\textbf{E-Factors for the Period Determinants of Curves}

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To Spencer Bloch

The myriad beings of the six worlds –  
gods, humans, beasts, ghosts, demons, and devils –  
are our relatives and friends.

Tesshu “Bushido”, translated by J. Stevens.

\textbf{Introduction}

0.1. Let $X$ be a smooth compact complex curve, $M$ be a holonomic $D$-module on $X$ (so outside a finite subset $T \subset X$, our $M$ is a vector bundle with a connection $\nabla$). Denote by $dR(M)$ the algebraic de Rham complex of $M$ placed in degrees $[-1,0]$; this is a complex of sheaves on the Zariski topology $X_{\text{Zar}}$. Its analytic counterpart $dR^{an}(M)$ is a complex of sheaves on the classical topology $X_{\text{cl}}$. Viewed as an object of the derived category of $\mathbb{C}$-sheaves, this is a perverse sheaf, which we denote by $B(M)$; outside $T$, it is the local system $M_{X,T}^\nabla$ of $\nabla$-horizontal sections (placed in degree $-1$). Set $H_{dR}(X,M) := H^{\cdot}(X_{\text{Zar}},dR(M))$, $H_B(X,M) := H^{\cdot}(X_{\text{cl}},B(M))$; these are the de Rham and Betti cohomology. We have the \textbf{period isomorphism} $\rho : H_{dR}(X,M) \sim H_B(X,M)$.

The cohomology $H_{dR}$ and $H_B$ have, respectively, algebraic and topological nature that can be tasted as follows. Let $k,k' \subset \mathbb{C}$ be subfields. Then:

- For $(X,M)$ defined over $k$, we have \textbf{de Rham $k$-structure} $H_{dR}(X_k,M_k)$ on $H_{dR}(X,M)$;
- A $k'$-structure on $B(M)$, i.e., a perverse $k'$-sheaf $B_{k'}$ on $X_{\text{cl}}$ together with an isomorphism $B_{k'} \otimes \mathbb{C} \sim B(M)$, yields Betti $k'$-structure $H^{\cdot}(X_{\text{cl}},B_{k'})$ on $H_B(X,M)$.

If both $(X_k,M_k)$ and $B_{k'}$ are at hand, then, computing $\det \rho$ with respect to rational bases, one gets a number whose class $[\det \rho]$ in $\mathbb{C}^\times/k^\times k'^\times$ does not depend on the choice of the bases. In his farewell seminar at Bures [Del], Deligne, guided
by an analogy between \( \det \rho \) and the constant in the functional equation of an 
\( L \)-function, asked if \( \det \rho \) can be expressed, in presence of an extra datum of a 
rational 1-form \( \nu \), as the product of certain factors of local origin at points of \( T \) and 
\( \div(\nu) \). He also suggested the existence of a general geometric format which would 
yield the product formula (see 0.3 below). Our aim is to establish such a format.

0.2. \textbf{Remarks}. (i) A natural class of \( k' \)-structures on \( B(M) \) comes as follows. 
Suppose for simplicity that \( M \) equals the (algebraic) direct image of \( M_X \) by \( X \setminus T \to X \). Let \( \pi : \tilde{X} \to X \) be the real blow-up of \( X \) at \( T \) (so \( \tilde{X} \) is a real-analytic 
surface with boundary \( \partial \tilde{X} = \pi^{-1}(T) \), and \( \pi \) is an isomorphism over \( X \setminus T \)). Then 
\( M_{\tilde{X}} \) extends uniquely to a local system \( M_{\tilde{X}} \) on \( \tilde{X} \). Following Malgrange [M], 
consider the constructible subsheaf \( M_{\tilde{X}} \) of sections of moderate growth (so \( M_{\tilde{X}} \) coincides with \( M_{\tilde{X}} \) off \( \partial \tilde{X} \), and \( M_{\tilde{X}} \) equals \( M_X \) if and only if \( M \) has regular 
singularities). By [M] 3.2, one has a canonical isomorphism 
\[ R\pi_* M_{\tilde{X}} \cong B(M). \] (0.2.1)
Therefore a \( k' \)-structure on \( M_{\tilde{X}} \) yields a \( k' \)-structure on \( B(M) \). Notice that 
the former is the same as a \( k' \)-structure on the local system \( M_{\tilde{X}} \), i.e., on \( M_{\tilde{X}} \), such 
that the subsheaf \( M_{\tilde{X}} \) is defined over \( k' \).

(ii) By (0.2.1), one has \( H_{B}(X, M) = H'(\tilde{X}, M_{\tilde{X}}) \). The dual vector space equals 
\( H'(\tilde{X}, D M_{\tilde{X}}) \), where \( D \) is the Verdier duality functor, which is the homology group 
of cycles with coefficients in \( M_{\tilde{X}} \) on \( X \setminus T \), having rapid decay at \( T \). So \( \rho \), viewed 
as a pairing \( H_{dR}(X, M) \times H'(\tilde{X}, D M_{\tilde{X}}) \to \mathbb{C} \), is the matrix of periods of \( M \)-valued 
forms along the cycles of rapid decay. See [BE] for many examples.

(iii) The setting of 0.1 makes sense for proper \( X \) of any dimension. The passage 
\( B \) to perverse sheaves commutes with direct image functors for proper morphisms 
\( X \to Y \), so the data \((X, M, B_{k'})\) are functorial with respect to direct image.

0.3. The next format, which yields the product formula, was suggested in the 
last exposé of [Del]:

(i) There should exist \( \varepsilon \)-factorization formalisms for \( \det H_{dR} \) and \( \det H_{B} \). These 
are natural rules which assign to every non-zero meromorphic 1-form \( \nu \) on \( X \) two 
collections of lines \( \mathcal{E}_{dR}(M)(x, \nu) \) and \( \mathcal{E}_{B}(M)(x, \nu) \) labeled by points \( x \in X \). The lines 
\( \mathcal{E}_{dR}(M)(x, \nu) \) have local nature; if \( x \notin \bigcup T \cup \div(\nu) \), then \( \mathcal{E}_{dR}(M)(x, \nu) \) is naturally 
trivialized. Finally, one has \( \varepsilon \)-factorization, alias \textit{product formula}, isomorphisms 
\[ \gamma : \bigotimes_{x \in \bigcup T \cup \div(\nu)} \mathcal{E}_{dR}(M)(x, \nu) \cong \det H_{dR}(X, M). \] (0.3.1)

(ii) The de Rham and Betti \( \varepsilon \)-factorizations should have, respectively, algebraic 
and topological origin. Thus, if \( X, M, \nu \) are defined over \( k \), then the datum 
\( \{\mathcal{E}_{dR}(M)(x, \nu)\} \) is defined over \( k \), and a \( k' \)-structure on \( B(M) \) yields a \( k' \)-structure 
on every \( \mathcal{E}_{dR}(M)(x, \nu) \). One wants these structures to be compatible with the trivi-
alizations of \( \mathcal{E}_{dR}(M)(x, \nu) \) off \( T \cup \div(\nu) \), and \( \gamma_{dR}, \gamma_{B} \) to be defined over \( k, k' \).
(iii) There should be natural $\varepsilon$-period isomorphisms $\rho^\varepsilon = \rho^\varepsilon_{(x,\nu)} : E_{\text{dR}}(M)_{(x,\nu)} \sim E_B(M)_{(x,\nu)}$ of $x$-local origin such that the next diagram commutes:

$$
\begin{array}{ccc}
\otimes E_{\text{dR}}(M)_{(x,\nu)} & \xrightarrow{\text{min}} & \det H_{\text{dR}}(X, M) \\
\otimes \rho^\varepsilon_{(x,\nu)} & \downarrow & \rho \downarrow \\
\otimes E_B(M)_{(x,\nu)} & \xrightarrow{\text{min}} & \det H_B(X, M)
\end{array}
$$

(0.3.2)

Suppose $(X, M, \nu)$ is defined over $k$. The points in $T \cup \text{div}(\nu)$ are algebraic over $k$; let $\{O_\alpha\}$ be their partition by the Galois orbits. By (ii), the lines $\otimes \chi \in O_\alpha$ carry $k$-structure. If $B(M)$ is defined over $k'$, then, by (ii), $E_B(M)_{(x,\nu)}$ carry $k'$-structure. Writing $\otimes \rho^\varepsilon_{(x,\nu)}$ in $k$-$k'$-bases, we get numbers $[\rho^\varepsilon_{(O_\alpha,\nu)}] \in C^x/k^x k'^x$.

Now (0.3.2) yields the promised product formula

$$
[\det \rho] = \prod_\alpha [\rho^\varepsilon_{(O_\alpha,\nu)}].
$$

(0.3.3)

We will show that the above picture is, indeed, true.

0.4. Parts of this format were established earlier: the de Rham $\varepsilon$-factorization was constructed already in Deligne (and reinvented later in BBD); the Betti counterpart was presented (in the general context of “animation” of Kashiwara’s index formula) in B [8]. It remains to construct $\rho^\varepsilon$. The point is that $E_\varepsilon$ satisfy several natural constraints, and compatibility with them determines $\rho^\varepsilon$ almost uniquely. Notice that we work completely over $\mathbb{C}$: the $k$- and $k'$-structures are irrelevant.

The principal constraints are the global product formula (0.3.1) and its next local counterpart. For $\nu'$ close to $\nu$, the points of $T \cup \text{div}(\nu')$ cluster around $T \cup \text{div}(\nu)$. Now the isomorphism $\bigotimes_{x \in T \cup \text{div}(\nu')} E_\varepsilon^!(M)_{(x',\nu')} \sim \bigotimes_{x \in T \cup \text{div}(\nu)} E_\varepsilon^!(M)_{(x,\nu)}$ that comes from global identifications (0.3.1) can be written as the tensor product of natural isomorphisms of local origin at points of $T \cup \text{div}(\nu)$. This local factorization structure (which is a guise, with an odd twist, of the geometric class field theory) is fairly rigid: $E_\varepsilon(M)$ is determined by a rank 1 local system det $M_{X,T}$ and a collection of lines labeled by elements of $T$.

The rest of constraints for $M \mapsto E_\varepsilon(M)$ are listed in 5.1. We show that there is an isomorphism $\rho^\varepsilon : E_{\text{dR}} \sim E_B$ compatible with them, which is determined uniquely up to a power of a simple canonical automorphism of $E_\varepsilon$, i.e., $\rho^\varepsilon$ form a $\mathbb{Z}$-torsor $E_B/\text{dR}$. First we recover $\rho^\varepsilon$ from $\eta$-compatibility (0.3.2) for $(\mathbb{P}^1, \{0, \infty\})$, $M$ with regular singularities at $\infty$, and $(\mathbb{P}^1, \{0, 1, \infty\})$, $M$ of rank 1 with regular singularities. Having $\rho^\varepsilon$ at hand, one has to prove that it is compatible with the constraints for all $(X, T, M)$, of which (0.3.2) is central. The core of the argument is global: we use a theorem of Goldman [C] and Pickrell-Xia [PX1], [PX2], which asserts that the action of the Teichmüller group on the moduli space of unitary local systems with fixed local monodromies is ergodic. As in [C], this implies that, when the genus of $X$ and the order of $T$ are fixed, the possible discrepancy of (0.3.2) depends only on the local datum of monodromies at singularities of $M$. An observation that this discrepancy does not change upon quadratic degenerations of $X$ reduces the proof to a few simple computations.

1In Deligne it was suggested that in case when $\text{Re}(\nu)$ is exact, $\text{Re}(\nu) = df$, the Betti $\varepsilon$-factorization comes from the Morse theory of $f$; see 4.6 or [H] 3.8 for a proof.

2In the same manner as the $\nu$-dependence of the classical $\varepsilon$-factor of a Galois module $V$ is controlled, via the class field theory, by det $V$. 

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0.5. One can ask for an explicit formula for $\rho^\varepsilon(M)$. An analytic approach as in [PS] or [SW] shows that the de Rham $\varepsilon$-factors of a $\mathcal{D}$-module $M$ can be recovered from the $D^\infty$-module $M^\infty := D^\infty \otimes \mathcal{D}$. Thus the ratio between the $\varepsilon$-factors of $M$ and of another $\mathcal{D}$-module $M'$ (say, with regular singularities) with $B(M) = B(M')$, is certain Fredholm determinant (a variant of $\tau$-function). If $x$ is a regular singular point of $M$, then $[\rho^\varepsilon_{x,\nu}]$ can be written explicitly using the $\Gamma$-function, see 6.3 (which is similar to the fact that the classical $\varepsilon$-factors of tamely ramified Galois modules are essentially products of Gauß sums). An example of the product formula is the Euler identity
\[ \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \]

0.6. Plan of the article: §1 presents a general story of factorization lines (i.e., of the local factorization structure); in §2–4 the algebraic and analytic de Rham $\varepsilon$-factors, and their Betti counterpart are defined; §5 treats the $\varepsilon$-period map; in §6 the $\varepsilon$-periods are written explicitly in terms of the $\Gamma$-function.

A different approach to product formula (0.3.3), based on Fourier transform, was developed by Bloch, Deligne, and Esnault [BDE], [E] (some essential ideas go back to [Del1] and [L]; the case of regular singularities was considered earlier, and for $X$ of arbitrary dimension, in [A], [LS], [ST], and [T]).

The two constructions are fairly complementary; the relation between them remains to be understood.

Questions & hopes.

(a) For Verdier dual $M$, $M^\vee$ the lines $E^\varepsilon(M)(x,\nu)$ and $E^\varepsilon(M^\vee)(x,-\nu)$ should be naturally dual, and $\rho^\varepsilon$ should be compatible with duality.

(i) The period story should exist for $X$ of any dimension, with mere lines replaced by finer objects (the homotopy points of $K$-theory spectra). For the Betti side, see [B]; for the de Rham one, see [P].

The meaning of local factorization structure for $\dim X > 1$ is not clear (as of the more general notion of factorization sheaves in the setting of algebraic geometry). Is there an agebro-geometric analog of the recent beautiful work of Lurie on the classification of TQFT?

(ii) There should be a geometric theory of $\varepsilon$-factors (cf. 5.1) for étale sheaves; for an étale sheaf of virtual rank 0 on a curve over a finite field, the corresponding trace of Frobenius function should be equal to the classical $\varepsilon$-factors. Notice that Laumon’s construction [L] (which is the only currently available method to establish the product formula for classical $\varepsilon$-factors) has different arrangement: its input is more restrictive (the forms $\nu$ are exact), while the output is more precise (the $\varepsilon$-lines are realized as determinants of true complexes).

(iii) What would be a motivic version of the story?

(iv) $\Gamma$-function appears in Deninger’s vision [Den] of classical local Archimedean $\varepsilon$-factors. Are the two stories related on a deeper level?

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1 Factorization lines

This section is essentially an exposition of geometric class field theory (mostly) in its algebraic de Rham version.

1.1. We live over a fixed ground field $k$ of characteristic 0; “scheme” means “separated $k$-scheme of finite type”. The category $\mathcal{S}ch$ of schemes is viewed as a site for the étale topology (so “neighborhood” means “étale neighborhood”, etc.), “space” means a sheaf on $\mathcal{S}ch$; for a space $F$ and a scheme $S$ elements of $F(S)$ are referred to as $S$-points of $F$. All Picard groupoids are assumed to be commutative and essentially small. For a Picard groupoid $\mathcal{L}$, we denote by $\pi_0(\mathcal{L})$, $\pi_1(\mathcal{L})$ the group of isomorphism classes of its objects and the automorphism group of any of its object; for $L \in \mathcal{L}$ its class is $[L] \in \pi_0(\mathcal{L})$.

Let $X$ be a smooth (not necessary proper or connected) curve, $T$ its finite subscheme, $K$ a line bundle on $X$. For a test scheme $S$, we write $X_S := X \times S$, $T_S := T \times S$, $K_S := K \boxtimes \mathcal{O}_S$; $\pi : X_S \to S$ is the projection. For a Cartier divisor $D$ on $X_S$ we denote by $|D|$ the support of $D$ viewed as a reduced closed subscheme.

Consider the next spaces:

(a) $\text{Div}(X)$: its $S$-points are relative Cartier divisors $D$ on $X_S/S$ such that $|D|$ is finite over $S$;

(b) $2^T$ is a scheme whose $S$-points $c$ are idempotents in $\mathcal{O}(T_S)$. Such $c$ amounts to an open and closed subscheme $T^c_S$ of $T_S$ (the support of $c$);

(c) $\mathfrak{D} = \mathfrak{D}(X,T) \subset \text{Div}(X) \times 2^T$ consists of those pairs $(D,c)$ that $D \cap T_S \subset T^c_S$;

(d) $\mathfrak{D}^0 = \mathfrak{D}^0(X,T;K)$ is formed by triples $(D,c,\nu_P)$ where $(D,c) \in \mathfrak{D}$ and $\nu_P$ is a trivialization of the restriction of the line bundle $K(D) := K_S(D)$ to the subscheme $P = F_{D,c} := T^c_S \cup |D|$.

Denote by $\pi_0(X)$ the scheme of connected components of $X$. One has projection $\deg : \text{Div}(X) \to \mathbb{Z}^{\pi_0(X)}$, hence the projections $\mathfrak{D}^0 \to \mathfrak{D} \to \mathbb{Z}^{\pi_0(X)} \times 2^T$. Notice that the component $\mathfrak{D}_{c=0}$ equals $\text{Div}(X \setminus T)$, and $\mathfrak{D}_{c=1}$ equals $\text{Div}(X)$.

Remarks. (i) Every $S$-point of $\mathfrak{D}$ can be lifted $S$-locally to $\mathfrak{D}^0$.

(ii) Every $\nu_P$ as in (d) can be extended $S$-locally to a trivialization $\nu$ of $K(D)$ on a neighborhood $V \subset X_S$ of $P$. One can view $\nu_P$ as an equivalence class of $\nu$’s, where $\nu$ and $\nu'$ are equivalent if the function $\nu/\nu'$ equals 1 on $P$. We often write $(D,c,\nu)$ for $(D,c,\nu_P)$.

(iii) Each space $F$ of the list (a)–(d) is smooth in the next sense: for every closed embedding $S \hookrightarrow S'$, a geometric point $s \in S$, and $\phi \in F(S)$ one can find an neighborhood $U'$ of $s$ in $S'$ and $\phi' \in F(U')$ such that $\phi|_{U'_s} = \phi'|_{U'_s}$.

(iv) The geometric fibers of $\text{Div}(X)$ over $\mathbb{Z}^{\pi_0(X)}$ and of $\mathfrak{D}$, $\mathfrak{D}^0$ over $\mathbb{Z}^{\pi_0(X)} \times 2^T$, are connected (i.e., every two geometric points of any fiber are members of one connected family).

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6Starting from §2, our $K$ equals $\omega_X$.

7Which is the spectrum of the integral closure of $k$ in the ring of functions on $X$. 
A comment about the fiber $\mathfrak{D}^\circ(D,c)$ of $\mathfrak{D}^\circ/\mathfrak{D}$ over $(D,c) \in \mathfrak{D}(S)$: Suppose $S$ is smooth, so $P_{D,c}$ is a relative Cartier divisor in $X_S/S$. Denote by $O^\circ_{D,c}$ the Weil $P_{D,c}/S$-descent of $\mathbb{G}_{m,p}$, and by $K(D)^\circ_{D,c}$ the Weil $P_{D,c}/S$-descent of the $\mathbb{G}_{m,p}$-torsor of trivializations of the line bundle $K(D)|_{P_{D,c}}$. Then $O^\circ_{D,c}$ is a smooth group $S$-scheme, and $K(D)^\circ_{D,c}$ is an $O^\circ_{D,c}$-torsor; for any $S$-scheme $S'$ an $S'$-point of $K(D)^\circ_{D,c}$ is the same as a trivialization of $K(D)$ over $(P_{D,c})_{S'}$. The latter relative divisor contains $P_{D',c',S'}$ (the corresponding reduced schemes coincide), so we have a canonical surjective morphism $K(D)^\circ_{D,c} \to \mathfrak{D}^\circ_{(D,c)}$, hence a canonical $(D,c,\nu_P) \in \mathfrak{D}^\circ(K(D)^\circ_{D,c})$. The map $K(D)^\circ_{D,c}(S') = \mathfrak{D}^\circ_{(D,c)}(S')$ is bijective if $S'$ is smooth over $S$, but not in general.

**Examples.** (i) Suppose $S = X \setminus T$, $(D,c) = (\ell \Delta, 0)$ where $\Delta$ is the diagonal divisor, $\ell$ is any integer. Then $O^\circ_{D,c} = \mathbb{G}_{m,S}$, and $K(D)^\circ_{D,c}$ is the $\mathbb{G}_{m}$-torsor $K(\ell)$ of trivializations of the line bundle $K(D)|_\Delta = K \otimes \omega^\ell_X|_S$. For any $S'/S$ an $S'$-point of $\mathfrak{D}^\circ_{(\ell \Delta, 0)}$ is the same as an $S'_{red}$-point of $K(\ell)$, i.e., $\mathfrak{D}^\circ_{(\ell \Delta, 0)}$ is the quotient of $K(\ell)$ modulo the action of the formal multiplicative group $\mathbb{G}_{m}$. (ii) For a point $b \in T$ let $k_b$ be its residue field, $T_b \subset T$ be the component of $b$, and $m_b$ its multiplicity. Consider $(D,c) = (nb, 1_b) \in \mathfrak{D}(S)$, where $S = \text{Spec } k_b$, $n$ is any integer, $1_b$ is the characteristic function of $b \in T(S)$. Then $P_{D,c} = T_b$, so $O^\circ_{D,c} = O_{T_b}^\circ$ is an extension of $\mathbb{G}_{m,S}$ by the unipotent radical. One has $K(nb)^\circ_{T_b} := K(D)^\circ_{D,c} \simeq K^\circ_{(D,c)}$. We set $K^\circ_{T_b} := K((nb))^\circ_{T_b}$.

### 1.2.

Let $\mathcal{V}$ be a stack, alias a sheaf of categories, on $\mathcal{S}ch$. For a space $F$ we denote by $\mathcal{V}(F)$ the category of Cartesian functors $V : F \to \mathcal{V}$. Explicitly, such $V$ is a rule that assigns to every test scheme $S$ and $\phi \in F(S)$ an object $V_\phi \in \mathcal{V}(S)$ together with a base change compatibility constraint. If $\mathcal{V}$ is a Picard stack, alias a sheaf of Picard groupoids, then $\mathcal{V}(F)$ is naturally a Picard groupoid.

Below we denote by $\hat{F}$ the space with $\hat{F}(S) := F(S_{red})$. The stack of $\mathcal{V}$-crystals $\mathcal{V}_{crys}$ is defined by formula $\mathcal{V}_{crys}(S) := \mathcal{V}(\hat{S})$. If $F$ is formally smooth (i.e., satisfies the property from Remark (iii) in 1.1 for every nilpotent embedding $S \to S'$), then $\hat{F}$ is the quotient of $F$ modulo the evident equivalence relation; therefore objects of $\mathcal{V}_{crys}(F) = \mathcal{V}(\hat{F})$ are the same as objects $V \in \mathcal{V}(F)$ equipped with a de Rham structure, i.e., a natural identification $\alpha : V_\phi \simeq V_{\phi'}$ for every $\phi, \phi' \in F(S)$ such that $\alpha|_{S_{red}} = \phi'|_{S_{red}}$, which is transitive and compatible with base change. E.g. if $F$ is a smooth scheme, then a vector bundle crystal on $F$ is the same as a vector bundle on $F$ equipped with a flat connection.

**Key examples:** Let $\mathcal{L}_k$ be the Picard groupoid of $\mathbb{Z}$-graded $k$-lines (with “super” commutativity constraint for the tensor structure). Below we call them simply “lines” or “$k$-lines”; the degree of a line $G$ is denoted by $\deg(G)$. An $O$-line on $S$ (or $O_S$-line) is an invertible $\mathbb{Z}$-graded vector bundle on $S$. These objects form a Picard groupoid $\mathcal{L}_O(S)$; the usual pull-back functors make $\mathcal{L}_O$ a Picard stack. Below $\mathcal{L}_O$-crystals are referred to as de Rham lines; they form a Picard stack $\mathcal{L}_{dr}$. Instead of $\mathbb{Z}$-graded lines, we can consider $\mathbb{Z}/2$-graded ones; the corresponding Picard stacks are denoted by $\mathcal{L}_O', \mathcal{L}_{dr}'$. We mostly consider $\mathbb{Z}$-graded setting; all the results remain valid, with evident modifications, for $\mathbb{Z}/2$-graded one.

**Remarks.** (i) Let $(X', T')$ be another pair as in 1.1, and $\pi : (X', T') \to (X, T)$ be a finite morphism of pairs, i.e., $\pi : X' \to X$ is a finite morph of curves such that $\pi(T') \subset T$. It yields a morphism of spaces $\pi^* : \mathfrak{D}^\circ(X, T; K) \to \mathfrak{D}^\circ(X', T'; \pi^* K)$,
(D, c, ν) ↦ (π∗D, π∗c, π∗ν), hence the pull-back functor π*: \mathcal{V}(\mathcal{O}^p(X', T'; \pi^∗K)) → \mathcal{V}(\mathcal{O}^p(X, T; K)) denoted by π*; if \mathcal{V} is a Picard stack, then π* is a morphism of Picard groupoids. If X = X, T ⊂ T, we refer to π* as “restriction to (X, T)”. Exercise. If T ⊂ T, \text{Tred} = \text{Tred}, then the restriction \mathcal{L}_{\text{dr}}(\mathcal{O}^p(X, T; K)) → \mathcal{L}_{\text{dr}}(\mathcal{O}^p(X, T; K)) is a fully faithful embedding.

We denote the union of the Picard groupoids \mathcal{L}_{\text{dr}}(\mathcal{O}^p(X, T''; K)) for all T'' with \text{Tred} = \text{Tred} by \mathcal{L}_{\text{dr}}(\mathcal{O}^p(X, \hat{T}; K)) (here \hat{T} is the formal completion of X at T).

(ii) The space \mathcal{O}^p(X, T; K), hence \mathcal{L}_r(\mathcal{O}^p(X, T; K)), actually depends only on the restriction of K to X \setminus T. Indeed, for any divisor D_{(T)} supported on T, there is a canonical identification \mathcal{O}^p(X, T; K) \sim \mathcal{O}^p(X, T; K(D_{(T)})). (D, c, ν) ↦ (D − D_{(T)}, ν), where D_{(T)} is D(T) on T\_Σ and to 0 outside. We keep K to be a line bundle on X for future notational convenience.

(iii) If U is any open subset of X, then \mathcal{O}^p(U, T_U; K_U) \subset \mathcal{O}^p(X, T; K), hence we have the restriction functor \mathcal{V}(\mathcal{O}^p(X, T; K)) → \mathcal{V}(\mathcal{O}^p(U, T_U; K_U)).

(iv) Remark (iv) in 1.1 implies that π₁(\mathcal{L}_{\text{dr}}(\mathcal{O}^p)) = \mathcal{O}^p(\mathbb{Z}_n(X) × 2^T).

1.3. Let Sm ⊂ Sch be the full subcategory of smooth schemes. For \mathcal{V}, F as in 1.2 we denote by \mathcal{V}^{\text{sm}}(F) the Picard groupoid of Cartesian functors F|_{\text{Sm}} → \mathcal{V}|_{\text{Sm}}. One has a restriction functor \mathcal{V}(F) → \mathcal{V}^{\text{sm}}(F). If F is smooth in the sense of Remark (iii) in 1.1, then this is a faithful functor.

Exercise. Suppose we have \mathcal{E}, \mathcal{E}' ∈ \mathcal{L}_{\text{dr}}(\mathcal{O}^p) and a morphism \phi: \mathcal{E} → \mathcal{E}' in \mathcal{L}_{\text{O}}(\mathcal{O}^p). Then \phi is a morphism in \mathcal{L}_{\text{dr}}(\mathcal{O}^p), if (and only if) the corresponding morphism in \mathcal{L}_{\text{sm}}^{\text{sm}}(\mathcal{O}^p) lies in \mathcal{L}_{\text{dr}}^{\text{sm}}(\mathcal{O}^p).8

Lemma. \mathcal{L}_{\text{dr}}(F) \sim \mathcal{L}_{\text{dr}}^{\text{sm}}(F).

Proof. This follows from the fact that \mathcal{L}_{\text{dr}} is a stack with respect to the h-topology, and h-locally every scheme is smooth. □

Remark. By the lemma and 1.1, one can view \mathcal{E} ∈ \mathcal{L}_{\text{dr}}(\mathcal{O}^p) as a rule that assigns to every smooth S and (D, c) ∈ \mathcal{O}(S) a de Rham line \mathcal{E}_{(D, c)} := \mathcal{E}_{(D, c, ν)} on K(D)_{(D, c)} in a way compatible with the base change.

1.4. For this subsection, X is proper. Let Rat(X, K) = Rat(X) be a space whose S-points are rational sections ν of the line bundle KS such that |div(ν)| does not contain a connected component of any geometric fiber of X\_S/S. There is a natural morphism Rat(X) → \mathcal{O}^p_{\nu=1}, ν ↦ (−div(ν), 1, ν), so every \mathcal{E} ∈ \mathcal{L}_{\text{O}}(\mathcal{O}^p) yields naturally an object of \mathcal{L}_{\text{O}}(\text{Rat}(X)), which we denote again by \mathcal{E}.

The next fact is a particular case of [RD] 4.3.13:

Proposition. Every function on Rat(X) is constant. All O- and de Rham lines on Rat(X) are constant.

Proof. Let L be an auxiliary ample line bundle on X; set \nu^{(m)} := Γ(X, K ⊗ L^{⊗ m}), \nu^{(m)} := Γ(X, K ⊗ L^{⊗ m}). Let U^{(m)} ⊂ \text{P}(V^{(m)} × V^{(m)}) be the open subset of those (φ₁, φ₂) that neither φ₁ nor φ₂ vanishes on any connected component of X. Consider a map \theta^{(m)}: U^{(m)} → Rat(X), (φ₁, φ₂) ↦ φ₁/φ₂. We will check that for large the \theta^{(m)}-pull-back of any O- or de Rham line on Rat(X) is trivial, and every

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8Hint: Use Remark (iii) in 1.1 for embeddings S → S′ where S′ is smooth.
function on $U^{(m)}$ is constant. This implies the proposition, since geometric fibers of $\theta^{(m)}$ are connected and the images of $\theta^{(m)}$ form a directed system of subspaces whose inductive limit equals $\text{Rat}(X)$ (i.e., every $\nu \in \text{Rat}(X)(S)$ factors $S$-locally through $\theta^{(m)}$ for sufficiently large $m$).

Notice that the complement to $U^{(m)}$ in $\mathbb{P}(V_1^{(m)} \times V_2^{(m)})$ has codimension $\geq 2$ for $m$ large. Therefore every function and every $\mathcal{O}$- and de Rham line extend to $\mathbb{P}(V_1^{(m)} \times V_2^{(m)})$. Thus for $m$ large every function on $U^{(m)}$ is constant and every de Rham line is trivial.

The case of an $\mathcal{O}$-line requires an extra argument. Any $(\psi_1, \psi_2) \in U^{(n)}$ yields an embedding $\mathbb{P}(\Gamma(X, L\otimes k)) \hookrightarrow \mathbb{P}(V_1^{(m)} \times V_2^{(m)}), \gamma \mapsto (\gamma \psi_1, \gamma \psi_2)$; here $m = n + k$. For $k$ large, the preimage of $U^{(m)}$ in $\mathbb{P}(\Gamma(X, L\otimes k))$ is an open dense subset of codimension $\geq 2$, and $\theta^{(m)}$ is constant on it. Thus for any $\mathcal{O}$-line $\mathcal{L}$ on $\text{Rat}(X)$ the restriction of the corresponding line on $\mathbb{P}(V_1^{(m)} \times V_2^{(m)})$ to $\mathbb{P}(\Gamma(X, L\otimes k))$ is trivial, hence the line itself is trivial, so $\theta^{(m)} \ast \mathcal{L}$ is trivial, and we are done.

Therefore for any $\mathcal{E}$ in $\mathcal{L}_\mathcal{O}(\mathfrak{D}^\circ)$ or $\mathcal{L}_{\text{dR}}(\mathfrak{D}^\circ)$ the lines $\mathcal{E}_\nu$ for all rational non-zero $\nu$ are canonically identified. We denote this line simply by $\mathcal{E}(X)$.

### 1.5

A finite subset of $\{(D_\alpha, c_\alpha, \nu_\alpha)\}$ of $\mathfrak{D}^\circ(S)$, is said to be disjoint if the subschemes $P_{D_\alpha, c_\alpha}$ are pairwise disjoint. Then we have $\Sigma(D_\alpha, c_\alpha, \nu_\alpha) := (\Sigma_{D_\alpha}, \Sigma_{c_\alpha}, \Sigma_{\nu_\alpha}) \in \mathfrak{D}^\circ(S)$, where $\Sigma_{\nu_\alpha} equals \nu_\alpha$ on $P_{D_\alpha, c_\alpha}$.

For $\mathcal{E}$ in $\mathcal{L}_{\mathfrak{D}^\circ}$, where $\mathfrak{D}^\circ$ is a Picard stack, a factorization structure on $\mathcal{E}$ is a rule which assigns to every disjoint family as above a factorization isomorphism

$$\otimes \alpha \mathcal{E}(D_\alpha, c_\alpha, \nu_\alpha) \sim \mathcal{E}(\Sigma(D_\alpha, c_\alpha, \nu_\alpha)).$$

These isomorphisms should be compatible with base change and satisfy an evident transitivity property. One defines factorization structure on objects of $\mathcal{L}_{\mathfrak{D}^\circ}(X, T)$, $\mathcal{L}_{\mathfrak{D}^\circ}(\text{Div}(X))$, $\mathcal{L}_{\mathfrak{D}^\circ}(\text{Div}^{\text{dR}}(X))$, or $\mathcal{L}_{\mathfrak{D}^\circ}(\mathcal{L}^2(2^T))$ in the similar way.

Objects of $\mathcal{L}_{\mathfrak{D}^\circ}$ equipped with a factorization structure are called $(K$-twisted) factorization objects of $\mathcal{L}_{\mathfrak{D}^\circ}$ on $(X, T; K)$; they form a Picard groupoid $\mathcal{L}_{\mathfrak{D}^\circ}(X, T; K)$. In particular, we have Picard groupoids $\mathcal{L}_{\mathfrak{D}^\circ}(X, T; K)$, $\mathcal{L}^{\text{dR}}_{\mathfrak{D}^\circ}(X, T; K)$ of $\mathcal{O}$- and de Rham factorization lines.

**Proposition.** Factorization objects have local nature: $U \mapsto \mathcal{L}_{\mathfrak{D}^\circ}(U, T_U; K_U)$ is a Picard stack on $X_\mathfrak{D}$.

**Proof.** Let $\pi : U \to X$ be an étale map. For $\mathcal{E} \in \mathcal{L}_{\mathfrak{D}^\circ}(X, T; K)$ one defines its pull-back $\pi^* \mathcal{E}$ as follows. Take any $(D_\alpha, c_\alpha, \nu_\alpha) \in \mathfrak{D}^\circ(U, T_U; K_U)$. It suffices to define $\mathcal{E}(D_\alpha, c_\alpha, \nu_\alpha)$ étale locally on $S$. Write $(D_\alpha, c_\alpha, \nu_\alpha) = (\Sigma(D_\alpha, c_\alpha, \nu_\alpha))$ with connected $P_\alpha = P_{D_\alpha, c_\alpha}$. Then there is a uniquely defined $(D'_\alpha, c'_\alpha, \nu'_\alpha) \in \mathfrak{D}^\circ(X, T; K)$ such that $D_\alpha$ is a connected component of the pull-back of $D'_\alpha$ to $U$ and $\pi$ yields an isomorphism $P_\alpha \sim P'_\alpha$ which identifies $\nu_\alpha P_\alpha$ with $\nu'_\alpha P'_\alpha$.

Set $\pi^* \mathcal{E}(D_\alpha, c_\alpha, \nu_\alpha) := \otimes \mathcal{E}(D'_\alpha, c'_\alpha, \nu'_\alpha)$. Due to factorization structure on $\mathcal{E}$, this definition is compatible with base change, and $\pi^* \mathcal{E} \in \mathcal{L}_{\mathfrak{D}^\circ}(U, T_U; K_U)$ so defined has an evident factorization structure. Thus $\mathcal{L}_{\mathfrak{D}^\circ}$ is a presheaf of Picard groupoids on $X_{\mathfrak{D}}$. We leave it to the reader to check the gluing property.

**NB:** The pull-back functor for open embeddings is defined regardless of factorization structure (see Remark (iii) in 1.2).

**Remarks.** (i) The evident forgetful functor $\mathcal{L}_{\mathfrak{D}^\circ}(X, T; K) \to \mathcal{L}_{\mathfrak{D}^\circ}(X, T; K)$ is faithful. By 1.2 (and Remark (iii) in 1.1), for $\mathcal{E} \in \mathcal{L}_{\mathfrak{D}^\circ}(X, T; K)$ a de Rham structure
on $E$, i.e., a lifting of $E$ to $\mathcal{L}^0_{\text{dR}}(X, T; K)$, amounts to a rule which assigns to every scheme $S$ and a pair of points $(D, c, \nu p), (D', c', \nu p') \in \mathcal{D}^0(S)$ which coincide on $S_{\text{red}}$, a natural identification (notice that $c = c'$)

$$\alpha^\varepsilon : \mathcal{E}_{(D, c, \nu p)} \sim \mathcal{E}_{(D', c, \nu p')}.$$  

(1.5.2)

The $\alpha^\varepsilon$ should be transitive and compatible with base change and factorization.

(ii) Remarks in 1.3 and (i)--(ii) in 1.2 remain valid for factorization lines. Thus we have a Picard groupoid $\mathcal{L}^\rho_{\text{dR}}(X, T; K)$, etc.

(iii) There is a natural Picard functor

$$\prod_{b \in T_{\text{red}}} \mathcal{L}_c(b) \rightarrow \mathcal{L}^\rho_{\text{dR}}(X, T; K),$$

(1.5.3)

which assigns to $E = (E_b) \in \prod \mathcal{L}_c(b)$ a factorization object $E$ with $\mathcal{E}_{(D, c, \nu p)} = \text{Nm}^E_{\text{red}}/S(E)$; here $T_{\text{red}} = S$ is the preimage of $T_{\text{red}}$ by the projection $p : T_S \rightarrow T$.

(iv) By Remark (iv) in 1.2 there is a natural isomorphism

$$\mathcal{O}^\times(T_{\text{red}}) \times \mathcal{O}^\times(\pi_0(X)) \sim \pi_1(\mathcal{L}^\rho_{\text{dR}}(X, T; K)).$$

(1.5.4)

Here $(\alpha, \beta) \in \mathcal{O}^\times(T_{\text{red}}) \times \mathcal{O}^\times(\pi_0(X))$ acts on $\mathcal{E}_{(D, c, \nu p)}$ as multiplication by the locally constant function $\text{Nm}_{T_{\text{red}}/S}(\alpha)\text{Nm}_{\text{red}}(\nu p)/S(\beta^{\deg(D)})$. Notice that the embedding $\mathcal{O}^\times(T_{\text{red}}) \rightarrow \pi_1(\mathcal{L}^\rho_{\text{dR}}(X, T; K))$ comes from (1.5.3).

1.6. As in 1.1, every $(D, c) \in \mathcal{D}(S)$, $S$ is smooth, yields a morphism $K(D)_{\text{dR}}^{\times} \rightarrow \mathcal{D}^0$, hence a Picard functor $\mathcal{L}^\rho_{\text{dR}}(\mathcal{D}^0) \rightarrow \mathcal{L}_c(K(D)^{\times}D, c) \rightarrow \mathcal{E}_{(D, c)}$. In particular, following Examples in 1.1, for $\ell \in \mathbb{Z}$ we have $\mathcal{E}^{(\ell)} := \mathcal{E}_{(\Delta, 0)}^{(\ell)} \in \mathcal{L}_c(K^{(\ell)})$, and for $b \in T$, $n \in \mathbb{Z}$, we have $\mathcal{E}^{(n)}(b) := \mathcal{E}_{(nb, 1_b)} \in \mathcal{L}_c(K(nb)^{\times}T_b)$. Set $\mathcal{E}^{(0)} := \mathcal{E}^{(0)}(b) \in \mathcal{L}_c(K^{(0)}T_b)$. Notice that $\mathcal{E}^{(0)} \in \mathcal{L}_c(K^{(0)})$ is canonically trivialized.

If $E \in \mathcal{L}^\rho_{\text{dR}}(\mathcal{D}^0)$, then the $\mathcal{O}$-lines $\mathcal{E}^{(\ell)}$ carry a canonical connection along the fibers of the projection $\mathcal{E}^{(\ell)} \rightarrow X \setminus T$ (see Example (i) in 1.1). A de Rham structure on $E$ provides a flat connection $\nabla^\varepsilon$ on $\mathcal{E}^{(\ell)}$ that extends this relative connection. Since the degrees of lines are locally constant, the factorization implies

$$\deg(\mathcal{E}^{(\ell)}) = \ell \deg(\mathcal{E}^{(1)}) = \deg(\mathcal{E}^{(n+1)}) = \deg(\mathcal{E}^{(n)}) + \deg(\mathcal{E}^{(1)}).$$

(1.6.1)

Let $\mathcal{L}_{\text{dR}}(X, T) \subset \mathcal{L}_{\text{dR}}(X \setminus T)$ be the Picard subgroupoid of those de Rham lines whose connection at every $b \in T_{\text{red}}$ has pole of order less or equal the multiplicity of $T$ at $b$.

For a $\mathbb{G}_m$-torsor $K$ over a scheme $Y$ we denote by $\mathcal{L}_{\text{dR}}(Y; K)$ the Picard groupoid of de Rham lines $G$ on $K$ such that $G^{\otimes 2}$ is constant along the fibers (i.e., comes from a de Rham line on $Y$) and the fiberwise monodromy of $G$ equals $(-1)^{\deg G}$. Thus we have Picard groupoids $\mathcal{L}_{\text{dR}}(X \setminus T)^{(\ell)} := \mathcal{L}_{\text{dR}}(X \setminus T; K^{(\ell)})$, $\ell \in \mathbb{Z}$; let $\mathcal{L}_{\text{dR}}(X, T)^{(\ell)} \subset \mathcal{L}_{\text{dR}}(X \setminus T)^{(\ell)}$ be the Picard subgroupoids of those $G$ that $G^{\otimes 2} \in \mathcal{L}_{\text{dR}}(X, T)$.

Choose a trivialization $\nu_T$ of the restriction of $K$ to $T$, i.e., a collection $\{\nu_T^b\}$ of $k_T$-points in $K^{\times}_T$. For a factorization line $E$ set $\mathcal{E}_{\nu_T^b} := \mathcal{E}_{(0, 1_b, \nu_T^b)} = \mathcal{E}_b$ at $\nu_T^b$. The next theorem is the main result of this section. The proof for $T = \emptyset$ is in 1.7–1.9; the general case is treated in 1.10–1.11. In 1.11 one finds its reformulation free from the auxiliary $\nu_T$. 

\vspace{1em}
Theorem. For \( E \in \mathcal{L}_{dR}^\Phi(X, T; K) \) one has \( E^{(1)} \in \mathcal{L}_{dR}(X, T)^{(1)} \), and the functor
\[
\mathcal{L}_{dR}^\Phi(X, T; K) \to \mathcal{L}_{dR}(X, T)^{(1)} \times \prod_{b \in T_{\text{red}}} \mathcal{L}_{k_b}, \quad E \mapsto (E^{(1)}, \{ \nu_{T_b} \}),
\]
(1.6.2)
is an equivalence of Picard groupoids.

If \( K = \omega_X \), then the \( \mathbb{G}_m \)-torsor \( K^{(1)} \) is trivialized by a canonical section \( \nu_1 \) (its value at \( x \in X \setminus T \) is the element in \( \omega(x)/\omega \) with residue 1). The functor \( \nu_1^* : \mathcal{L}_{dR}(X, T)^{(1)} \to \mathcal{L}_{dR}(X, T) \) is evidently an equivalence, so the theorem can be reformulated as follows (here \( E^{(1)}_{X,T} := \nu_1^* E^{(1)} = E_{(\Delta, 0, \nu_1)}^\Phi \)):

Theorem'. One has a Picard groupoid equivalence
\[
\mathcal{L}_{dR}^\Phi(X, T; \omega_X) \simeq \mathcal{L}_{dR}(X, T) \times \prod_{b \in T_{\text{red}}} \mathcal{L}_{k_b}, \quad E \mapsto (E^{(1)}_{X,T}, \{ \nu_{T_b} \}).
\]
(1.6.3)

Variant. More generally, we can fix a divisor \( \Sigma_b \) supported on \( T \) and take for \( \nu_T \) a trivialization of \( K(\Sigma_b \omega) \) on \( T \). The corresponding assertion is equivalent to the above theorem by Remark (ii) in 1.2.

Example. If \( K = \omega_X \) and \( T = T_{\text{red}} \), then a convenient choice is \( \omega_k \equiv 1 \), for \( K(b \omega_k) \) is canonically trivialized by \( \nu_1 \) as above. We denote the fiber of \( E^{(1)}_{T_b} \) at \( \nu_1 \) by \( E^{(1)}_b \).

