Exact analytical expression for magnetoresistance using quantum groups.

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We obtain an exact analytical expression for magnetoresistance using noncommutative geometry and quantum groups. Then we will show that there is a deep relationship between magnetoresistance and the quantum group $su_q(2)$, from which we understand the quantum interpretation of the quantum corrections to the conductivity.

Introduction. Quantum groups [1] and deformed algebras have proved to be so rich and powerful that it seems natural to apply them to different problems in physics and mathematical physics. Many physicists have studied the notion of quantum groups and deformed algebras from different points of view and applied them to a variety of physical theories from nuclear and high energy physics to condensed matter physics [See e.g. 2-15].

Quantum transport phenomena and magnetotransport in a two-dimensional electron gas (2DEG) has been attracting much attention for scores of years. This attention has been motivated by the progress in preparing high-quality semiconductor heterostructures which has opened up new areas in both fundamental and applied physics. The quantum corrections to the Drude conductivity in disordered metals and doped semiconductors has been most intensively studied for the last 20 years, see [16] for a review. The negative magnetoresistance induced by the suppression of the quantum interference correction by magnetic field is a famous manifestation of weak localization.
It gives simple analytical expression for quantum correction to conductivity which allowed to determine the phase breaking time experimentally. A standard fitting procedure has been used to analyze the experimental data and determine the phase breaking time [17,18]. The phase breaking time is the fitting parameter. Another approach to study the negative magnetoresistance due to weak localization has been presented in Ref [19], which is based on a quasi-classical treatment of the problem [20,21,22], and an analysis of the statistics of closed paths of a particle moving over 2D plane with randomly distributed scatterers. This method has been used to study the weak localization in InGaAs/GaAs heterostructures with single [23] and double [24] quantum wells. As mentioned above there is an analytical expression which gives the dependence of the conductivity correction on the magnetic field [17].

$$\Delta \sigma(B) = \delta \sigma(B) - \delta \sigma(0) = aG_0[\Psi(0.5 + \frac{B_{tr}}{B}\gamma) - \ln(\frac{B_{tr}}{B}\gamma)],$$  \hspace{1cm} (1)

where $G_0 = \frac{e^2}{2\pi^2\hbar}, B_{tr} = \frac{h}{2e\gamma}, \gamma = \frac{\tau}{\tau_\phi} = \frac{\ell}{\ell_\phi}, \ell$ is the mean free path, $\ell_\phi$ is the phase breaking length, $\Psi(x)$ is a digamma function, $\tau$ and $\tau_\phi$ are elastic and phase breaking time respectively. The value of prefactor “$a$” is theoretically equal to unity. Using this expression physicists have been attempting to analyse the experimental data and to determine phase breaking time or phase breaking length and their temperature dependence through the fitting of the experimental curves. It is shown in Ref [23] that Eq.(1) describes the magnetic field dependence of the magnetoresistance relatively well but with prefactor $a < 1$.

In order to give an expression for conductivity correction in a magnetic field Minkov et al. [19], introduced the distribution of closed random paths of area ”$s$” $w_N(s)$ such that $w_N(s)ds$ gives the probability density of return after $N$ collisions following a trajectory which enclosed the area in the range $(s, s + ds)$, this gives:

$$\delta \sigma(b) = -2\pi G_0 \{ \ell^2 \sum_{N=3}^{\infty} \int_{-\infty}^{+\infty} ds e^{\frac{-B}{(1+\gamma)B_{tr}}} w_N(s) \cos \frac{(1+\gamma)^2bs}{\ell^2} \},$$ \hspace{1cm} (2)

where $b = \frac{B}{(1+\gamma)B_{tr}}$. They took into account inelastic processes destroying the phase coherence by including the factor $e^{\frac{-B}{(1+\gamma)B_{tr}}}$, where $L$ is the path length. Note that the expression in the bracket is dimensionless, and it doesn’t make any difference what units of measurement we choose for length and area. Therefore we can treat $\ell, \ell_\phi, L$ and $s$ as dimensionless variables. We introduce new dimensionless variables $\alpha = \ell^2$ and $\beta = \ell_\phi$ which we will use them later.

In this paper which is in close relation with [12], we provide an exact expression for $w_N(s)$ using noncommutative geometry and quantum groups. Then by use of it we present an exact analytical expression for magnetoresistance. We will also discuss the relationship between magnetoresistance and the $su_q(2)$ algebra.
2. Exact area distributions of closed random paths.

