9 GENERATORS OF THE SKEIN SPACE OF THE 3-TORUS

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Abstract. We show that the skein vector space of the 3-torus is finitely generated. We show that it is generated by 9 elements: the empty set, some simple closed curves representing the non null elements of the first homology group with coefficients in $\mathbb{Z}_2$, and a link consisting of two parallel copies of one of the previous non empty knots.

1. Introduction

An alternative approach to representation theory for quantum invariants is provided by skein theory. The word “skein” and the notion were introduced by Conway in 1970 for his model of the Alexander polynomial. This idea became really useful after the work of Kauffman [K] which redefined the Jones polynomial in a very simple and combinatorial way passing through the Kauffman bracket. These combinatorial techniques allow us to reproduce all quantum invariants arising from the representations of $U_q(\mathfrak{sl}_2)$ without any reference to representation theory. This also leads to many interesting and quite easy computations. This skein method was used by various authors [L1, L2, L3, L4, BHMV, KL] to re-interpret and extend some of the methods of representation theory.

The first notion in skein theory is the one of “skein vector space” (or skein module). These are vector spaces ($R$-modules) associated to oriented 3-manifolds, where the base field is equipped with a fixed invertible element $A$. These were introduced independently in 1988 by Turaev [Tu] and in 1991 by Hoste and Przytycki [HP1]. We can think of them as an attempt to get an algebraic topology for knots: they can be seen as homology spaces obtained using isotopy classes instead of homotopy or homology classes. In fact they are defined taking a vector space generated by sub-objects (framed links) and then quotienting them by some relations. In this framework, the following questions arise naturally and are still open in general:

Question 1.1.

- Are skein spaces (modules) computable?
- How powerful are them to distinguish 3-manifolds and links?
- Do the vector spaces (modules) reflect the topology/geometry of the 3-manifolds (e.g. surfaces, geometric decomposition)?
- Does this theory have a functorial aspect? Can it be extended to a functor from a category of cobordisms to the category of vector spaces (modules) and linear maps?
Skein spaces (modules) can also be seen as deformations of the ring of the $SL_2(\mathbb{C})$-character variety of the 3-manifold $B_2$. Moreover they are useful to generalize the Kauffman bracket, hence the Jones polynomial, to manifolds other than $S^3$. Thanks to result of Hoste-Przytycki [HP4, P2] and (with different techniques) to Costantino [C], now we can define the Kauffman bracket also in the connected sum $\#_g(S^1 \times S^2)$ of $g \geq 0$ copies of $S^1 \times S^2$.

Until now there are only few 3-manifolds whose skein space (module) is known, see for instance [B1, HP2, HP3, HP4, Ma, Mr1, Mr2, MrD, P1, P2, P3, TL]. Another natural question is:

**Question 1.2.** Is the skein vector space of a closed oriented 3-manifold always finite generated?

In this paper we take as base field the set $\mathbb{Q}(A)$ of all rational functions with rational coefficients and abstract variable $A$, and then we note that every result in this work holds also for the field $\mathbb{C}$ of complex numbers with $A \in \mathbb{C}$ a non null number such that $A^n \neq 1$ for every $n > 0$.

**Theorem 1.3.** The skein space $K(T^3)$ of the 3-torus $T^3 = S^1 \times S^1 \times S^1$ is finitely generated.

A set of 9 generators is given by the empty set $\emptyset$, some simple closed curves representing the non null elements of the first homology group $H_1(T^3; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^3$ with coefficients in $\mathbb{Z}_2$, and a skein element $\alpha$ that is equal to the link consisting of two parallel copies of any previous non empty knots.

Our main tool is the algebraic work of Frohman and Gelca [FG]. The skein space (module) of a (thickened) surface has a natural structure of algebra obtained by overlap of framed links. In their work Frohman and Gelca gave a nice formula that describes the product in the skein space (algebra) $K(T^2)$ of the 2-torus $T^2 = S^1 \times S^1$. A standard embedding of $T^2$ in $T^3$ makes this product commutative, hence we can get further relations from the formula of Frohman-Gelca.