1.7. For the subsections 1.7–1.9 we assume that \( T = \emptyset \), so \( \mathcal{D} = \text{Div}(X) \).

For a Picard stack \( \mathcal{L}_\gamma \) and a commutative monoid space \( \mathcal{D} \) denote by \( \text{Hom}(\mathcal{D}, \mathcal{L}_\gamma) \) the Picard groupoid of symmetric monoidal morphisms \( \mathcal{D} \to \mathcal{L}_\gamma \) (we view \( \mathcal{D} \) as a “discrete” symmetric monoidal stack). Thus an object of \( \text{Hom}(\mathcal{D}, \mathcal{L}_\gamma) \) is \( \mathcal{F} \in \mathcal{L}_\gamma(\mathcal{D}) \) together with a multiplication structure which is a rule that assigns to every finite collection \( \{ D_a \} \) of \( S \)-points of \( \mathcal{D} \) a multiplication isomorphism \( \otimes D_a \to \mathcal{F}_{\mathcal{D}_a} \) (where \( \Sigma \) is the operation in \( \mathcal{D} \)); the isomorphisms should be compatible with base change and satisfy an evident transitivity property. If \( \mathcal{D}^{\text{gr}} \) is the group completion of \( \mathcal{D} \), then \( \text{Hom}(\mathcal{D}^{\text{gr}}, \mathcal{L}_\gamma) \to \text{Hom}(\mathcal{D}, \mathcal{L}_\gamma) \).

We are interested in \( \mathcal{D} \) equal to the monoid space of effective divisors \( \text{Div}^{\text{eff}}(X) = \sqcup \text{Sym}^a(X) \subset \text{Div}(X) \); one has \( \text{Div}^{\text{eff}}(X)^{\text{gr}} = \text{Div}(X) \). A multiplication structure on \( \mathcal{F} \) being restricted to disjoint divisors makes a factorization structure. Pulling \( \mathcal{F} \) back to \( \mathcal{D}^{\text{gr}} \) is a Picard functor
\[
\text{Hom}(\text{Div}(X), \mathcal{L}_\gamma) \to \mathcal{L}_\gamma^{\text{gr}}(X; K).
\]
(1.7.1)
Let \( \mathcal{L}_{dR}^\Phi \subset \mathcal{L}_{dR} \) be the Picard stack of degree 0 de Rham lines.

Proposition. One has \( \text{Hom}(\text{Div}(X), \mathcal{L}_{dR}^\Phi) \simeq \mathcal{L}_{dR}^\Phi(X; K) \).

Proof. (a) Let us show that for any \( E \in \mathcal{L}_{dR}^\Phi(X; K) \) the de Rham lines \( E^{(t)} \) on \( \mathcal{K}^{(t)} \) come from de Rham lines on \( X \).

The claim is \( X \)-local, so we trivialize \( K \) by section a \( \nu_0 \) and pick a function \( t \) on \( X \) with \( dt \) invertible. Then \( \mathcal{K}^{(t)} \) is trivialized by section \( \nu_0^{(t)} := \nu_0 dt_{0(t)}^{-t} \); let \( z \) be the corresponding fiberwise coordinate on \( \mathcal{K}^{(t)} \). Choose \( X \)-locally a trivialization \( e^{(t)} \) of \( E^{(t)} \); let \( \theta^{(t)} \in \omega_{\mathcal{K}^{(t)}}/X \) be the restriction of \( \nabla(e^{(t)})/e^{(t)} \) to the fibers. Then \( \theta^{(t)} = \sum f_k^{(t)}(x) z^k d \log z \), where \( f_k^{(t)}(x), k \in \mathbb{Z} \), are functions on \( X \); we want to show that \( f_k^{(t)}(x) = 0 \) for \( k \neq 0 \bmod 2 \) and \( f_0^{(t)}(x) \in \mathbb{Z} \).

The proof uses only the factorization \( \mathcal{O} \)-line structure on \( E \).
Let $S \subset X \times X$ be a sufficiently small neighborhood of the diagonal, $x_1, x_2$ be the coordinate functions on $S$ that correspond to $t, D^{(t, t)} \in \text{Div}(X)(S)$ be the divisor $t \Delta_1 + t \Delta_2$. Then $(t-x_1)^{t_1}(t-x_2)^{-t_2} \nu_0$ is a trivialization of $K(D^{(t, t)})$ near $|D^{(t, t)}|$. Denote by $\nu^{(t, t)}$ the corresponding section of $K(D^{(t, t)}) \times_{D^{(t, t)}, 0}$ (see 1.1); set $\mathcal{K}^{(t, t)} := \mathbb{G}_{m, 0}^{(t, t)} \subset K(D^{(t, t)}) \times_{D^{(t, t)}, 0}, \mathcal{E}^{(t, t)} := \mathcal{E}_{D^{(t, t)}, 0}|_{\mathcal{K}^{(t, t)}}$.

Outside the diagonal in $S$, one has an embedding $i^{(t, t)} : \mathcal{K}^{(t, t)} \hookrightarrow pr_1^* \mathcal{K}^{(t)} \times pr_2^* \mathcal{K}^{(t)}$ defined by the factorization; explicitly, it identifies $\nu^{(t, t)}(x_1, x_2)$ with $(z(x_1 - x_2)^{-t_2} \nu^{(t_1)}(x_1), z(x_2 - x_1)^{-t_1} \nu^{(t_2)}(x_2))$. Restricting to $\mathcal{K}^{(t, t)}$ the image of $e^{(t_1)} \otimes e^{(t_2)}$ by the factorization isomorphism (1.5.1), we get a trivialization $e^{(t, t)}$ of $\mathcal{E}^{(t, t)}$ outside the diagonal in $S$. Let $m(t_1, t_2)$ be its order of pole at the diagonal, so $(x_1 - x_2)^{m(t_1, t_2)} e^{(t_1, t_2)}$ is a trivialization of $\mathcal{E}^{(t_1, t_2)}$ on $S$. Therefore the restriction $\theta^{(t_1, t_2)}$ of $\nabla(e^{(t_1, t_2)})/e^{(t_1, t_2)}$ to the fibers is a regular relative form, which equals $\theta^{(t_1, t_2)}(\theta^{(t_1)} + \theta^{(t_2)}) = (f_1^{(t_1)}(x_1)(x_1 - x_2)^{-t_1} + f_2^{(t_2)}(x_1)(x_2 - x_1)^{-t_2}) \log z$.

Since $\theta^{(t_1, t_2)}$ has no pole at the diagonal, the above formula implies that $f_k^{(t)} = 0$ for $k \ell < 0$. Similarly, the formula for $\theta^{(t_2, t)}$ shows that $f_k^{(t)} = 0$ for $k \ell > 0$. To see that $f_k^{(t)} \in \mathbb{Z}$, notice that the above picture for $t_1 = t_2 = \ell$ is symmetric with respect to the transposition involution $\sigma$ of $X \times X$, hence descends to $S/\sigma \subset \text{Sym}^2 X$.

Thus $m(t, \ell)$ is even. One has $\nabla(e^{(t)})/e^{(t)} = f_0^{(t)} d \log z + g^{(t)}(x) dx$ where $g^{(t)}(x)$ is a regular function. Then $\nabla(e^{(t)})/e^{(t)} + d \log(x_1 - x_2)^{m(t, \ell)}$ is a regular 1-form. It equals $(f_0^{(t)}(x_1) + f_1^{(t)}(x_2)) (d \log z - d \log(x_1 - x_2)) + m(t, \ell) d \log(x_1 - x_2) + g^{(t)}(x_1) dx_1 + g^{(t)}(x_2) dx_2$. Therefore $f_0^{(t)}(x) = m(t, \ell)/2 \in \mathbb{Z}$, and we are done.

(b) The next properties of de Rham lines will be repeatedly used. Let $\pi : K \to S$ be a smooth morphism of smooth schemes with dense image.

**Lemma.** (i) The functor $\pi^* : \mathcal{L}_{dR}(S) \to \mathcal{L}_{dR}(K)$ is faithful. If the geometric fibers of $\pi$ are connected (say, $\pi$ is an open embedding), then $\pi^*$ is fully faithful.

(ii) If, in addition, $\pi$ is surjective, then a de Rham line $E$ on $K$ comes from $S$ if (and only if) this is true over a neighborhood $U$ of the generic point(s) of $S$. □

(c) As was mentioned, $\text{Div}(X)$ is the group completion of $\text{Div}^\text{eff}(X) = \sqcup \text{Sym}^n(X)$. So we have the projection $\text{Div}^\text{eff}(X) \times \text{Div}^\text{eff}(X) \to \text{Div}(X), (D_1, D_2) \mapsto D_1 - D_2$, which identifies $\text{Div}(X)$ with the quotient of $\text{Div}^\text{eff}(X) \times \text{Div}^\text{eff}(X)$ with respect to the diagonal action. Therefore a line $E$ on $\text{Div}(X)$ is the same as a collection of lines $E^{n_1, n_2}$ on $\text{Sym}^{n_1, n_2}(X) := \text{Sym}^{n_1}(X) \times \text{Sym}^{n_2}(X)$ together with $\text{Div}(X)$-equivariance structure, which is the datum of identifications of their pull-backs by $\text{Sym}^{n_1, n_2}(X) \rightarrow \text{Sym}^{n_1, +n_2}(X) \times \text{Sym}^{n_3}(X) \rightarrow \text{Sym}^{n_1+n_2+n_3}(X), (D_1, D_2) \mapsto (D_1 + D_2, D_3) \rightarrow (D_1 + D_3, D_2 + D_3)$ that satisfy a transitivity property.

Let us prove the proposition. We need to show that any $E \in \mathcal{L}_{dR}(X; K)$, viewed as a mere de Rham line on $\mathcal{D}^\circ$, is the pull-back by $\mathcal{D}^\circ \to \mathcal{D}(X)$ of a uniquely defined line in $\mathcal{L}_{dR}(\mathcal{D}(X))$, which we denote by $E$ or $E_{\text{div}}$, and that the factorization structure on $E$ comes from a uniquely defined multiplication structure on $E_{\text{div}}$.

We use the fact that for any $D \in \text{Div}(X)(S)$, $S$ is smooth, the projection $K(D)^\times := K(D^\times)_{S, 0} \to S$ satisfies the conditions of (i), (ii) of the lemma.
To define $\mathcal{E}_{\text{Div}}$ on $S = \text{Sym}^{n_1,n_2}(X)$, we apply (ii) of the lemma to $\mathcal{E}$ on $K = K(D_1 - D_2)\times_{D_1 + D_2}$. Let $U$ be the complement to the diagonal divisor in $X^{n_1} \times X^{n_2}$. Over $U$ our $\mathcal{E}$ equals $(\mathcal{E}(1))^{\otimes n_1} \otimes (\mathcal{E}(-1))^{\otimes n_2}$, and we are done by (a). The $\text{Div}_{\text{eff}}(X)$-equivariance structure on the datum of $\mathcal{E}^{n_1,n_2}_{\text{Div}}$ is automatic by (i) of the lemma (applied to $K(D_1 - D_2)\times_{D_1 + D_2 + 2D_0}$).

The factorization structure on $\mathcal{E}$ yields one on $\mathcal{E}_{\text{Div}}$. Let us show that it extends uniquely to a multiplication structure. It suffices to define the multiplication $\otimes \mathcal{E}_{D_{\alpha}} \to \mathcal{E}_{D_{\alpha}}$ over each $\Pi \text{Sym}^{n_1,n_2}_{\text{eff}}(X)$ in a way compatible with the $\text{Div}_{\text{eff}}(X)$-equivariance structure. Our multiplication equals factorization over the open dense subset where all $D_{\alpha}$ are disjoint, so we have it everywhere by (i) of the lemma. The compatibility with $\text{Div}_{\text{eff}}(X)$-equivariance holds over the similar open subset of $\Pi(\text{Sym}^{n_1,n_2}_{\text{eff}}(X) \times \text{Sym}^{n_2}_{\text{eff}}(X))$, hence everywhere, and we are done. 

\[\square\]

**Corollary.** The functor $L^{\Phi}_{\text{dR}}(X; K) \to L^{0}_{\text{dR}}(X)$, $E \mapsto E^{(1)}$, is an equivalence.

**Proof.** Its composition with the equivalence of the proposition is a functor $\text{Hom}(\text{Div}_{\text{eff}}(X), L^{\Phi}_{\text{dR}}) \to L^{0}_{\text{dR}}(X)$ which assigns to $\mathcal{F}$ its restriction to the component $X = \text{Sym}^{1}(X)$ of $\text{Div}_{\text{eff}}(X)$. This functor is clearly invertible: its inverse assigns to $P \in L^{0}_{\text{dR}}(X)$ the de Rham line $\text{Sym}(P)$ on $\text{Div}_{\text{eff}}(X) = \sqcup \text{Sym}^{n}_{\text{eff}}(X)$, $\text{Sym}(P)\text{Sym}^{1}(X)$ equipped with an evident multiplication structure. $\square$

**1.8.** An example of a de Rham factorization line $\mathcal{E}$ with $\deg(E^{(1)}) = 1$:

Suppose $X = \mathbb{A}^1$, $T = \emptyset$, $K = \mathcal{O}_X$. We construct $\mathcal{E}$ in the setting of Remark 1.3. For a smooth $S$ and $D \in \text{Div}(S)$ the line $\mathcal{O}_X(D)$ is naturally trivialized by a section $\nu_D, \nu_{\Sigma_n x_i} = \Pi(t - x_i)^{-n_i}$. Then $\nu_D$ trivializes the $\mathcal{O}^{\times}_{D,0}$-torsor $K(D)^{\times}_{D,0}$, so the canonical character $f \mapsto f(D)$ of $O^{\times}_{D,0}$ yields an invertible function $\phi_D \in \mathcal{O}^{\times}(K(D)^{\times}_{D,0})$, $\phi_D(n) := (n/\nu_D)(D)$. Our $\mathcal{E}_{D,0}$ comes from the Kummer torsor for $\phi_D^{-1/2}$ placed in degree $\deg(D)$, i.e., it equals $\mathcal{O}_{K(D)^{\times}_{D,0}}[\deg(D)]$ as an $\mathcal{O}$-line, the connection is given by the $1$-form $d\log \phi_D$.

Notice that if $D' \in \text{Div}(S)$ is another divisor such that $|D| \cap |D'| = \emptyset$, then the invertible function $\nu_{D'}$ on $X_S \setminus |D|$ yields $(D, D') := \nu_D(D') \in \mathcal{O}^{\times}(S)$. One has

\[(D, D') = (-1)^{\deg(D)\deg(D')} (D', D).\]  

\(1.8.1\)

Let us define the factorization structure on $\mathcal{E}$. Suppose $D = \sqcup D_{\alpha}$, so $K(D)^{\times}_{D,0} = \Pi K(D_{\alpha})^{\times}_{D_{\alpha},0}$. Any linear order on the set of indices $\alpha$ yields an evident identification of the “constant” $\mathcal{O}$-lines $\otimes \mathcal{E}_{D_{\alpha},0} \rightarrow \mathcal{E}_{D,0}$. The factorization isomorphism (1.5.1) is its product with $\Pi (D_{\alpha}, D_{\alpha'})$. The choice of order is irrelevant due to the “super” commutativity constraint and (1.8.1). Both transitivity property and horizontality follow since $\Pi \phi_{D_{\alpha}} = \phi_D \Pi_{\alpha \neq \alpha'} (D_{\alpha}, D_{\alpha'})$.

**1.9. Proof of the theorem in 1.6 in case $T = \emptyset$.** Let us check that for $\mathcal{E} \in L^{\Phi}_{\text{dR}}(X; K)$ one has $\mathcal{E}^{(1)} \in L^{0}_{\text{dR}}(X)^{(1)}$. The claim is $X$-local, so we can assume that $K$ is trivialized and there is a function $t$ on $X$ with $dt$ invertible, i.e., $t : X \to \mathbb{A}^1$ is étale. Let $\mathcal{E}' \in L^{\Phi}_{\text{dR}}(X; K)$ be the pull-back of the factorization line from 1.8. Then $L^{\Phi}_{\text{dR}}(X; K)$ is generated by $L^{\Phi}_{\text{dR}}(X; K)$ and $\mathcal{E}'$. Our claim holds for $\mathcal{E} \in L^{\Phi}_{\text{dR}}(X; K)$ by 1.7 and it is evident for $\mathcal{E}'$; we are done.
Let us show that $\mathcal{L}_{dR}^{\nu}(X; K) \to \mathcal{L}_{dR}(X)^{(1)}$, $\mathcal{E} \mapsto \mathcal{E}^{(1)}$, is an equivalence. Notice that the preimage of $\mathcal{L}_{dR}^{\nu}(X) \subset \mathcal{L}_{dR}(X)$ equals $\mathcal{L}_{dR}^{\nu}(X; K)$, and, by the corollary in 1.7, $\mathcal{L}_{dR}^{\nu}(X; K) \cong \mathcal{L}_{dR}^{\nu}(X)$. Since $X$-locally there is $\mathcal{E}$ with $\deg(\mathcal{E}^{(1)}) = 1$ by 1.8 and $\deg : \pi_0(\mathcal{L}_{dR}(X))/\pi_0(\mathcal{L}_{dR}^{\nu}(X)) \cong \mathbb{Z}$, we are done. 

1.10. Suppose now $T \neq \emptyset$. Pick $b \in T_{\text{red}}$, and consider the $O_{T_b}^{\times}$-torsor $K_{T_b}^{\times}$ (see Example (ii) in 1.1; we follow the notation of loc.cit.). Let $(\omega)_b := \omega_X(\infty b)/\omega_X$ be the $k_b$-vector space of polar parts of rational 1-forms at $b$, $(\omega)_b^{\leq n}$ be the subspace of polar parts of order $\leq n$. The Lie algebra of $O_{T_b}^{\times}$ equals $\mathcal{O}(T_b)$. The space $\Omega^1(K_{T_b}^{\times})^{\text{inv}}$ of translation invariant 1-forms on $K_{T_b}^{\times}$ is its dual. The residue pairing $(\omega)_b^{\leq m_b} \times \mathcal{O}(T_b) \to k_b$, $(\psi, f) \mapsto \text{Res}_b(f\psi)$, identifies it with $(\omega)_b^{\leq m_b}$. So one has

$$\Omega^1(K_{T_b}^{\times})^{\text{inv}} \cong (\omega)_b^{\leq m_b}. \tag{1.10.1}$$

Let $U$ be a smooth affine curve over $k_b$, $u \in U$ a closed point; as above, we set $(\omega)_u := \omega_U(\infty u)/\omega_U$. Let $\xi : U^o := U \setminus \{u\} \to K_{T_b}^{\times}$ be a $k_b$-morphism, which amounts to a trivialization $\nu^\xi$ of $K$ on $T_bU^o \subset X_{U^o}$. Denote by $(\xi)$ the composition $(\omega)_b^{\leq m_b} \cong \Omega^1(K_{T_b}^{\times})^{\text{inv}} \xrightarrow{\xi^*} \omega(U^o) \to \omega(U)/\omega(U) = (\omega)_u$.

**Lemma.** (i) After a possible localization of $U$ at $u$, one can find $D \in \text{Div}(X)(U)$ and a trivialization $\nu$ of $K(D)$ on a neighborhood $V \subset X_U$ of $|D| \cup T_{BU}$ such that $|D| \cap X_u$ is supported at $b$, $|D| \cap T_{BU} = \emptyset$, and $\nu|_{T_{BU}} = \nu^\xi$.

(ii) Suppose $U$ is a neighborhood of $b$, i.e., we have an ´etale $\pi : U \to X$, $\pi(u) = b$. Then one can find $(D, \nu)$ as in (i) with $D$ equal to (the graph of) $\pi$ if and only if $-(\xi)$ equals $\pi^*$, i.e., the composition $(\omega)_b^{\leq m_b} \subset (\omega)_b^{\pi^*} \to (\omega)_u$.

**Proof.** (i) Let us extend $\nu^\xi$ to a rational section $\nu$ of $K$ on an open subset $V$ of $X_U$, $V \supset T_{BU}$, which is defined at $T_{BU}$. Shrinking $U$ and $V$, one can find $\nu$ with $D := \text{div}(\nu)$ prime to $X_u$ (if $n$ is the multiplicity of $X_u$ in $D$, then we replace $\nu$ by $f^{-n}\nu$, where $f$ is any rational function which equals 1 on $T_{BU}$ and whose divisor contains $X_u$ with multiplicity 1). After further localization of $U$ and shrinking of $V$, we get $D \in \text{Div}(X)(U)$ and $|D| \cap X_u$ is supported at $b$; we are done.

(ii) A map $\phi : U^o \to K_{T_b}^{\times}$ extends to $U$ if and only if for every $\beta \in \Omega^1(K_{T_b}^{\times})^{\text{inv}}$ the form $\phi^*(\beta) \in \omega(U^o)$ is regular at $b$. Thus either of the properties of $\xi$ in the assertion of (ii) determines $\nu^\xi$ uniquely up to multiplication by an invertible function on $T_{BU}$. It remains to present a trivialization $\nu$ of $K(\pi)$ such that the corresponding $\xi$ satisfies $-(\xi) = \pi^*$.

Shrinking $X$, we trivialize $K$ and pick a function $t$ with $dt$ invertible; set $x := \pi^*(t) \in \mathcal{O}(U)$. Our $\nu$ is $(t - x)^{-1}$. The differential of the corresponding $\xi$ is the Lie($O_{T_b}^{\times}$) = $\mathcal{O}(T_b)$-valued 1-form $\nu^{-1}d_x\nu = -(1 + t/x + (t/x)^2 + \ldots)dx/x$. So if $\beta \in \Omega^1(K_{T_b}^{\times})^{\text{inv}}$ is identified with $\psi(t) \in (\omega)_b$ by (1.10.1), then $\xi^*(\beta) = -(\text{Res}_b(1 + t/x + (t/x)^2 + \ldots)\psi(t))dx/x = -\psi(x)$, q.e.d.

\footnote{To find such $f$ (after possible shrinking of $U$), pick local coordinate $t$ on $X$ at $b$, and $x$ on $U$ at $u$ (so $t(b) = 0 = x(u)$, $dt(b) \neq 0 \neq dx(u)$); set $f = x(x - t^{m_b})^{-1}$.}
11. A de Rham line \( \mathcal{F} \) on \( K_{T_a}^\chi \) is said to be translation invariant if the de Rham line \( \cdot \mathcal{F} \circ pr_2^* \mathcal{F}^{\otimes -1} \) lies in \( pr_1^* \mathcal{L}_{dr}(O_{T_a}^\chi) \subset \mathcal{L}_{dr}(O_{T_a}^\chi \times K_{T_a}^\chi) \); here \( \cdot : O_{T_a}^\chi \times K_{T_a}^\chi \rightarrow K_{T_a}^\chi \) is the action map, \( pr_i \) are the projections to the factors. Such \( \mathcal{F} \)'s form a Picard subgroupoid \( \mathcal{L}_{inv}^{dr}(K_{T_a}^\chi) \) of \( \mathcal{L}_{dr}(K_{T_a}^\chi) \).

**Lemma.** (i) We trivialize \( K_{T_a}^\chi \), i.e., identify it with \( O_{T_a}^\chi \). The translation invariance of \( \mathcal{F} \) is equivalent to the next properties:

(a) The de Rham line \( pr_1^* \mathcal{F} \otimes pr_2^* \mathcal{F} \otimes \cdot \mathcal{F}^{\otimes -1} \) is constant;

(b) For any smooth curve \( U \), a point \( u \in U \), and two maps \( \xi_1, \xi_2 : U^\circ := U \setminus \{ u \} \rightarrow K_{T_a}^\chi \), the de Rham line \( \xi_1^* \mathcal{F} \otimes \xi_2^* \mathcal{F} \otimes (\xi_1 \xi_2)^* \mathcal{F}^{\otimes -1} \) on \( U^\circ \) extends to \( U \).

(c) For some (or every) invertible section \( e_\mathcal{F} \) of \( \mathcal{F} \) on \( K_{T_a}^\chi \) one has \( \nabla(e_\mathcal{F})/e_\mathcal{F} \in \Omega^1(K_{T_a}^\chi)^{inv} \). (ii) There is a natural isomorphism \( \pi_0(\mathcal{L}_{inv}^{dr}(K_{T_a}^\chi)) \cong \mathbb{Z} \times (\omega_b)^{\leq m_b}/\mathbb{Z} \) where \( \mathbb{Z} \subset (\omega_b)^{\leq m_b} \) are polar parts of 1-forms with simple pole and integral residue.

**Proof.** (a) (i) is evidently equivalent to invariance of \( \mathcal{F} \). Since \( K_{T_a}^\chi \) is a rational variety, (a) amounts to the fact that \( \cdot \mathcal{F} \circ pr_2^* \mathcal{F}^{\otimes -1} \otimes pr_2^* \mathcal{F}^{\otimes -1} \) extends to a compactification of \( K_{T_a}^\chi \). This can be tested on curves, which is (b). Finally (c) is equivalent to the translation invariance since every invertible function on \( O_{T_a}^\chi \) is the product of a character by a constant, and every line bundle on \( K_{T_a}^\chi \) is trivial.

(ii) One assigns to \( \mathcal{F} \) the pair \((n, \psi)\) where \( n = \deg(\mathcal{F}) \) and \( \psi \) is the class of the image of \( \nabla(e_\mathcal{F})/e_\mathcal{F} \) by (1.10.1).

We say that \( \mathcal{G} \in \mathcal{L}_{dr}(X \setminus T)^{(1)} \) is compatible with \( \mathcal{F} \in \mathcal{L}_{inv}^{dr}(K_{T_a}^\chi) \) for some neighborhood \( U \) of \( b \), a trivialization \( \nu \) of \( K \) on \( U \), and a map \( \xi : U^\circ := U \setminus \{ b \} \rightarrow K_{T_a}^\chi \) as in (ii) of the lemma in 1.10, the \( \mathcal{F} \) line \( \nu^* \mathcal{G} \otimes \xi^* \mathcal{F} \) on \( U^\circ \) extends to \( U \) (the validity of this does not depend on the choice of \( U, \nu, \) and \( \xi \)). By loc.cit., compatibility is equivalent to the next condition: Pick \( U, \nu \), and \( e_\mathcal{F} \) above; let \( e_\mathcal{G} \) be a non-zero rational section of \( \nu^* \mathcal{G} \). Then the image of \( \nabla(e_\mathcal{F})/e_\mathcal{F} \) by (1.10.1) is \( (\omega_b)^{\leq m_b}/\mathbb{Z} \subset (\omega_b)/\mathbb{Z} \) equals the class of \( \nabla(e_\mathcal{G})/e_\mathcal{G} \).

Let \( \mathcal{L}_{dr}^{\Phi}(X, T; K) \) be the Picard subgroupoid of \( \mathcal{L}_{dr}(X \setminus T)^{(1)} \times \prod_{b \in T_{red}} \mathcal{L}_{inv}^{dr}(K_{T_b}^\chi) \) formed by those collections \((\mathcal{G}, \{ \mathcal{F}_{T_b} \})\) that \( \mathcal{G} \) is compatible with every \( \mathcal{F}_{T_b} \). Then \( \mathcal{G} \) lies automatically in \( \mathcal{L}_{dr}(X, T)^{(1)} \). By (ii) of the lemma, the functor \( \mathcal{L}_{dr}^{\Phi}(X, T; K) \rightarrow \mathcal{L}_{dr}(X, T)^{(1)} \times \prod \mathcal{L}_{k_b}, (\mathcal{G}, \{ \mathcal{F}_{T_b} \}) \mapsto (\mathcal{G}, \{ \mathcal{F}_{\nu_{T_b}} \}) \), where \( \mathcal{F}_{\nu_{T_b}} \) is the fiber of \( \mathcal{F}_{T_b} \) at \( \nu_{T_b} \) from 1.6, is an equivalence of categories. Thus the theorem in 1.6 follows from the next one:

**Theorem.** For every \( \mathcal{E} \in \mathcal{L}_{dr}^{\Phi}(X, T; K) \) one has \( \mathcal{E}^{(1)}, \{ \mathcal{E}_{T_b} \} \in \mathcal{L}_{dr}^{(1)}(X, T; K) \), and the functor

\[
\mathcal{L}_{dr}^{\Phi}(X, T; K) \rightarrow \mathcal{L}_{dr}^{(1)}(X, T; K); \quad \mathcal{E} \mapsto (\mathcal{E}^{(1)}, \{ \mathcal{E}_{T_b} \}),
\]

is an equivalence of the Picard groupoids.

**Proof.** The assertion is \( X \)-local, and we have proved it for \( T = \emptyset \). So we can assume that \( T_{red} \) is a single \( k \)-point \( b \). Thus \( T_b = T \) and \( D \) is the disjoint sum of \( D_{c=0} \) equal to \( \text{Div}(X \setminus T) \) and \( D_{c=1} \) equal to \( \text{Div}(X) \). If needed, we can assume that \( K \) is trivialized and there is an étale map of \( X \rightarrow \mathbb{A}^1 \).

(a) Let us show that \((\mathcal{E}^{(1)}, \mathcal{E}_{T_b}) \in \mathcal{L}_{dr}^{(1)}(X, T; K) \). Notice that \( \mathcal{L}_{dr}^{\Phi}(X, T; K) \) is generated by \( \mathcal{L}_{dr}^{\Phi}(X, T; K) \), the image of (1.5.3), and the pull-back by \( t \) of the
factorization line on $\mathbb{A}^1$ from 1.8. Since the assertion is evident for factorization lines of the latter two types, it suffices to consider the case of $E \in \mathcal{L}^{0\phi}_{\text{dR}}(X; T; K)$.

We know that $\mathcal{E}^{(1)}$ comes from a de Rham line on $X \setminus T$ (see 1.7). Let us check that $\mathcal{E}_T$ is translation invariant using the criterion of (i)(b) in the lemma. For $U$, $u$, $\xi$, as in loc.cit., let us choose $D_i$, $\nu_i$ as in (i) of the lemma in 1.10; then $D_3 := D_1 + D_2$, $\nu_3 := \nu_1 \nu_2$ serves $\xi_3 := \xi_1 \xi_2$. The lines $\mathcal{E}_{D, 1}(X)$ on $U$ are equal to $\mathcal{E}_D \times \xi^*_D \mathcal{E}_T$ on $U^\circ$ by factorization; here $\mathcal{E}_D := \mathcal{E}_{D, 0}$. Since $\mathcal{E}_D = \mathcal{E}_{D_1} \otimes \mathcal{E}_{D_2}$ by 1.7, the de Rham line $\xi_1^* \mathcal{E}_T \otimes \xi_2^* \mathcal{E}_T \otimes (\xi_1 \xi_2)^* \mathcal{E}_{T^+}^{\otimes -1}$ on $U^\circ$ extends to $U$, q.e.d.

It remains to check that $\mathcal{E}^{(1)}$ is compatible with $\mathcal{E}_T$. Let $\nu$ be a trivialization of $K(\Delta)$ on an open $V \subset X \times X$ that contains $(b, b)$, $U := V \cap \{b\} \times X$, $\xi : U^\circ \to K_X^\times$ is translation invariant using the criterion of (i)(b) in the lemma.

As in 1.7, the de Rham line $\xi \mathcal{E}_T$ extends to $U \times U^\circ$. The de Rham line $\mathcal{E}_T$ comes from a uniquely defined de Rham line on $K_X^\times \times \text{Div}(X)$ which we denote by $\mathcal{E}_1$ or $\mathcal{E}_{\text{Div1}}$.

We use the fact that for every $D \in \text{Div}(X)(S)$, $S$ smooth, the projection $K(D)_{D, 1} : K_X^\times \times S$ satisfies the conditions of (i), (ii) of the lemma in 1.7.

As in part (c) of the proof in 1.7, we need to define $\mathcal{E}_{\text{Div1}}$ on every $K_X^\times \times \text{Sym}^{n_1, n_2}(X)$ and provide the $\text{Div}^{\text{eff}}(X)$-equivariance structure. Consider our $\mathcal{E}$ on $K(D_1 - D_2)^{\times}_{D_1 + D_2}$. Over $K^\times_X \times \text{Sym}^{n_1, n_2}(X \setminus T)$ it equals $\mathcal{E}_T \boxtimes \mathcal{E}_{(D_1 - D_2, 0)}$ by factorization, hence it descends to $K^\times_X \times \text{Sym}^{n_1, n_2}(X \setminus T)$ by 1.7. By (ii) of the lemma in 1.7, we have $\mathcal{E}_{\text{Div1}}$ on the whole $K^\times_X \times \text{Sym}^{n_1, n_2}(X)$. The $\text{Div}^{\text{eff}}(X)$-equivariance structure is unique determined by (i)(b) of the lemma (applied to $K(D_1 - D_2)^{\times}_{D_1 + D_2}$). As in (b), $\mathcal{E}_1 := \mathcal{E}_{\text{Div1}}$ comes from $K^\times_X \times \text{Div}(X)$. By factorization, its restriction to $K^\times_X \times \text{Div}(X \setminus T)$ equals $\mathcal{E}_T \boxtimes \mathcal{E}_0$. It remains to show that $\mathcal{E}_T \boxtimes \mathcal{E}_0$ extends in a unique way to a de Rham line $\mathcal{E}_1$ on $K^\times_X \times \text{Div}(X)$.

As in (c) of the proof in 1.7, we should define $\mathcal{E}_1$ on every $K^\times_X \times \text{Sym}^{n_1, n_2}(X)$ and provide the $\text{Div}^{\text{eff}}(X)$-equivariance structure. Our $\mathcal{E}_1$ is defined on an open dense subset $U$ of triples $(\xi, D_1, D_2)$, $D_i \in \text{Sym}^{n_i}(X \setminus T)$. Let $U' \supset U$ be the open subset of those $(\xi, D_1, D_2)$ that $D_1 + D_2$ contains $b$ with multiplicity at most 1. Then $\mathcal{E}$ extends to $U'$ due to compatibility of $\mathcal{E}^{(1)}$ and $\mathcal{E}_T$. Since the complement to $U'$ has codimension $\geq 2$, $\mathcal{E}$ extends to $K^\times_X \times \text{Sym}^{n_1, n_2}(X)$, and we are done.

As in loc.cit., the $\text{Div}^{\text{eff}}(X)$-equivariance structure extends naturally to the pull-backs of our line by $K^\times_X \times \text{Sym}^{n_1, n_2}(X) - K^\times_X \times \text{Sym}^{n_1, n_2}(X) \times \text{Sym}^{n_3}(X) \to K^\times_X \times \text{Sym}^{n_1 + n_3, n_2 + n_3}(X)$. The two de Rham lines coincide on the dense open subset $K^\times_X \times \text{Sym}^{n_1, n_2}(X \setminus T) \times \text{Sym}^{n_3}(X \setminus T)$, so they are canonically identified everywhere, and we are done. The factorization structure on $\mathcal{E}$ is evident.
(d) By (e), the theorem is reduced to the claim that our functor yields an isomorphism between the quotients
\[ \pi_0(\mathcal{L}^\phi_{\text{dR}}(X,T;K))/\pi_0(\mathcal{L}^\phi_{\text{dR}}(X,T;K)) \simeq \pi_0(\mathcal{L}^\phi_{\text{dR}}(X,T;K))/\pi_0(\mathcal{L}^\phi_{\text{dR}}(X,T;K)). \]
The degree map identifies the right group with \( \mathbb{Z} \times \mathbb{Z} \). Our map is evidently injective; looking at the image of (1.5.3) and the pull-back by \( t \) of the factorization line on \( \mathbb{A}^1 \) from 1.8, we see that it is surjective, q.e.d.

1.12. A complement. A connection on a trivialized line bundle amounts to a 1-form; multiplying the trivialization by \( f \), we add to the form \( d \log f \). Here is a similar fact in the factorization story.

Consider the group \( \pi_1(\mathcal{L}^\phi_{\text{dR}}(X,T;K)) \) of invertible functions on \( \mathcal{D}^\circ \) that satisfy factorization property. One has evident embeddings \( \mathcal{O}^\times(T_{\text{red}}) \rightarrow \pi_1(\mathcal{L}^\phi_{\text{dR}}(X,T;K)) \), \( \mathcal{L}^\phi_{\text{dR}}(X,T;K) \) see Remarks (iii), (iv) in 1.5. Let \( \mathcal{L}^\phi_{\text{dR}}(X,T;K)^{\text{O-triv}} \) be the kernel of the Picard functor \( \mathcal{L}^\phi_{\text{dR}}(X,T;K) \rightarrow \mathcal{L}^\phi_{\text{dR}}(X,T;K) \). This is a mere abelian group (since the functor is faithful); its elements are pairs \( (\mathcal{E}, e) \) where \( \mathcal{E} \) is a factorization de Rham line, \( e \) is a trivialization of \( \mathcal{E} \) as a factorization \( \mathcal{O} \)-line. Let \( \omega(X,T) \) be the space of 1-forms on \( X \setminus T \) whose order of pole at any \( b \in T \) is less or equal to the multiplicity of \( T \) at \( b \).

**Proposition.** There is a natural commutative diagram
\[ \begin{array}{ccc}
\pi_1(\mathcal{L}^\phi_{\text{dR}}(X,T;K))/\mathcal{O}^\times(T_{\text{red}}) & \cong & \mathcal{O}^\times(X \setminus T) \\
\downarrow & & \downarrow \\
\mathcal{L}^\phi_{\text{dR}}(X,T;K)^{\text{O-triv}} & \cong & \omega(X,T).
\end{array} \] (1.12.1)

**Proof.** (a) The connection on \( \mathcal{E}^{(t)} \) along the fibers of \( \mathcal{K}^{(t)}/X \setminus T \) is determined solely by the \( \mathcal{O} \)-line structure. So the action of any \( h \in \pi_1(\mathcal{L}^\phi_{\text{dR}}(X,T)) \) on \( \mathcal{E}^{(t)} \) is fiberwise horizontal, i.e., it is multiplication by a function \( h^{(t)} \in \mathcal{O}^\times(X \setminus T) \). The top horizontal arrow is \( h \mapsto h^{(1)} \).

For the same reason, for \( (\mathcal{E}, e) \in \mathcal{L}^\phi_{\text{dR}}(X,T;K)^{\text{O-triv}} \) the trivializations \( e^{(t)} \) of \( \mathcal{E}^{(t)} \) are fiberwise horizontal, i.e., \( \nabla(e^{(1)})(\xi,e^{(1)}) \in \omega(X \setminus T) \). By the theorem in 1.6, \( \nabla(e^{(1)})/e^{(1)} \in \omega(X,T) \). The bottom horizontal arrow is \( (\mathcal{E}, e) \mapsto (\nabla^{(1)})(e^{(1)})/e^{(1)} \).

The map \( \pi_1(\mathcal{L}^\phi_{\mathcal{O}}(X,T;K)) \rightarrow \mathcal{L}^\phi_{\text{dR}}(X,T;K)^{\text{O-triv}}, f \mapsto (\mathcal{O}^\circ,f1) \), with kernel \( \pi_1(\mathcal{L}^\phi_{\text{dR}}(X,T;K)) \) yields the left vertical arrow. The right one is the \( d \log \) map.

The diagram is evidently commutative. It remains to check that its horizontal arrows are isomorphisms.

(b) For every \( h \in \pi_1(\mathcal{L}^\phi_{\text{dR}}(X,T)) \) its restriction to \( \mathcal{D}^\circ_{c=0} \) comes from a multiplicative function \( h_0 \) on \( \text{Div}(X \setminus T) \). Similarly, if \( T_{\text{red}} \) is a single \( k \)-point \( b \), then the restriction of \( h \) to \( \mathcal{D}^\circ_{c=1} \) comes from a function \( h_1 \) on \( K^X_T \times \text{Div}(X) \) such that for \( \xi \in K^X_T, D \in \text{Div}(X \setminus T) \) one has \( h_1(\xi,D) = h_1(\xi)h_0(D) \). This follows by a simple modification of the argument from, respectively, part (c) of the proof in 1.7 and part (b) of the proof in 1.11. The details are left to the reader.

Let us show that the map \( \pi_1(\mathcal{L}^\phi_{\mathcal{O}}(X,T))/\mathcal{O}^\times(T_{\text{red}}) \rightarrow \mathcal{O}^\times(X \setminus T) \) is injective. Suppose we have \( h \) such that \( h^{(1)} = 1 \). Since the group space \( \text{Div}(X \setminus T) \) is generated by effective divisors of degree 1, one has \( h_0 = 1 \). It remains to check that \( h \) is locally constant on other components of \( \mathcal{D}^\circ \). The assertion is \( X \)-local, so we can assume that \( T_{\text{red}} \) is a single \( k \)-point \( b \), and we look at \( \mathcal{D}^\circ_{c=1} \). By above, it suffices to check that the restriction \( h_T \) of \( h_1 \) to \( K^X_T \) is constant. We use (i) of the lemma 1.10; we follow the notation of loc.cit. For \( \xi : U^\circ \rightarrow K^X_T \), consider \( h_{(D,1,\nu)}(U) \in \mathcal{O}^\times(U); \)
by factorization, its restriction to $U^o$ equals $\xi^*h_Th_{(D,0,\nu)} = \xi^*h_T$. Since $\xi^*h_T$ is regular at $u$ for every $\xi$, $h_T$ is constant, q.e.d.

A similar argument shows that the bottom horizontal arrow in (1.12.1) is injective. The details are left to the reader.

(c) Let us construct a section $\mathcal{O}^\times (X \setminus T) \to \pi_1(\mathcal{L}_{S}^\times(X,T), f \mapsto \tilde{f}$, of the map $h \mapsto h^{(1)}$. Fix a trivialization $\nu_0$ of $K$ on an open subset $V_0$ of $X$ that contains $T$. For $(D, c, \nu_P) \in \mathcal{D}^c(S)$ let us define $\tilde{f}_{(D, c, \nu_P)} \in \mathcal{O}^\times(S)$. Pick $\nu, V$ corresponding to $(D, c, \nu_P)$ as in Remark (ii) in 1.1; we can assume that $V \cap T_S = T_S^\times$. Localizing $S$, we can decompose $D$ in a disjoint sum of $D'$ and $D''$ such that $D' \subset V \setminus T_S$ and $D'' \subset V_0$. Set

$$\tilde{f}_{(D, c, \nu_P)} := f(D')\{f, \nu_0/\nu\}_{|D''\setminus T_S^\times}. \quad (1.12.2)$$

Here $\{f, \nu_0/\nu\}_{|D''\setminus T_S^\times} \in \mathcal{O}^\times(S)$ is the Contou-Carrère symbol at $|D''| \cup T_S^\times$ (see [CC] or [BBE] 3.3). One readily checks that (1.12.2) does not depend on the auxiliary choices of $\nu$ and the decomposition $D = D' + D''$; its compatibility with the factorization is evident. So $\tilde{f} \in \pi_1(\mathcal{L}_{S}^\times(X,T))$; clearly $\tilde{f}^{(1)} = f$.

Remark. If $\nu_0'$ is another trivialization of $K$ near $T$, $f \mapsto \tilde{f}'$ the corresponding section, then $\tilde{f}/\tilde{f}'$ is an element of $\mathcal{O}^\times(T_{red}) \subset \pi_1(\mathcal{L}_{S}^\times(X,T))$ whose value at $b \in T_{red}$ equals $(\nu_0/\nu_0')(b)^{\nu_0}$. Where $\div(f) = \Sigma_{n,b}b$.

(d) To finish the proof, let us construct explicitly a section of the bottom horizontal arrow in (1.12.1). For $\phi \in \omega(X,T)$, we construct the corresponding $\mathcal{E}^\phi = (\mathcal{E}^\phi, e) \in \mathcal{L}_{\mathcal{D}^c}(X,T;K)^{\mathcal{O}\text{-triv}}$ using Remark (i) in 1.5. Since $\mathcal{E}^\phi$ is trivialized as an $\mathcal{O}$-line, $\alpha^\phi$ of (1.5.2) is multiplication by a function $\alpha^\phi = \alpha_{(D, c, \nu_P)}(D, c, \nu_P) \in \mathcal{O}^\times(S)$. To determine it, we extend $\nu_P, \nu_P'$ to $\nu, \nu'$ as in Remark (ii) in 1.1 such that $\nu$ equals $\nu'$ on $V_{red}$. Then $\nu/\nu' \in \mathcal{O}^\times(V\setminus P)$ equals 1 on $V_{red}$, so we have a function $\log(\nu/\nu') \in \mathcal{O}(V\setminus P)$ that vanishes on $V_{red}$. The residue $\Res_{P/S}(\log(\nu/\nu')\phi) \in \mathcal{O}(S)$ vanishes on $S_{red}$, and we set

$$\alpha^\phi := \exp(\Res_{P/S}(\log(\nu/\nu')\phi)). \quad (1.12.3)$$

Our $\alpha^\phi$ does not depend on the auxiliary choice of $\nu$ and $\nu'$: Indeed, $\nu, \nu'$ can be changed to $f\nu, f'\nu'$ with $f, f' \in \mathcal{O}^\times(V)$ that coincide on $V_{red}$ and equal 1 on $T_S^\times$ (see Remark (ii) in 1.1); then $\log(f/f')$ is a regular function on $V$ that vanishes on $T_S^\times$, so $\Res_{P/S}(\log(f/f')\phi) = 0$, and we are done. The transitivity of $\alpha^\phi$ and compatibility with base change and factorization are evident; we have defined $\mathcal{E}^\phi$.

Remark. Suppose we have $(D, c, \nu_P) \in \mathcal{D}^c(S)$ where $S$ is smooth. The de Rham structure on $\mathcal{E}^\phi_{(D,c,\nu_P)}$ amounts to a flat connection $\nabla^\phi$ on our line bundle, which is the same as a closed 1-form $\theta^\phi = \nabla^\phi(e)/e$ on $S$. Choose $\nu$ as in Remark (ii) in 1.1; then (1.12.3) implies that

$$\theta^\phi = \Res_{P/S}((d_S(\nu)/\nu) \otimes \phi). \quad (1.12.4)$$

Here $d_S$ means derivation along the fibers of the projection $V \subset X \to S$, so $d_S(\nu)$ is a section of $\Omega^1_S \boxtimes K$ over $V \setminus P$, and $d_S(\nu)/\nu$ is a section of the pull-back of $\Omega^1_S$ to $V \setminus P$. Of course, due to the lemma in 1.3, one can use (1.12.4) as an alternative definition of $\mathcal{E}^\phi$.

