Chiral sum rule on the light front

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We obtain a sum rule for the valence sector light-front wave function of the pion from analyzing the conserved axial-vector current and the general structures of the wave functions. This chiral sum rule provides a synergy between the quark-antiquark bound-state structure of the lightest hadron and its role as the Goldstone boson. Using an analytic model motivated by holography, we show this sum rule is consistent with requirements of chiral symmetry breaking in AdS/QCD. Within the same model, we find some remarkable feature of the pion: it is mostly uniform within.

Introduction

Pion, the lightest hadron, is a dual citizen in quantum chromodynamics (QCD). On one hand, it is the bound state of a quark-antiquark pair. One the other hand, it is the Nambu-Goldstone boson of the spontaneously broken chiral symmetry [1]. It is therefore curious to know how these two fundamental aspects of the pion are unified in its inner structure. The chiral symmetry becomes relevant in QCD because the $u,d$ quark masses are much smaller than $\Lambda_{\text{QCD}}$, the characteristic scale of QCD.

Chiral perturbation theory ($\chi$PT) puts constraints on the dynamics of the pion [2]. Alas, $\chi$PT as a low-energy effective theory treats the pion as a fundamental particle. No direct information on the inner structure of the pion is provided. Lattice gauge theory (LGT) simulates pions from the quark and gluon degrees of freedom [3]. However, it’s computationally formidable to work with chiral fermions, although Lattice simulations with a physical pion mass are on the rise [4]. No simple picture is drawn for the inner structure of the pion as a Goldstone boson either. Polylog analysis in Dyson-Schwinger equations (DSEs) provides rigorous constraints on the covariant structure of the pion [5]. Indeed, models based on this analysis are very successful in describing spectroscopy and structures of light mesons [6].

The light-front wave functions (LFWFs) describe the partonic structure of pion, which is how the pion is “seen” in high-energy scattering experiments [7]. These wave functions are frame-independent and provide the intrinsic information of the subatomic particle. They are directly related to the light-like bi-local hadronic matrix elements, as parametrized by the parton distributions and the parton distribution amplitudes, which play central roles in hard inclusive and exclusive processes, respectively [8].

Note that the LFWFs are Minkowskian in nature. Methods based on Euclidean quantum field theories, e.g. LGT and DSEs, have to perform a Wick rotation to obtain the LFWFs. Despite recent strides, the direct access to the pion LFWFs remains elusive [9]. In principle, the LFWFs can be directly obtained from diagonalizing the QCD Hamiltonian operator on a null plane $\omega \cdot x = 0$ with $\omega_\mu \omega^\mu = 0$ [7]. The resulting problem to be solved is a relativistic quantum many-body problem, one of the most challenging problems in physics.

In light of these challenges in the direct computation of the pion LFWFs, constraints from the chiral symmetry on these quantities are desired. In this work, we show that the valence sector $(q\bar{q})$ pion wave function satisfies,

$$\int \frac{dx}{\sqrt{x^3(1-x)^3}} \int \frac{d^2k_\perp}{(2\pi)^3} k_\perp^2 \psi_\pi(x, \vec{k}_\perp) = 0,$$

in the chiral limit $(m_q \to 0)$. Note that here we neglected the difference in the bare (current) $u,d$ quark masses and consider the unflavored case with $m_q = \frac{1}{2}(m_u + m_d)$. We have suppressed the explicit dependence on the renormalization scheme and scale $\mu$. The pion valence sector LFWF depends on boost invariants, $x = p_1^+/p^+$ and $\vec{k}_\perp = \vec{p}_1 - x\vec{p}_1$, where $p_1$ ($p_2$) is the 4-momentum of the quark (antiquark) and $p$ is the
vertically coordinate space pion wave function. Com-

resulting in a plateau at the origin of the trans-

to adopt a Gaussian ansatz for the transverse

distribution amplitude (LCDA). It is customary

to adopt a Gaussian ansatz for the transverse

distribution of the pion \[10\], where the compo-

4-momentum of the pion. We adopt the stan-

dard light-front coordinates, where the compo-

ents of a 4-vector \(\vec{v}\) are written as \(v^\pm = v^0 \pm v^3\)

and \(\vec{v}_\perp = (v^1, v^2)\), corresponding to \(\omega^- = 2\),

\(\omega^+ = \omega_\perp = 0\) [7].

