Full counting statistics for voltage and dephasing probes

S. Pilgram\textsuperscript{1}, P. Samuelsson\textsuperscript{2}, H. Förster\textsuperscript{3} and M. Büttiker\textsuperscript{3}

\textsuperscript{1}Theoretische Physik, ETH Zürich, CH-8093 Zürich, Switzerland
\textsuperscript{2}Division of Solid State Theory, Lund University, Sölvegatan 14 A, S-223 62 Lund, Sweden and
\textsuperscript{3}Département de Physique Théorique, Université de Genève, CH-1211 Genève 4, Switzerland

(Dated: March 23, 2022)

We present a stochastic path integral method to calculate the full counting statistics of conductors with energy conserving dephasing probes and dissipative voltage probes. The approach is explained for the experimentally important case of a Mach-Zehnder interferometer, but is easily generalized to more complicated setups. For all geometries where dephasing may be modeled by a single one-channel dephasing probe we prove that our method yields the same full counting statistics as phase averaging of the cumulant generating function.

PACS numbers: 73.23.-b, 03.65.Yz, 72.70.+m

Introduction – Voltage probes are essential elements of many small conductors. In mesoscopic structures such probes permit to obtain information deep within the phase-coherent volume. An ideal voltage probe is an infinite impedance terminal with zero net current flow, i.e. any electron leaving the conductor through the probe is thermalized by dissipation and later on fed back into the conductor. Early theory used dissipative voltage probes as simple and successful means to investigate the transition from quantum coherent conduction to the classical series addition of resistances. Later on it was realized that theoretically dissipation at a probe can be suppressed by demanding that each electron exiting into the probe is replaced by an electron incident from the probe at the same energy. Such a dephasing probe can serve as a simple model to describe dephasing in mesoscopic conduction processes. The power of this phenomenological approach resides in the fact that the theoretical modelling may be split into two tasks, first the coherent part of the problem is treated by the well-developed scattering theory, second dephasing is taken into account by evaluating the response of the dephasing probe.

Voltage and dephasing probes also play an important role in the investigation of noise and current correlations in mesoscopic conductors. As a prominent example it has been predicted and recently measured that inelastic scattering generated by a voltage probe can render the sign of cross-correlations positive, while to the contrary anti-bunching of quantum coherent electrons causes always negative cross-correlations of currents. Nevertheless, despite the wide use of voltage and dephasing probes, the theory remains in many respects not well developed.

The central aim of this work is to provide a unified description of electrical transport in the presence of voltage or dephasing probes using the framework of full counting statistics (FCS). We employ the stochastic path integral approach and calculate explicitly the generating function of the FCS. Focusing on the dephasing probe, we apply the theory to a Mach-Zehnder interferometer (MZI). We compare our results to averaging over a phase distribution and to classical exclusion models. For the case of the MZI we find exact agreement among the models, however this coincidence does not hold for more complicated scatterers.

Dephasing Probe – The electronic MZI is shown in Fig. 1. It consists of two arms connected to four electronic reservoirs 1 to 4 via reflectionless beamsplitters $A$ and $B$ and to one dephasing probe $\phi$. Transport in the single mode arms is unidirectional (indicated by arrows), corresponding to transport along edge-states in the quantum Hall regime (such a setup was recently realized experimentally). A bias $eV$ is applied to reservoir 1, the other reservoirs are kept at ground. An Aharonov-Bohm flux $\Phi$ threads the interferometer. The transmission (reflection) probability of the beamsplitters is $T_A$ and $T_B$ ($R_A$ and $R_B$) respectively. The coupling to the probe is $\varepsilon$, ranging from 0 for an uncoupled to 1 for a fully coupled probe. The scattering matrix $S$ for the total system, including the probe, is found along the lines of e.g. Ref. 20. We consider for simplicity zero temperature, the electron occupation function is thus unity in reservoir 1 while zero in reservoirs 2 to 4 in the energy interval $0 \leq E \leq eV$ of interest. The occupation function

![FIG. 1: Mach-Zehnder interferometer coupled to a dephasing probe $\phi$, $\varepsilon$, $\lambda_\phi$, and $n_\phi$ are the coupling strength, counting field, and the electron distribution of the dephasing probe.](image)
$n_φ(E)$ describes the state of the dephasing probe.