Example. Consider the $\mathcal{O}$-trivialized de Rham line $(\mathcal{E}^\phi_{T_b}, e)$ on the $O_{T_b}^\times$-torsor $K_{T_b}^\times$ (see 1.6). Its 1-form $\theta^\phi = \nabla(e)/e$ is translation invariant and corresponds to the functional $f \mapsto \Res_{T_b}(f\phi)$ on the Lie algebra $\mathcal{O}(T_b)$ of $O_{T_b}^\times$ (cf. 1.11).
It remains to check that the bottom horizontal arrow in (1.12.1) sends $(E^0, e)$ to $\phi$, i.e., that $\nabla(e^{(1)})/e^{(1)} = \phi$. Pick a local trivialization of $K$ and a local function $t$ on $X \setminus T$ with non-vanishing $dt$; let $x$ be the corresponding local function on $S = X \setminus T$. Then $\nu = (t - x)^{-1}$ is a trivialization of $K^*_S(\Delta)$ near the diagonal, so we have the de Rham line $E^\phi_{(\Delta, \nu, \nu)}$ on $S$. Then $d_S(\nu)/\nu = (t - x)^{-1} dx$, so, by (1.12.4), one has $\nabla(e^{(1)})/e^{(1)} = \theta^\phi = (\Res_{t=x}((t - x)^{-1})\phi) dx = \phi$, q.e.d. \hfill \Box

**Corollary.** For $E, E' \in \mathcal{L}^\phi_{\text{dR}}(X, T; K)$, a morphism $E \to E'$ in $\mathcal{L}^\phi_{\text{dR}}(X, T; K)$ is horizontal, i.e., is a morphism in $\mathcal{L}^\phi_{\text{dR}}(X, T; K)$, if (and only if) the corresponding morphism $E^{(1)} \to E'^{(1)}$ of $\mathcal{O}$-lines on $K^{(1)}$ is horizontal. \hfill \Box

**Remark.** If $K = \omega_X$, then in the corollary one can replace $\phi^{(1)}$ by the morphism $\phi^{(1)}_{X|T} : E^{(1)}_{X|T} \to E'^{(1)}_{X|T}$ of $\mathcal{O}$-lines on $X \setminus T$ (see 1.6).

1.13. The next lemma will be used in 2.12. Assume that $X \setminus T$ is affine and $K = \omega_X$. The Lie algebra $\Theta(X \setminus T)$ of vector fields on $X \setminus T$ acts naturally on $\mathcal{D}^\omega(X, \bar{T}; \omega) := \lim_{\to} \mathcal{D}^\omega(X, nT; \omega)$. Therefore we have the notion of $\Theta(X \setminus T)$-action on any $\mathcal{O}$-line $E$ on $\mathcal{D}^\omega(X, \bar{T}; \omega)$. If $E$ carries a factorization structure, then one can ask our action to be compatible with it. Ditto for a de Rham structure.

Suppose that $E$ is a de Rham line. The flat connection yields then a $\Theta(X \setminus T)$-action on $E$, which we refer to as the standard action. It is evidently compatible with the de Rham structure.

**Lemma.** Any $\Theta(X \setminus T)$-action $\tau$ on $E$ compatible with the de Rham structure equals the standard one.

**Proof.** Let $\tau_0$ be the standard action. Then $\theta \mapsto \tau(\theta) - \tau_0(\theta)$ is a Lie algebra homomorphism from $\Theta(X \setminus T)$ to the Lie algebra of de Rham line endomorphisms of $E$. The latter Lie algebra is commutative; the former one is perfect. Thus our homomorphism is $0$, i.e., $\tau = \tau_0$. \hfill \Box

1.14. The whole story makes sense in the relative setting. The input is a smooth (not necessary proper) $Q$-family of curves $q : X \to Q$ (where $Q$ is a scheme), a relative divisor $T \subset X$ such that $T_{\text{red}}$ is finite and étale over $Q_{\text{red}}$, and a line bundle $K$ on $X$. It yields a space $\mathcal{D} = \mathcal{D}^\omega(X/Q, T; K)$ over $Q$. If $L\mathcal{C}$ is any sheaf of Picard groupoids on the category $\mathcal{S}ch_{/Q}$ of $Q$-schemes (equipped with the étale topology), then we have the Picard groupoid $L\mathcal{C}(\mathcal{D})$ and $L\mathcal{C}(\mathcal{D}^\omega)$ defined as in 1.2, and the Picard groupoid of factorization objects $L\mathcal{C}(\mathcal{D}) (X/Q, T; K)$ as in 1.5. The $L\mathcal{C}$’s we need are $L\mathcal{O}, L\mathcal{O}/Q$, and $L\mathcal{dR}$, where $L\mathcal{O}, L\mathcal{dR}$ are as in 1.2, and $L\mathcal{dR}/Q(S)$ is formed by $\mathcal{O}$-lines equipped with an action of the universal relative formal groupoid on $S/Q$ (if $S/Q$ is smooth, then this is the same as a flat relative connection, cf. 1.1). All the results above immediately generalize to the relative setting. Thus, as in 1.4, for proper $q$ every $E$ in $L\mathcal{O}(\mathcal{D})$ or in $L\mathcal{dR}/Q(\mathcal{D})$ yields an $\mathcal{O}$-line $E(X/Q)$ on $Q$; if $E$ lies in $L\mathcal{dR}(\mathcal{D})$, then $E(X/Q) \in L\mathcal{dR}(Q)$. The theorem in 1.6 remains valid both in the setting of $L\mathcal{dR}/Q$ and $L\mathcal{dR}$, etc.
1.15. Suppose $k = \mathbb{C}$, and $X$ is any complex smooth curve. All the above definitions and results render into the analytic setting without problems. In fact, the story simplifies since $\mathcal{L}_{dR}^\varphi(X, T_{red}) \cong \mathcal{L}_{dR}^{\varphi}(X, T)$ It is equivalent to the Betti version of the story with de Rham lines replaced by local systems of $\mathbb{C}$-lines. And the Betti version works if we replace this $\mathbb{C}$ by any commutative ring $R$ of coefficients.

Remark. The fact that $\mathcal{L}_{dR}^\varphi(X, T; K) = \mathcal{L}_{dR}^{\varphi}(X, T_{red}; K)$ permits to consider in case $K = \omega_X$ a canonical equivalence (1.6.3) as in Example 1.6.

Every de Rham factorization line $\mathcal{E}$ in the analytic setting carries a canonical automorphism $\mu$ which acts on $\mathcal{E}_{(d,c,\nu)}$ as multiplication by the (counterclockwise) monodromy of the de Rham line $\mathcal{E}_{(d,c,\nu)}$, $z \in \mathbb{C}^\times$, around $z = 0$. Notice that $\mu$ is multiplicative, i.e., we have a homomorphism $\mu : \pi_0(\mathcal{L}_{dR}^\varphi(X, T)) \to \pi_1(\mathcal{L}_{dR}^\varphi(X, T))$. Same is true for the Betti factorization lines.

2 The de Rham $\varepsilon$-lines: algebraic theory

This section recasts the story of [Del] and [BBE] in the factorization line format.

2.1. We follow the setting and notation of 1.1, so $X$ is a smooth curve over $k$, $T \subset X$ is a finite subscheme. From now on we assume that $K$ from 1.1 equals $\omega = \omega_X$, so $\mathcal{D}^\omega = \mathcal{D}^\omega(X; \omega)$.

Let $M$ be a (left) holonomic $\mathcal{D}$-module on $(X, T)$, i.e., a holonomic module on $X$ which is smooth on $X \setminus T$. We say that $T$ is compatible with $M$ if $\det M_{X \setminus T} \in \mathcal{L}_{dR}(X, T)$ (see 1.6).

Theorem-construction. $M$ defines naturally a de Rham factorization line $\mathcal{E}_{dR}(M) \in \mathcal{L}_{dR}(X, T)$ with $\mathcal{E}_{dR}(M)^{(1)} = (\det M_{X \setminus T})^{\otimes -1}$. It is functorial with respect to isomorphisms of $M$’s, has local origin, and lies in $\mathcal{L}_{dR}(X, T)$ if $T$ is compatible with $M$. For proper $X$ there is a canonical isomorphism of $\varepsilon$-lines $\eta_{dR} : \mathcal{E}_{dR}(M)(X) \cong \det H_{dR}(X, M)$. The construction is compatible with base change of $k$, filtrations on $M$, and direct images for finite morphisms of $X$’s.

We construct $\mathcal{E}_{dR}(M)$ as a factorization $\mathcal{O}$-line in 2.5, and define a de Rham structure on it in 2.10. The identification $\mathcal{E}_{dR}(M)^{(1)} \cong (\det M_{X \setminus T})^{\otimes -1}$ is established in (2.6.1); we check that it is horizontal in 2.11. $\eta_{dR}$ is defined in 2.7, the compatibilities are discussed in 2.8. The compatibility of $T$ with $M$ becomes relevant only in 2.10. Let us begin with necessary preliminaries.

2.2. $\mathcal{L}$-groupoids and $\mathcal{L}$-torsors: a dictionary. Let $\mathcal{L}$ be a Picard groupoid with the product operation $\otimes$. Below “$\mathcal{L}$-groupoid” means “enriched category over $\mathcal{L}$”. Thus this is a collection of objects $J$ and a rule which assigns to every $j, j' \in J$ an object $\lambda(j/j') \in \mathcal{L}$, and to every $j, j', j'' \in J$ a composition isomorphism $\lambda(j/j') \otimes \lambda(j'/j'') \cong \lambda(j/j'')$ which satisfies associativity property. Then $J$ is automatically a mere groupoid with $\text{Hom}(j', j) := \text{Hom}(1_{\mathcal{L}}, \lambda(j/j'))$.

Let $J_1, J_2$ be $\mathcal{L}$-groupoids. Their tensor product $J_1 \otimes J_2$ is an $\mathcal{L}$-groupoid whose objects $j_1 \otimes j_2$ correspond to pairs $j_1 \in J_1, j_2 \in J_2$, $\lambda(j_1 \otimes j_2/j'_1 \otimes j'_2) := \lambda(j_1/j'_1) \otimes \lambda(j_2/j'_2)$, and the composition $\lambda(j_1 \otimes j_2/j'_1 \otimes j'_2) \otimes \lambda(j'_1 \otimes j'_2/j''_1 \otimes j''_2) \rightarrow \lambda(j_1 \otimes j_2/j''_1 \otimes j''_2)$ equal to $(\lambda(j_1/j'_1) \otimes \lambda(j_2/j'_2) \otimes \lambda(j'_1/j''_1) \otimes \lambda(j'_2/j''_2)) \rightarrow (\lambda(j_1/j'_1) \otimes \lambda(j_1/j'_1) \otimes \lambda(j_2/j'_2) \otimes \lambda(j_2/j'_2))$ where the first arrow is the

\footnote{Since for any $\mathcal{L}$-torsor $K/S$ the pull-back functor $\mathcal{L}_{dR}(S) \rightarrow \mathcal{L}_{dR}(K)$ is an equivalence.}
functor is the composition \( \text{Hom}_L(J_1, J_2) \to L \) identification of \( J \mapsto \lambda \). The composition \( \lambda(j/j') \) is compatible with the composition. A \( \phi \) yields a morphism of mere groupoids \( J_1 \to J_2 \). All \( \mathcal{L} \)-morphisms form naturally an \( \mathcal{L} \)-groupoid \( \text{Hom}_L(J_1, J_2) \). Precisely, there is an \( \mathcal{L} \)-groupoid structure on \( \text{Hom}_L(J_1, J_2) \) together with an \( \mathcal{L} \)-morphism \( \text{Hom}_L(J_1, J_2) \otimes J_1 \to J_2 \) that lifts the action map \( \text{Hom}_L(J_1, J_2) \times J_1 \to J_2 \) of mere groupoids, and such pair is unique (up to a unique 2-isomorphism). The composition \( \text{Hom}_L(J_2, J_3) \times \text{Hom}_L(J_1, J_2) \to \text{Hom}_L(J_1, J_3) \) lifts naturally to a morphism of \( \mathcal{L} \)-groupoids \( \text{Hom}_L(J_2, J_3) \otimes \text{Hom}_L(J_1, J_2) \to \text{Hom}_L(J_1, J_3), \) etc.

For an \( \mathcal{L} \)-groupoid \( J \) its inverse \( J^{\otimes -1} \) is an \( \mathcal{L} \)-groupoid whose objects are in bijection \( j \mapsto j^{\otimes -1} \) with elements of \( J \), and \( \lambda(j^{\otimes -1}/j^{\otimes -1}) = \lambda(j'/j) \).

There are two equivalent ways to define the notion of \( \mathcal{L} \)-torsor: (a) This is a mere groupoid \( F \) equipped with a \( \mathcal{L} \)-action, i.e., a functor \( \circ : \mathcal{L} \times F \to F \) together with an associativity constraint, such that for some (hence every) object \( j \in F \) the functor \( \mathcal{L} \to F, \ell \mapsto \ell \circ j \), is an equivalence of groupoids; (b) This is an \( \mathcal{L} \)-groupoid such that the image of \( \lambda \) meets every isomorphism class in \( \mathcal{L} \). To pass from (a) to (b), we lift the groupoid structure on \( F \) to \( \mathcal{L} \)-groupoid one with \( \lambda(f(f')) := f \otimes f'^{\otimes -1} \) (the latter is an object of \( \mathcal{L} \) together with an isomorphism \( f \otimes f'^{\otimes -1} \otimes f' \simeq f \); the pair is defined uniquely up to a unique isomorphism).

For a non-empty \( \mathcal{L} \)-groupoid \( J \) and an \( \mathcal{L} \)-torsor \( F \) both \( \mathcal{L} \)-groupoids \( F \otimes J \) and \( \text{Hom}_L(J, F) \) are \( \mathcal{L} \)-torsors. In particular, we have the product and ratio of \( \mathcal{L} \)-torsors (with \( \mathcal{L} \) being a unit). Notice that there is a natural equivalence \( F_1 \otimes F_2^{\otimes -1} \simeq \text{Hom}_L(F_2, F_1) \) which assigns to \( f_1 \otimes f_2^{\otimes -1} \) the \( \mathcal{L} \)-morphism \( f_1 \mapsto \lambda(f_1/f_2) \otimes f_1 \).

Remarks. (i) For any non-empty \( \mathcal{L} \)-groupoid \( J \) the \( \mathcal{L} \)-morphism \( J \to \mathcal{L} \otimes J, j \mapsto 1_{\mathcal{L}} \otimes j \), is a universal \( \mathcal{L} \)-morphism to an \( \mathcal{L} \)-torsor.

(ii) Every \( \mathcal{L} \)-morphism between \( \mathcal{L} \)-torsors is an equivalence. Thus for non-empty \( J \)'s every \( \mathcal{L} \)-morphism \( J_1 \to J_2 \) yields an equivalence of \( \mathcal{L} \)-torsors \( \text{Hom}_L(J_2, \mathcal{L}) \simeq \text{Hom}_L(J_1, \mathcal{L}) \) and \( \mathcal{L} \otimes J_1 \simeq \mathcal{L} \otimes J_2 \); in particular, the \( \mathcal{L} \)-torsor \( \text{Hom}_L(J, \mathcal{L}) \) does not change if we replace \( J \) by its non-empty subset. E.g. every \( j \in J \) yields an identification of \( \mathcal{L} \)-torsors \( \text{Hom}_L(J, \mathcal{L}) \simeq \mathcal{L}, \lambda \mapsto \lambda(j) \), and \( \mathcal{L} \simeq \mathcal{L} \otimes J, \ell \mapsto \ell \otimes J \).

(iii) If \( F_1 = \text{Hom}_L(J_1, \mathcal{L}) \) where \( J_1 \) are any \( \mathcal{L} \)-groupoids, then \( F_1 \otimes F_2^{\otimes -1} \) identifies naturally with the \( \mathcal{L} \)-torsor whose objects are maps \( \mu : J_1 \times J_2 \to \mathcal{L}, (j_1, j_2) \mapsto \mu(j_1/j_2) \), together with identifications \( \lambda(j_1/j_1) \otimes \mu(j_1/j_2) \otimes \lambda(j_2/j_2') \simeq \mu(j_1/j_2') \) which are associative with respect to the composition of \( \lambda \) on both \( J_1 \).

Here \( f_1 \otimes f_2^{\otimes -1} \) corresponds to \( \mu(j_1/j_2) := f_1(j_1) \otimes f_2(j_2)^{\otimes -1} \).

Suppose a group \( G \) acts on a non-empty \( \mathcal{L} \)-groupoid \( J \). Then it acts on the \( \mathcal{L} \)-torsor \( \mathcal{L} \otimes J \) by transport of structure. We can view this action as a monoidal functor \( g \to \lambda_g \) from \( G \) (considered as a discrete monoidal category) into the monoidal category \( \text{End}_L(\mathcal{L} \otimes J) \), which is naturally equivalent to \( \mathcal{L} \). Explicitly, \( \lambda_g := \lambda(g(1_{\mathcal{L}})/j), j \in J \); the product isomorphism \( \lambda_g \otimes \lambda_g' \simeq \lambda_{g_1g_2} \) is the composition \( \lambda(g_1(g_2(j))/g_2(j)) \otimes \lambda(g_2(j)/j) \simeq \lambda((g_1g_2)(j)/j) \). A monoidal functor \( G \to \mathcal{L} \) is sometimes called (central) \( \mathcal{L} \)-extension of \( G \) of \( \mathcal{L} \).

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\[12\] If \( \mathcal{L} \) is the Picard groupoid of \( A \)-torsors, \( A \) is an abelian group, then \( G^\circ \) amounts to a central extension of \( G \) by \( A \), which is the reason for the terminology.
The group $G$ acts on $G^2$ in adjoint way. Namely, for $h \in G$ the isomorphism $\text{Ad}_h : \lambda_g \overset{\sim}{\rightarrow} \lambda_{gh^{-1}}$ is the composition $\lambda_g \overset{\sim}{\rightarrow} \lambda_g \otimes \lambda_h \otimes \lambda_{h^{-1}} \overset{\sim}{\rightarrow} \lambda_h \otimes \lambda_g \otimes \lambda_{h^{-1}} \overset{\sim}{\rightarrow} \lambda_{gh^{-1}}$, the first arrow is the tensoring with the inverse to the composition map $1_L \overset{\sim}{\rightarrow} \lambda_h \otimes \lambda_{h^{-1}}$, the second one is the commutativity constraint. Equivalently, $\text{Ad}_h$ is determined by the condition that the composition $\lambda(g(j)/j) \overset{\sim}{\rightarrow} \lambda_g \text{Ad}_h \lambda_{gh^{-1}} \overset{\sim}{\rightarrow} \lambda(hgh^{-1}(h(j))/h(j)) = \lambda(h(g(j))/h(j))$ coincides with the action of $h$.

For commuting $g, h \in G$ denote by $\{g, h\} \in \pi_1(L) := \text{Aut}_L(1_L)$ their commutant in $G^2$, i.e., the action of $\text{Ad}_g$ on $\lambda_h$ or the ratio of $\lambda_g \otimes \lambda_h \to \lambda_{gh}$ and $\lambda_g \otimes \lambda_h \to \lambda_h \otimes \lambda_g \to \lambda_{gh} = \lambda_{gh}$ where the first and the last arrows are the product, the middle one is the commutativity constraint (cf. [BBE] A5).

### 2.3. A digression on lattices and relative determinants (see e.g. [Dr] §5).

Let $S$ be a scheme, $P$ be a relative effective divisor in $X_S/S$ finite over $S$. Let $E$ be any quasi-coherent $\mathcal{O}_{X_S}$-module such that for some neighborhood $V$ of $P$ the restriction of $E$ to $V \setminus P$ is coherent and locally free.

A $P$-lattice in $E$ is an $\mathcal{O}_{X_S}$-submodule $L$ of $E$ which is locally free (hence coherent) on a neighborhood of $P$, and equals $E$ on $X_S \setminus P$. Denote by $\Lambda_P(E)$ the set of $P$-lattices in $E$. We assume that it is non-empty. Then $\Lambda_P(E)$ is directed by the inclusion ordering. Since $P$ is finite over $S$, for every $P$-lattices $L \supset L'$ the $\mathcal{O}_S$-module $\pi_*(L/L')$ is locally free of finite rank.

$\Lambda_P(E)$ carries a natural $\mathcal{O}_S$-groupoid structure. Namely, for $P$-lattices $L, L'$ one has their relative determinant $\lambda_P(L/L') \in \mathcal{L}_\mathcal{O}(S)$; for $L, L', L''$ there is a canonical composition isomorphism

$$
\lambda_P(L/L') \otimes \lambda_P(L'/L'') \overset{\sim}{\rightarrow} \lambda_P(L/L'')
$$

which satisfies associativity property. This datum is uniquely determined by a demand that for $L \supset L'$ one has $\lambda_P(L/L') := \det \pi_*(L/L')$, and for $L \supset L' \supset L''$ the composition is the standard isomorphism defined by the short exact sequence $0 \to \pi_*(L'/L'') \to \pi_*(L/L'') \to \pi_*(L/L') \to 0$.

We denote by $\text{Det}_{P/S}(E)$ the $\mathcal{L}_\mathcal{O}(S)$-torsor $\text{Hom}_{\mathcal{O}_S}(\Lambda_P(E), \mathcal{L}_\mathcal{O}(S))$. Its objects, referred to as determinant theories on $E$ at $P$, are rules $\lambda$ that assigns to every $L \in \Lambda_P(E)$ a line $\mathcal{L}(L) \in \mathcal{L}_\mathcal{O}(S)$ together with identifications $\lambda_P(L/L') \otimes \lambda(L') \overset{\sim}{\rightarrow} \lambda(L)$ compatible with (2.3.1). Here one can replace $\Lambda_P(E)$ by any its non-empty subset. $\text{Det}_{P/S}(E)$ is compatible with base change change. For quasi-coherent $\mathcal{O}_X$-modules $E_1, E_2$ as above set

$$
\text{Det}_{P/S}(E_1/E_2) := \text{Det}_{P/S}(E_1) \otimes \text{Det}_{P/S}(E_2) \otimes \text{Det}_{P/S}(E_2)^{-1}.
$$

By Remark (iii) in 2.2, objects of this $\mathcal{L}_\mathcal{O}(S)$-torsor, referred to as relative determinants theories on $E_1/E_2$ at $P$, can be viewed as rules $\mu$ which assign to every $P$-lattices $L_1$ in $E_1$ and $L_2$ in $E_2$ a line $\mu(L_1/L_2) \in \mathcal{L}_\mathcal{O}(S)$ together with natural identifications

$$
\lambda_P(L'_1/L_1) \otimes \mu(L_1/L_2) \otimes \lambda_P(L_2/L'_2) \overset{\sim}{\rightarrow} \mu(L'_1/L'_2)
$$

which are associative with respect to composition (2.3.1) on the two sides. We can also restrict ourselves to $L_i$ in any non-empty subsets of $\Lambda_P(E_i)$.

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13Let us show that $\pi_*(L/L')$ is $\mathcal{O}_S$-flat. We can assume that $X$ is affine, $L, L'$ are locally free. Since $\pi_*(L/L') = \pi_*(L)/\pi_*(L')$ and $\pi_*L_*L'$ are $\mathcal{O}_S$-flat, it suffices to check that for any geometric point $s$ of $S$ the map $\pi_*(L')_s \to \pi_*(L)_s$ is injective. This is clear, since $\pi_*(L''')_s = \Gamma(X_s, L''')$ and $L'_s \to L_s$ is injective (being an isomorphism at the generic points of $X_s$).
Remarks. (i) Let \( E(\infty P) := \varprojlim E(nP) \) be the localization of \( E \) with respect to \( P \). An evident morphism of \( \mathcal{L}_S(S) \)-groupoids \( \Lambda_P(E) \to \Lambda_P(E(\infty P)) \) yields an equivalence \( \text{Det}_{P/S}(E) \cong \text{Det}_{P/S}(E(\infty P)) \); same for relative determinant theories. Thus \( \text{Det}_{P/S}(E) \) depends only on the restriction of \( E \) to the complement of \( P \). Notice that \( \Lambda_P(E(\infty P)) \) is directed by both inclusion ordering and the opposite one.

(ii) Let us order the set of pairs of \( P \)-lattices \((L_1, L_2)\) by the product of either of the inclusion orderings. Let \( I \) be any of its directed subsets. Suppose we have a rule \( \lambda \) that assigns to every \((L_1, L_2) \in I \) a line \( \lambda(L_1/L_2) \in \mathcal{L}_O(S) \) together with natural identifications \((2.3.2)\) defined for \((L_1, L_2) \geq (L'_1, L'_2)\) and associative for \((L_1, L_2) \geq (L'_1, L'_2) \geq (L''_1, L''_2)\). Then \( \lambda \) extends uniquely to a relative determinant theory.

(iii) The group \( \text{Aut}(E_{V\setminus P}) \) acts on \( \Lambda_P(E(\infty P)) \) as on an \( \mathcal{L}_O(S) \)-groupoid. As in 2.2, this defines an \( \mathcal{L}_O(S) \)-extension \( \text{Aut}(E_{V\setminus P})^\bullet \) of \( \text{Aut}(E_{V\setminus P}) \). Thus for commuting \( g', g \in \text{Aut}(E_{V\setminus P}) \) we have \( \{g', g\}^y = \{g', g\}_P \in \mathcal{O}^\times(S) \). Example: if \( g' \) is multiplication by a function \( f \in \mathcal{O}^\times(V \setminus P) \), then \( \{g', g\}_P \) equals the Contou-Carrére symbol \( \{\text{det} g, f\}_P \) at \( P \) (see e.g. [BBE] 3.3). In particular, if \( f \in \mathcal{O}^\times(V) \), then \( \{g', g\}_P = f(\text{div}(\text{det} g)) \); here \( \text{div}(\text{det} g) \) is the part of the divisor supported on \( V \), i.e., at \( P \).

\( P \)-lattices have local nature with respect to \( P \). In particular, if \( P \) is the disjoint sum of components \( P_\alpha \), then a \( P \)-lattice \( L \) amounts to a collection of \( P_\alpha \)-lattices \( L_\alpha \), and there is an evident canonical factorization isomorphism

\[
\otimes \lambda_{P_\alpha}(L_\alpha/L'_\alpha) \sim \lambda_P(L/L')
\quad \tag{2.3.3}
\]

compatible with composition \((2.3.1)\). Therefore \( \text{Det}_{P/S}(E) \) is the \( \mathcal{L}_O(S) \)-torsor product of \( \text{Det}_{P_\alpha/S}(E) \). So every collection of determinant theories \( \lambda_\alpha \) on \( E \) at \( P_\alpha \) yields a determinant theory \( \otimes \lambda_\alpha \) on \( M \) at \( P \), \( (\otimes \lambda_\alpha)(L) := \otimes \lambda_\alpha(L_\alpha) \). Same for relative determinant theories.

2.4. We return to the story of 1.1. Suppose we have \((D, c) \in \mathcal{D}(S) \). Let us apply the format of 2.3 to \( E_1 = M_S = M \otimes \mathcal{O}_S, E_2 = \omega M_S := (\omega \otimes M)_S \), and \( P = P_{D,c} \). The connection \( \nabla = \nabla_M : M \to \omega M \) yields a relative determinant theory \( \mu_M = (\mu(M))_P \in \text{Det}_{P/S}(M/\omega M) := \text{Det}_{P/S}(M_S/\omega M_S) \) defined as follows.

Let \( L, L_\omega \) be \( P \)-lattices in \( M_S, \omega M_S \) such that \( \nabla(L) \subset L_\omega \). Let \( \mathcal{C}(L, L_\omega) = \mathcal{C}(L, L_\omega)_{M,P} \) be the complex \( M_S/L \nabla \omega M_S/L_\omega \) in degrees \(-1 \) and \( 0 \), i.e., it is the quotient \( dR(M) \otimes \mathcal{O}_S/\text{Cone}(L \nabla \omega M_S/L_\omega) \). Then \( \pi_* \mathcal{C}(L, L_\omega) \) is a complex of quasi-coherent \( \mathcal{O}_S \)-modules.

Lemma. \( \pi_* \mathcal{C}(L, L_\omega) \) has \( \mathcal{O}_S \)-coherent cohomology.

Proof. We can assume that \( T_S = T_S \), since the assertion is \( S \)-local. Its validity does not depend on the choice of \( L, L_\omega \). Take them to be “constant” \( T \)-lattices equal to \( M, \omega M \) on \( X \setminus T \), and we are reduced to the case of \( S = \text{Spec } k, P = T \).

Let \( X \) be the smooth projective curve that contains \( X, T \infty := X \setminus X \). Let us extend \( M \) to a holonomic \( P \)-module on \( X \), which we denote also by \( M \); let \( L, L_\omega \) be

\footnote{A short proof: Both expressions are compatible with base change. Since the datum of \( E, V, P, g, f \) can be extended \( S \)-locally to a smooth base, we can assume that \( S \) is smooth. Then it suffices to check the equality at the generic point of \( S \), where the Contou-Carrére symbol is the usual tame symbol. The rest is a standard computation.}
T \cup T^\infty\text{-lattices as above. Since } \mathcal{C}(L, L_\omega)_{M,T^\cup T^\infty} = \mathcal{C}(L, L_\omega)_{M,T} \oplus \mathcal{C}(L, L_\omega)_{M,T^\infty}, \text{ it suffices to check that } \pi_* \mathcal{C}(L, L_\omega)_{M,T^\cup T^\infty} \text{ has finite-dimensional cohomology. Therefore we are reduced to the case of projective } X.\]

Now the lemma follows since \( \mathcal{C}(L, L_\omega) = dR(M)/\text{Cone}(L \circlearrowright L_\omega) \), and the cohomology of \( X \) with coefficients in \( dR(M) \), \( L, L_\omega \) are finite dimensional. \( \square \)

Pairs \((L, L_\omega)\) as above form a directed set \( I_C \) as in Remark (ii) in 2.3. Set

\[
\mu^\nu_P(L/L_\omega) := \det \pi_* \mathcal{C}(L, L_\omega). \tag{2.4.1}
\]

If \((L', L'_\omega) \in I_C \) is such that \( L \supset L', L_\omega \supset L'_\omega \), then the short exact sequence

\[
0 \to \text{Cone}(L'/L' \circlearrowright L_\omega/L'_\omega) \to \mathcal{C}(L', L'_\omega) \to \mathcal{C}(L, L_\omega) \to 0
\]

together with the identification

\[
\det \pi_* \text{Cone}(L/L' \circlearrowright L_\omega/L'_\omega) = \det \pi_* (L_\omega/L'_\omega) \otimes \det \pi_* (L/L')^{-1}
\]

yields an isomorphism

\[
\lambda_P(L'/L) \otimes \mu^\nu_P(L/L_\omega) \otimes \lambda_P(L_\omega/L'_\omega) \sim \mu^\nu_P(L'/L'_\omega)
\]

which evidently satisfies associativity property. By Remark (ii) in 2.3, we have defined \( \mu^\nu_P = \mu(M/\omega M) \mid_P \in \text{Det}_{P/S}(M/\omega M). \)

Remarks. (i) Sometimes it is convenient to consider a smaller directed set formed by pairs \((L, L_\omega)\) that equal \( M, \omega M \) outside \( T_\infty \), or its subset of those \((L, L_\omega)\) that are locally constant with respect to \( S \).

(ii) Denote by \( j_T \) the embedding \( X \setminus T \hookrightarrow X \). Then \( j_{T*} M := j_{T*} M_{X \setminus T} \) is holonomic \( D \)-module as well; by Remark (i) from 2.3, one has \( \text{Det}_{P/S}(M/\omega M) = \text{Det}_{P/S}(j_{T*} M/\omega j_{T*} M) \). Suppose \( T_\infty = T_S \). Then \( L, L_\omega \) are \( P \)-lattices in \( j_{T*} M, \omega j_{T*} M \), and \( \text{Cone}(\mathcal{C}(L, L_\omega)_{M,P}, \mathcal{C}(L, L_\omega)_{j_{T*} M,P}) \sim dR(\text{Cone}(M \to j_{T*} M)) \otimes O_S \).

Thus \( \mu(M/\omega M) \mid_P = \mu(j_{T*} M/\omega j_{T*} M) \mid_P \otimes \det Rl_{\text{dR}}(X, M) \).

The above construction has local nature with respect to \( P \). If \( P \) is disjoint sum of components \( P_\alpha \), and \( L, L_\omega \) correspond to collections of \( P_\alpha \)-lattices \( L_\alpha, L_\omega \), then \( \mathcal{C}(L, L_\omega) = \oplus \mathcal{C}(L_\alpha, L_\omega) \). Passing to the determinants, we see that

\[
\mu(M/\omega M) \mid_P = \otimes \mu(M/\omega M) \mid_P. \tag{2.4.3}
\]

2.5. Now we can construct the promised \( \mathcal{E}_{\text{dR}}(M) \) as a factorization \( O \)-line.

For \((D,c) \in \mathcal{S}(S)\), any trivialization \( \nu \) of \( \omega(D) \) on a neighborhood \( V \) of \( P = P_{D,c} \) yields naturally \( \mu^\nu_P = \mu(M/\omega M) \mid_P \in \text{Det}_{P/S}(M/\omega M) \). Namely, the multiplication by \( \nu \) isomorphism \( M_V \otimes \omega \overset{\sim}{\to} \omega M_V \otimes \nu \) identifies the \( L \mathcal{O}(S) \)-groupoids \( \Lambda_P(M(\infty P)) \sim \Lambda_P(\omega(M(\infty P))) \); passing to \( \text{Det}_{P/S} \), we get \( \mu^\nu_P \).

Explicitly, for every \( P \)-lattices \( L, L_\omega \) one has a canonical identification

\[
\mu^\nu_P(L/L_\omega) \sim \lambda_P(\nu L/L_\omega) = \lambda_P(\omega(L(D)/L_\omega)), \tag{2.5.1}
\]

and identifications (2.3.2) are (2.3.1) combined with the isomorphism

\[
\nu_{L/L'} : \lambda_P(L/L') \sim \lambda_P(\nu L/L')
\]

that comes from multiplication by \( \nu \).

The r.h.s. of (2.5.1) does not depend on \( \nu \). Thus every \( L \) provides an identification \( e_L \) between \( \mu^\nu_P \) for all trivializations \( \nu \) of \( \omega(D) \) near \( P \); it is characterized by property that (2.5.1) transforms \( e_L(L/L_\omega) \) into the identity map for \( \lambda_P(\omega(L(D)/L_\omega)) \).

Exercise. Let \( \nu_1, \nu_2 \) be any trivializations of \( \omega(D) \) on \( V \); set \( f := \nu_2/\nu_1 \in \mathcal{O}^X(V) \). Consider the identifications \( e_{L'}, e_{L'} : \mu^\nu_{P'} \sim \mu^\nu_{P'} \). Show that

\[
e_{L'} = f(\text{div}(L/L'))e_L. \tag{2.5.2}
\]
Here \( \text{div}(L/L') := \text{div}(\phi_{L'}/\phi_L) \), where \( \phi_L, \phi_{L'} \) are local trivializations of the line bundles \( \det(L), \det(L') \) at \( P \); this is a relative Cartier divisor supported at \( P \).

If \( \nu_1|_P = \nu_2|_P \), i.e., \( f|_P = 1 \), then we define a canonical identification
\[
e : \mu^\nu_1_P \sim \mu^\nu_2_P
\]
as follows. The function \( f(D) \in \mathcal{O}^\times(S) \) equals 1 on \( S_{\text{red}} \) since \( f|_P = 1 \). Let \( f(D)^{\frac{1}{2}} \) be the branch of the root that equals 1 on \( S_{\text{red}} \). Pick a lattice \( L_0 \) which equals \( M \) off \( T^\nu_S \) and is \( S \)-locally constant. Then
\[
e := f(D)^{\frac{1}{2}} \in \mathcal{E}_{\text{dr}}(L_0). \quad \text{By (2.5.2), the auxiliary choice of \( L_0 \) is irrelevant: indeed, if \( L_0, L'_0 \) that satisfy our condition, then \( f(\text{div}(L_0/L'_0)) = 1 \), for the divisor \( \text{div}(L_0/L'_0) \) is \( S \)-locally constant and supported on \( T^\nu_S \), and \( f|_{T^\nu_S} = 1 \). The identifications \( e \) are evidently transitive.

Remark. Suppose \( L \) is an arbitrary lattice. Then
\[
e = f(\text{div}(L/L_0)) f(D)^{\text{rk}(M)/2} \in \mathcal{E}_{\text{dr}}(L_0)
\]
where \( L_0 \) is any lattice as above (see (2.5.2) and (2.5.3)).

Our \( e \) provides a canonical identification of \( \mathcal{O}_S \)-lines
\[
r := \text{id}_{\mu^\nu_P} \otimes e^{\otimes -1} : \mu^\nu_P \otimes (\mu^\nu_P)^{\otimes -1} \sim \mu^\nu_P \otimes (\mu^\nu_P)^{\otimes -1}.
\]
Therefore for \( (D, c, \nu_P) \in \mathcal{D}^\circ(S) \) the \( \mathcal{O}_S \)-line \( \mu^\nu_P \otimes (\mu^\nu_P)^{\otimes -1} \) does not depend on the choice of \( \nu \) such that \( \nu|_P = \nu_P \). Set
\[
\mathcal{E}_{\text{dr}}(M)_{(D, c, \nu_P)} := \mu^\nu_P \otimes (\mu^\nu_P)^{\otimes -1}.
\]
The construction is compatible with base change, so \( \mathcal{E}_{\text{dr}}(M) \) is an \( \mathcal{O} \)-line on \( \mathcal{D}^\circ \).

By (2.4.3) and similar property of \( \mu^\nu_P \), it carries a factorization structure. So we have defined \( \mathcal{E}_{\text{dr}}(M) \in \mathcal{L}^\circ_\mathcal{O}(X,T) \).

Example. If \( M \) is supported at \( T \), then \( \mathcal{E}_{\text{dr}}(M)_{(D, c, \nu_P)} = \det \Gamma_{\text{dr}T^\nu}(X,M) \).

Summary. Suppose we have \( (D, c, \nu_P) \in \mathcal{D}^\circ(S) \). By (2.5.1), every \( L \) and \( \nu \) such that \( \nu|_P = \nu_P \) yields an isomorphism
\[
r_{L, \nu} : \mathcal{E}_{\text{dr}}(M)_{(D, c, \nu_P)} \sim \mu^\nu_P(L/\omega L(D)).
\]
If \( f|_P = 1 \), then, by Remark,
\[
r_{L, f\nu} = f(\text{div}(L_0/L)) f(D)^{-\text{rk}(M)/2} a_{L, \nu}.
\]
In particular, \( r_{L, f\nu} = r_{L, \nu} \) for reduced \( S \).

2.6. Lemma. (i) On \( X \setminus T \) there is a canonical isomorphism
\[
\mathcal{E}_{\text{dr}}(M)_{X \setminus T}^{(1)} \sim (\det M_{X \setminus T})^{\otimes -1}.
\]
(ii) Suppose \( T' \subset T \) is such that \( M \) is smooth at \( T \setminus T' \). Let \( \mathcal{E}_{\text{dr}}(M)' \) be the restriction to \( (X,T) \) of the \( \varepsilon \)-line of \( M \) on \( (X,T') \). Then there is a canonical identification \( \mathcal{E}_{\text{dr}}(M) \sim \mathcal{E}_{\text{dr}}(M)' \).

\footnote{The reason for the normalization will become clear in 2.10.}
Proof. (i) Recall that \( E^{(1)}_{X,T} = E_{(D,0,\nu)} \) where \( S = X \setminus T, D = \Delta = P \) is the diagonal divisor, and \( \nu \) is (the principal part of) a form with logarithmic singularity at \( \Delta \) with residue 1. Take \( L = M_{X,T} \). Then \( C(L,\omega_L) = 0 \), hence \( E^{(1)}_{dR}(M) \) is an \( R \)-isomorphism followed by the residue map \( \lambda'_P(\omega_L/\omega_L(D)) \) at \( \Delta \) with residue 1. Take \( L = M_{X,T} \).

(ii) Evident.

2.7. Proposition. For proper \( X \) there is a canonical isomorphism

\[
\eta_{dR} : \mathcal{E}_{dR}(M)(X) \rightarrow \det H_{dR}(X, M).
\] (2.7.1)

Proof. By 1.4, (2.7.1) amounts to a natural identification \( \eta_{dR} : \mathcal{E}_{dR}(M) \) which freely generates \( \mathcal{E}_{dR}(M, \nu) \) for each \( \nu \) as above. The complex of \( \mathcal{O}_S \)-modules \( \mathcal{O}_S(L) \) is perfect; set \( \lambda(L) := \det R\pi_*\mathcal{O}_S(L) \). Then \( \lambda \) is a determinant theory on \( M \) at \( P \) in an evident manner. Replacing \( M \) by \( M \otimes \mathcal{O}_S(L) \), we get \( \lambda \otimes \lambda^{-1} \in \mathcal{O}_S(M/\omega_M) \). One has an isomorphism

\[
\mu'_\nu \sim \lambda \otimes \lambda^{-1},
\] (2.7.2)

namely, \( \mu'_\nu(L/L_\omega) := \lambda'_{\nu}(L/L_\omega) = \lambda_\omega(\nu L) \otimes \lambda_\omega(L_\omega)^{-1} \sim \lambda(L) \otimes \lambda(L_\omega)^{-1} = (\lambda \otimes \lambda_\omega^{-1})(L/L_\omega) \) where \( \sim \) comes from isomorphism \( \nu^{-1} : \nu L \sim L \).

For \( L \) in \( M, L_\omega \) in \( \omega_M \) with \( \nabla(L) \subset L_\omega \) (see 2.4), set \( dR(L, L_\omega) := \mathcal{C}one(L \nabla L_\omega) \subset dR(M) \). Since \( dR(M)/dR(L, L_\omega) = \mathcal{C}(L, L_\omega) \), we see that \( dR(M) \) carries a 3-step filtration with successive quotients \( L_\omega, L[1], \mathcal{C}(L, L_\omega) \). Applying \( \det \mathcal{C}(L, L_\omega) \) on \( \det R\pi_* \), we get an isomorphism

\[
\det R\pi_* \mathcal{C}(L, L_\omega) \otimes \lambda(L)^{-1} \otimes \lambda(L_\omega) \sim \det H_{dR}(X, M) \otimes \mathcal{O}_S.
\] (2.7.3)

To get \( \eta_{dR} \), we combine it with (2.7.2) (and (2.5.5)). The construction does not depend on the auxiliary choice of \( L, L_\omega \).

Example. Suppose \( M \) is a \( \mathcal{D} \)-module on \( \mathbb{P}^1 \) with regular singularities at 0 and \( \infty \), where it is, respectively, the \( + \) and the \( \! - \) extension. Since \( R\Gamma_{dR}(\mathbb{P}^1, M) = 0 \), our \( \eta_{dR} \) is an isomorphism

\[
\eta_{dR} : \mathcal{E}_{dR}(M)_{(0,t^{-1})} \otimes \mathcal{E}_{dR}(M)_{(\infty,t^{-1})} \sim k.
\] (2.7.4)

To compute it explicitly, pick a \( t\partial_t \)-invariant vector subspace \( V \) of \( \Gamma(\mathbb{P}^1 \setminus \{0, \infty\}, M) \), which freely generates \( M_{\mathbb{P}^1 \setminus \{0, \infty\}, \mathcal{O}} \) as an \( \mathcal{O} \)-module and such that the only possible integral eigenvalue of \( t\partial_t \) on \( V \) is 0. Set \( L := \mathcal{O}_{\mathbb{P}^1}(0) \otimes V, L_\omega := t^{-1}dt \otimes L = \omega_{\mathbb{P}^1}(0) \otimes L \). The condition on \( V \) implies that there are \( \mathcal{O} \)-linear embeddings \( i : L \hookrightarrow M, i_\omega : L_\omega \hookrightarrow \omega M \), which extend the evident isomorphisms on \( \mathbb{P}^1 \setminus \{0, \infty\} \), such that \( \nabla(L) \subset L_\omega \) and \( i_\omega(\phi t) = \phi \ell(t) \) for any \( \phi \in \omega_{\mathbb{P}^1}, \ell \in L \). Such \( i, i_\omega \) are unique. The complex \( \mathcal{C}(L, L_\omega) = \mathcal{C}(L, L_\omega)_{0} \oplus \mathcal{C}(L, L_\omega)_{\infty} \) is acyclic, so (2.5.6) provides trivializations \( \iota_0 \) of \( \mathcal{E}_{dR}(M)_{(0,t^{-1})} \) and \( \iota_{\infty} \) of \( \mathcal{E}_{dR}(M)_{(\infty,t^{-1})} \).

Lemma. One has \( \eta_{dR}(\iota_0 \otimes \iota_{\infty}) = 1 \).
Proof. The determinant of the complex $R\Gamma(\mathbb{P}^1, dR(L, L_\omega))$ has two natural trivializations $\alpha_1, \alpha_2$: the first one comes since the complex is acyclic, the second one from the identification $\det R\Gamma(\mathbb{P}^1, dR(L, L_\omega)) = \det R\Gamma(\mathbb{P}^1, L_\omega) \otimes \det R\Gamma(\mathbb{P}^1, L) \otimes^{-1}$ and the multiplication by $t^{-1}dt$ isomorphism $L \xrightarrow{\sim} L_\omega$. Now (2.7.3) identifies $\iota_0 \otimes \iota_\infty \otimes \alpha_1$ with $1$, and $\iota_0 \otimes \iota_\infty \otimes \alpha_2$ with $\eta_{dR}(\iota_0 \otimes \iota_\infty)$. Since $R\Gamma(\mathbb{P}^1, L) = R\Gamma(\mathbb{P}^1, L_\omega) = 0$, one has $\alpha_1 = \alpha_2$; we are done. \hfill \Box

2.8. The next constraints follow directly from the construction:

(i) For a finite filtration $M$ on $M$, there is a canonical isomorphism

$$E_{dR}(M) \xrightarrow{\sim} E_{dR}(\gr M)$$

which satisfies transitivity property with respect to refinement of the filtration.

