Inverse spectral analysis for a class of finite band symmetric matrices

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Abstract

In this note, we solve an inverse spectral problem for a class of finite band symmetric matrices. We provide necessary and sufficient conditions for a matrix valued function to be a spectral function of the corresponding operators and give an algorithm for recovering the matrix from this spectral function. Our approach to the inverse problem is based on the rational interpolation theory developed in a previous paper.

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1. Introduction

This work deals with the direct and inverse spectral analysis of a class of finite symmetric band matrices with emphasis in the inverse problems of characterization and reconstruction. Inverse spectral problems for band matrices have been studied extensively in the particular case of Jacobi matrices (see for instance [5] [4] [10] [15] [18] [19] [20] [27] [28] [29] for the finite case and [7] [8] [9] [10] [16] [17] [30] [31] for the infinite case). Works dealing with band matrices non-necessary tridiagonal are not so abundant (see [3] [13] [14] [22] [23] [26] [32] [33] for the finite case and [2] [12] for the infinite case).

Let $\mathcal{H}$ be a finite dimensional Hilbert space and fix an orthonormal basis $\{\delta_k\}_{k=1}^N$ in it. Consider the operators $D_j$ ($j = 0, 1, \ldots, n$ with $n < N$), whose matrix representation with respect to $\{\delta_k\}_{k=1}^N$ is a diagonal matrix, i.e., $D_j \delta_k = d_k^{(j)} \delta_k$ for all $k = 1, \ldots, N$, where $d_k^{(j)}$ is a real number. Also, let $S$ be the shift operator, that is, $S \delta_k = \begin{cases} \delta_{k+1} & k = 1, \ldots, N-1 \\ 0 & k = N. \end{cases}$

The object of our considerations in this note is the symmetric operator

$$A := D_0 + \sum_{j=1}^n S^j D_j + \sum_{j=1}^n D_j (S^*)^j. \quad (1.1)$$

Hence, the matrix representation of $A$ with respect to $\{\delta_k\}_{k=1}^N$ is an Hermitian band matrix which is denoted by $\mathcal{A}$.

We assume that the diagonals satisfy the following conditions.

The diagonal farthest from the main one, which is given by the diagonal matrix $\text{diag}\{d_k^{(n)}\}_{k=1}^N$ representing $D_n$ with respect to $\{\delta_k\}_{k=1}^N$, is such that all the numbers $d_1^{(n)}, \ldots, d_{m_1-1}^{(n)}$ are different from zero and $d_{m_1}^{(n)} = \cdots = d_{N-n}^{(n)} = 0$ with

$$1 < m_1 \leq N - n + 1. \quad (1.2)$$

Now, the elements $d_{m_1+1}^{(n-1)}, \ldots, d_{N-(n-1)}^{(n-1)}$ of $D_{n-1}$ behave in the same way as the elements of $D_n$, that is, there is $m_2$, satisfying

$$m_1 < m_2 \leq N - (n-1) + 1, \quad (1.3)$$

such that $d_{m_1+1}^{(n-1)}, \ldots, d_{m_2-1}^{(n-1)} \neq 0$ and $d_{m_2}^{(n-1)} = \cdots = d_{N-(n-1)}^{(n-1)} = 0$. We continue applying this rule up to the diagonal given by $D_2$ which gives the number $m_{n-1}$, and assume that the numbers $d_{m_{n-1}+1}^{(1)}, \ldots, d_{N-1}^{(1)}$ are all different from zero. Note that, by construction, the elements of the diagonal matrix representing
$D_j$ satisfy
\[ d_{m_j}^{(n-j+1)} = 0 \]
for $j = 1, \ldots, n - 1$. For convenience, we define $m_n := N$.

**Definition 1.** Fix the natural numbers $n$ and $N$ such that $n < N$. All the matrices satisfying the above properties for a given set of numbers $\{m_i\}_{i=1}^n$ are denoted by $\mathcal{M}(n, N)$. Note that in this notation, $N$ represents the dimension and $2n + 1$ is the number of diagonals of the matrices.

An example of a matrix in $\mathcal{M}(3, 7)$, when $m_1 = 3$ and $m_2 = 5$, is the following.

\[
\mathcal{A} = \begin{pmatrix}
  d_1^{(0)} & d_1^{(1)} & d_1^{(2)} & d_1^{(3)} & 0 & 0 & 0 \\
  d_1^{(1)} & d_2^{(0)} & d_2^{(1)} & d_2^{(2)} & d_2^{(3)} & 0 & 0 \\
  d_2^{(2)} & d_3^{(0)} & d_3^{(1)} & d_3^{(2)} & 0 & 0 \\
  d_3^{(1)} & d_4^{(0)} & d_4^{(1)} & d_4^{(2)} & 0 \\
  0 & d_4^{(2)} & d_5^{(0)} & d_5^{(1)} & 0 \\
  0 & 0 & 0 & d_5^{(2)} & d_6^{(0)} & d_6^{(1)} \\
  0 & 0 & 0 & 0 & d_6^{(2)} & d_7^{(0)}
\end{pmatrix}
\] (1.4)

Here we say that the matrix $\mathcal{A}$ underwent a degeneration of the diagonal $D_3$ in $m_1 = 3$ and a degeneration of $D_2$ in $m_2 = 5$.

We remark that when $m_i = N - n + i$ (see (1.2) and (1.3)) there is no degeneration of the diagonals $D_j$ for all $j \in \{n - i + 1, \ldots, 1\}$.

It is known that the dynamics of a finite linear mass-spring system is characterized by the spectral properties of a finite Jacobi matrix [11, 25] (see Fig. 1) when the system is within the regime of validity of the Hooke law. The entries of the Jacobi matrix are determined by the masses and spring constants of the system [6, 7, 8, 11, 25]. The movement of the mechanical system of Fig 1 is a superposition of harmonic oscillations whose frequencies are the square roots of absolute values of the elements of the Jacobi operator’s spectrum. Analogously,

Figure 1: Mass-spring system corresponding to a Jacobi matrix

one can deduce that a matrix in $\mathcal{M}(n, N)$ models a linear mass-spring system where the interaction extends to all the $n$ neighbors of each mass. For instance, if the matrix is in $\mathcal{M}(2, 10)$ and no degeneration of the diagonals occurs, viz. $m_1 = 9$, the corresponding mass-spring system is given in Fig. 2. If for another
matrix in \( M(2, 10) \), one has degeneration of the diagonals, for instance \( m_1 = 4 \), the corresponding mass-spring system is given in Fig. 3.

![Figure 3: Mass-spring system of a matrix in \( M(2, 10) \): degenerated case](image)

In this work, the approach to the inverse spectral analysis of the operators whose matrix representation belongs to \( M(n, N) \) is based on the one used in [22, 23], but it allows to treat the case of arbitrary \( n \). An important ingredient of the methods used here is the linear interpolation of \( n \)-dimensional vector polynomials, recently developed in [24].

This paper is organized as follows. The next section deals with the direct spectral analysis of the operators under consideration. In this section, a family of spectral functions is constructed for each element in \( M(n, N) \). In Section 3, the connection of the spectral analysis and the interpolation problem is established. Section 4 treats the problem of reconstruction and characterization. In the Appendix, we present an algorithm for computing the heights of the vector polynomials involved in the spectral analysis. The algorithm is constructed heuristically and there are possible applications of it in numerical methods.

### 2. The spectral function

Consider \( \varphi = \sum_{k=1}^{N} \varphi_k \delta_k \in \mathcal{H} \) and the equation

\[
(A - zI)\varphi = 0, \quad z \in \mathbb{C}.
\]  

We know that the equation has nontrivial solutions only for a finite set of \( z \).

From (2.1) one obtains a system of \( N \) equations, where each equation, given
by a fixed $k \in \{1, \ldots, N\}$, is of the form

$$
\sum_{i=0}^{n-1} d^{(n-i)}_{k-n+i} \varphi_{k-n+i} + d^{(0)}_k \varphi_k + \sum_{i=1}^{n} d^{(i)}_k \varphi_{k+i} = z \varphi_k, \quad (2.2)
$$

where it has been assumed that

$$
\varphi_k = 0, \quad \text{for } k < 1, \quad (2.3a)
$$

$$
\varphi_k = 0, \quad \text{for } k > N. \quad (2.3b)
$$

One can consider (2.3) as boundary conditions where (2.3a) is the condition at the left endpoint and (2.3b) is the condition at the right endpoint.

The system (2.2) and (2.3a) restricted to $k \in \{1, 2, \ldots, N-1\} \setminus \{m_i\}_{i=1}^{n-1}$ can be solved recursively whenever the first $n$ entries of the vector $\varphi$ are given.

Let $\varphi^{(j)}(z) (j \in \{1, \ldots, n\})$ be a solution of (2.2) for all $k \in \{1, 2, \ldots, N-1\} \setminus \{m_i\}_{i=1}^{n-1}$ such that

$$
\langle \delta_i, \varphi^{(j)}(z) \rangle = t_{ji}, \quad \text{for } i = 1, \ldots, n, \quad (2.4)
$$

where $\mathcal{T} = \{t_{ji}\}_{j,i=1}^{n}$ is an upper triangular real matrix and $t_{jj} \neq 0$ for all $j \in \{1, \ldots, n\}$.

The condition given by (2.4) can be seen as the initial conditions for the system (2.2) and (2.3a). We emphasize that given the boundary condition at the left endpoint (2.3a) and the initial condition (2.4), the system restricted to $k \in \{1, 2, \ldots, N-1\} \setminus \{m_i\}_{i=1}^{n-1}$ has a unique solution for any fixed $j \in \{1, \ldots, n\}$ and $z \in \mathbb{C}$.

**Remark 1.** Note that the properties of the matrix $\mathcal{T}$ guarantee that the collection of vectors $\{\varphi^{(j)}(z)\}_{j=1}^{n}$ is a fundamental system of solutions of (2.2) restricted to $k \in \{1, 2, \ldots, N-1\} \setminus \{m_i\}_{i=1}^{n-1}$ with the boundary condition (2.3a).