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2. The result

2.1. **Definition of skein module.** Let $M$ be an oriented 3-manifold, $R$ a commutative ring with unit and $A \in R$ an invertible element of $R$. Let $V$ be the abstract free $R$-module generated by all framed links in $M$ (considered up to isotopies) including the empty set $\emptyset$.

**Definition 2.1.** The $(R, A)$-Kauffman bracket skein module of $M$, or the $R$-skein module, or simply the KBSM, is sometimes indicated with $KM(M; R, A)$,
and is the quotient of $V$ by all the possible skein relations:

\[
\begin{align*}
L \cup \bigcirc &= A + A^{-1} \\
\bigcirc &= (-A^2 - A^{-2})D \\
\bigcirc &= (-A^2 - A^{-2})\emptyset
\end{align*}
\]

These are local relations where the framed links in an equation differ just in the pictured 3-ball that is equipped with a positive trivialization. An element of $KM(M; R, A)$ is called a skein or a skein element. If $M$ is the oriented $I$-bundle over a surface $S$ (this is $M = S \times [-1, 1]$ if $S$ is oriented) we simply write $KM(S; R, A)$ and call it the skein module of $S$.

Let $\mathbb{Q}(A)$ be field of all rational function with rational coefficients and abstract variable $A$. We set

\[
K(M) := KM(M; \mathbb{Q}(A), A)
\]

and we call it the skein vector space, or simply the skein space, of $M$.

**Remark 2.2.** It is easy to verify that if we modify the framing of a component of a framed link, the skein changes by the multiplication of an integer power of $-A^3$:

\[
\begin{align*}
\bigcirc &= -A^3 \\
\bigcirc &= -A^{-3}
\end{align*}
\]

### 2.2. The skein algebra of the 2-torus.

**Definition 2.3.** Let $S$ be a surface the skein module $KM(S; R, A)$ has a natural structure of $R$-algebra that is given by the linear extension of the multiplication defined on framed links. Given two framed links $L_1, L_2 \subset S \times [-1, 1]$, the product $L_1 \cdot L_2 \subset S \times [-1, 1]$ is obtained by putting $L_1$ above $L_2$, $L_1 \cdot L_2 \cap S \times [0, 1] = L_1$ and $L_1 \cdot L_2 \cap S \times [-1, 0] = L_2$.

Look at the 2-torus $T^2$ as the quotient of $\mathbb{R}^2$ modulo the standard lattice of translations generated by $(1,0)$ and $(0,1)$, hence for any non null pair $(p,q)$ of integers we have the notion of $(p,q)$-curve: the simple closed curve in the 2-torus that is the quotient of the line passing trough $(0,0)$ and $(p,q)$.

**Definition 2.4.** Let $p$ and $q$ be two co-prime integers, hence $(p,q) \neq (0,0)$. We denote by $(p,q)_T$ the $(p,q)$-curve in the 2-torus $T^2$ equipped with the black-board framing. Given a framed knot $\gamma$ in an oriented 3-manifold $M$ and an integer $n \geq 0$, we denote by $T_n(\gamma)$ the skein element defined by induction as follows:

\[
\begin{align*}
T_0(\gamma) &:= 2 \cdot \emptyset \\
T_1(\gamma) &:= \gamma \\
T_{n+1}(\gamma) &:= \gamma \cdot T_n(\gamma) - T_{n-1}(\gamma)
\end{align*}
\]
where \( \gamma \cdot T_n(\gamma) \) is the skein element obtained adding a copy of \( \gamma \) to all the framed links that compose the skein \( T_n(\gamma) \). For \( p, q \in \mathbb{Z} \) such that \((p, q) \neq (0, 0)\), we denote by \((p, q)_T\) the skein element
\[
(p, q)_T := T_{\text{MCD}(p,q)} \left( \frac{p}{\text{MCD}(p,q)} \frac{q}{\text{MCD}(p,q)} \right)_T,
\]
where \(\text{MCD}(p,q)\) is the maximum common divisor of \(p\) and \(q\). Finally we set
\[
(0, 0)_T := 2 \cdot \varnothing.
\]

It is easy to show that the set of all the skein elements \((p, q)_T\) with \(p, q \in \mathbb{Z}\) generates \(KM(T^2; R, A)\) as \(R\)-module.