Remark. If $M = \oplus M_n$, then every linear ordering of the indices yields a filtration on $M$, hence an isomorphism $E_{dR}(M) \xrightarrow{\sim} E_{dR}(M_n)$. This isomorphism does not depend on the ordering. Thus $E_{dR}$ is a symmetric monoidal functor.

(ii) Let $\pi : (X', T') \rightarrow (X, T)$ be a finite morphism of pairs (see Remark (i) in 1.2) which is étale over $X \setminus T$. As in loc. cit., we have a morphism of Picard groupoids $\pi_* : \mathcal{L}^\pi(X', T') \rightarrow \mathcal{L}^\pi(X, T)$. We also have the $D$-module direct image functor $\pi_*$ which is exact. If $M'$ is a holonomic $D$-module on $(X', T')$, then $\pi_* M'$ is holonomic $D$-module on $(X, T)$. Notice that $\pi_* M'$ coincides with the “naive” direct image $\pi_* M'$ outside $T$, and $dR(\pi_* M')$ is canonically quasi-isomorphic to $\pi dR(M')$ as a dg module over the de Rham dg algebra of $X$. Therefore one has a canonical isomorphism

$$E_{dR}(\pi_* M') \xrightarrow{\sim} \pi_* E_{dR}(M')$$

compatible with the composition of $\pi$’s and with (2.8.1).

Exercise. Consider the standard isomorphism $H_{dR}(X, \pi_* M') \xrightarrow{\sim} H_{dR}(X', M')$. If $X$ is proper, then (2.8.2) yields an isomorphism $E_{dR}(\pi_* M')(X) \xrightarrow{\sim} E_{dR}(M')(X')$. Show that $\eta_{dR}$’s identify the second isomorphism with the determinant of the first one.

2.9. Another digression on lattices and relative determinants. For a Clifford algebra explanation of the next constructions, see [BBE] 2.14–2.17. In this subsection and the next one we use $\mathbb{Z}/2$-graded lines instead of $\mathbb{Z}$-graded ones; as in 1.2, the corresponding Picard groupoids are marked by $\check{}$.

Let $S, P$ be as in 2.3. Let $E, E^\circ$ be $\mathcal{O}_X$-modules as in 2.3, $V$ a neighborhood of $P$ such that both $E, E^\circ$ are locally free over $V \setminus P$, $\psi : E_{V \setminus P} \otimes E^\circ_{V \setminus P} \rightarrow \mathcal{O}_{V \setminus P}/S$ be a non-degenerate pairing. For a $P$-lattice $L$ in $E(\infty P)$ its $\psi$-dual $L^\psi$ is the $P$-lattice in $E^\circ(\infty P)$ such that $\psi$ is a non-degenerate $\mathcal{O}_S$-valued pairing between $L_V$ and $L^\psi_V$. The map $\tau_\psi : \Lambda_P(E(\infty P)) \rightarrow \Lambda_P(E^\circ(\infty P)), L \mapsto \tau_\psi(L) := L^\psi$ is an order-reversing bijection. It lifts to an isomorphism of $\mathcal{L}^\pi_\mathcal{O}(S)$-groupoids:

Lemma. For every $L, L' \in \Lambda_P(E(\infty P))$ there is a canonical isomorphism

$$\tau_\psi : \Lambda_P(L/L') \xrightarrow{\sim} \Lambda_P(L^\psi/L^\psi)$$

of $\mathbb{Z}/2$-graded lines compatible with the composition.
Proof. It suffices to define (2.9.1) for \( L' \supset L \) and check the compatibility with composition for \( L'' \supset L' \supset L \).

The pairing \( \ell, \ell' \mapsto (\ell, \ell')_\psi := \text{Res}_{P/S} \psi(\ell, \ell') \) yields a duality between the vector bundles \( \pi_*(L'/L) \) and \( \pi_*(L''/L') \), hence a duality between the determinant lines \( \lambda_P(L'/L) \otimes \lambda_P(L''/L') \sim \mathcal{O}_S, (\wedge \ell_i) \otimes (\wedge \ell_j') \mapsto (-1)^{n(n-1)/2} \det(\ell_i, \ell_j') \psi \) where \( n = \text{rk} \pi_*(L'/L) \). Then (2.9.1) is characterized by the property that \( \text{id}_{\lambda_P(L'/L)} \otimes \tau_\psi \) identifies the latter pairing with the composition \( \lambda_P(L'/L) \otimes \lambda_P(L'/L') \sim \mathcal{O}_S \). The compatibility of \( \tau_\psi \) with composition is left to the reader.

Remark. Here is a sketch of a Clifford algebra interpretation of \( \tau_\psi \). Consider the Clifford \( \mathcal{O}_S \)-algebra generated by \( \pi_*E_{V \setminus P} \oplus \pi_*E_{V \setminus P}^\vee \) equipped with the hyperbolic form \( \text{Res}_{P/S} \psi \). Let \( N \) be any “invertible” continuous Clifford module. For any \( P \)-lattice \( L \) the \( \mathcal{O}_S \)-submodule \( N^{L \oplus L^\vee} \) of vectors killed by \( L \oplus L^\vee \) lies in \( \mathcal{L}_P(S) \), and \( \lambda_P(L'/L') = N^{L \oplus L^\vee} \otimes (N^{L \oplus L^\vee})^{-1} = \lambda_P(L'/L') \); the composition is \( \tau_\psi \).

Passing to \( \text{Det}'_{P/S} \), our \( \tau_\psi \) yields \( \mu^\psi \in \text{Det}'_{P/S}(E/E^\circ) \). Thus for \( L \in \Lambda_P(E(\infty P)) \), \( L_\omega \in \Lambda_P(E^\circ(\infty P)) \) one has \( \mu^\psi(L/L_\omega) = \lambda_P(L'/L_\omega) \), and identifications (2.3.2) are (2.3.1) combined with \( \tau_\psi \).

Exercise. Suppose we have another non-degenerate pairing \( E_{V \setminus P} \times E_{V \setminus P}^\vee \rightarrow \omega_{V \setminus P/S} \); one can write it as \( \psi_\nu(g, \cdot) = \psi(g, \cdot \, g) \) where \( g \in \text{Aut}(E_{V \setminus P}) \) and \( \psi g \in \text{Aut}(E_{V \setminus P}) \) is the \( \nu \)-adjoint to \( g \). Then \( L^\psi \nu = \psi^{-1}(L^\psi) = (g(L))^\psi \) and

\[
\tau_{\psi g} = \psi^{-1} \tau_\psi = \tau_\psi g.
\]  

(2.9.2)

E.g. for \( f \in O^\times(V) \) one has \( L^\psi = L^\psi \) and \( \tau f_\psi = f(\text{div}(L/L')) \psi : \lambda_P(L'/L') \sim \lambda_P(L'/L^\psi) \); here \( \text{div}(L'/L') \) was defined in 2.5.

As in 2.2 and Remark (iii) in 2.3, we denote by \( g \mapsto \lambda_\nu_g \) the \( \mathcal{L}'_P(S) \)-extension \( \text{Aut}(\mathcal{L}'_P(S))(\Lambda_P(E(\infty P)))^\nu \) of the group \( \text{Aut}(\mathcal{L}_P(S))(\Lambda_P(E(\infty P))) \); same for \( E \) replaced by \( E^\circ \). Passing to \( \text{Det}'_{P/S} \), (2.9.2) yields then an isomorphism

\[
\mu^\psi \sim \lambda_\nu \otimes \mu^\psi \sim \mu^\psi \otimes \lambda_\nu.
\]  

(2.9.3)

Suppose now that \( E^\circ = E \) and \( \psi \) is symmetric. Then \( \tau_\psi \) is an involution of the \( \mathcal{L}'_P(S) \)-groupoid \( \Lambda_P(E(\infty P)) \). Therefore, since \( \mu^\psi = \lambda_\tau \psi \in \text{Det}'_{P/S}(E/E) = \mathcal{L}'_P(S) \), the composition yields a canonical identification

\[
a_\psi : \mu^\psi \otimes \mu^\psi \sim \mathcal{O}_S.
\]  

(2.9.4)

Explicitly, the isomorphism \( \mu^\psi \sim \lambda_P(\tau_\psi(L)/L) \) identifies \( a_\psi \) with the pairing \( \lambda_P(\tau_\psi(L)/L) \otimes \lambda_P(\tau_\psi(L)/L) \rightarrow \mathcal{O}_S, l_1 \otimes l_2 \mapsto \tau_\psi(l_1)l_2 = l_1 \tau_\psi(l_2) \).

2.10. Let us construct on \( \mathcal{E} = \mathcal{E}_{M}(M) \in \mathcal{L}_P(S,X,T) \) a de Rham structure such that the identification of (2.6.1) is horizontal. By the corollary in 1.12, it is uniquely defined by this property, and the constraints from 2.8 are automatically compatible with the de Rham structure. As in 2.1, we assume that \( T \) is compatible with \( M \).

As in Remark (i) in 1.2, we need to present for every scheme \( S \) and a pair of points \( (D, c, \nu_p), (D', c', \nu'_p) \in \mathfrak{D}^\circ(S) \) which coincide on \( S_{\text{red}} \), a natural identification (notice that \( c = c' \))

\[
\alpha^\psi : \mathcal{E}_{(D, c, \nu_p)} \sim \mathcal{E}_{(D', c, \nu'_p)}.
\]  

(2.10.1)

The \( \alpha^\psi \) should be transitive and compatible with base change and factorization.

Set \( P := T_S^\circ \cup |D| \cup |D'| = P_{D,c} \cup P_{D',c} \). Localizing \( S \), we find an open neighborhood \( V \) of \( P \), \( V \cap T_S = T_S^\circ \), together with a datum \( \nu, \nu', \kappa \), where:
(a) \( \nu, \nu' \) are trivializations of \( \omega_{V/S}(D) \), \( \omega_{V/S}(D') \) which coincide on \( V_{\text{red}} \) and such that \( \nu|_{\text{red}} = \nu' \), \( \nu'|_{\text{red}} = \nu' \) (cf. Remark (ii) in 1.1);
(b) \( \kappa : M_{V \times P} \times M_{V \times P} \rightarrow \mathcal{O}_{V \times P} \) is a non-degenerate symmetric bilinear form.

We construct \( \alpha^* \) explicitly using this datum.

The notation from 2.9 are in use. One can view \( \kappa \) as a non-degenerate pairing \( M_{V \times P} \times \omega M_{V \times P} \rightarrow \omega_{V \times P/S} \), which yields \( \mu_p^\kappa = \mu_p^\kappa \in \text{Det}^*_{P/S}(M/\omega M) \). We get

\[
\mu_p^{\kappa/\nu} := \mu_p^\kappa \otimes (\mu_p^\nu)^{-1}, \quad \mu_p^{\kappa/\nu} := \mu_p^\kappa \otimes (\mu_p^\nu)^{-1} \in \mathcal{L}^*_p(S); \tag{2.10.2}
\]

recall that \( \mu_p^\kappa \in \text{Det}^*_{P/S}(M/\omega M) \) corresponds to the multiplication by \( \nu \) identification of \( M(\infty P) \) and \( \omega M(\infty P) \). Let us rewrite (2.5.5) as an identification

\[
\mathcal{E}_{(D,c,\nu)} \cong \mu_p^\kappa \otimes (\mu_p^\nu)^{-1} \otimes \mu_p^{\kappa/\nu} = \mu_p^{\kappa/\nu} \otimes \mu_p^{\kappa/\nu}. \tag{2.10.3}
\]

There is a similar identification for \( \mathcal{E}_{(D',c,\nu')} \).

Notice that \( \mu_p^{\kappa/\nu} \) is the line that corresponds to the symmetric pairing \( \kappa/\nu = \nu^{-1} \kappa : \omega M_{V \times P} \times \omega M_{V \times P} \rightarrow \omega_{V \times P/S} \). So, by (2.9.4), one has a canonical trivialization \( \alpha_{\kappa/\nu} : \mu_p^{\kappa/\nu} \otimes \mu_p^{\kappa/\nu} \rightarrow \mathcal{O}_S \).

Let \( \beta : \mathcal{E}_{(D,c,\nu)} \cong \mathcal{E}_{(D',c,\nu')} \) be an isomorphism obtained by means of (2.10.3) from the tensor product of \( \beta_1 = \text{id}, \nu/\kappa \) and an identification \( \beta_2 : \mu_p^{\kappa/\nu} \cong \mu_p^{\kappa/\nu} \). Let \( \beta_2 \) be such that \( \beta_2 = a_{\kappa/\nu} \otimes a_{\kappa/\nu} \) and \( \beta_2 \) equals identity on \( S_{\text{red}} \). We set \( \alpha^* := \gamma \beta \), where

\[
\gamma := \exp \text{Res}_{P/S}(\log(\nu'/\nu) \phi_\kappa) \in \mathcal{O}_S^\times(S). \tag{2.10.4}
\]

Here \( \nu'/\nu \) is an invertible function on \( V \setminus P \) that equals 1 on \( V_{\text{red}} \), so \( \log(\nu'/\nu) \) is a nilpotent function on \( V \setminus P \), and \( \phi_\kappa := \frac{1}{2} \nabla_M(\det \kappa^{-1})/\det \kappa^{-1} \in \Gamma(V \setminus P, \omega_{V/S}) \) where \( \det \kappa^{-1} \) is the trivialization of \( \det M_{V \times P}^{\kappa/\nu} \) defined by \( \kappa \).

**Proposition.** \( \alpha^* \) does not depend on the auxiliary choice of \( \nu, \nu', \) and \( \kappa \).

**Proof.** (a) Let us show that \( \alpha^* \) does not depend on \( \kappa \) for fixed \( \nu, \nu' \). Suppose we have two forms \( \kappa, \kappa_g \), so \( \kappa_g(\cdot, \cdot) = \kappa(\cdot, \cdot) \) for a \( \kappa \)-self-adjoint \( g \in \text{Aut}(M_{V \times P}) \). Consider the corresponding \( \beta, \gamma, \beta_g, \gamma_g \). Then \( \beta_g/\beta, \gamma/\gamma_g \) are functions on \( S \) that equal 1 on \( S_{\text{red}} \); we want to check that they are equal.

We have an \( \omega \)-valued symmetric bilinear form \( \psi := \psi/\nu \) on \( \omega M_{V \times P} \). Our \( g \), viewed as an automorphism of \( \omega M_{V \times P} \), is \( \psi \)-self-adjoint and \( \kappa_g/\psi = \psi_\gamma \). Since \( \text{Ad}_{\tau_\psi}(g) = g^{-1} \), we have the isomorphism \( \text{Ad}_{\tau_\psi} : \lambda_{g^{-1}} \rightarrow \lambda_g \) (see 2.2). Consider the identification \( \mu_p^{\psi \kappa} \cong \lambda_{g^{-1}} \otimes \mu_p^{\psi \kappa} \) of (2.9.3). The next lemma follows directly from the definition of \( a_{\psi_\gamma} \) and \( a_{\psi_\gamma} \) as the composition in the \( \psi \)-extension (see (2.9.4)):

**Lemma.** \( a_{\psi_\gamma} \) equals the composition \( \mu_p^{\psi \kappa} \otimes \mu_p^{\psi \kappa} \rightarrow \lambda_{g^{-1}} \otimes \mu_p^{\psi \kappa} \otimes \mu_p^{\psi \kappa} \rightarrow \mathcal{O}_S \). Here the second arrow is the commutativity constraint, the third one is tensor product of a map \( \beta_1 : \lambda_{g^{-1}} \otimes \mathcal{E}_{(D,c,\nu)} \rightarrow \mathcal{O}_S, \ell_1 \otimes \ell_2 \mapsto \ell_1 \text{Ad}_{\tau_\psi}(\ell_2) \), and \( a_{\psi_\gamma} \).

There is a similar assertion for \( \psi \) replaced by \( \psi' := \psi/\nu' \). Combining them, we see that \( (\beta_g/\beta)^2 = \psi_\gamma/\psi'_\gamma \). Since \( \tau_\psi = h\tau_\psi \) where \( h \) is the multiplication by \( \nu'/\nu \) automorphism (see (2.9.2)), one has \( \text{Ad}_{\tau_\psi} = \{ h, g \} \text{Ad}_{\tau_\psi} : \lambda_{g^{-1}} \rightarrow \lambda_g \) (see 2.2). Therefore \( (\beta_g/\beta)^2 = \{ h, g \}^{-1} \); by Remark (iii) in 2.3, this equals the Contou-Carrère symbol \( \{ \nu'/\nu, \det g \}_P \).
Now \( \phi_κ - \phi_{κ/ν} = \frac{1}{2} d \log(\det g) \), hence \( γ/γ_0 = \exp \text{Res}_{P/S}(\frac{1}{2} \log(ν'/ν) d \log(\det g)) \) = \{ν'/ν\}^2,\det g \}_P, and we are done.

(b) It remains to show that \( α^ ε \) does not depend on the choice of the lifts \( ν, ν' \) of \( ν_0, ν'_0 \). One can change \( ν, ν' \) to \( f ν, f ν' \) where \( f, f' ∈ \mathcal{O}^*(V) \) are such that \( f \) equals \( f' \) on \( V_{\text{red}} \), \( f \) equals 1 on \( P_{D,c} \), \( f' \) equals 1 on \( P_{D,c'} \).

By (a), in the computation we are free to use \( κ \) of our choice. We work \( X \)-locally, so we can assume that \( c = 1 \) and pick \( κ \) to be a non-degenerate symmetric form on \( MX \setminus T \). Then \( \phi_κ \in \omega(X,T) \) due to the compatibility of \( T \) with \( M \), see 2.1. The function \( \log(f'/f) \) is regular on \( V \) and vanishes on \( T_0^* \). Therefore \log(f'/f)φ_κ is regular at \( P \), hence its residue vanishes. Thus \( γ(f'/f, ν) = γ(ν', ν) \). It remains to show that \( β(f'/f, ν) = β(ν', ν) \).

Let \( L_0 \) be any \( T \)-lattice in \( M \), \( L_0^\kappa \) be the \( κ \)-orthogonal \( T \)-lattice. Since \( \omega L_0^\kappa = \tau_{κ/ν}(\omega L_0(D)) = \tau_{ν/ν}(\omega L_0(D)) \), one has \( μ^{κ}/ν \sim \lambda_π(ω L_0^\kappa/ω L_0(D)) \sim μ^{κ}/ν \). Let \( e^\kappa_{L_0} : μ^{κ}/ν \sim μ^{κ'/ν} \) be the composition, and \( r_L : E_{(D,c,ν)} \sim E_{(D,c,ν')} \) be the tensor product of \( e^\kappa_{L_0} \) and \( e^\kappa_{L_0} \) (we use identifications (2.10.3) for \( ν \) and \( ν' \)). We see that

\[
E_{(D,c,ν)} \sim E_{(D,c,ν')} = \text{the canonical isomorphism (2.5.4)}.
\]

We want to check that \( r \) and the similar isomorphism for \( f'^{\ast}, ν' \) identify \( β(f'/f, ν) \) with \( β(ν', ν) \). Indeed, \( e^\kappa_{L_0} \) identifies \( a_κ/f_ν \) with \( f(D)^{\kappa/ν} a_κ/ν \) (see Exercise in 2.9), so \( r_L \) identifies \( β(f'/f, ν) \) with \( f(D)^{\kappa/ν} f(D) β(ν', ν) \). By above, this implies the assertion for \( r \).

The isomorphisms \( α^ ε \) are transitive (since such are \( α^ ε \) with fixed \( κ \) and evidently compatible with base change and factorization, so we have defined a de Rham structure on \( E \). The horizontality of (2.6.1) is checked in Example (1) of 2.11.

Remark. Let us fix a non-degenerate symmetric bilinear form \( κ \) on \( MX \setminus T \) (like in part (b) of the proof of the lemma). Then the above construction can be reformulated as follows. There is a canonical isomorphism

\[
E_{dR}(M) \sim E_1 ⊗ E_2 ⊗ E_3,
\]

where \( E_i \) are the next de Rham factorization lines:
- \( E_1(D,c,ν) := μ^{κ}/ν \) in (2.10.2); it depends only on \( c \), i.e., \( E_1 \) comes from \( T^2 \).
- \( E_2(D,c,ν) := μ^{κ'/ν} \) in (2.10.2), so \( E_2 ⊗ E_2 \) is canonically trivialized (as a de Rham factorization line) by \( a_κ/ν \). Notice that \( E_2 \) does not depend on the connection \( \nabla_ν \).
- \( E_3 := E^{-φ_κ} ∈ L_0^dR(X,T)^{O-\text{triv}} \) (see 1.12), i.e., it is a de Rham factorization line equipped with an \( O \)-trivialization \( e \) with \( \nabla(e^{(1)})/e^{(1)} = -φ_κ \), where \( φ_κ := \frac{1}{2} \nabla(\det κ^{-1})/\det κ^{-1} ∈ \omega(X,T) \) (see 1.12.3). Isomorphism (2.10.5) is (2.10.3) ⊗ \( e \).

2.11. As in 1.3, the de Rham structure on \( E \) can be viewed as a datum of integrable connections \( \nabla^ε \) on \( E_{(D,c,ν)} \) for \( S \) smooth. The next explicit construction of \( \nabla^ε \) is a paraphrase of the above:

Our problem is \( X \)-local, so we can fix \( κ \) as in the above remark and a \( T \)-lattice \( L \) in \( M \); we can assume that \( c = 1 \). Choose any \( ν \) as in Remark (ii) in 1.1. We have identification \( r_{L,ν} : E_{(D,c,ν)} \sim μ^{κ}/ν(L/L(D)) = μ^{κ}/ν(L/ω L^κ/L(D)) \) of (2.5.6). Let \( \nabla^1 = \nabla^1_{L,ν,κ} \) be the “constant” connection on \( μ^{κ}/ν(L/ω L^κ) \), and \( \nabla^2 = \nabla^2_{L,ν,κ} \) be the connection on \( \lambda_π(ω L^κ/L(D)) \) for which the pairing \( a_κ/ν :
\[ \lambda_p(\omega^k/L(D)) \otimes \lambda_p(\omega^k/L(D)) \to \mathcal{O}_S, \ell_1 \otimes \ell_2 \mapsto \tau_{\kappa/\nu}(\ell_1)\ell_2, \] of (2.9.4) is horizontal. We get a connection \( \nabla^\epsilon = \nabla^1 \otimes \nabla^2 \) on \( \mathcal{E}(D,x,\nu) \). Then

\[ \nabla^\epsilon = \nabla_{\nu,\kappa} - \theta_{\nu,\kappa} \]  

(2.11.1)

where \( \theta_{\nu,\kappa} := \text{Res}_{p/S}(d_S\nu/\nu \otimes \phi_\kappa) \in \Omega^1_S \). Here \( d_S \) is the derivation along the fibers of \( X_S/X \), so \( d_S\nu/\nu \) is a section of \( \pi^*_S \Omega^1_S \) on \( V \setminus P \).

**Examples.** (i) Let us compute the connection on \( \mathcal{E}(t) \) (see 1.6). So let \( S \) be a copy of \( X \setminus T, P = \Delta, D = t\Delta \). Let \( t \) be a local coordinate on \( X \setminus T, x \) be the corresponding coordinate on \( S, z \) be the coordinate on \( \mathbb{G}_m, \nu := z(t-x)^{-1}dt \). Our \( \mathcal{E}(t) \) is the de Rham line \( \mathcal{E}(D,0,\nu) \) on \( S \times \mathbb{G}_m \).

We take \( L = M_{X \setminus T}, \) so \( L^k = L \) and \( \mu^\mathbb{P}(L/\omega^k) \) is trivialized. Therefore \( \mathcal{E}(t) = \lambda_p(\omega M/\omega M(D)) \). The choice of \( t \) trivializes \( \omega \) and all the vector bundles \( \pi_*\mathcal{O}_{X \times S}(m\Delta)/\mathcal{O}_{X \times S}(n\Delta) \), which provides an identification \( \mathcal{E}(t) \encircledast \mathcal{E}(t)^{-1} \). The pairing \( \kappa/\nu \) is equal (up to sign) to \( (z^{-1} \det \kappa)^{-1}, n := \text{rk}(M), \) so \( \nabla_{\nu,\kappa} = \nabla_{(\det M)\otimes -1} + \ell \phi_\kappa + \frac{\ell n}{2} z^{-1}dz \). One has \( d_S\nu/\nu = z^{-1}dz + \ell(t-x)^{-1}dx \). Hence \( \theta_{\nu,\kappa} = \ell \phi_\kappa, \) and

\[ \nabla^\epsilon = \nabla_{(\det M)\otimes -1} + \frac{\ell n}{2} z^{-1}dz. \]  

(2.11.2)

In case \( \ell = 1 \) and \( z = 1, (2.11.2) \) says that (2.6.1) is horizontal with respect to \( \nabla^\epsilon \) and the connection on \( (\det M_{X \setminus T})^\otimes -1 \).

(ii) Suppose \( M \) has regular singularities at \( b \in T \). Let \( t \) be a parameter at \( b \). Consider \( P = b, D = t\ell b, \) and a family of 1-forms \( \nu_z := z(t-x)^{-1}dt, z \in \mathbb{G}_m \). Let us compute the connection \( \nabla^\epsilon \) on the line bundle \( \mathcal{E}(b,\nu,\nu) := \mathcal{E}(D,1,\nu) \) on \( \mathbb{G}_m \).

Let \( L \) be any \( t\ell \)-invariant \( P \)-lattice in \( M(\infty b); \) denote by \( r \) the trace of \( t\ell \) acting on \( L/tL, n := \text{rk}(M) \). Let \( \nabla_0 \) be a connection on \( \mathcal{E}(b,\nu,\nu) \) such that \( r_{L,\nu} \) of (2.5.6) identifies it with the “constant” connection on \( z \)-independent line \( \mu^\mathbb{P}(L/\omega L(D)) \). Then

\[ \nabla^\epsilon = \nabla_0 + \left( \frac{\ell n}{2} - r \right) z^{-1}dz. \]  

(2.11.3)

Indeed, consider the above construction with \( \nu = \nu_z \) and \( \kappa = t^\ell \kappa_0 \) where \( \kappa_0 \) is a non-degenerate symmetric bilinear form on \( L \). Then \( L^k = L(D), \) so \( \nabla_{\nu,\kappa} = \nabla_0 \). The trivialization \( \det \kappa^{-1} \) of \( \det M_{X \setminus T}^{\otimes 2} \) has pole of order \( \ell n \) at \( b \), thus the form \( \phi_\kappa \) has logarithmic singularity at \( b \) with residue \( r - \ell n/2 \). Since \( d_z(\nu_z)/\nu_z = z^{-1}dz \), one has \( \theta_{\nu,\kappa} = (r - \ell n/2) z^{-1}dz, \) and we are done by (2.11.1).

2.12. Let \( q : X \to Q, T \) be as in 1.14. Let \( M \) be a coherent \( \mathcal{D}(X/Q) \)-module which is \( \mathcal{O}_Q \)-flat and is a vector bundle on \( X \setminus T \). We call such \( M \) a flat \( Q \)-family of holonomic \( D \)-modules on \( (X/Q, T) \). The notion of compatibility of \( T \) and \( M \) is defined as in 5.1.

Let \( dR_{X/Q}(M) = \text{Cone}(\nabla) \) be the relative de Rham complex. If for some (hence every) \( T \)-lattices \( L \in M, L_\omega \) in \( \omega M := \omega_{X/Q} \otimes M \) with \( \nabla(L) \subset L_\omega \) the complex \( q_*(dR_{X/Q}(M)/\text{Cone}(L)) \) has \( \mathcal{O}_Q \)-coherent cohomology, then we call \( M \) a nice \( Q \)-family of \( \mathcal{D} \)-modules.

**Exercises.** Suppose \( Q = \text{Spec} \mathbb{C}[s], X = \text{Spec} \mathbb{C}[t, s], T \) is the divisor \( t = 0 \).

(i) Show that \( M \) generated by a section \( m \) subject to the relation \( t\partial_t m = sm \) is nice, and that the \( \mathcal{D}(X/Q) \)-module \( j_* \mathcal{D}(M[t^{-1}] \) is not coherent (cf. [BG]).

(ii) Show that \( M \) generated by section \( m \) subject to the relation \( \tau^\ell \partial_t m = sm, n > 1, \) is nice over the subset \( s \neq 0 \).
By a straightforward relative version of the constructions of this section, every nice family compatible with $T$ gives rise to a relative factorization line $E_{\text{dR}}(M) \in \mathcal{L}_{\text{dR}}(X/Q, T)$ (see 1.14). The construction is compatible with base change. For proper $X/Q$, the $\mathcal{O}_Q$-complex $Rq_{\text{dR}}(M) := Rq_{dR_{X/Q}}(M)$ is perfect, and we have an isomorphism of $\mathcal{O}$-lines $\eta_{\text{dR}} : E_{\text{dR}}(M)(X/Q) \cong \det Rq_{\text{dR}}(M)$.

Suppose that $Q$ is smooth and the relative connection on $M$ is extended to a flat connection (so our nice family is isomonodromic).

**Proposition.** The relative connection on $E_{\text{dR}}(M)$ extends naturally to a flat absolute connection which has local origin and is compatible with base change and constraints from 2.8. Thus $E_{\text{dR}}(M) \in \mathcal{L}^\phi_{\text{dR}}(X/Q, T)$. For proper $X/Q$, $\eta_{\text{dR}}$ is horizontal (for the Gauß-Manin connection on the target).

**Proof.** Let $L = L((X, T)/Q)$ be the Lie algebra of infinitesimal symmetries of $(X, T)/Q$; its elements are pairs $(\theta_X, \theta_Q)$ where $\theta_X$, $\theta_Q$ are vector fields on $X$, $Q$ such that $\theta_X$ preserves $T$ and $d\eta(\theta_X) = \theta_Q$. Our $L$ acts on $\mathcal{D}^\omega(X/Q, T; \omega)$ and on $\mathcal{E} := E_{\text{dR}}(M)$ by transport of structure. This action is compatible with constraints from 2.8 and the relative de Rham structure.

**Variant.** Let $T' \subset T$ be a component of $T$; let $L' := L((X \setminus T', T \setminus T')/Q) \supset L$. Then $L'$ acts naturally on $\mathcal{D}^\omega(X/Q, T; \omega)$ (see 1.13 where we considered the “vertical” part of this action). If $M = j_{T*}M$, then this action lifts naturally to $E_{\text{dR}}(M)$ (as follows directly from the construction of $E_{\text{dR}}(M)$). The $L'$-action extends the $L$-action (pulled back to $\mathcal{D}^\omega(X/Q, T; \omega)$) and satisfies similar compatibilities.

**Lemma.** The Lie ideals $L_0 \subset L$, $L'_0 \subset L'$ act on $\mathcal{E}$ via $\nabla^\varepsilon$.

**Proof of Lemma.** It suffices to check this $Q$-pointwise, so, due to compatibility with the base change, we can assume that $Q$ is a point. By the compatibility with the first constraint in 2.8, it suffices to consider the cases when $M$ is supported at $T$ and $M = j_{T*}M$. In the first situation the lemma is evident. If $M = j_{T*}M$, then it suffices to consider the case of $L'_0$ for $T' = T$, i.e., $L'_0 = \Theta(X \setminus T)$. The $L'_0$-action is compatible with the de Rham structure, so we are done by the lemma in 1.13. □

We want to define the connection in a manner compatible with the localization of $X$, so it suffices to do it in case when $X$ and $Q$ are affine. Then $L/L_0$ is the Lie algebra of vector fields on $Q$. Therefore, by the lemma, $\nabla^\varepsilon$ extends in a unique manner to an absolute flat connection such that $L$ acts via this connection.

**Remark.** In the situation with $T'$ the Lie algebra $L'$ acts on $\mathcal{E}$ via the connection as well (by the same lemma).

All the properties stated in the proposition, except the last one, are evident from the construction. Let us show that $\eta_{\text{dR}}$ is horizontal. We work $Q$-locally, so we can assume that $Q$ is affine and $X \setminus T$ admits a section $s$. We can enlarge $T$ to $T^+ := T \sqcup T'$, $T^+ := s(Q)$. By the first constraint from 2.8, the assertion for $M$ reduces to that for $s_*s^*M$ and $j_{T^*}M = M(\infty T')$. The first case is evident. In the second case the Gauß-Manin connection comes from the action of the Lie algebra $L'$ (for $T^+$ and $T'$), and we are done by the remark.

2.13. **Compatibility of $\eta_{\text{dR}}$ with quadratic degenerations of $X$.** We will show that $\eta_{\text{dR}}$ remains constant (in some sense) when $X$ degenerates quadratically and $M$ stays constant outside the node. Notice that the family is not isomonodromic.
(so 2.12 is not applicable). We will need the result in §5; the reader can presently skip the subsection. Consider the next data (a), (b):

(a) A smooth proper curve $Y$, a finite subscheme $T \subset Y$, two points $b_+, b_- \in (Y \setminus T)(k)$, a rational 1-form $\nu$ on $Y$ invertible off $T \cup \{b_+\}$ and having poles of order 1 at $b_\pm$ with $\text{Res}_{b_\pm} \nu = \pm 1$. Let $t_\pm$ be formal coordinates at $b_\pm$ such that $d \log t_\pm = \pm \nu$.

(b) A $D$-module $N$ on $(Y, T \cup b_\pm)$ which has regular singularities at $b_\pm$, is the $\pm$-extension at $b_\pm$ and the !-extension at $b_-$. We also have a $t_\pm \partial_\pm$-invariant $b_\pm$-lattice $L$ in $M$, and an identification of the $b_\pm$-fibers $\alpha : L_{b_\pm} \sim L_{b_-}$. Let $A_\pm \in \text{End}(L_{b_\pm})$ be the action of $\pm t_\pm \partial_\pm$ on the fibers; we ask that $\alpha A_+ = A_- \alpha$, and that the eigenvalues of $A_+$ (or $A_-$) and their pairwise differences cannot be non-zero integers.

Then the restriction of $L$ to the formal neighborhoods $Y_\pm = \text{Spf} k[[t_\pm]]$ of $b_\pm$ can be identified in a unique way with $L_{b_\pm}[[t_\pm]]$ so that $L_{b_\pm} \subset \Gamma(Y_\pm, L)$ is $t_\pm \partial_\pm$-invariant.

Datum (a) yields a proper family of curves $X$ over $Q = \text{Spf} k[[q]] = \varprojlim Q_n$, $Q_n = \text{Spec} k[q]/q^n$, which has quadratic degeneration at $q = 0$. The 0-fiber $X_0$ is $Y$ with $b_\pm$ glued to a single point $b_0 \in X_0(k)$; let $j_{b_0}$ be the embedding $X \setminus \{b_+, b_-\} = X_0 \setminus \{b_0\} \hookrightarrow X_0$. Outside $b_0$ our $X$ is trivialized, i.e., $\mathcal{O}_X \setminus \{b_0\} = \mathcal{O}_{X_0 \setminus \{b_0\}}[[q]]$.

The formal completion of the local ring at $b_0$ equals $k[[t_+, t_-]]$ with $q = t_+ t_-$, and the glueing comes from the embedding $k[[t_+, t_-]] \hookrightarrow k((t_+))[[q]] \times k((t_-))[[q]]$, $t_+ \mapsto (q/t_+, t_-)$. Set $\mathcal{R} := k((t_+))[[q]] \times k((t_-))[[q]]/k[[t_+, t_-]]$. We have a short exact sequence (R is viewed as a skyscraper at $b_0$)

$$0 \rightarrow \mathcal{O}_X \rightarrow j_{b_0*} \mathcal{O}_{X \setminus \{b_0\}}[[q]] \rightarrow \mathcal{R} \rightarrow 0,$$

where the right projection assigns to $f = \Sigma f_n q^n$ the image of $(f_+, f_-)$ in $\mathcal{R}$, $f_\pm = \Sigma f_n(t_\pm)q^n \in k((t_\pm))[[q]]$ are the expansions of $f$ at $b_\pm$.

Our families of curves has standard nodal degeneration, so we have the dualizing line bundle $\omega_{X/Q}$. Our $\nu$ defines a rational section $\nu_0$ of $\omega_{X/Q}$, which is “constant” on $X \setminus \{b_0\}$ with respect to the above trivialization, and is invertible near $b_0$.

Below for an $\mathcal{O}_X$-module $F$ a relative connection on $F$ means a morphism $\nabla : F \rightarrow \omega_{X/Q} \otimes F$ such that $\nabla(f \phi) = d(f) \otimes \phi + f d(\phi)$, where $d$ is the canonical differentiation $d : \mathcal{O}_X \rightarrow \omega_{X,Q}$. Set $dR_{X/Q}(F) := \text{Conc}(\nabla)$. $R_{Q, \mathcal{R}} : := R_{Q, dR_{X/Q}(F)}$.

Datum (b) yields an $\mathcal{O}$-module $M$ on $X$ equipped with a relative connection $\nabla$. Our $M$ is locally free over $X \setminus TQ$. Outside $b_0$ it is constant with respect to the above trivialization: one has $M\mid_{X \setminus \{b_0\}} = N\mid_{X \setminus \{b_+, b_-\}}[[q]] = L\mid_{X \setminus \{b_+, b_-\}}[[q]]$.

The restriction $M_0$ of $M$ to $X_0$ equals $L$ with fibers $L_{b_\pm}$ identified by $\alpha$. On the formal neighborhood of $b_0$ our $M$ equals $M_0\mid_{X_0 \setminus \{b_+, b_-\}}$, and the glueing comes from the trivializations of $L$ on $Y_\pm$ (see (b)) and the gluing of functions. Therefore we have a short exact sequence

$$0 \rightarrow M \rightarrow (j_{b_0*} L\mid_{Y \setminus \{b_+, b_-\}})[[q]] \rightarrow M_0 \otimes \mathcal{R} \rightarrow 0,$$

where the right projection assigns to $f = \Sigma f_n q^n \in (j_{b_0*} L\mid_{Y \setminus \{b_+, b_-\}})[[q]]$ the image in of $(\ell_+, \ell_-) \in \ell_{b_0} \otimes \mathcal{R}$, $\ell_\pm = \Sigma \ell_n(t_\pm)q^n \in L_{b_\pm}((t_\pm))[[q]] = M_{b_\pm}((t_\pm))[[q]]$ are the expansions of $\ell$ at $b_\pm$ with respect to the formal trivializations of $L$ on $Y_\pm$. On $X \setminus \{b_0\}$ the relative connection $\nabla$ comes from the $D$-module structure on $N$; on the formal neighborhood of $b_0$ this is the relative trivializations on $M_{b_\pm}[t_+, t_-]$ with potential $At_\pm^{-1} dt_\pm = -At_\pm^{-1} dt_\pm$.

Remarks. (2.13.2) is an exact sequence of $\mathcal{O}_X$-modules equipped with relative connections. The projection $(t_\pm^{-1} k[t_\pm^{-1}] \otimes k[t_\pm^{-1}])[q] \rightarrow \mathcal{R}$ is an isomorphism. The relative connection on $M_0 \otimes \mathcal{R}$ is $\nabla(m \otimes t_\pm^a q^b) = \nu(A(m) \pm am) t_\pm^a q^b$. 

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Set $D = -\text{div}(\nu) - b_+ - b_-$. Then $\text{div}(\nu_Q) = -D_Q$ does not intersect $b_0$, so we have an $\mathcal{O}_Q$-line $E_{\text{dR}}(M)_{\nu_Q} := E_{\text{dR}}(M)_{(D_Q,1,\nu_Q)}$. An immediate modification of the construction in 2.7 (to be spelled out in the proof of the proposition below) yields an isomorphism

$$
\eta_{\text{dR}} : E_{\text{dR}}(M)_{\nu_Q} \cong \det R_{q_{\text{dR}*}}M.
$$

(2.13.3)

Our aim is to compute it explicitly. Notice that since our family $(X,T_Q,M,\nu_Q)$ is trivialized outside $b_0$, one has a canonical identification

$$
E_{\text{dR}}(M)_{\nu_Q} \cong E_{\text{dR}}(N)_{(D,1,\nu)}[[q]].
$$

(2.13.4)

(i) Consider an embedding $\text{Cone}(A)[[q]] \hookrightarrow dR_{X/Q}(M_{b_0} \otimes \mathcal{R})$ whose components are $M_{b_0}[[q]] \hookrightarrow M_{b_0} \otimes \mathcal{R}$, $m \otimes f(q) \mapsto m \otimes (0,f(q))$, and $M_{b_0}[[q]] \hookrightarrow \omega_X \otimes M_{b_0} \otimes \mathcal{R}$. By the condition on $A$, this is a quasi-isomorphism. Thus (2.13.2) yields an isomorphism

$$
\det R_{q_{\text{dR}*}}M \cong \det R_{\Gamma_{\text{dR}}(Y, j_{b_+}*N)} \otimes \det \text{Cone}(A)[[q]].
$$

(2.13.5)

Let $L^{-}$ be a $b_{\pm}$-lattice in $N$ that equals $L$ outside $b_-$ and $t_-L$ at $b_+$. Set $C_1 := \text{Cone}(\nabla : L^{-} \rightarrow \omega_L (\log b_+L))$, $C_* := \text{Cone}(\nabla : L \rightarrow \omega_L (\log b_+)L)$. Recall that $N$ is the $!$-extension at $b_-$ and the $*$-extension at $b_+$, so the condition on $A$ assures that the embeddings $C_1 \hookrightarrow dR(N)$ and $C_* \hookrightarrow dR(j_{b_+}N)$ are quasi-isomorphisms. Since $C_*/C_1$ equals $\text{Cone}(A)$ (viewed as a skyscraper at $b_-$), we see that $\text{Cone}(dR(N) \rightarrow dR(j_{b_+}N)) \cong \text{Cone}(A)$, hence

$$
\det \Gamma_{\text{dR}}(Y,N) \cong \det \Gamma_{\text{dR}}(Y,j_{b_+}*N) \otimes \det \text{Cone}(A).
$$

(2.13.6)

Combining it with (2.1.5), we get an isomorphism

$$
\det R_{q_{\text{dR}*}}M \cong \det R_{\Gamma_{\text{dR}}(Y,N)}[[q]].
$$

(2.13.7)

(ii) Consider a $\mathcal{D}$-module on $\mathbb{P}^1 \setminus \{0, \infty\}$ which equals $\mathcal{O}_{\mathbb{P}^1 \setminus \{0, \infty\}} \otimes M_{b_0}$ as an $\mathcal{O}$-module, $\nabla(f \otimes m) = df \otimes m + f \otimes A(m)$. Let $\bar{N}$ be the $!$-extension to $\infty$ and the $*$-extension to 0 of $N$. Consider the embeddings $t_+: Y_+ \hookrightarrow \mathbb{P}^1$, $t_- : Y_- \hookrightarrow \mathbb{P}^1$ which identify $Y_{\pm}^\vee$ with the formal neighborhoods of 0 and $\infty$. The trivializations of $L$ on $Y_{\pm}^\vee$ from (b) identify the pull-back of $N$ with $N|_{Y_{\pm}^\vee}$. Since the pull-back of $t^{-1}dt$ equals $\nu|_{Y_{\pm}^\vee}$, we get the identifications $E_{\text{dR}}(N)_{(b_+*,\nu)} \cong E_{\text{dR}}(\bar{N})_{(0,t^{-1}dt)}$, $E_{\text{dR}}(N)_{(b_-,\nu)} \cong E_{\text{dR}}(\bar{N})_{(\infty,t^{-1}dt)}$. Combined with (2.7.4) (for $M$ in loc. cit. equal to $\bar{N}$), they produce an isomorphism

$$
E_{\text{dR}}(N)_{(b_+*,\nu)} \otimes E_{\text{dR}}(N)_{(b_-,\nu)} \cong k.
$$

(2.13.8)

Since $E_{\text{dR}}(N)_\nu = E_{\text{dR}}(N)_{(D,1,\nu)} \otimes E_{\text{dR}}(N)_{(b_+*,\nu)} \otimes E_{\text{dR}}(N)_{(b_-,\nu)}$, we rewrite it as

$$
E_{\text{dR}}(N)_{(D,1,\nu)} \cong E_{\text{dR}}(N)_\nu.
$$

(2.13.9)

**Proposition.** The diagram

$$
\begin{array}{ccc}
E_{\text{dR}}(M)_{\nu_Q} & \xrightarrow{\eta_{\text{dR}}} & \det R_{q_{\text{dR}*}}M \\
\downarrow & & \downarrow \\
E_{\text{dR}}(N)_\nu[[q]] & \xrightarrow{\eta_{\text{dR}}} & \det \Gamma_{\text{dR}}(Y,N)[[q]],
\end{array}
$$

where the vertical arrows are (2.13.9)$\circ$(2.13.4) and (2.13.7), commutes.
Proof. We check the assertion modulo $q^{n+1}$. Thus we restrict our picture to $Q_n := \text{Spec } R_n$, $R_n := k[q]/q^{n+1}$; we get a $Q_n$-curve $X_n$, $M_n = M \otimes R_n$, etc.

