Limits Under Conjugacy of the Diagonal Subgroup in $SL_n(\mathbb{R})$

December 18, 2014

Abstract

We give a quadratic lower bound on the dimension of the space of conjugacy classes of subgroups of $SL_n(\mathbb{R})$ that are limits under conjugacy of the diagonal subgroup. We give the first explicit examples of abelian $n-1$ dimensional subgroups of $SL_n(\mathbb{R})$ which are not such a limit, however all such abelian groups are limits of the diagonal group iff $n \leq 4$.

1 Introduction

Let $G$ be a Lie group and $H$ a closed subgroup. A sequence of closed subgroups $H_n \leq G$ converges to $H$ in the Chabauty topology if the following two conditions are satisfied:

(a) For every $h \in H$ there is a sequence $h_n \in H_n$ converging to $h$
(b) For every sequence $h_n \in H_n$, if there is a subsequence which converges to $h$, then $h \in H$.

A subgroup $L \leq G$ is called a conjugacy limit of a subgroup $H$, if there is a sequence of conjugating matrices, $P_n$, such that $P_nHP_n^{-1}$ converges to $L$.

Let $C \leq SL_n(\mathbb{R})$ be the group of diagonal matrices, a Cartan subgroup. The conjugacy limits of $C$ are classified for $n \leq 4$, in [4], [7], [8]. It is an open problem to classify the conjugacy limits of $C$ when $n > 4$.

The set of all closed subgroups of a group is a Hausdorff topological space with the Chabauty topology on closed sets: [2], [3]. Following notation in [6], the set of all closed abelian subgroups $\hat{Ab}(n) = \{G \leq SL_n(\mathbb{R}) : G \cong (\mathbb{R}^{n-1}, +)\}$, is then a subspace, as is the set of conjugacy limit groups $\hat{Red}(n) = \{G \leq SL_n(\mathbb{R}) : G$ is a limit of $C\}$. Taking the quotients by conjugacy, we have two topological spaces with the quotient topology: $\hat{Ab}(n) = \hat{Ab}(n)/$conjugacy and $\hat{Red}(n) = \hat{Red}(n)/$conjugacy. In general these are not Hausdorff.

Since every conjugacy limit of $C$ is isomorphic to $\mathbb{R}^{n-1}$, we have $\hat{Red}(n) \subset \hat{Ab}(n)$, see [6]. From [4], [7], and [8], we know $\hat{Ab}(3) = \hat{Red}(3)$, which has 5 points corresponding to 5 conjugacy classes of groups, and $\hat{Ab}(4) = \hat{Red}(4)$, which has 15 points. When $n \leq 6$, Suprenko and Tyshkevitch, [11], have classified maximal commutative nilpotent (i.e. $ad_x$ is nilpotent for all $x \in X$) subalgebras of $\mathfrak{sl}_n(\mathbb{C})$. Their results imply $\hat{Ab}(5)$ has finitely many points, so $\hat{Red}(5)$ has finitely many points. Iliev and Manivel, [6], ask if $\hat{Red}(n)$ is finite when $n \geq 6$ (question C). The answer follows for $n \geq 7$ from our main result:

Theorem 1. $\frac{n^2 - 6n}{8} \leq \dim \hat{Red}(n) \leq n^2 - n$.

The upper bound is given in [6]. This leaves the case $n = 6$ open. Haettel, and Iliev and Manivel show $\dim \hat{Red}(n) < \dim \hat{Ab}(n)$ for $n > 6$. We also give the first explicit examples of elements of $\hat{Ab}(n) - \hat{Red}(n)$ for $n = 5, 6, 8$ by describing certain properties of limit groups. In particular, we show

Theorem 2. If $n \leq 4$, then $\hat{Ab}(n) = \hat{Red}(n)$. If $n \geq 5$, then $\hat{Red}(n) \subset \hat{Ab}(n)$.

2 A Family of Conjugacy Limit Groups

In this section, we define a family of groups, $L_T$, and show each is a conjugacy limit of the Cartan subgroup.
Definition 3. Let $T$ be an $m$ by $n$ matrix, and $\rho_T : \mathbb{R}^{m+n} \to SL_{m+n+1}(\mathbb{R})$ be the homomorphism given by

$$
\rho_T : (a_1, ..., a_m, b_1, ..., b_n) \mapsto \begin{pmatrix}
1 & 0 & \ldots & 0 & T_{11}a_1 & T_{12}a_1 & \ldots & T_{1n}a_1 \\
0 & 1 & \ldots & 0 & T_{21}a_2 & T_{22}a_2 & \ldots & T_{2n}a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & T_{m1}a_m & T_{m2}a_m & \ldots & T_{mn}a_m \\
0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1
\end{pmatrix}_{(m+1) \times (m+1)}.
$$

The image of $\rho_T$ is a group, $L_T \leq SL_{m+n+1}(\mathbb{R})$.

One may easily check that $\rho_T$ is a homomorphism and $L_T$ is a group, since matrix multiplication is given by

$$
\begin{pmatrix}
I & P \\
I & Q
\end{pmatrix} \begin{pmatrix}
I & P \\
I & Q
\end{pmatrix} = \begin{pmatrix}
I & P+Q \\
I & P+Q
\end{pmatrix}.
$$

Lemma 4. For any $m$ by $n$ matrix $T$, with at least one nonzero entry in every row, the group $L_T$ is a conjugacy limit of the diagonal Cartan subgroup.

