A GENERALIZATION OF EULER TOTIENT FUNCTION AND TWO MENON-TYPE IDENTITIES

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Abstract. Menon's identity is a classical identity involving gcd sums and Euler totient function. In this paper, we propose a natural generalization of the Euler totient function and discuss various properties of this function. Using these properties, we derive the identity \( \sum_{m=1}^{n} \frac{(m-1,n)}{\gcd(m,n)} = \phi_s(n)\tau_s(n) \) where \( n, s \in \mathbb{N} \) which becomes the Menon’s identity when \( s = 1 \). Using the same techniques, we also offer an alternate approach to prove the Menon-type identity given by B. Sury in [Rend. Circ. Mat. Palermo 58, 99-108(2009)]. Towards the end of this paper, a generalization of the Sury’s identity in terms of the generalized Euler totient function is also given.

1. Introduction

The classical Euler totient function \( \phi \) has been generalized in various ways. The Jordan function \( J_s(n) \) and the von Sternetck’s function \( H_s(n) \) are two such important extensions (see definitions in the next section). Analogous to Jordan function, we here suggest a generalization of the Euler function based on the generalized gcd suggested by Cohen in [2]. It turns out that \( J_s \) and \( H_s \) are same. But there is a slight
difference in the generalization we propose here. The precise definition of our function is given in the next section.

The classical Menon’s identity which originally appeared in [6] is a gcd sum turning out to be equivalent to a product of the Euler function and the number of divisors function $\tau$. If $(m, n)$ denotes the gcd of $m$ and $n$, the identity is precisely the following:

$$\sum_{\substack{m=1 \\
\frac{n}{(m,n)}=1}}^{n} (m - 1, n) = \phi(n)\tau(n). \quad (1)$$

It has been generalized and extended by many authors. Many of the proofs are based on elementary number theory techniques. For example, in a recent paper, Zhao and Kao [12] suggested a generalization involving Dirichlet characters mod $n$ using elementary number theoretic methods. Their identity was

$$\sum_{\substack{m=1 \\
\frac{n}{(m,n)}=1}}^{n} (m - 1, n)\chi(m) = \phi(n)\tau \left(\frac{n}{d}\right),$$

where $\chi$ is a Dirichlet character mod $n$ with conductor $d$. When one takes $\chi$ as the principal character mod $n$, this identity turns to be equal to the Menon’s identity. After this, a similar type of identity in terms of even functions mod $n$ was given by L. Tóth [11]. An arithmetical function $f$ is $n$–even (or even mod $n$) if $f(r) = f((r, n))$. Tóth also used elementary number theory techniques and properties of arithmetical functions to prove his identity. A different approach was used by B. Sury in [9]. He used the method of group actions to prove the following identity.

$$\sum_{1 \leq m_1, m_2, \ldots, m_k \leq n \atop \frac{n}{(m_1, n)}=1} (m_1 - 1, m_2, \ldots, m_k, n) = \phi(n)\sigma_{k-1}(n) \quad (2)$$
where $\sigma_k(n) = \sum_{d|n} d^k$. Various other generalizations of the Menon’s identity were provided by many authors, see for example [3], [4], [7] and [10].

The natural question arising is if the gcd appearing in the Menon’s identity (1) is changed to the generalized gcd (which we define in the next section), what could be the possible change that can happen to this identity. We propose a very natural generalization of this identity towards the end of this paper using many of the properties we discuss about the generalization of the Euler function we develop. We also present an alternate proof of Menon-type identity (2) given by B. Sury. Further we generalize the identity (2) by replacing gcd with the generalized gcd function.