The entries of the vector $\varphi^{(j)}(z)$ are polynomials, so we denote $P^{(j)}_k(z) := \varphi^{(j)}_k(z)$, for all $k \in \{1, \ldots, N\}$. And, define

$$
Q^{(j)}_i(z) := (z - d^{(0)}_{m_i}) P^{(j)}_{m_i}(z) - \sum_{k=0}^{n-1} d^{(n-k)}_{m_i-n+k} P^{(j)}_{m_i-n+k}(z) - \sum_{k=1}^{n-i} d^{(k)}_{m_i} P^{(j)}_{m_i+k}(z)
$$

for $i \in \{1, \ldots, n\}$ (it is assumed that the last sum is zero when $i = n$).

It is worth remarking that the polynomials $\{P^{(j)}_k(z)\}_{k=1}^{N}$ and $\{Q^{(j)}_i(z)\}_{i=1}^{n}$ depend on the initial conditions given by the matrix $\mathcal{T}$.

Define the matrix
\[ Q(z) := \begin{pmatrix} Q_1^{(1)}(z) & \cdots & Q_1^{(n)}(z) \\ \vdots & \ddots & \vdots \\ Q_n^{(1)}(z) & \cdots & Q_n^{(n)}(z) \end{pmatrix}. \]

It turns out that for any \( z \) where there exists a solution of (2.1), there also exists a solution of the homogeneous equation
\[ Q(z) \begin{pmatrix} \beta_1(z) \\ \vdots \\ \beta_n(z) \end{pmatrix} = 0. \quad (2.5) \]

Indeed, since \( \{\varphi^{(j)}(z)\}_{j=1}^n \) is a fundamental system for any \( z \in \mathbb{C} \), the vector \( \beta(z) \), given by
\[ \beta(z) = \sum_{j=1}^n \beta_j(z)\varphi^{(j)}(z), \quad (2.6) \]
is a solution of (2.2), (2.3a). Thus, using the difference equations (2.2), one verifies that
\[ (A - zI)\beta(z) = \sum_{k=1}^N c_k(z)\delta_k, \]
where
\[ c_k(z) := \begin{cases} \sum_{j=1}^n \beta_j(z)Q_i^{(j)}(z) & \text{if } k = m_i, \text{ for all } i = 1, \ldots, n, \\ 0 & \text{otherwise}. \end{cases} \]

Therefore, (2.6) is a solution of (2.1) if
\[ \sum_{j=1}^n \beta_j(z)Q_i^{(j)}(z) = 0 \quad (2.7) \]
for all \( i \in \{1, \ldots, n\} \), which is equivalent to (2.5).

**Lemma 2.1.** Let \( \bar{n}(z) := \dim \ker(A - zI) \). Then,
\[ \text{rank}(Q(z)) = n - \bar{n}(z). \]

**Proof.** The proof is straightforward. Having fixed \( z \in \mathbb{C} \), one recurs to the Kronecker-Capelli-Rouché Theorem (see [21, Chap.3 Secs.1-2]) to obtain that the dimension of the space of solutions of (2.5) is equal to \( n - \text{rank}(Q(z)). \) \( \square \)
Immediately from Lemma 2.1 it follows that
\[ \text{spec}(A) = \{ z \in \mathbb{C} : \det Q(z) = 0 \} . \]

Fix \( j \in \{1, \ldots, n\} \). For \( \varphi^{(j)}(z_0) \) to be a solution of (2.1), the equation
\[ Q_i^{(j)}(z_0) = 0, \quad (2.8) \]
should be satisfied for any \( i \in \{1, \ldots, n\} \). The conditions (2.8) can be seen as inner boundary conditions (of the right endpoint type) for the difference equations (2.2). Note that the degeneration of diagonals gives rise to inner boundary conditions.

Let \( \{x_k\}_{k=1}^{N} \) be a real sequence such that \( x_k \in \text{spec}(A) \) and the elements of this sequence have been enumerated taking into account the multiplicity of eigenvalues. Also, let \( \alpha(x_k) \) be the corresponding eigenvectors such that
\[ \langle \alpha(x_k), \alpha(x_l) \rangle = \delta_{kl} \text{, with } k, l \in \{1, \ldots, N\} . \]

It follows from Remark 1 that
\[ \alpha(x_k) = \sum_{j=1}^{n} \alpha_j(x_k) \varphi^{(j)}(x_k) \quad (2.9) \]
for any \( k \in \{1, \ldots, N\} \). Clearly, by construction
\[ \sum_{j=1}^{n} |\alpha_j(x_k)| > 0 \text{ for all } k \in \{1, \ldots, N\} . \quad (2.10) \]

Note also that, since \( \{\alpha(x_k)\}_{k=1}^{N} \) is a basis of \( \mathcal{H} \) and \( \{\varphi^{(j)}(z)\}_{j=1}^{n} \) is a linearly independent system for each \( z \in \mathbb{C} \), it holds true that
\[ \sum_{k=1}^{N} |\alpha_j(x_k)| > 0 \text{ for all } j \in \{1, \ldots, n\} . \quad (2.11) \]

By (2.7) and the fact that \( \alpha(x_k) \in \ker(A - x_kI) \), it follows that
\[ \sum_{j=1}^{n} \alpha_j(x_k) Q_i^{(j)}(x_k) = 0 \text{ for all } i \in \{1, \ldots, n\} \quad (2.12) \]
is true.
Now, define the matrix valued function

$$\sigma(t) := \sum_{x_k < t} \sigma_k,$$  \hspace{1cm} (2.13)

where

$$\sigma_k = \begin{pmatrix} |\alpha_1(x_k)|^2 & \bar{\alpha}_1(x_k)\alpha_2(x_k) & \cdots & \bar{\alpha}_1(x_k)\alpha_n(x_k) \\ \alpha_2(x_k)\bar{\alpha}_1(x_k) & |\alpha_2(x_k)|^2 & \cdots & \bar{\alpha}_2(x_k)\alpha_n(x_k) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n(x_k)\bar{\alpha}_1(x_k) & \bar{\alpha}_n(x_k)\alpha_2(x_k) & \cdots & |\alpha_n(x_k)|^2 \end{pmatrix}$$  \hspace{1cm} (2.14)

is a rank-one, nonnegative matrix (cf. [22, Sec. 1]).

**Remark 2.** The matrix valued function $\sigma(t)$ has the following properties:

i) It is nondecreasing monotone step function.

ii) Each jump is a matrix of rank not greater than $n$.

iii) The sum of the ranks of all jumps is equal to $N$, i.e., the dimension of the space $\mathcal{H}$.

For any matrix valued function $\sigma(t)$ satisfying properties i)-iii), there is a collection of vectors $\{\alpha(x_k)\}_{k=1}^N$ satisfying (2.10) such that $\sigma(t)$ is given by (2.13) and (2.14) (cf. [22, Thm. 2.2]).

If $\mathcal{F} = I$, then $\sigma_{ij}(t) = \langle \delta_i, E(t)\delta_j \rangle$ ($i, j \in \{1, \ldots, n\}$) where $E(t)$ is the spectral resolution of $A$. Indeed,

$$\langle \delta_i, E(t)\delta_j \rangle = \left\langle \delta_i, \sum_{x_k < t} \langle \alpha(x_k), \delta_j \rangle \alpha(x_k) \right\rangle = \sum_{x_k < t} \langle \alpha(x_k), \delta_j \rangle \langle \delta_i, \alpha(x_k) \rangle = \sum_{x_k < t} \alpha_j(x_k)\bar{\alpha}_i(x_k) = \sigma_{ij}(t).$$

Therefore, in this case, the matrix valued function $\sigma(t)$ is the spectral function of the operator $A$.

**Definition 2.** The set of all matrix valued functions $\sigma(t)$ given by (2.13) and (2.14), where the collection of vectors $\{\alpha(x_k)\}_{k=1}^N$ satisfies (2.10) and (2.11) is denoted by $\mathfrak{M}(n, N)$.

Consider the Hilbert space $L_2(\mathbb{R}, \sigma)$, where $\sigma$ is the spectral function corresponding to operator $A$ given by (2.13) and (2.14) (see [1, Sec. 72]). Clearly, the property iii) implies that $L_2(\mathbb{R}, \sigma)$ is an $N$-dimensional space and in each
equivalence class there is an $n$-dimensional vector polynomial. Define the vector polynomials in $L_2(\mathbb{R}, \sigma)$

\[ q_i(z) := (Q_i^{(1)}(z), \ldots, Q_i^{(n)}(z))^t \quad (2.15) \]

for all $i \in \{1, \ldots, n\}$, and

\[ p_k(z) := (P_k^{(1)}(z), \ldots, P_k^{(n)}(z))^t \quad (2.16) \]

for all $k \in \{1, \ldots, N\}$.

**Lemma 2.2.** The vector polynomials \( \{p_k(z)\}_{k=1}^N \), defined by (2.16), satisfy

\[ \langle p_j, p_k \rangle_{L_2(\mathbb{R}, \sigma)} = \delta_{jk} \]

for $j, k \in \{1, \ldots, N\}$.

**Proof.**

\[
\langle p_j, p_k \rangle_{L_2(\mathbb{R}, \sigma)} = \sum_{l=1}^{N} \langle p_j(x_l), \sigma_l p_k(x_l) \rangle \\
= \sum_{l=1}^{N} \left( \sum_{s=1}^{n} \alpha_s(x_l) P_j^{(s)}(x_l) \right) \sum_{s=1}^{n} \alpha_s(x_l) P_k^{(s)}(x_l) \\
= \sum_{l=1}^{N} (\delta_{jl}, \alpha(x_l)) \langle \alpha(x_l), \delta_{lk} \rangle = \delta_{jk},
\]

where it has been used that $\delta_l = \sum_{i=1}^{n} \langle \alpha(x_i), \delta_l \rangle \alpha(x_i)$.

