Mathematics Education in Digital Era: Utilizing Spatialized Instrumentation in Digital Learning Tools to Promote Conceptual Understanding

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Abstract. Although digital technology is widely used in education, there is still little explanation on how to design and utilized a digital learning tool to promote conceptual learning. To overcome this gap, we propose the idea of spatial instrumentation as a theoretical framework in designing (or developing) and implementing digital learning tools to promote conceptual understanding of mathematics learning through progressive mathematics. The reason, students can interact rich and meaningful and encourage them to use their basic understanding and intuition to support the learning process. Here, we outline theories that underlie ideas about spatial instrumentation, give examples of digital learning tools developed under these ideas, and show empirical illustrations of how students will interact with learning tools during the conceptualization process.

1. Introduction

Although the utilization of digital technology is prevalent in education and changes the face of classroom learning, the impacts of the technology in improving educational practices are widely criticized due to its disfunction. The international comparative study, for example, indicates that students who frequently used computers to support their learning in schools show worse learning outcomes comparing to those who have fewer interactions with computers [1]. Moreover, although investing a tremendous amount of budget to support the attachment of information and communication technology (ICT) in education, many countries experience a minor improvement in students’ achievement [1]. It seems that the impact of ICT in education is far behind the general expectation.

So, what is wrong with the use of digital technology in education? We argue that if a technology does not support the emergence of conceptual understanding constructively and progressively and dismiss the power of basic skills, intuitions and interactions, the dysfunction of technology will continuously prevalence since it overlooks the nature of human learning. However, there is still little explanation on how to design and utilized digital learning tools to promote such valuable learning [2,3] although digital technology is widely used in education. To address the gap, we propose the idea of Spatialized Instrumentation (SI) as the theoretical foundation in designing (or developing) and implementing learning tools (e.g. digital learning tools) to promote conceptual understanding and meaningful learning. Here, we are going to elaborate the underpinning theories of the notion of the spatialized instrumentation, provide an example of a digital learning tool developed under the notion, and show an
empirical illustration on how students will interact with the learning tool during the process of conceptualization.

Although many experts propose several theoretical frameworks for integrating digital technology in education, no framework yet or little has been known about a framework that explains the mental mechanism underpinning the reciprocal interactions between the mental affairs of the human being (e.g. thinking and reasoning) and the use of digital learning tools to promote meaningful learning. Promote the idea of embodied cognition by utilizing the power of the bodily experience as the physical mechanism to promote valuable learning (i.e. mental mechanism) through digital learning tools [2]. Propose the notion of sensorimotor-based digital tools as a means to actualize the embodied cognition [4]. However, none of them yet discuss the mental mechanism (e.g. thinking and reasoning) underpinning the mutual interplay between physical experiences (e.g. bodily or sensorimotor experience) and learning tools (e.g. digital learning tools) to promote constructive, progressive and meaningful mental experiences (e.g. the emergence of conceptual understanding). To address the gap, we propose a learning mechanism called spatialized instrumentation. In the learning mechanism, we utilized the power of spatial skills as the underpinning thinking and reasoning (the mental mechanism) of the meaningful interplay between the physical experience and the used learning tools to promote constructive, progressive and conceptual understanding.

2. Method

It is a systematic literature study with the primary goal to clarify and formulating an alternative theoretical foundation in designing, developing and utilizing digital learning tools to promote conceptual understanding in the context of mathematics education.

Based on our initial literature studies, we established four characteristics of high quality of digital learning tools, namely; first, the tools are designed to foster conceptual understanding instead of merely performing or doing mathematical tasks [3,5,6]. The second, the tools encourage users to utilize their basic understanding and intuitions instead of diminishing it [2,3,6]. Moreover, the tools provoke meaningful and productive interactions for the users either mentally or bodily [2–4]. Finally, to promote the conceptual understanding, the tools not only serve as learning tools but also serve as didactical phenomena to promote students’ experience of progressive mathematizations under a guided reinvention and model emergence process of mathematical ideas as they interact with the tools meaningfully [2,3,6].

