Nonparametric Regression for Locally Stationary Functional Time Series

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Abstract. In this study, we develop an asymptotic theory of nonparametric regression for a locally stationary functional time series. First, we introduce the notion of a locally stationary functional time series (LSFTS) that takes values in a semi-metric space. Then, we propose a nonparametric model for LSFTS with a regression function that changes smoothly over time. We establish the uniform convergence rates of a class of kernel estimators, the Nadaraya-Watson (NW) estimator of the regression function, and a central limit theorem of the NW estimator.

1. Introduction

In an increasing number of situations, the collected data appear as functional or curve time series coming from different research fields such as biometrics (Chiu and Müller (2009)), environmetrics (Aue, Dubart Norinho, and Hörmann (2015)), econometrics (Bugni et al. (2009), Bugni and Horowitz (2021)), and finance (Kokoszka and Zhang (2012), Chen, Lei, and Tu (2016), Li, Robinson, and Shang (2020)). We refer to Ferraty and Vieu (2006) and Horváth and Kokoszka (2012) as standard references for functional time series analysis.

In the literature on functional time series analysis, most studies are based on (linear) stationary models (e.g., Bosq (2000, 2002), Dehling and Sharipov (2005), Antoniadis, Paparoditis, and Sapatinas (2006), Aue, Dubart Norinho, and Hörmann (2015)). However, many functional time series exhibit a nonstationary behavior. For example, in the financial industry, implied volatility of an option as a function of moneyness changes over time. We can also find other examples of possibly nonstationary functional time series in van Delft and Eichler (2018). One way to model nonstationary behavior is provided by the theory of locally stationary processes.

Locally stationary processes, as proposed by Dahlhaus (1997), are nonstationary time series that allow parameters of the time series to be time-dependent. They can be approximated by a stationary time series locally in time, which enables asymptotic theories to be established for the estimation of time-dependent characteristics. In time series analysis, locally stationary models are mainly considered in a parametric framework with time-varying coefficients. For example, we refer to Dahlhaus and Subba Rao (2006), Fryzlewicz, Sapatinas, and Subba Rao (2008), Koo and Linton (2012), Zhou (2014) and Truquet (2017). Moreover, nonparametric methods for stationary and nonstationary time series models have also been developed. We refer to, among others, Masry (1996), Fan and Yao (2003) and Hansen (2008) for stationary time series as well as Kristensen (2009), Vogt (2012), Zhang and Wu (2015), and Truquet (2019) for contributions on locally stationary time series. Recently, the notion of local stationarity has been extended to spatial data by Matsuda and Yajima (2018), Pezo (2018), and Kurisu (2022). We refer to

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Dahlhaus, Richter, and Wu (2019) for a general theory in the literature on locally stationary processes.

In contrast with the abovementioned studies regarding (nonparametric) time series analysis, studies pertaining to nonparametric methods for locally stationary functional time series are scarce despite empirical interest in modeling time-varying dependence structures of functional data. We refer to the works of van Delft and Eichler (2018) and Aue and van Delft (2020) as important recent contributions on the theoretical development for statistical methods for locally stationary functional time series. We also mention Kurisu (2021) who investigates functional principal component analysis for locally stationary functional data.

The objective of this study is to develop a framework of nonparametric regression for a locally stationary functional time series that takes values in a semi-metric space (e.g., Banach and Hilbert spaces). In particular, we first propose a nonparametric regression model for a locally stationary functional time series with a regression function that is allowed to change smoothly over time. Then, we (1) derive the uniform convergence rate and (2) establish the point-wise asymptotic normality of a kernel estimator for the regression function. To attain the first objective, we derive uniform convergence rates for a class of estimators based on kernel averages, which are crucial for demonstrating our main results. As these estimators include a wide range of kernel-based estimators such as the Nadaraya-Watson (NW) estimators, the general results are of independent interest. Our results extend the results of Vogt (2012), who studies univariate locally stationary time series, and of Masry (2005) and Ferraty and Vieu (2006), who study stationary functional time series to our framework. To the best of our knowledge, this is the first study to develop an asymptotic theory of nonparametric regression for locally stationary functional time series.

The organization of this paper is as follows. In Section 2, we introduce the notion of local stationarity for functional time series that takes values in a semi-metric space and explain the dependence structure of the functional time series. In Section 3, we present the main results including the uniform convergence rates of kernel estimators and asymptotic normality of the NW estimator of the regression function. All proofs are included in the Appendix.

1.1. Notations. For any positive sequences \(a_n\) and \(b_n\), we write \(a_n \lesssim b_n\) if a constant \(C > 0\) independent of \(n\) exists such that \(a_n \leq C b_n\) for all \(n\), \(a_n \sim b_n\) if \(a_n \lesssim b_n\) and \(b_n \lesssim a_n\), and \(a_n \ll b_n\) if \(a_n/b_n \to 0\) as \(n \to \infty\). For any \(a, b \in \mathbb{R}\), we write \(a \vee b = \max\{a, b\}\) and \(a \wedge b = \min\{a, b\}\). We use the notation \(\overset{d}{\to}\) to denote convergence in distribution. For \(x \in \mathbb{R}\), \([x]\) denotes the integer part \(x\).

2. Settings

In this section, we introduce the notion of a locally stationary functional time series that extends the notion of local stationarity introduced by Dahlhaus (1997). Furthermore, we discuss dependence structures of the functional time series.

2.1. Model. Let \(\{Y_{t,T}, X_{t,T}\}_{t=1}^T\) be random variables where \(Y_{t,T}\) is real-valued and \(X_{t,T}\) takes values in some semi-metric space \(\mathcal{H}\) with a semi-metric \(d(\cdot, \cdot)\). In most applications, \(\mathcal{H}\) is a Banach or Hilbert space with norm \(\|\cdot\|\) so that \(d(u, v) = \|u - v\|\).

In this study, we consider the following model:

\[
Y_{t,T} = m\left(\frac{t}{T}, X_{t,T}\right) + \sigma\left(\frac{t}{T}, X_{t,T}\right) \varepsilon_t, \quad t = 1, \ldots, T, \tag{2.1}
\]
where \( \{\varepsilon_t\}_{t \in \mathbb{Z}} \) is a sequence of independent and identically distributed random variables that is independent of \( \{X_{t,T}\}_{t=1}^T \). We write \( \sigma(\hat{f}, X_{t,T}) \varepsilon_t \) as \( \varepsilon_{t,T} \) for notational convenience. We also assume that \( \{X_{t,T}\} \) is a locally stationary functional time series, and the regression function \( m \) is allowed to change smoothly over time.

2.2. Local stationarity. Intuitively, a functional time series, \( \{X_{t,T}\}_{t=1}^T \) \((T \to \infty)\), is locally stationary if it behaves approximately stationary in local time. We refer to Dahlhaus and Subba Rao (2006) and Dahlhaus, Richter, and Wu (2019) for the idea of a locally stationary time series and its general theory, as well as to van Delft and Eichler (2018) and Aue and van Delft (2020) for the notion of local stationarity for a Hilbert space-valued time series. To ensure that it is locally stationary around each rescaled time point \( u \), a process \( \{X_{t,T}\} \) can be approximated by a stationary functional time series \( \{X^{(u)}_t\} \) stochastically. This concept can be defined as follows.

Definition 2.1. The \( \mathcal{H} \)-valued stochastic process \( \{X_{t,T}\}_{t=1}^T \) is locally stationary if for each rescaled time point \( u \in [0, 1] \), there exists an associated \( \mathcal{H} \)-valued process \( \{X^{(u)}_t\}_{t \in \mathbb{Z}} \) with the following properties:

(i) \( \{X^{(u)}_t\}_{t \in \mathbb{Z}} \) is strictly stationary.

(ii) It holds that

\[
d \left( X_{t,T}, X^{(u)}_t \right) \leq \left( \left| \frac{t}{T} - u \right| + \frac{1}{T} \right) U_{t,T} \text{ a.s.},
\]

for all \( 1 \leq t \leq T \), where \( \{U^{(u)}_{t,T}\} \) is a process of positive variables satisfying \( E[(U^{(u)}_{t,T})^{\rho}] < C \) for some \( \rho > 0 \) and \( C < \infty \) that are independent of \( u, t, \) and \( T \).

