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ABSTRACT. We prove global Sobolev regularity and pointwise upper bounds for the gradient of transition densities associated with second order differential operators in $\mathbb{R}^d$ with unbounded diffusion, drift and potential terms.

1. Introduction

In this article, we are concerned with elliptic operators of the form

$$A\varphi = \text{div}(Q \nabla \varphi) + F \cdot \nabla \varphi - V \varphi, \quad \varphi \in C^2(\mathbb{R}^d),$$

where the diffusion coefficients $Q$, the drift $F$ and the potential $V$ are typically unbounded functions. Moreover, we write $A_0$ for the operator $A + V$. Throughout, we make the following assumptions on $Q$, $F$ and $V$.

**Hypothesis 1.1.** We have $Q = (q_{ij})_{i,j=1,...,d} \in C^{1+\zeta}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^{d\times d})$, $F = (F_j)_{j=1,...,d} \in C^{1+\zeta}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ and $0 \leq V \in C^1_{\text{loc}}(\mathbb{R}^d)$ for some $\zeta \in (0,1)$. Moreover,

(a) The matrix $Q$ is symmetric and uniformly elliptic, i.e. there is $\eta > 0$ such that

$$\sum_{i,j=1}^d q_{ij}(x)\xi_i \xi_j \geq \eta |\xi|^2 \quad \text{for all } x, \xi \in \mathbb{R}^d;$$

(b) there is $0 \leq Z \in C^2(\mathbb{R}^d)$ and a constant $M \geq 0$ such that $\lim_{|x| \to \infty} Z(x) = \infty$, $AZ(x) \leq M$ and $\eta \Delta Z(x) + F \cdot \nabla Z(x) - V(x)Z(x) \leq M$ for all $x \in \mathbb{R}^d$.

(c) there is $0 \leq Z_0 \in C^2(\mathbb{R}^d)$ and a constant $M \geq 0$ such that $\lim_{|x| \to \infty} Z_0(x) = \infty$, $A_0 Z_0(x) \leq M$ and $\eta \Delta Z_0(x) + F \cdot \nabla Z_0(x) \leq M$ for all $x \in \mathbb{R}^d$.

We immediately see that (c) implies (b), but in the applications will be useful to distinguish between $Z$ and $Z_0$.

It is well known (see [10], Theorem 2.2.5] and [12]) that, assuming Hypothesis 1.1, a suitable realization of the above operator $A$ generates a (typically not strongly continuous) semigroup $T = (T(t))_{t \geq 0}$ on the space $C_b(\mathbb{R}^d)$ that is given through an integral kernel $p$, i.e.

$$T(t)f(x) = \int_{\mathbb{R}^d} p(t,x,y)f(y) \, dy, \quad t > 0, \, x \in \mathbb{R}^d, \, f \in C_b(\mathbb{R}^d),$$

where the kernel $p$ is positive, $p(t, \cdot, \cdot)$ and $p(t, x, \cdot)$ are measurable for any $t > 0$, $x \in \mathbb{R}^d$, and for a.e. fixed $y \in \mathbb{R}^d$, $p(\cdot, \cdot, y) \in C^{1+\frac{d+2+\zeta}{2}}_{\text{loc}}((0, \infty) \times \mathbb{R}^d)$.

An important aspect in the study of elliptic operators is to have estimates for the kernel $p$ and, consequently, this question has received a lot of attention in the literature. We mention here [3], [5], [6], [7], [8], [9], [10], [11], where specific operators were considered. We refer also to the manuscript [12] and the references therein. There is also a general approach to establish estimates for $p$ making use of so called Lyapunov functions, initiated in [10] and later refined in [11]. We point out that
these results are assuming the diffusion coefficients $Q$ to be uniformly bounded. However, using an approximation procedure, these results were subsequently extended to unbounded diffusion coefficients, see [7, 8].

In this article, we are concerned to establish not only estimates for $p$ but also for $\nabla p$, the gradient of $p$. An important tool to obtain such estimates is the square integrability of the logarithmic gradient of $p$. Such integrability property plays an important role to obtain regularity results for $p$, cf. [5, Section 7.4]. Moreover, as in [11, Thm. 5.3], once estimates for $\nabla p$ are obtained, one can repeat the same procedure to get estimates for $D^2 p$ and hence estimates for $\partial_t p$. This allows us to obtain the differentiability of the semigroup $T(t)$. Estimates for the gradient of $p$ were obtained in [11] in the case of bounded diffusion coefficients. As in [7, 8], we use approximation to extend this to unbounded diffusion coefficients. We point out that the constant in the estimate for $\nabla p$ obtained in [11, Thm. 5.3] depend on $\|Q\|_\infty$ and thus this estimate cannot be used in an approximation result. Therefore, we establish these estimates in a different way. Theorem 3.7 which provides an estimate of approximating them with bounded ones (see Section 3.1). In this way, we can prove our main result where the constant in the estimate does not depend on $\|Q\|_\infty$.

With this estimate at hand, we can then tackle the case of unbounded diffusion coefficients by approximating them with bounded ones (see Section 3.2). In this way, we can prove our main result Theorem 3.7 which provides an estimate of $\nabla p$ in the general case. We illustrate our results by applying them to the prototype operator

$$\text{div}((1 + |x|^\alpha)\nabla u) - |x|^{p-1}x \cdot \nabla u - |x|^\alpha,$$

where $x \mapsto |x|_\alpha$ is a $C^2$-function satisfying $|x|_\alpha = |x|$ for $|x| \geq 1$.

**Notation.** $B_r$ denotes the open ball of $\mathbb{R}^d$ of radius $r$ and center 0. For $0 \leq a < b$, we write $Q(a, b)$ for $(a, b) \times \mathbb{R}^d$.

If $u : J \times \mathbb{R}^d \to \mathbb{R}$, where $J \subset [0, \infty)$ is an interval, we use the following notation:

$$\partial_t u = \frac{\partial u}{\partial t}, \quad D_i u = \frac{\partial u}{\partial x_i}, \quad D_{ij} u = D_i D_j u,$$

$$\nabla u = (D_1 u, \ldots, D_d u), \quad \text{div}(F) = \sum_{i=1}^d D_i F_i \text{ for } F : \mathbb{R}^d \to \mathbb{R}^d,$$

and

$$|\nabla u|^2 = \sum_{j=1}^d |D_j u|^2, \quad |D^2 u|^2 = \sum_{i,j=1}^d |D_{ij} u|^2.$$
We shall also use parabolic Sobolev spaces. We denote by $W^{1,2}_k(Q(a,b))$ the space of functions $u \in L^k(Q(a,b))$ having weak space derivatives $D^2 u \in L^k(Q(a,b))$ for $|\alpha| \leq 2$ and weak time derivative $\partial_t u \in L^k(Q(a,b))$ equipped with the norm

$$
\|u\|_{W^{1,2}_k(Q(a,b))} := \|u\|_{L^k(Q(a,b))} + \|\partial_t u\|_{L^k(Q(a,b))} + \sum_{1 \leq |\alpha| \leq 2} \|D^\alpha u\|_{L^k(Q(a,b))}.
$$

Let $\mathcal{H}^{k,1}(Q(a,b))$ denote a space of all functions $u \in W^{0,1}_k(Q(a,b))$ with $\partial_t u \in (W^{0,1}_k(Q(a,b)))'$, the dual space of $W^{0,1}_k(Q(a,b))$, endowed with the norm

$$
\|u\|_{\mathcal{H}^{k,1}(Q(a,b))} := \|\partial_t u\|_{(W^{0,1}_k(Q(a,b)))'} + \|u\|_{W^{0,1}_k(Q(a,b))},
$$

where $1/k + 1/k' = 1$.

## 2. Results for bounded diffusion coefficients

Throughout this section we assume that the coefficients $q_{ij}$ and their spatial derivatives $\partial_k q_{ij}$ are bounded on $\mathbb{R}^d$ for all $i, j, k = 1, \ldots, d$.

As in [1, 14] and [5], we introduce time dependent Lyapunov functions for $L := \partial_t + A$.

**Definition 2.1.** We say that a function $W : [0, T] \times \mathbb{R}^d \to [0, \infty)$ is a time dependent Lyapunov function for $L$ if $W \in C^{1,2}([0, T] \times \mathbb{R}^d) \cap C([0, T] \times \mathbb{R}^d)$ such that $\lim_{|x| \to \infty} W(t, x) = \infty$ uniformly for $t$ in compact subsets of $(0, T)$, $W \leq Z$ and there is $h \in [0, T] \to [0, \infty)$ integrable near 0 such that

$$
(2.1) \quad LW(t, x) \leq h(t)W(t, x)
$$

and

$$
(2.2) \quad \partial_t W(t, x) + \eta \Delta W(t, x) + F(x) \cdot \nabla W(t, x) - V(x)W(t, x) \leq h(t)W(t, x)
$$

for all $(t, x) \in (0, T) \times \mathbb{R}^d$. To emphasize the dependence on $Z$ and $h$, we also say that $W$ is a time dependent Lyapunov function for $L$ with respect to $Z$ and $h$.

Moreover, given a time dependent Lyapunov function $W$ for $L$, we define the function

$$
\xi_W(t, x) := \int_{\mathbb{R}^d} p(t, x, y)W(t, y)\, dy.
$$

This is finite due to the following result which is true also for possibly unbounded diffusion coefficients.

**Proposition 2.2.** If $W$ is a time dependent Lyapunov function for $L$ with respect to $h$, then for $\xi_W(t, x) := \int_{\mathbb{R}^d} p(t, x, y)W(t, y)\, dy$, we have

$$
\xi_W(t, x) \leq e^{\int_t^0 h(s)\, ds}W(0, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d.
$$

**Proof.** The proof is similar to the one given in [7, Proposition 12.1].

Making use of time dependent Lyapunov functions, we start by establishing pointwise upper bounds for the kernel $p$. The following result can be deduced as in [7, Theorem 12.4] and [5, Theorem 4.2].

**Theorem 2.3.** Fix $T > 0$, $x \in \mathbb{R}^d$ and $0 < a_0 < a < b < b_0 < T$. Let us consider two time dependent Lyapunov functions $1 \leq W_1$, $W_2$ with $W_1 \leq W_2$ and a weight function $1 \leq w \in C^{1,2}((0, T) \times \mathbb{R}^d)$ such that

(a) the functions $w^{-2}\partial_t w$ and $w^{-2}\nabla w$ are bounded on $Q(a_0, b_0)$;

(b) there exist $k > d+2$ and constants $c_1, \ldots, c_6 \geq 1$, possibly depending on the interval $(a_0, b_0)$, with

(i) $w \leq c_1 w^{-\frac{k-2}{k-1}}W_1^{\frac{2}{k-1}}$,

(ii) $|Q\nabla w| \leq c_2 w^{-\frac{k-1}{k-2}}W_1^{\frac{1}{k-1}}$,

(iii) $|\text{div}(Q\nabla w)| \leq c_3 w^{-\frac{k-2}{k-1}}W_1^{\frac{2}{k-1}}$,

(iv) $|\partial_t w| \leq c_4 w^{-\frac{k-2}{k-1}}W_1^{\frac{2}{k-1}}$,

(v) $V^\frac{2}{k} \leq c_5 w^{-\frac{k-1}{k-2}}W_2^{\frac{1}{k-1}}$,

(vi) $|F| \leq c_6 w^{-\frac{k-2}{k-1}}W_2^{\frac{2}{k-1}}$,

on $[a_0, b_0] \times \mathbb{R}^d$. 

Then there is a constant \( C > 0 \) depending only on \( d, k \) and \( \eta \) such that

\[
  w(t, y)p(t, x, y) \leq C \left( c^k_1 \sup_{t \in (a_0, b_0)} \xi_{W_1}(t, x) + \left( c^k_2 + \frac{c^k_4}{(b_0 - b)^{\frac{k}{2}}} + c^k_3 \right) \int_{a_0}^{b_0} \xi_{W_1}(t, x) \, dt \right.
  \left. + \left( c^k_5 + c^k_6 + c^k_2 c^k_6 \right) \int_{a_0}^{b_0} \xi_{W_2}(t, x) \, dt \right),
\]

(2.3)

for all \((t, y) \in (a, b) \times \mathbb{R}^d\) and any fixed \( x \in \mathbb{R}^d \).

**Remark 2.4.** If one assumes \(|Q\nabla w| \leq c_2 W^{\frac{k}{2}}, |QD^2w| \leq c_3 W^{\frac{k}{2}}\) and \(|\nabla Q| \leq c_7 w^{-\frac{k}{2}} W^{\frac{k}{2}}\), for some positive constants \( c_2, c_3, c_7 \), then, since \( w \geq 1 \), we have

\[
  |\text{div}(Q\nabla w)| \leq d (|\nabla Q| |\nabla w| + |QD^2w|) \\
  \leq d \left( c_2 c_7 \eta^{-1} w^{-\frac{k}{2}} W^{\frac{k}{2}} + c_3 W^{\frac{k}{2}} \right) \\
  \leq d \left( c_2 c_7 \eta^{-1} + c_3 \right) w^{-\frac{k}{2}} W^{\frac{k}{2}}.
\]

(2.4)

So, the assumption \((iii)\) of the above theorem is satisfied with \( c_3 = d (c_2 c_7 \eta^{-1} + c_3) \), since \( 1 \leq W_1 \).

For further purposes, we obtain from the above remark the following corollary.