2. Notations and basic results

Most of the notations, functions, and identities we mention below are standard and can be found in [1]. $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ will denote the set of all natural numbers, integers, real numbers and complex numbers respectively. $\mu$ will be used to denote the Möbius function on $\mathbb{N}$. Note that for any given $k \in \mathbb{N}$, we have $\sum_{d|k} \mu(d) = 0$. If $[x]$ denotes the greatest integer $\leq x$, we have the identity $\sum_{d|n} \mu(n) = \left[ \frac{1}{n} \right]$. For two arithmetical functions $f$ and $g$, $f \ast g$ denotes their Dirichlet convolution (Dirichlet product). This product is commutative. We also recall the Möbius inversion [1, Theorem 2.9] which states that for two arithmetical functions $f, g$, $f(n) = \sum_{d|n} g(d) \iff g(n) = \sum_{d|n} f(d)\mu\left(\frac{n}{d}\right)$. For a finite set $A$, by $\#A$ we mean the number of elements in $A$.

For $m, n \in \mathbb{N}$, $(m, n)$ will denote the the gcd of $m$ and $n$. Generalizing this notion, for positive integer $s$, integers $a, b$, not both zero, the largest
Following Cohen \cite{2} we call this function on $\mathbb{N} \times \mathbb{N}$ as the generalized gcd function. When $s = 1$ this will be equal to the usual gcd function. Like the gcd function, $(a, b)_s = (b, a)_s$. $a \in \mathbb{N}$ is said to be $s$–power free or $s$–free if no $l^s$ where $l \in \mathbb{N}$ divides $a$.

**Definition 2.1.** For positive integers $s$ and $n$, define $\phi_s(n)$ to be the cardinality of the set $\{m \in \mathbb{N} : 1 \leq m \leq n, (m, n)_s = 1\}$.

For $s = 1$, the above function turns out to be the usual Euler totient function on $\mathbb{N}$. The Jordan totient function $J_s(n)$ defined for positive integers $s$ and $n$ is the number of ordered sets of $s$ elements from a complete residue system (mod $n$) such that the greatest common divisor of each set is prime to $n$ \cite{5} pp 95-97. By $H_s$ we mean the function $H_s(n) = \sum_{n=[d_1, d_2, \ldots, d_s]} \phi(d_1)\phi(d_2) \cdots \phi(d_s)$, where the summation ranges over all ordered set of $s$ positive integers $d_1, d_2, \ldots, d_s$ with least common multiple equal to $n$. Note that $[d_1, d_2]$ denotes the lcm of these two integers.

The values of $J_s(n)$ and $\phi_s(n^s)$ are the same by \cite{2} Theorem 5. However, one should not be confused with the notation $\phi_s$ we use here and the one appearing in \cite{2} as in \cite{2}, $\phi_s$ was used to denote $J_s$.

Now we have an obvious result. Though we feel that a proof of the following might be available somewhere else, since we couldn’t locate it, we give a short justification here itself.

**Lemma 2.2.** $(a, b)_s$ is multiplicative in the first variable.

**Proof.** For coprime positive integers $a$ and $c$ write $(ac, b)_s = l^s$. If the prime factorization of $l$ is $p_1^{r_1}p_2^{r_2} \cdots p_n^{r_n}$, then $l^s = p_1^{sr_1}p_2^{sr_2} \cdots p_n^{sr_n}$. Now $l^s | ac$ and $l^s | b$. Since $(a, c) = 1$, Some of these prime powers
must appear in the prime factorization of $a$ and the rest in $c$. We may assume that $p_1^{sr_1}, p_2^{sr_2}, \ldots, p_t^{sr_t}$ are in the prime factorization of $a$ and the remaining primes $P_t^{sr_{t+1}}, \ldots, P_n^{sr_n}$ are in $c$. Then $(a, b)_s = p_1^{sr_1}p_2^{sr_2}\cdots p_t^{sr_t}$ and $(c, b)_s = p_t^{sr_{t+1}}\cdots p_n^{sr_n}$.

Note that $(a, b)_s = (b, a)_s$ so that $(a, b)_s$ is multiplicative in the second variable also. But $(a, b)_s$ is not completely multiplicative in first (or second) variable. For example, if we take $a = p^{s-1}, b = p^s, c = p$, we have $(ac, b)_s = p^s$ and $(a, b)_s(c, b)_s = 1$.