Definition 2.1 is a natural extension of the notion of local stationarity for the real-valued time series introduced by Dahlhaus (1997). Moreover, our definition corresponds to that of van Delft and Eichler (2018) (Definition 2.1) when \( \mathcal{H} \) is the Hilbert space \( L^2_{\mathbb{F}}([0, 1]) \) of all real-valued functions that are square integrable with respect to the Lebesgue measure on the unit interval \([0, 1]\) with the \( L^2 \) norm given by

\[
\|f\|_2 = \sqrt{\langle f, f \rangle}, \quad \langle f, g \rangle = \int_0^1 f(t)g(t)dt,
\]

where \( f, g \in L^2_{\mathbb{F}}([0, 1]) \). The authors also give sufficient conditions so that an \( L^2_{\mathbb{F}}([0, 1]) \)-valued stochastic process \( \{X_{t,T}\} \) satisfies \((2.2)\) with \( d(f, g) = \|f - g\|_2 \) and \( \rho = 2 \).

2.3. Mixing condition. Let \( (\Omega, \mathcal{F}, P) \) be a probability space, and let \( \mathcal{A} \) and \( \mathcal{B} \) be subfields of \( \mathcal{F} \). Define

\[
\alpha(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(A \cap B) - P(A)P(B)|.
\]

Moreover, for an array \( \{Z_{t,T} : 1 \leq t \leq T\} \), define the coefficients

\[
\alpha(k) = \sup_{t, T : 1 \leq t \leq T - k} \alpha(\sigma(Z_{s,T} : 1 \leq s \leq t), \sigma(Z_{s,T} : t + k \leq s \leq T)),
\]

where \( \sigma(Z) \) is the \( \sigma \)-field generated by \( Z \). The array \( \{Z_{t,T}\} \) is said to be \( \alpha \)-mixing (or strongly mixing) if \( \alpha(k) \to 0 \) as \( k \to \infty \).
3. Main results

In this section, we consider general kernel estimators and derive their uniform convergence rates. Based on the result, we derive the uniform convergence rate and asymptotic normality of the NW estimator for the regression function in model (2.1).

3.1. Kernel estimation for regression functions. We consider the following kernel estimator for \( m(u,x) \) in model (2.1):

\[
\hat{m}(u,x) = \frac{\sum_{t=1}^{T} K_{1,h}(u - t/T)K_{2,h}(d(x,X_{t,T}))Y_{t,T}}{\sum_{t=1}^{T} K_{1,h}(u - t/T)K_{2,h}(d(x,X_{t,T}))},
\]

where \( K_1 \) and \( K_2 \) denote one-dimensional kernel functions, and we used the notations \( K_{j,h}(v) = K_j(v/h), \ j = 1,2 \). Here, \( h = h_T \) is a bandwidth satisfying \( h \to 0 \) as \( T \to \infty \).

Before we state the main results, we summarize the assumptions made for model (2.1) and the kernel functions. These assumptions are standard, and similar assumptions are made by Masry (2005) and van Delft and Eichler (2018).

Assumption 3.1. (M1) The process \{\( X_{t,T} \)\} is locally stationary, that is, \{\( X_{t,T} \)\} satisfies Definition (2.7).

(M2) Let \( B(x,h) = \{y \in \mathcal{H} : d(x,y) \leq h\} \) denote the ball of radius \( h \) centered in \( x \in \mathcal{H} \). We assume that there exist positive constants \( c_d < C_d \), such that for all \( u \in [0,1] \), all \( x \in \mathcal{H} \), and all \( h > 0 \),

\[
0 < c_d \phi(h)f_1(x) \leq P(X_{t}^{(u)} \in B(x,h)) =: F_u(h; x) \leq C_d \phi(h)f_1(x),
\]

where \( \phi(h) \to 0 \) as \( h \to \infty \), and \( f_1(x) \) is a nonnegative functional in \( x \in \mathcal{H} \). Moreover, there exist constants \( C_\phi > 0 \) and \( \varepsilon_0 > 0 \) such that for any \( 0 < \varepsilon < \varepsilon_0 \),

\[
\int_0^\varepsilon \phi(u)du > C_\phi \varepsilon \phi(\varepsilon).
\]

(M3) \( \sup_{s,T} \sup_{x \neq y} P((X_{s,T}, X_{t,T}) \in B(x,h) \times B(x,h)) \leq \psi(h)f_2(x) \), where \( \psi(h) \to 0 \) as \( h \to 0 \), and \( f_2(x) \) is a nonnegative functional in \( x \in \mathcal{H} \). We assume that the ratio \( \psi(h)/\phi^2(h) \) is bounded.

(M4) \( m(u,x) \) is twice continuously partially differentiable with respect to \( u \). We also assume that

\[
\sup_{u \in [0,1]} |m(u,x) - m(u,y)| \leq c_m d(x,y)^\beta
\]

for all \( x,y \in \mathcal{H} \) for some \( c_m > 0 \) and \( \beta > 0 \).

Condition (3.3) is satisfied for fractal-type processes (i.e., \( \phi(h) \sim \varepsilon^\tau \) as \( \varepsilon \to 0 \) for some \( \tau > 0 \)). In particular the condition (3.2) is consistent with the assumption made by Gasser, Hall, and Presnell (1998). If the space \( \mathcal{H} \) is a separable Hilbert space, it is possible to choose a semi-metric for which condition (3.3) is fulfilled with \( \phi(\varepsilon) \sim \varepsilon^{\tau_0} \) as \( \varepsilon \to 0 \) for some \( \tau_0 > 0 \) (Lemma 13.6 in Ferraty and Vieu (2006)). We refer to Bogachev (1998) and Ferraty and Vieu (2006) for detailed discussions on fractal-type processes and the effect of a semi-metric on the small ball probability \( F_u(h; x) \). Condition (M3) gives the behavior of the joint distribution of \( (d(X_{s,T}, x), d(X_{t,T}, x)) \) near the origin. Assumptions (M1), (M2) and (M3) can be satisfied by a class of random coefficient models: For each \( u \in [0,1] \), let \( \{H(u,t) = (H_1(u,t), \ldots, H_d(u,t))\}'_{t \in \mathbb{Z}} \) be a \( d \)-variate (\( \alpha \)-mixing)
stationary time series with independent components. Assume that there exist random variables \( H_k(u,t,T), k = 1, \ldots, d \) such that

\[
|H_k(t/T,t) - H_k(u,t)| \leq \left( \left| \frac{t}{T} - u \right| + \frac{1}{T} \right) \tilde{H}_k(u,t,T), \quad E[\tilde{H}_k(u,t,T)]^2 \leq C < \infty
\]

(3.4) for some constant \( C \) independent of \( u, t, \) and \( T \). Note that we can construct locally stationary time series \( H_k(t/T,t), k = 1, \ldots, d \) that satisfy (3.4) (see Vogt (2012) for example). Let \( \{b_k(s)\}_{k=1}^{\infty} \) be an orthogonal basis of \( L_2^2([0,1]) \). Define \( X_t^{(u)}(s) = \sum_{k=1}^{d} H_k(u,t)b_k(s) \), \( X_{t,T}(s) = X_t^{(t/T)}(s) \). Then we have

\[
\|X_{t,T} - X_t^{(u)}\| = \left( \sum_{k=1}^{d} (H_k(t/T,t) - H_k(u,t))^2 \right)^{1/2} \leq \left( \left| \frac{t}{T} - u \right| + \frac{1}{T} \right) \left( \sum_{k=1}^{d} \tilde{H}_k^2(u,t,T) \right)^{1/2}.
\]

This implies that \( X_{t,T} \) is a locally stationary functional time series. Further, the distributions of the functional time series \( X_t^{(u)} \) and \( X_{t,T} \) are completely determined by the multivariate time series \( H(u,t) \) and \( H(t/T,t) \), respectively. In this case, the functions \( \phi(h) \) and \( \psi(h) \) in Assumptions (M2) and (M3) can be \( \phi(h) \sim h^d \) and \( \psi(h) \sim h^{2d} \). Condition (M4) is concerned with the smoothness and continuity of the regression function \( m(u,x) \) with respect to \( u \) and \( x \), respectively. Conditions (M3) and (M4) are consistent with the assumptions made by Masry (2005) and Ferraty and Vieu (2006).

