MORITA’S THEORY FOR THE SYMPLECTIC GROUPS

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Abstract. We construct and study the holomorphic discrete series representations and the principal series representations of the symplectic group $\text{Sp}(2n, F)$ over a $p$-adic field $F$ as well as a duality between some sub-representations of these two representations. The constructions of these two representations generalize those defined in Morita and Murase’s works. Moreover, Morita built a duality for $\text{SL}(2, F)$ defined by residues. We view the duality we defined as an algebraic interpretation of Morita’s duality in some extent and its generalization to the symplectic groups.

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Notations

Let $p$ be a prime, $F$ a finite extension of $\mathbb{Q}_p$, $\mathfrak{o}$ the ring of integers of $F$, $\varpi$ the uniformizer of $\mathfrak{o}$, $||$ the normalized absolute value, and $F^{\text{alg}}$ an algebraic closure of $F$. Let $K$ be an extension of $F$ with an absolute value extending $|\cdot|$, and $K$ is complete under this absolute value. Because Hahn-Banach theorem is applied, we assume that $K$ is spherically complete in §2 and §3.

0. Introduction

Morita and Murase constructed and studied in [5] the $p$-adic holomorphic discrete series representations of $\text{SL}(2, F)$. In [8], Schneider introduced the holomorphic discrete series of $\text{SL}(n + 1, F)$ associated to a rational representation of $\text{GL}(n, F)$. He showed that, as a $\text{SL}(n + 1, F)$-representation, the space of holomorphic exterior differential $r$-forms on the Drinfel’d’s space belongs to the holomorphic discrete series.

Morita started the systematic study of the principal series (parabolic induced representations) of $\text{SL}(2, F)$ in [6] and [7]. In order to prove the irreducibility conjectures on the holomorphic discrete series, Morita constructed a duality pairing via residues between the holomorphic discrete series and the principal series ([6]).

In the first paragraph, we generalize Morita’s constructions to the symplectic groups. After recollecting some notions on the symplectic groups, in §1.2, following [6], we construct another interpretation of the parabolic induced representation, which is conventionally called the principal series. General results of Féaux de Lacroix on the induced representations of the $F$-Lie groups ([3]) are applied for our purpose. In §1.3, we introduce a $p$-adic analogue of the Siegel upper half-space, and on which we define an $F$-rigid analytic structure. The method is similar to the one utilized to study the Drinfel’d’s space in [10]. In §1.4, we introduce the notion of the holomorphic discrete series of $\text{Sp}(2n, F)$ associated to a $K$-rational representation of $\text{GL}(n, F)$ and prove that the space of rigid analytic exterior differential $r$-forms on the Siegel upper half-space can be realized as a holomorphic discrete series representation.

In the second paragraph, in a purely algebraic way, we construct two invariant closed subspaces of the principal series and the holomorphic discrete series respectively and establish a duality operator between them. We remark that, since the two spaces are of compact type and nuclear $K$-Fréchet respectively, the duality fits into the framework of Schneider and Teitelbaum’s theory (cf. [11]).

In the last paragraph, in the case of $\text{SL}(2, F)$, we analyze the relations between the duality constructed in the second paragraph and Morita’s duality: composing with Casselman’s intertwining operator defined by taking derivation, Morita’s duality equals our duality up to a constant.

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1. Symplectic groups and their representations

1.1. The symplectic group $\text{Sp}(2n, F)$. Let $n$ be a positive integer and

$$J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$ 

The symplectic group $\text{Sp}(2n, F)$ is the subgroup of $\text{GL}(2n, F)$ consisting of $g$ satisfying

$$g^t J_n g = J_n.$$

If we write $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then $g \in \text{Sp}(2n, F)$ if and only if either of the following two conditions hold:

\begin{align*}
(1.1) & \quad tAD - tCB = I_n, \quad tAC = tCA, \quad tBD = tDB; \\
(1.2) & \quad D^tA - C^tB = I_n, \quad D^tC = C^tD, \quad B^tA = A^tB.
\end{align*}

Then we introduce two homogeneous spaces $\mathcal{P}(n)$ and $\mathcal{L}(n)$ associated to $\text{Sp}(2n, F)$. $\mathcal{P}(n)$ denotes the set of pairs $(X, Y)$, $X, Y \in M(n, F)$, such that

$$X^t Y = Y^t X, \quad \text{rank}(XY) = n.$$ 

We define a right action of $\text{Sp}(2n, F)$ and a left action of $\text{GL}(n, F)$ on $\mathcal{P}(n)$ via

$$(X, Y)g := (XA +YC, XB + YD), \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2n, F),$$

$$h(X, Y) := (hX, hY), \quad h \in \text{GL}(n, F),$$

respectively. Let

$$U := \left\{ \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \in \text{Sp}(2n, F) \right\}.$$ 

We identify the $\text{Sp}(2n, F)$-homogeneous space $U \setminus \text{Sp}(2n, F)$ and $\mathcal{P}(n)$ via $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto (C, D)$ (the inverse map comes from the symplectic Gram-Schmidt process).

$\mathcal{L}(n)$ denotes the set of transposed Lagrangian subspaces. Let

$$P := \left\{ \begin{pmatrix} D^{-1} & B \\ 0 & D \end{pmatrix} \in \text{Sp}(2n, F) \right\}.$$ 

We identify $\mathcal{L}(n)$ with the $\text{Sp}(2n, F)$-homogeneous space $P \setminus \text{Sp}(2n, F)$. Since $P$ is a parabolic subgroup, $P \setminus \text{Sp}(2n, F)$ is a smooth projective variety over $F$.

Because $P \cong U \rtimes \text{GL}(n, F)$, we have a natural $\text{Sp}(2n, F)$-equivariant isomorphism $\mathcal{P}(n) \cong \mathcal{L}(n)$; the projection from $\mathcal{P}(n)$ to $\mathcal{L}(n)$ maps $(X, Y)$ to the transposed Lagrangian subspace spanned by the row vectors of $(X, Y)$.

Finally we define open subsets which define the coordinates on $\text{Sp}(2n, F)$, $\mathcal{P}(n)$ and $\mathcal{L}(n)$.

Let

$$U_0 := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2n, F) \mid \det(C) \neq 0 \right\}.$$
We have the following unique decomposition in $\text{Sp}(2n, F)$ for matrices in $U_0$:

\begin{equation}
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
I_n & AC^{-1} \\
0 & I_n
\end{pmatrix} \begin{pmatrix}
C^{-1} & 0 \\
0 & C
\end{pmatrix} \begin{pmatrix}
I_n & -I_n \\
0 & C^{-1}D
\end{pmatrix}
\end{equation}

$AC^{-1}$ and $C^{-1}D$ are symmetric ((1.1) and (1.2)); we identify $U_0$ with $\text{Sym}(n, F) \times \text{GL}(n, F) \times \text{Sym}(n, F)$.

Let $U_0$ be the open subset of $\mathcal{P}(n)$:

$$\{(h, hz) \mid h \in \text{GL}(n, F), z \in \text{Sym}(n, F)\}.$$  

Under the identification $\mathcal{P}(n) \equiv U \backslash \text{Sp}(2n, F)$ we have $U_0 \equiv U \backslash U_0$.

Also we identify $\text{Sym}(n, F)$ with the open subset $P \backslash U_0$ of $\mathcal{L}(n)$.

To lighten notations, hereafter we let $G = \text{Sp}(2n, F)$, $G_o = \text{Sp}(2n, o)$, $H = \text{GL}(n, F)$, $H_o = \text{GL}(n, o)$ and abbreviate $\mathcal{P}(n)$ and $\mathcal{L}(n)$ to $\mathcal{P}$ and $\mathcal{L}$ respectively. $\text{pr}_{\mathcal{P}}^G$, $\text{pr}_{\mathcal{L}}^G$ and $\text{pr}_{\mathcal{L}}^D$ denote the canonical projections.

1.2. $\text{Ind}_p^G \sigma$ and the principal series ($C^\text{an}_\sigma(\mathcal{P}, V), T_\sigma$). Let $(V, \sigma)$ be a locally analytic representation (cf. [3] 3.1.5 and [11] §3) of $H$ on a barreled locally convex Hausdorff $K$-vector space $V$, which means that the orbit maps are $V$-valued locally analytic functions; more precisely, for any $v \in V$ there exists a BH-space $W$ of $V$ (that is, a Banach space $W$ together with a continuous injection $W \hookrightarrow V$) such that $g \mapsto \sigma(g)v$ expands (in a neighborhood of the unit element) to a power series with $W$-coefficients (cf. [3]).

$\sigma$ extends to a representation of $P$ via the projection

$$P \to H, \quad \begin{pmatrix}
D^{-1} & B \\
0 & D
\end{pmatrix} \mapsto D.$$  

We consider the parabolic induced representation $\text{Ind}_p^G \sigma$ whose underlying space is the space of $V$-valued locally analytic functions $f$ on $G$ satisfying

$$f(pg) = \sigma(p)f(g), \quad \text{for all } g \in G, p \in P;$$

$G$ acts by the right translation.

Because the homogeneous space $\mathcal{L}$ is compact, $\text{Ind}_p^G \sigma$ is a locally analytic representation of $G$ ([3] 4.1.5).

Next, we give another description of $\text{Ind}_p^G \sigma$. Let $C^\text{an}_\sigma(\mathcal{P}, V)$ be the space of $V$-valued locally analytic functions $\varphi$ on $\mathcal{P}$ satisfying

$$\varphi(hX, hY) = \sigma(h)\varphi(X, Y), \quad \text{for all } (X, Y) \in \mathcal{P} \text{ and } h \in H.$$  

We define the principal series representation ($C^\text{an}_\sigma(\mathcal{P}, V), T_\sigma$) of $G$:  

\begin{equation}
(T_\sigma(g)\varphi)(X, Y) := \varphi((X, Y)g).
\end{equation}

**Lemma 1.1.**

1. The representation $\text{Ind}_p^G \sigma$ is (naturally) isomorphic to $(C^\text{an}_\sigma(\mathcal{P}, V), T_\sigma)$.

2. $\text{Ind}_p^G \sigma$ is isomorphic to $C^\text{an}(\mathcal{L}, V)$.  

