Invariants of Mappings
Between Nonsymmetric Affine Connection Spaces

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Abstract
Invariants of geometric mappings between nonsymmetric affine connection spaces are studied in this paper. Thomas projective parameter and Weyl projective tensor as invariants of geodesic mappings are the invariants which are generalized for a random mapping in here. Formulae, which are obtained, are tested on the example of geodesic mappings between non-symmetric affine connection spaces.

1 Introduction
Invariants of geodesic mappings are very important for different applications of differential geometry. Weyl projective tensor of a symmetric affine connection space is studied in papers [6,13,15] and many other ones. Following the research methods about generalizations of Weyl projective tensor used in a lot of papers [18,20,21,23], we are going to obtain invariants of geometric mappings in this paper.

Many research papers and monographs [1–23] are focused on development of affine connection spaces theory, the theory of mappings between these spaces and the theory of invariants of these mappings. S. Bochner [1], E. Einstein [2–4], G. Hall [6], J. Mikeš with his research group [7–9,17], S. M. Minčić [10,12], G. P. Pokhariyal and K. C. Mishra [14,15], N. S. Sinyukov [16] and many other authors gave significant approaches to the development of the theory of affine connection spaces.

E. Einstein was the first scientist who applied a non-symmetric affine connection in his research about the theory of relativity. Before Einstein, basics of the non-symmetric affine connection are presented in the excellent Eisenhart’s book [5]:

Definition 1. An \( N \)-dimensional differentiable manifold \( \mathcal{M}_N \) endowed with an affine connection \( \nabla \) such that \( \nabla_Y X - \nabla_X Y \neq [X,Y] \), i.e. coordinately \( L^1_{jk} \neq L^1_{kj} \), is the non-symmetric affine connection space \( \mathcal{GA}_N \).

Remark 1. In the tensor form, the space \( \mathcal{GA}_N \) is determined with affine connection whose coefficients \( \Gamma^i_{jk} \) and \( \Gamma^i_{kj} \) are different.

Remark 2. A generalized Riemannian space \( \mathcal{GR}_N \) is example of a non-symmetric affine connection space.

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The previous definition of non-symmetric affine connection space $GA_N$ has motivated a lot of researchers to develop the theory of non-symmetric affine connection spaces. A lot of papers (see [10–12, 18–21, 23]) are dedicated to the developments of this theory. The definitions and results from previously mentioned works are the main subjects of our following research.

Because of the non-symmetry $L^i_{jk} \neq L^i_{kj}$, the symmetric and the anti-symmetric part of the affine connection coefficient $L^i_{jk}$ are:

$$L^i_{jk} = \frac{1}{2} (L^i_{jk} + L^i_{kj}) \quad \text{and} \quad L^i_{jk} = \frac{1}{2} (L^i_{jk} - L^i_{kj}).$$

(1.1)

The anti-symmetric part $L^i_{jk}$ is called the torsion tensor. Affine connection spaces with the vanished torsion-tensor are the torsion-free spaces.

**Definition 2.** Let $GA_N$ be a non-symmetric affine connection space endowed with affine connection $L^i_{jk}$. An $N$-dimensional manifold $M_N$ endowed with the torsion-free affine connection $L^i_{jk}$ is the associated space $\mathbb{A}_N$ of the space $GA_N$.

Because of the torsion-freeness of affine connection of the associated space $\mathbb{A}_N$, it exists only one type of covariant differentiation with regard to the affine connection of this space:

$$a^i_{j,k} = a^i_{j,k} + L^i_{\alpha k}a^\alpha_j - L^i_{\alpha j}a^\alpha_k,$$

(1.2)

for a tensor $a^i_j$ and a partial derivative denoted by comma. For this reason, it exists only one Ricci-type identity with regard to the affine connection of the space $\mathbb{A}_N$. Because of that, it exists only one curvature tensor

$$R^i_{jmn} = L^i_{j;m:n} - L^i_{j;n:m} = L^i_{j,m,n} - L^i_{j,n,m} + L^\alpha_{j,m}L^i_{\alpha,n} - L^\alpha_{j,n}L^i_{\alpha,m},$$

(1.3)

of the associated space $\mathbb{A}_N$.

Motivated by Einstein’s work [2–4] and Eisenhart’s framework [5], S. Minčić obtained a lot of significant results in the non-symmetric affine connection space theory (see [10–12]). It exists four kinds of covariant differentiation with regard to a non-symmetric affine connection. Consequently, it exists twelve Ricci-type identities with regard to this affine connection. For this reason, it exists twelve curvature tensors of the space $GA_N$. These curvature tensors are:

$$K^i_{jmn} = R^i_{jmn} + uL^i_{j;m:n} + u'L^i_{j;n:m} + vL^\alpha_{j,m}L^i_{\alpha,n} + v'L^\alpha_{j,n}L^i_{\alpha,m} + wL^\alpha_{m,n}L^i_{\alpha,j},$$

(1.4)

for real constants $u, u', v, v', w$ and a covariant derivation with regard to the affine connection of the associated space $\mathbb{A}_N$ denoted by semicolon.

### 1.1 Motivation

Mappings between affine connection spaces have an important role in applications of differential geometry. Definitions and a lot of properties of mappings between symmetric affine connection spaces $\mathbb{A}_N$ and $\overline{\mathbb{A}}_N$ are presented in books [9,16].

Mappings $\varphi : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ [9,16] which any geodesic line of the space $\mathbb{A}_N$ turn into a geodesic line of the space $\overline{\mathbb{A}}_N$ (named the geodesic mappings) are important for different researches
in differential geometry and for different applications [6–9,14,15]. A generalization of invariants
of geodesic mappings [12,14,15,20,21,23] is the main motivation for further research presented
below.

Geodesic mappings defined onto the associated space \( \mathbb{A}_N \) are determined with the equation
\[
L^i_{jk} = L^i_{jk} + \psi_j \delta_k^i + \psi_k \delta_j^i,
\]
(1.5)
for a covariant vector \( \psi_i \). Based on this equation, it is directly obtained that the generalized
Thomas projective parameter [12], defined as
\[
T^i_{jk} = L^i_{jk} - \frac{1}{N+1} (\delta_j^i L^\alpha_{k\alpha} + \delta_k^i L^\alpha_{j\alpha}),
\]
(1.6)
is an invariant of this mapping. Furthermore, based on the equation (1.5) it is obtained a low
of transformation of the curvature tensor \( R^i_{jmn} \) under this mapping:
\[
\overline{R}^i_{jmn} = R^i_{jmn} + \eta^i_{jmn} - \overline{\tau}^i_{jmn},
\]
for a geometrical object \( \eta^i_{jmn} \in \mathbb{A}_N \). From this equation, it directly holds that the Weyl projective
tensor
\[
W^i_{jmn} = R^i_{jmn} + \frac{1}{N+1} \delta_j^i R_{[mn]} + \frac{N}{N^2 - 1} \delta_{[m} R_{n]i} + \frac{1}{N^2 - 1} \delta_{[m} R_{n]i},
\]
(1.7)
for anti-symmetrization without division denoted by square brackets \([m...n]\) and Ricci-tensor \( R_{ij} = R^\alpha_{ij\alpha} \) is an invariant of the mapping \( \phi \).

