EFFICIENT SIXTH ORDER ITERATIVE METHOD FREE FROM HIGHER DERIVATIVES FOR NONLINEAR EQUATIONS

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Abstract. In this paper, we proposed new iterative sixth order convergence method for solving nonlinear equations. The combination of the Taylor series and composition approach is used to derive the new method. Numerous methods have been developed by many researchers whenever the function’s second and higher order derivatives exist in the neighbourhood of the root. Computing the second and higher derivative of a function is a very cumbersome and time consuming task. In terms of low computation cost, the newly proposed method finds the best approximation to the root of non-linear equations by evaluating the function and its first derivative. The proposed method has been theoretically demonstrated to have sixth-order convergence. The proposed method has an efficiency index of 1.56. Several comparisons of the proposed method with the various existing iterative method of the same order have been performed on the number of problems. Finally, the computational results suggest that the newly proposed method is efficient compared to the well-known existing methods.

Keywords: iterative method; non-linear equations; efficiency index.

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1. INTRODUCTION

The solution of nonlinear equations governing the natural phenomena of real word problems is a very important and pertinent problem in computational sciences. It is almost impossible to
find exact solutions to many non-linear equations. For such problems, it would be important
to develop methods to obtain approximate solutions. Therefore, we investigate an iterative
technique for obtaining an approximate solution of the non linear equations of the form:

\[ \varphi(x) = 0 \]  

where \( \varphi : D \subset R \rightarrow R \) is substantially differential function in the interval D. The well-known
Newton’s method to find the approximate solution of the non-linear equation (1) is given by

\[ x_{n+1} = x_n - \frac{\varphi(x_n)}{\varphi'(x_n)}, \quad n = 0, 1, 2, \ldots. \]  

It is one point second-order method and requires evaluation of one function and one derivative at
each iteration. The one point iteration method based on the k-function evaluation has maximum
order k \([1, 2]\). The multipoint iterative approach has surpassed the theoretical limit of the one-
point methods in terms of computational efficiency and convergence order. If \( n \) be the number
of function evaluations per iteration of the method and \( \rho \) represents the order of convergence,
then the efficiency index \([2]\) of the method can be measured by \( \rho^\frac{1}{n} \). In the last few decades, the
problems of finding an approximation to the root of non-linear equations have been extensively
studied. Some surveys and complete literatures in this direction could be found in Argyros et
al. \([3]\), Abbasbandy \([4]\), Chun \([5, 6]\), Kogan et al. \([7]\), Kumar et al. \([8]\), Thota \([9]\), Mahesh et
al. \([10]\), Sabharwal \([11]\), Sharma et al. \([12]\), Chanu et al. \([13]\) and the references therein. Some
of the well-known methods developed recently for solving non-linear equations are detailed
below:

In 2007, Noor et al. \([14]\), proposed following modified Housholder iterative method (NM1 for
short) for solving nonlinear equations with order six:

\[ y_n = x_n - \frac{\varphi(x_n)}{\varphi'(x_n)} \]

\[ x_{n+1} = y_n - \frac{\varphi(y_n)}{\varphi'(y_n)} - \frac{[\varphi(y_n)]^2 \varphi''(y_n)}{2[\varphi'(y_n)]^3} \]  

In 2007, Noor et al. \([15]\), devised the following Predictor-Corrector Halley scheme for nonlinear
equations (NM2 for short) with order six:
\begin{equation}
  y_n = x_n - \frac{\varphi(x_n)}{\varphi'(x_n)}
\end{equation}

\begin{equation}
  x_{n+1} = y_n - \frac{2\varphi(y_n)\varphi'(y_n)}{2\varphi'(y_n) - \varphi(y_n)\varphi''(y_n)}
\end{equation}

In 2008, Parhi [16], presented a following new method for nonlinear equations (PM for short) with order six:

\begin{align*}
  y_n &= x_n - \frac{\varphi(x_n)}{\varphi'(x_n)} \\
  z_n &= x_n - \frac{2\varphi(x_n)}{\varphi'(x_n) + \varphi'(y_n)} \\
  x_{n+1} &= z_n - \frac{\varphi(z_n)}{\varphi'(x_n)} \frac{\varphi'(x_n) + \varphi'(y_n)}{3\varphi'(y_n) - \varphi'(x_n)}
\end{align*}