Proof. (1) From a locally analytic section \(\iota\) of \(pr_L^G\) we obtain an isomorphism \(\overline{\iota} : \text{Ind}^G_B(I) \cong C^{\text{an}}(\mathcal{P}, V), f \mapsto f \circ \overline{\iota}\) ([3] 4.3.1); by restriction, \(\overline{\iota}\) induces an isomorphism between \(\text{Ind}^G_B(\sigma)\) and \(C^{\text{an}}_\sigma(\mathcal{P}, V)\) which is independent of \(\iota\). G-equivariance is evident.

(2) A locally analytic section \(\iota\) of \(pr_L^P\) induces an isomorphism \(\iota^\circ : \text{Ind}^G_B(\sigma) \cong C^{\text{an}}(\mathcal{L}, V)\) (ibid.).

Because \(\mathcal{L}\) is compact, \(C^{\text{an}}(\mathcal{L}, V)\) is of compact type ([11] Lemma 2.1). By [11] Proposition 1.2, Theorem 1.3 and [9] Proposition 16.10, we have

**Corollary 1.2.** Let \(B\) be a closed subspace of \(C^{\text{an}}(\mathcal{P}, V)\), then \(B\) and \(C^{\text{an}}(\mathcal{P}, V)\) are of compact type, in particular, they are reflexive, bornological, and complete; \(B^*_p\) and \(C^{\text{an}}_\sigma(\mathcal{P}, V)/B\) are nuclear Fréchet spaces.

For technical needs, we fix a finite disjoint open covering \(\{\overline{U}_k\}_k\) of \(\mathcal{L}\) satisfying:

1. \(\text{Sym}(n, \sigma) \subset \{\overline{U}_k\}_k\);
2. each \(\overline{U}_k\) is translated into \(\text{Sym}(n, \sigma)\) by an element \(g_k\) in \(G\);
3. Let \(U_k := (pr_L^P)^{-1}(\overline{U}_k)\). We define the analytic local section \(\iota_k : \overline{U}_k \to U_k\) of \(pr_L^P\) to be the \(g_k^{-1}\)-translation of the section

\[
\iota_0 : \text{Sym}(n, F) \to U_0, \quad z \mapsto (1, -z).
\]

Let \(\iota\) be the locally analytic section of \(pr_L^P\) defined by \(\iota_k\), and define \(K := \iota(\mathcal{L})\).

If the locally analytic sections \(\iota\) and \(\overline{\iota}\) in the proof of Lemma 1.1 are compatible with \(\iota\), that is, if \(\overline{\iota} = \iota \circ \overline{\iota}\), then Lemma 1.1 implies that \(\iota\) induces an isomorphism

\[
(1.5) \quad \iota^\circ : C^{\text{an}}(\mathcal{P}, V) \to C^{\text{an}}(\mathcal{L}, V) \quad \varphi \mapsto \varphi \circ \iota.
\]

1.3. The p-adic Siegel upper half-space. In this section, we define a \(p\)-adic analogue of the Siegel upper half-space which also generalize the \(p\)-adic upper half-plane (cf. [2]), and we start the discussion of some basic properties.

Let \(S\) be the \(F\)-rigid analytic variety \(\text{Sym}(n)\) which is isomorphic to the affine space \(A_{\text{div}}(n+1)/F^\text{alg}\). The underlying space of \(S\) is \(\text{Sym}(n, F^\text{alg})\) (Strictly speaking, \(\text{Sym}(n, F^\text{alg})/\text{Gal}(F^\text{alg}/F)\) (cf. [1]), but it is more convenient not to consider the Galois action in our situation).

**Definition 1.3.** Let

\[
\Sigma := \{Z \in S \mid \det(XZ + Y) \neq 0 \text{ for any pair } (X, Y) \in \mathcal{P}\}.
\]

\(\Sigma\) is called the \(p\)-adic Siegel upper half-space.

First, we show that \(\Sigma\) is nonempty.

**Lemma 1.4.** If \(Z\) is a diagonal matrix in \(S\) whose diagonal entry \(Z_{ii}\) is of absolute value \(|Z_{ii}|^{1/(n+1)}\), with distinct positive integers \(k_i\), then \(Z \in \Sigma\).

**Proof.** We show that \(\det(XZ + Y) \neq 0\) for any pair \((X, Y) \in \mathcal{P}\). By multiplying a matrix of \(H\) on the left and a permutation matrix on the right of \(X\) and \(Y\), and conjugating \(Z\) by the permutation matrix, we may assume that \(X = \begin{pmatrix} I_r & \tilde{X} \\ 0 & 0 \end{pmatrix}\), where...
Therefore
\[ \det(XZ + Y) = \det(Z_1 + \bar{X}Z_2 \bar{Y} + Y_1 + Y_2 \bar{Y}) \det Y_4. \]
\[ \text{rank}(X Y) = n \text{ and } Y_3 = -Y_4 \bar{Y} \text{ implies that } Y_4 \text{ is invertible, } \det Y_4 \neq 0. \]
Clearly, the first determinant on the right is a nonzero polynomial in $Z_{ii}$ with coefficients in $F$, and the degree of $Z_{ii}$ in each term $\leq n$; by the assumptions on $Z_{ii}$, the terms appeared in the polynomial are of distinct absolute values, and therefore the determinant is nonzero. In conclusion, $\det(XZ + Y) \neq 0$.

Q.E.D.

In the following, we endow $\Sigma$ with a structure of $F$-rigid analytic variety and show that it is an admissible open subset of $S$ and consequently an open rigid analytic subspace of $S$ (compare [10] §1 Proposition 1).

We define $P_o = \text{pr}^G_p(G_o), P_o$ is compact. By Iwasawa’s decomposition, $G = P \cdot G_o$, and $P = H \cdot P_o$, and therefore
\[ \Sigma = \{ Z \in S \mid \det(XZ + Y) \neq 0 \text{ for any pair } (X, Y) \in P_o \}. \]

For $Z \in S$, let
\[ |Z| := \max_{1 \leq i,j \leq n} \left\{ 1, |Z_{ij}| \right\}. \]

For a nonnegative integer $m$ and a pair $(X, Y) \in P_o$, we define
\[ B^-(m; X, Y) := \{ Z \in S \mid |\det(XZ + Y)| < |Z|^n |\sigma|^{mn} \} \text{ for some } h \in H_o, \]
\[ B^-(m; X, Y) = B^-(m; X', Y'). \]

**Lemma 1.5.** If $m$ is a nonnegative integer and $(X, Y), (X', Y') \in P_o$ such that $(X, Y) \equiv (hX', hY') \mod \sigma^{mn+1}$ for some $h \in H_o$, then
\[ B^-(m; X, Y) = B^-(m; X', Y'). \]

**Proof.** Obviously $B^-(m; X, Y) = B^-(m; hX, hY)$. We may assume $(X, Y) \equiv (X', Y') \mod \sigma^{mn+1}$.

We choose $\lambda \in (F^{alg})^\times$ such that $|\lambda| = |Z|$. We note that $|\lambda^{-1}| \leq 1$ and $|\lambda^{-1}Z_{ij}| \leq 1$, then
\[ X \cdot \lambda^{-1}Z + Y \cdot \lambda^{-1} \equiv X' \cdot \lambda^{-1}Z + Y' \cdot \lambda^{-1} \mod \sigma^{mn+1}, \]
\[ \det(XZ + Y) \cdot \lambda^{-n} \equiv \det(X'Z + Y') \cdot \lambda^{-n} \mod \sigma^{mn+1}, \]
whence
\[ |\det(XZ + Y)| |Z|^{-n} < |\sigma|^{mn} \iff |\det(X'Z + Y')| |Z|^{-n} < |\sigma|^{mn}. \]

Therefore $B^-(m; X, Y) = B^-(m; X', Y')$.

Q.E.D.
We define

$$\Sigma(m; X, Y) := S - B^{-1}(m; X, Y) = \{Z \in S \mid |det(XZ + Y)| \geq |Z|^m |\omega|^{mn}\}.$$ 

Let

$$\Sigma(m) := \bigcap_{(X,Y) \in \mathcal{P}_0} \Sigma(m; X, Y)$$

$$= \left\{ Z \in S \mid \frac{|\omega|^{mn}}{|det(XZ + Y)|} \leq 1, \frac{|\omega|^{mn}Z_{ij}^n}{|det(XZ + Y)|} \leq 1 \text{ for any } (X,Y) \in \mathcal{P}_0 \right\}.$$ 

Let $\mathcal{P}^{(m)}$ be any finite subset of $\mathcal{P}_0$ containing $(0, I_n)$ and a set of representatives in $\mathcal{P}_0$ for $H_0 \mathcal{P}_0$ (mod $\omega^{mn+1}$). Then Lemma 1.5 implies that

$$\Sigma(m) = \bigcap_{(X,Y) \in \mathcal{P}^{(m)}} \Sigma(m; X, Y).$$

Let $\mathcal{P}^{(m)}_0 = \mathcal{P}^{(m)} - \{(0, I_n)\}$.

$$\Sigma(m; 0, I_n) = \{Z \in S \mid |Z_{ij}| \leq |\omega|^{-m}, 1 \leq i \leq j \leq n\}$$

$$= \text{Sp} \left( F \left( \left( \frac{|\omega|^{mn}Z_{ij}}{|det(XZ + Y)|} \right) \leq 1 \right) \right),$$

is an admissible open affinoid subset of $S$. $\Sigma(m)$ is the intersection of a finite number of rational sub-domains of $\Sigma(m; 0, I_n)$:

$$\left\{ Z \in \Sigma(m; 0, I_n) \mid \frac{|\omega|^{mn}}{|det(XZ + Y)|} \leq 1, \frac{|\omega|^{mn}Z_{ij}^n}{|det(XZ + Y)|} \leq 1 \right\},$$

with $(X, Y)$ running through $\mathcal{P}^{(m)}_0$. Therefore $\Sigma(m)$ is the affinoid variety:

$$\text{Sp} \left( F \left( \left( \frac{|\omega|^{mn}Z_{ij}}{|det(XZ + Y)|}, \frac{|\omega|^{mn}Z_{ij}^n}{|det(XZ + Y)|} \right) \leq 1 \right) \right).$$

$$\{\Sigma(m)\}_{m=0}^{\infty}$$

forms an admissible affinoid covering of $\Sigma$: $\Sigma$ admits a rigid analytic variety structure (see [1] 9.3). According to [1] 9.1.2 Lemma 3 (compare [1] 9.1.4 Proposition 2), the following Proposition implies that $\Sigma$ is an admissible open subset of $S$.