We assume the following conditions on \( \sigma \) and kernel functions. Similar assumptions are made by Masry (2005) and Ferraty and Vieu (2006), and Vogt (2012).

**Assumption 3.2.** (\( \Sigma 1 \)) \( \sigma : [0,1] \times \mathcal{H} \rightarrow \mathbb{R} \) is bounded by some constant \( C_\sigma < \infty \) from above and by some constant \( c_\sigma > 0 \) from below, that is, \( 0 < c_\sigma \leq \sigma(u,x) \leq C_\sigma < \infty \) for all \( u \) and \( x \).

(\( \Sigma 2 \)) \( \sigma \) is Lipschitz continuous with respect to \( u \).

(\( \Sigma 3 \)) \( \sup_{u \in [0,1]} \sup_{d(x,y) \leq h} |\sigma(u,x) - \sigma(u,y)| = o(1) \) as \( h \rightarrow 0 \).

**Assumption 3.3.** (KB1) The kernel \( K_1 \) is symmetric around zero, bounded, and has a compact support, that is, \( K_1(v) = 0 \) for all \( |v| > C_1 \) for some \( C_1 < \infty \). Moreover, \( \int K_1(z)dz = 1 \) and \( K_1 \) is Lipschitz continuous, that is, \( |K_1(v_1) - K_1(v_2)| \leq C_2 |v_1 - v_2| \) for some \( C_2 < \infty \) and all \( v_1, v_2 \in \mathbb{R} \).

(KB2) The kernel \( K_2 \) is nonnegative, bounded, and has support in \( [0,1] \) such that \( 0 < K_2(0) \) and \( K_2(1) = 0 \). Moreover, \( K_2'(v) = dK_2(v)/dv \) exists on \( [0,1] \) and satisfies \( C'_1 \leq K_2'(v) \leq C'_2 \) for two real constants \( -\infty < C'_1 < C'_2 < 0 \).

Conditions (\( \Sigma 1 \)) and (\( \Sigma 2 \)) are consistent with the assumption made by Vogt (2012). Condition (\( \Sigma 3 \)) is required for investigating the asymptotic property of the variance of \( \hat{m}(u,x) \) to establish its asymptotic normality. The assumption on kernel functions \( K_1 \) and \( K_2 \) are standard in the literature and satisfied by popular kernel functions, such as the (asymmetric) triangle and quadratic kernels.

### 3.2 Uniform convergence rates for kernel estimators.

As a first step to study the asymptotic properties of the estimator (3.1), we analyze the following general kernel estimator:

\[
\hat{\psi}(u,x) = \frac{1}{Th\phi(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) K_{2,h}(d(x,X_{t,T})) W_{t,T}, \quad (3.5)
\]
where \( \{W_{t,T}\} \) is an array of one-dimensional random variables. Some kernel estimators, such as NW estimators, can be represented by (3.5). In this study, we use the results with \( W_{t,T} = 1 \) and \( W_{t,T} = \varepsilon_{t,T} \).

Next, we derive the uniform convergence rate of \( \hat{\psi}(u, x) - E[\hat{\psi}(u, x)] \). We assume the following for the components in (3.5).

**Assumption 3.4.**

(E1) It holds that \( \sup_{t,T} \sup_{x \in \mathcal{X}} E[|W_{t,T}|^\zeta |X_{t,T} = x] \leq C \) for some \( \zeta > 2 \) and \( C < \infty \).

(E2) The \( \alpha \)-mixing coefficients of the array \( \{X_{t,T}, W_{t,T}\} \) satisfy \( \alpha(k) \leq A k^{-\gamma} \) for some \( A > 0 \) and \( \gamma > 2 \). We also assume that \( \delta + 1 < \gamma (1 - \frac{2}{\nu}) \) for some \( \nu > 2 \) and \( \delta > 1 - \frac{2}{\nu} \), and

\[
\lambda_T^{2(1+\beta) - 1} \left( \phi(h) \lambda_T + \sum_{k=\lambda_T}^\infty k^\delta (\alpha(k))^{1-\frac{2}{\nu}} \right) \to 0, \quad (3.6)
\]

as \( T \to \infty \), where \( \lambda_T = \left( \phi(h) \right)^{-\left(1-\frac{2}{\nu}\right)/\delta} \).

We need (3.6) to establish the asymptotic normality of \( \hat{m}(u, x) \). We also need Condition (3.6) to show the asymptotic negligibility of the bias of \( \hat{m}(u, x) \). Condition (E2) is consistent with assumptions made in [Masry (2005)] and [Vogt (2012)].

Furthermore, we assume the following regularity conditions on \( h \) and \( \phi(h) \).

**Assumption 3.5.**

As \( T \to \infty \),

(R1) \( \frac{(\log T)^{\zeta_0(1+1)}}{T^{\frac{1}{2} - \frac{\zeta_0(1+1)}{2}} h^{\zeta_0(1+1) \phi(h)^{\frac{1}{2}}} + \phi(h)^{\frac{1}{2}}} \to 0 \) for some \( \zeta_0 > 0 \), and

(R2) \( Th^3, Th^3 \phi(h) \to \infty \),

where \( \zeta \) and \( \gamma \) are positive constants that appear in Assumption 3.4.

Condition (R1) is required to apply an exponential inequality for \( \alpha \)-mixing sequence to establish the uniform convergence rate of the general estimator and \( \hat{m}(u, x) \). Condition (R2) is concerned with the bias and the convergence rate of the general estimator \( \hat{\psi}(u, x) \).

The next theorem generalizes the uniform convergence results of [Vogt (2012)] to a functional time series.

**Proposition 3.1.** Assume that Assumptions 3.1 (M1), (M2), 3.3, 3.4, and 3.5 are satisfied. Then the following result holds for any \( x \in \mathcal{H} \):

\[
\sup_{u \in [0,1]} |\hat{\psi}(u, x) - E[\hat{\psi}(u, x)]| = O_p \left( \sqrt{\frac{\log T}{Th^3 \phi(h)}} \right).
\]

Apart from \( h \), which comes from the smoothing in time direction, the convergence rate in the above proposition is the same as the point-wise convergence rate of the (nonparametric) regression function obtained in [Ferraty and Vieu (2006)] for a strictly stationary functional time series. The next theorem provides the uniform convergence rate of the kernel estimator \( \hat{m}(u, x) \).

**Theorem 3.1.** Assume that Assumptions 3.1, 3.2, 3.3, 3.4, and 3.7 are satisfied and that Assumption 3.4 is satisfied with \( W_{1,T} = 1 \) and \( W_{t,T} = \varepsilon_{t,T} \). Then, the following result holds for any \( x \in \mathcal{H} \):

\[
\sup_{u \in [C_1 h, 1 - C_1 h]} |\hat{m}(u, x) - m(u, x)| = O_p \left( \sqrt{\frac{\log T}{Th^3 \phi(h)}} + h^{2\lambda} \right). \quad (3.7)
\]
Theorem 3.1 generalizes the results on point-wise convergence in Ferraty and Vieu (2006) and the results in Masry (2005) for a strictly stationary functional time series case to our setting. Using Proposition 3.1, the stochastic part is shown to be of order $O_p(\sqrt{\log T/Th\phi(h)})$. Compared with Theorem 4.2 in Vogl (2012), we do not have the bias term that comes from the approximation error of $X_{1,T}$ by $X_{1}^{(u)}$. Indeed, under our assumptions, the approximation error is $O\left( T^{-1}h^{(1+\beta)/2}-1\phi^{-1}(h) \right) \ll h^{2\beta}$. 

Remark 3.1. For a fractal-type process $\{X_{t}^{(u)}\}$, the right-hand side of (3.4) with $\beta \leq 2$ is optimized by choosing $h \sim \left( \frac{\log T}{T} \right)^{\frac{1}{2\beta+1}}$, and the optimized rate is

$$\sup_{u \in [C_{1}h,1-C_{1}h]} |\hat{m}(u, x) - m(u, x)| = O_p \left( \frac{\log T}{T} \right)^{\frac{1}{2\beta+1}}.$$ 

3.3. Asymptotic normality for kernel estimators. In this section, we provide a central limit theorem for the kernel estimator $\hat{m}(u, x)$. To establish the asymptotic normality of the NW estimator $\hat{m}(u, x)$, we additionally make the following assumption, which is used to employ Bernstein’s big-block and small-block procedure.