To provide a more intuitive picture, we adopt

an analytic model for the pion \[10\],

\[
\tilde{\psi}_\pi(x, \vec{\zeta}_\perp) = N \frac{\phi_\pi(x)}{\sqrt{x(1-x)}} \tilde{\varphi}_\pi(\vec{\zeta}_\perp),
\]  

where \(\tilde{\psi}_\pi(x, \vec{\zeta}_\perp)\) is the pion valence wave

function in the coordinate space \(\vec{\zeta}_\perp = \sqrt{x(1-x)}(\vec{r}_\perp - \vec{r}_\perp')\). \(\phi_\pi\) is the pion light-cone

distribution amplitude (LCDA). It is customary

to adopt a Gaussian ansatz for the transverse

wave function \(\tilde{\varphi}_\pi(\vec{\zeta}_\perp)\) \[10–13\]. On the other

hand, the sum rule Eq. (1) requires,

\[
\nabla^2_\perp \tilde{\varphi}_\pi(\vec{\zeta}_\perp) \bigg|_{\zeta_\perp=0} = 0,
\]  

resulting in a plateau at the origin of the trans-

verse coordinate space pion wave function. Com-

bining this result with the light-front holography

(LFH) prediction for the LCDA \(\phi_\pi \propto \sqrt{x(1-x)}\) \[10\], we obtain a remarkable 3D picture of the

pion \(\psi\) shown in Fig. 1.

**Formalism** The key point for obtaining this

sum rule is the covariant decomposition of the

pion LFWFs. In Fock space, the state vector of

a pseudoscalar \(P\) can be decomposed as,

\[
|P(p)\rangle = \sum_{s,\bar{s}} \int \frac{dx}{2x(1-x)} \int \frac{d^2k_\perp}{(2\pi)^2} \tilde{\psi}_{s\bar{s}/P}(x, \vec{k}_\perp) \\
\times \frac{1}{\sqrt{N_C}} \sum_i \bar{u}_s(p_1)d^i_{s\bar{s}}(p_2)|0\rangle + \cdots
\]  

where \(N_C = 3\) is the number of colors. \(p_1\) and

\(p_2\) are the quark and antiquark momenta, re-

spectively. The longitudinal momentum fraction

of the quark (antiquark) is \(x = p_1^+/p^+\)

\((1 - x = p_2^+/p^+\). The relative transverse mo-

mentum \(\vec{k}_\perp = \vec{p}_1 - \vec{x}\vec{p}_\perp\). Momentum conserva-

tion implies, \(p^+ = p_1^+ + p_2^+\) and \(\vec{p}_\perp = \vec{p}_1 + \vec{p}_2\).

The ellipsis represents higher Fock sector wave

functions. The most general form of the pion va-

lence LFWF that satisfies all kinematical sym-

metries on the light front is given by the covari-

ant light-front dynamics (CLFD) \[14\],

\[
\psi_{s\bar{s}/P}(x, \vec{k}_\perp) = \bar{u}_s(p_1) \left[ \gamma_5 \phi_1(x, k_\perp) + \frac{\gamma_5 \tilde{\phi}}{\omega \cdot \vec{p}} \tilde{\phi}_2(x, k_\perp) \right] v_\bar{s}(p_2),
\]  

where \(\omega\) is the null vector indicating the orienta-

tion of the quantization surface, \(\tilde{\phi} = \gamma^+\). With

our choice of light front coordinates, \(\tilde{\phi} = \gamma^+\).

