Anisotropic Variable Hardy-Lorentz Spaces and Their Real Interpolation

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Abstract Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a variable exponent function satisfying the globally log-Hölder continuous condition, $q \in (0, \infty]$ and $A$ be a general expansive matrix on $\mathbb{R}^n$. In this article, the authors first introduce the anisotropic variable Hardy-Lorentz space $H^{p(\cdot)}_A(\mathbb{R}^n)$ associated with $A$, via the radial grand maximal function, and then establish its radial or non-tangential maximal function characterizations. Moreover, the authors also obtain characterizations of $H^{p(\cdot)}_A(\mathbb{R}^n)$, respectively, in terms of the atom and the Lusin area function. As an application, the authors prove that the anisotropic variable Hardy-Lorentz space $H^{p(\cdot)}_A(\mathbb{R}^n)$ sever as the intermediate space between the anisotropic variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ and the space $L^\infty(\mathbb{R}^n)$ via the real interpolation. This, together with a special case of the real interpolation theorem of H. Kempka and J. Vybíral on the variable Lorentz space, further implies the coincidence between $H^{p(\cdot)}_A(\mathbb{R}^n)$ and the variable Lorentz space $L^{p(\cdot)}(\mathbb{R}^n)$ when $\text{essinf}_{x \in \mathbb{R}^n} p(x) \in (1, \infty)$.

1 Introduction

As a generalization of the classical Lebesgue spaces $L^p(\mathbb{R}^n)$, the variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$, in which the constant exponent $p$ is replaced by an exponent function $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, were studied by Musielak [53] and Nakano [55, 56], which can be traced back to Orlicz [59, 60]. But the modern theory of function spaces with variable exponents was started with the articles [45] of Kovářík and Rákosník and [32] of Fan and Zhao as well as [21] of Cruz-Uribe and [24] of Diening, and nowadays has been widely used in harmonic analysis (see, for example, [22, 25, 79]). In addition, the theory of variable function spaces also has interesting applications in fluid dynamics [2], image processing [17], partial differential equations and variational calculus [3, 30, 40, 65].

Recently, Nakai and Sawano [54] and, independently, Cruz-Uribe and Wang [23] with some weaker assumptions on $p(\cdot)$ than those used in [54], extended the theory of variable Lebesgue spaces via investigating the variable Hardy spaces on $\mathbb{R}^n$. Later, Sawano [66], Zhuo et al. [85] and Yang et al. [81] further completed the theory of these variable Hardy spaces. For more developments of function spaces with variable exponents, we refer the reader to [6, 26, 44, 57, 58, 75, 76, 77, 78, 82] and their references. In particular, Kempka and Vybíral [44] introduced the variable Lorentz spaces which were a generalization of both the variable Lebesgue spaces and

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the classical Lorentz spaces and obtained some basic properties of these spaces including several embedding conclusions. The real interpolation result that the variable Lorentz space serves as the intermediate space between the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ and the space $L^{\infty}(\mathbb{R}^n)$ was also presented in [44].

Very recently, Yan et al. [78] first introduced the variable weak Hardy spaces on $\mathbb{R}^n$ and established various real-variable characterizations of these spaces; as application, the boundedness of some Calderón-Zygmund operators in the critical case was also presented. Based on these results, via establishing a very interesting decomposition for any distribution of the variable weak Hardy space, Zhuo et al. [86] proved the following real interpolation theorem between the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ and the space $L^{\infty}(\mathbb{R}^n)$:

$$ (H^{p(\cdot)}(\mathbb{R}^n), L^{\infty}(\mathbb{R}^n))_{\theta, \infty} = WH^{p(\cdot)/(1-\theta)}(\mathbb{R}^n), \quad 0 < \theta < 1, $$

where $WH^{p(\cdot)/(1-\theta)}(\mathbb{R}^n)$ denotes the variable weak Hardy space and $(\cdot, \cdot)_{\theta, \infty}$ the real interpolation.

As was well known, Fefferman et al. [33] showed that the Hardy-Lorentz space $H^{p, q}(\mathbb{R}^n)$ was actually the intermediate space between the classical Hardy space $H^p(\mathbb{R}^n)$ and the space $L^{\infty}(\mathbb{R}^n)$ under the real interpolation, which is the main motivation to develop the real-variable theory of $H^{p, q}(\mathbb{R}^n)$. Thus, it is natural and interesting to ask whether or not the variable Hardy-Lorentz space also serves as the intermediate space between the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ and the space $L^{\infty}(\mathbb{R}^n)$ via the real interpolation, namely, if $(\cdot, \cdot)_{\theta, \infty}$ in (1.1) is replaced by $(\cdot, \cdot)_{\theta, q}$ with $q \in (0, \infty]$, what happens?

On the other hand, as the series of works (see, for example, [1, 5, 7, 33, 35, 49, 61]) reveal, the Hardy-Lorentz spaces (as well the weak Hardy spaces) serve as a more subtle research object than the usual Hardy spaces when studying the boundedness of singular integrals, especially, in some critical cases, due to the fact that these function spaces own finer structures. Moreover, after the celebrated articles [14, 15, 16] of Calderón and Torchinsky on parabolic Hardy spaces, there has been an enormous interest in extending classical function spaces arising in harmonic analysis from Euclidean spaces to some more general underlying spaces; see, for example, [28, 36, 38, 39, 67, 68, 69, 71, 72, 80]. The function spaces in the anisotropic setting have proved of wide generality (see, for example, [10, 11, 12]), which include the classical isotropic spaces and the parabolic spaces as special cases. For more progresses about this theory, we refer the reader to [46, 47, 50, 51, 52, 31, 73, 74] and their references. In particular, the authors recently introduced the anisotropic Hardy-Lorentz spaces $H^{p, q}_A(\mathbb{R}^n)$, associated with some dilation $A$, and obtained their various real-variable characterizations (see [50, 51]). Also, very recently, Zhuo et al. [84] developed the real-variable theory of the variable Hardy space $H^{p(\cdot)}(\mathbb{X})$ on an RD-space $\mathbb{X}$. Recall that a metric measure space of homogeneous type $\mathbb{X}$ is called an RD-space if it is a metric measure space of homogeneous type in the sense of Coifman and Weiss [19, 20] and satisfies some reverse doubling property, which was originally introduced by Han et al. [39] (see also [83] for some equivalent characterizations).

To further study the intermediate space between the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ and the space $L^{\infty}(\mathbb{R}^n)$ via the real interpolation and also to give a complete theory of variable Hardy-Lorentz spaces in anisotropic setting, in this article, we first introduce the anisotropic variable Hardy-Lorentz space, via the radial grand maximal function, and then establish its several real-variable characterizations, respectively, in terms of the atom, the radial or the non-tangential maximal functions, and the Lusin area function. As an application, we prove that the anisotropic
variable Hardy-Lorentz space $H^{p,q}_A(\mathbb{R}^n)$ serves as the intermediate space between the anisotropic variable Hardy space $H^{p}(\mathbb{R}^n)$ and the space $L^\infty(\mathbb{R}^n)$ via the real interpolation. This, together with a special case of the real interpolation theorem of Kempka and Vybíral in [44] on the variable Lorentz space, further implies the coincidence between $H^{p,q}_A(\mathbb{R}^n)$ and the variable Lorentz space $L^{p,q}(\mathbb{R}^n)$ when $\text{essinf}_{x\in \mathbb{R}^n} p(x) \in (1, \infty)$.

To be precise, this article is organized as follows.

In Section 2, we first recall some notation and notions on Euclidean spaces, with anisotropic dilations, and variable Lebesgue spaces as well as some basic properties of these spaces to be used in this article. Then we introduce the anisotropic variable Hardy-Lorentz space $H^{p,q}_A(\mathbb{R}^n)$ via the radial grand maximal function.

Section 3 is aimed to characterize $H^{p,q}_A(\mathbb{R}^n)$ by means of the radial or the non-tangential maximal functions (see Theorem 3.8 below). To this end, via the Aoki-Rolewicz theorem (see [8, 63]), we first prove that the $L^{p,q}_A(\mathbb{R}^n)$ quasi-norm of the tangential maximal function $T^N(\mathbb{R}^n)$ can be controlled by that of the non-tangential maximal function $M^N(\mathbb{R}^n)$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$ (see Lemma 3.5 below), where $K$ is the truncation level, $L$ is the decay level and $\mathcal{S}'(\mathbb{R}^n)$ denotes the set of all tempered distributions on $\mathbb{R}^n$. Then, by the boundedness of the Hardy-Littlewood maximal function as in (3.1) below on $L^{p,q}(\mathbb{R}^n)$ (see Lemma 3.3 below) with $p(\cdot)$ satisfying the so-called globally log-Hölder continuous condition (see (2.6) and (2.7) below) and $1 < p_- \leq p_+ < \infty$, where $p_-$ and $p_+$ are as in (2.4) below, we obtain the boundedness of the Hardy-Littlewood maximal function on $L^{p,q}(\mathbb{R}^n)$ (see Lemma 3.4 below) with $p(\cdot)$ satisfying the same condition as that in Lemma 3.3 and $q \in (0, \infty)$. We point out that the monotone convergence property for increasing sequences on $L^{p,q}(\mathbb{R}^n)$ (see Proposition 2.8 below) as well as Lemmas 3.3 and 3.4 play a key role in proving Theorem 3.8.

In Section 4, via borrowing some ideas from [50, Theorem 3.6] and [78, Theorem 4.4], we establish the atomic characterization of $H^{p,q}_A(\mathbb{R}^n)$. Indeed, we first introduce the anisotropic variable atomic Hardy-Lorentz space $H^{p,q}_{A}(\mathbb{R}^n)$ in Definition 4.2 below and then prove

$$H^{p,q}_A(\mathbb{R}^n) = H^{p,q}_{A}(\mathbb{R}^n)$$

with equivalent quasi-norms (see Theorem 4.8 below). To prove that $H^{p,q}_{A}(\mathbb{R}^n)$ is continuously embedded into $H^{p,q}_A(\mathbb{R}^n)$, motivated by [66, Lemma 4.1], we first conclude that some estimates related to $L^{p,q}(\mathbb{R}^n)$ norms for some series of functions can be reduced into dealing with the $L'(\mathbb{R}^n)$ norms of the corresponding functions (see Lemma 4.5 below), which actually is an anisotropic version of [66, Lemma 4.1]. Then, by using this key lemma and the Fefferman-Stein vector-valued inequality of the Hardy-Littlewood maximal operator $M_{HL}$ on $L^{p}(\mathbb{R}^n)$ (see Lemma 4.3 below), we prove that $H^{p,q}_{A}(\mathbb{R}^n) \subset H^{p,q}_A(\mathbb{R}^n)$ and the inclusion is continuous. The method used in the proof for the converse embedding is different from that used in the proof for the corresponding embedding of variable Hardy spaces $H^{p}(\mathbb{R}^n)$ (or, resp., anisotropic Hardy spaces $H^{p}_A(\mathbb{R}^n)$). Recall that $L^{1}_{\text{loc}}(\mathbb{R}^n) \cap H^{p}(\mathbb{R}^n)$ (or, resp., $L^{1}_{\text{loc}}(\mathbb{R}^n) \cap H^{p}_A(\mathbb{R}^n)$) is dense in $H^{p}(\mathbb{R}^n)$ (or, resp., $H^{p}_A(\mathbb{R}^n)$), which plays a key role in the atomic decomposition of $H^{p}(\mathbb{R}^n)$ (or, resp., $H^{p}_A(\mathbb{R}^n)$). However, this standard procedure is invalid for the space $H^{p,\infty}_A(\mathbb{R}^n)$, due to its lack of a dense function subspace. To overcome this difficulty, we borrow some ideals from [50, Theorem 3.6] (see also [27]), in which the authors directly obtained an atomic decomposition for convolutions of distributions in $H^{p,\infty}_A(\mathbb{R}^n)$ and Schwartz functions instead of some dense function subspace.
As an application of the atomic characterization of $H_A^{p(\cdot),q}(\mathbb{R}^n)$ obtained in Theorem 4.8, in Section 5, we establish the Lusin area function characterization of $H_A^{p(\cdot),q}(\mathbb{R}^n)$ (see Theorem 5.2 below). In the proof of Theorem 5.2, the anisotropic Calderón reproducing formula and the method used in the proof of the atomic characterization of $H_A^{p(\cdot),q}(\mathbb{R}^n)$ play a key role. However, when we decompose a distribution into a sum of atoms, the dual method used in estimating the norm of each atom in the classic case does not work anymore in the present setting. Instead, a strategy, used in [51], originated from Fefferman [34], that obtains a subtle estimate (see, for example, [51, (3.23)]) plays a key role here; see the estimate (5.16) below.

In Section 6, as another application of the atomic characterization of $H_A^{p(\cdot),q}(\mathbb{R}^n)$, we prove the following real interpolation result between the anisotropic variable Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$ and the space $L^\infty(\mathbb{R}^n)$:

\begin{equation}
(H_A^{p(\cdot)}(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\theta,q} = H_A^{p(\cdot)/(1-\theta),q}(\mathbb{R}^n), \quad \theta \in (0,1), \; q \in (0,\infty]
\end{equation}

(see Theorem 6.2 below). To prove this result, via borrowing some ideas from [86], we first obtain a decomposition for any distribution of the anisotropic variable Hardy-Lorentz space into “good” and “bad” parts (see Lemma 6.5 below), which is of independent interest. We point out that, as a special case of [84, Theorem 4.3(i)], we know that the atomic characterization of $H_A^{p(\cdot)}(\mathbb{R}^n)$ holds true. This, together with the vector-valued inequality of the Hardy-Littlewood maximal function on the variable Lebesgue space (see Lemma 4.3 below), plays a key role in the proof of Lemma 6.5. Applying (1.2), together with [84, Corollary 4.20] on the coincidence between $H_A^{p(\cdot)}(\mathbb{R}^n)$ and $L^{p(\cdot)}(\mathbb{R}^n)$ as well as [44, Remark 4.2(ii)], we further obtain the coincidence between $H_A^{p(\cdot),q}(\mathbb{R}^n)$ and $L^{p(\cdot),q}(\mathbb{R}^n)$ when $\text{essinf}_{x \in \mathbb{R}^n} p(x) \in (1,\infty)$; see Corollary 6.3 below.

We should point out that, if $A := d I_{n \times n}$ for some $d \in \mathbb{R}$ with $|d| \in (1,\infty)$, here and hereafter, $I_{n \times n}$ denotes the $n \times n$ unit matrix and $| \cdot |$ denotes the Euclidean norm in $\mathbb{R}^n$, then the space $H_A^{p(\cdot),q}(\mathbb{R}^n)$ becomes the classical isotropic variable Hardy-Lorentz space. In this case, the results in this article are also independently obtained by Jiao et al. [43] via some slight different methods.

Finally, we make some conventions on notation. Throughout this article, we always let $\mathbb{N} := \{1,2,\ldots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. For any multi-index $\beta := (\beta_1,\ldots,\beta_n) \in \mathbb{Z}_+^n$, let $|\beta| := \beta_1 + \cdots + \beta_n$. We denote by $C$ a positive constant which is independent of the main parameters, but its value may change from line to line. Moreover, we use $f \lesssim g$ to denote $f \leq C g$ and, if $f \lesssim g \lesssim f$, we then write $f \sim g$. For any $q \in [1,\infty]$, we denote by $q'$ its conjugate index, namely, $1/q + 1/q' = 1$. In addition, for any set $E \subset \mathbb{R}^n$, we denote by $E^{\complement}$ the set $\mathbb{R}^n \setminus E$, by $\chi_E$ its characteristic function and by $|E|$ the cardinality of $E$. The symbol $[s]$, for any $s \in \mathbb{R}$, denotes the largest integer not greater than $s$.

## 2 Preliminaries

In this section, we introduce the anisotropic variable Hardy-Lorentz space via the radial grand maximal function. To this end, we first recall some notation and notions on spaces of homogeneous type associated with dilations and variable Lebesgue spaces as well as some basic conclusions of these spaces to be used in this article. For an exposition of these concepts, we refer the reader to the monographs [10, 22, 25].

We begin with recalling the notion of expansive matrices in [10].
**Definition 2.1.** A real $n \times n$ matrix $A$ is called an **expansive matrix** (shortly, a dilation) if

$$\min_{\lambda \in \sigma(A)} \lambda > 1,$$

here and hereafter, $\sigma(A)$ denotes the **collection of all eigenvalues of $A$**.

Throughout this article, $A$ always denotes a fixed dilation and $b := | \det A |$. Then we easily find that $b \in (1, \infty)$ by [10, p. 6, (2.7)]. Let $\lambda_- \text{ and } \lambda_+$ be two positive numbers satisfying that

$$1 < \lambda_- < \min(|\lambda| : \lambda \in \sigma(A)) \leq \max(|\lambda| : \lambda \in \sigma(A)) < \lambda_+.$$

In the case when $A$ is diagonalizable over $\mathbb{C}$, we can even take $\lambda_- := \min(|\lambda| : \lambda \in \sigma(A))$ and $\lambda_+ := \max(|\lambda| : \lambda \in \sigma(A))$. Otherwise, we need to choose them sufficiently close to these equalities according to what we need in the arguments below.

It was proved in [10, p. 5, Lemma 2.2] that, for a given dilation $A$, there exist an open ellipsoid $\Delta$ and $r \in (1, \infty)$ such that $\Delta \subset r\Delta \subset A\Delta$, and one may additionally assume that $|\Delta| = 1$, where $|\Delta|$ denotes the $n$-dimensional Lebesgue measure of the set $\Delta$. For any $k \in \mathbb{Z}$, let $B_k := A^k\Delta$. Obviously, $B_k$ is open, $B_k \subset rB_k \subset B_{k+1}$ and $|B_k| = b^k$. An ellipsoid $x + B_k$ for some $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ is called a **dilated ball**. Denote by $\mathcal{B}$ the set of all such dilated balls, namely,

$$\mathcal{B} := \{ x + B_k : x \in \mathbb{R}^n, k \in \mathbb{Z} \}. \quad (2.1)$$

Throughout this article, let $\tau$ be the **minimal integer** such that $r^\tau \geq 2$. Then, for any $k \in \mathbb{Z}$, it holds true that

$$B_k + B_k \subset B_{k+\tau} \quad (2.2)$$

and

$$B_k + (B_{k+\tau})^C \subset (B_k)^C, \quad (2.3)$$

where $E + F$ denotes the algebraic sum $\{ x + y : x \in E, y \in F \}$ of sets $E, F \subset \mathbb{R}^n$.

The notion of the homogeneous quasi-norm induced by $A$ was introduced in [10, p. 6, Definition 2.3] as follows.