Proof. Let $C = \text{diag}(x_1, x_2, ..., x_{m+n+1}) \leq SL_{m+n+1}(\mathbb{R})$, be the diagonal Cartan subgroup, so $x_1 \cdot x_2 \cdot \ldots \cdot x_{m+n+1} = 1$. Let $\{P_r\}_{r=0}^{\infty}$ be the sequence of matrices

$$
P_r = \begin{pmatrix}
1 & 0 & \ldots & 0 & T_{11}r & T_{12}r & \ldots & T_{1n}r \\
0 & 1 & \ldots & 0 & T_{21}r & T_{22}r & \ldots & T_{2n}r \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & T_{m1}r & T_{m2}r & \ldots & T_{mn}r \\
0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1
\end{pmatrix}
$$

Conjugating $C$ by $P_r$, we have $P_mCP_r^{-1} =$
Assume for simplicity that the entries in the first column of $T$ are non-zero. Given an element $l_T \in L_T$, we will find a sequence of elements in $P_r C P_{r}^{-1}$ which converges to $l_T$. Then the definition of convergence implies that the entire group $P_r C P_{r}^{-1}$ converges to $L_T$.

Given $x_{m+1}$ for $1 \leq i \leq n$ define

$$x_{m+1+i} = -r^{-2}b_i + x_{m+1}.$$ 

This ensures row $m + 1$ of $l_T$ and of $P_r C P_{r}^{-1}$ are equal since

$$r^2(x_{m+1} - x_{m+1+i}) = b_i.$$ 

(1)

For $i \leq m$ we define $x_i$ in terms of $x_{m+1}$ by

$$x_i = r^{-1}a_i - r^{-2}b_1 + x_{m+1}.$$ 

It follows that column $m + 2$ of $l_T$ and of $P_r C P_{r}^{-1}$ are equal because

$$x_i - x_{m+2} = (r^{-1}a_i - r^{-2}b_1 + x_{m+1}) - (-r^{-2}b_1 + x_{m+1}) = r^{-1}a_i.$$ 

(2)

The determinant condition $x_1 \cdot \cdots x_{m+n+1} = 1$ determines $x_{m+1}$. Observe that $x_i \to x_{m+1}$ as $r \to \infty$, so the determinant is approximately $(x_{m+1})^{m+n+1}$. Thus every $x_i \to 1$.

We have now determined $x_i$ for $1 \leq i \leq m + n + 1$. It remains to show convergence in the remainder of the entries. Using equation (1) since $r \to \infty$,

$$r(x_{m+1} - x_{m+1+i}) \to 0.$$ 

By taking the difference of any two of these terms, we see

$$r(x_{m+1+j} - x_{m+1+k}) \to 0,$$

and, in particular

$$r(x_{m+2} - x_{m+1+k}) \to 0.$$ 

(3)

Consider the $(j, m + 1 + k)$ entry, for $1 \leq j, k \leq n$. Using (2) and (3), we have

$$T_{jk}a_j - T_{jk}0 = T_{jk}a_j.$$ 

This completes the proof when the entries in the first column of $T$ are non-zero. Suppose some entries in the first column of $T$ are zero. By hypothesis, $T$ has a nonzero entry in every row, say $T_{jk}$. Pick $x_i$ for $1 \leq i \leq m$ so that $T_{jk}m(x_j - x_{m+1+k}) \to a_j T_{jk}$. Since $T_{jk} \neq 0$, we proceed as in the rest of the proof. Thus we have found a sequence diag$(x_1, \ldots, x_{m+n+1})$ such that $P_r C P_{r}^{-1} \to l_T$.

This shows $L_T$ is contained in the limit of $P_r C P_{r}^{-1}$. For dimension reasons (see proposition 3.1 in [3]) $L_T$ is the limit.

### 3 A Continuum of Conjugacy Classes of Limit Groups in $SL_7(\mathbb{R})$

In this section we find some conjugacy invariants of the group $L_T$ and use them to produce a family of conjugacy classes of dimension at least $(n^2 - 6n)/8$. We first illustrate this when $n = 7$.

A subgroup $G \leq SL_n(\mathbb{R})$ acts on $\mathbb{R}P^{n-1}$. The orbit of a point, $x \in \mathbb{R}P^{n-1}$ is $G.x = \{ y \in \mathbb{R}P^{n-1} : g.x = y \}$ for some $g \in G$. Denote by $G.x$ the orbit closure of $x$.

**Lemma 5.** If $G, H \leq SL_n(\mathbb{R})$ are conjugate by a similarity matrix $Q$, then $Q$ is a projective transformation taking the orbit closures of $G$ to the orbit closures of $H$. 


Proof. Suppose $G = QHQ^{-1}$, where $H \leq SL_n(\mathbb{R})$. View $Q$ as a projective transformation $Q : \mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^{n-1}$. We claim $Q$ maps the orbit closures of $G$ to the orbit closures of $H$. Let $X$ be a projective subspace that is an orbit closure of $G$. So $G.x \in X$ for all $x \in X$, and given $x, y \in X$, there exists $g \in G$ such that $g.x = y$. Now, for any $g \in G$, we have $(QgQ^{-1}).Q(x) \in Q(X)$ for all $Q(x) \in Q(X)$, so $(QgQ^{-1}).Q(x) \in Q(X)$ for all $Q(x) \in Q(X)$. Finally, for any $Q(x), Q(y) \in Q(X)$, there exists $g \in G$ such that $QgQ^{-1}.Q(x) = Q(y)$. Thus $Q(X)$ is an orbit closure of $H$. 