Note that the null vector \(\omega\) is only defined up to

a scaling factor. Therefore, the scale invariant

structure \(\tilde{\phi}/(\omega \cdot p)\) is mandatory. As a result,

we need to introduce a constant \(\tilde{f}_X\) with mass
dimension to balance the dimension. In CLFD,

this constant is usually taken as the quark mass

\(m_q\) (or the pion mass \(M_P\) which vanishes in the

chiral limit \[13, 14\]. We instead take a constant

non-vanishing in the chiral limit. Later, it will be

shown that a convenient choice is the pion decay

constant. Equation (5) can be written explicitly as,

\[
\psi_{\uparrow\uparrow/P}(x, \vec{k}_\perp) = \psi_{\downarrow\downarrow/P}(x, \vec{k}_\perp) = -\frac{k_1 e^{-i \arg \vec{k}_\perp}}{\sqrt{x(1-x)}} \phi_1(x, k_\perp);
\]  

\[
\psi_{\downarrow\uparrow-\uparrow/P}(x, \vec{k}_\perp) = \frac{\sqrt{2} m_q}{\sqrt{x(1-x)}} \phi_1(x, k_\perp) - \tilde{f}_X \sqrt{8x(1-x)} \phi_2(x, k_\perp);
\]  

where \(\psi_{\uparrow\downarrow-\uparrow/P} = [\psi_{\uparrow\uparrow/P} - \psi_{\downarrow\uparrow/P}] / \sqrt{2}.\)
Chiral sum rule  The sum rule (1) follows directly from examining the partially conserved axial-vector current (PCAC) \([15]\),

\[ \partial_\mu J_5^\mu = 2i m_q \bar{q} \gamma_5 q. \]  

(8)

Here, \(J_5^\mu = \bar{q} \gamma_\mu \gamma_5 q\) is the axial-vector current, and \(J_5 = \bar{q} \gamma_5 q\) is known as the pseudoscalar current. We have neglected the chiral anomaly term, which is irrelevant to our discussion here. Consider the local vacuum-to-pseudoscalar matrix elements of the operators,

\[ \langle 0 | J_5^\mu(x) | P(p) \rangle = e^{-ip \cdot x} i p^\mu f_P, \]  

(9)

\[ \langle 0 | J_5(x) | P(p) \rangle = e^{-ip \cdot x} g_P. \]  

(10)

The constant \(f_P\) is known as the decay constant. Applying the PCAC (8), we obtain the Gell-Mann-Oakes-Renner (GMOR) relation \([17]\):

\[ M_P^2 f_P = 2 m_q g_P, \]  

(11)

where \(M_P\) is the mass of the pseudoscalar, \(M_P^2 = p^2\). In the LFWF representation, these local matrix elements are,

\[ \langle 0 | J_5^\mu(0) | P(p) \rangle = \sqrt{NC} \sum_{s,\bar{s}} \int \frac{dx}{2x(1-x)} \int \frac{d^2 k_\perp}{(2\pi)^3} \psi_{s\bar{s}/P}(x, \vec{k}_\perp) \bar{v}_{\bar{s}}(p_2)^\gamma \gamma_5 u_{s}(p_1), \]  

(12)

\[ \langle 0 | J_5(0) | P(p) \rangle = \sqrt{NC} \sum_{s,\bar{s}} \int \frac{dx}{2x(1-x)} \int \frac{d^2 k_\perp}{(2\pi)^3} \psi_{s\bar{s}/P}(x, \vec{k}_\perp) \bar{v}_{\bar{s}}(p_2) i \gamma_5 u_{s}(p_1) \]  

(13)

Note that these expressions are exact, viz. the vacuum-to-pion matrix elements only depend on the valence wave function. Applying the PCAC (8) and the covariant decomposition of the LFWF (5), we obtain,

\[ \int \frac{dx}{2x(1-x)} \int \frac{d^2 k_\perp}{(2\pi)^3} \left( \frac{k_\perp^2 + m_q^2}{x(1-x)} - M_P^2 \right) \left( 2 m_q \phi_1(x, k_\perp) - 4 x(1-x) \hat{f}_\chi \phi_2(x, k_\perp) \right) = 0, \]  

(14)