Let $U : \mathcal{H} \to L_2(\mathbb{R}, \sigma)$ be the isometry given by $U \delta_k \mapsto p_k$, for all $k \in \{1, \ldots, N\}$. Under this isometry the operator $A$ becomes the operator of mul-
ultiplication by the independent variable in \( L_2(\mathbb{R}, \sigma) \). Indeed,

\[
\langle \delta_k, A\delta_j \rangle = \left\langle \sum_{l=1}^{N} \langle \alpha(x_l), \delta_k \rangle \alpha(x_l), A \sum_{s=1}^{N} \langle \alpha(x_s), \delta_j \rangle \alpha(x_s) \right\rangle
= \sum_{l=1}^{N} \langle \delta_k, \alpha(x_l) \rangle \left\langle \alpha(x_l), \sum_{s=1}^{N} x_s \langle \alpha(x_s), \delta_j \rangle \alpha(x_s) \right\rangle
= \sum_{l=1}^{N} \langle \delta_k, \alpha(x_l) \rangle x_l \langle \alpha(x_l), \delta_j \rangle
= \langle p_k, t p_j \rangle.
\]

If the matrix \( T \) in (2.4) turns out to be the identity matrix, i.e., \( T = I \), then it can be shown that \( U^{-1} \) is the isomorphism corresponding to the canonical representation of the operator \( A \) [1 Sec. 75], that is,

\[
U^{-1}p_k = \sum_{j=1}^{n} P^{(j)}_k(A)\delta_j = \delta_k
\]

for all \( k \in \{1, \ldots, N\} \).

**Remark 3.** The matrix representation of the multiplication operator in \( L_2(\mathbb{R}, \sigma) \) with respect to the basis \( \{p_1(z), \ldots, p_N(z)\} \) is again the matrix \( A \). Thus

\[
\sum_{i=0}^{n-1} d_k^{(n-i)} p_{k-n+i}(z) + d_k^{(0)} p_k(z) + \sum_{i=1}^{n} d_k^{(i)} p_{k+i}(z) = z p_k(z),
\]

and one verifies that

\[
q_j(z) = (z - d_{m_j}^{(0)}) p_{m_j}(z) - \sum_{i=0}^{n-1} d_{m_j-n+i}^{(n-i)} p_{m_j-n+i}(z) - \sum_{i=1}^{n-j} d_{m_j}^{(i)} p_{m_j+i}(z)
\]

for all \( j \in \{1, \ldots, n\} \), where the last sum vanishes when \( j = n \).

### 3. Connection with a linear interpolation problem

Motivated by (2.12), we consider the following interpolation problem. Given a collection of complex numbers \( \{z_k\}_{k=1}^{N} \) and \( \{\alpha_j(k)\}_{j=1}^{n} \) \( (k = 1, \ldots, N) \), find
the scalar polynomials \( R_j(z) \) \((j = 1, \ldots, n)\) which satisfy the equation

\[
\sum_{j=1}^{n} \alpha_j(k) R_j(z_k) = 0, \quad \forall k \in \{1, \ldots, N\}. \tag{3.1}
\]

The polynomials satisfying (3.1) are the solutions to the interpolation problem and the numbers \( \{z_k\}_{k=1}^{N} \) are called the interpolation nodes.

In [24], this interpolation problem is studied in detail. Let us introduce some of the notions and results given in [24].

**Definition 3.** For a collection of complex numbers \( z_1, \ldots, z_N \), and matrices \( \sigma_k := \{\alpha_i(k)\alpha_j(k)\}_{i,j=1}^{n} \) \((k \in \{1, \ldots, N\})\), let us consider the equations

\[
\langle r(z_k), \sigma_k r(z_k) \rangle_{C^n} = 0 \tag{3.2}
\]

for \( k = 1, \ldots, N \), where \( r(z) \) is a nonzero \( n \)-dimensional vector polynomial. We denote by \( \mathcal{S} = \mathcal{S}(\{\sigma_k\}_{k=1}^{N}, \{z_k\}_{k=1}^{N}) \) the set of all vector polynomials \( r(z) \) that satisfy (3.2) (c.f. [24, Def. 3]).

It is worth remarking that solving (3.2) is equivalent to solving the linear interpolation problem (3.1), whenever \( r(z) = (R_1(z), \ldots, R_n(z))^t \).

**Definition 4.** Let \( r(z) = (R_1(z), R_2(z), \ldots, R_n(z))^t \) be an \( n \)-dimensional vector polynomial. The height of \( r(z) \) is the number

\[
h(r) := \max_{j \in \{1, \ldots, n\}} \{n \deg(R_j) + j - 1\},
\]

where it is assumed that \( \deg 0 := -\infty \) and \( h(0) := -\infty \).

In [24, Thm. 2.1] the following proposition is proven.

**Proposition 3.1.** Let \( \{g_1(z), \ldots, g_{m+1}(z)\} \) be a sequence of vector polynomials such that \( h(g_i) = i - 1 \) for all \( i \in \{1, \ldots, m+1\} \). Any vector polynomial \( r(z) \) with height \( m \neq -\infty \) can be written as follows

\[
r(z) = \sum_{i=1}^{m+1} c_i g_i(z),
\]

where \( c_i \in \mathbb{C} \) for all \( i \in \{1, \ldots, n\} \) and \( c_{m+1} \neq 0 \).

**Definition 5.** Let \( \mathcal{S} \) be an arbitrary subset of the set of all \( n \)-dimensional vector polynomials. We define the height of \( \mathcal{S} \) by

\[
h(\mathcal{S}) := \min \{h(r) : r \in \mathcal{S}, r \neq 0\}.
\]
We say that $r(z)$ in the set $S$ is a first generator of $S$ when

$$h(r) = h(S).$$

**Definition 6.** Let $M(r)$ be the subset of vector polynomials given by

$$M(r) := \{ p(z) : p(z) = S(z)r(z), S(z) \text{ is an arbitrary scalar polynomial} \}.$$

Note that for all $p(z) \in M(r)$, there is a $k \in \mathbb{N} \cup \{0\}$ such that

$$h(p) = nk + h(r).$$

In this case $k = \deg S$, where $p(z) = S(z)r(z)$.

**Proposition 3.2.** ([24, Lem. 4.3]) Fix a natural number $m$ such that $1 \leq m < n$. If the vector polynomials $r_1(z), \ldots, r_m(z)$ are arbitrary elements of $S$, then

$$S \setminus [M(r_1) + \cdots + M(r_m)] \neq \emptyset$$

and

$$h(S \setminus [M(r_1) + \cdots + M(r_m)]) \neq h(r_j) + nk$$

for any $j \in \{1, \ldots, m\}$ and $k \in \mathbb{N} \cup \{0\}$. In other words,

$$h(S \setminus [M(r_1) + \cdots + M(r_m)]) \quad \text{and} \quad h(r_j)$$

are different elements of the factor space $\mathbb{Z}/n\mathbb{Z}$ for any $j \in \{1, \ldots, m\}$.

Due to Proposition 3.2, the following definition makes sense.

**Definition 7.** One defines recursively the $j$-th generator of $S$ as the vector polynomial $r_j(z)$ in $S \setminus [M_1 + \cdots + M_{j-1}]$ such that

$$h(r_j) = h(S \setminus [M_1 + \cdots + M_{j-1}]),$$

where the notation $M_j := M(r_j)$ has been used.

**Remark 4.** Immediately from the definition, one verifies that the heights of the generators of $S$ are different elements of the factor space $\mathbb{Z}/n\mathbb{Z}$ (see [24, Rem. 3]).

In [24] Thm. 5.3, the following results were obtained.

**Proposition 3.3.** There are exactly $n$ generators of $S$. Moreover, if the vector polynomials $r_1(z), \ldots, r_n(z)$ are the generators of $S$, then

$$S = M(r_1) + \cdots + M(r_n).$$
Proposition 3.4. Let \( r_j(z) \) be the \( j \)-th generator of \( S(n, N) \). It holds true that
\[
\sum_{j=1}^{n} h(r_j) = Nn + \frac{n(n-1)}{2} .
\] (3.3)

Now, let us apply these results to the spectral analysis of the operator \( A \). To this end, consider the solution of (2.12) as elements of \( S(\{\sigma_k\}_{k=1}^{N}, \{x_k\}_{k=1}^{N}) \), where \( \sigma_k \) is given by (2.14).

Lemma 3.1. Let \( q_j(z) \) be the vector polynomials given in (2.15), then
\[
q_j(z) \in S(\{\sigma_k\}_{k=1}^{N}, \{x_k\}_{k=1}^{N})
\]
for all \( j \in \{1, \ldots, n\} \) and any \( x_k \in \text{spec}(A) \).

Proof. The assertion follows by comparing (2.7) with (2.12). \(\square\)

From this lemma, taking into account the definition of the inner product in \( L_2(\mathbb{R}, \sigma) \) (see the proof of Lemma 2.2 and Definition 3) one arrives at the following assertion.

Corollary 3.1. For all \( j \in \{1, \ldots, n\} \) the vector polynomials \( q_j(z) \) are in the equivalence class of the zero in \( L_2(\mathbb{R}, \sigma) \), that is,
\[
\langle q_j, q_j \rangle_{L_2(\mathbb{R}, \sigma)} = 0
\]
and, for all \( r \in L_2(\mathbb{R}, \sigma) \),
\[
\langle r, q_j \rangle_{L_2(\mathbb{R}, \sigma)} = 0 .
\] (3.4)

In the following assertions we use the results obtained in the Appendix by means of an algorithm constructed heuristically. We remark that these results can be proven analytically (see Lemma 4.4).

Theorem 3.1. The vector polynomial \( q_j(z) \) is a \( j \)-generator of
\[
S(\{\sigma_k\}_{k=1}^{N}, \{x_k\}_{k=1}^{N})
\]
for all \( j \in \{1, \ldots, n\} \).

