converges to some element \( (f'', g'') \) in \( PSL(2, \mathbb{R}) \times G \). Notice that

\[
\gamma_n(f, g')d_n = \gamma_n \gamma^{-1} \gamma(f, g')d_n = \gamma_n \gamma^{-1} (fu d_n, gg') = \gamma_n \gamma^{-1} (f d_n, d_n^{-1} ud_n, gg')
\]
and
\[ \lim_{n \to +\infty} d_n^{-1}ud_n = \lim_{n \to +\infty} \begin{pmatrix} 1 & t \lambda_n^{-2} \\ 0 & 1 \end{pmatrix} = Id. \]

We deduce that the sequence \( \gamma_n \gamma^{-1}(fd_n, gg') \) also converges to \((f'', g'')\). Now, since
\[
\gamma_n \gamma^{-1}(fd_n, gg') = \gamma_n \gamma^{-1} \gamma^{-1} \left( \gamma_n(f, g')d_n \right) \left( \text{Id}, (g')^{-1}gg' \right)
\]
and
\[
\lim_{n \to +\infty} \gamma_n(f, g')d_n = (f'', g''),
\]
it follows that \( \gamma_n \gamma^{-1} \gamma_n^{-1} \) converges to \((\text{Id}, g''(g')^{-1}g^{-1}g''(g')^{-1})\) in \( \text{PSL}(2, \mathbb{R}) \times G \). Since \( \Gamma \) is discrete, for \( n \) large enough, we have \( p_1(\gamma_n \gamma^{-1} \gamma_n^{-1}) = \text{Id} \) and therefore \( u = \text{Id} \) contradicting the hypothesis. \( \square \)

Let us assume \( X \) is compact. From Lemma 2.3, we have immediately:

**Proposition 2.4.** If \( X = \Gamma \backslash \text{PSL}(2, \mathbb{R}) \times G \) is compact, then \( \Gamma \) does not contain semi-parabolic elements. \( \square \)

Before we reformulate Proposition 2.1 in the cocompact case, let us recall the following classification lemma:

**Classification Lemma 2.5.** Let \( \Delta \) be a subgroup of \( \text{PSL}(2, \mathbb{R}) \) and denote by \( \overline{\Delta^0} \) the connected component of the identity of its closure \( \overline{\Delta} \). If \( \Delta \) is neither discrete nor dense, then \( \overline{\Delta^0} \) is conjugated to \( \text{PSO}(2, \mathbb{R}) \) or a Lie subgroup of \( B \). \( \square \)

**Proposition 2.6.** Let \( \Gamma \) be a cocompact discrete subgroup of \( \text{PSL}(2, \mathbb{R}) \times G \). Denote by \( X \) the compact quotient by left translation. Then the right \( B \)-action on \( X \) is minimal if and only if \( p_2(\Gamma) = G \), or equivalently if the right \( \text{PSL}(2, \mathbb{R}) \)-action on \( X \) is minimal.

**Proof.** We have only to prove the ‘if’ part. Now, according to Proposition 2.1, it is enough to show that \( \Delta = p_1(\Gamma) \) of \( \text{PSL}(2, \mathbb{R}) \) acts minimally on \( \partial \mathbb{H} \). We distinguish two cases, depending on whether this group is discrete or not.

If \( \Delta = p_1(\Gamma) \) is discrete, the surface \( \Delta \backslash \mathbb{H} \) is compact because \( X \) is compact too. It follows that any orbit of \( p_1(\Gamma) \) in \( \mathbb{H} \) accumulates to its full boundary \( \partial \mathbb{H} \). In other words, the limit set of \( \Delta \) is equal to \( \partial \mathbb{H} \). Since the limit set of any non-elementary Fuchsian group is minimal, we deduce that the action of \( \Delta \) on \( \partial \mathbb{H} \) is minimal.

If \( \Delta = p_1(\Gamma) \) is non-discrete but dense, then the action of \( \Delta \) on \( \partial \mathbb{H} \) is still minimal. Otherwise, according to Classification Lemma 2.5 and using the fact that \( \Delta \) normalises \( \overline{\Delta^0} \), we deduce that \( \Delta \) is conjugated to a subgroup of \( \text{PSO}(2, \mathbb{R}) \) or \( B \). Since \( X \) is compact, the first case is excluded. Assuming \( \Delta \subset fBf^{-1} \) for some \( f \in \text{PSL}(2, \mathbb{R}) \), we have \( [\Delta, \Delta] \subset fUf^{-1} \) and therefore \( \Delta \) is abelian as a consequence of Proposition 2.1. It follows that \( \Delta \) is conjugated to a subgroup of \( D \), which contradicts the compactness of \( X \). \( \square \)

**Proof of Theorem 7.** According to Proposition 2.6, it is enough to prove that if the right \( B \)-action is minimal, then the right \( U \)-action is minimal too. By compactness of \( X \), the \( U \)-action has a non-empty minimal set \( M \). Let us prove \( M \) is \( B \)-invariant so that \( M = X \) and the right \( U \)-action minimal.
Let \( h = (f, g) \) be an element of \( H = \text{PSL}(2, \mathbb{R}) \times G \) such that \( x = \Gamma h \in \mathcal{M} \). Since \( \mathcal{M} \) is closed, there are elements \( \gamma_n = (\gamma_{1n}, \gamma_{2n}) \in \Gamma \) and \( t_n \) with \( t_n \to +\infty \) such that
\[
\lim_{n \to +\infty} \gamma_n(f, g)u_n = \lim_{n \to +\infty} (\gamma_{1n}fu_n, \gamma_{2n}g) = (f, g) = h
\]
If we write \( f_n = f^{-1}\gamma_{1n}fu_n, g_n = g^{-1}\gamma_{2n}g \), and \( h_n = (f_n, g_n) \), the sequence
\[
h_n = (ff_n, gg_n) = \gamma_nhu_n
\]
converges to \( h \) so that
\[
\lim_{n \to +\infty} h_n = \lim_{n \to +\infty} (f_n, g_n) = (Id, e).
\]
Notice that the sequence \( \{\gamma_{1n}\}_{n \geq 0} \) does not admit any convergent subsequence because \( t_n \to +\infty \). On the other hand, since \( hh_n = \gamma_nh u_n \) represents the class \( xu_n \) in the orbit \( xU \), the element \( h_n = (f_n, g_n) \in H \) belongs to the set
\[H_M = \{h' \in H/\mathcal{M}h' \cap \mathcal{M} \neq \emptyset\}\]
having the following properties:

**Lemma 2.7.** The set \( H_M \) is a closed subset of \( H = \text{PSL}(2, \mathbb{R}) \times G \) which is invariant under the right and left \( U \)-actions on \( H \).

**Proof.** Let \( h'_n \in H_M \) be a sequence that converges to some element \( h' \in H \). By definition, for any \( n \in \mathbb{N} \), there is \( x_n = \Gamma h_n \in \mathcal{M} \) such that \( x_nh'_n = \Gamma h_nh'_n \in \mathcal{M} \). By compactness of \( \mathcal{M} \) and replacing the sequence with some subsequence if necessary, we may assume that the sequence \( x_n \) converges to a class \( x = \Gamma h \in \mathcal{M} \). Then \( xh' = \lim_{n \to +\infty} x_nh'_n \in \mathcal{M} \) and hence \( h' \in H_M \).