The FCS is the probability distribution $P_c(Q_3, Q_4) = (1/4π^2) \int dλ_3 dλ_4 \exp[-i(λ_3 Q_3 + λ_4 Q_4) + S_0(λ_3, λ_4)]$ that charges $Q_3$ and $Q_4$ are transmitted to reservoirs 3 and 4 during the measurement time $τ$. The FCS is expressed in terms of a cumulant generating function $S_0(λ_3, λ_4)$ yielding all irreducible cumulants by consecutive derivatives $(⟨Q_3⟩^m ⟨Q_4⟩^n) = \partial^{m+n} S_0/(\partial λ_3^m \partial λ_4^n)$. To derive $S_0$ we first obtain the generating function $S_0$ for a fixed value of the occupation function $n_φ$ in the dephasing probe. This is achieved by using the scattering matrix expression introduced by Levitov and Lesovik

$S_0 = \frac{1}{\hbar} \int dE \int_0^τ dt \ H_0(λ_φ, n_φ)$

with

$H_0 = \ln \det [1 + n(Λ^1 S^1 Λ S − 1)]$  \hspace{1cm} (1)

and $n = \text{diag}(1, 0, 0, 0, n_φ)$, $Λ = \text{diag}(1, 1, e^{iλ_3}, e^{iλ_4}, e^{iλ_5})$.

It is clear that this generating function $S_0$ is not the solution of the full problem, since current and fluctuations at the dephasing probe obtained from derivatives with respect to $λ_φ$ are non-zero. This violates the defining property of the dephasing probe, that for each energy not only average current but also low-frequency current fluctuations are zero. Therefore we introduce fluctuations of the distribution function $n_φ$ on the slow time scale of $τ_d$ which is set by the delay time of particles in the probe. These fluctuations respond in such a way that the incoming particle current in each energy interval is compensated exactly by an induced outgoing current. As a consequence, there is no charge pile-up in the probe and all carriers end finally up in terminal 3 or 4. This idea can be formalized under the assumption that $τ_d$ is much longer than the time spread $τ_s = \hbar/eV$ of individual electron wave packets. This high-voltage assumption allows us to derive the generating function $S_c$ of the current fluctuations from $S_0$ employing the stochastic path integral approach.

The condition of charge conservation in the probe formally enters the total generating function as a factor $iQ_φe^{iφ} = iτ_d/\hbar \int dE dφ n_φ$ where $Q_φ$ can be seen as a charge stored in the dephasing probe. Moreover, being interested in the charge distribution independent on the particular realization of $n_φ$ we integrate over the occupation function. As a consequence the total generating function $S_c$ is expressed as a path integral

$e^{S_c} = \int Dn_φ Dλ_φ \exp \left\{ \int_0^τ dE dt \left[ -iτ_d λ_φ n_φ + H_0 \right] \right\}$.  \hspace{1cm} (2)

We evaluate this path integral in the saddle point approximation (Gaussian corrections are discussed below). The saddle point equations are

$i\dot{λ}_φ = \frac{1}{τ_d} ∂H_0/∂λ_φ, \quad i\dot{n}_φ = −\frac{1}{τ_d} ∂H_0/∂n_φ$.  \hspace{1cm} (3)

In what follows we will only consider the stationary limit, $τ ≫ τ_d$, and can therefore omit the time derivatives in Eq. (3). The solutions $λ_φ$ and $n_φ$ are readily obtained and substituted back into $S_c$. This then gives the stationary generating function $(ψ = 2πΦ/(h/e) and N = eVτ/h)$,