Proof. For any fixed \( j \in \{1, \ldots, n\} \), suppose that there is an element \( r(z) \in S(\{\sigma_k\}_{k=1}^{N}, \{x_k\}_{k=1}^{N}) \setminus (M(q_1) + \cdots + M(q_{j-1})) \), where \( q_0(z) := 0 \). Write \( r \) as (A.2). If one assumes that \( h(q_{j-1}) < h(r) < h(q_j) \), then by Corollary 3.1,
\[
0 = \langle r, r \rangle_{L_2(\mathbb{R}, \sigma)} = \left\langle \sum_{k=1}^{N} c_k p_k, \sum_{k=1}^{N} c_k p_k \right\rangle_{L_2(\mathbb{R}, \sigma)} = \sum_{k=1}^{N} |c_k|^2 .
\]
This implies that \( c_k = 0 \) for all \( k \in \{1, \ldots, N\} \). In turn, again by (A.2), one has
\[
\mathbf{r}(z) \in \mathcal{M}^{(q_1)} + \cdots + \mathcal{M}^{(q_{j-1})}
\]
for \( j > 1 \), and \( \mathbf{r}(z) \equiv 0 \) for \( j = 1 \). This contradiction yields that \( q_j(z) \) satisfies the definition of \( j \)-generator for any \( j \in \{1, \ldots, n\} \).

The following assertion is a direct consequence of Theorem 3.1 and Proposition 3.4.

**Theorem 3.2.** Let \( \{q_1(z), \ldots, q_n(z)\} \) be the \( n \)-dimensional vector polynomials defined by (2.13). Then, \( h(q_1), \ldots, h(q_n) \) are different elements of the space \( \mathbb{Z}/n\mathbb{Z} \). Also,
\[
\sum_{j=1}^{n} h(q_j) = Nn + \frac{n(n - 1)}{2}.
\]

**4. Reconstruction**

In this section, we take as a starting point a matrix valued function \( \tilde{\sigma}(t) \in \mathcal{M}(n, N) \) and construct a matrix \( \mathcal{A} \) in \( \mathcal{M}(n, N) \) from this function. Moreover, we verify that, for some matrix \( \mathcal{T} \) giving the initial conditions, the function \( \sigma \) generated by the matrix \( \mathcal{A} \) (see Section 2) coincides with \( \tilde{\sigma} \). Thus, the results of this section show that any matrix in \( \mathcal{M}(n, N) \) can be reconstructed from its function in \( \mathcal{M}(n, N) \).

Let \( \tilde{\sigma}(t) \) be a matrix valued function in \( \mathcal{M}(n, N) \). Thus, one can associate an interpolation problem (3.1) which is equivalent to (3.2) (with \( \tilde{\sigma}_k \) instead of \( \sigma_k \)). Then, by Proposition 3.3 there are \( n \) generators \( \tilde{q}_1(z), \ldots, \tilde{q}_n(z) \) of \( \mathbb{S}(\{\tilde{\sigma}_k\}_{k=1}^{N}, \{\tilde{z}_k\}_{k=1}^{N}) \).

For the study of the inverse problem, we construct a sequence of vector polynomials on the basis of certain elements in the sets \( \mathcal{M}(\tilde{q}_j) \) (\( j \in \{1, \ldots, n\} \)).

Let \( \{e_i(z)\}_{i \in \mathbb{N}} \) be a sequence of \( n \)-dimensional vector polynomials defined by

\[
e_{nk+1}(z) := \begin{pmatrix} z^k \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_{nk+2}(z) := \begin{pmatrix} 0 \\ z^k \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n(k+1)}(z) := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ z^k \end{pmatrix} \quad (4.1)
\]
Clearly, \( h(e_i) = i - 1 \). Now, for \( j \in \{2, \ldots, n\} \), define recursively the sets

\[
\mathcal{B}_j := \{ m \in \mathbb{N} : m = h(\tilde{q}_j) + nl + 1, \ \text{where} \ i < j \ \text{and} \ l \in \mathbb{N} \cup \{0\} \},
\]

\[
\mathcal{A}_1 := \{1, 2, \ldots, h(\tilde{q}_1)\},
\]

\[
\mathcal{A}_j := \{h(\tilde{q}_{j-1}) + 1, \ldots, h(\tilde{q}_j)\} \setminus \mathcal{B}_j.
\]

Note that \( \mathcal{B}_{j-1} \subset \mathcal{B}_j \) and the sets \( \mathcal{A}_j \) are finite. Also, for any vector polynomial \( r \) being in

\[
\mathcal{M}(\tilde{q}_1) + \cdots + \mathcal{M}(\tilde{q}_{j-1})
\]

it holds that \( h(r) + 1 \) is in \( \mathcal{B}_j \). On the other hand, the sets \( \mathcal{A}_j \) account for the heights non related to the set \( (4.2) \). Define the sequence \( \{g_i(z)\}_{i \in \mathbb{N}} \) as follows

\[
g_i(z) := \begin{cases} e_i(z) & \text{for } i \in \bigcup_{j=1}^n \mathcal{A}_j, \\ z^j \tilde{q}_j(z) & \text{for } i = h(\tilde{q}_j) + nl + 1 \ (l \in \mathbb{N} \cup \{0\}, \ j \in \{1, \ldots, n\}). \end{cases}
\]

(4.3)

Note that \( h(g_i) = i - 1 \), that is, when \( i \) runs through \( \mathbb{N} \), the heights of the vector polynomials \( g_i(z) \) cover the set \( \mathbb{N} \cup \{0\} \). Furthermore, the sequence \( \{g_i(z)\}_{i=1}^\infty \) contains all the vectors of the form \( z^j \tilde{q}_j(z) \ (l \in \mathbb{N} \cup \{0\}, \ j \in \{1, \ldots, n\}) \) and the elements of \( (4.1) \) whose heights do not coincide with the heights of the elements of \( (4.2) \) (\( j = n + 1 \)).

Consider the space \( L_2(\mathbb{R}, \tilde{\sigma}) \). Clearly, the only elements of the sequence \( \{g_i(z)\}_{i=1}^\infty \) that are not in the equivalence class of zero are the ones where \( g_i(z) = e_i(z) \). Since any element in \( L_2(\mathbb{R}, \tilde{\sigma}) \) can be written as a finite linear combination of the elements of \( \{g_i(z)\}_{i=1}^\infty \), it follows that there is exactly \( N \) values of \( i \), where \( g_i(z) = e_i(z) \).

Let \( \{\tilde{p}_1(z), \ldots, \tilde{p}_N(z)\} \) be the orthonormal set of \( n \)-dimensional vector polynomials obtained after applying the Gram-Schmidt process in \( L_2(\mathbb{R}, \tilde{\sigma}) \) to the set \( \{g_i(z)\}_{i \in \bigcup_{j=1}^n \mathcal{A}_j} \). Then, \( \{\tilde{p}_1(z), \ldots, \tilde{p}_N(z)\} \) forms an orthonormal basis for this space. Consider the monotone increasing function \( a(k) \) defined on \( \{1, \ldots, N\} \), taking values in \( \bigcup_{j=1}^n \mathcal{A}_j \) such that \( h(\tilde{p}_k) = h(e_{a(k)}) \).

Denote

\[
\tilde{g}_i(z) := \begin{cases} \tilde{p}_{a^{-1}(i)}(z) & \text{for } i \in \bigcup_{j=1}^n \mathcal{A}_j, \\ g_i(z) & \text{otherwise}. \end{cases}
\]

(4.4)

Then, \( \{\tilde{g}_i(z)\}_{i=1}^\infty \) is a basis of the space of all \( n \)-dimensional vector polynomial. Indeed, the following table illustrates that the heights of the elements of the sequence \( \{\tilde{g}_i(z)\}_{i \in \mathbb{N}} \) cover all the set \( \{0\} \cup \mathbb{N} \).

| \( h(\tilde{q}_i) \) | \( 0 \) | 1 | 2 | \ldots | \( h_1 - 1 \) | \( h_1 \) | \( h_1 + 1 \) | \( h_1 + 2 \) | \ldots | \( h_1 + n \) | \ldots |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( \{\tilde{g}_i(z)\}_{i \in \mathbb{N}} \) | \( \tilde{p}_1 \) | \( \tilde{p}_2 \) | \( \tilde{p}_3 \) | \ldots | \( \tilde{p}_{h_1} \) | \( q_1 \) | \( \tilde{p}_{h_1+1} \) | \( \tilde{p}_{h_1+2} \) | \ldots | \( zq_1 \) | \ldots |

(4.5)
where $h_j := h(q_j)$ for all $j \in \{1, \ldots, n\}$. Here we have used that, according to Theorem \ref{thm3.2}, \(\{h(q_j)\}_{j=1}^n\) are different elements of $\mathbb{Z}/n\mathbb{Z}$. For instance, if $N = 8$, $n = 3$, $h_1 = 4$, $h_2 = 9$ and $h_3 = 14$, the table is the following

| $\{g_i\}_{i \in \mathbb{N}}$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | $q_1$ | $q_2$ | $q_3$ | $zq_1$ | $zq_2$ | $z^2q_1$ | $z^2q_2$ | $q$ |
|-----------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $h(g_i)$                   | 0     | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     | 10    | 11    | 12    | 13    | 14    |

**Remark 5.** For $k = 1, \ldots, h_1$. We have that

$$h(p_k) = a(k) - 1 = k - 1,$$

and, for $k > h_1$

$$h(p_k) = a(k) - 1 = k - 1 + b_k,$$

where $b_k$ is the number of elements of the sequence $\{\tilde{g}_i\}_{i=1}^\infty$ in the set $\mathbb{M}(\tilde{q}_1) + \cdots + \mathbb{M}(\tilde{q}_n)$ which are before $\tilde{p}_k(z)$ in the table (4.3). Observe that $b_k < b_{n+k}$, which in turn implies

$$h(\tilde{p}_{n+k}) > h(\tilde{p}_k) + n. \quad (4.6)$$

**Remark 6.** Any $n$-dimensional vector polynomial can be expanded as a finite linear combination of the elements of $\{g_i(z)\}_{i \in \mathbb{N}}$. In particular, for $k \in \{1, \ldots, N\}$,

$$z \tilde{p}_k(z) = \sum_{l=1}^{N} c_{kl} \tilde{p}_l(z) + \sum_{j=1}^{n} S_{kj}(z) \tilde{q}_j(z), \quad (4.7)$$

where $c_{kl} \in \mathbb{C}$ and $S_{kj}(z)$ is scalar polynomial. Moreover it holds true that

i) $c_{kl} = 0$ if $h(z \tilde{p}_k) < h(\tilde{p}_l)$,

ii) $S_{kl}(z) = 0$ if $h(z \tilde{p}_k) < h(S_{kl}(z) \tilde{q}_j)$,

iii) $c_{kl} \neq 0$ if $h(z \tilde{p}_k) = h(\tilde{p}_l)$ for any $l \in \{1, \ldots, N\}$.

