Computing Robust Controlled Invariant Sets of Linear Systems

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Abstract. We consider controllable linear discrete-time systems with bounded perturbations and present two methods to compute robust controlled invariant sets. The first method tolerates an arbitrarily small constraint violation to compute an arbitrarily precise outer approximation of the maximal robust controlled invariant set, while the second method provides an inner approximation. The outer approximation scheme is δ-complete, given that the constraint sets are formulated as finite unions of polytopes.

1. Introduction

Let us consider two matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ with $m \leq n$ and a nonempty set $W \subseteq \mathbb{R}^n$. Throughout this note, we analyze linear, time-invariant, discrete-time systems with additive perturbations described by the difference inclusion

$$\xi(t + 1) \in A\xi(t) + Bu(t) + W, \quad W \neq \emptyset$$

(1)

where $\xi(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ is the state signal, respectively, input signal and $W$ is the set of disturbances. Here we slightly abuse notation and use $x + W$ instead of $\{x\} + W$ to denote the Minkowski set addition defined for two sets $P, Q \subseteq \mathbb{R}^n$ by $Q + P = \{y \in \mathbb{R}^n \mid \exists q \in Q, \exists p \in P \ y = q + p\}$.

In addition to the dynamics, we consider state constraints and input constraints given by the compact sets

$$X \subseteq \mathbb{R}^n \quad \text{and} \quad U \subseteq \mathbb{R}^m.$$  

(2)

We are interested in the computation of feedbacks that map states to admissible inputs

$$\mu : \mathbb{R}^n \Rightarrow U$$

(3)

which force the trajectories of (1) to evolve inside the state constraint set $X$. The double-arrow notation $\Rightarrow$ indicates that $\mu$ is set-valued, i.e., for $x \in \mathbb{R}^n$, the image $\mu(x)$ is a subset of $U$, see Ch. 5. Subsequently, we use $\mathcal{F}(U)$ to denote the set of all feedbacks that satisfy for all $x \in X$: $u \in \mu(x)$ implies for all $x' \in Ax + Bu + W : \mu(x') \neq \emptyset$.

A trajectory of (1) and $\mu \in \mathcal{F}(U)$, with initial state $x \in \mathbb{R}^n$, is a sequence $\xi : \mathbb{Z}_{\geq 0} \to \mathbb{R}^n$ that satisfies $\xi(0) = x$ and for which there exists $\nu : \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$ so that $\nu(t) \in \mu(\xi(t))$ and (1) hold for all $t \in \mathbb{Z}_{\geq 0}$.

It is well-known that the feedbacks of interest, i.e., the maps $\mu$ that force every trajectory of (1) and $\mu$ to evolve inside $X$ for all time, are characterized by the maximal robust controlled invariant set [3, 4], also known as infinite reachable set [2] or discriminating kernel [5, 6], contained in $X$.

A set $R \subseteq \mathbb{R}^n$ is called robust controlled invariant w.r.t. (1) and $U$, if there exists a feedback $\mu \in \mathcal{F}(U)$ so that for every trajectory $\xi$ of (1) and $\mu$ with initial state $\xi(0) \in R$ we have $\xi(t) \in R$ for all times $t \in \mathbb{N}$. We use $R(X)$ to denote the maximal robust controlled invariant set of (1) and $U$ defined as the largest robust controlled invariant subset of $X$.

Key words and phrases. Invariance, Viability, Infinite Reachability, Safety Properties, Finite Termination, δ-Decidability.

1Note that the invariance property of a set is closed under union so that the maximal controlled invariant set is well-defined.

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Given $R(X)$, the map $C : \mathbb{R}^n \rightarrow U$ defined by
\[ C(x) = \{ u \in U \mid Ax + Bu + W \subseteq R(X) \} \] characterizes all feedbacks of interest in the following sense: Suppose that a feedback $\mu \in \mathcal{F}(U)$ enforces the constraints $X$ on the system (1), i.e., every trajectory $\xi$ of (1) and $\mu$ satisfies $\xi(t) \in X$ for all $t \in \mathbb{Z}_{\geq 0}$, then we have for all $x \in \mathbb{R}^n$ the inclusion $\mu(x) \subseteq C(x)$, see e.g. [7, Thm. 1]. Therefore, it is sufficient to determine $R(X)$, whenever one is interested in feedbacks that enforce the constraints $X$ and $U$ on (1).

Even though, set invariance has a rich history, see e.g. [2, 6, 8, 9], the computation of $R(X)$ for most types of constraint sets $X$ and $U$, e.g. when $X$ and $U$ are given as a union of polytopes, is still an open problem. In this note, we propose two algorithms to compute an outer, respectively, inner approximation of the maximal robust controlled invariant set. Both algorithms are obtained as modifications of the well-known dynamic programming approach to the computation of the infinite reachable set [2, 4, 9]. Before we provide a more detailed description of our contribution, we review the state of the art on the computation of invariant sets of linear systems.

Let $\text{pre}(R) = \{ x \in \mathbb{R}^n \mid \exists u \in U Ax + Bu + W \subseteq R \}$ denote the set of states that are mapped into $R$ by the dynamics when the input is appropriately chosen. In [2], Bertsekas introduced the iteration
\[ R_0 = X, \quad R_{i+1} = \text{pre}(R_i) \cap X \] and showed, that for every open set $\Omega$ that contains $R(X)$, there exists $j$ so that $R_j \subseteq \Omega$ for all $j \geq i$, provided that the sets $R_i$ are nonempty. In our case, this implies the convergence
\[ R(X) = \lim_{i \to \infty} R_i \] with respect to the Hausdorff distance. See also [8].