Also the function $(a, b)_s$ is not multiplicative in $s$. For example, take $s_1 = 2$ and $s_2 = 3$ and choose $a = 24$ and $b = 36$. Then

$$(a, b)_{s_1s_2} = 1,$$

$$(a, b)_{s_1}(a, b)_{s_1} = 4.$$ 

By $\tau_s(n)$ where $s, n \in \mathbb{N}$, we mean the number of $l^s$ with $l \in \mathbb{N}$ dividing $n$. The function $\tau_s(n)$ is multiplicative in $n$, because for $(m, n) = 1$, $\tau_s(mn) = \sum_{d^s|m} 1 = \sum_{d_1^s|m} \sum_{d_2^s|n} 1 = \tau_s(m)\tau_s(n)$. But $\tau_s(n)$ is not completely multiplicative, because for $m = p_1^sp_2$ and $n = p_2^{s-1}p_3^s$, $\tau_s(mn) \neq \tau_s(m)\tau_s(n)$.

**Definition 2.3.** The generalized divisor function $\sigma_{k,s}(n)$ is defined as the $k^{th}$ power sum of the $s^{th}$ power divisors of $n$, thus $\sigma_{k,s}(n) = \sum_{d^s|n} (d^n)^k$.

Note that $\sigma_{k,s}(n) \neq \sigma_{ks}(n)$.

The principle of cross-classification [1, Theorem 5.31] is about counting number of elements in certain sets. Since we use it in our proofs, we state it below.
Theorem 2.4. If $A_1, A_2, \ldots, A_n$ are given subsets of a finite set $A$, then

$$\#(A - \bigcup_{i=1}^{n} A_i) = \#A - \sum_{1 \leq i \leq n} \#A_i + \sum_{1 \leq i < j \leq n} \#(A_i \cap A_j) - \sum_{1 \leq i < j < k \leq n} \#(A_i \cap A_j \cap A_k) + \ldots + (-1)^n \#(A_1 \cap A_2 \cap \ldots \cap A_n).$$

3. Main Results

We state below the main results we prove in this paper. The proofs of these results are included in the next section. Our initial results generalize certain properties enjoyed by the Euler totient function $\phi$ giving analogous properties of $\phi_s$. First of them is a relation $\phi$ enjoys with the Möbius function which is precisely $\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$. $\phi_s$ holds a similar relation with the Möbius function.

**Theorem 3.1.** For $n, s \in \mathbb{N}$ we have

$$\phi_s(n) = \sum_{d^n|n} \mu(d) \frac{n}{ds}.$$  

Similar to the totient function $\phi$, $\phi_s$ also has a product formula in terms of prime divisors of $n$.

**Theorem 3.2** (Product formula for $\phi_s$). For $n, s \in \mathbb{N}$

$$\phi_s(n) = n \prod_{p \mid n \text{ prime}} \left(1 - \frac{1}{p^s}\right)$$

where by convention, empty product is taken to be equal to 1.

Note that the above result is different from [2, Theorem 3].

As a consequence of the principle of cross-classification, we prove the following.
Theorem 3.3. Let \( n, d, s, r \in \mathbb{N}, d^s | n \). Let \((r, d^s)_s = 1\). Number of elements in \( A = \{ r + td^s : t = 1, 2, \ldots, \frac{n}{d^s} \} \) such that \((r + td^s, n)_s = 1\) is \( \frac{\phi(n)}{\phi_s(d^s)} \).

We next suggest a generalization to the original version of Menon’s identity [6] in terms of our generalized gcd function.

Theorem 3.4 (Generalization of Menon’s Identity). For \( n, s \in \mathbb{N} \),

\[ \sum_{m=1}^{n} (m - 1, n)_s = \phi_s(n)\tau_s(n). \]

B. Sury in [9] proved the following identity.

Theorem 3.5 (Menon-Sury identity). Let \( m_1, m_2, \cdots m_k, n \in \mathbb{N} \). Then

\[ \sum_{1 \leq m_1, m_2, \cdots , m_k \leq n_{\text{(m,n)=1}}} (m_1 - 1, m_2, \cdots, m_k, n) = \phi(n)\sigma_{k-1}(n), \text{ where } \phi \text{ is the Euler totient function and } \sigma_k(n) = \sum_{d|n} d^k. \]

We give an alternate proof for the above identity and propose the following generalization.