**Definition 2.2.** A measurable mapping $\rho : \mathbb{R}^n \to [0, \infty)$ is called a **homogeneous quasi-norm**, associated with a dilation $A$, if

(i) $x \neq \vec{0}_n$ implies that $\rho(x) \in (0, \infty)$, here and hereafter, $\vec{0}_n$ denotes the **origin** of $\mathbb{R}^n$;

(ii) $\rho(Ax) = b\rho(x)$ for any $x \in \mathbb{R}^n$;

(iii) $\rho(x + y) \leq H(\rho(x) + \rho(y))$ for any $x, y \in \mathbb{R}^n$, where $H \in [1, \infty)$ is a constant independent of $x$ and $y$.

In the standard dyadic case $A := 2I_{2^n \times 2^n}$, $\rho(x) := |x|^n$ for any $x \in \mathbb{R}^n$ is an example of the homogeneous quasi-norm associated with $A$. In [10, p. 6, Lemma 2.4], it was proved that all homogeneous quasi-norms associated with $A$ are equivalent. Therefore, for a given dilation $A,$
in what follows, we always use the step homogeneous quasi-norm \( \rho \) defined by setting, for any \( x \in \mathbb{R}^n \),
\[
\rho(x) := \sum_{j \in \mathbb{Z}} b_j^i \chi_{B_{j+1} \setminus B_j}(x) \quad \text{if} \ x \neq 0_n, \quad \text{or else} \quad \rho(0_n) := 0
\]
for convenience. Obviously, for any \( k \in \mathbb{Z}, B_k = \{ x \in \mathbb{R}^n : \rho(x) < b_k \} \). Observe that \((\mathbb{R}^n, \rho, dx)\) is a space of homogeneous type in the sense of Coifman and Weiss [19, 20], here and hereafter, \( dx \) denotes the \( n \)-dimensional Lebesgue measure, and, moreover, \((\mathbb{R}^n, \rho, dx)\) is indeed an RD-space (see [39, 83]).

Recall that a measurable function \( p(\cdot) : \mathbb{R}^n \to (0, \infty) \) is called a \textit{variable exponent}. For any variable exponent \( p(\cdot) \), let
\[
(2.4) \quad p_- := \text{ess inf}_{x \in \mathbb{R}^n} p(x) \quad \text{and} \quad p_+ := \text{ess sup}_{x \in \mathbb{R}^n} p(x).
\]
Denote by \( \mathcal{P}(\mathbb{R}^n) \) the \textit{set of all variable exponents} \( p(\cdot) \) satisfying \( 0 < p_- \leq p_+ < \infty \).

Let \( f \) be a measurable function on \( \mathbb{R}^n \) and \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \). Then the \textit{modular} \( \varrho_p(\cdot) \), associated with \( p(\cdot) \), is defined by setting
\[
\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx
\]
and the Luxemburg (also called Luxemburg-Nakano) \textit{quasi-norm} \( \| f \|_{L^{p(\cdot)}(\mathbb{R}^n)} \) by
\[
\| f \|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \varrho_{p(\cdot)}(f/\lambda) \leq 1 \right\}.
\]
Moreover, the \textit{variable Lebesgue space} \( L^{p(\cdot)}(\mathbb{R}^n) \) is defined to be the set of all measurable functions \( f \) satisfying that \( \varrho_{p(\cdot)}(f) < \infty \), equipped with the quasi-norm \( \| f \|_{L^{p(\cdot)}(\mathbb{R}^n)} \).

\textbf{Remark 2.3.} Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \).

(i) Obviously, for any \( r \in (0, \infty) \) and \( f \in L^{p(\cdot)}(\mathbb{R}^n) \),
\[
\| f^r \|_{L^{p(\cdot)}(\mathbb{R}^n)} = \| f \|_{L^{p(\cdot)}(\mathbb{R}^n)}^r.
\]
Moreover, for any \( \mu \in \mathbb{C} \) and \( f, g \in L^{p(\cdot)}(\mathbb{R}^n) \), \( |\mu| \| f \|_{L^{p(\cdot)}(\mathbb{R}^n)} = |\mu| \| f \|_{L^{p(\cdot)}(\mathbb{R}^n)} \) and
\[
\| f + g \|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \| f \|_{L^{p(\cdot)}(\mathbb{R}^n)}^p + \| g \|_{L^{p(\cdot)}(\mathbb{R}^n)}^p,
\]
here and hereafter,
\[
(2.5) \quad \underline{p} := \min\{p_-, 1\}
\]
with \( p_- \) as in (2.4). In particular, when \( p_- \in [1, \infty) \), \( L^{p(\cdot)}(\mathbb{R}^n) \) is a Banach space (see [25, Theorem 3.2.7]).

(ii) It was proved in [22, Proposition 2.21] that, for any function \( f \in L^{p(\cdot)}(\mathbb{R}^n) \) with \( \| f \|_{L^{p(\cdot)}(\mathbb{R}^n)} > 0 \), \( \varrho_{p(\cdot)}(f/\| f \|_{L^{p(\cdot)}(\mathbb{R}^n)}) = 1 \) and, in [22, Corollary 2.22] that, if \( \| f \|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1 \), then \( \varrho_{p(\cdot)}(f) \leq \| f \|_{L^{p(\cdot)}(\mathbb{R}^n)} \).
A function $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ is said to satisfy the *globally log-Hölder continuous condition*, denoted by $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, if there exist two positive constants $C_{\log}(p)$ and $C_\infty$, and $p_\infty \in \mathbb{R}$ such that, for any $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \leq \frac{C_{\log}(p)}{\log(e + 1/\rho(x - y))}$$

and

$$|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + \rho(x))}.$$ 

The following variable Lorentz space $L^{p(\cdot),q}(\mathbb{R}^n)$ is known as a special case of the variable Lorentz space $L^{p(\cdot),q}(\mathbb{R}^n)$ investigated by Kempka and Vybiral in [44].

**Definition 2.4.** Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The *variable Lorentz space* $L^{p(\cdot),q}(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ such that

$$\|f\|_{L^{p(\cdot),q}(\mathbb{R}^n)} := \left\{ \int_0^\infty \lambda^q \left\|\chi_{\{x \in \mathbb{R}^n : |f(x)| > \lambda\}}\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}^q \frac{d\lambda}{\lambda} \right\}^{1/q} \quad \text{when } q \in (0, \infty),$$

and

$$\|f\|_{L^{p(\cdot),q}(\mathbb{R}^n)} := \sup_{\lambda \in (0, \infty)} \left[ \lambda \left\|\chi_{\{x \in \mathbb{R}^n : |f(x)| > \lambda\}}\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \right]\quad \text{when } q = \infty,$$

is finite.

From [44, Lemma 2.4 and Theorem 3.1], we deduce the following Lemmas 2.5 and 2.6, respectively.

**Lemma 2.5.** Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $q \in (0, \infty)$. Then, for any measurable function $f$,

$$\|f\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \sim \sum_{k \in \mathbb{Z}} 2^{kq} \left\|\chi_{\{x \in \mathbb{R}^n : |f(x)| > 2^k\}}\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}^{1/q}$$

with the usual interpretation for $q = \infty$, where the equivalent positive constants are independent of $f$.

**Lemma 2.6.** Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $q \in (0, \infty)$. Then $\|\cdot\|_{L^{p(\cdot),q}(\mathbb{R}^n)}$ defines a quasi-norm on $L^{p(\cdot),q}(\mathbb{R}^n)$.

It is easy to obtain the following result, the details being omitted.

**Lemma 2.7.** Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $q \in (0, \infty)$. Then, for any $r \in (0, \infty)$ and $f \in L^{p(\cdot),q}(\mathbb{R}^n)$,

$$\|f\|_{L^{p(\cdot),q}(\mathbb{R}^n)} = \|f\|_{L^{p(\cdot),q}(\mathbb{R}^n)}.$$ 

From the monotone convergence theorem of $L^{p(\cdot),q}(\mathbb{R}^n)$ (see [22, Corollary 2.64]), we easily deduce the following monotone convergence property of $L^{p(\cdot),q}(\mathbb{R}^n)$, the details being omitted.

**Proposition 2.8.** Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\{f_k\}_{k \in \mathbb{N}} \subset L^{p(\cdot),q}(\mathbb{R}^n)$ be some sequence of non-negative functions satisfying that $f_k$, as $k \to \infty$, increases pointwisely almost everywhere to $f \in L^{p(\cdot),q}(\mathbb{R}^n)$ in $\mathbb{R}^n$. Then

$$\|f - f_k\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \to 0 \quad \text{as } k \to \infty.$$
Throughout this paper, denote by $S(\mathbb{R}^n)$ the space of all Schwartz functions, namely, the set of all $C^\infty(\mathbb{R}^n)$ functions $\varphi$ satisfying that, for every integer $\ell \in \mathbb{Z}_+$ and multi-index $\alpha \in (\mathbb{Z}_+)^n$,

$$\|\varphi\|_{\alpha, \ell} := \sup_{x \in \mathbb{R}^n} |\partial^\alpha \varphi(x)| < \infty.$$ 

These quasi-norms $\{\| \cdot \|_{\alpha, \ell}\}_{\alpha \in (\mathbb{Z}_+)^n, \ell \in \mathbb{Z}_+}$ also determine the topology of $S(\mathbb{R}^n)$. We use $S'(\mathbb{R}^n)$ to denote the dual space of $S(\mathbb{R}^n)$, namely, the space of all tempered distributions on $\mathbb{R}^n$ equipped with the weak-* topology. For any $N \in \mathbb{Z}_+$, let

$$S_N(\mathbb{R}^n) := \{\varphi \in S(\mathbb{R}^n) : \|\varphi\|_{\alpha, \ell} \leq 1, |\alpha| \leq N, \ell \leq N\};$$

equivalently,

$$\varphi \in S_N(\mathbb{R}^n) \iff \|\varphi\|_{S_N(\mathbb{R}^n)} := \sup_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} \left\{ |\partial^\alpha \varphi(x)| \max\{1, |\varphi(x)|^N\} \right\} \leq 1.$$ 

In what follows, for any $\varphi \in S(\mathbb{R}^n)$ and $k \in \mathbb{Z}$, let $\varphi_k(\cdot) := b^{-k}\varphi(b^{-k} \cdot)$.

**Definition 2.9.** Let $\varphi \in S(\mathbb{R}^n)$ and $f \in S'(\mathbb{R}^n)$. The non-tangential maximal function $M_\varphi(f)$ and the radial maximal function $M^0_\varphi(f)$ of $f$ with respect to $\varphi$ are defined, respectively, by setting, for any $x \in \mathbb{R}^n$,

$$(2.8) \quad M_\varphi(f)(x) := \sup_{y \in x + B_1, k \in \mathbb{Z}} |f \ast \varphi_k(y)|$$

and

$$(2.9) \quad M^0_\varphi(f)(x) := \sup_{k \in \mathbb{Z}} |f \ast \varphi_k(x)|.$$ 

For any given $N \in \mathbb{N}$, the non-tangential grand maximal function $M_N(f)$ and the radial grand maximal function $M^0_N(f)$ of $f \in S'(\mathbb{R}^n)$ are defined, respectively, by setting, for any $x \in \mathbb{R}^n$,

$$(2.10) \quad M_N(f)(x) := \sup_{\varphi \in S_N(\mathbb{R}^n)} M_\varphi(f)(x)$$

and

$$M^0_N(f)(x) := \sup_{\varphi \in S_N(\mathbb{R}^n)} M^0_\varphi(f)(x).$$ 

We now introduce anisotropic variable Hardy-Lorentz spaces as follows.

**Definition 2.10.** Let $p(\cdot) \in C^\text{log}(\mathbb{R}^n)$, $q(\cdot) \in (0, \infty]$ and $N \in \mathbb{N} \cap [(\frac{1}{p} - 1) \frac{\ln n}{\ln m}, + \infty)$, where $p$ is as in (2.5). The anisotropic variable Hardy-Lorentz space, denoted by $H^{p(\cdot), q(\cdot)}_A(\mathbb{R}^n)$, is defined by setting

$$H^{p(\cdot), q(\cdot)}_A(\mathbb{R}^n) := \left\{f \in S'(\mathbb{R}^n) : M^0_N(f) \in L^{p(\cdot), q(\cdot)}_A(\mathbb{R}^n) \right\}$$

and, for any $f \in H^{p(\cdot), q(\cdot)}_A(\mathbb{R}^n)$, let $\|f\|_{H^{p(\cdot), q(\cdot)}_A(\mathbb{R}^n)} := \|M^0_N(f)\|_{L^{p(\cdot), q(\cdot)}_A(\mathbb{R}^n)}$. 

Remark 2.11.  (i) Even though the quasi-norm of $H_A^{p,q}(\mathbb{R}^n)$ in Definition 2.10 depends on $N$, it follows from Theorem 3.8 below that the space $H_A^{p,q}(\mathbb{R}^n)$ is independent of the choice of $N$ as long as $N \in \left(\left[1, \frac{1}{p} - 1\right] \frac{\ln p}{\ln e} \right] + 2, \infty)$. If $p(\cdot) \equiv p \in (0, \infty)$, then the space $H_A^{p,q}(\mathbb{R}^n)$ is just the anisotropic Hardy-Lorentz space $H_A^{p,q}(\mathbb{R}^n)$ investigated by Liu et al. in [50] and, if $A := d I_{\text{tang}}$ for some $d \in \mathbb{R}$ with $|d| \in (1, \infty)$ and $q = \infty$, then the space $H_A^{p(\cdot),\infty}(\mathbb{R}^n)$ becomes the variable weak Hardy space introduced by Yan et al. in [78].

(ii) Very recently, via the variable Lorentz spaces $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in [29], where

$$p(\cdot), q(\cdot) : (0, \infty) \to (0, \infty)$$

are bounded measurable functions, Almeida et al. [4] investigated the anisotropic variable Hardy-Lorentz spaces $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ on $\mathbb{R}^n$. As was mentioned in [44, Remark 2.6], the space $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in [29] never goes back to the space $L^{p(\cdot)}(\mathbb{R}^n)$, since the variable exponent $p(\cdot)$ in $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is only defined on $(0, \infty)$ while not on $\mathbb{R}^n$. On the other hand, the space $H_A^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, in this article, is defined via the variable Lorentz space $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ (with $q(\cdot) \equiv q$ or a constant $\in (0, \infty)$) from [44], which is not covered by the space $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ in [4]. Moreover, as was pointed out in [4, p.6], the key tool of [4] is the fact that the set $L^{1}_{\text{loc}}(\mathbb{R}^n) \cap H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ is dense in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. Therefore, the method used in [4] does not work for $H_A^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in the present article, due to the lack of a dense function subspace of $H_A^{p(\cdot),\infty}(\mathbb{R}^n)$ even when $p(\cdot) \equiv q(\cdot)$ or a constant $\in (0, \infty)$ and $A := d I_{\text{tang}}$ for some $d \in \mathbb{R}$ with $|d| \in (1, \infty)$.

3 Maximal function characterizations of $H_A^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$

In this section, we characterize $H_A^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in terms of the radial maximal function $M^0_\phi$ (see (2.9)) or the non-tangential maximal function $M_\phi$ (see (2.8)). We begin with the following Definitions 3.1 and 3.2 from [10].

Definition 3.1. For any function $F : \mathbb{R}^n \times \mathbb{Z} \to [0, \infty)$, $K \in \mathbb{Z} \cup \{\infty\}$ and $\ell \in \mathbb{Z}$, the maximal function $F^\ast_K$ of $F$ with aperture $\ell$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$F^\ast_K(x) := \sup_{k \in \mathbb{Z}} \sup_{k \leq K, y \in i + B_k} F(y, k).$$

Definition 3.2. Let $K \in \mathbb{Z}$, $L \in [0, \infty)$ and $N \in \mathbb{N}$. For any $\phi \in \mathcal{S}$, the radial maximal function $M^0(K,L)(\phi)$, the non-tangential maximal function $M^N(K,L)(\phi)$ and the tangential maximal function $T^N(K,L)(\phi)$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ are, respectively, defined by setting, for any $x \in \mathbb{R}^n$,

$$M^0(K,L)(\phi)(x) := \sup_{k \in \mathbb{Z}, k \leq K} |(f \ast \phi_k)(x)| \left[ \max \left\{ 1, \rho \left( A^{-K} x \right) \right\} \right]^{-L} (1 + b^{-K-K})^{-L},$$

$$M^N(K,L)(\phi)(x) := \sup_{k \in \mathbb{Z}, k \leq K} \sup_{y \in x + B_k} |(f \ast \phi_k)(y)| \left[ \max \left\{ 1, \rho \left( A^{-K} y \right) \right\} \right]^{-L} (1 + b^{-K-K})^{-L},$$

$$T^N(K,L)(\phi)(x) := \sup_{k \in \mathbb{Z}, k \leq K} \sup_{y \in x + B_k} |(f \ast \phi_k)(y)| \left[ \max \left\{ 1, \rho \left( A^{-K} y \right) \right\} \right]^{-L} (1 + b^{-K-K})^{-L}.$$
and
\[ T_{\phi}^{N(K,L)}(f)(x) := \sup_{k \in \mathbb{Z}} \sup_{k \leq k \in \mathbb{R}^n} \frac{|(f \ast \phi_k)(y)|}{[\max \{1, \rho(A^{-k}(x-y))\}]^N [\max \{1, \rho(A^{-k}y)\}]^L}. \]

Furthermore, the radial grand maximal function \( M_N^{(K,L)}(f) \) and the non-tangential grand maximal function \( M_N^{(K,L)}(f) \) of \( f \in S'(\mathbb{R}^n) \) are, respectively, defined by setting, for any \( x \in \mathbb{R}^n \),
\[ M_N^{(K,L)}(f)(x) := \sup_{\phi \in \mathcal{S}(\mathbb{R}^n)} M_{\phi}^{(K,L)}(f)(x) \]
and
\[ M_N^{(K,L)}(f)(x) := \sup_{\phi \in \mathcal{S}(\mathbb{R}^n)} M_{\phi}^{(K,L)}(f)(x). \]

For any \( r \in (0, \infty) \), denote by \( L^r_{loc}(\mathbb{R}^n) \) the set of all locally \( r \)-integrable functions on \( \mathbb{R}^n \) and, for any measurable set \( E \subset \mathbb{R}^n \), by \( L^r(E) \) the set of all measurable functions \( f \) such that
\[ \|f\|_{L^r(E)} := \left[ \int_E |f(x)|^r \, dx \right]^{1/r} < \infty. \]

Recall that the Hardy-Littlewood maximal operator \( M_{HL}(f) \) is defined by setting, for any \( f \in L^1_{loc}(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \),
\[ M_{HL}(f)(x) := \sup_{k \in \mathbb{Z}} \sup_{y \in \mathbb{R}^n} \frac{1}{|B_k|} \int_{y+B_k} |f(z)| \, dz = \sup_{x \in B \in \mathcal{B}} \frac{1}{|B|} \int_B |f(z)| \, dz, \]
where \( \mathcal{B} \) is as in (2.1).