**Proposition 1.6.** Any morphism from an affinoid variety to $S$ with image in $\Sigma$ factors through some $\Sigma(m)$.

**Proof:** The argument is similar to the third proof of [10] §1 Proposition 1.

Let $X$ be an affinoid variety, $\phi : X \to S$ a morphism from $X$ to $S$ with image in $\Sigma$. For any $(X, Y) \in \mathcal{P}_0$,

$$x \mapsto \frac{1}{|det(X\phi(x) + Y)|}, \quad x \mapsto \frac{(\phi(x))_{ij}^n}{|det(X\phi(x) + Y)|}.$$
are $F$-rigid analytic functions on $X$. By the maximum modulus principle ([1] §6.2 Proposition 4 (i)), there exists a positive integer $m_{(X,Y)}$ such that

$$\max_{1 \leq i \leq j \leq n} \max_{x \in X} \left\{ \frac{1}{\det(X\phi(x) + Y)}, \left| \frac{(\phi(x))_{ij}^m}{\det(X\phi(x) + Y)} \right| \right\} \leq |\sigma|^{-m_{(X,Y)}}.$$  

In other words, $\phi(X) \subset \Sigma(m_{(X,Y)}; X, Y)$. In view of Lemma 1.5, $m_{(X,Y)}$ can be chosen locally constant; because $P_0$ is compact, there exists a positive integer $m$ such that $\phi(X) \subset \Sigma(m)$. Q.E.D.

$\mathcal{O}(\Sigma(m))$ denotes the space of $F$-rigid analytic functions on $\Sigma(m)$; it is an $F$-affinoid algebra with the supremum norm. We see from (1.6) that $\psi \in \mathcal{O}(\Sigma(m))$ has an expansion in the form:

$$\psi(Z) = \sum_{(k, (X,Y)) \in (\mathbb{N}_0)^{p(m)}} P_{(k, (X,Y))}(Z) \prod_{(X,Y) \in P^{(m)}} \det(XZ + Y)^{-k_{(X,Y)}},$$

where $\mathbb{N}_0$ denotes the set of non-negative integers, $P_{(k, (X,Y))}(Z)$ are polynomials in $Z_{ij}$ with coefficients in $F$, and the expansion converges with respect to the supremum norm $\| \cdot \|_{\mathcal{O}(\Sigma(m))}$. In particular, $\det(XZ + Y)^{-1} \in \mathcal{O}(\Sigma(m))$ for any $(X, Y) \in P$. Let $\mathcal{O}(\Sigma)$ be the $F$-algebra of $F$-rigid analytic functions on $\Sigma$, which is the projective limit of $\mathcal{O}(\Sigma(m))$,

$$\mathcal{O}(\Sigma) := \varprojlim_m \mathcal{O}(\Sigma(m)).$$

We endow $\mathcal{O}(\Sigma)$ with the projective limit topology.

Let $\mathcal{O}_K(\Sigma(m))$ and $\mathcal{O}_K(\Sigma)$ denote $\mathcal{O}(\Sigma(m)) \otimes_F K$ and $\mathcal{O}(\Sigma) \otimes_K K$ respectively. If we let $\Sigma_K(m)$ and $\Sigma_K$ denote the extension of the ground field $K/F$ of $\Sigma(m)$ and $\Sigma$ respectively ([1] §9.3.6), then $\mathcal{O}_K(\Sigma(m))$ and $\mathcal{O}_K(\Sigma)$ are the $K$-rigid analytic functions on $\Sigma_K(m)$ and $\Sigma_K$ respectively.

**Proposition 1.7.** $\mathcal{O}_K(\Sigma)$ is a nuclear $K$-Fréchet space.

**Proof.** By [9] Corollary 16.6 and Proposition 19.9, it suffices to prove that $\mathcal{O}_K(\Sigma(m))$ form a compact projective system.

$\mathcal{O}_K(\Sigma(m))$ is generated by

$$\sigma^m Z_{ij}, \frac{\sigma^m Z_{ij}}{\det(XZ + Y)}, \frac{\sigma^m Z_{ij}}{\det(XZ + Y)}, \ 1 \leq i \leq j \leq n, (X, Y) \in P^{(m)}_0.$$  

Since

$$\sup_{Z \in \Sigma(m-1)} \sup_{(X,Y) \in P^{(m)}_0} \left\{ |\sigma^m Z_{ij}|, \left| \frac{\sigma^m Z_{ij}}{\det(XZ + Y)} \right|, \left| \frac{\sigma^m Z_{ij}}{\det(XZ + Y)} \right| \right\} \leq |\sigma|.$$  

By [12] Lemma 1.5, the transition homomorphism from $\mathcal{O}_K(\Sigma(m))$ to $\mathcal{O}_K(\Sigma(m-1))$ is compact. Q.E.D.

Clearly the compact projective system passes to closed subspaces. Moreover, a $K$-Fréchet space is the strong dual of a space of compact type if and only if it is nuclear ([11] Theorem 1.3).
Corollary 1.8. Let $\mathcal{N}$ be a closed subspace of $\mathcal{O}_K(\Sigma)$, then $\mathcal{N}$ is a nuclear Fréchet space; $\mathcal{N}^*$ is of compact type.

Remark 1.9. If $K$ is spherically complete, Theorem 1.3 and Proposition 1.2 in [11] imply that $\mathcal{O}_K(\Sigma)/\mathcal{N}$ is also a nuclear Fréchet space.

Since all the generators (1.8) of $\mathcal{O}(\Sigma(m))$ are $F$-rigid analytic functions on $\Sigma(m')$ for any $m' \geq m$ and therefore on $\Sigma$, we obtain

Proposition 1.10.

(1) $\Sigma$ is a Stein space ([4]), that is, the image of $\mathcal{O}(\Sigma(m+1))$ under the transition homomorphism in $\mathcal{O}(\Sigma(m))$ is dense for any nonnegative integer $m$.

(2) The image of $\mathcal{O}(\Sigma)$ under the transition homomorphism in $\mathcal{O}(\Sigma(m))$ is dense.

Finally, we define a rigid analytic G-action (need (1.1) to check) on $\Sigma$:

$$gZ := (AZ + B)(CZ + D)^{-1}, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G, Z \in \Sigma.$$  

We define the automorphy factor

$$j(g, Z) := (CZ + D).$$

From a straightforward computation, we have the automorphy (cocycle) relation

$$j(g_1g_2, Z) = j(g_1, g_2Z)j(g_2, Z).$$

Lemma 1.11. Let $m$ be a nonnegative integer. Then for any $g \in G_o$,

$$g\Sigma(m) \subset \Sigma(nm).$$

Proof. Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_o, Z \in \Sigma(m)$ and $(X, Y) \in P_o$. Then $(XA + YC, XB + YD) \in P_o$, whence

$$\frac{|Z|^n}{|\det(CZ + D)|} \leq |\sigma|^{-nm},$$

$$\frac{|Z|^n}{|\det((XA + YC)Z + (XB + YD))|} \leq |\sigma|^{-nm}.$$  

By Cramer’s rule,

$$gZ = (AZ + B)(CZ + D)^{-1}$$

$$= (AZ + B) \cdot \text{adj}(CZ + D) \frac{1}{\det(CZ + D)},$$

where $\text{adj}(CZ+D)$ denotes the adjugate matrix of $CZ+D$. It is clear that $\det(CZ+D)$ and all the entries of $(AZ + B) \cdot \text{adj}(CZ + D)$ are polynomials in $Z_{ij}$ with coefficients in $0$ and degree $\leq n$, hence

$$|\det(CZ + D)| \leq |Z|^n,$$

$$|(AZ + B) \cdot \text{adj}(CZ + D)|_{ij} \leq |Z|^n.$$
Finally,
\[
\frac{|gZ|^n}{|\det(X(gZ) + Y)|} = \frac{|\det(CZ + D)|}{|\det((XA + YC)Z + (XB + YD))|} \max \left\{ 1, \frac{|(AZ + B) \cdot \text{adj}(CZ + D))_{ij}|^n}{|\det(CZ + D)|^n} \right\}
\leq \frac{|Z|^n}{|\det((XA + YC)Z + (XB + YD))|} \max \left\{ 1, \frac{|Z|^{n-1}}{|\det(CZ + D)|^{n-1}} \right\}
\leq |\sigma|^{-n^2m}.
\]
Therefore \(g\Sigma(m) \subset \Sigma(nm)\).

Q.E.D.

1.4. Holomorphic discrete series \((\mathcal{O}_\sigma(\Sigma), \pi_\sigma)\). We abbreviate \(\Sigma_k(m)(K)\) and \(\Sigma_k(K)\) to \(\Sigma(m)\) and \(\Sigma\) respectively. Conventionally, \(\mathcal{O}_K(\Sigma(m))\) is described as the space of \(K\)-valued functions on \(\Sigma(m)\) with expansions in the form (1.7) which converge in the supremum norm of the \(K\)-valued function space on \(\Sigma_k(m)\), and \(\mathcal{O}_K(\Sigma)\) as the space of \(K\)-valued functions on \(\Sigma\) whose restrictions on \(\Sigma(m)\) are functions in \(\mathcal{O}_K(\Sigma(m))\). We abbreviate \(\mathcal{O}_K(\Sigma(m))\) and \(\mathcal{O}_K(\Sigma)\) to \(\mathcal{O}(\Sigma(m))\) and \(\mathcal{O}(\Sigma)\) respectively.

