Families of one-point interactions resulting from the squeezing limit of the sum of two- and three-delta-like potentials

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Abstract
Several families of one-point interactions are derived from the system consisting of two and three δ-potentials which are regularized by piecewise constant functions. In physical terms such an approximating system represents two or three extremely thin layers separated by some distance. The two-scale squeezing of this heterostructure to one point as both the width of δ-approximating functions and the distance between these functions simultaneously tend to zero is studied using the power parameterization through a squeezing parameter \( \varepsilon \to 0 \), so that the intensity of each δ-potential is \( c_j = a_j \varepsilon^{1-\mu} \), \( a_j \in \mathbb{R}, j = 1, 2, 3 \), the width of each layer \( l = \varepsilon \) and the distance between the layers \( r = c \varepsilon^{\tau}, c > 0 \). It is shown that at some values of the intensities \( a_1, a_2 \) and \( a_3 \), the transmission across the limit point potentials is non-zero, whereas outside these (resonance) values the one-point interactions are opaque splitting the system at the point of singularity into two independent subsystems. Within the interval \( 1 < \mu < 2 \), the resonance sets consist of two curves on the \( (a_1, a_2) \)-plane and three surfaces in the \( (a_1, a_2, a_3) \)-space. As the parameter \( \mu \) approaches the value \( \mu = 2 \), three types of splitting the one-point interactions into countable families are observed.

Keywords: splitting of point interactions, multiple-resonant tunnelling, resonance curves and surfaces

(Some figures may appear in colour only in the online journal)

1. Introduction

The models described by the Schrödinger operators with singular zero-range potentials have widely been discussed in both the physical and mathematical literature (see books [1–4] for details and references). These models admit exact closed analytical solutions which describe
realistic situations using different approximations via Hamiltonians describing point interactions [5–10]. Currently, because of the rapid progress in fabricating nanoscale quantum devices, of particular importance is the point modelling of different structures like quantum waveguides [11, 12], spectral filters [13, 14] or infinitesimally thin sheets [15–17]. A whole body of literature (see, e.g. [18–30], a few to mention), including the very recent studies [31–38] with references therein, has been published where the one-dimensional Schrödinger operators with potentials given in the form of distributions are shown to exhibit a number of peculiar features with possible applications to quantum physics. A detailed list of references on this subject can also be found in the recent review [39]. On the other hand, using some particular regular approximations of the potential expressed in the form of the derivative of Dirac’s delta function, a number of interesting resonance properties of quantum particles tunnelling through this point potential has been observed [8, 40–42]. Particularly, it was found that at some values of the potential strength of the $\delta'$-potential the transmission across this barrier is non-zero, whereas outside these values the barrier is fully opaque. In general terms, the existence of such resonance sets in the space of potential intensities has rigorously been established for a whole class of approximations of the $\delta'$-potential by Golovaty with coworkers [43–49]. This type of point interactions may be referred to as ‘resonant-tunnelling $\delta'$-potentials’. These results differ from those obtained within Kurasov’s theory [21] which was developed for the distributions defined on the space of functions discontinuous at the point of singularity. Here the limit point interaction is also called a $\delta'$-potential. The common feature of Kurasov’s point potential and the resonant-tunnelling $\delta'$-potential is that the transmission matrices of both these interactions are of the diagonal form, but the elements of these matrices are different. It is of interest therefore to find a way where it would be possible to describe both these types in a unique regularization scheme starting from the same initial regularized potential profile.

In the present work we address the problem on the relation between the point interactions realized within Kurasov’s theory and the resonant-tunnelling $\delta'$-potentials studied in [8, 40–50]. We show that Kurasov’s $\delta'$-potential emerges from the realistic heterostructure consisting of two or three extremely thin parallel plane layers separated by some distance. This system is studied in the limit as both the width of layers and the distance between them simultaneously tend to zero. In such a squeezing limit, the limit one-point interactions are proved to depend crucially on relative approaching zero the width and the distance. As a result, different types of one-point interactions appear in this limit depending on the way of convergence. Surprisingly, within the same regularization scheme, it is possible to realize both the Kurasov $\delta'$-potential and the $\kappa\delta'$-potentials with countable sets in the $\kappa$-space at which a non-zero resonant tunnelling occurs. Another surprising point is that the $\delta$-potential discovered by Šeba in [18] can also be realized together with its countable splitting at some critical point.

Consider the system consisting of $N$ parallel sheets arranged successively with their planes perpendicular to the $x$-axis. The sheets are assumed to be homogeneous, so that one can explore the one-dimensional stationary Schrödinger equation

$$-
\frac{d^2\psi(x)}{dx^2} + V_\varepsilon(x)\psi(x) = E\psi(x)$$

where $\psi(x)$ is the wavefunction and $E$ the energy of a particle. The potential $V_\varepsilon(x)$ with a squeezing parameter $\varepsilon > 0$ shrinks to one point, say $x = 0$, as $\varepsilon \to 0$. One of the ways to realize limit point interactions is to choose the potential $V_\varepsilon(x)$ in the form of a sum of several Dirac’s delta functions as [31, 51, 52]

$$V_\varepsilon(x) = \sum_{j=1}^{N} c_j(\varepsilon)\delta(x - r_j(\varepsilon)),$$  

(2)
where the constants \( c_j \in \mathbb{R} \) and the distances between the \( \delta \)-functions \( r_j(\varepsilon) \) tend to zero as \( \varepsilon \to 0 \). The particular case of the three-delta \((N = 3)\) spatially symmetric potential \((2)\), in the squeezing limit has been studied by Cheon and Shigehara [51], and Albeverio and Nizhnik [52]. In this limit a whole four-parameter family of point interactions has been constructed, independently on whether or not potential \((2)\) has a distributional limit. Here we follow the approach developed by Exner, Neidhardt and Zagrebnov [7], who have approximated the \( \delta \)-potentials by regular functions and constructed a one-point \( \delta' \)-interaction. In particular, they have proved that the limit takes place if the distances between the ‘centers’ of regularized potentials tend to zero sufficiently slow relatively to shrinking the \( \delta \)-approximating potentials.

A similar research [6] concerns about the convergence of regularized \( \delta \)-approximating structures to point potentials in higher dimensions.

In this paper we focus on the two cases when potential \((2)\) consists of two \((N = 2)\) and three \((N = 3)\) \( \delta \)-potentials separated equidistantly by a distance \( r(\varepsilon) \) that tends to zero as \( \varepsilon \to 0 \). The transmission matrices for the two- and three-delta potentials are the products \( \Lambda_{\varepsilon} = \Lambda_2 \Lambda_0 \Lambda_1 \) and \( \Lambda_{\varepsilon} = \Lambda_3 \Lambda_0 \Lambda_2 \Lambda_0 \Lambda_1 \), respectively, where \((j = 1, 2, 3)\)

\[
\Lambda_0 = \begin{pmatrix}
\cos(kr) & k^{-1} \sin(kr) \\
-k \sin(kr) & \cos(kr)
\end{pmatrix}, \quad \Lambda_j = \begin{pmatrix} 1 & 0 \\ c_j & 1 \end{pmatrix}, \quad k := \sqrt{E}.
\] (3)

We restrict ourselves to the most simple approximation of the \( \delta \)-potentials by piecewise constant functions resulting in a three- \((\text{for } N = 2)\) and a five- \((\text{for } N = 3)\) layered potential profile. In the limit as both the width of \( \delta \)-approximating functions and the distance between them tends to zero simultaneously we obtain different families of one-point interactions. We observe that, starting from the same profile of the three- and five-layered structure that approximates potential \((2)\), the limit point interactions crucially depend on the relative rate of tending the width of layers and the distance between them to zero.

