A paradigm for entanglement theory based on quantum communication

Jonathan Oppenheim

1Department of Applied Mathematics and Theoretical Physics, University of Cambridge U.K.

Here it is shown that the squashed entanglement has an operational meaning – it is the fastest rate at which a quantum state can be sent between two parties who share arbitrary side-information. Likewise, the entanglement of formation and entanglement cost is shown to be the fastest rate at which a quantum state can be sent when the parties have access to side-information which is maximally correlated. A further restriction on the type of side-information implies that the rate of state transmission is given by the quantum mutual information. This suggests a new paradigm for understanding entanglement and other correlations. Different types of side-information correspond to different types of correlations with the squashed entanglement being one extreme. The paradigm also allows one to classify states not only in terms of how much quantum communication is needed to transfer half of it, but also in terms of how much entanglement is needed. Furthermore, there is a dual paradigm: if one distributes the side-information as maliciously as possible so as to make the sending of the state as difficult as possible, one finds maximum rates which give interpretations to known quantities (such as the entanglement of assistance), as well as new ones. The infamous additivity questions can also be recast and receive an operational interpretation in terms of maximally correlated states.

There are two main paradigms for understanding entanglement between two parts of a system. We say that a system is in an entangled state if measurements on it cannot be simulated by a local hidden variable model. Another (and perhaps inequivalent way) to understand entanglement is to say that we do not know what entanglement is, but we do know that it is a type of correlation which cannot increase under local operations and classical communication (LOCC) [1]. Entanglement measures are thus monotones (they must go down under LOCC).

Here, we will show that two of the most prominent entanglement measure and one more general correlation measure have an operational meaning. They give the rate at which one share of a state can be sent to a receiver – broadly speaking the more entangled a state is, the harder it is to send it. Here, the rate is how much quantum communication is needed to send each state, and which correlation measure corresponds to this rate is determined by what resource is given to the sender and receiver. In particular, the resource that is given to the sender and receiver is side-information – i.e. we give the sender and receiver quantum states which contain information about the state which is being sent. Different restrictions on the states which contain the side information give different rates at which quantum states can be sent, and these rates turn out to correspond to different correlation measures.

Two of the correlation measures (E, the entanglement cost [2, 3] and I(A : B) the quantum mutual information) had other operational meanings, but the squashed entanglement [4, 5] E_{sq} had thus far been a purely formal quantity as the quantum analog of the intrinsic mutual information [6]:

\[ E_{sq} \equiv \inf_{\Lambda} \frac{1}{2} I(A : B|\Lambda(E)) \]  \hspace{1cm} (1)

with \( \Lambda \) a completely positive trace preserving map, \( |\psi\rangle_{ABE} \) a pure state, \( I(A : B|\tilde{E}) = S(\tilde{A}\tilde{E}) + S(\tilde{B}\tilde{E}) - S(\tilde{AB}\tilde{E}) - S(\tilde{E}) \) the conditional mutual information of a tripartite state \( \rho_{ABE} \) and \( S(A) \) the entropy of the reduced state \( Tr_{BE} \rho_{ABE} \). Here, we find that the squashed entanglement is an extreme case: it is the rate at which a share of the state can be sent when the sender and receiver are given the best possible side information. The squashed entanglement is a remarkable entanglement measure because of it’s additivity, elegance and simple expression. We now see that it has a very intuitive and operational meaning as well.

This inspires the introduction of a third paradigm in which to understand entanglement and other correlations. We can in general consider the rate at which states can be sent to a receiver given different types of restrictions on the side-information states. This gives a relationship between sets of states and correlation measures. Furthermore, there is a dual paradigm. Instead of considering the best rate at which information can be sent, one can find the worst possible rate. i.e. the side-information is distributed as maliciously as possible in order to make the information as unhelpful as possible to the sender and receiver. This leads us to discover a set of operational quantities, some of which had already been known to be of significance, and some of which were previously unknown. We also are able to recast well known additivity questions [7] concerning the entanglement of formation and other quantities into an operational question about the utility of certain types of side information.