The symmetric part \( L^i_{jk} \) of affine connection coefficients covers the theory gravity, but the
torsion-tensor \( L^i_{jk} \) covers the theory of electro-magnetism [2–4]. That motivated different authors
to start the researches in the theory of mappings between non-symmetric affine connection
spaces.

To find invariants of mappings \( \hat{f} : \mathbb{G}\mathbb{A}_N \to \mathbb{G}\mathbb{A}_N \) determined with an equality
\[
\overline{T}^i_{jk} = L^i_{jk} + P^i_{jk} + \xi^i_{jk},
\]
(1.8)
for tensors \( P^i_{jk} \) and \( \xi^i_{jk} \) such that \( P^i_{jk} = P^i_{kj} \) and \( \xi^i_{jk} = -\xi^i_{kj} \) is the main purpose of this paper.
In [22], it is obtained general formulas for generalizations of Thomas projective parameter and
Weyl projective tensor of a mapping between symmetric affine connection spaces. We will
continue this work in this paper.

Using the equation (1.8), different authors have attempted (and succeeded in the most cases)
to find a law of transformation of the general curvature tensor \( K^i_{jmn} \) given in the equation (1.4)
under these mappings. They generalized the Weyl projective tensor \( W^i_{jmn} \) from the equation
(1.7) (see [18,20,21,23]) in this way, inter alia.

The applied methods in these researches are different depending on a class where the mapping
\( \hat{f} \) is classified. We will discover a universal form of invariants of mappings defined on a space
\( \mathbb{G}\mathbb{A}_N \) in here. Some novel invariants of geodesic mappings will be obtained in this paper.
2 Thomas-projective invariants of spaces $A_N$ and $GA_N$

For a mapping $f : A_N \to A_N$ determined with the equation

\[ L_{ijk}^{i} = L_{ijk}^{j} + \omega_{ijk}^{i} - \omega_{ijk}^{j}, \]

it is obtained in [22] that the geometrical object (named the s-generalized Thomas projective parameter)

\[ T_{jk}^{i} = L_{jk}^{i} - \omega_{jk}^{i} \]  \hspace{1cm} (2.1)

is an invariant of the mapping $f$. We will generalize this invariant below.

Let $\hat{f} : GA_N \to GA_N$ be a mapping between non-symmetric affine connection spaces $GA_N$ and $GA_N$ determined with the equation

\[ L_{ijk}^{i} = L_{ijk}^{j} + P_{ijk}^{i} + \xi_{ijk}^{i}, \]  \hspace{1cm} (2.2)

for tensors $P_{ijk}^{i}$ and $\xi_{ijk}^{i}$ such that $P_{ijk}^{i} = P_{kj}^{i}$ and $\xi_{ijk}^{i} = -\xi_{kij}^{i}$. After the symmetrization and anti-symmetrization of this equation by indices $j$ and $k$, we obtain that it is satisfied the equations

\[ T_{jk}^{i} = L_{jk}^{i} + P_{jk}^{i}, \]  \hspace{1cm} (2.3)

\[ \xi_{jk}^{i} = T_{jk}^{i} - L_{jk}^{i}. \]  \hspace{1cm} (2.4)

Let the tensor $P_{ijk}^{i}$ be factorized, i.e. let be

\[ P_{jk}^{i} = \omega_{jk}^{i} - \omega_{jk}^{j}, \]  \hspace{1cm} (2.5)

From $P_{jk}^{i} = T_{jk}^{i} - L_{jk}^{i} = \omega_{jk}^{i} - \omega_{jk}^{j}$, it directly holds that is

\[ T_{jk}^{i} = T_{jk}^{j} \]

for

\[ T_{jk}^{i} = L_{jk}^{i} - \omega_{jk}^{i} \quad \text{and} \quad T_{jk}^{i} = L_{jk}^{i} - \omega_{jk}^{j}. \]  \hspace{1cm} (2.6)

In this way, it is confirmed that $T_{jk}^{i}$ is an invariant of the mapping $\hat{f}$.

Let the torsion-tensor $L_{jk}^{i}$ be factorized, i.e. let be

\[ \tilde{T}_{jk}^{i} = L_{jk}^{i} + \dot{T}_{jk}^{i} - \dot{\tau}_{jk}^{i}, \]  \hspace{1cm} (2.7)

for some geometric objects $\dot{\tau}_{jk}^{i}$ and $\dot{\tau}_{jk}^{i}$. From this equation, we have that it is satisfied

\[ \dot{T}_{jk}^{i} = \dot{T}_{jk}^{i}, \]

for

\[ \dot{T}_{jk}^{i} = L_{jk}^{i} - \dot{\tau}_{jk}^{i} \quad \text{and} \quad \dot{T}_{jk}^{i} = L_{jk}^{i} - \dot{\tau}_{jk}^{j}. \]  \hspace{1cm} (2.8)

It holds the following proposition.
Proposition 1. Let $\hat{f} : GA_N \to GA_N$ be a mapping between non-symmetric affine connection spaces $GA_N$ and $GA_N$. The geometrical object $T_{jk}^i \in AN$ defined into the equation (2.6) is an invariant of the mapping $\hat{f}$. The geometrical object $\hat{T}_{jk}^i \in GA_N$ defined into the equation (2.8) is an invariant of the mapping $\hat{f}$. The geometrical object

$$T_{jk}^i = T_{jk}^i + \hat{T}_{jk}^i$$

(2.9)

is an invariant of the mapping $\hat{f}$.

The invariant $T_{jk}^i$ is the $\hat{f}$-symmetric Thomas projective parameter. The invariant $\hat{T}_{jk}^i$ is the $\hat{f}$-antisymmetric Thomas projective parameter. The invariant $T_{jk}^i$ is the $\hat{f}$-generalized Thomas projective parameter.

Remark 3. In the case of equitorsion mappings (the mappings which preserve the torsion tensor), the $\hat{f}$-generalized Thomas projective parameter is equal to the $\hat{f}$-symmetric Thomas projective parameter.

3 Weyl-projective invariants of spaces $AN$ and $GA_N$

Let $\hat{f} : GA_N \to GA_N$ be a mapping between non-symmetric affine connection spaces $GA_N$ and $GA_N$. As it was proved above, the $\hat{f}$-symmetric Thomas projective parameter $T_{jk}^i = L_{jk}^i - \omega_{jk}^i$ is an invariant of the mapping $\hat{f}$.