This is not the standard way to color framed links in a skein module. The colorings \(JW_n(\gamma)\), \(n \geq 0\), with the Jones-Wenzl projectors are defined in the same way of \(T_n(\gamma)\) but at the 0-level we have \(JW_0(\gamma) = \varnothing\).

**Theorem 2.5** (Frohman-Gelca). For any \(p, q, r, s \in \mathbb{Z}\) the following holds in the skein module \(KM(T^2; R, A)\) of the 2-torus \(T^2\):
\[
(p, q)_T \cdot (r, s)_T = A^{\left| \begin{array}{cc} p & q \\ r & s \end{array} \right|} (p + r, q + s)_T + A^{\left| \begin{array}{cc} p & q \\ r & s \end{array} \right|} (p - r, q - s)_T,
\]
where \(\left| \begin{array}{cc} p & q \\ r & s \end{array} \right|\) is the determinant \(ps - qr\).

*Proof. See [FG].* \(\square\)

### 2.3. The abelianization.

**Definition 2.6.** Let \(B\) be a \(R\)-algebra for a commutative ring with unity \(R\). We denote by \(C(B)\) the \(R\)-module defined as the following quotient:
\[
C(B) := \frac{B}{[B, B]}
\]
where \([B, B]\) is the sub-module of \(B\) generated by all the elements of the form \(ab - ba\) for \(a, b \in B\). We call \(C(B)\) the abelianization of \(B\).

**Remark 2.7.** Usually in non-commutative algebra the abelianization is the \(R\)-algebra defined as the quotient of \(B\) modulo the sub-algebra (sub-module and ideal) generated by all the elements of the form \(ab - ba\). In our definition the denominator is just a sub-module and we only get a \(R\)-module. We use the word “abelianization” anyway.

Now we work with \(C(K(T^2))\) and we still use \((p, q)_T\) and \((p, q)_T \cdot (r, s)_T\) to denote the class of \((p, q)_T \in K(T^2)\) and \((p, q)_T \cdot (r, s)_T \in K(T^2)\) in \(C(K(T^2))\).

**Lemma 2.8.** Let \((p, q)\) be a pair of integers different from \((0, 0)\). Then in the abelianization \(C(K(T^2))\) of the skein algebra \(K(T^2)\) of the 2-torus \(T^2\)
we have
\[(p, q)_T = \begin{cases} (1, 0)_T & \text{if } p \in 2\mathbb{Z} + 1, \ q \in 2\mathbb{Z} \\ (0, 1)_T & \text{if } p \in 2\mathbb{Z}, \ q \in 2\mathbb{Z} + 1 \\ (1, 1)_T & \text{if } p, q \in 2\mathbb{Z} + 1 \\ (2, 0)_T & \text{if } p, q \in 2\mathbb{Z} \end{cases}.\]

Hence \(C(K(T^2))\) is generated as a \(\mathbb{Q}(A)\)-vector space by the empty set \(\emptyset\), the framed knots \((1, 0)_T\), \((0, 1)_T\), \((1, 1)_T\), and a framed link consisting of two parallel copies of \((1, 0)_T\).

**Proof.** By Theorem 2.5 for every \(p, q \in \mathbb{Z}\) we have
\[A^{-q}(p + 2, q)_T + A^q(p, q)_T = (p + 1, q)_T \cdot (1, 0)_T = (1, 0)_T \cdot (p + 1, q)_T = A^q(p + 2, q)_T + A^{-q}(-p, -q)_T.\]

Since \((p, q)_T = (-p, -q)_T\) we have \((A^q - A^{-q})(p, q)_T = (A^q - A^{-q})(p + 2, q)_T.\) Hence if \(q \neq 0\) we get \((p, q)_T = (p + 2, q)_T\) (here we use the fact that the base ring is a field and \(A^n \neq 1\) for every \(n > 0\)). Thus
\[(p, q)_T = \begin{cases} (0, q)_T & \text{if } p \in 2\mathbb{Z}, \ q \neq 0 \\ (1, q)_T & \text{if } p \in 2\mathbb{Z} + 1, \ q \neq 0 \end{cases}.
\]