Theorem 3.6 (Generalization of Menon-Sury identity). Let \( m_1, m_2, \cdots m_k, n, s \in \mathbb{N} \). Then

\[ \sum_{1 \leq m_1, m_2, \cdots , m_k \leq n^s_{\text{(m,n)^s)=1}}} (m_1 - 1, m_2, \cdots m_k, n^s)_s = \phi_s(n^s)\sigma_{k-1,s}(n^s), \]

where \( \phi_s \) is the generalized Euler totient function and \( \sigma_{k-1,s} \) is the generalized divisor function.

4. Proofs of Results

To prove theorem 3.1, we require the following two lemmas.

Lemma 4.1. For \( n, s \in \mathbb{N} \), \( \phi_s(n) = \sum_{m=1}^{n} \left[ \frac{1}{(m,n)_s} \right]. \)
Proof. Since

\[
\left[ \frac{1}{(m, n)_s} \right] = \begin{cases} 
1 & \text{if } (m, n)_s = 1 \\
0 & \text{otherwise}
\end{cases}
\]

the lemma quickly follows from the definition of \( \phi_s \). \( \square \)

Lemma 4.2. For \( m, n, s \in \mathbb{N} \)

\[
\sum_{d^s|(m,n)_s} \mu(d) = \sum_{d|(m,n)_s} \mu(d).
\]

Proof. If \( (m, n)_s = 1 \), this is obvious. So assume \( (m, n)_s = l^s > 1 \).
Write the prime factorization of \( l \) as \( l = p_1^{r_1}p_2^{r_2}\cdots p_t^{r_t} \). Then

\[
\sum_{d^s|l^s} \mu(d) = \mu(1) + \mu(p_1) + \cdots + \mu(p_t)
\]

\[+ \mu(p_1p_2) + \cdots + \mu(p_{t-1}p_t) + \cdots + \mu(p_1p_2\cdots p_t)\]

\[= \sum_{d|l^s} \mu(d).\]

\( \square \)

We now have enough tools to prove theorem 3.1.

Proof of theorem 3.1. The identity \( \sum_{d|n} \mu(d) = \left[ \frac{1}{n} \right] \) and the above lemma gives

\[
\sum_{d^s|(m,n)_s} \mu(d) = \sum_{d|(m,n)_s} \mu(d) = \left[ \frac{1}{(m, n)_s} \right].
\]
Using it with lemma 3.1 we get

\[ \phi_s(n) = \sum_{m=1}^{n} \sum_{d^s | (m,n)} \mu(d) \]

\[ = \sum_{m=1}^{n} \sum_{d^s | m} \mu(d) \]

\[ = \sum_{d^s | n} \mu(d) \sum_{q=1}^{n/d^s} 1 \]

\[ = \sum_{d^s | n} \mu(d) \frac{n}{d^s}. \]

□

Next we prove theorem 3.2, the product formula for \( \phi_s \).

**Proof of theorem 3.2.** If \( n \) is an \( s \)-power free integer, then \( \phi_s(n) = n \).

So we move to the case when \( n \) is not an \( s \)-power free integer. Let \( p_1, \ldots, p_t \) be the primes such that \( p_i^s \) divides \( n \). Then

\[ \phi_s(n) = \sum_{d^s | n} \mu(d) \frac{n}{d^s} \]

\[ = n \sum_{d^s | n} \frac{\mu(d)}{d^s} \]

\[ = n \left( \frac{1}{1^s} + \frac{\mu(p_1^s)}{p_1^s} + \frac{\mu(p_2^s)}{p_2^s} + \ldots + \frac{\mu(p_1^s p_2^s)}{(p_1 p_2)^s} + \ldots + \frac{\mu(p_1^s p_2^s \ldots p_t^s)}{(p_1 p_2 \ldots p_t)^s} \right) \]

\[ = n \left( 1 - \sum \frac{1}{p_i^s} + \sum \frac{1}{(p_i p_j)^s} + \ldots + \frac{(-1)^t}{(p_1 p_2 \ldots p_t)^s} \right) \]

\[ = n \left( (1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_t}) \right) \]

\[ = n \prod_{\substack{p | n \\text{ prime}}} \left( 1 - \frac{1}{p^s} \right). \]

□
When $n$ is a prime power, we have the following result.