To establish a heuristic of an instructional-tool design to meet the four criteria of a high-quality digital learning tool, we study many kinds of literature from various domain and perspectives. Each literature in each domain or perspective is critically analyzed to identify its potential to inspire the criteria. As the result of the analysis, we consolidate four different perspectives as the basis to formulate the heuristic for designing and implementing digital learning tools in mathematics education, namely the Realistic Mathematics Education (RME), the Embodiment Cognition Theory (ECT), the Instrumentation Theory (IT), and Spatial Intelligence Theory (SIT). We take RME as the heuristic for effective mathematics learning and instructional design. While considering the ECT as the nature of semantic mechanism underpinning the conceptualization of mathematical ideas, we regard SIT as the mental process of the ECT. Finally, the IT is used as the theoretical foundation to promote valuable and productive interactions between tools and users during the conceptualization. We name the formulated heuristic as Spatialized Instrumentation (SI). Hence, in general, the spatialized instrumentation is a heuristic of how to design, implement and evaluate a digital learning tool for mathematics learning by incorporating the power of human spatial intelligence as an instrument to process information and gain understanding.

3. Result and Discussion

3.1. Realistic Mathematics Education view toward learning tools

Realistic Mathematics Education (RME) is learning theory in the domain of mathematics [7]. The fundamental idea of this theory is the belief that mathematics will be best learnt if it is regarded as a human activity that is the process of reinventing mathematical ideas in which students actively do or explore the mathematics instead of passively receiving ready-mate mathematical ideas from teachers.
Under the notion of mathematics as a human activity, RME suggests four crucial principles to promote conceptual understanding of mathematics [7–10]. First, RME promotes the idea of guided reinvention that learning mathematics is considered as the process of reinventing mathematical ideas under guidance. Here, students should be allowed to experience the invention of mathematical ideas similar to how the ideas were invented [11].

Second, to support the process of the guided reinvention, RME pinpoints the notion of progressive mathematization [7,8]. Through this notion, RME stresses the importance of allowing students to mathematize phenomena as a means to reinvent or rediscover mathematical ideas. The progressive mathematization shows students experiences of moving from concrete and specific dimensions of mathematical ideas to more abstract and general ideas, or from the horizontal mathematization to the vertical mathematization [12].

Third, as students experience progressive mathematization, RME promotes the idea of emergent modelling [13,14]. This notion stresses the importance for students to develop their model or representation of thinking or reasoning (the model of and the model for mathematical ideas) as they experience the transitions during the progressive mathematization.

Fourth, to provoke the active role of students during the reinvention, mathematization and modelling, RME advocates the notion of the didactical phenomenology [15]. This notion suggests that mathematic learning should be started from phenomena or problems that are imaginable and meaningful for students, that beg to be organized and stimulates the reinvention of mathematical ideas through progressive mathematization and emergent modelling [11].

Since RME is a well-established theory of mathematic learning for conceptual understanding, we take the benefit of this theory to formulate the characteristics of a high-quality digital learning tool. In this idea, we intend to rearticulate the RME principles in the context of digital learning tools, instead of learning trajectories. How a digital learning tool will be looked like that is in line with the idea of RME? To address the question, the tools should guide and allow students to actively construct their mathematical ideas, promote the mathematical reinvention through a progressive mathematization, and trigger the emergent mathematical modelling as they are dealing with meaningful mathematical problems or phenomena.

As the RME view of mathematics learning inspires it, digital learning tools should be intended to promote conceptual understanding [2,3,6]. Here, the tool can be treated as a meaningful medium for students to develop mathematical concepts actively or to enhance their mathematics understanding or skills. The utilized learning tools should effectively foster students’ experience of progressive mathematizations [2,4,6]. Here, the learning tools provide meaningful phenomena, tasks and experience for students who promote the exploration and the shifting from the horizontal to the vertical mathematization [7,11] and the emergent of mathematical models of thinking and reasoning as well as representations from model-of to model-for [7,9,13]. Once students meaningfully experience the progressive mathematizations, they will develop their meta-representational competences, the ability to express mathematical ideas meaningfully through appropriate and multiple representations that effectively communicate the notions [16].

Moreover, to support students to experience the progressive mathematization and the emergence of their mathematical models, the digital learning tools should not get rid of the power of mathematical intuitions and the use of basic mathematical understanding [6] as they play an essential role in establishing initial perceptions once students deal with discovering or inventing mathematical ideas [17,18]. An excellent digital learning tool encourages students to use such a mental skill, instead of diminishing it.