We follow the notations and the classification of one-point interactions given by Brasche and Nizhnik [31]. Thus, we denote

\[
\psi_j(0) := \psi(+0) - \psi(-0), \quad \psi'_j(0) := \psi'(+0) - \psi'(-0),
\]

\[
\psi_r(0) := \eta \psi(+0) + (1 - \eta) \psi(-0), \quad \psi'_r(0) := \eta \psi'(0) + (1 - \eta) \psi'(+0),
\] (4)

where \( \eta \in \mathbb{R} \) is an arbitrary parameter (this is a generalization of the generally accepted case with \( \eta = 1/2 \), see, e.g. [20, 21, 31, 33]). Then the \( \delta \)-interaction, or \( \delta \)-potential, with intensity \( \alpha \) is defined by the boundary conditions \( \psi_j(0) = 0 \) and \( \psi'_j(0) = \alpha \psi_j(0) \), so that the \( \Lambda \)-matrix in this case has the form

\[
\Lambda = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}.
\] (5)

The dual interaction is called a \( \delta' \)-interaction (the notation has been suggested in [3, 19] and adopted in the literature). This point interaction with intensity \( \beta \) defined by the boundary conditions \( \psi'_j(0) = 0 \) and \( \psi_j(0) = \beta \psi'_j(0) \) has the \( \Lambda \)-matrix of the form

\[
\Lambda = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}.
\] (6)

As follows from formulae (5) and (6), the usage of the parameter \( \eta \) for both the \( \delta \) - and \( \delta' \)-interactions does not play any role. However, for the \( \delta' \)-potential with intensity \( \gamma \) the potential part in equation \((1)\) is given by \( \gamma \delta'(x) \psi(x) \) where the wavefunction \( \psi(x) \) must be discontinuous at \( x = 0 \). Therefore, due to the ambiguity of the product \( \delta'(x) \psi(x) \), one can suppose the following generalized (asymmetric) averaging in the form
\( \delta'(x)\psi(x) = [(1 - \eta)\psi(-0) + \eta\psi(+0)] \delta'(x) + [\eta\psi'(-0) + (1 - \eta)\psi'(+0)] \delta(x). \)  

(7)

This suggestion is also motivated by the studies [53–55] which demonstrate that the plausible averaging with \( \eta = 1/2 \) at the point of singularity in general does not work. The \( \delta' \)-potential with intensity \( \gamma \) is defined by the boundary conditions \( \psi_\pm(0) = \gamma \psi_\pm(0) \) and \( \psi'_\pm(0) = -\gamma \psi'_\pm(0) \) [31]. An equivalent form of these conditions is given by the \( \Lambda \)-matrix in the diagonal form

\[ \Lambda = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix} \]

with \( \theta = [1 + (1 - \eta)\gamma]/(1 - \eta\gamma) \).

Finally, instead of the fourth type of point interactions defined in [31] as \( \delta \)-magnetic potentials, in this paper we shall be dealing with potentials which at some (resonant) values of intensities are fully transparent, whereas outside these values they are completely opaque satisfying the Dirichlet boundary conditions \( \psi(\pm 0) = 0 \). At the resonance sets the boundary conditions are given by one of the unit matrices \( \Lambda = \pm I, I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

2. A piecewise constant approximation of the \( \delta \)-potentials

Let us approximate the \( \delta \)-potentials in (2) by piecewise constant functions. Then potential (2) is replaced by the rectangular function

\[ V_\epsilon(x) = \begin{cases} 0 & \text{for } -\infty < x < 0, \ l < x < l + r, \\ \epsilon_j & \text{for } (j - 1)(l + r) < x < j(l + r) - r, \ j = 1, 2, 3, \end{cases} \]

(9)

and, as a result, all the matrices \( \Lambda_j, j = 1, 2, 3, \) in the product for \( \Lambda_\epsilon \) are replaced by

\[ \Lambda_{j,\epsilon} = \begin{pmatrix} \cos(k_j l) & k_j^{-1} \sin(k_j l) \\ -k_j \sin(k_j l) & \cos(k_j l) \end{pmatrix}, \]

(10)

where

\[ k_j := \sqrt{k^2 - \epsilon_j}, \ j = 1, 2, 3. \]

(11)

In other words, the regularized transmission matrix \( \Lambda_\epsilon \) defined by the relations

\[ \begin{pmatrix} \psi(x_2) \\ \psi'(x_2) \end{pmatrix} = \Lambda_\epsilon \begin{pmatrix} \psi(x_1) \\ \psi'(x_1) \end{pmatrix}, \quad \Lambda_\epsilon = \Lambda_{3,\epsilon} \Lambda_{2,\epsilon} \Lambda_{1,\epsilon} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}, \]

(12)

connects the boundary conditions for the wavefunction \( \psi(x) \) and its derivative \( \psi'(x) \) at \( x = x_1 = 0 \) and \( x = x_2 = 3l + 2r \) \( (N = 3) \). For the case of the two-delta potential \( (N = 2) \) we set \( a_3 \equiv 0 \) in potential (9), so that the boundary conditions are \( x_1 = 0 \) and \( x_2 = 2l + r \). The matrix elements in (12), denoted by overhead bars, depend on the shrinking parameters \( l \) and \( r \), whereas in the limit matrix elements, if they exist, the bars are omitted, i.e. we write

\[ \lim_{l,r \to 0} \Lambda_\epsilon = \Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}. \]

Having accomplished the limit procedure, we set \( x_1 = -0 \) and \( x_2 = +0 \).

One can compute the matrix products \( \Lambda_\epsilon \) explicitly both for \( N = 2 \) and \( N = 3 \). Using that \( k_j \to \infty, k_j/l \) and \( k_j/k_j, i, j = 1, 2, 3, \) are finite and \( r \to 0 \), it is sufficient to write their truncated
expressions. As a result, we find the asymptotic behaviour of the elements of the matrix \( \Lambda_{lr} = \Lambda_{2l}^{1/2} \Lambda_{01} \) (\( N = 2 \)):

\[
\hat{\lambda}_{11} \to \cos(k_1 l) \cos(k_2 l) - (k_1/k_2) \sin(k_1 l) \sin(k_2 l) - k_1 r \sin(k_1 l) \cos(k_2 l),
\]

(13)

\[
\hat{\lambda}_{12} \to 0,
\]

(14)

\[
\hat{\lambda}_{21} \to -k_1 \sin(k_1 l) \cos(k_2 l) - k_2 \cos(k_1 l) \sin(k_2 l) + k_1 k_2 r \sin(k_1 l) \sin(k_2 l),
\]

(15)

\[
\hat{\lambda}_{22} \to \cos(k_1 l) \cos(k_2 l) - (k_2/k_1) \sin(k_1 l) \sin(k_2 l) - k_2 r \cos(k_1 l) \sin(k_2 l).
\]

(16)

Similarly, for the three-delta potential the asymptotic representations of the elements of the matrix product \( \Lambda_{lr} = \Lambda_{3l}^{1/2} \Lambda_{02} \Lambda_{01} \) become

\[
\hat{\lambda}_{11} \to \cos(k_1 l) \cos(k_3 l) \cos(k_2 l) - (k_1/k_2) \sin(k_1 l) \sin(k_2 l) \sin(k_3 l)
- (k_1/k_3) \sin(k_1 l) \sin(k_3 l) - (k_2/k_3) \cos(k_1 l) \sin(k_2 l) \sin(k_3 l)
- 2k_1 r \sin(k_1 l) \cos(k_2 l) \cos(k_3 l) - k_2 r \cos(k_1 l) \sin(k_2 l) \sin(k_3 l)
+ k_1 k_2 r^2 \sin(k_1 l) \sin(k_2 l) \cos(k_3 l) + (k_1 k_2 r/k_3) \sin(k_1 l) \sin(k_2 l) \sin(k_3 l),
\]

(17)

\[
\hat{\lambda}_{12} \to -k_2 r^2 \cos(k_1 l) \sin(k_2 l) \cos(k_3 l),
\]

(18)

\[
\hat{\lambda}_{21} \to -k_1 \sin(k_1 l) \cos(k_2 l) \cos(k_3 l) - k_2 \cos(k_1 l) \sin(k_2 l) \sin(k_3 l)
- k_3 \cos(k_1 l) \cos(k_3 l) \sin(k_2 l) + k_1 k_2 r \sin(k_1 l) \sin(k_2 l) \cos(k_3 l)
+ 2k_1 k_3 r \sin(k_1 l) \cos(k_2 l) \sin(k_3 l) + k_3 k_2 r \cos(k_1 l) \sin(k_2 l) \sin(k_3 l)
+ k_1 k_2 (k_2^{-1} - k_2 r^2) \sin(k_1 l) \sin(k_2 l) \sin(k_3 l) + k_2 r^2 \cos(k_2 l) |k_1 | \sin(k_1 l) \cos(k_3 l)
+ k_3 \cos(k_1 l) \sin(k_3 l)),
\]

(19)

\[
\hat{\lambda}_{22} \to \cos(k_1 l) \cos(k_2 l) \cos(k_3 l) - (k_2/k_1) \sin(k_1 l) \sin(k_2 l) \sin(k_3 l)
- (k_3/k_1) \sin(k_1 l) \cos(k_2 l) \sin(k_3 l) - (k_3/k_2) \cos(k_1 l) \sin(k_2 l) \sin(k_3 l)
- 2k_3 r \cos(k_1 l) \cos(k_2 l) \sin(k_3 l) - k_2 r \cos(k_1 l) \sin(k_2 l) \sin(k_3 l)
+ k_2 k_3 r^2 \cos(k_1 l) \sin(k_2 l) \sin(k_3 l) + (k_2 k_3 r/k_1) \sin(k_1 l) \sin(k_2 l) \sin(k_3 l).
\]

(20)

The convergence of these elements in the limit as both the parameters \( l \) and \( r \) tend to zero will be analysed below. Depending on the relative degree of coming up these parameters to zero, quite different limit transmission matrices will be obtained realizing various one-point interactions.