Let \((V, \sigma)\) be a \(d\)-dimensional \(K\)-rational representation of \(H\). Let
\[
\sigma(h) = \det(h)^{-\frac{d}{2}} P(h), \quad s \in \mathbb{N}_0, P \in M(d, K[h_{ij}]).
\]
Let \(\mathcal{O}_\sigma(\Sigma(m)) := \mathcal{O}(\Sigma(m)) \otimes_{K} V\) and \(\mathcal{O}_\sigma(\Sigma) := \mathcal{O}(\Sigma) \otimes_{K} V\). We define the holomorphic (rigid analytic) discrete series representation \((\mathcal{O}_\sigma(\Sigma), \pi_\sigma)\) of \(G\):
\[
(\pi_\sigma(g)\psi)(Z) := \sigma(j(g^{-1}, Z))^{-1} \psi(g^{-1}Z), \quad \psi \in \mathcal{O}_\sigma(\Sigma), g \in G.
\]
Proposition 1.6 implies that \(g^{-1}\) translates \(\Sigma(m)\) into some \(\Sigma(m')\), it is not difficult to see that \(\pi_\sigma(g)\psi \in \mathcal{O}_\sigma(\Sigma)\) by checking that its coordinates have expansions of the form (1.7) and they are bounded under the supremum norms \(\|\cdot\|_{\mathcal{O}(\Sigma(m))}\). By the automorphy relation (1.9), we verify that \(\pi_\sigma\) is a \(G\)-representation.

**Proposition 1.12.** \((\mathcal{O}_\sigma(\Sigma), \pi_\sigma)\) is continuous.

**Proof.** Since \(\mathcal{O}_\sigma(\Sigma)\) is the projective limit of \(\mathcal{O}_\sigma(\Sigma(m))\), it suffices to prove, for each \(m\), the continuity of
\[
G_0 \times \mathcal{O}_\sigma(\Sigma) \rightarrow \mathcal{O}_\sigma(\Sigma(m))
\]
\[
(g, \psi) \mapsto (\pi_\sigma(g)\psi)_{\Sigma(m)}.
\]
Moreover, according to Lemma 1.11, \(G_0\Sigma(m) \subset \Sigma(nm)\), whence the above map factors through \(G_0 \times \mathcal{O}_\sigma(\Sigma(nm))\). Thus we only need to consider the continuity of the map:
\[
G_0 \times \mathcal{O}_\sigma(\Sigma(nm)) \rightarrow \mathcal{O}_\sigma(\Sigma(m))
\]
\[
(g, \psi) \mapsto (\pi_\sigma(g)\psi)_{\Sigma(m)}.
\]
For \(g \in G_0\), the entries of \(\sigma(j(g^{-1}, Z))^{-1}\) are polynomials with coefficients in \(\mathfrak{o}\) and variables \(Z_{ij}, \det(j(g^{-1}, Z))^{-1}\) and the coefficients of \(P\). We note that, for \(Z \in \Sigma(m)\),
\[ |Z_{ij}| \leq |\sigma|^{-m} \text{ and } |\det(j(g^{-1}, Z))^{-1}| \leq |\sigma|^{-nm}, \text{ then there is a constant } c > 0 \text{ such that} \]
\[
\max_{g \in G_o} \max_{Z \in \Sigma(m)} \|\sigma(j(g^{-1}, Z))^{-1}\|_{\text{End}(U)} \leq c.
\]
Therefore
\[
\max_{g \in G_o} \|\pi_o(g)\psi\|_{\sigma_o(\Sigma(m))} = \max_{g \in G_o} \max_{Z \in \Sigma(m)} \|\pi_o(g)\psi(Z)\|_V
\]
\[
\leq \max_{g \in G_o} \max_{Z \in \Sigma(m)} \|\sigma(g^{-1}, Z)^{-1}\|_{\text{End}(V)} \cdot \max_{g \in G_o} \max_{Z \in \Sigma(m)} \|\psi(g^{-1}Z)\|_V
\]
\[
\leq c \cdot \max_{Z \in \Sigma(m)} \|\psi(Z)\|_V
\]
\[
= c\|\psi\|_{\sigma_o(\Sigma(m))}.
\]
So the map (1.11) is continuous. Q.E.D.

Now let \( U_0(o) \) denote the parameterized open neighborhood of the unit element, \( \text{Sym}(n, o) \times H_o \times \text{Sym}(n, o) \subset U_0 \cap G_o. \)

**Proposition 1.13.** For any \( \psi \in \sigma_o(\Sigma(nm)) \), the orbit map
\[
U_0(o) \to \sigma_o(\Sigma(m))
\]
\[ g \mapsto (\pi_o(g)\psi)_{\Sigma(m)} \]
is an \( \sigma_o(\Sigma(m)) \)-valued analytic function (that is, can be expanded as a convergent power series with variables the coordinate parameters of \( U_0(o) \) and coefficients in the Banach space \( \sigma_o(\Sigma(m)) \)).

**Proof.** We first prove the following

**Lemma 1.14.** Let \( \psi \in \sigma_o(\Sigma(nnm)), z \in \text{Sym}(n, o) \) and \( h \in H_o. \)

1. \( \pi_o\begin{pmatrix} I_n & z \\ 0 & I_n \end{pmatrix} \psi(Z) = \psi(Z - z) \) expands into a convergent power series in \( z_{ij} \) (1 \( \leq i \leq j \leq n \)) with coefficients in \( \sigma_o(\Sigma(m)) \);

2. \( \pi_o\begin{pmatrix} h^{-1} & 0 \\ 0 & h \end{pmatrix} \psi(Z) = \sigma(h)\psi(hZ) \) expands into a convergent power series in \( h_{ij} - \delta_{ij} \) (1 \( \leq i, j \leq n \)) with coefficients in \( \sigma_o(\Sigma(m)) \), where \( \delta_{ij} \) is the Kronecker delta.

**Proof.** (1) We consider the ring \( \sigma(\Sigma(m))[z] \) of formal power series \( \varphi(z) \) in \( z_{ij} \) with coefficients in \( \sigma(\Sigma(m)) \); \( \varphi(z) \) is expressed as
\[
\varphi(z) = \sum_{\alpha \in \text{Sym}(n,H_o)} \alpha \cdot z^\alpha, \quad \alpha \in \sigma(\Sigma(m)), \quad z^\alpha := \prod_{1 \leq i \leq j \leq n} z_{ij}^{r_{ij}}.
\]
If the constant term \( \alpha_0 \) is invertible in \( \sigma(\Sigma(m)) \), then \( \varphi(z) \in \sigma(\Sigma(m))[z] \). In particular, for \( (X, Y) \in \mathcal{P} \), the constant term in the expansion of \( \det(X(Z - z) + Y) \) is \( \det(XZ + Y) \) which is invertible in \( \sigma(\Sigma(m)) \), whence \( \det(X(Z - z) + Y)^{-1} \) belongs to \( \sigma(\Sigma(m))[z] \).
In view of the expansion form (1.7), each coordinate of \( \psi(Z - z) \) expands into a formal power series in \( z_j \) with coefficients series in \( \mathcal{O}(\Sigma(m)) \), but it follows from (1.12) that

1. the coefficients are indeed convergent series in \( \mathcal{O}(\Sigma(m)) \) so that each coordinate of \( \psi(Z - z) \) belongs to \( \mathcal{O}(\Sigma(m))[\![z]\!] \),

2. the \( \mathcal{O}(\Sigma(m)) \)-coefficient formal power series expansion of \( \psi(Z - z) \) converges in \( \mathcal{O}_\sigma(\Sigma(m)) \) for all \( z \in \text{Sym}(n, o) \).

(2) is proved similarly. Q.E.D.

From (1.3), we see that \( g \in U_0(o) \) decomposes in \( G_o \) into

\[
\begin{pmatrix} I_h & z_1 \\ I_n & I_n \end{pmatrix}
\begin{pmatrix} I_n & 0 \\ h^{-1} & I_n \end{pmatrix}
\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}
\begin{pmatrix} I_n & z_2 \\ 0 & I_n \end{pmatrix}
\]

where \( z_1, z_2 \in \text{Sym}(n, o) \) and \( h \in H_o \). Lemma 1.14 and (1.12) imply that \( \pi_\sigma(g) \psi \) expands into a convergent power series with variables the coordinate parameters of \( U_0(o) \) and coefficients in \( \mathcal{O}_\sigma(\Sigma(m)) \).

Q.E.D.

**Corollary 1.15.** The power series expansion of \( \det(Z - z)^{-1} \) on \( \text{Sym}(n, o) \) converges in \( \mathcal{O}(\Sigma(m)) \). Or equivalently, \( \det(Z - z)^{-1} \) expands into a power series

\[
\sum_{\xi \in \text{Sym}(n, 0)} \alpha_\xi(Z) \cdot \mathcal{O}(\Sigma(m)),
\]

such that \( \lim_{|z| \to 0} \|\alpha_\xi\|_{\mathcal{O}(\Sigma(m))} = 0 \) (\( |z| = \sum_{1 \leq i \leq j \leq n} r_{ij} \)).

Next, we consider the adjoint representation \( \pi_\sigma^* \) of \( G \) on \( \mathcal{O}_\sigma(\Sigma(m))^* = \lim_{m} \mathcal{O}_\sigma(\Sigma(m))^*_m \).