As a consequence of the notion of the didactical phenomenology, digital learning tools should provide a meaningful experience and productive interactions for students [2,4]. The rich and meaningful interactions can be between the students and the tools or between students and teachers or other students. Meaningful and productive interactions between users and the tools are necessary to provoke students to be engaged in the learning process and to stimulate them to utilize their basic mathematical understanding and intuitions to promote the progressive mathematization and the emergence of mathematical models. Here, the learning phenomena (e.g. the tasks, the features, and the interactions)
promoted through the tools should be meaningful for users and help users to develop their understanding constructively.

Hence, a high-quality digital learning tool promotes a conceptual understanding of mathematical ideas by triggering students to actively experience the process of a reinvention of mathematical ideas through a progressive mathematization and emergent mathematical model by exploring a meaningful phenomenon or problem emerging as the students actively interact with the tool.

To satisfy such a quality of digital learning tools, we propose an integrative view from various perspectives to established a theoretical base of how to design and utilize learning tools in general and a digital learning tools in particular that potentially promote conceptual understanding of mathematical ideas. To address the goal, we propose the idea called Spatialized Instrumentation (SI) as the theoretical bases and approach in designing and integrating digital learning tools in education, especially in mathematics education. We first elaborate three theoretical foundations and then discuss how these foundations shape the idea of the spatialized instrumentation.

3.2. The embodied cognition: the embodied nature of understanding

Piaget stresses that knowledge is gain and developed through two main mental phenomena, namely assimilation and accommodation [19]. Assimilation is the process of perceiving and adapting new information or knowledge within the current structure of knowledge (i.e. the schema). In contrast, accommodation is the process of altering the pre-existing structure of knowledge or information (the pre-existing schemas) as other new relevant knowledge or information is founded. Assimilation and accommodation occur as people interact with their environment. One of the primary forms of such interactions is sensorimotor-based interaction, an interaction involving our senses (i.e. sight, hearing, touch, smell, and taste) together with our body movement. The sensorimotor-based interaction is used frequently by children in the early ages to understand their world. However, such an interaction is still relevant for adults to extend their understanding of their environment [4]. The process of the assimilation and accommodation involving the senses (a sensory experience) and bodily experiences (motoric experience) to gain understanding is an example of the embodied cognition.

Roughly speaking, embodied cognition stresses that our cognition is shaped by our experience bodily or physically with our environment. Human understanding of numbers, for example, is the result of the interactions between the human brain and their physical experience of counting, measuring, and pointing [20]. Moreover, our understanding of geometry is formed by our bodily interactions with objects in our space [21]. These imply that human cognition is not only about mental affairs, but it is also the product of meaningful physical interactions between human brains and their physical and social world [22,23]. Here, human cognition, to some extent, is conditioned and shaped by the capabilities and the constraints of the human body in navigating their physical world [24]. These facts reveal the embodied nature of our cognition.

The embodied nature of our cognition implies that our bodily experience could shape and form our cognitive scheme, including what and how we gain an understanding. Reference [4] claims that the development of our cognitive scheme is the result of rich and meaningful interactions between our perception (thinking), sensory impressions (senses) and motoric experience (body). It implies that the origin of our understanding is not only from our mental affairs (e.g. thinking and reasoning) but also rooted in our sensorimotor experiences. The objects of mathematics, furthermore, are recognized to be rooted in sensorimotor experience, although they are acknowledged as highly abstract ideas of thinking and reasoning [2,20,22,24].

Acknowledging the embodied nature of cognition, we suggest that learning mathematics should not be based only on mental activities (e.g. thinking, reasoning, and conjecturing) but it should be connected to and encourage students to experience bodily doing the mathematics (e.g. splitting, collecting, measuring, and transforming) which involves their sensory and motoric skills. Such an integrative approach of learning mathematics potentially promotes conceptual understanding of mathematics since the fact that mathematical understanding is shaped by sensorimotor experiences together with mental experiences.

However, how to actualize the embodied nature of cognition in mathematics learning? What are the mental mechanisms underpinning the embodied cognition? To address the question, in this study, we
propose the idea of spatialized cognition as the mental mechanism or reasoning underpinning the embodied cognition.