3. The power parameterization of the matrix \( \Lambda_{lr} \)

The convergence of the transmission matrix \( \Lambda_{lr} \) as \( l, r \to 0 \) both for \( N = 2 \) and \( N = 3 \) can be parameterized through the parameter \( \varepsilon > 0 \) using the positive powers \( \mu \) and \( \tau \) from the \( \{\mu > 0, \tau > 0\} \)-quadrant as follows.
Here each coefficient $a_i \in \mathbb{R}$ may be called an ‘intensity’ or a ‘charge’ of the $j$th $\delta$-approximating layer. According to (11), we have the following asymptotic relations:

$$k_j \to \sqrt{-a_i} e^{-\mu/2}, \quad k_l \to \sqrt{-a_i} e^{-\mu/2}, \quad k_j^2 l \to -a_i e^{1-\mu}.$$  

(22)

Using these relations in which $\mu \in (0, 2)$ and $\tau \in (0, \infty)$, we obtain that in the limit as $\varepsilon \to 0$ asymptotic expressions (13), (15)–(17), (19) and (20) are reduced to

$$\lambda_{11} \to 1 + c a_1 e^{1-\mu+\tau}, \quad \lambda_{22} \to 1 + c a_2 e^{1-\mu+\tau};$$  

(23)

$$\lambda_{11} \to (a_1 + a_2) e^{1-\mu+\tau} + c a_1 a_2 e^{2(1-\mu+\tau)}, \quad N = 2;$$  

(24)

$$\lambda_{11} \to 1 + c (2 a_1 + a_2) e^{1-\mu+\tau} + c^2 a_1 a_2 e^{2(1-\mu+\tau)}, \quad N = 2.$$  

(25)

$$\lambda_{21} \to (a_1 + a_2 + a_3 + a_1 a_3 e^{2-\mu}) e^{1-\mu} + c (a_1 a_2 + 2 a_1 a_3 + a_2 a_3) e^{2(1-\mu+\tau)} + c^2 a_1 a_2 a_3 e^{3(1-\mu+\tau)} - k^2 c^2 (a_1 + a_3) e^{1-\mu+2\tau}, \quad N = 3.$$  

(26)

Next, in the case when $\mu = 2$ and $0 < \tau < \infty$, asymptotic relations (13), (15) and (16) become

$$\lambda_{11} \to \left(\cos \sqrt{-a_1} - c \sqrt{-a_1} \sin \sqrt{-a_1} e^{\tau-1}\right) \cos \sqrt{-a_2} - \sqrt{-a_1/a_1} \sin \sqrt{-a_1} \sin \sqrt{-a_2},$$  

(27)

$$\lambda_{21} \to - \left(\sqrt{-a_1} \sin \sqrt{-a_1} \cos \sqrt{-a_2} + \sqrt{-a_1} \sin \sqrt{-a_2}\right) e^{-1} + c \sqrt{-a_1/a_2} \sin \sqrt{-a_1} \sin \sqrt{-a_2} e^{\tau-2}.$$  

(28)

for $N = 2$. Similarly, in the case with $N = 3$ asymptotic expressions (17), (19) and (20) are transformed to

$$\lambda_{11} \to \left[\cos \sqrt{-a_1} \cos \sqrt{-a_2} - \sqrt{-a_1/a_2} \sin \sqrt{-a_1} \sin \sqrt{-a_2} - 2 c \sqrt{-a_1} \sin \sqrt{-a_1} \cos \sqrt{-a_2} \cos \sqrt{-a_1} \sin \sqrt{-a_2} \varepsilon^{-1} + c \sqrt{-a_1/a_2} \sin \sqrt{-a_1} \sin \sqrt{-a_2} e^{2(\tau-1)} \cos \sqrt{-a_1} \right.$$  

$$\left.- \left[\sqrt{-a_1} \sin \sqrt{-a_1} \cos \sqrt{-a_2} + \sqrt{-a_1} \cos \sqrt{-a_2} \cos \sqrt{-a_1} \sin \sqrt{-a_2} \right] \sin \sqrt{-a_1} \sin \sqrt{-a_2}, \right.$$  

$$\lambda_{22} \to \left[\cos \sqrt{-a_1} \cos \sqrt{-a_2} - \sqrt{-a_1/a_2} \sin \sqrt{-a_1} \sin \sqrt{-a_2} - 2 c \sqrt{-a_1/a_2} \sin \sqrt{-a_1} \cos \sqrt{-a_2} \cos \sqrt{-a_1} \sin \sqrt{-a_2} \right.$$  

$$\left.- \left[\sqrt{-a_1} \sin \sqrt{-a_1} \cos \sqrt{-a_2} + \sqrt{-a_1} \cos \sqrt{-a_2} \cos \sqrt{-a_1} \sin \sqrt{-a_2} \right] \sin \sqrt{-a_1} \sin \sqrt{-a_2} \sin \sqrt{-a_2}, \right.$$  

$$\lambda_{21} \to - \left[\sqrt{-a_1} \sin \sqrt{-a_1} \cos \sqrt{-a_2} + \sqrt{-a_1} \sin \sqrt{-a_2}\right] e^{-1} + c \sqrt{-a_1/a_2} \sin \sqrt{-a_1} \sin \sqrt{-a_2} e^{\tau-1}$$  

$$+ \left[\sqrt{-a_1} \sin \sqrt{-a_1} \cos \sqrt{-a_2} + \sqrt{-a_1} \cos \sqrt{-a_2} \cos \sqrt{-a_1} \sin \sqrt{-a_2} \right] \sin \sqrt{-a_1} \sin \sqrt{-a_2} \sin \sqrt{-a_2} \varepsilon^{-1} + c \sqrt{-a_1/a_2} \sin \sqrt{-a_1} \sin \sqrt{-a_2} \varepsilon^{-1} \right] \sin \sqrt{-a_1} / \sqrt{-a_1},$$  

$$+ k^2 c^2 \cos \sqrt{-a_2} \left(\sqrt{-a_1} \sin \sqrt{-a_1} \cos \sqrt{-a_2} + \sqrt{-a_1} \sin \sqrt{-a_2}\right) \varepsilon^{\tau-1}.$$  

(29)

(30)
4. Admissible sets of the parameters \( \mu \) and \( \tau \) for realizing one-point interactions

For the realization of (both connected and separated) interactions in the squeezing limit, the elements \( \lambda_{11} \) and \( \lambda_{22} \) given by asymptotic formulae (23), (25), (27) and (29) must be finite as \( \varepsilon \to 0 \). It follows from these relations that the region \( Q := \{ 0 < \mu \leq 1, 0 < \tau < \infty \} \cup \{ 1 < \mu \leq 2, \mu - 1 \leq \tau < \infty \} \) is admissible for the finiteness of the limit elements \( \lambda_{11} \) and \( \lambda_{22} \). Concerning the relation for \( \lambda_{12} \) in the case with \( N = 3 \), we have in the region \( Q \) the limits \( \lambda_{12} \to c^2 \bar{a}_2 \varepsilon^{1-\mu+2\tau} \to 0 \) for \( 0 < \mu < 2 \) and \( \bar{a}_2 \to -c^2 \sqrt{-\bar{a}_2} \cos \sqrt{-\bar{a}_1} \sin \sqrt{-\bar{a}_2} \cos \sqrt{-\bar{a}_3} \varepsilon^{2\tau-1} \to 0 \) for \( \mu = 2 \). Therefore the convergence of the matrix \( \Lambda_{12} \) in the limit as \( \varepsilon \to 0 \) has to be analysed in the region \( Q \) shadowed in figure 1. While the matrix element \( \lambda_{12} \) for both \( N = 2 \) and \( N = 3 \) has the zero limit, the elements \( \lambda_{31} \) given by asymptotic representations (24), (26), (28) and (30) are in general divergent as \( \varepsilon \to 0 \). In the region \( Q \), the highest divergence is of the order \( \varepsilon^{1-\mu} \), \( 1 < \mu \leq 2 \). However, on some sets of \( Q \) and at some conditions on the intensities \( a_1 \), the total coefficient in front of the divergent term \( \varepsilon^{1-\mu} \) can be zero. These conditions will be analysed below in detail, but now we single out those sets of \( Q \) where this analysis will be carried out. Thus, we split the region \( Q \) into the sets as shown in figure 1 using the following definitions:

region \( Q_0 := \{ 0 < \mu < 1, 0 < \tau < \infty \} \);
lines \( L_0 := \{ \mu = 2, 2 < \tau < \infty \} \), \( L_D := \{ \mu = 2, 1 < \tau < 2 \} \), \( L_1 := \{ \mu = 1, 0 < \tau < \infty \} \), \( L_2 := \{ \mu = 2, 0 < \tau < \infty \} \), \( L_K := \{ 1 < \mu < 2, \tau = \mu - 1 \} \), \( L_S := \{ 1 < \mu < 2, \tau = 2(\mu - 1) \} \);
regions \( Q_1 := \{ 1 < \mu < 2, \mu - 1 < \tau < 2(\mu - 1) \} \), \( Q_2 := \{ 1 < \mu < 2, 2(\mu - 1) < \tau < \infty \} \);
points \( P_1 := \{ \mu = 2, \tau = 1 \} \), \( P_2 := \{ \mu = \tau = 2 \} \).

