The method of similar operators in the study of the spectra of the adjacency matrices of graphs

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Abstract. The method of similar operators [1, 2, 3] is used to investigate spectral properties of a certain class of matrices in the context of graphs [4, 5]. Specifically, we consider the adjacency matrix of an “almost-complete graph”. Then we generalize the result to allow the matrices obtained as combinations of the Kronecker products [6, 7] and the small-norm perturbations. We derive the estimates of the spectra and the eigenvectors of such matrices.

1. Introduction
Consider a directed graph on $N$ vertices. It is uniquely defined by its adjacency matrix $A = (a_{ij})$ of the size $N \times N$, in which $a_{ij}$ is the number of edges from the vertex $i$ to the vertex $j$. The simple spectrum of a graph is defined as the spectrum of its adjacency matrix. Different spectra of a graph are defined in terms of different matrices corresponding to this graph. It is customary to consider combinations of the matrices $A$, $D$ (the out-degree matrix), $E$ (the identity matrix) and $J_N$ (the all-ones matrix). These matrices arise naturally in some stochastic models [8, p. 184]. Spectral properties of these matrices often play a vital role in such models. For instance, the Markovian random walk on a graph yields the notion of eigenvector centrality in a network [9, 10]. The score of a vertex $i$ is defined as the $i$th coordinate of a dominating (left) eigenvector of the transition matrix of such a process. This eigenvector defines the only stationary distribution of this random walk. The PageRank [9] algorithm originally used by Google to compute the eigenvector centrality relies on the Power-Method. Its speed of convergence depends on the ratio of the two largest absolute eigenvalues. Stability of a stationary distribution is determined [11] by the condition number which is bounded from below by the spectral gap — the distance between the two largest eigenvalues of the transition matrix of a process. The method of estimation of almost-invariant sets proposed in [12] also relies on spectral decompositions of such matrices. In the Susceptible-Infected-Susceptible model a viral spread in a network is modeled [13, 14] as a Markov process with $2^N$ states. An asymptotic (endemic or epidemic) behaviour of such a system is determined by the spectral radius (the largest absolute eigenvalue) of the adjacency matrix and the rates of curage and infection.

We should also note that the Kronecker products of the adjacency matrices are themselves of interest as they correspond to the adjacency matrices of the non-complete extended $p$-sums (NEPS) of graphs [8, p. 44].

For more details and comprehensive description of the graph spectra theory and its applications refer to [5, 8, 15].
2. The method of similar operators

In some applications it is infeasible to compute the exact eigenvalues. Then one needs to make reasonable estimates. We use the abstract method of similar operators to estimate the eigenvalues and eigenvectors for a certain class of matrices. This method originates from Friedrichs [16] and was later developed in abstract setting by Baskakov [1, 2, 3]. It relies on contraction mappings in Banach spaces and the Banach fixed-point theorem. This approach is often superior to usual methods of perturbation analysis that use series expansions. We are only concerned with finite-dimensional problems so we will only state the required notation and theorems in a simplified form.

Let $K \in \{\mathbb{R}, \mathbb{C}\}$ be a field of either real or complex numbers. We consider the vector space $K^n$, $n \in \mathbb{N}$ supplied with Euclidean structure:

$$(x, y) = \sum_{k=1}^{n} x_k y_k, \quad x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n) \in K^n$$

and the $L_2$-norm: $\|x\|_2^2 = (x, x)$. We also consider the canonical basis $e_1, \ldots, e_n$ in $K^n$ given by $(e_i)_j = \delta_{ij}$, $i, j = 1, n$ ($\delta_{ij}$ is the Kronecker symbol). When $V_1, V_2$ are normed vector spaces we denote by $L(V_1, V_2)$ the space of bounded linear mappings from $V_1$ to $V_2$. An algebra of bounded linear endomorphisms from a Banach space $V$ into itself is denoted by $L(V) = L(V, V)$. It is a Banach algebra with the operator norm:

$$\|A\|_{\text{op}} = \sup_{\|x\| = 1, x \in V} \|Ax\|, \quad A \in L(V).$$

Together with $L(K^n, K^n)$ we consider its isomorphic space $K^{m \times n}$ of matrices of the size $m \times n$ with entries from the field $K$. The space $K^{m \times n} \sim L(K^n)$ forms a Banach algebra when supplied with a submultiplicative norm $\|\cdot\|$, e.g.: $\|A\|_{\text{op}} = \sup_{\|x\|_2 = 1, x \in K^n} \|Ax\|_2$, $\|A\|_F = \sqrt{\sum_{i,j}|a_{ij}|^2}$, for $A= (a_{ij}) \in K^{n \times n}$. Finally we will also be dealing with the isomorphic spaces $L(L(K^n))$ and $L(K^{m \times n})$ with the operator norm. We will follow Krein and refer to elements of $L(K^{m \times n})$ as “transformers”.