3.3. Spatialized cognition: Spatializing as a mental mechanism to gain understanding

We quote the name spatialized cognition to refer to the notion of spatializing mental or physical objects or phenomena to gain understanding. Spatializing is the process of transforming, manipulating, and representing non-spatial objects into spatial affairs (i.e. situate the objects in space). It is mainly based on the notion of spatial intelligence and skills. Here, we argue that incorporating spatial skills in dealing with an abstract idea (e.g. establishing mental simulations or visualization of the idea) lead to a better understanding of the idea.

In general, spatial intelligence is mental and physical ability to deal with space either imaginatively (spatial imagination), interpretatively (spatial interpretation), or representatively (spatial representation) [25–28] (see Figure 1). Spatial imagination is the ability to visualize or simulate mentally of what is said, stated, or depicted. In reverse, spatial representation is the ability to express or represent spatially of what is imagined, seen, visualize or simulated. Meanwhile, spatial interpretation is the ability to construct meaning, analyze, modify the meaning spatially that bridge the interplay between spatial imagination and spatial representation.

**Figure 1. The Three Dimensions of Spatial Intelligence**

The spatial intelligence underpins spatial reasoning and skills, such as visualization, mental rotation, and spatial orientation. Spatial intelligence can be categorized into two main components, namely, internal and external skills [25,27,28]. Internal spatial skills refer to the ability to spatially create, interpret, analyze, manipulate, and represent abstract ideas mentally, for example, creating mental simulations or visualization of an object or idea. Meanwhile, the external spatial skills refer to DiSessa ideas about meta-representational competence [16] that is the ability to represent similarity in multiple ways (e.g. graphs, ratios, equations) and to conceptually understand the relationships among these various forms of representations [16,28]. In line with the view of the internal and external component of spatial intelligence, Devis and colleagues classify spatial reasoning into two dimensions, namely mental and physical dimension. The mental dimension of spatial reasoning refers to the internal representations or abstract conceptions (i.e. the understanding) of spatial intelligence. Meanwhile, the physical dimension relates to the internal representations or the tangible action (i.e. the transformation) of spatial skills. The mental dimension (the understanding) includes the skills relating to interpreting (e.g. diagramming, modelling, symmetrizing, comparing, and relating), sensating (e.g. perspective-taking, visualizing, and imagining), dan constructing (e.g. de/re/composing, un/re/packing, re/arranging, sectioning, and fitting). Meanwhile, the physical dimension (the spatial transformation) consists of the following skills: moving (e.g. rotating, sliding, reflecting, and balancing), situating (e.g. locating, dimension shifting, orienting, pathfinding, and intersecting) and altering (e.g. dilating/contracting, distorting/morphing, scaling, folding, and shearing).

Spatial reasoning, the mental mechanism underpinning spatial intelligence, plays an essential role in understanding our environment. It is the nature of human thinking and reasoning, including in mathematics [20,21,25]. Moreover, spatial reasoning provides hooks for students’ thinking while they are making the transition from concrete reasoning to abstract reasoning, and encourage students to make sense of their reasoning [13,29–34]. It is known that spatial reasoning is a powerful approach of reasoning in seeing the abstract ideas of mathematics, engaging with mathematical concepts and meaning, promoting self-evidence and immediacy, and serving the process of problem-solving as well as establishing mathematical proofs [25]. It promotes the emergence of representations for expressing, conjecturing, testing, and refining mathematical thinking and argumentations that facilitate the development of generalizable ideas [30], for example, in establishing conceptions of the abstract symbols of mathematics and their manipulation [31].
Hence, spatializing or reasoning spatially is the nature of reasoning to gain understanding. However, Differences in the way of spatializing objects (or the objects being spatialized) shapes relatively different perceptions and understanding (i.e. cognitive schema). For instance, students who spatialize numbers in the form of tallies will have different conceptions about numbers to those who spatialize the numbers in the number line, or the idea of multiplication that is spatialized in forms of linear groups will lead to different understanding to that of area model. It implies that spatializing mathematical ideas are not only a mechanism to gain an understanding but it also the mechanism that shapes the understanding. This fact reveals the essential role of spatializing in structuring our cognition. We call this mechanism as spatialized cognition.