**Corollary 2.5.** Assume all the assumptions of Theorem 2.3 except \((ii)\) and \((iii)\). If \(|Q\nabla w| \leq c_2 W^{\frac{k}{2}}, |QD^2w| \leq c_3 W^{\frac{k}{2}}\) and \(|\nabla Q| \leq c_7 w^{-\frac{k}{2}} W^{\frac{k}{2}}\) hold for some positive constants \( c_2, c_3, c_7 \), then there is a constant \( C > 0 \) depending only on \( d, k \) and \( \eta \) such that

\[
  w(t, y)p(t, x, y) \leq C \left( A_1 \sup_{t \in (a_0, b_0)} \xi_{W_1}(t, x) + A_2 \Xi_1(a_0, b_0) + A_3 \Xi_2(a_0, b_0) \right),
\]

(2.5)

with

\[
  A_1 = c^k_1, \\
  A_2 = c^k_2 + \frac{c^k_4}{(b_0 - b)^{\frac{k}{2}}} + c^k_3 + c^k_5, \\
  A_3 = c^k_5 + c^k_6 + c^k_2 c^k_6,
\]

(2.6)

where \( c_3 \) is as in Remark 2.4 and

\[
  \xi_{W_i}(t, x) := \int_{\mathbb{R}^d} p(t, x, y) W_i(t, y) \, dy, \quad \Xi_i(a_0, b_0) := \int_{a_0}^{b_0} \xi_{W_i}(t, x) \, dt
\]

for \( i = 1, 2 \).

We aim to establish estimates for the derivatives of the kernel \( p \). To this purpose we make the following assumptions.

**Hypothesis 2.6.** Fix \( T > 0, x \in \mathbb{R}^d \) and \( 0 < a_0 < a < b < b_0 < T \). Let us consider two time dependent Lyapunov functions \( 1 \leq W_1, W_2 \) with \( W_1 \leq W_2 \) and a weight function \( 1 \leq w \in C^{1,3}((0, T) \times \mathbb{R}^d) \) with \( \partial_t \nabla w \in C((0, T) \times \mathbb{R}^d) \) such that for some \( \varepsilon \in (0, 1) \) and \( k > 2(d + 2) \) the following hold true:

\((a)\) \( \int_{\mathbb{R}^d} \left( \frac{1}{w(t, y)} \right)^{1-\varepsilon} \, dy < \infty \) and \( \int_{Q(a,b)} \left( \frac{1}{w(t, y)} \right)^{1-\varepsilon} \, dt \, dy < \infty \);

\((b)\) the functions \( w^{-2}\nabla w, w^{-2}\partial_t w, w^{-2}D^2w, w^{-3}\nabla w \cdot \nabla w, w^{-2}\partial_t \nabla w, w^{-3}\partial_t w \nabla w, (\nabla w)^{-k-1} D^2w \) and \( (\nabla w)^{-k-1}\partial_t \nabla w \) are bounded on \( Q(a_0, b_0) \);

\((c)\) there exist constants \( c_1, \ldots, c_{11} \geq 1 \), possibly depending on the interval \((a_0, b_0)\), such that
Theorem 2.8. (a), (b) and (c) can be deduced adapting respectively [11, Theorem 5.1], [11, Lemma 5.1] to infer integrability of the functions and [11, Theorem 5.2] to operators with potential term. We note that Hypothesis 2.6(a) is used.

Theorem 2.7. Assume Hypothesis 2.6. Then the following statements hold.

(a) The functions $p \log p$ and $p \log^2 p$ are integrable in $Q(a, b)$ and in $\mathbb{R}^d$ for all fixed $t \in [a, b]$ and

$$
\int_{Q(a, b)} \frac{|\nabla p(t, x, y)|^2}{p(t, x, y)} \, dt \, dy \leq \frac{1}{\eta^2} \int_{Q(a, b)} (|F(y)|^2 + V^2(y)) \rho(t, x, y) \, dt \, dy
$$

$$
+ \int_{Q(a, b)} \rho(t, x, y) \log^2 \rho(t, x, y) \, dt \, dy
$$

$$
- \frac{2}{\eta} \int_{\mathbb{R}^d} \rho(t, x, y) \log \rho(t, x, y) |_{t=a}^{t=b} \, dy < \infty.
$$

In particular, $p^{\frac{1}{2}}$ belongs to $W^{0,1}_2(Q(a, b))$.
(b) $\nabla p \in L^s(Q(a_1, b_1))$ for all $1 \leq s \leq \infty$.
(c) $p \in W^{1,2}_{k/2}(Q(a_1, b_1))$.

Proof. (a), (b) and (c) can be deduced adapting respectively [11, Theorem 5.1], [11, Lemma 5.1] and [11, Theorem 5.2] to operators with potential term. We note that Hypothesis 2.6(a) is used to infer integrability of the functions $p \log p$ and $p \log^2 p$. □

It is possible to prove even more regularity on $\nabla p$, as the following result shows.

Theorem 2.8. Assume Hypothesis 2.6. Then $\nabla p \in \mathcal{H}^{\frac{1}{k}}(Q(a_1, b_1))$.

Proof. In view of Theorem 2.7, we are left to show that

$$
\partial_t \nabla p(t, x, y) \in (W^{0,1}_{(k/2)}(Q(a_1, b_1)))'.
$$

Let $\vartheta \in C^\infty(\mathbb{R})$ such that $\vartheta(t) = 1$ for $t \in [a_1, b_1]$, $\vartheta(t) = 0$ for $t \leq a$, $t \geq b$, $0 \leq \vartheta \leq 1$. We define, for fixed $x \in \mathbb{R}^d$,

$$
q(t, y) := \vartheta^{k/2}(t) p(t, x, y).
$$

Consider $\varphi \in C^1_c(Q(a, b))$. By [11, Lemma 2.1], we have

$$
\int_{Q(a, b)} (\partial_t \varphi(t, y) + A \varphi(t, y)) p(t, x, y) \, dt \, dy = \int_{\mathbb{R}^d} (p(b, x, y) \varphi(b, y) - p(a, x, y) \varphi(a, y)) \, dy.
$$

Substituting $\vartheta^{\frac{1}{k}} \varphi$ instead of $\varphi$ in the previous equation, we get

$$
\int_{Q(a, b)} \left( q \partial_t \varphi - \langle Q \nabla \varphi, \nabla q \rangle + \langle F, \nabla \varphi \rangle q - V \varphi q + p \varphi \partial_t \vartheta^{\frac{1}{k}} \right) \, dt \, dy = 0.
$$

We replace again $\varphi$ by the difference quotients with respect to the variable $y$

$$
\tau_y \varphi(t, y) = \frac{\varphi(t, y + he_j) - \varphi(t, y)}{|h|},
$$
for \((t, y) \in Q(a, b), 0 \neq h \in \mathbb{R}\) and we obtain
\[
\int_{Q(a, b)} q \partial_t (\tau_{-h} \varphi) \, dt \, dy - \int_{Q(a, b)} \langle Q \nabla (\tau_{-h} \varphi), \nabla q \rangle \, dt \, dy + \int_{Q(a, b)} \langle F, \nabla (\tau_{-h} \varphi) \rangle q \, dt \, dy
- \int_{Q(a, b)} V q (\tau_{-h} \varphi) \, dt \, dy + \int_{Q(a, b)} p(\tau_{-h} \varphi) \partial_t \varphi \, dt \, dy = I_1 - I_2 + I_3 - I_4 + I_5 = 0,
\]
where
\[
I_1 = \int_{Q(a, b)} q \partial_t (\tau_{-h} \varphi) \, dt \, dy, \quad I_2 = \int_{Q(a, b)} \langle Q \nabla (\tau_{-h} \varphi), \nabla q \rangle \, dt \, dy,
I_3 = \int_{Q(a, b)} \langle F, \nabla (\tau_{-h} \varphi) \rangle q \, dt \, dy, \quad I_4 = \int_{Q(a, b)} V q (\tau_{-h} \varphi) \, dt \, dy, \quad I_5 = \int_{Q(a, b)} p(\tau_{-h} \varphi) \partial_t \varphi \, dt \, dy.
\]

By a change of variables we have
\[
I_1 = \int_{Q(a, b)} (\tau_h q) \partial_t \varphi \, dt \, dy
\]
and
\[
I_2 = \frac{1}{|h|} \int_{Q(a, b)} \left( \langle Q(y + he_j) \nabla \varphi(t, y), \nabla q(t, y + he_j) \rangle - \langle Q(y) \nabla \varphi(t, y), \nabla q(t, y) \rangle \right) \, dt \, dy.
\]

Summing and subtracting \(|h|^{-1} \int_{Q(a, b)} \langle Q(y + he_j) \nabla \varphi(t, y), \nabla q(t, y) \rangle \, dt \, dy \) in the previous expression yields
\[
I_2 = \int_{Q(a, b)} \left( \langle Q(y + he_j) \nabla \varphi(t, y), \nabla \tau_h q(t, y) \rangle + \langle \tau_h Q(y) \nabla \varphi(t, y), \nabla q(t, y) \rangle \right) \, dt \, dy.
\]

Similarly, we find that
\[
I_3 = \int_{Q(a, b)} (\tau_h q(t, y)) \langle F(y + he_j), \nabla \varphi(t, y) \rangle + q(t, y) \langle \tau_h F(t, y), \nabla \varphi(t, y) \rangle \, dt \, dy,
I_4 = \int_{Q(a, b)} (\tau_h V(y) q(t, y) + V(y + he_j) \tau_h q(t, y)) \varphi(t, y) \, dt \, dy
\]
and
\[
I_5 = \int_{Q(a, b)} (\tau_h p) \varphi \partial_t \varphi \, dt \, dy.
\]

Since \(q_{ij} \in C^1_b(\mathbb{R}^d)\), applying the Cauchy-Schwarz inequality and Hölder’s inequality we deduce that
\[
|I_2| \leq c \left( \left\| \nabla \tau_h q \right\|_{L^{k/2}(Q(a, b))} + \left\| \nabla q \right\|_{L^{k/2}(Q(a, b))} \right) \left\| \varphi \right\|_{W^{0,1}_{(k/2)'}}(Q(a, b)).
\]

Moreover,
\[
|I_3| \leq \left( \int_{Q(a, b)} |\tau_h q(t, y)|^{\frac{k}{k-2}} |F(y + he_j)|^{\frac{k}{k-2}} \, dt \, dy \right)^{\frac{2}{k}} \left\| \varphi \right\|_{W^{0,1}_{(k/2)'}}(Q(a, b))
+ \left( \int_{Q(a, b)} q^{\frac{k}{k-2}} |\tau_h F|^{\frac{k}{k-2}} \, dt \, dy \right)^{\frac{2}{k}} \left\| \varphi \right\|_{W^{0,1}_{(k/2)'}}(Q(a, b))
\leq \left\| \tau_h q \right\|_{L^{\infty}(Q(a, b))} \left( \int_{Q(a, b)} \frac{|\tau_h p(t, y)|}{p} \, dt \, dy \right)^{\frac{2}{k}} \left( \int_{Q(a, b)} |F(y + he_j)|^{k} p \, dt \, dy \right)^{\frac{1}{k}} \left\| \varphi \right\|_{W^{0,1}_{(k/2)'}}(Q(a, b))
+ \left\| q \right\|_{L^{\infty}(Q(a, b))} \left( \int_{Q(a, b)} |\tau_h F|^{\frac{k}{k-2}} \, dt \, dy \right)^{\frac{2}{k}} \left\| \varphi \right\|_{W^{0,1}_{(k/2)'}}(Q(a, b)).
\]
Similarly, we have
\[ |I_4| \leq \|\tau_h q\|_{L^\infty} \left( \int_{Q(a,b)} |V(y + he_j)|^k p \, dt \, dy \right)^{\frac{1}{k}} \left( \int_{Q(a,b)} \frac{|\tau_h p(t, y)|^2}{p} \, dt \, dy \right)^{\frac{1}{2}} \|\varphi\|_{W^{0,1}_{(k/2)}(Q(a,b))} \]
\[ + \|q\|_{L^\infty(Q(a,b))} \left( \int_{Q(a,b)} |\tau_h V|^\frac{p}{2} \, dt \, dy \right)^{\frac{1}{2}} \|\varphi\|_{W^{0,1}_{(k/2)}(Q(a,b))}. \]

Finally,
\[ |I_5| \leq c \|\tau_h p\|_{L^{k/2}(Q(a,b))} \|\varphi\|_{W^{0,1}_{(k/2)}(Q(a,b))}. \]

Hence,
\[
\left| \int_{Q(a,b)} (\tau_h q) \partial_i \varphi \, dt \, dy \right| \leq c \left[ \|\nabla \tau_h q\|_{L^{k/2}(Q(a,b))} + \|\nabla q\|_{L^{k/2}(Q(a,b))} \right]
\[ + \|\tau_h q\|_{L^\infty} \left( \int_{Q(a,b)} |F(y + he_j)|^k p(t, y) \, dt \, dy \right)^{\frac{1}{k}} \]
\[ + \|\tau_h p\|_{L^2(Q(a,b))} \left( \int_{Q(a,b)} |V(y + he_j)|^k p(t, y) \, dt \, dy \right)^{\frac{1}{2}} \]
\[ + \|q\|_{L^\infty(Q(a,b))} \left\{ \left( \int_{Q(a,b)} |\tau_h F|^\frac{p}{2} q(t, y) \, dt \, dy \right)^{\frac{1}{2}} + \left( \int_{Q(a,b)} |\tau_h V|^\frac{p}{2} q(t, y) \, dt \, dy \right)^{\frac{1}{2}} \right\}
\[ + \|\tau_h p\|_{L^{k/2}(Q(a,b))} \|\varphi\|_{W^{0,1}_{(k/2)}(Q(a,b))}. \]

As \( p \in W^{1,2}_{k/2}(Q(a,b)) \) by Theorem 2.7(c), it follows that \( \nabla \tau_h q \to \nabla D_j q \) in \( L^\infty(Q(a,b)) \) which implies the boundedness of \( \|\nabla \tau_h q\|_{L^\infty(Q(a,b))} \). Similarly, we may infer the boundedness of \( \|\nabla q\|_{L^\infty(Q(a,b))} \) from Theorem 2.7(a). As \( \nabla p \in L^\infty(Q(a,b)) \) by Theorem 2.7(b), the difference quotients \( \tau_h q \) converge weak* in \( L^\infty(Q(a,b)) \) to \( D_j q \), where also \( \|\tau_h q\|_\infty \) is bounded. Boundedness of the integrals involving \( F \) can easily be deduced from the fact that \( F \in C^{1+c}_0 \), \( V \in C^\infty \) and the mean value theorem. All together, we see that for a certain constant \( C \), we have
\[
\left| \int_{Q(a,b)} (\tau_h q) \partial_i \varphi \, dt \, dy \right| \leq C \|\varphi\|_{W^{0,1}_{(k/2)}(Q(a,b))},
\]
for all \( \varphi \in C^{1,2}_0(Q(a,b)) \). By density, this estimate extends to \( \varphi \in W^{0,1}_{(k/2),(Q(a,b))} \) and it follows that the elements \( \tau_h q \) are uniformly bounded in \( W^{0,1}_{(k/2),r}(Q(a,b)) \). Thus, by reflexivity, we see that as \( h \to 0 \) we find cluster-points in \( W^{0,1}_{(k/2),r}(Q(a,b)) \). But testing against functions in \( C^\infty_c(Q(a,b)) \), we find that the only possible cluster point is \( D_j q \). This yields \( \partial_i D_j p \in (W^{0,1}_{(k/2),r}(Q(a,b)))' \) and finishes the proof. 

