THE TATE CONJECTURE FOR POWERS OF ORDINARY CUBIC FOURFOLDS OVER FINITE FIELDS

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Abstract. Recently N. Levin proved the Tate conjecture for ordinary cubic fourfolds over finite fields. In this paper we prove the Tate conjecture for self-products of ordinary cubic fourfolds. Our proof is based on properties of so-called polynomials of K3 type introduced by the author about a dozen years ago.

1. Introduction.

Let $X$ be a smooth projective variety over a finite field $k$ of characteristic $p$. We write $X_a$ for $X \times k(a)$ where $k(a)$ is an algebraic closure of $k$. For each non-negative integer $m$ and each rational prime $l$ different from $p$ let us consider the $2m$-th twisted $\ell$-adic cohomology group $H^{2m}(X_a, \mathbb{Q}_\ell)(m)$ of $Y_a$. The Galois group $G(k)$ of $k$ acts on $H^{2m}(X_a, \mathbb{Q}_\ell)(m)$ in a natural way. In [23] Tate conjectured that the subspace fixed under the Galois action is spanned by cohomology classes of codimension $m$ algebraic cycles on $X$. The famous conjecture of Serre and Grothendieck [23, 24, 12, 20] asserts that the action of $G(K)$ on $H^{2m}(X_a, \mathbb{Q}_\ell)(m)$ is semisimple, i.e., the Frobenius automorphism acts on $H^{2m}(X_a, \mathbb{Q}_\ell)(m)$ as a semisimple linear operator.

The Tate conjecture is known to be true in certain cases, e.g., Fermat varieties satisfying certain numerical conditions [23, 21, 26]; abelian varieties for $m = 1$ [24], certain classes of abelian varieties with arbitrary $m$ [22, 27, 28, 10, 14], “almost all” K3 surfaces [3, 16, 17].

Let $Y$ be a cubic fourfold, i.e., a smooth projective hypersurface of degree 3 in $\mathbb{P}^5$ defined over $k$. It is well-known that all odd $\ell$-adic Betti numbers of $Y_a$ do vanish; it is also known that the second and sixth Betti numbers of $Y_a$ are equal to 1 while its fourth Betti number is 23 [18]. Let $l \in H^2(Y_a, \mathbb{Q}_\ell)(1)$ be the class of a hyperplane section of $Y$. One may easily check that

$$H^2(Y_a, \mathbb{Q}_\ell)(1) = \mathbb{Q}_\ell l = H^2(Y_a, \mathbb{Q}_\ell)(1)^{G(k)}; H^6(Y_a, \mathbb{Q}_\ell)(1) = \mathbb{Q}_\ell l^3 = H^6(Y_a, \mathbb{Q}_\ell)(1)^{G(k)}.$$

It was proven by Rapoport [18] that the Frobenius automorphism acts on $H^4(X_a, \mathbb{Q}_\ell)(2)$ as a semisimple linear operator.

Recently N. Levin [11] proved the Tate conjecture when $X = Y$ is an ordinary cubic fourfold. In this case the only non-trivial case is $m = 2$ and the theorem of Levin asserts that $H^4(Y_a, \mathbb{Q}_\ell)(2)^{G(k)}$ is generated as a $\mathbb{Q}_\ell$-vector subspace by algebraic classes of codimension 2 (defined over $k$).

The aim of the present paper is to prove the Tate conjecture for self-products $X = Y^r$ of an ordinary K3 cubic fourfold $Y$. For example, if $r = 2$ and $X = Y^2$ then the most interesting case is $m = 8$ and we prove, assuming $k$ “sufficiently large”, that $H^8(X_a, \mathbb{Q}_\ell)(2)^{G(k)}$ is generated as a $\mathbb{Q}_\ell$-vector subspace by algebraic
classes of the following four types: 1) products of codimension 2 algebraic cycles on
factors \(Y\) of \(Y^2\); 2) the classes of graphs of Frobenius and its iterates; 3) the classes
of \(Y \times \{\text{point}\}\) and \(\{\text{point}\} \times Y\); 4) \(I \times I^0\) and \(I^0 \times I\). The proofs are based on results
and ideas of our previous papers \[29, 30\] where the Tate conjecture was proven for
self-products of ordinary K3 surfaces.

The article is organized as follows. In Section 2 we discuss tensor invariants
of certain \(\ell\)-adic representations of \(G(k)\). Section 3 treats cohomological \(\ell\)-adic representations. The Tate conjecture is discussed in Section 4. In Section 5 we will
prove the Tate conjecture for powers of an ordinary cubic fourfold.

We write \(\Z_+\) for the set \(\{0, 1, 2, \ldots\\}\) of non-negative integers. Recall that \(G(k)\) is procyclic
and has a canonical generator, namely, the arithmetic Frobenius automorphism
\[\sigma_k : k(a) \to k(a), x \to x^q\]
where \(q\) is the number of elements of \(k\). Clearly, \(q\) is an integral power of \(p\). Another
canonical generator of \(G(k)\) is the geometric Frobenius automorphism \(\varphi_k = \sigma_k^{-1}\).

2. Finite fields and \(\ell\)-adic representations

Let \(k\) be a finite field of characteristic \(p\) consisting of \(q\) elements. We keep all
notations of the Introduction connected with \(k\). If \(k' \subset k(a)\) is a finite overfield
of \(k\), then \(k'(a) = k(a)\), the Galois group \(G(k')\) of \(k'\) is an open subgroup of finite
index \([k' : k]\) in \(G(k)\) and \(\varphi_{k'} = \varphi_k^{[k' : k]}\).