We define the spatialized cognition as the idea that the way we spatialize an object or the way the object being spatialized shapes the way we understand the object. There are two points that we stress in the definition. First, how we spatialize an object (i.e. mental or physical object) will influence our conception or understanding of the object. Second, at the same time, the way the object being represented spatially will shape our understanding of the object.

So, how the spatialized cognition serves the embodied cognition? Mental imagination or visualization of an object will not be created well if we have no related physical or space-related experience (e.g. sensorimotor experience) regarding the objects. For example, we will have no idea how to perceive the notion of “three” if we have no physical relations about it, such as three apples or three steps. The physical relations help us to understand and modify the concepts, such as seeing “three” with “two” and “one” as seeing three apples as adding one apple to two apples or seeing three steps as move forward one step from the second step to reach three-steps distance. Here, adding or moving forward is an example of embodied experiences that require spatial reasoning to conceptualize the embodied experience. The reasoning helps to see spatially how adding one apple to two apples resulting three apples, or why moving forward one steps after two steps reaching three-steps distance. In this context, spatializing serves as the underpinning reasoning that helps us to make sense of bodily experiences (i.e. the embodied experience). For example, the embodied experience of rotating an object 90 degree clockwise after 30 degrees anti-clockwise from the initial north face will be difficult to be defined mathematically if we have no idea about the spatial aspect of the embodied experience. Here, spatial reasoning or spatializing the embodied experience provides us with insight and meaning physically and potentially mathematically about the embodied experience.

Although spatial perceptions provide meaning to the embodied experience, the spatial perceptions are constructed as the result of the generalization of embodied experience. For example, we gain the spatial meaning of rotation through the generalization of bodily doing the rotation. It is in line with the view of the embodied cognition theory where it clarifies how we gain spatial conceptions of a particular bodily experience. Hence, there is a strong reciprocal relationship between spatial reasoning and embodied experience in shaping our cognition. While reasoning spatially provides meaning to the embodied experience, the conceptions underpinning the spatial reasoning are constructed as the result of the generalized embodied experience. In other words, spatial reasoning is the mental mechanism underpinning the embodied cognition, and the mechanism is formulated as the result of the generalization of embodied experiences.

3.4. The instrumentation theory: the interplay between tools and understanding

A learning tool is a critical aspect of learning to promote students’ understanding and skills. It is the mediator between students’ cognition and the phenomena being investigated [35] or between students’ learning activity and the context of the learning [36]. How a learning tool shapes the learning activity, and students’ cognition is explained in theory called the instrumentation theory. The theory explains the interplay interactions between a cognitive user schema and a tool used to develop the schema [2,4]. The central claim of the theory is that the constraints of tools shape the cognitive schema of the users, and by the same time, the existing cognitive schema of the users influences how the tools being utilized. Artefact, cognitive scheme, instrument, instrumental genesis, and instrumented technique are several key ideas in the instrumentation theory. While an artefact refers to the object that is used as a tool, a cognitive scheme represents the structure of users’ knowledge. An instrument is formed from an artefact together with the cognitive scheme that users apply and develop once they interact with the artefact.
Meanwhile, the instrumental genesis is the process once an artefact becomes a part of an instrument. During the instrumental genesis, users apply a particular cognitive scheme to deal with the artefact. Although the cognitive scheme is mental affairs, it can be observed once users interact with the artefact. The observable scheme is then called as the instrumented technique. The technique represents how users interact with the artefact during the instrumental genesis. The emergence of instrumented technique is triggered by the constraints of the artefact and users’ existing scheme [2]. Such a mechanism potentially leads to the emergence of new knowledge (a new scheme) together with the instrumented technique. The co-emergence of the new scheme and technique reflects the didactical potential of using purposively designed tools to reach a particular learning goal.

During the instrumental genesis, users experience developing appropriate cognitive schemes to deal with the tools. The developed scheme is the result of the reciprocal relationships between the tools and users’ pre-existing cognitive schemes. The relationships promote two fundamental mechanisms of cognitive development, namely instrumentalization and instrumentation. The instrumentalization occurs once users’ pre-existing cognitive schemes or knowledge shape the way the tools being utilized or used. Meanwhile, the instrumentation is the process once the constraints of the tools influence users’ existing cognitive schemes which may trigger the emergence of relatively new schemes or adapting the currently existing schemes. During the instrumentalization and instrumentation, users potentially experience cognitive development through Piaget’s assimilation or accommodation mechanism.