The spectrum of a matrix $A$ (the set of its eigenvalues) will be denoted as $\sigma(A)$. We call two matrices $A_1, A_2$ similar if there is an invertible matrix $U$ (the similarity matrix) such that $A_1U = UA_2$. Similar matrices share some spectral properties: they are isospectral ($\sigma(A_1) = \sigma(A_2)$) and $U$ maps the eigenvectors of one to another’s: $A_2x = \lambda x \implies A_1Ux = \lambda Ux$.

The most important notion in the abstract method of similar operators is that of an admissible triple. For our specific purposes it suffices to say that $(K^{n \times n}, J, \Gamma)$ forms an admissible triple for a matrix $A \in K^{n \times n}$ if the following conditions are met:

- $J, \Gamma \in L(K^{n \times n})$ are transformers;
- $J$ is a projection ($J^2 = J$);
- $\Gamma$ satisfies the equations:

$$A\Gamma X - (\Gamma X)A = X - JX,$$
$$J\Gamma X = 0, \quad X \in K^{n \times n}.$$

The main theorem of the method may be now formulated as follows:

**Theorem 1.** Consider a matrix $A - B$ with $A, B \in K^{n \times n}$. Suppose $(K^{n \times n}, J, \Gamma)$ is an admissible triple for the matrix $A$ and suppose the following inequality holds:

$$\|B\| \|\Gamma\|_{\text{op}} \leq \frac{1}{4}.$$
Then there exists such a matrix $X^o \in \mathbb{K}^{n \times n}$ that $A - B$ is similar to $A - JX^o$; the similarity matrix is $E + \Gamma X^o$; the following estimates are valid:

$$
\|X^o - B\| \leq 3\|B\|,
$$
$$
\text{spr}(X^o) \leq \|X^o\| \leq 4\|B\|,
$$

where $\text{spr}(X^o)$ is the spectral radius of $X^o$ (the largest absolute eigenvalue). Such $X^o$ can be found as the limit of a convergent sequence $(\Phi^k(0); k \in \mathbb{N})$ in a Banach algebra $\mathbb{K}^{n \times n}$. Here $\Phi$ is a nonlinear contraction mapping defined on the ball $\{X \in \mathbb{K}^{n \times n}; \|X - B\| \leq 3\|B\|\}$ and given by

$$
\Phi(X) = B\Gamma X - (\Gamma X)J(B + B\Gamma X) + B
$$

and $\Phi^k = \underbrace{\Phi \circ \cdots \circ \Phi}_{k \text{ copies}}$ denotes the composition.

3. Results and discussion

Almost-complete graph example.

Consider a digraph defined by the following adjacency matrix:

$$
A = J_N - B = \begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{pmatrix} - B,
$$

Here $J_N$ is the all-ones matrix. The unity on the intersection of $i$'th row and $j$'th column of $B$ corresponds to an edge from $i$ to $j$ being absent in the graph. This example has already been considered in [17] and here we will reproduce some steps to demonstrate the technique.

One can easily find the minimal annihilating polynomial of $J_N$ to be $(\lambda - N)^2$. This comes from the fact that $J_N^2 = NJ_N$. Consequently the spectrum of $J_N$ is

$$
\sigma(J_N) = \{0, N\}.
$$

The only non-zero eigenvalue $N$ has the corresponding eigenvector

$$
h_N = \frac{1}{\sqrt{N}} (1, \ldots, 1) \in \mathbb{R}^N.
$$

The null-space of $J_N$ is orthogonal to $h_N$ and allows the orthonormal basis:

$$
h_k = \frac{1}{\sqrt{k(k+1)}} \begin{pmatrix}
1, \ldots, 1, -k, 0, \ldots, 0 \\
k \text{ copies}
\end{pmatrix}, \quad k = 1, \ldots, N - 1.
$$