Let $F$ be a $(b_+, b_-)$-lattice in $j_{b_+, N}$ such that $\nabla(F) \subset \nu F = \omega \nu (\log b_+) \otimes F$; set $dR(F) := \text{Cone}(\nabla : F \to \nu F)$, $\text{RT}_{dR}(Y, F) := \Gamma(Y, dR(F))$. Then there is a canonical isomorphism

$$\eta_{dR}(F) : \mathcal{E}_{dR}(N)_{(D, 1_T, \nu)} \xrightarrow{\sim} \det \text{RT}_{dR}(Y, F) \quad (2.13.11)$$

defined as in the proposition in 2.7. Precisely, pick any $T \cup |D|$-lattices $E, E_\omega$ in $N$, $\omega N$ such that $\nabla(E) \subset E_\omega$. Denote by $FE$ the $T \cup \{b_+, b_-\} \cup |D|$-lattice in $j_{b_+, N}$ that equals $F$ off $T \cup |D|$ and $E$ off $b_+$; similarly, $F_{E_\omega}$ equals $E_\omega$ off $b_+$ and $\nu F$ off $T \cup |D|$. Now follow the construction from the proposition in 2.7, with $L, L_\omega, dR(M)$ from loc. cit. replaced by $FE, F_{E_\omega}$ and $dR(F)$. Namely, $dR(F)$ carries a 3-step filtration with successive quotients $F_{E_\omega}, F_{E_\omega}$, and $\eta_{dR}(F)$ is the composition $\mathcal{E}_{dR}(N)_{(D, 1_T, \nu)} \xrightarrow{\sim} \det \Gamma(Y, C(E, E_\omega)) \otimes \lambda(F_{E_\omega}/\nu FE) \xrightarrow{\sim} \det \Gamma(Y, C(E, E_\omega)) \otimes \det \text{RT}(Y, F_{E_\omega}) \otimes (\text{det } \text{RT}(Y, \nu FE))^{-1} \xrightarrow{\sim} \det \Gamma(Y, C(E, E_\omega)) \otimes \det \text{RT}(Y, F_{E_\omega}) \otimes \det \Gamma(Y, F_{E_\omega}[1]) \xrightarrow{\sim} \det \Gamma(Y, dR(F))$.

For example, for $F = L^- \neq (i)$ above one has $dR(L^-) = C_1$, so $\text{RT}_{dR}(Y, L^-)) = \text{RT}_{dR}(Y, N)$, and we get $\eta_{dR}(L^-) : \mathcal{E}_{dR}(N)_{(D, 1_T, \nu)} \xrightarrow{\sim} \det \text{RT}_{dR}(Y, N)$. We can also view $L^-$ as a lattice in $N$, and compute $\eta_{dR} : \mathcal{E}_{dR}(N)_{\nu} \xrightarrow{\sim} \text{RT}_{dR}(Y, N)$ using it (as in the proposition in 2.7). Now the lemma in 2.7 implies that $\eta_{dR}(L^-)$ equals the composition $\mathcal{E}_{dR}(N)_{(D, 1_T, \nu)} \xrightarrow{(2.13.9)} \mathcal{E}_{dR}(N)_{\nu} \xrightarrow{\eta_{dR}} \text{RT}_{dR}(Y, N)$.

Exercise. If $F' \subset F$ is a sublattice with $\nabla(F') \subset \nu F'$, then $dR(F)/dR(F') = \text{Cone}(\nabla : F/F' \to \nu F/\nu F')$. Thus $dR(F)$ carries a 3-step filtration with successive quotients $dR(F'), \nu F/\nu F', F/F'$, hence $\det \text{RT}_{dR}(Y, F') \xrightarrow{\sim} \det \text{RT}_{dR}(Y, F) \otimes \det \Gamma(Y, F/F') \otimes \det \Gamma(Y, \nu F/\nu F') \otimes (\text{det } \text{RT}(Y, \nu FE))^{-1}$. The multiplication by $\nu$ isomorphism $F/F' \xrightarrow{\sim} \nu F/\nu F'$ cancels the last two factors, i.e., we have $\det \text{RT}_{dR}(Y, F) \xrightarrow{\sim} \Gamma_{dR}(Y, F')$. Show that this isomorphism equals $\eta_{dR}(F') \eta_{dR}(F)^{-1}$.

One can repeat the above story with $Y$ replaced by $X_n, j_{b_+, N}$ by $j_{b_+, N} \otimes R_n$, and $E, E_\omega$ by $E \otimes R_n, E_\omega \otimes R_n$. For a $b_0$-lattice $G$ in $j_{b_0, N} \otimes R_n$ (i.e., an $\mathcal{O}_{X_n}$-submodule, which is $R_n$-flat and equals $j_{b_0, N} \otimes R_n$ outside $b_0$) such that $\nabla(G) \subset \nu G$, we get an isomorphism

$$\eta_{dR}(G) : \mathcal{E}_{dR}(N)_{(D, 1_T, \nu)} \otimes R_n \xrightarrow{\sim} \det \text{RT}_{dR*}(G) \quad (2.13.12)$$

For $G = M_n$, this is (2.13.3) combined with (2.13.4). If $G$ is a “constant” lattice, $G = F \otimes R_n$, then $\text{RT}_{dR*}G = \text{RT}_{dR}(Y, F) \otimes R_n$ and $\eta_{dR}(G) = \eta_{dR}(F) \otimes \text{id } R_n$.

By above, the proposition means that $\eta_{dR}(L^-) \eta_{dR}(M_n)^{-1} : \det \text{RT}_{dR*}M \xrightarrow{\sim} \det \text{RT}_{dR}(Y, N) \otimes R_n$ coincides with (2.13.7). Set $L^{(n)} := \mathcal{I}^{n+1} L$, where $\mathcal{I}$ is the ideal of $\{b_+, b_-\} \subset \mathcal{O}_Y$. Then $L^{(n)}$ lies in both $L^-$ and $M_n$. Set $P_N := L^-/L^{(n)}$ and $P_M := M_n/L^{(n)} \otimes R_n$; let $B_N, B_M$ be their endomorphisms $\nu^{-1}\nabla$. One has evident isomorphisms $\text{Cone}(B_M) \xrightarrow{\sim} \Gamma(X_n, dR(M_n)/dR(L^{(n)} \otimes R_n), \text{Cone}(B_N) \xrightarrow{\sim} \Gamma(Y, dR(L^-)/dR(L^{(n)}))$. Thus det $\text{RT}_{dR*}M = \det \text{RT}_{dR}(Y, L^{(n)}) \otimes \text{det } \text{Cone}(B_M)$, det $\text{RT}_{dR}(Y, N) \xrightarrow{\sim} \det \text{RT}_{dR}(Y, L^{(n)}) \otimes \text{det } \text{Cone}(B_N)$, so both $\eta_{dR}(L^-) \eta_{dR}(M_n)^{-1}$ and (2.13.7) can be rewritten as isomorphisms $\text{det } \text{Cone}(B_M) \xrightarrow{\sim} \text{det } \text{Cone}(B_M) \otimes R_n$.

Both $\text{det } \text{Cone}(B_M)$ and $\text{det } \text{Cone}(B_N)$ are naturally trivialized (since $\text{det } \text{Cone}(B_M) = \text{det } (P_M) \otimes \text{det } (P_M[1])$, etc.). By Exercise, $\eta_{dR}(L^-) \eta_{dR}(M_n)^{-1}$ identifies these trivializations. Isomorphism (2.13.7) comes due to the fact that
components \( B_0 \) immediately into the complex-analytic setting of 1.15. Thus every triple \((X,T,M)\), where \( T \) is a coordinate disc, and \( M \) is a holonomic \( D \)-module, yields a factorization line \( E_{\text{dR}}(M) \in \mathcal{L}_{\text{dR}}(X,T) \) in the complex-analytic setting of 1.15. If \( X \) and \( M \) came from an algebraic setting, then \( E_{\text{dR}}(M) \) is an analytic factorization line produced by the algebraic one (defined previously). If an algebraic family of \( D \)-modules is nice (see 2.12), then the corresponding analytic family is nice.

We work in the analytic setting. Let \( q : X \to Q \), \( i : T \to X \) be as in 1.14; we assume that \( T \) is étale over \( Q \) (see 1.15). Let \( M \) be a flat family of \( D \)-modules on \((X/Q,T)\) which admits locally a \( T \)-lattice, see 2.12. Consider the sheaf-theoretic restriction \( F := i \cdot dR_{X/Q}(M) \) of the relative de Rham complex to \( T \). Since \( q|_T \mathcal{O}_Q = \mathcal{O}_T \), this is a complex of \( \mathcal{O}_T \)-modules.

**Lemma.** \( M \) is nice if and only if \( F \) has \( \mathcal{O}_T \)-coherent cohomology.

*Proof.* The assertion is \( Q \)-local, so we can assume that \( T \) is a disjoint sum of several copies of \( Q \). Since \( q_* (dR_{X/Q}(M)/\text{Cone}(L \overset{\nabla}{\to} L_\omega)) \) is the direct sum of pieces corresponding to the components of \( T \), we are reduced to the situation when \( X \) equals \( U \times Q \), where \( U \subset \mathbb{A}^1 \) is a coordinate disc, and \( T = \{0\} \times Q \).

\( M \) extends in a unique manner to a \( D_{\mathbb{A}^1/Q} \)-module on \( \mathbb{A}^1_Q \) which is smooth outside \( T \); denote it also by \( M \). So we can assume that \( X = \mathbb{A}^1_Q \). Set \( \tilde{X} := \mathbb{P}^1_Q \); let \( \tilde{q} : \tilde{X} \to Q \) be the projection, so \( X = \tilde{X} \setminus T^\infty, T^\infty_\infty := \{\infty\} \times Q \).

Let us extend \( M \) to an \( \mathcal{O}_\tilde{X} \)-module \( \tilde{M} \) such that the relative connection has logarithmic singularity at \( T^\infty \). Such an \( \tilde{M} \) exists locally on \( Q \). Replacing \( \tilde{M} \) by some \( \tilde{M}(nT^\infty) \), we can assume that the eigenvalues of \(-t\partial_t\) in the fiber of \( \tilde{M} \) over \( T^\infty \) do not meet \( \mathbb{Z}_{\geq 0} \). Let \( dR_{\tilde{X}/Q}(\tilde{M}) := \text{Cone}(\tilde{M} \overset{\nabla}{\to} \omega \tilde{M}(T^\infty)) \) be the relative de Rham complex of \( \tilde{M} \) with logarithmic singularities at \( T^\infty \). One has the usual quasi-isomorphisms

\[
R\tilde{q}_*(dR_{\tilde{X}/Q}(\tilde{M})) \sim q_* (dR_{X/Q}(M)) \sim i_T dR_{X/Q}(M). \tag{2.14.1}
\]

Let \( \tilde{L} \subset \tilde{M} \), \( \tilde{L}_\omega \subset \omega \tilde{M}(T^\infty) \) be \( \mathcal{O}_{\tilde{X}} \)-submodules that equal \( L \), \( L_\omega \) on \( X \) and coincide with \( \tilde{M} \), \( \omega \tilde{M}(T^\infty) \) outside \( T \). Now \( dR_{X/Q}(M)/\text{Cone}(L \to L_\omega) \) equals \( dR_{\tilde{X}/Q}(\tilde{M})/\text{Cone}(\tilde{L} \to \tilde{L}_\omega) \), so (2.14.1) yields an exact triangle

\[
R\tilde{q}_* \text{Cone}(\tilde{L} \to \tilde{L}_\omega) \to i_T dR_{X/Q}(M) \to q_* dR_{X/Q}(M)/\text{Cone}(L \to L_\omega). \tag{2.14.2}
\]

Its left term is \( \mathcal{O}_Q \)-coherent, so the other two are coherent simultaneously, q.e.d. \( \square \)
3 The de Rham ε-lines: analytic theory

From now on we work in the analytic setting over \( \mathbb{C} \) using the classical topology.

3.1. Let \( X \) be a smooth (not necessary compact) complex curve, \( T \) its finite subset. For a holonomic \( \mathcal{D} \)-module \( M \) we denote by \( B(M) \) the de Rham complex \( dR(M) \) viewed as mere perverse \( \mathbb{C} \)-sheaf on \( X \), and set \( H^{\mathbb{R}}_B(X, M) := H(X, B(M)) \); thus one has an evident period isomorphism \( \rho : H^{\mathbb{R}}_B(X, M) \sim H^{dR}_d(X, M) \). Here is the principal result of this section:

**Theorem-construction.** Let \( M, M' \) be holonomic \( \mathcal{D} \)-modules on \( (X,T) \). Then every isomorphism \( \phi : B(M) \sim B(M') \) yields naturally an identification of the de Rham factorization lines \( \phi^* : \mathcal{E}^{dr}_d(M) \sim \mathcal{E}^{dr}_d(M') \). The construction has local origin, and is compatible with constraints from 2.8. If \( X \) is compact, then the next diagram of isomorphisms commutes:

\[
\begin{array}{ccc}
\mathcal{E}^{dr}_d(M)(X) & \xrightarrow{\phi^*} & \mathcal{E}^{dr}_d(M')(X) \\
\eta^{dr}\downarrow & & \eta^{dr}\downarrow \\
det H^{dr}_d(X, M) & \xrightarrow{\rho} & det H^{dr}_d(X, M') \\
\end{array}
\]

(3.1.1)

The idea of the proof: By a variant of Riemann-Hilbert correspondence, \( B(M) \) amounts to the \( \mathcal{D}^{\infty} \)-module \( M^{\infty} \). Thus what we need is to render the story of \( \S 2 \) into the analytic setting of \( \mathcal{D}^{\infty} \)-modules, which is done using a version of constructions from [PS, SW].

An alternative proof of the theorem, which uses 2.13 and \( \S 4 \) instead of analytic Fredholm determinants, is presented in 5.8. Thus the reader can skip the rest of the section and pass directly to \( \S 4 \).

3.2. A digression on \( \mathcal{D}^{\infty} \)-modules and Riemann-Hilbert correspondence. For the proofs of the next results, see [Bj] III 4, V 5.5, or [Mc].

For a complex variety \( X \) we denote by \( \mathcal{D}^{\infty} \) or \( \mathcal{D}^{\infty}_X \) the sheaf of differential operators of infinite order on \( X \). If \( X \) is a curve and \( U \) is an open subset with a coordinate function \( t \), then \( \mathcal{D}^{\infty}(U) \) consists of series \( \sum_{n \geq 0} a_n \partial^n_t \), where \( a_n \) are holomorphic functions on \( U \) such that for every \( \epsilon > 0 \) the series \( \sum a_n \epsilon^{-n}n! \) converges absolutely on any compact subset. \( \mathcal{D}^{\infty}_X \) is a sheaf of rings that acts on \( \mathcal{O}_X \) in an evident manner. It contains \( \mathcal{D}_X \), and \( \mathcal{D}^{\infty}_X \) is a faithfully flat \( \mathcal{D}_X \)-module.

By Grothendieck and Sato, one can realize \( \mathcal{D}^{\infty}(U) \) as \( H^{dimX}_{\Delta(U)}(U \times U, \mathcal{O} \boxtimes \omega) \) where \( \Delta \) is the diagonal embedding. If \( X \) is a curve and \( U \) has no compact components, this means that

\[
\mathcal{D}^{\infty}(U) = (\mathcal{O} \boxtimes \omega)(U \times U \setminus \Delta(U))/(\mathcal{O} \boxtimes \omega)(U \times U).
\]

(3.2.1)

Here \( k(x,y) \in (\mathcal{O} \boxtimes \omega)(U \times U \setminus \Delta(U)) \) acts on \( \mathcal{O}(U) \) as \( f \mapsto k(f), k(f)(x) := \text{Res}_{y=x} k(x,y)f(y) \). For a \( (\mathcal{L}) \) \( \mathcal{D} \)-module \( M \) set \( M^{\infty} := \mathcal{D}^{\infty} \otimes \mathcal{D} \). The embedding \( dR(M) \hookrightarrow dR(M^{\infty}) \) is a quasi-isomorphism, hence \( H^{dR}_d(X, M) \sim H^{dR}_d(X, M^{\infty}) \). If \( M \) is smooth, then \( M \sim M^{\infty} \).

\[\text{By [I], } \mathcal{D}^{\infty}_X \text{ coincides with the sheaf of all } \mathbb{C} \text{-linear continuous endomorphisms of } \mathcal{O}_X.\]
For a $D^\infty$-module $N$ a $D$-structure on $N$ is a $D$-module $M$ together with a $D^\infty$-isomorphism $M^\infty \sim N$. Our $N$ is said to be holonomic if it admits a $D$-structure with holonomic $M$.

For a holonomic $D$-module $M$ the de Rham complex $dR(M)$ is a perverse $\mathbb{C}$-sheaf, which we denote, as above, by $B(M)$; same for a holonomic $D^\infty$-module. Therefore $B(M) = B(M^\infty)$. The functor $B$ is an equivalence between the category of holonomic $D^\infty$-modules and that of perverse $\mathbb{C}$-sheaves. The inverse functor assigns to a perverse sheaf $F$ the $D^\infty$-module $[17]_{\mathbb{C}} \mathcal{O}_X \oplus F[\dim X]$. Thus for a holonomic $M$ the $D^\infty$-module $M^\infty$ carries the same information as $B(M)$.

The functor $M \mapsto M^\infty$ yields an equivalence between the category of holonomic $D$-modules with regular singularities and that of holonomic $D^\infty$-modules. Its inverse assigns to a holonomic $D^\infty$-module $N$ its maximal $D$-submodule $N^s$ with regular singularities, so one has
\[(N^s)^\infty = N. \tag{3.2.2}\]

Therefore every holonomic $D^\infty$-module admits a unique $D$-structure with regular singularities.

**Exercises.** Let $U$ be a coordinate disc, $t$ be the coordinate, $j$ be the embedding $U^o := U \setminus \{0\} \hookrightarrow U$.

(i) Recall a description of indecomposable $D_U$-modules which are smooth of rank $n$ on $U^o$ and have regular singularity at $0$. For $s \in \mathbb{C}$ denote by $M_{s,n}$ a $D$-module whose sections are collections of functions $(f_i) = (f_1, \ldots, f_n)$ having meromorphic singularity at $0$, and $\nabla_{\partial_t}((f_i)) = (\partial_t (f_i) + s f_i + f_{i-1})$. Let $M_{0,n}^1$ be a $D$-submodule of $M_{0,n}$ formed by $(f_i)$ with $f_1$ regular at $0$. Consider an embedding $O_U \hookrightarrow M_{0,n+1}$, $f \mapsto (0, \ldots, 0, f)$; set $M_{0,n}^2 := M_{0,n+1}/\mathcal{O}$, $M_{3,0}^0 := M_{0,n+1}/\mathcal{O}$. E.g., $M_{0,1}^1 = O_U$, and $M_{0,0}^2 = \delta$ (the $\delta$-function $D$-module). Then any indecomposable $D_U$-module $M$ as above is isomorphic to either some $M_{s,n}$, or one of $M_{0,n}^a$, $a = 1,2,3$.

Show that the corresponding $D^\infty$-module $M^\infty$ has the same explicit description with “meromorphic singularity” replaced by “arbitrary singularity”.

(ii) For $n > 0$ let $E_{(n)}$ be a $D$-module of rank $1$ generated by $\exp(t^{-n})$, i.e., $E_{(n)}$ is generated by a section $e$ subject to the (only) relation $t^{n+1} \partial_t(e) = -ne$. Show that there is an isomorphism of $D^\infty$-modules
\[E_{(n)}^\infty \sim M_{0,1}^{2\infty} \oplus (\delta^\infty)^{n-1}. \tag{3.2.3}\]

Here is an explicit formula for (3.2.3). Let $g(z)$, $h_1(z), \ldots, h_{n-1}(z)$ be entire functions such that $(\partial_z - nz^{n-1})g(z) = z^{-2}(\exp(z^n) - 1 - z^n)$ and $(\partial_z - n z^{n-1})h_i(z) = z^{i-1}$. Then (3.2.3) assigns $e$ a vector whose $(M_{0,1}^2)^\infty$-component is $(\exp(t^{-n}), g(t^{-1}))$ and the $\delta^\infty$-components are $h_i(t^{-1}) \in (M_{0,0}^2)^\infty = \delta^\infty$.

### 3.3. A digression on Fredholm determinants (cf. [FS] 6.6).

Recall that a *Fréchet space* is a complete, metrizable, locally convex topological $\mathbb{C}$-vector space. The category $\mathcal{F}$ of those is a quasi-abelian (hence exact) Karoubian $\mathbb{C}$-category. A morphism $\phi : F \to F'$ is said to be Fredholm if it is Fredholm as a morphism of abstract vector spaces, i.e., if $\text{Ker} \phi$ and $\text{Coker} \phi$ have finite dimension. Then $\phi$ is a split morphism, i.e., $\text{Ker} \phi$, $\text{Im} \phi$ are direct summands of, respectively, $F$ and $F'$, $\phi$ acts via the $\mathcal{O}_X$-factor, and $F \otimes^G G := R\Delta^* F \boxtimes G$. $M_{s,n}$ depends only on $s$ modulo $\mathbb{Z}$-translation: one has $M_{s,n} \sim M_{s-1,n}, (f_i) \mapsto (t f_i)$.}$
and $F/Ker\phi \sim \text{Im}\phi$. Denote by $\mathcal{F} \subset \mathcal{F}$ the subcategory of Fredholm morphisms $\mathcal{F}(F, F') \subset \text{Hom}(F, F')$.

A Fredholm $\phi$ yields the determinant line $\lambda_{\phi} := \text{det}(\text{Coker}\phi) \otimes \text{det}^{\otimes -1}(\text{Ker}\phi) \in \mathcal{L} := \mathcal{L}_{\mathcal{E}}$ (see 1.2). Sometimes we denote $\lambda_{\phi}$ by $\lambda(F' \overset{\phi}{\to} F)$ or, if $F = F'$, by $\lambda(F)$. If $\phi$ is invertible, then $\lambda_{\phi}$ has an evident trivialization; denote it by $\text{det}(\phi) \in \lambda_{\phi}$.

For any Fredholm $\phi$ one can find finite-dimensional $F_0 \subset F$, $F_0' \subset F'$ such that $\phi(F_0) \subset F_0$ and the induced map $F'/F_0' \to F/F_0$ is an isomorphism (equivalently, $F_0 + (\phi(F')) = F$, $F_0' = \phi^{-1}(F_0)$). Then the exact sequence $0 \to \text{Ker}\phi \to F_0' \to F_0 \to \text{Coker}\phi \to 0$ yields a natural isomorphism

$$\lambda_{\phi} \sim \text{det}(F_0) \otimes \text{det}^{\otimes -1}(F_0').$$

If $\phi$ is invertible, then (3.3.1) identifies $\text{det}(\phi) \in \lambda_{\phi}$ with the usual determinant of $\phi|_{F_0'} : F_0' \sim F_0$ in $\text{Hom}(\text{det} F_0', \text{det}(F_0)) = \text{det}(F_0) \otimes \text{det}^{\otimes -1}(F_0')$.

For Fredholm $F'' \overset{\phi}{\to} F' \overset{\psi}{\to} F$ there is a canonical “composition” isomorphism

$$\lambda_{\psi} \otimes \lambda_{\phi} \sim \lambda_{\phi \circ \psi}, \quad a \otimes b \mapsto ab,$n

which satisfies the associativity property. Therefore $\phi \mapsto \lambda_{\phi}$ is a (central) $\mathcal{L}$-extension $\mathcal{F}\sslash_{\mathcal{E}}$ of $\mathcal{F}$ (see e.g. [BBE], Appendix to §1, for terminology). To construct (3.3.2), choose $F_0$, $F_0'$, $F_0''$ for $\phi$, $\phi'$ as above; then (3.3.1) identifies the composition with an evident map $\text{det}(F_0') \otimes \text{det}^{\otimes -1}(F_0') \otimes \text{det}(F_0) \otimes \text{det}^{\otimes -1}(F_0') \sim \text{det}(F_0) \otimes \text{det}^{\otimes -1}(F_0')$. For invertible $\phi$, $\phi'$ one has $\text{det}(\phi) \text{det}(\phi') = \text{det}(\phi \phi')$.

Suppose $F$, $F'$ are equipped with finite split filtrations $F$, $F'$, $\phi : F' \to F$ preserves the filtrations, and $\text{gr}\phi : \text{gr}F' \to \text{gr}F$ is Fredholm. Then $\phi$ is Fredholm, and there is a canonical isomorphism

$$\lambda_{\phi} \sim \otimes \lambda_{\text{gr},\phi}.$$n

The identification is transitive with respect to refinement of the filtration. If $\text{gr}\phi$ is invertible, it identifies $\text{det}(\text{gr}\phi) = \otimes \text{det}(\text{gr},\phi)$ with $\text{det}(\phi)$. For example, for a finite collection of Fredholm morphisms $\{\phi_\alpha\}$, every linear ordering of indices $\alpha$ produces a filtration, hence an isomorphism $\lambda_{\phi_\alpha} \sim \otimes \lambda_{\phi_\alpha}$; it does not depend on the ordering.

Let $\mathcal{T}^\text{fin} \subset \mathcal{T}^\text{tr} \subset \mathcal{T}^\text{com}$ be the two-sided ideals of finite rank, nuclear, and compact morphisms in $\mathcal{F}$. We have the quotient categories $\mathcal{F}/\mathcal{T}^\text{tr}$: their objects are Fréchet spaces, and morphisms $\text{Hom}_{/\mathcal{T}}(F, F')$ equal $\text{Hom}(F, F')/\mathcal{T}^\text{tr}(F, F')$. A morphism $\phi$ is Fredholm if and only if $\phi$ is invertible in either $\mathcal{F}/\mathcal{T}^\text{tr}$. Therefore the groupoids $\text{Isom}(\mathcal{F}/\mathcal{T}^\text{tr})$ of isomorphisms in $\mathcal{F}/\mathcal{T}^\text{tr}$ are quotients of $\mathcal{F}$ modulo the $\mathcal{T}^\text{tr}$-equivalence relation $\phi \sim \phi'$ in $\mathcal{T}^\text{tr}$.

Exercise. Let $G^\text{tr}(F) \subset \text{Aut}(F)$ be the (normal) subgroup of automorphisms $\psi$ of $F$ that are $\mathcal{T}^\text{tr}$-equivalent to $\text{id}_F$. The next sequence is exact:

$$1 \to G^\text{tr}(F)/G^\text{fin}(F) \to \text{Aut}_{/\mathcal{T}^\text{fin}}(F) \to \text{Aut}_{/\mathcal{T}^\text{tr}}(F) \to 1$$

(3.3.4)

**Proposition.** $\mathcal{F}\sslash_{\mathcal{E}}$ descends naturally to an $\mathcal{L}$-extension $\text{Isom}\sslash_{\mathcal{E}}(\mathcal{F}/\mathcal{T}^\text{tr})$ of the groupoid $\text{Isom}(\mathcal{F}/\mathcal{T}^\text{tr})$.

**Proof.** We first descend $\mathcal{F}\sslash_{\mathcal{E}}$ to $\text{Isom}(\mathcal{F}/\mathcal{T}^\text{fin})$, and then to $\text{Isom}(\mathcal{F}/\mathcal{T}^\text{tr})$.

(i) To descend $\mathcal{F}\sslash_{\mathcal{E}}$ to $\text{Isom}(\mathcal{F}/\mathcal{T}^\text{fin})$, means to define for every $\mathcal{T}^\text{fin}$-equivalent $\phi, \psi \in \mathcal{F}\sslash_{\mathcal{E}}(F, F')$ a natural identification $\tau = \tau_{\phi, \psi} : \lambda_{\phi} \sim \lambda_{\psi}$ which satisfies the transitivity property and is compatible with composition.
The condition on \( \phi, \psi \) means that we can find finite-dimensional \( F_0 \subset F, F'_0 \subset F' \) such that \( \phi(F_0), \phi(F'_0) \subset F_0, \phi(F'/F'_0) \subset F/F_0 \) induced by \( \phi \) (or \( \psi \)) is an isomorphism. Then \( \tau \) is the composition \( \lambda_\phi \circ \det(F_0) \circ \det(\phi^{-1}(F'_0)) \circ \lambda_\psi \) of isomorphisms (3.3.1) for \( \phi, \psi \). The construction does not depend on the choice of auxiliary datum, and satisfies the necessary compatibilities.

(ii) Recall that for \( \psi \in EndF \) that is \( T^1 \)-equivalent to \( id_F \), its Fredholm determinant \( \det_{\mathcal{F}}(\psi) \in \mathbb{C} \) is defined (see e.g. [Gr2]) as the sum of a rapidly converging series

\[
\det_{\mathcal{F}}(\psi) := \sum_{k \geq 0} \text{tr} \Lambda^k (\psi - id_F),
\]

where \( \Lambda^k (\psi - id_F) \) is the \( k \)-th exterior power of \( \psi - id_F \). If \( \psi - id_F \) is of finite rank, then the sum is finite, and \( \det_{\mathcal{F}}(\psi) \) is the usual determinant \( \psi \).

The central \( C^x \)-extension \( Aut^x(F) \) of \( Aut(F) \) is trivialized by the section \( \psi \mapsto \det(\psi) \). The Fredholm determinant is multiplicative and invariant with respect to the adjoint action of \( Aut(F) \). We get a trivialization \( \psi \mapsto \tau_{\mathcal{F}}^\psi := \det_{\mathcal{F}}^{-1} (\psi) \det(\psi) \) of the \( C^x \)-extension \( G^x(F)^b \) which is invariant for the adjoint \( Aut(F) \)-action.

Since for \( \psi \in G^{\text{fin}}(F) \) one has \( \tau_{\mathcal{F}}^\psi = \tau_{id_F,\psi} \in \lambda_\psi \), our \( \tau_{\mathcal{F}}^\psi \) can be viewed as a trivialization of the extension \( Aut^x(F) \) over the normal subgroup \( G^x(F)/G^{\text{fin}}(F) \). It is invariant with respect to the adjoint action of \( Aut_{/T^1}(F) \). Thus, by (3.3.4), \( \tau_{\mathcal{F}}^\psi \) defines a descent of \( Aut^x_{/T^1}(F) \) to an \( L \)-extension \( Aut^x_{/T^1}(F) \) of \( Aut_{/T^1}(F) \).

More generally, for every \( F, F' \in \mathcal{F} \), the set \( \text{Isom}_{/T^1}(F, F') \) is a \((G^x(F)/G^{\text{fin}}(F'), G^x(F)/G^{\text{fin}}(F))\)-bitorsor over \( \text{Isom}_{/T^1}(F, F') \), and we define the \( L \)-extension \( \text{Isom}_{/T^1}(F, F') \) as the quotient of \( \text{Isom}_{/T^1}(F, F') \) by the \( \tau_{\mathcal{F}}^\psi \)-lifting of either \( G^x(F)/G^{\text{fin}}(F) \)- or \( G^x(F)/G^{\text{fin}}(F') \)-action. \( \square \)

Remark. The above constructions are compatible with constraint (3.3.3).

3.4. For a topological space \( X \) whose topology has countable base, a Fréchet sheaf on \( X \) means a sheaf of Fréchet vector spaces. A Fréchet algebra \( A \) is a sheaf of topological algebras which is a Fréchet sheaf; a Fréchet \( A \)-module is a Fréchet sheaf equipped with a continuous (left) \( A \)-action.

The problem of finding Fréchet structures on a given \( A \)-module \( M \) is delicate. Here is a simple uniqueness assertion. Suppose that \( M \) satisfies the next condition: those open subsets \( U \) of \( X \) that \( M(U) \) is a finitely generated \( A(U) \)-module form a base of the topology of \( X \).

Lemma. Every morphism of \( A \)-modules \( \phi : M \to N \) is continuous with respect to any Fréchet structures on \( M, N \). Thus \( M \) admits at most one Fréchet structure.

Proof. It suffices to check that the maps \( \phi_U : M(U) \to N(U) \) are continuous for all \( U \) as above. Thus there is a surjective \( A(U) \)-linear map \( \pi_U : A(U)^n \to M(U) \). The maps \( \pi_U \) and \( \phi_U \pi_U \) are evidently continuous. Since \( A(U)^n/\text{Ker}(\pi_U) \to M(U) \) is a continuous algebraic isomorphism of Fréchet spaces, it is a homeomorphism, and we are done. \( \square \)

Example. Every locally free \( A \)-module of finite rank is a Fréchet \( A \)-module.

From now on our \( X \) is a complex curve. The two basic examples of Fréchet algebras on \( X \) are \( \mathcal{O}_X \) and \( \mathcal{D}_X^\infty \). For an open \( U \subset X \) the topology on the space of holomorphic functions \( \mathcal{O}(U) \) is that of uniform convergence on compact subsets of \( F_0 \) is any finite-dimensional subspace containing the image of \( \psi - id_F \).
If $t$ is a coordinate function on $U$, then the topology on $\mathcal{D}^\infty(U)$ is given by a collection of semi-norms $||\sum a_n \partial^\alpha_t||_{K\infty}$: here $K$ is any compact subset of $U$ and $\epsilon$ is any small positive real number. Equivalently, one can use (3.2.1): then $(\mathcal{O} \boxtimes \omega)(U \times U)$ is a closed subspace of $(\mathcal{O} \boxtimes \omega)(U \times U \setminus \Delta(U))$, and the topology on $\mathcal{D}^\infty(U)$ is the quotient one.

**Proposition.** Any holonomic $\mathcal{D}^\infty$-module $N$ on $X$ admits a unique structure of a Fréchet $\mathcal{D}^\infty_X$-module.

**Proof.** Uniqueness: As follows easily from Exercise (i) in 3.2, $N$ satisfies the condition of the previous lemma. Existence: The problem is local, so it suffices to define some Fréchet structure compatible with the $\mathcal{D}^\infty(U)$-action on $N(U)$, where $U$ is a disc and $N$ is smooth outside the center 0 of $U$. Then $N \sim M^\infty$ where $M$ is a $\mathcal{D}$-module with regular singularities; we can assume that $M$ is indecomposable.

If $M \sim M_{x,n}$ (see Exercise (i) in 3.2), then, by loc. cit., $M^\infty(U) \sim M^\infty(U^o) \sim \mathcal{O}(U^o)^n$, and we equip it with the topology of $\mathcal{O}(U^o)^n$. Otherwise $M^\infty(U)$ is a subquotient of some $M_{0,n}(U)$, and we equip it with the corresponding Fréchet structure.

**Question.** Can one find a less ad hoc proof (that would not use (3.2.2))? Is the assertion of the proposition remains true for all perfect $\mathcal{D}^\infty$-modules (or perfect $\mathcal{D}^\infty$-complexes) on $X$ of any dimension?\!

**3.5.** Let $K \subset X$ be a compact subset which does not contain a connected component of $X$; denote by $j_K$ the embedding $X \setminus K \hookrightarrow X$. Let $E$ be a Fréchet $\mathcal{O}_X$-module. Suppose that for some open neighborhood $U$ of $K$, $E|_{U \setminus K}$ is a locally free $\mathcal{O}_{U \setminus K}$-module of finite rank. A $K$-lattice in $E$ is an $\mathcal{O}_X$-module $L$, which is locally free on $U$, together with an $\mathcal{O}_X$-linear morphism $L \to E$ such that $L|_{X \setminus K} \sim E|_{X \setminus K}$. Then $L$ is a Fréchet $\mathcal{O}_X$-module, and $L \to E$ is a continuous morphism. Set $\Gamma(E/L) := H^0\Gamma(X,\text{Cone}(L \to E)) = H^0\Gamma(U,\text{Cone}(L \to E))$.

Shrinking $U$ if needed, we can assume that the closure $\bar{U}$ of $U$ is compact with smooth boundary $\partial U$. We denote by $\partial U$ a contour in $U \setminus K$ homologous to the boundary of $U$ in $U \setminus K$.

**Remarks.** (i) For all our needs it suffices to consider the situation when $U$ is a disjoint union of several discs.

(ii) If $\text{Int}(K) \neq \emptyset$, then the morphism $L \to E$ need not be injective. The map $L(U) \to E(U)$ is injective though, i.e., $H^{-1}\Gamma(U,\text{Cone}(L \to E)) = 0$.

**Example.** If $L$ is any locally free $\mathcal{O}_X$-module of finite rank, then $L$ is a $K$-lattice in $j_K^{-1}L := j_K^{-1}(L|_{X \setminus K})$.

**Proposition.** (i) $L(U)$ is a direct summand of the Fréchet space $E(U)$.

(ii) If $H^1(U,L) = 0$ (which happens, e.g., if none of the connected components of $U$ is compact), then $E(U)/L(U) \sim \Gamma(E/L)$.

(iii) Let $(E',L')$ be a similar pair, and $\phi : E' \to E$ be any morphism of Fréchet sheaves. Then the map $E'(U) \to E(U)$, viewed as a morphism in $\mathcal{D}/\mathcal{D}_U$, sends the subobject $L'(U)$ to $L(U)$.

\footnote{For a perfect $\mathcal{D}^\infty$-complex $N$, [PSCH] define a natural ind-Banach structure on its complex of solutions $R\text{Hom}_{\mathcal{D}^\infty}(N,\mathcal{O}_X)$. It is not clear if this result helps to see the topology on $N$.}
The composition $L'(U) \rightarrow E'(U) \rightarrow E(U) \rightarrow E(U') \rightarrow (U')/L(U')$ is continuous and hence a homeomorphism, and $U'$ is a directed set. The first assertion in (b) follows from (iii): for the second one, consider $\phi = \text{id}_E$.

\[ \lambda_K(L/L') := \lambda_{E/L,E/L'} \]

The composition

\[ \lambda_K(L/L') \otimes \lambda_K(L'/L'') \sim \lambda_K(L/L'') \]

comes from (3.3.2).

Suppose $\Lambda_K(E)$ is non-empty. We get an $\mathcal{L}$-torsor $\mathcal{D}et_K(E) := \text{Hom}_\mathcal{L}(\Lambda_K(E), \mathcal{L})$ of determinant theories on $E$ at $K$. For $E_1$, $E_2$ we get an $\mathcal{L}$-torsor $\mathcal{D}et_K(E_1/E_2) := \mathcal{D}et_K(E_1) \otimes \mathcal{D}et_K(E_2)^{-1}$ of relative determinant theories on $E_1/E_2$ at $K$.

If $K = \cup K_\alpha$, then a $K$-lattice $L$ amounts to a collection of $K_\alpha$-lattices $L_\alpha$, and one has an evident canonical isomorphism

\[ \otimes \lambda_{K_\alpha}(L_\alpha/L'_\alpha) \sim \lambda_K(L/L') \]

compatible with composition isomorphisms (3.6.2). Thus one has a canonical identification $\otimes \mathcal{D}et_{K_\alpha}(E) \sim \mathcal{D}et_K(E)$, $(\otimes \lambda_{K_\alpha})(L) = \otimes \lambda_{K_\alpha}(L_\alpha)$.

Below we fix a neighborhood $U$ of $K$ as in 3.5; we assume that it has no compact components, so for every $K$-lattice $L$ the $\mathcal{O}(U)$-module $L(U)$ is free.

$\Lambda_K(E)$ carries a natural topology of compact convergence on $U \setminus K$. Namely, to define a neighborhood of $L$ we pick an $\mathcal{O}(U)$-base $\{\ell_i\}$ of $L(U)$, a compact

\[ \text{if } U \text{ is compact, then } L(U) \text{ is finite dimensional, and the assertion follows from the Hahn-Banach theorem.} \]

\[ \text{Proof. (i) We want to construct a left inverse to the morphism of Fréchet spaces } L(U) \rightarrow E(U). \text{ It suffices to define a left inverse to the composition } L(U) \rightarrow E(U) \rightarrow E(U \setminus K) = L(U \setminus K), \text{ i.e., to the restriction map } L(U) \rightarrow L(U \setminus K). \]

We can assume that $U$ is connected and non compact. Then one can find a Cauchy kernel on $U \times U$, which is a section $\kappa$ of $L \otimes \omega L^*(\Delta)$ with residue at the diagonal equal to $-\text{id}_E$. The promised left inverse is $f \mapsto \kappa(f)$, $\kappa(f)(x) = \int \kappa(x,y) f(y)$.

(ii) Follows from the exact cohomology sequence.

(iii) We want to check that the composition $L'(U) \rightarrow E'(U) \rightarrow E(U) \rightarrow E(U') \rightarrow (U')/L(U')$ is connected and non compact. Choose an open $V \supset K$ whose closure $\bar{V}$ is compact and lies in $U$. Our map equals the composition $L'(U) \rightarrow L'(V) \rightarrow E'(V) \rightarrow E(V) \rightarrow E(V)/L(V) \sim E(U)/L(U)$ (for $\sim$, see (ii)). We are done, since the first arrow is nuclear (see e.g. [Gr1]).

Corollary. (a) The isomorphism from (ii) yields a natural Fréchet space structure on $\Gamma(E/L)$, which does not depend on the auxiliary choice of $U$.

(b) Every $\phi$ as in (iii) yields naturally a morphism $\phi_{E'/L',E/L} : \Gamma(E'/L') \rightarrow \Gamma(E/L)$ in $\mathcal{F}/\mathcal{T}^\mathcal{U}$. In particular, the spaces $\Gamma(E/L)$ for all $K$-lattices $L$ in $E$ are canonically identified as objects of $\mathcal{F}/\mathcal{T}^\mathcal{U}$.

Proof. (a) follows since, for $U' \subset U$ as in (ii), the restriction map $E(U)/L(U) \rightarrow E(U')/L(U')$ is a continuous algebraic isomorphism, hence a homeomorphism, and $U'$ is a directed set. The first assertion in (b) follows from (iii); for the second one, consider $\phi = \text{id}_E$.

\[ \text{Corollary. (a) The isomorphism from (ii) yields a natural Fréchet space structure on } \Gamma(E/L), \text{ which does not depend on the auxiliary choice of } U. \]

(b) Every $\phi$ as in (iii) yields naturally a morphism $\phi_{E'/L',E/L} : \Gamma(E'/L') \rightarrow \Gamma(E/L)$ in $\mathcal{F}/\mathcal{T}^\mathcal{U}$. In particular, the spaces $\Gamma(E/L)$ for all $K$-lattices $L$ in $E$ are canonically identified as objects of $\mathcal{F}/\mathcal{T}^\mathcal{U}$. □
$C \subset U \setminus K$ and a number $\epsilon > 0$. The neighborhood is formed by those $L'$ which admit a base $\{\ell_i'\}$ of $L'(U)$, $\ell_i' = \sum a_{ij} \ell_i$, with $(a_{ij})$ $\epsilon$-close to the unit matrix on $C$.

The $\mathcal{L}$-groupoid structure on $\Lambda_K(E)$ is continuous, i.e., $\lambda_K$ form naturally a line bundle on $\Lambda_K(E) \times \Lambda_K(E)$, and the composition is continuous (cf. [PS] 6.3, 7.7). Namely, if $P$ is a closed linear subspace of $E(U)$, then the subset $\Lambda_K^P(E)$ of those $K$-lattices $L$ that $L(U)$ is complementary to $P$, i.e., $P \supset \Gamma(E/L)$, is open in $\Lambda_K(E)$. For $L, L' \in \Lambda_K^P(E)$ the morphism $\text{id}_{E/L,E/L'}$ in $\mathcal{F}/\mathcal{T}^\Gamma$ is represented by $\text{id}_P$, hence the restriction of $\lambda_K$ to $\Lambda_K^P(E) \times \Lambda_K^P(E)$ is trivialized by the section $\delta^P := \det(\text{id}_P)$. The topology on $\lambda_K$ is uniquely determined by the condition that all these local trivializations are continuous, and the composition is continuous.

Remarks. (i) If $K'$ is a compact such that $K \subset K' \subset U$, then every $K$-lattice $L$ in $E$ is a $K'$-lattice, and $\lambda_K(L/L') = \lambda_K(L/L')$. Hence $\text{Det}_K(E) = \text{Det}_{K'}(E)$.

(ii) By (iii) of the proposition in 3.5, the subobjects $L(U), L'(U)$ of $E(U)$ coincide if viewed in $\mathcal{F}/\mathcal{T}^\Gamma$. Let $\phi_U : L(U) \xrightarrow{\sim} L'(U)$ be the corresponding isomorphism in $\mathcal{F}/\mathcal{T}^\Gamma$. Then there is a canonical identification $\lambda_K(L/L') \xrightarrow{\sim} \lambda_{\phi^{-1}_U}$

\[(3.6.4)\]

compatible with the composition maps. Indeed, by (3.3.3) and Remark at the end of 3.3 applied to the filtrations $L(j) \subset E(U)$, one has $\lambda_{\text{id}_{E/L,E/L'}} \otimes \phi_U \xrightarrow{\sim} \lambda_{\text{id}_{E(U)}} = C$.

(iii) Every $K$-lattice $L$ in $E$ can be viewed as a $K'$-lattice in $j_K^{-1}L := j_K^{-1}K(E)$ (via $L \to j_K(E)$). Then (3.6.4) shows that $\lambda_K(L/L')$ does not depend on whether we consider $L, L'$ as $K$-lattices in $E$ or in $j_K(E)$. Thus $\text{Det}_K(E) = \text{Det}_{K'}(j_K(E))$.

(iv) Let $S$ be an analytic space, $L_S$ be an S-family of $K$-lattices in $E$. Then the pull-back of $\lambda_K$ to $S \times S$ is naturally a holomorphic line bundle, so that the pull-back to $S$ of any local trivialization $\delta^P$ is holomorphic.

(v) If $L, L'$ are meromorphically equivalent, then $\lambda_K(L/L')$ coincides with the relative determinant line from 2.3 (where $P$ is a finite subset in $K$ such that $L$ equals $L'$ off $P$). Indeed, in view of (3.6.2), it suffices to identify the lines in case $L \supset L'$, where the identification comes from $L(U)/L(U) \xrightarrow{\sim} \Gamma(U,L/L')$. If $L, L'$ vary holomorphically as in (iv), then this identification is holomorphic.