Let \(\ell\) be a rational prime different from \(p\). We refer to \[13, 23, 29\] for the
definition of the one-dimensional \(\Q_\ell\)-vector space \(\Q_\ell(1)\) and the corresponding
cyclotomic character
\[\chi_\ell : G(k) \to \Z_+^\times \subset \Q_\ell^\times = \Aut \Q_\ell(1).\]
This character defines the Galois action on \(\Q_\ell(1)\). Notice, that
\[\chi_\ell(\sigma_k) = q, \quad \chi_\ell(\varphi_k) = q^{-1}.\]
We write \(\Q_\ell(-1)\) for the one-dimensional dual vector space \(\Hom_{\Q_\ell}(\Q_\ell(1), \Q_\ell)\) with
a natural structure of the dual Galois module defined by the character \(\chi_\ell^{-1}\). To
each integer \(i\) one may attach a certain one-dimensional \(\Q_\ell\)-vector space \(\Q_\ell(i)\) with
Galois action defined by \(\chi_\ell^i\). Namely, \(\Q_\ell(0) = \Q_\ell, \Q_\ell(i) = \Q_\ell(1)^{\otimes i}\) if \(i\) is positive
and \(\Q_\ell(i) = \Q_\ell(-1)^{\otimes (-i)}\) if \(i\) is negative.

For all integers \(i, j\) there are natural isomorphisms of Galois modules
\[\Q_\ell(-i) = \Hom_{\Q_\ell}(\Q_\ell(i), \Q_\ell), \quad \Q_\ell(i) \otimes_{\Q_\ell} \Q_\ell(j) = \Q_\ell(i + j).\]
Let
\[\rho : G(k) \to \Aut(V)\]
be an \(\ell\)-adic representation of \(G(k)\), i.e., \(V\) is a finite-dimensional \(\Q_\ell\)-vector space
and \(\rho\) is a continuous homomorphism \[19\]. Clearly,
\[V^{G(k)} = V^{\rho(\varphi_k)}\].
To each integer \(i\) one may attach the twisted \(\ell\)-adic representation
\[\rho[i] : G(k) \to \Aut(V(i))\]
where
\[V(i) = V \otimes_{\Q_\ell} \Q_\ell(i), \quad \rho[i](\sigma)(v \otimes a) = \chi_\ell^i(\sigma)((\sigma v) \otimes a).\]
For example, 
\[
\rho[i](\sigma_k)(v \otimes a) = q^i(\rho(\sigma_k)(v) \otimes a), \quad \rho[i](\varphi_k)(v \otimes a) = q^{-i}(\rho(\varphi_k)(v) \otimes a).
\]
We have \( V(0) = V, \mathbb{Q}_\ell(i)(j) = \mathbb{Q}_\ell(i+j) \). If \( \rho^* : G(K) \to \text{Aut}(V^*) \) is the dual \( \ell \)-adic representation then there are natural isomorphisms of Galois modules \( (V(i))^* = V^*(-i) \).

**Remark 2.1.** Let \( \rho' : G(k) \to \text{Aut}(W) \) be (may be, another) \( \ell \)-adic representation of \( G(K) \). Then we obtain natural isomorphisms of Galois modules 
\[
V(i) \otimes_{\mathbb{Q}_\ell} W(j) = (V \otimes_{\mathbb{Q}_\ell} W)(i+j), \quad \rho[i] \otimes \rho'[j] = (\rho \otimes \rho')(i+j),
\]

\[
\text{Hom}_{\mathbb{Q}_\ell}(V,W) = V^* \otimes_{\mathbb{Q}_\ell} W = V^*(-i) \otimes_{\mathbb{Q}_\ell} W(i)
\]

\[
= V(i)^* \otimes_{\mathbb{Q}_\ell} W(i) = \text{Hom}_{\mathbb{Q}_\ell}(V(i),W(i)).
\]

**Example 2.2.** If \( u = \rho(\sigma) \) for some \( \sigma \in G(k) \) then one may easily check that 
\[
u[i] = \chi^{-1}_i(\sigma) \rho[i](\sigma) \in \text{End}_{\mathbb{Q}_\ell}(V(i)).
\]
In particular, if \( u = \rho(\sigma_k) \) then \( u[i] = q^{-i} \rho[i](\sigma_k) \). If \( u = \rho(\varphi_k) \) then \( u[i] = q^i \rho[i](\varphi_k) \).

**Theorem 2.3** (Theorem 3.1.4 of [29]). Let us assume that the \( \ell \)-adic representation \( \rho[i] \) is semisimple and consider the characteristic polynomial 
\[
P_{\rho[i]}(t) = \det(\text{id} - t \rho[i](\varphi_k),V(i)).
\]
Let \( R \) be the set of reciprocal roots of \( P_{\rho[i]}(t) \). Assume that either 1 is the unique element of \( R \), i.e., \( R = \{1\} \) is the one-element set or there exists a non-empty subset \( B \subset R \) such that:

1. \( B \) consists of multiplicatively independent elements; in particular, it does not contain 1 and does not meet \( B^{-1} = \{\alpha^{-1} | \alpha \in B\} \);
2. Either \( R \) coincides with the disjoint union of \( B \) and \( B^{-1} \) or \( R \) coincides with the disjoint union of \( B, B^{-1} \) and the one-element set \( \{1\} \).

Then for each even natural number \( 2n \) all elements of \( (V(i)^{\otimes 2n})^{G(k)} \) can be presented as a linear combination of tensor products of \( n \) elements of \( (V(i)^{\otimes 2})^{G(k)} \). Each element of \( (V(i)^{\otimes (2n+1)})^{G(k)} \) can be presented as a linear combination of tensor products of an element of \( V(i)^{G(k)} \) and \( n \) elements of \( (V(i)^{\otimes 2})^{G(k)} \).

**Definition 2.4.** Recall [29, 3.2] that \( \rho \) is called semistable if it enjoys the following property. If \( u = \rho(\sigma) \in \text{Aut}(V) \) for some \( \sigma \in G(k) \) and an eigenvalue \( \alpha \) of \( u \) is a root of unity, then \( \alpha = 1 \). In fact, in order to make sure that \( \rho \) is semistable, it suffices to inspect the eigenvalues only for \( u = \rho(\varphi_k) \) [29, 3.2.1].

**Remark 2.5.** If \( \rho \) is semistable and \( k' \) is a finite overfield of \( k \) then the restriction of \( \rho \) to \( G(k') \) is also a semistable \( \ell \)-adic representation of \( G(k') \) and the invariants of \( G(k) \) and \( G(k') \) coincide, i.e., \( V^{G(k)} = V^{G(k')} \).