The following result, which is a version of [3] Thm. 3.7, is the key to prove the main theorem of this section.

**Theorem 2.9.** Let \( q_{ij} \in C^\infty_{loc}(\mathbb{R}^d) \cap C^1(\mathbb{R}^d) \) be such that \( q_{ij} = q_{ji} \) for \( i, j = 1, \ldots, d \) and such that \( \langle Q(x), \xi, \xi \rangle \geq \eta |\xi|^2 \) for a certain \( \eta > 0 \) and any \( x, \xi \in \mathbb{R}^d \).

Further, let \( 0 \leq a_0 < b_0 \leq 1, k > d + 2 \) and let functions \( f \in L^\frac{k}{2}(Q(a_0, b_0)), h = (h_i) \in L^k(Q(a_0, b_0), \mathbb{R}^d) \) and \( u \in H^{p,1}(Q(a_0, b_0)) \cap L^\infty(a_0, b_0; L^2(\mathbb{R}^d)) \), for some \( p > d + 2 \), be given such
that \( u(a_0) = 0 \) and
\[
(2.7) \quad \int_{Q(a_0, b_0)} [(Q \nabla u, \nabla \psi) + \psi \partial_t u] \, dt \, dx = \int_{Q(a_0, b_0)} f \psi \, dt \, dx + \int_{Q(a_0, b_0)} (h, \nabla \psi) \, dt \, dx,
\]
for all \( \psi \in C_c^\infty(Q(a_0, b_0)) \). Then, \( u \) is bounded and there exists a constant \( C > 0 \), depending only on \( \eta, \delta \) and \( k \) (but not depending on \( \|Q\|_\infty \)) such that
\[
\|u\|_\infty \leq C(\|u\|_2 + \|f\|_\frac{k}{2} + \|h\|_k).
\]

**Proof.** The difference of [8, Thm. 3.7] and the above theorem is that in [8, Thm. 3.7], the norm \( \|u\|_\infty,2 \) in the right-hand side is replaced with \( \|u\|_\infty,2 \). However, inspecting the proof of [8, Theorem 3.7], we see that basically the same proof works. Indeed, in the proof it is initially assumed that \( \|u\|_\infty,2 \leq 1 \). This assumption is needed to prove that for \( A_\ell := \{u(t) \geq \ell\} \) and \( A_\ell := \{u \geq \ell\} \) we have \( |A_\ell| \leq 1 \). However, this is still true under the weaker assumption \( \|u\|_2 \leq 1 \):
\[
|A_\ell| < \int_{A_\ell} \ell^2 \, dt \, dx \leq \int_{A_\ell} |u(t, x)|^2 \, dt \, dx \leq \|u\|_2 \leq 1.
\]
As this is the only place where the \( \|\cdot\|_{\infty,2} \)-norm appears, the rest of the proof carries over verbatim.

With the help of Theorem 2.9 we can now prove an upper bound for \( |w \nabla p| \) that does not depend on the \( \|\cdot\|_\infty \)-bound of the diffusion coefficients.

**Theorem 2.10.** Assume Hypothesis 2.6. Then there is a constant \( C > 0 \) depending only on \( d, k \) and \( \eta \) (but not depending on \( \|Q\|_\infty \)) such that
\[
|w(t, y) \nabla p(t, x, y)| \leq C \left\{ B_1 \Xi_1(a_0, b_0)^{\frac{k}{2}} \|wp\|_{L\infty(Q(a,b))}^\frac{k}{2} + \left( B_2 \Xi_2(a_0, b_0)^{\frac{k}{2}} + B_3 \Xi_1(a_0, b_0)^{\frac{k}{2}} \right) \|wp\|_{L\infty(Q(a,b))}^\frac{k-1}{2} \right\}
\]
for all \( (t, y) \in (a_1, b_1) \times \mathbb{R}^d \) and fixed \( x \in \mathbb{R}^d \), where \( B_i, i = 1, \ldots, 7 \) are positive constants depending only on \( c_i, i = 1, \ldots, 11, b_1 \) and \( k \).

**Proof.** We first prove the theorem assuming that the weight function \( w \), along with its first order partial derivatives and its second order partial derivatives of the form \( D_i w \) and \( \partial_i D_i w \) are bounded. We fix \( a_0 < a < a_2 < a_1 < b_1 < b_2 < b < b_0 \).

We show that
\[
\nabla(wp) \in H^{\frac{k}{2},1}(Q(a_2, b_2)) \cap L^\infty(Q(a_2, b_2)).
\]
We apply Theorem 2.7(b) and Theorem 2.8 to infer that \( \nabla p \in H^{\frac{k}{2},1}(Q(a_2, b_2)) \cap L^\infty(Q(a_2, b_2)) \). Moreover, by [7, Lemma 12.4] and Theorem 2.3 we have that \( p \in H^{\frac{k}{2},1}(Q(a_2, b_2)) \cap L^\infty(Q(a_2, b_2)) \). Thus, we get (2.8).

Let \( \vartheta \in C_c(\mathbb{R}) \) such that \( \vartheta(t) = 1 \) for \( t \in [a_1, b_1] \), \( \vartheta(t) = 0 \) for \( t \leq a_2, t \geq b_2 \), \( 0 \leq \vartheta \leq 1 \) and \( |\vartheta'| \leq \frac{2}{|a_2 - b_1|} \). We define
\[
q(t, y) := \vartheta^{k/2}(t)p(t, x, y)
\]
and we note that \( \nabla(q) \in H^{\frac{k}{2},1}(Q(a_2, b_2)) \cap L^\infty(Q(a_2, b_2)) \). Moreover, given \( \varphi \in C_c(Q(a_2, b_2)) \), we write
\[
\psi(t, y) := \vartheta^{k/2}(t)w(t, y)D_h \varphi(t, y),
\]
with \( h = 1, \ldots, d \). For each \( h = 1, \ldots, d \) we apply \([11, \text{Lemma 2.1}]\), which remains valid for operators with potential term. Hence

\[
\int_{Q(a_2, b_2)} (\partial_t \psi(t, y) + A \psi(t, y)) p(t, x, y) \, dt \, dy = 0.
\]

Integrating by parts, we get

\[
\int_{Q(a_2, b_2)} [p \partial_t \psi - \langle Q \nabla \psi, \nabla p \rangle + \langle F, \nabla \psi \rangle p - V \psi \psi] \, dt \, dy = 0.
\]

Replacing the expression of the functions \( \psi \) and \( q \), after some computations we derive that

\[
\int_{Q(a_2, b_2)} \left[ \frac{k}{2} \theta' \theta^\frac{k-2}{2} w p(D_h \varphi) + w q(\partial_t D_h \varphi) - \langle Q \nabla w, \nabla q \rangle (D_h \varphi) - \langle Q \nabla D_h \varphi, w \nabla q \rangle + \langle F, q \nabla w \rangle (D_h \varphi) + \langle F, \nabla D_h \varphi \rangle w q - V w q(D_h \varphi) + q(\partial_t w)(D_h \varphi) \right] \, dt \, dy = 0.
\]

Integrating by parts again in order to remove the derivative \( D_h \) in front of \( \varphi \), we have that

\[
\int_{Q(a_2, b_2)} \left[ -\frac{k}{2} \theta' \theta^\frac{k-2}{2} w (D_h p) \varphi - \frac{k}{2} \theta' \theta^\frac{k-2}{2} p (D_h w) \varphi + (\partial_t D_h(wq)) \varphi + \langle (D_h Q) \nabla w, \nabla q \rangle \varphi + \langle Q(D_h \nabla w), \nabla q \rangle \varphi + \langle Q \nabla w, \nabla q \rangle \varphi + \langle (D_h Q) \nabla w, \nabla q \rangle \varphi + \langle Q \nabla D_h \varphi, \nabla w q \rangle \varphi - q \langle F, D_h \varphi \rangle \varphi - q \langle D_h F, \nabla \varphi \rangle + q \langle D_h q, \nabla \varphi \rangle - V w(D_h q) \varphi + V q(D_h \varphi) + (D_h V) w q \varphi \right] \, dt \, dy = 0.
\]

(2.10)

Since

\[
\int_{Q(a_2, b_2)} \langle Q \nabla w, D_h \nabla \varphi \rangle \, dt \, dy = -\int_{Q(a_2, b_2)} \left[ (D_h q) \text{div}(Q \nabla w) \varphi + (D_h q) \langle Q \nabla w, \nabla \varphi \rangle \right] \, dt \, dy
\]

and

\[
\int_{Q(a_2, b_2)} \langle Q D_h(w q), \nabla \varphi \rangle \, dt \, dy = \int_{Q(a_2, b_2)} \left[ \langle Q \nabla D_h(w q), \nabla \varphi \rangle - q \langle Q D_h(\nabla w), \nabla \varphi \rangle + (D_h q) \langle Q \nabla w, \nabla \varphi \rangle \right] \, dt \, dy,
\]

we can adjust the terms in (2.10) to obtain that

\[
\int_{Q(a_2, b_2)} \left[ \langle Q \nabla u, \nabla \varphi \rangle + \varphi \partial_t u \right] \, dt \, dy = \int_{Q(a_2, b_2)} f \varphi \, dt \, dy + \int_{Q(a_2, b_2)} \langle h, \nabla \varphi \rangle \, dt \, dy,
\]

where

\[
u = D_h(wq), \quad f = \frac{k}{2} \theta' \theta^\frac{k-2}{2} w(D_h p) + \frac{k}{2} \theta' \theta^\frac{k-2}{2} p(D_h w) - \langle (D_h Q) \nabla w, \nabla q \rangle - \langle Q(D_h \nabla w), \nabla q \rangle + \langle (D_h q) \text{div}(Q \nabla w) \rangle q + q(D_h \nabla w, F) + q(\nabla w, D_h F) + \langle D_h q, \nabla w \rangle F - V w(D_h q) - V q(D_h w) - w q(D_h V) + (\partial_t D_h w) q + (\partial_t w)(D_h q), \quad h = 2(D_h q) Q \nabla w - w(D_h Q) \langle \nabla q \rangle + q D_h \nabla w + w F(D_h q) + q F(D_h w) + w q(D_h F).
\]
We now want to apply Theorem [2.9] to the function \( u \) and infer that there exists a constant \( C \), depending only on \( d, \eta \) and \( k \), but not on \( \|Q\|_\infty \), such that

\[
\|D_h(wq)\|_\infty \leq C \left[ \|D_h(wq)\|_2 + \frac{k}{b_2 - b_1} \left\| \partial^{\frac{b_2}{b_1}} w(D_h p) \right\|^\frac{b_2}{b_1} + \frac{k}{b_2 - b_1} \left\| \partial^{\frac{b_2-2}{b_1}} p(D_h w) \right\|^\frac{b_2-2}{b_1} \right.
\]

\[
+ \left. \|\langle \nabla Q \nabla w, \nabla q \rangle \|_\frac{b_2}{2} \right] \] 

\[
+ \left[ \left( \|Q(D_h \nabla w, \nabla q)\|_\frac{b_2}{2} + \|\langle Q(D_h \nabla w, \nabla q)\rangle \|_\frac{b_2}{2} + \|\langle Q(D_h \nabla w, \nabla q)\rangle \|_\frac{b_2}{2} + \|\langle Q(D_h \nabla w, \nabla q)\rangle \|_\frac{b_2}{2} \right) \right]
\]

\[
+ \left[ \|QD_2 w \nabla q\|_\frac{b_2}{2} + \|\nabla Q \nabla w, \nabla q\|_\frac{b_2}{2} \right]
\]

\[
+ \left[ \|D_h q \nabla w\|_k + \|w(D_h Q) \nabla q\|_k + \|qQD_h \nabla w\|_k + \|wF(D_h q)\|_k \right] + \|Q \nabla w\|_\infty .
\]

Summing over \( h = 1, \ldots, d \) and since \( \|\nabla (wq)\|_\infty \geq \|w \nabla q\|_\infty - \|q \nabla w\|_\infty \) yields

\[
\|w \nabla q\|_\infty \leq C \left[ \left[ \|w \nabla q\|_2 + \|q \nabla w\|_2 + \frac{k}{b_2 - b_1} \left\| \partial^{\frac{b_2}{b_1}} w \nabla p \right\|^\frac{b_2}{b_1} + \frac{k}{b_2 - b_1} \left\| \partial^{\frac{b_2-2}{b_1}} p \nabla w \right\|^\frac{b_2-2}{b_1} \right]
\]

\[
+ \left[ \|\nabla Q \nabla w, \nabla q\|_\frac{b_2}{2} \right]
\]

\[
+ \left[ \left( \|QD_2 w \nabla q\|_\frac{b_2}{2} + \|\langle QD_2 w \nabla q\rangle \|_\frac{b_2}{2} + \|\langle QD_2 w \nabla q\rangle \|_\frac{b_2}{2} + \|\langle QD_2 w \nabla q\rangle \|_\frac{b_2}{2} \right) \right]
\]

\[
+ \left[ \|Q \nabla w, \nabla q\|_\frac{b_2}{2} \right]
\]

\[
+ \left[ \|QD_2 w \nabla q\|_\frac{b_2}{2} + \|w(D_h Q) \nabla q\|_\frac{b_2}{2} \right]
\]

\[
+ \left[ \|D_h q \nabla w\|_k + \|w(D_h Q) \nabla q\|_k + \|qQD_h \nabla w\|_k + \|wF(D_h q)\|_k \right] + \|Q \nabla w\|_\infty .
\]

(2.11)

We set

\[
P := \int_{Q(a_2,b_2)} \frac{|\nabla p|^2}{p} \, dt \, dy
\]

and, for a sake of simplicity, we write \( \Xi_1 \) instead of \( \Xi_1(a_2, b_2) \) to refer to \( \int_a^b \xi_{W_1}(t, x) \, dt \) for \( i = 1, 2 \).