In the chiral limit \(m_q \to 0\), this expression reduces to,

\[ \int \frac{dx}{2x(1-x)} \int \frac{d^2 k_\perp}{(2\pi)^3} k_\perp^2 \phi_2^0(x, k_\perp) = 0, \]  

(15)

due to the GMOR relation (11), where we have used the superscript (0) to indicate quantities in the chiral limit. From Eq. (7), the leading-twist \((L_z = 0)\) wave function \(\psi_\pi \equiv \psi_\pi^0(x \downarrow \downarrow \uparrow \uparrow) \propto \sqrt{x(1-x)} \phi_2^0\) in the chiral limit. Hence, we obtain the chiral sum rule Eq. (1).

In the above derivation, there is an explicit assumption that \(\hat{f}_\chi\) does not vanish in the chiral limit. If, instead, we take \(\hat{f}_\chi \to 0\) as usually done in CLFD (cf. Ref. \([16]\)), Eq. (14) will be automatically satisfied in the chiral limit – for arbitrary pion LFWF. However, the pion decay constant (18) also vanishes, and the pion mass is not required to be zero (11). Indeed, this is the case when the chiral symmetry is not spontaneously broken, and the pion is not a Goldstone boson regardless of its mass.

In the vicinity of the chiral limit, we can expand the wave functions and related quantities in terms of the quark mass \(m_q\),

\[ \phi_{1,2} = \phi_{1,2}^{(0)} + m_q \phi_{1,2}^{(1)} + \cdots. \]  

(16)
Substitute this expansion into Eq. (14) and take the \( O(m_q) \) term, we obtain,
\[
-2m_q \int \frac{dx}{2x(1-x)} \int \frac{d^2 k_\perp}{(2\pi)^3} \frac{2k_\perp^2}{x(1-x)} \phi_1^{(0)}(x, k_\perp) = M_P^2 f_\chi \int dx \int \frac{d^2 k_\perp}{(2\pi)^3} \phi_2^{(0)}(x, k_\perp).
\]  
(17)
The r.h.s. is just the wave function representation of the pseudoscalar decay constant in the chiral limit [8]:
\[
f_P^{(0)} = 4i f_\chi \sqrt{N_C} \int dx \int \frac{d^2 k_\perp}{(2\pi)^3} \phi_2^{(0)}(x, k_\perp),
\]  
(18)
where the decay constant is extracted from the local matrix element \( \langle 0 | J_5^\mu | P(p) \rangle \equiv ip^+ f_P \). Similarly, the l.h.s. is the wave function representation of the pseudoscalar amplitude in the chiral limit [8]:
\[
g_P^{(0)} = -2i \sqrt{N_C} \int \frac{dx}{2x(1-x)} \int \frac{d^2 k_\perp}{(2\pi)^3} \frac{k_\perp^2}{x(1-x)} \phi_1^{(0)}(x, k_\perp).
\]  
(19)
Therefore, this is consistent with the GMOR relation (11),
\[
M_P^2 f_P^{(0)} = 2m_q g_P^{(0)} + O(m_q^2).
\]  
(20)
Note that the above relation also implies that the decay constants of the excited pions vanish in the chiral limit [5].

**Analytic model** The effect of the chiral sum rule is better capitalized in a simple analytic model. In the chiral limit, the light-front kinetic energy depends only on a transverse vector \( \tilde{k}_\perp = \tilde{k}_\perp / \sqrt{x(1-x)} \). In some semi-classical approaches, e.g. LFH [10], it is assumed the dynamics is the chiral limit is solely dictated by the transverse direction, and the wave function is separable [18]. Taking this ansatz, we can rewrite \( \phi_2 \) for a pseudoscalar \( P \) as,
\[
\phi_2,P(x, k_\perp) = \frac{\pi}{2\sqrt{N_C} x(1-x)f_\pi} \varphi_P(k_\perp).
\]  
(21)
Here, \( \phi_P(x) \) is the light-cone distribution amplitude with \( \int dx \phi_P(x) = f_P \). \( \varphi_P(k_\perp) \) is the transverse function, whose Fourier transform,
\[
\tilde{\varphi}_P(k_\perp) = \int \frac{d^2 k_\perp}{(2\pi)^2} e^{ik_\perp \cdot \zeta} \varphi_P(k_\perp).
\]  
(22)
The chiral sum rule (15) implies,
\[
f_P \nabla_\perp^2 \tilde{\varphi}_P(\zeta_\perp = 0) = 0.
\]  
(23)
N.B. this relation is applicable to both the ground state, i.e. pion, and the excited states, i.e. excited pions. For pion, Eq. (3) is required. For the excited pions, the decay constants vanish in the chiral limit \( f_P = 0 \), as required by the GMOR relation (11) [5].