Analogously by using \((0, 1)_T\) instead of \((1, 0)_T\) for \(p \neq 0\) we get
\[(p, q)_T = \begin{cases} (p, 0)_T & \text{if } q \in 2\mathbb{Z}, \ q \neq 0 \\ (p, 1)_T & \text{if } q \in 2\mathbb{Z} + 1, \ q \neq 0 \end{cases}.
\]

Therefore if \(p, q \in 2\mathbb{Z} + 1\), \((p, q)_T = (1, 1)_T\). If \(p \neq 0\) we get
\[(p, 0)_T = (p, 2)_T = \begin{cases} (0, 2)_T & \text{if } p \in 2\mathbb{Z} \\ (1, 2)_T & \text{if } p \in 2\mathbb{Z} + 1 \end{cases} = \begin{cases} (0, 2)_T & \text{if } p \in 2\mathbb{Z} \\ (1, 0)_T & \text{if } p \in 2\mathbb{Z} + 1 \end{cases}.
\]

In the same way for \(q \neq 0\) we get
\[(0, q)_T = (2, q)_T = \begin{cases} (2, 0)_T & \text{if } p \in 2\mathbb{Z} \\ (2, 1)_T & \text{if } p \in 2\mathbb{Z} + 1 \end{cases} = \begin{cases} (2, 0)_T & \text{if } p \in 2\mathbb{Z} \\ (0, 1)_T & \text{if } p \in 2\mathbb{Z} + 1 \end{cases}.
\]

In particular we have
\[(2, 0)_T = (2, 2)_T = (2, -2)_T = (0, 2)_T = (p, q)_T \text{ for } (p, q) \neq (0, 0), \ p, q \in 2\mathbb{Z}.\]

\[\square\]

2.4. The \((p, q, r)\)-type curves. As for the 2-torus \(T^2\), we look at the 3-torus \(T^3\) as the quotient of \(\mathbb{R}^3\) modulo the standard lattice of translations generated by \((1, 0, 0), \ (0, 1, 0)\) and \((0, 0, 1)\).

**Definition 2.9.** Let \((p, q, r)\) be a triple of co-prime integers, that means \(\text{MCD}(p, q, r) = 1\), where \(\text{MCD}(p, q, r)\) is the maximum common divisor of \(p\), \(q\) and \(r\), in particular we have \((p, q, r) \neq (0, 0, 0)\). The \((p, q, r)\)-curve is the simple closed curve in the 3-torus that is the quotient (under the standard
Proof. Every embedding $e : T^2 \to T^3$ of the 2-torus in the 3-torus is standard if it is the quotient (under the standard lattice) of a plane in $\mathbb{R}^3$ that is the image of the plane generated by $(1, 0, 0)$ and $(0, 1, 0)$ under a linear map defined by a matrix of $SL_3(\mathbb{Z})$ (a $3 \times 3$ matrix with integer entries and determinant 1).

Remark 2.11. There are infinitely many standard embeddings even up to isotopies. A standard embedding of $T^2$ in $T^3$ is the quotient under the standard lattice of the plane generated by two columns of a matrix of $SL_3(\mathbb{Z})$.

Lemma 2.12. Let $(p, q, r)$ be a triple of co-prime integers. Then the skein element $[p, q, r] \in K(T^3)$ is equal to $[x, y, z]$, where $x, y, z \in \{0, 1\}$ and have respectively the same parity of $p, q$ and $r$.