Therefore, if we take the inner product of (4.7) with $\tilde{p}_l(z)$ in $L_2(\mathbb{R}, \tilde{\sigma})$, we obtain

$$c_{kl} = \langle \tilde{p}_l, z \tilde{p}_k \rangle_{L_2(\mathbb{R}, \tilde{\sigma})} = \langle z \tilde{p}_l, \tilde{p}_k \rangle_{L_2(\mathbb{R}, \tilde{\sigma})} = c_{lk}, \quad (4.8)$$

where (3.4) has been used. Hence, the matrix $\{c_{lk}\}_{l,k=1}^N$ is symmetric and it is the matrix representation of the operator of multiplication by the independent variable in $L_2(\mathbb{R}, \tilde{\sigma})$ with respect to the basis $\{\tilde{p}_k(z)\}_{k=1}^N$. 

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The following results shed light on the structure of the matrix \( \{c_{lk}\}_{l,k=1}^N \).

**Lemma 4.1.** If \(|l-k| > n\). Then,
\[
c_{kl} = c_{lk} = 0.
\]

*Proof.* For \(l-k > n\), we obtain of (4.6) that
\[
h(\tilde{p}_l) > h(\tilde{p}_{k+n}) > h(\tilde{p}_k) + n = h(z\tilde{p}_k).
\]
Therefore by Remark [5]
\[
c_{kl} = \langle \tilde{p}_l, z\tilde{p}_k \rangle_{L^2(\mathbb{R}, \sigma)} = 0.
\]
And similarly for \(k-l > n\). \(\square\)

Lemma 4.1 shows that \( \{c_{lk}\}_{l,k=1}^N \) is a band matrix. Let us turn to the question of characterizing the diagonals of \( \{c_{lk}\}_{l,k=1}^N \). It will be shown that they undergo the kind of degeneration given in the Introduction.

For a fixed number \(i \in \{0, \ldots, n\}\), we define the numbers
\[
d^{(i)}_k := c_{k+i,k} = c_{k,k+i}
\]
for \(k = 1, \ldots, N - i\).

**Lemma 4.2.** For a fixed number \(j \in \{0, \ldots, n-1\}\), it holds true that
\[
d^{(n-j)}_k \begin{cases} 
\neq 0, & \text{if } h(\tilde{q}_j) < h(z\tilde{p}_k) < h(\tilde{q}_{j+1}), \\
= 0, & \text{if } h(z\tilde{p}_k) \geq h(\tilde{q}_{j+1}),
\end{cases}
\]
when \(k\) runs through \(\{1, \ldots, N - (n - j)\}\) and where \(h(q_0) := n - 1\).

*Proof.* Fix number \(j \in \{0, \ldots, n-1\}\), then any vector polynomial of the basis \(\{\tilde{p}_k(z)\}_{k=1}^N\) satisfies either
\[
h(\tilde{q}_j) < h(z\tilde{p}_k) < h(\tilde{q}_{j+1})
\]
(4.11)

or
\[
h(z\tilde{p}_k) \geq h(\tilde{q}_{j+1}).
\]
(4.12)

Suppose that \(k \in \{1, \ldots, N\}\) is such that (4.11) holds, then there is \(l \in \{1, \ldots, N\}\) such that
\[
h(\tilde{p}_l) = h(\tilde{p}_k) + n = h(z\tilde{p}_k).
\]
Indeed, if there is no vector polynomial \(\tilde{p}_l(z)\) such that \(h(\tilde{p}_l) = h(z\tilde{p}_k)\), then \(h(z\tilde{p}_k) = h(z^s\tilde{q}_i)\) for any \(i \leq j\) and \(s \geq 1\). Therefore \(h(\tilde{p}_k) = h(z^{s-1}\tilde{q}_j)\), which contradicts the definition of \(\tilde{p}_k(z)\).
Let $f_k$ be the number of elements of the sequence $\{\tilde{g}_i\}_{i=1}^\infty$ in $M(\tilde{q}_i) + \cdots + M(\tilde{q}_j)$ whose heights lies between $h(\tilde{p}_k)$ and $h(\tilde{p}_k) + n$. If one assumes that \(^{(4.11)}\) holds, then
\[ f_k = j. \tag{4.13} \]
This is so because there are $n - 1$ “places” between $h(\tilde{p}_k)$ and $h(\tilde{p}_k) + n$ and, for each generator $\tilde{q}_j(z)$ ($j \in \{1, \ldots, n\}$) the heights of the elements of $M(\tilde{q}_j)$ fall into the same equivalence class of $\mathbb{Z}/n\mathbb{Z}$ (see Proposition \[3.2\]). By \(4.13\), one has
\[ h(z\tilde{p}_k) = h(\tilde{p}_k) + n = h(\tilde{p}_{k+n-f_k}) = h(\tilde{p}_{k+n-f_k}). \]
Therefore, Remark \(5 \ iii\) implies that $d_k^{(n-j)} \neq 0$.

Now, suppose that \(4.12\) takes place. In this case, one verifies that
\[ f_k \geq j + 1. \tag{4.14} \]
Let $\tilde{f}_k$ be the number of elements in $\{\tilde{p}_k(z)\}_{k=1}^N$ whose heights lies between $h(\tilde{p}_k)$ and $h(\tilde{p}_k) + n$. Then
\[ h(\tilde{p}_{k+\tilde{f}_k}) < h(\tilde{p}_k) + n \leq h(\tilde{p}_{k+\tilde{f}_k+1}) . \]
Also, it follows from \(4.14\) and $n - 1 = f_k + \tilde{f}_k$ that
\[ h(\tilde{p}_{k+\tilde{f}_k+1}) \leq h(\tilde{p}_{k+n-j-1}) < h(\tilde{p}_{k+n-j}) . \]
Thus $h(\tilde{p}_k) + n < h(\tilde{p}_{k+n-j})$. This implies that $\langle \tilde{p}_{k+n-j}, z\tilde{p}_k \rangle_{L_2(\mathbb{R}, \tilde{\sigma})} = 0$, which yields that $d_k^{(n-j)} = 0$ whenever \(4.12\) holds.

Taking into account \(4.9\), it follows from Lemma \(4.1\) and \(4.2\) that the matrix $\{c_{kl}\}_{k,l=1}^N$ whose entries are given by \(4.8\) is in the class $\mathcal{M}(n, N)$. Thus, the matrix representation of the operator of multiplication by the independent variable in $L_2(\mathbb{R}, \tilde{\sigma})$ with respect to the basis $\{\tilde{p}_k\}_{k=1}^N$ is a matrix in $\mathcal{M}(n, N)$.

**Remark 7.** Since the matrix $\{c_{kl}\}_{k,l=1}^N$ is in $\mathcal{M}(n, N)$, there are numbers $\{m_i\}_{i=1}^n$ associated with it (see Introduction). This numbers can be found from \(4.10\) when there exists $k \in \{1, \ldots, N\}$ such that $h(z\tilde{p}_k) = h(\tilde{q}_{j+1})$ (this happens for each $j \in \{1, \ldots, n - 1\}$). Thus,
\[ h(z\tilde{p}_{m_j}) = h(\tilde{q}_j), \quad \forall j \in \{1, \ldots, n\}. \tag{4.15} \]

It is straightforward to verify that \(2.11\) is equivalent to the fact that $e_i(z)$ is not in the equivalence class of zero in $L(\mathbb{R}, \tilde{\sigma})$ for $i \in \{1, \ldots, n\}$. Therefore, the first $n$ elements of $\{\tilde{p}_k(z)\}_{k=1}^N$ are obtained by applying Gram-Schmidt to
the set \( \{e_i(z)\}_{i=1}^{n} \). Thus, if one defines
\[
t_{ij} := \langle \delta_i, \tilde{p}_j \rangle, \quad \forall i, j \in \{1, \ldots, n\},
\]
the matrix \( T = \{t_{ij}\}_{i,j=1}^{n} \) turns out to be upper triangular. Now, for this matrix \( T \) and \( A \) construct the solutions \( \varphi^{(j)}(z) \) satisfying (2.4). Hence, the vector polynomials \( \{p_1(z), \ldots, p_n(z)\} \) defined by (2.16) satisfy (4.16). In other words
\[
p_j(z) = \tilde{p}_j(z), \quad \forall j \in \{1, \ldots, n\}.
\]