(5)

In 2012, Chun et al. [17], suggested a following new scheme for nonlinear equation (CM for short) with order six:

\begin{align*}
  y_n &= x_n - \frac{\varphi(x_n)}{\varphi'(x_n)} \\
  z_n &= y_n - \frac{\varphi(y_n)}{\varphi'(x_n)} \frac{1}{1 - \frac{\varphi(y_n)}{\varphi(x_n)}^2} \\
  x_{n+1} &= z_n - \frac{\varphi(z_n)}{\varphi'(x_n)} \frac{1}{1 - \frac{\varphi(y_n)}{\varphi(x_n)}^2 - \frac{\varphi(z_n)}{\varphi(x_n)}^2}
\end{align*}

(6)

In 2014, Singh et al. [18], presented the following new method for nonlinear equations (SM for short) with order six:

\begin{align*}
  z_n &= x_n - \frac{\varphi(x_n)}{\varphi'(x_n)} \\
  \tilde{x}_{n+1} &= z_n - \frac{\varphi(z_n)}{\varphi'(z_n)} \frac{\varphi(x_n)}{\varphi(x_k) - 2\varphi(z_n)} \\
  x_{n+1} &= \tilde{x}_{n+1} - \frac{\varphi(\tilde{x}_{n+1})(\tilde{x}_{n+1} - z_n)}{\varphi(\tilde{x}_{n+1}) - \varphi(z_n)}
\end{align*}

(7)

In 2019, Shengfen Li [19], proposed the following fourth order iterative method (LM for short) using Thiele’s continued fraction:
(8) \[ x_{n+1} = x_n - \frac{\varphi(x_n)(6\varphi'(x_n)^2\varphi''(x_n) - 3\varphi(x_n)\varphi'''(x_n)^2 + 2\varphi(x_n)\varphi'(x_n)\varphi'''(x_n))}{2\varphi'(x_n)(3\varphi'(x_n)^2\varphi''(x_n) - 3\varphi(x_n)\varphi''(x_n)^2 + \varphi(x_n)\varphi'(x_n)\varphi''''(x_n))} \]

Inspired by the recent activities in this direction, we present a new sixth-order approach for solving non-linear equations by altering the Taylor’s series expansion and composition techniques. The proposed method uses two evaluations of the function and two evaluations of the first derivative in each iteration. The proposed method is tested through numerical experimentation to support the theory on various non-linear equations. Finally, the newly proposed method is efficient in approximating the roots of non-linear equations. The remaining segment of the present study is organized as follows: The development of the new method is presented in Section 2. The theoretical result about the order of convergence of the proposed method is also established in Section 2. In Section 3, the numerical implementation of the proposed method is presented and the comparison of the results of the new method with other existing techniques of identical orders are summarized in tables. The concluding remarks are presented in Section 4.

2. CONSTRUCTION OF THE PROPOSED METHOD WITH ANALYSIS OF CONVERGENCE

Let \( \varphi(x) \) be a differentiable real valued function defined on the interval \( D \subset R \). Suppose that \( \alpha \in D \) is a simple zero for non-linear equation \( \varphi(x) = 0 \) and let \( x_n \) be a initial guess sufficiently close to \( \alpha \). By Taylor series quadratic approximation of \( \varphi(x) \) about the point \( x_n \), we get

(9) \[ \varphi(x) = \varphi(x_n) + (x - x_n)\varphi'(x_n) + \frac{(x - x_n)^2}{2!}\varphi''(x_n) \]

Assuming \( \varphi(x_{n+1}) = 0 \) to obtain the next approximation \( x_{n+1} \) of the root \( \alpha \) of \( \varphi(x) \) in the above equation, we get

(10) \[ \varphi(x_n) + (x_{n+1} - x_n)\varphi'(x_n) + \frac{(x_{n+1} - x_n)^2}{2!}\varphi''(x_n) = 0 \]

Reordering above equation, we get

(11) \[ x_{n+1} = x_n - \frac{\varphi(x_n)}{\varphi'(x_n)} - \frac{(x_{n+1} - x_n)^2}{2\varphi'(x_n)}\varphi''(x_n) \]

Substituting the value of \( x_{n+1} - x_n \) from equation (8) on the right side of equation (11), we get