The transition homomorphisms \( \mathcal{O}_\sigma(\Sigma(m))^*_m \to \mathcal{O}_\sigma(\Sigma)^*_b \) are injective (Proposition 1.10 (2)). Lemma 1.11 implies that, for any \( g \in G_o \), \( \pi_\sigma^*(g) \) maps \( \mathcal{O}_\sigma(\Sigma(m))^*_m \) into \( \mathcal{O}_\sigma(\Sigma(nm))^*_b \) via

\[
\langle \psi, \pi_\sigma^*(g)\mu \rangle = \langle (\pi_\sigma(g^{-1})\psi)|_{\Sigma(m)}, \mu \rangle, \quad \mu \in \mathcal{O}_\sigma(\Sigma(m))^*, \psi \in \mathcal{O}_\sigma(\Sigma(nm)).
\]

It is easy to deduce from Proposition 1.13 that, for any \( \mu \in \mathcal{O}_\sigma(\Sigma(m))^* \), the orbit map

\[
U_0(o)^{-1} \to \mathcal{O}_\sigma(\Sigma(nm))^*_b
\]

\[
g \mapsto \pi_\sigma^*(g)\mu
\]

is an \( \mathcal{O}_\sigma(\Sigma(nm))^*_b \)-valued analytic function. Therefore we have

**Corollary 1.16.** \( \mathcal{O}_\sigma(\Sigma)^*_b, \pi_\sigma^* \) is locally analytic.

Finally we study the *de Rham complex* \( \Omega(\Sigma) \) of rigid analytic exterior differential forms. Explicitly, let \( 0 \leq r \leq n(n + 1)/2 \),

\[
\begin{align*}
\Omega^1_K &= \bigoplus_{1 \leq i \leq j \leq n} Kd\zeta_{ij}, \\
\Omega^r_K &= \bigwedge^r \Omega^1_K(\Sigma), \\
\Omega^r(\Sigma) &= \mathcal{O}(\Sigma) \otimes K \Omega^r_K.
\end{align*}
\]
As interesting examples, we show that the spaces \( \Omega'(\Sigma) \) as \( \mathbb{G} \)-representations belong to the holomorphic discrete series of \( \mathbb{G} \) (compare [8] §3).

We define a \( K \)-rational representation \( \sigma_1 \) of \( \mathbb{H} \) on \( \Omega^1_K \):

\[
\sigma_1(h) dZ_{ij} := \sum_{1 \leq k < l \leq n} (h_{ik} h_{jl} + h_{il} h_{jk}) dZ_{kl} + \sum_{k=1}^n h_{ik} h_{jk} dZ_{kk}.
\]

Or succinctly,

\[
\sigma_1(h) dZ = h \cdot dZ \cdot \phi, \quad dZ := (dZ_{ij}).
\]

Let \( \sigma_r := \wedge^r \sigma_1 \).

1. For \( g = \begin{pmatrix} I_n & z \\ 0 & I_n \end{pmatrix} \), \( g \cdot dZ = d(Z - z) = dZ \).
2. For \( g = \begin{pmatrix} I_n & z \\ 0 & I_n \end{pmatrix} \), \( g \cdot dZ = d(hZ \cdot \phi) = h \cdot dZ \cdot \phi = \sigma_1(h) dZ \).
3. For \( g = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \), from the identity \( d(Z^{-1}) \cdot Z + Z^{-1} \cdot dZ = 0 \), we have \( g \cdot dZ = d(-Z^{-1}) = Z^{-1} \cdot dZ \cdot Z^{-1} = \sigma_1(Z^{-1}) dZ \).

In view of the decomposition (1.3) of \( U_0 \), the discussions above implies that the action of \( \mathbb{G} \) on \( \Omega'(\Sigma) \) coincides with \( \pi_{\sigma_r} \) on \( U_0 \); as \( U_0 \) is dense in \( \mathbb{G} \), they coincide on \( \mathbb{G} \). We have proved the following result.

**Proposition 1.17.** Let \( 1 \leq r \leq n(n+1)/2 \) and \( \sigma_r \) defined above. The \( \mathbb{G} \)-action on \( \Omega'(\Sigma) \) coincides with \( \pi_{\sigma_r} \).

2. **Duality**

In the following, we assume that \( \mathbb{K} \) is spherically complete. Let \( (V, \sigma) \) be a \( d \)-dimensional \( K \)-rational representation of \( \mathbb{H} \). We choose a basis \( v_1, \ldots, v_d \) of \( V \); we denote by \( v_1^*, \ldots, v_d^* \) the corresponding dual basis of the dual space \( V^* \). \( (V^*, \sigma^*) \) denotes the dual representation of \( (V, \sigma) \).

### 2.1 Duality operator \( I_{\sigma} \). For \( Z \in \Sigma \) and \( v^* \in V^* \), let \( \varphi_{Z,v^*} \) be the \( V^* \)-valued locally analytic function on \( \mathcal{P} \):

\[
\varphi_{Z,v^*}(X,Y) := \sigma^*(XZ + Y)v^*.
\]

Let \( B^0_{\sigma^*}(\mathcal{P}, V^*) \) be the subspace of \( C^0_{\sigma^*}(\mathcal{P}, V^*) \) spanned by \( \varphi_{Z,v^*} \), \( B_{\sigma^*}(\mathcal{P}, V^*) \) the closure of \( B^0_{\sigma^*}(\mathcal{P}, V^*) \). Clearly \( B_{\sigma^*}(\mathcal{P}, V^*) \) is \( \mathbb{G} \)-invariant.

For any continuous linear functional \( \xi \in B_{\sigma^*}(\mathcal{P}, V^*)^* \), we define a \( V \)-valued function on \( \Sigma \):

\[
I_{\sigma}(\xi)(Z) := \sum_{k=1}^d \langle \varphi_{Z,v_k}, \xi \rangle v_k, \quad Z \in \Sigma.
\]

We check that \( I_{\sigma}(\xi) \) is independent of the choice of the basis \( \{v_k\}_{k=1}^d \). Evidently, \( I_{\sigma} \) is injective.
Lemma 2.1. \( I_\sigma \) is \( G \)-equivariant, that is,
\[
I_\sigma(T^*_\sigma(g)\xi) = \pi_\sigma(g)I_\sigma(\xi),
\]
for any \( g \in G \).

Proof. Let \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G \). We have
\[
I_\sigma(T^*_\sigma(g)\xi)(Z) = \sum_{k=1}^d \langle \varphi_{Z,v_k^*}, T^*_\sigma(g)\xi \rangle v_k
\]
\[
= \sum_{k=1}^d \langle T_\sigma^*(g^{-1})\varphi_{Z,v_k^*}, \xi \rangle v_k
\]
\[
= \sum_{k=1}^d \langle (\sigma^*(X^*g^{-1}Z + Y)v_{k,g}, \xi) v_k \rangle v_k
\]
\[
= \sigma(j(g^{-1}, Z))^{-1}(\sum_{k=1}^d \langle (\sigma^*(X^*g^{-1}Z + Y)v_{k,g}), \xi \rangle v_k)
\]
\[
= (\pi_\sigma(g)I_\sigma(\xi))(Z),
\]
where \( v_{k,g} = \sigma(j(g^{-1}, Z))v_k \).

Q.E.D.

Proposition 2.2.

(1) For any continuous linear functional \( \xi \in B_{\sigma^*}(\mathcal{P}, V^*)^* \), \( I_\sigma(\xi) \) is a \( V \)-valued rigid analytic function on \( \Sigma \).

(2) \( I_\sigma \) is a continuous homomorphism of \( G \)-representations from \( (B_{\sigma^*}(\mathcal{P}, V^*)^*_b, T^*_\sigma) \) to \( (\mathcal{O}_{\sigma}(\Sigma), \pi_\sigma) \).

Proof. We denote by \( i \) the inclusion: \( B_{\sigma^*}(\mathcal{P}, V^*) \hookrightarrow C^\text{an}_{\sigma^*}(\mathcal{P}, V^*) \), \( i^* \) its adjoint operator. Because of our assumption that \( K \) is spherically complete, Hahn-Banach Theorem ([9] Corollary 9.4) implies that \( i^* \) is surjective. Because \( C^\text{an}_{\sigma^*}(\mathcal{P}, V^*)^*_b \) and \( B_{\sigma^*}(\mathcal{P}, V^*)^*_b \) are Fréchet spaces (Corollary 1.2), \( i^* \) is open (from the open mapping theorem ([9] Proposition 8.6)). Consequently, the continuity of \( I_\sigma \circ i^* \) implies that of \( I_\sigma \). Therefore, (1) and (2) are equivalent to:

(1') \( I_\sigma \circ i^*(\xi) \in \mathcal{O}_{\sigma}(\Sigma) \) for any \( \xi \in C^\text{an}_{\sigma^*}(\mathcal{P}, V^*)^* \);

(2') \( I_\sigma \circ i^* : (C^\text{an}_{\sigma^*}(\mathcal{P}, V^*)^*_b, T^*_\sigma) \to (\mathcal{O}_{\sigma}(\Sigma), \pi_\sigma) \) is a continuous homomorphism of \( G \)-representations.
We still denote \( I_\sigma \circ \iota \) by \( I_\sigma \). For \( \xi \in C^\text{an}_{\sigma^*}(\mathcal{P}, V^*)_b \), we write \( I_\sigma(\xi) \) in integral:

\[
I_\sigma(\xi)(Z) = \sum_{k=1}^{d} \int_{U} \varphi_{Z,k^d} d\xi \cdot v_k
\]

\[
= \sum_{k=1}^{d} \sum_{k} \int_{U_k} \varphi_{Z,k} d\xi \cdot v_k
\]

\[
= \sum_{k} \pi_\sigma(g_k) \left( \sum_{k=1}^{d} \int_{U_k} \varphi_{Z,(v_k g_k)^*} d(T_{\sigma^*}(g_k^{-1})\xi) \cdot v_{k g_k} \right),
\]

where the disjoint open covering \( \{U_k\}_k \) of \( \mathcal{P} \) and \( g_k \) are defined in Sec. 1.2, and \( v_{k g_k} \) is defined in the proof of Lemma 2.1. Therefore it suffices to consider

\[
(2.3) \quad \sum_{k=1}^{d} \int_{U} \varphi_{Z,k^d} d\xi' \cdot v_k.
\]

where \( U \) are taken to be \( U_k \cdot g_k \) and \( \xi' \) is the image of \( \xi \) under \( C^\text{an}_{\sigma^*}(\mathcal{P}, V^*)_b \to C^\text{an}(U, V^*)_b \).

For the open subset \( \overline{U} = \text{pr}_U(U) \) of \( \text{Sym}(n, 0) \), we have the isomorphism induced from the section \( t_0 \) (compare (1.5)):

\[
C^\text{an}_{\sigma^*}(\mathcal{U}, V^*)_b \cong C^\text{an}(\overline{U}, V^*)_b.
\]

Then (2.3) equals to

\[
\overline{T}_{\sigma^*}(\overline{U})(\overline{\xi})(Z) := \sum_{k=1}^{d} \int_{U} (\sigma^*(Z - z) v_k^*) d\overline{\xi}(z) \cdot v_k,
\]

where \( \overline{\xi} \) is the image of \( \xi' \) in \( C^\text{an}(\overline{U}, V^*)_b \) via the isomorphism (2.4). It suffices to prove that \( \overline{T}_{\sigma^*}(\overline{U})(\overline{\xi}) \) is rigid analytic on \( \Sigma(m) \), and that the map

\[
C^\text{an}(\overline{U}, V^*)_b \to \Theta_{\sigma}(\Sigma(m))
\]

\[
\overline{\xi} \mapsto \overline{T}_{\sigma^*}(\overline{U})(\overline{\xi})|_{\Sigma(m)}
\]

is continuous (G-equivariance is proved in Lemma 2.1).