Let $G \leq SL_n(\mathbb{R})$. Define a function $R_G : \mathbb{R}P^{n-1} \rightarrow \mathbb{N}$ by $R_G(x) = \text{dim}(G.x)$. As a corollary of lemma 5, $R_G(Q(x)) = R_G(QgQ^{-1}(x))$ for all $x \in \mathbb{R}P^{n-1}$.

Next we define some conjugacy invariants of the action of a group on $\mathbb{R}P^{n-1}$. To do this we need an invariant, the unordered generalized cross ratio, of a collection of points in general position in projective space, which generalizes the cross ratio of 4 points on a projective line. This invariant is a finite subset of a product of projective spaces. Let $P(S)$ denote the power set of $S$.

Let $\{e_1, ..., e_n\}$ be the standard basis in $\mathbb{R}^n$. The standard projective basis in $\mathbb{R}P^{n-1}$ is $\{(e_1, ..., [e_n], [e_1 + \cdots + e_n]\}$, and an augmented basis in $\mathbb{R}P^{n-1}$ is a set of $m \geq n + 2$ points in general position.

Definition 6. 1. The ordered generalized cross ratio is the function, $C : (\mathbb{R}P^{n-1})^m \rightarrow (\mathbb{R}P^{n-1})^{m-(n+1)}$ defined as follows. Given any (ordered) augmented basis $(y_1, y_2, ..., y_m)$ in $\mathbb{R}P^{n-1}$, there is a unique projective transformation with the property, $C : (y_1, ..., y_n) \rightarrow ([e_1], [e_n], [e_1 + \cdots + e_n])$. Define $C(y_1, y_2, ..., y_m) := (Q(y_{m-(n+1)}), Q(y_{m+1-(n+1)}), ..., Q(y_m))$.

2. Given an (unordered) augmented basis in $\mathbb{R}P^{n-1}$, the unordered generalized cross ratio, $UC : (\mathbb{R}P^{n-1})^m \rightarrow P((\mathbb{R}P^{n-1})^{m-(n+1)})$ is the set of all generalized cross ratio tuples, $UC(y_1, ..., y_m) := \{C(y_{\sigma(1)}, ..., y_{\sigma(m)}) : \sigma \in S_m\}$.

Proposition 7. Let $\{y_1, ..., y_m\}$ and $\{x_1, ..., x_m\}$ be unordered augmented bases in $\mathbb{R}P^{n-1}$, so $m \geq n + 2$. Then $UC(y_1, ..., y_m) = UC(x_1, ..., x_m)$, if and only if there is a projective transformation, $Q : \mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^{n-1}$, such that $Q : \{y_1, ..., y_m\} \rightarrow \{x_1, ..., x_m\}$.

Proof. Suppose $UC(y_1, ..., y_m) = UC(x_1, ..., x_m)$. For the generalized cross ratio tuple coming from the identity permutation, $C(x_1, ..., x_m) \in UC(x_1, ..., x_m)$, there is some reordering, $\sigma \in S_m$ such that $C(x_1, ..., x_m) = \sigma(C(y_{\sigma(1)}, ..., y_{\sigma(m)})).$ That is, there exist projective transformations $Q_1, Q_2 : \mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^{n-1}$ such that $Q_1 : (x_1, ..., x_{n+1}) \rightarrow ([e_1], [e_n], [e_1 + \cdots + e_n])$ and $Q_2 : (y_{\sigma(1)}, ..., y_{\sigma(n+1)}) \rightarrow ([e_1], [e_n], [e_1 + \cdots + e_n])$, and also $Q_1(x_{n+1+i}) = z_i = Q_2(y_{\sigma(n+1+i)})$, for $1 \leq i \leq m - (n + 1)$. Set $Q := Q_2^{-1}Q_1$, so $Q$ is a projective transformation such that $Q : (x_1, ..., x_m) \rightarrow (y_{\sigma(1)}, ..., y_{\sigma(m)}).

Suppose there exists a projective transformation $Q_0 : \mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^{n-1}$ such that $Q_0 : \{x_1, ..., x_m\} \rightarrow \{y_1, ..., y_m\}$. Recall $UC(x_1, ..., x_m) = \{C(x_{\sigma(1)}, ..., x_{\sigma(m)}) : \sigma \in S_m\}$. Set $Q_1 : \mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^{n-1}$ to be the unique projective transformation such that $Q_1 : (x_{\sigma(1)}, ..., x_{\sigma(n+1)}) \rightarrow ([e_1], [e_n], [e_1 + \cdots + e_n])$. Then $UC(x_1, ..., x_m) = \{Q_1(x_{\sigma(m)}) : \sigma \in S_m\}$. Since $Q_1Q_2^{-1} : (y_{\sigma(1)}, ..., y_{\sigma(n+1)}) \rightarrow ([e_1], [e_n], [e_1 + \cdots + e_n])$, and any such projective transformation is unique, we have $UC(y_1, ..., y_m) = \{Q_1Q_2^{-1}(y_{\sigma(m)}) : \sigma \in S_m\} = UC(x_1, ..., x_m)$.

Proposition 7 shows that unordered cross ratio of an unordered augmented basis is a complete projective invariant. The cross ratio on $\mathbb{R}P^1$ is a special case of the ordered generalized cross ratio. We will compute a motivating example in $SL_2(\mathbb{R})$ to show $Red(7)$ contains a subspace homeomorphic to an interval.