(12) \[ x_{n+1} = x_n - \frac{\varphi(x_n)}{\varphi'(x_n)} - \left( \frac{\varphi(x_n)}{\varphi'(x_n)}\left( \frac{6J_1 - 3J_2 + 2J_3\varphi'''(x_n)}{3J_1 - 3J_2 + J_3\varphi'''(x_n)} \right) \right) \]
where
\[ J_1 = \varphi'(x_n)^2 \varphi''(x_n), \quad J_2 = \varphi(x_n) \varphi''(x_n)^2, \quad J_3 = \varphi(x_n) \varphi'(x_n) \]

Using Newton’s method as the predictor and equation (12) as the corrector, we get the following method

\[
y_n = x_n - \frac{\varphi(x_n)}{\varphi'(x_n)}
\]

\[
x_{n+1} = y_n - \frac{\varphi(y_n)}{\varphi'(y_n)} - \left( \frac{\varphi(y_n)}{\varphi'(y_n)} \left( \frac{6K_1 - 3K_2 + 2K_3 \varphi'''(y_n)}{3K_1 - 3K_2 + K_3 \varphi'''(y_n)} \right) \right)^2 \frac{\varphi''(y_n)}{2\varphi'(y_n)}
\]

where
\[ K_1 = \varphi'(y_n)^2 \varphi''(y_n), \quad K_2 = \varphi(y_n) \varphi''(y_n)^2, \quad K_3 = \varphi(y_n) \varphi'(y_n) \]

The third order derivative evaluation is required to implement the method given in equation (13). We introduce an approximation of third derivative to overcome this drawback. Let \( y_n = x_n - \varphi(x_n)/\varphi'(x_n) \) and using the Taylor series about the point \( x_n \), we get

\[
\varphi(y_n) = \varphi(x_n) + \varphi'(x_n)(y_n - x_n) + \frac{1}{2} \varphi''(x_n)(y_n - x_n)^2 + \frac{1}{6} \varphi'''(x_n)(y_n - x_n)^3
\]

Using \( y_n - x_n = -\varphi(x_n)/\varphi'(x_n) \) in above equation, we get

\[
\varphi(y_n) = \frac{\varphi(x_n)^2 \varphi''(x_n)}{2\varphi'(x_n)^2} - \frac{\varphi(x_n)^3 \varphi'''(x_n)}{6\varphi'(x_n)^3}
\]

\[
\varphi''(y_n) = \varphi''(x_n) - \frac{\varphi(x_n) \varphi'''(x_n)}{\varphi(x_n)}
\]

\[
\varphi'''(y_n) = \varphi'''(x_n) = \frac{6\varphi(y_n) \varphi'(x_n)^3 - 3\varphi(x_n)^2 \varphi'(x_n) \varphi''(y_n)}{2\varphi(x_n)^3}
\]

Substituting the value of \( \varphi'''(y_n) \) from equation (17) into the equation (13), we get the following method free from third derivative evaluation of the function:

\[
y_n = x_n - \frac{\varphi(x_n)}{\varphi'(x_n)}
\]

\[
x_{n+1} = y_n - \frac{\varphi(y_n)}{\varphi'(y_n)} - \left( \frac{\varphi(y_n)}{\varphi'(y_n)} \left( \frac{-2L_1 + L_2 + L_3 \cdot L_7}{2L_1 - 2L_2 - L_3 \cdot L_4} \right) \right)^2 \frac{\varphi''(y_n)}{2\varphi'(y_n)}
\]
where

\[ L_1 = \varphi(x_n)^3 \varphi'(y_n)^2 \varphi''(y_n), \quad L_2 = \varphi(x_n)^3 \varphi(y_n) \varphi''(y_n)^2, \quad L_3 = \varphi'(x_n) \varphi(y_n) \varphi'(y_n) \]

and \( L_4 = \varphi(x_n)^2 \varphi''(y_n) - 2 \varphi(y_n) \varphi'(x_n)^2 \).