Assumption 3.6. There exists a sequence of positive integers $\{v_T\}$ satisfying $v_T \to \infty$, $v_T = o(\sqrt{Th\phi(h)})$ and $\sqrt{\frac{T}{Th\phi(h)}} o(v_T) \to 0$ as $T \to \infty$.

Observe that

$$\hat{m}(u, x) - m(u, x) = \frac{1}{\hat{m}_1(u, x)} \left( \hat{g}_1(u, x) + \hat{g}_2(u, x) - m(u, x) \hat{m}_1(u, x) \right) = \frac{1}{\hat{m}_1(u, x)} \left( \hat{g}_1(u, x) + \hat{g}^B(u, x) \right),$$

where

$$\hat{m}_1(u, x) = \frac{1}{Th\phi(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) K_{2,h} \left( d(x, X_{t,T}) \right),$$

$$\hat{g}_1(u, x) = \frac{1}{Th\phi(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) K_{2,h} \left( d(x, X_{t,T}) \right) \varepsilon_{t,T},$$

$$\hat{g}_2(u, x) = \frac{1}{Th\phi(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) K_{2,h} \left( d(x, X_{t,T}) \right) m \left( \frac{t}{T}, X_{t,T} \right).$$

Under the same assumption in Theorem 3.1, we can show that $\text{Var}(\hat{g}^B(u, x)) = o\left( \frac{1}{Th\phi(h)} \right)$ and $1/\hat{m}_1(u, x) = O_p(1)$. See the proof of Theorem 3.2 for details. Then, we have

$$\hat{m}(u, x) - m(u, x) = \frac{\hat{g}_1(u, x)}{\hat{m}_1(u, x)} + B_T(u, x) + o_p \left( \sqrt{\frac{1}{Th\phi(h)}} \right),$$

where $B_T(u, x) = E[\hat{g}^B(u, x)]/E[\hat{m}_1(u, x)]$ is the “bias” term and $\frac{\hat{g}_1(u, x)}{\hat{m}_1(u, x)}$ is the “variance” term.

In the following result, we set $K_2$ as the asymmetrical triangle kernel, that is, $K_2(x) = (1 - x)I(x \in [0, 1])$ to simplify the proof.
Theorem 3.2. Assume that Assumptions $3.1$, $3.2$, $3.3$, $3.5$, and $3.6$ are satisfied and that Assumption $3.4$ is satisfied for both $W_{1,T} = 1$ and $W_{t,T} = \varepsilon_{l,T}$. Then as $T \to \infty$, the following result holds for any $x \in \mathcal{H}$:

$$\sqrt{T}h\phi(h)(\hat{m}(u, x) - m(u, x) - B_T(u, x)) \xrightarrow{d} N(0, V(u, x)),$$

where $B_T(u, x) = O(h^{2\gamma^\beta})$ and

$$V(u, x) = \lim_{T \to \infty} Th\phi(h) \frac{\text{Var}(\hat{g}_1(u, x))}{E[\hat{m}_1(u, x)]} > 0.$$

Theorem 3.2 is an extension of the results in Masry (2005) and Vogt (2012) to a locally stationary functional time series. In particular, the bias and variance expressions $B_T(u, x)$ and $V(u, x)$ are very similar to those in Masry (2005). By requiring that $Th^{1+2(2\gamma^\beta)}\phi(h) \to 0$, the bias $B_T(u, x)$ term is asymptotically negligible.

4. Concluding remarks

In this paper, we have developed an asymptotic theory for nonparametric regression models with time-varying regression function with locally stationary functional covariate. In particular, we derived uniform convergence rates of general kernel estimators and the NW estimator of the regression function. We also established a central limit theory of the NW estimator.

As discussed in Vogt (2012), it would be possible to provide the uniform convergence rate of $\hat{m}(u, x)$ over $(1 - C_1h, 1] \times \{x\}$, which is important for forecasting purposes by using boundary-corrected kernels or one-sided kernels. In both cases, we have to ensure that the kernels are compactly supported and they are Lipschitz continuous to get the theory to work.

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Appendix A. Proofs

A.1. Proofs for Section

Proof of Proposition 3.1. Define $B = [0, 1]$, $a_T = \sqrt{\log T / Th\phi(h)}$ and $\tau_T = \rho_T T^{1/\zeta}$ with $\rho_T = (\log T)^{\zeta_0}$ for some $\zeta_0 > 0$. Define

$$
\hat{\psi}_1(u, x) = \frac{1}{Th\phi(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) K_{2,h} (d(x, X_{t,T})) W_{t,T} I(|W_{t,T}| \leq \tau_T),
$$

$$
\hat{\psi}_2(u, x) = \frac{1}{Th\phi(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) K_{2,h} (d(x, X_{t,T})) W_{t,T} I(|W_{t,T}| > \tau_T).
$$

Note that $\hat{\psi}(u, x) - E[\hat{\psi}(u, x)] = \hat{\psi}_1(u, x) - E[\hat{\psi}_1(u, x)] + \hat{\psi}_2(u, x) - E[\hat{\psi}_2(u, x)]$.

(Step 1) First we consider the term $\hat{\psi}_2(u, x) - E[\hat{\psi}_2(u, x)]$.

$$
P \left( \sup_{u \in B} |\hat{\psi}_2(u, x)| > a_T \right) \leq P \left( |W_{t,T}| > \tau_T \text{ for some } t = 1, \ldots, T \right)
$$

$$
\leq \tau_T^{-\zeta} \sum_{t=1}^{T} E[|W_{t,T}|^\zeta] \leq T \tau_T^{-\zeta} = \rho_T^{-\zeta} \to 0.
$$

$$
E \left[ |\hat{\psi}_2(u, x)| \right] \leq \frac{1}{Th\phi(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) E \left[ K_{2,h} (d(x, X_{t,T})) |W_{t,T}| I(|W_{t,T}| > \tau_T) \right].
$$

Since

$$
K_{2,h} (d(x, X_{t,T})) \leq \left| K_{2,h} (d(x, X_{t,T})) - K_{2,h} \left( d \left( x, X_{t/(t/T)}^{(t/T)} \right) \right) \right| + K_{2,h} \left( d \left( x, X_{t/(t/T)}^{(t/T)} \right) \right)
$$

$$
\leq h^{-1} \left| d(x, X_{t,T}) - d \left( x, X_{t/(t/T)}^{(t/T)} \right) \right| + K_{2,h} \left( d \left( x, X_{t/(t/T)}^{(t/T)} \right) \right)
$$

$$
\leq h^{-1} d \left( X_{t,T}, X_{t/(t/T)}^{(t/T)} \right) + K_{2,h} \left( d \left( x, X_{t/(t/T)}^{(t/T)} \right) \right)
$$

$$
\leq \frac{1}{Th} I_{t,T}^{(t/T)} + K_{2,h} \left( d \left( x, X_{t/(t/T)}^{(t/T)} \right) \right)
$$

and

$$
E \left[ K_{2,h} (d(x, X_{t,T})) |W_{t,T}| I(|W_{t,T}| > \tau_T) \right] \leq \tau_T^{-\zeta-1} E \left[ K_{2,h} (d(x, X_{t,T})) |W_{t,T}|^\zeta \right]
$$

$$
\leq \tau_T^{-\zeta-1} E \left[ K_{2,h} (d(x, X_{t,T})) \right],
$$

we have

$$
E \left[ K_{2,h} (d(x, X_{t,T})) |W_{t,T}| I(|W_{t,T}| > \tau_T) \right]
$$

$$
\leq \frac{1}{Th} \tau_T^{-\zeta-1} E \left[ I_{t,T}^{(t/T)} \right] + \tau_T^{-\zeta+1} E \left[ K_{2,h} \left( d \left( x, X_{t/(t/T)}^{(t/T)} \right) \right) \right]
$$

$$
\leq \frac{1}{Th} \tau_T^{-\zeta-1} + \tau_T^{-\zeta+1} E \left[ I(d(x, X_{t/(t/T)}^{(t/T)}) \leq h) \right] \leq \frac{1}{Th} \tau_T^{\zeta} + \tau_T^{-\zeta+1} \phi(h).
$$
Then we have

\[ E \left[ |\hat{\psi}_2(u, x)| \right] \leq \tau_T^{-\xi + 1} \phi(h) \frac{1}{Th\phi(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \]

\[ \leq \frac{1}{\tau_T^{-1}} \frac{1}{T h} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \leq \frac{1}{\tau_T^{-1}} = \rho_T^{-(\xi - 1)} T^{-\xi} \leq a_T. \]