Proof. Every embedding $e : T^2 \to T^3$ of the 2-torus $T^2$ in $T^3$ defines a linear map between the skein spaces

$$e_* : K(T^2) \to K(T^3).$$

The map $e_*$ factorizes with the quotient map $K(T^2) \to C(K(T^2))$. In fact we can slide the framed links in $e(T^2 \times [-1, 1])$ from above to below getting $e_*(L_1 \cdot L_2) = e_*(L_2 \cdot L_1)$ for every two framed links, $L_1$ and $L_2$, in $T^2 \times [-1, 1]$. As said in Remark 2.11, a standard embedding $e : T^2 \to T^3$ corresponds to the plane generated by two columns $(p_1, q_1, r_1), (p_2, q_2, r_2) \in \mathbb{Z}^3$ of a matrix in $SL_3(\mathbb{Z})$. In this correspondence $e_*((a, b)_T) = [ap_1 + bp_2, ap_1 + bq_2, ar_1 + br_2]$ for every co-prime $a, b \in \mathbb{Z}$. Therefore by Lemma 2.8 we get

$$[a'p_1 + b'p_2, a'q_1 + b'q_2, a'r_1 + b'r_2] = e_*((a', b')_T) = e_*((a, b)_T) = [ap_1 + bp_2, ap_1 + bq_2, ar_1 + br_2]$$

for every two pairs $(a, b), (a', b') \in \mathbb{Z}^2$ of co-prime integers such that $a + a', b + b' \in 2\mathbb{Z}$.

Let $(p, q, r)$ be a triple of co-prime integers. By permuting $p, q, r$ we get either $(p, q, r) = (1, 0, 0)$ or $p, q \neq 0$. Consider the case $p, q \neq 0$. Let $d$ be the maximum common divisor of $p$ and $q$, and let $\lambda, \mu \in \mathbb{Z}$ such that $\lambda p + \mu q = d$. The following matrix belongs in $SL_3(\mathbb{Z})$:

$$M_1 := \begin{pmatrix} \frac{p}{d} & -\mu & 0 \\ \frac{q}{d} & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
Let \( v_1^{(1)} \) and \( v_3^{(1)} \) be the first and the third columns of \( M_1 \). We have \( (p, q, r) = dv_1^{(1)} + rv_3^{(1)} \). Hence

\[
[p, q, r] = \begin{cases}
\left[ \frac{p}{d}, \frac{q}{d}, 0 \right] & \text{if } d \in 2\mathbb{Z} + 1, \ r \in 2\mathbb{Z} \\
[0, 0, 1] & \text{if } d \in 2\mathbb{Z}, \ r \in 2\mathbb{Z} + 1 \\
\left[ \frac{p}{d}, \frac{q}{d}, 1 \right] & \text{if } d, r \in 2\mathbb{Z} + 1
\end{cases}
\]

The integers \( p, q, r \) can not be all even because they are co-prime, hence \( d \) and \( r \) can not be both even. Therefore we just need to study the cases where \( r \in \{0, 1\} \).

If \( r = 0 \) we consider the trivial embedding of \( T^2 \) in \( T^3 \). The corresponding matrix of \( SL_3(\mathbb{Z}) \) is the identity. We have \( (p/d, q/d, 0) = p/d(1, 0, 0) + q/d(0, 1, 0) \), hence

\[
[p, q, 0] = \left[ \frac{p}{d}, \frac{q}{d}, 0 \right] = \begin{cases}
[1, 0, 0] & \text{if } \frac{p}{d} \in 2\mathbb{Z} + 1, \ \frac{q}{d} \in 2\mathbb{Z} \\
[0, 1, 0] & \text{if } \frac{p}{d} \in 2\mathbb{Z}, \ \frac{q}{d} \in 2\mathbb{Z} + 1 \\
[1, 1, 0] & \text{if } \frac{p}{d}, \frac{q}{d} \in 2\mathbb{Z} + 1
\end{cases}
\]

If \( r = 1 \) we take the following matrix of \( SL_3(\mathbb{Z}) \):

\[
M_2 := \begin{pmatrix} 0 & 0 & 1 \\ q & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\]

Let \( v_1^{(2)} \) and \( v_3^{(2)} \) be the first and the third columns of \( M_2 \). We have \( (p, q, 1) = pv_3^{(2)} + v_1^{(2)} \), hence

\[
[p, q, 1] = \begin{cases}
[1, q, 1] & \text{if } p \in 2\mathbb{Z} + 1 \\
[0, q, 1] & \text{if } p \in 2\mathbb{Z}
\end{cases}
\]

By permuting \( p, q, r \) we reduce the case \( (p, q, r) = (0, q, 1) \) to the case \( p, q \neq 0, r = 0 \) that we studied before.