5. Families of one-point interactions realizing on the sets with \( 0 < \mu < 2 \):
\( Q_0, L_1, L_K, Q_1, L_S, Q_2 \)

Below, starting up from the most simple situations (successively from the left to the right in the diagram shown by figure 1), we shall analyse all the possible one-point interactions on the open set \( Q_0 \cup L_1 \cup L_K \cup Q_1 \cup L_S \cup Q_2 \).

5.1. Non-resonant point interactions on the set \( Q_0 \cup L_1 \) \( (0 < \mu \leq 1) \)

Since in the region \( Q_0 \) we have \( 0 < \mu < 1 \) and \( 0 < \tau < \infty \), it follows immediately from asymptotic representation (23)–(26) that the limit transmission matrix \( \Lambda \) is identically one, so that this point interaction is trivial (with full transmission). At the boundary of this region, i.e. on the line \( L_1 \) \((\mu = 1)\), the \( \varepsilon \to 0 \) limit of representation (23)–(26) results in the \( \Lambda \)-matrix of type (5) that describes the \( \delta \)-potential. The total intensity of this potential is the algebraic sum of the layer intensities, i.e. \( \alpha = a_1 + a_2 \) or \( \alpha = a_1 + a_2 + a_3 \) in (5) (see equations (24) and (26)). Note that no constraints are imposed on the layer intensities in the whole region \( Q_0 \cup L_1 \), so that \( (a_1, a_2) \in \mathbb{R}^2 \) and \( (a_1, a_2, a_3) \in \mathbb{R}^3 \) and these interactions may be called ‘non-resonant’ ones.
In the following we turn to the analysis of point interactions which are realized in the strip region \( \{1 < \mu < 2, 0 < \tau < \infty\} \) (see figure 1). For the existence of (connected or separated) point interactions in this region the diagonal elements of the limit matrix \( \Lambda \) must be finite in the limit as \( \varepsilon \to 0 \). To this end, as follows from the asymptotic representation given by (23) and (25), we have to impose the inequality \( \tau \geq \mu - 1 \) for all \( \mu \in (1, 2) \). The region below the boundary line \( \tau = \mu - 1 \) is forbidden for the existence of point interactions.

5.2. Point interactions on the line \( L_K \)

Consider first the boundary line \( L_K \). As follows from asymptotic relations (23) and (25), on this line the \( \varepsilon \to 0 \) limits of the diagonal elements \( \lambda_{11} \) and \( \lambda_{22} \) are finite. However, the elements \( \lambda_{21} \) given asymptotically by (24) and (26) are in general divergent in the limit as \( \varepsilon \to 0 \). Nevertheless, in this limit, at some conditions on the intensities \( a_1, a_2 \) and \( a_3 \), the terms in \( \lambda_{21} \) may cancel out. One of the ways to remove the divergence in asymptotic expressions (24) and (26) as \( \varepsilon \to 0 \) is the requirement that the coefficient in front of the divergent term \( \varepsilon^{1-\mu}, 1 < \mu < 2 \), must be zero, i.e. \( \lim_{\varepsilon \to 0} (\varepsilon^{\mu-1}\lambda_{21})_{\tau=\mu-1} = 0 \). Indeed, as follows from (24) and (26), this requirement can be satisfied on the line \( L_K \) if the conditions

\[
K_2(a_1, a_2; c) := a_1 + a_2 + ca_1a_2 = 0 \quad \text{for } N = 2,
K_3(a_1, a_2, a_3; c) := a_1 + a_2 + a_3 + c(a_1a_2 + 2a_1a_3 + a_2a_3) + c^2a_1a_2a_3 = 0 \quad \text{for } N = 3,
\]

are fulfilled. At fixed coefficient \( c \), these conditions may be considered as equations with respect to the intensities \( a_1, a_2 \) and \( a_3 \). The first of these equations \( (N = 2) \) has been obtained earlier by Brasche and Nizhnik [31]. In what follows we shall also be dealing with other conditions like (32). They may be referred to as ‘resonance equations’ and their solutions ‘resonance sets’. The solutions of equations (32) are plotted in figures 2 and 3, respectively.

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**Figure 1.** Diagram of the \((\mu, \tau)\)-sets arranged in the correspondence with different families of one-point interactions to be realized from two- and three-delta-like layers. Notations of regions, lines and points are defined by equations (31).
for $N = 2$ and $3$. In each of these two cases, the curve and the surface marked in these figures by $0$ appear to be ‘pinned’ to the origins $a_1 = a_2 = 0$ and $a_1 = a_2 = a_3 = 0$. They are considered as background branches of the corresponding resonance sets and therefore we call them the ‘zeroth resonance’ curve ($N = 2$, figure 2) and the ‘zeroth resonance’ surface ($N = 3$, figure 3). In the limit as $c \to 0$, the curve $1’$ in figure 2 and the surfaces $1’$ and $2’$ in figure 3 vanish ‘escaping’ to infinity. At the same time, the zeroth branches 0 straighten to the line $a_1 + a_2 = 0$ and to the plane $a_1 + a_2 + a_3 = 0$, respectively.

Using next equations (32) in asymptotic representations (23) and (25), we obtain the diagonal elements of the limit $\Lambda$-matrix:

$$\lim_{c \to 0} \lambda_{11} =: \theta = \begin{cases} 1 + ca_1 & \text{for } N = 2, \\ 1 + c(a_1 + a_2) + c^2 a_1 a_2 & \text{for } N = 3, \end{cases}$$

$$\lim_{c \to 0} \lambda_{22} =: \rho = \begin{cases} 1 + ca_2 & \text{for } N = 2, \\ 1 + c(a_2 - 2a_3) + c^2 a_2 a_3 & \text{for } N = 3. \end{cases}$$

In virtue of equations (32), we have $\rho = \theta^{-1}$ and therefore on the line $L_K$ the limit transmission matrix becomes of diagonal form (8). In particular, setting here $\theta = (2 + \gamma)/(2 - \gamma)$ and $\rho = (2 - \gamma)/(2 + \gamma)$, we obtain Kurasov’s $\delta'$-potential with intensity $\gamma \in \mathbb{R} \setminus \{\pm 2\}$ defined in the distributional sense on the space of discontinuous at $x = 0$ test functions. For this case one can find the resonance values of $a_1$, $a_2$ and $a_3$ as functions of the strength $\gamma$:

$$a_1 = \frac{2\gamma}{c(2 - \gamma)}, \quad a_2 = -\frac{2\gamma}{c(2 + \gamma)}$$

for $N = 2$ and

$$a_1 = \frac{1}{c(2 + ca_2)} \left( \frac{2\gamma}{2 - \gamma} - ca_2 \right), \quad a_3 = -\frac{1}{c(2 + ca_2)} \left( \frac{2\gamma}{2 + \gamma} + ca_2 \right).$$

with arbitrary $a_2 \in \mathbb{R} \setminus \{-a_2/c\}$, for $N = 3$. In particular, for $N = 2$ the barrier-well structure corresponds to the interval $-2 < \gamma < 2$ ($a_1 > 0$, $a_2 < 0$ for $-2 < \gamma < 0$ and $a_1 < 0$, $a_2 > 0$ for $0 < \gamma < 2$), whereas beyond this interval ($2 < |\gamma| < \infty$), we have the double-well configuration. We call the vectors $(a_1, a_2) \in \mathbb{R}^2$ and $(a_1, a_2, a_3) \in \mathbb{R}^3$ that satisfy equations (32) the resonance sets $K_2$ and $K_3$, respectively. Next, the point interactions realized on the line $L_K$ are referred in the following to as ‘resonant-tunnelling $\delta'$-potentials of the $K$-type’, despite that the double-well case is involved here as well, together with a barrier-well structure.

Finally, it should be noticed that there exists a particular subfamily of the intensities $a_1, a_2$ and $a_3$ from the $K_2, K_3$-sets for which $\theta = \pm 1$ in (8), realizing the point interactions with full transmission. Thus, for $N = 2$ these values are $a_1 = a_2 = -2/c$ (this point is indicated in figure 2) resulting in the matrix $\Lambda = -I$. In the three-delta case the two conditions $a_1 = a_3$ and $2a_1 + a_2 + ca_1 a_2 = 0$ provide the matrix $\Lambda = I$, whereas the other two conditions $a_1 + a_3 = -2/c$ (in general, an asymmetric structure) and $2a_1 + a_2 + ca_1 a_2 = -2/c$ lead to the matrix $\Lambda = -I$.