For the third inequality, we used Lemma 13.2 below. As a result, \( \sup_{u \in B} |\hat{\psi}_2(u, x) - E[\hat{\psi}_2(u, x)]| = O_p(a_T). \)

(Step 2) Now we show \( \sup_{u \in B} |\hat{\psi}_1(u, x) - E[\hat{\psi}_1(u, x)]| = O_p(a_T). \) Cover the region \( B \) with \( N \leq h^{-1}a_T^{-1} \) balls \( B_{k,T} = \{ u \in \mathbb{R} : |u - u_k| \leq a_T h \} \) and use \( u_k \) to denote the mid point of \( B_{k,T}. \) In addition, let \( K^*(w, v) = CI(|w| \leq 2C_1 K^2(v)) \) for \( (w, v) \in \mathbb{R}^2. \) Note that for \( u \in B_{k,T} \) and sufficiently large \( T, \)

\[ \left| K_{1,h} \left( u - \frac{t}{T} \right) - K_{1,h} \left( u_k - \frac{t}{T} \right) \right| \]

\[ \leq a_T K^*_h \left( u_k - \frac{t}{T}, d(x, X_{t,T}) \right) \]

with \( K^*_h(v) = K^*(v/h). \) Define

\[ \bar{\psi}_1(u, x) = \frac{1}{Th\phi(h)} \sum_{t=1}^{T} K^*_h \left( u - \frac{t}{T}, d(x, X_{t,T}) \right) |W_{t,T}| I(|W_{t,T}| \leq \tau_T). \]

Note that \( E \left[ |\bar{\psi}_1(u, x)| \right] \leq M < \infty \) for some sufficiently large \( M. \) Then we obtain

\[ \sup_{u \in B_{k,T}} \left| \hat{\psi}_1(u, x) - E[\hat{\psi}_1(u, x)] \right| \]

\[ \leq \left| \bar{\psi}_1(u_k, x) - E[\bar{\psi}_1(u_k, x)] \right| + a_T \left( \left| \bar{\psi}_1(u_k, x) \right| + E \left[ \left| \bar{\psi}_1(u_k, x) \right| \right] \right) \]

\[ \leq \left| \bar{\psi}_1(u_k, x) - E[\bar{\psi}_1(u_k, x)] \right| + \left| \bar{\psi}_1(u_k, x) - E[\bar{\psi}_1(u_k, x)] \right| + 2Ma_T. \]

Hence we have

\[ P \left( \sup_{u \in B} \left| \hat{\psi}_1(u, x) - E[\hat{\psi}_1(u, x)] \right| > 4Ma_T \right) \]

\[ \leq N \max_{1 \leq k \leq N} P \left( \sup_{u \in B_{k,T}} \left| \hat{\psi}_1(u, x) - E[\hat{\psi}_1(u, x)] \right| > 4Ma_T \right) \leq Q_{1,T} + Q_{2,T} \]

where

\[ Q_{1,T} = N \max_{1 \leq k \leq N} P \left( \left| \bar{\psi}_1(u_k, x) - E[\bar{\psi}_1(u_k, x)] \right| > Ma_T \right), \]

\[ Q_{2,T} = N \max_{1 \leq k \leq N} P \left( \left| \bar{\psi}_1(u_k, x) - E[\bar{\psi}_1(u_k, x)] \right| > Ma_T \right). \]

We focus on the analysis of \( Q_{1,T} \) since \( Q_{2,T} \) can be analyzed in almost the same way. Define

\[ Z_{t,T}(u, x) = K_{1,h} \left( u - \frac{t}{T} \right) \{ K_{2,h} (d(x, X_{t,T})) W_{t,T} I(|W_{t,T}| \leq \tau_T) \}

\[ - E \{ K_{2,h} (d(x, X_{t,T})) W_{t,T} I(|W_{t,T}| \leq \tau_T) \}. \]
Note that the array \( \{Z_{t,T}(u,x)\} \) is \( \alpha \)-mixing for each fixed \((u,x)\) with mixing coefficients \( \alpha_{Z,T} \) such that \( \alpha_{Z,T}(k) \leq \alpha(k) \). We apply Lemma [B.3] below with \( \varepsilon = M\alpha_T \Phi(h), b_T = C \tau_T \) for sufficiently large \( C > 0 \) and \( S_T = a_T^{-1} \tau_T^{-1} \). Furthermore, a straightforward extension of Theorem 2 in [Masyr (2005)] yields that \( \sigma_{Z_{t,T}}^2 \leq C'S_T \Phi(h) \) with a constant \( C' \) independent of \((u,x)\). Note that we can take \( M > 0 \) sufficiently large such that \( C' < M \). Therefore, for any fixed \((u,x)\) and sufficiently large \( T \), we have

\[
P \left( \left| \sum_{t=1}^T Z_{t,T}(u,x) \right| \geq M a_T \Phi(h) \right) \leq 4 \exp \left( \frac{\varepsilon^2}{64 \sigma_{S_{t,T}}^2 T/ST} + \frac{8T\varepsilon b_T S_T}{C M} \right) + \frac{T}{S_T} \alpha(S_T) \\
\leq 4 \exp \left( \frac{-M^2 \log T}{64 C' + \frac{8}{3} CM} \right) + \frac{T}{S_T} \alpha(S_T) \\
\lesssim \exp \left( \frac{-M \log T}{64 C' + \frac{8}{3} C} \right) + TS_T^{-\gamma-1} \\
\leq \exp \left( \frac{-M \log T}{64 + \frac{8}{3} C} \right) + TS_T^{-\gamma-1} \\
= T^{-\frac{M}{64+3C}} + T a_T^{\gamma+1} \tau_T^{\gamma+1}.
\]

Observe that

\[
R_{1,T} = h^{-1} a_T^{-1} T^{-\frac{M}{64+3C}} = o(1) \quad \text{(for sufficiently large } M > 0),
\]
\[
R_{2,T} = h^{-1} a_T^{-1} T a_T^{\gamma+1} \tau_T^{\gamma+1} = h^{-1} T \left( \sqrt{\frac{\log T}{\Phi(h)}} \right)^{\frac{2}{T}} \rho_T^{\gamma+1} T^{-\frac{\gamma+1}{T}} \\
= \frac{(\log T)^{\frac{2}{T} + o(1)}}{T^\frac{2}{T} - 1 - \frac{\gamma+1}{T} h^{\frac{2}{T} + 1} \Phi(h)} = o(1)
\]

Therefore, we have \( Q_{1,T} \lesssim O(R_{1,T}) + O(R_{2,T}) = o(1) \).

**Proof of Theorem [Z4]** Recall that

\[
\hat{m}(u,x) - m(u,x) = \frac{1}{\hat{m}_1(u,x)} (\hat{g}_1(u,x) + \hat{g}_2(u,x) - m(u,x)\hat{m}_1(u,x))
\]

where

\[
\hat{m}_1(u,x) = \frac{1}{T \Phi(h)} \sum_{t=1}^T K_{1,h} \left( u - \frac{t}{T} \right) K_{2,h} (d(x,X_{t,T})),
\]
\[
\hat{g}_1(u,x) = \frac{1}{T \Phi(h)} \sum_{t=1}^T K_{1,h} \left( u - \frac{t}{T} \right) K_{2,h} (d(x,X_{t,T})) \varepsilon_{t,T},
\]
\[
\hat{g}_2(u,x) = \frac{1}{T \Phi(h)} \sum_{t=1}^T K_{1,h} \left( u - \frac{t}{T} \right) K_{2,h} (d(x,X_{t,T})) m \left( \frac{t}{T}, X_{t,T} \right).
\]

(Step1) First we give a sketch of the proof. In Steps 1 and 2, we show the following four results:

(i) \( \sup_{u \in [0,1]} |\hat{g}_1(u,x)| = O_p \left( \sqrt{\log T}/\Phi(h) \right) \).
(ii) 
\[
\sup_{u \in [0,1]} \left( \hat{g}_2(u, x) - m(u, x) \hat{m}_1(u, x) - E \left[ \hat{g}_2(u, x) - m(u, x) \hat{m}_1(u, x) \right] \right) = O_p \left( \sqrt{(\log T)/Th(h)} \right).
\]

(iii) \( \sup_{u \in [C_1 h, 1-C_1 h]} |E \left[ \hat{g}_2(u, x) - m(u, x) \hat{m}_1(u, x) \right] = O(h^2) + O(h^3). \)

(iv) \( 1/\inf_{u \in [C_1 h, 1-C_1 h]} \hat{m}_1(u, x) = O_p(1). \)

(i) can be shown by applying Proposition 3.1 with \( W \) and \( \hat{u} \) uniformly in \( u \). Observe that \( \hat{m}_1(u, x) \) is the solution of the equation \( \hat{m}_1(u, x) = \hat{m}_1(u, x) + \hat{m}_1(u, x) \)

where

\[
\hat{m}_1(u, x) = \frac{1}{Th(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) K_{2,h} \left( d \left( x, X^{(t/T)}_t \right) \right),
\]

\[
\hat{m}_1(u, x) = \frac{1}{Th(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \left\{ K_{2,h} \left( d \left( x, X^{(t/T)}_t \right) \right) - K_{2,h} \left( d \left( x, X^{(t/T)}_t \right) \right) \right\}.
\]

Applying Proposition 3.1 with \( W_{t,T} = \varepsilon_{t,T} \), we have that \( \sup_{u \in [0,1]} |\hat{m}_1(u, x) - E[\hat{m}_1(u, x)]| = o_p(1) \) uniformly in \( u \). Moreover,

\[
E \left[ |\hat{m}_1(u, x)| \right] \leq \frac{1}{Th(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \frac{1}{Th} E[U^{(t/T)}_{t,T}]
\]

\[
\leq \frac{o(\phi(h))}{Th(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) = o(1) \text{ (from Lemma B.2)}
\]

uniformly in \( u \). Then we have

\[
\hat{m}_1(u, x) = \hat{m}_1(u, x) - E[\hat{m}_1(u, x)] + E[\hat{m}_1(u, x)] = o_p(1) + E[\hat{m}_1(u, x)] + E[\hat{m}_1(u, x)]
\]

\[
= E[\hat{m}_1(u, x)] + o_p(1) + o(1)
\]

uniformly in \( u \). Observe that

\[
E[\hat{m}_1(u, x)] = \frac{1}{Th(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) E \left[ K_{2,h} \left( d \left( x, X^{(t/T)}_t \right) \right) \right]
\]

\[
= \frac{1}{Th(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \int_{0}^{h} K_{2,h}(y) dF_{t/T}(y; x)
\]

\[
\geq \frac{1}{Th(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \phi(h)f_1(x) f_1(x) > 0
\]
We first consider $Q_u$. Note that the last inequality can be obtained by slightly extending Lemma 4.4 in Ferraty and Vieu (2006). Therefore, we obtain
\[
\inf_{u \in [C_1 h, 1 - C_1 h]} m_1(u, x) = \inf_{u \in [C_1 h, 1 - C_1 h]} E[m_1(u, x)] + o_p(1) + o(1) = O_p(1).
\]
Combining the results (i), (ii), (iii) and (iv), we have that
\[
\sup_{u \in [C_1 h, 1 - C_1 h]} |\hat{m}(u, x) - m(u, x)| \leq \inf_{u \in [C_1 h, 1 - C_1 h]} m_1(u, x) \left( \sup_{u \in [C_1 h, 1 - C_1 h]} |\hat{g}_1(u, x)| + \sup_{u \in [C_1 h, 1 - C_1 h]} |\hat{g}_2(u, x) - m(u, x)\hat{m}_1(u, x)| \right)
\]
\[
\leq \inf_{u \in [C_1 h, 1 - C_1 h]} m_1(u, x) O_p \left( \sqrt{\frac{\log T}{Th\phi(h)}} + h^2 + h^\beta \right) = O_p \left( \sqrt{\frac{\log T}{Th\phi(h)}} + h^2 + h^\beta \right).
\]
Therefore, we complete the proof.

(Step 2) In this step, we show (iii). Let $K_0 : [0, 1] \to \mathbb{R}$ be a Lipschitz continuous function with support $[0, q]$ for some $q > 1$. Assume that $K_0(x) = 1$ for all $x \in [0, 1]$ and write $K_{0,h}(x) = K_0(x/h)$. Observe that
\[
E[g_2(u, x) - m(u, x)\hat{m}_1(u, x))] = \sum_{i=1}^{4} Q_i(u, x),
\]
where
\[
Q_i(u, x) = \frac{1}{nh\phi(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) q_i(u, x)
\]
and
\[
q_1(u, x) = E \left[ K_{0,h}(d(x, X_{t,T})) \left\{ K_{2,h}(d(x, X_{T,T})) - K_{2,h} \left( d(x, X_{t,T}) \right) \right\} \right]
\times \left\{ m \left( \frac{t}{T}, X_{t,T} \right) - m(u, x) \right\},
\]
\[
q_2(u, x) = E \left[ K_{0,h}(d(x, X_{t,T})) K_{2,h} \left( d(x, X_{t,T}) \right) \left\{ m \left( \frac{t}{T}, X_{t,T} \right) - m \left( \frac{t}{T}, X_{t,T} \right) \right\} \right],
\]
\[
q_3(u, x) = E \left[ \left\{ K_{0,h}(d(x, X_{t,T})) - K_{0,h} \left( d(x, X_{t,T}) \right) \right\} \right]
\times K_{2,h} \left( d(x, X_{t,T}) \right) \left\{ m \left( \frac{t}{T}, X_{t,T} \right) - m(u, x) \right\},
\]
\[
q_4(u, x) = E \left[ K_{2,h} \left( d(x, X_{t,T}) \right) \left\{ m \left( \frac{t}{T}, X_{t,T} \right) - m(u, x) \right\} \right].
\]
We first consider $Q_1(u, x)$. Observe that
\[
Q_1(u, x) \leq \frac{1}{Th\phi(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) E \left[ \left| K_{2,h}(d(x, X_{T,T})) - K_{2,h} \left( d(x, X_{t,T}) \right) \right| \right]
\times K_{0,h}(d(x, X_{t,T})) \left\{ m \left( \frac{t}{T}, X_{t,T} \right) - m(u, x) \right\}.
\]
Note that \( K_{0,t}(d(x, X_{t,T})) |m \left( \frac{t}{T}, X_{t,T} \right) - m(u, x) | \lesssim h^{1,\beta} \). Since \( K_2 \) is Lipschitz and \( d \left( X_{t,T}, X_{t,T}^{t/T} \right) \leq \frac{1}{T} U_{t,T}^{(t/T)} \), we have that

\[
Q_1(u, x) \leq \frac{h^{1,\beta}}{T h \phi(h)} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) E \left[ K_{2,h}(d(x, X_{t,T})) - K_{2,h} \left( d \left( X_{t,T}, X_{t,T}^{t/T} \right) \right) \right]
\]

uniformly in \( u \). Using similar arguments, we can also show that

\[
\sup_{u \in [C_1 h, 1 - C_1 h]} |Q_2(u, x)| \lesssim \frac{1}{T h^{1-(1-\beta)} \phi(h)}, \quad \sup_{u \in [C_1 h, 1 - C_1 h]} |Q_3(u, x)| \lesssim \frac{1}{T h^{1-(1-\beta)} \phi(h)}.
\]

Finally, applying Lemma B.1 below and using the assumptions on the smoothness of \( m \), we have that \( \sup_{u \in [C_1 h, 1 - C_1 h]} |Q_4(u, x)| \lesssim h^2 + h^\beta \).

**Proof of Theorem 3.2.** Recall that

\[
\hat{m}(u, x) - m(u, x) = \frac{1}{\hat{m}_1(u, x)} \left( \hat{g}_1(u, x) + \hat{g}_2(u, x) - m(u, x)\hat{m}_1(u, x) \right).
\]

Define \( \hat{g}^B(u, x) = (\hat{g}_2(u, x) - m(u, x)\hat{m}_1(u, x)) \).