Consider the recurrence equation, which is obtained from (4.7), but only for the case \( \tilde{p} \) of the Remark 6 taking into account (4.9) and (4.10). That is,
\[
d^{(0)}_1 p_1 + \cdots + d^{(n)}_1 p_{n+1} = z p_1
\]
\[
d^{(0)}_1 \tilde{p}_1 + d^{(0)}_2 \tilde{p}_2 + \cdots + d^{(n)}_2 \tilde{p}_{n+2} = z \tilde{p}_2
\]
\[
\vdots
\]
\[
d^{(n)}_{m_1-1-n} \tilde{p}_{m_1-1-n} + \cdots + d^{(0)}_{m_1-1} \tilde{p}_{m_1-1} + d^{(1)}_{m_1-1} \tilde{p}_{m_1} + \cdots + d^{(n)}_{m_1-1} \tilde{p}_{m_1+n} = z \tilde{p}_{m_1-1}
\]
\[
\vdots
\]
\[
\cdots + d^{(0)}_{m_2-1} \tilde{p}_{m_2-1} + d^{(1)}_{m_2-1} \tilde{p}_{m_2} + \cdots + d^{(n)}_{m_2-1} \tilde{p}_{m_2+n} + S_{m_2+1,1} q_1 = z \tilde{p}_{m_2-1}
\]
\[
\vdots
\]
\[
\cdots + d^{(0)}_{m_2+1} \tilde{p}_{m_2+1} + d^{(1)}_{m_2+1} \tilde{p}_{m_2+2} + \cdots + d^{(n-2)}_{m_2+1} \tilde{p}_{m_2+n} + \sum_{i=1}^{2} S_{m_2+1,1} q_i = z \tilde{p}_{m_2+n}
\]
\[
\vdots
\]
\[
(4.17)
\]

Since \( p_k(z) \) and \( \tilde{p}_k(z) \) satisfy the same recurrence equation for any \( k \in \{1, \ldots, m_1-1+n\} \), one has
\[
p_k(z) = \tilde{p}_k(z), \quad \forall k \in \{1, \ldots, m_1-1+n\}.
\]

From the recurrence equations (4.17), take the equation containing the vector polynomial \( z \tilde{p}_{m_1+1}(z) \). By comparing this equation with the corresponding one from (2.17), one concludes
\[
p_{m_1+n}(z) = \tilde{p}_{m_1+n}(z) + S(z) \tilde{q}_1(z),
\]
where \( S(z) \) is a scalar polynomial, so \( S(z) \tilde{q}_1(z) \) is in the equivalence class of zero of \( L_2(\mathbb{R}, \tilde{\sigma}) \). Observe that \( h(\tilde{p}_{m_1+n}) > h(S \tilde{q}_1) \) since \( h(\tilde{p}_{m_1+n}) = h(z \tilde{p}_{m_1+1}) \) in the equation containing \( z \tilde{p}_{m_1+1}(z) \) and the height of \( z \tilde{p}_{m_1+1}(z) \) does not continue...
coincide with the height of $S(z)\tilde{q}_1(z)$. Recursively, for $k > m_1 + n$, one obtains the following lemma.

**Lemma 4.3.** The vector polynomials $\{p_k(z)\}_{k=1}^N$ and $\{\tilde{p}_k(z)\}_{k=1}^N$ defined above (see the text below (4.16) and above (4.4), respectively) satisfy that

$$p_k(z) = \tilde{p}_k(z) + \tilde{r}_k(z)$$

for all $k \in \{1, \ldots, N\}$, where $\tilde{r}_k(z)$ is in the equivalence class of zero of $L_2(\mathbb{R}, \tilde{\sigma})$ and $h(\tilde{r}_k) < h(p_k)$. Therefore,

$$h(p_k) = h(\tilde{p}_k), \quad \forall k \in \{1, \ldots, N\}.$$ 

On the other hand, for the particular case $k = m_1$, (4.7) and (4.15) imply that

$$z\tilde{p}_{m_1} = d_{m_1-n}^{(n)}\tilde{p}_{m_1-n} + \cdots + d_{m_1}^{(0)}\tilde{p}_{m_1} + d_{m_1}^{(1)}\tilde{p}_{m_1+1} + \cdots + d_{m_1+n-1}^{(n-1)}\tilde{p}_{m_1+n-1} + \gamma_1\tilde{q}_1,$$

where $\gamma_1 \neq 0$.

In general, one verifies that for all $j \in \{1, \ldots, n\}$

$$z\tilde{p}_m = d_{m-j}^{(n)}\tilde{p}_{m-j} + \cdots + d_{m}^{(0)}\tilde{p}_{m} + d_{m+1}^{(1)}\tilde{p}_{m+1} + \cdots + d_{m+n-j}^{(n-j)}\tilde{p}_{m+n-j} + \sum_{i<j} S_{m,j,i}\tilde{q}_i + \gamma_j\tilde{q}_j,$$

where $\gamma_j \neq 0$ and $S_i(z)$ is a scalar polynomial. Hence,

$$\gamma_j\tilde{q}_j = \left( z - d_{m_1}^{(0)} \right) \tilde{p}_{m_1} - \left( d_{m_1-n}^{(n)}\tilde{p}_{m_1-n} + \cdots + d_{m_1-1}^{(1)}\tilde{p}_{m_1-1} + \\
+ d_{m_1}^{(1)}\tilde{p}_{m_1+1} + \cdots + d_{m_1+n-j}^{(n-j)}\tilde{p}_{m_1+n-j} + \sum_{i<j} S_{m,j,i}\tilde{q}_i \right)$$

(4.19)

for all $j \in \{1, \ldots, n\}$.

Let us define the set of vector polynomials $\{q_1(z), \ldots, q_n(z)\}$ by means of (2.18) using $\{p_1(z), \ldots, p_N(z)\}$, as was done in Section 2.

**Lemma 4.4.** Let $\tilde{q}_j(z)$ be $j$-generator of $S(\tilde{\sigma}_k)_{k=1}^N, \{\tilde{\sigma}_k\}_{k=1}^N$, and $q_j(z)$ be defined as above. Then $h(q_j) = h(\tilde{q}_j)$ for all $j \in \{1, \ldots, n\}$ and

$$q_j(z) = \sum_{i \leq j} S_i(z)\tilde{q}_i(z), \quad S_j \neq 0,$$

(4.20)

where $S_i(z)$ are scalar polynomials.
Proof. It follows from (2.18), (4.18) and (4.19) that
\[ q_j(z) = \gamma_j \tilde{q}_j(z) + \tilde{s}_j(z), \quad \text{for all } j \in \{1, \ldots, n\}, \quad (4.21) \]
where \( \tilde{s}_j(z) \) is in the equivalence class of the zero of \( L_2(\mathbb{R}, \tilde{\sigma}) \) and its height is
strictly less that the height of \( \tilde{q}_j(z) \) since, due to (4.15), the height of \( \tilde{q}_j(z) \) is
strictly greater than the height of any other term in the equation with \( k = m_j \) in the system (4.7). Thus, \( h(q_j) = h(\tilde{q}_j) \) for all \( j \in \{1, \ldots, n\} \).

Equation (4.21) also shows that
\[ q_i(z) \in S(\{\tilde{\sigma}_k\}_{k=1}^N, \{\tilde{x}_k\}_{k=1}^N) \]
and, due to Proposition 3.3, (4.20) is satisfied. \( \square \)

Lemma 4.5. The set of the \( n \)-dimensional vector polynomials \( \{p_k(z)\}_{k=1}^N \cup \{z^l q_j(z)\} \) (\( j \in \{1, \ldots, n\} \) and \( l \in \mathbb{N} \)) is a basis of the linear space of \( n \)-dimensional vector polynomials.

Proof. In view of Lemma 4.3 and 4.4 the assertion follows directly from Proposition 3.1 taking into account (4.3), (4.4), and that \( \{\tilde{g}_i(z)\}_{i=1}^\infty \) is a basis. \( \square \)

Lemma 4.6. Let \( r(z) \) and \( s(z) \) be any two \( n \)-dimensional vector polynomials. Then,
\[ \langle r, s \rangle_{L_2(\mathbb{R}, \sigma)} = \langle r, s \rangle_{L_2(\mathbb{R}, \tilde{\sigma})}. \]

Proof. Any vector polynomial \( r(z) \) can written as
\[ r(z) = \sum_{k=1}^N c_k p_k(z) + \sum_{j=1}^n S_j(z) q_j(z), \]
where \( c_k = \langle r, p_k \rangle_{L_2(\mathbb{R}, \sigma)} \) and \( S_j(z) \) are scalar polynomials. Thus,
\[ \langle r, \tilde{p}_k \rangle_{L_2(\mathbb{R}, \tilde{\sigma})} = \left\langle \sum_{l=1}^N c_l \tilde{p}_l + \sum_{j=1}^n S_j(z) q_j, \tilde{p}_k \right\rangle_{L_2(\mathbb{R}, \tilde{\sigma})} \]
\[ = \left\langle \sum_{l=1}^N c_l (\tilde{p}_l + \tilde{r}_l) + \sum_{j=1}^n S_j \left( \sum_{i \leq j} S_i \tilde{q}_i \right), \tilde{p}_k \right\rangle_{L_2(\mathbb{R}, \tilde{\sigma})} \]
\[ = \left\langle \sum_{l=1}^N c_l \tilde{p}_l, \tilde{p}_k \right\rangle_{L_2(\mathbb{R}, \tilde{\sigma})} = c_k. \]

For the functions \( \sigma(t) \) and \( \tilde{\sigma}(t) \) in \( \mathcal{M}(n, N) \) consider the points \( x_k \) and \( \tilde{x}_k \),
where, respectively, \( \sigma(t) \) and \( \tilde{\sigma}(t) \) have jumps \( \sigma_k \) and \( \tilde{\sigma}_k \). By definition, \( k \) takes all the values of the set \( \{1, \ldots, N\} \).