(vi) For every $f \in \mathcal{O}(U \setminus K)$ the lines $\lambda_K(fL/L)$ for all $L \in \Lambda_K(j_K(E)$ are canonically identified. Namely, one defines the isomorphism $\lambda_K(fL'/L') \xrightarrow{\sim} \lambda_K(fL/L)$ as $\lambda_K(fL'/L') \xrightarrow{\sim} \lambda_K(fL'/L) \otimes \lambda_K(fL/L) \otimes \lambda_K(U/L')^{-1} \xrightarrow{\sim} \lambda_K(fL/L)$ where the first arrow is inverse to the composition, and the second comes from the multiplication by $f$ identification $\lambda_K(L'/L) \xrightarrow{\sim} \lambda_K(fL'/fL)$. \hspace{100em}

(vii) Let $g \in \mathcal{O}(U)$ be an invertible function. The multiplication by $g$ automorphism of $E|_U$ preserves every $K$-lattice. Let $g(L/L') \in C^\times$ be the corresponding automorphism of $\lambda_K(L/L')$.

Example. Suppose that $\lambda_K(L/L')$ has degree 0. Choose a Fréchet isomorphism $\alpha : \Gamma(E/L) \xrightarrow{\sim} \Gamma(E/L')$ which represents $\text{id}_{E/L,E/L'}$. Then $g(L/L')$ is the Fredholm determinant $\det(g^{-1}g, g^{-1}_E/L')$, where $g_E/L, g_E/L'$ are multiplication by $g$ automorphisms of $\Gamma(E/L), \Gamma(E/L')$.

Here is a formula for $g(L/L')$ (cf. [PS] 6.7, [SW] 3.6). Consider the line bundle $\det E|_{U \setminus K}$. Then $L \in \Lambda_K(j_K(E)$ yields a $K$-lattice $\det L|_U \in \Lambda_K(j_K, \det E|_{U \setminus K})$. We can assume that $U$ has no compact components. Then the line bundle $\det L|_U$ is an $\mathcal{O}_X$-module together with a morphism $L \to E_S$ such that $L_S$ is locally free on $U_S$, and $L \to E_S$ is an isomorphism off $K_S$.\footnote{I.e., $L_S$ is an $\mathcal{O}_X$-module together with a morphism $L \to E_S$ such that $L_S$ is locally free on $U_S$, and $L \to E_S$ is an isomorphism off $K_S$.}
is trivial; let \( \theta_L \) be any its trivialization. For two lattices \( L, L' \) we get a function \( \theta_{L'}/\theta_L \in \mathcal{O}^\times(U \setminus K) \). Consider the analytic symbol \( \{g, \theta_{L'}/\theta_L\} \in H^1(U \setminus K, \mathbb{C}^\times) \). Then (see 3.5 for the notation \( \partial U \))

\[
g(L/L') = \{g, \theta_{L'}/\theta_L\}(\partial U).
\]

(3.6.5)

To check (3.6.5), consider first the case when \( L, L' \) are meromorphically equivalent. Then \( \theta_{L'}/\theta_L \) is a meromorphic function on \( U \), and both parts of (3.6.5) evidently coincide with \( g(\text{div}(\theta_{L'}/\theta_L)) \). The general case follows since for any \( L, L' \) one can find (possibly enlarging \( K \), as in (ii)) an \( L'' \) meromorphically equivalent to \( L \) which is arbitrary close to \( L' \), and both parts of (3.6.5) depend continuously on \( L' \).

3.7. Let \( N \) be a holonomic \( D^\infty \)-module on \( X \). By 3.4, it carries a canonical Fréchet structure. For \( K \) as in 3.5, 3.6, let us define a relative determinant theory

\[
\mu_K^\nabla = \mu(N/\omega N)^\nabla_K \in \text{Det}_K(N/\omega N) \quad \text{(cf. 2.4).}
\]

If \( L, L_\omega \) are \( K \)-lattices in \( N, \omega N \) such that \( \nabla(L) \subset L_\omega \), then \( \nabla \) yields a morphism of sheaves \( N/L \rightarrow \omega N/L_\omega \), and we denote by \( \mathcal{C}(L, L_\omega)_{N,K} \) its cone.

Example. Let \( M^\infty \rightarrow N \) be a \( D \)-structure on \( N \) (see 3.2), and \( P \) be any finite subset of \( K \) such that \( M \) is smooth on \( K \setminus P \). Every \( P \)-lattices \( L, L_\omega \) in \( M, \omega M \) can be viewed as \( K \)-lattices in \( N, \omega N \). If \( \nabla(L) \subset L_\omega \), then we get an evident morphism of complexes of sheaves (see 2.4)

\[
\mathcal{C}(L, L_\omega)_{M,P} \rightarrow \mathcal{C}(L, L_\omega)_{N,K}.
\]

(3.7.1)

Proposition. (3.7.1) is a quasi-isomorphism.

Proof. Our complexes are supported on a finite set \( P \), so it suffices to check that \( R\Gamma(X, (3.7.1)) \) is a quasi-isomorphism. Notice that \( dR(L, L_\omega) := \text{Cone}(L \rightarrow L_\omega) \) is a subcomplex of both \( dR(M) \) and \( dR(N) \), and \( \mathcal{C}(L, L_\omega)_{M,P} = dR(M)/dR(L, L_\omega), \mathcal{C}(L, L_\omega)_{N,P} = dR(N)/dR(L, L_\omega) \). Since \( dR(M) \rightarrow dR(N) \) is a quasi-isomorphism (see 3.2), we are done.

By the corollary in 3.5, for \( K \)-lattices \( L, L_\omega \) in \( N, \omega N \) one has a \( \mathcal{F}/\mathcal{F}^\omega \)-morphism

\[
\nabla_{N/L,\omega N/L_\omega} : \Gamma(N/L) \rightarrow \Gamma(\omega N/L_\omega).
\]

(3.7.2)

Corollary. (3.7.2) is a Fredholm map.

Proof. By loc. cit., the validity of the assertion does not depend on the choice of \( L, L_\omega \). So we can assume to be in the situation of Example, and we are done by 2.4 and the proposition.

We define \( \mu_K^\nabla \in \text{Det}_K(N/\omega N) \) as a relative determinant theory such that for any \( L \in \Lambda_K(N), L_\omega \in \Lambda_K(\omega N) \) one has

\[
\mu_K^\nabla(L/L_\omega) := \lambda_{N/L,\omega N/L_\omega},
\]

(3.7.3)

and the structure isomorphisms \( \lambda_{K}(L'/L) \otimes \mu_K^\nabla(L/L_\omega) \otimes \lambda_{K}(L_\omega/L'_\omega) \sim \mu_K^\nabla(L'/L'_\omega) \) are compositions (3.3.2) for \( \text{id}_{N/L,\omega N/L_\omega} \nabla_{N/L,\omega N/L_\omega} \text{id}_{\omega N/L_\omega,\omega N/L_\omega} = \nabla_{N/L',\omega N/L'_\omega} \).
3.8. Let $\nu$ be any invertible holomorphic 1-form defined on $U \setminus K$, where $U$ as in 3.5. The multiplication by $\nu$ isomorphism $j_K \cdot N|_U \sim j_K \cdot \omega N|_U$ yields an identification of the $\mathcal{L}$-groupoids $\Lambda_K(j_K \cdot N) \sim \Lambda_K(j_K \cdot \omega N)$, hence a relative determinant theory (see Remark (iii) in 3.6)

$$\mu_K^{\nu} = \mu(N/\nu N)_K^* \in \text{Det}_K(j_K \cdot N/j_K \cdot \omega N) = \text{Det}_K(N/\omega N).$$

(3.8.1)

Set

$$\mathcal{E}_{\text{DR}}(N)_{(K,\nu)} := \mu_K^{\nu} \otimes (\mu_K^{\nu})^{-1} \in \mathcal{L}.$$  

(3.8.2)

Thus for every $L \in \Lambda_K(j_K \cdot N)$ one has a canonical isomorphism

$$\mathcal{E}_{\text{DR}}(N)_{(K,\nu)} \sim \mu_K^{\nu}(L/\nu L);$$

(3.8.3)

for two lattices $L, L'$ the corresponding identification $\mu_K^{\nu}(L/\nu L) \sim \mu_K^{\nu}(L'/\nu L')$ is

$$\mu_K^{\nu}(L/\nu L) \sim \lambda_K(L/L') \otimes \mu_K^{\nu}(L'/\nu L') \otimes \lambda_K(\nu L/\nu L')^{-1} \sim \mu_K^{\nu}(L'/\nu L'),$$

(3.8.4)

where the first arrow is the composition, the second one comes from the multiplication by $\nu$ identification $\lambda_K(L'/\nu L') \sim \lambda_K(\nu L/\nu L')$.

The construction does not depend on the auxiliary choice of $U$. When $\nu$ varies holomorphically, $\mathcal{E}_{\text{DR}}(N)_{(K,\nu)}$ form a holomorphic line bundle on the parameter space (by (3.8.2) and Remark (vi) in 3.6). If $K = \sqcup K_\alpha$, then (3.6.3) yields a factorization (here $\nu_\alpha$ are the restrictions of $\nu$ to neighborhoods of $K_\alpha$)

$$\otimes \mathcal{E}_{\text{DR}}(N)_{(K,\nu_\alpha)} \sim \mathcal{E}_{\text{DR}}(N)_{(K,\nu)}.$$

(3.8.5)

3.9. The above constructions are compatible with those from 2.5. Precisely, let $M, P$ be as in Example in 3.7, and suppose that $\nu$ is meromorphic on $U$ with $D := \text{div}(\nu)$ supported on $P$. Then $\Lambda_P(M) \subset \Lambda_K(N)$, $\Lambda_P(M(\infty P)) \subset \Lambda_K(j_K \cdot N)$ (see loc. cit.). These embeddings are naturally compatible with the $\mathcal{L}$-groupoid structures, so

$$\text{Det}_P(M) = \text{Det}_K(M), \quad \text{Det}_P(M/\omega M) = \text{Det}_K(N/\omega N).$$

(3.9.1)

By the proposition in 3.7, (3.7.1) provides an identification

$$\mu_K^{\nu}_P(M/\omega M) \sim \mu_K^{\nu}(N/\omega N).$$

(3.9.2)

Joint with an evident isomorphism $\mu_K^{\nu}(M/\omega M) \sim \mu_K^{\nu}(N/\omega N)$, it yields

$$\mathcal{E}_{\text{DR}}(M)_{(D,c,\nu P)} \sim \mathcal{E}_{\text{DR}}(N)_{(K,\nu)}.$$  

(3.9.3)

If $\nu$ varies holomorphically, then (3.9.3) is holomorphic.

3.10. Proof of the theorem in 3.1. By 3.2, $\phi : B(M) \sim B(M')$ amounts to an isomorphism of $\mathcal{D}^\infty$-modules $M^\infty \sim M'^\infty$. Therefore we can view $M$ and $M'$ as two $\mathcal{D}$-structures on a holonomic $\mathcal{D}^\infty$-module $N$. Choose the multiplicity of $T$ to be compatible with both $M$ and $M'$ (see 2.1). We want to define an isomorphism $\phi^* : \mathcal{E}_{\text{DR}}(M) \sim \mathcal{E}_{\text{DR}}(M')$ in $\mathcal{L}^\phi_{\text{DR}}(X, T)$.

Let $S$ be an analytic space, $(D,c,\nu P) \in \mathcal{D}^\phi(S)$ (see 1.1). We work locally on $S$, so we have $T^c \subset T$. Choose a compact $K$, its open neighborhood $U$, and a meromorphic $\nu$ on $U$ such that $K, U$ satisfy the conditions from 3.5, $P \subset K_S$, $D = -\text{div}(\nu)$, and $\nu|_P = \nu|_P$. We define $\phi^*$ at $(D,c,\nu P)$ as the composition $\mathcal{E}_{\text{DR}}(M)_{(D,c,\nu P)} \sim \mathcal{E}_{\text{DR}}(N)_{(K,\nu)} \sim \mathcal{E}_{\text{DR}}(M')_{(D,c,\nu P)}$; here $\sim$ are (3.9.3). Equivalently, choose $L \in \Lambda_P(M)$, $L' \in \Lambda_P(M')$; then $\phi^*$ is the composition $\mathcal{E}_{\text{DR}}(M)_{(D,c,\nu P)} \sim \mu_K^{\nu}(L/\nu L) \sim \mu_K^{\nu}(L'/\nu L') \sim \mathcal{E}_{\text{DR}}(M')_{(D,c,\nu P)}$, the first and the last arrows are compositions of (3.9.2) and (2.5.6), the middle one is (3.8.4).
The last description shows that $\phi^\epsilon$ does not depend on the auxiliary choice of $\nu$. Indeed, $\nu$ is defined up to multiplication by an invertible function $g$ on $U$ which equals 1 on $P$. By (3.8.4), replacing $\nu$ by $g\nu$ multiplies the isomorphism $\mu_K^\nu(L/\nu L) \simeq \mu_K^L(L/\nu L')$ by $g(L/L')$ (see Remark (vii) in 3.6). Since $T$ is compatible with $M$ and $M'$, the 1-form $d\log(\theta_L/\theta_L)$ (see loc. cit.) has pole at $P$ of order $\leq$ the multiplicity of $P$, so (3.6.5) implies that $g(L/L') = 1$.

The construction is compatible with factorization, so we have defined $\phi^\epsilon$ as isomorphism in $L^\Phi_{\nu}(X,T)$. One has $M|_{X\setminus T} = M'|_{X\setminus T}$, and $\phi^\epsilon|_{X\setminus T}$ is the corresponding evident identification. By the corollary in 1.12, this implies that $\phi^\epsilon$ is horizontal, i.e., it is an isomorphism in $L^\Phi_{\nu}(X,T)$.

The compatibility of $\phi^\epsilon$ with constraints from 2.8 is evident. Finally, the commutativity of (3.1.1) follows from the next proposition:

3.11. **Proposition.** Suppose that $X$ is compact, $N$ is a holonomic $D^\infty$-module smooth on $X\setminus K$, and $\nu$ is a holomorphic invertible 1-form on $X\setminus K$. Then there is a canonical isomorphism

$$\eta_{\nu} : \mathcal{E}_{\nu}(N)_{(K,\nu)} \simeq \det R\Gamma_{\nu}(X, N).$$

(3.11.1)

If $\nu$ is meromorphic and $M$ is a $D$-structure on $N$, then the next diagram of isomorphisms commutes (see 1.4 and 2.7 for the left column, the top arrow is (3.9.2)):

$$\begin{array}{ccc}
\mathcal{E}_{\nu}(M) & \simeq & \mathcal{E}_{\nu}(N)_{(K,\nu)} \\
\eta_{\nu} & \downarrow & \eta_{\nu} \\
\det H_{\nu}(X, M) & \simeq & \det H_{\nu}(X, N).
\end{array}$$

(3.11.2)

**Proof** (cf. 2.7). For $L \in \Lambda_K(jK, N)$ set $\lambda(L) := \det R\Gamma(X, L)$. Then $\lambda$ is a determinant theory on $jK, N$ at $K$ in an evident way. Replacing $N$ by $\omega N$, we get $\lambda_\omega \in \mathcal{D}et_K(jK, \omega N)$, hence $\lambda \otimes \lambda_\omega^{-1} \in \mathcal{D}et_K(jK, N/jK, \omega N)$. One has an isomorphism

$$\mu_K^\nu \simeq \lambda \otimes \lambda_\omega^{-1},$$

(3.11.3)

namely, $\mu_K^\nu(L/\nu L) := \lambda_\omega(\nu L/L_{\omega}) \otimes \lambda_\omega(L_{\omega})^{-1} \simeq \lambda(\nu L/L_{\omega})^{-1} = (\lambda \otimes \lambda_\omega^{-1})(L/\nu L)$ where $\simeq$ comes from the isomorphism $\nu^{-1} : \nu L \simeq L$.

For $K$-lattices $L$ in $N$, $L_{\omega}$ in $N_\omega$ such that $\nabla(L) \subset L_{\omega}$, set $dR(L, L_{\omega}) := \mathcal{C}one(L, \nabla L_{\omega})$. Since $dR(N)/dR(L, L_{\omega}) = \mathcal{C}(L, L_{\omega})$, our $dR(N)$ carries a 3-step filtration with successive quotients $L_{\omega}$, $L$, $\mathcal{C}(L, L_{\omega})$. Applying $\det R\Gamma$, we get an isomorphism

$$\det R\Gamma(X, \mathcal{C}(L, L_{\omega})) \otimes \lambda(L)^{-1} \otimes \lambda(L_{\omega}) \simeq \det R\Gamma_{\nu}(X, N).$$

(3.11.4)

To get $\eta_{\nu}$, we combine (3.11.4) with (3.11.3) (and (3.7.3)). The construction does not depend on the auxiliary choice of $L, L_{\omega}$. \hfill \Box

4 The Betti $\epsilon$-line

We present a construction from [B] in a format adapted for the current subject. In 4.2–4.5 $X$ is considered as a mere real-analytic surface.
4.1. Let $\mathcal{L}$ be any Picard groupoid. For a (non-unital) Boolean algebra $\mathcal{C}$, an $\mathcal{L}$-valued measure $\lambda$ on $\mathcal{C}$ is a rule that assigns to every $S \in \mathcal{C}$ an object $\lambda(S) \in \mathcal{L}$, and to every finite collection $\{S_\alpha\}$ of pairwise disjoint elements of $\mathcal{C}$ an identification $\otimes \lambda(S_\alpha) \sim \lambda(S)$ (referred to as integration); the latter should satisfy an evident transitivity property. Such $\lambda$ form naturally a Picard groupoid $\mathcal{M}(\mathcal{C}, \mathcal{L})$.

Remarks. (i) For an abelian group $A$ denote by $\mathcal{M}(\mathcal{C}, A)$ the group of $A$-valued measures on $\mathcal{C}$. Then $\pi_1(\mathcal{M}(\mathcal{C}, \mathcal{L})) = \mathcal{M}(\mathcal{C}, \pi_1(\mathcal{L}))$, and one has a map $\pi_0(\mathcal{M}(\mathcal{C}, \mathcal{L})) \to \mathcal{M}(\mathcal{C}, \pi_0(\mathcal{L}))$ which assigns to $|\lambda|$ a $\pi_0(\mathcal{L})$-valued measure $|\lambda|$, $|\lambda|(S) := |\lambda(S)|$ (see 1.1 for the notation).

(ii) Let $I \subset \mathcal{C}$ be an ideal. Then $\mathcal{M}(\mathcal{C}/I, \mathcal{L})$ identifies naturally with the Picard groupoid of pairs $(\lambda, \tau)$ where $\lambda \in \mathcal{M}(\mathcal{C}, \mathcal{L})$ and $\tau$ is a trivialization of its restriction $\lambda|_I$ to $I$, i.e., an isomorphism $1_{\mathcal{M}(\mathcal{C}, \mathcal{L})} \sim \lambda|_I$ in $\mathcal{M}(I, \mathcal{L})$.

(iii) Suppose $\mathcal{C}$ is finite, i.e., $\mathcal{C} = (\mathbb{Z}/2)^T := \text{the Boolean algebra of all subsets of a finite set } T$. Then an $\mathcal{L}$-valued measure $\lambda$ on $\mathcal{C}$ is the same as a collection of objects $\lambda_i = \lambda(\{t\})$, $t \in T$. Thus $\mathcal{M}(T, \mathcal{L}) := \mathcal{M}(\mathbb{Z}/2^T, \mathcal{L}) \sim T^\vee$.

4.2. For an open $U \subset X$ we denote by $\mathcal{C}(U)$ the (non-unital) Boolean algebra of relatively compact subanalytic subsets of $U$. For $U' \subset U$ one has $\mathcal{C}(U') \subset \mathcal{C}(U)$, and $U \mapsto \mathcal{M}(\mathcal{C}(U), \mathcal{L})$ is a sheaf of Picard groupoids on $X$.

For a commutative ring $R$, let $\mathcal{L}_R$ be the Picard groupoid of $\mathbb{Z}$-graded super $R$-lines. Its objects are pairs $L = (L, \deg(L))$ where $L$ is an invertible $R$-module, $\deg(L)$ a locally constant $\mathbb{Z}$-valued function on $\text{Spec} R$; the commutativity constraint is “super” one. Every perfect $R$-complex $F$ yields a graded super line $\det F \in \mathcal{L}_R$. For a finite filtration $F$ on $F$ by perfect subcomplexes, one has a canonical isomorphism $\det F \sim \otimes \det F_i$; it satisfies transitivity property with respect to refinement of the filtration. For a finite collection $\{F_{i}\}$, every linear ordering of $\alpha$'s yields a filtration on $\oplus F_{i}$, and the corresponding isomorphism $\det(\oplus F_{i}) \sim \otimes \det F_{i}$ is independent of the ordering; thus $\det F$ is a symmetric monoidal functor.

Let $F = F_U$ be a perfect constructible complex of $R$-sheaves on $U$. Then for every locally closed subanalytic subset $i_{C} : C \hookrightarrow U$ the $R$-complex $\Gamma(C, Ri_{C}^! F)$ is perfect, so we have $\det \Gamma(C, Ri_{C}^! F) \in \mathcal{L}_R$.

Suppose we have a finite closed subanalytic filtration $\{C_{>i}\}$ on $C$ (therefore $C_{i} : = C_{>i} \setminus C_{>i-1}$ are locally closed and form a partition of $C$). It yields a finite filtration $\{C_{>i}\}$ on $\Gamma(C, Ri_{C}^! F)$ with $\det \Gamma(C, Ri_{C}^! F) = \Gamma(C, Ri_{C}^! F)$, hence an identification

$$\otimes \det \Gamma(C, Ri_{C}^! F) \sim \det \Gamma(C, Ri_{C}^! F). \quad (4.2.1)$$

It satisfies transitivity property with respect to refinement of the filtration.

Lemma. There is a unique (up to a unique isomorphism) pair $(\lambda(F), i)$, where $\lambda(F) \in \mathcal{M}(\mathcal{C}(U), \mathcal{L}_R)$, $i$ is a datum of isomorphisms

$$i_C : \lambda(F)(C) \sim \det \Gamma(C, Ri_{C}^! F)$$

23Recall that a Boolean algebra is the same as a commutative $\mathbb{Z}/2$-algebra each of whose elements is idempotent; the basic Boolean operations are $x \cap y = xy$, $x \cup y = x + y + xy$; elements $x$, $y$ are said to be disjoint if $x \cap y = 0$. The Boolean algebras we meet are already realized as Boolean algebras of subsets of some set.

24A filtration on an object $C$ of a derived category is an object of the corresponding filtered derived category identified with $C$ after the forgetting of the filtration.
defined for any locally closed \( C \), such that for every filtration \( C \leq i \) on \( C \) as above, \( i \) identifies (4.2.1) with the integration \( \otimes \lambda(F)(C) \overset{\sim}{\rightarrow} \lambda(F)(C) \).

**Proof.** Suppose we have a compact subanalytic subset of \( U \) equipped with a subanalytic stratification whose strata \( C_{\alpha} \) are smooth and connected. The strata generate a Boolean subalgebra \( C(\{C_{\alpha}\}) \) of \( C(U) \): call a subalgebra of such type nice. Every finite subset of \( C(U) \) lies in a nice subalgebra; in particular, the set of nice subalgebras is directed. To prove the lemma, it suffice to define the restriction of \( \lambda(F, \iota) \) to every nice \( C(\{C_{\alpha}\}) \); their compatibility is automatic.

By Remark (iii) in 4.1, \( \lambda(F)|_{C(\{C_{\alpha}\})} \) is the measure defined by condition \( \lambda(F)(C) = \text{det} \Gamma(C_{\alpha}, R_{C_{\alpha}}^i F) \). For a locally closed \( C \) in \( C(\{C_{\alpha}\}) \) one defines \( \iota_C \) using (4.2.1) for a closed filtration on \( C \) whose layers are strata of increasing dimension; its independence of the choice of filtration follows since \( \text{det} \) is a symmetric monoidal functor in the way described above. The compatibility with (4.2.1) is checked by induction on the number of strata involved.

**Example.** Suppose \( C', C'' \in C(U) \) are such that \( C', C'' \), and \( C := C' \cap C'' \) are locally closed, \( C' \cap C'' = \emptyset \). By the lemma, there is a canonical isomorphism

\[
\text{det} \Gamma(C', R_{C_{\alpha}}^i F) \otimes \text{det} \Gamma(C'', R_{C_{\alpha}}^i F) \overset{\sim}{\rightarrow} \text{det} \Gamma(C, R_{C_{\alpha}}^i F). \tag{4.2.2}
\]

If, say, \( C' \) is closed in \( C \), then this is (4.2.1) for the filtration \( C' \subset C \). To construct (4.2.2) when neither \( C' \) nor \( C'' \) are closed (e.g. \( C = X \) is a torus, and \( C', C'' \) are non-closed annuli), consider a 3-step closed filtration \( C' \subset C' \subset C \), where \( C' \) is the closure of \( C' \) in \( C \); set \( P := \bar{C}' \setminus C' = C' \cap C'' \), \( Q := C \setminus C' = C'' \setminus P \). By (4.2.1),

\[
\text{det} \Gamma(C', R_{C_{\alpha}}^i F) \otimes \text{det} \Gamma(P, R_{P_{\alpha}}^i F) \overset{\sim}{\rightarrow} \text{det} \Gamma(C', R_{C_{\alpha}}^i F), \quad \text{det} \Gamma(P, R_{P_{\alpha}}^i F) \otimes \text{det} \Gamma(Q, R_{Q_{\alpha}}^i F) \overset{\sim}{\rightarrow} \text{det} \Gamma(C'', R_{C_{\alpha}}^i F), \quad \text{det} \Gamma(C', R_{C_{\alpha}}^i F) \otimes \text{det} \Gamma(Q, R_{Q_{\alpha}}^i F) \overset{\sim}{\rightarrow} \text{det} \Gamma(C, R_{\alpha}^i F). \]

Combining them, we get (4.2.2).

4.3. Let \( U \subset X \) be an open subset, and \( N = N_U \subset T_U \) be a continuous family of proper cones in the tangent bundle (so for each \( x \in U \) the fiber \( N_x \) is a proper closed sector with non-empty interior in the tangent plane \( T_x \)). For an open \( V \subset U \) we denote by \( N_V \) the restriction of \( N \) to \( V \).

One calls \( C \in C(U) \) an \( N \)-lens if it satisfies the next two conditions:

(a) Every point \( v \in U \) has a neighborhood \( V \) such that \( C \cap V = C_1 \cap C_2 \) where \( C_1, C_2 \) are closed subsets of \( V \) that are invariant with respect to some family of proper cones \( N_{C_1} \equiv N_V \).

(b) There is a \( C \)-function \( f \) defined on a neighborhood \( V \) of the closure \( C \) such that for every \( x \in V \) and a non-zero \( \tau \in N_x \) one has \( \tau(f) > 0 \).

Let \( \mathcal{I}(U, N) \subset C(U) \) be the Boolean subalgebra generated by all \( N \)-lenses.

Basic properties of lenses (see [B] 2.4, 2.7): (i) Every \( N \)-lens \( C \) is locally closed, and \( Int \) of \( C \) is dense in \( C \); the intersection of two \( N \)-lenses is an \( N \)-lens.

(ii) Every point in \( U \) admits a base of neighborhoods formed by \( N \)-lenses.

(iii) Suppose we have an \( N \)-lens \( C \) and a (finite) partition \( \{C_{\alpha}\} \) of \( C \) with \( C_{\alpha} \in \mathcal{I}(U, N) \). Then there exists a finer partition \( \{C_1, \ldots, C_n\} \) of \( C \) such that \( C_i \) are \( N \)-lenses and each subset \( C_{\leq i} := C_1 \cup C_2 \cup \ldots \cup C_i \) is closed in \( C \).

**Exercise.** Every \( C \in C(U) \) that satisfies (a) lies in \( \mathcal{I}(U, N) \).

---

\( ^{26} \) Here \( \supseteq \) means that \( \text{Int} N_{C_{\alpha}}^i \supseteq N_V \setminus \{0\} \), and \( C \)-invariance of \( C \) means that every \( C \)-arc \( \gamma : [0, 1] \to V \) such that \( \gamma(0) \in C_i \) and \( \gamma(t) \in N_{C_{\alpha}} \setminus \{0\} \) for every \( t \), lies in \( C_i \).
Suppose $F$ from 4.2 is locally constant (say, a local system of finitely generated projective $R$-modules). Then for any $\mathcal{N}$-lens $C$ one has $R\Gamma(C, R\mathcal{I}_C^* F) = 0$ (see [18 2.5]). Let $\tau_C : 1 \xrightarrow{\sim} \lambda(F)(C)$ be the corresponding trivialization of the determinant.

**Proposition.** The restriction of $\lambda(F)$ to $\mathcal{I}(U, \mathcal{N})$ admits a unique trivialization $\tau_{\mathcal{N}} : 1_{\mathcal{M}(\mathcal{I}(U, \mathcal{N}), \mathcal{L}_R)} \xrightarrow{\sim} \lambda(F)|_{\mathcal{I}(U, \mathcal{N})}$ such that for every $\mathcal{N}$-lens $C$ the trivialization $\tau_{\mathcal{N}C}$ coincides with $\tau_C$.

**Proof.** By (iii) above, every finite subset of $\mathcal{I}(U, \mathcal{N})$ lies in the Boolean subalgebra generated by a finite subset of pairwise disjoint $\mathcal{N}$-lenses. This implies uniqueness. To show that $\tau_{\mathcal{N}}$ exists, it suffices to check the next assertion: For any $\mathcal{N}$-lens $C$ and any finite partition $\{C_\alpha\}$ of $C$ by $\mathcal{N}$-lenses the integration $\otimes \lambda(F)(C_\alpha) \xrightarrow{\sim} \lambda(F)(C)$ identifies $\otimes \tau_{C_\alpha}$ with $\tau_C$.

Choose $\{C_1, \ldots, C_n\}$ as in (iii) above. Since $C_{\leq \alpha}$ is a closed filtration, the integration $\otimes \lambda(F)(C_i) \xrightarrow{\sim} \lambda(F)(C)$ identifies $\otimes \tau_{C_{\alpha}}$ with $\tau_C$ (see the lemma in 4.2), and for each $\alpha$ the integration $\otimes \lambda(F)(C_i \cap C_\alpha) \xrightarrow{\sim} \lambda(F)(C_\alpha)$ identifies $\otimes \tau_{C_i \cap C_\alpha}$ with $\tau_{C_\alpha}$. The partition $\{C_i\}$ is finer than $\{C_\alpha\}$, so we are done by the transitivity of integration. \qed

**4.4.** Let $K$ be a compact subset of $X$, $W$ be an open subset that contains $K$, $U := W \setminus K$. For $\mathcal{N} = \mathcal{N}_U$ as above, let $\mathcal{C}(W, \mathcal{N})$ be the set of $C \in \mathcal{C}(W)$ that satisfy the next two conditions:

(a) For every $C' \in \mathcal{I}(U, \mathcal{N})$ one has $C \cap C' \in \mathcal{I}(U, \mathcal{N})$.

(b) One has $\text{Int}(C) \cap K = \tilde{C} \cap K$.

Then $\mathcal{C}(W, \mathcal{N})$ is a Boolean subalgebra of $\mathcal{C}(W)$, and $\mathcal{I}(U, \mathcal{N})$ is an ideal in it.

**Exercise.** Let $K'$ be a subset of $K$ which is open and closed in $K$. Choose an open relatively compact subset $V$ of $W$ such that $V \cap K = \tilde{V} \cap K = K'$, and $C' \in \mathcal{I}(U, \mathcal{N})$ such that $C' \supset \partial V := \tilde{V} \setminus V$. Then $C := V \setminus C' \in \mathcal{C}(W, \mathcal{N})$ and $C \cap K = K'$.

Denote by $\mathcal{C}[K]$ the Boolean algebra of subsets of $K$ which are open and closed in $K$. By (b), we have a morphism of Boolean algebras $\mathcal{C}(W, \mathcal{N}) \rightarrow \mathcal{C}[K]$, $C \mapsto C \cap K$. It yields an identification

$$\mathcal{C}(W, \mathcal{N})/\mathcal{I}(U, \mathcal{N}) \xrightarrow{\sim} \mathcal{C}[K].$$

(4.4.1)

Let $F = F_W$ be a perfect constructible complex of $R$-sheaves on $W$ whose restriction to $U$ is locally constant. By the proposition in 4.3, we have a trivialization $\tau_{\mathcal{N}}$ of the restriction of $\lambda(F)$ to $\mathcal{I}(U, \mathcal{N})$. By Remark (ii) in 4.1, (4.4.1), the pair $(\lambda(F)|_{\mathcal{C}(W, \mathcal{N})}, \tau_{\mathcal{N}})$ can be viewed as a measure $\mathcal{E}(F)|_{\mathcal{C}[K]} \in \mathcal{M}(\mathcal{C}[K], \mathcal{L}_R)$. If $K$ is finite, then, by Remark (iii) in 4.1, it amounts to a collection of lines $\mathcal{E}(F)_{b, \mathcal{N}} := \mathcal{E}(F)_{\mathcal{N}b}$, $b \in K$.

If $C \in \mathcal{C}(W, \mathcal{N})$ is locally closed, then we have identifications

$$\mathcal{E}(F)_{\mathcal{N}}(C \cap K) \xrightarrow{\sim} \lambda(F)(C) \xrightarrow{\sim} \text{det} R\Gamma(C, R\mathcal{I}_C^* F).$$

(4.4.2)

In particular, if $X$ is compact and $W = X$, then $X \in \mathcal{C}(X, \mathcal{N})$, and

$$\mathcal{E}(F)_{\mathcal{N}}(K) \xrightarrow{\sim} \text{det} R\Gamma(X, F).$$

(4.4.3)

If $K$ is finite, this is a product formula

$$\otimes_{b \in K} \mathcal{E}(F)_{b, \mathcal{N}} \xrightarrow{\sim} \text{det} R\Gamma(X, F).$$

(4.4.4)
4.5. Suppose we have another datum of $W' \supset U'$, $\mathcal{N}' = \mathcal{N}'_{U'}$ as above, such that $W' \subset W$, $U' \subset U$, and $\mathcal{N}' \supset \mathcal{N}_U$. Then $\mathcal{I}(U', \mathcal{N}') \subset \mathcal{I}(U, \mathcal{N})$, $\mathcal{C}(W', \mathcal{N}') \subset \mathcal{C}(W, \mathcal{N})$, and (4.4.1) identifies the morphism of Boolean algebras $\mathcal{C}(W', \mathcal{N}')/\mathcal{I}(U', \mathcal{N}') \to \mathcal{C}(W, \mathcal{N})/\mathcal{I}(U, \mathcal{N})$ with a morphism $r : \mathcal{C}[K'] \to \mathcal{C}[K]$, $Q \mapsto r(Q) := Q \setminus U = Q \cap K$. Since $\tau_{\mathcal{N}'}$ equals the restriction of $\tau_{\mathcal{N}}$ to $\mathcal{I}(U', \mathcal{N}')$, one has

$$\mathcal{E}(F)_{\mathcal{N}'} = r^*\mathcal{E}(F)_{\mathcal{N}}. \quad (4.5.1)$$

Remarks. (i) Taking for $W'$ a small neighborhood of a component $K'$ of $K$, $U' = W' \cap U$, $\mathcal{N}' = \mathcal{N}|_{W'}$, we see that $\mathcal{E}(F)_{\mathcal{N}'}$ has local nature with respect to $K$.

(ii) By (4.5.1), $\mathcal{E}(F)_{\mathcal{N}}(W' \cap K)$ depends only on the restriction of $\mathcal{N}$ to $U'$.

4.6. Suppose now $X$ is a complex curve, $T \subset X$ a finite subset, $F$ is a constructible sheaf on $X$ which is smooth on $X \setminus T$. Let us define a constructible factorization $R$-line $\mathcal{E}_B(F)$ on $(X, T)$ (see 1.15).

Let $S$ be an analytic space, $(D, c, \nu_p) \in \mathfrak{D}^e(S)$. Let us define a local system of $R$-lines $\mathcal{E}(F)_{(D, c, \nu_p)}$ on $S$. Consider a datum $(W, K, \mathcal{N}, \nu_S)$, where $W$ is an open subset of $X$, $K$ a compact subset of $W$, $\mathcal{N} = \mathcal{N}_U$ is a continuous family of proper cones in the tangent bundle to $U := W \setminus K$ (viewed as a real-analytic surface, see 4.3), $\nu_S$ is an $S$-family of meromorphic 1-forms on $W$. We say that our datum is compatible if $P = P_{D, c} \subset K_S$, $\text{div}(\nu) = -D$, $\nu|_P = \nu_P$, and the 1-forms $\text{Re}(\nu_s)$ are negative on $\mathcal{N}$. As in 4.4, every compatible datum yields the $R$-line $\mathcal{E}(F)_{\mathcal{N}}(K)$.

Lemma. Locally on $S$ compatible data exist; the lines $\mathcal{E}(F)_{\mathcal{N}}(K)$ for all compatible data are naturally identified.

Proof. The existence statement is evident. Suppose that we fix an open subset $W_0$ of $X$ and an $S$-family of meromorphic forms $\nu_S$ on $W_0$ such that $P \subset W_0$, $\text{div}(\nu_S) = D$, and $\nu_p = \nu|_P$. Let us consider compatible data with $W \subset W_0$ and the above $\nu_S$. The identification of the lines for these data comes from 4.5. Thus our line depends only on $\nu_S$; in fact, by Remark (i) in 4.5, on the germ of $\nu_S$ at $P$. If we move $\nu_S$, it remains locally constant. Since the space of germs of $\nu_S$ is contractible, we are done.

Locally on $S$, we define $\mathcal{E}(F)_{(D, c, \nu_p)}$ as $\mathcal{E}(F)_{\mathcal{N}}(K)$ for a compatible datum. The factorization structure is evident. For $X$ compact, we have, by (4.4.3), a canonical identification

$$\eta : \mathcal{E}(F)(X) \sim \text{det} \Gamma(X, F). \quad (4.6.1)$$

Exercise. Check that $\mathcal{E}$ satisfies the constraints of 2.8.

Remark. Suppose $X$ is compact and a rational form $\nu$ has property that $\text{Re}(\nu)$ is exact, $\text{Re}(\nu) = df$. Then the isomorphism $\eta : \mathcal{E}(F)_{\nu} \sim \text{det} \Gamma(X, F)$ can be computed using Morse theory: indeed, if $a < a'$ are non-critical values of $f$, then $f^{-1}((a', a)] \subset C(\mathcal{N})$ for $\mathcal{N}$ compatible with $\nu$.

In §5 we apply this construction to $F = B(M)$, the de Rham complex of a holonomic $D$-module $M$, and write $\mathcal{E}_B(M) := \mathcal{E}(B(M))$. We use the same notation for the corresponding de Rham factorization line (in the analytic setting).

4.7. For $b \in X$ and a meromorphic $\nu$ on a neighborhood of $b$, $\nu_b(\nu) = -\ell$, set $\mathcal{E}(F)_{(b, \nu)} := \mathcal{E}(F)_{(b, \nu)}$. Let $F_b^{(\ell)} := R_i\nu_b^*F = \Gamma_b(\ell)(X, F) = \Gamma(X_b, F)$, $F_b^{(\ast)} := i_b^*F = \Gamma(X_b, F)$ be the fibers of $F$ at $b$ in !- and *-sense (here $X_b$ is a small open disc around $b$). Let $t$ be a local parameter at $b$. 

Proposition. with (4.7.2) shows that its monodromy around \( z \)

A simple computation (or a reference to the compatibility property in 1.11) together with the Euler vector field \( \text{Re}(\partial_t) \), then \( X_b \in C(W,\mathcal{N}) \) and \( X_b \in C(W,-\mathcal{N}) \). The data \((W,K,N,-t^{-1}dt)\) and \((W,K,N',t^{-1}dt)\) are compatible, and we are done. □

Thus if \( F \) is the *-extension at \( b \), i.e., \( F_b^{(\ast)} = 0 \), then \( \mathcal{E}(F)_{(b,-t^{-1}dt)} \) is canonically trivialized. Denote these trivializations by \( 1_b^\ast \in \mathcal{E}(F)_{(b,-t^{-1}dt)} \), \( 1_b^! \in \mathcal{E}(F)_{(b,t^{-1}dt)} \).

Exercise. Suppose \( X = \mathbb{P}^1 \) and \( F \) is smooth outside \( 0,\infty \). Then the composition \( \det F_0^{(1)} \otimes \det F_\infty^{(1)} \simeq \mathcal{E}(F)_{(0,t^{-1}dt)} \otimes \mathcal{E}(F)_{(\infty,t^{-1}dt)} \) comes from the standard triangle \( \mathcal{R} \Gamma(\mathbb{P}^1,F) \to \mathcal{R} \Gamma(\mathbb{P}^1,F) \to \mathcal{R} \Gamma(\mathbb{P}^1 \setminus \{0\}, F) = F_\infty^{(1)} \).

In particular, if \( F \) is *-extension at \( 0 \) and !-extension at \( \infty \), then \( \eta(1_b^\ast \otimes 1_\infty^\ast) = 1 \) identifies the trivialization of \( \det F_\infty^{(1)} \) that comes since \( \mathcal{R} \Gamma(\mathbb{P}^1,F) = 0 \).

For \( x \in X \setminus T \) one has \( F_x^{(1)} = F_x^{(1)}(-1)[-2] \), so (4.7.1) yields a natural identification

\[
\mathcal{E}(F)^{(1)}_{X \setminus T} \simeq \det F(-1)_{X \setminus T}.
\] (4.7.2)

If \( R = \mathbb{C} \), then the Tate twist acts as identity. If \( M \) is a holonomic \( D \)-module, then \( B(M)_{X \setminus T} = M_{X \setminus T}[1] \), and (4.7.2) can be rewritten as

\[
\mathcal{E}(B)^{(1)}_{X \setminus T} \simeq (\det M_{X \setminus T})^{\otimes -1}.
\] (4.7.3)

Remark. For any \( \ell \in \mathbb{Z} \) we have the local system of lines \( \mathcal{E}(F)_{(b,zt^{-\ell}dt)} \), \( z \in \mathbb{C}^\times \). A simple computation (or a reference to the compatibility property in 1.11) together with (4.7.2) shows that its monodromy around \( z = 0 \) equals \((-1)^{\text{rk}(F)}m_b\) where \( \text{rk}(F) = \deg \det F_{X \setminus T} \) is the rank of \( F \) and \( m_b \) is the monodromy of \( F \) around \( b \). Thus (4.7.1) provides two descriptions of this local system for \( \ell = 1 \) (using the fibers at \( z = \pm 1 \)). For a relation between them, see below.

4.8. We are in the situation of 4.7. Suppose \( R \) is a field, outside singular points our \( F \) is a local system of rank 1 placed in degree \(-1\), and \( F_b^{(1)} = 0 \). Let \( m \) be the monodromy of \( F \) around \( b \); suppose \( m \neq 1 \). Then \( F_b^{(1)} \) vanishes as well, so we have \( (m - 1)^{-1} b^* \in \mathcal{E}(F)_{(b,\pm 1,dt)} \), \( 1_b^* \in \mathcal{E}(F)_{(b,-t^{-1}dt)} \).

Proposition. The (counterclockwise) monodromy from \( t^{-1}dt \) to \( -t^{-1}dt \) identifies \( 1_b^* \) with \( (1 - m)^{-1} 1_b^* \).

Proof. Consider an annulus around \( b \) (which lies in \( U \)), and cut it like this:
Let $D_+$ be the larger open disc, $D_-$ the smaller one, $\tilde{D}_\pm$ be their closures. Let $\bigtriangleup$ be the image of a 2-simplex with one face removed, $P$ a cell decomposition of the drawing. The graded vector space $d\Gamma_C(X,F)$ carries a differential $d_A$ such that $(P_A,d_A)$ is the chain complex of the simplex modulo the face. One defines $d_B$ in a similar way. Both $(P_A,d_A)$ and $(P_B,d_B)$ are acyclic, and the corresponding trivializations of $\det P_A = \lambda(F)(A)$, $\det P_B = \lambda(F)(B)$ equal $\tau_{N_A}, \tau_{N_B}$. Let $P' = P_A \oplus P_B$ be sum of $P_\alpha$’s for $\alpha$ in the boundary of $\bigtriangleup$. Then $P'$ is a subcomplex with respect to both $d$ and $d_A \oplus d_B$, on $P/P'$ the differentials $d$ and $d_A \oplus d_B$ coincide, and the complex $P/P'$ is acyclic. We see that $P'$ is acyclic with respect to both $d|_{P'}$ and $(d_A \oplus d_B)|_{P'}$, and $\tau_{N_A}, \tau_{N_B}$ are the trivializations of $\det P'$ that correspond to these differentials. Our $P'$ sits in degrees 0, 1. Choose base vectors $e_A \in P_A^0$, $f_A \in P_A^1$, $e_B \in P_B^0$, $f_B \in P_B^1$ such that $d_A(e_A) = f_A$, $d_B(e_B) = f_B$, and $d(e_A) = f_A - f_B$. Then $d(e_B) = -m f_A + f_B$. Therefore $\tau_{N_A}, \tau_{N_B}/\tau = 1 - m$, q.e.d.

**4.9.** Let $b, \nu, \ell$ be as in the beginning of 4.7; suppose $\ell \neq 1$. Denote by $\nu_b$ the principal term of $\nu$ at $b$. Let $f$ be any holomorphic function defined near $b$ and vanishing at $b$ such that $\nu_b = (df)_b$ if $\ell < 1$, and $\nu_b = (df^{-1})_b$ if $\ell > 1$. For a small $\epsilon \in \mathbb{C}, \epsilon \neq 0$, the set $f^{-1}(z)$ is finite of order $|\ell - 1|$. For a finite subset $Z$ of $X$, denote by $F_Z^{(t)}$, $F_Z^{(\ast)}$ the direct sum of $!$, resp. $\ast$-fibers of $F$ at points of $Z$. Let $\epsilon$ be a small positive real number.