![Figure 2. A broken calculator and decomposing under multiplication](image)

Hence, the instrumentalization and instrumentation are the essences of the instrumentation theory depicting users’ cognitive development as the result of the interplay between the constraints of used tools and the limitation of users’ cognitive schemes during the instrumental genesis. Regarding these mechanisms, a learning tool needs to be designed purposively to let students actively and constructively develop an intended cognitive scheme. The relationships promote two fundamental mechanisms of cognitive development, namely instrumentalization and instrumentation. The instrumentalization occurs once users’ pre-existing cognitive schemes or knowledge shape the way the tools being utilized or used. Meanwhile, the instrumentation is the process once the constraints of the tools influence users’ existing cognitive schemes which may trigger the emergence of relatively new schemes or adapting the currently existing schemes. During the instrumentalization and instrumentation, users potentially experience cognitive development through Piaget’s assimilation or accommodation mechanism.

3.5. The spatialized instrumentation: an alternative approach to gain understanding through tools

The term “instrumentation” is inspired by the idea of instrumentation of the instrumentation theory that reflects how the constraints of an artefact or a tools influence and shape the cognitive structure of the
users. For example, the constraints of the broken calculator (see Figure 2) promote students to come up with the idea of decomposing numbers under multiplication, in which the constraints together with the learning tasks lead the students to realize about how to simplify or reformulation a multiplication problem. Meanwhile, the terms “spatialized” refers to the notion of the spatialized cognition asserting the spatialized nature of cognition under the embodied cognition theory. It implies that an individual’s cognition is influenced or shaped by the bodily experience of the individual (i.e. sensorimotor experience). To conceptualize the bodily experience, the individual utilizes his/her spatial intelligence as a means to imagine, analyze, interpret, and represent the experience. This mechanism contributes to the development of cognitive scheme called the spatialized cognition.

Hence, spatialized instrumentation can be defined as a mechanism of developing human cognitive scheme by incorporating the power of spatialized cognition and the instrumentation theory. In spatialized instrumentation, we utilized the power of spatializing and instrumentation process as the mechanisms to develop users’ cognitive scheme (e.g. knowledge or understanding) by providing purposively designed learning tools that allow users to experience the instrumental genesis (i.e. the instrumentation and the instrumentalization) under spatialized and embodied cognition theory.

Other than the three theoretical foundations (i.e. the embodied cognition, the spatialized cognition, and the instrumentation theory), the notion of spatialized instrumentation is inspired by the theory about the didactical functionality of learning tools in mathematics education. Regarding their didactical functionality, learning tools can be classified into two types, namely tools for doing mathematics and tools for learning mathematics \[2–5,37,38\]. Tools for doing mathematics refer to the use of learning tools to help students in doing mathematics, such as calculating, graphing, and simulating, for example, calculator for calculating and graphic calculator for graphing and simulating. Meanwhile, tools for learning mathematics refer to the learning tools that are used to either enhance mathematical skills or develop an understanding of particular mathematics concepts. So, there are two types of learning tools for learning mathematics, namely tools for improving mathematics skills and tools for developing mathematics concepts. Although software programs have been designed for practising mathematics concept, there are still few learning tools that are intended for developing mathematics concepts. However, it is the essence of the use of learning tools in mathematics education \[2–5,38\]. Therefore, the spatialized instrumentation intends to provide a theoretical basis for instructional design of learning tools that contributes to the construction of learning tools for developing mathematics concepts.