**Corollary 4.3.** \( \phi_s(p^a) = \begin{cases} p^a - p^{a-s} & \text{if } a \geq s \\ p^a & \text{otherwise.} \end{cases} \)

Since \( \phi_s \) is a generalization of the Euler totient function \( \phi \), it would be interesting to check which properties of the \( \phi \) function can be extended to \( \phi_s \). The above theorem helps us to list out the following observations.

1. \( \phi_s(n) \) is multiplicative in \( n \), for if \( (a, b) = 1 \) then

\[
\phi_s(ab) = ab \prod_{p^s \mid ab} (1 - \frac{1}{p^s}) = ab \prod_{p_1 \mid a, p_1 \text{ prime}} (1 - \frac{1}{p_1^s}) \prod_{p_2 \mid b, p_2 \text{ prime}} (1 - \frac{1}{p_2^s}) = \phi_s(a)\phi_s(b).
\]

2. \( \phi_s(n) \) is not completely multiplicative in \( n \). This is because

\[
\phi_s(ab) = ab \prod_{p^s \mid ab} (1 - \frac{1}{p^s}) = ab \prod_{p \mid a, p \text{ prime}} (1 - \frac{1}{p^s}) \prod_{p \mid b, p \text{ prime}} (1 - \frac{1}{p^s}) \prod_{p \mid (a, b)_s} (1 - \frac{1}{p^s}) = \phi_s(a)\phi_s(b) \frac{\delta^s}{\phi_s(\delta^s)} \prod_{p \mid (a, p^a p^b) \mid ab } (1 - \frac{1}{p^s}), \text{ where } \delta^s = (a, b)_s.
\]

The right handside of the above fails to be equal to \( \phi_s(a)\phi_s(b) \) when, for example \( a = p^{s-1}, b = p \).

3. If \( a \) divides \( b \) and \( (a, \frac{b}{a}) = 1 \), then \( \phi_s(a) \) divides \( \phi_s(b) \). It follows from \( \phi_s(b) = \phi_s(a\frac{b}{a}) = \phi_s(a)\phi_s(\frac{b}{a}) \).
(4) For a prime $p$, $\phi_s(p) = p$. So $\phi_s(n)$ need not be even where as $\phi(n)$ is even for $n > 2$.

(5) If $2^{s+1}$ divides $n$ or $2^s \mid n$ and $2^s \nmid n$, then $\phi_s(n)$ is even.

(6) If $p$ is an odd prime such that $p^s$ divides $n$, then $\phi_s(n)$ is even.

(7) If $n = 2^s a$, where $a$ is odd and $a$ is $s$–free, then $\phi_s$ is odd.

We now prove theorem 3.3, which is essential in the proof of the generalized version of Menon’s identity (theorem 3.4).

**Proof of theorem 3.3.** This result is a generalization of Theorem 5.32 appearing in [1]. We use the same techniques used there to justify our claim.