To factorize the difference $\bar{R}_{jmn}^i - R_{jmn}^i$, we have two approaches (as same as in [22]):

1. Because $\bar{T}_{jk}^i = T_{jk}^i$, we have that

$$\bar{T}_{jm,n}^i - \bar{T}_{jn,m}^i + \bar{T}_{jm}^\alpha \bar{T}_{an}^i - \bar{T}_{jn}^\alpha \bar{T}_{am}^i = T_{jm,n}^i - T_{jn,m}^i + T_{jm}^\alpha T_{an}^i - T_{jn}^\alpha T_{am}^i.$$  

From this equation, together with the equations (1.3, 2.6), one obtains that

$$\bar{R}_{jmn}^i = R_{jmn}^i + \omega_{jmn}^i - \omega_{jm}^i \omega_{mn}^i - \omega_{jn}^i \omega_{mm}^i + \omega_{jm}^\alpha \omega_{am}^i - \omega_{jn}^\alpha \omega_{am}^i.$$

(3.1)

From this equation, it directly holds that

$$\overline{\mathcal{W}}_{(1)}^{i_{1}m_{1}n_{1}} = \mathcal{W}_{(1)}^{i_{1}m_{1}n_{1}},$$

for

$$\mathcal{W}_{(1)}^{i_{1}m_{1}n_{1}} = R_{jmn}^i - \omega_{jmn}^i + \omega_{jm}^i \omega_{mn}^i + \omega_{jn}^i \omega_{am}^i - \omega_{jn}^\alpha \omega_{am}^i$$

(3.2)

and the corresponding $\overline{\mathcal{W}}_{(1)}^{i_{1}m_{1}n_{1}}$.

2. We also have that

$$\bar{T}_{jm,n}^i - T_{jm,n}^i = \overline{L}_{\alpha n}^i \bar{T}_{jm}^\alpha - \overline{L}_{\alpha m}^\alpha \bar{T}_{jm}^i - \overline{L}_{\alpha m}^{\alpha} \bar{T}_{ja}^i - \overline{L}_{\alpha m}^i \bar{T}_{jm}^\alpha + \overline{L}_{\alpha j}^\alpha \bar{T}_{am}^i + \overline{L}_{\alpha m}^\alpha \bar{T}_{ja}^i.$$
\[ T_{jmn}^i = L_{jm;n}^i + \omega^i_{jmn} + (T_{jm}^\alpha - \omega^\alpha_{jm})\omega^i_{\alpha mn} - (T_{\alpha mn}^i - \omega^i_{\alpha mn})\omega^\alpha_{jm} - (L_{jmn}^i - \omega^i_{jmn})\omega^\alpha_{mn} \]

\[ \omega^i_{jmn} = -(L_{jm}^\alpha - \omega^\alpha_{jm})\omega^i_{\alpha mn} + (T_{\alpha mn}^i - \omega^i_{\alpha mn})\omega^\alpha_{jm} + (L_{jmn}^i - \omega^i_{jmn})\omega^\alpha_{mn}. \]

Because \( \overline{T}_{jmn}^i = T_{jmn}^i - T_{jnm}^i \) and \( \overline{R}_{jmn}^i = L_{jm;n}^i - L_{jnm;n}^i \), from the last equation and \( \overline{T}_{jk}^i - L_{jk}^i = \omega^i_{jk} - \omega^i_{jk} \), based on the last equation, it directly holds that it is satisfied

\[ \overline{R}_{jmn}^i = R_{jmn}^i + \omega^i_{jmn} - \omega^i_{jnm} + (L_{jm}^\alpha - \omega^\alpha_{jm})\omega^i_{\alpha mn} - (L_{\alpha mn}^i - \omega^i_{\alpha mn})\omega^\alpha_{jm} \]

\[ - (L_{jmn}^i - \omega^i_{jmn})\omega^\alpha_{mn} + (L_{jm}^\alpha - \omega^\alpha_{jm})\omega^i_{\alpha mn} - (L_{\alpha mn}^i - \omega^i_{\alpha mn})\omega^\alpha_{jm} + (L_{jmn}^i - \omega^i_{jmn})\omega^\alpha_{mn}. \]

From this equation, we obtain that is

\[ \overline{W}_{jmn}^i = W_{jmn}^i \]

for

\[ W_{jmn}^i = R_{jmn}^i - \omega^i_{jmn} + \omega^i_{jnm} - (L_{jm}^\alpha - \omega^\alpha_{jm})\omega^i_{\alpha mn} + (L_{\alpha mn}^i - \omega^i_{\alpha mn})\omega^\alpha_{jm} \]

\[ + (L_{jmn}^i - \omega^i_{jmn})\omega^\alpha_{mn} - (L_{jm}^\alpha - \omega^\alpha_{jm})\omega^i_{\alpha mn} - (L_{\alpha mn}^i - \omega^i_{\alpha mn})\omega^\alpha_{jm}. \]

and the corresponding \( \overline{W}_{jmn}^i \).

It holds the following lemma:

**Lemma 1.** [22] Let \( f : \mathbb{GA}_N \to \mathbb{GA}_N \) be a mapping between nonsymmetric affine connection spaces \( \mathbb{GA}_N \) and \( \mathbb{GA}_N \).

a) The geometrical objects \( W_{jmn}^i \) and \( W_{jmn}^i \) defined in the equations \((3.2, 3.4)\) are invariants of the mapping \( f \).

b) The invariants \( W_{jmn}^i \) and \( W_{jmn}^i \) of the mapping \( f \) satisfy the correlation

\[ W_{jmn}^i = W_{jmn}^i - L_{jm}^\alpha \omega^\alpha_{jm} + L_{\alpha mn}^i \omega^\alpha_{jm} - L_{jmn}^i \omega^\alpha_{jm} + \omega^i_{jm} \omega^\alpha_{jm} - \omega^i_{jm} \omega^\alpha_{jm}. \]

(3.5)

c) The invariants \( W_{jmn}^i \) and \( W_{jmn}^i \) of the mapping \( f \) are linearly independent. \( \square \)

The invariants \( W_{jmn}^i \) and \( W_{jmn}^i \) are the first and the second associated generalized Weyl projective tensors.
Theorem 1. Let \( f : \mathcal{G}_N \rightarrow \mathcal{G}_{\mathcal{N}} \) be a mapping between nonsymmetric affine connection spaces \( \mathcal{G}_N \) and \( \mathcal{G}_{\mathcal{N}} \) which affine connection coefficients \( L_{jk}^i \) and \( \mathcal{T}_{jk}^i \) satisfy the equation

\[
\mathcal{T}_{jk}^i = L_{jk}^i + \tau_{jk}^i - \tau_{jk}^i,
\]