Conversely, for each \( \rho \) there exists a positive integer \( r \) such that if an eigenvalue \( \alpha \) of \( \rho(\varphi_k) \) is a root of unity then \( \alpha^r = 1 \). Now if \( k_r \subset k(\alpha) \) is the degree \( r \) extension of \( k \) then every eigenvalue \( \beta \) of \( \rho(\varphi_k) = \rho(\varphi_k)^r \) that is a root of unity is equal to 1. This means that the restriction of \( \rho \) to \( G(k_r) \) is semistable.
3. \(\ell\)-adic cohomology

Let \(Y\) be a smooth projective variety over \(k\) and \(Y_a = Y \times_k k(a)\). Let \(\ell\) be a rational prime \(\neq p\). The Galois group \(G(k)\) acts on \(Y_a = Y \times_k k(a)\) through the second factor. For each non-negative integer \(i\) this action induces by functoriality the Galois action on the \(i\)th \(\ell\)-adic étale cohomology group \(H^i(Y_a, \mathbb{Q}_\ell)\). We write \(\rho_{Y,i}\) for the corresponding cohomological \(l\)-adic representation \([20]\)

\[
\rho_{Y,i} : G(k) \to \text{Aut}(H^i(Y_a, \mathbb{Q}_\ell)).
\]

Let \(F_{Y/k} : Y \to Y\) be the Frobenius endomorphism of the \(k\)-scheme \(Y\) \([23, 13]\). It is defined as the identity map on points, together with the map \(f \mapsto f^q\) in the structure sheaf. Let \(\text{Frob}_Y = F_{Y/k} \times \text{id}_{k(a)}\) be the corresponding \(k(a)\)-endomorphism of \(Y_a\). We write \((\text{Frob}_Y)_i\) for the endomorphism of \(H^i(Y_a, \mathbb{Q}_\ell)\) induced by \(\text{Frob}_Y\).

**Remark 3.1.** If \(k' \subset k(a)\) is a finite overfield of \(k\) and \(Y' = Y \times_k k'\) is the corresponding smooth projective variety over \(k'\), then

\[
Y'_a = Y_a, \quad H^i(Y'_a, \mathbb{Q}_\ell) = H^i(Y_a, \mathbb{Q}_\ell)
\]

and \(\rho_{Y',i}\) coincides with the restriction of \(\rho_{Y,i}\) to \(G(k')\). In particular,

\[
\rho_{Y',i}(\varphi_{k'}) = \rho_{Y,i}(\varphi_k)^{[k':k]}.
\]

**Remark 3.2.** It is well-known \([23, 13]\) that

\[
(\text{Frob}_Y)_i = \rho_{Y,i}(\varphi_k).
\]

Let us consider the characteristic polynomial

\[
P_{Y,i}(t) = \det(\text{id} - t\rho_{Y,i}(\varphi_k), H^i(Y_a, \mathbb{Q}_\ell)).
\]

A famous theorem of Deligne \([4, 6]\) (conjectured by Weil) asserts that \(P_{Y,i}(t)\) lies in \(1 + t\mathbb{Z}[t]\), does not depend on the choice of \(l\) and all its (complex) reciprocal roots have absolute values equal to \(q^i/2\).

Notice, that in the case of cubic fourfolds this result was proven earlier than the general case by Rapoport \([13]\) (inspired by ideas of \([1]\)). His paper also contain the proof of semisimplicity of the action of Frobenius in the case of cubic fourfolds. For Abelian varieties the semisimplicity was proven by Weil (see \([1]\)).

Let \(i = 2m\) be an even non-negative integer. Let us consider the twisted cohomological \(l\)-adic representation

\[
\rho_{Y,2m}[m] : G(k) \to \text{Aut}(H^{2m}(Y_a, \mathbb{Q}_\ell)(m)).
\]

One may easily check \([20, 4.2]\) that

\[
P_{Y,[m]}(t) = \det(\text{id} - t\rho_{Y,2m}[m](\varphi_k), H^{2m}(Y_a, \mathbb{Q}_\ell)(m)) = P_{Y,2m}(t/q^m).
\]

Now, the theorem of Deligne implies that \(P_{Y,[m]}(t)\) lies in \(1 + t\mathbb{Z}[1/q][t]\), does not depend on the choice of \(l\) and all its (complex) reciprocal roots have absolute values equal to 1. In other words, \(P_{Y,[m]}(t)\) is a \(q\)-admissible polynomial in the sense of \([20]\). In particular, if \(\alpha\) is a (reciprocal) root of \(P_{Y,[m]}(t)\) then \(\alpha^{-1}\) is also one.

**Remark 3.3.** Clearly, there exists a positive integer \(r\) such for all roots \(\alpha\) of \(P_{Y,[m]}(t)\) we have \(\alpha^r = 1\) if \(\alpha\) is a root of unity. It follows from Remarks \([23, 4.4]\) and \([13]\) that if \(k'\) is the degree \(r\) extension \(k_r\) of \(k\) and \(Y' = Y \times_k k'\) then \(\rho_{Y',2m}[m]\) is semistable for all \(\ell \neq p\).
Let $d = \dim(Y)$. Let us consider the Künneth decomposition
\[ H^{2d}(Y \times Y, \mathbb{Q}_\ell)(d) = \bigoplus_{i=0}^{2d} H^{2d-i}(Y, \mathbb{Q}_\ell)(d) \otimes H^i(Y, \mathbb{Q}_\ell) \]
of the $d$-twisted middle $l$-adic cohomology group of $Y \times Y = (Y \times Y)^{\alpha}$. Clearly, each element $c \in H^{2d}(Y \times Y, \mathbb{Q}_\ell)(d)$ can be presented uniquely as a sum
\[ c = \bigoplus_{i=0}^{2d} c_i \]
with $c_i \in H^{2d-i}(Y, \mathbb{Q}_\ell)(d) \otimes H^i(Y, \mathbb{Q}_\ell)$. Notice that the Künneth decomposition is $G(k)$-equivariant. In particular, $c \in H^{2d}(Y \times Y, \mathbb{Q}_\ell)(d)$ is a $G(k)$-invariant if and only if all $c_i$ are $G(k)$-invariant.

**Remark 3.4.** Let $c \in H^{2d}(Y \times Y, \mathbb{Q}_\ell)(d)$ be an algebraic cohomology class, i.e., a linear combination of cohomology classes of closed irreducible codimension $d$ subvarieties on $Y \times Y$. It follows from results of [8] that all $c_i$ are also algebraic cohomology classes.