5.3. Point interactions on the set $Q_1 \cup L_S \cup Q_2$: Šeba’s ‘transition’

Other types of one-point interactions can be realized in the regions $Q_1$ and $Q_2$ including the line $\tau = 2(\mu - 1)$, $1 < \mu < 2$, denoted by $L_S$, that separates these regions (see figure 1). In fact, as shown below, the point interactions on these three sets are related to those studied by
Šeba in [18] (see Theorem 3 therein). First, as follows from asymptotic relations (23) and (25), we have the limits
\[
\bar{\lambda}_{11}, \bar{\lambda}_{22} \to 1
\]
as \(\varepsilon \to 0\) everywhere on the sets \(Q_1, Q_2\) and \(\mathcal{LS}\). Next, if we impose in (24) and (26) the (resonance) conditions
\[
K_2(a_1, a_2; c)\big|_{c=0} = L_2(a_1, a_2) = a_1 + a_2 = 0 \quad \text{for } N = 2,
\]
\[
K_2(a_1, a_2, a_3; c)\big|_{c=0} = L_3(a_1, a_2, a_3) = a_1 + a_2 + a_3 = 0 \quad \text{for } N = 3,
\]
being just a 'linearized' version of equations (32), the element \(\lambda_{21}\) becomes infinite in the region \(Q_1\), finite on the line \(L_2\) and zero in the region \(Q_2\). Thus, under the resonance conditions (36), in the limit as \(\varepsilon \to 0\) we have on the set \(Q_1\) the separated interactions satisfying the Dirichlet conditions \(\psi(\pm 0) = 0\), the \(\delta\)-potentials realizing on the line \(L_\delta\) and described by the \(\Lambda\)-matrix of type (5) with the intensity
\[
\alpha = -c \begin{cases} 
    a_1^2 = a_2^2 & \text{for } N = 2, \\
    a_1^2 + (a_1 + a_2)^2 = a_2^2 + a_3^2 = (a_2 + a_3)^2 + a_3^2 & \text{for } N = 3
\end{cases}
\]
and the reflectionless potentials on the set \(Q_2\) for which \(\Lambda = I\).

In physical terms, the point interactions realized in the region \(Q_1 \cup Q_2 \cup L_\delta\) exhibit the transition of transmission that occurs on the resonance \(\mathcal{L}_{2,3}\)-sets while varying the rate of increasing the distance \(r\) between the \(\delta\)-approximating potentials in (9). For sufficiently slow squeezing \([\mu - 1 < \tau < 2(\mu - 1)]\) this distance, the limit point interactions are opaque, for intermediate shrinking \([\tau = 2(\mu - 1)]\) the interactions become partially transparent (\(\delta\)-well)
and for fast shrinking \([2(\mu - 1) < \tau < \infty]\) the interactions appear to be fully transparent. In other words, the line \(L_S\) separates the regions \(Q_1\) of full reflection and \(Q_2\) of perfect transmission.

In what follows we refer the line \(a_1 + a_2 = 0\) (green in figure 2) and the plane \(a_1 + a_2 + a_3 = 0\) (figure 3) to as \(L_{2\Sigma}\) and \(L_{3\Sigma}\)-sets, respectively. In its turn, the interactions realized on the line \(L_2\) are called ‘resonant-tunnelling \(\delta\)-potentials of the \(L\)-type’ and ones in the region \(Q_2\) ‘resonant-tunnelling reflectionless potentials of the \(L\)-type’.

Let us consider now potential (2) with \(N = 3\) rewritten in the form

\[
V_3(x) = (c/\epsilon)^\sigma [a_1 \delta(x) + a_2 \delta(x - \epsilon) + a_3 \delta(x - 2\epsilon)], \quad 0 < \sigma \leq 1,
\]

with a new squeezing parameter \(\epsilon > 0\). For the case with \(N = 2\) (\(a_3 \equiv 0\) in (38)), Šeba has proved (see Theorem 3 in [18]) that at \(\sigma = 1/2\) the limit point interaction is the \(\delta\)-potential described by the \(\Lambda\)-matrix of form (5) with the intensity \(\alpha\) given by the first equation (37) if \(a_1 + a_2 = 0\), being in fact a resonance condition for intensities \(a_1\) and \(a_2\). At this condition for all \(\sigma \in (0, 1/2)\) the limit \(\Lambda\)-matrix is the unit, while for \(\sigma \in (1/2, 1)\) the limit point interactions are separated satisfying the two-sided Dirichlet conditions \(\psi(\pm 0) = 0\). In physical terms, the value \(\sigma = 1/2\) is a ‘transition’ point (at which the transmission is partial) separating the opaque potentials from those with full transmission.

In fact, Šeba’s theorem can be extended to the case with \(N = 3\). Indeed, by a straightforward calculation of the matrix product \(\Lambda_s = \Lambda_3 \Lambda_0 \Lambda_2 \Lambda_0 \Lambda_1\) with matrices (3) in which \(c_j = a_j (c/\epsilon)^\sigma\)
and \( r = \epsilon \), we find the limits \( \tilde{\lambda}_{11}, \tilde{\lambda}_{22} \to 1 \) and \( \tilde{\lambda}_{12} \to 0 \) for \( \sigma \in (0, 1) \). The singular element \( \tilde{\lambda}_{21} \) has the asymptotic representation

\[
\tilde{\lambda}_{21} \to (a_1 + a_2 + a_3)(c/\epsilon)^\sigma + (a_1a_2 + 2a_1a_3 + a_2a_3)\epsilon(c/\epsilon)^{2\sigma} + a_1a_2a_3\epsilon^2(c/\epsilon)^{3\sigma}
\]  
(39)

as \( \epsilon \to 0 \). It follows from this relation that under the resonance condition \( a_1 + a_2 + a_3 = 0 \) the limit \( \delta \)-potential is realized at \( \sigma = 1/2 \) with the intensity \( \alpha \) given by the second formula (37).

All these results obtained above for potential (38) appear to be in agreement with those obtained for both the lines \( L_K \) and \( L_5 \). Indeed, the comparison of expression (38) with potential (2) in which \( N = 2, 3 \) leads to the equations \( e_j = a_0\epsilon^{1-\mu} \) and \( r = c\epsilon^\tau \). As a result, we find the relation \( \tau = \sigma^{-1}(\mu - 1) \), so that at \( \sigma = 1 \) we have the resonance sets defined by equations (32) on the line \( L_K \), while at \( \sigma = 1/2 \), i.e. on the line \( L_5 \), we find that formulae (37) hold true.

Thus, all the interactions realized at \( \sigma = 1 \) or the same on the line \( L_K \) are of Kurasov’s \( \delta' \)-potential type defined via the test functions which are discontinuous at \( x = 0 \). It is of interest the problem whether or not the standard \( \delta' \)-potentials defined on the \( C_0^\infty \) functions can be found among the \( K \)-families defined by conditions (32). This problem has been examined by Brascie and Nizhnik [31] including also the case when the fourth \( \delta \)-potential, i.e. the term \( a_4\delta(x - 3\epsilon) \) is added in the square brackets of (38). In our notations these results read as follows. For \( N = 2 \), in virtue of the first equation (32), the distributional limit \( V_n(x) \to \kappa\delta'(x) \) cannot exist because it is necessary that \( a_1 + a_2 = 0 \). However, for \( N = 3 \) and 4, such a limit can be realized on the planes \( a_1 + a_2 + a_3 = 0 \) and \( a_1 + a_2 + a_3 + a_4 = 0 \), respectively. For instance, in the case with \( N = 3 \) we have \( \kappa = c(a_1 - a_3) \) where \( a_1 \) and \( a_3 \) are connected through the quadratic equation

\[
(1 + ca_1)a_3^2 + (1 + ca_3)a_1^2 = 0.
\]  
(40)

Solving the last equation with respect to \( a_1 \) or \( a_3 \), one can be convinced that there exists a non-empty set of solutions to this equation. The existence of these solutions is illustrated in figure 3 as the intersection of the resonance surface \( \Gamma' \) with the \( L_3 \)-plane.

To conclude this section, note that, as follows from asymptotic representation (26), the finite \( \epsilon \to 0 \) limit of \( \tilde{\lambda}_{11} \) can also be realized on the line \( \tau = 3(\mu - 1)/2 \) if the two equations \( a_1 + a_2 + a_3 = 0 \) and \( a_1a_2 + 2a_1a_3 + a_2a_3 = 0 \) are fulfilled simultaneously. However, excluding \( a_3 \) from these equations, we find the condition \( a_1^2 + (a_1 + a_2)^2 = 0 \) which is valid only if \( a_1 = a_2 = 0 \) and therefore \( a_3 = 0 \).