**Lemma 4.7.** The points where the jumps of the matrices \( \sigma(t) \) and \( \tilde{\sigma}(t) \) take place coincide, i.e.,

\[
x_k = \tilde{x}_k, \quad \text{for all } k \in \{1, \ldots, N\}.
\]

**Proof.** Define the \( n \)-dimensional vector polynomial

\[
r(z) := \prod_{l=1}^{N} (z - x_l)e_1(z)
\]

(see (4.1)). Therefore,

\[
\langle r, r \rangle_{L^2(\mathbb{R}, \sigma)} = \sum_{k=1}^{N} \langle r(x_k), \sigma_k r(x_k) \rangle_{\mathbb{C}^n} = 0.
\]

Now, if one assumes that \( \{\tilde{x}_k\}_{k=1}^{N} \setminus \{x_k\}_{k=1}^{N} \neq \emptyset \), then

\[
\langle r, r \rangle_{L^2(\mathbb{R}, \tilde{\sigma})} = \sum_{k=1}^{N} \langle r(\tilde{x}_k), \tilde{\sigma}_k r(\tilde{x}_k) \rangle_{\mathbb{C}^n} > 0
\]

due to (2.11). In view of Lemma 4.6 our assumption has lead to a contradiction, so \( \{\tilde{x}_k\}_{k=1}^{N} \subset \{x_k\}_{k=1}^{N} \). Analogously, one proves that \( \{x_k\}_{k=1}^{N} \subset \{\tilde{x}_k\}_{k=1}^{N} \). \( \square \)

**Lemma 4.8.** The jumps of the matrix valued functions \( \sigma(t) \) and \( \tilde{\sigma}(t) \) coincide, namely, for all \( k \in \{1, \ldots, N\} \),

\[
\sigma_k = \tilde{\sigma}_k.
\]

**Proof.** Define for each \( i \in \{1, \ldots, n\} \), the \( n \)-dimensional vector polynomial by

\[
r_{ki}(z) := \prod_{\substack{l=1 \atop l \neq k}}^{N} (z - x_l)e_i(z).
\]
Thus, for all $i,j \in \{1, \ldots, n\}$
\[
\langle r_{ki}, r_{kj}\rangle_{L_2(\mathbb{R}, \sigma)} = \sum_{s=1}^{N} \langle r_{ki}(x_s), \sigma_s r_{kj}(x_s) \rangle_{C_n} = \sum_{s=1}^{N} \prod_{t=1 \atop t \neq k}^{N} (x_s - x_t)^2 \alpha_i(x_s) \alpha_j(x_s) \]
\[
= \prod_{l=1 \atop l \neq k}^{N} |x_k - x_l|^2 \alpha_i(x_k) \alpha_j(x_k) .
\]

Analogously,
\[
\langle r_{ki}, r_{kj}\rangle_{L_2(\mathbb{R}, \tilde{\sigma})} = \prod_{l=1 \atop l \neq k}^{N} |x_k - x_l|^2 \tilde{\alpha}_i(x_k) \tilde{\alpha}_j(x_k) ,
\]
where Lemma 4.7 was used together with the fact that the numbers $\tilde{\alpha}_i(x_k)$ define the entries of the matrix $\tilde{\sigma}_k$ (see (2.14)). Therefore, by Lemma (4.6)
\[
\sigma_k = \tilde{\sigma}_k , \quad \text{for all } k \in \{1, \ldots, N\} .
\]

Thus, with the help of the above results, one can assert the following theorem.

**Theorem 4.1.** Let $\tilde{\sigma}(t)$ be an element of $\mathfrak{M}(n, N)$ and \( \{c_{kl}\}_{k,l=1}^{N} \in \mathcal{M}(n, N) \) be the corresponding matrix that results from applying the method of reconstruction to the matrix valued function $\tilde{\sigma}(t)$. If $A$ is the operator whose matrix representation with respect to the basis $\{\delta_1, \ldots, \delta_N\}$ in $\mathcal{H}$, is $\{c_{kl}\}_{k,l=1}^{N}$, then there is an upper triangular matrix $\mathcal{T}$ such that the corresponding spectral function $\sigma(t)$ for the operator $A$ coincides with $\tilde{\sigma}(t)$.

Paraphrasing the previous theorem, we assert that the spectral function of a matrix in $\mathcal{M}(n, N)$ uniquely determine the matrix itself. In other words, a matrix in $\mathcal{M}(n, N)$ can be uniquely recovered from its spectral function.

**Appendix**

In this section, for a given matrix in the class $\mathcal{M}(n, N)$, we calculate degrees and heights of the vector polynomials $\{p_k(z)\}_{k=1}^{N}$ and $\{q_j(z)\}_{j=1}^{n}$ on the basis of a technique obtained heuristically using tables. Although most of the results can be proven analytically (see Section 1), the method presented here has advantages in doing concrete computations. An algorithm for calculating degrees
and heights from the numbers \(n\), \(N\), and \(\{m_i\}_{i=1}^n\) is provided. This algorithm may have applications in numerical methods.

Recall that from (2.2) one obtains scalar polynomials \(P_k^{(j)}\) and \(Q_i^{(j)}\) which depend on the initial condition \(\mathcal{T}\) (see (2.4)). According to (2.4) and our convention on degrees (see Definition 4), the following holds:

\[
\text{deg} P_k^{(j)}(z) = \begin{cases} 
\leq 0 & k > j, \\
= 0 & k = j, \\
= -\infty & k < j
\end{cases}
\]

for \(k = 1, \ldots, n\). Therefore, \(h(p_k) = k - 1\) for all \(k = 1, \ldots, n\). Consider the matrix \(A\) given in (1.4). For this matrix, one can construct the following table:

| \(P_k^{(j)} \mid Q_i^{(j)}\) | \(j = 1\) | \(j = 2\) | \(j = 3\) | \(l\) |
|---|---|---|---|---|
| \(k = 4\) | 1 | \(\leq 0\) | \(\leq 0\) | 1 |
| \(k = 5\) | \(\leq 1\) | 1 | \(\leq 0\) | 2 |
| \(i = 1\) | \(\leq 1\) | \(\leq 1\) | 1 | \(m_1 = 3\) |
| \(k = 6\) | \(\leq 2\) | 2 | \(\leq 1\) | 4 |
| \(i = 2\) | \(\leq 2\) | 2 | \(\leq 1\) | \(m_2 = 5\) |
| \(k = 7\) | \(\leq 3\) | 3 | \(\leq 2\) | 6 |
| \(i = 3\) | \(\leq 4\) | \(\leq 3\) | \(\leq 3\) | \(m_3 = 7\) |

In this table the last column gives the number of the recurrence equation in (2.2) (which we denote by \(l\)). The first column shows the numbers \(k \in \{1, \ldots, N\}\) or \(i \in \{1, \ldots, n\}\) corresponding to the polynomials \(P_k^{(j)}\) or \(Q_i^{(j)}\) which are determined by the corresponding \(l\)-th equation. The next (inner) columns give the degrees of the polynomials that appear in the first column for different values of \(j\). The degrees depend on \(\mathcal{T}\), but, by the conditions imposed to this matrix \(\mathcal{T}\), the first \(n\) rows are always filled in the same way.

This table gives all the necessary information for calculating the heights of the polynomials \(p_k(z)\) and \(q_i(z)\) (this information appears in bold face). Indeed, by Definition 4 taking into account (2.15) and (2.16), one verifies

\[
\begin{align*}
    h(p_4) &= 3(1) + 0 = 3 & h(p_6) &= 3(2) + 0 = 6 & h(p_7) &= 3(3) + 0 = 9 \\
    h(p_5) &= 3(1) + 1 = 4 & h(q_2) &= 3(2) + 1 = 7 & h(q_3) &= 3(4) + 0 = 12 \\
    h(q_1) &= 3(1) + 2 = 5
\end{align*}
\]

Remarkably, the information relevant to us can be obtained for any matrix by filling in a table as the one above using the following simple algorithm:
1. First fill in, according to the matrix entries, the first and last column. Note that the in first column \( k = n + 1, \ldots, N \) and we skip the rows where \( m_1, \ldots, m_{n-1} \) appear in the last column.

2. We fill the columns corresponding to \( j = 1, \ldots, n \) by putting only one number in each row as follows:
   
   (a) Put 1’s in the intersection of the \( j \)-th column and \( j \)-th row for \( j = 1, \ldots, n \).
   
   (b) The \( l \)-th row \( (l = n + 1, \ldots, N) \) is filled as follows: Take note of the value of \( l \) (last column) and look for the row in the first column where \( k = l \). In the found row look for the column that contains a number. Pick that number plus one and write it down in the intersection of this column and the \( l \)-th row.

Having filled the table, it is easy to find the heights of the vector polynomials \( q_j(z) \) and \( p_k(z) \) for \( j \in \{1, \ldots, n\} \) and \( k \in \{1, \ldots, N\} \), respectively. By the formula

\[
\begin{align*}
    h(p_k) &= ns + (j - 1) \quad \text{for any } k = n + 1, \ldots, N, \\
    h(q_i) &= ns + (j - 1) \quad \text{for any } i = 1, \ldots, n, \\
\end{align*}
\]

(A.1)

where \( s \) is the number that one has put in the \( k \)-th or \( i \)-th row.

For instance, if \( n = 5 \), \( N = 15 \), \( m_1 = 2 \), \( m_2 = 3 \), \( m_3 = 7 \) and \( m_4 = 11 \), we have the following table, filled in with our algorithm:

| \( P_k(j) \) | \( Q_i(j) \) | \( j = 1 \) | \( j = 2 \) | \( j = 3 \) | \( j = 4 \) | \( j = 5 \) | \( l \) |
|-------------|-------------|-------------|-------------|-------------|-------------|-------------|---------|
| \( k = 6 \) |             | 1           |             |             |             |             | 1       |
| \( i = 1 \) |             | 1           |             |             |             |             | \( m_1 = 2 \) |
| \( i = 2 \) |             | 1           |             |             |             |             | \( m_2 = 3 \) |
| \( k = 7 \) |             |             | 1           |             |             |             | 4       |
| \( k = 8 \) |             |             | 1           |             |             |             | 5       |
| \( k = 9 \) |             |             |             | 2           |             |             | 6       |
| \( i = 3 \) |             |             |             | 2           |             |             | \( m_3 = 7 \) |
| \( k = 10 \) |             |             |             | 2           |             |             | 8       |
| \( k = 11 \) |             |             |             | 3           |             |             | 9       |
| \( k = 12 \) |             |             |             | 3           |             |             | 10      |
| \( i = 4 \) |             |             |             | 4           |             |             | \( m_4 = 11 \) |
| \( k = 13 \) |             |             |             | 4           |             |             | 12      |
| \( k = 14 \) |             |             |             |             | 5           |             | 13      |
| \( k = 15 \) |             |             |             |             | 6           |             | 14      |
| \( i = 5 \) |             |             |             |             |             | 7           | \( m_5 = 15 \) |
Now, the heights of the vector polynomials are given by \((A.1)\):

\[
\begin{align*}
    h(p_6) &= 5(1) + 0 = 5 & h(p_9) &= 5(2) + 0 = 10 & h(q_1) &= 5(4) + 0 = 20 \\
    h(q_1) &= 5(1) + 1 = 6 & h(q_3) &= 5(2) + 3 = 13 & h(p_{13}) &= 5(4) + 4 = 24 \\
    h(q_2) &= 5(1) + 2 = 7 & h(p_{10}) &= 5(2) + 4 = 14 & h(p_{14}) &= 5(5) + 4 = 29 \\
    h(p_7) &= 5(1) + 3 = 8 & h(p_{11}) &= 5(3) + 0 = 15 & h(p_{15}) &= 5(6) + 4 = 34 \\
    h(p_8) &= 5(1) + 4 = 9 & h(p_{12}) &= 5(3) + 4 = 19 & h(q_5) &= 5(7) + 4 = 39
\end{align*}
\]

**Remark 8.** The algorithm shows that there are exactly \(N\) vector polynomials \(p_k(z)\) and all of these vectors form an orthonormal set on \(L^2(\mathbb{R}, \sigma)\).