Let $u$ be a $k(a)$-endomorphism of $Y(a)$. We write $\text{Graph}_u$ for the graph of $u$; it is a $d$-dimensional irreducible closed subvariety of $Y \times Y$. We write $\text{cl}(\text{Graph}_u)$ for its $\ell$-adic cohomology class: it is an element of $H^{2d}(Y \times Y, \mathbb{Q}_\ell)(d)$. By functoriality, $u$ induces an endomorphism of $H^i(Y, \mathbb{Q}_\ell)$ which will be denoted by
\[ u_i \in \text{End}(H^i(Y, \mathbb{Q}_\ell)) = \text{End}(V_i). \]

**Example 3.5.** If $u = \text{id}$ then its graph is the diagonal $\Delta$ and therefore $u_i$ is the identity endomorphism of $H^i(Y, \mathbb{Q}_\ell)$.

**Example 3.6.** If $u = \text{Frob}_Y$ then according to [23, 32] we have
\[ u_i = (\text{Frob}_Y)_i = \rho_{Y,i}(\varphi_k). \]
For each positive $j$ the $j$th power $\text{Frob}_Y^j$ of $\text{Frob}_Y$ is defined. As usual, if $j = 0$, we put $\text{Frob}_Y^0 = \text{id}$. Let us put
\[ f_{r,i} = (\text{cl}(\text{Graph}_{\text{Frob}_Y^j}))_i. \]
It is known [7] that $f_{r,i}$ can be presented as a linear combination of $\text{cl}(\text{Graph}_{\text{Frob}_Y^j})$ with rational coefficients ($j \in \mathbb{Z}_+$). Notice, that it is well-known [23, 32] that all $\text{cl}(\text{Graph}_{\text{Frob}_Y^j})$ are $G(k)$-invariants.

**Remark 3.7.** If $d = \dim(Y) = 2m$ is even then there is the canonical isomorphism
\[ H^d(Y, \mathbb{Q}_\ell)(d) \otimes H^d(Y, \mathbb{Q}_\ell) = H^{2m}(Y, \mathbb{Q}_\ell)(m) \otimes H^{2m}(Y, \mathbb{Q}_\ell)(m). \]

**Remark 3.8.** If $d = \dim(Y) = 2m$ is even then one may easily deduce from [24, 32] and [32] that $q^m \rho_{Y, 2m}[m](\varphi_k) = ((\text{Frob}_Y)_i)[m]$, i.e.,
\[ g := \rho_{Y, 2m}[m](\varphi_k) = q^{-m}((\text{Frob}_Y)_i)[m]. \]

**Theorem 3.9** (Theorem 4.4 of [30]). Assume that $d = \dim(Y) = 2m$ is even, $g = \rho_{Y, 2m}[m](\varphi_k)$ is a semisimple linear operator and all its eigenvalues different from $1$ are simple. Then the vector subspace of Galois invariants
\[ (H^d(Y, \mathbb{Q}_\ell)(d) \otimes H^d(Y, \mathbb{Q}_\ell))_{G(k)} \subset H^{2m}(Y, \mathbb{Q}_\ell)(m) \otimes H^{2m}(Y, \mathbb{Q}_\ell)(m) \]
is generated by $(H^{2m}(Y, \mathbb{Q}_\ell)(m))_{G(k)} \otimes (H^{2m}(Y, \mathbb{Q}_\ell)(m))_{G(k)}$ and all $(\text{cl}(\text{Graph}_{\text{Frob}_Y^j}))_d$ with $j \in \mathbb{Z}_+$. In particular, $(H^d(Y, \mathbb{Q}_\ell)(d) \otimes H^d(Y, \mathbb{Q}_\ell))_{G(k)}$ is contained in the vector subspace of $H^{2d}(Y \times Y, \mathbb{Q}_\ell)(d)_{G(k)}$ generated by
\[ (H^{2m}(Y, \mathbb{Q}_\ell)(m))_{G(k)} \otimes (H^{2m}(Y, \mathbb{Q}_\ell)(m))_{G(k)}. \]
and all \( \text{cl}(\text{Graph}_{\mathbb{F}_{\text{Frob}}}^j) \) with \( j \in \mathbb{Z}_+ \).

4. The Tate conjecture

We write \( \text{Al}^m(Y) \) for the \( \mathbb{Q}_\ell \)-vector subspace of \( H^{2m}(Y, \mathbb{Q}_\ell)(m) \) spanned by the cohomology classes of all algebraic cycles of codimension \( m \) on \( Y \). It is well-known [20] that

\[
\text{Al}^m(Y) \subset H^{2m}(Y, \mathbb{Q}_\ell)(m)^{G(k)}.
\]

Elements of \( H^{2m}(Y, \mathbb{Q}_\ell)(m)^{G(k)} \) are called Tate classes on \( Y \).

Tate [19] conjectured that the following assertion holds true.

\[
T(Y, m, k, l) : \quad \text{Al}^m(Y) = H^{2m}(Y, \mathbb{Q}_\ell)(m)^{G(k)}.
\]

**Remark 4.1.** Let \( k' \) be a finite algebraic extension of \( k \) and \( Y' := Y \times_k k' \). It is known [19] that if the assertion \( T(Y', m, k', l) \) holds true then the assertion \( T(Y, m, k, l) \) also holds true.

Recall [29], that \( Y \) is called to be of K3 type in dimension \( 2m \) if the characteristic polynomial \( P_{Y,[m]}(t) \) is of K3 type, i.e. its \( p \)-adic Newton polygon [9] enjoys the following properties. There exists a non-zero rational number \( c \) such that the set of slopes is either \( \{c, -c\} \) or \( \{c, -c, 0\} \). In both cases slopes \( c \) and \( -c \) must have length 1.

For example, a K3 surface is of K3 type in dimension 2 if and only if it is ordinary [29]. An ordinary Abelian surface is of K3 type in dimension 2.