In LFH, the transverse wave function satisfies the semi-classical light-front Schrödinger wave equation (LFSWE),
\[
\left[ -\nabla_\perp^2 + U(\zeta_\perp) \right] \tilde{\varphi}_P(\zeta_\perp) = M_P^2 \tilde{\varphi}_P(\zeta_\perp).
\]  
(24)
The sum rule (23) leads to \( U(\zeta_\perp = 0) = 0 \). It was shown that Eq. (24) can be identified with the string equation of motion in the fifth dimension in AdS/QCD if \( \zeta_\perp \) is identified with the fifth coordinate \( z \) in AdS\(_5\) [20]. The effective potential \( U = (1/4)\Phi' - (1/2)\Phi'' - (3/2)\Phi \) is related to the dilaton profile \( \Phi(z) \) introduced in AdS/QCD to generate confinement by breaking the conformal invariance [20, 21]. Phenomenologically, the Regge scaling requires \( \Phi(z \to \infty) \to z^2 \) [22]. The chiral condensate \( \Sigma = \langle \bar{q}q \rangle \) is encoded in the background scalar field \( X \) at the UV boundary \( (z = 0) \), viz. \( \langle X(z \to 0) \rangle \to \frac{1}{2} \Sigma z^3 \) in the chiral limit [23]. The former produces a quadratic confining potential, \( U(\zeta_\perp \to \infty) \sim \zeta_\perp^2 \), while the latter requires \( U(\zeta_\perp \to 0) \sim -\Sigma^2 \zeta_\perp^4 \), consistent with the chiral sum rule.

The chiral sum rule does not uniquely determine the effective potential. Nevertheless, it can be shown that such potentials have the shape of a sombrero (see Fig. 2). From the AdS/QCD perspective, this shape is inherited from the Higgs potential through the condensate. The exact shape is dictated by QCD. Instead of solving for such a potential, we can construct it from some model pion wave function, \( U = \nabla_\perp^2 \tilde{\varphi}_\pi / \tilde{\varphi}_\pi \). For example, the following harmonic oscillator wave function satisfies the chiral sum rule,
\[
\tilde{\varphi}_\pi(\zeta_\perp) = (1 + \frac{1}{2} \zeta_\perp^2 + \frac{1}{8} \zeta_\perp^4) e^{-\frac{1}{2} \zeta_\perp^2}.
\]  
(25)
The scale is arbitrary. The corresponding potential is in Padé form,
\[
U(\zeta_\perp) = \frac{\zeta_\perp^4 (\zeta_\perp^2 - 6)}{\zeta_\perp^4 + 4 \zeta_\perp^2 + 8}.
\]  
(26)
FIG. 2. (Colors online) Comparison of the pion wave functions obtained from (a) a purely quadratic potential and (b) a sombrero potential that implements chiral symmetry breaking. The plateau of the wave function near \( \zeta_\perp = 0 \) is required by the chiral sum rule. This feature resembles a structureless particle up to the confinement scale.

Figure 2 compares wave function (25) with a Gaussian wave function obtained from the pure quadratic potential. The former has a distinctive plateau as required by the chiral sum rule. Of course, this potential can be improved to accommodate better light meson phenomenology [24].