**Proposition.** For $\ell < 1$, one has canonical identifications

$$
\mathcal{E}(F)_{(b,\nu)} \simeq \det f_b^{(t)} \otimes (\det F_{f^{-1}(\epsilon)}^{(t)})^{\otimes -1} \simeq \det f_b^{(\ast)} \otimes (\det F_{f^{-1}(\epsilon)}^{(\ast)})^{\otimes -1}.
$$

(4.9.1)

For $\ell > 1$, one has

$$
\mathcal{E}(F)_{(b,\nu)} \simeq \det f_b^{(\ast)} \otimes \det F_{g^{-1}(\epsilon)}^{(t)} \simeq \det f_b^{(t)} \otimes \det F_{g^{-1}(\epsilon)}^{(\ast)}.
$$

(4.9.2)

**Proof.** Since $\mathcal{E}(F)_{(b,\nu)}$ depends only on the principal term of $\nu$ at $b$, we can assume that $\nu$ equals $(df)$ or $(df^{-1})$. Set $\lambda := \lambda(F)$.

Case $\ell = 0$: Then $f$ is a local coordinate at $b$. Let $Q$ be an open romb around $b$ with vertices at $f = \pm \epsilon, \pm i\epsilon, I$ and $I'$ be the parts of its boundary
where Re(f) ≤ 0, resp. Re(f) > 0; set C := \overline{Q} \setminus I = Q \cup I'. Then one has $E(F)_{(b,\nu)} = \lambda(C) \overset{\sim}{\otimes} \lambda(I) \otimes \lambda(I')^{-1} \overset{\sim}{\otimes} \lambda(Q) \otimes \lambda(I')$. This yields (4.9.1) due to the next identifications:

(a) $\lambda(Q) \overset{\sim}{\otimes} \det R\Gamma_f(X,F) \overset{\sim}{\otimes} \det F^1_b$ and $\lambda(Q) \overset{\sim}{\otimes} \det R\Gamma(Q,i_Q^*F) \overset{\sim}{\otimes} \det F^1_b$;
(b) $\lambda(I) \overset{\sim}{\otimes} \det R\Gamma_f(X,F)$, and $R\Gamma_{\rightarrow(-\epsilon)}(X,F) \overset{\sim}{\otimes} R\Gamma_f(X,F)$;
(c) $\lambda(I') \overset{\sim}{\otimes} \det R\Gamma(I',\tilde{r}_i^*F[-1]) \overset{\sim}{\otimes} \det F^{(\epsilon)}_{\rightarrow(-\epsilon)} \overset{\sim}{\otimes} \det F^1_{\rightarrow(-\epsilon)} \overset{\sim}{\otimes} \det F^{\ast}_b$, where the second isomorphism comes from $F^1_b$.

Case $\ell = 2$: Then $f$ is a local coordinate at $b$. Define $Q$, etc., as above. Then $E(F)_{(b,\nu)} = \lambda(Q \cup I) = \lambda(Q \setminus I')$, so (a)–(c) yield (4.9.2).

If $\ell \neq 1$ is arbitrary, then $f$ is a $|\ell - 1|$-sheeted cover of a neighborhood of $b$ over a coordinate disc. The projection formula compatibility $E(F)_{(b,\nu)} \overset{\sim}{\otimes} E(f_\ast F)_{(a,dt)}$ for $\ell < 1$ and $E(F)_{(b,\nu)} \overset{\sim}{\otimes} E(f_\ast F)_{(a,dt)}$ for $\ell > 1$ reduces the assertion to the cases of $\ell$ equal to 0 and 2, and we are done. □

5 The torsor of $\varepsilon$-periods.

5.1. We consider triples $(X, T, M)$ where $X$ is a complex curve, $T$ its finite subset, and $M$ is a holonomic $\mathcal{D}$-module on $(X, T)$ (i.e., a $\mathcal{D}$-module on $X$ smooth off $T$). For us, a weak theory of $\varepsilon$-factors is a rule $\mathcal{E}$ that assigns to every such triple a de Rham factorization line $\mathcal{E}(M)$ on $(X, T)$ (in complex-analytic sense). Our $\mathcal{E}$ should be functorial with respect to isomorphisms of triples, and have local nature, i.e., be compatible with pull-backs by open embeddings. We ask that:

(i) For a nice flat family $(X/Q, T, M)$ (see 2.12) with reduced $Q$ the factorization lines $\mathcal{E}(M_q)$ vary holomorphically, i.e., we have $\mathcal{E}(M) \in \mathcal{L}^\mathcal{D}_{\text{dr}}(X/Q, T)$. If the family is isomonodromic, then $\mathcal{E}(M) \in \mathcal{L}^\mathcal{D}_{\text{dr}}(X/Q, T)$.

(ii) $\mathcal{E}(M)$ is multiplicative with respect to finite filtrations of $M$’s: for a finite filtration $M$. on $M$ there is a natural isomorphism $\mathcal{E}(M) \overset{\sim}{\otimes} \mathcal{E}(\text{gr}_1 M)$.

(iii) (projection formula) Let $\pi : (X', T') \rightarrow (X, T)$ be a finite morphism of pairs étale over $X \setminus T$, so, as in Remarks (i), (ii) in 1.2, (ii) in 1.5, we have a morphism $\pi_* : \mathcal{L}^\mathcal{D}_{\text{dr}}(X', T') \rightarrow \mathcal{L}^\mathcal{D}_{\text{dr}}(X, T)$. Then for any $M'$ on $(X', T')$ one has a natural identification $\mathcal{E}(\pi_* M') \overset{\sim}{\otimes} \pi_* \mathcal{E}(M')$ compatible with composition of $\pi$'s.

(iv) (product formula) For compact $X$ there is a natural identification (see 1.4 for the notation) $\eta = \eta(M) : \mathcal{E}(M)(X) \overset{\sim}{\otimes} \det R\Gamma_{\text{dr}}(X, M)$.

The constraints should be pairwise compatible in the evident sense. (i) should be compatible with the base change. (ii) should be transitive with respect to refinements of the filtration, and the isomorphism $\mathcal{E}(\oplus M_a) \overset{\sim}{\otimes} \mathcal{E}(M_a)$ should not depend on the linear ordering of the indices $a$ (which makes $\mathcal{E}$ a symmetric monoidal functor). (iii) should be compatible with the composition of $\pi$’s.

Weak theories of $\varepsilon$-factors form naturally a groupoid which we denote by $\mathcal{E}_0$. Its key objects are $\mathcal{E}^\mathcal{D}_{\text{dr}}$ and $\mathcal{E}_B$.

Replacing $\det R\Gamma_{\text{dr}}(X, M)$ in (iv) by the trivial line $\mathcal{C}$ and leaving the rest of the story unchanged, we get a groupoid $\mathcal{E}_0^\mathcal{D}$. It has an evident Picard groupoid structure, and $\mathcal{E}_0$ is naturally a $\mathcal{E}_0^\mathcal{D}$-torsor. Below we denote by (i)–(iv) the structure constraints in $\mathcal{E}_0^\mathcal{D}$ that correspond to (i)–(iv) above.

5.2. Compatibility with quadratic degenerations of $X$. Let us formulate an important property of $\mathcal{E} \in \mathcal{E}_0$.
Suppose we have data 2.13(a),(b) in the analytic setting; we follow the notation of loc. cit. In 2.13 we worked in the formal scheme setting. Now the whole story of (2.13.1)–(2.13.7) makes sense analytically: we have a proper family of curves $X$ over a small coordinate disc $Q$ and an $\mathcal{O}_X$-module $M$ equipped with a relative connection $\nabla$, etc., so that the picture of 2.13 coincides with the formal completion of the present one at $q = 0$. To construct $X$, notice that $t_\pm$ from 2.13(a) converge on some true neighborhoods $U_\pm$ of $b_+$. Suppose that $t_\pm$ identify $U_\pm$ with coordinate discs of radii $r_\pm$ and $U_+ \cap U_- = U_\pm \cap (T \cup \{D\}) = \emptyset$. Then $Q$ is the coordinate disc of radius $r_+ r_-$. Let $W$ be an open subset of $Y \times Q$ formed by those pairs $(y, q)$ that if $y \in U_+$, then $r_+(y) > |q|r_+$, and if $y \in U_-$, then $r_-(y) > |q|r_-$. Our $X$ is the union of two open subsets $V := U_+ \times U_-$ and $W$: we glue $(y, q) \in W$ such that $y \in U_\pm$ with $(y_+, y_-) \in V$ such that either $y_+$ or $y_-$ equals $y$ and $t_+(y_+) t_-(y_-) = q$. The projection $q : X \to Q$ is $(y, q) \mapsto q$ on $W$ and $q(y_+, y_-) = t_+(y_+) t_-(y_-)$ on $V$. Set $K_{WQ} := X \setminus W = \{(y_+, y_-) \in V : r_+(y_+) r_-(y_-) = 0\}$. The formal trivializations of $L$ at $b_+$ from 2.13(b) converge on $U_\pm$, and we define $M$ by gluing $M_{b_+} \otimes \mathcal{O}_V$ and the pull-back $N_{W}$ of $N$ by the projection $W \to Y$.

Let us define an analytic version of (2.13.7), which is an isomorphism of $\mathcal{O}_Q$-lines

$$
det R_{dR} M \sim \det R_{dR} (Y, N) \otimes \mathcal{O}_Q. \tag{5.2.1}
$$

Let $i, j$ be the embeddings $K_{WQ} \hookrightarrow X \hookrightarrow W$. Since $V \setminus K_{WQ}$ is disjoint union of two open subsets $V_\pm, V_+ := \{(y_+, y_-) : r_+^{-1} |t_+(y_+)| > r_-^{-1} |t_-(y_-)|\}$, the complex $F := i^* j_* dR_{W/Q}(N_W)$ is the direct sum of the two components $F_\pm$. Both maps $i^* dR_{X/Q}(M) \to F_\pm$ are quasi-isomorphisms, so $i^* F_+ \sim Cone(dR_{X/Q}(M) \to j_* dR_{W/Q}(N_W))$, hence $\det R_{dR} M \sim (\det R_{dR} M_{W}) \otimes (\det R_{dR} N_{W})^{\otimes -1}$. Let $N_X \subset j_* N_W$ be a $-\text{extension}$ from $V_+$ side and $+\text{-extension}$ from $V_-$ side. Then $i^* dR_{X/Q}(N_X) = F_+ \subset F_+ \oplus F_-$, hence $\det R_{dR} N_X \sim (\det R_{dR} N_{W}) \otimes (\det R_{dR} N_{W})^{\otimes -1}$. Thus we get a canonical identification $\alpha : \det R_{dR} M \sim \det R_{dR} N_X$. Now $N_W$, hence $N_X$, are $D$-modules, i.e., they carry an absolute connection, so $R_{dR} N_X, R_{dR} N_W$ carry a natural connection. It is clear from the topology of the construction that the cohomology are smooth, hence constant, $D_Q$-modules. Since the fiber of $R_{dR} N_X$ at $q = 0$ equals $R_{dR} (Y, N)$, we get (5.2.1).

For any $\mathcal{E} \in \mathcal{U}E$ we have an $\mathcal{O}_Q$-line $\mathcal{E}(M)_{\nu Q} := \mathcal{E}(M)_{(D_Q, 1_{T, \nu Q})}$ and a natural isomorphism $\mathcal{E}(M)_{\nu Q} \sim \mathcal{E}(N)_{(D_{1_{T, 1}}, \nu Q)} \otimes \mathcal{O}_Q$, cf. (2.13.4). There is a canonical isomorphism $\mathcal{E}(N)_{(D_{1_{T, 1}}, \nu Q)} \sim \mathcal{E}(N)_{\nu}$ defined in the same way as (2.13.9) (using $\eta$ on $\mathbb{P}^1$). Since $X$ is smooth over $Q^o := Q \setminus \{0\}$, we have $\eta : \mathcal{E}(M)_{\nu Q} |_{Q^o} \sim \det R_{dR} M |_{Q^o}$. Thus comes a diagram

$$
\begin{array}{ccc}
\mathcal{E}(M)_{\nu Q} |_{Q^o} & \xrightarrow{\eta} & \det R_{dR} M |_{Q^o} \\
\downarrow & & \downarrow \\
\mathcal{E}(N)_{\nu} \otimes \mathcal{O}_{Q^o} & \xrightarrow{\eta} & \det R_{dR} (Y, N) \otimes \mathcal{O}_{Q^o}.
\end{array} \tag{5.2.2}
$$

We say that $\mathcal{E}$ is a theory of $\varepsilon$-factors if (5.2.1) commutes for all data 2.13(a),(b). Such $\mathcal{E}$ form a subgroupoid $\mathcal{E}$ of $\mathcal{U}E$ called the $\varepsilon$-gerbe; see 5.4 for the reason.

For $\mathcal{E}^0 \in \mathcal{U}E^0$ there is a similar diagram
Those $\mathcal{E}$ for which (5.2.3) commutes for every datum 2.13(a),(b) form a Picard subgroupoid $\mathbb{E}^0$ of $\mathbb{E}^0$. Our $\mathcal{E}$ is an $\mathbb{E}^0$-torsor.

**Proposition.** $\mathcal{E}_{\text{dR}}$ and $\mathcal{E}_B$ are theories of $\varepsilon$-factors.

**Proof.** Compatibility of $\mathcal{E}_{\text{dR}}$ with quadratic degenerations follows from 2.13. Namely, the construction from loc. cit., spelled analytically as above, provides $\eta_{\text{dR}} : \mathcal{E}_{\text{dR}}(M)_{\nu Q} \xrightarrow{\sim} \det R_{q_{\text{dR}}}{M}$ over the whole $Q$ (not only on $Q^0$), and the proposition in 2.13 says that our diagram commutes on the formal neighborhood of $q = 0$. Hence it commutes everywhere, q.e.d.

Let us treat $\mathcal{E}_B$. Let $K_{\infty}$ be a compact neighborhood of $T \cup |D|$ in $Y$ that does not intersect $U_{\pm}$, $K := K_{\infty} \cup \{b_+, b_-\}$. Let $\mathcal{N}$ be a continuous family of proper cones in the tangent bundle to $Y \setminus K$ such that $\text{Re}(\nu)$ is negative on it. Set $K_{\infty Q} := K_{\infty} \times Q$, $K_Q := K_{\infty Q} \cup K_{\infty Q} \subset X$; let $\mathcal{N}_W$ be the pull-back of $\mathcal{N}$ by the projection $W \to Y$. Then $(X, K_Q, \mathcal{N}_W, \nu Q)$ form a $Q$-family of compatible data as in 4.6. Consider isomorphisms $\eta$ of (4.7.1) for $M$ and $\mathcal{N}_X$. Our $Q$-family is constant near $K_{\infty Q}$, so $\mathcal{E}_B(M)|_{\mathcal{N}(K_{\infty Q})} = \mathcal{E}_B(N_X)|_{\mathcal{N}(K_{\infty Q})} = \mathcal{E}_B(N)|_{\mathcal{N}(K_{\infty})} \otimes \mathcal{O}_Q$. Let $C$ be a locally closed subset of $V$ which consists of those $(y_+, y_-)$ that $2|t_+(y_+)|/r_+ - |t_-(y_-)|/r_- \leq 1$ and $2|t_-(y_-)|/r_- - |t_+(y_+)|/r_+ < 1$; set $R_{q_{\text{dR}}}(?) := R_{q_{\text{dR}}}(M)_{\nu Q}$. If $\mathcal{N}$ is sufficiently tight, then $C \subset C(X, \mathcal{N}_W)$ (see the proof of the lemma in 4.7), thus $\mathcal{E}_B(?)|_{\mathcal{N}(K_{\infty Q})} = \det R_{q_{\text{dR}}}(?)$.

Since the construction of (5.2.1) was local at $K_{\infty Q}$, the composition of $\mathcal{E}_B(M) \xrightarrow{\eta_{\text{dR}}} \det R_{q_{\text{dR}}}{M}$ in (5.2.2) can be rewritten as $\mathcal{E}_B(N)|_{\mathcal{N}(K_{\infty})} \otimes \det R_{q_{\text{dR}}}(?) \xrightarrow{\sim} \mathcal{E}_B(N)|_{\mathcal{N}(K_{\infty})} \otimes \det R_{q_{\text{dR}}}(?) \xrightarrow{\sim} \det R_{q_{\text{dR}}}(?)$. Here $\sim$ comes from the identification

$$\alpha_C : \det R_{q_{\text{dR}}}(?) \xrightarrow{\sim} \det R_{q_{\text{dR}}}(?)$$

defined by the same construction as (5.2.1) with $R_{q_{\text{dR}}}$ replaced by $R_{q_{\text{dR}}}$.

The composition $\mathcal{E}_B(M) \xrightarrow{\sim} \mathcal{E}_B(N)|_{\mathcal{N}(K_{\infty})} \otimes \det R_{q_{\text{dR}}}(?) \xrightarrow{\sim} \det R_{q_{\text{dR}}}(?) \xrightarrow{\sim} \mathcal{E}_B(N)|_{\mathcal{N}(K_{\infty})} \otimes \det R_{q_{\text{dR}}}(?) \xrightarrow{\sim} \det R_{q_{\text{dR}}}(?) \xrightarrow{\sim} \det R_{q_{\text{dR}}}(?) \xrightarrow{\sim} \det R_{q_{\text{dR}}}(?)$. Here $\sim$ comes since $R_{q_{\text{dR}}}(?) = R_{q_{\text{dR}}}(?)$; hence their determinant lines are trivialized (notice that the trivialization of $R_{q_{\text{dR}}}(?)$ is horizontal, and at $q = 0$ it equals the Betti version of (2.13.8) due to Exercise in 4.7).

We see that commutativity of (5.2.2) means that $\alpha_C$ identifies the above trivializations. To see this, consider the open subspace $V_\subset V$, and the corresponding 2-step filtration $\mathcal{E}_{\text{dR}}M|_{V_\subset M|V}$, and notice that $N_X|V = \text{gr} M|V$. The assertion follows now from the construction of $\alpha_C$.

$$\mathcal{E}^0(M)_{\nu Q}|Q^0 \xrightarrow{\sim} \mathcal{O}_{Q^0}$$
5.3. Let $(X,T,M)$ be as in 5.1. For $E \in \mathbb{E}, b \in X,$ and a meromorphic form $\nu$ on a neighborhood of $b$, $v_b(\nu) = -\ell$, we write $E(M)_{(b,\nu)} := E(M)_{(\theta b,\nu)}$.

Remark. If $b$ is a smooth point of $M$, then $E(M)_{(b,\nu)}$ does not depend on whether we view $b$ as a point of $T$ or $X \setminus T$ (by 5.1(iii) with $\pi = \text{id}_X$, $T = T' \cup \{b\}$).

Let $\delta(M)_{b,\nu} \in \mathbb{Z}$ be the degree of $E(M)_{(b,\nu)}$, and $\mu(M)_{b,\nu} \in \mathbb{C}^\times$ be the value of $\mu \in \text{Aut}(E(M))$ (see 1.15) at $(b,\nu)$.

**Lemma.** (i) One has $\delta(M)_{b,\nu} = \dim(B(M)_{b}^{(1)}) + (1-\ell)\text{rk}(M)$.

(ii) For $c \in \mathbb{C}^\times$ the multiplication by $c$ automorphism of $M$ acts on $E(M)_{(b,\nu)}$ as multiplication by $c^{\delta(M)_{b,\nu}}$.

(iii) One has $\mu(M)_{b,\nu} = (-1)^{\ell} \mu_b(M)^{-1}$ where $\mu_b(M)$ is the monodromy of $det M_{X \setminus T}$ around $b$.

(iv) For smooth $M$, there is an isomorphism $E(M)^{(1)} \xrightarrow{\sim} (\text{det} M)^{\otimes -1}$ compatible with constraint 5.1(ii) and pull-backs by open embeddings. In particular, this is an isomorphism of symmetric monoidal functors.

(v) Suppose we have $M, M'$ over discs $U, U'$ which have regular singularity and are either $*$- or $!$-extension at $b, b'$. Let $\phi : U \rightarrow U'$ be any open embedding, $\phi(b) = b'$, and $\phi : M \rightarrow \phi^* M'$ be any its lifting. Then the isomorphism $E(\tilde{\phi}) : E(M)^{(1)} \xrightarrow{\sim} E(M')^{(1)}$ does not depend on the choice of $(\phi, \tilde{\phi})$.

**Proof.** (i) Let $t$ be a local coordinate at $b$, $t(b) = 0$. We can view $t$ as an identification of a small disc $X_b$ at $b$ with a neighborhood of $0 \in \mathbb{P}^1$. Let us extend $M|_{X_b}$ to a $D$-module $M^{(1)}$ on $\mathbb{P}^1$ which is smooth outside $\{0, \infty\}$, and is the $*$-extension with regular singularities at $\infty$. Such $M^{(1)}$ is unique.

By continuity, $\delta(M)_{b,\nu}$ is the same for all $\nu$ with fixed $\ell$. We can assume that $\nu$ is meromorphic on $\mathbb{P}^1$ with $\text{div}(\nu) \subset \{0, \infty\}$, so $v_0(\nu) = -2 - v_\infty(\nu)$. By 5.1(iv), one has $\delta(M^{(1)})_{0,\nu} + \delta(M^{(1)})_{\infty,\nu} = \chi_{\text{dR}}(\mathbb{P}^1, M^{(1)}) = \dim(B(M^{(1)})_{b})$. Thus the assertion for $(X_b, M, \nu)$ amounts to that for $(\mathbb{P}^1_{\infty}, M^{(1)}, \nu)$, i.e., we are reduced to the case when $M$ is the $*$-extension with regular singularities. By 5.1(ii), it suffices to treat the case of $\text{rk}(M) = 1$; then, by continuity, it suffices to consider $M = O_X$ (the trivial $D$-module). By 5.1(iv) applied to $\mathbb{P}^1$ and $t^{-1}dt$, we see that $\delta(O_{\mathbb{P}^1})_{0,-1dt} + \delta(O_{\mathbb{P}^1})_{\infty,-1dt} = -2$, hence, since $v_\infty(t^{-1}dt) = v_0(t^{-1}dt)$, one has $\delta(O_{\mathbb{P}^1})_{b,-1dt} = -1$.

By factorization, $\delta(O_{\mathbb{P}^1})_{0,-1dt} = \ell \delta(O_{\mathbb{P}^1})_{0,1dt} = \ell t \delta(O_{\mathbb{P}^1})_{0,1dt} = -\ell$, q.e.d.

(ii) The $\mathbb{C}^\times$-action on $E(M)_{(b,\nu)}$, which comes from the action of homotheties on $M$, is a holomorphic character of $\mathbb{C}^\times$. Thus $c$ acts as multiplication by $c^{\delta(M)_{b,\nu}}$ for some $\delta'(M)_{b,\nu} \in \mathbb{Z}$. The argument of (i) works for $\delta$ replaced by $\delta'$, so $\delta$ and $\delta'$ are given by the same formula, q.e.d.

(iv) By (i), $E^1(O_X)$ is a de Rham line of degree $-1$, which has local origin. Thus there is a line $E$ of degree $-1$ and an isomorphism $E \otimes O_X \xrightarrow{\sim} E^1(O_X)$ compatible with the pull-backs by open embeddings of $X$’s; such a datum is uniquely defined.

The set of isomorphisms $\alpha : E \xrightarrow{\sim} \mathbb{C}[1]$ identifies naturally with the set of isomorphisms of symmetric monoidal functors $\alpha_M : E(M)^{(1)} \xrightarrow{\sim} (\text{det} M)^{\otimes -1}$ (where $M$ is smooth) that are compatible with the pull-backs by open embeddings. Namely, $\alpha_M$ is a unique isomorphism that equals $\alpha \otimes \text{id}_{O_X}$ for $M = O_X$. To see this, notice that a matrix $g \in \text{GL}_n(\mathbb{C}) \xrightarrow{\sim} \text{Aut}(O^*_X)$ acts on $E^1(O^*_X)$ as multiplication by $\text{det}(g)^{-1}$ (which follows from (ii) and 5.1(ii)).

(iii) Use (iv) and the compatibility property from 1.11.
(v) Let us show that Aut(M) acts trivially on $E(M)^{(1)}_0$. Pick any $g \in \text{Aut}(M)$. Since $M_{U \setminus \{b\}}$ admits a $g$-invariant filtration with successive quotients of rank 1, we are reduced, by 5.1(ii), to the case of $M$ of rank 1. Here $g$ is multiplication by some $c \in \mathbb{C}^*$, and we are done by (ii) (since, by (i), $E(M)^{(1)}_b$ has degree 0).

Thus for given $\phi$ the isomorphism $\mathcal{E}(\hat{\phi})$ does not depend on the choice of $\hat{\phi}$. The space of $\hat{\phi}$'s is connected, so it suffices to show that the map $\phi \mapsto \mathcal{E}(\hat{\phi})$ is locally constant. If $\phi$ varies in a disc $Q$, then we can find $\hat{\phi}$ which is an isomorphism of $D$-modules on $U \times Q$, hence our map is horizontal (see 5.1(i)), q.e.d.

5.4. For $\mathcal{E} \in \mathcal{E}$ and $(X,T,M)$ as in 5.1 the canonical automorphism $\mu$ of $\mathcal{E}(M)$ (see 1.15) is evidently compatible with constraints 5.1(i)–(iv), i.e., $\mu$ is an automorphism of $\mathcal{E}$. Here is the main result of this section:

**Theorem.** Aut($\mathcal{E}$) is an infinite cyclic group generated by $\mu$. All objects of $\mathcal{E}$ are isomorphic. Thus $\mathcal{E}$ is a $\mathbb{Z}$-gerbe.

We call $\rho : \mathcal{E}_\text{dR} \to \mathcal{E}_B$ an $\varepsilon$-period isomorphism. By the theorem, $\varepsilon$-period isomorphisms form a $\mathbb{Z}$-gerbe $\mathcal{E}_B/\text{dR}$ referred to as the $\varepsilon$-period torsor.

Since $\mathcal{E}$ is an $\mathbb{E}^0$-torsor, the theorem can be reformulated as follows. By (iii) of the lemma in 5.3, every $\mathbb{E}^0 \in \mathbb{E}^0$ carries a natural automorphism $\mu^0$ that acts on $\mathcal{E}^0(M)^{b,\nu}$ as multiplication by $\mu(M)^{b,\nu} \in \mathbb{C} \times \mathbb{E}$.

**Theorem'.** The map $\mathbb{Z} \to \pi_1(\mathbb{E}^0)$, $1 \mapsto \mu^0$, is an isomorphism, and $\pi_0(\mathbb{E}^0) = 0$.

The proof occupies the rest of the section.

5.5. Pick any $\mathcal{E}^0 \in 
\text{u} \mathbb{E}^0$. Then for $(X,T,M)$ as in 5.1 one has:

**Lemma.** (i) The factorization line $\mathcal{E}^0(M) \in \mathcal{E}_\text{dR}^0(X,T)$ is trivial.

(ii) Every automorphism of $M$ acts trivially on $\mathcal{E}^0(M)$.

(iii) For $M$ of rang 0, the factorization line $\mathcal{E}^0(M)$ is canonically trivialized. The trivialization has local nature and is compatible with constraints 5.1(i)$^0$–(iv)$^0$; it is uniquely defined by this property.

**Proof.** (i) One checks that the de Rham line $\mathcal{E}^0(M)^{(1)}$ on $X \setminus T$ is trivial by modifying the argument in the proof of 5.3(iv) in the evident manner (or one can use 5.3(iv) directly, noticing that $\mathcal{E}^0$ is the ratio of two objects of $\text{u} \mathbb{E}$). Similarly, $\mathcal{E}(M)$ has zero degree by 5.3(i). Now use the theorem in 1.6.

(ii) Let us show that $g \in \text{Aut}(M)$ acts trivially on $\mathcal{E}^0(M)^{(b,\nu)}$. Let $M^{(t)}$ be as in the proof of 5.3(i); $g$ acts on it. We can assume that $\nu$ is meromorphic on $\mathbb{P}^1$ with div($\nu$) $\subset \{0,\infty\}$. The action of $g$ on $\mathcal{E}^0(M^{(t)})^{(b,\nu)} = \mathcal{E}^0(M)^{(0,\nu)} \otimes \mathcal{E}^0(M^{(t)})^{(\infty,\nu)}$ is trivial by 5.1(iv)$^0$, it suffices to check that it acts trivially on $\mathcal{E}^0(M^{(t)})^{(\infty,\nu)}$. Thus we are reduced to the situation when our $M$ is the $*$-extension with regular singularities. By constraint 5.1(ii)$^0$, it suffices to consider the case when the monodromy of $M$ around $b$ is multiplication by a constant. Then $\text{Aut}(M)$ is generated by the diagonal matrices, and by 5.1(ii)$^0$ we are reduced to the case when $M$ has rank 1, where we are done by 5.3(ii).

(iii) is left to the reader. □

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$^{27}\mu^0$ does not equal the canonical automorphism $\mu$ of $\mathcal{E}^0(M)$ (which is identity by 5.5(i)).
Remarks. (i) By (i) of the lemma, the degree 0 lines $E^0(M)_{(b,\nu)}$ for all $\nu$ with fixed $v_0(\nu) = -\ell$ are canonically identified; we denote this line by $E^0(M)_{(b,\ell)}$. By (ii) of loc. cit., it depends only on the isomorphism class of $M$, and by (iii) there is a canonical identification $E^0(M)_{(b,\ell)} \cong E^0(j_{b*}M)_{(b,\ell)}$.

(ii) Suppose we have $M, M'$ on discs $U, U'$ which have regular singularity at $b(t) \in U(t)$. Let $\phi : U \to U'$ be an open embedding, $\phi(b) = b'$, and $\tilde{\phi} : M \to \phi'^*M'$ be any its lifting. Then the isomorphisms $E^0(\tilde{\phi}) : E^0(M)_{(b,\ell)} \cong E^0(M')_{(\nu,\ell)}$ do not depend on the choice of $(\phi, \tilde{\phi})$. To see this, we can assume that $M, M'$ are $*$-extensions at $b, b'$, and then repeat the second part of the proof of 5.3(v).

5.6. For $m \in \mathbb{C}^\times$ let $M_m$ be a $D$-module of rank 1 on a disc $U$, which has regular singularity at $b \in U$ with the monodromy $m$ and is $*$-extension at $b$. By the remark in 5.5, the degree 0 line $\mathcal{G}_{(m,\ell)} := E^0(M_m)_{(b,\ell)}$ depends only on $m$ and $\ell$. By 5.1(i), $\mathcal{G}_{(m,\ell)}$ form a holomorphic line bundle $\mathcal{G} = \mathcal{G}(E^0)$ over $\mathbb{G}_m \times \mathbb{Z}$. The factorization structure on $E^0(M_m)$ provides, by 5.5(i), canonical isomorphisms

$$\otimes \mathcal{G}_{(1,\ell_a)} \cong \mathcal{G}_{(1,\Sigma_{\ell_a})}, \quad \mathcal{G}_{(1,\ell)} \otimes \mathcal{G}_{(m,\ell')} \cong \mathcal{G}_{(m,\ell+\ell')}.$$  

(5.6.1)

Suppose we have a finite collection $\{(m_\alpha, \ell_\alpha)\}$ with $\Pi m_\alpha = 1, \Sigma \ell_\alpha = 2$. Then for any choice of a subset $\{b_\alpha\} \subset \mathbb{P}^1$ one can find a $D$-module $M$ on $\mathbb{P}^1$ of rank 1 which is smooth off $\{b_\alpha\}$ and is $*$-extension with regular singularity at $b_\alpha$ with monodromy $m_\alpha$, and a rational form $\nu$ with $\text{div}(\nu) = -\Sigma \ell_\alpha b_\alpha$. Writing $E(M)(\mathbb{P}^1) = E(M)$, in 5.1(iv), we get

$$\eta : \otimes \mathcal{G}_{(m_\alpha,\ell_\alpha)} \cong \mathbb{C}.$$  

(5.6.2)

It does not depend on the auxiliary choices (for $\eta$ is locally constant, and the datum of $\{b_\alpha\}, \nu$ forms a connected space; $M$ is unique up to an isomorphism).

Now assume that $E^0 \in \mathbb{E}$. Consider the holomorphic $\mathbb{G}_m$-torsor $\mathbb{G}^\times = \mathbb{G}^\times(E^0)$ over $\mathbb{G}_m \times \mathbb{Z}$ that corresponds to $\mathcal{G}$.

Lemma. $\mathbb{G}^\times$ has a unique structure of holomorphic commutative group $\mathbb{G}_m$-extension of $\mathbb{G}_m \times \mathbb{Z}$ such that for any $g_i \in \mathbb{G}_{(m_i,\ell_i)}^\times, 1 \leq i \leq n, g_1 \in \mathbb{G}_{(m_1,\ell_1)}^\times, \ldots, g_n \in \mathbb{G}_{(m_n,\ell_n)}^\times, 1 \leq i \leq n$, one has $\eta(g \otimes (g_1 \cdots g_n)) = \eta(g \otimes g_1 \otimes \cdots \otimes g_n)$.

Proof. The above formula defines commutative $n$-fold product maps $\mathbb{G}^\times \to \mathbb{G}^\times, (g_1, \ldots, g_n) \mapsto g_1 \cdots g_n$, which lift the $n$-fold products on $\mathbb{G}_m \times \mathbb{Z}$. We need to check the associativity property, which says that for any $g_1, \ldots, g_n \in \mathbb{G}^\times$ and any $k, 1 < k < n$, one has $g_1 \cdots g_n = (g_1 \cdots g_k)g_{k+1} \cdots g_n$.

For $m \in \mathbb{C}^\times$ set $\mathbb{G}^\times_m := \sqcup \mathcal{G}_{(m,\ell)} \subset \mathbb{G}^\times$. The maps $(\mathbb{G}_{\ell}^\times)^n \to \mathbb{G}_{\ell}^\times, \mathbb{G}^\times \times \mathbb{G}^\times \to \mathbb{G}^\times$ coming from the arrows in (5.6.1) are evidently associative and commutative, i.e., they define a commutative group structure on $\mathbb{G}^\times_1$ and a $\mathbb{G}^\times_1$-torsor structure on $\mathbb{G}^\times_m$. Thus $\mathbb{G}^\times$ is a $\mathbb{G}^\times_1$-torsor over $\mathbb{G}_m$.

Since (5.6.2) comes from a trivialization of $E(M)$, the above maps are restrictions of the multiple products maps in $\mathbb{G}^\times$. Moreover, the $n$-fold product on $\mathbb{G}^\times$ is compatible with the $\mathbb{G}^\times_{\ell}$-action on $\mathbb{G}^\times$: for $h \in \mathbb{G}^\times_{\ell}$ one has $(h g_1)g_2 \cdots g_n = h(g_1, \ldots, g_n)$. So, while checking the associativity, we have a freedom to change $g_i$ in its $\mathbb{G}^\times_{\ell_i}$-orbit. Thus we can assume that $g_i \in \mathbb{G}_{(m_i,\ell_i)}^\times$ are such that $\ell_1 + \ldots + \ell_k = 1$.

Then one can find a quadratic degeneration picture as in 5.2 such that $\check{T} = b_{1Q} \sqcup \ldots \sqcup b_{nQ}, M$ of rank 1 has regular singularities at $b_{1Q}$ with monodromy $m_1$, $\text{div}(\nu) = -\Sigma \ell b_{1Q}$; the fiber $\check{X}_1$ is $\mathbb{P}^1$, and $\check{X}_0$ is the union of two copies of $\mathbb{P}^1$ with
{b_1, \ldots, b_k} \) in the first copy and \( \{b_{k+1}, \ldots, b_n\} \) in the second. The compatibility with quadratic degeneration yields the promised associativity, q.e.d.

\begin{equation}
\text{Theorem} (5.7.3) \text{ is left inverse to (5.7.4), so the theorem amounts to the next statement: leave it to the reader to check that } E \text{ of } \bar{G}(m, \ell) \text{ comes from the product in } \bar{G}_{m} \text{ of } \det b \text{ and } G_{m}. \end{equation}

This follows from compatibility of \( \eta \) with \( \pi \), applied to a covering \( \mathbb{P}^1 \to \mathbb{P}^1, t \mapsto t^2 \), the trivial \( D \)-module \( \mathcal{O}_{\mathbb{P}^1} \) on the source, and the form \( t^{-1} dt \) on the target.

Let \( \text{Ext}(G_m, G_m) \) be the Picard groupoid of holomorphic commutative group extensions of \( G_m \) by \( G_m \). One has \( \pi_0(\text{Ext}(G_m, G_m) = 0 \) and \( \pi_1(\text{Ext}(G_m, G_m) = \text{Hom}(G_m, G_m) = \mathbb{Z}. \)

The quotient of \( G_m \times \mathbb{Z} \) modulo the subgroup generated by \((-1,1)\) identifies with \( G_m \) by the projection \((m, \ell) \mapsto (-1)^\ell m \). Thus the quotient \( \bar{G}^\times(\mathcal{E}^0) \) of \( G^\times(\mathcal{E}^0) \) modulo the subgroup generated by \( e(-1,1) \) is an object of \( \text{Ext}(G_m, G_m) \).

\[ \bar{G}^\times : \mathbb{E}^0 \to \text{Ext}(G_m, G_m) \] (5.7.3)

is a Picard functor. It assigns to \( \mu^0 \in \text{Aut}(\mathcal{E}^0) \) (see 5.4) the generator \(-1\) of \( \mathbb{Z} = \text{Aut}(\bar{G}^\times(\mathcal{E}^0)) \). Therefore we can reformulate the theorem from 5.4 as follows:

\textbf{Theorem.} \( \bar{G}^\times \) is an equivalence of Picard groupoids.

Let us define a Picard functor

\[ \text{Ext}(G_m, G_m) \to \mathbb{E}^0 \] (5.7.4)

right inverse to (5.7.3). We need to assign to an extension \( \bar{G}^\times \) an object \( \mathcal{E}^0 = \mathcal{E}^0(\bar{G}^\times) \) of \( \mathbb{E}^0 \). Suppose we have \((X, T, M)\) as in 5.1. For \( b \in T \) let \( m(M)_b \) be the monodromy of \( \det M|_{X \setminus T} \) around \( b \); for \( c \in 2^T \) we denote by \( m(M)_c \) the product of \( m(M)_b \) for \( b \in T^c \) (see 1.1 for the notation). Then

\[ \mathcal{E}^0(M)_{(D, c, v)} := \bar{G}(-1)^{\deg(D) / \text{rk}(M)} m_{(M)_c}. \] (5.7.5)

Here \( \bar{G} \) is the degree 0 line that corresponds to the \( G_m \)-torsor \( \bar{G}^{\times} \). The factorization structure comes from the product in \( \bar{G}^{\times} \). Constraints 5.1(i)\( ^0 \), 5.1(ii)\( ^0 \) are evident. The identification \( \mathcal{E}^0(\pi \cdot M^\prime) \sim \pi \cdot \mathcal{E}(M^\prime) \) of 5.1(iii)\( ^0 \) comes since both lines are fibers of \( \bar{G} \) over the same point of \( \mathbb{C}^{\times} \). To see this, it suffices to consider the situation of (5.7.1): there the assertion is clear since \( \prod \prod \) \( (-1)^m = \prod m^{m_1} \). Finally, for compact \( X \) one has \( m(M)_1 = 1 \) and \( \deg(D) \) is even, hence \( \mathcal{E}^0(M)_{(D, c, v)} = \bar{G}_1 = \mathbb{C} \), which is 5.1(iv)\( ^0 \). The constraints are mutually compatible by construction. We leave it to the reader to check that \( \mathcal{E}^0 \) is compatible with quadratic degenerations of \( X \), so we have defined (5.7.4). Due to an evident identification \( \bar{G}^{\times}(\mathcal{E}^0) = \mathcal{G}^{\times} \), (5.7.3) is left inverse to (5.7.4), so the theorem amounts to the next statement:

\textbf{Theorem’.} \textit{For any } \mathcal{E}^0 \in \mathbb{E}^0 \text{ there is a natural isomorphism } \iota : \mathcal{E}^0 \sim \mathcal{E}^0(\bar{G}^{\times}(\mathcal{E}^0)).
5.8. The next step reduces us to the setting of \(D\)-modules with regular singularities. For a holonomic \(D\)-module \(M\) we denote by \(M^{rs}\) the holonomic \(D\)-module with regular singularities such that \(B(M^{rs}) = B(M)\), or, equivalently, \(M^∞ = M^{rs∞}\), see 3.2. The functor \(M \mapsto M^{rs}\) sends nice families of \(D\)-modules to nice families (as follows from 2.14), it is exact, commutes with \(\pi_∗\), and one has an evident identification \(R\Gamma_{DR}(X,M) \simeq R\Gamma_{DR}(X,M^{rs})\). Thus for any theory of \(\varepsilon\)-factors \(\mathcal{E}\) the rule \(M \mapsto \mathcal{E}(M) := \mathcal{E}(M^{rs})\) is again a theory of \(\varepsilon\)-factors. Clearly \(r\) is an endofunctor of \(\mathcal{E}\). The same formula defines an endofunctor \(r^0\) of \(\mathcal{E}^0\). It is naturally a Picard endofunctor, and \(r\) is a companion \(\mathcal{E}^0\)-torsor endofunctor: one has \(r^0\mathcal{E}^0 \otimes \mathcal{E}^0 \simeq r^0(\mathcal{E}^0 \otimes \mathcal{E}^0)\), \(r^0\mathcal{E}^0 \otimes \mathcal{E}^0 \simeq \mathcal{E}(\mathcal{E}^0 \otimes \mathcal{E}^0)\).

**Proposition.** The endofunctor \(r\) of \(\mathcal{E}\) is naturally isomorphic to \(\text{id}_\mathcal{E}\). Namely, there is a unique \(\kappa : \text{id}_\mathcal{E} \simeq r\) such that \(\kappa_{\mathcal{E},M} : \mathcal{E}(M) \to \mathcal{E}(M^{rs})\) is the identity map if \(M\) has regular singularities. Same is true for \(\mathcal{E}\) replaced by \(\mathcal{E}^0\). Here \(\kappa^0 : \text{id}_{\mathcal{E}^0} \simeq r^0\) is an isomorphism of Picard endofunctors, and \(\kappa\) is an isomorphism of the companion \(\mathcal{E}^0\)-torsor endofunctors.

**Proof.** (i) Let us define a canonical isomorphism of factorization lines

\[
\kappa = \kappa_{\mathcal{E},M} : \mathcal{E}(M) \xrightarrow{\sim} \mathcal{E}(M^{rs}). \tag{5.8.1}
\]

One has \(M|_{X \setminus T} = M^{rs}|_{X \setminus T}\), and over \(X \setminus T\) our \(\kappa\) is the identity map. It remains to define \(\kappa^{(1)} : \mathcal{E}(M)_{b}^{(1)} \xrightarrow{\sim} \mathcal{E}(M^{rs})_{b}^{(1)}\) for \(b \in T\) (see 1.6 and 1.15).

Pick a local parameter \(t\) at \(b\). As in the proof of 5.3(i), \(M\) yields a \(D\)-module \(M^{(t)}\) on \(\mathbb{P}^1\) with \(\mathcal{E}(M)_{b}^{(1)} = \mathcal{E}(M^{(t)})_{(0,t^{-1}dt)}\). Ditto for \(M^{rs}\). Since \(M^{rs(t)} = M^{(t)rs}\) equals \(M^{(t)}\) outside \(0\), constraints 5.1(iv) for \(M^{(t)}\) and \(M^{rs(t)}\) yield isomorphisms

\[
\mathcal{E}(M^{(t)})_{(0,t^{-1}dt)} \otimes \mathcal{E}(M^{(t)})_{(∞,t^{-1}dt)} \xrightarrow{\sim} \text{det } R\Gamma_{DR}(\mathbb{P}^1, M^{(t)}) = \text{det } R\Gamma_{DR}(\mathbb{P}^1, M^{rs(t)}) \xrightarrow{\sim} \mathcal{E}(M^{rs(t)})_{(0,t^{-1}dt)} \otimes \mathcal{E}(M^{rs(t)})_{(∞,t^{-1}dt)}. \tag{5.8.2}
\]

Factoring out \(\mathcal{E}(M^{(t)})_{(∞,t^{-1}dt)}\), we get \(\kappa^{(1)}\).

It remains to show that \(\kappa^{(1)}\) does not depend on the auxiliary choice of \(t\). The space of local parameters \(t\) is connected, so we need to check that \(\kappa\) is locally constant with respect to it. Let \(t_s\) be a family of local parameters at \(b\) that are defined on the same disc \(X_b\) and depend holomorphically on \(s \in S\); then \(t_s\) identify \(X_b\) with a neighborhood \(U\) of \(\{0\} \times S\) in \(\mathbb{P}^1_S\). Let \(M^{(t)}_U\) be the pull-back of \(M\) by the projection \(U \to X_b, (v,s) \mapsto t_s^{-1}(v)\). This is a holonomic \(D_U\)-module; let \(M^{(t)}\) be a holonomic \(D\)-module on \(\mathbb{P}^1\) which equals \(M^{(t)}_U\) on \(U\), is smooth outside \(\{0,∞\} \times S\), and is the \(s\)-extension with regular singularities at \(\{∞\} \times S\). The restriction of \(M^{(t)}\) to any fiber equals \(M^{(t_s)}\), i.e., \(M^{(t_s)}\) form a nice isomonodromic family. The identifications \(\mathcal{E}(M^{(t)})_{b}^{(1)} \xrightarrow{\sim} \mathcal{E}(M^{(t_s)})_{(0,t_s^{-1}dt)}\) are horizontal, ditto for \(M^{rs}\). We are done the compatibility with \(\eta\) by the Gauß-Manin connection.