The implementation of the proposed method (NPM1) given by equation (18) required the evaluation of second derivative. The evaluation of second derivative of the function is very cumbersome and time-consuming. So, the approximation of second derivative is obtained by applying Hermite’s interpolation. We consider that \( T(t) = q_1 + q_2 (t - y_n) + q_3 (t - y_n)^2 + q_4 (t - y_n)^3 \), where \( q_1, q_2, q_3 \) and \( q_4 \) are unknowns that can be established from the following conditions [20]:

\[ \varphi(x_n) = T(x_n), \quad \varphi'(y_n) = T'(y_n), \quad \varphi'(x_n) = T'(x_n), \]

\[ \varphi''(y_n) = T''(y_n), \quad \varphi''(y_n) = T''(y_n) \]

(19)

The above condition will create a four linear equations with four unknown variable \( q_1, q_2, q_3 \) & \( q_4 \). The following expression can be obtained by solving those four linear equations [20]:

\[ S(x_n, y_n) = \varphi''(y_n) = \frac{2}{(x_n - y_n)} \left[ 3 \frac{\varphi(x_n) - \varphi(y_n)}{(x_n - y_n)} - 2 \varphi'(y_n) - \varphi'(x_n) \right] \]

(20)

Substituting the value of \( \varphi''(y_n) \) from equation (20) into equation (18), we get the following iterative scheme free from the evaluation of higher derivatives:

\[ y_n = x_n - \frac{\varphi(x_n)}{\varphi'(x_n)} \]

\[ x_{n+1} = y_n - \frac{\varphi(y_n)}{\varphi'(y_n)} - \left( \frac{\varphi(y_n)}{\varphi'(y_n)} \left( \frac{-2M_1 + M_2 + M_3 \cdot M_4}{2M_1 - 2M_2 - M_3 \cdot M_4} \right) \right)^2 \]

(21)

where

\[ M_1 = \varphi(x_n)^3 \varphi'(y_n)^2 S(x_n, y_n), \quad M_2 = \varphi(x_n)^3 \varphi(y_n) S(x_n, y_n)^2, \quad M_3 = \varphi'(x_n) \varphi(y_n) \varphi'(y_n) \]

and \( M_4 = \varphi(x_n)^2 S(x_n, y_n) - 2 \varphi(y_n) \varphi'(x_n)^2 \).

The iterative schemes given by equation (21) has the sixth order of convergence and are denoted as NPM2. The newly proposed method (NPM2) requires two function evaluations and two evaluations of the first derivative per iteration. So, the efficiency index of new proposed method
given by equation (21) is $(6)^{\frac{1}{2}} \approx 1.56$. The order of convergence of the preceding method is analyzed in the following Theorem 2.1.

**Theorem 2.1.** Let $\alpha \in D$ be a simple root of a substantially differentiable function $\varphi : I \subset R \to R$ in an open interval $D$. If $x_0$ be initial guesses substantially nearby $\alpha$, then the iterative method defined by equation (21) has sixth-order convergence and satisfies the following error equation: 

$$e_{n+1} = -c_2^3 c_3 e_n^6 + O(e_n)^7,$$

where $e_n = x_n - \alpha$ is the error at $n$th iteration.

**Proof:** Since $\alpha$ be a root of $\varphi(x)$ and $e_n = x_n - \alpha$ is the error at $n$th iteration. So, we can expand $\varphi(x_n)$ in powers of $e_n$ by Taylor’s series expansion as follows:

$$\varphi(x_n) = \varphi(\alpha) + e_n \varphi'(\alpha) + \frac{e_n^2}{2!} \varphi''(\alpha) + \frac{e_n^3}{3!} \varphi'''(\alpha) + \ldots + \frac{e_n^7}{7!} \varphi^{(7)}(\alpha) + O(e_n^8)$$

Substituting $\varphi(\alpha) = 0$ and simplifying, we have

$$\varphi(x_n) = \varphi'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + O(e_n)]$$

where, $c_n = \frac{\varphi^{(n)}(\alpha)}{(n!)\varphi'(\alpha)}$ for $j = 2, 3, 4, \ldots$.