(Step 1) First, we will show that

\[
\hat{g}^B(u, x) - E[\hat{g}^B(u, x)] = o_p \left( \sqrt{\frac{1}{T h \phi(h)}} \right). \tag{A.1}
\]

Define \( \Delta_{t,T}(u, x) = K_{2,h}(d(x, X_{t,T})) \left( m \left( \frac{t}{T}, X_{t,T} \right) - m(u, x) \right) \). Observe that

\[
\text{Var}(\hat{g}^B(u, x)) = \frac{1}{(Th \phi(h))^2} \sum_{t=1}^{T} K_{1,h}^2 \left( u - \frac{t}{T} \right) \text{Var}(\Delta_{t,T}(u, x))
\]

\[
+ \sum_{t_1, t_2 = 1, t_1 \neq t_2} K_{1,h} \left( u - \frac{t_1}{T} \right) K_{1,h} \left( u - \frac{t_2}{T} \right) \text{Cov}(\Delta_{t_1,T}(u, x), \Delta_{t_2,T}(u, x))
\]

\[
=: V_{1,T}^B + V_{2,T}^B.
\]

For \( V_{1,T}^B \),

\[
|V_{1,T}^B| \leq \frac{h^{2(1,\beta)}}{(Th \phi(h))^2} \sum_{t=1}^{T} K_{1,h}^2 \left( u - \frac{t}{T} \right) E \left[ K_{2,h}(d(x, X_{t,T})) \right]
\]

\[
\leq \frac{h^{2(1,\beta)}}{(Th \phi(h))^2} \sum_{t=1}^{T} K_{1,h}^2 \left( u - \frac{t}{T} \right) \left\{ E \left[ K_{2,h}^2 \left( d \left( x, X_{t,T}^{t/T} \right) \right) \right] + \frac{1}{Th} E[U_{t,T}^{(t/T)}] \right\}
\]

\[
\leq \frac{h^{2(1,\beta)}}{(Th \phi(h))^2} \sum_{t=1}^{T} K_{1,h}^2 \left( u - \frac{t}{T} \right) \leq \frac{h^{2(1,\beta)}}{(Th \phi(h))^2} \lesssim \frac{1}{Th \phi(h)}.
\]
For \( V_{2,T}^B \),
\[
V_{2,T}^B = \frac{1}{(T h \phi(h))^2} \sum_{t_1,t_2=1}^T \left( \sum_{1 \leq |t_1-t_2| \leq \lambda_T} K_{1,h} \left( u - \frac{t_1}{T} \right) K_{1,h} \left( u - \frac{t_2}{T} \right) \text{Cov}(\Delta_{t_1,T}(u,x), \Delta_{t_2,T}(u,x)) \right) \\
+ \frac{1}{(T h \phi(h))^2} \sum_{t_1,t_2=1}^T \left( \sum_{|t_1-t_2| > \lambda_T} K_{1,h} \left( u - \frac{t_1}{T} \right) K_{1,h} \left( u - \frac{t_2}{T} \right) \text{Cov}(\Delta_{t_1,T}(u,x), \Delta_{t_2,T}(u,x)) \right)
\]
\[=: V_{21,T}^B + V_{22,T}^B \]
where \( \lambda_T = o(T) \) at a rate specified in the sequel. For \( V_{21,T}^B \),
\[
|V_{21,T}^B| \leq \frac{1}{(T h \phi(h))^2} \sum_{t_1,t_2=1}^T \left( \sum_{1 \leq |t_1-t_2| \leq \lambda_T} K_{1,h} \left( u - \frac{t_1}{T} \right) K_{1,h} \left( u - \frac{t_2}{T} \right) \right) \\
\times \left( E[\Delta_{t_1,T}(u,x)\Delta_{t_2,T}(u,x)] + E[\Delta_{t_1,T}(u,x)] E[\Delta_{t_2,T}(u,x)] \right) \\
\leq \frac{h^{2(1/\beta)}}{(T h \phi(h))^2} \sum_{t_1,t_2=1}^T \left( \sum_{1 \leq |t_1-t_2| \leq \lambda_T} K_{1,h} \left( u - \frac{t_1}{T} \right) K_{1,h} \left( u - \frac{t_2}{T} \right) (\psi(h) + \phi^2(h)) \right) \\
\leq \frac{h^{2(1/\beta)}(\psi(h) + \phi^2(h))}{(T h \phi(h))^2} T \lambda_T \leq \frac{1}{T h \phi(h)} \times h^{2(1/\beta)-1} \phi(h) \lambda_T.
\]
We shall subsequently select \( \lambda_T \) to make the right hand side of (A.2) tends to zero as \( T \to \infty \). By Davydov’s Lemma [Hall and Heyde (1980), Corollary A.2],
\[
\text{Cov}(\Delta_{t_1,T}(u,x), \Delta_{t_2,T}(u,x)) \\
\leq E[\Delta_{t_1,T}(u,x)^{\nu}]^{1/\nu} E[\Delta_{t_2,T}(u,x)^{\nu}]^{1/\nu} (\alpha(|t_1-t_2|))^{1-\frac{2}{\nu}} \\
\leq h^{2(1/\beta)} E[K_{2,h}(d(x,X_{t_1,T}))^\nu]^{1/\nu} E[K_{2,h}(d(x,X_{t_2,T}))^\nu]^{1/\nu} (\alpha(|t_1-t_2|))^{1-\frac{2}{\nu}} \\
\leq h^{2(1/\beta)} E[K_{2,h}(d(x,X_{t_1,T}))^2]^{1/\nu} E[K_{2,h}(d(x,X_{t_2,T}))^2]^{1/\nu} (\alpha(|t_1-t_2|))^{1-\frac{2}{\nu}} \\
\leq h^{2(1/\beta)} \phi^{2/\nu}(h)(\alpha(k))^{1-\frac{2}{\nu}}.
\]
For the third inequality, we used the boundedness of \( K_2 \). Then for \( V_{22,T}^B \),
\[
|V_{22,T}^B| \leq \frac{h^{2(1/\beta)} \phi^{2/\nu}(h)}{(T h \phi(h))^2} \sum_{t_1,t_2=1}^T \left( \sum_{1 \leq |t_1-t_2| > \lambda_T} (\alpha(|t_1-t_2|))^{1-\frac{2}{\nu}} \right) \\
\leq \frac{1}{T h \phi(h)} \times \frac{h^{2(1/\beta)-1}}{\lambda_T^{\delta}(\phi(h))^{1-\frac{2}{\nu}}} \sum_{k=\lambda_T+1}^\infty k^{\delta}(\alpha(k))^{1-\frac{2}{\nu}}.
\]
Now we select \( \lambda_T \) as \( \lambda_T = [(\phi(h))^{-1-\frac{2}{\nu}}]^{1/\delta} \). Then by Assumption 3.4,
\[
\text{Var}(\hat{g}^B(u,x)) \leq |V_{1,T}^B| + |V_{2,T}^B| = o \left( \frac{1}{T h \phi(h)} \right).
\]
This yields (A.1). From the argument in (Step1) of the proof of Theorem 3.1, we have \( E[\hat{g}^B(u,x)] = O(h^{2/\beta}) \), \( \hat{m}_1(u,x) = E[\hat{m}_1(u,x)] + o_p(1) \) and \( \lim_{T \to \infty} E[\hat{m}_1(u,x)] > 0 \). Therefore,
\[
\hat{m}(u,x) - m(u,x) = \frac{\hat{g}_1(u,x)}{\hat{m}_1(u,x)} + B_T(u,x) + o_p \left( \frac{1}{T h \phi(h)} \right).
\]
(Step 2) In this step, we will show

\[ Th\phi(h) \text{Var}(\tilde{g}_1(u,x)) \sim E[\varepsilon_1^2]\sigma^2(u,x) \int K_t^2(w)dw > 0 \text{ as } T \to \infty. \]