To get a deeper understanding of what we mean by spatialized instrumentation. We provide you with an example of a digital learning tool that is designed under the notion of the spatialized instrumentation to illustrate the idea. The learning tool is designed to support students in developing their understanding of the principle of constant difference in subtraction. More detail illustration of the tool can be traced in the following link: https://youtu.be/w9rWNNziz4o. The tool is designed to be used on touch screen devices which allows users to have bodily experiences with the tool. As the learning task, students are asked to determine the appropriate locations of the green and the blue button using the tool such that the light bar turns to be red (see Figure 3b). Once students realize the pattern, they will have a list of pair corresponding numbers (e.g. the locations of the green button concerning the location of the blue one) that situate the light bar to be red (such as 10 and 4, 18 and 12, 25 and 19, etc.) Once they got the list, the students are then asked to discuss the relationships of each of the corresponding numbers. It is conjectured that the students will go back to physically explore or critically examine the visualization of the tool to trace for the pattern. Here, the bodily experience together with the spatial visualization of the tool triggers the students to be aware of the constant distance between each corresponding number, for example, the distance between 10 and 4 and between18 and 12 is always 6. Considering the fact, students realize that 18 – 12 is 10 – 4 and so on. Finally, based on the pattern, students potentially generate a generalization that \(a – b = (a\pm c) – (b\pm c)\), which is the idea of the constant difference.
In the process of recognizing the principle of the constant difference through the learning tool, students will physically slide the buttons to trace the appropriate locations while seeing the changes of the light bar simultaneously. Such bodily experiences (i.e. sensory and motoric experiences) spark the use of spatial reasoning to identify the pattern in which through the reasoning students gain understanding. Here, as they bodily slide the buttons, students start to realize the pattern in which the locations of the buttons satisfy the condition. The emergence of the pattern at least is due to two potentials namely: first, the motoric experience once students sliding the buttons, and the second visual experience once they see the locations of the buttons concerning the changes in the light bar. The motoric, together with the visual experience, contribute to the students’ sensorimotor experience. As they are dealing with such experiences, their spatial reasoning is stimulated and activated that assists them in analyzing and identifying the changes in which later plays as a mental instrument to realize the intended mathematics principles.

4. Conclusion
Although digital technology is widely used in education, there is still little explanation on how to design and utilize a digital learning tool to promote conceptual learning. Moreover, even though many experts propose theoretical frameworks for integrating digital technology in education, no framework yet or still little known framework that provides an explanation of mental mechanism underpinning the reciprocal interactions between the mental affairs of the human being (e.g. thinking and reasoning) and the use of digital learning tools to promote meaningful learning. To address the gap, we propose the ideas of spatialized instrumentation as an instructional framework in designing (or developing) and implementing digital learning tools to promote conceptual understanding of mathematical ideas through progressive mathematization as it provides students with rich and meaningful either mental and physical interactions between the users and the tools and encourages the users to use their basic understanding and intuitions to support their learning process.

The term “instrumentation” in the spatialized instrumentation reflects the idea that learning tools influence and shape the cognitive structure of users. Meanwhile, the terms “spatialized” refers to the spatialized nature of cognition that an individual’s cognition is influenced or shaped by their spatial reasoning through bodily experiences (e.g. sensorimotor experience) involving spatially information processing. In the spatialized instrumentation, the power of spatial skills is utilized as the underpinning thinking and reasoning (the mental mechanism) of the meaningful interplay between the physical experience and the learning tools to promote constructive, progressive and conceptual understanding. Therefore, the spatialized instrumentation can be viewed as the mechanism of developing cognition (e.g. understanding) by incorporating the power of spatialized cognition and the instrumentation theory.

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6. References
[1] OECD 2015 Students, Computers and Learning
[2] Drijvers P 2019 Embodied instrumentation: combining different views on using digital technology in mathematics education Eleventh Congress of the European Society for
Research in Mathematics Education (Freudenthal Group; Freudenthal Institute; ERME)

[3] Putrawangsa S and Hasanah U 2018 Integrasi Teknologi Digital Dalam Pembelajaran Di Era Industri 4.0 J. Tatsqif 16 42–54

[4] Putrawangsa S and Hasanah U 2020 Sensorimotor-Based Digital Media: An Alternative Design of Digital Tools in Mathematics Education 2nd International Conference on Islam, Science and Technology (ICONIST 2019) (Atlantis Press) pp 160–5

[5] Drijvers P, Tabach M and Vale C 2018 Uses of Technology in K–12 Mathematics Education: Concluding Remarks Uses of technology in primary and secondary mathematics education (Springer) pp 421–35

[6] NCTM 2000 Principles and Standards for School Mathematics (Reston: The National Council of Teachers of Mathematics, Inc.)

