On the use of Multigrid Preconditioners for Topology Optimization

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Abstract Topology optimization for large scale problems continues to be a computational challenge. Several works exist in the literature to address this topic, and all make use of iterative solvers to handle the linear system arising from the Finite Element Analysis (FEA). However, the preconditioners used in these works vary, and in many cases are notably suboptimal. A handful of works have already demonstrated the effectiveness of Geometric Multigrid (GMG) preconditioners in topology optimization. Here, we show that Algebraic Multigrid (AMG) preconditioners offer superior robustness with only a small overhead cost. The difference is most pronounced when the optimization develops fine-scale structural features or multiple solutions to the same linear system are needed. We thus argue that the expanded use of AMG preconditioners in topology optimization will be essential for the optimization of more complex criteria in large-scale 3D domains.

Keywords Topology Optimization, Multigrid

1 Introduction

In nearly every form of continuum topology optimization, the bulk of the computational cost is incurred in either solving the linear system arising from the Finite Element Analysis (FEA) or in solving another system defined by the same linear operator (such as an adjoint equation for sensitivities). Numerous approaches have been developed in an effort to alleviate some of this cost, such as multiresolution topology optimization (Kim and Yoon 2000, Nguyen et al. 2010), adaptively restricting/expanding the design space (Kim and Kwak 2002), or developing efficient, scalable methods to solve the system of equations (e.g. multigrid-preconditioned conjugate gradient (Amir et al. 2014). The last approach differs in that the optimization procedure itself is unaltered, and changes are isolated to only the associated finite element analysis. This paper will focus on the last approach and what improvements can remain available.

There exist numerous papers in the literature exploring either 3D optimization (Aage and Lazarov 2013, Liu and Tovar 2014, Aage et al. 2015, 2017) or large scale 2D optimization (Amir et al. 2014, Jang and Kim 2010), both of which generally require iterative solvers for the solution of the linear system in the finite element analysis. The efficiency of these solvers is more dependent on the choice of preconditioner than the iterative solver itself. However, even recent papers may make use of suboptimal preconditioners (Benzi and Tůma 1999, Benzi 2002) such as weighted Jacobi (Mahdavi et al. 2006) or incomplete Cholesky factorizations (Liao et al. 2019). While these preconditioners are often easy to set up, their performance is less scalable than multilevel methods such as multigrid preconditioning. Nonetheless, they still have value as smoothers within these multilevel methods (Benzi 2002). A few papers have explored the use of multigrid preconditioners in topology optimization (Aage and Lazarov 2013, Amir et al. 2014, Aage et al. 2015) with very promising results. However, these studies are limited to only geometric multigrid (GMG) on uniform FEM grids.

To the best of our knowledge, no works exploring the use of AMG preconditioners in topology optimization exist in the literature. This may be at least partially attributed to the ease of implementing GMG for...
topology optimization. The vast majority of optimization implementations use uniform Q4 meshes in 2D or Hex8 elements in 3D. These uniform grid structures are easy to coarsen geometrically; however, the geometric approach ignores the evolution of the underlying topology. The few examples with topology-agnostic preconditioners (such as GMG) available in the literature do not suffer a major reduction in performance as the topology evolves, but we will demonstrate cases where a topology-aware preconditioner (such as algebraic multi-grid, AMG) demonstrates significant improvement over the GMG approach.

This paper compares the use of AMG vs. GMG preconditioners for topology optimization and what factors influence their relative performance. We start in Section 2 by outlining the topology optimization framework in which the comparisons are performed. Section 3 details the methods used to solve the linear systems and generalized eigenvalue problem, including a comparison between the basic features of AMG and GMG. In Section 4 we present a variety of example problems to provide a numerical comparison of the performance of AMG and GMG in different scenarios. We conclude with a discussion of the findings in Section 4 and recommendations for the appropriate use of AMG and GMG preconditioners in topology optimization.

2 Topology Optimization

To demonstrate the performance of the various preconditioners in topology optimization we consider two standard problems: compliance minimization and stability maximization. The first problem demonstrates the performance of the preconditioners when only a single solution to the linear system is needed and the second demonstrates the performance when multiple solutions to the same linear system are needed. In both cases we use the modified solid isotropic material with penalization (SIMP) approach (Sigmund and Torquato 1997) with the linear density filter (Bruns and Tortorelli 2001).