We conclude that the adjacency matrix $A = J_N - B$ of this graph is similar to $A - B$ where $A$ is a block matrix (subscripts denote block sizes; in what follows throughout this subsection block sizes are the same and will be omitted):

$$
A = \begin{pmatrix}
N & 0_{1 \times (N-1)} \\
0_{(N-1) \times 1} & 0_{(N-1) \times (N-1)}
\end{pmatrix} \in \mathbb{R}^{N \times N}
$$

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and $\mathcal{B}$ is obtained with a similarity transform: $\mathcal{B} = U^{-1}BU \in \mathbb{R}^{N \times N}$. The similarity matrix $U$ is given by stacking the eigenvectors in columns:

$$U = \text{columns}(h_N, h_1, \ldots, h_{N-1}) = \begin{pmatrix} h_N & h_1 & \cdots & h_{N-1} \end{pmatrix}.$$ 

Following the general method, we should first construct an admissible triple. Since $A$ is block-diagonal it is natural to set $J$ with the formula

$$JX = \begin{pmatrix} x_{11} & 0 \\ 0 & X_{22} \end{pmatrix} \in \mathbb{K}^{N \times N}.$$ 

Then $A - JX$ is block-diagonal for any $X$ and its spectrum is the union of its diagonal blocks’ spectra: $\sigma(A - JX) = \{N, \sigma(-X_{22})\}$.

Now we can find the corresponding $X$. Suppose it is defined by the formula

$$X = \begin{pmatrix} x_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \mathbb{K}^{N \times N}.$$ 

Then the equations

$$A\Gamma X - (\Gamma X)A = N\begin{pmatrix} 0 & \Gamma_{12}(X) \\ -\Gamma_{21}(X) & 0 \end{pmatrix} = X - JX,$$

and $J\Gamma X = 0$ yield the result

$$\Gamma X = \frac{1}{N} \begin{pmatrix} 0 & X_{12} \\ -X_{21} & 0 \end{pmatrix}, \quad X \in \mathbb{K}^{N \times N}.$$ 

The last step is to estimate the norms and apply the theorem. One can easily check that $\|\Gamma\|_{\text{op}} = \frac{1}{N}$. It is also apparent that $\|B\|_{\text{op}} \leq \|B\|_{F}$ since multiplication by the orthogonal matrix $U$ is an isometry in Euclidean space. The Frobenius norm $\|B\|_{F} = \sqrt{\sum_{ij} b_{ij}^2} = M$ of $B$ reduces to the square root of the number of absent edges. This directly implies the following

**Theorem 2.** Suppose the number of absent edges is

$$M < \frac{1}{16}N^2.$$ 

Then the spectrum of the adjacency matrix $A = J_N - B$ can be represented as disjoint union

$$\sigma(A) = \{N - x_{11}^o\} \cup \sigma_2.$$ 

The dominating eigenvector of $A$ is

$$\hat{h}_N = U(E + \Gamma X^o)e_1 = h_N - \frac{1}{N}(X_{21,(i)}^o h_1 + \cdots + X_{21,(N-1)}^o h_{N-1}),$$

where $X_{21,(i)}^o, \ i = 1, N-1$ are the coordinates of the vector $X_{21}^o$. Moreover $\hat{h}_N \in \mathbb{R}^N$, $x_{11}^o \in \mathbb{R}$ and $\sigma_2 \subset \mathbb{C}$ satisfy the following inequalities:

$$\|\hat{h}_N - h_N\|_2 \leq 4\sqrt{M},$$

$$|x_{11}^o|, \sup_{\lambda \in \sigma_2} |\lambda| \leq 4\sqrt{M}.$$
A-tiled matrix example.
Now let $A \in \mathbb{K}^{M \times M}$ and consider the following ("A-tiled") block-matrix

$$A = \begin{pmatrix} A & \cdots & A \\ \vdots & \ddots & \vdots \\ A & \cdots & A \end{pmatrix} \in \mathbb{K}^{MN \times MN}$$

and the perturbed matrix

$$A - B, B \in \mathbb{K}^{MN \times MN}.$$ 

**Lemma 1.** Suppose $A$ is an invertible self-adjoint matrix. Then it has $M$ orthonormal eigenvectors $h_1, \ldots, h_M$ ($\|h_i\|_2 = 1$, $i=1,M$) with the corresponding eigenvalues $\lambda_1, \ldots, \lambda_M \neq 0$. The spectrum of $A$ is

$$\sigma(A) = \{0\} \cup N \sigma(A) = \{0\} \cup \{N \lambda; \lambda \in \sigma(A)\}.$$ 