Let us imagine that Alice and Bob share many copies of a quantum system in state \( \rho_{AB} \), and we ask how many qubits are required for Alice to send her share of the state to a third party, Charlie. Everything we discuss will be symmetric under exchanging the roles of Alice and Bob. Here, Bob will be completely passive, his only role being to hold his part of the state \( \rho_B \). However, for Alice to
successfully send her share of the state, one requires that her protocol work for all possible pure states she may hold. Here, Alice knows the statistics of the state (i.e. she knows $\rho_{AB}$), but not which particular pure state she holds.

We are interested in maximising the rate at which the state can be sent, and so we allow Alice and Charlie to have as much help as possible in the form of ancillas. I.e we give them as much additional information as is allowed by the laws of quantum mechanics. In the classical world, this would be a lot, since there is nothing to prevent Charlie from knowing exactly what Alice has and therefore nothing would need to be sent from Alice to Charlie. However, quantumly, the amount of information Charlie can have is limited by the fact that Alice’s state is entangled with Bob’s. In particular, if we imagine the purification of $\rho_{AB}$, i.e. a pure state $|\psi\rangle_{ABE}$ such that $\text{Tr}_E[|\psi\rangle_{ABE}\langle\psi|_{ABE}] = \rho_{AB}$, then the most information that Charlie could possibly have would be to possess the share $\rho_E$. This might not be much information – for example, if Alice and Bob share a pure state, than Charlie’s holdings are completely uncorrelated from Alice’s.

Now if Charlie has $\rho_E$ then the amount of quantum communication that is required when no classical communication is allowed is given by the mutual information $I(A : B)/2$ [8], with the protocol for doing this given by a coherent version of merging [9]. Note that the usual convention is for Bob to be a reference system $R$ and for the receiver to be called Bob, but here this conflicts with the convention of quantifying the entanglement between two parties called Alice and Bob.

Now the rate might not be maximised by giving Charlie all the side information – it may be beneficial for the sender Alice to have some as well. One may therefore want to split $\rho_E$ into two shares, some of which is given to the receiver Charlie (call this $\rho_C$), and some of which is given to Alice (call this $\rho_{A'}$). What is the best rate we can achieve if we optimally give side information to the sender and receiver?

It is simple to show that the answer is the squashed entanglement. Given a noiseless quantum channel and arbitrary shared entanglement between a sender and receiver let $Q_{AC}$ be the rate of quantum communication that is required to send $\rho_A$ to a receiver who holds state $\rho_C$ with $\rho_{A'}$ being held by the sender. Then

**Theorem 1** Given a state $\rho_{AB}$ and arbitrary side-information $\rho_{AC}$, $\inf Q_{AC} = E_{sq}(A : B)$ where the infimum is taken over states $\rho_{AC}$ and $E_{sq}$ is the squashed entanglement.

The problem of finding the rate for state merging when both the sender and receiver have side information was found in [10] and was called state redistribution. Namely Alice only wants to send part of her state to Bob who shares part of her state. The situation is depicted in figure 1.

**FIG. 1:** Redistribution: $\rho_A$ is sent to $C$ retaining its correlations with $\rho_{A'BC}$. By optimising over how we split $\rho_{AC}$ we isolate the entanglement shared between $A$ and $B$ (dashed line)

The minimum amount of quantum communication required is given by the optimal cost pair

$$Q_{AC} = \frac{1}{2}I(A : B|C)$$
$$E = \frac{1}{2}I(A : A') - \frac{1}{2}I(A : C).$$

with $E$ being the amount of entanglement that is required. A simple explanation for this rate in terms of state merging is given in [11].