2.1 Compliance minimization

The compliance minimization problem takes the following form

\[
\begin{align*}
\min_{\alpha} \quad & \mathcal{F}(\alpha) = f^T u \\
n\text{s.t.:} \quad & \sum_{e=1}^{N_{el}} v_e \rho_e \leq V \\
& 0 \leq \alpha_e \leq 1 \quad e = 1, \ldots, N_{el}
\end{align*}
\]

where \( \alpha \) represents the design variables, \( S \) is the filtering matrix, \( \rho \) represents the filtered densities, \( K \) is the stiffness matrix, \( u \) is the vector of displacements, \( f \) is the vector of external forces, \( v_e \) is the volume of element \( e \), and \( V \) is the total allowable volume of the structure. To prevent the stiffness matrix from becoming singular, the element stiffnesses used to construct the local stiffness matrices are calculated using the modified SIMP rule:

\[
E(\rho) = E_{\text{min}} + (E_{\text{max}} - E_{\text{min}}) \rho^p
\]

where the penalty, \( p \) is gradually increased from 1 to 4 and \( E_{\text{min}} \) is set as \( 1e - 10E_{\text{max}} \).

Using the adjoint method, the sensitivities of the objective function are calculated as

\[
\begin{align*}
\frac{\partial \mathcal{F}}{\partial \rho_e} &= -\frac{\partial E_e}{\partial \rho_e} u^T \frac{\partial K}{\partial E_e} u \\
\frac{\partial \mathcal{F}}{\partial \alpha} &= S^T \frac{\partial \mathcal{F}}{\partial \rho}
\end{align*}
\]

For the sake of generality we will use the method of moving asymptotes (MMA) (Svanberg 1987) for design updates.

2.2 Stability maximization

The problem of optimizing for structural stability takes the following form

\[
\begin{align*}
\min_{\alpha} \quad & \mathcal{F}(\alpha) = \frac{1}{P_{\text{critical}}} = \lambda_{\text{max}} \\
n\text{s.t.:} \quad & \sum_{e=1}^{N_{el}} v_e \rho_e \leq V \\
& 0 \leq \alpha_e \leq 1 \quad e = 1, \ldots, N_{el}
\end{align*}
\]

where:

\[
\begin{align*}
\rho &= S\alpha \\
K(\rho)u &= f \\
K_e\Phi &= \lambda K\Phi
\end{align*}
\]
where $K_\sigma$ is the stress stiffness matrix, $\Phi$ is the eigenvector of the generalized system, and $\lambda$ is the corresponding eigenvalue. Because $K_\sigma$ is potentially indefinite, it is most natural to write the generalized eigenvalue equation in this form so that the matrix on the right-hand-side is positive definite, which is assumed for many eigenvalue solvers. To prevent critical buckling modes from appearing in non-structural regions of the domain we use another modified version of the SIMP formula to interpolate the values of stiffness for the stress stiffness matrix (Bendsøe and Sigmund 2003; Gao and Ma 2015; Thomsen et al. 2018).

\[ E_\sigma(\rho) = \begin{cases} E_{\text{max}} \cdot \rho^p, & \text{if } \rho \geq 0.1 \\ 0, & \text{if } \rho < 0.1 \end{cases} \]  

(5)

Again using the adjoint method, we can derive the sensitivities of the stability problem as

\[ \frac{\partial F}{\partial \rho_e} = \Phi^T \left( \frac{\partial E_{\sigma,e}}{\partial \rho_e} \frac{\partial K_{\sigma,e}}{\partial E_e} - \lambda_{\text{max}} \frac{\partial E_e}{\partial \rho_e} \frac{\partial K}{\partial E_e} \right) \Phi + v^T \frac{\partial E_{\sigma,e}}{\partial \rho_e} \frac{\partial K}{\partial E_e} u \]  

(6)

where $v$ is the solution to the adjoint equation defined by

\[ K_v = \Phi^T \frac{\partial K_\sigma}{\partial \rho} \Phi \]  

(7)

Here we see that the adjoint equation requires another solution to a linear system defined by $K$ for each eigenvalue calculated, in addition to the multiple solutions required inside any solver for the generalized eigenvalue problem. While the formulation shown here includes only the maximal eigenvalue in the optimization, in practice it is necessary to aggregate a subset of the largest eigenvalues (Ferrari and Sigmund 2019). For the purposes of this paper we optimize the 1-norm of the set of $n_{\text{max}}$ largest eigenvalues such that any two consecutive eigenvalues in this set are separated by no more than 1% of each other, possibly changing the number of eigenvalues considered at every iteration.