Furthermore, we get

$$\varphi'(x_n) = \varphi'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_7 e_n^6 + O(e_n^7)]$$

Using equation (23) and equation (24), we obtain

$$\frac{\varphi(x_n)}{\varphi'(x_n)} = e_n - (2c_3 + 2c_2^2) e_n^3 + (-3c_4 + 7c_2 c_3 - 4c_2^3) e_n^4 + (8c_2^4 - 20c_2^2 c_3 - 6c_3^2 + 10c_2 c_4 - 4c_5) e_n^5 + (-16c_2^5 + 52c_3^2 c_3 - 33c_2 c_3^2 - 28c_2^2 c_4 + 17c_3 c_4 + 13c_2^2 c_5 - 5c_6) e_n^6 + O(e_n^7)$$

By utilizing the equation (25) in the first step of equation (21), we get

$$y_n = c_2 e_n^2 + (2c_3 - 2c_2^2) e_n^3 + (3c_4 - 7c_2 c_3 + 5c_3^2) e_n^4 + (12c_4^2 - 24c_2^2 c_3 + 6c_3^2 + 10c_2 c_4 - 4c_5) e_n^5 + (28c_2^5 - 73c_2^3 c_3 + 37c_2 c_3^2 + 34c_2^4 c_4 - 17c_3 c_4 + 13c_2^2 c_5 + 5c_6) e_n^6 + O(e_n^7)$$
We expand $\varphi(y_n)$ and $\varphi'(y_n)$ in powers of $e_n$ by Taylor’s series expansion using equation (26), we get

$$\varphi(y_n) = \varphi'(\alpha)[c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_n^4 + (12c_2^4 - 24c_2^2c_3 + 6c_3^2 + 10c_2c_4 - 4c_5)e_n^5 + (28c_2^5 - 73c_2^3c_3 + 37c_2c_3^2 + 34c_2^2c_4 - 17c_3c_4 - 13c_2c_5 + 5c_6)e_n^6 + O(e_n^7)]$$

and

$$\varphi'(y_n) = \varphi'(\alpha)[1 + 2c_2 e_n^2 + (4c_2c_3 - 4c_2^3)e_n^3 + (6c_2c_4 - 11c_2^2c_3 + 8c_2^4)e_n^4 + 4c_2(-4c_2^4 + 7c_2^2c_3 - 5c_2c_4 + 2c_5)e_n^5 + 2(16c_2^6 - 34c_2^4c_3 + 6c_3^3 + 30c_3^3c_4 - 8c_2c_3c_4 - 13c_2c_5 + 5c_2c_6)e_n^6 + O(e_n^7)]$$

Using equations (23), (24), (26), (27) and (28) in equation (20), we get

$$S(x_n, y_n) = \varphi'(\alpha)[2c_2 + (6c_2c_3 - 2c_4)e_n^2 - 4(3c_2^2 - 3c_3^2 - c_2c_4 + c_5)e_n^3 + 2(12c_3^3c_3 + c_2^2c_4 + 13c_3c_4 + c_2(-21c_3^2 + c_5) - 3c_6)e_n^4 + 4(12c_2^4c_3 + 9c_3^3 + 6c_3^2c_4 + 12c_2c_3c_4 - 3c_2^2 - 7c_3c_5 - c_2^2(30c_3^2 + c_5) + 2c_7)e_n^5 + 2(48c_2^5c_3 + 44c_2^4c_4 - 43c_2^3c_4 + 11c_4c_5 - c_2^2(156c_3^2 + 5c_5) + 15c_3c_6 + c_2^2(14c_3c_4 + 3c_6) + c_2(99c_3^2 + 10c_4^2 - 30c_3c_5 - c_7) - 5c_8)e_n^6 + O(e_n^7)]$$

Using equations (24), (25), (28), (29) and (30) in the second step of method given by equation (21), we get

$$x_{n+1} = \alpha - c_2^3c_3e_n^6 + O(e_n^7)$$

Then we can write equation (30) as

$$e_{n+1} = -c_2^3c_3e_n^6 + O(e_n^7)$$

Hence, the method given by equation (21) has sixth order convergence.
3. Numerical Experimentation

In this section, we present the efficiency of the new proposed methods by applying the method on some nonlinear test function. The test functions and their initial guesses are listed in Table 1.

| Test Function $\varphi(x)$ | Initial guesses ($x_0$) |
|---------------------------|-------------------------|
| $\varphi_1(x) = (x)^2 - e^x - 3x + 2$ | 2 |
| $\varphi_2(x) = (x - 1)^3 - 1$ | 2.5 |
| $\varphi_3(x) = x^3 - 10$ | 2 |
| $\varphi_4(x) = \cos(x) - 10$ | 1.7 |
| $\varphi_5(x) = \sin(x)^2 - x^2 + 1$ | 1 |