Observe that \((\mathbb{R}^n, \rho, dx)\) is a space of homogeneous type in the sense of Coifman and Weiss [19, 20]. From this and [41, Theorems 5.2 and 4.3], we deduce the following lemma, which is an anisotropic version of [22, Theorem 3.16], the details being omitted.

**Lemma 3.3.** Let \( p(\cdot) \in C^{\log}(\mathbb{R}^n) \).

(i) If \( 1 \leq p_- \leq p_+ < \infty \), then, for any \( s \in [1, \infty) \), \( sp(\cdot) \in C^{\log}(\mathbb{R}^n) \) and, for any \( f \in L^{sp(\cdot)}(\mathbb{R}^n) \),
\[ \sup_{s \in (0,\infty)} \|M_{HL}(f)(x)\|_{L^{sp(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{sp(\cdot)}(\mathbb{R}^n)}, \]
where \( C \) is a positive constant independent of \( f \);

(ii) If \( 1 < p_- \leq p_+ < \infty \), then, for any \( s \in [1, \infty) \), \( sp(\cdot) \in C^{\log}(\mathbb{R}^n) \) and, for any \( f \in L^{sp(\cdot)}(\mathbb{R}^n) \),
\[ \|M_{HL}(f)\|_{L^{sp(\cdot)}(\mathbb{R}^n)} \leq \widetilde{C}\|f\|_{L^{sp(\cdot)}(\mathbb{R}^n)}, \]
where \( \widetilde{C} \) is a positive constant independent of \( f \).

Moreover, as a simple consequence of [44, Theorem 4.1], [78, Theorem 3.1] and Lemma 3.3(ii), we immediately obtain the following boundedness of \( M_{HL} \) on \( L^{p(\cdot)}(\mathbb{R}^n) \), which is of independent interest, the details being omitted.
Lemma 3.4. Let $p(\cdot) \in C^\log(\mathbb{R}^n)$ satisfy $1 < p_- \leq p_+ < \infty$, where $p_-$ and $p_+$ are as in (2.4), and $q \in (0, \infty]$. Then the Hardy-Littlewood maximal operator $M_{\text{HL}}$ is bounded on $L^{p(\cdot),q}(\mathbb{R}^n)$.

Lemma 3.5. Let $p(\cdot) \in C^\log(\mathbb{R}^n)$, $N \in (1/p_-, \infty) \cap \mathbb{N}$ and $\phi \in S(\mathbb{R}^n)$. Then there exists a positive constant $C$ such that, for any $K \in \mathbb{Z}$, $L \in [0, \infty)$ and $f \in S'(\mathbb{R}^n)$,

$$\left\| T_{\phi}^{N(K,L)}(f) \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \leq C \left\| M_{\phi}^{(K,L)}(f) \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}.$$  

Proof. We first prove that, for any $p(\cdot) \in C^\log(\mathbb{R}^n)$, $K \in \mathbb{Z}$ and $\ell \in [\ell', \infty) \cap \mathbb{Z}$,

$$\left\| F_{\ell}^{+K} \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \leq b^{(\ell-\ell')/p_-} \left\| F_{\ell'}^{+K} \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)},$$

where $F_{\ell}^{+K}$ is as in Definition 3.1 and, for any $\phi \in S(\mathbb{R}^n)$, $f \in S'(\mathbb{R}^n)$, $k \in \mathbb{Z}$ and $y \in \mathbb{R}^n$,

$$F(y,k) := \left\{ (f \ast \phi_k)(y) \right\} \max \left[ 1, \rho \left( A^{-k} y \right) \right]^{-\ell} \left( 1 + b^{-k-k} \right)^{-\ell}.$$  

Indeed, by a proof similar to that of [10, p. 42, Lemma 7.2], we easily find that, for any $\ell$, $\ell' \in \mathbb{Z}$ with $\ell \geq \ell'$ and $\lambda \in (0, \infty)$,

$$\left\{ x \in \mathbb{R}^n : F_{\ell}^{+K}(x) > \lambda \right\} \subset \left\{ x \in \mathbb{R}^n : M_{\text{HL}}(\chi_{\Omega_k})(x) \geq b^{\ell-\ell'} \right\},$$

where, for any $\lambda \in (0, \infty)$, $\Omega_k := \{ x \in \mathbb{R}^n : F_{\ell'}^{+K}(x) > \lambda \}$. Then, by (3.4), Lemma 3.3(i) and Remark 2.3(i), we know that

$$\left\| X_{\lambda \in \mathbb{R}^n : F_{\ell}^{+K}(x) > \lambda} \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \leq \left\| X_{\lambda \in \mathbb{R}^n : M_{\text{HL}}(\chi_{\Omega_k})(x) \geq b^{\ell-\ell'}} \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}$$

$$= \left\| X_{\lambda \in \mathbb{R}^n : M_{\text{HL}}(\chi_{\Omega_k})(x) \geq b^{\ell-\ell'}} \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}^{1/p_-} \left\| X_{\lambda \in \mathbb{R}^n : M_{\text{HL}}(\chi_{\Omega_k})(x) \geq b^{\ell-\ell'}} \right\|_{L^{\infty}(\mathbb{R}^n)}$$

$$\sim b^{\ell-\ell'} \left\| X_{\Omega_k} \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}^{1/p_-} \sim b^{\ell-\ell'} \left\| X_{\Omega_k} \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)},$$

which, together with the definition of $\Omega_k$ and Definition 2.4, further implies (3.3).

On the other hand, by [50, (4.7)], for any $N \in \mathbb{N}$, $K \in \mathbb{Z}$, $L \in [0, \infty)$, $f \in S'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we have

$$T_{\phi}^{N(K,L)}(f)(x) \leq \sum_{j=0}^{\infty} F_{j+1}^{+K}(x) b^{-jN}.$$  

Now we show (3.2). By (3.6), the Aoki-Rolewicz theorem (see [8, 63]), (3.5) and the fact that $N \in (1/p_-, \infty) \cap \mathbb{N}$, it is easy to see that there exists $\nu \in (0, 1]$ such that

$$\left\| T_{\phi}^{N(K,L)}(f) \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}^{\nu} \leq \sum_{j=0}^{\infty} b^{-jN\nu} \left\| F_{j+1}^{+K} \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}^{\nu} \leq \sum_{j=0}^{\infty} b^{-jN\nu} b^{j+1} \left\| F_{j}^{+K} \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}^{\nu} \leq \left\| M_{\phi}^{(K,L)}(f) \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}^{\nu},$$

which implies (3.2) and hence completes the proof of Lemma 3.5.  

□
The following Lemmas 3.6 and 3.7 are just [10, p.45, Lemma 7.5 and p.46, Lemma 7.6], respectively.

**Lemma 3.6.** Let \( \phi \in S(\mathbb{R}^n) \) and \( \int_{\mathbb{R}^n} \phi(x) \, dx \neq 0 \). Then, for any given \( N \in \mathbb{N} \) and \( L \in [0, \infty) \), there exist an \( I \in \mathbb{N} \) and a positive constant \( C_{(N,L)} \), depending on \( N \) and \( L \), such that, for any \( K \in \mathbb{Z}_+ \), \( f \in S'_{\mathbb{R}^n} \) and \( x \in \mathbb{R}^n \),
\[
M^{0(K,L)}_I(f)(x) \leq C_{(N,L)} T^{N(K,L)}_\phi(f)(x).
\]

**Lemma 3.7.** Let \( \phi \in S(\mathbb{R}^n) \) and \( \int_{\mathbb{R}^n} \phi(x) \, dx \neq 0 \). Then, for any given \( M \in (0, \infty) \) and \( K \in \mathbb{Z}_+ \), there exist \( L \in (0, \infty) \) and a positive constant \( C_{(K,M)} \), depending on \( K \) and \( M \), such that, for any \( f \in S'_{\mathbb{R}^n} \) and \( x \in \mathbb{R}^n \),
\[
M^{(K,L)}_\phi(f)(x) \leq C_{(K,M)} \left[ \max \{1, \rho(x)\} \right]^{-M}.
\]

Now we state the main result of this section as follows.

**Theorem 3.8.** Suppose that \( p(\cdot) \in C^{\log}(\mathbb{R}^n) \) and \( \phi \in S(\mathbb{R}^n) \) with \( \int_{\mathbb{R}^n} \phi(x) \, dx \neq 0 \). Then, for any \( f \in S'_{\mathbb{R}^n} \), the following statements are mutually equivalent:
\[
\begin{align*}
(3.8) & \quad f \in H_{\mathcal{A}}^{p(\cdot),q}(\mathbb{R}^n); \\
(3.9) & \quad M_\phi(f) \in L^{p(\cdot),q}(\mathbb{R}^n); \\
(3.10) & \quad M^{0}_\phi(f) \in L^{p(\cdot),q}(\mathbb{R}^n).
\end{align*}
\]

In this case, it holds true that
\[
\|f\|_{H_{\mathcal{A}}^{p(\cdot),q}(\mathbb{R}^n)} \leq C_1 \left\| M_\phi(f) \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \leq C_1 \left\| M^{0}_\phi(f) \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \leq C_2 \|f\|_{H_{\mathcal{A}}^{p(\cdot),q}(\mathbb{R}^n)},
\]

where \( C_1 \) and \( C_2 \) are two positive constants independent of \( f \).

**Proof.** Obviously, (3.8) implies (3.9) and (3.9) implies (3.10). Thus, to prove Theorem 3.8, it suffices to show that (3.9) implies (3.8) and that (3.10) implies (3.9).

We first prove that (3.9) implies (3.8). To this end, notice that, by Lemma 3.6 with \( N \in (1/p_-, \infty) \cap \mathbb{N} \) and \( L = 0 \), we find that there exists an \( I \in \mathbb{N} \) such that \( M^{0(K,0)}_I(f)(x) \leq T^{N(K,0)}_\phi(f)(x) \) for any \( x \in \mathbb{R}^n \), \( f \in S'_{\mathbb{R}^n} \) and \( K \in \mathbb{Z}_+ \). From this and Lemma 3.5, we further deduce that, for any \( K \in \mathbb{Z}_+ \) and \( f \in S_{\mathbb{R}^n} \),
\[
(3.11) \quad \left\| M^{0(K,0)}_I(f) \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \leq \left\| M^{(K,0)}_\phi(f) \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}.
\]

Letting \( K \to \infty \) in (3.11), by Proposition 2.8, we know that
\[
\left\| M^{0}_\phi(f) \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \leq \left\| M_\phi(f) \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)},
\]

which shows that (3.9) implies (3.8).
Now we show that (3.10) implies (3.9). Assume now that $M^0_\phi(f) \in L^{p,-}\delta(\mathbb{R}^n)$. By Lemma 3.7 via choosing $M \in (1/p_-, \infty)$, we find that there exists $L \in (0, \infty)$ such that (3.7) holds true, which further implies that $M^{(K,L)}_\phi(f) \in L^{p,-}\delta(\mathbb{R}^n)$ for any $K \in \mathbb{Z}_+$. Indeed, when $q \in (0, \infty)$ and $M \in (1/p_-, \infty)$, we have

$$
\|M^{(K,L)}_\phi(f)\|_{L^{p,-}\delta(\mathbb{R}^n)}^q \leq \sum_{k=0}^{\infty} b^{-kMq} B^{(\kappa_1)q} \sim 1.
$$

Clearly, when $q = \infty$, (3.12) still holds true. Thus, $M^{(K,L)}_\phi(f) \in L^{p,-}\delta(\mathbb{R}^n)$.

On the other hand, by Lemmas 3.6 and 3.5, we know that, for any $L \in (0, \infty)$, there exists some $I \in \mathbb{N}$ such that, for any $K \in \mathbb{Z}_+$ and $f \in S'(\mathbb{R}^n)$,

$$
\|M_I^{(K,L)}(f)\|_{L^{p,-}\delta(\mathbb{R}^n)} \leq C_3 \|M^{(K,L)}_\phi(f)\|_{L^{p,-}\delta(\mathbb{R}^n)},
$$

where $C_3$ is a positive constant independent of $K$ and $f$. For any fixed $K \in \mathbb{Z}_+$, let

$$
\Omega_K := \{x \in \mathbb{R}^n : M_I^{(K,L)}(f)(x) \leq C_4 M^{(K,L)}_\phi(f)(x)\}
$$

with $C_4 := 2C_3$. Then

$$
\|M^{(K,L)}_\phi(f)\|_{L^{p,-}\delta(\mathbb{R}^n)} \leq \|M^{(K,L)}_\phi(f)\|_{L^{p,-}\delta(\Omega_K)},
$$

because

$$
\|M^{(K,L)}_\phi(f)\|_{L^{p,-}\delta(\Omega_K)} \leq C_4^{-1} \|M_I^{(K,L)}(f)\|_{L^{p,-}\delta(\Omega_K)} \leq C_3/C_4 \|M^{(K,L)}_\phi(f)\|_{L^{p,-}\delta(\mathbb{R}^n)}.
$$

For any given $L \in [0, \infty)$, by an argument similar to that used in the proof of [50, (4.17)], we conclude that, for any $t \in (0, p_-)$, $K \in \mathbb{Z}_+$, $f \in S'(\mathbb{R}^n)$ and $x \in \Omega_K$,

$$
M^{(K,L)}_\phi(f)(x) \leq M_{HL}(\left[\|M^{(K,L)}_\phi(f)\|^{1/t}\right](x))^{1/t}.
$$

Then, by (3.13), Lemma 2.7, (3.14) and Lemma 3.4, for any $K \in \mathbb{Z}_+$ and $f \in S'(\mathbb{R}^n)$, we further find that

$$
\|M^{(K,L)}_\phi(f)\|_{L^{p,-}\delta(\mathbb{R}^n)} \leq \|M^{(K,L)}_\phi(f)\|_{L^{p,-}\delta(\Omega_K)} \sim \|M^{(K,L)}_\phi(f)\|_{L^{p,-}\delta(\Omega_K)} \|M^{(K,L)}_\phi(f)\|^{\frac{p_1}{p_2}}_{L^{p,\frac{q}{2}}(\Omega_K)}
$$
which, together with the fact that $M^\phi(f)$ and $M^\phi_0(f)$ converge pointwisely and monotonically, respectively, to $M_\phi(f)$ and $M^\phi_0(f)$ as $K \to \infty$ and Proposition 2.8, implies that

$$
\left\|M_\phi(f)\right\|_{L^{p,q}(\mathbb{R}^n)} \leq \left\|M^\phi_0(f)\right\|_{L^{p,q}(\mathbb{R}^n)}.
$$

This shows that (3.10) implies (3.9) and hence finishes the proof of Theorem 3.8. \hfill \Box

4 Atomic characterization of $H^{p(\cdot),q}_A(\mathbb{R}^n)$

In this section, we establish the atomic characterization of $H^{p(\cdot),q}_A(\mathbb{R}^n)$. We begin with the following notion of anisotropic $(p(\cdot), r, s)$-atoms.

**Definition 4.1.** Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $r \in (1, \infty]$ and

$$
(4.1) \quad s \in \left(\frac{1}{p_+} - 1\right) \frac{\ln b}{\ln \lambda} \cap \mathbb{Z}_+.
$$

A measurable function $a$ on $\mathbb{R}^n$ is called an anisotropic $(p(\cdot), q, s)$-atom if

(i) \quad $\text{supp } a \subset B$, where $B \in \mathcal{B}$ and $\mathcal{B}$ is as in (2.1);

(ii) \quad $\|a\|_{L^r(\mathbb{R}^n)} \leq \|b\|_{L^r(\mathbb{R}^n)}^{1/r}$;

(iii) \quad $\int_{\mathbb{R}^n} a(x)\alpha^s \, dx = 0$ for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq s$.

For the presentation simplicity, throughout this article, we call an anisotropic $(p(\cdot), r, s)$-atom simply by a $(p(\cdot), r, s)$-atom. Now, via $(p(\cdot), r, s)$-atoms, we introduce the notion of the anisotropic variable atomic Hardy-Lorentz space $H^{p(\cdot),r,s}_A(\mathbb{R}^n)$ as follows.

**Definition 4.2.** Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (0, \infty]$, $r \in (1, \infty]$, $s$ be as in (4.1) and $A$ be a dilation. The anisotropic variable atomic Hardy-Lorentz space $H^{p(\cdot),r,s}_A(\mathbb{R}^n)$ is defined to be the set of all distributions $f \in S'(\mathbb{R}^n)$ satisfying that there exist a sequence of $(p(\cdot), r, s)$-atoms, $\{a^k_i\}_{i \in \mathbb{N}, k \in \mathbb{Z}_+}$, supported, respectively, on $\{B^k_i\}_{i \in \mathbb{N}, k \in \mathbb{Z}_+} \subset \mathcal{B}$ and a positive constant $\widetilde{C}$ such that

$$
\sum_{i \in \mathbb{N}, k \in \mathbb{Z}_+} \chi^k_i x^s \leq \widetilde{C}
$$

for any $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, with some $j_0 \in \mathbb{Z} \setminus \mathbb{N}$, and

$$
(4.2) \quad f = \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}} \lambda^k_i a^k_i \quad \text{in} \quad S'(\mathbb{R}^n),
$$

where $\lambda^k_i \sim 2^k \|\chi^k_i x^s\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ for any $k \in \mathbb{Z}$ and $i \in \mathbb{N}$ with the equivalent positive constants independent of $k$ and $i$.

Moreover, for any $f \in H^{p(\cdot),r,s}_A(\mathbb{R}^n)$, define

$$
\|f\|_{H^{p(\cdot),r,s}_A(\mathbb{R}^n)} := \inf \left\{ \sum_{k \in \mathbb{Z}} \left\| \sum_{i \in \mathbb{N}} \left( \frac{\lambda^k_i \chi^k_i x^s}{\|\chi^k_i x^s\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right)^{1/p} \right\|_q^{1/q} \right\}
$$

with the usual interpretation for $q = \infty$, where the infimum is taken over all decompositions of $f$ as above.
In order to establish the atomic decomposition of $H^{p; q, \rho}_{A}(\mathbb{R}^n)$, we need several technical lemmas as follows. First, observe that $(\mathbb{R}^n, \rho, dx)$ is an RD-space (see [39, 83]). From this and [84, Theorem 2.7], we deduce the following Fefferman-Stein vector-valued inequality of the maximal operator $M_{HL}$ on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$, the details being omitted.