Bellissard et al. [25] obtained the exact area distributions of closed random walks using noncommutative geometry. Their method was based upon Harper model [26]. Let us consider a spinless electron on a two dimensional lattice and submitted to a uniform magnetic field along the z-direction and perpendicular to the plane of motion. The system is not invariant under translations but there is an invariance under the so-called magnetic translation operators \( w(a) \). The Harper model Hamiltonian is [26]:

\[
H = \sum_{|a|=1} w(\vec{a}) = w(\vec{a}) + w(\vec{b}) + w(-\vec{a}) + w(-\vec{b}),
\]

(3)

Here \( \vec{a} \) and \( \vec{b} \) are two perpendicular unit vectors which build the lattice unit cell, \( |a| = |b| = 1 \). After some calculations they showed that the probability distribution of closed random paths is given by [25]:

\[
P(A, N) = \left( \frac{2 \pi N}{4N+1} \right)^{1/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iax} T(H_N(x)) dx,
\]

(4)

where \( a = \frac{A}{N} \) is the renormalized area, \( A \) is the algebraic area and \( N \) is the number of collisions. \( \eta = \frac{a}{N} \) is a dimensionless variable and \( \eta = 2\pi \frac{\Phi}{\Phi_0} \). \( \Phi \) and \( \Phi_0 \) are the magnetic flux through the unit cell and the quantum of flux respectively. \( H \) is the Harper’s model Hamiltonian and its trace is given by [25]:

\[
T(H_N(x)) = \frac{4^{N+1}}{2\pi N \sinh(\frac{\pi}{4})} \left[ 1 - \frac{1}{2N} \frac{(\frac{\pi}{4})^2}{\sinh^2(\frac{\pi}{4})} + O\left( \frac{1}{N^2} \right) \right],
\]

(5)

In the limit of large \( N \), when the diffusion approximation is valid, we can neglect the second term, and equ.(4), (5) and the normalization condition (which ensures that the total probability is equal to unity) give:

\[
P_N(A) = \frac{1}{N \cosh^2(\frac{\pi A}{2N})}.
\]

(6)

To make connection with equ.(2), note that:

\[
w_N(A) = \frac{1}{\pi N} P_N(A).
\]

(7)

Compare with the equ.(13) and (15) in [19].

3. Exact analytical expression for magnetoresistance.

Now we can write equ.(2) as follows:

\[
\delta \sigma(b) = -2\pi G_0 \left\{ \alpha \sum_{N=3}^{\infty} \int_{-\infty}^{\infty} dA e^{-\frac{2\pi}{N} w_N(A) \cos\left( \frac{(1 + \gamma)^2 b A}{\alpha} \right)} \right\}.
\]

(8)
Using eqns.(4),(5),(7) and (8) we will obtain the following expression for magnetoresistance:

\[
\Delta \sigma(B) = \delta \sigma(B) - \delta \sigma(0) = -\sqrt{\frac{2}{\pi}} \alpha G_0 \sum_{N=3}^{\infty} e^{-\frac{N}{2\pi}} \left\{ [1 - \frac{1}{2N} \frac{\left( \frac{B N}{4 \alpha B_{c1}} \right)^2}{\sinh\left( \frac{BN}{4 \alpha B_{c1}} \right)} \frac{\left( \frac{B}{4 \alpha B_{c1}} \right)}{\sinh\left( \frac{BN}{4 \alpha B_{c1}} \right)} - \frac{1}{N} \left( 1 - \frac{1}{2N} \right) \right\}. \quad (9)
\]

Our appropriate candidate for exact distribution of closed random walks using quantum groups presented in [12] leads to the following exact expression for magnetoresistance:

\[
\Delta \sigma(B) = \delta \sigma(B) - \delta \sigma(0) = -\sqrt{\frac{2}{\pi}} \alpha G_0 \sum_{N=3}^{\infty} e^{-\frac{N}{2\pi}} \left\{ \cos^N \left( \frac{BN}{4 \alpha B_{c1}} \right) \frac{\left( \frac{B}{4 \alpha B_{c1}} \right)}{\sinh\left( \frac{BN}{4 \alpha B_{c1}} \right)} - \frac{1}{N} \cos^N \left( \frac{1}{N} \right) \right\}. \quad (10)
\]

which strongly supports the equ.(9). For comparison we have plotted \( \Delta \sigma(B)/G_0 \) as given by equ.(1), (9) and equ.(10) for a given values of \( \ell \) and \( \ell_\phi \) in fig.1. We have taken into account \( a = 1 \). In the summation of equ.(9) and equ.(10) we have allowed \( N \) to range from 3 to 900. As it is seen the difference between equ.(9) and equ.(10) is negligibly small and equ.(9) is also exact. From physical point of view they are the same but from mathematical point of view there is a small difference between them, note that:

\[
\cos^N\left( \frac{x}{N} \right) = 1 - \frac{1}{2N} x^2 + O(x^4). \quad (11)
\]

