Mathematical model and numerical algorithms to analyze gas filtration process in a porous medium

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Abstract. The paper deals with the problem of gas filtration process in a porous medium using a mathematical model of an object; the model is described by a nonlinear partial differential equation and the corresponding boundary and internal conditions. The paper presents the main stages in construction of mathematical model of the process of gas filtration in porous media, taking into account the changes in hydrodynamic parameters of the object of study. To solve the above problem, the following numerical methods have been used: local-one-dimensional schemes and longitudinal-transverse direction schemes. To solve the problem of nonlinear gas filtration in a porous medium, several methods of constructing an iterative process have been checked. To study the responses of the principal parameters of the process, a series of computational experiments on a computer, their analyses and conclusions are given in the paper.

1. Introduction

One of the main natural sources of energy production is oil and gas fields, which require great attention, study, rational and effective management to increase their oil and gas output and the volume of the final product.

The problems of production, processing, transportation and saving of energy and the creation of management systems for rational use of energy resources, in particular, gas, oil and petroleum products are the priority directions on a global scale.

To accelerate the process of designing and developing new oil and gas fields, to improve technical-economic indices of oil and gas production of reservoir systems and to provide for the most complete extraction of products from old oil and gas deposits, it is necessary to conduct comprehensive studies using efficient, easily implemented mathematical tools, methods, computing systems and software based on new developments of information and communication technology.

One of the main stages in the analysis of operation, prediction and operational management of complex hydrodynamic processes in porous media is an account of non-homogeneity of the reservoir system and the structure of the reservoir and the properties of the non-homogeneity of reservoir systems that determine the structure of filtration flows in the process of field operation.
It should be noted that many researchers are engaged in the problems of modeling the process of mass transfer in porous media. To date, a number of significant theoretical and applied results have been obtained.

In particular, P.J. Monteiro, Ch.H. Rycroft, and G.I. Barenblatt [1] have developed a mathematical model for filtering fluids in nano-porous rocks. In deriving the process model, it is assumed that the filtration layer consists of two components: a fractured porous medium and specific organic inclusions consisting of carbogenes; the model is based on the hypothesis that the permeability of the inclusions substantially depends on the pressure gradient.

In [2] F. Boyer described some aspects of the simulation of a diffuse flow of incompressible media consisting of three immiscible components, without phase transformations, where the simulation process of three-phase flows includes the Cahn-Hillard and Navier-Stokes equations.

An improved mathematical model for non-equilibrium two-component (for example, water-oil) flows in porous media is given in [3]. The calculations results obtained have been compared to experimental data.

A. Raeini, M. Blunt and B. Bijeljic [4] have presented a stable numerical scheme for modeling multiphase flow in porous media, when the characteristic size of the flow region is from micron to millimeters.

M. Chraibi [5] have considered a mathematical model of the filtering process of a multicomponent medium. In the proposed model, the authors replaced the relative phase permeability of the gas phase with a new expression that takes into account the influence of viscosity, density, and capillary effect of the mixture.

The authors in [6] have proposed a mathematical model for the development of oil and gas fields, taking into account the probability distribution of the parameters of the process under study.

The study in [7] is focused on the derivation of a numerical model of gas filtration in porous inhomogeneous media based on the finite element method using fractional derivatives with respect to time. The authors have considered the Caputo and Riemann-Liouville fractional derivatives. A numerical analysis using experimental input data has been done.

2. Statement of the problem

Mathematical statement of the considered boundary value problem of the theory of filtration in the general case has the following form

It is necessary to solve the differential equation of unsteady filtering of real gas in a deformable porous medium.

\[
\frac{\partial}{\partial x} \left[ \frac{K(p)}{\mu(p)} z(p) \frac{\partial}{\partial x} P(x,y,t) \right] + \frac{\partial}{\partial y} \left[ \frac{K(p)}{\mu(p)} z(p) \frac{\partial}{\partial y} P(x,y,t) \right] = \frac{\partial}{\partial t} \left[ \frac{m(p)}{z(p)} P(x,y,t) \right] + \delta Q.
\]

(1)

In a multiply connected region of filtration \( G - \sum \gamma_l \) at \( t > t_0 \) an outer boundary of which is \( \Gamma, \gamma_l \) is the inner contour of the \( l \)-th well.