6. Families of one-point interactions with countable resonant-tunnelling sets

This section is devoted to the analysis of point interactions which are realized on the line \( L_2 \). The starting point for this analysis is the asymptotic representation of the \( \Lambda \)-matrix at \( \mu = 2 \) given by relations (27)–(30). As shown in figure 1 and defined by equations (31), the line \( L_2 \) includes the points \( P_1 \) and \( P_2 \) as limiting right-hand sets of the lines \( L_K \) and \( L_5 \) as well as the lines \( L_D \) and \( L_0 \) as limiting right-hand sets of the regions \( Q_1 \) and \( Q_2 \), respectively. The existence of (connected and separated) point interactions on these limiting sets, where \( \tau \geq 1 \), immediately follows from (27) and (29), resulting in finite \( \epsilon \to 0 \) limits of the diagonal elements \( \lambda_{11} \) and \( \lambda_{22} \). Concerning the divergent terms \( \tilde{\lambda}_{21} \) given asymptotically by (28) and (30), their \( \epsilon \to 0 \) limits appear to be finite under some conditions on the intensities \( a_1, a_2 \) and \( a_3 \) which are listed below. Thus, at \( \tau = 1 \) these terms become zero if the (resonance) equations

...
\[ F_2(a_1, a_2; c) := \sqrt{-a_1} \sin \sqrt{-a_1} \cos \sqrt{-a_2} + \left( \cos \sqrt{-a_1} - c \sqrt{-a_1} \sin \sqrt{-a_1} \right) \sqrt{-a_2} \sin \sqrt{-a_2} = 0 \]  
(41)

for \( N = 2 \) and

\[ F_3(a_1, a_2, a_3; c) := \left[ \sqrt{-a_1} \sin \sqrt{-a_1} \cos \sqrt{-a_2} + \sqrt{-a_2} \cos \sqrt{-a_1} \sin \sqrt{-a_2} \right. \\
- c \sqrt{-a_1} \sqrt{-a_2} \sin \sqrt{-a_1} \sin \sqrt{-a_2} \left] \cos \sqrt{-a_3} + \left[ \cos \sqrt{-a_1} \cos \sqrt{-a_2} \\
- 2c \sqrt{-a_1} \sin \sqrt{-a_1} \cos \sqrt{-a_2} - c \sqrt{-a_2} \cos \sqrt{-a_1} \sin \sqrt{-a_2} \\
+ \left( c^2 \sqrt{-a_1} \sqrt{-a_2} - \sqrt{-a_1} / \sqrt{-a_2} \right) \sin \sqrt{-a_1} \sin \sqrt{-a_2} \right] \sqrt{-a_3} \sin \sqrt{-a_3} = 0 \]  
(42)

for \( N = 3 \) hold true. Note that \( F_3(a_1, a_2) = F_2(a_2, a_1) \) and \( F_3(a_1, a_2, a_3) = F_3(a_3, a_2, a_1) \).

The diagonal elements of the limit \( \Lambda \)-matrix will be given below in terms of the following four functions:

\[ f_2(a_1, a_2; c) := \frac{\cos \sqrt{-a_1} - c \sqrt{-a_1} \sin \sqrt{-a_1}}{\cos \sqrt{-a_2}}, \quad g_2(a_1, a_2) := -\frac{\sqrt{-a_1} \sin \sqrt{-a_1}}{\sqrt{-a_2} \sin \sqrt{-a_2}}; \]  
(43)

\[ f_3(a_1, a_2, a_3; c) := \left[ \cos \sqrt{-a_1} \cos \sqrt{-a_2} - 2c \sqrt{-a_1} \sin \sqrt{-a_1} \cos \sqrt{-a_2} \\
- \sqrt{-a_1} \cos \sqrt{-a_1} \sin \sqrt{-a_2} \right. \\
+ \left( c^2 \sqrt{-a_1} \sqrt{-a_2} - \sqrt{-a_1} / \sqrt{-a_2} \right) \sin \sqrt{-a_1} \sin \sqrt{-a_2} \left] / \cos \sqrt{-a_3}, \right. \\
g_3(a_1, a_2, a_3) := -\left( \sqrt{-a_1} \sin \sqrt{-a_1} \cos \sqrt{-a_2} + \sqrt{-a_2} \cos \sqrt{-a_1} \sin \sqrt{-a_2} \\
- \sqrt{-a_1} \sqrt{-a_2} \sin \sqrt{-a_1} \sin \sqrt{-a_2} \right) / \sqrt{-a_3} \sin \sqrt{-a_3}. \]  
(44)

where \( c \geq 0 \). By straightforward calculations one can check the following properties of these functions:

\[ f_2(a_1, a_2; c) = g_2(a_1, a_2), \quad f_3(a_1, a_2, a_3; c) = g_3(a_1, a_2, a_3; c), \]  
\[ f_2(a_2, a_1; c) = f_2^{-1}(a_1, a_2; c), \quad f_3(a_2, a_1; a_1; c) = f_3^{-1}(a_1, a_2, a_1; c). \]  
(45)

if resonance equations (41) and (42) are fulfilled.

Below we describe the scenario of 'splitting' the point interactions defined on the sets \( L_K, K_S \) and \( Q_2 \) into countable families realized on the sets \( P_1, P_2 \) and \( L_0 \), respectively. For visualization purposes figures 4 and 5, as numerical solutions of equations (41) and (42), are present. The corresponding resonance sets are determined by the solutions of transcendental equations (41) and (42), with \( c > 0 \) for \( P_1 \) and \( c = 0 \) for \( P_2 \cup L_0 \).

6.1. Splitting of the \( \mathcal{K} \)-type interactions at the limiting point \( P_1 \)

Consider first the case \( N = 2 \), where the resonance set is determined by the solutions to equation (41). In this case figure 4 (see red curves, \( c > 0 \)) clearly illustrates the detachment of the first four resonance curves \((n = 1, 4)\) on the \((a_1, a_2)\)-plane from the zeroth resonance curve \( 0 \) which is deformed a bit in comparison with that shown in figure 2 by red curve. In order to display in the figure the maximum number of resonance curves, instead of \( a_1 \) and \( a_2 \), the new rescaled coordinates \( X := \text{sign}(a_1)|a_1|^{1/2} \) and \( Y := \text{sign}(a_2)|a_2|^{1/2} \) have been used.

The similar situation takes place for the case \( N = 3 \), where the set of solutions to equation (42) describes resonance surfaces in the \((a_1, a_2, a_3)\)-space. These surfaces are not plotted for \( c > 0 \), but their topology is the same as for the case \( c = 0 \) shown in figure 5. We refer the
resonance curves and surfaces, being the solutions of equations (41) and (42) with $c > 0$, to as $F_2$- and $F_3$-sets, respectively. Using next the resonance equations (41) and (42) in asymptotic representations (27) and (29) at $\tau = 1$, the limit diagonal elements of the $\Lambda$-matrix can be rewritten in terms of functions (43) and (44). As a result, in virtue of properties (45), the family of point interactions realized at the point $P_1$ is described by the $\Lambda$-matrix of type (8) with

$$\theta = \begin{cases} f_2(a_1, a_2; c) = g_2(a_1, a_2) & \text{for } N = 2, \\ f_2(a_1, a_2, a_3; c) = g_2(a_1, a_2, a_3) & \text{for } N = 3, \end{cases} \quad (46)$$

where the intensities $a_1$, $a_2$ and $a_3$ satisfy the resonance equations (41) and (42). Therefore the point interactions of this countable family may be called ‘multiple-resonant-tunnelling $\delta’$-potentials of the $F$-type’.

6.2. Splitting of the $L$-type interactions at the limiting point $P_2$

Another way of the cancellation of divergences in the terms $\lambda_{21}$ can occur at the point $P_2$ if the coefficients in front of $\varepsilon^{-1}$ in asymptotic relations (28) and (30) are set zero. This assumption immediately leads to the same equations (41) and (42) in which $c$ is formally set zero, i.e. to the equations

$$J_2(a_1, a_2) := F_2(a_1, a_2; c)|_{c=0} = 0 \quad \text{and} \quad J_3(a_1, a_2, a_3) := F_3(a_1, a_2, a_3; c)|_{c=0} = 0. \quad (47)$$

Then, at $\tau = 2$, the $\varepsilon \to 0$ limit of the $\lambda_{21}$-terms becomes finite. The solutions of resonance equations (47) yield the countable set of curves on the $(a_1, a_2)$-plane (shown by green curves in figure 4) and surfaces in the $(a_1, a_2, a_3)$-space (three of which are depicted in figure 5).
Similarly to the $F_{2,3}$-sets, the sets determined by the solutions of equations (47) may be referred to as $J_2$- and $J_3$-sets, respectively.