3 Solvers

3.1 Linear Solvers

First we discuss the methods available to solve a single linear system, either for calculating displacements or solving the adjoint equation. The system of equations takes the form $KU = F$, though $U$ and $F$ may represent something other than the displacements and external forces in the case of the adjoint equation. $K$ is an arbitrarily large, sparse, symmetric, and positive definite matrix with an approximately constant number of nonzeros per row. In this case, the sparse Cholesky factorization operates in $O(n^{3/2})$ time ($n$ being the size of the stiffness matrix) (Davis and Society for Industrial and Applied Mathematics. 2006). For small 2D problems this scaling is generally satisfactory for solving the linear system, and offers the additional bonus that subsequent solutions to the same linear system can make use of the same factorization and operate in closer to $O(n)$ time.

The advantages of the Cholesky factorization, and direct solvers in general, begin to fade for large problems in 2D and even for smaller problems in 3D. The factorization requires substantially more memory than the matrix itself and as $n$ grows, the difference in $O(n^{3/2})$ and $O(n)$ time becomes significant. In these cases, iterative solvers (Saad 2003) are more attractive as their cost is dominated by $O(n)$ matrix-vector operations, and storage requirements beyond the matrix itself are limited to a small set of vectors of length $n$. The Conjugate Gradient method (Hestenes and Stiefel 1952) is preferred for sparse, symmetric, positive-definite (SPD) matrices because it most effectively makes use of these properties, as opposed to more general methods like Generalized Minimum Residual (GMRES) (Saad and Schultz 1986). While the CG method is guaranteed to converge in $n$ iterations, in practice a certain amount of error is allowed in the solution and iterations are stopped when a convergence tolerance is reached. Thus, the effectiveness of the method for solving a linear system is governed (roughly) by the convergence rate

\[ \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \]  

(8)

where

\[ \kappa = \text{cond}(K) = \frac{\lambda_{\text{max}}(K)}{\lambda_{\text{min}}(K)} \]  

(9)

In the case of topology optimization $\kappa \gg 1$, especially after the optimization begins to produce a structure, and void regions of the design domain assume a stiffness several orders of magnitude smaller than the solid regions. For a fully converged solid-void structure, the lowest energy modes are almost always confined to the void regions while the high energy modes exist primarily in the solid regions, meaning that $\kappa$ scales roughly with $E_{\text{min}}$.

The ill-conditioning of the system is not unique to topology optimization, but is instead very common when using iterative solvers for any linear system. The poor
conditioning is overcome through the use of a quality preconditioner, \( \mathbf{M} \). The preconditioner, which may itself be another matrix or simply an operator, is chosen so that \( \kappa(\mathbf{M}^{-1}\mathbf{K}) \ll \kappa(\mathbf{K}) \). Most of the literature on topology optimization uses direct methods to solve the linear system, where \( \mathbf{M} = \mathbf{K} \) explicitly. There are a handful of works that explore the use of other preconditioners, particularly for optimizations in 3D (Aage and Lazarov 2013 [Aage et al. 2015]) or with a large number of degrees of freedom in 2D (Mahdavi et al. 2006). Some of these works have shown great promise using multigrid preconditioners (Amir et al. 2014 [Kennedy 2015]), though they focus exclusively on geometric versions.

### 3.2 Multigrid Methods

Multigrid methods are a class of multilevel methods that replicate the discretization of the original partial differential equation (PDE) on increasingly coarser grids to improve performance. They are based on the premise that while smoothers such as Gauss-Seidel or Jacobi may be ineffective at reducing all of the error in a solution approximation, they are very effective at removing error with large eigenvalues (high-energy errors) (Saad 2003). There are various methods available to perform this hierarchy assembly, going back to the classical Ruge-Stüben (Ruge and Stüben 1987). All methods follow a general procedure of identifying which nodes (or degrees of freedom) are strongly connected to each other and lumping for each level of the multigrid. The operator on the fine grid is given from the original linear system of equations for displacements, and each subsequent operator is defined by the Galerkin projection as

\[
\mathbf{A}^{i+1} = \mathbf{P}^T \mathbf{A} \mathbf{P}
\]  

If a direct solver is used on the coarsest level, the coarse operator should also be factorized at this time.