The set convergence [6] shows that the maximal robust controlled invariant set $R(X)$ can, in principle, be outer approximated by the sequence $(R_i)_{i \in \mathbb{Z}_{\geq 0}}$, with arbitrary precision. Nevertheless, even if the sets $R_i$ are computable, the approximation is not very useful since in general the sets $R_i$ are not robust controlled invariant and it is not possible to derive a feedback from any $R_i$ that ensures that the system always evolves inside the state constraint set.

However, in some cases it is possible to determine the maximal robust controlled invariant set by the iteration [5]. If there exists $i \in \mathbb{Z}_{\geq 0}$ so that two consecutive iterations in (5) result in equal sets, i.e., $R_{i+1} = R_i$, then $R_i = R(X)$. In this case, we say that $R(X)$ is finitely determined [10, Lem. 2.1]. Depending on the dynamics $(A, B)$ and the shape of $X$, $U$ and $W$ there exist conditions which ensure that $R(X)$ is finitely determined, see [7]. A large class of cases is covered by the following conditions. Suppose that $(A, B)$ is controllable, i.e., the controllability matrix $[B, AB, \ldots, A^{n-1}B]$ has full rank [11], then without loss of generality, we may assume that the system is in Brunovsky normal form, also known as Controller Form, see [11, Sec. 6.4.1]. In this representation, if $W = \{0\}$ and the sets $X$ and $U$ are given by a finite union of hyper-rectangles, then the maximal control invariant set is finitely determined, see [7, 12, 13].

Unfortunately, for one of the most popular settings, where $(A, B)$ is assumed to be controllable, $W = \{0\}$ and the sets $X$ and $U$ are assumed to be polytopes with the origin in the interior, $R(X)$ is not finitely determined. Nevertheless, in this case, one can modify the iteration [5] and set $R_0 = \{0\}$ (instead of $R_0 = X$). As a result, each set $R_i$ is controlled invariant and in fact $R_i$ is the $i$-step null-controllable set [14, 15] and the union of the sets $R_i$ converges to the largest null-controllable set $N(X)$, i.e., the set of all initial states from which the system can be forced to the origin in finite time without violating the constraints. As $R_i$ converges to the maximal null controllable set $N(X)$ and the closure of $N(X)$ equals $R(X)$, see [15, Prop. 1], the iteration [5] with $R_0 = \{0\}$ provides an algorithm for the arbitrarily precise (inner) approximation of $R(X)$, with the considerable advantage that the approximation is robust controlled invariant. Moreover, this approach provides a so-called anytime algorithm, i.e., for each iteration $i \in \mathbb{Z}_{\geq 0}$ the set $R_i$ is controlled invariant and a feedback can be derived, which enforces the trajectories of (1) with initial state in $R_i$ to evolve inside the constraint set.
X. Additionally, due to the convergence of $R_i$, the mismatch between $R_i$ and $R(X)$ decreases as the computation continues.

An alternative modification of the iteration [5], which also provides an invariant approximation of $R(X)$ and is not restricted to $W = \{0\}$, is presented in [9] [16] and [4] Sec. 5.2. The set iteration, with initial set $X$, is given by

$$R_0 = X, \quad R_{i+1} = \text{pre}(\lambda R_i) \cap X$$

for some contraction factor $\lambda \in [0, 1]$, where $\lambda P$ for $\lambda \in \mathbb{R}_{\geq 0}$ and $P \subseteq \mathbb{R}^n$ is defined by $\lambda P = \{x \in \mathbb{R}^n \mid \exists \rho \in \rho \ x = \lambda p\}$. The computation of $(R_i)_{i \in \mathbb{Z}_{\geq 0}}$ terminates, once the inclusion $R_i \subseteq \lambda / \lambda R_{i+1}$ holds for $\lambda \in ]\lambda, 1[$. Given that $X$ contains a $\lambda$-contractive convex set (see [4] Def. 4.18) with the origin in its interior, it is shown in [4] Prop. 5.9 that there exists $i \in \mathbb{Z}_{\geq 0}$ so that the termination condition is satisfied $R_i \subseteq \lambda / \lambda R_{i+1}$ and $R_i$ is robust controlled invariant, in fact $R_i$ is $\lambda$-contractive, see also [9] Thm. 3.2.

In this note, we assume that the dynamics $(A, B)$ are controllable and the constraint sets $X$ and $U$ are compact. Under these assumptions, we provide two novel results for the outer as well as inner approximation of $R(X)$. For the outer invariant approximation of $R(X)$, we use the set iteration [5] and modify the stopping criterion in [4] Eq. (5.17) to

$$R_i \subseteq R_{i+n} + \varepsilon \mathbb{B},$$

where $\mathbb{B}$ denotes the closed unit ball in $\mathbb{R}^n$ w.r.t. to the infinity norm $\cdot \cdot$. We show that for every $\varepsilon \in \mathbb{R}_{>0}$ there exists an $i \in \mathbb{Z}_{\geq 0}$ so that (9) holds. Based on the set $R_{i+n}$, we derive a $\delta$-relaxed robust control invariant set $R$, i.e., $R(X) \subseteq R \subseteq X + \delta \mathbb{B}$ and $R$ is robust controlled invariant w.r.t. (7) and $U + \delta \mathbb{B}$. Here $\delta = \varepsilon c$, where $c \in \mathbb{R}_{>0}$ is a constant that is known a-priori and the relaxation of the constraints can be made arbitrarily small by choosing an appropriate $\varepsilon \in \mathbb{R}_{>0}$. Moreover, we show that the set $R$ converges to $R(X)$ as $\varepsilon$ decreases to zero. Note that this approach can also be used in an anytime scheme. In that situation, at each iteration $i \geq n$, we determine $\varepsilon \in \mathbb{R}_{>0}$ so that (9) holds. If the constraint relaxation $\delta$ is tolerable, we stop the computation, otherwise, we continue with $R_{i+1}$.