Let us prove \( H_M \) is invariant under the right and left \( U \)-actions on \( \text{PSL}(2, \mathbb{R}) \times G \). Indeed, since \( \mathcal{M}u^{-1} = \mathcal{M} \), we have:
\[
\mathcal{M}uh \cap \mathcal{M} = \mathcal{M}u^{-1}uh \cap \mathcal{M} = \mathcal{M} \cap \mathcal{M} \neq \emptyset
\]
for all \( u \in U \) and for all \( h \in H_M \). Likewise, we have:
\[
\mathcal{M}hu \cap \mathcal{M} = (\mathcal{M}h \cap \mathcal{M}u^{-1})u = (\mathcal{M}h \cap \mathcal{M})u \neq \emptyset
\]
proving the right invariance. \(\square\)

Returning to the proof of Theorem 1, we have:

**Lemma 2.8.** There exists \( k \in \mathbb{N} \) such that \( f_n \not\in B \) for \( n \geq k \).

**Proof.** Let us assume on the contrary that for every \( k \in \mathbb{N} \), there exists \( n_k \geq k \) such that \( f_{n_k} \in B \). Then \( f^{-1}\gamma_{1n_k}f = f_{n_k}u_{n_k}^{-1} \in B \) and hence \( \gamma_{1n_k} \in fBf^{-1} \). It follows that \( [\gamma_{1n_k}, \gamma_{1n_k'}] \in \Gamma \cap fUf^{-1} \) for all \( k, k' \geq 0 \). But according to Proposition 2.4, \( \Gamma \) does not contain semi-parabolic elements and therefore \( [\gamma_{1n_k}, \gamma_{1n_k'}] = Id \). Then there exists \( u \in U \) such that
\[
f^{-1}\gamma_{1n_k}f = u \left( \begin{array}{cc} \lambda_{n_k} & 0 \\ 0 & \lambda_{n_k}^{-1} \end{array} \right) u^{-1}
\]
for all \( k \geq 0 \). Since the sequence \( \{\gamma_{1n_k}\}_{k \geq 0} \) does not converge, the sequence \( \{\lambda_{n_k}\}_{k \geq 0} \) is not bounded, which is impossible because the matrices \( f_{n_k} = f^{-1}\gamma_{1n_k}f \) converge to \( Id \) and hence the vectors \( f_{n_k}e_1 = (\lambda_{n_k}, 0) \) converge to \( e_1 = (1, 0) \). \(\square\)
To conclude, let us put
\[ f_n = \left( \begin{array}{cc} a_n & b_n \\ c_n & d_n \end{array} \right) \]
where \( c_n \neq 0 \) according to Lemma 2.8. For every \( \alpha \in \mathbb{R}_+ \), take
\[ u'_n = \left( \begin{array}{c} 1 \\ \frac{\alpha - a_n}{c_n} \end{array} \right) \quad \text{and} \quad u''_n = \left( \begin{array}{c} 1 \\ -\frac{1}{\alpha}(b_n + d_n \frac{\alpha - a_n}{c_n}) \end{array} \right) \]
in \( U \). From Lemma 2.7 as \( h_n = (f_n, g_n) \in H_M \), we have:
\[ u'_n h_n u''_n = (u'_n f_n u''_n, g_n) = \left( \begin{array}{c} \alpha \\ 0 \alpha^{-1} \end{array} \right), g_n) \in H_M \]
Since \( \lim_{n \to +\infty} c_n = 0 \) and \( \lim_{n \to +\infty} g_n = e \), we deduce that
\[ \left( \begin{array}{c} \alpha \\ 0 \alpha^{-1} \end{array} \right), e) \in H_M. \]
This means that
\[ M_\alpha = M \left( \begin{array}{cc} \alpha & 0 \\ 0 & \alpha^{-1} \end{array} \right) \cap M \neq \emptyset \]
for all \( \alpha \in \mathbb{R}_+ \). Since
\[ \left( \begin{array}{cc} \alpha & 0 \\ 0 & \alpha^{-1} \end{array} \right) \left( \begin{array}{cc} 1 & t \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} 1 & \alpha^2 t \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \alpha & 0 \\ 0 & \alpha^{-1} \end{array} \right), \]
the set \( M_\alpha \) is a \( U \)-invariant closed subset of \( M \). By minimality, we have \( M_\alpha = M \) and therefore \( M \) is \( D \)-invariant, i.e.
\[ M \left( \begin{array}{cc} \alpha & 0 \\ 0 & \alpha^{-1} \end{array} \right) = M \]
for all \( \alpha \in \mathbb{R}_+ \). So \( M \) is also \( B \)-invariant and hence \( M = X \) from Proposition 2.6.

In the particular case where \( G \) is trivial, we have just given a simple proof of Hedlund’s Theorem, which is essentially that Ghys gave in [15]. We illustrate the general situation with two examples:

**Examples 2.9.** (i) As mentioned in the introduction, according to Theorem C of [4], if \( G = PSL(2, \mathbb{R}) \), then \( H = PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R}) \) admits discrete uniform subgroups \( \Gamma \). If \( \Gamma \) is irreducible, then \( p_1(\Gamma) \) and \( p_2(\Gamma) \) are dense in \( PSL(2, \mathbb{R}) \). In particular, the natural right \( PSL(2, \mathbb{R}) \)-action on \( X = \Gamma \backslash H \) is minimal. From Theorem 1, the natural right \( U \)-action is minimal too.

(ii) In [6], the authors proved that any torsion-free cocompact Fuchsian group \( \Gamma \) can be realised as a dense subgroup of \( G = SO(3, \mathbb{R}) \). Let \( h \) be an injective representation of \( \Gamma \) into \( SO(3, \mathbb{R}) \) and consider the free and properly discontinuous action of \( \Gamma \) on \( H = PSL(2, \mathbb{R}) \times SO(3, \mathbb{R}) \) given by \( \gamma.(f, g) = (\gamma f, h(\gamma) g) \) for all \( \gamma \in \Gamma \) and for all \((f, g) \in H \). This allows us to see \( \Gamma \) as a cocompact discrete subgroup of \( H \). Since \( h(\Gamma) \) is dense in \( SO(3, \mathbb{R}) \), by applying Theorem 1 we conclude that the natural right \( U \)-action on \( X = \Gamma \backslash H \) is minimal.
3. Proof of Theorem 2

Let $G$ be a connected Lie group and let $\mathfrak{g}$ be its Lie algebra. Right $\text{PSL}(2, \mathbb{R})$-actions on homogeneous manifolds $X = \Gamma \backslash \text{PSL}(2, \mathbb{R}) \times G$ are examples of smooth $G$-Lie foliations. This type of foliations has been classically defined using smooth foliated cocycles with values in $G$ or smooth differential 1-forms with values in $\mathfrak{g}$, see [17], [24] and [25]. However, in our context, it is more convenient to use the following criterium as definition:

**Theorem 3.1** ([12]). A smooth foliation $\mathcal{F}$ on a compact connected manifold $M$ is a $G$-Lie foliation if and only if there are

(i) a discrete group $\Gamma$ acting freely and properly discontinuously on a manifold $\widetilde{M}$,
(ii) a group homomorphism $h : \Gamma \to G$,
(iii) a $\Gamma$-equivariant locally trivial smooth fibration $\rho : \widetilde{M} \to G$ with connected fibres,

such that $M = \Gamma \backslash \widetilde{M}$ and $\mathcal{F}$ is induced by the foliation $\widetilde{\mathcal{F}}$ of $\widetilde{M}$ whose leaves are the fibres of $\rho$. The group $\Gamma$ is called the holonomy group of $\mathcal{F}$.

Assume that the leaves of $\mathcal{F}$ are 2-dimensional. Given a complete Riemannian metric $g_0$ on $M$, $\mathcal{F}$ is said to be a foliation by hyperbolic surfaces if the restriction of $g_0$ to each leaf has hyperbolic conformal type. Actually, according to the Uniformisation Theorem of [8] and [30] which remains valid for any foliation by surfaces, there exists a (leaf-wise smooth) continuous function $u : M \to \mathbb{R}$ such that the restriction of the conformal Riemann metric $g = ug_0$ to each leaf has constant negative curvature equal to $-1$. In this case, there is a natural continuous $\text{PSL}(2, \mathbb{R})$-action on $X = T^1 \mathcal{F}$ whose orbits are the unitary tangent bundle to the leaves. Reciprocally, if the unitary tangent bundles to the leaves of a foliation by surfaces arise as orbits of a $\text{PSL}(2, \mathbb{R})$-action on $X$, then $\mathcal{F}$ is a foliation by hyperbolic surfaces. In the case of the $G$-Lie foliations, we have also the following additional property:

**Proposition 3.2.** Let $\mathcal{F}$ be a $G$-Lie foliation by hyperbolic surfaces of a compact connected manifold $M$. Then the developing map $\rho$ is trivial, so $\widetilde{M}$ is homeomorphic to a product $L \times G$. Moreover, the homeomorphism becomes a diffeomorphism if and only if $\mathcal{F}$ admits a smooth uniformisation.