We observe that \( \Xi_1, \Xi_2 < \infty \) by Proposition [2.2] Moreover, thanks to Theorem [2.2] (a), we know that \( P < \infty \). Finally, we estimate the terms in the right hand side of (2.11). We start with

\[
\|w \nabla q\|_2^2 = \int_{Q(a_2,b_2)} w^2 |\nabla q|^2 \, dt \, dy \leq \|w \nabla q\|_\infty \int_{Q(a_2,b_2)} \frac{|\nabla q|^2}{\sqrt{q}} w \, dt \, dy
\]

\[
\leq \|w \nabla q\|_\infty \left( \int_{Q(a_2,b_2)} \frac{|\nabla q|^2}{q} \, dt \, dy \right)^\frac{1}{2} \left( \int_{Q(a_2,b_2)} w^2 q \, dt \, dy \right)^\frac{1}{2}
\]

\[
\leq c_1^\frac{k}{k} \|w \nabla q\|_\infty \left( \int_{Q(a_2,b_2)} \frac{|\nabla p|^2}{p} \, dt \, dy \right)^\frac{1}{2} \left( \int_{Q(a_2,b_2)} \xi_{W_1}(t, x) \, dt \right)^\frac{1}{2}
\]

\[
= c_1^\frac{k}{k} \|w \nabla q\|_\infty P^{\frac{k}{k}} \Xi_1^\frac{k}{k}.
\]

Hence, we have

\[
\|w \nabla q\|_2 \leq c_1^\frac{k}{k} P^{\frac{k}{k}} \Xi_1^\frac{k}{k} \|w \nabla q\|_\infty .
\]

Similarly, we get

\[
\|\partial^{\frac{b_2}{b_1}} w \nabla p\|_\frac{b_2}{b_1} \leq c_1 P^\frac{k}{k} \Xi_1^\frac{k}{k} \|w \nabla q\|_\infty .
\]

\[
\|\langle \nabla Q \nabla w, \nabla q\rangle \|_\frac{b_2}{2} \leq \eta^{-1} c_2 c_7 P^\frac{k}{k} \Xi_1^\frac{k}{k} \|w \nabla q\|_\infty .
\]

\[
\|QD_2 w \nabla q\|_\frac{b_2}{2} \leq c_3 P^\frac{k}{k} \Xi_1^\frac{k}{k} \|w \nabla q\|_\infty ,
\]

\[
\]
\[ \|(\nabla q)\text{div}(Q \nabla w)\|_\frac{2}{q} \leq d(\eta^{-1}c_2c_7 + c_3)P^{\frac{1}{2}}_\Xi \|w \nabla q\|_\infty^{\frac{1}{q} - 2}, \]

where we have applied here (2.4). Moreover,

\[ \|(\nabla q)\langle \nabla w, F \rangle\|_\frac{2}{q} \leq \eta^{-1}c_2c_6P^{\frac{1}{2}}_\Xi \|w \nabla q\|_\infty^{\frac{1}{q} - 2}, \]

\[ \|V w \nabla q\|_\frac{2}{q} \leq c_2^2P^{\frac{1}{2}}_\Xi \|w \nabla q\|_\infty^{\frac{1}{q} - 2}, \]

\[ \|\partial_t w)(\nabla q)\|_\frac{2}{q} \leq c_4P^{\frac{1}{2}}_\Xi \|w \nabla q\|_\infty^{\frac{1}{q} - 2}, \]

\[ \|Q \nabla w, \nabla q\|_k \leq c_2P^{\frac{1}{2}}_\Xi \|w \nabla q\|_\infty^{\frac{1}{q} - 1}, \]

\[ w(\nabla Q)(\nabla q)\|_k \leq c_7P^{\frac{1}{2}}_\Xi \|w \nabla q\|_\infty^{\frac{1}{q} - 1}, \]

Moreover, we estimate \(\|q \nabla w\|_2^2\) as follows:

\[ \|q \nabla w\|_2^2 = \int_{Q(a_2, b_2)} q^2 |\nabla w|^2 \, dt \, dy \leq \eta^{-2}c_2^2 \|w q\|_\infty \int_{Q(a_2, b_2)} W^{\frac{1}{q}}_1 q \, dt \, dy \leq \eta^{-2}c_2^2 \|w q\|_\infty \Xi_1. \]

Thus, we have

\[ \|q \nabla w\|_2 \leq \eta^{-1}c_2 \Xi_1^\frac{1}{q} \|w q\|_\infty^\frac{1}{q}. \]

In a similar way, we obtain

\[ \|q(D^2 w)F\|_\frac{2}{q} \leq \eta^{-1}c_3c_6 \Xi_1^\frac{1}{q} \|w q\|_\infty^{\frac{1}{q} - 2}, \]

\[ \|q(\nabla w, \nabla F)\|_\frac{2}{q} \leq \eta^{-1}c_2c_8 \Xi_1^\frac{1}{q} \|w q\|_\infty^{\frac{1}{q} - 2}, \]

\[ \|V q \nabla w\|_\frac{2}{q} \leq \eta^{-1}c_2c_5 \Xi_1^\frac{1}{q} \|w q\|_\infty^{\frac{1}{q} - 2}, \]

\[ \|w q \nabla V\|_\frac{2}{q} \leq c_9 \Xi_1^\frac{1}{q} \|w q\|_\infty^{\frac{1}{q} - 2}, \]

\[ \|\partial_t q w)\|_\frac{2}{q} \leq c_1 \Xi_1^\frac{1}{q} \|w q\|_\infty^{\frac{1}{q} - 2}, \]

\[ \|qQ D^2 w\|_k \leq c_3 \Xi_1^\frac{1}{q} \|w q\|_\infty^{\frac{1}{q} - 2}, \]

\[ \|q(\nabla w, F)\|_k \leq \eta^{-1}c_2c_6 \Xi_1^\frac{1}{q} \|w q\|_\infty^{\frac{1}{q} - 1}, \]

Finally, we get

\[ \|q \nabla w\|_\infty \leq \|q\|_\infty^{\frac{1}{q} - 1} \|q(1 + |\nabla w|^2)^\frac{1}{q}\|_\infty^{\frac{1}{q}}. \]

We now estimate \(\|q(1 + |\nabla w|^2)^\frac{1}{q}\|_\infty\) by applying Theorem \[2.6\] with \(w\) replaced by \(\hat{w} = (1 + |\nabla w|^2)^\frac{1}{q}\). First, we check the assumptions using Hypothesis \[2.6(c)\]:

\[ \hat{w}^{\frac{1}{rac{1}{q}}} = 1 + |\nabla w|^2 \leq 1 + \eta^{-2}c_2^2 W^{\frac{1}{q}}_1 \leq (1 + \eta^{-2}c_2^2) W^{\frac{1}{q}}_1, \]
\[ |Q \nabla \bar{w}| = k(1 + |\nabla w|^2)^{\frac{\kappa - 2}{2}} |Q D^2 w| |\nabla w| \leq k \bar{w}^{\frac{\kappa - 2}{2}} |Q D^2 w| |\nabla w| \leq k c_2 \bar{w}^{\frac{\kappa - 2}{2}} W_1^{\frac{\kappa}{2}}, \]

\[ |\text{div}(Q \nabla \bar{w})| \leq |d(\nabla Q \nabla \bar{w})| \leq d |\nabla Q||\nabla \bar{w}| + d |Q D^2 \bar{w}| \]
\[ \leq d |\nabla Q| k \bar{w}^{\frac{\kappa - 2}{2}} |Q D^2 w||\nabla w| + d [(k - 2) \bar{w}^{-\frac{2}{2}} |Q D^2 w||\nabla w|] \]
\[ + k \bar{w}^{-\frac{\kappa - 2}{2}} |Q \nabla w| + k \bar{w}^{-\frac{\kappa - 2}{2}} |Q D^2 w||D^2 w| \]
\[ \leq k d [\eta^{-2} c_2 c_3 c_7 + (k - 1) \eta^{-1} c_3^2 + c_2 c_10] \bar{w}^{\frac{\kappa - 2}{2}} W_1^{\frac{\kappa}{2}}, \]

\[ |\partial_t \bar{w}| \leq k(1 + |\nabla w|^2)^{\frac{\kappa - 2}{2}} |\nabla w| |\partial_t \nabla w| \leq k \eta^{-1} c_2 c_11 \bar{w}^{\frac{\kappa - 2}{2}} W_1^{\frac{\kappa}{2}}, \]

\[ \bar{w}^{\frac{\kappa}{2}} V^{\frac{\kappa}{2}} \leq (1 + |\nabla w|) V^{\frac{\kappa}{2}} \leq (c_5 + \eta^{-1} c_2 c_5) W_2^{\frac{\kappa}{2}}, \]

\[ \bar{w}^{\frac{\kappa}{2}} |F| \leq (1 + |\nabla w|) |F| \leq (c_6 + \eta^{-1} c_2 c_6) W_2^{\frac{\kappa}{2}}. \]

Moreover, \( \bar{w}^{-\frac{2}{2}} \nabla \bar{w} \) and \( \bar{w}^{-\frac{2}{2}} \partial_t \bar{w} \) are bounded on \( Q(a_0, b_0) \) as we assume in Hypothesis 2.6 that the functions \( (\nabla w)^{-k-1} D^2 w \) and \( (\nabla w)^{-k-1} \partial_t \nabla w \) are bounded. Hence, the assumptions of Theorem 2.3 hold true with \( \bar{w} \) replaced by \( \bar{w} \) and with the constants \( c_1, \ldots, c_6 \) replaced, respectively, by \( 1 + \eta^{-2} c_2^2 \), \( c_3, c_4, k d [\eta^{-2} c_2 c_3 c_7 + (k - 1) \eta^{-1} c_3^2 + c_2 c_10], k \eta^{-1} c_2 c_11, c_5 + \eta^{-1} c_2 c_5 \) and \( c_6 + \eta^{-1} c_2 c_6 \). Thus, we obtain that

\[ \left\| q (1 + |\nabla w|^2)^{\frac{\kappa}{2}} \right\|_\infty \leq C \left[ c_2^{k} \sup_{t \in (a_2, b_2)} \xi_{W_1(t,x)} + \left( c_3 + \frac{c_2}{(b_2 - b_1)^{\frac{\kappa}{2}}} + c_4 \xi_{c_7} + c_2 \xi_{c_10} + c_2 \xi_{c_11} \right) \xi_1 \right. \]
\[ + \left( c_6 + c_2 c_6 + c_4 \xi_{c_6} + c_2 \xi_{c_6} + c_6 \xi_{c_6} + c_6 \right) \xi_2 \right]. \]

Consequently, if we set

\[ \overline{\xi} := \sup_{t \in (a_2, b_2)} \xi_{W_1(t,x)}, \]

we estimate the last term in the right hand side of 2.11 as follows

\[ \|q \nabla w\|_\infty \leq C \left[ c_2 \overline{\xi} + \left( c_3 + \frac{c_2}{(b_2 - b_1)^{\frac{\kappa}{2}}} + c_4 \xi_{c_7} + c_2 \xi_{c_10} + c_2 \xi_{c_11} \right) \xi_1 \right. \]
\[ + \left( c_6 + c_2 c_6 + c_4 \xi_{c_6} + c_2 \xi_{c_6} + c_6 \xi_{c_6} + c_6 \right) \xi_2 \right] \left\| w q \right\|^{\frac{\kappa}{2}}. \]

Combining 2.11 with the above estimates yields

\[ \|w \nabla q\|_\infty \leq C c_1^{\frac{\kappa}{2}} P^{\frac{\kappa}{2}} \left\| w \nabla q \right\|^{\frac{\kappa}{2}} + C \left( c_2 + c_7 \right)^{\frac{\kappa}{2}} \left\| w \nabla q \right\|^{\frac{\kappa}{2}} \]
\[ + C P^{\frac{\kappa}{2}} \left( \frac{c_1}{b_2 - b_1} + c_2 c_7 + c_4 \right) \left( c_2 + c_7 \right)^{\frac{\kappa}{2}} + C \left( c_2 + c_7 \right)^{\frac{\kappa}{2}} \left\| w \nabla q \right\|^{\frac{\kappa}{2}} + C \left( c_2 + c_7 \right)^{\frac{\kappa}{2}} \left\| w q \right\|^{\frac{\kappa}{2}} \]
\[ + C \left( \frac{c_2}{b_2 - b_1} + c_11 \right) \left( c_2 + c_7 \right)^{\frac{\kappa}{2}} + C \left( c_2 + c_7 \right)^{\frac{\kappa}{2}} \left\| w q \right\|^{\frac{\kappa}{2}} + C \left( c_2 + c_7 \right)^{\frac{\kappa}{2}} \left\| w q \right\|^{\frac{\kappa}{2}} \]
\[ + C \left( c_2 + c_7 \right)^{\frac{\kappa}{2}} \left( c_3 + \frac{c_2}{(b_2 - b_1)^{\frac{\kappa}{2}}} + c_4 + \xi_{c_7} + c_2 \xi_{c_10} + c_2 \xi_{c_11} \right) \xi_1 \]
\[ + \left( c_6 + c_2 c_6 + c_4 + \xi_{c_6} + c_2 \xi_{c_6} + c_6 \xi_{c_6} + c_6 \right) \xi_2 \right] \left\| w q \right\|^{\frac{\kappa}{2}}. \]
We observe that, by Young’s inequality, we find
\[ Cc_1^k P_{11}^{1/k} \|w \nabla q\|_\infty^{1/k} \leq C^2 c_1^k P_{11}^{1/k} + \frac{1}{4} \|w \nabla q\|_\infty. \]