**Lemma 4.3.** Let $r \in (1, \infty]$. Assume that $p(\cdot) \in C^{0, \infty}(\mathbb{R}^n)$ satisfies $1 < p_- \leq p_+ < \infty$. Then there exists a positive constant $C$ such that, for any sequence $\{f_k\}_{k \in \mathbb{N}}$ of measurable functions,

$$\left\| \left\{ \sum_{k \in \mathbb{N}} [M_{HL}(f_k)]^r \right\}^{1/r} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \left\| \left( \sum_{k \in \mathbb{N}} |f_k|^r \right)^{1/r} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

with the usual modification made when $r = \infty$, where $M_{HL}$ denotes the Hardy-Littlewood maximal operator as in (3.1).

**Remark 4.4.**
(i) Let $p(\cdot) \in C^{0, \infty}(\mathbb{R}^n)$ and $i \in \mathbb{Z}_+$. Then, by Lemma 4.3 and the fact that, for any dilated ball $B \in \mathcal{B}$ and $r \in (0, p)$, $X_{A_i B_i} \leq b^+ [M_{HL}(\chi_B)]^r$, we conclude that there exists a positive constant $C$ such that, for any sequence $\{B^{(k)}\}_{k \in \mathbb{N}} \subset \mathcal{B}$,

$$\left\| \sum_{k \in \mathbb{N}} X_{A_i B_{i(k)}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C b^+ \left\| \sum_{k \in \mathbb{N}} \chi_{B^{(k)}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

(ii) Let $f \in H^{p(\cdot), r, s, q}_{A}(\mathbb{R}^n)$. By the definition of $H^{p(\cdot), r, s, q}_{A}(\mathbb{R}^n)$, we know that there exists a sequence $\{d^{(k)}_n\}_{k \in \mathbb{N}, n \in \mathbb{Z}_+}$ of $(\rho(\cdot), r, s)$-atoms supported, respectively, on $\{B^{(k)}\}_{k \in \mathbb{N}, n \in \mathbb{Z}_+} \subset \mathcal{B}$, satisfying that, for any $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, $\sum_{i \in \mathbb{N}} X_{A_i n B_i} (x) \leq C$ with some $j_0 \in \mathbb{Z} \setminus \mathbb{N}$ and $C$ being a positive constant independent of $k$ and $x$ such that $f$ admits a decomposition as in (4.2) with $\|f\|_{H^{p(\cdot), r, s, q}_{A}(\mathbb{R}^n)} \sim \|X_{\mathcal{B}^n}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ for any $k \in \mathbb{Z}$ and $i \in \mathbb{N}$, where the equivalent positive constants are independent of $k$ and $i$.

$$\|f\|_{H^{p(\cdot), r, s, q}_{A}(\mathbb{R}^n)} \sim \left\| \sum_{k \in \mathbb{Z}} 2^{kj_n} \left\| \sum_{j \in \mathbb{N}} \frac{X_{\mathcal{B}^j}}{\|X_{\mathcal{B}^j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{1/p} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{1/q} \frac{1}{q}$$

with the equivalent positive constants independent of $f$. Moreover, by $\sum_{i \in \mathbb{N}} X_{A_i n B_i} \leq C$ for any $k \in \mathbb{Z}$, the definition of $X_{\mathcal{B}^n}$ and (i) of this remark, we further conclude that

$$\|f\|_{H^{p(\cdot), r, s, q}_{A}(\mathbb{R}^n)} \sim \left\| \sum_{k \in \mathbb{Z}} 2^{kj_n} \left\| \sum_{j \in \mathbb{N}} X_{B_i} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{1/p} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{1/q} \frac{1}{q}$$

where the equivalent positive constants are independent of $f$. 
We also need the following useful technical lemma, whose proof is similar to that of [66, Lemma 4.1], the details being omitted.

**Lemma 4.5.** Let \( p(\cdot) \in C^{\log}(\mathbb{R}^n) \), \( t \in (0, p] \) and \( r \in [1, \infty] \cap (p, \infty] \). Then there exists a positive constant \( C \) such that, for any sequence \( \{B(k)\}_{k \in \mathbb{N}} \subset \mathcal{B} \) of dilated balls, numbers \( \{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{C} \) and measurable functions \( \{a_k\}_{k \in \mathbb{N}} \) satisfying that, for each \( k \in \mathbb{N} \), \( a_k \subset B(k) \) and \( \|a_k\|_{L^r(\mathbb{R}^n)} \leq |B(k)|^{1/r} \), it holds true that

\[
\left\| \left( \sum_{k \in \mathbb{N}} |\lambda_k a_k|^t \right)^{1/t} \right\|_{L^r(\mathbb{R}^n)} \leq C \left\| \left( \sum_{k \in \mathbb{N}} |\lambda_k 1_{B(k)}|^t \right)^{1/t} \right\|_{L^r(\mathbb{R}^n)}.
\]

The following Proposition 4.6 and Lemma 4.7 are just [10, p. 17, Proposition 3.10 and p. 9, Lemma 2.7], respectively.

**Proposition 4.6.** For any given \( N \in \mathbb{N} \), there exists a positive constant \( C_{(N)} \), depending only on \( N \), such that, for any \( f \in \mathcal{S}'(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \),

\[
M_N^0(f)(x) \leq M_N(f)(x) \leq C_{(N)} M_N^0(f)(x),
\]

where \( M_N^0(f) \) and \( M_N(f) \) are as in Definition 2.9.

**Lemma 4.7.** Let \( \Omega \subset \mathbb{R}^n \) be an open set with \( |\Omega| < \infty \). Then, for any \( m \in \mathbb{Z}_+ \), there exist a sequence of points, \( \{x_k\}_{k \in \mathbb{N}} \subset \Omega \), and a sequence of integers, \( \{\ell_k\}_{k \in \mathbb{N}} \subset \mathbb{Z} \), such that

(i) \( \Omega = \bigcup_{k \in \mathbb{N}} (x_k + B(\ell_k)) \);

(ii) \( \{x_k + B(\ell_k)\}_{k \in \mathbb{N}} \) are pairwise disjoint, where \( \tau \) is as in (2.2) and (2.3);

(iii) for each \( k \in \mathbb{N} \), \( (x_k + B(\ell_k + m)) \cap \Omega^c = \emptyset \), but \( (x_k + B(\ell_k + m + 1)) \cap \Omega^c \neq \emptyset \);

(iv) if \( (x_i + B(\ell_i + m - 2\tau)) \cap (x_j + B(\ell_j + m - 2\tau)) \neq \emptyset \), then \( |i - j| \leq \tau \);

(v) for any \( i \in \mathbb{N} \), \( \# \{j \in \mathbb{N} : (x_i + B(\ell_i + m - 2\tau)) \cap (x_j + B(\ell_j + m - 2\tau)) \neq \emptyset \} \leq R \), where \( R \) is a positive constant independent of \( \Omega \) and \( i \).

Now, it is a position to state the main result of this section.

**Theorem 4.8.** Let \( p(\cdot) \in C^{\log}(\mathbb{R}^n) \), \( q \in (0, \infty] \), \( r \in (\max\{p_+, 1\}, \infty] \) with \( p_+ \) as in (2.4) and \( s \) be as in (4.1). Then \( H_A^{p(\cdot),r,s,q}(\mathbb{R}^n) = H_A^{p(\cdot),r,s,q}(\mathbb{R}^n) \) with equivalent quasi-norms.

**Proof.** First, we show that

(4.3) \[
H_A^{p(\cdot),r,s,q}(\mathbb{R}^n) \subset H_A^{p(\cdot),r,s,q}(\mathbb{R}^n).
\]

To this end, for any \( f \in H_A^{p(\cdot),r,s,q}(\mathbb{R}^n) \), by Remark 4.4(ii), we find that there exists a sequence of \( (p(\cdot), r, s) \)-atoms, \( \{a_l^k\}_{l \in \mathbb{N}, k \in \mathbb{Z}} \), supported, respectively, on \( \{B_l^k\}_{l \in \mathbb{N}, k \in \mathbb{Z}} \subset \mathcal{B} \) such that

\[
f = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{N}} \lambda_l^k a_l^k \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n),
\]
where $\lambda_i^k \sim 2^k \|X_{B_i^k}\|_{L^p(\mathbb{R}^n)}$ for any $k \in \mathbb{Z}$ and $i \in \mathbb{N}$, $\sum_{i} \lambda_i^k B_i^k(x) \lesssim 1$ with some $j_0 \in \mathbb{Z} \setminus \mathbb{N}$ for any $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, and

$$\|f\|_{H^p_N^{\alpha,\beta,q}(\mathbb{R}^n)} \lesssim \left[ \sum_{k \in \mathbb{Z}} 2^{kq} \left( \sum_{i} \lambda_i^k \|X_{B_i^k}\|_{L^p(\mathbb{R}^n)} \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}. \tag{4.4}$$

For the notational simplicity, in what follows of this proof, for any $f \in S'(\mathbb{R}^n)$, we denote $M^0_N(f)$ simply by $M(f)$ when $N \in \mathbb{N} \cap \{(\frac{1}{p} - 1)\ln b_{\mathbb{A},\mathbb{B}} + 2, \infty\}$. To prove $f \in H^p_A^{\alpha,\beta,q}(\mathbb{R}^n)$, by Definition 2.10 and Lemma 2.5, it suffices to show that

$$\left[ \sum_{k \in \mathbb{Z}} 2^{kq} \|X_{\{x \in \mathbb{R}^n: M(f)(x) > 2^k\}}\|_{L^p(\mathbb{R}^n)} \right]^{\frac{1}{q}} \leq \|f\|_{H^p_N^{\alpha,\beta,q}(\mathbb{R}^n)}.$$  

For any fixed $k_0 \in \mathbb{Z}$, we write

$$f = \sum_{k=-\infty}^{k_0-1} \sum_{i} \lambda_i^k a_i^k + \sum_{k=k_0}^{\infty} \sum_{i} \cdots =: f_1 + f_2.$$ 

Then, by Remark 2.3(i), we know that

$$\|X_{\{x \in \mathbb{R}^n: M(f)(x) > 2^k\}}\|_{L^p(\mathbb{R}^n)} \leq \|X_{\{x \in \mathbb{R}^n: M(f_1)(x) > 2^k\}}\|_{L^p(\mathbb{R}^n)} + \|X_{\{x \in \mathbb{R}^n: M(f_2)(x) > 2^k\}}\|_{L^p(\mathbb{R}^n)}$$

where $E_{k_0} := \bigcup_{k=k_0}^{\infty} \bigcup_{i} A_i^k B_i^k$.

For $J_1$, clearly, we have

$$J_1 \lesssim \left[ \sum_{k=-\infty}^{k_0-1} \sum_{i} \lambda_i^k M(a_i^k)(x)X_{A_i^kB_i^k}(x) \|X_{\{x \in \mathbb{R}^n: M(f)(x) > 2^k\}}\|_{L^p(\mathbb{R}^n)} \right]^{\frac{1}{q}}$$

$$+ \left[ \sum_{k=-\infty}^{k_0-1} \sum_{i} \lambda_i^k M(a_i^k)(x)X_{A_i^kB_i^k}(x) \|X_{\{x \in \mathbb{R}^n: M(f)(x) > 2^k\}}\|_{L^p(\mathbb{R}^n)} \right]^{\frac{1}{q}}$$

$$=: J_{1,1} + J_{1,2}.$$ 

To deal with $J_{1,1}$, by the Hölder inequality, we find that, for any given $t \in (0, \min\{p, q\})$, $\bar{q} \in (1, \min\{r, \frac{1}{\max(p, q)}\}, \frac{1}{t})$, $\delta \in (0, 1 - \frac{1}{\bar{q}})$ and any $x \in \mathbb{R}^n$, 

$$\sum_{k=-\infty}^{k_0-1} \sum_{i} \lambda_i^k M(a_i^k)(x)X_{A_i^kB_i^k}(x)$$
On the other hand, since
\[ J_{1,1} \leq \left( \sum_{k=-\infty}^{k_0-1} \right) \begin{cases} 2^{k_0-1} \left( \sum_{k=-\infty}^{k_0-1} 2^{-k\delta q} \left( \sum_{j \in \mathbb{N}} \lambda_j^k M(a_j^k)(x) \chi_{A^r B_i^t}(x) \right) \right)^{\frac{1}{q'}} \\ \left( \sum_{k=-\infty}^{k_0-1} 2^{-k\delta q} \left( \sum_{j \in \mathbb{N}} \lambda_j^k M(a_j^k)(x) \chi_{A^r B_i^t}(x) \right) \right)^{\frac{1}{q'}} \end{cases} , \]

where \( q' \) denotes the conjugate index of \( q \), namely, \( \frac{1}{q} + \frac{1}{q'} = 1 \). By this, the facts that \( \widetilde{q} t < 1 \) and \( M(f)(x) \leq M_{HL}(f)(x) \) for any \( x \in \mathbb{R}^n \), Remark 2.3(i) and [22, Theorem 2.61], we further conclude that

\[ J_{1,1} \leq \left\| \chi \right\|_{L^p(\mathbb{R}^n)} \leq 2^{-k_0 \tilde{q}(1-\delta)} \left\| \sum_{k=-\infty}^{k_0-1} 2^{-k\delta q} \left( \sum_{j \in \mathbb{N}} \lambda_j^k M(a_j^k)(x) \chi_{A^r B_i^t}(x) \right) \right\|_{L^p(\mathbb{R}^n)} \]

\[ \leq 2^{-k_0 \tilde{q}(1-\delta)} \left\| \sum_{k=-\infty}^{k_0-1} 2^{-k\delta q} \sum_{j \in \mathbb{N}} \lambda_j^k M(a_j^k)(x) \chi_{A^r B_i^t}(x) \right\|_{L^p(\mathbb{R}^n)} \]

\[ \leq 2^{-k_0 \tilde{q}(1-\delta)} \left\| \sum_{k=-\infty}^{k_0-1} 2^{-k\delta q} \sum_{j \in \mathbb{N}} M_{HL}(a_j^k)(x) \chi_{A^r B_i^t}(x) \right\|_{L^p(\mathbb{R}^n)} \]

\[ \leq 2^{-k_0 \tilde{q}(1-\delta)} \left\| \sum_{k=-\infty}^{k_0-1} 2^{k_0 \tilde{q}(1-\delta)} \left\| \sum_{j \in \mathbb{N}} M_{HL}(a_j^k)(x) \chi_{A^r B_i^t}(x) \right\|_{L^p(\mathbb{R}^n)} \right\|_{L^p(\mathbb{R}^n)} \]

On the other hand, since \( r > 1 \), then, from the boundedness of \( M_{HL} \) on \( L^{1/q} \), we deduce that, for any \( k \in \mathbb{Z} \) and \( i \in \mathbb{N} \),

\[ \left\| \left\| \chi \right\|_{L^p(\mathbb{R}^n)} M_{HL}(a_j^k)(x) \chi_{A^r B_i^t}(x) \right\|_{L^{1/q}(\mathbb{R}^n)} \leq \left\| \chi \right\|_{L^p(\mathbb{R}^n)} \left\| M_{HL}(a_j^k)(x) \chi_{A^r B_i^t}(x) \right\|_{L^{1/q}(\mathbb{R}^n)} \leq \left\| B_i^t \right\|_{L^{1/q}(\mathbb{R}^n)} \]

which, combined with Lemma 4.5, Remark 4.4(i), the fact that \( t < q \) and the Hölder inequality, implies that

\[ J_{1,1} \leq 2^{-k_0 \tilde{q}(1-\delta)} \left\| \sum_{k=-\infty}^{k_0-1} 2^{k_0 \tilde{q}(1-\delta)} \left\| \sum_{j \in \mathbb{N}} \chi_{A^r B_i^t}(x) \right\|_{L^p(\mathbb{R}^n)} \right\|_{L^p(\mathbb{R}^n)} \]

\[ \leq 2^{-k_0 \tilde{q}(1-\delta)} \left\| \sum_{k=-\infty}^{k_0-1} \left( \sum_{j \in \mathbb{N}} \chi_{A^r B_i^t}(x) \right) \right\|_{L^p(\mathbb{R}^n)} \]

\[ \leq 2^{-k_0 \tilde{q}(1-\delta)} \left\| \sum_{k=-\infty}^{k_0-1} \left( \sum_{j \in \mathbb{N}} \chi_{A^r B_i^t}(x) \right) \right\|_{L^p(\mathbb{R}^n)} \]
where \( \varepsilon \in (1, (1 - \delta)q) \). From this and (4.4), we deduce that

\[
(4.7) \quad \left[ \sum_{k_0 \in \mathbb{Z}} 2^{k_0 q} (J_{1,1})^q \right]^\frac{1}{q} \leq \left[ \sum_{k_0 \in \mathbb{Z}} 2^{k_0 q} 2^{-\varepsilon k_0 q} \sum_{k = -\infty}^{k_0 - 1} 2^{\varepsilon k} \left\| X_{B_i^k} \right\|_{L^p(\mathbb{R}^n)}^q \right]^\frac{1}{q} \\
\sim \left[ \sum_{k \in \mathbb{Z}} \sum_{k_0 = k + 1}^{\infty} 2^{(1-\varepsilon)k_0 q} 2^{\varepsilon k} \left\| X_{B_i^k} \right\|_{L^p(\mathbb{R}^n)}^q \right]^\frac{1}{q} \sim \| f \|_{H^p_{\lambda_1, \ldots, \lambda_q}(\mathbb{R}^n)}.
\]

To estimate \( J_{1,2} \), for any \( i \in \mathbb{N}, k \in \mathbb{Z} \) and \( x \in (A^* B_i^k)^\mathbb{C} \), by an argument similar to that used in the proof of [50, (3.27)], we have

\[
M(a_i^k(x)) \leq \left\| X_{B_i^k} \right\|_{L^p(\mathbb{R}^n)}^{-1} \left[ \frac{|B_i^k|}{\rho(x - x_i^k)^q} \right]^\alpha \left\| X_{B_i^k} \right\|_{L^p(\mathbb{R}^n)}^{-1} \left[ M_{H^p \lambda_1, \ldots, \lambda_q}(X_{B_i^k}(x)) \right]^\beta,
\]

where, for any \( i \in \mathbb{N}, k \in \mathbb{Z}, x_i^k \) denotes the centre of the dilated ball \( B_i^k \) and

\[
(4.9) \quad \beta := \left( \frac{\ln b}{\ln \lambda_-} + s + 1 \right) \frac{\ln \lambda_-}{\ln b} > \frac{1}{p}.
\]