Definition 8. Let $\alpha \in \mathbb{R} - \{0, 1, 2\}$ be fixed, and let $\rho_\alpha : \mathbb{R}^6 \rightarrow SL_2(\mathbb{R})$ be the homomorphism defined by

$$
\rho_\alpha : (a, b, c, d, s, t) \mapsto \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & a \\
0 & 1 & 0 & 0 & 0 & b \\
0 & 0 & 1 & 0 & c & 2c \\
0 & 0 & 0 & 1 & 0 & d + \alpha d \\
0 & 0 & 0 & 0 & 1 & s + \alpha t \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$
The image of $\rho_\alpha$ is a group, $L_\alpha \leq SL_7(\mathbb{R})$.

An application of lemma 4 shows that $L_\alpha$ is a conjugacy limit group. The unordered generalized cross ratio may be used to distinguish conjugacy classes of limit groups. We showed in lemma 9 that if two groups are conjugate, there is a projective transformation taking the orbit closures of the first group to the orbit closures of the second. The group $L_\alpha$ partitions $\mathbb{R}P^6$ into orbit closures, and we will use the cross ratio to give an invariant of such a partition.

Let $\{e_1, \ldots, e_7\}$ be the standard basis for $\mathbb{R}^7 := V$. Let $U = \langle e_1, \ldots, e_5 \rangle$, and $W = \langle e_6, e_7 \rangle$. Then $V = U \oplus W$, and denote the quotient map $q : V \rightarrow V/U \cong W$. Given $[te_6 + e_7] \in \mathbb{P}(W)$, define the 5 dimensional projective subspace $H_t = \mathbb{P}(e_1, \ldots, e_5, te_6 + e_7) = \langle q^{-1}(t) \rangle$. We show the orbit closure of a typical point $x \in \mathbb{R}P^6$ is $H_t$, but there are 4 exceptional $H_t$, which are the pre-images of 4 points in $\mathbb{P}(W)$. The generalized cross ratio gives an invariant of these points in $\mathbb{P}(W) \cong \mathbb{P}^1$.

For convenience, we will denote $R_\alpha := R_{L_\alpha}$. Let $x = [x_1 : \cdots : x_7] \in \mathbb{R}P^6$. The action of $L_\alpha$ is given by

$$L_\alpha x = [x_1 + ax_6 : x_2 + b(x_6 + x_7) : x_3 + c(x_6 + 2x_7) : x_4 + d(x_6 + \alpha x_7) : x_5 = sx_6 + tx_7 : x_6 : x_7]. \quad (4)$$

If $x \in \mathbb{P}(U)$, then $R_\alpha(x) = 0$, since $\mathbb{P}(U) = \text{Fix}(L_\alpha)$. From (1), we see if $x \in \mathbb{P}(V - U)$, then $R_\alpha(x) = 5$, unless the coefficients on $a, b, c, d$ are zero, i.e., $x$ satisfies one of the equations

$$x_6 = 0, x_6 + x_7 = 0, x_6 + 2x_7 = 0, x_6 + \alpha x_7 = 0. \quad (5)$$

Since $x \in V - U$, at least one of $x_6, x_7$ is not zero, and $x$ satisfies at most one equation in (5). Consequently,

$$R_\alpha(x) = \begin{cases} 0 & \text{if } x \in \mathbb{P}(U) \\ 4 & \text{if } x \in H_t \text{ and } t \in \{0, 1, 2, \alpha\} \\ 5 & \text{if } x \in H_t \text{ and } t \not\in \{0, 1, 2, \alpha\}. \end{cases}$$

Set $A := \{[1 : t] : t = 0, 1, 2, \alpha\}$, an augmented basis in $\mathbb{R}P^1$. The unordered generalized cross ratio of $A$ is the set of cross ratios of $A$, permuting the order of the points. Thus

$$\mathcal{U}(A) = \left\{ \frac{2(\alpha - 1)}{\alpha}, \frac{\alpha}{2(\alpha - 1)}, \frac{\alpha}{2 - \alpha}, \frac{2 - \alpha}{\alpha}, \frac{2(\alpha - 1)}{\alpha - 2}, \frac{\alpha - 2}{2(\alpha - 1)} \right\} \subset \mathbb{R}P^1$$

Therefore $L_\alpha$ is conjugate to $L_\beta$ if and only if $\beta \in \mathcal{U}(A)$. Thus we have shown the map $\mathbb{R} \rightarrow \text{Red}(7)$ given by $\alpha \mapsto L_\alpha$ is at most 6 to 1. So $\text{Red}(7)$ contains a continuum of non-conjugate limits.

Recall the covering dimension of a topological space, $X$, is smallest number, $n$, such that any open cover has a refinement in which no point is included in more than $n + 1$ sets in the open cover. (See [9].) Denote the covering dimension of $X$ by dim $X$. Covering dimension is a topological invariant. We will show later that dim $\text{Red}(7) \geq 1$.

4 The General Case: Bounds for dim $\text{Red}(n)$

Motivated by this example which uses the generalized cross ratio of dual points to some orbit closures to produce a projective invariant, we may make the following definitions:

Definition 9. Let $G \leq SL_n(\mathbb{R})$ and $x \in \mathbb{R}P^{n-1}$. Let $H$ be a projective subspace of $\mathbb{R}P^{n-1}$.

1. Set $M_G := \max \{ R_G(x) : x \in \mathbb{R}P^{n-1} \}$.

2. We say $x$ is **typical** if $R_G(x) = M_G$. The subspace $H$ is **typical** if $H$ is the orbit closure of a typical point.

3. We say $x$ is **exceptional** if $0 < R_G(x) < M_G$. The subspace $H$ is **exceptional** if $H$ is the union of orbit closures of exceptional points, and dim $H = M_G$. 