The same construction (with \(R\Gamma_{DR}(\mathbb{P}^1, \cdot)\) replaced by \(C\)) yields for \(\mathcal{E}^0 \in \mathcal{E}^0\) a canonical isomorphism

\[
\kappa^0 : \mathcal{E}^0(M) \xrightarrow{\sim} \mathcal{E}^0(M^{rs}). \tag{5.8.2}
\]
Let \( b \in T \) be a point where \( M \) has non-regular singularity, and \( M^{rs} \) be the \( D \)-module which equals \( M \) outside of \( b \) and \( M^{rs} \) near \( b \). Let \( \kappa_{b}^{0} : \mathcal{E}^{0}(M) \cong \mathcal{E}^{0}(M^{rs}) \) be equal to \( \kappa^{0} \) near \( b \) and the identity morphism off \( b \).

Lemma. The composition \( \mathcal{E}^{0}(M)(X) \xrightarrow{\kappa_{b}^{0}} \mathcal{E}^{0}(M^{rs})(X) \xrightarrow{\eta(M^{rs})} \mathbb{C} \) equals \( \eta(M) \).

The lemma implies the proposition: since \( (M^{rs})^{rs} = M^{rs} \) and the composition \( \mathcal{E}^{0}(M) \xrightarrow{\kappa_{b}^{0}} \mathcal{E}^{0}(M^{rs}) \xrightarrow{\kappa^{0}} \mathcal{E}(M^{rs}) \) equals \( \kappa^{0} \), we are done by induction by the number of points of \( T \) where \( M \) has non-regular singularity.

Proof of Lemma. Let \( \nu \) be a rational form on \( X \) such that \( \text{Res}_{b} \nu = 1 \). Let \( t_{b} \) be a local parameter at \( b \) such that \( t_{b}^{-1} dt_{b} = \nu \).

Consider a datum 2.13(a) with \( Y = \mathbb{P}^{1} \sqcup X, b_{+} = b \in X, b_{-} = \infty \in \mathbb{P}^{1} \) and \( \nu_{\nu} \) equal to \( t^{-1} dt \) on \( \mathbb{P}^{1} \) and \( \nu \) on \( X \). Let \( t_{\pm} \) be the parameter \( b_{\pm} \), \( t_{-} \) be the parameter \( t^{-1} \) at \( \infty \). The corresponding family of curves \( X' \) as defined in 5.2 (it was denoted by \( X \) there) over \( Q = \mathbb{A}^{1} \) is the blow-up of \( X \times \mathbb{A}^{1} \) at \( (b,0) \). We have a datum 2.13(b) with \( N \) equal to \( M^{(t_{b})} \) on \( \mathbb{P}^{1} \setminus \{ \infty \} \) and to \( M \) on \( X \setminus \{ b \} \) (this determines \( N \) since it is \( ! \)-extension with regular singularities at \( b_{-} \) and \( * \)-extension with regular singularities at \( b_{+} \)). Let \( L \) be any \( t_{\pm} \partial t_{\pm} \) -invariant \( b_{\pm} \)-lattice in \( N \) such that the eigenvalues of \( \pm t_{\pm} \partial t_{\pm} \) on \( L_{b_{\pm}} \) and their pairwise differences do not contain non-zero integers. Then the spectra of the \( \pm t_{\pm} \partial t_{\pm} \) actions on \( L_{b_{\pm}} \) coincide, and there is a canonical identification \( \alpha : L_{b_{-}} \sim L_{b_{-}} \) characterized by the next property: Consider the \( t_{\pm} \partial t_{\pm} \) -invariant embeddings \( L_{b_{\pm}} \subset \Gamma(\mathbb{P}^{1} \setminus \{ 0 \}, L) \) and \( L_{b_{\pm}} \subset \Gamma(U, L) \) as in 2.13(b). By the definition of \( M^{(t_{b})} \), its sections over a punctured neighborhood of \( 0 \) are identified with sections of \( M \) over \( U \setminus \{ b \} \); by this identification the subspaces \( L_{b_{\pm}} \) correspond to one another, and \( \alpha \) is the corresponding isomorphism. Let \( M' \) be the corresponding family of \( \mathcal{O}_{X} \)-modules with relative connection on \( X/Q \) (which was denoted by \( M \) in 5.2).

At \( q = 1 \) our \( M' \) equals \( M \), so the top arrow in (5.2.3) at \( q = 1 \) equals \( \eta(M) \). And the composition of its lower arrows equals the composition from the statement of our lemma. Since \( \mathcal{E}^{0} \in \mathcal{E}^{0} \), the diagram commutes; we are done.

5.9. Let us turn to the proof of Theorem’ in 5.7. For \( \mathcal{E}^{0} \in \mathcal{E}^{0} \) let \( \tilde{\mathcal{G}}^{\times} = \mathcal{G}^{\times} \) be the corresponding extension of \( G_{\mathbb{R}}^{n} \) (see (5.7.3)) and \( \mathcal{E}^{0'} \in \mathcal{E}^{0} \) be the object defined by \( \mathcal{G}^{\times} \) (see (5.7.4)). We want to define a natural isomorphism \( \iota : \mathcal{E}^{0} \sim \mathcal{E}^{0'} \).

For \( (X, T, M) \) as in 5.1 we define a canonical isomorphism of factorization lines

\[
\iota : \mathcal{E}^{0}(M) \sim \mathcal{E}^{0'}(M)
\]  

(5.9.1)

as follows. Due to isomorphism (5.8.2), we can assume that \( M = M^{rs} \). Now \( \iota \) is a unique isomorphism of local nature which is compatible with constraints 5.1(ii)\(^{0} \) and 5.5(iii), and is the identity map for \( M = M_{rn} \) (see 5.6). Indeed, we can assume that \( X \) is a disc, \( T = \{ b \} \), and, by 5.5(iii), that \( M \) is \( * \)-extension at \( b \). Then \( M = \oplus M(m) \), where the monodromy around \( b \) acts on \( M(m) \) with eigenvalues \( m \). By compatibility with 5.1(ii)\(^{0} \), we can assume that \( M = M^{(m)} \). Pick any filtration on \( M \) with successive quotients of rank 1 and define \( \iota \) as the composition \( \mathcal{E}^{0}(M) \sim \otimes \mathcal{E}(\text{gr}_{\mathbb{R}} M) = \otimes \mathcal{E}^{0}(\text{gr}_{\mathbb{R}} M) \sim \mathcal{E}^{0'}(M) \) where \( \sim \) are constraints 5.1(ii)\(^{0} \). The choice of the filtration is irrelevant by 5.1\(^{0} \) (for the space of filtrations is connected).

Our \( \iota \) is compatible with constraints 5.1(i)\(^{0} \), 5.1(ii)\(^{0} \); its compatibility with 5.1(iii)\(^{0} \) will be checked in 5.13. We treat 5.1(iv)\(^{0} \) first; this takes 5.10–5.12. For
$(X, T, M)$ with compact $X$ let $\xi(X, M) \in \mathbb{C}^\times$ be the ratio of $\eta(M)$ for $E^0$ and the composition of $\eta(M)$ for $E^0$ with $\iota$. We want to show that $\xi(X, M) \equiv 1$.

5.10. By 5.8 and the construction of $\iota$, it suffices to consider $M$ with regular singularities. By 5.5(iii), $\xi(X, M)$ depends only on $M|_{X \setminus T}$. Therefore, by compatibility with 5.10, $\xi(X, M)$ depends only on the purely topological datum of (the isomorphism class of) a punctured oriented surface $X \setminus T$ (that can be replaced by a compact surface with boundary) and a local system on it.

The compatibility with quadratic degenerations implies that if $Y$ is obtained from $X$ by cutting along a disjoint union of embedded circles and $N = M|_Y$, then $\xi(X, M) = \xi(Y, N)$. Here is an application:

**Lemma.** (i) If $M$ admits a filtration such that $gr\, M$ are $D$-modules of rank 1, then $\xi(X, M) = 1$.

(ii) For every $(X, T, M)$ with $X$ connected one can find $(X', T, M')$ such that the restriction of $M$ to a neighborhood of $T$ is isomorphic to that of $M'$, $X'$ is connected of any given genus $g \geq g(X)$, and $\xi(X, M) = \xi(X', M')$.

(iii) For every $(X, T, M)$ one can find $(X', T, M')$ with $X'$ connected such that $\xi(X, M) = \xi(X', M')$ and for every $b \in T$ the restriction of $M'$ to a neighborhood of $b$ is isomorphic to that of $M$ plus a direct sum of copies of a trivial $D$-module.

**Proof.** (i) By compatibility with 5.1(ii)$^0$, we can assume that $M$ is a $D$-module of rank 1. Our assertion is true if $X$ has genus 0 by the construction. An arbitrary $X$ can be cut into a union of genus 0 surfaces, and we are done.

(ii) Consider $Y = X \sqcup Z$ where $Z$ is a compact smooth connected curve of genus $g_0 = g(X)$; let $N$ be a $D_Y$-module such that $N|_X = M$ and $N|_Z$ is a trivial $D$-module of the same rank as $M$. Pick $x \in X \setminus T$, $z \in Z$, cut off small discs around $x$, $z$ and connect their boundaries by a tube. This is $X'$. Take for $M'$ any extension of $M_Y$ (restricted to the complement of the cut discs) to a local system on $X'$. Since $\xi(Z, N|_Z) = 1$ by (i), one has $\xi(Y, N) = \xi(X, M)$, hence $\xi(X', M') = \xi(X, M)$.

(iii) Let us construct $(X', M')$. First, add to $M$ on different components of $X$ an appropriate number of copies of the trivial $D$-module to assure that the rank of $M$ is constant; this does not change $\xi(X, M)$ by (i). Let $X_1, \ldots, X_n$ be the connected components of $X$. On each $X_i \setminus T$, choose a pair of distinct points $x_i, y_i$. Cut off small discs around $x_1, \ldots, x_{n-1}$ and $y_2, \ldots, y_n$, and connect the boundary circle at $x_i$ with that at $y_{i+1}$ by a tube. This is our $X'$. Take for $M'$ any extension of $M$ to a local system on $X'$.

5.11. **Proposition.** For $X$ connected, $\xi(X, M)$ depends only on the datum of conjugacy classes of local monodromies of $M$ (the rank of $M$ is fixed).

**Proof.** According to [PX1], [PX2], the action of the mapping class group on the moduli of unitary local systems of given rank with fixed conjugacy classes of local monodromies, is ergodic (provided that the genus of the Riemann surface is $> 1$). As in Theorem 1.4.1 in [G], this implies that for connected $X$ with $g(X) > 1$ our $\xi(X, M)$ depends only on $g(X)$, the rank of $M$, and the datum of conjugacy classes of local monodromies of $M$ (indeed, $\xi$ is invariant with respect to the action of the mapping class group by the compatibility with 5.1$^0$, and is holomorphic; by the ergodicity, its restriction to the real points of the moduli space of local systems is
constant, and we are done). Use 5.10(ii) to eliminate the dependence on \( g(X) \) (and the condition on \( g(X) \)).

5.12. For any \((X, M)\), let \( Sp(M) \) be the datum of other than 1 eigenvalues (with multiplicity) of the direct sum of local monodromies. We write it as an element \( \Sigma n_i z_i \) (\( z_i \) are the eigenvalues, \( n_i \) are the multiplicities) of the quotient of \( \text{Div}(\mathbb{C}^\infty) \) modulo the subgroup of divisors supported at \( 1 \in \mathbb{C}^\infty \).

**Lemma.** \( \xi(X, M) \) depends only on \( Sp(M) \).

**Proof.** (i) By 5.10(iii), it suffices to check this assuming that \( X \) is connected, and by 5.10(i) we can assume that the rank of \( M \) is fixed. By 5.11, it suffices to find for any \((X, M)\) some \((X', M')\) such that \( \xi(X, M) = \xi(X', M') \), \( Sp(M) = Sp(M') \), and each local monodromy of \( M' \) has at most one eigenvalue different from 1. Take any \( b \in T \); let \( m_b \) be the local monodromy at \( b \). Then one can find a local system \( K(b) \) on \( \mathbb{P}^1 \) with ramification at \( \infty \) and \( n \) other points, \( n = \text{rk}(M) \), such that its local monodromy at \( \infty \) is conjugate to \( m_b^{-1} \), and \( K(b) \) admits a flag of local subsystems such that each \( \text{gr}_i K(b) \) has rank 1 and ramifies at \( \infty \) and only one other point.

Cut a small disc around \( b \) in \( X \) and that around \( \infty \) in \( \mathbb{P}^1 \), and connect the two boundary circles by a tube; we get a surface \( X'(b) \). Let \( M'_b \) be a local system on it that extends \( M \) and \( K(b) \). By 5.10(i), \( \xi(\mathbb{P}^1, K(b)) = 1 \), so \( \xi(X'(b), M'_b) = \xi(X, M) \) by the compatibility with quadratic degenerations. Repeating this construction for each point of \( T \), we get \( (X', M') \).

The lemma implies that \( \xi(X, M) = 1 \). Indeed, if \( Sp(M) = \Sigma n_i z_i \), then \( \Pi z_i^{n_i} = 1 \) (for the product does not change if we replace \( M \) by \( det M \), where it equals 1 by the Stokes formula). Therefore one can find a \( \mathcal{D} \)-module \( M' \) of rank 1 on \( \mathbb{P}^1 \) with \( Sp(M') = Sp(M) \). Since \( \xi(X, M') = 1 \), we are done by the lemma.

5.13. It remains to check that \( \iota \) of (5.9.1) is compatible with 5.1(iii)\(^0 \). We want to show that for \( \pi : X' \to X \) and a \( \mathcal{D} \)-module \( M' \) the diagram

\[
\begin{array}{ccc}
\mathcal{E}^0(\pi_* M') & \to & \mathcal{E}^0(\pi_* M') \\
\downarrow & & \downarrow \\
\pi_* \mathcal{E}^0(M') & \to & \pi_* \mathcal{E}^0(M'),
\end{array}
\]

(5.13.1)

where the vertical arrows are constraints 5.1(iii)\(^0 \) for \( \mathcal{E}^0, \mathcal{E}^0', \) commutes. For \( b \in X \) let \( \psi(M', \pi, b) \) be the ratio of the morphisms \( \mathcal{E}^0(\pi_* M'_b) \to \pi_* \mathcal{E}^0(M'_b) \) that come from the two sides of the diagram. We want to show that \( \psi(M', \pi, b) \equiv 1 \) (see 1.6). It is clear that \( \psi(M', \pi, b) = 1 \) if \( \pi \) is unramified at \( b \).

Our \( \psi \) has \( X \)-local nature and it is multiplicative with respect to disjoint unions of \( X' \), so it suffices to consider the case when \( X, X' \) are discs and \( \pi = \pi^{(n)} \) is ramified of index \( n \) at \( b \). Choosing a local coordinate \( t \) at \( b \), we identify \( X' \) and \( X \) with neighborhoods of 0 in \( \mathbb{P}^1 \) so that our covering is the restriction of \( \pi : \mathbb{P}^1 \to \mathbb{P}^1, \ t \mapsto t^n \), to \( X \). Let us extend \( M' \) to a \( \mathcal{D} \)-module on \( \mathbb{P}^1 \), which we again denote by \( M' \), such that it is smooth outside 0 and 1. We know that 5.1(iii)\(^0 \) is compatible with 5.1(iv)\(^0 \) for both \( \mathcal{E}^0 \) and \( \mathcal{E}^0' \). Since \( \iota \) is compatible with 5.1(iv)\(^0 \) and \( \pi \) is ramified only at 0 and \( \infty \), we know that \( \psi(M', \pi, 0) \psi(M', \pi, \infty) = 1 \). Since \( M' \) is smooth over \( \infty \), this means that \( \psi(M', \pi^{(n)}, b) \psi(\mathcal{O}_{X'}, \pi^{(n)}, b)^{\text{rk}(M)} = 1 \). We are reduced to the case \( M' = \mathcal{O}_{X'} \).
Set $\psi_n := \psi(O_{X'}, \pi^{(n)}, b)$. By above, $\psi_n^2 = 1$, i.e., $\psi_n = \pm 1$. By the construction of $\mathcal{E}^{\nu}$ (see 5.7), one has $\psi_2 \equiv 1$. Due to compatibility with the composition, one has $\psi_{mn} = \psi_n \psi_m = \psi_m \psi_m$, i.e., $\psi_m^{-1} = \psi_m^{-1}$. For $m = 2$ we get $\psi_n \equiv 1$, q.e.d.

6 The $\Gamma$-function.

6.1. Let us describe explicitly the $\varepsilon$-period map $\rho^\varepsilon = \rho^\varepsilon(M) : \mathcal{E}_{\text{dR}}(M) \subset \mathcal{E}_B(M)$ for $\rho^\varepsilon \in \mathcal{E}_{\text{dR}}/\mathcal{E}$ (see 5.4).

By 5.8, $\rho^\varepsilon(M)$ equals the composition $\mathcal{E}_{\text{dR}}(M) \subset \mathcal{E}_{\text{dR}}(M^{\nu}) \cong \mathcal{E}_B(M^{\nu}) = \mathcal{E}_B(M)$, where $\kappa$ is the canonical isomorphism of (5.8.1) and $\cong$ is $\rho^\varepsilon(M^{\nu})$. A different construction of the same $\kappa$ was given in (3.1.1) in terms of certain analytical Fredholm determinant (a version of $\tau$-function).

From now on we assume that $M$ has regular singularities.

By (1.6.3), Example in 1.6, and Remark in 1.15, $\mathcal{E}_{\varepsilon}(M)$ amounts to a datum $(\mathcal{E}_{\varepsilon}((M^{(1)}),\{\mathcal{E}_{\varepsilon}(M^{(b)}_{n})\})$. Thus $\rho^\varepsilon$ is completely determined by the isomorphisms $\mathcal{E}_{\text{dR}}(M^{(1)}_{n}) \cong \mathcal{E}_B(M^{(1)}_{n})$ and $\mathcal{E}_{\text{dR}}(M^{(b)}_{t}) \cong \mathcal{E}_B(M^{(b)}_{t})$, $b \in T$.

Let us write a formula for $\rho^\varepsilon = \rho^\varepsilon_b : \mathcal{E}_{\text{dR}}(M^{(b)}_{t}) \cong \mathcal{E}_B(M^{(b)}_{t})$, $b \in T$. Below $t$ is a local parameter at $b$, and $t_b, X_b$ are the embeddings $\{b\} \hookrightarrow X_b \hookrightarrow X^{g} := X_b \setminus \{b\}$.

If $M$ is supported at $b$, then $\mathcal{E}_{\text{dR}}(M^{(b)}_{t}) = \mathcal{E}_B(M^{(b)}_{t}) = \det R\mathcal{E}_{\text{dR}}(X, M)$ and $\rho^\varepsilon_b$ is the identity map. Thus for arbitrary $M$ one has $\mathcal{E}_{\varepsilon}(M^{(b)}_{t}) = \mathcal{E}_{\varepsilon}(j_{bs}M^{(b)}_{t}) \otimes \det R\mathcal{E}_{\text{dR}}(X, M)$ and

$$\rho^\varepsilon_b(M) = \rho^\varepsilon_b(j_{bs}M) \otimes \text{id}_{\det R\mathcal{E}_{\text{dR}}(X, M)}. \tag{6.1.1}$$

So it suffices to define $\rho^\varepsilon_b$ for $M = j_{bs}M$. Then we have a canonical trivialization $1_b$ of $\mathcal{E}_B(M^{(b)}_{t}) := \mathcal{E}_B(M^{(b)}_{t-1 dt})$, see 4.7.

Let $L$ be a $t\partial_t$-invariant $b$-lattice in $M$. Denote by $\Lambda(L)$ the spectrum of the operator $t\partial_t$ acting on on $L_b = L/tL$. Suppose that it does not contain positive integers. Then the complex $\mathcal{C}(L, \omega L(b))$ (see 2.4) is acyclic; here $L(b) := t^{3}L$, i.e., $\omega L(b) = t^{-3}dtL$. Denote by $\iota(L)_{t-1 dt}$ the corresponding trivialization of $\mathcal{E}_{\text{dR}}(M^{(b, z^{(-1)} dt)} \cong \det \mathcal{C}(L, \omega t(b))$, $z \neq 0$ (see (2.5.6)); it does not depend on the choice of $t$. If $L' \supset L$ is another lattice, then $\iota(L'_{t-1 dt}) = \iota(L)_{z^{(-1)} dt}$ is the determinant of the action of $z^{-1}t\partial_t$ on $L'/L$ (see 2.5). In particular, we have a trivialization $\iota(L)_{t-1 dt}$ of $\mathcal{E}_{\text{dR}}(M^{(b)}_{t})$. We write $\rho^\varepsilon_b(\iota(L)_{t-1 dt}) = \gamma_{\rho^\varepsilon_b}(L)1_b$. For example, if $M = M^{n}_{t}$ is the $D$-module $M^{n}_{t}$ generated by $t^{n}$, $t\partial_t(t^{n}) = \lambda t^{n}$, where $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{>0}$, and $L = L^{n}_{t}$ is the lattice generated by $t^{n}$, then $\Lambda(L^{n}_{t}) = \{\lambda\}$, and we write $\gamma_{\rho^\varepsilon_b}(\lambda) := \gamma_{\rho^\varepsilon_b}(L^{n}_{t})$.

Theorem. (i) One has $\gamma_{\rho^\varepsilon_b}(L) = \prod_{\lambda \in \Lambda(L)} \gamma_{\rho^\varepsilon_b}(\lambda)$.

(ii) For $a \in \mathbb{Z}$ one has \[ \gamma_{\rho^\varepsilon_b}(\lambda) = (-1)^{a} \exp(-2\pi i \lambda a) \gamma_{\rho^\varepsilon_b}(\lambda). \]

(iii) For one $\rho^\varepsilon$ in $\mathcal{E}_{\text{dR}}/\mathcal{E}$ one has

$$\gamma_{\rho^\varepsilon_b}(\lambda) = (2\pi)^{-1/2} \left(1 - \exp(2\pi i \lambda)\right) \Gamma(\lambda), \tag{6.1.2}$$

where $\Gamma$ is the Euler $\Gamma$-function and $(2\pi)^{1/2}$ is the positive square root.

\[ \text{\footnote{Recall that } E_{\text{dR}}/\mathcal{E} \text{ is a } L\text{-torsor.} } \]
Proof. (i) By above, for $L' \supset L$ one has $\gamma_{\rho,\mu}^1(L)/\gamma_{\rho,\mu}^1(L') = \det(t\partial_t; L'/L)$. Therefore the validity of (i) does not depend on the choice of $L$. So we can assume that $(M, L)$ is a successive extension of some $(M^I, L^I)$, and we are done since all our objects are multiplicative with respect to extensions.

(ii) By 5.3(iii), $\mu(M^I_{\lambda})$ acts on $E(M^I_{\lambda})_b^{(1)}$ as multiplication by $-\exp(-2\pi i\lambda)$.

(iii) The claim follows from (ii) and the next lemma:

Lemma. (i) The function $\gamma_{\rho,\mu}^1(\lambda)$ is holomorphic and invertible for $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{>0}$, and satisfies the next relations: (a) $\gamma_{\rho,\mu}^1(\lambda + 1) = \lambda \gamma_{\rho,\mu}^1(\lambda)$; (b) For every positive integer $n$ one has $\gamma_{\rho,\mu}^1(\frac{\lambda}{n})\gamma_{\rho,\mu}^1(\frac{\lambda+1}{n}) \cdots \gamma_{\rho,\mu}^1(\frac{\lambda+n-1}{n}) = n^{2-\lambda}\gamma_{\rho,\mu}^1(\lambda)$.

(ii) Any function $\gamma$ that satisfies the properties from (i) equals one of the functions $\gamma_a(\lambda) = (2\pi)^{-1/2}(-1)^a \exp(2\pi i\alpha)(1 - \exp(2\pi i\lambda))\Gamma(\lambda)$ for some integer $a$.

Proof of Lemma. (i) We check (b); the rest is clear. Let $\pi : X' \to X$ is a covering of a disc completely ramified of index $n$ at $b$, so for a parameter $t'$ at $b'$ one has $\pi^*(t) = t^n$, hence $\pi^*(t^{-1}dt') = nt'^{-1}dt'$. Let $M'$ be a $\mathcal{D}$-module on $X'$ which is the $*$-extension with regular singularities at $b'$. Consider isomorphisms

$$\mathcal{E}_{dr}(\pi, M')_{(x', t', -1, dt')} \overset{\alpha}{\longrightarrow} \mathcal{E}_{dr}(M')_{(x', ntv^{-1}, dt')} \overset{\beta}{\longrightarrow} \mathcal{E}_{dr}(M')_{(x', t', -1, dt')}$$

(6.1.3)

where $\alpha$ is the projection formula identification and $\beta$ is the $\nabla^*$-parallel transport along the interval $[m, 1]t'^{-1}dt'$. By the construction of $1_b^\lambda$, the Betti version of $\beta \alpha$ transforms $1_b^\lambda$ to $1_b^{\lambda+1}$.

Suppose $M'$ equals $M^I_{\lambda}$ for some $\lambda \in \mathbb{C}$. Then $\pi_*M'$ equals $M^I_{\lambda/n} \oplus M^I_{(\lambda+1)/n} \oplus \cdots \oplus M^I_{(\lambda+n-1)/n}$. If $L' = L^I_\lambda \subset M^I_{\lambda}$, then $\pi_*L' = L^I_{\lambda/n} \oplus L^I_{(\lambda+1)/n} \oplus \cdots \oplus L^I_{(\lambda+n-1)/n}$. It is clear that $\alpha$ sends $\iota(L', t', -1, dt')$ to $\iota(L')_{nt'^{-1}, dt'}$. Since $\iota(L')$ is horizontal for the connection $\nabla_0$ of (2.11.3) (with $t$ and $n$ in loc. cit. equal to 1), $\beta$ sends $\iota(L')_{nt'^{-1}, dt'}$ to $n^{2-\lambda}\iota(L')_{t'^{-1}, dt'}$. Now $\rho^*\iota(L')_{t'^{-1}, dt'} = \gamma_{\rho,\mu}^1(\lambda)1_b^\lambda$ and $\rho^*\iota(\pi_*L', t', -1, dt') = \gamma_{\rho,\mu}^1(\pi_*L')1_b^\lambda = \gamma_{\rho,\mu}^1(\frac{\lambda}{n})\gamma_{\rho,\mu}^1(\frac{\lambda+1}{n}) \cdots \gamma_{\rho,\mu}^1(\frac{\lambda+n-1}{n})1_b^\lambda$, and we are done since $\rho^*$ is compatible with 5.1(iii).

(ii) Denote by $E$ the set of functions $\gamma$ that satisfy properties from (i). Let $E^0$ be the set of functions $e(\lambda)$ which are invertible and holomorphic on the whole $\mathbb{C}$ and satisfy the relations (a) $e(\lambda + 1) = e(\lambda)$; (b) $e(\frac{\lambda}{n})e(\frac{\lambda+1}{n}) \cdots e(\frac{\lambda+n-1}{n}) = e(\lambda)$ for any positive integer $n$. Then $E^0$ is a group with respect to multiplication, and $E$ is an $E^0$-torsor.

Notice that the function $\mu(\lambda) := -\exp(-2\pi i\lambda)$ belongs to $E^0$, and $\{\lambda_a\}_{a \in \mathbb{Z}}$ is a $\mu^2$-torsor. Recall that $\Gamma(\lambda)$ is holomorphic and invertible for $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, and satisfies the next relations: (a) $\Gamma(\lambda + 1) = \lambda \Gamma(\lambda)$; (b) $\Gamma(\frac{\lambda}{n}) \cdots \Gamma(\frac{\lambda+n-1}{n}) = (2\pi)^{-\frac{n}{2}} n^{\frac{\lambda}{2}} \Gamma(\lambda)$ for any positive integer $n$. This implies that $\gamma_a$ belong to $E$. To prove the lemma, it remains to check that $\mu$ generates $E^0$.

Pick any $e \in E^0$, let $a$ be the index of the holomorphic map $e : \mathbb{C}/\mathbb{Z} \to \mathbb{C}^\times$. Let us check that $e\mu_a^\mu = 1$. Indeed, $e\mu_a^\mu$ has index 0, so $e\mu_a^\mu(\lambda) = \exp(f(\lambda))$ for some holomorphic $f : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$. Notice that for any $n \in \mathbb{Z}_{>0}$ the function $\lambda \mapsto f(\lambda/n) + f(\lambda+1/n) + \cdots + f(\lambda+n-1/n) - f(\lambda)$ takes values in $2\pi i\mathbb{Z}$, hence constant. Consider the coefficients of the Laurent series $f(\lambda) = S_m \exp(2\pi i m\lambda)$. The above property implies that $(n - 1)b_{\pm n} = 0$ for $n > 1$, i.e., $b_m = 0$ for $|m| > 1$. The case $n = 2$ shows that $b_{\pm 1} = 0$. Finally the fact that $(n - 1)b_0 \in 2\pi i\mathbb{Z}$ for any $n > 0$ implies that $b_0 \in 2\pi i\mathbb{Z}$, and we are done.
Corollary. For $\rho^\ast$ as in (6.1.2) the isomorphism $\rho^\ast : \mathcal{E}_{dR}(M)^{(1)}_{X\setminus T} \xrightarrow{\sim} \mathcal{E}_B(M)^{(1)}_{X\setminus T}$ equals the composition $\mathcal{E}_{dR}(M)^{(1)}_{X\setminus T} \xrightarrow{(2.6.1)\text{ (det } M_{X\setminus T})^{\otimes -1}} \mathcal{E}_B(M)^{(1)}_{X\setminus T}$ multiplied by $((2\pi)^{1/2})^{\text{rk}(M)}$. Replacing $\rho^\ast$ by $\rho^\ast + a$ multiplies it by $(-1)^{\text{rk}(M)a}$.

Proof. Suppose $M$ is smooth at $b$. Compatibility with 5.1(iii) implies, as in Remark 5.3, that $\rho_b^\ast : \mathcal{E}_{dR}(M)_b^{(1)} \xrightarrow{\sim} \mathcal{E}_B(M)_b^{(1)}$ does not depend on whether $b$ is viewed as a point of $T$ or not. The exact sequence $0 \to M \to j_b^*j_b^!M \to i_b^*M_b \to 0$ shows that $R\Gamma_{dR}(b, X, M) = M_b[-1]$, hence $\mathcal{E}_\gamma(M)_b^{(1)} = \mathcal{E}(j_b^*M_1^{(1)} \otimes (\text{det } M_b)^{\otimes -1})$. The isomorphisms $\mathcal{E}_\gamma(M)_b^{(1)} \xrightarrow{\sim} (\text{det } M_b)^{\otimes -1}$ come from trivializations of $\mathcal{E}_\gamma(j_b^*M_1^{(1)})$, which are $\epsilon(M)_t^{-1}dt$ in the de Rham and $1_b^*$ in the Betti case. Since $\gamma_{\rho^\ast}^1(M) = ((2\pi)^{1/2})^{\text{rk}(M)}$ by the theorem, we are done. □

The corollary together with the theorem completely determines $\rho^\ast(M)$.

6.2. Here is another explicit formula for

$$\rho^\ast : \mathcal{E}_{dR}(M)_{(b, t^{-1}dt)} \xrightarrow{\sim} \mathcal{E}_B(M)_{(b, t^{-1}dt)}.$$ 

Recall that $\mathcal{E}_B(M)_{(b, t^{-1}dt)} \xrightarrow{\sim} \det R\Gamma_{dR}(X_b, M)$ by (4.7.1); here $X_b$ is a small open disc at $b$. Let $L$ be a $t\partial_b$-invariant $b$-lattice in $M$ such that $\Lambda(L)$ does not contain non-positive integers, $L_\omega$ be the $\mathcal{O}$-submodule of $\omega M$ generated by $\nabla(L)$ (this is a $b$-lattice). Then the projection

$$\Gamma(X_b, dR(M)) \to \mathcal{C}(L, L_\omega) \quad (6.2.1)$$

is a quasi-isomorphism. Together with isomorphism $r_{L, t^{-1}dt} : \mathcal{E}_{dR}(M)_{(b, t^{-1}dt)} \xrightarrow{\sim} \det \mathcal{C}(L, L_\omega)$ from (2.5.6), it yields an identification $e(L) : \mathcal{E}_{dR}(M)_{(b, t^{-1}dt)} \xrightarrow{\sim} \mathcal{E}_B(M)_{(b, t^{-1}dt)}$. Thus $\rho^\ast_{(b, t^{-1}dt)} = \gamma^\ast_{\rho^\ast}(L)e(L)$ for some $\gamma^\ast_{\rho^\ast}(L) \in \mathbb{C}^\times$.

Proposition. One has $\gamma^\ast_{\rho^\ast}(L) = \prod_{\lambda \in \Lambda(L)} \gamma^\ast_{\rho^\ast}(\lambda)$, and for $\rho^\ast$ as in (6.1.2)

$$\gamma^\ast_{\rho^\ast}(\lambda) = (2\pi)^{-1/2}\exp(\pi i(\lambda - 1/2))\Gamma(\lambda). \quad (6.2.2)$$

Proof. If $L' \supset L$ is another lattice as above, then $e(L')/e(L)$ is the determinant of the action of $-t\partial_b$ on $tL'/tL$, so the validity of the assertion does not depend on the choice of $L$. It is compatible with filtrations, and holds for $M$ supported at $b$, so we can assume that $M$ has rank 1 and is the $*$-extension at $b$. Thus $\Lambda(L) = \{\lambda\}$; by continuity, it suffices to consider the case of $\lambda \notin \mathbb{Z}$. Then the complexes in (6.2.1) are acyclic. The corresponding trivializations of $\mathcal{E}_{dR}(M)_{(b, t^{-1}dt)}$ and $\mathcal{E}_B(M)_{(b, t^{-1}dt)}$ are $e(L)_{t^{-1}dt}$ from 6.1 and $1_b^*$ from 4.7; by construction, $e(L)(i(L)_{t^{-1}dt}) = 1_b^*$.

By (2.11.3), the counterclockwise monodromy from $t^{-1}dt$ to $-t^{-1}dt$ sends $\epsilon(L)_{t^{-1}dt}$ to $\exp(\pi i(\lambda - 1/2))\epsilon(L)_{t^{-1}dt}$. According to 4.8, the same monodromy sends $1_b^*$ to $\epsilon_{\rho^\ast}^\ast(1 - \exp(-2\pi i\lambda))^{-1}1_b^*$. Since $\rho^\ast$ is horizontal, one has $\gamma^\ast_{\rho^\ast}(\lambda) = (1 - \exp(-2\pi i\lambda))^{-1}\gamma^\ast_{\rho^\ast}(\lambda)\exp(\pi i(1/2 - \lambda)) = \exp(\pi i(\lambda - 1/2))(1 - \exp(2\pi i\lambda))^{-1}\gamma^\ast_{\rho^\ast}(\lambda)$, and we are done by (6.1.2).

Example. If $M$ is smooth at $b$ and $L = tM$, then isomorphism (6.2.1) is $\Gamma(X_b, M^\vee) \xrightarrow{\sim} M_b$, $m \mapsto m_b$, and $\gamma^\ast_{\rho^\ast}(tM) = ((2\pi)^{-1})^{\text{rk}(M)}e(L)$.

Exercise. Deduce Euler’s reflection formula $\Gamma(\lambda)\Gamma(1 - \lambda) = \pi \sin^{-1}(\pi \lambda)$ from (6.1.2), (6.2.2), the lemma in 2.7, and Exercise in 4.7.
6.3. Let us write down a formula for the factors $[\rho^\epsilon_{(O,\nu)}]$ from 0.3.  

Recall that we have $X$, $M$ and $\nu$ defined over a subfield $k$ of $C$, and $B(M)$ is defined over a subfield $k'$. The finite set of singular points of $M$ and $\nu$ is defined then over $k$; it is partitioned by $\text{Aut}(C/k)$-orbits. Let $O$ be such an orbit. The $C$-line $E_{\text{dR}}(M)_{(O,\nu)} = \otimes_{x \in O} E_{\text{dR}}(M)_{(x,\nu)}$ is defined over $k$ by $\Sect{29}$ and $E_{B}(M)_{(O,\nu)} = \otimes_{x \in O} E_{B}(M)_{(x,\nu)}$ is defined over $k'$ by $\Sect{4}$. Computing $\rho^\epsilon : E_{\text{dR}}(M)_{(O,\nu)} \rightarrow E_{B}(M)_{(O,\nu)}$, $\rho^\epsilon \in E_{B}/dR$, in $k$- and $k'$-bases, we get a number whose class $[\rho^\epsilon_{(O,\nu)}]$ in $C^\times/k^\times k^\times$ does not depend on the choice of the bases and the choice of $\rho^\epsilon$ in $E_{B}/dR$. Let us compute $[\rho^\epsilon_{(O,\nu)}]$ explicitly assuming that $M$ has regular singularities.

For $b \in O$ let $k_b \subset C$ be its field of definition; let $X_b$ be a small disc around $b$. Choose an auxiliary datum on the de Rham side: it is $(t, L, u, v)$, where $t$ is a parameter at $b$, $L$ is a $t\partial_t$-invariant $b$-lattice in $M$, $u$ is a non-zero vector in $\det L_b$, and $v$ is a non-zero vector in $\det C(L, L_\omega)$ (see $\Sect{2}$); we assume that $(t, L, u, v)$ are defined over $k_b$. Let $\Lambda_b$ be the spectrum (with multiplicities) of $t\partial_t$ acting on the fiber $L_{x_b}$; we assume that $\Lambda_b$ does not contain non-positive integers. An auxiliary datum on the Betti side is $(\phi, w)$, where $\phi$ is a non-zero horizontal section of $\det M$ over the half-disc $\text{Re}(t) > 0$, which is defined over $k'$ (with respect to the Betti $k'$-structure on the sheaf of horizontal sections), $w$ is a non-zero vector in $\det R\text{dR}(X_b, M)$ defined over $k'$.

The data yield numbers: The leading term of $\nu$ at $b$ is $\alpha_b e^{-t} dt$, $\alpha_b \in k_b^\times$; let $r_b \in k_b$ be the trace of $t\partial_t$ acting on $L_b$. Notice that $m_b := \exp(-2\pi i r_b)$ is the monodromy of $\det M_b^\nu$ around $b$, so $m_b \in k^\times$. Then the section $t^{r_b}\phi$ on the half-disc extends to an invertible holomorphic section of $\det L$ on $X_b$; set $\beta_b := (t^{r_b}\phi)_b/u \in C^\times$. Let $\delta_b \in C^\times$ be the ratio of $v$ and the image of $w$ by the determinant of $\Sect{2.1}$.  

Let us compute the numbers $\alpha_b$, $\beta_b$, $\delta_b$ and the spectrum $\Lambda_b$ for each $b \in O$ using Galois-conjugate de Rham side data. Set $n := \text{rk}(M)$.

**Proposition.** One has

$$[\rho^\epsilon_{(O,\nu)}] = \prod_{b \in O} (2\pi)^{-\frac{n}{2}} t^n \frac{(t^11)}{m_b^{-1} \alpha_b^{-r_b} \beta_b \delta_b} \prod_{\lambda \in \Lambda_b} \Gamma(\lambda). \tag{6.3.1}$$

**Proof.** For the sake of clarity, we do the computation assuming that $b$ is a $k$-point, leaving the general case to the reader.

As follows from $\Sect{2.1.3}$, the validity of formula does not depend on $\alpha_b$. Notice that the class of $\alpha_b^{-r_b} := \exp(-r_b \log(\alpha_b))$ in $C^\times/k^\times$ is well defined: adding $2\pi i t$ to the logarithm multiplies the exponent by $m_b \in k^\times$.

If $\ell = 1$ and $\alpha_b = -1$, then the formula follows from $\Sect{6.2.2}$.

To finish the proof, it remains to check that

$$[\rho^\epsilon_{(b, t-\ell dt)}] = (2\pi)^{-\frac{n}{2}} t^n \beta_b \rho^\epsilon_{(b, t-\ell dt)}].$$

Consider a family of forms $\nu_x := t^{-\ell}(x-t)^{-1} dt$. Then $E_{\text{dR}}(M)_{(x,1b,\nu_x)} = \mu^\nu(t^{l-1}(t-x) L/L_\omega) = \det C(L, L_\omega) \otimes \lambda(L/t^{l-1}(x-t)L)^{\otimes -1} = C(L, L_\omega) \otimes \lambda(t^{l-1}L/t^{l-1}(x-t)L)^{\otimes -1} \otimes \lambda(\ell-1).$ We fix a non-zero $t$ in $\lambda(L/t^{l-1}L)$ defined over $k$. Any local trivialization $g$ of $\det L$ yields then a trivialization $e(g)_x := v \otimes t^{-1} \otimes t^{n(1-\ell)} g_x^{-1}$ of $E_{\text{dR}}(M)_{(x,1b,\nu_x)}$; if $g$ is defined over $k$, then so is $e(g)$.

$\Sect{29}$The group $\text{Aut}(C/k)$ acts on $E_{\text{dR}}(M)_{(O,\nu)}$ by transport of structure; its fixed points is the $k$-structure on $E_{\text{dR}}(M)_{(O,\nu)}$. 


The leading terms of $\nu_x$ at $t = 0$ and $t = x$ are $x^{-1}t^{-t}dt$ and $x^{-\ell}(x-t)^{-1}dt$. Applying (2.11.2) to the $x$-lattice $(x-t)L$ and (2.11.3) to the $b$-lattice $t^{-1}L$, we see that $\epsilon(t^k\phi_x) = \epsilon\left(x^{-\frac{\omega(t)}{2\pi} - t^{-1} - \frac{1}{2}} \phi - x^{-\frac{\omega(t)}{2\pi}}\phi\right)$ is a horizontal (with respect to $x$) section of $E_{\text{dr}}(M)_{(x_1, b, \nu_x)}$. Since the value at $b$ of $\beta_b t^{n(1-t)}(t^\phi - 1)$ is a generator of $\det(t^{-1}L/tL)^{\otimes^{-1}}$ defined over $k$, we see that $\beta_b \epsilon(t^\phi_b) \in E_{\text{dr}}(M)_{(b, -t^{-1} - \text{dt})}$ is defined over $k$. If $s$ a horizontal section of $E_{\text{dR}}(M)_{(x_1, b, \nu_x)}$ over $X_b$ defined over $k'$, then $\rho^s(\epsilon(t^\phi_b)/s)$ is a constant function. Its value at $x = 0$, i.e., at $b$, belongs to $\beta_b^{-1} [\rho^s_{(b, -t^{-1} - \text{dt})}]$. By factorization and Example in 6.2, its value at $x = 1$ belongs to $\rho^s_{(t^{-1} - \text{dt})}(2\pi)^{-\frac{1}{2}} \phi^s$, and we are done.

\[\square\]

**Notation.** $a_\psi$ 2.9; $\mathcal{C}(U)$ 4.2; $\mathcal{C}(W, \mathcal{N})$ 4.4; $\mathcal{C}(L, L_\infty)$ 2.4; $\mathcal{D}$, $\mathcal{D}^\circ$, $(D, c, \nu, p)$, $(D, c, \nu)$ 1.1; $[D]$ 1.1; $dR(L, L_\infty)$ 2.7; $\text{Det}_{P/S}(E)$, $\text{Det}_{P/S}(E_1/E_2)$ 2.3; Div($X$) 1.1; $\psi_{\infty}$, $\psi_{\infty}^\circ$ 5.1; $\mathcal{E}_{\infty}$ 5.2; $\mathcal{E}(\ell)$ 1.6; $E_{\text{dR}}/\mathbb{B}$ 5.4; $e$, $e_\infty$ 2.5; $E_B(M)$ 4.6; $E_{\text{dR}}(M)$ 2.5; $I(U, \mathcal{N})$ 4.3; $G$ 2.2; $\text{Hom}_C(J_1, J_2)$ 2.2; $j_{\infty}$ 2.4; $K(D)$, $K(D)_D^{\emptyset}$ 1.1; $K^\ell$ 1.1; $K^\ell_{\text{dr}}$ 1.1; $L^\ell_{\text{dr}}$, $L^\ell_{\text{dr}}$, $L^\ell_{\text{dr}}$ 1.2; $L^\ell_{\text{dr}}(K^\ell_{\text{dr}})$ 1.11; $L^0_{\text{dR}}$ 1.7; $L^\ell_{\text{dR}}(X, T^\ell)$, $L^\ell_{\text{dR}}(X, T)$ 1.11; $L_{\text{dR}}(X, T)$ 1.6; $L_{\text{dR}}(X, T)$ 1.5; $L^\ell_{\text{dR}}(X, T)$ 1.11; $L^\ell_{\text{dR}}(X, T)$ 1.7; $L^\ell_{\text{dR}}(X, T)$ 1.5; $L^\ell_{\text{dR}}(X, T)$ 1.1; $M(C(U))$ 4.1; $O^\ell_{\text{dR}}$ 1.1; $O^\ell_{\text{dR}}$ 1.1; $P_{D,c}$ 1.1; $P_{D,c}$ 1.1; $r_L, v$ 2.5; $\text{Vcrys}$ 1.2; $\mathcal{Y}^\infty$ 1.3; $\tau_{\psi}$ 2.9; $\tau_N$ 4.3; $T$, $T^\ell_S$ 1.1; $\gamma^\ell_{\rho, \ell}(L)$, $\gamma^\ell_{\rho, \ell}(L)$ 6.2.1; $\gamma^\ell_{\rho, \ell}(L)$ 6.1; $\lambda_P$ 2.3; $\lambda(S)$ 4.2; $\Lambda(L)$ 6.1; $\mu^\psi_\infty$ 2.4; $\mu^\psi_\infty$ 2.5; $\mu^\psi_\infty$ 2.9; $\mu^\psi_\infty$, $\mu^\psi_\infty$ 2.10; $\pi_0(X)$, $\pi_0(L)$ 1.1; $\phi^0_\infty$ 2.10; $\nabla^\infty_\ell$ 2.11; $\omega(X, T)$ 1.12; $W^\infty_\ell(K^\ell_{\text{dR}})$ 1.10; $\tau_{\psi}$ 2.9; $I_b^1$ 4.7; $2^f$ 1.1.

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