As \( \sigma^* \) is algebraic, there is a nonnegative integer \( t \) and polynomials \( Q_k \) \( (1 \leq k, \ell \leq d) \) in \( h_{ij} \) \( (1 \leq i, j \leq n) \) with coefficients in \( K \), such that

\[
\sigma^*(h)v_k^* = \sum_{\ell=1}^{d} \det(h)^{-t} Q_k(h) v_{\ell}^*.
\]

We expand

\[
\det(Z - z)^{-t} Q_k(Z - z) = \sum_{\ell} \alpha_{\ell k}(Z) \cdot z_{\ell}^k
\]

Evidently from Corollary 1.15, \( \alpha_{\ell k} \in \Theta(\Sigma(m)) \) and

\[
(2.5) \quad \lim_{|U| \to \infty} ||\alpha_{\ell k}||_{\Theta(\Sigma(m))} = 0.
\]
Moreover, there is a constant $c_m > 0$, depending only on $m$, $\sigma$ and $\{v_k\}_{k=1}^d$, such that

$$\|\alpha_{f,k}\|_{\mathcal{O}(\Sigma(m))} \leq c_m. \tag{2.6}$$

Then

$$\begin{aligned}
I_{\sigma;U}(\vec{\xi})(Z) &= \sum_{k=1}^d \int_U \det(Z - z)^{-1} Q_{k\ell}(Z - z) d\vec{\xi}(z) \cdot v_k \\
&= \sum_{k=1}^d \left( \sum_{\ell=1}^d \left( \int_U \vec{\xi} \cdot v_\ell^* d\vec{\xi}(z) \right) \cdot \alpha_{f,k}(Z) \right) v_k. \tag{2.7}
\end{aligned}$$

Since $\|\vec{\xi}\|_{C^m(\Omega)} \leq 1$,

$$\begin{aligned}
\left| \int_U \vec{\xi} \cdot v_\ell^* d\vec{\xi}(z) \right| &\leq \|v_\ell^*\|_V \cdot \|\vec{\xi}\|_{C^m(\Omega;V^*)}. \tag{2.8}
\end{aligned}$$

(2.5) and (2.8) imply that the expansion (2.7) of $I_{\sigma;U}(\vec{\xi})$ converges in $\mathcal{O}_\sigma(\Sigma(m))$. (2.6) and (2.8) imply

$$\|I_{\sigma;U}(\vec{\xi})\|_{\mathcal{O}(\Sigma(m))} \leq \max_{1 \leq k, \ell \leq d} c_m \|v_\ell^*\|_V \|v_k\|_V \|\vec{\xi}\|_{C^m(\Omega;V^*)}. \tag{2.9}$$

The continuity follows. Q.E.D.

2.2. **Duality operator** $J_{\sigma}$ and the image of $I_{\sigma}$. Let $\mathcal{N}_\sigma(\Sigma)$ denote the image of $I_{\sigma}$. In this section, we propose to determine $\mathcal{N}_\sigma(\Sigma)$. For this, we need to introduce $J_{\sigma}$, the adjoint operator of $I_{\sigma}$: an injective continuous linear operator from $\mathcal{N}_\sigma(\Sigma)_b^*$ to $(B_{\sigma}(\mathcal{P}, V^*)_b^*)^* \cong B_{\sigma}(\mathcal{P}, V^*)$ ($B_{\sigma}(\mathcal{P}, V^*)$ is reflexive according to Corollary 1.2). First, we find the formula for $J_{\sigma}$.

For any $\mu \in \mathcal{N}_\sigma(\Sigma)^*$ and $\xi \in B_{\sigma}(\mathcal{P}, V^*)$, we have

$$\langle J_{\sigma}(\mu), \xi \rangle = \langle I_{\sigma}(\xi), \mu \rangle. \tag{2.10}$$

For $(X, Y) \in \mathcal{P}$ and $v \in V$, we define the Dirac distribution $\xi_{(X,Y),v}$ as follows, which is a continuous linear functional of $B_{\sigma}(\mathcal{P}, V^*)$:

$$\langle \varphi, \xi_{(X,Y),v} \rangle = \langle v, \varphi(X, Y) \rangle_v, \quad \varphi \in B_{\sigma}(\mathcal{P}, V^*),$$

and a $V$-valued rigid analytic function $\psi_{(X,Y),v}$ on $\Sigma$:

$$\psi_{(X,Y),v}(Z) := \sigma(XZ + Y)^{-1} v. \tag{2.11}$$

**Lemma 2.3.**

$$I_{\sigma}(\xi_{(X,Y),v}) = \psi_{(X,Y),v}. \tag{2.12}$$

**Proof.** By definition (2.2),

$$\begin{aligned}
(I_{\sigma}(\xi_{(X,Y),v}))(Z) &= \sum_{k=1}^r \langle \varphi Z v_k^*, \xi_{(X,Y),v} \rangle v_k \\
&= \sum_{k=1}^r \langle v, \sigma^* (XZ + Y) v_k^* \rangle_v \cdot v_k
\end{aligned}$$
valued locally analytic on structure is naturally induced onto Banach algebra is surjective (Hahn-Banach Theorem),

Moreover, the dual space of \( \Sigma \) is an isomorphism from \( B \).

\( \sum k=1^n \langle \psi(X,Y),v \rangle v_k \)

\( \sigma(XZ + Y)^{-1}v = \psi(X,Y,v) \).

Q.E.D.

Let \( \mathcal{N}_0^0(\Sigma) \) denote the subspace of \( \mathcal{O}(\Sigma) \) spanned by \( \psi(X,Y,v) \) for all \( (X, Y) \in \mathcal{P} \) and \( v \in V. \) Clearly \( \mathcal{N}_0^0(\Sigma) \) is G-invariant. Lemma 2.3 implies \( \mathcal{N}_0^0(\Sigma) \subset \mathcal{N}_0^0(\Sigma). \)

**Proposition 2.4.** For any continuous linear functional \( \mu \in \mathcal{N}_0^0(\Sigma)^* \), we have

\[
J_\sigma(\mu)(X, Y) = \sum_{k=1}^d \langle \psi(X,Y,v_k), \mu \rangle v_k^*.
\]

**Proof.** We have

\[
\sum_{k=1}^r \langle \psi(X,Y,v_k), \mu \rangle v_k^* = \sum_{k=1}^r \langle J_\sigma(\xi(X,Y),v_k), \mu \rangle v_k^*
\]

(Lemma 2.3)

\[
= \sum_{k=1}^r \langle J_\sigma(\mu), \xi(X,Y,v_k) \rangle v_k^*
\]

(Duality formula (2.9))

\[
= \sum_{k=1}^r \langle v_k, J_\sigma(\mu)(X,Y) \rangle v_k^* = J_\sigma(\mu)(X,Y).
\]

Q.E.D.

From (2.11), we see that \( J_\sigma \) factors through \( \mathcal{N}_0^0(\Sigma)^* \), and (2.11) defines an injection from \( \mathcal{N}_0^0(\Sigma)^*_b \) to \( B_0(\mathcal{P}, V^*) \). Because \( J_\sigma \) is injective and \( \mathcal{N}_0^0(\Sigma)^*_b \to \mathcal{N}_0^0(\Sigma)^*_b \) is surjective (Hahn-Banach Theorem), \( \mathcal{N}_0^0(\Sigma)^*_b = \mathcal{N}_0^0(\Sigma)^*_b \). Therefore by Hahn-Banach theorem, we have

**Lemma 2.5.** \( \mathcal{N}_0^0(\Sigma) \) is dense in \( \mathcal{N}_0^0(\Sigma) \).

**Theorem 2.6.**

1. \( J_\sigma \) is an isomorphism from \( B_0(\mathcal{P}, V^*) \) to \( \mathcal{N}_0^0(\Sigma) \).
2. \( \mathcal{N}_0^0(\Sigma) \) is the closure of \( \mathcal{N}_0^0(\Sigma) \) in \( \mathcal{O}(\Sigma) \).

**Proof.** Let \( B(L, V^*) := \iota(\sigma) \mathcal{B}_0(\mathcal{P}, V^*) \), and still denote \( \iota|_{\mathcal{B}_0(\mathcal{P}, V^*)} \) by \( \iota \).

Let \( I \) be any (finite) disjoint open chart covering \( \{\overline{U}_i\} \) of \( L \), then we recall that \( \mathcal{C}^0(L, V^*) \) is defined as the inductive limit, indexed with all the \( I \), of the \( K \)-Banach algebra \( F_I(L, V^*) = \prod_i \mathcal{O}(U_i, V^*) \), where \( \mathcal{O}(\overline{U}_i, V^*) \) denotes the space of \( K \)-analytic functions on \( \overline{U}_i \) (cf. [3] 2.1.10 and [11] \S 2). The inductive limit structure is naturally induced onto \( B(L, V^*) \), that is, \( B(L, V^*) = \lim_i E_I(L, V^*) \).

Moreover, the dual space \( B(L, V^*)^* \) is the projective limit of \( E_I(L, V^*) \).

Let \( \mathcal{N}_0^0(\Sigma(m)) \) be the image of \( \mathcal{N}_0^0(\Sigma) \) in \( \mathcal{O}(\Sigma(m)) \).