By this, Remark 2.3(i), Lemma 4.3, Remark 4.4(i) and the Hölder inequality, we find that, for any \( t \in (0, \min\{q, 1/\beta\}), q_1 \in (\frac{1}{2q_t}, \frac{1}{q_t}) \) and \( \delta \in (0, 1 - \frac{1}{q_t}) \),

\[
J_{1,2} \leq \left\| X_{(x \in \mathbb{R}^n : \sum_{k=-\infty}^{k_0-1} 2^{k_0 q} 2^{2kq_t} [\sum_{i \in \mathbb{N}} a_i^k M(a_i^k(x)(A^* B_i^k)^\mathbb{C}(x) q_1)^{1/q_1} 2^{(1-\varepsilon)q_1}]^{1/2} \right\|_{L^p(\mathbb{R}^n)}
\]

\[
\leq 2^{-k_0 q t (1-\delta)} \sum_{k = -\infty}^{k_0 - 1} 2^{-k_0 q t} \left\| \sum_{i \in \mathbb{N}} a_i^k M(a_i^k(x)(A^* B_i^k)^\mathbb{C}(x) q_1)^{1/q_1} \right\|_{L^p(\mathbb{R}^n)}
\leq 2^{-k_0 q t (1-\delta)} \left\{ \sum_{k = -\infty}^{k_0 - 1} 2^{(1-\delta)q_1 t} \left\| \sum_{i \in \mathbb{N}} \left[ M_{H^p \lambda_1, \ldots, \lambda_q}(X_{B_i^k}(x)) \right]^{\beta q_1} \right\|_{L^p_{\lambda_1, \ldots, \lambda_q}(\mathbb{R}^n)} \right\}^\frac{1}{\beta q_1}
\leq 2^{-k_0 q t (1-\delta)} \left\{ \sum_{k = -\infty}^{k_0 - 1} 2^{(1-\delta)q_1 t} \left\| X_{B_i^k} \right\|_{L^p_{\lambda_1, \ldots, \lambda_q}(\mathbb{R}^n)} \right\}^\frac{1}{\beta q_1}
\leq 2^{-k_0 q t (1-\delta)} \left\{ \sum_{k = -\infty}^{k_0 - 1} 2^{(1-\delta)q_1 t} \left\| X_{A_{k_0} B_i^k} \right\|_{L^p_{\lambda_1, \ldots, \lambda_q}(\mathbb{R}^n)} \right\}^\frac{1}{\beta q_1}.
\]
\[ \leq 2^{-k_0 q_1 (1 - \delta)} \sum_{k = -\infty}^{k_0 - 1} 2^{(1 - \delta) q_1 \epsilon |k|^{1/p}} \sum_{k = -\infty}^{k_0 - 1} 2^{\epsilon k q} \left\| \sum_{i \in \mathbb{N}} X_{B_i^k}^{q} \right\|_{L^{\infty}((\mathbb{R}^n))}^{1 \over q}, \]

where \( \epsilon \in (1, (1 - \delta) q_1) \). This, together with a calculation similar to that of (4.7), further implies that

\begin{equation}
\left\| \sum_{k_0 \in \mathbb{Z}} 2^{k_0 q} (J_{1, 2})^{q} \right\|_{2} \leq \left\| \sum_{k \in \mathbb{Z}} 2^{k q} \left\| \sum_{i \in \mathbb{N}} X_{B_i^k}^{q} \right\|_{L^{\infty}((\mathbb{R}^n))} \right\|_{2} \sim \| f \|_{H^{(\alpha, \beta, \delta) q}_A(\mathbb{R}^n)}.
\end{equation}

By this, (4.6) and (4.7), we have

\begin{equation}
\left\| \sum_{k_0 \in \mathbb{Z}} 2^{k_0 q} (J_{1})^{q} \right\|_{2} \leq \left\| \sum_{k \in \mathbb{Z}} 2^{k q} \left\| \sum_{i \in \mathbb{N}} X_{B_i^k}^{q} \right\|_{L^{\infty}((\mathbb{R}^n))} \right\|_{2} \sim \| f \|_{H^{(\alpha, \beta, \delta) q}_A(\mathbb{R}^n)}.
\end{equation}

For \( J_2 \), by Remark 4.4(i), the Hölder inequality, we know that, for any \( t \in (0, \min \{ p, q \}) \) and \( \delta \in (0, 1) \),

\[ J_2 \leq \| X_{E_k^d} \|_{L^{\alpha}((\mathbb{R}^n))} \leq \left\| \sum_{k = k_0}^{\infty} \sum_{i \in \mathbb{N}} X_{A_i^k B_i^k} \right\|_{L^{\alpha}((\mathbb{R}^n))} \leq \left\| \sum_{k = k_0}^{\infty} \sum_{i \in \mathbb{N}} X_{B_i^k} \right\|_{L^{\alpha}((\mathbb{R}^n))}^{1/t} \leq \left( \sum_{k = k_0}^{\infty} 2^{-k_0 q} \right)^{1/t} \left\| \sum_{k = k_0}^{\infty} 2^{k_0 q} \sum_{i \in \mathbb{N}} X_{B_i^k} \right\|_{L^{\alpha}((\mathbb{R}^n))}^{1/q} \sim 2^{-k_0 \delta} \left( \sum_{k = k_0}^{\infty} 2^{k_0 q} \sum_{i \in \mathbb{N}} X_{B_i^k} \right)^{1/q} \sim 2^{-k_0 \delta} \sum_{i \in \mathbb{N}} X_{B_i^k} \left\| \sum_{i \in \mathbb{N}} X_{B_i^k} \right\|_{L^{\infty}((\mathbb{R}^n))}^{1/q}.
\]

By this, similarly to (4.7), we obtain

\begin{equation}
\left\| \sum_{k_0 \in \mathbb{Z}} 2^{k_0 q} (J_{2})^{q} \right\|_{2} \leq \left\| \sum_{k \in \mathbb{Z}} 2^{k q} \left\| \sum_{i \in \mathbb{N}} X_{B_i^k}^{q} \right\|_{L^{\infty}((\mathbb{R}^n))} \right\|_{2} \sim \| f \|_{H^{(\alpha, \beta, \delta) q}_A(\mathbb{R}^n)}.
\end{equation}

For \( J_3 \), since \( \beta > 1 - \delta \), it follows that there exist \( 0 < \gamma < 1 \) and \( L \in (1, \infty) \) such that \( \beta t > {1 \over 2} \) and \( q \beta t L > 1 \). By this, the value of \( \lambda_{t, \alpha}^{k} \), (4.8), Lemma 4.3 and the Hölder inequality, we find that

\[ J_3 \leq \| X_{(x \in E_k^d) : \Sigma_{k = k_0}^{\infty} \sum_{i \in \mathbb{N}} X_{B_i^k}^{q} \lambda_{t, \alpha}^{k} (x) > 2^{k_0 q - 1)} \|_{L^{\gamma}((\mathbb{R}^n))} \leq 2^{-k_0} \left\| \sum_{k = k_0}^{\infty} \sum_{i \in \mathbb{N}} \left[ X_{B_i^k}^{q} \lambda_{t, \alpha}^{k} (x) \right] X_{E_k^d} \right\|_{L^{\gamma}((\mathbb{R}^n))} \leq 2^{-k_0} \left\| \sum_{k = k_0}^{\infty} \sum_{i \in \mathbb{N}} \left[ X_{B_i^k}^{q} \lambda_{t, \alpha}^{k} (x) \right] X_{E_k^d} \right\|_{L^{\gamma}((\mathbb{R}^n))} \leq \left\| X_{E_k^d} \right\|_{L^{\gamma}((\mathbb{R}^n))} \left\| \sum_{i \in \mathbb{N}} \left[ X_{B_i^k}^{q} \lambda_{t, \alpha}^{k} (x) \right] X_{E_k^d} \right\|_{L^{\gamma}((\mathbb{R}^n))} \leq 2^{-k_0} \left\| \sum_{k = k_0}^{\infty} \sum_{i \in \mathbb{N}} \left[ X_{B_i^k}^{q} \lambda_{t, \alpha}^{k} (x) \right] X_{E_k^d} \right\|_{L^{\gamma}((\mathbb{R}^n))} \| f \|_{H^{(\alpha, \beta, \delta) q}_A(\mathbb{R}^n)}.
\]
Step 1. Now we prove (4.14) by three steps.

We now prove that
\[ \|f\|_{H^A_p(\mathbb{R}^n)} \leq \left( \sum_{k_0 \in \mathbb{Z}} 2^{k_0\beta} \right)^{1/q} \left( \sum_{k_0 \in \mathbb{Z}} 2^{k_0q} \right)^{1/q} \leq \left( \sum_{k_0 \in \mathbb{Z}} 2^{k_0q} \right)^{1/q}, \]
which, combined with (4.5), (4.11) and (4.12), implies that
\[ \|f\|_{H^A_p(\mathbb{R}^n)} \sim \left( \sum_{k_0 \in \mathbb{Z}} 2^{k_0q} \right)^{1/q} \left( \sum_{k_0 \in \mathbb{Z}} 2^{k_0q} \right)^{1/q} = \|f\|_{H^A_p(\mathbb{R}^n)}^{1/q}, \]
This further implies that \( f \in H^A_{p}^{(\cdot),q}(\mathbb{R}^n) \) and hence finishes the proof of (4.3).

We now prove that \( H^A_{p}^{(\cdot),q}(\mathbb{R}^n) \subset H^A_{p}^{(\cdot),r,s,q}(\mathbb{R}^n) \). To this end, it suffices to show that
\[ (4.14) \quad H^A_{p}^{(\cdot),q}(\mathbb{R}^n) \subset H^A_{p}^{(\cdot),\infty,s,q}(\mathbb{R}^n), \]
due to the fact that each \( (p(\cdot), \infty, s) \)-atom is also a \( (p(\cdot), r, s) \)-atom and hence
\[ H^A_{p}^{(\cdot),\infty,s,q}(\mathbb{R}^n) \subset H^A_{p}^{(\cdot),r,s,q}(\mathbb{R}^n). \]

Now we prove (4.14) by three steps.

Step 1. To show (4.14), for any \( f \in H^A_{p}^{(\cdot),q}(\mathbb{R}^n) \), \( \phi \in \mathcal{S}(\mathbb{R}^n) \) with \( \int_{\mathbb{R}^n} \phi(x) \, dx = 1 \) and \( \nu \in \mathbb{N} \), let \( f^{(\nu)} := f \ast \phi_{-\nu} \). By this and [10, p. 15, Lemma 3.8], we know that \( f^{(\nu)} \to f \) in \( \mathcal{S}'(\mathbb{R}^n) \) as \( \nu \to \infty \). Moreover, by [10, p. 39, Lemma 6.6], we conclude that there exists a positive constant \( C_{(N,\phi)} \), depending on \( N \) and \( \phi \), but independent of \( f \), such that, for any \( \nu \in \mathbb{N} \) and \( x \in \mathbb{R}^n \),
\[ M_{N+2}(f^{(\nu)})(x) \leq C_{(N,\phi)}M_N(f)(x). \]
Thus, by Definition 2.10 and Proposition 4.6, we find that \( f^{(v)} \in H^{p,q}_{\lambda}(\mathbb{R}^n) \) and
\[
\|f^{(v)}\|_{H^{p,q}_{\lambda}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p,q}_{\lambda}(\mathbb{R}^n)}.
\]

The aim of this step is to prove that, for any \( v \in \mathbb{N} \),
\[
(4.15) \quad f^{(v)} = \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}} h_i^{v,k} \text{ in } S'(\mathbb{R}^n),
\]
where, for each \( v, i \in \mathbb{N} \) and \( k \in \mathbb{Z} \), \( h_i^{v,k} \) is a \((p(\cdot), \infty, \sigma)\)-atom multiplied by a constant depending on \( k \) and \( i \), but independent of \( f \) and \( v \).

To show (4.15), we borrow some ideas from the proofs of [10, p. 38, Theorem 6.4] and [50, Theorem 3.6]. For any \( k \in \mathbb{Z} \) and \( N \in \mathbb{N} \cap \{[(\frac{1}{2} - 1) \ln h_i^{v,k}] + 2, \infty\} \), let
\[
\Omega_k := \{x \in \mathbb{R}^n : M_N(f)(x) > 2^k\}.
\]
Clearly, \( \Omega_k \) is open. From this and Lemma 4.7 with \( m = 6 \tau \), we deduce that there exist a sequence \( \{x_i^k\}_{i \in \mathbb{N}} \subset \Omega_k \) and a sequence \( \{\ell_i^k\}_{i \in \mathbb{N}} \subset \mathbb{Z} \) such that
\[
(4.16) \quad \Omega_k = \bigcup_{i \in \mathbb{N}} (x_i^k + B_{\ell_i^k});
\]
\[
(4.17) \quad (x_i^k + B_{\ell_i^k-\tau}) \cap (x_j^k + B_{\ell_j^k-\tau}) = \emptyset \text{ for any } i, j \in \mathbb{N} \text{ with } i \neq j;
\]
\[
(x_i^k + B_{\ell_i^k + 6\tau}) \cap \Omega_k^C = \emptyset, \quad (x_i^k + B_{\ell_i^k + 6\tau + 1}) \cap \Omega_k^C \neq \emptyset \text{ for any } i \in \mathbb{N};
\]
\[
\text{if } (x_i^k + B_{\ell_i^k + 4\tau}) \cap (x_j^k + B_{\ell_j^k + 4\tau}) \neq \emptyset, \text{ then } |\ell_i^k - \ell_j^k| \leq \tau;
\]
\[
(4.18) \quad \# \{j \in \mathbb{N} : (x_i^k + B_{\ell_i^k + 4\tau}) \cap (x_j^k + B_{\ell_j^k + 4\tau}) \neq \emptyset\} \leq R \text{ for any } i \in \mathbb{N},
\]
where \( \tau \) and \( R \) are as same as in Lemma 4.7.

Fix \( \eta \in S(\mathbb{R}^n) \) satisfying \( \supp \eta \subset B_r, 0 \leq \eta \leq 1 \) and \( \eta \equiv 1 \) on \( B_0 \). For any \( i \in \mathbb{N}, k \in \mathbb{Z} \) and \( x \in \mathbb{R}^n \), let \( \eta_i^k(x) := \eta(A^{-\ell_i^k}(x - x_i^k)) \) and
\[
\theta_i^k(x) := \frac{\eta_i^k(x)}{\sum_{j \in \mathbb{N}} \eta_j^k(x)}.
\]
Then \( \theta_i^k \in S(\mathbb{R}^n), \supp \theta_i^k \subset x_i^k + B_{\ell_i^k + \tau}, 0 \leq \theta_i^k \leq 1, \theta_i^k \equiv 1 \text{ on } x_i^k + B_{\ell_i^k - \tau} \) by (4.17), and \( \sum_{i \in \mathbb{N}} \theta_i^k = \chi_{\Omega_k} \). From this, it follows that \( \{\theta_i^k\}_{i \in \mathbb{N}} \) forms a smooth partition of unity of \( \Omega_k \).