On the outer boundary of the filtration region the boundary conditions can be as follows:

\[
\begin{aligned}
\frac{K(p)}{\mu(p)} z(p) \frac{\partial}{\partial n} P(x,y,t) &= \eta(P-f(x,y,t)), \quad (x,y) \in \Gamma \\
P(x,y,t) &= f(x,y,t).
\end{aligned}
\]

(2)
The first condition means that, generally speaking, there may be a hydrodynamic interaction of the reservoir through the outer boundary of the filtration region. In particular, at $\eta = 0$ the filtration rate at the boundary is zero. The second condition means that, at the corresponding points of the boundary, the gas pressure regions in time are known.

On the wells, the boundary conditions depend on the accepted modes of operation. If the specified flow rate $q_l(t)$ is maintained in time, then we have

$$\int_0^K \frac{K(p)}{\mu(p)} \frac{P}{z(p)} \frac{\partial P}{\partial n} ds = c q_l(t)$$

(3)

where $c = P_{in}T_{in} / T_{in}v$; $P_{in}, T_{in}$ are the pressure and gas temperature in normal conditions, $T_{in}$ is the reservoir temperature, $v$ is the reservoir capacity.

If to maintain a given bottomhole pressure in time, then

$$P(x, y, t) = f_l, \ (x, y) \in \gamma_l$$

(4)

In addition to the boundary conditions, it is necessary to set an initial condition characterizing the reservoir pressure before the next number of wells is put into operation:

$$P(x, y, t_0) = \varphi(x, y), \ (x, y) \in G + \Gamma$$

(5)

For the numerical solution of the boundary value problem, it is convenient to proceed to new dimensionless variables

$$P^* = \frac{P}{P_0}, \ x^* = \frac{x}{L}, \ y^* = \frac{y}{L}, \ K^* = \frac{K}{K_0}, \ \mu^* = \frac{\mu}{\mu_0}, \ z^* = \frac{z}{z_0}, \ m^* = \frac{m}{m_0}, \ t^* = \frac{K_0 P_0 Z_0 t}{m_0 \mu_0 L^2}, \ q_v^* = \frac{c \mu_0 z_0 q_v}{K_0 P_0}$$

Here $P_0$ is the reservoir pressure prior to development; $L$ is the length of the reservoir; $K_0, \mu_0, z_0, m_0$ are the relevant values of permeability, dynamic viscosity, over-compressibility and porosity under normal conditions, respectively.

In the new variables, the form of differential equation and the corresponding boundary and initial conditions do not change; in what follows, the dimensionless index is omitted and denoted by

$$\overline{K} = \frac{K}{\mu z}, \ M = \frac{m}{z P},$$

and then, omitting the “dash” above $K$, we get:

$$\frac{\partial}{\partial x} \left[ K \frac{\partial P^2(x, y, t)}{\partial x} \right] + \frac{\partial}{\partial y} \left[ K \frac{\partial P^2(x, y, t)}{\partial y} \right] = \frac{\partial}{\partial t} \left[ MP^2(x, y, t) \right] + \delta Q.$$
or vice versa. For approximation of differential equations the homogeneous difference schemes are applied, given in detail in the studies of Samarsky A.A. and his students.

In economical difference schemes, the greatest attention is paid to two schemes: a local-one-dimensional scheme and a longitudinal-transverse direction scheme. Samarsky A.A. gave theoretical substantiations of both methods, which have the characteristic feature of economical schemes (a scheme of total approximation and additivity).

Using a local-one-dimensional scheme or a scheme of longitudinal-transverse directions, one can obtain a system of equations in finite differences for the internal nodes of the discrete filtering region.

In the first case, the system has the form:

on the straight lines $c_{1,j}$, $j=1, N_2$ [8, 9]

$$a_{i,j} P_{i+1,j} + b_{i,j} P_{i,j} + c_{i,j} P_{i-1,j} = -\ddot{d}_{i,j}$$

where

$$\ddot{d}_{i,j} = \frac{\sigma K_{i+0.5,j}}{h_i h_{i+1}}, \quad \ddot{d}_{i,j} = \frac{\sigma K_{i-0.5,j}}{h_i h_{i-1}}, \quad \ddot{b}_{i,j} = \frac{\sigma K_{i+0.5,j}}{h_i h_{i+0.5}} + \frac{\sigma K_{i-0.5,j}}{h_i h_{i-0.5}} + \frac{M_{i,j}}{0.5\tau_k},$$