It remains to consider the case \( p = r = 1 \). We consider the following matrix of \( SL_3(\mathbb{Z}) \):

\[
M_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}
\]

Let \( v_1^{(3)} \) and \( v_2^{(3)} \) be the first and the second columns of \( M_3 \). We have \( (1, q, 1) = v_1^{(3)} + qv_2^{(3)} \). Hence

\[
[1, q, 1] = \begin{cases}
[1, 0, 1] & \text{if } q \in 2\mathbb{Z} \\
[1, 1, 1] & \text{if } q \in 2\mathbb{Z} + 1
\end{cases}
\]

\( \square \)

**Lemma 2.13.** The intersection of any two different standard embedded 2-tori in \( T^3 \) contains a \( (p, q, r) \)-type curve.
Proof. Let $T_1$ and $T_2$ be two standard embedded tori in the 3-torus, and let $\pi_1$ and $\pi_2$ be two planes in $\mathbb{R}^3$ whose projection under the standard lattice is respectively $T_1$ and $T_2$. The intersection $T_1 \cap T_2$ contains the projection of $\pi_1 \cap \pi_2$. We just need to prove that in $\pi_1 \cap \pi_2$ there is a point $(p, q, r) \neq (0, 0, 0)$ with integer coordinates $p, q, r \in \mathbb{Z}$. Every plane defining a standard embedded torus is generated by two vectors with integer coordinates, and hence it is described by an equation $ax + by + cz = 0$ with integer coefficients $a, b, c \in \mathbb{Z}$. Applying a linear map described by a matrix of $SL_3(\mathbb{Z})$ we can suppose that $\pi_1$ is the trivial plane $\{z = 0\}$. Let $a, b, c \in \mathbb{Z}$ such that $\pi_2 = \{ax + by + cz = 0\}$. The vector $(-b, a, 0)$ is non null and lies on $\pi_1 \cap \pi_2$. □

2.5. Diagrams. Framed links in $T^3$ can be represented by diagrams in the 2-torus $T^2$. These diagrams are like the usual link diagrams but with further oriented signs on the edges (see Fig. 1-(left)). Fix a standard embedded 2-torus $T$ in $T^3$. After a cut along a parallel copy $T'$ of $T$, the 3-torus becomes diffeomorphic to $T \times [-1, 1]$ and framed links in $T^3$ correspond to framed tangles of $T \times [-1, 1]$. These diagrams are generic projections on $T$ of the framed tangles in $T \times [-1, 1]$ via the natural projection $(x, t) \mapsto x$. The further signs on the diagrams represent the intersection of the framed links with the boundary $T \times \{-1, 1\}$, in other words they represent the passages of the links along the $(p, q, r)$-type curve that in the Euclidean metric is orthogonal to $T$ (see Fig. 1-(right)). If $T$ is the trivial torus $S^1 \times S^1 \times \{x\}$, the further signs represent the passages through the third $S^1$-factor. We use the proper notion of blackboard framing.

2.6. Generators for the 3-torus. The following is the main theorem proved in this paper. We use all the previous lemmas to get a set of 9 generators of $K(T^3)$.
Theorem 2.14. The skein space $K(T^3)$ of the 3-torus $T^3$ is generated by the empty set $\emptyset$, $[1,0,0]$, $[0,1,0]$, $[0,0,1]$, $[1,1,0]$, $[1,0,1]$, $[0,1,1]$, $[1,1,1]$ and a skein $\alpha$ that is equal to the framed link consisting of two parallel copies of any $(p,q,r)$-type curve.