(3.6)

for the tensors \( \tau_{jk}^i, \tau_{jk}^i \) symmetric by \( j \) and \( k \) as well tensors \( \tau_{jk}^i, \tau_{jk}^i \) antisymmetric by \( j \) and \( k \). Following families of geometrical objects:

\[
\mathcal{W}_{jmn}^{i (1)} = K_{jmn}^i - \omega_{jmn}^i + \omega_{jmn}^i \alpha \omega_{jmn}^i - \omega_{jmn}^i \alpha \omega_{jmn}^i
\]

(3.7)

\[
\mathcal{W}_{jmn}^{i (2)} = K_{jmn}^i + \omega_{jmn}^i + \omega_{jmn}^i \alpha \omega_{jmn}^i - \omega_{jmn}^i \alpha \omega_{jmn}^i
\]

(3.8)

\[
\mathcal{W}_{jmn}^{i (3)} = K_{jmn}^i - \omega_{jmn}^i + \omega_{jmn}^i \alpha \omega_{jmn}^i - \omega_{jmn}^i \alpha \omega_{jmn}^i
\]

(3.9)

\[
\mathcal{W}_{jmn}^{i (4)} = K_{jmn}^i + \omega_{jmn}^i + \omega_{jmn}^i \alpha \omega_{jmn}^i - \omega_{jmn}^i \alpha \omega_{jmn}^i
\]

(3.10)

\[
\mathcal{W}_{jmn}^{i (2.1)} = K_{jmn}^i - \omega_{jmn}^i + \omega_{jmn}^i - (L_{jm}^i - \omega_{jm}^i) \omega_{jm}^i + \omega_{jm}^i \alpha \omega_{jm}^i
\]

(3.11)

\[
\mathcal{W}_{jmn}^{i (2.2)} = K_{jmn}^i - \omega_{jmn}^i + \omega_{jmn}^i - (L_{jm}^i - \omega_{jm}^i) \omega_{jm}^i + \omega_{jm}^i \alpha \omega_{jm}^i
\]

(3.12)

\[
\mathcal{W}_{jmn}^{i (2.3)} = K_{jmn}^i - \omega_{jmn}^i + \omega_{jmn}^i - (L_{jm}^i - \omega_{jm}^i) \omega_{jm}^i + \omega_{jm}^i \alpha \omega_{jm}^i
\]

(3.13)

\[
\mathcal{W}_{jmn}^{i (2.4)} = K_{jmn}^i - \omega_{jmn}^i + \omega_{jmn}^i - (L_{jm}^i - \omega_{jm}^i) \omega_{jm}^i + \omega_{jm}^i \alpha \omega_{jm}^i
\]

(3.14)

for the corresponding real constants \( u, u', v, v', w \) and
we obtain that symmetric and antisymmetric parts of $f$ are families of invariants of the mapping $f$.

Proof. After symmetrization and antisymmetrization of the equation (3.6) by indices $j$ and $k$, we obtain that symmetric and antisymmetric parts of $L_{jk}^i$ and $T_{jk}^i$ satisfy the equations

\[ \bar{T}_{jk}^i = L_{jk}^i - \tau_{jk}^i \quad \text{and} \quad \bar{T}_{jk}^i = L_{jk}^i - \tau_{jk}^i, \]  

(3.15)

(3.16)

The $f$-antisymmetric Thomas projective parameters of the spaces $G_A$ and $G_{\bar{A}}$ are

\[ \hat{T}_{jk}^i = L_{jk}^i - \tau_{jk}^i \quad \text{and} \quad \hat{T}_{jk}^i = L_{jk}^i - \tau_{jk}^i, \]  

(3.17)

respectively.

From $\hat{T}_{jm}^i \hat{T}_{an}^i = \hat{T}_{jm}^i \hat{T}_{an}^i$, we obtain that is

\[ \bar{T}_{jm}^i \bar{T}_{an}^i = L_{jm}^i \tau_{an}^i + L_{jm}^i \tau_{an}^i - L_{jm}^i \tau_{an}^i - L_{jm}^i \tau_{an}^i + L_{jm}^i \tau_{an}^i + L_{jm}^i \tau_{an}^i. \]  

(3.18)

Moreover, from the invariance $\hat{T}_{jk}^i = \hat{T}_{jk}^i$, $\hat{T}_{jk}^i = \hat{T}_{jk}^i$, and $\bar{T}_{jk}^i = \tau_{jk}^i$, we obtain that is

\[ \bar{T}_{jm}^i = L_{jm}^i \tau_{jn}^i - L_{jm}^i \tau_{an}^i - L_{jm}^i \tau_{an}^i - L_{jm}^i \tau_{an}^i + L_{jm}^i \tau_{an}^i + L_{jm}^i \tau_{an}^i. \]  

(3.19)

(3.20)

From the equality (3.19), we obtain that is

\[ \bar{T}_{jm}^i = L_{jm}^i \tau_{jn}^i + L_{jm}^i \tau_{an}^i + L_{jm}^i \tau_{an}^i + L_{jm}^i \tau_{an}^i + L_{jm}^i \tau_{an}^i + \tau_{jm}^i. \]  

(3.21)

From the equality (3.20), it directly holds that is

\[ \bar{T}_{jm}^i = L_{jm}^i \tau_{jn}^i + \tau_{jm}^i + \omega_{jm}(L_{jm}^i - \tau_{jm}^i) - \omega_{jn}(L_{jm}^i - \tau_{jm}^i) - \omega_{am}(L_{jm}^i - \tau_{jm}^i) - \omega_{an}(L_{jm}^i - \tau_{jm}^i). \]  