**Proof.** Firstly, by replacing $\widetilde{M}$ and $G$ with the universal coverings of $M$ and $G$, we can assume that $M$ and $G$ are simply connected. Furthermore, since the second homotopy group of the Lie group $G$ is trivial [9], we can use the homotopy sequence of $\rho$ to deduce that the fibre $L$ is also simply connected and hence $L = \mathbb{H}$. Then the natural right $\text{PSL}(2, \mathbb{R})$-action on $X = T^1 \mathcal{F}$ lifts to a free and proper $\text{PSL}(2, \mathbb{R})$-action on $\widetilde{X}$ with orbit space $G$ (since their orbits are the unitary tangent bundles to the fibres of $\rho$). It follows that $\widetilde{X}$ is a continuous principal $\text{PSL}(2, \mathbb{R})$-bundle over $G$, which becomes smooth if and only if $\mathcal{F}$ has a smooth uniformisation. By construction, the bundle map $\tilde{\rho} : \widetilde{X} \to G$ is the developing map of the $G$-Lie foliation on $X$ whose leaves are the $\text{PSL}(2, \mathbb{R})$-orbits.

On the other hand, since the structure group $\text{PSL}(2, \mathbb{R})$ retracts by deformation on the stabiliser $\text{PSO}(2, \mathbb{R})$ of $z = i$ in $\mathbb{H}$, the $\text{PSL}(2, \mathbb{R})$-bundle $\widetilde{X}$ admits a reduction to $\text{PSO}(2, \mathbb{R})$. This means that there exists a continuous principal $\text{PSO}(2, \mathbb{R})$-bundle $P$ over $G$ such that $\widetilde{X}$ is isomorphic to the continuous principal $\text{PSL}(2, \mathbb{R})$-bundle associated to $P$, that is, $\widetilde{X}$ is homeomorphic to the quotient of $P \times \text{PSL}(2, \mathbb{R})$.
by the diagonal $PSO(2, \mathbb{R})$-action that is given by $(p, f)r = (pr, r^{-1}f)$ for all $(p, f) \in P \times PSL(2, \mathbb{R})$ and all $r \in PSO(2, \mathbb{R})$.

Finally, let us recall that the principal $PSO(2, \mathbb{R})$-bundles over $G$ are classified by its Euler class in the integer cohomology group $H^2(G, \mathbb{Z})$, see for example [3]. Actually, according to the universal coefficient theorem (see also [3]), this group $H^2(G, \mathbb{Z}) = \text{Hom}(H_2(G, \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}(H_1(G, \mathbb{Z}), \mathbb{Z})$ is trivial because the homotopy groups $\pi_1(G)$ and $\pi_2(G)$ are trivial. Briefly, the principal $PSO(2, \mathbb{R})$-bundle $P$ is trivial, so there is a homeomorphism $\varphi : P \to PSL(2, \mathbb{R}) \times G$ which is equivariant for the natural right $PSO(2, \mathbb{R})$-actions. By sending each $PSO(2, \mathbb{R})$-orbit represented by $(p, f) \in P \times PSL(2, \mathbb{R})$ with $\phi(p) = (r, g)$ to the point $\Phi((p, f)PSO(2, \mathbb{R})) = (rf, g)$ in $PSL(2, \mathbb{R}) \times G$, we obtain a well-defined $PSL(2, \mathbb{R})$-equivariant homeomorphism $\Phi : \widetilde{X} \to PSL(2, \mathbb{R}) \times G$ such that $\rho = p_{20}\Phi$. Now, by passing to the quotient by the corresponding $PSO(2, \mathbb{R})$-action, $\Phi$ induces a homeomorphism $\overline{\Phi} : \widetilde{M} \to \mathbb{H} \times G$ such that $\rho = p_{20}\overline{\Phi}$. From the previous discussion, it is also clear that $\overline{\Phi}$ is a diffeomorphism if and only if $F$ admits a smooth uniformisation. □

Now, we restrict our attention to the notion of homogeneous $G$-Lie foliation as defined in the introduction and illustrated by Examples [23]. Recall that a $G$-Lie foliation by hyperbolic surfaces is said to be **homogeneous** if there is connected Lie group $H$ equipped with a surjective morphism $\rho : H \to G$ with kernel $K$ and a cocompact discrete subgroup $\Gamma$ of $H$ such that $X$ is diffeomorphic to $\Gamma \backslash H$ and the right $PSL(2, \mathbb{R})$-action on $X$ is conjugated to the right action of $K$ on $\Gamma \backslash H$. Our interest is justified by the following result:

**Proposition 3.3.** Let $F$ be $G$-Lie foliation by hyperbolic surfaces of a compact connected manifold $M$. Then the following conditions are equivalent:

(i) The foliation $F$ is homogeneous.

(ii) The right $PSL(2, \mathbb{R})$-action on $X$ is conjugated to the natural right $PSL(2, \mathbb{R})$-action on some quotient of the Lie group $H = PSL(2, \mathbb{R}) \times G$ by a cocompact discrete subgroup.

(iii) Up to conjugation by a diffeomorphism between $\widetilde{M}$ and $\mathbb{H} \times G$, the holonomy group $\Gamma$ acts diagonally on $\mathbb{H} \times G$, that is, $\gamma.(z, g) = (\gamma(z), \rho(\gamma)g)$ for all $\gamma \in \Gamma$ and for all $(z, g) \in \mathbb{H} \times G$.

**Proof.** As in the proof of Proposition [32], we are assuming that $\widetilde{M}$ and $G$ are simply connected. We also keep the notation just described. Now we prove the proposition through the following cycle of implications:

(i) $\Rightarrow$ (ii) According to a result of H. Cartan, see for example [28], the Lie algebra $\mathfrak{g}$ of the Lie group $H$ split into the direct sum $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{g}$ of the Lie algebras of $PSL(2, \mathbb{R})$ and $G$. Then the simply connected Lie group $\widetilde{H}$ integrating $\mathfrak{g}$ split into the product $\widetilde{PSL}(2, \mathbb{R}) \times G$ where $\widetilde{PSL}(2, \mathbb{R})$ is the universal covering of $PSL(2, \mathbb{R})$. Moreover, the fundamental group of $H$ is isomorphic to the fundamental group of $PSL(2, \mathbb{R})$. If follows that $H$ is isomorphic to $PSL(2, \mathbb{R}) \times G$.

(ii) $\Rightarrow$ (iii) By hypothesis, the action of the holonomy group $\Gamma$ on $\widetilde{X}$ is conjugated to the action of some discrete cocompact subgroup of $H = PSL(2, \mathbb{R}) \times G$. Then the $\Gamma$-action on $\widetilde{M}$ is conjugated to a diagonal action on $H/PSO(2, \mathbb{R}) \cong \mathbb{H} \times G$. 
(iii) ⇒ (i) If the holonomy group $\Gamma$ acts diagonally on the universal covering $\tilde{M}$, up to conjugation by a diffeomorphism between $\tilde{M}$ and $\mathbb{H} \times G$, then the $\Gamma$-action on $\tilde{X} = T^1\tilde{F}$ is conjugated to the natural left action of a discrete cocompact subgroup of $H = PSL(2, \mathbb{R}) \times G$ and therefore $X = T^1F$ becomes diffeomorphic to the corresponding quotient of $H$, endowed with the natural $PSL(2, \mathbb{R})$-action. $\square$

In [18], G. Hector, S. Matsumoto and G. Meigniez constructed an example of minimal $PSL(2, \mathbb{R})$-Lie foliation by hyperbolic surfaces which is not homogeneous. Comparing with Propositions 3.2 and 3.3, the universal covering $\tilde{M}$ is diffeomorphic to $\mathbb{H} \times PSL(2, \mathbb{R})$, but the holonomy group $\Gamma$ does not act diagonally. However, in the homogeneous setting, Theorem 2 can be immediately deduced as a corollary of Proposition 3.3 and Theorem 1.