Since the terms with $c \varepsilon^{\tau-1}$ and $c^2 \varepsilon^{2(\tau-1)}$ in formulae (27) and (29) vanish at $\tau = 2$ in the limit as $\varepsilon \to 0$, one can put in this case $c = 0$. As a result, on the resonance sets $J_{2,3}$, the $\varepsilon \to 0$ limit of asymptotic representation (27)–(30) produces the family of one-point interactions described by the transmission matrix of the type

$$\Lambda = \begin{pmatrix} \theta & 0 \\ \frac{\alpha}{\theta} & \theta^{-1} \end{pmatrix},$$

where

$$\alpha = c \begin{cases} \sqrt{a_1 a_2} \sin a_1 \sin a_2, & N = 2, \\ \sqrt{a_1 a_3} \sin a_1 \sin a_3 \cos a_2 \sin a_3 + 2 \sqrt{a_1 a_3} \sin a_1 \cos a_2 \sin a_3 \sqrt{a_1 a_3}, & N = 3, \\ \sqrt{a_2 a_3} \cos a_1 \sin a_2 \sin a_3 + \sqrt{a_2 a_3} \cos a_1 \sin a_2 \sqrt{a_3}, & N = 3, \end{cases}$$

and the diagonal elements are given by

$$\theta = \begin{cases} f_2(a_1, a_2; c) |_{c=0} = g_2(a_1, a_2) & \text{for } N = 2, \\ f_2(a_1, a_2, a_3; c) |_{c=0} = g_2(a_1, a_2, a_3; c) |_{c=0} & \text{for } N = 3. \end{cases}$$

Figure 5. The first three $(n = 0, 1, 2)$ resonance surfaces as solutions of equation (42) with $c = 0$. While approaching the critical value $\mu = 2$, the zeroth resonance surface 0 in figure 3 is deformed remaining to be ‘pinned’ to the origin $a_1 = a_2 = a_3 = 0$, while surfaces 1' and 2' vanish. Instead of the latter surfaces, the countable set of surfaces numbered by $n = 1, 2, \ldots$ detaches from the surface 0 involving in addition triple-well configurations.
The intensities $a_1$, $a_2$ and $a_3$ in (49) and (50) satisfy the resonance equations (47), resulting in a countable set of values of $\Lambda$-matrix (48). Similarly, this family of one-point interactions may be called ‘multiple resonant-tunnelling $(\delta^\prime + \delta)$-potentials of the $J$-type’.

6.3. Splitting of the $J$-type interactions on the limiting set $L_0$

Finally, for $\tau > 2$ (on the line $L_0$) the divergent terms in (28) and (30) become zero if resonance conditions (47) are fulfilled. In the same way as in the previous section, from asymptotic relations (27) and (29) we conclude that the limit $\Lambda$-matrix becomes of form (8) with the element $\theta$ defined by formulae (50). Similarly, this family of point interactions may be called ‘resonant-tunnelling $\delta^\prime$-potentials of the $J$-type’. For $N = 2$ this type of interactions coincides with that studied earlier in [8, 43].

Thus, we have defined in this section the resonance $F_{2,1}$, $F_{2,2}$- and $F_{2,3}$-sets as solutions of transcendental equations (41), (42) and (47), respectively. These sets consist of the infinite number of curves (for $N = 2$) and surfaces (for $N = 3$) which are numbered by $0, 1, 2, \ldots$. On the other hand, in the open sets $L_2$, $L_2^\prime$ and $Q_2$, we have defined above the resonance $k_{2,1}$- and $k_{2,3}$-sets as solutions of equations (32) and (36), respectively, consisting of two curves ($N = 2$, numbered by $0$ and $1'$) and three surfaces ($N = 3$, numbered by $0$, $1'$ and $2'$). The phenomenon of ‘splitting’ these resonance sets is observed while approaching the right-hand limiting sets: the point $P_1$ and the line $P_2 \cup L_0$. The mechanism of this splitting can be explained as follows.

The curve $1'$ in figure 2 and the surfaces $1'$ and $2'$ in figure 3, which are ‘unpinned’ to the origins $a_1 = a_2 = 0$ and $a_1 = a_2 = a_3 = 0$, vanish while approaching the limiting sets. At the same time, the zeroth (background) resonance subsets marked in figures 4 and 5 by 0 remain to be ‘pinned’ to the origins, modifying their shape and becoming to be embedded in the angles with the vertices at $a_1 = a_2 = -b_0$ where $b_0$ is a solution of the equation $\cot \sqrt{b} = c \sqrt{b}$ and $a_1 = a_2 = a_3 = -(\pi/2)^2$, respectively for $N = 2$ and 3. Instead of the curve $1'$ and the surfaces $1'$ and $2'$, the detachment of the resonances numbered by $n = 1, 2, \ldots$ from the zeroth resonances 0 occurs forming countable sets. Thus, figure 4 illustrates the resonance curves with $n = 0, 1, 2, 3, 4$ for the two cases with $c > 0$ ($F_{2,2}$-set) and $c = 0$ ($F_{2,3}$-set). In figure 5 the three resonance surfaces with $n = 0, 1, 2$ from the $F_{2,3}$-set are plotted.

6.4. Reflectionless point interactions realizing on the whole line $L_2$

Finally, as follows from asymptotic relations (27)–(30), on the whole line $L_2$, there are the point subsets of the intensities $a_1$, $a_2$ and $a_3$ for which $\sin \sqrt{-a_j} = 0$, $j = 1, 2$ ($N = 2$) and $j = 1, 2, 3$ ($N = 3$). For this case we have $\Lambda = \pm \mathbf{I}$ and therefore the limit interactions are reflectionless. They are ‘non-interacting’ wells because in the case of a single well ($N = 1$) the full transmission across this well occurs if $\sin \sqrt{-a_1} = 0$. These resonance sets are denoted as

$$P_2 := \{a_1, a_2 \mid \sin \sqrt{-a_j} = 0, \ j = 1, 2 \} \quad \text{for } N = 2,$$

$$P_3 := \{a_1, a_2, a_3 \mid \sin \sqrt{-a_j} = 0, \ j = 1, 2, 3 \} \quad \text{for } N = 3.$$  \hspace{1cm} (51)

To conclude this section, note that everywhere beyond the resonance sets $F_{2,3}$ and $F_{2,3}$, the point interactions are separated, similarly to the region $Q_1$ and the line $L_{03}$, where they satisfy the Dirichlet conditions $\psi'(\pm 0) = 0$. The point $P_2$ can also be considered as Šeba’s transition between fully reflecting interactions realizing on the line $L_0$ and the splitting family with the diagonal $\Lambda$-matrix of type (8). At the transition point $P_2$, because of the presence of the non-zero $\alpha$ in matrix (48), the transmission coefficient appears less than on the line $L_0$. 

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Table 1. The \((a_1, a_2, a_3)\)-sets of the existence of resonant-tunnelling and non-resonant one-point interactions realized on all the sets defined by equations (31) as well as the corresponding transmission matrices or boundary conditions.

| \((\mu, \nu)\)-sets | \((a_1, a_2, a_3)\)-sets of the existence of one-point interactions | \(\Lambda\)-matrices or boundary conditions |
|-----------------------|---------------------------------------------------------------|----------------------------------|
| \(Q_0\)               | \(\mathbb{R}^3\)                                             | \(\Lambda = I\)                 |
| \(L_1\)               | \(\mathbb{R}^3\)                                             | \(\Lambda\) of type (5) with \(\alpha = a_1 + a_2 + a_3\) |
| \(L_K\)               | set \(K_3\) defined by (32)                                  | \(\Lambda\) of type (8) with \(\theta\) given by (33) |
| \(L_5\)               | plane \(a_1 + a_2 + a_3 = 0\) (set \(L_3\))                 | \(\psi(\pm 0) = 0\)             |
| \(Q_1\)               | \(\mathbb{R}^3\)                                             | \(\psi(\pm 0) = 0\)             |
| \(Q_2\)               | plane \(a_1 + a_2 + a_3 = 0\) (set \(L_3\))                 | \(\Lambda = I\)                 |
| \(P_1\)               | set \(F_3\) defined by (42) and obtained from splitting set \(K_3\) | \(\Lambda\) of type (8) with \(\theta\) given by (46) |
|                      | \(\mathbb{R}^3 \setminus F_3\)                              | \(\psi(\pm 0) = 0\)             |
| \(P_2\)               | set \(F_3\) defined by (47) and obtained from splitting set \(L_3\) | \(\Lambda\) of type (48)         |
|                      | \(\mathbb{R}^3 \setminus F_3\)                              | \(\psi(\pm 0) = 0\)             |
| \(L_0\)               | set \(F_3\) defined by (47) and obtained from splitting set \(L_3\) | \(\Lambda\) of type (8) with \(\theta\) given by (50) |
|                      | \(\mathbb{R}^3 \setminus F_3\)                              | \(\psi(\pm 0) = 0\)             |
| \(L_D\)               | \(\mathbb{R}^3 \setminus P_3\)                              | \(\psi(\pm 0) = 0\)             |
| \(L_2\)               | \(\mathbb{R}^3 \setminus P_3\)                              | \(\Lambda = \pm I\)             |

7. Concluding remarks

In this paper the system consisting of two or three \(\delta\)-potentials (with intensities \(a_j \in \mathbb{R}, j = 1, 2\) if \(N = 2\) and \(j = 1, 2, 3\) if \(N = 3\)) has been approximated in the most simple way, namely by piecewise constant functions and then the convergence of the corresponding transmission matrices has been studied in the squeezing limit as both the width of \(\delta\)-approximating functions \(l\) and the distance between them \(r\) tend to zero simultaneously. The admissible rates of shrinking the parameters \(l\) and \(r\) have been controlled through power parameterization (21), involving the two powers \(\mu\) and \(\tau\) as well as the squeezing parameter \(\varepsilon \to 0\). For convenience of the presentation, we have used the diagram of admissible values for \(\mu\) and \(\tau\) plotted in figure 1. Using this parameterization as well as the piecewise constant approximation of the \(\delta\)-functions in potential (2), it was possible to get the explicit expressions for the corresponding \(\Lambda\)-matrices and to treat thus the reflection-transmission properties of the one-point interactions directly.