For the inner invariant approximation of $R(X)$, we modify the iteration (7) to

$$R_0 = X, \quad R_{i+1} = \text{pre}_\rho(R_i) \cap X$$

where the map $\text{pre}_\rho$ is defined for $\rho \in \mathbb{R}_{\geq 0}$ by

$$\text{pre}_\rho(R) = \{x \in \mathbb{R}^n \mid \exists u \in U \ Ax + Bu + W + \rho \mathbb{B} \subseteq R\}.$$  

Given $\rho \in \mathbb{R}_{>0}$, we show that there exists $i \in \mathbb{Z}_{\geq 0}$ so that $R_i \subseteq R_{i+1} + \rho \mathbb{B}$ holds and that $R_{i+1}$ is robust controlled invariant. Moreover, we provide conditions which ensure that $R_{i+1}$ is nonempty. Although, the modification from $\text{pre}(P)$ to $\text{pre}_\rho(P)$ is rather straightforward, it has substantial effects. Not only allows this modification to extend the idea of $\lambda$-contractive sets [9] from convex sets to non-convex sets, but it also removes the requirement that $X$ contains a convex $\lambda$-contractive set that contains the origin in its interior.

In summary, compared to existing approaches, we do not assume that the state constraint set contains a $\lambda$-contractive convex set with the origin in its interior [4] [9] [16], nor do we impose any restrictions on the shape of the constraint sets [7] [12] [13], neither do we assume $W = \{0\}$ [14] [15], but simply consider compact constraint sets and general disturbance sets. Specifically, we allow sets given by finite unions of polytopes, i.e., the sets $X_i \subseteq \mathbb{R}^n$, $U_j \subseteq \mathbb{R}^m$, $W_k \subseteq \mathbb{R}^n$ with $i \in [1; J]$, $j \in [1; J]$, $k \in [1; K]$ and $I, J, K \in \mathbb{N}$ are polytopes and

$$X = \bigcup_{i \in [1; I]} X_i, \quad U = \bigcup_{j \in [1; J]} U_j, \quad W = \bigcup_{k \in [1; K]} W_k.$$ 

In this case, the sets $(R_i)_{i \in \mathbb{Z}_{\geq 0}}$ are computable [17] Sec. III.B and the proposed scheme for the outer invariant approximation is $\delta$-complete [18]: Let $\delta \in \mathbb{R}_{>0}$, $(A, B)$ be controllable and $X, U, W \neq \emptyset$ be defined in [11], then the proposed algorithm either returns an empty set $R_{i+n}$, in which case the set $R(X)$ is empty, or we obtain a $\delta$-relaxed robust controlled invariant set $R$. 

\[ \text{pre}_\rho(R) = \{x \in \mathbb{R}^n \mid \exists u \in U \ Ax + Bu + W + \rho \mathbb{B} \subseteq R\}. \]
We would like to point out that constrains sets in the form of \( (11) \) arise in a variety of different situations, see e.g. [19], and are particularly important in the synthesis problems with respect to safe linear temporal logic specifications [15].

2. Outer Invariant Approximation

We begin with a lemma which shows that the stopping criterion \( (8) \) is valid.

**Lemma 1.** Consider the system \( (1) \) and the compact constraint sets in \( (2) \). Let \( (R_i)_{i \in \mathbb{Z}_{>0}} \) be defined according to \( (5) \). Then for any \( \varepsilon \in \mathbb{R}_{>0} \) there exists \( i \in \mathbb{Z}_{>0} \) so that \( (8) \) holds.

**Proof.** Since \( R_i = \emptyset \) implies \( R_j = \emptyset \) for all \( j \geq i \), the assertion trivially holds since \( (8) \) results in \( \emptyset \subseteq \emptyset \) for \( i \in \mathbb{Z}_{>0} \) with \( R_i = \emptyset \) and subsequently we assume \( R_i \neq \emptyset \) for all \( i \in \mathbb{Z}_{>0} \). From \( (6) \) follows that there exists \( i' \in \mathbb{Z}_{>0} \) so that for all \( i \geq i' \) we have \( R_i \subseteq R(X) + \varepsilon \mathbb{B} \) and we obtain \( R_i \subseteq R(J) + \varepsilon \mathbb{B} \) for any \( j \in \mathbb{Z}_{>0} \) which shows \( (8) \). \( \square \)

In the following, we make use of \( \delta \)-constraint \( i \)-step null-controllable sets \( N^\delta_i \subseteq \mathbb{R}^n \), i.e., the set of initial states from which the unperturbed system \( \xi(t + 1) = A\xi(t) + B\nu(t) \) can be forced to the origin while satisfying the input and state constraints \( U = \delta \mathbb{B} \) and \( X = \delta \mathbb{B} \). Let \( \delta \in \mathbb{R}_{>0} \), then we define the sequence of sets \( (N^\delta_i)_{i \in \mathbb{Z}_{>0}} \) recursively by

\[
N^\delta_0 = \{0\},
N^\delta_{i+1} = \{ x \in \mathbb{R}^n \mid \exists u \in \delta \mathbb{B} \ Ax + Bu \in N^\delta_i \} \cap \delta \mathbb{B}.
\]  

(12)

Note that for a fixed \( \delta \in \mathbb{R}_{>0} \) it is straightforward to compute the sets \( (N^\delta_i) \) by polyhedral projection and intersection \( (4) \). We use the following technical lemma about \( \delta \)-constraint \( i \)-step null-controllable sets.