Proof of Theorem 2. Let $F$ be a $G$-foliation of a compact connected manifold $M$ whose leaves are hyperbolic surfaces. The natural right $PSL(2, \mathbb{R})$-action on the unitary tangent bundle $X = T^1F$ is minimal if and only if $F$ is minimal because they have the same holonomy representation $h : \Gamma \to G$. Assuming $F$ is homogeneous and using Proposition 3.3 and Theorem 1, we can apply Theorem 1 to deduce that $PSL(2, \mathbb{R})$-minimality and $U$ minimality are equivalent on $X = T^1F$. $\square$

As we already mentioned in the introduction, when we replace the Lie group $G$ with the quotient $G/G_0$ by a compact Lie subgroup $G_0$, we obtain an example of Riemannian foliation where the distance between two leaves (deduced from a left-invariant Riemannian metric on $G$ that is also invariant by the right $G_0$-action) remains locally constant. In general, a foliation $F$ is said to be *Riemannian* when the distance between two leaves verifies this property, see [17], [24] and [25].

Examples 3.4. (i) Consider the $PSL(2, \mathbb{R})$-Lie foliation constructed in Examples 2.9 (i) by quotienting the Lie group $H = PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ by an irreducible cocompact discrete subgroup $\Gamma$. Assuming $\Gamma$ torsion-free, the $\Gamma$-action on $\mathbb{H} \times \mathbb{H}$ induces a minimal Riemannian foliation $\mathcal{F}$ on the quotient manifold $M = \Gamma \backslash \mathbb{H} \times \mathbb{H}$. The foliation $\mathcal{F}$ lifts to a minimal $PSL(2, \mathbb{R})$-Lie foliation $\mathcal{F}_T$ on $\mathcal{F}_T = \Gamma \backslash \mathbb{H} \times \mathbb{H}$ defined by the representation of $\Gamma$ onto the dense subgroup $\phi_2(\Gamma)$ of $PSL(2, \mathbb{R})$. Notice that $\mathcal{F}_T$ is a principal $PSO(2, \mathbb{R})$-bundle on $M$ whose elements are positively-oriented orthonormal frames for the normal bundle to the foliation. If $\Gamma$ is the product of two torsion-free cocompact Fuchsian groups, we have again a Riemannian foliation $\mathcal{F}$ on $M = \Gamma \backslash \mathbb{H} \times \mathbb{H}$, but the lifted foliation on $\mathcal{F}_T = \Gamma \backslash \mathbb{H} \times PSL(2, \mathbb{R})$ is not longer minimal (since the leaves closures are parametrized by the compact manifold $p_2(\Gamma) \backslash PSL(2, \mathbb{R})$).

(ii) According to the construction given in [6], let $h$ be an injective group homomorphism of a torsion-free discrete subgroup $\Gamma$ of $PSL(2, \mathbb{R})$ into $SO(3, \mathbb{R})$ such that $\overline{h}(\Gamma) = SO(3, \mathbb{R})$. As observed in Examples 2.9 (ii), the horizontal foliation of $H = PSL(2, \mathbb{R}) \times SO(3, \mathbb{R})$ induces a minimal $SO(3)$-Lie foliation on the quotient of $H$ by the image of the injective group homomorphism $i : \Gamma \to PSL(2, \mathbb{R}) \times SO(3, \mathbb{R})$ deduced from $h$. Thus $\Gamma$ acts freely and properly on the product $\mathbb{H} \times S^2$ and the quotient manifold $M = \Gamma \backslash \mathbb{H} \times S^2$ admits a minimal Riemannian foliation $\mathcal{F}$, which can be directly defined by the suspension of the representation of $\Gamma$ as a group of orientation-preserving isometries of $S^2$. As before, the foliation $\mathcal{F}$ lifts to a minimal
$SO(3, \mathbb{R})$-Lie foliation $\mathcal{F}_T$ on $E_T = \Gamma \backslash \mathbb{H} \times SO(3, \mathbb{R})$ defined by the representation $h : \Gamma \to SO(3, \mathbb{R})$.

In both examples, the leaves of $\mathcal{F}$ are dense hyperbolic planes and cylinders. By replacing the unitary tangent bundle $X = T^1\Gamma \times \mathfrak{g}^*$ by $X_T = T^1\mathcal{F}_T = \Gamma \backslash PSL(2, \mathbb{R}) \times G$ where $G = PSL(2, \mathbb{R})$ or $G = SO(3, \mathbb{R})$, we can derive $U$-minimality on $X$ from $U$-minimality on $X_T$. The same strategy can be applied to general Riemannian foliations by using Molino’s theory and more specifically the following important result:

**Molino’s Structure Theorem.** If $\mathcal{F}$ is a smooth Riemannian foliation of a compact connected manifold, then $\mathcal{F}$ lifts to a smooth foliation $\mathcal{F}_T$ on the transverse orthonormal frame bundle $E_T$ of $\mathcal{F}$ such that

(i) the closures of the leaves of $\mathcal{F}_T$ are the fibres of a locally trivial smooth fibration $\pi_T : E_T \to B_T$;

(ii) there is a Lie group $G$ such that $\mathcal{F}_T$ induces a $G$-Lie foliation with dense leaves on each fibre of $\pi_T$.

Let us explain how construct the lifted foliation $\mathcal{F}_T$. Assume $\mathcal{F}$ is given by foliated charts $\varphi_i : U_i \to P_i \times T_i$ from open subsets $U_i$ that covers $M$ to the product of open discs $P_i$ and $T_i$ in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. If we can endow each local transversal $T_i$ with a Riemannian metric $g_i$ that is invariant by changes of chart, the foliation $\mathcal{F}$ is Riemannian. From this local point of view, it is clear that each canonical projection $\pi_i = p_2 \circ \varphi_i : U_i \to T_i$ becomes a Riemannian submersion, so the lifted foliation $\mathcal{F}_T$ is defined by the projection $\pi_i : E_T|_{U_i} = p_T^{-1}(U_i) \to E_i$ where $p_T : E_T \to M$ is the bundle map and $E_i$ is the orthonormal frame $O(q, \mathbb{R})$-bundle over $T_i$. By construction, if $\mathcal{F}$ is a foliation by hyperbolic surfaces, then $\mathcal{F}_T$ is also a foliation by hyperbolic surfaces. As in Examples 3.4, the $U$-minimality problem for Riemannian foliations by hyperbolic surfaces can be reduced to the simpler case of Lie foliations by hyperbolic surfaces:

**Proposition 3.5.** Let $\mathcal{F}$ be a minimal Riemannian foliation by hyperbolic surfaces of a compact connected manifold $M$. Let $X = T^1\mathcal{F}$ and $X_T = T^1\mathcal{F}_T$ be the unitary tangent bundles of $\mathcal{F}$ and $\mathcal{F}_T$. For $F = U$, $B$ or $PSL(2, \mathbb{R})$, if the right $F$-action on $X_T$ is minimal, then the right $F$-action on $X$ is minimal.

**Proof.** We first see that, under the conditions above, $\mathcal{F}_T$ is a minimal $G$-Lie foliation by hyperbolic surfaces. Let $L_T$ be any fibre of the basic fibration $\pi_T : E_T \to B_T$, and let $\mathcal{F}_T|_{L_T}$ be the $G$-Lie foliation induced by $\mathcal{F}_T$ on $L_T$. Any closed subset $C \subset L_T$ saturated by $\mathcal{F}_T|_{L_T}$ is also a closed subset of $E_T$ saturated by $\mathcal{F}_T$. Since $O(q, \mathbb{R})$ is compact, its image $p_T(C)$ is a closed subset of $M$ saturated by $\mathcal{F}$. Now, since $\mathcal{F}$ minimal, we have $p_T(C) = M$ and hence the fibre $F_T$ projects on the whole manifold $M$. In other words, $F_T = E_T$ and $B_T$ reduces to one point. Thus, according to Molino’s theorem, $\mathcal{F}_T$ is a minimal $G$-Lie foliation. By construction, its unitary tangent bundle $X_T = T^1\mathcal{F}_T$ is a $O(q, \mathbb{R})$-principal bundle over $X = T^1\mathcal{F}$ and the right $PSL(2, \mathbb{R})$-action on $X$ is induced by the right $PSL(2, \mathbb{R})$-action on $X_T$. Finally, if the right $F$-action on $X_T$ is minimal for $F = U$, $B$ or $PSL(2, \mathbb{R})$, then the right $F$-action on $X$ is minimal too.