Considering \( \pi_m(g^{-1})v_k \), we see that the map \( (X, Y) \mapsto \psi(X,Y,v_k) \) is an \( \mathcal{O}(\Sigma(m)) \)-valued locally analytic on \( \mathcal{P} \) (see Proposition 1.13). Since \( \mathcal{K} \) is compact,

\[
r_m = \min_{1 \leq k \leq n} \inf_{d(X,Y) \in \mathcal{K}} \|\psi(X,Y,v_k)\|_{\mathcal{O}(\Sigma(m))}
\]
is positive. Let \( \mathcal{L} \) be the lattice \( \sum_{k=1}^{d} \sum_{(X,Y) \in \mathcal{K}} \mathfrak{a}_k \cdot \psi_{(X,Y),v} \) in \( \mathcal{M}_{\sigma_\sigma^0}(\Sigma) \), then, for each \( m \), the image of \( \mathcal{L} \) in \( \mathcal{M}_{\sigma_\sigma^0}(\Sigma(m)) \) contains the ball of radius \( r_m \) centered at zero, and therefore the interior of \( \mathcal{L} \) is a nontrivial open lattice.

Consider

\[
(\iota^{\sigma^{-1}})^* \circ I_{\sigma,1}^{-1}|_{\mathcal{M}_{\sigma_\sigma^0}(\Sigma)} : \mathcal{M}_{\sigma_\sigma^0}(\Sigma) \rightarrow B(\mathcal{L}, V^*)^*_b
\]

\[
\psi_{(X,Y),v} \mapsto (\iota^{\sigma^{-1}})^*(\xi_{(X,Y),v})
\]

For \( (X,Y) \in \mathcal{K} \),

\[
\| (\iota^{\sigma^{-1}})^*(\xi_{(X,Y),v}) \|_{B(\mathcal{L}, V^*)^*_b} = \max_{\varphi \in E_{\mathcal{L}}(V^*)} \frac{\langle \varphi, (\iota^{\sigma^{-1}})^*(\xi_{(X,Y),v}) \rangle}{\| \varphi \|_{E_{\mathcal{L}}(V^*)}}
\]

\[
= \max_{\varphi \in \iota^{\sigma^{-1}}(E_{\mathcal{L}}(V^*))} \frac{\langle \varphi, \xi_{(X,Y),v} \rangle}{\| \varphi \|_{E_{\mathcal{L}}(V^*)}}
\]

\[
\leq \max_{(X,Y) \in \mathcal{K}} \| \varphi(X', Y') \|_{V^*}
\]

Therefore the image of \( \mathcal{L} \) under \( (\iota^{\sigma^{-1}})^* \circ I_{\sigma,1}^{-1}|_{\mathcal{M}_{\sigma_\sigma^0}(\Sigma)} \) in \( B(\mathcal{L}, V^*)^*_b \) is bounded, since its image in \( E_{\mathcal{L}}(V^*)^* \) are all norm-bounded by \( \max_{1 \leq l \leq d} \| v_l \|_V \). Because \( \mathcal{M}_{\sigma_\sigma^0}(\Sigma) \) is metrizable, it is bornological ([9] Proposition 6.14), and therefore \( I_{\sigma,1}^{-1}|_{\mathcal{M}_{\sigma_\sigma^0}(\Sigma)} \) is continuous ([9] Proposition 6.13). \( \mathcal{M}_{\sigma_\sigma^0}(\Sigma) \) is isomorphic to \( I_{\sigma,1}^{-1}(\mathcal{M}_{\sigma_\sigma^0}(\Sigma)) \), then their completions are isomorphic, which, in view of Lemma 2.5, must be \( \mathcal{M}_{\sigma}(\Sigma) \) and \( B_{\sigma^*}(P, V^*)^*_b \) respectively.

**Corollary 2.7.** \( J_{\sigma} \) is an isomorphism of G-representations from \( (\mathcal{M}_{\sigma}(\Sigma), \pi_{\sigma}) \) to \( (B_{\sigma^*}(P, V^*), T_{\sigma^*}) \).

**Remark 2.8.** We conjecture that \( (\mathcal{M}_{\sigma}(\Sigma), \pi_{\sigma}) \) and \( (B_{\sigma^*}(P, V^*), T_{\sigma^*}) \) are topologically irreducible G-representations if \( \sigma \) is irreducible. These are conjectured and claimed by Morita for \( \text{SL}(2, F) \) ([6] Corollary after Theorem 3 and [7] Theorem 1 (i).) However, there is a serious gap in his proof of [7] Proposition 3. Schneider and Teitelbaum gave the first valid proof of [7] Theorem 1 (i) in [11] when \( F = \mathbb{Z}_p \).

3. Morita’s theory for \( \text{SL}(2, F) \)

In this section, we study Morita’s theory for \( \text{Sp}(2, F) = \text{SL}(2, F) \). We study the duality established in \( \S 2 \) for \( \text{SL}(2, F) \) and show its relations with Morita’s duality and Casselman’s intertwining operator.

First, we review the construction of the holomorphic discrete series and principal series for \( \text{SL}(2, F) \) from [5], [6] and [7] in accordance with our notations.

3.1. The \( p \)-adic upper half-plane. For more details, we refer the readers to [5] \( \S 2 \) and [2] 1.2.

In the following, let \( G = \text{SL}(2, F) \) and \( G_o = \text{SL}(2, o) \).
Let $\Sigma := K - F$ be the $p$-adic upper half-plane, $\mathcal{P}$ the set of nonzero pairs $(x, y) \in F \times F$, $\mathcal{L} = F^x \setminus \mathcal{P} = \mathbb{P}^1(F)$, $\mathcal{P}_o \subset \mathcal{P}$ the set of pairs $(x, y) \in o \times o$ such that $(x, y) \not\equiv (0,0)$ mod $\mathfrak{p}$. As usual, we define a $G$-action on $\Sigma$ via
\[ g \cdot Z := (aZ + b)(cZ + d)^{-1}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G. \]

Let $m$ be a nonnegative integer. For a pair $(x, y) \in \mathcal{P}_o$, we define
\[ B^-(m; x, y) := \{ Z \in K \mid |xZ + y| < \max\{1, |Z|\} |\sigma^m| \}. \]

Let $\Sigma(m) := \bigcap_{(x, y) \in \mathcal{P}_o} K - B^-(m; x, y)$
\[ = \{ Z \in \Sigma \mid |xZ + y| \geq \max\{1, |Z|\} |\sigma^m| \text{ for any } (x, y) \in \mathcal{P}_o \}. \]

It is not hard to verify that the admissible affinoid covering $\{\Sigma(m)\}_{m=0}^{\infty}$ of $\Sigma$ coincide with that defined in [2] 1.2.

Let $\mathcal{O}(\Sigma(m))$ be the space of $K$-valued rigid analytic functions on $\Sigma(m)$. Explicitly, by taking partial fractional expansion of each summand in (1.7), we see that $\psi \in \mathcal{O}(\Sigma(m))$ is a $K$-valued functions on $\Sigma(m)$ which has an expansion in the form:
\[ \psi(Z) = \sum_{i=0}^{\infty} a^{(\infty)}_i Z^i + \sum_{j=1}^{\ell} \sum_{i=-s}^{\infty} a^{(j)}_i (Z - z_j)^i, \]
where $\ell \geq 0$, $a_i^{(s)} \in K$, $z_j \in F$, and the expansion converges with respect to the supremum norm. The space of $K$-rigid analytic functions on $\Sigma$ is the projective limit of $\mathcal{O}(\Sigma(m))$.

3.2. Holomorphic discrete series of $\text{SL}(2, F)$. Let $s$ be an integer. We define the holomorphic discrete series $(\mathcal{O}(\Sigma), \pi_s)$ of $G$ (see (1.10); compare [5] §3-1.):

\[ \pi_s(g)\psi(Z) := (-cZ + a)^{-s} \psi((dZ - b)(-cZ + a)^{-1}) \]
where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $\psi \in \mathcal{O}(\Sigma)$. $\pi_s$ is a continuous representation of $G$.

Let $\mathcal{N}_s^0(\Sigma)$ be the subspace of $\mathcal{O}(\Sigma)$ spanned by 1 and $\psi_s^{(s)}(Z) := (Z - z)^{-s}$ (see (2.10)) for all $z \in F$, and $\mathcal{N}_s^0(\Sigma)$ the closure of $\mathcal{N}_s^0(\Sigma)$. $\mathcal{N}_s^0(\Sigma)$ is $G$-invariant.

If $s \leq 0$, obviously $\mathcal{N}_s^0(\Sigma)$ is the space of polynomial functions $\psi(Z)$ of degree $\leq -s$.

If $s$ is a positive integer, let $\mathcal{N}_s^0(\Sigma)$ be the subspace of $\mathcal{O}(\Sigma)$ consisting of all rational functions $\psi$ which has a partial fractional expansion of the form
\[ \psi(Z) = \sum_{i=0}^{\infty} a^{(\infty)}_i Z^i + \sum_{j=1}^{\ell} \sum_{i=-s}^{\infty} a^{(j)}_i (Z - z_j)^i, \]
where the sum is finite with $\ell \geq 0$, $z_j \in F$ and $a_i^{(s)} \in K$. Clearly $\mathcal{N}_s^0(\Sigma)$ is $G$-invariant. Let $\mathcal{N}_s(\Sigma)$ be the closure of $\mathcal{N}_s^0(\Sigma)$ in $\mathcal{O}(\Sigma)$.

The next lemma follows immediately from [5] Theorem 2 (i).
Lemma 3.1. Let \( s \) be a positive integer. The smallest \( G \)-invariant closed subspace of \( \mathcal{O}(\Sigma) \) containing 1 is \( \mathcal{N}_s(\Sigma) \).

We note that \( 1 \in \mathcal{N}_s(\Sigma) \) and \( \mathcal{N}_s^0(\Sigma) \subset \mathcal{N}_s^0(\Sigma) \), and therefore we have

Proposition 3.2. Let \( s \) be a positive integer. \( \mathcal{N}_s(\Sigma) = \mathcal{N}_s^0(\Sigma) \).