$$\ddot{d}_{i,j} = \left[ \frac{M_{i,j}}{0.5\tau_k} - \frac{1-\sigma}{h_j} \left( \frac{K_{i+0.5,j}}{h_{i+1}} + \frac{K_{i-0.5,j}}{h_i} \right) \right] \frac{\ddot{p}_{i,j}}{h_i} + \frac{1-\sigma}{h_j} \left( \frac{K_{i+0.5,j}}{h_{i+1}} + \frac{K_{i-0.5,j}}{h_i} \right) \frac{\ddot{p}_{i,j+1}}{h_{i+1}}.$$

On the straight lines $c_{2,i}$, $i=1, N_1$

$$a_{i,j} P_{i+1,j} + b_{i,j} P_{i,j} + c_{i,j} P_{i-1,j} = -\ddot{d}_{i,j}$$

where

$$\ddot{d}_{i,j} = \frac{\sigma K_{i,j+0.5}}{h_i h_{i+1}}, \quad \ddot{d}_{i,j} = \frac{\sigma K_{i,j-0.5}}{h_i h_{i-1}}, \quad \ddot{b}_{i,j} = \frac{\sigma K_{i,j+0.5}}{h_i h_{i+0.5}} + \frac{\sigma K_{i,j-0.5}}{h_i h_{i-0.5}} + \frac{M_{i,j}}{0.5\tau_k},$$

$$\ddot{d}_{i,j} = \left[ \frac{M_{i,j}}{0.5\tau_k} - \frac{1-\sigma}{h_j} \left( \frac{K_{i,j+0.5}}{h_{i+1}} + \frac{K_{i,j-0.5}}{h_i} \right) \right] \frac{\ddot{p}_{i,j}}{h_i} + \frac{1-\sigma}{h_j} \left( \frac{K_{i,j+0.5}}{h_{i+1}} + \frac{K_{i,j-0.5}}{h_i} \right) \frac{\ddot{p}_{i,j+1}}{h_{i+1}}.$$

In the second case the system is presented in the following form: on the straight lines $c_{1,j}$, $j=1, N_2$

$$a_{i,j} P_{i+1,j} + b_{i,j} P_{i,j} + c_{i,j} P_{i-1,j} = -\ddot{d}_{i,j}$$

where

$$\ddot{d}_{i,j} = \frac{\sigma K_{i+0.5,j}}{h_i h_{i+1}}, \quad \ddot{d}_{i,j} = \frac{\sigma K_{i-0.5,j}}{h_i h_{i-1}}, \quad \ddot{b}_{i,j} = \frac{\sigma K_{i+0.5,j}}{h_i h_{i+0.5}} + \frac{\sigma K_{i-0.5,j}}{h_i h_{i-0.5}} + \frac{M_{i,j}}{0.5\tau_k},$$

$$\ddot{d}_{i,j} = \left[ \frac{M_{i,j}}{0.5\tau_k} - \frac{1}{h_j} \left( \frac{K_{i+0.5,j}}{h_{i+1}} + \frac{K_{i-0.5,j}}{h_i} \right) \right] \frac{\ddot{p}_{i,j}}{h_i} + \frac{1}{h_j} \left( \frac{K_{i+0.5,j}}{h_{i+1}} + \frac{K_{i-0.5,j}}{h_i} \right) \frac{\ddot{p}_{i,j+1}}{h_{i+1}}.$$
On the straight lines  \( c_{2i} \), \( i=1, N \)

\[
a_{i,j} p_{i+1,j}^2 - b_{i,j} p_{i,j}^2 + c_{i,j} p_{i-1,j}^2 = -d_{i,j}
\]

(10)

where

\[
a_{i,j} = \frac{K_{i,j} + 0.5}{\bar{h}_i h_{i+1}}, \quad c_{i,j} = \frac{K_{i,j} - 0.5}{\bar{h}_i h_{i}}, \quad b_{i,j} = \frac{K_{i,j} + 0.5}{\bar{h}_i h_{i+0.5}}, \quad M_{i,j} = \frac{K_{i,j} - 0.5}{0.5 h_i},
\]

\[
v_{d_{i,j}} = \left[ \frac{M_{i,j}}{0.5 \tau_k} - \frac{1}{h_i} \left( \frac{K_{i+0.5,j} + v_{K_{i-0.5,j}}}{h_{i+1}} \right) \right] \frac{p_{i+1,j}^2}{h_{i+1}} + \frac{1}{h_i} \left( \frac{K_{i+0.5,j} + v_{K_{i-0.5,j}}}{h_{i+1}} \right) \frac{p_{i+1,j}^2}{h_{i+1}} + \frac{v_{K_{i+0.5,j}}}{h_i} \frac{p_{i-1,j}^2}{h_{i+1}}.
\]