For any \( m \in \mathbb{Z}_+ \), define \( \mathcal{P}_m(\mathbb{R}^n) \) to be the linear space of all polynomials on \( \mathbb{R}^n \) with degree not greater than \( m \). Moreover, for each \( i \) and \( P \in \mathcal{P}_m(\mathbb{R}^n) \), let
\[
(4.19) \quad \|P\|_{i,k} := \left[ \frac{1}{\int_{\mathbb{R}^n} \theta_i^k(x) dx} \int_{\mathbb{R}^n} |P(x)|^2 \theta_i^k(x) dx \right]^{1/2}
\]
Then \( (P_m(\mathbb{R}^n), \| \cdot \|_{i,k}) \) is a finite dimensional Hilbert space. For each \( i \), since \( f^{(v)} \) induces a linear functional on \( P_m(\mathbb{R}^n) \) via
\[
Q \mapsto \frac{1}{\int_{\mathbb{R}^n} \theta_i^k(x) dx} \left\langle f^{(v)}, Q \theta_i^k \right\rangle, \quad Q \in P_m(\mathbb{R}^n),
\]
by the Riesz lemma, it follows that there exists a unique polynomial \( P_i^{v,k} \in P_m(\mathbb{R}^n) \) such that, for any \( Q \in P_m(\mathbb{R}^n) \),
\[
\frac{1}{\int_{\mathbb{R}^n} \theta_i^k(x) dx} \left\langle f^{(v)}, Q \theta_i^k \right\rangle = \frac{1}{\int_{\mathbb{R}^n} \theta_i^k(x) dx} \left\langle P_i^{v,k}, Q \theta_i^k \right\rangle = \frac{1}{\int_{\mathbb{R}^n} \theta_i^k(x) dx} \int_{\mathbb{R}^n} P_i^{v,k}(x) Q(x) \theta_i^k(x) dx.
\]
For each \( i \in \mathbb{N} \), \( k \in \mathbb{Z} \) and \( v \in \mathbb{N} \), let \( b_i^{v,k} := [f^{(v)} - P_i^{v,k}] \theta_i^k \) and
\[
g^{v,k} := f^{(v)} - \sum_{i \in \mathbb{N}} b_i^{v,k} = f^{(v)} - \sum_{i \in \mathbb{N}} [f^{(v)} - P_i^{v,k}] \theta_i^k = f^{(v)} \chi_{\Omega_k} + \sum_{i \in \mathbb{N}} P_i^{v,k} \theta_i^k.
\]
Then, by an argument similar to that used in [50, p. 1679], we conclude that, for any \( v \in \mathbb{N} \),
\[
\| g^{v,k} \|_{L^p(\mathbb{R}^n)} \leq 2^k \text{ and } \| g^{v,k} \|_{L^p(\mathbb{R}^n)} \to 0 \text{ as } k \to -\infty.
\]
For any \( k \in \mathbb{Z} \), let
\[
\Omega_k := \left\{ x \in \mathbb{R}^n : M_N(f^{(v)}(x)) > 2^k \right\}.
\]
Then, for any \( \epsilon \in (0, 1) \), by \( f^{(v)} \in H^{(\lambda,p)}_A(\mathbb{R}^n) \), we know that there exists an integer \( k(\epsilon) \) such that, for any \( k \in [k(\epsilon), \infty) \cap \mathbb{Z} \), \( |\Omega_k| < \epsilon \). Since, for any \( k \in [k(\epsilon), \infty) \cap \mathbb{Z} \) and \( \alpha \in (0, \infty) \),
\[
|\Omega_k \cap \{ x \in \mathbb{R}^n : M_N(f^{(v)}(x)) > \alpha \}| \leq \min \left\{ |\Omega_k|, \alpha^{-p_-} \left\| M_N(f^{(v)}) \right\|_{L^{p_-}(m(\Omega_k))} \right\}
\]
\[
\leq \min \left\{ |\Omega_k|, \alpha^{-p_-} \left\| M_N(f^{(v)}) \right\|_{L^{p_-}(m(\Omega_k))} \right\},
\]
it follows that, for any \( p_0 \in (0, p_-) \) satisfying \( [(1/p_0 - 1) \ln b/ \ln \lambda_-] \leq s \),
\[
(4.20) \quad \int_{\Omega_k} \left[ M_N(f^{(v)}(x)) \right]^{p_0} dx = \int_0^\infty p_0 \alpha^{p_0-1-1} \left\| \chi_{\Omega_k} \right\|_{L^{p_0}(m(\Omega_k))} d\alpha \leq \int_\gamma^\infty p_0 \alpha^{p_0-1} d\alpha + \int_0^\infty p_0 \alpha^{p_0-1-p_-} \left\| M_N(f^{(v)}) \right\|_{L^{p_-}(m(\Omega_k))} d\alpha = \frac{p_- - p_0}{p_- - p_0} \left[ \frac{1}{p_-} \right] \left\| M_N(f^{(v)}) \right\|_{L^{p_-}(m(\Omega_k))} \left[ \frac{1}{p_-} \right] \left\| M_N(f^{(v)}) \right\|_{L^{p_-}(m(\Omega_k))},
\]
where \( \gamma := \left\| M_N(f^{(v)}) \right\|_{L^{p_-}(m(\Omega_k))} \left[ \frac{1}{p_-} \right] \). On the other hand, by [50, (3.9)], we know that, for any \( k \in \mathbb{Z} \) and \( p_0 \) as in (4.20),
\[
\int_{\mathbb{R}^n} \left[ M_N \left( \sum_{i \in \mathbb{N}} b_i^{v,k} \right) \right]^{p_0} dx \leq \int_{\Omega_k} \left[ M_N(f^{(v)}(x)) \right]^{p_0} dx,
\]
which, together with (4.20), further implies that
\[
\left\| \sum_{i \in \mathbb{N}} b_i^{\nu,k} \right\|_{H^0_A(\mathbb{R}^n)} := \left\| M_N \left( \sum_{i \in \mathbb{N}} b_i^{\nu,k} \right) \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \Omega \right\|_p^{1/p} \left\| M_N(f^{(v)}) \right\|_{L^p(\mathbb{R}^n)} \to 0
\]
as \( k \to \infty \). Here and hereafter, for any \( p \in (0, \infty) \), \( H^0_A(\mathbb{R}^n) \) denotes the anisotropic Hardy space introduced by Bownik in [10]. By this and the fact that, for any \( \nu \in \mathbb{N} \), \( \|g^{\nu,k}\|_{L^\infty(\mathbb{R}^n)} \to 0 \) as \( k \to -\infty \), we have
\[
\left\| f^{(v)} - \sum_{k=-N}^{N} (g^{\nu,k+1} - g^{\nu,k}) \right\|_{H^0_A(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)} \leq \left\| \sum_{i \in \mathbb{N}} b_i^{\nu,N+1} \right\|_{H^0_A(\mathbb{R}^n)} + \left\| g^{\nu,-N} \right\|_{L^\infty(\mathbb{R}^n)} \to 0 \text{ as } N \to \infty,
\]
where, for any \( f \in H^0_A(\mathbb{R}^n) + L^\infty(\mathbb{R}^n) \),
\[
\left\| f \right\|_{H^0_A(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)} := \inf \left\{ \|f_1\|_{H^0_A(\mathbb{R}^n)} + \|f_2\|_{L^\infty(\mathbb{R}^n)} : f = f_1 + f_2, \ f_1 \in H^0_A(\mathbb{R}^n), \ f_2 \in L^\infty(\mathbb{R}^n) \right\}
\]
with the infimum being taken over all decompositions of \( f \) as above. From this and a proof similar to the proofs of [50, (3.12), (3.13), (3.14) and (3.16)], we deduce that, for any \( \nu \in \mathbb{N} \),
\[
f^{(v)} = \sum_{k \in \mathbb{Z}} (g^{\nu,k+1} - g^{\nu,k}) = \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}} \left[ b_i^{\nu,k} - \sum_{j \in \mathbb{N}} \left( b_i^{\nu,k+1} \theta_i^j - P_{i,j}^{\nu,k+1} \theta_i^j \right) \right] =: \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}} h_i^{\nu,k} \text{ in } S'(\mathbb{R}^n),
\]
where, for any \( \nu, i, j \in \mathbb{N} \) and \( k \in \mathbb{Z} \), \( P_{i,j}^{\nu,k+1} \) is the orthogonal projection of \( f^{(v)} - P_{i,j}^{\nu,k+1} \theta_i^j \) on \( \mathcal{P}_m(\mathbb{R}^n) \) with respect to the norm defined by (4.19) and \( h_i^{\nu,k} \) is a multiple of a \( (p(\cdot), \infty, s) \)-atom satisfying
\[
(4.21) \quad \int_{\mathbb{R}^n} h_i^{\nu,k}(x) Q(x) \, dx = 0 \quad \text{for any } Q \in \mathcal{P}_m(\mathbb{R}^n),
\]
\[
(4.22) \quad \text{supp } h_i^{\nu,k} \subset (x_i^k + B_{\ell_i^k + 4r})
\]
and
\[
(4.23) \quad \left\| h_i^{\nu,k} \right\|_{L^\infty(\mathbb{R}^n)} \leq 2^k.
\]
This finishes the proof of (4.15).

Step 2. From (4.23) and the Alaoglu theorem (see, for example, [64, Theorem 3.17]), it follows that there exists a subsequence \( \{\nu_i \}_{i=1}^\infty \subset \mathbb{N} \) such that, for each \( i \in \mathbb{N} \) and \( k \in \mathbb{Z} \), \( h_i^{\nu_i,k} \to h_i^k \) as
$t \to \infty$ in the weak-* topology of $L^\infty(\mathbb{R}^n)$. Moreover, supp $h^k_i \subset (x^k_i + B_{t_i+4r}^0)$, $\|h^k_i\|_{L^\infty(\mathbb{R}^n)} \leq 2^k$ and $\int_{\mathbb{R}^n} h^k_i(x) Q(x) \, dx = 0$ for any $Q \in \mathcal{P}_m(\mathbb{R}^n)$. Therefore, $h^k_i$ is a multiple of a $(p(\cdot), \infty, s)$-atom $a^k_i$. Let $h^k_i := \lambda^k_i a^k_i$, where $\lambda^k_i \sim 2^k \|X^k_i + B_{t_i+4r}^0\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. Then, by (4.16), (4.18) and Lemma 2.5, we have

$$\left[ \sum_{k \in \mathbb{Z}} \left( \sum_{i \in \mathbb{N}} \left[ \frac{\lambda^k_i X^k_i + B_{t_i+4r}^0}{\|X^k_i + B_{t_i+4r}^0\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\frac{1}{p}} \right)^{\frac{1}{q}} \right]^{\frac{1}{q}} \sim \left( \sum_{k \in \mathbb{Z}} 2^k \|X^k_i\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{\frac{1}{q}} \sim \|M_N(f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \sim \|f\|_{H^p_\Phi(\mathbb{R}^n)}$$

with the usual modification made when $q = \infty$.

**Step 3.** By Step 2, we conclude that, to prove (4.14), it suffices to show that $f = \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}} h^k_i$ in $\mathcal{S}'(\mathbb{R}^n)$.

To this end, for any $k \in \mathbb{Z}$, let

$$f^k := \sum_{i \in \mathbb{N}} h^k_i$$

and, for any $\nu \in \mathbb{N}$ and $k \in \mathbb{Z}$, $f^{(\nu)}_k := g^{\nu,k+1} - g^{\nu,k}$. Then $f_k^{(\nu)} \to f_k$ in $\mathcal{S}'(\mathbb{R}^n)$ as $t \to \infty$. Indeed, by (4.18) and the support conditions of $h^k_i$ and $h^{sk}_i$, we find that, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle f^{(\nu)}_k, \varphi \rangle = \sum_{i \in \mathbb{N}} h^{(\nu,k)}_i, \varphi \rangle = \sum_{i \in \mathbb{N}} h^{(\nu,k)}_i, \varphi \rangle$$

$$\to \sum_{i \in \mathbb{N}} \langle h^k_i, \varphi \rangle$$

$$= \langle f_k, \varphi \rangle$$

as $t \to \infty$.

Next we aim to prove that, for any $\nu \in \mathbb{N}$,

$$\sum_{|k| \geq K_1} f^{(\nu)}_k \to 0 \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n) \quad \text{as} \quad K_1 \to \infty.$$  

(4.24)

Indeed, for any $p_0 \in (0, p_-)$ with $[(1/p_0 - 1) \ln b/ \ln \lambda_-] \leq s$, by (4.21), (4.22) and (4.23), we know that $(2^k|B_{t_i+4r}^0|^{1/p_0})^{-1}h^k_i$ is a $(p_0, \infty, s)$-atom multiplied by a constant. From this, [10, p. 19, Theorem 4.2], (4.18) and (4.16) and Lemma 2.5, we deduce that, as $K_2 \to \infty$,

$$\sum_{|k| \geq K_2} f^{(\nu)}_k \leq \sum_{|k| \geq K_2} \sum_{i \in \mathbb{N}} h^{(\nu,k)}_i \|h^{(\nu,k)}_i\|_{H^p_{\Phi}(\mathbb{R}^n)} \leq \sum_{|k| \geq K_2} \sum_{i \in \mathbb{N}} 2^{kp_0} \|B_{t_i+4r}^0\|_{H^p_{\Phi}(\mathbb{R}^n)}$$

$$\leq \sum_{|k| \geq K_2} 2^{kp_0} \left[ \|\chi\|_{L^p(\mathbb{R}^n)} + \|\chi\|_{L^{p\ast}(\mathbb{R}^n)} \right]$$

(4.25)
as we conclude that
Then, by (4.21), (4.22) and (4.23) again, we find that \( (2^k |B_{(j+1)|}^{(−1)k}) \) is a \((1, \infty, s)\)-atom multiplied by a constant. Therefore, by [10, p. 19, Theorem 4.2], (4.18), (4.16) and Lemma 2.5 again, we conclude that

\[
\sum_{k \leq K_3} \left| \sum_{j \in \mathbb{N}} h_{jk}^{(v)} \right|_{H^1_k(E_1)} \leq \sum_{k \leq K_3} \left| \sum_{j \in \mathbb{N}} h_{jk}^{(v)} \chi_{E_1} \right|_{H^1_k(\mathbb{R}^n)} \leq \sum_{k \leq K_3} 2^k |\Omega_k \cap E_1|,
\]

as \( K_3 \to -\infty \). Similarly, for any \( \tilde{p}_+ \in (p_+(E_2), \infty) \), we have

\[
\left\| \sum_{k \leq K_4} f_k^{(v)} \right\|_{L^{\tilde{p}_+}(E_2)} \leq \sum_{k \leq K_4} \left\| \sum_{j \in \mathbb{N}} h_{jk}^{(v)} \chi_{E_2} \right\|_{L^{\tilde{p}_+}(\mathbb{R}^n)} \leq \sum_{k \leq K_4} 2^k |\Omega_k \cap E_2|^{1/\tilde{p}_+},
\]

as \( K_4 \to -\infty \). This, combined with (4.25) and (4.26), implies that (4.24) holds true.

By an argument similar to that used in the proof of (4.24), we know that \( \sum_{k \geq K_1} f_k \to 0 \) in \( S'(\mathbb{R}^n) \) as \( K_1 \to \infty \). From this, (4.24) and a proof similar to that used in [50, pp. 1682-1683], we further deduce that

\[
f = \sum_{k \in \mathbb{Z}} f_k = \lim_{K_1 \to \infty} \sum_{k \geq K_1} f_k \rightarrow 0 \text{ in } S'(\mathbb{R}^n),
\]

which completes the proof of (4.14). This shows \( H^{p, (\cdot), r, s, q}(\mathbb{R}^n) \subset H^{p, (\cdot), r, s, q}(\mathbb{R}^n) \) and hence finishes the proof of Theorem 4.8.
5 Lusin area function characterizations of $H_A^{p(\cdot),q}(\mathbb{R}^n)$

In this section, using the atomic characterization of $H_A^{p(\cdot),q}(\mathbb{R}^n)$ obtained in Theorem 4.8, we establish the Lusin area function characterization of $H_A^{p(\cdot),q}(\mathbb{R}^n)$ in Theorem 5.2. To this end, we first recall the notion of the anisotropic Lusin area function (see [48, 50]).

**Definition 5.1.** Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \psi(x)x^q \, dx = 0$ for any multi-index $\gamma \in (\mathbb{Z}_+)^n$ with $|\gamma| \leq s$, where $s \in \mathbb{N} \cap \{(\frac{1}{p} - 1)\ln b/\ln \lambda, \infty\}$ and $p_-$ is as in (2.4). Then, for any $f \in \mathcal{S}'(\mathbb{R}^n)$, the anisotropic Lusin area function $S(f)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$S(f)(x) := \left[ \sum_{k \in \mathbb{Z}} b^{-k} \int_{x+B_k} |f * \psi_k(y)|^2 \, dy \right]^{1/2},$$

Recall also that a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to vanish weakly at infinity if, for each $\psi \in \mathcal{S}(\mathbb{R}^n)$, $f * \psi_k \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $k \to \infty$. Denote by $\mathcal{S}'_0(\mathbb{R}^n)$ the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ vanishing weakly at infinity.

The main result of this section is the following Theorem 5.2.

**Theorem 5.2.** Let $p(\cdot) \in C^\log(\mathbb{R}^n)$ and $q \in (0, \infty]$. Then $f \in H_A^{p(\cdot),q}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_0(\mathbb{R}^n)$ and $S(f) \in L^{p(\cdot),q}(\mathbb{R}^n)$. Moreover, there exists a positive constant $C$ such that, for any $f \in H_A^{p(\cdot),q}(\mathbb{R}^n)$,

$$C^{-1}\|S(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \leq \|f\|_{H_A^{p(\cdot),q}(\mathbb{R}^n)} \leq C\|S(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)}.$$

To prove Theorem 5.2, we need several technical lemmas. First, by an argument similar to that used in the proof of [78, Lemma 6.5], it is easy to see that the following lemma holds true, the details being omitted.

**Lemma 5.3.** Let $p(\cdot) \in C^\log(\mathbb{R}^n)$ and $q \in (0, \infty]$. Then $H_A^{p(\cdot),q}(\mathbb{R}^n) \subset \mathcal{S}'_0(\mathbb{R}^n)$.

Via borrowing some ideas from [85, Lemma 2.6] and [54, Lemma 2.2], we obtain the following result, which is an anisotropic version of [85, Lemma 2.6].

**Lemma 5.4.** Let $p(\cdot) \in C^\log(\mathbb{R}^n)$ and $p_- \in (1, \infty)$. Then there exists a positive constant $C$ such that, for all subsets $E_1$, $E_2$ of $\mathbb{R}^n$ with $E_1 \subset E_2$,

$$C^{-1} \left( \frac{\|E_1\|}{\|E_2\|} \right)^{\frac{1}{p_-^*}} \leq \frac{\|\chi_{E_1}\|_{L^{p(\cdot),q}(\mathbb{R}^n)}}{\|\chi_{E_2}\|_{L^{p(\cdot),q}(\mathbb{R}^n)}} \leq C \left( \frac{\|E_1\|}{\|E_2\|} \right)^{\frac{1}{p_-^*}}.$$

**Proof.** In view of similarity, we only show that

$$\frac{\|\chi_{E_1}\|_{L^{p(\cdot),q}(\mathbb{R}^n)}}{\|\chi_{E_2}\|_{L^{p(\cdot),q}(\mathbb{R}^n)}} \leq \left( \frac{\|E_1\|}{\|E_2\|} \right)^{\frac{1}{p_-^*}}. \tag{5.1}$$

To this end, for any $i \in \{1, 2\}$, let

$$p_-(E_i) := \text{ess inf}_{x \in E_i} p(x) \quad \text{and} \quad p_+(E_i) := \text{ess sup}_{x \in E_i} p(x).$$
If $|E_2| \leq 1$, then, by (2.6) and $p_\infty \in (1, \infty)$, we know that, for any $i \in \{1, 2\}$ and $x \in E_i$,

$$|E_i|^{\frac{1}{p_i}} \sim |E_i|^{\frac{1}{p_i-1}} \sim |E_i|^{\frac{1}{p_i+1}}|E_i|^{\frac{1}{p_i}}$$

which, combined with

$$|E_i|^{\frac{1}{p_i}} \leq \|\chi_{E_i}\|_{L^{p_i}(\mathbb{R}^n)} \leq |E_i|^{\frac{1}{p_i+1}}|E_i|^{\frac{1}{p_i}},$$

implies that

$$\|\chi_{E_i}\|_{L^{p_i}(\mathbb{R}^n)} \sim |E_i|^{\frac{1}{p_i}} \sim |E_i|^{\frac{1}{p_i-1}} \sim |E_i|^{\frac{1}{p_i+1}}|E_i|^{\frac{1}{p_i}}.$$  

By this, we conclude that, for any $x \in E_1$,

$$\|\chi_{E_1}\|_{L^{p_i}(\mathbb{R}^n)} \sim \left(\frac{|E_1|}{|E_2|}\right)^{\frac{1}{p_i}} \leq \left(\frac{|E_1|}{|E_2|}\right)^{\frac{1}{p_i}}. $$

If $|E_1| \geq 1$, let $\{Q_{ij}\}_{j \in \mathbb{N}}$ be a partition of $\mathbb{R}^n$ such that, for any $i, j \in \mathbb{N}$, $|Q_{ij}| = |Q_{ij}| = 1$ and, when $\text{dist}(\overline{Q_{ij}}, Q_{ij}) > \text{dist}(\overline{Q_{ij}}, Q_{ij})$, $i > j$, where, for any $i \in \mathbb{N}$, $\text{dist}(\overline{Q_{ij}}, Q_{ij}) := \inf\{|x| : x \in Q_{ij}\}$. Then, by [42, Theorem 2.4] and (5.2), we find that, for any $i \in \{1, 2\}$,

$$\|\chi_{E_1}\|_{L^{p_i}(\mathbb{R}^n)} \sim \left(\frac{|E_1|}{|E_i|}\right)^{\frac{1}{p_i}} \leq \left(\frac{|E_1|}{|E_i|}\right)^{\frac{1}{p_i}}.$$  

where $p_\infty$ is as in (2.7). From this, it follows that

$$\|\chi_{E_1}\|_{L^{p_i}(\mathbb{R}^n)} \sim \left(\frac{|E_1|}{|E_i|}\right)^{\frac{1}{p_i}} \leq \left(\frac{|E_1|}{|E_i|}\right)^{\frac{1}{p_i}}. $$

If $|E_1| < 1 < |E_2|$, then, from (5.2) and (5.4), we deduce that

$$\|\chi_{E_1}\|_{L^{p_i}(\mathbb{R}^n)} \sim \left(\frac{|E_1|}{|E_2|}\right)^{\frac{1}{p_i}} \leq \left(\frac{|E_1|}{|E_i|}\right)^{\frac{1}{p_i}}.$$  

Combining (5.3), (5.5) and (5.6), we obtain (5.1). This finishes the proof of Lemma 5.4. 