**Lemma 2.** Consider the system \( (1) \) with \( W = \{0\} \). Let \( N^\delta_n \) be defined according to \( (12) \). Suppose that \( (A, B) \) is controllable, then

\[
\exists \varepsilon \in \mathbb{R}_{>0} \forall \varepsilon \in \mathbb{R}_{>0} : \varepsilon \mathbb{B} \subseteq N^\delta_n \text{ with } \delta = c \varepsilon.
\]  

(13)

**Proof.** We show that there exists \( c \in \mathbb{R}_{>0} \) such that for every \( x \in \mathbb{R}^n \) there exists \( \nu : [0; n] \rightarrow \mathbb{R}^m \) so that the trajectory of \( \xi(t + 1) = A\xi(t) + B\nu(t) \) with \( \xi(0) = x \) satisfies \( \xi(n) = 0 \), and for all \( t \in [0; n] \) we have \( |\xi(t)| \leq c|x| \) and \( |\nu(t)| \leq c|x| \). This implies the assertion of the lemma, since it is easy to see that \( \xi(t) \in N^\delta_{n-t} \) with \( \delta \geq c|x| \) holds for all \( t \in [0; n] \). The trajectory at time \( n \) is given by \( \xi(n) = A^n x + C V \), where \( C \) is the controllability matrix \( [B, AB \ldots A^{n-1}B] \) and \( V \) is a vector in \( \mathbb{R}^{mn} \) with \( V = [v(n - 1)^T, \ldots, v(0)^T]^T \). Let \( C' \in \mathbb{R}^{nxn} \) denote a matrix containing \( n \) linearly independent columns of \( C \). Such a matrix always exists, since \( (A, B) \) is controllable and hence \( C \) has full rank.

Given \( x \in \mathbb{R}^n \), we determine the input sequence \( V \) by setting the entries \( V' \) of \( \mathbb{V} \) associated with \( C' \) to \( V' = - (C')^{-1} A^n x \) and the remaining entries of \( V \) to zero. It follows that \( \xi(n) = A^n x + CV = 0 \). Moreover, \( |V'| \leq c'|x| \) with \( c' = |(C')^{-1} A^n| \) holds and \( |\nu(t)| \leq c|x| \) for all \( t \in [0; n] \) follows. From \( \xi(t) = A^t + \sum_{s=0}^{t-1} A^{t-s-1} B \nu(s) \) follows that \( |\xi(t)| \leq (|A^t| + \sum_{s=0}^{t-1} |A^{t-s-1} B| c')|x| \) holds and the assertion follows. \( \square \)

**Corollary 1.** Let \( z_j \in \mathbb{R}^n, j \in [1; 2^n] \) denote the vertices of the unit cube \( \mathbb{B} \). A constant \( c \in \mathbb{R}_{>0} \) that satisfies \( (13) \) is given by \( c = \max_{j \in [1; 2^n]} c_j \) where \( c_j \) is obtained by solving the linear program

\[
\min_{c_j, u_0, \ldots, u_{n-1}} c_j
\text{ s.t. } \begin{align*}
A^n z_j + \sum_{k=0}^{n-1} A^{n-k-1} B u_k &= 0 \\
\forall i \in [0; n-1] &|u_i| \leq c_j \\
\forall i \in [1; n-1] &|A^i z_j + \sum_{k=0}^{i-1} A^{i-k-1} B u_k| \leq c_j \end{align*} \]  

(14)
Note that $|x|$ denotes the infinite norm of $x \in \mathbb{R}^n$ and the corollary follows by the linearity of the trajectories of (1).

We proceed with the main result related to the outer invariant approximation.

**Theorem 1.** Consider the system (1) and compact constraint sets (2). Let $(A, B)$ be controllable and consider the sequences of sets $(R_i)_{i \in \mathbb{Z}_{\geq 0}}$ and $(N^\delta_i)_{i \in \mathbb{Z}_{\geq 0}}$ given according to (5), respectively (12), with $\varepsilon \in \mathbb{R}_{>0}$, $\delta = c\varepsilon$ and $c$ satisfying (13). Let $i^* \in \mathbb{Z}_{\geq 0}$ be the smallest index, so that (8) holds. The set
\begin{equation}
R := \bigcup_{j \in [1:n]} R_{i^* + j} + N^\delta_j
\end{equation}
is a subset of $X + \delta \mathbb{B}$ and is robust controlled invariant w.r.t. (1) and $U + \delta \mathbb{B}$.

**Proof.** Consider the set $R$ defined in (15). If $R = \emptyset$ the assertion trivially holds (since the empty set is robust controlled invariant) and subsequently we consider $R \neq \emptyset$. Due to the choice of $\delta = c\varepsilon$ with $c$ satisfying (13) we have $\varepsilon \mathbb{B} \subseteq \bigcup_{j \in [1:n]} N^\delta_j \subseteq \delta \mathbb{B}$, which together with $R_i \subseteq X$ implies that $R \subseteq X + \delta \mathbb{B}$.