#### Algorithm 1 V-cycle multigrid with weighted Jacobi smoothing for the system \( \mathbf{Ax} = \mathbf{b} \)

```
procedure MG(\( \mathbf{A}, \mathbf{P}, \mathbf{b}, k, n_{\text{levels}}, n_{\text{pre}}, n_{\text{post}}, w \))
    \( x = 0 \)
    for \( i = 1..n_{\text{pre}} \) do
        \( r = \mathbf{b} - \mathbf{A}^i x \)
        \( x^+ = w * (\text{diag}(\mathbf{A}^i)^{-1} r) \)
    end for
    \( b = (\mathbf{P}^T \mathbf{b}) - \mathbf{A}^i x \)
    if \( k < n_{\text{levels}} - 1 \) then
        \( x^+ = \mathbf{P}^k * \text{MG} (\mathbf{A}, \mathbf{P}, \mathbf{b}, k + 1, n_{\text{levels}}, n_{\text{post}}, w) \)
    else
        \( x^+ = \mathbf{P}^k * (\text{diag}(\mathbf{A}^k)^{-1} \mathbf{b}) \)
    end if
    for \( i = 1..n_{\text{post}} \) do
        \( r = \mathbf{b} - \mathbf{A}^i x \)
        \( x^+ = w * (\text{diag}(\mathbf{A}^i)^{-1} r) \)
    end for
    return \( x \)
end procedure
```

Algebraic methods differ from geometric methods in that the levels in the multigrid are constructed not based on the mesh, but instead are constructed directly from the linear operator itself. This gives the method flexibility to be applied to problems where a regular mesh may not be available, or grid restriction does not accurately capture smooth modes of the operator (as in the case of anisotropic diffusion (Saad 2003)). There are various methods available to perform this hierarchy assembly, going back to the classical Ruge-Stüben (Ruge and Stüben 1987). All methods follow a general procedure of identifying which nodes (or degrees of freedom) are strongly connected to each other and lumping...
them into “supernodes” as part of the restriction operation. The definition of “strongly connected” is somewhat heuristic, but in the case of topology optimization it ensures that nodes attached to structural elements are not lumped with nodes in the void regions. For this work we will focus on the smoothed aggregation method (Vanek et al. 1996) due to its superior performance and wide availability in software packages.

The assembly of the operator hierarchy in AMG works similarly to GMG, but with a few additional steps. Starting with the fine grid operator $A^0$, the off-diagonal elements of the matrix are compared to diagonal elements to determine which degrees of freedom are strongly connected. A restriction operator, $R$ is then constructed based on the type of AMG used, which projects smooth error from the fine system onto a coarsened system where multiple strongly connected degrees of freedom are represented with only a few reduced degrees of freedom. Again taking advantage of the symmetry of the system and using the Galerkin projection to define coarse grid operators using Equation 10 and factorize the coarsest operator.

### 3.3 Eigenvalue solvers

The performance of eigenvalue solvers, particularly for the generalized eigenvalue problem, is more complicated than that of linear solvers. While all practical eigensolvers must be iterative methods (Abel 1824), generalized eigensolvers internally require a solution to a linear system at each iteration, which may be solved using any of the previously described methods for solving linear systems of equations. Some methods, such as Arnoldi or Lanczos, (Arnoldi 1951) make use of a factorization of one of the matrices ($K$ in this case as it is positive definite) to get a high-precision solution each time. Other methods, so-called preconditioned eigensolvers, make use of an iterative method to solve (or approximate the solution to) a linear system. As in the case of simple linear systems, matrix factorizations are really only feasible for smaller problems and the preconditioned eigensolvers are necessary for large-scale problems. It should be noted that in the eigenvalue problem only one factorization is needed, but the system must be solved with many right-hand-sides. This suggests that the tradeoff in efficiency between factorized and preconditioned methods occurs at a larger system size in the eigenvalue problem than that of the simple linear system.

Of the preconditioned eigensolvers, three are widely used and demonstrate good performance in the problem type we are examining. Two are variations of Davidson’s method: generalized Davidson and Jacobi-Davidson (Saad 2011), and the other is the locally optimal block preconditioned conjugate gradient (LOBPCG) (Knyazev 2001). While all three methods use a preconditioner to solve a linear system, they differ in how that system is defined and in how accurately it needs to be solved. The Jacobi-Davidson method, with its internal correction equation, is more suited to target interior eigenvalues (Sleijpen et al. 1996), whereas generalized Davidson and LOBPCG are more effective for problems where exterior eigenvalues are needed (as in the case of stability optimization). In this paper we use the generalized Davidson’s method because the larger search space it uses gives it better performance in the context of stability optimization than LOBPCG.

### 4 Numerical Results

To demonstrate the relative performance of AMG and GMG we will look at four different topology optimization cases. We use the PyAMG library (Olson and Schroder 2018) to provide the framework for both multigrid schemes, though the projection/restriction operators for the GMG preconditioner is set up manually. For every case the AMG and GMG preconditioners are set up with enough levels that the coarsest grid has no more than 80 nodes. At each level of the preconditioner a single pre- and post-smoothing pass is applied using weighted Jacobi, and the coarsest level is solved with an LU decomposition. Within the AMG preconditioner we also compare the performance of a blocked (one block per node) and unblocked weighted Jacobi smoother. Our results indicated no difference in performance between blocked and unblocked versions for the GMG preconditioner, so only the results for the block smoother are shown here. For brevity, the three schemes will be referred to as GMG, AMG, and block AMG for the remainder of this manuscript.