(3.22)
Let us prove the invariance of $\mathcal{W}^i_{(1.1)}{}_{jmn}^\prime$. All other invariants may be obtain in the similar way. Because $K^i_{jmn} = \mathbf{R}^i_{jmn} + u\mathbf{T}^i_{jv}{}_{v}{}_{m} + u\mathbf{T}^i_{jv}{}_{v}{}_{n} + v\mathbf{T}^i_{jm}{}_{v}{}_{\alpha} + v\mathbf{T}^i_{jm}{}_{v}{}_{\alpha} + w\mathbf{T}^i_{mn}{}_{v}{}_{\alpha}$ and based on the equations (3.1) (3.19) (3.18), we obtain that is

$$K^i_{jmn} = K^i_{jmn} + \sigma^i_{jmn} - \sigma^i_{jmn} - \omega^i_{jmn} + \omega^i_{jmn} + \omega^i_{jmn} - \omega^i_{jmn} - \omega^i_{jmn}$$

for the above defined $\sigma^i_{jmn}, \sigma^i_{jmn}, \sigma^i_{jmn}, t^i_{jmn}, \sigma^i_{jmn}, t^i_{jmn}$. In this way, it is proved that is

$$\mathcal{W}^i_{(1.1)}{}_{jmn}^\prime = \mathcal{W}^i_{(1.1)}{}_{jmn},$$

which proves this theorem. □

**Corollary 1.** Let $f : \mathbb{G} \rightarrow \mathbb{G}$ be an equitorsion mapping of the space $\mathbb{G}$ (the case of $\tau^i_{jk} = \tau^i_{jk}$). Following families of geometrical objects:

$$\mathcal{W}^i_{(1.1)}{}_{jmn}^\prime = K^i_{jmn} - \omega^i_{jmn} + \omega^i_{jmn} + \omega^i_{jmn} - \omega^i_{jmn}$$

$$+ u\left(-L^i_{qm}L^i_{jm} + L^i_{jm}L^i_{qm} + L^i_{mn}L^i_{jn}\right) + u\left(-L^i_{qm}L^i_{jn} + L^i_{jm}L^i_{qm} + L^i_{mn}L^i_{jn}\right),$$

$$\mathcal{W}^i_{(1.2)}{}_{jmn}^\prime = K^i_{jmn} - \omega^i_{jmn} + \omega^i_{jmn} + \omega^i_{jmn} - \omega^i_{jmn}$$

$$+ u\left(-\omega^i_{jmn}L^i_{qm} + \omega^i_{jmn}L^i_{jm} + \omega^i_{jmn}L^i_{mn}\right) + u\left(-\omega^i_{jmn}L^i_{jm} + \omega^i_{jmn}L^i_{qm} + \omega^i_{jmn}L^i_{mn}\right),$$

$$\mathcal{W}^i_{(1.3)}{}_{jmn}^\prime = K^i_{jmn} - \omega^i_{jmn} + \omega^i_{jmn} + \omega^i_{jmn} - \omega^i_{jmn}$$

$$+ u\left(-L^i_{qm}L^i_{jm} + L^i_{jm}L^i_{qm} + L^i_{mn}L^i_{jn}\right) + u\left(-L^i_{qm}L^i_{jn} + L^i_{jm}L^i_{qm} + L^i_{mn}L^i_{jn}\right),$$

$$\mathcal{W}^i_{(1.4)}{}_{jmn}^\prime = K^i_{jmn} - \omega^i_{jmn} + \omega^i_{jmn} + \omega^i_{jmn} - \omega^i_{jmn}$$

$$+ u\left(-\omega^i_{jmn}L^i_{qm} + \omega^i_{jmn}L^i_{jm} + \omega^i_{jmn}L^i_{mn}\right) + u\left(-L^i_{qm}L^i_{jm} + L^i_{jm}L^i_{qm} + L^i_{mn}L^i_{jn}\right),$$

$$\mathcal{W}^i_{(2.1)}{}_{jmn}^\prime = K^i_{jmn} - \omega^i_{jmn} + \omega^i_{jmn} - \left(L^i_{jm} - \omega^i_{jn}\right)\omega^i_{jn} + \left(L^i_{qm} - \omega^i_{nm}\right)\omega^i_{nm}$$

$$+ \left(L^i_{jm} - \omega^i_{jn}\right)\omega^i_{jn} + \left(L^i_{qm} - \omega^i_{nm}\right)\omega^i_{nm},$$

$$\mathcal{W}^i_{(2.2)}{}_{jmn}^\prime = K^i_{jmn} - \omega^i_{jmn} + \omega^i_{jmn} - \left(L^i_{jm} - \omega^i_{jn}\right)\omega^i_{jn} + \left(L^i_{qm} - \omega^i_{nm}\right)\omega^i_{nm},$$

$$+ \left(L^i_{jm} - \omega^i_{jn}\right)\omega^i_{jn} + \left(L^i_{qm} - \omega^i_{nm}\right)\omega^i_{nm}.$$
\[
\tilde{W}_{(2,3)i \ j \ m \ n}^i = K_{j \ m \ n}^i - \omega_{j \ m \ n}^i + \omega_{j \ n \ m}^i - (L_{j \ m}^\alpha - \omega_{j \ m}^\alpha)\omega_{\alpha \ n}^i + (L_{j \ m}^\alpha - \omega_{j \ m}^\alpha)\omega_{\alpha \ n}^i
\]
\[
+ (L_{j \ n}^\alpha - \omega_{j \ n}^\alpha)\omega_{\alpha \ m}^i - (L_{j \ m}^\alpha - \omega_{j \ m}^\alpha)\omega_{\alpha \ n}^i
\]
\[
+ u(-L_{j \ m}^\alpha L_{j \ n}^\alpha + L_{j \ n}^\alpha L_{i \ m}^\alpha + L_{i \ m}^\alpha L_{j \ n}^\alpha) + u'(-\omega_{j \ m}^\alpha \omega_{j \ n}^\alpha + \omega_{j \ m}^\alpha \omega_{j \ n}^\alpha + \omega_{m \ n} L_{j \ n}^\alpha),
\]
\[
\tilde{W}_{(2,4)i \ j \ m \ n}^i = K_{j \ m \ n}^i - \omega_{j \ m \ n}^i + \omega_{j \ n \ m}^i - (L_{j \ m}^\alpha - \omega_{j \ m}^\alpha)\omega_{\alpha \ n}^i + (L_{j \ m}^\alpha - \omega_{j \ m}^\alpha)\omega_{\alpha \ n}^i
\]
\[
+ (L_{j \ n}^\alpha - \omega_{j \ n}^\alpha)\omega_{\alpha \ m}^i - (L_{j \ m}^\alpha - \omega_{j \ m}^\alpha)\omega_{\alpha \ n}^i
\]
\[
+ u(-\omega_{j \ n}^\alpha L_{j \ m}^\alpha + \omega_{j \ m}^\alpha L_{i \ n}^\alpha + \omega_{m \ n} L_{j \ n}^\alpha) + u'(-L_{j \ m}^\alpha L_{j \ n}^\alpha + L_{j \ m}^\alpha L_{i \ n}^\alpha + L_{i \ m}^\alpha L_{j \ n}^\alpha)
\]

are families of invariants of the mapping \( f \).

The invariant \( \tilde{W}_{(i}^i j m \ n) \), as well the invariant \( \tilde{W}_{(i}^i j m \ n} \), \( i = 1, 2, j = 1, 2, 3, 4 \), is the \( i \)-th \textit{f}-

generalized Weyl projective tensor of the \( k \)-th kind.