Moreover, (8) and (13) imply $R_{i^*} \subseteq R$. We show that for every $x \in R$ there exists $u \in U + \delta \mathbb{B}$ so that $Ax + Bu + W \subseteq R$ which shows that $R$ is robust controlled invariant [17 Prop. 1, ii]). Let $x \in R$, then there exists $j \in [1:n]$ so that $x \in R_{i^* + j} + N^\delta_j$. Let $x = x_r + x_n$ so that $x_r \in R_{i^* + j}$ and $x_n \in N^\delta_j$. Then there exists $u_r \in U$ and $u_n \in \delta \mathbb{B}$ so that $Ax_r + Bu_r + W \subseteq R_{i^* + j}$ and $Ax_n + Bu_n \in N^\delta_j$ and it follows that $Ax + Bu + W \subseteq R_{i^* + j - 1} + N^\delta_{j - 1}$ where $u = u_r + u_n \in U + \delta \mathbb{B}$.

If $j \geq 2$, it follows from the definition of $R$ that $Ax + Bu + W \subseteq R$. If $j = 1$, we use (8) and (13) to get $Ax + Bu + W \subseteq R_{i^*} \subseteq R_{i^* + n} + \varepsilon \mathbb{B} \subseteq R$. \hfill $\square$

By decreasing the stopping parameter $\varepsilon \in \mathbb{R}_{>0}$ the set $R$ defined in (15) converges to $R(X)$ w.r.t. the Hausdorff distance $d_H(P, Q) := \inf \{ \eta \in \mathbb{R}_{>0} \mid Q \subseteq P + \eta \mathbb{B} \land P \subseteq Q + \eta \mathbb{B} \}$.

**Corollary 2.** Consider the hypothesis of Theorem 1. Let $R_{\varepsilon}$ denote the set $R$ defined in (15) for parameter $\varepsilon \in \mathbb{R}_{>0}$ and let $R(X)$ be the maximal robust controlled invariant set of (1) and $U$. For any sequence $(\varepsilon_j)_{j \geq 0}$ in $\mathbb{R}_{>0}$ with limit 0 we either have $R_{\varepsilon_j} = \emptyset$ for some $j$ so that $R(X) = \emptyset$ follows, or we have $\lim_{j \to \infty} d_H(R(X), R_{\varepsilon_j}) = 0$.

**Proof.** Consider the sequence $(R_i)_{i \in \mathbb{Z}_{\geq 0}}$ according to (5). Let $i^*(\varepsilon)$ denote the smallest $i^* \in \mathbb{Z}_{\geq 0}$ such that (8) holds for a fixed $\varepsilon \in \mathbb{R}_{>0}$. Consider a sequence $(\varepsilon_j)_{j \geq 0}$ in $\mathbb{R}_{>0}$ that converges to zero. If $R_{\varepsilon_j} = \emptyset$ for some $j \geq 0$, it follows from (15) that $R_{i^*(\varepsilon_j)+j'} = \emptyset$ for all $j' \in [1:n]$, and $R(X) = \emptyset$ follows. Subsequently we consider $R_{\varepsilon_j} \neq \emptyset$ for all $j \in \mathbb{Z}_{\geq 0}$. From $\delta_j = c\varepsilon_j$ and $\varepsilon_j \mathbb{B} \subseteq \bigcup_{j \in [1:n]} N^\delta_j \subseteq \delta \mathbb{B}$ follows $R_{i^*(\varepsilon_j)} \subseteq R_{\varepsilon_j} \subseteq R_{i^*(\varepsilon_j)} + c\varepsilon_j \mathbb{B}$ and it is sufficient to show that $R_{i^*(\varepsilon_j)}$ converges to $R(X)$. We use the fact that $\varepsilon_j < \varepsilon_j$ implies $i^*(\varepsilon_j') \geq i^*(\varepsilon_j)$ and distinguish two cases: 1) if $i^*(\varepsilon_j) \to \infty$ as $j \to \infty$ we use (6) to conclude $\lim_{j \to \infty} d_H(R_{i^*(\varepsilon_j)}, R(X)) = 0$; 2) otherwise we can assume that $i^*(\varepsilon_j) \to i'$ for some $i' \in \mathbb{Z}_{\geq 0}$. Hence, there exists $j' \in \mathbb{Z}_{\geq 0}$ such that $i^*(\varepsilon_j) = j$ for all $j \geq j'$ and by (8) we have $R_{i} \subseteq R_{i + n} + \varepsilon_j \mathbb{B}$ for all $j \geq j'$, which implies $R_i = R_{i + n}$ and we get $R_i = R(X)$. \hfill $\square$

**Remark 1.** Consider the system (1) and the compact constraint sets (2). Let $(A, B)$ be controllable and fix $\varepsilon \in \mathbb{R}_{>0}$. Suppose that we have an algorithm to iteratively compute $R_i$ and check the inclusion (8), as it is the case e.g. for sets given by (11) see [17 20]. Then it follows from Lemma 1 that there exists $i \in \mathbb{Z}_{\geq 0}$ so that (8) holds. If $R_{i+n} = \emptyset$, then there does not exist a feedback to enforce the constraints $X$ and $U$, in particular $R(X) = \emptyset$. If $R_{i+n} \neq \emptyset$, due to the controllability of $(A, B)$ we can solve the linear program (14) and compute the sets $(N^\delta_j)_{j \in [1:n]}$ with which we construct the set $R$ according to (15). Then it follows from Theorem 1 that $R$ is robust controlled invariant and a feedback to enforce the constraints $X + \delta \mathbb{B}$ and $U + \delta \mathbb{B}$ is derived from the map

$$K(x) = \{ u \in U + \delta \mathbb{B} \mid Ax + Bu + W \subseteq R \}.$$ 

Since $R(X) \subseteq R$ it is straightforward to see that the map defined in (4) satisfies $C(x) \subseteq K(x)$ for all $x \in R$. 