For the AMG preconditioner, strength of connection is calculated in a block fashion (one value per node) using the symmetric strength of connection formulation with $\alpha = 0.003$.

$$S = K(i, j)^2 > \alpha K(i, i)K(j, j)$$ (11)

To construct the prolongation operator connecting the first two levels of multigrid, candidate vectors (the rigid body modes before boundary conditions are applied) are smoothed with 4 iterations of block Gauss-Seidel (1 block per node). No such candidate smoothing is applied to any of the subsequent prolongation operators connecting lower levels of the hierarchy. However, once assembled, each tentative prolongation operator is improved with a single pass of a weighted Jacobi
The system of linear equations for displacements is solved with preconditioned conjugate gradients, to a relative residual tolerance of 1e-8. The initial guess for the iterative solver is set to the displacements calculated in the last topology iteration.

For the optimization itself we use continuation on the penalty parameter, with the penalty initially set to 1 and gradually increased to 4. The number of iterations at each penalty value and the step size for the penalty values varies slightly between the different examples. Penalization is performed according to the modified SIMP method (Sigmund and Torquato 1997), with minimum stiffness set to 1e-10. The design variable updates are calculated using the Method of Moving Asymptotes (MMA) (Svanberg 1987). In each case we use a density filter with radius equal to 1.5 times the element dimensions. The near constant coarse grid size and decreasing feature size will help to demonstrate how both multigrid approaches fare at representing finer features on coarse grids.

4.1 2D cantilever beam

The first example is compliance minimization for a cantilever domain in 2D with an aspect ratio of 2:1. The design domain and result of the highest resolution optimization are illustrated in Figure 2. We run the problem at four different resolutions: 96x48 elements, 192x96 elements, 384x192 elements, and 768x384 elements. We perform 80 optimization iterations for each penalty value and the penalty is increased in increments of 0.25.

Figure 3a shows the time to set up each of the preconditioners for varying grid sizes, measured by the number of global degrees of freedom. Data for each optimization iteration is shown as a scatterplot, and a moving average trendline for each grid size is shown for clarity. The times are fairly consistent between preconditioners for a given mesh size, although the GMG setup costs about 20% less than the AMG setup and 30% less than the block AMG setup. Figure 3b similarly shows the time to solve the system of equations using preconditioned conjugate gradient with the respective preconditioners. The trend here is reversed, the block AMG preconditioner is the fastest and GMG is the slowest. The difference in performance also increases with problem size, for the smallest problem the block AMG preconditioner is only about 30% faster than GMG, but for the largest problem it is nearly 300% faster (standard AMG is pretty consistently 50% more expensive than block AMG). Figure 3c shows the combined time to assemble the preconditioners and solve the linear system.

For the smallest problem size the performance of all three preconditioners is nearly identical (no more than 2% difference between all three); however, the AMG preconditioner scales slightly worse than the block AMG preconditioner with problem size, and GMG scales much worse than either of them. For the largest problem size AMG runs in about half the time of GMG and block AMG is almost three times faster than GMG.

It is important to further note that the GMG preconditioner performs similarly to the AMG preconditioner at the beginning and end of the optimization, but significantly worse in between. As Figure 3d shows, this is directly attributable to worsened convergence of the preconditioned conjugate gradient solver (increased number of iterations) in the intermediate stages of the optimization process. The decreased performance also coincides with the penalty increasing from the value of unity, prompting the optimization to begin developing well-defined structural features and steeper gradients of material stiffness (Figure 4). Prior to this stage the element densities vary smoothly across the domain, making the system amenable to geometric multigrid. This is further explained as follows.

As structural features develop, the GMG preconditioner tends to “blur” them together, leading to worsened performance. These features appear as soon as the penalty parameter is increased from unity, and after several more increments of the penalty parameter the majority of the domain has become either solid or void. At this point the optimization has difficulty adding new features and primarily evolves the structure through moving or removing features incrementally. This means that changes to the structure are more modest and the displacement field changes much more gradually through the optimization iterations. As a result, the displacement field from the previous iteration serves as an excellent initial guess for the iterative solver in the next iteration. As the optimizer finalizes the structure and the displacement field “converges,” the iterative solver needs to do less work to correct the displacements at every optimization iteration. For this reason, performance of the iterative solver begins to improve regardless of the preconditioner used, though it rarely matches the performance at the beginning of the optimization process.