We compare the novel methods with the existing sixth order method given in (3), (4), (5), (6) and (7) denoted by NM1, NM2, PM, CM and SM respectively. We denote the methods given in equation (18) and (21) by NPM1 and NPM2 respectively. The results of numerical comparison on the test functions with their roots are summarized in Table 2 to Table 6. The absolute residual error ($|\varphi(x_n)|$) of the corresponding functions, the approximated root ($x_n$) and the total number of function evaluation (TNFE) after completion of four full iterations of methods is presented from Table 2 to Table 6. From the results available in Table 2 to Table 6, we conclude that the newly proposed method given by equation (21) provides a better estimation of roots than other existing methods.
**Table 2. Convergence Behaviour for $\phi_1$**

| Method | $x_n$   | $|\phi_1(x_n)|$   | TNFE |
|--------|---------|-----------------|------|
| NM1    | 0.257502| $3.7207 \times 10^{-565}$ | 20   |
| NM2    | 0.257502| $2.3754 \times 10^{-559}$ | 20   |
| PM     | 0.257502| $1.0256 \times 10^{-384}$ | 20   |
| CM     | 0.257502| $9.5776 \times 10^{-676}$ | 16   |
| SM     | 0.257502| $2.0833 \times 10^{-635}$ | 20   |
| NPM1   | 0.257502| $1.7670 \times 10^{-557}$ | 20   |
| NPM2   | 0.257502| $8.7548 \times 10^{-779}$ | 16   |

**Table 3. Convergence Comparison for $\phi_2$**

| Method | $x_n$ | $|\phi_2(x_n)|$   | TNFE |
|--------|-------|-----------------|------|
| NM1    | 2     | $1.3866 \times 10^{-518}$ | 20   |
| NM2    | 2     | $2.8571 \times 10^{-604}$ | 20   |
| PM     | 2     | $1.7753 \times 10^{-686}$ | 20   |
| CM     | 2     | $1.1237 \times 10^{-474}$ | 16   |
| SM     | 2     | $1.1700 \times 10^{-596}$ | 20   |
| NPM1   | 2     | $1.6772 \times 10^{-693}$ | 20   |
| NPM2   | 2     | $1.6772 \times 10^{-693}$ | 16   |

**Table 4. Convergence Comparison for $\phi_3$**

| Method | $x_n$   | $|\phi_3(x_n)|$   | TNFE |
|--------|---------|-----------------|------|
| NM1    | 2.154434| $1.1007 \times 10^{-1398}$ | 20   |
| NM2    | 2.154434| $5.8619 \times 10^{-1501}$ | 20   |
| PM     | 2.154434| $2.0492 \times 10^{-1652}$ | 20   |
| CM     | 2.154434| $2.8285 \times 10^{-1272}$ | 16   |
| SM     | 2.154434| $2.0386 \times 10^{-1503}$ | 20   |
| NPM1   | 2.154434| $4.7520 \times 10^{-1579}$ | 20   |
| NPM2   | 2.154434| $4.7520 \times 10^{-1579}$ | 16   |
Table 5. Convergence Comparison for $\phi_4$

| Method | $x_n$     | $| \phi_4(x_n) |$ | TNFE |
|--------|-----------|-----------------|------|
| NM1    | 0.739085  | $1.1007 \times 10^{-1181}$ | 20   |
| NM2    | 0.739085  | $5.8619 \times 10^{-1217}$ | 20   |
| PM     | 0.739085  | $2.0492 \times 10^{-888}$  | 20   |
| CM     | 0.739085  | $5.1947 \times 10^{-857}$  | 16   |
| SM     | 0.739085  | $6.8174 \times 10^{-1099}$ | 20   |
| NPM1   | 0.739085  | $1.2111 \times 10^{-1276}$ | 20   |
| NPM2   | 0.739085  | $4.7520 \times 10^{-1241}$ | 16   |