For polyhedral disturbances and constraints sets \( \{1\} \), the set iterates \( R_i \) can be effectively computed and the inclusion can be effectively tested, see \[\{17\} \text{ Sec. III.B} \] and \[\{20\} \]. In the worst case, the computational complexity of these operations grows exponentially with \( i \), see \[\{20\} \]. Nevertheless, we present in Section V a nontrivial example where the proposed algorithm can be executed until termination.

3. Inner Invariant Approximation

For the inner approximation of \( R(X) \) we fix \( \rho \in \mathbb{R}_{>0} \) and analyze the sequence

\[
R_i^\rho = X, \quad R_i^{\rho+1} = \text{pre}_\rho(R_i^\rho) \cap X \tag{16}
\]

where \( \text{pre}_\rho \) is defined in \[\{10\} \]. The stopping criterion, as proposed in (5.17) in \[\{4\} \], is given by

\[
R_i^\rho \subseteq R_i^{\rho+1} + \rho \mathbb{B}. \tag{17}
\]

**Theorem 2.** Consider the system \( \{1\} \) and compact constraint sets \( \{2\} \). Let \( (R_i^\rho)_{i \in \mathbb{Z}_{\geq 0}} \) be defined in \[\{16\} \]. For every \( \rho \in \mathbb{R}_{>0} \) there exists an index \( i \in \mathbb{Z}_{\geq 0} \) such that \( \tag{17} \) holds and \( R_i^{\rho+1} \) is robust controlled invariant w.r.t. \( \{1\} \) and \( U \).

**Proof.** The proof of the existence of \( i \in \mathbb{Z}_{\geq 0} \) so that \( \tag{17} \) holds, follows by the same arguments as the proof of Lemma \[\{1\} \] and is omitted here.

If \( R_i^{\rho+1} = \emptyset \) the assertion trivially holds and subsequently we consider \( R_i^{\rho+1} \neq \emptyset \). Let \( x \in R_i^{\rho+1} = \text{pre}_\rho(R_i^\rho) \cap X \). There exists \( u \in U \) such that \( Ax + Bu + W + \rho \mathbb{B} \subseteq R_i^\rho \subseteq R_i^{\rho+1} + \rho \mathbb{B} \) which implies that \( Ax + Bu + W \subseteq R_i^{\rho+1} \) and it follows that \( R_i^{\rho+1} \) is robust controlled invariant \( \tag{17} \) [\[17\] Prop. 1, ii)].

Let \( \varepsilon \in \mathbb{R}_{>0} \), in the following theorem we consider the strengthened constraint sets

\[
\tilde{X}_\varepsilon = \{ x \in \mathbb{R}^n \mid x + \varepsilon \mathbb{B} \subseteq X \}
\]

\[
\tilde{U}_\varepsilon = \{ u \in \mathbb{R}^m \mid u + \varepsilon \mathbb{B} \subseteq U \}
\]

and show that there exists a parameter \( \rho \in \mathbb{R}_{>0} \) so that any robust controlled invariant set \( \tilde{R}_\varepsilon \subseteq \tilde{X}_\varepsilon \) w.r.t. \( \{1\} \) and \( \tilde{U}_\varepsilon \) is a subset of \( R_i^{\rho+1} \).

**Theorem 3.** Consider the system \( \{1\} \), \( (A,B) \) being controllable and compact constraint sets \( \{2\} \). Let \( (R_i^\rho)_{i \in \mathbb{Z}_{\geq 0}} \) be defined in \[\{16\} \]. Let \( \varepsilon \in \mathbb{R}_{>0} \), and consider the sets \( \tilde{X}_\varepsilon \) and \( \tilde{U}_\varepsilon \) in \[\{18\} \]. There exists \( \rho \in \mathbb{R}_{>0} \) so that for any set \( \tilde{R}_\varepsilon \subseteq \tilde{X}_\varepsilon \) that satisfies

\[
x \in \tilde{R}_\varepsilon \Rightarrow \exists u \in \tilde{U}_\varepsilon : Ax + Bu + W \subseteq \tilde{X}_\varepsilon \tag{19}
\]

we have \( \tilde{R}_\varepsilon \subseteq R_i^{\rho+1} \), where \( i \in \mathbb{Z}_{\geq 0} \) satisfies \( \tag{17} \).

**Proof.** Let us consider the system

\[
\xi(t+1) = A\xi(t) + B\nu(t) + W + \rho \mathbb{B}. \tag{20}
\]

Let \( R^\rho(X) \) be the maximal robust controlled invariant set of \( \tag{20} \) and \( U \). From the definition of \( \text{pre}_\rho \) in \[\{16\} \] we see that \( R^\rho(X) \subseteq R_i^\rho \) for every \( i \in \mathbb{Z}_{\geq 0} \). Let \( \varepsilon \in \mathbb{R}_{>0} \). In the following, we consider \( \tilde{R}_\varepsilon \neq \emptyset \) (otherwise the assertion trivially holds) and show that there exists \( \rho \in \mathbb{R}_{>0} \) and a set \( K \) with \( \tilde{R}_\varepsilon \subseteq K \subseteq R^\rho(X) \), which proves the theorem.