4.2 3D cantilever beam

The next example is compliance minimization for a cantilever domain in 3D with an aspect ratio of 2:1:1. The design domain and result of the highest resolution optimization are shown in Figure 5. The problem is run at three different resolutions: 16x8x8 elements, 32x16x16 elements, and 64x32x32 elements. We perform 80 optimization iterations for each penalty value and the
penalty is increased in increments of 0.5 (the penalty increment is higher than in 2D due to the higher cost of 3D simulations).

The overall performance of the various preconditioners on the 3D cantilever problem are shown in Figure 6. The relative performance is similar to the 2D case; however, now the GMG preconditioner outperforms AMG for the smaller two problem sizes. For these problem sizes, the number of iterations to converge is similar between AMG and GMG, both requiring less than 20 iterations at a majority of the optimization steps. For the larger problem size though, as soon as the structure begins to develop, the average number of iterations for the GMG preconditioner jumps to 75, while the AMG preconditioner only requires 30 iterations. Again breaking the cost down into the setup and solve phase, the AMG preconditioner takes 2-2.5 times longer to setup than GMG and the block AMG preconditioner takes 2-3 times longer than GMG. In the solve phase the relative performance is drastically different depending on the problem size. For the smaller two problems AMG takes about 40% longer than GMG, and block AMG only 15% longer. However, for the larger problem size AMG and block AMG takes 40% and 50% less time than GMG, respectively. Combined, the total time to setup and solve the system is 50-80% longer for either AMG solver in the smaller problems, but about 20% shorter for the larger problem.

To understand the spike in number of iterations for the GMG preconditioner at the highest resolution, we compare the result of the larger two optimizations, shown in Figure 7. The two figures represent the optimized structure on the same domain with the same boundary conditions, but the structure on the right was optimized on a mesh with exactly twice as many elements in each direction. Both structures also have the same number of nodes at the coarsest level of the GMG hierarchy. At this coarsest resolution, there are only 3 nodes in each of the shorter dimensions, and 5 along the primary axis of the cantilever, corresponding to a 4x2x2 mesh. Note that the structure on the left consists of one main feature along the primary axis, and the structure on the right consists of two. When projected to the coarsest level of the GMG hierarchy, both structures have a very similar representation. Most importantly, as this mesh contains only two elements along either transverse axis, the two features in the second result are effectively fused together, eliminating many smooth deformation modes of the structure where the features deform independently. The loss of these modes is the primary reason for the decrease in performance of the GMG preconditioner shown in Figure 6c. This phenomenon is further examined with the final example.

4.3 Column Stability

The next problem is less common in the literature, but will help us to further differentiate the performance of the two multigrid approaches. Here we perform stability optimization of a column as described in (Bendsøe and Sigmund 2003). The design domain has aspect ratio 4:1 and we run the problem at three different resolutions: 32x128 elements, 64x256 elements, and 96x392 elements. We perform 70 optimization iterations for each penalty value and the penalty is increased in increments of 0.25. The design domain and boundary conditions, as well as a sample optimized shape from the largest problem, are shown in Figure 8.

When optimizing for stability we have to solve both an eigenvalue problem for the structural performance and an adjoint problem for the sensitivities. Whereas compliance optimization requires only a single solution to a linear system each time the preconditioner is constructed, stability optimization requires multiple solutions to the same linear system with different right-hand-sides. Thus, we use this example to demonstrate how the preconditioners perform when the improvement to the conditioning of the system is more important relative to the setup cost.

When solving for displacements we use the same procedure as before, however the procedure to solve the
Fig. 3: Performance of the different preconditioners in the 2D cantilever optimization on varying mesh sizes. I) GMG, II) AMG, III) Block AMG.
adjoint problem is slightly different. We retain the same iterative solver and preconditioners, but now we use an all-zero initial condition and the residual tolerance is also relaxed to 1e-5. To solve the eigenvalue problem we use the generalized Davidson method with the same preconditioners as before. At every optimization step we calculate six eigenvalues to a relative residual tolerance of 1e-8 for a maximum of 1e3 eigenvalue iterations.

The performance when solving for displacements is shown in Figure 9. The trends are again consistent with what we identified in the 2D cantilever example, namely that the AMG solvers have a higher setup cost, but overall the time to solve for displacements is lower thanks to a significant reduction in the number of CG iterations. The main difference for the stability optimization problem is that the AMG preconditioner performs noticeably worse than the block AMG preconditioner. This is due to the fact that the block smoothing allows the preconditioner to more accurately solve for the displacements of supernodes on the coarse grids. Quantitatively, the time to setup the preconditioner and solve for displacements is consistently twice as much for GMG than block AMG and 50% higher for AMG than block AMG.