Non-zero eigenvalues of $A$ have corresponding block eigenvectors:

$$f_j = \frac{1}{\sqrt{N}}(h_j, \ldots, h_j) \in \mathbb{K}^{MN}, j = 1,M.$$ 

The null-space of $A$ has an orthonormal basis:

$$f_{j,k} = \frac{1}{\sqrt{k(k+1)}}(e_j, \ldots, e_j, -ke_j, 0, \ldots, 0) \in \mathbb{K}^{MN \times MN}$$

The matrix $A$ is similar to a block-diagonal matrix:

$$A = \begin{pmatrix} \text{diag}(N \lambda_1, \ldots, N \lambda_M) & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{K}^{MN \times MN};$$

the similarity transform matrix is

$$U = \text{columns}(f_1, \ldots, f_M, f_{1,1}, \ldots, f_{1,N-1}, \ldots, f_{M,N-1}).$$

Throughout this subsection we will consider block matrices of the size $MN \times MN$ in the form

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1M} & x_{1,M+1} \\ \vdots & \ddots & \vdots & \vdots \\ x_{M1} & \cdots & x_{MM} & x_{M,M+1} \\ x_{M+1,1} & \cdots & x_{M+1,M} & x_{M+1,M+1} \end{pmatrix},$$

where $X_{ij} = x_{ij}$, $X_{M+1,j} = x_{M+1,j}$, $X_{i,M+1} = x_{i,M+1} \in \mathbb{K}$ for $1 \leq i,j \leq M$, and $X_{M+1,M+1}$ is a block of the size $M(N-1) \times M(N-1)$.

Just like before we are going to investigate the spectral behaviour of $A$ under perturbations. We begin with constructing an admissible triple. A natural choice for $J$ in this case is

$$JX = \begin{pmatrix} x_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_{MM} \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$
Lemma 2. Suppose $A$ has a simple spectrum, i.e. its eigenvalues are pairwise distinct: $\lambda_i \neq \lambda_j$ for all $1 \leq i \neq j \leq M$.

Then a tuple $(\mathcal{X}^{MN \times MN}, J, \Gamma)$ forms an admissible triple if we define

$$\Gamma X = \frac{1}{n} \left( \begin{array}{cccc}
0 & \gamma_{12} x_{12} & \cdots & \gamma_{1M} x_{1M} \\
\gamma_{21} x_{21} & 0 & \cdots & \gamma_{2M} x_{2M} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{M1} x_{M1} & \gamma_{M2} x_{M2} & \cdots & 0 \\
\gamma_{M+1,1} x_{M+1,1} & \gamma_{M+1,2} x_{M+1,2} & \cdots & \gamma_{M+1,M} x_{M+1,M} \\
\end{array} \right).$$

$$\gamma_{ij} = \begin{cases} 
\frac{1}{\lambda_i - \lambda_j}, & 1 \leq i \neq j \leq M+1, \\
0, & i = j 
\end{cases}$$

and we use the convention:

$$\lambda_{M+1} = 0.$$

The operator norm of $\Gamma$ is:

$$\|\Gamma\|_{\text{op}} = \frac{1}{N} \min_{1 \leq i \neq j \leq M+1} |\lambda_i - \lambda_j| =$$

$$= \frac{1}{N} \max \left\{ \frac{1}{\min_{1 \leq i \neq j \leq M} |\lambda_i - \lambda_j|}, \frac{1}{\min_{1 \leq j \leq M} |\lambda_j|} \right\}.$$

Theorem 3. Suppose $A$ has simple spectrum and the following inequality holds:

$$\|B\|_{\text{op}} \leq \frac{N}{4} \min \left\{ \min_{1 \leq i \neq j \leq M} |\lambda_i - \lambda_j|, \min_{1 \leq j \leq M} |\lambda_j| \right\}.$$ 

Then the spectrum of disturbed matrix $A - B$ is

$$\sigma(A) = \{ N\lambda_1 - x_{11}^0, \ldots, N\lambda_M - x_{MM}^0 \} \cup \sigma_{M+1}.$$

The eigenvectors $\hat{f}_j$, $f_{j,k}$, $j = 1, M$, $k = 1, N-1$ of the matrix $A - B$, the values $x_{jj}^0$, $j = 1, M$ and the set $\sigma_{M+1}$ are in the following bounds:

$$|x_{jj}^0|, \max_{\lambda \in \sigma_{M+1}} |\lambda| \leq 4\|B\|,$$

$$\|\hat{f}_j - f_j\|_2, \|f_{j,k} - f_{j,k}\|_2 \leq \frac{4}{N}\|B\| \max \left\{ \frac{1}{\min_{1 \leq i \neq j \leq M} |\lambda_i - \lambda_j|}, \frac{1}{\min_{1 \leq i \leq M} |\lambda_i|} \right\}$$

for all $j = 1, M$, $k = 1, N-1$. 