Let \( \delta = \varepsilon/n \) and \( \rho \in \mathbb{R}_{>0} \) so that \( c\rho = \delta \), where the constant \( c \) is chosen according to Lemma \[\{2\} \] (which is applicable, since \( (A,B) \) is controllable). Consider \( N_i^\delta \), \( i \in [0;n] \) defined according to \[\{12\} \]. Note that \( \tag{13} \) implies that \( \rho \mathbb{B} \subseteq N_i^\delta \). We define the set \( K := \tilde{R}_\varepsilon + \sum_{i=1}^n N_i^\delta \). Note that \( N_i^\delta \subseteq \delta \mathbb{B} \) holds for every \( i \in [1;n] \), which together with \( \tilde{R}_\varepsilon + \varepsilon \mathbb{B} \subseteq X \) and \( \delta = \varepsilon/n \), implies \( K \subseteq X \). We show that \( K \) is robust controlled invariant w.r.t. \( \{20\} \) and \( U \). Let \( x \in K \), and pick \( x_r \in \tilde{R}_\varepsilon \) and \( x_i \in N_i^\delta \), \( i \in [1;n] \) so that \( x = x_r + \sum_{i=1}^n x_i \). Since \( \tilde{R}_\varepsilon \) satisfies \( \tag{19} \), we can pick \( u_r \in \tilde{U}_\varepsilon \) so that \( Ax_r + Bu_r + W \subseteq \tilde{X}_\varepsilon \), which implies that \( Ax_r + Bu_r + W + \rho \mathbb{B} \subseteq \tilde{R}_\varepsilon + N_i^\delta \). Moreover, for \( x_i \in N_i^\delta \), we pick \( u_i \in \delta \mathbb{B} \) so that \( Ax_i + Bu_i \in N_i^{\delta+1} \). Let \( u = u_r + \sum_{i=1}^n u_i \). As \( u_r \in \tilde{U}_\varepsilon \) and \( \delta \leq \varepsilon/n \) we have \( u \in U \). We see that \( Ax + Bu + W + \rho \mathbb{B} \subseteq K \) holds, which shows \( K \subseteq R^\rho(X) \).
4. An illustrative example

We proceed with a simple example taken from [21] to illustrate our results. We consider the system (1) with parameters

\[
A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad W = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mid \alpha \in \mathbb{R}^2 \right\} \cap \mathbb{R}^2, \quad \alpha \in [-1, 1].
\]

The constraint sets are given by \( U = [-100, 100] \) and \( X = \{ x \in \mathbb{R}^2 \mid Hx \leq h_0 \} \) with

\[
H = \begin{bmatrix} 1 & 1 \\ -3 & 1 \\ 0 & -1 \end{bmatrix}, \quad h_0 = \begin{bmatrix} 100 \\ -50 \\ -26 \end{bmatrix}.
\]

For this particular example we are able to analytically compute the set iterations \((R_i)_{i \in \mathbb{Z}_{>0}}\) defined in (3). Specifically, the sets \((R_i)_{i \in \mathbb{Z}_{>0}}\) and \(W\) are polytopes and we follow the approach in [2] to compute \(\text{pre}(R_i)\) in terms of the Pontryagin set difference \(R_i \sim W = \{ x \in R_i \mid x + W \subseteq R_i \}\), i.e.,

\[
\text{pre}(R_i) = \{ x \in \mathbb{R}^2 \mid \exists u \in U Ax + Bu \in (R_i - W) \}.
\]

See also [22, Sec. 3.3]. For \(R_0 = X\), we apply [23, Thm. 2.4], and obtain the difference \(R_0 \sim W = \{ x \in \mathbb{R}^2 \mid Hx \leq h_0' \}\) with \(h_0' = [98, -52, -27]^T\) and \(\text{pre}(R_0)\) follows simply by projecting the polytope

\[
\left\{ (x, u) \in \mathbb{R}^3 \mid \begin{bmatrix} HA & HB \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} h_0' \\ 100 \\ 100 \end{bmatrix} \right\}
\]

onto its first two coordinates. After the intersection of \(\text{pre}(R_0)\) with \(R_0\) we obtain \(R_1 = \{ x \in \mathbb{R}^2 \mid Hx \leq h_1 \}\) with

\[
h_1 = [100, -50, -26 - \frac{1}{3}]^T.
\]

We repeat this computation and obtain the sequence of sets by \(R_i = \{ x \in \mathbb{R}^2 \mid Hx \leq h_i \}\) with

\[
h_i = [100, -50, -25 - \sum_{j=0}^{i} \frac{1}{3^j}]^T
\]

whose limit is given by \(R(X) = \{ x \in \mathbb{R}^2 \mid Hx \leq h \}\) with

\[
h = [100, -50, -26.5]^T.
\]

The boundary of the maximal robust controlled invariant set \(R(X)\) is illustrated in Figure 2 and 3 by the dotted line.

Note that \(R(X)\) is not finitely determined, \(X\) does contain the origin in its interior, nor is \(W = \{ 0 \}\). Hence, it is not possible to apply any of the methods in [9, 12, 14, 21], to invariantly approximate the maximal robust controlled invariant set. In the following we apply the results from Sections 2 and 3 to compute outer and inner invariant approximations of \(R(X)\).

**Outer approximation.** We start by solving the linear program [14] to determine the constant \(c = 2\) which satisfies (13). The \(\delta\)-constraint \(i\)-step null controllable sets \(N_i^\delta\) for \(i \in [1; 2]\) are illustrated in Figure 1. From the previous consideration it is straightforward to see that \(R_i \subseteq R_{i+2} + \frac{1}{3^i} B\) holds.