A minimal Riemannian foliation $\mathcal{F}$ is said to be homogeneous if the lifted foliation $\mathcal{F}_T$ is homogeneous. In this case, using Proposition 3.5, we obtain the following corollary of Theorem 2.
Corollary 3.6. Let \( X = T^1F \) be the unitary tangent bundle of a minimal Riemannian foliation \( F \) by hyperbolic surfaces of a compact manifold \( M \). Assume \( F \) is homogeneous. Then the right \( U \)-action on \( X \) is minimal.

In fact, any minimal Riemannian foliation is transversely homogeneous, like the transversely hyperbolic and transversely elliptic foliations described in Examples 3.3 see [17]. Now it is a natural question to ask if the generalisation of Hedlund’s theorem holds for transversely homogeneous foliations.

4. Proof of Theorem 4

In this section, we prove that Theorem 2 fails when we consider a transversely homogeneous foliation instead a \( G \)-Lie foliation. We start by exhibiting a first example of transversely projective counter-example:

Example 4.1. Let \( \Gamma \) be a torsion-free discrete subgroup of \( \text{PSL}(2, \mathbb{R}) \). Consider its diagonal action on \( \text{PSL}(2, \mathbb{R}) \times \partial \mathbb{H} \) given by \( \gamma(f, \xi) = (\gamma f, \gamma(\xi)) \) for all \( \gamma \in \Gamma \) and for all \( (f, \xi) \in \text{PSL}(2, \mathbb{R}) \times \partial \mathbb{H} \). If \( \Gamma \) is cocompact, then \( \Gamma \) acts minimally on \( \partial \mathbb{H} \) and hence the right \( \text{PSL}(2, \mathbb{R}) \)-action on \( X = \Gamma \backslash \text{PSL}(2, \mathbb{R}) \times \partial \mathbb{H} \) is minimal. However, the right \( B \)-action is not minimal because the dual \( \Gamma \)-action on \( \partial \mathbb{H} \times \partial \mathbb{H} \) is not minimal. More precisely, the diagonal set \( \Delta \) consisting of all pairs \((\xi, \xi)\) is a non-trivial \( \Gamma \)-invariant closed subset of \( \partial \mathbb{H} \times \partial \mathbb{H} \). This means that neither Proposition 2.1 nor Proposition 2.6 can be extended to this more general context. In fact \( \Delta \) is the unique non-empty \( \Gamma \)-minimal subset of \( \partial \mathbb{H} \times \partial \mathbb{H} \). By duality,

\[
\mathcal{M} = \{ \Gamma \left( \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) / \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{R}) \} \]

is the unique non-empty \( B \)-minimal subset of \( X \), proving the first part of Theorem 4. However, we have the following result:

Proposition 4.2. The set \( \mathcal{M} \) is the unique non-empty \( U \)-minimal subset of \( X \).

Proof. According to Hedlund’s theorem, the \( U \)-action on \( \Gamma \backslash \text{PSL}(2, \mathbb{R}) \). By duality, the \( \Gamma \)-action on \( E = \mathbb{R}^2 - \{0\} / \{\pm I\} \) is minimal too. This implies that the diagonal \( \Gamma \)-action on the subset

\[
\mathcal{K} = \{ (v, \xi) \in E \times \partial \mathbb{H} / v \text{ is collinear to } (1, 0) \text{ if } \xi \neq \infty \text{ and collinear to } (1, 1) \text{ if } \xi = \infty \}
\]

is minimal. Coming back to \( X \) and using again duality, we obtain that \( \mathcal{M} \) is \( U \)-minimal. Let \( \gamma_1 \) and \( \gamma_2 \) two hyperbolic isometries in \( \Gamma \) generating a Schottky group, that is, the fundamental group of a pair of pants. Since \( \Gamma \) acts minimal on \( E \), for each point \( (v, \xi) \in E \times \partial \mathbb{H} \), there exists \( (v_1, \xi_1) \in \Gamma(v, \xi) \) such that \( \gamma_1 v_1 = \lambda_1 v_1 \) with \( |v_1| > 1 \). Moreover, since \( \gamma_1 \) and \( \gamma_2 \) have not common fixed points, for some sequences \( \{p_n\}_{n \geq 0} \) and \( \{q_n\}_{n \geq 0} \) in \( \mathbb{Z} \), we have:

\[
\lim_{n \to +\infty} \gamma_2 p_n \gamma_1 q_n v_1 = v_2 \quad \text{and} \quad \lim_{n \to +\infty} \gamma_2 p_n \gamma_1 q_n \xi_1 = \xi_2
\]

where \( \gamma_2 v_2 = \lambda_2 v_2 \) with \( |v_2| > 1 \) and \( v_2 \) is collinear to \( (\xi_2, 1) \) or \( (1, 1) \). It follows that \( \Gamma(v, \xi) \cap \mathcal{K} \neq \emptyset \) and hence \( \Gamma(v, \xi) = \mathcal{K} \).

Notice also that the dynamics of the \( B \)-action on \( X - \mathcal{M} \) is related to the dynamics of the geodesic flow on \( \Gamma \backslash \text{PSL}(2, \mathbb{R}) \) since each point \( (\xi^-, \xi^+) \in \partial \mathbb{H} \times \partial \mathbb{H} - \Delta \) represents a geodesic in \( \mathbb{H} \). Like in Examples 3.3 \( \Gamma \) acts freely and properly discontinuously on \( \mathbb{H} \times \partial \mathbb{H} \), so the horizontal foliation of \( \mathbb{H} \times \partial \mathbb{H} \) induces a foliation \( F \) on the quotient manifold \( M = \Gamma \backslash \mathbb{H} \times \partial \mathbb{H} \) whose unitary tangent bundle is \( X \).
Proposition 4.3. The transversely homographic foliation $F$ is defined by a locally free $B$-action whose orbit are dense hyperbolic planes and cylinders.

Proof. By construction, since $\partial \mathbb{H}$ is identified to homogeneous space $PSL(2, \mathbb{R})/B$, $F$ is a minimal transversely homographic foliation whose leaves are dense hyperbolic planes and cylinders. To prove that they are the orbits of a smooth $B$-action on $M$, we use an idea of Martínez and Verjovsky from [23]. Indeed, the bundle map $\pi : X = T^1F \to M$ becomes a diffeomorphism from the unique $B$-minimal set $M$ onto the quotient manifold

$$M = \{ \Gamma \left( \begin{array}{cc} a_i + b & a \\ ci + d & c \end{array} \right) / \pm \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in PSL(2, \mathbb{R}) \}. $$

It follows that $F$ is defined by a locally free $B$-action, which is conjugated to the $B$-action on $M$. □

We are now interested to provide another counter-example (locally modelled by $PSL(2, \mathbb{R}) \times \mathbb{R}$) having a non-trivial $B$-minimal set which is not $U$-minimal. Although the construction is classical, see [16], we recall some details. Thus, any matrix

$$A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z})$$

defines an orientation-preserving automorphism of the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The Lie group automorphism $(z, t) \in T^2 \times \mathbb{R} \mapsto (A(z), t + 1) \in T^2 \times \mathbb{R}$ generates a free and properly discontinuous $\mathbb{Z}$-action on the product $T^2 \times \mathbb{R}$. Its orbit space is a compact 3-manifold $T^3_A$ admitting a natural structure of fibre bundle over $S^1 = \mathbb{R}/\mathbb{Z}$.