Summary and discussions Starting from the axial-vector current conservation and the general structures of the LFWF, we derived a sum rule for the valence wave function of the pion. Taking advantage of holographic LFQCD, we showed that this sum rule is consistent with the chiral symmetry breaking in AdS/QCD.

Using a separable analytic model, we showed that the valence partons are uniformly distributed near the center of the pion. In other words, pion is almost a hard disk on the transverse plane \( \zeta_\perp \). Furthermore, the longitudinal function is shown to be \( \chi(x) = \phi_\pi(x) / \sqrt{x(1-x)} \rightarrow \text{const.} \), corresponding to a light-cone distribution amplitude \( \phi_\pi(x) = (8f_\pi/\pi)\sqrt{x(1-x)} \) [19]. The valence density, \( \bar{\rho}(x, \zeta_\perp) = N|\bar{\varphi}_\pi(\zeta_\perp)|^2 \). Therefore, in this simplistic model, pion is remarkably simple and uniform.

The pion 3D coordinate-space wave functions are also of interests, which are defined as the 3D Fourier transform of the LFWFs shown in (27), taking advantage of the Ioffe time formalism introduced by Miller and Brodsky [25]. In the analytic model we employ, the pion is infinitely long as shown in Fig. 1 [26].

The use of the general Lorentz structure of the LFWF is essential to obtain this sum rule. In particular, the sum rule constrains the part of wave function associated with the Lorentz structure \( \gamma^+ \gamma_5 / p^+ \). Since this part does not appear to be covariant, it is often neglected in the literature (see, e.g., Refs. [12, 16]). In fact, it can be written in a more covariant form: \( \not{p} \gamma_5 \). The emergence of this term signifies the spontaneous chiral symmetry breaking. It is also related to the leading-twist pion wave function as well as the associated bi-local matrix element [27],

\[
\langle 0 | \bar{\psi}(z^-) \not{p} \gamma_5 \psi(z^-) | P(p) \rangle_{z^- = 0} = 2\sqrt{2N_C} \sum_{s, \bar{s}} \int \frac{dx}{2\sqrt{x(1-x)}} \int \frac{d^2k_\perp}{(2\pi)^3} \times e^{i(x\tau - k_\perp \cdot \zeta_\perp)} \psi_{\uparrow \downarrow - \downarrow \uparrow}(x, k_\perp). \tag{27}
\]

Here, \( \tau = p \cdot z \) is a generalization of the Ioffe time of Miller and Brodsky [25]. Only recently, it was realized that this part is important for modeling the pion [13, 28, 29].

A proper renormalization is needed for the bare quark mass \( m_q \) and the local operators, e.g. \( J_5^\mu \). The actual implementation of the renormalization is beyond the scope of this work. However, it is important for practical \textit{ab initio} or model calculations. It appears the perturbation type of UV renormalization, taking advantage of the asymptotic freedom, is successful [6, 33].

The next important issue is the dynamical mass generation. In DSEs, the dynamical quark mass (self-energy) is constrained by the axial-vector Ward-Takahashi identity (AV-WTI) [5]. How the AV-WTI is realized on the light front is an interesting question. The problem is that the light-front axial charge \( Q_5 \) does not create
the pion pole [30, 31]. It is likely that the proper relation that constrains the dynamical mass generation on the light front comes from additional light-front chiral current algebras [31].

There is a long-standing myth that the vacuum in light-front QCD is trivial hence pion on the light front is not a Nambu-Goldstone boson. This myth has been debunked many times (see, e.g. Ref. [31] for a recent review). The key point is that chiral condensate on the light front is non-local, viz. \( \langle 0 | \bar{q} q | 0 \rangle = \frac{1}{2} m_q \langle 0 | \frac{\alpha_s}{\pi} q | 0 \rangle \neq 0 \) in the chiral limit \( (m_q \rightarrow 0) \). However, as Ref. [32] pointed out, the relevant question is how the chiral condensate can be understood in terms of the structure of pion expressed in its light-front wave functions. We hope this work provides a preliminary answer.

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