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Thus there are three types of points: fixed points with $R_G(x) = 0$, exceptional points when $0 < R_G(x) < M_G$, and typical points where $R_G(x) = M_G$. In our previous example, $M_{L_3} = 5$, the dimension of a typical subspace, and $H_t$ is the orbit closure of a typical point. There are 4 exceptional subspaces $\{H_t : t = 0, 1, 2, \alpha\}$ that break into orbit closures of smaller dimension. Next we generalize this example.

**Definition 10.** An $m \times n$ matrix, $T$, is generic if all collections of $n$ row vectors of $T$ are linearly independent. When $m \geq n + 2$, the rows of a generic matrix, $T$, determine an augmented basis, $\hat{T} \subset \mathbb{R}^{m-1}$.

Define an equivalence relation $T \sim S$ if $\mathcal{U}(\hat{T}) = \mathcal{U}(\hat{S})$. Set $\hat{T} := \{T : T$ is generic $\}$, and $\mathcal{T} := \hat{T}/\sim$. Let $[T] \in \mathcal{T}$ denote the equivalence class of $T$.

Notice $\mathcal{T}$ is a topological space: take the subspace topology on $\hat{T} \subset \mathbb{R}^{m \times n}$, so $\mathcal{T}$ is a topological space with the quotient topology. Since $\hat{T}$ is an open subset of $\mathbb{R}^{n \times m}$, we know $\dim \hat{T} = nm$. To find $\dim \mathcal{T}$, take $\dim \hat{T}$, and subtract 1 for projectivizing. We also subtract the dimensions corresponding to the quotient by the equivalence relation $\sim$, which allows us to map the projectivization of the first $n+1$ rows of $T \in \hat{T}$ to a projective basis, and reorder rows. This shows:

**Proposition 11.** $\dim \mathcal{T} = nm - n(n+1) - 1$.

In our earlier example of $SL_7(\mathbb{R})$, we had a 2 $\times$ 4 matrix $T$, so $n = 2$. We normalized by sending the first three rows to a projective basis of $\mathbb{R}^1$, so $\dim \text{Red}(\hat{T}) \geq 2 \cdot 4 - 2 \cdot 3 - 1 = 1$.

We say a set of hyperplanes in $\mathbb{P}$ is in *general position* in $\mathbb{P}^n$, if the set of points dual to these hyperplanes is in general position. Let $[L]$ denote the conjugacy class of a group $L$, and $T^t$ denote the transpose of $T$.

**Proposition 12.** Suppose $m \geq n + 2$, and $n \geq 2$. The function $f : \mathcal{T} \rightarrow \text{Red}(m+n+1)$ given by $f([T]) = [L_T]$ is well defined and injective.

**Proof.** First we show $f$ is well defined. Suppose $[S] = [T]$. Then there is a linear map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $Q$ maps the rows of $T$ to the rows of $S$. That is, $Q(T^t) = S^t$, and taking the transpose of both sides, $TQ^t = S$. Set $Q^t = P$. Then $L_T$ is conjugate to $L_S$ by $I_{m+1} \oplus P^{-1}$. We have:

$$
\begin{bmatrix}
I & 0 \\
0 & P^{-1}
\end{bmatrix}
\begin{bmatrix}
I & T \\
0 & I
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
0 & P^{-1}
\end{bmatrix}^{-1}
= 
\begin{bmatrix}
I & TP \\
0 & I
\end{bmatrix}
= 
\begin{bmatrix}
I & S \\
0 & I
\end{bmatrix}
$$

So if $[T] = [S]$ then $[L_T] = [L_S]$.

To prove $f$ is injective, we show the action of $L_T$ partitions $\mathbb{R}^{n+m+1}$ into orbit closures. We give an invariant of such a partition, which shows if $[T] \neq [S]$ then $[L_T] \neq [L_S]$.

Let $\{e_1, \ldots, e_{m+n+1}\}$ be the standard basis for $\mathbb{R}^{n+m+1}$. Let $V = \langle e_1, \ldots, e_{m+n+1} \rangle$, $U = \langle e_1, \ldots, e_{m+1} \rangle$, and $W = \langle e_{m+2}, \ldots, e_{m+n+1} \rangle$. Then $V = U \oplus W$, and let $q : V \rightarrow V/U \cong W$ be the quotient map. Given $[v] \in \mathbb{P}(W)$, let $H_v$ be the $m+1$ dimensional projective subspace $H_v = \langle e_1, \ldots, e_{m+1}, v \rangle = \langle q^{-1}(v) \rangle$. We show the orbit closure of a typical point $x \in \mathbb{P}^{n+m+1}$ is $H_v$, and the exceptional subspaces are the pre-image of $m$ hyperplanes in $\mathbb{P}(W)$, which determine an invariant of $L_T$.

For convenience, denote $\mathcal{R}_T := \mathcal{R}_{L_T}$, and $M_T := M_{L_T}$. The action of $L_T$ on $\mathbb{P}^{n+m+1}$ is given by

$$L_T[x_1 : \cdots : x_{m+n+1}] = [x_1 + a_1(\sum_{i=1}^{n} T_{1i}x_{m+i+1}) : x_2 + a_2(\sum_{i=1}^{n} T_{2i}x_{m+i+1}) : \cdots : x_m + a_m(\sum_{i=1}^{n} T_{mi}x_{m+i+1}) : x_{m+1} + \sum_{i=1}^{n} x_{m+i+1}b_i : x_{m+2} : \cdots : x_{n+m+1}].$$