3.3. Principal series of \( SL(2, F) \). The references for this section are [6] §2, 3 and [7] §2.

Let \( s \) be an integer and \( \chi_s \) the character of \( F^\times \), \( \chi_s(z) = z^s \). Let \( \mathcal{C}^\text{an}(\mathcal{P}) \) be the space of \( K \)-valued locally analytic functions \( \varphi \) on \( \mathcal{P} \) satisfying

\[
\varphi(hx, hy) = \chi_s(h)\varphi(x, y), \quad (x, y) \in \mathcal{P}, h \in F^\times.
\]

In the following, we identify \((1, F)\) with \((1, -z) \to z\) and write \( \varphi(z) = \varphi(1, -z) \) and \( \varphi(\infty) = \varphi(0, 1) \). Then \( \varphi(z) \) is a locally analytic function on \( F \) which has Laurent expansion at infinity of the form:

\[
\varphi(z) = \sum_{i=s}^{\infty} b_i^{(\infty)} z^i, \quad b_i^{(\infty)} \in K.
\]

Clearly, \( \varphi(\infty) = (-1)^i b_i^{(\infty)} \).

Let \( D_s \) denote the space of all such functions \( \varphi(z) \) on \( F \). We have a \( K \)-linear bijective \( D_s \equiv \mathcal{C}^\text{an}_{\chi_s}(\mathcal{P}) \); we endow \( D_s \) the topology which makes this map into an isomorphism. Then the representation of \( G \), \((\mathcal{C}^\text{an}_{\chi_s}(\mathcal{P}), T_{\chi_s})\), defined by (1.4) is realized as the representation \((D_s, T_s)\):

\[
(3.2) \quad T_s(g)\varphi(z) := (-cz + a)^s \varphi((dz - b)(-cz + a)^{-1}), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \varphi \in D_s.
\]

Let \( B_s^0 \) be the subspace of \( D_s \) spanned by the \( K \)-valued locally analytic functions \( \varphi^{(s)}_z(z) := (Z - z)^s \) (see (2.1)) for all \( Z \in \Sigma \), and \( B_s \) the closure of \( B_s^0 \). \( B_s \) is \( G \)-invariant.

If \( s \) is a nonnegative integer, let \( P_s^\text{loc} \) be the subspace of \( D_s \) consisting of \( K \)-valued functions \( \varphi(z) \) on \( F \) such that the local Taylor expansion at each point of \( F \) and the Laurent expansion at infinity of \( \varphi(z) \) are both given by polynomials of degree \( \leq s \). \( P_s^\text{loc} \) is \( G \)-invariant. \( \varphi \in P_s^\text{loc} \) if and only if its \((s + 1)\)-th derivation \((d/dz)^{s+1} \varphi(z) \equiv 0 \). Let \( P_s \) be the space of all polynomial functions on \( F \) of degree \( \leq s \). \( P_s^\text{loc} \) and \( P_s \) are both closed \( G \)-invariant subspaces. Clearly \( B_s \supset P_s \).

The subspace of \( \mathcal{C}^\text{an}_{\chi_s}(\mathcal{P}) \) corresponding to \( P_s \) (resp. \( P_s^\text{loc} \)) is the space of (resp. locally) homogeneous polynomial functions \( \varphi(x, y) \) on \( \mathcal{P} \) of degree \( s \).

In addition, we define \( F_s^\text{loc} = P_{-1} = 0 \).

Proposition 3.3 (Casselman’s intertwining operator). Let \( s \geq -1 \).

\[
S_s : D_s \to D_{-s-2}
\]

\[
(3.3) \quad \begin{array}{c}
\varphi(z) \mapsto (d/dz)^{s+1} \varphi(z)
\end{array}
\]

induces a \( G \)-isomorphism \( D_s / P_s^\text{loc} \) onto \( D_{-s-2} \).
3.4. Morita’s duality for $\text{SL}(2, F)$.

**Definition 3.4** (cf. [6] §5.). Let $s$ be an integer.

1. We call the following $K$-linear pairing $\langle , \rangle_{M}^{(s)} : D_{s-2} \times \mathcal{O}(\Sigma) \to K$ Morita’s pairing:

$$\langle \varphi, \psi \rangle_{M}^{(s)} := \text{the sum of residues of the 1-form } \varphi(z)\psi(z) \text{ d}z \text{ on } L,$$

where $\varphi \in D_{s-2}$ and $\psi \in \mathcal{O}(\Sigma)$.

2. For $\psi \in \mathcal{O}(\Sigma)$, let $\mathcal{M}_{s}(\psi)$ be the linear functional of $D_{s-2}$ defined by

$$\langle \varphi, \mathcal{M}_{s}(\psi) \rangle = \langle \varphi, \psi \rangle_{M}^{(s)}, \quad \varphi \in D_{s-2}.$$

$\mathcal{M}_{s} : \mathcal{O}(\Sigma) \to (D_{s-2})^{*}$ is called Morita’s duality operator.

By the explicit computations of $\langle , \rangle_{M}^{(s)}$ (ibid.), we obtain:

**Proposition 3.5** (Compare ibid. Theorem 3). Let $s$ be an integer.

1. If $s$ is a positive integer, then $\mathcal{M}_{s}$ induces isomorphisms of $G$-representations

$$\mathcal{O}(\Sigma)/\mathcal{N}_{s}(\Sigma, \pi_{s}) \mathcal{M}_{s} \cong (D_{s-2}/P_{s-2})_{b}^{*}, T_{s-2}^{*}$$

and

$$\mathcal{N}_{s}(\Sigma, \pi_{s}) \mathcal{M}_{s} \cong (D_{s-2}/P_{s-2}^{\text{loc}})_{b}^{*}, T_{s-2}^{*}.$$

2. If $s \leq 0$, then $\mathcal{M}_{s}$ induces isomorphisms of $G$-representations

$$\mathcal{O}(\Sigma)/\mathcal{N}_{s}(\Sigma, \pi_{s}) \mathcal{M}_{s} \cong ((D_{s-2})^{*}_{b}, T_{s-2}^{*}).$$

We still denote these isomorphisms by $\mathcal{M}_{s}$.

3.5. Duality operator $I_{s}$. We define a continuous linear operator $I_{s}$ from $(B_{s})_{b}^{*}$ to $\mathcal{N}_{s}(\Sigma)$ (see §2.1)

$$I_{s}(\xi)(Z) := \langle \varphi_{Z}^{(s)}, \xi \rangle.$$

**Theorem 3.6.** If $s$ is a positive integer. We have a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{N}_{s}(\Sigma, \pi_{s}) & \xrightarrow{(s-1)! \mathcal{M}_{s}} & (D_{s-2}/P_{s-2}^{\text{loc}})_{b}^{*}, T_{s-2}^{*} \\
\uparrow I_{s} & \& \downarrow S_{s-2}^{*} \\
(D_{s})_{b}^{*}, T_{s}^{*} & \cong & ((D_{s})_{b}^{*}, T_{s}^{*})
\end{array}
$$

Proof. $i^{*} : (D_{s})_{b}^{*} \to (B_{s})_{b}^{*}$ is surjective, where $i$ is the inclusion: $B_{s} \hookrightarrow D_{s}$ (Hahn-Banach Theorem). According to Theorem 2.6, Proposition 3.5 (1) and Proposition 3.3, $I_{s}$, $(s-1)! \mathcal{M}_{s}$ and $S_{s-2}^{*}$ in the diagram are all isomorphisms of $G$-representations. Therefore it suffices to prove the commutativity of the following diagram:

$$
\begin{array}{ccc}
\mathcal{N}_{s}(\Sigma, \pi_{s}) & \xrightarrow{(s-1)! \mathcal{M}_{s}} & (D_{s-2}/P_{s-2}^{\text{loc}})_{b}^{*}, T_{s-2}^{*} \\
\downarrow I_{s} & \& \downarrow i^{*}(S_{s-2}^{*}) \\
(D_{s})_{b}^{*}, T_{s}^{*} & \cong & ((B_{s})_{b}^{*}, T_{s}^{*})
\end{array}
$$
We define \( \xi_\infty \in (B_{-s})^* \) by \( \langle \varphi_{Z}^{(-s)}, \xi_\infty \rangle = \varphi_{Z}^{(-s)}(\infty) = 1 \), then \( I_s(\xi_\infty)(Z) = 1 \) by definition (3.5).

Since \( \pi_s(g)1 \), for all \( g \in G \), topologically spans \( \mathcal{N}_s \), we require the equality

\[
(s - 1)! (S_{s-2}^{-1})^* \circ M_s(1) = \xi_\infty.
\]

For any \( Z \in \Sigma \), we have \( S_{s-2} \left( \varphi_{Z}^{(-1)} \right) = (s - 1)! \varphi_{Z}^{(-s)} \), hence

\[
\langle \varphi_{Z}^{(-s)}, (s - 1)! (S_{s-2}^{-1})^* \circ M_s(1) \rangle = (s - 1)! S_{s-2}^{-1}(\varphi_{Z}^{(-s)}), M_s(1))
\]

\[
= \langle \varphi_{Z}^{(-1)}, (1)^{(s)}_M \rangle
\]

\[
= \text{Res}_\infty (Z - z)^{-1} dz
\]

\[
= 1
\]

\[
= \langle \varphi_{Z}^{(-s)}, \xi_\infty \rangle.
\]

Since \( \varphi_{Z}^{(-s)} \), for all \( Z \in \Sigma \), topologically spans \( B_{-s} \), (3.6) follows. Q.E.D.

If \( s \leq 0 \), then \( I_s : (B_{-s})^*_b \to \mathcal{N}_s(\Sigma) \) is an isomorphism between two \((-s + 1)\)-dimensional \( G \)-representations.

4. Concluding remarks

Professor P. Schneider pointed out that the \( p \)-adic Siegel upper half-space \( \Sigma \) was constructed in M. van der Put and H. Voskuil’s paper [13] as the symmetric space associated to the symplectic group \( G = \text{Sp}(2n, F) \). In fact, if we let \( P^- \) denote the transpose of \( P \), and \( G, U \) and \( P^- \) the \( F \)-rigid analytifications of \( G, U \) and \( P^- \) respectively, then \( \Sigma \) can be realized as the complement of all the \( G \)-translations of \((G - U \cdot P^-)/P^- \) in \( G/P^- \). However, the construction of the affinoid covering using the Bruhat-Tits building in [13] is different from ours.

We claim that this observation enables us to generalize most of the constructions and results in this article to the split reductive groups.

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