$S_c = N \ln \left[ b \left( 1 - \frac{ε}{2} \right) + c\sqrt{1 - ε} \cos ψ + \frac{ε}{2} \sqrt{b^2 - ε^2} \right]$  \hspace{1cm} (4)

The main result of this work. We note that the first and second derivatives of $S_c$ reproduce known results for current and noise calculated with the dephasing probe and that the generating function becomes flux independent in the limit of strong dephasing, $ε → 1$. The first term in $S_c$ is proportional to $b = (T_A T_B + R_A R_B)e^{iλ_3} + (T_A R_B + R_A T_B)e^{iλ_4}$ representing the classical contribution due to particles which go either along the upper or the lower arm, the second term is proportional to $c = 2\sqrt{R_A T_A R_B T_B} (e^{iλ_4} - e^{iλ_3})$ giving the coherent quantum interference contribution. The last term proportional to $b^2 - ε^2$ has an exchange, or two-particle interference origin: the dephasing probe gives rise to two-particle processes such as $1 → 3, φ → φ$ and $1 → φ, φ → 3$ which are indistinguishable.

To determine the range of validity of the saddle point approximation, we introduce fluctuations of the occupation number $n_φ = ˜n_φ + δn_φ(t)$ and of the Lagrange multiplier $λ_φ = ˜λ_φ + δλ_φ(t)$ and expand the exponent of Eq. (4) up to second order in the fluctuations. Carrying out the gaussian integrals we obtain the contribution

$δ^2 S_c = -\frac{τ_s N}{τ_d} \frac{ε}{4R_A T_A \exp(S_c/N)} \left[ 2R_A (R_B e^{iλ_3} + T_B e^{iλ_4}) - b + (T_A - R_A) \sqrt{b^2 - ε^2} \right]$.  \hspace{1cm} (5)

The saddle point approximation is thus valid if $τ_s/τ_d ≪ 1$. Note that the delay time can be expressed in terms of a density of states $τ_d = hN_F$ of the dephasing probe. One may thus equivalently state that our approximation is valid if the number of states in the probe participating in transport is large, $N_F eV ≫ 1$. It should also be emphasized that the path integral calculation is easily extended to more than one probe: it is sufficient to replace $n_φ$ and $λ_φ$ by corresponding vector quantities.

\textbf{Phase averaging} – It is interesting to compare the dephasing probe approach with the most elementary model for dephasing, a phase averaging. Consider the upper arm of the MZI coupled with strength $ε$ not to a dephasing probe, but to an elastic (coherent) scatterer reflecting with a phase factor $e^{iφ}$ (see Fig. 2(a)). Guided by random matrix theory, we assume that the phase $φ$ is uniformly distributed and therefore simply average the cumulant generating function over the scattering phase $2[2]$. An electron traversing the upper path can enter the scatterer and after multiple internal reflections continue on its path with an additional total phase factor $e^{iφ}$. The electron phase $φ$ is related to the scattering phase by $φ = φ + π + 2 \text{arctan} [√1 - ε \sin φ/(1 - √1 - ε \cos φ)]$. In
terms of the new variable $\tilde{\varphi}$ the phase average becomes
\[
\langle S_c \rangle_{\varphi} = N \int_0^{2\pi} d\tilde{\varphi} f(\tilde{\varphi}) H_0(\psi + \tilde{\varphi})|_{\tilde{\varphi} = 0},
\] (6)
where the $2\pi$ periodic distribution of $\tilde{\varphi}$ is given by
\[
f(\tilde{\varphi}) = \frac{1}{2\pi} \frac{d\varphi}{d\tilde{\varphi}} = \frac{1}{2\pi} \frac{\varepsilon}{2 - \varepsilon + 2\sqrt{1 - \varepsilon \cos(\tilde{\varphi})}}.
\] (7)
The outcome of the phase average agrees exactly with the result (4) obtained from the dephasing probe model. Therefore one can argue that the dephasing probe acts like a disordered mirror that reflects with the phase distribution $f(\tilde{\varphi})$. We remark that a microscopic model for the scatterer is a single mode chaotic cavity with a dwell time much longer than $\tau_0$ (recently, the influence of dephasing on noise was modelled using a many mode chaotic cavity). It is possible to show that the agreement between the dephasing probe and phase averaging holds for an arbitrary mesoscopic conductor connected to a single probe. For a conductor characterized by a scattering matrix $S$ of size $M + 1$ (last index denotes the probe terminal) the generating function is given by
\[
S_c = N \ln \left[ A + \sqrt{A^2 - 4B} \right]
\] (8)
with $A = F_{00} + F_{11} - F_{10}$ and $B = F_{10}F_{01} - F_{00}F_{11}$ where $F_{\alpha\beta} = \det[1 + \tilde{n}_\alpha (\Lambda[S\Lambda \beta S - 1])]$ with $\tilde{n}_\alpha = \text{diag}(n_1, n_2, ..., n_M, \alpha)$ and $\Lambda_\alpha = \text{diag}(e^{i\lambda_1}, e^{i\lambda_2}, ..., e^{i\lambda_M}, \alpha)$. Note that for any geometry containing two or more probes, the dephasing probes destroy interference between paths that pass a given set of probes in different chronological order. Such an interference is not removed by phase averaging. Therefore, it is generally not possible to describe the effect of many dephasing probes with phase averaging.