It is worth to mention that there are two other theoretical results [18,22] for quantum corrections to the conductivity. In the diffusion approximation i.e. when the number of collisions is much greater than unity, we have [18]:

\[
\Delta \sigma(b) = \delta \sigma(b) - \delta \sigma(0) = aG_0 [\Psi(0.5 + \frac{x}{2}) - \Psi(0.5 + \frac{1}{2}) - \ln(\gamma)]
\]

For \( x >> 1, \Psi(0.5+x) \simeq \ln(x) \), and we get equ.(1). The calculations of \( \Delta \sigma(B) \) beyond the diffusion approximation show that \( \Delta \sigma(B) \) deviates from this equation if the number of collisions for actual trajectories is not very large [18]. The role of nonbackscattering contribution to magnetoresistance has been studied in Ref.[22]. They showed that the enhancement of backscattering responsible for the weak localization is accompanied by a reduction of the scattering in other directions. The reduction of the scattering at the arbitrary angles leads to the decrease of the quantum correction to the conductivity. Within the diffusion approximation this decrease is small, but it should be taken into account in the case of a relatively strong magnetic field. They have performed numerical calculations which shows that the inclusion of nonbackscattering contribution
leads to a decrease in magnetoconductance.

4. Comparison with experimental data.
We compared equ.(10)(or equ.(9)) with the experimental data of structure 1 samples in fig.2. The heterostructures with $200\,\text{A}^{-}\text{In}_{0.07}\text{Ga}_{0.93}$, as quantum well, $\delta$-doped by Si in the centre. It should be mentioned that we have considered $\beta$ as fitting parameter and we have $\ell_{\varphi} = \frac{\beta}{\sqrt{\alpha}}$. As it is seen, exact expression is in good agreement with experiment and on the other hand in our approach there is no need to introduce the imposed prefactor "$a$", and the number of fitting parameters is reduced. The temperature dependence of $\ell_{\varphi}$ is plotted in fig.3. As is observed the exact analytical expression suggests $\ell_{\varphi} \propto T^{p}$ with $p = -1$.
As mentioned in Ref[27], at low temperatures the phase-breaking time is determined by inelasticity of the electron-electron interaction and is:

$$\tau_{\varphi} = \frac{\hbar}{kT} \frac{\sigma_{0}}{2\pi G_{0}} \frac{1}{\ln\left(\frac{\sigma_{0}}{2\pi G_{0}}\right)},$$

(12)

where $\sigma_{0} = \frac{e^{2}k_{F}\ell}{2\pi\hbar}$, $G_{0} = \frac{e^{2}}{2\pi\hbar}$, and therefore: $\frac{\sigma_{0}}{2\pi G_{0}} = k_{F}\ell$, then we have:

$$\tau_{\varphi} = \left\{ \frac{k_{F}\ell}{\ln\left(\frac{k_{F}\ell}{2}\right)} \right\} \frac{\hbar}{2kT}.$$  

(13)

The expression in the bracket is not important, because it is a number and can be omitted by changing the scale of measurement. The results of the fitting of $\tau_{\varphi}$ obtained from exact and approximate(i.e. equ.1) expressions with $\frac{\hbar}{T}$ (fig.3), are as follows:

$$c_{\text{exact}} = 2.3764352$$

(14)

$$c_{\text{approx}} = 2.2059803$$

(15)

The one obtained from equ.(12) is:

$$c = 2.3687788$$

(16)

As it is seen the exact expression is in very good agreement with the equ.(12).

5. Connection between magnetoresistance and the quantum groups.
Quantum algebras are the q-deformation of the ordinary Lie algebras [1]. Our argument is based on the quantum algebra $su_{q}(2)$. The generators of the $su_{q}(2)$ algebra satisfy the commutation relations:

$$[j_{3}, j_{\pm}] = \pm j_{\pm}.$$  

(17)

$$[j_{+}, j_{-}] = \frac{q^{2j_{3}} - q^{-2j_{3}}}{q - q^{-1}}.$$  

(18)
One can show that the following combinations of the magnetic translations:

\[ j_+ = \frac{w(\vec{a}) + w(\vec{b})}{q - q^{-1}}, \]  
\[ j_- = -\frac{w(-\vec{a}) + w(-\vec{b})}{q - q^{-1}}, \]

where \( w(\vec{b} - \vec{a}) = q^j \) satisfy the \( su_q(2) \) algebra. \( q \) is the parameter of deformation, \( q = \exp(i \frac{e}{2\hbar} \vec{B}.(\vec{a} \times \vec{b})) = e^{2\pi i \frac{q}{q_0}} \). The magnetic translation operators \( w(\vec{a}) \) satisfy the following relation:

\[ w(\vec{a})w(\vec{b}) = \exp(i \frac{e}{2\hbar} \vec{B}.(\vec{a} \times \vec{b}))w(\vec{a} + \vec{b}). \]  