Kronecker products

Now we consider Kronecker product

\[ A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1N}B \\ \vdots & \ddots & \vdots \\ a_{N1}B & \cdots & a_{NN}B \end{pmatrix} \in \mathbb{K}^{MN \times MN} \]

of squared matrices \( A = (a_{ij}) \in \mathbb{K}^{N \times N}, \ B = (b_{ij}) \in \mathbb{K}^{M \times M} \). We will analyze its spectral properties under small-norm perturbations:

\[ A \otimes B - F. \]  

(1)

Kronecker product has several appealing properties [6].

- It is associative:
  \[ A \otimes (B \otimes C) = (A \otimes B) \otimes C. \]

- It is distributive with respect to addition:
  \[ (A + B) \otimes (C + D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D. \]

- The Kronecker product of matrix products has the following property:
  \[ (AB) \otimes (CD) = (A \otimes C)(B \otimes D) \]
  whenever products \( AB \) and \( CD \) make sense.

- The trace of \( A \otimes B \) is
  \[ \text{tr}(A \otimes B) = \text{tr}A\text{tr}B. \]

- If \( A \) and \( B \) are both symmetric matrices, then \( A \otimes B \) is symmetric as well.

Note also that the “tiled” matrix from the last example can be represented as a Kronecker product:

\[ A = \begin{pmatrix} A & \cdots & A \\ \vdots & \ddots & \vdots \\ A & \cdots & A \end{pmatrix} = J_N \otimes A. \]

Lemma 3. Suppose \( A \) and \( B \) have simple structure, i.e. \( A \) has \( N \) eigenvectors \( f_1, \ldots, f_N \) whose corresponding eigenvalues are \( \mu_1, \ldots, \mu_N \) and \( B \) has eigenvectors \( h_1, \ldots, h_M \) with the eigenvalues \( \lambda_1, \ldots, \lambda_M \). Then \( A \otimes B \) also has simple structure; it has \( MN \) independent eigenvectors \( f_i \otimes h_j, \ i=1, \ldots, N, \ j=1, \ldots, M \) and the corresponding eigenvalues are \( \mu_i \lambda_j \).

Now suppose that among these pairwise products \( \mu_i \lambda_j \) there are only \( s \) distinct values \( \nu_1, \ldots, \nu_s \). To each eigenvalue \( \nu_k \) \( (k=1, s) \) there corresponds an eigenspace

\[ E_k = \text{span}(f_i \otimes h_j; \ \mu_i \lambda_j = \nu_k, \ i=1, N, \ j=1, M) \subset \mathbb{K}^{MN}. \]

These eigenspaces form a direct-sum decomposition of \( \mathbb{K}^{MN} \):

\[ \mathbb{K}^{MN} = E_1 \oplus \cdots \oplus E_s. \]

Any vector \( x \in \mathbb{K}^{MN} \) can be uniquely represented in the form

\[ x = x_1 + \cdots + x_s, \ x_k \in E_k, \ k=1, s. \]  

(2)
This direct-sum decomposition of $K^{MN}$ corresponds to a decomposition of the identity matrix $E \in K^{MN\times MN}$ (which defines an identity operator) into a sum of matrices of the spectral projections:

$$E = P_1 + \cdots + P_s.$$ 

The spectral projection $P_k$ ($k=1, s$) is given by the formula

$$P_k x = x_k \in E_k \subset K^{MN}$$

with respect to the decomposition (2) of $x$.

For any matrix $X \in K^{MN\times MN}$ the following trivial equality holds:

$$X = \sum_{i,j=1}^s P_i X P_j.$$ 

The matrix $A \otimes B$ can be decomposed into

$$A = \sum v_j P_j.$$ 

Now we should be able to reproduce the same steps as before to retrieve the estimates.