We have to find the number of elements $r + td^a$ such that $(n, r + td^a)_s = 1$. Hence we need to remove elements from $A$ that have $(r + td^a, n)_s > 1$. If for an element $r + td^a$ of $A$, $p^s | n$ and $p^s | r + td^a$, then since $(r, n)_s = 1$, $p^s \nmid d^a$. Therefore the number we require is the number of elements in $A$ with $p^s | n$ and $p^s \nmid d^a$ for some prime $p$. Let these primes be $p_1, p_2, \ldots, p_m$. Write $l = p_1^s p_2^s \cdots p_m^s$. Let $A_i = \{ x : x \in A$ and $p_i^s | x \}$, $i = 1, 2, \cdots, m$. If $x \in A_i$ and $x = r + td^a$, then $r + td^a \equiv 0 \pmod{p_i^s}$. This means that $td^a \equiv -r \pmod{p_i^s}$. Since $p_i^s \nmid d^a$ (which is, if and only if $p_i \nmid d$), there is a unique $t \mod{p_i^s}$ satisfying this congruence equation. Therefore there exists exactly one $t$ in each of the intervals $[1, p_i^s], [p_i^s + 1, 2p_i^s], \ldots, [(q - 1)p_i^s + 1, qp_i^s]$ where $qp_i^s = \frac{n}{d^a}$. Therefore, $\#(A_i) = q = \frac{n/d^a}{p_i^s}$. Similarly, $\#(A_i \cap A_j) = \frac{n/d^a}{p_i^s p_j^s}$, $\#(A_i \cap A_j \cap A_k) = \frac{n/d^a}{p_i^s p_j^s p_k^s}$.
Hence by cross classification principle, the number of elements we seek is equal to

\[
\#(A - \bigcup_{i=1}^{m} A_i) = \#(A) - \sum_{i=1}^{m} \#(A_i) + \sum_{1 \leq i < j \leq m} \#(A_i \cap A_j) - \cdots \\
+ (-1)^m \#(A_1 \cap A_2 \cap \cdots \cap A_m)
\]

\[
= \frac{n}{d^s} - \sum \frac{n/d^s}{p_i^s} + \sum \frac{n/d^s}{p_i^s p_j^s} + \cdots + (-1)^m \frac{n/d^s}{p_1^s p_2^s \cdots p_m^s}
\]

\[
= \frac{n}{d^s} \left(1 - \sum \frac{1}{p_i^s} + \sum \frac{1}{p_i^s p_j^s} + \cdots + \frac{(-1)^m}{p_1^s p_2^s \cdots p_m^s}\right)
\]

\[
= \frac{n}{d^s} (1 - \frac{1}{p_1^s})(1 - \frac{1}{p_2^s}) \cdots (1 - \frac{1}{p_m^s})
\]

\[
= \frac{n}{d^s} \prod_{p^s \mid n} (1 - \frac{1}{p^s})
\]

\[
= \frac{n}{d^s} \prod_{p^s \mid n} (1 - \frac{1}{p^s})
\]

\[
= \frac{\phi_s(n)}{\phi_s(d^s)}.
\]

□

We now prove the identity given in theorem 3.4

Proof of theorem 3.4. It is known that [8, Section V.3] \( n^s = \sum_{d \mid n} J_s(d) \).
Recall that \( J_s(n) = \phi_s(n^s) \). Therefore \( n^s = \sum_{d \mid n} \phi_s(d^s) = \sum_{d^s \mid n^s} \phi_s(d^s) \).
Now \((m, n)\) is an \(s\)th power of some integer. So

\[
\sum_{m=1}^{n} (m - 1, n)_s = \sum_{m=1}^{n} \sum_{d^s|(m-1, n)_s} \phi_s(d^s)
\]

\[
= \sum_{m=1}^{n} \sum_{d^s|n} \phi_s(d^s)
\]

\[
= \sum_{d^s|n} \phi_s(d^s) \sum_{m=1}^{n} 1
\]

\[
= \sum_{d^s|n} \phi_s(d^s) \frac{\phi_s(n)}{\phi_s(d^s)} \quad \text{(using theorem 3.3)}
\]

\[
= \sum_{d^s|n} \phi_s(n)
\]

\[
= \phi_s(n) \sum_{d^s|n} 1
\]

\[
= \phi_s(n) \tau_s(n).
\]

Now we proceed to give an alternate proof to Menon-type identity given in theorem (2) due to B. Sury. Here we use some elementary number theoretic techniques that we employed in the above proof. Note that the method of group action was used by B. Sury in [9] to prove this identity.
Proof of theorem 3.5.