**Figure 1.** The \(\delta\)-constraint 1-step (thick black bar) and 2-step (dark gray polytope) null controllable sets \(N_j^\delta\) containing the ball \(\frac{1}{2} B\) (light gray box).
for all $i \in \mathbb{Z}_{\geq 0}$. Hence, in each iteration the stopping parameter is given by $\varepsilon = 4/3^i + 2$. We illustrate the robust controlled invariant set defined in (15) for $i = 0$ and $i = 3$ relative to $R(X)$ in Figure 2. For $i = 3$, $\delta = 8/243$ and $R$ in Figure 2 is indistinguishable form $R(X)$.

**Figure 2.** Invariant outer approximations of $R(X)$ given according to (15) for $i = 0$ (left) and $i = 3$ (right). The dotted line indicates $R(X)$.

**Inner approximation.** In order to obtain an inner approximation of $R(X)$, we compute the sequence of sets $\{R^\rho_i\}_{i \in \mathbb{Z}_{\geq 0}}$ defined in (16). Similar as before, we compute $\text{pre}_\rho(R^\rho_i)$ by using the Pontryagin set difference, i.e.,

$$\text{pre}_\rho(R^\rho_i) = \{x \in \mathbb{R}^2 \mid \exists u \in U \ Ax + Bu \in (R_i \sim (W + \rho B))\}.$$  

We apply again [23, Thm. 2.4] to compute $R_i \sim (W + \rho B)$. Two invariant inner approximations of $R(X)$ with parameters $\rho = 1$ and $\rho = 1/10$ are illustrated in Figure 3.

**Figure 3.** Invariant inner approximations of $R(X)$ with parameters $\rho = 1$ (left) and $\rho = 1/10$ (right). The dotted line indicates $R(X)$.

5. **Numerical Experiments**

We continue with the approximation of the maximal robust controlled invariant set for a more complex system. To this end, we consider the linear dynamics used in [24] to model a rotor craft. The system consists of four states. The first two states represent the position and the last two states represent the velocity of the rotor craft. The acceleration is considered as the input of the system. The parameters of the differential inclusion are given by

$$A = \begin{bmatrix} I_2 & \tau I_2 \\ 0 & I_2 \end{bmatrix}, \quad B = \begin{bmatrix} \tau^2 I_2 \\ \tau I_2 \end{bmatrix},$$

$$W = ([-\tau/2, \tau/2] \times [-\tau, \tau]^2)w_{\max}$$

where $I_2$ denotes the 2-dimensional identity matrix and $\tau = 2.6$ sec. The state constraint set is given by

$$X = ([-35, 5] \times [-10, 10] \times [-v_{\max}, v_{\max}]^2) \setminus O$$
with \( O = \bigcup_{i=1}^p o_i + [-4, 4] \times [-1, 1] \times [-v_{\text{max}}, v_{\text{max}}]^2 \) representing some obstacles. The first two coordinates of the centers of the obstacles are randomly generated integer values
\[
\begin{bmatrix}
-8 & -5 & -13 & -22 & -11 & -10 & -2 & -15 & -17 \\
1 & 5 & -8 & -2 & 4 & 3 & -1 & 7 & -6 & 5
\end{bmatrix}
\]
while the last two coordinates of \( o_i \) are set to zero. The input is constrained to \( U = [-a_{\text{max}}, a_{\text{max}}] \). We follow [24] and set
\[
v_{\text{max}} = 0.5, \quad a_{\text{max}} = 0.17 \quad \text{and} \quad w_{\text{max}} \in \bar{w} \cdot a_{\text{max}}
\]
where the perturbation level ranges over \( \bar{w} \in \{0, 1, 2\} \). Using the control input \( u = -[1/\tau^2 I_2 3/2\tau^2 I_2]^T \), all states in the unit cube of the unperturbed system can be steered to the origin in two steps without leaving the unit cube. Hence, a constant which satisfies (13) is given by \( c = 1 \). Moreover, in the subsequent computation of the outer approximation of the maximal controlled invariant set we can use \( R_i \subseteq R_i + 2 \varepsilon \) as stopping criterion.

In the conducted experiments, in addition to the different perturbation levels \( \bar{w} \in \{0, 1, 2\} \), we vary the number of obstacles \( p \in \{0, 5, 10\} \). The approximation accuracy is set to \( \varepsilon = 0.01 \). For each computation, we display in Figure 4 the run-times of the computation to determine the set \( R_i \) as well as the numbers of halfspaces \#\( R_i \) used to represent the set \( R_i \). The number of iterations until termination can be deduced by the last shown data-point. For example, for \( \bar{w} = 0.1 \) and \( p = 5 \) ( ), we see only three data-points in the upper, middle subplot, which indicates that at time \( i = 3 \) the termination criterion \( R_3 \subseteq R_1 + 0.01 \varepsilon \) holds. Although, the worst case estimates predict that the number of halfspaces necessary to represent the sets \( R_i \) increases exponentially with the number of iterations, see e.g. [20], we do not observe such an increase in our experiments and are therefore able to successfully approximate \( R(X) \) for this example.

All the computations were conducted on a single core of an Intel i7 3.5GHz CPU with 32GB memory, using MATLAB and the freely available Multi-Parametric Toolbox [http://people.ee.ethz.ch/~mpt/2/] which provides all the polyhedral operations, necessary to compute the set iterates \( R_i \) and to check the set inclusion [8].
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