**Classical exclusion models** – The FCS in the limit of complete dephasing ($\varepsilon = 1$) may also be understood from a completely classical point of view. In exclusion (classical ball) models electrons enter the interferometer through terminal 1 as a regular noiseless stream of classical particles (see Fig. 2b). Time and space are discretized and in each time step the particles in the interferometer arms propagate one site forward. At the beamsplitters $A$ and $B$ the particles scatter with the same probabilities, $T_A, R_A, T_B$ and $R_B$, as in the quantum case. The Pauli principle is introduced by hand by excluding scattering events which would lead to two electrons occupying the same site.

To calculate the noise of the MZI in the classical limit ($\varepsilon = 1$), Marquardt and Bruder [21] consider an exclusion model with both arms of the interferometer having equal length (the same number of sites). Consequently, two electrons can not arrive simultaneously at beamsplitter $B$ and the Pauli principle never comes into play. The generating function for this model is given by
\[
S_c = N \ln b.
\] (9)
Surprisingly, there is a disagreement between the exclusion model result and the dephasing probe (and phase averaging) result in Eq. 4. The fact that the dephasing probe model does not reproduce a seemingly obvious classical result lead the authors of Refs. [11] to strongly question the reliability of the dephasing probe.

As it turns out, this contradiction can be resolved by considering a model with unequal arm lengths [21]. Such a model takes into account that the dephasing probe effectively delays the particle a time $\sim \tau_d$. Particles that enter the interferometer at different times might thus arrive at the second beamsplitter $B$ simultaneously and consequently, the Pauli principle comes into play. The generating function for an interferometer with finite arm length difference can be calculated with known methods, see e.g. [28] and is found to be
\[
S_c = N \ln [b/2 + 1/2\sqrt{b^2 - c^2}] (10)
\]
which completely agrees with Eq. 4 in the limit of $\varepsilon = 1$. The term $\sqrt{b^2 - c^2}$ identified above as an exchange term is in the classical ball model a direct consequence of the Pauli principle. The dephasing probe model thus captures correctly two essential ingredients of the FCS in the presence of dephasing: it is able to destroy phase information in the fermionic wavefunction and to keep at the same time its antisymmetry, the Pauli principle.