Then, setting
\[
X := \|w \nabla q\|_\infty^{1/k},
\alpha := C^2 c_1^k P_{11}^{1/k} + Cc_2^2 \|w \nabla q\|_\infty^{1/k} + C \left[ \left( \frac{c_2}{b_2 - b_1} + c_1 \right) \Xi_1^{1/k} \right. \\
+ \left. \left( c_2 c_3 + c_3 c_6 + c_2 c_8 + c_6 \right) \Xi_2^{1/k} \right] \|w \nabla q\|_\infty^{1/k},
\]
\[
+ C \left[ \left( c_2^2 M_1^{1/k} + (c_2 c_4 + c_3) \Xi_1^{1/k} + c_2 c_6 + c_2 c_8 + c_2 c_10 + c_2 c_11 \right) \Xi_1^{1/k} \\
+ \left( c_6 + c_2 c_6 + c_2 c_7 + c_2 c_8 + c_2 c_10 + c_2 c_11 \right) \Xi_2^{1/k} \right] \|w \nabla q\|_\infty^{1/k},
\]
\[
\beta := C P_{11}^{1/k} \left[ (c_2 + c_7) \Xi_1^{1/k} + c_6 \Xi_2^{1/k} \right],
\gamma := C P_{11}^{1/k} \left[ \left( \frac{c_1}{b_2 - b_1} + c_2 c_7 + c_3 + c_4 \right) \Xi_1^{1/k} + (c_2 c_6 + c_2 c_8) \Xi_2^{1/k} \right],
\]
we derive that
\[ X^k \leq \frac{4}{3} \alpha + \frac{4}{3} \beta X^{k-1} + \frac{4}{3} \gamma X^{k-2}. \]

We now prove that it leads to
\[ X \leq \frac{4}{3} \beta + \sqrt{\frac{4}{3} \gamma + \left( \frac{4}{3} \alpha \right)^2}. \]

We consider the function
\[ f(r) := r^k - \frac{4}{3} \beta r^{k-1} - \frac{4}{3} r^{k-2} - \frac{4}{3} \alpha = r^{k-2} \left( r^2 - \frac{4}{3} \beta r - \frac{4}{3} \gamma \right) - \frac{4}{3} \alpha =: r^{k-2} g(r) - \frac{4}{3} \alpha. \]

First, we show that \( f \) is increasing in \( \left( \frac{4}{3} \beta + \sqrt{\frac{4}{3} \gamma + \left( \frac{4}{3} \alpha \right)^2}, \infty \right) \). This can be seen by computing the first derivative:
\[ f'(r) = (k - 2) r^{k-3} g(r) + r^{k-2} g'(r). \]

Since the function \( g \) is positive and increasing in \( \left( \frac{4}{3} \beta + \sqrt{\frac{4}{3} \gamma + \left( \frac{4}{3} \alpha \right)^2}, \infty \right) \), it follows that \( f'(r) \geq 0 \) in the given interval, so \( f \) is increasing.
Second, we have that
\[
\begin{align*}
    f \left( \frac{4}{3} \beta + \sqrt[\frac{3}{2}]{3} \gamma + \left( \frac{4}{3} \alpha \right)^{\frac{\beta}{2}} \right) &= \left( \frac{4}{3} \beta + \sqrt[\frac{3}{2}]{3} \gamma + \left( \frac{4}{3} \alpha \right)^{\frac{\beta}{2}} \right)^{k-2} \left[ \left( \frac{4}{3} \beta + \sqrt[\frac{3}{2}]{3} \gamma + \left( \frac{4}{3} \alpha \right)^{\frac{\beta}{2}} \right)^{2} \right. \\
    &\quad - \frac{4}{3} \beta \left( \frac{4}{3} \beta + \sqrt[\frac{3}{2}]{3} \gamma + \left( \frac{4}{3} \alpha \right)^{\frac{\beta}{2}} \right) - \frac{4}{3} \gamma - \frac{4}{3} \alpha \\
    &= \left( \frac{4}{3} \beta + \sqrt[\frac{3}{2}]{3} \gamma + \left( \frac{4}{3} \alpha \right)^{\frac{\beta}{2}} \right)^{k-2} \left[ \left( \frac{4}{3} \alpha \right)^{\frac{\beta}{2}} + \frac{8\sqrt{3}}{9} \sqrt[\frac{3}{2}]{\beta} \right] \\
    &\quad + \frac{4\sqrt{3}}{3} \left( \frac{4}{3} \alpha \right)^{\frac{\beta}{2}} \left( \sqrt[\frac{3}{2}]{\beta} + \sqrt{\gamma} \right) - \frac{4}{3} \alpha \\
    &> \left( \frac{4}{3} \alpha \right)^{\frac{\beta}{2}} \left( \frac{4}{3} \alpha \right)^{\frac{\beta}{2}} - \frac{4}{3} \alpha = 0.
\end{align*}
\]

On one hand, from the previous observations we deduce that \(f(r) \geq 0\) if \(r > \frac{1}{3} \beta + \sqrt[\frac{3}{2}]{3} \gamma + \left( \frac{4}{3} \alpha \right)^{\frac{\beta}{2}}\).

On the other hand, by (2.12), \(f(X) \leq 0\). Thus, we conclude that (2.13) holds true. Consequently, there exists a positive constant \(K_1\) such that
\[
\|w \nabla q\|_{\infty} \leq K_1 \left( \alpha + \beta^k + \gamma^k \right).
\]

By plugging in the previous inequality the definition of \(\alpha, \beta, \gamma\) we get
\[
\begin{align*}
    \|w \nabla q\|_{L^\infty(Q(a_2, b_2))} &\leq C \left\{ c_2 \Xi_1 \|wq\|_{L^\infty(Q(a_2, b_2))} \right. \left[ \left( \frac{c_2}{b_2 - b_1} + c_11 \right) \Xi_1 \right. \\
    &\quad + \left( c_2 c_1^2 + c_3 c_6 + c_2 c_8 + c_9 \right) \Xi_2 \left. \right] \|wq\|_{L^\infty(Q(a_2, b_2))} \\
    &\quad + \left[ c_2 M_1^\frac{\beta}{2} + \left( c_3 + \frac{c_2}{(b_2 - b_1)^2} + \frac{c_1}{c_2} c_1^2 + \frac{c_2}{c_2} c_2 + \frac{c_2}{c_2} c_2 + \frac{c_2}{c_2} c_2 \right) \Xi_1 \right. \\
    &\quad + \left( c_6 + c_2 c_6 + c_2 c_6 + c_2 c_2 c_6 + c_5 + c_2 c_5 + c_8 \right) \Xi_1 \left. \right] \|wq\|_{L^\infty(Q(a_2, b_2))} \\
    &\quad + \left[ \left( \frac{c_2}{c_1} + \frac{c_1}{(b_2 - b_1)^2} + c_2 + \frac{c_2}{c_2} c_2 + \frac{c_2}{c_2} c_2 + c_2 + c_2 \right) \Xi_1 \right. \\
    &\quad + \left( c_6 + c_2 c_6 + c_2 \right) \Xi_2 \right] \|wq\|_{L^\infty(Q(a_2, b_2))} \\
\end{align*}
\]

Letting \(a_2 \downarrow a\) and \(b_2 \uparrow b\) and considering that \(\int_{a}^{b} \xi_{W_j} (t, x) \, dt \leq \int_{a_0}^{b_0} \xi_{W_j} (t, x) \, dt\) for \(j = 1, 2\), we gain
\[
\begin{align*}
    |w(t, y) \nabla p(t, x, y)| &\leq C \left\{ c_2 \Xi_1(a_0, b_0)^{\frac{\beta}{2}} \|wq\|_{L^\infty(Q(a, b))} \right. \left[ \left( \frac{c_2}{b_2 - b_1} + c_11 \right) \Xi_1(a_0, b_0)^{\frac{\beta}{2}} \\
    &\quad + \left( c_2 c_1^2 + c_3 c_6 + c_2 c_8 + c_9 \right) \Xi_2(a_0, b_0)^{\frac{\beta}{2}} \right] \|wq\|_{L^\infty(Q(a, b))} + \left[ c_2 \left( \sup_{t \in (a_0, b_0)} \xi_{W_1} (t, x) \right)^{\frac{\beta}{2}} \\
    &\quad + \left( c_3 + \frac{c_2}{(b_2 - b_1)^2} + \frac{c_2}{c_2} c_2 + \frac{c_2}{c_2} c_2 + \frac{c_2}{c_2} c_2 + \frac{c_2}{c_2} c_2 \right) \Xi_1(a_0, b_0)^{\frac{\beta}{2}} \\
    &\quad + \left( c_6 + c_2 c_6 + c_2 \right) \Xi_2(a_0, b_0)^{\frac{\beta}{2}} \right] \Xi_1(a_0, b_0)^{\frac{\beta}{2}} + \left( c_6 + c_2 c_6 + c_2 c_6 + c_2 c_2 c_6 + c_5 \right).
\end{align*}
\]
we now check Hypothesis 2.6(c) we have that

\[
D_{i} \text{derivatives and its second order partial derivatives of the form}
\]

\[
\frac{\partial}{\partial x_{i}} + \left( c_{i}^{n} + c_{i}^{b} c_{i}^{d} + c_{i}^{f} \right) \Xi_{2}(a_{0}, b_{0})^{\frac{1}{2}}
\]

(2.14)

\[
\times \Xi_{1}(a_{0}, b_{0})^{\frac{1}{2}} + \left( c_{i}^{n} + c_{i}^{b} c_{i}^{d} + c_{i}^{f} \right) \Xi_{2}(a_{0}, b_{0})^{\frac{1}{2}}
\]

\[
\left( \int_{Q(a,b)} \frac{|\nabla p|^{2}}{p} \, dt \, dy \right)^{\frac{1}{2}}
\]

for all \((t, y) \in (a_{1}, b_{1}) \times \mathbb{R}^{d}\) and fixed \(x \in \mathbb{R}^{d}\).

To finish the proof, it remains to remove the additional assumption on the weight \(w\). We define

\[
w_{\varepsilon} := \frac{w}{1 + \varepsilon w}, \quad \varepsilon > 0.
\]

Since

\[
D_{i} w_{\varepsilon} = (1 + \varepsilon w)^{-2} D_{i} w
\]

and

\[
D_{ij} w_{\varepsilon} = (1 + \varepsilon w)^{-2} D_{ij} w - 2\varepsilon (1 + \varepsilon w)^{-3} (D_{i} w)(D_{j} w),
\]

for all \(i, j = 1, \ldots, d\), then by Hypothesis 2.6(b) it follows that \(w_{\varepsilon}\), along with its first order partial derivatives and its second order partial derivatives of the form \(D_{ij} w\) and \(\partial_{t} D_{i} w\) are bounded. If we now check Hypothesis 2.6(c) we have that

\[
w_{\varepsilon} \leq w \leq c_{1}^{\frac{1}{p}} W_{1}^{\frac{1}{p}},
\]

\[
|Q \nabla w_{\varepsilon}| = (1 + \varepsilon w)^{-2} |Q \nabla w| \leq c_{2} W_{1}^{\frac{1}{p}},
\]

\[
|Q D^{2} w_{\varepsilon}| \leq (1 + \varepsilon w)^{-2} |Q D^{2} w| + 2\varepsilon (1 + \varepsilon w)^{-3} |Q \nabla w| |\nabla w| \leq (c_{3} + 2\eta^{-1} c_{2}^{2}) W_{1}^{\frac{1}{p}},
\]

\[
|D^{3} w_{\varepsilon}| \leq (1 + \varepsilon w)^{-2} |D^{3} w| + 6\varepsilon (1 + \varepsilon w)^{-3} |\nabla w| |D^{2} w| + 6\varepsilon^{2} (1 + \varepsilon w)^{-4} |\nabla w|^{3}
\]

\[
\leq (c_{10} + 6\eta^{-2} c_{2} c_{3} + 6\eta^{-3} c_{2}^{3}) W_{1}^{\frac{1}{p}},
\]

\[
|\partial_{t} w_{\varepsilon}| = (1 + \varepsilon w)^{-2} |\partial_{t} w| \leq c_{4} w^{\frac{k-2}{k}} W_{1}^{\frac{1}{p}}
\]

and

\[
|\partial_{t} \nabla w_{\varepsilon}| \leq (1 + \varepsilon w)^{-2} |\partial_{t} \nabla w| + 2\varepsilon (1 + \varepsilon w)^{-3} |\nabla w| |\partial_{t} w| \leq (c_{11} + 2\eta^{-1} c_{2} c_{4}) W_{1}^{\frac{1}{p}}.
\]

This shows that \(w_{\varepsilon}\) satisfies Hypothesis 2.6(c) with the same constants \(c_{1}, c_{2}, c_{4}, c_{5}, c_{6}, c_{7}, c_{8}, c_{9}\) and with the constants \(c_{3}, c_{10}, c_{11}\) replaced, respectively, by \(c_{3} + 2\eta^{-1} c_{2}^{2}, c_{10} + 6\eta^{-2} c_{2} c_{3} + 6\eta^{-3} c_{2}^{3}\) and \(c_{11} + 2\eta^{-1} c_{2} c_{4}\).