The natural way to define $J$ is as follows:

$$J X = \sum_{j=1}^s P_j X P_j.$$ 

The system of equations

$$\begin{cases} A \Gamma X - (\Gamma X) A = X - JX, \\ J \Gamma X = 0, \quad X \in K^{MN\times MN} \end{cases}$$

has the unique solution:

$$\Gamma X = \sum_{1 \leq i \neq j \leq s} \frac{1}{\nu_i - \nu_j} P_i X P_j.$$ 

The norm of $\Gamma$ is:

$$\|\Gamma\|_{op} = \gamma = \frac{1}{\min_{1 \leq i \neq j \leq s} |\nu_i - \nu_j|}.$$ 

**Theorem 4.** Consider the perturbed matrix (1)

$$A \otimes B - F.$$ 

Let $A \in K^{N\times N}$ and $B \in K^{M\times M}$ be diagonalizable matrices. Let $f_1, \ldots, f_N$ be the eigenvectors of $A$ corresponding to the eigenvalues $\mu_1, \ldots, \mu_N$ and let $h_1, \ldots, h_M$ be the eigenvectors of $B$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_M$. The spectrum of their Kronecker product $A \otimes B$ is composed of all the possible pairwise products $\mu_i \lambda_j$ and the corresponding eigenvectors are $f_i \otimes h_j$. Suppose that out of these $MN$ eigenvalues only $s$ are distinct: $\nu_1, \ldots, \nu_s$.

Suppose

$$\|F\| \leq \frac{1}{4} \gamma^{-1} = \frac{1}{4} \min_{1 \leq i \neq j \leq s} |\nu_i - \nu_j|.$$ 

Then $A \otimes B - F$ is similar to

$$\sum_{k=1}^s \nu_k P_k - JX^o \approx \sum_{k=1}^s (\nu_k P_k - P_k X^o P_k)$$
for some $X^o \in \mathbb{R}^{MN \times MN}$, $\|X^o - F\| \leq 3\|F\|$. All the eigenvalues of $A \otimes B - F$ are contained in the circles

$$\Omega_k = \{ \lambda \in \mathbb{C}; |\lambda - \nu_k| \leq 4\|F\| \}, \ k = 1, \ldots, s.$$ 

There is at least one eigenvalue in each of these circles.

Suppose the eigenvalue $\nu_k = \mu_k \lambda_j$ of $A \otimes B$ has multiplicity 1, that is it has the only eigenvector $v_k = f_i \otimes h_j$. It is equivalent to the statement that the eigenvalue $\mu_k$ of $A$ and the eigenvalue $\lambda_j$ of $B$ are both of multiplicity 1. Then $A \otimes B - F$ has eigenvalue in the circle $\Omega_k$ and the corresponding eigenvector $v_k$ is within bounds

$$\|v_k - \nu_k\| \leq 4\gamma\|F\|.$$ 

If $\nu_k$ is well separated from all the other eigenvalues of $A \otimes B$:

$$\min_{i \neq k} |\nu_k - \nu_i| \geq 4\|F\|,$$

then $\nu_k$ is the only eigenvalue of $A \otimes B - F$ in that circle.

For example, in the case of a “tiled” matrix

$$J_N \otimes B = \begin{pmatrix} B & \cdots & B \\ \vdots & \ddots & \vdots \\ B & \cdots & B \end{pmatrix}$$

we would have $\nu_1 = N$, $\nu_2 = 0$. Let $\lambda_1, \ldots, \lambda_M$ be the eigenvalues of $B$. Spectrum of $J_N \otimes B$ is

$$\sigma(J_N \otimes B) = \{ \mu_i \lambda_j; i=1, 2, j=1, M \} = \{ 0 \} \cup N \sigma(B).$$

All of these eigenvalues except for 0 are of multiplicity 1 and well-separated for sufficiently large $N$. Then $\gamma = \frac{1}{N}$. This directly implies the theorem of the previous section.

These results might be refined with the use of the theorem on splitting an operator [18] which allows to consider each eigenvalue individually and obtain more precise estimates for the corresponding eigenvalue of the perturbed matrix.

**Conclusion**

We have derived bounds for perturbations of Kronecker products of matrices using the method of similar operators which is a powerful tool in perturbation theory. Developed in the context of Banach spaces this method turns out to be just as useful in finite-dimensional problems.

The problem of perturbation analysis for the Kronecker products despite being old still has a whole open field for future research. An example of more general yet quite similar problem is that of an analysis of tensor products of finite-dimensional operators (defined by matrices) and abstract operators in Banach spaces.

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