Starting from the same three-layer (for \(N = 2\)) and five-layer (for \(N = 3\)) potential profile given by (9), a family of limit one-point interactions with resonant-tunnelling behaviour has been realized. For the interactions realized on the line \(L_5\), the resonance sets referred to as \(K_2,3\) consist of two curves on the \((a_1, a_2)\)-plane \((N = 2)\) and three surfaces in the \((a_1, a_2, a_3)\)-space \((N = 3)\). In its turn, for the interactions realized on \(L_5 \cup Q_2\), the resonance sets called \(L_2,3\) are the line \(a_1 + a_2 = 0\) \((N = 2)\) and the plane \(a_1 + a_2 + a_3 = 0\) \((N = 3)\). When the \((\mu, \tau)\)-points
from the sets $L_K$, $L_S$ and $Q_2$ approach correspondently the right-hand limiting sets $P_1$, $P_2$ and $L_0$, the $K$- and $L$-sets split into infinite but countable sets resulting in the set transformations $K_{2,3} \rightarrow F_{2,3}$ and $L_{2,3} \rightarrow J_{2,3}$. These transformations are illustrated by figures from 2 to 5. In other words, on the limiting sets, the detachment of the countable $F$- and $J$-sets from the $K$- and $L$-sets takes place, respectively. All the point interactions analysed above together with the sets on which they are realized, including the corresponding resonance sets and $\Lambda$-matrices or boundary conditions are summarized in table 1.

In this paper we have restricted ourselves to the case of the squeezed-potential approximation of the $\delta$-potentials using parameterization (21). Within this approach, keeping things simple, we are able to (i) observe the phenomenon of splitting one-point interactions into countable families and to (ii) find the way how to ‘connect’ Šeba’s $\delta$-potential [18] and Kurasov’s $\delta'$-potential [21] with the resonant-tunnelling point interactions studied recently in a number of publications [8, 41–50]. However, it should be emphasized that the families of point interactions realised within specific parameterization (21) do not cover the whole four-parameter family of point interactions [22]. For instance, the case of the $\delta'$-interaction with the $\Lambda$-matrix of type (6), studied in the norm-resolvent sense [7], cannot be adapted to parameterization scheme (21).

In principle, a similar straightforward analysis could be carried out for higher $N$ resulting in the same types of one-point interactions with resonance sets $K_N$, $L_N$, $F_N$ and $J_N$ being $(N-1)$-dimensional hypersurfaces, however, the corresponding formulae appear to be quite complicated. To conclude, it should be noticed that the approach developed in this paper can be a starting point for further studies on regular approximations of point interactions and understanding the resonance mechanism.

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References

[1]Demkov Y N and Ostrovskii V N 1975 Zero-Range Potentials and Their Applications in Atomic Physics (Leningrad: Leningrad University Press)
[2]Demkov Y N and Ostrovskii V N 1988 Zero-Range Potentials and Their Applications in Atomic Physics (New York: Plenum)
[3]Albeverio S, Gesztesy F, Høegh-Krohn R and Holden H 2005 Solvable Models in Quantum Mechanics (With an Appendix by Pavel Exner) 2nd revised edn (Providence, RI: American Mathematical Society)
[4]Albeverio S and Kurasov P 1999 Singular Perturbations of Differential Operators: Solvable Schrödinger-Type Operators (Cambridge: Cambridge University Press)
[5]Perez J F and Coutinho F A B 1991 Am. J. Phys. 59 52
[6]Brasche J F, Figari R and Teta A 1998 Potential Anal. 8 163
[7]Exner P, Neidhardt H and Zagrebnov V A 2001 Commun. Math. Phys. 224 593
[8]Christiansen P L, Arnabak N C, Zolotaryuk A V, Ermakov V N and Gaididei Y B 2003 J. Phys. A: Math. Gen. 36 7589
[9]Coutinho F A B and Amaku M 2009 Eur. J. Phys. 30 1015
[10]Exner P and Manko S S 2014 Lett. Math. Phys. 104 1079
[11] Albeverio S, Cacciapuoti C and Finco D 2007 J. Math. Phys. 48 032103
[12] Cacciapuoti C and Exner P 2007 J. Phys. A: Math. Theor. 40 F511
[13] Turek O and Cheon T 2012 Europhys. Lett. 98 50005
[14] Turek O and Cheon T 2013 Ann. Phys., NY 330 104
[15] Zolotaryuk A V 2013 Phys. Rev. A 87 052121
[16] Zolotaryuk A V and Zolotaryuk Y 2015 Phys. Lett. A 379 511
[17] Zolotaryuk A V and Zolotaryuk Y 2015 J. Phys. A: Math. Theor. 48 035302
[18] Šeba P 1986 Rep. Math. Phys. 24 111
[19] Gesztesy F and Holden H 1987 J. Phys. A: Math. Gen. 20 5157
[20] Griffiths D J 1993 J. Phys. A: Math. Gen. 26 2265
[21] Kurasov P 1996 J. Math. Anal. Appl. 201 297
[22] Albeverio S, Dąbrowski L and Kurasov P 1998 Lett. Math. Phys. 45 33
[23] Coutinho F A B, Nogami Y and Perez J F 1997 J. Phys. A: Math. Gen. 30 3937
[24] Coutinho F A B, Nogami Y and Tomio L 1999 J. Phys. A: Math. Gen. 32 4931
[25] Albeverio S and Nizhnik L 2003 Lett. Math. Phys. 65 11
[26] Nizhnik L N 2003 J. Funct. Anal. Appl. 40 74
[27] Gadella M, Negro J and Nieto L M 2009 Phys. Lett. A 373 1310
[28] Arnbak H, Christiansen P L and Guadalupe Y B 2011 Phil. Trans. R. Soc. A 369 1228
[29] Lange R-J 2012 J. High Energy Phys. JHEP11(2012)1
[30] Brasche J F and Nizhnik L P 2013 Methods Funct. Anal. Topol. 19 4
[31] Gadella M, García-Ferrero M A, González-Martín S and Maldonado-Villamizar F H 2014 Int. J. Theor. Phys. 53 1614
[32] Lange R-J 2015 J. Math. Phys. 56 122105
[33] Zolotaryuk A V 2015 J. Phys. A: Math. Theor. 48 255304
[34] Kalinski V and Panchenko D Y 2015 Physica B 472 78
[35] Dias N C, Jorge C and Prata J N 2016 J. Differ. Equ. 260 6548
[36] Gadella M, Mateos-Guilarte J, Muñoz-Castañeda J M and Nieto L M 2016 J. Phys. A: Math. Theor. 49 015204
[37] Konno K, Nagasawa T and Takahashi R 2016 Ann. Phys., NY 375 91
[38] Kostenko A and Malamud M 2013 Spectral Analysis, Differential Equations and Mathematical Physics—Proc. of Symp. in Pure Mathematics vol 87, ed H Holden et al (Providence: RI: American Mathematical Society) p 235
[39] Zolotaryuk A V, Christiansen P L and Iermakova S V 2006 J. Phys. A: Math. Gen. 39 9329
[40] Golovaty Y D 2009 Ukr. Math. Bull. 6 169
[41] Golovaty Y D and Hryniv R O 2010 J. Phys. A: Math. Theor. 43 155204
[42] Golovaty Y D and Hryniv R O 2011 J. Phys. A: Math. Theor. 44 049802
[43] Man’ko S S 2010 J. Phys. A: Math. Theor. 43 445304
[44] Golovaty Y 2012 Methods Funct. Anal. Topol. 18 243
[45] Man’ko S S 2012 J. Math. Phys. 53 123521
[46] Golovaty Y D and Hryniv R O 2013 Proc. R. Soc. Edinburgh A 143 791
[47] Golovaty Y 2013 Integr. Equ. Oper. Theor. 75 341
[48] Zolotaryuk A V 2010 Phys. Lett. A 374 1636
[49] Cheon T and Shigehara T 1998 Phys. Lett. A 243 111
[50] Albeverio S and Nizhnik L 2000 Ukr. Mat. Zh. 52 582
[51] Albeverio S and Nizhnik L 2001 Ukr. Mat. J. 52 664 (translation)
[52] Griffiths D and Walborn S 1999 Am. J. Phys. 67 446
[53] De Vincenzo S and Sanchez C 2010 Can. J. Phys. 88 809
[54] Coutinho F A B, Nogami Y and Toyama F M 2012 Can. J. Phys. 90 383