An ordinary cubic fourfold \( Y \) is of K3 type in dimension 4. Indeed, the Hodge numbers of a cubic fourfold (in dimension 4) are as follows [18].

\[
h^{4,0} = h^{0,4} = 0, h^{3,1} = h^{1,3} = 1, h^{2,2} = 21.
\]

Since the Hodge polygon of an ordinary cubic fourfold coincides with the Newton polygon, the \( p \)-adic Newton polygon of \( P_{Y,4}(t) \) admits the following description. Its set of slopes is \( \{1, 2, 3\} \); the length of both slopes 1 and 3 is 1 while the length of slope 2 is 21. This implies easily that the Newton polygon of \( P_{Y,[4]}(t) \) is of K3 type with the set of slopes \( \{-1, 0, 1\} \). (The length of its slopes 1 and \(-1\) is 1 while the length of slope 0 is 21.)

**Remark 4.2.** One may easily define motives of K3 type. The paper [7] contains examples of motives of K3 type arising from Fermat varieties.

**Theorem 4.3.** Let \( d = \dim(Y) = 2m \) be even. Assume that \( \rho_{Y,2m}[m] \) is semistable. Assume also \( Y \) is of K3 type in dimension \( 2m \). We write \( a(m, Y) \) for the multiplicity of 1 viewed as a root of \( P_{Y,[m]}(t) \). Then

\[
P_{Y,[m]}(t) = (1 - t)^{a(m, Y)} P_{Y,\text{tr}}(t)
\]

where the polynomial \( P_{Y,\text{tr}}(t) \in \mathbb{Q}[t] \) enjoys the following properties:

(i) \( P_{Y,\text{tr}}(t) \) is irreducible over \( \mathbb{Q} \);

(ii) The set \( R_{Y,\text{tr}} \) of reciprocal roots of \( P_{Y,\text{tr}}(t) \) enjoys the following properties: if \( \alpha \in R_{Y,\text{tr}} \) then \( \alpha^{-1} \) (= complex conjugate of \( \alpha \)) also belongs to \( R_{Y,\text{tr}} \). In addition, \( R_{Y,\text{tr}} \) does not contain roots of unity.

(iii) One may choose a subset \( B \subset R_{Y,\text{tr}} \) such that \( B \) does not meet \( B^{-1} := \{\alpha^{-1} | \alpha \in B\} \) and \( R_{Y,\text{tr}} \) is the (disjoint) union of \( B \) and \( B^{-1} \).

(iv) In addition, the set \( B \) consists of multiplicatively independent elements.
Proof. The semistability means that none of the roots of $P_{Y, \text{tr}}(t)$ is a root of unity. Taking into account, that all the coefficients of $P_{Y, \text{tr}}(t)$ are real, we obtain (ii). Now the assertion (iii) follows readily.

Recall that $P_{Y, \text{tr}}(t)$ is of K3 type. It follows easily that $P_{Y, \text{tr}}(t)$ is also of K3 type. Recall that none of roots of $P_{Y, \text{tr}}(t)$ is a root of unity. Now the assertions (i) and (iv) follow easily from general results about polynomials of K3 type \cite[th. 2.4.3 and 2.4.4]{20}.

**Theorem 4.4** (Theorem 5.3.1 of \cite{20}). Let $d = \dim(Y) = 2m$ be even. Assume that $\rho_{Y, 2m}[m]$ is semistable and semisimple. If $Y$ is of K3 type in dimension $2m$ then $(H^d(Y_a, \mathcal{Q}_l)(d) \otimes H^d(Y_a, \mathcal{Q}_l))^\mathbb{G}(k)$ is generated as a vector subspace of $H^{2m}(Y_a, \mathcal{Q}_l)(m) \otimes H^{2m}(Y_a, \mathcal{Q}_l)(m)$ by $(H^{2m}(Y_a, \mathcal{Q}_l)(m))^\mathbb{G}(k) \otimes (H^{2m}(Y_a, \mathcal{Q}_l)(m))^\mathbb{G}(k)$ and all $\text{cl}((\text{Graph}_{\text{Frob}})_{\mathcal{Q}_l})_d$ with $j \in \mathbb{Z}_+$. In particular, $(H^d(Y_a, \mathcal{Q}_l)(d) \otimes H^d(Y_a, \mathcal{Q}_l))^\mathbb{G}(k)$ is contained in the vector subspace of $H^{2d}(Y_a \times Y_a, \mathcal{Q}_l)(n)^\mathbb{G}(k)$ generated by

$$(H^{2m}(Y_a, \mathcal{Q}_l)(m))^\mathbb{G}(k) \otimes (H^{2m}(Y_a, \mathcal{Q}_l)(m))^\mathbb{G}(k)$$

and all $\text{cl}((\text{Graph}_{\text{Frob}})_{\mathcal{Q}_l})_d$ with $j \in \mathbb{Z}_+$.

**Corollary 4.5.** Let $Y$ be an ordinary cubic fourfold over a finite field $k$ of characteristic $p$. Then there exists a finite overfield $k' \supset k$ such that the ordinary cubic fourfold $Y' = Y \times_k k'$ over $k'$ enjoys the following properties for all primes $l \neq p$:

(i) $\rho_{Y, 4}[2]$ is semistable;

(ii) $(H^4(Y'_a, \mathcal{Q}_l)(2) \otimes \mathcal{Q}_l, H^4(Y'_a, \mathcal{Q}_l)(2))^\mathbb{G}(k')$ is contained in the $\mathcal{Q}_l$-vector subspace of $H^8(Y'_a \times Y'_a, \mathcal{Q}_l)(4)^\mathbb{G}(k')$ generated by $\mathcal{A}^2(Y') \otimes \mathcal{Q}_l, \mathcal{A}^2(Y')$ and all $\text{cl}((\text{Graph}_{\text{Frob}})_{\mathcal{Q}_l})_d$ with $j \in \mathbb{Z}_+$.