[7] Van den Heuvel-Panhuizen M and Drijvers P 2020 Realistic mathematics education Encycl. Math. Educ. 713–7

[8] Gravemeijer K 1997 Instructional design for reform in mathematics education role Context. Model. Dev. Math. Strateg. Proced.

[9] Putrawangsa S 2013 Educational Design Research: Developing students’ understanding of the multiplication strategy in area measurement

[10] Putrawangsa S 2017 Desain Pembelajaran Matematika Realistik Mataram CV Reka Karya Amerta

[11] Gravemeijer K and Doorman M 1999 Context problems in realistic mathematics education: A calculus course as an example Educ. Stud. Math. 39 111–29

[12] Treffers A 2012 Three dimensions: A model of goal and theory description in mathematics instruction—The Wiskobas Project vol 3 (Springer Science & Business Media)

[13] Van Den Heuvel-Panhuizen M 2003 The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage Educ. Stud. Math. 54 9–35

[14] Streefland L 1993 The design of a mathematics course. A theoretical reflection Educ. Stud. Math. 25 109–35

[15] Freudenthal H 1986 Didactical phenomenology of mathematical structures vol 1 (Springer Science & Business Media)

[16] Disessa A A 2004 Metarepresentation: Native competence and targets for instruction Cogn. Instr. 22 293–331

[17] Burton L 1999 Why is intuition so important to mathematicians but missing from mathematics education? Learn. Math. 19 27–32

[18] Tirosh D and Tsamir P 2020 Intuition in mathematics education Encycl. Math. Educ. 428–33

[19] Slavin R E 2006 Educational Psychology: Theory and Practice (Boston: Pearson Education, Inc.)

[20] Sriraman B and Wu K 2020 Embodied cognition Encycl. Math. Educ. 266–8

[21] Freudenthal H 2012 Mathematics as an educational task (Springer Science & Business Media)

[22] De Freitas E and Sinclair N 2014 Mathematics and the body: Material entanglements in the classroom (Cambridge University Press)

[23] Radford L 2009 Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings Educ. Stud. Math. 70 111–26

[24] Alibali M W and Nathan M J 2012 Embodiment in mathematics teaching and learning: Evidence from learners’ and teachers’ gestures J. Learn. Sci. 21 247–86

[25] Arcavi A 2003 The role of visual representations in the learning of mathematics Educ. Stud. Math. 52 215–41

[26] Davis B and Group S R S 2015 Spatial reasoning in the early years: Principles, assertions, and speculations (Routledge)

[27] Tahta D 1989 Is there a geometric imperative Math. Teach. 129 20–9

[28] Hegarty M 2010 Components of spatial intelligence Psychology of learning and motivation vol 52 (Elsevier) pp 265–97

[29] Kho T H, Yeo S M and Fan L 2014 Model method in Singapore primary mathematics
textbooks Dev. 275

[30] Chan E C M 2008 Using model-eliciting activities for primary mathematics classrooms Math. Educ. 11 47–66

[31] Cheong Y K and Consultancy M 2002 The model method in Singapore Math. Educ. 6 47–64

[32] Ng S F and Lee K 2009 The model method: Singapore children’s tool for representing and solving algebraic word problems J. Res. Math. Educ. 282–313

[33] Fong N G S and Lee K 2009 Model method: A visual tool to support algebra word problem solving at the primary level Mathematics education: the Singapore journey (World Scientific) pp 169–203

[34] Tolar T D, Lederberg A R and Fletcher J M 2009 A structural model of algebra achievement: computational fluency and spatial visualization as mediators of the effect of working memory on algebra achievement Educ. Psychol. 29 239–66

[35] Hoyles C and Noss R 2003 What can digital technologies take from and bring to research in mathematics education? Second international handbook of mathematics education (Springer) pp 323–49

[36] Vygotsky L S 1980 Mind in society: The development of higher psychological processes (Harvard university press)

[37] Drijvers P, Goddijn A and Kindt M 2011 Algebra education: Exploring topics and themes Secondary algebra education (Brill Sense) pp 5–26

[38] Arcavi A, Drijvers P and Stacey K 2016 The learning and teaching of algebra: Ideas, insights and activities (Routledge)