Proof. Let $T$ be the trivial embedded 2-torus: the one containing the $(p,q,r)$-type curves with $r = 0$. Use $T$ to project the framed links and make diagrams. By using the first skein relation on these diagrams we can see that $K(T^3)$ is generated by the framed links described by diagrams on $T$ without crossings. These diagrams are union of simple closed curves on $T$ equipped with some signs as the one with $+$ and $-$ in Fig. [1]. These simple closed curves are either parallel to a $(p,q)$-curve or homotopically trivial. The framed links described by these diagrams lie in the standard embedded tori that are the projection (under the standard lattice) of the planes generated by $(0,0,1)$ and $(p,q,0)$ for some $p$ and $q$. Therefore $K(T^3)$ is generated by the images of $K(T^2)$ under the linear maps induced by the standard embeddings of $T^2$ in $T^3$.

As said in the proof of Lemma 2.12 the linear map $e_*$ induced by any standard embedding $e : T^2 \to T^3$ factorizes with the quotient map $K(T^2) \to C(K(T^2))$. Lemma 2.8 applied on the standard embedding $e$ shows that the image $e_*(K(T^2))$ is generated by $\emptyset$, three $(p,q,r)$-type curves lying on $e(T^2)$, and the skein $\alpha_e$ that is equal to the framed link consisting of two parallel copies of any $(p,q,r)$-type curve lying on $e(T^2)$.

Let $e_1, e_2 : T^2 \to T^3$ be two standard embeddings. By Lemma 2.13 $e_1(T^2) \cap e_2(T^2)$ contains a $(p,q,r)$-type curve $\gamma$, hence $\alpha_{e_1}$ and $\alpha_{e_2}$ coincide with the framed link that is two parallel copies of $\gamma$. Therefore the skein element $\alpha_e$ does not depend on the embedding $e$.

We conclude by using Lemma 2.12 that says that the skein of any $(p,q,r)$-type curve is equal to the one of a standard representative of a non null element of the first homology group $H_1(T^3; \mathbb{Z}_2)$ with coefficient in $\mathbb{Z}_2$, namely a $(p,q,r)$-type curve with $p, q, r \in \{0, 1\}$. □

Remark 2.15. Theorem 2.14 Lemma 2.8 and Lemma 2.12 work for every base pair $(R, A)$ such that $A^n - 1$ is an invertible element of $R$ for any $n > 0$. In particular they work for $(\mathbb{C}, A)$, where $A^n \neq 1$ for any $n > 0$.

2.7. Linear independence. Here we talk about the linear independence of generators of $K(T^2)$ we have shown. The following proposition shows a decomposition in direct sum of $K(T^3)$.

Proposition 2.16. The skein space $K(T^3)$ is the direct sum of 8 sub-spaces

$$K(T^3) = V_0 \oplus V_1 \oplus \ldots \oplus V_7$$

such that:

1. $V_0$ is generated by the empty set $\emptyset$ and the skein $\alpha$ (see Theorem 2.14);
2. every $(p,q,r)$-type curve generates a $V_j$ with $j > 0$ and every $V_j$ with $j > 0$ is generated by one such curve.
Proof. The skein relations relates framed links holding in the same $\mathbb{Z}_2$-homology class. Hence for every oriented 3-manifold $M$ we have a decomposition in direct sum

$$KM(M; R, A) = \bigoplus_{h \in H_1(M; \mathbb{Z}_2)} V_h,$$

where $V_h$ is generated by the framed links whose $\mathbb{Z}_2$-homology class is $h$. The statement follow by this observation and the fact that if $[p, q, r]$ and $[p', q', r']$ represent the same $\mathbb{Z}_2$-homology class, then $[p, q, r] = [p', q', r'] \in K(T^3)$. □

Remark 2.17. Given a triple of integers $(x, y, z) \neq (0, 0, 0)$ such that $x, y, z \in \{0, 1\}$, we can easily find an orientation preserving diffeomorphism of the 3-torus $T^3$ sending $[x, y, z]$ to $[1, 0, 0]$. Hence if the skein of one such curve $[x, y, z]$ is null then also all the others skein elements of such curves are null. Therefore by Proposition 2.16 the possibly dimensions of the skein space $K(T^3)$ are 0, 1, 2, 7, 8 and 9.

Question 2.18. Is 9 the dimension of the skein vector space $K(T^3)$ of the 3-torus?

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