(iii) Let $I \in H^2(Y'_a, \mathcal{Q}_l)(1)$ be the class of a hyperplane section. Let $a$ be an effective 0-cycle on $Y'$. (For instance, if $Y'(k') = Y(k')$ is non-empty then one may take as any $k'$-rational point on $Y'$.) Then $H^8(Y'_a \times Y'_a, \mathcal{Q}_l)(4)^\mathbb{G}(k')$ is generated as a $\mathcal{Q}_l$-vector subspace by $\mathcal{A}^2(Y'_a) \otimes \mathcal{Q}_l, \mathcal{A}^2(Y'_a) \otimes I, I \otimes I, \mathcal{A}^2(Y'_a) \otimes I, \text{cl}((\text{Graph}_{\text{Frob}})_{\mathcal{Q}_l})_d(j \in \mathbb{Z}_+)$ and the classes of $Y' \times a$ and $a \times Y'$. In particular, the Tate conjecture holds true for for $Y' \times Y'$ and therefore also for $Y \times Y'$.

Proof. Let us choose a finite overfield $k'$ of $k$ such that all $\rho_{Y', 4}[2]$ are semistable: its existence follows from Remark \cite{33}.

It was already mentioned that, thanks to the theorem of Rapoport \cite{43}, $\rho_{Y', 4}[2]$ is semisimple. Since, thanks to the theorem of Levine \cite{11}, $H^4(Y'_a, \mathcal{Q}_l)(2)^\mathbb{G}(k') = \mathcal{A}^2(Y')$, the assertion (ii) follows from Theorem \cite{43}.

The assertion (iii) follows from (ii) combined with the Galois-invariance of the Künneth decomposition for $H^8(Y'_a \times Y'_a, \mathcal{Q}_l)(2)$ and obvious equalities

$$H^8(Y'_a, \mathcal{Q}_l)(4) = H^8(Y'_a, \mathcal{Q}_l)(4)^\mathbb{G}(k') = \mathcal{Q}_l \cdot \text{cl}(a),$$

$$H^2(Y'_a, \mathcal{Q}_l)(1) = H^2(Y'_a, \mathcal{Q}_l)(1)^\mathbb{G}(k') = \mathcal{Q}_l \cdot I,$$

$$H^6(Y'_a, \mathcal{Q}_l)(3) = H^6(Y'_a, \mathcal{Q}_l)(3)^\mathbb{G}(k') = \mathcal{Q}_l \cdot I^3$$

(where $\text{cl}(a)$ is the cohomology class of $a$).

**Remark 4.6.** If $Y$ is an ordinary cubic fourfold over a finite field $k$ then the already mentioned results of Levin and Rapoport imply that

$$\mathcal{A}^2(Y) = \text{Num}_2(Y) \otimes \mathcal{Q}_l$$
where $\text{Num}_2(Y)$ is the group of numerical equivalence classes of cycles of codimension 2 on $Y$. In particular, the rank of $\text{Num}_2(Y)$ coincides with the multiplicity of 1 viewed as a root of $P_{Y,2}(t)$ \[24\] Th. 2.9, pp. 74–75. Clearly, if $\rho_{Y,4}[2]$ if semistable then the rank of $\text{Num}_2(Y)$ is $a(2,Y) = 23 - \deg(P_{Y,\text{tr}}(t))$.

**Corollary 4.7.** Let $Y$ and $Z$ be ordinary cubic fourfolds over a finite field $k$ of characteristic $p$, enjoying the following properties:

(i) $\rho_{Y,4}[2]$ and $\rho_{Z,4}[2]$ are semistable;
(ii) $a(2,Y) \neq a(2,Z)$.

Let $l_Y \in H^2(Y,\mathbb{Q}_l)(1)$ and $l_Z \in H^2(Z,\mathbb{Q}_l)(1)$ are classes of hyperplane sections of $Y$ and $Z$ respectively. Let $a_Y$ and $a_Z$ be effective zero cycles on $Y$ and $Z$ respectively.

Let us put $W = Y \times Z$. Then $(H^8(W,\mathbb{Q}_l)(4))^{G(k)}$ is generated as a $\mathbb{Q}_l$-vector subspace by $\text{Al}_2(Y_a) \otimes \mathbb{Q}_l, \text{Al}_2(Z_a), I_Y \otimes l_Y^3, I_Z \otimes l_Z$ and the classes of $Y \times a_Z$ and $a_Y \times Z$. In particular, the Tate conjecture holds true for for $Y \times Z$.

**Proof.** It suffices to check that
\[
(H^4(Y_a,\mathbb{Q}_l)(2) \otimes H^4(Y_t,\mathbb{Q}_l)(2))^{G(k)} = (H^4(Y_a,\mathbb{Q}_l)(2) \otimes (H^4(Y_a,\mathbb{Q}_l)(2)))^{G(k)}.
\]
In order to do that it suffices to check that if $\alpha$ is a root of $P_{Y,2}(t)$ and $\beta$ is a root of $P_{Z,2}(t)$ then $\alpha \beta = 1$ if and only if $\alpha = 1, \beta = 1$.

Let us prove it. Suppose $\alpha \neq 1$ and $\beta \neq 1$. Then $\alpha$ and $\beta^{-1}$ are roots of $P_{Y,\text{tr}}(t)$ and $P_{Z,\text{tr}}$ respectively. But $P_{Y,\text{tr}}(t)$ and $P_{Z,\text{tr}}$ are $\mathbb{Q}$-irreducible polynomials with different degrees and therefore cannot have common roots. Hence $\alpha \neq \beta^{-1}$, i.e., $\alpha \beta \neq 1$. \hfill \square

**Remark 4.8.** Similar arguments prove the Tate conjecture for $Y \times S$ where $S$ is an ordinary K3 surface over $k$ with semistable $\rho_{Y,2}[1]$ and $a(2,Y) \neq a(1,S) + 1$. (The second Betti number of a K3 surface is 22 while the fourth Betti number of a cubic fourfold is 23.) Also by the same token one may prove the Tate conjecture for the product of two ordinary K3 surfaces with different Picard numbers.

5. **Powers of fourfolds**

**Theorem 5.1.** Let $Y$ be a smooth geometrically irreducible 4-dimensional projective variety over $k$ such that the first and third Betti numbers of $Y_a$ are zero and the second Betti number of $Y_a$ is 1. Let us assume that $\rho_{Y,4}[2]$ is semisimple and semistable. Assume, in addition, that $Y$ is of K3 type in dimension 4. Let $r > 1$ be an integer and let us put $X := Y^r$. Then each cohomology class in $(H^{2m}(X_a,\mathbb{Q}_l)(m))^{G(k)}$ can be presented as a linear combination of products of pullbacks of Tate classes on $Y$ and $Y^2$ with respect to the projection maps $X = Y^r \to Y, X = Y^r \to Y^2$. In particular, if for some prime $\ell$ the Tate conjecture holds true for $Y$ and $Y^2$ then it is also true for $X$ with the same $\ell$.