The following lemma is just [13, Lemma 2.3], which is a slight modification of [18, Theorem 11].

**Lemma 5.5.** Let $A$ be a dilation. Then there exists a collection

$$Q := \{Q^k_\alpha \subset \mathbb{R}^n : k \in \mathbb{N}, \alpha \in I_k\}$$

of open subsets, where $I_k$ is certain index set, such that

(i) for any $k \in \mathbb{N}$, $|\mathbb{R}^n \setminus \bigcup_\alpha Q^k_\alpha| = 0$ and, when $\alpha \neq \beta$, $Q^k_\alpha \cap Q^k_\beta = \emptyset$;

(ii) for any $\alpha, \beta, k, \ell$ with $\ell \geq k$, either $Q^k_\alpha \cap Q^\ell_\beta = \emptyset$ or $Q^k_\alpha \subset Q^\ell_\beta$. 


(iii) for each \((\ell, \beta)\) and each \(k < \ell\), there exists a unique \(\alpha\) such that \(Q^\ell_{\beta} \subset Q^k_{\alpha}\);

(iv) there exist some negative integer \(v\) and positive integer \(u\) such that, for any \(Q^k_{\alpha}\) with \(k \in \mathbb{Z}\) and \(\alpha \in I_k\), there exists \(x_{Q^k_{\alpha}} \in Q^k_{\alpha}\) satisfying that, for any \(x \in Q^k_{\alpha}\),

\[
x_{Q^k_{\alpha}} + B_{vk-u} \subset Q^k_{\alpha} \subset x + B_{vk+u}.
\]

In what follows, we call \(Q := \{Q^k_{\alpha}\}_{k \in \mathbb{Z}, \alpha \in I_k}\) from Lemma 5.5 dyadic cubes and \(k\) the level, denoted by \(\ell(Q^k_{\alpha})\), of the dyadic cube \(Q^k_{\alpha}\) with \(k \in \mathbb{Z}\) and \(\alpha \in I_k\).

**Remark 5.6.** In the definition of \((p(\cdot), r, s)\)-atoms (see Definition 4.1), if we replace dilated balls \(B\) by dyadic cubes, then, from Lemma 5.5, we deduce that the corresponding anisotropic variable atomic Hardy-Lorentz space coincides with the original one (see Definition 4.2) in the sense of equivalent quasi-norms.

The following Calderón reproducing formula is just [13, Proposition 2.14].

**Lemma 5.7.** Let \(s \in \mathbb{Z}_+\) and \(A\) be a dilation. Then there exist \(\varphi, \psi \in S(\mathbb{R}^n)\) such that

(i) \(\text{supp } \varphi \subset B_0, \int_{\mathbb{R}^n} x^\gamma \varphi(x) \, dx = 0\) for any \(\gamma \in (\mathbb{Z}_+)^n\) with \(|\gamma| \leq s\), \(\hat{\varphi}(\xi) \geq C\) for any \(\xi \in \{x \in \mathbb{R}^n : a \leq \rho(x) \leq b\}\), where \(0 < a < b < 1\) and \(C\) are positive constants;

(ii) \(\text{supp } \hat{\psi}\) is compact and bounded away from the origin;

(iii) \(\sum_{j \in \mathbb{Z}} \hat{\psi}(A^j) \hat{\varphi}(A^j) = 1\) for any \(\xi \in \mathbb{R}^n \setminus \{0\}\), where \(A^\ast\) denotes the adjoint of \(A\).

Moreover, for any \(f \in S_0'(\mathbb{R}^n)\), \(f = \sum_{j \in \mathbb{Z}} f \ast \psi_j \ast \varphi_j\) in \(S'(\mathbb{R}^n)\).

Now we prove Theorem 5.2.

**Proof of Theorem 5.2.** We first show the necessity of this theorem. Let \(f \in H^p(\mathbb{R}^n)\). It follows from Lemma 5.3 that \(f \in S_0'(\mathbb{R}^n)\). On the other hand, for any \(k_0 \in \mathbb{Z}\), due to Theorem 4.8 and Remark 5.6, we can decompose \(f\) as follows

\[
f = \sum_{k=-\infty}^{k_0-1} \sum_{i \in \mathbb{N}} \lambda_i^k a_i^k + \sum_{k=k_0}^{\infty} \sum_{i \in \mathbb{N}} \cdots =: f_1 + f_2,
\]

where \(\{\lambda_i^k\}_{i \in \mathbb{N}, k \in \mathbb{Z}}\) and \(\{a_i^k\}_{i \in \mathbb{N}, k \in \mathbb{Z}}\) are as in Theorem 4.8 satisfying (4.2). Let \(v, u\) be as in Lemma 5.5 and \(w := u - v + 2r\). Then we have

\[
\|X_{\lambda \in \mathbb{R}^n; \; S(f)(x) > 2^k}\|_{L^p(\mathbb{R}^n)} \\
\leq \|X_{\lambda \in \mathbb{R}^n; \; S(f_1)(x) > 2^{k_0-1}}\|_{L^p(\mathbb{R}^n)} + \|X_{\lambda \in \mathbb{R}^n; \; S(f_2)(x) > 2^{k_0-1}}\|_{L^p(\mathbb{R}^n)} \\
+ \|X_{\lambda \in \mathbb{R}^n; \; S(f_3)(x) > 2^{k_0-1}}\|_{L^p(\mathbb{R}^n)} \\
=: I_1 + I_2 + I_3,
\]

(5.7)
where $E_{k_0} := \bigcup_{k=k_0}^{\infty} \bigcup_{i \in \mathbb{N}} A^w Q_i^k$ and $(Q_j^k)_{j \in \mathbb{N}, k \in \mathbb{Z}} \subset Q$ are as in Lemma 5.5.

Obviously,

$$I_1 \lesssim \left\| \chi_{\{x \in \mathbb{R}^n : \sum_{k=-\infty}^{\infty} \sum_{i \in \mathbb{N}} |x|^{\beta} S(a_i^k(x)) (x) > 2^{k0-2} \} \right\|_{L^{p}({\mathbb{R}^n})}$$

$$= : I_{1,1} + I_{1,2}.$$

For $I_{1,1}$, by [13, Theorem 3.2], Lemma 4.5, Remark 4.4(i) and a proof similar to that of (4.7), we conclude that

$$\left( \sum_{k \in \mathbb{Z}} 2^{kq} (I_{1,1})^q \right)^{1/q} \lesssim \left( \sum_{k \in \mathbb{Z}} 2^{kq} \left\| \sum_{i \in \mathbb{N}} \chi_{Q_i^k} \right\|_{L^{p}({\mathbb{R}^n})}^q \right)^{1/q} \sim \|f\|_{H^t_{\lambda, s, a,q}^{p}({\mathbb{R}^n})}.$$  \hfill (5.8)

To deal with $I_{1,2}$, assume that $a$ is a $(\rho(-), r, s)$-atom supported on a dyadic cube $Q$. For any $j \in \mathbb{N}$, let

$$U_j := x_Q \left( B_{\lambda(\ell(Q)-j-1)+u+2r} \setminus B_{\lambda(\ell(Q)-j)+u+2r} \right).$$

Then, by Lemma 5.5(iv), we know that, for any $x \in (A^w Q)^C$, there exists some $j_0 \in \mathbb{N}$ such that $x \in U_{j_0}$. For this $j_0$, choose $N \in \mathbb{N}$ large enough such that

$$(N - \beta) r j_0 + (1 - v) \left( \frac{1}{r} - \beta \right) \ell(Q) < 0$$

with $\beta$ as in (4.9). By this and an argument similar to that used in the proof [51, (3.3)], we find that

$$S(a(x)) \lesssim b^{N \rho j_0} b^{\frac{-\ell(Q)}{r}} \|d\|_{L^{r}({\mathbb{R}^n})}.$$  \hfill (5.9)

From this and the size condition of $a$, we deduce that

$$S(a(x)) \lesssim b^{(N - \beta) \rho j_0 + (1 - v)(1 - \beta) \ell(Q)} \left\| \chi_{Q} \right\|_{L^{p}({\mathbb{R}^n})}^{-1} \left\| Q \right\|_{B(\ell(Q)-j_0)}^{B(\ell(Q)-j_0)} \|a\|_{L^{r}({\mathbb{R}^n})}^{\beta} \|\chi_{Q}\|_{L^{p}({\mathbb{R}^n})}^{-1} \left[ M_{H\ell}(\chi_{Q})(x) \right]^{\beta}.$$  \hfill (5.10)

By (5.10), similarly to (4.10), we conclude that

$$\left( \sum_{k \in \mathbb{Z}} 2^{kq} (I_{1,2})^q \right)^{1/q} \lesssim \left( \sum_{k \in \mathbb{Z}} 2^{kq} \left\| \sum_{i \in \mathbb{N}} \chi_{Q_i^k} \right\|_{L^{p}({\mathbb{R}^n})}^q \right)^{1/q} \sim \|f\|_{H^t_{\lambda, s, a,q}^{p}({\mathbb{R}^n})},$$

which, together with (5.8) and (5.9), further implies that

$$\left( \sum_{k_0 \in \mathbb{Z}} 2^{k_0 q} (I_1)^q \right)^{1/q} \lesssim \|f\|_{H^t_{\lambda, s, a,q}^{p}({\mathbb{R}^n})}.  \hfill (5.11)$$
For $I_2$ and $I_3$, from (5.10) and a proof similar to those of (4.12) and (4.13), it follows that

\[
\left( \sum_{k_0 \in \mathbb{Z}} 2^{k_0 q}(I_2)^q \right)^{\frac{1}{q}} \leq \|f\|_{H^p_{\alpha,\delta,q}(\mathbb{R}^n)} \quad \text{and} \quad \left( \sum_{k_0 \in \mathbb{Z}} 2^{k_0 q}(I_3)^q \right)^{\frac{1}{q}} \leq \|f\|_{H^p_{\alpha,\delta,q}(\mathbb{R}^n)}.
\]

Combining (5.7), (5.11) and (5.12), we obtain

\[
\|S(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \sim \left( \sum_{k_0 \in \mathbb{Z}} 2^{k_0 q} \|x_{|x|>2^{k_0}} \|_{L^{p(\cdot),q}(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} \\
\leq \left( \sum_{k_0 \in \mathbb{Z}} 2^{k_0 q}(I_1)^q \right)^{\frac{1}{q}} + \left( \sum_{k_0 \in \mathbb{Z}} 2^{k_0 q}(I_2)^q \right)^{\frac{1}{q}} + \left( \sum_{k_0 \in \mathbb{Z}} 2^{k_0 q}(I_3)^q \right)^{\frac{1}{q}} \\
\leq \|f\|_{H^p_{\alpha,\delta,q}(\mathbb{R}^n)} \sim \|f\|_{H^p_{\alpha,\delta,q}(\mathbb{R}^n)}
\]

with the usual modification made when $q = \infty$, which implies that $S(f) \in L^{p(\cdot),q}(\mathbb{R}^n)$ and

\[
\|S(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \leq \|f\|_{H^p_{\alpha,\delta,q}(\mathbb{R}^n)}.
\]

This shows the necessity of Theorem 5.2.

Next we prove the sufficiency of Theorem 5.2. Let $f \in S'_0(\mathbb{R}^n)$ and $S(f) \in L^{p(\cdot),q}(\mathbb{R}^n)$. Then we need to prove that $f \in H^p_{\alpha,\delta,q}(\mathbb{R}^n)$ and

\[
\|f\|_{H^p_{\alpha,\delta,q}(\mathbb{R}^n)} \leq \|S(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)}.
\]

To this end, for any $k \in \mathbb{Z}$, let $\Omega_k := \{x \in \mathbb{R}^n : S(f)(x) > 2^k\}$ and

\[
Q_k := \left\{ Q \in Q : |Q \cap \Omega_k| > \frac{|Q|}{2} \quad \text{and} \quad |Q \cap \Omega_{k+1}| \leq \frac{|Q|}{2} \right\}.
\]

Clearly, for each $Q \in Q$, there exists a unique $k \in \mathbb{Z}$ such that $Q \in Q_k$. Let $\{Q_i^k\}$ be the set of all maximal dyadic cubes in $Q_k$, namely, there exists no $Q \in Q_k$ such that $Q_i^k \subsetneq Q$ for any $i$.

Let $u, v$ be as in Lemma 5.5 and, for any $Q \in Q$,

\[
\widehat{Q} := \{(y, t) \in \mathbb{R}^n \times \mathbb{R} : y \in Q, \, v\ell(Q) + u + \tau \leq t < v(\ell(Q) - 1) + u + \tau\}.
\]

Then $\{\widehat{Q}\}_{Q \in Q}$ are mutually disjoint and

\[
\mathbb{R}^n \times \mathbb{R} = \bigcup_{k \in \mathbb{Z}} \bigcup_{Q \in Q_k} B_{k,l} =: \bigcup_{k \in \mathbb{Z}} \bigcup_{l \in \mathbb{Z}} B_{k,l}.
\]

Obviously, $\{B_{k,l}\}_{k \in \mathbb{Z}}$ are mutually disjoint by Lemma 5.5(ii).

Let $\psi$ and $\varphi$ be as in Lemma 5.7. Then, by Lemma 5.7, the properties of the tempered distributions (see [37, Theorem 2.3.20] or [70, Theorem 3.13]) and (5.14), we find that, for any $f \in S'_0(\mathbb{R}^n)$ with $S(f) \in L^{p(\cdot),q}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

\[
f(x) = \sum_{k \in \mathbb{Z}} f * \psi_k * \varphi_k(x) = \int_{\mathbb{R}^n \times \mathbb{R}} f * \psi_t(y) * \varphi_t(x-y) \, dy \, dm(t)
\]
This finishes the proof of (5.13) and hence Theorem 5.2.

\[
\sum_{k \in \mathbb{Z}} \sum_{i} \int_{B_{k,i}} f \ast \psi_j(y) \ast \varphi_i(x-y) \, dy \, dm(t) =: \sum_{k \in \mathbb{Z}} \sum_{i} h_i^k(x)
\]

in \(S'(\mathbb{R}^n)\), where \(m(t)\) is the counting measure on \(\mathbb{R}\), namely, for any set \(E \subset \mathbb{R}\), \(m(E) := \#E\) if \(E\) has only finite elements, or else \(m(E) := \infty\). Moreover, by an argument similar to that used in the proofs of [51, (3.24), (3.29) and (3.30)], it is easy to see that there exists some \(C_0 \in (0, \infty)\) such that, for any \(r \in (\max\{p_+, 1\}, \infty), k \in \mathbb{Z}, i\) and \(\alpha \in (\mathbb{Z}_+)^n\) as in Definition 4.1,

\[
(5.15) \quad \text{supp } h_i^k \subset x_{Q_i^k} + B_{\max\{Q_i^k\}}^{+r} =: B_i^k,
\]

\[
(5.16) \quad \|h_i^k\|_{L^s(\mathbb{R}^n)} \leq C_0 2^k |B_i^k|^{1/r}
\]

and

\[
(5.17) \quad \int_{\mathbb{R}^n} h_i^k(x)x^\alpha \, dx = 0,
\]

namely, \(h_i^k\) is a \((p(\cdot), r, s)\)-atom multiplied by a constant.

For any \(k \in \mathbb{Z}\) and \(i\), let \(\lambda_i^k := C_0 2^k \|\chi_{B_i^k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\) and \(a_i^k := (\lambda_i^k)^{-1} h_i^k\), where \(C_0\) is a positive constant as in (5.16). Then we obtain

\[
f = \sum_{k \in \mathbb{Z}} \sum_{i} h_i^k = \sum_{k \in \mathbb{Z}} \sum_{i} \lambda_i^k a_i^k \quad \text{in } S'(\mathbb{R}^n).
\]

By (5.15) and (5.17), we know that \(\text{supp } a_i^k \subset B_i^k\) and \(a_i^k\) also has the vanishing moments up to \(s\). From (5.16) and Lemma 5.5(iv), it follows that \(\|a_i^k\|_{L^s(\mathbb{R}^n)} \leq \|\chi_{B_i^k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} |B_i^k|^{1/r}\). Therefore, \(a_i^k\) is a \((p(\cdot), r, s)\)-atom for any \(k \in \mathbb{Z}\) and \(i\). Moreover, by Theorem 4.8, the mutual disjointness of \(\{Q_i^k\}_{k \in \mathbb{Z}, i}\), Lemma 5.5(iv) again, \(|Q_i^k \cap \Omega_k| \geq \frac{|Q_i^k|}{2}\), Lemmas 5.4 and 2.5, we conclude that

\[
\|f\|_{H^m_{\lambda}(\mathbb{R}^n)} \sim \left\{ \sum_{k \in \mathbb{Z}} \left\| \left( \sum_{i \in \mathbb{N}} \left( \frac{\lambda_i^k \chi_{B_i^k}}{\|\chi_{B_i^k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right)^{1/p} \right)^{1/q} \right\|^q_{L^p(\mathbb{R}^n)} \right\}^{1/q}
\]

\[
\sim \left\{ \sum_{k \in \mathbb{Z}} 2^{kq} \left\| \left( \sum_{i \in \mathbb{N}} \chi_{B_i^k} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \right\}^{1/q} \sim \left\{ \sum_{k \in \mathbb{Z}} 2^{kq} \left\| \left( \sum_{i \in \mathbb{N}} \chi_{Q_i^k} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \right\}^{1/q}
\]

\[
\sim \left\{ \sum_{k \in \mathbb{Z}} 2^{kq} \left\| \chi_{\Omega_k} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{1/q} \right\}^{1/q} \sim \|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}
\]

with the usual modification made when \(q = \infty\), which implies that \(f \in H^m_{\lambda}(\mathbb{R}^n)\) and

\[
\|f\|_{H^m_{\lambda}(\mathbb{R}^n)} \leq \|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\]

This finishes the proof of (5.13) and hence Theorem 5.2. \(\square\)
6 Real interpolation

As another application of the atomic characterization of $H^{p(\cdot)}_{\lambda}(\mathbb{R}^n)$, in this section, we obtain a real interpolation result between $H^{p(\cdot)}_{\lambda}(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$. Moreover, using this result, together with [84, Corollary 4.20] and [44, Remark 4.2(ii)], we then show that the anisotropic variable Hardy-Lorentz space $H^{p(\cdot),q}_{\lambda}(\mathbb{R}^n)$ with $p_-=1, \infty$ coincides with the variable Lorentz space $L^{p(\cdot),q}(\mathbb{R}^n)$.