## 4 Invariants of geodesic mappings

Let \( f : \mathbb{A}_N \rightarrow \mathbb{A}_N \) be an equitorsion geodesic mapping. The basic equation of \( f \) is

\[
\mathcal{T}_{jk}^i = L_{jk}^i + \psi_j \delta_k^i + \psi_k \delta_j^i,
\]

for a covariant vector \( \psi_j \). After symmetrizing of this equation by indices \( j \) and \( k \), we obtain that is

\[
\mathcal{T}_{jk}^i = L_{jk}^i + \psi_j \delta_k^i + \psi_k \delta_j^i.
\]

After contracting of this equation by \( i \) and \( k \), one concludes that is

\[
\psi_j = \frac{1}{N + 1}(\mathcal{T}_{j\alpha}^\alpha - L_{j\alpha}^\alpha),
\]

i.e.

\[
\mathcal{T}_{jk}^i = L_{jk}^i + \frac{1}{N + 1}(\mathcal{T}_{j\alpha}^\alpha \delta_k^i + \mathcal{T}_{k\alpha}^\alpha \delta_j^i) - \frac{1}{N + 1}(L_{j\alpha}^\alpha \delta_k^i + L_{k\alpha}^\alpha \delta_j^i).
\]

From this equation, we have that is

\[
\omega_{jk}^i = \frac{1}{N + 1}(L_{j\alpha}^\alpha \delta_k^i + L_{k\alpha}^\alpha \delta_j^i).
\]

The \( f \)-symmetric Thomas projective parameter of this mapping is

\[
\mathcal{T}_{jk}^i = L_{jk}^i - \frac{1}{N + 1}(L_{j\alpha}^\alpha \delta_k^i + L_{k\alpha}^\alpha \delta_j^i).
\]

This invariant coincides with Thomas projective parameter.

Ricci-tensor \( R_{ij} \) of the associated space \( \mathbb{A}_N \) is

\[
R_{ij} = L_{ij\alpha}^\alpha - L_{i\alpha j}^\alpha.
\]
For this reason and based on the equation (4.4), we have that is

\[-\omega_{jm;n}^j + \omega_{jm;m}^j = \frac{1}{N + 1} \delta^j_m R_{[mn]} - \frac{1}{N + 1} \delta^j_m L_{\alpha m}^\alpha + \frac{1}{N + 1} \delta^j_m L_{\alpha m}^\alpha,\]

\[\omega_{jm;m}^\alpha - \omega_{jm;n}^\alpha = \frac{1}{(N + 1)^2} \delta^\alpha_m L_{\alpha m}^\alpha L_{m\beta}^\beta - \frac{1}{(N + 1)^2} \delta^\alpha_m L_{\alpha m}^\alpha L_{m\beta}^\beta,\]

\[-L_{jm;}^i \omega_{jm;m}^i - L_{jm;}^i \omega_{jm;n}^i = \frac{1}{N + 1} \delta^i_m L_{jm;\alpha m}^\alpha - \frac{1}{N + 1} \delta^i_m L_{jm;\alpha m}^\alpha.\]

From these equations, we obtain that the first and the second associated generalized Weyl projective tensor of the space \(\mathbb{G}_A N\) are:

\[\mathcal{W}^i_{(1)jm;n} = R_{jm;n}^i + \frac{1}{N + 1} \delta^i_m R_{[mn]} - \frac{1}{(N + 1)^2} \delta^i_m ((N + 1)L_{\alpha m}^\alpha + L_{\alpha m}^\alpha)\]

\[+ \frac{1}{(N + 1)^2} \delta^i_m ((N + 1)L_{\alpha m}^\alpha + L_{\alpha m}^\alpha),\]

\[\mathcal{W}^i_{(2)jm;n} = R_{jm;m}^i + \frac{1}{N + 1} \delta^i_m R_{[mn]} - \frac{1}{(N + 1)^2} \delta^i_m ((N + 1)L_{\alpha m}^\alpha + 2L_{\alpha m}^\alpha)\]

\[+ \frac{1}{(N + 1)^2} \delta^i_m ((N + 1)L_{\alpha m}^\alpha + 2L_{\alpha m}^\alpha).\]  

Remark 4. No one of the invariants (4.7, 4.8) coincides with Weyl projective tensor. However, from the equality \(\mathcal{W}^i_{(1)jm;n} = \mathcal{W}^i_{(1)jm;m}\) (contraction of this equality by \(i\) and \(n\) we conclude that is

\[\frac{1}{(N + 1)^2} \left( (N + 1)L_{\alpha m}^\alpha + L_{\alpha m}^\alpha L_{m\beta}^\beta - (N + 1)L_{\alpha m}^\alpha + L_{\alpha m}^\alpha L_{m\beta}^\beta \right)\]

\[= \frac{1}{N^2 - 1} (R_{jm;n} - R_{jm}) + \frac{1}{N^2 - 1} (R_{jm;m} - R_{jm};),\]

which involved in the equality \(\mathcal{W}^i_{(1)jm;n} = \mathcal{W}^i_{(1)jm;m}\) confirms that Weyl projective tensor is an invariant of the geodesic mapping \(f\).

At the end, let us obtain families of invariants of the mapping \(f\) from the changes of curvature tensor \(K_{jm;n} = R_{jm;n} + uL_{jm;n}^\nu + vL_{jm;n}^\nu L_{\alpha m}^\alpha + v'L_{jm;n}^\nu L_{\alpha m}^\alpha + wL_{jm;n}^\nu L_{\alpha m}^\alpha.\) We want to these invariants be functions of \(K_{jm;n}^\alpha\) and

\[K_{jm} = K_{jm;\alpha} = R_{jm} + uL_{jm;\alpha m}^\nu + u'L_{jm;\alpha m}^\nu L_{\alpha m}^\alpha + vL_{jm;\alpha m}^\nu L_{\alpha m}^\alpha + (u' + w)L_{jm;\alpha m}^\nu L_{\alpha m}^\alpha.\]