In fact, we consider only the hyperbolic case where $tr A > 2$ and hence $A$ has two real eigenvalues $\lambda > 1$ and $1/\lambda < 1$.

Lemma 4.4. If $A$ is hyperbolic, then the eigenvectors $u$ and $v$ associated to the eigenvalues $\lambda > 1$ and $1/\lambda < 1$ generate two different eigenlines with irrational slope.

Proof. Assume on the contrary that $w = (p, q)$ is an eigenvector of $A$ where $p, q \in \mathbb{Z}$ are relatively prime (including the cases where $p = 0$ and $q = 1$ or $p = 1$ and $q = 0$). Then there exists $w' = (p', q') \in \mathbb{Z}$ such that $pq' - qp' = 1$ and then the matrix

$$B = \left( \begin{array}{cc} p & p' \\ q & q' \end{array} \right) \in SL(2, \mathbb{Z})$$

satisfies $Be_1 = w$ and $Be_2 = w'$, This implies that $e_1$ is an eigenvector of $B^{-1} AB$ so that $B^{-1} AB$ is an upper triangular matrix. Since $B^{-1} AB$ belongs to $SL(2, \mathbb{Z})$, we have $\pm B^{-1} AB \in U$. Then the eigenvalues of $A$ are equal to $\lambda = \pm 1$, which contradicts the hyperbolicity of $A$. □

Example 4.5. The foliation of $\mathbb{R}^2$ by parallel $u$-lines induces a minimal flow on $T^2$. The product of this foliation with the vertical factor defines a 2-dimensional foliation of $T^2 \times \mathbb{R}$ which is invariant by the $\mathbb{Z}$-action described above. Thus, by passing to the quotient, we obtain a foliation $F$ of $T^3_A$ whose leaves are planes and cylinders. Indeed, according to [3] and denoting by $\pi$ the projection from $\mathbb{R}^2$ onto $T^2$, for each point $(x, y) \in \mathbb{Q}^2$, there is a positive integer $p \geq 1$ such that $A^p \pi (x, y) = \pi (x, y)$, and so $F$ contains infinite countable many cylindrical leaves. We will see that all
the leaves are hyperbolic surfaces. Denote by Aff$_+(\mathbb{R})$ the group of orientation-preserving affine transformations of $\mathbb{R}$, which is isomorphic to $B$. Let $\Gamma$ be the discrete subgroup of $SL(2, \mathbb{R}) \times$ Aff$_+(\mathbb{R})$ generated by

$$
(4.1) \quad T_1(x, y, t) = (x + 1, y, t)
$$

$$
(4.2) \quad T_2(x, y, t) = (x, y + 1, t)
$$

$$
(4.3) \quad h_A(x, y, t) = (A\left(\frac{x}{y}\right), t + 1).
$$

acting on $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$. The foliated manifold $T^3_\mathbb{A}$ is the quotient of $\mathbb{R}^3$ by the action of $\Gamma$, so is endowed with a complete affine structure. Let $u$ and $v$ be the eigenvectors of $A$ verifying $Au = \lambda u$ and $Av = \lambda^{-1}v$. Assume $det(u|v) = 1$. By changing the canonical affine frame $(0, e_1, e_2, e_3)$ by $(0, u, v, e_3)$ in $\mathbb{R}^3$, the transformations (4.1), (4.2) and (4.3) can be written as follows:

$$
(4.4) \quad T_1(x', y', t') = (x' + a', y' + b', t')
$$

$$
(4.5) \quad T_2(x', y', t') = (x' + c', y' + d', t')
$$

$$
(4.6) \quad h_A(x', y', t') = (\lambda x', \lambda^{-1} y', t' + 1)
$$

where $u = (d', -b')$ and $v = (c', -a')$. Thus, from Lemma 4.4, the entries $a'$ and $c'$, and the entries $b'$ and $d'$ are linear independent over $\mathbb{Z}$. In fact, the universal covering $\tilde{\mathbb{R}}^3$ of $T^3_\mathbb{A}$ can be identified with the product $\mathbb{H} \times \mathbb{R}$ by sending each point $(x', y', t') \in \mathbb{R}^3$ to the point $(z', y') = (x' + i \lambda t', y') \in \mathbb{H} \times \mathbb{R}$. In this model, the transformations (4.4), (4.5) and (4.6) can be written

$$
(4.7) \quad T_1(z', y') = (z' + a', y' + b')
$$

$$
(4.8) \quad T_2(z', y') = (z' + c', y' + d')
$$

$$
(4.9) \quad h_A(z', y') = (\lambda z', \lambda^{-1} y')
$$

Moreover, the foliation $\mathcal{F}$ lifts to the horizontal foliation of $\mathbb{H} \times \mathbb{R}$.

**Proposition 4.6 ([16]).** The transversely affine foliation $\mathcal{F}$ is defined by a locally free $B$-action whose orbits are dense hyperbolic planes and cylinders.

**Proof.** Firstly, we remark that $\mathcal{F}$ admits an affine transverse structure because the $\Gamma$-action on $\mathbb{H} \times \mathbb{R}$ defined by (4.7), (4.8) and (4.9) induces an affine action on the $\mathbb{R}$-factor generated by

$$
(4.10) \quad \overline{T}_1(y') = y' + b'
$$

$$
(4.11) \quad \overline{T}_2(y') = y' + d'
$$

$$
(4.12) \quad \overline{T}_A(y') = \lambda^{-1} y'
$$

We know that the leaves of $\mathcal{F}$ are planes or cylinders. Since $b'$ and $d'$ are linear independent over $\mathbb{Z}$, $\overline{T}_1$ and $\overline{T}_2$ generate a dense subgroup of translations of $\mathbb{R}$ and hence all leaves are dense. On the other hand, there is a natural right $B$-action on $\mathbb{H} \times \mathbb{R}$ where each element

$$
\begin{pmatrix}
\sqrt{\alpha} & \beta / \sqrt{\alpha} \\
0 & 1 / \sqrt{\alpha}
\end{pmatrix}
$$

of $B$ acts by homographies on the first factor $\mathbb{H}$ sending $z$ to $az + \beta$, and trivially on the second factor $\mathbb{R}$. Since this free $B$-action commutes with the $\Gamma$-action, it induces a locally free $B$-action on $T^3_\mathbb{A}$ whose orbits are just the leaves of $\mathcal{F}$.  \[\square\]
We first observe that \( \{\infty\} \times \text{Verjovsky} \) remains open: is it true that the horocycle flow on the unitary tangent \( \mathbb{R} \) and \( \text{action induced on } (\partial \text{PSL})^2 \)?

Remark 4.7. The group law \( (x', y', t')(x'', y'', t'') = (x' + \lambda^t x'', y' + \lambda^{-t} y'', t' + t'') \) defines a group structure on \( \mathbb{R}^3 \) that becomes a Lie group isomorphic to the solvable Lie group \( \text{Sol}^3 \). Each horizontal leaf \( \mathbb{H} \times \{y'\} \) is the orbit of any point \( (x', y', t') \) by the right \( B \)-action determined by the inclusion
\[
i(\begin{pmatrix} \sqrt{\alpha} & \beta \sqrt{\alpha} \\ 0 & 1/\sqrt{\alpha} \end{pmatrix}) = (\beta, 0, \frac{\log \alpha}{\log \lambda})
\]
of \( B \) as closed subgroup of \( \text{Sol}^3 \). The orbits of the corresponding \( U \)-action are the horizontal \( x' \)-lines in \( \mathbb{R}^3 \), which correspond to the parallel \( u \)-lines before changing the affine frames.