To state the main result of this section, we first recall some basic notions about the real interpolation (see [9]). Assume that $(X_1, X_2)$ is a compatible couple of quasi-normed spaces, namely, $X_1$ and $X_2$ are two quasi-normed linear spaces which are continuously embedded in some larger topological vector space. Let

$$X_1 + X_2 := \{ f_1 + f_2 : f_1 \in X_1, f_2 \in X_2 \}.$$  

For any $t \in (0, \infty)$, the Peetre $K$-functional on $X_1 + X_2$ is defined by setting, for any $f \in X_0 + X_1$,

$$K(t; f; X_1, X_2) := \inf \{ \| f_1 \|_{X_1} + t \| f_2 \|_{X_2} : f = f_1 + f_2, f_1 \in X_1 \text{ and } f_2 \in X_2 \}.$$  

Moreover, for any $\theta \in (0, 1)$ and $q \in (0, \infty]$, the real interpolation space $(X_1, X_2)_{\theta,q}$ is defined as

$$(X_1, X_2)_{\theta,q} := \left\{ f \in X_1 + X_2 : \| f \|_{\theta,q} := \left[ \int_0^\infty \left( t^\theta K(t; f; X_1, X_2) \right)^q \frac{dt}{t} \right]^{1/q} < \infty \right\}.$$  

Definition 6.1. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $N \in \mathbb{N} \cap [(\frac{1}{p(\cdot)} - 1) \frac{\ln p(\cdot)}{\ln M(\cdot)} + 2, \infty)$, where $p$ is as in (2.5). The anisotropic variable Hardy space, denoted by $H^{p(\cdot)}_{\lambda}(\mathbb{R}^n)$, is defined by setting

$$H^{p(\cdot)}_{\lambda}(\mathbb{R}^n) := \{ f \in \mathcal{S}'(\mathbb{R}^n) : M_N^\lambda(f) \in L^{p(\cdot)}(\mathbb{R}^n) \}$$  

and, for any $f \in H^{p(\cdot)}_{\lambda}(\mathbb{R}^n)$, let $\| f \|_{H^{p(\cdot)}_{\lambda}(\mathbb{R}^n)} := \| M_N^\lambda(f) \|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

The main result of this section is stated as follows.

Theorem 6.2. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (0, \infty]$ and $\theta \in (0, 1)$. Then it holds true that

$$(H^{p(\cdot)}_{\lambda}(\mathbb{R}^n), L^q(\mathbb{R}^n))_{\theta,q} = H^{p(\cdot)}_{\lambda}(\mathbb{R}^n),$$  

where $\frac{1}{p(\cdot)} = \frac{1}{p_+} + \frac{1}{p_-} - \theta$.

As a consequence of Theorem 6.2, [84, Corollary 4.20] and [44, Remark 4.2(ii)], we immediately obtain the following conclusion, the details being omitted.

Corollary 6.3. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$. If $p_- \in (1, \infty)$ and $q \in (0, \infty]$, then $H^{p(\cdot),q}_{\lambda}(\mathbb{R}^n)$ is anisotropic quasi-norm.

Remark 6.4. (i) When $p(\cdot) \equiv p \in (0, 1]$, Theorem 6.2 goes back to [50, Lemma 6.3], which states that, for any $\theta \in (0, 1)$ and $q \in (0, \infty]$,

$$(H^{p(\cdot)}_{\lambda}(\mathbb{R}^n), L^q(\mathbb{R}^n))_{\theta,q} = H^{p(1-\theta),q}_{\lambda}(\mathbb{R}^n).$$
Lemma 6.5. Let \( \theta \in (0, 1) \), \( q \in (0, \infty) \), \( p(\cdot) \) and \( \overline{p}(\cdot) \) be as in Theorem 6.2. Then, for any \( f \in H^{\overline{p}(\cdot), q}_A(\mathbb{R}^n) \) and \( k \in \mathbb{Z} \), there exist \( g_k \in L^{\overline{p}}(\mathbb{R}^n) \) and \( b_k \in S'(\mathbb{R}^n) \) such that \( f = g_k + b_k \) in \( S'(\mathbb{R}^n) \), \( \| b_k \|_{L^{\overline{p}}(\mathbb{R}^n)} \leq C 2^k \) and

\[
\tag{6.1}
\| b_k \|_{H^{\overline{p}(\cdot)}_A(\mathbb{R}^n)} \leq \overline{C} \| M_N(f) \chi_{\{x \in \mathbb{R}^n : M_N(f)(x) > 2^k\}} \|_{L^{\overline{p}(\cdot)}(\mathbb{R}^n)} < \infty,
\]

where, for any \( N \in \mathbb{N} \cap [\lceil \frac{1}{p} - 1 \rceil \frac{\ln b}{\ln \lambda}, 2 + \infty) \) with \( p \) as in (2.5), \( M_N(f) \) is as in (2.10), \( C \) and \( \overline{C} \) are two positive constants independent of \( f \) and \( k \).

**Proof.** Let all the notation be the same as those used in the proof of Theorem 4.8. For any \( f \in H^{\overline{p}(\cdot), q}_A(\mathbb{R}^n) \), \( k \in \mathbb{Z} \) and \( N \in \mathbb{N} \cap [\lceil \frac{1}{p} - 1 \rceil \frac{\ln b}{\ln \lambda}, 2 + \infty) \), let

\[
\Omega_k := \left\{ x \in \mathbb{R}^n : M_N(f)(x) > 2^k \right\}.
\]

Then, for any \( k_0 \in \mathbb{Z} \), by an argument similar to that used in the proof of (4.14), we have

\[
f = \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}} h_i^k = \sum_{k = -\infty}^{k_0} \sum_{i \in \mathbb{N}} h_i^k + \sum_{k = k_0 + 1}^{\infty} \sum_{i \in \mathbb{N}} \cdots =: g_{k_0} + b_{k_0} \quad \text{in } S'(\mathbb{R}^n),
\]

where, for any \( k \in \mathbb{Z} \) and \( i \in \mathbb{N} \), \( h_i^k \) is a \((p(\cdot), \infty, s)\)-atom multiplied by a constant and satisfies that

\[
\tag{6.2}
\text{supp } h_i^k \subset (x_i^k + B^s_{i+4}),
\]

\[
\tag{6.3}
\| h_i^k \|_{L^\infty(\mathbb{R}^n)} \leq 2^k
\]

and

\[
\tag{6.4}
\int_{\mathbb{R}^n} h_i^k(x) Q(x) dx = 0 \quad \text{for any } Q \in \mathcal{P}_m(\mathbb{R}^n).
\]
By the finite intersection property of \( \{x_i^k + B_i^r\}_{i \in \mathbb{N}} \) for each \( k \in \mathbb{Z} \) (see (4.18)), (6.2) and (6.3), we conclude that, for any \( k_0 \in \mathbb{Z} \),

\[
\|S_{k_0}\|_{L^\infty(\mathbb{R}^n)} \leq \sum_{k=-\infty}^{k_0} 2^k \sim 2^{k_0}.
\]

On the other hand, for any \( k \in \mathbb{Z} \) and \( i \in \mathbb{N} \), let \( a_i^k := (\lambda_i^k)^{-1} h_i^k \), where

\[
\lambda_i^k \sim 2^k \left\| X_i^k + B_i^r \right\|_{L^{p/(\rho')}(\mathbb{R}^n)}.
\]

Then, by this, (6.2), (6.3) and (6.4), we know that, for any \( k \in \mathbb{Z} \) and \( i \in \mathbb{N} \), \( a_i^k \) is a \((\rho(\cdot), \infty, s)\)-atom. Therefore, by the finite intersection property of \( \{x_i^k + B_i^r\}_{i \in \mathbb{N}} \) for each \( k \in \mathbb{Z} \) again and the fact that \( \Omega_k = \bigcup_{i \in \mathbb{N}} (x_i^k + B_i^r) \) (see (4.16)), we find that

\[
\| S_{k_0} \|_{L^\infty(\mathbb{R}^n)} \leq \sum_{k=-\infty}^{k_0} 2^k \left\| X_i^k + B_i^r \right\|_{L^p(\mathbb{R}^n)}.
\]

Moreover, from Remark 2.3(i) and Lemma 2.5, it follows that

\[
\| M_N(f) \chi_{\{|x| \geq 2^{k_0}\}} \|_{L^p(\mathbb{R}^n)} \\
\leq \left\{ \sum_{j \in \mathbb{Z}} \left\| M_N(f) \right\|_{L^p(\mathbb{R}^n)}^{2^{-j}} \left\| \chi_{\{|x| \leq 2^{j+k_0}\}: M_N(f)(x) \leq 2^{j+k_0+1}} \right\|_{L^p(\mathbb{R}^n)} \right\}^{\frac{1}{2}}
\]

\[
\sim \left\{ \sum_{j \in \mathbb{Z}} (2^{j+k_0})^{\frac{p}{2}} \left\| \chi_{\{|x| \leq 2^{j+k_0}\}: M_N(f)(x) \leq 2^{j+k_0+1}} \right\|_{L^p(\mathbb{R}^n)} \right\}^{\frac{1}{2}}
\]


Proof of Theorem 6.2. We first prove that

\[
\lambda_{R} \approx (2^{k_{0}})^{-\theta/(1-\theta)} \| f \|_{H^{(\lambda)}_{R}((\mathbb{R}^{n})^{d})} \leq (2^{k_{0}})^{-\theta/(1-\theta)} \| f \|_{H^{(\lambda)}_{A}(\mathbb{R}^{n})} < \infty.
\]

Observe that \((\mathbb{R}^{n}, \rho, dx)\) is an RD-space (see [39, 83]). From this, [84, Theorem 4.3(i)], (6.5) and (6.6), we further deduce that

\[
\| b_{k_{0}} \|_{H^{(\lambda)}_{A}(\mathbb{R}^{n})} \leq \left\| \sum_{k=k_{0}+1}^{\infty} \sum_{j=\infty}^{n} \frac{A_{A}(x)_{k} + B_{A}(x)_{k}}{L_{A}(x)} \right\|_{L^{\infty}(\mathbb{R}^{n})}.
\]

This finishes the proof of Lemma 6.5. \(\square\)

Now we prove Theorem 6.2.

**Proof of Theorem 6.2.** We first prove that

\[
(6.7) \quad H^{(\lambda)}_{A}(\mathbb{R}^{n}) \subset (H^{(\lambda)}_{A}(\mathbb{R}^{n}), L^{\infty}(\mathbb{R}^{n}))_{\theta,q}.
\]

To this end, let \( f \in H^{(\lambda)}_{A}(\mathbb{R}^{n}) \). Then, by Lemma 6.5, we know that, for any \( k \in \mathbb{Z} \), there exist \( g_{k} \in L^{\infty}(\mathbb{R}^{n}) \) and \( b_{k} \in H^{(\lambda)}_{A}(\mathbb{R}^{n}) \) such that \( f = g_{k} + b_{k} \) in \( S'(\mathbb{R}^{n}) \), \( \| g_{k} \|_{L^{\infty}(\mathbb{R}^{n})} \leq 2^{k} \) and \( b_{k} \) satisfies (6.1). By this and a proof similar to that of [86, (3.3)], we find that, for any \( t \in (0, \infty) \),

\[
(6.8) \quad K(t, f; H^{(\lambda)}_{A}(\mathbb{R}^{n}), L^{\infty}(\mathbb{R}^{n})) \leq 2^{k(t)} t,
\]

where, for any \( t \in (0, \infty) \),

\[
k(t) : = \inf \left\{ \ell \in \mathbb{Z} : \left[ \sum_{j=\infty}^{n} \left( 2^{j} h(2^{j-\ell}) \right) \right]^{1/2} \leq t \right\}
\]

and, for any \( \lambda \in (0, \infty) \),

\[
h(\lambda) : = \left\| X_{\{x \in \mathbb{R}^{n} : M_{w}(f)(x) > \lambda \}} \right\|_{L^{\infty}(\mathbb{R}^{n})}.
\]

On the other hand, by (6.8), it is easy to see that, for any \( \theta \in (0, 1) \) and \( q \in (0, \infty) \),

\[
(6.9) \quad \left\{ \int_{0}^{\infty} t^{-\theta q} \left[ K(t, f; H^{(\lambda)}_{A}(\mathbb{R}^{n}), L^{\infty}(\mathbb{R}^{n})) \right]^{1/2} dt \right\}^{q} \lesssim \left[ \sum_{\ell \in \mathbb{Z}} 2^{\ell} \int_{\{t \in (0, \infty) : 2^{\ell} < 2^{\ell+1} t \}} t^{(1-\theta) q} dt \right]^{1/2} \lesssim \left[ \sum_{\ell \in \mathbb{Z}} 2^{\ell} \int_{0}^{\infty} \left( \int_{2^{\ell} \leq t < 2^{\ell+1}} t^{(1-\theta) q} dt \right) dt \right]^{1/2}.
\]
where $C$ is a positive constant independent of $f$. Next we estimate $I$ by considering two cases.

**Case 1.** $\frac{(1-\theta)q}{p} \in (0, 1]$. For this case, by the well-known inequality that, for any $d \in (0, 1]$ and $\{a_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$,

$$\left( \sum_{i \in \mathbb{N}} |a_i| \right)^d \leq \sum_{i \in \mathbb{N}} |a_i|^d,$$

we obtain

$$J^q \leq \sum_{t \in \mathbb{Z}} 2^q \left[ \sum_{j \in \mathbb{Z}^+} 2^j \left[ h(2^{j+\ell}) \right]^{(1-\theta)q} \right] = \sum_{t \in \mathbb{Z}} 2^q \left[ \sum_{j \in \mathbb{Z}^+} 2^{j+\ell} \left[ h(2^{j+\ell}) \right]^{(1-\theta)q} \right].$$

**Case 2.** $\frac{(1-\theta)q}{p} \in (1, \infty]$. For this case, let $\eta := \frac{(1-\theta)q}{\ell q}$. Then, from the Hölder inequality, it follows that, for any $\delta \in (0, 1)$,

$$J^q \leq \sum_{t \in \mathbb{Z}} 2^q \left[ \sum_{j \in \mathbb{Z}^+} 2^{-\delta q} \left[ h(2^{j+\ell}) \right]^{(1-\theta)q} \right]^{1/q} \left( \sum_{j \in \mathbb{Z}^+} 2^{\delta q} \left[ h(2^{j+\ell}) \right]^{(1-\theta)q} \right)^{1/q}.$$

Combining (6.9), (6.10) and (6.11), we conclude that

$$\left\{ \int_0^1 t^{-\theta q} K(t, f; H_A^{p, \delta q}(\mathbb{R}^n), L^\infty(\mathbb{R}^n)) \frac{dt}{t} \right\}^{1/q} \leq \sum_{t \in \mathbb{Z}} 2^q \left\| \chi_{\{x \in \mathbb{R}^n: M_N(f)(x) > 2^f\}} \right\|_{L^p(\mathbb{R}^n)}^{1/q} \sim \sum_{t \in \mathbb{Z}} 2^q \left\| \chi_{\{x \in \mathbb{R}^n: M_N(f)(x) > 2^f\}} \right\|_{L^{p, \delta q}(\mathbb{R}^n)}^{1/q} \sim \| M_N(f) \|_{L^{p, \delta q}(\mathbb{R}^n)} \sim \| f \|_{H_A^{p, \delta q}(\mathbb{R}^n)}$$

with the usual modification made when $q = \infty$, which implies that

$$f \in (H_A^{p, \delta q}(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\theta, q}$$

and hence completes the proof of (6.7).
Conversely, we need to show that

\[
(H_A^{p(\cdot)}(\mathbb{R}^n), L^{\infty}(\mathbb{R}^n))_{\theta,q} \subset H_A^{p(-\cdot)}(\mathbb{R}^n).
\]

To this end, we claim that $M_N$ is bounded from the space $(H_A^{p(\cdot)}(\mathbb{R}^n), L^{\infty}(\mathbb{R}^n))_{\theta,q}$ to the space $(L^{p(\cdot)}(\mathbb{R}^n), L^{\infty}(\mathbb{R}^n))_{\theta,q}$. Indeed, let $f \in (H_A^{p(\cdot)}(\mathbb{R}^n), L^{\infty}(\mathbb{R}^n))_{\theta,q}$. Then, by definition, we know that there exist $f_1 \in H_A^{p(\cdot)}(\mathbb{R}^n)$ and $f_2 \in L^{\infty}(\mathbb{R}^n)$ such that

\[
\left\{ \int_0^\infty r^{-\theta q} \left[ \|f_1\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} + t \|f_2\|_{L^{\infty}(\mathbb{R}^n)} \right]^q \frac{dt}{t} \right\}^{1/q} \leq \|f\|_{(H_A^{p(\cdot)}(\mathbb{R}^n), L^{\infty}(\mathbb{R}^n))_{\theta,q}}.
\]

On the other hand, since $M_N$ is bounded from $H_A^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$ and also from $L^{\infty}(\mathbb{R}^n)$ to $L^{\infty}(\mathbb{R}^n)$, it follows that $M_N(f_1) \in L^{p(\cdot)}(\mathbb{R}^n)$ and $M_N(f_2) \in L^{\infty}(\mathbb{R}^n)$. For any $i \in \{1, 2\}$, let

\[
E_i := \left\{ x \in \mathbb{R}^n : \frac{1}{2} M_N(f)(x) \leq M_N(f_i)(x) \right\}.
\]

Then $\mathbb{R}^n = E_1 \cup E_2$. Thus, we have

\[
M_N(f) = M_N(f)_{E_1} + M_N(f)_{E_2 \setminus E_1} \in L^{p(\cdot)}(\mathbb{R}^n) + L^{\infty}(\mathbb{R}^n),
\]

which, combined with (6.13), further implies that

\[
\|M_N(f)\|_{(L^{p(\cdot)}(\mathbb{R}^n), L^{\infty}(\mathbb{R}^n))_{\theta,q}} \leq \left\{ \int_0^\infty r^{-\theta q} \left[ \|M_N(f)_{E_1}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + t \|M_N(f)_{E_2 \setminus E_1}\|_{L^{\infty}(\mathbb{R}^n)} \right]^q \frac{dt}{t} \right\}^{1/q}
\]

with the usual modification made when $q = \infty$. Therefore, the above claim holds true.

By this claim and [44, Remark 4.2(ii)], we find that, if $f \in (H_A^{p(\cdot)}(\mathbb{R}^n), L^{\infty}(\mathbb{R}^n))_{\theta,q}$, then $M_N(f)$ belongs to $L^{p(\cdot)}(\mathbb{R}^n)$, namely, $f \in H_A^{p(-\cdot)}(\mathbb{R}^n)$. Thus, (6.12) holds true. This finishes the proof of Theorem 6.2.

\[\square\]

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