Set

$$\phi_j(x_{m+2}, \ldots, x_{m+n+1}) = \sum_{i=1}^{n} T_{ji}x_{m+i+1}, \quad 1 \leq j \leq m,$$
so we have a collection of linear functionals \( \phi_j : \mathbb{R}^n \to \mathbb{R} \). Then we may rewrite

\[
L_T[x_1 : \cdots : x_{m+n+1}] = [x_1 + a_1 \phi_1(x_{m+2}, \ldots, x_{m+n+1}) : x_2 + a_2 \phi_2(x_{m+2}, \ldots, x_{m+n+1}) : \cdots : x_m + a_m \phi_m(x_{m+2}, \ldots, x_{m+n+1}) : x_{m+1} + \sum_{i=1}^{n} x_{m+1+i} \phi_i : x_{m+2} : \cdots : x_{m+n+1}].
\]

Since \( T \in \mathcal{T} \), any \( n \) rows of \( T \) are linear independent, and in particular, \( M_T = m+1 \). It is easy to see from definition 3 that if \( x \in \mathbb{P}(U) \), then \( \mathcal{R}_T(x) = 0 \), since \( L_T \) acts as the identity on \( \mathbb{P}(U) = \text{Fix}(L_T) \). We want to find the exceptional points. We see from (7) that \( \mathcal{R}_T(x) < m+1 \) if and only if the coefficient on some \( a_i \) is zero. Since the \( \phi_i \) are linear independent, and no \( \phi_i \) is zero, the coefficient on \( a_i \) is zero if and only if \( (x_{m+2}, \ldots, x_{m+n+1}) \) is in the kernel of some of the linear functionals, \( \phi_j \).

Set \( W_j := \ker(\phi_j) \subset W \), a hyperplane, then \( \mathcal{R}_T(x) < n+3 \) if and only if \( x \in q^{-1}(W_j) \equiv U \oplus W_j \), for some \( 1 \leq j \leq m \). Thus, the set of exceptional points is the pre-image of \( m \) hyperplanes, \( \mathbb{P}(W_j) \subset \mathbb{P}(W) \cong \mathbb{R}^{m-1} \). Let \( w_j \in \mathbb{P}(W^*) \) denote the point in the dual projective space determined by the hyperplane \( W_j \subset W \).

By hypothesis, \( T \) is generic, so these hyperplanes are in general position. The dual points are in general position. The dual points are in general position, and form an augmented basis,

\[
\delta(T) = \{ w_1, \cdots, w_m \} \subset \mathbb{P}(W^*) \cong \mathbb{R}^{m-1}
\]

We are now able finish the proof that \( f \) is injective. Suppose \( T, S \in \mathcal{T} \) with \( f(S) = f(T) \). That is, \( L_S \) is conjugate to \( L_T \), so lemma 5 implies this conjugacy takes the exceptional hyperplanes in the orbit closures of \( L_T \), to the exceptional hyperplanes in the orbit closures of \( L_S \). The dual conjugacy takes the dual augmented basis, \( \delta(T) \), to the dual augmented basis, \( \delta(S) \). By proposition 14, we have \( \mathcal{U}C(\delta(T)) = \mathcal{U}C(\delta(S)) \), so there is a projective transformation taking \( \delta(T) \) to \( \delta(S) \). A row of \( T \) determines a dual vector, \( \phi_i \), with \( \ker \phi_i = W_i \), dual to \( w_i = [\phi_i] \in \delta(T) \). So the dual transformation takes the rows of \( T \) to the rows of \( S \). Thus \( [T] = [S] \), and \( f \) is injective.

Proposition 12 shows that there are infinitely many non-conjugate limits of the diagonal Cartan subgroup in \( SL_n(\mathbb{R}) \). We want to give bounds for \( \dim \text{Red}(n) \).

**Theorem 13.** Let \( m \geq 2 \geq n \geq 2 \). The function \( f : \mathcal{T} \to \text{Red}(m+n+1) \) given by \( f([T]) = [L_T] \) restricted to any compact set \( K \) is a homeomorphism onto its image.

**Proof.** We claim the map \( f : \mathcal{T} \to \text{Red}(m+n+1) \subset \text{End}(\mathbb{R}^{m+n+1}) \) is continuous. Consider the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\hat{f}} & \text{End}(\mathbb{R}^{m+n+1}) \\
\downarrow q_1 & & \downarrow q_2 \\
\mathcal{T} & \xrightarrow{f} & \text{End}(\mathbb{R}^{m+n+1})/\text{conjugacy}
\end{array}
\]

Both quotient maps \( q_1, q_2 \) are continuous. Recall \( L_T \) is the image of \( p_T : \mathbb{R}^{n-1} \to SL_n(\mathbb{R}) \) which is an affine map, see definition 3. So, \( \hat{f} : T \mapsto L_T = \text{image}(p_T) \) is a continuous polynomial map. The diagram commutes, so \( f \) is continuous and has image in the subspace \( \text{Red}(m+n+1) \subset \text{End}(\mathbb{R}^{m+n+1})/\text{conjugacy} \).

To show \( f \) is a homeomorphism, we find a continuous inverse. Consider the composition:
\[ \text{Red}(m + n + 1) \]

\begin{align*}
\mathcal{T} & \xrightarrow{f} f(\mathcal{T}) \xrightarrow{g} (\mathbb{R}P^{n-1})^m \\
\cup & \xrightarrow{\cup} \cup \\
K & \xrightarrow{f} f(K) \xrightarrow{g} (g \circ f)(K)
\end{align*}

where \( g : [L_T] \to \delta(T) \), takes dual points to the exceptional hyperplanes of \( L_T \). We claim \( g \) is continuous. Consider the map \( \hat{g} : L_T \to (\phi_1, \ldots, \phi_m) \to \delta(T) \). The first map is projection onto the rows of \( L_T \), which is continuous. The second map is given by projecting a set of (dual) vectors into projective space, which is also continuous. Notice \( \hat{g} \) factors through the quotient map, so \( g : [L_T] \to \delta(T) \) is continuous.