Proof of Theorem \[5. By construction, the unitary tangent bundle \( X = T^1\mathcal{F} \) is the quotient of \( T^1\mathcal{H} \times \mathbb{R} \) by the \( \Gamma \)-action generated by \( (4.7), (4.8) \), and \( (4.9) \). By duality, the right \( B \)-action on \( X = T^1\mathcal{F} \) has the same dynamics as the \( \Gamma \)-action on \( \partial \mathcal{H} \times \mathbb{R} \) generated by the transformations
\[(4.13)\quad T_{1\ast}(\xi, y') = (\xi + a', y' + b')\]
\[(4.14)\quad T_{2\ast}(\xi, y') = (\xi + c', y' + d')\]
\[(4.15)\quad h_{\lambda\ast}(\xi, y') = (\lambda \xi, \lambda^{-1} y')\]
We first observe that \( \{\infty\} \times \mathbb{R} \) is a closed \( \Gamma \)-invariant subset of \( \partial \mathcal{H} \times \mathbb{R} \). Minimality and uniqueness arise from \( \lim_{n \to +\infty} h_{\lambda_n \ast}^n(\xi, y') = (\infty, 0) \) for all \( (\xi, y') \in (\partial \mathcal{H} \setminus \{0\}) \times \mathbb{R} \) and \( \Gamma(0, y') = \partial \mathcal{H} \times \mathbb{R} \) for all \( y' \in \mathbb{R} \). Therefore, there is an unique minimal set \( \mathcal{M} \) for the right \( B \)-action on \( X = T^1\mathcal{F} \), obtained as the \( \Gamma \)-quotient of the pre-image of \( (\infty) \times \mathbb{R} \) by the canonical projection of \( T^1\mathcal{H} \times \mathbb{R} \) onto \( \partial \mathcal{H} \times \mathbb{R} \). But the closure of each \( U \)-orbit reduces to a toroidal fibre of the bundle structure of \( T^1\mathcal{A} \) over \( S^1 \).

An important difference between this example and all the previous ones is that the discrete subgroup \( \Gamma \) of \( \text{PSL}(2, \mathbb{R}) \times \text{Aff}_+(\mathbb{R}) \) projects onto a subgroup \( p_1(\Gamma) \) of \( \text{PSL}(2, \mathbb{R}) \) which is neither discrete, nor dense. Moreover, by construction, the \( \Gamma \)-action induced on \( (\partial \mathcal{H} \setminus \{\infty\}) \times \mathbb{R} \) is conjugated to the action of the group of affine transformations of \( \mathbb{R}^2 \) generated by the linear automorphism \( A \) and the translations \( t_1(x, y) = (x + 1, y) \) and \( t_2(x, y) = (x, y + 1) \). Thus, the \( B \)-action induced on \( X - \mathcal{M} \) is dual to the \( \mathbb{Z} \)-action on \( T^2 \) generated by \( A \), whose topological dynamics has been carefully described by R. Adler \[1\]. It follows that there are \( B \)-orbits which are dense in \( X \), and others whose closures are not manifolds.

5. Final comments

As we already mentioned, Example \[4.4\] shows that neither Proposition \[2.1\] nor Proposition \[2.6\] are valid in the non-Riemannian case. Nevertheless, even if Hedlund’s Theorem cannot be generalised, the question formulated by Martínez and Verjovsky remains open: is it true that the horocycle flow on the unitary tangent bundle \( X = T^1\mathcal{F} \) of a minimal foliation \( \mathcal{F} \) of a compact manifold \( M \) by hyperbolic surfaces is minimal if and only the \( B \)-action is minimal? Example \[4.7\] proves that this conjecture cannot be strengthened by establishing an equivalence between \( U \)-minimal and \( B \)-minimal sets.

As proved in Proposition \[4.6\] the homogeneous manifold \( T^3_A \) admits a locally free \( B \)-action. In fact, Ghys proved in \[13\] that any volume preserving locally free \( B \)-action on a compact 3-manifold is conjugated to either this action or the classical \( B \)-action on \( \Gamma \setminus \text{PSL}(2, \mathbb{R}) \), which are called homogeneous. Recently, M. Asaoka \[2\]
constructed examples of non-homogeneous $B$-actions on compact 3-manifolds. In Example 4.1, according to Proposition 1.3, we have just described an example of non-homogeneous $B$-action on a compact manifold $M$ whose geometry is modelled by $\tilde{PSL}(2, \mathbb{R})$ like in Asaoka’s examples.

In [20], Martínez and Verjovsky show that the properties exhibited by these examples are equivalent in general. This allows them to reformulate their conjecture asking if is it true that for any compact manifold foliated by dense hyperbolic surfaces, either the foliation is defined by a $B$-action or the $U$-action on $X$ is minimal? Notice that the non-homogeneous $PSL(2, \mathbb{R})$-Lie foliation constructed in [18] (as well as any Riemannian foliation) cannot be defined by a $B$-action, since it admits a transverse invariant volume, according to Proposition 3.1 of [20]. It is an open question to know if the $U$-action is minimal or not.

Progress on this issue are interesting but very restricted in the non-homogeneous case. We place in an appendix some results, which are related to those of Sections 2 in this more general context.

**APPENDIX: $U$-MINIMALITY FOR SOME NON-HOMOGENEOUS FOLIATIONS**

Let us introduce now the group of orientation-preserving $C^r$-diffeomorphisms $\text{Diff}^r_+(F)$ of some orientable $C^r$-manifold $F$, $0 \leq r \leq +\infty$ or $r = \omega$, and give some remarks for the case where $M$ is a compact manifold obtained as the quotient of $\mathbb{H} \times F$ by a subgroup $\Gamma \subset PSL(2, \mathbb{R}) \times \text{Diff}^r_+(F)$ acting freely and properly discontinuously on $\mathbb{H} \times F$. Like in Section 2 we denote by $p_1$ and $p_2$ the first and second projection of $PSL(2, \mathbb{R}) \times \text{Diff}^r_+(F)$ onto $PSL(2, \mathbb{R})$ and $\text{Diff}^r_+(F)$ respectively. Recall that $M$ admits a foliation $\mathcal{F}$ induced by the horizontal foliation of $\mathbb{H} \times F$ and $\mathcal{F}$ is minimal if and only if $p_2(\Gamma)$ acts minimally on $F$. Denote by $p$ and $q$ the canonical projections

$$p : PSL(2, \mathbb{R}) \times F \to \partial \mathbb{H} \times F = PSL(2, \mathbb{R})/B \times F$$

and

$$q : PSL(2, \mathbb{R}) \times F \to X = \Gamma \backslash PSL(2, \mathbb{R}) \times F$$

corresponding to the natural right $B$-action and left $\Gamma$-action on $PSL(2, \mathbb{R}) \times F$. The first result generalises Example 4.1.

**Proposition A.1.** If $p_1(\Gamma)$ is solvable, then the natural right $B$-action on $X$ is not minimal. More precisely, there is a $B$-minimal set homeomorphic to $M$.

**Proof.** Since $p_1(\Gamma)$ is solvable, but it is not included in $PSO(2, \mathbb{R})$ by the compactness of $M$, this group fixes a point $\xi \in \partial \mathbb{H}$. Then $Z = \{\xi\} \times F$ is a $B$-minimal closed subset of $\partial \mathbb{H} \times F$ because $p_2(\Gamma)$ acts minimally on $F$. It follows that $\tilde{Y} = p^{-1}(Z)$ is a $\Gamma$-invariant and $B$-invariant closed subset of $PSL(2, \mathbb{R}) \times F$, which is homeomorphic to $\mathbb{H} \times F$. Clearly, we deduce that $\tilde{Y}$ projects onto a $B$-minimal closed set $Y = q(\tilde{Y}) \subset X$, which is homeomorphic to $M$. \hfill \Box

Suppose now that $p_1(\Gamma)$ is not solvable, so $p_1(\Gamma)$ is discrete cocompact or dense. In particular, its action on $\partial \mathbb{H}$ is minimal. Assuming $F$ compact and $\Gamma$ torsion-free, we have the following result:

**Proposition A.2.** Assume that $F$ is compact, $\Gamma$ is torsion-free, and $p_1(\Gamma)$ is not solvable. If $p_2$ is not injective, then the natural right $U$-action on $X$ is minimal.
Proof. Assume the projection $p_2$ is not injective. Since the kernel $N$ is normalised by $p_1(\Gamma)$, the group $p_1(\Gamma)$ is discrete cocompact and $N$ is not cyclic. It follows that $\mathcal{F}$ admits leaves which are not homeomorphic to the plane or the cylinder. Moreover, the foliated manifold $M$ is the quotient of $\mathbb{R} \times F$ by the diagonal action $\gamma(z, y) = (\gamma_1(z), \gamma_2(y))$ where $\gamma_1$ is an element of the cocompact discrete subgroup $p_1(\Gamma)$ of $PSL(2, \mathbb{R})$ and $\gamma_2$ is the corresponding element of $Diff^+_+(F)$. So $\mathcal{F}$ is obtained as suspension of the representation $h : \gamma_1 \in p_1(\Gamma) \mapsto \gamma_2 \in Diff^+_+(F)$. Now, let us prove that the corresponding $U$-action is minimal. By duality, it is enough to prove that $\Gamma$ acts minimally on the product $E \times F$ where $E = \mathbb{R}^2 - \{0\}/\{\pm Id\}$. Let $v$ be an element of $E$ such that $\gamma_1 v = \lambda_1 v$ for some $\gamma_1 \in N$ with $|\lambda_1| \neq 1$. Since $N$ acts minimally on $\partial \mathfrak{H}$, it is known [11] that $\mathcal{N}v = E$. It follows $\Gamma(v, y)$ contains $E \times \{y\}$ for all $y \in F$. Using the minimality of the action of $p_2(\Gamma)$ on $F$, we deduce that $\Gamma(v, y) = E \times F$ for all $y \in F$. Indeed, for each point $(w, z) \in E \times F$, there is a sequence $\{(\gamma_n)_{n \geq 0} = \{(\gamma_1 n, \gamma_2 n)\}_{n \geq 0}\}$ in $\Gamma$ such that $z = \lim_{n \to +\infty} \gamma_2 n(y)$. Since $(\gamma_1^{-1} n, w, y) \in E \times \{y\} \subseteq \Gamma(v, y)$, we have:

$$
(w, z) = \lim_{n \to +\infty} (w, n \gamma_2 n(y)) = \lim_{n \to +\infty} \gamma_n(\gamma_1^{-1} n, w, y) \in \Gamma(v, y).
$$

Finally, since $p_1(\Gamma)$ is discrete cocompact, given any point $(w, z) \in E \times F$, there is another sequence $\{(\gamma'_n)_{n \geq 0} = \{(\gamma'_1 n, \gamma'_2 n)\}_{n \geq 0}\}$ in $\Gamma$ such that $z = \lim_{n \to +\infty} \gamma'_1 n w$. By compactness of $F$, extracting a subsequence if necessary, we may assume that $\gamma'_2 n(y)$ converges to a point $y' \in F$. Thus

$$
\lim_{n \to +\infty} \gamma'_n(w, y) = \lim_{n \to +\infty} \left((\gamma'_1 n, w, \gamma'_2 n(y)) = (w, y')\right)
$$

Since $\Gamma(v, y') \subseteq \Gamma(w, y)$ and $\Gamma(v, y') = E \times F$, we obtain that $\Gamma(w, y) = E \times F$. □

References

[1] R. L. Adler, Symbolic dynamics and Markov partitions. Bull. Amer. Math. Soc., 35 (1997), 1-56.
[2] M. Asaoka, Non-homogeneous locally free actions of the affine group. Ann. of Math., 175 (2012), 1-21.
[3] T. Banchoff, M. I. Rosen. Periodic points of Anosov diffeomorphisms. Proc. Sympos. Pure Math., XIV (1970), 17-21.
[4] A. Borel, Compact Clifford-Klein forms of symmetric spaces. Topology, 2 (1963), 111–122.
[5] R. Bott, L. W. Tu, Differential Forms in Algebraic Topology. Springer-Verlag, New York, 1982.
[6] E. Breuillard, T. Gelander, J. Souto, P. Storm, Dense embeddings of surface groups. Geom. Topol., 10 (2006), 1373–1389.
[7] R. A. Blumenthal, Transversely homogeneous foliations. Ann. Inst. Fourier, 29 (1979), 143–158.
[8] A. Candel. Uniformization of surface laminations. Ann. Sci. École Norm. Sup., 26 (1993) 489–516.
[9] E. Cartan, La topologie des espaces représentatifs des groupes de Lie. L’Enseignement Mathematique, 35 (1936), 177–200. Reprinted in Oeuvres, Part I, vol. 2, Gauthier-Villars, Paris, 1952-1955, 1307–1330.
[10] F. Dal’Bo, Topologie du feuilletage fortement stable. Ann. Inst. Fourier, 50 (2000), 981–993.
[11] F. Dal’Bo, Geodesic and Horocyclic Trajectories. Springer-Verlag, London, 2011.
[12] E. Fédida, Sur les feuilletages de Lie. C.R. Acad. Sci. Paris, 272 (1971), 999–1002.
[13] E. Ghys, Actions localement libres du groupe affine. Invent. Math., 82 (1985), 479–526.
[14] E. Ghys, Riemannian foliations: examples and problems, Appendix to Riemannian foliations by P. Molino. Progr. Math. 73, Birkhäuser, Boston, 1988, 297–314.
[15] E. Ghys, Dynamique des flots unipotents sur les espaces homogènes. Astérisque, 206 (1992), 93–136.
DYNAMICS OF THE HOROCYCLE FLOW FOR HOMOGENEOUS FOLIATIONS

[16] É. Ghys, V. Sergiescu, Stabilité et conjugaison différentiable pour certains feuilletages. *Topology*, 19 (1980), 179–197.

[17] C. Godbillon, *Feuilletages: études géométriques*. Progr. Math. 98, Birkhäuser, Basel, 1991.

[18] G. Hector, S. Matsumoto, G. Meigniez, Ends of leaves of Lie foliations. *J. Math. Soc. Japan*, 57 (2005), 753–779.

[19] G. A. Hedlund, Fuchsian groups and transitive horocycles. *Duke Math. J.*, 2 (1936), 530–542.

[20] R. Hermann, A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle. *Proc. Amer. Math. Soc.*, 11 (1960), 236–242.

[21] D. Kleinbock, N. Shah, A. Starkov, Dynamics of subgroup actions on homogeneous spaces of Lie groups and applications to number theory, in *Handbook on Dynamical Systems*, Vol. 1A. North Holland, 2002, 813–930.

[22] M. Kulikov, The horocycle flow without minimal sets. *C. R. Acad. Sci. Paris, Sér. I*, 338 (2004), 477–480.

[23] M. Martínez, A. Verjovsky, Horocyclic flows for laminations by hyperbolic Riemann surfaces and Hedlunds theorem. Preprint arXiv:0711.2307v3.

[24] P. Molino, Géométrie globale des feuilletages riemanniens. *Proc. Kon. Ned. Akad. Wetensch. Ser. A*, 85 (1982), 45–76.

[25] P. Molino, *Riemannian foliations*. Progr. Math. 73, Birkhäuser, Boston, 1988.

[26] S. Petite, On invariant measures of finite affine type tilings. *Ergodic Theory Dynam. Systems*, 26 (2006), 1159–1176.

[27] M. Ratner, Invariant measures and orbit closures for unipotent actions on homogeneous spaces. *Geom. Funct. Anal.*, 4 (1994), 236–257.

[28] H. Samuelson *Notes on Lie Algebras*. Universitext, Springer-Verlag, New York, 1990.

[29] N. A. Shah, B. Weiss, On actions of epimorphic subgroups on homogeneous spaces. *Ergodic Theory Dynam. Systems*, 20 (2000), 567–592.

[30] A. Verjovsky, A uniformization theorem for holomorphic foliations, in *Proc. Lefschetz Centen. Conf. (México D.F., 1984)*. *Contemp. Math.*, 58 (1987), 233-253.

[31] B. Weiss, Unique Ergodicity on Compact Homogeneous Spaces. *Proc. Amer. Math. Soc.*, 129 (2001), 585–592.

GeoDynApp - ECSING group (Spain)

E-mail address: fernando.alcalde@usc.es

Institut de Recherche Mathématiques de Rennes, Université de Rennes 1, F-35042 Rennes (France)

E-mail address: francoise.dalbo@univ-rennes1.fr