Now observe that because $\rho_C$ is a splitting of $\rho_E$ into two parts, $\rho_C$ and $\rho_{A'}$, we can treat the system $A'$ as an environment for a channel $\Lambda$ (i.e. completely trace preserving positive map) which acts on $\rho_E$ and produces the output $\rho_C$. Now, because we give Alice and Charlie all side information they require, this includes shared entanglement, and therefore the rate of Equation 2 can be obtained. If we now minimise the amount of quantum communication required as a function of all possible splittings of the ancillas, we have that this minimal communication $\overline{Q}_{all} \equiv \inf Q_{AC}$ is given by

$$\overline{Q}_{all} = \inf_{\rho_{A'C}} \frac{1}{2}I(A : B|C)$$
$$= \inf_{\Lambda} \frac{1}{2}I(A : B|\Lambda(E))$$
$$= E_{sq}(A : B).$$

Here, the last line follows from the definition of the squashed entanglement.

This gives an operational meaning to the squashed entanglement. The amount of pure state entanglement, $E$, that is gained given the ancilla splitting which minimises $Q_{AC}$ gives us a second parameter $E$ which would appear to characterise the state. Using the purity of the total state, and the fact that $S(F) = S(G)$ for a total pure state $\psi_{FG}$, we can convert Equation 3 into

$$E = S(A|C) - Q$$

with $S(A|C)$ the conditional entropy $S(AC) - S(C)$ and $E$ being the entanglement that is consumed during the
protocol (negative values indicate that entanglement is gained). Not surprisingly, the total amount of shared states i.e. \( Q_{AC} + E \) gives the amount of entanglement required to merge \( \rho \) \( A \) with the ancilla \( \rho_C \). Even a state with zero squashed entanglement (thus \( Q_{all} = 0 \)) may consume \( S(A|C) \) bits of pure state entanglement to share (or may result in a gain of ebits). Minimising this quantity as well
\[
E_{all} = \inf_{C} S(A|C) - Q_{all} \tag{6}
\]
where the infimum is taken over all \( C \) which minimise \( Q_{all} \) gives a way to differentiate even between states with the same squashed entanglement. We will see that this quantity is zero for separable states, although there may be some other states for which the squashed entanglement is zero, but for which \( E_{all} \) is non-zero. We will also find that this quantity is zero for any state where the squashed entanglement is equal to the entanglement cost.

It is worthwhile trying to understand why entanglement between Alice and Bob should effect the rate at which Alice’s share can be sent to a receiver. We noted previously that monogamy of entanglement plays a crucial role here. If Alice’s system is entangled with Bob, then it will be more difficult to find side-information on the ancillas \( A’ \) and \( C \) which can be correlated with \( \rho_A \) in such a way as to make the task of sending \( \rho_A \) easier. There is another reason why the procedure above quantifies correlation. Sending \( \rho_A \) is deemed successful if Alice’s share of the state is at the receiver, and the fidelity of the total pure state \( |\psi\rangle_{ABE} \) is kept during the protocol. This is equivalent to the protocol working for any pure state decomposition of \( \rho_{AB} \) (since measurements on \( \rho_E \) will induce a pure state ensemble on \( \rho_{AB} \). To preserve the overall pure state, Alice must thus send her state while keeping it correlated with Bob’s. The more correlation there is with Bob, the more she has to send. We can think of Bob as being the referee who checks to make sure that Alice has actually sent her state to Charlie (he would check this by bringing his share together with the other shares and measuring the total state). The more correlation Bob has with Alice, the more she must send.

Given this intuition, we can see if it leads to a more general paradigm – quantifying a state’s correlation with a another system by how much has to be sent to transfer the state and maintain the correlation. If we allow arbitrary side-information at the sender and receiver, then the best rate of sending is given by the squashed entanglement \( E_{sq} \). We can generalise this to consider more restricted types of side-information. \( E_{sq} \) thus represents one possible extreme. We will see that another correlation measure, the mutual information, is the other extreme. In between, we have other measures such as entanglement cost. We will also see that there is a dual paradigm. Table 1 gives a summary of the results.