**Dephasing versus voltage probes** – While in a dephasing probe the occupation number $n_\phi$ fluctuates to conserve current at each energy, in a voltage probe $n_\phi$ is Fermi distributed and the voltage $V_\phi$ fluctuates to conserve the total, energy integrated current [10, 14, 15, 27]. The generating function in the limit of zero temperature is
\[
S_V = \frac{\tau}{\hbar} \left[ \int_{0}^{eV_\phi} dE H_0|_{n_\phi = 1} + \int_{-eV_\phi}^{eV_\phi} dE H_0|_{n_\phi = 0} \right].
\] (11)
In analogy to the procedure for the dephasing probe, we get saddle point equations for the long time limit $dS_V/d\lambda_\phi = 0$ and $dS_V/dV_\phi = 0$. Importantly, for a single probe in the linear voltage regime the saddle points for dephasing and voltage probes are equivalent, i.e. the FCS does not depend on the presence of dissipation.

Differences between dephasing and voltage probes become visible when two or more probes are attached. As
an illustrative example, we consider the MZI with a second probe attached to the lower arm of the interferometer, similar to Fig. 1. The corresponding unconstraint generating function is nonlinear in occupation numbers and counting variables of the probes, leading to more complicated saddle point equations. We limit our discussion to the first two cumulants, mean current and noise in terminal 3, which already differ for the two probe types.

We first consider the case where the two probes are disconnected, characterized by independent distribution function $n_{\phi_1}$ and $n_{\phi_2}$ or voltages $V_{\phi_1}$ and $V_{\phi_2}$ respectively. For the dephasing probe the current and noise are obtained from the single probe result, Eq. (14) by replacing $\sqrt{1 - \varepsilon}$ with $1 - \varepsilon$. The same result is found for the voltage probe current but the noise differs by a term proportional to $\varepsilon^2$. Interestingly, the dephasing probe result is also obtained from phase averaging by considering, in analogy to the single probe case, two uniformly and independently distributed phases $\phi_1$ and $\phi_2$, one for each probe. This equivalence, a result of the simple MZI geometry, does not hold for more complicated systems.

For the case with the two connected probes, characterized by a single distribution function $n_{\phi}$ and a single voltage $V_{\phi}$ respectively, the situation is quite different. A particle entering the probe in e.g. the upper arm can be reemitted in the lower arm, thus only the sum of the currents into the two probes is conserved. In the limit of complete dephasing, $\varepsilon = 1$, the problem simplifies considerably: all particles injected from 1 are first emitted into one of the two probes and then reemitted again before exiting into terminal 3 or 4. The mean current is simply $I = (e^2V/h)/2$, independent of the beamsplitter scattering probabilities. For a voltage probe the noise is suppressed to zero because the outgoing streams from the two probes are noiseless, due to unit occupation in the probes up to energy $eV/2$. For a dephasing probe the distribution function is instead a two-step function with $n_{\phi} = 1/2$ in the energy interval $[0, eV]$. Thus, the streams incident on $B$ are occupied with probability $1/2$, leading to a phase $S_{\text{deph}} = (2e^2V/h)/8$, again independent on beamsplitter scattering probabilities. Note that there is no natural extension of the phase average to the case with two connected probes.

Conclusion – We have extended the discussion of voltage probes and dephasing probes for current and noise to the level of full counting statistics, valid for the case that the dephasing mechanism is slow and averages over many electron wave packets. For conductors connected to a single dephasing probe we find that there exists a phase distribution such that the phase average of the generating function of FCS is identical to the generating function of the conductor with the dephasing probe. Contrary to statements in the literature, this proves that dephasing probes correctly account for the Pauli principle and exchange effects. The stochastic path integral approach, which correctly describes exchange effects in the multichannel limit [18], can readily be extended to the multiple (or multichannel) probe case. No such simple extension is possible for phase averaging. The physics of voltage probes, an essential element of electrical transport, and the closely related dephasing probe approach will likely also in the future remain an important element in discussions of the statistics of the conduction process.

We thank E. V. Sukhorukov for discussions. This work is supported by MaNEP and the Swiss and Swedish NSF.