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\[
\begin{align*}
\tilde{W}_{(1)jmn}^i &= K_{jmn}^i + \frac{1}{N+1} \delta_j^i K_{[mn]} - \frac{1}{N+1} \delta_j^i (2u L_{jm;m}^\alpha + u' (L_{j;m}^\alpha - L_{m;j}^\alpha) + 2v L_{jm}^\alpha L_{\alpha j}^\beta) \\
&\quad - \frac{1}{N+1} \left( (\delta_m^i L_{j;m}^\alpha - \delta_n^i L_{j;m}^\alpha) + u(-L_{am}^\alpha L_{jm}^\alpha + L_{jm}^\alpha L_{am}^\alpha + L_{am}^\alpha L_{ma}^\alpha + L_{am}^\alpha L_{ja}^\alpha) + u'(-L_{am}^\alpha L_{jm}^\alpha + L_{jm}^\alpha L_{ma}^\alpha + L_{am}^\alpha L_{ma}^\alpha) \right) \\
\tilde{W}_{(2)jmn}^i &= K_{jmn}^i + \frac{1}{N+1} \delta_j^i K_{[mn]} - \frac{1}{N+1} \delta_j^i (2u L_{jm;m}^\alpha + u' (L_{j;m}^\alpha - L_{m;j}^\alpha) + 2v L_{jm}^\alpha L_{\alpha j}^\beta) \\
&\quad - \frac{1}{N+1} \left( (\delta_m^i L_{j;m}^\alpha - \delta_n^i L_{j;m}^\alpha) + u(-L_{am}^\alpha L_{jm}^\alpha + L_{jm}^\alpha L_{am}^\alpha + L_{am}^\alpha L_{ma}^\alpha + L_{am}^\alpha L_{ja}^\alpha) + u'(-L_{am}^\alpha L_{jm}^\alpha + L_{jm}^\alpha L_{ma}^\alpha + L_{am}^\alpha L_{ma}^\alpha) \right) \\
\tilde{W}_{(3)jmn}^i &= K_{jmn}^i + \frac{1}{N+1} \delta_j^i K_{[mn]} - \frac{1}{N+1} \delta_j^i (2u L_{jm;m}^\alpha + u' (L_{j;m}^\alpha - L_{m;j}^\alpha) + 2v L_{jm}^\alpha L_{\alpha j}^\beta) \\
&\quad - \frac{1}{N+1} \left( (\delta_m^i L_{j;m}^\alpha - \delta_n^i L_{j;m}^\alpha) + u(-L_{am}^\alpha L_{jm}^\alpha + L_{jm}^\alpha L_{am}^\alpha + L_{am}^\alpha L_{ma}^\alpha + L_{am}^\alpha L_{ja}^\alpha) + u'(-L_{am}^\alpha L_{jm}^\alpha + L_{jm}^\alpha L_{ma}^\alpha + L_{am}^\alpha L_{ma}^\alpha) \right) \\
\tilde{W}_{(4)jmn}^i &= K_{jmn}^i + \frac{1}{N+1} \delta_j^i K_{[mn]} - \frac{1}{N+1} \delta_j^i (2u L_{jm;m}^\alpha + u' (L_{j;m}^\alpha - L_{m;j}^\alpha) + 2v L_{jm}^\alpha L_{\alpha j}^\beta) \\
&\quad - \frac{1}{N+1} \left( (\delta_m^i L_{j;m}^\alpha - \delta_n^i L_{j;m}^\alpha) + u(-L_{am}^\alpha L_{jm}^\alpha + L_{jm}^\alpha L_{am}^\alpha + L_{am}^\alpha L_{ma}^\alpha + L_{am}^\alpha L_{ja}^\alpha) + u'(-L_{am}^\alpha L_{jm}^\alpha + L_{jm}^\alpha L_{ma}^\alpha + L_{am}^\alpha L_{ma}^\alpha) \right).
\end{align*}
\]
\[ \tilde{W}_{(2.3)}^i_{jmn} = K_{jmn}^i + \frac{1}{N+1}\delta_j^i K_{[mn]} - \frac{1}{N+1}\delta_j^i (2uL_{j\alpha m}^\alpha + u'(L_{j\alpha m}^\alpha - L_{m\alpha j}^\alpha + 2vL_{j\alpha}^\beta L_{\alpha \beta}^\beta) \\
- \frac{1}{N+1}\left(\delta_m^i L_{j\alpha n}^\alpha - \delta_n^i L_{j\alpha m}^\alpha\right) - \frac{1}{(N+1)^2}\left(\delta_m^i L_{\alpha j\beta}^\beta - \delta_n^i L_{j\alpha m}^\beta + \delta_j^i L_{j\alpha m}^\beta\right) \\
+ u(-L_{j\alpha m}^\alpha L_{j\alpha}^\alpha + L_{j\alpha m}^\alpha L_{\alpha m}^\alpha + L_{j\alpha m}^\alpha L_{j\alpha}^\alpha) \\
+ \frac{u'}{N+1}\left(-\delta_m^i L_{j\alpha}^\alpha L_{\alpha m}^\beta + L_{j\alpha m}^\alpha L_{\alpha m}^\beta + L_{j\alpha m}^\alpha L_{j\alpha m}^\beta + L_{j\alpha m}^\alpha L_{\alpha m}^\alpha\right) \\
+ u'(-L_{j\alpha m}^\alpha L_{j\alpha}^\alpha + L_{j\alpha m}^\alpha L_{\alpha m}^\alpha + L_{j\alpha m}^\alpha L_{j\alpha}^\alpha). \]

Moreover, if we use the mapping \( f \) is equitors, i.e. \( L_{j\alpha}^\alpha = \tilde{L}_{j\alpha}^\alpha \), we transform previous invariants to:

\[ \tilde{W}_{(1.1)}^i_{jmn} = K_{jmn}^i + \frac{1}{N+1}\delta_j^i K_{[mn]} - \frac{1}{N+1}\delta_j^i (2uL_{j\alpha m}^\alpha + u'(L_{j\alpha m}^\alpha - L_{m\alpha j}^\alpha + 2vL_{j\alpha}^\beta L_{\alpha \beta}^\beta) \\
+ u(-L_{j\alpha m}^\alpha L_{j\alpha}^\alpha + L_{j\alpha m}^\alpha L_{\alpha m}^\alpha + L_{j\alpha m}^\alpha L_{j\alpha}^\alpha) \\
+ \frac{u'}{N+1}\left(-\delta_m^i L_{j\alpha}^\alpha L_{\alpha m}^\beta + L_{j\alpha m}^\alpha L_{\alpha m}^\beta + L_{j\alpha m}^\alpha L_{j\alpha m}^\beta + L_{j\alpha m}^\alpha L_{\alpha m}^\alpha\right) \\
+ u'(-L_{j\alpha m}^\alpha L_{j\alpha}^\alpha + L_{j\alpha m}^\alpha L_{\alpha m}^\alpha + L_{j\alpha m}^\alpha L_{j\alpha}^\alpha). \]