We consider the quantity \( Q^2 \) to be the minimum qubit rate for sending \( \rho_A \) to a receiver, using side information at the encoder and decoder \( \rho_{A’C} \) chosen from the set \( S \), and the allowed class of operations being \( O \). In this paper we will mostly consider \( O \) to be local operations and the sending of qubits which is quantified, so we will thus drop the superscript. However, one might also want to consider \( O \) which include free entanglement or classical communication.

**Definition 2**
\[
Q_S = \inf_{\rho_{A’C} \in S} Q_{AC}(A : B) \tag{7}
\]
However, in general one can consider any class of operations, and one can even count some other resource (e.g. private communication) rather than sent qubits. We have so far considered the case when the set \( S \) is unlimited and contains all possible states, including possible extensions of \( \rho_{AB} \).

Let us now consider the case where the set \( S \) is restricted to the set of maximally correlated states (MCS). I.e. states of the form
\[
\rho_{A’C} = \sum_{ij} \sigma_{ij} |ii\rangle\langle jj| . \tag{8}
\]
With such a restriction on the side information \( \rho_{A’C} \), we find that the maximum rate for sending \( \rho_A \) is given by the entanglement cost \( E_c \).

**Theorem 3** Given a state \( \rho_{AB} \), \( Q_S = E_c(A : B) \) when \( S \) is the set of MCS.

\( E_c \) is defined as the amount of pure state entanglement which is required to create a copy of the state in the limit that many copies of the state are created, and it is given by
\[
E_c = \lim_{n\to\infty} \inf \sum_{i} p_i S(\rho_{A}^{i}) / n \tag{9}
\]
where the infimum is taken over decompositions
\[
\rho_{AB}^{\otimes n} = \sum_{i} p_i |\psi_{AB}^{i}\rangle\langle \psi_{AB}^{i}| \tag{10}
\]
(the entanglement of formation is defined as the entanglement cost, but for single copies of \( \rho_{AB} \), i.e. \( n = 1 \)).

To prove Theorem 3 we note that for any decomposition (not just the one which minimises the entanglement of formation), we can write the purification as
\[
|\psi\rangle_{ABAC}^{\otimes n} = \sum \sqrt{p_i} |\psi^{i}\rangle_{AB}|ii\rangle_{AC} . \tag{11}
\]
One can verify that the reduction of this state on \( A’C \) is a maximally correlated state with \( \sigma_{ij} = \sqrt{p_i} \sqrt{p_j} |\psi^{i}\rangle\langle \psi^{j}| \).

Furthermore for MCS we find,
\[
I(A : B|C)/2 = \sum p_i S(\rho_{A}) . \tag{12}
\]
We thus have $E_{Q} = \min \{S(A), S(B)\}$. $\ast E_{pu}$ is the supremum rate if pure entanglement is an allowable resource.

This quantity is equal to the rate for sending Alice’s share to Bob when entanglement is allowed as a resource, however, because pure state entanglement can be in the form of a MCS, it gives the rate in this case as well. Taking the infimum of this expression over decompositions of Eq. (11) gives $\inf I(A : B | C) / 2 = n E_c$. Since all maximally correlated states can be written as the reduction of a purification of the form in Eq. (11) we have that the infimum over decompositions (10) is equivalent to an infimum over maximally correlated states. We thus have another interpretation of the entanglement cost: it is the fastest rate at which $\rho_A$ can be sent to a receiver using side-information composed of maximally correlated states.

Note that the entanglement of formation has a similar interpretation if we restrict ourselves to side-information which is an extension of each single copy. While this doesn’t matter for the squashed entanglement which is additive, we do not know whether it matters for the entanglement of formation. Here, the additivity question is whether using maximally correlated states which are the purification of two states is no better for state transfer, than maximally correlated states which are the purification of each single state. We thus have another additivity question related to the others – one that is operational.