Notice \( \mathcal{T} \) is an increasing union of compact sets. Let \( K \subset \mathcal{T} \) be compact. Since \( K \) is compact, \( (\mathbb{R}P^{n-1})^m \) is Hausdorff, and \( g \circ (f|_K) \) is continuous, then \( g \circ (f|_K) \) is a homeomorphism. Let \( h : g \circ f|_K(K) \to K \) be the inverse of this homeomorphism. Since \( h \) and \( g \) are continuous, \( h \circ g \) is continuous. Finally \( \text{id} = h(g \circ f|_K) = (h \circ g)f|_K \). Thus \( f|_K \) is a homeomorphism on any compact subset \( K \).

**Corollary 14.** \( \dim \text{Red}(k) \geq \frac{k^2 - 6k}{8} \).

**Proof.** Set \( k = m + n + 1 \). Covering dimension is a topological invariant, preserved by homeomorphism. Proposition \( 14 \) says \( \dim T = nm - n(n + 1) - 1 \), and theorem \( 14 \) shows \( T \) is homeomorphic to a subspace of \( \text{Red}(m + n + 1) \) if \( m - 2 \geq n \geq 2 \). The bounds on \( m \) and \( n \) imply \( k \geq 7 \). If \( k \leq 6 \), the result is vacuous.

We may change the size of the \( m \) by \( n \) matrix (as long as \( m - 2 \geq n \geq 2 \)), so \( \dim \text{Red}(n) \) is bounded below by the maximum of \( nm - n(n + 1) - 1 \). Since \( m + n + 1 = k \), and \( k \) is fixed, we want to maximize \( g(n) = n(k - n) - n(n + 1) - 1 \). The maximum occurs at \( n = \frac{k - 2}{4} \), so \( m = \frac{3k - 6}{4} \), and the maximum of \( mn - n(n + 1) - 1 \) is \( \frac{k^2 - 6k}{8} \).

We present a proof for an upper bound of \( \dim \text{Red}(k) \), given in \( 12 \) for (Krull) dimension of \( \text{Red}(n) \).

**Theorem 15.** \( \dim \text{Red}(k) \leq k^2 - k \).

**Proof.** Let \( C \) denote the diagonal Cartan subgroup, and let \( P \in GL_k(\mathbb{R}) \). The dimension of the set of all conjugates of \( C \) is \( k^2 - k \), since \( \text{PCP}^{-1} = C \) if and only if \( P \) is a diagonal matrix. (Alternatively, see \( 10 \) theorem 1, or \( 12 \) theorem 2.9.7.) The set of conjugates of \( C \) is a semi-algebraic set, and the set of conjugacy limits of \( C \) is the boundary of the Zariski closure of the set of conjugates. We may apply propositions 2.8.2 and 2.8.13 from \( 11 \), so \( \dim(\text{Red}(k)) \leq k^2 - k \).

**Corollary 14** and **Theorem 15** imply **Theorem 1**.

### 5 Abelian Groups which are Not Conjugacy Limit Groups

In this section, we give examples of elements of \( \text{Ab}(n) - \text{Red}(n) \). We discuss two properties of conjugacy limit groups of the diagonal Cartan subgroup, \( C \), which are not universal amongst abelian groups. The first property is a conjugacy limit group is flat, and the second is that it contains a one parameter subgroup with a particular Jordan block structure.

Suppose \( L \) is a conjugacy limit of \( C \) in \( SL_n(\mathbb{R}) \). Then we claim \( L \) is the intersection of a vector space with \( SL_n(\mathbb{R}) \). The diagonal Cartan subgroup is of this form, and conjugacy is a linear map, so it preserves this property. We call such a group a flat group. Conjugacy limits of \( C \) are flat groups.
Definition 16. Let $\mu_k : \mathbb{R}^{k-1} \to SL_k(\mathbb{R})$ be the representations below for $k = 5, 6$.

$$
\mu_5 : (a, b, c, d) \mapsto \begin{pmatrix} 1 & a & 0 & \frac{a^2}{2} & b \\ 0 & 1 & 0 & a & 0 \\ 0 & 0 & 1 & c & d \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

and $\mu_6 : (a, b, c, d, e) \mapsto \begin{pmatrix} 1 & a & \frac{a^2}{2} & 0 & b & c \\ 0 & 1 & a & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & d & e \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Set $M_k \leq SL_k(\mathbb{R})$ to be the respective images of $\mu_k$.

It is easy to check that $M_k$ is an abelian group of dimension $k - 1$. Moreover, neither is a limit of $C$, since they are not flat groups.

Thus we have given examples of elements in $Ab(n) - Red(n)$ for $n = 5, 6$. This shows $Ab(n) \neq Red(n)$ when $n = 5, 6$, which answers question A in [6]. Haettel, [4], proves in lemma 3.4 that for $n \geq 7$, there is an abelian subalgebra of dimension $n - 1$ which is not a conjugacy limit of diagonal Cartan subalgebras in $SL_n(\mathbb{R})$, following the argument in [6] for the complex case. Combined with Haettel’s result we see $Ab(n) = Red(n)$ if and only if $n \leq 4$. For $n = 5, 6$, we have shown $Red(n) \subsetneq Ab(n)$. Combined with Haettel’s result, this completes the proof of theorem [2].