Thus, the estimate (2.14), shown in the first part of the proof holds true with the function \(w_{\varepsilon}\) instead of \(w\) and with the constants on the right hand side that do not depend on \(\varepsilon\). We finally let \(\varepsilon \to 0\) to gain the desired inequality (2.3).
Approximation of the coefficients.

Remark 2.11. From the above proof one can see that the constants $B_i$, $i = 1, \ldots, 6$ are given by

$$
B_1 = c_2,
B_2 = \frac{c_2}{b - b_1} + c_2c_4 + c_{11},
B_3 = c_2c_2^2 + c_3c_6 + c_2^2c_6 + c_2c_8 + c_9,
B_4 = c_3 + c_2^2 + \frac{c_2}{(b - b_1)^2} + \frac{c_2}{3} + \frac{c_2}{2}c_7 + \frac{c_2}{2}c_7 + \frac{c_2}{2}c_1 + c_2^2c_7 + c_2c_3c_7 + c_2c_4c_7 + c_2^2c_4c_7,
B_5 = c_6 + c_2c_6 + c_2c_6 + c_2^2c_6 + c_2c_6c_6 + c_2c_6c_6 + c_2c_6 + c_5 + c_2c_5 + c_8,
B_6 = c_1^k + \frac{c_1^k}{(b - b_1)^2} + c_2^k + c_2^k + c_3 + c_7 + c_4,
B_7 = c_k^k + c_2^k c_6^k + c_5^k.
$$

(2.15)

3. Results for general diffusion coefficients

3.1. Approximation of the coefficients. We approximate the operator $A$ with a family of operators $A_n$ with bounded diffusion coefficients in order to apply the results from the previous section.

To that end, we approximate the diffusion matrix $Q$ as follows. Picking a function $\varphi \in C^\infty_0(\mathbb{R})$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $(-1, 1)$, $\varphi \equiv 0$ in $\mathbb{R} \setminus (-2, 2)$ and $|s\varphi'(s)| \leq 2$ for all $s \in \mathbb{R}$, we set

$$
\varphi_n(x) := \varphi(W_1(t_0, x)/n),
$$

where $W_1$ is the Lyapunov function from Hypothesis 2.6. The constant $t_0 \in (0, T)$ will be chosen later on. We then put

$$
q_{ij}^{(n)}(x) := \varphi_n(x)q_{ij}(x) + (1 - \varphi_n(x))\eta\delta_{ij},
$$

where $\delta_{ij}$ is the Kronecker delta. Replacing $Q$ with $Q_n := (q_{ij}^{(n)})$ we approximate $A$ with the operators $A_n$ defined by

$$
A_n = \text{div}(Q_n \nabla) + F \cdot \nabla - V.
$$

Lemma 3.1. For every $n \in \mathbb{N}$ the diffusion coefficients $q_{ij}^{(n)}$ and their first order spatial derivatives are bounded on $\mathbb{R}^d$. Moreover, the operator $A_n$ satisfies Hypothesis 1.1 and if $W$ is a time dependent Lyapunov function for the operator $\partial_t + A$ such that $|\nabla W| = 0$ on $[0, T] \times B_R$ for all $R > 0$, then $W$ is a time dependent Lyapunov function for $\partial_t + A_n$.

Proof. Clearly, since $\lim_{|x| \to \infty} W_1(t_0, x) = +\infty$, the functions $\varphi_n$ vanish outside a compact set. As a consequence, the coefficients $q_{ij}^{(n)}$ and their spatial derivatives $D_k q_{ij}^{(n)}$ are bounded on $\mathbb{R}^d$ for all $i, j, k = 1, \ldots, d$. We now check Hypothesis 1.1. First, we observe that $Q_n$ is symmetric and, thanks to the uniformly ellipticity of $Q$, we get

$$
\sum_{i,j=1}^d q_{ij}^{(n)}(x)\xi_i \xi_j = \varphi_n(x) \sum_{i,j=1}^d q_{ij}(x)\xi_i \xi_j + \eta(1 - \varphi_n(x))|\xi|^2 \geq \eta|\xi|^2
$$

for all $x, \xi \in \mathbb{R}^d$. It remains to prove that $A_n Z(x) \leq M_1$ for a certain constant $M_1 \geq 0$ and for all $x \in \mathbb{R}^d$. Let $x \in \mathbb{R}^d$. Then

$$
A_n Z(x) = \text{div}(Q_n \nabla Z(x)) + F(x) \cdot \nabla Z(x) - V(x)Z(x)
= \varphi_n(x) \text{div}(Q \nabla Z(x)) + Q \nabla \varphi_n(x) \cdot \nabla Z(x) - \eta \nabla \varphi_n(x) \cdot \nabla Z(x)
+ \eta(1 - \varphi_n(x))\Delta Z(x) + F(x) \cdot \nabla Z(x) - V(x)Z(x)
= \varphi_n(x)A Z(x) + (1 - \varphi_n(x))((\eta A Z(x) + F(x) \cdot \nabla Z(x) - V(x)Z(x))
+ Q \nabla \varphi_n(x) \cdot \nabla Z(x) - \eta \nabla \varphi_n(x) \cdot \nabla Z(x).
$$
For the first and the second term in the right hand side we apply Hypothesis 1.1 that holds true for the operator $A$:

$$A_n Z(x) \leq M + Q \nabla \varphi_n(x) \cdot \nabla Z(x) - \eta \nabla \varphi_n(x) \cdot \nabla Z(x).$$

We can find a bound also for the last two terms since the functions $\varphi_n$ vanish outside a compact set. As a result we find a constant $M_1 \geq 0$ such that

$$A_n Z(x) \leq M_1 \quad \forall x \in \mathbb{R}^d.$$ 

Finally, we check that if $W$ is a time dependent Lyapunov function for the operator $\partial_t + A$ such that $|\nabla \Psi|_1$ is bounded on $[0, T] \times B_R$ for all $R > 0$, then $W$ is a time dependent Lyapunov function for $\partial_t + A_n$. This can be seen by computing $\partial_t W(t, y) + A_n W(t, y) = (\partial_t W(t, y) + \text{div}(Q_n \nabla W(t, y)) + F(y) \cdot \nabla W(t, y) - V(y) W(t, y)$

$$= \varphi_n(y) LW(t, y) + (1 - \varphi_n(y)) (\partial_t W(t, y) + \eta \Delta W(t, y) + F(y) \cdot \nabla W(t, y) - V(y) W(t, y)) + Q\nabla \varphi_n(y) \cdot \nabla W(t, y) - \eta \nabla \varphi_n(y) \cdot \nabla W(t, y).$$

Since $W$ is a time dependent Lyapunov function for the operator $\partial_t + A$, the first two terms in the right hand side are bounded by $b(t) W(t, y)$:

$$\partial_t W(t, y) + A_n W(t, y) \leq b(t) W(t, y) + Q \nabla \varphi_n(y) \cdot \nabla W(t, y) - \eta \nabla \varphi_n(y) \cdot \nabla W(t, y),$$

where $b(t) \in L^1([0, T])$. Furthermore, the last terms are bounded by a nonnegative constant because $\varphi_n$ vanishes outside a compact set and $|\nabla \Psi|$ is bounded on $[0, T] \times B_R$ for all $R > 0$. Hence, there is a function $\tilde{h}(t) \in L^1([0, T])$ such that

$$\partial_t W(t, y) + A_n W(t, y) \leq \tilde{h}(t) W(t, y)$$

for all $(t, y) \in (0, T) \times \mathbb{R}^d$. Then $W$ is a time dependent Lyapunov function also for $\partial_t + A_n$. 

As a consequence of the previous lemma, for every $n \in \mathbb{N}$ the semigroup generated by $A_n$ in $C_b(\mathbb{R}^d)$ is given by a kernel $p_n(t, x, y)$. In order to show further properties about the operators $A_n$, we make the following assumptions.

**Hypothesis 3.2.** Fix $T > 0$, $x \in \mathbb{R}^d$ and $0 < a_0 < a < b < b_0 < T$. Let us consider two time dependent Lyapunov functions $W_1, W_2$ with $W_1 \leq W_2$ and $|\nabla W_1|, |\nabla W_2|$ bounded on $[0, T] \times B_R$ for all $R > 0$ and a weight function $1 \leq w \in C^{1,2}((0, T) \times \mathbb{R}^d)$ such that

(a) there is $t_0 \in (0, T)$ such that

$$|Q| |\nabla W_1(t_0, \cdot)| \leq c_{12} W_1(t_0, \cdot) w^{-1/k} W_1^{1/2k};$$

(b) there are $c_0 > 0$ and $\sigma \in (0, 1)$ such that

$$W_2 \leq c_0 Z^{1-\sigma}.$$

We now prove that if the operator $A$ satisfies Hypothesis 2.6 then the same is true for the operators $A_n$ assuming further Hypothesis 3.2.

**Lemma 3.3.** Assume that the operator $A$ satisfies Hypotheses 2.6(c) and 3.2(a). Then the operator $A_n$ satisfies Hypothesis 2.6(c) with the same constants $c_1, c_4, c_5, c_6, c_8, c_9, c_{10}, c_{11}$ and with $c_2, c_3, c_7$ being replaced, respectively, by $2 c_2, 2 c_3$ and $\sqrt{3}(c_2 + 2(1 + \sqrt{d})c_{12})$.

**Proof.** The constants $c_1, c_4, c_5, c_6, c_8, c_9, c_{10}, c_{11}$ remain the same because the correspondent inequalities do not depend on the diffusion coefficients. Let us note that Hypothesis 2.6(c)-(ii) implies that

$$|\nabla w| \leq \eta^{-1} c_2 W_1^{\frac{1}{k}}.$$ 

So, it follows that

$$|Q_n \nabla w| = |\varphi_n Q \nabla w + (1 - \varphi_n) \eta \nabla w| \leq |Q \nabla w| + \eta |\nabla w| \leq 2 c_2 W_1^{\frac{1}{k}}.$$ 

Similarly, we get

$$|Q_n D^2 w| = |\varphi_n Q D^2 w + (1 - \varphi_n) \eta D^2 w| \leq |Q D^2 w| + \eta |D^2 w| \leq 2 c_3 W_1^{\frac{1}{k}}.$$
Finally, for \((t, y) \in [a_0, b_0] \times \mathbb{R}^d\), given that \(|s\varphi'(s)| \leq 2\) and using Hypothesis 3.2(a), we have

\[
|\nabla Q_n(t, y)|^2 = \sum_{i,j,h=1}^{d} \left| \varphi_n D_hq_{ij} + \frac{\varphi'(W_1(t_0, y)/n)}{n} D_h W_1(t_0, y)(q_{ij} - \eta \delta_{ij}) \right|^2
\]

\[
\leq 3 \sum_{i,j,h=1}^{d} \left[ |\varphi_n D_hq_{ij}|^2 + |\frac{\varphi'(W_1(t_0, y)/n)}{n}|^2 |D_h W_1(t_0, y)|^2 (q_{ij}^2 + \eta^2 \delta_{ij}) \right]
\]

\[
\leq 3|\varphi_n|^2 |\nabla Q|^2 + 3(W_1(t_0, y)/n)^2 |\varphi'(W_1(t_0, y)/n)|^2 (W_1(t_0, y))^{-2} \sum_{i,j,h=1}^{d} |q_{ij} D_h W_1(t_0, y)|^2
\]

\[
+ 3(W_1(t_0, y)/n)^2 |\varphi'(W_1(t_0, y)/n)|^2 (W_1(t_0, y))^{-2} \sum_{h=1}^{d} |D_h W_1(t_0, y)|^2 \eta^2 \sum_{i,j=1}^{d} \delta_{ij}
\]

\[
\leq 3(c_2^2 + 4c_{12}^2 + 4dc_{12}) w^{-\frac{3}{2}} W_1^{\frac{7}{2}} .
\]

Then,

\[
|\nabla Q_n(t, y)| \leq \sqrt{3}(c_7 + 2c_{12} + 2\sqrt{dc_{12}}) w^{-\frac{3}{2}} W_1^{\frac{7}{2}} .
\]

We can now obtain estimates for the gradients of the kernels \(p_n\).