Also, the entanglement cost $E$ when the side-information is restricted to MCS is always zero. This can be seen from expression (3) and the fact that MCS are symmetric. It is for these reason that $E$ is zero whenever the squashed entanglement is equal to the entanglement cost.

The most simple example of the paradigm is when the set $\mathcal{S}$ are states such that all the side information is at the sender or the receiver. In such case, it is trivial to see that the communication rate is the mutual information $I(A : B) / 2$ which is a measure of total correlations (both classical and quantum). Note that if we bend the paradigm slightly, and allow for the restriction on $\mathcal{S}$ to allow for no distribution of side-information then the best rate that Alice can achieve is through simple compression, giving a rate of $S(A)$.

Thus far, we have been distributing the side information so as to minimise the amount of communication required. There is a dual paradigm – instead we arrange the side-information in the worst possible manner. I.e.

**Definition 4** the dual to $Q_S$ is

$$Q_S \equiv \sup_{\rho_{A'C} \in \mathcal{S}} Q_{A'C}(A : B)$$

Here, the distributor of the side information is forced to distribute all the side information, so the only choice they have is how to arrange it between Alice and Charlie. When we do this, we will find that the rates for state transfer are given by dual correlation measures. For example, the entanglement of assistance [12] is the maximum amount of pure state entanglement that can be created between Alice and Bob given a measurement on the purification of their state. Here, we find another interpretation – it is the amount of information which must be sent if the side-information $\rho_{A'C}$ is maliciously chosen from the set of maximally correlated states.

**Theorem 5** Given a state $\rho_{AB}$, $Q_S = E_{ass}(A : B)$ when $\mathcal{S}$ is the set of MCS.

The proof is similar to that of the entanglement cost except that one needs to use the fact that $E = 0$ for MCS since a malicious distributor of the side information is unlikely to give the parties any entanglement. The entanglement of assistance is known to be equal to $\min \{S(A), S(B)\}$ [9, 13].

It is tempting to look at maximising the amount of communication required over all possible side-information states. This quantity would be a dual to the squashed entangled but with the infimum changed to a supremum.

**Definition 6** The puffed entanglement [17] of the state $\rho_{AB}$ is $E_{pu} = \sup \frac{1}{2} I(A : B | E)$, with the supremum taken over all extensions $E$ of $\rho_{AB}$.

One can show that it is not an entanglement measure, but it’s operational meaning is that it is the rate required to send the state $\rho_A$ if the distributor of the side information is as malicious as possible, and distributes $\rho_{A'C}$ in as unhelpful a way possible. However, here one does have to allow the parties free entanglement.

| side-information | infimum rate | supremum rate |
|------------------|--------------|---------------|
| no restriction   | $E_{sq}$ (squashed entanglement) | $E_{pu}$ (puffed entanglement) |
| maximally correlated states (MCS) | $E_c$ (entanglement cost) | $E_{ass}$ (entanglement of assistance) |
| all of it at the receiver or sender | $I(A : B) / 2$ (mutual information) | $I(A : B) / 2$ (mutual information) |

TABLE I: Summary of relationships between the restrictions on side-information and the rate at which half the state can be sent to the receiver. Note that $E_{pu} = E_{ass} = \min \{S(A), S(B)\}$. *$E_{pu}$ is the supremum rate if pure entanglement is an allowable resource.
Remarkably, the puffed entanglement is equal to the entanglement of assistance.

**Theorem 7** Given a state $\rho_{AB}$, $Q_S = E_{pu}(A : B) = \min\{S(A), S(B)\}$ when $S$ is the set of all possible states.