\[\sum_{1 \leq m_1, m_2, \cdots, m_k \leq n \atop (m_1, n) = 1} (m_1 - 1, m_2, \cdots, m_k, n) = \sum_{m_1 = 1}^{n} \ldots \sum_{m_k = 1}^{n} (m_1 - 1, m_2, \cdots, m_k, n)\]

\[= \sum_{m_1 = 1}^{n} \ldots \sum_{m_k = 1}^{n} \sum_{d \mid (m_1 - 1, m_2, \cdots, m_k, n)} \phi(d)\]

\[= \sum_{d \mid n} \phi(d) \sum_{m_1 = 1}^{n} \ldots \sum_{m_k = 1}^{n} \frac{\phi(n)}{d} \phi(d) \quad \text{(using [1, Theorem 5.32])}\]

\[= \sum_{d \mid n} \phi(n) \sum_{m_2 = 1}^{n} \ldots \sum_{m_k = 1}^{n} \frac{1}{d}\]

\[= \phi(n) \sum_{d \mid n} \frac{(\frac{n}{d})^{k-1}}{d}\]

\[= \phi(n) \sum_{d \mid n} (d)^{k-1}\]

\[= \phi(n) \sigma_{k-1}(n),\]

which completes the proof. \(\square\)

Now we prove theorem 3.6, which is the generalization of the above identity. The proof is similar to the one given above.
Proof of theorem 3.6. Consider the sum

\[
\sum_{1 \leq m_1, m_2, \ldots, m_k \leq n^s, \ (m_1, n^s) = 1} (m_1 - 1, m_2, \ldots, m_k, n^s)_s = \sum_{m_1 = 1}^{n^s} \sum_{m_2 = 1}^{n^s} \cdots \sum_{m_k = 1}^{n^s} (m_1 - 1, m_2, \ldots, m_k, n^s)_s
\]

\[
= \sum_{m_1 = 1}^{n^s} \sum_{m_2 = 1}^{n^s} \cdots \sum_{m_k = 1}^{n^s} \sum_{d \mid (m_1 - 1, m_2, \ldots, m_k, n^s)} \phi_s(d^s)
\]

\[
= \sum_{m_1 = 1}^{n^s} \sum_{m_2 = 1}^{n^s} \cdots \sum_{m_k = 1}^{n^s} \sum_{d \mid m_1 - 1, d \mid m_2, \ldots, d \mid m_k, d \mid n^s} \phi_s(d^s)
\]

\[
= \sum_{d^s \mid n^s} \phi_s(d^s) \sum_{m_2 = 1}^{n^s} \sum_{m_k = 1}^{n^s} \phi_s(n^s) (d^s) \phi_s(n^s) (d^s)
\]

\[
= \sum_{d^s \mid n^s} \phi_s(n^s) \sum_{m_2 = 1}^{n^s} \sum_{m_k = 1}^{n^s} 1
\]

\[
= \phi_s(n^s) \sum_{d^s \mid n^s} \frac{n^s}{d^s} (d^s)^{k-1}
\]

\[
= \phi_s(n^s) \sum_{d^s \mid n^s} (d^s)^{k-1}
\]

\[
= \phi_s(n^s) \sigma_{k-1, s}(n^s).
\]

This completes the proof. \[\square\]

5. Further directions

Since we feel that this is the first time Menon’s identity is revisited through the generalized gcd concept, it would be interesting to see what possible results can be obtained if one tries to apply our techniques to
other generalizations of the identity. In particular, we would like to investigate in the future how does the identity of Zhao and Cao change if one uses the generalized gcd, $\phi_s$ and $\tau_s$.

6. Acknowledgements

The first author thanks the University Grants Commission of India for providing financial support for carrying out research work through their Junior Research Fellowship (JRF) scheme. The second author thanks the Kerala State Council for Science, Technology and Environment, Thiruvananthapuram, Kerala, India for providing financial support for carrying out research work.

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