**Corollary 5.2.** Let $Y$ be an ordinary cubic fourfold over a finite field $k$. Let $r > 1$ be an integer and let us put $X := Y^r$. Then the Tate conjecture holds true for $X$.

**Proof of Corollary 5.2.** According to Corollary 4.7 there exists a finite overfield $k' \supset k$ such that if $Y' = X \times k'$ then $\rho_{Y',4}[2]$ is semistable and the Tate conjecture is true for $Y'^2$. Recall [11] that the Tate conjecture is valid for $Y'$. Applying Theorem 5.1 to $Y'/k'$, we conclude that the Tate conjecture is valid for $Y'^r$. Since
Proof of Theorem 5.1. Let $l \in H^2(Y_a, \mathbb{Q}_l)$ be the class of a hyperplane section of $Y$. Clearly,

\[
\begin{align*}
H^8(Y_a', \mathbb{Q}_l)(4) &= (H^8(Y_a', \mathbb{Q}_l)(4))^G(k') = \mathbb{Q}_l \cdot \text{cl}(a), \\
H^2(Y_a', \mathbb{Q}_l)(1) &= (H^2(Y_a', \mathbb{Q}_l)(1))^G(k') = \mathbb{Q}_l \cdot l, \\
H^6(Y_a', \mathbb{Q}_l)(3) &= (H^6(Y_a', \mathbb{Q}_l)(3))^G(k') = \mathbb{Q}_l \cdot l^3
\end{align*}
\]

(where $\text{cl}(a)$ is the cohomology class of an effective 0-cycle $a$) on $Y$. In particular, the cohomology spaces $H^2(Y_a', \mathbb{Q}_l)(1), H^6(Y_a', \mathbb{Q}_l)(3), H^8(Y_a', \mathbb{Q}_l)(4)$ consist of Tate classes.

We say that $c \in H^{2m}(X_a, \mathbb{Q}_l)(m)$ is a decomposable cohomology class if it can be presented as a linear combination of products of pullbacks of Tate classes on $Y$ and $Y^2$ with respect to the projection maps $X = Y^r \to Y$, $X = Y^r \to Y^2$.

Clearly, linear combinations and $\cup$–products of decomposable cohomology classes are also decomposable ones.

Let $r' < r$ be a positive integer, $Y^r \to Y^{r'}$ any projection map. If $c \in H^{2m}(Y_a', \mathbb{Q}_l)(m)$ is a decomposable cohomology class on $Y_a'$, then its pullback is a decomposable cohomology class in $H^{2m}(Y_a'^r, \mathbb{Q}_l)(m) = H^{2m}(X_a, \mathbb{Q}_l)(m)$. If $r = 1$ or $r = 2$ then each Tate class on $X_a = Y_a'^r$ is decomposable by obvious reasons. Clearly, in order to prove Theorem 5.1 we have to check that each Galois-invariant cohomology class in $H^{2m}(X_a, \mathbb{Q}_l)(m)$ is decomposable.

Let us look more thoroughly at the cohomology of $X_a = Y_a'^r$. First, notice that the Künneth formula combined with Poincaré duality implies that under our assumptions all odd-dimensional cohomology groups of $X_a$ vanish. In order to describe explicitly the even-dimensional cohomology groups of $X_a$ let us fix a non-negative integer $m$ and consider the set $\mathcal{M}(r, m)$ of maps

\[
j : \{1, 2, \ldots, r\} \to \{0, 1, 2, 3, 4\} \quad \text{with} \quad \sum_{i=1}^{r} j(i) = m.
\]

Then the Künneth formula for $X_a = Y_a'^r$ implies easily that

\[
H^{2m}(X_a, \mathbb{Q}_l) = \bigoplus_{j \in \mathcal{M}(r, m)} \otimes_{i=1}^{r} H^{2j(i)}(Y_a, \mathbb{Q}_l).
\]

After the proper twist we obtain a canonical isomorphism of Galois modules

\[
H^{2m}(X_a, \mathbb{Q}_l)(m) = \bigoplus_{j \in \mathcal{M}(r, m)} \otimes_{i=1}^{r} H^{2j(i)}(Y_a, \mathbb{Q}_l)(j(i))
\]

compatible with $\cup$–products. In particular,

\[
(H^{2m}(X_a, \mathbb{Q}_l)(m))^G(k) = \sum_{j \in \mathcal{M}(r, m)} H_j^{G(k)} \quad \text{where} \quad H_j := \otimes_{i=1}^{r} H^{2j(i)}(Y_a, \mathbb{Q}_l)(j(i)).
\]

The symmetric group $S_r$ of permutation in $r$ letters acts on $X = Y^r$ in a natural way. By factoriality, it acts on $H^{2m}(X_a, \mathbb{Q}_l)(m)$ and this action commutes with Galois action. Clearly, if $s \in S_r$ and $c \in H^{2m}(X_a, \mathbb{Q}_l)(m)$ then the cohomology class $c$ is decomposable if and only if $s^*c$ is decomposable. Notice also that

\[
s^*H_j = H_{js^{-1}} \quad \forall j \in \mathcal{M}(r, m)
\]
with the map $js^{-1} : \{1, 2, \ldots, r\} \to \{0, 1, 2, 3, 4\}$, $js^{-1}(i) := j(s^{-1}(i))$. Of course, the latter formula defines an obvious action of $S_r$ on $\mathcal{M}(r, m)$. It follows that $s^*(H_j^{G(k)}) = (H_{js^{-1}})^{G(k)}$ for all $j \in \mathcal{M}(r, m)$; in particular, all Galois-invariant cohomology classes in $H_j$ are decomposable if and only if all Galois-invariant cohomology classes in $H_{js^{-1}}$ are decomposable. This implies that in order to prove Theorem 5.1 it suffices to check that all Galois-invariant cohomology classes in $H_j$ are decomposable for each non-decreasing maps $j : \{1, 2, \ldots, r\} \to \{0, 1, 2, 3, 4\}$ from $\mathcal{M}(r, m)$.