The performance of the preconditioners in the eigen-solver and for solving the adjoint equations are shown in Figures 10 and 11, respectively. As expected based on previous results, the GMG preconditioner again exhibits the worst performance and the block AMG preconditioner performs the best. Looking first at the eigensolver performance, the GMG preconditioner struggled to find all six eigenvalues for the larger two problem sizes. For the majority of the optimization iterations in these two problems the eigensolver was stopped after 1,000 iterations even though as few as two eigenvalues were found. In comparison, the AMG preconditioned eigensolver found all six eigenvalues in the majority of the optimization iterations and the block AMG preconditioned eigensolver found all six eigenvalues for all but four iterations of the largest problem and all but...
Fig. 6: Performance of the different preconditioners in the 3D cantilever optimization on varying mesh sizes. I) GMG, II) AMG, III) Block AMG.
one iteration of the second largest problem. Even in the cases where all six eigenvalues were found for each preconditioner, the reduced iteration counts with the block AMG preconditioned solver meant that the time spent in the eigensolver was greatly reduced.

The trend for the adjoint equation solver (Figure 11) is somewhat harder to identify. In general, AMG outperformed GMG slightly, and block AMG offered additional improvement. However, in all three cases the results have a large number of outlier values where the iteration counts are much different than the median values. Nonetheless, when comparing the moving average the performance trend remains the same, with block AMG providing the best results. Quantitatively, block AMG takes about 80% less time than GMG and 30% less than AMG.

4.4 Bridge Domain

The final problem is somewhat contrived, but serves to illustrate the exact situations where AMG outperforms GMG for topology optimization. Here, we use the same domain from the previous 2D cantilever example with aspect ratio of 2:1. However, now we fix the top 5% of the domain to be a solid structure and apply a uniform distributed load to the top edge. Fixed supports of width 0.08 are applied with various uniform spacings to the bottom of the domain. In the case of just 2 supports at either end, we approximate the bridge problem described in [Zegard and Paulino 2016], albeit only in 2D. As the number of supports increases, we see longer and thinner columns spanning from the top of the domain all the way to the supports at the bottom (Figure 13). These columns and their associated low-energy bending modes prove troublesome for the GMG preconditioner, while the AMG preconditioner handles them without trouble. We run the optimization with 2, 4, 8, 16, and 25 supports. The case of 25 supports is equivalent to fixing every node on the bottom of the domain. The optimized structures from each case are shown in Figure 13.

The performance of each preconditioner as the number of supports is changed is shown in Figure 14. Once again, the AMG and block AMG preconditioners are much more expensive to set up (both about 50% more expensive to setup than GMG); however, the cost is offset by the decreased number of iterations necessary.

Fig. 7: Optimized 3D cantilever beams on 2 different meshes.

Fig. 8: Column stability problem domain and result.

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Fig. 9: Performance of the different preconditioners when solving for displacements in the stability optimization on varying mesh sizes. I) GMG, II) AMG, III) Block AMG.
Fig. 10: Performance of the different preconditioners in the stability optimization eigenvalue computation for varying mesh sizes. I) GMG, II) AMG, III) Block AMG.

Fig. 11: Performance of the different preconditioners in the stability optimization adjoint computation for varying mesh sizes. I) GMG, II) AMG, III) Block AMG.
The bottom edge of the domain is fixed in small patches with varying spacing, and a uniform distributed load is applied to the top of the domain.

The time to setup and solve with the GMG preconditioner is consistently 50% more than that for both AMG preconditioners, except for the case of 16 supports where the AMG preconditioners require roughly half the time of GMG.

The major difference in performance of the AMG and GMG preconditioners for the case of 16 supports is a result of the columns becoming longer and/or thinner as the number of supports is increased, making them more flexible. The more flexible a column is, the lower the energy of its principle bending mode. For structural members with a sufficiently high aspect ratio, the bending modes will become smooth enough that they cannot effectively be removed from the solution by a smoother, even after restriction to coarser grids. If the columns are also close enough together, the coarse levels of the geometric multigrid may blur them together, eliminating their bending modes before reaching the coarse grid (the same phenomena discussed for the 3D cantilever). AMG preconditioners avoid this problem through the strength of connection measure, which effectively prevents features from connecting across void regions.