Table 6. Convergence Comparison for $\phi_5$

| Method | $x_n$     | $| \phi_5(x_n) |$ | TNFE |
|--------|-----------|-----------------|------|
| NM1    | 1.404491  | $1.4731 \times 10^{-452}$ | 20   |
| NM2    | 1.404491  | $5.4945 \times 10^{-512}$ | 20   |
| PM     | 1.404491  | $1.1839 \times 10^{-566}$ | 20   |
| CM     | 1.404491  | $1.6063 \times 10^{-285}$ | 16   |
| SM     | 1.404491  | $2.8223 \times 10^{-538}$ | 20   |
| NPM1   | 1.404491  | $3.4810 \times 10^{-748}$ | 20   |
| NPM2   | 1.404491  | $1.1600 \times 10^{-612}$ | 16   |
4. **Concluding Remarks**

We have introduced a new sixth-order root-finding scheme for solving non-linear equations with four function evaluations. Taylor’s series and composition technique are used to build the proposed scheme. The new iterative approach has an efficiency index of 1.56. Compared to other existing well-known sixth-order schemes, numerical experimentation has shown that the newly proposed method is faster, uses fewer total number of function evaluations, and has a very low absolute residual error.

**Conflict of Interests**

The author(s) declare that there is no conflict of interest.

**References**

[1] H. T. Kung and J. F. Traub, Optimal order of one-point and multipoint iteration. J. ACM. 21(4) (1974), 643-651.

[2] J. F. Traub, Iterative methods for the solution of equations, Prentice-Hall, Englewood Cliffs. NJ. USA (1964).

[3] I. K. Argyros, A note on the Halley method in Banach spaces, Appl. Math. Comput. 58 (2-3) (1993), 215-224.

[4] S. Abbasbandy, Improving Newton-Raphson method for non-linear equations by modified Adomian decomposition method, Appl. Math. Comput. 145 (2003), 887-893.

[5] C. Chun, Iterative methods improving Newton’s methods by the decomposition method, Comput. Math. Appl. 50 (10) (2005), 1559-1568.

[6] C. Chun. A new iterative method for solving non-linear equations, Appl. Math. Comput. 187 (2006), 415-422.

[7] T. Kogan, L. Sapir and A. Sapir. To the question of efficiency of iterative methods, Appl. Math. Lett. 66 (2017), 40-46.

[8] A. Kumar, P. Maroju, R. Behl, D. K. Gupta and S. S. Motsa, A family of higher order iterations free from second derivative for non-linear equations in R, J. Comput. Appl. Math. 330 (2018), 676-694.

[9] S. Thota, A New Root–Finding Algorithm Using Exponential Series, Ural Math. J. 5(1) (2019), 83-90.

[10] G. Mahesh, G. Swapna, G. and K. Venkateshwarlu, An iterative method for solving non-linear transcendental equations. J. Math. Comput. Sci. 10(5) (2020), 1633-1642.

[11] C. L. Sabharwal, An iterative hybrid algorithm for roots of non-linear equations. Eng. 2(1) (2021), 80-98.

[12] E. Sharma, S. Panday and M. Dwivedi, New optimal fourth order iterative method for solving nonlinear equations, Int. J. Emerging Technol. 11(3) (2020), 755-758.
[13] W. H. Chanu, S. Panday and M. Dwivedi, New fifth order iterative method for finding multiple root of nonlinear function, IAENG Eng. Lett. 29(3) (2021), 942-947.

[14] K. I. Noor, M. A. Noor and S. Momani. Modified Householder iterative method for non-linear equations, Appl. Math. Comput. 190 (2007), 1534-1539.

[15] K. I. Noor and M. A. Noor, Predictor-corrector Halley method for non-linear equations, Appl. Math. Comput. 188 (2007), 1587-1591.

[16] S. K. Parhi and D. K. Gupta, A sixth-order method for nonlinear equations, Appl. Math. Comput. 203 (2008), 50-55.

[17] C. Chun and B. Neta, A new sixth-order scheme for nonlinear equations, Appl. Math. Lett. 25 (2012), 185-189.

[18] S. Singh and D. K. Gupta, A new sixth order method for nonlinear equations in R, Sci. World J. 2014 (2014), Article ID 890138.

[19] S. Li, Fourth order iterative method without calculating the higher derivatives for non-linear equation, J. Algorithms Comput. Technol. 13 (2019), 1-8.

[20] O. S. Solaiman and I. Hashim, Two new efficient sixth order iterative methods for solving non-linear equations, J. King Saud Univ. Sci. 31 (2019), 701-705.