So, Theorem 5.1 follows from the following assertion.

**Lemma 5.3.** Let $j$ be a non-decreasing map

$$j : \{1, 2, \ldots, r\} \to \{0, 1, 2, 3, 4\} \quad \text{with} \quad \sum_{i=1}^{r} j(i) = m.$$ 

Let $H_j = \otimes_{i=1}^{r} H^{2j(i)}(Y_a, Q_r)(j(i))$ be the correspondent Künneth chunk of $H^{2m}(X_a, Q_r)(m) = H^{2m}(Y_r, Q_r)(m)$. Then each Galois-invariant cohomology class in $H_j$ is decomposable.

**Proof of Lemma 5.3.** We use induction by $r$. We already know that the Lemma is true for $r = 1$ and $r = 2$. So, we may assume that $r > 2$.

**Case 1.** Assume that $j(1) < 2$, i.e., $j(1) = 0$ or $1$. Let us consider the the projection map $\phi_1 : X \to Y$ on the first factor, the projection map $\phi : X \to Y$ on the product of last $(r - 1)$ factors and a non-decreasing map

$$j' : \{1, 2, \ldots, r - 1\} \to \{0, 1, 2, 3, 4\}, \quad j'(i) := j(i + 1) \quad \text{with} \quad \sum_{i=1}^{r-1} j'(i) = m - j(1).$$

Notice that $H^{2j(1)}(Y_a, Q_r)(j(1))$ consists of Tate classes and therefore $\phi_1^*(H^{2j(1)}(Y_a, Q_r)(j(1)))$ consists of decomposable classes. Clearly,

$$H_j = H^{2j(1)}(Y_a, Q_r)(j(1)) \otimes \otimes_{i=2}^{r} H^{2j(i)}(Y_a, Q_r)(j(i))$$

$$= H^{2j(1)}(Y_a, Q_r)(j(1)) \otimes H_{j'} = \phi^*(H_{j'}),$$

$$H_j^{G(k)} = H^{2j(1)}(Y_a, Q_r)(j(1)) \otimes (H_{j'})^{G(k)} = \phi^*((H_{j'})^{G(k)})$$

where $H_{j'} = \otimes_{i=1}^{r-1} H^{2j'(i)}(Y_a, Q_r)(j'(i))$ is a Künneth chunk of $H^{2m-2j(1)}(Y_a, Q_r)(m-j(1))$. By induction assumption, all cohomology classes in $(H_{j'})^{G(k)}$ are decomposable and, therefore, their pullbacks with respect to $\phi$ are also decomposable. Recall that $\phi_1^*(H^{2j(1)}(Y_a, Q_r)(j(1)))$ consists of decomposable classes. This ends the proof of the Lemma in the case of $j(1) < 2$.

**Case 2.** Assume that $j(r) > 2$, i.e. $j(r) = 3$ or $4$. Let us consider the projection map $\phi : X \to Y$ on the product of first $(r - 1)$ factors, the projection map $\phi_r : X \to Y$ on the last factor and a non-decreasing map

$$j' : \{1, 2, \ldots, r - 1\} \to \{0, 1, 2, 3, 4\}, \quad j'(i) := j(i) \quad \text{with} \quad \sum_{i=1}^{r-1} j'(i) = m - j(r).$$
Notice that \(H^{2j(r)}(Y_a, Q_r)(j(r))\) consists of Tate classes and therefore \(\phi^r_*(H^{2j(r)}(Y_a, Q_r)(2j(r)))\) consists of decomposable classes. This implies that
\[
H_j = \bigotimes_{r=1}^{r-1} H^{2j(i)}(Y_a, Q_r)(j(i)) \otimes H^{2j(i)}(Y_a, Q_i)(j(r))
\]
\[
= \phi^r_*(H_{j'}) \otimes \phi^r_*(H^{2j(r)}(Y_a, Q_r)(2j(r))),
\]
\[
H_j^{G(k)} = (H_{j'})^{G(k)} \otimes H^{2j(r)}(Y_a, Q_r)(4) = \phi^r_*(((H_{j'})^{G(k)}) \otimes \phi^r_*(H^{2j(r)}(Y_a, Q_r)(j(r))
\]
where \(H_{j'} = \bigotimes_{r=1}^{r-1} H^{2j'(i)}(Y_a, Q_r)(j'(i))\) is a Künneth chunk of \(H^{2m-2j(r)}(Y_a, Q_r)(m-j(r))\). By induction assumption, all cohomology classes in \((H_{j'})^{G(k)}\) are decomposable and, therefore, their pullbacks with respect to \(\phi\) are also decomposable. Recall that \(\phi^r_*(H^{2j(r)}(Y_a, Q_r)(2j(r)))\) consists of decomposable classes. This ends the proof of the Lemma in the case of \(j(r) > 2\).

\textbf{Case 3.} Assume that \(j(1) > 1\) and \(j(r) < 3\). Since \(j\) is non-decreasing and may take on only values \(0, 1, 2, 3, 4\), this implies that \(j(i) = 2\) for all \(i\) and, therefore,
\[
H_j = \bigotimes_{i=1}^{r-1} H^{2j(i)}(Y_a, Q_r)(j(i)) = \bigotimes_{i=1}^{r-1} H^4(Y_a, Q_i)(2).
\]
So, we have to prove that all cohomology classes in
\[
H_j^{G(k)} = (\bigotimes_{i=1}^{r-1} H^4(Y_a, Q_i)(2))^{G(k)}
\]
are decomposable. Notice that semistability of \(\rho_{Y,2}[1]\) implies that each (reciprocal) root of \(P_{Y,2}(t)\) which is a root of unity must be equal to \(1\). Now, the decomposability property of elements of \((\bigotimes_{i=1}^{r-1} H^2(Y_a, Q_i)(1))^{G(k)}\) follows from the combination of Theorem 4.3 applied to \(Y\) and \(m = 2\) and Theorem 2.3 applied to \(V = H^4(Y_a, Q_i)\), \(\rho = \rho_{Y,2}\) and \(i = 2\).

This ends the proof of Theorem 5.1.

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