On the other hand from equs.(3), (19) and (20) we have:

\[ H = (q - q^{-1})(j_+ - j_-) = 2i(q - q^{-1})j_y. \]

Hence Harper’s Hamiltonian is a generator of the \( su_q(2) \) algebra. From equs.(4),(7) and (8) we have:

\[ \delta\sigma(B) = -\sqrt{\frac{2}{\pi}} G_0 \alpha \sum_{N=1}^{\infty} \frac{1}{N} e^{-\frac{N}{N_0}} T(\mathcal{H}^N(x)) \big|_{x=\lambda N}, \]

where \( N_0 = 2\beta \). We have changed the lower limit in the summation from 3 to 1, because the paths with \( N = 1, 2 \) have zero areas and therefore have zero contributions to the conductivity corrections. We can omit \( \frac{1}{N} \) because it will be eliminated if we calculate \( T(\mathcal{H}^N(x)) \) at \( x = \lambda N \), where \( \lambda = \frac{B}{\alpha B_r} \). Therefore we have:

\[ \delta\sigma(B) = -\sqrt{\frac{2}{\pi}} G_0 \alpha T\{\sum_{N=1}^{\infty} [(q - q^{-1})e^{-\frac{N}{N_0}}(j_+ + j_-)]^N\}, \]

or

\[ \delta\sigma(B) = -\sqrt{\frac{2}{\pi}} G_0 \alpha T\{\frac{1}{1 - (q - q^{-1})e^{-\frac{1}{N_0}}(j_+ - j_-)}\}. \]

Expression in the bracket is a Green function, and this is a key point to understand the quantum interpretation of the quantum correction to the conductivity. This can be written as:

\[ \delta\sigma(B) = -\sqrt{\frac{2}{\pi}} G_0 \rho \alpha T\{\frac{1}{(j_+ - j_0^2) - (j_+ - j_0^2)}\}, \]
with: \( \rho = e^{\frac{\tau_{\phi}}{\alpha}}(q - q^{-1})^{-1} \). This equation shows the relationship between conductivity corrections in a magnetic field and the \( su_q(2) \) algebra and reflects the quantum symmetry of magnetoresistance.

In conclusion the quantum group is a key symmetry of the magnetoresistance problem and has as its consequence the exact analytical expression for magnetoresistance. It if found here that(see also [12]), the Harper Hamiltonian is a generator of the \( su_q(2) \) algebra and there is a closed connection between random walks problem in two dimensions and the \( su_q(2) \) algebra.

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Figure Captions

Fig1. Magnetic field dependence of \( \Delta \sigma(B)/G_0 \) as given by equ.(1)(dashed lines), equ.(10)(solid line) and equ.(9)(dots). \( B_{tr} = 0.52T \) and the values of dimensionless variables \( \alpha = \ell^2 \) and the fitting parameter \( \beta = \ell \ell_{\phi} \)(i.e. \( \ell_{\phi} = \frac{\beta}{\sqrt{\alpha}} \)) are \( \alpha = 1.33 \) and \( \beta = 12.7 \).

Fig2. Magnetic field dependence of \( \Delta \sigma(B)/G_0 \) for temperatures \( T = 1.5K, 2K, 2.5K, 3K \), and \( 4.2K \). Solid lines are the results of fitting by exact expression(equ.(10) or equ.(9)). Circles are experimental data for structures 1. Dashed line shows the result of fitting by equ.(1). \( B_{tr} = 0.52T, \alpha = 1.33 \) and \( \beta = 26.71, 22.5, 19.2, 16.5, 12.7 \) for the temperatures \( T = 1.5K, 2K, 2.5K, 3K \) and \( 4.2K \) respectively.

Fig3. Temperature dependence of \( \tau_{\phi} \). Solid lines are the dependencies \( \tau_{\phi} = \frac{C}{T} \) with \( C_{\text{exact}} \) and \( C_{\text{approx}} \) respectively. Circles and plus are the results of fitting by equ.(10)(or equ.(9)) and equ.(1) respectively. Open squares are obtained from equ.(12).

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