\[ \tilde{W}_{(1.2)}^i_{jmn} = K_{jmn}^i + \frac{1}{N+1}\delta_j^i K_{[mn]} - \frac{1}{N+1}\delta_j^i (2uL_{j\alpha m}^\alpha + u'(L_{j\alpha m}^\alpha - L_{m\alpha j}^\alpha + 2vL_{j\alpha}^\beta L_{\alpha \beta}^\beta) \\
+ u(-L_{j\alpha m}^\alpha L_{j\alpha}^\alpha + L_{j\alpha m}^\alpha L_{\alpha m}^\alpha + L_{j\alpha m}^\alpha L_{j\alpha}^\alpha) \\
+ \frac{u'}{N+1}\left(-\delta_m^i L_{j\alpha}^\alpha L_{\alpha m}^\beta + L_{j\alpha m}^\alpha L_{\alpha m}^\beta + L_{j\alpha m}^\alpha L_{j\alpha m}^\beta + L_{j\alpha m}^\alpha L_{\alpha m}^\alpha\right) \\
+ u'(-L_{j\alpha m}^\alpha L_{j\alpha}^\alpha + L_{j\alpha m}^\alpha L_{\alpha m}^\alpha + L_{j\alpha m}^\alpha L_{j\alpha}^\alpha). \]

\[ \tilde{W}_{(1.3)}^i_{jmn} = K_{jmn}^i + \frac{1}{N+1}\delta_j^i K_{[mn]} - \frac{1}{N+1}\delta_j^i (2uL_{j\alpha m}^\alpha + u'(L_{j\alpha m}^\alpha - L_{m\alpha j}^\alpha + 2vL_{j\alpha}^\beta L_{\alpha \beta}^\beta) \\
+ u(-L_{j\alpha m}^\alpha L_{j\alpha}^\alpha + L_{j\alpha m}^\alpha L_{\alpha m}^\alpha + L_{j\alpha m}^\alpha L_{j\alpha}^\alpha) \\
+ \frac{u'}{N+1}\left(-\delta_m^i L_{j\alpha}^\alpha L_{\alpha m}^\beta + L_{j\alpha m}^\alpha L_{\alpha m}^\beta + L_{j\alpha m}^\alpha L_{j\alpha m}^\beta + L_{j\alpha m}^\alpha L_{\alpha m}^\alpha\right) \\
+ u'(-L_{j\alpha m}^\alpha L_{j\alpha}^\alpha + L_{j\alpha m}^\alpha L_{\alpha m}^\alpha + L_{j\alpha m}^\alpha L_{j\alpha}^\alpha). \]

\[ \tilde{W}_{(1.4)}^i_{jmn} = K_{jmn}^i + \frac{1}{N+1}\delta_j^i K_{[mn]} - \frac{1}{N+1}\delta_j^i (2uL_{j\alpha m}^\alpha + u'(L_{j\alpha m}^\alpha - L_{m\alpha j}^\alpha + 2vL_{j\alpha}^\beta L_{\alpha \beta}^\beta) \\
+ u(-L_{j\alpha m}^\alpha L_{j\alpha}^\alpha + L_{j\alpha m}^\alpha L_{\alpha m}^\alpha + L_{j\alpha m}^\alpha L_{j\alpha}^\alpha) \\
+ \frac{u'}{N+1}\left(-\delta_m^i L_{j\alpha}^\alpha L_{\alpha m}^\beta + L_{j\alpha m}^\alpha L_{\alpha m}^\beta + L_{j\alpha m}^\alpha L_{j\alpha m}^\beta + L_{j\alpha m}^\alpha L_{\alpha m}^\alpha\right) \\
+ u'(-L_{j\alpha m}^\alpha L_{j\alpha}^\alpha + L_{j\alpha m}^\alpha L_{\alpha m}^\alpha + L_{j\alpha m}^\alpha L_{j\alpha}^\alpha). \]
\[
\widetilde{W}^{i}_{(2.2)jmn} = K^i_{jmn} + \frac{1}{N+1} \delta^i_j K^i_{mn} + \frac{2u}{N+1} \delta^i_j (L^\alpha_{ja} L^i_{\beta m} + L^\beta_{ja} L^\alpha_{m i}) \\
- \frac{1}{N+1} (\delta^i_m L^\alpha_{ja, n} - \delta^i_n L^\alpha_{ja, m}) - \frac{1}{(N+1)^2} (\delta^i_m L^\alpha_{ja} L^\beta_{n \alpha} - \delta^i_n L^\alpha_{ja} L^\beta_{m \alpha}) \\
+ \frac{u}{N+1} (-\delta^i_n L^\alpha_{jm, \alpha} + L^i_{na} L^\alpha_{ja} + L^i_{jm} L^\alpha_{na} + L^i_{jn} L^\alpha_{ma}) \\
+ \frac{u'}{N+1} (-\delta^i_m L^\alpha_{ja, \alpha} + L^i_{na} L^\alpha_{ja} + L^i_{jm} L^\alpha_{na} + L^i_{jn} L^\alpha_{ma}),
\]

\[
\widetilde{W}^{i}_{(2.3)jmn} = K^i_{jmn} + \frac{1}{N+1} \delta^i_j K^i_{mn} + \frac{2u}{N+1} \delta^i_j (L^\alpha_{ja} L^i_{\beta m} + L^\beta_{ja} L^\alpha_{m i}) \\
- \frac{1}{N+1} (\delta^i_m L^\alpha_{ja, n} - \delta^i_n L^\alpha_{ja, m}) - \frac{1}{(N+1)^2} (\delta^i_m L^\alpha_{ja} L^\beta_{n \alpha} - \delta^i_n L^\alpha_{ja} L^\beta_{m \alpha}) \\
+ \frac{u}{N+1} (-L^i_{na} L^\alpha_{jm} + L^i_{jn} L^\alpha_{na} + L^i_{jm} L^\alpha_{ma} + L^i_{jn} L^\alpha_{ma}) \\
+ \frac{u'}{N+1} (-L^i_{na} L^\alpha_{jm} + L^i_{jn} L^\alpha_{na} + L^i_{jm} L^\alpha_{ma} + L^i_{jn} L^\alpha_{ma}),
\]

\[
\widetilde{W}^{i}_{(2.4)jmn} = K^i_{jmn} + \frac{1}{N+1} \delta^i_j K^i_{mn} + \frac{2u}{N+1} \delta^i_j (L^\alpha_{ja} L^i_{\beta m} + L^\beta_{ja} L^\alpha_{m i}) \\
- \frac{1}{N+1} (\delta^i_m L^\alpha_{ja, n} - \delta^i_n L^\alpha_{ja, m}) - \frac{1}{(N+1)^2} (\delta^i_m L^\alpha_{ja} L^\beta_{n \alpha} - \delta^i_n L^\alpha_{ja} L^\beta_{m \alpha}) \\
+ \frac{u}{N+1} (-\delta^i_n L^\alpha_{jm, \alpha} + L^i_{na} L^\alpha_{ja} + L^i_{jm} L^\alpha_{na} + L^i_{jn} L^\alpha_{ma}) \\
+ \frac{u'}{N+1} (-L^i_{am} L^\alpha_{jm} + L^i_{jm} L^\alpha_{na} + L^i_{am} L^\alpha_{ja} + L^i_{jn} L^\alpha_{ma}).
\]

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