Define \( \tilde{g}_1(u,x) = \sqrt{Th\phi(h)}g_1(u,x) \). Observe that

\[
\text{Var}(\tilde{g}_1(u,x)) = \frac{1}{Th\phi(h)} \sum_{t=1}^{T} K_{t,h}^{2} \left( u - \frac{t}{T} \right) E \left[ K_{t,h}^2 (d(x,X_{t,T})) \varepsilon_t^2 \right]
\]
\[
= \frac{\sigma^2(u,x) + o(1)}{Th\phi(h)} \sum_{t=1}^{T} K_{t,h}^{2} \left( u - \frac{t}{T} \right) E \left[ K_{t,h}^2 (d(x,X_{t,T})) \varepsilon_t^2 \right]
\]
\[
= \frac{E[\varepsilon_1^2](\sigma^2(u,x) + o(1))}{Th\phi(h)} \sum_{t=1}^{T} K_{t,h}^{2} \left( u - \frac{t}{T} \right) E \left[ K_{t,h}^2 (d(x,X_{t,T})) \right].
\]

Since

\[
E \left[ K_{t,h}^2 (d(x,X_{t,T})) - K_{t,h}^2 \left( d(x,X_{t^{(t/T)})}) \right) \right] \leq E \left[ K_{t,h}^2 (d(x,X_{t,T})) - K_{t,h} \left( d(x,X_{t^{(t/T)})}) \right) \right]
\]
\[
\leq \frac{1}{Th} E[U_{t,T}'] = o(\phi(h)),
\]

we have

\[
\text{Var}(\tilde{g}_1(u,x)) = \frac{E[\varepsilon_1^2](\sigma^2(u,x) + o(1))}{Th\phi(h)} \sum_{t=1}^{T} K_{t,h}^{2} \left( u - \frac{t}{T} \right) E \left[ K_{t,h}^2 (d(x,X_{t^{(t/T)})}) \right]
\]
\[
+ \frac{E[\varepsilon_1^2]o(\phi(h))(\sigma^2(u,x) + o(1))}{Th\phi(h)} \sum_{t=1}^{T} K_{t,h}^{2} \left( u - \frac{t}{T} \right)
\]
\[
= \frac{E[\varepsilon_1^2](\sigma^2(u,x) + o(1))}{Th\phi(h)} \sum_{t=1}^{T} K_{t,h}^{2} \left( u - \frac{t}{T} \right) E \left[ K_{t,h}^2 (d(x,X_{t^{(t/T)})}) \right] + o(1).
\]

By the integration by parts and change of variables,

\[
E \left[ K_{t,h}^2 \left( d\left(x,X_{t^{(t/T)})}\right) \right) \right] = -\frac{2}{h} \int_{0}^{h} K_{t,h}(y)K_{t,h}'(y)F_{l/T}(y;x)dy
\]
\[
\sim -\frac{2}{h} \int_{0}^{h} K_{t,h}(y)K_{t,h}'(y)\phi(y)dy
\]
\[
= \frac{2}{h} \int_{0}^{h} \left( 1 - \frac{y}{h} \right) \phi(y)dy
\]
\[
= \frac{2}{h^2} \int_{0}^{h} \left( \int_{0}^{y} \phi(z)dz \right) dy
\]
\[
\sim \frac{2}{h^2} \int_{0}^{h} y\phi(y)dy \sim \frac{1}{h^2} h^2 \phi(h) \sim \phi(h).
\]
Therefore, we have

$$\text{Var}(\bar{g}_1(u, x)) \sim E[\varepsilon_1^2] \left( \frac{\sigma^2(u, x) + o(1)}{Th} \right) \sum_{t=1}^T K_{1,h}^2 \left( u - \frac{t}{T} \right)$$

$$\sim E[\varepsilon_1^2] \sigma^2(u, x) \int K_{1,h}^2(w) dw.$$  

(Step 3) Moreover, $\bar{g}_1(u, x)$ is asymptotically normal. In particular,

$$\bar{g}_1(u, x) \xrightarrow{d} N(0, V(u, x)) \text{ as } T \to \infty.$$  

We can show (A.3) by applying blocking arguments of Bernstein (1927) and Volkonskii and Rozanov inequality (cf. Proposition 2.6 in Fan and Yao (2003)). Assumption 3.6 implies that there exists a sequence of positive integers $\{q_T\}$ such that as $T \to \infty, q_T \to \infty,$

$$q_T v_T = o(\sqrt{Th\phi(h)}), \quad q_T \sqrt{\frac{T}{h\phi(h)}} \alpha(v_T) \to 0.$$  

Decompose $\bar{g}_1(u, x)$ into big-blocks and small-blocks as follows:

$$\bar{g}_1(u, x) = \frac{1}{\sqrt{Th\phi(h)}} \sum_{j=1}^{k_T} \xi_j(u, x) + \frac{1}{\sqrt{Th\phi(h)}} \sum_{j=1}^{k_T} \eta_j(u, x) + \zeta(u, x)$$

$$= \bar{g}_{11}(u, x) + \bar{g}_{12}(u, x) + \bar{g}_{13}(u, x),$$  

where

$$\xi_j(u, x) = \sum_{t=(j-1)s_T}^{j\ell_T+(j-1)s_T} K_{1,h}(u - \frac{t}{T}) K_{2,h}(d(x, X_{t,T})) \varepsilon_{t,T},$$

$$\eta_j(u, x) = \sum_{t=j\ell_T+(j-1)s_T+1}^{(j+1)\ell_T+(j-1)s_T} K_{1,h}(u - \frac{t}{T}) K_{2,h}(d(x, X_{t,T})) \varepsilon_{t,T},$$

$$\zeta(u, x) = \sum_{t=k_T(\ell_T+s_T)+1}^{T} K_{1,h}(u - \frac{t}{T}) K_{2,h}(d(x, X_{t,T})) \varepsilon_{t,T},$$

and where $\ell_T = [\sqrt{Th\phi(h)/q_T}], s_T = v_T, k_T = [T/(\ell_T+s_T)].$ We can neglect the sum of small blocks $\bar{g}_{12}(u, x)$ and $\bar{g}_{13}(u, x),$ and exploit the mixing conditions to replace the big blocks $\xi_j(u, x)$ by independent random variables. This allows us to apply a Lindeberg theorem to get the result. We omit the details as the proof is similar to that of Theorem 4 in Masry (2005). Combining (A.3) and the results in Steps 1 and 2, we obtain the conclusion.

**Appendix B. Technical Tools**

In this section we provide some lemmas used in the proofs of main results. The proofs of following Lemmas B.1 and B.2 are straightforward and thus omitted. Let $I_h = [C_1 h, 1 - C_1 h].$

**Lemma B.1.** Suppose that kernel $K_1$ satisfies Assumption 3.3 (KB1). Then for $k = 0, 1, 2,$

$$\sup_{u \in I_h} \left| \frac{1}{Th} \sum_{t=1}^T K_{1,h} \left( u - \frac{t}{T} \right) \left( \frac{u - t/T}{h} \right)^k - \int_0^{1/h} \frac{1}{h} K_{1,h} (u - v) \left( \frac{u - v}{h} \right)^k \ dv \right| = O \left( \frac{1}{Th^2} \right).$$


Lemma B.2. Suppose that kernel $K_1$ satisfies Assumption 3.3 (KB1) and let $g : [0, 1] \times \mathcal{H} \to \mathbb{R}$, $(u, x) \mapsto g(u, x)$ be continuously differentiable with respect to $u$. Then,

$$
\sup_{u \in I_h} \left| \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) g \left( \frac{t}{T}, x \right) - g(u, x) \right| = O \left( \frac{1}{Th^2} \right) + o(h).
$$

The following result is an exponential inequality for strongly mixing sequences given in Liebscher (1996).

Lemma B.3 (Theorem 2.1 in Liebscher (1996)). Let $\{Z_{t,T}\}$ be a zero-mean triangular array such that $|Z_{t,T}| \leq b_T$ with $\alpha$-mixing coefficients $\alpha(k)$. Then for any $\varepsilon > 0$ and $S_T \leq T$ with $\varepsilon > 4S_T b_T$,

$$
P \left( \sum_{t=1}^{T} Z_{t,T} \geq \varepsilon \right) \leq 4 \exp \left( -\frac{\varepsilon^2}{64\sigma_{S_T,T}^2 S_T^2} + \frac{4}{S_T} \alpha(S_T) \varepsilon b_T S_T \right) + 4 \frac{T}{S_T} \alpha(S_T)
$$

where $\sigma_{S_T,T}^2 = \sup_{0 \leq j \leq T-1} E \left[ \left( \sum_{t=j+1}^{T} Z_{t,T} \right)^2 \right]$.

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