**Lemma 3.4.** Assume Hypotheses 3.2 hold and that the operator \(A\) satisfies Hypotheses 2.6. For \(i = 1, 2\), we set

\[
\xi_{W_{i,n}}(t, x) := \int_{\mathbb{R}^d} p_n(t, x, y) W_i(t, y) \, dy \quad \text{and} \quad \Xi_{i,n}(a_0, b_0) := \int_{a_0}^{b_0} \xi_{W_{i,n}}(t, x) \, dt.
\]

Then for any \(n \in \mathbb{N}\) we have

\[
|w(t, y) \nabla p_n(t, x, y)| \leq K_n
\]

for all \((t, y) \in (a_1, b_1) \times \mathbb{R}^d\) and fixed \(x \in \mathbb{R}^d\), where

\[
K_n = C \left\{ \begin{array}{l}
B_1 A_1^{\frac{1}{n-1}} \left( \sup_{t \in (a_0, b_0)} \xi_{W_{1,n}}(t, x) + (B_1 A_2^{\frac{1}{3}} + B_2 A_3^{\frac{1}{3}} + B_3 A_4^{\frac{1}{3}} + B_4 A_5^{\frac{1}{3}} + B_5 A_6^{\frac{1}{3}} + B_6 B_7 + B_7 B_8 + B_8 B_9 + B_9 B_{10}) \Xi_{2,n}(a_0, b_0) + B_1 A_1^{\frac{1}{3}} \Xi_{1,n}(a_0, b_0) ^{\frac{1}{3}} \left( \sup_{t \in (a_0, b_0)} \xi_{W_{2,n}}(t, x) \right) ^{\frac{1}{3}} 
+ B_1 A_1^{\frac{1}{3}} \left( B_2 \Xi_{1,n}(a_0, b_0) ^{\frac{1}{3}} + B_3 \Xi_{2,n}(a_0, b_0) ^{\frac{1}{3}} \right) \left( \sup_{t \in (a_0, b_0)} \xi_{W_{2,n}}(t, x) \right) ^{\frac{1}{3}} 
+ B_1 \left( A_2^{\frac{1}{3}} \Xi_{1,n}(a_0, b_0) ^{\frac{1}{3}} + A_3^{\frac{1}{3}} \Xi_{2,n}(a_0, b_0) ^{\frac{1}{3}} \right) \left( \sup_{t \in (a_0, b_0)} \xi_{W_{2,n}}(t, x) \right) ^{\frac{1}{3}} 
+ A_1^{\frac{1}{1-n}} \left( \int_{\mathbb{R}^d} p_n(t, x, y) \log^2 p_n(t, x, y) \, dt \, dy \right) ^{\frac{1}{3}} - \left( \int_{\mathbb{R}^d} [p_n(t, x, y) \log p_n(t, x, y)]_{t=a}^{b} \, dy \right) ^{\frac{1}{3}} \right\}
\right.
\]

and the constants \(A_1, A_3, B_1, \ldots, B_8, A_2, B_4, B_6\) are defined as in (2.0), (2.13), (3.4), (3.2) and (5.3).
Proof: Since the operator $A$ satisfies Hypotheses 2.6 and 3.2 then for any $n \in \mathbb{N}$ the operator $A_n$ satisfies Hypotheses 1.1 and 2.3 with slightly different constants by Lemmas 3.1 and 3.3 Consequently, applying (2.25) to $p_n$ we get

$$w(t,y)p_n(t,x,y) \leq C \left( A_1 \sup_{t \in (a_0,b_0)} \xi_{W_1,n}(t,x) + \tilde{A}_2 \Xi_{1,n}(a_0,b_0) + A_3 \Xi_{2,n}(a_0,b_0) \right),$$

where

$$\tilde{A}_2 = c_2^k + \frac{c_1^k}{(b_0-b)^{\frac{k}{2}}} + c_2^3 c_3^3 + c_2^3 c_1^12 + c_3^k + c_4^k + c_3^3.$$

Moreover, applying (2.28) to $p_n$, we obtain

$$|w(t,y)\nabla p_n(t,x,y)| \leq C \left\{ B_1 \Xi_{1,n}(a_0,b_0)^{\frac{k}{2}} \|w p_n\|^{\frac{k}{2}}_{L^\infty(Q(a,b))} + \left( B_2 \Xi_{1,n}(a_0,b_0)^{\frac{k}{2}} + B_3 \Xi_{2,n}(a_0,b_0)^{\frac{k}{2}} \right) \|w p_n\|^{\frac{k}{2}}_{L^\infty(Q(a,b))} + \left[ B_4 \Xi_{1,n}(a_0,b_0)^{\frac{k}{2}} + B_5 \Xi_{2,n}(a_0,b_0)^{\frac{k}{2}} \right] \left( \int_{Q(a,b)} \frac{\|\nabla p_n\|^2}{p_n} dt \right)^{\frac{1}{2}} \right\},$$

where

$$B_4 = c_3 + c_2 + \frac{c_2^2}{(b_0-b)^{\frac{k}{2}}} + c_2^3 c_3^3 + c_2^3 c_1^12 + c_3^k + c_3^3 + c_4^k + c_4^2,$$

$$B_6 = c_1^k + \frac{c_2^k}{(b_0-b)^{\frac{k}{2}}} + c_3^k + c_2^2 c_3^3 + c_2^3 c_1^12 + c_3^k + c_4^k + c_4^2.$$

Finally, by Theorem 2.7(a) we have

$$\int_{Q(a,b)} \frac{\|\nabla p_n\|^2}{p_n} dt \ dy \leq \left[ (c_0^k + c_1^k) \Xi_{2,n}(a_0,b_0) + \int_{Q(a,b)} p_n(t,x,y) \log^2 p_n(t,x,y) dt \ dy \right. - \left. \int_{R^d} [p_n(t,x,y) \log p_n(t,x,y)]^{\frac{k}{2}} dt \ dy \right].$$

Combining them yields

$$|w(t,y)\nabla p_n(t,x,y)| \leq C \left\{ B_1 \Xi_{1,n}(a_0,b_0)^{\frac{k}{2}} \left[ A_1^{\frac{k}{2}} \left( \sup_{t \in (a_0,b_0)} \xi_{W_1,n}(t,x) \right) \right]^{\frac{1}{2}} + \tilde{A}_2 \Xi_{1,n}(a_0,b_0)^{\frac{k}{2}} + \tilde{A}_2 \Xi_{1,n}(a_0,b_0)^{\frac{k}{2}} + A_3 \Xi_{2,n}(a_0,b_0)^{\frac{k}{2}} \right\} + \left[ B_1 \left( \sup_{t \in (a_0,b_0)} \xi_{W_1,n}(t,x) \right) \right]^{\frac{1}{2}} + \left( B_2 \Xi_{1,n}(a_0,b_0)^{\frac{k}{2}} + B_3 \Xi_{2,n}(a_0,b_0)^{\frac{k}{2}} \right) \left[ A_1^{\frac{k}{2}} \left( \sup_{t \in (a_0,b_0)} \xi_{W_1,n}(t,x) \right) \right]^{\frac{k}{2}} + A_4^{\frac{k}{2}} \Xi_{1,n}(a_0,b_0)^{\frac{k}{2}} + A_3^{\frac{k}{2}} \Xi_{2,n}(a_0,b_0)^{\frac{k}{2}} \right\} + \left[ B_4 \Xi_{1,n}(a_0,b_0)^{\frac{k}{2}} + B_5 \Xi_{2,n}(a_0,b_0)^{\frac{k}{2}} \right] \left[ A_1^{\frac{k}{2}} \left( \sup_{t \in (a_0,b_0)} \xi_{W_1,n}(t,x) \right) \right]^{\frac{k}{2}} + A_4^{\frac{k}{2}} \Xi_{1,n}(a_0,b_0)^{\frac{k}{2}}.$$
Assume that the operator theorem 3.7.

Making use of hypothesis 2.6(a), we find integrable majorants for corollary 3.6. Assume hypotheses 2.6 and 3.2. Then (c) follows from (a) by means of the dominated convergence theorem.

\[ B_8 = c_6 + c_2. \]

Considering that \( \Xi_{1,n}(a_0, b_0) \leq \Xi_{2,n}(a_0, b_0) \), we gain the desired estimate.

3.2. Estimates for the derivatives of the kernel.

Lemma 3.5. Assume hypotheses [3.6] hold and that the operator \( A \) satisfies hypothesis [3.6]. Then the following statements hold.

(a) \( p_n(t, x, \cdot) \to p(t, x, \cdot) \) locally uniformly in \( \mathbb{R}^d \) as \( n \to \infty \).
(b) \( \xi_{W_j,n}(\cdot, x) \to \xi_{W_j}(\cdot, x) \) uniformly in \( (a_0, b_0) \) as \( n \to \infty \) for \( j = 1, 2 \),
(c) \( \int_{\mathbb{R}^d} |p_n(t, x, y) \log p_n(t, x, y)|_{l=a}^{l=b} dy \to \int_{\mathbb{R}^d} |p(t, x, y) \log p(t, x, y)|_{l=a}^{l=b} dy \) as \( n \to \infty \) and \( \int_{Q(a,b)} p_n(t, x, y) \log^2 p_n(t, x, y) dt dy \to \int_{Q(a,b)} p(t, x, y) \log^2 p(t, x, y) dt dy \) as \( n \to \infty \). In particular, the latter integrals are finite.

Proof. (a) follows as in [7, lemma 12.7] and (b) as in the proof of [7, theorem 12.6]. It follows from equation (3.1), that \( p_n \leq C_n w^{-1} \) for a certain constant \( C_n \). But by (a) and (b) sup \( C_n < \infty \). Making use of hypothesis 2.6(a), we find integrable majorants for \( p_n \log p_n \) and \( p_n \log^2 p_n \). At this point, (c) follows from (a) by means of the dominated convergence theorem.

Corollary 3.6. Assume hypotheses [3.6] and [3.8] Then \( \sqrt{p} \in W^0,1_{2}(Q(a, b)) \).

Proof. As a consequence of hypothesis 3.6(b) and lemma 3.5

\[
C := \sup_{n \in \mathbb{N}} \frac{1}{\eta} \int_{Q(a,b)} (|F(y)|^2 + V^2(y)) p_n(t, x, y) dt dy \\
+ \int_{Q(a,b)} p_n(t, x, y) \log^2 p_n(t, x, y) dt dy \\
- \frac{2}{\eta} \int_{\mathbb{R}^d} [p_n(t, x, y) \log p_n(t, x, y)]_{l=a}^{l=b} dy < \infty.
\]

It follows from theorem 2.7 that \( \sqrt{p} \) is bounded in \( W^0,1_{2}(Q(a, b)) \). As this space is reflexive, a subsequence of \( p_n \) converges weakly to some element \( q \) of \( W^0,1_{2}(Q(a, b)) \). However, as \( p_n \to p \) pointwise and with an integrable majorant, testing against a function in \( C_c^\infty(\mathbb{R}^d) \), we see that \( q = p \).

We can now prove our main result.

Theorem 3.7. Assume that the operator \( A \) satisfies hypotheses [3.6] and [3.8] Then we have

\[ |w(t, y) \nabla p(t, x, y)| \leq K, \]
for all \((t,y) \in (a,b) \times \mathbb{R}^d\) and fixed \(x \in \mathbb{R}^d\), where

\[
K = \mathcal{C} \left\{ B_1A_1^{\frac{k-2}{k}} \sup_{t \in (a,b)} \xi_{W_1}(t,x) + \left( B_1A_2^{\frac{k}{k}} + B_2A_3^{\frac{k}{k}} + B_2A_4^{\frac{k}{k}} \right) \Xi_1(a_0,b_0) \right. \\
+ \left[ B_1A_3^{\frac{k-2}{k}} + (B_2 + B_3)A_4^{\frac{k}{k}} + B_3A_5^{\frac{k}{k}} + (B_4 + B_5)A_6^{\frac{k}{k}} + B_5A_6^{\frac{k}{k}} \right. \\
+ \tilde{B}_6B_8 + B_7B_9 \right] \Xi_2(a_0,b_0) + B_1A_7^{\frac{1}{k}} \Xi_1(a_0,b_0) \frac{1}{2} \left( \sup_{t \in (a,b)} \xi_{W_1}(t,x) \right) \right. \\
+ A_1^{\frac{k-2}{k}} \left( B_2 \Xi_1(a_0,b_0) \frac{1}{k} + B_3 \Xi_2(a_0,b_0) \frac{1}{k} \right) \left( \sup_{t \in (a,b)} \xi_{W_1}(t,x) \right)^{\frac{k-2}{k}} \\
+ B_1 \left( A_2^{\frac{k}{k}} \Xi_1(a_0,b_0) \frac{1}{k} + A_3^{\frac{k}{k}} + \Xi_2(a_0,b_0) \frac{1}{k} \right) \left( \sup_{t \in (a,b)} \xi_{W_1}(t,x) \right) \right. \\
+ A_1^{\frac{k-2}{k}} \left( \tilde{B}_2 \Xi_1(a_0,b_0) \frac{1}{k} + B_5 \Xi_2(a_0,b_0) \frac{1}{k} \right) \left( \sup_{t \in (a,b)} \xi_{W_1}(t,x) \right) \right. \\
+ \left( \tilde{B}_6 \Xi_1(a_0,b_0) \frac{1}{k} + B_7 \Xi_2(a_0,b_0) \frac{1}{k} \right) \left( \int_{Q(a,b)} p(t,x,y) \log^2 p(t,x,y) \, dt \, dy \right)^{\frac{1}{2}} \\
- \left( \tilde{B}_6 \Xi_1(a_0,b_0) \frac{1}{k} + B_7 \Xi_2(a_0,b_0) \frac{1}{k} \right) \left( \int_{\mathbb{R}^d} [p(t,x,y) \log p(t,x,y)]_{t=a}^{t=b} \, dy \right)^{\frac{1}{2}} \right\},
\]

(3.6)

and the constants \(A_1,A_3,B_1,\ldots,B_8,\tilde{A}_2,\tilde{A}_4,\tilde{B}_6\) are defined as in (2.0), (2.14), (3.4), (3.2) and (3.3).

Proof. By Lemmas 3.4 and 3.5 we infer that

\[
\limsup_{n \to \infty} |w(t,y)\nabla p_n(t,x,y)| \leq K.
\]

Then, for \(|h|\) small, we have

\[
|w(t,y) - p(t,x,y + h) - p(t,x,y)| \leq \limsup_{n \to \infty} |w(t,y)| \left| \frac{p_n(t,x,y + h) - p_n(t,x,y)}{h} \right| \\
\leq \limsup_{n \to \infty} |w(t,y)| \int_0^1 |\nabla p_n(t,x,y + sh)| \, ds \\
\leq K \int_0^1 \frac{|w(t,y)|}{|w(t,y + sh)|} \, ds.
\]

If we now let \(|h| \to 0\), we obtain the desired inequality. \(\square\)

As a simple consequence one obtains the following Sobolev regularity for \(p\).

Corollary 3.8. Assume in addition to Hypotheses 3.0 and 3.2 that \(\int_{Q(a,b)} w(t,x)^{-r} \, dt \, dx < \infty\) for some \(r \in (1, \infty)\). Then \(p \in W^{0,1}_r(Q(a,b))\).

3.3. Polynomially growing coefficients. Here we apply the results of the previous sections to the case of operators with polynomial diffusion coefficients, drift and potential terms.