To further clarify this behavior, Figure 15 shows the number of iterations required for the preconditioned conjugate gradient PCG solver as the number of levels in the multigrid hierarchies is increased for the case of continuous supports. While the AMG preconditioners require a near constant number of iterations, the GMG preconditioners see a sharp increase in iteration counts when the number of multigrid levels is increased to 4 or 5. The effective representation of the structure at each level of the multigrid is shown in Figure 16. Note that level 2 captures all the features of the original structure, and level 3 only introduces a slight blurring in the middle of the domain. However, by level 4 nearly all of the features have blurred together, and at level 5 the structure is completely unrecognizable. As structural features are blended together, their associated deformation modes (which in this case are very low-energy) are lost in the projection. In these situations, the capacity for AMG preconditioners to retain these modes grants them superior performance.

5 Discussion

We have analyzed the relative performance of geometric (GMG) and algebraic (AMG) preconditioners in the context of topology optimization. For topology optimization at large scales it is necessary to use iterative solvers, which rely on effective preconditioners for their performance. AMG and GMG preconditioners both use the same basic procedure for solving or preconditioning a linear system; however, the methods differ in how the preconditioners are constructed. In GMG, the coarser grids are constructed directly from the mesh that discretizes the problem domain, ignoring the evolution of structural features throughout the design optimization. In contrast, AMG methods perform grid coarsening based on the stiffness matrix alone, without any direct knowledge of the underlying discretization of the partial differential equation (PDE).

In all of our examples we have seen that the AMG preconditioner is more expensive to construct due to the nature of the interpolation/restriction operators. However, the setup cost is generally offset by the fact that the AMG preconditioner more actively adapts to changing structural topology. For some simple cases seen often in the literature, the performance of GMG preconditioners is similar to or slightly better than AMG preconditioners. However, we have also demonstrated several cases where the AMG preconditioners are much more robust due to the extra work in their assembly. The increased robustness comes from the inherent capacity for AMG to identify where structural features exist while constructing the hierarchy. In addition, AMG preconditioners are much better suited for optimization problems on irregular meshes, for example when polygonal meshes are used (Talischi et al. 2012).

The relative merit of GMG or AMG preconditioners is also highly dependent on the type of optimization being performed. In the simple case of compliance minimization, where only a single solution to the linear system is needed, the GMG preconditioner is often just as effective as the AMG preconditioner. In cases where additional solutions to the linear system are needed, for example to solve adjoint problems or evaluating structural stability using the generalized eigenvalue problem, AMG methods are much more effective. In these cases the better performance of the iterative solver with
an AMG preconditioner is more important than the cheaper setup cost of the GMG preconditioner.

We conclude that for simpler optimization formulations such as compliance minimization (only one solution to the linear system is needed) on uniform grids, the relative simplicity of GMG makes it a very appealing preconditioner. However, for problems where multiple solutions to the same system are needed or the topology exhibits certain characteristics, such as tightly-packed, highly flexible features, AMG offers a substantial performance improvement. In addition, AMG readily extends to problems on irregular domains or non-uniform meshes, and may be combined with other cost saving measures, such as design space optimization, to further improve performance. Algebraic multigrid has also experienced sufficient development that near black-box functions are available in most scientific computing environments [Balay et al. 2016; Olson and Schroder 2018; Falgout and Yang 2002].

A possible extension of this study would be to combine a GMG preconditioner at the finest levels of the hierarchy and an AMG preconditioner at the coarsest levels. We have already shown that even if the full GMG preconditioner is inefficient, it still may perform well on the first few levels of the hierarchy. The performance of GMG only begins to falter when structural features are sufficiently fine to the point that the grid coarsening begins to blur them together. At this point the coarsening strategy could be switched to make use of an AMG scheme to keep structural features distinct. This hybrid approach would make use of GMG’s cheap setup and excellent performance on fine grids with a large number of degrees of freedom while allowing AMG to extend the hierarchy even further without suffering as much from the increased setup costs at the fine levels.

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**Conflict of interest** The authors declare that they have no conflict of interest.

**Replication of results** The code for this paper is available at [https://github.com/darinpeetz/PyOpt](https://github.com/darinpeetz/PyOpt).
Fig. 14: Performance of the different preconditioners in the bridge optimization for varying number of supports. I) GMG, II) AMG, III) Block AMG.
Fig. 15: Number of iterations to solve the system of equations for displacements for various preconditioners and multigrid levels for the bridge problem with continuous supports. A) GMG, B) AMG, C) Block AMG.

Fig. 16: Bridge result with continuous supports projected onto each level of the GMG hierarchy.
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