The second equality comes because standard entropic inequalities imply that $I(A : B|C)/2 \leq \min\{S(A), S(B)\}$, while a protocol exists (using an MCS) to get equality. Thus the same side information which maximises the entanglement of assistance also maximises the puffed entanglement. It is perhaps surprising that getting to choose the side information over all possible states but with free entanglement for the parties, is equivalent to choosing the side information from maximally correlated states. One can also show that another way of selecting the worst possible side-information is to perform a random unitary on $\rho_{AB}^{\otimes n}$ and then choose $\rho'_A$ of size just over $\min\{nI(E : A)/2, nI(E : B)/2\}$ qubits. This means that $\rho_C$ will be decoupled from both $\rho_A$ and $\rho_B$. It is an open question what the supremum sending rate is if entanglement is not a free resource.

Both the entanglement of assistance, and puffed entanglement are super-additive because they involve an infimum over possible extensions. However, neither are additive i.e. one can have

$$E_{pu}(\rho_{AB}^1 \otimes \rho_{AB}^2) > E_{pu}(\rho_{AB}^1) + E_{pu}(\rho_{AB}^2) \quad (14)$$

a situation which can occur when, for example, $S(A_1) < S(B_1)$ and $S(B_2) > S(B_2)$.

For the case of side-information all at the sender or receiver, the rate is half the mutual information regardless of whether we want to minimise or maximise the communication rate, so in this paradigm, the mutual information is self-dual.

It would be interesting to look at the correlation measures and rates under other restrictions of the set $S$. For example, one could consider side-information which is separable, or whose states have positive partial transpose, or which are only classically correlated. Likewise, one might also consider other types of operations. As one potential example, there appears to be a strong link between the intrinsic mutual information and the maximum rate of sending a private distribution using only a private channel. Here, the intrinsic information is a measure of the maximum rate for sending classical distributions if one stays within a certain class of distributions.

It would be interesting to know what the conditions are on the set $S$ such that $\overline{Q}_S$ is an entanglement measure (or correlation measure). Some small step towards this is given by the theorem

**Theorem 8** If $S$ includes pure entanglement for all $\sigma_0, \sigma_1 \in S$ and all purifications $|\psi_0\rangle, |\psi_1\rangle$, $T_X(|\psi_0\rangle|00) + |\psi_1\rangle|11\rangle)/\sqrt{2} \in S$ then $\overline{Q}_S$ is an entanglement monotone.

Here the trace is taken over the Hilbert space which is complement to $\sigma_0, \sigma_1$ (i.e. the purification).

The above theorem is enough to imply that $\overline{Q}_S$ is convex. One then only needs to show that $\overline{Q}_S$ cannot increase on average after application of a local CPT map. The adaptation of the proof of Theorem 11.15 of related in [4] is sufficient for our purposes.

In general, we have found a relationship between various sets of states and correlation measures. Here, the squashed entanglement arises as a more quantum version of the entanglement cost. While traditionally, these correlation measures arise as a monotone under some class of operations, here everything is determined in terms of the set of states. Remarkably, the notion of classical communication or classicality does not enter into the discussion, yet it gives rise to at least two entanglement measures.

There is a sense in which the paradigm is reversible: rates are the same regardless of who gets what part of the side-information state, i.e. if we swap $A'$ with $C$. This means that the same amount of communication is needed to send the state from Alice to Charlie as from Charlie to Alice, and we can send the state back and forth using and recovering the pure state entanglement.

It would also be interesting to explore the significance of the amount of entanglement which is consumed at the maximal (or dual) sending rate. It may even shed some light on the relationship between the paradigm considered here, and that of LOCC. In this case, a key question is whether a state that doesn’t show its entanglement here (in the sense that it requires no quantum communication to send half of it to a receiver), is also unentangled in the LOCC case. We know that if $E_c = 0$ then $E_{eq} = 0$, but the converse could well be false. It would also be interesting to know what the dual to the squashed entanglement is in the case where pure entanglement is not an allowable resource.

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[17] I apologise for this name, but what exactly would you expect the dual of the squashed entanglement to be called?