[1] A. D. Benoit, S. Washburn, C. P. Umbach, R. B. Laibowitz, and R. A. Webb, Phys. Rev. Lett. 57, 1765 (1986).
[2] R. de Picciotto, H. L. Stormer, L. N. Pfeiffer, K. W. Baldwin, K. W. West, Nature 411, 51 (2001).
[3] B. Gao, Y. F. Chen, M. S. Fuhrer, D. C. Glattli, A. Bachtold, Phys. Rev. Lett. 95, 196802 (2005).
[4] M. Büttiker, IBM J. Res. Develop., 32, 63-75 (1988).
[5] J. D’Amato and H. M. Pastawski, Phys. Rev. B 41, 7411 (1990).
[6] T. Ando, Surface Science 361/362, 267 (1996).
[7] M. J. M. de Jong, and C. W. J. Beenakker, Physica A 230, 219 (1996).
[8] S. A. van Lanen, and M. Büttiker, Phys. Rev. B 56, R1680 (1997).
[9] C. W. J. Beenakker, Rev. Mod. Phys. 69, 000731 (1997).
[10] C. W. J. Beenakker, and M. Büttiker, Phys. Rev. B 46, 1889 (1992).
[11] F. Marquardt and C. Bruder, Phys. Rev. Lett. 92, 56805 (2004); Phys. Rev. B 70, 125305 (2004).
[12] A. A. Clerk and A. D. Stone, Phys. Rev. B 69, 245303 (2004); E. Prada, F. Taddei, and R. Fazio, Phys. Rev. B 72, 125333 (2005).
[13] H. Förster, S. Pilgram and M. Büttiker, Phys. Rev. B 72, 075301 (2005).
[14] C. Texier and M. Büttiker, Phys. Rev. B 62, 7454 (2000).
[15] S. Oberholzer, E. Bieri, C. Schoenenberger, M. Giovannini, J. Faist, Phys. Rev. Lett. 96, 046804 (2006).
[16] Ya. M. Blanter and M. Büttiker, Phys. Rep. 336 1 (2000).
[17] For a recent review see Quantum noise in Mesoscopic Physics, edited by Yu. V. Nazarov (Kluwer, 2003).
[18] S. Pilgram, A. Jordan, E. V. Sukhorukov and M. Büttiker, Phys. Rev. Lett. 90, 206801 (2003); A. N. Jordan, E. V. Sukhorukov and S. Pilgram, J. Math. Phys. 45, 4386 (2004); for related work see also D. B. Gutman, A. D. Mirlin, and Y. Gefen, Phys. Rev. B 71,85118 (2005).
[19] Y. Ji, Y. Chung, D. Sprinzak, M. Heiblum, D. Mahalu and H. Shtrikman, Nature 422, 415 (2003).
[20] V. S.-W. Chung, P. Samuelsson, M. Büttiker, Phys. Rev. B 72, 125320 (2005).
[21] G. Seelig, M. Büttiker, Phys. Rev. B 64, 245313 (2001).
[22] F. Marquardt, Europhys. Lett. 72, 788 (2005).
[23] L. S. Levitov and G. Lesovik, JETP Lett. 58, 230 (1993).
[24] The phase average $\langle S_\phi \rangle$ excludes modulation contributions which appear in an experiment with external phase fluctuations. There one rather measures the logarithm of the averaged moment generating function $\ln \langle e^{iS_\phi} \rangle$, which is discussed in Ref. [18]. The two averages coincide only to linear order in the voltage $eV$.
[25] C. W. J. Beenakker and B. Michaelis, J. Phys. A: Math. Gen. 38, 10639 (2005).
[26] P.-E. Roche, B. Derrida and B. Doucot, Eur. Phys. J. B 43, 529 (2005).

[27] FCS in the presence of voltage fluctuations in networks of two-terminal conductors is discussed in M. Kindermann, Yu. V. Nazarov, and C. W. J. Beenakker, Phys. Rev. B 69, 035336 (2004).