4. Discussion

Show the results of computer calculations of a one-dimensional nonlinear boundary value problem of gas filtration theory. The following boundary conditions \( \frac{dP}{dx} \) at \( t = 0, P = 1 \) at \( 0 \leq x \leq 1 \).

The analysis of consumed numerical calculations shows that the iteration method, based on the expansion of nonlinear expressions into a linear form, does not give any improvements as compared to the simple iteration method for various parameters of numerical method: \( q = 2; 1; 0.5 \) and \( \tau = 0.01; 0.005; 0.001; h = 0.05; 0.02 \). So, we can conclude that for a fairly wide range of parameters of numerical method the simple iteration method can be used. Its convergence is observed after two or three iterations for the problem solution accuracy \( \varepsilon = 10^{-6} \). So, further, the method of simple iteration is used for systems of one-dimensional equations in finite differences, generated by using the considered economical difference schemes.

To study the practical properties of a local-one-dimensional scheme and a scheme of longitudinal-transverse directions, consider the following example of computer calculation of the boundary value problem of filtration theory.

Assume that in the square region of filtration of \( L = 10 \) km there is an unsteady gas filtration to 5 wells with \( q_c = 4 \) mln.\( m^3 \) daily flow rate assigned to them. One well is located in the center, and the others are symmetrical to it, in the vertices of a concentric square at a distance of 2.5 km from the center. There is no normal motion on the outer boundary of the filtration region. Values of \( \mu \) filtration parameters under normal conditions are as follows: \( k_0 = 0.6 \) darcy; \( \mu_0 = 0.38 \) cP; \( z_0 = 1; m_0 = 2 \). The initial reservoir pressure before the introduction of gas wells into operation is 300 atm. in the entire filtration region. The thickness of the gas reservoir is 25m.

From the point of view of gas dynamics, it is clear that the results in the second, third, fourth and fifth lines should be the same, as the results in the sixth, seventh, eighth and ninth lines.

Computer numerical results shows that the use of local-one-dimensional systems and longitudinal-transverse directions gives a significant scatter of results. This is also evidenced by control comparisons of average pressure \( P_{cp}(t) \). While the scheme of longitudinal-transverse directions has an error of solution at such a comparison of about 0.1% and does not increase with time, local-one-dimensional scheme already after 20 days has an error of about 2%, after 3 years it is 25%, and after 10 years - 35%. It is obvious that such discrepancies between numerical and exact solutions in the average-integral norm cannot be accepted.
Computational experiment established that the scheme of longitudinal-transverse directions gives good results, the symmetry of the filtration process pattern is completely reached, and this symmetry is observed in all 9 positions. This could be expected due to the fact that the fitting conditions of boundary conditions are satisfied for the system of equations in finite differences.

The results of calculations for the local-one-dimensional scheme, neither satisfy the material balance equation nor give complete symmetry pattern of the filtration process, and the symmetry is maintained only local-one-dimensionally: along the $OX$ axis and along the $OY$ axis.

5. Conclusion
From the analysis of conducted numerical calculations, it can be seen that the iteration method, based on the expansion of nonlinear expressions into a linear form, does not provide any improvements as compared to the simple iteration method for various parameters of numerical method. It follows that for a fairly wide range of parameters of numerical method, the method of simple iteration can be used. Its convergence is observed already after two or three iterations for the problem solution accuracy $\varepsilon = 10^{-6}$.

Computational experiments conducted have established that local-one-dimensional systems and longitudinal-transverse direction schemes give significantly different results, which is testified by control comparisons of the average pressure value.

Analysis of numerical calculations has shown that the longitudinal-transverse direction scheme has an error of the order of 0.1%, and the local-one-dimensional scheme has an error of about 2%, and it increases over time (a forecast after 10 years) up to 35%.

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