Another example that is useful for understanding the algorithm, is when \(n = 3, N = 12, m_1 = 4, m_2 = 9\)

| \(P_k^{(j)}|Q_i^{(j)}\) |  \(j = 1\) |  \(j = 2\) |  \(j = 3\) |  \(l\) |
|-----------------|----------|----------|----------|------|
| \(k = 4\)      |  1       |          |          | 1    |
| \(k = 5\)      |          |  1       |          | 2    |
| \(k = 6\)      |          |          |  1       | 3    |
| \(i = 1\)      |          |          |          | 2    |
| \(k = 7\)      |          |          |          | 2    |
| \(k = 8\)      |          |          |  2       | 5    |
| \(k = 9\)      |          |          |  3       | 7    |
| \(k = 10\)     |          |          |  3       | 8    |
| \(i = 2\)      |          |          |  4       | 6    |
| \(k = 11\)     |          |          |  4       | 9    |
| \(k = 12\)     |          |  5       |          | 10   |
| \(i = 3\)      |  6       |          |          | 11   |
|                 |          |          |          |      |

So, the heights are given by

\[
\begin{align*}
    h(p_4) &= 3 & h(p_7) &= 7 & h(q_2) &= 13 \\
    h(p_5) &= 4 & h(p_8) &= 8 & h(p_{11}) &= 14 \\
    h(p_6) &= 5 & h(p_9) &= 10 & h(p_{12}) &= 17 \\
    h(q_1) &= 6 & h(p_{10}) &= 11 & h(q_3) &= 20
\end{align*}
\]

**Remark 9.** Observe that the heights of the set \(\{p_k(z)\}^N_{k=1} \cup \{z^i q_i(z)\}^n_{i=1}\) are in one-to-one correspondence with the set \(\{0\} \cup \mathbb{N}\). Thus, due to Proposition 3.1 one can write any \(n\)-dimensional vector polynomial \(r\) as

\[
r(z) = \sum_{k=1}^N c_k p_k(z) + \sum_{j=1}^n S_j(z) q_j(z), \quad (A.2)
\]

25
where $c_k \in \mathbb{C}$, $S_j(z)$ are scalar polynomials. Also, $c_k = 0$, respectively $S_j(z) = 0$, if $h(r) > h(p_k)$, respectively $h(r) > h(q_j)$.

References

[1] N. I. Akhiezer and I. M. Glazman. *Theory of linear operators in Hilbert space*. Dover Publications Inc., New York, 1993. Translated from the Russian and with a preface by Merlynd Nestell, Reprint of the 1961 and 1963 translations, Two volumes bound as one.

[2] B. Beckermann and A. Osipov. Some spectral properties of infinite band matrices. *Numer. Algorithms*, 34(2-4):173–185, 2003. International Conference on Numerical Algorithms, Vol. II (Marrakesh, 2001).

[3] F. W. Biegler-König. Construction of band matrices from spectral data. *Linear Algebra Appl.*, 40:79–87, 1981.

[4] M. T. Chu and G. H. Golub. *Inverse eigenvalue problems: theory, algorithms, and applications*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2005.

[5] C. de Boor and G. H. Golub. The numerically stable reconstruction of a Jacobi matrix from spectral data. *Linear Algebra Appl.*, 21(3):245–260, 1978.

[6] R. del Rio and M. Kudryavtsev. Inverse problems for Jacobi operators: I. Interior mass-spring perturbations in finite systems. *Inverse Problems*, 28(5):055007, 18, 2012.

[7] R. del Rio, M. Kudryavtsev, and L. O. Silva. Inverse problems for Jacobi operators III: Mass-spring perturbations of semi-infinite systems. *Inverse Probl. Imaging*, 6(4):599–621, 2012.

[8] R. del Rio, M. Kudryavtsev, and L. O. Silva. Inverse problems for Jacobi operators II: Mass perturbations of semi-infinite mass-spring systems. *Zh. Mat. Fiz. Anal. Geom.*, 9(2):165–190, 2013.

[9] M. G. Gasymov and G. S. Guseinov. On inverse problems of spectral analysis for infinite Jacobi matrices in the limit-circle case. *Dokl. Akad. Nauk SSSR*, 309(6):1293–1296, 1989.

[10] F. Gesztesy and B. Simon. $m$-functions and inverse spectral analysis for finite and semi-infinite Jacobi matrices. *J. Anal. Math.*, 73:267–297, 1997.
[11] G. M. L. Gladwell. *Inverse problems in vibration*, volume 119 of *Solid Mechanics and its Applications*. Kluwer Academic Publishers, Dordrecht, second edition, 2004.

[12] L. Golinskii and M. Kudryavtsev. Inverse spectral problems for a class of five-diagonal unitary matrices. *Dokl. Akad. Nauk*, 423(1):11–13, 2008.

[13] L. Golinskii and M. Kudryavtsev. Rational interpolation and mixed inverse spectral problem for finite CMV matrices. *J. Approx. Theory*, 159(1):61–84, 2009.

[14] L. Golinskii and M. Kudryavtsev. An inverse spectral theory for finite CMV matrices. *Inverse Probl. Imaging*, 4(1):93–110, 2010.

[15] L. J. Gray and D. G. Wilson. Construction of a Jacobi matrix from spectral data. *Linear Algebra and Appl.*, 14(2):131–134, 1976.

[16] G. Š. Guseinov. The determination of the infinite Jacobi matrix from two spectra. *Mat. Zametki*, 23(5):709–720, 1978.

[17] R. Z. Halilova. An inverse problem. *Izv. Akad. Nauk Azerbaidžan. SSR Ser. Fiz.-Tehn. Mat. Nauk*, 1967(3-4):169–175, 1967.

[18] H. Hochstadt. On some inverse problems in matrix theory. *Arch. Math. (Basel)*, 18:201–207, 1967.

[19] H. Hochstadt. On the construction of a Jacobi matrix from spectral data. *Linear Algebra and Appl.*, 8:435–446, 1974.

[20] H. Hochstadt. On the construction of a Jacobi matrix from mixed given data. *Linear Algebra Appl.*, 28:113–115, 1979.

[21] V. A. Ilyin and È. G. Poznyak. *Linear algebra*. “Mir”, Moscow, 1986. Translated from the Russian by Irene Aleksanova.

[22] M. Kudryavtsev. The direct and the inverse problem of spectral analysis for five-diagonal symmetric matrices. I. *Mat. Fiz. Anal. Geom.*, 5(3-4):182–202, 1998.

[23] M. Kudryavtsev. The direct and the inverse problem of spectral analysis for five-diagonal symmetric matrices. II. *Mat. Fiz. Anal. Geom.*, 6(1-2):55–80, 1999.

[24] M. Kudryavtsev, S. Palafox, and L. O. Silva. On a linear interpolation problem for n-dimensional vector polynomials. arXiv:1401.5384v1 [math.CA] 2014.
[25] V. A. Marchenko. *Introduction to the theory of inverse problems of spectral analysis*. Universitetski Lekcii. Akta, Kharkov, 2005. In Russian.

[26] M. P. Mattis and H. Hochstadt. On the construction of band matrices from spectral data. *Linear Algebra Appl.*, 38:109–119, 1981.

[27] P. Nylen and F. Uhlig. Inverse eigenvalue problem: existence of special spring-mass systems. *Inverse Problems*, 13(4):1071–1081, 1997.

[28] P. Nylen and F. Uhlig. Inverse eigenvalue problems associated with spring-mass systems. In *Proceedings of the Fifth Conference of the International Linear Algebra Society (Atlanta, GA, 1995)*, volume 254, pages 409–425, 1997.

[29] Y. M. Ram. Inverse eigenvalue problem for a modified vibrating system. *SIAM J. Appl. Math.*, 53(6):1762–1775, 1993.

[30] L. O. Silva and R. Weder. On the two spectra inverse problem for semi-infinite Jacobi matrices. *Math. Phys. Anal. Geom.*, 9(3):263–290 (2007), 2006.

[31] L. O. Silva and R. Weder. The two-spectra inverse problem for semi-infinite Jacobi matrices in the limit-circle case. *Math. Phys. Anal. Geom.*, 11(2):131–154, 2008.

[32] S. M. Zagorodnyuk. Direct and inverse spectral problems for $(2N + 1)$-diagonal, complex, symmetric, non-Hermitian matrices. *Serdica Math. J.*, 30(4):471–482, 2004.

[33] S. M. Zagorodnyuk. The direct and inverse spectral problems for $(2N + 1)$-diagonal complex transposition-antisymmetric matrices. *Methods Funct. Anal. Topology*, 14(2):124–131, 2008.