We give another property satisfied by conjugacy limit groups of $C$, and an example of an element of $Ab(8) - Red(8)$, which is flat group, but does not satisfy this additional property. Thus to determine if a group is a conjugacy limit of $C$, it is necessary but not sufficient for the group to be a flat group.

Suppose $\mathbb{R}^{n-1} \cong G \leq SL_n(\mathbb{R})$. Define the rank of $G$ to be $\text{rank}(G) = \text{rk}(G) := \max_{g \in G} \text{rk}(g - I_n)$. In the special case when $G$ is a unipotent group, one may compute the rank from the JNF of each group element, by counting the number of off diagonal entries.

Proposition 17. Suppose $G \leq SL_n(\mathbb{R})$, a unipotent group, and $L$ is a conjugacy limit of $G$. Then $\text{rk}(L) \leq \text{rk}(G)$.

Proof. Proposition 3.2 in [3] shows that the dimension of the normalizer must always increase under taking a conjugacy limit. The dimension of the normalizer of a group depends on the size of the blocks of the JNF of a typical element. The normalizer has largest dimension when typical elements have JNF closest to the identity, which is when the size of the blocks is smallest.

By proposition 3.4 in [3], the set of characteristic polynomials of elements of $L$ is contained in the set of characteristic polynomials of elements of $G$. Putting these results together, we see the size of the Jordan blocks of a generic element must remain constant or break down. Since the rank of a unipotent group may be computed by counting the sizes of the blocks in the JNF of a typical element, the rank cannot increase.

Let $G \leq SL_n(\mathbb{R})$ be the image of a representation $\rho : \mathbb{R}^{n-1} \to SL_n(\mathbb{R})$. We say $G$ contains a flag of subgroups, $H_i \leq G$, if the following conditions are satisfied: $H_{i-1} \leq H_i$, and each $H_i$ is the image of $\mathbb{R}^i$ under $\rho$.

Corollary 18. If $L$ is a conjugacy limit of $C$, then $L$ contains a flag of subgroups with rank less than or equal to $1, \ldots, rk(L)$. In particular, every conjugacy limit of $C$ contains a 1 parameter subgroup with rank 1.

Proof. The conjugacy limit of $\text{diag}(a, 1, 1, \ldots, 1)$ is a rank 1 subgroup by an immediate application of proposition [17]. In general, $C$ has a flag of subgroups with rank $1, \ldots, n - 1$, as more of the entries on the diagonal are allowed to vary. The conjugacy limits of this flag of subgroups of $C$ give a flag of conjugacy limits.
Set $E \leq SL_8(\mathbb{R})$ to be the image of the representation $\rho : \mathbb{R}^7 \to SL_8(\mathbb{R})$:

$$
\rho : (a, b, c, d, e, f, g) \mapsto \begin{pmatrix}
1 & 0 & 0 & 0 & c & g & f \\
0 & 1 & 0 & 0 & c & b & f & e \\
0 & 0 & 1 & 0 & a & b & f & e \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

It is easy to check that $E$ is an abelian subgroup, since matrix multiplication is given by

$$
\left( \begin{array}{c|c}
I & A \\
0 & I
\end{array} \right) \left( \begin{array}{c|c}
I & B \\
0 & I
\end{array} \right) = \left( \begin{array}{c|c}
I & A + B \\
0 & I
\end{array} \right).
$$

**Proposition 19.** The group $E$ has no 1 parameter subgroups of rank 1.

**Proof.** A matrix has rank 1 if and only if every $2 \times 2$ minor is zero. We will show that $\rho(a, b, c, d, e, f) - I_8$ has rank 1 if and only if $(a, b, c, d, e, f) = (0, ..., 0)$. Consider the $2 \times 2$ minors of

$$
\begin{pmatrix}
0 & c & g & f \\
c & b & f & e \\
b & a & e & d \\
a & g & d & 0
\end{pmatrix}.
$$

Since the upper left minor must be zero, we see $c = 0$. Looking at the minor directly below, we see $b = 0$. Continuing in this fashion, $b = 0, a = 0, d = 0, e = 0, f = 0$ and $g = 0$. (Alternatively, we could take all of the minors, and check $(0, 0, ..., 0)$ is the only solution.) Thus $\rho(a, b, c, d, e, f) - I_8$ has rank 1 if and only if $(a, b, c, d, e, f) = (0, ..., 0)$. But if $(a, b, c, d, e, f) = (0, ..., 0)$ then $\rho(0, ..., 0) - I_8$ is the zero matrix, with rank 0. Therefore $E$ (the image of $\rho$) contains no rank 1 subgroups.\[\square\]

Combining corollary[18] and proposition[19] we have shown the abelian group, $E$, is not a conjugacy limit of $C$. Thus we have shown two necessary conditions for a group to be a limit group: the group must be a flat group, and contain a rank 1 subgroup. Are these conditions sufficient?

Further, there are many more questions we might ask about the spaces $Red(n)$ and $Ab(n)$. For example: are they connected? Does every component of $Ab(n)$ contain a component of $Red(n)$, and is it possible to retract from $Ab(n)$ to $Red(n)$? What properties characterize $Red(n)$ that are not inherited by $Ab(n)$?

The author would like to thank Daryl Cooper for many helpful discussions, and Thomas Haettel, whose work inspired the paper. The author was partially supported by NSF grants DMS0706887, 1207068 and 1045292. The author acknowledges support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 RNMS: GEometric structures And Representation varieties (the GEAR Network).

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