Consider \(Q(x) = (1 + |x|^m)I, F(x) = -|x|^{p-1} x\) and \(V(x) = |x|^s\) with \(p > (m-1) \vee 1, s > |m-2|\) and \(m > 0\). To apply Theorem 3.7 we set

\[
w(t,x) = e^{\varepsilon |x|^\beta} \text{ and } W_j(t,x) = e^{\varepsilon_j |x|^\beta},
\]

for \((t,y) \in (0,1) \times \mathbb{R}^d\), where \(j = 1, 2, \beta = \frac{m+2}{2} - 2k\varepsilon < \varepsilon_1 < \varepsilon_2 < \frac{1}{\beta}\) and \(\alpha > \frac{\beta}{\beta + m-2}\).
Theorem 3.9. Let $p$ be the integral kernel associated with the operator $A$ with $Q(x) = (1 + |x|^m)I$, $F(x) = -|x|^{p-1}x$ and $V(x) = |x|^s$, where $p > (m - 1) \lor 1$, $s > |m - 2|$ and $m > 0$. Then

$$\left| \nabla p(t, x, y) \right| \leq C(1 - \log t) t^{\frac{3\alpha(m + \epsilon) + \gamma + \alpha}{2}} e^{-\epsilon|x|^\beta}$$

for $k > d + 2$ and any $t \in (0, 1)$, $x, y \in \mathbb{R}^d$.

Proof. Step 1. We show that $W_1$ and $W_2$ are time dependent Lyapunov functions for $L = \partial_t + A$ with respect to the function

$$Z(x) = e^{\varepsilon_1|x|^\beta}.$$

For that, we apply [6, Proposition 3.3]. Let $|x| \geq 1$ and set $G_j = \sum_{i=1}^d D_i q_{ij} = m|x|^{m-2}x_j$. Since $s > |m - 2|$, we have $\beta > (2 - m) \lor 0$. Moreover,

$$|x|^{1-\beta-m} \left( (G + F) \cdot x - \frac{V}{\varepsilon_j \beta |x|^\beta} \right) = |x|^{1-\beta-m} \left( m|x|^{m-1} - \frac{|x|^s}{\varepsilon_j \beta |x|^\beta} \right) \leq m|x|^{-\beta} - \frac{1}{\varepsilon_j \beta}.$$

If $|x|$ is large enough, for example $|x| \geq K$ with

$$K > \left( \frac{m}{\varepsilon_j \beta - 1} \right),$$

we get

$$|x|^{1-\beta-m} \left( (G + F) \cdot x - \frac{V}{\varepsilon_j \beta |x|^\beta} \right) \leq m|x|^{-\beta} - \frac{1}{\varepsilon_j \beta} \leq mK^{-\beta} - \frac{1}{\varepsilon_j \beta} < -1,$$

where we have used that $\varepsilon_j < 1$. In addition, we have

$$\lim_{|x| \to \infty} V(x) |x|^{2-2\beta-m} = \lim_{|x| \to \infty} |x|^{2-2\beta-m+s} = 1.$$ 

Hence, $\lim_{|x| \to \infty} V(x) |x|^{2-2\beta-m} > c$ for any $c < 1$. Consequently, by [6, Proposition 3.3] we obtain that $W_1$ and $W_2$ are time dependent Lyapunov functions for $L = \partial_t + A$. Similar computations show that the functions $Z(x)$ and $Z_0(x) = \exp(\varepsilon_2|x|^{p+1-m})$ satisfy, respectively, Hypothesis (1) (b) and (c).

Step 2. We now show that $A$ satisfies Hypotheses [2,4] and [5,6]. Fix $T = 1$, $x \in \mathbb{R}^d$, $0 < a_0 < b < b_0 < T$ and $k > 2(d + 2)$. Let $(t, y) \in [a_0, b_0] \times \mathbb{R}^d$. Clearly, Hypothesis [2,4] (a)-(b) and Hypothesis [5,6] (b) are satisfied. We assume that $|y| \geq 1$; otherwise, in a neighborhood of the origin, all the quantities we are going to estimate are obviously bounded.

First, since $2\varepsilon < \varepsilon_1$, we infer that

$$w \leq c_1 w_{1} \frac{\varepsilon_2}{\varepsilon_1} W_1^\frac{1}{p}$$

with $c_1 = 1$. Second, we have

$$\frac{|Q(y) \nabla w(t, y)|}{W_1(t, y)^\frac{1}{p}} = \varepsilon \beta t^\alpha |y|^\beta - (1 + |y|^m)t e^{-\frac{\gamma}{\beta}(\varepsilon_1 - 2k\varepsilon)t^\alpha |y|^\beta} \leq 2\varepsilon \beta t^\alpha |y|^\beta e^{-\frac{\gamma}{\beta}(\varepsilon_1 - 2k\varepsilon)t^\alpha |y|^\beta}.$$

We make use of the following remark: since the function $t \mapsto t^\alpha e^{-t}$ on $(0, \infty)$ attains its maximum at the point $t = p$, then for $\tau, \gamma, z > 0$ we have

$$z^\gamma e^{-\tau z^{-\beta}} = \tau^{-\frac{\gamma}{\beta}}(\tau z^{-\beta})^\frac{\gamma}{\beta} e^{-\tau z^{-\beta}} \leq \tau^{-\frac{\gamma}{\beta}} \left( \frac{\gamma}{\beta} \right) \frac{\gamma}{\beta} e^{-\frac{\gamma}{\beta}} = C(\gamma, \beta) \tau^{-\frac{\gamma}{\beta}}.$$

Applying (3.8) to the inequality (3.7) with $z = |y|$, $\tau = \frac{1}{2k}(\varepsilon_1 - 2k\varepsilon)t^\alpha$, $\beta = \beta$ and $\gamma = \beta + m - 1 > 0$ yields

$$\frac{|Q(y) \nabla w(t, y)|}{W_1(t, y)^\frac{1}{p}} \leq 2C(\beta + m - 1, \beta)\varepsilon \beta t^\alpha \left[ \frac{1}{2k}(\varepsilon_1 - 2k\varepsilon)t^\alpha \right]^{-\frac{\beta + m - 1}{\alpha}} \leq c t^{-\frac{\alpha(m-1)+}{\beta}} \leq c a_0^{-\frac{\alpha(m-1)+}{\beta}}.$$
Thus, we choose \( c_2 = \overline{c}_0 \alpha^{\frac{(m-2)^+}{\beta}} \), where \( \overline{c} \) is a universal constant. In a similar way,

\[
\frac{|Q(y)D^2w(t,y)|}{W_1(t,y)^\frac{1}{2}} \leq (1 + |y|^m)|D^2w(t,y)| \leq 2\sqrt{3}e^{\beta t^\alpha} \left[ (\beta - 2)^+ + \sqrt{d} \right] |y|^{\beta + m - 2} + \varepsilon \beta t^\alpha |y|^{2\beta + m - 2} e^{-\frac{1}{2}(\varepsilon - k)\varepsilon^\alpha |y|^\beta}.
\]

Applying (3.8) to each term, we get

\[
\frac{|Q(y)D^2w(t,y)|}{W_1(t,y)^\frac{1}{2}} \leq C(\beta, m) \varepsilon \beta t^\alpha \left\{ (\beta - 2)^+ + \sqrt{d} \right\} \left[ \frac{1}{k}(\varepsilon_1 - k\varepsilon) t^\alpha \right]^{-\frac{\beta + m - 2}{\beta}}
\]

\[
+ \varepsilon \beta t^\alpha \left[ \frac{1}{k}(\varepsilon_1 - k\varepsilon) t^\alpha \right]^{-\frac{2\beta + m - 2}{\beta}} \leq \overline{c}_0 \alpha^{\frac{(m-2)^+}{\beta}} \frac{\alpha^{(m-1)^+}}{\beta} + \overline{c}_0 \alpha^{\frac{(m-2)^+}{\beta}}.
\]

Therefore, we pick \( c_3 = \overline{c}_0 \alpha^{\frac{(m-2)^+}{\beta}} \). Furthermore, if we consider \( t_0 \in (0, t) \), we have

\[
\frac{|Q(y)||\nabla W_1(t_0, y)|}{W_1(t_0, y)w(t,y)^{1/k}W_1(t,y)^{1/2k}} = \sqrt{d} \varepsilon t_0^\alpha (1 + |y|^m)|y|^{\beta - 1} e^{-\frac{1}{2}(\varepsilon - k)\varepsilon^\alpha |y|^\beta}
\]

\[
\leq \overline{c}_0^{-\frac{(m-1)^+}{\beta}} \leq \overline{c}_0^{-\frac{(m-1)^+}{\beta}} =: c_{12},
\]

where we used (3.8). We can proceed in the same way to check the remaining inequalities. To sum up, the constants \( c_1, \ldots, c_{12} \) are the following:

\[
c_1 = 1, \quad c_2 = c_7 = c_{12} = \overline{c}_0 \alpha^{\frac{(m-1)^+}{\beta}}, \quad c_3 = \overline{c}_0 \alpha^{\frac{(m-2)^+}{\beta}},
\]

\[
c_4 = c_{11} = \overline{c}_0 \alpha^{\frac{1}{\beta}}, \quad c_5 = \overline{c}_0 \alpha^{\frac{1}{2}}, \quad c_6 = \overline{c}_0 \alpha^{\frac{1}{2}},
\]

\[
c_8 = \overline{c}_0 \alpha^{\frac{(m-1)^+}{\beta}}, \quad c_9 = \overline{c}_0 \alpha^{\frac{1}{2}}, \quad c_{10} = \overline{c}.
\]

Step 3. We are now ready to apply Theorem 3.7. To that end, we choose \( a_0 = \frac{t_0}{2}, a = t, b = \frac{3}{2}t \) and \( b_0 = 2t \). If we now set \( \lambda = m \lor \frac{s}{2} \), since \( \alpha > \frac{\beta}{\beta + m - 2} \), \( s > |m - 2| \) and \( \beta = \frac{s + m - 2}{2} \), we have

\[
\frac{\alpha \lambda}{\beta} > \frac{s}{2(\beta + m - 2)} = \frac{s}{s + m - 2} > \frac{1}{2}.
\]

Hence we can estimate the constants \( A_1 \) and \( A_3 \) in (2.6) as follows

\[
A_1 = c_1^{\frac{s}{2}} = 1,
\]

\[
A_3 = c_6^{\frac{s}{2}} + c_8^{\frac{s}{2}} + c_5^{\frac{s}{2}} = \overline{c} \left( t^{\frac{\alpha \lambda}{\beta}} + t^{\frac{\alpha \lambda}{\beta} + \frac{p k}{2}} \right) \leq \overline{c}^{-\frac{\alpha \lambda}{\beta}} t^{\frac{\alpha \lambda}{\beta}}.
\]

Similarly, if we consider the remaining constants in the right hand side of (2.6) we obtain that

\[
\tilde{A}_2 \leq \overline{c} \left( t^{\frac{\alpha \lambda}{\beta}} + t^{\frac{\alpha \lambda}{\beta}} \right), \quad B_1 \leq \overline{c}^{-\frac{\alpha \lambda}{\beta}}, \quad B_2 \leq \overline{c}^{-\frac{\alpha \lambda}{\beta}},
\]

\[
B_3 \leq \overline{c}^{-\frac{\alpha \lambda}{\beta}}, \quad B_4 \leq \overline{c}^{-\frac{\alpha \lambda}{\beta}}, \quad B_5 \leq \overline{c}^{-\frac{\alpha \lambda}{\beta}},
\]

\[
\tilde{B}_6 \leq \overline{c}^{-\frac{\alpha \lambda}{\beta}}, \quad B_7 \leq \overline{c}^{-\frac{\alpha \lambda}{\beta}}, \quad B_8 \leq \overline{c}^{-\frac{\alpha \lambda}{\beta}}.
\]

Moreover, by [6, Proposition 3.3], there are two constants \( H_1 \) and \( H_2 \) not depending on \( a_0 \) and \( b_0 \) such that \( \xi_{W_j}(t, x) \leq H_j \) for all \((s, x) \in [0, 1] \times \mathbb{R}^d \), so for \( j = 1, 2 \) we have

\[
(3.11) \quad \Xi_j(a_0, b_0) = \int_{a_0}^{b_0} \xi_{W_j}(t, x) dt \leq H_j(b_0 - a_0) = \frac{3t}{2} H_j.
\]

Furthermore, by Corollary 2.5 we obtain

\[
p(t, x, y) \leq Ct^{1 - \frac{\alpha \lambda}{\beta}} e^{-\varepsilon t^\alpha |y|^\beta},
\]
Then,
\[
p(t, x, y) \log p(t, x, y) \leq C t^{1 - \frac{\alpha k + a}{\beta}} \left[ \log C + \left( 1 - \frac{\alpha k}{\beta} \right) \log t - \varepsilon t^\alpha |y|^2 \right] e^{-\varepsilon t^\alpha |y|^2}.
\]

Considering that \(a = t\) and \(b = \frac{3}{2} t\), it leads to
\[
\int_{\mathbb{R}^d} [p(t, x, y) \log p(t, x, y)]_{t=a}^{t=b} dy \leq C (1 - \log t) t^{1 - \frac{\alpha k + a}{\beta}} \int_{\mathbb{R}^d} e^{-\varepsilon |z|^2} dz \leq C (1 - \log t) t^{1 - \frac{\alpha k + a}{\beta}},
\]
(3.12)
where in the integrals we performed the change of variables \(z = \frac{a}{2} y\) and \(z = \frac{b}{2} y\). We also get
\[
\int_{Q(a,b)} p(t, x, y) \log^2 p(t, x, y) dt dy \leq C (1 - \log t)^2 t^2 - \frac{\alpha k + a}{\beta}.
\]
(3.13)
Putting (3.12) in (3.11) yields
\[
K \leq C (1 - \log t)^2 t^2 - \frac{\alpha k + a}{\beta}.
\]
\(\square\)

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